Additive Lie derivations on the algebras of locally measurable operators

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Abstract

Let $M$ be a von Neumann algebra without central summands of type $I$. We are studying conditions that an additive map $L$ on the algebra of locally measurable operators has the standard form, that is equal to the sum of an additive derivation and an additive center-valued trace.

Key words: von Neumann algebras, locally measurable operator, derivation, additive derivation, additive Lie derivation, center-valued trace.
INTRODUCTION

The structure of Lie derivations on $C^*$-algebras and on more general Banach algebras has attracted some attention over the past years. Let $A$ be an algebra over the complex number. An additive (linear) operator $D : A \rightarrow A$ is called an additive derivation (linear derivation) if $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$ (Leibniz rule). Each element $a \in A$ defines linear associative a derivation $D_a$ on $A$ given as $D_a(x) = ax - xa$, $x \in A$. Such derivations $D_a$ are said to be inner derivations. If the element $a$ implementing the derivation $D_a$ on $A$, belongs to a larger algebra $B$, containing $A$ (as a proper ideal as usual) then $D_a$ is called a spatial derivation. An additive (linear) operator $L : A \rightarrow A$ is called an additive Lie derivation (linear Lie derivation) if $L([x, y]) = [L(x), y] + [x, L(y)]$, for all $x, y \in A$, where $[x, y] = xy - yx$.

Denote by $Z(A)$ the center of $A$.

An additive (linear) operator $\tau : A \rightarrow Z(A)$ is called an additive centred-valued trace (a linear center-valued trace) if $\tau(xy) = \tau(yx)$, $\forall x, y \in A$. The problem of the standard decomposition for a Lie derivation in rings theory was studied in work by W. S. Martindale [9]. W. S. Martindale solved this problem for primitives rings containing nontrivial idempotents and with the characteristic unequal to 2. Following these results obtained for rings, C. Robert Miers in [11] solved the problem of the standard decomposition for the case of von Neumann algebras. In the present work we are studying conditions that an additive map $L$ on $LS(M)$ has the standard form, that is equal to the sum of an additive derivation and an additive center-valued trace.

Development of the theory of algebras measurable operators $S(M)$ and the algebra of locally measurable operators $LS(M)$ affiliated with von Neumann algebra or $AW^*$ algebras [6], [10] provided an opportunity to construct and learn new interesting examples of $^*$-algebras unbounded operators.

We use terminology and notations from the von Neumann algebra theory [7] and the theory of locally measurable operators from [10].

Let $H$ be a complex Hilbert space, $B(H)$ be the algebra of all bounded linear operators acting in $H$, $M$ be a von Neumann algebra in $B(H)$, $P(M)$ be a complete lattice of all orthoprojections in $M$.

Let $H$ be a Hilbert space, $B(H)$ be the algebra of all bounded linear operators acting in $H$, $M$ be a von Neumann subalgebra in $B(H)$, $P(M)$ be a complete lattice of all orthoprojections in $M$.

A linear subspace $D$ on $H$ is said to be affiliated with $M$ (denoted as $D \eta M$), if $uD \subseteq D$ for every unitary operator $u$ from the commutant $M = \{ y \in B(H) : xy = yx, \forall x \in M \}$ of the algebra $M$.

A linear operator $x$ on $H$ with the domain $D(x)$ is said to be affiliated with $M$ (denoted as $x \eta M$), if $ux(\xi) = xu(\xi)$ for every unitary operator $u \in M$, and all $\xi \in D(x)$.

A linear subspace $D$ in $H$ is said to be strongly dens in $H$ with respect to the von Neumann algebra $M$, if

1) $D \eta M$,

2) there exists a sequence of projections $\{p_n\}_{n=1}^\infty \subseteq P(M)$, such that $p_n \uparrow 1$, $p_n(H) \subseteq D$, and $p_n^* = 1 - p_n$ is finite in $M$ for all $n \in \mathbb{N}$, where $1$ is the identity $M$.

A closed linear operator $x$, on $H$, is said to be measurable with respect to the von Neumann algebra $M$, if $x \eta M$, and $D(x)$ is strongly dens in $H$. Denote by $S(M)$ the set of all measurable operators affiliated with $M$ (see. [5,11]) and the center of an algebra $S(M)$ by $Z(S(M))$.

A closed linear operator $x$ in $H$ is said to be locally measurable with respect to $M$ if $x \eta M$, and there exists a sequence $\{z_n\}_{n=1}^\infty$ of central projections in $M$ such that $z_n \uparrow 1$ and $xz_n \in S(M)$ for all $n \in \mathbb{N}$. It is well-known [11] that the set $LS(M)$ of all locally measurable operators with respect to $M$ is a unital $^*$-algebra when equipped with the algebraic operations of strong addition and multiplication and taking the adjoint of an operator. Note that if $M$ is a finite von Neumann algebra then $S(M) = LS(M)$.
Denote by \( Z(\text{LS}(M)) \) the center of \( \text{LS}(M) \).

Let \( M \) be a von Neumann algebra without central summands of type \( I_1 \).

Let \( L : \text{LS}(M) \to \text{LS}(M) \) is an additive map. If \( p_i, p_j \) are projectors in \( S(M) \), then
\[
p_i \text{LS}(M) p_j = \{ p_i A p_j : A \in \text{LS}(M) \}, \quad i, j = 1, 2.
\]
Set \( p_1 = p \) and \( p_2 = 1 - p \). Then
\[
\text{LS}(M) = \sum_{i=1}^{2} \sum_{j=1}^{2} p_i (\text{LS}(M)) p_j.
\]
Let further \( S_{ij} = p_i \text{LS}(M) p_j, i, j = 1, 2 \). Recall that \( S_{ij} = S_{ik} S_{kj} \) for \( i, j = 1, 2 \).

