Existence of solutions for a problem of resonance road space with weight ♠

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Abstract

In this work we use variational methods to show the existence of weak solutions for a nonlinear problem of the type elliptic. This problem was initially study by the authors Ahmad, Lazer and Paul (see [1]) considering the space $\Omega \subset \mathbb{R}^n$ a bounded domains. In this work we extend your result now considering the domain $\mathbb{R}^n$. Indeed, the main theorems in this paper constitute an extension to $\mathbb{R}^n$ of your previous results in bounded domains.

Key words: Resonance, weight space, variational methods, elliptic equation

Introduction

In this work we obtain a result of existence of weak solution for the problem

$$\begin{cases}
-\Delta u + u = \lambda h(x)u + g(x, u), & \text{on } \mathbb{R}^n, \ n \geq 3 \\
u \in H^1(\mathbb{R}^n)
\end{cases}$$

(1)

where $h : \mathbb{R}^n \rightarrow \mathbb{R}_+$, $g : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$, are continuous functions satisfying the following conditions:
(i) $h \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$;
(ii) There is a function $Z \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $Z(x) > 0$, $\forall x \in \mathbb{R}^n$, and
\[
|g(x, t)| \leq Z(x), \forall (x, t) \in \mathbb{R}^n \times \mathbb{R}.
\] (2)
Furthermore, $\lambda_k$ is the k-th eigenvalue associated with the problem
\[
\left\{
\begin{array}{ll}
-\Delta u + u = \lambda h(x)u, & \text{on } \mathbb{R}^n, \ n \geq 3 \\
u \in H^1(\mathbb{R}^n)
\end{array}
\right.
\] (3)
characterizing the problem (1) as a problem of resonance (see Definition 1.2 pg. 2 of [4]).

Furthermore, we assume that $g$ satisfies one of the conditions $(g^+_2)$ or $(g^-_2)$, this is,
\[
\int_{\mathbb{R}^n} G(x, v(x)) \, dx \xrightarrow{||v|| \rightarrow \infty} \pm \infty,
\] (4)
where $v \in N_{\lambda_k}$ the eigenspace associated with the eigenvalue $\lambda_k$ and
\[
G(x, \cdot) = \int_0^s g(x, \tau) \, d\tau
\]
is the primitive of the function $g(x, \cdot)$.

We find a weak solution for the problem (1) determining critical points of the energy functional associated $\Phi : H^1(\mathbb{R}^n) \to \mathbb{R}$ defined by
\[
\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^n} (|\Delta u|^2 + |u|)^2 \, dx - \frac{\lambda_k}{2} \int_{\mathbb{R}^n} u^2 h \, dx - \int_{\mathbb{R}^n} G(x, u) \, dx.
\]

In this work we generalized the result from Lazer-Ahmad-Paul in [1]. For this was necessary some theorics results once again that the domain considered here is no bounded. In a space no bounded no exists immersion compact of the Soboleve space $H^1(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$. We contorted this working in spaces with weight.

Problems at resonance have been of interest to researchers ever since the pioneering work of Landesman and Lazer [2] in 1970 for second order elliptic operators in bounded domains. The literature on resonance problems in bounded domains is quite vast; of particular interest to this paper are the works of Ahmad, Lazer and Paul [1] in 1976 and of Rabinowitz in 1978, in which critical point methods are applied. Recently, using other tecnic, Garza and Rumbos [4] do a result that is a extension to $\mathbb{R}^n$ of the Ahmad, Lazer
and Paul result. In this paper, furthermore of resonance they obtain result on strong resonance.

Resonance problems on unbounded domains, and in particular in $\mathbb{R}^n$, have been studied recently by Costa and Tehrani [2] and by Jeanjean [6], and by Stuart and Zhou [9] for radially symmetric solutions for asymptotically linear problems in $\mathbb{R}^n$. In all these references variational methods were used. Hetzer and Landesman [5] for resonant problems for a class of operators which includes the Schrödinger operator. The problem of resonance that we work can be see in [10] for bounded domain.

We write in general the direction of our proposal, In the Section we define spaces with weight and show that this space is Banach. In the Section 2 we show two important results, a of immersions continuous and the other of immersion compact space of the space $H^1(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n, h \, dx)$. Finally, in the Section 3 we show the main result of this work that guarantees the existence of weak solution to the problem (1).

1. Spaces with weight

In this section we define spaces with weight and show some properties involving these spaces. For this, start with the following:

**Definition 1.** Let $h : \mathbb{R}^N \rightarrow (0, +\infty)$ be a measurable function and $1 < p < \infty$. We define the space $L^p(\mathbb{R}^N, h \, dx)$ the space of all the measurable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\int_{\mathbb{R}^n} |f(x)|^p h(x) \, dx < \infty$, i.e,

$$L^p \left( \mathbb{R}^N, h \, dx \right) := \left\{ f : \mathbb{R}^n \rightarrow \mathbb{R} \text{ measurable} \mid \int_{\mathbb{R}^n} |f(x)|^p h(x) \, dx < \infty \right\}.$$

In the that follow we denoted by

$$\|u\| = \left( \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) \, dx \right)^{1/2} \quad \text{and} \quad \|f\|_{p,h} = \left( \int_{\mathbb{R}^N} |f|^p h \, dx \right)^{1/p}$$

the norms in $H^1(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N, h \, dx)$, respectively.

