Flocking in one dimension: effect of update rules

R. K. Singh\textsuperscript{1}† and Shradha Mishra\textsuperscript{2}∗

\textsuperscript{1}The Institute of Mathematical Sciences, CIT Campus, 4th Cross Street, Taramani, Chennai 600113, India
\textsuperscript{2}Department of Physics, Indian Institute of Technology(BHU), Varanasi- 221005, India

In this study the effect of parallel and random-sequential updates on the dynamical properties of flocks in one dimension is considered. It is found that the frequency of directional switching is increased for random-sequential updates as compared to a parallel update. The nature of disorder to order transition is also affected by the difference of updating mechanism: discontinuous for parallel and continuous for random-sequential updates.

I. INTRODUCTION

Collective motion is a ubiquitous phenomena observed in active systems driven out of equilibrium across widely separated length scales from single cells \textsuperscript{1}, to unicellular organisms \textsuperscript{2}, to bird flocks \textsuperscript{3}, and humans \textsuperscript{4}. Emergence of such a motion in a group of self-propelled units is termed as a flocking transition \textsuperscript{5} and was reported by Vicsek and coworkers for a system of particles in two dimensions \textsuperscript{6}. Although most of the natural systems of interest are generally two or three dimensional, emergence of collective motion in one dimension has attracted attention in recent times \textsuperscript{7}. \textsuperscript{9}. Such one dimensional flocks exhibit the interesting property of direction switching \textsuperscript{8}. \textsuperscript{10} and recent theoretical and experimental studies have proven the usefulness of the study of collective motion in one dimension, in particular relevance to the phenomena of directional switching \textsuperscript{11}. \textsuperscript{12}.

Most such models studying flocking in one dimension generally employ discrete time evolutions of the system which are closer in nature to the sense of time as represented in digital simulations. However, it is known that generally employ discrete time evolutions of the system as has been observed in equilibrium \textsuperscript{13} and nonequilibrium systems \textsuperscript{14}. Such differences in update rules have lead to the appearance of transient and steady-state properties of the flocks. The paper is organized as follows: in the next section we compare the two update rules followed by the conclusion.

II. COMPARISON OF THE TWO UPDATE RULES

We start with a collection of active spins moving in the unit interval [0, 1] with a fixed speed $v_0$ and periodic boundary conditions. The position $x^i$ of the $i$th spin evolves as:

$$x^i_{n+1} = x^n_i + s^i_nv_0.$$  \hspace{1cm} (1)

with $s^n_i$ being the spin state at time $n$. A spin $i$ interacts with all other spins in the interval $[x^i_n - r, x^i_n + r]$ by flipping its orientation in accordance with the Metropolis algorithm \textsuperscript{15}. If $f$ is the net spin in the interaction radius $r$, then depending on the product $s^i_nf$ the spin flips certainly if the product is negative and with probability $\exp(-\beta s^i_nf)$ when the product is positive, where the inverse temperature $\beta$ measures global randomness.

To study long-range order in the system, we define:

$$m = \frac{1}{N} \sum_{i} s^i$$ \hspace{1cm} (2)

as the average orientation of the system. The magnitude $|m|$ serves as an appropriate order-parameter and takes values in the interval $[0, 1]$ with 0 representing completely disordered state and 1 the state of complete long-range order. The above system of active spins can evolve either by a parallel update or by a random-sequential update which we now define.

In a parallel update rule, $\forall i = 1, \ldots, N$, $s^i$ is modified to $\tilde{s}^i$ based on the local interactions of each $s^i$. Position vector of the system then evolves according to: $x^i_{n+1} = x^n_i + v_0\tilde{s}_i$, followed by $\tilde{s}_i \rightarrow s_i$. In a random-sequential update, on the other hand, an active spin $i$ is chosen at random from the collection of $N$ spins and its spin $s^i$ is modified according to the Metropolis algorithm. The difference lies in the step that the updated value of spin $s^i$ is used immediately to modify the position of the $i^{th}$ particle according to (1). This process is repeated $N$ times so that each spin gets an equal chance of update and this process of $N$ random flips constitutes one unit of time equivalent to a parallel update of $N$ spins. It is evident from the definition of the two update rules that a parallel update is synchronous whereas a random-sequential update is intrinsically asynchronous, as there

\textsuperscript{†} smishra.phy@itbhu.ac.in
\textsuperscript{∗} rksingh@ims.res.in
is a randomness inherent in the very nature of the update rule. Such randomness has implications in the dynamics of the spin system and reflects in both the transient and steady-state properties.

