THE EXISTENCE OF BOUND STATES IN A SYSTEM OF THREE PARTICLES IN AN OPTICAL LATTICE

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ABSTRACT. We consider the hamiltonian $H_{\mu}, \mu \in \mathbb{R}$ of a system of three-particles (two identical fermions and one different particle) moving on the lattice $\mathbb{Z}^d, d = 1, 2$ interacting through repulsive ($\mu > 0$) or attractive ($\mu < 0$) zero-range pairwise potential $\mu V$. We prove for any $\mu \neq 0$ the existence of bound state of the discrete three-particle Schrödinger operator $H_{\mu}(K), K \in T^d$ being the three-particle quasi-momentum, associated to the hamiltonian $H_{\mu}$.

1. INTRODUCTION

The main goal of the paper is to prove the existence of bound states of the three-particle Schrödinger operator $H_{\mu}(K), K \in T^d$ associated to a system of three particles (two identical fermions and one different particle) for all non-zero interactions $\mu \in \mathbb{R}$.

Throughout physics, stable composite objects are usually formed by way of attractive forces, which allow the constituents to lower their energy by binding together. Repulsive forces separate particles in free space. However, in structured environment such as a periodic potential and in the absence of dissipation, stable composite objects can exist even for repulsive interactions that arise from the lattice band structure [10].

The Bose-Hubbard model which is used to describe the repulsive pairs is the theoretical basis for applications. The work [10] exemplifies the important correspondence between the Bose-Hubbard model [5, 6] and atoms in optical lattices, and helps pave the way for many more interesting developments and applications [11]. Stable repulsively bound objects should be viewed as a general phenomenon and their existence will be ubiquitous in cold atoms lattice physics. They give rise also to new potential composites with fermions [12] or Bose-Fermi mixtures [13], and can be formed in an analogous manner with more than two particles.

Cold atoms loaded in an optical lattice provide a realization of a quantum lattice gas. The periodicity of the potential gives rise to a band structure for the dynamics of the atoms. The dynamics of the ultracold atoms loaded in the lower band is well described by the Bose-Hubbard hamiltonian.

In the continuum case due to rotational invariance the hamiltonian separates in a free hamiltonian for the center of mass $H_{\text{free}}$ and in a hamiltonian $H_{\text{rel}}$ for the relative motion. Bound states are eigenstates of $H_{\text{rel}}$.

The fundamental difference between the discrete and continuous multiparticle Schrödinger operators is that in the discrete case the kinetic energy operator is not rotationally invariant.
One can rather resort to a Floquet-Bloch decomposition. The three-particle Hilbert space $\mathcal{H} \equiv l^2(\mathbb{Z}^d)^3$ is represented as a direct integral associated to the representation of the discrete group $\mathbb{Z}^d$ by shift operators

$$L^2([\mathbb{Z}^d]^3) = \int_{K \in \mathbb{T}^d} \oplus L^2([\mathbb{Z}^d]^3)\eta(dK),$$

where $\eta(dp) = \frac{dp}{(2\pi)^d}$ is the (normalized) Haar measure on the torus $\mathbb{T}^d$. Hence the total three-body Hamiltonian appears to be decomposable

$$H = \int_{\mathbb{T}^d} \oplus H(K)\eta(dK).$$

The fiber hamiltonian $H(K)$ depends parametrically on the quasi-momentum $K \in \mathbb{T}^d \equiv \mathbb{R}^d/(2\pi\mathbb{Z}^d)$. It is the sum of a free part and an interaction term, both bounded and the dependence on $K$ is continuous. Bound states $\psi_{E,K}$ are solutions of the Schrödinger equation

$$H(K)\psi_{E,K} = E\psi_{E,K}, \quad \psi_{E,K} \in l^2(\mathbb{Z}^d)^2.$$

In this work we consider the hamiltonian $h_\mu$ of a system of two-particles (a fermion and different particle) and the hamiltonian $H_\mu$ of a system of three-particles (two identical fermions and one different particle) on the lattice $\mathbb{Z}^d$, $d = 1, 2$ interacting through zero-range potential $\mu V$.

We denote by $h_\mu(k), k \in \mathbb{T}^d$ and $H_\mu(K), K \in \mathbb{T}^d, d = 1, 2$ the Schrödinger operators corresponding to the hamiltonians $h_\mu$ and $H_\mu$, respectively.

First we show the existence of a unique two-particle bound state of $h_\mu(k), k \in \mathbb{T}^d$ with energy lying above the top of the essential spectrum in the case of repulsive ($\mu > 0$) interaction and below the bottom in the attractive ($\mu < 0$) case.

Second for any non-zero interaction we establish that the essential spectrum of the three-particle operator $H_\mu(K), K \in \mathbb{T}^d$ consists of the three-particle branch, spectrum of the non-perturbed operator $H_0(K)$ and a non-empty two-particle branches arising due to the eigenvalues of the two-particle operator $h_\mu(k), k \in \mathbb{T}^d$.

Third for any interacting particles ($\mu \neq 0$) we prove the existence of three-particle bound states with energy lying above the top resp. below the bottom of the essential spectrum for repulsive ($\mu > 0$) resp. attractive ($\mu < 0$) interactions.

In addition, we derive some important properties of bound states as well as their energies: The two-particle and three-particle bound states $\psi_{e_\mu(k),k}$ and $\psi_{E_\mu(K),K}$ in position representation exponentially vanishes at infinity. Moreover the bound states $\psi_{e_\mu(k),k}$ and $\psi_{E_\mu(K),K}$ and associated energy functions $e_\mu(k)$ and $E_\mu(K)$ are holomorphic in $k \in \mathbb{T}^d$ and $K \in \mathbb{T}^d$ respectively.

Our third result is quite surprising, because the three identical bosonic bound states with energy lying above the top resp. below the bottom of the essential spectrum exist also only for repulsive ($\mu > 0$) resp. attractive ($\mu < 0$) interactions.

To our knowledge analogous results have not been published yet even for a system of three particles interacting via attractive potentials on Euclidean space $\mathbb{R}^d$.

The results for a system of two identical fermions and one different particle theoretically predict the existence of stable attractively and repulsively bound objects for two fermionic and one bosonic atoms. Hopefully it can be experimentally confirmed as is done for pair of atoms with repulsive interaction in [10].
Notice that these results are characteristic to the Schrödinger operators associated to a system of three particles moving in one- or two-dimensional lattice $\mathbb{Z}^d$ and Euclidean space $\mathbb{R}^d$.

We remark that the results of the paper should be hold for a system of three particles interacting via short range attractive and repulsive potentials.

