Can a hidden-variable-based system violate Bell’s inequality?

Mahmoud Bahnasy
Huawei Technologies Canada Co., Ltd, Waterloo, Ontario, Canada
Email: mmbahnasy@gmail.com

Abstract. Quantum mechanics predicts faster-than-light information transportation in order to satisfy quantum entanglement experimental results. Such a prediction violates the locality principle and Bell’s inequality. Quantum mechanisms also state that the world is intrinsically probabilistic. Such an idea was not welcome by many scientists including Einstein, Podolsky and Rosen. To resolve such a dilemma, the ”Many-world” theory was presented by Hugh Everett in 1975. Such a theory asserts that the universal wave function is real and that there is no wave function collapse. Also, it states that all possible outcomes of quantum measurements are physically realized in other parallel universes. Besides, in quantum mechanics, the observer effect defines a state transition of quantum objects. Such a phenomenon affirms that quantum objects behave differently depending on being observed or not.

In this paper, we present few assumptions based on which we build a classical hidden-variable-based system that can violate Bell’s inequality. Therefore, adopting similar assumptions in explaining quantum entanglement could lead to a better explanation of quantum mechanics without violating the locality principle or adopting such weird assumptions. Hence, one can conclude that extending Quantum Mechanism by a set of similar hidden variables can resolve such a paradox and predict the quantum entanglement experimental results without violating the locality principle.

1 Introduction
For a pair of particles that are generated with a total spin equal to zero, if the spin is measured on a certain axis, the total spin of the whole system must be zero. I.e., if the first particle is found to be spinning clockwise, the other particle must be spinning counterclockwise. This property is due to their entanglement. Figure 1 depicts a source that generates two entangled particles and sends each particle in one direction. The measured spinning angle of each particle must add up to zero. Therefore, the particles are entangled.
In quantum mechanisms, the observation causes the particle to collapse and change irreversibly its state [1]. For entangled particles, changing the state of one particle changes the state of the other particle which involves transferring information faster than the speed of light. This paradoxical phenomenon is known as Einstein, Podolsky, and Rosen (EPR) paradox [2, 3, 4]. Other theories, such as the ”many-world” theory, was emerged to explain the weird uncertainties principle in quantum mechanics by asserting that all possible outcomes of quantum measurements are physically realized in other worlds or universes [5]. Moreover, quantum mechanics states that observation changes the state of quantum objects. Thus, one must ask what defines an observer? and can quantum particles be considered as observers?. Such a long-standing debate is known as Wigner’s friend paradox [6]. According to classical mechanics, faster-than-light information transformation is impossible as it violates the locality principle [7]. Several interpretations are emerged based on the hidden-
variable concept to explain quantum entanglement. The most notably known theory that adopts such a concept is Pilot wave theory [8, 9] where the two particles contain a predetermined set of information about the spinning property. Therefore, no need for a faster-than-light information transformation.

John Bell presented an analogy where any physical theory that incorporates local realism cannot reproduce all the predictions of quantum mechanical theory [3]. However, in [10] the spinning of entangled particles was measured at separate locations and found that it statistically violates Bell’s inequality [11] while matches quantum mechanics prediction. Hence, the hidden local variable theorem can not correctly interpret the behavior of quantum particles.

On the other hand, several research projects have shown that quantum phenomena are explainable at macroscopic levels. E.g. Gatti et al. have shown that wave-particle duality features could be demonstrated in macroscopic level [12]. In addition, Couder et al. have depicted experimentally that a droplet bouncing on a surface of oscillating liquid could introduce similar behavior of that introduced by quantum particles [13].

In Section 3, we show that quantum entanglement could also be explained with the hidden variable concept because the three binary properties that were used in Bell’s inequality are based on one fundamental non-binary property; namely the spinning angle. Therefore, it is better to understand such binary variables as three separate tests that depend on one non-binary property rather than three binary properties. Taking that into consideration, we present a hidden-variable-based system that can pass or fail three different tests. We show numerically that such a system can violate Bell’s inequality. Hence, we conclude that quantum entanglement can be modeled using a set of hidden variables without violating the locality principle. However, a full representation of such a system requires a more complex set of equations which is left for future work.

2 Bell’s inequality

Bell’s inequality is best explained for a system with three binary properties such as a three-coin set. When such correlated coin sets are prepared, the values of the hidden variables are not completely specified. Therefore, any coin is as likely to be ahead (H) as it is to be tail (T). Measuring the probability of such three properties must sum up to one ($\sum P(x, y, z) = 1$).

