GENERIC CONTINUOUS SPECTRUM FOR ERGODIC SCHRÖDINGER OPERATORS

MICHAEL BOSHERNITZAN AND DAVID DAMANIK

ABSTRACT. We consider discrete Schrödinger operators on the line with potentials generated by a minimal homeomorphism on a compact metric space and a continuous sampling function. We introduce the concepts of topological and metric repetition property. Assuming that the underlying dynamical system satisfies one of these repetition properties, we show using Gordon’s Lemma that for a generic continuous sampling function, the associated Schrödinger operators have no eigenvalues in a topological or metric sense, respectively. We present a number of applications, particularly to shifts and skew-shifts on the torus.

1. Introduction

In this paper we study Schrödinger operators acting in $\ell^2(\mathbb{Z})$ by

$$(H_\omega \psi)(n) = \psi(n + 1) + \psi(n - 1) + V_\omega(n) \psi(n),$$

where the potential $V_\omega$ is generated by some homeomorphism $T$ of a compact metric space $\Omega$ and a continuous sampling function $f : \Omega \to \mathbb{R}$ as follows:

$$V_\omega(n) = f(T^n \omega), \quad \omega \in \Omega, \ n \in \mathbb{Z}. \tag{2}$$

We assume throughout this paper that $(\Omega, T)$ is minimal, that is, the $T$-orbit of every $\omega \in \Omega$ is dense in $\Omega$. A strong approximation argument shows that the spectrum of $H_\omega$ does not depend on $\omega$. Since the case of periodic potentials is well understood, we assume in addition that $\Omega$ is not finite. This ensures that $T$ does not have any periodic points.

If $\mu$ is a $T$-ergodic Borel probability measure, it is known that the spectral type of $H_\omega$ is $\mu$-almost surely independent of $\omega$ (see, e.g., Carmona-Lacroix [18]). In general, however, the spectral type is not globally independent of $\omega$ (see Jitomirskaya-Simon [26] for counterexamples).

Popular examples include shifts, (3)

$$\Omega = \mathbb{T}^d, \ T \omega = \omega + \alpha$$

and skew-shifts

$$(4) \quad \Omega = \mathbb{T}^d, \ T(\omega_1, \omega_2) = (\omega_1 + 2\alpha, \omega_1 + \omega_2)$$
on the torus. Here, we write $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Operators with potentials generated by shifts on the torus are called quasi-periodic.

In these examples, minimality of $(\Omega, T)$ holds if and only if the coordinates of $\alpha$ together with 1 are independent over the rational numbers. Moreover, in these cases, normalized Lebesgue measure on the torus in question is the unique $T$-ergodic.
Borel probability measure and hence one is particularly interested in identifying the spectrum and the spectral type of $H$ for Lebesgue almost every $\omega$.

We refer the reader to Bourgain’s recent book [14] for the current state of the art, especially for these two classes of models.

While the quasi-periodic case has been studied heavily for several decades and many fundamental results have been obtained, the case of the skew-shift is much less understood. It is of interest as it naturally arises in the study of the quantum kicked rotor model; compare [13]. The general expected picture is that, while the skew-shift model seems to be formally close to a quasi-periodic model, on the operator level one observes much different behavior – similar to that found for random potentials. Namely, while any spectral type occurs naturally for quasi-periodic models and their spectra have a tendency to be Cantor sets, it is expected that skew-shift models “almost always” have pure point non-Cantor spectrum. Thus, interest in Schrödinger operators generated by the skew-shift is also triggered by the question of how much one can reduce the randomness of the potentials until one fails to observe complete localization phenomena. Such a transition could occur at or near skew-shift models.

To illustrate this point, let us recall some conjectures from [14, p. 114]. Consider the potential

$$V(n) = \lambda \cos(2\pi(\omega_1 + \omega_2 n + \alpha(n - 1))).$$

Then, one expects that for every $\lambda \neq 0$, Diophantine $\alpha \in \mathbb{T}$, and Lebesgue almost every $\omega \in \mathbb{T}^2$, the operator (1) with potential (5) has positive Lyapunov exponents\(^1\) pure point spectrum with exponentially decaying eigenfunctions, and its spectrum has no gaps. Generally, one expects further that these properties persist even when the cosine is replaced by a sufficiently regular function $f$.

There are very few positive results in this direction. For example, Bourgain, Goldstein, and Schlag proved a localization result for analytic $f = \lambda g$ and sufficiently large $\lambda$ and Bourgain proved the existence of some point spectrum for (5) with small $\lambda$ and certain $(\alpha, \omega) \in \mathbb{T}^3$; see [11, 16]. Some results (that, however, do not determine the spectral type) assuming weaker regularity can be found in the paper [19] by Chan, Goldstein, and Schlag.

Naturally, negative results that point out limitations to the scope in which the expected properties actually hold are of interest as well. One result of this kind is obtained in the work [11] by Avila, Bochi, and Damanik, where it is shown that for the skew-shift model (and generalizations thereof), the spectrum is a Cantor set for a residual set of continuous sampling functions. Another result showing that expected phenomena may not occur can be found in the paper [5] by Bjerklöv. He showed that even for (large) analytic sampling functions, the Lyapunov exponent may vanish. Further negative results, concerning the spectral type, will be established in the present paper. Namely, we will show under reasonably weak assumptions that for a generic continuous sampling function $f$, the point spectrum is empty.

