Chapter

Influence of Input Parameters on the Solution of Inverse Heat Conduction Problem

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Abstract

A one-dimensional transient heat conduction equation is solved using analytical and numerical methods. An iterative technique is employed which estimates unknown boundary conditions from the measured temperature time history. The focus of the present chapter is to investigate effects of input parameters such as time delay, thermocouple cavity, error in the location of thermocouple position and time- and temperature-dependent thermophysical properties. Inverse heat conduction problem IHCP is solved with and without material conduction. A two-time level implicit finite difference numerical method is used to solve nonlinear heat conduction problem. Effects of uniform, nonuniform and deforming computational grids on the estimated convective heat transfer are investigated in a nozzle of solid rocket motor. A unified heat transfer analysis is presented to obtain wall heat flux and convective heat transfer coefficient in a rocket nozzle. A two-node exact solution technique is applied to estimate aerodynamic heating in a free flight of a sounding rocket. The stability of the solution of the inverse heat conduction problem is sensitive to the spatial and temporal discretization.

Keywords: analytical solution, inverse heat conduction problem, numerical analysis, deforming grid, heat transfer coefficient, heat flux, random search method

1. Introduction

The basic theory of heat and structure of solid body is associated with the internal energy of matter which in the first law of thermodynamics is referred to as the internal energy concerned with the physical state of the material. The first law of thermodynamics defines that the flowing heat energy is conserved in the absence of heat sources and sinks. It is, therefore, important to study the influence of thermocouple lead wires and distortion due to the thermocouple cavity in solution of the inverse heat conduction problem. According to the second law of thermodynamics, the heat will be transferred from one body to another body only when the bodies are at two different temperatures level and the heat will flow from the point of higher to the point of lower temperature.

A direct solution of transient heat conduction equation with prescribed initial and boundary conditions yields temperature distribution inside a slab of finite thickness. The direct solution is mathematically considered as well-posed because the solution exists, unique and continuously depends on input data. The estimation
of unknown parameters from the measured temperature history is called as inverse problem of heat conduction. It is mathematically known as an ill-posed problem since the solution now does not continuously depend on the input data. Measurement data error in temperature, thermal lagging, thermocouple’s cavity, signal noise, etc. makes stability problem in the estimation of unknown parameters.

Numerical inversion of the integral solution [1], exact solution [2], numerical techniques [3], least-squares method [4], transform methods [5], different series approach [6], variable time-step size [7] have been applied to solve inverse heat conduction problems. Solutions of the ill-posed inverse heat conduction problem have been presented in detail by Beck et al. [8] and Özisik et al. [9]. Tikhonov regularization method [10] has been described for cross-validation criterion for selecting the regularization parameter to obtain a stable approximation to the solution. Kurpisz et al. [11] have presented series with derivatives with temperature to solve inverse thermal problem. Hensel [12] has described space marching numerical methods to solve inverse heat transfer problem. Various mathematical methods and numerical algorithms for solving inverse heat conduction problems are described and compared by Alifanov [13]. Taler and Duda [14] have presented solutions of direct and inverse heat conduction problems.

Inverse heat conduction analysis provides an efficient tool for estimating the thermophysical properties of materials, the boundary conditions, or the initial conditions. Estimation of surface heat flux has been carried out without [15] and with [16] heat conduction and comparison between them shows discrepancies as high as about 27% [17]. Moving window optimization method [18] has been applied to predict the aerodynamic heating in a free-flight of sounding rocket by comparing numerically calculated and measured temperature history. Howard [19] developed a numerical procedure for estimating the heat flux with variable thermal properties using a single embedded thermocouple. Simultaneous identification of the temperature-dependent thermal conductivity and the asymmetry parameter of the Heneyy-Greenstein scattering phase function have been shown by Zmywaczyk and Koniorczyk [20].

The conjugate gradient method with adjoint problem for function estimation iterative technique is used to solve IHCP to estimate heat flux and internal wall temperature of the throat section of the rocket nozzle [21]. Heisler [22] have reported supplementary “short-time” temperature-time charts for the center, mid-location and surface of large plates, long cylinders and spheres for the dimensionless time sub-domain. Convective heat transfer coefficient and combustion temperature in a rocket nozzle is determined using transient-temperature response chart [23].

