A Linear-Quadratic Stackelberg Differential Game with Mixed Deterministic and Stochastic Controls

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Abstract: This paper is concerned with a linear-quadratic (LQ) Stackelberg differential game with mixed deterministic and stochastic controls. Here in the game, the follower is a random controller which means that the follower can choose adapted random processes, while the leader is a deterministic controller which means that the leader can choose only deterministic time functions. An open-loop Stackelberg equilibrium solution is considered. First, an optimal control process of the follower is obtained by maximum principle of controlled stochastic differential equation (SDE), which is a linear functional of optimal state variable and control variable of the leader, via a classical Riccati equation. Then an optimal control function of the leader is got via a direct calculation of derivative of cost functional, by the solution to a system of mean-field forward-backward stochastic differential equations (MF-FBSDEs). And it is represented as a functional of expectation of optimal state variable, together with solutions to a two-point boundary value problem of ordinary differential equation (ODE), by a system consisting of two coupled Riccati equations. The solvability of this new system of Riccati equation is discussed.

Keywords: Stackelberg differential game, mixed deterministic and stochastic controls, linear-quadratic control, feedback representation of optimal control, mean-field forward-backward stochastic differential equation

Mathematics Subject Classification: 93E20, 49K45, 49N10, 49N70, 60H10

1 Introduction

In this paper, we use $\mathbb{R}^n$ to denote the Euclidean space of $n$-dimensional vectors, $\mathbb{R}^{n \times d}$ to denote the space of $n \times d$ matrices, $\mathbb{S}^n$ to denote the space of $n \times n$ symmetric matrices. For a matrix-
valued function \( R : [0, T] \to \mathbb{S}^n \), we denote by \( R \geq 0 \) that \( R_t \) is uniformly positive semi-definite for any \( t \in [0, T] \). For a matrix-valued function \( R : [0, T] \to \mathbb{S}^n \), we denote by \( R \gg 0 \) that \( R_t \) is uniformly positive definite, i.e., there is a positive real number \( \alpha \) such that \( R_t \geq \alpha I \) for any \( t \in [0, T] \). \( \langle \cdot, \cdot \rangle \) and \(|\cdot|\) are used to denote the scalar product and norm in some Euclidean space, respectively. \( A^T \) appearing in the superscript of a matrix, denotes its transpose. \( \text{trace}[A] \) denotes the trace of a square matrix \( A \). \( f_x, f_{xx} \) denote the first- and second-order partial derivatives with respect to \( x \) for a differentiable function \( f \), respectively.

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space, on which an \( \mathbb{R}^d \)-valued standard Brownian motion \( \{W_t\}_{t \geq 0} = \{W_t^1, W_t^2, \cdots, W_t^d\}_{t \geq 0} \) is defined. \( \{\mathcal{F}_t\}_{t \geq 0} \) is the natural filtration generated by \( W(\cdot) \) which is augmented by all \( \mathbb{P} \)-null sets, and \( T > 0 \) is a fixed finite time duration. \( \mathbb{E} \) denotes the expectation with respect to the probability measure \( \mathbb{P} \).

We will use the following notations. \( L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n) \) denotes the set of \( \mathbb{R}^n \)-valued, \( \mathcal{F}_T \)-measurable random vectors \( \xi \) with \( \mathbb{E}[|\xi|^2] < \infty \), \( L^2_{\mathcal{F}}(0, T; \mathbb{R}^n) \) denotes the set of \( \mathbb{R}^n \)-valued, \( \mathcal{F}_t \)-adapted processes \( f \) on \([0, T]\) with \( \mathbb{E}\left[\int_0^T |f(t)|^2dt\right] < \infty \), \( L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n \times d}) \) denotes the set of \( n \times d \)-matrix-valued, \( \mathcal{F}_t \)-adapted processes \( \Phi \) on \([0, T]\) with \( \mathbb{E}\left[\int_0^T |\Phi(t)|^2dt\right] = \mathbb{E}\left[\int_0^T \text{trace}[\Phi(t)^\top \Phi(t)]dt\right] < \infty \), and \( L^2(0, T; \mathbb{R}^n) \) denotes the set of \( \mathbb{R}^n \)-valued functions \( f \) on \([0, T]\) with \( \int_0^T |f(t)|^2dt < \infty \).

We consider the state process \( x^{u,w} : \Omega \times [0, T] \to \mathbb{R}^n \) satisfies a linear SDE

\[
\begin{align*}
\left\{ \begin{array}{ll}
\frac{dx_t^{u,w}}{dt} &= (A_t x_t^{u,w} + B_t^1 u_t + B_t^2 w_t)dt + (C_t x_t^{u,w} + D_t^1 u_t + D_t^2 w_t)dw_t, & t \in [0, T], \\
\end{array} \right.
\end{align*}
\] (1.1)

Here for simplicity, we denote \( (C_t x_t^{u,w} + D_t^1 u_t + D_t^2 w_t)dw_t = \sum_{j=1}^d (C_t^j x_t^{u,w} + D_t^{1j} u_t + D_t^{2j} w_t)dw_t^j \) with \( A, B^1, B^2, C^j, D^{1j} \) and \( D^{2j} \) being all bounded Borel measurable functions from \([0, T]\) to \( \mathbb{R}^{n \times n}, \mathbb{R}^{n \times k_1}, \mathbb{R}^{n \times k_2}, \mathbb{R}^{n \times n}, \mathbb{R}^{n \times k_1} \) and \( \mathbb{R}^{n \times k_2} \), respectively. Similar notations are used in the rest of this paper. In the above, \( u : \Omega \times [0, T] \to \mathbb{R}^{k_1} \) is the follower’s control process and \( w : [0, T] \to \mathbb{R}^{k_2} \) is the leader’s control function. Let \( \mathcal{U}^1_{ad} = L^2_{\mathcal{F}}(0, T; \mathbb{R}^{k_1}) \) and \( \mathcal{U}^2_{ad} = L^2(0, T; \mathbb{R}^{k_2}) \) be the admissible control sets of the follower and the leader, respectively. That is to say, the control process \( u \) of the follower is taken from \( \mathcal{U}^1_{ad} \) and the control function \( w \) of the leader is taken from \( \mathcal{U}^2_{ad} \).

