Lorentz surfaces with constant curvature and their physical interpretation

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SUMMARY- In recent years it has been recognized that the hyperbolic numbers (an extension of complex numbers, defined as \( z = x + h y; \) \( h^2 = 1 \) \( x, y \in \mathbb{R}, h \notin \mathbb{R} \)) can be associated to space-time geometry as stated by the Lorentz transformations of special relativity.

In this paper we show that as the complex numbers had allowed the most complete and conclusive mathematical formalization of the constant curvature surfaces in the Euclidean space, in the same way the hyperbolic numbers allow a representation of constant curvature surfaces with non-definite line elements (Lorentz surfaces).

The results are obtained just as a consequence of the space-time symmetry stated by the Lorentz group, but, from a physical point of view, they give the right link between fields and curvature as postulated by general relativity.

This mathematical formalization can open new ways for application in the studies of field theories.

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1 Introduction

The Riemann surfaces with constant curvature (definite line element) form one of the essential topics of geometry, while the Lorentz surfaces (non-definite line element) are relevant for physics [1, 2]. A milestone in the study of the former is the application of complex numbers theory, that yields the best mathematical formalization [3, 4, 5]. Indeed complex analysis allows to use the most suitable formalism, because the groups associated with complex numbers are the same as those of the Euclidean geometry. More precisely, as it is well known, the finite additive and unimodular multiplicative groups of complex numbers correspond to roto-translation in the Euclidean geometry.

The infinite dimensional conformal group of functional transformations can be fruitfully applied for the formalization of the extensions of the Euclidean geometry as those represented by the non-Euclidean geometries and the differential geometry [4, § 13].

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The physical importance of constant curvature surfaces, represented by a non-definite quadratic form (also called Lorentz surfaces or hyperbolic), derives from special relativity which states that a spatial coordinate and time are linked in the framework of a new geometry, the space-time geometry, formulated by the Lorentz transformations of special relativity.

In recent years a system of numbers, previously introduced by S. Lie as a two-dimensional example of hypercomplex numbers \([6]\), has been associated with space-time geometry \([3]\). This system of numbers, called hyperbolic \([7]\) and defined as:

\[
\{ z = x + h y; \ h^2 = 1; \ x, y \in \mathbb{R}, \ h \not\in \mathbb{R} \}
\]

could play the same role for the pseudo-Euclidean geometry as that played by complex numbers in the Euclidean geometry. However, in spite of their potentialities \([8, 9]\), their applications are not yet comparable with those of complex numbers.

In the present work we use these numbers and the methods of classical differential geometry for studying the surfaces described by non-definite line elements. We shall see that the expressions of the line elements and the equations of the geodesics written as functions of the hyperbolic variable are the same as the analogous expressions for constant curvature Riemann surfaces written as functions of the complex variable. In a similar way motions on these surfaces may be expressed by hyperbolic bilinear transformations.

If we give to the variables \(x, y\) the physical meaning of time and a space variable, respectively, the geodesics on constant curvature Lorentz surfaces represent the relativistic hyperbolic motion \([3]\). This result has been obtained just as a consequence of the space-time symmetry stated by Lorentz group, but it goes beyond the special relativity, because the constant acceleration is linked to the curvature of the surfaces, in agreement with the postulate that Einstein stated for the formalization of the general relativity.

The paper is organized in the following way: in sect. 2 the basic concepts on hyperbolic numbers and on geometry of the pseudo-Euclidean plane are briefly summarized. For a more complete introduction the reader is referred to \([3, 9]\). In sect. 3, by applying the results obtained in \([9, 10]\) and by using hyperbolic numbers, the classical results obtained for constant curvature Riemann surfaces \([4, 5, 11, 12]\) are extended to the constant curvature Lorentz surfaces. In sect. 4 a method is proposed for determining the geodesic for two points and the geodesic distance between the same points, as a function of the point coordinates. The method may be applied to negative as well as positive constant curvature surfaces described by definite or non-definite line elements. In sect. 5 the physical meaning of the obtained results is given. An appendix is added for recalling some concepts of classical differential geometry used in this work that do not appear in recent books.

## 2 Some concepts of the pseudo-Euclidean plane geometry

### 2.1 Hyperbolic numbers: basic concepts

Here we briefly summarize some fundamental properties of hyperbolic numbers. This number system has been introduced by S. Lie \([6]\) as a two-dimensional example of the more general class of the hypercomplex number systems. Today the hyperbolic numbers are also included in the Clifford algebras \([13]\).

Let us now introduce a hyperbolic plane by analogy with the Gauss-Argand plane of the complex variable. In this plane we associate the points \(P \equiv (x, y)\) with hyperbolic numbers \(z = x + h y\). If we represent these numbers on a Cartesian plane, in this plane the square distance \((D)\) of the point

\[
D = x^2 - h^2 y^2
\]
from the origin of the coordinate axes ($O$) is defined as:

$$D = z \tilde{z} \equiv x^2 - y^2,$$  \hspace{1cm} (1)

where, in analogy with complex analysis, we have indicated by $\tilde{z} = x - h y$ the hyperbolic conjugate of $z$.

The definition of distance (metric element) is equivalent to introduce the bilinear form of the scalar product. The scalar product and the properties of hypercomplex numbers allow to state suitable axioms \[3, p. 245\] and to give the pseudo-Euclidean plane the structure of a vector space.

Let us consider the multiplicative inverse of $z$ that, if it exists, is given by: $1/z \equiv \tilde{z}/z\tilde{z}$. This implies that $z$ does not have an inverse when $z\tilde{z} \equiv x^2 - y^2 = 0$, i.e., when $y = \pm x$, or alternatively when $z = x \pm h x$. In the language of algebras, these numbers are called divisors of zero. The two straight-lines $y = \pm x$ divide the hyperbolic plane in four sectors, and, this property is the same as that of the special relativity representative plane and this correspondence gives a physical meaning (space-time interval) to the definition of distance. Given two points $P_j \equiv (x_j, y_j), P_k \equiv (x_k, y_k)$ we define the square distance between them by extending eq. (1):

$$D_{j,k} = (z_j - z_k)(\tilde{z}_j - \tilde{z}_k).$$

The hyperbolic plane has the same characteristics of the pseudo-Euclidean flat plane [8, 9] described by the line element:

$$ds^2 = e (dx^2 - dy^2),$$

where $e = \pm 1$, in a way that $ds^2 \geq 0$.

