A Frobenius homomorphism for Lusztig’s quantum groups over arbitrary roots of unity

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Abstract. For a finite dimensional semisimple Lie algebra and a root of unity, Lusztig defined an infinite dimensional quantum groups of divided powers. Provided the root of unity has order not divisible by 2 (and 3 for $G_2$), he constructed a Frobenius homomorphism with image the enveloping algebra.

In this article we define and study a Frobenius homomorphism for arbitrary roots of unity; we largely restrict ourselves to the Borel part. We explicitly calculate kernel and image of the new Frobenius homomorphism in rank 2. In general, it switches short and long roots and hence maps to the enveloping algebra with the dual root system.

We expect this pattern to hold in higher rank as well.

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1. Introduction

Fix a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ and a primitive $\ell$-th root of unity $q$. For this data, Lusztig defined in 1989 an infinite-dimensional complex Hopf algebra $U_q^\mathbb{C}(\mathfrak{g})$ called “restricted specialization” [Lusz90a, Lusz90b]. He conjectured that for $\ell$ prime the representation theory of $U_q^\mathbb{C}(\mathfrak{g})$ is deeply connected to the one of the respective adjoint Lie group over $\overline{\mathbb{F}}_\ell$ and of the respective affine Lie algebra. The former was proven in 1994 by Andersen, Jantzen & Soergel [AJS94] for large $\ell$.

For $\ell$ odd (and in case $\mathfrak{g} = G_2$ not divisible by 3) he obtained a remarkable Hopf algebra homomorphism to the classical universal enveloping algebra $U(\mathfrak{g})$, which was for $\ell$ prime related to the Frobenius homomorphism over the finite field $\mathbb{F}_\ell$. The Hopf algebra kernel (more precisely the coinvariants) of this map turned out to be a finite-dimensional Hopf algebra, called the small quantum group or Frobenius-Lusztig-kernel $u_q(\mathfrak{g})$. It’s representation theory is connected to the one of the respective Lie group over $\mathbb{F}_p$.

The discovery of this finite-dimensional Hopf algebra triggered among others the development of the theory of finite-dimensional pointed Hopf algebras that culminated in the classification of pointed Hopf algebras by Andruskiewitsch & Schneider [AS10] (for small prime divisors see [AnI11]) and the more general classification of possible quantum Borel parts, so-called Nichols algebras [Heck09].

The aim of this article is to consider a more general short exact sequence of Hopf algebras without restrictions on the root of unity and study it in detail for Lie algebras of rank 2. The cases with $2, 3 \mid \ell$ exhibit new phenomena, most prominently the new Frobenius homomorphism maps to the Lie algebra enveloping with dual root system $\mathfrak{g}^\vee$ instead of the initial $\mathfrak{g}$. Moreover, we consider from the outset arbitrary rational form of the Lusztig quantum group $U_q^\mathbb{C}(\mathfrak{g}, \Lambda)$ instead of just the adjoint form.

This paper is organized as follows:

In Section 2 we fix the Lie-theoretic notation and prove some technical preliminaries.
In Section 3 we review the construction of the Lusztig quantum group $U_q^\mathbb{C}(\mathfrak{g})$ via rational and integral forms and some basic properties.
In Section 4 we slightly improve some results in [Lusz90a, Lusz90b] to account for arbitrary rational forms and arbitrary roots of unity.
In Section 5 we introduce the more general Frobenius homomorphism by reversing the order of Lusztig’s approach: We yields an indirect quotient and show by the theorem of Kostant-Cartier that it is a Lie algebra enveloping $U(g^{(ℓ)})$. The Lie algebra $g^{(ℓ)}$ can then be determined by explicit calculation (see below). More precisely, we consider the short exact sequence, that arrises from dividing out the normal hull $N^L$ of the coradical, which we prove to contain the small quantum group $u_q^L(g)$:

$$0 \to N^L \to U_q^L(g, \Lambda) \xrightarrow{Frob} U(g^{(ℓ)}) \to 0$$

Unfortunately there exist cases, where this approach yields the trivial result $N^L = U_q^L$; this effect disappears when we consider a second short exact sequence for the Borel part:

$$0 \to N^L, + \to U_q^L(g, \Lambda, +) \xrightarrow{Frob} U(g^{(ℓ)}, +) \to 0$$

This is a short exact sequence of Hopf algebras in the category of $\Lambda$-Yetter-Drinfel’d modules. We give Lie-theoretic criteria when the image $U(g^{(ℓ)}, +)$ is in fact an ordinary Lie algebra enveloping. The other “braided cases” are where the first exact sequence fails.

In Section 6 we explicitly calculate the terms of the short exact sequence for Borel algebras for all non-braided cases of rank 2, i.e. $ℓ \not\equiv 2 \pmod{4}$ for $g = A_2, G_2$. We find in each instance (and conjecture in general) that $N^{L, +} = u_q^L(g)^+$ is the Borel part of the small quantum group defined by Lusztig as subalgebra of $U_q^L(g)$. In general, however, neither the kernel $u_q^L(g)^+$ nor the image $U(g^{(ℓ)}, +)$ have to be the ones associated to the Lie algebra $g$. A complete summary of all non-braided cases is as follows:

| $\ell$ = ord($q$) | $u_q^L(g)^+$ | $U_q^L(g)^+$ | $Frob$ | $U(g^{(ℓ)}, +)$ |
|-------------------|-------------|-------------|--------|----------------|
| $= 1, 3 \mod 4$   | $\ell = 1$  | 0           | $A_2$  | $A_2$          | trivial          |
|                   | $\ell \not\equiv 1$ | $A_2$    | $A_2$  | $A_2$          | generic          |
| $= 0 \mod 4$      | $\ell = 1$  | 0           | $B_2$  | $B_2$          | trivial          |
|                   | $\ell \not\equiv 1$ | $B_2$    | $B_2$  | $B_2$          | generic          |
| $= 2 \mod 4$      | $\ell = 2$  | 0           | $B_2$  | $B_2$          | trivial          |
|                   | $\ell \not\equiv 2$ | $B_2$    | $B_2$  | $B_2$          | generic          |
| $= 0 \mod 4$      | $\ell = 4$  | $A_1 \times A_1$ | $B_2$ | $B_2^\vee$ | dual             |
|                   | $\ell \not\equiv 4$ | $B_2$    | $B_2$  | $B_2^\vee$ | dual             |
| $= 1, 5, 7, 11 \mod 12$ | $\ell = 1$  | 0           | $G_2$  | $G_2$          | trivial          |
|                   | $\ell \not\equiv 1$ | $G_2$    | $G_2$  | $G_2$          | generic          |
| $= 4, 8 \mod 12$  | $\ell = 4$  | $A_3$       | $G_2$  | $G_2$          | exotic           |
|                   | $\ell \not\equiv 4$ | $G_2$    | $G_2$  | $G_2$          |                 |
| $= 0, 3, 9 \mod 12$ | $\ell = 3$  | $A_2$       | $G_2$  | $G_2^\vee$ | dual             |
|                   | $\ell \not\equiv 3$ | $G_2$    | $G_2$  | $G_2^\vee$ | dual             |
We now discuss the cases in the previous table:

For $\ell = 1$ we recover the well-known trivial case $U_L^G(g)^+ \cong U(g)^+$. For $2 \mid \ell$ (and $3 \mid \ell$ for $g = G_2$) we recover Lusztig’s results. Several more cases look alike. In other cases, the Frobenius homomorphism shows curious new phenomena, due to mainly two reasons:

- For small $\ell$, the small quantum group $u_{L_q}(g)^+$ may be only generated by a subset of roots (either none for trivial cases or $A_1 \times A_1 \subset B_2$ or $A_2 \subset G_2$). In the exotic case $g = G_2, \ell = 4$ the small quantum group has more algebra generators than expected and is of type $A_3$ (both $G_2, A_3$ have 12 roots). See section 6.4.

- Due to different nilpotency degrees $\ell_i$ of the simple root vectors $E_i$, the image of the Frobenius homomorphism may have a different root system than $g$: We find that whenever $\ell_i \neq \ell_j$ the new Frobenius homomorphism interchanges short and long roots and the image has the dual root system. See for example Section 6.2.

We conjecture in [6.2] that this pattern holds for arbitrary rank. In Example [6.1] we sketch the case $g = B_n$ for $\ell = 4$, yielding $g^{(\ell)} = g^\vee = C_n$.

2. Preliminaries

2.1. Lie Theory. Let $g$ be a finite-dimensional, semisimple complex Lie algebra with simple roots $\alpha_i$ indexed by $i \in I$ and a set of positive roots $\Phi^+$. Denote the Killing form by $(,)$, normalized such that $(\alpha, \alpha) = 2$ for the short roots. The Cartan matrix is

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

Be warned that there are different conventions for the index order of $a$, here we use the convention usual in the theory of quantum groups.

It is custom to call $d_\alpha := (\alpha, \alpha)/2$ with $d_\alpha \in \{1, 2, 3\}$, especially $d_i := d_{\alpha_i}$, which only depends on the orbit of $\alpha$ under the Weyl group. In this notation $(\alpha_i, \alpha_j) = d_i a_{ij}$.

**Definition 2.1.** The root lattice $\Lambda_R = \Lambda_R(g)$ is the abelian group with rank $\text{rank}(\Lambda_R) = |I|$ and is generated by $K_{\alpha_i}$ for each simple root $\alpha_i$. We denote general elements in $\Lambda_R$ by $K_\alpha$ for elements $\alpha$ in the root lattice of $g$. The Killing form induces an integral pairing of abelian groups, turning $\Lambda_R$ into an integral lattice:

$$(_,_): \Lambda_R \times \Lambda_R \rightarrow \mathbb{Z}$$

$$(K_\alpha, K_\beta) := (\alpha, \beta)$$

**Definition 2.2.** The weight lattice $\Lambda_W = \Lambda_W(g)$ is the abelian group with rank $\text{rank}(\Lambda_W) = \text{rank}(g)$ generated by $K_\lambda$, for each fundamental dominant weight $\lambda_i$. We denote general elements in $\Lambda_W$ by $K_\lambda$ with $\lambda$ in the weight lattice of $g$. It is a standard fact of Lie theory (cf. [Hum72], Section 13.1) that the root lattice is contained in the weight lattice and we
shall in what follows tacitly identify $\Lambda_R \subset \Lambda_W$. Moreover it is known that the pairing on $\Lambda_R$ can be extended to a non-degenerate integral pairing:

$$(\_,\_): \Lambda_W \times \Lambda_R \to \mathbb{Z}$$

$$(K_\lambda, K_\beta) := (\lambda, \beta)$$

Note that for multiply-laced $g$ the group $\Lambda_W$ is no integral lattice.

Example 2.3.

- For $\Lambda = \Lambda_R$ a set of generators is the set of simple roots $K_\alpha$.
- For $\Lambda = \Lambda_W$ a set of generators is the set of fundamental weights $K_\lambda$.

For later use, we also define the following sublattice of the root lattice $\Lambda_R$:

**Definition 2.4.** The $\ell$-lattice $\Lambda^{(\ell)}_R \subset \Lambda_R$ for any positive integer $\ell$ is defined as follows

$$\Lambda^{(\ell)}_R := \langle K^{\ell_i}_\alpha, i \in I \rangle$$

where $\ell_i = \ell / \gcd(\ell, 2d_i)$ is the order of $q^{2d_i}$ for $q$ a primitive $\ell$-th root of unity. More generally we define for any root $\ell_\alpha = \ell / \gcd(\ell, 2d_\alpha)$, which only depends on the orbit of $\alpha$ under the Weyl group.

**Example 2.5.** In the case where $g$ is simply-laced (hence all $d_\alpha = 1$) we have

$$\ell_i = \begin{cases} \ell, & \ell \text{ odd} \\ \frac{\ell}{2}, & \ell \text{ even} \end{cases}$$

$$\Lambda^{(\ell)}_R = \begin{cases} \ell \cdot \Lambda_R, & \ell \text{ odd} \\ \frac{\ell}{2} \cdot \Lambda_R, & \ell \text{ even} \end{cases}$$

Frequently, later statements can be simplified if all $\ell_i = \ell$, which is equivalent to the “generic case” $2 \nmid \ell$ (and $3 \nmid \ell$ for $g = G_2$). Moreover for small $\ell$ the set of roots with $\ell_\alpha = 1$ will be important. For later use we prove

**Lemma 2.6.** For all $\alpha, \beta \in \Lambda^{(\ell)}_R$ we have

$$(\alpha, \alpha) \in \ell \mathbb{Z} \quad (\alpha, \beta) \in \frac{\ell}{2} \mathbb{Z}$$

Moreover we have $(\alpha, \beta) \in \ell \mathbb{Z}$ except in the following cases:

| $g$       | Exceptions                      |
|-----------|---------------------------------|
| $A_n, D_n, E_6, E_7, E_8, G_2$ | $\ell = 2 \mod 4$               |
| $B_n, n \geq 3$ | $\ell = 4 \mod 8$              |
| $C_n, n \geq 3$ | $\ell = 2 \mod 4$              |
| $F_4$      | $\ell = 2, 4, 6 \mod 8$        |

The exceptions will correspond to braided cases of the short exact sequence Thm. 5.8.
Proof. It is sufficient to check the condition $\ell | (\alpha, \beta)$ on the lattice basis $\ell_i \alpha_i, i \in I$. We check for each $i, j$ whether the quotient $X$ is an integer:

$$X := \frac{(\ell_i \alpha_i, \ell_j \alpha_j)}{\ell} = \frac{\ell_i \ell_j (\alpha_i, \alpha_j)}{\ell} = \frac{\ell \cdot (\alpha_i, \alpha_j)}{\gcd(\ell, 2d_i) \cdot \gcd(\ell, 2d_j)}$$

We start by checking the cases $i = j$ where we find indeed $X = \frac{\ell}{\gcd(\ell, 2d_i)} \cdot \frac{2d_i}{\gcd(\ell, 2d_i)} \in \mathbb{Z}$. To check the cases $i \neq j$ we can restrict ourselves to Lie algebras of rank 2, where we check the claim case by case:

- For type $A_1 \times A_1$ we have $(\alpha_i, \alpha_j) = 0$.
- For type $A_2$ we have $d_i = d_j = d$ and $(\alpha_i, \alpha_j) = -d$, hence

$$X = \frac{\ell \cdot (-d)}{\gcd(\ell, 2d) \cdot \gcd(\ell, 2d)}$$

If $2 \nmid \ell$ we have $\gcd(\ell, 2d) = \gcd(\ell, d)$ and hence $X = \frac{\ell}{\gcd(\ell, d)} \cdot \frac{-d}{\gcd(\ell, d)} \in \mathbb{Z}$. If $d \mid \ell$ but $2d \nmid \ell$ we have $\gcd(\ell, 2d) = d$ and hence $X = \frac{\ell}{d} \cdot \frac{-d}{d} \in \mathbb{Z}$. If $4d \mid \ell$ we have $X = \frac{\ell}{2d} \cdot \frac{-d}{2d} \in \mathbb{Z}$. If however $2d \mid \ell$ but $4d \nmid \ell$ we have $X = \frac{\ell}{2d} \cdot \frac{-d}{d} \in \mathbb{Z} + \frac{1}{2}$.
- For type $B_2$ we have $d_i = 1, d_j = 2$ and $(\alpha_i, \alpha_j) = -2$. Hence

$$X = \frac{\ell \cdot (-2)}{\gcd(\ell, 2) \cdot \gcd(\ell, 4)}$$

If $2 \nmid \ell$, then we have $X = -2 \ell$. If $2 \mid \ell$ but $4 \nmid \ell$, then we have $X = -\frac{\ell^2}{2}$. If $4 \mid \ell$, then we have $X = -\frac{\ell^2}{4}$. In all cases $X \in \mathbb{Z}$.
- For type $G_2$ we have $d_i = 1, d_j = 3$ and $(\alpha_i, \alpha_j) = -3$. Hence

$$X = \frac{\ell \cdot (-3)}{\gcd(\ell, 2) \cdot \gcd(\ell, 6)}$$

If $2 \nmid \ell$ or $4 \nmid \ell$ we have as for $A_2$ that $X \in \mathbb{Z}$, while for $2 \mid \ell, 4 \nmid \ell$ we have $X \in \mathbb{Z} + \frac{1}{2}$.

The assertion follows now from considering all pairs of simple roots $(i, j)$:

- For $g$ simply-laced, all $(i, j)$ are either $A_1 \times A_1$ or $A_2$ for short roots $d = 1$. The exceptional cases are hence $\ell = 2 \mod 4$.
- For $g = C_n$, all $(i, j)$ are either $A_1 \times A_1$ or $A_2$ for short roots $d = 1$ or $B_2 = C_2$. The exceptional cases are hence $\ell = 2 \mod 4$.
- For $g = B_n$, all $(i, j)$ are either $A_1 \times A_1$ or $A_2$ for long roots $d = 2$ or $B_2 = C_2$. The exceptional cases are hence $\ell = 4 \mod 8$.
- For $g = E_4$, all $(i, j)$ are either $A_1 \times A_1$ or $A_2$ for short roots $d = 1$ or long roots $d = 2$ or $B_2 = C_2$. The exceptional cases are hence $\ell = 2 \mod 4$ as well as $\ell = 4 \mod 8$.
- For $G_2$ we already calculated the exceptional cases to be $\ell = 2 \mod 4$.

□
2.2. Quantum binomial coefficient for even order. In the following we briefly dis-
cuss \( q \)-symbols for \( q \) an even order of unity and especially shows the factorization of
the quantum binomial coefficients. Its proof is entirely as in \[Lusz89\] Prop. 3.2a for odd
order. As already remarked there, we only have some additional signs to consider.

Definition 2.7. For \( q \in \mathbb{C}^\times \) and \( n \leq k \in \mathbb{N}_0 \) we define

\[
[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} \quad [n]_q! := [1]_q[2]_q \cdots [n]_q \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \frac{[n]_q}{[k]_q[n-k]_q} & 0 \leq k \leq n \\ 0 & \text{else} \end{cases} 
\]

Lemma 2.8. Let \( q \) be a primitive \( \ell \)-th root of unity and \( \ell_0 \) be the order of \( q^2 \),

hence \( \ell_0 = \ell, \frac{\ell}{2} \) for odd/even \( \ell \).

• \([n]_q = 0 \) iff \( \ell_0 > 1 \) and \( \ell_0 | n \).

• The quantum binomial coefficients have the following generating function: Let \( X \)

be an indeterminant, then

\[
\prod_{j=0}^{n-1} (1 + q^{2j} X) = \sum_{k=0}^{m} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(n-1)} X^k
\]

• For \( n = n' + n'' \ell_0 \) and \( k = k' + k'' \ell_0 \) with \( 0 \leq n', k' < \ell_0 \) we have

\[
\begin{bmatrix} n \\ k \end{bmatrix}_q = (-1)^{(\ell+1)(k'n''+k'n')} \begin{bmatrix} n' \\ k' \end{bmatrix}_q \begin{bmatrix} n'' \\ k'' \end{bmatrix}_q
\]

Especially \([n]_q = 0 \) iff \( k' > n' \).

Proof.

• By definition

\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{-n+1} \frac{(q^2)^n - 1}{q^2 - 1}
\]

This is zero iff \( q^2 \neq 1 \) and \( (q^2)^n = 1 \).

