CHARACTERIZING JORDAN DERIVATIONS OF MATRIX RINGS THROUGH ZERO PRODUCTS

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Abstract. Let $M_n(R)$ be the ring of all $n \times n$ matrices over a unital ring $R$, let $M$ be a 2-torsion free unital $M_n(R)$-bimodule and let $D : M_n(R) \to M$ be an additive map. We prove that if $D(ab) + D(ba) + bD(a) = 0$ whenever $a, b \in M_n(R)$ are such that $ab = ba = 0$, then $D(a) = \delta(a) + aD(1)$, where $\delta : M_n(R) \to M$ is a derivation and $D(1)$ lies in the centre of $M$. It is also shown that $D$ is a generalized derivation if and only if $D(ab) + D(ba) + bD(a) - aD(1)b - bD(1)a = 0$ whenever $ab = ba = 0$. We apply this results to provide that any (generalized) Jordan derivation from $M_n(R)$ into a 2-torsion free $M_n(R)$-bimodule (not necessarily unital) is a (generalized) derivation. Also, we show that if $\varphi : M_n(R) \to M_n(R)$ is an additive map satisfying $\varphi(ab + ba) = a\varphi(b) + \varphi(b)a$ (a, b $\in M_n(R)$), then $\varphi(a) = a\varphi(1)$ for all $a \in M_n(R)$, where $\varphi(1)$ lies in the centre of $M_n(R)$. By applying this result we obtain that every Jordan derivation of the trivial extension of $M_n(R)$ by $M_n(R)$ is a derivation.

1. Introduction

Throughout this paper all rings are associative. Let $A$ be a unital ring and $M$ be an $A$-bimodule. Recall that an additive map $D : A \to M$ is said to be a Jordan derivation (or generalized Jordan derivation) if $D(ab + ba) = D(a)b + aD(b) + D(b)a + bD(a)$ (or $D(ab + ba) = D(a)b + aD(b) + D(b)a + bD(a) - aD(1)b - bD(1)a$) for all $a, b \in A$. It is called a derivation (or generalized derivation) if $D(ab) = D(a)b + aD(b)$ (or $D(ab) = D(a)b + aD(b) - aD(1)b$) for all $a, b \in A$. Each map $I_m : A \to M$ given by $I_m(a) = am - ma$ ($m \in M$) is a derivation which will be called an inner derivation. Clearly, each (generalized) derivation is a (generalized) Jordan derivation. The converse is, in general, not true.

Remark 1. Let $A$ be a unital ring, $M$ be an $A$-bimodule and $D : A \to M$ be an additive mapping. Then the following are equivalent:

(i) $D$ is a generalized derivation,
(ii) there is a derivation \( \delta : \mathcal{A} \rightarrow \mathcal{M} \) such that \( D(a) = \delta(a) + aD(1) \) for \( a \in \mathcal{A} \).

If (i) holds, define \( \delta : \mathcal{A} \rightarrow \mathcal{M} \) by \( \delta(a) = D(a) - aD(1) \). It is easily seen that \( \delta \) is a derivation, so (ii) obtain. Conversely, if (ii) holds we have

\[
D(ab) = \delta(ab) + abD(1) = \delta(a)b + a\delta(b) + abD(1)
\]
\[
= (D(a) - aD(1))b + a(D(b) - bD(1)) + abD(1)
\]
\[
= D(a)b + aD(b) - aD(1)b.
\]

Thus \( D \) is a generalized derivation.

The question under what conditions a map becomes a (generalized or Jordan) derivation attracted much attention of mathematicians. Herstein [11] proved that every Jordan derivation from a 2-torsion free prime ring into itself is a derivation. Brešar [6] showed that every Jordan derivation from a 2-torsion free semiprime ring into itself is a derivation. By a classical result of Jacobson and Rickart [6] every Jordan derivation on a full matrix ring over a 2-torsion free unital ring is a derivation and Alizadeh in [4] generalized this result. For more studies concerning Jordan derivations we refer the reader to [5, 10, 12, 16, 17, 18, 19] and the references therein. Also, there have been a number of papers concerning the study of conditions under which (generalized or Jordan) derivations of rings can be completely determined by the action on some sets of points [1, 2, 3, 7, 9, 13, 14, 15, 21, 22].

In this paper, following [3], we consider the subsequent condition on an additive map \( D \) from a ring \( \mathcal{A} \) into an \( \mathcal{A} \)-bimodule \( \mathcal{M} \):

\[
a, b \in \mathcal{A}, \quad ab = ba = 0 \Rightarrow D(a)b + aD(b) + D(b)a + bD(a) = 0. \quad (\ast)
\]

Our purpose is to investigate whether the condition (\ast) characterizes Jordan derivations. A similar question is concerned with generalized Jordan derivations. So we consider the following condition on an additive map \( D : \mathcal{A} \rightarrow \mathcal{M} \) to the context of generalized Jordan derivations, where \( \mathcal{A} \) is unital and \( \mathcal{M} \) is unital \( \mathcal{A} \)-bimodule:

\[
ab = ba = 0 \Rightarrow D(a)b + aD(b) + D(b)a + bD(a) - aD(1)b - bD(1)a = 0. \quad (\ast\ast)
\]