The solution of transient IHCP can be obtained using analytical or numerical schemes in conjunction with measured temperature-time history. The estimation of the unknown parameters can be carried out by employing gradient or non-gradient methods to predict the unknown parameters in a prescribed tolerance limit. The focus of the present work is to investigate the influence of various parameters on the solution of inverse heat conduction problem.

2. Measurement errors

Experimental difficulties [24] are noticed in implanting thermocouples at the surface for temperature measurements. Temperature response delays have been studied to solve IHCP applied to cooled rocket thrust chamber [25]. The temperature measured inside the slab may delay and damp depending on $x_m$ as illustrated in Figure 1. A thermocouple indicates temperature lag behind the actual temperature. The effect of the thermocouple sensor dynamics on prediction of a triangular heat
flux history has been analyzed with simulated data in a one-dimensional domain by Woodbury [26].

Chen and Danh [27] have carried out experimental studies to obtain transient temperature distortion and thermal delay in a slab due to presence of thermocouple cavity. The distortion of temperature profiles inside the slab may be influenced by the dissimilar thermophysical properties of thermocouple and surrounding materials and by the diameter and depth of the cavity. The temperature distortion [28] inside a slab is a function of the thermocouple cavity diameter $d$ and location $x_m$.

Standard statistical analysis consists of error in the measurement as an additive of true plus random, in zero mean, in constant variance, uncorrelated, normal, bell shaped probability density function, constant variance known, errors in the dependent variables and no-prior information about the parameters. The error in measurement can be obtain using exact analytical solution [29] as

$$E = \cos \left(\frac{x}{L}\right) (\cos \varepsilon - 1) e^{-\delta \tau} \quad (1)$$

$$E = \sin \left(\frac{x}{L}\right) (\cos \varepsilon - 1) e^{-\delta \tau} \quad (2)$$

where $\varepsilon$ and $\delta \tau$ refer to error in measurement of thermocouple location and in time recording, respectively. One of the important points that must be mentioned, here, is the use of a starting solution. In the case of a solid rocket motor where boundary conditions are suddenly imposed by a wall, there will be high intensity of heat flux on cold wall, and the heat flux during the first few steps in time may not be very accurate. The numerical solution is initiated using an exact analytical solution instead of starting from the initial constant condition. Such solutions can be obtained from to exact analytical solution [30] of transient heat conduction equation. Heat transfer rates to the calorimetric probe are estimated from measurements of temperature and rate of temperature change using energy conservation considerations [31].
An optimization method based on a direct and systematic search region reduction optimization method [32] can be employed to estimate the unknown convective heat transfer coefficient in a typical rocket nozzle. The most attractive feature of the direct search scheme is the simplicity of computer programming. The pseudo-random algorithm, an effective tool for optimization, does not require computation of derivatives but depends only on function evaluation. It works even when the differentiability requirements cannot be ensured in the feasible domain. For initiating the search only an estimate of the feasible domain is needed. Therefore, another advantage of the method is that the starting condition is not crucial; any reasonable value will do.

3. Heat conduction equation

3.1 Analytical solution

The computation of the turbulent convective heat transfer coefficient from combustion gases to the rocket nozzle wall is based on the Bartz’s equation [33] incorporating the effects of compressibility, throat curvature and variation of transport properties in the boundary layer. The transient heat conduction in a one-dimensional Cartesian coordinate system having two parallel plane surfaces \( S_n \) \((n = 1, 2)\) of a slab may be written in dimensional form [34] as.

\[
\frac{\partial \theta(X, \tau)}{\partial \tau} + \frac{\partial}{\partial X} \left[ K(\theta) \frac{\partial \theta(X, \tau)}{\partial X} \right], \text{ in region } R, \tau > 0
\]  

with following initial and boundary conditions:

\[
\theta(X, 0) = f_i(X), \text{ in region } R, \quad (4)
\]

\[
-\frac{\partial \theta|_{S_n}(\tau)}{\partial X} = Bi_n \left[ \theta|_{S_n}(\tau) \right], \text{ or } -\frac{q_w L}{(T_g - T_i)} \text{ on boundary } S_n, \tau > 0
\]  

where \( f_i \) is initial temperature distribution in the region \( R \) of the slab. Eq. (4) represents both convective heat transfer or heat flux condition as applied to the inner surface.

