Analogue Gravity on a Superconducting Chip

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We describe how analogues of a Hawking evaporating black hole as well as the Unruh effect for an accelerating photodetector in vacuum may be realized using superconducting, microwave circuits that are fashioned out of Josephson tunnel junction and film bulk acoustic resonator elements.
1. Introduction

In an earlier publication [1], we proposed a possible way to demonstrate a Hawking radiation analogue by utilizing a superconducting microwave transmission line comprising an array of direct-current superconducting quantum interference devices (dc-SQUID’s). This proposal was directly inspired by related analogue Hawking effect schemes involving electromagnetic waveguides [2] and non-linear optical fibres [3]. One advantage to utilizing superconducting capacitor and inductor elements for the waveguide is that they are low noise, with quantum zero-point fluctuations dominating over thermal radiation fluctuations at standard, operating temperatures of tens of milliKelvins and below for ultrahigh vacuum-dilution fridge set-ups. Furthermore, with Josephson junctions functioning effectively as strongly non-linear inductors, the quantum fluctuations can be converted into real microwave photons analogously to the Hawking and dynamical Casimir effects [4–6]. Finally, by incorporating also micrometre scale, mechanically vibrating crystals in the gigahertz frequency range, close analogues of both the dynamical Casimir [7] and Unruh effects [8] may be realized.

Since the publication of our proposal [1], there have been significant advances in superconducting microwave circuit technology, largely driven by the push to realize quantum computers implementing superconducting qubits; in particular, the recently developed Josephson traveling-wave parametric amplifier (JTWPA) [9] is ideally suited (with some modifications) for demonstrating the analogue Hawking effect with detectable microwave photon production.

In this paper, we begin with a description of our analogue Hawking effect scheme in light of recent superconducting circuit technology developments. In the second part, we outline a possible scheme for realizing a close analogue of the Unruh effect for an oscillating (actual accelerating) superconducting qubit photodetector by utilizing a so-called film bulk acoustic resonator (FBAR) element in addition to capacitor and Josephson junction elements. The following discussions will emphasize near-future, possible directions for theoretical and experimental exploration.

2. Analogue Hawking Effect

We begin our discussion by considering a simple, infinitely long one-dimensional transmission line comprising lumped element inductors in series and capacitors in parallel (Fig.1). The inductors have inductance value $L_0$ and capacitors have capacitance value $C_0$. The transmission line is made out of repeating unit cells of length $a$. Applying Kirchhoff’s laws, which state that

the current entering a circuit node must equal the current exiting the node (current conservation)
and that the sum over all voltages around a closed circuit loop must equal the time rate of change of the magnetic flux threading the loop (Faraday’s law of induction) [10], we have

\[ I_n - I_{n+1} = \frac{dQ_{n+1}}{dt}, \]  
\[ V_n - V_{n+1} = \frac{d\Phi_n}{dt}, \]  

where \( Q_{n+1} \) is the charge on the capacitor attached to node \( n + 1 \) and \( \Phi_n \) is the magnetic flux threading the inductor linking nodes \( n \) and \( n + 1 \). Utilizing the constitutive relations between voltage and charge for the capacitor, \( V_n = Q_n/C_0 \), and between current and flux for the inductor, \( I_n = \Phi_n/L_0 \), Eqs. (2.1) and (2.2) become

\[ \Phi_n - \Phi_{n+1} = L_0 \frac{dQ_{n+1}}{dt}, \]  
\[ Q_n - Q_{n+1} = C_0 \frac{d\Phi_n}{dt}, \]  

These equations can be obtained from the Hamiltonian

\[ H = \sum_{n=-\infty}^{+\infty} \left[ \frac{p_n^2}{2C_0} + \frac{(\Phi_{n+1} - \Phi_n)^2}{2L_0} \right], \]  
via Hamilton’s equations, where the momentum conjugate to the flux coordinate \( \Phi_n \) is \( p_n = C_0 \frac{d\Phi_n}{dt} = Q_n - Q_{n+1} \).

Equations (2.3) and (2.4) support coupled, propagating flux and charge waves. Restricting to long wavelength solutions \( \lambda \gg a \), we can approximate the flux and current differences as \( (\Phi_{n+1} - \Phi_n)/a \approx \partial \Phi/\partial x \) and \( (Q_{n+1} - Q_n)/a \approx \partial Q/\partial x \) respectively, to obtain

\[ \partial_x (\Phi(x,t)) = -L \frac{\partial Q(x,t)}{\partial t}, \]  
\[ \partial_x (Q(x,t)) = -C \frac{\partial \Phi(x,t)}{\partial t}, \]  

where \( L = L_0/a \) and \( C = C_0/a \) denote the inductance and capacitance per unit length respectively of the transmission line. These equations follow from the continuum limit of Hamiltonian (2.5):

\[ H = \int_{-\infty}^{+\infty} dx \left[ \frac{(\partial \Phi(x,t))^2}{2C} + \frac{(\partial_x \Phi(x,t))^2}{2L} \right], \]  

