A survey on mining and analysis of uncertain graphs

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Abstract
An uncertain graph (also known as probabilistic graph) is a generic model to represent many real-world networks from social to biological. In recent times, analysis and mining of uncertain graphs have drawn significant attention from the researchers of the data management community. Several noble problems have been introduced, and efficient methodologies have been developed to solve those problems. Hence, there is a need to summarize the existing results on this topic in a self-organized way. In this paper, we present a comprehensive survey on uncertain graph mining focusing on mainly three aspects: (i) different problems studied, (ii) computational challenges for solving those problems, and (iii) proposed methodologies. Finally, we list out important future research directions.

Keywords
Uncertain graph · Reliability · Clustering · Classification · Node Similarity · Reachability Query

1 Introduction

Graphs are often used to represent real-world networks, such as social networks (nodes represent users and edges represent social ties) [18], transportation networks (nodes represent cities and edges represent roads) [11], protein-protein interaction networks (nodes represent proteins and edges represent interaction relationship) [17] and so on. In many realistic applications, uncertainty is intrinsic in graph data due to many practical reasons, which include noisy measurement [2], inferences and prediction error [1], explicit manipulation, and so on. As an example, in case of protein-protein interaction (PPI) networks, as high throughput interaction detection methods are often erroneous, and hence, interaction of two proteins is probabilistic in nature. These kinds of graphs are often called uncertain graphs or probabilistic graphs, where each edge of the network is associated with a probability value. The interpretation of this probability may vary from context to context. In case of PPI Network,
the assigned probability to an edge is basically its existential probability. On the other hand, in case of a social network, the assigned probability to an edge may signify the probability with which one can influence other [23]. In a generic sense, an uncertain graph is basically a graph whose edges are marked with a probability value. Due to the wider applicability of uncertain graphs in different domains such as social network analysis [59], computational biology [132], crowdsourcing [57, 119, 120], recommender systems [112], wireless networks [28] analysis of mining of such networks become key research area in recent times.

Extensive studies on uncertain graphs lead to many interesting problems [56, 63], which includes frequent pattern matching, yuan2016efficient, chen2018efficient, subgraph extraction, chen2010continuous, yuan2011efficient, clustering of uncertain graphs [19, 48], motif counting, ma13linc, reliability computation [62] and many more. Also, over the years several solution methodologies have been developed. So there is a need to organize the existing results in a self-contained manner. In this paper, we serve this purpose by surveying the existing literature. First, we report the goals of this survey.

1.1 Focus and goal of the survey

In this survey, the main focus is three folded and they are listed below:
– Different problems studies on uncertain graph mining and analysis,
– Major challenges for solving these problems,
– Proposed solution methodologies.

The goals of the survey are as follows:
– to provide a comprehensive background on uncertain graph mining and different problems introduced and studied in this domain.
– to propose a taxonomy for classifying the existing literature and brief it concisely.
– to summarize the existing literature and points out future research directions.

1.2 Proposed taxonomy

Broadly, the problems that have been studied in uncertain graph mining domain can be classified into three main categories: (i) computational problems (e.g., computation of $u - v$ reliability, clustering, etc.), (ii) querying problems (e.g., reachability queries, queries regarding the existence of a particular combinatorial structures, etc.), and (iii) graph algorithmic problems (e.g., construction of spanning tree, link prediction, information flow maximization, etc.). Also, there are some problems such as sparsification and node classification which do not come under any of the three heads, and we put them under miscellaneous problems. Figure 1 gives a diagrammatic view for the proposed classification of the existing literature on uncertain graph mining.

1.3 Organization of the survey

The rest of the paper is organized as follows: Sect. 2 lists required preliminary definitions. Sect. 3 describes different problems studied on uncertain graph mining. In Section 4, we report the existing challenges for solving the problems. Available solution methodologies for these problems are described in Sect. 5. In Sect. 6, we describe the current research trends and point out existing research gaps. Section 7 lists out existing future research directions. Finally, Sect. 8 concludes the survey.

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2 Preliminaries

In this section, we describe the required preliminary concepts. Initially, we start with the uncertain graph.

**Definition 1 (Uncertain Graph)** An uncertain graph is denoted as $G(V, E, P)$, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ denotes the set of vertices, $E(G) = \{e_1, e_2, \ldots, e_m\}$ denotes the set of edges, and $P$ is the edge weight function that assigns each edge to its probability, i.e., $P : E(G) \rightarrow (0, 1]$.

If the uncertain graph is weighted, along with $V$, $E$, and $P$, there is also an edge weight function $W$ that assigns each edge to a real number, i.e., $W : E(G) \rightarrow \mathbb{R}$. We denote the number of vertices and edges of $G$ as $n$ and $m$, respectively. Depending upon the situation, an uncertain graph may be directed or undirected. As in most of the existing studies, the considered uncertain graph is undirected, in this paper also unless otherwise stated by the uncertain graph we mean it is undirected. Also, in some situations instead of a single probability value, there may be multiple such values associated with an edge. Take the example of a social network, where probability associated with each edge is the influence probability between two users. Now, influence probability may vary from context to context [24]. This means a sportsman can influence his friends and followers regarding any news related to sports with a higher probability compared to the others. Let, $C = \{c_1, c_2, \ldots, c_k\}$ be the set of $k$ different contexts. In this case, the social network can be modeled as an uncertain graph, where each edge of the network is associated with $k$ number of probability values. In that case, the probability function can be defined as $P : E \times C \rightarrow (0, 1]$. Standard graph theoretic terminologies such as degree, neighborhood, path, subgraph, subgraph isomorphism, spanning tree with their definitions and notations have been adopted from [34] and not described here. For any arbitrary edge $e \in E(G)$, $P(e)$ denotes the probability associated

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1 Please don’t confuse between uncertain graph and random graph [14]. Uncertain graph and probabilistic graph are same, however, random graph is completely different and noting to do in this paper. Hence, we have not defined it in this paper.
Possible World Semantic is widely used to represent an uncertain graph as a probability distribution over a set of deterministic graphs, which is defined next.

**Definition 2** (Possible World Semantic) By this model, an uncertain graph with \( m \) edges is represented as a probability distribution over \( 2^m \) number of deterministic graphs by keeping an edge with probability \( P(e) \) and removing it with probability \( (1 - P(e)) \). Hence, given an uncertain graph \( G(V, E, P) \), the probability that the deterministic graph \( G(V, E) \) (Here, \( V(G) = V(G) \), and \( E(G) \subseteq E(G) \)) will be generated can be computed by the following equation:

\[
P_{G \subseteq G} = \prod_{e \in E(G)} P(e) \prod_{e \in E(G) \setminus E(G)} (1 - P(e))
\]  

(1)

Figure 2 shows an example of an uncertain graph with its possible world. In the example \( |E(G)| = 3 \), hence \( 2^3 = 8 \) deterministic graphs will be in the possible world of \( G \).

Given a graph and two of its vertices distance between them is defined as the length of the shortest path. However, in case of uncertain graphs distance between two vertices can be defined in many ways, which is stated next.

**Definition 3** (Distance in Uncertain Graphs) [100] Given an uncertain graph \( G(V, E, P) \), and two of its vertices \( u, v \in V(G) \), there are four different distance measure available in the literature:

- **Majority Distance**: It is defined as the most probable shortest path distance between \( u \) and \( v \).
- **Expected Distance**: It is defined as the expected shortest path distance between \( u \) and \( v \) among all the possible world graphs.
- **Expected Reliable Distance**: It is defined as the expected shortest path distance between \( u \) and \( v \) among all the possible world graphs in which \( u \) and \( v \) are connected.
- **Median Distance**: It is defined as the median shortest path distance among all the possible world graphs.
Table 1 Different distance measures in uncertain graphs

| Distance Metric      | Mathematical Expression                      |
|----------------------|---------------------------------------------|
| Majority Distance    | \( \text{dist}_{MD} = \arg\max_d P_{uv}(d) \) |
| Expected Distance    | \( \text{dist}_{ED} = \sum_d d \cdot P_{uv}(d) \) |
| Expected Reliable Distance | \( \text{dist}_{ERD} = \sum_{d|d<\infty} d \cdot \frac{P_{uv}(d)}{1 - P_{uv}(d)} \) |
| Median Distance      | \( \text{dist}_{MD} = \arg\max_D \left\{ \sum_{d=0}^D 1 - P_{uv}(d) < \frac{1}{2} \right\} \) |

Mathematical expressions for computing these distance measures are given in Table 1. Here, \( P_{uv}(d) \) denotes the probability that the distance between \( u \) and \( v \) is \( d \).

In a deterministic graph, two vertices are always reachable from each other if they belong to the same connected component. However, in case of uncertain graphs, reachability between two vertices is always probabilistic, which is called as ‘reliability’ and it is defined next.

**Definition 4** \((u-v\) Reliability) Given an uncertain graph \( G(V,E,P) \), and two vertices \( u,v \in V(G) \), let \( I_G(u,v) \) be the indicator Boolean variable which takes the value 1 if \( u \) and \( v \) are reachable from each other in the deterministic graph \( G \), otherwise it is 0. Now, reliability between \( u \) and \( v \) is basically the expected value of reachability, which can be given by Equation 2.

\[
R(u,v) = \sum_{G \in L(G)} I_G(u,v) \cdot P_{G \subseteq G} \tag{2}
\]

Now, the notion of \( u-v \) reliability for an uncertain graph can be extended for a subset of the vertices (called ‘terminal vertices’) which is stated in Definition 5.

**Definition 5** (Terminal Reliability) For a given uncertain graph \( G(V,E,P) \), and a set of terminal vertices \( T \subseteq V(G) \), the network reliability is defined as the probability that the graph induced by the vertex set \( T \) is connected in \( G \). This can be computed by Equation 3. Here, \( I(G,T) \) is an indicator variable whose value is 1 when \( T \) is connected and 0 otherwise.

\[
R(G,T) = \sum_{G \in L(G)} I(G,T) \cdot P_{G \subseteq G} \tag{3}
\]

Subsequently, the notion of \( u-v \) reliability has been generalized by Khan et al. [62] when the uncertain graph has more than one probability value in every edge and this is defined next.

**Definition 6** (Conditional \( u-v \) Reliability) [62] Let, \( G(V,E,P) \) be an uncertain graph, and \( C = \{ c_1, c_2, \ldots, c_k \} \) be the \( k \) different contexts. Here, \( P \) is defined as \( P : E \times C \rightarrow (0,1] \). Now, for any given \( C_1 \subseteq C \) and with the assumption that the contexts are independently aggregated, the effective probability for any edge \( e \in E(G) \) can be computed by Equation 4.

\[
P(e|C_1) = 1 - \prod_{c \in C_1} (1 - P(e|c)) \tag{4}
\]

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Similarly, by considering the contexts $C_1$, the probability that the deterministic graph $G$ will be generated can be computed by Equation 6.

$$P(G|C_1) = \prod_{e \in E(G)} P(e|C_1) \prod_{e \in E(G) \setminus E(G)} (1 - P(e|C_1)) \quad (5)$$

Now, given two vertices $u, v \in V(G)$, and given the set of contexts $C_1$, the conditional reliability between $u$ and $v$ can be defined by Equation 6.

$$R((u, v)|C_1) = \sum_{G \subseteq G|C_1} I_G(u, v). P(G|C_1) \quad (6)$$

Here, $I_G(u, v)$ is an indicator variable whose value is 1 if $u$ and $v$ are connected in $G$. $L(G|C_1)$ denotes the set of possible worlds for the given set of contexts $C_1$.

Given two deterministic graphs $G$ and $H$, a well-known computational task is to determine whether $G$ contains a subgraph isomorphic to $H$. This is the popular subgraph isomorphism problem [69] which is known to be NP-Complete [70]. However, this problem can be translated in case of uncertain graphs by incorporating the concept of support, which is defined next.

**Definition 7** (Support) Given an uncertain graph $G(V, E, P)$, and a subgraph $S$, the support of $S$ is defined as the fraction of the possible worlds that contain $S$ as subgraph. Mathematically, this can be expressed as follows:

$$\text{sup}(S; G) = \frac{|G \in L(G) : S \subseteq G|}{2^m} \quad (7)$$

It is easy to follow that the value of support lies in between 0 and 1. Now, given a support value $\alpha \in (0, 1)$, an uncertain graph $G(V, E, P)$ and a deterministic graph $G(V, E')$, one immediate question is that: ‘Does the uncertain graph $G$ contains the $G$ as a subgraph with support greater than or equal to $\alpha$? This kind of querying problem is called structural pattern-based queries. Section 3.1 discusses it in more detail.

For a given graph, computation of node similarity (how similar two nodes are?) remains a central question in graph mining [26]. Among many, one of the popularly adopted node similarity measure is SimRank which was initially proposed by Jeh and Widom [53]. The intuition behind this similarity measure is that two vertices are similar if they are referenced by similar vertices.

**Definition 8** (SimRank) Given a deterministic directed graph $G(V, E)$, and two vertices $u, v \in V(G)$, the SimRank score between these two vertices is given by the following equation:

$$\sigma(u, v) = \begin{cases} 1, & \text{if } u = v \\ \frac{c}{|N^{in}(u)|.|N^{in}(v)|} \sum_{w \in N^{in}(u)} \sum_{w' \in N^{in}(v)} \sigma(w, w'), & \text{if } u \neq v \end{cases}$$

where $N^{in}(u)$, and $N^{in}(v)$ denotes the set of incoming neighbors of the nodes $u$ and $v$, respectively. $c \in (0, 1)$ is the decaying factor whose value is chosen as 0.8 [71]. Section 5.1.2 summarizes the existing literature for SimRank computation on uncertain graphs.

