Radiation reaction, renormalization and conservation laws in six-dimensional classical electrodynamics

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Abstract

A self-action problem for a point-like charged particle arbitrarily moving in flat space-time of six dimensions is considered. A consistent regularization procedure is proposed which relies on energy-momentum and angular momentum balance equations. Structure of the angular momentum tensor carried by the retarded ”Liénard-Wiechert” field testifies that a point-like source in six dimensions possesses an internal angular momentum. Its magnitude is proportional to the square of acceleration. It is the so-called rigid relativistic particle; its motion is determined by the higher-derivative Lagrangian depending on the curvature of the world line. It is shown that action functional contains, apart from usual ”bare” mass, an additional renormalization constant which corresponds to the magnitude of ”bare” internal angular momentum of the particle.

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1 Introduction

Recently [1, 2], there has been considerable interest in renormalization procedure in classical electrodynamics of a point particle moving in flat space-time of arbitrary dimensions. The main task is to derive the analogue of the well-known Lorentz-Dirac equation [3]. The Lorentz-Dirac equation is an equation of motion for a charged particle under the influence of an external force as well as its own electromagnetic field. (For a modern review see [4, 5, 6].) In earlier paper [7] the Lorentz-Dirac equation in six dimensions is obtained via the consideration of energy-momentum conservation.

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All the authors [1, 2, 7] deal with an obvious generalization of the standard variational principle used in four dimensions

\[ I = I_{\text{particle}} + I_{\text{int}} + I_{\text{field}}, \]  

(1.1)

with

\[ I_{\text{field}} = \frac{1}{4\Omega_{D-2}} \int d^D y F^{\mu\nu} F_{\mu\nu}, \quad I_{\text{particle}} = -m \int d\tau \sqrt{-z'^2}, \]  

(1.2)

and the interaction term given by

\[ I_{\text{int}} = e \int d\tau A_\mu \dot{z}^\mu. \]  

(1.3)

By \( \Omega_{D-2} \) the area of a \((D-2)\)-dimensional sphere of unit radius is denoted:

\[ \Omega_{D-2} = 2\pi^{(D-1)/2} \frac{\Gamma\left(\frac{D-1}{2}\right)}{\Gamma(D-1)}. \]  

(1.4)

Strictly speaking, the action integral (1.1) may be used to derive trajectories of the test particles, when the field is given \textit{a priori}. It may also be used to derive \( D \)-dimensional Maxwell equations, if the particle trajectories are given \textit{a priori}. Simultaneous variation with respect to both field and particle variables is incompatible since the Lorentz force will always be ill defined in the immediate vicinity of the particle’s world line.

The elimination of the divergent self-energy of a point charge is the key to the problem. In four-dimensional space-time one usually assumes that the parameter \( m \) involving in \( I_{\text{particle}} \) is the unphysical bare mass. It absorbs the inevitable infinity within the renormalization procedure and becomes the observable rest mass of the particle. In \( D \) dimensions the Coulomb potential of a charge scales as \(|x|^{3-D} \) [8]. Inevitable infinities arising in higher-dimensional electrodynamics are stronger than in four dimensions.

All the authors [1, 2, 7] agree, that in even dimensions higher than four divergences cannot be removed by the renormalization of mass included in the initial action integral (1.1). To make classical electrodynamics in six dimensions a renormalizable theory, in [7] the six-dimensional analogue of the relativistic particle with rigidity [10, 11, 16] is substituted for the structureless point charge whose action term is proportional to worldline length. Corresponding Lagrangian involves, apart from usual "bare mass", an additional regularization constant which absorbs one extra divergent term. In [2] the procedure of regularization in any dimensions is elaborated. It allows to remove the infinities coming from the particle’s self-action by introducing new counterterms in the particle action.

On the contrary, Gal’tsov [1] states that the theory is nonrenormalizable in dimensions higher than four. Introduction of higher derivatives in the particle action term seems for him not reasonable enough.

In the present paper the problem of renormalizability will be reformulated within the problem of Poincaré invariance of a closed particle plus field system. The conservation laws are an immovable fulcrum about which tips the balance of truth regarding renormalization and radiation reaction. Either nonrenormalizable theory or renormalizable one should be compatible with the Poincaré symmetry.
In [12] a strict geometrical sense of divergent terms arising in four dimensions is made. Teitelboim calculates how many energy-momentum of the retarded electromagnetic field of an arbitrarily moving point charge flows across a thin tube around the world line. The energy-momentum contains two quite different terms: (i) the bound part which is permanently "attached" to the charge and is carried along with it; (ii) the radiation part detaches itself from the charge and leads an independent existence. The former is divergent while the latter is finite (the integral of the Larmor relativistic rate of radiated energy-momentum over particle's world line is meant). Hence, a charged particle can not be separated from its bound electromagnetic "cloud", so that the four-momentum of the particle is the sum of the mechanical momentum and the electromagnetic bound four-momentum. The electromagnetic part contains, apart from divergent self-energy which is linked with the bare mass, also a finite structureless term which is proportional to the particle acceleration. On rearrangement, Teitelboim's expression for the four-momentum of accelerated point-like charge looks as follows

\[ p_{\text{part}}^\mu = m u^\mu - \frac{2}{3} e^2 a^\mu, \]

where \( m \) is already renormalized rest mass.

Similar decomposition of the angular momentum tensor of the retarded Liénard-Wiechert field into the bound and the radiative components is performed in [13]. It is shown, that the bound electromagnetic "cloud" possesses its own angular momentum. It has precisely the same form as the mechanical angular momentum of a "bare" charge. Therefore, the regularization of angular momentum can be reduced to the renormalization of mass.

Conserved quantities place stringent requirements on the dynamics of the system. They demand that the change in radiative energy-momentum and angular momentum should be balanced by a corresponding change in the already renormalized momentum and angular momentum of the particle. It is shown [14] that energy-momentum balance equation gives the relativistic generalization of Newton’s second law where loss of energy due to radiation is taken into account. The angular momentum balance equation explains how four-momentum of charged particle depends on its velocity and acceleration (see eq.(1.5)). So, a careful analysis with the use of regularization procedure compatible with the Poincaré symmetry leads to the Lorentz-Dirac equation in four-dimensional case.