### 2.2 Motions in the pseudo-Euclidean plane

It is well known that motions (translations, rotations and inversions) in the Euclidean plane are well described by complex numbers \[4, p. 45\]. In the same way motions in the pseudo-Euclidean plane can be described by a mapping $z \rightarrow w$ such that \[3, p. 276\]:

\[\begin{align*}
(I) & \quad w = az + b \\
(II) & \quad w = a\tilde{z} + b \quad (a\tilde{a} = 1 \Rightarrow a = \cosh \theta + h \sinh \theta)
\end{align*}\]

where $w = p + h q$ and $z = x + h y$ are hyperbolic variables and $a, b$ are hyperbolic constants. Eq. (3, I) represents a hyperbolic rotation about $(O)$ through the hyperbolic angle $\theta$ \[3, 8, 9\], followed by a translation by $b$. Eq. (3, II) represents the same motion and a reflection in the line $y = 0$. It is easy to check that the mappings (I) and (II) in eq. (3) are square-distance-preserving. From a geometrical point of view, because the curves $z\tilde{z} = \text{const}$ represent the four arms of the equilateral hyperbolas centered in $O$, the mappings (3) preserve these hyperbolas. Each of the above-mentioned arms is inside one of the four sectors in which the hyperbolic plane is divided by the straight lines (null lines) $x \pm y = 0$ \[3, 7\].

### 2.3 The geodesics in the pseudo-Euclidean plane

By means of the method summarized in appendix A we can find the geodesics in the pseudo-Euclidean plane, described by the line element [2]. At first we have to solve the partial differential equation with constant coefficients:

$$\Delta_1 \tau \equiv \left(\frac{\partial \tau}{\partial x}\right)^2 - \left(\frac{\partial \tau}{\partial y}\right)^2 = e \equiv \pm 1.$$
The elementary solution is given by \( \tau = Ax + By + C \) with the condition \( A^2 - B^2 = \pm 1 \). Then we can put:

\[
A = \cosh \theta, \quad B = \sinh \theta, \quad \text{if} \quad \Delta_1 \tau = +1 ; \quad (4)
\]
\[
B = \cosh \theta, \quad A = \sinh \theta, \quad \text{if} \quad \Delta_1 \tau = -1 . \quad (5)
\]

The equations of the geodesics are obtained equating \( \partial \tau / \partial \theta \) to a constant \( c \), then:

\[
\text{if} \quad \Delta_1 \tau = +1 \quad \Rightarrow \quad x \sinh \theta + y \cosh \theta = c
\]
\[
\text{if} \quad \Delta_1 \tau = -1 \quad \Rightarrow \quad x \cosh \theta + y \sinh \theta = c.
\]

In the pseudo-Euclidean plane it is more appropriate to write the straight line equations by means of hyperbolic trigonometric functions instead of circular trigonometric functions as it is done in the Euclidean plane.

We note that the integration of the equations of the geodesics in the Euler form (see eq. 32 in App. A) would not give the condition on the integration constants of eqs. (4, 5) that allows one to introduce the hyperbolic trigonometric functions in a natural way.

Following [3, p. 179], a segment or a straight line is said to be of the first (second) kind if it is parallel to a line through the origin located in the sectors containing the axis Ox (Oy). For a physical interpretation (special relativity), we give the \( x \) variable the physical meaning of the normalized (speed of light \( c = 1 \)) time variable and to \( y \) the physical meaning of the space variable: the lines of the first (second) kind are called, in special relativity, timelike (spacelike) \[1, 3, 14\]. As far as the hyperbolas are concerned, we may extend the definition for the segments and straight lines and call hyperbolas of first (second) kind those ones for which the tangent straight lines are of the first (second) kind \[1, p. 9\]. Following our procedure for calculating the equations of the geodesics, we automatically obtain lines of the first kind in the case of \( \Delta_1 \tau = +1 \) and lines of the second kind in the case of \( \Delta_1 \tau = -1 \).

### 3 Constant curvature Lorentz surfaces

#### 3.1 Line element

The sphere in three-dimensional Euclidean space and the two-sheet hyperboloid in three-dimensional Minkowski space-time are considered as examples of positive and negative constant curvature Riemann surfaces, respectively \[4\].

These results are examples of a more general theorem \[15, p. 117\], \[16, p. 201\]. Before introducing this theorem, let us point out that in a flat space with line element

\[
ds^2 = \sum_{\alpha=0}^{N} c_{\alpha} (dx^\alpha)^2 \quad \text{with } c_{\alpha} \text{ constants} \quad (6)
\]

the surfaces:

\[
\sum_{\alpha=0}^{N} c_{\alpha} (x^\alpha)^2 = \pm R^2 , \quad \text{with the same } c_{\alpha} \text{ of eq. (6),}
\]

are called fundamental hyperquadrics \[15, 16\]. For these surfaces the following theorem holds: the fundamental hyperquadrics are \( N \)-dimensional spaces of constant Riemannian curvature and are the only surfaces of constant Riemannian curvature of a \((N + 1)\)-dimensional flat semi-Riemannian space.
In particular, the constant curvature is positive if in the equation of the fundamental hyperquadric there is \( +R^2 \) and negative if there is \( -R^2 \).

This theorem could be used for finding the line element of Lorentz constant curvature surfaces, but here we follow a direct approach.