where the momentum conjugate to the flux field coordinate \( \Phi(x,t) \) is \( p(x,t) = C \partial_t \Phi(x,t) = -\partial_x Q(x,t) \). Combining Eqs. (2.6) and (2.7), we choose to eliminate the charge ‘field’ coordinate, obtaining the Klein-Gordon equation for a massless scalar field in 1 + 1 spacetime dimensions:

\[ \partial_{\mu} \partial^{\mu} \varphi = 0, \]  

where \( \partial_{\mu} = (\partial/\partial t, \partial/\partial x) \), \( \varphi(x,t) = 2\pi \Phi(x,t)/\Phi_0 \) is a dimensionless flux field coordinate with \( \Phi_0 = \hbar/(2e) \) the magnetic flux quantum, and we have introduced an effective Minkowski spacetime metric:

\[ \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & c^2 \end{pmatrix}, \]  

with \( \partial^\mu = \eta^{\mu\nu} \partial_\nu \) and \( c = 1/\sqrt{LC} \) the electromagnetic wave speed in the transmission line. The advantages to working with the dimensionless flux field coordinate \( \varphi(x,t) \) will become apparent when we introduce the Josephson junction element below.

With the transmission line made from a common superconducting element such as aluminium or niobium, cooling below the critical temperature results in low loss-low noise electromagnetic wave propagation in the microwave frequency regime; quantum behaviour such as vacuum fluctuations can become apparent and hence the transmission line must be modeled as a quantum
system. The equal time canonical commutation relation for the continuum field operator and its conjugate momentum operator in the Heisenberg picture is:

$$[\hat{\varphi}(x,t), \hat{p}(x',t)] = i\hbar \delta(x-x'),$$  \hspace{1cm} (2.11)

where \(\hat{p}(x,t) = \mathcal{C}(\Phi_0/2\pi)^2 \partial_x \hat{\varphi}(x,t)\). The field operator equations of motion in the Heisenberg picture are \(\partial_t \hat{\varphi} = 0\) and can be solved in terms of photon creation and annihilation operators as

$$\hat{\varphi}(x,t) = \frac{2\pi}{\Phi_0} \sqrt{\frac{\hbar}{4\pi c^2}} \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{|k|}} \left[ e^{-i(\omega t - kx)} \hat{a}(k) + e^{+i(\omega t - kx)} \hat{a}^\dagger(k) \right],$$  \hspace{1cm} (2.12)

with linear dispersion relation \(\omega = ck\). From Eqs. (2.11) and (2.12), the creation and annihilation operators satisfy the commutation relations

$$[\hat{a}(k), \hat{a}^\dagger(k')] = \delta(k - k').$$  \hspace{1cm} (2.13)

Photon states are created from the vacuum defined through \(\hat{a}(k,0)|0\rangle = 0\), with single photon states \(\hat{a}^\dagger(k)|0\rangle\) having energy \(E = h\omega\) and momentum \(p = h k\). This quantum field model describes photons propagating in an effective 1 + 1 dimensional Minkowski spacetime. Such a continuum approximation is a good one provided photon energies are much smaller than the ‘Planck’ energy \(hc/a\) set by the transmission line unit cell size \(a\).

Is it possible however to modify the superconducting transmission line circuit in some way so as to recover a continuum effective quantum field theory on a spacetime with an event horizon? The answer to this question is indeed ‘yes’! We require a distinct type of circuit element called a Josephson tunnel junction (JJ) [10], which comprises two overlapping, superconducting metal electrodes separated by an electrically insulating, nanometre thick oxide layer, such that Cooper pairs—the superconducting charge carriers—are able to quantum tunnel between the electrodes. A lumped element model of a JJ satisfies the following respective relations for the voltage drop across the junction and current flow through the junction:

$$V_J = \frac{\Phi_0}{2\pi} \frac{d\varphi_J}{dt},$$

$$I_J = I_c \sin \varphi_J,$$  \hspace{1cm} (2.14)

where the coordinate \(\varphi_J\) is the gauge invariant phase difference across the tunnel junction of the macroscopic wave function describing the Cooper pairs and \(I_c\) is the critical current of the tunnel junction, i.e., the maximum possible dc Cooper-pair current. Comparing the voltage expression with that for an inductor [see, Eq. (2.2)], we can define an effective flux variable for the Josephson junction: \(\Phi_J = \frac{\Phi_0}{2\pi} \varphi_J\). For small phase magnitudes \(|\varphi| \text{mod} 2\pi n \ll 1\), Eq. (2.15) can be approximately expressed as \(I_J \approx \frac{2\pi}{2\Phi_0} \Phi_J\), and comparing with the constitutive relation given above between current and flux for the inductor, we see that the JJ functions as an effective inductor with inductance \(L_J \approx \frac{2\pi}{2\Phi_0}\). For not necessarily small phase magnitudes, we can interpret the JJ as an effective phase coordinate-dependent inductor:

$$L_J(\varphi_J) = \frac{\Phi_J}{I_c \sin(2\pi \Phi_J/\Phi_0)} = \frac{\Phi_0 \varphi_J}{2\pi I_c \sin \varphi_J}.$$  \hspace{1cm} (2.16)

It is this phase/flux dependent effective inductance of the JJ that will enable the realization of an analogue 1 + 1 dimensional spacetime with event horizon as we now show.