In this paper we calculate the energy-momentum and angular momentum of the retarded electromagnetic field generated by a point-like charge in six dimensions. The form of (divergent) bound parts reveals the structure of a "bare" core "dressed" in the electromagnetic cloud. Does a consistent classical electrodynamics in spacetimes of dimensions \( D > 4 \) lead inevitably to the rigid particle? If so, that the bound characteristics possess the specific features, e.g. the internal angular momentum.

Further we analyse the radiative parts. We see that the energy-momentum and angular momentum balance equations allow us to establish the radiation reaction force in four dimensions [14]. Does a consistent classical electrodynamics in spacetimes of dimensions \( D > 4 \) lead inevitably to the rigid particle? If so, that the Poincaré conservation laws give the corresponding radiation reaction force.
2 General setting

Let \( M_6 \) be 6-dimensional Minkowski space with coordinates \( y^\mu \) and metric tensor \( \eta_{\mu\nu} = \text{diag}(-1,1,1,1,1,1) \). We use the natural system of units with the velocity of light \( c = 1 \). Summation over repeated indices is understood throughout the paper; Greek indices run from 0 to 5, and Latin indices from 1 to 5.

We consider an electromagnetic field \( F_{\alpha\beta} \) produced by a point-like particle of charge \( e \). The particle moves in flat space-time \( M_6 \) on an arbitrary world line

\[
\zeta : \mathbb{R} \to M_6 \\
u \mapsto (z^\mu(u)),
\]

where \( u \) is proper time. The Maxwell field equation are

\[
F^{\alpha\beta},\beta = \frac{8\pi^2}{3}j^\alpha\tag{2.2}
\]

where current density \( j^\alpha \) is given by

\[
j^\alpha = e\int d\tau u^\alpha(u)\delta(y - z(u)).\tag{2.3}
\]

\( u^\alpha(u) \) denotes the (normalized) six-velocity vector \( dz^\alpha(u)/du \) and the factor \( 8\pi^2/3 \) is the area of 4-dimensional unit sphere embedded in \( M_6 \) (see eq.(1.4) for \( D = 6 \)).

We express the electromagnetic field in terms of a vector potential, \( \hat{F} = d\hat{A} \). In the Lorentz gauge \( A^\alpha,\alpha = 0 \) the Maxwell field equations become

\[
\Box A^\alpha(y) = -\frac{8\pi^2}{3}j^\alpha(y)\tag{2.4}
\]

where \( \Box := \eta_{\alpha\beta}\partial_\alpha\partial_\beta \) is the wave operator. Using the retarded Green function [1, eq.(3.4)] associated with the D’Alembert operator \( \Box \) and the charge-current density vector (2.3) we construct the retarded Liénard-Wiechert potential in six dimensions:

\[
A^\mu(y) = e\int d\sigma u^\mu(u)\left(\frac{1}{2\pi R} \frac{d}{dR} \frac{\delta(T - R)}{R}\right).\tag{2.5}
\]

Here \( T := y^0 - z^0(u) \) and \( R := |y - z(u)| \).

We suppose that the dynamics of our composite particle plus field system is governed by the conservation laws which arise from the invariance of the closed system under time and space translations as well as space and mixed space-time rotations. The components of momentum 6-vector carried by the electromagnetic field are [7]

\[
p^\nu_{\text{em}}(\tau) = P \int_{\Sigma} d\sigma\mu T^{\mu\nu}\tag{2.6}
\]

where \( d\sigma\mu \) is the vectorial surface element on an arbitrary space-like hypersurface \( \Sigma \). The components of the electromagnetic field’s stress-energy tensor

\[
\frac{8\pi^2}{3}T^{\mu\nu} = F^{\mu\lambda}F_{\nu\lambda} - 1/4\eta^{\mu\nu}F^{\kappa\lambda}F_{\kappa\lambda}\tag{2.7}
\]
Figure 1: Integration region considered in the evaluation of the bound and emitted conserved quantities produced by all points of the world line up to the end point \( E \). Retarded spheres \( S(z(u), r), u \in [-\infty, \tau] \), of constant radii \( r \) constitute a thin world tube \( \Sigma_r \) enclosing the world line \( \zeta \). The sphere \( S(z(u), r) \) is the intersection of the future light cone with vertex at point \( z^\mu(u) \in \zeta \) and \( r \)-shifted hyperplane \( \Sigma(z(u), r) \) which is orthogonal to six-velocity \( u^\mu(u) \).

have singularities on a particle trajectory (2.1). In eq.(2.6) capital letter \( P \) denotes the principal value of the singular integral, defined by removing from \( \Sigma \) an \( \varepsilon \)-sphere around \( 0 \) and then passing to the limit \( \varepsilon \rightarrow 0 \).

The angular momentum tensor of the electromagnetic field is written as [4]

\[
M_{\text{em}}^{\mu
u}(\tau) = P \int_{\Sigma} d\sigma_\alpha (y^\mu T^\alpha\nu - y^\nu T^\alpha\mu).
\] (2.8)

Kosyakov [7] calculates the radiative part of the energy-momentum (2.6) which flows across a world tube of constant radius \( r \) enclosing the world line \( \zeta \). This integration hypersurface, say \( \Sigma_r \), is a disjoint union of (retarded) spheres of constant radii \( r \) centered on a world line of the particle (see Fig.1). The sphere \( S(z(u), r) \) is the intersection of future light cone generated by null rays emanating from \( z(u) \in \zeta \) in all possible directions

\[
C(z(u)) = \{ y \in M_6 : (y^0 - z^0(u))^2 = \sum_i (y^i - z^i(u))^2, y^0 - z^0(u) > 0 \},
\] (2.9)

and tilted hyperplane

\[
\Sigma(z(u), r) = \{ y \in M_6 : u_\alpha(u)(y^\alpha - z^\alpha(u) - u^\alpha(u)r) = 0 \}.
\] (2.10)
Figure 2: Integration region considered in the evaluation of the energy-momentum and angular momentum produced by the upper segment EA of the world line. Corresponding electromagnetic field conserved quantities flow across the cap covering the top of world tube $\Sigma_r$. The cap is the part of the observation hyperplane $\Sigma(z(\tau), r)$. Point E has coordinates $(z^\mu(\tau))$ and point $A = \zeta \cup \Sigma(z(\tau), r)$.