Let us consider a Lorentz surface with a line element given by:

\[
\text{ds}^2 = du^2 - r^2(u) \, dv^2. \tag{7}
\]

For surfaces with the line element given by eq. (7) the Gauss curvature \( K \) is given by \[4\]:

\[
K = -\frac{1}{r(u)} \frac{d^2 r(u)}{du^2}. \]

In particular if we put \( K = \pm R^{-2} \), with a suitable choice of the initial conditions, we obtain: for positive constant curvature surfaces (PCC):

\[
\text{ds}^2 = du^2 - R^2 \sin^2(u/R) \, dv^2, \tag{8}
\]

for negative constant curvature surfaces (NCC):

\[
\text{ds}^2 = du^2 - R^2 \sinh^2(u/R) \, dv^2. \tag{9}
\]

### 3.2 Isometric forms of the line elements

Following the same procedure generally used for the definite line elements also the non-definite line elements can be transformed in the isometric form \[4\]:

\[
\text{ds}^2 = f(\rho, \phi)(d\rho^2 - d\phi^2). \tag{10}
\]

In particular by means of the transformation:

\[
\rho = -\int \frac{du}{R \sin(u/R)} \equiv \ln \cot(u/2R); \quad \phi = v
\]

for the line element (8) and the transformation:

\[
\rho = -\int \frac{du}{R \sinh(u/R)} \equiv \ln \tanh(u/2R); \quad \phi = v
\]

for the line element (9), we obtain the line elements in the isometric form:

\[
\text{ds}^2 = R^2 \frac{d\rho^2 - d\phi^2}{\cosh^2 \rho} \quad \text{for PCC}, \quad \text{ds}^2 = R^2 \frac{d\rho^2 - d\phi^2}{\sinh^2 \rho} \quad \text{for NCC}. \tag{11}
\]

Moreover these isometric forms are preserved by transformations with functions of the hyperbolic variable. In fact by introducing the hyperbolic variable, eq. (10) becomes \( ds^2 = g(z, \bar{z})dz d\bar{z} \) and, by a hyperbolic transformation \( z = F(w) \), we obtain \( ds^2 = g_1(w, \bar{w})|F'(w)|^2 dw d\bar{w} \[10\].

Then, the transformations by means of functions of the hyperbolic variable are the conformal transformations for the non-definite differential quadratic forms \[10\].

Now by means of the hyperbolic exponential transformation \[9, 17\]:

\[
x + hy = R \exp[\rho + h \phi] \equiv R \exp[\rho](\cosh \phi + h \sinh \phi), \tag{12}
\]

eqs. (11) can be rewritten in the form:

\[
\text{ds}^2 = 4R^4 \frac{dx^2 - dy^2}{(R^2 + x^2 - y^2)^2} \quad \text{for PCC}, \quad \text{ds}^2 = 4R^4 \frac{dx^2 - dy^2}{(R^2 - x^2 + y^2)^2} \quad \text{for NCC}, \tag{13}
\]

where \( x, y \) can be considered the coordinates in a Cartesian representation.
3.3 The equations of the geodesics

By means of the method summarized in appendix A, we can find the equations of the geodesics for a surface represented by the line element:

\[ ds^2 = f^2(\rho)(d\rho^2 - d\phi^2). \]

The first step is the resolution of the partial differential equation:

\[ \Delta_1 \tau \equiv \left( \frac{\partial \tau}{\partial \rho} \right)^2 - \left( \frac{\partial \tau}{\partial \phi} \right)^2 = f^2(\rho). \]

Here we follow a direct procedure, without using the general solution given in the appendix A. By the substitution \( \tau = A\phi + \tau_1(\rho) + C \), where \( A, C \) are constants, we obtain the solution in integral form:

\[ \tau = A\phi + \int \sqrt{f^2(\rho) + A^2} \, d\rho + C. \]

The equation of the geodesic is given by:

\[ \frac{\partial \tau}{\partial A} \equiv \phi + \int \frac{A \, d\rho}{\sqrt{f^2(\rho) + A^2}} = B. \]

From eqs. (14, 15) we obtain a relation between \( \rho \) and the line parameter \( \tau \):

\[ \tau - A \frac{\partial \tau}{\partial A} \equiv \tau - AB = \int \frac{f^2(\rho) \, d\rho}{\sqrt{f^2(\rho) + A^2}}. \]

3.3.1 Application to constant curvature surfaces

Let us consider a positive constant curvature surface with the line element given by eq. (11). For this case we have \( f^2(\rho) = R^2 \cosh^{-2} \rho \) and eqs. (14, 15, 16) become, respectively:

\[ \tau = A\phi + \int \sqrt{R^2 + A^2 \cosh^2 \rho} \, d\rho + C \]

\[ \sqrt{(R/A)^2 + 1} \sinh(B - \phi) = \sinh \rho \]

\[ \tau - AB = R \sin^{-1} \left[ \frac{\tanh \rho}{\sqrt{1 + (A/R)^2}} \right]. \]

In order to obtain simplified final expressions we put in eqs. (17, 18) \( A = R \sin \epsilon, \; B = \sigma \).

In a similar way we obtain the equations of the geodesics for the negative constant curvature surfaces represented by eq. (11). In this case we put \( A = R \sin \epsilon, \; B = \sigma \).

It is well known from differential geometry [15] that if we transform a line element, the same transformation holds for the equations of the geodesics. Therefore by substituting eq. (12) in eq. (17) we obtain the equations of the geodesics in a Cartesian \( x, y \) plane. In fact eq. (17) can be written

\[ 2 \exp[\rho](\sinh \phi \cosh \sigma - \cosh \phi \sinh \sigma) = \tanh \epsilon(\exp|2\rho| - 1). \]

Substituting the \( x, y \) variables from eq. (12) the expressions in tab. 1 are obtained.