At $n = 0$, positions of the spins are chosen uniformly from the unit interval. Initial spin states are also chosen $\pm 1$ at random in all the following observations unless explicitly stated. Fig. 1 shows the variation of the order-parameter $\langle |m| \rangle$ with time $n$ for a system of $N = 1000$ spins moving in the unit interval $[0, 1]$ for parallel(a) and random-sequential(b) updates respectively. For large inverse temperature, e.g., $\beta = 4$, when the system exhibits long-range order, the $\langle |m| \rangle$ vs $n$ curves are similar for the two update rules: increased local interactions lead to a reduced time to achieve long-range order starting from complete disorder. However, for $\beta = 1$ we observe large fluctuations in the order-parameter for random-sequential updates as compared to that for parallel updates (inset in (a-b)). The fluctuations are intrinsic to the random-sequential update rule and later we will show that such fluctuations are prominent not only for low values of $\beta$ but also for higher values.

Fig. 2 shows snapshots of the probability distribution of positions $x^i$ of the spins for the two update rules for $\beta = 4$. Starting with a uniform distribution of the positions of spins at $n = 0$ (black), the spins tend to move in close neighborhoods when the system exhibits long-range order implying that the emergence of long-range order for high $\beta$ values is a mean-field effect. The cause for this effect is the propagation of the local interaction amongst the spins across the interval due to the finite speed of movement of the spins, i.e., $v_0 > 0$. Such a propagation of the local interaction leads to the emergence of a long-range order when global fluctuations are less (high $\beta$ values). This is because for high $\beta$, the probability of flipping against the majority $\exp(-\beta s_n^i f)$ is less at any instant $n$ and hence the alignment of all the spins along the interval is achieved. On the other hand, for low values of $\beta$, the enhanced magnitude of global fluctuations increases the chance of any given spin $s^i$ to flip against the majority. As a result, even when the spins are moving with a constant speed $v_0$, a long-range order is not established because of the increased strength of global fluctuations which tends to disrupt the established local order at every instant. Now, random-sequential updates have an additional randomness due to the asynchronous updating mechanism which is reflected in the higher spread in comparison to the parallel counterpart. The clustered movement of the spins along the interval also implies towards the stability of the flocking state. For example, if a fraction of spins is flipped from their present state to reduce the value of the order-parameter $\langle |m| \rangle$, or in the extreme case, if the spins are completely randomized such that $\langle |m| \rangle \approx 0$, the flocking state of the system is restored to the previous value of $\langle |m| \rangle$. In addition, the time taken for the restoration of the flocking state after perturbation is less as compared to the time taken from $n = 0$. The reason for this reduction in time to restoring the flocking state is the proximity of the spins at the time of destabilization.

Next we report the directional switching behavior of the flocking state for parallel and random-sequential updates in Fig. 3. Starting with the initial state in which all the spins are in $s^i = 1$ state, we find that the average

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FIG. 1: Variation of the order-parameter $\langle |m| \rangle$ with time $n$ (in multiples of 100) for a system of $N = 1000$ spins for parallel(a) and random-sequential(b) updates. Parameter values for the observations are: $(v_0, r) = (0.001, 0.01)$ (black), $(0.001, 0.05)$ (red) and $(0.003, 0.01)$ (blue) respectively for inverse temperature $\beta = 4$. The inset in the two panels show the time-dependence of $\langle |m| \rangle$ for $\beta = 1$ and $(v_0, r) = (0.001, 0.01)$ for the two update rules. The data are averaged over 30 ensembles.

FIG. 2: Distribution of the positions $x$ of the spins for different snapshots of time $n$ for parallel(a) and random-sequential(b) updates respectively. The distributions are calculated over a single trajectory of the system starting with a uniform distribution of spins over the unit interval at $n = 0$. The systems consist of $N = 1000$ spins moving with speed $v_n = 0.001$ at inverse temperature $\beta = 4$. Interaction radius of the spins $r = 0.01$. 
spin $m$ fluctuates between $m = 1$ and $m = -1$ states for the two update rules for $\beta = 4$. The frequency of such flipping is, however, dependent on the size $N$ of the system as well as on the nature of update. For example, using the data for the evolution of the system over 50000 iterations and 30 ensembles, we find that for a parallel update rule the $N = 500$ size system fluctuates 60 times between the two states but no flipping is observed for $N = 1000$. On the contrary, for a random-sequential update the $N = 500$ size system exhibits 290 flips which is reduced to 146 for $N = 1000$. In parts (c-d) of the Fig. 3 we report the residence-time $\tau$ statistics of the two states. The probability $p(\tau)$ of the residence-time of a given $m$ vs $n$ trajectory in $m = 1$ and $m = -1$ states shows that in the long-time limit, the trajectories tend to spend longer times in their initial state $m = 1$ for parallel updates whereas for a random-sequential updates the trajectories exhibit a true bistable behavior. These observations are a derivative of the local interactions and the intrinsic differences of the two update rules. The reduction in the alternating frequency of steady-state values of $m$ with increasing $N$ is consequent of the increased magnitude of local interactions against the same intensity of global noise $\beta$. In addition, the differences of the flipping frequency for the two update rules is attributed to the intrinsic fluctuations in the random-sequential updates. Our observation of the alternating steady-states is similar to a previous study for a lattice based model in one dimension $\delta$.

