Generalized fuzzy variable precision rough sets based on bisimulations and the corresponding decision-making

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Abstract
Recently, the classical rough set has been extended in many ways. However, some of them are based on binary relations which only excavate “one step” information to distinguish objects. The “one step” in the binary relation means that the ordered pair of the starting and end points of the step belongs to the relation. Faced with some complex data sets, the “one step” information may be not feasible. Motivated by the notion of bisimulation in computer science, three types of bisimulation-based generalized fuzzy variable precision rough set (BGFVPRS) models are constructed. Different from many existed rough set models which are based on binary relations, the BGFVPRS models can distinguish objects by excavating the “multi-step” information of underlying relations. The related properties and relationships of BGFVPRS models are investigated. The uncertainty measure of BGFVPRS models and the reduction of fuzzy bisimulations are also discussed. Furthermore, learning from the PROMETHEE II method and combining it with our presented BGFVPRS models, a novel multiple-attribute decision-making method is provided. This method can effectively deal with complex problems including attribute data and relational data. The flexibility and effectiveness of our decision-making method are illustrated by comparative analysis and sensitivity analysis in the Zachary karate club network.

Keywords Multi-attribute decision-making · Bisimulation · PROMETHEE II method · Relational data · Fuzzy variable precision rough set · Fuzzy logical operator

1 Introduction

Multi-attribute decision-making (MADM) methods are well-known decision-making methods that aim to select the best alternative from the ranking order for all alternatives under the multiple attributes data tables. With the increasing complexity of the research problem, decision-makers (DMs) have to consider the relational data among the studied objects. For example, during the outbreak of a novel coronavirus, experts must consider whether people infected with the virus come into contact with others. In this case, some popular MADM methods only considering the attribute data seem unavailable. In view of this, an approach based on the generalized fuzzy variable precision rough set models with fuzzy bisimulations is presented in this paper. In the following, the development of MADM methods and the classical rough set (RS) model is briefly reviewed, respectively.

The MADM methods are widely used in many fields, such as investment decision-making and project evaluation. The essence of MADM method in [16] is that using the existing decision-making information to sort a group of (finite) schemes or select the optimal object in a certain way. It consists of two parts: obtaining decision information; gathering decision information in a certain way, and ranking the alternatives and selecting the best alternative. In [3, 19, 31, 33, 34, 45], many MADM methods are provided. The outranking approaches are a widely used class of MADM methods. These approaches are taken to identify whether the alternatives under consideration are preferable, neutral, or incomparable with attributes than other alternatives.

It is well known that ELECTRE [49, 59] and PROMETHEE [6, 29] methods are two main methods of outranking approaches. The PROMETHEE method, as a new sorting method in multi-attribute analysis, is stable, simple, and clear. Brans [6] first introduced the concept of PROMETHEE. Hereafter, many scholars developed it in many
The PROMETHEE I and PROMETHEE II methods are two versions of the PROMETHEE method. Through a comparison between them, PROMETHEE II method gets a complete ranking of all alternatives while PROMETHEE I obtains a partial sort of all alternatives. Then, PROMETHEE II method [51] has been widely applied in practical applications. Because of the complexity of the environment in real life, the data tends to be vague or imprecise. It is hard to calculate or rank directly by means of the classical PROMETHEE II method [5] because of the lack of the relational data between objects. For example, these MADM methods in [6, 20, 59, 63, 64] seem invalid for solving social network problems. The reason is that they only consider the attribute data and ignore the relational data between objects.

Figure 1 shows the development of the RS model. For effectively extracting useful information and making a clear decision from much uncertain knowledge in information systems, the RS model was early put forward by Pawlak [40]. Although this model can handle many uncertain problems well, it is hindered by some shortcomings in the development process. For example, the equivalence relation of RS model is too idealistic to be applied in real life. Because most of the relations in real life are general, they cannot always meet the harsh conditions of equivalence relations. At present, RS model is developed from the aspects of universe, equivalence relation, intersection and union, division and so on. The equivalence relations are extended to many forms such as neighborhood relations [32, 53, 55, 58], tolerance relations [18, 26, 43], similarity relations [1, 44], and even arbitrary binary relations [48, 54]. Based on the extensions of equivalence relations, some generalized RS models were born in [62–65]. On the foundation of the predecessor or successor neighborhoods, Yao [55] discussed the significant properties of the lower and upper approximations. Based on the similarity relations, two definitions of rough approximations were compared in [41]. In a relational structure, Fan [14] considered three key concepts of indiscernibility: congruence, bisimulation, and exact equivalence. By these indiscernibility relations, she also investigated rough approximations and knowledge reduction. Concerning a fuzzy relational structure, Du and Zhu took the bisimulation in [36] as indiscernibility and constructed a pair of fuzzy approximation operators in [12]. Zhu et al. pointed out that many generalized rough approximations which are on the basis of binary relations, only rely on “one step” information of potential relations in [68]. The “one step” information could not be effective for discerning objects in some complex environments. Inspired by the concept of bisimulation, they put forward a generalized rough set based on bisimulation (GRSB). Subsequently, Du and Zhu [13] focused on the labeled fuzzy approximation space and took the bisimulation into consideration, and they provided a new pair of approximation operators in the fuzzy rough set based on the largest fuzzy bisimulation (FRSLFB).

A large number of researches on existed RS models find that some of them are sensitive to misclassification and perturbations. In order to make up this defect, the variable precision rough set (VPRS) in [35, 69] was proposed. Since then, many researchers have expanded the VPRS models and applied them in many complicated areas. The measure, which the VPRS is based on, can be written as the form of the conditional probabilities in a decision-theoretic rough set (DTRS) [56]. In this time, the VPRS is an exception to DTRS. The VPRS has been further studied under the fuzzy environment, and some achievements have been made. Combining the fuzzy rough set (FRS) [46] and VPRS [69], a powerful tool named fuzzy variable precision rough set (FVPRS) was constructed in [67]. It cannot only deal with numerical data but also is insensitive to misclassification and disturbance. In 2020, Zhan et al. [62] modified Ma’s fuzzy neighborhood operator and proposed a novel β-neighborhood operator that satisfies the reflexivity. Based on their defined neighborhood operator, a covering-based variable precision fuzzy rough set (CVFPRS) was presented. Meanwhile, Jiang et al. [22] presented a new type of CVFPRS models by means of fuzzy neighborhoods and applied it in the context of medical diagnosis.

From the above discussions, combining the ideas of bisimulation and VPRS models, some types of bisimulations-based...
generalized fuzzy variable precision rough set (BGFVPRS) models are constructed in the paper. Considering PROMETHEE II method and on the basis of BGFVPRS models, we hope to design a decision-making method for solving some complicated problems including not only the attribute data (or relational data) but also the mixed data including both the attribute data and relational data.

Next, the motivations of the paper are briefly introduced as follows.

- The classical PROMETHEE methods (including PROMETHEE I and PROMETHEE II) [5, 6] aim at dealing with the problems which only include the attribute data. However, they are invalid for solving the problems which contain relational data. For example, faced with the social network analysis problems, the classical PROMETHEE II method may be infeasible. In view of this, motivated by the idea of the PROMETHEE II method, an approach is constructed for settling these complex problems.

- Through observations, some MADM methods in [20, 62–64, 66] based on the extension of RS are effective for the multiple-attribute problems. However, faced with the problems involving relational data, they may be impossible to make a feasible decision, or even choose an optimal object. For this reason, a novel decision-making method based on BGFVPRS models and PROMETHEE II method is provided.

- On the basis of bisimulations relations, some approximation operators were constructed in [12–14]. It is worth noting that these models are only deeply studied in theory, and rarely studied in application. Inspired by this, based on our proposed BGFVPRS models, we hope to put forward an approach and apply it in real life.

- Some extended rough set models based on bisimulation [14, 68] and other generalized rough set models are sensitive to disturbance or misclassification, and no errors are allowed. To enrich the theoretical research of the RS models and apply it in wider field, three types of BGFVPRS models are raised. These models are combing the VPRS models, FRS models, and bisimulations that can help DMs extract “multi-step” information.

- Because of the different requirements of different DMs during different periods, the decision-making method is required to have high flexibility. Through the research of some algorithms, we find that they cannot change flexibly with the change of DMs’ preferences. Then, it forces us to search for a flexible decision-making method to help DMs make a clear decision. DMs can adopt different fuzzy logical operators, negator operators, and variable precision values, according to the actual situation and their preferences. It also shows the flexibility of our decision-making method.

Furthermore, the main contribution of the paper is shown as follows.

- Our proposed BGFVPRS models can depend on the “multi-step” information of underlying relationship and can be applied in the complex environment including complex databases.

- By adopting different fuzzy logic operators and changing the value of θ, the decision-making method proposed in the paper can help DMs make a clear decision according to the actual situation and their own preferences.

- Compared with many existing RS models, the BGFVPRS models can be used as a powerful tool. They can solve the relational data or mixed data which contains relational data and attribute data. Besides, they are less sensitive for misclassification and disturbance.

- The method based on BGFVPRS models extends the application range of some classical MADM methods, such as the classical PROMETHEE II method which is ineffective for social network analysis and other relational data issues.

The remainder of the paper is as follows. Some correlated preliminary concepts are given in Sect. 2. Section 3 defines three types of BGFVPRS models and investigates the relationships among these models and some existing RS models. Besides, this section also discusses the related properties of BGFVPRS models. In Sect. 4, the uncertainty measure of BGFVPRS models and the reduction of fuzzy bisimulations are put forward. Section 5 gives a decision-making method based on the BGFVPRS models and the principle of PROMETHEE II. Through an example of selecting the best alternative in Zachary karate club network, from the comparative analysis and sensitivity analysis, the flexibility and effectiveness of our raised method are illustrated. In Sect. 6, a conclusion of our work is made and some further researches are given.

2 Preliminary

Some related preliminary concepts and properties of fuzzy logical operators, fuzzy relations, and fuzzy bisimulations are briefly introduced in this section.

2.1 Fuzzy logical operators

In the paper, S denotes the universe of the discourse, \( \mathcal{F}(S) \) represents the set of all fuzzy sets, and \( \gamma \) indicates a constant fuzzy set in which \( \hat{\gamma}(a) = \gamma \) for each \( a \in S \). Without special explanation, \( S \) is finite.

Firstly, some significant fuzzy logical operators in [41] and the related properties of them are introduced as follows.
Definition 1 [41] If a mapping $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfies the following conditions:

1. $T(\mu_1, \mu_2) = T(\mu_2, \mu_1)$;
2. $T(T(\mu_1, \mu_2), \mu_3) = T(\mu_1, T(\mu_2, \mu_3))$;
3. if $\mu_1 \leq \mu_2, \mu_2 \leq \mu_4$, then $T(\mu_1, \mu_2) \leq T(\mu_3, \mu_4)$;
4. $T(1, \mu_1) = \mu_1$;

for all $\mu_1, \mu_2, \mu_3, \mu_4 \in [0, 1]$, then $T$ is named as a $t$-norm on $[0, 1]$. Three widely known continuous $t$-norms are the min operator $T_\min$, the algebraic product $T_\prod$ and the $\&$ ukasiewicz $t$-norm $T_\&$. where $T_\min(\mu_1, \mu_2) = \mu_1 \wedge \mu_2, T_\prod(\mu_1, \mu_2) = \mu_1 \cdot \mu_2$ and $T_\&(\mu_1, \mu_2) = 0 \vee (\mu_1 + \mu_2 - 1)$ for each $\mu_1, \mu_2 \in [0, 1]$.

Definition 2 [41] Let a mapping $\mathcal{I} : [0, 1] \times [0, 1] \rightarrow [0, 1]$. $\mathcal{I}$ satisfies the following conditions:

1. $\mathcal{I}(\mu_1, \mu_2) = \mathcal{I}(\mu_2, \mu_1)$;
2. $\mathcal{I}(\mathcal{I}(\mu_1, \mu_2), \mu_3) = \mathcal{I}(\mu_1, \mathcal{I}(\mu_2, \mu_3))$;
3. if $\mu_1 \leq \mu_3, \mu_2 \leq \mu_4$, then $\mathcal{I}(\mu_1, \mu_2) \leq \mathcal{I}(\mu_3, \mu_4)$;
4. $\mathcal{I}(0, \mu_1) = \mu_1$;

for any $\mu_1, \mu_2, \mu_3, \mu_4 \in [0, 1]$, then $\mathcal{I}$ is called an $S$-norm on $[0, 1]$. The three most popular $S$-norms are the standard max operator $\mathcal{L}_\max$, the probabilistic sum $\mathcal{L}_\prod$ and the bounded sum $\mathcal{L}_\&$ where $\mathcal{L}_\max(\mu_1, \mu_2) = \mu_1 \cup \mu_2, \mathcal{L}_\prod(\mu_1, \mu_2) = \mu_1 \cdot \mu_2$ and $\mathcal{L}_\&(\mu_1, \mu_2) = 1 \wedge (\mu_1 + \mu_2 - 1)$ for each $\mu_1, \mu_2 \in [0, 1]$. It is worth noting that when $T$-norm and $S$-norm are two continuous functions on $[0, 1]$, they are called the continuous $T$-norm and continuous $S$-norm, respectively.

Definition 3 [41] Suppose that $\mathcal{N} : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a decreasing mapping. If $\mathcal{N}$ meets the condition that $\mathcal{N}(1) = 0$ and $\mathcal{N}(0) = 1$, then it is a negator operator. For each $\mu_1 \in [0, 1]$, $\mathcal{N}$ is a standard negator operator if $\mathcal{N}(\mu_1) = 1 - \mu_1$. In the following, $\mathcal{N}$ is used to represent the standard negator operator. Furthermore, $\mathcal{N}$ is involutive when $\mathcal{N}(\mathcal{N}(\mu_1)) = \mu_1$.

In the paper, using symbol $\text{co}_\mathcal{N}$ denotes the fuzzy complement. In other words, $\text{co}_\mathcal{N}(K(t_1)) = \mathcal{N}(K(t_1))$ for each $K \in \mathcal{F}(S), t_1 \in S$.

Definition 4 [41] Given a mapping $I : [0, 1] \times [0, 1] \rightarrow [0, 1], if I contents the requirement that $I(0, 0) = I(0, 1) = I(1, 1) = 1$ as well as $I(1, 1) = 0$, then it is referred as a fuzzy implicator operator.

A border implicator is an implicator $I$ if it satisfies $I(1, \mu_1) = 1$, for each $\mu_1 \in [0, 1]$ For any $\mu_1 \in [0, 1]$, if implicator $I$ meets that $I(\cdot, \mu_1)$ is decreasing (resp. $I(\mu_1, \cdot)$ is increasing), $I$ is left monotonic (resp. right monotonic). An implicator $I$ is hybrid monotonic if it is both left monotonic and right monotonic.

Three most popular fuzzy implicators are $R$-implicator, $S$-implicator and $QL$-implicator in [41]. An $R$-implicator $I$ based on a continuous $t$-norm $T$ iff for each $\mu_1, \mu_2 \in [0, 1]$, $I(\mu_1, \mu_2) = \sup \{ \gamma \in [0, 1] : T(\mu_1, \gamma) \leq \mu_2 \}$. An $S$-implicator $I$ based on $T, \mathcal{L}$ and $\mathcal{N}$ iff $I(\mu_1, \mu_2) = \mathcal{L}(\mathcal{N}(\mu_1), \mu_2)$ for every $\mu_1, \mu_2 \in [0, 1]$. A $QL$-implicator $I$ based on $T, \mathcal{L}$ and $\mathcal{N}$ iff for each $\mu_1, \mu_2 \in [0, 1], I(\mu_1, \mu_2) = \mathcal{L}(\mathcal{N}(\mu_1), T(\mu_1, \mu_2))$.

2.2 Fuzzy relations

In this section, some important concepts and properties of fuzzy relations are introduced. Firstly, we show the concept of a binary fuzzy relation and the corresponding properties. A binary relation is a subset of $S \times S$ where $S \times S$ is the product set of $S$ and $S$. The binary fuzzy relation is the natural extension of the binary relation. Compared with binary relation, the application of binary fuzzy relation is wider. With the help of binary fuzzy relation, some problems in a fuzzy environment can be solved well.

Definition 5 [24, 39] A binary fuzzy relation is a mapping $\psi : S \times S \rightarrow [0, 1]$. The binary fuzzy relation is called:

1. reflexive if for each $t_1 \in S, \psi(t_1, t_1) = 1$;
2. symmetric if for each $t_1, t_2 \in S, \psi(t_1, t_2) = \psi(t_2, t_1)$;
3. $T$-transitive if for each $t_1, t_2, t_3 \in S, T(\psi(t_1, t_2), \psi(t_2, t_3)) \leq \psi(t_1, t_3)$.

For each $t, z \in S$, when $\psi(t, z) = 1$ or $\psi(t, z) = 0$, the binary fuzzy relation is a binary relation. In other words, the binary relation is a special case of the binary fuzzy relation. In Definition 5, the $\psi$ is a $T$-similarity relation if it satisfies the conditions (1), (2), and (3). If $T = \&$ and $\psi$ satisfies the condition (3), then $\psi$ is transitive. In this case, if $\psi$ meets the above three conditions, $\psi$ is a fuzzy equivalence relation.

The $n$-ary fuzzy relation is a natural extension of the binary fuzzy relation. A fuzzy set $S_1 \times S_2 \times \ldots \times S_n$ is named as $n$-ary fuzzy relation between $S_1, S_2, \ldots, S_n$. If $S_1 = S_2 = \ldots = S_n = S, S_1 \times S_2 \times \ldots \times S_n$ is an $n$-ary fuzzy relation on a set $S$. For simplicity, the set of all $n$-ary fuzzy relation on $S$ is denoted by $S^n$. In the paper, the $n$-ary fuzzy relation is called the fuzzy relation.

In [9, 25], the composition of binary fuzzy relations is given. For non-empty sets $S_1, S_2, S_3$ and two binary fuzzy relations $\psi_1 \in \mathcal{F}(S_1 \times S_2), \psi_2 \in \mathcal{F}(S_2 \times S_3)$, their composition $\psi_1 \circ \psi_2$ is a binary fuzzy relation from $\mathcal{F}(S_1 \times S_3)$ defined by $(\psi_1 \circ \psi_2)(a, c) = \bigvee_{b \in S_2} \psi_1(a, b) \otimes \psi_2(b, c)$, where $\otimes$ (called multiplication) is applied to model the conjunction. Subsequently, the composition of $n$-ary fuzzy relations is provided.
Assume that $\psi_1$ is a $(h_1 + h)$-ary fuzzy relation and $\psi_2$ is a $(h + h_2)$-ary fuzzy relation where $\psi_1 \in S^{(h_1 + h)}$ and $\psi_2 \in S^{(h + h_2)}$. Then, the $h$-composition of $\psi_1$ and $\psi_2$ is a $(h_1 + h_2)$-ary fuzzy relation defined as:

$$ (\psi_1 \circ_h \psi_2)(a, c) = \bigvee_{b=(b_1, b_2)} \psi_1(a, b) \otimes \psi_2(b, c), $$

for each $(a, c) \in S^{h_1 + h_2}$ where $a = (a_1, a_2, \ldots, a_{h_1})$ and $c = (c_1, c_2, \ldots, c_{h_2})$. Especially, if $h=1$, the subscript is omitted and the formula is written as $\psi_1 \circ \psi_2$.

