SPECTRAL ISOLATION OF NATURALLY REDUCTIVE METRICS ON SIMPLE LIE GROUPS

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Abstract. We show that within the class of left-invariant naturally reductive metrics \( M_{\text{Nat}}(G) \) on a compact simple Lie group \( G \), every metric is spectrally isolated. We also observe that any collection of isospectral compact symmetric spaces is finite; this follows from a somewhat stronger statement involving only a finite part of the spectrum.

1. Introduction

Given a connected closed Riemannian manifold \((M, g)\) its spectrum, denoted \(\text{Spec}(M, g)\), is defined to be the sequence of eigenvalues \(0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \rightarrow +\infty\) of the associated Laplacian \(\Delta\) acting on smooth functions. Two Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\) are said to be isospectral if their spectra (counting multiplicities) agree. Inverse spectral geometry is the study of the extent to which geometric properties of a Riemannian manifold \((M, g)\) are encoded in its spectrum.

One typically expects that distinguished metrics, e.g., metrics of constant curvature or more generally symmetric spaces, should be spectrally distinguishable from other metrics. However, only a very few metrics, such as flat metrics on two-dimensional tori and Klein bottles as well as the round metric on the two-sphere, are actually known to be spectrally unique. In fact, except among orientable manifolds in low dimensions \([T1]\), it is not known whether one can tell from the spectrum whether a metric has constant sectional curvature or even whether it is flat. Moreover, examples \([GSz]\) show that the spectrum does not determine whether a closed manifold has constant scalar curvature or whether a manifold with boundary has constant Ricci curvature.

For metrics \(g\) that are not spectrally determined or not known to be so, one may ask about the size or structure of the isospectral set of \(g\), i.e., of the set of isometry classes of metrics that are isospectral to \(g\). Types of results include:

- **Spectral finiteness**: showing that a metric is determined up to finitely many possibilities, at least within some class of metrics. For example, isospectral sets of Riemann surfaces \([M]\) and isospectral sets of flat tori \([Wo, Pes]\) are always finite.

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• **Spectral isolation**: showing that within the space of all metrics on a given manifold $M$, a punctured neighborhood of a metric $g$ (in a suitable topology on the space of metrics) contains no metrics isospectral to $g$. For instance, in all dimensions, the round metric on a sphere is spectrally isolated [T2], and any flat metric [Ku] or any metric of constant negative curvature [Sh] on a manifold is spectrally isolated.

• **Spectral rigidity**: showing that a metric cannot be isospectrally deformed. For example, negatively curved metrics [GK, CS] and metrics of constant positive curvature [T2] are spectrally rigid.

For arbitrary metrics, a major open question is whether isospectral sets are always compact. More precisely, is it the case that every isospectral set of closed Riemannian manifolds contains only finitely many diffeomorphism types $M_1, \ldots, M_k$ with corresponding sets of metrics that are compact in the $C^\infty$-topology? Osgood, Phillips and Sarnak [OPS] gave a positive answer in the case of surfaces. Under various restrictions, some progress has been made in higher dimensions, but the general question remains elusive. Spectral isolation or rigidity results are not sufficient for compactness, although they lend support to the conjecture.

Other than metrics of constant curvature, we are not aware of any special metrics that are known to be spectrally isolated. This article was initially motivated by the difficult question of whether the spectrum determines whether a compact Riemannian manifold is symmetric. As a first step, we consider the following questions:

1. Is a symmetric Riemannian metric on a compact manifold spectrally isolated within some larger natural class of metrics?
2. Is every collection of mutually isospectral compact symmetric spaces finite?

We answer the second question affirmatively in Corollary 2.5. In fact, we prove the following stronger statement:

**Result 1.** Within the class of compact symmetric spaces of a given dimension, each compact symmetric space is finitely determined by a lower volume bound and a finite part of its spectrum.

See Corollary 2.5 for a more precise statement. Note that the various known examples of isospectral flat tori show that the result cannot be strengthened to spectral uniqueness in general.

For the first question, we consider the class of naturally reductive left-invariant metrics on compact simple Lie groups. The symmetric metrics within this class are the bi-invariant metrics. The naturally reductive metrics are the left-invariant metrics that in many ways most closely resemble the symmetric ones. For example, the geodesic symmetries are volume preserving, and every geodesic is an orbit of a one-parameter group of isometries. Moreover, while the condition of natural reductivity does not imply the Einstein condition, the class of naturally reductive metrics do provide a rich source of Einstein metrics (see, for example, [DZ] or [WZ]). Thus one might expect that if the bi-invariant metric is not uniquely determined
by its spectrum, the naturally reductive metrics would be the most likely source of counterexamples. We in fact find that not only is the bi-invariant metric spectrally isolated among the left-invariant naturally reductive metrics, but that every metric in this class is spectrally isolated within the class.

The class of left-invariant metrics on a compact Lie group $G$ of dimension $n$ may be identified with $\text{Symm}_+(n, \mathbb{R}) \subset GL(n, \mathbb{R})$, the space of $n \times n$ positive definite symmetric matrices. We can give the space $\mathcal{M}_{\text{Nat}}(G)$ of naturally reductive left-invariant metrics the subspace topology from $\text{Symm}_+(n, \mathbb{R})$.

**Result 2 (Main Result, Theorem 4.1).** Let $G$ be a compact simple Lie group. Then every metric $g$ in the space $\mathcal{M}_{\text{Nat}}(G)$ of naturally reductive left-invariant metrics on $G$ is spectrally isolated within $\mathcal{M}_{\text{Nat}}(G)$. That is, there exists an open neighborhood $U$ of $g$ in $\mathcal{M}_{\text{Nat}}(G)$ such that if $g' \in U$ and $\text{Spec}(G, g') = \text{Spec}(G, g)$, then $(G, g')$ and $(G, g)$ are isometric.

We do not know whether the spectral isolation statement of Theorem 4.1 can be strengthened to uniqueness or finiteness. Currently, there are no known examples of pairs of isospectral left-invariant naturally reductive metrics on Lie groups, but the second author has constructed examples of isospectral naturally reductive (in fact, normal homogeneous) metrics on simply-connected quotients of $\text{SU}(n)$, where $n$ is extremely large [Sut].

Theorem 4.1 cannot be generalized to arbitrary left-invariant metrics. Indeed, D. Schueth [Sch] and E. Proctor [Pro] have shown the existence of non-trivial isospectral deformations of left-invariant metrics on every classical simple compact Lie group except for a few low-dimensional ones. On the other hand, the authors, along with D. Schueth, demonstrate in a forthcoming article [GSS], that each bi-invariant metric on a connected compact Lie group is spectrally isolated within the class of all left-invariant metrics.

Presently, establishing the conjecture that the bi-invariant metric is spectrally isolated among all metrics on a compact Lie group appears to be a rather formidable task. However, in light of the fact that most counterexamples in spectral geometry exploit metrics with “large” symmetry groups, Theorem 4.1 and the results of [GSS] lend strong support to this conjecture.

The outline of this paper is as follows. In Section 2 we address the finiteness of isospectral sets of symmetric spaces. In Section 3 we review the definition of naturally reductive metrics and the classification of left-invariant naturally reductive metrics on compact simple Lie groups due to D’Atri and Ziller [DZ], and we establish a structural result. Section 4 contains the proof of our main result. We conclude our paper in Section 5 by describing how to compute explicitly the spectra of certain left-invariant naturally reductive metrics on Lie groups.

**Acknowledgments.** We consulted several colleagues concerning Proposition 3.6 (1) and would like to thank G. Prasad for making us aware of R.W. Richardson’s paper “A rigidity theorem for subalgebras of Lie and associative algebras” [R], which provides an elegant proof.
2. ISOSPECTRAL SETS OF SYMMETRIC SPACES

2.1. Definition. Let \((M, g)\) be a closed Riemannian manifold.

(i) The eigenvalue set, \(\mathcal{E}(M, g)\), of \((M, g)\) is the collection of eigenvalues of the Laplace-Beltrami operator \(\Delta\) of \((M, g)\) ignoring multiplicities.

(ii) The fundamental tone of \((M, g)\), denoted by \(\lambda_1(M, g)\), is the lowest non-zero eigenvalue of the Laplace-Beltrami operator \(\Delta\) of \((M, g)\).

2.2. Remark. We note that for any Riemannian manifold \((M, g)\) of dimension \(n\) the quantity \(\lambda_1(M, g)\frac{n}{2} \text{vol}(M, g)\) is invariant under homotheties of \(g\).