All the results are summarized in tab. 1. In the same table the data for constant curvature surfaces with definite line elements are also reported. As far as the Cartesian representation is concerned we report the line elements and the equations of the geodesics as functions of complex (definite line elements 1-2) or hyperbolic (non-definite line elements 3-4) variables. These expressions for definite and non-definite line elements are the same if they are written as functions of the \( z \) variable [3].
3.3.2 The “limiting curves”

The results reported in tab. 1 for constant curvature surfaces with definite line element are well known and the geodesics are represented in the $x,y$ plane by circles limited by the “limiting circle”. The equation of the limiting circle is obtained equating to zero the denominator of the line element in Cartesian coordinates. For negative constant curvature surfaces this equation is given by $x^2 + y^2 = R^2$, while for positive constant curvature surfaces it is given by $x^2 + y^2 = -R^2$, representing a circle with imaginary points. Then for positive constant curvature surfaces the geodesics are complete circles. Here let us show that the same situation applies to constant curvature surfaces with non definite line elements.

For positive as well as for negative constant curvature surfaces the geodesics are hyperbolas of the form:

$$(y - y_0)^2 - (x - x_0)^2 = d^2$$

where $x_0$, $y_0$, $d$ depend on two parameters, as it can be obtained from the equations of tab. 1. From the same equations we see that for $\epsilon \to 0$ ($A \to 0$) the geodesics are given by straight lines through the coordinate axes origin, as it happens for Riemann constant curvature surfaces.

The expressions of $x_0$, $y_0$, $d$ as functions of $A$ and $B$ can be obtained from the equations reported in tab. 1. In particular, the half diameter $d$ of the geodesic hyperbolas, as well as the radius of the geodesic circles on the constant curvature Riemann surfaces are:

$$d = \frac{R^2}{A} \equiv \frac{1}{A|K|}.$$  \hspace{1cm} (20)

The limiting hyperbolas are:

$$y^2 - x^2 = R^2 \quad \text{for PCC;} \quad x^2 - y^2 = R^2 \quad \text{for NCC.}$$

The former limiting hyperbola does not intersect the geodesic hyperbolas. For the latter we find the intersecting points by subtracting the equation of the limiting hyperbola ($\Phi_2 \equiv [x^2 - y^2 - R^2 = 0]$) multiplied by $\tan \epsilon$ from the geodesic hyperbolas ($\Phi_1 \equiv [(x^2-y^2+R^2) \tan \epsilon - 2R(x \sinh \sigma - y \cosh \sigma) = 0]$). In this way we have the system:

$$x^2 - y^2 - R^2 = 0$$

$$R \tan \epsilon - x \sinh \sigma + y \cosh \sigma = 0$$  \hspace{1cm} (21)

Let us calculate the crossing angle between $\Phi_1$ and $\Phi_2$: the cosine of this angle is proportional to the scalar product (in the metric of the hyperbolic plane) of the gradients to $\Phi_1$ and $\Phi_2$ in their crossing points:

$$\frac{\partial \Phi_1}{\partial x} \frac{\partial \Phi_2}{\partial x} - \frac{\partial \Phi_1}{\partial y} \frac{\partial \Phi_2}{\partial y} = 4 \left[ \tan \epsilon (x^2 - y^2) - R x \sinh \sigma + R y \cosh \sigma \right].$$

Since in the crossing points $x^2 - y^2 = R^2$, due to eq. (21), this product is zero. Then, as for definite line elements, $\Phi_1$ and $\Phi_2$ are pseudo-orthogonal [9, 14, 16].

We note that this property, as many others, can be considered a direct consequence of the fact that the equilateral hyperbolas represented in the pseudo-Euclidean plane satisfy the same theorems as the circles in the Euclidean plane, as shown in [3, 9]. From an algebraic point of view this also follows from the fact that the equations of circles and equilateral hyperbolas are the same, if they are expressed in terms of complex and hyperbolic variables, respectively.
3.4 Motions

In this subsection and in the next sect. 4 we follow [4, pag. 120], and use normalized variables 
\( x/R, y/R \Rightarrow x, y \). The line elements of eqs. (13) become:

\[
\frac{\sqrt{ds^2}}{4R^2} = \frac{dx^2 - dy^2}{(1 + x^2 - y^2)^2} \quad \text{for PCC}, \quad \frac{\sqrt{ds^2}}{4R^2} = \frac{dx^2 - dy^2}{(1 - x^2 + y^2)^2} \quad \text{for NCC}.
\]

It is well known that motions (transformations that leave unchanged the expression of the line 
element) on constant curvature surfaces with definite line element are described by means of bilinear 
complex transformations [4, 11]. Here we will show that the same result is obtained using hyperbolic 
numbers for constant curvature surfaces with non definite line elements. As it can be seen in tab. 
the expressions of the line elements of the Euclidean and pseudo-Euclidean constant curvature 
surfaces are the same, if they are written in terms of complex or hyperbolic variables, respectively. 
Then, also the transformations that leave unchanged the line elements are the same, if they are 
written in terms of complex or hyperbolic variables. On the other hand, one can find directly 
the bilinear hyperbolic transformations that represent motions on the constant curvature Lorentz 
surfaces, following the procedure of [4, p. 121] for the constant curvature Riemann surfaces. In 
conclusion, calling \( z = x + iy \) and \( w = p + hq \) two hyperbolic variables and \( \alpha, \beta \) two hyperbolic 
constants, one obtains the following expressions for the motions:

\[
\begin{align*}
  w &= \frac{\alpha z + \beta}{-\beta z + \alpha} ; \quad (\alpha \bar{\alpha} + \beta \bar{\beta} \neq 0) \quad \text{for PCC}; \\
  w &= \frac{\alpha z + \beta}{\beta z + \alpha} ; \quad (\alpha \bar{\alpha} - \beta \bar{\beta} \neq 0) \quad \text{for NCC}.
\end{align*}
\]

These transformations are also reported in [3, p. 288], without demonstration. They depend on two 
hyperbolic constants linked by a relation, i.e., they actually depend on three real constants.