**Theorem 1.** The space $(L^p(\mathbb{R}^N, h \, dx); \| \cdot \|_{p,h})$ with $1 \leq p < \infty$ is a Banach’s space.
Proof. Let \( \{u_n\} \subset L^p(\mathbb{R}^N, hdx) \) be a Cauchy’s sequence. Then, give \( \epsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that
\[
\|u_n - u_m\|_{p,h} < \epsilon, \quad \forall \ n, m \geq n_0. \tag{5}
\]
Define \( v_n = h^{\frac{1}{p}} u_n \), it follows from (5) that
\[
\|v_n - v_m\|_{L^p(\mathbb{R}^N)} < \epsilon, \quad \forall \ n, m \geq n_0.
\]
Since \( L^p(\mathbb{R}^N) \) is a Banach space, there exists \( w \in L^p(\mathbb{R}^N) \) with
\[
v_n \xrightarrow{n \to \infty} w \tag{6}
\]
Defining, \( u(x) = \frac{w(x)}{h(x)^{\frac{1}{p}}} \), note that
\[
\|u_n - u\|_{p,h} = \left( \int_{\mathbb{R}^N} |u_n - u|^p hdx \right)^{\frac{1}{p}} = \left( \int_{\mathbb{R}^N} \left| h^{\frac{1}{p}} u_n - h^{\frac{1}{p}} u \right|^p dx \right)^{\frac{1}{p}}
\]
which implies,
\[
\|u_n - u\|_{p,h} = \left( \int_{\mathbb{R}^N} |v_n - w|^p dx \right)^{\frac{1}{p}} = \|v_n - w\|_{L^p(\mathbb{R}^N)}.
\]
From (6), we have \( \|u_n - u\|_{p,h} \xrightarrow{n \to \infty} 0 \), that is, \( u_n \xrightarrow{n \to \infty} u \) in \( L^p(\mathbb{R}^N, hdx) \). Therefore, \( L^p(\mathbb{R}^N, hdx) \) is a Banach’s space.

From Theorem we can conclude that \( (L^2(\mathbb{R}^N, hdx), \| \cdot \|_{2,h}) \) is a Hilbert’s space with the inner product \( \langle f, g \rangle_{2,h} = \int_{\mathbb{R}^N} hfg \, dx \).

2. Auxiliary results

In this section we show two auxiliary results important for we show that the problem \( (H) \) have weak solution. The first is a result of continuous immersion and the second a result of compact immersion, both of the space \( H^1(\mathbb{R}^n) \) in \( L^p(\mathbb{R}^n, h \, dx) \).
2.1. A result of continuous immersion

**Theorem 2.** If \( h \in L^\infty(\mathbb{R}^N) \), then applies continuous immersion

\[
H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N, hdx)
\]

for all \( p \in [1, 2^*] \) (\( 2^* = 2N/(N - 2) \)) if \( N \geq 3 \).

**Proof.** For \( u \in H^1(\mathbb{R}^N) \), we have

\[
\left( \int_{\mathbb{R}^N} |u|^p h dx \right)^{\frac{1}{p}} \leq \|h\|_\infty \left( \int_{\mathbb{R}^N} |u|^p dx \right)^{\frac{1}{p}} = \|h\|_\infty \|u\|_{L^p(\mathbb{R}^N)}.
\]

Since \( h \in L^\infty(\mathbb{R}^N) \). From Sobolev’s continuous immersion, we have

\[
H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N), \quad \forall \ p \in [2, 2^*],
\]

if \( N \geq 3 \). Thus, there exists \( C > 0 \) such that

\[
\|u\|_{L^p(\mathbb{R}^N)} \leq C \||u||, \quad \forall \ u \in H^1(\mathbb{R}^N).
\]

Therefore,

\[
\left( \int_{\mathbb{R}^N} |u|^p h dx \right)^{\frac{1}{p}} \leq C \|h\|_\infty \|u\|, \quad \forall \ u \in H^1(\mathbb{R}^N).
\]

Considering \( C_1 = C \|h\|_\infty \), we have

\[
\left( \int_{\mathbb{R}^N} |u|^p h dx \right)^{\frac{1}{p}} \leq C_1 \|u||,
\]

that is,

\[
\|u\|_{p,h} \leq C_1 \|u||, \quad \forall \ u \in H^1(\mathbb{R}^N)
\]

showing the continuous immersion. \( \square \)
2.2. A result of Compact immersion

**Lemma 1.** Give $\epsilon > 0$, there exists $R > 0$ such that

$$\|u_{n_j} - u\|_{L^p(B^c_R(0), h dx)} < \frac{\epsilon}{2}, \forall n_j.$$ 

**Proof.** Indeed, to $R > 0$ and using the Hölder inequality with exponents $\frac{1}{\gamma} + \frac{1}{p} = 1$, we get

$$\int_{B^c_R(0)} |u_{n_j} - u|^p h \, dx \leq \|h\|_{L^\gamma(B^c_R(0))} \cdot \|u_{n_j} - u\|^p_{L^{2\ast}(\mathbb{R}^N)}$$

which implies,

$$\int_{B^c_R(0)} |u_{n_j} - u|^p h \, dx \leq \|h\|_{L^\gamma(B^c_R(0))} \cdot \|u_{n_j} - u\|^p_{L^{2\ast}(\mathbb{R}^N)}$$

and from the Sobolev’s continuous immersions, we have

$$\int_{B^c_R(0)} |u_{n_j} - u|^p h \, dx \leq C \|h\|_{L^\gamma(B^c_R(0))} \cdot \|u_{n_j} - u\|^p.$$

Consequently,

$$\int_{B^c_R(0)} h \, |u_{n_j} - u|^p \, dx \leq C_1 \|h\|_{L^\gamma(B^c_R(0))}. \quad (7)$$

