CONDITIONAL GLOBAL REGULARITY OF SCHRÖDINGER MAPS: SUBTHRESHOLD DISPERSED ENERGY

PAUL SMITH

ABSTRACT. We consider the Schrödinger map initial value problem

\[ \begin{align*}
\partial_t \varphi &= \varphi \times \Delta \varphi \\
\varphi(x, 0) &= \varphi_0(x),
\end{align*} \]

with \( \varphi_0 : \mathbb{R}^2 \to \mathbb{S}^2 \hookrightarrow \mathbb{R}^3 \) a smooth \( H^\infty \) map from the Euclidean space \( \mathbb{R}^2 \) to the sphere \( \mathbb{S}^2 \) with subthreshold (\(< 4\pi\)) energy. Assuming an a priori \( L^4 \) boundedness condition on the solution \( \varphi \), we prove that the Schrödinger map system admits a unique global smooth solution \( \varphi \in C(\mathbb{R} \to H^\infty) \) provided that the initial data \( \varphi_0 \) is sufficiently energy-dispersed, i.e., sufficiently small in the critical Besov space \( \dot{B}^{1/2}_{4, \infty} \). Also shown are global-in-time bounds on certain Sobolev norms of \( \varphi \). Toward these ends we establish improved local smoothing and bilinear Strichartz estimates, adapting the Planchon-Vega approach to such estimates to the nonlinear setting of Schrödinger maps.

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1. INTRODUCTION

We consider the Schrödinger map initial value problem

\[ \begin{align*}
\partial_t \varphi &= \varphi \times \Delta \varphi \\
\varphi(x, 0) &= \varphi_0(x),
\end{align*} \]

with \( \varphi_0 : \mathbb{R}^d \to \mathbb{S}^2 \hookrightarrow \mathbb{R}^3 \).

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The system (1.1) enjoys conservation of energy

\[ E(\varphi(t)) := \frac{1}{2} \int_{\mathbb{R}^d} |\partial_x \varphi(t)|^2 dx \]  

and mass

\[ M(\varphi(t)) := \int_{\mathbb{R}^d} |\varphi(t) - Q|^2 dx, \]

where \( Q \in S^2 \) is some fixed base point. When \( d = 2 \), both (1.1) and (1.2) are invariant with respect to the scaling

\[ \varphi(x,t) \rightarrow \varphi(\lambda x, \lambda^2 t), \quad \lambda > 0, \]  

in which case we call (1.1) energy-critical. In this article we restrict ourselves to the energy-critical setting.

For the physical significance of (1.1), see [9, 38, 39, 32]. The system also arises naturally from the (scalar-valued) free linear Schrödinger equation

\[ (\partial_t + i \Delta)u = 0 \]

by replacing the target manifold \( \mathbb{C} \) with the sphere \( S^2 \hookrightarrow \mathbb{R}^3 \), which then requires replacing \( \Delta u \) with \( (u^* \nabla)_j \partial_j u = \Delta u - \perp (\Delta u) \) and \( i \) with the complex structure \( u \times \cdot \). Here \( \perp \) denotes orthogonal projection onto the normal bundle, which, for a given point \( (x,t) \), is spanned by \( u(x,t) \). For more general analogues of (1.1), e.g., for Kähler targets other than \( S^2 \), see [12, 34, 37]. See also [25, 26, 3] for connections with other spin systems. The local theory for Schrödinger maps is developed in [50, 9, 12, 34]. For global results in the \( d = 1 \) setting, see [9, 42]. For \( d \geq 3 \), see [6, 7, 2, 20, 22, 4]. Concerning the related modified Schrödinger map system, see [23, 24, 37].

The small-energy (take \( d = 2 \)) theory for (1.1) is now well-understood: building upon previous work (e.g., see below or [4, §1] for a brief history), global wellposedness and global-in-time bounds on certain Sobolev norms are shown in [4] given initial data with sufficiently small energy. The high-energy theory, however, is still very much in development. One of the main goals is to establish what is known as the threshold conjecture, which asserts that global wellposedness holds for (1.1) given initial data with energy below a certain energy threshold, and that finite-time blowup is possible for certain initial data with energy above this threshold. The threshold is directly tied to the nontrivial stationary solutions of (1.1), i.e., maps \( \phi \) into \( S^2 \) that satisfy

\[ \phi \times \Delta \phi = 0 \]

and that do not send all of \( \mathbb{R}^2 \) to a single point of \( S^2 \). Therefore we identify such stationary solutions with nontrivial harmonic maps \( \mathbb{R}^2 \rightarrow S^2 \), which we refer to as solitons for (1.1). It turns out that there exist no nontrivial harmonic maps into the sphere \( S^2 \) with energy less than \( 4\pi \), and that the harmonic map given by the inverse of stereographic projection has energy precisely equal to \( 4\pi =: E_{\text{crit}} \). We therefore refer to the range of energies \([0, E_{\text{crit}}]\) as subthreshold, and call \( E_{\text{crit}} \) the critical or threshold energy.
Recently, an analogous threshold conjecture was established for wave maps (see [31, 43, 47, 48] and, for hyperbolic space, [30, 55, 56, 57, 58, 59]). When $\mathcal{M}$ is a hyperbolic space, or, as in [47, 48], a generic compact manifold, we may define the associated energy threshold $E_{\text{crit}} = E_{\text{crit}}(\mathcal{M})$ as follows. Given a target manifold $\mathcal{M}$, consider the collection $\mathcal{S}$ of all non-constant finite-energy harmonic maps $\phi : \mathbb{R}^2 \to \mathcal{M}$. If this set is empty, as is, for instance, the case when $\mathcal{M}$ is equal to a hyperbolic space $H^m$, then we formally set $E_{\text{crit}} = +\infty$. If $\mathcal{S}$ is nonempty, then it turns out that the set $\{E(\phi) : \phi \in \mathcal{S}\}$ has a least element and that, moreover, this energy value is positive. In such case we call this least energy $E_{\text{crit}}$. The threshold $E_{\text{crit}}$ depends upon geometric and topological properties of the target manifold $\mathcal{M}$; see [33, Chapter 6] for further discussion. This definition yields $E_{\text{crit}} = 4\pi$ in the case of the sphere $S^2$. For further discussion of the critical energy level in the wave maps setting, see [48, 55].

We now summarize what is known for Schrödinger maps in $d = 2$. Some of these developments postdate the submission of this article. Asymptotic stability of harmonic maps of topological degree $|m| \geq 4$ under the Schrödinger flow is established in [17]. The result is extended to maps of degree $|m| \geq 3$ in [18]. A certain energy-class instability for degree-1 solitons of (1.1) is shown in [5], where it is also shown that global solutions always exist for small localized equivariant perturbations of degree-1 solitons. Finite-time blowup for (1.1) is demonstrated in [36, 35], using less-localized equivariant perturbations of degree-1 solitons, thus resolving the blowup assertion of the threshold conjecture. Global wellposedness given data with small critical Sobolev norm (in all dimensions $d \geq 2$) is shown in [4]. Recent work of the author [45] extends the result of [4] and the present conditional result to global regularity (in $d = 2$) assuming small critical Besov norm $\dot{B}^1_{2,\infty}$. In the radial setting (which excludes harmonic maps), [16] establishes global wellposedness at any energy level. Most recently, [1] establishes global existence and uniqueness as well as scattering given 1-equivariant data with energy less than $4\pi$. Although stating the results only for data with energy less than $4\pi$, [1] also notes that their proofs remain valid for maps with energy slightly larger than $4\pi$, suggesting that the “right” threshold conjecture for Schrödinger maps should be stated also in terms of homotopy class, leading to a threshold of $8\pi$ rather than $4\pi$ in the case where the target is $S^2$. See the Introduction of [1] for further discussion of this point.

The main purpose of this paper is to show that (1.1) admits a unique smooth global solution $\varphi$ given smooth initial data $\varphi_0$ satisfying appropriate energy conditions and assuming a priori boundedness of a certain $L^4$ spacetime norm of the spatial gradient of the solution $\varphi$. In particular, we admit a restricted class of initial data with energy ranging over the entire subthreshold range. As such, our main result is a small step toward a large data regularity theory for (1.1) and the attendant threshold conjecture.
In order to go beyond the small-energy results of [4], we introduce physical-space proofs of local smoothing and bilinear Strichartz estimates, in the spirit of [31, 40, 60], that do not heavily depend upon perturbative methods. The local smoothing estimate that we establish is a nonlinear analogue of that shown in [20]. The bilinear Strichartz estimate is a nonlinear analogue of the improved bilinear Strichartz estimate of [5]. These proofs more naturally account for magnetic nonlinearities, and the technique developed here we believe to be of independent interest and applicable to other settings. For local smoothing in the context of Schrödinger equations, see [27, 28, 29, 19, 20, 22]. For other Strichartz and smoothing results for magnetic Schrödinger equations, see [46, 10, 11, 12, 13, 14, 15] and the references therein. We also use in a fundamental way the subthreshold \textit{caloric gauge} of [44], which is an extension of a construction introduced in [53].

To make these statements more precise, we now turn to some basic definitions and elementary observations.

1.1. Preliminaries. First we establish some basic notation. The boldfaced letters $\mathbb{Z}$ and $\mathbb{R}$ respectively denote the integers and real numbers. We use $\mathbb{Z}_{+} = \{0, 1, 2, \ldots\}$ to denote the nonnegative integers. Usual Lebesgue function spaces are denoted by $L^{p}$, and these sometimes include a subscript to indicate the variable or variables of integration. When function spaces are iterated, e.g., $L^{\infty}_{t}L^{2}_{x}$, the norms are applied starting with the rightmost one. When we use $L^{4}$ without subscripts, we mean $L^{4}_{t,x}$.

We use $S^{2}$ to denote the standard 2-sphere embedded in 3-dimensional Euclidean space: $\{x \in \mathbb{R}^{3} : |x| = 1\}$. The ambient space $\mathbb{R}^{3}$ carries the usual metric and $S^{2}$ the inherited one. Throughout, $S^{1}$ denotes the unit circle.

We use $\partial_{x} = (\partial_{x_{1}}, \partial_{x_{2}}) = (\partial_{1}, \partial_{2})$ to denote the gradient operator, as throughout “$\nabla$” will stand for the Riemannian connection on $S^{2}$. As usual, “$\Delta$” denotes the (flat) spatial Laplacian.

The symbol $|\partial_{x}|^{\sigma}$ denotes the Fourier multiplier with symbol $|\xi|^{\sigma}$. We also use standard Littlewood-Paley Fourier multipliers $P_{k}$ and $P_{\leq k}$, respectively denoting restrictions to frequencies $\sim 2^{k}$ and $\lesssim 2^{k}$; see section §3 for precise definitions. We use $\hat{f}$ to denote the Fourier transform of a function $f$ in the spatial variables.

We also employ without further comment (finite-dimensional) vector-valued analogues of the above.

We use $f \lesssim g$ to denote the estimate $|f| \leq C|g|$ for an absolute constant $C > 0$. As usual, the constant is allowed to change line-to-line. To indicate dependence of the implicit constant upon parameters (which, for instance, can include functions), we use subscripts, e.g. $f \lesssim_{k} g$. As an equivalent alternative we write $f = O(g)$ (or, with subscripts, $f = O_{k}(g)$, for instance)
to denote $|f| \leq C|g|$. If both $f \lesssim g$ and $g \lesssim f$, then we indicate this by writing $f \sim g$.

Now we introduce the notion of Sobolev spaces of functions mapping from Euclidean space into $S^2$. The spaces are constructed with respect to a choice of base point $Q \in S^2$, the purpose of which is to define a notion of decay: instead of decaying to zero at infinity, our Sobolev class functions decay to $Q$.

For $\sigma \in [0, \infty)$, let $H^\sigma = H^\sigma(\mathbb{R}^2)$ denote the usual Sobolev space of complex-valued functions on $\mathbb{R}^2$. For any $Q \in S^2$, set

$$H^\sigma_Q := \{ f : \mathbb{R}^2 \to \mathbb{R}^3 \text{ such that } |f(x)| \equiv 1 \text{ a.e. and } f - Q \in H^\sigma \}.$$  

This is a metric space with induced distance $d^\sigma_Q(f, g) = \|f - g\|_{H^\sigma}$. For $f \in H^\sigma_Q$ we set $\|f\|_{H^\sigma_Q} = d^\sigma_Q(f, Q)$ for short. We also define the spaces

$$H^\infty := \bigcap_{\sigma \in \mathbb{Z}^+} H^\sigma \quad \text{and} \quad H^\infty_Q := \bigcap_{\sigma \in \mathbb{Z}^+} H^\sigma_Q.$$  

For any time $T \in (0, \infty)$, the above definitions may be extended to the spacetime slab $\mathbb{R}^2 \times (-T, T)$ (or $\mathbb{R}^2 \times [-T, T]$). For any $\sigma, \rho \in \mathbb{Z}^+$, let $H^{\sigma, \rho}(T)$ denote the Sobolev space of complex-valued functions on $\mathbb{R}^2 \times (-T, T)$ with the norm

$$\|f\|_{H^{\sigma, \rho}(T)} := \sup_{t \in (-T, T)} \sum_{\rho' = 0}^{\rho} \|\partial_t^{\rho'} f(\cdot, t)\|_{H^\sigma}.$$  

and for $Q \in S^2$ endow

$$H^{\sigma, \rho}_Q := \{ f : \mathbb{R}^2 \times (-T, T) \to \mathbb{R}^3 \text{ such that } |f(x, t)| \equiv 1 \text{ a.e. and } f - Q \in H^{\sigma, \rho}(T) \}$$  

with the metric induced by the $H^{\sigma, \rho}(T)$ norm. Also, define the spaces

$$H^{\infty, \infty}(T) = \bigcap_{\sigma, \rho \in \mathbb{Z}^+} H^{\sigma, \rho}(T) \quad \text{and} \quad H^{\infty, \infty}_Q(T) = \bigcap_{\sigma, \rho \in \mathbb{Z}^+} H^{\sigma, \rho}_Q(T).$$  

For $f \in H^\infty$ and $\sigma \geq 0$ we define the homogeneous Sobolev norms as

$$\|f\|_{H^\sigma} = \|\hat{f}(\xi) \cdot |\xi|^\sigma\|_{L^2}.$$  

We mention two important conservation laws obeyed by solutions of the Schrödinger map system \(\square\). In particular, if $\varphi \in C((T_1, T_2) \to H^\infty_Q)$ solves \(\square\) on a time interval $(T_1, T_2)$, then both

$$\int_{\mathbb{R}^2} |\varphi(t) - Q|^2 \, dx \quad \text{and} \quad \int_{\mathbb{R}^2} |\partial_x \varphi(t)|^2 \, dx$$

are conserved. Hence the Sobolev norms $H^0_Q$ and $H^1_Q$ are conserved, as well as the energy \(\square\). Note also the time-reversibility obeyed by \(\square\), which in particular permits the smooth extension to $(-T, T)$ of a smooth solution on $[0, T)$.  

According to our conventions,
\[ |\partial_x \varphi(t)|^2 := \sum_{m=1,2} |\partial_m \varphi(t)|^2.\]

We can now give a precise statement of a key known local result.

Theorem 1.1 (Local existence and uniqueness). If the initial data \( \varphi_0 \) is such that \( \varphi_0 \in H^\infty_Q \) for some \( Q \in S^2 \), then there exists a time \( T = T(\|\varphi_0\|_{L^2_Q}) > 0 \) for which there exists a unique solution \( \varphi \in C([-T,T] \to H^\infty_Q) \) of the initial value problem (1.1). 

Proof. See [50, 9, 12, 34] and the references therein. \( \square \)

1.2. Global theory. Theorem 1.1 yields short-time existence and uniqueness as well as a blow-up criterion; as such it is central to the continuity arguments used for global results. In the small-energy setting, global regularity (and more) was proved for (1.1) by Bejenaru, Ionescu, Kenig, and Tataru [4]. We now state a special case of their main result, omitting for the sake of brevity the consideration of higher spatial dimensions and continuity of the solution map.

Theorem 1.2 (Global regularity). Let \( Q \in S^2 \). Then there exists an \( \varepsilon_0 > 0 \) such that, for any \( \varphi_0 \in H^\infty_Q \) with \( \|\partial_x \varphi_0\|_{L^2_x} \leq \varepsilon_0 \), there is a unique solution \( \varphi \in C(\mathbb{R} \to H^\infty_Q) \) of the initial value problem (1.1). Moreover, for any \( T \in [0, \infty) \) and \( \sigma \in \mathbb{Z}^+ \),
\[ \sup_{t \in (-T,T)} \|\varphi(t)\|_{H^\sigma_Q} \lesssim_{\sigma,T,\|\varphi_0\|_{H^\sigma_Q}} 1.\]

Also, given any \( \sigma_1 \in \mathbb{Z}_+ \), there exists a positive \( \varepsilon_1 = \varepsilon_1(\sigma_1) \leq \varepsilon_0 \) such that the uniform bounds
\[ \sup_{t \in \mathbb{R}} \|\varphi(t)\|_{H^\sigma_Q} \lesssim_{\sigma} \|\varphi_0\|_{H^\sigma_Q} \]
hold for all \( 1 \leq \sigma \leq \sigma_1 \), provided \( \|\partial_x \varphi_0\|_{L^2_x} \leq \varepsilon_1 \).

A complete proof may be found in [4]. Among the key contributions of their work are the construction of the main function spaces and the completion of the linear estimate relating them, which includes an important maximal function estimate. A significant observation of [4], emphasized in their work, is that it is important that these spaces take into account a local smoothing effect; [4] crucially uses this effect to help bring under control the worst term of the nonlinearity. Another novelty of [4] is its implementation of the caloric gauge, which was first introduced by Tao [53] and subsequently recommended by him for use in studying Schrödinger maps [51]. As the caloric gauge is defined using harmonic map heat flow, it can be thought
of as an intrinsic and nonlinear analogue of classical Littlewood-Paley theory. In [1], both the intrinsic caloric gauge and the extrinsic (and modern) Littlewood-Paley theory are used simultaneously.

Our main result extends Theorem 1.2.

**Theorem 1.3.** Let $T > 0$ and $Q \in S^2$. Let $\varepsilon_0 > 0$ and let $\varphi \in H_Q^{\infty, \infty}(T)$ be a solution of the Schrödinger map system (1.1) whose initial data $\varphi_0$ has energy $E_0 := E(\varphi_0) < E_{\text{crit}}$ and satisfies the energy dispersion condition

$$\sup_{k \in \mathbb{Z}} \| P_k \partial_x \varphi_0 \|_{L^2_y} \leq \varepsilon_0. \quad (1.4)$$

Let $I \supset (-T, T)$ denote the maximal time interval for which there exists a smooth (necessarily unique) extension of $\varphi$ satisfying (1.1). Suppose a priori that

$$\sum_{k \in \mathbb{Z}} \| P_k \partial_x \varphi \|_{L^4_t L^4_x(I \times \mathbb{R}^2)}^2 \leq \varepsilon_0^2. \quad (1.5)$$

Then, for $\varepsilon_0$ sufficiently small,

$$\sup_{t \in (-T, T)} \| \varphi(t) \|_{H_Q^\infty} \lesssim_{\sigma, T} \| \varphi_0 \|_{H_Q^\infty}, \quad (1.6)$$

for all $\sigma \in \mathbb{Z}_+$. Additionally, $I = \mathbb{R}$, so that, in particular, $\varphi$ admits a unique smooth global extension $\varphi \in C(\mathbb{R} \to H_Q^\infty)$. Moreover, for any $\sigma_1 \in \mathbb{Z}_+$, there exists a positive $\varepsilon_1 = \varepsilon_1(\sigma_1) \leq \varepsilon_0$ such that

$$\| \varphi \|_{L^\infty_t H^\sigma_Q(\mathbb{R} \times \mathbb{R}^2)} \leq \sigma \| \varphi \|_{H^\sigma_Q(\mathbb{R}^2)} \quad (1.7)$$

holds for all $0 \leq \sigma \leq \sigma_1$ provided (1.4) and (1.5) hold with $\varepsilon_1$ in place of $\varepsilon_0$.

Note that the energy dispersion condition (1.4) holds automatically in the case of small energy. In such case, our proofs may be modified (essentially by collapsing to or reverting to the arguments of [4]) so that the a priori $L^4$ bound is not required. Such an $L^4$ bound, however, can then be seen to hold a posteriori.

Using time divisibility of the $L^4$ norm, we can replace (1.5) with

$$\sum_{k \in \mathbb{Z}} \| P_k \partial_x \varphi \|_{L^4_t L^4_x(I \times \mathbb{R}^2)}^2 \leq K$$

for any $K > 0$ provided we allow the threshold for $\varepsilon_0$ and the implicit constant in (1.7) to depend upon $K > 0$. We work with (1.5) as stated so as to avoid the additional technicalities that would arise otherwise.

We now turn to a very rough sketch of the proof of Theorem 1.3 for a detailed outline, see [4].

**Basic setup and gauge selection.**
It suffices to prove homogeneous Sobolev variants of (1.6) and (1.7) over a suitable range. Thanks to mass and energy conservation, we need only consider $\sigma > 1$. For $\sigma \geq 1$, controlling $\|\varphi(t)\|_{H^\sigma_{loc}}$ is equivalent to controlling $\|\partial_x \varphi(t)\|_{L^\sigma_{loc}}$. We therefore consider the time evolution of $\partial_x \varphi$, which may be written entirely in terms of derivatives of the map $\varphi$. A more intrinsic way of expressing these equations is to select a \textit{gauge} rather than an extrinsic embedding and coordinate system. We employ the caloric gauge, which is geometrically natural and is analytically well-suited for studying Schrödinger maps. See [44] for the complete details of the construction. It turns out that Sobolev bounds for the gauged derivative map imply corresponding Sobolev bounds for the ungauged derivative map. We schematically write the gauged equation as

$$(\partial_t - \Delta) \psi = \mathcal{N},$$

where $\psi$ is $\partial_x \varphi$ placed in the caloric gauge and $\mathcal{N}$ is a nonlinearity constructed in part from $\psi$ and $\partial_x \psi$.

**Function spaces and their interrelation.**

To prove global results in the energy-critical setting, we of course must look for bounds other than energy estimates to control the solution. Local smoothing estimates and Strichartz estimates will be among the most important required. Our goal is to prove control over $\psi$ within a suitable space through the use of a bootstrap argument. A standard setup requires a space, say $G$, for the functions $\psi$ and a space, say $N$, for the nonlinearity $\mathcal{N}$. In fact, we work with stronger, frequency-localized spaces, $G_k$ and $N_k$, to respectively hold $P_k \psi$ and $P_k \mathcal{N}$. We want them to be related at least by the linear estimate

$$\|P_k \psi\|_{G_k} \lesssim \|P_k \psi(t = 0)\|_{L^2_x} + \|P_k \mathcal{N}\|_{N_k}.$$  

The hope, then, is to control $\|P_k \mathcal{N}\|_{N_k}$ in terms of $\|P_k \psi(t = 0)\|_{L^2_x}$ and $\varepsilon \|P_k \psi\|_{G_k}$ (with $\varepsilon$ small), so that, by proving (under a bootstrap hypothesis) a statement such as

$$\|P_k \psi\|_{G_k} \lesssim \|P_k \psi(t = 0)\|_{L^2_x} + \varepsilon \|P_k \psi\|_{G_k},$$

we may conclude

$$\|P_k \psi\|_{G_k} \lesssim \|P_k \psi(t = 0)\|_{L^2_x}.$$  

Once (1.8) is proved, showing (1.6) and (1.7) is reduced to the comparatively easy tasks of unwinding the gauging and frequency localization steps so as to conclude with a standard continuity argument.

**Controlling the nonlinearity.**

In this context, the main contribution of this paper lies in showing that we may conclude (1.8) without assuming small energy. The most difficult-to-control terms in the nonlinearity $P_k \mathcal{N}$ are those involving a derivative landing on high-frequency pieces of the derivative fields; we represent them schematically as $A_{hi} \partial \psi_{hi}$. Local smoothing estimates controlling the linear
evolution (introduced in [20, 22]) were successfully used in [4] to handle \(A_b \partial_x \psi_{hi}\). These are not strong enough to control \(A_b \partial_x \psi_{hi}\) in the sub-threshold energy setting. We instead pursue a more covariant approach, working directly with a certain covariant frequency-localized Schrödinger equation (see §5). Our approach is also physical-space based, in the vein of [41, 40, 60], and modular.

2. Gauge field equations

In §2.1 we pass to the derivative formulation of the Schrödinger map system (1.1). All of the main arguments of our subsequent analysis take place at this level. The derivative formulation is at once both overdetermined, reflecting geometric constraints, and underdetermined, exhibiting gauge invariance. §2.2 introduces the caloric gauge, which is the gauge we select and work with throughout. Both [51] and [4] give good explanations justifying the use of the caloric gauge in our setting as opposed to alternative gauges. The reader is referred to [41] for the requisite construction of the caloric gauge for maps with energy up to \(E_{crit}\). §2.3 deals with frequency localizing components of the caloric gauge. Proofs are postponed to §6 so that we can more quickly turn our attention to the gauged Schrödinger map system.

2.1. Derivative equations. We begin with some constructions valid for any smooth function \(\phi : \mathbb{R}^2 \times (-T,T) \to S^2\). For a more general and extensive introduction to the gauge formalism we now introduce, see [53].

Space and time derivatives of \(\phi\) are denoted by \(\partial_\alpha \phi(x,t)\), where \(\alpha = 1, 2, 3\) ranges over the spatial variables \(x_1, x_2\) and time \(t\) with \(\partial_3 = \partial_t\).

Select a (smooth) orthonormal frame \((v(x,t), w(x,t))\) for \(T_{\phi(x,t)}\mathbb{S}^2\), i.e., smooth functions \(v, w : \mathbb{R}^2 \times (-T,T) \to T_{\phi(x,t)}\mathbb{S}^2\) such that at each point \((x, t)\) in the domain the vectors \(v(x,t), w(x,t)\) form an orthonormal basis for \(T_{\phi(x,t)}\mathbb{S}^2\). As a matter of convention we assume that \(v\) and \(w\) are chosen so that \(v \times w = \phi\).

With respect to this chosen frame we then introduce the derivative fields \(\psi_\alpha\), setting

\[
\psi_\alpha := v \cdot \partial_\alpha \phi + iw \cdot \partial_\alpha \phi. \tag{2.1}
\]

Then \(\partial_\alpha \phi\) admits the representation

\[
\partial_\alpha \phi = v \text{ Re}(\psi_\alpha) + w \text{ Im}(\psi_\alpha) \tag{2.2}
\]

with respect to the frame \((v, w)\). The derivative fields can be thought of as arising from the following process: First, rewrite the vector \(\partial_\alpha \phi\) with respect to the orthonormal basis \((v, w)\); then, identify \(\mathbb{R}^2\) with the complex numbers \(\mathbb{C}\) according to \(v \leftrightarrow 1, w \leftrightarrow i\). Note that this identification respects the complex structure of the target manifold.
Through this identification the Riemannian connection on $S^2$ pulls back to a covariant derivative on $C$, which we denote by

$$D_\alpha := \partial_\alpha + iA_\alpha.$$  

The real-valued connection coefficients $A_\alpha$ are defined via

$$A_\alpha := w \cdot \partial_\alpha v,$$  

so that in particular

$$\partial_\alpha v = -\phi \text{ Re}(\psi_\alpha) + wA_\alpha$$  

$$\partial_\alpha w = -\phi \text{ Im}(\psi_\alpha) - vA_\alpha.$$  

Due to the fact that the Riemannian connection on $S^2$ is torsion-free, the derivative fields satisfy the relations

$$D_\beta \psi_\alpha = D_\alpha \psi_\beta.$$  

or equivalently,

$$\partial_\beta A_\alpha - \partial_\alpha A_\beta = \text{ Im}(\psi_\beta \overline{\psi_\alpha}) =: q_{\beta\alpha}.$$  

The curvature of the connection is therefore given by

$$[D_\beta, D_\alpha] := D_\beta D_\alpha - D_\alpha D_\beta = iq_{\beta\alpha}. (2.5)$$  

Assuming now that we are given a smooth solution $\varphi$ of the Schrödinger map system (1.1), we derive the equations satisfied by the derivative fields $\psi_\alpha$. The system (1.1) directly translates to

$$\psi_t = iD_\ell \psi_\ell$$  

because

$$\varphi \times \Delta \varphi = J(\varphi)(\varphi^* \nabla)_j \partial_j \varphi,$$

where $J(\varphi)$ denotes the complex structure $\varphi \times$ and $(\varphi^* \nabla)_j$ the pullback of the Levi-Civita connection $\nabla$ on the sphere.

Let us pause to note the following conventions regarding indices. Roman typeface letters are used to index spatial variables. Greek typeface letters are used to index the spatial variables along with time. Repeated lettered indices within the same subscript or occurring in juxtaposed terms indicate an implicit summation over the appropriate set of indices.

Using (2.4) and (2.5) in (2.6) yields

$$D_t \psi_m = iD_\ell D_\ell \psi_m + q_{\ell m} \psi_\ell,$$

which is equivalent to the nonlinear Schrödinger equation

$$(i\partial_t + \Delta) \psi_m = N_m,$$  

where the nonlinearity $N_m$ is defined by the formula

$$N_m := -iA_\ell \partial_\ell \psi_m - i\partial_\ell (A_\ell \psi_m) + (A_t + A_x^2) \psi_m - i\psi_\ell \text{ Im}(\overline{\psi_\ell} \psi_m).$$
We split this nonlinearity as a sum $N_m = B_m + V_m$ with $B_m$ and $V_m$ defined by

$$B_m := -i\partial_t (A_t \psi_m) - i A_t \partial_t \psi_m$$

and

$$V_m := (A_t + A^2_x)\psi_m - i\psi_t \text{Im}(\overline{\psi_t} \psi_m),$$

thus separating the essentially semilinear magnetic potential terms and the essentially semi-linear electric potential terms from each other.

We now state the gauge formulation of the differentiated Schrödinger map system:

$$\begin{align*}
D_t \psi_m &= i D_t D_t \psi_m + \text{Im}(\psi_t \psi_m) \psi_t \\
D_\alpha \psi_\beta &= D_\beta \psi_\alpha \\
\text{Im}(\psi_\alpha \psi_\beta) &= \partial_\alpha A_\beta - \partial_\beta A_\alpha.
\end{align*}$$

A solution $\psi_m$ to (2.10) cannot be determined uniquely without first choosing an orthonormal frame $(v, w)$. Changing a given choice of orthonormal frame induces a gauge transformation and may be represented as

$$\psi_m \to e^{i\theta} \psi_m, \quad A_m \to A_m + \partial_m \theta$$

in terms of the gauge components. The system (2.10) is invariant with respect to such gauge transformations.

The advantage of working with this gauge formalism rather than the Schrödinger map system or the derivative equations directly is that a carefully selected choice of gauge tames the nonlinearity. In particular, when the caloric gauge is employed, the nonlinearity in (2.7) is nearly perturbative.

2.2. Introduction to the caloric gauge. In this section we introduce the caloric gauge, which is the gauge we shall employ throughout the remainder of the paper. Gauges were first used to study (1.1) in [9]. We note here that the while the Coulomb gauge would seem an attractive choice, it turns out that this gauge is not well-suited to the study of Schrödinger maps in low dimension, as in low dimension parallel interactions of waves are more probable than in high dimension, resulting in unfavorable high x high $\to$ low cascades. See [51] and [4] for further discussion and a comparison of the Coulomb and caloric gauges. Also see [54, Chapter 6] for a discussion of various gauges that have been used in the study of wave maps.

The caloric gauge was introduced by Tao in [53] in the setting of wave maps into hyperbolic space. In a series of unpublished papers [55, 56, 57, 58, 59], Tao used this gauge in establishing global regularity of wave maps into hyperbolic space. In his unpublished note [51], Tao also suggested the caloric gauge as a suitable gauge for the study of Schrödinger maps. The caloric gauge was first used in the Schrödinger maps problem by Bejenaru, Ionescu, Kenig, and Tataru in [4] to establish global well-posedness in the setting of initial data with sufficiently small critical norm. We recommend [53, 56, 51, 4] for background on the caloric gauge and for helpful heuristics.
Theorem 2.1 (The caloric gauge). Let \( T \in (0, \infty) \), \( Q \in S^2 \), and let \( \phi(x,t) \in H^{\infty,\infty}_Q(T) \) be such that \( \sup_{t \in (-T,T)} E(\phi(t)) < E_{\text{crit}} \). Then there exists a unique smooth extension \( \phi(s,x,t) \in C([0,\infty) \to H^{\infty,\infty}_Q(T)) \) solving the covariant heat equation

\[
\partial_s \phi = \Delta \phi + \phi \cdot |\partial_x \phi|^2
\]  

and with \( \phi(0,x,t) = \phi(x,t) \). Moreover, for any given choice of a (constant) orthonormal basis \((v_\infty, w_\infty)\) of \( T \) of \( S^2 \), there exist smooth functions \( v, w : [0,\infty) \times \mathbb{R}^2 \times (-T,T) \to S^2 \) such that at each point \((s,x,t)\), the set \( \{v, w, \phi\} \) naturally forms an orthonormal basis for \( \mathbb{R}^3 \), the gauge condition

\[
w \cdot \partial_s v \equiv 0,
\]

is satisfied, and

\[
|\partial^\rho_s f(s)| \lesssim (s)^{-((|\rho|+1)/2}
\]

for each \( f \in \{\phi - Q, v - v_\infty, w - w_\infty\} \), multiindex \( \rho \), and \( s \geq 0 \).

Proof. This is a special case of the more general result [44, Theorem 7.6]. Whereas in [44] everything is stated in terms of the category of Schwartz functions, in fact this requirement may be relaxed to \( H^{\infty,\infty}_Q(T) \) without difficulty (at least in the case of compact target manifolds) since weighted decay in \( L^2\)-based Sobolev spaces is not used in any proofs.

In our application in this paper, \( E(\varphi(t)) \) is conserved. Therefore, here and elsewhere, we set \( E_0 := E(\varphi_0) \).

Having extended \( v, w \) along the heat flow, we may likewise extend \( A_x \) along the flow. We record here for reference a technical bound from [44] that proves useful.

Theorem 2.2. Assume the conditions of Theorem 2.1 are in force. Then the following bound holds:

\[
\|A_x(s)\|_{L^2_x(\mathbb{R}^2)} \lesssim_{E_0} 1.
\]

Proof. See [44, §§7, 7.1].

Corollary 2.3 (Energy bounds for the frame). Suppose that \( \varphi \) is a Schrödinger map with energy \( E_0 < E_{\text{crit}} \). Then it holds that

\[
\|\partial_x v\|_{L^\infty_x L^2_x} \lesssim_{E_0} 1.
\]

Proof. Because \( |v| \equiv 1 \), it holds that \( v \cdot \partial_m v \equiv 0 \). Therefore, with respect to the orthonormal frame \((v, w, \varphi)\), the vector \( \partial_m v \) admits the representation

\[
\partial_m v = A_m \cdot w - \text{Re}(\psi_m) \cdot \varphi.
\]

The bound (2.15) then follows from using \( |w| \equiv 1 \equiv |\varphi| 
, \|\psi_m\|_{L^2_x} \equiv \|\partial_m \varphi\|_{L^2_x} 
, 
energy conservation, and (2.14) all in (2.16).
Adopting the convention $\partial_0 = \partial_s$, and now and hereafter allowing all Greek indices to range over heat time, spatial variables, and time, we define for all $(s, x, t) \in [0, \infty) \times \mathbb{R}^2 \times (-T, T)$ the various gauge components

$$
\psi_\alpha := v \cdot \partial_\alpha \varphi + iw \cdot \partial_\alpha \varphi,
A_\alpha := w \cdot \partial_\alpha v,
D_\alpha := \partial_\alpha + A_\alpha,
q_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha.
$$

For $\alpha = 0, 1, 2, 3$ it holds that

$$
\partial_\alpha \varphi = v \text{Re}(\psi_\alpha) + w \text{Im}(\psi_\alpha).
$$

The parallel transport condition $w \cdot \partial_s v \equiv 0$ is equivalently expressed in terms of the connection coefficients as

$$
A_s \equiv 0. \tag{2.17}
$$

Expressed in terms of the gauge, the heat flow (2.11) lifts to

$$
\psi_s = D_\ell \psi_\ell. \tag{2.18}
$$

Using (2.4) and (2.5), we may rewrite the $D_m$ covariant derivative of (2.18) as

$$
\partial_s \psi_m = D_\ell D_\ell \psi_\ell + i\text{Im}(\psi_m \overline{\psi_\ell}) \psi_\ell, \tag{2.19}
$$

or equivalently

$$
(\partial_s - \Delta) \psi_m = iA_\ell \partial_\ell \psi_\ell + i\partial_\ell (A_\ell \psi_m) - A_\ell^2 \psi_m + i\psi_\ell \text{Im}(\overline{\psi_\ell} \psi_\ell). \tag{2.20}
$$

More generally, taking the $D_\alpha$ covariant derivative, we obtain

$$
(\partial_s - \Delta) \psi_\alpha = U_\alpha, \tag{2.21}
$$

where we set

$$
U_\alpha := iA_\ell \partial_\ell \psi_\alpha + i\partial_\ell (A_\ell \psi_\alpha) - A_\ell^2 \psi_\alpha + i\psi_\ell \text{Im}(\overline{\psi_\ell} \psi_\alpha), \tag{2.22}
$$

which admits the alternative representation

$$
U_\alpha = 2iA_\ell \partial_\ell \psi_\alpha + i(\partial_\ell A_\ell) \psi_\alpha - A_\ell^2 \psi_\alpha + i\psi_\ell \text{Im}(\overline{\psi_\ell} \psi_\alpha). \tag{2.23}
$$

From (2.5) and (2.17) it follows that

$$
\partial_s A_\alpha = \text{Im}(\psi_s \overline{\psi_\alpha}).
$$

Integrating back from $s = \infty$ (justified using (2.13)) yields

$$
A_\alpha(s) = -\int_s^\infty \text{Im}(\overline{\psi_\alpha} \psi_s)(s') ds'. \tag{2.24}
$$

At $s = 0$, $\varphi$ satisfies both (1.1) and (2.11), or equivalently, $\psi_t(s = 0) = i\psi_s(s = 0)$. While for $s > 0$ it continues to be the case that $\psi_s = D_\ell \psi_\ell$ by construction, we no longer necessarily have $\psi_t(s) = iD_\ell(s) \psi_\ell(s)$, i.e., $\varphi(s, x, t)$ is not necessarily a Schrödinger map at fixed $s > 0$. In the following lemma we derive an evolution equation for the commutator $\Psi = \psi_t - i\psi_s$. 

