On the connection between the theorems of Gleason and of Kochen and Specker

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We present an elementary proof of a reduced version of Gleason’s theorem and the Kochen-Specker theorem to provide a novel perspective on the relation between both theorems. The proof is based on a set of linear equations for the values of a function $m$ on the unit sphere. In the case of Gleason’s theorem the entire unit sphere needs to be considered, while a finite set of points suffices to prove the Kochen-Specker theorem.

\section{Introduction}

Quantum theory is a spectacularly successful description of the dynamics of atoms and molecules and has been confirmed in countless experiments. 90 years after de Broglie proposed matter waves, quantum mechanics still fascinates us because it is so profoundly different from classical mechanics and sometimes seems to defy common sense. Two of the most famous theorems that point the differences between classical and quantum theory are those of Gleason \cite{1} and of Kochen and Specker \cite{2}. In a nutshell, Gleason proved that the probability $p(\psi) = |\langle\psi|\sigma|\psi\rangle|^2$ to find a system in state $|\psi\rangle$ when it has been prepared in state $|\sigma\rangle$ follows from a small number of rather general assumptions. Kochen and Specker showed that it is impossible to assign a value to all observables simultaneously. This is in contrast to classical theories, where observables always assume a specific value, even if we may not know this value. The physical and philosophical implications of both theorems have been described in many publications. An overview can be found in Refs. \cite{3, 4}.

It is well-known that both theorems are connected and that the Kochen-Specker theorem may be considered as a corollary of Gleason’s theorem \cite{5}. However, their proofs are of very different nature. The proof by Kochen and Specker can be reduced to showing that it is impossible to color the unit sphere with two colours in a particular way. Gleason’s proof, on the other hand, has been described as “famously difficult” \cite{6}. The theorem has since been proven in different ways \cite{7–11} and has been extended to open quantum systems \cite{12–15} and to quantum information \cite{16, 17}.

If both theorems are closely connected, why is the result of Gleason so much more difficult to obtain? The purpose of this paper is to answer this question in a simple way that is also accessible to undergraduate students.

\section{Gleason’s theorem}

We consider a variant of Gleason’s theorem that has been discussed by Gudder (corollary 5.17 of Ref. \cite{18}).

**Reduced Gleason Theorem:** Let $\mathcal{H}$ be a real separable Hilbert space of dimension $\geq 3$ and $\mathcal{P}(\mathcal{H})$ the lattice of projectors (see App. A) on $\mathcal{H}$. Let $m$ be a map $\mathcal{P}(\mathcal{H}) \to [0, 1]$ which satisfies

\begin{equation}
 m(\mathbf{1}) = 1
\end{equation}

\begin{equation}
 m\left(\sum_i \hat{P}_i\right) = \sum_i m(\hat{P}_i) \text{ for mutually orthogonal } \hat{P}_i.
\end{equation}

Furthermore, we assume that a rank-1 projector $\hat{P}_\sigma$ exists such that $m(\hat{P}_\sigma) = 1$. Then $m(\hat{P}) = \text{Tr}(\hat{P}\hat{P}_\sigma)$ for all $\hat{P} \in \mathcal{P}(\mathcal{H})$.

We begin by discussing the main differences to Gleason’s full theorem. First, we have chosen to consider a real Hilbert space because it is suitable for our purpose. Below we show that if the Kochen-Specker theorem holds for a real Hilbert space, then it also holds for a complex Hilbert space. To establish a connection between Gleason’s theorem and the real Kochen-Specker theorem, the reduced form of Gleason’s Theorem is sufficient. In addition, a real Hilbert space is advantageous for pedagogical purposes. We remark that a real Hilbert space can still capture many, though not all, aspects of quantum mechanics. For instance, (nonlocal) violations of the Bell inequality \cite{19, 20}, which are often used in quantum information to test entanglement \cite{21}, can be obtained on a real Hilbert space \cite{3}. Another example is quantum chemistry, where the vast majority of calculations employ real superpositions of real electron orbitals \cite{22}.

A second difference to Gleason’s full theorem is that we assume the existence of a rank-1 projector $\hat{P}_\sigma$ for which $m(\hat{P}_\sigma) = 1$. In the literature such an $m$ is called an atomic state. Gleason showed that such a function $m$ represents the same information as the (pure) state $|\sigma\rangle \in \mathcal{H}$ on which $\hat{P}_\sigma$ projects. Furthermore, Gleason proved the existence of $|\sigma\rangle$, rather than assuming it, and thus showed that the usual expression for quantum mechanical mean values, $m(\hat{P}) = \langle\sigma|\hat{P}|\sigma\rangle$, is unique under the assumptions of his theorem. Gleason also considered mixed states, but for the purpose of a comparison of the two theorems we will concentrate our efforts on pure states.