We now consider the constant thermal property solution and can be written in terms of eigen function \( \psi_{\lambda m}(X) \) as

\[
\theta(X, \tau) = \sum_{m=1}^{\infty} \exp \left[ -\lambda_m^2 \tau \right] \psi(\lambda_m, X) \int_R \psi(\lambda_m X) f(X) dX \]  

In the above Eq. (5), \( Bi \) or \( q_w \) is the unknown parameter to be determined using measured temperature time history at location \( x_m \) as depicted in Figure 1. In estimating the unknown condition, one has to minimize the absolute difference between the calculated and measured temperature at specified location and time \((x_m, \tau)\) in a prescribed tolerance value using an iteration procedure. The iteration scheme is described in the following sections.

3.2 Inverse algorithm

The IHCP is solved by comparing calculated and measured temperature using an iterative technique [30]. In estimating \( q_w \), one minimizes
where $\theta_c$ and $\theta_m$ are the calculated and measured temperatures at $(X_m, \tau)$, respectively. The computed temperature is a nonlinear function of unknown parameters such as wall heat flux or convective heat transfer coefficient. Temperature is calculated using Eq. (6) and compared with the measured temperature as expressed in Eq. (7). The inverse problem starts with initial guess value of the unknown parameter. The second step is to correct the previous guessed unknown parameter using the Newton-Raphson method. The sensitivity coefficient can be obtained by differentiating temperature with respect to wall heat flux $q_w$. The iteration procedure will continue until $|F(q_w)| \leq 10^{-4}$. This iterative scheme estimates the component of the $q_w$ at a time and thus may be considered on-line method.

The inverse method for solving a value of $q_w(0, \tau)$ is as follows. Initiate with an initial guess value of $q_w$, satisfy the convergence criterion, and implement the Newton-Raphson to obtain the estimate value.

Now, it is possible to estimate convective heat transfer coefficient and combustion gas temperature in conjunction with measured temperature history [35]. The equation for converting the calculated heat flux to the heat transfer coefficient is

$$Bi = \frac{Lq_w}{K(\theta)(T_g - T_i)}$$

In the foregoing equation, $T_g$ is an unknown quantity and can be estimated using again the above-mentioned minimization and iteration methods. The convergence criterion for the iterative scheme remains same as mentioned above.

3.3 Numerical methods

It is not always feasible to obtain analytical solution of temperature-dependent thermal conductivity and radiation boundary condition. The Crank-Nicolson finite difference method with two-time level implicit numerical scheme [36] has been employed to solve the nonlinear conduction problem with the Newton-Raphson method to consider the radiation boundary condition.

Deforming or moving finite elements method [37] is used to solve linear heat conduction equation. The moving finite element [38] is used to consider the time delay in the measurement of back wall temperature.

3.4 Two-nodes system of transient heat conduction equation

For only two nodes the system of [39] equations reduce to the following pair of equations:

$$\frac{d\theta_0}{d\tau} = \frac{1}{(\Delta X)^2}[-2(\theta_1 + Bi\Delta X)\theta_0 + 2\theta_1] \quad (9)$$

$$\frac{d\theta_1}{d\tau} = \frac{1}{(\Delta X)^2}[2\theta_0 - 2\theta_1] \quad (10)$$

where 0 and 1 represent node in a slab of finite thickness. These are the exact solutions to the system of two ordinary differential equations which resulted from a two-node finite-difference approximation to the original problem.
\[ \theta_0 = \frac{1}{\lambda_2 - \lambda_1} \left[ (\lambda_2 + 2Bi)e^{\lambda_1 \tau} - (\lambda_1 + 2Bi)e^{\lambda_2 \tau} \right] \] 

(11)

\[ \theta_1 = \frac{1}{\lambda_2 - \lambda_1} \left[ \lambda_2 e^{\lambda_1 \tau} - \lambda_1 e^{\lambda_2 \tau} \right] \] 

(12)

where \( \lambda_1 = -2 - Bi + \sqrt{Bi^2 + 4} \) and \( \lambda_2 = -2 - Bi - \sqrt{Bi^2 + 4} \).