In this paper is established the standard form of additive Lie derivation, acting on algebra of \( \text{LS}(M) \) when \( M \) be a von Neumann algebra without central summands of type \( I_1 \).

In particular, it follows that the properly infinite von Neumann algebras \( M \), all additive Lie derivations operations on the arbitrarily algebras \( \text{LS}(M) \), is the linear derivations

**RESULTS**

**Lemma 1.** If \( x \in S_{ij} \) and \( xy = 0 \) for all \( y \in S_{jk} \), then \( x = 0 \).

**Lemma 2.** \( pL(p)p + (1-p)L(p)(1-p) \in Z(\text{LS}(M)) \).

Let \( \delta : \text{LS}(M) \to S(M) \) defined as follows: \( \delta(x) = L(x) + sx - xs \) for each \( x \in \text{LS}(M) \).

We have the following

**Lemma 3.** \( p\delta(1)(1-p) = (1-p)\delta(1)p = 0 \).

**Lemma 4.** \( L(S_{ij}) \subseteq S_{ij} \), where \( i, j = 1, 2 \).

**Lemma 5.** There exists a map \( f : S_{ij} \to Z(\text{LS}(M)) \) such that \( \delta(x_i) \in S_{ij} + f_i(x_i) \) all \( x_i \in S_{ij}, i, j = 1, 2 \).

Now defined the mappings \( f : \text{LS}(M) \to Z(\text{LS}(M)) \) and \( d : \text{LS}(M) \to S(M) \) as follows:
\[
f(x) = f_1(x_{11}) + f_2(x_{22}) \quad \text{and} \quad d(x) = \delta(x) - f(x) \quad \text{all} \quad x_{11} + x_{12} + x_{21} + x_{22} \in \text{LS}(M) .
\]

Then by Lemma 4 and 5, we obtain \( d(S_{ij}) \subseteq S_{ij}, d(S_{ij}) \subseteq S_{ij}, d(S_{ij}) = \delta(S_{ij}), 1 \leq i \neq j \leq 2 \).

**Lemma 6.** \( d \) and \( f \) are additive.

**Lemma 7.** The mapping \( d \) is derivation.

**Lemma 8.** \( f([x, y]) = 0 \) for all \( x, y \in \text{LS}(M) \), where \( xy = 0 \)

Now we can formulate the main theorem.

**Theorem 1.** Let \( \text{LS}(M) \) be of all locally measurable operators affiliated with a von Neumann algebra \( M \) without central summands of type \( I_1 \). Let \( L : \text{LS}(M) \to \text{LS}(M) \) additive mapping. Then
\[
L([x, y]) = [L(x), y] + [x, L(y)] , \quad \text{for all} \quad x, y \in \text{LS}(M) , \quad \text{where} \ xy = 0 , \quad \text{if and only if there exists an additive derivation} \ \phi : \text{LS}(M) \to \text{LS}(M) \ \text{and an additive map} \ f : \text{LS}(M) \to Z(\text{LS}(M)) \ \text{where} \ f([x, y]) = 0 , \quad \text{such that} \ L(x) = \phi(x) + f(x) , \ x \in \text{LS}(M) , \ \text{where} \ Z(\text{LS}(M)) \ \text{center of} \ \text{LS}(M) .
\]

Now Theorem 1 implies the following

**Corollary:** Let \( \text{LS}(M) \) be of all locally measurable operators affiliated with a von Neumann algebra \( M \) without central summands of type \( I_1 \). Suppose that \( L : \text{LS}(M) \to \text{LS}(M) \) is an additive map. Then is a Lie derivation if and only if
there exists an additive derivation
\[
\varphi : \text{LS}(M) \to \text{LS}(M) \quad \text{and an additive map } f : \text{LS}(M) \to Z(\text{LS}(M)), \quad \text{where } f([x, y]) = 0, \quad \text{such that}
\]
\[
L(x) = \varphi(x) + f(x) \quad \text{for all } x \in \text{LS}(M), \quad \text{where}
\]
\[
Z(\text{LS}(M)) \quad \text{center of } \text{LS}(M).
\]

**Theorem 2.** Let \( \text{LS}(M) \) be of all locally measurable operators affiliated with a von Neumann algebra \( M \) without central summands of type \( I_1 \). Then any additive Lie derivation \( L : \text{LS}(M) \to \text{LS}(M) \) can be represented in the form
\[
L = \varphi + f, \quad \text{where } \varphi - \text{additive derivation on the algebra } \text{LS}(M) \text{ and } f - \text{additive } Z(\text{LS}(M))-\text{valued trace on the } \text{LS}(M).
\]

**Theorem 3.** If \( M \) is a type \( I \) or \( III \) von Neumann algebra, then any additive Lie derivation \( L : \text{LS}(M) \to \text{LS}(M) \) is linear Lie derivation and has the form \( L = D_a + f \), where \( D_a \) - is inner derivation on the algebra \( \text{LS}(M) \) and \( f \) - is linear \( Z(\text{LS}(M)) \)-valued trace on the \( \text{LS}(M) \).

**Corollary.** Let \( M \) be a von Neumann algebra of type \( I_\infty \). Then any additive Lie derivation \( L : \text{LS}(M) \to \text{LS}(M) \) is linear derivation.

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