Convergence of the Weighted Yamabe Flow

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Abstract

We introduce the weighted Yamabe flow

$$\begin{cases}
\frac{\partial g}{\partial t} &= (r^m_\phi - R^m_\phi) g \\
\frac{\partial \phi}{\partial t} &= \frac{m}{n}(R^m_\phi - r^m_\phi)
\end{cases}$$

on a smooth metric measure space \((M^n, g, e^{-\phi}dvol_\phi, m)\), where \(R^m_\phi\) denotes the associated weighted scalar curvature, and \(r^m_\phi\) denotes the mean value of the weighted scalar curvature. We prove long-time existence and convergence of the weighted Yamabe flow if the dimension \(n\) satisfies \(n \geq 3\).

Key points: Yamabe flow, Convergence, Smooth measure metric spaces.

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1 Introduction

The Yamabe flow was first introduced by Richard Hamilton in [Ham88]. Hamilton conjectured that, for every initial metric, the flow converges to a conformal metric of constant scalar curvature. In case \(Y(M^n, g_0) \leq 0\), it is not difficult to show that the conformal factor is uniformly bounded above and below. Moreover, the flow converges to a metric of constant scalar curvature as \(t \to \infty\).

The case \(Y(M^n, g_0) > 0\) is more interesting. Chow [Cho92] proved the convergence of the flow for locally conformally flat metrics with positive Ricci curvature. Ye [Ye94] later extended the result to all locally conformal flat metrics. Later, Brendle [Bre05] proved convergence of the flow for all conformal classes and arbitrary initial metrics, and extended the results to higher dimensions [Bre07].

In this paper, we generalize the Yamabe flow to smooth metric measure spaces.
To explain the results of this article requires some terminology. A smooth metric measure space is a four-tuple \((M^n, g, e^{-\phi}dV_g, m)\) of a Riemannian manifold \((M^n, g)\), a smooth measure \(e^{-\phi}dV_g\) determined by \(\phi \in C^\infty(M)\) and the Riemannian volume element of \(g\), and a dimensional parameter \(m \in [0, \infty]\). In the case \(m = 0\), we require \(\phi = 0\). We frequently denote a smooth metric measure space by the triple \((M^n, g, \nu, m)\), where \(\nu\) and \(\phi\) will denote throughout this article functions which are related by \(\nu^m = e^{-\phi}\); when \(m = \infty\), this is to be interpreted as the formal definition of the symbol \(\nu^\infty\).

Conformal equivalence between smooth metric measure spaces are defined as the following, see [Cas15] for more details.

**Definition 1.1.** Smooth metric measure spaces \((M^n, g, e^{-\phi}dV_g, m)\) and \((\hat{M}^n, \hat{g}, e^{-\hat{\phi}}dV_{\hat{g}}, \hat{m})\) are conformally equivalent if there is a smooth function \(\sigma \in C^\infty(M)\) such that

\[
(M^n, \hat{g}, e^{-\hat{\phi}}dV_{\hat{g}}, \hat{m}) = (M^n, e^{\frac{m+n-2\sigma}{m+n-2}} g, e^{\frac{m+n-2\sigma}{m+n-2}} e^{-\phi}dV_g, m).
\]

(1.1)

In the case \(m = 0\), conformal equivalence is defined in the classical sense.

If we denote \(e^{\frac{1}{2}\sigma}\) by \(w\), (1.1) is equivalent to

\[
(M^n, \hat{g}, e^{-\hat{\phi}}dV_{\hat{g}}, \hat{m}) = (M^n, w^{\frac{4}{m+n-2}} g, w^{\frac{2(m+n)}{m+n-2}} e^{-\phi}dV_g, m),
\]

(1.2)

which is an alternative way to formulate the conformal equivalence of smooth metric measure spaces.

The weighted scalar curvature \(R^m_\phi\) of a smooth metric measure space is

\[
R^m_\phi := R + 2\Delta \phi - \frac{m+1}{m} |\nabla \phi|^2,
\]

where \(R\) and \(\Delta\) are the scalar curvature and the Laplacian associated to the metric \(g\), respectively.

In this article, we study a Yamabe-type flow on the smooth metric measure space \((M^n, g, e^{-\phi}dV_g, m)\), \(m \in (0, \infty)\), called the weighted Yamabe flow. The definition of the weighted Yamabe flow arises from the following observation.

We consider a family of pairs \((g(t), \phi(t)) \in \mathcal{M}_+ \times C^\infty(M)\), where \(\mathcal{M}_+\) is the space of Riemannian metrics on \(M^n\). The fact \((g(t), \phi(t))\) varies within the conformal class \(\[(g_0, \phi_0)\]\) implies that the metric \(e^{\phi(t)}\hat{g}(t)\) will be fixed. Denoting \(\frac{\partial \phi(t)}{\partial t} = \psi(t)\) and \(\frac{\partial \phi(t)}{\partial t} = h(t)g(t)\), we conclude that

\[
\frac{2}{m} \psi(t) + h(t) = 0.
\]
Based on this observation, we define the (normalized) weighted Yamabe flow as:

\[
\begin{align*}
\frac{\partial g}{\partial t} &= (r^m_m - R^m_m) g, \\
\frac{\partial R^m_m}{\partial t} &= \frac{m}{2} (R^m_m - r^m_m),
\end{align*}
\]  

(1.4)

where \( r^m_m \) is the mean value of \( R^m_m \); i.e.

\[
r^m_m = \frac{\int_M R^m_m e^{-\phi} dV_g}{\int_M e^{-\phi} dV_g}.
\]

By formulas of first variations in [Bes08, Theorem 1.174], direct calculation shows that evolution equations for \( R^m_m \), \( \Delta \phi \) and \( |\nabla \phi|^2 \) are

\[
\begin{align*}
\frac{\partial R^m_m}{\partial t} &= (n - 1) \Delta R^m_m + (R^m_m - r^m_m) R^m_m \\
\frac{\partial \Delta \phi}{\partial t} &= (R^m_m - r^m_m) \Delta \phi - \frac{n - 2}{2} g(dR^m_m, d\phi) + \frac{m}{2} \Delta R^m_m \\
\frac{\partial g(\nabla \phi, \nabla \phi)}{\partial t} &= (R^m_m - r^m_m) g(\nabla \phi, \nabla \phi) + mg(dR^m_m, d\phi).
\end{align*}
\]

Hence, by the definition of \( R^m_m \),

\[
\frac{\partial R^m_m}{\partial t} = (n + m - 1) \Delta_{\phi(t)} R^m_m + R^m_m (R^m_m - r^m_m),
\]

(1.5)

where \( \Delta_{g(t),\phi(t)} \) is the weighted Laplacian on \((M^n, g(t), e^{-\phi(t)} dV_{g(t)}), m\).

**Remark 1.2.** The weighted Laplacian: Let \((M^n, g, e^m dV_g)\) be a smooth metric measure space. The weighted Laplacian \( \Delta_\phi : C^\infty(M) \to C^\infty(M) \) is the operator

\[
\Delta_\phi := \Delta - \nabla \phi.
\]

It is formally self-adjoint with respect to the measure \( e^{-\phi} dV_g \), see [Cas15] for more details.

Equation (1.5) is analogous to the evolution of the scalar curvature along the normalized Yamabe flow. Noting that the flow (1.4) is subcritical in the sense that \( \frac{2(n + m)}{n + m - 2} < \frac{2m}{n - 2} \). As a result, we can establish the sequential compactness in Proposition 4.2, which is the main difference between the classical Yamabe flow and (1.4). Moreover, we adapt an argument of Brendle [Bre05] to establish long-time existence and convergence of the weighted Yamabe flow.

**Theorem 1.3.** On a smooth metric measure spaces \((M^n, g, e^{-\phi} dV_g, m)\), where \((M^n, g)\) is a closed Riemannian manifold of dimension \( n \geq 3 \), for every choice of the initial metric and the measure, the weighted Yamabe flow (1.4) exists for all time and converges to a metric with constant weighted scalar curvature.
This article is organized as follows. As mentioned above, we first deal with the positive case.

In Section 2 we prove that the conformal factor \( w(t) \) can not blow up in finite time by controlling \( w(t) \) from above and below on the interval \([0, T]\). The long-time existence follows from this.

Convergence of the weighted Yamabe flow (1.4) will be based on the following crucial proposition.

**Proposition 3.3.** Let \( \{t_i : i \in \mathbb{N}\} \) be a sequence of times such that \( t_i \to \infty \) as \( i \to \infty \). Then, we can find a real number \( 0 < \gamma < 1 \) and a constant \( C \) such that, after passing to a subsequence, we have

\[
| r^{m}_{\phi}(t_i) - r^{m}_{\phi} | \leq C \left( \int_{M} w(t_i)^{\frac{2(n+m)}{n+2(n+m)}} | R^{m}_{\phi}(t_i) - r^{m}_{\phi} |^{\frac{2(n+m)}{n+2(n+m)}} e^{-\phi_0} dV_{g_0} \right)^{\frac{n+m+2}{2(n+m)}} (1+\gamma)
\]

for all integers \( i \) in that subsequence.

In Section 3 under Proposition 3.3, we can obtain decay rates of \( r^{m}_{\phi}(t) \) and the uniform upper bound of \( | R^{m}_{\phi}(t) - r^{m}_{\phi}(t) | \) in \( L^2 \) norm. Together with the interior regularity theorem, \( w(t) \) is uniformly bounded above and below on \([0, \infty)\), such that the weighted Yamabe flow can converge smoothly.

In Section 4 we complete the proof of Proposition 3.3 using the spectral theorem of self-adjoint operators and asymptotic analysis.

In Section 5 in the same spirit as [Ye94], we refine the argument in Section 2 to obtain the uniform bound on \( w(t) \) and prove the long-time existence and smooth convergence in the negative case. Besides, in the zero case, we obtain the Harnack inequality such that uniform smooth estimates hold.

## 2 Longtime existence

In this section we collect some basic facts for smooth metric measure spaces and prove various properties of the weighted Yamabe flow that will be used throughout this article.

**Definition 2.1.** On a smooth metric measure space \((M^n, g, e^{-\phi} dV_g, m)\), which is conformal to \((M^n, g_0, e^{-\phi_0} dV_{g_0}, m)\) in the sense of Definition 1.1,

\[
(M^n, g, e^{-\phi} dV_g, m) = (M^n, w^{\frac{4}{n+2(n+m)}} g_0, w^{\frac{2(n+m)}{n+2(n+m)}} e^{-\phi_0} dV_{g_0}, m),
\]

analogous to the classical Yamabe problem, we define the normalized energy
functional $E(w)$ as

$$E_{(g_0, \phi_0)}(w) = \frac{\int_M \left( \frac{4(n+m-1)}{n+m-2} \right) \ell_m \cdot w \cdot e^{-\phi_0} dV_{g_0}}{\left( \int_M \frac{2(n+m-1)}{n+m-2} e^{-\phi_0} dV_{g_0} \right)^{\frac{n+m-2}{n+m}}},$$

(2.1)

where $L^m_{\phi_0}$ is the weighted conformal Laplacian on $(M^n, g_0, e^{-\phi_0} dV_{g_0}, m)$

$$L^m_{\phi_0} = -\Delta_{\phi_0} + \frac{n + m - 2}{4(n + m - 1)} R^m_{\phi_0}.$$  

**Remark 2.2.** The weighted scalar curvature: More generally, the weighted scalar curvature is defined as

$$R^m_{\phi} := R + 2\Delta_{\phi} - \frac{m + 1}{m} |\nabla_{\phi}|^2 + m(m-1)\mu e^{\frac{-2\phi}{m}}.$$  

(2.2)

The smooth metric measure space $(M^n, g, e^{\phi} dV_g, m)$ can be regarded as the base of the warped product

$$\left( M^n \times F^m(\mu), g \oplus e^{-\frac{2\phi}{m}} h \right)$$

(2.3)

where $(F^m(\mu), h)$ is the $m$-dimensional simply connected spaceform with constant sectional curvature $\mu$ [Cas19]. $R^m_{\phi}$ defined in (2.2) is the natural analogue of the scalar curvature because it is the scalar curvature of the warped product (2.3) and plays the role of the scalar curvature in various geometric problems (see [Lot07, Per02, Cas13b, Cas15] for more details). The constancy of $R^m_{\phi}$ defined in (1.3) is equivalent to the fact that the warped product $(M^n \times F^m(0), g \oplus e^{-\frac{2\phi}{m}} h)$ has constant scalar curvature. Throughout this article, we will adopt (1.3) as the definition of the weighted scalar curvature.