In most applications, the map $m$ represents the probability distribution for observables represented by projectors, and $|\sigma\rangle$ describes the state in which the system is prepared. Clearly, the probability to find the system in the state $|\sigma\rangle$ in which it has been prepared must be unity, so that $m(\hat{P}_\sigma) = 1$, where $\hat{P}_\sigma$ is the projector on the sub-
space spanned by $|\sigma\rangle$. Also, the probability to find any state at all must be 1, which is the statement of Eq. (1).

Projectors that project on orthogonal subspaces are commuting and can therefore be measured simultaneously. Eq. (2) expresses the fact that such measurements are statistically independent, so that the respective probabilities can be added.

Gleason’s theorem is an extremely powerful result. The axioms of quantum mechanics include the statement that if a system is prepared in state $|\sigma\rangle$, then the probability to find it in state $|\psi\rangle$ is given by $p(\psi) = |\langle \psi | \sigma \rangle|^2$. If $P_\psi$ denotes the projector on vector $|\psi\rangle$, then this probability can also be expressed in the form $p(\psi) = \langle \sigma | P_\psi | \sigma \rangle$.

What Gleason achieved is to reduce the axiomatic framework of quantum theory: if we accept that the probability to find the system in a state $|\psi\rangle$ is somehow related to $P_\psi$, then his theorem completely fixes $p(\psi)$.

There is one physical assumption behind Gleason’s theorem that is not obvious from its mathematical statement: non-contextuality. To understand what this means, imagine we try to measure whether the spin of an electron points in the positive $z$-direction. Mathematically, this measurement can be described by a projector $P_z$. Physically, the Zeeman effect implies that we should employ a magnetic field $\mathbf{B}$ pointing in the $z$-direction for this experiment. On the other hand, if we measure whether the spin points in the positive $x$-direction (projector $P_x$), a magnetic field $\mathbf{B}_x$ pointing in the $x$-direction would be needed.

The map $m(\hat{P})$ in Gleason’s theorem is non-contextual in the sense that it does not depend on how the measurement is performed: we use the same map $m$ regardless of whether we consider $P_x$ or $P_z$. However, in quantum physics measuring non-commuting observables requires a different experimental setup, so that there is no compelling reason why $m$ should be the same. In a contextual theory, $m$ would depend both on the projector and on all physical parameters needed to perform the experiment. In our spin example, a contextual theory would consider a map $m(\hat{P}, \mathbf{B})$ rather than $m(\hat{P})$. Such a change would ruin the proof of Gleason’s theorem as presented below.

This point may seem a bit meticulous, but it has important consequences. Much work has been devoted to the question whether quantum mechanics can be interpreted in the terms of classical probability theories by introducing “hidden variables” (HV), i.e., parameters that may affect an experiment but to which we have no access. If $m(\hat{P})$ represents the probability to find the system in the subspace associated with $\hat{P}$, then Gleason’s theorem can be used to show that non-contextual HV theories cannot be in agreement with the results predicted by quantum theory [23, 24]. However, it does not exclude contextual HV theories [3, 4, 20]. In the discussion of the Kochen-Specker theorem below, we will return to contextuality and provide a refined definition that is more amenable for quantum theory.

### III. PROVING THE REDUCED GLEASON THEOREM

The fundamental idea behind the proof is to find a set of orthogonal vectors such that assumptions (1) and (2) can only be fulfilled for a unique function $m(\hat{P})$. We will do this in several steps: (A) show that working in a 3D space is sufficient, (B) show that $m$ can only depend on the scalar product $\langle \psi | \sigma \rangle$ between a vector $|\psi\rangle$ and the prepared state $|\sigma\rangle$, and (C) show that this function of the overlap must take the form given in the theorem. Our proof starts in a similar way as that of Gudder [18] and is inspired by some of the techniques used in Refs. [5, 25].

#### A. Reduction to 3D and some Lemmas

Our goal is to derive the value of $m(\hat{P}_x)$ for a specific vector $|\chi\rangle \in \mathcal{H}$. If $|\chi\rangle$ is proportional to $|\sigma\rangle$, we have $m(\hat{P}_x) = 1$. In all other cases, $|\chi\rangle$ and $|\sigma\rangle$ span a two-dimensional subspace of $\mathcal{H}$, for which we can use a basis consisting of the two vectors $|\sigma\rangle$ and $|\sigma_\perp\rangle$. For technical reasons we will need a third dimension [26] and therefore introduce a third orthonormal normalized vector $|\sigma'\rangle$ that is perpendicular to both $|\sigma\rangle$ and $|\sigma_\perp\rangle$. Any normalized vector $|\psi\rangle$ in this 3D subspace of $\mathcal{H}$ can then be written as

$$|\psi(\theta, \varphi)\rangle = \cos \theta |\sigma\rangle + \sin \theta \cos \varphi |\sigma_\perp\rangle + \sin \theta \sin \varphi |\sigma'\rangle. \quad (3)$$

Hence, $\cos \theta$ corresponds to the overlap $\langle \sigma | \psi \rangle$ between $|\psi(\theta, \varphi)\rangle$ and $|\sigma\rangle$.