Solution of the above simultaneous equation calculates the temperature with a given value of \( Bi \). The solution is now solving simultaneously Eqs. (11) and (12) to determine the unknown parameter.

4. Inverse problem of heat conduction applied to a rocket nozzle

The influence of constant (average) thermal conductivity, temperature-dependent thermal conductivity, computational grid in numerical solver, nonlinear boundary condition, cylindrical coordinate and the estimation of the wall heat flux and convective heat transfer is carried out by employing measured temperature history of a rocket nozzle of a solid motor. Solution of linear heat conduction equation is used to estimate the convective heat transfer coefficient with the measured temperature data of outer wall of a rocket nozzle. The running time of rocket motor is 16 s. The nozzle wall thickness \( L = 0.0211 \) m. The thermo-physical properties of the material are: \( \rho = 7900 \) kg m\(^{-3}\), \( C_p = 545 \) J kg\(^{-1}\) K\(^{-1}\), \( K \) (average) = 35 Wm\(^{-1}\) K\(^{-1}\). Initial temperature \( T_i = 300 \) K and combustion gas temperature \( T_g = 2946.2 \) K are used in the solution of the heat conduction equation.

4.1 Average thermal conductivity

Prediction of convective heat transfer coefficient is carried out in conjunction with the calculated and measured temperature history at outer surface of nozzle divergent in a solid rocket motor static test. The constant thermal conductivity solution of the linear transient heat conduction problem [30] is

\[ \theta(X, \tau) = 1 - 2 \sum_{n=1}^{\infty} \frac{Bi}{(Bi^2 + \lambda_n^2 + Bi)} \cos \left( \frac{\lambda_n(1-X)}{\cos \lambda_n} \right) e^{-\lambda_n^2 \tau} \] 

(13)

\[ \lambda \tan \lambda = Bi \] 

(14)

For estimating unknown boundary condition, the heat conduction equation is and solved with the following boundary and initial conditions.

\[ \frac{\partial \theta(0, \tau)}{\partial X} = Bi[\theta(0, \tau) - 1], \tau > 0 \] 

(15)

\[ \frac{\partial \theta(1, \tau)}{\partial X} = 0, \tau > 0 \] 

(16)

and

\[ \theta(X, 0) = 0, \text{for all} \ X \] 

(17)

Exact analytical solution of transient heat conduction as written in Eq. (13) is used to estimate convective heat transfer on the inner surface of the rocket nozzle.
An iterative scheme is used to solve inverse problem [30]. The iteration is carried out till the absolute difference between calculated and measured temperature is less than or equal to $10^{-4}$. Table 1 exhibits the comparison between the estimated values of the convective heat transfer coefficient based on the exact solution of heat conduction equation with the calculated values of Bartz [33]. Bartz’s equation calculates conservative estimates for the convective heat transfer to the wall [40].

### 4.2 Temperature-dependent thermal conductivity

An iteration procedure [41] is employed in conjunction with exact solution to predict convective heat transfer coefficient from the measured temperature-time data at the outer wall of the nozzle as shown in Table 2. The expression for temperature-dependent conductivity is $K(T) = k_0 - \beta T$. The value of $k_0$ and $\beta$ are 57 Wm$^{-1}$K$^{-1}$ and 2.718 Wm$^{-1}$K$^{-2}$, respectively. The advantage of using the exact solution is found directly at specified location and time as compared to the numerical method which needs the computation from the initial state.

### 4.3 Numerical solution with various computational grids

Deforming or moving finite element is used to consider the time delay in temperature at the outer wall of the slab [37]. Estimated values of wall heat flux and heat transfer coefficient are tabulated in Table 3. It can be observed from the table that the estimated wall quantities are having significant influence on the predicted unknown boundary conditions. This example is extended to consider spatial grid changed and temporal dependence on the numerical solution using moving finite element method [38].