### 2.3 Fuzzy bisimulations

Before the introduction of fuzzy bisimulation, we firstly bring into the concept of bisimulation. The fuzzy-language equivalence is a tool to compare the behavior of fuzzy systems. However, at this level, the comparison is too coarse. By means of this tool, the behavior of some fuzzy systems cannot be compared well. In view of this, bisimulation as a finer behavior measure has been introduced in fuzzy-finite automata. Cao et al. [7] considered this bisimulation for general fuzzy systems and obtained some results which are useful to compare the behavior of these systems. In the following, the concept of bisimulation defined in [68] is introduced.

**Definition 6** [68] Assume that $(S, R)$ is a generalized approximation space. A binary relation $B \subseteq S \times S$ is called a bisimulation if for all $(z, w) \in B$,

1. $(z, z') \in R$ implies that $(w, w') \in R$ for some $w \in S$ with $(z', w) \in B$,
2. $(w, w') \in R$ implies that $(z, z') \in R$ for some $w' \in S$ with $(z', w') \in B$.

If $B$ is a bisimulation and $(z, w) \in B$, then states $z$ and $w$ are bisimilar, denoted by $z \sim w$. In this paper, the relation $\sim$ is named as bisimilarity.

A bisimulation is a binary relation between discrete event systems. It relates systems with the same behavior. That is, one system simulates another, and vice versa. Intuitively, if the actions of the two systems match, they are bisimilar. In order to understand this relation well, an example is given as follows.

**Example 1** Figure 2 represents a generalized approximation space where $S = \{s_1, s_2, s_3, s_4, t_1, t_2, t_3\}$ and the solid arrows represent the relation $R$. Assume that a binary relation $R_1 = \{(s_2, t_2), (s_4, t_3)\}$. From the Fig. 2, we can find that when the state $s_2$ moves one step, then the state $t_2$ moves one step. Similarly, when the state $t_2$ moves one step, the state $s_2$ also moves one step. After moving one step, the subsequent states of $s_2$ and $t_2$ are $s_4$ and $t_3$, respectively. In this time, $(s_4, t_3) \in R_1$. Furthermore, state $s_4$ stops when state $t_3$ stops, and vice versa. Then, the relation $R_1$ is called a bisimulation.

Compared with some binary relations which only obtain “one step” information, the bisimulation can excavate “multi-step” information. It is worth noting that the “one step” information could be not effective for discerning objects in some complex environments. In the following, an example is given to illustrate this. This example is adapted from [68].

**Example 2** Assume that two reactive systems as two black boxes with one red button each, shown as Fig. 3. Through pressing the buttons, the interaction between persons and black boxes has been done. Sometimes the red button goes down, and sometimes it does not. The button going down means the persons succeed while the button not going down means the persons don’t succeed. By this way, the difference between the two black boxes can be told. Assume that the button of the first black box can successively go down two times from its initial state and after that, it does not continue; for the second one, it can always go down. The right state transition graph in Fig. 3 shows their behaviors.
Given a generalized approximation space \((S, R)\) where \(S = \{a, b, c, d\}\), \(R = \{(a, b), (b, c), (d, d)\}\). Assume that the successor neighborhood \(R(z) = \{w \in S \mid (z, w) \in R\}\) of \(z\). Furthermore, let us approximate a concept \(W = \{a, d\}\). Using the approximation operators based on the successor neighborhoods, we have that
\[
apr_{R_i}(W) = \{z \in S \mid R(z) \subseteq W = \{d\}\},
\]
\[
\overline{apr}_{R_i}(W) = \{z \in S \mid R(z) \cap W \neq \emptyset\} = \{d\}.
\]

It is known to us that the basic idea of rough set is that using a pair precise concept to describe the incomplete or inexact concepts. Through our research, the four states are different from one another. For example, in state \(a\) the left red button can successively go down two times, while in state \(d\) the right red button can only go down one time. Therefore, each state should be regarded as a granule of knowledge. In view of this, approximating the concept \(W\) by the union of two knowledge granules \([a]\) and \([d]\) i.e., \([a, d]\) is much better than using \([d]\) and \([d]\).

But, considering the approximations based on bisimilarity \(\sim\) which is the largest bisimulation, we have that:
\[
\sim = \{(a, a), (b, b), (c, c), (d, d)\}, \quad \Pi = \{y \in S \mid (t, y) \in \sim\},
\]
\[
apr_{\sim}(W) = \{t \in S \mid \Pi \subseteq W = \{a, d\}\},
\]
\[
\overline{apr}_{\sim}(W) = \{t \in S \mid \Pi \cap W \neq \emptyset\} = \{a, d\}.
\]

In this case, by the basic idea of rough set, we should use \([a, d]\) to describe the concept \(W\). Through the comparison between the two rough models, we find that the results of the second model does follow our intuition. This shows that the bisimulation plays an important role in some situations.

However, the bisimulation just can distinguish whether two states belong to bisimulation relation. If the relation between states is vague or fuzzy, it seems infeasible. Then, extending this relation in a fuzzy environment is necessary. In the following, a concept of fuzzy bisimulation which is the natural extension of bisimulation is given.

According to [14], a fuzzy relational structure is a binary tuple \((S, (R_i)_{i \in \Lambda})\) where \(S\) is a universe, \(\Lambda\) is an index set and \(R_i\) is a fuzzy relation on \(S\). Based on this structure, the concept of fuzzy bisimulation is defined as follows.

**Definition 7** [12] Assume that \((S, (R_i)_{i \in \Lambda})\) is a fuzzy relational structure. A binary fuzzy relation \(\psi\) is referred to as a fuzzy bisimulation on \(S\), for every \((R_i)_{i \in \Lambda}\) if it meets the following conditions:

1. \(\psi^{-1} \circ R_i \subseteq R_i \circ \psi^{-1}\),
2. \(\psi \circ R_i \subseteq R_i \circ \psi\),

where \(\psi\) is the reverse of the fuzzy relation \(\psi\). For each \(s_1, s_2 \in S\), \(\psi^{-1}(s_1, s_2) = \psi(s_2, s_1)\).

Note that if a fuzzy bisimulation \(\varphi\) satisfies the condition (1) in Definition 5, it is a reflexive fuzzy bisimulation. If \(\varphi\) meets the condition (2) in Definition 5, \(\varphi\) is called a symmetric fuzzy bisimulation. When \(\varphi\) fits the condition (3) in Definition 5, \(\varphi\) is a \(T\)-transitive fuzzy bisimulation. Especially, if \(\varphi\) meets the conditions (1), (2), and (3) in Definition 5, \(\varphi\) is a fuzzy \(T\)-similarity bisimulation relation. Similarly, if \(\varphi\) is reflective, symmetric, and transitive, it is a fuzzy equivalence bisimulation relation. The largest bisimulation [36] is an equivalence relation which is a special fuzzy equivalence bisimulation relation.

For better understand the concept of the fuzzy bisimulation, an example is given as follows.

**Example 3** Let \((S, R_1 \cup \overline{R_2})\) be a fuzzy relational structure and \(\emptyset = \emptyset\). Here \(S = \{t_0, t_1\}\) and \(R_1 = \{0.5, 0.5\} + \{0.5, 0.5\}\) and \(R_2 = \{0.8, 0.6\} + \{0.5, 0.5\} + \{0.5, 0.5\}\) and \(0.5\) = \{0.5, 0.5\} + \{0.5, 0.5\}\). Assume that a binary fuzzy relation \(\varphi = \{0.5, 0.5\} + \{0.5, 0.5\} + \{0.5, 0.5\}\). By Definition 7, it is easy to verify that \(\varphi\) is a fuzzy bisimulation.

The bisimulation is a special case of fuzzy bisimulation. If the fuzzy bisimulation \(\varphi\) is a bisimulation on \(S \times S\), \(\varphi(a, b) \in \{0, 1\}\) for each \(a, b \in S\). Compared with bisimulation, the application of fuzzy bisimulation is wider than the bisimulation. For example, if the relation between states of objects is vague, the bisimulation seems infeasible. But, the fuzzy bisimulation is still effective. In a word, the fuzzy bisimulation can be applied for dealing with vague relations in a fuzzy environment while the bisimulation cannot.

### 3 Generalized fuzzy variable precision rough sets based on bisimulations

The fuzzy logic operator [41] plays an important role in fuzzy set theory. In order to overcome some defects of the classical RS model, Radzikowska and Kerre proposed a general FRS model [41] based on the fuzzy logical operators. However, the FRS models and the classical RS model [40] are easily disturbed by noise data sets. To make up for this shortcoming, the VP model [69] is proposed. Based on logical operators, some extensions of FVPRS models [20, 21, 62, 67] are constructed. Their work further improves the theoretical basis of classical RS model. A study shows that these models are based on the binary relations which are only dependent on “one step” information. In the face of some more complex data sets, it is not
enough to distinguish objects only by “one step” information. Therefore, we need an effective tools to obtain “multi-step” information for distinguishing objects. The fuzzy bisimulations can excavate “multi-step” information for underlying relations. By means of the fuzzy bisimulations, objects can be well distinguished. At present, there is rare research on RS models based on fuzzy bisimulations. Then, we want to develop the RS model by means of these relations.

To improve the RS model, enrich its theoretical research, and make it applicable to a wider range of fields, we want to present some generalized RS models. Motivated by fuzzy bisimulations and the concept of VP models, three types of BGFVPRS models are defined in this section. The relationships among these BGFVPRS models and other types of existing RS models are investigated. Besides, some feasible properties of these BGFVPRS models are discussed. In the paper, since some properties are easy to obtain, then we omit some trivial proof.

Definition 8 Assume \((S, (R_i)_{i \in A})\) is a fuzzy relational structure and \(\varphi\) is a fuzzy bisimulation. Suppose that \(I_1, I_2\) are two fuzzy implicators and \(T_1, T_2\) are two continuous \(t\)-norms. For each \(K \in \mathcal{F}(S)\), the first type of bisimulation-based generalized fuzzy variable precision \(I_1, I_2\)-lower approximation \((1-\text{BGFVPRSI}_1, I_2)\) \(\text{apr}^{I_1, I_2}(K)\) and \(I_1, I_2\)-upper approximation \((1-\text{BGFVPRSI}_1, I_2)\) \(\text{apr}^{I_1, I_2}(K)\) of \(K\), with negator \(\mathcal{N}\) and the variable precision \(\theta \in [0, 1]\), are defined as:

\[
\text{apr}^{I_1, I_2}(K)(a) = \bigvee_{b \in S} T_2(\varphi(b, a), \bigwedge_{c \in S} I_1(\varphi(b, c), \theta \lor K(c))),
\]

(1)

\[
\overline{\text{apr}}^{I_1, I_2}(K)(a) = \bigvee_{b \in S} \bigwedge_{c \in S} I_1(\varphi(b, c), \mathcal{N}(\theta) \land K(c))\bigvee_{c \in S} T_1(\varphi(b, c), N(\theta) \land K(c)).
\]

(2)

If \(\text{apr}^{I_1, I_2}(K) \neq \overline{\text{apr}}^{I_1, I_2}(K)\), we call \(K\) the first type of bisimulation-based generalized fuzzy variable precision rough set \((1-\text{BGFVPRS})\), otherwise we call it definable.

To better understand Definition 8, an example is given to illustrate it in the following. It is noted that the labeled (or unlabeled) fuzzy transition systems [23] are special fuzzy relational structures.

**Example 4** Let \((S, R)\) be a fuzzy relational structure. Assume that \(S_1 \subseteq S\), and \(S_1 = \{t_0, t_1, t_2, t_3, t_4\}\). Figure 4a is an unlabeled transition system. In the figure, \(t_1 \rightarrow t_2\) represents \(R(t_1, t_2) = \alpha\) where \((i, j) = 0, 1, 2, 3, 4\) and \(\alpha \in [0, 1]\). Here, \(R(t_1, t_2) = \alpha\) denotes that the membership degree of \(t_1\) and \(t_2\) belonging to the binary fuzzy relation \(R\) is \(\alpha\). Assume that \(\varphi\) is a binary fuzzy relation where \(\varphi(t_1, t_2) = 1\) and \(\varphi(t_1, t_3) = 0\) \((l \neq j)\).

**Fig. 4** a Is a fuzzy transition system and b is a social network for 5 persons

According to Fig. 4a, \(R = \{(t_0, t_1), (t_0, t_2), (t_0, t_3), (t_0, t_4), (t_1, t_2), (t_1, t_3), (t_1, t_4), (t_2, t_3), (t_2, t_4), (t_3, t_4)\}\) can be got. Let \(\mathcal{G} = \land\). Since \(\varphi \land R \subseteq R \circ \varphi\) and \(\varphi \land R \subseteq R \circ \varphi\), then \(\varphi\) is a fuzzy bisimulation by Definition 7. Furthermore, assume that \(I_1(k_1, k_2) = \min(1, 1 - k_1 + k_2)\), for each \(k_1, k_2 \in [0, 1]\). Let \(I_1 = I_2 = I_1, T_1 = T_2 = T_3 = T_4\) and \(\mathcal{N} = \mathcal{N}_1\). Suppose that \(\theta = 0.6\) and \(K = 0.1 + 0.3 + 0.3 + 0.5 + 0.6 + 0.8 + 0.3 + 0.3\), and by Definition 8, the following results are got:

\[
\text{apr}^{I_1, T_1}(K) = \frac{0.8}{t_0} + \frac{0.6}{t_1} + \frac{0.6}{t_2} + \frac{0.6}{t_3} + \frac{0.3}{t_4},
\]

\[
\overline{\text{apr}}^{I_1, T_1}(K) = \frac{0.4}{t_0} + \frac{0.3}{t_1} + \frac{0.3}{t_2} + \frac{0.4}{t_3} + \frac{0.4}{t_4}.
\]

Thus \(K\) is a 1-BGFVPRS.

In the following, another example is given to help us understand Definition 8 better.

**Example 5** The center for disease control and prevention gets a message that a man has a novel coronavirus pneumonia test and the result of the test is positive. The institution immediately finds that the person \(t_1\) who had dinner together John H. recently. A social network map for 5 persons is shown as Fig. 4b. In the figure, \(t_i \rightarrow t_j\) means the contact degree between person \(t_i\) and \(t_j\) is \(\alpha \in [0, 1]\) where \((i, j) = 1, 2, 3, 4, 5\). Here, \(t_i \rightarrow t_j\) is equivalent to \(R(t_i, t_j) = \alpha\). Note that the contact relationship is bidirectional or symmetric. Then the binary fuzzy relation \(R\) is symmetric. Even though the first testing results of 5 persons are negative, the experts need to observe and treat them further. Through observation, they obtain an assessment result as \(K = \frac{0.75}{t_1} + \frac{0.1}{t_2} + \frac{0.3}{t_3} + \frac{0.35}{t_4} + \frac{0.1}{t_5}\). The value of \(K(t_i)\) is the
possibility of infection for person \( t_i \). Based on experience, experts choose a regulatory threshold \( \theta = 0.6 \), a fuzzy relation \( \varphi = R \), and \( \otimes = \wedge \). The \( \varphi \) is a fuzzy bisimulation by Definition 7. According to experts’ preferences and the complexity of the environment, they choose \( I_1 = I_2 = I_1, T_1 = T_2 = T_1, N = N_1 \). Then, by Definition 8, some results can be obtained as follows:

\[
\overline{apr}_{I_1, I_2}^{0.4, 0.6}(K) = \frac{0.75}{t_1} + \frac{0.4}{t_2} + \frac{0.6}{t_3} + \frac{0.6}{t_4} + \frac{0.8}{t_5},
\]

\[
\overline{apr}_{T_1, T_2}^{0.4, 0.6}(K) = \frac{0.4}{t_1} + \frac{0.65}{t_2} + \frac{0.4}{t_3} + \frac{0.4}{t_4} + \frac{0.4}{t_5}.
\]

Here \( K \) is a 1-BGFVPRS due to \( \overline{apr}_{I_1, I_2}^{0.4, 0.6}(K) \neq \overline{apr}_{T_1, T_2}^{0.4, 0.6}(K) \).

The institution uses the comprehensive disease degree of person \( t_i \). Furthermore, according to the comprehensive disease degree of each person, \( t_5 > t_1 > t_2 > t_3 = t_4 \) is obtained. Thus, \( t_5 \) is most likely to get sick and should be treated further.

Furthermore, the relations among some existed RS models in \([10, 20, 27, 41, 55, 58, 62, 63, 65]\) and 1-BGFVPRS model are discussed as follows.

**Remark 1**

1. When \( \theta = 0 \), formulas (1) and (2) become the following forms:

\[
\overline{apr}_{I_1, I_2}^{0, 0}(K)(a) = \bigvee_{b \in S} T_2(\varphi(b, a), \bigwedge_{c \in S} I_1(\varphi(b, c), K(c))),
\]

\[
\overline{apr}_{T_1, T_2}^{0, 0}(K)(a) = \bigvee_{b \in S} I_2(\varphi(b, a), \bigvee_{c \in S} T_1(\varphi(b, c), K(c))).
\]

2. When \( \theta \neq 0, \) if for each \( c \in S, \theta \leq K(c) \leq N(\theta) \), the following model is obtained:

\[
\overline{apr}_{I_1, I_2}^{\theta, 0}(K)(a) = \bigvee_{b \in S} T_2(\varphi(b, a), \bigwedge_{c \in S} I_1(\varphi(b, c), K(c))),
\]

\[
\overline{apr}_{T_1, T_2}^{\theta, 0}(K)(a) = \bigvee_{b \in S} I_2(\varphi(b, a), \bigvee_{c \in S} T_1(\varphi(b, c), K(c))).
\]

(i) Since the fuzzy bisimulation \( \varphi \) is a special fuzzy relation which can extract “multi-step” information, then formulas (3) and (4) become the models defined in \([10]\). Formulas (3) and (4) can help DMs solve more complex uncertain problems that some RS models in \([55, 58, 62, 63]\) cannot solve.

(ii) If \( \varphi \) is a fuzzy \( \beta \)-neighborhood \( \tilde{N} \), the above model turns into the model defined in \([63]\):

\[
\overline{apr}_{I_1, I_2}^{\beta, 0}(N)(a) = \bigvee_{b \in S} T_2(\tilde{N}(b, a), \bigwedge_{c \in S} I_1(\tilde{N}(b, c), K(c))),
\]

\[
\overline{apr}_{T_1, T_2}^{\beta, 0}(N)(a) = \bigvee_{b \in S} I_2(\tilde{N}(b, a), \bigwedge_{c \in S} T_1(\tilde{N}(b, c), K(c))).
\]

(iii) Let \( \varphi(b, a) = \varphi_0(a) \). If for each \( a \in S \), there at least exist one element \( t \in S \) s.t. \( \varphi_0(t) = 1 \), i.e., \( S \) forms a fuzzy covering, models (5) and (6) boil down the models built in \([27]\).