In this section, we focus on the class of all compact symmetric spaces, including not only symmetric spaces of the so-called compact type but also those with Euclidean (toral) factors, and we prove the following rigidity result:

2.3. Theorem. Given a positive integer \(n\) and positive real numbers \(\lambda\) and \(v\), there exists a constant \(A > 1\) depending only on \(n\) and on \(\lambda^2\sqrt{v}\) such that for any finite subset \(E\) of the interval \([\lambda, A\lambda]\), there exist up to isometry at most finitely many \(n\)-dimensional compact symmetric spaces \((M, g)\) satisfying the following conditions:

(1) \(\lambda_1(M, g) \geq \lambda\)
(2) \(\text{vol}(M, g) \geq v\)
(3) \(\mathcal{E}(M, g) \cap [\lambda, A\lambda] \subset E\).

2.4. Remark. The choice of \(A\) in the proof of the theorem will guarantee that \(\mathcal{E}(M, g) \cap [\lambda, A\lambda] \neq \emptyset\) for any symmetric space \((M, g)\) satisfying conditions (1) and (2) of the theorem.

2.5. Corollary. In any given dimension, each compact symmetric space is finitely determined by a lower volume bound and a finite part of its spectrum. More specifically, let \(M\) be a compact symmetric space of dimension \(n\). Then, up to isometry, there exist only finitely many \(n\)-dimensional compact symmetric spaces \(M'\) such that

(1) \(\text{vol}(M') \geq \text{vol}(M)\)
(2) \(\mathcal{E}(M') \cap [0, A\lambda_1(M)] = \mathcal{E}(M) \cap [0, A\lambda_1(M)]\)

where \(A\) is the constant given in Theorem 2.3.

Since volume and dimension are spectral invariants, Corollary 2.5 gives us the following result.

2.6. Corollary. A compact symmetric space \((M, g)\) is finitely determined by its spectrum within the class of compact symmetric spaces.

We now turn to the proof of Theorem 2.3.

2.7. Notation and Remarks.
(i) Every irreducible simply-connected symmetric space $M$ of compact type is either a simple compact Lie group $G$ with a bi-invariant metric or it is of the form $G/K$, where $G$ is a simple compact Lie group and $K$ is one of only finitely many connected Lie subgroups of $G$ for which $G/K$ admits a symmetric metric. We will consider both cases together by taking $K = \{e\}$ in the first case. The symmetric metric on $G/K$ is unique up to scaling.

(ii) Every compact symmetric space is of the form

$$M = \Gamma \backslash (M_0 \times M_1 \times \cdots \times M_k)$$

where $M_0$ is a torus (viewed as an abelian Lie group), $M_i$ is an irreducible symmetric space of compact type, say $M_i = G_i/K_i$ as above, and $\Gamma$ is a finite subgroup of the center of $M_0 \times G_1 \times \cdots \times G_k$. We may assume that $\Gamma \cap M_0$ is trivial; otherwise, replace $M_0$ by the torus $M_0/\langle \Gamma \cap M_0 \rangle$. Ignoring the metric, we will refer to the structure $\Gamma \backslash (M_0 \times G_1/K_1 \times \cdots \times G_k/K_k)$ as the homogeneity type of $M$. Thus the homogeneity type of $M$ is its structure as a quotient of a compact Lie group.

(iii) Given $M$ with homogeneity type as above, let $\mathcal{M}$ be the collection of symmetric metrics on $M$. Each $g \in \mathcal{M}$ lifts to a symmetric metric on $M_0 \times M_1 \times \cdots \times M_k$ of the form $g_0 \times g_1 \times \cdots \times g_k$ where $g_i$ is a symmetric metric on $M_i$. In particular, $g_0$ is a flat metric on the torus $M_0$ and $g_i$ is given by a multiple $a_i B_{g_i}$ of the Killing form $B_{g_i}$ of the Lie algebra $g_i$ of $G_i$, where $a_i > 0$. Thus, the data defining the metric $g$ consists of the positive scalars $a_1, \ldots, a_k$ and the flat metric $g_0$. Letting $\Gamma_0$ denote the image of $\Gamma$ under the homomorphic projection $M_0 \times G_1 \times \cdots \times G_k \to M_0$, then two metrics $g, g' \in \mathcal{M}$ are isometric if the associated scalars satisfy $a_i = a'_i$, $1 \leq i \leq k$ and if there exists a $\Gamma_0$-equivariant isometry $(M_0, g_0) \to (M_0, g'_0)$.

2.8. Lemma. In any given dimension $n$, there are only finitely many homogeneity types of compact symmetric spaces.

Proof. There are only finitely many homogeneity types of irreducible simply-connected compact symmetric spaces of dimension at most $n$. The torus $M_0$ (viewed as a Lie group) depends up to isomorphism only on its dimension. Moreover for a given choice of $M_0 \times M_1 \times \cdots \times M_k$ of dimension $n$, there are only finitely many choices of $\Gamma$ as in 2.7. (Recall our standing assumption that $\Gamma \cap M_0 = \{0\}$.) Indeed, the group $G := G_1 \times \cdots \times G_k$ has finite center $Z(G)$ and there are only finitely many homomorphisms from subgroups of $Z(G)$ into $M_0$. The lemma follows.

2.9. Notation and Remarks. Up to isometry, a flat torus may be expressed as $T = \mathcal{L} \backslash \mathbb{R}^m$ with metric given by the standard Euclidean inner product, where $\mathcal{L}$ is a lattice of full rank in $\mathbb{R}^m$. Two tori $\mathcal{L} \backslash \mathbb{R}^m$ and $T = \mathcal{L}' \backslash \mathbb{R}^m$ are isometric if and only if $\mathcal{L}$ and $\mathcal{L}'$ are congruent. Given $T = \mathcal{L} \backslash \mathbb{R}^m$, let $\mathcal{L}^*$ be the dual lattice to $\mathcal{L}$. We will refer to the flat torus $T^* = \mathcal{L}^* \backslash \mathbb{R}^m$ (again with the standard Euclidean metric) as the dual torus. Recall that:
(1) $\text{vol}(T^*) = \frac{1}{\text{vol}(T)}$.

(2) The spectrum of $T$ is given by the collection $4\pi^2\|\gamma\|^2$ as $\gamma$ varies over $\mathcal{L}^*$.

(3) The systole $\text{syst}(T)$, i.e., the length of the shortest non-contractible loop in $T$, is given by the length of the shortest non-zero element of $\mathcal{L}$. In particular, $\lambda_1(T) = 4\pi^2\text{ syst}(T^*)^2$.

2.10. Minkowski’s Theorem. (See [Gro] and [GL].) There exist constants $C_m$ and $C'_m$ depending only on $m$ such that for all flat tori $T = \mathcal{L} \setminus \mathbb{R}^m$, we have:

(1) $\text{syst}(T) \leq C_m \text{ vol}(T)^{\frac{1}{m}}$, where $C_m$ is a constant depending only on the dimension $m$.

(2) There exists at least one choice of lattice basis $\{\delta_1, \ldots, \delta_m\}$ of $\mathcal{L}$ satisfying the conditions that $\|\delta_1\| = \text{syst}(T)$ and that

$$\prod_{i=1}^{m} \|\delta_i\| \leq C'_m \text{ vol}(T).$$

The first statement and the remarks in 2.9 imply:

2.11. Corollary. Let $T$ be a flat $m$-torus. Then

$$\lambda_1(T) \leq C_m \frac{1}{\text{vol}(T)^{\frac{1}{m}}},$$

where $C_m$ is a constant depending only on the dimension $m$.

2.12. Remark. In [U] it is shown that the corollary above fails for a non-abelian compact Lie group equipped with a left-invariant metric.

2.13. Lemma. Theorem 2.3 holds with “compact symmetric spaces” replaced by “flat tori”.

Our proof of Lemma 2.13 closely follows H. Pesce’s proof [Pes] that there are at most finitely many tori with any given spectrum.