• This seems well known and follows by easily (and without assumptions on the

order) by induction from the formula

\[
\begin{bmatrix} n + 1 \\ k \end{bmatrix}_q = q^{-k} \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{-k+m+1} \begin{bmatrix} n \\ k - 1 \end{bmatrix}_q
\]

which in turn immediately follows from

\[
[n + 1]_q = q^{-k}[m - k + 1]_q + q^{-k+m+1}[k]_q
\]
Using $q^{2\ell_0} = 1$ we rearrange the generating function as follows:

$$n' \prod_{j=0}^{n-1} (1 + q^{2j} X) = \prod_{j=0}^{n'} (1 + q^{2(n''\ell_0+j)} X) \prod_{i=0}^{n''-1} \prod_{j=0}^{\ell_0} (1 + q^{2(i\ell_0+j)} X)$$

$$= \prod_{j=0}^{n'} (1 + q^{2j} X) \cdot \left( \prod_{j=0}^{\ell_0} (1 + q^{2j} X) \right)^{n''}$$

Using $\prod_{j=0}^{\ell_0} (1 + q^{2j} X) = 1 + X^{\ell_0}$

$$= \prod_{j=0}^{n'} (1 + q^{2j} X) \cdot \left( \sum_{k'=0}^{n''} \left( \begin{array}{c} n'' \\ k' \end{array} \right) X^{k\ell_0} \right)$$

We know that $\prod_{j=0}^{n-1} (1 + q^{2j} X) = \sum_{k=0}^{m} \left( \begin{array}{c} n \\ k \end{array} \right) q^{k(n-1)} X^k$ so comparing coefficients of $X^k = X^{k'+k''\ell_0}$ on both sides yields

$$\left( \begin{array}{c} n \\ k \end{array} \right) q^{k(n-1)} = \sum_{k'=0}^{n''} \left( \begin{array}{c} n'' \\ k' \end{array} \right) q^{k'(n'-1)} \left( \begin{array}{c} n'' \\ k'' \end{array} \right)$$

We discuss the factor $q^{k'(n'-1)-k(n-1)}$: Since $q$ has order $\ell$ and $q^2$ has order $\ell_0$, we have $q^{\ell_0} = -(-1)\ell$ and $q^{2\ell_0} = 1$. Hence

$$q^{k'(n'-1)-k(n-1)} = q^{-k''\ell_0(n'-1)-k'n'\ell_0-k''\ell_0n''\ell_0} = (-1)^{(\ell+1)(k'n''+k''n')}$$

\[\square\]

2.3. Extension of scalars by abelian groups. We introduce the following tool without referring to quantum groups. It will later allow us to quickly transport results about the adjoint rational form ($\Lambda = \Lambda_R$) in literature to arbitrary $\Lambda$.

Suppose $H$ a Hopf algebra over a commutative ring $k$ with group of grouplikes $G(H)$ and fix some subgroup the group of grouplikes $\Lambda_R \subset G(H)$. Let $\Lambda \triangleright \Lambda_R$ be a group containing $\Lambda_R$ normally and let $\rho : k[\Lambda] \otimes H \to H$ be an action, such that

- The action $\rho$ turns $H$ into a $k[\Lambda]$-module Hopf algebra.
- The action $\rho$ restricts on $k[\Lambda_R] \subset k[\Lambda]$ to the adjoint representation $\rho_R$ of the Hopf subalgebra $k[\Lambda_R] \subset H$.
- The action $\rho$ restricts on $k[\Lambda_R] \subset H$ to the adjoint representation $\rho_{\Lambda}$ of $k[\Lambda]$ on the Hopf subalgebra $k[\Lambda_R]$, given by conjugacy action of the group $\Lambda$ on the normal subgroup $\Lambda_R$. 
We have to show that the structures by:

structure of the tensor product and the multiplication action $\rho$

Proof. The smash-product of two Hopf algebras $H'$ is the vector spaces $k[\Lambda] \otimes H$ with respect to the inclusions $k[\Lambda_R] \subset k[\Lambda]$ and $k[\Lambda_R] \subset G(H) \subset H$

Especially the choice $\Lambda = \Lambda_R$ recovers $H_{\Lambda} := k[\Lambda] \otimes_{k[\Lambda_R]} H$. We have to show that the structures $1_{H_{\Lambda}'} : \mu_{H_{\Lambda}}, \Delta_{H_{\Lambda}}, \epsilon_{H_{\Lambda}}$ factorize over the surjection $\phi : H_{\Lambda} := k[\Lambda] \otimes_k H \to k[\Lambda] \otimes_{k[\Lambda_R]} H =: H_{\Lambda}$

The multiplication $\mu_{H_{\Lambda}'}$ factorizes as follows: For all $g,h,t \in \Lambda, x,y \in H$ we have $g,h,t$ by assumption commuting and hence

$$((\phi \circ \mu_{H_{\Lambda}'})((g \otimes x) \cdot (h \otimes y)) = ght \otimes_{k[\Lambda_R]} \rho(h^{-1} \otimes x)y$$

On the other hand we have

$$((\phi \circ \mu_{H_{\Lambda}'})((g \otimes x) \cdot (ht \otimes y)) = ght \otimes_{k[\Lambda_R]} \rho((ht)^{-1} \otimes x)y$$

The unit $1_{H_{\Lambda}'}$ maps to $\phi(1_{H_{\Lambda}'}) \in H_{\Lambda}$.
• The comultiplication $\Delta_{H'}$ factorizes as follows: For all $g, h \in \Lambda, t \in \Lambda_R, x, y \in H$ we have $t$ grouplike and hence

$$
(\phi \circ \Delta_{H'})(gt \otimes x) = \left( gt \otimes_{k[\Lambda_R]} x^{(1)} \right) \otimes \left( gt \otimes_{k[\Lambda_R]} x^{(2)} \right) = \left( g \otimes_{k[\Lambda_R]} tx^{(1)} \right) \otimes \left( gt \otimes_{k[\Lambda_R]} tx^{(2)} \right) = (\phi \circ \Delta_{H'})(g \otimes tx)
$$

• The counit $\epsilon_{H'}$ factorizes as follows: For all $g \in \Lambda, x \in H$ we have

$$
(\phi \circ \epsilon_{H'})(gt \otimes x) = \epsilon_{H'}(gt) \cdot \epsilon_{H'}(x) = \epsilon_{H'}(g) \cdot \epsilon_{H'}(tx) = (\phi \circ \epsilon_{H'})(gt \otimes x)
$$

3. Different Forms of Quantum Groups

We recall several Hopf algebras associated to $\mathfrak{g}$ over various commutative rings $k$.

**Remark 3.1.** The following notion is added for completeness and not used in the sequel: There is a so-called topological Hopf algebra $U^C[[q]](\mathfrak{g})$ over the ring of formal power series $k = \mathbb{C}[[q]]$. cf. [Chari95] 6.5.1. It was defined by Drinfel’d (1987) and Jimbo (1985).

**3.1. The rational forms.** We next define the rational form $U^Q(q)(\mathfrak{g})$. There are in fact several rational forms $U^Q(q)(\mathfrak{g}, \Lambda)$ associated to the $U^C[[q]](\mathfrak{g})$ that differ by a choice of a subgroup $\Lambda_R \subset \Lambda \subset \Lambda_W$ resp. a choice of a subgroup in the fundamental group $\pi_1 := \Lambda_W/\Lambda_R$. This corresponds to choosing a Lie group associated to the Lie algebra $\mathfrak{g}$; we call the two extreme cases $\Lambda = \Lambda_W$ the simply-connected form and $\Lambda = \Lambda_R$ the usual adjoint form (e.g. $SL_2$ vs. $PSL_2$), see e.g. [Chari95] Sec. 9.1.

**Definition 3.2.** For each abelian group $\Lambda$ with $\Lambda_R \subset \Lambda \subset \Lambda_W$ we define the rational form $U^Q(q)(\mathfrak{g}, \Lambda)$ over the ring of rational functions $k = \mathbb{Q}(q)$ as follows:

As algebra, let $U^Q(q)(\mathfrak{g}, \Lambda)$ be generated by the group ring $k[\Lambda]$ spanned by $K_\lambda, \lambda \in \Lambda$ and additional generators $E_{\alpha_i}, F_{\alpha_i}$ for each simple root $\alpha_i, i \in I$ with relations:

- $K_\lambda E_{\alpha_i} K_\lambda^{-1} = q^{(\lambda, \alpha_i)} E_{\alpha_i}, \forall \lambda \in \Lambda$ (group action)
- $K_\lambda F_{\alpha_i} K_\lambda^{-1} = \bar{q}^{(\lambda, \alpha_i)} F_{\alpha_i}, \forall \lambda \in \Lambda$ (group action)
- $[E_{\alpha_i}, F_{\alpha_i}] = \delta_{i,j} \cdot \frac{K_{\alpha_i} - K_{\alpha_i}^{-1}}{q_{\alpha_i} - \bar{q}_{\alpha_i}}$ (linking)
and two identical sets of Serre-relations for any $i \neq j \in I$

$$
\begin{align*}
\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \frac{1 - a_{ij}}{r} \right] q^{a_{ij}r} E_{\alpha_i}^{1-a_{ij}r} E_{\alpha_j} E_{\alpha_i} &= 0 \\
\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \frac{1 - a_{ij}}{r} \right] q^{a_{ij}r} F_{\alpha_i}^{1-a_{ij}r} F_{\alpha_j} F_{\alpha_i} &= 0
\end{align*}
$$

where $\bar{q} := q^{-1}$, the quantum binomial coefficients are defined in Definition 2.7 and by definition $q^{(\alpha_i, \alpha_j)} = (q^{a_{ij}})^{a_{ij}}$.

As a coalgebra, let the counit $\epsilon$ and the antipode $S$ be defined on the group-Hopf-algebra $k[\Lambda]$ as usual

$$
\Delta(K_\lambda) = K_\lambda \otimes K_\lambda \quad \epsilon(K_\lambda) = 1 \quad S(K_\lambda) = K_\lambda^{-1} = K_{-\lambda}
$$

and on the additional generators $E_{\alpha_i}, F_{\alpha_i}$ for each simple root $\alpha_i, i \in I$ as follows:

$$
\Delta(E_{\alpha_i}) = E_{\alpha_i} \otimes K_{\alpha_i} + 1 \otimes E_{\alpha_i} \quad \Delta(F_{\alpha_i}) = F_{\alpha_i} \otimes 1 + K_{\alpha_i}^{-1} \otimes F_{\alpha_i}
$$

$$
S(E_{\alpha_i}) = -E_{\alpha_i} K_{\alpha_i}^{-1} \quad S(F_{\alpha_i}) = -K_{\alpha_i} F_{\alpha_i}
$$

$$
\epsilon(E_{\alpha_i}) = 0 \quad \epsilon(F_{\alpha_i}) = 0
$$

**Theorem 3.3 (Rational Form).** $U_q^Q(\mathfrak{g}, \Lambda)$ is a Hopf algebra over the field $k = \mathbb{Q}(q)$. For arbitrary $\Lambda$ using the construction in Theorem 2.9 we have

$$
U_q^Q(\mathfrak{g}, \Lambda) = k[\Lambda] \ltimes k[\Lambda_R] U_q^Q(\mathfrak{g}, \Lambda_R)
$$

Moreover, we have a triangular decomposition: Consider the subalgebras $U_q^Q(\mathfrak{g}, \Lambda)_+ \quad \text{generated by the } E_{\alpha_i} \quad \text{and } U_q^Q(\mathfrak{g}, \Lambda)_- \quad \text{generated by the } F_{\alpha_i} \quad \text{and } U_q^Q(\mathfrak{g}, \Lambda)_0 = k[\Lambda] \quad \text{spanned by the } K_\lambda.$

Then multiplication in $U_q^Q(\mathfrak{g}, \Lambda)$ induces an isomorphism of vector spaces:

$$
U_q^Q(\mathfrak{g}, \Lambda)_+ \otimes U_q^Q(\mathfrak{g}, \Lambda)_0 \otimes U_q^Q(\mathfrak{g}, \Lambda)_- \cong U_q^Q(\mathfrak{g})
$$

**Proof.** The case of the adjoint form $\Lambda = \Lambda_R$ is classical, see e.g. [Jan03] II, H.2 & H.3. In principle, this and later proofs work totally analogous for arbitrary $\Lambda_R \subset \Lambda \subset \Lambda_W$, but to connect them directly to results in literature without repeating everything, we deduce the case of arbitrary $\Lambda$ from $\Lambda = \Lambda_R$ and the construction in Section 2.3.

Let $k = \mathbb{Q}(q)$, take $\Lambda_R \subset \Lambda \subset \Lambda_W$ an abelian group and let $H = U_q^Q(\mathfrak{g}) := U_q^Q(\mathfrak{g}, \Lambda_R)$ be the adjoint form with smallest $\Lambda = \Lambda_R$. Define an action $\rho$ of $k[\Lambda]$ on $H$ given by

$$
\rho(K_\lambda \otimes E_{\alpha_i}) = q^{\langle \lambda, \alpha_i \rangle} E_{\alpha_i}
$$

$$
\rho(K_\lambda \otimes F_{\alpha_i}) = \bar{q}^{\langle \lambda, \alpha_i \rangle} F_{\alpha_i}
$$
Then certainly the restriction of this action to $k[\Lambda_R] \subset k[\Lambda]$ is the adjoint action in Definition 3.2 for $H = U_q^\mathbb{Q}(g, \Lambda_R)$ and the restriction to $k[\Lambda_R] \subset H$ is trivial ($\Lambda$ is an abelian group). Hence we can apply extension of scalars by an abelian group in Theorem 2.9 and yield a Hopf algebra

$$H_\Lambda := k[\Lambda] \ltimes_k k[\Lambda_R] H$$

Denote the elements $K_\lambda \otimes k[\Lambda_R] 1$ by $K_\lambda$, especially for $\alpha \in \Lambda_R$ we have $K_\alpha = 1 \otimes k[\Lambda_R] K_\alpha$ with $K_\alpha \in U_q^\mathbb{Q}(g, \Lambda_R)$. Then it is clear that these elements fulfill the relations given in the previous Definition of $U_q^\mathbb{Q}(g, \Lambda)$ for general $\Lambda$. It follows from the triangular decomposition of $H$ that this is an isomorphism of vector spaces as

$$k[\Lambda] \otimes k[\Lambda_R] k[\Lambda_R] \cong k[\Lambda]$$

Especially, $U_q^\mathbb{Q}(g, \Lambda)$ defined above is a Hopf algebra with a triangular decomposition as vector spaces

$$U_q^\mathbb{Q}(q)^+, U_q^\mathbb{Q}(q)^0 \otimes U_q^\mathbb{Q}(q)^- \rightarrow U_q^\mathbb{Q}(q)$$

with $U_q^\mathbb{Q}(q)^0 \cong k[\Lambda]$ and $U_q^\mathbb{Q}(q)^\pm$ independent of the choice of $\Lambda$. □

A tool of utmost importance has been introduced by Lustzig, see [Jan03] H.4:

**Definition 3.4.** Fix a reduced expression $s_{i_1} \cdots s_{i_t}$ of the longest element in the Weyl group $W(g)$ in terms of reflections $s_i$ on simple roots $\alpha_i$.

1. There exist algebra automorphisms $T_i : U_q^\mathbb{Q}(g, \Lambda) \rightarrow U_q^\mathbb{Q}(g, \Lambda)$, such that the action restricted to $K_\lambda \in \Lambda \subset \Lambda_W$ is the reflection of the weight $\lambda$ on $\alpha_i$.
2. Every positive root $\beta$ has a unique expression $\beta = s_{i_1} \cdots s_{i_{k-1}} 2 \alpha_{i_k}$ for some index $k$. This defines a total ordering on the set of positive roots $\Phi^+$ and the reversed ordering on $\Phi^-$. Define the root vectors for a root $\beta \in \Phi^+$ by

$$E_\beta := T_{i_1} \cdots T_{k-1} E_{\alpha_k}$$

$$F_\beta := T_{i_1} \cdots T_{k-1} F_{\alpha_k}$$

For simple roots $\beta = \alpha_i$ this coincides with the generator $E_{\alpha_i}, F_{\alpha_i}$.

With these definitions, Lusztig establishes in a remarkable PBW-basis:

**Theorem 3.5 (PBW-basis).** Multiplication in $U_q^\mathbb{Q}(q)$ induces an isomorphism of $k$-vector spaces for the field $k = \mathbb{Q}(q)$

$$k[\Lambda] \bigotimes_{\alpha \in \Phi^+} k[E_\alpha] \bigotimes_{-\alpha \in \Phi^-} k[F_\alpha] \overset{\cong}{\rightarrow} U_q^\mathbb{Q}(q)(g, \Lambda)$$

where the orderings on $\Phi^+, \Phi^-$ are as above.
Proof. The adjoint case $\Lambda = \Lambda_R$ is classical and in [Jan03] H.4. Note that using the relations between $K_\alpha, E_\alpha$ all $K$’s can be sorted to the left side. The case of arbitrary $\Lambda$ could be derived totally analogously, but it also follows directly from the presentation as extension in Theorem 3.3. Namely, we have by construction isomorphisms of vector spaces

$$U_q^{\mathbb{Z}[q^{-1}],\mathcal{K}}(\mathfrak{g}, \Lambda) \cong k[\Lambda] \otimes_{k[\Lambda_R]} U_q^{\mathbb{Z}[q^{-1}],\mathcal{K}}(\mathfrak{g}, \Lambda_R)$$

$$\cong k[\Lambda] \otimes_{k[\Lambda_R]} k[\Lambda_R] \bigotimes_{\alpha \in \Phi^+} k[E_\alpha] \bigotimes_{\alpha \in \Phi^-} k[F_\alpha]$$

$$\cong k[\Lambda] \bigotimes_{\alpha \in \Phi^+} k[E_\alpha] \bigotimes_{\alpha \in \Phi^-} k[F_\alpha]$$

$\square$

3.2. Two integral forms. Next we define two distinct integral forms $U_q^{\mathbb{Z}[q^{-1}],\mathcal{K}}(\mathfrak{g}, \Lambda)$ and $U_q^{\mathbb{Z}[q^{-1}],\mathcal{L}}(\mathfrak{g}, \Lambda)$. These are $\mathbb{Z}[q, q^{-1}]$-subalgebras of $U_q^{\mathbb{Z}[q^{-1}],\mathcal{K}}(\mathfrak{g}, \Lambda)$, such that the extension of scalars $\otimes_{\mathbb{Z}[q^{-1}]}\mathbb{Q}(q)$ is an isomorphism of $\mathbb{Q}(q)$-algebras.

Definition 3.6. (cf. [Char95] Sec. 9.2 and 9.3) Recall $q_\alpha := q^{s_\alpha} = q^{(\alpha, \alpha)/2}$.

- The so-called unrestricted integral form $U_q^{\mathbb{Z}[q^{-1}],\mathcal{K}}(\mathfrak{g}, \Lambda)$ is generated as a $\mathbb{Z}[q, q^{-1}]$-algebra by $\Lambda$ and the following elements in $U_q^{\mathbb{Z}[q^{-1}],\mathcal{K}}(\mathfrak{g}, \Lambda)^+$:

  $$E_\alpha, F_\alpha, K_\alpha - K_\alpha^{-1} \quad \forall \alpha \in \Phi^+, i \in I$$

  We use the superscript $\mathcal{K}$ in honor of Victor Kac, who has defined and studied it in 1967 with Weisfeiler and the present form in 1990–1992 with De Concini and Procesi.