For given initial value \( x \in \mathbb{R}^n \) and \((u, w) \in \mathcal{U}_{ad} \times \mathcal{U}_{ad}^2 \), it is classical that there exists a unique solution \( x^{u,w} \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n) \) to (1.1). Thus, we could define the cost functionals of the players as follows:

\[
\begin{align*}
J_1(x; u, w) &= \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle Q^1_t x_t^{u,w}, x_t^{u,w} \rangle + \langle S^1_t x_t^{u,w}, u_t \rangle + \langle R^1_t u_t, u_t \rangle \right) dt + \langle G^1 x_T^{u,w}, x_T^{u,w} \rangle \right], \\
J_2(x; u, w) &= \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle Q^2_t x_t^{u,w}, x_t^{u,w} \rangle + \langle S^2_t x_t^{u,w}, w_t \rangle + \langle R^2_t w_t, w_t \rangle \right) dt + \langle G^2 x_T^{u,w}, x_T^{u,w} \rangle \right],
\end{align*}
\] (1.2) (1.3)
where $Q^1, Q^2, S^1, S^2, R^1, R^2$ are bounded Borel measurable functions from $[0, T]$ to $S^n, S^n, \mathbb{R}^{k_1 \times n}$, $\mathbb{R}^{k_2 \times n}, S^{k_1}, S^{k_2}$, respectively, and $G^i$ are $S^n$-valued matrices for $i = 1, 2$.

We formulate the Stackelberg game by two steps. In the first step, for any chosen $w \in U^2_{ad}$ and a fixed initial state $x \in \mathbb{R}^n$, the follower would like to choose a $u^* \in U^1_{ad}$ such that $J_1(x; u^*, w)$ is the minimum of the cost functional $J_1(x; u, w)$ over $U^1_{ad}$. In a more rigorous way, the follower wants to find a map $\alpha^*: U^2_{ad} \times [0, T] \rightarrow U^1_{ad}$, such that

$$J_1(x; \alpha^*[w, x], w) = \min_{u \in U^1_{ad}} J_1(x; u, w), \text{ for all } w \in U^2_{ad}. \quad (1.4)$$

In the second step, knowing that the follower would take $u^* \equiv \alpha^*[w, x_0]$, the leader wishes to choose some $w^*$ to minimize $J_2(x_0; \alpha^*[w, x], w)$ over $U^2_{ad}$. That is to say, the leader wants to find a control function $w^*$ such that

$$J_2(x; \alpha^*[w^*, x], w^*) = \min_{w \in U^2_{ad}} J_2(x; \alpha^*[w, x], w). \quad (1.5)$$

If $(\alpha^*, w^*)$ exists, we refer to it as an open-loop Stackelberg equilibrium solution to the above LQ Stackelberg differential game with mixed deterministic and stochastic controls. In this paper, we will make a great effort to find a state feedback representation for the open-loop Stackelberg equilibrium solution.

The Stackelberg differential game is also known as leader-follower differential game, which attracts more and more research attention recently, since it has wide practical backgrounds, especially in economics and finance. The earliest work about the game can be traced back to Stackelberg [11], where the concept of Stackelberg equilibrium solution was defined for economic markets when some firms have power of domination over others. Bagchi and Başar [1] discussed an LQ stochastic Stackelberg differential game, where state and control variables do not enter diffusion coefficient in state equation. Yong [16] considered an LQ Stackelberg differential game in a rather general framework, with random coefficient, control dependent diffusion and weight matrix for controls in cost functional being not necessarily nonnegative definite. Øksendal et al. [8] obtained a maximum principle for Stackelberg differential game in the jump-diffusion case, and applied the result to a newsvendor problem. Bensoussan et al. [2] investigated several information structures for stochastic Stackelberg differential game, whereas diffusion coefficient does not contain control variable. Shi et al. [9] introduced a new explanation for the asymmetric information feature of Stackelberg differential game, and an LQ stochastic Stackelberg differential game with noisy observation was solved, where not all the diffusion coefficients contain control variables. Shi et al. [10] studied an LQ stochastic Stackelberg differential game with asymmetric information, where control variables enter both diffusion coefficients of state equation. Xu and Zhang [14] and Xu et al. [13] addressed a Stackelberg differential game with time-delay. Li and Yu [4] applied FBSDE with a multilevel self-similar domination-monotonicity structure, to characterize the unique equilibrium of an LQ generalized Stackelberg game with hierarchy.
Moon and Başar [7] investigated an LQ mean field Stackelberg differential game with adapted open-loop information structure of the leader where there are only one leader but arbitrarily large number of followers. See also Lin et al. [5], Wang et al. [12] for recent developments on open-loop LQ Stackelberg game of mean-field type stochastic systems.

Recently, an interesting paper by Hu and Tang [3], considered a mixed deterministic and random optimal control problem of linear stochastic system with quadratic cost functional, with two controllers—one can choose only deterministic time functions which is called the deterministic controller, while the other can choose adapted random processes which is called the random controller. The optimal control is characterized via a system of fully coupled FBS-DEs of mean-field type, whose solvability is proved by solutions to two (not coupled) Riccati equations. Inspired by [3], here in this paper we consider an LQ Stackelberg differential game with mixed deterministic and random controls, where the follower is a random controller and the leader is a deterministic controller. In practical applications such as in Stackelberg’s type financial market, some securities investor is the follower and the government who makes macro policies is the leader. The novelty and contribution of this paper can be summarized as follows.

• The game problem is new. To the best of our knowledge, it is the first paper to consider the mixed deterministic and random controls in the study of Stackelberg games. So this paper can be regarded as a continuation of [3], from control to game problems.

• The problem of the leader is related with a system of MF-FBSDEs, via a direct calculation of derivative of cost functional. This interesting feature is different from [16].

• A feedback representation of optimal control function of the leader with respect to the expectation of optimal state variable, is obtained by solutions to a system of two coupled Riccati equations and a two-point value problem of ODEs. This is also different from [16], where a dimensional-expansion technique is applied.

The rest of this paper is organized as follows. In Section 2, the game problem is solved in two subsections. The problem of the follower is discussed in Subsection 2.1, and that of the leader is studied in Subsection 2.2. First, an optimal control process of the follower is obtained by maximum principle of controlled SDE, which is a linear functional of optimal state variable and control variable of the leader, via a classical Riccati equation. Then an optimal control function of the leader is got via a direct calculation of derivative of cost functional, via the solution to a system of MF-FBSDEs. And it is represented as a functional of expectation of optimal state variable, together with solutions to a two-point boundary value problem of ODEs, by a system consisting of two coupled Riccati equations. The solvability of this new system of Riccati equation is discussed. Finally, Section 3 gives some concluding remarks.
2 Main Result

We split this section into two subsections, to deal with the problems of the follower and the leader, respectively.