Proof of Lemma 2.13 A lattice $\mathcal{L}$ is determined up to congruence by $\|\alpha_i\|$, $i = 1, \ldots, n$ and $\|\alpha_i + \alpha_j\|$, $1 \leq i < j \leq n$, where $\{\alpha_1, \ldots, \alpha_n\}$ is any lattice basis. That is, if $\mathcal{L}'$ is another lattice, then $\mathcal{L}'$ is congruent to $\mathcal{L}$ if and only if it admits a basis $\{\beta_1, \ldots, \beta_n\}$ with $\|\alpha_i\| = \|\beta_i\|$ and $\|\alpha_i + \alpha_j\| = \|\beta_i + \beta_j\|$ for all $i, j$. Since a pair of tori are isometric if and only if the dual tori are isometric, a torus $T = \mathcal{L} \setminus \mathbb{R}^n$ is determined up to isometry by the values $4\pi\|\delta_i\|^2$, $1 \leq i \leq n$ and $4\pi\|\delta_i + \delta_j\|^2$, $1 \leq i < j \leq n$ where $\{\delta_1, \ldots, \delta_n\}$ is any lattice basis of $\mathcal{L}^*$. Each of these values lies in the spectrum of $T$. Choose the basis $\{\delta_1, \ldots, \delta_n\}$ of $\mathcal{L}^*$ as in Minkowski’s Theorem so that $4\pi\|\delta_1\|^2 = \lambda_1(T)$ and

$$\prod_{i=1}^{n} \|\delta_i\| \leq C'_n \text{ vol}(T^*) = \frac{C'_n}{\text{vol}(T)}.$$

Since $\|\delta_j\| \geq \|\delta_1\|$ for $j = 1, \ldots, n$, we then have for each $i$ that

$$\|\delta_i\| \leq \frac{C'_n}{\|\delta_1\|^n \text{vol}(T)}.$$
and hence
\[ 4\pi^2 \|\delta_1\|^2 \leq C \frac{1}{\lambda_1(T)^{n-1} \text{vol}(T)^2}, \]
where \( C \) is a constant depending only on \( n \). Replacing \( C \) by \( 4C \), then the same bound is satisfied by \( 4\pi^2 \|\delta_i + \delta_j\|^2 \), \( 1 \leq i < j \leq n \). In particular, if \( \lambda_1(T) \geq \lambda \) and \( \text{vol}(T) \geq v \), then the elements \( 4\pi^2 \|\delta_i\|^2 \) and \( 4\pi^2 \|\delta_i + \delta_j\|^2 \), \( 1 \leq i, j \leq n \) of \( \text{Spec}(T) \) all lie in the interval \([\lambda, A\lambda]\) where
\[ (2.14) \quad A = C \frac{1}{\lambda^nv^2} = C \frac{1}{(\lambda^{n/2}v^2)^2}. \]
In particular, if \( T \) satisfies the conditions of Theorem 2.3, then each of \( 4\pi^2 \|\delta_i\|^2 \) and \( 4\pi^2 \|\delta_i + \delta_j\|^2 \) must lie in the finite set \( E \). Thus, these values are determined up to finitely many possibilities and hence so is the isometry class of \( T \).

**Proof of Theorem 2.3.** By Lemma 2.8, it suffices to prove the theorem with the homogeneity type of \( M \) fixed. We use the notation of 2.7. Thus \( M = \Gamma \setminus (M_0 \times M_1 \times \cdots \times M_k) \). Writing \( M_i = G_i/K_i \), let \( \Gamma_i \) be the homomorphic image of \( \Gamma \subset M_0 \times G_1 \times \cdots \times G_k \) in \( G_i \), \( 1 \leq i \leq k \), and as in 2.7 let \( \Gamma_0 \) be the homomorphic image of \( \Gamma \) in \( M_0 \). Set \( M_i = \Gamma_i \setminus M_i \), \( 0 \leq i \leq k \).

In the notation of 2.7 each symmetric metric on \( M \) is specified by scaling factors \( a_i \) (defining a metric \( g_i \) on \( M_i \)), \( i = 1, \ldots, k \), along with a flat metric \( g_0 \) on \( M_0 \). Given positive numbers \( v \) and \( \lambda \), let \( M(\lambda, v) \) be the set of all such metrics satisfying \( \lambda_1(M, g) \geq \lambda \) and \( \text{vol}(M, g) \geq v \). Let \( g \in M(\lambda, v) \) and let \( g_i \) denote the corresponding metric on \( M_i \). Then the metric \( g_i \) on \( M_i \) induces a unique metric, denoted \( \overline{g}_i \), on \( \overline{M}_i \), \( 0 \leq i \leq k \), so that \( (M_i, g_i) \to (\overline{M}_i, \overline{g}_i) \) is a Riemannian covering. Since the projection \( (M, g) \to (\overline{M}_i, \overline{g}_i) \) is a Riemannian submersion with totally geodesic fibers it follows that \( \mathcal{E}(\overline{M}_i, \overline{g}_i) \subseteq \mathcal{E}(M, g) \); hence, we see that
\[ (2.15) \quad \lambda_1(\overline{M}_i, \overline{g}_i) \geq \lambda_1(M_i, g) \geq \lambda. \]

Write \( V_i = \text{vol}(\overline{M}_i, \overline{g}_i), m_i = \dim(M_i), \) and \( n = \dim(M) = m_0 + m_1 + \cdots + m_k \). For \( i > 0 \), the metric \( \overline{g}_i \) is uniquely determined by the scaling factor \( a_i \) that defines \( g_i \). By Remark 2.2, there exists a constant \( c_i \), independent of the scaling factor \( a_i \) such that
\[ (2.16) \quad \lambda_1(\overline{M}_i, \overline{g}_i) \frac{m_i}{n} V_i = c_i. \]
Thus by Equation 2.15 we have
\[ (2.17) \quad V_i \leq \frac{c_i}{\lambda \frac{m_i}{n}}. \]
By Corollary 2.11 we similarly have
\[ (2.18) \quad V_0 \leq \frac{c_0}{\lambda \frac{m_0}{n}}, \]
where \( c_0 \) is a constant depending only on the dimension \( m_0 \) of the torus \( M_0 \).

Now
\[ (2.19) \quad \text{vol}(M, g) = aV_0V_1 \cdots V_k, \]
where $a = \frac{1}{|\Gamma|} |\Gamma_1| \cdots |\Gamma_k|$. Since $\text{vol}(M, g) \geq v$, Equations 2.17, 2.18 and 2.19 imply for $0 \leq i \leq k$ that

$$V_i \geq \frac{v}{aV_1 \cdots V_i \cdots V_k} \geq \frac{c_i v \lambda_i^2}{ac \lambda^{\frac{m_i}{2}}},$$

where $c = c_0 c_1 \cdots c_k$.

For $1 \leq i \leq k$, it follows from Equations 2.16 and 2.20 that

$$\frac{\lambda_1(M_i, \overline{g}_i)}{\lambda} \leq \frac{ac}{v \lambda^{\frac{m_i}{2}}}$$

and thus

$$\frac{\lambda_1(M_i, \overline{g}_i)}{\lambda} \leq \left(\frac{ac}{v \lambda^{\frac{m_i}{2}}}\right)^{\frac{2}{m_i}}.$$

Let $A_i$ be the expression on the right hand side of Equation 2.21. Then

$$\lambda_1(M_i, \overline{g}_i) \in [\lambda, A_i \lambda].$$

Next consider $(\overline{M}_0, \overline{g}_0)$. We apply Lemma 2.13 with $v$ replaced by the lower bound for the volume of $V_0$ given in Equation 2.20 (with $i = 0$). Note that $m_0$ plays the role of $n$ in the lemma. We continue to use $\lambda$ for the lower bound on $\lambda_1$. Letting $A_0$ be the expression in Equation 2.14 with our new lower bound on volume, we have

$$A_0 = \overline{C} \frac{\lambda^{m_0}}{\lambda^{m_0 / 2} \lambda^n} = \overline{C} \left(\frac{\lambda^{n/2} v}{\lambda^n}\right)^{\frac{2}{m_n}},$$

where $\overline{C}$ is a constant depending only on the homogeneity type of $M$.

Letting

$$A = \max\{A_0, A_1, \ldots, A_k\},$$

then the discussion above shows us that for any $g \in M(\lambda, v)$ we have $\mathcal{E}(M, g) \cap [\lambda, A \lambda] \neq \emptyset$. Now, let $E$ be as in the statement of the theorem and let $\mathcal{M}(\lambda, v, E)$ be the collection of all symmetric metrics $g \in M(\lambda, v)$ such that $\mathcal{E}(M, g) \cap [\lambda, A \lambda] \subset E$. For each such $g$, we have by Equation 2.22 that $\lambda_1(M_i, \overline{g}_i) \in E$, $1 \leq i \leq k$. Since $E$ is finite, there are only finitely many possibilities for $\lambda_1(M_i, \overline{g}_i)$ and thus only finitely many possibilities for $a_i$. Next by Lemma 2.13 and the fact that $A \geq A_0$, the isometry class of $\overline{g}$ is determined up to finitely many possibilities. An elementary argument then shows that the $\Gamma_0$-equivariant isometry class of $(M_0, g_0)$ is determined up to finitely many possibilities. The theorem now follows from the remarks in 2.7(iii).
3. Naturally Reductive Metrics

