Standard transmutation operators for the one dimensional Schrödinger operator with a locally integrable potential

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Abstract

We study a special class of operators $T$ satisfying the transmutation relation

$$(\frac{d^2}{dx^2} - q)Tu = T\frac{d^2}{dx^2}u$$

in the sense of distributions, where $q$ is a locally integrable function, and $u$ belongs to a suitable space of distributions depending on the smoothness properties of $q$.

A method which allows one to construct a fundamental set of transmutation operators of this class in terms of a single particular transmutation operator is presented. Moreover, following [27], we show that a particular transmutation operator can be represented as a Volterra integral operator of the second kind.

We study the boundedness and invertibility properties of the transmutation operators, and use these to obtain a representation for the general distributional solution of the equation $\frac{d^2}{dx^2} - qu = \lambda u$, $\lambda \in \mathbb{C}$, in terms of the general solution of the same equation with $\lambda = 0$.

Keywords: Transmutation, Transformation operator, Schrödinger operator, Goursat problem, Weak solution, Spectral parameter power series.

1 Introduction

If $L$ and $M$ are differential operators, then an operator $T$ satisfying the relation $LT = TM$, in a suitable functional space, is called a transmutation operator. Transmutation operators are a useful tool in the study of differential operators with variable coefficients, see [1], [9], [25], [26], and [27] for their classical theory.
In the present paper we deal mainly with transmutations corresponding to the pair
\[ L = \frac{d^2}{dx^2} - q(x) \]  
and \[ M = \frac{d^2}{dx^2}. \]  It is well known that if \( q \in C[-a,a], a > 0, \) then a transmutation operator for such pair on \( C^2[-a,a] \) can be represented in the form
\[
Tu(x) = u(x) + \int_{-x}^{x} K(x,t)u(t)dt,  \tag{1}
\]
where \( K(x,t) \) is the unique solution of a certain Goursat problem \[27].

Recent work \[8\] shows an interesting connection between the transmutation operators, and the concept of \( L\)-base introduced in \[11\]. There, a parametrized family of operators \( \{T_c\} \) was introduced possessing the following properties: i) for each \( c \in \mathbb{C} \), \( T_c \) is a Volterra integral operator and \( T_0 = T \); ii) if \( q \in C^1[-a,a] \), then \( T_c \) is a transmutation operator on \( C^2[-a,a] \) for suitable values of \( c \); iii) the family of functions \( \{\varphi_k\} \), where \( \varphi_k = T_c[x^k] \), \( k = 0, 1, 2, ... \), is an \( L\)-base.

It is worth mentioning that the \( L\)-base \( \{\varphi_k\} \) is the main ingredient to obtain the so-called spectral parameter power series (SPPS) representation, a useful representation of the general solution of the Sturm–Liouville equation \[17\], \[19\], and it also arises in Bers’s theory of pseudoanalytic functions \[3\], \[18\] in the study of a special kind of Vekua equations. These connections have led to new applications of the transmutation operators to the study of different problems arising in mathematical physics \[7\], \[8\], \[13\], \[14\], \[20\], \[24\].

Regarding these applications, it is desirable to have the transmutation property of \( T_c \) under the most general scenario, which, to the best of our knowledge, is valid for a continuous potential \( q \) and for all parameters \( c \in \mathbb{C} \) (see Remark \[24\] for further details).

The first goal of this paper is the study of transmutations for the Schrödinger operator with integrable coefficients. Our approach is not based on merely extending existing results from continuous coefficients, but instead we make our study in the spirit of standard \( L\)-bases (see Definition \[7\]). We will consider the class of transmutation operators, here named standard, which have the property of mapping the nonnegative integer powers of the independent variable into an \( s\)-\( L\)-base. This class includes the transmutation operators having the form of a Volterra integral of the second kind, and others.

This paper is organized as follows. In Section 2 we fix some notation and we give some auxiliary results needed throughout the paper.

In Section 3, based on the properties of the \( s\)-\( L\)-bases, we arrive at several results concerning the \( s\)-transmutation operators, and we establish the first main result of the paper, Theorem \[10\] that if a bounded operator \( T \) on \( L^1[a,b] \) maps the powers of the independent variable into an \( s\)-\( L\)-base, then \( T \) is a transmutation operator on \( W^{2,1}[a,b] \). This result represents another proof for the transmutation property of the operator \( T_c \) and also suggests that some properties or relations on the \( s\)-transmutations might be studied first on smaller spaces, like the linear space of polynomials \( \mathcal{P}(\mathbb{R}) \), and then extended to suitable Banach spaces.

In Section 4 we present a fairly simple method for constructing new transmutation operators on \( \mathcal{P}(\mathbb{R}) \) when one single \( s\)-transmutation is known (Theorem \[17\]) and we also provide explicit formulas relating any two \( s\)-transmutations on \( \mathcal{P}(\mathbb{R}) \), allowing the construction of one of them in terms of the other (Proposition \[18\]).
From the viewpoint of applications, it is desirable to have the transmutation property \( \textup{(5)} \) on larger spaces than \( P(\mathbb{R}) \). Section 5 is devoted to the study of s-transmutations on Sobolev spaces. Extending the results of [27], we show that a particular s-transmutation operator on \( W^{2,1}[-a, a] \) can be represented in the form \( \textup{(1)} \), where \( K(x,t) \) is a weak solution of a Goursat problem (Theorem 21). Combining this result with those of Section 4, we arrive at the main result of the paper, Theorem 22 an explicit representation of the general s-transmutation operator on Sobolev space. As an application, we get the SPPS representation for the general solution of the equation \( \frac{d^2u}{dx^2} - qu = \lambda u \) in a slightly more general setting than was known previously (Corollary 23).

Even in the case of continuous coefficients, the transmutation relation \( \textup{(5)} \) has been studied mainly in the space of twice differentiable functions, and assuming the existence of all the involved derivatives in the classical sense. However, as always, for the best understanding of differential operators, the framework of distributions is the most suitable.

The second goal of the paper is to study the s-transmutations on spaces of distributions, which is done in Section 6. Assuming that the potential of the Schrödinger operator is locally integrable, we show that each s-transmutation operator admits a continuous extension as a linear operator and as a transmutation operator to a suitable space of distributions, depending on the smoothness of the potential \( q \). Also, this procedure leads to the construction of all the distributional s-transmutation operators.

Lastly, in Section 7 we provide some conclusions and generalizations.

## 2 Preliminaries

First, following [2], [3], and [28], let us introduce the spaces we are going work with. Let \((a,b) \subset \mathbb{R}\) be a bounded open interval and denote by \( F(a,b) \) any one of the following classical spaces: \( L^p(a,b) \), \( C^k(a,b) \), or \( C^\infty(a,b) \), for \( k \in \mathbb{N}_0 = \{0,1,2,\ldots\} \) and \( 1 \leq p \leq \infty \). Note that \( L^p(a,b) \) and \( C^k(K) \) are Banach spaces with the corresponding natural norms whenever \( K \subset (a,b) \) is compact, and \( C^\infty(K) \) is a Fréchet space when equipped with the topology of uniform convergence on \( K \) in each derivative \( \textup{[2]} \).

Let us denote by \( F_{loc}(a,b) \) the subspace of \( F(a,b) \) formed by those functions \( \psi \in F(K) \), for each compact set \( K \subset (a,b) \). By \( F_c(a,b) \) we denote the set of functions \( \psi \in F(a,b) \) such that \( \textup{supp} \psi \subset (a,b) \), here \( \textup{supp} \psi \) is defined as the complement of the largest open set on which \( \psi \) vanishes a.e. We endow \( F_c(a,b) \) with the inductive limit topology, turning \( F_c(a,b) \) into a locally convex topological vector space \( \textup{[2]}, \textup{[28]} \). A sequence of functions \( \psi_n \in F_c(a,b) \) converges to \( \psi \in F_c(a,b) \) with respect to this topology if there exists a compact set \( K \subset (a,b) \) such that \( \textup{supp} \psi_n \subset K \) for all \( n \) and \( \psi_n \to \psi \) in \( F(K) \) as \( n \to +\infty \). Let \( (F_c(a,b))^\prime \) be the dual space of \( F_c(a,b) \), consisting of all continuous functionals \( u : F_c(a,b) \to \mathbb{C} \). The value of a functional \( u \) on \( \psi \in F_c(a,b) \) is denoted by \( \langle u, \psi \rangle \) and continuity means that \( \langle u, \psi_n \rangle \to \langle u, \psi \rangle \) provided that \( \psi_n \to \psi \) in \( F_c(a,b) \) as \( n \to \infty \). Convergence in \( (F_c(a,b))^\prime \) is defined to be pointwise convergence. Obviously, for a smaller set of \( F_c(a,b) \), the set \( (F_c(a,b))^\prime \) will be bigger.