4 The equation of the geodesic and geodesic distance between two points

Let us remind from differential geometry that two points on a surface generally determine a geodesic 
and the distance between these points can be calculated by a line integral of the linear line element. 
It is known that for the constant curvature surfaces here considered, the equation of the geodesic 
as well as the geodesic distance can be determined in an algebraic way as functions of the point 
coordinates. In what follows we determine these expressions for all the four constant curvature 
Lorentz and Riemann surfaces. For the Lorentz surfaces just points for which the geodesic exists [15, 
p. 150] are considered.
The method we propose is based on the results already obtained, i.e., the motions that transform 
geodesic lines in geodesic lines are given by bilinear transformations. We proceed in the following way: 
we take the points \( P_1, P_2 \) in the complex (hyperbolic) representative plane \( z = x + iy \) (\( z = x + hy \)) 
and look for the parameters of the bilinear transformation (22) or (23) that maps these points on the 
geodesic straight line \( q = 0 \) of the complex (hyperbolic) plane \( w = p + iq \) (\( w = p + hq \)). The inverse 
mapping of this straight line will give the equation of the geodesic determined by the given points. 
Moreover this approach allows to obtain the distance between two points as a function of the point 
coordinates.
As far as the distance is concerned we shall show that the proposed method works for positive 
constant curvature surfaces as well as for negative ones.
4.1 The equation of the geodesic

Here we consider just positive constant curvature Lorentz surfaces. Results for the other Lorentz and Riemann constant curvature surfaces, obtained in a similar way, are summarized in tab. 2. Let us consider the two points:

\[ P_1 : (z_1 = x_1 + h y_1 \equiv \rho_1 \exp[h \theta_1]) ; \quad P_2 : (z_2 = x_2 + h y_2 \equiv \rho_2 \exp[h \theta_2]) \]

and look for the parameters \((\alpha, \beta)\) of the bilinear transformation that maps \(P_1, P_2\) in the points \(P_O \equiv (0, 0)\) and \(P_l \equiv (l, 0)\), respectively, of the plane \(w = p + h q\). The \(\alpha, \beta\) parameters are obtained by solving the system:

\[
\begin{align*}
    w_1 &= 0 \rightarrow \alpha z_1 = -\beta; \\
    w_2 &= l \rightarrow \alpha z_2 + \beta = l(-\tilde{\beta} z_2 + \tilde{\alpha}).
\end{align*}
\]

By putting \(\alpha = \rho_\alpha \exp[h \theta_\alpha], \beta = \rho_\beta \exp[h \theta_\beta]\) we obtain:

\[
\begin{align*}
    \rho_\beta \exp[h \theta_\beta] &= -\rho_\alpha \rho_1 \exp[h(\theta_\alpha + \theta_1)] \quad (24) \\
    (z_2 - z_1) \exp[2h \theta_\alpha] &= l(1 + \tilde{\alpha} z_2). \quad (25)
\end{align*}
\]

Eq. (25) can be rewritten as:

\[
l = \frac{|z_2 - z_1|}{|1 + \tilde{\alpha} z_2|} \exp \left[ h \left( 2\theta_\alpha + \arg \left( \frac{z_2 - z_1}{1 + \tilde{\alpha} z_2} \right) \right) \right].
\]

Since \(l\) is real, we obtain:

\[
2\theta_\alpha + \arg \left( \frac{z_2 - z_1}{1 + \tilde{\alpha} z_2} \right) = 0, \quad l = \frac{|z_2 - z_1|}{|1 + \tilde{\alpha} z_2|}.
\]

The constants of the bilinear transformations are given but for a multiplicative constant; therefore putting \(\rho_\alpha = 1\), we obtain from eq. (24):

\[
\theta_\beta = \theta_\alpha + \theta_1, \quad \rho_\beta = -\rho_1. \quad (26)
\]

Now the equation of the geodesic between the points \(P_1, P_2\) is derived by the transformation of the geodesic straight line \(w - \tilde{w} = 0\) by means of eq. (22):

\[
(x^2 - y^2 - 1)\rho_1 \sinh(\theta_\alpha + \theta_\beta) + x(\sinh 2\theta_\alpha - \rho_1^2 \sinh 2\theta_\beta) + y(\cosh 2\theta_\alpha + \rho_1^2 \cosh 2\theta_\beta) = 0. \quad (27)
\]

Similarly, for the positive constant curvature surfaces with definite line element, by putting \(\alpha = \rho_\alpha \exp[i \psi_\alpha], \beta = \rho_\beta \exp[i \psi_\beta]\), eq. (24) becomes \(\rho_\beta \exp[i \psi_\beta] = -\rho_\alpha \rho_1 \exp[i(\psi_\alpha + \psi_1)]\). Substituting, in the right-hand side \(-1 = \exp[i \pi]\) we obtain:

\[
\psi_\beta = \pi + \psi_\alpha + \psi_1, \quad \rho_\beta = \rho_1.
\]

The parameters of the transformations and the equations of the geodesics for the four constant curvature surfaces are reported in tab. 2.
4.2 Geodesic distance

The transformations discussed above allow one to find the geodesic distance $\delta(z_1, z_2)$ between two points $P_1$ and $P_2$ as a function of the point coordinates. If we take a negative constant curvature Lorentz surface, using the line element (4) in tab. I, the distance between $P_0 \equiv (0, 0)$ and $P_l \equiv (l, 0)$ is given by

$$\delta(0, l) = 2R \int_0^l \frac{dp}{1 - p^2} = 2R \cdot \tanh^{-1} l \equiv R \ln \frac{1 + l}{1 - l}.$$ 

This equation can be written as a cross ratio [12, p. 182], i.e., in a form that is invariant with respect to bilinear transformations [3, p. 263], [5, p. 57]. In fact we have

$$\frac{1 + l}{1 - l} \equiv (1, -1, 0, l).$$

The points 1 and -1 are the intersecting points between the geodesic straight line $q = 0$ and the limiting hyperbola $x^2 - y^2 = 1$.