The present paper should be regarded as a companion piece to work by Avila and Damanik [2]. They proved that the absolutely continuous spectrum is generically empty. Putting these two results together, it follows that for a large class of ergodic

\(^1\)Lyapunov exponents measure the averaged rate of exponential growth of the so-called transfer matrices. Since we will not need them in our study, we omit the exact definition.
Schrödinger operators, the generic spectral type is singular continuous. This will be discussed in more detail in Subsection 4.3.

A bounded potential \( V : \mathbb{Z} \to \mathbb{R} \) is called a Gordon potential if there are positive integers \( q_k \to \infty \) such that

\[
\max_{1 \leq n \leq q_k} |V(n) - V(n \pm q_k)| \leq k^{-q_k}
\]

for every \( k \geq 1 \). Equivalently, there are positive integers \( q_k \to \infty \) such that

\[
\forall C > 0 : \lim_{k \to \infty} \max_{1 \leq n \leq q_k} |V(n) - V(n \pm q_k)| C^{q_k} = 0.
\]

This condition is of interest because it ensures the absence of point spectrum. That is, if \( V \) is a Gordon potential, then the Schrödinger operator in \( L^2(\mathbb{Z}) \) with potential \( V \) has no eigenvalues. In fact, it is even true that for every \( E \in \mathbb{C} \), the difference equation

\[
u(n + 1) + u(n - 1) + V(n) = Eu(n)
\]

has no non-trivial solution \( u \) with \( \lim_{|n| \to \infty} u(n) = 0 \); compare \([21, 23, 25]\).

For potentials generated by a shift on the circle, \( T : \mathbb{T} \to \mathbb{T} \), \( \omega \mapsto \omega + \alpha \), it is not hard to show that for \( f \) having a fixed modulus of continuity and \( \alpha \) from an explicit residual subset of \( T \) (which depends only on the modulus of continuity and has zero Lebesgue measure), \( V_\omega \) is a Gordon potential for every \( \omega \). For the specific case of cosine, this observation was used by Avron and Simon \([3]\) to exhibit explicit quasi-periodic operators with purely singular continuous spectrum; see also \([21]\).

Our goal is to exhibit a variety of situations, in the general context we consider, where the potentials satisfy the Gordon condition, either on a residual subset of \( \Omega \) or on a subset of \( \Omega \) that has full \( \mu \) measure.

**Definition 1.** A sequence \( \{\omega_k\}_{k \geq 0} \) in the compact metric space \( \Omega \) has the repetition property if for every \( \varepsilon > 0 \) and \( r > 0 \), there exists \( q \in \mathbb{Z}_+ \) such that \( \text{dist}(\omega_k, \omega_{k+q}) < \varepsilon \) for \( k = 0, 1, 2, \ldots, \lfloor rq \rfloor \).

Of course, this definition makes sense in any metric space; but we remark that the compactness of \( \Omega \) implies that the validity of the repetition property for any given sequence in \( \Omega \) is independent of the choice of the metric.

**Definition 2.** We denote the set of points in the compact metric space \( \Omega \) whose forward orbit with respect to the minimal homeomorphism \( T \) has the repetition property by \( \text{PRP}(\Omega, T) \). That is,

\[
\text{PRP}(\Omega, T) = \{ \omega \in \Omega : \{T^k \omega\}_{k \geq 0} \text{ has the repetition property} \}.
\]

We say that \( (\Omega, T) \) satisfies the topological repetition property (TRP) if \( \text{PRP}(\Omega, T) \neq \emptyset \).

It is not hard to see that since \( (\Omega, T) \) is minimal, \( \text{PRP}(\Omega, T) \) being non-empty actually implies that it is residual. One can of course define \( \text{PRP}(\Omega, T) \) also for a non-minimal topological dynamical system \( (\Omega, T) \). In general, \( \text{PRP}(\Omega, T) \) is always a \( G_\delta \) set and so, in particular, a dense \( G_\delta \) set in the closure of the orbit of any of its elements.

\(^2\)Here, \( |V(n) - V(n \pm q_k)| \leq k^{-q_k} \) is shorthand for having both \( |V(n) - V(n + q_k)| \leq k^{-q_k} \) and \( |V(n) - V(n - q_k)| \leq k^{-q_k} \).
Theorem 1. Suppose \((\Omega, T)\) satisfies (TRP). Then there exists a residual subset \(F\) of \(C(\Omega)\) such that for every \(f \in F\), there is a residual subset \(\Omega_f \subseteq \Omega\) with the property that for every \(\omega \in \Omega_f\), \(V_\omega\) defined by (2) is a Gordon potential.

Combining this theorem with Gordon’s result, we find that \(H_\omega\) has purely continuous spectrum for topologically generic \(f \in C(\Omega)\) and \(\omega \in \Omega\) whenever \((\Omega, T)\) satisfies (TRP).

Let us now fix some \(T\)-ergodic measure \(\mu\). As mentioned above, given the general theory of ergodic Schrödinger operators (cf. [18]), it is a natural goal to identify the almost sure spectral type with respect to \(\mu\). By almost sure independence, it is sufficient to consider sets of positive \(\mu\) measure. Thus, we are seeking a criterion that implies the applicability of the Gordon’s lemma at least on a set of positive \(\mu\) measure.

Definition 3. We say that \((\Omega, T, \mu)\) satisfies the metric repetition property (MRP) if \(\mu(\text{PRP}(\Omega, T)) > 0\).

The following result shows that (MRP) implies the desired Gordon property on a full measure set.