In Section 2 we prove that, in the case when \( \mathcal{A} \) is a full matrix ring \( M_n(\mathcal{R}) \) over a unital ring \( \mathcal{R} \) and \( \mathcal{M} \) is a 2-torsion free unital \( M_n(\mathcal{R}) \)-bimodule, conditions (\ast) and (\ast\ast) imply that \( D \) is of the form \( D(a) = \delta(a) + aD(1) \) for each \( a \in M_n(\mathcal{R}) \), where \( \delta : M_n(\mathcal{R}) \rightarrow \mathcal{M} \) is a derivation and \( 1 \) is the identity matrix. In the case (\ast) we have \( D(1) \in Z(\mathcal{M}) \), where \( Z(\mathcal{M}) \) is the centre of \( \mathcal{M} \). In section 3 our previous results are applied to characterize (generalized) Jordan derivations from \( M_n(\mathcal{R}) \) into a 2-torsion free \( M_n(\mathcal{R}) \)-bimodule \( \mathcal{M} \) which is not necessarily a unital \( M_n(\mathcal{R}) \)-bimodule. Indeed, we show that each (generalized) Jordan derivation from \( M_n(\mathcal{R}) \) into \( \mathcal{M} \) is a (generalized) derivation. This generalizes the main
result of [4]. In section 4 we get some related results. In particular, by applying results from section 2 we obtain that if \( \varphi : M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R}) \) is an additive map satisfying \( \varphi(ab + ba) = a\varphi(b) + \varphi(b)a \) for all \( a, b \in M_n(\mathbb{R}) \), then \( \varphi(a) = a\varphi(1) \) for all \( a \in M_n(\mathbb{R}) \), where \( \varphi(1) \in Z(M_n(\mathbb{R})) \). As applications of the above results, we show that every Jordan derivation of the trivial extension of \( M_n(\mathbb{R}) \) by \( M_n(\mathbb{R}) \) is a derivation.

**Remark 2.** Each of the following conditions on an additive map \( D : A \rightarrow M \) implies \((*)\), which have been considered by a number of authors (see, for instance, [13, 20]):

\[
\begin{align*}
& a, b \in A, \quad ab + ba = 0 \Rightarrow D(ab) + D(b)a + bD(a) = 0. \\
& a, b \in A, \quad ab = 0 \Rightarrow D(ab) + D(b)a + bD(a) = D(ab + ba).
\end{align*}
\]

Therefore, Theorem [2.1] still holds with each of the above conditions replaced by \((*)\).

The following notations will be used in this paper.

We shall denote the elements of \( M_n(\mathbb{R}) \) by bold letters and the identity matrix by \( \mathbf{1} \). Also, \( e_{ij} \) for \( 1 \leq i, j \leq n \) is the matrix unit, \( a\mathbf{e}_{ij} \) is the matrix whose \((ij)\)th entry is \( a \) and zero elsewhere, where \( a \in \mathbb{R} \) and \( 1 \leq i, j \leq n \), and \( a_{ij} \) is the \((ij)\)th entry of \( a \in M_n(\mathbb{R}) \).

2. **Characterizing Jordan derivations through zero products**

From this point up to the last section \( M_n(\mathbb{R}) \), for \( n \geq 2 \), is the ring of all \( n \times n \) matrices over a unital ring \( \mathbb{R} \) and \( M \) is a 2-torsion free unital \( M_n(\mathbb{R}) \)-bimodule. In this section, we discuss the additive maps from \( M_n(\mathbb{R}) \) into \( M \) satisfying \((*)\).

**Theorem 2.1.** Let \( D : M_n(\mathbb{R}) \rightarrow M \) be an additive map satisfying

\[
\text{ab} \in M_n(\mathbb{R}, \quad ab = ba = 0 \Rightarrow D(ab) + aD(b) + D(b)a + bD(a) = 0.}
\]

Then there exist a derivation \( \delta : M_n(\mathbb{R}) \rightarrow M \) such that \( D(a) = \delta(a) + aD(1) \) for each \( a \in M_n(\mathbb{R}) \) and \( D(1) \in Z(M) \).

**Proof.** Set \( e = e_11 \) and \( f = 1 - e_{11} = \sum_{j=2}^{n} e_{jj} \). Then \( e \) and \( f \) are nontrivial idempotents such that \( e + f = 1 \) and \( ef = fe = 0 \). Let \( m = eD(e)f - fD(e)e \).

Define \( \Delta : M_n(\mathbb{R}) \rightarrow M \) by \( \Delta(a) = D(a) - I_m(a) \). Then \( \Delta \) is an additive mapping which satisfies \((*)\). Moreover \( e\Delta(e)f = f\Delta(e)e = 0 \).

We complete the proof by checking some steps.

**Step 1.** \( \Delta(\text{eae}) = \text{e}\Delta(\text{eae})e \) and \( \Delta(\text{faf}) = \text{f}\Delta(\text{faf})f \) for all \( a \in M_n(\mathbb{R}) \).