Integration surface $\Sigma_r$ is time-like. For the sake of completeness, the narrow cylindrical tube surrounding the world line should be covered by a space-like "cap" (see Fig.2). To evaluate the part of energy-momentum and angular momentum produced by the upper segment EA of the world line, we integrate (2.6) and (2.8) over space-like hyperplane $\Sigma(z(\tau), r)$. For the sake of simplicity we make such a Lorentz transformation that this tilted hyperplane becomes $\Sigma_{\tau'} = \{ y \in \mathbb{M}_6 : y^{0'} = t' \}$. The Lorentz matrix, $\Lambda(\tau)$, determines the transformation to the particle’s momentarily comoving Lorentz frame (MCLF) where the particle is momentarily at rest at observation instant $\tau$. On rearrangement, energy-momentum (2.6) and angular momentum (2.8) take the form

$$p_{\text{em}}^{\nu}(\tau) = \Lambda^{\nu}_{\nu'}(\tau) P \int_{\Sigma_{\tau'}} d\sigma_{0'} T^{0'\nu'}$$  \hspace{1cm} (2.11)$$
$$M_{\text{em}}^{\mu
u}(\tau) = \Lambda^{\mu}_{\mu'}(\tau)\Lambda^{\nu}_{\nu'}(\tau) P \int_{\Sigma_{\tau'}} d\sigma_{0'} \left( y^{\mu'} T^{0'\nu'} - y^{\nu'} T^{0'\mu'} \right).$$  \hspace{1cm} (2.12)$$

3 Coordinate systems

An appropriate coordinate system is a very important for the integration. We use an obvious generalization of a coordinate system centered on an accelerated world line\cite{17, 5}. The set of curvilinear coordinates for flat spacetime $\mathbb{M}_6$ involves the retarded time $u$ and the retarded distance $r$ introduced in previous section (see eqs.(2.9) (2.10)). To understand the situation more thoroughly, we pass to the MCLF where the particle is momentarily
Figure 3: In MCLF the retarded distance is the distance between any point on spherical light front $S(0, r) = \{y \in \mathbb{M}_6 : (y^0)'^2 = \sum_i (y^i)'^2, y^0' = r > 0\}$ and the particle. The charge is placed in the coordinate origin; it is momentarily at the rest. The point $C \in S(0, r)$ is linked to the coordinate origin by a null ray characterized by the angles $\vartheta^A$ specifying its direction on the cone. The vector with components $n^\alpha'$ is tangent to this null ray.

at the rest at the retarded time $u$. The retarded distance $r$ is the distance between an observer event $C \in \Sigma_r$ and the particle, as measured at $u$ in the MCLF (see Fig.3). Points on the sphere $S(0, r) \subset \Sigma_r$ are distinguished by four spherical angles $(\phi, \vartheta_1, \vartheta_2, \vartheta_3)$:

\[
y^0' = r, \quad y^1' = r \cos \phi \sin \vartheta_1 \sin \vartheta_2 \sin \vartheta_3 \\
y^2' = r \sin \phi \sin \vartheta_1 \sin \vartheta_2 \sin \vartheta_3 \\
y^3' = r \cos \vartheta_1 \sin \vartheta_2 \sin \vartheta_3 \\
y^4' = r \cos \vartheta_2 \sin \vartheta_3 \\
y^5' = r \cos \vartheta_3 
\] (3.1)

In the laboratory frame the points on this sphere have the following coordinates:

\[
y^\alpha = z^\alpha(u) + r\Lambda^{\alpha \alpha'}(u)n^{\alpha'} \\
= z^\alpha(u) + rk^{\alpha} 
\] (3.2)

where $n^{\alpha'} = (1, n^i)$ is null vector with space components given by eqs.(3.1).

We see that flat space-time $\mathbb{M}_6$ becomes a disjoint union of world tubes $\Sigma_r, r > 0$, enclosing the particle trajectory (2.1). A world tube is a disjoint union of (retarded) spheres of constant radii $r$ centered on a world line of the particle. Points on a sphere are distinguished by four spherical polar angles.
If we take the integration hypersurface $\Sigma_r$ being a surface of constant $r$, these parametrization of $\mathbb{M}_6$ is suitable. If we integrate over hyperplane $\Sigma_t = \{ y \in \mathbb{M}_6 : y^0 = t \}$, we need another curvilinear coordinates. We substitute the expression $(t-u)/k^0$ for the retarded distance $r$ in eqs.(3.2). Particle trajectory now is meant as a local section

$$\zeta : \mathbb{R} \rightarrow \mathbb{M}_6 \quad u \mapsto (u, z^i(u))$$

of trivial bundle $(\mathbb{M}_6, i, \mathbb{R})$ where the projection [19]

$$i : \mathbb{M}_6 \rightarrow \mathbb{R} \quad (y^0, y^i) \mapsto y^0$$

defines the instant form of dynamics [18]. The final coordinate transformation $(y^\alpha) \mapsto (t, u, \vartheta_1, \vartheta_2, \vartheta_3, \phi)$ looks as follows:

$$y^0 = t, \quad y^i = z^i(u) + (t-u) \frac{k^i}{k^0}$$

The integration hyperplane $\Sigma_t$ is a surface of constant $t$.