2.1 Problem of the Follower

For given control function \( w \in U^1_{ad} \), assume that \( u^* \) is an optimal control process of the follower and the corresponding optimal state is \( x^{u^*,w} \). Define the Hamiltonian function \( H_1 : [0, T] \times \mathbb{R}^n \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R} \) of the follower as

\[
H_1(t, x, u, w, q, k) = \langle q, Ax + B_1 u + B_2 w \rangle + \langle k, C x + D_1 u + D_2 w \rangle - \frac{1}{2} \langle Q^1 x, x \rangle - \langle S^1 x, u \rangle - \frac{1}{2} \langle R^1 u, u \rangle.
\]  

(2.1)

By the maximum principle (see, e.g., Chapter 6 of Yong and Zhou [17]), there exists a unique pair of processes \((q, k) \equiv (k^1, k^2, \ldots, k^d)\) \(\in L^2_x(0, T; \mathbb{R}^n) \times (L^2_x(0, T; \mathbb{R}^n))^d\) satisfying the backward SDE (BSDE)

\[
\begin{cases}
-dq_t = [A^\top_t q_t + C^\top_t k_t - (S^1_t)^\top u_t - Q^1_t x^{u^*,w}_t] \, dt - k_t \, dW_t, \, t \in [0, T], \\
q_T = -G^1_x u^*_T, w_T,
\end{cases}
\]  

(2.2)

and the optimality condition holds true

\[
0 = R^1_t u^*_t + S^1_t x^{u^*,w}_t - (B^1_t)^\top q_t - (D^1_t)^\top k_t, \, t \in [0, T].
\]  

(2.3)

We wish to obtain a state feedback representation of \( u^* \). Noticing the terminal condition of (2.2) and the appearance of the control function \( w \), we set

\[
q_t = -P_t x^{u^*,w}_t - \varphi_t, \, t \in [0, T],
\]  

(2.4)

for some differentiable function \( P \) and \( \varphi \) from \([0, T]\) to \( \mathbb{S}^n \) and \( \mathbb{R}^n \), respectively, satisfying \( P_T = G^1 \) and \( \varphi_T = 0 \).

Applying Itô’s formula to (2.4), we have

\[
-dq_t = (\dot{P}_t x^{u^*,w}_t + P_t A_t x_t^{u^*,w}_t + \dot{\varphi}_t + P_t B^1_t u^*_t + P_t B^2_t w_t) \, dt + P_t (C_t x^{u^*,w}_t + D^1_t u^*_t + D^2_t w_t) \, dW_t.
\]  

(2.5)

Comparing the \( dW_t \) term in (2.5) with that in (2.2), we arrive at

\[
k_t = -P_t (C_t x^{u^*,w}_t + D^1_t u^*_t + D^2_t w_t), \, t \in [0, T].
\]  

(2.6)

Plugging (2.4) and (2.6) into optimality condition (2.3), and supposing that (A2.1) \( R^1_t + (D^1_t)^\top P_t D^1_t \) is convertible, for all \( t \in [0, T] \),
we immediately arrive at
\[ u_t^* = -(R_t^1 + (D_t^1)^\top P_t D_t^1)^{-1} \left\{ [(B_t^1)^\top P_t + (D_t^1)^\top P_tC_t + S_t^1] x_t^{u^*,w} ight. \\
+ (D_t^1)^\top P_tD_t^2 w_t + (B_t^1)^\top \varphi_t \right\}, \ t \in [0, T]. \quad (2.7) \]
Comparing the \( dt \) term in (2.5) with that in (2.2), noting (2.4), (2.6) and (2.7), we can obtain that if
\[
\begin{align*}
\dot{P}_t + A_t^\top P_t + P_t A_t + C_t^\top P_tC_t + Q_t^1 - [P_t B_t^1 + C_t^\top P_tD_t^1 + (S_t^1)^\top] \\
\times (R_t^1 + (D_t^1)^\top P_tD_t^1)^{-1} [(B_t^1)^\top P_t + (D_t^1)^\top P_tC_t + S_t^1] = 0, \ t \in [0, T], \quad (2.8)
\end{align*}
\]
adsmits a unique differentiable solution \( P \in S^n \), then
\[
\begin{align*}
\dot{\varphi}_t + [A_t^\top - (P_t B_t^1 + C_t^\top P_tD_t^1 + (S_t^1)^\top)(R_t^1 + (D_t^1)^\top P_tD_t^1)^{-1}(B_t^1)^\top] \varphi_t \\
+ [P_t B_t^2 + C_t^\top P_tD_t^2 - (P_t B_t^1 + C_t^\top P_tD_t^1 + (S_t^1)^\top)] \\
\times (R_t^1 + (D_t^1)^\top P_tD_t^1)^{-1}(D_t^1)^\top P_tD_t^2 w_t = 0, \ t \in [0, T], \quad (2.9)
\end{align*}
\]
For the solvability of Riccati equation (2.8), in the following standard assumption that
\[(A2.2) \quad R^1 \gg 0, \ G^1 \gg 0, \ Q^1 - S^1(R^1)^{-1}(S^1)^\top \gg 0, \]
(2.8) admits a unique differentiable solution \( P \geq 0 \) by Theorem 7.2, Chapter 6 of [17]. For given \( w \in U_{ad}^2 \), the solvability of ODE (2.9) is obvious.
Under \((A2.2)\), the map \( u \mapsto J_1(x;u,w) \) is uniformly convex, thus (2.7) is also sufficient for \( (u^*, x^{u^*,w}) \) being a unique optimal pair of the follower.
Now, inserting (2.7) into the state equation of (1.1), we have
\[
\begin{align*}
\frac{dx_t^{u^*,w}}{dt} = \left\{ \left[ A_t - B_t^1 (R_t^1 + (D_t^1)^\top P_tD_t^1)^{-1} ((B_t^1)^\top P_t + (D_t^1)^\top P_tC_t) + S_t^1 \right] x_t^{u^*,w} \right. \\
+ \left[ B_t^2 - B_t^1 (R_t^1 + (D_t^1)^\top P_tD_t^1)^{-1}(D_t^1)^\top P_tD_t^2 \right] w_t \\
- B_t^1 (R_t^1 + (D_t^1)^\top P_tD_t^1)^{-1}(B_t^1)^\top \varphi_t \right\} \ dt \\
+ \left\{ \left[ C_t - D_t^1 (R_t^1 + (D_t^1)^\top P_tD_t^1)^{-1} ((B_t^1)^\top P_t + (D_t^1)^\top P_tC_t) + S_t^1 \right] x_t^{u^*,w} \right. \\
+ \left[ D_t^2 - D_t^1 (R_t^1 + (D_t^1)^\top P_tD_t^1)^{-1}(D_t^1)^\top P_tD_t^2 \right] w_t \\
- D_t^1 (R_t^1 + (D_t^1)^\top P_tD_t^1)^{-1}(B_t^1)^\top \varphi_t \right\} \ dW_t, \ t \in [0, T], \quad (2.10)
\end{align*}
\]
which admits a unique solution \( x^{u^*,w} \in L^2_F(0,T;\mathbb{R}^n) \), for given \( w \in U_{ad}^2 \).
Moreover, we have the result.
Theorem 2.1 Let (A2.1), (A2.2) hold, \( P \succeq 0 \) satisfy (2.8). For chosen control function \( w \in \mathcal{U}_{ad}^2 \) of the leader, there is a unique optimal control process \( u^* \in \mathcal{U}_{ad}^1 \) of the follower, whose state feedback representation is given by (2.7), where \( x^{u^*,w} \in L^2_T(0,T;\mathbb{R}^n) \) is the optimal state satisfying (2.10) and the differential function \( \varphi \) satisfy (2.9). The optimal value is given by