Let \( a \in M_n(\mathbb{R}) \). Since \( e(\text{faf}) = (\text{faf})e = 0 \), we have

\[
\Delta(\text{e})\text{faf} + e\Delta(\text{faf}) + \Delta(\text{faf})e + \text{faf}\Delta(\text{e}) = 0.
\]

(2.1)
Multiplying this identity by $e$ both on the left and on the right we see that $2e\Delta(faf)e = 0$ so $e\Delta(faf)e = 0$. Now, multiplying the Equation (2.1) from the left by $e$, from the right by $f$ and by the fact that $e\Delta(e)f = 0$, we get $e\Delta(faf)f = 0$. Similarly, from Equation (2.1) and the fact that $f\Delta(e)e = 0$, we see that $f\Delta(faf)e = 0$. Therefore, from above equations we arrive at 

$$\Delta(faf) = f\Delta(faf)f$$

We have $(eae)f = f(eae) = 0$. Thus

$$\Delta(eae)f + e\Delta(f) + \Delta(f)(eae) + f\Delta(eae) = 0. \quad (2.2)$$

By $\Delta(faf) = f\Delta(faf)f$, Equation (2.2) and using similar methods as above we obtain

$$\Delta(eae) = e\Delta(eae)e.$$ 

**Step 2.** $\Delta(efaf) = e\Delta(efaf)f$ for all $a \in M_n(\mathbb{R})$.

Let $a, b \in M_n(\mathbb{R})$. Since $(eaf)(ebf) = (ebf)(eaf) = 0$ we have

$$\Delta(efaf)ebf + eaf\Delta(ebf) + \Delta(ebf)efaf + ebf\Delta(efaf) = 0. \quad (2.3)$$

Multiplying Equation (2.3) by $e$ both on the left and on the right, we get

$$e\Delta(efaf)ebf + ebf\Delta(efaf)e = 0. \quad (2.4)$$

Similarly, multiplying Equation (2.3) by $f$ both on the left and on the right, we find

$$f\Delta(efaf)ebf + f\Delta(ebf)efaf = 0. \quad (2.5)$$

We have $(eae + eaebf)(f - ebf) = (f - ebf)(eae + eaebf) = 0$ and so

$$\Delta(eae + eaebf)(f - ebf) + (eae + eaebf)\Delta(f - ebf) + \Delta(f - ebf)(eae + eaebf) + (f - ebf)\Delta(eae + eaebf) = 0. \quad (2.6)$$

Multiplying Equation (2.6) by $e$ both on the left and on the right and replacing $a$ by $e$, from Step 1 and Equation (2.4), we get $e\Delta(ebf)e = 0$. Now multiplying Equation (2.6) by $f$ both on the left and on the right, by Equation (2.5) and a similar arguments as above we find $f\Delta(ebf)f = 0$.

Multiplying Equation (2.6) by $f$ on the left and by $e$ on the right. By Step 1, we arrive at

$$f\Delta(eaebf)e = f\Delta(ebf)eae. \quad (2.7)$$

For any $a \in M_n(\mathbb{R})$ and $2 \leq j \leq n$, let $e_{1j}ae_{jj} = ae_{1j}$. By Equation (2.3) we have

$$f\Delta(ae_{1j})e_{1j} = f\Delta(e(ae_{1j})f)f = -f\Delta(ee_{1j}f)e( ae_{1j})f$$

$$= -f\Delta( ee_{1j}f)e(e_{11})e_{1j}.$$ 

Also from Equation (2.7) we see that

$$f\Delta(ee_{1j}f)e(ae_{11})e_{1j} = f\Delta(eae_{11}ee_{1j}f)e_{1j} = f\Delta(ae_{1j}e_{1j}).$$
So $f \Delta (ae_{1j})e_{1j} = -f \Delta (ae_{1j})e_{1j}$ and hence $f \Delta (ae_{1j})e_{1j} = 0$. Multiplying this identity on the right by $e_{j1}$, we get $f \Delta (ae_{1j})e = 0$. Therefore $f \Delta (ae_{11}ae_{jj})e = f \Delta (ae_{1j})e = 0$. So

$$f \Delta (eaf)e = f \Delta \left( \sum_{j=2}^{n} e_{1j}ae_{j} \right)e = \sum_{j=2}^{n} f \Delta (e_{1j}ae_{j})e = 0.$$ 

Now from previous equations it follows that

$$\Delta (eaf) = e \Delta (eaf)f.$$ 

**Step 3.** $\Delta (fae) = f \Delta (fae)e$ for all $a \in M_n(\mathbb{R})$.

Let $a, b \in M_n(\mathbb{R})$. Applying $\Delta$ to $(fae)(fbe) = (fbe)(fae) = 0$, we get

$$\Delta (fae)fbe + fae \Delta (fbe) + \Delta (fbe)fae + fbe \Delta (fae) = 0. 
(2.8)$$