Our results allow us to formulate the following two hypothesis on existence of bound states for a system of three particles (two fermions and one different particle) moving on the lattice $\mathbb{Z}^d$, $d \geq 1$ and interacting via a short-range attractive or repulsive forces:

**In dimension $d = 1, 2$ this system has finite number of three-particle bound states, with energies lying as above the essential spectrum, as well as below its bottom.**

**In dimension $d = 3$ we propose the existence of infinitely many bound states (Efimov’s effect) with energies lying as above, as well as below the essential spectrum for special repulsive or attractive interaction.**

This paper is organized as follows. Section 1 is introduction. In section 2 we describe the hamiltonian of the two-body and the three-body case in the Schrödinger representation. It corresponds to the Hubbard hamiltonian in the number of particles representation. In section 3 we introduce the Floquet-Bloch (von Neumann) decomposition and choose relative coordinates to express the discrete Schrödinger operator $H_\mu(K)$, $K \in \mathbb{T}^d$ explicitly. In section 4 we state our main results. In section 5 we introduce channel operators and describe the essential spectrum of $H_\mu(K)$, $K \in \mathbb{T}^d$ by means discrete spectrum of $h_\mu(k)$, $k \in \mathbb{T}^d$. We prove the existence of bound states in section 6.

## 2. Hamiltonian of a System of Three Particles (Two Identical Fermions and One Different Particle) on Lattices

### 2.1. The coordinate representation.
Let $\mathbb{Z}^d$, $d \geq 1$ be the $d$-dimensional lattice. Let $\ell^2((\mathbb{Z}^d)^m)$, $m = 2, 3, \ldots$ be Hilbert space of square-summable functions $\hat{\varphi}$ defined on the Cartesian power of $(\mathbb{Z}^d)^m$, $d = 1, 2$ and $\ell^{2,\alpha}((\mathbb{Z}^d)^m) \subset \ell^2((\mathbb{Z}^d)^m)$ be the subspace of functions antisymmetric with respect to the permutation of the first two coordinates of the particles. Let $\Delta$ be the lattice Laplacian, i.e., the operator which describes the transport of a particle from one site to another site:

$$(\Delta \hat{\psi})(x) = -\sum_{|s|=1} [\hat{\psi}(x) - \hat{\psi}(x+s)].$$

The free Hamiltonian $\hat{h}_{0,\gamma}$ of a system of two arbitrary particles (a fermion and different particle) on the $d$-dimensional lattice $\mathbb{Z}^d$, $d = 1, 2$ acts on $\ell^2((\mathbb{Z}^d)^2)$ and is of the form

$$\hat{h}_{0,\gamma} = \Delta \otimes I + I \otimes \gamma \Delta,$$

where $\gamma > 0$ is the ratio of the masses of fermion and different particle.

Respectively, the free Hamiltonian $\hat{h}_0$ of a system of two identical fermions acts on $\ell^{2,\alpha}((\mathbb{Z}^d)^2)$ and is of the form

$$\hat{h}_0 = \Delta \otimes I + I \otimes \Delta.$$

The total Hamiltonian of a system of two arbitrary particles $\hat{h}_\mu$, resp. identical fermions $\hat{h}_\mu$ with zero-range pairwise interaction $\mu \neq 0$ is a bounded perturbation of the free Hamiltonian $\hat{h}_{0,\gamma}$ resp. $\hat{h}_0$ acts on the Hilbert space $\ell^2((\mathbb{Z}^d)^2)$ resp. $\ell^{2,\alpha}((\mathbb{Z}^d)^2)$ and is of the form

$$\hat{h}_\mu \equiv \hat{h}_{0,\gamma} + \mu \hat{v} \text{ resp. } \hat{h}_\mu = \hat{h}_0 + \mu \hat{v}$$

(2.1)
where
\[(\hat{\psi}(x_1, x_2) = \delta_{x_1, x_2} \hat{\psi}(x_1, x_2), \quad \hat{\psi} \in \ell^2([\mathbb{Z}^2])^2, \quad \text{resp.} \quad \hat{\psi} \in \ell^2, [([\mathbb{Z}^3])]^2\]\nand \(\delta_{x_1, x_2}\) is the Kronecker delta.

The free Hamiltonian \(\hat{H}_{0,\gamma}\) of a system of three particles (two identical fermions and one different particle) on the \(d\)-dimensional lattice \(\mathbb{Z}^d\) acts in \(\ell^2, [([\mathbb{Z}^3])]^3\) can be represented as
\[
\hat{H}_{0,\gamma} = \Delta \otimes I \otimes I + I \otimes \Delta \otimes I + I \otimes I \otimes \gamma \Delta.
\]

The total Hamiltonian \(\hat{H}_\mu\) of a system of three-particles with the pairwise zero-range interaction is a bounded perturbation of the free Hamiltonian \(\hat{H}_0\)
\[
\hat{H}_\mu = \hat{H}_{\mu,\gamma} = \hat{H}_{0,\gamma} + \mu (\hat{V}_{12} + \hat{V}_{13} + \hat{V}_{23}),
\]
where \(\hat{V}_{1,2}\) and \(\hat{V}_{\alpha,3}\), \(\alpha = 1, 2\) are multiplication operators
\[(\hat{V}_{1,2}\hat{\psi})(x_1, x_2, x_3) = \delta_{x_1, x_2} \hat{\psi}(x_1, x_2, x_3), \quad \hat{\psi} \in \ell^2, [([\mathbb{Z}^3])]^3,\]
and
\[(\hat{V}_{\alpha,3}\hat{\psi})(x_1, x_2, x_3) = \delta_{x_\alpha, x_3} \hat{\psi}(x_1, x_2, x_3), \quad \hat{\psi} \in \ell^2, [([\mathbb{Z}^3])]^3.\]

**Remark 2.1.** It can be easily seen that the equalities
\[
(\hat{V}_{1,2}\hat{\psi})(x_1, x_2) = 0, \quad \text{for all} \quad \hat{\psi} \in \ell^2, [([\mathbb{Z}^3])]^2,
\]
\[
(\hat{V}_{\alpha,3}\hat{\psi})(x_1, x_2, x_3) = 0, \quad \text{for all} \quad \hat{\psi} \in \ell^2, [([\mathbb{Z}^3])]^3
\]
hold. It means that two interacting fermions cannot coexist on the same site of the lattice, this is Pauli’s exclusion principle for identical fermions. Further, we deal only with the Hamiltonian \(\hat{H}_\mu\).

2.2. **The momentum representation.** Let us rewrite our operators in the momentum representation. Let \(\mathbb{T}^d\) be the \(d\)-dimensional torus (Brillouin zone)
\[
\mathbb{T}^d = (\mathbb{R}/2\pi \mathbb{Z})^d \equiv [\pi, \pi)^d,
\]
the Pontryagin dual group of \(\mathbb{Z}^d\) and \(\eta(dp) = \frac{d^d}{(2\pi)^d}\) is the (normalized) Haar measure on the torus. Let \(L^2, [([\mathbb{T}^d])]^3 \subset L^2([\mathbb{T}^d]^3)\) be the subspace of the functions antisymmetric with respect to the permutation of the first two coordinates of particles.