(iv) If \( \varphi \) is a \( T \)-similarity relation on \( S \) and \( I_1, I_2 \) are two \( R \)-implicators, formulas (3) and (4) degenerate into the models in \([41]\). Note that with the condition given by \([63]\), the two pairs of generalized approximation operators in \([63]\) cannot degenerate into the models defined in \([41]\). But, if the fuzzy relation is a \( T \)-similarity relation, both pairs operators in \([63]\) degenerate into the models in \([41]\).

The reference \([65]\) gives examples and proof to show why the equality relationship in Theorem 5.3 in \([27]\) does not hold. But it does not give the condition for the equality relationship in Theorem 5.3 in \([27]\). Here we give the condition of the equality relationship in the theorem. The condition is that if the fuzzy relation is a \( T \)-similarity relation.

The proof is easy to prove and thus we omit it.

(v) Assume that \( I_1, I_2 \) are \( R \)-implicators and \( T_1, T_2 \) are continuous \( t \)-norms while \( \varphi \) is a fuzzy equivalence relation. If \( \varphi \) is a fuzzy \( \beta \)-neighborhood, formulas (3) and (4) degenerate into the models defined in \([20]\).

(2) When \( \theta \neq 0 \), if for each \( c \in S, \theta \leq K(c) \leq N(\theta) \), the following model is obtained:

\[
\overline{apr}_{I_1, I_2}^{\theta, 0}(K)(a) = \bigvee_{b \in S} T_2(\varphi(b, a), \bigwedge_{c \in S} I_1(\varphi(b, c), K(c))),
\]

\[
\overline{apr}_{T_1, T_2}^{\theta, 0}(K)(a) = \bigvee_{b \in S} I_2(\varphi(b, a), \bigvee_{c \in S} T_1(\varphi(b, c), K(c))).
\]

(i) When \( \varphi \) is replaced by the covering-neighborhood defined in \([63]\), formulas (7) and (8) turn into the models provided in \([63]\).

(ii) If \( \varphi \) is a general fuzzy relation, formulas (7) and (8) become the formulas defined in \([10]\).

(3) If \( \theta \neq 0, \) \( \varphi \) is a fuzzy equivalence relation, using the new fuzzy \( \beta \)-neighborhood defined in \([62]\) to substitute \( \varphi \), then formulas (1) and (2) turn into the formulas built in \([62]\).

(4) If \( K \) is a crisp set, formule (1) and (2) become the following forms:

\[
\overline{apr}_{I_1, I_2}^{\theta, 0}(K)(a) = \bigvee_{b \in S} T_2(\varphi(b, a), \bigwedge_{c \in S, c \notin K} I_1(\varphi(b, c), \theta)),
\]

\[
\overline{apr}_{T_1, T_2}^{\theta, 0}(K)(a) = \bigvee_{b \in S} I_2(\varphi(b, a), \bigvee_{c \in S, c \notin K} T_1(\varphi(b, c), N(\theta)))).
\]

Furthermore, another type of BGFVPRS model is defined in the following.

**Definition 9** Under the condition of Definition 8, for each \( K \in F(S) \), the second type of bisimulation-based generalized...
fuzzy variable precision \(I_1I_2\)-lower approximation (2-BGFVP) \(I_3\) and \(T_1T_2\)-upper approximation (2-BGFVPT) \(T_3\) of \(K\), with negator \(N\) and the variable precision \(\theta \in [0, 1]\), are defined as:

\[
apr_{I_1I_2}^\theta(K)(a) = \bigcap_{b \in S} I_2(\varphi(b, a)) \cap I_1(\varphi(b, c), \theta \lor K(c)),
\]

\[
apr_{T_1T_2}^\theta(K)(a) = \bigcup_{b \in S} T_2(\varphi(b, a)) \cup T_1(\varphi(b, c), \theta \lor K(c)).
\]  

If \(apr_{I_1I_2}^\theta(K) \neq apr_{T_1T_2}^\theta(K)\), we call \(K\) the second type of bisimulation-based generalized fuzzy variable precision rough set (2-BGFVPRS), otherwise we call it definable.

Example 6 An unlabeled fuzzy transition system is shown as Fig. 5 where \(S_1 = \{t_0, t_1\} \subseteq S\). From this figure, we have that 
\(R = \frac{0.5}{0.5} + \frac{0.3}{0.3} + \frac{0.3}{0.3} + \frac{0.5}{0.5}\). Assume that \(\varphi\) is a binary fuzzy relation in which \(\varphi = \frac{0.5}{0.5} + \frac{0.3}{0.3} + \frac{0.3}{0.3} + \frac{0.5}{0.5}\). When \(\varnothing = \land\), we have that \(\varphi^* = R \subseteq R \varphi = \varphi R \subseteq R \varphi\). Then \(\varphi\) is a fuzzy bisimulation. Furthermore, suppose that \(I_1 = I_2 = I_3\), \(T_1 = T_2 = T_3\), \(N = N_\theta, \theta = 0.58\) and \(K = \frac{0.8}{0.8} + \frac{0.8}{0.8}\). By Definitions 8 and 9, some results will be obtained as follows:

\[
apr^{0.58}_{I_1I_2}(K) = apr^{0.58}_{I_1I_2}(K) = \frac{0.8}{0.8} + \frac{0.58}{0.58},
\]

\[
apr^{0.58}_{T_1T_2}(K) = apr^{0.58}_{T_1T_2}(K) = \frac{0.42}{0.42} + \frac{0.42}{0.42}.
\]

Obviously, \(K\) is both 1-BGFVPRS and 2-BGFVPRS.

Let \(I_{KD}(k_1, k_2) = \max(1 - k_1, k_2)\), for each \(k_1, k_2 \in [0, 1]\). When \(t_1 = t_2 = I_{KD}, T_1 = T_2 = T_1\), a comparison between 1-BGFVPRS and 2-BGFVPRS is given as Table 1.

\[
\text{Table 1 A comparison between 1-BGFVPRS and 2-BGFVPRS}
\]

| \(apr^{0.58}_{I_1I_2}(K)\) | \(apr^{0.58}_{I_1I_3}(K)\) | \(apr^{0.58}_{T_1I_2}(K)\) | \(apr^{0.58}_{T_1I_3}(K)\) |
|--------------------------|--------------------------|--------------------------|--------------------------|
| \(t_0\) | 0.58 | 0.7 | 0.42 | 0.42 |
| \(t_1\) | 0.58 | 0.58 | 0.42 | 0.42 |

Through Table 1, we find that \(apr^{0.58}_{I_1I_2}(K) \subseteq apr^{0.58}_{I_1I_2}(K), apr^{0.58}_{T_1I_2}(K) = apr^{0.58}_{T_1I_2}(K)\). The relationship between 1-BGFVPRS and 2-BGFVPRS will be investigated deeply later, and thus it is not discussed here.

Besides, the the relations among some existing RS models in [27, 41, 50, 62, 63] and 2-BGFVPRS models are discussed as follows.

Remark 2 (1) When \(\theta = 0\), formulas (11) and (12) will be the following forms:

\[
apr_{I_1I_2}^\theta(K)(a) = \bigcap_{b \in S} I_2(\varphi(b, a)) \cap I_1(\varphi(b, c), K(c)),
\]

\[
apr_{T_1I_2}^\theta(K)(a) = \bigcup_{b \in S} T_2(\varphi(b, a)) \cup T_1(\varphi(b, c), K(c)).
\]

(i) If \(I_1 = I_2\) is an \(R\)-implicator, \(T_1 = T_2\) is a continuous \(t\)-norm, \(\varphi\) is replaced by the fuzzy covering-based fuzzy neighborhood operator defined in [63], formula (13) and (14) turn into the models defined in [63].

(ii) If \(I_1 = I_2\) is an \(R\)-implicator, \(T_1 = T_2\) is a continuous \(t\)-norm, \(\varphi\) is a fuzzy equivalence relation, models (13) and (14) degenerate into the models built in [41, 50].

(iii) Let \(\varphi\) be a successor fuzzy relation. For each \(a, b \in S\), using \(\varphi_{\theta}(a)\) denotes \(\varphi(b, a)\). For each \(b \in S\), if there at least exists one element \(t \in S\) s.t. \(\varphi_{\theta}(t) = 1\) i.e., \(S\) forms a fuzzy covering. In this situation, formulae (13) and (13sps14) boil down the models built in [27].

(2) When \(\theta \neq 0\), for each \(c \in S, \theta \leq K(c) \leq N(\theta)\), the following model is obtained:

\[
apr_{I_1I_2}^\theta(K)(a) = \bigcap_{b \in S} I_2(\varphi(b, a)) \cap I_1(\varphi(b, c), K(c)),
\]

\[
apr_{T_1I_2}^\theta(K)(a) = \bigcup_{b \in S} T_2(\varphi(b, a)) \cup T_1(\varphi(b, c), K(c)).
\]

Especially, when \(\varphi\) is the covering-neighborhood defined in [63], formulas (15) and (16) become the models defined in [63].
(3) Suppose that \( \theta \neq 0, \phi \) is a fuzzy equivalence relation, \( I_1, I_2 \) are \( R \)-implicators and \( T_1, T_2 \) are continuous \( t \)-norms. If using the new fuzzy \( \beta \)-neighborhood defined in [62] to substitute \( \varphi \), then formulas (11) and (12) turn into the models built in [62].

(4) If \( K \) is a crisp set, then formulae (11) and (12) become the following forms:

\[
\text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K)(a) = \bigvee_{b \in S} I_2(\varphi(b, a), \bigwedge_{c \in S, c \in K} I_1(\varphi(b, c), \theta), \theta)), \\
\text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K)(a) = \bigvee_{b \in S} I_2(\varphi(b, a), \bigwedge_{c \in S, c \in K} I_1(\varphi(b, c), \theta(N(\theta)))).
\]

(17) \hspace{1cm} (18)

To better grasp 1-BGFVPRS and 2-BGFVPRS models, the related axiomatic properties of them are investigated in the following.

**Proposition 1** For the approximation operators of 1-BGFVPRS and 2-BGFVPRS models, some important properties are listed as follows.

1. \( \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(S) = S \), if \( \varphi \) is a reflexive fuzzy bisimulation and \( I_1 \) is a border implicator which is left monotonic.
2. \( \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(S) = S \), when \( \varphi \) is a reflexive fuzzy bisimulation and \( I_1, I_2 \) are border implicators which are left monotonic.
3. \( \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(\emptyset) = \emptyset, \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(S) = co_N(\emptyset) \), if \( \varphi \) is a reflexive fuzzy bisimulation and \( I_1 \) is a left monotonic and border implicator.
4. \( \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(\emptyset) = \emptyset, \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(S) = co_N(\emptyset) \), while \( \varphi \) is a reflexive fuzzy bisimulation.
5. Let \( I_1, I_2 \) be \( R \)-implicators. The \( \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K) \subseteq K \cup \hat{\theta} \) and \( K \cap co_N(\hat{\theta}) \subseteq \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K) \), for each \( K \in \mathcal{F}(S) \).
6. If \( I_1, I_2 \) are two border implicators which satisfy the hybrid monotonicity, then \( \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K) \subseteq K \cup \hat{\theta} \).
7. \( K \cap co_N(\hat{\theta}) \subseteq \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K) \), when \( \varphi \) is a reflexive fuzzy bisimulation.
8. If \( K \cup \hat{\theta} \subseteq K \cap co_N(\hat{\theta}) \) and \( I_1, I_2 \) are two \( R \)-implicators, then

\[
\text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K) \subseteq K \cup \hat{\theta} \subseteq K \cap co_N(\hat{\theta}) \subseteq \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K), \hspace{1cm} (K) \subseteq K \cap co_N(\hat{\theta}) \subseteq \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K). \\
\]

9. When \( I_1, I_2 \) are two \( R \)-implicators with \( \theta = 0 \), then

\[
\text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K) \subseteq K \subseteq \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K), \hspace{1cm} (K) \subseteq K \subseteq \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K). \\
\]

10. \( \forall K_1, K_2 \in \mathcal{F}(S) \), if \( K_1 \subseteq K_2 \) and \( I_1, I_2 \) are two right monotonic implicators, then

\[
\text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K_1) \cup \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K_2) \subseteq \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K_1 \cup K_2), \hspace{1cm} (K_1) \cap \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K_2) \subseteq \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K_1 \cap K_2), \hspace{1cm} (K_1) \cup \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K_2) \subseteq \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K_1 \cup K_2), \hspace{1cm} (K_1) \cap \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K_2) \subseteq \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K_1 \cap K_2).
\]

It is noted that the upper approximation operators of the 1-BGFVPRS and 2-BGFVPRS models have the same properties with the assertion (10) and thus we omit them.

(11) When \( I_1, I_2 \) are two border implicators satisfying the left monotonicity, for each \( \hat{\gamma} \in \mathcal{F}(S) \), we have

\[
\text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(\hat{\gamma}) = \hat{\gamma} \vee \theta, \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(\hat{\gamma}) = co_N(\hat{\gamma} \wedge \gamma). \\
\text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(\hat{\gamma}) = \hat{\gamma} \vee \theta, \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(\hat{\gamma}) = co_N(\hat{\gamma} \wedge \gamma). \\
\]

(12) If \( \theta_1 \leq \theta_2 \), implicators \( I_1, I_2 \) are right monotonic, then for each \( K \in \mathcal{F}(S) \), we have

\[
\text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta_1}(K) \subseteq \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta_1}(K), \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta_1}(K) \subseteq \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta_1}(K). \\
\text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta_1}(K) \subseteq \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta_1}(K), \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta_1}(K) \subseteq \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta_1}(K). \\
\]

**Proof** In the paper, we only prove (5), (11), and (12). The proof of other assertions is similar to them and thus we omit it.

(5) Suppose that \( I_1, I_2 \) are \( R \)-implicators. Due to Lemma 2.1 in [27], for each \( K \in \mathcal{F}(S), \theta \in [0, 1) \) and \( a \in S \), we have

\[
\text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K)(a) = \bigvee_{b \in S} T_2(\varphi(b, a), \bigwedge_{c \in S} I_1(\varphi(b, c), \theta \cup K(c))) \leq \bigvee_{b \in S} T_2(\varphi(b, a), I_1(\varphi(b, a), \theta \cup K(c))) \leq \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K)(a).
\]

That is, \( \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K) \subseteq K \cup \hat{\theta} \).

In the similar way, due to Lemma 2.1 in [27], for each \( K \in \mathcal{F}(S), \theta \in [0, 1) \) and \( a \in S \), we have

\[
\text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K)(a) = \bigvee_{b \in S} T_2(\varphi(b, a), \bigwedge_{c \in S} I_1(\varphi(b, c), \theta \cup K(c))) \leq \bigvee_{b \in S} T_2(\varphi(b, a), I_1(\varphi(b, a), \theta \cup K(c))) \leq \text{apr}_{\hat{\gamma}, T_2}^{\varphi, \theta}(K)(a).
\]
Theorem 11 Assume that $I_1, I_2$ are two border implicators satisfying the left monotonicity. For each $a \in S$, we have

$$apr_{I_1,I_2}^\theta(\gamma)(a) = \bigvee_{b \in S} T_2(\varphi(b,a), \bigwedge_{c \in S} I_1(\varphi(b,c), \theta \vee \gamma(c)))$$

$$\geq \bigwedge_{b \in S} T_2(\varphi(b,a), \bigwedge_{c \in S} I_1(\varphi(b,c), \theta \vee \gamma))$$

$$= \bigwedge_{b \in S} T_2(\varphi(b,a), \theta \vee \gamma)$$

$$= \theta \vee \gamma.$$

That is, $apr_{I_1,I_2}^\theta(\gamma) = \gamma \lor \theta$. The other equations can be proved similarly, and their proof is omitted here. \(\square\)

Note that the assertions (1)–(4) and (7) in Proposition 1 for the general binary fuzzy relation cannot be always true. In Example 4, for the Early Zadeh implicator $I_2$ (a border implicator) in [41], $apr_{I_1,I_2}^{R,0.6}(S) = \frac{1}{5} + \frac{9}{10} + \frac{7}{10} + \frac{9}{10}$ and $apr_{I_1,I_2}^{R,0.58}(S) = \frac{1}{5} + \frac{9}{10} + \frac{7}{10} + \frac{9}{10}$. In Example 6, for the general fuzzy relation $R$, through computation we have that $apr_{I_1,I_2}^{R,0.58}(S) = apr_{I_1,I_2}^{R,0.58}(S) = \frac{1}{5} + \frac{9}{10} + \frac{7}{10} + \frac{9}{10}$. This phenomenon shows that for the general binary fuzzy relation $R$, the assertions (1)–(4) and (7) cannot hold. However, the above proposition for the largest fuzzy bisimulation $apr^\theta$ i.e., the fuzzy bisimilarity holds.

It is worth noting that the $S$-implicator, $R$-implicator and $QL$-implicator are three known and widely used logical operators. Because of their characteristic properties, our provided approximation operators also have some feasible properties which are listed in the following.

Corollary 10

1. If $\varphi$ is a reflexive fuzzy bisimulation and $I_1$ is an $S$-implicator or $R$-implicator, then Proposition 1(1) holds.
2. If $\varphi$ is a reflexive fuzzy bisimulation, $I_1$ is an $S$-implicator or $R$-implicator while $I_2$ is an $S$-implicator or $R$-implicator, then Propositions 1(2), (6) and (11) are true.
3. If $\varphi$ is reflexive, and $I_2$ is an $S$-implicator or $R$-implicator, then Proposition 1(3) is true.
4. Assume that $I_1, I_2$ are two $QL$-implicators or $R$-implicators or $S$-implicators, Propositions 1(10) and (12) hold.

Note that the largest fuzzy bisimulation is a special binary fuzzy relation that satisfies the reflexivity. Under the conditions of Corollary 10, the same properties can be obtained similarly. For a general binary fuzzy relation, the assertion (4) in Corollary 10 is true. But, the assertions (1)–(3) cannot always be true for the general binary fuzzy relation.

We have studied the properties of approximation operators of 1-BGFVPRS and 2-BGFVPRS models. Furthermore, the relationships among $apr_{I_1,I_2}^\theta(K), apr_{I_1,I_2}^{R,0.58}(K), apr_{I_1,I_2}^{R,0.58}(K), apr_{I_1,I_2}^{R,0.58}(K)$ are discussed in the following. Through the relations of these approximation operators, the DMs according to their needs choose the satisfied RS model and apply it to make a clear decision.

