Identification and control of SARS-CoV-2 epidemic model parameters  
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Abstract. We propose a mathematical model with five compartments for the SARS-CoV-2 transmission: susceptible $S$, undetected infected asymptomatic $A$, undetected infected symptomatic $I$, confirmed positive and isolated $L$, and recovered $R$, for which we have a twofold objective. First, we formulate and solve an inverse problem focusing mainly on the identification of the values $A_0$ and $I_0$ of the undetected asymptomatic and symptomatic individuals, at a time $t_0$, by available measurements of the isolated and recovered individuals at two succeeding times, $t_0$ and $T > t_0$. Simultaneously, we identify the rate standing for the average number of individuals infected in unit time by an infective symptomatic individual. Then, we propose a control problem aiming at controlling the infected classes by improving the actions in view of isolating as much as possible the populations $A$ and $I$ in the class $L$. These objectives are formulated as minimization problems, the second one including a state constraint, which are treated by an optimal control technique. The existence of optimal controllers is proved and the first order necessary conditions of optimality are determined. For the second problem, they are deduced by passing to the limit in the conditions of optimality calculated for an appropriately defined approximating problem. In this case, the dual system is singular and has a component in the space of measures. The discussion of the asymptotic stability of the system done for the case when life immunity is gained reveals an asymptotic extinction of the disease, with a well determined reproduction number.

Key words: inverse problems, control with state constraints, necessary conditions of optimality, epidemics, SARS-CoV-2

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1 Introduction

The current pandemics of COVID-19 disease caused by the severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) has undergone an accentuated exponential increase of cases all over the world. As in other transmissible diseases, the infected with SARS-CoV-2 may have in the incubation period mild forms or even no symptoms such that they can be not aware of the fact that are carrying the virus. They are known as exposed. But, the particular and the worst aspect of COVID-19 disease is that the exposed, called here asymptomatic, are highly contagious and can transmit the disease (see [5]). Hence,
the early identification of individuals infected with SARS-CoV-2 and the necessity of isolation of the people found infected is crucial for reducing the virus spread. Mathematical modeling can help to estimate some relevant parameters of this epidemic, which can allow the prediction of the disease evolution and the preparation of the necessary measures for the disease containment.

The study of various aspects of the SARS-CoV-2 has led to an extremely rich article production since the debut of the pandemics. Mathematical modelling of various aspects of the disease has been addressed as well. Many of them are based on the SIR model which describes the transmission of the disease through three stages of infection, susceptible, infected and recovered. However, taking into account the previous considerations, the SIR model cannot adequately characterize the COVID-19 disease. Several more complex models have been also considered to illustrate the spread of this disease. We cite here a few studies: a SEIR (susceptible, exposed, infectious, removed) model considering risk perception and the cumulative number of cases has been developed in [9]; a discrete-time SIR model including dead individuals was proposed in [1], a control-oriented SIR model that puts into evidence the effects of delays and compares the outcomes of different containment policies was discussed in [6]. In [13] a mathematical method was developed to deduce the evolution over time of the new coronavirus infection and to establish the effect of isolation strategies from the accumulated data, such as the number of deaths and hospitalizations. A more detailed model of transmission in Italy that extends the classic SEIR model was presented and analyzed in [8]. This model, called SIDARTHE, involves many compartments, such as: susceptible, non-life-threatening cases, asymptomatic with minor and moderate infection, symptomatic, for each of them being separate classes of detected and undetected individuals, symptomatic with a severe situation, dead and recovered. The model omits the probability rate of becoming susceptible again after having recovered from the infection. The model parameters were estimated by a best-fit approach, namely by finding the parameters that locally minimize the sum of the squares of the errors. The computations were based on data measured in Italy between the beginning of the outbreak and early April and were updated over time to reflect the progressive introduction of increased restrictions.

In this paper we introduce a mathematical model for SARS-CoV-2 epidemic, involving five compartments considered to be essential to depict the feature of the epidemic. A first goal is to approach by an optimal control technique the identification of some parameters such as the average number of individuals infected in unit time by an infected symptomatic and the number of asymptomatic and symptomatic individuals still undetected at a moment of time. These parameters can be further used to calculate the reproduction rate and are relevant for predicting the evolution of the epidemic and for planning effective control policies. These justifies the second objective in the paper which refers to the control of classes $A$ and $I$, by finding optimal control coefficients related to actions as screening, testing, tracing, which may lead to the reduction of the infected population by moving it into the isolated compartment $L$. 

1.1 Mathematical model

We introduce a mathematical model of SEIR type with five compartments, represented at time $t$ by: susceptible $S(t)$ (a healthy individual which can acquire the disease), infected asymptomatic $A(t)$ (individuals who have acquired the disease, have no symptoms or only mild ones and have been not tested yet), infected with symptoms but not confirmed yet $I(t)$, detected by tests and isolated (by hospitalization, quarantine, isolation) $L(t)$ and recovered $R(t)$. As said before, the individuals in the infected classes $A$ and $I$, meaning those not tested yet, and whose number is not known, can be in circulation and infect other people. The people tested and found infectious are supposed to be removed from circulation until healing and introduced in the class $L$. Since the identification of the parameters we propose here is considered to be performed on short intervals of time, we skip in this model the extinct (from any reason) and the newborns, because their number is negligible compared with the large number of people in the relevant compartments for this disease.