**Remark 2.3.** The normalized total weighted scalar curvature: Under the setting of Definition 2.1, by the transformation law of the weighted scalar curvature in [Cas13a],

$$\int_M R^m_{\phi} e^{-\phi} dV_g$$

(2.4)

the normalized energy $E_{(g_0, \phi_0)}(w)$ is exactly the normalized total weighted scalar curvature of $(M^n, g, e^{-\phi} dV_g, m)$; i.e.

$$E_{(g_0, \phi_0)}(w) = E_{(g, \phi)}(1) = \frac{\int_M R^m_{\phi} e^{-\phi} dV_g}{\text{Vol}(M^n, e^{-\phi} dV_g)^{\frac{n+m-2}{n+m}}}.$$  

We set

$$Y_{n,m}[(g, \phi)] = \inf \{ E_{(g_0, \phi_0)}(w) | w \in C^\infty(M; \mathbb{R}_+) \}.$$  

By the transformation law in (2.4), $Y_{n,m}[(g, \phi)]$ is conformal invariant.
Remark 2.4. The limit smooth metric measure space \((M, g_{x}, e^{-\phi_{x}}dV_{g_{x}}, m)\) of (1.4) has constant weighted scalar curvature; i.e.,

\[
L_{\phi_{0}}^{m}w_{x} = \frac{n + m - 2}{4(n + m - 1)} \lambda w_{x}^{m + n + 2}
\]

for some constant \(\lambda\), where \(w_{x}\) is determined by

\[
g_{x} = w_{x}^{-\frac{4}{2(m-n-2)}}g_{0},
\]

\[
e^{-\phi_{x}} = w_{x}^{-\frac{2m}{2(m-n-2)}}e^{-\phi_{0}}.
\]

Since the energy functional \(E\) is subcritical in the sense that \(\frac{2(n+m)}{n+m+2} < \frac{2n}{n-2}\), the equation (2.6) can be solved directly by minimizing the energy. Therefore, we are interested in the weighted Yamabe flow itself, instead of solving (2.6).

As usual in this article, all integrals are computed with respect to the weighted measure \(e^{-\phi}dV_{g}\). We denote \(W^{1,2}_{p}(M, e^{-\phi}dV_{g})\) and \(L^{p}(M, e^{-\phi}dV_{g})\) respectively the closure of \(C_{c}^{\infty}(M)\) with respect to the norm

\[
||w||_{W^{1,2}_{p}(M, e^{-\phi}dV_{g})} := \left( \int_{M} (|\nabla w|^{2} + w^{2}) e^{-\phi}dV_{g} \right)^{\frac{1}{2}}
\]

\[
||w||_{L^{p}(M, e^{-\phi}dV_{g})} := \left( \int_{M} |w|^{p} e^{-\phi}dV_{g} \right)^{\frac{1}{p}}.
\]

As observed in [Cas15], the sign of \(Y_{n,m}((g, \phi))\) is the same as the sign of the weighted conformal Laplacian.

Proposition 2.5 ([Cas15, Proposition 3.5]). Let \((M^{n}, g, e^{-\phi}dV_{g}, m)\) be a compact smooth metric measure space and denote

\[
\lambda_{1}(L_{\phi}^{m}) := \inf \left\{ \frac{(L_{\phi}^{m}w, w)}{||w||_{L^{2}_{2}}^{2}} \mid 0 \neq w \in W^{1,2}(M, e^{-\phi}dV_{g}) \right\}.
\]

Then exactly one of the three following statements is true:

- \(\lambda_{1}(L_{\phi}^{m})\) and \(Y_{n,m}((g, \phi))\) are both positive.
- \(\lambda_{1}(L_{\phi}^{m})\) and \(Y_{n,m}((g, \phi))\) are both zero.
- \(\lambda_{1}(L_{\phi}^{m})\) and \(Y_{n,m}((g, \phi))\) are both negative.

In light of the discussion in [Ye94], in case \(Y_{n,m}((g, \phi)) \leq 0\), it is not difficult to show convergence of the weighted Yamabe flow (1.4) as \(t \to \infty\). We leave the proof to section 5 and deal with the positive case firstly.

In the following, without further comment, we choose \((M^{n}, g_{0}, e^{-\phi_{0}}dV_{g_{0}}, m)\) to be the initial metric measure space with \(Y_{n,m}((g_{0}, \phi_{0})) > 0\). Since the
weighted Yamabe flow preserves the conformal structure, we may write

\[
\begin{aligned}
g(t) &= w(t)^{\frac{4}{n+m-2}} g_0, \\
e^{-\phi(t)} &= w(t)^{\frac{4m}{n+m-2}} e^{-\phi_0},
\end{aligned}
\]  

(2.7)

as the solution of (1.4) with \((g(0), \phi(0)) = (g_0, \phi_0)\). Hence, the weighted Yamabe flow reduces to the following evolution equation for the conformal factor:

\[
\frac{\partial}{\partial t} w(t)^{\frac{n+m+2}{n+m-2}} = \frac{n + m + 2}{4} \left( \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} w - R^m_{\phi_0} w + r^m_{\phi_0} w^{\frac{n+m+2}{n+m-2}} \right).
\]

(2.8)

Since

\[
\frac{d}{dt} \int_M e^{-\phi(t)} dV_{g(t)} = \frac{n + m}{2} \int_M (r^m_{\phi} - R^m_{\phi}) e^{-\phi} dV_g = 0,
\]

(2.9)

we may assume that

\[
\int_M e^{-\phi(t)} dV_{g(t)} = 1
\]

(2.10)

for all \(t \geq 0\). With this normalization, the mean value of the weighted scalar curvature can be written as

\[
r^m_{\phi}(t) = \int_M R^m_{\phi}(t) e^{-\phi(t)} dV_{g(t)}.
\]

Using the evolution equation (1.5), we obtain

\[
\frac{d}{dt} r^m_{\phi}(t) = -\frac{n + m - 2}{2} \int_M (r^m_{\phi} - R^m_{\phi})^2 e^{-\phi} dV_g \leq 0.
\]

(2.11)

Observe that \(r^m_{\phi}(t) > 0\) since \(Y_{n,m}[(g, \phi)] > 0\). Hence, \(r^m_{\phi}(t)\) can be bounded above and below i.e.

\[
0 < r^m_{\phi}(t) \leq r^m_{\phi}(0).
\]

(2.12)

In particular, the function \(t \mapsto r^m_{\phi}(t)\) is decreasing.

**Proposition 2.6.** The weighted scalar curvature of the pair \((g(t), \phi(t))\) satisfies

\[
\inf_M R^m_{\phi}(t) \geq \min \left\{ \inf_M R^m_{\phi}(0), 0 \right\}
\]

(2.13)

for all \(t \geq 0\).

**Proof.** According to the evolution equation (1.5),

\[
\frac{\partial R^m_{\phi}}{\partial t} = (n + m - 1) \Delta_{\phi(t)} R^m_{\phi} + R^m_{\phi} (R^m_{\phi} - r^m_{\phi}) \geq (n + m - 1) \Delta_{\phi(t)} R^m_{\phi} - r^m_{\phi} R^m_{\phi}.
\]

Since \(r^m_{\phi}(t) > 0\), the assertion follows from the maximum principle. \(\square\)
Remark 2.7. Positivity along the flow: In particular, if \( R_\phi^m(0) > 0 \), we have \( R_\phi^m(t) > 0 \) for all \( t \geq 0 \).

Remark 2.8. Lower bound of \( \phi \): From above discussion, we obtain that

\[
\frac{d\phi}{dt} \geq \frac{m}{2}\left[ \inf_M R_\phi^m(0) - r_\phi^m(0) \right],
\]

which provides a lower bound of \( \phi \).

For abbreviation, let

\[
\sigma = \max \left\{ \sup_M (1 - R_\phi^m(0)), 1 \right\},
\]

so that \( R_\phi^m(t) + \sigma \geq 1 \) for all \( t \geq 0 \).

The following result is similar to that in [Bre05]. Our arguments mostly follow those of Brendle, but in our setting, all integrals are computed using the weighted measure.

Lemma 2.9. For every \( p > 2 \), we have

\[
\frac{d}{dt} \int_M \left( R_\phi^m + \sigma \right)^{p-1} e^{-\phi(t)} dV_{g(t)}
\]

\[
= -\frac{4(n + m - 1)(p - 2)}{p - 1} \int_M |d(R_\phi^m + \sigma)|_g(t) e^{-\phi(t)} dV_{g(t)}
\]

\[
- \frac{n + m + 2 - 2p}{2} \int_M ((R_\phi^m + \sigma)^{p-1} - (r_\phi^m + \sigma)^{p-1})(R_\phi^m - r_\phi^m) e^{-\phi(t)} dV_{g(t)}
\]

\[
- (p - 1) \int_M \sigma((R_\phi^m + \sigma)^{p-2} - (r_\phi^m + \sigma)^{p-2})(R_\phi^m - r_\phi^m) e^{-\phi(t)} dV_{g(t)}.
\]

Proof. This follows immediately from the evolution equation (1.5) and integration by parts. Notice that \( r_\phi^m(t) \) and \( \sigma \) are independent of points in \( M \).
Lemma 2.10. For every $p > \max\{\frac{n}{2} + m, 2\}$, we have

\[
\frac{d}{dt} \int_M \left| R^m_{\phi(t)} - r^m_{\phi(t)} \right|^{p-\phi(t)} dV_{g(t)} \leq C \left( \int_M \left| R^m_{\phi(t)} - r^m_{\phi(t)} \right|^{p-\phi(t)} dV_{g(t)} \right)^{\frac{2p-(n+m)+2}{p-(n+m)}} + C \int_M \left| R^m_{\phi(t)} - r^m_{\phi(t)} \right|^{p-\phi(t)} dV_{g(t)}
\]

(2.15)

for some uniform constant $C$ independent of $t$.

Proof. Using (1.5) and (2.11), we obtain

\[
\frac{d}{dt} \int_M \left| R^m_{\phi(t)} - r^m_{\phi(t)} \right|^{p-\phi(t)} dV_{g(t)} = p(n + m - 1).
\]

\[
\int_M \left| R^m_{\phi(t)} - r^m_{\phi(t)} \right|^{p-2} (R^m_{\phi(t)} - r^m_{\phi(t)}) \left( \Delta_{\phi(t)} R^m_{\phi} + R^m_{\phi} (R^m_{\phi} - r^m_{\phi}) \right) e^{-\phi(t)} dV_{g(t)}
\]

\[
- \frac{n + m}{2} \int_M (R^m_{\phi(t)} - r^m_{\phi(t)}) |R^m_{\phi(t)} - r^m_{\phi(t)}|^{p-2} e^{-\phi(t)} dV_{g(t)}
\]

\[
+ \frac{(n + m - 2)p}{2} \int_M (R^m_{\phi(t)} - r^m_{\phi(t)}) |R^m_{\phi(t)} - r^m_{\phi(t)}|^{p-2} e^{-\phi(t)} dV_{g(t)}
\]

Moreover, we have

\[
\frac{d}{dt} \int_M \left| R^m_{\phi(t)} - r^m_{\phi(t)} \right|^{p-\phi(t)} dV_{g(t)}
\]

\[
= -\frac{4(p - 1)(n + m - 1)}{p} \int_M \left( L^m_{\phi} R^m_{\phi(t)} - r^m_{\phi(t)} \right)^{\frac{p}{2}} e^{-\phi(t)} dV_{g(t)}
\]

\[
+ \left( \frac{(n + m - 2)(p - 1)}{p} + p - \frac{n + m}{2} \right) \int_M \left( R^m_{\phi(t)} - r^m_{\phi(t)} \right)^{\frac{p}{2}} e^{-\phi(t)} dV_{g(t)}
\]

\[
+ \left( \frac{(n + m - 2)(p - 1)}{p} + p \right) \int_M \left( R^m_{\phi(t)} - r^m_{\phi(t)} \right)^{\frac{p}{2}} e^{-\phi(t)} dV_{g(t)}
\]

\[
+ \frac{(n + m - 2)p}{2} \int_M \left( R^m_{\phi(t)} - r^m_{\phi(t)} \right)^{\frac{p}{2}} e^{-\phi(t)} dV_{g(t)}
\]

\[
\times \int_M \left| R^m_{\phi(t)} - r^m_{\phi(t)} \right|^2 e^{-\phi(t)} dV_{g(t)}.
\]
Since \( Y_{n,m}[(g_0, \phi_0)] > 0 \) and the function \( t \mapsto r_{m,\phi}^m(t) \) is decreasing, we obtain

\[
\frac{d}{dt} \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^p e^{-\phi(t)} dV_{g(t)} \\
\leq - \frac{(n + m - 2)(p - 1)}{p} Y_{n,m} \left( \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^\frac{p(n + m)}{n + m - 2} e^{-\phi(t)} dV_{g(t)} \right) \frac{n + m - 2}{n + m - 2} \\
+ \frac{(n + m - 2)(p - 1)}{p} \left( \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^p e^{-\phi(t)} dV_{g(t)} \right) \\
+ \frac{(n + m - 2)(p - 1)}{p} \left( \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^p e^{-\phi(t)} dV_{g(t)} \right) \\
+ \frac{(n + m - 2)p}{2} \left( \int_M (R_{\phi(t)}^m - r_{\phi(t)}^m)^2 e^{-\phi(t)} dV_{g(t)} \right) \\
\times \left( \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^2 e^{-\phi(t)} dV_{g(t)} \right).
\]

By Hölder’s inequality in \( L^p(M, e^{-\phi(t)} dV_{g(t)}) \) and (2.10), we have

\[
\int_M (R_{\phi(t)}^m - r_{\phi(t)}^m)(R_{\phi(t)}^m - r_{\phi(t)}^m)^{p-2} e^{-\phi(t)} dV_{g(t)} \times \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^2 e^{-\phi(t)} dV_{g(t)} \\
\leq \left( \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^p e^{-\phi(t)} dV_{g(t)} \right)^\frac{p+1}{p},
\]

and

\[
\int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^{p+1} e^{-\phi(t)} dV_{g(t)} \leq \left( \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^{p+1} e^{-\phi(t)} dV_{g(t)} \right)^\frac{2p-(n+m)+2}{2p} \\
\times \left( \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^\frac{p(n+m)}{n+m-2} e^{-\phi(t)} dV_{g(t)} \right)^\frac{n+m-2}{2p}.
\]

Moreover, since \( p > \max\{\frac{n+m}{2}, 2\} \), we denote \( \frac{2p-(n+m)+2}{2p-(n+m)} \) by \( \tilde{p} \). By Young’s inequality, we obtain

\[
\int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^{p+1} e^{-\phi(t)} dV_{g(t)} \leq C_1 \left( \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^{p} e^{-\phi(t)} dV_{g(t)} \right)^\tilde{p} \\
+ C_2 \left( \int_M |R_{\phi(t)}^m - r_{\phi(t)}^m|^\frac{p(n+m)}{n+m-2} e^{-\phi(t)} dV_{g(t)} \right)^\frac{n+m-2}{n+m-2}.
\]

From this, the assertion follows.