Proposition 2 For each $K \in F(S)$ and $\theta \in [0, 1]$, if $\varphi$ is a reflexive fuzzy bisimulation and $I_2$ is a border implicator, then we get the following results.

1. $apr_{I_1,I_2}^\theta(K) \subseteq apr_{I_1,I_2}^{R,0.58}(K),$
2. $apr_{I_1,I_2}^{R,0.58}(K) \subseteq apr_{I_1,I_2}^{R,0.58}(K).$

Proof

1. Assume that $\varphi$ is a reflexive fuzzy bisimulation and $I_2$ is a border implicator. For each $a \in S$ and $\theta \in [0, 1],$

$$apr_{I_1,I_2}^\theta(K)(a) = \bigvee_{b \in S} T_2(\varphi(b,a), \bigwedge_{c \in S} I_1(\varphi(b,c), \theta \vee K(c)))$$

$$\geq \bigwedge_{b \in S} T_2(\varphi(b,a), \bigwedge_{c \in S} I_1(\varphi(b,c), \theta \vee K(c)))$$

$$= \bigwedge_{c \in S} I_1(\varphi(a,c), \theta \vee K(c)),$$

and

$$apr_{I_1,I_2}^{R,0.58}(K)(a) = \bigwedge_{b \in S} I_2(\varphi(b,a), \bigwedge_{c \in S} I_1(\varphi(b,c), K(c)))$$

$$\leq I_2(\varphi(a,a), \bigwedge_{c \in S} I_1(\varphi(a,c), \theta \vee K(c)))$$

$$= \bigwedge_{c \in S} I_1(\varphi(a,c), \theta \vee K(c)).$$
we have \( \text{apr}^{\varphi,0}(K)(a) \leq \text{apr}^{\varphi,0}(K)(a) \).

Thus \( \text{apr}^{\varphi,0}(K) \subseteq \text{apr}^{\varphi,0}(K) \) for each \( K \in \mathcal{F}(S) \) and \( \theta \in [0, 1) \).

(2) If \( \varphi \) is a reflexive fuzzy bisimulation and \( I_2 \) is a border implicator, for any \( a \in S \) and \( \theta \in [0, 1) \),

\[
\text{apr}^{\varphi,0}(K)(a) = \bigvee_{b \in S} I_2((\varphi(b, a), \bigvee_{c \in S} T_1((\varphi(c, a), N(\theta) \land K(c)))) \\
\leq I_2((\varphi(a, a), \bigvee_{c \in S} T_1((\varphi(c, a), N(\theta) \land K(c)))) \\
= \bigvee_{c \in S} T_1((\varphi(a, c), N(\theta) \land K(c))),
\]

and

\[
\text{apr}^{\varphi,0}(K)(a) = \bigvee_{b \in S} I_2((\varphi(b, a), \bigvee_{c \in S} T_1((\varphi(c, a), N(\theta) \land K(c)))) \\
\geq T_2((\varphi(a, a), \bigvee_{c \in S} T_1((\varphi(c, a), N(\theta) \land K(c)))) \\
= \bigvee_{c \in S} T_1((\varphi(a, c), \theta \land K(c))).
\]

Thus \( \text{apr}^{\varphi,0}(K) \subseteq \text{apr}^{\varphi,0}(K) \) is proved.

**Corollary 11** If \( I_2 \) is an \( S \)-implicator (or \( R \)-implicator or \( Q \)-implicator), then the following results are obtained.

(1) When \( \varphi \) is a reflexive fuzzy bisimulation, \( \text{apr}^{\varphi,0}(K) \subseteq \text{apr}^{\varphi,0}(K) \) and \( \text{apr}^{\varphi,0}(K) \subseteq \text{apr}^{\varphi,0}(K) \); \( \text{apr}^{\varphi,0}(K) \subseteq \text{apr}^{\varphi,0}(K) \), \( \text{apr}^{\varphi,0}(K) \subseteq \text{apr}^{\varphi,0}(K) \).

(2) If \( \varphi = \varphi^0 \) is the largest fuzzy bisimulation.

**Proof** Since \( R \)-implicator, \( S \)-implicator and \( Q \)-implicator are three border implicators and \( \varphi^0 \) the largest fuzzy bisimulation is reflexive, the assertions (1) and (2) in Corollary 11 can be easily obtained by Proposition 2.

**Proposition 3** Assume that \( \varphi \) is a reflexive fuzzy bisimulation, \( I_1, I_2 \) are two \( R \)-implicators and \( K \cup \bar{\theta} \subseteq K \cap co_N(\bar{\theta}) \), we have that

\[
\text{apr}^{\varphi,0}(K) \subseteq \text{apr}^{\varphi,0}(K) \subseteq K \cup \bar{\theta} \\
\subseteq K \cap co_N(\bar{\theta}) \subseteq \text{apr}^{\varphi,0}(K) \subseteq \text{apr}^{\varphi,0}(K).
\]

**Proof** Since an \( R \)-implicator is a special border implicator that is hybrid monotonic, by Proposition 1(8) and Proposition 2, the proof of this proposition is easily verified.

When the value of fuzzy variable precision is 0, the relationship between 1-BGFVPRS and 2-BGFVPRS models is discussed in the following.

**Proposition 4** Under the condition in Proposition 3, if \( \theta = 0 \), for each \( K \in \mathcal{F}(S) \), we have

\[
\text{apr}^{\varphi,0}(K) \subseteq \text{apr}^{\varphi,0}(K) \subseteq K \\
\subseteq \text{apr}^{\varphi,0}(K) \subseteq \text{apr}^{\varphi,0}(K).
\]

The identity relation \( \chi_S = \{(a, a) \mid a \in S\} \) and empty relation \( \phi_S \) are two bisimulations. In the paper, \( \chi_S \) is a special fuzzy set in which \( \chi_S(a, a) = 1, \forall a \in S \). The \( \phi_S \) is also a special fuzzy set where \( \phi_S(a, b) = 0 \) for each \( a, b \in S \).

**Proposition 5** If \( I_2 \) is an \( R \)-implicator and \( \varphi \) is a fuzzy bisimulation, the following statements are true.

(1) \( \text{apr}^{\varphi,0}(K) = \text{apr}^{\varphi,0}(K) = \emptyset \), if \( \varphi = \phi_S \).

(2) \( \text{apr}^{\varphi,0}(K) = \text{apr}^{\varphi,0}(K) \subseteq K \cap co_N(\bar{\theta}) \), if \( \varphi = \chi_S \).

From the above propositions, \( (\text{apr}^{\varphi,0}(K), \text{apr}^{\varphi,0}(K)) \) is tighter than \( (\text{apr}^{\varphi,0}(K), \text{apr}^{\varphi,0}(K)) \). As we all know, the equivalence relation acts a pivotal part in the RS model. The effects of fuzzy bisimulation \( \varphi \) on 1-BGFVPRS and 2-BGFVPRS models are studied as follows. As the reflexivity of \( \varphi \) has already been mentioned above, we don’t discuss it next.

**Proposition 6** If \( \varphi \) is a symmetric fuzzy bisimulation, then for each \( K \in \mathcal{F}(S) \), the following statements are got.

(1) \( \text{apr}^{\varphi,0}(K) = \text{apr}^{\varphi,0}(K) = \text{apr}^{\varphi,0}(K) \).

(2) \( \text{apr}^{\varphi,0}(K) = \text{apr}^{\varphi,0}(K) = \text{apr}^{\varphi,0}(K) \).

**Corollary 12** If \( \varphi = \varphi^0 \), for each \( K \in \mathcal{F}(S) \), the following statements are obtained.

(1) \( \text{apr}^{\varphi,0}(K) = \text{apr}^{\varphi,0}(K) \).

(2) \( \text{apr}^{\varphi,0}(K) = \text{apr}^{\varphi,0}(K) \).

**Remark 3** The models defined in [62] are generalized into the following forms with fuzzy bisimulation relation \( \varphi \) as follows:

\[
\text{apr}^{\varphi}_I(K)(a) = \bigvee_{b \in S} I_2((\varphi(a, b), \theta \lor K(b)), \quad (19)
\]
Here \( q^\theta_t (K), \bar{q}^\theta_t (K) \) are called the third type of bisimulation-based generalized fuzzy variable precision \( I \)-lower approximation (3-BGFVPI LA) and \( T \)-fuzzy upper approximation (3-BGFVFPI UA) of \( K \). When \( q^\theta_t (K) \neq \bar{q}^\theta_t (K), K \) is called the third type of bisimulation-based generalized fuzzy variable precision rough set (3-BGFVPRS). Otherwise it is definable.

(1) It is remarkable that if fuzzy bisimulation \( \varphi \) is replaced by the largest fuzzy bisimulation \( \varphi^b, \theta = 0 \) and \( I \) is an \( S \)-implicator [41] based on the involutive negator \( \mathcal{N} \), the 3-BGFVPRS model degenerates into the model defined in [12].

(2) When \( \varphi \) is a fuzzy \( \beta \)-neighborhood relation, the 3-BGFVPRS model turns out to be the model constructed in [20].

(3) If \( \theta = 0 \), \( \varphi \) is a general binary fuzzy relation and \( I \) is an \( S \)-implicator, formulas (19) and (20) will be the \( S \)-lower and \( T \)-upper approximation operators in [67], respectively.

Let \( \varphi_t (i \in \Lambda) \) be a fuzzy bisimulation. Then, \( \bar{q}^\theta_t (\bigcup_{i \in \Lambda} \varphi_i) \) and \( \bigcup_{i \in \Lambda} \varphi_i \) are also fuzzy bisimulations. The union of all fuzzy bisimulations is the largest fuzzy bisimulation \( \varphi^b \). Based on the properties of fuzzy bisimulations, some results of our models are listed as follows.

**Proposition 7** Let \( \varphi_t (i \in \Lambda) \) be a fuzzy bisimulation. For any \( \theta \in [0,1) \) and \( K \in \mathcal{F}(S) \), the following statements are true.

1. \( \overline{\text{apr}_{I \cap J} \varphi^\theta(K)} \subseteq \text{apr}_{I \cup J} \varphi^\theta(K) \) with two left monotonic implicators \( I_1, I_2 \).
2. \( \overline{\text{apr}_{I \cap J} \varphi^\theta(K)} \subseteq \bigcup_{i \in \Lambda} \text{apr}_{I \cup J} \varphi^\theta(K) \) with two \( R \)-implicators \( I_1, I_2 \).
3. \( \overline{\text{apr}_{I_1 \cap J} \varphi^\theta(K)} = \bigcup_{i \in \Lambda} \text{apr}_{I_1 \cup J} \varphi^\theta(K) \) with \( \varphi^t_i \).
4. \( \overline{\text{apr}_{I_1 \cap J} \varphi^\theta(K)} \subseteq \bigcup_{i \in \Lambda} \text{apr}_{I \cup J} \varphi^\theta(K) \) where \( I \) is a left monotonic implicator.
5. \( \overline{\text{apr}_{I_1 \cap J} \varphi^\theta(K)} \subseteq \bigcup_{i \in \Lambda} \text{apr}_{I \cup J} \varphi^\theta(K) \) where \( I \) is an \( R \)-implicator.
6. \( \bigcup_{i \in \Lambda} \text{apr}_{I_1 \cap J} \varphi^\theta(K) = \bigcup_{i \in \Lambda} \text{apr}_{I_1 \cup J} \varphi^\theta(K) \) (6).

If \( \varphi_1, \varphi_2 \) are two fuzzy bisimulations, then \( \varphi_1 \preceq \varphi_2 \) is a fuzzy bisimulation. Due to \( (\varphi_1 \preceq \varphi_2) = \varphi_2 \preceq \varphi_1 \) and \( (\bigcup_{i \in \Lambda} \varphi_i)^{\sim} = \bigcup_{i \in \Lambda} \varphi_i^{\sim} \), the following statements are true.

**Proposition 8** Assume that \( \varphi_t (i \in \Lambda) \) is a fuzzy bisimulation. For any \( \theta \in [0,1) \) and \( K \in \mathcal{F}(S) \), the following statements are true.

1. \( \overline{\text{apr}_{I \cap J} \varphi^\theta(K)} = \overline{\text{apr}_{I_1 \cap J} \varphi^\theta(K)} \).
2. \( \overline{\text{apr}_{I \cap J} \varphi^\theta(K)} = \overline{\text{apr}_{I_1 \cap J} \varphi^\theta(K)} \).
3. Since the BGFVPRS models have the same properties of Proposition 8, only the relevant properties of 1-BGFVPRS model are shown.

Next, the relation between 2-BGFVPRS and 3-BGFVPRS models is researched as follows.

**Proposition 9** Let \( I_1 = I_2 = I_3 = I \) and \( T_1 = T_2 = T = T \). If \( \varphi \) is a \( T \)-transitive and symmetric fuzzy bisimulation, \( I \) is an \( R \)-implicator based on continuous \( \tau \)-norm \( T \), for each \( K \in \mathcal{F}(S) \) with \( \theta \in [0,1) \), the following statements are true.

1. \( \varphi^\theta(K) \subseteq \bigcup_{i \in \Lambda} \varphi^\theta(K) \subseteq K \cup \hat{\theta} \).
2. \( \overline{\text{apr}_{I \cap J} \varphi^\theta(K)} \subseteq \overline{\text{apr}_{I_3 \cap J} \varphi^\theta(K)} \).

**Proof**

(1) Due to \( \varphi \) is a \( T \)-transitive and symmetric fuzzy bisimulation, by Lemma 2.1 in [27], for each \( a \in S \) and \( \theta \in [0,1) \), we have

\[
\overline{\text{apr}_{I_3 \cap J} \varphi^\theta(K)} \equiv \bigwedge_{a \in S} I(\varphi(b, a), \bigwedge_{c \in S} I(\varphi(b, c), \theta \vee K(c)))
\]

(2) For each \( K \in \mathcal{F}(S), \theta \in [0,1) \) and \( a \in S \),
The relationship among the approximation operators of 2-BGFVPRS and 3-BGFVPRS models is investigated in Proposition 9. Figure 6 gives the relationship of them clearly.

**Proposition 10** Let \( I_1 = I_2 = I_3 = I, T_1 = T_2 = T_3 = T \) be an R-implicator, a continuous t-norm, respectively. For each \( K \in \mathcal{F}(S) \) and \( \theta \in [0, 1) \), we have the following results.

1. \( \text{apr}_{\bar{T},T}(K) = \varphi^\theta(K) \), if \( \varphi \) is a T-similarity fuzzy bisimulation.

2. \( \text{apr}_{\bar{T},T}(K) = \varphi^\theta(K) \), when \( \varphi \) is a reflexive fuzzy bisimulation.

3. \( \text{apr}_{\bar{T},T}(K) = (\varphi^\theta)_{\bar{T}}(K) \).

**Proof** (1) For each \( a \in S, \theta \in [0, 1) \) and \( K \in \mathcal{F}(S) \), \( \text{apr}_{\bar{T},T}(K) \) holds.

(2) For any \( a \in S, \theta \in [0, 1) \) and \( K \in \mathcal{F}(S) \), \( \text{apr}_{\bar{T},T}(K) \) holds.

**Remark 4** Under the condition in Proposition 9, if \( \varphi \) is a T-similarity fuzzy bisimulation, for each \( K \in \mathcal{F}(S) \) and \( \theta \in [0, 1) \), some statements are obtained as follows.

1. \( \varphi^\theta(K) = \text{apr}_{\bar{T},T}(K) \subseteq \text{apr}_{\bar{T},T}(K) \subseteq K \cup \hat{\theta} \subseteq K \cap \text{co}_N(\hat{\theta}) \).

Fig. 6 The relationship between the 2-BGFVPRS and 3-BGFVPRS models
(2) \[ \varphi^0_1(K) = \overline{apr_{\theta, T_1}}(K) = \overline{apr_{\theta, T_2}}(K) \subseteq K \subseteq \overline{apr_{\theta, T_1}}(K) = \varphi^0_3(K), \text{ if } \theta = 0. \]

The statements in Propositions 7 and 8 for the general binary fuzzy relation are also true. It is worth noting that the assertions in Propositions 2, 3, 6, 9, 10, in Corollaries 11, 12 and in Remark 4 cannot always hold if \( \varphi \) is a general binary fuzzy relation. But, for the largest fuzzy bisimulation \( \varphi^b \), the above statements are always true.

According to Remark 4, under some special circumstances, the relationship among the approximation operators of 1-BGFVPRS, 2-BGFVPRS, and 3-BGFVPRS models is obtained, as shown in Fig. 7. Through the research, the approximation operators of 1-BGFVPRS and 2-BGFVPRS models are tighter than the approximation operators of 3-BGFVPRS in some special cases. The approximation operator of 1-BGFVPRS is the most compact approximation operator among the three models.

It is noticed that the duality between rough approximation operators plays an important role in RS models. Many researchers usually take advantage of the duality to build a dual pair of approximation operators. Based on this, the duality of approximation operators provided in the paper is studied as follows.

**Proposition 11** Let \( I_1, I_2 \) be two \( S \)-implicators based on continuous \( t \)-norm \( T_1, T_2 \), respectively. If \( N \) is an involutive negator i.e., for any \( k_1, k_2 \in [0, 1], I(k_1, k_2) = N(T(k_1, N(k_2))), \) then \( \forall K \in F(S) \) and \( \theta \in [0, 1] \), the following statements are true.

\[
\begin{align*}
\text{(1) } & \overline{apr_{\theta, T_1}}(K) = co_N(\overline{apr_{\theta, T_1}}(co_N(K))). \\
\text{(2) } & \overline{apr_{\theta, T_2}}(K) = co_N(\overline{apr_{\theta, T_2}}(co_N(K))). \\
\text{(3) } & \overline{apr_{\theta, T_2}}(K) = co_N(\overline{apr_{\theta, T_2}}(co_N(K))). \\
\text{(4) } & \overline{apr_{\theta, T_2}}(K) = co_N(\overline{apr_{\theta, T_2}}(co_N(K))).
\end{align*}
\]

**Proof** (1) Since \( N \) is an involutive negator, then for any \( k_1, k_2 \in [0, 1], I(k_1, k_2) = N(T(k_1, N(k_2))). \) Due to \( I_1, I_2 \) are two \( S \)-implicators, for each \( a \in S \) and \( \theta \in [0, 1] \),

\[
\begin{align*}
co_N(\overline{apr_{\theta, T_1}}(co_N(K)) & (a) \\
& = N(\overline{apr_{\theta, T_1}}(co_N(K)))(a) \\
& = N(\bigcap_{b \in S} I_2(\varphi(b, a), \bigvee_{c \in S} T_1(\varphi(b, c), N(\theta \land N(K(c)))))) \\
& = N(\bigcap_{b \in S} I_2(\varphi(b, a), \bigvee_{c \in S} T_1(\varphi(b, c), N(\theta \lor K(c)))))) \\
& = N(\bigcap_{b \in S} (N(T_2(\varphi(b, a), \bigcap_{c \in S} N(T_1(\varphi(b, c), N(\theta \lor K(c))))))) \\
& = N(\bigcap_{b \in S} I_1(\varphi(b, c), \theta \lor K(c)))) \\
& = N(\bigcap_{b \in S} (N(T_2(\varphi(b, a), \bigcup_{c \in S} I_1(\varphi(b, c), \theta \lor K(c))))))) \\
& = \overline{apr_{\theta, T_2}}(K)(a).
\end{align*}
\]
Thus, \( \text{appr}^{\theta} (K) = \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) \) can be obtained.