\[
J_1(x; u^*, w) = \frac{1}{2} \langle P_0 x, x \rangle + \langle \varphi_0, x \rangle + \int_0^T \left( \langle (B_t^2)\varphi_t, w_t \rangle + \langle (D_t^2)\varphi_t, w_t \rangle \right)
- \frac{1}{2} \left[ (R_t^1 + (D_t^1)^TP_tD_t^2)^{-\frac{1}{2}} \langle (B_t^2)\varphi_t + (D_t^2)^TP_tD_t^2w_t \rangle \right]^2 dt.
\]

(2.11)

Proof. We only need to prove (2.11). However, it can be easily obtained by applying Itô’s formula to \( \langle P_{x^{u^*,w}}, x^{u^*,w} \rangle + \langle \varphi, x^{u^*,w} \rangle \), together with the completion of squares technique. We omit the detail. \( \square \)

The results in this subsection is a special case of those in Section 2 of Yong [16], but with the cross term. We display those here with some refined derivation for the self-integrity of this paper.

2.2 Problem of the Leader

Since the leader knows that the follower will take his optimal control process \( u^* \in \mathcal{U}_{ad}^1 \) by (2.7), the state equation of the leader now writes

\[
\begin{align*}
\begin{cases}
 dx^{w}_t = (\bar{A}_t x^{w}_t + \bar{B}_t^1 \varphi_t + \bar{B}_t^2 w_t) dt + (\bar{C}_t x^{w}_t + \bar{D}_t^1 \varphi_t + \bar{D}_t^2 w_t) dW_t, \\
 d\varphi_t = -(\bar{A}_t \varphi_t + \Gamma_t w_t) dt, \quad t \in [0,T], \\
x^{w}_0 = x, \quad \varphi_T = 0,
\end{cases}
\end{align*}
\]

(2.12)

where we have denoted \( x^{w} \equiv x^{u^*,w} \) and

\[
\begin{align*}
\bar{R}_1 := \bar{R}^1(P) := R^1 + (D^1)^TPD^1, \\
\bar{A} := \bar{A}(P) := A - B^1(\bar{R}_1)^{-1}[(B^1)^TP + (D^1)^TPC + S^1], \\
\bar{B}^1 := \bar{B}^1(P) := -B^1(\bar{R}_1)^{-1}(B^1)^T, \\
\bar{B}^2 := \bar{B}^2(P) := B^2 - B^1(\bar{R}_1)^{-1}(D^1)^TPD^2, \\
\bar{C} := \bar{C}(P) := C - D^1(\bar{R}_1)^{-1}[(B^1)^TP + (D^1)^TPC + S^1], \\
\bar{D}^1 := \bar{D}^1(P) := -D^1(\bar{R}_1)^{-1}(B^1)^T, \\
\bar{D}^2 := \bar{D}^2(P) := D^2 - D^1(\bar{R}_1)^{-1}(D^1)^TPD^2, \\
\Gamma := \Gamma(P) := PB^2 + C^TPD^2 - [PB^1 + C^TPD^1 + (S^1)^T](\bar{R}_1)^{-1}(D^1)^TPD^2.
\end{align*}
\]

The problem of the leader is to choose an optimal control function \( w^* \in \mathcal{U}_{ad}^2 \) such that

\[ J_2(x; u^*, w^*) = \min_{w \in \mathcal{U}_{ad}^2} J_2(x; u^*, w). \]
We first have the following result.

**Theorem 2.2** Suppose that \( w^* \) is an optimal control function of the leader, and the corresponding optimal state is \( x^* = x^{w^*} \) together with \( \varphi^* \) being solution to (2.12). Then we have

\[
0 = R_t^2 w_t^* + (\tilde{B}_t^*)^\top \mathbb{E} y_t + (\tilde{D}_t^2) y_t + S_t^2 \mathbb{E} x_t^* + \Gamma_t^\top \mathbb{E} p_t, \quad t \in [0, T],
\]

where the triple of processes \( (y, z, p) \in \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^n \) satisfy the FBSDE

\[
\begin{cases}
    dp_t = [A_t^\top p_t + (\tilde{B}_t^*)^\top y_t + (\tilde{D}_t^2) z_t] dt, \\
    -dy_t = [\tilde{A}_t^\top y_t + \tilde{C}_t^\top z_t + (S_t^2) w_t^* + Q_t^2 x_t^*] dt - z_t dW_t, \quad t \in [0, T], \\
    p_0 = 0, \quad y_T = G^2 x_T^*.
\end{cases}
\]

Moreover, if we assume that

(A2.3) \( G^2 \geq 0, \quad Q^2 - S^2 (R^2)^{-1} (S^2)^\top \geq 0, \quad R^2 \gg 0, \)

then the above optimality condition becomes sufficient for the unique existence of the optimal control function \( w^* \) of the leader.