In order to bound the solution \( w(t) \) above and below in the interval \([0, T]\), we need the following two lemmas.
Lemma 2.11. Let $P$ be a smooth function on $(M^n, g, e^{-\phi}dV_g, m)$. Moreover, assume that $u$ is a positive function on $M$ such that

$$-rac{4(n + m - 1)}{n + m - 2} \Delta_{\phi} u + Pu \geq 0.$$ 

Then, there exists a constant $C$, depending only on $g, \phi$ and $P$, such that

$$\int_M u e^{-\phi} dV_g \leq C \inf_M u. \quad (2.16)$$

Moreover, we have

$$\int_M u^{\frac{2(n + m)}{n + m - 2}} e^{-\phi} dV_g \leq C \inf_M u \left(\sup_M u\right)^{\frac{n + m + 2}{n + m - 2}}. \quad (2.17)$$

Proof. Fix $r > 0$ sufficiently small. Notice that the weighted Laplacian $\Delta_{\phi}$ has the same second-order terms as the classical Laplacian. The difference only occurs on lower order terms. Therefore, the weak Harnack inequality for linear elliptic equations [GT01, Theorem 8.18] can still hold in the weighted case, i.e. we obtain

$$\int_{B_2(x)} u e^{-\phi} dV_g \leq e^{-\inf_{B_r(x)} \phi} \int_{B_2(x)} u dV_g \leq e^{-\inf_{B_r(x)} \phi} L_0 \inf_{B_r(x)} u$$

for some constant $L_0$. The assertion follows from the same argument as that in [Bre05, Proposition A.2].

Proposition 2.12. Given any $T > 0$, we can find positive constants $C(T)$ and $c(T)$ such that

$$\sup_M w(t) \leq C(T)$$

and

$$\inf_M w(t) \geq c(T)$$

for all $0 \leq t \leq T$.

Proof. The function $w(t)$ satisfies

$$\frac{\partial}{\partial t} w(t) = -\frac{n + m - 2}{4} (R^m_{\phi(t)} - r^m_{\phi(t)}) w(t) \leq \frac{n + m - 2}{4} \left(r^m_{\phi_0} + \sigma\right) w(t).$$

Hence,

$$\frac{\partial}{\partial t} \ln(w(t)) \leq \frac{n + m - 2}{4} (r^m_{\phi_0} + \sigma).$$

We conclude that $\sup_M w(t) \leq C(T)$ for all $0 \leq t \leq T$. Hence, if we define

$$P = R^m_{\phi_0} + \sigma \left(\sup_{0 \leq t \leq T} \sup_M w(t)\right)^{\frac{4}{n + m - 2}},$$

then $w(t) \leq P$ for all $0 \leq t \leq T$. The assertion follows from the same argument as in [Bre05, Proposition A.2].
then we obtain
\[
-\frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} w(t) + Pw(t) \\
\geq -\frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} w(t) + R_{\phi_0}^m w(t) + \sigma w(t) \frac{n + m + 2}{n + m} \\
= (R_{\phi(t)}^m + \sigma) w(t) \frac{n + m + 2}{n + m} \geq 0
\] (2.18)
for all \(0 \leq t \leq T\). By Lemma 2.11 and (2.10), we can find a positive constant \(c(T)\) such that
\[
\inf_M \frac{w(t)}{w(t)} \frac{n + m + 2}{n + m} \geq c(T)
\]
for all \(0 \leq t \leq T\). Since \(\sup_M w(t) \leq C(T)\), the assertion follows. \(\square\)

**Proposition 2.13.** Let \(0 < \alpha < \min\{\frac{2m}{n+m}, 1\}\). Given any \(T > 0\), there exists a constant \(C(T)\) such that
\[
|w(x_1, t_1) - w(x_2, t_2)| \leq C(T)((t_1 - t_2)^{\frac{\alpha}{2}} + d(x_1, x_2)^\alpha)
\] (2.19)
for all \(x_1, x_2 \in M\) and all \(t_1, t_2 \in [0, T]\) satisfying \(0 < t_1 - t_2 < 1\).

**Proof.** Using Lemma 2.9 with \(p = \frac{n + m + 2}{2}\), we obtain for all \(0 \leq t \leq T\)
\[
\frac{d}{dt} \int_M (R_{\phi(t)}^m + \sigma) \frac{n + m}{2} e^{-\phi(t)} dV_{\phi(t)} \leq 0,
\]
which implies for all \(0 \leq t \leq T\)
\[
\int_M (R_{\phi(t)}^m + \sigma) \frac{n + m + 2}{n + m} e^{-\phi(t)} dV_{\phi(t)} \leq C.
\]
Hence,
\[
\left(\int_M (R_{\phi(t)}^m + \sigma) \frac{n + m + 2}{n + m} e^{-\phi(t)} dV_{\phi(t)}\right) \frac{n + m}{2} \\
\leq \left(\int_M (R_{\phi(t)}^m + \sigma) \frac{n + m + 2}{n + m} e^{-\phi(t)} dV_{\phi(t)}\right) \frac{n + m}{2} + (R_{\phi(t)}^m + \sigma) \\
\leq C.
\] (2.20)
Let \(\alpha = 2 - \frac{m}{p}\), where \(\frac{2}{p} < p < \frac{n + m}{2}, m > 0\). Using (2.8), (2.20) and Proposition 2.12, we obtain
\[
\int_M \left| -\frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} w(t) + R_{\phi_0}^m w(t) \right|^p e^{-\phi_0} dV_{\phi_0} \leq C(T)
\] (2.21)
and
\[
\int_M \left| \frac{\partial}{\partial t} w(t) \right|^p e^{-\phi(t)} dV_{\phi(t)} \leq C(T)
\] (2.22)
for all $t \in [0, T]$. By the embedding $W^{2,p}(M) \hookrightarrow C^{0,\alpha}(M)$, the first inequality implies that

$$|w(x_1, t) - w(x_2, t)| \leq C(T)d(x_1, x_2)^\alpha$$

for all $x_1, x_2 \in M$ and all $t \in [0, T]$. Using the second inequality, we obtain

$$|w(x, t_1) - w(x, t_2)|$$

$$\leq C(t_1 - t_2)^{-\frac{\alpha}{2}} \int_{B_{\sqrt{t_1 - t_2}}(x)} |w(x, t_1) - w(x, t_2)| e^{-\phi_0} dV_{g_0}$$

$$\leq C(t_1 - t_2)^{\frac{\alpha}{2}} \int_{B_{\sqrt{t_1 - t_2}}(x)} |w(t_1) - w(t_2)| e^{-\phi_0} dV_{g_0} + C(T)(t_1 - t_2)^\frac{\alpha}{2}$$

$$\leq C(t_1 - t_2)^\frac{\alpha}{2} \sup_{t_2 \leq t \leq t_1} \left( \int_{B_{\sqrt{t_1 - t_2}}(x)} \left| \frac{\partial}{\partial t} w(t) \right|^p e^{-\phi_0} dV_{g_0} \right)^{\frac{1}{p}} + C(T)(t_1 - t_2)^\frac{\alpha}{2}$$

$$\leq C(T)(t_1 - t_2)^\frac{\alpha}{2},$$

for all $x \in M$ and all $t_1, t_2 \in [0, T]$ satisfying $0 < t_1 - t_2 < 1$. This proves the assertion. \hfill \Box

We can now use the standard regularity theory for parabolic equations [Fri64, Section 3, Theorem 5] to show that all higher order derivatives of $w$ are uniformly bounded on every fixed time interval $[0, T]$. Therefore, the flow exists for all time.

## 3 Proof of the main result assuming Proposition 3.3

In this section, we will prove Theorem 1.3 based on Proposition 3.3. In the following, $c$ and $C$ are positive constants whose value are independent of $t$ and may change from line to line.

**Proposition 3.1.** Fix $\max\left\{ \frac{n+m}{2}, 2 \right\} < p < \frac{n+m+2}{2}$. Then, we have

$$\lim_{{t \to \alpha}} \int_M |P^m_{\phi(t)} - r^m_{\phi(t)}|^{p} e^{-\phi(t)} dV_{g(t)} = 0.$$

**Proof.** It follows from Lemma 2.9 that

$$\frac{d}{dt} \int_M (R^m_{\phi(t)} + \sigma)^{p-1} e^{-\phi(t)} dV_{g(t)} \leq - \left( \frac{n+m+2}{2} - p \right).$$

$$\int_M ((R^m_{\phi(t)} + \sigma)^{p-1} - (r^m_{\phi(t)} + \sigma)^{p-1})(R^m_{\phi(t)} - r^m_{\phi(t)}) e^{-\phi(t)} dV_{g(t)}.$$
Since $p > 2$, we have
\[
((R^m_{\phi(t)} + \sigma)^{p-1} - (r^m_{\phi(t)} + \sigma)^{p-1})(R^m_{\phi(t)} - r^m_{\phi(t)}) \geq c|R^m_{\phi(t)} - r^m_{\phi(t)}|^p
\]
for a suitable constant $c > 0$. Since $\frac{n+m+2}{2} > \frac{n+m}{2}$, it follows that
\[
\frac{d}{dt}\int_M (R^m_{\phi(t)} + \sigma)^{p-1}e^{-\phi(t)}dV \leq -c\int_M |R^m_{\phi(t)} - r^m_{\phi(t)}|^p e^{-\phi(t)}dV.
\]
Integrating with respect to $t$ yields
\[
\int_0^\infty \int_M |R^m_{\phi(t)} - r^m_{\phi(t)}|^p e^{-\phi(t)}dV \leq C,
\]
hence,
\[
\liminf_{t \to \infty} \int_M |R^m_{\phi(t)} - r^m_{\phi(t)}|^p e^{-\phi(t)}dV = 0.
\]
On the other hand, since $p > \max\{\frac{n+m}{2}, 2\}$, by Lemma 2.10, we have
\[
\frac{d}{dt}\int_M |R^m_{\phi(t)} - r^m_{\phi(t)}|^p e^{-\phi(t)}dV \leq C \left( \int_M |R^m_{\phi(t)} - r^m_{\phi(t)}|^p e^{-\phi(t)}dV \right)^{\frac{2p-(n+m+2)}{2p-(n+m)}}
\]
\[
+ C \int_M |R^m_{\phi(t)} - r^m_{\phi(t)}|^p e^{-\phi(t)}dV.
\]
(3.1)
From this, the assertion follows. \[\square\]

Hence, if we define
\[
r^m_{\infty} = \lim_{t \to \infty} r^m_{\phi(t)},
\]
then we obtain the following result:

**Corollary 3.2.** For every $1 < p < \frac{n+m+2}{2}$, we have
\[
\lim_{t \to \infty} \int _M |R^m_{\phi(t)} - r^m_{\infty}|^p e^{-\phi(t)}dV = 0.
\]
(3.3)

The proof of the main result will be based on the following proposition. The proof of this critical proposition will occupy Section 4.

**Proposition 3.3.** Let $\{t_i : i \in \mathbb{N}\}$ be a sequence of times such that $t_i \to \infty$ as $i \to \infty$. Then, we can find a real number $0 < \gamma < 1$ and a constant $C$ such that, after passing to a subsequence, we have
\[
r^m_{\phi(t_i)} - r^m_{\infty} \leq C \left( \int _M w(t_i)^{\frac{2(n+m)}{n+m+\gamma}} |R^m_{\phi(t_i)} - r^m_{\infty}|^\frac{2(n+m)}{n+m+\gamma} e^{-\phi_0}dV \right)^{\frac{n+m+2}{2(n+m+\gamma)}(1+\gamma)}
\]
(3.4)
for all integers $i$ in that subsequence.
Note that \( \gamma \) and \( C \) may depend on the sequence \( \{t_i : i \in \mathbb{N}\} \). The following result is an immediate consequence of Proposition 3.3.

**Proposition 3.4.** There exists real numbers \( 0 < \gamma < 1 \) and \( t_0 > 0 \) such that

\[
\left| r_{\phi(t)} - R_{\phi(t)} \right| \leq \left( \int_M w(t) \frac{2(n+m)}{n+m+2} \left| R_{\phi(t)}^m \right| \frac{2(n+m)}{n+m+2} e^{-\phi_0} dV_{g_0} \right)^{\frac{n+m+2}{2(n+m+2)}} (1+\gamma) \tag{3.5}
\]

for all \( t \geq t_0 \).