(2) For each \( a \in S \) and \( \theta \in [0, 1] \),
\[
\text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))(a) = N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))(a))
\]
\[
= N(\bigwedge_{b \in S} T_2(\varphi(b, a), \bigwedge_{c \in S} I_1(\varphi(b, c), \theta \vee N(\text{I}(K(c)))))
\]
\[
= \bigwedge_{b \in S} N(\bigwedge_{c \in S} I_1(\varphi(b, c), \theta \vee N(\text{I}(K(c))))
\]
\[
= \bigwedge_{b \in S} \bigwedge_{c \in S} I_1(\varphi(b, c), \theta \vee N(\text{I}(K(c))))
\]
\[
= \bigwedge_{b \in S} \bigwedge_{c \in S} I_1(\varphi(b, c), \theta \vee N(\text{I}(K(c))))
\]
\[
= \text{appr}^{\theta}_{\text{I}, T_1, T_2} (K).
\]

Thus, \( \text{appr}^{\theta}_{\text{I}, T_1, T_2} (K) = \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) \) is proved. Similarly, the assertions (3) and (4) can be proved and thus we omit them here.

**Proposition 12** Assume that \( I_1, I_2 \) are two R-implicators based on continuous \( T_1, T_2 \), respectively. If \( N \) is a negator induced by implicator \( I \), for each \( k_1, k_2 \in [0, 1] \), \( N(\text{I}(k_1)) \geq k_1 \) and \( N(\text{I}(k_1, k_2)) \geq T(k_1, N(\text{I}(k_2))) \), \( \forall K_1, K_2 \in \mathcal{F}(S) \) and \( \theta \in [0, 1] \), we have the following statements.

1. \( \text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K)) \subseteq \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) \).
2. \( \text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K)) \subseteq \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) \).
3. \( \text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K)) \subseteq \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) \).
4. \( \text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K)) \subseteq \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) \).

**Proof** Here, we only prove the first assertion in this proposition, others can be verified similarly. Under the condition given in this proposition, for each \( a \in S \),
\[
\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))(a) = \bigwedge_{b \in S} \bigwedge_{c \in S} I_1(\varphi(b, a), \bigwedge_{d \in S} I_2(\varphi(b, c), N(\theta) \wedge N(K(c))))
\]
\[
\leq \bigwedge_{b \in S} \bigwedge_{c \in S} I_1(\varphi(b, a), \bigwedge_{d \in S} I_2(\varphi(b, c), \theta \vee K(c)))
\]
\[
\leq \bigwedge_{b \in S} \bigwedge_{c \in S} N(\bigwedge_{d \in S} I_2(\varphi(b, c), \theta \vee K(c)))
\]
\[
\leq \bigwedge_{b \in S} \bigwedge_{c \in S} I_1(\varphi(b, a), \bigwedge_{d \in S} I_2(\varphi(b, c), \theta \vee K(c)))
\]
\[
= \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))(a)).
\]

Hence, \( \text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K)) \subseteq \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) \) holds.

**Proposition 13** Let \( I_1, I_2 \) be two R-implicators based on continuous \( T_1, T_2 \), respectively. For each \( K \in \mathcal{F}(S) \) and \( k_1, k_2 \in [0, 1] \), if \( N(\text{I}(k_1)) = k_1 \) and \( N(\text{I}(k_1, k_2)) = T(k_1, N(\text{I}(k_2))) \), the following statements hold.

1. \( \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) \leq \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) \).
2. \( \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) \leq \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) \).
3. \( \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) \leq \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) \).
4. \( \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) \leq \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) \).

**Proposition 14** Let \( I_1, I_2 \) be two R-implicators based on continuous \( T_1, T_2 \), respectively. If \( N(\text{I}(k_1)) = k_1 \) and \( N(\text{I}(k_1, k_2)) = T(k_1, N(\text{I}(k_2))) \) for each \( k_1, k_2 \in [0, 1] \), then the following results are obtained.

1. \( \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) = \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) \).
2. \( \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) = \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) \).
3. \( \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) = \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) \).
4. \( \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) = \text{co}_N(\text{appr}^{\theta}_{\text{I}, T_1, T_2} (\text{co}_N(K))) \).

4 The uncertainty measure and reduction

The uncertainty measure and reduction are two important concepts in classical RS model. In this section, the uncertainty measure of BGFVPRS models and reduction of fuzzy bisimulations are introduced. Furthermore, the related properties of them are discussed. In the following, we firstly investigate the uncertainty measure of BGFVPRS models.

4.1 The uncertainty measure

The inaccuracy of a set is caused by the existence of the boundary domain. To express this point more accurately, we introduce the concept of accuracy measures of BGFVPRS models of the fuzzy set \( K \). Obviously that 1-BGFVPRS, 2-BGFVPRS, and 3-BGFVPRS are three fuzzy sets on \( S \). Then \( \forall K_1, K_2 \in \mathcal{F}(S) \), the distance function of fuzzy sets is used to define the closeness measure between \( K_1 \) and \( K_2 \). As we all know that the Euclid distance function satisfies the conditions of distance. In general, it is used as a similarity. In the paper, by using the Euclid distance, we can compute the accuracy measure between fuzzy sets \( K_1 \) and \( K_2 \).
Suppose that \( K_1 = \frac{K_1(a_1)}{a_1} + \frac{K_1(a_2)}{a_2} + \cdots + \frac{K_1(a_n)}{a_n} \),
\( K_2 = \frac{K_2(a_1)}{a_2} + \frac{K_2(a_2)}{a_2} + \cdots + \frac{K_2(a_n)}{a_n} \),
\( S = \{a_1, a_2, \ldots, a_n\}, |S| = n \). Assume that \( K_1(a_i) \) is the membership degree that element \( a_i \) belongs to fuzzy set \( K_i \), where \( i \in \{1, 2\} \) and \( i \in \{1, 2, \ldots, n\} \). The Euclidean distance function between fuzzy sets \( K_1 \) and \( K_2 \) is defined as:
\[
d_{E}(K_1, K_2) = 1 - \frac{1}{\sqrt{n}} \left( \sum_{i=1}^{n} (K_1(a_i) - K_2(a_i))^2 \right)^{\frac{1}{2}}.
\]

Based on the Euclidean distance function, a concept of the accuracy measures of the BGFVPRS models is given in the following.

**Definition 13** Let \((S, \varphi)\) be a fuzzy relational structure where \( \theta \in [0, 1], S_1 = \{a_1, a_2, \ldots, a_n\} \subseteq S \). Then for each \( \theta \in \mathcal{F}(S) \), the \( \theta \)-accuracy measures of upper and lower approximation operators of 1-BGFVPRS, 2-BGFVPRS, and 3-BGFVPRS for fuzzy set \( K \) are defined as follows:
\[
d_{E,1}^{p_{\theta}}(K) = 1 - \frac{1}{\sqrt{n}} \left( \sum_{i=1}^{n} (ap_{\theta,T_1}^{p_1}(K)(a_i) - \overline{ap}_{\theta,T_2}^{p_1}(K)(a_i))^2 \right)^{\frac{1}{2}},
\]
\[
d_{E,2}^{p_{\theta}}(K) = 1 - \frac{1}{\sqrt{n}} \left( \sum_{i=1}^{n} (ap_{\theta,T_1}^{p_2}(K)(a_i) - \overline{ap}_{\theta,T_2}^{p_2}(K)(a_i))^2 \right)^{\frac{1}{2}},
\]
\[
d_{E,3}^{p_{\theta}}(K) = 1 - \frac{1}{\sqrt{n}} \left( \sum_{i=1}^{n} (\overline{ap}_{\theta,T_1}^{p_3}(K)(a_i) - \overline{ap}_{\theta,T_2}^{p_3}(K)(a_i))^2 \right)^{\frac{1}{2}}.
\]

Meanwhile, another concept of the roughness measure of the BGFVPRS models is defined as follows.

**Definition 14** Under the condition in Definition 13, for each \( \theta \in \mathcal{F}(S) \), the \( \theta \)-roughness measure of upper and lower approximation operators of \( \lambda \)-BGFVPRS for the fuzzy set \( K \) is defined as \( \rho_{\lambda}^{p_{\theta}}(K) = 1 - d_{E,\lambda}^{p_{\theta}}(K) \), where \( \lambda = 1, 2, 3 \).

For better understand the above two definitions, an example is given in the following.

**Example 7** Assume that in the condition of Example 4, if \( I_1 = I_2 = I_3 = I_4 \), \( T_1 = T_2 = T_3 = T_4 \), through Definitions 13 and 14, we have that \( d_{E,1}^{p_{0.58}}(K) = d_{E,2}^{p_{0.58}}(K) = d_{E,3}^{p_{0.58}}(K) = 0.7085 \),
\( \rho_{1}^{p_{0.58}}(K) = \rho_{2}^{p_{0.58}}(K) = \rho_{3}^{p_{0.58}}(K) = 0.2915 \). In this case, the 0.58-accuracy measures of 1-BGFVPRS, 2-BGFVPRS, and 3-BGFVPRS are identical meanwhile the 0.58-roughness measures of them are the same.

In the following, the relationships of the \( \theta \)-accuracy measures (or the \( \theta \)-roughness measures) of the BGFVPRS models are discussed.

**Proposition 15** If \( K \cup \tilde{\theta} \subseteq K \cap co_{\lambda}(\tilde{\theta}) \), \( \varphi \) is reflexive and \( I_1 \), \( I_2 \) are R-implicators, for each \( K \in \mathcal{F}(S) \) and \( \tilde{\theta} \in [0, 1] \), we have \( d_{E,2}^{p_{\theta}}(K) \geq d_{E,3}^{p_{\theta}}(K), \rho_{1}^{p_{\theta}}(K) \leq \rho_{2}^{p_{\theta}}(K) \).

**Proposition 16** Suppose that \( K \cup \tilde{\theta} \subseteq K \cap co_{\lambda}(\tilde{\theta}) \), \( I_1 = I_2 = I_3 = I \) and \( T = T_2 = T_3 = T \) where \( I \) is an R-implicator based on a continuous t-norm \( T \). For each \( K \in \mathcal{F}(S) \) and \( \tilde{\theta} \in [0, 1] \), we have the following statements.

1. If \( \varphi \) is symmetric and \( T \)-transitive fuzzy bisimulation, then \( d_{E,3}^{p_{\theta}}(K) \leq d_{E,2}^{p_{\theta}}(K), \rho_{3}^{p_{\theta}}(K) \geq \rho_{2}^{p_{\theta}}(K) \).
2. If \( \varphi \) is a fuzzy \( T \)-similarity bisimulation relation, then \( d_{E,3}^{p_{\theta}}(K) \leq d_{E,2}^{p_{\theta}}(K), \rho_{3}^{p_{\theta}}(K) \geq \rho_{2}^{p_{\theta}}(K) \).
3. If \( \varphi \) is a fuzzy equivalence bisimulation relation, then \( d_{E,3}^{p_{\theta}}(K) \leq d_{E,2}^{p_{\theta}}(K), \rho_{3}^{p_{\theta}}(K) \geq \rho_{2}^{p_{\theta}}(K) \).
4. If \( \varphi \) is a largest fuzzy bisimulation relation, then \( d_{E,3}^{p_{\theta}}(K) \leq d_{E,2}^{p_{\theta}}(K), \rho_{3}^{p_{\theta}}(K) \geq \rho_{2}^{p_{\theta}}(K) \).

5. Under the condition (2) (or (3), or (4)), if \( \theta = 0 \), then \( d_{E,3}^{p_{\theta}}(K) \leq d_{E,2}^{p_{\theta}}(K) \leq 1, \rho_{3}^{p_{\theta}}(K) \geq \rho_{2}^{p_{\theta}}(K) \geq 0 \).

6. For each \( \theta_1, \theta_2 \in [0, 1] \), if \( \theta_1 \leq \theta_2 \), then \( d_{E,3}^{p_{\theta}}(K) \leq d_{E,2}^{p_{\theta}}(K) \) where \( i = 1, 2, 3 \).

The above Propositions show that we can study RS models from a numerical point of view. Through Propositions 15 and 16, when describing a fuzzy set at the same time, the accuracy (or roughness) measures of 1-BGFVPRS, 2-BGFVPRS, and 3-BGFVPRS may be different. The 1-BGFVPRS is the most accurate, 2-BGFVPRS is the second, and 3-BGFVPRS is the worst. Conversely, 3-BGFVPRS is the roughest, 2-BGFVPRS is the next, and 1-BGFVPRS is the least rough.

### 4.2 The reduction

In two models (or the model and itself), the largest fuzzy bisimulation is the union of all fuzzy bisimulations. If there are some redundant or unnecessary fuzzy bisimulations, when we compute the largest fuzzy bisimulation, we will spend a lot of unnecessary manpower and material resources. Especially when dealing with big data, these redundant data even make our work impossible because of the limited equipment and manpower.

To deal with these redundant data, some related concepts are described below.
**Definition 15** Let $(S, (R_i)_{i \in \Lambda})$ be a fuzzy relational structure. Assume that $H = \{\varphi_1, \varphi_2, \ldots, \varphi_k\}$ which is the set of some fuzzy bisimulations in this structure. Then $\varphi_i \in H$ is called dispensable to $\varphi_j$ if $\varphi_j - \varphi_i = \varphi_j$; otherwise, it is called indispensable to $\varphi_j$. The set of all indispensable fuzzy bisimulations to $\varphi_j$ in $H$ is named the reduction of $\varphi_j$, which is written as $\varphi_j^\ast$. Note that fuzzy transition system in [7] is a special case of fuzzy relational structure with labels. In general, the labels are regarded as attributes. The following example is given to demonstrate the feasibility of the above definition.

**Example 8** Assume $\mathcal{X} = \{T = \{t_1, t_2, t_3\}, A = \{m\}, \delta, t_1\}$ is a labeled fuzzy transition system where $T$ is a set of states, $A$ is a set of labels, $\delta$ is the fuzzy transition relation and $t_1$ is the initial state. The fuzzy transition system is shown as Fig. 8 where $\delta(t_1, m, t_2) = 0.8$ and $\delta$ takes values 0 for all other cases.

Suppose that

\[
\begin{align*}
R_1 &= \frac{1}{(t_1, t_1)} + \frac{1}{(t_2, t_2)} + \frac{1}{(t_3, t_3)}, \\
R_2 &= \frac{1}{(t_1, t_1)} + \frac{1}{(t_2, t_2)} + \frac{1}{(t_3, t_3)}, \\
R_3 &= \frac{1}{(t_1, t_1)} + \frac{1}{(t_2, t_2)} + \frac{1}{(t_3, t_3)} + \frac{1}{(t_3, t_2)}.
\end{align*}
\]

It is easy to prove that $R_1$, $R_2$, $R_3$ are three fuzzy bisimulations.

In this case, since $\varphi^b = \bigcup_{i=1}^3 R_i = R_3$, $\varphi^b - R_3 \neq \varphi^b$, then $R_3$ is an indispensable fuzzy bisimulation to $\varphi^b$. Due to $\varphi^b - R_1 = \varphi^b - R_2 = \varphi^b$, we have $R_1$, $R_2$ are two dispensable fuzzy bisimulations to $\varphi^b$. Hence, $R_3$ is the reduction to $\varphi^b$.

**Proposition 17** In a fuzzy relational structure $(S, (R_i)_{i \in \Lambda})$, if $H = \{\varphi_1, \varphi_2, \ldots, \varphi_k\}$, $\psi = \bigcup_{i=1}^m \varphi_i$, and $\varphi_i \in H$ is a reduction of $\varphi^b$, then for each $K \in \mathcal{F}(S)$ and $\theta \in [0, 1)$ the following results are true.

\[
\begin{align*}
(1) & \quad \text{ap}^\theta\varphi_i\varphi_j(K) = \text{ap}^\theta\varphi_i\varphi_j(K), \text{ap}^\theta_{T_1}T_2(K) = \text{ap}^\theta_{T_1}T_2(K). \\
(2) & \quad \text{ap}^\theta\varphi_i\varphi_j(K) = \text{ap}^\theta\varphi_i\varphi_j(K), \text{ap}^\theta_{T_1}T_2(K) = \text{ap}^\theta_{T_1}T_2(K). \\
(3) & \quad (\varphi^b)^{\theta}_K(K) = \psi_{T_1}^{\theta}_K(K), (\varphi^b)^{\theta}_T(K) = \psi_{T_1}^{\theta}_T(K). \\
(4) & \quad d_{E,1}^\theta\varphi_i\varphi_j(K) = d_{E,1}^\theta\varphi_i\varphi_j(K), d_{E,1}^\theta\varphi_i\varphi_j(K) = d_{E,1}^\theta\varphi_i\varphi_j(K). \\
(5) & \quad \rho_1^\theta\varphi_i\varphi_j(K) = \rho_1^\theta\varphi_i\varphi_j(K), \rho_2^\theta\varphi_i\varphi_j(K) = \rho_2^\theta\varphi_i\varphi_j(K), \rho_3^\theta\varphi_i\varphi_j(K) = \rho_3^\theta\varphi_i\varphi_j(K).
\end{align*}
\]

Of course, for the general fuzzy bisimulation, the reduction is very important. In order to maintain some properties, we can also propose some reduction methods of fuzzy bisimulation from other angles, such as keeping the positive region unchanged. The paper from keeping the distance between the lower and the upper approximation operator invariable provides a method to depict the reduction of the general fuzzy bisimulation. Using our method, in the era of big data, human and material resources can be greatly saved. The reduction forms of the three models are similar. Next, only the reduction of 1-BGFVPRS model is discussed.

**Definition 16** Let $(S, (R_i)_{i \in \Lambda})$ be a finite fuzzy relational structure. Assume that $H = \{\varphi_1, \varphi_2, \ldots, \varphi_k\}$ which is the set of all fuzzy bisimulations. If $d_{E,1}^{\theta-\varphi_1,\varphi_2}(K) \neq d_{E,1}^{\theta-\varphi_1,\varphi_2}(K)$, then $\varphi_i$ is relative indispensable to $\varphi_2$; otherwise it is relative dispensable to $\varphi_2$. The relative reduction of $H$ is all relative indispensable fuzzy bisimulations for $H$.

To help us understand the above definition well, the following example is given.