We continue the proof in a similar fashion as Gudder. Obviously we have $m(\hat{P}_x) = 1$ if $|\psi\rangle = |\sigma\rangle$. If $|\psi\rangle$ is orthogonal to $|\sigma\rangle$ then $\hat{P}_x$ is orthogonal to $\hat{P}_\sigma$. Hence

$$m(\hat{P}_x + \hat{P}_\psi) = m(\hat{P}_x) + m(\hat{P}_\psi) \quad (4)$$

$$= 1 + m(\hat{P}_\psi) \quad (5)$$

$$\leq 1. \quad (6)$$

From this we can infer

**Lemma 1:** if $|\psi\rangle$ is orthogonal to $|\sigma\rangle$ then $m(\hat{P}_\psi) = 0$.

The general state $|\psi(\theta, \varphi)\rangle$ is completely determined by the two angles $\theta, \varphi$. Because $\hat{P}_\psi$ is in turn completely specified by the state $|\psi\rangle$, we can consider the function $m$ as a function of these angles, $m(\hat{P}_\psi) = m(\theta, \varphi)$. We now derive a set of conditions on this function of two angles.

**Lemma 2:**

$$m\left(\frac{\pi}{2} - \theta, \varphi + \pi\right) = 1 - m(\theta, \varphi).$$

To prove this we refer to Fig. 1, where $|\psi(\theta, \varphi)\rangle$ and $|\psi\left(\frac{\pi}{2} - \theta, \varphi + \pi\right)\rangle$ span a 2D subspace that is also spanned by $|\sigma\rangle$ and a vector $|\zeta\rangle$ that is orthogonal to $|\sigma\rangle$. We therefore have $m(\hat{P}_\zeta) = 0$ and $\hat{P}_\psi + \hat{P}_\psi = \hat{P}_\sigma + \hat{P}_\zeta$, so...
conclude that

Because

that

\begin{align}
m(\hat{P}_x) + m(\hat{P}_\zeta) = 1 = m(\hat{P}_\psi) + m(\hat{P}_\psi'),
\end{align}

which proves Lemma 2.

Lemma 3: \(m(\pi - \theta, \varphi + \pi) = m(\theta, \varphi)\).

This can be proven by looking at Fig. 2, where \(\psi = |\psi(\theta, \varphi)\rangle\) and \(\psi' = |\psi(\pi - \theta, \varphi + \pi)\rangle\). We then have

\begin{align}
m(\hat{P}_\psi) + m(\hat{P}_\zeta) = m(\hat{P}_\psi') + m(\hat{P}_\zeta),
\end{align}

with a vector

\begin{align}
|\zeta\rangle = -\sin \varphi|\sigma_\perp\rangle + \cos \varphi|\sigma'_\perp\rangle,
\end{align}

that is orthogonal to both \(|\sigma\rangle\) and \(|\psi(\theta, \varphi)\rangle\), so that \(m(\hat{P}_\zeta) = 0\).

Lemma 3 implies that we can restrict our considerations to angles \(0 < \theta < \pi/2\). Because of Lemma 2 we can further reduce this range to \(0 < \theta < \pi/4\).

**B. \(m(\theta, \varphi)\) cannot depend on \(\varphi\)**

We now introduce the states

\begin{align}
|\varphi\rangle &= \cos(\beta)|\psi(\varphi, \varphi)\rangle + \sin(\beta)|\zeta\rangle, \\
|\varphi'\rangle &= \sin(\beta)|\psi(\theta, \varphi)\rangle - \cos(\beta)|\zeta\rangle,
\end{align}

which are orthogonal to each other and span the same 2D subspace as \(|\psi(\theta, \varphi)\rangle\) and \(|\zeta\rangle\). We therefore have

\begin{align}
\hat{P}_\psi + \hat{P}_\zeta = \hat{P}_x + \hat{P}_y.
\end{align}

Because \(m(\hat{P}_\zeta) = 0\) and \(\hat{P}_x \hat{P}_y = \hat{P}_y \hat{P}_x = 0\), we can conclude that

\begin{align}
m(\hat{P}_\psi) = m(\hat{P}_x) + m(\hat{P}_y).
\end{align}

This is a key relation in Gudder’s proof, but from this point on we will deviate from his line of reasoning.