**Proof.** Suppose this is not true. Then, there exists a sequence of times \( \{t_i : i \in \mathbb{N}\} \) such that \( t_i \geq i \) and

\[
r_{\phi(t_i)} - R_{\phi(t_i)} \geq \left( \int_M w(t_i) \frac{2(n+m)}{n+m+2} \left| R_{\phi(t_i)}^m \right| \frac{2(n+m)}{n+m+2} e^{-\phi_0} dV_{g_0} \right)^{\frac{n+m+2}{2(n+m+2)}} (1+\gamma) \.
\]

We now apply Proposition 3.3 to this sequence \( \{t_i : i \in \mathbb{N}\} \). Hence, there exists an infinite subset \( I \subset \mathbb{N} \), real numbers \( 0 < \gamma < 1 \) and \( C \) such that

\[
r_{\phi(t_i)} - R_{\phi(t_i)} \leq C \left( \int_M w(t_i) \frac{2(n+m)}{n+m+2} \left| R_{\phi(t_i)}^m \right| \frac{2(n+m)}{n+m+2} e^{-\phi_0} dV_{g_0} \right)^{\frac{n+m+2}{2(n+m+2)}} (1+\gamma)
\]

for all \( i \in I \). Thus, we conclude that

\[
1 \leq C \left( \int_M w(t_i) \frac{2(n+m)}{n+m+2} \left| R_{\phi(t_i)}^m \right| \frac{2(n+m)}{n+m+2} e^{-\phi_0} dV_{g_0} \right)^{\frac{n+m+2}{2(n+m+2)}} (1+\gamma)
\]

for all \( i \in I \).

On the other hand, using Corollary 3.2 with \( p = \frac{2(n+m)}{n+m+2} < \frac{n+m+2}{2} \) and

\[
w(t_i) \frac{2(n+m)}{n+m+2} e^{-\phi_0} dV_{g_0} = e^{-\phi(t_i)} dV_{g(t_i)},
\]

we have

\[
\lim_{i \to \infty} \int_M \left| R_{\phi(t_i)}^m \right| \frac{2(n+m)}{n+m+2} e^{-\phi(t_i)} dV_{g(t_i)} = 0.
\]

Therefore, if \( i \) is sufficiently large,

\[
\lim_{i \to \infty} \left( \int_M \left| R_{\phi(t_i)}^m \right| \frac{2(n+m)}{n+m+2} e^{-\phi(t_i)} dV_{g(t_i)} \right)^{\frac{n+m+2}{2(n+m+2)}} (1+\gamma)
\leq \lim_{i \to \infty} \left( \int_M \left| R_{\phi(t_i)}^m \right| \frac{2(n+m)}{n+m+2} e^{-\phi(t_i)} dV_{g(t_i)} \right)^{\frac{n+m+2}{2(n+m+2)}} (1+\gamma) = 0.
\]

This is a contradiction. \( \square \)

**Proposition 3.5.** We have

\[
\int_0^\infty \left( \int_M w(t) \frac{2(n+m)}{n+m+2} \left| R_{\phi(t)}^m \right| \frac{2(n+m)}{n+m+2} e^{-\phi_0} dV_{g_0} \right)^{\frac{1}{2}} dt \leq C. \tag{3.6}
\]
Proof. It follows from Proposition 3.4 that

\[ r^m_{\phi(t)} - r^m_{\infty} \leq \left( \int_M w(t) \frac{2(n+m)}{n+m-2} |R^m_{\phi(t)} - r^m_{\infty}| \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_0 \right)^{\frac{n+m-2}{n+m}} (1+\gamma) \]

hence,

\[ r^m_{\phi(t)} - r^m_{\infty} \leq \left( \int_M w(t) \frac{2(n+m)}{n+m-2} |R^m_{\phi(t)} - r^m_{\infty}| \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_0 \right)^{\frac{n+m-2}{n+m}} (1+\gamma) \tag{3.7} \]

if \( t \) is sufficiently large. Therefore, by Hölder’s inequality and (2.10), we have

\[
\frac{d}{dt} (r^m_{\phi(t)} - r^m_{\infty}) = -\frac{n+m-2}{2} \int_M (R^m_{\phi(t)} - r^m_{\infty})^2 w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_0 \\
\leq -\frac{n+m-2}{2} \left( \int_M w(t) \frac{2(n+m)}{n+m-2} |R^m_{\phi(t)} - r^m_{\infty}| \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_0 \right)^{\frac{n+m-2}{n+m}} \\
\leq -\frac{n+m-2}{2} (r^m_{\phi(t)} - r^m_{\infty})^{\frac{n+m-2}{n+m}}. \tag{3.8} \]

This implies

\[
\frac{d}{dt} (r^m_{\phi(t)} - r^m_{\infty})^{1+\gamma} \geq c. \tag{3.9} \]

From this, it follows that if \( t \) is sufficiently large,

\[ r^m_{\phi(t)} - r^m_{\infty} \leq Ct^{-\frac{\gamma}{1+\gamma}}. \tag{3.10} \]

Moreover, integrating the first equality in (3.8) from \( T \) to \( 2T \) yields

\[ r^m_{\phi(T)} - r^m_{\phi(2T)} = \frac{n+m-2}{2} \int_T^{2T} \int_M (R^m_{\phi(t)} - r^m_{\phi(t)})^2 w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_0 dt. \tag{3.11} \]

Using Hölder’s inequality, we obtain

\[
\int_T^{2T} \left( \int_M (R^m_{\phi(t)} - r^m_{\phi(t)})^2 w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_0 \right)^{\frac{1}{2}} dt \\
\leq \left( \int_T^{2T} \int_M (R^m_{\phi(t)} - r^m_{\phi(t)})^2 w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_0 dt \right)^{\frac{1}{2}} \\
\leq \left( \frac{2}{n+m-2} T(r^m_{\phi(T)} - r^m_{\phi(2T)}) \right)^{\frac{1}{2}} \\
\leq CT^{-\frac{\gamma}{1+\gamma}}. \tag{3.12} \]
if $T$ is sufficiently large. Since $0 < \gamma < 1$, we conclude that
\[
\int_0^\infty \left( \int_M (R^m_{\phi(t)} - r^m_{\phi(t)})^2 w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_{g_0} \right)^{\frac{\gamma}{2}} dt
\]
\[
= \int_0^1 \left( \int_M (R^m_{\phi(t)} - r^m_{\phi(t)})^2 w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_{g_0} \right)^{\frac{\gamma}{2}} dt
\]
\[
+ \sum_{k=0}^{\infty} \int_1^{2^{k+1}} \left( \int_M (R^m_{\phi(t)} - r^m_{\phi(t)})^2 w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_{g_0} \right)^{\frac{\gamma}{2}} dt
\]
\[
\leq C \left( 1 + \sum_{k=0}^{\infty} 2^{-\frac{\gamma}{2}k} \right) \leq C.
\]
This proves the assertion.

**Proposition 3.6.** Given any $\eta_0 > 0$, we can find a real number $r > 0$ such that
\[
\int_{B_r(x)} w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_{g_0} \leq \eta_0
\]
for all $x \in M$ and $t \geq 0$.

**Proof.** We can find a real number $T > 0$ such that
\[
\int_T^\infty \left( \int_M (R^m_{\phi(t)} - r^m_{\phi(t)})^2 w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_{g_0} \right)^{\frac{\gamma}{2}} dt \leq \frac{\eta_0}{n}.
\]
We now choose a real number $r > 0$ such that
\[
\int_{B_r(x)} w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_{g_0} \leq \frac{\eta_0}{2}
\]
for all $x \in M$ and $0 \leq t \leq T$. Using the evolution equation (2.10) and Hölder’s inequality, we have
\[
\frac{d}{dt} \int_{B_r(x)} e^{-\phi(t)} dV_{g(t)} = \frac{n + m}{2} \int_{B_r(x)} (r^m_{\phi(t)} - R^m_{\phi(t)}) e^{-\phi(t)} dV_{g(t)}
\]
\[
\leq \frac{n + m}{2} \left( \int_M (R^m_{\phi(t)} - r^m_{\phi(t)})^2 e^{-\phi(t)} dV_{g(t)} \right)^{\frac{1}{2}}.
\]
Integrating (3.17) from $T$ to $t$ yields
\[
\int_{B_r(x)} w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_{g_0} \leq \int_{B_r(x)} w(T) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_{g_0}
\]
\[
+ \frac{n + m}{2} \int_T^t \left( \int_M (R^m_{\phi(t)} - r^m_{\phi(t)})^2 w(t) \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_{g_0} \right)^{\frac{1}{2}} dt \leq \eta_0
\]
for all $x \in M$ and $t \geq T$. This proves the assertion.

$\square$
Lemma 3.7. Let $p = \frac{2(n+m)}{n+m-2}$ and $q > \frac{n}{2}$. There are positive constants $\eta_1, C$ such that if

$$
\begin{align*}
g &= w^{\frac{1}{n+m-2}}g_0, \\
e^{-\phi} &= w^{\frac{2m}{n+m-2}}e^{-\phi_0},
\end{align*}
$$

is conformal to $(M^n, g_0, e^{-\phi_0}dV_{g_0}, m)$ and

$$
\begin{align*}
\int_{B_r(x)} e^{-\phi}dV_g &\leq 1, \\
\int_{B_r(x)} |R^m_{\phi_0}|^q e^{-\phi}dV_g &\leq \eta_1,
\end{align*}
$$

where $B_r(x)$ is the ball with respect to $g_0$ with $r < 1$, then

$$
w(x) \leq Cr^{-\frac{\mu}{p}} \left( \int_{B_r(x)} e^{-\phi}dV_g \right)^{\frac{1}{p}}. 
$$

Proof. By the smoothness of the conformal factor $w(t)$, there exists $r_0$ a real number such that $r_0 < r$ and

$$
(r - s)^{\frac{\mu}{p}} \sup_{B_r(x)} w \leq (r - r_0)^{\frac{\mu}{p}} \sup_{B_{r_0}(x)} w
$$

for all $s < r$. Moreover, we choose a point $x_0 \in B_{r_0}(x)$ such that

$$
\sup_{B_{r_0}(x)} w = w(x_0).
$$

Notice that in (2.4), the conformal weighted Laplacian $L^m_{\phi_0}$ has the same second-order terms as the classical Laplacian $\Delta_{g_0}$. The difference only occurs on lower order terms. Using a standard interior estimate for linear elliptic equations in [GT01, Theorem 8.17], we obtain

$$
\begin{align*}
s^{\frac{\mu}{p}} w(x_0) &\leq C \left( \int_{B_s(x_0)} w^p e^{-\phi_0}dV_{g_0} \right)^{\frac{1}{p}} \\
&\quad + C s^{\frac{\mu+2-\frac{\mu}{p}}{p}} \left( \int_{B_s(x_0)} \frac{4(n+m-1)}{(n+m-2)} L^m_{\phi_0} w |e^{-\phi_0}dV_{g_0}|^q \right)^{\frac{1}{p}}
\end{align*}
$$

for $s \leq \frac{r-r_0}{2}$. From this, it follows that

$$
\begin{align*}
s^{\frac{\mu}{p}} w(x_0) &\leq C \left( \int_{B_s(x_0)} e^{-\phi}dV_g \right)^{\frac{1}{p}} \\
&\quad + C s^{\frac{\mu+2-\frac{\mu}{p}}{p}} \left( \int_{B_s(x_0)} w^{(p-1)q-p} |R^m_{\phi_0}|^q e^{-\phi}dV_g \right)^{\frac{1}{p}}
\end{align*}
$$
for $s \leq \frac{r-r_0}{2}$. By definition of $r_0$ and $x_0$, we have

$$\sup_{B_{r-r_0}(x_0)} w \leq \sup_{B_{r+r_0}(x)} w \leq 2 \hat{p} \sup_{B_{r_0}(x)} w = 2 \hat{p} w(x_0). \tag{3.23}$$

Notice that $2 - n + \frac{2n}{p} > 0$ for $p = \frac{2(n+m)}{n+m-2} < \frac{2n}{n-2}$, and $s < r < 1$. Hence, we can find a fixed constant $K$ such that

$$s \hat{p} w(x_0) \leq K \left( \int_{B_{s}(x_0)} e^{-\phi} dV_g \right)^{\frac{1}{p}}$$

$$+ K \left( s \hat{p} w(x_0) \right)^{(p-1)-\frac{n}{p}} \left( \int_{B_{s}(x_0)} |R^m \eta e^{-\phi} dV_g \right)^{\frac{1}{q}}, \tag{3.24}$$

for $s \leq \frac{r-r_0}{2}$. We now choose $\eta_1 > 0$ such that

$$(2K)^{(p-1)-\frac{n}{p}} \eta_1^\frac{1}{q} \leq \frac{1}{2}.$$ 

We claim that $\left( \frac{r-r_0}{2} \right)^{\frac{p}{q}} w(x_0) \leq 2K$. Indeed, if $\left( \frac{r-r_0}{2} \right)^{\frac{p}{q}} w(x_0) \geq 2K$, then we may apply inequality (3.24) with $s = \left( \frac{2K}{w(x_0)} \right)^{\frac{p}{q}} \leq \frac{r-r_0}{2}$. This yields

$$2K \leq K \left( \int_{B_{r}(x_0)} e^{-\phi} dV_g \right)^{\frac{1}{p}}$$

$$+ K \left( 2K \right)^{(p-1)-\frac{n}{p}} \left( \int_{B_{r}(x_0)} |R^m \eta e^{-\phi} dV_g \right)^{\frac{1}{q}},$$

hence

$$2K \leq K + (2K)^{(p-1)-\frac{n}{p}} \eta_1^\frac{1}{q}.$$ 

Using (3.24) with $s = \frac{r-r_0}{2}$, we obtain

$$\left( \frac{r-r_0}{2} \right)^{\frac{p}{q}} w(x_0) \leq K \left( \int_{B_{r}(x_0)} e^{-\phi} dV_g \right)^{\frac{1}{p}}$$

$$+ K \left( 2K \right)^{(p-2)-\frac{n}{p}} \left( \int_{B_{r}(x_0)} |R^m \eta e^{-\phi} dV_g \right)^{\frac{1}{q}} \cdot \left( \frac{r-r_0}{2} \right)^{\frac{p}{q}} w(x_0). \tag{3.25}$$