Replacing $l$ with the expression as a function of the point coordinates, given in tab. 2, we obtain:

$$\delta(z_1, z_2) = 2R \tanh^{-1} \left| \frac{z_1 - z_2}{1 - \bar{z}_1 \bar{z}_2} \right|. \quad (28)$$

Eq. (28) is also valid for negative constant curvature Riemann surfaces. For these surfaces the same result has been obtained in a different way, as reported in [5, p. 57].

These results can be extended to positive constant curvature surfaces: also for the positive constant curvature surfaces the geodesic distance between two points is a function of a cross ratio.

By means of the line element (1) of tab. I we calculate the distance between the points $P_0$ and $P_l$ for definite line element:

$$\delta(0, l) = 2R \int_0^l \frac{dp}{1 + p^2} = 2R \arctan l \equiv R \ln \frac{i - l}{i + l} \equiv R \ln(-i, i, 0, l).$$

Now $-i, i$ are the intersecting points between the geodesic straight line $q = 0$ and the limiting circle $x^2 + y^2 = -1$. Since the properties of the cross ratio are valid also for imaginary elements, we can again substitute for $l$ its expression reported in tab. 2, obtaining:

$$\delta(z_1, z_2) = 2R \arctan \left| \frac{z_1 - z_2}{1 + \bar{z}_1 \bar{z}_2} \right|. \quad (29)$$

If we apply the same procedure for non definite line elements, we obtain the same expression (29) where $z_1$ and $z_2$ are hyperbolic variables.

5 Physical interpretation of geodesics on Riemann and Lorentz surfaces with positive constant curvature

5.1 The sphere

The relationship existing between the curvature of bidimensional surfaces and the Laplace equation is known [11, p. 118], so we can look for a physical interpretation in the light of this equation. A geodesic circle in the $x, y$ representation (tab. I, row (1)) can be considered as an equipotential curve generated by a point source in its center. On the other hand the geodesic circles have the geometrical
meaning of stereographic projections, from the northern pole to the equatorial plane, of the geodesic great circles on the sphere [4, p. 96]. This projection induces on the plane \( x, y \) a Gauss metric, so that the radius of the geodesic circle on the plane depends both on the radius of the sphere and on the position (connected with the constant \( A \)) of the great circle on the sphere. In fact from the data in table 1 we obtain that the radiuses of these circles are inversely proportional to the constant \( A \) times the curvature of the starting surface: \( r = 1/(A K) \). Then the parametric equations of these circles are given by:

\[
x = x_0 + (A K)^{-1} \cos[A K s], \quad y = y_0 + (A K)^{-1} \sin[A K s],
\]

where \( s \) indicate the line element and \( x_0, y_0 \) the center coordinates.

5.2 The Lorentz surfaces

Following the positions in subsect. 2.3 we give the \( x \) variable the physical meaning of a normalized (speed of light \( c = 1 \)) time variable and the \( y \) variable the physical meaning of a space variable. The geodesics of eq. (19), taking into account eq. (20), are given by the hyperbolas: \((x - x_0)^2 - (t - t_0)^2 = 1/(A K)^2\) where \( K \) is the Gauss curvature. By following the same approach used for writing eqs. (30), we can write the equations of hyperbolas in a parametric form as functions of the line element:

\[
t = t_0 + (A K)^{-1} \sinh[A K s], \quad x = x_0 + (A K)^{-1} \cosh[A K s].
\]

Comparing eqs (31) with those of the hyperbolic motion [19, p. 166], i.e., \( t = t_0 + (g)^{-1} \sinh g \tau; \quad x = x_0 + (g)^{-1} \cosh g \tau \) (\( g \) is the constant acceleration), we see that the geodesics in a plane with a constant curvature metric (eq. (31)) are the same as those resulting from a motion with constant acceleration. Here we note that the result of eq. (31) has been obtained using the space-time symmetry as stated by Lorentz transformations.

It is known that Einstein, for formalizing the general relativity, started from the equivalence principle and postulated that the gravitational field would be described by the curvature tensor in a non-flat space. Here we have obtained the relation between (gravitational) fields and the space curvature without the need for this postulate.

6 Conclusions

By using the methods of differential geometry we have extended the studies on constant curvature Riemann surfaces to constant curvature Lorentz surfaces. In this paper we have shown that hyperbolic numbers are a very efficient tool for studying constant curvature Lorentz surfaces, in the same way as complex numbers are for the Euclidean space. Then the suitable mathematics for describing space-time is the mathematics of hyperbolic numbers [8, 9, 17]. Moreover the similarities between complex and hyperbolic numbers establish an important link between the Euclidean and pseudo-Euclidean geometries. Added to this we have shown that the obtained results are important from a physical point of view. In fact if we give to the \( x \) variable the physical meaning of a normalized time variable, the geodesics on a constant curvature Lorentz surfaces, represented in a Cartesian \( t, x \) plane, give the hyperbolic motion of special relativity. Moreover acceleration in this motion is proportional to the surface curvature and, in this way, we have obtained a confirmation of a postulate of the general relativity.
A Differential parameter and the equations of the geodesics

We recall some concepts of classical differential geometry [11, 16] that do not appear in recent books [15, 18]. In particular we report a method for obtaining the equations of the geodesics firstly obtained by E. Beltrami [20, vol. I, p. 366] who extended to the line elements the Hamilton-Jacobi integration method, that is used for integrating the dynamics equations. In general, it is not convenient to consider a partial differential equation instead of the ordinary Euler equation:

\[
\frac{d^2 x^i}{ds^2} + \sum_{i, k=1}^{N} \Gamma^i_{ik} \frac{dx^i}{ds} \frac{dx^k}{ds} = 0,
\]

except for particular line elements as, for instance, the examples considered in this paper or the Schwarzschild (general relativity) metric.