The space \( C^\infty_c(a,b) \), called the space of test functions, plays an important role in the
theory of distributions. As defined above, \( \phi_n \to \phi \) in \( C_c^\infty(a,b) \) if there exists a compact set \( K \subset (a,b) \) such that \( \text{supp } \phi_n \subset K \) for all \( n \) and \( \frac{d^n \phi_n}{dx^n} \to \frac{d^n \phi}{dx^n} \) uniformly on \( K \) as \( n \to \infty \) for each \( m \in \mathbb{N}_0 \). A distribution is an element from the dual of \( C^\infty_c(a,b) \), which from now on we denote by \( \mathcal{D}'(a,b) \), i.e., \( \mathcal{D}'(a,b) \equiv (C^\infty_c(a,b))' \). A distribution \( u \) is called regular if there exists an \( f \in L^1_{\text{loc}}(a,b) \) such that \( \langle u, \phi \rangle = \int_a^b f(x) \phi(x) dx \) for all \( \phi \in C^\infty_c(a,b) \). We usually identify a regular distribution \( u \) with the corresponding function \( f \), \( u \equiv f \).

**Definition 1** Let \( T \) and \( T^\square \) be linear operators from \( C^\infty_c(a,b) \) to \( L^1_{\text{loc}}(a,b) \) such that \( \int_a^b (T \psi) \phi dx = \int_a^b \psi(T^\square \phi) dx \) for all \( \psi, \phi \in C^\infty_c(a,b) \). The operator \( T^\square \), if it exists, is called the transpose of \( T \).

The most common way to extend linear operations from functions to distributions is the following.

**Proposition 2** Let \( T \) be a linear operator defined on \( C^\infty_c(a,b) \). If \( T^\square : C^\infty_c(a,b) \to C^\infty_c(a,b) \) is continuous, then the operator \( T \) can be extended to a continuous linear operator from \( \mathcal{D}'(a,b) \) to itself by the formula

\[
\langle Tu, \phi \rangle := \langle u, T^\square \phi \rangle , \quad u \in \mathcal{D}'(a,b), \phi \in C^\infty_c(a,b).
\]

**Proof.** This is a special case of [12, Th. 3.8]. ■

As \( \frac{d^2}{dx^2} = -\frac{d}{dx} \) is continuous on \( C^\infty_c(a,b) \), the derivative of \( u \in \mathcal{D}'(a,b) \) is defined by employing the previous proposition: \( \langle u', \phi \rangle := -\langle u, \phi' \rangle \). It is again an element of \( \mathcal{D}'(a,b) \).

The Sobolev space \( W^{k,p}(a,b) \), \( k \in \mathbb{N}, 1 \leq p \leq \infty \) consists of all functions \( f \in L^p(a,b) \) having distributional derivatives \( f^{(n)} \in L^p(a,b) \) up to order \( n \leq k \).

Denote the set of complex-valued absolutely continuous functions on \( [a,b] \) by \( AC[a,b] \). We recall that a function \( V \) belongs to \( AC[a,b] \) iff there exists \( v \in L^1(a,b) \) and \( x_0 \in [a,b] \) such that \( V(x) = \int_{x_0}^x v(t) dt + V(x_0), \forall x \in [a,b] \) (see, e.g., [2], [16]). Thus, if \( V \in AC[a,b] \), then \( V \in C[a,b] \) and \( \text{V is a.e. differentiable with } \text{V}'(x) = v(x) \text{ on } (a,b) \). Moreover, the usual derivative of a function from \( AC[a,b] \) coincides a.e. with its distributional derivative [16], thus \( AC[a,b] \subseteq W^{1,1}(a,b) \). In fact, the linear spaces \( W^{1,1}(a,b) \) and \( AC[a,b] \) coincide (see [9], Theorem 8.2) in the following sense: \( u \in W^{1,1}(a,b) \) iff there exists \( V \in AC[a,b] \) such that \( u = V \text{ a.e. on } (a,b) \).

Consider the differential equation

\[
u'' + qu = h,
\]

in the sense of distributions, where \( q, h \in L^1(a,b) \) and the distribution \( qu \) is defined by \( \langle qu, \varphi \rangle := \langle u, q \varphi \rangle \). Note that if \( q \notin C^\infty_c(a,b) \), then it might happen that \( u \in \mathcal{D}'(a,b) \) but \( qu \notin \mathcal{D}'(a,b) \). To overcome this situation, we first will look for solutions of (2) in \( W^{1,1}(a,b) \), which are called weak solutions.

**Proposition 3** Let \( q \) and \( h \) be functions in \( L^1(a,b) \). Then a function \( u \in W^{1,1}(a,b) \) is a weak solution of (2) iff \( u \in W^{2,1}(a,b) \) and (2) is satisfied a.e. on \( (a,b) \).
The last result reveals the density of polynomials in $W^{2,1}(a,b)$.

**Proposition 4** Let $u \in W^{2,1}(a,b)$. Then there exists a sequence of polynomials $P_n$ such that $P_n \to u$, $P'_n \to u'$ uniformly on $[a,b]$ and $P''_n \to u''$ in $L^1(a,b)$, as $n \to \infty$.

**Proof.** Let $u \in W^{2,1}(a,b)$. Then $u'' \in L^1(a,b)$, and because polynomials are dense in $L^1(a,b)$ there exists a sequence of polynomials $Q_n$ such that $Q_n \to u''$ in $L^1(a,b)$, as $n \to \infty$.

Integrating (3) twice we see that the sequence of polynomials $P_n(x) := u(0) + u'(0)x + \int_0^x \int_0^t Q_n(s) ds dt$ fulfills the conclusion of the proposition.

### 3 s-transmutations and $L$-bases. First results

Consider the one-dimensional Schrödinger operator

$$L = \frac{d^2}{dx^2} - q(x),$$

with domain of definition $D_L = W^{2,1}(a,b)$, where $q \in L^1(a,b)$.

**Definition 5** A linear map $T$ is called a transmutation operator on $\chi \subseteq D_L$ if for every $u \in \chi$ we have $u'' \in D_T$, $Tu \in D_L$ and the following equality holds

$$\left( \frac{d^2}{dx^2} - q \right) Tu = T \frac{d^2}{dx^2} u.$$  

Let $\mathcal{P}(\mathbb{R})$ denote the linear space of polynomials. We will be mainly interested in operators satisfying the transmutation property (5) on sets $\chi \supseteq \mathcal{P}(\mathbb{R})$. Let $T$ be a transmutation operator and define $\varphi_k := T [x^k]$, $k \in \mathbb{N}_0$. From the transmutation property (5), we obtain

$$\left( \frac{d^2}{dx^2} - q \right) \varphi_k = \left( \frac{d^2}{dx^2} - q \right) T x^k = T \frac{d^2}{dx^2} x^k.$$  

Thus,

$$\left( \frac{d^2}{dx^2} - q \right) \varphi_k = 0, \quad k = 0, 1;$$  

and

$$\left( \frac{d^2}{dx^2} - q \right) \varphi_k = k(k-1)\varphi_{k-2}, \quad k \geq 2.$$  

As in [11], we define the following:
Definition 6 A system of functions \( \{ \varphi_k \} \), \( k \in \mathbb{N}_0 \), satisfying (6) and (7) in the sense of weak solutions, is called an L-base.

We have shown that if \( T \) is a transmutation operator on \( \mathcal{P}(\mathbb{R}) \), then \( \varphi_k := T [x^k] \) is an L-base. Moreover, given an L-base \( \{ \varphi_k \} \), there exists a transmutation operator \( T_\varphi \) on \( \mathcal{P}(\mathbb{R}) \) such that \( \varphi_k = T_\varphi [x^k] \) for all \( k \in \mathbb{N}_0 \). To see this, define \( T_\varphi \) on powers to be \( T_\varphi [x^k] := \varphi_k \) and extend it to \( \mathcal{P}(\mathbb{R}) \) by linearity. It follows from Proposition 3 that each member of an L-base is a function from the space \( W^{2,1}(a,b) \), hence \( T_\varphi \) maps \( \mathcal{P}(\mathbb{R}) \) into \( W^{2,1}(a,b) \). Also, if \( P = \sum_{k=0}^{m} \alpha_k x^k \), from (6) and (7) we get

\[
\left( \frac{d^2}{dx^2} - q \right) T_\varphi P = \sum_{k=0}^{m} \alpha_k \left( \frac{d^2}{dx^2} - q \right) \varphi_k = \sum_{k=2}^{m} \alpha_k k(k-1) \varphi_{k-2} = T_\varphi P''.
\]

This proves the existence of a transmutation operator on \( \mathcal{P}(\mathbb{R}) \) satisfying \( T_\varphi [x^k] = \varphi_k \).