In particular, we replace each inductor element in the transmission line circuit of Fig. 1 with two JJ’s in parallel—called a dc-SQUID (Fig. 2). The JJ elements (symbolized by the crossed boxes) making up the dc-SQUID’s are assumed to have identical critical current values \(I_c\) and capacitance values \(C_J\). The dc-SQUID’s are threaded with an external magnetic flux that is allowed to vary from unit cell to unit cell, as well as depend on time. The dc-SQUID array transmission line is an example of a ‘metamaterial’, which enables microwave propagation dynamics that would not be possible in a simple co-planar transmission line fashioned from a centre superconductor strip and ground plane. The circuit shown in Fig. 2 is also closely
related to Josephson junction traveling-wave parametric amplifiers (JTWPA's), which can now be realised with very small deviations in the repeating unit cell JJ and capacitor values \cite{9,11}. Not shown is the circuitry necessary for providing the space-time dependent flux bias, which would involve another, current biased transmission line metamaterial running in parallel to the indicated dc-SQUID array \cite{11}.

Applying Kirchhoff's laws to the circuit shown in Fig. 2, we have [c.f. Eqs. (2.1) and (2.2)]:

\[ I_n - I_{n+1} = \frac{dQ_n + Q_{n+1}}{dt}, \]  
\[ V_n - V_{n+1} = \frac{\Phi_0}{2\pi} \frac{d\phi_{Jn}}{dt}, \]  
\[ I_n = 2C_J \frac{\phi_0}{2\pi} \frac{d^2\phi_{Jn}}{dt^2} + 2I_c \cos \left( \frac{\pi\phi_{ext}}{\Phi_0} \right) \sin \phi_{Jn}, \]

while voltage across each capacitor is related to the charge as \( V_n = Q_n/C_0 \). In contrast to the above analysis for the inductor-capacitor transmission line, it is more convenient to express the dynamics in terms of the phase coordinates \( \phi_{Jn} \) across the \( C_0 \) capacitors instead of the JJ phase coordinates \( \phi_J \), where from Eq. (2.18) we have \( \phi_{Jn} = \phi_n - \phi_{n+1} \) (up to an inessential constant), with \( \phi_n = 2\pi V_n/\Phi_0 \); in particular, the system Lagrangian is more straightforward to write down. Substituting Eq. (2.19) into Eq. (2.17) gives for the dc-SQUID array equations of motion:

\[ (C_0 + 4C_J) \left( \frac{\phi_0}{2\pi} \right)^2 \frac{d^2\phi_n}{dt^2} = 2C_J \left( \frac{\phi_0}{2\pi} \right)^2 \frac{d^2\phi_{n-1}}{dt^2} + \frac{d^2\phi_{n+1}}{dt^2} \]
\[ = -2E_J \cos \left( \frac{\pi\phi_{ext}}{\Phi_0} \right) \sin \left( \phi_n - \phi_{n+1} \right) + 2E_J \cos \left( \frac{\pi\phi_{ext}}{\Phi_0} \right) \sin \left( \phi_n - \phi_{n+1} \right), \]

(2.20)

where \( E_J = \Phi_0 I_c/(2\pi) \) is the so-called 'Josephson energy'. Equation (2.20) follows via Lagrange's equations from the Lagrangian

\[ L = \sum_{n=-\infty}^{\infty} \left[ \frac{1}{2} C_0 \left( \frac{\phi_0}{2\pi} \right)^2 \left( \frac{d\phi_n}{dt} \right)^2 + C_J \left( \frac{\phi_0}{2\pi} \right)^2 \left( \frac{d\phi_{n-1}}{dt} - \frac{d\phi_{n+1}}{dt} \right)^2 \right] \]
\[+2E_J \cos \left( \frac{\pi \phi_{n+1}}{\phi_0} \right) \cos (\phi_n - \phi_{n+1}) \],
\]
where we assume for an ideal, infinitely long dc-SQUID array; an actual dc-SQUID array will comprise several thousand unit cells [9].