- The so-called restricted integral form $U_q^{\mathbb{Z}[q^{-1}],\mathcal{L}}(\mathfrak{g}, \Lambda)$ is generated as a $\mathbb{Z}[q, q^{-1}]$-algebra by $\Lambda$ and the following elements in $U_q^{\mathbb{Z}[q^{-1}],\mathcal{L}}(\mathfrak{g}, \Lambda)^\pm$ called divided powers

  $$E^{(r)}_\alpha := \frac{E^r_\alpha}{\prod_{s=1}^r q^{\alpha_s}_{\alpha_s} - q^{\alpha_s}_{\alpha_s}}, \quad F^{(r)}_\alpha := \frac{F^r_\alpha}{\prod_{s=1}^r q^{\alpha_s}_{\alpha_s} - q^{\alpha_s}_{\alpha_s}} \quad \forall \alpha \in \Phi^+, r > 0$$

  and by the following elements in $U_q^{\mathbb{Z}[q^{-1}],\mathcal{L}}(\mathfrak{g}, \Lambda)^0$:

  $$K^{(r)}_{\alpha_i} = \begin{bmatrix} K_{\alpha_i}; 0 \\ r \end{bmatrix} := \prod_{s=1}^r \frac{K_{\alpha_i}q^{1-s}_{\alpha_i} - K_{\alpha_i}^{-1}q^{s-1}_{\alpha_i}}{q^s_{\alpha_i} - q^{-s}_{\alpha_i}} \quad i \in I$$

  We use the superscript $\mathcal{L}$ in honor of Georg Lusztig, who has defined and studied it in 1988–1990.
Theorem 3.7. The Lusztig quantum group $U_q[\mathfrak{g}^{-1}]$ is Hopf algebra over the ring $k = \mathbb{Z}[q^{-1}]$ and is integral for $U_q[\mathfrak{g}]$. Hereby for arbitrary $\Lambda$ we have using the construction in Theorem 2.7

$$U_q[\mathfrak{g}^{-1}] \cong k[\Lambda] \otimes_k U_q[\mathfrak{g}]$$

A similar result holds for the Kac integral form, see Chari [95] Sec. 9.2. Generators and relations for simply-laced $\mathfrak{g}$ are discussed in [95] Thm. 9.3.4. The proof, that the $U_q[\mathfrak{g}^{-1}]$, $U_q[\mathfrak{g}]$ are integral forms for $U_q[\mathfrak{g}]$ follows immediately from the remarkable knowledge of a PBW-basis:

Theorem 3.8 (PBW-Basis). For the Lusztig integral form $U_q[\mathfrak{g}^{-1}]$ over the commutative integral domain $k = \mathbb{Z}[q^{-1}]$, multiplication induces an isomorphism of $k$-modules

$$k[\Lambda/2\Lambda] \otimes_{\mathbb{Z}[q,q^{-1}]} K^{(r)}_{\alpha_i} k \otimes_{\mathbb{Z}[q,q^{-1}]} E^{(r)}_{\alpha} k \otimes_{\mathbb{Z}[q,q^{-1}]} F^{(r)}_{\alpha} k \rightarrow U_q[\mathfrak{g}^{-1}] \cong k[\Lambda/2\Lambda] \otimes_{\mathbb{Z}[q,q^{-1}]} K^{(r)}_{\alpha_i} k$$

Especially, the Lusztig integral form is free as $k$-module. Note that the group algebra $k[\Lambda/2\Lambda]$ is not contained in $U_q[\mathfrak{g}^{-1}] \cong k[\Lambda/2\Lambda] \otimes_{\mathbb{Z}[q,q^{-1}]} K^{(r)}_{\alpha_i} k$. Note that the PBW-basis theorem does not follows from the PBW-basis of the rational form. Rather, the proof proceeds parallel and roughly uses that the $T_i$ preserve the chosen generator set of $U_q[\mathfrak{g}^{-1}]$.

The case of arbitrary $\Lambda$ could be derived totally analogously, it also follows directly from the presentation in Theorem 3.7 and is proven as in the proof of Theorem 3.5.

$$U_q[\mathfrak{g}^{-1}] \cong k[\Lambda] \otimes_{k[\Lambda/2\Lambda]} U_q[\mathfrak{g}^{-1}] \cong k[\Lambda/2\Lambda] \otimes_{\mathbb{Z}[q,q^{-1}]} K^{(r)}_{\alpha_i} k \otimes_{\mathbb{Z}[q,q^{-1}]} E^{(r)}_{\alpha} k \otimes_{\mathbb{Z}[q,q^{-1}]} F^{(r)}_{\alpha} k$$

3.3. Specialization to roots of unity. Next we define the restricted specialization $U_q^{\mathbb{C}}[\mathfrak{g}, \Lambda]$. It is a complex Hopf algebra depending on a specific choice $q \in \mathbb{C}^\times$. 

Definition 3.9. (cf. [Chari95] Sec. 9.2 and 9.3) The infinite-dimensional complex Hopf algebra $U_L^\mathcal{L}(g, \Lambda)$ is defined by

$$U_L^\mathcal{L}(g, \Lambda) := U_q^{\mathbb{Z}[q,q^{-1}].\mathcal{L}} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}_q$$

where (by slight abuse of notation) $\mathbb{C}_q = \mathbb{C}$ with the $\mathbb{Z}[q,q^{-1}]$-module structure defined by the specific value $q \in \mathbb{C}^\times$.

Note that we have a PBW-basis in Theorem 3.8, which especially shows $U_q^{\mathbb{Z}[q,q^{-1}].\mathcal{L}}$ is free as a $\mathbb{Z}[q,q^{-1}]$-module. Hence the specialization has an induced vector space basis, the impact of the specialization is to severely modify the algebra structure, such that e.g. former powers may become new algebra generators.

Corollary 3.10. For the Lusztig quantum group $U_L^\mathcal{L}$ over $\mathbb{C}$, multiplication induces an isomorphism of $\mathbb{C}$-vector spaces:

$$\mathbb{C}[\Lambda/2\Lambda_R] \bigotimes_{i \in I, r > 0} K^{(r)}_{\alpha_i} \mathbb{C} \bigotimes_{\alpha \in \Phi^+, r > 0} E_{\alpha}^{(r)} \mathbb{C} \bigotimes_{-\alpha \in \Phi^-, r > 0} F_{\alpha}^{(r)} \mathbb{C} \xrightarrow{\cong} U_L^\mathcal{L}(g, \Lambda)$$

This PBW-basis will we refined in Theorem 4.4.

Example 3.11. For $q = 1$ we have a cosmash-product

$$U_L^\mathcal{L}(g, \Lambda) \cong \mathbb{C}[\Lambda/2\Lambda] \ltimes U(g)$$

4. First Properties of the specialization

For the rest of the article we assume $q \in \mathbb{C}^\times$ a primitive $\ell$-th root of unity without restrictions on $\ell$. We study the infinite-dimensional Lusztig quantum group $U_L^\mathcal{L}(g, \Lambda)$ from Definition 3.9 which is a Hopf algebra over $\mathbb{C}$. It was defined as a specialization of the Lusztig integral form in Definition 3.6 and hence shares the explicit vector space basis given by Theorem 3.8.

4.1. The zero-part. The zero-part $u_L^\mathcal{L}(g, \Lambda)^0$ in the triangular decomposition uses different arguments as the quantum Borel parts. Recall from Corollary 3.10 that multiplication in $U_q^\mathcal{L}$ induces an isomorphism of vector spaces

$$\bigotimes_{i \in I} \mathbb{C}[K_{\alpha_i}]/(K_{\alpha_i}^{2\ell_i}) \bigotimes_{i \in I, r > 0} K^{(r)}_{\alpha_i} \mathbb{C} = \mathbb{C}[\Lambda/2\Lambda_R] \bigotimes_{i \in I, r > 0} K^{(r)}_{\alpha_i} \mathbb{C} \xrightarrow{\cong} U_L^\mathcal{L}(g, \Lambda)^0$$

We want to determine the algebra structure of $U_q^\mathcal{L,0}$. It is clear from the definition, that $U_q^\mathcal{L,0}$ is a commutative, cocommutative complex Hopf algebra. Note that by the theorem of Kostant-Cartier (see e.g. [Mont93] Sec. 5.6) this already implies it is of the form $\mathbb{C}[G] \otimes U(\mathfrak{h})$ with group of grouplikes $G = G(U_q^\mathcal{L,0})$ and $\mathfrak{h}$ an abelian Lie algebra.
Theorem 4.1. With $\ell_i = \text{ord}(q_{a_i}^2)$ as always we have an isomorphism of Hopf algebras

$$U_q^L(g, \Lambda)^0 \cong \mathbb{C}[\Lambda/2\Lambda(\ell)] \otimes U(h) \quad \mathfrak{h} = \bigoplus_{i \in I} K_{a_i}^{-\ell_i} K_{a_i}^{(\ell_i)} \mathbb{C}$$

$$\Delta(K_{\lambda}) = K_{\lambda} \otimes K_{\lambda} \quad \Delta(K_{a_i}^{-\ell_i} K_{a_i}^{(\ell_i)}) = 1 \otimes K_{a_i}^{-\ell_i} K_{a_i}^{(\ell_i)} + K_{a_i}^{-\ell_i} K_{a_i}^{(\ell_i)} \otimes 1$$

(the term $K_{a_i}^{-\ell_i} K_{a_i}^{(r)}$ is denoted $K_{i,r}$ in \cite{Lusz90a} Section 6)

Proof. The elements $K_{a_i}, K_{a_i}^{(\ell_i)}$ lay in $U_q^L(0)$. It is proven in \cite{Lusz90a} Thm. 8.3 (without restrictions on $\ell$) that in $U_q^L(g, \Lambda R)$ the elements $K_{a_i}$ generate the group $\Lambda R/(K_{a_i}^{(2)})$, $i \in I = \Lambda R/2\Lambda(\ell)$. The presentation in Theorem 3.7 easily extends this result to arbitrary $\Lambda$.

We show first that the coproduct of $K_{a_i}^{-\ell_i} K_{a_i}^{(\ell_i)}$ for each $i \in I$ is as prescribed. This follows from the following equation in $U_q^L(\mathbb{C}, A_{\alpha_i})$

$$K_{a_i}^{-\ell_i} K_{a_i}^{(\ell_i)} = \prod_{s=1}^{\ell_i} K_{a_i} q_{a_i}^{-s} - K_{a_i}^{-1} q_{a_i}^{-s} = \prod_{s=1}^{\ell_i} q_{a_i}^{-s} - q_{a_i}^{-s}$$

$$\Delta(K_{a_i}^{-\ell_i} K_{a_i}^{(\ell_i)}) = \frac{1}{q_{a_i}} \left( 1 \otimes 1 - K_{a_i}^{-2\ell_i} \otimes 1 + K_{a_i}^{-2\ell_i} \otimes 1 - K_{a_i}^{-2\ell_i} \otimes K_{a_i}^{-2\ell_i} \right)
$$

In the third line we used that by definition $\ell_i = \text{ord}(q_{a_i}^2)$. The asserted coproduct follows from the previously shown relation $K_{a_i}^{2\ell_i} = 1$ in the specialization $U_q^L$.

It hence remains to show, that without restrictions on $\ell$ these elements generate all of $U_q^L(0)$. Following Lusztig, we do so by reducing a generator $K_{a_i}^{(r)}$, $r > 0$ to these generators for any fixed $i \in I$. Note however that the factorization formula \cite{Lusz90a} Lemma 6.4 does no hold for arbitrary $\ell$ and a direct expression would be significantly more complicated!

We hence proof the claim $K_{a_i}^{(r)} \in \mathbb{k}[\Lambda]\mathbb{k}[\Lambda]$ by induction: For $r < \ell_i$ we have certainly that $K_{a_i}^{(r)} \in \mathbb{k}[\Lambda]$ and for $r = \ell_i$ we recover the generator $K_{a_i}^{(\ell_i)}$. Suppose inductively we have shown the claim for all $r' < r$. We have two cases:
• Let $\ell_i \nmid r$, then we have by definition and induction

$$K^{(r)}_{\alpha_i} = K^{(r-1)}_{\alpha_i} \frac{K_{\alpha_i}^{1-r} - K_{\alpha_i}^{-1} q_{\alpha_i}^r}{q_{\alpha_i}^r - q_{\alpha_i}^{-r}} \in k[\Lambda][k[K^{(\ell_i)}_{\alpha_i}]]$$

since the denominator $q_{\alpha_i}^r - q_{\alpha_i}^{-r} \neq 0$.

• Let $k\ell_i = r$, then we have in $U_q\mathbb{Z}[q^{-1}].\mathcal{L}$

$$[k]_{q_{\alpha_i}} K^{(\ell_i-1)}_{\alpha_i} . K^{(r)}_{\alpha_i} = \frac{q_{\alpha_i}^r - q_{\alpha_i}^{-r}}{q_{\alpha_i}^r - q_{\alpha_i}^{-r}} . K^{(r-1)}_{\alpha_i} K_{\alpha_i} q_{\alpha_i}^{1-k\ell_i} - K_{\alpha_i}^{-1} q_{\alpha_i}^{k\ell_i-1} = \frac{q_{\alpha_i}^r - q_{\alpha_i}^{-r}}{q_{\alpha_i}^r - q_{\alpha_i}^{-r}} . K^{(r-1)}_{\alpha_i} K_{\alpha_i} q_{\alpha_i}^{1-k\ell_i} - K_{\alpha_i}^{-1} q_{\alpha_i}^{k\ell_i-1}$$

$$= K^{(r-1)}_{\alpha_i} \left( q^{-(k-1)\ell_i} K_{\alpha_i} q_{\alpha_i}^{1-k\ell_i} - K_{\alpha_i}^{-1} q_{\alpha_i}^{-(k-1)\ell_i} \right) + q^{\ell_i-1} K_{\alpha_i}^{-1} q_{\alpha_i}^{-(k-1)\ell_i} - K_{\alpha_i}^{-1} q_{\alpha_i}^{k\ell_i-1}$$

$$= K^{(r-1)}_{\alpha_i} \left( q^{-(k-1)\ell_i} K_{\alpha_i} q_{\alpha_i}^{1-k\ell_i} - K_{\alpha_i}^{-1} q_{\alpha_i}^{-(k-1)\ell_i} \right) + q^{\ell_i-1} K_{\alpha_i}^{-1} q_{\alpha_i}^{-(k-1)\ell_i} - K_{\alpha_i}^{-1} q_{\alpha_i}^{k\ell_i-1}$$

$$= K^{(r-1)}_{\alpha_i} \left( q^{-(k-1)\ell_i} K_{\alpha_i} q_{\alpha_i}^{1-k\ell_i} - K_{\alpha_i}^{-1} q_{\alpha_i}^{-(k-1)\ell_i} \right) + q^{\ell_i-1} K_{\alpha_i}^{-1} q_{\alpha_i}^{-(k-1)\ell_i} - K_{\alpha_i}^{-1} q_{\alpha_i}^{k\ell_i-1}$$

In the specialization $U_q^\mathcal{L}$ we have $[k]_{q_{\alpha_i}} = \pm k$ invertible, moreover $K^{(\ell_i-1)}_{\alpha_i} \in [k[\Lambda]]$ invertible, hence $K^{(r)}_{\alpha_i} \in [k[\Lambda][k[K^{(\ell_i)}_{\alpha_i}]]$ as claimed.

□

4.2. The coradical. The following assertion is known under various restrictions and follows from a standard argument, see e.g. [Mont93] Lemma 5.5.5. We include it for completeness in the case of arbitrary $\ell$:

**Lemma 4.2.** The coradical of the infinite-dimensional Hopf algebra $U_q^\mathcal{L}(\mathfrak{g}, \Lambda)$ is $\mathbb{C}[\Lambda]$. Especially the Hopf algebra is pointed with group of grouplikes $\Lambda$.

**Proof.** Consider the (very coarse) coalgebra $\mathbb{N}$-grading induced by setting $\deg(E^{(r)}_{\alpha}) = \deg(F^{(r)}_{\alpha}) = r$ and $\deg(x) = 0$ for $x \in U_q^{\mathcal{L},0}(\mathfrak{g}, \Lambda)$. By [Sw69] Prop. 11.1.1 this already implies that the coradical is contained in $U_q^{\mathcal{L},0}(\mathfrak{g}, \Lambda)$. We have shown in Theorem 4.1 that

$$U_q^{\mathcal{L},0}(\mathfrak{g}, \Lambda) \cong \mathbb{C}[\Lambda] \otimes U(\mathfrak{h})$$

Hence the coradical is indeed the group algebra $\mathbb{C}[\Lambda]$. This especially shows that there are no other grouplikes than $\Lambda$. □
4.3. More on the PBW-basis. We give in the following preliminary results on the PBW-basis of $U_q^\mathbb{L}(\mathfrak{g}, \Lambda)$. From the PBW-basis in $U_q^\mathbb{L}(\mathfrak{g}, \Lambda)$ we have already determined in Corollary 3.10 that multiplication induces an isomorphism of vector spaces

$$\mathbb{C}[\Lambda/2\Lambda_\ell] \bigotimes_{i \in I, r > 0} K_{\alpha_i}^{(r)} \mathbb{C} \bigotimes_{\alpha \in \Phi^+, r > 0} E_{\alpha}^{(r)} \mathbb{C} \bigotimes_{-\alpha \in \Phi^-, r > 0} F_{\alpha}^{(r)} \mathbb{C} \xrightarrow{\cong} U_q^\mathbb{L}(\mathfrak{g}, \Lambda)$$

The aim of the next theorem is to use the knowledge of the zero-part in the previous section and a straightforward calculation to incorporate at least part of the algebra relations that hold specifically in the specialization, without any restrictions on $\ell$:

**Theorem 4.4.** Let $q$ be a primitive $\ell$-th root of unity and again $\mathfrak{h} = \bigoplus_{i \in I} K_{\alpha_i}^{-\ell_i} K_{\alpha_i}^{(\ell_i)} \mathbb{C}$. Then multiplication in $U_q^\mathbb{L}(\mathfrak{g}, \Lambda)$ induces an isomorphism of vector spaces, which restricts on each tensor factor to an injection of algebras:

$$\mathbb{C}[\Lambda/2\Lambda_\ell] \otimes U(\mathfrak{h}) \bigotimes_{\alpha \in \Phi^+} \left( \mathbb{C}[E_{\alpha}]/(E_{\alpha}^{(\ell_\alpha)}) \otimes \mathbb{C}[E_{\alpha}^{(\ell_\alpha)}] \right) \bigotimes_{-\alpha \in \Phi^-} \left( \mathbb{C}[F_{\alpha}]/(F_{\alpha}^{(\ell_\alpha)}) \otimes \mathbb{C}[F_{\alpha}^{(\ell_\alpha)}] \right) \xrightarrow{\cong} U_q^\mathbb{L}(\mathfrak{g}, \Lambda)$$

**Proof.** By Corollary 3.10 multiplication in $U_q^\mathbb{L}$ induces an isomorphism of vector spaces

$$\mathbb{C}[\Lambda/2\Lambda_\ell] \bigotimes_{i \in I, r > 0} K_{\alpha_i}^{(r)} \mathbb{C} \bigotimes_{\alpha \in \Phi^+, r > 0} E_{\alpha}^{(r)} \mathbb{C} \bigotimes_{-\alpha \in \Phi^-, r > 0} F_{\alpha}^{(r)} \mathbb{C} \xrightarrow{\cong} U_q^\mathbb{L}(\mathfrak{g}, \Lambda)$$

We clarified in Theorem 4.1 the zero-part $U_q^{\mathbb{L}, 0}$, so we get an isomorphism

$$\mathbb{C}[\Lambda/2\Lambda_\ell^0] \otimes U(\mathfrak{h}) \bigotimes_{\alpha \in \Phi^+, r > 0} E_{\alpha}^{(r)} \mathbb{C} \bigotimes_{-\alpha \in \Phi^-, r > 0} F_{\alpha}^{(r)} \mathbb{C} \xrightarrow{\cong} U_q^\mathbb{L}(\mathfrak{g}, \Lambda)$$

with the abelian Lie algebra $\mathfrak{h} = \bigoplus_{i \in I} K_{\alpha_i}^{-\ell_i} K_{\alpha_i}^{(\ell_i)} \mathbb{C}$.