This implies

$$\left( \frac{r-r_0}{2} \right)^{\frac{p}{q}} w(x_0) \leq K \left( \int_{B_{r}(x_0)} e^{-\phi} dV_g \right)^{\frac{1}{p}}$$

$$+ \frac{1}{2} (2K)^{(p-1)-\frac{n}{p}} \eta_1^{\frac{1}{q}} \cdot \left( \frac{r-r_0}{2} \right)^{\frac{p}{q}} w(x_0), \tag{3.26}$$
hence

\[ \left( \frac{r - r_0}{2} \right)^2 w(x_0) \leq 2K \left( \int_{B_r(x_0)} e^{-\phi} dV \right)^{\frac{1}{p}}. \]

Thus, we conclude that

\[ r^2 w(x) \leq (r - r_0)^2 w(x_0) \leq 2^{\frac{p+1}{p}} K \left( \int_{B_r(x_0)} e^{-\phi} dV \right)^{\frac{1}{p}}. \]

This proves the assertion. \qed

**Proposition 3.8.** The function \( w(t) \) satisfies

\[ \sup_M w(t) \leq C \] \hspace{1cm} (3.27)

and

\[ \inf_M w(t) \geq c \] \hspace{1cm} (3.28)

for all \( t \geq 0 \). Here, \( C \) and \( c \) are positive constants independent of \( t \).

**Proof.** Fix \( \frac{2}{p} < q < p < \frac{n+m+2}{2} \). By Corollary 3.2, we have

\[ \int_M |R^m_{\phi(t)}|^p e^{-\phi(t)} dV_g(t) \leq C, \]

for some constant \( C \) independent of \( t \). By Proposition 3.6, we can find a constant \( r > 0 \) independent of \( t \) such that

\[ \int_{B_r(x)} e^{-\phi(t)} dV_g(t) \leq \eta_0 \]

for all \( x \in M \) and \( t \geq 0 \). Using H"older's inequality, we obtain

\[ \int_{B_r(x)} |R^m_{\phi(t)}|^q e^{-\phi(t)} dV_g(t) \leq \left( \text{Vol}(B_r(x)) \right)^{\frac{p-q}{p}} \left( \int_{B_r(x)} |R^m_{\phi(t)}|^p e^{-\phi(t)} dV_g(t) \right)^{\frac{q}{p}}. \]

Hence, if we choose \( \eta_0 \) sufficiently small, then we have

\[ \int_{B_r(x)} |R^m_{\phi(t)}|^q e^{-\phi(t)} dV_g(t) \leq \eta_1 \]

for all \( x \in M \) and \( t \geq 0 \). Here, \( \eta_1 \) is the constant appearing in Lemma 3.7. Applying Lemma 3.7 at the maximum point of \( w(t) \) over \( M \),

\[ \sup_M w(t) \leq Cr^{-\frac{q}{p}} \left( \int_{B_r(x)} e^{-\phi(t)} dV_g(t) \right)^{\frac{q}{p}}. \] \hspace{1cm} (3.29)
By (2.10), we conclude that \( w(t) \) is uniformly bounded above. Hence, if we define
\[
P = R_{\phi_0}^m + \sigma (\sup_{t \geq 0} \sup_M w(t))^{\frac{1}{n+m-2}},
\]
then we obtain
\[
-\frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} w(t) + Pw(t) \geq -\frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} w(t) + R_{\phi_0}^m w(t) + \sigma w(t)^\frac{n+m+2}{n+m-2} \tag{3.30}
\]
\[
= (R_{\phi(t)}^n + \sigma) w(t)^\frac{n+m+2}{n+m-2} \geq 0.
\]

By Lemma 2.11 and (2.10), we can find a positive constant \( c \) independent of \( t \) such that
\[
\inf_M w(t)(\sup_M w(t))^{\frac{n+m+2}{n+m-2}} \geq c
\]
for all \( t \geq 0 \). Since \( \sup_M w(t) \leq C \), the assertion follows.

Proposition 3.9. Let \( 0 < \alpha < \min\{\frac{2m}{n+m}, 1\} \). Then, the function \( w(t) \) satisfies
\[
|w(x_1, t_1) - w(x_2, t_2)| \leq C((t_1 - t_2)^{\frac{\alpha}{2}} + d(x_1, x_2)^{\alpha}) \tag{3.31}
\]
for all \( x_1, x_2 \in M \) and \( 0 < t_1 - t_2 < 1 \). Here, \( C \) is a positive constant independent of \( t_1 \) and \( t_2 \).

Proof. Let \( \alpha = 2 - \frac{m}{p} \), where \( \frac{\alpha}{2} < p < \frac{n+m}{2} \), \( m > 0 \). Using (3.2) and Proposition 3.8, we obtain
\[
\int_M \left| -\frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} w(t) + R_{\phi_0}^m w(t) \right|^p e^{-\phi_0} dV_0 \leq C \tag{3.32}
\]
and
\[
\int_M \left| \frac{\partial}{\partial t} w(t) \right|^p e^{-\phi(t)} dV_{g(t)} \leq C \tag{3.33}
\]
where \( C \) is a positive constant independent of \( t \). By the embedding \( W^{2,p}(M) \hookrightarrow C^{0,\alpha}(M) \), the first inequality implies that
\[
|w(x_1, t) - w(x_2, t)| \leq Cd(x_1, x_2)^{\alpha}
\]
for all $x_1, x_2 \in M$ and all $t \geq 0$. Using the second inequality, we obtain
\[
|w(x, t_1) - w(x, t_2)| 
\leq C(t_1 - t_2)^{-\frac{m}{p}} \int_{B_{\sqrt{t_1 - t_2}}(x)} |w(x, t_1) - w(x, t_2)| e^{-\phi_0} dV_{g_0}
\leq C(t_1 - t_2)^{-\frac{m}{p}} \int_{B_{\sqrt{t_1 - t_2}}(x)} |w(t_1) - w(t_2)| e^{-\phi_0} dV_{g_0} + C(t_1 - t_2)^{\frac{m}{p}}
\leq C(t_1 - t_2)^{-\frac{m}{p}} \frac{\sup_{t_2 \leq t \leq t_1} \int_{B_{\sqrt{t_1 - t_2}}(x)} |\partial_t w(t)| e^{-\phi_0} dV_{g_0}}{t_1 - t_2} + C(t_1 - t_2)^{\frac{m}{p}}
\leq C(t_1 - t_2)^{\frac{m}{p}}.
\]
for all $x \in M$ and all $t_1, t_2$ satisfying $0 < t_1 - t_2 < 1$. This proves the assertion. 

In light of the foregoing argument at the end of Section 2, we derive uniform estimates for all higher order derivatives of $w(t)$, $t \geq 0$. The uniqueness of the asymptotic limit follows from Proposition 3.5. This completes the proof of the main result.

4 Proof of the Critical Proposition

Let $\{t_i : i \in \mathbb{N}\}$ be a sequence of times such that $t_i \to \infty$ as $i \to \infty$. For abbreviation, let $w_i = w(t_i)$. The normalization condition implies that
\[
\int_M e^{-\phi_i} dV_{g_i} = 1,
\]
where
\[
\left\{ \begin{array}{l}
  g_i = u_i^{\frac{4}{n + m - 2}} g_0, \\
  e^{-\phi_i} = u_i^{\frac{2(n + m)}{n + m - 2}} e^{-\phi_0}.
\end{array} \right. \tag{4.1}
\]
Hence
\[
\int_M u_i^{\frac{2(n + m)}{n + m - 2}} e^{-\phi_0} dV_{g_0} = 1 \tag{4.2}
\]
for all $i \in \mathbb{N}$. Moreover, it follows from Corollary 3.2 that
\[
\int_M |R_{\phi_i(t_i)}^m - m \nabla_{g_i} e^{-\phi_i(t_i)} dV_{g_i(t_i)} | \to 0,
\]
hence
\[
\int_M \left| \frac{4(n + m - 1)}{n + m - 2} \Delta_{g_0} w_i - R_{\phi_0}^m w_i + m \nabla_{g_i} \frac{2(n + m)}{n + m - 2} e^{-\phi_0} dV_{g_0} \right| \to 0 \tag{4.3}
\]
as \( i \to \infty \).

Using the standard elliptic theory, we have the following compactness result.

**Proposition 4.1.** Let \( \{w_i : i \in \mathbb{N}\} \) be a sequence of positive functions satisfying (4.2) and (4.3). After passing to a subsequence if necessary, \( \{w_i : i \in \mathbb{N}\} \) converges to a positive smooth function \( w_\infty \) satisfying the equation:

\[
\frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} w_\infty - R^m_{\phi_0} (0) w_\infty + r^m_{\phi_0} w_\infty^{\frac{n + m + 2}{n + m - 2}} = 0.
\]

**Proof.** Noticing that \( \frac{n + m + 2}{n + m - 2} < \frac{n + 2}{n - 2} \), the assertion follows from the standard elliptic theory [Eva10, Section 8, Theorem 3]. \( \square \)

In order to prove the critical proposition, we need the following results.

**Proposition 4.2.** The exists a sequence of smooth functions \( \{\psi_a : a \in \mathbb{N}\} \) and a sequence of positive real numbers \( \{\lambda_a : a \in \mathbb{N}\} \) with the following properties:

(i) For every \( a \in \mathbb{N} \), the function \( \psi_a \) satisfies the equation

\[
\frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} \psi_a - R^m_{\phi_0} \psi_a + \lambda_a w_\infty^{\frac{4}{n + m - 2}} \psi_a = 0. \tag{4.4}
\]

(ii) For all \( a, b \in \mathbb{N} \), we have

\[
\int_M \frac{4}{w_{\infty}^{n + m - 2}} \psi_a \psi_b e^{-\phi_0} dV_{g_0} = \begin{cases} 0, & a \neq b \\ 1, & a = b \end{cases}. \tag{4.5}
\]

(iii) The span of \( \{\psi_a : a \in \mathbb{N}\} \) is dense in \( L^2(M, e^{-\phi_0} dV_{g_0}) \).

(iv) \( \lambda_a \to \infty \) as \( a \to \infty \).

**Proof.** Consider the linear operator

\[
\psi \mapsto \frac{4}{w_{\infty}^{n + m - 2}} \left( \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} \psi - R^m_{\phi_0} \psi \right).
\]

This operator is symmetric with respect to the inner product

\[
(\psi_1, \psi_2) \mapsto \int_M \frac{4}{w_{\infty}^{n + m - 2}} \psi_1 \psi_2 e^{-\phi_0} dV_{g_0}
\]

on \( L^2(M, e^{-\phi_0} dV_{g_0}) \). Hence, the assertion follows from the spectral theorem. \( \square \)

Let \( A \) be a maximal finite subset of \( \mathbb{N} \) such that \( \lambda_a \leq \frac{n + m + 2}{n + m - 2} r^m_{\phi_0} \) for all \( a \in A \). We denote by \( \Pi \) the projection operator

\[
\Pi f = \sum_{a \in A} \left( \int_M \psi_a f e^{-\phi_0} dV_{g_0} \right) \frac{4}{w_{\infty}^{n + m - 2}} \psi_a = f - \sum_{a \in A} \left( \int_M \psi_a f e^{-\phi_0} dV_{g_0} \right) \frac{4}{w_{\infty}^{n + m - 2}} \psi_a \tag{4.6}
\]

\[23\]
In the rest of this section, for simplicity, we denote $W^{1,2}(M, e^{-\phi_0} dV_{g_0})$ and $L^p(M, e^{-\phi_0} dV_{g_0})$ by $W^{1,2}(M)$ and $L^p(M)$, respectively.