However, regarding possible applications, it is important to establish the existence of transmutations on larger domains than \( \mathcal{P}(\mathbb{R}) \), for example on \( W^{2,1}(a,b) \), and investigate when they are bounded or invertible. Looking for the existence of such operators, we will work with a special class of L-bases, which we introduce below. We will assume from now on, without loss of generality, that \( 0 \in [a,b] \).

Definition 7 A standard L-base (or, simply, an s-L-base) is an L-base \( \{ \varphi_k \} \) such that \( \varphi_k(0) = \varphi_k'(0) = 0 \) for all \( k \geq 2 \). A transmutation operator is called standard if \( \varphi := T [x^k] \) is an s-L-base.

Example 8 The system of powers \( \{ x^k \} \), \( k \in \mathbb{N}_0 \), is an s-L-base corresponding to the operator \( L = \frac{d^2}{dx^2} \). The shift operator \( Eu(x) = u(x-x_0) \), \( x_0 \neq 0 \) is a transmutation operator (on \( W^{2,1}(\mathbb{R}) \)) corresponding to \( L = \frac{d^2}{dx^2} \) but it is not an s-transmutation operator.

Remark 9 We stress that an s-L-base \( \{ \varphi_k \} \) is completely determined by its first two elements \( \varphi_0 \) and \( \varphi_1 \). For instance, from Proposition 3 and by the variation of parameters formula, we see that the unique weak solution of (7) satisfying \( \varphi_k(0) = \varphi'_k(0) = 0 \) is given by

\[
\varphi_k(x) = k(k-1) \int_0^x G(x,s) \varphi_{k-2}(s) ds, \quad k \geq 2,
\]

where

\[
G(x,s) = \frac{\psi_0(s)\psi_1(x) - \psi_0(x)\psi_1(s)}{W(\psi_0, \psi_1)},
\]

\( W(\psi_0, \psi_1) = \psi_0(x)\psi_1'(x) - \psi_0'(x)\psi_1(x) \), and \( \psi_0, \psi_1 \) are any two linearly independent weak solutions of (6). Note that \( G(x,s) \) does not depend on the choice of \( \psi_0 \) and \( \psi_1 \).

Theorem 10 Suppose \( \{ \varphi_k \} \) is an s-L-base and suppose \( T : L^1(a,b) \to L^1(a,b) \) is a bounded linear operator such that \( T[x^k] = \varphi_k, k \in \mathbb{N}_0 \). Then \( T \) is an s-transmutation operator on \( W^{2,1}(a,b) \).
Proof. Let us prove that

\[ T[u] = u(0)\varphi_0(x) + u'(0)\varphi_1(x) + \int_0^x G(x, s) T[u''](s) ds \] (10)

for any \( u \in W^{2,1}(a, b) \), where \( G(x, s) \) is given by (9). Then (5) will follow from the application of \( L = \frac{d^2}{dx^2} + q(x) \) to both sides of (11).

As \( T \) is linear and \( T[x^k] = \varphi_k \) is an \( s \)-\( L \)-base, the application of \( T \) to \( P = \sum_{k=0}^{m} \alpha_k x^k \) gives

\[ T[P] = \sum_{k=0}^{m} \alpha_k T[x^k] = \sum_{k=0}^{m} \alpha_k \varphi_k(x) \]

\[ = \alpha_0 \varphi_0(x) + \alpha_1 \varphi_1(x) + \int_0^x G(x, s) \sum_{k=2}^{m} k(k-1) \alpha_k \varphi_{k-2}(s) ds \]

\[ = P(0)\varphi_0(x) + P'(0)\varphi_1(x) + \int_0^x G(x, s) T[u''](s) ds, \]

where we have used (8). According to Proposition 4, given \( u \in W^{1,2}(a, b) \) there exists a sequence of polynomials \( P_n \) such that \( P_n \to u, \ P'_n \to u' \) uniformly on \([a, b]\) and \( P''_n \to u'' \) in \( L_1(a, b) \), as \( n \to \infty \). Then (11) follows from a simple limiting procedure using the continuity of \( T \) and the above relations,

\[ T[u] = \lim_{n \to \infty} T[P_n] = \lim_{n \to \infty} \left[ P_n(0)\varphi_0(x) + P'_n(0)\varphi_0(x) + \int_0^x G(x, s) T[u''](s) ds \right] \]

\[ = u(0)\varphi_0(x) + u'(0)\varphi_1(x) + \int_0^x G(x, s) T[u''](s) ds. \]

Remark 11 It is easy to see that the above theorem is still true if we replace the boundedness of \( T \) on \( L_1(a, b) \) by the boundedness from \( W^{1,2}(a, b) \) to \( L_1(a, b) \). However, in this case the transmutation property (5) holds on \( W^{3,1}(a, b) \), instead of on \( W^{2,1}(a, b) \).

The following proposition establishes conditions under which the limit of a sequence of \( s \)-transmutation operators is an \( s \)-transmutation operator as well.

Proposition 12 Let \( q_n \) be a convergent sequence in \( L^1(a, b) \) with limit \( q \). Let \( T \) be a bounded operator on \( L^1(a, b) \) and \( T_n \) be a sequence of \( s \)-transmutation operators corresponding to \( L_n = \frac{d^2}{dx^2} + q_n \). If for each \( k = 0, 1, 2, \ldots \)

\[ T_n[x^k] \to T[x^k] \text{ uniformly on } [a, b], \text{ as } n \to \infty, \]

then \( T \) is an \( s \)-transmutation operator on \( W^{2,1}(a, b) \) corresponding to \( L = \frac{d^2}{dx^2} + q \).
Proof. Write \( \varphi_{k,n}(x) = T_n \left[ x^k \right] \) and \( \varphi_k(x) = T \left[ x^k \right] \). To prove the theorem, it suffices to show that \( \{ \varphi_k \} \) is an s-L-base. Indeed, using the boundedness of \( T \) together with the mapping property \( T \left[ x^k \right] = \varphi_k(x) \), we conclude from Theorem \( 10 \) that \( T \) is an s-transmutation operator on \( W^{2,1}(a,b) \).

Let us start by proving that \( L [ \varphi_0 ] = \varphi''_0(x) - q(x)\varphi_0(x) = 0 \). Consider the functions \( r_n(x) \) and \( r(x) \) defined by

\[
\begin{align*}
r_n(x) &= \varphi_{0,n}(x) - \int_0^x (x-t)q_n(t)\varphi_{0,n}(t)dt, \\
r(x) &= \varphi_0(x) - \int_0^x (x-t)q(t)\varphi_0(t)dt.
\end{align*}
\]

It is easy to see from (11) that \( r''(x) = 0 \) iff \( L [ \varphi_0 ] = 0 \). In this way, as \( L_n [ \varphi_{0,n} ] = \varphi''_{0,n}(x) - q_n(x)\varphi_{0,n}(x) = 0 \) from (11), we get \( r''_n(x) = 0 \). Thus \( r_n(x) = c_1 + c_2x \), and taking into account the boundary values at \( x = 0 \), we see that

\[
r_n(x) = \varphi_{0,n}(0) + \varphi'_0(0)x.
\]

Since, by assumption, \( \varphi_{0,n}(x) = T_n[1] \to T[1] = \varphi_0(x) \) uniformly on \([a,b]\) as \( n \to \infty \), we can see from (11) and (12) that \( r_n(x) \to r(x) \) uniformly on \([a,b]\). From this and (13) we must have that \( \varphi'_{0,n}(0) \to \varphi'_0(0) \) as \( n \to \infty \), and \( r(x) = \varphi_0(0) + \varphi'_0(0)x \). Thus \( r''(x) = 0 \) and from (12) we conclude that \( L [ \varphi_0 ] = 0 \). In the same manner we can see that \( L [ \varphi_1 ] = 0 \).

Let \( G(x,s) \) be the function in (9) where \( \psi_0 \) and \( \psi_1 \) are linearly independent solutions of \( L [ \psi ] = 0 \) and denote by \( G_n(x,s) \) the analogous function corresponding to the operator \( L_n \).