Considering long wavelength \(\lambda \gg a\) dynamics, where \(\phi_n\) changes little from one unit cell to the next, we can make the approximation \((\phi_{n+1} - \phi_n)/a \approx \partial \phi/\partial x\). The resulting continuum approximation for the Lagrangian (2.21) is
\[L = \int_{-\infty}^{\infty} dx \left[ \frac{1}{2} \mathcal{C} \left( \frac{\phi_0}{\phi} \right)^2 \left( \frac{\partial \phi}{\partial x} \right)^2 + C_J a \left( \frac{\phi_0}{\phi} \right)^2 \left( \frac{\partial^2 \phi}{\partial t \partial x} \right)^2 - E_J a \cos \left( \frac{\pi \phi_{\text{ext}}(x,t)}{\phi_0} \right) \left( \frac{\partial \phi}{\partial x} \right)^2 \right],
\]
where \(\mathcal{C} = C_0/a\). We can furthermore neglect the second term in the Lagrangian provided the JJ capacitances satisfy \(C_J \ll C_0(\lambda/a)^2\), and the resulting approximate Hamiltonian is
\[H = \int_{-\infty}^{+\infty} dx \left[ \frac{1}{2} \mathcal{C} \left( \frac{\phi_0}{\phi} \right)^2 \left( \frac{\partial \phi}{\partial x} \right)^2 \left( \frac{\partial^2 \phi}{\partial t \partial x} \right)^2 \right] + E_J a \cos \left( \frac{\pi \phi_{\text{ext}}(x,t)}{\phi_0} \right) \left( \frac{\partial \phi}{\partial x} \right)^2,
\]
where the momentum conjugate to the phase field coordinate \(\phi(x,t)\) is \(\Pi(x,t) = \mathcal{C} \left( \frac{\phi_0}{\phi} \right)^2 \partial_t \phi\).

Hamilton’s equations give the following wave equation
\[-\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial x} \left( c^2(x,t) \frac{\partial \phi}{\partial x} \right) = 0,
\]
where the spacetime dependent electromagnetic wave phase speed is
\[c(x,t) = \frac{1}{\sqrt{\mathcal{L}(x,t)}},
\]
with the effective inductance per unit length given by
\[\mathcal{L}(x,t) = \frac{\phi_0}{4\pi I_C a} \sec \left( \frac{\pi \phi_{\text{ext}}(x,t)}{\phi_0} \right).
\]
The phase speed inherits its spacetime dependence from the varying magnetic flux threading the dc-SQUID loops of the array via the effective inductance of the dc-SQUIDs. The wave equation (2.24) can also be directly obtained by taking the continuum limit of the dc-SQUID array equations (2.20).

Notice from Eqs. (2.25) and (2.26) that increasing the external flux bias \(\phi_{\text{ext}}\) from zero increases the effective inductance and hence decreases the electromagnetic wave phase speed. This is made more transparent by combining Eqs. (2.26) and (2.25) in the form
\[c(x,t) = c_0 \sqrt{\cos \left( \frac{\pi \phi_{\text{ext}}(x,t)}{\phi_0} \right)},
\]
where \(c_0\) is the phase speed that follows from Eqs. (2.25) and (2.26) with zero external flux bias \((\phi_{\text{ext}} = 0)\):
\[c_0 = a \sqrt{\frac{4\pi I_C}{\phi_0 C_0}}.
\]
The ability to utilize an external flux bias to lower the phase speed suggests a way to make an effective event horizon [2,3]. In particular, consider as an example a flux step of magnitude \(\phi_{\text{ext}} = 0.2\phi_0\) with front that moves with speed \(u = 0.95c_0\), and where the flux bias is zero ahead of the propagating front (Fig. 3–top plot). Substituting the flux step function into Eq. (2.27) gives the resulting electromagnetic phase speed function shown in the bottom plot of Fig. 3. Well to the left of the location \(x_h(t)\) for which \(c(x_h, t) = u = 0.95c_0\), electromagnetic waves travel with speed \(0.95c_0 < u\), while well to the right of the location \(x_h(t)\), electromagnetic waves travel with speed \(c_0 > u\). Thus the location \(x_h(t)\) constitutes an effective moving event horizon.
Figure 3. Propagating external flux bias step with front speed $u = 0.95c_0$ (top plot) and the resulting electromagnetic wave phase speed in the dc-SQUID array (bottom plot). The marked location $x_h(t)$ on the phase speed plot where $c(x_h, t) = u = 0.95c_0$ constitutes an effective moving event horizon.

Transforming to the comoving frame of the propagating flux front, $x' = x - ut, t' = t$, the wave equation (2.24) becomes

$$
\left( \frac{\partial^2}{\partial t'^2} + 2u \frac{\partial^2}{\partial x \partial t'} + \frac{\partial}{\partial x} \left( c^2 - u^2 \right) \frac{\partial}{\partial x} \right) \phi = 0,
$$

where we have dropped the primes on the comoving coordinates. Equation (2.29) can be expressed in the ‘general covariant’ form

$$
\frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu\nu} \partial_{\nu} \phi \right) = 0,
$$

with $g = \text{det}g_{\mu\nu}$ and the effective spacetime metric defined as

$$
g^{\mu\nu} = \frac{1}{c(x)} \begin{pmatrix} -1 & u \\ u & c^2(x) - u^2 \end{pmatrix}.
$$

This effective metric displays an event horizon where $c(x) = u$.