We next turn our attention to the the algebra generated for a fixed root $\alpha \in \Phi^+$ by all elements $E_{\alpha}^{(r)} = E_{\alpha}^r/[r]_{q_\alpha}$ in the specialization to a primitive $\ell$-th root of unity (respectively
for $F$). Since $[r]_{q_{a}} = 0$ iff $\ell_{a} := \text{ord}(q_{a}^{2})/r$, it is clearly isomorphic to

$$\bigoplus_{r \geq 1} E_{\alpha}^{(r)} \cong \begin{cases} \mathbb{C}[E_{\alpha}]/(E_{\alpha}^{(\ell_{\alpha})}) \otimes \mathbb{C}[E_{\alpha}^{(\ell_{\alpha})}] & \ell_{\alpha} > 1 \\ \mathbb{C}[E_{\alpha}] & \ell_{\alpha} = 1 \end{cases}$$

This yields the asserted isomorphism. $\square$

5. A SHORT EXACT SEQUENCE

Lusztig has in [Lusz90b] Thm 8.10. discover a remarkable homomorphism by to the classical universal enveloping Hopf algebra $U(g)$

$$U_{q}^{\ell}(g, \Lambda) \xrightarrow{\text{Frob}} U(g)$$

whenever $\ell$ is odd and for $g = G_{2}$ not divisible by 3. He described the kernel in terms of an even more remarkable finite-dimensional Hopf algebra $u_{q}(g)$, that has under the name “Frobenius-Lusztig-kernel” triggered the development of the theory and far-reaching classification results of finite-dimensional pointed Hopf algebras and Nichols algebras in the past $\sim$20 years.

The following is a more systematic construction, using more recent techniques, that provides a new proof idea and generalizes to the situation of small prime divisors (which has been excluded by Lusztig and throughout the following literature). The theorem shows, what is “abstractly” true and what has to be checked by more explicit calculation, and fail or is severely modified for small primes.

The following crucial definition and theorem is essentially in [Lusz90b] Sec. 8.2, without restrictions on $\ell$:

**Theorem 5.1.** Let $u_{q}^{\ell} \subset U^{\ell}(g, \Lambda)$ be the Hopf subalgebra generated by $\Lambda$ and all $E_{\alpha}, F_{\alpha}$ with $\alpha \in \Phi^{+}$ such that $\ell_{\alpha} > 1$. Then multiplication in $u_{q}^{\ell}$ defines an isomorphism of vector spaces:

$$\mathbb{C}[\Lambda/2\Lambda_{R}^{\ell_{\alpha}}] \otimes_{\alpha \in \Phi^{+}, \ell_{\alpha} > 1} \mathbb{C}[E_{\alpha}]/(E_{\alpha}^{(\ell_{\alpha})}) \otimes_{-\alpha \in \Phi^{-}, \ell_{\alpha} > 1} \mathbb{C}[F_{\alpha}]/(F_{\alpha}^{(\ell_{\alpha})}) \xrightarrow{\cong} u_{q}^{\ell}$$

Especially $u_{q}^{\ell}$ is of finite dimension $|\Lambda| \cdot \prod_{\alpha \in \Phi^{+}} \ell_{\alpha}^{2}$

**Proof.** In [Lusz90b] Thm 8.3 iii) Lusztig proved for $\Lambda = \Lambda_{R}$ and without restrictions on $\ell$ that $u_{q}^{\ell}$ has a PBW-basis consisting of $\Lambda_{R}$ and all $E_{\alpha}^{(r)}, F_{\alpha}^{(r)}$ with $r < \ell_{\alpha}$. Note that this set is empty for $\ell_{\alpha} = 1$. We’ve proven as part of Theorem 4.4 the simple fact that $\mathbb{C}[E_{\alpha}]$ consists precisely of all $E_{\alpha}^{(r)}$ with $r < \ell_{\alpha}$. This shows the claim for $\Lambda = \Lambda_{R}$. The extension to arbitrary $\Lambda$ via $\mathbb{C}[\Lambda] \otimes \mathbb{C}[\Lambda_{R}]$ follows again from the presentation in Theorem 3.7 $\square$
Remark 5.2. Note that for $g = G_2, \ell = 4$, the Hopf algebra $u^G_4$ is not generated as algebra by simple root vectors $E_{\alpha_i}$. This has already been noticed in in the last sentence of [Lusz90b] Sec. 8.3. It can be understood in terms of Nichols algebras, see Section 6.3.

The following abstract notion is straight-forward, but in our specific case gives a curious new characterization of the Frobenius-Lusztig kernel $u_q^L$ defined above:

Lemma 5.3.

(1) Let $K \subset H$ be Hopf algebras. Then there is a unique minimal normal Hopf algebra $N$ with $K \subset N \subset H$. We call $N$ the normal hull of $K$ in $H$.

(2) For the normal hull $N^L$ of the coradical $\mathbb{C}[\Lambda]$ in $U_q^L(g, \Lambda)$ we have $u_q^L \subset N^L$.

Proof. (1) There are two methods to construct such a Hopf subalgebra $N$, both resembling the respective notions in group theory: Either, let $N$ be the intersection of all normal Hopf-subalgebras $N_i$ containing $K$, then $N$ is again a normal Hopf subalgebra. Alternatively, consider $H$ as the adjoint $H$-representation and consider the $H$-submodule generated by $K$. Because the adjoint representation $H$ is an $H$-module-Hopf-algebra, this will again yield a Hopf subalgebra, which is normal by construction.

(2) The proof that $N$ contains the simple root vectors $E_{\alpha_i}, F_{\alpha_i}$ with $\ell_{\alpha_i} > 1$ proceeds as follows: We calculate the adjoint action on $E_{\alpha_i}$ on the element $K_{\alpha_i} \in \Lambda$:

$$E_{\alpha_i}^{[1]} K_{\alpha_i} S(E_{\alpha_i}^{[2]}) = E_{\alpha_i} K_{\alpha_i} K_{\alpha_i}^{-1} + K_{\alpha_i} (-E_{\alpha_i} K_{\alpha_i})$$

$$= E_{\alpha_i} - q_{\alpha_i}^2 E_{\alpha_i} = E_{\alpha_i} (1 - q_{\alpha_i}^2)$$

which is hence in $N^L$. Since we assumed $\text{ord}(q_{\alpha_i}^2) =: \ell_{\alpha_i} > 1$, we have proven $E_{\alpha_i} \in N^L$. The same reasoning shows $F_{\alpha_i} \in N$. For non-simple root vectors $E_\alpha$ note, that the used terms $E_\alpha \otimes K_\alpha$ and $1 \otimes E_\alpha$ are the only summands in $\Delta(E_\alpha)$ containing $E_\alpha$(the other are products of root vectors $E_\beta$ for smaller roots), so by induction such summands are already in $N$. By definition $u_q^L$ is generated by these root vectors $E_\alpha, F_\alpha$ for all $\alpha \in \Phi^+$ with $\ell_\alpha > 1$ together with $\Lambda$, so we conclude $u_q^L \subset N^L$.

\[\square\]

Theorem 5.4. Let $N^L$ be the normal hull of the coradical $k[\Lambda]$ from Lemma 5.3, which is a normal Hopf subalgebra with $u_q^L \subset N^L \subset U_q^L(g, \Lambda)$. We have a short exact sequence of Hopf algebras

$$0 \rightarrow N^L \rightarrow U_q^L(g, \Lambda) \rightarrow \text{Frob} U(g^{(\ell)}) \rightarrow 0$$

where $g^{(\ell)}$ is a complex Lie algebra generated by the images of $E_\alpha^{(\ell)}, F_\alpha^{(\ell)}, K_\alpha^{(\ell)}$. They form a PBW-basis of $U(g^{(\ell)})$ iff $u_q^L$ is normal.
Example 5.5. For odd $\ell$ (and for $g = G_2$ not divisible by 3) the theorem returns $g^{(\ell)} = g$ and $N^g = u^g_q$ the small quantum group associated to $g$ as in [Lusz90b].

Remark 5.6. The general case exhibits additional phenomena:

- $N^g$ may be significantly larger than $u^g_q$, see the example in section 6.4. This seems to only happen at the exceptions in Lemma 2.6.
- $u^g_q$ may be smaller than expected due to the condition $\ell_\alpha > 1$, yielding a Lie subalgebra. Moreover for $g = G_2, \ell = 4$ we have an additional algebra generator $E_1^{\ell_2}$ and $u^g_q$ is of type $A_3$, see the example in section 6.3.
- $U(g^{(\ell)})$ may be a different Lie algebra enveloping than $U(g)$, due to $\ell_\alpha \neq \ell_\beta$ for short and long roots. In these cases we obtain the dual root system $g^{(\ell)} = g^\vee$. See section 6.2.

Proof. (Theorem 5.4) The proof reverses the order of Lusztig’s approach: It yields an indirect quotient and shows that it is a Lie algebra enveloping without determining it.

By definition $N^g$ is a normal Hopf subalgebra of $U^g$, so consider the Hopf algebra quotient

$$U := U^g / \ker(\epsilon(N^g)) U^g = U^g / U^g \ker(\epsilon(N^g))$$

In order for this to form an exact sequence of Hopf algebras, we need to show cleftness resp. normal basis [Schau02] Sec. 3.2. This means that in addition to the injection $\iota : N^g \rightarrow U^g$ there exists a unital, counital $U$-comodule map $j : U \rightarrow U^g$, such that multiplication induces an isomorphism of vector spaces:

$$N^g \otimes U \xrightarrow{\iota \otimes 1} U^g \otimes U^g \rightarrow U^g$$

This could be formulated as the existence of an alternative PBW-basis in $U^g$ sorted along $N^g$ and $U$. The cleftness follows from Schneider’s criterion, see [Mont93] Thm. 8.4.4. We hence obtain a short exact sequence of Hopf algebras:

$$0 \rightarrow N^g \xrightarrow{\iota} U^g_q(g, \Lambda) \xrightarrow{Frob} U \rightarrow 0$$

The PBW-basis of $U^g$ in the form given in Theorem 4.4 and the PBW-basis of $u^g_q \subset N^g$ proves that the image is generated by the images of the PBW-generators of $U^g$ not contained in $u^g_q$, namely as asserted

$$E^{\ell_\alpha}_{\alpha}, F^{\ell_\alpha}_{\alpha}, K^{\ell_\alpha i}_{\alpha i}$$

It is also clear from the normal basis above, that as asserted they form a PBW-basis of $U$ iff $u^g_q = N^g$, which is by definition of $N^g$ the case iff $u^g_q$ is normal.

We next claim that $U$ is a cocommutative Hopf algebra, which is sufficient to check on the above algebra generators. The generators $K^{\ell_\alpha i}_{\alpha i}$ are cocommutative in $U^g$, so we focus on...
This is precisely why we defined $N^\ell_L$ as the normal hull of the coradical $C[\Lambda]$: Namely, the elements $K_\Lambda - 1 \in \ker(C(\Lambda)) \subset \ker(N^\ell_L)$ are by definition in the kernel of the quotient map $U^\ell_L \to U$, hence the group $\Lambda \subset U^\ell_L$ is sent to $1_U$. Thus the claim follows for simple root vectors from formula $[Lusz90b]$ Sec. 1.3 a)

$$\Delta E_{\alpha_i}^n = \sum_{b=0}^{n} q^{d_i(b(n-b))} E_{\alpha_i}^{n-b} K_{\alpha_i}^b \otimes E_{\alpha_i}^b$$

This assertion is again true for all root vectors by successive application of Weyl reflections $T_i$. This proves the claim that $U$ is a cocommutative Hopf algebra.

The theorem of Kostant-Cartier (see $[Mont93]$ Sec. 5.6) states, that a cocommutative Hopf algebra over an algebraically closed field of characteristic 0 is of the form $U = k[\Gamma] \rtimes U(\mathfrak{g}(\ell))$ for some group $G$ and some Lie algebra, say $\mathfrak{g}(\ell)$. In Lemma 4.2 we determined the set of grouplikes of $U^\ell_L$ to be $\Lambda$, hence by construction these elements are sent to 1 in $U$ and thus $\Gamma = 1$. This proves $U \cong U(\mathfrak{g}(\ell))$ and thus the theorem.

A very inconvenient discovery during the work on this article was the fact, that already in easy cases the normal hull $N^\ell_L$ of $C[\Lambda]$ may be all of $U^\ell_L$. This seems at first sight not directly related to the structural changes to the root system mentioned in Example 5.5; rather it is due to the fact that for even $\ell$ some group elements $K_\alpha$ act nontrivial on some $E_{\alpha_j}^\ell$. The author would speculate, that this still allows for the construction of a Galois extension. In this article, we instead shall from now on concentrate on the Borel parts $U^\ell_L, U^\ell_L^+$ that capture the essential combinatorics of the root system. In all our examples in Section 6.1 we indeed find $u^\ell_L q$ to be normal. See Conjecture 5.10.

Consider the Hopf-subalgebra $U' = C[\Lambda] u^\ell_L q$ generated by $\Lambda$ and all positive root vectors $E_\alpha$. There is a projection $\pi$ to the Hopf subalgebra $C[\Lambda]$. Hence by the Radford projection theorem we may decompose $U'$ as a Radford biproduct

$$U' \cong C[\Lambda] \rtimes U^\ell_L q$$

where the Borel part $U^\ell_L q$ a Hopf algebra in the category of $\Lambda$-Yetter-Drinfel’d modules.

Remark 5.7. Recently Angiono has in $[An14]$ characterized (dually) the Borel part of the Kac-Procesi-DeConcini-Form $U^K_q(\mathfrak{g})$ purely in terms of so-called distinguished Pre-Nichols algebra in the braided category of $\Lambda$-Yetter-Drinfel’d Modules. These algebras are however much more general and all come with a version of a Frobenius homomorphism.

If we repeat the previous construction for the Borel part, the split exact sequence returns a-priori a $\Lambda$-Yetter-Drinfel’d Hopf algebra. We are interested in cases, where the resulting
Hopf algebra is in fact an ordinary complex Hopf algebra. This requires precisely the Lie-theoretic assumption in Lemma 2.6.

**Theorem 5.8.** Let $N_{\ell}^L +$ be the normal hull of $u_q^{\ell} \subset U_q^{\ell} +$ and assume:

- $A_n, D_n, E_6, E_7, E_8, G_2 \quad \ell \neq 2 \bmod 4$
- $B_n, n \geq 3 \quad \ell \neq 4 \bmod 8$
- $C_n, n \geq 3 \quad \ell \neq 2 \bmod 4$
- $F_4, \ell \neq 2, 4, 6 \bmod 8$

Then we have short exact sequence of $\Lambda$-Yetter-Drinfel’d Hopf algebras

$$0 \longrightarrow N_{\ell}^L + \longrightarrow U_q^{\ell} + (g, \Lambda) \xrightarrow{\text{Frob}} U(g^{\ell}) + \longrightarrow 0$$

where $g^{\ell}$ is an ordinary complex Lie algebra generated by the images of $E_\alpha^{(\ell)}$.

**Remark 5.9.** In the excluded cases, $g^{(\ell)}$ is a Lie algebra in a symmetric braided category, but not a super-Lie-algebra (the self-braidings are all +1). See the proof for details.

**Conjecture 5.10.** We expect that $u_q^{\ell}(g) +$ is always normal in $U_q^{\ell}(g, \Lambda)$, hence $N_{\ell}^L = u_q^{\ell}(g)^{\ell}$. This is in contrast to $u_q^{\ell}(g, \Lambda)$ in Thm. 5.4, see the counterexample in Sec. 6.4.

**Proof.** By definition $N_{\ell}^L$ is a normal Hopf subalgebra of $U_{\ell}$ in the braided category of $\Lambda$-Yetter-Drinfel’d modules, so consider the Hopf algebra quotient

$$U^+ := U_{\ell} / \ker(\epsilon(N_{\ell}^L))U_{\ell} + = U_{\ell} / \ker(\epsilon(N_{\ell}^L))$$

A-priori, $U^+$ is a $\Lambda$-Yetter-Drinfel’d Hopf algebra. We calculate that the braiding in the quotient $U^+$ is trivial, so $U^+$ is an ordinary complex Hopf algebra:

$$\tau(E\alpha^{(\ell)} \otimes E\beta^{(\ell)}) = K_{\ell\alpha\beta} E\beta^{(\ell)} \otimes E\alpha^{(\ell)} = q^{(\ell\alpha, \ell\beta)} E\beta^{(\ell)} \otimes E\alpha^{(\ell)}$$

We’ve proven in Lemma 2.6 that except in the excluded cases we have $(\ell_\alpha, \ell_\beta) \in \ell\mathbb{Z}$, hence the braiding is trivial. Note that in the excluded cases we have $(\ell_\alpha, \ell_\beta) \notin \ell\mathbb{Z}$, hence $\tau^2 = 1$ and the braiding is still symmetric. Moreover we’ve generally shown that $(\ell_\alpha, \ell_\alpha) \in \ell\mathbb{Z}$, hence the self-braiding is trivial and $U^+$ is a domain (no truncations).

In order for this to form an exact sequence of Hopf algebras, we need to show cleftness resp. normal basis [Schau02] Sec. 3.2. This means that in addition to the injection $\iota : N_{\ell} \rightarrow U_{\ell}$ there exists a unital, counital $U$-comodule map $j : U \rightarrow U_{\ell}$, such that multiplication induces an isomorphism of vector spaces:

$$N_{\ell} \otimes U \xrightarrow{\iota \otimes j} U_{\ell} \otimes U_{\ell} \rightarrow U_{\ell}$$

This could be formulated as the existence of an alternative PBW-basis in $U_{\ell}$ sorted along $N_{\ell}$ and $U$. The cleftness follows from Schneider’s criterion, see [Mont93] Thm. 8.4.4. Note that in the excluded cases one could use the Radford-biprodut to reduce the
statement to complex Hopf algebra \( \mathbb{C} \left[ \Lambda \right] \rtimes U^+ \). We hence obtain a short exact sequence of Hopf algebras:

\[
0 \to N^\mathcal{L} \subseteq U^\mathcal{L}_q (g, \Lambda) \xrightarrow{\text{Frob}} U \to 0
\]

The PBW-basis of \( U^\mathcal{L} \) in the form given in Theorem 4.4 and the PBW-basis of \( u^\mathcal{L}_q \subset N^\mathcal{L} \) proves that the image is generated by the images of the PBW-generators of \( U^\mathcal{L}_q \) not contained in \( u^\mathcal{L}_q \), namely as asserted

\[
E_{\alpha_i}^{\ell_{\alpha_i}}, F_{\alpha_i}^{\ell_{\alpha_i}}, K_{\alpha_i}^{\ell_{\alpha_i}}
\]

It is also clear from the normal basis above, that as asserted they form a PBW-basis of \( U \) iff \( u^\mathcal{L}_q = N^\mathcal{L} \), which is by definition of \( N^\mathcal{L} \) the case iff \( u^\mathcal{L}_q \) is normal.

We next claim that \( U \) is a cocommutative Hopf algebra, which is sufficient to check on the above algebra generators. The generators \( K_{\alpha_i}^{\ell_{\alpha_i}} \) are cocommutative in \( U^\mathcal{L} \), so we focus on \( E_{\alpha_i}^{\ell_{\alpha_i}} \) (and analogously \( F_{\alpha_i}^{\ell_{\alpha_i}} \)). This is precisely why we defined \( N^\mathcal{L} \) as the normal hull of the coradical \( \mathbb{C} \left[ \Lambda \right] \); Namely, the elements \( K_{\lambda} - 1 \in \ker(\mathbb{C} \left( \Lambda \right)) \subset \ker(\mathbb{N}^\mathcal{L}) \) are by definition in the kernel of the quotient map \( U^\mathcal{L} \to U \), hence the group \( \Lambda \subset U^\mathcal{L} \) is sent to \( 1_U \). Thus the claim follows for simple root vectors from formula \[\text{[Lusz90b] Sec. 1.3 a)}\]

\[
\Delta E_{\alpha_i}^{n_{\alpha}} = \sum_{b=0}^{n} q^{d_{b}(n-b)} E_{\alpha_i}^{n-b} K_{\alpha_i}^{b} \otimes E_{\alpha_i}^{b}
\]

This assertion is again true for all root vectors by successive application of Weyl reflections \( T_i \). This proves the claim that \( U \) is a cocommutative Hopf algebra.