**Example 9** Under the condition of Example 8, let $I_1 = I_2 = I_3, T_1 = T_2 = T_3, N = N', \theta = 0.58$. Assume that $K = 0.8 + 0.75 + 0.75$. By Definition 8 and formula (3.21), we have the following results:

\[
\begin{align*}
&d_{E,1}^{\theta-\varphi_1,\varphi_2}(K) = d_{E,1}^{\theta-\varphi_1,\varphi_2}(K) = 0.74; \\
&d_{E,1}^{\theta-\varphi_1,\varphi_2}(K) = d_{E,1}^{\theta-\varphi_1,\varphi_2}(K) = 0.74; \\
&0 = d_{E,1}^{\theta-\varphi_1,\varphi_2}(K) \neq d_{E,1}^{\theta-\varphi_1,\varphi_2}(K) = 0.74.
\end{align*}
\]

Through Definition 16, we obtain that $R_3$ is relative indispensible while $R_1$, $R_2$ are relative dispensable to $\varphi^b$. Thus, $R_3$ is a relative reduction to $\varphi^b$. 

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In the following, a method to compute the relative reduction for a general fuzzy bisimulation is given. Through this method, the reduction to the largest fuzzy bisimulation \( q^b \) can be easily obtained.

Let \( H = \{ \varphi_1, \varphi_2, \ldots, \varphi_k \} \) be a set of \( k \) fuzzy bisimulations in the finite fuzzy relational structure \((S, (R_i)_{i \in \Lambda})\). The fuzzy bisimulation \( \sigma \) is a relative reduction to the fuzzy bisimulation \( q \) if it satisfies the following conditions:

1. \( d_{E,1}^{\sigma, \varphi}(K) = d_{E,1}^{\varphi, \theta}(K) \),
2. \( \forall \varphi_i \in H, \varphi_i \subseteq \sigma, d_{E,1}^{\sigma, \varphi_i}(K) \neq d_{E,1}^{\varphi_i, \theta}(K) \).

In the following, an example is given to illustrate the above method.

**Example 10** From Example 9, through Definition 8 and formula (3.21), we have that \( d_{E,1}^{\sigma, 0.58}(K) = 0.74, d_{E,1}^{\varphi, 0.58}(K) = 0.71, d_{E,1}^{R_1, 0.58}(K) = 0.15, d_{E,1}^{R_2, 0.58}(K) = 0.74 \). Based on this, we find that \( d_{E,1}^{\sigma, 0.58}(K) = d_{E,1}^{R_1, 0.58}(K), d_{E,1}^{\varphi, 0.58}(K) = d_{E,1}^{R_2, 0.58}(K), d_{E,1}^{\sigma, 0.58}(K) \neq d_{E,1}^{R_3, 0.58}(K) \).

Since \( R_1, R_2 \subseteq R_3 \), according to the above method, \( R_3 \) is the relative reduction to \( q^b \) while \( R_1 \) and \( R_2 \) are not the relative reduction.

The relationship between the reduction and relative reduction is discussed as follows.

**Remark 5** Through the analysis of the above definitions and propositions, we have the following statements.

1. If \( \psi \) is the relative reduction of \( \varphi^b \), it may not the reduction for \( q^b \).
2. If \( \psi \) is the reduction of \( q^b \), it is the relative reduction for \( q^b \).

## 5 The PROMETHEE II method based on BGFVPRS models

The classical PROMETHEE methods in [5, 6] and some MADM methods in [20, 21, 62–64] based on the generalized RS models have achieved many great results in applications. However, they seem powerless for the multiple relationship structure problems such as community promotion problems and social network analysis problems. In view of this, a novel MADM method is needed. In this section, based on the concept of fuzzy bisimulation and the thought of PROMETHEE II method, we want to provide an effective decision-making method based on the BGFVPRS models.

The BGFVPRS models are based on the fuzzy bisimulations. With them, the theoretical research of the classical rough set model has been enriched. These relations can excavate “multi-step” information for underlying relations in a fuzzy environment. By means of the fuzzy bisimulations, objects can be well distinguished. Based on this, we can approximate the uncertain concept well and make a better result.

From Fig. 7 in Section 3 and Propositions 15 and 16 in Section 4, 1-BGFVPRS is most accurate than the 2-BGFVPRS and 3-BGFVPRS. That is to say, the results by using 1-BGFVPRS are closer to the uncertainty concept than the other BGFVPRS models provided in the paper. In order to make the decision results closer to the actual needs of DMs, the 1-BGFVPRS is selected to be applied. Of course, the other two types of BGFVPRS models can also be applied in this decision-making method, and then some different results may be obtained.

To make the decision-making method more general and be applied in widely areas, the general fuzzy logic operators are introduced. A threshold parameter \( \theta \) where \( \theta \in [0, 1) \), is given to control the influence of environmental noise data on our decision results. In decision-making, DMs can give the appropriate threshold \( \theta \) according to the actual situation of the environment, which will make the decision-making result better. Because the BGFVPRS models are not easily disturbed by noise data, the decision-making method in the paper is robust.

The subjective weight is important in some decision-making methods. However, if the experts have different opinions, then the subjective weights are difficult to obtain. Or, if the DMs are lack of experience, the subjective weights obtained may not be right. Compared with it, the objective weight can avoid errors caused by human factors. In this section, by means of the concept of \( \theta \)-accuracy measure which is uncertainty measure, an objective weight formula is proposed.

In classical rough set theory, the attribute reduction can help remove redundant attributes and improve the decision-making efficiency of the algorithm. In view of this, the reduction defined in Section 4 is applied in our decision-making method. When making a decision, we don’t need to compute the decision-making result of every fuzzy bisimulation. Then, we will save a lot of time and human resources. In other words, it will improve the decision-making efficiency of our proposed algorithm.

Above, our method is robust, flexible, and effective.

### 5.1 The PROMETHEE II method based on 1-BGFVPRS model

Recently, MADM methods have drawn many scholars’ attention and their contribution is enormous. Some scholars relied on RS models and presented many decision-making methods. However, some MADM methods in [5, 6, 20, 21, 62–64] have some defects in relational structure
decision-making problems such as graph mining or social network analysis. Based on this, we hope to look for an effective approach to solve these problems.

On the basis of the 1-BGFVPRS model, combing the principle of the PROMETHEE II method, a novel decision-making method is provided. Furthermore, an example for selecting the optimal alternative by mass organizations in the Zachary karate club network is given to illustrate this method. Through the example, the flexibility and effectiveness are illustrated from two aspects: comparative analysis and sensitivity analysis.

5.2 Problem statement

An organization or a community wants to choose talent leaders. It is necessary to consider not only the belief ability, planning ability, and goal ability of alternatives, but also their influence ability, interpersonal ability, and communication ability. If the managers not only have political integrity and ability, but also have high leadership ability, the organization will make brilliant achievements under their leadership.

Let $S = \{t_0, t_1, \ldots, t_{n-1}\}$ be the universe of $n$ alternatives in an organization or a community, $\mathcal{M} = \{m_0, m_1, \ldots, m_{q-1}\}$ be the set of $q$ labels, $\Psi = \{R_0, R_1, \ldots, R_{n-1}\}$ be a set of fuzzy relations, and $\Psi = \{\varphi_0, \varphi_1, \ldots, \varphi_{q-1}\}$ be a set of binary fuzzy relations. Here, $R_h(t_i, m_j, t_k) = h_{ijk}$ denotes the degree value of alternative $t_i$ to alternative $t_k$ with respect to the relation $R_h \in \mathcal{R}$ for label $m_j$ where $t_i, t_k \in S$, $m_j \in \mathcal{M}$, and $h_{ijk} \in [0, 1]$. Here, the $n, q, \gamma, x$ are natural numbers.

The information of $R_h(t_i, m_j, t_k)$ is provided by a lot of experts and shown as Table 2. The value of $\varphi_f(t_i, t_j) = f_{ijk}$ shows that the membership degree for alternative $t_i$ to alternative $t_k$ with respect to binary fuzzy relation $\varphi_f$. Table 3 presents the information of the fuzzy set $\varphi_f$. Note that the labels can represent a certain identity, a certain characteristic, or a certain aspect, etc. Generally speaking, labels can be considered as attributes. Therefore, this kind of problem is an MADM with multiple relationships.

To select the optimal member, a decision-making method is given as follows.

5.3 Decision-making methodology

First of all, through the experience of experts, professional evaluation, and statistical analysis, we can get real data from a community or a social network. Through normalization, we get the fuzzy data sets $\mathcal{Y}$ and $\Psi$.

Secondly, according to Definition 7 and the reduction of fuzzy bisimulation, some fuzzy bisimulations are selected. These fuzzy bisimulations constitute a new family of relation sets $\Xi = \{\psi_0, \psi_1, \ldots, \psi_{q-1}\} \subseteq \Psi$.

| Table 2 The information of $R_h(t_i, m_j, t_k)$ |
|-----------------|-----------------|-----------------|-----------------|
| $R_h$ | $t_0$ | $t_1$ | $\cdots$ | $t_{n-1}$ |
| $t_0$ | $h_{000}$ | $h_{010}$ | $\cdots$ | $h_{0(n-1)}$ |
| $t_1$ | $h_{100}$ | $h_{110}$ | $\cdots$ | $h_{1(n-1)}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\cdots$ | $\vdots$ |
| $t_{n-1}$ | $h_{(n-1)0}$ | $h_{(n-1)1}$ | $\cdots$ | $h_{(n-1)(n-1)}$ |

| Table 3 The information of $\varphi_f(t_i, t_j)$ |
|-----------------|-----------------|-----------------|-----------------|
| $\varphi_f$ | $t_0$ | $t_1$ | $\cdots$ | $t_{n-1}$ |
| $t_0$ | $f_{00}$ | $f_{01}$ | $\cdots$ | $f_{0(n-1)}$ |
| $t_1$ | $f_{10}$ | $f_{11}$ | $\cdots$ | $f_{1(n-1)}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\cdots$ | $\vdots$ |
| $t_{n-1}$ | $f_{(n-1)0}$ | $f_{(n-1)1}$ | $\cdots$ | $f_{(n-1)(n-1)}$ |

Thirdly, based on the principle of PROMETHEE II, through Definition 8, the preference function is given as follows:

$$P_f(t_i, t_j) = D_f(t_i, t_j),$$

where $D_f(t_i, t_j) = \frac{\text{appr}^{\psi_{ij}}(K)(t_i) + \text{appr}^{\psi_{ji}}(K)(t_j)}{\text{appr}^{\psi_{ij}}(K)(t_i) + \text{appr}^{\psi_{ji}}(K)(t_j)}$ and $\psi_f \in \Xi$.

Fourthly, by the following formula, the overall preference index $\pi$ is obtained:

$$\pi(t_i, t_j) = \sum_{f=0}^{n-1} P_f(t_i, t_j) \cdot \omega_f,$$

where $\omega_f = \frac{D_f^{\psi_{ij}}(K)}{\sum_{f=0}^{n-1} D_f^{\psi_{ij}}(K)}$. Especially, when $\sum_{f=0}^{n-1} D_f^{\psi_{ij}}(K) = 0$ and $w_f = 0$.

Furthermore, the leaving flow, the entering flow, and net flow are defined as:

$$\phi^+(t_i) = \sum_{j=1}^{n-1} \pi(t_j, t_i),$$

$$\phi^-(t_i) = \sum_{j=1}^{n-1} \pi(t_j, t_i),$$

$$\phi(t_i) = \phi^+(t_i) - \phi^-(t_i).$$

Finally, the alternative $t_i$ is ranked by the value of $\phi(t_i)$. The best alternative is selected according to the ranking order.

In the following, an algorithm of our proposed method is given as follows.
5.4 Procedure for decision-making method

**Algorithm 1** The algorithm of the PROMETHEE II method based on 1-BGFVPRS method in fuzzy relational structure

**Input:** A fuzzy relational structure \((S, \Upsilon \cup \Psi)\)

**Output:** The ranking of all members and the optimal member

1. begin for \(\Xi = \emptyset, R_h \in \Upsilon, \varphi_f \in \Psi, h \in \{0, 1, \cdots, y - 1\}, f \in \{0, 1, \cdots, x - 1\}\) do
   2. if \((\varphi_f) \circ R_h \subseteq R_h \circ (\varphi_f)\) and \(\varphi_f \circ R_h \subseteq R_h \circ \varphi_f\) then \(\Xi = \varphi_f \cup \Xi\) end
2. Given a parameter \(\theta\) and a fuzzy set \(K\)
   for \(t_i \in S, \varphi_f \in \Xi\)
   compute: \(\text{apr}_{\pi_f, \pi_f}^\varphi (K)(t_i) = \bigvee_{b \in S} T_2(\varphi_f(b, t_i), \bigwedge_{c \in S} I_1(\varphi_f(b, c), \theta \vee K(c)))\)
   compute: \(\text{apr}_{\pi_f, \pi_f}^\varphi (K)(t_i) = \bigwedge_{b \in S} I_2(\varphi_f(b, t_i), \bigvee_{c \in S} T_1(\varphi_f(b, c), N(\theta) \land K(c)))\)
3. for \(t_i, t_j \in S\)
   compute: \(D_f(t_i, t_j) = [\text{apr}_{\pi_f, \pi_f}^\varphi (K)(t_i) + \text{apr}_{\pi_f, \pi_f}^\varphi (K)(t_j)] - [\text{apr}_{\pi_f, \pi_f}^\varphi (K)(t_j) + \text{apr}_{\pi_f, \pi_f}^\varphi (K)(t_j)]\)
4. for \(t_i, t_j \in S\)
   compute: \(P_f(t_i, t_j) = D_f(t_i, t_j)\)
5. for \(\varphi_f \in \Xi\)
   if \(\sum_{v=0}^{z-1} d_{\varphi_f}^\varphi (K) = 0\), then \(\omega_f = 0\)
   else \(\omega_f = \sum_{v=0}^{z-1} d_{\varphi_f}^\varphi (K)\)
6. for \(t_i, t_j \in S\)
   compute: \(\pi(t_i, t_j) = \sum_{j=0}^{z-1} P_f(t_i, t_j) \cdot \omega_f\)
7. for \(t_i \in S\)
   compute: \(g^+(t_i) = \sum_{j=0}^{n-1} \pi(t_i, t_j)\)
   compute: \(g^-(t_i) = \sum_{j=0}^{n-1} \pi(t_j, t_i)\)
   compute: \(g(t_i) = g^+(t_i) - g^-(t_i)\)
8. for \(t_i, t_j \in S\)
   if \(g(t_i) > g(t_j)\), then \(t_i > t_j\)
   else \(t_i \leq t_j\)
9. return: The ranking of all members and the optimal member.

In this subsection, Algorithm 1 is the algorithm of our proposed method. The complexity of Algorithm 1 is \(O(z \ast n^2 + zxy)\). In this algorithm, its computational complexity is analyzed as follows. Step 1 is to obtain fuzzy bisimulations by Definition 7 and the computational complexity is \(O(xy)\). Step 2 is to get the approximation operators of the fuzzy set \(K\) and the computational complexity is \(O(n^2)\). Steps 3 and 4 are to calculate the preference between alternatives and the computational complexity is \(O(2n^2)\). Steps 5 and 6 are to obtain the overall preference between alternatives and the computational complexity is \(O(zn^2 + z)\). Step 7 is to get the net flow and the computational complexity is \(O(n^2)\).
Step 8 is to rank the alternatives and the computational complexity is $O(n^2)$. In a word, the computational complexity of Algorithm 1 is $O(xy + zn^2)$.

In the following, an illustrative example is given for the above algorithm.

### 5.5 An illustrative example

**Example 11** From reference [60], we observe that the karate club network shown as Fig. 9, was divided into two small subgroups by the conflict between the club president, John A., and Mr. Hi about the price of karate lessons. Mr. Hi wanted to raise the prices and asked the mechanism to give him the right to set his lesson fees due to that he was the instructor. However, John A, the chief executive of the club, insisted on stabilizing prices and claimed that he had the right to set the tuition.

As time went by, this contradiction split the original community into two small groups. However, these factions are only ideological groups, which have never materialized in the organization. It is worth noting that there are no political differences, and the club members do not name their names or even deny the existence of these factions. No one has tried to arrange or guide the political tactics of these groups.

To make the club better and stronger, the top management of the company came forward to make John A and Mr. Hi reconcile. For increasing the cohesion and competitiveness of the company came forward to make John A and Mr. Hi about the price of karate lessons. Because in workouts, club meetings, and karate classes, they interact with each other almost identically. The labels considered here for all members are the same, which can be considered as having no labels. In other words, this is an unlabeled approximation space. The line can be regarded as an edge. The edge in Fig. 9 is bidirectional which makes that the fuzzy relation $R$ is symmetrical.

Anthropologists have investigated and counted the frequency of interaction between club members, the depth of conversation, the influence of the conversation, and the mutual trust after communication. The degree of communication between members is shown in Tables 4 and 5. In the paper, $R(t_i, t_j) \neq 0$ denotes that there is one line between the colored numbered balls $i$ and $j$, which shows that the degree of communication between the members $t_i$ and $t_j$ is not 0, vice versa. For the sake of simplicity, some unnecessary data representations are omitted. For example $R(t_{14}, t_j) = 0$ \( (j \in \{0, 1, \ldots, 8\}) \) is omitted.