From the fact that \( q_n \to q \) in \( L^1(a,b) \) it is easy to see that \( G_n(x,s) \to G(x,s) \) uniformly on \([a,b] \times [a,b]\) as \( n \to \infty \). Hence, as \( \varphi_{k,n}(x) \to \varphi_k(x) \) uniformly, as \( n \to \infty \), for \( k \geq 2 \) we finally obtain

\[
\varphi_k(x) = \lim_{n \to \infty} \varphi_{k,n}(x) = k(k-1) \lim_{n \to \infty} \int_0^x G_n(x,s)\varphi_{k-2,n}(s)ds
\]

\[
= k(k-1) \int_0^x G(x,s)\varphi_{k-2}(s)ds.
\]

This shows that \( \{ \varphi_k \} \) is an s-L-base. The theorem is proved. \( \blacksquare \)

4 New s-transmutations

In this section a method for constructing new s-transmutation operators on \( P(\mathbb{R}) \) in terms of one single s-transmutation operator is presented.

Let us introduce the operators

\[
P_{\pm}u(x) = \frac{u(x) \pm u(-x)}{2}, \quad Au(x) = \int_0^x u(s)ds,
\]

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mapping $\mathcal{P}(\mathbb{R})$ into itself. Note that the $P_\pm$ are pairwise projection operators, that is, $P_+^2 = P_+$, $P_-^2 = P_-$, $P_+ + P_- = I$, and $P_+ P_- = P_- P_+ = 0$.

**Proposition 13** The relations $\frac{d}{dx} P_\pm = P_\pm \frac{d}{dx}$ and $A P_\pm = P_\pm A$ hold on $\mathcal{P}(\mathbb{R})$.

**Proof.** Straightforward. ■

**Proposition 14** Let $T$ be an $s$-transmutation operator on $\mathcal{P}(\mathbb{R})$. Then

i) $T[1] = 0$ if and only if $T = TP_-$ on $\mathcal{P}(\mathbb{R})$;

ii) $T[x] = 0$ if and only if $T = TP_+$ on $\mathcal{P}(\mathbb{R})$;

iii) $T[1] = 0$ and $T[x] = 0$ if and only if $T \equiv 0$ on $\mathcal{P}(\mathbb{R})$.

**Proof.** We prove the direct assertions corresponding to i), ii) and iii). The reciprocal assertions are obvious. First, denote by $\varphi_k = T[x^k]$ the $s$-$L$-base corresponding to $T$. Let us prove i). Since $\varphi_0 = T[1] = 0$, it follows from Remark 9 that $\varphi_k = T[x^k] = 0$ for $k \geq 0$ even. Thus it is easy to see that $T[x^k] = TP_[-x^k]$ for all $k \geq 0$. From this and from the linearity of $T$ and $TP_-$ we conclude that $T[p(x)] = TP_-[p(x)]$ for every polynomial $p(x)$. Part ii) is proved by analogy. Part iii) is a consequence of i) and iii). For instance, since $T = TI = T(P_+ + P_-) = TP_+ + TP_- = 2T$, we have $T = 0$. ■

**Corollary 15** Let $T$ and $M$ be $s$-transmutation operators on $\mathcal{P}(\mathbb{R})$ such that $T[1] = M[1]$ and $T[x] = M[x]$. Then $T = M$ on $\mathcal{P}(\mathbb{R})$.

**Proof.** This follows at once from part iii) of the above proposition. Setting $S = T - M$, we have that $S$ is an $s$-transmutation operator and $S[1] = S[x] = 0$. Thus, $S \equiv 0$ on $\mathcal{P}(\mathbb{R})$. ■

**Proposition 16** Let $T$ be an $s$-transmutation operator on $\mathcal{P}(\mathbb{R})$. Then:

i) $T \frac{d}{dx}$ is an $s$-transmutation operator on $\mathcal{P}(\mathbb{R})$ if and only if $T[x] = 0$, and in this case $T \frac{d}{dx} P_-$ on $\mathcal{P}(\mathbb{R})$;

ii) $TA$ is an $s$-transmutation operator on $\mathcal{P}(\mathbb{R})$ if and only if $T[1] = 0$, and in this case $TA = TAP_+$ on $\mathcal{P}(\mathbb{R})$.

**Proof.** Let $T$ be an $s$-transmutation operator and denote by $\varphi_k = T[x^k]$ the corresponding $s$-$L$-base. Let us prove i). The transmutation property of $T \frac{d}{dx}$ is a straightforward calculation. Analyzing the corresponding $L$-base $\psi_k = T \frac{d}{dx} [x^k] = kT [x^{k-1}] = k \varphi_{k-1}$, it is easy to conclude that $\psi_k$ is standard iff $\varphi_1 = T[x] = 0$. Thus, from Proposition 14 $T = TP_+$ and therefore $T \frac{d}{dx} = TP_+ \frac{d}{dx} = T \frac{d}{dx} P_-$.

(ii) From the equalities $(\frac{d^2}{dx^2} - q)Ta = T \frac{d^2}{dx^2}Au = T \frac{d}{dx}u$ and $TA \frac{d^2}{dx^2}u(x) = T [u'(x) - u'(0)] = T \frac{d}{dx}u - u'(0)T[1]$ we see that $TA$ is a transmutation operator on $\mathcal{P}(\mathbb{R})$ whenever $T[1] = 0$. In this case, Proposition 14 tell us that $T = TP_+$, from which follows the equality $TA = TPA = TAP_-$.
Theorem 17 Let $T$ be an $s$-transmutation operator on $\mathcal{P}(\mathbb{R})$. Then the operators $TP_+$, $TP_-$, $TAP_+$ and $T+\frac{d}{dx}P_-$ are $s$-mutations on $\mathcal{P}(\mathbb{R})$ as well.

Proof. Using the relation $\frac{d^2}{dx^2}P_+ = P_+\frac{d^2}{dx^2}$ (Proposition 13) it is easy to see that $TP_+$ is a transmutation operator. As the corresponding $L$-base $\psi_k = TP_+ x^k$ is such that $\psi_k = T[x^k] = \varphi_k$ for $k$ even, and $\psi_k = 0$ for $k$ odd, we easily see that $\psi_k(0) = \psi_k(0) = 0$, $k \geq 2$. Thus $\psi_k$ is an s-L-base and $TP_+$ is an s-transmutation operator.

Let us now consider the operator $T\frac{d}{dx}P_-$. As $TP_+$ is an s-transmutation and $TP_+ x = 0$, then, in accordance with Proposition 18 part i), the operator $TP_+ \frac{d}{dx} = T\frac{d}{dx}P_-$ is an s-transmutation as well.

By analogous means we can prove that $TP_-$ and $TAP_+$ are s-transmutation operators.

\section*{5 s-Transmutation operators on Sobolev spaces}

\subsection*{5.1 Existence of s-transmutation operators on $W^{2,1}(-a, a)$}

Let $q \in C[-a, a]$, $a > 0$. Then, according to [27], a transmutation operator on $C^2[-a, a]$ corresponding to $L = \frac{d^2}{dx^2} - q(x)$ can be represented in the form

$$Tu(x) = u(x) + \int_{-x}^{x} K(x, t)u(t)dt$$
where $K(x,t)$ is constructed as follows. Consider the transformation $2u = x + t$, $2v = x - t$, which maps the square $\Omega : -a \leq x, y \leq a$ onto the square $\Omega_1$ in the $(u,v)$-plane with vertices $(-a,0), (0,a), (a,0)$ and $(0,-a)$. Then, $K(x,t) = H(\frac{x+t}{2}, \frac{x-t}{2}) = H(u,v)$ where $H(u,v)$ is the unique solution of the integral equation

$$H(u,v) = \frac{1}{2} \int_0^u q(s)ds + \int_0^u \int_0^v q(\alpha + \beta)H(\alpha, \beta)d\beta d\alpha, \text{ in } \Omega_1. \quad (16)$$

If $q$ is $n$ times differentiable, then $K(x,t)$ is $n+1$ times continuously differentiable [27]. Moreover, if $q \in C^1[-a,a]$, then $K(x,t) \in C^2(\Omega)$ is the unique classical solution of the following Goursat problem

$$\left( \frac{\partial^2}{\partial x^2} - q(x) \right)K(x,t) = \frac{\partial^2}{\partial t^2}K(x,t), \quad \text{in } \Omega. \quad (17)$$

$$K(x,x) = \frac{1}{2} \int_0^x q(s)ds, \quad K(x,-x) = 0. \quad (18)$$

We stress that if $q$ is differentiable, then the transmutation property of $T$ follows from straightforward calculations, using (17) and (18), [23]. However, for a continuous $q$, since $K(x,t) \notin C^2(\Omega_1)$, the proof of this result is not trivial. In [27], a proof involving the use of Riemann’s function was presented. Another method was mentioned in [25, pag.9] without presenting a rigorous proof: if we approximate the continuous potential $q$ by differentiable potentials $q_n$, then the transmutation property of $T$ should follow by a limit extrapolation of the transmutations $T_n$, say $T = \lim T_n$. This idea was also used in [21, proof of Th. 3.4] and [22, proof of Th. 6].