Let $(M, g)$ be a connected homogeneous Riemannian manifold. Choose a base point $o \in M$. Let $H$ be a transitive group of isometries of $(M, g)$, and let $K$ be the isotropy subgroup at $o$. The Lie algebra $\mathfrak{h}$ of $H$ decomposes into a sum $\mathfrak{h} = \mathfrak{r} + \mathfrak{q}$, where $\mathfrak{r}$ is the Lie algebra of $K$ and $\mathfrak{q}$ is an $\text{Ad}(K)$-invariant complement of $\mathfrak{r}$. Given a vector $X \in \mathfrak{h}$ we obtain a Killing field $X^*$ on $M$ by $X^*_p \equiv \frac{d}{dt} |_{t=0} \exp(tX) \cdot p$ for $p \in M$. The map $X \mapsto X^*$ is an antihomomorphism of Lie algebras. We may identify $\mathfrak{q}$ with $T_0M$ by the linear map $X \mapsto X^*_o$. Thus the homogeneous Riemannian metric $g$ on $M$ corresponds to an inner product $\langle \cdot , \cdot \rangle$ on $\mathfrak{q}$. For $X \in \mathfrak{g}$, write $X = X_K + X_q$ with $X_K \in \mathfrak{r}$ and $X_q \in \mathfrak{q}$. Recall that for $X, Y \in \mathfrak{q}$,

\[
(\nabla_X Y^*)_o = -\frac{1}{2}([X,Y]^*_o) + U(X,Y)^*_o,
\]

where $U : \mathfrak{q} \times \mathfrak{q} \to \mathfrak{q}$ is defined by

\[
2 < U(X,Y), Z > = \langle [Z,X]_q, Y \rangle + \langle X, [Z,Y]_p \rangle.
\]

3.1. Definition. Let $(M, g)$ be a Riemannian homogeneous space and let $H$ be a transitive group of isometries of $(M, g)$. $(M, g)$ is said to be naturally reductive (with respect to $H$), if there exists an $\text{Ad}(K)$-invariant complement $\mathfrak{q}$ of $\mathfrak{r}$ (as above) such that

\[
\langle [Z,X]_q, Y \rangle + \langle X, [Z,Y]_q \rangle = 0,
\]

or equivalently $U \equiv 0$. That is, for any $Z \in \mathfrak{q}$ the map $[Z,\cdot]_q : \mathfrak{q} \to \mathfrak{q}$ is skew symmetric with respect to $\langle \cdot, \cdot \rangle$.

3.2. Remark. Naturally reductive metrics are a generalization of symmetric metrics. Although the geodesic symmetries of naturally reductive metrics need not be isometries, they are (up to sign) volume preserving. Moreover, every geodesic is an orbit of a one-parameter subgroup of the transitive group $H$. In particular, the geodesics through the base point $o$ are precisely the curves $\exp(tX) \cdot o$, $X \in \mathfrak{q}$.

3.3. Theorem ([DZ] Theorems 3 and 7). Let $G$ be a connected compact simple Lie group and let $g_0$ be the bi-invariant Riemannian metric on $G$ given by the negative of the Killing form. Let $K \leq G$ be a connected subgroup with Lie algebra $\mathfrak{r}$, and let $\mathfrak{p}$ be a $g_0$ orthogonal complement of $\mathfrak{r}$ in $\mathfrak{g}$. Given any $\alpha \in \mathbb{R}$ and any bi-invariant Riemannian metric $h$ on $K$, define a left-invariant metric $g$ on $G$ by the orthogonal direct sum

\[
g = e^\alpha g_0 \upharpoonright \mathfrak{p} \oplus h \upharpoonright \mathfrak{r}.
\]

Then:

1. $g$ is $\text{Ad}(K)$-invariant and is naturally reductive with respect to $G \times K$, where $G$ acts by left translations and $K$ by right translations on $G$.

2. $g$ is normal homogeneous if and only if $h \leq e^\alpha g_0 \upharpoonright \mathfrak{r}$.  


Let $N_G(K)$ denote the normalizer of $K$ in $G$ and $N_G(K)^0$ denote the connected component of the identity. Then the full connected compact isometry group of $(G, g)$ is given by $G \times N_G(K)^0$, where $G$ acts via left translations and $N_G(K)^0$ acts via right translations on $G$. The metric $g$ is therefore $N_G(K)$ bi-invariant and we conclude that the left cosets of $N_G(K)^0$ are totally geodesic submanifolds of $(G, g)$.

Every left-invariant naturally reductive metric on $G$ is of the form given in Equation (3.4) for some $K$, $\alpha$, and $h$.

3.5. NOTATION AND REMARKS.

(1) We will denote a metric $g$ of the form given in Equation (3.4) by $g_{\alpha, h}$.

(2) In the setting of Theorem 3.3 it follows from the second statement of the theorem that the metric $g$ is also naturally reductive with respect to $G \times L$ where $L$ is any connected subgroup of $N_G(K)$ containing $K$.

(3) If $g$ is naturally reductive with respect to $G \times K$, then it is also naturally reductive with respect to $G \times aKa^{-1}$ for any $a \in G$. Conjugating $K$ corresponds to changing the choice of base point for which $K$ is the isotropy group.

(4) A Lie group $G$ can admit metrics naturally reductive with respect to $H \times K$ where $H, K < G$, but which are not left-invariant [DZ, p. 12–14]. Such metrics are sometimes called semi-invariant.

(5) If $G$ is an arbitrary connected compact Lie group it is known that left-invariant metrics of the form given by Equation (3.4) are naturally reductive, where we allow $g_0$ to denote any bi-invariant metric on $G$. (In the case that $G$ is simple as in Theorem 3.3 all bi-invariant metrics are multiples of each other. Since $\alpha$ is arbitrary, no greater generality is achieved by varying $g_0$.) However, it is unknown whether (up to isometry) this list is exhaustive (see [DZ, Theorem 1 and p. 20]).

3.6. PROPOSITION. Let $G$ be a compact Lie group. Then

(1) There are only finitely many conjugacy classes of semisimple subgroups of $G$.

(2) There are only finitely many conjugacy classes of subgroups of $G$ of the form $TK$ where $K$ is trivial or connected and semisimple, and $T$ is a maximal torus in the centralizer of $K$ in $G$.

Proof. The first statement follows directly from the fact that up to conjugacy a real (or complex) Lie algebra contains finitely many semi-simple Lie algebras (cf. [R, Prop. 12.1]). The second statement follows from the first statement, the fact that the centralizer of any closed subgroup $K$ of $G$ is compact, and the conjugacy of all maximal tori in any compact Lie group.

3.7. COROLLARY. Let $G$ be a connected simple compact Lie group. Then there exists a finite collection $K$ of connected subgroups of $G$ such that (up to isometry) every naturally reductive left-invariant metric $g$ on $G$ is naturally reductive with respect to $G \times K$ for some $K \in K$. The collection $K$ consists of a choice of representative from each of the conjugacy classes of...
subgroups of \( G \) given in Proposition \ref{prop:connected}, hence, the moduli space of naturally reductive metrics \( \mathcal{M}_{\text{Nat}}(G) \) is given by

\[
\mathcal{M}_{\text{Nat}}(G) = \cup_{K \in \mathfrak{K}} \mathcal{M}_K
\]

where \( \mathcal{M}_K \) is the space of metrics on \( G \) naturally reductive with respect to \( G \times K \).

**Proof.** Choose a connected subgroup \( H \) of \( G \) such that \( g \) is of the form given in Theorem \ref{thm:naturally_reductive} with respect to \( G \times H \). Write \( H = H_z H_{ss} \), where \( H_z \) is abelian and \( H_{ss} \) is semi-simple or trivial. Let \( T \) be a maximal torus in the centralizer of \( H_{ss} \) containing \( H_z \), and let \( K = TH_{ss} \). Note that \( K \) is a connected subgroup such that \( H \subset K \subset N_G(H) \). Thus by Remark \ref{rem:naturality}(2), \( g \) is naturally reductive with respect to \( G \times K \). Finally, observe that, up to conjugacy, \( K \) lies in \( \mathfrak{K} \), so the corollary follows from Remark \ref{rem:naturality}(3).

\[ \square \]

### 4. Proof of the Main Theorem

We are ready to prove the following theorem.

#### 4.1. Theorem. Let \( G \) be a connected compact simple Lie group and let \( \mathcal{M}_{\text{Nat}}(G) \) denote the class of left-invariant naturally reductive metrics on \( G \), where \( \mathcal{M}_{\text{Nat}}(G) \subset \text{GL}(n,\mathbb{R}) \) has the subspace topology. Then every metric \( g \in \mathcal{M}_{\text{Nat}}(G) \) is spectrally isolated within this class. That is, there exists an open neighborhood \( U \) of \( g \) in \( \mathcal{M}_{\text{Nat}}(G) \) such that if \( g' \in U \) and \( \text{Spec}(G,g') = \text{Spec}(G,g) \), then \( (G,g') \) and \( (G,g) \) are isometric.