Lemma 2.4 (Flows do not commute). Set $\Psi := \psi_t - i\psi_s$. Then
\[
\partial_s \Psi = D_\xi D_\xi \Psi + i\text{Im}(\psi_t \bar{\psi}_t)\psi_t - \text{Im}(\psi_s \bar{\psi}_s)\psi_t \\
=D_\xi D_\xi \Psi + i\text{Im}(\Psi \bar{\Psi})\psi_t + i\text{Im}(i\psi_s \bar{\psi}_s)\psi_t - \text{Im}(\psi_s \bar{\psi}_t)\psi_t. \tag{2.24}
\]

Proof. We prove (2.24), since (2.25) is a trivial consequence of it.

Applying (2.19) and (2.20) to $\psi_s$ and $\psi_t$ and collapsing the covariant derivative terms yields
\[
\partial_s \psi_t = D_\xi D_\xi \psi_t + i\text{Im}(\psi_t \bar{\psi}_t)\psi_t \tag{2.26}
\]
\[
\partial_s \psi_s = D_\xi D_\xi \psi_s + i\text{Im}(\psi_s \bar{\psi}_s)\psi_t. \tag{2.27}
\]

Multiply (2.27) by $i$ to obtain the $s$-evolution of $i\psi_s$. Multiplication by $i$ commutes with $D_\xi$, but fails to do so with $\text{Im}(\cdot)$, and thus we obtain
\[
\partial_s i\psi_t = D_\xi i\psi_t - \text{Im}(\psi_s \bar{\psi}_t)\psi_t. \tag{2.28}
\]

Together (2.26) and (2.28) imply (2.24). \hfill $\square$

2.3. Frequency localization. Frequency localization plays an indispensable role in our analysis. In this subsection we establish some basic concepts and then state some basic results for the caloric gauge.

Our notation for a standard Littlewood-Paley frequency localization of a function $f$ to frequencies $\sim 2^k$ is $P_k f$ and to frequencies $\lesssim 2^k$ is $P_{\lesssim k} f$. The particular localization chosen is of course immaterial to our analysis, but for definiteness is specified in the next section and chosen for convenience to coincide with that in \cite{4}.

We shall frequently make use of the following standard Bernstein inequalities for $\mathbb{R}^2$ with $\sigma \geq 0$ and $1 \leq p \leq q \leq \infty$:
\[
||P_{\lesssim k}|\partial_x|^\sigma f||_{L^p_v(\mathbb{R}^2)} \lesssim_p \sigma \quad ||P_{\lesssim k} f||_{L^p_v(\mathbb{R}^2)} \\
||P_k|\partial_x|^\pm\sigma f||_{L^p_v(\mathbb{R}^2)} \lesssim_p \sigma \quad ||P_k f||_{L^p_v(\mathbb{R}^2)} \\
||P_{\lesssim k} f||_{L^{q_v}(\mathbb{R}^2)} \lesssim_{p,q} 2^{k(1/p-1/q)} ||P_{\lesssim k} f||_{L^p_v(\mathbb{R}^2)} \\
||P_k f||_{L^{q_v}(\mathbb{R}^2)} \lesssim_{p,q} 2^{k(1/p-1/q)} ||P_k f||_{L^p_v(\mathbb{R}^2)}.
\]

A particularly important notion for us is that of a frequency envelope, as it provides a way to rigorously manage the “frequency leakage” phenomenon and the frequency cascades produced by nonlinear interactions. We introduce a parameter $\delta$ in the definition; for the purposes of this paper $\delta = \frac{1}{40}$ suffices.

Definition 2.5 (Frequency envelopes). A positive sequence $\{a_k\}_{k \in \mathbb{Z}}$ is a frequency envelope if it belongs to $\ell^2$ and is slowly varying:
\[
a_k \leq a_j 2^{\delta|k-j|}, \quad j, k \in \mathbb{Z}. \tag{2.29}
\]
A frequency envelope \( \{a_k\}_{k \in \mathbb{Z}} \) is \( \varepsilon \)-energy dispersed if it satisfies the additional condition
\[
\sup_{k \in \mathbb{Z}} a_k \leq \varepsilon.
\]

Note in particular that frequency envelopes satisfy the following summation rules:
\[
\sum_{k' \leq k} 2^{pk'} a_{k'} \lesssim (p - \delta)^{-1} 2^{pk} a_k \quad p > \delta \tag{2.30}
\]
\[
\sum_{k' \geq k} 2^{pk'} a_{k'} \lesssim (p - \delta)^{-1} 2^{-pk} a_k \quad p > \delta. \tag{2.31}
\]

In practice we work with \( p \) bounded away from \( \delta \), e.g., \( p > 2\delta \) suffices, and iterate these inequalities only \( O(1) \) times. Therefore, in applications we drop the factors \( (p - \delta)^{-1} \) appearing in (2.30) and (2.31).

Finally, pick a positive integer \( \sigma_1 \) and hold it fixed throughout the remainder of this section. Results in this section hold for any such \( \sigma_1 \), though implicit constants are allowed to depend upon this choice.

Given initial data \( \varphi_0 \in H^\infty_Q \), define for all \( \sigma \geq 0 \) and \( k \in \mathbb{Z} \)
\[
c_k(\sigma) := \sup_{k' \in \mathbb{Z}} 2^{-|k-k'|} 2^{\sigma k'} \|P_{k'} \partial_x \varphi_0\|_{L^2_x}. \tag{2.32}
\]

Set \( c_k := c_k(0) \) for short. For \( \sigma \in [0, \sigma_1] \) it then holds that
\[
\|\partial_x \varphi_0\|_{H^2_x}^2 \sim \sum_{k \in \mathbb{Z}} c_k^2(\sigma) \quad \text{and} \quad \|P_k \partial_x \varphi_0\|_{L^2_x} \leq c_k(\sigma) 2^{-\sigma k}. \tag{2.33}
\]

Similarly, for \( \varphi \in H^\infty_{Q}(T) \), define for all \( \sigma \geq 0 \) and \( k \in \mathbb{Z} \)
\[
\gamma_k(\sigma) := \sup_{k' \in \mathbb{Z}} 2^{-|k-k'|} 2^{\sigma k'} \|P_{k'} \varphi\|_{L^\infty_T L^2_x}. \tag{2.34}
\]

Set \( \gamma_k := \gamma_k(1) \).

**Theorem 2.6** (Frequency-localized energy bounds for heat flow). Let \( f \in \{\varphi, v, w\} \). Then for \( \sigma \in [1, \sigma_1] \) the bound
\[
\|P_k f(s)\|_{L^\infty_T L^2_x} \lesssim 2^{-\sigma k} \gamma_k(\sigma)(1 + s 2^{2k})^{-20} \tag{2.35}
\]
holds and for any \( \sigma, \rho \in \mathbb{Z}_+ \) it holds that
\[
\sup_{k \in \mathbb{Z}} \sup_{s \in [0, \infty)} (1 + s)^{\sigma/2} 2^{\sigma k} \|P_k \partial_t^\rho f(s)\|_{L^\infty_T L^2_x} < \infty. \tag{2.36}
\]

**Corollary 2.7** (Frequency-localized energy bounds for the caloric gauge). For \( \sigma \in [0, \sigma_1 - 1] \), it holds that
\[
\|P_k \psi_x(s)\|_{L^\infty_T L^2_x} + \|P_k A_m(s)\|_{L^\infty_T L^2_x} \lesssim 2^k 2^{-\sigma k} \gamma_k(\sigma)(1 + s 2^{2k})^{-20}. \tag{2.37}
\]
Moreover, for any $\sigma \in \mathbb{Z}_+$,
\[
\sup_{k \in \mathbb{Z}} \sup_{s \in [0, \infty)} (1 + s)^{\sigma/2} 2^{\sigma k} 2^{-k} \left( \| P_k (\partial_t^\sigma \psi_x (s)) \|_{L^\infty_t L^2_x} + \| P_k (\partial_t^\sigma A_x (s)) \|_{L^\infty_t L^2_x} \right) < \infty
\]  
(2.38)
and
\[
\sup_{k \in \mathbb{Z}} \sup_{s \in [0, \infty)} (1 + s)^{\sigma/2} 2^{\sigma k} \left( \| P_k (\partial_t^\sigma \psi_t (s)) \|_{L^\infty_t L^2_x} + \| P_k (\partial_t^\sigma A_t (s)) \|_{L^\infty_t L^2_x} \right) < \infty.
\]  
(2.39)

We prove Theorem 2.6 and its corollary in §6.

Note that Corollary 2.7 has as an elementary consequence the following:

**Corollary 2.8.** For $\sigma \in [0, \sigma_1 - 1]$, it holds that
\[
\| P_k \psi_x (0, \cdot) \|_{L^2_x} \lesssim 2^{-\sigma k} c_k (\sigma).
\]  
(2.40)

3. Function spaces and basic estimates

3.1. Definitions.

**Definition 3.1** (Littlewood-Paley multipliers). Let $\eta_0 : \mathbb{R} \to [0, 1]$ be a smooth even function vanishing outside the interval $[-8/5, 8/5]$ and equal to 1 on $[-5/4, 5/4]$. For $j \in \mathbb{Z}$, set
\[
\chi_j (\cdot) = \eta_0 (\cdot / 2^j) - \eta_0 (\cdot / 2^{j - 1}), \quad \chi_{\leq j} (\cdot) = \eta_0 (\cdot / 2^j).
\]

Let $P_k$ denote the operator on $L^\infty (\mathbb{R}^2)$ defined by the Fourier multiplier $\xi \to \chi_k (|\xi|)$. For any interval $I \subset \mathbb{R}$, let $\chi_I$ be the Fourier multiplier defined by $\chi_I = \sum_{j \in I \cap \mathbb{Z}} \chi_j$ and let $P_I$ denote its corresponding operator on $L^\infty (\mathbb{R}^2)$. We shall denote $P_{(-\infty, k]}$ by $P_{\leq k}$ for short. For $\theta \in \mathbb{S}^1$ and $k \in \mathbb{Z}$, we define the operators $P_{k, \theta}$ by the Fourier multipliers $\xi \to \chi_k (\xi \cdot \theta)$.

Some frequency interactions in the nonlinearity of (2.7) can be controlled using the following Strichartz estimate:

**Lemma 3.2** (Strichartz estimate). Let $f \in L^2_x (\mathbb{R}^2)$ and $k \in \mathbb{Z}$. Then the Strichartz estimate
\[
\| e^{it \Delta} f \|_{L^4_t L^4_x} \lesssim \| f \|_{L^2_x}
\]
holds, as does the maximal function bound
\[
\| e^{it \Delta} P_k f \|_{L^\infty_t L^2_x} \lesssim 2^{k/2} \| f \|_{L^2_x}.
\]
The first bound is the original Strichartz estimate (see [49]) and the second follows from scaling. These will be augmented with certain lateral Strichartz estimates to be introduced shortly. Strichartz estimates alone are not sufficient for controlling the nonlinearity in (2.7). The additional control required comes from local smoothing and maximal function estimates. Certain local smoothing spaces localized to cubes were introduced in [27] to study the local wellposedness of Schrödinger equations with general derivative nonlinearities. Stronger spaces were introduced in [21] to prove a low-regularity global result. In the Schrödinger map setting, local smoothing spaces were first used in [20] and subsequently in [22, 2, 6]. The particular local smoothing/maximal function spaces we shall use were introduced in [4].

For a unit length \( \theta \in S^1 \), we denote by \( H_\theta \) its orthogonal complement in \( \mathbb{R}^2 \) with the induced measure. Define the lateral spaces \( L^{p,q}_\theta \) as those consisting of all measurable \( f \) for which the norm
\[
\| h \|_{L^{p,q}_\theta} = \left( \int_{\mathbb{R}} \left( \int_{H_\theta \times \mathbb{R}} |h(x_1 \theta + x_2, t)|^q dx_2 dt \right)^{p/q} dx_1 \right)^{1/p},
\]
is finite. We make the usual modifications when \( p = \infty \) or \( q = \infty \). The most important spaces for our analysis are the local smoothing space \( L^{\infty,2}_\theta \) and the inhomogeneous local smoothing space \( L^{1,2}_\theta \). To move between these spaces we use the maximal function space \( L^{2,\infty}_\theta \).

The following two estimates were shown in [20] and [22]:

**Lemma 3.3 (Local smoothing).** Let \( f \in L^2_\theta(\mathbb{R}^2), k \in \mathbb{Z}, \) and \( \theta \in S^1 \). Then
\[
\| e^{it\Delta} P_k f \|_{L^{\infty,2}_\theta} \lesssim 2^{-k/2} \| f \|_{L^2_\theta}.
\]

For \( f \in L^2_\theta(\mathbb{R}^d) \), the maximal function space bound
\[
\| e^{it\Delta} P_k f \|_{L^{2,\infty}_\theta} \lesssim 2^{k(d-1)/2} \| f \|_{L^2_\theta}
\]
holds for dimension \( d \geq 3 \).

In \( d = 2 \), the maximal function bound fails due to a logarithmic divergence. In order to overcome this, we exploit Galilean invariance as in [4] (the idea goes back to [61] in the setting of wave maps).

For \( p, q \in [1, \infty], \theta \in S^1, \lambda \in \mathbb{R} \), define \( L^{p,q}_{\theta,\lambda} \) using the norm
\[
\| h \|_{L^{p,q}_{\theta,\lambda}} = \| T_\lambda(h) \|_{L^{p,q}_\theta} = \left( \int_{\mathbb{R}} \left( \int_{H_\theta \times \mathbb{R}} |h((x_1 + t\lambda)\theta + x_2, t)|^q dx_2 dt \right)^{p/q} dx_1 \right)^{1/p},
\]
where \( T_w \) denotes the Galilean transformation
\[
T_w(f)(x, t) = e^{-ix \cdot w/2} e^{-it|w|^2/4} f(x + tw, t).
\]
With $W \subset \mathbb{R}$ finite we define the spaces $L^{p,q}_{\theta,W}$ by

$$ L^{p,q}_{\theta,W} = \sum_{\lambda \in W} L^{p,q}_{\theta,\lambda}, \quad \|f\|_{L^{p,q}_{\theta,W}} = \inf_{f=\sum_{\lambda \in W} f_{\lambda}} \sum_{\lambda \in W} \|f_{\lambda}\|_{L^{p,q}_{\theta,\lambda}}. $$

For $k \in \mathbb{Z}$, $K \in \mathbb{Z}_+$, set

$$ W_k := \{\lambda \in [-2^k, 2^k] : 2^k + 2K\lambda \in \mathbb{Z}\}. $$

In our application we shall work on a finite time interval $[-2^{2K}, 2^{2K}]$ in order to ensure that the $W_k$ are finite. This still suffices for proving global results so long as our effective bounds are proved with constants independent of $T,K$. As discussed in [4, §3], restricting $T$ to a finite time interval avoids introducing additional technicalities.

**Lemma 3.4** (Local smoothing/maximal function estimates). Let $f \in L^2_x(\mathbb{R}^2)$, $k \in \mathbb{Z}$, and $\theta \in S^1$. Then

$$ \|e^{it\Delta} P_k \theta f\|_{L^2_{\theta,\lambda}} \lesssim 2^{-k/2} \|f\|_{L^2_{\theta,\lambda}} \quad |\lambda| \leq 2^{k-40}, $$

and moreover, if $T \in (0,2^{2K}]$, then

$$ \|1_{[-T,T]}(t) e^{i t \Delta} P_k f\|_{L^2_{\theta,\lambda}} \lesssim 2^{k/2} \|f\|_{L^2_{\theta,\lambda}}. $$

**Proof.** The first bound follows from Lemma 3.3 via a Galilean boost. The second is more involved and proven in [4, §7].

**Lemma 3.5** (Lateral Strichartz estimates). Let $f \in L^2_x(\mathbb{R}^2)$, $k \in \mathbb{Z}$, and $\theta \in S^1$. Let $2 < p \leq \infty, 2 \leq q \leq \infty$ and $1/p + 1/q = 1/2$. Then

$$ \|e^{it\Delta} P_k f\|_{L^p_{\theta,\lambda}} \lesssim 2^{k(2/p-1/2)} \|f\|_{L^2_{\theta,\lambda}}, \quad p \geq q, $$

$$ \|e^{it\Delta} P_k f\|_{L^p_{\theta,\lambda}} \lesssim_p 2^{k(2/p-1/2)} \|f\|_{L^2_{\theta,\lambda}}, \quad p \leq q. $$

**Proof.** Informally speaking, these bounds follow from interpolating between the $L^4$ Strichartz estimate and the local smoothing/maximal function estimates of Lemma 3.4. See [4, Lemma 7.1] for the rigorous argument.

We now introduce the main function spaces. Let $T > 0$. For $k \in \mathbb{Z}$, let $I_k = \{\xi \in \mathbb{R}^2 : |\xi| \in [2^{k-1}, 2^{k+1}]\}$. Let

$$ L^2_k(T) := \{f \in L^2(\mathbb{R}^2 \times [-T,T]) : \text{supp} \hat{f}(\xi,t) \subset I_k \times [-T,T]\}. $$

For $f \in L^2(\mathbb{R}^2 \times [-T,T])$, let

$$ \|f\|_{F^k(T)} := \|f\|_{L^\infty_{k,T}L^2_{\theta,\lambda}} + \|f\|_{L^4_{k,T}L^\infty_{\theta,\lambda}} + 2^{-k/2} \|f\|_{L^4_{k,T}L^\infty_{\theta,\lambda}} + 2^{-k/6} \sup_{\theta \in S^1} \|f\|_{L^3_{k,\theta}} + 2^{-k/2} \sup_{\theta \in S^1} \|f\|_{L^2_{k,\theta,\lambda}}. $$
We then define, similarly to as in \[4\], $F_k(T)$, $G_k(T)$, $N_k(T)$ as the normed spaces of functions in $L^2_k(T)$ for which the corresponding norms are finite:

$$
\|f\|_{F_k(T)} := \inf_{J,m_1,\ldots,m_J \in \mathbb{Z}^+} \inf_{f = f_{m_1} + \cdots + f_{m_J}} \sum_{j=1}^{J} 2^{m_j} \|f_{m_j}\|_{F_k}^0
$$

$$
\|f\|_{G_k(T)} := \|f\|_{F_k}^0 + \sup_{|j-k| \leq 20} \sup_{\theta \in S^1} \|P_{j,\theta} f\|_{L^{5,3}}^\infty
$$

$$
\|f\|_{N_k(T)} := \inf_{f=f_1+\cdots+f_6} \|f_1\|_{L^{4/3}} + 2^{k/6} \|f_2\|_{L^{3/2,6/5}} + 2^{k/6} \|f_3\|_{L^{3/2,6/5}}
$$

$$
+ 2^{-k/2} \|f_4\|_{L^{6/5,3/2}} + 2^{-k/6} \|f_5\|_{L^{6,5/3,2}} + 2^{-k/2} \sup_{\theta \in S^1} \|f_6\|_{L_\theta^{1,2,1,40}}
$$

where $(\hat{\theta}_1, \hat{\theta}_2)$ denotes the canonical basis in $\mathbb{R}^2$.

There are a few minor differences between these spaces and those appearing in \[4\]. The space $F_k$ now includes the lateral Strichartz space $L^{3,6}_\theta$, whereas in \[4\], only $G_k$ was endowed with this norm. The net effect on the space $G_k$ is that it is left unchanged. The space $F_k$, however, now explicitly incorporates this particular lateral Strichartz structure. Note though, that for fixed $\theta \in S^1$, we have by enough applications of Young’s and Hölder’s inequalities that

$$
2^{-k/6} \|f\|_{L^{3,6}_\theta} = 2^{-k/6} \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^2} |f(x_1 \theta + x_2, t)|^6 dx_2 dt \right)^{1/2} dx_1 \right)^{1/3}
$$

$$
\lesssim 2^{-k/6} \left( \int_{\mathbb{R}} \|f\|_{L^{1,6}_\theta}^2 \|f\|_{L^{\infty}_\theta} dx_1 \right)^{1/3}
$$

$$
\lesssim 2^{-k/6} \left( \int_{\mathbb{R}} \|f\|_{L^{6,5/3,2}}^4 dx_1 \right)^{1/6} \left( \int_{\mathbb{R}} \|f\|_{L^{\infty}_\theta}^2 dx_1 \right)^{1/6}
$$

$$
\lesssim \|f\|_{L^{4/3}_\theta} \cdot 2^{-k/6} \|f\|_{L^{1,2,1,40}_\theta}^{1/3} \|f\|_{L^{6,5/3,2}_\theta}^{1/6}
$$

$$
\lesssim \|f\|_{L^{4/3}_\theta} \cdot 2^{-k/2} \|f\|_{L^{2,1,\infty}_\theta}^{1/3} \|f\|_{L^{6,5/3,2}_\theta}^{1/6}
$$

We also make one change to the $N_k$ space: We explicitly incorporate $L^{6/5,3/2}_\theta$.

Incorporating these extra lateral Strichartz spaces affords us greater flexibility in certain estimates: We can avoid having to use local smoothing/maximal function spaces if we are willing to give up some decay. This tradeoff pays off in \[3\], where as a consequence we can prove a stronger local smoothing estimate for a certain magnetic nonlinear Schrödinger equation in the one regime where this improvement is absolutely essential.

**Proposition 3.6 (Main linear estimate).** Assume $K \in \mathbb{Z}^+$, $T \in (0, 2^k]$ and $k \in \mathbb{Z}$. Then for each $u_0 \in L^2$ that is frequency-localized to $I_k$ and for
any $h \in N_k(T)$, the solution $u$ of
\[(i\partial_t + \Delta_x)u = h, \quad u(0) = u_0\]
satisfies
\[
\|u\|_{G_k(T)} \lesssim \|u(0)\|_{L_2^2} + \|h\|_{N_k(T)}.
\]

Proof. See \[4, Proposition 7.2\] for details. Our changes to the spaces necessitate only minor changes in their proof, as we must incorporate $L^{6/5, 3/2}_{\hat{\theta}_1}$ and $L^{6/5, 3/2}_{\hat{\theta}_2}$ into the space $N^0_k(T)$. □

The spaces $G_k(T)$ are used to hold projections $P_k\psi_m$ of the derivative fields $\psi_m$ satisfying (2.7). The main components of $G_k(T)$ are the local smoothing/maximal function spaces $L^{\infty, 2}_{\theta, \lambda}, L^{2, \infty}_{\theta, W^{k+40}}$, and the lateral Strichartz spaces. The local smoothing and maximal function space components play an essential role in recovering the derivative loss that is due to the magnetic nonlinearity.

The spaces $N_k(T)$ hold frequency projections of the nonlinearities in (2.7). Here the main spaces are the inhomogeneous local smoothing spaces $L^{1, 2}_{\theta, W_k-40}$ and the Strichartz spaces, both chosen to match those of $G_k(T)$.

The spaces $G_k(T)$ clearly embed in $F_k(T)$. Two key properties enjoyed only by the larger spaces $F_k(T)$ are
\[
\|f\|_{F_k(T)} \approx \|f\|_{F_{k+1}(T)},
\]
for $k \in \mathbb{Z}$ and $f \in F_k(T) \cap F_{k+1}(T)$, and
\[
\|P_k(uv)\|_{F_k(T)} \lesssim \|u\|_{F_{k'}(T)}\|v\|_{L_\infty^\infty}
\]
for $k, k' \in \mathbb{Z}$, $|k - k'| \leq 20$, $u \in F_{k'}(T)$, $v \in L^\infty(\mathbb{R}^2 \times [-T, T])$. Both of these properties follow readily from the definitions.

In order to bound the nonlinearity of (2.7) in $N_k(T)$, it is important to gain regularity from the parabolic heat-time smoothing effect. The desired frequency-localized bounds do not (or at least not so readily) propagate in heat-time in the spaces $G_k(T)$, whereas these bounds do propagate with decay in the larger spaces $F_k(T)$. Note that since the $F_k(T)$ norm is translation invariant, it holds that
\[
\|e^{s\Delta}h\|_{F_k(T)} \lesssim (1 + s2^{2k})^{-20}\|h\|_{F_k(T)} \quad s \geq 0,
\]
for $h \in F_k(T)$. In certain bilinear estimates we do not need the full strength of the spaces $F_k(T)$ and instead can use the bound
\[
\|f\|_{F_k(T)} \lesssim \|f\|_{L_2^2 L_\infty^\infty} + \|f\|_{L_4^4 L_\infty^\infty},
\] (3.1)
which follows from
\[ \|f\|_{L^2 \rightarrow \infty}^{k,\infty} \leq \|f\|_{L^2 \rightarrow \infty} \lesssim 2^{k/2} \|f\|_{L^2 \rightarrow L^\infty}. \]

We introduce one more class of function spaces. These can be viewed as a refinement of the Strichartz part of \( F_k(T) \). For \( k \in \mathbb{Z} \) and \( \omega \in [0, 1/2] \) we define \( S_k^\omega(T) \) to be the normed space of functions belonging to \( L^2(T) \) whose norm
\[ \|f\|_{S_k^\omega(T)} = 2^{\omega k} \left( \|f\|_{L^2} + \|f\|_{L^4 L^6} + 2^{-k/2} \|f\|_{L^\infty} \right) \]
is finite, where the exponents \( 2_\omega \) and \( p_\omega \) are determined by
\[ \frac{1}{2_\omega} - \frac{1}{2} = \frac{1}{p_\omega} - \frac{1}{4} = \frac{\omega}{2}. \]

Note that \( F_k(T) \hookrightarrow S_k^0(T) \) and that by Bernstein we have
\[ \|f\|_{S_k^\omega(T)} \lesssim \|f\|_{S_k^0(T)}, \quad \omega' \leq \omega. \]

3.2. Bilinear estimates.

**Lemma 3.7** (Bilinear estimates on \( N_k(T) \)). For \( k, k_1, k_3 \in \mathbb{Z} \), \( h \in L^2(T) \), \( f \in F_{k_1}(T) \), and \( g \in G_{k_3}(T) \), we have the following inequalities under the given restrictions on \( k, k_1 \),
\[ |k_1 - k| \leq 80 : \|P_k(hf)\|_{N_k(T)} \leq \|h\|_{L^2} \|f\|_{F_{k_1}(T)} \] \[ k_1 \leq k - 80 : \|P_k(hf)\|_{N_k(T)} \leq 2^{-|k-k_1|/6} \|h\|_{L^2} \|f\|_{F_{k_1}(T)} \] \[ k_1 \leq k_3 - 80 : \|P_k(hg)\|_{N_k(T)} \leq 2^{-|k-k_3|/6} \|h\|_{L^2} \|g\|_{G_{k_3}(T)}. \]

**Proof.** Estimate (3.3) follows from Hölder’s inequality and the definition of \( F_k(T), N_k(T) \):
\[ \|Ff\|_{L^{4/3}} \leq \|F\|_{L^2} \|f\|_{L^4}. \]
For (3.4) and (3.5), we use an angular partition of unity in frequency to write
\[ f = f_1 + f_2, \quad \|f_1\|_{L^{3.6}_{\delta_1}} + \|g_1\|_{L^{3.6}_{\delta_2}} \lesssim 2^{k_1/6} \|f\|_{F_{k}(T)} \] and
\[ g = g_1 + g_2, \quad \|g_1\|_{L^{6.3}_{\delta_1}} + \|g_1\|_{L^{6.3}_{\delta_2}} \lesssim 2^{-k_1/6} \|g\|_{G_{k}(T)}. \]
Then
\[ \|P_k(fg)\|_{N_k(T)} \lesssim 2^{-k/6} \left( \|Ff_1\|_{L^{6/5,3/2}_{\delta_1}} + \|Ff_2\|_{L^{6/5,3/2}_{\delta_2}} \right) \lesssim 2^{-k/6} \|F\|_{L^2} \left( \|f_1\|_{L^{3.6}_{\delta_1}} + \|f_1\|_{L^{3.6}_{\delta_2}} \right) \lesssim 2^{(k_1-k)/6} \|F\|_{L^2} \|f\|_{F_{k_1}(T)}. \]
Lemma 3.8 (Bilinear estimates on $L^2_{t,x}$). For $k_1, k_2, k_3 \in \mathbb{Z}$, $f_1 \in F_{k_1}(T)$, $f_2 \in F_{k_2}(T)$, and $g \in G_{k_3}(T)$, we have

$$
\|f_1 \cdot f_2\|_{L^2_{t,x}} \lesssim \|f_1\|_{F_{k_1}(T)} \|f_2\|_{F_{k_2}(T)} \quad (3.6)
$$

$$
k_1 \leq k_3 : \quad \|f \cdot g\|_{L^2_{t,x}} \lesssim 2^{-|k_1 - k_3|/6} \|f\|_{F_{k_1}(T)} \|g\|_{G_{k_3}(T)} \quad (3.7)
$$

Proof. It suffices to show

$$
\|fg\|_{L^2} \lesssim \|f\|_{F^0_{k_1}(T)} \|g\|_{G_{k_2}(T)}, \quad k_1 \geq k_2 - 100 \quad (3.8)
$$

and

$$
\|fg\|_{L^2} \lesssim 2^{(k_1 - k_2)/6} \|f\|_{F^0_{k_1}(T)} \|g\|_{G_{k_2}(T)}, \quad k_1 < k_2 - 100. \quad (3.9)
$$

Estimate (3.8) follows from estimating each factor in $L^4$. For (3.9), we first observe that, using a smooth partition of unity in frequency space, we may assume that $\hat{g}$ is supported in the set

$$
\big\{ \xi : |\xi| \in [2^{k_2-1}, 2^{k_2+1}] \text{ and } \xi \cdot \theta_0 \geq 2^{k_2-5} \big\}
$$

for some direction $\theta_0 \in \mathbb{S}^1$. Then

$$
\|fg\|_{L^2} \lesssim \|f\|_{L^4_{\theta_0}} \|g\|_{L^{6,3}_{\theta_0}} \lesssim 2^{(k_1 - k_2)/6} \|f\|_{F^0_{k_1}(T)} \|g\|_{G_{k_2}(T)}
$$

We also have the following stronger estimates, which rely upon the local smoothing and maximal function spaces.

Lemma 3.9 (Bilinear estimates using local smoothing/maximal function bounds). For $k, k_1, k_2 \in \mathbb{Z}$, $h \in L^2_{t,x}$, $f \in F_{k_1}(T)$, $g \in G_{k_2}(T)$, we have the following inequalities under the given restrictions on $k_1, k_2$.

$$
k_1 \leq k - 80 : \quad \|P_k(hf)\|_{N_{\theta_0}(T)} \lesssim 2^{-|k_1 - k_2|/2} \|h\|_{L^2_{\theta_0}} \|f\|_{F_{k_1}(T)} \quad (3.10)
$$

$$
k_1 \leq k_2 : \quad \|f \cdot g\|_{L^2_{t,x}} \lesssim 2^{-|k_1 - k_2|/2} \|f\|_{F_{k_1}(T)} \|g\|_{G_{k_2}(T)}. \quad (3.11)
$$
Proof. Estimate (3.10) follows from the definitions since
\[ \|P_k(hf)\|_{N_k(T)} \lesssim 2^{-k/2} \sup_{\theta \in \mathbb{S}^1} \|hf\|_{L^1_{t,x} |_{\theta \in \mathbb{S}^1, k \leq k-60}} \lesssim 2^{-k/2} \sup_{\theta \in \mathbb{S}^1} \|f\|_{L^2_{t,x} |_{\theta \in \mathbb{S}^1, k \leq k}} \|h\|_{L^2_{t,x}}. \]
The proof of (3.11) parallels that of (3.7) and is omitted (see [4, Lemma 6.5] for details.)

3.3. Trilinear estimates and summation. We combine the bilinear estimates to establish some trilinear estimates. As we do not control local smoothing norms along the heat flow, we will oftentimes be able to put only one term in a \(G_k\) space. Nonetheless, such estimates still exhibit good off-diagonal decay.

Define the sets \(Z_1(k), Z_2(k), Z_3(k) \subset \mathbb{Z}^3\) as follows:
\[ Z_1(k) := \{(k_1, k_2, k_3) \in \mathbb{Z}^3 : k_1, k_2 \leq k - 40 \text{ and } |k_3 - k| \leq 4\}, \]
\[ Z_2(k) := \{(k_1, k_2, k_3) \in \mathbb{Z}^3 : k, k_3 \leq k_1 - 40 \text{ and } |k_2 - k_1| \leq 45\}, \]
\[ Z_3(k) := \{(k_1, k_2, k_3) \in \mathbb{Z}^3 : k_3 \leq k \text{ and } |k - \max\{k_1, k_2\}| \leq 40 \]
or \(k_3 > k \text{ and } |k_3 - \max\{k_1, k_2\}| \leq 40\).\] (3.12)

In our main trilinear estimate, we avoid using local smoothing / maximal function spaces.

**Lemma 3.10 (Main trilinear estimate).** Let \(C_{k,k_1,k_2,k_3}\) denote the best constant \(C\) in the estimate
\[ \|P_k(P_{k_1}f_1 P_{k_2}f_2 P_{k_3}g)\|_{N_k(T)} \lesssim C\|P_{k_1}f_1\|_{F_1(T)}\|P_{k_2}f_2\|_{F_2(T)}\|P_{k_3}g\|_{G_3(T)}. \] (3.13)
The best constant \(C_{k,k_1,k_2,k_3}\) satisfies the bounds
\[ C_{k,k_1,k_2,k_3} \lesssim \begin{cases} 2^{-|(k_1+k_2)/2-k_3/3|} & (k_1, k_2, k_3) \in Z_1(k) \\ 2^{-|k-k_3|/6} & (k_1, k_2, k_3) \in Z_2(k) \\ 2^{-|k-k_3|/6} & (k_1, k_2, k_3) \in Z_3(k) \\ 0 & (k_1, k_2, k_3) \in \mathbb{Z}^3 \not\subset \{Z_1(k) \cup Z_2(k) \cup Z_3(k)\}, \end{cases} \]
where \(\Delta k = \max\{k_1, k_2, k_3\} - \min\{k_1, k_2, k_3\} \geq 0\).

**Proof.** After placing the term \(P_k(P_{k_1}f_1 P_{k_2}f_2 P_{k_3}g)\) in \(L^4_{t,x}\) and then using Hölder’s inequality to bound each factor in \(L^1_{t,x}\), it follows from Bernstein that
\[ C_{k,k_1,k_2,k_3} \lesssim 1, \] (3.14)
and so, in particular, for any choice of integers \(k, k_1, k_2, k_3\), such a constant \(C_{k,k_1,k_2,k_3}\) exists.

Frequencies not represented in one of \(Z_1(k), Z_2(k), Z_3(k)\) cannot interact so as to yield a frequency in \(I_k\). Over \(Z_1(k)\), we apply (3.14) and (3.7).
On $\mathbb{Z}_2(k)$ we apply (3.4) if $k > k_3$ and (3.5) if $k \leq k_3$. We conclude with (3.6).

On $\mathbb{Z}_3(k)$ we may assume without loss of generality that $k_1 \leq k_2$. First suppose that $k_3 \leq k$ and $|k - k_2| \leq 40$. If $k_1 \leq k_3$, then use (3.4), applying (3.6) to $P_{k_2}f_2P_{k_3}g$. If $k_3 < k_1$, then use (3.6) on $P_{k_1}f_1P_{k_3}g$. If $k_{\min} = k$, then use (3.5) and (3.6).

Now suppose that $k_3 > k$ and $|k_3 - k_2| \leq 40$. If $k_1 \leq k$, then use (3.3), applying (3.7) to $P_{k_1}f_1P_{k_3}g$. If $k_{\min} = k$, then use (3.5) and (3.6).

**Corollary 3.11.** Let $\{a_k\}, \{b_k\}, \{c_k\}$ be $\delta$-frequency envelopes. Let $C_{k,k_1,k_2,k_3}$ be as in Lemma 3.10. Then

$$\sum_{(k_1,k_2,k_3) \in \mathbb{Z}_3 \setminus \mathbb{Z}_2(k)} C_{k,k_1,k_2,k_3} a_{k_1} b_{k_2} c_{k_3} \lesssim a_k b_k c_k.$$

**Proof.** By Lemma 3.10, it suffices to restrict the sum to $(k_1, k_2, k_3)$ lying in $Z_1(k) \cup Z_3(k)$. On $Z_1(k)$, the sum is bounded by

$$\sum_{(k_1,k_2,k_3) \in Z_1(k)} 2^{-\frac{(k_1+k_2)}{6}-\frac{k_3}{3}} a_{k_1} b_{k_2} c_{k_3} \leq \sum_{k_1,k_2 \leq k-40} 2^{-\frac{(k_1+k_2)}{6}-\frac{k_3}{3}} \cdot 2^{\frac{k}{6} - k_2} a_k b_k c_k \lesssim a_k b_k c_k.$$

On $Z_3$, we may assume without loss of generality that $k_2 \leq k_1$. The sum is then controlled by

$$\sum_{(k_1,k_2,k_3) \in Z_3(k)} 2^{-\frac{|k_3-k_1|}{6}} a_{k_1} b_{k_2} c_{k_3} \lesssim \sum_{k_2 \leq k, k_3 \leq k} 2^{-\frac{|k - \min\{k_2,k_3\}|}{6}} a_{k_1} b_{k_2} c_{k_3} + \sum_{k_2 \leq k_1 \leq k, k_3 \geq k} 2^{-\frac{|k_3 - k_1|}{6}} a_{k_1} b_{k_2} c_{k_3} \lesssim \sum_{k_2 \leq k, k_3 \leq k} 2^{-\frac{|k - \min\{k_2,k_3\}|}{6}} a_{k_1} b_{k_2} c_{k_3} + \sum_{k_2 \geq k_1 \leq k, k_3 \geq k} 2^{-\frac{|k_3 - k_1|}{6}} a_{k_1} b_{k_2} c_{k_1}.$$
The first of these summands is controlled by
\[
\sum_{k_3 \leq k_2 \leq k} 2^{-|k-k_3|/6} a_{k_2} b_{k_2} c_{k_3} + \sum_{k_2 < k_3 \leq k} 2^{-|k-k_2|/6} a_k b_{k_2} c_{k_3}
\]
\[\lesssim \sum_{k_3 \leq k_2 \leq k} 2^{-|k-k_3|/6} 2^{\delta |k-k_2|} a_{k_2} b_{k} c_{k_3} + \sum_{k_2 < k_3 \leq k} 2^{-|k-k_2|/6} 2^{\delta |k-k_3|} a_k b_{k_2} c_k \]
\[\lesssim \sum_{k_3 \leq k} 2^{(2\delta-1/6)|k-k_3|} a_k b_{k_2} c_{k_3} + \sum_{k_2 < k} 2^{(2\delta-1/6)|k-k_2|} a_k b_k c_k \]
\[\lesssim a_k b_k c_k.\]

The second is controlled by
\[
\sum_{k \leq k_2 \leq k_1} 2^{-|k-k_2|/6} a_{k_2} b_{k_2} c_{k_1} + \sum_{k_2 < k \leq k_1} 2^{-|k-k_2|/6} a_k b_{k_2} c_{k_1}
\]
\[\lesssim \sum_{k \leq k_2 \leq k_1} 2^{-|k-k_2|/6} 2^{\delta |k-k_2|} a_{k_2} b_k c_{k_1} + \sum_{k_2 < k \leq k_1} 2^{-|k-k_2|/6} 2^{\delta |k-k_1|} a_k b_{k_2} c_{k_1} \]
\[\lesssim \sum_{k \leq k_1} 2^{(2\delta-1/6)|k-k_1|} a_k b_{k_2} c_{k_1} + \sum_{k_2 < k \leq k_1} 2^{(2\delta-1/6)|k-k_2|} a_k b_k c_k \]
\[\lesssim a_k b_k c_k.\]

**Corollary 3.12.** Let \( \{a_k\}, \{b_k\} \) be \( \delta \)-frequency envelopes. Let \( C_{k,k_1,k_2,k_3} \) be as in Lemma 3.11. Then
\[
\sum_{(k_1,k_2,k_3) \in Z_2(k) \cup Z_3(k)} 2^{\max\{k,k_3\} - \max\{k_1,k_2\}} C_{k,k_1,k_2,k_3} a_{k_1} b_{k_2} c_{k_3} \lesssim a_k b_k c_k
\]

**Proof.** On \( Z_3(k) \), \( \max\{k_1,k_2\} \sim \max\{k,k_3\} \), and so the bound on \( Z_3(k) \) follows from Corollary 3.11.