In this section, we give a rigorous justification of the procedure proposed in [25], and combining it with Proposition 12, we prove the existence of a transmutation operator on $W^{2,1}(-a,a)$, in the case when $q \in L^1(-a,a)$.

**Proposition 19** Let $q \in L^1(-a,a)$. Then the integral equation (16) has a unique solution $H(u,v)$. Moreover, $H(u,v) \in C(\Omega_1) \cap W^{1,1}(\Omega_1)$ and

$$|H(u,v)| \leq \|q\|_{L^1} e^{a\|q\|_{L^1}}. \quad (19)$$

**Proof.** The proof is analogous to the one given in [27, Th. 1.2.2] for a continuous potential $q$. For convenience, we sketch the main steps. First, using the method of successive approximations, we write a hypothetical series representation for the solution of (16), say

$$H(u,v) = \sum_{n=0}^{+\infty} H_n(u,v) \text{ where } H_0(u,v) = \frac{1}{2} \int_0^u q(s)ds \text{ and } n \geq 1$$

$$H_n(u,v) = \int_0^u \int_0^v q(\alpha + \beta)H_{n-1}(\alpha, \beta)d\beta d\alpha. \quad (20)$$

Next, we prove the uniform convergence of the above series, which in turn will imply the existence and the unicity of a continuous solution of (16). From [20], and following [27], we arrive at the following estimate

$$|H_n(u,v)| \leq \|H_0\|_{C(\Omega_1)} \frac{1}{n!} \left| \int_0^{u+v} \int_0^\beta |q(\alpha)| \, d\alpha \right| d\beta \leq \|q\|_{L^1} \frac{(a\|q\|_{L^1})^n}{n!}$$

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From this and by the Weierstrass M-test it follows that the series defining \( H \) is uniformly convergent and \( |H(u, v)| \leq \sum_{n=0}^{+\infty} |H_n(u, v)| \leq \|q\|_{L^1} e^{a|q|_{L^1}} \). Lastly, from (16) we see that \( H \in W^{1,1}(\Omega_1) \) as it is the sum of the absolutely continuous function \( \frac{1}{2} \int_0^u H(q)ds \) with a function in \( C^1(\Omega_1) \). \( \blacksquare \)

**Proposition 20** Let \( q \in L^1(-a, a) \) and let \( H(u, v) \) be the unique solution of (16). Then the mapping \( q \to H(u, v) \) is continuous from \( L^1(-a, a) \) to \( C(\Omega_1) \).

**Proof.** Denote by \( \tilde{H}(u, v) \) the solution of (16) with \( q \) replaced by \( \tilde{q} \in L^1(-a, a) \). It suffices to prove that the inequality

\[
|\tilde{H}(u, v) - H(u, v)| \leq C \|q - \tilde{q}\|_{L^1} e^{a|q|_{L^1}}
\]

holds for all \( q, \tilde{q} \in L^1(-a, a) \) with some constant \( C > 0 \) not depending on \( \tilde{q} \). In order to prove this, we first observe that \( M(u, v) := \tilde{H}(u, v) - H(u, v) \) is the unique solution of the integral equation

\[
M(u, v) = M_0(u, v) + \int_0^u \tilde{q}(\alpha + \beta)M(\alpha, \beta)d\beta d\alpha \quad \text{where} \quad M_0(u, v) = \frac{1}{2} \int_0^u (\tilde{q}(s) - q(s)) ds + \int_0^u \int_0^v H(\alpha, \beta) (\tilde{q}(\alpha + \beta) - q(\alpha + \beta)) d\beta d\alpha. \tag{21}
\]

Now, applying the same reasoning as in the proof of the above proposition, one can check that \(|M(u, v)| \leq \|M_0\|_{C(\Omega_1)} e^{a|q|_{L^1}}\). On the other hand, from (21) we get

\[
|M_0(u, v)| \leq \frac{1}{2} \|q - \tilde{q}\|_{L^1} + a \sup_{\Omega_1} |H(\alpha, \beta)| \|q - \tilde{q}\|_{L^1} = C \|q - \tilde{q}\|_{L^1}.
\]

Now, combining the last two inequalities, the desired result follows. \( \blacksquare \)

**Theorem 21** Let \( q \in L^1(-a, a) \) and let \( H(u, v) \) be the unique solution of (16). Then, \( K(x, t) = H(\frac{x+t}{2}, \frac{x-t}{2}) \) is a weak solution of the Goursat problem (17), (18) and

\[
Tu(x) = u(x) + \int_{-x}^{x} K(x, t)u(t)dt \tag{22}
\]

is an \( s \)-transmutation operator on \( W^{2,1}(-a, a) \), corresponding to \( L = \frac{d^2}{dx^2} - q(x) \). The corresponding \( L \)-base \( \alpha_k(x) := T \left[ x^k \right] \) is such that

\[
\alpha_0(0) = \alpha'_0(0) = 1 \quad \text{and} \quad \alpha'_0(0) = \alpha_1(0) = 0. \tag{23}
\]

Moreover, \( T \) is invertible and \( T^{-1}u(x) = u(x) - \int_{-x}^{x} K(t, x)u(t)dt \).
Theorem 22 Let \( q_n \in C^1[-a, a] \) be a sequence of functions that converges to \( q \) in \( L^1(-a, a) \) and let \( H_n(u, v) \) be the unique solution of the integral equation

\[
H_n(u, v) = \frac{1}{2} \int_0^u q_n(s)ds + \int_0^v \int_0^\infty q_n(\alpha + \beta)H_n(\alpha, \beta)d\beta d\alpha. \tag{24}
\]

Since \( q_n \in C^1[-a, a] \), we have that \( K_n(x, t) := H_n(\frac{x+t}{2}, \frac{x-t}{2}) \) is a classical solution of the equation \((\partial_t^2 - \partial_x^2 - q_n(x))K_n(x, t) = 0\) (see the beginning of this section). In particular, the equality

\[
\int \int [\partial_t K_n(\varphi) - (\partial_x K_n)(\partial_x \varphi - q_n K_n \varphi)] dx dt = 0, \tag{25}
\]

holds for all \( \varphi \in C_c^\infty(\Omega) \). As \( q_n \to q \) in \( L^1(-a, a) \), it follows from Proposition 20 that \( H_n \to H \) uniformly on \( \Omega_1 \). Moreover, differentiating (24) and using the uniform convergence of \( H_n \), we see that \( \partial_n H_n \to \partial_n H \) and \( \partial_t H_n \to \partial_t H \) in \( L^1(\Omega_1) \) as \( n \to \infty \). Thus \( K_n \to K \) uniformly on \( \Omega \) and \( \partial_x K_n \to \partial_x K, \partial_y K_n \to \partial_y K \) in \( L^1(\Omega) \) as \( n \to \infty \). Then, letting \( n \to +\infty \) in (25), we obtain

\[
\int \int [\partial_t K(\varphi) - (\partial_x K)(\partial_x \varphi - q K \varphi)] dx dt = 0
\]

for all \( \varphi \in C_c^\infty(\Omega) \). Thus \( K \) is a weak solution of (17). The boundary conditions (18) for \( K \) are obtained by taking the limit of the corresponding boundary conditions for \( K_n \), \( K_n(x, x) = \frac{1}{2} \int_0^x q_n(s)ds, K_n(x - x) = 0 \). Thus \( K \) is a weak solution of (17), (18).

The operator \( T \) is an s-transmutation because it is the uniform limit of the s-transmutation operators \( T_n u(x) = u(x) + \int_x^\infty K_n(x, t)u(t)dt \), see Proposition 12. The initial conditions (23) of the \( L \)-base \( \{\alpha_k\} \) follow by differentiating (22) and then using (18) at \( x = 0 \).