#### 4.2. Lemma. Let \( G \) be a connected compact simple Lie group. Using the notation of Theorem \ref{thm:naturally_reductive} and of Corollary \ref{cor:naturally_reductive}, let \( K \in \mathfrak{K} \) and let \( g = g_{\alpha,h} \in \mathcal{M}_K \). Then there exists a neighborhood \( W \) of \( g \) in \( \mathcal{M}_K \) such that if \( g' = g_{\alpha',h'} \in \mathcal{M}_K \) is isospectral to \( g \), then \( \alpha' = \alpha \).

**Proof.** Let \( \pi : G \to G/K \) be the canonical projection. For \( g' \in \mathcal{M}_K \), the metric \( \overline{g'} \) on \( G/K \) induced by \( g' \) depends only on \( \alpha' \) and is a scalar multiple of the metric \( \overline{g} \) induced by \( g \). Since \( g' \) is \( K \)-bi-invariant, the Riemannian submersion \( \pi : (G,g') \to (G/K,\overline{g'}) \) has totally geodesic fibers. Thus the spectrum of \( (G/K,\overline{g'}) \) is contained in the spectrum of \( (G,g') \). Let \( \lambda \) be the lowest non-zero eigenvalue of \( \text{Spec}(G/K,\overline{g}) \). Choose \( \epsilon > 0 \) so that \( \text{Spec}(G,g) \) has no entries in the interval \( (\lambda - \epsilon, \lambda + \epsilon) \) other than \( \lambda \). For \( g' \in \mathcal{M}_K \), the spectrum of \( \overline{g'} \) is just a rescaling of \( \text{Spec}(G/K,\overline{g}) \) with the scale factor depending non-trivially on \( \alpha' \). Thus, we can find \( \delta > 0 \) so that if \( 0 < |\alpha' - \alpha| < \delta \), then the lowest non-zero eigenvalue of \( \overline{g'} \) lies in the interval \( (\lambda - \epsilon, \lambda + \epsilon) \) and is distinct from \( \lambda \). But then \( g' \) cannot be isospectral to \( g \). The lemma follows. \( \square \)

#### 4.3. Notation. Let \( G \) be a compact, simple, connected Lie group, and let \( g_0 \) be the negative of the Killing form. Given a connected compact Lie subgroup \( K \), let \( \mathcal{B}(K) \) denote the set of all bi-invariant metrics on \( K \). For \( h \in \mathcal{B}(K) \), write \( h(X,Y) = g_0(A_h X,Y) \). The map \( h \mapsto A_h \) is a bijection between \( \mathcal{B}(K) \) and the set of all linear transformations of \( \mathfrak{g} \) that commute with \( \text{Ad}(K) \) and are positive-definite symmetric (with respect to the restriction of \( g_0 \) to \( \mathfrak{g} \)). We give \( \mathcal{B}(K) \) the subspace topology from the space of positive-definite symmetric linear transformations.
For $h \in \mathcal{B}(K)$, we will say that $\beta \in \mathbb{R}$ is \textit{admissible} for $h$ if $e^\beta$ is strictly larger than each of the eigenvalues of $A_h$. Given $\beta \in \mathbb{R}$, we let $\text{Adm}(K, \beta)$ be the set of all $h \in \mathcal{B}(K)$ such that $\beta$ is admissible for $h$. Note that $\text{Adm}(K, \beta)$ is an open subset of $\mathcal{B}(K)$.

4.4. \textbf{Proposition.} We use the notation of Corollary 3.7 and Notation 4.3. Let $K \in \mathcal{K}$ and let $\beta \in \mathbb{R}$. Define $\Phi : K \times K \to K$ by $\Phi(k_1, k_2) = k_1 k_2^{-1}$. Then:

(1) For each $h \in \text{Adm}(K, \beta)$, there exists a unique metric $\tilde{h} \in \mathcal{B}(K)$ for which

$$\Phi : (K \times K, e^\beta g_0 \times \tilde{h}) \to (K, h)$$

is a Riemannian submersion.

(2) The mapping $F : \text{Adm}(K, \beta) \to \mathcal{B}(K)$ given by $h \mapsto \tilde{h}$ is a homeomorphism from $\text{Adm}(K, \beta)$ to an open subset of $\mathcal{B}(K)$.

\textit{Proof.} For $\Phi$ as in the Proposition, the kernel of $\Phi_*$ is given by the diagonal subspace $\Delta \mathfrak{R} = \{(X, X) : X \in \mathfrak{R}\}$. For $h \in \mathcal{B}(K)$, the orthogonal complement $\mathcal{H}$ of $\Delta \mathfrak{R}$ in $\mathfrak{R} \times \mathfrak{R}$ with respect to $e^\beta g_0 \oplus \tilde{h}$ is given by

$$\mathcal{H} = \{(X, Y) \in \mathfrak{R} \times \mathfrak{R} : e^\beta X + A_h Y = 0\} = \{\tilde{X} : = (X, -e^\beta A_h^{-1} X) : X \in \mathfrak{R}\}.$$

Denoting the left-invariant metric $e^\beta g_0 \oplus \tilde{h}$ by $q$, a simple computation shows that for $\tilde{X}, \tilde{Y} \in \mathcal{H}$, we have

$$q(\tilde{X}, \tilde{Y}) = g_0(X, e^\beta D Y)$$

where $D = \text{Id} + e^\beta A_h^{-1}$.

Next $\Phi_*(\tilde{X}) = X + e^\beta A_h^{-1}(X) = DX$. Thus $\Phi_* : (\mathcal{H}, q) \to (\mathfrak{R}, h)$ is an inner product space isometry if and only if $DA_h D = e^\beta D$, i.e., $D = e^\beta A_h^{-1}$. The admissibility of $\beta$ implies that this transformation $D$ has all eigenvalues strictly larger than one. Thus given $h$, we may define $\tilde{h}$ by the condition $A_{\tilde{h}} = e^\beta (D - \text{Id})^{-1} = e^\beta (e^\beta A_h^{-1} - \text{Id})^{-1}$. Note that $A_{\tilde{h}}$ is positive-definite and symmetric and it commutes with $\text{Ad}(K)$, so $\tilde{h}$ is a well-defined element of $\mathcal{B}(K)$. This proves statement (1). Statement (2) is also immediate. $\square$

4.5. \textbf{Corollary.} Let $G$ be a connected compact simple Lie group. Using the notation of Theorem 3.3 and of Proposition 4.4, let $K \in \mathcal{K}$, let $\mathfrak{p}$ be an $\text{Ad}(K)$-invariant complement of $\mathfrak{R}$ in $\mathfrak{g}$, and let $\alpha, \beta \in \mathbb{R}$. Let $\tilde{g} = \tilde{g}_{\alpha, \beta}$ be the left-invariant naturally reductive Riemannian metric on $G$ given by

$$\tilde{g} = e^\alpha g_0 \restriction \mathfrak{p} \oplus e^\beta g_0 \restriction \mathfrak{R}.$$ 

For $h \in \text{Adm}(K, \beta)$, define $\tilde{h} = F(h)$ as in Proposition 4.4. Then the map

$$\Phi : (G \times K, \tilde{g} \times \tilde{h}) \to (G, g_0, h)$$

given by $(g, k) \mapsto gk^{-1}$ is a Riemannian submersion. Moreover, the fibers of this submersion are totally geodesic.
**Proof.** The first statement follows easily from Proposition 4.4. The assertion that the fibers are totally geodesic follows from the facts that the metric on \( G \times K \) is bi-invariant with respect to the subgroup \( K \times K \) and that the fibers are cosets in \( G \times K \) of \( \Delta K \leq K \times K \).

4.6. Review of representation theory. We briefly review the representation theory needed in the proof of the main theorem.

(i) If \( G \) is a compact Lie group let \( \widehat{G} \) denote the set of equivalence classes of irreducible unitary representations of \( G \). Since \( G \) is compact we see that all of its irreducible representations are finite dimensional. For \( \mu \in \widehat{G} \) and \( \sigma \) any unitary representation of \( G \), let \([\sigma : \mu]\) denote the multiplicity of \( \mu \) in \( \sigma \). (More generally, for any representations \( \rho : G \rightarrow U(\mathcal{H}_\rho) \) and \( \tau : G \rightarrow U(\mathcal{H}_\tau) \), one defines

\[
[\rho : \tau] = \dim \{ T \in \text{Hom}(\mathcal{H}_\tau, \mathcal{H}_\rho) : T \circ \tau(g) = \rho(g) \circ T \text{ for all } g \in G \}.
\]

The right regular representation of \( G \) is the mapping \( \rho_G : G \rightarrow U(L^2(G)) \), where

\[
\rho_G(g)f(x) = f(xg)
\]

for any \( f \in L^2(G) \). The Peter-Weyl theorem (see [Fo, p. 133]) states that every irreducible unitary representation \( \pi : G \rightarrow U(\mathcal{H}_\pi) \) occurs in the right regular representation \( \rho_G \) of \( G \) on \( L^2(G) \); in fact \( [\rho_G : \pi] = d_\pi \) where \( d_\pi \) is the dimension of \( \pi \).