Quantizing the continuum model Hamiltonian (2.23) and assuming a propagating external flux bias step as just described, then photon pair production from the electromagnetic vacuum should occur in the vicinity of the event horizon, in direct analogy to Hawking radiation. The effective Hawking temperature is [2]

$$
T_H = \frac{\hbar}{2\pi k_B} \frac{\left| \partial c(x) / \partial x \right|_{x_h}}{c_0}.
$$

Let us now estimate possible Hawking temperatures that can be realized. Assuming the phase speed step length to be about ten times the unit cell length $a$ and the step height to be $0.1c_0$ as considered above, we have $|\partial c / \partial x|_{x_h} \approx 0.01c_0/a$. Substituting in Eq. (2.28), we obtain

$$
T_H \approx \frac{1}{100\pi k_B} \sqrt{\frac{\hbar e I}{C_0}},
$$

(2.33)
where recall that $C_0$ is the unit cell lumped element capacitance to ground (see Fig. 2) [not to be confused with the zero bias flux phase speed $c_0$]. For the realizable example circuit values $I_c = 5 \mu A$, $C_0 = 1 \text{ fF}$ [9], we have $T_h \approx 70 \text{ mK}$. Verification of such correlated photon pairs should be possible with existing superconducting circuit microwave photon detection techniques [12,13]. An experimental run would require the repeated launching of propagating flux bias steps down the dc-SQUID array, with the photon correlation signal for Hawking radiation recovered through sufficiently long time-averaging along the lines of the acoustic Hawking radiation counterpart Bose-Einstein condensate experiment of Ref. [14]. The ability to realize dc-SQUID arrays comprising thousands of unit cells [9], as well as operate at a few tens of mK temperatures and with low noise quantum limited microwave photon detectors, shows promise for demonstrating strong microwave Hawking radiation signals.

Looking forward, much remains to be explored; the JTWPA design [9] needs to be adapted so as to enable propagating flux pulse/step biasing [11], and microwave photon detection circuitry suitable for verifying correlated photon pairs needs to be developed.

In terms of theory modeling, an analysis is required of the photon production from vacuum starting with the discrete Lagrangian (2.21). This will allow us to determine the effect of the ‘Planck’ scale physics set by the unit cell length $a$ on the Hawking radiation prediction following from the approximate continuum model [2,15]. For such an analysis, we can assume in the first instance that the phase field fluctuations are small in magnitude, hence harmonically approximating the nonlinear cosine potential in Eq. (2.21).

Moving beyond, an interesting question to be addressed concerns the consequence of applying a non-zero, uniform (i.e. constant) external flux bias ‘floor’ over the length of the dc-SQUID array [16]. This reduces the effective Josephson energy $E_J \cos (\pi \Phi_{\text{ext}} / \Phi_0)$, somewhat analogous to weakening the spring constant of a harmonic oscillator mass. Zero-point fluctuations (i.e., quantum uncertainty) in the phase coordinate operators $\hat{\varphi}_n$ will correspondingly increase and the full, nonlinear cosine potential in the discrete Lagrangian (2.21) may need to be taken into account. If we now add this constant flux bias floor to the propagating flux bias front, what will be the consequence for photon production from vacuum? In an analogous sense, increasing quantum uncertainty in the phase coordinate operators may be interpreted as increasing fluctuations in the effective metric and hence fluctuating event horizon location. The ability to tunably control such fluctuations through the externally applied flux is a unique feature of the present superconducting circuit analogue.

3. Analogue Unruh Effect

In the preceding section, we saw how Josephson junction elements enhance the functionality of superconducting microwave transmission lines, and in particular enable the realization of an effective event horizon and accompanying Hawking radiation. In this section, we consider a further enhancement in the functionality of superconducting circuits by bringing into play a special type of mechanical oscillator called a ‘film bulk acoustic resonator’ (FBAR) [7,17]. Such crystal (or crystalline) mechanical oscillators may be fashioned out of silicon or other materials and undergo dilatational (i.e., breathing) mode oscillations in the few to tens of GHz frequency range depending on their thickness (typically a few hundred nm). These mechanical frequencies are large enough to match those of microwave cavities; with the upper and lower FBAR surfaces metalized and having areas of the order of a few hundreds of $\mu \text{m}^2$, the dilational mechanical motion can strongly couple capacitively to microwave cavity fields. This then allows the possibility to realize close analogues of the dynamical Casimir effect for mechanically oscillating (i.e., accelerating) mirrors [7] and the Unruh effect for mechanically oscillating photodetectors [8,18].