Let us consider a N-dimensional space with a semi-Riemannian line element given by [16, p. 39]:

\[
ds^2 = e \sum_{i,k=1}^{N} g_{ik} dx^i dx^k,
\]

where \(e = \pm 1\) in a way that \(ds^2 \geq 0\).

If \(\tau\) is any function of the \(x^i\), the function defined by:

\[
\Delta_1 \tau = \sum_{i, k=1}^{N} g^{ik} \frac{\partial \tau}{\partial x_i} \frac{\partial \tau}{\partial x_k}
\]

is called Beltrami differential parameter of the first order [16, p. 41], [21, p. 476]; in eq. \(34\) \(g^{ik}\) are the reciprocal elements of the metric tensor \(g_{ik}\).

Let us now consider the non-linear partial differential equation of the first order:

\[
\Delta_1 \tau = e,
\]

where \(e\) is the same as in eq. \(33\).

The solution of this equation depends on an additive constant and on \(N - 1\) essential constants \(A_i\) [16]. Now if we know the complete solution of eq. \(35\), we can obtain the equations of the geodesics by the following theorem [11 p. 299], [16 p. 59]: when a complete solution of eq. \(35\) is known, the equations of the geodesics are given by \(\partial \tau / \partial A_i = B_i\) where \(A_i, B_i\) are arbitrary constants, and the arc of the geodesics is given by the value of \(\tau\).

In particular if the line element has the generalized Liouville form, namely:

\[
ds^2 = e \left[ X_1(x_1) + X_2(x_2) + ... + X_N(x_N) \right] \sum_{i=1}^{N} e_i (dx^i)^2,
\]

where \(e_i = \pm 1\), a complete integral of eq. \(35\) is

\[
\tau = C + \sum_{i=1}^{N} \int \sqrt{e_i (e X_i + A_i)} dx_i,
\]

where \(C\) and \(A_i\) are constants, the latter being subject to the condition \(\sum_{i=1}^{N} A_i = 0\) [11 Vol. II, p. 426], [16 p. 60], [18 p. 263]. The equations of the geodesics are given, in an integral form, by:

\[
\frac{\partial \tau}{\partial A_i} \equiv \frac{1}{2} \int \frac{e_i dx_i}{\sqrt{e_i (e X_i + A_i)}} = B_i.
\]
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Table 1: Line elements and equations of the geodesics for constant curvature surfaces with definite (1, 2) and non-definite (3, 4) line elements.

In the rows (1) and (2) $z = x + iy$ is a complex variable obtained by means of the complex exponential transformation of the variable $\rho + i\phi$; in the rows (3) and (4) $z = x + hy$ is a hyperbolic variable obtained by means of the hyperbolic exponential transformation of the variable $\rho + h\phi$. $x, y$ are the coordinates in the Cartesian representation; $\epsilon, \sigma$ are constants connected with the integration constants (A and B) of the equations of the geodesics as it follows: $\sigma = B$ and $\epsilon = \sin^{-1}(A/R)$ in the rows (1) and (4), $\epsilon = \sinh^{-1}(A/R)$ in the rows (2) and (3). The constants are determined by setting two conditions that fix the geodesic (two points or one point and the tangent in the point). $\tau$ is the linear element on the arc of geodesics. $\tau_0 = AB$.

| Isometric forms | Cartesian isometric forms |
|-----------------|---------------------------|
| $d\bar{s}^2$    | $d\bar{s}$                |
| $\frac{R^2 \, d\rho^2 + d\phi^2}{\cosh^2 \rho}$ | $4R^4 \frac{dx^2 + dy^2}{(R^2 + x^2 + y^2)^2}$ |
| $\sin(\phi - \sigma) = \tan \epsilon \sinh \rho$ | $\frac{x^2 + y^2}{R^2} + 2 \frac{\sin \sigma x - \cos \sigma y}{R \tan \epsilon} - 1 = 0$ |
| tanh $\rho = \cos \epsilon \sin \left( \frac{\tau - \tau_0}{R} \right)$ | $4R^4 \frac{dz \, dz}{(R^2 + z^2)^2}$ |
| $\frac{R^2 \, d\rho^2 - d\phi^2}{\sinh^2 \rho}$ | $\frac{2 \frac{\sin \sigma x - \cos \sigma y}{R \tan \epsilon} - 1 = 0}{\frac{R^2}{R^2}} + \frac{z \frac{\sin \sigma x - \cos \sigma y}{R \tan \epsilon} + 1 = 0}{\frac{R^2}{R^2}} + \frac{z \frac{\sin \sigma x - \cos \sigma y}{R \tan \epsilon} + 1 = 0}{\frac{R^2}{R^2}} + \frac{z \frac{\sin \sigma x - \cos \sigma y}{R \tan \epsilon} + 1 = 0}{\frac{R^2}{R^2}} + \frac{z \frac{\sin \sigma x - \cos \sigma y}{R \tan \epsilon} + 1 = 0}$ |
| $\sinh(\sigma - \phi) = \tanh \epsilon \sinh \rho$ | $4R^4 \frac{dx^2 - dy^2}{(R^2 + x^2 - y^2)^2}$ |
| tanh $\rho = \cos \epsilon \sin \left( \frac{\tau - \tau_0}{R} \right)$ | $\frac{x^2 - y^2}{R^2} - 2 \frac{\sinh \sigma x - \cosh \sigma y}{R \tan \epsilon} - 1 = 0$ |
| $\frac{R^2 \, d\rho^2 - d\phi^2}{\sinh^2 \rho}$ | $4R^4 \frac{dz \, dz}{(R^2 + z^2)^2}$ |
| $\sin(\sigma - \phi) = \tan \epsilon \cosh \rho$ | $\frac{x^2 - y^2}{R^2} - 2 \frac{\sinh \sigma x - \cosh \sigma y}{R \tan \epsilon} + 1 = 0$ |
| coth $\rho = \cos \epsilon \cosh \left( \frac{\tau - \tau_0}{R} \right)$ | $4R^4 \frac{dz \, dz}{(R^2 + z^2)^2}$ |
| $\frac{R^2 \, d\rho^2 - d\phi^2}{\sinh^2 \rho}$ | $\frac{z \frac{\sinh \sigma x - \cosh \sigma y}{R \tan \epsilon} + 1 = 0}{\frac{R^2}{R^2}} + \frac{z \frac{\sinh \sigma x - \cosh \sigma y}{R \tan \epsilon} + 1 = 0}{\frac{R^2}{R^2}} + \frac{z \frac{\sinh \sigma x - \cosh \sigma y}{R \tan \epsilon} + 1 = 0}$ |
Table 2: Equations of the geodesics between two points expressed as a function of the points coordinates.