Let \(\hat{\mathcal{F}}_m : L^2([\mathbb{T}^d]^m) \rightarrow L^2([\mathbb{Z}^d])^m\), \(m \in \mathbb{N}\) be the standard Fourier transform and \(\hat{\mathcal{F}}_3\) be the restriction of \(\hat{\mathcal{F}}_3\) on the subspace \(L^2, [([\mathbb{T}^d])]^3\).

In the momentum representation the two-and three-particle Hamiltonian \(h_\mu\) and \(H_\mu\) is given by the bounded self-adjoint operator
\[
h_\mu = \hat{\mathcal{F}}^{-1}_2 \hat{H}_\mu \hat{\mathcal{F}}_2
\]
and
\[
H_\mu = [\hat{\mathcal{F}}_3^{-1}] \hat{H}_\mu \hat{\mathcal{F}}_3
\]
respectively. The operator \(h_\mu\) acts in \(L^2([\mathbb{T}^d]^2)\) and is of the form
\[
h_\mu = h_0 + \mu v,
\]
where
\[
(h_0 f)(k_\alpha, k_3) = \left[\varepsilon(k_\alpha) + \gamma \varepsilon(k_3)\right] f(k_\alpha, k_3).
\]
The interaction operator $v$ acts in $L^2(\mathbb{T}^d)^2$ as

$$
(vf)(k_\alpha, k_\beta) = \int_{(\mathbb{T}^d)^2} \delta(k_\alpha + k_\beta - k'_\alpha - k'_\beta) f(k'_\alpha, k'_\beta) dk'_\alpha dk'_\beta
$$

and

$$
\epsilon = \int_{(\mathbb{T}^d)^2} f(k_\alpha + k_\beta - k'_\alpha - k'_\beta) dk'_\alpha dk'_\beta,
$$

where $\epsilon$ is of the form

$$
\epsilon(p) = 2 \sum_{i=1}^{d} (1 - \cos p^{(i)}), \quad p = (p^{(1)}, ..., p^{(d)}) \in \mathbb{T}^d
$$

and $\delta(k)$ denotes the $d$- dimensional Dirac delta-function.

The three-particle Hamiltonian $H_\mu$ is of the form

$$
H_\mu = H_0 + \mu (V_{1,3} + V_{2,3}),
$$

where non-perturbed operator $H_0 = H_{0,\gamma}$ acts in $L^{2,\alpha}(\mathbb{T}^d)^3$ as

$$
(H_0 f)(k_1, k_2, k_3) = [\epsilon(k_1) + \epsilon(k_2) + \gamma \epsilon(k_3)] f(k_1, k_2, k_3),
$$

and the interaction operator $V_{\alpha,3}, \alpha = 1, 2$ is given by

$$
(V_{1,3}f)(k_1, k_2, k_3)
$$

and

$$
(V_{2,3}f)(k_1, k_2, k_3)
$$

3. Decomposition of the energy operators into von Neumann direct integrals. Quasi-momentum and coordinate systems

Denote by $k = k_1 + k_2 \in \mathbb{T}^d$ and $K = k_1 + k_2 + k_3 \in \mathbb{T}^d$ the two- and three-particle quasi-momenta. Define the sets $Q_k$ and $Q_K$ as follows

$$
Q_k = \{ (k_1, k - k_1) \in (\mathbb{T}^d)^2 : k_1 \in \mathbb{T}^d, k - k_1 \in \mathbb{T}^d \}
$$

and

$$
Q_K = \{ (k_1, k_2, K - k_1 - k_2) \in (\mathbb{T}^d)^3 : k_1, k_2 \in \mathbb{T}^d, K - k_1 - k_2 \in \mathbb{T}^d \}.
$$

We introduce the maps

$$
\pi_2 : (\mathbb{T}^d)^2 \to \mathbb{T}^d, \quad \pi_2(k_1, k_2) = k_1
$$

and

$$
\pi_3 : (\mathbb{T}^d)^3 \to (\mathbb{T}^d)^2, \quad \pi_3(k_1, k_2, k_3) = (k_1, k_2).
$$
Denote by $\pi_k, k \in \mathbb{T}^d$ and $\pi_K, K \in \mathbb{T}^d$ the restrictions of $\pi_2$ and $\pi_3$ onto $\mathbb{Q}_k \subset (\mathbb{T}^d)^2$ and $\mathbb{Q}_K \subset (\mathbb{T}^d)^3$ respectively, i.e.,

$$\pi_k = \pi_2|_{\mathbb{Q}_k} \quad \text{and} \quad \pi_K = \pi_3|_{\mathbb{Q}_K}.$$ 

**Remark 3.1.** We note that $\mathbb{Q}_k, k \in \mathbb{T}^d$ and $\mathbb{Q}_K, K \in \mathbb{T}^d$ are $d-$ and $2d-$ dimensional manifolds isomorphic to $\mathbb{T}^d$ and $(\mathbb{T}^d)^2$ respectively: The maps $\pi_k, k \in \mathbb{T}^d$ and $\pi_K, K \in \mathbb{T}^d$ are bijective from $\mathbb{Q}_k \subset (\mathbb{T}^d)^2$ and $\mathbb{Q}_K \subset (\mathbb{T}^d)^3$ onto $\mathbb{T}^d$ and $(\mathbb{T}^d)^2$ with

$$(\pi_k)^{-1}(k_1) = (k_1, k - k_1)$$

and

$$(\pi_K)^{-1}(k_1, k_2) = (k_1, k_2, K - k_1 - k_2).$$

Decomposing the Hilbert spaces $L^2((\mathbb{T}^d)^2)$ and $L^{2,\alpha}((\mathbb{T}^d)^3)$ into the direct integrals

$$L^2((\mathbb{T}^d)^2) = \int_{k \in \mathbb{T}^d} \oplus L^2(\mathbb{Q}_k) \eta(dk)$$

and

$$L^{2,\alpha}((\mathbb{T}^d)^3) = \int_{K \in \mathbb{T}^d} \oplus L^{2,\alpha}(\mathbb{Q}_K) \eta(dK)$$

yield the decompositions of the Hamiltonians $h_\mu$ and $H_\mu$ into the direct integrals

$$h_\mu = \int_{k \in \mathbb{T}^d} \oplus \tilde{h}_\mu(k) \eta(dk)$$

and

$$H_\mu = \int_{K \in \mathbb{T}^d} \oplus \tilde{H}_\mu(K) \eta(dK).$$

### 3.1. The discrete Schrödinger operators.

The fiber operator $\tilde{h}_\mu(k), k \in \mathbb{T}^d$ from the direct integral decomposition (3.1) acts in $L^2(\mathbb{Q}_k)$ and is unitarily equivalent to the operator $h_\mu(k), k \in \mathbb{T}^d$ given by

$$h_\mu(k) = h_{\gamma,0}(k) + \mu \nu, \mu \neq 0.$$ 

The operator $h_0 = h_{\gamma,0}(k)$ is the multiplication operator by the function $\mathcal{E}_{\gamma,k}(p)$:

$$(h_{\gamma,0}f)(p) = \mathcal{E}_{\gamma,k}f(p), \quad f \in L^2(\mathbb{T}^d),$$

where

$$(\mathcal{E}_{\gamma,k}(p) = \varepsilon(q) + \gamma \varepsilon(k-q),$$

and

$$(v f)(p) = \int_{\mathbb{T}^d} f(q) d\eta(q), \quad f \in L^2(\mathbb{T}^d).$$

The fiber operator $\tilde{H}_\mu(K), K \in \mathbb{T}^d$ from the direct integral decomposition (3.2) acts in $L^{2,\alpha}(\mathbb{Q}_K)$ and is unitarily equivalent to the operator $H_\mu(K), K \in \mathbb{T}^d$ given by

$$H_\mu(K) = H_0(K) + \mu (V_{13} + V_{23}).$$

The operator $H_0(K) = h_{\gamma,0}(K)$ acts in the Hilbert space $L^{2,\alpha}((\mathbb{T}^d)^2)$ as

$$(H_{\gamma,0}(K)f)(p,q) = E(K; p, q)f(p, q),$$

where

$$E(K; p, q) = \varepsilon(p) + \varepsilon(q) + \gamma \varepsilon(K - p - q).$$
The perturbation operator $\mathcal{V} = V_{13} + V_{23}$ in coordinates $(p, q) \in (\mathbb{T}^d)^2$ can be written in the form

\[(3.5) \quad (\nabla f)(p, q) = \int_{\mathbb{T}^d} f(p, t)\eta(dt) + \int_{\mathbb{T}^d} f(t, q)\eta(dt), \quad f \in L^2,\]

4. STATEMENT OF THE MAIN RESULTS

According to the Weyl theorem [9] the essential spectrum $\sigma_{\text{ess}}(h_\mu(k))$ of the operator $h_\mu(k)(k), k \in \mathbb{T}^d$ coincides with the spectrum $\sigma(h_0(k))$ of $h_0(k)$. More specifically, since for any $k \in \mathbb{T}^d$ the function $E_k(p)$ is continuous in $p \in \mathbb{T}^d$ the equality

$$\sigma_{\text{ess}}(h_\mu(k)) = [\sigma_{\text{min}}(k), \sigma_{\text{max}}(k)]$$

holds, where

$$E_{\text{min}}(k) = \min_{p \in \mathbb{T}^d} E_k(p) = \varepsilon(p_{\text{min}}(k)) + \gamma\varepsilon(K - p_{\text{min}}(k)), \quad p_{\text{min}} \in \mathbb{T}^d,$$

$$E_{\text{max}}(k) = \max_{p \in \mathbb{T}^d} E_k(p) = \varepsilon(p_{\text{max}}(k)) + \gamma\varepsilon(K - p_{\text{max}}(k)), \quad p_{\text{max}} \in \mathbb{T}^d.$$

The spectrum $\sigma(H_0(K))$ of the non-perturbed operator $H_0(K), K \in \mathbb{T}^d$ coincides with the segment $[E_{\text{min}}(K), E_{\text{max}}(K)]$. Since for each $K \in \mathbb{T}^d$ the function $E(K; p, q)$ is continuous and symmetric on $(\mathbb{T}^d)^2$, $d = 1, 2$ the equalities

$$E_{\text{min}}(K) = \min_{p, q \in \mathbb{T}^d} E(K; p, q) = E(K; p_{\text{min}}(K), q_{\text{min}}(K))$$

and

$$E_{\text{max}}(K) = \max_{p, q \in \mathbb{T}^d} E(K; p, q) = E(K; p_{\text{max}}(K), q_{\text{max}}(K))$$

hold, where $(p_{\text{min}}(K), q_{\text{min}}(K)), (p_{\text{max}}(K), q_{\text{max}}(K)) \in (\mathbb{T}^d)^2$.

**Note 4.1.** We remark that the essential spectrum $[E_{\text{min}}(k), E_{\text{max}}(k)]$ strongly depends on the quasi-momentum $k \in \mathbb{T}^d$; when $k = \vec{\pi} = (\pi_1, \ldots, \pi_d) \in \mathbb{T}^d$ the essential spectrum of $h_\mu(k)$ degenerated to the set consisting of a unique point $\{E_{\text{min}}(\vec{\pi}) = E_{\text{max}}(\vec{\pi}) = 2d\}$ and hence the essential spectrum of $h_\mu(k)$ is not absolutely continuous for all $k \in \mathbb{T}^d$. Similar arguments should be true for the spectrum of $H_0(K)$.

The following theorem asserts the existence of a unique eigenvalue $e_\mu(k)$ of the operator $h_\mu(k)$, which lays above the top $E_{\text{max}}(k)$ resp. below the bottom $E_{\text{min}}(k)$ of the essential spectrum $\sigma_{\text{ess}}(h_\mu(k))$ for repulsive ($\mu > 0$) resp. attractive ($\mu < 0$) interactions.

**Theorem 4.2.** Let $d = 1, 2$. For any $\mu \neq 0$ the operator $h_\mu(k), k \in \mathbb{T}^d$ has a unique eigenvalue $e_\mu(k)$, which satisfies the relations:

$$e_\mu(k) > E_{\text{max}}(k), \quad k \in \mathbb{T}^d \text{ and } e_\mu(0) > e_\mu(k), \quad k \in \mathbb{T}^d \setminus \{0\} \text{ for } \mu > 0$$

and

$$e_\mu(k) < E_{\text{min}}(k), \quad k \in \mathbb{T}^d \text{ and } e_\mu(0) < e_\mu(k), \quad k \in \mathbb{T}^d \setminus \{0\} \text{ for } \mu < 0.$$

The eigenvalue $e_\mu(k)$ is holomorphic function in $k \in \mathbb{T}^d$ and for any $k \in \mathbb{T}^d$ the associated eigenfunction $f_{\mu, e_\mu(k)}(p)$ is holomorphic in $p \in \mathbb{T}^d$ and is of the form

$$f_{\mu, e_\mu(k)}(\cdot) = \frac{\mu c(k)}{e_\mu(k) - E_k(\cdot)} \text{ resp. } f_{\mu, e_\mu(k)}(\cdot) = \frac{\mu c(k)}{\sigma_{\text{max}}(\cdot) - e_\mu(k)}.$$
where $c(k) \neq 0$ is a normalizing constant. Moreover, the vector valued mapping
\[ f_\mu : \mathbb{T}^d \rightarrow L^2[\mathbb{T}^d, \eta(dk); L^2(\mathbb{T}^d)], k \rightarrow f_\mu, e_\mu(k) \]
is holomorphic on $\mathbb{T}^d$.

Theorem 4.2 can be proven in the same way as Theorem 4.2 in [4].

The essential spectrum of the three-particle operator $H_\mu(K), K \in \mathbb{T}^d$ is described by
the spectrum of the non perturbed operator $H_0(K)$ and the discrete spectrum of the two-
particle operator $h_\mu(k), k \in \mathbb{T}^d$.