Follows the theory of measure that if $h \in L^\gamma(\mathbb{R}^N), \forall \gamma \in [1, \infty)$, give $\epsilon > 0$ there exists $R > 0$ such that

$$\|h\|_{L^\gamma(B^c_R(0))} < \left(\frac{\epsilon}{2}\right)^p \frac{1}{C_1}.$$ 

Now, from result above and from (7), we obtain

$$\int_{B^c_R(0)} h \, |u_{n_j} - u|^p \, dx \leq \left(\frac{\epsilon}{2}\right)^p \forall \, n_j.$$

Therefore,

$$\|u_{n_j} - u\|_{L^p(B^c_R(0), h dx)} < \frac{\epsilon}{2} \forall \, n_j \quad (8)$$

which show the lemma. \qed
Theorem 3. If $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, has the compact immersion

$$H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N, h \, dx), \ \forall \ p \in [1, 2^*)$$

if $N \geq 3$.

Proof. Let $(u_n) \subset H^1(\mathbb{R}^N)$ be a bounded sequence, using the fact that $H^1(\mathbb{R}^N)$ is a reflexive space there exists $(u_{nj}) \subset (u_n)$ such that

$$u_{nj} \rightharpoonup u \text{ in } H^1(\mathbb{R}^N).$$

Furthermore,

$$\|u_{nj} - u\|_{L^p(B_R(0), h \, dx)} = \left( \int_{B_R(0)} h \left| u_{nj} - u \right|^p \, dx \right)^{\frac{1}{p}}$$

$$\leq \|h\|_\infty \|u_{nj} - u\|_{L^p(B_R(0))}.$$ 

From Sobolev’s compact immersions, we have $H^1(\mathbb{R}^N) \hookrightarrow L^p(B_R(0))$, of where follows

$$u_{nj} \rightharpoonup u \text{ in } L^p(B_R(0)).$$

Thus, give $\epsilon > 0$ there exists $n_{j_0}$ such that

$$\|u_{nj} - u\|_{L^p(B_R(0))} < \frac{\epsilon}{2\|h\|_\infty}, \ \forall \ n_j \geq n_{j_0},$$

thus, we have

$$\|u_{nj} - u\|_{L^p(B_R(0), h \, dx)} < \frac{\epsilon}{2}, \ \forall \ n_j \geq n_{j_0}. \quad (9)$$

From Lemma 1 and (9), we have

$$\left\|u_{nj} - u\right\|_{p,h} < \epsilon, \ \forall \ n_j \geq n_{j_0}$$

which implies,

$$u_{nj} \rightharpoonup u \text{ in } L^p(\mathbb{R}^N, h \, dx)$$

showing the compactness. \qed
3. Main result

3.1. Preliminaries

We consider the functional $\Phi : H^1(\mathbb{R}^N) \to \mathbb{R}$ such that

$$
\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) \, dx - \frac{\lambda_k}{2} \int_{\mathbb{R}^N} u^2 h \, dx - \int_{\mathbb{R}^N} G(x,u) \, dx. \quad (10)
$$

This functional is well defined in $H^1(\mathbb{R}^N)$, is of class $C^1(H^1(\mathbb{R}^N), \mathbb{R})$ (see [3]) and its critical points are weak solutions of (1).

We set the linear operator $Lu = u - \lambda_k S(u)$, we have

$$
(Lu, u)_{H^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (\nabla (Lu) \nabla u + Lu u) \, dx,
$$

thus,

$$
(Lu, u)_{H^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (\nabla (u - \lambda_k S(u)) \nabla u + (u - \lambda_k S(u)) u) \, dx,
$$

this is,

$$
(Lu, u)_{H^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) \, dx - \lambda_k \int_{\mathbb{R}^N} (\nabla S(u) \nabla u + S(u) u) \, dx. \quad (11)
$$

Moreover, considering $S(u) = w$, we have

$$
\begin{align*}
-\Delta w + w &= hu, \\
w &\in H^1(\mathbb{R}^N) \\
\mathbb{R}^N
\end{align*}
$$

which implies,

$$
\int_{\mathbb{R}^N} (\nabla \phi \nabla w + \phi w) \, dx = \int_{\mathbb{R}^N} uh \phi \, dx, \quad \forall \, \phi \in H^1(\mathbb{R}^N).
$$

Fixing $u = \phi$,

$$
\int_{\mathbb{R}^N} (\nabla S(u) \nabla u + S(u) u) \, dx = \int_{\mathbb{R}^N} u^2 h \, dx. \quad (12)
$$

Therefore, from (11) and (12), we have

$$
\frac{1}{2} (Lu, u)_{H^1(\mathbb{R}^N)} = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) \, dx - \frac{\lambda_k}{2} \int_{\mathbb{R}^N} u^2 h \, dx
$$

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from which follows
\[ \Phi(u) = \frac{1}{2} (Lu, u)_{H^1(\mathbb{R}^N)} - \int_{\mathbb{R}^N} G(x, u) \, dx. \]

Now, we fix the following orthogonal decomposition of \( X = H^1(\mathbb{R}^N) \),

\[ X = X_- \oplus X_0 \oplus X_+, \]

where

\[ X_0 = N_{\lambda_k}, \quad X_- = N_{\lambda_1} \oplus N_{\lambda_2} \oplus \cdots N_{\lambda_{k-1}} \quad \text{and} \quad X_+ = N_{\lambda_{k+1}} \oplus N_{\lambda_{k+2}} \oplus \cdots \]