Multiplying Equation (2.8) by $e$ both on the left and on the right, we get

$$e \Delta (fae)fbe + e \Delta (fbe)fae = 0. 
(2.9)$$

Similarly, multiplying Equation (2.8) by $f$ both on the left and on the right, we have

$$fae \Delta (fbe)f + fbe \Delta (fae)f = 0. 
(2.10)$$

We have $(f + fae)(faebe - ebe) = (faebe - ebe)(f + fae) = 0$ and so

$$\Delta (f + fae)(faebe - ebe) + (f + fae)\Delta (faebe - ebe) + \Delta (faebe - ebe)(f + fae) + (faebe - ebe)\Delta (f + fae) = 0. 
(2.11)$$

Multiplying Equation (2.11) by $e$ both on the left and on the right and replacing $b$ by $e$, from Step 1 and Equation (2.9), we get $e \Delta (fae)e = 0$. Now multiplying Equation (2.11) by $f$ both on the left and on the right, by Equation (2.10) and a similar arguments as above we find $f \Delta (fae)f = 0$.

Multiplying Equation (2.11) by $e$ on the left and by $f$ on the right. By Step 1, we arrive at

$$e \Delta (faebe)f = ebe \Delta (fae)f. 
(2.12)$$

For any $a \in M_n(\mathbb{R})$ and $2 \leq j \leq n$, let $e_{jj}ae_{11} = ae_{j1}$. By Equation (2.11), we have

$$e_{j1} \Delta (ae_{j1})f = fe_{j1}e \Delta (f(ae_{j1})e)f = -f(ae_{j1})e \Delta (fe_{j1}e)f = -e_{j1}e(ae_{11})e \Delta (fe_{j1}e)f.$$ 

Also from Equation (2.12) we see that

$$e_{j1}e(ae_{11})e \Delta (fe_{j1}e)f = e_{j1}e \Delta (fe_{j1}e(ae_{11})e)f = e_{j1} \Delta (ae_{j1})f.$$ 

So $e_{j1} \Delta (ae_{j1})f = -e_{j1} \Delta (ae_{j1})f$ and hence $e_{j1} \Delta (ae_{j1})f = 0$. Therefore

$$e \Delta (e_{jj}ae_{11})f = e \Delta (ae_{j1})f = 0.$$
So
\[ e\Delta(fae)f = e\Delta(\sum_{j=2}^{n} e_{jj}ae_{11})f = \sum_{j=2}^{n} e\Delta(e_{jj}ae_{11})f = 0. \]

Now from previous equations it follows that
\[ \Delta(fae) = f\Delta(fae)e. \]

**Step 4.**
\[ e\Delta(eaebf)f = eae\Delta(ebf)f + e\Delta(eae)ebf - eaebf\Delta(f)f \]
and
\[ e\Delta(eabf)f = e\Delta(eaf)fbf + eaf\Delta(fbf)f - eaf\Delta(f)fbf \]
for all \(a, b \in M_n(\mathbb{R}).\)

Let \(a, b \in M_n(\mathbb{R}).\) Multiplying Equation\(2.6\) by \(e\) on the left and by \(f\) on the right, from Step 1 and 2 we obtain
\[ e\Delta(eaebf)f = eae\Delta(ebf)f + e\Delta(eae)ebf - eaebf\Delta(f)f. \]

Replacing \(a\) by \(e\) in above equation, we get
\[ e\Delta(e)ebe = ebf\Delta(f)f \quad (2.13) \]

Since \((e + eaf)(fbf - eabf) = (fbf - eabf)(e + eaf) = 0, we have\n\[ \Delta(e + eaf)(fbf - eabf) + (e + eaf)\Delta(fbf - eabf) \]
\[ + \Delta(fbf - eabf)(e + eaf) + (fbf - eabf)\Delta(e + eaf) = 0 \]

Multiplying this identity by \(e\) on the left and by \(f\) on the right, from Equation\(2.13\) and Step 1 and 2 we arrive at
\[ e\Delta(eabf)f = e\Delta(eaf)fbf + eaf\Delta(fbf)f - eaf\Delta(f)fbf. \]

**Step 5.**
\[ f\Delta(faebe)e = f\Delta(fae)ebe + fae\Delta(ebe)e - f\Delta(f)faebe \]
and
\[ f\Delta(fafbe)e = faf\Delta(fbe)e + f\Delta(faf)fbe - faf\Delta(f)fbe \]
for all \(a, b \in M_n(\mathbb{R}).\)

Let \(a, b \in M_n(\mathbb{R}).\) Multiplying Equation\(2.11\) by \(f\) on the left and by \(e\) on the right, from Step 1 and 3 we obtain
\[ f\Delta(faebe)e = f\Delta(fae)ebe + fae\Delta(ebe)e - f\Delta(f)faebe \]