**Definition 9** (Random Walk on Graphs) [43] This is a discrete-time process defined on graphs and also it is one kind of graph search technique. Suppose, at time $t = 0$ an object is placed on the vertex $v$ of an undirected graph $G(V, E)$. At every discrete time steps $t = 1, 2, \ldots$ the
object must move from one vertex to one of its neighboring ones. So, if the object is on the vertex \( v_i \) at time \( t \), then the probability that the object will be on the vertex \( v_j \) at time \( t + 1 \) is given by the following equation.

\[
m_{ij} = \begin{cases} 
\frac{1}{\deg(v_i)}, & \text{if } (v_i, v_j) \in E(G) \\
0, & \text{if } (v_i, v_j) \notin E(G)
\end{cases}
\]

These probability values are reported as a \( n \times n \) matrix \( M \).

Existing solution methodologies for the SimRank computation on uncertain graphs use the concept of random walk as described in Sect. 5.1.2.

Table 2 describes the notations and symbols with their interpretations used in this paper. Many of the symbols have not been introduced yet and done as and when needed.

3 Problems studied on uncertain graphs

In this section, we describe the problems that have been studied in the domain of uncertain graph mining. We devote one subsection to each category of sub-problems as shown in Figure 1.

3.1 Querying uncertain graphs

Different kinds of querying problems have been studied in the context of uncertain graph such as \textit{k-nearest neighbor queries} (kNN queries), \textit{reliability queries}, \textit{queries for structural pattern} and so on. We describe them one by one.

3.1.1 Reliability-Based Queries

Analysis of network reliability is an old still active research area [9, 10, 16, 45, 61]. Given an uncertain graph \( G(V, E, P, W) \), a probability threshold \( \eta \in (0, 1) \), and a set of source nodes \( S \subseteq V(G) \), \textit{Reliability-Based Query Problem} asks to return maximum set \( T \subseteq V(G) \setminus S \), such that the probability of reachability from the set \( S \) to \( u \), \( \forall u \in T \) is more than \( \eta \) [61]. From the computational point of view, the problem can be posed as follows:

**RELIABILITY-BASED QUERY PROBLEM**

**Input:** Uncertain Graph \( G(V, E, P, W) \), a vertex subset \( S \subseteq V(G) \), and probability threshold \( \eta \).

**Question:** Return the maximum set \( T \subseteq V(G) \setminus S \), such that \( \forall u \in T \), \( P(S, u) \geq \eta \).

Here, \( P(S, u) \) denotes the probability of reachability from the nodes in \( S \) to \( u \).

3.1.2 Conditional reliability maximization queries

Recently, Khan et al. [62] studied the conditional reliability maximization problem. In case of single source single terminal variant of this problem along with an uncertain graph \( G(V, E, P) \), and set of different contexts \( C = \{c_1, c_2, \ldots, c_k\} \) with \( P : E(G) \times C \rightarrow (0, 1] \), two specified vertices \( s, t \in V(G) \), and a positive integer \( \ell \leq k \), this problem asks
Table 2 Notations used in this study

| Notation       | Meaning                                                                 |
|----------------|-------------------------------------------------------------------------|
| $G(V, E, P)$   | An uncertain graph                                                      |
| $G(V, E, P, W)$| A weighted uncertain graph                                              |
| $V(G), E(G)$  | Vertex set and edge set of $G$                                           |
| $n, m$         | The size of the vertex set and edge set of $G$                          |
| $d$            | Diameter of $G$                                                         |
| $P$            | Edge weight function, i.e., $P : E(G) \rightarrow (0, 1]$              |
| $W$            | Edge weight function, $W : E(G) \rightarrow \mathbb{R}^+$              |
| $A$            | Adjacency matrix of $G$                                                |
| $u, v$         | Any two arbitrary vertices of $V(G)$                                    |
| $N^i(u)$       | Set of incoming neighbors of $u$                                        |
| $e/(uv)$       | An arbitrary edge from $E(G)$                                           |
| $P(e)/P(uv)$   | Edge probability of the edge $e = (uv)$                                  |
| $L(G)$         | Possible world set of $G$                                               |
| $P_G \subseteq G$ | Generation probability of $G$ from $G$                                      |
| $\mathcal{G}(V, \mathcal{E})$ | A deterministic graphs from $L(G)$                                      |
| $\sigma(u, v)$ | SimRank between $u$ and $v$ in $G$                                      |
| $T$            | Set of terminal vertices from $V(G)$                                    |
| $\eta$         | Reachability/Reliability threshold                                       |
| $C$            | Set of different contexts/ tags                                         |
| $P(e|c)$       | The edge probability of the edge $e$ for the tag $c$                    |
| $C_1$          | A subset of contexts $C_1 \subseteq C$                                |
| $P(e|C_1)$     | Edge probability of $e$ after the aggregating the tags in $C_1$         |
| $R((s, t)|C_1)$ | Reliability between $s$ and $t$ for the tags in $C_1$                  |
| $D_\beta(u, v)$ | Distance between $u$ and $v$ in $G$ under distance measure $\beta$ |
| sup$(S; G)$   | Support of $S$ on $G$                                                  |
| $\alpha$      | Specification Ratio                                                    |
| $S$            | The SimRank matrix                                                     |
| $q$            | An $s-t$ query                                                         |
| $G(q)$         | Uncertain graph corresponding to the $s-t$ query $q$                    |
| $g$            | Query Subgraph                                                         |
| $k$            | The set $\{1, 2, \ldots, k\}$                                         |

To choose a subset of the contexts $C^* \subseteq C$ such that $|C^*| = \ell$ such that conditional $u - v$ reliability $R((s, t)|C^*)$ as described in Definition 6 gets maximized. Mathematically, this problem can be expressed by the following equation.

$$C^* = \arg\max_{C^* \subseteq C; |C^*| = \ell} R((s, t)|C^*)$$  \hspace{1cm} (8)

From the computational perspective, this problem can be posed as follows:
3.1.3 Nearest neighbor (NN) queries

This is a very generic data mining task, where given a set of data points and a specific one of them (say \( d \)) the question is to return \( k \) most similar data points as \( d \) from the remaining [103]. This problem has been studied extensively in graph data [84]. Recently, the \( k \)-NN Query Problem has also been studied on uncertain graph as well [101]. Given a weighted uncertain graph \( G(V, E, P, W) \), a specific vertex \( v \in V(G) \), a probabilistic distance function \( D_P \) (anyone of four as shown in Table 1), and a positive integer \( k \), the \( k \)NN query on uncertain graph problem asks to return the set \( \{u_1, u_2, \ldots, u_k\} \subseteq V(G) \setminus \{v\} \) such that for any vertex in \( w \in V(G) \setminus \{v, u_1, u_2, \ldots, u_k\} \), \( D_P(v, w) \geq D_P(v, u_i) \) for any \( i \in [k] \). From the computational perspective, the problem can be posed as follows:

\[
\textit{k-NN Query on Uncertain Graph Problem}
\]

**Input:** Weighted Uncertain Graph \( G(V, E, P, W) \), a specific vertex \( v \in V(G) \), a probabilistic distance function \( D_P \), and \( k \in \mathbb{Z}^+ \).

**Question:** Return the vertex set \( \{u_1, u_2, \ldots, u_k\} \subseteq V(G) \setminus \{v\} \) such that for any vertex in \( w \in V(G) \setminus \{v, u_1, u_2, \ldots, u_k\} \), \( D_P(v, w) \geq D_P(v, u_i) \) for any \( i \in [k] \).

For \( k = 1 \), we call this problem as the nearest neighbor query. This problem can also be defined on unweighted uncertain graph as well. In that case \( \forall e \in E(G), W(e) = 1 \). Here, we have described the weighted version only.

3.1.4 Structural pattern-based queries

Finding or counting a given structure (also known as query graph) in a graph database is a fundamental graph mining task [67, 121]. This problem has also been studied in the context of an uncertain graph as well. Particularly, the structure that has been studied extensively is the frequent subgraph, zou2010discovering,li2012mining, densest subgraph [138], Clique [92, 94, 142] etc.

\[
\textit{Structural Pattern-Based Query Problem}
\]

**Input:** Uncertain Graph \( G(V, E, P) \), a structure \( X \), and a probability threshold \( \alpha \).

**Question:** Is there a pattern \( X \) whose support value is greater than or equal to \( \alpha \).

Here, \( X \) may be a \( k \)-clique, \( k \)-truss and many more.

3.2 Computation on uncertain graphs

Different computational problems have been studied in the context of uncertain graphs. Here, we list them one by one.
3.2.1 $u - v$ reliability computation

Given an uncertain graph $G(V, E, P)$, and two vertices $u$ and $v$, this problem asks to compute the probability that $v$ is reachable from $u$. It can be considered as the fundamental reachability problem in the uncertain graph context. From the computational perspective, the problem can be listed as follows:

\[
\begin{array}{ll}
\text{Input:} & \text{An Uncertain Graph } G(V, E, P), \text{ two specific vertices } u, v \in V(G). \\
\text{Task:} & \text{Compute the reliability between } u \text{ and } v, \text{i.e., } R(u, v) \text{ in } G.
\end{array}
\]

A variant of this problem has been introduced by Jin et al. \[55\], where along with $G(V, E, P)$, $u, v \in V(G)$, we are also given with a distance $d$ and the question is that what is the probability that $u$ is connected with $v$ within the distance $d$.

3.2.2 $k$-Terminal Reliability Computation

This is a more generalized version of the $u-v$ reliability computation problem. In this problem, we are given with an uncertain graph $G(V, E, P)$ and a set of $k$ terminal vertices $T$, the network reliability $R(G, T)$ can be given by the following equation:

\[
R(G, T) = \sum_{\mathcal{G} \in \mathcal{L}(G)} I(\mathcal{G}, T).P_{\mathcal{G} \subseteq G}
\]

(9)

where $I(\mathcal{G}, T)$ is an indicator variable whose value will be 1, if the vertices in $T$ are connected in $\mathcal{G}$ and 0 otherwise. This problem asks to compute the quantity $R(G, T)$ \[106\]. From the computational point of view, this problem can be posed as follows:

\[
\begin{array}{ll}
\text{Input:} & \text{An Uncertain Graph } G(V, E, P), \text{ a subset of vertices } T. \\
\text{Task:} & \text{Compute the } k\text{-terminal reliability } R(T, G) \text{ in } G.
\end{array}
\]

3.2.3 SimRank Computation

SimRank is a popular similarity measure between any two vertices of a graph due to its applications in different problems including entity resolution \[73, 122\], similar protein identification \[116\], and so on. Recently, SimRank computation problem has also been studied for uncertain graph as well \[35, 135, 136\]. For a given uncertain graph $G(V, E, P)$, this problem asks to compute an $n \times n$ similarity matrix $S_{sim}$, where the $(i, j)$-th entry contains the SimRank similarity value. Computationally, the problem looks like the following:

\[
\begin{array}{ll}
\text{Input:} & \text{An Uncertain Graph } G(V, E, P). \\
\text{Task:} & \text{Compute the } n \times n \text{ similarity matrix } S_{sim} \text{ for } G.
\end{array}
\]
3.2.4 Clustering of Uncertain Graphs

Clustering is a very generic data mining task. Given a set of data points, this problem asks to partition them into a number (may be given or may be dataset dependent) of partitions (may be overlapping) such that data points belong to the same partitions should be similar in some sense and data points belongs to the different partition should be different [52]. Clustering has also been studied extensively on graph data [3]. This problem has also been studied in the context of uncertain graph as well [48, 80]. The problem can be summarized as follows:

**CLUSTERING OF UNCERTAIN GRAPH**

**Input:** An Uncertain Graph $G(V, E, P)$, and the number of clusters $K$.

**Task:** Partition $V(G)$ into $C_1, C_2, \ldots, C_K$ based on certain similarity measure.

3.3 Graph algorithmic problems

3.3.1 Spanning tree problem

Given a weighted graph, computing the minimum cost spanning tree is a classic problem in Graph Algorithms [29]. This problem has been studied when the edge weights are uncertain [36, 88]. Recently, this problem has also been studied in the context of uncertain graphs, where the weight of the edge is fixed; however, there is a probability of existence associated with every edge [130]. From the computational perspective, this problem can be posed as follows:

**MOST RELIABLE SPANNING TREE PROBLEM ON UNCERTAIN GRAPHS**

**Input:** An Weighted Uncertain Graph $G(V, E, P, W)$.

**Task:** Compute the minimum cost spanning tree having the highest probability.

3.3.2 Link Prediction Problem

Link prediction is a classic graph mining task, where the snapshot of the network in different times, say $t = 1, 2, \ldots, T$, i.e., $G_1, G_2, \ldots, G_T$ is given and the task is to predict the network snapshot at time $T + 1$, i.e., $G_{T+1}$ [131]. This problem has lot of applications in social network analysis [79], recommender systems [77], and so on. Recently, this problem has been studied considering the edge uncertainty [4, 86]. From the computational point of view, this problem can be posed as follows:

**LINK PREDICTION PROBLEM ON UNCERTAIN GRAPHS**

**Input:** Snapshots of an Uncertain Graph $G_t(V, E_t, P_t), \forall t \in [T]$.

**Task:** Output a $n \times n$ similarity matrix, where its $(i, j)$-th denotes the likelihood of $(v_i, v_j) \in E(G_{T+1})$. 

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3.3.3 Information flow maximization problem

Network flow is a classic problem in network algorithms, where given a directed graph with each edge is marked with its capacity and a source and target vertex, the goal here is to decide how much flow need to pass through each of the links such that total flow from the source to the target vertex is maximized [5]. Recently, this problem has been studied in the uncertain graph as well, however in a different setting [39, 40]. Here, the problem is defined as follows: Given a vertex weighted uncertain graph $G(V, E, P)$, a query vertex $Q \in V(G)$, and a positive integer $k$, this problem asks to find out the subgraph $\mathcal{G}$ that can contain at most $k$ edges that maximizes the expected information flow toward $Q$. From the computational point of view, this problem can be posed as follows:

**Information Flow Maximization Problem on Uncertain Graph**

**Input:** A Vertex Weighted Uncertain Graph $G(V, E, P, W)$, a query vertex $Q \in V(G)$, a positive integer $k$.  
**Task:** Find out a subgraph containing $Q$ with at most $k$ edges such that information flow toward $Q$ gets maximized.