Theorem 2. Suppose \((\Omega, T, \mu)\) satisfies (MRP). Then there exists a residual subset \(F\) of \(C(\Omega)\) such that for every \(f \in F\), there is a subset \(\Omega_f \subseteq \Omega\) of full \(\mu\) measure with the property that for every \(\omega \in \Omega_f\), \(V_\omega\) defined by (2) is a Gordon potential.

Naturally, the strongest repetition property, defined in terms of the size of the set \(\text{PRP}(\Omega, T)\), that can hold is the following:

Definition 4. We say that \((\Omega, T)\) satisfies the global repetition property (GRP) if \(\text{PRP}(\Omega, T) = \Omega\).

It is an interesting question whether (GRP) implies a stronger statement for the associated potentials. Namely, when (GRP) holds, is it true that for generic continuous \(f\), \(V_\omega\) is a Gordon potential for every \(\omega \in \Omega\)? We believe the answer is no but have been unable to prove this.

Let us explore when (TRP), (MRP), and (GRP) hold for our two classes of examples. It will turn out that, in these examples, (TRP), (MRP), and (GRP) always hold or fail simultaneously. Trivially, we have that (GRP) \(\Rightarrow\) (MRP) \(\Rightarrow\) (TRP), but one can ask whether some reverse implication holds in general. This turns out not to be the case, see the discussion in Subsection 4.5.

For minimal shifts of the form (3), the situation is particularly nice as the following result shows.

Theorem 3. Every minimal shift \(T_\omega = \omega + \alpha\) on the torus \(\mathbb{T}^d\) satisfies (GRP), and hence also (TRP) and (MRP).

As a consequence, we find that for every minimal shift on \(\mathbb{T}^d\) and a generic function \(f \in C(\mathbb{T}^d)\), the operator \(H_\omega\) has empty point spectrum for almost every \(\omega\). This is especially surprising if a coupling constant is introduced. Generally, one expects pure point spectrum at large coupling, but our proof excludes point spectrum for all values of the coupling constant at once! This is discussed in more detail in Subsection 4.2. We also want to mention that recent results indicate that point spectrum should become more and more prevalent as the dimension of the torus increases [12]. Our result on the absence of point spectrum, on the other hand, holds for all torus dimensions.
Let us now turn to minimal skew-shifts of the form (4). Recall that \( \alpha \in \mathbb{T} \) is called badly approximable if there is a constant \( c > 0 \) such that
\[
\langle \alpha q \rangle > \frac{c}{q}
\]
for every \( q \in \mathbb{Z} \setminus \{0\} \). Here, we write \( \langle x \rangle = \text{dist}_T(x, 0) \) (\( = \min\{|x - p| : p \in \mathbb{Z}\} \)), where \( x \) denotes any representative in \( \mathbb{R} \). The set of badly approximable \( \alpha \)'s has zero Lebesgue measure; see, for example, [28, Theorem 29 on p. 60]. In terms of the continued fraction expansion of \( \alpha \) (cf. [28]), being badly approximable is equivalent to having bounded partial quotients.

**Theorem 4.** For a minimal skew-shift \( T(\omega_1, \omega_2) = (\omega_1 + 2 \alpha, \omega_1 + \omega_2) \) on the torus \( \mathbb{T}^2 \), the following are equivalent:

(i) \( \alpha \) is not badly approximable.

(ii) \((\Omega, T)\) satisfies (GRP).

(iii) \((\Omega, T, \text{Leb})\) satisfies (MRP).

(iv) \((\Omega, T)\) satisfies (TRP).

Thus, for Lebesgue almost every \( \alpha \), the operator \( H_\omega \) generated by the corresponding skew-shift and a generic function \( f \in C(\mathbb{T}^2) \) has empty point spectrum for Lebesgue almost every \( \omega \). This is surprising given that the expected spectral type for operators generated by the skew shift is pure point. Again, one can introduce a coupling constant and absence of point spectrum then holds for all values of the coupling constant simultaneously. Let us also emphasize that, to the best of our knowledge, our result provides the first examples of Schrödinger operators with potentials defined by a skew-shift that have empty point spectrum. In particular, the expected localization result for such operators will need a suitable regularity assumption for the sampling function.

The paper is organized as follows. In Section 2, we establish the relation between the topological or metric repetition property for the underlying dynamical system and the Gordon property for the associated potentials when a generic continuous sampling function is chosen; that is, we prove Theorems 1 and 2. The validity of the topological, metric, or global repetition property for the two classes of examples is then explored in Section 3 where we prove Theorems 3 and 4. We conclude the paper with some further results and comments in Section 4.

2. Repetition Properties and Gordon Potentials

In this section we prove Theorems 1 and 2.