We will now outline a possible scheme for realizing the oscillatory Unruh effect that utilizes a microwave cavity, a dc-SQUID, and an FBAR (Fig. 4). The scheme is somewhat related to our recent proposal [8], but with the key difference that the dc-SQUID functions as a qubit [10] photodetector instead of using another coupled microwave cavity as photodetector. This enables
the placing of the photodetector atop the FBAR (Fig. 4b); when the FBAR is mechanically driven at say its fundamental transverse breathing mode frequency $\omega_m$, then the qubit undergoes actual oscillatory acceleration at this frequency, modulating its coupling to the microwave cavity via the FBAR capacitance $C_m(t)$ (where the ‘m’ subscript denotes ‘mechanical’). Under the frequency matching condition $\omega_m = \omega_c + \Delta E_{qb}/\hbar$, where $\omega_c$ is the fundamental mode frequency of the cavity and $\Delta E_{qb}$ is the qubit energy level spacing, we can have resonant enhancement of correlated microwave photon pair production from vacuum for realizable large, superconducting cavity quality factors, with one photon in the pair appearing in the cavity and the other photon absorbed by the detector, inducing a transition to its excited state. By utilizing non-piezoelectric materials such as silicon for the FBAR, oscillatory dipole radiation due to induced capacitor plate surface charges is minimized; the photon pair production from vacuum may then be viewed as a consequence of the photodetector’s actual acceleration.

Applying Kirchhoff’s laws to the circuit shown in Fig. 4a, we obtain the following dynamical equations in terms of the dc-SQUID average phase coordinate $\varphi = (\varphi_1 + \varphi_2)/2$, (with $\varphi_1$, $\varphi_2$ the gauge invariant phases across each JJ), the phase coordinate $\varphi_m$ across the FBAR capacitance $C_m(t)$, and the cavity phase field coordinate $\varphi_c(x, t)$ between the centre conductor $x$ location and ground:

1. $2C_J \frac{\Phi_0}{2\pi} \bar{\varphi} + 2I_c \cos \left( \frac{\pi \varphi_{\text{ext}}}{\Phi_0} \right) \sin \varphi - \frac{\Phi_0}{2\pi} \frac{d}{dt} \left( C_m(t) \frac{d\varphi_m}{dt} \right) = 0$, \hspace{1cm} (3.1)
2. $\frac{\partial^2 \varphi_c}{\partial x^2} - L_c C_c \frac{\partial^2 \varphi_c}{\partial t^2} = 0$, \hspace{1cm} (3.2)

with the boundary conditions

1. $\frac{\Phi_0}{2\pi L_c} \frac{\partial \varphi_c(0, t)}{\partial x} = \frac{\Phi_0}{2\pi} \frac{d}{dt} \left( C_m(t) \frac{d\varphi_m}{dt} \right)$, \hspace{1cm} (3.3)
2. $\frac{\partial \varphi_c(L, t)}{\partial x} = 0$. \hspace{1cm} (3.4)

Figure 4. (a) Circuit diagram of the analogue oscillatory Unruh effect scheme. The dc-SQUID serves as a superconducting qubit photodetector, which couples to a co-planar waveguide cavity via an FBAR capacitor (blue). (b) Side view of the scheme indicating the placement of the photodetector atop the FBAR. Not shown are the qubit-cavity probe circuitry and means of FBAR mechanical actuation.
The latter condition follows from the vanishing of the current at the right, terminated end of the center conductor of length $L$. The FBAR capacitance satisfies

$$C_m(t) = C_m^{(0)} (1 + z(t)/D),$$

where $z(t) = A \cos(\omega_m t)$ is the driven dilatational displacement of the FBAR, with $A$ the amplitude, and $D$ the FBAR thickness; we assume that $|z(t)| \ll D$. The FBAR capacitance phase coordinate can be eliminated through the following phase coordinate relation that follows from the voltage Kirchhoff law:

$$\varphi_m(t) = \varphi_c(0, t) - \varphi(t).$$

We approximate the cavity phase field $\varphi_c(x, t)$ as a series expansion in terms of eigenfunctions $\phi_n(x)$:

$$\varphi_c(x, t) = \sum_n q_n(t) \phi_n(x).$$

Assuming that $C_m^{(0)} \ll C_c L$, we replace Eq. (3.3) with the approximate boundary condition $\varphi_c'(0, t) \approx 0$, and taking into account the opposite end boundary condition (3.4), the orthogonal eigenfunctions are

$$\phi_n(x) = \cos \left( \frac{n\pi x}{L} \right), \quad n = 1, 2, \ldots$$

From Eqs. (3.7) and (3.8), the to be determined cavity phase mode coordinates $q_n(t)$ are given as

$$q_n(t) = \frac{2}{L} \int_0^L dx \varphi_n(x, t) \phi_n(x).$$

Differentiating (3.9) twice with respect to time and applying the cavity wave equation (3.2), we have

$$\ddot{q}_n(t) = \frac{2}{L c_c} \int_0^L dx \varphi''_n(x, t) \phi_n(x).$$

Integrating (3.10) by parts twice, applying the eigenvalue equation $\phi''_n(x) = -k_n^2 \phi_n$ (with $k_n = n\pi/L$) and the boundary conditions (3.3) and (3.4), we obtain

$$\ddot{q}_n(t) = -\frac{2}{C_c L} \frac{d}{dt} \left[ C_m(t) (\dot{\varphi}_c(0, t) - \dot{\varphi}) \right] - \frac{k_n^2}{L c_c} q_n(t).$$