![Figure 1. Flowchart of the SAILR model](image)

Thus, we assume that both $A$ and $I$ are infectious and so the susceptible can be infected by the infective with symptoms with the rate $\beta_I(t)$, as well as by the infective asymptomatic with a rate $\beta_A(t)$. We also assume that the infective $A$ and $I$ detected by various actions are quickly isolated, in order to help the disease containment. Then, the individuals $A$ have three possibilities: can become symptomatic with the rate $\sigma$, can be detected by testing and isolated with the rate $l_A$ or can recover with the rate $\mu_A$. At their turn, the infected
symptomatic can be detected by testing and isolated with the rate \( l_I \) and can recover with the rate \( \mu_I \). The individuals isolated recover with the rate \( \mu_L \). Existing evidence (see e.g., [10]) shows that the immunity is gained for a short time only, so that the recovered can lose it after some time and go back into the susceptible compartment with the rate \( \xi(t) \). Of course, the model can be completed with other classes of population but in this paper we keep only this formulation. We call the model SAILR and depict its flowchart in Fig. 1.

The model is going be written for fractions of the total population (that is \( S, A, I, L, R \) represent the real number of the individuals in these classes divided by the total population \( N(t) \)). For the SIR model the form of the normalized system is described in the monograph [11]. There, \( \beta_I \) stands for the average number of individuals infected in unit time by an infective symptomatic. This is given by the number of contacts an infective \( I \) has in the time unit, multiplied by the probability that a contact produces an infective, when one of the two individuals is susceptible and the other is infective symptomatic. For the SAILR model, we derive in a similar way as in [11] the normalized equations, by simple calculations, but we do no longer write them. Thus, we can analogously characterize \( \beta_A \) as being the average number of individuals infected in unit time by an infective asymptomatic. The rates \( \beta_I, \beta_A \) and \( \xi \) can vary in time. For example, \( \xi \) can be zero for some time and then begin to increase, and \( \beta_I \) and \( \beta_A \) may have a periodic increasing-decreasing behavior due to the particularities of the transmission, the medical or social measures that are imposed or the variation of the virus virulence. The removal rates \( \mu_I, \mu_A \) and also \( \mu_L \) are considered constant because they generally depend on the interaction between the pathogen agents and the immune system of an infected individual (see e.g. [11]). The parameters \( \tau_I = 1/\mu_I, \tau_A = 1/\mu_A \) and \( \tau_L = 1/\mu_L \) are the average durations of the infection for an infective \( I \), an infective \( A \) (which may follow a treatment or not) and an infective isolated \( L \), respectively. The last one is supposed to receive a treatment.

The rates \( l_A \) and \( l_I \) are related to the probability rate of detection, relative to asymptomatic and symptomatic cases, respectively. They may reflect for instance the number of tests performed over the population and they can be modified by enforcing sustained actions, as for example a massive testing campaign (see [12]). The value \( l_I \) may be larger than \( l_A \), because a symptomatic individual is more likely to be tested. We assume that \( l_I \) and \( l_A \) are constant.

The values of the model parameters are used to compute the expression of the reproduction rate, denoted in this paper by \( R^0 \), which represents the average number of secondary cases produced by one infected individual introduced into a population of susceptible individuals. This is the crucial indicator in a transmissible disease. Detailed approaches of the determination of the reproduction rate in particular models are found in the literature, see e.g., [11], [7] and in the references indicated there.

Thus, the mathematical model we propose here is

\[
S'(t) = -\beta_I(t)S(t)I(t) - \beta_A(t)S(t)A(t) + \xi(t)R(t), \tag{1.1}
\]

\[
A'(t) = \beta_I(t)S(t)I(t) + \beta_A(t)S(t)A(t) - (\sigma + \mu_A + l_A)A(t), \tag{1.2}
\]

\]
\[ I'(t) = \sigma A(t) - (\mu_I + l_I)I(t), \quad (1.3) \]
\[ L'(t) = l_A A(t) + l_I I(t) - \mu_L L(t), \quad (1.4) \]
\[ R'(t) = \mu_A A(t) + \mu_I I(t) + \mu_L L(t) - \xi(t) R(t), \quad (1.5) \]

for a.a. \( t > 0 \), with the initial conditions

\[ S(0) = S_0, \ A(0) = A_0, \ I(0) = I_0, \ L(0) = L_0, \ R(0) = R_0. \quad (1.6) \]

The model is given here in a normalized form, that is the real number of individuals at time \( t \) in each class is divided to the total population \( N(t) \).