**Proof of Theorem 2.** By assumption, there is a point \( \omega \in \Omega \) whose forward orbit has the repetition property. For each \( k \in \mathbb{Z}_+ \), consider \( \varepsilon = \frac{1}{k}, \ r = 3 \), and the associated \( q_k = q(\varepsilon, r) \). This ensures
\[
q_k \to \infty.
\]
Take an open ball \( B_k \) around \( \omega \) with radius small enough so that
\[
T^n(B_k), \ 1 \leq n \leq 4q_k
\]
are disjoint and, for every \( 1 \leq j \leq q_k \),
\[
\bigcup_{l=0}^{3} T^{j+lq_k}(B_k)
\]
is contained in some ball of radius $4\varepsilon$. Define
\[ C_k = \left\{ f \in C(\Omega) : f \text{ is constant on each set } \bigcup_{l=0}^{3} T^{j+4l} (B_k), \ 1 \leq j \leq q_k \right\} \]
and let $F_k$ be the open $k^{-q_k}$ neighborhood of $C_k$ in $C(\Omega)$. Notice that for each $m$,
\[ \bigcup_{k \geq m} F_k \]
is an open and dense subset of $C(\Omega)$. This follows since every $f \in C(\Omega)$ is uniformly continuous and the diameter of the set $\bigcup_{l=0}^{3} T^{j+4l} (B_k)$ goes to zero, uniformly in $j$, as $k \to \infty$. Thus,
\[ F = \bigcap_{m \geq 1} \bigcup_{k \geq m} F_k \]
is a dense $G_\delta$ subset of $C(\Omega)$.

Consider some $f \in F$. Then, $f \in F_{k_l}$ for some sequence $k_l \to \infty$. Observe that for every $m \geq 1$,
\[ \bigcup_{l \geq m} \bigcup_{j=1}^{q_{k_l}} T^{j+4k_l} (B_{k_l}) \]
is an open and dense subset of $\Omega$ since $T$ is minimal and $q_{k_l} \to \infty$. Thus,
\[ \Omega_f = \bigcap_{m \geq 1} \bigcup_{l \geq m} \bigcup_{j=1}^{q_{k_l}} T^{j+4k_l} (B_{k_l}) \]
is a dense $G_\delta$ subset of $\Omega$.

It now readily follows that for every $f \in F$ and $\omega \in \Omega_f$, $V_\omega$ is a Gordon potential. Explicitly, since $\omega \in \Omega_f$, $\omega$ belongs to $\bigcup_{j=1}^{q_{k_l}} T^{j+4k_l} (B_{k_l})$ for infinitely many $l$. For each such $l$, we have by construction that
\[ \max_{1 \leq j \leq q_{k_l}} |f(T^{j \omega}) - f(T^{j+4k_l \omega})| < 2k_l^{-q_{k_l}} \]
and
\[ \max_{1 \leq j \leq q_{k_l}} |f(T^{j \omega}) - f(T^{j-4k_l \omega})| < 2k_l^{-q_{k_l}} \]
This shows that $V_\omega(n) = f(T^n \omega)$ is a Gordon potential. \hfill $\square$

**Proof of Theorem 2.** Notice that, by ergodicity, the assumption $\mu(PR\{\Omega, T\}) > 0$ implies $\mu(PR\{\Omega, T\}) = 1$.

Let us inductively define a sequence of positive integers, $\{n_i\}_{i \geq 1}$, and a sequence of subsets of $\Omega$, $\{\Omega_i\}_{i \geq 1}$.

Since $\mu(PR\{\Omega, T\}) = 1$ we can choose $n_1 \in \mathbb{Z}_+$ large enough so that $3^3$
\[ \Omega_1 = \left\{ \omega \in \Omega : \text{there exists } n \in [1, n_1] \text{ such that for } k_1, k_2 \in [1, n] \right\}
\text{ with } k_1 \equiv k_2 \pmod{n}, \text{ we have } \text{dist}(T^{k_1 \omega}, T^{k_2 \omega}) < 1 \}
obeys
\[ \mu(\Omega_1) > 1 - 2^{-1}. \]

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3The definition of $\Omega_1$ contains redundancies whose purpose is to motivate the definition of the subsequent sets $\Omega_i$. 
Once $n_{i-1}$ and $\Omega_{i-1}$ have been determined, we can define $n_i$ and $\Omega_i$ as follows. It is possible to find $n_i > \max\{n_{i-1}, 2^i\}$ such that

$$\Omega_i = \left\{ \omega \in \Omega : \text{there exists } n \in [n_{i-1}, n_i) \text{ such that for } k_1, k_2 \in [1, in] \right. \left. \text{with } k_1 \equiv k_2 \mod n, \text{ we have } \dist(T^{k_1}\omega, T^{k_2}\omega) < 2^{-i} \right\}$$

obeys

$$\mu(\Omega_i) > 1 - 2^{-i}. \label{eq:6}$$

For $i \geq 5$, we set $m_i = (in_i)^2$ and choose (using the Rokhlin-Halmos Lemma [20, Theorem 1 on p. 242]) $O_i \subset \Omega$ in a way that $T^j O_i$, $1 \leq j \leq m_i$ are disjoint and

$$\mu \left( \bigcup_{j=1}^{m_i} T^j O_i \right) > 1 - 2^{n_i-1}. \label{eq:7}$$

Next we further partition $O_i$ into sets $S_{i,l}$, $1 \leq l \leq s_i$ such that for every $0 \leq u \leq m_i$,

$$\text{diam}(T^u S_{i,l}) < \frac{1}{m_i}. \label{eq:8}$$

Choose $K_{i,l} \subseteq S_{i,l}$ compact with

$$\mu(K_{i,l}) > \mu(S_{i,l}) \left( 1 - 2^{n_i-1} \right).$$

Then, $T^u K_{i,l}$, $0 \leq u \leq m_i$, $1 \leq l \leq s_i$ are disjoint and their total measure is

$$\mu \left( \bigcup_{u,l} T^u K_{i,l} \right) \geq (1 - 2^{n_i-1})^2 > 1 - 2^{-n_i}. \label{eq:9}$$