**Proposition 1.** If \( u \in X_0 \), then \((Lu, u)_{H^1(\mathbb{R}^N)} = 0\).

**Proof.** As we have seen that
\[ (Lu, u)_{H^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) \, dx - \lambda_k \int_{\mathbb{R}^N} u^2 h \, dx, \]

since \( u \in X_0 \), it is solution of the problem
\[ \begin{cases} -\Delta u + u &= \lambda_k h u, \quad \mathbb{R}^N \\ u &\in H^1(\mathbb{R}^N) \end{cases} \]

of where, we have
\[ \int_{\mathbb{R}^N} (\nabla \phi \nabla u + \phi u) = \lambda_k \int_{\mathbb{R}^N} u \phi h \, dx, \quad \forall \phi \in H^1(\mathbb{R}^N). \]

Fixing \( \phi = u \),
\[ \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) \, dx = \lambda_k \int_{\mathbb{R}^N} u^2 h \, dx, \]

showing that
\[ (Lu, u)_{H^1(\mathbb{R}^N)} = 0, \]

if \( u \in X_0 \).
\[ \square \]

**Proposition 2.** If \( u \in X_- \), then there exists \( \alpha > 0 \) such that
\[ (Lu, u)_{H^1(\mathbb{R}^N)} \leq -\alpha \|u\|^2, \]

this is, \( L \) is negative defined in \( X_- \).

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Proof. Let \( u \in X_\pm = N_{\lambda_1} \oplus N_{\lambda_2} \oplus \ldots \oplus N_{\lambda_k-1} \), be, then

\[
  u = \phi_1 + \phi_2 + \ldots + \phi_{k-1} \quad \text{and} \quad \nabla u = \nabla \phi_1 + \nabla \phi_2 + \ldots + \nabla \phi_{k-1}.
\]

Note that from (11)

\[
  (Lu, u)_{H^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) \, dx - \lambda_k \int_{\mathbb{R}^N} (\nabla S(u) \cdot \nabla u + uS(u)) \, dx
\]

thus,

\[
  (Lu, u)_{H^1(\mathbb{R}^N)} = (u, u)_{H^1(\mathbb{R}^N)} - \lambda_k (S(u), u)_{H^1(\mathbb{R}^N)}. \tag{13}
\]

Provided that \( \phi_j \) satisfies

\[
  \begin{cases}
    -\Delta \phi_j + \phi_j = \lambda_j h \phi_j, & \mathbb{R}^N \\
    \phi_j \in H^1(\mathbb{R}^N)
  \end{cases}
\]

From definition of the solution operator

\[
  S(\lambda_j \phi_j) = \phi_j
\]

therefore,

\[
  S(\phi_j) = \frac{1}{\lambda_j} \phi_j \quad \text{and} \quad \nabla S(\phi_j) = \frac{1}{\lambda_j} \nabla \phi_j.
\]

From linearity of \( S \), we have

\[
  S(u) = S(\phi_1) + S(\phi_2) + \ldots + S(\phi_{k-1})
\]

which implies,

\[
  S(u) = \frac{1}{\lambda_1} \phi_1 + \frac{1}{\lambda_2} \phi_2 + \ldots + \frac{1}{\lambda_{k-1}} \phi_{k-1}.
\]

Thus,

\[
  (S(u), u)_{H^1(\mathbb{R}^N)} = (S(\phi_1) + S(\phi_2) + \ldots + S(\phi_{k-1}), \phi_1 + \phi_2 + \ldots + \phi_{k-1})_{H^1(\mathbb{R}^N)}
\]

and provided that \( (\phi_j, \phi_k)_{H^1(\mathbb{R}^N)} = 0 \), if \( j \neq k \), we obtain

\[
  (S(u), u)_{H^1(\mathbb{R}^N)} = \frac{1}{\lambda_1} (\phi_1, \phi_1)_{H^1(\mathbb{R}^N)} + \ldots + \frac{1}{\lambda_{k-1}} (\phi_{k-1}, \phi_{k-1})_{H^1(\mathbb{R}^N)}.
\]
It follows from definition of inner product in $H^1(\mathbb{R}^N)$ that

$$(S(u), u)_{H^1(\mathbb{R}^N)} = \frac{1}{\lambda_1} \int_{\mathbb{R}^N} (|\nabla \phi_1|^2 + |\phi_1|^2) \, dx + \cdots + \frac{1}{\lambda_{k-1}} \int_{\mathbb{R}^N} (|\nabla \phi_{k-1}|^2 + |\phi_{k-1}|^2) \, dx$$

which implies

$$\lambda_k (S(u), u)_{H^1(\mathbb{R}^N)} = \sum_{j=1}^{k-1} \int_{\mathbb{R}^N} \frac{\lambda_k}{\lambda_j} (|\nabla \phi_j|^2 + |\phi_j|^2) \, dx,$$

this is,

$$\lambda_k (S(u), u)_{H^1(\mathbb{R}^N)} = \sum_{j=1}^{k-1} \frac{\lambda_k}{\lambda_j} \|\phi_j\|^2.$$

Furthermore,

$$(u, u)_{H^1(\mathbb{R}^N)} = (\phi_1 + \phi_2 + \cdots + \phi_{k-1}, \phi_1 + \phi_2 + \cdots + \phi_{k-1})_{H^1(\mathbb{R}^N)}$$

hence,

$$(u, u)_{H^1(\mathbb{R}^N)} = \sum_{j=1}^{k-1} \int_{\mathbb{R}^N} (|\nabla \phi_j|^2 + |\phi_j|^2) \, dx = \sum_{j=1}^{k-1} \|\phi_j\|^2.$$