Note that \( \max\{k_1,k_2\} > \max\{k,k_3\} \) on \( Z_2 \), where the sum is controlled by
\[
\sum_{(k_1,k_2,k_3) \in Z_2(k)} 2^{\max\{k,k_3\} - \max\{k_1,k_2\}} 2^{-|k-k_3|/6} a_{k_1} b_{k_2} c_{k_3}
\]
\[\lesssim \sum_{k,k_3 \leq k_1 - 40} 2^{\max\{k,k_3\} - k_1} 2^{-|k-k_3|/6} a_{k_1} b_k c_{k_3}.\]

Restricting the sum to \( k_3 \leq k \), we get
\[
\sum_{k_3 \leq k \leq k_1 - 40} 2^{-|k-k_1|} 2^{-|k-k_3|/6} a_{k_1} b_{k} c_{k_3} \lesssim a_k b_k c_k
\]
Over the complementary range $k \leq k_3 \leq k_1 - 40$, we have
\[
\sum_{k \leq k_3 \leq k_1 - 40} 2^{-|k_3-k_1|}2^{-|k-k_3|/6}a_k b_k c_k
\lesssim a_k b_k c_k \sum_{k \leq k_3 \leq k_1 - 40} 2^{-|k_3-k_1|}2^{-|k-k_3|/6}2^2|k-k|2^δ|k-k-3|.
\]
Performing the change of variables $j := k_1 - k_3$, $ℓ := k_3 - k$, we control the sum by
\[
\sum_{j,ℓ \geq 0} 2^{-j}2^{-ℓ/6}2^{2δ(j+ℓ)}2^{δℓ} \lesssim \sum_{j,ℓ \geq 0} 2^{(2δ-1)j}2^{(3δ-1/6)ℓ} \lesssim 1.
\]

Taking advantage of the local smoothing/maximal function spaces, we can obtain the following improvement.

**Lemma 3.13 (Main trilinear estimate improvement over $Z_1$).** The best constant $C_{k,k_1,k_2,k_3}$ in (3.13) satisfies the improved estimate
\[
C_{k,k_1,k_2,k_3} \lesssim 2^{-|(k_1+k_2)/2-k|}
\]
when $\{k_1, k_2, k_3\} \in Z_1(k)$.

4. **Proof of Theorem 1.3**

In this section we outline the proof of Theorem 1.3, taking as our starting point the local result stated in Theorem 1.1.

For technical reasons related to the function space definitions of the last section, it will be convenient to construct a solution $ϕ$ on a time interval $(-2^{2K}, 2^{2K})$ for some given $K \in \mathbb{Z}^+$ and proceed to prove bounds that are uniform in $K$. We assume $1 \ll K \in \mathbb{Z}^+$ is chosen and hereafter fixed. Invoking Theorem 1.1, we assume that we have a solution $ϕ \in C([T, T] \to H^σ_Q)$ of (1.1) on the time interval $[-T, T]$ for some $T \in (0, 2^{2K})$. In order to extend $ϕ$ to a solution on all of $(-2^{2K}, 2^{2K})$ with uniform bounds (uniform in $T, K$), it suffices to prove uniform a priori estimates on

\[
\sup_{t \in (-T,T)} \|ϕ(t)\|_{H^σ_Q}
\]

for, say, $σ$ in the interval $[1, σ_1]$, with $σ_1 \gg 1$ chosen sufficiently large (e.g., $σ_1 = 25$ will do).

The first step in our approach, carried out in [2] is to lift the Schrödinger map system (1.1) to the tangent bundle and view it with respect to the caloric gauge. Recall that the lift of (1.1) expressed in terms of the caloric gauge takes the form (2.7), or, equivalently,
\[
(i∂_t + ∆)ψ_m = B_m + V_m,
\]
with initial data $\psi_m(0)$. Here $B_m$ and $V_m$ respectively denote the magnetic and electric potentials (see (2.3) and (2.9) for definitions).

The goal then becomes proving a priori bounds on $\|\psi_m\|_{L^\infty_t H^\sigma_x}$. Herein lies the heart of the argument, and the purpose of this section is not only to give a high level description of the proof of Theorem 1.3, but also to outline the proof of the key a priori bounds. To establish these bounds, we in fact prove stronger frequency-localized estimates. The argument naturally splits into several components, and we consider each individually below.

Finally, to complete the proof of Theorem 1.3 we must transfer the a priori bounds on the derivative fields $\psi_m$ back to bounds on the map $\varphi$, thereby allowing us to close a bootstrap argument. Once the derivative field bounds are established, this is, comparatively speaking, an easy task, and we take it up in the last subsection.

We return now to (4.1), projecting it to frequencies $\sim 2^k$ using the Littlewood-Paley multiplier $P_k$. Applying the linear estimate of Lemma 3.6 then yields

$$
\|P_k\psi_m\|_{G_k(T)} \lesssim \|P_k\psi_m(0)\|_{L_2^k} + \|P_kV_m\|_{N_k(T)} + \|P_kB_m\|_{N_k(T)}.
$$

(4.2)

In order to express control of the $G_k(T)$ norm of $P_k\psi_m$ in terms of the initial data, we introduce the following frequency envelopes. Let $\sigma_1 \in \mathbb{Z}_+$ be positive. For $\sigma \in [0, \sigma_1 - 1]$, set

$$
b_k(\sigma) = \sup_{k' \in \mathbb{Z}} 2^{\sigma k'} 2^{-|k-k'|} \|P_{k'}\psi_x\|_{G_k(T)}.
$$

(4.3)

By (2.38), these envelopes are finite and in $\ell^2$. We abbreviate $b_k(0)$ by setting $b_k := b_k(0)$.

We now state the key result for solutions of the gauge field equation (4.1).

**Theorem 4.1.** Assume $T \in (0, 2^{2K})$ and $Q \in S^2$. Choose $\sigma_1 \in \mathbb{Z}_+$ positive. Let $\varepsilon_1 > 0$ and let $\varphi \in H^{\infty, \infty}_Q(T)$ be a solution of the Schrödinger map system (1.1) whose initial data $\varphi_0$ has energy $E_0 := E(\varphi_0) < E_{\text{crit}}$ and satisfies the energy dispersion condition

$$
\sup_{k \in \mathbb{Z}} c_k \leq \varepsilon_1.
$$

(4.4)

Assume moreover that

$$
\sum_{k \in \mathbb{Z}} \|P_k\psi_x\|_{L_{\frac{2}{1-k}}(I \times \mathbb{R}^2)}^2 \leq \varepsilon_1^2
$$

(4.5)

for any smooth extension $\varphi$ on $I$, $[-T, T] \subset I \subset (-2^{2K}, 2^{2K})$. Suppose that the bootstrap hypothesis

$$
b_k \leq \varepsilon_1^{-1/10} c_k
$$

(4.6)

is satisfied. Then, for $\varepsilon_1$ sufficiently small,

$$
b_k(\sigma) \lesssim c_k(\sigma)
$$

(4.7)

holds for all $\sigma \in [0, \sigma_1 - 1]$ and $k \in \mathbb{Z}$. 
Proof. We use a continuity argument to prove Theorem 4.1. For $T' \in (0, T]$, let
\[
\Psi(T') = \sup_{k \in \mathbb{Z}} c_k^{-1} \| P_k \psi_m(s = 0) \|_{G_k(T')},
\]
Then $\psi : (0, T] \to [0, \infty)$ is well-defined, increasing, continuous, and satisfies
\[
\lim_{T' \to 0} \psi(T') \lesssim 1.
\]
The critical implication to establish is
\[
\Psi(T') \leq \varepsilon_1^{-1/10} \implies \Psi(T') \lesssim 1,
\]
which in particular follows from
\[
b_k \lesssim c_k. \tag{4.8}
\]
We also must similarly establish
\[
b_k(\sigma) \lesssim c_k(\sigma) \tag{4.9}
\]
for $\sigma \in (0, \sigma_1 - 1]$. The next several subsections describe the main steps of the proof of (4.8) and (4.9), to which the bulk of the remainder of this paper is dedicated. In §4.5 we complete the high level argument used to prove (4.8) and (4.9).

Corollary 4.2. Given the conditions of Theorem 4.1,
\[
\| P_k |\partial_x|^\sigma \partial_m \varphi \|_{L^\infty_t L^2_x([-T,T] \times \mathbb{R}^2)} \lesssim c_k(\sigma) \tag{4.10}
\]
holds for all $\sigma \in [0, \sigma_1 - 1]$.

The proof we defer to §4.6.

Together Theorem 1.1, Theorem 4.1, and Corollary 4.2 are almost enough to establish Theorem 1.3. The next lemma provides the final piece. We also defer its proof to §4.6.

Lemma 4.3. It holds that
\[
\sum_{k \in \mathbb{Z}} \| P_k \psi_x \|_{L^4_{t,x}}^2 \sim \sum_{k \in \mathbb{Z}} \| P_k \partial_x \varphi \|_{L^4_{t,x}}^2.
\]

Note that this lemma affords us a condition equivalent to (4.5) whose advantage lies in the fact that it is not expressed in terms of gauges.

Proof of Theorem 1.3. Fix $\sigma_1 \in \mathbb{Z}_+$ positive and let $\varepsilon_1 = \varepsilon_1(\sigma_1) \geq 0$. It suffices to prove (1.7) on the time interval $[-T, T]$ provided the estimate is uniform in $T$. In view of Theorem 1.1 and mass-conservation, proving
\[
\| \partial_x \varphi \|_{L^\infty_t \hat{H}^\sigma_0([-T,T] \times \mathbb{R}^2)} \lesssim \| \partial_x \varphi \|_{\hat{H}^\sigma_0(\mathbb{R}^2)} \tag{4.11}
\]
for $\sigma \in [0, \sigma_1 - 1]$ with $\sigma_1 = 25$ is enough to establish (1.6).
By virtue of Lemma 4.3, the assumptions of Theorem 1.3 are equivalent to those of 4.1. Therefore we have access to Corollary 4.2, which states that (4.10) holds $\sigma \in [0, \sigma_1 - 1]$. Using (2.33) and the Littlewood-Paley square function completes the proof of (4.11).

Global existence and (1.7) then follow via a standard bootstrap argument from Theorem 1.1 and from the fact that the constants in (4.11) are uniform in $T$. □

The remainder of this section is organized as follows. In §4.1 we state the key lemmas of parabolic type that are used to control the electric and magnetic nonlinearities. In §4.2 we state bounds that rely principally upon local smoothing, including a bilinear Strichartz estimate; they find application in controlling the worst magnetic nonlinearity terms.

In §4.3 we piece together the parabolic estimates to control the electric potential. In §4.4 we decompose the magnetic potential into two main pieces and demonstrate how to control one of these pieces.

In §4.5 we close the bootstrap argument proving Theorem 4.1. Here the remaining piece of the magnetic potential is addressed using a certain nonlinear version of a bilinear Strichartz estimate.

Finally, in §4.6 we prove Corollary 4.2 and Lemma 4.3.

4.1. Parabolic estimates. By “parabolic estimates” we mean those that principally rely upon the smoothing effect of the harmonic map heat flow. We include here only those that play a direct role in controlling the nonlinearity $\mathcal{N}$. These are proved in §7 where a host of auxiliary parabolic estimates are included as well. As the proofs rely upon a bootstrap argument that takes advantage of energy dispersion (4.4), these bounds rely upon this smallness constraint implicitly. On the other hand, $L^4$ smallness (4.5) is not used in the proofs of these bounds, but rather only in their application in this paper.

Lemma 4.4. For $\sigma \in [0, \sigma_1 - 1]$, the derivative fields $\psi_m$ satisfy

$$\|P_k \psi_m(s)\|_{F^\sigma_k(T)} \lesssim (1 + s^{2k})^{-\frac{4}{\sigma} - 2^{-\sigma k} b_k(\sigma)}$$

(4.12)

for $s \geq 0$.

This estimate is used in §4.4 in controlling the magnetic nonlinearity, which schematically looks like $A \partial_x \psi$. To recover the loss of derivative, it is important to take advantage of parabolic smoothing by invoking representation (2.23) of $A$. Within the integral we schematically have $\psi(s) D_x \psi(s)$, and hence (4.12) allows us to take advantage of (3.3)–(3.7) in bounding this term. We prove (4.12) in §7.4.
Lemma 4.5. For $\sigma \in [0, \sigma_1 - 1]$, the derivative fields $\psi_\ell$ and connection coefficients $A_m$ satisfy

\[
\|P_k(A_m(s)\psi_\ell(s))\|_{F_k(T)} \lesssim (s2^{2k})^{-3/8}(1 + s2^{2k})^{-2}2^{-(\sigma-1)k}b_k(\sigma).
\] (4.13)

Like the previous estimate, this estimate is also used in §4.4 in controlling the magnetic nonlinearity. Its proof is given in §7.2. The need for this estimate arises from the need to control $D_x\psi$ appearing in representation (2.23) of $A$.

The next several estimates are used in §4.3 to control the electric potential. In particular, they provide a source of smallness crucial here for closing the bootstrap argument. They are proved in §7.2.

Lemma 4.6. For $\sigma \in [2\delta, \sigma_1 - 1]$, the connection coefficient $A_x$ satisfies

\[
\|A_x^2\|_{L^2_{t,x}} \lesssim \sup_{j \in \mathbb{Z}} b_j^2 \sum_{k \in \mathbb{Z}} b_k^2
\] (4.14)

and

\[
\|P_k A_x^2(0)\|_{L^2_{t,x}} \lesssim 2^{\sigma k} b_k(\sigma) \cdot \sup_j b_j \sum_{\ell \in \mathbb{Z}} b_{\ell}^2.
\] (4.15)

Lemma 4.7. For $\sigma \in [2\delta, \sigma_1 - 1]$, the connection coefficient $A_t$ satisfies

\[
\|A_t\|_{L^2_{t,x}} \lesssim (1 + \sum_{j \in \mathbb{Z}} b_j^2)^2 \sum_{k \in \mathbb{Z}} \|P_k \psi_x(0)\|^2_{L^4_{t,x}}
\] (4.16)

and

\[
\|P_k A_t\|_{L^2_{t,x}} \lesssim (1 + \sum_{\ell \in \mathbb{Z}} b_{\ell}^2)b_{\ell}2^{-\sigma k}b_k(\sigma).
\] (4.17)

In subsequent estimates the following shorthand will be useful:

\[
\epsilon := (1 + \sum_{j \in \mathbb{Z}} b_j^2)^2 \sum_{\ell \in \mathbb{Z}} \|P_\ell \psi_x(0)\|^2_{L^4_{t,x}} + (1 + \sum_{\ell \in \mathbb{Z}} b_{\ell}^2) \sup_k b_k^2.
\] (4.18)

Under the assumptions of Theorem 4.1, $\epsilon$ is a very small quantity, being at least as good as $O(\varepsilon_1^{1/2})$.

4.2. Smoothing and Strichartz. The key result of §5 is the following frequency-localized bilinear Strichartz estimate.

Theorem 4.8. Suppose that $\psi_m$ satisfies (2.7) on $[-T, T]$. Assume $\sigma \in [0, \sigma_1 - 1]$. Let the frequency envelopes $b_j$ and $c_j$ be defined as in (4.3) and (2.32). Let $\epsilon$ be given by (4.18). Suppose also that $2^{-j} \ll 1$. Then

\[
2^{k-j}(1 + s2^{j})^8 \|P_\ell \psi_x(s) \cdot P_k \psi_m(0)\|^2_{L^2_{t,x}} \lesssim 2^{-2\sigma k} c_j^2 c_k^2(\sigma) + \epsilon^2 b_j^2 b_k^2(\sigma).
\] (4.19)
In §5.2 we split the proof into two cases: $s = 0$ and $s > 0$, the more involved being the $s = 0$ case. In either case, if instead we only were to appeal to the local smoothing-based estimate (3.11) and the frequency envelope definition (4.3), then we would get the bound

$$2^k - j (1 + s 2^{2j})^8 \| P_j \psi_\ell(s) \cdot P_k \psi_m(0) \|_{L^2_t L^\infty_x} \lesssim b_j^2 b_k^2.$$  

In practice this sort of bound must needs be summed over $j \ll k$. When initial energy is assumed to be small, as is done in [4], the sum $\sum_j b_j^2 \ll 1$ is small, and consequently the resulting term perturbative. In our subthreshold energy setting this is no longer the case, as in fact the sum may be large. What (4.19) reveals, though, is that any $b_j$ contributions come with a power of $\epsilon$. In view of additional work which we present in due course, this turns out to be sufficient for establishing $b_k \lesssim c_k$.

An interesting related bound is the following local smoothing estimate, also proved in §5.2. It arises as an easy corollary of our proof of Theorem 4.8.

**Theorem 4.9.** Suppose that $\psi_m$ satisfies (2.7) on $[-T,T]$. Assume $\sigma \in [0,\sigma_1 - 1]$. Let the frequency envelopes $b_j(\sigma)$ and $c_j(\sigma)$ be defined as in (4.3) and (2.32). Also, let $\epsilon$ be given by (4.18). Then

$$2^k \sup_{|j-k| \leq 20} \sup_{\theta \in S^1} \| P_{j,\theta} P_k \psi_m \|_{L^\infty_{\theta,\lambda}}^2 \lesssim 2^{-2\sigma k} c_k^2(\sigma) + \epsilon 2^{-2\sigma k} b_k^2(\sigma)$$

(4.20) holds for each $k \in \mathbb{Z}$.

We note that (4.20) likely extends to $L^\infty_{\theta,\lambda}$ for $\lambda$ satisfying $|\lambda| < 2^{k-40}$, though we do not prove this. For comparison, note that from the definition of (4.3) we have

$$2^k \sup_{|j-k| \leq 20} \sup_{\theta \in S^1} \| P_{j,\theta} P_k \psi_m \|_{L^\infty_{\theta,\lambda}}^2 \lesssim 2^{-2\sigma k} b_k^2(\sigma).$$

(4.21)

On the other hand, while the right hand side of (4.20) may indeed be large, it so happens thanks to our hypotheses of energy dispersion and $L^4$ smallness that the $b_k(\sigma)$ term is perturbative. For our purposes, this is a substantial improvement over (4.21). However, it can be seen from the argument in §4.5 that even an extension of (4.20) to $L^\infty_{\theta,\lambda}$ spaces is not sufficient for proving $b_k(\sigma) \lesssim c_k(\sigma)$: it is important that we can replace two “$b_j$” terms with corresponding “$c_j$” terms as in (4.19).

### 4.3. Controlling the electric potential $V$

**Lemma 4.10.** Suppose that $\sigma < \frac{1}{6} - 2\delta$. Then the electric potential term $V_m$ satisfies the estimate

$$\| P_k V_m \|_{N_k(T)} \lesssim \left( \| A_x^2 \|_{L^2_{t,x}} + \| A_t \|_{L^2_{t,x}} + \| \psi_\ell^2 \|_{L^4_{t,x}} \right) 2^{-\sigma k} b_k(\sigma).$$

(4.22)
Proof. Letting \( f \in \{ A_1, A^2, \psi^2 \} \), we bound \( P_k(f \psi_x) \) in \( N_k(T) \). Begin with the following Littlewood-Paley decomposition of \( P_k(f \psi_x) \):

\[
P_k(f \psi_x) = P_k(P_{< k-80} f P_{k-5<k+5} \psi_x) + \sum_{k_2 \leq k-80} P_k(P_{k_1} f P_{k_2} \psi_x) + \sum_{k_1 \leq 4, k_2 > k-80} P_k(P_{k_1} f P_{k_2} \psi_x).
\]

The first term is controlled using Hölder’s inequality:

\[
\| P_k(P_{< k-80} f P_{k-5<k+5} \psi_x) \|_{N_k(T)} \leq \| P_k(P_{< k-80} f P_{k-5<k+5} \psi_x) \|_{L^1_{t,x}^{1/3}} \\
\leq \| P_{< k-80} f \|_{L^3_{t,x}} \| P_{k-5<k+5} \psi_x \|_{L^3_{t,x}}.
\]

To control the second term we apply (3.4):

\[
\| P_k(P_{k_1} f P_{k_2} \psi_x) \|_{N_k(T)} \lesssim 2^{(k_2-k)/6} \| P_{k_1} f \|_{L^3_{t,x}} \| P_{k_2} \psi_x \|_{G_{k_2}(T)}.
\]

Using (1.3), (2.30), and \( \sigma < 1/6 - 2\sigma \), we conclude

\[
\| \sum_{k_1 \leq 4, k_2 \leq k-80} P_k(P_{k_1} f P_{k_2} \psi_x) \|_{N_k(T)} \lesssim 2^{-\sigma k} b_k(\sigma) \| P_{k_1} f \|_{L^3_{t,x}}.
\]

To control the high-high interaction, apply (3.5):

\[
\| P_k(P_{k_1} f P_{k_2} \psi_x) \|_{N_k(T)} \lesssim 2^{(k-k_2)/6} \| P_{k_1} f \|_{L^3_{t,x}} \| P_{k_2} \psi_x \|_{G_{k_2}(T)}.
\]

Therefore, by (1.3),

\[
\sum_{k_1 \leq 4, k_2 \leq k-80} \| P_k(P_{k_1} f P_{k_2} \psi_x) \|_{N_k(T)} \lesssim \sum_{k_1 \leq 4, k_2 \leq k-80} 2^{(k-k_2)/6} \| P_{k_1} f \|_{L^3_{t,x}} 2^{-\sigma k} b_k(\sigma).
\]

Using Cauchy-Schwarz and (2.31) yields

\[
\sum_{k_1 \leq 4, k_2 \leq k-80} \| P_k(P_{k_1} f P_{k_2} \psi_x) \|_{N_k(T)} \lesssim 2^{-\sigma k} b_k(\sigma) \left( \sum_{k_1 \geq k-80} \| P_{k_1} f \|_{L^3_{t,x}}^2 \right)^{1/2},
\]

and so, by switching the \( L^2_{t,x} \) and \( \ell^2 \) norms, we get from the standard square function estimate that

\[
\sum_{k_1 \leq 4, k_2 \leq k-80} \| P_k(P_{k_1} f P_{k_2} \psi_x) \|_{N_k(T)} \lesssim \| f \|_{L^2_{t,x}} 2^{-\sigma k} b_k(\sigma).
\]

□
Corollary 4.11. For \( \sigma \in [0, \sigma_1 - 1] \) it holds that
\[
\| P_k V_m \|_{N_k(T)} \lesssim \epsilon 2^{-\sigma k} b_k(\sigma).
\]

Proof. Given (4.22), this is a direct consequence of (4.14), (4.16), and the fact that
\[
\| f \|_{L^4_{t,x}}^2 \lesssim \sum_{k \in \mathbb{Z}} \| P_k f \|_{L^4_{t,x}}^2.
\]
Therefore the result holds for \( \sigma < 1/6 - 2\delta \).

To extend the proof to larger \( \sigma \), we may mimic the proof of Lemma 4.10 by performing the same Littlewood-Paley decomposition and then, with regard to the first and third terms of the decomposition, proceeding as before in the proof of that lemma. The argument, however, must be modified in handling the term
\[
\sum_{|k_1 - k| \leq 4 \atop k_2 \leq k - 80} P_k(P_{k_1} f P_{k_2} \psi_x),
\]
where \( f \in \{ A_t, A^2_x, \psi^2_x \} \). We take different approaches according to the choice of \( f \).

When \( f = A^2_x \), we apply (3.4) and invoke (4.15) to obtain
\[
\| \sum_{|k_1 - k| \leq 4 \atop k_2 \leq k - 80} P_k(P_{k_1} A^2_x P_{k_2} \psi_x) \|_{N_k(T)} \lesssim \sum_{|k_1 - k| \leq 4 \atop k_2 \leq k - 80} 2^{(k_2 - k)/6} \| P_{k_1} A^2_x \|_{L^4_{t,x}} \| P_{k_2} \psi_x \|_{C_{k_2}(T)}
\]
\[
\lesssim \sum_{|k_1 - k| \leq 4 \atop k_2 \leq k - 80} 2^{(k_2 - k)/6} 2^{-\sigma k_1 b_{k_1}} b_{k_2} \sup_j b_j \sum_{\ell} b^2_{\ell}
\]
\[
\lesssim 2^{-\sigma k} b_{k}(\sigma) \cdot b_k \cdot \sup_j b_j \cdot \sum_j b^2_j.
\]
In the case where \( f = A_t \), we apply (3.4) and use (4.17) to conclude
\[
\| \sum_{|k_1 - k| \leq 4 \atop k_2 \leq k - 80} P_k(P_{k_1} A_t P_{k_2} \psi_x) \|_{N_k(T)} \lesssim 2^{-\sigma k} b_{k}(\sigma) \tilde{b}_{k} b_{k}(1 + \sum_p b^2_p),
\]
which suffices by Cauchy-Schwarz.

Finally we turn to \( f = \psi^2_x \), which we further decompose as
\[
f = 2 \sum_{|j_1 - k| \leq 4 \atop j_2 < k - 80} P_{j_1} \psi_x P_{j_2} \psi_x + \sum_{|j_1 - j_2| \leq 8 \atop j_1, j_2 \geq k - 80} P_{j_1} \psi_x P_{j_2} \psi_x.
\]
To control the high-low term, we apply estimate \([3.7]\):
\[
\sum_{|j_1 - k| \leq 4 \atop j_2 < k \leq 80} \| P_{j_1} \psi_x P_{j_2} \psi_x \|_{L^2} \lesssim \sum_{|j_1 - k| \leq 4 \atop j_2 < k \leq 80} 2^{(j_2 - j_1)/6} | b_{j_2} | \lesssim 2^{-\sigma k} b_k \sigma b_k (\sigma).
\]

We turn to the high-high case. The full trilinear expression is given by
\[
\sum_{|k_1 - k| \leq 4 \atop k_2 < k \leq 80} P_{k_1} \left( \sum_{|j_1 - k| \leq 4 \atop j_1, j_2 \geq k_1 \leq 80} P_{j_1} \psi_x P_{j_2} \psi_x \cdot P_{k_2} \psi_x \right).
\]

We can drop the \(P_{k_1}\) factor because of the summation ranges:
\[
\sum_{|k_1 - k| \leq 4 \atop j_1 - j_2 \leq 8 \atop j_1, j_2 \geq k_1 \leq 80} P_{k_1} \left( \sum_{|j_1 - k| \leq 4 \atop j_1, j_2 \geq k_1 \leq 80} P_{j_1} \psi_x P_{j_2} \psi_x \cdot P_{k_2} \psi_x \right).
\]

We apply estimate \([3.4]\) with \(h = \psi_x P_{k_2} \psi_x\) to get
\[
\sum_{|k_1 - k| \leq 4 \atop j_1 - j_2 \leq 8 \atop k_2 < k \leq 80 \atop j_1, j_2 \geq k_1 \leq 80} \left\| P_{k_1} \left( \sum_{|j_1 - k| \leq 4 \atop j_1, j_2 \geq k_1 \leq 80} P_{j_1} \psi_x P_{j_2} \psi_x \cdot P_{k_2} \psi_x \right) \right\|_{N_k(T)} \lesssim \sum_{|k_1 - k| \leq 4 \atop j_1 - j_2 \leq 8 \atop k_2 < k \leq 80 \atop j_1, j_2 \geq k_1 \leq 80} 2^{-|j_1 - k|/6} \| P_{j_1} \psi_x \|_{G_{j_1} (T)} \| P_{j_2} \psi_x P_{k_2} \psi_x \|_{L^2}.
\]

Next we use \([3.7]\) to control the \(L^2\) norm:
\[
\sum_{|k_1 - k| \leq 4 \atop j_1 - j_2 \leq 8 \atop k_2 < k \leq 80 \atop j_1, j_2 \geq k_1 \leq 80} 2^{-|j_1 - k|/6} \| P_{j_1} \psi_x \|_{G_{j_1} (T)} \| P_{j_2} \psi_x P_{k_2} \psi_x \|_{L^2} \lesssim \sum_{|k_1 - k| \leq 4 \atop j_1 - j_2 \leq 8 \atop k_2 < k \leq 80 \atop j_1, j_2 \geq k_1 \leq 80} 2^{-|j_1 - k|/6} 2^{-|j_2 - k_2|/6} 2^{-\sigma j_1 b_j_1 (\sigma) b_{j_2} b_{k_2}}
\]

In this sum we can replace the factor \(2^{-|j_2 - k_2|/6}\) by the larger factor \(2^{-|k - k_2|/6}\), from which it is seen that the whole sum is controlled by
\[
2^{-\sigma k} b_k (\sigma) b_k \sum_{k_2 < k \leq 8} 2^{-|k - k_2|/6} b_{k_2} \lesssim 2^{-\sigma k} b_k^2 (\sigma).
\]

4.4. **Decomposing the magnetic potential.** We begin by introducing a paradifferential decomposition of the magnetic nonlinearity, splitting it into two pieces. This decomposition depends upon a frequency parameter \(k \in \mathbb{Z}\), which we suppress in the notation; this same \(k\) will also be the output frequency whose behavior we are interested in controlling. The decomposition also depends upon the frequency gap parameter \(\varpi \in \mathbb{Z}_+\). How \(\varpi\) is chosen and the exact role it plays are discussed in \([5.2]\). There it is shown that \(\varpi\)
may be set equal to a sufficiently large universal constant (independent of \(\varepsilon, \varepsilon_1, k, \) etc.).

Define \(A_{\text{lo} \lor \text{lo}}\) as

\[
A_{m, \text{lo} \lor \text{lo}}(s) := - \sum_{k_1, k_2 \leq k - \omega} \int_{s}^{\infty} \text{Im}(\overline{P_{k_1}} \psi_m P_{k_2} \psi_s)(s') ds'
\]

and \(A_{\text{hi} \lor \text{hi}}\) as

\[
A_{m, \text{hi} \lor \text{hi}}(s) := - \sum_{\max\{k_1, k_2\} > k - \omega} \int_{s}^{\infty} \text{Im}(\overline{P_{k_1}} \psi_m P_{k_2} \psi_s)(s') ds'
\]

so that \(A_m = A_{m, \text{lo} \lor \text{lo}} + A_{m, \text{hi} \lor \text{hi}}.\) Similarly define \(B_{\text{lo} \lor \text{lo}}\) as

\[
B_{m, \text{lo} \lor \text{lo}} := - i \sum_{k_3} (\partial \ell (A_{\ell, \text{lo} \lor \text{lo}} P_{k_3} \psi_m) + A_{\ell, \text{lo} \lor \text{lo}} \partial \ell P_{k_3} \psi_m)
\]

and \(B_{\text{hi} \lor \text{hi}}\) as

\[
B_{m, \text{hi} \lor \text{hi}} := - i \sum_{k_3} (\partial \ell (A_{\ell, \text{hi} \lor \text{hi}} P_{k_3} \psi_m) + A_{\ell, \text{hi} \lor \text{hi}} \partial \ell P_{k_3} \psi_m)
\]

so that \(B_m = B_{m, \text{lo} \lor \text{lo}} + B_{m, \text{hi} \lor \text{hi}}.\)

Our goal is to control \(P_k B_m\) in \(N_k(T).\) We consider first \(P_k B_{m, \text{hi} \lor \text{hi}},\) performing a trilinear Littlewood-Paley decomposition. In order for frequencies \(k_1, k_2, k_3\) to have an output in this expression at a frequency \(k,\) we must have \((k_1, k_2, k_3) \in Z_2(k) \cup Z_3(k) \cup Z_0(k),\) where

\[
Z_0(k) := Z_1(k) \cap \{ (k_1, k_2, k_3) \in \mathbb{Z}^3 : k_1, k_2 > k - \omega \} \quad (4.24)
\]

and the other \(Z_j(k)\)’s are defined in (3.12). We apply Lemma 3.10 to bound \(P_k B_{m, \text{hi} \lor \text{hi}}\) in \(N_k(T)\) by

\[
\sum_{(k_1, k_2, k_3) \in Z_2(k) \cup Z_3(k) \cup Z_0(k)} \int_0^{\infty} 2^{\max\{k, k_3\}} C_{k, k_1, k_2, k_3} \| P_{k_1} \psi_2(s) \|_{F_{k_1}} \times \\| P_{k_2} (D \ell \psi_{t}(s)) \|_{F_{k_2}} \| P_{k_3} \psi_m(0) \|_{C_{k_3}} ds,
\]

which, thanks to (4.12) and (4.13), is controlled by

\[
\sum_{(k_1, k_2, k_3) \in Z_2(k) \cup Z_3(k) \cup Z_0(k)} 2^{\max\{k, k_3\}} C_{k, k_1, k_2, k_3} b_{k_1} b_{k_2} b_{k_3} \times \int_0^{\infty} (1 + s 2^{2k_1})^{-4} 2^{k_2} (s 2^{2k_2})^{-3/8} (1 + s 2^{2k_2})^{-2} ds.
\]

As

\[
\int_0^{\infty} (1 + s 2^{2k_1})^{-4} 2^{k_2} (s 2^{2k_2})^{-3/8} (1 + s 2^{2k_2})^{-2} ds \lesssim 2^{-\max\{k_1, k_2\}}, \quad (4.25)
\]
we reduce to
\[ \sum_{(k_1,k_2,k_3) \in Z_2(k) \cup Z_3(k) \cup Z_0(k)} 2^{\max\{k,k_3\} - \max\{k_1,k_2\}} C_{k,k_1,k_2,k_3} b_{k_1} b_{k_2} b_{k_3}. \]  
(4.26)

To estimate \( P_k B_{m,hi\cup hi} \) on \( Z_2 \cup Z_3 \), we apply Corollary 3.12 and use the energy dispersion hypothesis. As for \( Z_0(k) \), we note that its cardinality \( |Z_0(k)| \) satisfies \( |Z_0(k)| \lesssim \omega \) independently of \( k \). Hence for fixed \( \omega \) summing over this set is harmless given sufficient energy dispersion. We obtain a bound of
\[ \| P_k B_{m,hi\cup hi} \|_{N_k(T)} \lesssim b_k^2 b_k \lesssim e b_k. \]  
(4.27)

Consider now the leading term \( P_k B_{m,lo\cup lo} \). Bounding this in \( N_k \) with any hope of summing requires the full strength of the decay that comes from the local smoothing/maximal function estimates. Such bounds as are immediately at our disposal (i.e., (3.10) and (3.11), however, do not bring \( B_{m,lo\cup lo} \) within the perturbative framework, instead yielding a bound of the form
\[ \sum_{k_1,k_2 \leq k - \omega, |k_3 - k| \leq 4} b_{k_1} b_{k_2} b_{k_3}, \]

which is problematic since even \( \sum_{j < k} \epsilon_j^2 \sim \epsilon_j^2 = O(1) \) for \( k \) large enough. This stands in sharp contrast with the small energy setting.

In the next section, however, we are able to capture enough improvement in such estimates so as to barely bring \( B_{m,lo\cup lo} \) back within reach of our bootstrap approach.

Finally, we need for \( \sigma > 0 \) an estimate analogous to (4.27). Returning to the proof of (1.26), we remark that any \( b_{k_j} \) may be replaced by \( 2^{-\sigma k_j} b_{k_j} \); in order to obtain an analogue of (4.27), we must make replacements judiciously so as to retain summability. In particular, for any \( (k_1,k_2,k_3) \in Z_2(k) \cup Z_3(k) \cup Z_0(k) \), we replace \( b_{k_{\max}} \) with \( 2^{-\sigma k_{\max}} b_{k_{\max}}(\sigma) \) so that (4.26) becomes
\[ \sum_{(k_1,k_2,k_3) \in Z_2(k) \cup Z_3(k) \cup Z_0(k)} 2^{\max\{k,k_3\} - \max\{k_1,k_2\}} C_{k,k_1,k_2,k_3} b_{k_{\min}} b_{k_{mid}} b_{k_{\max}} 2^{-\sigma k_{\max}} b_{k_{\max}}(\sigma), \]

where \( k_{\min}, k_{mid}, k_{\max} \) denote, respectively, the min, mid, and max of \( \{k_1,k_2,k_3\} \). Over the set \( Z_2(k) \cup Z_3(k) \cup Z_0(k) \) (see (3.12) and (4.24) for definitions), we have \( k_{\max} \gtrsim k \), which guarantees summability due to straightforward modifications of Corollaries 3.11 and 3.12. Therefore
\[ \| P_k B_{m,hi\cup hi} \|_{N_k(T)} \lesssim b_k^2 2^{-\sigma k} b_k(\sigma), \]
which, combined with (4.27) and the definition (4.18) of \( \epsilon \), implies

**Corollary 4.12.** Assume \( \sigma \in [0,\sigma_1 - 1] \). The term \( B_{m,hi\cup hi} \) satisfies the estimate
\[ \| P_k B_{m,hi\cup hi} \|_{N_k(T)} \lesssim \epsilon 2^{-\sigma k} b_k(\sigma). \]  
(4.28)
4.5. **Closing the gauge field bootstrap.** We turn first to the completion of the proof of Theorem 4.1 as we now have in place all of the estimates that we need to prove (4.3).