Flat space-time $\mathbb{M}_6$ becomes a disjoint union of fibres $i^{-1}(t) := \Sigma_t$ of the trivial bundle (3.4). A fibre $\Sigma_t$ is a disjoint union of (retarded) spheres centered on a world line of the particle. The sphere

$$S(z(u), t-u) = \{ y \in \mathbb{M}_6 : (y^0 - u)^2 = \sum_i (y^i - z^i(u))^2, y^0 = t, t-u > 0 \}$$

is the intersection of future light cone, generated by null rays emanating from $z(u) \in \zeta$ in all possible directions, and hyperplane $\Sigma_t$. For the fixed instant $t$ the retarded time parameter $u \in ]-\infty, t[ \subset \mathbb{R}$. Points on the sphere are distinguished by spherical polar angles (see eqs.(3.1)).

4 Electromagnetic potential and electromagnetic field in six dimensions

In even space-time dimensions the Green function associated with the D’Alembert operator is localized on the light cone [1, 2]. Having integrated (2.5) we obtain

$$A_\mu = \frac{e}{2\pi r} \frac{1}{du} \left( \frac{u_\mu(u)}{r} \right) = \frac{e}{2\pi} \left[ \frac{a_\mu(u)}{r^2} + \frac{u_\mu(u)}{r^3} (1 + ra_k) \right]$$

where $a_k = a_\alpha k^\alpha$ is the component of the acceleration $a_\alpha = du_\alpha/du$ in the direction of $k^\alpha$. It is understood that in eq.(4.1), all world line quantities (such as $u_\mu$ and $a_\mu$) are to be evaluated at the retarded time $u$. 
The potential (4.1) differs from that of [7, eq.(13)] just by an overall coefficient $e/2\pi$.

The direct particle field [20] is defined in terms of this potential by $F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}$.

Having used the differentiation rule [7, eqs.(2),(3)]

$$\frac{\partial u}{\partial x^\mu} = -k_\mu, \quad \frac{\partial r}{\partial x^\mu} = -u_\mu + (1 + ra_k) k_\mu,$$

we obtain

$$F = \frac{e}{2\pi} \left( \frac{u \wedge a}{r^3} + V \wedge k \right)$$

where

$$V_\mu = \frac{3u_\mu}{r^4} + \frac{3(a_\mu + 2u_\mu a_k)}{r^3} + \frac{\dot{a}_\mu + u_\mu \dot{a}_k + 3a_\mu a_k + 3u_\mu a_k^2}{r^2}.$$ (4.4)

The overdot means the derivative with respect to retarded time $u$. Liénard-Wiechert field (4.3) coincides with the field obtained in [7, eq.(14)] where the ”mostly minus” metric signature should be replaced by the ”mostly plus” one.

## 5 Energy-momentum of the retarded Liénard-Wiechert field in six dimensions

It is straightforward to substitute the components (4.3) into eq.(2.7) to calculate the electromagnetic field’s stress-energy tensor. Following [7], we present $T^{\alpha\beta}$ as a sum of radiative and bound components,

$$T^{\alpha\beta} = T^{\alpha\beta}_{\text{rad}} + T^{\alpha\beta}_{\text{bnd}}.$$ (5.1)

The radiative part scales as $r^{-4}$:

$$\frac{8\pi^2}{3} T^{\alpha\beta}_{\text{rad}} = \frac{e^2}{4\pi^2} \frac{k^{\alpha}k^{\beta}}{r^4} V^\mu_{(-2)} V^{\mu(-2)}$$ (5.2)

where the components $V^{\mu(-2)}$ of six-vector $V_{(-2)}$ is defined by eq.(4.4). The others $T_{(-\kappa)}$ constitute the bound part of the Maxwell energy-momentum tensor density:

$$T^{\alpha\beta}_{\text{bnd}} = T_{(-8)} + T_{(-7)} + T_{(-6)} + T_{(-5)}.$$ (5.3)

(Each term has been labelled according to its dependence on the distance $r$.)

According [7], the outward-directed surface element $d\sigma_\mu$ of a five-cylinder $r = \text{const}$ in $\mathbb{M}_6$ is

$$d\sigma_\mu = [-u_\mu + (1 + ra_k) k_\mu] r^4 d\Omega_4 du$$ (5.4)

where $d\Omega_4 = d\vartheta_1 d\vartheta_2 d\vartheta_3 d\phi \sin \vartheta_1 \sin^2 \vartheta_2 \sin^3 \vartheta_3$ is the element of solid angle in five dimensions. The angular integration can be handled via the relations

$$\int d\Omega_4 = \frac{8\pi^2}{3}, \quad \int d\Omega_4 n^\alpha n^\beta = \frac{8\pi^2}{15} (\eta^{\alpha\beta} + u^\alpha u^\beta),$$

$$\int d\Omega_4 n^\alpha n^\beta n^\gamma n^\kappa = \frac{8\pi^2}{105} \left[ (\eta^{\alpha\beta} + u^\alpha u^\beta) (\eta^{\gamma\kappa} + u^\gamma u^\kappa) + (\eta^{\alpha\gamma} + u^\alpha u^\gamma) (\eta^{\beta\kappa} + u^\beta u^\kappa) + (\eta^{\alpha\kappa} + u^\alpha u^\kappa) (\eta^{\beta\gamma} + u^\beta u^\gamma) \right]$$ (5.5)
The integral of polynomial in odd powers of $n^\alpha := k^\alpha - u^\alpha$ vanishes.

We are now concerned with volume integration of (2.6). Although the surface element (5.4) contains the term which is proportional to $r$, the radiative part of electromagnetic-field six-momentum $p_{\text{rad}}$ does not depend on the distance:

$$p^\mu_{\text{rad}} = \frac{e^2}{4\pi^2} \int_{-\infty}^\tau du \left( \frac{4}{5} u^\mu \dot{a}^2 - \frac{6}{35} a^2 a^\mu + \frac{3}{7} a^\mu (a^2') + 2 a^4 u^\mu \right)$$

(We denote $(a^2)'$ the derivative $da^2/du$.) The reason is that $k^\alpha T_{\alpha\beta}^{\text{rad}} = 0$. Since $k^\alpha T_{\alpha\beta}^{\text{rad}}(-5) = 0$, this term does not produce a change in radiation flux.