The vectors \(|x\rangle, |y\rangle\) can be expressed in the form

\begin{align}
|x\rangle = |\psi(\theta_x, \varphi + \delta \varphi_x)\rangle, \\
|y\rangle = |\psi(\theta_y, \varphi + \delta \varphi_y)\rangle
\end{align}

with

\begin{align}
\theta_x &= \arccos(\cos \theta \cos \beta) \\
\theta_y &= \arccos(\cos \theta \sin \beta) \\
\delta \varphi_x &= \arctan(\csc \theta \tan \beta) \\
\delta \varphi_y &= -\arctan(\csc \theta \cot \beta).
\end{align}

We can use these vectors for any value of \(\beta\), but we are particularly interested in one arbitrary but fixed value \(0 < \beta < \pi/2\) and a second value \(\beta' = \pi - \beta\), which corresponds to a second orthogonal pair of vectors \(|x'\rangle, |y'\rangle\).

A sketch of all of these vectors for \(\beta = \pi/8\) is presented in Fig. 3. It is not hard to see that

\begin{align}
\theta_x &= \theta_y, \\
\theta_y &= \theta_x \\
\delta \varphi_x &= -\delta \varphi_y, \\
\delta \varphi_y &= -\delta \varphi_x.
\end{align}

Eq. (13) can be evaluated for both pairs \(x, y\) and \(x', y'\) of orthogonal vectors so that we arrive at two equations

\begin{align}
m(\theta, \varphi) &= m(\theta_x, \varphi + \delta \varphi_x) + m(\theta_y, \varphi + \delta \varphi_y) \\
m(\theta, \varphi) &= m(\theta_y, \varphi - \delta \varphi_y) + m(\theta_x, \varphi - \delta \varphi_x).
\end{align}

These equations are valid for all choices of \(\varphi\). We can therefore replace \(\varphi\) by \(\varphi - \delta \varphi_y\) in Eq. (21) and by \(\varphi + \delta \varphi_y\) in Eq. (22) and then eliminate \(m(\theta_y, \varphi)\) from Eq. (21) to obtain

\begin{align}
m(\theta, \varphi - \delta \varphi_y) &= m(\theta, \varphi + \delta \varphi_y) + m(\theta_x, \varphi + \delta \varphi_x - \delta \varphi_y) \\
&- m(\theta_x, \varphi - \delta \varphi_x + \delta \varphi_y).
\end{align}
For $0 < \theta \leq \theta_x < \pi/2$, the angle $\beta$ is uniquely determined by $\theta$ and $\theta_x$ through $\beta = \arccos(\cos \theta_x / \cos \theta)$, which can be derived from Eq. (15). A little algebra with inverse trigonometric functions then enables us to express $\delta \varphi_x$ and $\delta \varphi_y$ through $\theta$ and $\theta_x$ as

$$\delta \varphi_y = -\arctan \left( \frac{\cos \theta_x}{\sin \theta \sqrt{\cos^2 \theta - \cos^2 \theta_x}} \right),$$

$$\delta \varphi_y - \delta \varphi_x = \arctan \left( \frac{\sin \theta}{\cos \theta_x \sqrt{\cos^2 \theta - \cos^2 \theta_x}} \right).$$

What we have accomplished in Eq. (26) is to establish a relation expressing $m_n(\theta_x)$ through $m_n(\theta)$ for an arbitrary pair of angles $0 < \theta \leq \theta_x < \pi/2$. Hence, if we know $m_n(\theta)$ for one value of $\theta$ we also know it for angles $\theta_x > \theta$. We are now going to use this to show that $m_n(\theta) = 0$ for $n \neq 0$.

To do so, we start by considering the Fourier transform of Lemma 2, which for $n \neq 0$ reads

$$m_n \left( \frac{\pi}{2} - \theta \right) e^{in\pi} = -m_n(\theta),$$

or $m_n \left( \frac{\pi}{2} - \theta \right) = (-1)^{n+1} m_n(\theta)$. On the other hand, if for $\theta \leq \pi/4$ we set $\theta_x = \pi/2 - \theta$ we obtain

$$\delta \varphi_y - \delta \varphi_x = -\delta \varphi_y = -\arctan \left( \frac{1}{\sqrt{\cos(2\theta)}} \right),$$

so that Eq. (26) implies $m_n \left( \frac{\pi}{2} - \theta \right) = -m_n(\theta)$. Consequently, Lemma 2 and Eq. (26) can both be fulfilled only if $m_{2n+1}(\theta) = 0$.