The formula for the kernel of \( T^{-1} \) was obtained in (22) for a continuous potential \( q \) and, by using a limit procedure as above, it is easy to see that it remains true for an integrable \( q \). The theorem is proved. ■

5.2 A fundamental set of s-transmutation operators

Let \( T \) be the s-transmutation operator (22) studied in Proposition 21. In accordance with Theorem 17 each one of the operators

\[
T P_+, TP_-, TAP_+, T_\frac{d}{dx}P_-
\]

is an s-transmutation on \( P(\mathbb{R}) \) (the linear space of polynomials). Next we show that (26) is a fundamental set of s-transmutation operators in the sense that any s-transmutation operator is a linear combination of operators from (26). Set \( I = (-a, a), 0 < a < \infty \).

Theorem 22 Let \( \{\varphi_k\}, k \in \mathbb{N}_0, \) be an s-L-base. Then there exists \( T_\varphi \) an s-transmutation operator on \( W^{3,1}(I) \) such that \( T_\varphi[x^k] = \varphi_k \). The equality

\[
T_\varphi = \varphi_0(0)TP_+ + \varphi'_0(0)TAP_+ + \varphi_1(0)T_\frac{d}{dx}P_+ + \varphi'_1(0)TP_-
\tag{27}
\]
holds on \(W^{1,1}(I)\), where the operators \(T\), \(P_\pm\), and \(A\) are given by (22) and (14). Also, \(T_\varphi: W^{1,1}(I) \to L^1(I)\) is bounded. Moreover, if \(W(\varphi_0, \varphi_1) \neq 0\), then the operator

\[
M_\varphi := \frac{1}{W(\varphi_0, \varphi_1)} \left\{ \varphi_1'(0)P_+T^{-1} - \varphi_0'(0)AP_+T^{-1} - \varphi_1(0) \frac{d}{dx}P_-T^{-1} + \varphi_0(0)P_-T^{-1} \right\}
\]

satisfies \(T_\varphi M_\varphi(u) = M_\varphi T_\varphi(u) = u\) for all \(u \in W^{2,1}(I)\). In particular, if \(\varphi_1(0) = 0\), then \(T_\varphi: L^1(I) \to L^1(I)\) is bounded, and if in addition \(\varphi_0(0) \neq 0\), then \(T_\varphi\) is invertible with \(T^{-1}_\varphi = M_\varphi\).

**Proof.** As \(T\) and \(T_\varphi\) are both s-transmutations on \(\mathcal{P}(\mathbb{R})\) with \(T_\varphi[x^k] = \varphi_k\) and \(T[x^k] = \alpha_k\), from Proposition 18 and using (23) we see that (27) holds on \(\mathcal{P}(\mathbb{R})\). As \(T: L^1(I) \to L^1(I)\) is bounded, the operator defined on \(\mathcal{P}(\mathbb{R})\) by the right hand side of (27) can be extended continuously to \(W^{1,1}(I)\), from which we conclude that \(T_\varphi: W^{1,1}(I) \to L^1(I)\) is bounded. Thus employing Theorem 10 and Remark 11, \(T_\varphi\) is an s-transmutation operator on \(W^{3,1}(I)\). The relation \(T_\varphi M_\varphi(u) = M_\varphi T_\varphi(u) = u\) is obtained after a few but trivial calculations, and using the existence of \(T^{-1}_\varphi\) (see Proposition 21). All the remaining affirmations are obvious. \(\blacksquare\)

**Corollary 23** Let \(\{\varphi_k\}\) be an s-L-base such that \(W(\varphi_0, \varphi_1) \neq 0\) and \(\lambda \in \mathbb{C}\). Then the general weak solution of the equation

\[
v'' + qv = \lambda v
\]

on \(I\) has the form \(v_{\text{ger}} = c_1 v_1 + c_2 v_2\) where \(c_1\) and \(c_2\) are arbitrary complex constants,

\[
v_1 = \sum_{k=0}^{\infty} \lambda^k \frac{\varphi_2}{(2k)!}, \quad \text{and} \quad v_2 = \sum_{k=1}^{\infty} \lambda^k \frac{\varphi_{2k+1}}{(2k+1)!},
\]

and both series, as well as the corresponding series of derivatives, converge uniformly on \(\overline{I}\).

**Proof.** Let \(T_\varphi\) be the s-transmutation operator studied in Theorem 22 and satisfying \(T_\varphi[x^k] = \varphi_k\). From the transmutation property of \(T_\varphi\), we can see that \((\frac{d^2}{dx^2} - q - \lambda)T_\varphi u = T_\varphi(\frac{d^2}{dx^2} - \lambda)u\) for all \(\lambda \in \mathbb{C}\) and \(u \in W^{3,1}(I)\). Thus, the operator \(T_\varphi\) maps classical solutions of \(u'' = \lambda u\) to weak solutions of (28). Moreover, as \(W(\varphi_0, \varphi_1) \neq 0\), it follows that \(T_\varphi\) is one-to-one on \(W^{3,1}(I)\), and thus the general solution of (28) is obtained by applying \(T_\varphi\) to the general solution of \(u'' = \lambda u\), \(v_{\text{ger}} = c_1 \cosh(\lambda x) + c_2 (1/\sqrt{\lambda}) \sinh(\sqrt{\lambda} x)\). Thus, \(v_{\text{ger}} = T_\varphi[u_{\text{ger}}] = c_1 T_\varphi[\cosh(\sqrt{\lambda} x)] + c_2 T_\varphi[(1/\sqrt{\lambda}) \sinh(\sqrt{\lambda} x)]\) where

\[
T_\varphi[\cosh(\lambda x)] = T_\varphi \left[ \sum_{k=0}^{+\infty} \lambda^k \frac{x^{2k}}{(2k)!} \right] = \sum_{k=0}^{+\infty} \lambda^k \frac{\varphi_{2k}}{(2k)!} = v_1
\]

and \(T_\varphi[(1/\sqrt{\lambda}) \sinh(\sqrt{\lambda} x)] = T_\varphi \left[ \sum_{k=0}^{+\infty} \lambda^k \frac{x^{2k+1}}{(2k+1)!} \right] = \sum_{k=0}^{+\infty} \lambda^k \frac{\varphi_{2k+1}}{(2k+1)!} = v_2\). Lastly, the convergence of the series defining the solutions \(v_1\) and \(v_2\) follows from the fact that \(T_\varphi\) is bounded from \(C^2(\overline{I})\) into \(C^1(\overline{I})\). \(\blacksquare\)
Remark 24 Let us relate the results we have obtained so far with the ones appearing in the literature. Assume that \( q \in C(\mathcal{T}) \) and that equation \( (\mathbf{9}) \) possesses a nonvanishing solution \( f \in C^2(\mathcal{T}) \). Set \( \varphi_0 := f \) and note that a second linearly independent solution can be computed by the formula \( \varphi_1(x) = f(x) \int_0^x \frac{dt}{f(t)} \). Without loss of generality assume that \( f(0) = 1 \) and set \( c := f'(0) \). Thus, \( \varphi_0(0) = 1, \varphi_0'(0) = c, \varphi_1(0) = 0, \) and \( \varphi_1'(0) = 1 \). Let \( \{ \varphi_k \} \) be the s-L-base corresponding to this pair of solutions, and denote by \( T_c \) the (transmutation) operator such that \( T_c [x^k] = \varphi_k \). Notice that \( T_0 \equiv \mathbf{T} \). The operator \( T_c \) has been studied in several places in the literature. In \( \mathbf{[8]} \), assuming \( q \in C^1(\mathcal{T}) \), it was proved that \( T_c \) is a Volterra integral operator and also a transmutation operator on \( C^2(\mathcal{T}) \). Another proof of the transmutation property of \( T_c \), for any value of \( c \in \mathbb{C} \), was given in \( \mathbf{[22]} \) when assuming that \( q \in C^1(\mathcal{T}) \) and also on \( \mathbf{[23]} \) supposing that \( q \in C(\mathcal{T}) \) and that \( \varphi_0(x) \) is nonvanishing on \( \mathcal{T} \). In \( \mathbf{[8]} \) a formula for the kernel of the operator \( T_c \) was obtained in terms of the kernel of the same operator with \( c = 0 \). Starting from this result, the relation \( T_c u = \mathbf{T} \left[ u(x) + \frac{c}{2} \int_{-x}^x u(t) dt \right] \) was obtained in \( \mathbf{[23]} \). This equality is the one given by \( \mathbf{[27]} \), in this special case. In \( \mathbf{[21]} \), and with the help of the above relation, it was proved that \( T_c \) is a transmutation operator for any \( q \in C(\mathcal{T}) \) and for any value of \( c \). Perhaps at the same time, another proof for this result was presented in \( \mathbf{[8]} \), which is published here (the proof of Theorem \( \mathbf{11} \)) for the first time.