Now, we can write (13) of the following way

$$(Lu, u)_{H^1(\mathbb{R}^N)} = \sum_{j=1}^{k-1} \left(1 - \frac{\lambda_k}{\lambda_j}\right) \|\phi_j\|^2$$

and using the fact that

$$\lambda_j \leq \lambda_{k-1}, \quad \forall \ j \in \{1, 2, \ldots, k-1\}$$

which implies,

$$1 - \frac{\lambda_k}{\lambda_{k-1}} \geq 1 - \frac{\lambda_k}{\lambda_j}$$

thus,

$$(Lu, u)_{H^1(\mathbb{R}^N)} \leq \sum_{j=1}^{k-1} \left(1 - \frac{\lambda_k}{\lambda_{k-1}}\right) \|\phi_j\|^2 = \left(1 - \frac{\lambda_k}{\lambda_{k-1}}\right) \sum_{j=1}^{k-1} \|\phi_j\|^2.$$
Therefore,
\[(Lu, u)_{H^1(\mathbb{R}^N)} \leq \left(1 - \frac{\lambda_k}{\lambda_{k-1}}\right)\|u\|^2\]
from where we have
\[(Lu, u)_{H^1(\mathbb{R}^N)} \leq -\left(\frac{\lambda_k}{\lambda_{k-1}} - 1\right)\|u\|^2,
\]
this is,
\[(Lu, u)_{H^1(\mathbb{R}^N)} \leq -\alpha\|u\|^2 < 0, \ \forall \ u \in X_+ \setminus \{0\}\]
and \(\alpha > 0\).

**Proposition 3.** If \(u \in X_+\), then there exists \(\alpha > 0\) such that \((Lu, u)_{H^1(\mathbb{R}^N)} \geq \alpha\|u\|^2\), this is, \(L\) is positive defined in \(X_+\).

**Proof.** Suppose that \(u \in X_+ = N_{\lambda_{k+1}} \oplus N_{\lambda_{k+2}} \oplus \ldots\). Then \(u = \phi_{k+1} + \phi_{k+2} + \ldots\). We consider, \(u = \sum_{j=k+1}^{\infty} \phi_j = \lim_{N \to \infty} w_N\), where \(w_N = \sum_{j=k+1}^{N} \phi_j, w_N \in X_+\). We show that there exists \(\alpha > 0\) independent of \(N\) such that
\[(Lw_N, w_N)_{H^1(\mathbb{R}^N)} \geq \alpha\|w_N\|^2, \ \forall \ N \in \{k+1, k+2, \ldots\}.\]
Indeed, note that
\[(Lw_N, w_N)_{H^1(\mathbb{R}^N)} = \left(\sum_{j=k+1}^{N} L\phi_j, \sum_{j=k+1}^{N} \phi_j\right)_{H^1(\mathbb{R}^N)}\]
which implies,
\[(Lw_N, w_N)_{H^1(\mathbb{R}^N)} = \left(\sum_{j=k+1}^{N} (\phi_j - \lambda_k S(\phi_j)), \sum_{j=k+1}^{N} \phi_j\right)_{H^1(\mathbb{R}^N)}\]
consequently,
\[(Lw_N, w_N)_{H^1(\mathbb{R}^N)} = \left(\sum_{j=k+1}^{N} \phi_j, \sum_{j=k+1}^{N} \phi_j\right)_{H^1(\mathbb{R}^N)} - \left(\sum_{j=k+1}^{N} \lambda_k S(\phi_j), \sum_{j=k+1}^{N} \phi_j\right)_{H^1(\mathbb{R}^N)}\].
Now, using the same reasoning from Proposition 2, we have
\[(Lw_N, w_N)_{H^1(\mathbb{R}^N)} = \sum_{j=k+1}^{N} \|\phi_j\|^2 - \sum_{j=k+1}^{N} \frac{\lambda_k}{\lambda_j} \|\phi_j\|^2\]
and noting that \( \lambda_k \geq \lambda_{k+1} \) we find

\[
1 - \frac{\lambda_k}{\lambda_j} \geq 1 - \frac{\lambda_k}{\lambda_{k+1}}.
\]

Therefore,

\[
(Lw_N, w_N)_{H^1(\mathbb{R}^N)} \geq \left( 1 - \frac{\lambda_k}{\lambda_{k+1}} \right) \sum_{j=k+1}^{N} \|\phi_j\|^2.
\]

Thus,

\[
(Lw_N, w_N)_{H^1(\mathbb{R}^N)} \geq \left( 1 - \frac{\lambda_k}{\lambda_{k+1}} \right) \|w_N\|^2
\]

showing that

\[
(Lw_N, w_N)_{H^1(\mathbb{R}^N)} \geq \alpha \|w_N\|^2, \quad \forall \ N \in \{k + 1, k + 2, \ldots \},
\]

where \( \alpha = \left( 1 - \frac{\lambda_k}{\lambda_{k+1}} \right) \). Using the fact that \( w_N \overset{N \to \infty}{\longrightarrow} u \), we obtain

\[
\|w_N\|^2 \overset{N \to \infty}{\longrightarrow} \|u\|^2
\]

and from continuity of \( L \), we obtain \( L(w_N) \overset{N \to \infty}{\longrightarrow} L(u) \). Consequently,

\[
(Lw_N, w_N)_{H^1(\mathbb{R}^N)} \overset{N \to \infty}{\longrightarrow} (Lu, u)_{H^1(\mathbb{R}^N)}
\]

thus, passing to the limit at (14) follows that \( (Lu, u)_{H^1(\mathbb{R}^N)} \geq \alpha \|u\|^2 \)

\[\square\]

**Notation 1.** In the that follows, we denoted by \( P_- \), \( P_0 \) and \( P_+ \) the orthogonalais projections on \( X_- \), \( X_0 \) and \( X_+ \), respectively.