(ii) Let \( G \) and \( K \) be compact Lie groups. Given two representations \( \sigma : G \rightarrow U(\mathcal{H}_\sigma) \) and \( \tau : K \rightarrow U(\mathcal{H}_\tau) \) we may define the Kronecker product \( \sigma \otimes \tau \) to be the unitary representation of \( G \times K \) on \( \mathcal{H}_\sigma \otimes \mathcal{H}_\tau \) given by \( \sigma \otimes \tau(g, k)(v \otimes w) = \sigma(g)v \otimes \tau(h)w \). It is a standard fact that the irreducible representations of \( G \times K \) are precisely the representations of the form \( \sigma \otimes \tau \) where \( \sigma \in \widehat{G} \) and \( \tau \in \widehat{K} \) and we can identify \( \widehat{G} \times \widehat{K} \) with \( \widehat{G \times K} \) (see [Fo, Theorem 7.25]). Hence, by the Peter-Weyl theorem, \( \sigma \otimes \tau \) occurs in the right regular representation \( \rho_{G \times K} \) of \( G \times K \) on \( L^2(G \times K) \). In the case of finite-dimensional representations, we may identify \( \mathcal{H}_\sigma \otimes \mathcal{H}_\tau \) with \( \text{Hom}(\mathcal{H}_\sigma^*, \mathcal{H}_\tau) \) and then

\[
(4.7) \quad \sigma \otimes \tau(g, k)T := \sigma(g) \circ T \circ \tau(k^{-1}),
\]

for \( T \in \text{Hom}(\mathcal{H}_\sigma^*, \mathcal{H}_\tau) \), where \( \overline{\tau} : G \rightarrow U(\mathcal{H}_\sigma^*) \) is the contragredient of \( \tau \) given by \( \overline{\tau}(g)S = S \circ \tau(g^{-1}) \) for \( g \in G \) and \( T \in \mathcal{H}_\rho^* \). In the case of interest to us, \( K \) will be a compact subgroup of \( G \). Let \( \Delta K \) denote the diagonal subgroup of \( G \times K \). Observe that if \( T \in \text{Hom}(\mathcal{H}_\sigma^*, \mathcal{H}_\tau) \) is fixed by \( \sigma \otimes \tau(\Delta K) \), then Equation (4.7) says that \( T \) intertwines the contragredient \( \overline{\tau} \) of \( \tau \) with the restriction of \( \sigma \) to \( K \). In particular, if \( \tau \) is irreducible (and hence necessarily finite-dimensional), then \( \mathcal{H}_\sigma \otimes \mathcal{H}_\tau \) contains a non-trivial \( \Delta K \)-fixed vector if and only if the restriction of \( \sigma \) to \( K \) contains a copy of \( \overline{\tau} \); that is, \([\sigma \upharpoonright K : \overline{\tau}] \neq 0\).

(iii) Let \( K \) be a connected compact Lie subgroup of the connected compact Lie group \( G \). For \( \tau \in \widehat{K} \), let \( \text{Ind}_K^G(\tau) \) denote the representation of \( G \) induced by \( \tau \) (see [Fo, p. 152]), and for \( \sigma \in \widehat{G} \), let \( \sigma \upharpoonright K \) denote the restriction of \( \sigma \) to \( K \). Then Frobenius’ reciprocity theorem (see [Fo, p. 160]) shows us that

\[
[\sigma \upharpoonright K : \tau] = [\text{Ind}_K^G(\tau) : \sigma].
\]
(iv) Again letting $K$ be a connected compact Lie subgroup of the connected compact Lie group $G$, every irreducible unitary representation $\tau$ of $K$ occurs in $\rho_{G} \upharpoonright K$. Indeed, let $\sigma \in \hat{G}$ be an irreducible representation such that $[\text{Ind}^{G}_{K}(\sigma) : \tau] \neq 0$. Then by (iii), we have $[\sigma \upharpoonright K : \tau] \neq 0$ and thus by (i), $[\rho_{G} \upharpoonright K : \tau] \neq 0$.

(v) Given a left-invariant Riemannian metric $g$ on a compact Lie group $G$, then every $\rho_{G}$ invariant subspace of $L^{2}(G)$ is also invariant under the action of the Laplacian of $G$. If the metric is bi-invariant, then the Laplacian commutes with the action $\rho_{G}$ and thus each $\rho_{G}$-irreducible subspace of $L^{2}(G)$ lies in a single eigenspace. More generally, if the metric is left $G$-invariant and right $K$-invariant for some subgroup $K$ of $G$, then each $\rho_{G} \upharpoonright K$ irreducible subspace of $L^{2}(G)$ lies in a single eigenspace.

4.8. INVARlANTS OF BI-INVARIANT METRICS. Let $K$ be a compact connected Lie group, and let $\mathcal{B}(K)$ denote the set of all bi-invariant metrics on $K$. We define a complete set of invariants of the elements of $\mathcal{B}(K)$.

First, if $K$ is simple and $h \in \mathcal{B}(K)$, let $\lambda_{1}(h)$ denote the lowest nonzero eigenvalue of the Laplacian of $h$. Since all elements of $\mathcal{B}(K)$ are scalar multiples of each other, the map $\lambda_{1} : \mathcal{B}(K) \to \mathbb{R}_{+}$ is bijective.

Second, if $K$ is a torus, choose and fix a Lie group isomorphism of $K$ with $\mathbb{Z}^{m} \setminus \mathbb{R}^{m}$, thus realizing each bi-invariant (i.e., flat) metric $h \in \mathcal{B}(K)$ as an inner product on $\mathbb{R}^{m}$. (We note that this is different from our approach in Section 2, where we changed the geometry of the torus by varying the lattice). Let $h^{*}$ be the dual inner product on the dual space $(\mathbb{R}^{m})^{*}$, and let $Q^{*}$ be the associated quadratic form $Q^{*}(\gamma) = h^{*}(\gamma, \gamma)$. Letting $\{\delta_{1}, \ldots, \delta_{m}\}$ be the standard basis of $(\mathbb{R}^{m})^{*}$, set $b_{j}(h) := 4\pi^{2}Q^{*}(\delta_{j})$ and $c_{jk}(h) = 4\pi^{2}Q^{*}(\delta_{j} + \delta_{k})$ for $1 \leq j < k \leq m$. Then, as pointed out in [Peters], $Q^{*}$, and thus $h$, is uniquely determined by the invariants $b_{j}(h)$ and $c_{jk}(h)$, $j, k = 1, \ldots, m$.

For the general case, write $K = K_{0}K_{1}\ldots K_{r}$, where $K_{0}$ is a torus (the identity component of the center of $K$) and $K_{1}, \ldots, K_{r}$ are simple normal subgroups. The various $K_{i}$’s may intersect but only in finite central subgroups. The homomorphic projections $\mathfrak{R} \to \mathfrak{R}_{i}$ give rise to homomorphisms $\pi_{i} : K \to \overline{K}_{i}$, where $\overline{K}_{i}$ is a compact Lie group finitely covered by $K_{i}$ for $0 \leq i \leq r$. Each bi-invariant metric $h$ on $K$ induces bi-invariant metrics $\overline{h}_{i}$ on $\overline{K}_{i}$, and, conversely, the bi-invariant metrics $\overline{h}_{0}, \overline{h}_{1}, \ldots, \overline{h}_{r}$ uniquely determine $h$. For $i = 1, \ldots, r$, define $\lambda_{i}^{1} : \mathcal{B}(K) \to \mathbb{R}_{+}$ by $\lambda_{i}^{1}(h) = \lambda_{1}(\overline{h}_{i})$. Choose and fix a Lie group isomorphism of $\overline{K}_{0}$ with $\mathbb{Z}^{m} \setminus \mathbb{R}^{m}$ where $m = \dim(K_{0})$, and define $b_{j}(q) = b_{j}(\overline{h}_{0})$ and $c_{jk}(q) = c_{jk}(\overline{h}_{0})$ for $j, k = 1, \ldots, m$.

For notational convenience, we rename the full collection of invariants as $\gamma_{p} : \mathcal{B}(K) \to \mathbb{R}$, $p = 1, \ldots, s$, where $s = r + m + \binom{m}{2}$, $r$ is the number of simple factors in $K$ and $m$ is the dimension of the center of $K$.