**Proof.** Without loss of generality, let \( x \equiv 0 \), and set the perturbed optimal control function \( w^* + \lambda w \) for \( \lambda > 0 \) sufficiently small, with \( w \in \mathbb{R}^{k_x} \). Then it is easy to see from the linearity of (2.12), that the solution to (2.12) is \( x^* + \lambda x^w \). We first have

\[
\bar{J}(\lambda) := J_2(0; u^*, w^* + \lambda w)
\]

\[
= \frac{1}{2} \mathbb{E} \int_0^T \left[ \langle Q_t^2 (x_t^* + \lambda x_t^w), x_t^* + \lambda x_t^w \rangle + 2\langle S_t^2 (x_t^* + \lambda x_t^w), w_t^* + \lambda w_t \rangle \right. \\
\left. + \langle R_t^2 (w_t^* + \lambda w_t), w_t^* + \lambda w_t \rangle \right] dt + \frac{1}{2} \mathbb{E} \langle G^2 (x_T^* + \lambda x_T^w), x_T^* + \lambda x_T^w \rangle.
\]

Hence

\[
0 = \left. \frac{\partial \bar{J}(\lambda)}{\partial \lambda} \right|_{\lambda=0} = \mathbb{E} \int_0^T \left[ \langle Q_t^2 x_t^*, x_t^w \rangle + \langle S_t^2 x_t^*, w_t \rangle + \langle S_t^2 x_t^*, w_t^* \rangle \\
+ \langle R_t^2 w_t^*, w_t \rangle \right] dt + \mathbb{E} \langle G^2 x_T^*, x_T^w \rangle.
\]

Let the triple \( (p, y, z) \) satisfy (2.14). Then we have

\[
0 = \mathbb{E} \int_0^T \left[ \langle Q_t^2 x_t^*, x_t^{w^*} \rangle + \langle S_t^2 x_t^*, w_t \rangle + \langle S_t^2 x_t^{w^*}, w_t^* \rangle + \langle R_t^2 w_t^*, w_t \rangle \right] dt + \mathbb{E} \langle y_T, x_T^w \rangle.
\]

Applying Itô’s formula to \( \langle x_t^u, y_t \rangle - \langle \varphi_t, p_t \rangle \), noticing (2.12) and (2.14), we derive

\[
0 = \mathbb{E} \int_0^T \left( R_t^2 w_t^* + (\tilde{B}_t^*)^\top y_t + (\tilde{D}_t^2) y_t + S_t^2 \mathbb{E} x_t^* + \Gamma_t^\top \mathbb{E} p_t, w_t \right) dt
\]

\[
= \mathbb{E} \int_0^T \left( R_t^2 w_t^* + (\tilde{B}_t^*)^\top \mathbb{E} y_t + (\tilde{D}_t^2) \mathbb{E} y_t + S_t^2 \mathbb{E} x_t^* + \Gamma_t^\top \mathbb{E} p_t, w_t \right) dt.
\]

This implies (2.13). Further, if (A2.3) holds, then the functional \( w \rightarrow J_2(x; u^*, w) \) is uniformly convex. Thus the necessary condition becomes sufficient for the unique existence of \( w^* \). See the remark of Theorem 2.2 in Yong [16] for more details. The proof is complete. \( \square \)
Next, putting (2.12), (2.13) and (2.14) together, corresponding with the optimal triple \((w^*, x^*, \phi^*)\), we get

\[
\begin{aligned}
&dx_t^* = (\bar{A}_t x_t^* + \bar{B}_1^t \phi_t^* + \bar{B}_2^t w_t^*) dt + (\bar{C}_t x_t^* + \bar{D}_1^t \varphi_t^* + \bar{D}_2^t w_t^*) dW_t, \\
&d\varphi_t^* = -(\bar{A}_t^T \varphi_t^* + \Gamma_t w_t^*) dt, \\
&dp_t = [\bar{A}_t^T p_t + (\bar{B}_1^t)^T y_t + (\bar{D}_1^t)^T z_t] dt, \\
-dy_t = [\bar{A}_t^T y_t + (\bar{C}_t^T)^T z_t + (S_t^2)^T w_t^* + Q_t^2 x_t^*] dt - z_t dW_t, \\
&x_0^* = x, \quad \varphi_T^* = 0, \quad p_0 = 0, \quad y_T = G^2 x_T^*, \\
&0 = R_t^2 w_t^* + (\bar{B}_2^t)^T \mathbb{E} y_t + (\bar{D}_1^t)^T \mathbb{E} z_t + S_t^2 \mathbb{E} x_t^* + \Gamma_t^T \mathbb{E} p_t, \quad t \in [0, T], \tag{2.15}
\end{aligned}
\]

which is a system of coupled MF-FBSDEs. Note that it is different from that in Yong [16]. We need to decouple (2.15), and to study the solvability of it via some Riccati equations. For this target, for the optimal control function \(w^*\) of (2.13), we expect a state feedback representation of the form

\[
y_t = P_1^t x_t^* + P_2^t (x_t^* - \mathbb{E} x_t^*) + \phi_t, \tag{2.16}
\]

for some differentiable functions \(P_1, P_2\) and \(\phi\) from \([0, T]\) to \(\mathbb{S}^n, \mathbb{R}^{n \times n}\) and \(\mathbb{R}^n\), respectively, satisfying \(P_T^1 = G^2, P_T^2 = 0\) and \(\phi_T = 0\).

Noticing that

\[
\begin{aligned}
&d\mathbb{E} x_t^* = (\bar{A}_t \mathbb{E} x_t^* + \bar{B}_1^t \phi_t^* + \bar{B}_2^t w_t^*) dt, \quad t \in [0, T], \\
&\mathbb{E} x_0^* = x, \tag{2.17}
\end{aligned}
\]

and applying Itô’s formula to (2.16), we obtain

\[
\begin{aligned}
dy_t = &\left[\dot{\phi}_t + (P_1^1 + P_1^1 \bar{A}_t) x_t^* + (P_1^2 + P_2^2 \bar{A}_t) (x_t^* - \mathbb{E} x_t^*) + P_1^1 \bar{B}_1^t \phi_t^* + P_1^1 \bar{B}_2^t w_t^*\right] dt \\
+ &\left[(P_1^1 + P_2^1) \bar{C}_t x_t^* + (P_1^1 + P_2^2) \bar{D}_1^t \phi_t^* + (P_1^1 + P_2^2) \bar{D}_2^t w_t^*\right] dW_t \\
= &- \left[\bar{A}_t^T P_1^1 x_t^* + \bar{A}_t^T P_1^2 (x_t^* - \mathbb{E} x_t^*) + \bar{A}_t^T \phi_t + \bar{C}_t^T z_t + (S_t^2)^T w_t^* + Q_t^2 x_t^*\right] dt + z_t dW_t. \tag{2.18}
\end{aligned}
\]