Replacing \(b\) by \(e\) in above equation, we get
\[ fae\Delta(e)e = f\Delta(f)fae \quad (2.14) \]
Since \((e - fbe)(fafbe + faf) = (fafbe + faf)(e - fbe) = 0\), we have
\[
\Delta(e - fbe)(fafbe + faf) + (e - fbe)\Delta(fafbe + faf) + \Delta(fafbe + faf)(e - fbe) + (fafbe + faf)\Delta(e - fbe) = 0
\]
Multiplying this identity by \(f\) on the left and by \(e\) on the right, from Equation (2.14) and Step 1 and 3 we arrive at
\[
f\Delta(fafbe)e = faf\Delta(fbe)e + f\Delta(faf)fbe - faf\Delta(f)fbe.
\]
**Step 6.**
\[
e\Delta(eaebe)e = eae\Delta(ebe)e + e\Delta(eae)ebe - eae\Delta(e)ebe
\]
and
\[
f\Delta(fafbf)f = f\Delta(faf)fbf + faf\Delta(fbf)f - faf\Delta(f)fbf
\]
for all \(a, b \in M_n(R)\).

Let \(a, b \in M_n(R)\). For \(2 \leq j \leq n\), we have \(e_{1j} = ee_{1j}f\), so from Step 4 we see that
\[
e\Delta(eaebe_{1j})f = eaebe\Delta(e_{1j})f + e\Delta(eaebe)e_{1j} - eaebe_{1j}\Delta(f)f.
\]
On the other hand,
\[
e\Delta(eaebe_{1j})f = eae\Delta(ebe_{1j})f + e\Delta(eae)ebe_{1j} - eaebe_{1j}\Delta(f)f
\]
\[
= eaebe\Delta(e_{1j})f + eae\Delta(ebe)e_{1j} - eaebe_{1j}\Delta(f)f
\]
\[
+ e\Delta(eae)ebe_{1j} - eaebe_{1j}\Delta(f)f.
\]
By comparing the two expressions for \(e\Delta(eaebe_{1j})f\), Equation (2.13) and multiplying the resulting equation by \(e_{j1}\) on the right, yields
\[
e\Delta(eaebe)e = eae\Delta(ebe)e + e\Delta(eae)ebe - eae\Delta(e)ebe.
\]
We have \(e_{1j} = ee_{1j}f\) for \(2 \leq j \leq n\), so from Step 5 and a proof similar to above, we find
\[
e_{1j}\Delta(fafbf)f = e_{1j}\Delta(faf)fbf + e_{1j}af\Delta(fbf)f - e_{1j}af\Delta(f)fbf.
\]
Multiplying this identity from left by \(e_{j1}\) we get
\[
e_{j1}\Delta(fafbf)f = e_{j1}\Delta(faf)fbf + e_{j1}af\Delta(fbf)f - e_{j1}af\Delta(f)fbf.
\]
So
\[
f\Delta(fafbf)f = \sum_{j=2}^{n} e_{jj}\Delta(fafbf)f
\]
\[
= \sum_{j=2}^{n} (e_{jj}\Delta(faf)fbf + e_{jj}af\Delta(fbf)f - e_{jj}af\Delta(f)fbf
\]
\[
= f\Delta(faf)fbf + faf\Delta(fbf)f - faf\Delta(f)fbf.
\]
Step 7. \( a\Delta(1) = \Delta(1)a \) for all \( a \in M_n(\mathbb{R}) \).

Let \( a \in M_n(\mathbb{R}) \). By Equation (2.13) we have

\[
\begin{align*}
eae\Delta(e)e_{1j} &= eae_{1j}\Delta(f) = e\Delta(e)ea_{1j} \\
e_{1j}\Delta(f)f &= e\Delta(e)e_{1j}af = e_{1j}af\Delta(f)f
\end{align*}
\]

for \( 2 \leq j \leq n \). So

\[
\begin{align*}
eae\Delta(e)e &= e\Delta(e)eae, & e_{jj}\Delta(f)f &= e_{jj}af\Delta(f)f
\end{align*}
\]

and

\[
f\Delta(f)f = \sum_{j=2}^{n} e_{jj}\Delta(f)f = \sum_{j=2}^{n} e_{jj}af\Delta(f)f = f\Delta(f)f. \quad (2.15)
\]

By Step 1 we have \( \Delta(1) = e\Delta(e)e + f\Delta(f)f \). From this identity and Equations (2.13), (2.14), (2.15) we arrive at

\[
\begin{align*}
a\Delta(1) &= eae\Delta(1) + eaf\Delta(1) + fae\Delta(1) + f\Delta(1) \\
&= eae\Delta(1) + eaf\Delta(1) + fae\Delta(1) + f\Delta(1)f \\
&= eae\Delta(1) + eaf\Delta(1) + fae\Delta(1) + f\Delta(1)f \\
&= \Delta(1)eae + \Delta(1)eaf + \Delta(1)fae + \Delta(1)faf \\
&= \Delta(1)a.
\end{align*}
\]

Step 8.

\[
e\Delta(eafbe)e = e\Delta(eaf)fbe + eaf\Delta(fbe)e - eafbe\Delta(e)e
\]

and

\[
f\Delta(fbeaf)f = f\Delta(fbe)f \quad (2.16)
\]

for all \( a, b \in M_n(\mathbb{R}) \).