4.9. Proposition. In the notation of 4.8 we have:

(1) The collection of invariants $\gamma_{p}(h)$, for $1 \leq p \leq s$, depend continuously on $h$ and form a complete set of invariants of the elements of $\mathcal{B}(K)$.
(2) Each of these invariants lies in \( \text{Spec}(K, h) \). Moreover, for each \( p \), the \( \gamma_p(h) \)-eigenspace contains a \( \rho_K \)-invariant, irreducible subspace \( \mathcal{H}_p \subset L^2(K) \) that is independent of \( h \in \mathcal{B}(K) \).

(3) Each element of \( \mathcal{B}(K) \) is spectrally isolated within \( \mathcal{B}(K) \).

**Proof.** The first two statements are straightforward consequences of 4.8. The third statement follows from the first two.  

4.10. **Lemma.** Let \( G \) be a connected, compact simple Lie group and let \( K \in \mathcal{K} \). Let \( \alpha, \beta \in \mathbb{R} \). For each \( h \in \text{Adm}(K, \beta) \), let \( \tilde{h} = F(h) \) as in Proposition 4.4 so that we have a Riemannian submersion \( \Phi : (G \times K, \tilde{g}_{\alpha, \beta} \times \tilde{h}) \to (G, g_{\alpha, h}) \) given by \( \Phi(x, y) = xy^{-1} \). (See Corollary 4.3.) In the notation of 4.8, for each \( p = 1, \ldots, s \), there exists \( \zeta_p \in \text{Spec}(G, \tilde{g}_{\alpha, \beta}) \) such that \( \zeta_p + \gamma_p(\tilde{h}) \in \text{Spec}(G, g_{\alpha, h}) \) for all \( h \in \text{Adm}(K, \beta) \).

**Proof.** Since the Riemannian submersion \( \Phi : (G \times K, \tilde{g}_{\alpha, \beta} \times \tilde{h}) \to (G, g_{\alpha, h}) \) has totally geodesic fibers given by (left) translates of \( \Delta K = G \times K \), we see that the spectrum of \( (G, g_{\alpha, h}) \) coincides with the spectrum of the Laplacian of \( (G \times K, \tilde{g}_{\alpha, \beta} \times \tilde{h}) \) to \( (G, g_{\alpha, h}) \) restricted to \( \Delta K \)-invariant functions on \( L^2(G \times K) \). Let \( \tau_p \) be the irreducible representation of \( K \) on \( \mathcal{H}_p < L^2(K) \) given in Proposition 4.9 (2). By 4.9 (iv), the contragredient \( \tau \) occurs in \( \rho_K | K \), say with representation space \( \mathcal{H}_\tau < L^2(G) \). Since the metric \( \tilde{g}_{\alpha, \beta} \) on \( G \) is \( K \)-bi-invariant, we have by 4.9 (v) that \( \mathcal{H}_\tau \) lies in an eigenspace of the Laplacian of \( (G, \tilde{g}_{\alpha, \beta}) \), say with eigenvalue \( \zeta_p \). By 4.9 (ii), there is a non-trivial \( \Delta K \)-fixed vector in \( \mathcal{H}_\tau \otimes \mathcal{H}_p < L^2(G \times K) \). This vector is a \( (\zeta_p + \gamma_p(\tilde{h})) \)-eigenfunction on \( (G \times K, \tilde{g}_{\alpha, \beta} \times \tilde{h}) \); it is \( \Delta K \)-invariant and thus descends to an eigenfunction on \( (G, g_{\alpha, h}) \).

We now complete the proof of the Main Theorem. Let \( K \) be as in Corollary 3.7. In view of Corollary 3.7 it suffices to show that for each \( K \in \mathcal{K} \), each metric in \( \mathcal{M}_K \) is spectrally isolated within \( \mathcal{M}_K \). Let \( g = g_{\alpha, h} \in \mathcal{M}_K \). Choose a neighborhood \( W \) of \( g \) in \( \mathcal{M}_K \) as in Lemma 4.2 so that for \( g_{\alpha', h'} \in W \), isospectrality of \( g_{\alpha', h'} \) to \( g \) implies \( \alpha = \alpha' \). Choose \( \beta \in \mathbb{R} \) so that \( h \in \text{Adm}(K, \beta) \). Let \( \tilde{h} = F(h) \) as in Proposition 4.4. By Lemma 4.10 for each \( p = 1, \ldots, s \), there exists \( \zeta_p \in \text{Spec}(G, \tilde{g}_{\alpha, \beta}) \) so that \( \zeta_p + \gamma_p(\tilde{h}') \in \text{Spec}(G, g_{\alpha, h'}) \) for all \( h' \in \text{Adm}(K, \beta) \). Choose \( \epsilon > 0 \) so that for all \( p = 1, \ldots, s \), the interval \( (\zeta_p + \gamma_p(\tilde{h}) - \epsilon, \zeta_p + \gamma_p(\tilde{h}) + \epsilon) \) contains no eigenvalues in \( \text{Spec}(G, g_{\alpha, h}) \) other than \( \zeta_p + \gamma_p(\tilde{h}) \). Choose a neighborhood \( V \) of \( \tilde{h} \) in \( \mathcal{B}(K) \) so that for all \( p \), we have \( |\gamma_p(h') - \gamma_p(\tilde{h})| < \epsilon \) whenever \( h' \in V \). For \( h' \in V \) distinct from \( \tilde{h} \), we have \( \gamma_p(h') - \gamma_p(\tilde{h}) \neq 0 \) for at least one choice of \( p \), since the \( \gamma_p \)'s are a complete set of invariants. Thus letting \( U = F^{-1}(V) \), we have \( \text{Spec}(G, g_{\alpha, h}) \neq \text{Spec}(G, g_{\alpha, h'}) \) for all \( h' \in U \). Letting \( \mathcal{O} = W \cap \{ g_{\alpha', h'} : \alpha' \in \mathbb{R}, h' \in U \} \), then \( \mathcal{O} \) is an open neighborhood of \( g \) in \( \mathcal{M}_K \) all of whose elements other than \( g \) itself have spectrum different from that of \( g \). This completes the proof.

4.11. **Remark.** Although we have focused on left-invariant naturally reductive metrics on a compact simple Lie group, similar arguments demonstrate that for an arbitrary compact Lie
group $G$ metrics of the form given by Theorem 3.3 are spectrally isolated within this class (see Remark 3.5 (5)).

5. The Spectra of Semisimple Naturally Reductive Metrics

It is rare that one is able to explicitly compute the spectra of Riemannian manifolds. In this section we outline the computation of the spectrum of a class of left-invariant naturally reductive metrics on compact simple Lie groups. The class of metrics we consider are the “semisimply naturally reductive metrics” that we now define.

5.1. Definition. A left-invariant naturally reductive metric $g$ on a compact Lie group $G$ will be said to be semisimply naturally reductive if $g$ is naturally reductive with respect to $G \times K$ for some semisimple subgroup $K \leq G$.

5.2. Remark. In the notation of Theorem 3.3 suppose $G$ is a compact simple Lie group and the metric $g = g_{\alpha,h} = e^{\alpha}g_0 | \mathfrak{p} \oplus h | \mathfrak{k}$ is naturally reductive with respect to $G \times K$ for some semisimple subgroup $K \leq G$. Since the metric $h$ on $K$ is bi-invariant, we necessarily have

$$h = e^{\alpha_1}g_0 | \mathfrak{k}_1 \oplus \cdots \oplus e^{\alpha_r}g_0 | \mathfrak{k}_r,$$

where $\mathfrak{k}$, the Lie algebra of $K$, is the direct sum of simple factors $\mathfrak{k}_1, \ldots, \mathfrak{k}_r$.

If some $\alpha_i = \alpha$, then the factor $\mathfrak{k}_i$ can be absorbed into $\mathfrak{p}$. Thus we will always assume that $\alpha_i \neq \alpha$ for all $i = 1, \ldots, r$.

The next result shows that for a simple Lie group $G$ every left-invariant semisimply naturally reductive metric can be realized as the base space of a semi-Riemannian submersion (with totally geodesic fibers), where the total space is a compact Lie group equipped with a bi-invariant semi-Riemannian metric. This coupled with Theorem 5.5 below will allow us to explicitly compute the spectra of left-invariant semisimply naturally reductive metrics on simple Lie groups. Recall that a metric $g$ on a manifold $M$ is said to be semi-Riemannian if for each $p \in M$ we have that $g_p : T_pM \times T_pM \to \mathbb{R}$ is symmetric and non-degenerate.