Using the main linear estimate of Proposition 3.6 and the decomposition introduced in §4.4, we obtain

$$
\|P_k \psi_m\|_{G_k(T)} \lesssim \|P_k \psi_m(0)\|_{L^2_x} + \|P_k V_m\|_{N_k(T)} + \|P_k B_{m,\text{hivhi}}\|_{N_k(T)}.
$$

(4.29)

In §4.3, 4.4 it is shown that $P_k V_m$ and $P_k B_{m,\text{hivhi}}$ are perturbative in the sense that

$$
\|P_k V_m\|_{N_k(T)} + \|P_k B_{m,\text{hivhi}}\|_{N_k(T)} \lesssim \epsilon 2^{-\sigma k} b_k(\sigma),
$$

To handle $P_k B_{m,\text{lo\&\lo}}$, we first write

$$
P_k B_{m,\text{lo\&\lo}} = -i \partial_t (A_{\ell,\text{lo\&\lo}} P_k \psi_m) + R,
$$

where $R$ is a perturbative remainder (thanks to a slight modification of technical Lemma 5.11). Therefore

$$
\|P_k \psi_m\|_{G_k(T)} \lesssim 2^{-\sigma k} c_k(\sigma) + \epsilon 2^{-\sigma k} b_k(\sigma) + \|\partial_t (A_{\ell,\text{lo\&\lo}} P_k \psi_m)\|_{N_k(T)}.
$$

(4.30)

Thus it remains to control $-i \partial_t (A_{\ell,\text{lo\&\lo}} P_k \psi_m)$, which we expand as

$$
-i P_k \partial_t \sum_{k_1, k_2 \leq k - \infty} \int_0^\infty \text{Im}(\overline{P_{k_1} \psi_t} P_{k_2} \psi_s)(s') P_{k_3} \psi_m(0) ds',
$$

(4.31)

and whose $N_k(T)$ norm we denote by $N_{\text{lo}}$. In the $\sigma = 0$ case the key is to apply Theorem 4.3 to $\overline{P_{k_1} \psi_t}(s')$ and $P_{k_3} \psi_m(0)$, after first placing all of (4.31) in $N_k(T)$ using (3.10). We obtain

$$
N_{\text{lo}} \lesssim 2^k \sum_{k_1, k_2 \leq k - \infty} 2^{-|k-k_2|/2} 2^{-|k_1-k_3|/2} 2^{-\max\{k_1,k_2\} b_{k_2}} (c_{k_1} c_{k_3} + \epsilon^{1/2} b_{k_1} b_{k_3})
$$

\[ \lesssim 2^k \sum_{k_1, k_2 \leq k - \infty} 2^{(k_1+k_2)/2-k} 2^{-\max\{k_1,k_2\} b_{k_2}} (c_{k_1} c_{k} + \epsilon^{1/2} b_{k_1} b_{k}) \]

Without loss of generality we restrict the sum to $k_1 \leq k_2$:

$$
\sum_{k_1 \leq k_2 \leq k - \infty} 2^{(k_1+k_2)/2} b_{k_2} (c_{k_1} c_{k} + \epsilon^{1/2} b_{k_1} b_{k})
$$

Using the frequency envelope property to sum off the diagonal, we reduce to

$$
N_{\text{lo}} \lesssim \sum_{j \leq k - \infty} (b_j c_j c_k + \epsilon^{1/2} b_j^2 b_k).
$$

Combining this with (4.30) and the fact that $R$ is perturbative, we obtain

$$
b_k \lesssim c_k + \epsilon b_k + \sum_{j \leq k - \infty} (b_j c_j c_k + \epsilon^{1/2} b_j^2 b_k),
$$

(4.32)
which, in view of our choice of \( \epsilon \), reduces to
\[
b_k \lesssim c_k + c_k \sum_{j \leq k - \omega} b_j c_j.
\]
Squaring and applying Cauchy-Schwarz yields
\[
b_k^2 \lesssim (1 + \sum_{j \leq k - \omega} b_j^2) c_k^2.
\] (4.33)
Setting
\[
B_k := 1 + \sum_{j < k} b_j^2
\]
in (4.33) leads to
\[
B_{k+1} \leq B_k (1 + Cc_k^2)
\]
with \( C > 0 \) independent of \( k \). Therefore
\[
B_{k+m} \leq B_k \prod_{\ell=1}^{m} (1 + Cc_{k+\ell}^2) \leq B_k \exp(C \sum_{\ell=1}^{m} c_{k+\ell}^2) \lesssim E_0 \ B_k.
\]
Since \( B_k \to 1 \) as \( k \to -\infty \), we conclude
\[
B_k \lesssim E_0, \ 1
\]
uniformly in \( k \), so that, in particular,
\[
\sum_{j \in \mathbb{Z}} b_j^2 \lesssim 1, \quad (4.34)
\]
which, joined with (4.33), implies (4.8).

The proof of (4.9) is almost an immediate consequence. Instead of (4.32), we obtain
\[
b_k(\sigma) \lesssim c_k(\sigma) + \epsilon b_k(\sigma) + \sum_{j \leq k - \omega} (b_j c_j c_k(\sigma) + \epsilon^{1/2} b_j^2 b_k(\sigma)),
\]
which suffices to prove (4.9) in view of (4.34).

4.6. De-gauging. The previous subsections overcome the most significant obstacles encountered in proving conditional global regularity. All of the key estimates therein apply to the Schrödinger map system placed in the caloric gauge, and a bootstrap argument is in fact run and closed at that level. This final subsection justifies the whole approach, showing how to transfer these results obtained at the gauge level back to the underlying Schrödinger map itself.

Proof of (4.10). To gain control over the derivatives \( \partial_m \varphi \) in \( L_x^\infty L_t^2 \), we utilize representation (2.2) and perform a Littlewood-Paley decomposition. We
only indicate how to handle the term $v \cdot \text{Re}(\psi_m)$, as the term $w \cdot \text{Im}(\psi_m)$ may be handled similarly. Starting with

$$P_k(v\text{Re}(\psi_m)) = \sum_{|k_2-k| \leq 4} P_k(P_{\leq k-5}v \cdot P_{k_2}\text{Re}(\psi_m)) +$$

$$\sum_{|k_1-k| \leq 4} P_k(P_{k_1}v \cdot P_{k_2}\text{Re}(\psi_m)) +$$

$$\sum_{|k_1-k_2| \leq 8 \atop k_1, k_2 \geq k-4} P_k(P_{k_1}v \cdot P_{k_2}\text{Re}(\psi_m)),$$  \hspace{1cm} (4.35)

we proceed to bound each term in $L^\infty_t L^2_x$.

In view of the fact that $|v| \equiv 1$, the low-high frequency interaction is controlled by

$$\sum_{|k_2-k| \leq 4} \|P_k(P_{\leq k-5}v \cdot P_{k_2}\text{Re}(\psi_m))\|_{L^\infty_t L^2_x} \lesssim \|P_{\leq k-5}v\|_{L^\infty_t} \|P_k\psi_m\|_{L^\infty_t L^2_x}$$

$$\lesssim \|P_k\psi_m\|_{L^\infty_t L^2_x} \lesssim c_k. \hspace{1cm} (4.36)$$

To control the high-low frequency interaction, we use Hölder’s inequality, Bernstein’s inequality, \hspace{1cm} (2.33) \hspace{1cm} and Bernstein’s inequality again, and finally the bound \hspace{1cm} (2.15) \hspace{1cm} along with summation rule \hspace{1cm} (2.31):

$$\sum_{|k_1-k| \leq 4 \atop k_2 \leq k-4} \|P_k(P_{k_1}v \cdot P_{k_2}\text{Re}(\psi_m))\|_{L^\infty_t L^2_x} \lesssim \sum_{|k_1-k| \leq 4 \atop k_2 \leq k-4} \|P_{k_1}v\|_{L^\infty_t L^2_x} \|P_{k_2}\psi_m\|_{L^\infty_t}$$

$$\lesssim \sum_{|k_1-k| \leq 4 \atop k_2 \leq k-4} \|P_{k_1}v\|_{L^\infty_t L^2_x} \cdot 2^{k_2} \|P_{k_2}\psi_m\|_{L^\infty_t L^2_x}$$

$$\lesssim \sum_{|k_1-k| \leq 4 \atop k_2 \leq k-4} \|P_{k_1}\partial_x v\|_{L^\infty_t L^2_x} \cdot 2^{k_2-k} c_{k_2}$$

$$\lesssim c_k. \hspace{1cm} (4.37)$$
To control the high-high frequency interaction, we use Bernstein's inequality, Cauchy-Schwarz, Bernstein again, (2.15), and finally (2.31):

\[
\sum_{|k_1-k_2|\leq 8} \|P_k(P_{k_1} v \cdot P_{k_2} \text{Re}(\psi_m))\|_{L_t^\infty L_x^2} \lesssim \sum_{|k_1-k_2|\leq 8, k_1, k_2 \geq k-4} 2^k \|P_{k_1} v \cdot P_{k_2} \text{Re}(\psi_m)\|_{L_t^\infty L_x^2}
\]

\[
\lesssim \sum_{|k_1-k_2|\leq 8, k_1, k_2 \geq k-4} 2^k \|P_{k_1} v\|_{L_t^\infty L_x^2} \|P_{k_2} \psi_m\|_{L_t^\infty L_x^2}
\]

\[
\lesssim \sum_{|k_1-k_2|\leq 8, k_1, k_2 \geq k-4} 2^{k-k_1} \|P_{k_1} \partial_x \psi\|_{L_t^\infty L_x^2} \|P_{k_2} \psi_m\|_{L_t^\infty L_x^2}
\]

\[
\lesssim \sum_{k_2 \geq k-4} 2^{k-k_2} c_{k_2}
\]

\[
\lesssim c_k. \quad (4.38)
\]

Combining (4.36), (4.37), and (4.38) and applying them in (4.35), we obtain

\[
\|P_k(v \text{Re}(\psi_m))\|_{L_t^\infty L_x^2} \lesssim c_k.
\]

As the above calculation holds with \(w\) in place of \(v\), we conclude (recalling (2.21)) that

\[
\|P_k \partial_x \varphi\|_{L_t^\infty L_x^2} \lesssim c_k.
\]

Hence (4.10) holds for \(\sigma = 0\).

Now we turn to the case \(\sigma \in [0, \sigma_1 - 1]\). By using Bernstein's inequality in (4.36) and (4.38), we may obtain

\[
\sum_{|k_2-k|\leq 4} \|P_k(P_{k \leq k-5} v \cdot P_{k} \text{Re}(\psi_m))\|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k} c_k(\sigma) \quad (4.39)
\]

\[
\sum_{|k_1-k_2|\leq 8, k_1, k_2 \geq k-4} \|P_k(P_{k_1} v \cdot P_{k_2} \text{Re}(\psi_m))\|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k} c_k(\sigma), \quad (4.40)
\]

as well as analogous estimates with \(w\) in place of \(v\). Such a direct argument, however, does not yield the analogue of (4.37). We circumvent this obstruction as follows. Let \(C \in (0, \infty)\) be the best constant for which

\[
\|P_k \partial_x \varphi\|_{L_t^\infty L_x^2} \leq C 2^{-\sigma k} c_k(\sigma) \quad (4.41)
\]

holds for \(\sigma \in [0, \sigma_1 - 1]\). Such a constant exists by smoothness and the fact that the \(c_k(\sigma)\) are frequency envelopes. In view of definition (2.34) and estimate (2.35), we similarly have

\[
\|P_k \partial_x v(0)\|_{L_t^\infty L_x^2} \lesssim C 2^{-\sigma k} c_k(\sigma). \quad (4.42)
\]

Using (4.42) in (4.37), we obtain

\[
\sum_{|k_1-k|\leq 4} \|P_k(P_{k} v \cdot P_{k} \text{Re}(\psi_m))\|_{L_t^\infty L_x^2} \lesssim C 2^{-\sigma k} c_k c_k(\sigma). \quad (4.43)
\]
From the representations (2.2) and (4.35), and from the estimates (4.39), (4.40), and (4.43), along with the analogous estimates for \(w\), it follows that

\[
\|P_k \partial_x \varphi\|_{L_t^\infty L_x^2} \lesssim (1 + c_k C)^2 - \sigma k c_k \left(\frac{\sigma}{c_k}\right).
\]

In view of energy dispersion \((c_k \leq \varepsilon)\) and the optimality of \(C\) in (4.41), we conclude

\[
C \lesssim 1 + \varepsilon C
\]

so that \(C \lesssim 1\). Therefore

\[
\|P_k \partial_x^\sigma \partial_m \varphi\|_{L_t^\infty L_x^2} \sim 2^{\sigma k} \|P_k \partial_m \varphi\|_{L_t^\infty L_x^2} \lesssim c_k (\sigma),
\]

which completes the proof of (4.10). \(\square\)

It will be convenient in certain arguments to use the weaker frequency envelope defined by

\[
\tilde{b}_k = \sup_{k'' \in \mathbb{Z}} 2^{-|k - k''|} \|P_k' \psi_{x}\|_{L_t^4 \times \mathbb{R}^d}. \tag{4.44}
\]

**Proof of Lemma 4.3.** Let us first establish

\[
\sum_{k \in \mathbb{Z}} \|P_k \psi_{x}\|_{L_t^4 \times \mathbb{R}^d}^2 \lesssim \sum_{k \in \mathbb{Z}} \|P_k \partial_x \varphi\|_{L_t^4 \times \mathbb{R}^d}^2.
\]

We use (2.1), i.e., \(\psi_m = v \cdot \partial_m \varphi + iw \cdot \partial_m \varphi\), but for the sake of exposition only treat \(v \cdot \partial_m \varphi\). We start with the Littlewood-Paley decomposition

\[
P_k \psi_m(0) = \sum_{|k_2 - k| \leq 4} P_k (P_{\leq k-5} v \cdot P_{k_2} \partial_m \varphi) + \sum_{|k_1 - k| \leq 4} P_k (P_{k_1} v \cdot P_{k_2} \partial_m \varphi) + \sum_{|k_1 - k_2| \leq 8} P_k (P_{k_1} v \cdot P_{k_2} \partial_m \varphi).
\]

In view of \(|v| = 1\), the \(L_t^4 \times \mathbb{R}^d\) norm of the low-high interaction is controlled by \(\tilde{b}_k\) (see (4.44)). To control the high-low interaction, we use Hölder’s and
Bernstein’s inequalities along with (2.15):

\[
\sum_{|k_1-k| \leq 4 \atop k_2 \leq k-4} \| P_k(P_{k_1} v \cdot P_{k_2} \partial_m \varphi) \|_{L^4_t L^2_x} \lesssim \sum_{|k_1-k| \leq 4 \atop k_2 \leq k-4} \| P_{k_1} v \|_{L^\infty_t L^2_x} \cdot \| P_{k_2} \partial_m \varphi \|_{L^4_t L^\infty_x} \\
\lesssim \sum_{|k_1-k| \leq 4 \atop k_2 \leq k-4} 2^{k_1/2} \| P_{k_1} v \|_{L^\infty_t L^2_x} 2^{k_2/2} \| P_{k_2} \partial_m \varphi \|_{L^4_t L^4_x} \\
\lesssim \sum_{|k_1-k| \leq 4 \atop k_2 \leq k-4} 2^{k_1} \| P_{k_1} v \|_{L^\infty_t L^2_x} \tilde{b}_k \\
\lesssim \tilde{b}_k.
\]

To control the high-high interaction, we use Bernstein, Hölder, Bernstein again, and (2.15):

\[
\sum_{|k_1-k_2| \leq 8 \atop k_1, k_2 \geq k-4} \| P_k(P_{k_1} v \cdot P_{k_2} \partial_m \varphi) \|_{L^4_t L^2_x} \lesssim \sum_{|k_1-k_2| \leq 8 \atop k_1, k_2 \geq k-4} 2^{k/2} \| P_{k_1} v \cdot P_{k_2} \partial_m \varphi \|_{L^4_t L^2_x} \\
\lesssim \sum_{|k_1-k_2| \leq 8 \atop k_1, k_2 \geq k-4} 2^{k/2} \| P_{k_1} v \|_{L^\infty_t L^2_x} \| P_{k_2} \partial_m \varphi \|_{L^4_t L^4_x} \\
\lesssim \sum_{|k_1-k_2| \leq 8 \atop k_1, k_2 \geq k-4} 2^{(k+k_1)/2} \| P_{k_1} v \|_{L^\infty_t L^2_x} \| P_{k_2} \partial_m \varphi \|_{L^4_t L^4_x} \\
\lesssim \sum_{|k_1-k_2| \leq 8 \atop k_1, k_2 \geq k-4} 2^{(k-k_2)/2} \| P_{k_1} \partial_x v \|_{L^\infty_t L^2_x} \| P_{k_2} \partial_m \varphi \|_{L^4_t L^4_x} \\
\lesssim \sum_{k_2 \geq k-4} 2^{(k-k_2)/4} \tilde{b}_{k_2} \\
\lesssim \tilde{b}_k.
\]

Therefore

\[
\| P_k \psi_m(0) \|_{L^4_t L^2_x} \lesssim \tilde{b}_k
\]

and

\[
\sum_{k \in \mathbb{Z}} \| P_k \psi_m(0) \|_{L^4_t L^2_x}^2 \lesssim \sum_{k \in \mathbb{Z}} \tilde{b}_k^2 \sim \sum_{k \in \mathbb{Z}} \| P_k \partial_m \varphi(0) \|_{L^4_t L^2_x}^2.
\]

By using (2.2), creating an \( L^4 \) frequency envelope for \( P_k \partial_m \varphi(0) \), and reversing the roles of \( \psi_n \) and \( \partial_n \varphi \) in the preceding argument, we conclude the reverse inequality

\[
\sum_{k \in \mathbb{Z}} \| P_k \partial_m \varphi(0) \|_{L^4_t L^2_x}^2 \lesssim \sum_{k \in \mathbb{Z}} \| P_k \psi_m(0) \|_{L^4_t L^2_x}^2.
\]

□
5. Local smoothing and bilinear Strichartz

The main goal of this section is to establish the improved bilinear Strichartz estimate of Theorem 4.8. As a by-product we also obtain the frequency-localized local smoothing estimate of Theorem 4.9.

Our approach is to first establish abstract local smoothing and bilinear Strichartz estimates for solutions to certain magnetic nonlinear Schrödinger equations. These are in the spirit of [41, 40, 60]. We shall then apply these to Schrödinger maps, in particular to the parilinearized derivative field equations written with respect to the caloric gauge.

We introduce some notation. Let $I_k(\mathbb{R}^d)$ denote the set $\{\xi \in \mathbb{R}^d : |\xi| \in [-2^{k-1}, 2^{k+1}]\}$ and $I_{(-\infty, k]} := \bigcup_{j \leq k} I_j$. For a $d$-vector-valued function $B = (B_\ell)$ on $\mathbb{R}^d$ with real entries, define the magnetic Laplacian $\Delta_B$, acting on complex-valued functions $f$, via

$$\Delta_B f := (\partial_x + iB)(\partial_x + iB)f = \Delta f + i(\partial_t B_\ell)f + 2iB_\ell \partial_x f - B_\ell^2 f. \quad (5.1)$$

For a unit vector $e \in S^{d-1}$, denote by $\{x \cdot e = 0\}$ the orthogonal complement in $\mathbb{R}^d$ of the span of $e$, equipped with the induced measure. Given $e$, we can construct a positively oriented orthonormal basis $e, e_1, \ldots, e_{d-1}$ of $\mathbb{R}^d$ so that $e_1, \ldots, e_{d-1}$ form an orthonormal basis for $\{x \cdot e = 0\}$. For complex-valued functions $f$ on $\mathbb{R}^d$, define $E_e(f) : \mathbb{R} \to \mathbb{R}$ as

$$E_e(f)(x_0) := \int_{x \cdot e = 0} |f|^2 dx' = \int_{\mathbb{R}^{d-1}} |f(x_0 e + x_j e_j)|^2 dx', \quad (5.2)$$

where the implicit sum runs over $1, 2, \ldots, d - 1$, and $dx'$ is standard $d - 1$-dimensional Lebesgue measure. We also adopt the following notation for this section: for $z, \zeta$ complex,

$$z \wedge \zeta := z\bar{\zeta} - \bar{z}\zeta = 2i \text{Im}(z\bar{\zeta}).$$

5.1. The key lemmas.

**Lemma 5.1** (Abstract almost-conservation of energy). Let $d \geq 1$ and $e \in S^{d-1}$. Let $v$ be a $C^\infty_t(H^\infty_x)$ function on $\mathbb{R}^d \times [0, T]$ solving

$$(i\partial_t + \Delta_A)v = \Lambda_v \quad (5.3)$$

with initial data $v_0$. Take $A_\ell$ to be real-valued, smooth, and bounded, with $\Delta_A$ defined via (5.1). Then

$$\|v\|^2_{L^\infty_t L^2_x} \leq \|v_0\|^2_{L^2_x} + \left| \int_0^T \int_{\mathbb{R}^d} v \wedge \Lambda_v dx dt \right|. \quad (5.4)$$

**Proof.** We begin with

$$\frac{1}{2} \partial_t \int |v|^2 dx = \int \text{Im}(\overline{\partial_t v}) v dx,$$
which may equivalently be written as

\[ i\partial_t \int |v|^2 dx = -\int v \wedge i\partial_t v dx. \]

Substituting from (5.3) yields

\[ i\partial_t \int |v|^2 dx = \int v \wedge (\Delta_A v - \Lambda_v) dx. \]

Expanding \( \Delta_A \) using (5.1) and using the straightforward relations

\[ \partial_t (v \wedge i A) = v \wedge i (\partial_t A) v + v \wedge 2i A \partial_t v \]

and

\[ \partial (v \wedge \partial_t v) = v \wedge \Delta v, \]

we get

\[ i\partial_t \int |v|^2 dx = \int \partial (v \wedge \partial_t v) dx + \int \partial_t (v \wedge i A \partial_t v) dx \]

\[ - \int v \wedge A^2 v dx - \int v \wedge \Lambda_v dx. \]

The first two terms on the right hand side vanish upon integration in \( x \); the third is equal to zero because \( A^2 \) is real. Integrating in time and taking absolute values therefore yields

\[ \left| \int_{R^d} |v(T')|^2 - |v_0|^2 dx \right| = \left| \int_0^{T'} \int_{R^d} v \wedge \Lambda_v dx dt \right| \]

for any time \( T' \in (0, T] \).

\[ \square \]

**Lemma 5.2** (Local smoothing preparation). Let \( d \geq 1 \) and \( e \in S^{d-1} \). Let \( j, k \in \mathbb{Z} \) and \( j = k + O(1) \). Let \( \varepsilon_m > 0 \) be a small positive number such that \( \varepsilon_m 2^{O(1)} \ll 1 \). Let \( v \) be a \( C^\infty_t (H^\infty_x) \) function on \( R^d \times [0, T] \) solving

\[ (i\partial_t + \Delta_A) v = \Lambda_v, \]  \hspace{1cm} (5.5)

where \( A \) is real-valued, smooth, and satisfies the estimate

\[ \|A\|_{L^\infty_{t,x}} \leq \varepsilon_m 2^k. \]  \hspace{1cm} (5.6)

The solution \( v \) is assumed to have (spatial) frequency support in \( I_k \), with the additional constraint that \( e \cdot \xi \in [2^{j-1}, 2^{j+1}] \) for all \( \xi \) in the support of \( \hat{v} \). Then

\[ 2^j \int_0^T E_e(v) dt \lesssim \|v\|^2_{L^2_t L^2_x} + \left| \int_0^T \int_{x \leq 0} v \wedge \Lambda_v dx dt \right| + 2^j \int_0^T E_e(v + i2^{-j} \partial_e v) dt. \]  \hspace{1cm} (5.7)
Proof. We begin by introducing
\[ M_e(t) := \int_{x \in \mathbb{R}^d} |v(x,t)|^2 dx. \]
Then
\[ 0 \leq M_e(t) \leq \|v(t)\|_{L_x^2}^2 \leq \|v\|_{L_t^\infty L_x^2([-T,T] \times \mathbb{R}^d)}^2. \]  
Differentiating in time yields
\[ i \dot{M}_e(t) = \int_{x \in \mathbb{R}^d} v \wedge (i \partial_t v) dx \]
which may be rewritten as
\[ i \dot{M}_e(t) = \int_{x \in \mathbb{R}^d} v \wedge (\Delta_A v - \Lambda_v) dx, \]
and therefore (5.9) may be rewritten as
\[ -\int_{x \in \mathbb{R}^d} v \wedge (\partial_e v + i e \cdot A v) dx' = i \dot{M}_e(t) + \int_{x \in \mathbb{R}^d} v \wedge \Lambda_v dx. \]
On the one hand, we have the heuristic that \( \partial_e v \approx i 2^j v \) since \( v \) has localized frequency support. On the other hand, since \( A \) is real-valued, we have
\[ \int_0^T \int_{x \in \mathbb{R}^d} v \wedge i e \cdot A v dx' dt = 2 \int_0^T \int_{x \in \mathbb{R}^d} \Lambda_A |v|^2 dx' dt \]
and hence by assumption (5.6) also
\[ \int_0^T \int_{x \in \mathbb{R}^d} |A| |v|^2 dx' dt \leq \varepsilon_m 2^k \int_0^T \int_{x \in \mathbb{R}^d} |v|^2 dx' dt. \]
Together these facts motivate rewriting \( v \wedge \partial_e v \) as
\[ v \wedge \partial_e v = 2 \cdot i 2^j |v|^2 + v \wedge (\partial_e v - i 2^j v). \]
Using (5.11), (5.13), and the bounds (5.12) and (5.8) in (5.10), we obtain by time-integration that
\[ (1 - \varepsilon_m 2^{k-j}) 2^j \int_0^T E_e(v) dt \leq \|v\|_{L_t^\infty L_x^2}^2 \]
\[ + 2 \cdot 2^j \int_0^T \int_{x \in \mathbb{R}^d} |v + i 2^{-j} \partial_e v| |v| dx' dt. \]
Applying Cauchy-Schwarz to the last term yields
\[ 2^j \int_0^T \int_{x \in \mathbb{R}^d} |v + i 2^{-j} \partial_e v| |v| dx' dt \leq 8 \cdot 2^j \int_0^T E_e(v + i 2^{-j} \partial_e v) dt + \frac{1}{8} 2^j \int_0^T E_e(v) dt. \]
Therefore \(5.7\). \(\square\)

We now describe the constraints on the nonlinearity that we shall require in the abstract setting.

**Definition 5.3.** Let \(\mathcal{P}\) be a fixed finite subset of \(\{1 < p < \infty\}\). A bilinear form \(B(\cdot, \cdot)\) is said to be adapted to \(\mathcal{P}\) if it measures its arguments in Strichartz-type spaces, the estimate

\[
\left| \int_0^T \int_{\mathbb{R}^d} f \wedge g \, dx dt \right| \lesssim B(f, g)
\]

holds for all complex-valued functions \(f, g\) on \(\mathbb{R}^d \times [0, T]\), Bernstein’s inequalities hold in both arguments of \(B\), and these arguments are measured in \(L^p_x\) only for \(p \in \mathcal{P}\). Given \(B(\cdot, \cdot)\) and \(e \in \mathbf{S}^{d-1}\), we define \(B_e(\cdot, \cdot)\) via

\[
B_e(f, g) := B(f, \chi_{\{x \in \mathbb{R}^d : 0 < |x| < \epsilon\}} g).
\]

**Definition 5.4.** Let \(e \in \mathbf{S}^{d-1}\) and let \(A_\ell\) be real-valued and smooth. Let \(v\) be a \(C^\infty_t(H^\infty_x)\) function on \(\mathbb{R}^d \times [0, T]\) solving

\[
(i \partial_t + \Delta_A) v = \Lambda_v,
\]

Assume \(v\) is (spatially) frequency-localized to \(I_k\) with the additional constraint that \(e \cdot \xi \in [2^{j-1}, 2^{j+1}]\) for all \(\xi\) in the support of \(\hat{v}\). Define a sequence of functions \(\{v^{(m)}\}_{m=1}^\infty\) by setting \(v^{(1)} = v\) and

\[
v^{(m+1)} := v^{(m)} + i2^{-j} \partial_e v^{(m)}.
\]

By \(5.1\) and the Leibniz rule,

\[
(i \partial_t + \Delta_A) v^{(m)} = \Lambda_{v^{(m)}},
\]

where

\[
\Lambda_{v^{(m)}} := (1 + i2^{-j} \partial_e) \Lambda_e v^{(m-1)} + i2^{-j}(i \partial_e \partial_t A_\ell - \partial_e A_\ell^2) v^{(m-1)} - 2^{-j+1}(\partial_e A_\ell) \partial_t v^{(m-1)}.
\]

The sequence \(\{v^{(m)}\}_{m=1}^\infty\) is called the derived sequence corresponding to \(v\).

Suppose we are given a form \(B\) adapted to \(\mathcal{P}\). The derived sequence is said to be controlled with respect to \(B_e\) provided that \(B_e(v^{(m)}, \Lambda_{v^{(m)}}) < \infty\) for each \(m \geq 1\).

We remark that if the derived sequence \(\{v^{(m)}\}_{m=1}^\infty\) of \(v\) is controlled, then for all \(\ell \geq 1\), the derived sequences \(\{v^{(m)}\}_{m=1}^\infty\) are also controlled.

**Theorem 5.5** (Abstract local smoothing). Let \(d \geq 1\) and \(e \in \mathbf{S}^{d-1}\). Let \(j, k \in \mathbb{Z}\) and \(j = k + O(1)\). Let \(\varepsilon_m > 0\) be a small positive number such that \(\varepsilon_m 2^{O(1)} \ll 1\). Let \(\eta > 0\). Let \(\mathcal{P}\) be a fixed finite subset of \((1, \infty)\) with \(2 \in \mathcal{P}\), and let \(B_e\) be a form adapted to \(\mathcal{P}\). Let \(v\) be a \(C^\infty_t(H^\infty_x)\) function on \(\mathbb{R}^d \times [0, T]\) solving

\[
(i \partial_t + \Delta_A) v = \Lambda_v,
\]

(5.14)
where $A_\ell$ is real-valued, smooth, has spatial Fourier support in $I_{(-\infty,k]}$, and satisfies the estimate
\[ \|A\|_{L^\infty_t L^2_x} \leq \varepsilon_m 2^k. \] (5.15)

The solution $v$ is assumed to have (spatial) frequency support in $I_k$. We take $\Lambda_v$ to be frequency-localized to $I_{(-\infty,k]}$.

Assume moreover that
\[ \mathbf{e} \cdot \xi \in [(1 - \eta)2^j, (1 + \eta)2^j], \] (5.16)
for all $\xi$ in the support of $\hat{v}$.

If the derived sequence of $v$ is controlled with respect to $B_\mathbf{e}$, then there exist $\eta^* > 0$ such that, for all $0 \leq \eta < \eta^*$, the local smoothing estimate
\[ 2^j \int_0^T E_\mathbf{e}(v) dt \lesssim \|v\|^2_{L^\infty_t L^2_x} + B_\mathbf{e}(v, \Lambda_v) \] (5.17)
holds uniformly in $T$ and $j = k + O(1)$.

Proof. The foundation for proving (5.17) is (5.7), which for an adapted form $B_\mathbf{e}$ implies
\[ 2^j \int_0^T E_\mathbf{e}(v) dt \lesssim \|v\|^2_{L^\infty_t L^2_x} + B_\mathbf{e}(v, \Lambda_v) + 2^j \int_0^T E_\mathbf{e}(v + i2^{-j} \partial_\mathbf{e} v) dt. \] (5.18)

Therefore our goal is to control the last term in (5.18). This we do using a bootstrap argument that hinges upon the fact that $\tilde{v} := v + i2^{-j} \partial_\mathbf{e} v$ is the second term in the derived sequence of $v$, and that being “controlled” is an inherited property (in the sense of the comments following Definition 5.4).

By Bernstein’s and Hölder’s inequalities, we have
\[ 2^j \int_0^T E_\mathbf{e}(v) dt \lesssim 2^j T \|v\|^2_{L^\infty_t L^2_x}. \]
for any $v$. For fixed $T > 0$ and $k \in \mathbb{Z}$, let $K_{T,k} \geq 1$ be the best constant for which the inequality
\[ 2^j \int_0^T E_\mathbf{e}(v) dt \leq K_{T,k} \left( \|v\|^2_{L^\infty_t L^2_x} + B_\mathbf{e}(v, \Lambda_v) \right) \] (5.19)
holds for all controlled sequences. Applying (5.19) to $\tilde{v}$ results in
\[ 2^j \int_0^T E_\mathbf{e}(\tilde{v}) dt \leq K_{T,k} \left( \|\tilde{v}\|^2_{L^\infty_t L^2_x} + B_\mathbf{e}(\tilde{v}, \Lambda_{\tilde{v}}) \right), \] (5.20)
and thus we seek to control norms of $\tilde{v}$ in terms of those of $v$.

Let $\tilde{P}_k, \tilde{P}_{j,\mathbf{e}}$ denote slight fattenings of the Fourier multipliers $P_k, P_{j,\mathbf{e}}$. On the one hand, Plancherel implies
\[ \|(1 + i2^{-j} \partial_\mathbf{e}) \tilde{P}_{j,\mathbf{e}} \tilde{P}_k \|_{L^2_x \to L^2_x} \lesssim \eta. \] (5.21)
On the other hand, Bernstein’s inequalities imply
\[ \| (1 + i 2^{-j} \partial_e) \tilde{P}_j e \tilde{P}_k \|_{L^p_x \to L^p_x} \lesssim 1, \quad 1 \leq p \leq \infty. \]
Therefore it follows from Riesz-Thorin interpolation that
\[ \| (1 + i 2^{-j} \partial_e) \tilde{P}_j e \tilde{P}_k \|_{L^p_x \to L^p_x} \lesssim \left\{ \begin{array}{ll} \eta^{2/p} & 2 \leq p < \infty \\ \eta^{2-2/p} & 1 < p \leq 2. \end{array} \right. \]
Restricting to \( p \in \mathcal{P} \), we conclude that there exists a \( q > 0 \) such that
\[ \| (1 + i 2^{-j} \partial_e) \tilde{P}_j e \tilde{P}_k \|_{L^p_x \to L^p_x} \lesssim \eta^q \]  
for all \( p \in \mathcal{P} \) and all \( \eta \) small enough.

Applying (5.22) and Bernstein to \( \tilde{v} \) yields
\[ \| \tilde{v} \|_{L^2_x} \lesssim \eta^q \| v \|_{L^2_x}, \quad B_e(\tilde{v}, \Lambda \tilde{v}) \lesssim \eta^q B_e(v, \Lambda v), \]
which, combined with (5.20) and (5.18), leads to
\[ 2^j \int_0^T E_e(v) dt \lesssim (1 + \eta^q K_{T,k}) \left( \| v \|^2_{L^\infty_t L^2_x} + B_e(v, \Lambda v) \right) . \]
As \( K_{T,k} \) is the best constant for which (5.19) holds, it follows that
\[ K_{T,k} \lesssim 1 + \eta^q K_{T,k} \]
and hence that \( K_{T,k} \lesssim 1 \) for \( \eta \) small enough. \( \square \)

**Corollary 5.6.** Given the assumptions of Theorem 5.5, it holds that
\[ 2^j \int_0^T E_e(v) dt \lesssim \| v_0 \|^2_{L^2_x} + B(v, \Lambda v) + B_e(v, \Lambda v) \]

**Proof.** This is an immediate consequence of Theorem 5.5 and Lemma 5.1. \( \square \)

**Corollary 5.7** (Abstract bilinear Strichartz). Let \( d \geq 1 \) and \( e \in S^{d-1} \). Set \( \tilde{e} = (-e, e)/\sqrt{2} \). Let \( j, k \in \mathbb{Z} \) and \( j = k + O(1) \). Let \( \varepsilon_m > 0 \) be a small positive number such that \( \varepsilon_m 2^{O(1)} \ll 1 \). Let \( \eta > 0 \). Let \( \mathcal{P} \) be a fixed finite subset of \( (1, \infty) \) with \( 2 \in \mathcal{P} \), and let \( B_e \) be a form that is adapted to \( \mathcal{P} \).

Let \( w(x, t) \) be a \( C^\infty_t (H^\infty_{x,y}) \) function on \( \mathbb{R}^{2d} \times [0, T] \), equal to \( w_0 \) at \( t = 0 \) and solving
\[ (i\partial_t + \Delta_A)w = \Lambda_w, \]
where \( \Lambda_{Ak} \) is real-valued, smooth, has spatial Fourier support in \( I_{(-\infty, k]} \), and satisfies the estimate
\[ \| A \|_{L^\infty_t L^\infty_{x,y}} \leq \varepsilon_m 2^k. \]
Assume \( w \) has (spatial) frequency support in \( I_k \) and that
\[ \tilde{e} \cdot \xi \in [(1 - \eta)2^j, (1 + \eta)2^j] \]
for all \( \xi \) in the support of \( \tilde{w} \). Take \( \Lambda_w \) to be frequency-localized to \( I_{(-\infty, k]} \).
Suppose that \( w(x, y) \) admits a decomposition \( w(x, y) = u(x)v(y) \), where \( u \) has frequency support in \( I_\ell \), \( \ell \ll k \). Use \( u_0, v_0 \) to denote \( u(t = 0), v(t = 0) \).

If the derived sequence of \( w \) is controlled with respect to \( B_\ell \), then

\[
\| uv \|_{L_{x,y}^2}^2 \lesssim 2^{(d-1)2^{-j}} \left( \| u_0 \|_{L_x^2}^2 \| v_0 \|_{L_y^2}^2 + B(w, \Lambda_w) + B_\ell(w, \Lambda_w) \right) \tag{5.23}
\]

uniformly in \( T \) and \( j = k + O(1) \) provided \( \eta \) is small enough.

**Proof.** Taking into account that

\[

w_0 \|_{L_{x,y}^2} = \| u_0 \|_{L_x^2} \| v_0 \|_{L_y^2},
\]

we apply Corollary 5.6 to \( w \) at \( (x, y) = 0 \):

\[
2^j \int_0^T E_\ell(w) dt \lesssim \| u_0 \|_{L_x^2}^2 \| v_0 \|_{L_y^2}^2 + B(w, \Lambda_w) + B_\ell(w, \Lambda_w). \tag{5.24}
\]

We complete \((-e, e)/\sqrt{2}\) to a basis as follows:

\((-e, e)/\sqrt{2}, (0, e_1), \ldots, (0, e_{d-1}), (e, e)/\sqrt{2}, (e_1, 0), \ldots, (e_{d-1}, 0)\).

On the one hand, \( E_\ell(w)(0) \) is by definition (see (5.2)) equal to

\[
\int \int \int_{\mathbb{R}^{2d-2}} |u(0 \cdot e + re + xj e_j, t)v(0 \cdot e + re + yj e_j, t)|^2 dx' dy' dr.
\]

We rewrite it as

\[
\int \int \int_{\mathbb{R}^{d-1}} |v(re + yj e_j, t)|^2 dy' \int \int |u(re + xj e_j, t)|^2 dx' dr. \tag{5.25}
\]

On the other hand,

\[
\| uv \|_{L_y^2}^2 = \int \int |u(y, t)|^2 |v(y, t)|^2 dy
\]

\[= \int \int \int |u(re + yj e_j)|^2 |v(re + yj e_j)|^2 dy' dr,
\]

and by applying Bernstein to \( u \) in the \( y' \) variables, we obtain

\[
\| uv \|_{L_y^2}^2 \lesssim 2^{(d-1)} \int \int \int |v(re + yj e_j)|^2 dy' \int \int |u(re + xj e_j)|^2 dx' dr. \tag{5.26}
\]

Together (5.26), (5.25), and (5.24) imply (5.23). \( \square \)

### 5.2. Applying the abstract lemmas

We would like to apply the abstract estimates just developed to the evolution equation (2.7). We work in the caloric gauge and adopt the magnetic potential decomposition introduced in §4.3. Throughout we take \( \varepsilon \) as defined in (4.18).