Let \( a, b \in M_n(\mathbb{R}) \). By applying \( \Delta \) to

\[
(eafbe + eaf - fbe - f)(-e - eaf + fbe + fbeaf) = (-e - eaf + fbe + fbeaf)(eafbe + eaf - fbe - f) = 0
\]

and multiplying the resulting equation by \( e \) both on the left and on the right, from Steps 1–3 and Equations (2.15) we deduce that

\[
e\Delta(eafbe)e = e\Delta(eaf)fbe + eaf\Delta(fbe)e - eafbe\Delta(e)e.
\]

Also by applying \( \Delta \) to (2.16) and multiplying the resulting equation by \( f \) both on the left and on the right, from Steps 1–3 and Equations (2.15) we get

\[
f\Delta(fbeaf)f = f\Delta(fbe)f \quad (2.16)
\]

for all \( a, b \in M_n(\mathbb{R}) \).
We have $D(1) = \Delta(1)$ and hence from Step 7 we find that $D(1) \in Z(M)$. Since $ab = eab + eaf + eaf + ef + af + fa + ba + ab = 0$ for any $a, b \in M_n(R)$, by Steps 1–8, it follows that the mapping $d: M_n(R) \to M$ given by $d(a) = \Delta(a) - a\Delta(1)$ is a derivation. So the mapping $\delta: M_n(R) \to M$ given by $\delta(a) = d(a) + I_n(a)$ is a derivation and we have $D(a) = \delta(a) + aD(1)$ for all $a \in M_n(R)$. The proof is thus completed.

The following theorem is a consequence of Theorem 2.1.

**Theorem 2.2.** Let $D: M_n(R) \to M$ be an additive map satisfying

\[ab = ba = 0 \Rightarrow D(a)b + aD(b) + (b)a + bD(a) - aD(1)b - bD(1)a = 0.\]

Then there exist a derivation $\delta: M_n(R) \to M$ such that $D(a) = \delta(a) + aD(1)$ for each $a \in M_n(R)$.

**Proof.** Define $\delta: M_n(R) \to M$ by $\delta(a) = D(a) - aD(1)$. It is easy too see that $\delta$ is an additive map satisfying (*) and $\delta(1) = 0$. By Theorem 2.1 $\delta$ is a derivation. Thus $D(a) = \delta(a) + aD(1)$ for all $a \in M_n(R)$ and proof is completed.

Let $R$ be a unital ring and $N$ be a unital $R$-bimodule. Let $M_n(N)$ be the set of all $n \times n$ matrices over $N$, then $M_n(N)$ has a natural structure as unital $M_n(R)$-bimodule. Any derivation $d: R \to N$, induces a derivation $d: M_n(R) \to M_n(N)$ given by $d(a) = h$, where $a_{i,j} = d(a_{i,j})$. By similar method as in proof of [1] Theorem 3.1, we can show that if $\delta: M_n(R) \to M_n(N)$ is a derivation, then there is an inner derivation $I_g : M_n(R) \to M_n(N)$ and a derivation $d: R \to N$ such that $\delta = d + I_g$. So by Theorem 2.1 we have the following corollary.

**Corollary 2.3.** Let $R$ be a unital ring and $N$ be a 2-torsion free unital $R$-bimodule. Let $D: M_n(R) \to M_n(N)$ be an additive mapping.

(i) If $D$ satisfies (*), then there is an inner derivation $I_g : M_n(R) \to M_n(N)$ and a derivation $d: R \to N$ such that $D(a) = d(a) + I_g(a) + aD(1)$ for all $a \in M_n(R)$, where $D(1) \in Z(M_n(N))$.

(ii) If $D$ satisfies (**), then there is an inner derivation $I_g : M_n(R) \to M_n(N)$ and a derivation $d: R \to N$ such that $D(a) = d(a) + I_g(a) + aD(1)$ for all $a \in M_n(R)$.

3. **Jordan derivations of matrix rings**

In this section we characterize Jordan derivations of matrix rings into bimodules which are not necessarily unital bimodule. To prove the main result, we need the following lemma.

**Lemma 3.1.** Let $A$ be a unital ring. Then the following are equivalent:

(i) for every 2-torsion free unital $A$-bimodule $M$, each Jordan derivation $D : A \to M$ is a derivation.
(ii) for every 2-torsion free \(A\)-bimodule \(M\), each Jordan derivation \(D : A \to M\) is a derivation.

(iii) for every 2-torsion free \(A\)-bimodule \(M\), each generalized Jordan derivation \(D : A \to M\) is a generalized derivation.