We will now collect a large subfamily of $\{T^u K_{i,l}\}$. For each $l$, we let $u$ run from 0 upwards and ask for the corresponding $T^u K_{i,l}$ whether it has non-empty intersection with $\Omega_i$. If it does, then there is a point $\omega$ and a corresponding $n \in (n_{i-1}, n_i)$. We add the current $T^u K_{i,l}$ to the subfamily we construct, along with the sets $T^{u+h} K_{i,l}$, $1 \leq h \leq in$. Then we continue with $T^{u+in+1} K_{i,l}$ and do the same. We stop when we are within $n_i$ steps of the top of the tower. Let us denote the subfamily so constructed by $K_i$. By (6), (8), and $T$-invariance of $\mu$, we have that

$$\mu \left( \bigcup_{T^{n_i} K_{i,l} \in K_i} T^u K_{i,l} \right) > 1 - 2^{-n_i} - 2^{-i} - \frac{n_i}{m_i}. \label{eq:10}$$

The next step is to group each of these “runs” into arithmetic progressions. Notice that locally we have $n$ consecutive points that are $(i-1)$ times repeated up to some small error. Notice that this extends to the entire set if we make the allowed error a bit larger. Let us group these in sets into $n$ arithmetic progressions of length $i$. The union of each of these arithmetic progressions of sets will constitute a new set $C_{i,m}$. By the definition of $\Omega_i$ and (7), we have

$$\text{diam}(C_{i,m}) < \frac{2}{n_i} + 2^{-i}. \label{eq:11}$$

We will also consider the sets $\bar{C}_{i,m}$ that are defined similarly, but with the first and the last set in the corresponding sequence of $i$ sets deleted.
We can now continue as before. Define

\[ F_i = \{ f \in C(\Omega) : f \text{ is constant on each set } C_{i,m} \} \]

and let \( F_i \) be the \( i^{-m} \) neighborhood of \( F_i \) in \( C(\Omega) \). Notice that for each \( m \),

\[ \bigcup_{i \geq m} F_i \]

is an open and dense subset of \( C(\Omega) \). This follows from (9) and (10) since every \( f \in C(\Omega) \) is uniformly continuous. Thus,

\[ F = \bigcap_{m \geq 1} \bigcup_{i \geq m} F_i \]

is a dense \( G_\delta \) subset of \( C(\Omega) \).

If \( f \in F \), there is a sequence \( i_k \to \infty \) such that \( f \in F_{i_k} \). For each \( k \), \( f \) is within \( i_k^{-m_i_k} \) of being constant on each set \( C_{i_k,m} \). Recall that this set is the union of \( i \) sets in arithmetic progression relative to \( T \).

If we instead consider \( C_{i_k,m} \), we can go forward and backward one period and hence, by construction, this is exactly the Gordon condition at this level. Thus, it only remains to show that almost every \( \omega \in \Omega \) belongs to infinitely many \( C_{i_k,m} \).

This, however, follows from the measure estimates obtained above and the Borel-Cantelli Lemma. □

3. REPTITON PROPERTIES FOR SHIFTS AND SKEW-SHIFTS

In this section we consider minimal shifts and skew-shifts and identify those cases that obey (TRP), (MRP), and (GRP). That is, we prove Theorems 3 and 4.

Proof of Theorem 3. By assumption, the orbit of \( 0 \in \mathbb{T}^d \) is dense. In particular, we can define \( q_k \to \infty \) such that \( T^{q_k}(0) \) is closer to \( 0 \) than any point \( T^n(0) \), \( 1 \leq n < q_k \). In particular, for every \( \varepsilon > 0 \) and every \( r > 0 \), there is \( k(\varepsilon,r) \) such that for \( k \geq k(\varepsilon,r) \), \( T^{q_k} \) is a shift on \( \mathbb{T}^d \) with a shift vector of length bounded by \( \varepsilon \).

The repetition property now follows for the forward orbit of any choice of \( \omega \in \mathbb{T}^d \). Thus, (GRP) is satisfied. □

Remark. The proof only used that the shift on the torus is an isometry. Thus, the result extends immediately to any minimal isometry of a compact metric space.

Proof of Theorem 4. Iterating the skew-shift \( n \) times, we find

\[ T^n(\omega_1,\omega_2) = (\omega_1 + 2n\alpha, \omega_2 + 2n\omega_1 + n(n-1)\alpha) \]

Thus,

\[ T^{n+q}(\omega_1,\omega_2) - T^n(\omega_1,\omega_2) = (2q\alpha, 2q\omega_1 + q^2\alpha + 2nq\alpha - qa) \]

(i) ⇒ (ii): Assume that \( \alpha \) is not badly approximable. This means that there is some sequence \( q_k \to \infty \), such that

\[ \lim_{k \to \infty} q_k \langle \alpha q_k \rangle = 0. \]

Let \( (\omega_1,\omega_2) \in \mathbb{T}^2 \), \( \varepsilon > 0 \), and \( r > 0 \) be given. We will construct a sequence \( q_k \to \infty \) so that for \( 1 \leq n \leq r q_k \),

\[ (2q\alpha, 2q\omega_1 + q^2\alpha + 2nq\alpha - q_k\alpha) \]

is of size \( O(\varepsilon) \). Each \( q_k \) will be of the form \( m_k q_k \) for some \( m_k \in \{1, 2, \ldots, \lfloor \varepsilon^{-1} \rfloor + 1 \} \).
It follows from (12) that in (13), every term except $2q_k \omega_1$ goes to zero as $k \to \infty$, regardless of the choice of $m_k$, and hence is less than $\varepsilon$ for $k$ large enough. To treat the remaining term, we can just choose $m_k$ in the specified $\varepsilon$-dependent range so that $2q_k \omega_1 = m_k (2q_k \omega_1)$ is of size less than $\varepsilon$ as well. Consequently, by (11), the orbit of $(\omega_1, \omega_2)$ has the repetition property. Since $(\omega_1, \omega_2)$ was arbitrary, it follows that (GRP) holds.