**Theorem 4.3.** Let $d = 1, 2$. For any $\mu \neq 0$ the essential spectrum $\sigma_{\text{ess}}(H_\mu(K))$ of $H_\mu(K)$
satisfies the following relations
\[ \sigma_{\text{ess}}(H_\mu(K)) = \cup_{k \in \mathbb{T}^d} \{e_\mu(K-k) + \varepsilon(k)\} \cup \{E_{\min}(K), E_{\max}(K)\} \supset [E_{\min}(K), E_{\max}(K)], \]
where $e_\mu(k)$ is the unique eigenvalue of the operator $h_\mu(k), k \in \mathbb{T}^d$.

Theorem 4.3 can be proven in the same way as Theorem 4.3 in [3].

Let $\tau^t_{\text{ess}}(H_\mu(K))$ resp. $\tau^b_{\text{ess}}(H_\mu(K))$ be the top resp. the bottom of the essential spectrum $\sigma_{\text{ess}}(H_\mu(K))$.

Our main theorem asserts that the operator $H_\mu(k), K \in \mathbb{T}^d$ has eigenvalue for all repulsive ($\mu > 0$) and attractive ($\mu < 0$) forces.

**Theorem 4.4.** Let $d = 1, 2$. For all $\mu \neq 0$ and $K \in \mathbb{T}^d$ the operator $H_\mu(K)$ has
eigenvalue lying outside of the essential spectrum $\sigma_{\text{ess}}(H_\mu(K))$. Moreover, the eigenvalue $E_\mu(K)$ is lying above the top $\tau^t_{\text{ess}}(H_\mu(K))$ for repulsive ($\mu > 0$) interaction and below the bottom $\tau^b_{\text{ess}}(H_\mu(K))$ of $\sigma_{\text{ess}}(H_\mu(K))$ for attractive ($\mu < 0$).

Any eigenvalue $E_\mu(K)$ of $H_\mu(K)$ is a holomorphic function in $K \in \mathbb{T}^d$. The associated
eigenfunction (bound state) $f_{\mu, E_\mu(K)}(\cdot, \cdot) \in L^2, \theta([\mathbb{T}^d]^2)$ is holomorphic in $(p, q) \in (\mathbb{T}^d)^2$
and the vector valued mapping
\[ f_\mu : \mathbb{T}^d \rightarrow L^2[\mathbb{T}^d, \eta(dK); L^2([\mathbb{T}^d]^2)], K \rightarrow f_\mu, E_\mu(K) \]
is also holomorphic in $K \in \mathbb{T}^d$.

Theorem 4.4 yields the following corollary, which asserts the existence of a band spectrum
for two and three interacting particles on the lattice $\mathbb{Z}^d, d = 1, 2$.

**Corollary 4.5.** For any $\mu \neq 0$ the two- resp. three-particle hamiltonian $h_\mu$ resp. $H_\mu$ has a band spectrum
\[ [\min_{k \in \mathbb{T}^d} e_\mu(k), \max_{k \in \mathbb{T}^d} e_\mu(k)] \text{ resp. } [\min_{K \in \mathbb{T}^d} E_\mu(K), \max_{K \in \mathbb{T}^d} E_\mu(K)]. \]

**Note 4.6.** For any $\mu \neq 0$ Theorems 4.2 and 4.3 yield that the two-particle essential spectrum
\[ \sigma_{\text{ess two}}(H_\mu(K)) = \cup_{k \in \mathbb{T}^d} \{e_\mu(K-k) + \varepsilon(k)\} \]
of the operator $H_\mu(K), K \in \mathbb{T}^d$ is a non empty set and hence
\[ \tau^t_{\text{ess}}(H_\mu(K)) < E_{\min}(K) \text{ for } \mu < 0 \]
and
\[ \tau^t_{\text{ess}}(H_\mu(K)) > E_{\max}(K), \text{ for } \mu > 0 \]
which allows the existence of bound states of three (two identical fermions and one different particle) repulsively resp. attractively interacting particles on the lattice $\mathbb{Z}^d$ [1, 7].

We note that this result is characteristic to the Schrödinger operators associated to
a system of three particles moving in a one- or two-dimensional space.
5. The essential spectrum of the operator $H_\mu(K)$

Since the particles are identical there is only one channel operator $H_{\mu,\text{ch}}(K), K \in \mathbb{T}^d, d = 1, 2$ defined in the Hilbert space $L^2[(\mathbb{T}^d)^2]$ as

$$H_{\mu,\text{ch}}(K) = H_0(K) + \mu V.$$  

The operators $H_0(K)$ and $V = V_{1,3} = -V_{2,3}$ act as follows

$$(H_0(K)f)(p, q) = E(K; p, q)f(p, q), \quad f \in L^2[(\mathbb{T}^d)^2],$$

where

$$E(K; p, q) = \varepsilon(p) + \varepsilon(q) + \gamma \varepsilon(K - p - q)$$

and

$$(V f)(p, q) = \int_{\mathbb{T}^d} f(p, t)\eta(dt), \quad f \in L^2[(\mathbb{T}^d)^2].$$

The decomposition of the space $L^2[(\mathbb{T}^d)^2]$ into the direct integral

$$L^2[(\mathbb{T}^d)^2] = \int_{\mathbb{T}^d} \oplus L^2(\mathbb{T}^d)\eta(dp)$$

yields the decomposition

$$H_{\mu,\text{ch}}(K) = \int_{\mathbb{T}^d} \oplus h_{\mu}(K, p)\eta(dp).$$

The fiber operator $h_{\mu}(K, p)$ acts in the Hilbert space $L^2(\mathbb{T}^d)$ and is of the form

$$(5.1) \quad h_{\mu}(K, p) = h_{\mu}(K - p) + \varepsilon(p)I,$$

where $I = I_{L^2(\mathbb{T}^d)}$ is the identity operator and the operator $h_{\mu}(K - p)$ is defined by (3.3).

The representation (5.1) of the operator $h_{\mu}(K, p)$ and Theorem 4.2 yield the following description for the spectrum of $h_{\mu}(K, p)$

$$\sigma(h_{\mu}(K, p)) = [c_{\mu}(K - p) + \varepsilon(p)] \cup [E_{\min}(K, p), E_{\max}(K, p)],$$

where $E_{\min}(K, p) = \min_{q \in \mathbb{T}^d} E(K, p; q)$ and $E_{\max}(K, p) = \max_{q \in \mathbb{T}^d} E(K, p; q)$.

Notice that Theorem 4.2 yields the result, which states that for any $\mu \neq 0$ the essential spectrum $\sigma_{\text{ess}}(H_{\mu}(K))$ of the operator $H_{\mu}(K)$ is different from the spectrum of the non-perturbed operator $H_0(K)$.