The famous theorem of Konstat-Cartier states, that a cocommutative Hopf algebra over an algebraically closed field of characteristic \( 0 \) is of the form \( U = k[\Gamma] \rtimes U(g^\ell) \) for some group \( G \) and some Lie algebra, say \( g^\ell \). Note that in the excluded cases, one may alternatively invoke \[\text{[Ko77 Thm 3.3 resp. [Kh07] for the symmetrically braided setting.}}\]

In Lemma 4.2 we determined the set of grouplikes of \( U^\mathcal{L} \) to be \( \Lambda \), hence by construction these elements are sent to 1 in \( U \) and thus \( \Gamma = 1 \). This proves \( U \cong U(g^\ell) \) and thus the theorem.

\[\square\]

6. THE CASE OF RANK 2

We now present the explicit calculation of the short exact sequence in Theorem 5.8 for \( g = A_2, B_2, G_2 \) and arbitrary \( \ell \):

\[
0 \to N^\mathcal{L}_+ \subseteq U^\mathcal{L}_q^+ (g, \Lambda) \xrightarrow{\text{Frob}} U^\mathcal{L}_q^+(g^{\ell},+) \to 0
\]
where we exclude the case $g = A_2, \ell = 2 \mod 4$ since in this case the Theorem shows $U(g^{(\ell),+})$ is in a braided category (we will briefly discuss this case in Subsection 6.4). As it turns out, in all these cases $u_q^L(g, \Lambda)^+$ is normal and hence $N_L^+ = u_q^L$.

### 6.1. All non-braided cases.

**Theorem 6.1.** Let $g$ be of rank 2, then the short exact sequence of Hopf algebras in Theorem 5.8 takes the following form

$$\ell = \text{ord}(q) \quad \begin{array}{c|c|c|c|c|c} \ell = 1, 3 \mod 4 & \ell = 1 & 0 & A_2 & A_2 & \text{trivial} \\ \ell \neq 1 & A_2 & A_2 & A_2 & \text{generic} \\ = 0 \mod 4 \\ \ell = 1, 3 \mod 4 & \ell = 1 & 0 & B_2 & B_2 & \text{trivial} \\ \ell \neq 1 & B_2 & B_2 & B_2 & \text{generic} \\ = 2 \mod 4 \\ \ell = 2 & 0 & B_2 & B_2 & \text{generic} \\ \ell \neq 2 & B_2 & B_2 & B_2 & \text{generic} \\ = 0 \mod 4 \\ \ell = 4 & A_1 \times A_1 & B_2 & B_2^\vee \text{ dual} \\ \ell \neq 4 & B_2 & B_2 & B_2^\vee \text{ dual} \\ = 1, 5, 7, 11 \mod 12 & \ell = 1 & 0 & G_2 & G_2 & \text{trivial} \\ \ell \neq 1 & G_2 & G_2 & G_2 & \text{generic} \\ = 4, 8 \mod 12 & \ell = 4 & A_3 & G_2 & G_2 & \text{exotic} \\ \ell \neq 4 & G_2 & G_2 & G_2 & \text{generic} \\ = 0, 3, 9 \mod 12 & \ell = 3 & A_2 & G_2 & G_2^\vee & \text{dual} \\ \ell \neq 3 & G_2 & G_2 & G_2^\vee & \text{dual} \end{array}$$

**Proof.** For the explicit calculations see the appendix. The first columns distinguish different $\gcd(\ell, 2d_i)$ and hence $\ell_i$, as well as cases where some roots have $\ell_i = 1$. The proof proceeds case-wise and the calculations makes excessive use of [Lusz90] Sec. 5.

- In each case, we first determine the adjoint action of $E_{\ell_i}^L$ on $u_q(g)^+$ to verify the latter is a normal Hopf-subalgebra. Hence we show $N_L^+ = u_q^L$.
- Then we determine $u_q(g)^+$, which is generated by all $E_{\alpha}$ with $\ell_\alpha > 1$, as a Nichols algebra $B(M)$ of Cartan type using [Heck06]. In all cases except $g = G_2, \ell = 4$ (“exotic case”) we find that $u_q(g)^+$ is indeed generated by $E_{\alpha}$ for a subset of simple roots $\alpha$.
- Finally we calculate the commutators of the elements $E_{\alpha}^{\ell_i}$ modulo the kernel of the Frobenius homomorphism to determine the Lie algebra $g^{(\ell),+}$. In several cases, the Frobenius homomorphism picks up additional scalar factors. More severely, in all cases with $\ell_\alpha \neq \ell_\beta$ for short and long roots the Frobenius homomorphism switches short and long roots.
Conjecture 6.2. We expect for \( g \) of arbitrary rank and \( \ell \) as in Theorem 5.8 that either of the following case applies:

- If all \( \ell_\alpha \) are equal, then the image of the Frobenius homomorphism is \( g^{(\ell)} = g \).
- If \( \ell_\alpha \neq \ell_\beta \) for short and long roots, then the quotient is already \( \ell_\alpha/\ell_\beta = d_\alpha/d_\beta \).

In this case the image of the Frobenius homomorphism is the dual root system \( g^{(\ell)} = g^\vee \).

In the remaining sections we give three examples that illustrate the behaviour described above.

6.2. Dual example \( g = B_2, 4|\ell \).

\[ \Phi^+ = \{\alpha_2, \alpha_{12}, \alpha_{112}, \alpha_1\} \quad (\alpha_i, \alpha_j) = \begin{pmatrix} 2 & -2 \\ -2 & 4 \end{pmatrix} \]

We have short roots and long roots as follows:

\[
\begin{align*}
d_{12} &= d_1 = 1 \\
\ell_{12} &= \ell_1 = \frac{\ell}{2} \\
d_2 &= d_{112} = 2 \\
\ell_2 &= \ell_{112} = \frac{\ell}{4}
\end{align*}
\]

We first describe \( u_q^{L}(g) \).

- For \( \ell = 4 \) only the short root vectors \( E_{12}^{(k)}, E_{12}^{(k)}, E_{12}^{(k)} \) with \( k < \frac{\ell}{2} = 2 \) are contained in \( u_q^{L} \) and \( u_q^{L,+} \) is a Nichols algebra of type \( A_1 \times A_1 \).

- For \( \ell \neq 2 \) all root vectors \( E_{\alpha}^{(k)}, F_{\alpha}^{(k)}, k < \frac{\ell}{2} \) are contained in \( u_q^{L,+} \) and \( u_q^{L,+} \) is the Nichols algebra of type \( B_2 \).

Then, one checks normality, i.e. \( \ker_{e}(u_q^{L,+})U_q^{L,+} = U_q^{L,+} \ker_{e}(u_q^{L,+}) \) by explicit calculation (see appendix). Hence we may apply Theorem 5.8 with \( N^{L,+} = u_q^{L,+} \) yielding for \( \ell = 4 \) resp. \( \ell \neq 4 \) short exact sequences of \( \Lambda \)-Yetter-Drinfel’d Hopf algebras:

\[
\begin{align*}
B(A_1 \times A_1) &\longrightarrow U_q^{L}(B_2, \Lambda)^+ \xrightarrow{\text{Frob}} U^+ = U(g^{(\ell)},+) \\
B(B_2) &\longrightarrow U_q^{L}(B_2, \Lambda)^+ \xrightarrow{\text{Frob}} U^+ = U(g^{(\ell)},+)
\end{align*}
\]

We next give the relations in the quotient \( U^+ \), which is generated by the images of the remaining PBW-generators

\[
E^{(\frac{\ell}{2})}_2, E^{(\frac{\ell}{2})}_{12}, E^{(\frac{\ell}{2})}_{112}, E^{(\frac{\ell}{2})}_1
\]

Especially for \( \ell = 4 \) this contains the long root vectors \( E_2, E_{112} \). We directly compute the relations in the quotient \( U^+ \) and hence in \( g^{(\ell)},+ \).
\[ [E_1^{(\ell)}, E_2^{(\ell)}] = -E_{112}^{(\ell)} \]
\[ [E_{112}, E_2] = \left( \prod_{i=1}^{\ell} (q^{2-4i} - 1) \right) E_{12}^{(\ell)} \neq 0 \]
\[ [E_1^{(\ell)}, E_{12}^{(\ell)}] = [E_2^{(\ell)}, E_{12}^{(\ell)}] = [E_{112}^{(\ell)}, E_{12}^{(\ell)}] = [E_1^{(\ell)}, E_{112}^{(\ell)}] = 0 \]

Hence the root vectors \( E_2^{(\ell)}, E_{112}^{(\ell)}, E_1^{(\ell)} \) span a Borel algebra of type \( C_2 \), i.e. \( E_2^{(\ell)} \)
is the short root vector and \( E_1^{(\ell)} \) is the long root vector.

6.3. Exotic example \( g = G_2, \ell = 4 \).

\[ \Phi^+ = \{\alpha_2, \alpha_{112}, \alpha_{11122}, \alpha_{112}, \alpha_{1112}, \alpha_1\} \]

We have short roots and long roots as follows:
\[ d_{12} = d_{112} = d_1 = 1 \quad d_2 = d_{1122} = d_{1112} = 3 \]
\[ \ell_{12} = \ell_{112} = \ell_1 = \ell_2 = \ell_{1122} = \ell_{1112} = 2 \]

Again, explicit calculations in the appendix show that \( u_q^G(g)^+ \) is normal and we have a Frobenius homomorphism as usual

\[ u_q^G(g)^+ \rightarrow U_q^G(G_2, \Lambda)^+ \xrightarrow{Frob} U^+ = U(G_2^+) \]

and since \( \ell_\alpha \neq 1 \), all simple roots \( E_\alpha \) are contained in \( u^{\ell_+:} \).

However \( u_q^G(g)^+ \) is in this case most unusual: The \( G_2 \)-Nichols algebra in [Heck06] Figure 1 Row 11 has an excluded case. We make this more explicit: The Nichols algebra spanned by the simple root vectors \( E_\alpha \) depends on their braiding matrix (see proof of Thm. 5.8)

\[ q_{ij} := q^{(\alpha_i, \alpha_j)_0} \]

Usually this is of type \( g \) and hence generates the entire \( u_q^G \). If we spell out the braiding matrix for type \( G_2 \) for arbitrary \( q \) and especially for \( \ell = 4, q = \sqrt{-1} \) we get

\[
\begin{pmatrix}
q^2 & q^{-3} \\
q^{-3} & q^6
\end{pmatrix}
\begin{pmatrix}
-1 & \sqrt{-1} \\
\sqrt{-1} & -1
\end{pmatrix}
\]

The latter braiding matrix generates a Nichols algebra of type \( A_2 \) and dimension \( 2^{1|\Phi^+|} = 2^3 \), which is why the case \( q_{11} = -1 \) is excluded in Heckenberger list for \( G_2 \). On the other hand the dimension of \( u_q^G \) is by Lusztig’s result still \( \prod \ell_\alpha = 2^6 \), hence the two do not coincide as in all cases before. In fact, explicit calculations in the appendix show that

\[ u_q^G(g)^+ \cong B(A_3) \]
generated by “simple” root vectors $E_1, E_2, E_{112}$, yielding also the right dimension $2^6$. Note that $G_2, A_3$ have both 12 roots!

For $\ell \neq 4$ on the other hand we again have $u^{\ell, +} = B(G_2)$.

6.4. Braided example $g = A_2$, $\ell = 2$. We now demonstrate the case $g = A_2, \ell = 2 \mod 4$ where the Frobenius homomorphism in Theorem 5.8 maps to a Hopf algebra in a braided category. Moreover, the short exact sequence for the full $U^\ell_q(g)$ fails since $u^\ell_q$ is not normal. We take the easiest case $\ell = 2$, where $u^\ell_q(g)$ is trivial and yield a short exact sequence of A-Yetter-Drinfel’d Hopf algebras

$$\mathbb{C} \rightarrow U^\ell_q(A_2, \Lambda)^+ \xrightarrow{\text{Frob}} U^+$$

Here, $U^\ell_q(A_2, \Lambda)^+ \cong U$ has a PBW-basis in the generators $E_1, E_2, E_{12}$, but $E_{12} = E_1E_2 + E_2E_1$ and $U^+$ is an infinite-dimensional Nichols algebra with braiding matrix

$$q_{ij} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Especially it is not an ordinary Hopf algebra, but also no super-Lie-algebra enveloping!

We now turn our attention to the full quantum group $U_q^\ell(g)$. Explicit calculations in the appendix show:

$$[E_i, F_i] = K^{(1)}_i$$

$$[K^{(1)}_i, E_j] = -2E_i$$

$$[K^{(1)}_i, E_j] = -2E_j \left( K^{(1)}_i + \frac{1}{2} \right) \quad i \neq j$$

$$[K^{(1)}_i, E_i] = 2F_i$$

Especially the Hopf subalgebra $u_q(g, \Lambda)$ is not normal any more. It would be very interesting to understand this phenomenon further.

Appendix A. Calculations for $g = A_2$

Let $g = A_2$, then

$$\Phi^+ = \{\alpha_2, \alpha_{12}, \alpha_1\} \quad (\alpha_i, \alpha_j) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

All roots have equal length and

$$d_\alpha = (\alpha, \alpha)/2 = 1 \quad \ell_\alpha = \begin{cases} \ell & 2 \nmid \ell \\ \frac{\ell}{2} & 2 | \ell \end{cases}$$
We first describe what problems arise in the full quantum group $A$. The remaining PBW-generators $U_q^\mathbb{L}$ give the relations in the quotient

$$u_q^\mathbb{L} \rightarrow U_q^\mathbb{L}(A_2, A_R) \xrightarrow{\text{Frob}} U(A_2)$$

We have all $\ell_i = \ell$, so for $u_q^\mathbb{L}$ we get two cases $\ell = 1$ and $\neq 1$:

For $\ell = 1$ no root vectors $E_\alpha^{(k)}$, $F_\alpha^{(k)}$ are contained in $u_q^\mathbb{L}$, hence $u_q^\mathbb{L} = \mathbb{C}[A]$. Especially on the level of Borel subalgebras we get a short exact sequence of $\Lambda$-Yetter-Drinfel’d modules

$$\mathbb{C} \rightarrow U_q^\mathbb{L}(A_2)^+ \xrightarrow{\text{Frob}} U(A_2)^+$$

For $\ell \neq 1$ all root vectors $E_\alpha^{(k)}$, $F_\alpha^{(k)}$, $k < \ell$ are contained in $u_q^\mathbb{L}$ and $u_q^{\mathbb{L}^+}$ is the Nichols algebra of type $A_2$. We get a short exact sequence of $\Lambda$-Yetter-Drinfel’d modules

$$B(A_2) \rightarrow U_q^\mathbb{L}(A_2)^+ \xrightarrow{\text{Frob}} U(A_2)^+$$

A.0.2. $\ell$ even. For $2 \nmid \ell$ we have in Theorem 5.4 the condition $4|\ell$. In case $\ell = 2$ mod 4 the Frobenius homomorphism maps to a proper $\Lambda$-Yetter-Drinfel’d Hopf algebra. In the case $A_2$ we want to take the opportunity to describe both cases explicitly and also point out what problems arise in the full quantum group $U_q^\mathbb{L}$:

We first describe $u_q^\mathbb{L}$. We have $\ell_\alpha = \frac{\ell}{2}$, hence for $\ell = 2$ no root vectors $E_\alpha^{(k)}$, $F_\alpha^{(k)}$ are contained in $u_q^\mathbb{L}$, hence $u_q^\mathbb{L} = \mathbb{C}[A]$. For $\ell \neq 2$ all root vectors $E_\alpha^{(k)}$, $F_\alpha^{(k)}$, $k < \frac{\ell}{2}$ are contained in $u_q^{\mathbb{L}^+}$ and $u_q^{\mathbb{L}^+}$ is the Nichols algebra of type $A_2$. We check that $u_q^\mathbb{L} \subset U_q^\mathbb{L}$ is normal and thus $N^{\mathbb{L}^+} = u_q^{\mathbb{L}^+}$. Root vectors $E_\alpha^{(i)}$, $E_\alpha^{(i)}$ commute up to $q$, also the root vector $E_1^{(i)}$ commutes with all root vectors up to $q$. Hence the only case to be checked is

$$E_1^{(i)} E_2^{(j)} = \sum_{r,s,t \geq 0, r+s=j, s+t=i} q^{r} E_2^{(j)} E_1^{(i)} + \sum_{s=1}^{\min(i,j)} q^{(j-s)(i-s)+s} E_2^{(j-s)} E_1^{(i-s)}$$

where for $i < \frac{\ell}{2}$ we have $s, i - s < \frac{\ell}{2}$. This concludes the proof of $u^{\mathbb{L}^+} u_q^{\mathbb{L}^+} = U_q^{\mathbb{L}^+} u^{\mathbb{L}^+}$.

We next give the relations in the quotient $U^+$, which is generated by the images of the remaining PBW-generators $E_\alpha^{(2)}$. Especially we verify for $4|\ell$ that $U^+ = U(A_2)^+$, whereas for $\ell = 2$ mod 4 we get a braided version. Note that always $q^2 = -1$, whereas
\( q^{\frac{\ell^2}{4}} = (-1)^{\frac{\ell}{2}} \) distinguishes the ordinary from the braided case.

\[
E_1^{(\frac{\ell}{2})} E_2^{(\frac{\ell}{2})} = q^{\frac{\ell^2}{4}} E_1^{(\frac{\ell}{2})} E_2^{(\frac{\ell}{2})}
\]

\[
E_1^{(\frac{\ell}{2})} E_2^{(\frac{\ell}{2})} = \sum_{r,s,t \geq 0, r+s=\frac{\ell}{2}, s+t=\frac{\ell}{2}} q^{tr+st} E_2^{(r)} E_1^{(s)} E_2^{(t)} = q^{\frac{\ell^2}{4}} E_2^{(\frac{\ell}{2})} E_1^{(\frac{\ell}{2})} + q^{\frac{\ell}{2}} E_1^{(\frac{\ell}{2})}
\]

where we omitted monomials with \( r, s, t < \frac{\ell}{2} \), which are in the kernel. Thus in the quotient \( U \) we have

\[
\begin{array}{ccc}
4|\ell & & \ell = 2 \mod 4 \\
\left[ E_1^{(\frac{\ell}{2})}, E_2^{(\frac{\ell}{2})} \right] = -E_1^{(\frac{\ell}{2})} & & \left[ E_1^{(\frac{\ell}{2})}, E_2^{(\frac{\ell}{2})} \right]_+ = -E_1^{(\frac{\ell}{2})} \\
\left[ E_1^{(\frac{\ell}{2})}, E_1^{(\frac{\ell}{2})} \right] = 0 & & \left[ E_1^{(\frac{\ell}{2})}, E_1^{(\frac{\ell}{2})} \right]_+ = 0
\end{array}
\]

Especially for \( 4|\ell \) the Lie algebra \( g^{(\ell),+} \) in Theorem 5.8 is the Borel part \( A_2^+ \). Note that we have a nontrivial scaling \(-1\) for the root vector \( E_{12} \) in this identification. We get a short exact sequence of \( \Lambda \)-Yetter-Drinfel’d Hopf algebras:

\[
\mathcal{B}(A_2) \rightarrow U_q^{\mathcal{L}}(A_2, \Lambda)^+ \xrightarrow{\text{Frob}} U(A_2)^+
\]

Moreover, in the braided setting \( \ell = 2 \mod 4 \) we also just calculated explicitly short exact sequence of \( \Lambda \)-Yetter-Drinfel’d Hopf algebras to braided versions of \( U(A_2)^+ \), which we shall just indicate by \( \tau \) without further discussion. Depending on whether \( \ell = 2 \) or \( \ell \neq 2 \):

\[
\mathbb{C} \rightarrow U_q^{\mathcal{L}}(A_2, \Lambda)^+ \xrightarrow{\text{Frob}} U_\tau(A_2)^+
\]

\[
\mathcal{B}(A_2) \rightarrow U_q^{\mathcal{L}}(A_2, \Lambda)^+ \xrightarrow{\text{Frob}} U_\tau(A_2)^+
\]