When the resonance condition $\omega_m = \omega_n + \Delta E_{q_b}/\hbar$ is satisfied for some cavity mode frequency $\omega_n = k_n^2/L c_c$ and qubit level spacing $\Delta E_{q_b}$ (see below), we can apply the single-mode approximation to the cavity. Equation (3.11) then becomes

$$\ddot{q}_c + \frac{2}{L c_c} \frac{d}{dt} [C_m(t) (\dot{q}_c - \dot{\varphi})] + \omega_c^2 q_c = 0,$$

where $\omega_c^2 = \omega_n^2/L c_c$, $q_c(t) = q_n(t)$ and we have used $\phi_n(0) = 1$.

Using Eqs. (3.6) and (3.5), Eq. (3.1) becomes

$$\frac{\Phi_0}{2\pi} 2C_c J_0 \dot{\varphi} - \frac{\Phi_0}{2\pi} \frac{d}{dt} \left[ C_m(t) (\dot{q}_c - \dot{\varphi}) \right] + 2L c_c \cos \left( \frac{\pi \Phi_{\text{ext}}}{\Phi_0} \right) \sin \varphi = 0.$$  

The coupled cavity and dc-SQUID equations of motion (3.12) and (3.13) follow via the Euler-Lagrange equations from the following Lagrangian:

$$L = \left( \frac{\Phi_0}{2\pi} \right)^2 \left[ \frac{C_c + C_m(t)}{2} \dot{q}_c^2 - \frac{C_c \omega_c^2}{2} q_c^2 + \frac{2C_c J + C_m(t)}{2} \dot{\varphi}^2 - C_m(t) q_c \dot{\varphi} \right] + E_J \cos \varphi,$$
Performing the Legendre transformation on the Lagrangian, we obtain the following cavity dc-SQUID system Hamiltonian:

\[ H = \left( \frac{2\pi}{\Phi_0} \right)^2 \frac{p_c^2}{2C_c} + \frac{\Phi_0}{2\pi} \frac{2C_C}{\pi} \left( \frac{\Phi_0}{2\pi} + \frac{E_C}{\Phi_0^2} \right) \hat{p}_\varphi^2 - E_J \cos \varphi \]

where \( E_C = (2e)^2/(2C_\Sigma) \), \( C_\Sigma = 2C_J + C_m^{(0)} \), is the Cooper-pair charging energy of the dc-SQUID island formed by the two JJ’s and FBAR capacitor, and \( p_c \) and \( \hat{p}_\varphi \) are the momentum canonical conjugate to the \( q_c \) and \( q_\varphi \) coordinates, respectively. Expression (3.15) for the Hamiltonian assumes that \( C_m^{(0)} \ll C_c \).

Quantizing, Hamiltonian (3.15) can be expressed as the following corresponding operator:

\[ \hat{H} = \hbar \omega_c \hat{a} \hat{a} + E_C \hat{n}^2 - E_J \cos \varphi - i \sqrt{\frac{R_K}{4\pi Z_c}} E_C C_m^{(0)} \left( 1 + \frac{2C_J}{C_\Sigma} \frac{z(t)}{D} \right) \left( \hat{a} - \hat{a}^\dagger \right) \hat{n}, \]

where \( \hat{n} = \hat{p}_c/\hbar \) is the number operator for Cooper-pairs transferred to the dc-SQUID island, \( Z_c = 1/(\omega_c C_c) \) is the cavity mode impedance, and \( R_K = \hbar/e^2 \approx 25.8 \text{ K}\Omega \) is the von Klitzing resistance. In order to realize Hamiltonian (3.16), the dc-SQUID circuit shown in Fig. 4 should in practice also include a gate voltage bias in order to be able to cancel out any stray charge on the dc-SQUID island. Expressing the Hamiltonian operator (3.16) in terms of the decoupled dc-SQUID Hamiltonian energy eigenstates \( |i\rangle \) and eigenvalues \( E_i \), \( i = 0, 1, 2, \ldots \), we have approximately

\[ \hat{H} = \hbar \omega_c \hat{a} \hat{a} + \frac{\Delta E_{gb}}{2} \sigma_z - i\hbar \left( \sigma^+ \sigma^- \right) \left( \hat{a} - \hat{a}^\dagger \right) - i\hbar \cos (\omega_{m} t) \left( g_{m} \sigma^+ + g_{m}^* \sigma^- \right) \left( \hat{a} - \hat{a}^\dagger \right), \]

where the cavity-qubit coupling strength is

\[ g = \hbar^{-1} \sqrt{\frac{R_K}{4\pi Z_c}} E_C C_m^{(0)} (1| \hat{n} |0), \]

the driven, FBAR displacement cavity-qubit coupling strength is

\[ g_m = \frac{2C_J A}{C_\Sigma D} g. \]

\( \Delta E_{gb} = E_J - E_{gb} \), and we have assumed the resonance condition \( \hbar \omega_{m} = \hbar \omega_c + \Delta E_{gb} \).