The equations are reported for the four constant curvature surfaces, with definite (rows (1) and (2)) and non-definite (rows (3) and (4)) line elements. Upper part of the table: in the rows (1) and (2): \(z = x + iy, \bar{z} = x - iy\) are the complex variable and its conjugate in the \(x, y\) representation, \(\alpha = \rho_\alpha \exp[i \psi_\alpha], \beta = \rho_\beta \exp[i \psi_\beta]\) are the constants that define the motions, expressed in the complex polar form, \(z_1 = \rho_1 \exp[i \psi_1], z_2 = \rho_2 \exp[i \psi_2]\) are the complex coordinates of two fixed points expressed in the complex polar form. In the rows (3) and (4): \(z = x + hy, \bar{z} = x - hy\) are the hyperbolic variable and its conjugate in the \(x, y\) representation, \(\alpha = \rho_\alpha \exp[h \theta_\alpha], \beta = \rho_\beta \exp[h \theta_\beta]\) are the constants that define the motions, expressed in the hyperbolic polar form, \(z_1 = \rho_1 \exp[h \theta_1], z_2 = \rho_2 \exp[h \theta_2]\) are the hyperbolic coordinates of two fixed points expressed in the hyperbolic polar form. Using the parameter expressions given in the 5th, 6th and 7th columns (\(\rho_\alpha = 1\)), the motions given in the 2nd column map, in the four cases, the points of coordinate \(z_1\) and \(z_2\) into the points \((0,0)\) and \((l,0)\) \((l\) is given in the 4th column) of the \(w\) representation. The value of \(l\) allows one to calculate the geodesic distance between two given points. The equations of the geodesic in the \(z(x,y)\) representation (reported in the lower part of the table) are obtained by transforming the axis of the abscises of the \(w\) representation.

| Motions | \(l\) | \(\psi_\alpha, \theta_\alpha\) | \(\psi_\beta, \theta_\beta\) | \(\rho_\beta\) |
|---------|-------|-----------------|-----------------|-------|
| (1) \(w = \frac{\alpha z + \beta}{-\beta z + \alpha}\) \(z = \frac{\alpha w - \beta}{\beta w + \alpha}\) \([\frac{z_2 - z_1}{1 + z_1 z_2}]\) \(\frac{1}{2} \arg\left(\frac{1 + z_1 z_2}{z_2 - z_1}\right)\) \(\pi + \psi_\alpha + \psi_1\) \(\rho_1\) |
| (2) \(w = \frac{\alpha z + \beta}{\beta z + \alpha}\) \(z = -\frac{\alpha w + \beta}{\beta w - \alpha}\) \([\frac{z_2 - z_1}{1 - z_1 z_2}]\) \(\frac{1}{2} \arg\left(\frac{1 - z_1 z_2}{z_2 - z_1}\right)\) \(\pi + \psi_\alpha + \psi_1\) \(\rho_1\) |
| (3) \(w = \frac{\alpha z + \beta}{-\beta z + \alpha}\) \(z = \frac{\alpha w - \beta}{\beta w + \alpha}\) \([\frac{z_2 - z_1}{1 + z_1 z_2}]\) \(\frac{1}{2} \arg\left(\frac{1 + z_1 z_2}{z_2 - z_1}\right)\) \(\theta_\alpha + \theta_1\) \(-\rho_1\) |
| (4) \(w = \frac{\alpha z + \beta}{\beta z + \alpha}\) \(z = \frac{\alpha w + \beta}{\beta w - \alpha}\) \([\frac{z_2 - z_1}{1 - z_1 z_2}]\) \(\frac{1}{2} \arg\left(\frac{1 - z_1 z_2}{z_2 - z_1}\right)\) \(\theta_\alpha + \theta_1\) \(-\rho_1\) |

The equations of the geodesics

1. \((x^2 + y^2 - 1)\rho_1 \sin(\psi_\alpha + \psi_\beta) - x(\sin 2\psi_\alpha - \rho_1^2 \sin 2\psi_\beta) - y(\cos 2\psi_\alpha + \rho_1^2 \cos 2\psi_\beta) = 0\)
2. \((x^2 + y^2 + 1)\rho_1 \sin(\psi_\alpha + \psi_\beta) + x(\sin 2\psi_\alpha + \rho_1^2 \sin 2\psi_\beta) + y(\cos 2\psi_\alpha - \rho_1^2 \cos 2\psi_\beta) = 0\)
3. \((x^2 - y^2 - 1)\rho_1 \sin h(\theta_\alpha + \theta_\beta) + x(\sinh 2\theta_\alpha - \rho_1^2 \sinh 2\theta_\beta) + y(\cosh 2\theta_\alpha + \rho_1^2 \cosh 2\theta_\beta) = 0\)
4. \((x^2 - y^2 + 1)\rho_1 \sinh(\theta_\alpha + \theta_\beta) - x(\sinh 2\theta_\alpha + \rho_1^2 \sinh 2\theta_\beta) - y(\cosh 2\theta_\alpha - \rho_1^2 \cosh 2\theta_\beta) = 0\)