Our starting point is the equation

\[
(i\partial_t + \Delta)\psi_m = B_{m,lo\lambda_0} + B_{m,hi\lambda_0} + V_m. \tag{5.27}
\]
Applying Fourier multipliers $P_k$, $P_{j,k} P_k$, or variants thereof, we easily obtain corresponding evolution equations for $P_k \psi_m$, $P_{j,k} P_k$, etc. In rewriting a projection $P$ of (5.27) in the form (5.3), evidently $\Delta A \psi_m$ should somehow come from $\Delta P \psi_m - P B_{m,lo}/lo$, whereas $P B_{m,hi/hi} + P V_m$ ought to constitute the leading part of the nonlinearity $\Lambda$. Fourier multipliers $P$, however, do not commute with the connection coefficients $A$, and therefore in order to use the abstract machinery we must first track and control certain commutators. Toward this end we adopt some notation from [52].

Following [52, §1], we use $L_O(f_1, \ldots, f_m)(s, x, t)$ to denote any multi-linear expression of the form

$$L_O(f_1, \ldots, f_m)(s, x, t) := \int K(y_1, \ldots, y_{M(c)} f_1(s, x - y_1, t) \ldots f_m(s, x - y_{M(c)}, t) dy_1 \ldots dy_{M(c)},$$

where the kernel $K$ is a measure with bounded mass (and $K$ may change from line to line). Moreover, the kernel of $L_O$ does not depend upon the index $\alpha$. Also, we extend this notation to vector or matrices by making $K$ into an appropriate tensor. The expression $L_O(f_1, \ldots, f_m)$ may be thought of as a variant of $O(f_1, \ldots, f_m)$. It obeys two key properties. The first is a simple consequence of Minkowski’s inequality (e.g., see [52, Lemma 1]).

**Lemma 5.8.** Let $X_1, \ldots, X_m, X$ be spatially translation-invariant Banach spaces such that the product estimate

$$\|f_1 \cdots f_m\|_X \leq C_0 \|f_1\|_{X_1} \cdots \|f_m\|_{X_m}$$

holds for all scalar-valued $f_i \in X_i$ and for some constant $C_0 > 0$. Then

$$\|L_O(f_1, \ldots, f_m)\|_X \lesssim (C d)^{C m} C_0 \|f_1\|_{X_1} \cdots \|f_m\|_{X_m}$$

holds for all $f_i \in X_i$ that are scalars, $d$-dimensional vectors, or $d \times d$ matrices.

The next lemma is an adaption of [52 Lemma 2].

**Lemma 5.9** (Leibniz rule). Let $P_k'$ be a $C^\infty$ Fourier multiplier whose frequency support lies in some compact subset of $I_k(\mathbb{R}^d)$. The commutator identity

$$P_k'(fg) = f P_k' g + L_O(\partial_x f, 2^{-k} g)$$

holds.

**Proof.** Rescale so that $k = 0$ and let $m(\xi)$ denote the symbol of $P_0'$ so that

$$\hat{P_0'} h(\xi) := m(\xi) \hat{h}(\xi).$$
By the Fundamental Theorem of Calculus, we have
\[
(P_0'(fg) - fP_0'g)(s, x, t) = \int_{\mathbb{R}^d} \hat{m}(y) \left( f(s, x - y, t) - f(s, x, t) \right) g(s, x - y, t) dy.
\]
\[= -\int_0^1 \int_{\mathbb{R}^d} \hat{m}(y) y \cdot \partial_x f(s, x - ry, t) g(s, x - y, t) dy dr.
\]
The conclusion follows from the rapid decay of \(\hat{m}\).

We are interested in controlling \(P_{\theta,j}P_k\psi_m\) in \(L^\infty_t\) over all \(\theta \in \mathbb{S}^1\) and \(|j - k| \leq 20\). In the abstract framework, however, we assumed a much tighter localization than \(P_{\theta,j}\) provides. Therefore we decompose \(P_{\theta,j}\) as a sum
\[
P = \sum_{l = 1, \ldots, O((\eta^*)^{-1})} P_{\theta,j,l},
\]
and it suffices by the triangle inequality to bound \(P_{\theta,j,l}P_k\psi_m\). We note that this does not affect perturbative estimates since \(\eta^*\) is universal and in particular does not depend upon \(\varepsilon_1, \varepsilon\).

For notational convenience set \(P := P_{\theta,j,l}P_k\). Applying \(P\) to \(\psi_m\) yields
\[
(i\partial_t + \Delta)P\psi_m = P(B_{m,lo\land lo} + B_{m,hi\lor hi} + V_m).
\]
Now
\[
P B_{m,lo\land lo} = -i P \sum_{|k_3 - k| \leq 4} (\partial_t (A_{\ell,lo\land lo} P_{k_3} \psi_m) + A_{\ell,lo\land lo} \partial_t P_{k_3} \psi_m),
\]
as \(P\) localizes to a region of the annulus \(I_k\). Applying Lemma 5.9 we obtain
\[
P B_{m,lo\land lo} = -i(\partial_t (A_{\ell,lo\land lo} P\psi_m) - iA_{\ell,lo\land lo} \partial_t P\psi_m) + R
\]
where
\[
R := \sum_{|k_3 - k| \leq 4} \left( L_O(\partial_x \partial_t A_{\ell,lo\land lo}, 2^{-k} P_{k_3} \psi_m) + L_O(\partial_x A_{\ell,lo\land lo}, 2^{-k} P_{k_3} \partial_t \psi_m) \right).
\]
Set
\[
A_m := A_{m,lo\land lo}.
\]
Then
\[
(i\partial_t + \Delta_A)P\psi_m = P(B_{m,hi\lor hi} + V_m) + A^2 P\psi_m + R.
\]
It is this equation that we shall show fits within the abstract local smoothing framework.

First we check that Lemmas 5.1 and 5.2 apply. The main condition to check is (5.6). Key are (2.14) and Bernstein, which together with the fact that \(A\) is frequency-localized to \((\infty, k] \) provide the estimate
\[
\|A\|_{L^\infty_t} \lesssim 2^k.
\]
To achieve the $\varepsilon_m$ gain, we adjust $\varpi$, which forces a gap between $I_k$ and
the frequency support of $A$, i.e., we localize $A$ to $I_{(-\infty,k-\varpi]}$ instead. Thus
it suffices to set $\varpi \in \mathbb{Z}_+$ equal to a sufficiently large universal constant.

There is more to check in showing that (5.30) falls within the purview of
Theorem 5.5. Already we have $d = 2$, $e = \theta$, $\varepsilon_m \sim 2^{-\varpi}$, $A_m := A_{m,lo\land lo}$, $v = P_{\theta,j,t}P_k\psi_m$, and $\Lambda_v = P(B_{m,hv\lor hi} + V_m) + A^2_xP\psi_m + R$.

Next we choose $P$ based upon the norms used in $N_k$, with the exception of
the local smoothing/maximal function estimates. To be precise, define the new norms $\tilde{N}_k$ via
\[
\|f\|_{\tilde{N}_k(T)} := \inf_{f=f_1+f_2+f_3+f_4+f_5} \|\underline{f}_1\|_{L^4_{t,x}} + 2^{k/6}\|\underline{f}_2\|_{L^{3/2,6/5}_{t,y_1}} + 2^{k/6}\|\underline{f}_3\|_{L^{3/2,6/5}_{t,y_2}} + 2^{-k/6}\|\underline{f}_4\|_{L^{5/3,2}_{t,y_1}} + 2^{-k/6}\|\underline{f}_5\|_{L^{5/3,2}_{t,y_2}}
\]
and similarly $\tilde{G}_k$ via
\[
\|f\|_{\tilde{G}_k(T)} := \|f\|_{L^\infty_tL^4_x} \|f\|_{L^2_{t,x}} + 2^{-k/2}\|f\|_{L^4_{t,x}} + 2^{-k/6}\|f\|_{L^{3/2,6/5}_{t,y_1}} + 2^{-k/6}\|f\|_{L^{3/2,6/5}_{t,y_2}} + 2^{-k/6}\|f\|_{L^{3/2,6/5}_{t,y_3}} + 2^{-k/6}\|f\|_{L^{3/2,6/5}_{t,y_4}}.
\]

Set $P = \{2, 3, 3/2, 4, 4/3, 6, 5/6\}$. We define the form $B(\cdot, \cdot)$ via
\[
B(f, g) := \|f\|_{\tilde{G}_k(T)}\|g\|_{\tilde{N}_k(T)}
\]
and $B_{\theta}$ by
\[
B_{\theta}(f, g) := B(f, \chi_{\{\theta \geq 0\}} g)
\]
as in Definition 5.3. That $B_{\theta}$ is adapted to $P$ is a direct consequence of the
definition.

**Proposition 5.10.** Let $\eta > 0$ be a parameter to be specified later. Let $d = 2$, $e = \theta$, $\varepsilon_m \sim 2^{-\varpi}$, $A_m := A_{m,lo\land lo}$, $v = P_{\theta,j,t}P_k\psi_m$, $\Lambda_v = P(B_{m,hv\lor hi} + V_m) + A^2_xP\psi_m + R$, and $P = \{2, 3, 3/2, 4, 4/3, 6, 5/6\}$. Let $B, B_{\theta}$ be given by (5.31) and (5.32) respectively. Then the conditions of Theorem 5.5 are satisfied and the derived sequence of $v$ is controlled with respect to $B_{\theta}$ so that conclusion (5.17) holds for $v = P_{\theta,j,t}P_k\psi_m$ given $\eta$ sufficiently small.

**Proof.** The only claim of Proposition 5.10 that remains to be verified is that the derived sequence of $v = P_{\theta,j,t}P_k\psi_m$ is controlled with respect to $B_{\theta}$. In particular, we need to show that for each $q \geq 1$ we have
\[
B_{\theta}(v^{(q)}, \Lambda_{v^{(q)}}) < \infty,
\]
where $v^{(1)} := P_{\theta,j,t}P_k\psi_m$,
\[
v^{(q+1)} := v^{(q)} + i2^{-j}\partial_{\theta}v^{(q)}
\]
and
\[
\Lambda_{v^{(q+1)}} := (1 + i2^{-j}\partial_{\theta})\Lambda_{v^{(q)}} + i2^{-j}(i\partial_{\theta}\partial_{t}A_{t} - \partial_{\theta}A^2_{\theta})v^{(q)} - 2^{-j+1}(\partial_{\theta}A_{t})\partial_{t}v^{(q)}.
\]
We first prove the following lemma.

**Lemma 5.11.** Let \( \sigma \in [0, \sigma_1 - 1] \). The right hand side of (5.30) satisfies
\[
\| P(B_{m, \text{hi\textbackslash hi}} + V_m) + A_x^2 P \psi_m + R \| \lesssim e2^{-\sigma k} b_k(\sigma).
\]

**Proof.** We will repeatedly use implicitly the fact that the multiplier \( P_{\theta, j, l} \) is bounded on \( L^p, 1 \leq p \leq \infty \), so that in particular \( P \) obeys estimates that are at least as good as those obeyed by \( P_k \).

From Corollaries 4.11, 4.12 of \( PV \) it follows that \( P_k(B_{m, \text{hi\textbackslash hi}} + V_m) \) is perturbative and bounded in \( \tilde{N}_k(T) \) by \( e2^{-\sigma k} b_k(\sigma) \). The \( \tilde{N}_k(T) \) estimates on \( PV_m \) immediately imply the boundedness of \( A_x^2 P \psi_m \).

To estimate \( R \), we apply Lemma 3.10 to bound \( PB_{m, \text{lo\textbackslash lo}} \) by
\[
\sum_{(k_1, k_2, k_3) \in Z_1(k)} \int_0^\infty 2^{\max\{k_1, k_2\}} 2^{k_3 - k} C_{k, k_1, k_2, k_3} \| P \psi_x(s) \|_{F_{k_1} \times F_{k_2}} \times \| P_{k_2} (D \psi \ell(s)) \|_{F_{k_2}} \| P_{k_3} \psi_m(0) \|_{G_{k_3}} ds,
\]
which, in view of (4.12), (4.13), and (4.25), is controlled by
\[
\sum_{(k_1, k_2, k_3) \in Z_1(k)} C_{k, k_1, k_2, k_3} b_k b_{k_2} 2^{-\sigma k} b_{k_3} (\sigma).
\]
Summation is achieved thanks to Corollary 3.11. \( \square \)

We return to the proof of the proposition, and in particular to showing that \( B_\theta(v, \Lambda_v) < \infty \). With the important observation that the spatial multiplier \( \chi_{x, \theta \geq 0} \) is bounded on the spaces \( \tilde{N}_k(T) \), we may apply Lemma 5.11 to control \( \chi_{x, \theta \geq 0} \Lambda_v \) in \( \tilde{N}_k \). Since by assumption \( P \psi_m \) is bounded in \( \tilde{G}_k(T) \) (even in \( G_k(T) \)), we conclude that \( B_\theta(v, \Lambda_v) < \infty \).

Next we need to show \( B_\theta(v^q, \Lambda_{v^q}) < \infty \) for \( q > 1 \). By Bernstein,
\[
\| v^{(q)} \|_{\tilde{G}_k(T)} \lesssim \| v^{(q-1)} \|_{\tilde{G}_k(T)}.
\]
Similarly,
\[
\| (1 + i2^{-j}) \partial_{\theta} A_{v^{(q)}} \|_{\tilde{N}_k(T)} \lesssim \| A_{v^{(q-1)}} \|_{\tilde{N}_k(T)}.
\]

Thus it remains to control \( i2^{-j} (i \partial_{\theta} \partial_t A_{l} - \partial_{\theta} A_{l}^2) v^{(q)} \) and \( 2^{-j+1} (\partial_{\theta} A_{l}) \partial_{\theta} v^{(q)} \) in \( \tilde{N}_k \) for each \( q > 1 \). Both are consequences of arguments in Lemma 5.11: Boundness of \( 2^{-j} (\partial_{\theta} \partial_t A_{l}) v^{(q)} \) and \( 2^{-j+1} (\partial_{\theta} A_{l}) \partial_{\theta} v^{(q)} \) follows directly from the argument used to control \( R \) and from Bernstein’s inequality, whereas boundness of \( 2^{-j} (\partial_{\theta} A_{l}^2) v^{(q)} \) is a consequence of Bernstein and the estimates on \( A_x^2 P \psi_m \) from (4.13). \( \square \)
Combining Lemma 5.11 and Proposition 5.10, we conclude that Corollary 5.6 applies to \( v = P\psi_m \), with right hand side bounded by \( 2^{-2\sigma k_c} c_k(\sigma)^2 + \epsilon 2^{-2\sigma k} b_k(\sigma)^2 \). In view of the decomposition (5.28), we conclude

**Corollary 5.12.** Assume \( \sigma \in [0, \sigma_1 - 1] \). The function \( P_k\psi_m \) satisfies

\[
\sup_{|j-k| \leq 20} \sup_{\theta \in \mathbb{S}^1} \| P_{j,\theta} P_k\psi_m \|_{L^\infty_{x,y} L^2_{t}} \lesssim 2^{-k/2} \left( 2^{-\sigma k} c_k(\sigma) + \epsilon 1/2 2^{-\sigma k} b_k(\sigma) \right),
\]

thereby establishing Theorem 4.19.

Our next objective is to apply Corollary 5.7 to the case where \( w \) splits as a product \( u(x)v(y) \) where \( u, v \) are appropriate frequency localizations of \( \psi_m \) or \( \overline{\psi}_m \). First we must find function spaces suitable for defining an adapted form. We start with \( (i\partial_t + \Delta_A)w = \Lambda_w \) and observe how it behaves with respect to separation of variables. If \( w(x,y) = u(x)v(y) \), then the left hand side may be rewritten as \( u \cdot (i\partial_t + \Delta_A) v + v \cdot (i\partial_t + \Delta_A) u \). Let \( \Lambda_u := (i\partial_t + \Delta_A) u \) and \( \Lambda_v := (i\partial_t + \Delta_A) v \). Then

\[
(i\partial_t + \Delta_A)(uv) = u\Lambda_v + v\Lambda_u.
\]

We control

\[
\int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} u(x)v(y) (\Lambda_u(x)v(y) + u(x)\Lambda_v(y)) \, dx \, dy \, dt
\]

as follows: in the case of the first term \( u(x)v(y)\Lambda_u(x)v(y) \) we place each \( v(y) \) in \( L^\infty_{x} L^2_y \); we bound \( u(x)\Lambda_u(x) \) by placing \( u(x) \) in \( G_j \) and \( \Lambda_u(x) \) in \( \tilde{N}_j \). To control \( u(x)v(y)u(x)\Lambda_v(y) \), we simply reverse the roles of \( u \) and \( v \) (and of \( x \) and \( y \)). This leads us to the spaces \( \overline{N}_{k,\ell} \) defined by

\[
\|f\|_{\overline{N}_{k,\ell}(T)} := \inf_{J \in \mathbb{Z}^2 \setminus \{ f(x,y) = 0 \}} \left( \|g_{j}(x)\|_{\tilde{N}_j(T)} \|h_j\|_{L^\infty_{t} L^2_y} \right.
\]

\[
+ \|g_{j+1}\|_{L^\infty_{t} L^2_y} \|h_{j+1}\|_{\tilde{N}_{j+1}(T)} \right),
\]

and the spaces \( \overline{G}_{k,\ell} \) defined via

\[
\|f\|_{\overline{G}_{k,\ell}(T)} := \|f(x,y)\|_{\tilde{G}_{k}(T)(y)} \|\tilde{G}_{\ell}(T)(x)\|.
\]

We use these spaces to define the form \( \overline{B}(\cdot, \cdot) \) by

\[
\overline{B}(f, g) := \|f\|_{\overline{G}_{k,\ell}(T)} \|g\|_{\overline{N}_{k,\ell}(T)}
\]

and the form \( \overline{B}_\Theta \) by

\[
\overline{B}_\Theta(f, g) := \overline{B}(f, \chi_{\{x,y\}: \Theta \geq 0}) g,
\]

where \( \Theta := (-\theta, \theta) \).
Proposition 5.13. Let $\eta > 0$ be a small parameter and $\varpi \in \mathbb{Z}_+$ a large parameter, both to be specified later. Let $j, k, \ell \in \mathbb{Z}$, $j = k + O(1)$, $\ell \ll k$.

Let $d = 2$, $e = \theta$, $\varepsilon_m \sim 2^{-\infty}$, $A_x := A_{m, k \Delta k \varepsilon_{\alpha}}$, $v = P^j(\psi^m)$, $A_v = P(B_{m, \psi} + V_m) + A^2_j \psi^m + R$, and $\mathcal{P} = \{2, 3, 3/2, 4, 4/3, 6, 5/6\}$. Here $R$ is given by (5.29). Also, let $u = \overline{P_e \psi_p}$, $p \in \{1, 2\}$ and $\mathcal{P}_u = P(I_p(B_p, \psi_{hi} V_p) + A^2_j P \psi_{hi} + R')$, where $R'$ is given by (5.24), but defined in terms of derivative field $\psi_i$ and frequency $\ell$ rather than $\psi_m$ and $k$.

Let $w(x, y) := u(x) \psi(y)$, $A := (A_x, A_y)$, $A_w := \Lambda_u v + u \Lambda_u$. Then, for $\varpi$ sufficiently large and $\eta$ sufficiently small, the conditions of Corollary 5.12 are satisfied and (5.23) applies to $u(x) \psi(x)$.

Proof. The frequency support conditions on $A$ and $\Lambda_u$ are easily verified. That the $L^\infty$ bound on $A$ holds follows from (2.14) and Bernstein provided $\varpi$ is large enough (cf. discussion preceding Proposition 5.10). In order to guarantee the frequency support conditions on $w$, it is necessary to make the gap $\ell \ll k$ sufficiently large with respect to $\eta$.

That $\overline{\mathcal{P}_\varpi}$ is adapted to $\mathcal{P}$ is a straightforward consequence of its definition. To see that the derived sequence of $w$ is controllable, we look to the proof of Proposition 5.10 and the definitions of the $\overline{\mathcal{N}}_{k, \ell}, \overline{\mathcal{G}}_{k, \ell}$ spaces.

In a spirit similar to that of the proof of Corollary 5.12, we may combine Lemma 5.11 and the proof of Proposition 5.10 to control $B(w, \Lambda_w) + B\varphi(w, \Lambda_w)$; in fact, in measuring $\Lambda_w$ in the $\overline{\mathcal{N}}_{k, \ell}$ spaces, it suffices to take $J = 1$ (see (5.33)). Then we obtain $B(w, \Lambda_w) + B\varphi(w, \Lambda_w) \lesssim c_j 2^{-\sigma b_j} b_k(\sigma)$. Using decomposition (5.28) and the triangle inequality to bound $P_j \psi_m$ in terms of the bounds on $P_{j, \ell} \psi_m$, we obtain the bilinear Strichartz analogue of Corollary 5.12. In our application, however, the lower-frequency term will not simply be $P_j \psi_i$, but rather its heat flow evolution $P_j \psi_i(s)$.

Corollary 5.14 (Improved Bilinear Strichartz). Let $j, k \in \mathbb{Z}$, $j \ll k$, and $u \in \{P_j \psi_i, P_j \psi_i : j \leq k - \varpi, \ell \in \{1, 2\}\}$. Then for $s \geq 0$, $\sigma \in [0, \sigma_1 - 1]$,

$$
\|u(s) P_k \psi_m(0)\|_{L^2_{x, s}} \lesssim 2^{j-k}/2 (1 + s^{2j})^{-4k} \sigma^{-\alpha} \eta \left(c_j c_k(\sigma) + c_j b_k(\sigma)\right). \tag{5.37}
$$

Proof. It only remains to prove (5.37) when $s > 0$. Let $v := P_k \psi_m$. Using the Duhamel formula, we write

$$
u(s) = (e^{s \Delta} u(0)) v(0) + \int_0^s e^{s \Delta} U(s', s') ds' \cdot v(0), \tag{5.38}
$$

where $U$ is defined by (2.21) in terms of $u$.

To control the nonlinear term $\int_0^s e^{s \Delta} U(s', s') ds' \cdot v(0)$ in $L^2$, we apply local smoothing estimate (5.11), which places the nonlinear evolution in $F_j(T)$ and
Using Lemma 7.11 to bound the $F_j(T)$ norm, we conclude
\[
\| \int_0^t e^{(s-s')\Delta} \tilde{U}(s') \, ds' \cdot v(0) \|_{L^t_{t,x}} \lesssim \epsilon 2^{(j-k)/2} (1 + s 2^{2j})^{-4} 2^{-\sigma k} b_j b_k(\sigma). \tag{5.39}
\]
It remains to show
\[
\| (e^{s\Delta} u) v \|_{L^2_t L^2_x} \lesssim (1 + s 2^{2j})^{-4} 2^{(j-k)/2} 2^{-\sigma k} (c_j c_k(\sigma) + e b_j b_k(\sigma)), \tag{5.40}
\]
which is not a direct consequence of the time $s = 0$ bound. Let $T_a$ denote the spatial translation operator that acts on functions $f(x,t)$ according to
\[
T_a f(x,t) := f(x-a,t).
\]
Then, if
\[
\| (T_{x_1} u)(T_{x_2} v) \|_{L^2_t L^2_x} \lesssim 2^{(j-k)/2} 2^{-\sigma k} (c_j c_k(\sigma) + e b_j b_k(\sigma)) \tag{5.41}
\]
can be shown to hold for all $x_1, x_2 \in \mathbb{R}^2$, then (5.40) follows from Minkowski’s and Young’s inequalities.

Consider, then, a solution $w$ to
\[
(i \partial_t + \Delta_A(x,t)) w(x,t) = \Lambda w(x,t)
\]
satisfying the conditions of Theorem 5.5. The translate $T_{x_0} w(x,t)$ then satisfies
\[
(i \partial_t + \Delta_{T_{x_0} A}(x,t))(T_{x_0} w)(x,t) = (T_{x_0} \Lambda w)(x,t).
\]
The operator $T_{x_0}$ clearly does not affect $L^\infty_{t,x}$ bounds or frequency support conditions. The only possible obstruction to concluding (5.17) is this: whereas the derived sequence of $w$ is controlled with respect to $B_e$, in the abstract setting it may no longer be the case that the derived sequence of $T_{x_0} w$ is controlled. This is due to the presence of the spatial multiplier in the definition of $B_e$. Fortunately, as already alluded to in the proof of Proposition 5.10, in our applications we do enjoy uniform boundedness with respect to any spatial multipliers appearing in the second argument of an adapted form $B_e$. Therefore Proposition 5.13 holds for spatial translates of frequency projections of $\psi_m$, from which we conclude (5.41).

This establishes Theorem 4.8.

6. The caloric gauge

In §6.1 we briefly recall from [44] the construction of the caloric gauge and some useful quantitative estimates. In §6.2 we prove the frequency-localized estimates stated in §2.3.
6.1. Construction and basic results. In brief, the basic caloric gauge construction goes as follows. Starting with $H^\infty_Q$-class data $\varphi_0 : \mathbb{R}^2 \to S^2$ with energy $E(\varphi_0) < E_{\text{crit}}$, evolve $\varphi_0$ in $s$ via the heat flow equation (2.11). At $s = \infty$ the map trivializes. Place an arbitrary orthonormal frame $e(\infty)$ on $T_{\varphi(s=\infty)}S^2$. Evolving this frame backward in time via parallel transport in the $s$ direction yields a caloric gauge on $\varphi^*T_{\varphi(s=\infty)}S^2$.

For energies $E(\varphi_0)$ sufficiently small, global existence and decay bounds may be proven directly using Duhamel’s formula. In order to extend these results to all energies less than $E_{\text{crit}}$, we employ in [44] an induction-on-energy argument that exploits the symmetries of (2.11) via concentration compactness.

In [44] the following energy densities play an important role in the quantitative arguments.

**Definition 6.1.** For each positive integer $k$, define the energy densities $e_k$ of a heat flow $\varphi$ by

$$e_k := |(\varphi^n_x)^{k-1} \partial_x \varphi|^2$$

$$:= \langle ((\varphi^n_x)_{j_1} \cdots (\varphi^n_y)_{j_{k-1}} \partial_{j_k} \varphi), (\varphi^n_x)_{j_1} \cdots (\varphi^n_y)_{j_{k-1}} \partial_{j_k} \varphi \rangle,$$  \hspace{1cm} (6.1)

where $j_1, \ldots, j_k$ are summed over 1, 2 and $\nabla$ denotes the Riemannian connection on the sphere, i.e., for vector fields $X, Y$ on the sphere $\nabla_X Y$ denotes the orthogonal projection of $\partial_X Y$ onto the sphere.

A key quantitative result of [44] is the following

**Theorem 6.2.** For any initial data $\varphi_0 \in H^\infty_Q$ with $E(\varphi_0) < E_{\text{crit}}$ we have that there exists a unique global smooth heat flow $\varphi$ with initial data $\varphi_0$. Moreover, $\varphi$ satisfies the following estimates

$$\int_0^{\infty} \int_{\mathbb{R}^2} s^{k-1} e_{k+1}(s, x) \, dx \, ds \lesssim E_{0,k,1} \hspace{1cm} (6.2)$$

$$\sup_{0 < s < \infty} \int_{\mathbb{R}^2} s^{k-1} e_k(s, x) \, dx \lesssim E_{0,k,1} \hspace{1cm} (6.3)$$

$$\int_0^{\infty} \int_{\mathbb{R}^2} s^{k-1} \sup_{x \in \mathbb{R}^2} e_k(s, x) \, ds \lesssim E_{0,k,1} \hspace{1cm} (6.4)$$

for each $k \geq 1$, as well as the estimate

$$\int_0^{\infty} \int_{\mathbb{R}^2} e_1^2(s, x) \, ds \, ds \lesssim E_{0,1}.$$  \hspace{1cm} (6.5)

We employ (6.2), (6.3), and (6.4) below.
6.2. **Frequency-localized caloric gauge estimates.** The key estimates to establish are (2.35) for $\varphi$; most remaining estimates will be derived as a corollary. Our strategy is to exploit energy dispersion so that we can apply the Duhamel formula to a frequency localization of the heat flow equation (2.11), which for convenience we rewrite as

$$\partial_s \varphi = \Delta \varphi + \varphi e_1.$$  \hspace{1cm} (6.5)

**Proof of (2.35) for $\varphi$.** Let $\sigma_1 \in \mathbb{Z}_+$ be positive and let $S' \geq S \gg 0$. Let $K \in \mathbb{Z}_+$, $T \in (0, 2^{2K}]$ be fixed. Define for each $t \in (-T, T)$ the quantity

$$C(S, t) := \sup_{\sigma \in [2^6, \sigma_1]} \sup_{s \in [0, S]} \sup_{k \in \mathbb{Z}} \left(1 + s^{2k} \right)^{-\sigma_1} 2^{\sigma k} \gamma_k(\sigma)^{-1} \|P_k \varphi(s, \cdot, t)\|_{L^2_x(\mathbb{R}^2)}.$$  \hspace{1cm} (6.6)

For fixed $t$ the function $C(S, t) : [0, S'] \to (0, \infty)$ is well-defined, continuous, and non-decreasing. Moreover, in view of the definition (2.34) of $\gamma_k(\sigma)$, it follows that $\lim_{S \to 0} \mathcal{C}(S, t) \lesssim 1$. A simple consequence of (6.6) is

$$\|P_k \varphi(s, \cdot, t)\|_{L^2_x(\mathbb{R}^2)} \leq C(S, t)(1 + s^{2k})^{-\sigma_1} 2^{\sigma k} \gamma_k(\sigma)$$  \hspace{1cm} (6.7)

for $0 \leq s \leq S \leq S'$.

Our goal is to show $C(S, t) \lesssim 1$ uniformly in $S$ and $t$ and our strategy is to apply Duhamel’s formula to (6.5) and run a bootstrap argument. Beginning with the decomposition

$$P_k(\varphi e_1) = \sum_{|k_2 - k| \leq 4} P_k(P_{\leq k-5} \varphi \cdot P_{k_2} e_1) + \sum_{|k_2 - k| \leq 4} P_k(P_{k_1} \varphi \cdot P_{\leq k-5} e_1) + \sum_{k_1, k_2 \geq k-4 \atop |k_1 - k_2| \leq 8} P_k(P_{k_1} \varphi \cdot P_{k_2} e_1),$$

we proceed to place in $L^2_x$ each of the three terms on the right hand side; we then integrate in $s$ and consider separately the low-high, high-low, and high-high frequency interactions.

**Low-High interaction.** By Duhamel and the triangle inequality it suffices to bound

$$LH(s, t) := \int_0^s e^{-(s-s')^{2k-2}} \sum_{|k_2 - k| \leq 4} \|P_k(P_{\leq k-5} \varphi(s', \cdot, t) \cdot P_{k_2} e_1(s', \cdot, t))\|_{L^2_x} ds'.$$  \hspace{1cm} (6.8)
By Hölder’s inequality, $|\varphi| \equiv 1$, and $L^p$-boundedness of the Littlewood-Paley multipliers,

$$
\text{LH}(s, t) \lesssim \int_0^s e^{-(s-s')2^{k-2}} \sum_{|k_2-k| \leq 4} \|P_{\leq k-5}\varphi\|_{L^\infty} \|P_{k_2} e_1\|_{L^2} ds' 
\lesssim \int_0^s e^{-(s-s')2^{k-2}} \sum_{|k_2-k| \leq 4} \|P_{k_2} e_1(s', \cdot, t)\|_{L^2} ds'.
$$

To control the sum we further decompose $P_{\ell} e_1 = P_{\ell}(\partial_x \varphi \cdot \partial_x \varphi)$ into low-high and high-high frequency interactions:

$$
P_{\ell} e_1 = 2 \sum_{|\ell_1-\ell| \leq 4} P_{\ell}(P_{\leq \ell-5} \partial_x \varphi \cdot P_{\ell} \partial_x \varphi) + \sum_{\ell_1, \ell_2 \geq \ell-4 \atop |\ell_1-\ell_2| \leq 8} P_{\ell}(P_{\ell_1} \partial_x \varphi \cdot P_{\ell_2} \partial_x \varphi).
$$

**Low-High interaction (i).** We first attend to the low-high subcase. For convenience set $\Xi_{lh}$ equal to the first term of the right hand side of (6.9), i.e.,

$$
\Xi_{lh}(s, x, t) := \sum_{|\ell_1-\ell| \leq 4} P_{\ell}(P_{\leq \ell-5} \partial_x \varphi(s, x, t) \cdot P_{\ell} \partial_x \varphi(s, x, t)).
$$

By the triangle inequality, Hölder’s inequality, Berstein’s inequality, the definition (6.11) for $e_1(s, \cdot, t)$, and (6.7), it follows that

$$
\|\Xi_{lh}(s, \cdot, t)\|_{L^2} \lesssim \sum_{|\ell_1-\ell| \leq 4} \|P_{\ell}(P_{\leq \ell-5} \partial_x \varphi \cdot P_{\ell} \partial_x \varphi)\|_{L^2} 
\lesssim \sum_{|\ell_1-\ell| \leq 4} \|P_{\leq \ell-5} \partial_x \varphi\|_{L^\infty} \|P_{\ell_1} \partial_x \varphi\|_{L^2} 
\lesssim \sum_{|\ell_1-\ell| \leq 4} \|P_{\leq \ell-5} \partial_x \varphi\|_{L^\infty} 2^{\ell_1} \|P_{\ell_1} \varphi\|_{L^2} 
\lesssim \|\sqrt{e_1}\|_{L^\infty} 2^\ell \sum_{|\ell_1-\ell| \leq 4} \|P_{\ell_1} \varphi\|_{L^2} 
\lesssim \|\sqrt{e_1}(s, \cdot, t)\|_{L^\infty} 2^\ell 2^{-\sigma k} \gamma_k(\sigma) C(S, t)(1 + s2^{2\ell})^{-\sigma_1}.
$$

As we apply this inequality in the case where $\ell = k_2, |k_2 - k| \leq 4$, we have

$$
\int_0^s e^{-(s-s')2^{k-2}} \|\Xi_{lh}(s', \cdot, t)\|_{L^2} ds' 
\lesssim 2^{k_2-\sigma k} \gamma_k(\sigma) C(S, t) \int_0^s e^{-(s-s')2^{k-2}} \|\sqrt{e_1}(s', \cdot, t)\|_{L^\infty}(1 + s2^{2k})^{-\sigma_1} ds'.
$$

(6.10)

Apply Cauchy-Schwarz. Clearly

$$
\left(\int_0^s \|\sqrt{e_1}(s', \cdot, t)\|_{L^2}^2 ds'\right)^{1/2} \leq \|e_1(\cdot, \cdot, t)\|_{L^1_4 L^2}\cdot
$$

(6.11)
We postpone applying (6.3) with \( k = 1 \) to (6.11). As for the other factor, we have
\[
\left( \int_0^s e^{-(s-s')2^{k-1}} (1 + s'2^{2k})^{-2\sigma_1} ds' \right)^{1/2} \lesssim (s(1 + s2^{2k})^{-1})^{1/2}
\] (6.12)
since
\[
\int_0^s e^{-(s-s')\lambda} (1 + s'\lambda')^{-\alpha} ds' \lesssim s(1 + \lambda s)^{-\alpha}(1 + \lambda's)^{-1}
\]
for \( s \geq 0, 0 \leq \lambda \leq \lambda' \), and \( \alpha > 1 \). Hence, applying Cauchy-Schwarz to (6.10) and using (6.11) and (6.12), we get
\[
\int_0^s e^{-(s-s')2^{2k-2}} \| \Xi_{tt}(s', \cdot, t) \|_{L^2} ds' \\
\lesssim 2^{-\sigma k} \gamma_k(\sigma) C(S, t) 2^k s^{1/2} (1 + s2^{2k-1})^{-\sigma_1}(1 + s2^{2k})^{-1/2} \| e_1(t) \|_{L^1_t L^\infty_x([0, s] \times \mathbb{R}^2)}.
\]
Discarding \( s^{1/2} 2^k (1 + s2^{2k})^{1/2} \leq 1 \), we conclude
\[
\int_0^s e^{-(s-s')2^{2k-2}} \| \Xi_{tt}(s', \cdot, t) \|_{L^2} ds' \\
\lesssim 2^{-\sigma k} \gamma_k(\sigma) C(S, t) 2^k s^{1/2} (1 + s2^{2k-1})^{-\sigma_1} \| e_1(t) \|_{L^1_t L^\infty_x([0, s] \times \mathbb{R}^2)}.
\] (6.13)