Transforming to the interaction picture and dropping fast rotating terms (rotating wave approximation), we obtain the following time-independent Hamiltonian in the interaction picture:

\[ H_I = i\hbar \left( g_{m} \sigma^+ \hat{a}^\dagger - g_{m}^* \sigma^- \hat{a} \right), \]

where we assume that \( \omega_c \neq \Delta E_{gb}/\hbar \); in particular, the cavity and qubit frequencies must differ by more than their respective relaxation rates (see below). The latter condition is also required so that photon pairs are not generated directly in the cavity when \( \omega_{m} = 2\omega_c \), which would correspond to the dynamical Casimir effect; in deriving the Hamiltonian (3.15) above, we neglected direct mechanical modulation of the cavity by dropping the \( C_m^{(l)} \) contribution to the cavity kinetic energy term.

Hamiltonian (3.20) describes the correlated production of a photon in the cavity mode vacuum and the absorption of a photon by the qubit detector (i.e., excitation of the qubit from its ground state) due to the mechanically driven FBAR capacitance that couples the cavity and detector. In the presence of unavoidable cavity and qubit damping, the photon production will reach a steady state rate given approximately by

\[ \Gamma = \frac{4|g_m|^2}{\gamma_c + \gamma_{qb}}, \]

where we assume that \( |g_m| \ll \gamma_{gb}, \gamma_c \), with \( \gamma_{gb} \) and \( \gamma_c \) the qubit and cavity damping rates, respectively. The latter inequality implies that the average cavity photon number will be much
less than one and the average qubit energy will be close to $E_0$ in the steady state; photons are lost from the cavity and the qubit de-excites more rapidly than the mechanically driven photon production rate. To a good approximation then, the cavity approximately remains in its vacuum state and qubit photodetector in its ground state.

Utilizing Eqs. (3.18), (3.19), and (3.21) to estimate possible photon pair production rates, we consider a silicon FBAR with fundamental dilatational mode frequency $\omega_m \approx 2\pi \times 10$ GHz, corresponding to a thickness $D \approx 500$ nm and capacitance $C_m^{(0)} \sim 10$ fF assuming a 100 $\mu$m$^2$ FBAR capacitor plate area [8]. With a co-planar cavity characteristic impedance $Z_c \approx 50\,\Omega$ and for $\omega_c \approx 2\pi \times 5$ GHz, we have for the cavity lumped element capacitance $C_c \sim 1$ pF. Assuming $C_J \sim C_m^{(0)}$, we have approximately for the dc-SQUID charging energy $E_C/h \sim 1$ GHz. For a Josephson energy $E_J/h \sim 15$ GHz, we obtain $\Delta E_{qB}/h \sim 5$ GHz, ensuring that the resonance condition $\omega_m = \omega_c + \Delta E_{qB}/\hbar$ can be satisfied; fine tuning of $\Delta E_{qB}$ can be achieved by varying the external flux bias $\Phi_{\text{ext}}$. In order to avoid producing photon pairs directly in the cavity (dynamical Casimir effect), we require that $|\Delta E_{qB}/\hbar - \omega_c| \gg \gamma_c, \gamma_{qB}$, as mentioned above. For achievable FBAR dilatational amplitude $A \sim 10^{-11}$ m [17], the FBAR displacement cavity-qubit coupling (3.19) then becomes $|g_m| \sim 10$ kHz. Assuming realizable cavity and qubit damping rates $\gamma_c \approx \gamma_{qB} \approx 10^7$ s$^{-1}$, we obtain for the photon pair production rate (3.21) $\Gamma \sim 10^4$ s$^{-1}$; such a rate should be measurable.

Looking forward, a more complete analysis of the above, oscillatory Unruh effect scheme is required, in particular including the cavity mode and qubit state measurement circuitry, as well as the method of FBAR actuation. This will give a better idea of the feasibility of the scheme, including how to verify correlated photon production from vacuum, as opposed to typically unavoidable microwave radiation noise. It will also be interesting to consider a semi-infinite transmission line [7] instead of a finite length microwave cavity, such that the microwave vacuum to which the accelerating qubit photodetector couples comprises a dense spectrum of propagating modes. Such a set-up will facilitate a clearer picture of how close is the present scheme to an actual realization of an oscillatory, accelerating photodetector registering photons from the Minkowski vacuum [18].

4. Conclusion

In the present work, we have described possible schemes for realizing analogues of the Hawking and (oscillatory) Unruh effects utilizing superconducting microwave circuits with Josephson junction and film bulk acoustic resonator elements. Superconducting circuits are in principle ideally suited for demonstrating such microwave photon production from vacuum processes, due to their low noise, controllability and existing advanced microwave fabrication and quantum-limited detector capabilities. We also mapped out some possible directions for near future investigation; it is our view that analogue gravity phenomena involving superconducting circuits have yet to be comprehensively explored.

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