**Lemma 5.1.** For any $K \in \mathbb{T}^d$ the following inequalities hold

$$\tau_{\text{ess}}^b(H_{\mu}(K)) < \tau_{\text{ess,three}}(H_{\mu}(K)) = E_{\min}(K) \quad \text{for } \mu < 0$$

and

$$\tau_{\text{ess}}^b(H_{\mu}(K)) > \tau_{\text{ess,three}}(H_{\mu}(K)) = E_{\max}(K) \quad \text{for } \mu > 0.$$

**Proof.** Theorem 4.2 yields that for any $\mu < 0$ and $K \in \mathbb{T}^d$ the operator $h_{\mu}(K)$ has a unique eigenvalue $c_{\mu}(k) < E_{\min}(k)$. Set

$$Z_{\mu}(K, p) = c_{\mu}(K - p) + \varepsilon(p).$$

The definition of $\tau_{\text{ess}}(H_{\mu}(K))$ gives

$$\tau_{\text{ess}}(H_{\mu}(K)) = \inf_{p \in \mathbb{T}^d} Z_{\mu}(K, p)$$

$$\leq \inf_{p \in \mathbb{T}^d} [c_{\mu}(K - p_{\min}(K)) + \varepsilon(p_{\min}(K))] < E_{\min}(K - p_{\min}(K)) + \varepsilon(p_{\min}(K)) = E_{\min}(K).$$
which proves Lemma 6.1.

6. PROOF OF THE MAIN RESULTS

Let

\[ E_{\min}(K, k) = \min_{q \in \mathbb{T}^d} E(K, k; q) = \min_{q \in \mathbb{T}^d} \varepsilon_k(q) + \varepsilon(K - k), \]

\[ E_{\max}(K, k) = \max_{q \in \mathbb{T}^d} E(K, k; q) = \max_{q \in \mathbb{T}^d} \varepsilon_k(q) + \varepsilon(K - k). \]

For any \( \mu \in \mathbb{R} \) and \( K, k \in \mathbb{T}^d \), \( d = 1, 2 \) the determinant \( \Delta_{\mu}(K, k; z) \) associated to the operator \( h_{\mu}(K, k) \) can be defined as a real-analytic function in \( \mathbb{C} \setminus [E_{\min}(K, k), E_{\max}(K, k)] \) by

\[ \Delta_{\mu}(K, k; z) = 1 + \mu \int_{\mathbb{T}^d} \frac{\eta(dq)}{E(K; k, q) - z}. \]

**Lemma 6.1.** For any \( \mu \in \mathbb{R} \) and \( K, k \in \mathbb{T}^d \) the number \( z \in \mathbb{C} \setminus [E_{\min}(K, k), E_{\max}(K, k)] \) is an eigenvalue of the operator \( h_{\mu}(K, k) \) if and only if

\[ \Delta_{\mu}(K, k; z) = 0. \]

The proof of Lemma 6.1 is simple and can be proven in the same way as Lemma in [2].

**Remark 6.2.** We note that for each \( \mu \neq 0 \) and \( K, k \in \mathbb{T}^d \) there exist either \( z_l = z_l(K, k) < E_{\min}(K, k) \) or \( z_r = z_r(K, k) > E_{\max}(K, k) \) such that either for all

\[ z \in (-\infty, z_l) \cup [E_{\max}(K, k), +\infty) \]

or

\[ z \in (-\infty, E_{\max}(K, k)] \cup [z_r, +\infty) \]

the function \( \Delta_{\mu}(K, k; z) \) is non-negative and the square root function \( \Delta_{\mu}^{\frac{1}{2}}(K, k; z) \) is well defined.

We define for each \( \mu \in \mathbb{R} \) and \( z \in \mathbb{R} \setminus [\tau_{\text{ess}}^-(H_{\mu}(K)), \tau_{\text{ess}}^+(H_{\mu}(K))] \) the self-adjoint compact Birman-Schwinger operator \( L_{\mu}(K, z), K \in \mathbb{T}^d \) as

\[
L_{\mu}(K, z)\psi(p) = \mu \int_{\mathbb{T}^d} \frac{\Delta_{\mu}^{\frac{1}{2}}(K, p, z)\Delta_{\mu}^{\frac{1}{2}}(K, q, z)}{E(K; p, q) - z} \eta(dq) \psi(q), \psi \in L^2(\mathbb{T}^d).
\]

Notice that for \( \mu < 0 \) the operator \( L_{\mu}(K, z), z < \tau_{\text{ess}}^+(H_{\mu}(K)) \) has been introduced in [7] to investigate Efimov’s effect for the three-particle lattice Schrödinger operator \( H_{\mu}(K) \) associated to a system of two identical fermions and one different particle on the lattice \( \mathbb{Z}^3 \).

**Lemma 6.3.** Let \( \mu > 0 \) and \( z > \tau_{\text{ess}}^+(H_{\mu}(K)) \) resp. \( \mu < 0 \) and \( z < \tau_{\text{ess}}^+(H_{\mu}(K)) \). The following assertions (i)–(ii) hold true.

(i) If \( f \in L^2_{\alpha}[\mathbb{T}^d]^2 \) solves the equation \( H_{\mu}(K)f = zf \), then

\[
\psi(p) = \Delta_{\mu}^{\frac{1}{2}}(K, p; z) \int_{\mathbb{T}^d} f(p, t)\eta(dq) \in L^2(\mathbb{T}^d)
\]

solves \( L_{\mu}(K, z)\psi = \psi \).
(ii) If $\psi \in L^2(\mathbb{T}^d)$ solves $L_\mu(K, z)\psi = \psi$, then

$$f(p, q) = -\frac{\mu[\varphi(p) - \varphi(q)]}{E(K; p, q) - z} \in L^{2, \alpha}[(\mathbb{T}^d)^2]$$

solves the equation $H_\mu(K)f = zf$, where $\varphi(p) = \Delta^{-\frac{1}{2}}_\mu(K, p; z)\psi(p)$.

Proof.
(i) Let $\mu > 0$. Assume that for some $z > \tau^t_{\text{ess}}(H_\mu(K)), K \in \mathbb{T}^d$ the equation

$$(H_\mu(K)f)(p, q) = zf(p, q),$$

i.e., the equation

$$[E(K; p, q) - z]f(p, q) = -\mu \int_{\mathbb{T}^d} f(p, t)\eta(dt) + \mu \int_{\mathbb{T}^d} f(q, t)\eta(dt)$$

has a solution $f \in L^{2, \alpha}[(\mathbb{T}^d)^2]$. Write

$$\varphi(p) = \int_{\mathbb{T}^d} f(p, q)\eta(dq) \in L^2(\mathbb{T}^d).$$

Then we have the following representation

$$(6.3) \quad f(p, q) = -\frac{\mu[\varphi(p) - \varphi(q)]}{E(K; p, q) - z} \in L^{2, \alpha}[(\mathbb{T}^d)^2],$$

which gives the equation

$$(6.4) \quad \varphi(p)[1 + \mu \int_{\mathbb{T}^d} \frac{\eta(dp)}{E(K; p, q) - z}] = \mu \int_{\mathbb{T}^d} \frac{\varphi(q)\eta(dq)}{E(K; p, q) - z}.$$ 

Taking into account $\Delta_\mu(K, p; z) > 0$ for $z > \tau^t_{\text{ess}}(H_\mu(K)), K \in \mathbb{T}^d$ and denoting by $\psi(q) = \Delta^{-\frac{1}{2}}_\mu(K, q; z)\varphi(q)$ we get the equation

$$(6.5) \quad \mu \int_{\mathbb{T}^d} \frac{\Delta^{-\frac{1}{2}}_\mu(K, q, z)\Delta^{-\frac{1}{2}}_\mu(K, p, z)}{E(K; p, q) - z} \psi(p)\eta(dp) = \psi(q),$$

i.e., $L_\mu(k, z)\psi = \psi$.