Let us discuss the case \( \ell = 2 \) more thoroughly: On Borel algebras \( U_q^{\mathcal{L}}(A_2, \Lambda)^+ \) this is a rather trivial behaviour. This changes drastically when we explicitly calculate the relations in \( U_q^{\mathcal{L}}(A_2, \Lambda) \). As some require slightly more explicit calculations, be aware that we first calculate in \( U_q^{\mathcal{L}}(A_2, \Lambda) \), reduce the expressions to the appropriate generators
and then specialize to $U_q^{\mathbb{C}}$; we indicate this step by $q \mapsto 1$.

\[
E_i F_i = \sum_{0 \leq t \leq 1} E_i^{1-t} \left[ K_i; 2t - 2 \atop t \right] E_i^{1-t} = F_i E_i + K_i^{(1)}
\]

\[\Rightarrow \quad [E_i, F_i] = K_i^{(1)}\]

\[K_i^{(1)} E_i - E_i K_i^{(1)} = E_i \left[ K_i; 1 \cdot \frac{2}{1} \right] - E_i \left[ K_i; 0 \atop 1 \right] = E_i \left( K_i q^2 - K_i^{-1} q^{-2} - \frac{K_i - K_i^{-1}}{q - q^{-1}} \right)\]

\[= E_i \left( K_i q^2 - K_i^{-1} q^{-2} - \frac{K_i - K_i^{-1}}{q - q^{-1}} \right) = E_i (q K_i + q^{-1} K_i^{-1}) \mapsto 1 - 2E_i\]

\[\Rightarrow \quad [K_i^{(1)}, E_j] = -2E_j \left( K_i^{(1)} + \frac{1}{2} \right) \quad i \neq j\]

\[K_i^{(1)} F_i - F_i K_i^{(1)} = F_i \left[ K_i; -1 \cdot \frac{2}{1} \right] - F_i \left[ K_i; 0 \atop 1 \right] = F_i \left( K_i q^{-2} - K_i^{-1} q^2 - \frac{K_i - K_i^{-1}}{q - q^{-1}} \right)\]

\[= F_i \left( K_i q^{-2} - K_i^{-1} q^2 - \frac{K_i - K_i^{-1}}{q - q^{-1}} \right) = F_i (q^{-1} K_i - q K_i^{-1}) \mapsto 2F_i\]

\[\Rightarrow \quad [K_i^{(1)}, E_i] = 2F_i\]

**Remark A.1.** It is an important observation, that for fixed $i$ the generators $E_i, -F_i, -K_i^{(1)}$ form a set of generators for $A_1$, while the relation $[K_i^{(1)}, E_j] = -2E_j \left( K_i^{(1)} + \frac{1}{2} \right)$ for $i \neq j$ clearly shows that $\mathfrak{g}^{(1)}$ is not an ordinary Lie algebra. This implies that the Hopf subalgebra $u_q^{\mathbb{C}}(\mathfrak{g}, \Lambda)$ is not normal. The effect seems to correspond to the nontrivial braiding

\[K_i E_j^{(k)} = q^{(\alpha_i, \xi, \alpha_j)} E_j^{(k)} K_i = -E_j^{(k)} K_i\]
Appendix B. Calculations for $\mathfrak{g} = B_2$

Let $\mathfrak{g} = B_2$, then

$$\Phi = \{\alpha_2, \alpha_{12}, \alpha_{112}, \alpha_1\} \quad (\alpha_i, \alpha_j) = \begin{pmatrix} 2 & -2 \\ -2 & 4 \end{pmatrix}$$

We have short roots and long roots as follows:

$$d_{12} = d_1 = 1 \quad \quad d_2 = d_{112} = 2$$

$$\ell_{12} = \ell_1 = \begin{cases} \ell & 2 \nmid \ell \\ \ell / 2 & 2 | \ell \end{cases} \quad \quad \ell_2 = \ell_{112} = \begin{cases} \ell & 2 | \ell, 4 \nmid \ell \\ \ell / 4 & 4 | \ell \end{cases}$$

B.0.3. $\ell$ odd. For $2 \nmid \ell$ we get the generic case. Here, the Frobenius homomorphism in [Lusz90b] Thm 8.10 implies (for $\Lambda_R = \Lambda$) a short exact sequence of Hopf algebras

$$u_q^\ell \longrightarrow U_q^\ell(B_2, \Lambda_R) \xrightarrow{\text{Frob}} U(B_2)$$

We have all $\ell_i = \ell$, so for $u_q^\ell$ we get two cases $\ell = 1$ and $\neq 1$:

For $\ell = 1$ no root vectors $E_\alpha^{(k)}$, $F_\alpha^{(k)}$ are contained in $u_q^\ell$, hence $u_q^\ell = \mathbb{C}[\Lambda]$. Especially on the level of Borel subalgebras we get a short exact sequence of $\Lambda$-Yetter-Drinfel’d modules

$$\mathbb{C} \longrightarrow U_q^\ell(B_2, \Lambda)^+ \xrightarrow{\text{Frob}} U(B_2)^+$$

For $\ell \neq 1$ all root vectors $E_\alpha^{(k)}$, $F_\alpha^{(k)}$, $k < \ell$ are contained in $u_q^\ell$ and $u_q^{\ell, +}$ is the Nichols algebra of type $B_2$. We get a short exact sequence of $\Lambda$-Yetter-Drinfel’d modules

$$\mathcal{B}(B_2) \longrightarrow U_q^\ell(B_2, \Lambda)^+ \xrightarrow{\text{Frob}} U(B_2)^+$$

B.0.4. $\ell$ even, $4 \nmid \ell$. For $B_2$ we can apply Theorem 5.4 without restrictions on $\ell$. We first treat the case $4 \nmid \ell$. Then $\ell_\alpha = \ell / 2$ both for long and short roots.

We first describe $u_q^\ell$. For $\ell = 2$ no root vectors $E_\alpha^{(k)}$, $F_\alpha^{(k)}$ are contained in $u_q^\ell$, hence $u_q^\ell = \mathbb{C}[\Lambda]$. For $\ell \neq 2$ all root vectors $E_\alpha^{(k)}$, $F_\alpha^{(k)}$, $k < \ell / 2$ are contained in $u_q^{\ell, +}$ and $u_q^{\ell, +}$ is the Nichols algebra of type $B_2$. In the second case we check that $u_{q}^{\ell, +} \subset U_q^{\ell, +}$ is normal and thus $\mathcal{N}_q^{\ell, +} = u_{q}^{\ell, +}$ as in the case $A_2$. This is a generic argument whenever all $\ell_\alpha$ coincide: If we apply the commutation rules [Lusz90b] Sec. 5. to elements $E_\alpha^{(i)} E_\beta^{(j)}$ with $i < \ell / 2$, then the results has final exponents $< \ell / 2$, hence $u_{q}^{\ell, +} U_q^{\ell, +} = U_q^{\ell, +} u_{q}^{\ell, +}$. 
We next give the relations in the quotient $U^+$, which is generated by the images of the remaining PBW-generators $E^{(\frac{\ell}{2})}_\alpha$; especially we verify $U^+ = U(B_2)^+$. Note that by assumption $\ell$ is even, but $\frac{\ell}{2}$ is odd.

$$E_{112}^{(\frac{\ell}{2})} E_2^{(\frac{\ell}{2})} = q^{-2} E_2^{(\frac{\ell}{2})} E_{112}^{(\frac{\ell}{2})} = E_2^{(\frac{\ell}{2})} E_{112}^{(\frac{\ell}{2})}$$

$$E_{112}^{(\frac{\ell}{2})} E_1^{(\frac{\ell}{2})} = q^{-2} E_1^{(\frac{\ell}{2})} E_{112}^{(\frac{\ell}{2})} = E_1^{(\frac{\ell}{2})} E_{112}^{(\frac{\ell}{2})}$$

$$E_1^{(\frac{\ell}{2})} E_{112}^{(\frac{\ell}{2})} = q^{-2} E_{112}^{(\frac{\ell}{2})} E_1^{(\frac{\ell}{2})} = E_{112}^{(\frac{\ell}{2})} E_1^{(\frac{\ell}{2})}$$

For the more involved commutation rules, we again only calculate up to monomials ending with root vectors $E^{(n)}_\alpha$ with $0 < n < \frac{\ell}{2}$; such terms are in the kernel $U_q^{\ell^+} \ker_u(u^{\ell^+})$ of the quotient.

$$E_{112}^{(\frac{\ell}{2})} E_2^{(\frac{\ell}{2})} = \sum_{r,s,t \geq 0 \atop r+s=t=\frac{\ell}{2}} q^{-2sr-2st+2s} \left( \prod_{i=1}^{s} (q^{2-4i} - 1) \right) E_2^{(r)} E_{112}^{(s)} E_1^{(t)}$$

The product is vanishing, because $\frac{\ell}{2}$ is odd and hence there is a vanishing factor for $i = 2^{-1} \mod \frac{\ell}{2}$. Hence in the quotient $U^+$ we have

$$\Rightarrow [E_{112}^{(\frac{\ell}{2})}, E_2^{(\frac{\ell}{2})}] = 0$$

$$E_1^{(\frac{\ell}{2})} E_{12}^{(\frac{\ell}{2})} = \sum_{r,s,t \geq 0 \atop r+s=t=\frac{\ell}{2}} q^{-sr-st+s} \left( \prod_{i=1}^{s} (q^{2i} + 1) \right) E_{112}^{(r)} E_{112}^{(s)} E_1^{(t)}$$

$$\Rightarrow [E_1^{(\frac{\ell}{2})}, E_{12}^{(\frac{\ell}{2})}] = 0$$

The product is nonvanishing, because $\frac{\ell}{2}$, the order of $q^2$, is odd and hence no even power of $q$ is $-1$. Hence in the quotient $U$ we have

$$\Rightarrow [E_1^{(\frac{\ell}{2})}, E_{112}^{(\frac{\ell}{2})}] \neq 0$$
\[ E_1(\ell) E_2(\ell) = \sum_{r,s,t,u \geq 0} q^{2ru + 2rt + 2s + 2t} E_2^{(r)} E_1^{(s)} E_{12} E_1 \]

\[ = \sum_{r,s,t,u} E_2^{(r)} E_1^{(s)} + E_{12} + \ldots \]

Hence in the quotient \( U^+ \) we have

\[ [E_1(\ell), E_2(\ell)] = E_{12} \]

We have thus proven, that the Lie algebra in Theorem 5.8 \( U^+ = U(\ell,+) \) is the Borel part \( B_2^+ \) with \( E_1(\ell) \) the short root and \( E_2(\ell) \) the long root. Note that the identification has a nontrivial scalar factor \( \prod_{i=1}^{\ell} q^{2i} + 1 \) on the highest root vector \( E_{112} \). Depending on whether \( \ell = 2 \) or \( \ell \neq 2 \) we have short exact sequences of \( \Lambda \)-Yetter-Drinfel’d Hopf algebras

\[ \mathbb{C} \to U_q^C(B_2, \Lambda)^+ \xrightarrow{\text{Frob}} U(B_2)^+ \]

\[ B(B_2) \to U_q^C(B_2, \Lambda)^+ \xrightarrow{\text{Frob}} U(B_2)^+ \]

B.0.5. \( \ell \) even, \( 4|\ell \). For \( B_2 \) we can apply Theorem 5.4 without restrictions on \( \ell \). We now treat the case \( 4|\ell \). Then \( \ell_\alpha = \frac{\ell}{2} \) short roots and \( \ell_\alpha = \frac{\ell}{4} \) for long roots.

We first describe \( u_q^C \). For \( \ell = 4 \) only the short root vectors \( E_{12}^{(k)}, E_1^{(k)}, E_{12}^{(k)}, E_1^{(k)} \) with \( k < \frac{\ell}{2} = 2 \) are contained in \( u_q^C \). Even though its evident, we start by calculating the relations in \( u_q^C \) as follows:

\[ E_1 E_{12} = E_{12} E_1 + [2]_q E_{112} = E_{12} E_1 \]

\[ \Rightarrow [E_1, E_{12}] = 0 \]

\[ F_{12} F_1 = F_{12} F_1 + [2]_q F_{112} = F_{12} F_1 \]

\[ \Rightarrow [F_{12}, F_1] = 0 \]

\[ E_i F_i = F_i E_i + \begin{bmatrix} K_i; 0 \ 1 \end{bmatrix} = F_i E_i + \frac{K_i - K_i^{-1}}{q - q^{-1}} \]

\[ \Rightarrow [E_i, F_i] = \frac{K_i - K_i^{-1}}{q - q^{-1}} \]

Especially \( u_q^C \) is a small quantum group of different type:

\[ u_q^C = \langle \Lambda, E_{12}, E_1 \rangle_{\text{Alg}} = u(A_1 \times A_1) \]
For $\ell \neq 2$ all root vectors $E_{112}^{(k)}, F_u^{(k)}, k < \frac{\ell}{2}$ are contained in $u_+^{L,\ast}$ and $u_+^{L,\ast}$ is the Nichols algebra of type $B_2$.

We now show that $u_+^{L,\ast} \subset U_q^{L,\ast}$ is normal and thus $N^{L,\ast} = u_+^{L,\ast}$ as in the case $A_2$. Compared to the previous cases, this is more involved, due to the different $\ell_\alpha$. First note that root vectors $E_{1}^{(j)}, E_{12}^{(j)}$ commute up to $q$, as well as the pairs $E_{112}^{(j)}, E_{12}^{(j)}$ and $E_{1}^{(j)}, E_{112}^{(j)}$. The remaining cases depend on the commutation rules \cite[Sec. 5]{Lusz90b}.

\[
E_{112}^{(k)} E_2^{(k')} = \sum_{r,s,t \geq 0 \atop r+s=k'} q^{-2sr-2st+2s} \left( \prod_{i=1}^{s} (q^{2i} - 1) \right) E_2^{(r)} E_{12}^{(s)} E_{112}^{(t)}
\]

\[
E_1^{(k)} E_{12}^{(k')} = \sum_{r,s,t \geq 0 \atop r+s=k'} q^{-sr-st+s} \left( \prod_{i=1}^{s} (q^{2i} + 1) \right) E_{12}^{(r)} E_{112}^{(s)} E_1^{(t)}
\]

\[
E_1^{(k)} E_2^{(k')} = \sum_{r,s,t,u \geq 0 \atop r+s+t+u=k'} q^{2ru+2rt+us+2s+2t} E_2^{(r)} E_{12}^{(s)} E_{112}^{(t)} E_1^{(u)}
\]

To check normality, i.e. $\ker_e(u_+^{L,\ast}) U_q^{L,\ast} = U_q^{L,\ast}$ ker$_e(u_+^{L,\ast})$ we assume $0 < k < \frac{\ell}{4}$ for the short root $\alpha_1$ resp. $0 < k < \frac{\ell}{4}$ for the long root $\alpha_{112}$ (the latter set is empty for $\ell = 4$).

- Consider the first relation for $0 < \frac{\ell}{4}$: For $0 < t < \frac{\ell}{4}$ the last root vector $E_{112}^{(t)} \in \ker_e(u_+^{L,\ast})$ (it is long). For $t = 0$ we have $0 < s < \frac{\ell}{4}$ and the last root vector $E_{12}^{(s)} \in \ker_e(u_+^{L,\ast})$ (it is short).

- Consider the second relation for $0 < k < \frac{\ell}{2}$: For $0 < t < \frac{\ell}{2}$ the last root vector $E_1^{(t)} \in \ker_e(u_+^{L,\ast})$ (it is short). For $t = 0$ we have $0 < k < \frac{\ell}{2}$, but the last root vector $E_{112}^{(s)} \in \ker_e(u_+^{L,\ast})$ only for $s < \frac{\ell}{4}$, since it is long. We thus have to show the product vanishes for $s \geq \frac{\ell}{4}$, which follows from the factor

\[
q^{\frac{\ell}{4}} + 1 = q^{\frac{\ell}{2}} + 1 = (-1) + 1 = 0
\]

- Consider the third relation for $0 < k < \frac{\ell}{2}$: For $0 < u < \frac{\ell}{2}$ the last root vector $E_1^{(u)} \in \ker_e(u_+^{L,\ast})$ (it is short). For $u = 0$ we have $t \leq \frac{\ell}{4}$. For $0 < t < \frac{\ell}{4}$ the last root vector $E_{112}^{(u)} \in \ker_e(u_+^{L,\ast})$ (it is long). For $t = 0$ we have $0 < s < \frac{\ell}{4}$ and the last root vector $E_{112}^{(s)} \in \ker_e(u_+^{L,\ast})$ (it is long).

This concludes the proof of $u_+^{L,\ast} \subset U_q^{L,\ast}$ being normal. Hence we may apply Theorem 5.8 with $N^{L,\ast} = u_+^{L,\ast}$ yielding for $\ell = 4$ resp. $\ell \neq 4$ short exact sequences of A-Yetter-Drinfel’d Hopf algebras:
\[ B(A_1 \times A_1) \rightarrow U_q^\mathcal{L}(B_2, \Lambda)^+ \xrightarrow{\text{Frob}} U^+ = U(\mathfrak{g}^{(\ell),+}) \]

\[ B(B_2) \rightarrow U_q^\mathcal{L}(B_2, \Lambda)^+ \xrightarrow{\text{Frob}} U^+ = U(\mathfrak{g}^{(\ell),+}) \]

We next give the relations in the quotient \( U^+ \), which is generated by the images of the remaining PBW-generators

\[ E^{(\frac{\ell}{2})}_2, E^{(\frac{\ell}{2})}_{112}, E^{(\frac{\ell}{2})}_1 \]

Especially for \( \ell = 4 \) this contains the long root vectors \( E_2, E_{112} \). We directly compute the relations in the quotient \( U^+ \) and hence in \( \mathfrak{g}^{(\ell),+} \):

\[
E^{(\frac{\ell}{2})}_1 E^{(\frac{\ell}{2})}_2 = \sum_{\begin{array}{c} r,s,t,u \geq 0 \\ r+s+t=\frac{\ell}{4} \\ s+2t+u=\frac{\ell}{7} \end{array}} q^{2ru+2rt+us+2s+2t} E_2^{(r)} E_{112}^{(s)} E_1^{(u)}
\]

\[ = q^{\frac{\ell}{2}} E_2^{(\frac{\ell}{2})} E_1^{(\frac{\ell}{2})} + q^{\frac{\ell}{2}} E_{112}^{(\frac{\ell}{2})} \]

\[ \Rightarrow [E^{(\frac{\ell}{2})}_1, E^{(\frac{\ell}{2})}_2] = -E^{(\frac{\ell}{2})}_{112} \]

\[
E^{(\frac{\ell}{2})}_{112} E_2^{(\frac{\ell}{2})} = \sum_{\begin{array}{c} r,s,t \geq 0 \\ r+s+\frac{\ell}{2} = \frac{\ell}{2} \\ s+t=\frac{\ell}{4} \end{array}} q^{2sr-2st+2s} \left( \prod_{i=1}^{s} (q^{2-4i} - 1) \right) E_2^{(r)} E_{112}^{(2s)} E_1^{(t)}
\]

\[ = E_2^{(\frac{\ell}{2})} E_{112}^{(\frac{\ell}{2})} + \left( \prod_{i=1}^{\frac{\ell}{2}} (q^{2-4i} - 1) \right) E_{112}^{(\frac{\ell}{2})} \]

\[ \Rightarrow [E_{112}, E_2] = \prod_{i=1}^{\frac{\ell}{2}} \begin{pmatrix} -2 < 2-4i > 2-\ell \\ q^{2-4i} - 1 \end{pmatrix} E_{112}^{(\frac{\ell}{2})} \neq 0 \]

\[
E^{(\frac{\ell}{2})}_1 E^{(\frac{\ell}{2})}_{112} = \sum_{\begin{array}{c} r,s,t \geq 0 \\ r+s+\frac{\ell}{2} = \frac{\ell}{2} \\ s+t=\frac{\ell}{4} \end{array}} q^{-sr-st+s} \left( \prod_{i=1}^{s} q^{2i} + 1 \right) E_{112}^{(r)} E_{112}^{(s)} E_1^{(t)}
\]
Example B.1. \( \text{Evidences to this observation are given in the following example (g)} \)

\[
E_{12}^{(\frac{1}{2})} E_{1}^{(\frac{1}{2})} = E_{12}^{(\frac{1}{2})} + q^{\frac{1}{2}} \prod_{i=1}^{\ell} \left( q^{2i} + 1 \right) E_{112}^{(\frac{1}{2})},
\]

\[
\Rightarrow \left[ E_{1}^{(\frac{1}{2})}, E_{12}^{(\frac{1}{2})} \right] = 0
\]

\[
E_{12}^{(\frac{1}{2})} E_{2}^{(\frac{1}{2})} = q^{-\frac{1}{4}} E_{2}^{(\frac{1}{2})} E_{12}^{(\frac{1}{2})} \Rightarrow \left[ E_{2}^{(\frac{1}{2})}, E_{12}^{(\frac{1}{2})} \right] = 0
\]

\[
E_{112}^{(\frac{1}{2})} E_{12}^{(\frac{1}{2})} = q^{-\frac{1}{4}} E_{12}^{(\frac{1}{2})} E_{112}^{(\frac{1}{2})} \Rightarrow \left[ E_{112}^{(\frac{1}{2})}, E_{12}^{(\frac{1}{2})} \right] = 0
\]

\[
E_{1}^{(\frac{1}{2})} E_{112}^{(\frac{1}{2})} = q^{-\frac{1}{4}} E_{112}^{(\frac{1}{2})} E_{1}^{(\frac{1}{2})} \Rightarrow \left[ E_{1}^{(\frac{1}{2})}, E_{112}^{(\frac{1}{2})} \right] = 0
\]

Hence the root vectors \( E_{1}^{(\frac{1}{2})}, E_{12}^{(\frac{1}{2})}, E_{112}^{(\frac{1}{2})} \) span a Borel algebra of type \( C_{2} \), i.e. \( E_{2}^{(\frac{1}{2})} \)

is the short root vector and \( E_{1}^{(\frac{1}{2})} \) is the long root vector. Note we have again nontrivial factors in the identification.