**Lemma 2.** If \( u = P_0 u + P_+ u \in X_0 \oplus X_+ \), then \( (Lu, u)_{H^1(\mathbb{R}^N)} = (LP_+ u, P_+ u)_{H^1(\mathbb{R}^N)} \).

**Proof.** Suppose that \( (Lu, u)_{H^1(\mathbb{R}^N)} = (L(P_0 u + P_+ u), P_0 u + P_+ u)_{H^1(\mathbb{R}^N)} \). Then

\[
(Lu, u)_{H^1(\mathbb{R}^N)} = (LP_0 u + LP_+ u, P_0 u + P_+ u)_{H^1(\mathbb{R}^N)}.
\]

Using the Proposition[1] and the fact that \( L \) is symmetric operator, we obtain

\[
(Lu, u)_{H^1(\mathbb{R}^N)} = 2(LP_0 u, P_+ u)_{H^1(\mathbb{R}^N)} + (LP_+ u, P_+ u)_{H^1(\mathbb{R}^N)}.
\]
Note that \((LP_0u, P_+u)_{H^1(\mathbb{R}^N)} = (P_0u - \lambda_k S(P_0u), P_+u)_{H^1(\mathbb{R}^N)}\) hence,
\[
(LP_0u, P_+u)_{H^1(\mathbb{R}^N)} = \lambda_k (S(P_0u), P_+u)_{H^1(\mathbb{R}^N)}.
\]

Therefore, by the definition of operator solution, we obtain
\[
(LP_0u, P_+u)_{H^1(\mathbb{R}^N)} = (P_0u, P_+u)_{L^2(\mathbb{R}^N, dx)} = 0,
\]
where the last equality we use the orthogonality of the projections. Thus,
\[
(Lu, u)_{H^1(\mathbb{R}^N)} = (LP_0u, P_+u)_{H^1(\mathbb{R}^N)}.
\]

**Proposition 4.** Suppose valid the conditions \((2)\) and \((g^-)\). Then
\[
(a) \Phi(u) \xrightarrow{\|u\| \to +\infty} -\infty, \quad u \in X_-
\]
\[
(b) \Phi(u) \xrightarrow{\|u\| \to +\infty} +\infty, \quad u \in X_0 \oplus X_+.
\]

**Proof.** (a) Suppose that \(u \in X_-\). Using the fact that \(L\) is defined negative in \(X_-\), we obtain
\[
\Phi(u) \leq -\frac{1}{2} \alpha \|u\|^2 - \int_{\mathbb{R}^N} G(x, u) \, dx.
\]

Note that for the mean value theorem, we have \(|G(x, u) - G(x, 0)| = |G'(x, s)||u|, s(x) \in [0, u(x)]\) or \(s(x) \in [u(x), 0]\), thus from \((2)\)
\[
|G(x, u) - G(x, 0)| \leq Z(x)|u|
\]
therefore, \(- \int_{\mathbb{R}^N} G(x, u) \, dx \leq \|u\|_{1, Z}\), of where we have that
\[
\Phi(u) \leq -\frac{1}{2} \alpha \|u\|^2 + \|u\|_{1, Z}.
\]

From Theorem \(3\) follows that \(\|u\|_{2, Z} \leq C_1 \|u\|\), then
\[
\Phi(u) \leq -\frac{1}{2} \alpha \|u\|^2 + C_2 \|u\|
\]
consequently, \(\Phi(u) \xrightarrow{\|u\| \to +\infty} -\infty\), \(u \in X_-\).

(b) Suppose that \(u = P_0u + P_+u \in X_0 \oplus X_+\). Using the Lemma \(2\) has been
\[
\Phi(u) = \frac{1}{2} (LP_0u, P_+u)_{H^1(\mathbb{R}^N)} - \int_{\mathbb{R}^N} G(x, u) \, dx.
\]
Furthermore, since \(L\) defined positive in \(X_+\) follows that
\[
\Phi(u) \geq \frac{1}{2} \alpha \|P_+u\|^2 - \int_{\mathbb{R}^N} [G(x, u) - G(x, P_0u)] \, dx - \int_{\mathbb{R}^N} G(x, P_0u) \, dx.
\]
Using again the mean value Theorem and (2), we have

\[- \int_{\mathbb{R}^N} [G(x, u) - G(x, P_0 u)] \, dx \leq \int_{\mathbb{R}^N} |G(x, u) - G(x, P_0 u)| \, dx \leq \int_{\mathbb{R}^N} |u - P_0 u| Z(x) \, dx\]

which implies,

\[- \int_{\mathbb{R}^N} [G(x, u) - G(x, P_0 u)] \, dx \leq \|P_+ u\|_{1,Z}.\]

Therefore, follows from the continuous immersion (see Theorem 2) that

\[\Phi(u) \geq \frac{1}{2} \alpha \|P_+ u\|^2 - C\|P_+ u\| - \int_{\mathbb{R}^N} G(x, P_0 u) \, dx. \tag{15}\]

Now, using the condition \((g_2^-)\), the fact that \(\|u\|^2 = \|P_+ u\|^2 + \|P_0 u\|^2\) and analyzing the cases:

(i) \(\|P_0 u\| \to +\infty\) and \(\|P_+ u\| \leq M\). Using \((g_2^-)\) and doing the analysis in (15), we have \(\Phi(u) \to +\infty\).