The spectral parameter power series representation, like those representing the general solution of \( \mathbf{[22]} \) in Corollary \( \mathbf{23} \) was first introduced in \( \mathbf{[17]} \) as an application of pseudoanalytic function theory and nowadays represents a useful tool for the numerical solution of both regular and singular Sturm–Liouville problems, see \( \mathbf{[4, 21]} \) and the references therein. Originally, the coefficients of the power series \( \mathbf{[22]} \) were computed by slightly different recursive formulas, and its construction required the existence of a nonvanishing solution of \( \mathbf{[22]} \) with \( \lambda = 0 \). Here we have relaxed this restriction using the s-L-bases framework.

6 s-Transmutation operators on distribution spaces

Let \( I = (-a, a), 0 < a < \infty \), and let \( q \in L^1_{\text{loc}}(I) \). A linear operator \( T : D_T \subset \mathcal{D}'(I) \rightarrow \mathcal{D}'(I) \) is called a transmutation operator on \( \chi \subset \mathcal{D}(I) \) if for every \( u \in \chi \)

\[
\left( \frac{d^2}{dx^2} - q \right) Tu = T \left( \frac{d^2}{dx^2} u \right) \quad \text{in} \ \mathcal{D}'(I).
\]

Our next goal is to show that the transmutation operators constructed in Section 5 can be continuously extended from Sobolev spaces to suitable spaces of distributions, preserving the transmutation property. To this, let us consider the s-transmutation operator studied in Section 5.1 Theorem \( \mathbf{21} \)

\[
Tu(x) = u(x) + \int_{-x}^x K(x, t)u(t)dt.
\]

Since \( q \in L^1_{\text{loc}}(I) \), we have \( K(x, t) \in C(I \times I) \cap W^{1,1}_{\text{loc}}(I \times I) \) and thus \( T \) maps \( L^1_{\text{loc}}(I) \) into itself. In order to extend \( T \) to distributions, using Proposition \( \mathbf{2} \) we first have to look for
the existence of the corresponding transpose operator. An easy computation shows that
\[ T^\triangledown \psi(x) = \psi(x) - \int_{-\infty}^{-|x|} K(t, x)\psi(t)dt + \int_{|x|}^{\infty} K(t, x)\psi(t)dt, \quad \psi \in C_c(I), \quad (31) \]
with the equality \( \int_{|x|}^{\infty} (Tu)\phi dx = \int_{|x|}^{\infty} u(T^\triangledown \phi) dx \) valid for all \( u \in L^1_{loc}(I), \phi \in C_c(I) \).

Let us treat first the case when \( q \in C^\infty(I) \).

**Proposition 25** Suppose \( q \in C^\infty(I) \). If the operator \( T \) is continuous on \( D'(I) \), and a transmutation on \( C^\infty_c(I) \), then \( T \) is a transmutation operator on \( D'(I) \).

**Proof.** Let \( u \in D'(I) \). As \( C^\infty_c(I) \) is dense in \( D'(I) \) [12, Th. 3.18], there exists a sequence \( \phi_n \in C^\infty_c(I) \) such that \( \phi_n \to u \) in \( D'(I) \) as \( n \to \infty \). Hence,
\[
\left( \frac{d^2}{dx^2} - q \right) Tu = \lim_{n \to \infty} \left( \frac{d^2}{dx^2} - q \right) Tp_n = \lim_{n \to \infty} \frac{d^2}{dx^2} p_n = \frac{d^2}{dx^2} u, \quad \text{in } D'(I),
\]
where we have used the continuity in \( D'(I) \) of the involved operators and the transmutation property of \( T \) on \( C^\infty_c(I) \).

**Proposition 26** Assume that \( q \in C^\infty(I) \) and let \( T \) be the linear operator acting on distributions according to the rule
\[
\langle Tu, \phi \rangle := \langle u, T^\triangledown \phi \rangle, \quad u \in D'(I), \phi \in C^\infty_c(I) \quad (32)
\]
where \( T^\triangledown \) is given by (31). Then, \( T : D'(I) \to D'(I) \) is continuous and invertible. Also, \( T \) is a transmutation operator on \( D'(I) \).

**Proof.** First let us prove that \( T \) is well defined and continuous on \( D'(I) \). According to Proposition 22 it suffices to show that \( T^\triangledown \) is continuous on \( C^\infty_c(I) \). Suppose the sequence \( \phi_n \) converge in \( C^\infty_c(I) \) to \( \phi \). Then there exists a number \( b < a \) such that \( \text{supp} \phi_n \subseteq [-b, b] \subset I \) for all \( n \), and from (31) we see that \( \text{supp} (T^\triangledown \phi_n) \subseteq [-b, b] \subset I \) for all \( n \). On the other hand, as \( q \in C^\infty(I) \), we have \( K(x, t) \in C^\infty(I \times I) \) and thus the expression (31) defining \( T^\triangledown \phi \) can be differentiated infinitely many times whenever \( \phi \in C^\infty_c(I) \). For example,
\[
\frac{d}{dx} T^\triangledown \phi(x) = \frac{d}{dx} \phi(x) - \int_{-|x|}^{-|x|} \partial_x K(t, x)\phi(t)dt + \int_{|x|}^{\infty} \partial_x K(t, x)\phi(t)dt
\]
\[
+ K(-x, x)\phi(-x) + K(x, x)\phi(x). \quad (33)
\]
From the above facts it is easy to see that \( T^\triangledown \) maps \( C^\infty_c(I) \) into itself and that \( T^\triangledown \phi_n \to T^\triangledown \phi \) in \( C^\infty_c(I) \) whenever \( \phi_n \to \phi \) in \( C^\infty_c(I) \), as \( n \to \infty \). This proves the continuity of \( T^\triangledown \).

Since \( T \) is invertible on \( L^1_{loc}(I) \) and \( T^{-1} \) is a Volterra integral operator (Theorem 21), it follows that \( T^\triangledown \) is invertible as well, and \( (T^\triangledown)^{-1} = (T^{-1})^\triangledown \). Therefore the inverse operator of \( T \) on \( D'(I) \) is defined by the rule \( \langle T^{-1} u, \phi \rangle := \langle u, (T^\triangledown)^{-1} \phi \rangle \) for \( u \in D'(I) \) and \( \phi \in C^\infty(I) \).

Lastly, the transmutation property of \( T \) on \( D'(I) \) follows from Proposition 25 upon observing that \( T \) is continuous on \( D'(I) \) and a transmutation on \( C^\infty_c(I) \) (Theorem 21).
Corollary 27 Let $q \in C^\infty(I)$. Then the differential equation $u'' = qu$, $u \in \mathcal{D}'(I)$ has only classical solutions.

Proof. Let $u \in \mathcal{D}'(I)$ be a solution of $u'' = qu$. Then, from the transmutation property of $T$ it follows that $v = T^{-1}u$ is a solution of $v'' = 0$, $v \in \mathcal{D}'(I)$. The last equation has only classical solutions [2], thus $v = C_1 + C_2x$ for some constants $C_1$, $C_2$ and therefore $u = Tv = C_1T[1] + C_2T[x] \in C^\infty(I)$ is a classical solution of the first equation. 

Corollary 28 Let $q \in C^\infty(I)$. If $T$ is a transmutation operator on $\mathcal{D}'(I)$, then $T[x^k]$, $k \in \mathbb{N}_0$, is a regular distribution from the space $C^\infty(I)$.

Proof. Set $\varphi_k := T[x^k] \in \mathcal{D}'(I)$ for $k \in \mathbb{N}_0$. From the transmutation property of $T$ it follows that $\varphi_k$ is a distributional solution of (6) when $k = 0, 1$. For $k \geq 2$, it satisfies (7). However, from Corollary 27 we see that Equations (6) and (7) have only classical solutions, thus $T[x^k] \in C^\infty(I)$. 

The last result allows one to consider the notion of standard transmutation operator (Definition 7) on spaces of distributions.

In what follows, we obtain the analogue of some results of Section 4 for distributions.