**Lemma 4.3.** For every $1 < p < \infty$, we can find a constant $C$ such that

$$
\|f\|_{L^p(M)} \leq C \left\| \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} f - R_{\phi_0}^m f + \frac{n + m + 2}{n + m - 2} r_m^m W_{\infty}^{\frac{4}{n-m}} f \right\|_{L^p(M)} + C \sup_{a \in A} \left| \int_M w_{\infty}^{\frac{4}{n-m}} \psi_a f e^{-\phi_0} dV_{g_0} \right|. \tag{4.7}
$$

**Proof.** Assume that is not true. By compactness, we can find a function $f \in L^p(M)$ satisfying $\|f\|_{L^p(M)} = 1$,

$$
\int_M w_{\infty}^{\frac{4}{n-m}} \psi_a f e^{-\phi_0} dV_{g_0} = 0 \tag{4.8}
$$

for all $a \in A$ and

$$
\frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} f - R_{\phi_0}^m f + \frac{n + m + 2}{n + m - 2} r_m^m W_{\infty}^{\frac{4}{n-m}} f = 0 \tag{4.9}
$$

in the sense of distributions. Hence, if we use the function $\psi_a$ as a test function, then we obtain

$$
(\lambda_a - \frac{n + m + 2}{n + m - 2} r_m^m) \int_M w_{\infty}^{\frac{4}{n-m}} \psi_a f e^{-\phi_0} dV_{g_0} = 0
$$

for all $a \in \mathbb{N}$. In particular, we have

$$
\int_M w_{\infty}^{\frac{4}{n-m}} \psi_a f e^{-\phi_0} dV_{g_0} = 0
$$

for all $a \notin A$. Thus, we conclude that $f = 0$. This is a contradiction. \qed

**Lemma 4.4.**

(i) There exists a constant $C$ such that

$$
\|f\|_{L^{\frac{n+2}{n+1+m-2}}(M)} \leq C \sup_{a \in A} \left| \int_M w_{\infty}^{\frac{4}{n-m}} \psi_a f e^{-\phi_0} dV_{g_0} \right| + C \left\| \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} f - R_{\phi_0}^m f + \frac{n + m + 2}{n + m - 2} r_m^m W_{\infty}^{\frac{4}{n-m}} f \right\|_{L^p(M)}, \tag{4.10}
$$

where $s = \frac{n(n + m + 2)}{n(n + m - 2) + 2(n + m + 2)}$. 

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(ii) There exists a constant C such that

\[
\|f\|_{L^1(M)} \leq C \left\| \Pi \left( \frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} f - R_{\phi_0}^m f + \frac{n+m+2}{n+m-2} \frac{r_m}{w_{\infty}^{\frac{4}{m+2}}} f \right) \right\|_{L^1(M)} + C \sup_{a \in A} \left| \int_M \frac{4}{w_{\infty}^{\frac{4}{m+2}}} \psi_a f e^{-\phi_0} dV_{g_0} \right| .
\]

(4.11)

Proof. (i) It follows from the embedding \( W^{2,s}(M) \hookrightarrow L^{\frac{n+m+2}{n+m-2}}(M) \) that

\[
\|f\|_{L^{\frac{n+m+2}{n+m-2}}(M)} \leq C \|f\|_{L^s(M)}
\]

\[
+ C \left\| \frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} f - R_{\phi_0}^m f + \frac{n+m+2}{n+m-2} \frac{r_m}{w_{\infty}^{\frac{4}{m+2}}} f \right\|_{L^s(M)} .
\]

(4.12)

Using Lemma 4.3, we obtain

\[
\|f\|_{L^{\frac{n+m+2}{n+m-2}}(M)} \leq C \sup_{a \in A} \left| \int_M \frac{4}{w_{\infty}^{\frac{4}{m+2}}} \psi_a f e^{-\phi_0} dV_{g_0} \right| + C \left\| \frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} f - R_{\phi_0}^m f + \frac{n+m+2}{n+m-2} \frac{r_m}{w_{\infty}^{\frac{4}{m+2}}} f \right\|_{L^s(M)} .
\]

(4.13)

By definition of \( \Pi \), we have

\[
\frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} f - R_{\phi_0}^m f + \frac{n+m+2}{n+m-2} \frac{r_m}{w_{\infty}^{\frac{4}{m+2}}} f
\]

\[
= \Pi \left( \frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} f - R_{\phi_0}^m f + \frac{n+m+2}{n+m-2} \frac{r_m}{w_{\infty}^{\frac{4}{m+2}}} f \right)
\]

\[
- \sum_{a \in A} \left( \frac{n+m+2}{n+m-2} \frac{r_m}{w_{\infty}^{\frac{4}{m+2}}} \psi_a f e^{-\phi_0} dV_{g_0} \right) \frac{4}{w_{\infty}^{\frac{4}{m+2}}} \psi_a .
\]

(4.14)

This implies

\[
\left\| \frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} f - R_{\phi_0}^m f + \frac{n+m+2}{n+m-2} \frac{r_m}{w_{\infty}^{\frac{4}{m+2}}} f \right\|_{L^s(M)} \leq \left\| \Pi \left( \frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} f - R_{\phi_0}^m f + \frac{n+m+2}{n+m-2} \frac{r_m}{w_{\infty}^{\frac{4}{m+2}}} f \right) \right\|_{L^s(M)}
\]

(4.15)

+ C \sup_{a \in A} \left| \int_M \frac{4}{w_{\infty}^{\frac{4}{m+2}}} \psi_a f e^{-\phi_0} dV_{g_0} \right| .

Putting these facts together, the assertion follows.

Similar to (i), (ii) follows from Lemma 4.3 and the definition of \( \Pi \).
Lemma 4.5. There exists a positive real number $\xi$ such that for every vector $z \in \mathbb{R}^A$ with $|z| \leq \xi$, there exists a smooth function $\bar{w}_z$ such that

$$
\int_M w z \frac{4}{4+m} \psi_a (\bar{w}_z - w_\phi) e^{-\phi_0} dV_{g_0} = z_a \quad (4.16)
$$

for all $a \in A$ and

$$
\Pi \left( \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} \bar{w}_z - R_{\phi_0}^m \bar{w}_z + r_{\phi_0}^m \bar{w}_z \frac{n + m + 2}{n + m - 2} \right) = 0. \quad (4.17)
$$

Furthermore, the map $z \mapsto \bar{w}_z$ is real analytic.

Proof. This is a consequence of the implicit function theorem.

Lemma 4.6. There exists a real number $0 < \gamma < 1$ such that

$$
E(\bar{w}_z) - E(w_\phi) \leq C \sup_{a \in A} \left| \int_M \left( \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_0} \bar{w}_z - R_{\phi_0}^m \bar{w}_z + r_{\phi_0}^m \bar{w}_z \frac{n + m + 2}{n + m - 2} \right) \psi_a e^{-\phi_0} dV_{g_0} \right|^{1+\gamma}.
$$

(4.18)

if $z$ is sufficiently small.

Proof. Note that the function $z \mapsto E(\bar{w}_z)$ is real analytic. According to results of Lojasiewicz [Sim83, equation (2.4)], there exists a real number $0 < \gamma < 1$ such that

$$
|E(\bar{w}_z) - E(w_\phi)| \leq \sup_{a \in A} \left| \frac{\partial}{\partial z_a} E(\bar{w}_z) \right|^{1+\gamma} \quad (4.19)
$$

if $z$ is sufficiently small. For convenience, we define the energy functional $\tilde{E}_{(\gamma_0, \phi_0)}(w)$ as

$$
\tilde{E}_{(\gamma_0, \phi_0)}(w) = \frac{\int_M \left( \frac{4(n + m - 1)}{n + m - 2} \ell_{\phi_0}^m, w \right) e^{-\phi_0} dV_{g_0}}{\int_M w \frac{n + m + 2}{n + m - 2} e^{-\phi_0} dV_{g_0}}
$$

$$
\quad = \frac{\int_M R_{\phi_0}^m e^{-\phi} dV_g}{\text{Vol}(M^n, e^{-\phi} dV_g)}. \quad (4.20)
$$
The partial derivatives of the function $z \mapsto E(\bar{w}_z)$ are given by

$$
\frac{\partial}{\partial z_a} E(\bar{w}_z) = -2 \left( \frac{4(n+m-1)}{n+2m-2} \Delta_{\phi_0} \bar{w}_z - R_{\phi_0}^m \bar{w}_z + r_{\infty}^m \bar{w}_z^{\frac{n+m+2}{n+m-2}} \right) \bar{E}_a e^{-\phi_0} dV_g
$$

where $\bar{E}_a = \frac{\partial}{\partial z_a} \bar{w}_z$ for $a \in A$. The function $\bar{E}_a$ satisfies

$$
\int_M \bar{w}_z^{\frac{n+m}{n+m-2}} \bar{E}_a e^{-\phi_0} dV_g = \begin{cases} 0, & a \neq b \\ 1, & a = b \end{cases}
$$

for all $a \in A$ and

$$
\Pi \left( \frac{4(n+m-1)}{n+2m-2} \Delta_{\phi_0} \bar{w}_z - R_{\phi_0}^m \bar{w}_z + r_{\infty}^m \bar{w}_z^{\frac{n+m+2}{n+m-2}} \right) = 0.
$$

Using the identity

$$
\Pi \left( \frac{4(n+m-1)}{n+2m-2} \Delta_{\phi_0} \bar{w}_z - R_{\phi_0}^m \bar{w}_z + r_{\infty}^m \bar{w}_z^{\frac{n+m+2}{n+m-2}} \right) = 0,
$$

we obtain

$$
\frac{\partial}{\partial z_a} E(\bar{w}_z) = -2 \left( \frac{4(n+m-1)}{n+2m-2} \Delta_{\phi_0} \bar{w}_z - R_{\phi_0}^m \bar{w}_z + r_{\infty}^m \bar{w}_z^{\frac{n+m+2}{n+m-2}} \right) \bar{E}_a e^{-\phi_0} dV_g
$$

for all $a \in A$. Thus, we obtain that

$$
\sup_{a \in A} \left| \frac{\partial}{\partial z_a} E(\bar{w}_z) \right| \leq C \sup_{a \in A} \left| \int_M \left( \frac{4(n+m-1)}{n+2m-2} \Delta_{\phi_0} \bar{w}_z - R_{\phi_0}^m \bar{w}_z + r_{\infty}^m \bar{w}_z^{\frac{n+m+2}{n+m-2}} \right) \bar{E}_a e^{-\phi_0} dV_g \right|.
$$
From this, the assertion follows.

By Lemma 4.5, the function
\[ z \rightarrow \int_M \left( L_{\phi_0}^m (w_i - \bar{w}_z), (w_i - \bar{w}_z) \right) e^{-\phi_0} dV_{g_0} \]
is analytical and attains the infimum for \( |z| \leq \xi \). For every \( i \in \mathbb{N} \), we can find \( \bar{w}_{z_i} \) such that \( |z_i| \leq \xi \) and
\[
\int_M \left( L_{\phi_0}^m (w_i - \bar{w}_{z_i}), (w_i - \bar{w}_{z_i}) \right) e^{-\phi_0} dV_{g_0} \leq \int_M \left( L_{\phi_0}^m (w_i - w_z), (w_i - w_z) \right) e^{-\phi_0} dV_{g_0}
\]
for all \( |z| \leq \xi \).

**Proposition 4.7.** We have as \( i \to \infty \),
\[
\|w_i - \bar{w}_z\|_{W^{1,2}(M)} \to 0 \quad \text{and} \quad z_i \to 0.
\]

**Proof.** Notice that by assumptions in Lemma 4.5, we have \( \bar{w}_0 = w_x \). Combining this with the definition of \( \bar{w}_{z_i} \), yields
\[
\int_M \left( L_{\phi_0}^m (w_i - \bar{w}_{z_i}), (w_i - \bar{w}_{z_i}) \right) e^{-\phi_0} dV_{g_0} \leq \int_M \left( L_{\phi_0}^m (w_i - w_x), (w_i - w_x) \right) e^{-\phi_0} dV_{g_0}.
\]

By the compactness result in Proposition 4.2, the expression on the right-hand side tends to 0 as \( i \to \infty \), i.e. we have as \( i \to \infty \),
\[
\|w_i - \bar{w}_z\|_{W^{1,2}(M)} \to 0.
\]

We now decompose the function \( w_i \) as
\[ w_i = \bar{w}_{z_i} + u_i. \]
Note that the function \( u_i \) satisfies
\[
\int_M \left( \frac{4(n + m - 1)}{n + m - 2} L_{\phi_0}^m u_i, u_i \right) e^{-\phi_0} dV_{g_0} = o(1)
\]
by Proposition 4.7.
Proposition 4.8. The function $u_i$ satisfies the following two properties.

1. For every $a \in A$, we have

$$\left| \int_M \frac{n^m}{n^m + m \tau^m} \tilde{w}_a u_i e^{-\phi_0} dV_{g_0} \right| \leq o(1) \int_M \left| u_i \right| e^{-\phi_0} dV_{g_0}.$$  \hfill (4.30)

2. If $i$ is sufficiently large, then we have

$$\frac{n + m + 2}{n + m - 2} \int_M \frac{n^m}{n^m + m \tau^m} u_i^2 e^{-\phi_0} dV_{g_0} \leq \left( 1 - c \right) \int_M \left( \frac{4(n + m - 1)}{n + m - 2} L_{\phi_0}^m u_i, u_i \right) e^{-\phi_0} dV_{g_0} \hfill (4.31)$$

for some positive constant independent of $i$.