**Low-High interaction (ii).** We now move on to the high-high interaction subcase, setting \( \Xi_{hh} \) equal to the second term of the right hand side of (6.9):
\[
\Xi_{hh}(s, x, t) := \sum_{\ell_1, \ell_2 \geq \ell - 4, \ |\ell_1 - \ell_2| \leq 8} P_{\ell}(P_{\ell_1} \partial_x \varphi(s, x, t) \cdot P_{\ell_2} \partial_x \varphi(s, x, t)).
\]
By the triangle inequality, Bernstein, and Cauchy-Schwarz,
\[
\| \Xi_{hh} \|_{L^2} \lesssim \sum_{\ell_1, \ell_2 \geq \ell - 4, \ |\ell_1 - \ell_2| \leq 8} \| P_{\ell}(P_{\ell_1} \partial_x \varphi \cdot P_{\ell_2} \partial_x \varphi) \|_{L^2} \\
\lesssim \sum_{\ell_1, \ell_2 \geq \ell - 4, \ |\ell_1 - \ell_2| \leq 8} 2^\ell \| P_{\ell_1} \partial_x \varphi \cdot P_{\ell_2} \partial_x \varphi \|_{L^1} \\
\leq \sum_{\ell_1, \ell_2 \geq \ell - 4, \ |\ell_1 - \ell_2| \leq 8} 2^\ell \| P_{\ell_1} \partial_x \varphi \|_{L^2} \| P_{\ell_2} \partial_x \varphi \|_{L^2}.
\]
At this stage we apply Bernstein twice, exploiting \( |\ell_1 - \ell_2| \leq 8 \):
\[
\| P_{\ell_1} \partial_x \varphi \|_{L^2} \| P_{\ell_2} \partial_x \varphi \|_{L^2} \lesssim 2^{\ell_2} \| P_{\ell_1} \partial_x \varphi \|_{L^2} \| P_{\ell_2} \varphi \|_{L^2} \\
\lesssim \| P_{\ell_1} \partial_x \varphi \|_{L^2} \| P_{\ell_2} \varphi \|_{L^2}.
\]
So
\[
\|\Xi_{hh}\|_{L^2_x} \lesssim 2^\ell \sum_{\ell_1, \ell_2 \geq \ell - 4} \|P_{\ell_1} |\partial_x|^2 \varphi\|_{L^2_x} \|P_{\ell_2} \varphi\|_{L^2_x}.
\]
Applying Cauchy-Schwarz yields
\[
\|\Xi_{hh}\|_{L^2_x} \lesssim 2^\ell \left( \sum_{\ell_1 \geq \ell - 4} \|P_{\ell_1} |\partial_x|^2 \varphi\|_{L^2_x}^2 \right)^{1/2} \left( \sum_{\ell_2 \geq \ell - 4} \|P_{\ell_2} \varphi\|_{L^2_x}^2 \right)^{1/2} \lesssim \|\partial_x|^2 \varphi\|_{L^2_x} 2^\ell \left( \sum_{\ell_2 \geq \ell - 4} \|P_{\ell_2} \varphi\|_{L^2_x}^2 \right)^{1/2} \tag{6.14}
\]
As \(\varphi\) takes values in \(S^2\), which has constant curvature, we readily estimate ordinary derivatives by covariant ones:
\[
|\partial_x^2 \varphi| \lesssim \sqrt{\mathbf{e}_2} + \mathbf{e}_1. \tag{6.15}
\]
Applying \((6.15)\) in \((6.14)\) and using \((6.7)\), we arrive at
\[
\|\Xi_{hh}(s, \cdot, t)\|_{L^2_x} \lesssim \|\sqrt{\mathbf{e}_2} + \mathbf{e}_1\|_{L^2_x} 2^\ell \left( \sum_{\ell_2 \geq \ell - 4} \|P_{\ell_2} \varphi\|_{L^2_x}^2 \right)^{1/2} \lesssim \|((\sqrt{\mathbf{e}_2} + \mathbf{e}_1)(s, \cdot, t))\|_{L^2_x} 2^\ell \times C(S, t) \times (1 + s^2)^{2}\gamma_\ell^2(\sigma) \lesssim \|((\sqrt{\mathbf{e}_2} + \mathbf{e}_1)(s, \cdot, t))\|_{L^2_x} 2^\ell C(S, t)(1 + s^2)^{-\sigma_1} \times 2^{-2\sigma_1} \gamma_\ell^2(\sigma)^{1/2} \tag{6.16}
\]
As \(\sigma > \delta\) is bounded away from \(\delta\) uniformly, we may apply summation rule \((2.31)\) in \((6.16)\). Recalling \(\ell = k_2\) where \(|k_2 - k| \leq 4\), we conclude
\[
\|\Xi_{hh}(s, \cdot, t)\|_{L^2_x} \lesssim \|((\sqrt{\mathbf{e}_2} + \mathbf{e}_1)(s, \cdot, t))\|_{L^2_x} 2^k 2^{-\sigma k} \gamma_k(\sigma) C(S, t)(1 + s^2)^{-\sigma_1}.
\]
Integrating in \(s\) yields
\[
\int_0^s e^{-(s-s')2^k} \|\Xi_{hh}(s', \cdot, t)\|_{L^2_x} ds' \lesssim 2^k 2^{-\sigma k} \gamma_k(\sigma) C(S, t) \int_0^s e^{-(s-s')2^k} \|((\sqrt{\mathbf{e}_2} + \mathbf{e}_1)(s', \cdot, t))\|_{L^2_x}^2 (1 + s^2)^{-\sigma_1} ds'. \tag{6.17}
\]
We use the triangle inequality to write \(\|\sqrt{\mathbf{e}_2} + \mathbf{e}_1\|_{L^2_x} \leq \|\sqrt{\mathbf{e}_2}\|_{L^2_x} + \|\mathbf{e}_1\|_{L^2_x}\) and split the integral in \((6.17)\) into two pieces. By Cauchy-Schwarz and
By Duhamel and the triangle inequality it suffices to bound the high-low interaction. We now go on to bound the high-low interaction. Combining (6.13) and (6.20), we conclude in view of (6.8) and the decomposition (6.9) that
\[
\| P_k (P_{k+1} \varphi (s') \cdot t) \cdot P_{\leq k-5} e_1 (s', \cdot t) \|_{L^2} ds' \leq 2^{-\sigma_1} \gamma_k (\sigma) C (S, t) (1 + s^2) - \sigma_1 \left( \| e_1 (t) \|_{L^2_x} + \| e_2 (t) \|_{L^1_t L^2_x} \right) ^{1/2}.
\]

To the remaining integral we also apply Cauchy-Schwarz and (6.12):
\[
\int_0^s e^{-(s-s')^2} \| e_1 (s', \cdot t) \|_{L^2_x} (1 + s^2) - \sigma_1 ds'
\]
\[
\leq \left( \int_0^s \| e_1 (s', \cdot t) \|_{L^2_x} ds' \right)^{1/2} \left( \int_0^s e^{-(s-s')^2} (1 + s^2) - 2 \sigma_1 ds' \right)^{1/2}
\]
\[
\lesssim \| e_1 (t) \|_{L^2_x} (1 + s^2) - \sigma_1 (1 + s^2) - 1)^{1/2}.
\]

Hence using Cauchy-Schwarz, (6.18), and (6.19) in (6.17), we conclude
\[
\int_0^s e^{-(s-s')^2} \| \Xi_{h} (s', \cdot t) \|_{L^2_x} ds'
\]
\[
\lesssim 2^{-\sigma_1} \gamma_k (\sigma) C (S, t) (1 + s^2) - \sigma_1 \left( \| e_1 (t) \|_{L^2_x} + \| e_2 (t) \|_{L^1_t L^2_x} \right) ^{1/2}.
\]

**Low-High interaction conclusion.** Combining (6.13) and (6.20), we conclude in view of (6.8) and the decomposition (6.9) that
\[
\text{LH} (s, t) \lesssim 2^{-\sigma_1} \gamma_k (\sigma) C (S, t) (1 + s^2) - \sigma_1 \times
\]
\[
\left( \| e_1 (t) \|_{L^2_x} + \| e_1 (t) \|_{L^2_x} + \| e_2 (t) \|_{L^1_t L^2_x} \right) ^{1/2}.
\]

**High-Low interaction.** We now go on to bound the high-low interaction. By Duhamel and the triangle inequality it suffices to bound
\[
\text{HL} (s, t) := \int_0^s e^{-(s-s')^2} \sum_{|k_1 - k| \leq 4} \| P_k (P_{k+1} \varphi (s') \cdot t) \cdot P_{\leq k-5} e_1 (s', \cdot t) \|_{L^2_x} ds'.
\]
By Hölder’s inequality, \([6.7]\), and Bernstein’s inequality, we have
\[
\sum_{|k_1 - k| \leq 4} \| P_k (P_{k_1} \varphi(s, \cdot, t) \cdot P_{s, x, t} P_{k-5} e_1(s, \cdot, t)) \|_{L^2_x}
\]
\[
\lesssim \sum_{|k_1 - k| \leq 4} \| P_{k_1} \varphi \|_{L^2} \| P_{s, x, t} P_{k-5} e_1 \|_{L^\infty}
\]
\[
\lesssim \| P_{s, x, t} P_{k-5} e_1(s, \cdot, t) \|_{L^\infty} \sum_{|k_1 - k| \leq 4} (1 + s' 2^{k_1})^{-2 \sigma_1} 2^{-\sigma_1} 2^{\gamma_1} C(S, t)
\]
\[
\lesssim 2^k \| P_{s, x, t} P_{k-5} e_1(s, \cdot, t) \|_{L^2_x} 2^{-\sigma_1} 2^{\gamma_1} C(S, t)(1 + s' 2^{k})^{-\sigma_1}.
\]

Hence
\[
HL(s, t) \lesssim 2^k 2^{-\sigma_1} 2^{\gamma_1} C(S, t) \int_0^s e^{-(s-s') 2^{k-2}} (1 + s' 2^{k})^{-\sigma_1} \| e_1(s', \cdot, t) \|_{L^2_x} ds'.
\]

Bounding the integral as in \([6.18]\), we obtain
\[
HL(s, t) \lesssim 2^{-\sigma_1} 2^{\gamma_1} C(S, t)(1 + s 2^{k-1})^{-\sigma_1} \| e_1(t) \|_{L^2_x}.
\]

High-High interaction. We conclude with the high-high interaction. Set
\[
HHI(s, x, t) := \int_0^s e^{-(s-s') 2^{k-2}} \sum_{k_1, k_2 \geq k-4} \| P_k (P_{k_1} \varphi(s, x, t) \cdot P_{k_2} e_1(s, x, t)) \|_{L^2_x} ds'.
\]

By Bernstein, Cauchy-Schwarz, and \([6.7]\),
\[
\sum_{k_1, k_2 \geq k-4} \| P_k (P_{k_1} \varphi \cdot P_{k_2} e_1) \|_{L^2_x}
\]
\[
\lesssim \sum_{k_1, k_2 \geq k-4} 2^k \| P_{k_1} \varphi \|_{L^2_x} \| P_{k_2} e_1 \|_{L^2_x}
\]
\[
\lesssim 2^k \left( \sum_{k_1 \geq k-4} \| P_{k_1} \varphi \|_{L^2_x}^2 \right)^{1/2} \left( \sum_{k_2 \geq k-4} \| P_{k_2} e_1 \|_{L^2_x}^2 \right)^{1/2}
\]
\[
\lesssim 2^k \left( \sum_{k_1 \geq k-4} (1 + s' 2^{k_1})^{-2 \sigma_1} 2^{-2 \sigma_1} 2^{\gamma_1} C(S, t)^2 \right)^{1/2} \| e_1(s, \cdot, t) \|_{L^2_x}
\]
\[
= \| e_1(s, \cdot, t) \|_{L^2_x} 2^k C(S, t) \left( \sum_{k_1 \geq k-4} (1 + s' 2^{k_1})^{-2 \sigma_1} 2^{-2 \sigma_1} 2^{\gamma_1} C(S, t)^2 \right)^{1/2}.
\]

We handle the sum as in \([6.16]\), taking advantage of the frequency envelope summation rule \([2.31]\), and conclude
\[
HHI(s, x, t) \lesssim 2^{-\sigma_1} 2^{\gamma_1} C(S, t)(1 + s 2^{k-1})^{-\sigma_1} \| e_1(t) \|_{L^2_x}.
\]
Wrapping up. For the linear term \( e^{s \Delta} P_k \varphi \) we have
\[
\| e^{s \Delta} P_k \varphi \|_{L^2_x} \leq e^{-s2^{k-2}} \| P_k \varphi \|_{L^2_x} \\
\leq e^{-s2^{k-2}} 2^{-s \gamma_k(\sigma)}. \tag{6.24}
\]
Using (6.21), (6.22), (6.23), and (6.24) in Duhamel’s formula applied to the covariant heat equation (6.5), we have that for any \( s \in [0, S], t \in (-T, T) \),
\[
2^{s \gamma_k(\sigma)} P_k \varphi(s, t) \|_{L^2_x(1 + s2^{2k})^{\sigma_1}} \leq \gamma_k(\sigma) + LL(s, t) + LH(s, t) \leq \gamma_k(\sigma) + \gamma_k(\sigma)C(S, t) \left( \| e_1(t) \|_{L^1_{s,x}L^\infty_t} + \| e_2(t) \|_{L^1_{s,x}} + \| e_1(t) \|_{L^2_{s,x}} \right).
\]
In view of (6.3) with \( k = 1 \), (6.4), and (6.2), we may split up the \( s \)-time interval \((0, \infty)\) into \( O_{E_1}(1) \) intervals \( I_\rho \) on which \( \| e_1(t) \|_{L^1_{s,x}L^\infty_t(I_\rho \times \mathbb{R}^2)}, \| e_2(t) \|_{L^1_{s,x}L^2_t(I_\rho \times \mathbb{R}^2)} \) and \( \| e_1(t) \|_{L^2_{s,x}L^2_{s,x}(I_\rho \times \mathbb{R}^2)} \) are all simultaneously small uniformly in \( t \). By iterating a bootstrap argument \( O_{E_1}(1) \) times beginning with interval \( I_1 \), we conclude that \( C(s, t) \lesssim 1 \) for all \( s > 0 \), uniformly in \( t \). Therefore
\[
\| P_k \varphi(s) \|_{L^1_{s,x}L^2_t} \lesssim (1 + s2^{2k})^{-\sigma_1} 2^{-s \gamma_k(\sigma)} \tag{6.25}
\]
for \( s \in (0, \infty) \) and \( \sigma \geq 2\delta \).

Remark 6.3. Having proven the quantitative bounds (2.35) for \( \varphi \), one may establish as a corollary the qualitative bounds (2.36) for \( \varphi \) by using an inductive argument as in the proof of [4, Lemma 8.3]. We omit the proof, noting in particular that the argument deriving (2.36) from (2.35) does not require a small-energy hypothesis.

Proof of (2.36) for \( v, w \). We begin by introducing the matrix-valued function
\[
R(s, x, t) := \partial_s \varphi(s, x, t) \cdot \varphi(s, x, t)^\dagger - \varphi(s, x, t) \cdot \partial_s \varphi(s, x, t)^\dagger, \tag{6.26}
\]
where here \( \varphi \) is thought of as a column vector. The dagger “\( \dagger \)” denotes transpose. Using the heat flow equation (2.11) in (6.26), we rewrite \( R \) as
\[
R = \Delta \varphi \cdot \varphi^\dagger - \varphi \cdot \Delta \varphi^\dagger \\
= \partial_m(\partial_m \varphi \cdot \varphi^\dagger - \varphi \cdot \partial_m \varphi^\dagger), \tag{6.27}
\]
and proceed to bound its Littlewood-Paley projections \( P_k R \) in \( L^2_x \). Noting that by Bernstein we have
\[
\| P_k(\partial_m(\partial_m \varphi \cdot \varphi^\dagger)) \|_{L^2_x} \sim 2^k \| P_k(\partial_m \varphi \cdot \varphi^\dagger) \|_{L^2_x}, \tag{6.29}
\]
we further decompose the nonlinearity $P_k(\partial_m \varphi \cdot \varphi^\dagger)$:

$$
P_k(\partial_m \varphi \cdot \varphi^\dagger) = \sum_{|k_2-k|\leq 4} P_{\leq k-4} \partial_m \varphi \cdot P_{k_2} \varphi^\dagger + \sum_{|k_1-k|\leq 4} P_k \partial_m \varphi \cdot P_{\leq k-4} \varphi^\dagger + \sum_{k_1,k_2 \geq k-4 \atop |k_1-k_2| \leq 8} P_k(P_k \partial_m \varphi \cdot P_{k_2} \varphi^\dagger).
$$ (6.30)

By Hölder’s and Bernstein’s inequalities and by $|\varphi| \equiv 1$ and (6.25) with Bernstein,

$$
\sum_{|k_2-k|\leq 4} \|P_{\leq k-4} \partial_m \varphi \cdot P_{k_2} \varphi\|_{L^2_x} \lesssim \sum_{|k_2-k|\leq 4} 2^k \|P_{\leq k-4} \varphi\|_{L^\infty_x} \|P_{k_2} \varphi\|_{L^2_x} \lesssim 2^k (1 + s 2^{2k})^{-\gamma_1} 2^{-\gamma_k}(\sigma). \quad (6.31)
$$

Similarly,

$$
\sum_{|k_1-k|\leq 4} \|P_k \partial_m \varphi \cdot P_{\leq k-4} \varphi\|_{L^2_x} \lesssim \sum_{|k_1-k|\leq 4} \|P_k \partial_m \varphi\|_{L^2_x} \|P_{\leq k-4} \varphi\|_{L^\infty_x} \lesssim 2^k (1 + s 2^{2k})^{-\gamma_1} 2^{-\gamma_k}(\sigma). \quad (6.32)
$$

Finally, by Bernstein and Cauchy-Schwarz, energy decay, (6.25), and frequency envelope summation rule (2.31), we get

$$
\sum_{k_1,k_2 \geq k-4 \atop |k_1-k_2| \leq 8} \|P_k(P_k \partial_m \varphi \cdot P_{k_2} \varphi)\|_{L^2_x} \lesssim \sum_{k_1,k_2 \geq k-4 \atop |k_1-k_2| \leq 8} 2^k \|P_k \partial_m \varphi\|_{L^2_x} \|P_{k_2} \varphi\|_{L^2_x} \lesssim 2^k \sum_{k_2 \geq k-4} \|P_{k_2} \varphi\|_{L^2_x} \lesssim 2^k \sum_{k_1 \geq k-4} (1 + s 2^{2k_1})^{-\gamma_1} 2^{-\gamma_k}(\sigma) \lesssim 2^k (1 + s 2^{2k})^{-\gamma_1} 2^{-\gamma_k}(\sigma). \quad (6.33)
$$

Using the decomposition (6.30) and combining the cases (6.31), (6.32), and (6.33) to control (6.29), we conclude from the representation (6.28) of $R$ that for fixed $t \in (-T, T)$,

$$
2^{\sigma k} \|P_k R(s, \cdot, t)\|_{L^2_x} \lesssim 2^{2k} (1 + s 2^{2k})^{-\gamma_k}(\sigma).
$$

As this estimate is uniform in $T$, it follows that

$$
2^{\sigma k} \|P_k R(s)\|_{L^\infty_x L^2_t} \lesssim 2^{2k} (1 + s 2^{2k})^{-\gamma_k}(\sigma). \quad (6.34)
$$
By arguing as in [4] Lemma 8.4, one may obtain the qualitative estimate
\[
\sup_{s \geq 0} \left( (1 + s)^{(\sigma+2)/2} \| \partial_x^\sigma \partial_t^\rho R(s) \|_{L^1 L^2_x} \right) < \infty.
\]
From the Duhamel representation of \( \varphi \) and the explicit formula for the heat kernel, one can easily show that qualitative bound \[1\] implies
\[
\int_0^\infty \| R(s, \cdot, t) \|_{L_{\infty}^\varphi} \, ds \lesssim 1
\]
as in [44, §7]. Hence we may define \( v \) as the unique solution of the ODE
\[
\partial_s v = R(s) \cdot v \quad \text{and} \quad v(\infty) = Q',
\]
where \( Q' \in S^2 \) is chosen so that \( Q \cdot Q' = 0 \). This indeed coincides with the definition given in [44], since (6.36) is nothing other than the parallel transport condition \( (\varphi^* \nabla)_s v = 0 \) written explicitly in the setting \( S^2 \hookrightarrow \mathbb{R}^4 \). Smoothness and basic convergence properties follow as in [44], to which we refer the reader for the precise results and proofs. Our goal here is to exploit (6.36) and (6.34) to prove (2.35) for \( v \).

Using \( \int_0^\infty \| \partial_x^\sigma \partial_t^\rho R(s) \|_{L_{\infty}^\varphi} \, ds < \infty \) from (6.35), we conclude
\[
\sup_{s \geq 0} (1 + s)^{(\sigma+1)/2} \| \partial_x^\sigma \partial_t^\rho (v(s) - Q') \|_{L^\infty_{L^2_x}} < \infty
\]
for \( \sigma, \rho \in \mathbb{Z}_+ \). Integrating (6.36) in \( s \) from infinity, we get
\[
v(s) - Q' + \int_s^\infty R(s') \cdot Q' \, ds' = - \int_s^\infty R(s') \cdot (v(s') - Q') \, ds',
\]
which, combined with estimates (6.35) and (6.37), implies
\[
\sup_{s \geq 0} \sup_{k \in \mathbb{Z}} \| P_k v(s) \|_{L^\infty_{L^2_x}} < \infty,
\]
i.e., (2.36) for \( v \). Projecting (6.36) to frequencies \( \sim 2^k \) and integrating in \( s \), we obtain
\[
P_k(v(s)) = - \int_s^\infty P_k(R(s') \cdot v(s')) \, ds'.
\]
Set
\[
C_1(S, t) := \sup_{\sigma \in [2^k, \sigma]} \sup_{s \in [S, \infty)} \sup_{k \in \mathbb{Z}} \gamma_k(\sigma)^{-1} (1 + s 2^{2k})^{\sigma_1 - 2} 2^{\sigma k} \| P_k v(s, \cdot, t) \|_{L^2_x}.
\]
That \( C_1(S, t) < \infty \) follows from (6.39) and \( \sup_{k \in \mathbb{Z}} \gamma_k(\sigma)^{-1} 2^{-|\delta|} < \infty \). Consequently, for \( s \in [S, \infty) \),
\[
\| P_k v(s, \cdot, t) \|_{L^2_x} \lesssim C_1(S, t) (1 + s 2^{2k})^{-\sigma_1 + 1} 2^{-\sigma k} \gamma_k(\sigma).
\]
\[1\] We may alternately invoke (6.35) as in [4].
We perform the Littlewood-Paley decomposition

\[ P_k(R(s)v(s)) = \sum_{|k_2-k|\leq 4} P_k(P_{\leq k-4}R(s)P_{k_2}v(s)) + \]
\[ + \sum_{|k_1-k|\leq 4} P_k(P_{k_1}R(s)P_{\leq k-4}v(s)) + \]
\[ + \sum_{k_2\geq k-4} P_k(P_{\geq k-4}R(s)P_{k_2}v(s)) \]  \hspace{1cm} (6.42)

and proceed to consider individually the various frequency interactions. By Hölder’s inequality, Bernstein’s inequality, and (6.41),

\[ \sum_{|k_2-k|\leq 4} \| P_k(P_{\leq k-4}R(s)P_{k_2}v(s)) \|_{L^2_x} \leq \sum_{|k_2-k|\leq 4} \| P_{\leq k-4}R(s) \|_{L^2_x} \| P_{k_2}v(s) \|_{L^\infty_x} \]
\[ \leq \| R(s) \|_{L^2_x} \sum_{|k_2-k|\leq 4} 2^{k_2} \| P_{k_2}v(s) \|_{L^2_x} \]
\[ \leq \| R(s) \|_{L^2_x} 2^{k_2} 2^{-\sigma_2 k_2}(\sigma)(1 + s 2^{2k})^{-\sigma_1 + 1} C_1(S, t). \]  \hspace{1cm} (6.43)

By Hölder’s inequality, \(|v| \equiv 1\), and (6.31),

\[ \sum_{|k_1-k|\leq 4} \| P_k(P_{k_1}R(s)P_{\leq k-4}v(s)) \|_{L^2_x} \leq \| P_{\leq k-4}v(s) \|_{L^\infty_x} \sum_{|k_1-k|\leq 4} \| P_{k_1}R(s) \|_{L^2_x} \]
\[ \leq 2^{2k}(1 + s 2^{2k})^{-\sigma_1} 2^{-\sigma_2 k_2}(\sigma). \]  \hspace{1cm} (6.44)

From Bernstein’s inequality, Cauchy-Schwarz, (6.41), and \(\sigma > 2\delta\) with (2.31), it follows that

\[ \sum_{k_2\geq k-4} \| P_k(P_{\geq k-4}R(s)P_{k_2}v(s)) \|_{L^2_x} \leq \sum_{k_2\geq k-4} 2^{k} \| P_{\geq k-4}R(s)P_{k_2}v(s) \|_{L^1_x} \]
\[ \leq \| R(s) \|_{L^2_x} 2^{k} \sum_{k_2\geq k-4} \| P_{k_2}v(s) \|_{L^2_x} \]
\[ \leq \| R(s) \|_{L^2_x} 2^{k} \sum_{k_2\geq k-4} 2^{-\sigma_2 k_2}(\sigma)(1 + s 2^{2k})^{-\sigma_1 + 1} C_1(S, t) \]
\[ \leq \| R(s) \|_{L^2_x} 2^{k} 2^{-\sigma_2 k_2}(\sigma)(1 + s 2^{2k})^{-\sigma_1 + 1} C_1(S, t). \]  \hspace{1cm} (6.45)
Using the decomposition (6.42) in (6.40) and combining the estimates (6.43), (6.44), and (6.45) implies
\[
2^{\sigma k} \| P_k v(s) \|_{L_x^2} \leq \int_s^\infty 2^{\sigma k} \| P_k (R(s') v(s')) \|_{L_x^2} ds' \\
\lesssim \gamma_k(\sigma) \int_s^\infty 2^{2k} (1 + s'2^{2k})^{-\sigma_1} ds' \\
+ C_1(s,t)\gamma_k(\sigma) \int_s^\infty \| R(s') \|_{L_x^2} 2^{2k} (1 + s'2^{2k})^{-\sigma_1+1} ds'.
\]

Applying Cauchy-Schwarz in \( s \), we obtain
\[
2^{\sigma k} \| P_k v(s) \|_{L_x^2} \lesssim \gamma_k(\sigma) \int_s^\infty 2^{2k} (1 + s'2^{2k})^{-\sigma_1} ds' \\
+ C_1(s,t)\gamma_k(\sigma) \left( \int_s^\infty \| R(s') \|_{L_x^2}^2 ds' \right)^{1/2} \times \\
\times \left( \int_s^\infty 2^{2k} (1 + s'2^{2k})^{-2\sigma_1+2} ds' \right)^{1/2} \\
\lesssim \gamma_k(\sigma) + C_1(s,t)\gamma_k(\sigma) \left( \int_s^\infty \| R(s') \|_{L_x^2}^2 ds' \right)^{1/2}. \tag{6.46}
\]

As noted in (6.15), it holds that \( |\Delta \varphi| \leq \sqrt{e_2} + e_1 \), and so it follows from the representation (6.27) of \( R \) that
\[
|R(s,x,t)| \leq |e_1(s,x,t)| + |\sqrt{e_2}(s,x,t)|. \tag{6.47}
\]

As (6.47) implies
\[
\int_0^\infty \| R(s) \|^2_{L_x^2} ds \lesssim \| e_2 \|_{L^1_{s,x}} + \| e_1 \|^2_{L^2_{s,x}},
\]
we therefore in view of (6.2) with \( k = 1 \) and (6.4) may choose \( S \) large so that the integral of the \( R \) term in (6.46) is small, say \( \leq \varepsilon \). Then
\[
C_1(S,t) \lesssim 1 + \varepsilon C_1(S,t)
\]
so that \( C_1(S) \lesssim 1 \) for such \( S \). In fact, together (6.2) and (6.4) imply that we may divide the time interval \([0, \infty)\) into \( O_{E_0}(1) \) subintervals \( I_\rho \) so that on each such subinterval
\[
\int_{I_\rho} \| R(s) \|^2_{L_x^2} ds \leq \varepsilon^2.
\]

Hence by a simple iterative bootstrap argument we conclude
\[
C_1(0,t) \lesssim 1. \tag{6.48}
\]

As (6.48) is uniform in \( t \), we have
\[
\| P_k v(s, \cdot, t) \|_{L_x^2} \lesssim (1 + s2^{2k})^{-\sigma_1+1} 2^{-\sigma k} \gamma_k(\sigma). \tag{6.49}
\]
By repeating the above argument with \( w \) in place of \( v \) (and appropriately modifying the boundary condition at \( \infty \) in (6.36)), we get
\[
\|P_k w(s, \cdot, t)\|_{L^2_x} \lesssim (1 + s 2^{2k})^{-\sigma_1 + 1/2 - \sigma_k \gamma_k(\sigma)}.
\] (6.50)
and
\[
\sup_{s \geq 0} \sup_{k \in \mathbb{Z}} (1 + s)^{\sigma/2} 2^{\sigma k} \|P_k \partial_t^k w(s)\|_{L^\infty_t L^2_x} < \infty,
\]
i.e., (2.35) and (2.36) respectively for \( w \).

**Proof of (2.37).** Recall that
\[
\psi_m = v \cdot \partial_m \varphi + iw \cdot \partial_m \varphi = -\partial_m v \cdot \varphi - i \partial_m w \cdot \varphi.
\] (6.51)
Our first aim is to control \( \|P_k \psi_m\|_{L^\infty_t L^2_x} \). Toward this end, we perform a Littlewood-Paley decomposition of \( \partial_m v \cdot \varphi \):
\[
P_k (\partial_m v \cdot \varphi) = \sum_{|k_2-k| \leq 4} P_k (P_{\leq k-5} \partial_m v \cdot P_{k_2} \varphi) + \sum_{|k_1-k| \leq 4} P_k (P_{k_1} \partial_m v \cdot P_{\leq k-5} \varphi) + \sum_{k_1, k_2 \geq k-4 \atop |k_1 - k_2| \leq 8} P_k (P_{k_1} \partial_m v \cdot P_{k_2} \varphi).
\] (6.52)
To control the low-high frequency term we apply Hölder’s inequality, energy decay, and (6.25) with Bernstein’s inequality:
\[
\sum_{|k_2-k| \leq 4} \|P_k (P_{\leq k-5} \partial_m v \cdot P_{k_2} \varphi)\|_{L^2_x} \lesssim \sum_{|k_2-k| \leq 4} \|P_{\leq k-5} \partial_m v\|_{L^2_x} \|P_{k_2} \varphi\|_{L^\infty_x} \\
\lesssim (1 + s 2^{2k})^{-\sigma_1 2^{k/2} - \sigma_k \gamma_k(\sigma)}.
\] (6.53)
We control the high-low frequency term by using Hölder’s inequality, \( |\varphi| \equiv 1 \), and (6.49):
\[
\sum_{|k_1-k| \leq 4} \|P_k (P_{k_1} \partial_m v \cdot P_{\leq k-5} \varphi)\|_{L^2_x} \lesssim \sum_{|k_1-k| \leq 4} \|P_{k_1} \partial_m v\|_{L^2_x} \|P_{\leq k-5} \varphi\|_{L^\infty_x} \\
\lesssim (1 + s 2^{2k})^{-\sigma_1 2^{k/2} - \sigma_k \gamma_k(\sigma)}.
\] (6.54)
To control the high-high frequency term, we use Bernstein’s inequality and Cauchy-Schwarz, energy conservation and (6.29), and (2.31):
\[
\sum_{k_1, k_2 \geq k-4 \atop |k_1 - k_2| \leq 8} \|P_k (P_{k_1} \partial_m v \cdot P_{k_2} \varphi)\|_{L^2_x} \lesssim \sum_{k_1, k_2 \geq k-4 \atop |k_1 - k_2| \leq 8} 2^k \|P_{k_1} \partial_m v\|_{L^2_x} \|P_{k_2} \varphi\|_{L^2_x} \\
\lesssim 2^k \sum_{k_2 \geq k-4} (1 + s 2^{2k_2})^{-\sigma_1 2^{-\sigma_k \gamma_{k_2}}(\sigma)} \\
\lesssim (1 + s 2^{2k})^{-\sigma_1 2^{1/2} - \sigma_k \gamma_k(\sigma)}
\] (6.55)
We conclude using (6.52), (6.54), and (6.55) in representation (6.52) that
\[
\|P_k(\partial_m v \cdot \varphi)\|_{L^2_t L^2_x} \lesssim (1 + s2^{2k})^{-\sigma_1} 2^k 2^{-\sigma_k} \gamma_k(\sigma).
\]  
(6.56)

By repeating the argument with \( w \) in place of \( v \), it follows that (6.56) also holds with \( w \) in place of \( v \). Therefore, referring back to (6.51), we conclude
\[
\|P_k \psi_m\|_{L^2_t L^2_x} \lesssim (1 + s2^{2k})^{-\sigma_1} 2^k 2^{-\sigma_k} \gamma_k(\sigma).
\]

As this bound is uniform in \( t \), (2.37) holds for \( \psi_m \).

Recalling that
\[
A_m = \partial_m v \cdot w,
\]
and repeating the above argument with \( w \) in place of \( \varphi \) and (6.50) in place of (6.25), we conclude
\[
\|P_k A_x(s)\|_{L^\infty_t L^2_x} \lesssim (1 + s2^{2k})^{-\sigma_1 + 1} 2^k 2^{-\sigma_k} \gamma_k(\sigma).
\]

\[\square\]

7. Proofs of parabolic estimates

The purpose of this section is to prove the parabolic heat-time estimates stated in [4.1]. Many of these estimates have counterparts in [4]. Nevertheless, our proofs are more involved since we only require energy dispersion, which is weaker than the small-energy assumption made in [4]. Some of the \( L^p \) estimates in [7.2] are new.

Throughout we assume \( \varepsilon_1 \) energy dispersion on the initial data as stated in [4.1] and we assume that the bootstrap hypothesis (4.6) holds. Let \( \sigma_1 \in \mathbb{Z}_+ \) be positive and fixed. We work exclusively with \( \sigma \in [0, \sigma_1 - 1] \), even if this is not always explicitly stated. Set \( \varepsilon = \varepsilon_1^{7/5} \) for short.

In this section we extensively use the spaces defined via (3.2). They provide a crucial gain in high-high frequency interactions, which is captured in Lemmas 7.2 and 7.14.

**Lemma 7.1.** Let \( f \in L^2_{k_1}(T) \), where \( |k_1 - k| \leq 20 \), let \( 0 \leq \omega' \leq 1/2 \), and let \( h \in L^2_k(T) \). Then
\[
\|P_k(fg)\|_{F_k(T)} \lesssim \|f\|_{F_{k_1}(T)} \|g\|_{L^\infty_k}.
\]
\[
\|P_k(fg)\|_{S^\omega_k(T)} \lesssim \|f\|_{F_{k_1}(T)} 2^{k\omega'} \|g\|_{L^2_{\omega'} L^\infty_k}.
\]
\[
\|h\|_{L^\infty_k} + 2^{k\omega'} \|h\|_{L^2_{\omega'} L^\infty_k} \lesssim 2^k \|h\|_{F_k(T)}.
\]

Moreover, for \( f_{k_1}, g_{k_2} \) belonging to \( L^2_{k_1}(T), L^2_{k_2}(T) \) respectively, and with \( |k_1 - k_2| \leq 8 \), we have
\[
\|P_k(f_{k_1}g_{k_2})\|_{F_k(T) \cap S^{1/2}_{k'}(T)} \lesssim 2^{k/2} 2^{(k_2 - k)(1 - \omega)} \|f_{k_1}\|_{S^{1/2}_{k_1}(T)} \|g_{k_2}\|_{S^{1/2}_{k_2}(T)}.
\]
Proof. For the proofs, see [4, §3]. □

Lemma 7.2. Assume that $T \in (0, 2^{2K})$, $f, g \in H^{\infty, \infty}(T)$, $P_k f \in F_k(T) \cap S_\omega^\infty(T)$, $P_k g \in F_k(T)$ for some $\omega \in [0, 1/2]$ and all $k \in \mathbb{Z}$, and

$$\alpha_k = \sum_{|j-k| \leq 20} \|P_j f\|_{F_j(T) \cap S_\omega^\infty(T)}, \quad \beta_k = \sum_{|j-k| \leq 20} \|P_j g\|_{F_j(T)}.$$

Then, for any $k \in \mathbb{Z}$,

$$\|P_k(fg)\|_{F_k(T) \cap S_\omega^{1/2}(T)} \lesssim \sum_{j \leq k} 2^j (\beta_k \alpha_j + \alpha_k \beta_j) + 2^{k-j} (1 - \omega) \alpha_j \beta_j.$$

Proof. For the proof, see [4, §5]. □

7.1. Derivative field control. The main purpose of this subsection is to establish the estimate (4.12), which states

$$\|P_k \psi_m(s)\|_{F_k(T)} \lesssim (1 + s^2 2^{2k})^{-4} 2^{-\sigma k} b_k(\sigma).$$

In the course of the proof we shall also establish auxiliary estimates useful elsewhere. Estimate (4.12) plays a key role in controlling the nonlinear paradifferential flow, allowing us to gain regularity by integrating in heat time. The proof uses a bootstrap argument and exploits the Duhamel formula.

Recall that the fields $\psi_\alpha$, $A_\alpha$, $\alpha = 1, 2, 3$, $(\psi_3 \equiv \psi_t, A_3 \equiv A_t)$ satisfy (2.20), which states

$$(\partial_s - \Delta) \psi_\alpha = U_\alpha.$$

We use representation (2.22) of the heat nonlinearity:

$$U_\alpha := 2i A_\ell \psi_\alpha + i(\partial_\ell A_\ell) \psi_\alpha - A_\ell^2 \psi_\alpha + i \text{Im} (\psi_\alpha \overline{\psi_\ell}) \psi_\ell.$$  

Hence $\psi_\alpha$ admits the representation

$$\psi_\alpha(s) = e^{s \Delta} \psi_\alpha(s_0) + \int_{s_0}^s e^{(s-s') \Delta} U_\alpha(s') \, ds'$$  

(7.1)

for any $s \geq s_0 \geq 0$.

For each $k \in \mathbb{Z}$, set

$$a(k) := \sup_{s \in [0, \infty)} (1 + s^2 2^{2k})^4 \sum_{m=1, 2} \|P_k \psi_m(s)\|_{F_k(T)},$$

and for $\sigma \in [0, \sigma_1 - 1]$ introduce the frequency envelopes

$$a_k(\sigma) = \sup_{j \in \mathbb{Z}} 2^{-|k-j|} 2^{\sigma j} a(j).$$  

(7.2)

The frequency envelopes $a_k(\sigma)$ are finite and in $\ell^2$ by (2.38) and (3.1). Our goal is to show $a_k(\sigma) \lesssim b_k(\sigma)$, which in particular implies (4.12).
Lemma 7.3. Suppose that $\psi_x$ satisfies the bootstrap condition
\[ \| P_k\psi_x(s) \|_{F_k(T) \cap S_k^{1/2}(T)} \leq \varepsilon_p^{-1/2} b_k(1 + s2^{2k})^{-4}. \] (7.3)
Then (4.12) holds.

We can take $\varepsilon_p = \varepsilon_1^{1/10}$, for instance. As in [4], this result may be strengthened to

Corollary 7.4. The estimate (4.12) holds even when the bootstrap hypothesis (7.3) is dropped.

Proof. Directly apply the argument of [4, Corollary 4.4], which we omit. □

The sequence of lemmas we prove in order to establish Lemma 7.3 culminates in Lemma 7.11, which controls the nonlinear term of the Duhamel formula (7.1) by $2^{-\sigma k} a_k(\sigma)$ along with suitable decay and an epsilon-gain arising from energy dispersion. Its immediate predecessor, Lemma 7.10, controls $P_k U_m$ in $F_k(T)$.

Referring back to (2.22) and seeing as how $U_m$ contains the term $2i A_\ell \partial_\ell \psi_m$, we see that in order to apply the parabolic estimates of Lemma 7.1 toward controlling $P_k U_m$, it is necessary that we first control $P_k A_m$ in $F_k(T)$ in terms of the frequency envelopes $\{a_\ell(\sigma)\}$, and it is to this that we now turn.