It remains to show that $m_{2n}(\theta) = 0$ as well. To do so we consider a set of three orthonormal vectors given by

$$|\psi\rangle = |\psi \left( \frac{\pi}{4}, \varphi \right)\rangle,$$

$$|x\rangle = |\psi \left( \frac{\pi}{3}, \varphi + \pi + \arctan \sqrt{2} \right)\rangle,$$

$$|x'\rangle = |\psi \left( \frac{\pi}{3}, \varphi + \pi - \arctan \sqrt{2} \right)\rangle.$$

These vectors are illustrated in Fig. 4. Because the vectors are orthonormal we have $\hat{P}_x + \hat{P}_x' = 1$ and therefore

$$m(\hat{P}_x) + m(\hat{P}_x) + m(\hat{P}_x') = 1.$$ (34)

For $2n \neq 0$, the even Fourier components of this equation read

$$m_{2n} \left( \frac{\pi}{4} \right) + 2 \cos(2n \arctan \sqrt{2}) m_{2n} \left( \frac{\pi}{3} \right) = 0.$$ (35)

For $\theta = \frac{\pi}{4}$ and $\theta_x = \frac{\pi}{3}$ we have $\delta \varphi_y = -\arctan \sqrt{2}$ and $\delta \varphi_y - \delta \varphi_x = \arctan 2 \sqrt{2}$. Using Eq. (26) to express $m_{2n} \left( \frac{\pi}{4} \right)$ through $m_{2n} \left( \frac{\pi}{3} \right)$ in Eq. (35) we get

$$2 m_{2n} \left( \frac{\pi}{4} \right) = 0.$$ (36)

Hence, for $\theta = \pi/4$, all Fourier coefficients $n \neq 0$ are zero. Because of relation (26) this also holds for all angles $\pi/4 \leq \theta \leq \pi/2$. Because of Lemma 2 and 3, this conclusion must be true for arbitrary values of $\theta$. We therefore have shown that $m(\theta, \varphi)$ cannot depend on $\varphi$. 

FIG. 3. Sketch of the vectors involved in the derivation of Eq. (23) for $\beta = \pi/8$. The horizontal plane corresponds to all vectors orthogonal to $|\sigma\rangle$. The tilted plane corresponds to the plane spanned by $|\psi(\theta, \varphi)\rangle$ and $|\zeta\rangle$, or alternatively by $|x\rangle, |y\rangle$ or $|x', |y'\rangle$. 

Eq. (23) is central for our proof because it relates vectors with overlap $\cos \theta$ (with $|\sigma\rangle$) to vectors with a different overlap $\cos \theta_x$. It will also provide the connection between the proof of Gleason and that of Kochen and Specker. For special values of the angles (e.g., for $\delta \varphi_x = \delta \varphi_y$), one could use Eq. (23) to express $m(\theta_x, \varphi)$ directly in terms of $m(\theta, \varphi)$. However, to generally achieve such a relation we have to employ Fourier transformation.

The function $m(\theta, \varphi)$ is periodic in $\varphi$ and can therefore be expressed as a Fourier series

$$m(\theta, \varphi) = \sum_{n=-\infty}^{\infty} e^{in\varphi} m_n(\theta),$$

$$m_n(\theta) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \ e^{-in\varphi} m(\theta, \varphi).$$

Because $m(\theta, \varphi)$ is real we have the relation $m_{-n}(\theta) = m_n(\theta)$. Taking the Fourier transform of Eq. (23) and solving the resulting equation for $m_n(\theta_x)$ yields, for the case $n \neq 0$,

$$m_n(\theta_x) = \frac{\sin (n \delta \varphi_y)}{\sin (n(\delta \varphi_y - \delta \varphi_x))} m_n(\theta).$$

In the way we derived this equation, the angles $\theta_x, \delta \varphi_x$ and $\delta \varphi_y$ are functions of an arbitrary angle $\beta$. However, for $0 < \theta \leq \theta_x < \pi/2$, the angle $\beta$ is uniquely determined by $\theta$ and $\theta_x$ through $\beta = \arccos(\cos \theta_x / \cos \theta)$, which can be derived from Eq. (15). A little algebra with inverse trigonometric functions then enables us to express $\delta \varphi_x$ and $\delta \varphi_y$ through $\theta$ and $\theta_x$ as
C. Determining \( m(\theta) \)

Now that we know that \( m \) only depends on \( \theta \), relation (21) can be written as

\[
m(\theta) = m(\theta_x(\beta)) + m(\theta_y(\beta)).
\] (37)

We now make a change of variables from \( \theta \) to \( u = \cos^2 \theta \), with \( \tilde{m}(u) = m(\theta) \). Using Eqs. (15) and (16), relation (37) can then be written as

\[
\tilde{m}(u) = \tilde{m}(u \cos^2 \beta) + \tilde{m}(u \sin^2 \beta).
\] (38)

Setting \( u' = u \cos 2\beta \), this can be cast into the form

\[
\tilde{m}(u) = \tilde{m}(u') + \tilde{m}(u - u').
\] (39)

We can use this to show that

\[
\begin{align*}
\tilde{m}(2^{-n}) &= 2^{-n} \quad (40) \\
\tilde{m}(ku') &= k\tilde{m}(u'), \quad (41)
\end{align*}
\]

for \( k, n \in \mathbb{N} \). To do so, we set \( u' = u/2 \) in Eq. (39), so that

\[
\tilde{m}(u) = 2\tilde{m}(u/2). \quad (42)
\]

If we now set \( u = 1 \) we get \( \tilde{m}(1/2) = 1/2 \). Applying relation (42) \( n \) times yields Eq. (40).