5.3. Proposition. Let $G$ be a compact simple Lie group, let $K \leq G$ be a closed semisimple subgroup, let $g = g_{\alpha,h}$ be a left-invariant metric on $G$ naturally reductive with respect to $G \times K$, and let $\alpha, \alpha_1, \alpha_r$ be as in Remark 5.2. Let $\tilde{h}$ be the bi-invariant semi-Riemannian metric on $K$ given by

$$\tilde{h} = \beta_1g_0 | \mathfrak{k}_1 \oplus \cdots \oplus \beta_rg_0 | \mathfrak{k}_r$$

($\beta_i = \frac{e^{\alpha_i}}{1 - e^{(\alpha - \alpha_i)}} = \frac{e^\alpha}{e^{(\alpha - \alpha_i)} - 1}$).

Then the map

$$\Phi : (G \times K, e^\alpha g_0 \times \tilde{h}) \to (G, g_{\alpha,h})$$

given by

$$(g, k) \mapsto gk^{-1}$$

is a semi-Riemannian submersion with totally geodesic fibers.
5.4. Remarks. The submersion $\Phi$ will be Riemannian if $\alpha_i < \alpha$ for each $i$. Otherwise, it will be only semi-Riemannian.

In Corollary 4.5 we realized every naturally reductive metric $g_{\alpha,h}$ on a simple Lie group $G$ as the base space of a Riemannian submersion $\Phi : (G \times K, \tilde{g} \times \tilde{h}) \to (G, g)$, where the metric $\tilde{g} \times \tilde{h}$ is not necessarily bi-invariant. To obtain a bi-invariant metric on the total space, as in Proposition 5.3, we need to allow semi-Riemannian metrics.

Proof. Write $h(X,Y) = g_0(A_h(X,Y)$, where $A_h : K \to K$ is a $g_0$-self-adjoint linear transformation (with non-zero eigenvalues). Following the notation of 4.3 we let $B_{\text{semi}}(K)$ denote the space of bi-invariant semi-Riemannian metrics on $K$. Since $e^\alpha$ is not an eigenvalue of $A_h$, we may mimic the argument and notation of Proposition 4.4 and Corollary 4.5 to see that there is a unique $\tilde{h} \in B_{\text{semi}}(K)$ such that:

1. The semi-Riemannian metric $e^\alpha g_0 \times \tilde{h}$ on $K \times K$ is non-degenerate on $\Delta K \leq K \times K$.
2. The map $\Phi : (K \times K, e^\alpha g_0 \times \tilde{h}) \to (K, h)$ is a semi-Riemannian submersion.
3. The semi-Riemannian metric $e^\alpha g_0 \times \tilde{h}$ on $G \times K$ is non-degenerate when restricted to $\Delta K \leq g \times K$ and $\Phi : (G \times K, e^\alpha g_0 \times \tilde{h}) \to (G, g_{\alpha,h})$ is a semi-Riemannian submersion with totally geodesic fibers.

□

It is well known that the Laplace operator on a compact simple Lie group equipped with the bi-invariant metric is given by the Casimir operator. Consequently, in this case, we may explicitly compute the spectrum of the Laplacian in terms of the highest weights of irreducible representations. For convenience, given a compact Lie group $G$ we will identify $\hat{G}$, the space of irreducible unitary representations of $G$ (see 4.6), with the collection of highest weights of irreducible representations of $G$ with respect to some choice of positive Weyl chamber.

5.5. Theorem ([Fe]). Let $K$ be a simply-connected semisimple compact Lie group and let $B$ denote the Killing form on its Lie algebra $\mathfrak{g}$. We let $g_0$ denote the bi-invariant metric on $K$ determined by $-B$ and let $\| \cdot \|$ be the norm on $\mathfrak{g}^*$, the dual space to $\mathfrak{g}$, determined by $-B$. Then

$$\text{Spec}(K, g_0) = \{c(\lambda) : \lambda \in \hat{G}\}$$

where $c(\lambda) = \|\lambda + \rho\|^2 - \|\rho\|^2$ and $\rho$ is half the sum of the positive roots.

The preceding result allows us to compute the spectrum of any semisimple Lie group endowed with a bi-invariant metric. Indeed, let $K$ be a connected compact semisimple Lie group with Lie algebra $\mathfrak{k} = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_r$, where $\mathfrak{k}_i$ is simple for each $i = 1, \ldots, r$. We caution that $K$ is not necessarily a direct product of simple factors; the simple connected normal subgroups may have non-trivial, but necessarily finite, intersections. Let $\tilde{K} = \tilde{K}_1 \times \cdots \times \tilde{K}_r$ be the universal cover of $K$ with covering map $\pi$, here $\tilde{K}_i$ is the unique simply-connected Lie group with Lie algebra $\mathfrak{h}_i$. Then $K = \tilde{K}/\Gamma$ for some finite subgroup $\Gamma \leq Z(\tilde{K})$ of the center of $\tilde{K}$. It
follows that $\hat{K}$, the set of equivalence classes of irreducible unitary representations of $K$, can be identified with

$$\hat{K} \equiv \{ \sigma \in \hat{K} : \Gamma \leq \ker(\sigma) \}.$$ 

Since, moreover, $\hat{K}$ can be identified with $\hat{K}_1 \times \cdots \times \hat{K}_r$ (see 4.6 (2)), we may consider $\hat{K}$ to be a subset of $\hat{K}_1 \times \cdots \times \hat{K}_r$. Now, let $g = e^{\alpha_1}g_0 \mid \mathfrak{K}_1 \oplus \cdots \oplus e^{\alpha_r}g_0 \mid \mathfrak{K}_r$ be a bi-invariant metric on $K$ and let $\tilde{g}$ (given by the same expression as $g$) be the bi-invariant metric on $\hat{K}$ such that $\pi : (\hat{K}, \tilde{g}) \to (K, g)$ is a Riemannian covering. Then it follows from Theorem 5.5 that

$$\text{Spec}(K, g) = \{ \sum_{i=1}^{r} e^{-\alpha_i}c_i(\lambda_i) : (\lambda_1, \cdots, \lambda_r) \in \hat{K} \},$$

where for $\lambda_i \in \hat{K}_i$, $c_i(\lambda_i)$ is the expression defined in Theorem 5.5 with $K$ replaced by $\hat{K}_i$.

We now compute the spectrum of an arbitrary semisimply naturally reductive metric $g$ on a compact connected simple Lie group $G$. We express $g$ as in Remark 5.2. Let $(G \times K, e^\alpha g_0 \times \tilde{h})$ be the associated semi-Riemannian manifold given in Proposition 5.3. As in the Riemannian case, one defines the Laplacian $\Delta$ on a semi-Riemannian manifold as $\Delta f = -\text{Div}(\text{grad} f)$. In the semi-Riemannian setting the eigenvalues of the Laplace operator can be both positive and negative. It follows from Proposition 5.3 in particular the bi-invariance of the metric $e^\alpha g_0 \times \tilde{h}$, and the discussion above that the spectrum of $(G \times K, e^\alpha g_0 \times \tilde{h})$ is given by the collection of numbers

$$e^{-\alpha}\{\|\lambda + \rho\|^2 - \|\rho\|^2 + \sum_{i=1}^{r}(e^{\alpha_i} - 1)(\|\lambda_i + \rho_i\|^2 - \|\rho_i\|^2)\},$$

as $\lambda$ varies over $\hat{G}$ and $(\lambda_1, \cdots, \lambda_r)$ varies over $\hat{K} \subset \hat{K}_1 \times \cdots \times \hat{K}_r$. Since $\Phi : (G \times K, e^\alpha g_0 \times \tilde{h}) \to (G, g)$ is a semi-Riemannian submersion with totally geodesic fibers (isometric to $\Delta K$) it follows that the Laplacian of $(G, g)$ is given by the restriction of the Laplacian on $G \times K$ to the $\Delta K$-invariant functions. Hence, the discussion in 4.6 (2) implies that if we let $\hat{G} \times \hat{K}^* = \{ \sigma \otimes \tau \in \hat{G} \times \hat{K} : [\sigma \mid K : \tau] \neq 0 \} \subset \hat{G} \times \hat{K}$, then

$$\text{Spec}(G, g_0, h) = \{ e^{-\alpha}(e(\lambda) + \sum_{i=1}^{r}(e^{\alpha_i} - 1)c_i(\lambda_i)) : (\lambda; \lambda_1, \cdots, \lambda_r) \in \hat{G} \times \hat{K}^* \}.$$ 

5.6. Remark. Although we have concentrated on compact simple groups, the computation above can be generalized to obtain formulas for the known semisimply naturally reductive metrics on an arbitrary compact Lie group $G$.

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