3.4 Other problems

3.4.1 Uncertain graph sparsification

Due to the gigantic size of real-world networks, storing the entire network requires huge storage costs, and more importantly querying the entire graph requires huge computational time. The remedy to this problem is not to store the entire graph. One approach is to resolve this issue is that instead of storing all the edges, store a subset of them. This leads to the problem of graph sparsification. Here, the problem is to decide which edges to store such that some specific property of the graph will not change much in the sparsified graph [42, 110]. In this context, ‘sparcification ratio’ is defined as the ratio between the number of edges in the sparsified graph and the original graph. Several graph sparsification techniques have been proposed in the literature, such as spanner-based sparsifier [98], cut-based sparsifier [42], and many more. Recently, the cut-based uncertain graph sparsification problem has been studied by Parchas et al. [97]. This problem has been formalized as follows: Given an uncertain graph $G(V, E, P)$ and a vertex subset $S \subseteq V(G)$, the expected cut size is defined by Equation 10.

$$C_G(S) = \sum_{e=(u,v) \in E(G)} P(e)$$  \hspace{1cm} (10)

Now, the Absolute Discrepancy ($\delta_A(S)$) and Relative Discrepancy ($\delta_R(S)$) for the uncertain graph $G(V, E, P)$ with respect to the sparsified uncertain graph $G'(V', E', P')$ is defined in Equation 11 and 12, respectively.

$$\delta_A(S) = C_G(S) - C_{G'}(S)$$  \hspace{1cm} (11)
$$\delta_A(R) = \frac{C_G(S) - C_{G'}(S)}{C_G(S)}$$  \hspace{1cm} (12)
For a positive integer \( k \), the discrepancy of the sparsified uncertain graph \( G' \) is defined as the sum of the absolute values discrepancy values for all \( k \) sized subsets.

\[
\Delta_k(G') = \sum_{S \subseteq V(G), |S| = k} |\delta_A(S)|
\]

(13)

Now, for a given uncertain graph \( G(V, E, P) \), a positive integer \( k \), and a sparsification ratio \( \alpha \in (0, 1) \), the uncertain graph sparsification problem asks to construct another uncertain graph \( G'(V, E', P') \) such that \( |E(G')| = \alpha|E(G)| \) that minimizes the sum of the discrepancies \( \sum_{i \in [k]} \Delta_i(G') \). From the computational point of view, this problem can be posed as follows:

**Uncertain Graph Sparsification Problem**

**Input:** An Uncertain Graph \( G(V, E, P) \), a positive integer \( k \), a sparsification ratio \( \alpha \in (0, 1) \).

**Task:** Construct another uncertain graph \( G'(V, E', P') \) such that \( |E(G')| = \alpha|E(G)| \) to minimize \( \sum_{i \in [k]} \Delta_i(G') \).

### 3.4.2 Node classification in uncertain graphs

The problem of classifying the nodes of a network has been studied extensively [13]. Recently, this problem has also been studied in the context of uncertain graphs as well [30, 47, 66]. This problem can be posed as follows: Given an uncertain graph \( G(V, E, P) \), a set of labels \( T_0 \) and subset of its vertices \( S \subset V(G) \) labeled with a labeling function \( L : S \rightarrow T_0 \), predict the labels of the nodes in \( V(G) \setminus S \). Computationally, this problem can be posed as follows:

**Node Classification in Uncertain Graphs**

**Input:** An uncertain graph \( G(V, E, P) \), label set \( T_0 \), a subset of vertices \( S \subset V(G) \) with labeling function \( L : S \rightarrow T_0 \).

**Task:** Predict the labels of the nodes in \( V(G) \setminus S \).

There are several other problems that have been studied such as uncertain graph visualization [107, 108]. As the intended audience of this paper is the researchers of the data mining and data management community, hence we are not focusing on these problems. Also, it is important to note that the problems that have been studied on uncertain graphs may be classified in other way also. However, in our classification, the goal was to put the problems that are of similar kind under the same umbrella. Next, we proceed to describe the major challenges for solving these problems.

### 4 Challenges for solving these problems

As mentioned in the literature, there are mainly two major challenges described in the following two subsections.
4.1 Exponential number of possible worlds

As mentioned in Sect. 2, an uncertain graph with \( m \) edges will have \( 2^m \) number of possible worlds. Even for small values of \( m \) (say, \( m = 100 \)), the number of possible worlds is excessively large (more than the number of atoms in this universe). Almost all the problems that have been described in Sect. 3 require enumerating all the possible worlds to output the answer accurately. However, due to bounded computational resources, it is not possible to consider all the possible worlds. Now, here the challenge is that probabilistically how many samples to consider for the computation to output the result such that the error is bounded by at most \( \epsilon \). Consider \( u - v \) Reliability Computation Problem as described in Sect. 3. Let, \( R(u, v) \) and \( \hat{R}(u, v) \) be the \( u-v \) reliability values when all the sample graphs are considered and when all the graphs in the possible world are not considered, respectively. So the question here is that for a given \( \epsilon \) and \( \delta \) how many sample graphs to consider such that the following inequity holds: \( \Pr[|R(u, v) - \hat{R}(u, v)| \leq \epsilon] \geq 1 - \delta \). To address this issue, several effective sampling techniques have been developed such as recursive sampling [55], recursive stratified sampling [74, 75], lazy propagation sampling [75], and many more. Also, Parchas et al. [95, 96] proposed to generate deterministic representative instances such that underlying graph properties are preserved.

4.2 Gigantic size of real-world networks

As described, large size of possible world can be handled by approximately answering the result with high probability. However, if the size of the network is itself very large, then processing the graph becomes even further difficult. Recently, to address this issue a framework called simultaneously processing approach has been developed [143]. Basically, this method samples a number of possible worlds independently at random, efficiently stores them with compact data structures (as many of the sampled possible worlds have a common substructure), and simultaneously processes the query to generate the results. However, the literature in this category is very limited.

In the next section, we report the existing solution methodologies for the problems reported in Section 3.

5 Existing solution methodologies

In this section, we briefly describe the existing solution methodologies of the problems described in Section 3. First, we start with the computational problems on uncertain graphs.

5.1 Computational problems

5.1.1 Reliability computation

As mentioned previously, the reliability computation problem has been studied in different variants such as two vertex reliability computation (\( u - v \) reliability), \( k \)-terminal reliability computation etc. [63]. For a given uncertain graph, a trivial way to compute reliability is to enumerate all the possible worlds and process each of them sequentially. However, this approach leads to a huge computational burden. Recently, Sasaki et al. [106] proposed an efficient technique to compute the network reliability, which reduces
the number of required possible worlds and computes the bounds on the $k$-terminal reliability by efficiently constructing a binary decision diagram, which is basically a directed acyclic graph. The concept of binary decision diagram has been previously used to compute the network reliability on deterministic graphs as well \[49\]. Reported experimental results show that there proposed methodology leads to less variance and less error rate. Jin et al. \[55\] studied the distance-constrained $u$-$v$ reliability problem and their main contribution is to use the Horvitz-Thomson Type Estimator and Recursive Sampling Estimator that efficiently combines a deterministic computational procedure to boost up the estimation accuracy. Reported results show the superior performance of these two estimators. Recently, Ke et al. \[58\] performed an in-depth bench marking study for the $s-t$ reliability problem with different sampling strategies, i.e., Monte Carlo (MC) Sampling, B.F.S. with Indexing, zhu2015top, Recursive Sampling, jin2011distance, Recursive Stratified Sampling, li2014efficient, li2015recursive, Lazy Propagation Sampling, li2015recursive, Indexing via Probabilistic Trees, maniu2017indexing. Their goal was to make a comparative study of different sampling strategies to understand their estimator variance, memory usage, etc.

The simplest sampling technique is MC sampling. In this approach $k$ deterministic graphs $G_1, G_2, \ldots, G_k$ are sampled out from the possible world. Now, assume that $I_{G_i}(s, t)$ for all $i \in [k]$ denotes the Boolean variable whose value is 1 if $s$ and $t$ is reachable from each other and 0 otherwise. By this method, we have the following estimated reliability:

\[
R(s, t) \approx \hat{R}(s, t) = \frac{1}{k} \sum_{i \in [k]} I_{G_i}(s, t) \tag{14}
\]

Here, the estimator $\hat{R}(s, t)$ is an unbiased estimator of the $s-t$ reliability, i.e., $\mathbb{E}[\hat{R}(s, t)] = R(s, t)$. Now, it has been shown in \[100\] that the number of Monte Carlo Samples required to reach the $s-t$ reliability $\eta$ is greater than or equals to $\frac{3}{\epsilon^{2R(s,t)}} \ln(\frac{2}{\eta})$, i.e., $Pr[|\hat{R}(s, t) - R(s, t)| \geq \epsilon R(s, t)] \leq \eta$. Now, traversing each deterministic graph for checking required $O(m + n)$ time. Hence, the time requirement for reliability estimation using this technique is of $O(k(m + n))$.

Later, Zhu et al. \[134\] proposed an offline sampling scheme that is also space-efficient. In this sampling scheme, the given uncertain graph is entirely stored without the existence probability of the edges. However, each edge is associated with a bit vector of length $k$. For any arbitrary edge $e \in E(G)$, in its associated bit vector, the $i$-th entry is 1 if the edge $e$ is present in $G_i$. Now, it can be observed that the traversing in this compact graph structure is equivalent to traversing each of the sampled graphs in parallel. It is easy to follow that the index building scheme requires $O(km)$ time and space requirement is of $O(n + km)$.

Jin et al. \[55\] proposed the ‘recursive sampling’ approach that improves over the MC sampling due to the following two reasons. The first one is that: for a given $s-t$ query, if some edges are already missing in a possible world, then it may not be much relevant whether other sets of edges are present or not. Secondly, many possible worlds share a significant number of existing or missing edges. The working principle of this approach is as follows. Starting with the vertex $s$, an extendable edge $e$ incident on $s$ is randomly sampled for $k$ times. Now, the generated samples are divided into two groups: one group containing $e$ and the other one is not containing $e$. Assume, in the first group of samples by the edge $e$ the reachable node is $w$ and now more edges are expandable. This process is repeated for both the groups by picking up an randomly expandable edges and subdividing the groups into smaller ones. A very similar approach was developed by Zhu et al. \[133\] which is called as the dynamic MC sampling technique.
Now, assume that $E_1 \subseteq E(G)$ and $E_2 \subseteq E(G)$ be the set of present and non-present edges in one group. Let $G(E_1, E_2)$ denote the set of possible worlds, where the edges present in $E_1$ are present, though the edges of $E_2$ are not present. The generation probability of the deterministic graphs present in the group $G(E_1, E_2)$ is given by the following equation:

$$P_{G(E_1, E_2) \subseteq G} = \prod_{e \in E_1} P(e) \prod_{e' \in E_2} (1 - P(e'))$$

(15)

Now, the $s - t$ reliability of the group $G(E_1, E_2)$ is basically the probability that $s$ and $t$ are reachable when a deterministic graph $G$ from $G(E_1, E_2)$ are appearing and the following equation computes this.

$$R_{G(E_1, E_2)}(s, t) = \sum_{G \in G(E_1, E_2)} I_G(s, t) \times \frac{P_{G \subseteq G}}{P_{G(E_1, E_2) \subseteq G}}$$

(16)

Now, it is easy to verify that for any arbitrary edge $e \in E(G) \setminus (E_1 \cup E_2)$ the following holds:

$$R_{G(E_1, E_2)}(s, t) = P(e).R_{G(E_1 \cup \{e\}, E_2)}(s, t) + (1 - P(e))R_{G(E_1, E_2 \cup \{e\})}(s, t)$$

(17)

This process continues till the $E_1$ contains an $s - t$ path with $R_{G(E_1, E_2)}(s, t) = 1$ or $E_2$ contains an $s - t$ cut with $R_{G(E_1, E_2)}(s, t) = 0$.

Later, Li et al. [75] proposed the recursive stratified sampling technique that works based on the divide-and-conquer paradigm. In this method, the entire probability space is divided into $r + 1$ non-overlapping subspaces by selecting $r$ edges. Each division we call as ‘stratum.’ Let $X$ be the set of $r$ edges chosen via breadth first search from the source node $s$. Let, $Y$ is a Boolean matrix which stores the status value of the selected edges in different stratum. $Y_{ij}$ will be equal to 1 if the $j$-th ($1 \leq j \leq r$) edge belongs to the $i$-th stratum. Now, a deterministic graph $G$ from the possible world belongs to the $i$-th stratum is given by the following equation:

$$\pi_i = \prod_{e_j \in X \land Y_{ij} = 1} P(e_j) \prod_{e_j \in X \land Y_{ij} = 0} (1 - P(e_j))$$

(18)

The sample size of each stratum is fixed as $k_i = \pi_i.k$. Reliability is computed in each of the stratum and the expected value is returned as the $s - t$ reliability. It has been shown in [75] that the time requirement for recursive stratified sampling is same as the MC sampling, i.e., $O(k(m + n))$.

Li et al. [78] proposed the ‘lazy propagation sampling’ scheme. The basic working principle of this method is as follows: ‘If the existence probability of an edge is very low, then it quite natural that this edge will not be present in many possible worlds; hence, it is not to probe such edges.’ To describe formally, this method employs a geometric distribution in each edge and probes an edge if it is activated. It has been shown in [78] that the variance of the lazy propagation sampling is same as the MC sampling, though it improves the efficiency.