**Fig. 9** Zachary karate club network
Table 4  The information of fuzzy relation $R$  

| $R(t_i, t_j)$ | $t_0$     | $t_1$     | $t_2$     | $t_3$     | $t_4$     |
|--------------|-----------|-----------|-----------|-----------|-----------|
| $t_0$        | 0         | 0.5015    | 0.1381    | 0.6123    | 0.7508    |
| $t_1$        | 0.5014    | 0         | 0.4762    | 0.4228    | 0         |
| $t_2$        | 0.1381    | 0.4762    | 0         | 0.21      | 0         |
| $t_3$        | 0.6123    | 0.4228    | 0.21      | 0         | 0         |
| $t_4$        | 0.7508    | 0         | 0         | 0         | 0         |
| $t_5$        | 0.1661    | 0         | 0         | 0         | 0         |
| $t_6$        | 0.3146    | 0         | 0         | 0         | 0.4867    |
| $t_7$        | 0.4855    | 0.4447    | 0.2177    | 0.5053    | 0         |
| $t_8$        | 0.7258    | 0         | 0.4558    | 0         | 0         |
| $t_9$        | 0         | 0         | 0.4868    | 0         | 0         |
| $t_{10}$     | 0.5546    | 0         | 0         | 0         | 0.2545    |
| $t_{11}$     | 0.7263    | 0         | 0         | 0         | 0         |
| $t_{12}$     | 0.513     | 0         | 0         | 0.4034    | 0         |
| $t_{13}$     | 0.5766    | 0.7454    | 0.5876    | 0.8032    | 0         |
| $t_{14}$     | 0         | 0         | 0         | 0         | 0         |
| $t_{15}$     | 0.5547    | 0.4447    | 0.2177    | 0.5053    | 0         |
| $t_{16}$     | 0.7258    | 0         | 0.4558    | 0         | 0         |
| $t_{17}$     | 0         | 0         | 0.4868    | 0         | 0         |
| $t_{18}$     | 0         | 0         | 0         | 0         | 0         |
| $t_{19}$     | 0         | 0         | 0         | 0         | 0         |
| $t_{20}$     | 0         | 0         | 0         | 0         | 0         |
| $t_{21}$     | 0         | 0         | 0         | 0         | 0         |
| $t_{22}$     | 0         | 0         | 0         | 0         | 0         |
| $t_{23}$     | 0         | 0         | 0         | 0         | 0         |
| $t_{24}$     | 0         | 0         | 0         | 0         | 0         |
| $t_{25}$     | 0         | 0         | 0         | 0         | 0         |
| $t_{26}$     | 0         | 0         | 0         | 0         | 0         |
| $t_{27}$     | 0         | 0         | 0         | 0         | 0         |
| $t_{28}$     | 0         | 0         | 0         | 0         | 0         |
| $t_{29}$     | 0         | 0         | 0         | 0         | 0         |
| $t_{30}$     | 0         | 0         | 0         | 0         | 0         |
| $t_{31}$     | 0         | 0         | 0         | 0         | 0         |
| $t_{32}$     | 0         | 0         | 0         | 0         | 0         |
| $t_{33}$     | 0         | 0         | 0         | 0         | 0         |

| $R(t_i, t_j)$ | $t_5$     | $t_6$     | $t_7$     | $t_8$     |
|--------------|-----------|-----------|-----------|-----------|
| $t_0$        | 0.1662    | 0.3146    | 0.4855    | 0.7258    |
| $t_1$        | 0         | 0         | 0.4447    | 0         |
| $t_2$        | 0         | 0         | 0.2178    | 0.4558    |
| $t_3$        | 0         | 0         | 0.5053    | 0         |
| $t_4$        | 0         | 0.4867    | 0         | 0         |
| $t_5$        | 0         | 0.4867    | 0         | 0         |
| $t_6$        | 0.6181    | 0         | 0         | 0         |
| $t_7$        | 0.4849    | 0         | 0         | 0         |
| $t_8$        | 0.3639    | 0.5237    | 0         | 0         |
| $t_9$        | 0         | 0         | 0         | 0.3515    |
| $t_{10}$     | 0         | 0         | 0         | 0.3697    |
| $t_{11}$     | 0         | 0         | 0         | 0.2043    |

| $R(t_i, t_j)$ | $t_{12}$ | $t_{13}$ | $t_{14}$ | $t_{15}$ | $t_{16}$ | $t_{17}$ |
|--------------|----------|----------|----------|----------|----------|----------|
| $t_0$        | 0        | 0.5546   | 0.7263   | 0.513    | 0.5766   | 0.7454   |
| $t_1$        | 0        | 0        | 0        | 0        | 0.5876   | 0.8032   |
| $t_2$        | 0.4868   | 0        | 0        | 0        | 0.4034   | 0.8032   |
| $t_3$        | 0        | 0.2545   | 0        | 0        | 0        | 0        |
| $t_4$        | 0.2592   | 0        | 0        | 0        | 0        | 0        |
| $t_{13}$     | 0.5547   | 0.4447   | 0.2177   | 0.5053   | 0        | 0        |
| $t_{14}$     | 0        | 0        | 0        | 0        | 0        | 0        |
| $t_{15}$     | 0        | 0        | 0        | 0        | 0        | 0        |
| $t_{16}$     | 0        | 0        | 0        | 0        | 0        | 0        |
| $t_{17}$     | 0        | 0        | 0        | 0        | 0        | 0        |

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Table 4 (continued)

|       | \textit{t}_{14} | \textit{t}_{15} | \textit{t}_{16} | \textit{t}_{17} |
|-------|----------------|----------------|----------------|----------------|
| \textit{t}_5 | 0              | 0              | 0.3639         | 0              |
| \textit{t}_6 | 0              | 0              | 0.5237         | 0              |
| \textit{t}_{32} | 0.2446      | 0.7147         | 0              | 0              |
| \textit{t}_{33} | 0.6341      | 0.1818         | 0              | 0              |

|       | \textit{t}_{18} | \textit{t}_{19} | \textit{t}_{20} | \textit{t}_{21} | \textit{t}_{22} |
|-------|----------------|----------------|----------------|----------------|----------------|
| \textit{t}_0 | 0              | 0.7245         | 0              | 0.1618         | 0              |
| \textit{t}_1 | 0              | 0.1776         | 0              | 0.4564         | 0              |
| \textit{t}_2 | 0.4868         | 0              | 0              | 0              | 0.5876         |
| \textit{t}_{32} | 0.2025       | 0              | 0.7056         | 0              | 0.4584         |
| \textit{t}_{33} | 0.2754       | 0.5802         | 0.9985         | 0              | 0.5212         |

|       | \textit{t}_{27} | \textit{t}_{28} | \textit{t}_{29} | \textit{t}_{30} | \textit{t}_{31} | \textit{t}_{32} | \textit{t}_{33} |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \textit{t}_0 | 0              | 0              | 0.7245         | 0              | 0.1618         | 0              | 0.3578         |
| \textit{t}_1 | 0              | 0              | 0              | 0.6322         | 0              | 0              | 0              |
| \textit{t}_2 | 0.3637         | 0.6358         | 0              | 0              | 0              | 0              | 0              |
| \textit{t}_{27} | 0.6619       | 0.6736         | 0              | 0              | 0              | 0              | 0              |
| \textit{t}_{29} | 0.328        | 0              | 0              | 0              | 0.5642         | 0              | 0              |
| \textit{t}_{31} | 0            | 0.7191         | 0.3349         | 0              | 0              | 0              | 0              |
| \textit{t}_{32} | 0.568        | 0              | 0              | 0              | 0              | 0              | 0              |
| \textit{t}_{33} | 0.4903       | 0              | 0              | 0              | 0.6108         | 0              | 0              |

Table 5 The information of fuzzy relation \( R \)

|       | \textit{t}_{27} | \textit{t}_{28} | \textit{t}_{29} | \textit{t}_{30} | \textit{t}_{31} | \textit{t}_{32} | \textit{t}_{33} |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \textit{t}_0 | 0              | 0              | 0.7245         | 0              | 0.1618         | 0              | 0.3578         |
| \textit{t}_1 | 0              | 0              | 0              | 0.6322         | 0              | 0              | 0              |
| \textit{t}_2 | 0.3637         | 0.6358         | 0              | 0              | 0              | 0              | 0.2584         |
| \textit{t}_{33} | 0.6619       | 0.6736         | 0              | 0              | 0              | 0              | 0              |
| \textit{t}_{29} | 0.328        | 0              | 0              | 0              | 0.5642         | 0              | 0              |
| \textit{t}_{31} | 0            | 0.7191         | 0.3349         | 0              | 0              | 0              | 0              |
| \textit{t}_{32} | 0.568        | 0              | 0              | 0              | 0              | 0              | 0              |
| \textit{t}_{33} | 0.4903       | 0              | 0              | 0              | 0.6108         | 0              | 0              |
5.6 Comparative analysis

In this section, some comparative analyses are given. Firstly, a comparison of ranking results about our proposed method for different logical operators is demonstrated in the following.

5.6.1 A comparative analysis for different logical operators

Since $S$-implicator, $R$-implicator and $QL$-implicator are three kinds of popular and important implicators, by using these kinds of implicators and some usually used continuous $t$-norms in [41], we have 198 types of 1-BGFVPRS models, 198 types of 2-BGFVPRS models and 18 types of 3-BGFVPRS models. In the following, we random extract 15 types of 1-BGFVPRS models to apply our method for solving the problems. Table 11 gives the information of 15 cases and the optimal alternatives of each case. It should be noted that $I_i$ and $T_o$ ($\ast = L, G, \Delta, KD, \bullet, Z; \sigma = M, L, P$) [41] are an implicator and a continuous $t$-norm, respectively.

In order to compare the correlation of ranking results, we introduce Spearman’s rank correlation coefficient (SRCC). Suppose that the two random variables are $A$ and $B$ (can also be regarded as two sets), and the number of elements is $k$. The $ith$ ($1 \leq i \leq k$) value of the two random variables is represented by $X_i$ and $Y_i$ respectively. Sorting $A$ and $B$ (ascending or descending at the same time) to obtain two element ranking sets $X$ and $Y$ where elements $X_i$ and $Y_i$ are
Table 9 The results of the overall preference \( \pi(t_i, t_j) \)

| \( \pi(t_i, t_j) \) | \( t_0 \) | \( t_1 \) | \( t_2 \) | \( t_3 \) | \( t_4 \) | \( t_5 \) | \( t_6 \) |
|-------------------|------|------|------|------|------|------|------|
| \( t_0 \)        | 0    | 0.0541 | 0.3773 | \( \ldots \) | 0.5373 | 0.1141 | -0.0759 |
| \( t_1 \)        | -0.0541 | 0    | 0.3232 | \( \ldots \) | 0.4832 | 0.06  | -0.13  |
| \( t_2 \)        | -0.3773 | -0.3232 | 0  | \( \ldots \) | 0.16  | -0.2632 | -0.4532 |
| \( t_3 \)        | -0.4719 | -0.4178 | -0.0946 | \( \ldots \) | 0.0654 | -0.3578 | -0.5478 |
| \( t_4 \)        | -0.1677 | -0.1136 | 0.2096 | \( \ldots \) | 0.3696 | -0.0536 | -0.2436 |
| \( \vdots \)     | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( t_{16} \)     | -0.2543 | -0.2002 | 0.123 | \( \ldots \) | 0.283 | -0.1402 | -0.3302 |
| \( t_{17} \)     | -0.1611 | -0.1070 | 0.2162 | \( \ldots \) | 0.3762 | -0.047 | -0.237 |
| \( t_{18} \)     | -0.4141 | -0.36  | -0.0368 | \( \ldots \) | 0.1232 | -0.3  | -0.49  |
| \( \vdots \)     | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( t_{30} \)     | -0.2687 | -0.2146 | 0.1086 | \( \ldots \) | 0.2686 | -0.1546 | -0.3446 |
| \( t_{31} \)     | -0.5373 | -0.4832 | 0.016 | \( \ldots \) | 0  | -0.4232 | -0.6132 |
| \( t_{32} \)     | -0.1141 | -0.06  | 0.2632 | \( \ldots \) | 0.4232 | 0  | -0.19  |
| \( t_{33} \)     | 0.0759 | 0.13  | 0.4532 | \( \ldots \) | 0.6132 | 0.19  | 0  |

Table 10 The results of \( \rho^+(t_i) \), \( \rho^-(t_i) \) and \( \rho(t_i) \)

| \( t_i \) | \( t_0 \) | \( t_1 \) | \( t_2 \) | \( t_3 \) | \( t_4 \) | \( t_5 \) | \( t_6 \) |
|--------|------|------|------|------|------|------|------|
| \( \rho^+(t_i) \) | 10.0061 | 8.1667 | -2.8221 | -6.0385 | 4.3043 | 1.6421 | -9.7887 |
| \( \rho^-(t_i) \) | -10.0061 | -8.1667 | 2.8221 | 6.0385 | -4.3043 | -1.6421 | 9.7887 |
| \( \rho(t_i) \) | 20.0122 | 16.3334 | -5.6442 | -12.077 | 8.6086 | 3.2842 | -19.5774 |
| \( t_i \) | \( t_7 \) | \( t_8 \) | \( t_9 \) | \( t_{10} \) | \( t_{11} \) | \( t_{12} \) | \( t_{13} \) |
| \( \rho^+(t_i) \) | -2.4549 | 0.0679 | -9.1087 | -8.0207 | 4.5321 | 1.9413 | -7.4733 |
| \( \rho^-(t_i) \) | 2.4549 | -0.0679 | 9.1087 | 8.0207 | -4.5321 | -1.9413 | 7.4733 |
| \( \rho(t_i) \) | -4.9098 | 0.1358 | -18.2174 | -16.0414 | 9.0642 | 3.8826 | -14.9466 |
| \( t_i \) | \( t_{21} \) | \( t_{22} \) | \( t_{23} \) | \( t_{24} \) | \( t_{25} \) | \( t_{26} \) | \( t_{27} \) |
| \( \rho^+(t_i) \) | -1.0337 | -2.1965 | 3.7127 | 3.7127 | 2.4819 | 2.9715 | 2.7267 |
| \( \rho^-(t_i) \) | 1.0337 | 2.1965 | -3.7127 | -3.7127 | -2.4819 | -2.9715 | -2.7267 |
| \( \rho(t_i) \) | -2.0674 | -4.393 | 7.4254 | 7.4254 | 4.9638 | 5.943 | 5.4534 |
| \( t_i \) | \( t_{28} \) | \( t_{29} \) | \( t_{30} \) | \( t_{31} \) | \( t_{32} \) | \( t_{33} \) |
| \( \rho^+(t_i) \) | -9.1087 | 1.3905 | 0.8703 | -8.262 | 6.1267 | 12.5867 |
| \( \rho^-(t_i) \) | 9.1087 | -1.3905 | -0.8703 | 8.2621 | -6.1267 | -12.5867 |
| \( \rho(t_i) \) | -18.2174 | 2.781 | 1.7406 | -16.5242 | 12.2534 | 25.1734 |

the \( i \)th values in the ranking of \( X \) and \( Y \), respectively. The elements in sets \( X \) and \( Y \) are correspondingly subtracted to obtain a ranking difference set \( d = \{ d_i \mid i = 1, 2, \ldots, k \} \) where \( d_i = X_i - Y_i \). The SRCC (\( \rho \)) between random variables \( A \) and \( B \) can be calculated as follows:

\[
\rho = 1 - \frac{6 \sum_{i=1}^{k} d_i^2}{k(k^2 - 1)}.
\]

From Table 11, we find that in most cases, the best member is \( t_{33} \), while in Case 2, the best member is \( t_0 \). The reason is that the selection of different logical operators has a certain influence on sorting order. Although the optimal results of Case 15 and Case 2 are inconsistent, the SRCC of the two ranking results is 0.7406, which is highly correlated.

5.6.2 A comparative analysis for different variable precision values

In this section, we investigate how \( \theta \) affects the ranking orders of all considered members. When \( \theta \) changes its value, we make some comparisons among several cases as figures 10 and 11. The value range of \( \theta \) is \([0, 1]\), and the step length is 0.1. From Figs. 10 and 11, the following conclusions can be drawn.
With the change of variable precision value $\theta$, the ranking of each case will change, but the optimal members of them are consistent. For example, in Fig. 10, except for Case 2, when taking the same $\theta$, the optimal members of other five cases are highly consistent in most of the time.

In Fig. 10, for Case 2, when the variable precision is greater than a certain value, its sensitivity is not high. In particular, when $\theta = 0.9$, using Case 2 cannot help DMs to select the best member. Therefore, the DMs may not consider Case 2 when making decisions.

Through Fig. 11a, when $\theta = 0.95$, the best optimal member cannot be obtained in Cases 1–3. But, the optimal member can be selected in the other cases. In Cases 4 and 5, more than one optimal member is got while the best one can be obtained in Cases 6 and 15.

Figures 10 and 11a show that when $0.9 < \theta < 1$, the optimal member cannot be selected in some cases. This phenomenon is caused by human subjective factors interfere so strongly that many objective data sets are invalid. In fact, in these conditions the results obtained are unfair for the members under the test. Generally speaking, the variable precision value should not be too large or infinitely close to 1. If an irrational decision-maker has to make a decision in this situation, we can still help him make a decision and choose the best member, such as Case 15.

By Figs. 10 and 11a, the optimal member for Case 6 changes while the optimal member for Case 15 still stays the same i.e., $t_{33}$.

Above, the DMs can select different logical operators and variable precision values according to their actual needs. This shows the flexibility of the method in the paper.

### Table 11 The information of 15 cases and the optimal alternative of each case

| Case   | $I_1$ | $I_2$ | $T_1$ | $T_2$ | $N$  | $\theta$ | The optimal member |
|--------|-------|-------|-------|-------|------|----------|-------------------|
| Case 1 | $I_L$ | $I_L$ | $T_L$ | $T_L$ | $N_s$| 0.45     | $t_{33}$          |
| Case 2 | $I_{KD}$ | $I_L$ | $T_M$ | $T_L$ | $N_s$| 0.45     | $t_0$             |
| Case 3 | $I_L$ | $I_{KD}$ | $T_L$ | $T_M$ | $N_s$| 0.45     | $t_{33}$          |
| Case 4 | $I_L$ | $I_{KD}$ | $T_L$ | $T_P$ | $N_s$| 0.45     | $t_{33}$          |
| Case 5 | $I_L$ | $I_s$ | $T_L$ | $T_L$ | $N_s$| 0.45     | $t_{33}$          |
| Case 6 | $I_L$ | $I_s$ | $T_L$ | $T_M$ | $N_s$| 0.45     | $t_{33}$          |
| Case 7 | $I_L$ | $I_z$ | $T_L$ | $T_L$ | $N_s$| 0.45     | $t_{33}$          |
| Case 8 | $I_s$ | $I_z$ | $T_L$ | $T_L$ | $N_s$| 0.45     | $t_{33}$          |
| Case 9 | $I_L$ | $I_G$ | $T_L$ | $T_L$ | $N_s$| 0.45     | $t_{33}$          |
| Case 10 | $I_s$ | $I_z$ | $T_P$ | $T_L$ | $N_s$| 0.45     | $t_{33}$          |
| Case 11 | $I_{\Delta}$ | $I_L$ | $T_L$ | $T_L$ | $N_s$| 0.45     | $t_{33}$          |
| Case 12 | $I_{\Delta}$ | $I_s$ | $T_P$ | $T_L$ | $N_s$| 0.45     | $t_{33}$          |
| Case 13 | $I_{\Delta}$ | $I_z$ | $T_P$ | $T_L$ | $N_s$| 0.45     | $t_{33}$          |
| Case 14 | $I_{\Delta}$ | $I_{KD}$ | $T_P$ | $T_L$ | $N_s$| 0.45     | $t_{33}$          |
| Case 15 | $I_{\Delta}$ | $I_G$ | $T_P$ | $T_L$ | $N_s$| 0.45     | $t_{33}$          |

#### 5.6.3 A comparative analysis for different MADM methods

In this section, from a comparison among the classical PROMETHEE method (including PROMETHEE I and PROMETHEE II) in [5, 6], some popular MADM methods in [20, 21, 62–64] and our proposed method, the effectiveness and feasibility of our presented method are demonstrated. Through the analysis of the design principle and application environment of these methods, their differences are shown as Table 12. In this table, for simplicity, using FISs denotes functional information systems [42] while using RISs represents relational information systems [42].