Thus

\[
z_t = (P_1^1 + P_2^2) \bar{C}_t x_t^* + (P_1^1 + P_2^2) \bar{D}_1^t \phi_t^* + (P_1^1 + P_2^2) \bar{D}_2^t w_t^*, \quad t \in [0, T]. \tag{2.19}
\]

Plugging (2.10), (2.19) into (2.13), and supposing that

(A2.4) \(\bar{R}_t^2 := \bar{R}_t^2 (P_t^1, P_t^2, P_t^2) := \bar{R}_t^2 + (\bar{D}_t^2)^T (P_1^1 + P_2^2) \bar{D}_t^2\) is convertible, for all \(t \in [0, T]\),

we get

\[
w_t^* = -(\bar{R}_t^2)^{-1} \left\{\left[(\bar{B}_t^2)^T P_1^1 + (\bar{D}_t^2)^T (P_1^1 + P_2^2) \bar{C}_t + S_t^2\right] \mathbb{E} x_t^*ight. \\
+ \left. (\bar{D}_t^2)^T (P_1^1 + P_2^2) \bar{D}_1^t \phi_t^* + \Gamma_t^T \mathbb{E} p_t + (\bar{B}_t^2)^T \phi_t\right\}. \tag{2.20}
\]

Inserting (2.20) into (2.19), we have

\[
\begin{aligned}
z_t = &\left((P_1^1 + P_2^2) \bar{C}_t x_t^* - (P_1^1 + P_2^2) \bar{D}_t^2 (\bar{R}_t^2)^{-1} (\bar{B}_t^2)^T P_1^1 + (\bar{D}_t^2)^T (P_1^1 + P_2^2) \bar{C}_t + S_t^2\right) \mathbb{E} x_t^* \\
+ &\left[(P_1^1 + P_2^2) \bar{D}_t^2 (\bar{R}_t^2)^{-1} (\bar{B}_t^2)^T (P_1^1 + P_2^2) \bar{D}_t^2\right] \phi_t^* \\
- &\left(P_1^1 + P_2^2\right) \bar{D}_t^2 (\bar{R}_t^2)^{-1} \Gamma_t^T \mathbb{E} p_t - \left(P_1^1 + P_2^2\right) \bar{D}_t^2 (\bar{R}_t^2)^{-1} (\bar{B}_t^2)^T \phi_t. \tag{2.21}
\end{aligned}
\]
Comparing $dt$ terms in the fourth equation in (2.15) and (2.18) and substituting (2.20), (2.21) into them, we obtain

\[
\begin{align*}
0 &= \hat{P}_t^1 + P_t^1 \tilde{A}_t + \bar{A}_t^T P_t^1 + \bar{C}_t^T (P_t^1 + P_t^2) \bar{C}_t - [P_t^1 \bar{B}_t^2 + \bar{C}_t^T (P_t^1 + P_t^2) \bar{D}_t^2 + (S_t^2)^T] \\
&\quad \times (\bar{R}_t^2)^{-1} [(\bar{B}_t^2)^T P_t^1 + (\bar{D}_t^2)^T (P_t^1 + P_t^2) \bar{C}_t + S_t^2] + Q_t, \quad P_T^1 = G^2, \\
0 &= \hat{P}_t^2 + P_t^2 \tilde{A}_t + \bar{A}_t^T P_t^2 + [P_t^1 \bar{B}_t^2 + \bar{C}_t^T (P_t^1 + P_t^2) \bar{D}_t^2 + (S_t^2)^T] \\
&\quad \times (\bar{R}_t^2)^{-1} [(\bar{B}_t^2)^T P_t^1 + (\bar{D}_t^2)^T (P_t^1 + P_t^2) \bar{C}_t + S_t^2] + Q_t, \quad P_T^2 = 0,
\end{align*}
\]

(2.22)

and

\[
\begin{align*}
0 &= \hat{\phi}_t + \left\{ \bar{A}_t^T - \left[ P_t^1 \bar{B}_t^2 + \bar{C}_t^T (P_t^1 + P_t^2) \bar{D}_t^2 + (S_t^2)^T \right] (\bar{R}_t^2)^{-1} (\bar{B}_t^2)^T \right\} \phi_t + \left\{ P_t^1 \bar{B}_t^1 + \bar{C}_t^T (P_t^1 + P_t^2) \bar{D}_t^1 - \left[ P_t^1 \bar{B}_t^2 + \bar{C}_t^T (P_t^1 + P_t^2) \bar{D}_t^2 + (S_t^2)^T \right] (\bar{R}_t^2)^{-1} (\bar{D}_t^2)^T \right\} \phi_t \\
&\quad \times (P_t^1 + P_t^2) \bar{D}_1 \right\} \phi_t^* - \left[ P_t^1 \bar{B}_t^2 + \bar{C}_t^T (P_t^1 + P_t^2) \bar{D}_t^2 + (S_t^2)^T \right] (\bar{R}_t^2)^{-1} \bar{R}_t^2 E_{P_t}, \\
\phi_T &= 0.
\end{align*}
\]

(2.23)

Note that system (2.22) consists two coupled Riccati equations, which is entirely new and its solvability is interesting. In fact, adding the two equations in (2.22), it is obviously that $P^1 + P^2 \in \mathbb{R}^{n \times n}$ uniquely satisfies the ODE

\[
0 = \hat{P}_t + \mathcal{P}_t \tilde{A}_t + \bar{A}_t^T \mathcal{P}_t + \bar{C}_t^T \mathcal{P}_t \bar{C}_t + Q_t^2, \quad \mathcal{P}_T = G^2.
\]

(2.24)