Maniu et al. [85] proposed the ‘Indexing via Probabilistic trees’ methodology for efficiently computing the $s - t$ reliability in uncertain graphs. This method constructs a tree structure called as the ‘ProbTree’ from the given uncertain graph $G$. When a $s - t$ reliability query $q$ comes, an equivalent graph $G(q)$ is created from the ‘ProbTree’ and the MC sampling is done in $G(q)$ itself. If the size of $G(q)$ is smaller than $G$, then the query $q$ can be evaluated very efficiently. They developed three different indexing schemes, namely, $SPQR$ trees, $FWD$
Table 3  Time and Space Complexity of different sampling schemes

| Sampling Technique                  | Time Complexity | Space Complexity |
|-------------------------------------|-----------------|------------------|
| MC Sampling                         | $O(k(m + n))$   | $O(k(m + n))$    |
| B.F.S. with Indexing                | $O(k(m + n))$   | $O(n + km)$      |
| Recursive Sampling                  | $O(nd)$         | $O(k(m + n))$    |
| Recursive Stratified Sampling       | $O(k(m + n))$   | $O(rm)$          |
| Lazy Propagation Sampling           | $O(k(m + n))$   | $O(n)$           |
| Indexing via Probabilistic Trees    | $O(k(m + n))$   | $O(m)$           |

(Fixed Width Decomposition), and LIN ProbTrees (Linage Probabilistic Trees). As per their analysis, among these three, the FWD is the best because the both the build time and query time of this data structure is linear and the query quality is ‘lossless’ for $\omega \leq 2$, where $\omega$ is the width of the tree decomposition [7]. As per the analysis in [85], the time complexity of building the FWD ProbTree is linear in number of nodes of the graph. However, the query execution time is of $O(k(n' + m'))$. Here, $n'$ and $m'$ denotes the number of nodes and edges present in the graph $G(q)$. The space complexity of this sampling scheme is of $O(m)$. Table 3 shows the time and space complexity of different sampling strategies for the s – t reliability problem. Table 4 summarizes the pros and cons of different sampling techniques.

5.1.2 SimRank computation

To the best of our knowledge, Du et al. [35] first studied the problem of SimRank computation on uncertain graphs. They proposed a dynamic programming approach to compute the probability values of the probabilistic transition matrix, which works in linear time. To improve the efficiency further, they came up with incremental dynamic programming (IDP) approach. Reported results show that the IDP approach converges much faster than the naive dynamic programming.

Subsequently, Zhu et al. [135] also studied the same problem. They formalized the notion of random walk in uncertain graphs and also based on this notion they show how one can define SimRank on uncertain graphs. It is important to note that in case of uncertain graphs $k$ step transition probability matrix (i.e., $W^k$) is not equal to $k$-th power of the one step transition probability matrix ($W^1$). They defined the random walk on uncertain graphs by exploiting the all possible worlds. For a given uncertain graph $G(V, E, P)$, let $X_0, X_1, \ldots, X_n$ be the random variable associated with a random walk in $G$. Hence, $P_G(X_n = v_n | X_0 = v_0, X_1 = v_1, \ldots, X_{n-1} = v_{n-1})$ can be computed as follows:

$$P_G(X_n = v_n | X_0 = v_0, X_1 = v_1, \ldots, X_{n-1} = v_{n-1}) = \sum_{G \in L(G)} P_G(X_n = v_n | X_0 = v_0, X_1 = v_1, \ldots, X_{n-1} = v_{n-1}) . P_G \subseteq G.$$ 

Now, applying the Markovian property,

$$= \sum_{G \in L(G)} P_G(X_n = v_n | X_{n-1} = v_{n-1}) . P_G \subseteq G$$

$$= P_G(X_n = v_n | X_{n-1} = v_{n-1}).$$
| Sampling Technique                          | Advantages                                                                 | Disadvantages                                                                                   |
|--------------------------------------------|-----------------------------------------------------------------------------|------------------------------------------------------------------------------------------------|
| *Monte Carlo Sampling*                     | − simple to understand                                                      | − inefficient when the network is large                                                         |
|                                            | − easy to implement                                                         | − estimator variance is quite high                                                                |
|                                            | − only one sample is required to store in the main memory                   |                                                                                                |
| *B.F.S. with Indexing*, *zhu2015top*       | more efficient than MC Sampling                                             | − statistically same variance as the MC Sampling                                                 |
|                                            |                                                                            | − same accuracy as the MC Sampling                                                                |
|                                            |                                                                            | − memory requirement is more than MC sampling (to store the bit vectors of the edges)            |
| *Recursive Sampling*, *jin2011distance*    | more efficient than MC Sampling                                             | − same variance as MC Sampling                                                                   |
| *Recursive Stratified Sampling*, *li2014efficient*, *li2015recursive* | estimator variance is slightly reduced than MC Sampling.                     | running time is also more than MC and B.F.S. with Indexing.                                   |
| *Lazy Propagation Sampling*, *li2015recursive* | efficiency improves over the MC sampling                                   | same variance as the MC Sampling                                                                 |
| *Indexing via Probabilistic Trees*, *maniu2017indexing* | variance is less compared to MC Sampling                                   | Index construction is complicated and takes more time.                                          |
Now, k-step transition probability from the vertex \( u \) to \( v \) can be computed as follows:

\[
P_G(X_n = v | X_{n-k} = u) = \sum_{G \in L(G)} P_G(X_n = v | X_{n-k} = u) P_{G|\subseteq G} = \sum_{G \in L(G)} P_G(u \rightarrow_k v) P_{G|\subseteq G}.
\]

\( u \rightarrow_k v \) means that \( v \) is reachable from \( u \) in \( k \)-hop distance. Let, \( W^k \) be the \( k \)-step transition matrix of the uncertain graph \( G \). \( W^k \) can be computed by the following equation:

\[
W^k = \sum_{G \in L(G)} P_{G|\subseteq G} W^k_G
\]

where \( W^k_G \) denotes the \( k \)-step transition probability matrix of the possible world \( G \). Based on this random walk on uncertain graphs, they came up with four different approaches to compute SimRank, namely (i) baseline method, (ii) sampling technique, (iii) two-stage method, and (iv) speeding up techniques. Experimental results show that the methods have very high scalability.

### 5.1.3 Clustering of uncertain graphs

To the best of our knowledge, Liu et al. [80] first studied the clustering problem on uncertain graphs. Here, the problem is, given an uncertain graph \( G(V, E, P) \), we want to cluster the vertices of the graph into \( k \) clusters \( C_1, C_2, \ldots, C_k \). Consider any possible world \( \mathcal{G}_i \in L(G) \) and it has \( L_i \) many connected components denoted as \( CC^{i,1}_x, CC^{i,2}_x, \ldots, CC^{i,L_i}_x \). The vertices of each connected component are also divided into groups based on their cluster levels. As an example, the vertices of the connected component \( CC^{i,j}_x \) are denoted by \( CC^{i,j}_k \). First, they introduced the notion of purity by the cluster level entropy as follows:

\[
F_p = \sum_{G \in L(G)} P_{G|\subseteq G} \sum_{j=1}^{L_i} |CC^{i,j}_x| \cdot H(\bigcup_{x \in [k]} CC^{i,j}_x)
\]

where \( H(\bigcup_{x \in [k]} CC^{i,j}_x) = - \sum_{x \in [k]} \frac{|CC^{i,j}_x|}{|CC^{i,j}|} \log\left(\frac{|CC^{i,j}_x|}{|CC^{i,j}|}\right) \) is the entropy of cluster levels for fragment \( j \) for the \( i \)-th possible world of \( G \). As mentioned in [80], if purity is the only criteria then it may happen that the clustering process may leads to a single cluster containing maximum number of nodes. To address this issue, Liu et al. [80] considered the notion of size balance, which tells that two clusters cannot be too much imbalanced in terms of their sizes. Now, to make the clusters size balanced the following function could be maximized.

\[
F_e = \sum_{G \in L(G)} P_{G|\subseteq G} n \cdot H(\bigcup_{x \in [k]} C_x) = n \cdot H(\bigcup_{x \in [k]} C_k)
\]

Here, \( H(\bigcup_{x \in [k]} C_x) = - \sum_{x \in [k]} \frac{|C_x|}{n} \log\left(\frac{|C_x|}{n}\right) \). Now, Liu et al. [80] formulated the reliable clustering of an uncertain graph as to minimize the following function:

\[
F = F_p - F_e
\]

They developed a novel \( k \)-means clustering algorithm to optimize the function mentioned in Equation 22. Experimental results demonstrate the scalability of this method.

Later, Ceccarello et al. [20] also studied the uncertain graph clustering problem from a different perspective. Here, the goal is to partition the nodes of the uncertain graph into \( k \) cluster...
in such a way that each of the cluster features a node as a cluster center to maximize the minimum or average reliability (minimum connection probability (MCP) and average connection probability (ACP)) from the cluster center to the other nodes of the clusters. They proved that this problem is #P-Hard and came up with approximation algorithms for both the MCP and ACP variants of this problem. They showed that their proposed methodologies generate a $k$-clustering which gives $\Omega((P_{\text{opt-min}}(k))^2)$ and $\Omega((P_{\text{opt-avg}}(k))^2)$ lower bound in approximation guarantee, where $P_{\text{opt-min}}(k)$ and $P_{\text{avg-min}}(k)$ denote the maximum and average connection probability of any $k$-clustering. They compared their results with deterministic weighted graph clustering algorithms to show the efficiency of their proposed methodology.

Kollios et al. [65] extended the edit distance-based graph clustering technique for an uncertain graph. Given two deterministic graphs $\mathcal{G}(V, E_{\mathcal{G}})$ and $\mathcal{Q}(V, E_{\mathcal{Q}})$, their graph edit distance $ED(\mathcal{G}, \mathcal{Q})$ is defined by the following equation.

$$ED(\mathcal{G}, \mathcal{Q}) = |E_{\mathcal{G}} \setminus E_{\mathcal{Q}}| + |E_{\mathcal{Q}} \setminus E_{\mathcal{G}}|$$

Now, given an uncertain graph $G(V, E, P)$ and a deterministic graph $Q(V, E_{\mathcal{Q}})$, their graph edit distance $ED(G, Q)$ can be given by the following equation.

$$ED(G, Q) = \sum_{\mathcal{G} \in LG(G)} P_{\mathcal{G} \subseteq G}.ED(\mathcal{G}, \mathcal{Q})$$

Here, $ED(G, Q)$ can be computed using Equation 23. They introduced the notion of Cluster Graph, which is basically vertex disjoint disconnected cliques, denoted as $C(V, E_{C})$. That means $V$ is partitioned into $k$ disjoint sets $V_1, V_2, \ldots, V_k$, such that $\forall i \in [k], \forall v, v' \in V_i, (vv') \in E_{C}$, and $\forall i, j \in [k], i \neq j, v \in V_i$ and $v' \in V_j, (vv') \notin E_{C}$. Now, given an uncertain graph $G(V, E, P)$, its clustering problem basically asks to find out the cluster graph $C(V, E_{C})$ such that the edit distance $ED(G, C)$ is minimized. They exploited the connection between this problem with that of correlation clustering and showed that the randomized expected 5-approximation algorithm proposed by Ailon et al. [6] for weighted correlation clustering can be used for solving the uncertain graph clustering problem. Experimental evaluations show that this algorithm generates statistically significant clusters of an uncertain graph and also scales well on real-world networks.

Halim et al. [46] proposed a solution methodology for the uncertain graph clustering problem by exploiting the neighborhood information. By this method first, the input uncertain graph is converted into a deterministic graph by classification technique used for edge prediction, and finally, deterministic graph clustering technique can be used for clustering. They also performed an experimental study with different classification techniques for edge prediction.

Subsequently, Han et al. [48] studied the uncertain graph clustering problem with two different goals: divide the uncertain graph into $k$-clusters such that (i) the average reliability from cluster center to other nodes is maximized (similar to the $k$-median problem), and also (ii) the minimum reliability between any node of the cluster to its cluster center is maximized (similar to the $k$-center problem). Both $k$-center and $k$-median problems have been studied extensively by the researchers of theoretical computer scientists [89, 113]. For the $k$-median problem, they proposed an $(1 - \frac{1}{e})$-factor approximation algorithm, and also for the $k$-center problem they proposed an $(1 - \epsilon)OPT_k^{\log n}$ factor approximation (with high probability) algorithm where $OPT_k^{\log n}$ is the optimal value for the $k$-center objective function. Their experimental evaluation shows that the proposed approaches significantly outperforms the methods proposed by Ceccarello et al. [20].
It is important to notice that though there are several clustering techniques for uncertain graphs, the criteria are different across the methodologies. Like, in Liu et al.’s [80] study the goal is to generate size balanced clusters, whereas in Ceccarello et al.’s [20] study the goal is to cluster the uncertain graph to maximize the average/minimum reliability within each cluster. Table 5 briefly summarizes the uncertain graph clustering techniques.

Next, we describe solution methodologies of various graph algorithmic problems that have been studied in the context of uncertain graphs.

5.2 Querying uncertain graphs

5.2.1 Querying for subgraph

Subgraph searching and querying in a graph database in the deterministic setting is a very well-studied topic in the data management community. However, in the probabilistic setting, the amount of literature is very limited. Here, we briefly summarize the existing literature.

**Pattern Matching Queries** To the best of our knowledge (also as per the authors’ claim) Zou et al. [140] were the first to study the subgraph pattern matching queries on uncertain graphs. As the subgraph isomorphism problem is NP-Hard even in deterministic graphs, the same hardness result follows in the context of uncertain graphs as well. Now, as the reachability problem is #P-Hard in uncertain graphs, hence the subgraph pattern matching in an uncertain graph is also #P-Hard. So, the goal here was to find an approximate solution for this problem. They proposed an efficient approach to check whether a subgraph should be returned as a solution or not. They also derived a sample bound which is $16n\log(\frac{1}{\varepsilon})$. Here, $\minsup$ is the expected support value and $Pr(F)$ is the probability that the pattern graph $F$ is contained in $G$. $\hat{p}$ is an estimator of $Pr(F)$. This is the foundational work in this direction, and subsequently, many researchers gave deep dive in this direction.