For a three-coin system, we can measure the correlation between every two states as follows:

\[
\begin{align*}
P_{\text{same}}(1, 2) &= P(\text{HHH}) + P(\text{HHT}) + P(\text{THH}) + P(\text{THT}) \\
P_{\text{same}}(2, 3) &= P(\text{HHH}) + P(\text{TTH}) + P(\text{THT}) + P(\text{TTH}) \\
P_{\text{same}}(1, 3) &= P(\text{HHH}) + P(\text{HTT}) + P(\text{THH}) + P(\text{TTH})
\end{align*}
\]

Hence, the correlation of such a system can be expressed as follow:

\[
\sum_{x,y,z \in \{H,T\}} P(x, y, z) + 2P(\text{HHH}) + 2P(\text{TTH}) = 1 + 2P(\text{HHH}) + 2P(\text{TTH}) \geq 1
\]
Therefore, Bell’s inequality states that for any classical system with three binary property the probability of finding two similar states is always greater than one. To understand quantum entanglement based on such a concept, we plot in figure 2a, 2b and 2c the probability of spinning test measurement at any angle according to quantum theory, classical probability of a system with three intrinsic binary properties and our proposed deterministic model respectively.

![Figure 2a](quantum_mechanics_probability.png) ![Figure 2b](classical_probability.png) ![Figure 2c](p_up_vs_spinning_angle.png)

(a) Quantum mechanics Probability.  (b) Classical probability.  (c) $P_{Up}$ vs spinning angle $\phi$ at $\theta = 0$.

Figure 2: Probability of spinning up at test angle $\theta$.

Figure 2a shows the spinning up probability based on quantum mechanics predictions as $P_{Up} = (\cos((\theta - \phi)/2))^2$ where $\theta$ is the measurement angle and $\phi$ is the spinning angle. Several experimental studies shows that quantum system violates Inequality (2) at three different measurement angles $\theta$ [10, 14].

A numeral example that illustrate Bell’s inequality is expressed as follow: For a classical probability depicted in figure 2b, if the spinning angle of the particle is 0, and we measured the spinning up at angle $= 0$, we get $P(0) = 0.5$. If we also considered the spinning probability at angles $P_{Up}(\pi/2) = 1$ and $P_{Up}(\pi) = 0$ respectively. One can also notice that classical probability depicted in figure 2b obeys Bell’s inequality as explained in Equations (3) and (4).

$$P_{same}(1, 2) = P_{Up}(0) \cdot P_{Up}(\pi/2) + (1 - P_{Up}(0)) \cdot (1 - P_{Up}(\pi/2)) = 0.5 + 0,$$

$$P_{same}(2, 3) = P_{Up}(\pi/2) \cdot P_{Up}(\pi) + (1 - P_{Up}(\pi/2)) \cdot (1 - P_{Up}(\pi)) = 0 + 0.5,$$

$$P_{same}(1, 3) = P_{Up}(0) \cdot P_{Up}(\pi) + (1 - P_{Up}(0)) \cdot (1 - P_{Up}(\pi)) = 0.$$  

By substitution in Equation (2), we get:

$$P_{same}(1, 2) + P_{same}(2, 3) + P_{same}(1, 3) = 0.5 + 0 + 0.5 \geq 1 \quad \text{No violation} \quad (4)$$

One can see that such a classical probability always obeys Bell’s inequality. Therefore, quantum entanglement can’t be explained using such a model. For the deterministic model shown in figure 2c, such calculation is no longer straightforward as one can see the test results depend not only on the test angle $\theta$ but also on the spinning angle $\phi$ of the particle.

3 Classical system with hidden variables and deterministic test results

In this example, we demonstrate that a classical system with one non-binary property, namely spinning angle, can yield similar results as a quantum system. Such a system with deterministic test results can violate Bell’s inequality without violating the locality principle. The main three assumptions that we add to the original quantum model are:

(i) Three binary tests are performed on a quantum particle with one non-binary property.
(ii) Measurement causes the particle to collapse because of the interaction with another quantum system; namely the measurement device.
(iii) The test results are deterministic based on the particle state (spinning angle) and the measuring device (measuring angle).

As shown in figure 2c, the spinning angle is bounded between 0 and $2\pi$. The figure shows the result of measuring the spinning up at angle $\phi = 0$ is defined as finding the particle spinning within a range equals $\pi$ around the measurement angle $\theta$ and is represented in Equation (5), where 1 indicates spinning up and 0 indicates spinning down.

$$P(\theta) = \begin{cases} 
1 & \text{If } (\theta - \frac{\pi}{2}) < \phi \leq (\theta + \frac{\pi}{2}) \\
0 & \text{otherwise}
\end{cases}$$ (5)

For the two borders of the range, the probability of passing or failing the test is 50%. To accommodate for that, we include one border in the test and exclude the other; i.e., spinning test pass if $\phi \in (\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})$.