To prove Eq. (41) we set \( u = ku' \) in Eq. (39). We then obtain

\[
\tilde{m}(ku') = \tilde{m}(u') + \tilde{m}((k-1)u'). \quad (43)
\]

Because of Eq. (42), Eq. (41) is correct for \( k = 2 \). Assuming that it is correct for \( k - 1 \), Eq. (43) yields

\[
\tilde{m}(ku') = \tilde{m}(u') + (k-1)\tilde{m}(u') = k\tilde{m}(u'). \quad (44)
\]

By combining Eqs. (40) and (41) we have now shown that \( \tilde{m}(u) = u \) for all numbers of the form \( u = k2^{-n} \).

Now suppose that \( u' = k_12^{-n_1} \) and \( u - u' = k_22^{-n_2} \). Relation (39) then implies that this also holds for numbers of the form \( u = k_12^{-n_1} + k_22^{-n_2} \). By repeating this argument, we can show that \( \tilde{m}(u) = u \) for any number of the form \( u = \sum k_j2^{-n_j} \). However, this is the binary representation of real numbers in the interval \([0, 1]\), so that \( \tilde{m}(u) = u \) holds for all \( u \in [0, 1] \). Hence,

\[
m(\theta) = \cos^2 \theta = \langle \sigma | \hat{P}_\psi | \sigma \rangle.
\] (45)

This proves the theorem for rank-1 projectors \( \hat{P}_\psi \). Because any projector can be written as a sum of mutually orthogonal projectors of rank 1, assumption (2) ensures that the theorem holds for arbitrary projectors ☐

IV. THE KOCHEN-SPECKER THEOREM

The theorem of Kochen and Specker addresses an apparently very different question. It does not deal with probabilities but rather asks whether it is possible to assign specific values to all observables in a system that can be described using quantum theory.

Consider the observables corresponding to projectors on a three-dimensional real Hilbert space. In a measurement, all these observables would take values that are either 0 or 1. In a classical world, one would expect that observables take their values independently of whether one actually performs a measurement or not. For instance, if we throw a coin and do not look at the result, we would still be convinced that it would be either head (0) or tail (1). The value of the observables may not be known, but it would appear plausible that each possible set of values for an observable could be associated with a certain probability. The question is which sets of values are actually possible, and the answer given by Kochen and Specker is: none. In the language used here, their result can be stated as follows.

Adapted Kochen-Specker Theorem: Let \( \mathcal{H} \) be a real separable Hilbert space of dimension \( \geq 3 \) and \( \mathcal{P}(\mathcal{H}) \) the lattice of projectors. Then there is no homomorphism that maps \( \mathcal{P}(\mathcal{H}) \) to the set \( \{0, 1\} \).

In the original theorem, the lattice of projectors is replaced by a partial Boolean algebra, which includes all observables on \( \mathcal{H} \). In our case, the homomorphism is a map \( m : \mathcal{P}(\mathcal{H}) \rightarrow \{0, 1\} \) that preserves the lattice structure, i.e., it obeys assumptions (1) and (2) of Gleason’s theorem. However, it can only take the values 0 or 1.

The Kochen-Specker Theorem may be considered a corollary of Gleason’s theorem: because \( m(\hat{P}) \) is confined to map a projector to the discrete values 0 or 1, Gleason’s theorem tells us that it is impossible because the only possible map (45) takes continuous values. This connection between the two theorems is well known and
has been used by Hrushovski and Pitowsky to construct extensions of the Kochen-Specker Theorem [27].

However, direct proofs of the Kochen-Specker Theorem are much more intuitive than the proof of Gleason’s theorem. If one associates the values of 0 and 1 with the color blue and red, respectively, then one has to show that it is impossible to color the unit sphere (which is formed by the tips of all unit vectors $|\psi\rangle$ in red and blue in such a way that (i) all points separated from a red point by a right angle must be blue, and (ii) that any three points mutually separated by right angles must contain one red and two blue points. Condition (i) is similar to the statement of Lemma 1 above: if we know that $m(P_1) = 1$ for some vector $|\sigma\rangle$, then $m$ must vanish for all projectors on states that are orthogonal to $|\sigma\rangle$. Condition (ii) arises from the fact that, on a three-dimensional subspace of $H$, we have $m(1) = 1 = m(P_1) + m(P_2) + m(P_3)$ for three orthogonal rank-1 projectors $P_1, P_2, P_3$.

Kochen and Specker constructed a set of 117 vectors for which no consistent choice of colours could be made. The theorem has later been derived for larger Hilbert spaces and with fewer basis vectors [28–35], and has been generalized to open quantum systems [36, 37].