This example somewhat suggests that the Borel part of \( g^{(0,+)} \) is related to the dual of \( g \). We conjecture this pattern to hold for arbitrary rank, see Conjecture 6.2. Further evidences to this observation are given in the following example (\( C_{n} \) for \( q = \sqrt{-1} \)) and by the discussion of \( G_{2} \) in the next section, where the more complicated isomorphism \( G_{2}^{\vee} \cong G_{2} \) appears explicitly.

Example B.1. \( \text{We want to briefly sketch the case of } C_{n} \text{ for } q = \sqrt{-1}. \) We refrain from calculating all relations in \( U_{q}^{\vee} \left( g, \Lambda \right) \) (especially normality of \( u_{q}^{\vee} \)); we rather give a direct argument, why an assumed Lie algebra Borel part \( U^{+} = U_{q}^{\vee} / u_{q}^{\vee} \) has to be of type \( B_{n} \):

- Let \( \alpha_{1} \) denote the unique short root. Then \( \ell_{1} = 2 \) and \( \ell_{i} = 1 \) for \( i \neq 1 \). Hence Lustzig’s small quantum group \( u_{q}^{\vee} \subset U_{q}^{\vee} \) is generated only by \( \Lambda \) and the \( n \) short root vectors \( E_{\alpha}, F_{\alpha} \) (presumably of type \( A_{1} \cup \cdots \cup A_{1} \)). The remaining PBW-generators are \( E_{1}^{(2)}, F_{1}^{(2)}, K_{1}^{(2)} \) and \( E_{i}, F_{i}, K_{i}^{(1)} \) for \( i \neq n \).
- Consider the subalgebra of \( U_{q}^{\vee} \) associated to the sub-Lie algebra \( C_{2} \cong B_{2} \subset g \) generated by the short simple roots \( \alpha_{1} \) and the long simple root \( \alpha_{1} \). The explicit calculation in this section shows, that \( E_{1}^{(2)}, E_{2} \) generates a Lie algebra Borel part of type \( C_{2} \cong B_{2} \), where \( E_{1}^{(2)} \) is a long root vector and \( E_{2} \) is a short root vector.
- Consider the subalgebra of \( U_{q}^{\vee} \) associated to the sub-Lie algebra \( A_{n-1} \subset g \) generated by all short simple roots \( \alpha_{i}, i \neq 1 \). We have shown in section A that the root vectors \( E_{i} \) generate the Borel part of the Lie algebra \( A_{n-1} \).
Since the longer root vector \( E_2 \) in \( C_2 \) coincides with the root vector \( E_2 \) in \( A_{n-1} \), all root vectors \( E_i, i \neq 1 \) are long and hence generate a root system of type \( B_n \).

Hence, under assumption of normality of \( u^{L, +} \subset U^{L, +} \sqrt{-1} \) we have a short exact sequence of \( \Lambda \)-Yetter-Drinfel’d Hopf algebras:

\[
\begin{align*}
&u^{L, +}((A_1 \cup \cdots \cup A_1, \Lambda)^+) \longrightarrow U^{L, +}((C_n, \Lambda)^+) \xrightarrow{Frob} U((B_n)^+). \end{align*}
\]

**Appendix C. Calculations for \( \mathfrak{g} = G_2 \)**

Let \( \mathfrak{g} = G_2 \), then

\[
\Phi^+ = \{ \alpha_2, \alpha_{12}, \alpha_{1112}, \alpha_{112}, \alpha_{1122}, \alpha_1 \} \quad (\alpha_i, \alpha_j) = \begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix}
\]

We have short roots and long roots as follows:

\[
\begin{align*}
d_{12} &= d_{112} = d_1 = 1 & d_2 &= d_{1112} = d_{1112} = 3 \\
\ell_{12} &= \ell_{112} = \ell_1 = \begin{cases} \ell & 2 \nmid \ell \\ \frac{\ell}{2} & 2 | \ell \end{cases} & \ell_2 &= \ell_{1112} = \ell_{1112} = \begin{cases} \ell & 2 \nmid \ell \\ \frac{\ell}{2} & 2 | \ell, 3 \nmid \ell \\ \frac{\ell}{3} & 2 \nmid \ell, 3 | \ell \\ \frac{\ell}{6} & 6 | \ell \end{cases}
\end{align*}
\]

We will be less elaborate in this section and concentrate on deriving our results under heavy use of the fact that by applying Theorem 5.8 we know that \( \mathfrak{g}^{(\ell), +} \) is a Lie algebra. The cases are again subdivided with respect to \( \ell \alpha \), and to the condition \( \ell \neq 2 \mod 4 \) (otherwise we are in the braided case for \( \mathfrak{g}^{(\ell), +} \)). Hence we have three cases:

- \( \ell \) odd and \( 3 \nmid \ell \), hence \( \ell = 1, 5, 7, 11 \mod 12 \).
  This is the generic case.
- \( 2 | \ell \) and \( 3 \nmid \ell \), hence \( \ell = 4, 8 \mod 12 \).
  This will turn out to be the most exotic case with \( u^{L, +} = B(A_3) \) for \( \ell = 4 \).
- \( 2 \nmid \ell \) and \( 3 | \ell \) as well as \( 6 | \ell \), hence \( \ell = 0, 3, 9 \mod 12 \).
  We treat these two cases together, as both have \( \ell_{\text{short}}/\ell_{\text{long}} = 3 \). This will be similar to the case \( B_2, 2 | \ell \) and again recovers the dual \( G_2 \rightsquigarrow G_2' \).

**C.0.6. \( \ell \) odd and not divisible by \( 3 \).** For \( 2 \nmid \ell \) and \( 3 \nmid \ell \) we get the generic case. Here, the Frobenius homomorphism in [Lusz90b] Thm 8.10 implies (for \( \Lambda_R = \Lambda \)) a short exact sequence of Hopf algebras

\[
\begin{align*}
&u_q^L \longrightarrow U_q^L(G_2, \Lambda_R) \xrightarrow{\text{Frob}} U(G_2) \end{align*}
\]

We have all \( \ell_i = \ell \), so for \( u_q^L \) we get two cases \( \ell = 1 \) and \( \neq 1 \):
For $\ell = 1$ no root vectors $E^{(k)}_\alpha, F^{(k)}_\alpha$ are contained in $u^C_q$, hence $u^C_q = \mathbb{C}[\Lambda]$. Especially on the level of Borel subalgebras we get a short exact sequence of $\Lambda$-Yetter-Drinfel’d modules

$$\mathbb{C} \longrightarrow U^C_q(G_2, \Lambda) \overset{\text{Frob}}{\longrightarrow} U(G_2)^+$$

For $\ell \neq 1$ all root vectors $E^{(k)}_\alpha, F^{(k)}_\alpha, k < \ell$ are contained in $u^C_q$ and $u^{C,+}$ is the Nichols algebra of type $G_2$. We get a short exact sequence of $\Lambda$-Yetter-Drinfel’d Hopf algebras

$$B(G_2) \longrightarrow U^C_q(G_2, \Lambda) \overset{\text{Frob}}{\longrightarrow} U(G_2)^+$$

C.0.7. $\ell = 4,8 \mod 12$. This is the case $2|\ell$ and $3 \not| \ell$, together with the restriction $\ell \neq 2 \mod 4$ in Theorem 5.4. Especially $4|\ell$. We have $\ell_\alpha = \frac{\ell}{2}$ for short and long roots.

We first describe $u^{C,+}$, which shows a rather exotic phenomenon: By definition, it is generated by all $E^{(k)}_\alpha, k < \ell_\alpha$, which are precisely the root vectors $E_\alpha$ with self-braiding $q_{\alpha\alpha} = q^{2\ell_{\alpha}} = q^\ell_{\alpha} \neq 1$ and hence finite order. As always, $u^{C,+}$ contains the Nichols algebra generated by the braided vector space of all simple root vectors $E_\alpha$, and in all previous cases the both turned out to coincide.

In the present case, by $\ell_\alpha \neq 1$, all simple roots $E_\alpha$ are contained in $u^{C,+}$. However, the Nichols algebra in [Heck06] Figure 1 Row 11 has an excluded case for $q_{11} = -1$, i.e. $\ell = 4, \ell_\alpha = 2$. We make this more explicit: The Nichols algebra spanned by the simple root vectors $E_\alpha$ depends on their braiding matrix (see proof of Theorem 5.8)

$$q_{ij} := q^{(\alpha_i, \alpha_j)}$$

Usually this is of type $\mathfrak{g}$ and hence generates the entire $u^C_q$. If we spell out the braiding matrix for type $G_2$ for arbitrary $q$ and especially for $\ell = 4, q = \sqrt{-1}$ we get

$$\begin{pmatrix}
q^2 & q^{-3} \\
q^{-3} & q^6
\end{pmatrix} \begin{pmatrix}
-1 & \sqrt{-1} \\
\sqrt{-1} & -1
\end{pmatrix}$$

The latter braiding matrix generates a Nichols algebra of type $A_2$ and dimension $2^{|\Phi^+|} = 2^3$, which is why the case $q_{11} = -1$ is excluded in Heckenberger list for $G_2$. On the other hand the dimension of $u^C_q$ is by Lusztig’s result still $\prod_\alpha \ell_\alpha = 2^6$, hence the two do not coincide as in all cases before. As we shall explicitly calculate in a moment, we get the surprising result $u^C_q \cong B(A_3)$ generated by “simple” root vectors $E_1, E_2, E_{112}$, yielding also the right dimension $2^6$. For $\ell \neq 4$ on the other hand we again have $u^{C,+} = B(G_2)$.

The following discussions are independent of this inner structure of $u^{C,+}$. First, the fact that all $\ell = \frac{\ell}{2}$ immediately shows the normality of $u^C_q$ as in the case $B_2, 2|\ell, 4 \not| \ell$. We now calculate some relations between the $E^{(k)}_\alpha$. This is more tedious than in the previous cases, due to the more involved structure of $G_2$ and to the fact that Lusztig does not give
explicit formulae in this case. Hence we have to use solely the following commutation rule [Lusz90b] Sec. 5.:

$$E_1^{(k)} E_2^{(k')} = \sum_{\substack{p,q,r,s,t,u \geq 0 \\ p+q+2r+s+t=2k' \\ q+3r+2s+3t+u=k}} 3uq+2aq+3ar+ux+6tp+3tq+3tr+q +3sp+sq+3rp+3q+6r+4s+3t E_2^{(p)} E_1^{(q)} E_{1122} E_{112} E_{1112} E_1^{(u)}$$

We will need the following three relations in the quotient $U^+$, i.e. up to elements $U_q^{\mathcal{L}^+} \ker_\omega(\mathcal{L}^+)$. This means that (using normality) we can neglect monomials containing a factor $E_1^{(n)}$ with $0 < n < \frac{\ell}{2}$.

- Apply the commutation formula to the term $E_1^{(n)} E_2^{(n)}$. If $0 < u < \frac{\ell}{2}$, the summand is in the kernel.
  - Let $u = \frac{\ell}{2}$, then the second sum restriction $q + 3r + 2s + 3t + u = \frac{\ell}{2}$ shows $q = r = s = t = 0$ and the first sum restriction $p + q + 2r + s + t = \frac{\ell}{2}$ shows $p = \frac{\ell}{2}$. We hence yield a summand (note $4|\ell$ implies $q^2 \ell^2 = 1$)
    $$q^2 \ell^2 E_2^{(\frac{\ell}{2})} E_1^{(\frac{\ell}{2})} = E_2^{(\frac{\ell}{2})} E_1^{(\frac{\ell}{2})}$$
  - Let $u = 0$, then the second sum restriction $q + 3r + 2s + 3t + u = \frac{\ell}{2}$ shows $t \leq \frac{\ell}{6}$. The only summands not in the kernel have thus $t = 0$. Similarly, $s = 0$ and $r = 0$, hence $q = \frac{\ell}{2}$. The first sum restriction $p + q + 2r + s + t = \frac{\ell}{2}$ then shows also $p = 0$. We hence yield a summand
    $$q^3 \ell^2 E_2^{(\frac{\ell}{2})} E_1^{(\frac{\ell}{2})} = -E_1^{(\frac{\ell}{2})}$$
  \[\Rightarrow \quad E_1^{(\frac{\ell}{2})} E_2^{(\frac{\ell}{2})} = E_2^{(\frac{\ell}{2})} E_1^{(\frac{\ell}{2})} - E_1^{(\frac{\ell}{2})}\]

- Apply the commutation formula to the term $E_1^{(n)} E_2^{(n)}$. If $u \neq 0, \frac{\ell}{2}, \ell$, the summand is in the kernel.
  - Let $u = \ell$, then the second sum restriction $q + 3r + 2s + 3t + u = \ell$ shows $q = r = s = t = 0$ and the first sum restriction $p + q + 2r + s + t = \frac{\ell}{2}$ shows $p = \frac{\ell}{2}$. We hence yield a summand
    $$q^3 \ell^2 E_2^{(\frac{\ell}{2})} E_1^{(\frac{\ell}{2})} = E_2^{(\frac{\ell}{2})} E_1^{(\frac{\ell}{2})}$$
  - Let $u = \frac{\ell}{2}$, then the second sum restriction $q + 3r + 2s + 3t + u = \ell$ shows $t \leq \frac{\ell}{6}$. The only summands not in the kernel have thus $t = 0$. Similarly, $s = r = 0$, which shows $q = \frac{\ell}{2}$. The first sum restriction shows $p = 0$. We hence yield a summand
    $$q^2 \ell^2 + 3 \ell^2 E_1^{(\frac{\ell}{2})} E_2^{(\frac{\ell}{2})} = -E_1^{(\frac{\ell}{2})} E_2^{(\frac{\ell}{2})}$$
- Let \( u = 0 \), then the second sum restriction \( q + 3r + 2s + 3t + u = \ell \) shows \( t \leq \frac{\ell}{3} \). The only summands not in the kernel have thus \( t = 0 \). The second sum restriction shows \( s \leq \frac{\ell}{2} \). If \( s \neq \frac{\ell}{2} \), the summand is in the kernel. Let \( s = \frac{\ell}{2} \), then \( q = 0 \) and the first sum restriction shows \( p = 0 \). We hence yield a summand

\[
q^{4}E_{1112}^{(\frac{\ell}{2})} = E_{1112}^{(\frac{\ell}{2})}
\]

Let \( s = 0 \), then the second sum restriction shows \( r \leq \frac{\ell}{3} \). The only summands not in the kernel have thus \( r = 0 \) which shows \( q = \ell \). But this violates the first sum restriction \( p + q + 2r + s + t = \frac{\ell}{2} \).

\[
\Rightarrow \quad E_{1}^{(\ell)}E_{2}^{(\frac{\ell}{2})} = E_{1}^{(\frac{\ell}{2})}E_{1}^{(\ell)} - E_{12}^{(\frac{\ell}{2})}E_{1}^{(\ell)} + E_{1112}^{(\frac{\ell}{2})}
\]

- Apply the commutation formula to the term \( E_{1}^{(\frac{\ell}{3})}E_{2}^{(\frac{\ell}{2})} \). If \( u 
eq 0, \frac{\ell}{2}, \frac{3\ell}{2} \), the summand is in the kernel.

- Let \( u = \frac{3\ell}{2} \), then the second sum restriction shows \( q + 3r + 2s + 3t + u = \frac{\ell}{2} \) shows \( q = r = s = t = 0 \) and the first sum restriction \( p + q + 2r + s + t = \frac{\ell}{2} \) shows \( p = \frac{\ell}{2} \). We hence yield a summand

\[
q^{3}E_{2}^{(\frac{\ell}{2})}E_{1}^{(\frac{\ell}{2})} = E_{2}^{(\frac{\ell}{2})}E_{1}^{(\frac{\ell}{2})}
\]

- Let \( u = \ell \), then the second sum restriction shows \( t \leq \frac{\ell}{6} \). The only summands not in the kernel have thus \( t = 0 \). Similarly, \( r = s = 0 \) and \( q = \frac{\ell}{2} \), and the first sum restriction shows \( p = 0 \). We hence yield a summand

\[
q^{2}E_{12}^{(\frac{\ell}{2})}E_{1}^{(\ell)} = -E_{12}^{(\frac{\ell}{2})}E_{1}^{(\ell)}
\]

- Let \( u = \frac{\ell}{2} \), then the second sum restriction shows \( t \leq \frac{\ell}{3} \). The only summands not in the kernel have thus \( t = 0 \). The second sum restriction shows \( s \leq \frac{\ell}{2} \). If \( s \neq 0, \frac{\ell}{2} \), the summand is in the kernel. Let \( s = \frac{\ell}{2} \), then the first sum restriction shows \( q = r = 0 \) and the first sum restriction shows \( p = 0 \). We hence yield a summand

\[
q^{3}E_{112}^{(\frac{\ell}{2})}E_{1}^{(\ell)} = E_{112}^{(\frac{\ell}{2})}E_{1}^{(\ell)}
\]

Let \( s = 0 \), then the second sum restriction shows \( r < \frac{\ell}{3} \). The only summands not in the kernel have thus \( r = 0 \) and the second sum restriction shows \( q = \ell \), which violates the first sum restriction.