Proof. (i) \(\Rightarrow\) (ii) Let \(M\) be a 2-torsion free \(A\)-bimodule and 1 be the unity of \(A\). Define the following sets:
\[
M_1 = \{1m1 \mid m \in M\}, \quad M_2 = \{m - 1m1 \mid m \in M\}, \quad M_3 = \{m1 - 1m1 \mid m \in M\}, \quad M_4 = \{m - 1m - m1 + 1m1 \mid m \in M\}.
\]

Every \(M_j\) for \(1 \leq j \leq 4\) is an \(A\)-subbimodule of \(M\) such that \(M_1\) is unital and
\[M_2A = A M_3 = M_4 A = A M_4 = \{0\}.
\]

Also \(1m_2 = m_2\) for all \(m_2 \in M_2\), \(m_3 1 = m_3\) for all \(m_3 \in M_3\) and \(M = M_1 + M_2 + M_3 + M_4\) as sum of \(A\)-bimodules. Let \(D : A \to M\) be a Jordan derivation. So \(D = D_1 + D_2 + D_3 + D_4\), where each \(D_j\) is an additive map from \(A\) to \(M_j\). Since \(D(ab + ba) = D(a)b + aD(b) + D(b)a + bD(a)\) for all \(a, b \in A\), from the above results we get
\[
D_1(ab + ba) + D_2(ab + ba) + D_3(ab + ba) + D_4(ab + ba)
\]
\[= D_1(a)b + D_3(a)b + aD_1(b) + aD_2(b) + D_1(b)a + D_3(b)a + bD_1(a) + bD_2(a).
\]

Therefore
\[
D_1(ab + ba) = D_1(a)b + aD_1(b) + D_1(b)a + bD_1(a),
D_2(ab + ba) = aD_2(b) + bD_2(a),
D_3(ab + ba) = D_3(a)b + D_3(b)a \quad \text{and} \quad (3.1)
D_4(ab + ba) = 0
\]

So \(D_1\) is a Jordan derivation and by hypothesis it is a derivation since \(M_1\) is a 2-torsion free unital \(A\)-bimodule. Now taking \(b = 1\) in Equations (3.1), we arrive at \(D_2(a) = aD_2(1), D_3(a) = D_3(1)a\) and \(2D_4(a) = 0\). Hence \(D_4(a) = 0\) since \(M\) is 2-torsion free. By previous results it is obvious that \(D\) is a derivation.

(ii) \(\Rightarrow\) (iii) Let \(M\) be a 2-torsion free \(A\)-bimodule and \(D : A \to M\) be a generalized Jordan derivation. The mapping \(\delta : A \to M\) defined by \(\delta(a) = D(a) - aD(1)\) is a Jordan derivation and hence it is a derivation. So from Remark \(\Box\) \(D\) is a generalized derivation.

(iii) \(\Rightarrow\) (i) Let \(M\) be a 2-torsion free unital \(A\)-bimodule and \(D : A \to M\) be a Jordan derivation. So \(D(1) = 0\) since \(M\) is a unital \(A\)-bimodule. Hence from hypothesis it is clear that \(D\) is a derivation. \(\Box\)
If $M$ is a 2-torsion free unital $M_n(R)$-bimodule and $D : M_n(R) \to M$ is a Jordan derivation, then $D$ satisfies (*) and $D(1) = 0$, and hence $D$ is a derivation by Theorem 2.1. So from Lemma 3.1 we have the following theorem which is a generalization of [4, Theorem 3.1].

**Theorem 3.2.** Let $M$ be a 2-torsion free $M_n(R)$-bimodule and $D : M_n(R) \to M$ be an additive mapping.

(i) If $D$ is a Jordan derivation, then $D$ is a derivation.

(ii) If $D$ is a generalized Jordan derivation, then $D$ is a generalized derivation.

By Corollary 2.3, the following corollary is obvious.

**Corollary 3.3.** Let $R$ be a unital ring and let $N$ be a 2-torsion free unital $R$-bimodule. Let $D : M_n(R) \to M_n(N)$ be an additive mapping.

(i) If $D$ is a Jordan derivation, then there is an inner derivation $I_g : M_n(R) \to M_n(N)$ and a derivation $d : R \to N$ such that $D(a) = d(a) + I_g(a)$ for all $a \in M_n(R)$.

(ii) If $D$ is a generalized Jordan derivation, then there is an inner derivation $I_g : M_n(R) \to M_n(N)$ and a derivation $d : R \to N$ such that $D(a) = d(a) + I_g(a) + aD(1)$ for all $a \in M_n(R)$.

4. Some related results

In this section, by applying results in section 2, we obtain some results about matrix ring $M_n(R)$.

**Lemma 4.1.** Let $A$ be a 2-torsion free unital ring. Suppose that each additive mapping $D : A \to A$ satisfying (*) is a generalized derivation with $D(1) \in Z(A)$. Let $\varphi : A \to A$ be an additive map satisfying

$$\varphi(ab + ba) = a\varphi(b) + \varphi(b)a \quad (a, b \in A).$$

Then $\varphi(a) = a\varphi(1)$ for all $a \in A$, where $\varphi(1) \in Z(A)$.