Consider the operators $P_\pm u(x) = \frac{u(x) \pm u(-x)}{2}$ and $Au(x) = \int_0^x u(t)dt$ acting on $L^1_{loc}(I)$. Since $P_\pm = P_\pm$ is continuous on $C_c^\infty(I)$, it follows from Proposition 2 that $P_\pm$ is well defined and continuous on $\mathcal{D}'(I)$. A similar extension is not available for the operator $A$. Indeed,

$$A\psi(x) = \begin{cases} -\int_a^x \psi(t)dt, & -a \leq x \leq 0 \\ \int_x^a \psi(t)dt, & 0 < x \leq a \end{cases}.$$

Thus if $\psi \in C_c^\infty(I)$, we can not be assured of the continuity of the function $A\psi(x)$ at $x = 0$. However, it is easy to see that $(AP_+)^2 = (P_- A)^2 = A^2 P_+^2 = A^2 P_-$ is continuous on $C_c^\infty(I)$, and thus the operator $AP_+$ admits a continuous extension to $\mathcal{D}'(I)$.

Proposition 29 Let $q \in C^\infty(I)$ and let $T$ be the operator defined in $\mathcal{D}'(I)$ by (20)–(22). Then, a continuous linear operator $T : \mathcal{D}'(I) \to \mathcal{D}'(I)$ is an s-transmutation on $\mathcal{D}'(I)$ if and only if $T$ is a linear combination of operators from (20).

Proof. Once we have shown that each operator in (20) is a continuous s-transmutation on $\mathcal{D}'(I)$, then the same will be true for all their linear combinations. The continuity follows from the fact that each one of the operators $T$, $P_\pm$, $\frac{d}{dx}$, and $AP_+$ is continuous on $\mathcal{D}'(I)$, and the transmutation property on $\mathcal{D}'(I)$ is a consequence of Theorem 22 and Proposition 25.

Conversely, let $T$ be a continuous s-transmutation operator on $\mathcal{D}'(I)$. From Corollary 28 we see that the corresponding $L$-base, $\varphi_k := T[x^k]$ is such that $\varphi_k \in C^\infty(I)$. Let us set

$$T\varphi := \varphi_0(0)TP_+ + \varphi_0'(0)TAP_+ + \varphi_1(0)T\frac{d}{dx}P_+ + \varphi_1'(0)TP_-.$$

Note that by Theorem 22 we have $T\varphi[x^k] = \varphi_k$. Thus, $T\varphi[x^k] = T[x^k]$ for every $k \in \mathbb{N}_0$, and as both operators are linear they are equal on polynomials. In fact, as the set of
polynomials is dense in \( \mathcal{D}'(I) \) and both operators are continuous on \( \mathcal{D}'(I) \), we see that \( T_\varphi \) and \( T \) are equal on \( \mathcal{D}'(I) \). This shows that \( T \equiv T_\varphi \) is a linear combination of operators from Proposition 30. ■

Now assume that \( q \notin C^\infty(I) \). In such a case, the domain of the operator \( T \) defined in (30) is smaller than \( \mathcal{D}'(I) \) and it depends on the smoothness of the kernel \( K(t,x) \), which in turn depends on the smoothness of the potential \( q \). Denote by \( F(I) \), \( I = (-a,a) \), any one of the following spaces \( L^p(I) \), \( C^k(I) \), \( k \in \mathbb{N}_0 \), \( 1 \leq p \leq \infty \). We are going to consider \( F(I) \) with the convergence induced by the inductive limit topology (see Section 2). Define \( W^2(I;F) = \{ \varphi \in W^{1,1}(I) : \varphi'' \in F(I) \} \). If \( K \subset I \) is compact, then \( W^2(K;F) \) is a Banach space with norm \( \| u \|_{W^2(K,F)} = \| u \|_{W^{1,1}(K)} + \| u \|_{F(K)} \). We shall also consider \( W^2_c(I;F) \) with the inductive limit topology.

The extension of an operator \( T \) from functions to functionals will be based once more on the identity \( \langle Tu, \phi \rangle = \langle u, T^\square \phi \rangle \), whenever it makes sense.

**Proposition 30** Assume that \( q \in F_{loc}(I) \). Then the operator \( T \) defined in (30) is continuous and invertible on \( (W^2_c(I;F))' \) as well as on \( (F_c(I))' \). Moreover, \( T \) is a transmutation operator on \( (F_c(I))' \).

**Proof.** The continuity of \( T \) on \( (W^2_c(I;F))' \) follows from the continuity of \( T^\square \) on \( W^2_c(I;F) \), and its continuity on \( (F_c(I))' \) follows from the continuity of \( T^\square \) on \( F_c(I) \). Let us prove this when \( q \in L^1_{loc}(I) \), thus \( F(I) = L^p(I) \) and \( W^2_c(I;F) = W^2_c(I) \). The other cases will follow by analogy. From (33) and since \( K(t,x) \in C(I \times I) \cap W^1_{loc}(I \times I) \), it is clear that \( T^\square \) is continuous on \( L^1_c(I) \). Moreover, if \( \phi \in W^2_c(I) \), then formula (33) holds, and from this we see that \( T\phi \in W^1_{c,p}(I) \). On the other hand, if \( \phi, \psi \in W^2_c(I) \), then integration by parts gives

\[
\int_I (T^\square \phi)' \psi' \, dx = - \int_I (T^\square \phi) \psi'' \, dx = - \int_I \phi T \psi'' \, dx = - \int_I \phi \{ (T\psi)'' - qT\psi \} \, dx = - \int_I \{ T^\square (\phi'' - q\phi) \} \psi dx,
\]

where we have used the fact that the functions \( T^\square \phi \) and \( T^\square \psi \) have compact support and also the transmutation property of \( T \) on \( W^2_c(I) \). Thus we have shown that \( T^\square \phi \in W^2_c(I) \) and

\[
(T^\square \phi'') = T^\square (\phi'' - q\phi), \quad \phi \in W^2_c(I).
\]

Now, by (33) and (34) we see that \( T^\square \varphi_n \to T^\square \varphi \) in \( W^2_c(I) \), whenever \( \varphi_n \to \varphi \) in \( W^2_c(I) \), as \( n \to \infty \). This ensures the continuity of \( T^\square \) on \( W^2_c(I) \).

By analogy with the proof of Theorem 26, we arrive at the invertibility of \( T \).

Lastly, let us prove the transmutation property of \( T \). If \( u \in (F_c(I))' \subset (W^2_c(I;F))' \), then \( u'', Tu'', qTu, (Tu'') \in (W^2_c(I;F))' \) and using (33) we see that

\[
\langle (Tu'') - qTu, \phi \rangle = \langle u, T^\square (\phi'' - q\phi) \rangle = \langle u, (T^\square \phi)' \rangle = \langle Tu'', \phi \rangle
\]

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holds for all $\phi \in W^2_p(I)$. ■

**Remark 31** Let $q \in F_{\text{loc}}(I)$. Reasoning as in Corollary 27, we can conclude that the equation $u'' = qu$, $u \in (F_c(I))'$ has only weak solutions $u \in W^{2,1}(I)$. Thus, as before, we will be able to consider standard transmutation operators. Moreover, reasoning as in Proposition 29, we see that an $s$-transmutation operator on $(F_c(I))'$ that maps continuously $(F_c(I))'$ to $(W^1_c(I; F))'$ must have the form $T = c_1 TP_+ + c_2 TAP_+ + c_3 T \frac{d}{dx} P_- + c_4 TP_-$ where the $c_i$ are constants. Moreover, $T$ maps $(F_c(I))'$ to itself continuously if and only if $c_3 = 0$.

### 7 Conclusions and generalizations

A method to compute the general formula of the $s$-transmutation operators corresponding to $L = \frac{d^2}{dx^2} - q(x)$ has been presented. On the one hand, the method is based on fairly simple formulas, which allows one to compute a fundamental system of $s$-transmutation operators when one single (one-to-one) $s$-transmutation operator is known in closed form. On the other hand, with the aid of classical results we established that a particular $s$-transmutation operator can be represented in the form of a Volterra integral operator of the second kind.

Most of the results of Sections 3 and 4 can be extended to the second-order linear differential operator $L_1 = a_2(x) \frac{d^2}{dx^2} + a_1(x) \frac{d}{dx} + q(x)$ with smooth coefficients. This includes the explicit formulas leading to the construction of new $s$-transmutations and then to the general formula of the $s$-transmutations corresponding to $L_1$, once the closed form of one single $s$-transmutation corresponding to $L_1$ is known. If $a_2(x) \equiv 1$, then a closed form for such an $s$-transmutation can be obtained following [26, Chap. 1, Th. 7.1] and if $L_1 = \frac{d}{dx}p \frac{d}{dx} + q(x)$, by following [20].

It would also be interesting to develop this method for higher order linear differential operators. However, apart from the results of [11] and [15], there are, as far as our knowledge extends, no results known about the existence of transmutation operators for higher orders.

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