To simulate such a system, we implement algorithm 1 to define the spinning test result. As shown in Line 5 - 7, the test passes when the particle spinning angle is within the test range. Also, Line 8 - 13 specify the test result if the measuring period is reversed (i.e., $b > a$).

**Algorithm 1:** Test if a particle with spinning angle $\phi$ lies within the period $(a, b]$.

```
Function isWithinPeriod(\phi: int, a: int, b: int) : bool
1     \phi \leftarrow \phi \% 2\pi ;
2     a \leftarrow a \% 2\pi ;
3     b \leftarrow b \% 2\pi ;
4     if a < b then
5         if (\phi > a) && (\phi <= b) then
6             return True;
7         else
8             if (\phi > b) && (\phi <= a) then
9                 return False;
10            else
11                return True;
12         end if
13     end if
```

Additionally, we simulate the whole system using algorithm 2 where we randomly generate 1000 spinning angles $\phi$. We also calculate the measurement statistics using `getStats` function depicted in Line 10 - 20. Finally, the sum is calculated at Line 7. Full implementation of this algorithm is available on Github [15].

The results are shown in figure 3 where one can see that Bell’s inequality indeed has been violated (less than 1) for such a classical system because the test results are based on one fundamental property; namely spinning angle $\phi$, and deterministic behavior.
Algorithm 2: How to simulating Bell’s inequality.

Result: \( \text{stats} \)

1. Initialize \( \text{stats} = [] \);
2. for \( \theta \leftarrow 0 \) to \( \frac{\pi}{2} \) do
   3. for \( i \leftarrow 0 \) to 1000 particles do
      4. \( \phi = \text{uniform}(0, 2\pi) \);
      5. \( (P_{\text{same}(0, \theta)}, P_{\text{same}(\theta, 2\theta)}, P_{\text{same}(0, 2\theta)}) \leftarrow (0, 0, 0) \);
      6. \( (P_{\text{same}(0, \theta)}, P_{\text{same}(\theta, 2\theta)}, P_{\text{same}(0, 2\theta)}) + = \text{getStats}(\phi) \);
      7. \( \text{stats}.\text{append}(P_{\text{same}(0, \theta)} + P_{\text{same}(\theta, 2\theta)} + P_{\text{same}(0, 2\theta)}) \);
8. return \( \text{stats} \);
9. 

Function \( \text{getStats}(\phi: \text{int}) : (\text{int}, \text{int}, \text{int}) \)

10. \( P_{\text{same}(0, \theta)}, P_{\text{same}(\theta, 2\theta)}, P_{\text{same}(0, 2\theta)} = 0, 0, 0 \);
11. if \( \text{isWithinPeriod}(\phi, 0 - \frac{\pi}{2}, 0 + \frac{\pi}{2}) \) then
    12. if \( \text{isWithinPeriod}(\phi, \theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}) \) then
        13. \( P_{\text{same}(0, \theta)} = 1 \);
    14. if \( \text{isWithinPeriod}(\phi, 2\theta - \frac{\pi}{2}, 2\theta + \frac{\pi}{2}) \) then
        15. \( P_{\text{same}(0, 2\theta)} = 1 \);
    16. \( \text{otherwise} \) then
        17. \( P_{\text{same}(0, 2\theta)} = 1 \);
18. \( \text{otherwise} \) then
    19. \( \text{return} (P_{\text{same}(0, \theta)}, P_{\text{same}(\theta, 2\theta)}, P_{\text{same}(0, 2\theta)}) \);

![Figure 3: Validating \( P_{\text{same}(0, \theta)} + P_{\text{same}(\theta, 2\theta)} + P_{\text{same}(0, 2\theta)} \geq 1 \) inequality.](image)

4 Conclusion and future work

This paper depicts that a hidden-variable-based system could violate Bell’s inequality without violating the locality principle. The main assumption in such a system is it has one non-binary intrinsic property that depicts, in a deterministic way, three binary properties or test results. Based on such assumption, we conclude that quantum mechanics can be extended by adding a set of hidden variables to interpret quantum entanglement without violating the locality principle. However, a full representation of such a system requires a more complex set of equations which is left for future work.
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