(ii) Assume that $\psi$ is a solution of equation (6.5). Then the function

$$(6.6) \quad \varphi(p) = \Delta^{-\frac{1}{2}}_\mu(K, p; z)\psi(p)$$

is a solution of equation (6.4) and hence the function defined by (6.3) is a solution of the equation $H_\mu(K)f = zf$, i.e., is an eigenfunction of the operator $H_\mu(K)$ associated to the eigenvalue $z > \tau^t_{\text{ess}}(H_\mu(K))$.

The case $\mu < 0$ and $z < \tau^b_{\text{ess}}(H_\mu(K))$ of Lemma 6.3 can be proven in the same way. □

Proof of Theorem 4.3. The theorem can be proven, applying Lemma 5.1 in the same way as Theorem 3.2 in [1].
\textbf{Proof of Theorem 4.4.} Let $\mu > 0$ and $K \in \mathbb{T}^d$, $d = 1, 2$. For any non-negative $f \in L^2(\mathbb{T}^d)$ and $z > \tau^t_{\text{ess}}(H_\mu(K))$ the following relations

\begin{equation}
(L_\mu(K, z)f, f) = -\mu \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{f(p)f(q)\eta(dp)\eta(dq)}{\Delta^2_\mu(K, p, z)\Delta^2_\mu(K, q, z)(z - E(K; p, q))} \leq 0
\end{equation}

hold, where

$$\Delta_\mu(K, p, z) = 1 - \mu \int_{\mathbb{T}^d} \frac{\eta(dq)}{z - E(K; p, q)}.$$

Let

$$F_z(f) = \int_{\mathbb{T}^d} \frac{f(p)\eta(dp)}{\Delta^2_\mu(K, p, z)}, \quad z > \tau^t_{\text{ess}}(H_\mu(K))$$

be linear bounded functional defined on $L^2(\mathbb{T}^d)$. According the Riesz theorem

$$||F_z|| = \left[ \int_{\mathbb{T}^d} \frac{\eta(dp)}{\Delta_\mu(K, p, z)} \right]^{1/2} = ||\psi_z||.$$

Let $\mathcal{M}_+ \subset L^2(\mathbb{T}^d)$ be subset of all non-negative functions. Then

\begin{equation}
||F_z||_{L^2(\mathbb{T}^d)} \geq ||F_z||_{\mathcal{M}_+} \geq ||\psi_z||.
\end{equation}

Since for any $p \in \mathbb{T}^d$ the function $\Delta_\mu(K, p, z)$ is monotone decreasing in $z \in (\tau^t_{\text{ess}}(H_\mu(K)), +\infty)$ there exists a.e. the point-wise limit

$$\lim_{z \to \tau^t_{\text{ess}}(H_\mu(K))} \frac{1}{\Delta_\mu(K, p; \tau^t_{\text{ess}}(H_\mu(K)))} = \frac{1}{\Delta_\mu(K, p; \tau^t_{\text{ess}}(H_\mu(K)))}.$$

The Fatou’s theorem yields the inequality

$$\int_{\mathbb{T}^d} \frac{\eta(dp)}{\Delta_\mu(K, p; \tau^t_{\text{ess}}(H_\mu(K)))} \leq \liminf_{z \to \tau^t_{\text{ess}}(H_\mu(K))} \int_{\mathbb{T}^d} \frac{\eta(dp)}{\Delta_\mu(K, p; \tau^t_{\text{ess}}(H_\mu(K)))}.$$

Let $p_\mu(K) \in \mathbb{T}^d$, $K \in \mathbb{T}^d$ be a minimum point of the function $Z_\mu(K, p)$, $K \in \mathbb{T}^d$ on $\mathbb{T}^d$. Then $Z_\mu(K, p)$ has the following asymptotics

\begin{equation}
Z_\mu(K, p) = \tau^t_{\text{ess}}(H_\mu(K)) - (B(K)(p - p_\mu(K)), p - p_\mu(K)) + o(|p - p_\mu(K)|^2),
\end{equation}

as $|p - p_\mu(K)| \to 0$, where $\tau^t_{\text{ess}}(H_\mu(K)) = Z_\mu(K, p_\mu(K))$.

For any $K, p \in \mathbb{T}^d$ there exists a $\gamma = \gamma(K, p) > 0$ neighborhood $W_\gamma(Z_\mu(K, p))$ of the point $Z_\mu(K, p) \in C$ such that for all $z \in W_\gamma(Z_\mu(K, p))$ the following equality holds