(ii) \implies (iii): This is immediate.

(iii) \implies (iv): This is immediate.

(iv) \implies (i): Assuming (TRP), we see that there is a point $\omega$ such that $\{T^n \omega\}_{n \geq 0}$ has the repetition property. In particular, by (11), we see that for every $\varepsilon > 0$, there are $q_k \to \infty$ so that

\begin{equation}
\langle 2q_1 \omega_1 + q_k^2 \alpha + 2nq_k \alpha - q_k \alpha \rangle < \varepsilon \quad \text{for } 0 \leq n \leq q_k.
\end{equation}

Evaluating this for $n = 0$, we find that $\langle 2q_1 \omega_1 + q_k^2 \alpha - q_k \alpha \rangle < \varepsilon$. Now vary $n$. Each time we increase $n$, we shift in the same direction by $\langle 2q_k \alpha \rangle$. If $\varepsilon > 0$ is sufficiently small, it follows from the estimate (14) that we cannot go around the circle completely and hence we have $\langle 2nq_k \alpha \rangle = n \langle 2q_k \alpha \rangle$ for every $0 \leq n \leq q_k$. We find that $\langle q_k \rangle \lesssim \varepsilon q_k$, which shows that $\alpha$ is not badly approximable. \qed

4. Further Results and Comments

4.1. Interval Exchange Transformations. Of course, it is interesting to explore the validity of the various repetition properties for other underlying dynamical systems. In this subsection we will briefly discuss interval exchange transformations as these dynamical systems have been studied in the context of ergodic Schrödinger operators before; see [22] for references.

An interval exchange transformation is defined as follows. Let $m > 1$ be a fixed integer and denote

$$\Lambda_m = \{ \lambda \in \mathbb{R}^m : \lambda_j > 0, 1 \leq j \leq m \}$$

and, for $\lambda \in \Lambda_m$,

$$\beta_j(\lambda) = \begin{cases} 0 & j = 0 \\ \sum_{i=1}^{j} \lambda_i & 1 \leq j \leq m \end{cases}$$

$$I_j^\lambda = [\beta_{j-1}(\lambda), \beta_j(\lambda))$$

$$|\lambda| = \sum_{i=1}^{m} \lambda_i$$

$$I^\lambda = [0, |\lambda|).$$

Denote by $\mathcal{S}_m$ the group of permutations on $\{1, \ldots, m\}$, and set $\lambda^\pi_j = \lambda_{\pi^{-1}(j)}$ for $\lambda \in \Lambda_m$ and $\pi \in \mathcal{S}_m$. With these definitions, the $(\lambda, \pi)$-interval exchange map $T_{\lambda, \pi}$ is given by

$$T_{\lambda, \pi} : I^\lambda \to I^\lambda, \ x \mapsto x - \beta_j(\lambda) + \beta_{\pi(j)-1}(\lambda^\pi) \quad \text{for } x \in I_j^\lambda, 1 \leq j \leq m.$$ 

A permutation $\pi \in \mathcal{S}_m$ is called irreducible if $\pi(\{1, \ldots, k\}) = \{1, \ldots, k\}$ implies $k = m$. We denote the set of irreducible permutations by $\mathcal{S}_m^0$.  


Let $\pi \in S_0^0$. Then, for Lebesgue almost every $\lambda \in \Lambda_m$, $(I^\lambda, T_{\lambda, \pi})$ is strictly ergodic. The minimality statement follows from Keane’s work [27]. The unique ergodicity statement was shown by Masur [29] and Veech [32]; for a simpler proof of this result, see [6, 7, 34]. When unique ergodicity holds, it is clear that the unique $T_{\lambda, \pi}$-invariant Borel probability measure must be given by normalized Lebesgue measure on $I^\lambda$, denoted by $\text{Leb}$.

It is therefore natural to ask whether $(I^\lambda, T_{\lambda, \pi}, \text{Leb})$ satisfies (MRP). We have the following result:

**Theorem 5.** Let $\pi \in S_0^0$. Then for Lebesgue almost every $\lambda \in \Lambda_m$, $(I^\lambda, T_{\lambda, \pi})$ is strictly ergodic and $(I^\lambda, T_{\lambda, \pi}, \text{Leb})$ satisfies (MRP).

The proof of this theorem will rely on the following result due to Veech; see [33, Theorem 1.4].

**Theorem 6** (Veech 1984). Let $\pi \in S_0^0$. For Lebesgue almost every $\lambda \in \Lambda_m$ and every $\epsilon > 0$, there are $q \geq 1$ and an interval $J \subseteq I^\lambda$ such that

(i) $J \cap T_{\lambda, \pi}^l J = \emptyset$, $1 \leq l < q$,
(ii) $T_{\lambda, \pi}$ is linear on $T_{\lambda, \pi}^l J$, $0 \leq l < q$,
(iii) $\text{Leb}(\bigcup_{l=0}^{q-1} T_{\lambda, \pi}^l J) > 1 - \epsilon$,
(iv) $\text{Leb}(J \cap T_{\lambda, \pi}^q J) > (1 - \epsilon)\text{Leb}(J)$.