Volume integration of the bound part of the stress-energy tensor over the world tube $\Sigma_r$ of constant radius $r$ reveals that the bound energy-momentum is a function of the end points only:

$$p^\mu_{\text{bnd}} = \frac{e^2}{4\pi^2} \left[ \frac{3}{2} \frac{u^\mu(u)}{r^3} + \frac{12}{5} \frac{a^\mu(u)}{r^2} + 2 \frac{a^2 w^\mu(u)}{r} \right]_{u=\tau}^{u=-\infty}$$

(The matter is that the total (retarded) time derivatives arise from angular integration.) If the charged particle is asymptotically free at the remote past, we obtain the Coulomb-like self-energy of constant value. The upper limit drastically depends on the value of $r$. If it is finite, we have no problem. If $r$ tends to zero, $p^\mu_{\text{bnd}} \to \infty$. A subtle point in this integration is that a surface of constant $r$ is time-like. The space-like "cap" should be added to close the time-like tube $\Sigma_r$ (see Fig.2). The integration over this cap results $p^\mu_{\text{bnd}}$ finally.

Equation (2.11) shows that the volume integration over tilted hyperplane (2.10) taken at the observation instant $\tau$ can be reduced to the integration over hyperplane $\Sigma_t = \{ y \in M_6 : y^0 = t \}$. Direct calculation shows that the bound six-momentum depends on the state of the particle’s motion at the end points only:

$$p^\mu_{\text{bnd}} = \int_{\Sigma_t} d\sigma_0 T^\mu_{\text{bnd}}$$

$$= \frac{e^2}{4\pi^2} \left[ \frac{3}{5} \frac{u^\mu}{(t-u)^3} \right. \right.$$

$$+ \frac{3}{35} \frac{a^\mu}{(t-u)^2}$$

$$+ \frac{1}{35} \frac{37 a^0 a^\mu + 71 a^2 u^\mu + \eta^0 \eta^\mu u^\mu}{t-u} \bigg]_{u=-\infty}^{u=t}$$

The lower limit is equal to zero while the upper one tends to infinity.

Now we turn to the calculation over a piecewise surface which consists of the cap covering the world tube $\Sigma_r$ and $\Sigma_r$ itself. The cap is a part of the observation hyperplane. It is parametrized by the curvilinear coordinates (3.5) where the time parameter $u$ changes from $z^0(\tau)$ to $t$ (from the point $E$ to the point $A$, see Fig.2). According to the integration rule (2.11) we pass to MCLF where $u^\mu(\tau) = (1,0)$ and $a^\mu(\tau) = (0,a)$ and then performs the Lorenz transformation to the laboratory frame. The expression
in between the square brackets taken in MCLF and then transformed by corresponding Lorentz matrix $\Lambda$ becomes

$$\frac{e^2}{4\pi^2} \left[ \frac{3}{2} \frac{u^\mu(\tau)}{(t-z^0(\tau))^3} + \frac{12}{5} \frac{a^\mu(\tau)}{(t-z^0(\tau))^2} + 2 \frac{a^2 u^\mu(\tau)}{(t-z^0(\tau))} \right]. \quad (5.9)$$

This expression coincides with that (5.7) where the constant distance $r$ should be replaced by $t-z^0(\tau)$. Indeed, the retarded distance $r$ is equal to this value in MCLF (see Fig.3). We see that the bound of the cap is sewn with the upper bound of the world tube smoothly.

So far as the upper limit of integration is concerned, we obtain the divergent expression where particle’s characteristics are evaluated in the neighbourhood of point $A$ while the Lorentz matrix - at point $E$ (see Fig.2). It is too complicated. The best plan to be followed is to take the limit $r \to 0$ in (5.7). It allows us to evaluate the bound part of six-momentum in the neighbourhood of the particle without resort to the cap covering this thin tube.

The radiative part (5.6) of energy-momentum carried by the retarded electromagnetic field (4.3) does not contain the distance $r$ at all. Therefore, the radiation flux across the world tube of an arbitrary radius $r$ is equal to flux which flows across a space-like surface.

### 6 Angular momentum of the retarded Liénard-Wiechert field in six dimensions

We now turn to the calculation of the angular momentum tensor

$$M_{\mu\nu} = \int_{\Sigma_r} \, d\sigma_\alpha \left( y^\mu T^{\alpha\nu} - y^\nu T^{\alpha\mu} \right). \quad (6.1)$$

We calculate how much electromagnetic field angular momentum flows across a thin world tube enclosing particle’s trajectory (2.1) up to the observation time $\tau$ (see Fig.1).

Decomposition of the angular momentum tensor density into the bound and the radiative components is a very important for the calculation. Indeed, the former accounts for the angular momentum which remains bound the charge while the latter corresponds to the amount of angular momentum which escapes to infinity.

We put (5.1) into (6.1) where right-hand side of eq.(3.2) should be substituted for $y$. It contains the term which is proportional to distance $r$. Vector surface element (5.4) also depends on $r$. In general, terms scaling as $r^2$ may appear. Since $T_{(-4)}^{\alpha\beta}$ is proportional to $k^\alpha k^\beta$ and the equality $k_\alpha T^{\alpha\beta}_{(-5)} = 0$ is fulfilled, the radiative component $M_{\mu\nu}^{rad}$ does not depend on the distance:

$$M_{rad}^{\mu\nu} = \int_{\Sigma_r} \, d\sigma_\alpha \left( z^\mu T_{rad}^{\alpha\nu} - z^\nu T_{rad}^{\alpha\mu} \right) + \int_{\Sigma_r} \, d\sigma_\alpha \left[ (y^\mu - z^\mu) T_{(-5)}^{\alpha\nu} - (y^\nu - z^\nu) T_{(-5)}^{\alpha\mu} \right]$$
\[ + \int_{-\infty}^{\tau} du \int d\Omega_4 u_k \left( k^{\mu\nu} T_{-6}^{\alpha\beta} - k^{\nu\mu} T_{-6}^{\alpha\beta} \right) \]