### 4.4 Nonlinear boundary condition

Numerical analysis of nonlinear heat conduction with a radiation boundary condition [36] is carried out to estimate wall heat flux using temperature history on the back wall of the rocket nozzle. The high temperature variation alters thermophysical properties of the material of mild steel. Table 4 shows comparison

| $t$, s | $\theta_0$ at inner surface | $\theta_s$ at outer surface | $\theta_m$ at outer surface | $h$, W/m$^2$K | $h_B$, W/m$^2$K |
|-------|----------------------------|-----------------------------|-----------------------------|---------------|------------------|
| 6     | 0.2950                     | 0.0098                      | 0.0096                      | 1821.9        | 2254.2           |
| 7     | 0.3109                     | 0.0159                      | 0.0158                      | 1810.0        | 2254.2           |
| 8     | 0.2996                     | 0.0212                      | 0.0211                      | 1610.3        | 2254.2           |
| 9     | 0.3244                     | 0.0301                      | 0.0302                      | 1690.9        | 2254.2           |
| 10    | 0.3340                     | 0.0386                      | 0.0385                      | 1669.7        | 2254.2           |
| 11    | 0.3416                     | 0.0473                      | 0.0472                      | 1641.9        | 2254.2           |
| 12    | 0.3302                     | 0.0529                      | 0.0529                      | 1497.6        | 2254.2           |
| 13    | 0.3312                     | 0.0602                      | 0.0604                      | 1443.1        | 2254.2           |
| 14    | 0.3409                     | 0.0677                      | 0.0676                      | 1387.0        | 2254.2           |
| 15    | 0.3442                     | 0.0781                      | 0.0782                      | 1413.0        | 2254.2           |
| 16    | 0.3475                     | 0.0862                      | 0.0861                      | 1383.7        | 2254.2           |

Table 1. Solution of inverse heat conduction problem.
between the estimated convective heat transfer coefficients with the Bartz solution [33]. Effects of nonlinear IHCP with radiation boundary condition are investigated and results are presented in Table 4.

4.5 Heat conduction in a hollow cylinder

A grid point shift strategy [42] is adapted to solve inverse conduction problem in a radial coordinate of rocket nozzle with inner and outer radius of rocket nozzle. The inner and outer radius of the nozzle is 0.0839 m and 0.0105 m, respectively. The purpose of the present example to investigate the influence of radial coordinate
on the estimated values of heat transfer coefficient. Table 5 shows the effect of geometrical parameters on the predicted heat transfer coefficient.

### 4.6 Estimation of heat flux and heat transfer coefficient

The calculated convective heat transfer coefficients and inner wall temperature are used to determine the wall heat flux and the combustion temperature using Eq. (8). The iterative scheme is based on relation between wall heat flux and convective heat transfer coefficient [35]. Table 6 shows the predicted values of wall heat flux and convective heat transfer coefficient. The IHCP is extended to determine wall heat flux in conjunction with convective heat transfer coefficient. A similar IHCP but referring to the 122 mm medium-range missile during correction engine operation has been considered by Zmywaczyk et al. [43].

| $t, s$ | $T_w, K$ at $X = 0$ | $T_m, K$ at $X = 1$ | $q_c \times 10^6$ $W/m^2$ | $h, W/m^2K$ | $h_B, W/m^2K$ | $\theta_g, K$ | $\theta_{pr}, K$ |
|-------|------------------|------------------|------------------|-------------|-------------|-------------|-------------|
| 6     | 659.8            | 326              | 2.3547           | 950.0       | 2254.2      | 3137        | 2946.2      |
| 7     | 801.0            | 342              | 2.3899           | 1019.6      | 2254.2      | 3122        | 2946.2      |
| 8     | 900.7            | 356              | 2.2211           | 992.4       | 2254.2      | 3115        | 2946.2      |
| 9     | 996.3            | 380              | 2.6489           | 1237.1      | 2254.2      | 3113        | 2946.2      |
| 10    | 1050.5           | 402              | 2.3670           | 1135.5      | 2254.2      | 3108        | 2946.2      |
| 11    | 1066.4           | 425              | 1.7100           | 827.3       | 2254.2      | 3104        | 2946.2      |
| 12    | 1201.8           | 440              | 2.8144           | 1459.2      | 2254.2      | 3099        | 2946.2      |
| 13    | 1320.0           | 460              | 2.6559           | 1467.0      | 2254.2      | 3098        | 2946.2      |
| 14    | 1354.8           | 479              | 1.7595           | 991.7       | 2254.2      | 3095        | 2946.2      |
| 15    | 1383.4           | 507              | 1.3810           | 791.4       | 2254.2      | 3094        | 2946.2      |
| 16    | 1414.9           | 528              | 1.1684           | 681.8       | 2254.2      | 3094        | 2946.2      |