Proof. 1. As above, let $\tilde{\psi}_{a,z} = \frac{a}{\psi_{a,z}} \bar{w}_z$ for $a \in A$. By the definition of $z_i$, we have

$$\int_M \left( \frac{4(n + m - 1)}{n + m - 2} L_{\phi_0}^m \tilde{\psi}_{a,z}, u_i \right) e^{-\phi_0} dV_{g_0} = 0.$$ 

This implies that

$$\lambda_a \int_M \frac{n^m}{n^m + m \tau^m} \tilde{w}_a u_i e^{-\phi_0} dV_{g_0},$$

$$= - \int_M \left( \frac{4(n + m - 1)}{n + m - 2} L_{\phi_0}^m \psi_a, u_i \right) e^{-\phi_0} dV_{g_0},$$

$$= \int_M \left( \frac{4(n + m - 1)}{n + m - 2} L_{\phi_0}^m \left( \tilde{\psi}_{a,z} - \psi_a \right), u_i \right) e^{-\phi_0} dV_{g_0}.$$ 

Since $\lambda_a > 0$, we conclude that for all $a \in A$

$$\left| \int_M \frac{n^m}{n^m + m \tau^m} \tilde{w}_a u_i e^{-\phi_0} dV_{g_0} \right| \leq o(1) \int_M \left| u_i \right| e^{-\phi_0} dV_{g_0}.$$ 

2. Suppose this is not true. Upon rescaling, we obtain a sequence of function $\{ \tilde{u}_i : i \in \mathbb{N} \}$ such that

$$\int_M \left( \frac{4(n + m - 1)}{n + m - 2} L_{\phi_0}^m \tilde{u}_i, \tilde{u}_i \right) e^{-\phi_0} dV_{g_0} = 1 \hfill (4.32)$$

and

$$\lim_{i \to \infty} \frac{n + m + 2}{n + m - 2} \int_M \frac{n^m}{n^m + m \tau^m} \tilde{u}_i^2 e^{-\phi_0} dV_{g_0} \geq 1 \hfill (4.33)$$

Observe that

$$\int_M \left| \tilde{u}_i \right| \frac{n + m + 2}{n + m - 2} e^{-\phi_0} dV_{g_0} \leq Y_{n,m}^{-\frac{n + m + 2}{n + m - 2}}.$$
by (4.32). By (4.32) and (4.33), we conclude that
\[
\lim_{i \to \infty} \int_M \frac{4}{n+m-2} l^m \tilde{u}_i e^{-\phi_0} dV_{g_0} > 0
\]
and
\[
\lim_{i \to \infty} \int_M \left( \frac{4(n + m - 1)}{n + m - 2} l^m \tilde{u}_i \right) e^{-\phi_0} dV_{g_0} \\
\leq \lim_{i \to \infty} \frac{n + m + 2}{n + m - 2} \int_M \frac{4}{n+m-2} \tilde{u}_i e^{-\phi_0} dV_{g_0}.
\]
Let \( \tilde{u} \) be the weak limit of the sequence \( \{ \tilde{u}_i : i \in \mathbb{N} \} \). Then, the function \( \tilde{u} \) satisfies
\[
\int_M \frac{4}{n+m-2} \tilde{u}^2 e^{-\phi_0} dV_{g_0} > 0
\]
and
\[
\int_M \left( \frac{4(n + m - 1)}{n + m - 2} l^m \tilde{u} \right) e^{-\phi_0} dV_{g_0} \leq \frac{n + m + 2}{n + m - 2} \int_M \frac{4}{n+m-2} \tilde{u}^2 e^{-\phi_0} dV_{g_0}.
\]
This implies that
\[
\sum_{a \in \mathbb{N}} \lambda_a \left( \int_M \frac{4}{n+m-2} \psi_a \tilde{u} e^{-\phi_0} dV_{g_0} \right)^2 \\
\leq \sum_{a \in \mathbb{N}} \frac{n + m + 2}{n + m - 2} \left( \int_M \frac{4}{n+m-2} \psi_a \tilde{u} e^{-\phi_0} dV_{g_0} \right)^2.
\]
Using (4.30), we obtain that for all \( a \in A \)
\[
\int_M \frac{4}{n+m-2} \psi_a \tilde{u} e^{-\phi_0} dV_{g_0} = 0.
\]
Therefore, we conclude that \( \tilde{u} = 0 \) on \( M \). This is a contradiction.

\[
\square
\]

**Corollary 4.9.** If \( i \) is sufficiently large, then we have
\[
\frac{n + m + 2}{n + m - 2} \int_M \frac{4}{n+m-2} \tilde{u}_i e^{-\phi_0} dV_{g_0} \\
\leq (1 - c) \int_M \left( \frac{4(n + m - 1)}{n + m - 2} l^m u_i \right) e^{-\phi_0} dV_{g_0} \tag{4.34}
\]
for some positive constant independent of \( i \).

**Proof.** The assertion follows from Proposition 4.7 and Proposition 4.8. \( \square \)
Lemma 4.10. The function \( u_i \) satisfies
\[
\|u_i\|_{L^\frac{n+m+2}{n+m+2}(M)} \leq C \left\| \frac{n+m+2}{n+m+2} (R^m_{\phi_i} - r^m_\infty) \right\|_{L^\frac{2(n+m)}{2(n+m)+2}(M)} \tag{4.35}
\]
if \( i \) is sufficiently large.

Proof. Using the identities
\[
\frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_i} u_i - R^m_{\phi_i} u_i + r^m_\infty \frac{n+m+2}{n+m+2} = -u_i \frac{n+m+2}{n+m+2} (R^m_{\phi_i} - r^m_\infty)
\]
and
\[
\Pi \left( \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_i} \bar{u}_z - R^m_{\phi_0} \bar{u}_z + r^m_\infty \frac{n+m+2}{n+m+2} \right) = 0,
\]
we obtain
\[
\Pi \left( \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_i} u_i - R^m_{\phi_i} u_i + n + m + 2 \frac{n+m+2}{n+m+2} \frac{4}{4} \right) u_i
\]
\[
= \Pi \left( \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_i} u_i - R^m_{\phi_i} u_i + n + m + 2 \frac{n+m+2}{n+m+2} \right) u_i
\]
\[
+ r^m_\infty \left( u_i \frac{n+m+2}{n+m+2} \Pi \frac{4}{4} \right) u_i.
\]
Using the inequality,
\[
\|u_i\|_{L^\frac{n+m+2}{n+m+2}(M}) \leq C \sup_{a \in A} \left| \int_M w^\frac{4}{4} \psi_a u_i e^{-\phi_i} d\text{vol}_g \right| + C \left\| \Pi \left( \frac{4(n + m - 1)}{n + m - 2} \Delta_{\phi_i} u_i - R^m_{\phi_i} u_i + n + m + 2 \frac{n+m+2}{n+m+2} \frac{4}{4} \right) u_i \right\|_{L^*(M)}
\]
we conclude that
\[
\|u_i\|_{L^\frac{n+m+2}{n+m+2}(M)} \leq C \left\| w^\frac{n+m+2}{n+m+2} u_i \right\|_{L^*(M)} + C \left\| R^m_{\phi_i} u_i \right\|_{L^*(M)} + C \sup_{a \in A} \left| \int_M w^\frac{4}{4} \psi_a u_i e^{-\phi_i} d\text{vol}_g \right|.
\]
By the compactness of \((M, g)\), up to a subsequence, we can assume that
\[
w_i \to w_\infty \text{ and } w_{\phi_i} \to w_\infty \text{ a.e. in } M. \tag{4.36}
\]
When \( n \geq 3 \) and \( m > 0 \),
\[
s = \frac{n(n+m+2)}{n(n+m-2)+2(n+m+2)} < \frac{n+m+2}{n+m+2}.
\]
Combining (4.36) with Lebesgue’s dominated convergence theorem and Hölder’s inequality yields
\[
\left\| (w_{\phi_i} - w_\infty) u_i \right\|_{L^*(M)} = o(1) \left\| u_i \right\|_{L^\frac{n+m+2}{n+m+2}(M)}.
\]
Proof. The proof is totally similar to that of Lemma 4.10. We omit it.

Lemma 4.11. The difference \( u_i \) satisfies
\[
\|u_i\|_{L^1(M)} \leq C \left\| w_i^{n+m+2} (R_{\phi_i}^m - r_{\phi_i}^m) \right\|_{L^{\frac{4(n+m+2)}{n+m+1}}(M)} \quad (4.37)
\]
if \( i \) is sufficiently large.

Proof. The proof is totally similar to that of Lemma 4.10. We omit it.

Lemma 4.12. We have
\[
\sup_{\alpha \in A} \left( \left( \int_M \left( \frac{4(n+m-1)}{n+m-2} \Delta_{\phi_i} w_{z_i} - R_{\phi_i}^m w_{z_i} + r_{\phi_i}^m w_{z_i} \phi_i^m \right) \psi_0 e^{-\phi_i} dV_{g_0} \right) \right) 
\leq C \left( \int_M w(t_i) \phi_i^m |R_{\phi_i}^m - r_{\phi_i}^m|^{\frac{2(n+m+2)}{n+m+1}} e^{-\phi_i} dV_{g_0} \right) \quad (4.38)
\]
if \( i \) is sufficiently large.
Proof. Integration by parts yields
\[
\int_M \left( \frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} \bar{w}_{z_i} - R_{\phi_0}^m \bar{w}_{z_i} + r_{\infty}^m \bar{w}_{z_i}^{\frac{n+m+2}{n+m}} \right) \psi_a e^{-\phi_0} dV_{g_0}
\]
\[
\quad = \int_M \left( \frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} w_i - R_{\phi_0}^m w_i + r_{\infty}^m w_i^{\frac{n+m+2}{n+m}} \right) \psi_a e^{-\phi_0} dV_{g_0}
\]
\[
\quad + \lambda_a \int_M w_i^{\frac{n+m}{n+m-2}} (w_i - \bar{w}_{z_i}) \psi_a e^{-\phi_0} dV_{g_0}
\]
\[
\quad - r_{\infty}^m \int_M \left( w_i^{\frac{n+m+2}{n+m}} - \bar{w}_{z_i}^{\frac{n+m+2}{n+m}} \right) \psi_a e^{-\phi_0} dV_{g_0}.
\]
Using the identities
\[
\frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} w_i - R_{\phi_0}^m w_i + r_{\infty}^m w_i^{\frac{n+m+2}{n+m}} = -w_i^{\frac{n+m+2}{n+m}} (R_{\phi_0}^m - r_{\infty}^m),
\]
we obtain
\[
\int_M \left( \frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} \bar{w}_{z_i} - R_{\phi_0}^m \bar{w}_{z_i} + r_{\infty}^m \bar{w}_{z_i}^{\frac{n+m+2}{n+m}} \right) \psi_a e^{-\phi_0} dV_{g_0}
\]
\[
\quad = - \int_M w_i^{\frac{n+m+2}{n+m}} \left( R_{\phi_0}^m - r_{\infty}^m \right) \psi_a e^{-\phi_0} dV_{g_0}
\]
\[
\quad + \lambda_a \int_M w_i^{\frac{n+m}{n+m-2}} (w_i - \bar{w}_{z_i}) \psi_a e^{-\phi_0} dV_{g_0}
\]
\[
\quad - r_{\infty}^m \int_M \left( w_i^{\frac{n+m+2}{n+m}} - \bar{w}_{z_i}^{\frac{n+m+2}{n+m}} \right) \psi_a e^{-\phi_0} dV_{g_0}.
\]
Using the pointwise estimate
\[
\left| w_i^{\frac{n+m+2}{n+m}} - \bar{w}_{z_i}^{\frac{n+m+2}{n+m}} \right| \leq C \left| w_i - \bar{w}_{z_i} \right| + C \left| w_i - \bar{w}_{z_i} \right|^{\frac{n+m+2}{n+m-2}},
\]
we conclude that
\[
\sup_{a \in A} \left| \int_M \left( \frac{4(n+m-1)}{n+m-2} \Delta_{\phi_0} \bar{w}_{z_i} - R_{\phi_0}^m \bar{w}_{z_i} + r_{\infty}^m \bar{w}_{z_i}^{\frac{n+m+2}{n+m}} \right) \psi_a e^{-\phi_0} dV_{g_0} \right|
\]
\[
\quad \leq C \left( \int_M w(t_i)^{\frac{2(n+m)}{n+m-2}} \left| R_{\phi_0}^m - r_{\infty}^m \right|^{\frac{2(n+m)}{n+m+2}} e^{-\phi_0} dV_{g_0} \right)^{\frac{n+m+2}{2(n+m)}}
\]
\[
\quad + C \left\| w_i - \bar{w}_{z_i} \right\|_{L^1(M)} + C \left\| w_i - \bar{w}_{z_i} \right\|_{L^{\frac{n+m+2}{n+m-2}}(M)}.
\]
The assertion follows from Lemma 4.10 and 4.11.

Combining Lemma 4.6 and Lemma 4.12, we immediately obtain that
Proposition 4.13. $E(w_{z_1})$ satisfies the estimate

$$E(w_{z_1}) - E(w_{x}) \leq C \left( \int_M w(t_i)^{2/(n+m)} |R_{\phi_1} - r_{\phi X}^{m} (\frac{2(n+m)}{n+m-2})^2 e^{-\phi_0} dV_{g_0} \right)^{\frac{n+m+2}{n+m+2}} (1+\gamma) \quad (4.39)$$

if $i$ is sufficiently large.

Now, we can prove our critical proposition.