For $k, k_0 \in \mathbb{Z}$ and $s \in [2^{2k_0-1}, 2^{2k_0+1})$, set
\[ b_{k,s}(\sigma) = \begin{cases} \sum_{j=k}^{-k_0} a_j a_j(\sigma) & k + k_0 \leq 0 \\ 2^{k+k_0} a_{-k_0} a_k(\sigma) & k + k_0 \geq 0. \end{cases} \]

Let $C$ be the smallest number in $[1, \infty)$ such that
\[ \| P_k A_m(s) \|_{F_k(T) \cap S_k^{1/2}(T)} \leq C (1 + s2^{2k})^{-4} 2^{-\sigma k} b_{k,s}(\sigma) \] (7.4)
for all $s \in [0, \infty)$, $k \in \mathbb{Z}$, $m = 1, 2$, and $\sigma \in [0, \sigma_1 - 1]$. While this constant is indeed finite, it is not a priori controlled by energy. To show that $C$ is indeed controlled by energy, we use the integral representation
\[ A_m(s) = -\sum_{\ell=1,2} \int_s^\infty \text{Im}(\psi_m(\partial_\ell \psi_\ell + i A_\ell \psi_\ell))(r) \, dr \] (7.5)
and seek to control the Littlewood-Paley projection of the integrand in $F_k(T) \cap S_k^{1/2}(T)$. We treat differently the two types of terms in (7.5) that need to be controlled. In Lemma 7.6 we bound terms of the sort $P_k(\psi_x \bar{\psi}_x)$ and $P_k(\psi_x \bar{\partial_\ell \psi_\ell})$ in $F_k(T) \cap S_k^{1/2}(T)$. In Lemma 7.7 we combine the estimate on $P_k(\psi_x \bar{\psi}_x)$ with (7.4) to obtain control on $P_k(\psi_x \bar{\psi}_x A_x)$, gaining an epsilon from energy dispersion. Using (7.5) and exploiting the epsilon gain from energy dispersion will lead us to the conclusion of Lemma 7.8: $C \lesssim 1$. 

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We use the following bracket notation in the sequel:

\[ \langle f \rangle := (1 + f^2)^{1/2}. \]

**Lemma 7.5.** For any \( f, g \in \{ \psi_m, \overline{\psi_m} : m = 1, 2 \}, \ r \in [2^{2j-2}, 2^{2j+2}], \ j \in \mathbb{Z}, \ i = 1, 2, \) and \( \sigma \in [0, \sigma_1 - 1], \) we have the bounds

\[ \| P_k(f(r)g(r)) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \langle 2^{j+k} \rangle^{-82 - \sigma k} 2^{-j} a_{-j} \max(k, -j)(\sigma). \]  

(7.6)

and

\[ \| P_k(f(r)\partial_t g(r)) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \langle 2^{j+k} \rangle^{-82 - \sigma k} 2^{-j} a_{-j} (2^k a_k(\sigma) + 2^{-j} a_{-j}(\sigma)). \]  

(7.7)

**Proof.** By Lemma 7.2 with \( \omega = 0 \) we have

\[ \| P_k(fg) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \sum_{\ell \leq k} 2^\ell \alpha_k \beta_\ell + \sum_{\ell \geq k} 2^\ell \alpha_\ell \beta_\ell, \]  

(7.8)

where, due to the definition (7.2), \( \alpha_k \) and \( \beta_k \) satisfy

\[ \alpha_k \lesssim \langle 2^{j+k} \rangle^{-82 - \sigma k} a_k(\sigma), \quad \beta_k \lesssim \langle 2^{j+k} \rangle^{-8} a_k. \]  

(7.9)

Turning to the high-low frequency interaction first, we have using (7.9) and the frequency envelope property (2.29) that

\[ \sum_{\ell \leq k} 2^\ell \alpha_k \beta_\ell \lesssim \langle 2^{j+k} \rangle^{-82 - \sigma k} 2^{-j} a_{-j} \sum_{\ell \leq k} \langle 2^{j+\ell} \rangle^{-82 j^2 \delta[j+\ell]} a_\ell(\sigma). \]  

(7.10)

Thus it remains to show that

\[ \sum_{\ell \leq k} \langle 2^{j+\ell} \rangle^{-82 j^2 \delta[j+\ell]} a_\ell(\sigma) \lesssim a_{\max(k,-j)}(\sigma), \]  

(7.11)

which follows from pulling out a factor of \( a_k(\sigma) \) or \( a_{-j}(\sigma) \), according to whether \( k + j \geq 0 \) or \( k + j < 0 \), and then summing the remaining geometric series. In case \( k + j < 0 \) we pull out a factor of \( a_{-j}(\sigma) \) via (2.29).

Turning to the high-high frequency interaction term, we have

\[ \sum_{\ell \geq k} 2^\ell \alpha_\ell \beta_\ell \lesssim \langle 2^{j+k} \rangle^{-82 j^2 \delta[j+\ell]} a_{-j} \sum_{\ell \geq k} \langle 2^{j+\ell} \rangle^{-82 j^2 \delta[j+\ell]} a_\ell(\sigma), \]  

(7.12)

and so it remains to show that

\[ \sum_{\ell \geq k} \langle 2^{j+\ell} \rangle^{-82 j^2 \delta[j+\ell]} a_\ell(\sigma) \lesssim a_{\max(k,-j)}(\sigma). \]  

(7.13)

When \( k + j \geq 0 \), we have using (2.31)

\[ \sum_{\ell \geq k} \langle 2^{j+\ell} \rangle^{-82 j^2 \delta[j+\ell]} a_\ell(\sigma) \lesssim a_k(\sigma) \sum_{\ell \geq k} 2^{(2\delta - 1)(j+\ell)} \lesssim a_k(\sigma). \]

(2.30)

If \( k + j \leq 0 \), then we control the sum with (2.30) if \( \ell + j < 0 \) and with (2.31) if \( \ell + j \geq 0 \). Hence (7.13) holds.
Together (7.8)–(7.13) imply (7.6).

To establish (7.7) we follow a similar strategy. By Lemma 7.2 with \( \omega = 0 \) we have
\[
\|P_k(f \partial_i g)\|_{L^2(T) \cap S^1_k(T)} \lesssim \sum_{\ell \leq k} 2^\ell \alpha_\ell \beta_\ell + \sum_{\ell \geq k} 2^\ell \alpha_\ell \beta_\ell, \tag{7.14}
\]
where
\[
\alpha_k \lesssim (2^{j+k})^{-8} 2^{-\sigma_k} a_k(\sigma) \tag{7.15}
\]
for any \( \sigma \in [0, \sigma_1 - 1] \) and
\[
\beta_k \lesssim (2^{j+k})^{-8} 2^{-\sigma_k} a_k(\sigma) \tag{7.16}
\]
for any \( \sigma \in [0, \sigma_1 - 1] \).

Beginning with the low-high frequency interaction, we have
\[
\sum_{\ell \leq k} 2^\ell \alpha_\ell \beta_\ell \lesssim (2^{j+k})^{-8} 2^{-\sigma_k} 2^k a_k(\sigma) \sum_{\ell \leq k} (2^{j+\ell})^{-8} 2^\ell a_\ell, \tag{7.17}
\]
and so it remains to show that
\[
\sum_{\ell \leq k} (2^{j+\ell})^{-8} 2^\ell a_\ell \lesssim 2^{-j} a_{-j}. \tag{7.18}
\]

If \( k + j \leq 0 \), then (7.18) holds due to (2.30). If \( k + j \geq 0 \), then we apply (2.30) and (2.31) according to whether \( \ell + j \leq 0 \) or \( \ell + j > 0 \).

Turning now to the high-low frequency interaction, we have
\[
\sum_{\ell \geq k} 2^\ell \alpha_\ell \beta_\ell \lesssim (2^{j+k})^{-8} 2^{-\sigma_k} 2^{j} a_{-j} 2^k a_k(\sigma) \sum_{\ell \geq k} (2^{j+\ell})^{-8} 2^{\ell+2j} a_\ell. \tag{7.19}
\]
We need only check
\[
\sum_{\ell \leq k} (2^{j+\ell})^{-8} 2^{\ell+2j} a_\ell \lesssim 1,
\]
which can be seen to hold by considering separately the cases \( k + j \leq 0 \) and \( k + j \geq 0 \).

We conclude with the high-high frequency interaction:
\[
\sum_{\ell \geq k} 2^\ell \alpha_\ell \beta_\ell \lesssim (2^{j+k})^{-8} 2^{-\sigma_k} \sum_{\ell \geq k} (2^{j+\ell})^{-8} 2^{2\ell} a_\ell(\sigma) a_\ell \lesssim (2^{j+k})^{-8} 2^{-\sigma_k} 2^{-2j} a_j a_j(\sigma) \sum_{\ell \geq k} (2^{j+\ell})^{-8} 2^{2\ell+2j} a_\ell \tag{7.20}
\]
Here
\[
\sum_{\ell \geq k} (2^{j+\ell})^{-8} 2^{2\ell+2j} a_\ell \lesssim 1,
\]
which is seen to hold by considering separately the cases \( k + j \geq 0 \), \( k + j < 0 \).

Combining (7.17)–(7.1), we conclude (7.7). \( \square \)
Lemma 7.6. Let
\[ f(r) \in \{ \psi_m(r) \psi_\ell(r) : m, \ell = 1, 2 \}, \quad g(r) \in \{ A_m(r) : m = 1, 2 \}, \]
and \( r \in [2^{2j-2}, 2^{2j+2}] \). Then
\[
\| P_k(fg)(r) \|_{F_k(T) \cap S^{1/2}_k} \lesssim \begin{cases} 
\varepsilon C^{2-\sigma_k 2^{-2j} a_{-j} a_{-j}}(\sigma) & k + j \leq 0 \\
\varepsilon C^{2^{j+k} - 8 \sigma_k 2^{-2j} b_{k,r}(\sigma)} & k + j \geq 0.
\end{cases}
\]

Proof. We apply Lemma 7.2. By (7.6) and (7.4)
\[
\alpha_k(r) \lesssim 2^{-\sigma_k (2^{j+k}) - 8^{-j} a_{-j} a_{\max(k,-j)}(\sigma)}
\] (7.21)
and
\[
\beta_k(r) \lesssim C^{2^{-\sigma_k (2^{j+k})} - 8 b_{k,r}(\sigma)}
\] (7.22)
hold for any \( \sigma \in [0, \sigma_1 - 1] \).

We consider six cases, treating separately the low-high, high-low, and high-high frequency interactions, which we further divide according to whether \( k + j \geq 0 \) or \( k + j \leq 0 \).

Low-High frequency interaction with \( k + j \geq 0 \):

Using (7.21) and (7.22), we have
\[
\sum_{\ell \leq k} 2^\ell 2^j \alpha_\ell \lesssim C^{2^{j+k} - 8 \sigma_k 2^{-2j} b_{k,r}(\sigma)} \sum_{\ell \leq k} 2^\ell 2^j \alpha_\ell,
\] (7.23)
and so it remains to verify that
\[
\sum_{\ell \leq k} 2^\ell 2^j \alpha_\ell \lesssim \varepsilon.
\] (7.24)

Taking \( \sigma = 0 \) in the bounds (7.21) for \( \alpha_\ell \) and using (2.29), (2.31) yields
\[
\sum_{\ell \leq k} 2^\ell 2^j \alpha_\ell \lesssim \sum_{\ell \leq k} (2^{j+\ell} - 8 \sigma_k 2^{-2j} a_{-j} a_{\max(\ell,-j)})
\] = \sum_{\ell \leq k, j} 2^\ell 2^j a_{-j} + \sum_{j < \ell \leq k} (2^{j+\ell} - 8 \sigma_k 2^{-2j} a_{-j} a_{\ell})
\] \lesssim a_{-j} + a_{-j} \sum_{j < \ell \leq k} (2^{j+\ell} - 8 \sigma_k 2^{-2j} (1+\delta)(\ell+j) \lesssim \varepsilon,
which proves (7.24).

High-Low frequency interaction with \( k + j \geq 0 \):
Taking $\sigma = 0$ in the bounds for $b_{\ell,r}$, we have
\[
\sum_{\ell \leq k} 2^\ell \alpha_k \beta_\ell \lesssim C \langle 2^{j+k} \rangle^{2} 2^{-\sigma k} 2^{-2j} b_{k,r}(\sigma) \sum_{\ell \leq k} \langle 2^{j+\ell} \rangle^{2} 2^{-\ell-k} b_{\ell,r},
\]
and so it remains to show that
\[
\sum_{\ell \leq k} \langle 2^{j+\ell} \rangle^{2} 2^{-\ell-k} b_{\ell,r} \lesssim \varepsilon. \tag{7.26}
\]
Splitting the sum as follows,
\[
\sum_{\ell \leq k} \langle 2^{j+\ell} \rangle^{2} 2^{-\ell-k} b_{\ell,r} = \sum_{\ell \leq -j} \langle 2^{j+\ell} \rangle^{2} 2^{-\ell-k} a_q^2 + \sum_{-j < \ell \leq k} \langle 2^{j+\ell} \rangle^{2} 2^{-\ell-k} \sum_{-j < \ell \leq k} \langle 2^{j+\ell} \rangle^{2} 2^{-2\delta(j+q)}
\]
we note that the first summand is controlled by
\[
\sum_{\ell \leq -j} \langle 2^{j+\ell} \rangle^{2} 2^{-\ell-k} a_q^2 \lesssim a_{-j}^2 \sum_{\ell \leq -j} \langle 2^{j+\ell} \rangle^{2} 2^{-2\delta(j+q)} \lesssim a_{-j}^2 \lesssim \varepsilon.
\]
The second summand by may be handled similarly, thus proving \(7.26\).

**High-High frequency interaction with** \(k + j \geq 0\):

Taking $\sigma = 0$ in the bound \(7.22\) for $\beta_\ell$, we have
\[
\sum_{\ell \geq k} 2^\ell \alpha_\ell \beta_\ell \lesssim \langle 2^{j+\ell} \rangle^{2} 2^{-\sigma k} 2^{-2j} a_{-j} a_\ell (\sigma) C 2^{j+\ell} a_{-j} a_\ell
\]
\[
\lesssim C \langle 2^{j+k} \rangle^{2} 2^{-\sigma k} 2^{-2j} b_{k,r}(\sigma) \times \sum_{\ell \geq k} \langle 2^{j+\ell} \rangle^{2} 2^{-\ell-k} 2^{\delta(\ell-k)} 2^{j+\ell} a_{-j} a_\ell \tag{7.27}
\]
and so it remains to show that
\[
\sum_{\ell \geq k} \langle 2^{j+\ell} \rangle^{2} 2^{-\ell-k} 2^{\delta(\ell-k)} 2^{j+\ell} a_{-j} a_\ell \lesssim \varepsilon, \tag{7.28}
\]
which follows, for instance, from pulling out $a_{-j}^2$ via \(2.29\) and summing.

In view of \(7.23\)--\(7.28\), it follows from Lemma \(7.2\) with $\omega = 0$ that
\[
\| P_k(fg)(r) \|_{F_k(T) \cap S_1^{1/2}(T)} \lesssim \varepsilon C \langle 2^{j+k} \rangle^{2} 2^{-\sigma k} 2^{-2j} b_{k,r}(\sigma) \quad \text{for } k + j \geq 0 \tag{7.29}
\]
as required.

**Low-High frequency interaction with** \(k + j \leq 0\):
In this case it follows from (7.22) that
\[ \beta_k \lesssim C2^{-\sigma k} \sum_{p=k}^{k+j} a_p a_p(\sigma) \]
so that
\[
\sum_{\ell \leq k} 2^\ell \alpha_\ell \beta_k \lesssim C2^{-\sigma k} 2^{-j} \sum_{p=k}^{k+j} a_p a_p(\sigma) \sum_{\ell \leq k} (2^j + \ell)^{-8} 2^\ell \\
\lesssim C2^{-\sigma k} 2^{-2j} \sum_{p=k}^{k+j} a_p 2^{-\delta(j+p)} \sum_{\ell \leq k} 2^{\ell+j}. \tag{7.30}
\]
It remains to show
\[ a_{-j} \sum_{p=k}^{k+j} a_p 2^{-\delta(j+p)} \sum_{\ell \leq k} 2^{\ell+j} \lesssim \varepsilon, \]
which follows from pulling out \( a_p \) as an \( a_{-j} \) via (2.29) and summing.

**High-Low frequency interaction with \( k + j \leq 0 \):**

In this case
\[
\sum_{\ell \leq k} 2^\ell \alpha_\ell \beta_k \lesssim C2^{-\sigma k} 2^{-2j} \sum_{p=k}^{k+j} a_{-j} a_{-j}(\sigma) \sum_{\ell \leq k} \sum_{p=\ell}^{k+j} a_p^2, \tag{7.31}
\]
and so we need to show
\[ \sum_{\ell \leq k} 2^{\ell+j} \sum_{p=\ell}^{k+j} a_p^2 \lesssim \varepsilon, \]
which follows by pulling out \( a_{-j}^2 \) and summing.

**High-High frequency interaction with \( k + j \leq 0 \):**

As a first step we write
\[
2^k \sum_{\ell \geq k} 2^{(\ell-k)/2} \alpha_\ell \beta_\ell = 2^k \sum_{k \leq \ell < -j} 2^{(\ell-k)/2} \alpha_\ell \beta_\ell + 2^k \sum_{\ell \geq -j} 2^{(\ell-k)/2} \alpha_\ell \beta_\ell. \tag{7.32}
\]
The first summand is controlled by
\[
2^k \sum_{k \leq \ell < -j} 2^{(\ell-k)/2} \alpha \beta_\ell \lesssim C 2^{-\sigma k} 2^{-2j} a_{-j} a_{-j}(\sigma) \times \\
\times \sum_{k \leq \ell < -j} 2^{(\ell-k)/2} 2^{k+j} 2^{-\sigma(\ell-k)} \sum_{p=\ell}^{-j} a_p^2,
\] (7.33)
We have
\[
\sum_{k \leq \ell < -j} 2^{(\ell-k)/2} 2^{k+j} 2^{-\sigma(\ell-k)} \sum_{p=\ell}^{-j} a_p^2 \lesssim a_{-j}^2 2^{(k+j)/2} \sum_{k \leq \ell < -j} 2^{-2\delta(j+\ell)} \lesssim \varepsilon,
\]
which establishes the desired control on the first summand.

The second summand is controlled by
\[
2^k \sum_{\ell \geq -j} 2^{(\ell-k)/2} \alpha \beta_\ell \lesssim 2^k \sum_{\ell \geq -j} 2^{(\ell-k)/2} 2^{j+\ell} 2^{2j} 2^{-\sigma j} a_{-j} a_{-j}(\sigma) \times \\
\times C (2^{j+\ell}) 2^{2j} a_{-j} a_{-j} \lesssim C 2^{-\sigma k} 2^{-2j} a_{-j} a_{-j}(\sigma) \times \\
\times \sum_{\ell \geq -j} 2^{(\ell-k)/2} 2^{k+j} 2^{2(1+\delta)(\ell+j)} a_{-j} a_{-j},
\] (7.34)
and so it remains to show that
\[
\sum_{\ell \geq -j} 2^{(\ell-k)/2} 2^{k+j} 2^{2(1+\delta)(\ell+j)} a_{-j} a_{-j} \lesssim \varepsilon,
\] (7.35)
which follows from pulling out \(a_{-j}^2\) and summing.

Combining (7.30)–(7.35), we conclude from applying Lemma 7.2 with \(\omega = 1/2\) that
\[
\|P_k(\mathcal{F})(r)\|_{F_k(T) \cap S^{1/2}_k(T)} \lesssim \varepsilon C 2^{-\sigma k} 2^{-2j} a_{-j} a_{-j}(\sigma) \quad \text{for } k + j \leq 0,
\]
which, combined with (7.29) completes the proof of the lemma. \(\square\)

Lemma 7.7. For any \(k \in \mathbb{Z}\) and \(s \in [0, \infty)\) we have
\[
\|P_k A_m(s)\|_{F_k(T) \cap S^{1/2}_k(T)} \lesssim (1 + s^2)^{2k} 2^{-\sigma k} b_{k,s}(\sigma).
\]
Proof. From representation (7.3) for \( A_m \) it follows that

\[
\| P_k A_m(s) \|_{F_k(T) \cap S^k_1(T)} \lesssim \int_s^\infty \| P_k (\overline{\psi_m(r)} \partial_t \psi_r(r)) \|_{F_k(T) \cap S^k_1(T)} \, dr \\
+ \int_s^\infty \| P_k (\overline{\psi_m(r)} \psi_r A_r) \|_{F_k(T) \cap S^k_1(T)} \, dr.
\]

(7.36)

Taking \( k_0 \in \mathbb{Z} \) so that \( s \in [2^{2k_0-1}, 2^{2k_0+1}) \) and using (7.7), we have that the first term is dominated by

\[
\sum_{j \geq k_0} \int_{2^{j-1}}^{2^{j+1}} \| P_k (\overline{\psi_m(r)} \partial_t \psi_r(r)) \|_{F_k(T) \cap S^k_1(T)} \, dr \\
\lesssim 2^{-\sigma k} \sum_{j \geq k_0} \langle 2^{j+k} \rangle^{-8} (2^{j+k} a_{-j} a_k(\sigma) + a_{-j} a_{-j}(\sigma)).
\]

(7.37)

We claim that

\[
\sum_{j \geq k_0} \langle 2^{j+k} \rangle^{-8} (2^{j+k} a_{-j} a_k(\sigma) + a_{-j} a_{-j}(\sigma)) \lesssim (1 + s 2^{2k})^{-1} b_{k,s}(\sigma).
\]

(7.38)

When \( k + k_0 \geq 0 \), it follows from (2.29) that the left hand side of (7.38) is bounded by

\[
2^{k_0+k} a_{-k_0} a_k(\sigma) \sum_{j \geq k_0} \langle 2^{j+k} \rangle^{-8} \left( 2^{j-k_0} 2^{\delta(j-k_0)} + 2^{-k_0-k} 2^{\delta(j-k_0)} 2^{8(k+j)} \right) \\
\lesssim b_{k,s}(\sigma) \sum_{j \geq k_0} \langle 2^{j+k} \rangle^{-8} \left( 2^{(1+\delta)(j-k_0)} + 2^{(\delta-1)(k_0+k)} 2^{2\delta(j-k_0)} \right),
\]

(7.39)

and so it suffices to show

\[
\sum_{j \geq k_0} \langle 2^{j+k} \rangle^{-8} 2^{2(j-k_0)} \lesssim \langle 2^{j+k_0} \rangle^{-8},
\]

(7.40)

which follows from series comparison, for instance.

Together (7.40) and (7.39), show that (7.38) holds for \( k + k_0 \geq 0 \).

If, on the other hand, \( k + k_0 \leq 0 \), then we split the sum in (7.38) according to whether \( j + k \leq 0 \) or \( j + k > 0 \). In the first case,

\[
\sum_{k_0 \leq j \leq -k} \langle 2^{j+k} \rangle^{-8} (2^{j+k} a_{-j} a_k(\sigma) + a_{-j} a_{-j}(\sigma)) \\
\lesssim \langle 2^{k_0+k} \rangle^{-8} b_{k,s}(\sigma) + \sum_{k_0 \leq j \leq -k} \langle 2^{j+k} \rangle^{-8} 2^{j+k} a_{-j} a_k(\sigma).
\]

(7.41)
Then
\[ \sum_{k_0 \leq j \leq -k} (2^{j+k})^{-8} 2^{j+k} a_{-j} a_k(\sigma) \lesssim \sum_{k_0 \leq j \leq -k} (2^{j+k})^{-8} 2^{j+k} a_{-j-\sigma} 2^{-\delta(j+k)} \sim (1 + s 2^{2k})^{-4} b_{k,s}(\sigma). \quad (7.42) \]

When \( j + k > 0 \) we have
\[ \sum_{j > -k} (2^{j+k})^{-8} (2^{j+k} a_{-j} a_k(\sigma) + a_{-j} a_{-j}(\sigma)) \lesssim a_k a_k(\sigma) \sum_{j > -k} (2^{j+k})^{-8} (2^{j+k} 2^\delta(j+k) + 2^{2\delta(j+k)}) \lesssim b_{k,s}(\sigma). \quad (7.43) \]

Therefore (7.41) and (7.42) imply (7.38) holds when \( k + k_0 \leq 0 \) and \( j + k \leq 0 \) and (7.33) implies it holds when both \( k + k_0 \leq 0 \) and \( j + k > 0 \).

Having shown (7.38), we combine it with (7.37), concluding
\[ \int_s^\infty \| P_k(\psi_m(r) \partial_x \psi(r)) \|_{F_k(T) \cap S_k^{1/2}(T)} dr \lesssim (1 + s 2^{2k})^{-4} 2^{-\sigma k} b_{k,s}(\sigma). \quad (7.44) \]

We now move on to control the second term in (7.36). By Lemma 7.9 and (7.38), this term is bounded by
\[ \sum_{j \geq k_0} \int_{2^{j-1}}^{2^{j+1}} \| P_k(\psi_x(r) \psi_x(r) A_x(r)) \|_{F_k(T) \cap S_k^{1/2}(T)} dr \lesssim C 2^{-\sigma k} \sum_{j \geq k_0} (2^{j+k})^{-8} (1_{-}(k+j)a_{-j} a_{-j}(\sigma) + 1_{+}(k+j) b_{k,2j}(\sigma)) \lesssim C 2^{-\sigma k} (2^{k_0+k})^{-8} b_{k,2^{2k_0}}(\sigma). \quad (7.45) \]

Together (7.36), (7.44), and (7.45) imply
\[ \| P_k A_m(s) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim 2^{-\sigma k} (1 + s 2^{2k})^{-4} b_{k,s}(\sigma) (1 + C \varepsilon), \]
from which it follows that \( C \lesssim 1 + C \varepsilon \) and hence \( C \lesssim 1 \), proving the lemma. \( \square \)

**Lemma 7.8.** It holds that
\[ \| P_k A_\ell(r) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \begin{cases} \varepsilon 2^{-\sigma k} 2^{-j} a_{-j} a_{-j}(\sigma) & \text{if } k + j \leq 0 \\ \varepsilon 2^{-\sigma k} 2^{-j} b_{k,2j}(\sigma) & \text{if } k + j \geq 0. \end{cases} \]

**Proof.** We apply Lemma 7.2 with \( f = g = A_\ell \) and \( \omega = 0 \) so that
\[ \| P_k(A_\ell(r)) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \sum_{\ell \leq k} 2^\ell \alpha_k \beta_\ell + \sum_{\ell \geq k} 2^\ell \alpha_\ell \beta_\ell, \]
where
\[ \alpha_k \lesssim 2^{-\sigma_k} (2^{j+k})^{-8} b_{k,s}(\sigma), \quad \beta_k \lesssim (2^{j+k})^{-8} b_{k,s}. \]

**Case** \( k + j \leq 0; \)

We first consider the case \( k + j \leq 0 \) and proceed to control the high-low frequency interaction. We have
\[
\sum_{\ell \leq k} 2^\ell \alpha_k \beta_{\ell} \lesssim 2^{-\sigma_k} \sum_{\ell \leq k} 2^\ell b_{k,2j}(\sigma) b_{\ell,2j} \\
\lesssim 2^{-\sigma_k} \sum_{p=k}^{\ell - j} a_p a_p(\sigma) 2^\ell \sum_{\ell \leq q} a_q^2 \\
\lesssim 2^{-\sigma_k} a_{-j} a_{-j}(\sigma) \sum_{p=k}^{\ell - j} 2^{-2(\delta(j+p))} \sum_{\ell \leq q} a_q^2. \tag{7.46}
\]

It remains to show
\[
\sum_{p=k}^{\ell - j} 2^{-2(\delta(j+p))} \sum_{\ell \leq q} a_q^2 \lesssim \varepsilon, \tag{7.47}
\]
which follows from bounding \( a_{-j}^2 \) by \( \varepsilon \) and summing. To control the high-high interaction term we first split the sum as
\[
\sum_{\ell \geq k} 2^\ell \alpha_\ell \beta_\ell \lesssim \sum_{k \leq \ell < -j} 2^\ell \alpha_\ell \beta_\ell + \sum_{\ell \geq -j} 2^\ell \alpha_\ell \beta_\ell. \tag{7.48}
\]

The first summand is controlled by
\[
\sum_{k \leq \ell < -j} 2^\ell \alpha_\ell \beta_\ell \lesssim 2^{-\sigma_k} \sum_{k \leq \ell < -j} 2^\ell b_{k,2j}(\sigma) b_{\ell,2j} \\
\lesssim 2^{-\sigma_k} 2^{-j} \sum_{k \leq \ell < -j} 2^{\ell + j} \sum_{p=\ell}^{\ell - j} a_p a_p(\sigma) \sum_{q=\ell}^{\ell - j} a_q^2. 
\]

Pulling out \( a_{-j}^2 a_{-j}(\sigma) \) and summing implies
\[
\sum_{k \leq \ell < -j} 2^\ell \alpha_\ell \beta_\ell \lesssim \varepsilon 2^{-\sigma_k} 2^{-j} a_{-j} a_{-j}(\sigma), \tag{7.49}
\]
The second summand is controlled by
\[
\sum_{\ell \geq -j} 2^\ell \alpha_\ell \beta_\ell \lesssim 2^{-\sigma k} \sum_{\ell \geq -j} 2^\ell (2^{j+\ell})^{-8} b_{\ell,2^{2j}}(\sigma) b_{\ell,2^{2j}} \\
\lesssim 2^{-\sigma k} \sum_{\ell \geq -j} 2^\ell (2^{j+\ell})^{-8} 2^{2(\ell+j)} a_{-j} a_\ell(\sigma) \\
\lesssim \epsilon 2^{-\sigma k} 2^{-j} a_{-j} a_\ell(\sigma).
\] (7.50)

Combining (7.46)–(7.50), we conclude
\[
\| P_k A^2_r \|_{F_k(T \cap S^{1/2}_k(T))} \lesssim \epsilon 2^{-\sigma k} 2^{-j} a_{-j} a_\ell(\sigma) \quad \text{for} \quad k + j \leq 0. \quad (7.51)
\]

Case \( k + j \geq 0 \):

We now consider the case \( k + j \geq 0 \) and turn to the high-low frequency interaction, splitting it into two pieces:
\[
\sum_{\ell \leq k} 2^\ell \alpha_k \beta_\ell \leq \sum_{\ell \leq -j} 2^\ell \alpha_k \beta_\ell + \sum_{-j < \ell \leq k} 2^\ell \alpha_k \beta_\ell. \quad (7.52)
\]

The first summand is controlled by
\[
\sum_{\ell \leq -j} 2^\ell \alpha_k \beta_\ell \lesssim 2^{-\sigma k} 2^{-j} b_{k,2^{2j}}(\sigma) \sum_{\ell \leq -j} 2^{\ell+j} (2^{j+k})^{-8} b_{\ell,2^{2j}},
\] (7.53)
and so we need to show
\[
\sum_{\ell \leq -j} 2^{\ell+j} (2^{j+k})^{-8} b_{\ell,2^{2j}} \lesssim \epsilon,
\] (7.54)
which follows from
\[
\sum_{\ell \leq -j} 2^{\ell+j} b_{\ell,2^{2j}} \lesssim \sum_{\ell \leq -j} 2^{\ell+j} \sum_{p=\ell}^{-j} a_p^2 \\
\lesssim a_{-j}^2 \sum_{\ell \leq -j} 2^{(1-2\delta)(\ell+j)} \lesssim \epsilon.
\]

The second summand in (7.52) is controlled by
\[
\sum_{-j < \ell \leq k} 2^\ell \alpha_k \beta_\ell \lesssim 2^{-\sigma k} 2^{-j} b_{k,2^{2j}}(\sigma) \sum_{j < \ell \leq k} (2^{j+\ell})^{-8} 2^{(j+k)} a_{-j} a_\ell,
\] (7.55)
where we note
\[
\sum_{-j < \ell \leq k} (2^{j+\ell})^{-8} 2^{2j} a_{-j} a_\ell \lesssim a_{-j}^2 \sum_{-j < \ell \leq k} (2^{j+\ell})^{-8} 2^{(2+\delta)(\ell+j)} \\
\lesssim \epsilon.
\] (7.56)
We now turn to the high-high frequency interaction. We have
\[
\sum_{\ell \geq k} 2^\ell \alpha_\ell \beta_\ell \lesssim \sum_{\ell \geq k} 2^\ell 2^{-\sigma \ell} (2^j + \ell)^{-8} 2^{2(\ell + j)} a_{-j} a_\ell (\sigma)
\]
\[
\lesssim 2^{-\sigma k} 2^{-j} 2^{k+j} a_{-j} \sum_{\ell \geq k} 2^{\ell-k} 2^{-\sigma (\ell-k)} (2^j + \ell)^{-8} 2^{2(\ell + j)} a_{-j} a_\ell (\sigma)
\]
\[
\lesssim 2^{-\sigma k} 2^{-j} b_{k,2^j} (\sigma) \sum_{\ell \geq k} (2^j + \ell)^{-8} 2^{2(1+\delta)(\ell-k)} 2^{2(\ell + j)} a_{-j} a_\ell. \tag{7.57}
\]

It remains to show that
\[
\sum_{\ell \geq k} (2^j + \ell)^{-8} 2^{2(1+\delta)(\ell-k)} 2^{2(\ell + j)} a_{-j} a_\ell \lesssim \varepsilon, \tag{7.58}
\]
which follows from bounding \(a_{-j} a_\ell\) by \(\varepsilon\) and summing.

Together (7.52)–(7.58) imply
\[
\| P_k A^2_\ell (r) \|_{F_k(T) \cap S_k^{1/2} (T)} \lesssim \varepsilon 2^{-\sigma k} 2^{-j} b_{k,2^j} (\sigma) \quad \text{for } k + j \geq 0,
\]
which, combined with (7.51) implies the lemma. \(\square\)

Set
\[
c_{k,j} (\sigma) = \begin{cases} 
2^{-j} a_{-j} a_{-j} (\sigma) & \text{if } k + j \leq 0 \\
2^{k+j} a_{-j} a_k (\sigma) & \text{if } k + j \geq 0.
\end{cases} \tag{7.59}
\]

**Lemma 7.9.** Let \( r \in [2^{2j-2}, 2^{2j+2}] \) and let
\[
F \in \{ A^2_\ell, \partial_\ell A_\ell, f g : \ell = 1, 2; f, g \in \{ \psi_m, \overline{\psi}_m : m = 1, 2 \}\}.
\]

Then
\[
\| P_k F (r) \|_{F_k(T) \cap S_k^{1/2} (T)} \lesssim (2^j + \ell)^{-8} 2^{-\sigma k} c_{k,j} (\sigma). \tag{7.60}
\]

**Proof.** If \( F = A^2_\ell \), then (7.60) is an immediate consequence of Lemma 7.8 when \( k + j \leq 0 \). If \( k + j \geq 0 \), then Lemma 7.8 implies
\[
\| P_k A^2_\ell (r) \|_{F_k(T) \cap S_k^{1/2} (T)} \lesssim \varepsilon 2^{-\sigma k} 2^{-j} 2^{k+j} a_{-j} a_{-j} (\sigma),
\]
and multiplying the right hand side by \( 2^{k+j} \) yields the desired estimate.

Consider now the case where \( F = \partial_\ell A_\ell \). By Lemma 7.4 we have
\[
\| P_k (\partial_\ell A_\ell) (r) \|_{F_k(T) \cap S_k^{1/2} (T)} \lesssim 2^k (2^j + \ell)^{-8} 2^{-\sigma k} b_{k,2^j} (\sigma). \tag{7.61}
\]

When \( k + j \geq 0 \), we rewrite (7.61) as
\[
\| P_k (\partial_\ell A_\ell) (r) \|_{F_k(T) \cap S_k^{1/2} (T)} \lesssim (2^j + \ell)^{-8} 2^{-\sigma k} 2^k 2^{k+j} a_{-j} a_k (\sigma),
\]
which is the desired bound (7.60). If \( k + j \leq 0 \), then (7.61) becomes

\[
\| P_k(\partial_\ell A_\ell)(r) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \langle 2^{j+k} \rangle^{\sigma k} 2^{-j} a_k a_j(\sigma) = \langle 2^{j+k} \rangle^{\sigma k} c_{k,j}(\sigma).
\]

If \( F = fg \), as in the statement of the lemma, then (7.60) follows directly from (7.6) when \( k + j \leq 0 \). If \( k + j \geq 0 \), then to get (7.60) we multiply the right hand side of (7.6) by \( 2^{2j+2k} \). □

Set

\[
d_{k,j} := \varepsilon \langle 2^{j+k} \rangle^{\sigma k} 2^{-j} a_k a_j(\sigma).
\]

(7.62)

Lemma 7.10. It holds that

\[
\| P_k U_m(r) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \varepsilon \langle 2^{j+k} \rangle^{\sigma k} 2^{-j} a_k a_j(\sigma) =: d_{k,j}.
\]

Proof. Using now (2.21) instead of (2.22), i.e., taking now

\[
U_\alpha = iA_\ell \partial_\ell \psi_\alpha + i\partial_\ell (A_\ell \psi_\alpha) - A_\ell^2 \psi_\alpha + i\text{Im}(\psi_\alpha \psi_\ell)\psi_\ell,
\]

we have that it suffices to prove

\[
\| P_k(F(r)f(r)) \|_{F_k(T) \cap S_k^{1/2}(T)} + 2^k \| P_k(A_\ell f(r)) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim d_{k,j},
\]

where

\[
F \in \{ A_\ell^2, \partial_\ell A_\ell, gh : \ell = 1, 2; f, h \in \{ \psi_m, \overline{\psi}_m : m = 1, 2 \} \}
\]

and \( f \in \{ \psi_m, \overline{\psi}_m : m = 1, 2 \} \). We consider the terms \( P_k(Ff) \), and \( P_k(Af) \) separately.