Later, Chen et al. [21] studied the approximate subgraph search problem in an uncertain graph stream. Here, given an uncertain graph stream $<GS_1, GS_2, \ldots, GS_k>$, a set of query graphs $<Q_1, Q_2, \ldots, Q_{k_2}>$, and a probability threshold $\varepsilon$ the goal here is to report all joinable pairs $<GS_i, Q_j>$ in each timestamp $t$. This means that the subgraph $Q_j$ is contained in $GS_i$ with probability exceeding $\varepsilon$, where $1 \leq i \leq k_1, 1 \leq j \leq k_2$, and $t \geq 0$. They proposed two efficient pruning technique, namely ‘structural pruning’ and ‘probability pruning’ which makes their methodology efficient. Running time of their methodology is of $O(\sqrt{nm \log m})$.

Later, Yuan et al. [124] studied the same problem and proposed the ‘filtering and verifying’ strategy to speed up the search process. In the filtering phase, a probabilistic inverted index is maintained which can be used for probabilistic pruning. Next in the verification phase, the remaining candidates have been verified by an exact algorithm. This method is tested with both synthetic as well as real-world datasets. In a follow-up work by Yuan et al. [126] [128], they developed probabilistic match trees (PM Trees) based on match cuts and the cut selection process. Considering this index structure, they developed an effective pruning strategy to prune the unqualified matches. This makes the proposed methodology much more efficient, and hence, the sizes of the graphs that have been used in this study are much larger than the previous works.

Chen et al. [27] proposed the ‘enumeration–evaluation’ framework for this problem, where first they enumerate all the candidate subgraphs and then for each subgraph compute its
| Reference          | Criteria for clustering                                                                 | Main contributions                                                                                                                                 |
|-------------------|----------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------|
| Liu et al. [80]   | To generate size balanced clustering of an uncertain graph.                            | – Came up with a generalized reliability measurement based on purity and size balance.                                                           |
|                   |                                                                                        | – Proposed a novel k-means algorithm for uncertain graph clustering.                                                                                |
| Ceccarello et al. [20] | To cluster an uncertain graph such that the average/minimum reliability within each cluster is maximized. | – Showed that the problem is NP-Hard.                                                                                                                                 |
|                   |                                                                                        | – Proposed methodologies that gives approximation guarantee with respect to optimal k-Clustering.                                                     |
| Kollios et al. [65] | To minimize the edit distance between cluster graph and input uncertain graph          | – Formulated the uncertain graph clustering problem as the general case of Cluster Edit Problem.                                                      |
|                   |                                                                                        | – Showed that the randomized algorithm for proposed by Ailon et al. [6] can be used to solve the uncertain graph clustering problem                  |
| Halim et al. [46] | No specific criteria mentioned.                                                         | Proposed a methodology that converts the uncertain graph to a deterministic one by applying edge classification technique so that existing deterministic graph clustering techniques can be used. |
| Han et al. [48]   | To cluster an uncertain graph such that the average/minimum reliability within each cluster is maximized. | Proposed approximation algorithms for solving k-center and k-median problems that improves the study by Ceccarello et al. [20].                   |
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support and decide whether to output this as a result. They also showed that under the probabilistic semantic the computation of ‘Support’ is #P-Complete. Hence, their solutions are approximate in nature with an accuracy guarantee. Experimental results show the practical usability of the algorithms.

Recently, Ma et al. [81] studied the problem of counting a given motif present in an uncertain graph. Given a motif $M$, their goal was to evaluate the occurrence statistics of $M$ such as probability mass function, mean, and variance. Based on their sample size analysis, they showed that if the number of samples are more than $\frac{\ln \frac{2}{\delta}}{\epsilon^2}$ then the absolute error is bounded by $\epsilon$ with probability $(1 - \delta)$. The running time of their methodology is $O(nmd^{\frac{1}{2}}|V_M|^{-2})$, where $|V_M|$ denotes the number of vertices in the motif and $d$ denotes the maximum degree of the graph. In the experimental study of this work the size of the datasets is much larger than that has been used in previous studies.

Subgraph Similarity Search Yuan et al. [125, 127] studied the subgraph similarity search problem over uncertain graph databases. They showed that the problem is #P-Complete. They used their previously developed ‘filter and verify’ technique to gear up the search process. In the filtering phase, they develop lower and upper bound of subgraph similarity probability based on probabilistic matrix index. For the verification phase, they developed an efficient sampling approach to validate the remaining candidates. Experimental results show that their methods are scalable.

Gu et al. [44] studied the problem of similarity maximal all matches in an uncertain graph database. Given an uncertain graph $G$, a query graph $g$, distance threshold $d$, and probability threshold $\alpha$, a deterministic graph $G' \in L(G)$ is a similarity maximal match of $g$ in $G$ under the vertex mapping $F$, if there exists no other deterministic graph $G'' \in L(G)$ such that under the same vertex mapping $F$, $Pr(Dist(G'', g) \leq d) > Pr(Dist(G, g) \leq d)$. Here, $Dist(G', g)$ is the edit distance between $G'$ and $g$ [25]. They proposed different speed-up techniques such as partial graph evaluation, vertex pruning, probability upper bound-based filtering, and the incremental evaluation method. Experimental results show that they outperform baseline methods in orders of magnitude.

Table 6 summarizes the literature for pattern matching and subgraph similarity search queries on uncertain graphs.

Clique A Clique in a deterministic graph $G$ is defined as the subset of the vertices where every pair of them are adjacent to each other. A clique $C$ is said to be maximal if for any $v \in V(G) \setminus C, C \cup \{v\}$ is not a clique. A clique is said to be a $k$-clique if the number of vertices in the clique is $k$. Zou et al. [142] studied the top-$k$ maximal clique finding problem in uncertain graphs. They proposed a ‘branch and bound’ algorithm for this problem. Given the uncertain graph $G$, first, they construct the deterministic graph $\hat{G}$ by removing the uncertainties from $G$. Now, they defined a clique search tree where each node represents a distinct clique, the root is the trivial clique $\phi$, and if $C$ is a parent of a non-root clique $C'$ then $C \subset C'$ with $|C'| = |C| + 1$. Now, the problem reduces to a tree searching problem and their algorithm is divided into the following steps: pruning, computing maximal clique probability, expansion, and update. Experimental results show that the methods are scalable.

Later, Mukherjee et al. [93] introduced the notion of an $\alpha$-maximal clique in the context of uncertain graphs and studied the counting and enumeration problem of such cliques. They showed that this number of an $n$ vertex uncertain graph is exactly $\binom{n}{\frac{\alpha}{2}}$ and proposed an enumeration algorithm of such cliques having running time of $O(n.2^n)$. They also showed that the running time of their proposed algorithm is of $O(\sqrt{n})$ of an optimal algorithm. Exper-
Table 6 Summary of the literature of different structural pattern mining on uncertain graphs

| Type of subgraph | Author          | Major findings                                                                                     |
|------------------|-----------------|------------------------------------------------------------------------------------------------------|
| Any subgraph     | Zou et al. [140]| – first to study the subgraph pattern matching queries on uncertain graphs.                       |
|                  |                 | – derived a sample bound which is $16n \cdot \log(\frac{1}{\epsilon})$ for subgraph similarity search |
|                  | Chen et al. [21]| – proposed two efficient pruning technique, namely ‘structural pruning’ and ‘probability pruning’ for solving the problem. |
|                  | Yuan et al. [124, 126] [128] | – proposed ‘filtering and verifying’ strategy to speed up the search process |
|                  |                 | – developed probabilistic match trees (PM Trees) based on match cuts and cut selection process     |
|                  |                 | – developed effective pruning strategy to prune the unqualified matches                           |
|                  | Chen et al. [27]| – proposed the ‘enumeration–evaluation’ framework for this problem                                |
|                  |                 | – showed that under the probabilistic semantic the computation of ‘Support’ is #P-Complete        |
|                  | Ma et al. [81]  | – showed that if the number of samples are more than $\frac{ln^2 \delta}{2 \epsilon^2}$, the absolute error is bounded |
|                  |                 | – proposed a methodology with running time $O(nm d^V m|-2)$                                       |
| Subgraph Similarity | Yuan et al. [125, 127] | – showed that the subgraph similarity search problem is #P-Complete                               |
|                  |                 | – developed lower and upper bound of subgraph similarity probability based on probabilistic matrix index |
|                  | Gu et al. [44]  | – proposed different speed up techniques such as partial graph evaluation, vertex pruning, probability upper bound-based filtering, and the incremental evaluation method. |

Recent works have focused on studying the $(k, \tau)$-Clique search problem in an uncertain graph. A clique $C$ is said to be a $(k, \tau)$-Clique if the size of $C$ is $k$ with clique probability at least $\tau$, and also $C$ is maximal. They proposed two core-based pruning techniques and one cut-based pruning technique which makes the search process more effective. The running time of their algorithm is of $O(2^{n'} (m' + n'))$. Here, $m'$ and $n'$ denotes the number of edges and vertices of the largest connected component in $(Top_k, \tau)$-core which is much smaller than the original uncertain graph. They defined the notion of $(Top_k, \tau)$-core as the Rashid et al. [102].
**Truss** In a deterministic graph $G$, its subgraph $H$ is called a $k$-truss if every edge of this subgraph is incident with at least $(k - 2)$ triangles. Huang et al. [51] studied the truss decomposition problem in the context of uncertain graphs. They defined the notion of $(k, \gamma)$-truss as a maximal connected subgraph $H$, in which any edge, the probability that it is contained in at least $(k - 2)$ triangles is at least $\gamma$. They proposed a dynamic programming algorithm for decomposing an uncertain graph into maximal such $(k, \gamma)$-truss. They also defined the global $(k, \gamma)$-truss in which along with the previous conditions, it has to satisfy that the probability that $H$ contains a $k$-truss is at least $\gamma$.

Subsequently, Zou and Zhu [139] introduced the $\theta$-truss decomposition problem $(\theta \in [0, 1])$, where given an uncertain graph $G$, the goal is to decompose $G$ into the following set \( T_{i, \theta} : 1 \leq i \leq k \). Here, $k$ is the largest value for which $T_{k, \theta}$ is a probabilistic truss. They showed that like deterministic graphs truss decomposition in uncertain graphs also follows the ‘uniqueness’ and ‘hierarchy’ property. They proposed an $O(m^{1.5}Q)$ time algorithm to solve this problem. Here, $Q$ is the maximum number of common neighbors of the end vertices of an edge. In the worst case, it may be of $O(n)$. They also extend it to an external memory algorithm for this problem for larger size uncertain graphs.

**Core** In a deterministic graph $G$, a subgraph $\mathcal{J}$ is said to be a $k$-core of $G'$ if $\forall v \in V(\mathcal{J})$, $\text{deg}(u)$ in $\mathcal{J}$ is more than or equal to $k$ and also $\mathcal{J}$ is maximal. Bonchi et al. [15] introduced the notion of $(k, \eta)$-core, which is basically the maximal subgraph $\mathcal{J}$, such that every vertex $v \in V(\mathcal{J})$ has degree more than or equal to $k$ with probability at least $\eta$. First, they showed that given an $\eta$, the $(k, \eta)$-core decomposition of an uncertain graph is unique. They proposed a dynamic programming algorithm for this problem for larger size uncertain graphs. They have introduced the concept of ‘local’ ($l$), ‘weakly local’ ($wg$) nucleus, and ‘Weakly global’ ($gl$) nucleus. Given an uncertain graph $G$, a triangle $\Delta$, and $\mu \in \{l, gl, wg\}$, the probability that the random variable $X_{G, \Delta, \mu}$ takes the value $k$ with tail probability given...
in Equation 25.

\[ P[r|X \geq k] = \sum_{G \in L(G)} P[r|G \geq k] \]

(25)

Now, \( I_{G, \Delta, l} = 1 \) if \( \Delta \) is in \( G \) and the support of \( \Delta \) in \( G \) is at least \( k \). \( I_{G, \Delta, gl} = 1 \) if \( \Delta \) contained in \( G \), and \( G \) is a deterministic \( k \) nucleus. They showed that the local version of the problem of nucleus decomposition is polynomial time solvable, whereas the global version is NP-Hard. They proposed a dynamic programming approach for solving the local version. From the results, they concluded that compared to probabilistic core and truss decompositions, nucleus decomposition significantly outperforms in terms of density and clustering metrics.

Table 7 summarizes the literature related to different structural patterns in uncertain graph.

5.3 Graph algorithmic problem

As mentioned in Sect. 3, among the plethora of graph algorithmic problems, only a very few have been studied on uncertain graphs. The first one is the construction of the ‘minimum spanning tree (MST).’ Zhang et al. [130] introduced and studied the problem of finding the most reliable minimum spanning tree. The trivial solution for this problem is to enumerate all the possible worlds, compute the MST in each one of them and then choose one with the highest probability value. However, it requires exponential time. To get rid of this problem, Zhang et al. [130] proposed an incremental greedy approach for this problem. Starting with an empty MST, until all the vertices of the graph have been included in the MST this algorithm iteratively chooses an edge that connects the already build MST with maximum probability. To implement this, they have used the ‘max heap’ data structure. Their analysis shows that this algorithm is able to give a spanning tree having reliability \((1 - \left(\frac{1}{2}\right)^{d}n^{-1})\) and expected approximation ratio \(\left(1 - \left(\frac{1}{2}\right)^{d}n\right)\), where \(d\) is the maximum degree of the graph. The computational time requirement of this algorithm is of \(O(d^2 n^2)\). To the best of our knowledge, there is no other study on finding MST on uncertain graphs.