\[ = - \int_{-\infty}^{\tau} du \int d\Omega_4 u_\alpha \left( z^{\mu\nu} T_{-4}^{\alpha\beta} - z^{\nu\mu} T_{-4}^{\alpha\beta} \right) \]

\[ - \int_{-\infty}^{\tau} du \int d\Omega_4 u_\alpha \left( k^{\mu\nu} T_{-5}^{\alpha\beta} - k^{\nu\mu} T_{-5}^{\alpha\beta} \right) \]

\[ + \int_{-\infty}^{\tau} du \int d\Omega_4 u_k \left( k^{\mu\nu} T_{-6}^{\alpha\beta} - k^{\nu\mu} T_{-6}^{\alpha\beta} \right). \tag{6.2} \]

Having performed the angle integration we obtain

\[ M_{\mu\nu}^{\text{rad}} = \frac{e^2}{4\pi^2} \left\{ \int_{-\infty}^{\tau} du \left( z^{\mu\nu} P_{\text{rad}}^{\nu} - z^{\nu\mu} P_{\text{rad}}^{\mu} \right) \right. \]

\[ + \int_{-\infty}^{\tau} du \left[ \frac{4}{5} \left( a^\nu a^\nu - a^\mu a^\mu \right) + \frac{64}{35} a^2 \left( u^\mu a^\nu - u^\nu a^\mu \right) \right] \}

\[ \tag{6.3} \]

where \( P_{\text{rad}} \) denotes the integrand of eq.(5.6).

The remaining terms involved in the angular momentum tensor density constitute the bound part of the electromagnetic field’s angular momentum:

\[ M_{\mu\nu}^{\text{bnd}} = \int_{\Sigma_r} d\sigma_{\alpha} \left( z^{\mu\nu} T_{\text{bnd}}^{\alpha\beta} - z^{\nu\mu} T_{\text{bnd}}^{\alpha\beta} \right) \]

\[ + \int_{\Sigma_r} d\sigma_{\alpha} \left[ \left( y^{\mu} - z^{\mu} \right) T_{-8}^{\alpha\beta} - \left( y^{\nu} - z^{\nu} \right) T_{-8}^{\alpha\beta} \right] \]

\[ + \int_{\Sigma_r} d\sigma_{\alpha} \left[ \left( y^{\mu} - z^{\mu} \right) T_{-7}^{\alpha\beta} - \left( y^{\nu} - z^{\nu} \right) T_{-7}^{\alpha\beta} \right] \]

\[ + \int_{-\infty}^{\tau} du \int d\Omega_4 u_5 \left( -u_\alpha + k_\alpha \right) \left( k^{\mu\nu} T_{-6}^{\alpha\beta} - k^{\nu\mu} T_{-6}^{\alpha\beta} \right). \tag{6.4} \]

Volume integration shows that the decomposition is meaningful. Indeed, the bound angular momentum depends on the state of particle’s motion at the observation instant only:

\[ M_{\mu\nu}^{\text{bnd}} = \frac{e^2}{4\pi^2} \lim_{r \to 0} \left( z^{\mu\nu} P_{\text{bnd}}^{\nu} - z^{\nu\mu} P_{\text{bnd}}^{\mu} + \frac{12}{5} \frac{u^\mu a^\nu - u^\nu a^\mu}{r} \right). \tag{6.5} \]

By symbol \( P_{\text{bnd}} \) we mean the expression in between the squared brackets of eq.(5.7).

It is worth noting that \( M_{\mu\nu}^{\text{bnd}} \) contains, apart from the usual term of type \( z \wedge p_{\text{part}} \), also an extra term which can be interpreted as the "shadow" of internal angular momentum. It prompts that the bare "core" possesses a "spin".

## 7 Energy-momentum and angular momentum balance equations

To derive the radiation reaction force in six dimensions we study the energy-momentum and angular momentum balance equations.
We calculate how much electromagnetic-field momentum and angular momentum flow across hypersurface \( \Sigma_r \) up to the proper time \( \tau \). We can do it at a time \( \tau + \Delta \tau \). We demand that change in these quantities be balanced by a corresponding change in the particle’s ones, so that the total energy-momentum and angular momentum are properly conserved.

Expressions (5.7) and (6.5) show that a charged particle cannot be separated from its bound electromagnetic "cloud" which has its own energy-momentum and angular momentum. These quantities together with their "bare" mechanical counterparts constitute the six-momentum and angular momentum of "dressed" charged particle. We proclaim the finite characteristics as those of true physical meaning.

It would make no sense to disrupt the bonds between different powers of small parameter \( r \) in (5.7). It is sufficient to assume that a charged particle possesses its own (already renormalized) six-momentum \( p_{\text{part}} \) which is transformed as an usual six-vector under the Poincaré group. The total energy-momentum of a closed system of an arbitrarily moving charge and its electromagnetic field is equal to the sum

\[
P^\mu = p^\mu_{\text{part}} + p^\mu_{\text{rad}}
\]

where \( p_{\text{rad}} \) is the radiative part (5.6) of electromagnetic field’s energy-momentum which detaches itself from the charge and leads an independent existence.

With (6.5) in mind we assume that already renormalized angular momentum tensor of the particle has the form

\[
M_{\text{part}}^{\mu\nu} = z^\mu p_{\text{part}}^\nu - z^\nu p_{\text{part}}^\mu + u^\mu \pi_{\text{part}}^\nu - u^\nu \pi_{\text{part}}^\mu.
\]

In [10, 11, 15] the extra momentum \( \pi_{\text{part}} \) is due to additional degrees of freedom associated with acceleration involved in Lagrangian function for rigid particle.