(ii) \(\|P_+ u\| \to +\infty\) and \(\|P_0 u\| \leq K\). Note that

\[\left| - \int_{\mathbb{R}^N} G(x, P_0 u) \, dx \right| \leq \int_{\mathbb{R}^N} |G(x, P_0 u)| \, dx \leq \int_{\mathbb{R}^N} |P_0 u| Z(x) \, dx,\]

that is,

\[\left| - \int_{\mathbb{R}^N} G(x, P_0 u) \, dx \right| \leq K_1, \quad K_1 \in \mathbb{R}_+.\]

Thus, doing the analysis in (15), we have \(\Phi(u) \to +\infty\).

(iii) If \(\|P_+ u\| \to +\infty\) and \(\|P_0 u\| \to +\infty\). Again the condition \((g_2^-)\) and doing the analysis in (15) follows that \(\Phi(u) \to +\infty\),

which shows (b).

**Remark 1.** If we had assumed the condition \((g_2^+)\) to \((g_2^-)\) instead of the reasoning used was the same and the conclusions were the following:

(a) \(\Phi(u) \to -\infty, \quad u \in X_0 \oplus X_-\).

(b) \(\Phi(u) \to +\infty, \quad u \in X_+\).
3.2. Weak solution to the problem

We are ready to state and prove the main result of our work. This result guarantees the existence of a critical point to the functional $\Phi$, and therefore the existence of a weak solution to the problem (1).

For we show the existence of weak solution for the problem (1) we use the theorem from the saddle of Rabionowitz.

**Theorem 4** (Saddle Point Theorem [8]). Let $X = V \oplus W$ be a Banach’ space, of way that $\dim V < \infty$, and let $\phi \in C^1(X, \mathbb{R})$ be a maps satisfying the condition of Palais-Smale. If $D$ is a bounded neighborhood of the 0 in $V$ such that

$$a = \max_{\partial D} \phi < \inf_{W} \phi \equiv b,$$

then

$$c = \inf_{h \in \Gamma} \max_{u \in \partial D} \phi(h(u))$$

is a critical value of $\phi$ with $c \geq b$, where

$$\Gamma = \{ h \in C(\overline{D}, X) \mid h(u) = u, \ \forall u \in \partial D \}.$$

**Theorem 5.** If the conditions $(g^+_2)$ or $(g^-_2)$ are true, then (1) has a weak solution, i.e., there exists $u \in H^1(\mathbb{R}^N)$ such that $u$ is a weak solution of (1).

**Proof.** As we already know that $\Phi \in C^1(X, \mathbb{R})$, We show that $\Phi$ satisfies the Palais-Smale condition $(PS)$, i.e., give the sequence $(u_n) \subset H^1(\mathbb{R}^N)$ with

$$|\Phi(u_n)| \leq c$$

and $\Phi'(u_n) \to 0$

we have that $(u_n)$ has been a subsequence convergent. Indeed, since

$$\Phi(u_n) = \frac{1}{2}(Lu_n, u_n)_{H^1(\mathbb{R}^N)} - \int_{\mathbb{R}^N} G(x, u_n) \, dx,$$

then

$$(\Phi'(u_n))v = (Lu_n, v)_{H^1(\mathbb{R}^N)} - \int_{\mathbb{R}^N} g(x, u_n)v \, dx$$

which implies,

$$|\Phi'(u_n))v| = \left| (Lu_n, v) - \int_{\mathbb{R}^N} g(x, u_n)v \, dx \right|. \quad (17)$$
Furthermore,
\[ |(\Phi'(u_n))v| \leq \|\Phi'(u_n)\| \|v\| \]
where we have
\[ |(\Phi'(u_n))v| \leq \|v\|, \quad \forall \ v \in H^1(\mathbb{R}^N), \]
for \( n \) large enough, since
\[ \Phi'(u_n) \rightarrow 0 \Leftrightarrow \|\Phi(u_n)\| \rightarrow 0 \]
thus, \( \|\Phi'(u_n)\| \leq 1 \). Now, our next step is to show that
\[ u_n = P_0 u_n + P_- u_n + P_+ u_n \quad (18) \]
is bounded. Note that
(i) If \( v = P_+ u_n \), replacing in (17) follows that
\[ |(\Phi'(u_n))P_+ u_n| = |(Lu_n, P_+ u_n) - \int_{\mathbb{R}^N} g(x, u_n) P_+ u_n \, dx| . \]
Note that using the same reasoning from Lemma 2,
\[ (Lu_n, P_+ u_n) = (L(P_- u_n) + L(P_0 u_n) + L(P_+ u_n), P_+ u_n) \]
\[ = (L(P_+ u_n), P_+ u_n) . \]
Hence, using the fact that \( L \) is defined positive in \( X_+ \), we have that
\[ (Lu_n, P_+ u_n) = (L(P_+ u_n), P_+ u_n) \]
\[ \geq \alpha \|P_+ u_n\|^2 . \]
Furthermore,
\[ \|P_+ u_n\| \geq |\Phi'(u_n))P_+ u_n| \]
\[ \geq |(Lu_n, P_+ u_n)| - \left| \int_{\mathbb{R}^N} g(x, u_n) P_+ u_n \, dx \right| \]
which implies,
\[ \|P_+ u_n\| \geq \alpha \|P_+ u_n\|^2 - \|P_+ u_n\|_{1,Z} . \]
From Theorem 2, we have that
\[ \|P_+ u_n\| \geq \alpha \|P_+ u_n\|^2 - C \|P_+ u_n\| , \]
thus \( \|P_+ u_n\| \) is bounded.