Another important problem that has been studied in the context of uncertain graphs is the ‘link prediction problem.’ Ahmed et al. [4] studied this problem in uncertain social networks. They reduced the link prediction problem in uncertain network to the problem of random walk in a deterministic graph. They considered the ‘SimRank’ as the similarity metric between two nodes. Their method consisting of the following two steps:

– The first step is the computation of the transformation matrix. For the given probabilistic graph \( G(V, E, P) \) and a deterministic graph \( \mathcal{G}(V, E) \), the associated transformation matrix can be defined as follows:

\[ \overline{W}(u, u) = \prod_{(uq) \in E(\mathcal{G})} (1 - P(uq)) \]

\[ \overline{W}(u, v) = \sum_{G' \in \Omega(uv)} \sum_{(uq) \in E(G')} A(u, v) P[G' \supseteq G] \]

(26)

(27)

where \( A \) is the adjacency matrix of the uncertain graph \( G \), and \( \Omega(uv) \) is the subset of the possible worlds in which the edge \((u, v)\) is present.

– The second step is the computation of the SimRank matrix. Based on the computed value of \( \overline{W} \), this matrix is computed using the following equation:

\[ S = c \cdot \overline{W}^T \cdot S \cdot \overline{W} + (1 - c) \cdot I \]

(28)
Table 7 Summary of the literature of different structural pattern mining on uncertain graphs

| Type of subgraph | Author | Major findings |
|------------------|--------|----------------|
| Clique           | Zou et al. [142] | – introduced the top-k maximal clique finding problem  
|                  |        | – proposed a branch and bound algorithm for this problem |
|                  | Mukherjee et al. [93] | – introduced the notion of $\alpha$-maximal clique  
|                  |        | – showed that the number of such cliques are $2^n$  
|                  |        | – proposed an algorithm with running time $O(2^n)$  
|                  |        | – showed that their algorithm is $O(\sqrt{n})$ of optimal algorithm |
|                  | Li et al. [76] | – introduced the notion of $(k, \tau)$-clique  
|                  |        | – proposed core and cut-based pruning technique  
|                  |        | – proposed an algorithm with running time of $O(2^n (m' + n'))$ |
| Truss            | Huang et al. [51] | – introduced the truss decomposition problem in uncertain graphs  
|                  |        | – proposed a dynamic programming algorithm |
|                  | Zou and Zhu [139] | – introduced the $\theta$-truss decomposition problem  
|                  |        | – showed that the truss decomposition follow 'uniqueness' and 'hierarchy' property  
|                  |        | – proposed an $O(m^{1.5}Q)$ time algorithm |
| Core             | Bonchi et al. [15] | – introduced the core decomposition problem in uncertain graphs  
|                  |        | – proposed an algorithm with running time $O(m\Delta)$  
|                  |        | – applied there approach to solve ’influence maximization’ and ’task driven team formation’ |
|                  | Peng et al. [99] | – introduced the concept of $(k, \theta)$-core of an uncertain graph  
|                  |        | – gave a sample bound to compute the $k$-core probability with bounded error  
|                  |        | – proposed a $k$-core membership check algorithm for speeding up the procedure |
| Reliable         | Jin et al. [54] | – first defined the problem of reliable subgraph search on uncertain graphs  
|                  |        | – gave a sample bound to estimate the subgraph reliability with bounded error |
| Nucleus          | Esfahani et al. [37] | – introduced the notion of local, global, and weakly global nucleus  
|                  |        | – local version is polynomial time solvable and global version is NP-Hard |

where $c$ is a constant having value in between 0 and 1, and $I$ is the identity matrix.

Based on this computed SimRank value, for a given $k \in \mathbb{Z}^+$, vertex pairs (which are not currently linked) corresponding top-$k$ values are returned as the result.

Another algorithmic problem studied in context of the uncertain graph is the ‘information flow maximization’ problem. Frey et al. [39, 40] showed that this problem is $NP$-Hard by a reduction from the 0-1 Knapsack Problem, more efficient. To highlight, they study
this problem on vertex weighted graphs, where vertex weight signifies the weight on the
information in the corresponding vertex. For a vertex \( Q \in V(G) \), \( \text{flow}(Q, G) \) denotes the
random variable signifying the total vertex weights of all the nodes \( V(G) \) that are reachable
from the node \( Q \). As mentioned previously, the reachability problem in uncertain graphs is
\( \#P \)-Hard; hence, the complexity of the information flow estimation also falls in the same
category. To tackle this problem, they used traditional Monte Carlo Sampling to generate
a subset of possible worlds. It has been shown that the average value over these samples
is an unbiased estimator of the random variable \( \text{flow}(Q, G) \). Next, they have used the
concept of mono-connected nodes (two nodes are mono-connected if there exist a single
path connecting them and also for them the reliability can be computed efficiently), and bi-
connected nodes (two nodes are mono-connected if there exist two or more path connecting
them). Next, they proposed the Flow Tree data structure which is an adoption of Block Cut
Tree, hopcroft1973algorithm, westbrook1992maintaining. As finding the optimal solution of
this problem is \( NP \)-Hard, they proposed an incremental greedy algorithm that generates a
sub-optimal solution for this problem.

5.4 Miscellaneous problem

Among a few other problems, ‘graph sparsification’ has been studied in the context of uncer-
tain graphs. As mentioned previously, for a given sparsification ratio \( \alpha \in (0, 1) \) and an
uncertain graph \( G(V, E, P) \), this problem is all about find a subgraph \( G'(V, E', P') \) such
that \( E' \subset E \) and \( |E'| = \alpha.|E| \). For this problem, Parchas et al. [97] proposed a cut-based
sparsification technique for uncertain graphs. First, they generalized the notion of cut size
for a given set of vertices \( S \) in the context of an uncertain graph. Their proposed framework
begins by initializing a connected unweighted backbone graph \( G_b \) and after that, two tech-
niques are applied to \( G_b \) to generate the sparsified graph. First, they proposed an approach
to initialize the backbone graph \( G_b(V, E_b) \). As described in Sect. 3.4.1, the problem is to
minimize the discrepancy; hence, they formulated the following optimization problem:

\[
\min_{p'} |d - Y_b.p'|
\]

\[
\text{such that, } p' \in (0, 1]|E_b|
\]

where \( d \) is a vector of size \( |V(G)| \) containing the expected degree values of the nodes of \( G \)
and \( Y_b \) is the incidence matrix of size \( |V(G)| \times E_b \). Next, they showed that for the incidence
matrix \( A_b \) of \( G_b \), there exists a probability assignment \( p^* \) that minimizes the discrepancy
for which the expected degree of any node is less than its original degree. Using this result
they showed that the optimal probability distribution \( p^* \) for the degree discrepancy \( \delta_A \) is the
solution of the following linear programming problem (LPP):

\[
\max_{p'} |p'|
\]

\[
\text{such that, } A_b.p' \leq d
\]

\[
\text{and } p' \in (0, 1]|E_b|.\]

Now, this LPP can be solved using solver packages like Simplex, ficken2015simplex. They
also proposed an efficient methodology ‘gradient descent backbone’ which closely approxi-
mates the optimal probability assignment. The running time of this methodology is of
\( O(M_{\text{steps}}.\alpha.|E(G)|) \), where \( M_{\text{steps}} \) denotes the number of steps required to converge the
gradient descent method. This method only updates the edge probabilities without inserting or removing edges. They proposed another methodology called expectation maximization degree and this modifies both the backbone graph as well as the edge probabilities. This procedure is inspired by the expectation-maximization, which is an iterative optimization framework and also uses the gradient descent backbone as a subroutine. Running time of this methodology is that $O(E_{\text{steps}} \cdot \alpha \cdot |E(G)| \cdot (\log n + M_{\text{steps}}))$, where $E_{\text{steps}}$ denotes the number of iterations required to converge this process.

Another problem that has been studied in the context of uncertain graphs is the ‘node classification’ problem. Dallachiesa et al. [30] proposed an iterative probabilistic learning approach to solve this. For any unlabeled node $r$ in $G$ having neighbors $i_1, i_2, \ldots, i_s$ with labels $t_1, t_2, \ldots, t_s$, its probability is estimated by the Bayes’ rule over the adjacent nodes’ neighbors. This is formalized by the following equation:

$$P(L(r) = p | L(i_1) = t_1, \ldots, L(i_s) = t_s) \propto P(L(r) = p) \prod_x P(L(i_x) = t_x | L(r) = p)$$

(29)

However, if a significant number of edges have a low probability, then this method will have an impact on the classification process. To get rid of this problem, they proposed the iterative edge augmentation process. For this purpose, they split the already labeled nodes ($T_0$) into two parts ‘training’ ($T_{\text{train}}$) set and a ‘hold out’ set ($T_{\text{hold}}$). The ratio $T_{\text{train}}$ to $T_0$ is basically a user-defined parameter. Initially, starting with a small fraction of high probability edges, this edge set is expanded by incorporating the outside edges in an iterative manner. The ratio of active edges is denoted by $\theta$. The value of $\theta$ for which maximum accuracy is obtained is denoted by $\theta^*$. This can be obtained by evaluating the accuracy on a number of sampled graphs.

Han et al. [47] studied the uncertain graph classification problem using the extreme learning machine, which was originally proposed by Miche et al. [90]. Their method is broadly divided into three steps. First, they proposed a framework for classifying uncertain graph objects. Next, they extended the traditional algorithm used in the process of extracting frequent subgraphs to handle uncertain graph data. Finally, based on extreme learning machine (ELM) with fast learning speed, a classifier is constructed. They performed a number of experiments with biological datasets.

6 Current trends and existing research gaps

After analyzing the reported literature, here we list out the current research trends:

- **Bounds on required samples** As mentioned previously, for most of the problems on uncertain graph our main concern is that how we can sample few of them to answer the problem within bounded error with high probability. Recently, sample bounds are analyzed for different problems studied on uncertain graphs. Sadeh et al. [104] studied this problem for the influence maximization problem. However, the same study can be done for other problems as well.

- **Development of efficient indexing** Another way of reducing the computational time of processing any uncertain graph is to index it properly using the proposed indexing scheme. To the best of our knowledge, the only indexing scheme is available for uncertain graphs is probabilistic tree, maniu2017indexing. Much more indexing schemes can be developed which may lead to faster execution of the query problems on uncertain graphs.
Finding different structural patterns Recently several structural patterns finding algorithms are developed for analyzing the topology. As an example, Sun et al. [111] proposed a methodology for indexing the trusses of an uncertain graph. Such methods are not available in the literature for other structural patterns.

Uncertain graph sparsification A plethora of literature is available for the sparsification of a deterministic graph [8, 42, 98]. Recently, a cut-based sparsification technique has been developed for uncertain graphs [97]. Certainly, there is a gap in this area as the other principles of graph sparsification can be used for designing the same for the uncertain graph.

Graph algorithmic problems As mentioned previously, among the plethora of graph algorithmic problems, in the last five years very few of them have been studied such as spanning tree, link prediction, information flow maximization. There is a research gap as the remaining algorithmic graph theory problems can be studied under the uncertain graph framework.

Next, we proceed to describe the future research directions in this domain.

7 Future research directions

Based on the detailed analysis of the existing literature, in this section, we describe several future research directions.

Definition of uncertainty: In the existing literature, the notion of uncertainty is defined as the probabilistic existence of the edges of the graph. However, it can be observed that this not the only notion of uncertainty. The same thing applies to the vertices as well. Hence all the problems mentioned in Sect. 3, that have been studied under the edge uncertainty model, can also be studied under the node uncertainty model as well. Very recently, Fukunaga et al. [41] studied a well-studied graph theoretic problem CONNECTED DOMINATING SET PROBLEM under the node uncertainty setting.

Modeling the problem: Before applying any principle and approach of uncertain graph mining, it is important to represent the input network as an uncertain graph, and for that it is very much vital to decide which uncertain model is going to fit.

Studying existing graph theoretic problems in uncertain graph framework: There are tons of graph theoretic problems which has been studied by theoretical computer scientist. However, as mentioned in Section 3, very few of them have been studied in uncertain graph framework. So, naturally it appears that there are many problems which have not been studied yet in uncertain graph framework. It will be interesting to take an existing problem and study it on uncertain graph. Also, different models of uncertainty may give altogether different flavor to the problem.

Efficient sampling scheme: There are many problems in uncertain graph mining domain, such as reachability (i.e., reliability computation), subgraph enumeration and many more, where sampling of the deterministic graphs from the possible worlds plays a crucial role. It will be very important contribution to come up with an efficient sampling technique, that can beat the existing techniques in at least any one of the following two metrics: either (i) it should take a fewer number of samples compared to the existing methods, or (ii) it should be able to answer to the queries with higher probability.

Information flow maximization problem: In Frey et. al.’s [39] study of this problem, they have only considered the Monte-Carlo Sampling. However, as mentioned previously,
there are many other and more efficient sampling techniques [55, 134]. Those sampling techniques can be used to solve the problem even more efficiently.

- **Incorporation of temporal nature:** In almost entire existing literature, it has been considered that the assigned probability values to the edges are fixed over time. However, in reality, the case is not the same. As mentioned previously, the influence graph of a social network often represented as an uncertain graph. The links of a social network are time varying, and also if the probability values marked on the edges are considered as the diffusion probability that can also change with time [68].

- **Querying of combinatorial structures:** As mentioned in Section 3.1, finding and counting motifs in networks is an important problem. There are different kinds of motifs that are available in network science literature, such as Clan, Core, Club, and their different generalizations. However, it is surprising to observe that there are very few structures that have been studied in uncertain graph framework. So, it will be an interesting future work to develop efficient algorithms for finding remaining structural patterns in uncertain graphs.

- **Solution methodologies for modern hardware:** Recent times have witnessed a significant development in almost every aspect of computer architecture, starting from processor and accelerator design to memory system or subsystem organization [31, 64]. Also, several new computing paradigms have been introduced, such as map-reduce and many more. For deterministic graph mining there exist some literature, which developed algorithms for modern computing paradigms [22, 117]. Similarly, for uncertain graph mining also algorithms can be developed in the context of modern computing paradigms [32, 33].