Total angular momentum of our composite particle plus field system is written as

\[
M^{\mu\nu} = M_{\text{part}}^{\mu\nu} + M_{\text{rad}}^{\mu\nu}
\]

where \( M_{\text{rad}} \) is the radiative part (6.3) of electromagnetic field’s angular momentum which depends on all previous motion of a source. Our next task is to derive expressions which explains how six-momentum and angular momentum of charged particle depend on its velocity and acceleration etc. Via the differentiation of eq.(7.1) we obtain the following energy-momentum balance equation:

\[
\dot{p}_\mu^{\text{part}} = -\frac{e^2}{4\pi^2} \left( \frac{4}{5} u^\mu \dot{a}^2 - \frac{6}{35} \dot{a}^2 a^\mu + \frac{3}{7} a^\mu (a^2) \cdot + 2a^4 u^\mu \right).
\]

(All the particle characteristics are evaluated at the instant of observation \( \tau \).) Having differentiated (7.3) and taking into account (7.4) we arrive at the equality which does not contain \( \dot{p}_\text{part} \):

\[
\begin{align*}
&u^\mu \left( p_{\text{part}}^\nu + \pi_{\text{part}}^\nu \right) - u^\nu \left( p_{\text{part}}^\mu + \pi_{\text{part}}^\mu \right) + a^\mu \pi_{\text{part}}^\nu - a^\nu \pi_{\text{part}}^\mu \\
&= -\frac{e^2}{4\pi^2} \left[ \frac{4}{5} (a^\mu \dot{a}^\nu - a^\nu \dot{a}^\mu) + \frac{64}{35} a^2 (u^\mu a^\nu - u^\nu a^\mu) \right].
\end{align*}
\]
The rank of this system of fifteen linear equations in twelve variables $p_{\text{part}}^\alpha + \dot{\pi}_{\text{part}}^\alpha$ and $\pi_{\text{part}}^\alpha$ is equal to nine. It is convenient to rewrite them as follows

$$u \wedge (p + \dot{\pi}) + a \wedge \pi = -\frac{e^2}{4\pi^2} \left[ \frac{4}{5} a \wedge \dot{a} + \frac{64}{35} a^2 u \wedge a \right]$$  \hspace{1cm} (7.6)$$

where symbol $\wedge$ denotes the wedge product. Hence one has again

$$u \wedge (p + \dot{\pi} + \frac{e^2}{4\pi^2} \frac{64}{35} a^2 a) + a \wedge (\pi + \frac{e^2}{4\pi^2} \frac{4}{5} a) = 0.$$  \hspace{1cm} (7.7)$$

Their solutions involve three arbitrary scalar functions, say $M$, $\mu$ and $\nu$:

$$p_{\text{part}}^\beta + \dot{\pi}_{\text{part}}^\beta = MU^\beta + \nu a^\beta - \frac{e^2}{4\pi^2} \frac{64}{35} a^2 a^\beta \hspace{1cm} (7.8)$$

$$\pi_{\text{part}}^\beta = \mu a^\beta + \nu u^\beta - \frac{e^2}{4\pi^2} \frac{4}{5} \dot{a}^\beta \hspace{1cm} (7.9)$$

Scrupulous analysis of their consistency with six first-order differential equations (7.4) reveals, that six-momentum of charged particle contains two (already renormalized) constants (see Appendix A):

$$p_{\text{part}}^\beta = mu^\beta + \mu \left(-\dot{a}^\beta + \frac{3}{2} a^2 u^\beta\right) + a^2 \left[ \frac{4}{5} \dot{a}^\beta - \frac{8}{5} u^\beta (a^2) - \frac{64}{35} a^2 a^\beta \right]. \hspace{1cm} (7.10)$$

The first, $m$, looks as a rest mass of the charge. But the true rest mass is identical to the scalar product of the six-momentum and six-velocity [8]. Since the scalar product depends on the square of acceleration as well as its time derivative

$$m_0 = -(p_{\text{part}} \cdot u) = m + \mu \frac{e^2}{4\pi^2} \frac{2}{5} (a^2). \hspace{1cm} (7.11)$$

the renormalization constant $m$ is formal parameter and its physical sense is not clear.

The second, $\mu$, is intimately connected with the wedge product $u \wedge \pi_{\text{part}} := s_{\text{part}}$. With eq.(7.2) in mind we call

$$s_{\text{part}}^{\alpha\beta} = \mu \left(u^\alpha a^\beta - u^\beta a^\alpha\right) - \frac{e^2}{4\pi^2} \frac{4}{5} \left(u^\alpha \dot{a}^\beta - u^\beta \dot{a}^\alpha\right) \hspace{1cm} (7.12)$$

the internal angular momentum of the particle. But its magnitude is not constant:

$$s^2 = -\frac{1}{2} s_{\alpha\beta}^{\text{part}} s_{\text{part}}^{\alpha\beta} = \mu^2 a^2 + \mu \frac{e^2}{5\pi^2} (a^2) + \frac{e^4}{25\pi^4} (\dot{a}^2 + a^4). \hspace{1cm} (7.13)$$

Therefore, this name can not be understand literally.

Having substituted the right-hand side of eq.(7.10) for the particle’s six-momentum in eq.(7.4) we derive the Lorentz-Dirac equation of motion of a charged particle under the influence of its own electromagnetic field. The problem of including of an external device requires careful consideration.
When considering the system under the influence of an external device the time derivative $\dot{P}$ of total momentum $P$ is equal to external force $F_{\text{ext}}$. It changes the energy-momentum balance equation (7.4) as follows:

$$
\dot{p}_\text{part} + \frac{e^2}{4\pi^2} \left( \frac{4}{5} u^\mu a^2 - \frac{6}{35} a^2 \ddot{a}^\mu + \frac{3}{7} a^\mu (a^2) \dot{a}^\mu + 2a^4 u^\mu \right) = F_\mu^{\text{ext}}. \quad (7.14)
$$

Corresponding change of the total angular momentum $M^{\mu\nu}$ is defined by an external torque:

$$
\dot{M}^{\mu\nu} = z^\mu F_\nu^{\text{ext}} - z^\nu F_\mu^{\text{ext}}. \quad (7.15)
$$

Expression (7.10) was firstly obtained by Kosyakov in [7, eq. (37)]. The derivation is based upon consideration of energy-momentum conservation only. The author constructs an appropriate Schott term to ensure the orthogonality of the radiation reaction force to the particle six-velocity.