**Proof of Theorem 5.** Consider a $\lambda$ from the full measure subset of $\Lambda_m$ such that $(I^\lambda, T_{\lambda, \pi})$ is strictly ergodic and all the consequences listed in Theorem 6 hold. We claim that $(I^\lambda, T_{\lambda, \pi}, \text{Leb})$ satisfies (MRP).

For $\epsilon > 0$ and $r > 0$, consider the set $I_{\epsilon, r}^\lambda$ of points $\omega \in I^\lambda$ for which there exists $q \in \mathbb{Z}_+$ such that $\text{dist}(T_{\lambda, \pi}^k \omega, T_{\lambda, \pi}^{k+q} \omega) < \epsilon$ for $k = 0, 1, 2, \ldots, \lfloor rq \rfloor$. By Theorem 6 we have

$$\text{Leb}(I^\lambda \setminus I_{\epsilon, r}^\lambda) \lesssim r\epsilon.$$

Thus, by Borel-Cantelli, the set

$$\bigcap_{j, r \in \mathbb{Z}_+} I_{\epsilon, r}^\lambda$$

has full Lebesgue measure. Since this set is contained in $\text{PRP}(I^\lambda, T_{\lambda, \pi})$, it follows that $(I^\lambda, T_{\lambda, \pi}, \text{Leb})$ satisfies (MRP). □

**Remarks.** (a) The astute reader may point out that we have defined (MRP) only for homeomorphisms, and an interval exchange transformation is in general discontinuous. This, however, can be remedied in two ways. The first is to pass to a symbolic setting, where we code an orbit by the sequence of the exchanged intervals it hits. The standard shift transformation on this sequence space over a finite alphabet (of cardinality $m$) is then a homeomorphism and we can work in this representation; compare, for example, [27, Section 5]. Notice that in the strictly ergodic situation, the spectral consequences for the associated Schrödinger operator family are the same because the unique invariant measure in the symbolic setting is the push-forward of Lebesgue measure. The other way to circumvent this issue is to extend our result relating (MRP) to the almost sure absence of eigenvalues for the associated Schrödinger operators to certain discontinuous maps $T$.

(b) In light of the previous remark, it is interesting to point out the following. The
metric repetition property (MRP) is not invariant under the passage to the symbolic setting. Indeed, while we showed that every irrational rotation of the circle obeys (MRP), its symbolic counterpart (a Sturmian sequence, resulting from the symbolic coding of an exchange of two intervals) satisfies (MRP) if and only if the rotation number has unbounded partial quotients; see, for example, [9, 23, 30].

4.2. Uniformity in the Coupling Constant. One often introduces a coupling constant $\lambda$ and considers potentials of the form

$$V_{\omega}(n) = \lambda f(T^n \omega)$$

instead of (2). Since the regimes of small and large couplings can be regarded as small perturbations of simple models (the free Laplacian and a diagonal matrix, respectively), it is of especial interest to explore whether the spectral type of the limit model extends to the perturbation.

In this context it should be noted that if $V_{\omega}$ of the form (15) is a Gordon potential for one value of $\lambda$, it is a Gordon potential for all values of $\lambda$. In particular, the results on the absence of point spectrum above extend from potentials of the form (2) covered by our work to all potentials of the more general form (15).

4.3. Generic Singular Continuous Spectrum. Our work is closely related in spirit to the paper [2] by Avila and Damanik. It follows from [2] that there exists a residual set $F_{\text{sing}} \subseteq C(\Omega)$ such that for every $f \in F_{\text{sing}}$, the operator (11) with potential (2) has purely singular spectrum for almost every $\omega \in \Omega$, and moreover, for Lebesgue almost every $\lambda \in \mathbb{R}$, the operator (11) with potential (15) has purely singular spectrum for almost every $\omega \in \Omega$.

It is well known that discrete one-dimensional Schrödinger operators with periodic potentials have purely absolutely continuous spectrum. Thus, the results obtained in [2] and the present paper are rather strong implementations of the philosophy that absolutely continuous spectrum requires the presence of perfect repetition, while a rather weak repetition property already ensures the absence of eigenvalues.

Explicitly, the following consequence is obtained by combining the results on the absence of eigenvalues and the results on the absence of absolutely continuous spectrum.

**Theorem 7.** Suppose that $(\Omega, T, \mu)$ satisfies (MRP). Then, for $f$'s from a residual subset of $C(\Omega)$, the operator (11) with potential (2) has purely singular continuous spectrum for $\mu$ almost every $\omega \in \Omega$, and moreover, the operator (11) with potential (15) has purely singular continuous spectrum for $\mu$ almost every $\omega \in \Omega$ and Lebesgue almost every $\lambda \in \mathbb{R}$.

Recall that by Theorems 3 and 4, Theorem 7 applies to all minimal shifts of the form (3) and minimal skew-shifts of the form (4), where $\alpha$ is not badly approximable (which is satisfied by Lebesgue almost every $\alpha$).