- **Scalability of the methodologies:** Day by day the size of the real-world networks is increasing. As an example, it can be observed that for the social influence maximization problem (a problem in the domain of social network analysis, where the underlying social network is represented as an uncertain graph) the experimentation of the very first paper by Kempe et al. [59] uses a dataset of size 10748 nodes and 53000 edges. However, a very recent paper on the same problem by Zeng et al. [129] uses a dataset of size $4.17 \times 10^6$ nodes and $1.5 \times 10^9$ edges. So, to deal with this kind of gigantic network, it is essential that the solution methodologies should have high scalability. Hence, developing highly scalable methods for large-scale uncertain graphs will remain a major thrust in this area.

- **Privacy preserving uncertain graph mining:** As mentioned previously, the privacy preserving aspects has been deeply inducted with data mining in recent times [60]. However, in the context of uncertain graphs, this study is extremely limited [114, 118]. So, more research work can be carried out in this direction.

- **System oriented research:** In the last decade or so, a significant effort has been made to develop graph processing systems [83, 137]. Please look into [12, 87, 109]. However, none of the studies consider the uncertain nature of the graph. So, it will be an interesting future work to develop an uncertain graph processing and mining system, which includes efficient data structures for storing, efficient processing algorithm developed based on processor architecture, and also very effective data visualization, and interpretation component to represent the output.
8 Concluding remarks

In this paper, we have presented a concise and almost self-contained survey on the uncertain graph mining and analysis with a major focus on three aspects, namely different problems studied in this domain, challenges for solving those problems, and existing solution methodologies. After analyzing the existing literature, we have derived the current research trend. At the end, we have listed a number of future research directions. Hope that this survey will help the upcoming researchers and practitioners to have a finer understanding and better exposure in this field.