Despite being a corollary of Gleason’s theorem, the Kochen-Specker theorem makes a stronger statement about contextual HV theories. The reason is that in Gleason’s theorem $m$ represents a probability distribution, while in the Kochen-Specker theorem $m$ represents the allowed measurement values. One can distinguish two types of contextual HV theories: type I only allows the probability distribution to be context-dependent, while type II admits the possibility that both probability distribution and measurement values may depend on the experimental context.

The contextual measurement values in type II introduce a new challenge. Suppose we want to measure the sum $P_x + P_y$ of two non-commuting projectors. In each run of the experiment we would have to add the values measured for both observables, but since they cannot be measured simultaneously, this is not possible in practice. However, if quantum theory could be interpreted in terms of HV theories, both observables would need to take some value, regardless of whether we can actually measure it. One therefore had to introduce counterfactual values [3, 38, 39] that an observable would take even if the experiment is not set up to measure it.

Counterfactual values can be avoided if contextuality is defined in a refined way. Suppose $A$ is an observable that commutes with two other observables $B$ and $C$, but $[B, C] \neq 0$. In this case we can simultaneously measure $A$ and $B$, or $A$ and $C$. Then observable $A$ is non-contextual if the measurement outcomes do not depend on whether it is measured simultaneously with $B$ or $C$ [40]. This definition is well suited for projection measurements in quantum theory, but has been generalized by Spekkens [41] to include unsharp measurements and more general physical models.

Gleason’s theorem cannot exclude type I nor type II. The Kochen-Specker theorem can exclude type I because it does not depend on the probability distribution. Bell inequalities [19, 20, 42] can exclude “local” HV models of type I, where the probability distribution can only depend on local changes of the apparatus, not on distant changes that would require superluminal speed to affect the probability distribution [43].

We proceed by using the methods of Sec. III prove the Kochen-Specker theorem on a real three-dimensional subspace of $H$. If there can be no homomorphism for this subspace, then there can also be no homomorphism on $H$. Because of condition (ii) there must be at least one red point on the unit sphere, which we call $|\sigma\rangle$. Without loss of generality, we put this point at the North pole of the sphere. Lemma 1 then ensures that all points on the equator must be blue. Lemma 2 implies that if $|\psi(\theta, \phi)\rangle$ is red then $|\psi(\frac{\pi}{2} - \theta, \phi + \pi)\rangle$ must be blue, or vice versa.

Turning to Fig. 3 we can see that relation (21) connects the colours of $|\psi\rangle$, $|x\rangle$ and $|y\rangle$. If $|\psi\rangle$ is red, then one of $|x\rangle$ and $|y\rangle$ must be red. If $|\psi\rangle$ is blue, then both $|x\rangle$ and $|y\rangle$ must be blue as well. Because the choice of $|x\rangle$ and $|y\rangle$ is arbitrary, the entire plane spanned by $|\psi\rangle$ and $|\zeta\rangle$ must then be blue.

This observation enables us to construct a contradiction: suppose $|\psi\rangle$ is blue. We then know that both the equator and the plane spanned by $|\psi\rangle$ and $|\zeta\rangle$ must be blue. For a given vector $|x\rangle$ of Eq. (10), we can find another vector
\[
|\zeta_x\rangle = \frac{\sin \theta}{\sqrt{1 + \cot^2 \beta \sin^2 \theta}} (|\psi\rangle - \cot \beta |\zeta\rangle)
\]
that lies on the equator and is orthogonal to $|x\rangle$. The two vectors $|x\rangle$ and $|\zeta_x\rangle$ are both blue and therefore span a plane that must be blue. This is depicted in Fig. 5 a): if $|\sigma\rangle$ is red and $|\psi\rangle$ is blue, then the equator, the plane spanned by $|\psi\rangle$ and $|x\rangle$, and the plane spanned by $|x\rangle$ and $|\zeta_x\rangle$ must all be blue. Furthermore the vector $|x_1\rangle = |\zeta_x\rangle \times |x\rangle$ must be red, where $\times$ denotes the vector cross product in three dimensions.

Fig. 5 a) shows the blue plane spanned by $|x\rangle$ and $|\zeta_x\rangle$ for one particular choice of $|x\rangle$. However, by varying $\beta$ in Eq. (10) we can continuously change this plane from the plane spanned by $|\psi\rangle$ and $|\zeta\rangle$ (for $\beta = 0$) into the equatorial plane (for $\beta = \pi/2$). The set of all points that lie on any of these planes forms a blue area on the unit sphere that is shown in Fig. 5 b). Each of the planes also determines a red vector $|x_1\rangle$. As $\beta$ varies, this vector moves along a trajectory connecting $|\psi'\rangle$ (for $\beta = 0$) with $|\sigma\rangle$ (for $\beta = \pi/2$). This trajectory is shown in red in Fig. 5 b) for $\beta = \pi/10$.