Table 4.
Solution with nonlinear boundary condition.
5. Estimation of heat flux with two-nodes in a sounding rocket

A two-node exact solution is used to calculate the back-wall temperature as described in Section 3.4. The iterative method described above has been used for estimating aerodynamic heating for a sounding rocket in free flight test. Here, the wall heat flux is estimated using the measured temperature history in conjunction with the iterative technique [30]. The aerodynamic heating rate is estimated for a typical sounding rocket as depicted in Figure 2. The location of thermocouple is marked in the diagram. The thermophysical properties of Inconel and wall thickness are $k = 18 \text{ Wm}^{-1} \text{K}^{-1}$, $\alpha = 4.47 \times 10^{-6} \text{ m}^2/\text{s}$, $L = 0.7874 \times 10^{-3} \text{ m}$. Figure 3 depicts the measured temperature time history at different locations measured from the tip of the cone in the free flight of a sounding rocket as delineated in Figure 2. It can be observed from temperature history that the initial time delay in thermal response is 6 s. The unknown $q_w$ are estimated using an iterative technique which starts with an initial value of wall heat flux and is repeated until $|F(q_w)| \leq 10^{-4}$.

A two-node exact solution is used to calculate the wall temperature distribution. The unknown $q_w$ are estimated using an iterative technique which starts with an initial value of wall heat flux and is repeated until $|F(q_w)| \leq 10^{-4}$. Figure 4 displays the estimated variation of the wall heat flux as a function of flight time of the sounding rocket.
The wall heat flux variation depends on the sounding rocket speed. The increase and decrease of the aerodynamic heating are a function of flight Mach number.

The estimated wall heat flux is compared with Van Driest’s results [44]. Table 7 depicts the estimated values of wall heat flux as a function of flight time at thermocouple location 29 as shown in Figure 2. It can be observed from the table that highest aerodynamic heating occurs during 7–8 s, another significant peak wall heat flux was found at 22 s.
6. Conclusions

Analytical, transient numerical and two-node methods are used to compute temperature distribution in a finite slab. Numerical solution is carried out with temperature-dependent thermal conductivity. Implicit finite difference scheme with two-time level technique is implemented to solve nonlinear problem of heat conduction. Time delay is studied using finite element method with deforming grid strategy. A boundary shifting numerical scheme is used to solve transient heat conduction in radial coordinate. Evidence of temporal accuracy and dependence on time-step is demonstrated in the numerical solving of IHCP. Influence of thermocouple cavity and measurement errors in location and time are discussed. The IHCP is applied to predict the wall heat flux in a rocket nozzle of a solid motor. Wall heat flux is estimated in a free flight of a sounding rocket using the two-node method.

Nomenclature

| Symbol | Description |
|--------|-------------|
| Bi     | Biot number, $hL/k$ |
| $C_p$  | specific heat |

Table 7.
Comparison between calculated and Van Driest’s heat flux at location 29.
heat transfer coefficient

\( K(\theta) \) thermal conductivity, \( k(\theta)/k_0 \)

\( k_0 \) reference thermal conductivity at \( T_i \)

\( L \) slab thickness

\( p \) nondimensional parameter, \( \alpha \Delta T/(\Delta x)^2 \)

\( q_{w} \) wall heat flux

\( T \) temperature

\( t \) time

\( x \) distance from the inner surface

\( X \) dimensionless coordinate

\( \alpha \) thermal diffusivity

\( \rho \) density

\( \theta \) nondimensional temperature, \( = (T - T_g)/(T_g - T_i) \)

\( \tau \) nondimensional time, \( \alpha t/L^2 \)

Subscripts

\( B \) Bartz

\( c \) computed

\( g \) combustion gas temperature

\( i \) initial value

\( m \) measured

\( o \) outer wall

\( w \) wall

\( \beta \) constant thermal conductivity coefficient

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