Thus (2.22) becomes

\[
\begin{align*}
0 &= \hat{P}_t^1 + P_t^1 \tilde{A}_t + \bar{A}_t^T P_t^1 + \bar{C}_t^T \mathcal{P}_t \bar{C}_t - \left[ P_t^1 \bar{B}_t^2 + \bar{C}_t^T \mathcal{P}_t \bar{D}_t^2 + (S_t^2)^T \right] \\
&\quad \times (\bar{R}_t^2 + (\bar{D}_t^2)^T \mathcal{P}_t \bar{D}_t^2)^{-1} [(\bar{B}_t^2)^T P_t^1 + (\bar{D}_t^2)^T \mathcal{P}_t \bar{C}_t + S_t^2] + Q_t, \quad P_T^1 = G^2, \\
0 &= \hat{P}_t^2 + P_t^2 \tilde{A}_t + \bar{A}_t^T P_t^2 + [P_t^1 \bar{B}_t^2 + \bar{C}_t^T \mathcal{P}_t \bar{D}_t^2 + (S_t^2)^T] \\
&\quad \times (\bar{R}_t^2 + (\bar{D}_t^2)^T \mathcal{P}_t \bar{D}_t^2)^{-1} [(\bar{B}_t^2)^T P_t^1 + (\bar{D}_t^2)^T \mathcal{P}_t \bar{C}_t + S_t^2], \quad P_T^2 = 0,
\end{align*}
\]

(2.25)

and it is a decoupled one now. Let

\[
\tilde{Q}_t^2 := Q_t^2 + \bar{C}_t^T \mathcal{P}_t \bar{C}_t, \quad \tilde{S}_t^2 := S_t^2 + (\bar{D}_t^2)^T \mathcal{P}_t \bar{C}_t, \quad \forall t \in [0, T].
\]

Then the Riccati equation of $P^1$ can be written as

\[
\begin{align*}
0 &= \hat{P}_t^1 + P_t^1 \tilde{A}_t + \bar{A}_t^T \tilde{P}_t^1 - \left[ P_t^1 \bar{B}_t^2 + (S_t^2)^T \right] (\bar{R}_t^2)^{-1} [(\bar{B}_t^2)^T P_t^1 + \tilde{S}_t^2] + \tilde{Q}_t^2, \\
P_T^1 &= G^2,
\end{align*}
\]

(2.26)

If we assume that

(A2.5) \quad \tilde{Q}_t^2 - \tilde{S}_t^2 (\bar{R}_t^2)^{-1} (\bar{S}_t^2)^T \geq 0,

by (A2.3), (A2.4) and (A2.5), there is a unique solution $P^1 \geq 0$. Then there also exists a unique solution $P^2 = \mathcal{P} - P^1 \in \mathbb{R}^{n \times n}$. 


We discuss the solvability of equation (2.23) for the function \( \phi \). In fact, with some computation, we can obtain a two-point boundary value problem for coupled linear ODE for \((E_x^*, E_p, \varphi^*, \phi)\):

\[
\begin{cases}
\frac{dE_x^t}{dt} = [\bar{A}_t - \bar{B}_t^2(\bar{R}_1^2)^{-1}\bar{S}_2^2]E_x^t - \bar{B}_t^2(\bar{R}_2^2)^{-1}\Gamma_t^\top E_p t - \bar{B}_t^2(\bar{R}_2^2)^{-1}(\bar{B}_t^2)^\top \phi_t + \bar{B}_t^2 \varphi_t^t,
\frac{dE_p}{dt} = (\bar{A}_t^\top - \bar{G}_t^\top)E_p + (\bar{B}_t^2)^\top E_x^t + (\bar{B}_t^2)^\top \phi_t + \bar{B}_t^2 \varphi_t^t,
\frac{d\varphi_t^t}{dt} = (\Gamma_t - \bar{A}_t^\top)\varphi_t^t + \Gamma_t(\bar{R}_2^2)^{-1}\Gamma_t^\top E_p t + \Gamma_t(\bar{R}_2^2)^{-1}(\bar{B}_t^2)^\top \phi_t + \Gamma_t(\bar{R}_2^2)^{-1}\bar{S}_2^2 E_x^t,
\frac{d\phi_t}{dt} = -[(\bar{A}_t^\top - (\bar{S}_2^2)^\top (\bar{R}_2^2)^{-1}(\bar{B}_t^2)^\top)]\phi t - \bar{B}_t^2 \varphi_t^t + (\bar{S}_2^2)^\top (\bar{R}_2^2)^{-1}\Gamma_t^\top E_p t, \quad t \in [0, T],
\end{cases}
\]

\( E_x^0 = x, \ E_p^0 = 0, \ \varphi_T = 0, \ \phi_T = 0, \)

where for simplicity, we denote

\[
\begin{align*}
\bar{S}_2^2 &:= (\bar{B}_t^2)^\top P_1^1 + (\bar{D}_t^2)^\top (P_1^1 + P_2^1)\bar{C}_t + S_2^2, \\
\Gamma_t &:= \Gamma_t(\bar{R}_2^2)^{-1}(\bar{D}_t^2)^\top (P_1^1 + P_2^1)\bar{D}_t^1, \\
\bar{B}_t^1 &:= P_1^1 \bar{B}_t^1 + \bar{C}_t^\top (P_1^1 + P_2^1)\bar{D}_t^1 - (\bar{S}_2^2)^\top (\bar{R}_2^2)^{-1}(\bar{D}_t^2)^\top (P_1^1 + P_2^1)\bar{D}_t^1, \\
\bar{B}_t^2 &:= \bar{B}_t^1 - \bar{B}_t^2(\bar{R}_2^2)^{-1}(\bar{D}_t^2)^\top (P_1^1 + P_2^1)\bar{D}_t^1, \\
\bar{D}_t^1 &:= (\bar{D}_t^1)^\top [(P_1^1 + P_2^1)\bar{D}_t^1 - (P_1^1 + P_2^1)\bar{D}_t^2(\bar{R}_2^2)^{-1}(\bar{D}_t^2)^\top (P_1^1 + P_2^1)\bar{D}_t^1].
\end{align*}
\]