Controlling \( P_k(Ff) \):

We apply Lemma 7.2 to \( P_k(Ff) \), handling the different frequency interactions separately and according to cases. We record

\[
\alpha_k \lesssim \langle 2^{j+k} \rangle^{\sigma k} c_{k,j}(\sigma),
\]

a consequence of (7.60).
Let us begin by assuming $k + j \leq 0$. For the low-high frequency interaction, we have

\[
\sum_{\ell \leq k} 2^{\ell} \alpha_{\ell} \beta_{k} \lesssim 2^{-\sigma_{k}} a_{k}(\sigma) \sum_{\ell \leq k} 2^{\ell} c_{\ell,j} \\
\lesssim 2^{-\sigma_{k}} a_{k}(\sigma) \sum_{\ell \leq k} 2^{\ell-j} a_{-j}^{\ell} \\
\lesssim \varepsilon 2^{-\sigma_{k}} 2^{k-j} 2^{-\delta(k+j)} a_{-j}(\sigma).
\]

(7.63)

In a similar manner we control the high-low frequency interaction by

\[
\sum_{\ell \leq k} 2^{\ell} \alpha_{k} \beta_{\ell} \lesssim 2^{-\sigma_{k}} c_{k,j}(\sigma) \sum_{\ell \leq k} 2^{\ell} a_{\ell} \\
\lesssim 2^{-\sigma_{k}} 2^{-j} a_{-j} a_{-j}(\sigma) \sum_{\ell \leq k} 2^{\ell} a_{\ell} \\
\lesssim \varepsilon 2^{-\sigma_{k}} 2^{k-j} a_{-j}(\sigma).
\]

(7.64)

The high-high frequency interaction we split into two cases:

\[
2^{k} \sum_{\ell \geq k} 2^{(\ell-k)/2} \alpha_{\ell} \beta_{\ell} \lesssim 2^{k} \sum_{k \leq \ell < -j} 2^{(\ell-k)/2} \alpha_{\ell} \beta_{\ell} + 2^{k} \sum_{\ell \leq -j} 2^{(\ell-k)/2} \alpha_{\ell} \beta_{\ell}.
\]

(7.65)

We control the first summand using the definition (7.64) of $c_{k,j}(\sigma)$, the frequency envelope properties (2.29), (2.30), and energy dispersion:

\[
2^{k} \sum_{k \leq \ell < -j} 2^{(\ell-k)/2} \alpha_{\ell} \beta_{\ell} \lesssim 2^{k} \sum_{k \leq \ell < -j} 2^{(\ell-k)/2} 2^{\ell} c_{\ell,j}(\sigma) a_{\ell} \\
\lesssim 2^{-\sigma_{k}} 2^{k-j} a_{-j}(\sigma) a_{-j} \sum_{k \leq \ell < -j} 2^{(\ell-k)/2} a_{\ell} \\
\lesssim 2^{-\sigma_{k}} 2^{k-j} 2^{-(k+j)/2} a_{-j}(\sigma) a_{-j} \sum_{k \leq \ell < -j} 2^{(\ell-j)/2} a_{\ell} \\
\lesssim \varepsilon 2^{-\sigma_{k}} 2^{k-j} 2^{-(k+j)/2} a_{-j}(\sigma).
\]

(7.66)

In like manner we control the second summand:

\[
2^{k} \sum_{\ell \geq -j} 2^{(\ell-k)/2} \alpha_{\ell} \beta_{\ell} \lesssim 2^{k} \sum_{\ell \geq -j} (2^{j+\ell})^{-8} 2^{(\ell-k)/2} 2^{-\sigma_{\ell}} c_{\ell,j}(\sigma) a_{\ell} \\
\lesssim 2^{k} \sum_{\ell \geq -j} (2^{j+\ell})^{-8} 2^{(\ell-k)/2} 2^{-\sigma_{\ell}} 2^{2\ell+j} a_{-j} a_{\ell}(\sigma) a_{\ell} \\
\lesssim \varepsilon 2^{-\sigma_{k}} 2^{k-j} 2^{-(k+j)/2} a_{-j}(\sigma)
\]

(7.67)

Combining (7.63)–(7.67), we conclude

\[
\| P_{k}(F(r) f(r)) \|_{F_{k}(T) \cap S^{1/2}_{k}(T)} \lesssim \varepsilon 2^{-\sigma_{k}} 2^{k-j} 2^{-(k+j)/2} a_{-j}(\sigma), \quad k + j \leq 0.
\]

(7.68)
We now turn to the case \( k + j \geq 0 \). In the low-high frequency interaction case, we have

\[
\sum_{\ell \leq k} 2^\ell \alpha_k \beta_k \lesssim \langle 2^{j+k} \rangle^{-8} 2^{-\sigma k} a_k(\sigma) \sum_{\ell \leq k} \langle 2^{j+\ell} \rangle^{-8} 2^\ell c_{\ell,j} \\
\lesssim \langle 2^{j+k} \rangle^{-8} 2^{-\sigma k} 2^k a_k(\sigma) \times \\
\left( \sum_{\ell \leq -j} 2^\ell 2^{j+\ell-k} a_{-j}^2 + \sum_{-j<\ell \leq k} \langle 2^{j+\ell} \rangle^{-8} 2^{\ell-j} a_{-j} a_\ell \right) .
\]

(7.69)

To estimate the first term we use

\[
a_{-j}^2 \sum_{\ell \leq -j} 2^{\ell-k} 2^{-j-k} \lesssim \varepsilon 2^{-(j+k)} \cdot 2^{-(j+k)} \leq \varepsilon
\]

(7.70)

and for the second

\[
a_{-j} \sum_{-j<\ell \leq k} (2^{j+\ell})^{-8} 2^{3\ell-j-2k} a_\ell \\
= a_{-j} \sum_{-j<\ell \leq k} (2^{j+\ell})^{-8} 2^{j+2\ell-2k} a_\ell \\
\lesssim a_{-j} a_k \sum_{-j<\ell \leq k} (2^{j+\ell})^{-8} 2^{(2-\delta)(\ell-k)} \lesssim \varepsilon.
\]

(7.71)

In the high-low frequency interaction case, we have

\[
\sum_{\ell \geq k} 2^\ell \alpha_k \beta_k \lesssim \langle 2^{j+k} \rangle^{-8} 2^{-\sigma k} c_{\ell,j}(\sigma) \sum_{\ell \geq k} (2^{j+\ell})^{-8} 2^\ell a_\ell \\
\lesssim \langle 2^{j+k} \rangle^{-8} 2^{-\sigma k} 2^k a_k(\sigma) \sum_{\ell \geq k} (2^{j+\ell})^{-8} 2^\ell a_\ell \\
\lesssim \langle 2^{j+k} \rangle^{-8} 2^{-\sigma k} 2^k a_k(\sigma) a_{-j}^2.
\]

(7.72)

In the high-high frequency interaction case we have

\[
\sum_{\ell \geq k} 2^\ell \alpha_k \beta_k \lesssim \sum_{\ell \geq k} (2^{j+\ell})^{-8} 2^\ell 2^{-\sigma \ell} a_\ell(\sigma)c_{\ell,j} \\
\lesssim (2^{j+k})^{-8} 2^{-\sigma k} \sum_{\ell \geq k} (2^{j+\ell})^{-8} 2^\ell a_\ell(\sigma) 2^{2\ell+j} a_{-j} a_\ell \\
\lesssim (2^{j+k})^{-8} 2^{-\sigma k} 2^k a_k(\sigma) a_{-j}^2.
\]

(7.73)

From (7.69)–(7.73) we conclude

\[
\| P_k (F(r) f(r)) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \varepsilon (2^{j+k})^{-8} 2^{-\sigma k} 2^k a_k(\sigma), \quad k + j \geq 0.
\]

(7.74)
Controlling $2^k P_k (Af)$:

We now apply Lemma 7.2 to $P_k (A f)$. Note that
\[
\alpha_k \lesssim (2^{j+k} - 8) 2^{-\sigma_k} b_{k,r} (\sigma)
\]
because of Lemma 7.7 and that
\[
\beta_k \lesssim (2^{j+k} - 8) 2^{-\sigma_k} a_k (\sigma).
\]

We begin by assuming $k + j \leq 0$. The low-high frequency interaction is controlled by
\[
\sum_{\ell \leq k} 2^\ell \alpha_{\ell} \beta_k \lesssim 2^{-\sigma_k} \sum_{\ell \leq k} 2^\ell \sum_{p = \ell} \langle \eta \rangle_2 a_p^2 \lesssim 2^{-\sigma_k} 2^{-\delta(k+j)} a_{-j}^2 a_{-j} (\sigma) \sum_{\ell \leq k} 2^\ell \sum_{p = \ell} 2^{-2(j+p)}.
\]

Summing yields
\[
2^k \sum_{\ell \leq k} 2^\ell \alpha_{\ell} \beta_k \lesssim 2^{2k} 2^{-\sigma_k} 2^{-\delta(k+j)} a_{-j}^2 a_{-j} (\sigma). \tag{7.75}
\]

Control over the high-low frequency interaction follows from
\[
\sum_{\ell \leq k} 2^\ell \alpha_k \beta_{\ell} \lesssim 2^{-\sigma_k} \sum_{p = k} \langle \eta \rangle_2 a_p \sum_{\ell \leq k} 2^\ell a_\ell \lesssim 2^{k} 2^{-\sigma_k} 2^{-2\delta(k+j)} a_{-j} a_k a_{-j} (\sigma). \tag{7.76}
\]

We now turn to the high-high frequency interaction. We begin by splitting the sum:
\[
2^k \sum_{\ell \geq k} 2^\ell \alpha_{\ell} \beta_{\ell} \lesssim 2^k \sum_{k \leq \ell < -j} 2^\ell a_{-j} \sum_{k \leq \ell < -j} 2^\ell a_\ell + 2^k \sum_{\ell \geq -j} 2^\ell a_{-j} \sum_{k \leq \ell < -j} 2^\ell a_\ell. \tag{7.77}
\]

Then
\[
2^k \sum_{k \leq \ell < -j} 2^\ell a_{-j} \sum_{k \leq \ell < -j} 2^\ell a_\ell \lesssim 2^{2k} 2^{-\sigma_k} a_{-j} \sum_{k \leq \ell < -j} 2^\ell a_\ell \lesssim 2^{2k} 2^{-\delta(k+j)} a_{-j}^2 a_{-j} (\sigma). \tag{7.78}
\]

As for the second summand, we have
\[
2^k \sum_{\ell \geq -j} 2^\ell a_{-j} \sum_{k \leq \ell < -j} 2^\ell a_\ell \lesssim 2^{2k} 2^{-\delta(k+j)} a_{-j}^2 a_{-j} (\sigma). \tag{7.79}
\]
Combining (7.75)– (7.79) yields
\[ 2^k \| P_k(A_\ell(r)f(r)) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \varepsilon 2^{2k} 2^{-\sigma k} 2^{-(k+j)/2} a_{-j}(\sigma) \quad k + j \leq 0. \] (7.80)

Now let us assume that \( k + j \geq 0 \). The low-high frequency interaction we first split into two pieces.
\[ \sum_{\ell \leq k} 2^\ell \alpha_\ell \beta_k \lesssim \sum_{\ell \leq -j} 2^\ell \alpha_\ell \beta_k + \sum_{-j < \ell \leq k} 2^\ell \alpha_\ell \beta_k \] (7.81)

For the first term, we have
\[ \sum_{\ell \leq -j} 2^\ell \alpha_\ell \beta_k \lesssim (2^{j+k})^{-8} 2^{-\sigma k} a_k(\sigma) \sum_{\ell \leq -j} \sum_{p=\ell}^{-j} a_p^2 \]
\[ \lesssim (2^{j+k})^{-8} 2^{-\sigma k} a_{-j}^2 a_k(\sigma) \sum_{\ell \leq -j} 2^\ell \sum_{p=\ell}^{-j} 2^{-2\delta(j+p)}. \] (7.82)

Then
\[ \sum_{\ell \leq -j} 2^\ell \sum_{p=\ell}^{-j} 2^{-2\delta(j+p)} \lesssim \sum_{\ell \leq -j} 2^\ell 2^{-2\delta(j+\ell)} \lesssim 2^{-j} \lesssim 2^k. \] (7.83)

As for the second summand,
\[ \sum_{-j < \ell \leq k} 2^\ell \alpha_\ell \beta_k \lesssim (2^{j+k})^{-8} 2^{-\sigma k} a_{-j}^2 a_k(\sigma) \sum_{-j < \ell \leq k} (2^{j+\ell})^{-8} 2^{2j+\ell} a_{-j} a_\ell \]
\[ \lesssim (2^{j+k})^{-8} 2^{-\sigma k} 2^k a_{-j}^2 a_k(\sigma). \] (7.84)

The high-low frequency interaction is controlled by
\[ \sum_{\ell \leq k} 2^\ell \alpha_k \beta_\ell \lesssim (2^{j+k})^{-8} 2^{-\sigma k} 2^{j} a_{-j} a_k(\sigma) \sum_{\ell \leq k} (2^{j+\ell})^{-8} 2^\ell a_\ell \]
\[ \lesssim (2^{j+k})^{-8} 2^{-\sigma k} 2^k a_{-j}^2 a_k(\sigma). \] (7.85)

Finally, the high-high frequency interaction is controlled by
\[ \sum_{\ell \geq k} 2^\ell \alpha_\ell \beta_\ell \lesssim \sum_{\ell \geq k} (2^{j+\ell})^{-8} 2^\ell \alpha_\ell a_{-j} a_\ell 2^{-\sigma \ell} a_\ell(\sigma) \]
\[ \lesssim (2^{j+k})^{-8} 2^{-\sigma k} 2^k a_{-j} a_k(\sigma). \] (7.86)

Thus, in view of (7.81)– (7.86), we have shown that
\[ 2^k \| P_k(A_\ell(r)f(r)) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \varepsilon (2^{j+k})^{-8} 2^{-\sigma k} 2^k a_k(\sigma) \quad k + j \geq 0. \] (7.87)

Combining (7.68), (7.71), (7.80), and (7.87) proves the lemma. \( \square \)
Lemma 7.11. It holds that
\[ \| \int_0^s e^{(s-s')\Delta} P_k U_m(s') ds' \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \varepsilon (1 + s 2^{2k})^{-4} 2^{-\sigma k} a_k(\sigma). \]

Proof. Let \( k_0 \in \mathbb{Z} \) be such that \( s \in [2^{2k_0-1}, 2^{2k_0+1}) \). If \( k + k_0 \leq 0 \), then it follows from Lemma 7.10 that
\[ \| \int_0^s e^{(s-r)\Delta} P_k U_m(r) dr \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \sum_{j \leq k_0} \int_{[2^{2j} \cdot 2^{j+1}]} \| P_k U_m(r) \|_{F_k(T) \cap S_k^{1/2}(T)} dr \]
\[ \lesssim \sum_{j \leq k_0} 2^{2j} 2^{-\sigma k} 2^{k/2} (a_k(\sigma) + 2^{-3(k+j)/2} a_{j-1}(\sigma)) \]
\[ \lesssim \varepsilon 2^{-\sigma k} a_k(\sigma) \sum_{j \leq k_0} 2^{2k_2j} (1 + 2^{-3(k+j)/2} 2^{-\delta(k+j)}) \]
\[ \lesssim \varepsilon 2^{-\sigma k} a_k(\sigma). \]

On the other hand, if \( k + k_0 > 0 \), then
\[ \| \int_0^s e^{(s-r)\Delta} P_k U_m(r) dr \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \int_0^{s/2} \| e^{(s-r)\Delta} P_k U_m(r) \|_{F_k(T) \cap S_k^{1/2}(T)} dr + \]
\[ \int_{s/2}^s \| e^{(s-r)\Delta} P_k U_m(r) \|_{F_k(T) \cap S_k^{1/2}(T)} dr \]
\[ \lesssim \sum_{j \leq k_0} 2^{-20(k+k_0)} 2^{2j} \delta_{k,j} + 2^{k_0} \delta_{k,k_0} \]
\[ \lesssim 2^{-20(k_0+k)} \sum_{j \leq k_0} 2^{2j} \delta_{k,j} + 2^{-2k} \delta_{k,k_0}. \]

By Lemma 7.10 and the fact that \( k + k_0 > 0 \), it holds that
\[ 2^{-2k} \delta_{k,k_0} \lesssim \varepsilon (2^{k_0+k})^{-8} 2^{-\sigma k} a_k(\sigma) \]
and
\[ 2^{-20(k_0+k)} \sum_{j \leq k_0} 2^{2j} \delta_{k,j} \]
\[ \lesssim 2^{-20(k_0+k)} \sum_{j \leq k_0} \varepsilon (2^{j+k})^{-8} 2^{-\sigma k} 2^{k/2} (a_k(\sigma) + 2^{j/2} 2^{-3k/2} a_{j-1}(\sigma)) \]
\[ \lesssim \varepsilon 2^{-\sigma k} a_k(\sigma) 2^{-20(k_0+k)} \sum_{j \leq k_0} (2^{j+k})^{-8} (2^{2j+k} + 2^{j+k/2} 2^{j+k}) \]
\[ \lesssim \varepsilon (2^{k_0+k})^{-8} 2^{-\sigma k} a_k(\sigma), \]
which, combined with (7.88), completes the proof of the lemma. \qed
Lemma 7.12. The bound \((7.12)\) holds:
\[
\| P_k \psi_m(s) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim (1 + s2^{2k})^{-4} 2^{-\sigma k} b_k(\sigma).
\]

Proof. In view of \((7.11)\), we have
\[
P_k \psi_m(s) = e^{s \Delta} P_k \psi_m(0) + \int_0^s e^{(s-r) \Delta} P_k U_m(r) \, dr.
\]
Then it follows from Lemma \((7.11)\) that
\[
\| P_k \psi_m(s) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim 2^{-\sigma k}(1 + s2^{2k})^{-4}(b_k(\sigma) + \varepsilon a_k(\sigma)), \quad 0 \leq \sigma \leq \sigma_1 - 1.
\]
Therefore \(a_k(\sigma) \lesssim b_k(\sigma) + \varepsilon a_k(\sigma)\) and hence
\[
a_k(\sigma) \lesssim b_k(\sigma).
\]
\[\Box\]

7.2. Connection coefficient control. The main results of this subsection are the \(L^2_t L^2_x\) bounds \((4.14)\) and \((4.16)\), respectively proven in Corollary \((7.19)\) and Lemma \((7.21)\) and the frequency-localized \(L^2_t L^2_x\) bounds \((4.15)\) and \((4.17)\), respectively proven in Corollaries \((7.20)\) and \((7.22)\).

Lemma 7.13. Let \(s \in [2^{2j-2}, 2^{2j+2}]\). Then it holds that
\[
\| P_k(A_t(s) \psi_m(s)) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \varepsilon (1 + s2^{2k})^{-3}(s2^{2k})^{-3/8} 2^{k/2} 2^{-\sigma k} b_k(\sigma).
\]

Proof. Using \((7.80)\) and \((2.29)\), we have
\[
2^k \| P_k(A_t(s) \psi_m(s)) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \varepsilon 2^{2k} 2^{-\sigma k} 2^{-(1/2+\delta)(k+j)} a_k(\sigma).
\]
Combining \((7.90)\), \((7.87)\), and \((7.89)\) then yields
\[
\| P_k(A_t(s) \psi_m(s)) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \begin{cases} 
\varepsilon (s2^{2k})^{-3/8} 2^{k/2} 2^{-\sigma k} b_k(\sigma) & \text{if } k + j \leq 0 \\
\varepsilon (1 + s2^{2k})^{-3/8} 2^{k/2} 2^{-\sigma k} b_k(\sigma) & \text{if } k + j \geq 0,
\end{cases}
\]
which proves the lemma. \(\Box\)

Lemma 7.14. Assume that \(T \in (0, 2^K]\), \(f, g \in H^{\infty, \infty}(T)\), \(P_k f \in S^\omega_k(T)\), and \(P_k g \in L^4_{t,x}\) for some \(\omega \in \{0, 1/2\}\) and all \(k \in \mathbb{Z}\). Set
\[
\mu_k := \sum_{|j-k| \leq 20} \| P_j f \|_{S^\omega_j(T)}, \quad \nu_k := \sum_{|j-k| \leq 20} \| P_j g \|_{L^4_{t,x}}.
\]
Then, for any \(k \in \mathbb{Z}\),
\[
\| P_k(fg) \|_{L^4_{t,x}} \lesssim \sum_{j \leq k} 2^j \mu_j \nu_k + \sum_{j \leq k} 2^{(k+j)/2} \mu_k \nu_j + 2^k \sum_{j \geq k} 2^{-\omega(j-k)} \mu_j \nu_j.
\]
\(\Box\)
Proof. For the proof, see [4, §5]. □

**Lemma 7.15.** It holds that

\[ \|P_k \psi_s(0)\|_{L_t^4} + \|P_k \psi_t(0)\|_{L_t^4} \lesssim 2^k \tilde{b}_k (1 + \sum_j b_j^2). \]

**Proof.** We only treat \( \psi_t(0) \) since \( \psi_s(0) \) and \( \psi_t(0) \) differ only by a factor of \( i \). As \( \psi_t(0) = iD_\ell(0)\psi_\ell(0) \), we have

\[ \psi_t(0) = i\partial_\ell \psi_\ell(0) - A_\ell(0) \psi_\ell(0). \]

Clearly

\[ \|P_k \partial_\ell \psi_\ell(0)\|_{L_t^4} \lesssim 2^k \|P_k \psi_\ell(0)\|_{L_t^4} \lesssim 2^k \tilde{b}_k. \]

For the remaining term, we apply Lemma 7.14, bounding \( P_j A_\ell(0) \) in \( S_j^{1/2} \) by \( \sum_p b_p^2 \), which follows from Lemma 7.7. We get

\[ \|P_k (A_\ell(0) \psi_\ell(0))\|_{L_t^4} \lesssim \sum_{j \leq k} 2^j (\sum_p b_p^2) \tilde{b}_k + \sum_{j \leq k} 2^{(k+j)/2} (\sum_p b_p^2) \tilde{b}_j + 2^k \sum_{j \geq k} 2^{-(j-k)/2} \sum_p b_p^2 \tilde{b}_j. \]

Therefore

\[ \|P_k (A_\ell \psi_\ell(0))\|_{L_t^4} \lesssim 2^k \tilde{b}_k (\sum_j b_j^2). \]

□

**Corollary 7.16.** It holds that

\[ \|P_k \psi_s(0)\|_{L_t^4} + \|P_k \psi_t(0)\|_{L_t^4} \lesssim 2^k 2^{-\sigma k} b_k(\sigma) (1 + \sum_j b_j^2). \]

**Proof.** Without loss of generality, we prove the bound only for \( \psi_t \). We have

\[ \|P_k \partial_\ell \psi_\ell(0)\|_{L_t^4} \lesssim 2^k \|P_k \psi_\ell(0)\|_{L_t^4} \lesssim 2^k 2^{-\sigma k} b_k(\sigma). \]

It remains to control \( P_k (A_\ell(0) \psi_\ell(0)) \) in \( L_t^4 \). The obstruction to applying Lemma 7.14 as we did in Lemma 7.15 is the high-low interaction, for which summation can be achieved only for small \( \sigma \). If we restrict the range of \( \sigma \) to \( \sigma < 1/2 - 2\delta \), then we ensure the constant remains bounded and can apply Lemma 7.14 as in Lemma 7.15.

For \( \sigma \geq 1/2 - 2\delta \), we can still apply the bounds of Lemma 7.14 to the low-high and high-high interactions. For the remaining high-low interaction, we
bound $A_{\ell}(0)$ in $L_{t,x}^1$ and $\psi_\ell(0)$ in $L_{t,x}^\infty$. In particular, we have, thanks to (7.95) and Bernstein, that

$$
\sum_{\left| j_1-k \right| \leq 4 \atop j_2 \leq k+4} \| P_k(P_{j_1}A_{\ell}(0)P_{j_2}\psi_\ell(0)) \|_{L_{t,x}^1} \lesssim \sum_{\left| j_1-k \right| \leq 4 \atop j_2 \leq k+4} \| P_{j_1}A_{\ell}(0) \|_{L_{t,x}^1} \| P_{j_2}\psi_\ell(0) \|_{L_{t,x}^\infty}
$$

$$
\lesssim \sum_{\left| j_1-k \right| \leq 4 \atop j_2 \leq k+4} 2^{-\sigma j_1} b_{j_1} b_{j_1}(\sigma) 2^{j_2} \| P_{j_2}\psi_\ell(0) \|_{L_{t,x}^\infty}
$$

$$
\lesssim \sum_{j_2 \leq k+4} 2^{-\sigma k} b_k(\sigma) 2^{j_2} b_{j_2}
$$

$$
\lesssim 2^{-\sigma k} b_k^2(\sigma) \sum_{j_2 \leq k+4} 2^{k(2^{j_2-k}+(k-j_2)\delta)}
$$

$$
\lesssim 2^{-\sigma k} b_k^2(\sigma).
$$

\[ \text{Lemma 7.17.} \text{ It holds that} \]

$$
\| P_k\psi_a(s) \|_{L_{t,x}^1} + \| P_k\psi_\ell(s) \|_{L_{t,x}^1} \lesssim (1+s2^{2k})^{-2} 2^k b_k(1+\sum_j b_j^2).
$$

\[ \text{Proof.} \text{ We treat only } \psi_\ell(s) \text{ since the proof for } \psi_a(s) \text{ is analogous. From (7.11) we have}
$$

$$
\psi_\ell(s) = e^{s\Delta} \psi_\ell(0) + \int_0^s e^{(s-r)\Delta} U_t(r) \, dr.
$$

We claim that

$$
\| \int_0^s e^{(s-r)\Delta} P_k U_t(r) \, dr \|_{L_{t,x}^1} \lesssim \varepsilon (1+s2^{2k})^{-2} 2^k b_k(1+\sum_j b_j^2),
$$

(7.91)

which combined with Lemma 7.15 and a standard iteration argument proves the lemma.

As in the proof of Lemma 7.11 we take

$$
F \in \{ A_{\ell}^2, \partial_t A_{\ell}, f, g : \ell = 1, 2; f, g \in \{ \psi_m, \overline{\psi}_m : m = 1, 2 \} \}.
$$

By (7.60) and (7.89) we have

$$
\| P_k F(r) \|_{S_{k/2}^{1/2}(T)} \lesssim \varepsilon^{1/2} (1+s2^{2k})^{-2} (s2^{2k})^{-5/8} b_k.
$$

(7.92)

Moreover, by Lemma 7.14

$$
\| P_k A_{\ell}(r) \|_{S_{k/2}^{1/2}(T)} \lesssim \varepsilon^{1/2} (1+s2^{2k})^{-3} (s2^{2k})^{-1/8} b_k.
$$

(7.93)

Applying Lemma 7.14 with $\omega = 1/2$ yields

$$
\| P_k(F(r)\psi_\ell(r)) \|_{L_{t,x}^1} + 2^k \| P_k(A_{\ell}(r)\psi_\ell(r)) \|_{L_{t,x}^1} \lesssim \varepsilon (1+s2^{2k})^{-2} (s2^{2k})^{-7/8} 2^k b_k(1+\sum_j b_j^2).
$$

(7.94)
Integrating with respect to $s$ yields
\[
\int_0^s (1 + (s - r)2^{2k})^{-N} (1 + r2^{2k})^{-2} (r2^{2k})^{-7/8} dr \lesssim 2^{-2k} (1 + s2^{2k})^{-2},
\]
which, together with \eqref{7.94}, implies \eqref{7.91}.

\section*{Lemma 7.18. It holds that}
\[
\|P_k A_m(0)\|_{L^4_t, x} \lesssim 2^{-\sigma k} b_k(\sigma). \quad (7.95)
\]

\begin{proof}
We have
\[
\|P_k \psi_m(s)\|_{S^0_k} \lesssim (1 + s2^{2k})^{-4} 2^{-\sigma k} b_k(\sigma)
\]
and
\[
\|P_k (D_t \psi_t)(s)\|_{L^4_t, x} \lesssim (1 + s2^{2k})^{-3} (s2^{2k})^{-3/8} 2^{2k} - \sigma k b_k(\sigma).
\]
Applying Lemma \ref{lemma:7.14} with $\omega = 0$, we get
\[
\|P_k A_m(0)\|_{L^4_t, x} \lesssim \sum_{\ell = 1, 2} \int_0^\infty \|P_k (\psi_m(s) D_t \psi_t(s))\|_{L^4_t, x} ds
\]
\[
\lesssim 2^{-\sigma k} \sum_{j \leq k} b_j b_k(\sigma) 2^{j+k} \int_0^\infty (1 + s2^{2k})^{-3} (s2^{2k})^{-3/8} ds
\]
\[
+ 2^{-\sigma k} \sum_{j \leq k} b_k(\sigma) b_j 2^{(k+j)/2} 2^j \int_0^\infty (1 + s2^{2k})^{-4} (s2^{2j})^{-3/8} ds
\]
\[
+ \sum_{j \geq k} 2^{-\sigma j} b_j(\sigma) b_j 2^{k-j} 2^j \int_0^\infty (1 + s2^{2j})^{-7} (s2^{2j})^{-3/8} ds.
\]
Call the integrals $I_1$, $I_2$, and $I_3$, respectively. Clearly $I_1 \lesssim 2^{-2k}$ and $I_3 \lesssim 2^{-2j}$. By Cauchy-Schwarz, $I_2$ satisfies
\[
I_2 \lesssim \left( \int_0^\infty (1 + s2^{2k})^{-8} (1 + s2^{2j})^4 ds \right)^{1/2} \left( \int_0^\infty (1 + s2^{2j})^{-4} (s2^{2j})^{-3/8} ds \right)^{1/2}
\]
\[
\lesssim 2^{-j-k}.
\]
Therefore
\[
\|P_k A_m(0)\|_{L^4_t, x} \lesssim 2^{-\sigma k} b_k(\sigma) \sum_{j \leq k} (b_j 2^{j-k} + b_j 2^{(j-k)/2}) + 2^{-\sigma k} \sum_{j \geq k} b_j(\sigma) b_j 2^{k-j}
\]
\[
\lesssim 2^{-\sigma k} b_k b_k(\sigma).
\]
\end{proof}

\section*{Corollary 7.19. It holds that}
\[
\|A^2_x(0)\|_{L^2_t, x} \lesssim \sup_{j \in \mathbb{Z}} b_j^2 \sum_{k \in \mathbb{Z}} b_k^2.
\]
Proof. We have
\[
\|A_t^2(0)\|_{L_{t,x}^2} \lesssim \|A_x(0)\|_{L_{t,x}^4}^2 \\
\lesssim \sum_{k \in \mathbb{Z}} \|P_k A_x(0)\|_{L_{t,x}^4}^2 \\
\lesssim \sup_{j \in \mathbb{Z}} b_j^2 \cdot \sum_{k \in \mathbb{Z}} b_k^2.
\]

\[\square\]

Corollary 7.20. Let \(\sigma \geq 2\delta\). Then it holds that
\[
\|P_k A_x^2(0)\|_{L_{t,x}^2} \lesssim 2^{-\sigma k} b_k(\sigma) \cdot \sup_{j} b_j \cdot \sum_{k \in \mathbb{Z}} b_k^2.
\]

Proof. We perform a Littlewood-Paley decomposition and invoke (7.19).

Consider first the high-low interactions:
\[
\sum_{|j_2 - k| \leq 4 \atop j_1 \leq k - 5} \|P_k (P_{j_1} A_x P_{j_2} A_x)\|_{L^2} \lesssim \sum_{|j_2 - k| \leq 4 \atop j_1 \leq k - 5} \|P_{j_1} A_x\|_{L^4} \|P_{j_2} A_x\|_{L^4} \\
\lesssim 2^{-\sigma k} b_k(\sigma) \sum_{j_1 \leq k - 5} b_{j_1}^2.
\]

Next consider the high-high interactions:
\[
\sum_{j_1, j_2 \geq k - 4 \atop |j_1 - j_2| \leq 8} \|P_k (P_{j_1} A_x P_{j_2} A_x)\|_{L^2} \lesssim \sum_{j_1, j_2 \geq k - 4 \atop |j_1 - j_2| \leq 8} \|P_{j_1} A_x\|_{L^4} \|P_{j_2} A_x\|_{L^4} \\
\lesssim \sum_{j \geq k - 4} 2^{-\sigma j} b_j(\sigma) b_j^3.
\]

Using the frequency envelope property, we bound this last sum by
\[
\sum_{j \geq k - 4} 2^{-\sigma j} b_j(\sigma) b_j^3 \lesssim 2^{-\sigma k} b_k(\sigma) \sum_{j \geq k - 4} 2^{-\sigma (j-k)} 2^{\delta (j-k)} b_j^3 \\
\lesssim 2^{-\sigma k} b_k(\sigma) \sup_{j \geq k - 4} b_j \cdot \sum_{j \geq k - 4} b_j^2.
\]

It is in controlling this last sum that we use \(\sigma > \delta^+\). \[\square\]

Lemma 7.21. It holds that
\[
\|A_t(0)\|_{L_{t,x}^2} \lesssim (1 + \sum_j b_j^2)^2 \sum_k \|P_k \psi_x(0)\|_{L_{t,x}^4}^2.
\]
Proof. We begin with
\[ \|A_t(0)\|_{L_{t,x}^2} \lesssim \int_0^\infty \| (\overline{\psi_t} \cdot D \ell \psi_\ell) (s) \|_{L_{t,x}^2} \, ds. \]  
(7.96)

If we define
\[ \mu_k(s) := \sup_{k' \in \mathbb{Z}} 2^{-\delta |k-k'|} \| P_k \psi_t(s) \|_{L_{t,x}^4} \]  
and
\[ \nu_k(s) := \sup_{k' \in \mathbb{Z}} 2^{-\delta |k-k'|} \| P_k (D \ell \psi_\ell)(s) \|_{L_{t,x}^4}, \]  
then
\[ \| (\overline{\psi_t} \cdot D \ell \psi_\ell)(s) \|_{L_{t,x}^2} \lesssim \sum_k \mu_k(s) \sum_{j \leq k} \nu_j(s) + \sum_k \nu_k(s) \sum_{j \leq k} \mu_j(s). \]  
(7.98)

From Lemmas 7.15, 7.12, and 7.13, it follows that
\[ \mu_k(s), \nu_k(s) \lesssim (1 + s 2^{k})^{-2} 2^{\tilde{k}_b k} (1 + \sum_p b_p^2). \]  
(7.99)

Combining (7.96), (7.98), and (7.99), we have
\[ \|A_t(0)\|_{L_{t,x}^2} \lesssim \sum_k \mu_k(s) \sum_{j \leq k} \nu_j(s) \]
\[ \lesssim (1 + \sum_p b_p^2)^2 \sum_k 2^{\tilde{k}_b k} \sum_{j \leq k} 2^{j \tilde{b}_j} \int_0^\infty (1 + s 2^{2j})^{-2} (1 + s 2^{2k})^{-2} \, ds \]
\[ \lesssim (1 + \sum_p b_p^2)^2 \sum_k 2^{\tilde{k}_b k} \sum_{j \leq k} 2^{j \tilde{b}_j} \int_0^\infty (1 + s 2^{2k})^{-2} \, ds \]
\[ \lesssim (1 + \sum_p b_p^2)^2 \sum_k 2^{2\tilde{k}_b k} \int_0^\infty (1 + s 2^{2k})^{-2} \, ds \]
\[ \lesssim (1 + \sum_p b_p^2)^2 \sum_k \tilde{b}_k^2. \]
\[ \square \]

As a corollary of the proof, we also obtain

**Corollary 7.22.** Let \( \sigma \geq 2\delta \). It holds that
\[ \| P_k A_t \|_{L_{t,x}^2} \lesssim (1 + \sum_p b_p^2) \tilde{b}_k 2^{-\sigma k} b_k(\sigma). \]
Proof. We start by modifying the proof of Lemma 7.21 taking $\mu_k$ and $\nu_k$ as in (7.97). Then

$$
\|P_k A_t\|_{L^2} \lesssim \int_0^\infty \|P_k(\overline{\psi}_t \cdot D_t \psi_t)(s)\|_{L^2_{t,x}} ds
$$

$$
\lesssim \int_0^\infty \left( \mu_k(s) \sum_{j \leq k} \nu_j(s) + \nu_k \sum_{j \leq k} \mu_j(s) + \sum_{j \geq k} \mu_j(s) \nu_j(s) \right) ds.
$$

Combining Lemmas 7.12 and 7.13 gives a bound on $\nu_k$ of

$$
\|\nu_k(s)\|_{L^4} \lesssim (1 + s^{2k})^{-3/8} k^{-2\sigma_k b_k(\sigma),} \quad (7.100)
$$

which leads to

$$
\int_0^\infty \nu_k \sum_{j \leq k} \mu_j(s) ds \lesssim (1 + \sum_p b_p^2 b_k 2^{-\sigma_k b_k(\sigma)}).
$$

Also, by using (7.99) for $\mu_k$ and (7.100) for $\nu_k$ yields

$$
\int_0^\infty \sum_{j \geq k} \mu_j(s) \nu_j(s) ds \lesssim (1 + \sum_p b_p^2) \sum_{j \geq k} 2^{2j - 2\sigma_j b_j} \tilde{b}_j \int_0^\infty (1 + s^{2j})^{-3/8} k^{-2\sigma_k b_k(\sigma)} ds
$$

$$
\lesssim (1 + \sum_p b_p^2) \sum_{j \geq k} 2^{-\sigma_j b_j(\sigma)} \tilde{b}_j
$$

$$
\lesssim (1 + \sum_p b_p^2) 2^{-\sigma_k b_k(\sigma)} \sum_{j \geq k} 2^{(\delta - \sigma)(j-k)} \tilde{b}_j
$$

$$
\lesssim (1 + \sum_p b_p^2) 2^{-\sigma_k \tilde{b}_k b_k(\sigma)}.
$$

Here we have used $\sigma \geq 2\delta$. It remains to consider

$$
\int_0^\infty \mu_k(s) \sum_{j \leq k} \nu_j(s) ds.
$$

Suppose that

$$
\mu_k(s) \lesssim (1 + s^{2k})^{-2} k^{-2\sigma_k b_k(\sigma)} (1 + \sum_p b_p^2). \quad (7.101)
$$
Then
\[ \int_0^\infty \mu_k(s) \sum_{j \leq k} \nu_j(s) ds \]
\[ \lesssim (1 + \sum_p b_p^2 2^{-\sigma k} b_k(\sigma) 2^k) \sum_{j \leq k} \int_0^\infty (1 + s 2^{2j})^{-2} (1 + s 2^{2j})^{-2} 2^j \tilde{b}_j ds \]
\[ \lesssim (1 + \sum_p b_p^2 2^{-\sigma k} b_k(\sigma) 2^k) \sum_{j \leq k} 2^j \tilde{b}_j \int_0^\infty (1 + s 2^{2j})^{-2} ds \]
\[ \lesssim (1 + \sum_p b_p^2 2^{-\sigma k} b_k(\sigma) 2^k) \cdot 2^{-2k} \]
\[ = (1 + \sum_p b_p^2 2^{-\sigma k} b_k(\sigma) \tilde{b}_k). \]

Hence it remains to establish (7.101).

By Corollary 7.16, (7.101) holds when \( s = 0 \). To extend this estimate to \( s > 0 \), we proceed as in the proof of Lemma 7.17, replacing bounds (7.92) and (7.93) with their \( \sigma > 0 \) analogues as needed; that these analogues hold follows from the bounds referenced in establishing (7.92) and (7.93). To obtain the analogue of (7.94), we apply Lemma 7.14, choosing to use \( \sigma > 0 \) bounds only over the high frequency ranges.

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University of California, Berkeley

*E-mail address*: smith@math.berkeley.edu