The steady-state properties of the system are shown in Fig. 4 depicting the nature of the transitions. It is observed that the nature of the flocking transition is different for the two update rules: with a first-order transition for parallel update and second-order for random-sequential updates. These are reflected in the $\langle|m|\rangle$ vs $\beta$ curves and the variation of the Binder cumulant $G = 1 - \langle|m|^4\rangle/3\langle|m|^2\rangle^2$ against the inverse temperature. A jump in $\langle|m|\rangle$ and the strong negative values taken by $G$ imply that the disorder-to-order transition is first-order for parallel updates whereas it is second-order for random-sequential updates. It is to be noted that for random-sequential updates $G$ varies smoothly from zero to $2/3$ as the system goes from disordered state (small $\beta$) to long-ranged-ordered state (large $\beta$), a consequence of the Gaussian nature of fluctuations of $\langle|m|\rangle$ about the mean. But for parallel update $G > 0$ in the disordered state (small $\beta$) and goes to $2/3$ values for long-ranged ordered state (large $\beta$) with a strong negative value close to critical $\beta$. Although the mean magnetization for parallel update is zero in the disordered state, +ve value of

\(\text{FIG. 3: Variation of the order-parameter with sign } m \text{ with time } n \text{ (in multiples of } 10^2). \text{ Parts (a) and (b) represent a single trajectory of } m \text{ vs } n \text{ for parallel and random-sequential updates respectively for } N = 1000\text{(red)} \text{ and } N = 500\text{(black)} \text{ spins with } v_0 = 0.001, r = 0.01 \text{ and } \beta = 4. \text{ Starting with } m = 1, \text{ (c) and (d) represent the fraction of the residence-time } \tau \text{ spent by the sample trajectories in the } m = 1 \text{ and } m = -1 \text{ states. The histograms for } N = 1000\text{(red)} \text{ and } N = 500\text{(black)} \text{ are calculated by using data over 30 ensembles and shifted relative to each other for clarity.} \)

\(\text{FIG. 4: Comparison of the steady-state properties for parallel vs random-sequential updates. (a-b) show the variation of the order-parameter } \langle|m|\rangle \text{ and the Binder cumulant } G \text{ in the steady state for parallel and random-sequential updates respectively. The steady-state properties are calculated using 20000 iterations with 5000 iterations for the burn-in period and averaged over 10 ensembles. In parts (a) and (b), the symbols represent: parallel updates for } N = 500\text{(blue triangles)} \text{ and } N = 1000\text{(black squares)}; \text{ and random-sequential updates for } N = 500\text{(magenta inverted triangles)} \text{ and } N = 1000\text{(red circles). (c) and (d) show the distribution of the order-parameter } m \text{ in the neighborhood of transition } \beta \text{ for four different values for the two update rules. The bimodal nature of the order-parameter distribution(c) and unimodal character with mean shifting towards right with increasing } \beta\text{ are characteristic of discontinuous and continuous transitions respectively. The distributions are calculated using 100000 iterations over 10 ensembles. All the calculations are done with } v_0 = 0.001 \text{ and } r = 0.01 \text{ and the probabilities of the order-parameter } p(\langle|m|\rangle) \Delta m = 0.01 \text{ being the bin-width, are for a system of } N = 1000 \text{ spins.} \)

arise due to the deviation of the distribution of $\langle |m| \rangle$ from Gaussian about mean $\langle |m| \rangle = 0$. We also calculate the distribution of the order-parameter $|m|$ in the neighborhood of transition $\beta$ for four different values for the two update rules (parallel(c) and random-sequential(d) respectively). The distributions show their respective properties of bimodality and unimodality which are characteristic of the two natures of transition: discontinuous and continuous.

III. CONCLUSIONS

We have studied flocking in one dimension using a collection of active Ising spins moving in the unit interval. We find that the dynamical properties of the system: both transient and steady-state are intrinsically related to the nature of the update rules used to simulate the system. The magnitude of fluctuations in the disordered state of the spin system is more for random-sequential updates as against parallel updates. In the state of long-range order, the flocks alternate between the allowed orientations for the two update rules, the frequency of which is dependent on the strength of local interactions as well the type of update. For a fixed strength of local interactions, systems with parallel updates are less alternating in comparison to its random-sequential counterpart. The differences in the update rules also reflect in the transition from disorder to long-range order, with discontinuous for parallel updates whereas continuous for random-sequential updates. The differences arise due to intrinsic randomness in the random-sequential update which makes such an evolution asynchronous as opposed to parallel update which is inherently synchronous. The present study has implications towards the current understanding of collective motion in one dimension, in particular their modeling on digital computers.