4.4. A Remark on One-Frequency Quasi-Periodic Models. Here we discuss a technical point that appears to be remarkable in comparison with earlier studies of Gordon potentials generated by shifts on $\mathbb{T}$ (a.k.a. rotations of the circle). There are three truly distinct classes of sampling functions that lead to technically very different theories on the level of Schrödinger operators: piecewise constant functions with finitely many discontinuities (a.k.a. codings of rotations), continuous functions,
and smooth functions. Very roughly speaking, one-frequency quasi-periodic models with piecewise constant sampling functions seem to have, as a rule, purely singular continuous spectrum, whereas one-frequency quasi-periodic models with smooth sampling functions seem to have purely absolutely continuous spectrum for small coupling and pure point spectrum for large coupling; see, for example, [15, 17, 22] and references therein.

Continuous sampling functions fall between these two classes and in some sense, it is natural to use an approximation of a given continuous function from either side (i.e., by piecewise continuous functions or by smooth functions) in the study of such potentials. While it had been expected that models with continuous sampling function behave similarly to models with smooth sampling functions, recent work (especially [2] and the present paper) has shown that in fact they behave generically like models with piecewise continuous step functions.

The technical point we would like to make is the following. While [2] indeed used approximation by piecewise constant functions as a key tool in the proof, in this paper we did not! We were able to show a very general result: by Theorems 2 and 3 we see that for any irrational shift on $T$, a generic continuous sampling will almost surely generate a Gordon potential. It is clear from the proof that much stronger Gordon-type conditions (with more repetitions and smaller error estimates) can be obtained in the same way. This is surprising in the case of a badly approximable $\alpha$. It can be shown, [9], that for any piecewise constant function with finitely many discontinuities (that is not globally constant), such Gordon-type repetition properties do not hold for badly approximable $\alpha$. Consequently, the general result just described cannot be obtained by approximation with piecewise constant functions. Of course, for smooth functions, badly approximable $\alpha$ do not generate Gordon potentials either. Thus, for shifts on $T$ by a badly approximable $\alpha$, the generic Gordon property in the continuous category is a novel feature.

4.5. Some Remarks on the Repetition Properties. It is a natural question whether there are any non-trivial relations between the topological, metric, and global repetition properties. For example, as was pointed out earlier, in all the examples we have considered up to this point in the present paper, the properties (TRP), (MRP), and (GRP) either hold or fail simultaneously, so one could ask if there is a general principle. The following result answers this question.

**Proposition 1.** (a) There are strictly ergodic examples of $(\Omega, T)$ that satisfy (MRP) but not (GRP).
(b) There are strictly ergodic examples of $(\Omega, T)$ that satisfy (TRP) but not (MRP).

**Proof.** (a) The following general result holds: If $A$ is a finite set, $T$ is the standard shift-transformation on the sequence space $A^\mathbb{Z}$, and $\Omega$ is a closed, $T$-invariant subset of $A^\mathbb{Z}$ (a so-called subshift) that contains points that are not $T$-periodic, then $(\Omega, T)$ does not satisfy (GRP). This can be derived from [3, Proposition 2.1]. In particular, strictly ergodic examples can be constructed in this way.

To describe an explicit example, take $\alpha$ with unbounded partial quotients and consider the natural coding of the shift by $\alpha$ on $T$. The resulting Sturmian subshift satisfies (MRP) by [3, 29] and it does not satisfy (GRP) by the general result just

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4There, only minimal aperiodic subshifts are considered. If $(\Omega, T)$ is not minimal, we can just choose a minimal component of the given subshift and find an orbit without repetition property there.
(b) Here we only describe a procedure that generates the desired examples; we refer the reader to [10], where complete proofs and a much more detailed discussion can be found. Consider skew-products in the spirit of (4) on higher dimensional tori. That is, let $\Omega = \mathbb{T}^d$ and let, for example, $T : \Omega \rightarrow \Omega$ be given by

$$T(\omega_1, \omega_2, \omega_3, \ldots, \omega_d) = (\omega_1 + \alpha, \omega_1 + \omega_2, \omega_1 + \omega_2 + \omega_3, \ldots, \omega_1 + \cdots + \omega_d),$$

where $\alpha$ is irrational. Then, $(\Omega, T)$ is minimal and normalized Lebesgue measure is the unique invariant probability measure; see, for example, [24]. If $d \geq 4$, $(\Omega, T, \text{Leb})$ does not satisfy (MRP) for any $\alpha$. On the other hand, if $\alpha$ is sufficiently well approximated by rational numbers, $(\Omega, T)$ does satisfy (TRP). Again, see [10] for proofs of these two statements in a more general context.

We conclude with a brief discussion of how the repetition properties relate to notions of entropy. The following result establishes a connection:

**Proposition 2.** Suppose $(\Omega, T)$ is a subshift over a finite set $A$ and $\mu$ is an ergodic probability measure. If $(\Omega, T, \mu)$ satisfies (MRP), then it has metric entropy zero.

**Proof.** This follows from [31]; see also [8]. □

In view of Theorem 2, Proposition 2 is nicely in line with the general expectation that positive entropy strongly suggests positive Lyapunov exponents and localization for the associated Schrödinger operators, at least in an almost-everywhere sense. It would be interesting to see whether there is a topological analogue. That is, is it true that the validity of (TRP) implies zero topological entropy? Moreover, what conclusions (in the spirit of entropy) can be drawn when a dynamical system satisfies (GRP)?

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DEPARTMENT OF MATHEMATICS, RICE UNIVERSITY, HOUSTON, TX 77005, USA
E-mail address: michael@rice.edu
DEPARTMENT OF MATHEMATICS, RICE UNIVERSITY, HOUSTON, TX 77005, USA
E-mail address: damanik@rice.edu