Proof of Proposition 3.9: Using the transformation law (2.4), we obtain

$$r_{\phi}^m (t_i) = \int_M \left( \frac{4(n+m-1)}{n+m} L_{\phi_0}^m u_1, u_1 \right) e^{-\phi_0} dV_{g_0}$$

$$= \int_M \left( \frac{4(n+m-1)}{n+m} L_{\phi_0}^m \bar{w}_{z_1}, \bar{w}_{z_1} \right) e^{-\phi_0} dV_{g_0}$$

$$+ 2 \int_M \frac{n+m+2}{M} R_{\phi}^m (t_i) u_i e^{-\phi_0} dV_{g_0}$$

$$- \int_M \left( \frac{4(n+m-1)}{n+m} L_{\phi_0}^m u_1, u_1 \right) e^{-\phi_0} dV_{g_0}. $$

This implies that

$$r_{\phi}^m (t_i) = E(\bar{w}_{z_1}) \left( \int_M \frac{2(n+m)}{n+m+2} \bar{w}_{z_1}^{\frac{n+m+2}{n+m+2}} e^{-\phi_0} dV_{g_0} \right)^{\frac{n+m+2}{n+m+2}}$$

$$+ 2 \int_M \frac{n+m+2}{n+m+2} R_{\phi}^m (t_i) u_i e^{-\phi_0} dV_{g_0}$$

$$- \int_M \left( \frac{4(n+m-1)}{n+m} L_{\phi_0}^m u_1, u_1 \right) - \frac{n+m+2}{n+m-2} \frac{4}{\bar{w}_{z_1}} \frac{n+m+2}{n+m-2} u_1^2 e^{-\phi_0} dV_{g_0}$$

$$+ R_{\phi}^m \int_M \left( - \frac{n+m+2}{n+m-2} \frac{4}{\bar{w}_{z_1}^{n+m-2}} u_1^2 + 2 \bar{w}_{z_1}^{n+m-2} u_1 \right) e^{-\phi_0} dV_{g_0}. $$

In view of the volume normalization, we have

$$\int_M (\bar{w}_{z_1} + u_i)^{2(n+m)} e^{-\phi_0} dV_{g_0} = 1.$$ 

Furthermore, it is not difficult to show that

$$\left( \int_M \frac{2(n+m)}{n+m+2} \bar{w}_{z_1}^{n+m+2} e^{-\phi_0} dV_{g_0} \right)^{\frac{n+m+2}{n+m+2}} - 1$$

$$\leq \frac{n+m-2}{n+m} \left( \int_M \frac{2(n+m)}{n+m+2} \bar{w}_{z_1}^{n+m+2} e^{-\phi_0} dV_{g_0} \right) - \frac{n+m-2}{n+m},$$

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Moreover, it follows from Corollary 4.9 hence

\[
\left( \int_M \frac{2(n+m)}{n+m-2} \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_{g_0} \right)^{\frac{n+m-2}{n+m-2}} - 1
\]

\[
\leq \int_M \left( \frac{n + m - 2}{n + m} \frac{2(n+m)}{n+m-2} - \frac{n + m - 2}{n + m} \left( \frac{2(n+m)}{n+m-2} \right) \right) e^{-\phi_0} dV_{g_0}.
\]

It follows that

\[
r_{\phi}^{m} (t_1) \leq r_{\infty}^{m} + (E(\bar{w}_{z_i}) - r_{\infty}^{m}) \left( \int_M \frac{2(n+m)}{n+m-2} \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_{g_0} \right)^{\frac{n+m-2}{n+m-2}}
\]

\[
+ 2 \int_M \frac{n+m+2}{n+m-1} \left( R_{\phi}^{m} (t_1) - r_{\infty}^{m} \right) u_i e^{-\phi_0} dV_{g_0}
\]

\[
- \int_M \left( \frac{4(n+m-1)}{n+m-2} \frac{2(n+m)}{n+m-2} u_i \right) e^{-\phi_0} dV_{g_0}
\]

\[
+ r_{\infty}^{m} \int_M \left( - \frac{n+m+2}{n+m-1} \frac{2(n+m)}{n+m-2} u_i^2 + 2u_i \right) e^{-\phi_0} dV_{g_0}
\]

\[
+ r_{\infty}^{m} \int_M \left( \frac{n+m+2}{n+m-1} \frac{2(n+m)}{n+m-2} \right) e^{-\phi_0} dV_{g_0}.
\]

Using Hölder’s inequality, we obtain

\[
\int_M \frac{n+m+2}{n+m-1} \left( R_{\phi}^{m} (t_1) - r_{\infty}^{m} \right) u_i e^{-\phi_0} dV_{g_0}
\]

\[
\leq \left( \int_M w(t_1) \frac{2(n+m)}{n+m-2} R_{\phi}^{m} - r_{\infty}^{m} \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_{g_0} \right)^{\frac{n+m+2}{n+m-2}} \times (4.40)
\]

\[
\left( \int_M |u_i| \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_{g_0} \right)^{\frac{n+m+2}{n+m-2}}.
\]

Moreover, it follows from Corollary 4.9 that

\[
\int_M \left( \frac{4(n+m-1)}{n+m-2} \frac{2(n+m)}{n+m-2} u_i \right) e^{-\phi_0} dV_{g_0}
\]

\[
\geq c \int_M \left( \frac{4(n+m-1)}{n+m-2} \frac{2(n+m)}{n+m-2} u_i \right) e^{-\phi_0} dV_{g_0} \times (4.41)
\]

\[
\geq cY_{n,m} \left( \int_M |u_i| \frac{2(n+m)}{n+m-2} e^{-\phi_0} dV_{g_0} \right)^{\frac{n+m+2}{n+m-2}}.
\]

Finally, it follows from the pointwise estimate

\[
\left| \frac{n+m+2}{n+m-2} \frac{2(n+m)}{n+m-2} u_i \right| e^{-\phi_0} dV_{g_0}
\]

\[
\leq C \bar{w}_{z_i} \left| \frac{2(n+m)}{n+m-2} - 1 \right| \min \left( \frac{2(n+m)}{n+m-2} u_i \right) + C \max \left( \frac{2(n+m)}{n+m-2} u_i \right).
\]

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that
\[
\int_M \left( -\frac{n + m + 2}{n + m - 2} \overline{u}_z \overline{u}_t^2 + 2 \overline{w}_z \overline{u}_t \right) e^{-\phi_0} dV_{g_0} + \int_M \left( -\frac{n + m - 2}{n + m} \overline{w}_z \overline{u}_t^2 - \frac{n + m - 2}{n + m} \overline{w}_z \overline{u}_t \right) e^{-\phi_0} dV_{g_0} \\
\leq C \int_M \left( \max \{0, \frac{1}{n + m - 2} \} |u_t| \min \{ \frac{2(n + m)}{n + m - 2}, \frac{3}{2} \} + |u_t| \frac{2(n + m)}{n + m - 2} \right) e^{-\phi_0} dV_{g_0} \tag{4.42}
\]
\[
\leq C \left( \int_M |u_t| \frac{2(n + m)}{n + m - 2} e^{-\phi_0} dV_{g_0} \right) \frac{n + m + 2}{n + m}.
\]
Applying Cauchy–Schwarz inequality in (4.40) and combining this with (4.41) and (4.42), we conclude that
\[
r_m^\phi(t) \leq r_m^\phi + (E(\overline{u}_z) - r_m^\phi) \left( \int_M \frac{2(n + m)}{n + m - 2} \overline{w}_z \overline{u}_t^2 e^{-\phi_0} dV_{g_0} \right)^{\frac{n + m - 2}{n + m}} \\
+ C \left( \int_M |w| \frac{2(n + m)}{n + m - 2} |R_m^\phi - r_m^\phi| \frac{2(n + m)}{n + m - 2} e^{-\phi_0} dV_{g_0} \right)^{\frac{n + m + 2}{n + m}} \left( \frac{n + m + 2}{n + m} \right)^{\gamma}.
\]
This completes the proof. 

\[\Box\]

5 Nonpositive cases

In this section, we deal with nonpositive cases; i.e. \(Y_{n,m}((g_0, \phi_0)) \leq 0\).

5.1 Negative case

As discussed in [Cas15, Proposition 3.5], we can choose an initial metric measure space \((M^n, g_0, e^{-\phi_0} dV_{g_0}, m)\) such that \(R_m^\phi < 0\). Let \(w(t)\) be the solution of (2.8) on a maximal time interval \([0, T^*]\). Applying the maximal principle to (2.8) derives
\[
\frac{d}{dt} w_{\min}^N(t) \geq \frac{n + m + 2}{4} \left( \min_M |R_m^\phi| w_{\min}^N(t) + r_m^\phi w_{\min}^N(t) \right), \tag{5.1}
\]
where \(w_{\min}(t) = \min_M w(t)\) and \(N = \frac{n + m + 2}{n + m - 2}\). By the constancy of volume, we have
\[
r_m^\phi(t) \geq Y_{n,m}((g_0, \phi_0)). \tag{5.2}
\]

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By Hölder’s inequality, we know that $Y_{n,m}[(g_0, \phi_0)]$ is finite. Hence, integrating (5.1) yields
\[ w_{\min}^{N-1}(t) \geq C \cdot \min \left\{ w_{\min}^{N-1}(0), \frac{\min |R_m^n|}{|Y_{n,m}[(g_0, \phi_0)]|} \right\}, \tag{5.3} \]
for a uniform constant $C$. On the other hand, the maximum principle also implies
\[ \frac{d}{dt} w_{\max}^N(t) \leq \frac{n + m + 2}{4} \left( - \min_{M} R_m^n \right) w_{\max}(t) + r_m^m w_{\max}^N(t), \tag{5.4} \]
where $w_{\max}(t) = \max w(t)$. By (2.12), we conclude that
\[ w_{\max}^N(t) \leq (w_{\max}^N(0) + 1) e^{c(\min_{M} R_m^n + r_m^m)t}, \tag{5.5} \]
for some positive constant $c$. (5.3) and (5.5) imply that $w(t)$ will not blow up in finite time; i.e. $T^\star = \infty$.

Next we claim that $r_m^m(t)$ will eventually become negative, even if it may not be so at the start. Indeed, if $r_m^m(t)$ is always nonnegative for $t \geq 0$, (5.1) would imply
\[ \frac{d}{dt} w_{\min}^N(t) \geq \frac{n + m + 2}{4} \min |R_m^n| w_{\min}(t). \tag{5.6} \]
Hence $w_{\min}(t)$ approaches to infinity as $t \to \infty$. This contradicts the constancy of volume. Choosing a later time as the initial time, we may assume $r_m^m(0) < 0$. (2.12) and (5.4) yield
\[ w_{\max}^{N-1}(t) \leq C \cdot \max \left\{ w_{\max}^{N-1}(0), \frac{\max |R_m^n|}{|r_m^m(0)|} \right\}, \]
for a uniform constant $C$. Together with (5.3), we obtain that $w(t)$ is uniformly bounded from above and away from zero.

Moreover, by (1.5), we obtain that
\[ \frac{d}{dt} (R_m^n)_{\min} \geq (R_m^n)_{\min}((R_m^n)_{\min} - r_m^m) \geq r_m^m((R_m^n)_{\min} - r_m^m), \]
where $(R_m^n)_{\min}(t) = \min_{M} R_m^n(t)$. Combining this with (5.2), we can obtain a uniform lower bound on $R_m^n(t)$; i.e. for all $t \geq 0$
\[ R_m^n(t) \geq r_m^m(t) - C e^{r_m^m(t)} \geq Y_{n,m}[(g_0, \phi_0)] - C. \tag{5.7} \]

Similar to Proposition 2.6, the maximum principle also implies
\[ \sup_{M} R_m^n(t) \leq \max \left\{ \sup_{M} R_m^n(0), 0 \right\} \]
Therefore, we can generalize Lemma 2.9 and Proposition 3.9 to negative case. In light of the foregoing argument at the end of Section 2, we derive uniform estimates for all higher order derivatives of $w(t)$, $t \geq 0$.
5.2 Zero case

In the final subsection, we treat the zero case. Without loss of generality, we can fix a background metric measure space $(M^n, g^0, e^{-\phi^0} dV_{g^0}, m)$ such that $R_{g^0}^m \equiv 0$. Note that by [Cas15, Proposition 3.5], $r_{\phi(t)}^m$ can never be negative. Since the function $t \mapsto r_{\phi(t)}^m$ is nonincreasing, $r_{\phi(0)}^m = 0$ implies $r_{\phi(t)}^m \equiv 0$. Thus the solution of (1.4) is constant in time.

We next assume that $r_{\phi(0)}^m > 0$. We observe that

\[
\frac{w_{\min}^N(t)}{w_{\min}^N(0)} \geq c \int_0^t r_{\phi(t)}^m dt \quad \text{and} \quad \frac{w_{\max}^N(t)}{w_{\max}^N(0)} \leq c \int_0^t r_{\phi(t)}^m dt
\]

for some positive constant $c$. Hence we obtain the Harnack inequality

\[
\frac{w_{\min}^N(t)}{w_{\min}^N(0)} \geq \frac{w_{\max}^N(t)}{w_{\max}^N(0)}.
\]

It follows that $w(t)$ exists for all time.

By the same argument as Subsection 5.1, we can derive the smooth convergence.

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