We define

\[
X := \begin{pmatrix} E_x^* \\ E_p \end{pmatrix}, \quad Y := \begin{pmatrix} \varphi^* \\ \phi \end{pmatrix},
\]

\[
A_t := \begin{pmatrix} \bar{A}_t - \bar{B}_t^2(\bar{R}_1^2)^{-1}\bar{S}_2^2 & -\bar{B}_t^2(\bar{R}_2^2)^{-1}\Gamma_t^\top \\ (\bar{B}_t^2)^\top & \bar{A}_t^\top - \bar{G}_t^\top \end{pmatrix}, \quad B_t := \begin{pmatrix} \bar{B}_t^2 & -\bar{B}_t^2(\bar{R}_2^2)^{-1}(\bar{B}_t^2)^\top \\ \bar{D}_t^2 & (\bar{B}_t^2)^\top \end{pmatrix},
\]

\[
\hat{A}_t := \begin{pmatrix} \Gamma_t(\bar{R}_2^2)^{-1}\bar{S}_2^2 & \Gamma_t(\bar{R}_2^2)^{-1}\Gamma_t^\top \\ 0 & (\bar{S}_2^2)^\top (\bar{R}_2^2)^{-1}\Gamma_t^\top \end{pmatrix}, \quad \hat{B}_t := \begin{pmatrix} \Gamma_t - \hat{A}_t^\top & \Gamma_t(\bar{R}_2^2)^{-1}(\bar{B}_t^2)^\top \\ -\bar{D}_t^2 & -\hat{A}_t^\top + (\bar{S}_2^2)^\top (\bar{R}_2^2)^{-1}(\bar{B}_t^2)^\top \end{pmatrix},
\]

and denote

\[
A_t := \begin{pmatrix} A_t & B_t \\ \hat{A}_t & \hat{B}_t \end{pmatrix},
\]

thus (2.27) can be written as

\[
\begin{cases}
\frac{d}{dt} \begin{pmatrix} X_t \\ Y_t \end{pmatrix} = A_t \begin{pmatrix} X_t \\ Y_t \end{pmatrix} dt, \quad t \in [0, T]; \\
X_0 = (x^\top \ 0)^\top, \ Y_T = (0 \ 0)^\top.
\end{cases}
\]

From the theory by Yong [15], we know that (2.28) admits a unique solution \((X, Y) \in L^2(0, T; \mathbb{R}^{2n}) \times L^2(0, T; \mathbb{R}^{2n})\) if and only if

\[
\det \begin{pmatrix} (0 \ I) e^{A_t} & (0 \\ I) \end{pmatrix} > 0, \quad \forall t \in [0, T].
\]
In this case, (2.27) admits a unique solution \((\mathbb{E}x^*, \mathbb{E}p, \varphi^*, \phi) \in L^2(0, T; \mathbb{R}^n) \times L^2(0, T; \mathbb{R}^n) \times L^2(0, T; \mathbb{R}^n) \times L^2(0, T; \mathbb{R}^n)\). Some recent progress for the two-point boundary value problems associated with ODEs, refer to Liu and Wu [3].

We summarize the above process in the following theorem.

**Theorem 2.3** Let (A.2.1)–(A.2.5) and (2.22) hold, \((P^1, P^2)\) satisfy (2.22), and \((\mathbb{E}x^*, \mathbb{E}p, \varphi^*, \phi)\) satisfy (2.27). Then \(w^*\) given by (2.20) is the state feedback representation of the unique optimal control of the leader. Let \(x^*\) satisfy

\[
\begin{align*}
\frac{dx_t^*}{dt} = & \left\{ \tilde{A}_t x_t^* - \tilde{B}_t^2(\tilde{R}_t^2)^{-1}\mathbb{S}_t^2\mathbb{E}x_t^* - \tilde{B}_t^2(\tilde{R}_t^2)^{-1}\Gamma_t^\top \mathbb{E}p_t \\
& + \left[ \tilde{B}_t^1 - \tilde{B}_t^2(\tilde{R}_t^2)^{-1}(\tilde{D}_t^2)^\top (P_t^1 + P_t^2)\tilde{D}_t^1 \right] \varphi_t^* - \tilde{B}_t^2(\tilde{R}_t^2)^{-1}(\tilde{B}_t^2)^\top \phi_t \right\} dt \\
& + \left\{ \tilde{C}_t x_t^* - \tilde{D}_t^2(\tilde{R}_t^2)^{-1}\mathbb{S}_t^2\mathbb{E}x_t^* - \tilde{D}_t^2(\tilde{R}_t^2)^{-1}\Gamma_t^\top \mathbb{E}p_t \\
& + \left[ \tilde{D}_t^1 - \tilde{D}_t^2(\tilde{R}_t^2)^{-1}(\tilde{D}_t^2)^\top (P_t^1 + P_t^2)\tilde{D}_t^1 \right] \varphi_t^* - \tilde{D}_t^2(\tilde{R}_t^2)^{-1}(\tilde{B}_t^2)^\top \phi_t \right\} dW_t, \\
x_0^* = x,
\end{align*}
\]

and define \(y^*\) and \(z^*\) in (2.16) and (2.21), respectively, then \((x^*, y, z, p, \varphi)\) is the solution to the system of MF-FBSDEs (2.15).

Finally, from (2.7) and (2.20), we obtain

\[
\begin{align*}
u_t^* = & \left[ (\tilde{R}_t^1)^\top P_t + (\tilde{D}_t^1)^\top P_t C_t + S_t \right] x_t^* \\
& + (\tilde{R}_t^1)^\top P_t D_t^2(\tilde{R}_t^2)^{-1}(\tilde{D}_t^2)^\top (P_t^1 + P_t^2)\tilde{C}_t + S_t^2 \mathbb{E}x_t^* \\
& + (\tilde{R}_t^1)^\top P_t D_t^2(\tilde{R}_t^2)^{-1}(\tilde{D}_t^2)^\top (P_t^1 + P_t^2)\tilde{D}_t^1 - (\tilde{B}_t^1)^\top \varphi_t^* \\
& + (\tilde{R}_t^1)^\top P_t D_t^2(\tilde{R}_t^2)^{-1}\Gamma_t^\top \mathbb{E}p_t + (\tilde{R}_t^1)^\top (\tilde{D}_t^1)^\top P_t D_t^2(\tilde{R}_t^2)^{-1}(\tilde{B}_t^2)^\top \phi_t, \ t \in [0, T],
\end{align*}
\]

where \(x^*\) is given by the MF-SDE (2.30). Up to now, we obtain the state feedback representation for the open-loop Stackelberg equilibrium solution \((u^*, w^*)\).
3 Concluding Remarks

To conclude this paper, let us give some remarks. In this paper, we have considered a new kind of LQ Stackelberg differential game with mixed deterministic and stochastic controls. The open-loop Stackelberg equilibrium solution is represented as a feedback form of state variable and its expectation, via solutions to some new Riccati equations. Though the framework is a special case of Yong [16], some new ideas and interesting phenomena come out. We point out that it is possible for us to relax the assumptions in Section 2 of this paper. Possible extension of the results to those in an infinite time horizon with constant coefficients, is an interesting topic. In this case, some stabilizability problems need to be investigated first, and differential Riccati equations will become algebraic Riccati equations. The practical applications of the theoretic results to Stackelberg’s type financial market is another challenging problem. We will consider these problems in the near future.

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