By Table 12, some conclusions are made as follows:

A data table in RS consists of a set of objects and a set of attributes. In the data table, every attribute is considered as a function from a set of objects to the attributes’ domain of values. In [42], FISs are data tables that represent attribute data. In the FISs, the methods proposed in [5, 6, 20, 21, 62–64] only consider the information about individual objects but do not consider the relations between objects. Sometimes, it may be necessary to take into account the relations between objects. For example, in social network analysis, if you only consider the attribute data, you will lose much useful information since its main data types are attribute data [42] and relational data [42]. In this case, using the methods in [5, 6, 20, 21, 62–64] may be unavailable. The RISs [15] comprise a set of objects, a set of attributes and a set of relations between objects. In the RISs, using our proposed method can deal with the problems which the above mentioned
Fig. 10  The comparison of the ranking results for 6 cases with different $\theta$
methods cannot solve and helps DMs make a clear decision.

- From Table 12, we find that the weights in [6, 20, 21, 62–64] are given by experts or DMs while the weights in [5] and our proposed method are calculated from the raw data by a formula. In general, the weights given by experts or DMs are called subjective weights while the weights computed by a formula are named objective weights. For the methods in [6, 20, 21, 62–64], if the experts have different opinions, then the weights are difficult to obtain. Or, if the DMs are lack of experience, the weights obtained may not be suitable. That is, using these methods may be infeasible in these cases. But using objective weight formulas can effectively avoid these situations.

- From the above table, the two methods in [5, 6] are not based on relations. The methods in [20, 62, 64] are based on the fuzzy β-neighborhood operators while other methods in [21, 63] are based on the fuzzy neighborhood operators. Different from the aforementioned methods, our proposed method is on the basis of fuzzy bisimulations. It is noted that the fuzzy β-neighborhood operators and fuzzy neighborhood operators are binary fuzzy relations that only obtain “one step” information of the potential relations. However, for some complex problems, “one step” information may not be enough to discern objects. The fuzzy bisimulations which are natural generations of bisimulations can help us to characterize indiscernibility by exploiting “multi-step” information. With fuzzy bisimulations, the application scope of the classical PROMETHEE II has been expanded. For example, our proposed decision-making method can solve many MADM problems involving relational structure data where some popular MADM methods in [5, 6, 20, 21, 62–64] cannot solve.

**Table 12** The differences among some methods in [5, 6, 20, 21, 62–64] and our proposed method

| Methods                          | Information systems | Weights     | Relations                                      |
|----------------------------------|---------------------|-------------|------------------------------------------------|
| Brans and Vincke’s method [5]    | FISs                | Objective   | None                                           |
| Brans et al.’s method [6]        | FISs                | Subjective  | None                                           |
| Jiang et al.’s method [20]       | FISs                | Subjective  | Fuzzy β-neighborhood operators                 |
| Jiang et al.’s method [21]       | FISs                | Subjective  | Fuzzy neighborhood operators                   |
| Zhan et al.’s [62]              | FISs                | Subjective  | Fuzzy β-neighborhood operators                 |
| Zhang et al.’s [63]              | FISs                | Subjective  | Fuzzy β-neighborhood operators                 |
| Zhang et al.’s [64]              | FISs                | Subjective  | Fuzzy β-neighborhood operators                 |
| Our proposed method              | RISs                | Objective   | Fuzzy bisimulations                            |

**Table 13** A comparison among some methods in Example 11

| Methods                          | Ranking order | Optimal member |
|----------------------------------|---------------|----------------|
| Brans and Vincke’s method [5]    | ×             | ×              |
| Brans et al.’s method [6]        | ×             | ×              |
| Jiang et al.’s method [20]       | ×             | ×              |
| Jiang et al.’s method [21]       | ×             | ×              |
| Zhan et al.’s [62]              | ×             | ×              |
| Zhang et al.’s [63]              | ×             | ×              |
| Zhang et al.’s [64]              | ×             | ×              |
| Our proposed method              | ✓             | ✓              |

Fig. 11 A comparison for 7 cases when $\theta = 0.95$ and another comparison for Case 1 when $\theta \in [0, 1)$.
Furthermore, we apply some methods in [5, 6, 20, 21, 62–64] and our proposed method in Example 11. A comparison among them is given as Table 13. From this table, we have the following statements:

- Using the methods in [5, 6, 20, 21, 62–64] cannot get a ranking order for the considered members while using our proposed method can obtain a ranking order for them.
- By means of the methods in [5, 6, 20, 21, 62–64], the optimal member cannot be obtained while using our proposed method can get an optimal member.

Above all, using the methods in [5, 6, 20, 21, 62–64] cannot help DMs make a clear decision while using our proposed method can do it. This demonstrates the feasibility and effectiveness of our proposed method in the paper.

5.6.4 A comparative analysis for different BGFVPRS models

Different models may affect the results of decision-making results. The reason is that the different construction principles of different models will affect the decision-making results in a certain extent. In the following, an example is given to demonstrate this.

**Example 12** Let $I_1 = I_3 = I_\Delta, I_2 = I_G, T_1 = T_3 = T_p, T_2 = T_L, \Box = \wedge$ and $\theta = 0.45$. Furthermore, 1-BGFVPRS, 2-BGFVPRS, and 3-BGFVPRS models are applied in Algorithm 1, respectively. The information of K is given as Table 6. Under the same condition in Case 1, through computation, a comparison of the results for these three type of models is shown as Table 14. From this table, we find that the optimal member of 3-BGFVPRS is $t_{30}$ while the optimal members of 1-BGFVPRS and 2-BGFVPRS are $t_{33}$. The optimal member of 3-BGFVPRS is different from the other two types of BGFVPRS models. The ranking orders of three models are different. Above all, applying different models in Algorithm 1 may obtain different ranking orders of members and different optimal members. When making a decision, 1-BGFVPRS model plays a better performance than the other two models. The reason is that the accuracy measure of 1-BGFVPRS model is bigger than other two models. As the accuracy measure of the BGFVPRS model increases, the decision-making results get more accurate and closer to the actual needs. In order to make the decision results closer to the DMs’ actual needs, the 1-BGFVPRS is a better option than other two models.

5.7 Sensitivity analysis

The variable precision $\theta$ acts a critical role in BGFVPRS models, which is verified in Section 3. In subsection 5.6.2, the change of sorting results is studied when different variable precision values are taken in Case 1. Here, the value range of variable precision $\theta$ is [0, 1], where the step size is 0.05. The comparison among 20 ranking results in Case 1 when taking different variable precision values is given in Fig. 11b. From the figure, the optimal member is $t_{33}$ when $\theta \in [0, 0.8)$ while the optimal members are $t_{15}, t_{30}, t_{33}$ with $\theta \in [0.8, 0.95)$. Especially, if $\theta \in [0.95, 1)$, the optimal member cannot be selected.

**Remark 6** Through comparative analysis and sensitivity analysis, the following conclusions can be drawn.

(1) Our approach is highly flexible when solving complex problems by selecting different logical operators and changing the value of $\theta$.
(2) Although the ranking order will fluctuate after selecting different logic operators, the optimal results are consistent, which shows the effectiveness of the decision-making method on the other hand.
(3) Compared with some popular MADM methods in [5, 6, 20, 21, 62–64], our provided method can solve the complex problems which consist of attribute data and relational data. That is to say, this method can solve the MADM problems involving relational structure data.

6 Conclusion

Aiming at the complex MADM problems including attribute data and relational data, the paper has proposed a decision-making method. This method is based on BGFVPRS models which are the natural generalizations of the classical RS model. The BGFVPRS models are based on the fuzzy bisimulations. With these relations, the theoretical research and the application of the classical RS model has been enriched and expanded, respectively. The BGFVPRS models have many important properties such as duality. Using the concept of reduction, the redundant data can be removed quickly. According to the θ-accuracy measures (or θ-roughness measures) of BGFVPRS models defined in the paper, the compactness of them can be compared very well. Compared with many MADM methods, the method presented in the paper is effective to solve the problems including relational data. The comparative analysis and sensitivity analysis have shown that the flexibility and effectiveness of our decision-making method in the Zachary karate club network. For example, although the ranking order will fluctuate after selecting different logic operators and the values of θ, the optimal results are consistent.

The future research directions are shown as follows.

- Because the data in real life does not always exist in the form of numbers, it should be considered to extend our BGFVPRS models to solve linguistic problems [37, 38] and design an effective decision-making method for dealing with these problems by building a tool to minimize information loss.
- Motivated by the concept of weak bisimulations [17], the BGFVPRS models can be generalized and some related important properties about the models may be got.
- Learning from many popular three-way decision-making methods [11, 19, 31, 52], we will further modify our decision-making method and apply it to more complex environments such as the IF [2] environment.
- The granular computing is an important concept. Many scholars applied it in three-way decision-making methods [30, 47]. Motivated by this, integrating the concepts of fuzzy bisimulations, three-way decision-making methods and granular computing will provide a novel decision-making method.

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References

1. Abo-Tabl EA (2011) A comparison of two kinds of definitions of rough approximations based on a similarity relation. Inf Sci 181:2587–2596
2. Atanassov KT (1999) Intuitionistic fuzzy sets: theory and applications. Phys.-Verlag, Heidelberg, p 1999
3. Behret H (2014) Group decision making with intuitionistic fuzzy preference relations. Knowl Based Syst 70:33–43
4. Bai CZ, Zhang R, Qian LX, Liu LJ, Wu YN (2019) An ordered clustering algorithm based on fuzzy c-means and PROMETHEE. Int J Mach Learn Cybern 10:1423–1436
5. Brans JP, Vincke P (1985) A preference ranking organization: the promethee method for multiple criteria decision making. Manag Sci 31:647–656
6. Brans JP, Vincke P, Mareschal B (1986) How to select and how to rank projects: the PROMETHEE method. Eur J Oper Res 24:228–238
7. Cao YZ, Chen GQ, Kerre EE (2011) Bisimulations for fuzzy-transition systems. IEEE Trans Fuzzy Syst 19(3):540–552
8. Chen LH, Xu ZS, Wang H, Liu SS (2018) An ordered clustering algorithm based on K-means and the PROMETHEE method. Int J Mach Learn Cybern 9:917–926
9. Cirić M, Ignjatovic J, Damjanovic N, Basic M (2012) Bisimulations for fuzzy automata. Fuzzy Sets Syst 186:100–139
10. Cock MD, Cornelis C, Kerre EE (2004) Fuzzy rough sets: beyond the obvious. In: Proceedings of the 2004 IEEE international conference on fuzzy systems, vol 1. pp 103–108
11. Deng J, Zhan JM, Wu WZ (2021) A three-way decision methodology to multi-attributing decision-making in multi-scale decision information systems. Inf Sci 568:175–198
12. Du YB, Zhu P (2018) Fuzzy approximations of fuzzy relational structures. Int J Approx Reason 98:1–10
13. Du YB, Zhu P (2018) Labeled fuzzy approximations based on bisimulations. Int J Approx Reason 94:43–59
14. Fan TF (2013) Rough set analysis of relational structures. Inf Sci 221:230–244
15. Fan TF, Liu DR, Tzeng G (2006) Arrow decision logic for relational information systems. Transactions on Rough Sets V, pp 40–262
16. Figueira J, Greco S, Ehrgott M (2005) Multiple criteria decision analysis: state of the art surveys. Springer, New York
17. Jancic I (2014) Weak bisimulations for fuzzy automata. Fuzzy Sets Syst 249:49–72
18. Jarvinen J, Radeleczki S (2014) Rough sets determined by tolerances. Int J Approx Reason 55:1419–1438
19. Jiang HB, Hu BQ (2021) A novel three-way group investment decision model under intuitionistic fuzzy multi-attributed decision-making environment. Inf Sci 569:557–581
20. Jiang HB, Zhan JM, Chen DG (2019) Covering based variable precision (I, J)-fuzzy rough sets with applications to multi-attributed decision-making. IEEE Trans Fuzzy Syst 27(8):1558–1572
21. Jiang HB, Zhan JM, Chen DG (2021) PROMETHEE II method based on variable precision fuzzy rough sets with fuzzy neighborhoods. Artif Intell Rev 54:1271–1319
22. Jiang HB, Zhan JM, Sun BZ, Alcantud JCR (2020) An MADM approach to covering-based variable precision fuzzy rough sets: an application to medical diagnosis. Int J Mach Learn Cybern 11:2181–2207
23. Keller RM (1976) Formal verification of parallel programs. Commun ACM 19(7):371–384
24. Kerre EE (1988) Fuzzy sets and approximate reasoning, lecture notes, summer session, 1988. University of Nebraska, Lincoln
25. Klir GJ, Yuan B (1995) Fuzzy sets and fuzzy. Logic theory and application. Prentice-Hall, Englewood Cliffs
26. Kryszkiewicz M (1998) Rough set approach to incomplete information systems. Inf Sci 112:39–49
27. Li TJ, Leung Y, Zhang WX (2008) Generalized fuzzy rough fuzzy approximation operators based on fuzzy coverings. Int J Approx Reason 48(3):836–856
28. Li W, Li B (2010) An extension of the Promethee II method based on generalized fuzzy numbers. Experts Syst Appl 37:5314–5319
29. Liao HC, Xu ZS (2014) Multi-criteria decision making with intutionistic fuzzy PROMETHEE. J Intell Fuzzy Syst 27:1703–1717
30. Liu D, Yang X, Li TR (2020) Three-way decisions: beyond rough sets and granular computing. Int J Mach Learn Cybern 11:989–1002
31. Liu PD, Wang YM, Jia F, Fujita H (2020) A multiple attribute decision making three-way model for intutionistic fuzzy numbers. Int J Approx Reason 119:177–203
32. Luo S, Miao DQ, Zhang ZF, Zhang YJ, Hu SD (2020) A neighborhood rough set model with nominal metric embedding. Inf Sci 520:373–388
33. Mardani A, Jusoh A, Zavadskas EK (2015) Fuzzy multiple criteria decision-making techniques and applications-two decades review from 1994 to 2014. Expert Syst Appl 42:4126–4148
34. Merigo JM, Gil-Lafuente AM (2010) New decision-making techniques and their application in the selection of financial products. Inf Sci 180:2085–2094
35. Mi JS, Wu WZ, Zhang WX (2004) Approaches to knowledge reduction based on variable precision rough set model. Inf Sci 159(3–4):255–272
36. Milner R (1989) Communication and concurrency. Prentice-Hall, Englewood Cliffs, p 1989
37. Muhuri P, Gupta P (2019) Extended Tsukamoto’s inference method for solving multi-objective linguistic optimization problems. Fuzzy Sets Syst 377:102–124
38. Muhuri P, Gupta P (2020) A novel solution approach for multiobjective linguistic optimization problems based on the 2-tuple fuzzy linguistic representation model. Appl Soft Comput 95:106–395
39. Ovchinnikov S (1991) Similarity relations, fuzzy partitions, and fuzzy orderings. Fuzzy Sets Syst 40:107–126
40. Pawlak Z (1982) Rough sets. Int J Comput Inf Sci 11:341–356
41. Radzikowska AM, Kerre EE (2002) A comparative study of fuzzy rough sets. Fuzzy Sets Syst 126:137–155
42. Scott J (2000) Social network analysis: a handbook, 2nd edn. SAGE Publications, Thousand Oaks
43. Skowron A, Stepaniuk J (1996) Tolerance approximation spaces. Fundam Inf 27:245–253
44. Słowiński R, Vanderpooten D (2000) A generalized definition of rough approximations based on similarity. IEEE Trans Knowl Data Eng 12(2):331–336
45. Walczak D, Rutkowska A (2017) Project rankings for participatory budget based on the fuzzy TOPSIS method. Eur J Oper Res 260:706–714
46. Wang C, Huang Y, Ding W, Cao Z (2021) Attribute reduction with fuzzy rough self-information measures. Inf Sci 549:68–86
47. Wang XZ, Li JH (2020) New advances in three-way decision, granular computing and concept lattice. Int J Mach Learn Cybern 11:945–946
48. Wu H, Liu G (2020) The relationships between topologies and generalized rough sets. Int J Approx Reason 119:313–324
49. Wu MC, Chen TY (2011) The ELECTRE multicriteria analysis approach based on Atanassovs intutionistic fuzzy sets. Expert Syst Appl 38(10):12318–12327
50. Wu WZ, Leung Y, Mi JS (2005) On characterizations of (J,T)-fuzzy rough approximation operators. Fuzzy Sets Syst 154:76–102
51. Wu Y, Zhang B, Wu C, Zhang T, Liu F (2019) Optimal site selection for parabolic trough concentrating solar power plant using extended PROMETHEE method: a case in China. Renew Energy 143:1910–1927
52. Yang B, Li JH (2020) Complex network analysis of three-way decision researches. Int J Mach Learn Cybern 11:973–987
53. Yang XL, Chen HM, Li TR, Wan JH, Sang BB (2021) Neighborhood rough sets with distance metric learning for feature selection. Knowl Based Syst 224:107076
54. Yao YY (1998) On generalizing pawlak approximation operators. Lect Notes Artif Intell 1424:298–307
55. Yao YY (1998) Relational interpretations of neighborhood operators and rough set approximation operators. Inf Sci 111:239–259
56. Yao YY, Wong SKM (1992) A decision theoretic framework for approximating concepts. Int J Man Mach Stud 37:793–809
57. Yatsalo B, Korobov A, Oztaysi B, Kahraman C, Martinez L (2020) Fuzzy extensions of PROMETHEE: models of different complexity with different ranking methods and their comparison. Fuzzy Sets Syst. https://doi.org/10.1016/j.fss.2020.08.015
58. Ye J, Zhan JM, Ding W, Fujita H (2021) A novel fuzzy rough set model with fuzzy neighborhood operators. Inf Sci 544:266–297
59. Yu X, Zhang S, Liao X, Qi X (2017) ELECTRE methods in prioritized MCDM environment. Inf Sci 424:301–316
60. Zachary WW (1977) An information flow model for conflict and fission in small groups. J Anthropol Res 33:452–473
61. Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353
62. Zadeh LA (1969) Fuzzy sets and granular computing and concept lattice. Int J Mach Learn Cybern 11:945–946
63. Zadeh LA (1979) Fuzzy sets and self-information measures. Inf Sci 549:68–86
64. Zadeh LA (1979) Fuzzy sets and granular computing and concept lattice. Int J Mach Learn Cybern 11:945–946
65. Zadeh LA (1996) The role of fuzzy logic in the management of knowledge-based systems. IEEE Expert 11:945–946
66. Zhang L, Pu Z, Chen K, Yi J (2020) Sustainable maintenance decision-making techniques and applications-two decades review from 1994 to 2014. Expert Syst Appl 42:4126–4148
67. Zhao SY, Tsang ECC, Chen DG (2009) The model of fuzzy variable linguistic optimization problems. Fuzzy Sets Syst 159(3–4):255–272
68. Zhou G, Wang H (2020) An extension of the PROMETHEE II approach in petrochemical industry. J Clean Prod 273:122–771
69. Ziarko W (1993) Variable precision rough set model. J Comput Syst Sci 46(1):39–59
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