8 Conclusions

Our consideration is founded on the field and the interaction terms of the action (1.1). They constitute the action functional which governs the propagation of the electromagnetic field produced by a moving charge (i.e. the Maxwell equations with point-like source).

A surprising feature of the study of Poincaré-invariance of the dynamics of a closed particle plus field system is that the conservation laws determines the form of particle’s individual characteristics such as the momentum and the angular momentum. The matter is that a charged particle cannot be separated from its bound electromagnetic "cloud" which has its own momentum and angular momentum. These quantities together with corresponding characteristics of bare "core" constitute the momentum and angular momentum of "dressed" charged particle.

So, in four dimensions the momentum of a bare "core" is proportional to its four-velocity. An electromagnetic "cloud" renormalizes the bare mass and adds the term which is proportional to the four-acceleration (see eq.(1.5)). The extra term can be obtained from the angular momentum balance equation [14]. In six dimensions a bare charge should possess (non-conventional) internal angular momentum

$$
s_0^{\mu\nu} = \mu_0 (u^\mu a^\nu - u^\nu a^\mu) \quad (8.1)
$$

with magnitude which is proportional to the square of acceleration (see eq.(7.13)). Its six-momentum is not proportional to six-velocity:

$$
\dot{p}_0^\mu = m_0 u^\mu + \mu_0 \left( -\ddot{a}^\mu + \frac{3}{2} a^2 u^\mu \right) \quad (8.2)
$$

The energy-momentum and angular momentum balance equations give the six-momentum (7.10) of "dressed" charged particle which coincides with that obtained in [7].
We see that the particle part of initial action integral (1.1) which is proportional to the worldline length is inconsistent with the others $I_{\text{field}}$ and $I_{\text{int}}$. The action functional based on the higher-derivative Lagrangian for a "rigid" relativistic particle [10, 11, 15, 16] should be substituted for $I_{\text{particle}}$ in (1.2). It is sufficient to renormalize all the divergences arising in six-dimensional electrodynamics (these connected with bound six-momentum and those associated with bound angular momentum of the electromagnetic field). The variation of modified action with respect to particle variables results the appropriate equation of motion of a charged particle in response to the electromagnetic field.

It is interesting to consider the motion of test particles (i.e. point charges which themselves do not influence the field). In four dimensions the limit $e \to 0$ and $m \to 0$ with their ratio being fixed results the Maxwell-Lorentz theory of test particles. The momentum of test particle is proportional to its four-velocity, the loss of energy due to radiation is too small to be observed. In six dimensions the test particle is the rigid particle. Its momentum is not parallel to six-velocity. The problem of motion of such particles in an external electromagnetic field is considered in [21].

The results of the paper can be extended in an arbitrary even dimensions. We can limit our calculations to the radiative parts of energy-momentum and angular momentum carried by electromagnetic field. We are sure that the energy-momentum and angular momentum balance equations allow us to establish the radiation reaction force.

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Appendix

Since $(u \cdot a) = 0$, the scalar product of particle six-velocity on the first-order time derivative of particle six-momentum (7.4) is as follows:

$$
(p_{\text{part}} \cdot u) = \frac{e^2}{4\pi^2} \left( \frac{4}{5} \dot{a}^2 + \frac{64}{35} a^4 \right).
$$

Similarly, the scalar product of particle acceleration on the particle six-momentum (7.8) is given by

$$
(p_{\text{part}} \cdot a) = \nu a^2 - \frac{e^2}{4\pi^2} \frac{64}{35} a^4 - (\dot{p}_{\text{part}} \cdot a)
$$

Summing up (A.1) and (A.2) we obtain

$$
(p_{\text{part}} \cdot u) = \nu a^2 + \frac{e^2}{4\pi^2} \frac{4}{5} \dot{a}^2 - (\dot{p}_{\text{part}} \cdot a).
$$

From the other hand, the time derivative $(p_{\text{part}} \cdot u)'$ of scalar product of particle velocity on the momentum (7.8) is written as

$$
(p_{\text{part}} \cdot u)' = -\dot{M} - (\dot{p}_{\text{part}} \cdot u)'.
$$
Subtracting (A.4) from (A.3), one has again
\[ \dot{M} = -\nu a^2 - \frac{e^2}{4\pi^2} \frac{4}{5} \dot{a}^2 - (\ddot{\pi}_{\text{part}} \cdot u). \] (A.5)

Further we calculate the scalar product of the second-order derivative of (7.9) on particle’s velocity:
\[ (\ddot{\pi}_{\text{part}} \cdot u) = -2\mu a^2 - \frac{3}{2} \mu (a^2)^\prime - \ddot{\nu} - \nu a^2 + \frac{e^2}{4\pi^2} \left( \frac{8}{5} (a^2)^\prime - \frac{4}{5} \dot{a}^2 \right). \] (A.6)

Having substituted it into previous equation we arrive at the following differential equation:
\[ \dot{M} = 2\mu a^2 + \frac{3}{2} \mu (a^2)^\prime + \ddot{\nu} - \frac{e^2}{4\pi^2} \frac{8}{5} (a^2)^\prime. \] (A.7)

It can be solved iff the scalar \( \mu \) does not change with time:
\[ M = m + \frac{3}{2} \mu a^2 + \ddot{\nu} - \frac{e^2}{4\pi^2} \frac{8}{5} (a^2)^\prime. \] (A.8)

Having substituted it into (7.8) and taking into account the time derivative of (7.9), we derive the expression (7.10) for the components of six-momentum of charged particle. It depends on two renormalization constants, \( m \) and \( \mu \).

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