- Let \( u = 0 \), then the second sum restriction shows \( t \leq \frac{3\ell}{6} \). If \( t \neq 0, \frac{\ell}{2} \), the summand is in the kernel. Let \( t = \frac{\ell}{2} \), then the second sum restriction shows \( q = r = s = 0 \) and the first sum restriction shows \( p = 0 \). We hence yield a summand

\[
q^{3}E_{1112}^{(\frac{\ell}{2})} = -E_{1112}^{(\frac{\ell}{2})}
\]
Let $t = 0$, then the second sum restriction shows $s \leq \frac{3t}{4}$. If $s \neq 0$, $t$, the summand is in the kernel. Let $s = \frac{t}{2}$, then the second sum restriction shows $r < \frac{t}{6}$. The only summands not in the kernel have $r = 0$. The second sum restriction shows $q = \frac{t}{2}$, which violates the first sum restriction. Let $s = 0$, then the first sum restriction shows $r < \frac{3t}{6}$. The only summands not in the kernels are $r = \frac{t}{2}, q = 0$ and $r = 0, q = \frac{3t}{2}$, which both violate the first sum restriction.

$$E_1^{(I_1^r)} E_2^{(I_2)} = E_2^{(I_1^r)} E_1^{(I_2)} E_1^{(I_1^r)} - E_1^{(I_1^r)} E_1^{(I_1^r)} - E_1^{(I_1^r)} E_1^{(I_1^r)}$$

We calculate using $q^{t/2} = -1$ and $4|\ell$ that (see ??):

$$\left[ \frac{n \ell}{2} \right]_q = n! \cdot \left( \left[ \frac{\ell}{2} \right]_q \right)^n \quad E_n^{(I_1^r)} = \frac{1}{n!} \left( E_n^{(I_1^r)} \right)^n$$

This simplifies the above formulae further:

$$E_1^{(I_1^r)} E_2^{(I_2)} = E_2^{(I_2)} E_1^{(I_1^r)} - E_1^{(I_1^r)}$$

$$\left( E_1^{(I_1^r)} \right)^2 E_2^{(I_2)} = E_2^{(I_2)} \left( E_1^{(I_1^r)} \right)^2 - 2E_1^{(I_1^r)} E_1^{(I_1^r)} + 2E_1^{(I_1^r)}$$

These relations can be used as follows to prove $g^{(I_1^r)} = G_2^+$ with $E_1^{(I_1^r)}$ the short root and $E_2^{(I_1^r)}$ the long root.

$$[E_1^{(I_1^r)}, E_2^{(I_2)}] = E_1^{(I_1^r)} E_2^{(I_2)} - E_2^{(I_2)} E_1^{(I_1^r)}$$

$$= E_2^{(I_2)} E_1^{(I_1^r)} - E_2^{(I_2)} E_1^{(I_1^r)} - E_1^{(I_1^r)}$$

$$[E_1^{(I_1^r)}, [E_1^{(I_1^r)}, E_2^{(I_2)}]] = \left( E_1^{(I_1^r)} \right)^2 E_2^{(I_2)} - 2E_1^{(I_1^r)} E_2^{(I_2)} E_1^{(I_1^r)} + E_2^{(I_2)} \left( E_1^{(I_1^r)} \right)^2$$

$$= E_2^{(I_2)} \left( E_1^{(I_1^r)} \right)^2 - 2E_1^{(I_1^r)} E_1^{(I_1^r)} + 2E_1^{(I_1^r)}$$

$$- 2E_2^{(I_2)} \left( E_1^{(I_1^r)} \right)^2 + 2E_1^{(I_1^r)} E_1^{(I_1^r)} + E_2^{(I_2)} \left( E_1^{(I_1^r)} \right)^2 = 2E_1^{(I_1^r)}$$

$$[E_1^{(I_1^r)}, [E_1^{(I_1^r)}, [E_1^{(I_1^r)}, E_2^{(I_2)}]]] = \left( E_1^{(I_1^r)} \right)^3 E_2^{(I_2)} - 3 \left( E_1^{(I_1^r)} \right)^2 E_2^{(I_2)} E_1^{(I_1^r)}$$

$$+ 3E_1^{(I_1^r)} E_2^{(I_2)} \left( E_1^{(I_1^r)} \right)^2 - E_2^{(I_2)} \left( E_1^{(I_1^r)} \right)^3$$

$$= E_2^{(I_2)} \left( E_1^{(I_1^r)} \right)^3 - 3E_1^{(I_1^r)} E_1^{(I_1^r)} + 6E_1^{(I_1^r)} E_1^{(I_1^r)} - 6E_1^{(I_1^r)}$$
follows immediately because 

We finally check that modulo $u^{t+}$ we have $[E_1^{(2)}, E_{1112}] 
eq 0$ (actually $E_1^{(2)} E_{1112}$). But this follows immediately because $E_{1112}$ appears in the commutation formula of $E_1^{(2)} E_{1112}$ for $r = \frac{\ell}{2}$. This concludes the determination of $g^{(\ell)}$ in the case $\ell = 4, 8 \mod 12$ and shows that we get a short exact sequence of Yetter-Drinfel’d Hopf algebras for $\ell = 4$ resp. $\ell = 8$:

$$B(A_3) \longrightarrow U^{\mathbb{L}}_{\sqrt{-1}}(G_2, \Lambda)^+ \overset{\text{Frob}}{\longrightarrow} U(G_2)^+$$

$$B(G_2) \longrightarrow U^{\mathbb{L}}_{\mathbb{Q}}(G_2, \Lambda)^+ \overset{\text{Frob}}{\longrightarrow} U(G_2)^+$$

C.0.8. $\ell = 0, 3, 9 \mod 12$. This are the cases $2 \nmid \ell$, $3 | \ell$ as well as $6 | \ell$, together with the restriction $\ell \neq 2 \mod 4$ in Theorem 5.4. Especially $4 | \ell$. We have for short roots $\ell_\alpha = \ell$, resp. $\ell_\alpha = \frac{\ell}{2}$ and for long roots $\ell_\alpha = \frac{\ell}{6}$ resp. $\ell_\alpha = \frac{\ell}{5}$. In both cases $\ell_{short}/\ell_{long} = 3$. We again use solely the relation

$$E_1^{(k)} E_2^{(k')} = \sum_{p, q, r, s, t \geq 0} \sum_{u=0}^{3up+2rq+3ur+us+6tp+3q+3tr+q+3p+3q+3r+6s+3t} E_1^{(p)} E_2^{(q)} E_3^{(s)} E_4^{(t)} E_5^{(u)}$$

We first describe $u_q^{\mathbb{L}+}$: For $\ell_2 = 1$, i.e. $\ell = 3$ ($\ell = 6$ is excluded) $u_q^{\mathbb{L}+}$ contains only the short root vectors $E_1, E_{112}, E_{1112}$, generating clearly a Nichols algebra of type $A_3$. For $\ell \neq 3$ by Heckenberger’s list $u_q^{\mathbb{L}+}$ is a Nichols algebra of type $G_2$.

We first show normality of $u_q^{\mathbb{L}+}$ and in view of the later calculations it is sufficient to check the action of $E_1^{(l_1)}, E_2^{(l_2)}$ on $u_q^{\mathbb{L}+}$. We treat both cases separately:

- For $\ell \neq 3$ it suffices to check the action of $E_1^{(l_1)}, E_2^{(l_2)}$ on $E_1, E_2$ generating $u_q^{\mathbb{L}+}$: It is clear that $E_1^{(l_1)}$ normalizes $E_1$ as well as $E_2^{(l_2)}$ normalizes $E_2$. Also, $E_2^{(l_2)} E_1^{(k)}$ yields except $E_2^{(l_2)}$ only terms involving powers $\ell_2$; since $\ell_2$ is the minimal value of $\ell_\alpha$ there can be no terms $E_1^{(\ell_\alpha)}$. The case that has to be more thoroughly checked is $E_1^{(l_1)} E_2$, which could a-priori contain terms $E_1^{(\ell_\alpha)}$ for a long root $\alpha_1, \alpha_{1112}, \alpha_{1112}$.

- For $\ell = 3$ i.e. $\ell_1 = 3, \ell_2 = 1$ it suffices to check the action of $E_1^{(3)}, E_2$ on $E_1, E_{112}$ generating $u_q^{\mathbb{L}+}$: It is clear that $E_1^{(3)}$ normalizes $E_1$ as well as $E_2$ normalizes
We make excessive use of the commutation formula: that as in the case calculate the Lie algebra By normality Theorem 5.8 states that especially the adjoint action of \( E \) involving we hence find altogether: We rewrite \( E_{12} = E_1 E_2 - E_2 E_1 \) and evaluate \( E^{(3)}_1 E_{12} \) as follows. We first evaluate the commutation formula for \( E^{(3)}_2 E_2 \) and \( E^{(4)}_1 E_2 \):

\[
E^{(3)}_1 E_2 = \sum_{p=1, u=3}^{q=7} q^p E^p_{12} E_1^{(3)} + q^s E^{s, s}_1 E_1 + q^t E^{t, t}_1112
\]

\[
= E^2 E^{(3)}_1 + q E_{12} E^{(2)}_1 + q^2 E_{12} E_1 + E_{1112}
\]

\[
E^{(4)}_1 E_2 = \sum_{p=1, u=4}^{q=8} q^p E^p_{12} E_1^{(4)} + q^s E^{s, s}_1 E_1 + q^t E^{t, t}_1112
\]

\[
= E^2 E^{(4)}_1 + E_{12} E^{(3)}_1 + E_{1112} E_1 + E_{1112} E_1
\]

Then we calculate the following \( q \)-binomials:

\[
\binom{4}{3} q = 1 \quad \binom{3}{2} q = 0 \quad \binom{2}{1} q = -1
\]

We hence find altogether:

\[
E^{(3)}_1 E_{12} = E^{(3)}_1 E_1 E_2 - E^{(3)}_2 E_2 E_1
\]

\[
= E^2 E^{(3)}_1 + E_{12} E^{(3)}_1 + E_{1112} E_1 + E_{1112} E_1
\]

\[
- E^2 E^{(3)}_1 - q E_{12} \cdot 0 - q^2 E_{112} (-E^{(2)}_1) - E_{1112} E_1
\]

\[
= E_{12} E^3 - q E_{112} E^{2}_1
\]

Especially the adjoint action of \( E^{(3)}_1 \) sends \( E_{12} \) to \(-qE_{112}E^2_1 \in u_C^{++}\) (without involving \( E_{1112} \)). This concludes the proof of normality.

By normality Theorem 5.8 states that \( U_q(G_2)/u_C^{++} = U(\mathfrak{g}^{(3), +}) \). We finally want to calculate the Lie algebra \( \mathfrak{g}^{(3), +} \) generated by \( E^{(3)}_1, E^{(2)}_2 \) modulo \( u_C^{++} \). In fact, we shall verify that as in the case \( B_2, 2|6 \) we get a long root \( E^{(3)}_1 \) and a short root \( E^{(2)}_2 \) for \( G_2^\vee \cong G_2 \).

We make excessive use of the commutation formula:

- We calculate \( E^{(1)}_1 E^{(2)}_2 \) modulo \( u_C^{++} \), where by assumption \( \ell_1 = 3\ell_2 \). By the commutation formula we have monomials with

\[
p + q + 2r + s + t = \ell_2
\]

\[
q + 3r + 2s + 3t + u = \ell_1
\]
and these are nonzero in the quotient if \( \ell_2 | p, r, t \) and \( \ell_1 | q, s, u \). The first equation leaves only \( p = \ell_2 \) or \( t = \ell_2 \) and the other terms zero. The second equation returns unique solutions \( p = \ell_2, u = \ell_1 \) resp. \( t = \ell_2, u = 0 \). Hence:

\[
E_1^{(\ell_1)} E_2^{(\ell_2)} = q^{3\ell_1 \ell_2} E_2^{(\ell_2)} E_1^{(\ell_1)} \frac{E_1^{(\ell_1)} E_2^{(\ell_2)}}{p=\ell_2, u=\ell_1} + q^{3\ell_2} E_2^{(\ell_2)} E_1^{(\ell_1)} \frac{E_1^{(\ell_1)} E_2^{(\ell_2)}}{t=\ell_2, u=0}
\]

By assumption either \( \ell_1 = \ell, \ell_2 = \frac{\ell}{2} \) for odd \( \ell \) or \( \ell_1 = \frac{\ell}{2}, \ell_2 = \frac{\ell}{5} \) for \( 4 | \ell \). In both cases \( \ell | 3 \ell_1 \ell_2 \) (but not in the excluded braided case \( \ell = 2 \mod 4 \)) as well as \( q^{\ell_1} = (-1)^{\ell+1} \). Hence

\[
\Rightarrow \quad [E_1^{(\ell_1)}, E_2^{(\ell_2)}] = (-1)^{\ell+1} E_2^{(\ell_2)} E_1^{(\ell_1)} \mod uC^+.
\]

• We calculate \( E_1^{(\ell_1)} E_2^{(2\ell_2)} \) modulo \( uC^+ \). By the commutation formula we have monomials with

\[
p + q + 2r + s + t = 2\ell_2, \\
q + 3r + 2s + 3t + u = \ell_1
\]

and these are nonzero in the quotient if \( \ell_2 | p, r, t \) and \( \ell_1 | q, s, u \). The first equation leaves the possible solutions \( p = 2\ell_2 \) or \( t = 2\ell_2 \) or \( r = 2\ell_2 \) or \( t = 2\ell_2 \) or \( p = \ell_2, t = \ell_2 \) and the other terms respectively zero. The second equation yields unique solutions \( p = 2\ell_2, u = \ell_1 \) or \( r = \ell_2 \) or \( p = \ell_2, t = \ell_2 \), whereas \( t = 2\ell_2 \) yields a contradiction.

Altogether we find:

\[
E_1^{(\ell_1)} E_2^{(2\ell_2)} = \frac{q^{6\ell_1 \ell_2} E_2^{(2\ell_2)} E_1^{(\ell_1)}}{p=2\ell_2, u=\ell_1} + \frac{q^{6\ell_2} E_1^{(\ell_1)} E_2^{(2\ell_2)}}{r=\ell_2} + \frac{q^{6\ell_2 \ell_2 + 3\ell_2} E_2^{(\ell_2)} E_1^{(\ell_1)}}{p=\ell_2, t=\ell_2}
\]

\[
= E_2^{(2\ell_2)} E_1^{(\ell_1)} + E_1^{(\ell_1)} E_2^{(2\ell_2)} + (-1)^{\ell+1} E_2^{(\ell_2)} E_1^{(\ell_1)}
\]

Using Lemma \( \text{L} \), this allows us to calculate the double commutator

\[
[E_2^{(\ell_2)}, [E_2^{(\ell_2)}, E_1^{(\ell_1)}]] = \left( E_2^{(\ell_2)} \right)^2 E_1^{(\ell_1)} - 2 E_2^{(\ell_2)} E_1^{(\ell_1)} E_2^{(\ell_2)} + E_1^{(\ell_1)} \left( E_2^{(\ell_2)} \right)^2
\]

\[
= 2 E_2^{(2\ell_2)} E_1^{(\ell_1)} + 2(-1)^{\ell} E_2^{(\ell_2)} E_2^{(2\ell_2)} E_1^{(\ell_1)} - 4 E_2^{(\ell_2)} E_1^{(\ell_1)}
\]

\[
+ 2 E_2^{(2\ell_2)} E_1^{(\ell_1)} + 2 E_1^{(\ell_2)} E_2^{(2\ell_2)} E_1^{(\ell_1)} + 2(-1)^{\ell+1} E_2^{(\ell_2)} E_2^{(2\ell_2)} E_1^{(\ell_1)}
\]

\[
= 2 E_1^{(\ell_1)} E_2^{(2\ell_2)}
\]

• We calculate \( E_1^{(\ell_1)} E_2^{(3\ell_2)} \) modulo \( uC^+ \). By the commutation formula we have monomials with

\[
p + q + 2r + s + t = 3\ell_2 = \ell_1, \\
q + 3r + 2s + 3t + u = \ell_1
\]

and these are nonzero in the quotient if \( \ell_2 | p, r, t \) and \( \ell_1 | q, s, u \). The first equation leaves the possible solutions \( p = 3\ell_2 \) or \( p = \ell_2, r = \ell_2 \) or \( p = 2\ell_2, t = \ell_2 \) or
\[ p = \ell_2, t = 2\ell_2 \text{ or } r = \ell_2, t = 2\ell_2 \text{ or } t = 3\ell_2 \text{ or } q = \ell_1 \text{ or } s = \ell_1 \text{ and the other terms respectively zero. The second equation yields then 3 solutions, namely}

\[ p = 3\ell_2, u = \ell_1 \text{ or } p = \ell_2, r = \ell_2 \text{ or } p = 2\ell_2, t = \ell_2 \text{ or } q = \ell_1. \text{ Altogether we find:}

\[ E_1^{(\ell_1)} E_2^{(3\ell_2)} =
\]

\[ = q^{6\ell_2} E_2^{(3\ell_2)} E_1^{(\ell_1)} + q^{6\ell_2 + 3\ell_2} E_2^{(3\ell_2)} E_1^{(\ell_1)} + q^{12\ell_2 + 3\ell_2} E_2^{(3\ell_2)} E_1^{(\ell_1)} + q^{3\ell_1} E_2^{(1)}
\]

\[ = E_2^{(3\ell_2)} E_1^{(\ell_1)} + E_2^{(3\ell_2)} E_1^{(\ell_1)} + (-1)^{\ell_1 + 1} E_2^{(2\ell_2)} E_1^{(\ell_1)} + (-1)^{\ell_1 + 1} E_2^{(1)}
\]

Using Lemma 2.8 this allows us to calculate the triple commutator

\[ [E_2^{(\ell_2)}, [E_2^{(\ell_2)}, [E_2^{(\ell_2)}, E_1^{(\ell_1)}]] =
\]

\[ = (E_2^{(\ell_2)})^3 E_1^{(\ell_1)} - 3 (E_2^{(\ell_2)})^2 E_1^{(\ell_1)} E_2^{(\ell_2)} + 3 E_2^{(\ell_2)} E_1^{(\ell_1)} E_2^{(\ell_2)} - E_1^{(\ell_1)} (E_2^{(\ell_2)})^3
\]

\[ = 6 E_2^{(3\ell_2)} E_1^{(\ell_1)} - 6 E_2^{(2\ell_2)} E_1^{(\ell_1)} E_2^{(\ell_2)} + 6 E_2^{(2\ell_2)} E_1^{(\ell_1)} E_2^{(\ell_2)} - 6 E_1^{(\ell_1)} E_2^{(3\ell_2)}
\]

\[ = 6 E_2^{(3\ell_2)} E_1^{(\ell_1)} - 6 E_2^{(2\ell_2)} E_1^{(\ell_1)} E_2^{(\ell_2)} + 6 E_1^{(\ell_1)} E_2^{(3\ell_2)}
\]

\[ + 6 E_2^{(2\ell_2)} E_1^{(\ell_1)} + 6 E_2^{(\ell_2)} E_1^{(\ell_1)} E_2^{(\ell_2)} + 6 E_1^{(\ell_1)} E_2^{(3\ell_2)}
\]

\[ - 6 E_2^{(2\ell_2)} E_1^{(\ell_1)} - 6 E_2^{(\ell_2)} E_1^{(\ell_1)} E_2^{(\ell_2)} - 6 E_1^{(\ell_1)} E_2^{(3\ell_2)} - 6 (-1)^{\ell_1} E_2^{(2\ell_2)} E_1^{(\ell_1)}
\]

\[ = -6 (-1)^{\ell_1} E_2^{(2\ell_2)} E_1^{(\ell_1)}
\]

• The previous calculations established the triple root string in \( G_2 \) as (up to scaling)

\[ E_1^{(\ell_1)}, E_2^{(\ell_2)}, E_2^{(3\ell_2)}, E_1^{(\ell_1)}
\]

We finally check that modulo \( u^{\ell, +} \) we have \([E_1^{(\ell_1)}, E_1^{(\ell_1)}] \neq 0 \) (actually \( E_1^{(\ell_1)} \)).

But this follows immediately from \( E_{12} - E_1 E_2 - E_2 E_1 \). This concludes the determination of \( g^{(\ell, +)} \).

The above discussion of the case \( \ell = 0, 3, 9 \mod 12 \) shows that we get a short exact sequence of \( \Lambda \)-Yetter-Drinfel’d Hopf algebras for \( \ell = 3 \) resp. \( \ell \neq 3 \):

\[ \mathcal{B}(A_2) \longrightarrow U_q^\mathcal{L}(G_2, \Lambda)^+ \xrightarrow{\text{Frob}} U(G_2)^+
\]

\[ \mathcal{B}(G_2) \longrightarrow U_q^\mathcal{L}(G_2, \Lambda)^+ \xrightarrow{\text{Frob}} U(G_2)^+
\]

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