References

1. Adar E, Re C (2007) Managing uncertainty in social networks. IEEE Data Eng Bull 30(2):15–22
2. Aggarwal CC (2010) Managing and mining uncertain data, vol 35. Springer Science & Business Media
3. Aggarwal CC, Wang H (2010) A survey of clustering algorithms for graph data. In: Managing and mining graph data, Springer, pp 275–301
4. Ahmed NM, Chen L (2016) An efficient algorithm for link prediction in temporal uncertain social networks. Information Sciences 331:120–136
5. Ahuja RK, Magnanti TL, Orlin JB (1988) Network flows
6. Ailon N, Charikar M, Newman A (2008) Aggregating inconsistent information: ranking and clustering. Journal of the ACM (JACM) 55(5):23
7. Alber J, Niedermeier R (2002) Improved tree decomposition based algorithms for domination-like problems. In: Latin American Symposium on Theoretical Informatics, Springer, pp 613–627
8. Althöfer I, Das G, Dobkin D, Joseph D, Soares J (1993) On sparse spanners of weighted graphs. Discrete & Computational Geometry 9(1):81–100
9. Ball MO (1986) Computational complexity of network reliability analysis: An overview. IEEE Transactions on Reliability 35(3):230–239
10. Ball MO, Colbourn CJ, Provan JS (1995) Network reliability. Handbooks in operations research and management science 7:673–762
11. Bell MG, Iida Y (1997) Transportation network analysis
12. Besta M, Stanojevic D, Licht JDF, Ben-Nun T, Hoefer T (2019) Graph processing on fpgas: Taxonomy, survey, challenges. arXiv preprint arXiv:1903.06697
13. Bhagat S, Cormode G, Muthukrishnan S (2011) Node classification in social networks. In: Social network data analytics, Springer, pp 115–148
14. Bollobás B, Béla B (2001) Random graphs. 73, Cambridge university press
15. Bonchi F, Gullo F, Kaltenbrunner A, Volkovich Y (2014) Core decomposition of uncertain graphs. In: Proceedings of the 20th ACM SIGKDD international conference on Knowledge discovery and data mining, pp 1316–1325
16. Brecht TB, Colbourn CJ (1988) Lower bounds on two-terminal network reliability. Discrete applied mathematics 21(3):185–198
17. Brohee S, Van Helden J (2006) Evaluation of clustering algorithms for protein-protein interaction networks. BMC bioinformatics 7(1):488
18. Carrington PJ, Scott J, Wasserman S (2005) Models and methods in social network analysis, vol 28. Cambridge University Press
19. Ceccarello M, Fantozzi C, Pietracaprina A, Pucci G, Vandin F (2017a) Clustering uncertain graphs. Proceedings of the VLDB Endowment 11(4):472–484
20. Ceccarello M, Fantozzi C, Pietracaprina A, Pucci G, Vandin F (2017b) Clustering uncertain graphs. PVLDB 11(4):472–484. https://doi.org/10.1145/3186728.3164143 (http://www.vldb.org/pvldb/vol11/p472-ceccarello.pdf)
21. Chen L, Wang C (2010) Continuous subgraph pattern search over certain and uncertain graph streams. IEEE Transactions on Knowledge and Data Engineering 22(8):1093–1109
22. Chen Q, Fang C, Wang Z, Suo B, Li Z, Ives ZG (2016a) Parallelizing maximal clique enumeration over graph data. In: International Conference on Database Systems for Advanced Applications, Springer, pp 249–264
23. Chen W, Lakshmanan LV, Castillo C (2013) Information and influence propagation in social networks. Synthesis Lectures on Data Management 5(4):1–177
24. Chen W, Lin T, Yang C (2016b) Real-time topic-aware influence maximization using preprocessing. Computational social networks 3(1):8
25. Chen X, Huo H, Huan J, Vitter JS (2019) An efficient algorithm for graph edit distance computation. Knowledge-Based Systems 163:762–775
26. Chen X, Lai L, Qin L, Lin X (2021) Efficient structural node similarity computation on billion-scale graphs. The VLDB Journal 30(3):471–493
27. Chen Y, Zhao X, Lin X, Wang Y, Guo D (2018) Efficient mining of frequent patterns on uncertain graphs. IEEE Transactions on Knowledge and Data Engineering 31(2):287–300
28. Coon JP, Badu MA, Gündüz D (2018) On the conditional entropy of wireless networks. In: 2018 16th International Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks (WiOpt), IEEE, pp 1–6
29. Cormen TH, Leiserson CE, Rivest RL, Stein C (2009) Introduction to algorithms. MIT press
30. Dallachiesa M, Aggarwal C, Palpanas T (2014) Node classification in uncertain graphs. In: Proceedings of the 26th International Conference on Scientific and Statistical Database Management, ACM, p 32
31. Deng Y, Ni Y, Li Z, Mu S, Zhang W (2017) Toward real-time ray tracing: A survey on hardware acceleration and microarchitecture techniques. ACM Computing Surveys (CSUR) 50(4):1–41
32. Dhulipala L, Blelloch G, Shun J (2017) Julienne: A framework for parallel graph algorithms using work-efficient bucketing. In: Proceedings of the 29th ACM Symposium on Parallelism in Algorithms and Architectures, pp 293–304
33. Dhulipala L, Blelloch GE, Shun J (2021) Theoretically efficient parallel graph algorithms can be fast and scalable. ACM Transactions on Parallel Computing (TOPC) 8(1):1–70
34. Diestel R (2012) Graph theory, volume 173 of. Graduate texts in mathematics p 7
35. Du L, Li C, Chen H, Tan L, Zhang Y (2015) Probabilistic simrank computation over uncertain graphs. Information Sciences 295:521–535
36. Erlebach T, Hoffmann M, Krizanc D, Mihal’Ák M, Raman R (2008) Computing minimum spanning trees with uncertainty. arXiv preprint arXiv:0802.2855
37. Esfahani F, Srinivasan V, Thomo A, Wu K (2020) Nucleus decomposition in probabilistic graphs: Hardness and algorithms. arXiv preprint arXiv:2006.01958
38. Ficken FA (2015) The simplex method of linear programming. Courier Dover Publications
39. Frey C, Zülife A, Emrich T, Renz M (2017) Efficient information flow maximization in probabilistic graphs. IEEE Transactions on Knowledge and Data Engineering 30(5):880–894
40. Frey C, Zülife A, Emrich T, Renz M (2018) Efficient information flow maximization in probabilistic graphs (extended abstract). In: 34th IEEE International Conference on Data Engineering, ICDE 2018, Paris, France, April 16-19, 2018, pp 1801–1802, https://doi.org/10.1109/ICDE.2018.00258
41. Fukunaga T (2019) Adaptive algorithm for finding connected dominating sets in uncertain graphs. arXiv preprint arXiv:1912.12665
42. Fung WS, Hariharan R, Harvey NJ, Panigrahi D (2019) A general framework for graph sparsification. SIAM Journal on Computing 48(4):1196–1223
43. Göbel F, Jagers A (1974) Random walks on graphs. Stochastic processes and their applications 2(4):311–336
44. Gu Y, Gao C, Wang L, Yu G (2016) Subgraph similarity maximal all-matching over a large uncertain graph. World Wide Web 19(5):755–782
45. Guo H, Jerrum M (2019) A polynomial-time approximation algorithm for all-terminal network reliability. SIAM Journal on Computing 48(3):964–978
46. Halim Z, Waqas M, Baig AR, Rashid A (2017) Efficient clustering of large uncertain graphs using neighborhood information. International Journal of Approximate Reasoning 90:274–291
47. Han D, Hu Y, Ai S, Wang G (2015) Uncertain graph classification based on extreme learning machine. Cognitive Computation 7(3):346–358
48. Han K, Gui F, Xiao X, Tang J, He Y, Cao Z, Huang H (2019) Efficient and effective algorithms for clustering uncertain graphs. Proceedings of the VLDB Endowment 12(6):667–680
49. Hardy G, Lucet C, Limnios N (2007) K-terminal network reliability measures with binary decision diagrams. IEEE Transactions on Reliability 56(3):506–515
50. Hopcroft J, Tarjan R (1973) Algorithm 447: efficient algorithms for graph manipulation. Communications of the ACM 16(6):372–378
51. Huang X, Lu W, Lakshmanan LV (2016) Truss decomposition of probabilistic graphs: Semantics and algorithms. In: Proceedings of the 2016 International Conference on Management of Data, pp 77–90
52. Jain AK, Murty MN, Flynn PJ (1999) Data clustering: a review. ACM computing surveys (CSUR) 31(3):264–323
53. Jeh G, Widom J (2002) Simrank: a measure of structural-context similarity. In: Proceedings of the eighth ACM SIGKDD international conference on Knowledge discovery and data mining, ACM, pp 538–543
54. Jin R, Liu L, Aggarwal CC (2011a) Discovering highly reliable subgraphs in uncertain graphs. In: Proceedings of the 17th ACM SIGKDD international conference on Knowledge discovery and data mining, pp 992–1000
55. Jin R, Liu L, Ding B, Wang H (2011b) Distance-constraint reachability computation in uncertain graphs. Proceedings of the VLDB Endowment 4(9):551–562
56. Kassiano V, Gounaris A, Papadopoulos AN, Tsichlas K (2016) Mining uncertain graphs: An overview. In: Algorithmic Aspects of Cloud Computing - Second International Workshop, ALGOCLOUD 2016, Aarhus, Denmark, August 22, 2016, Revised Selected Papers, pp 87–116, https://doi.org/10.1007/978-3-319-57045-7_6
57. Ke X, Teo M, Khan A, Yalavarthi VK (2018) A demonstration of perc: probabilistic entity resolution with crowd errors. Proceedings of the VLDB Endowment 11(12):1922–1925
58. Ke X, Khan A, Quan LLH (2019) An in-depth comparison of st reliability algorithms over uncertain graphs. Proceedings of the VLDB Endowment 12(8):864–876
59. Kempe D, Kleinberg J, Tardos É (2003) Maximizing the spread of influence through a social network. In: Proceedings of the ninth ACM SIGKDD international conference on Knowledge discovery and data mining, ACM, pp 137–146
60. Kenthapadi K, Mironov I, Thakurta AG (2019) Privacy-preserving data mining in industry. In: Proceedings of the Twelfth ACM International Conference on Web Search and Data Mining, pp 840–841
61. Khan A, Bonchi F, Gionis A, Gullo F (2014) Fast reliability search in uncertain graphs. In: EDBT, pp 535–546
62. Khan A, Bonchi F, Gullo F, Nufer A (2018a) Conditional reliability in uncertain graphs. IEEE Transactions on Knowledge and Data Engineering 30(11):2078–2092
63. Khan A, Ye Y, Chen L (2018b) On uncertain graphs. Synthesis Lectures on Data Management 10(1):1–94
64. Kloda T, Solieri M, Mancuso R, Capodieci N, Valente P, Bertogna M (2019) Deterministic memory hierarchy and virtualization for modern multi-core embedded systems. In: 2019 IEEE Real-Time and Embedded Technology and Applications Symposium (RTAS), IEEE, pp 1–14
65. Kollios G, Potamias M, Terzi E (2011) Clustering large probabilistic graphs. IEEE Transactions on Knowledge and Data Engineering 25(2):325–336
66. Kong X, Yu PS, Wang X, Ragan AB (2013) Discriminative feature selection for uncertain graph classification. In: Proceedings of the 2013 SIAM International Conference on Data Mining, SIAM, pp 82–93
67. Kuramochi M, Karypis G (2005) Finding frequent patterns in a large sparse graph. Data mining and knowledge discovery 11(3):243–271
68. Laurent G, Saramäki J, Karsai M (2015) From calls to communities: a model for time-varying social networks. The European Physical Journal B 88(11):301
69. Lee J, Han WS, Kasperovics R, Lee JH (2012) An in-depth comparison of subgraph isomorphism algorithms in graph databases. Proceedings of the VLDB Endowment, VLDB Endowment 6:133–144
70. Lewis HR (1983) Computers and intractability. a guide to the theory of np-completeness
71. Li C, Han J, He G, Jin X, Sun Y, Yu Y, Wu T (2010a) Fast computation of simrank for static and dynamic information networks. In: Proceedings of the 13th International Conference on Extending Database Technology, ACM, pp 465–476
72. Li J, Zou Z, Gao H (2012) Mining frequent subgraphs over uncertain graph databases under probabilistic semantics. The VLDB Journal 21(6):753–777
73. Li L, Wang H, Gao H, Li J (2010b) Eif: a framework of effective entity identification. In: International conference on web-age information management, Springer, pp 717–728
74. Li RH, Yu JX, Mao R, Jin T (2014) Efficient and accurate query evaluation on uncertain graphs via recursive stratified sampling. In: 2014 IEEE 30th International Conference on Data Engineering, IEEE, pp 892–903
75. Li RH, Yu JX, Mao R, Jin T (2015) Recursive stratified sampling: A new framework for query evaluation on uncertain graphs. IEEE Transactions on Knowledge and Data Engineering 28(2):468–482
76. Li RH, Dai Q, Wang G, Ming Z, Qin L, Yu JX (2019) Improved algorithms for maximal clique search in uncertain networks. In: 2019 IEEE 35th International Conference on Data Engineering (ICDE), IEEE, pp 1178–1189
77. Li X, Chen H (2009) Recommendation as link prediction: a graph kernel-based machine learning approach. In: Proceedings of the 9th ACM/IEEE-CS joint conference on Digital libraries, ACM, pp 213–216
78. Li Y, Fan J, Zhang D, Tan KL (2017) Discovering your selling points: Personalized social influential tags exploration. In: Proceedings of the 2017 ACM International Conference on Management of Data, pp 619–634
79. Liben-Nowell D, Kleinberg J (2007) The link-prediction problem for social networks. Journal of the American society for information science and technology 58(7):1019–1031
80. Liu L, Jin R, Aggarwal C, Shen Y (2012) Reliable clustering on uncertain graphs. In: 2012 IEEE 12th International Conference on Data Mining. IEEE, pp 459–468
81. Ma C, Cheng R, Lakshmanan LV, Grubennmann T, Fang Y, Li X (2019) Linc: a motif counting algorithm for uncertain graphs. Proceedings of the VLDB Endowment 13(2):155–168
82. Ma C, Cheng R, Lakshmanan LV, Grubennmann T, Fang Y, Li X (2020) Linc: A motif counting algorithm for uncertain graphs. Proceedings of the VLDB Endowment 13(2)
83. Malewicz G, Austern MH, Bik AJ, Dehnert JC, Horn I, Leiser N, Czajkowski G (2010) Pregel: a system for large-scale graph processing. In: Proceedings of the 2010 ACM SIGMOD International Conference on Management of data, pp 135–146
84. Malkov Y, Ponomarenko A, Logvinov A, Krylov V (2014) Approximate nearest neighbor algorithm based on navigable small world graphs. Information Systems 45:61–68
85. Maniu S, Cheng R, Senellart P (2017) An indexing framework for queries on probabilistic graphs. ACM Transactions on Database Systems (TODS) 42(2):13
86. Martínez V, Berzal F, Cubero JC (2017) A survey of link prediction in complex networks. ACM Computing Surveys (CSUR) 49(4):69
87. McCune RR, Weninger T, Madey G (2015) Thinking like a vertex: a survey of vertex-centric frameworks for large-scale distributed graph processing. ACM Computing Surveys (CSUR) 48(2):1–39
88. Megow N, Meißner J, Skutella M (2017) Randomization helps computing a minimum spanning tree under uncertainty. SIAM Journal on Computing 46(4):1217–1240
89. Mettu RR, Plaxton CG (2003) The online median problem. SIAM Journal on Computing 32(3):816–832
90. Miche Y, Sorjamaa A, Bas P, Simula O, Jutten C, Lendasse A (2009) Op-elm: optimally pruned extreme learning machine. IEEE transactions on neural networks 21(1):158–162
91. Moradi N, Kayvanfar V, Rafiee M (2021) An efficient population-based simulated annealing algorithm for 0–1 knapsack problem. Engineering with Computers pp 1–20
92. Mukherjee AP, Xu P, Tirthapura S (2015) Mining maximal cliques from an uncertain graph. In: 2015 IEEE 31st International Conference on Data Engineering, IEEE, pp 243–254
93. Mukherjee AP, Xu P, Tirthapura S (2016) Enumeration of maximal cliques from an uncertain graph. IEEE Transactions on Knowledge and Data Engineering 29(3):543–555
94. Mukherjee AP, Xu P, Tirthapura S (2017) Enumeration of maximal cliques from an uncertain graph. IEEE Trans Knowl Data Eng 29(3):543–555. https://doi.org/10.1109/TKDE.2016.2527643
95. Parchas P, Gullo F, Papadis D, Bonchi F (2014) The pursuit of a good possible world: extracting representative instances of uncertain graphs. In: Proceedings of the 2014 ACM SIGMOD international conference on management of data, ACM, pp 967–978
96. Parchas P, Gullo F, Papadis D, Bonchi F (2015) Uncertain graph processing through representative instances. ACM Transactions on Database Systems (TODS) 40(3):20
97. Parchas P, Papailiou N, Papadis D, Bonchi F (2018) Uncertain graph sparsification. IEEE Transactions on Knowledge and Data Engineering 30(12):2435–2449
98. Peleg D, Schäffer AA (1989) Graph spanners. Journal of graph theory 13(1):99–116
99. Peng Y, Zhang Y, Zhang W, Lin X, Qin L (2018) Efficient probabilistic k-core computation on uncertain graphs. In: 2018 IEEE 34th International Conference on Data Engineering (ICDE), IEEE, pp 1192–1203
100. Potamias M, Bonchi F, Gionis A, Kollios G (2009) Nearest-neighbor queries in probabilistic graphs. Boston University Computer Science Department, Tech. rep
101. Potamias M, Bonchi F, Gionis A, Kollios G (2010) K-nearest neighbors in uncertain graphs. Proceedings of the VLDB 36(1–2):997–1008
102. Rashid A, Kamran M, Halim Z (2019) A top down approach to enumerate α-maximal cliques in uncertain graphs. Journal of Intelligent & Fuzzy Systems 36(4):3129–3141
103. Roussopoulos N, Kelley S, Vincent F (1995) Nearest neighbor queries. ACM sigmod record, ACM 24:71–79
104. Sadeh G, Cohen E, Kaplan H (2019) Sample complexity bounds for influence maximization. arXiv preprint arXiv:1907.13301
105. Sariyuce AE, Seshadri C, Pinar A, Catalyurek UV (2015) Finding the hierarchy of dense subgraphs using nucleus decompositions. In: Proceedings of the 24th International Conference on World Wide Web, pp 927–937
106. Sasaki Y, Fujiwara Y, Onizuka M (2019) Efficient network reliability computation in uncertain graphs. In: EDBT, pp 337–348
107. Schulz C, Nocaj A, Goertler J, Deussen O, Brandes U, Weiskopf D (2016) Probabilistic graph layout for uncertain network visualization. IEEE transactions on visualization and computer graphics 23(1):531–540
108. Sharara H, Sopan A, Namata G, Getoor L, Singh L (2011) G-pare: a visual analytic tool for comparative analysis of uncertain graphs. In: 2011 IEEE Conference on Visual Analytics Science and Technology (VAST), IEEE, pp 61–70
109. Shi X, Zheng Z, Zhou Y, Jin H, He L, Liu B, Hua QS (2018) Graph processing on gpus: A survey. ACM Computing Surveys (CSUR) 50(6):1–35
110. Spielman DA, Srivastava N (2011) Graph sparsification by effective resistances. SIAM Journal on Computing 40(6):1913–1926
111. Sun Z, Huang X, Xu J, Bonchi F (2021) Efficient probabilistic truss indexing on uncertain graphs. Proceedings of the Web Conference 2021:354–366
112. Taranto C, Di Mauro N, Esposito F (2012) Uncertain (multi) graphs for personalization services in digital libraries. In: Italian Research Conference on Digital Libraries, Springer, pp 141–152
113. Thorup M (2005) Quick k-median, k-center, and facility location for sparse graphs. SIAM Journal on Computing 34(2):405–432
114. Tian Y, Yan J, Hu J, Wu Z (2018) A privacy preserving model in uncertain graph mining. In: 2018 International Conference on Networking and Network Applications (NaNA), IEEE, pp 102–106
115. Westbrook J, Tarjan RE (1992) Maintaining bridge-connected and biconnected components on-line. Algorithmica 7(1–6):433–464
116. Whalen K, Sadkhin B, Davidson D, Gerlt J (2015) Sequence similarity networks for the protein universe. The FASEB Journal 29(1 supplement):573–17
117. Xiang J, Guo C, Aboulnaga A (2013) Scalable maximum clique computation using mapreduce. In: 2013 IEEE 29th International Conference on Data Engineering (ICDE), IEEE, pp 74–85
118. Xiao D, Eltabakh MY, Kong X (2018) Sharing uncertain graphs using syntactic private graph models. In: 2018 IEEE 34th International Conference on Data Engineering (ICDE), IEEE, pp 1336–1339
119. Yalavarthi VK, Ke X, Khan A (2017a) Probabilistic entity resolution with imperfect crowd. CoRR
120. Yalavarthi VK, Ke X, Khan A (2017b) Select your questions wisely: For entity resolution with crowd errors. In: Proceedings of the 2017 ACM on Conference on Information and Knowledge Management, ACM, pp 317–326
121. Yan X, Yu PS, Han J (2005) Substructure similarity search in graph databases. In: Proceedings of the 2005 ACM SIGMOD international conference on Management of data, ACM, pp 766–777
122. Yin X, Han J, Philip SY (2007) Object distinction: Distinguishing objects with identical names. In: 2007 IEEE 23rd International Conference on Data Engineering, IEEE, pp 1242–1246
123. Yuan Y, Wang G, Wang H, Chen L (2011a) Efficient subgraph search over large uncertain graphs. Proc VLDB Endowment 4(11):876–886
124. Yuan Y, Wang G, Wang H, Chen L (2011b) Efficient subgraph search over large uncertain graphs. Proc VLDB Endowment 4(11):876–886
125. Yuan Y, Wang G, Chen L, Wang H (2012) Efficient subgraph similarity search on large probabilistic graph databases. arXiv preprint arXiv:1205.6692
126. Yuan Y, Wang G, Chen L (2014) Pattern match query in a large uncertain graph. In: Proceedings of the 23rd ACM International Conference on Conference on Information and Knowledge Management, pp 519–528
127. Yuan Y, Wang G, Chen L, Wang H (2015) Graph similarity search on large uncertain graph databases. The VLDB Journal 24(2):271–296
128. Yuan Y, Wang G, Chen L, Ning B (2016) Efficient pattern matching on big uncertain graphs. Information Sciences 339:369–394
129. Zeng X, Zhang S, Tang B (2020) Rcelf: A residual-based approach for influence maximization problem. arXiv preprint arXiv:2001.06630
130. Zhang A, Zou Z, Li J, Gao H (2016) Minimum spanning tree on uncertain graphs. In: International Conference on Web Information Systems Engineering, Springer, pp 259–274
131. Zhang C, Zaïane OR (2019) Neighbor-based link prediction with edge uncertainty. In: Advances in Knowledge Discovery and Data Mining - 23rd Pacific-Asia Conference, PAKDD 2019, Macau, China, April 14-17, 2019, Proceedings, Part II, pp 462–474, https://doi.org/10.1007/978-3-030-16145-3_36
132. Zhao B, Wang J, Li M, Wu FX, Pan Y (2014) Detecting protein complexes based on uncertain graph model. IEEE/ACM Transactions on Computational Biology and Bioinformatics (TCBB) 11(3):486–497
133. Zhu K, Zhang W, Zhu G, Zhang Y, Lin X (2011) Bmc: an efficient method to evaluate probabilistic reachability queries. In: International Conference on Database Systems for Advanced Applications, Springer, pp 434–449
134. Zhu R, Zou Z, Li J (2015) Top-k reliability search on uncertain graphs. In: 2015 IEEE International Conference on Data Mining, IEEE, pp 659–668
135. Zhu R, Zou Z, Li J (2016a) Simrank computation on uncertain graphs. In: 2016 IEEE 32nd International Conference on Data Engineering (ICDE), IEEE, pp 565–576
136. Zhu R, Zou Z, Li J (2017) Simrank on uncertain graphs. IEEE Transactions on Knowledge and Data Engineering 29(11):2522–2536

137. Zhu X, Chen W, Zheng W, Ma X (2016b) Gemini: A computation-centric distributed graph processing system. In: 12th [USENIX] Symposium on Operating Systems Design and Implementation (OSDI 16), pp 301–316

138. Zou Z (2013) Polynomial-time algorithm for finding densest subgraphs in uncertain graphs. In: Proceedings of MLG Workshop

139. Zou Z, Zhu R (2017) Truss decomposition of uncertain graphs. Knowledge and Information Systems 50(1):197–230

140. Zou Z, Li J, Gao H, Zhang S (2009) Frequent subgraph pattern mining on uncertain graph data. In: Proceedings of the 18th ACM conference on Information and knowledge management, ACM, pp 583–592

141. Zou Z, Gao H, Li J (2010a) Discovering frequent subgraphs over uncertain graph databases under probabilistic semantics. In: Proceedings of the 16th ACM SIGKDD international conference on Knowledge discovery and data mining, ACM, pp 633–642

142. Zou Z, Li J, Gao H, Zhang S (2010b) Finding top-k maximal cliques in an uncertain graph. In: 2010 IEEE 26th International Conference on Data Engineering (ICDE 2010), IEEE, pp 649–652

143. Zou Z, Li F, Li J, Li Y (2017) Scalable processing of massive uncertain graph data: A simultaneous processing approach. In: 2017 IEEE 33rd International Conference on Data Engineering (ICDE), IEEE, pp 183–186

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