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(ii) Again, considering \( v = P_- u_n \) in (17) we obtain

\[
(\Phi'(u_n))(P_- u_n) \leq (L u_n, P_- u_n) + \int_{\mathbb{R}^N} |g(x, u_n)||P_+ u_n| \, dx,
\]

from which follow that

\[
(\Phi'(u_n))(P_- u_n) \leq -\alpha \|P_- u_n\|^2 + \|P_- u_n\|_1, Z
\]

thus,

\[
-\|P_- u_n\| \leq \Phi'(u_n)P_- u_n \leq -\alpha \|P_- u_n\|^2 + C_1 \|P_- u_n\|
\]

consequently,

\[
\|P_- u_n\| \geq \alpha \|P_- u_n\|^2 - C_1 \|P_- u_n\|.
\]

Thus, \( \|P_- u_n\| \) is bounded.

Now, from (i) and (ii) has been

\[
\|u_n - P_0 u_n\| = \|P_+ u_n + P_- u_n\| \leq K, \ K \in \mathbb{R}_+.
\]  \hspace{1cm} (19)

For another side, we can write

\[
\Phi(u_n) = \frac{1}{2}(L(u_n - P_0 u_n), u_n - P_0 u_n) - \int_{\mathbb{R}^N} G(x, u_n) - G(x, P_0 u_n) \, dx - \int_{\mathbb{R}^N} G(x, P_0 u_n) \, dx
\]

we have that,

\[
\int_{\mathbb{R}^N} G(x, P_0 u_n) \, dx = \frac{1}{2}(L(u_n - P_0 u_n), u_n - P_0 u_n) - \Phi(u_n) - \int_{\mathbb{R}^N} [G(x, u_n) - G(x, P_0 u_n)] \, dx.
\]

Now, since \( L = I - \lambda_k S \), we have that \( L \) is bounded, thus

\[
|(L(u_n - P_0 u_n), u_n - P_0 u_n)| \leq \|L(u_n - P_0 u_n)||u_n - P_0 u_n| \leq \|L\||u_n - P_0 u_n|^2,
\]

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and from (19), we obtain
\[ |(L(u_n - P_0u_n), u_n - P_0u_n)| \leq C, \ C \in \mathbb{R}_+. \]

Note that
\[ \left| \int_{\mathbb{R}^N} G(x, P_0u_n) \, dx \right| \leq \frac{1}{2} |(L(u_n - P_0u_n), u_n - P_0u_n)| + |\Phi(u_n)| + \left| \int_{\mathbb{R}^N} [G(x, u_n) - G(x, P_0u_n)] \, dx \right| \]
therefore,
\[ \left| \int_{\mathbb{R}^N} G(x, P_0u_n) \, dx \right| \leq C_1, \ C_1 \in \mathbb{R}_+. \]

Hence, using the condition \((g_2)\) we can conclude that \(|P_0u_n|\) is bounded.

Note that the orthogonality of the projection, we have that
\[ \|u_n\|^2 = \|P_0u_n\|^2 + \|P_-u_n\|^2 + \|P_+u_n\|^2, \]
showing that \((u_n)\) is bounded. We know that
\[ (\Phi'(u))v = \int_{\mathbb{R}^N} (\nabla u \nabla v + uv) \, dx - (\psi'(u))v, \ \forall \ u, v \in H^1(\mathbb{R}^N) \]
where
\[ \psi(u) = \int_{\mathbb{R}^N} \left( \frac{\lambda k}{2} u^2 h + G(x, u) \right) \, dx, \]
thus
\[ (\nabla(\Phi(u)), v)_{H^1(\mathbb{R}^N)} = (u, v)_{H^1(\mathbb{R}^N)} - (\nabla(\psi(u)), v)_{H^1(\mathbb{R}^N)}. \]
Therefore,
\[ (\nabla(\Phi(u)), v)_{H^1(\mathbb{R}^N)} = (u - \nabla(\psi(u)), v)_{H^1(\mathbb{R}^N)} \]
and consequently, \(\nabla(\Phi(u)) = u - \nabla(\psi(u))\). Considering \(T(u) = \nabla(\psi(u))\), we have that
\[ \nabla(\Phi(u)) = u - T(u). \]

Therefore,
\[ \nabla(\Phi(u_n)) = u_n - T(u_n) \]
which implies,
\[ u_n = \nabla(\Phi(u_n)) + T(u_n). \]
Now, since $T : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ compact (see Appendix C of [3]), there exists $(u_{n_j}) \subset (u_n)$ such that
\[ T(u_{n_j}) \xrightarrow{n_j \to +\infty} u \]
and using the fact that
\[ \Phi'(u_n) \to 0 \iff \|\Phi'(u_n)\| \to 0 \iff \|\nabla\phi(u_n)\| \to 0 \iff \nabla\Phi(u_n) \to 0, \]
passing to the limit in $u_{n_j} = \nabla(\Phi(u_{n_j})) + T(u_{n_j})$ we find
\[ u_{n_j} \xrightarrow{n_j \to \infty} u \]
showing that $\Phi$ satisfies the conditions (PS).

Finally, we have that $\Phi \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$, and $\Phi$ satisfies the condition of Palais-Smale and using the Proposition [4] we can apply the Theorem from the saddle of Rabionowitz with $V = X_-$ and $W = X_0 \oplus X_+$ and ensure the existence of a critical point for $\Phi$, therefore, a weak solution for the problem [1].

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