$$\Delta_\mu(K, p, z) = \sum_{n=1}^{\infty} C_n(\mu, K, p)[z - Z_\mu(K, p)]^n,$$

where

$$C_1(\mu, K, p) = \mu \int_{\mathbb{T}^d} \frac{\eta(dq)}{[Z_\mu(K, p) - E(K; p, q)]^2} > 0.$$
From here one concludes that for any $K \in U_3(0)$ there is $U_3(K)(\mu(K))$ so that for all $p \in U_3(K)(\mu(K))$ the equality
\[(6.10)\]
\[\Delta_{\mu}(K, p, \tau_{\text{ess}}^t(H_{\mu}(K)),\mu(K)) = (Z_{\mu}(K, p) - \tau_{\text{ess}}^t(H_{\mu}(K))\hat{\Delta}_{\mu}(K, p, \tau_{\text{ess}}^t(H_{\mu}(K)))\]
holds. Putting (6.9) into (6.10) yields the estimate
\[\Delta_{\mu}(K, p, \tau_{\text{ess}}^t(H_{\mu}(K)) \leq M(K)|p - \mu(K)|^2.\]
Hence, we have
\[\int_{\tau_d} \frac{\eta(dp)}{\Delta_{\mu}(K, p, \tau_{\text{ess}}^t(H_{\mu}(K)))} = +\infty.\]
Consequently, for any $P > 0$ there exists $z_0 > \tau_{\text{ess}}^t(H_{\mu}(K))$ such that the inequality
\[(6.11)\]
\[||F_{z_0}|| = \sup_{||\psi||=1} (F_{z_0}(\psi, \psi) = [\int_{\tau_d} \frac{\eta(dp)}{\Delta_{\mu}(K, p, z)}]^{\frac{1}{2}} > P\]
holds. Since for all $z > \tau_{\text{ess}}^t(H_{\mu}(K))$ the positive function $(z - E_{\text{min}})^{-1}$ is bounded above, the inequality (6.11) yields the existence of $\psi \in L^2(\tau_d)$, $||\psi||_{L^2(\tau_d)} = 1$ satisfying the inequality $(L_{\mu}(K, z_0)\psi, \psi) < 1$. At the same time
\[(L_{\mu}(K, z)\psi, \psi) \to 0\text{ as } z \to +\infty.\]
Therefor there exists $E_{\mu}(K) > z_0 > \tau_{\text{ess}}^t(H_{\mu}(K))$, such that
\[||(L_{\mu}(K, E_{\mu}(K))\psi, \psi)|| = 1\]
and hence the Hilbert-Schmidt theorem yields that the equation
\[(6.12)\]
\[L_{\mu}(K, E_{\mu}(K))\psi = \psi\]
has a solution $\psi \in L^2(\tau_d)$, $||\psi|| = 1$. Lemma (6.3) yields that $E_{\mu}(K) > \tau_{\text{ess}}^t(H_{\mu}(K))$ is an eigenvalue of the operator $H_{\mu}(K)$ and the associated eigenfunction $f_{E_{\mu}(K)}(K; p, q)$ takes the form
\[(6.13)\]
\[f_{E_{\mu}(K)}(K; p, q) = \frac{\mu c(K)[\varphi(p) - \varphi(q)]}{E_{\mu}(K) - E(K; p, q)} \in L^{2,\alpha}[\tau_d]^2\]
with $c(K) = ||f_{E_{\mu}(K)}(K; p, q)||^{-1}$, $K \in \tau_d$ being a normalizing constant.

Since for any $K \in \tau_d$ the functions $\Delta_{\mu}(K, p; E_{\mu}(K))$ and $E_{\mu}(K) - E(K; p, q) > 0$ are regular in $p, q \in \tau_d$ the solution $\psi$ of the equation (6.3) and the function $\varphi$ given in (6.4) are regular in $p \in \tau_d$. Hence, the eigenfunction (6.13) of the operator $H_{\mu}(K)$ associated to the eigenvalue $E_{\mu}(K) > \tau_{\text{ess}}^t(H_{\mu}(K))$ is also regular in $p, q \in \tau_d$.

For any $z > \tau_{\text{ess}}^t(H_{\mu}(K))$ the kernel function
\[(6.14)\]
\[L_{\mu}(K, z; p, q) = -\mu \frac{\Delta_{\mu}^{-\frac{1}{2}}(K, p, z)\Delta_{\mu}^{-\frac{1}{2}}(K, q, z)}{z - E(K; p, q)}\]
of the compact self-adjoint operator $L_{\mu}(K, z)$ is regular in $p, q \in \tau_d$. The Fredholm determinant $D_{\mu}(K, z) = \det[I - L_{\mu}(K, z)]$ associated to the kernel function (6.14) is real and regular in $z \in (\tau_{\text{ess}}^t(H_{\mu}(K)), +\infty)$. Lemma (6.3) and the Fredholm theorem yield that each eigenvalue of the operator $H_{\mu}(K)$ is a zero of the determinant $D_{\mu}(K, z)$ and vice versa. Consequently, the compactness of the torus $\tau_d$ and the implicit function theorem give that the eigenvalue $E_{\mu}(K)$ of $H_{\mu}(K), \mu > 0$ is a regular function in $K \in \tau_d$, $d = 1, 2$. 
Since for any \( p, q \in \mathbb{T}^d \) the functions \( \Delta_\mu(K, p ; E_\mu(K)) \) and \( E(K ; p, q) - E_\mu(K) \) are regular in \( K \in \mathbb{T}^d \) the solution \( \psi \) of \((6.5)\) and the function \( \varphi \) defined by \((6.4)\) are regular in \( K \in \mathbb{T}^d \). Hence, the eigenfunction \((6.13)\) of the operator \( H_\mu(K) \) associated to the eigenvalue \( E_\mu(K) > \tau^l_{\text{ens}}(H_\mu(K)) \) is also regular in \( K \in \mathbb{T}^d \). Consequently, the vector valued mapping

\[
 f_\mu : \mathbb{T}^d \to L^2[\mathbb{T}^d, \eta(dK); L^{2,\alpha}([\mathbb{T}^d]^2)], \quad K \to f_\mu(K)(\cdot, \cdot)
\]

is regular in \( \mathbb{T}^d \).

Now we prove that the operator \( H_\mu(K) \) has no eigenvalue lying below the bottom of the essential spectrum for \( \mu > 0 \) and \( K \in \mathbb{T}^d, d = 1, 2 \).

The operator \( H_\mu(K) \) acting in the Hilbert space \( L^{2,\alpha}([\mathbb{T}^d]^2) \) is of the form

\[
 (H_\mu(K)f)(p, q) = E(K ; p, q)f(p, q) + \mu\int_{\mathbb{T}^d} f(p, t)\eta(dt) + \int_{\mathbb{T}^d} f(t, q)\eta(dt)
\]

where

\[
 E(K ; p, q) = \varepsilon(p) + \varepsilon(q) + \gamma\varepsilon(K - p - q).
\]

Then

\[
 (H_\mu(K)f, f) = \int_{\mathbb{T}^d} E(K ; p, q)|f(p, q)|^2\eta(dp)\eta(dp) + \mu\int_{(\mathbb{T}^d)^2} \int_{\mathbb{T}^d} f(p, t)\eta(dt)\overline{f(p, q)}\eta(dp)\eta(dp) + \mu\int_{(\mathbb{T}^d)^2} \int_{\mathbb{T}^d} f(t, q)\eta(dt)\overline{f(p, q)}\eta(dp)\eta(dp).
\]

Fubini’s theorem gives us the following relations

\[
 (H_\mu(K)f, f)
 = \int_{(\mathbb{T}^d)^2} E(K ; p, q)|f(p, q)|^2\eta(dp)\eta(dp)
 + \mu\int_{(\mathbb{T}^d)^2} \int_{\mathbb{T}^d} f(p, t)\eta(dt)|f(p, q)|\eta(dp)\eta(dp)
 + \mu\int_{(\mathbb{T}^d)^2} \int_{\mathbb{T}^d} f(t, q)\eta(dt)|f(p, q)|\eta(dp)\eta(dp)
 = \int_{(\mathbb{T}^d)^2} E(K ; p, q)|f(p, q)|^2\eta(dp)\eta(dp)
 + \mu\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} |f(p, t)|^2\eta(dp)
 + \mu\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} |f(t, q)|^2\eta(dp)
 \geq \int_{(\mathbb{T}^d)^2} E(K ; p, q)|f(p, q)|^2\eta(dp)\eta(dp) \geq 0.
\]
The min-max principle completes the proof.

Note that the case \( \mu < 0 \) of Theorem 4.4 can be proven in the same way as above\(^3\). □

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