The size of the blue area and the red trajectory depends on the angle $\theta$ between $|\psi\rangle$ and $|\sigma\rangle$. For values of $\theta \approx \pi/2$, the blue area essentially corresponds to a blue ribbon around the equator and the red trajectory stays close to the north pole. However, the size of both the area and the trajectory grows as $\theta$ shrinks and they start to overlap for values of about $\theta \leq \pi/10$. More-
ciscely, we have numerically determined that the maximal angle for which the red trajectory and the blue area are overlapping is $\theta \approx 0.108 \pi$; we conjecture that the precise boundary is at $\theta = \arccos(\sqrt{8/9})$.

As no point can be both red and blue, the assumption that $|\psi\rangle$ is blue must therefore be wrong for $\theta \leq \pi/10$. Because we have not made any assumption about $|\sigma\rangle$ apart from being red, we have just shown that any point that is closer than $\pi/10$ from a red point must also be red. By repeated application of this principle to different red points we can infer that the entire unit sphere must be red, which would be in contradiction with the assumptions.

The above argument uses an infinite number of vectors (the blue area of the sphere), but it can easily be reduced to a finite number of vectors. For simplicity we consider a vector $|\psi(\theta, \varphi)\rangle$ of Eq. (3) for which $\theta = \pi/10$ and $\varphi = 0$. We then can pick one pair of vectors $|x\rangle, |\zeta_x\rangle$, characterized by an angle $\beta$, for which $|x_{\perp}\rangle$ is red at a specific point, and a second one, characterized by an angle $\beta'$, for which this point lies on the plane spanned by the pair and hence must be blue. To be specific, we numerically determined $\beta \approx 0.756\pi$, for which $|x_{\perp}\rangle \approx |\psi(0.24\pi, 0.599\pi)\rangle$, and $\beta' \approx 0.137\pi$. Because this leads to a contradiction, we can infer as before that $|\psi(\pi, 0)\rangle$ must be red. We then can repeat this procedure to show that all vectors $|\psi(n\pi, 0)\rangle$, with $n = 0, 1, 2, \ldots$ must be red. However, for $n = 5$ this vector is located on the equator and thus has to be blue, which proves the adapted Kochen-Specker theorem by using a finite set of vectors only. This argument is close to the original proof of the theorem.

V. DISCUSSION AND CONCLUSION

In this paper we have provided alternative proofs to a reduced version of Gleason’s theorem and the Kochen-Specker theorem. Both theorems are concerned with a function $m(P)$ that maps the lattice of projectors on a real Hilbert space $\mathcal{H}$ to real numbers. The main difference is the image of $m$, which is given by $[0, 1]$ for Gleason’s theorem and $\{0, 1\}$ for the Kochen-Specker theorem.

In our approach, both proofs utilize Lemma 1, Lemma 2, and Eq. (21), which establish algebraic relations between the values of $m$ for orthogonal projectors or, equivalently, orthogonal unit vectors. In particular, Eq. (21) relates $m$ for sets of unit vectors that are rotated around the vertical axis.

Mathematically, Gleason’s theorem is the stronger result and normally requires more powerful techniques for its proof. The methods developed here give further insight into this. It is well known that the Kochen-Specker theorem can be proven using only a finite number of unit vectors. To prove the reduced Gleason theorem, all vectors on the unit sphere are required. Technically, one may say that the Kochen-Specker theorem employs Eq. (21) for a finite number of discrete values of the rotation angle, while Gleason’s theorem requires continuous values. One can then use Fourier transformation to solve the corresponding algebraic relations.

Apart from providing a new perspective on the relation between both theorems, the techniques developed here may also be useful for extensions of the Kochen-Specker theorem that do not require the use of Gleason’s theorem. For instance, one may be able to find a set of observables that take more discrete values than just 0 and 1 and can be related by finite set of rotation angles in Eq. (21), but this is beyond the scope of this paper.

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Appendix A: Projector lattices

On a finite-dimensional Hilbert space $\mathcal{H}$, a projector $\hat{P}$ corresponds to a matrix that satisfies $\hat{P}^2 = \hat{P}$ and only has eigenvalues 0 and 1. It maps a state $|\psi\rangle \in \mathcal{H}$ to that part $\hat{P}|\psi\rangle$ of the state that lies in a given subspace of $\mathcal{H}$. Rank-1 projectors take the form $\hat{P}_m = |\phi\rangle\langle\phi|$ for some unit vector $|\phi\rangle$ and project on one-dimensional subspaces.

Projectors can be added, but the sum of two (or more) projectors is only a projector if they project on mutually orthogonal subspaces. The set of all projectors on a given Hilbert space, together with the rules how to add them to get new projectors, is called the lattice of projectors. More details can be found in Ref. [44], for instance.