**Proof.** Let $a, b \in A$ with $ab = ba = 0$. So $ab + ba = 0$ and hence

$$\varphi(ab + ba) = a\varphi(b) + \varphi(b)a = 0,$$

$$\varphi(ba + ab) = b\varphi(a) + \varphi(a)b = 0.$$

Therefore, $a\varphi(b) + \varphi(b)a + b\varphi(a) + \varphi(a)b = 0$ and $\varphi$ satisfies (*). Thus by hypothesis $\varphi$ is a generalized derivation with $\varphi(1) \in Z(A)$. So we have

$$\varphi(ab) = a\varphi(b) + \varphi(a)b - a\varphi(1)b$$

and

$$\varphi(ba) = b\varphi(a) + \varphi(b)a - b\varphi(1)a.$$
Let \(\phi : \mathcal{R} \to M_n(\mathcal{R})\) be an additive map satisfying
\[
\phi(ab + ba) = a\phi(b) + \phi(b)a \quad (a, b \in M_n(\mathcal{R})).
\]
Then \(\phi(a) = a\phi(1)\) for all \(a \in M_n(\mathcal{R})\), where \(\phi(1) \in Z(M_n(\mathcal{R}))\).

Given a ring \(\mathcal{A}\) and an \(\mathcal{A}\)-bimodule \(\mathcal{M}\), the **trivial extension** of \(\mathcal{A}\) by \(\mathcal{M}\) is the ring \(\mathcal{T}(\mathcal{A}, \mathcal{M}) = \mathcal{A} \oplus \mathcal{M}\) with the usual addition and the following multiplication:
\[
(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 + m_1a_2).
\]

**Lemma 4.3.** Let \(\mathcal{A}\) be a 2-torsion free unital ring. Suppose that each additive mapping \(D : \mathcal{A} \to \mathcal{A}\) satisfying \((*)\) is a generalized derivation with \(D(1) \in Z(\mathcal{A})\). Let \(\mathcal{T}(\mathcal{A}, \mathcal{A})\) be the trivial extension of \(\mathcal{A}\) by \(\mathcal{A}\). Then every Jordan derivation from \(\mathcal{T}(\mathcal{A}, \mathcal{A})\) into itself is a derivation.

**Proof.** Let \(\mathcal{T} = \mathcal{T}(\mathcal{A}, \mathcal{A})\) and \(\Delta : \mathcal{T} \to \mathcal{T}\) be a Jordan derivation. We have \(\Delta((a, b)) = (\delta_1(a) + \delta_2(b), \delta_3(a) + \delta_4(b))\) for each \(a, b \in \mathcal{A}\), where \(\delta_k : \mathcal{A} \to \mathcal{A}\) \((k = 1, 2)\) are additive maps. Applying \(\Delta\) to the equation \((ab + ba, 0) = (a, 0)(b, 0) + (b, 0)(a, 0)\) \((a, b \in \mathcal{A})\), we deduce that \(\delta_1, \delta_3\) are Jordan derivations. Hence \(\delta_1\) and \(\delta_3\) satisfy \((*)\) with \(\delta_1(1) = \delta_3(1) = 0\). So by hypothesis \(\delta_1\) and \(\delta_3\) are derivations.

Now by applying \(\Delta\) to
\[
(0, a)(0, b) + (0, b)(0, a) = (0, 0) \quad \text{and} \quad (a, 0)(0, b) + (0, b)(a, 0) = (0, ab + ba)
\]
for each \(a, b \in \mathcal{A}\), we get
\[
\delta_2(a)b + a\delta_2(b) + \delta_2(b)a + b\delta_2(a) = 0,
\]
and
\[
\delta_3(ab + ba) = a\delta_3(b) + \delta_3(b)a,
\]
\[
\delta_4(ab + ba) = \delta_1(a)b + a\delta_4(b) + b\delta_1(a) + \delta_4(b)a
\]
for all \(a, b \in \mathcal{A}\). By Equation\(4.2\), hypothesis and Lemma\(4.1\), we get \(\delta_2(a) = a\delta_2(1)\), for all \(a \in \mathcal{A}\), where \(\delta_2(1) \in Z(\mathcal{A})\). Now taking \(b = 1\) in Equation\(4.1\), it follows that \(\delta_2(a) = -a\delta_2(1)\), for each \(a \in \mathcal{A}\). So \(\delta_2(a) = 0\) for all \(a \in \mathcal{A}\). Define \(\phi : \mathcal{A} \to \mathcal{A}\) by \(\phi = \delta_1 - \delta_1\). Then by Equation\(4.2\), we get \(\phi(ab + ba) = a\phi(b) + \phi(b)a\) for all \(a, b \in \mathcal{A}\). Hence by Lemma\(4.1\) it follows that \(\phi(a) = a\phi(1)\) for all \(a \in \mathcal{A}\), where \(\phi(1) = \delta_1(1) \in Z(\mathcal{A})\) (since \(\delta_1\) is a derivation, \(\delta_1(1) = 0\)).
Thus $\delta_4(a) = \delta_1(a) + a\delta_4(1)$ for all $a \in \mathcal{A}$, where $\delta_4(1) \in Z(\mathcal{A})$. By this results it is obvious that $\Delta$ is a derivation. □

From Theorem 2.1 and Lemma 1.2 we get the following result.

**Theorem 4.4.** Let $R$ be a 2-torsion free unital ring. Then every Jordan derivation from $T(M_n(R), M_n(R))$ into itself is a derivation.

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