The sum of the individuals in all compartments gives the total population \( N(t) \) at time \( t \). By (1.1)-(1.5) we observe that

\[ \frac{d}{dt}(S(t) + A(t) + I(t) + L(t) + R(t)) = 0, \]

which implies that

\[ S(t) + A(t) + I(t) + L(t) + R(t) = S_0 + A_0 + I_0 + L_0 + R_0 = N, \]

for all \( t \geq 0 \), (1.7)

where the constant \( N \) is exactly the total population which, in this model, remains unchanged at all \( t \). It is clear that if each term in these sums is non-negative (as representing a fraction of population), then it is bounded by \( N \). Moreover, since we work with fractions of population, \( N \) defined before equals 1. However, we shall keep it written as \( N \) to precisely indicate where it occurs.

We assume the following conditions for the coefficients of the system:

\[ \beta_I, \ \beta_A, \ \xi \in L^\infty(0, \infty), \ \beta_I(t), \ \beta_A(t), \ \xi(t) \geq 0 \text{ a.e. } t \geq 0, \quad (1.8) \]

\[ \sigma, \ \mu_A, \ \mu_I, \ \mu_L, \ l_A, \ l_I \geq 0, \]

and denote

\[ k_1 := \sigma + \mu_A + l_A, \ k_2 := \mu_I + l_I, \ k_1 > 0, \ k_2 > 0. \quad (1.9) \]

At the end, we make a few comments on some different interpretations of the model, according to some possible modifications of the coefficients. Thus, if setting \( \xi = 0 \), and interpreting \( \mu_A, \ \mu_I \) and \( \mu_L \) as the mortality of the individuals in the classes \( A, I, L \), respectively, it follows that the class \( R \) turns out to correspond to the extinct population.

If \( l_A = l_I = \mu_L = 0 \) the class of isolated individuals disappear and so all asymptomatic and symptomatic remain in circulation. Thus, the class \( L \) is relevant as a control class for the disease containment. A larger isolation action can be modeled by larger coefficients \( l_A \) and \( l_I \).
1.2 Problem statement

We assume that two sets of measured values of the isolated and recovered people, at a time \( t = 0 \) and at a successive time \( t = T \), are available. Namely, it means that we know the nonnegative values \( L_0, R_0 \) at time \( t = 0 \) and \( L_T, R_T \), at time \( t = T \). As specified before, the number of undetected infected and of the susceptible at these times is not known, so that a first objective is to estimate \( A_0, I_0, S_0 \). As far as the parameters \( \sigma, \mu_A, \mu_I, \mu_L, l_A, l_I, \xi \) can be estimated by observations, a direct estimate of the rates \( \beta_I \) and \( \beta_A \) is less obvious. All these justify a study developed in the present paper, of identifying the rate \( \beta_I(t) \) and the number of the undetected infectious individuals \( A_0, I_0 \), relying on these available observations for the isolated and recovered people at times 0 and \( T \). Simultaneously, the number of susceptible \( S_0 \) is identified, too, because relation (1.7) implies

\[
S_0 + A_0 + I_0 = N_0 := N - (L_0 + R_0),
\]

whence \( S_0 = N_0 - (A_0 + I_0) \). Thus, it is sufficient to identify only \( A_0 \) and \( I_0 \).

Once the information about the size of the populations \( A, I \) and \( S \) is available at time 0, a prediction about their values at a further time \( t \) can be done.

The second objective is to control within a successive time interval the action of isolating more infected individuals by means of the controllers \( l_A \) and \( l_I \). More precisely, the target is to reduce the number of infected \( A \) and \( I \) by various actions which can lead to the isolation of those confirmed, by removing them from circulation and transferring in \( L \). This action is supposed however to be led such that \( L \) should not exceed an upper bound \( \hat{L} \).

These proposed objectives will be expressed by two minimization problems.

**Problem** \((P_0)\). We introduce the cost functional

\[
J(\beta_I, A_0, I_0) = \frac{1}{2} (L(T) - L_T)^2 + \frac{1}{2} (R(T) - R_T)^2 + \frac{\alpha_1}{2} \int_0^T \beta_I^2(t) dt + \frac{\alpha_0}{2} (A_0^2 + I_0^2 + (N_0 - A_0 - I_0)^2)
\]

and the minimization problem \((P_0)\) below

Minimize \( J(\beta_I, A_0, I_0) \); \( \beta_I(t) \geq 0 \) a.e. \( t \in (0, T) \), \n
\[
A_0 \geq 0, \quad I_0 \geq 0, \quad A_0 + I_0 \leq N_0, \quad A(t) \geq 0 \text{ for all } t \in [0, T]
\]

subject to (1.1)-(1.6), (1.7). Here, \( \alpha_0, \alpha_1 \) are positive constants which may give a larger or smaller weight to the terms they multiply. The constraint \( A_0 + I_0 \leq N_0 \) follows by the natural assumption that all data are should be nonnegative and so \( S_0 = N_0 - A_0 - I_0 \geq 0 \).

**Problem** \((P)\). For the second objective we introduce the cost functional

\[
J(l_A, l_I) = \frac{\alpha_0}{2} \int_0^T A^2(t) dt + \frac{\alpha_0}{2} \int_0^T I^2(t) dt + \frac{\alpha_1}{2} (l_A^2 + l_I^2)
\]

and formulate the optimal control problem

Minimize \( J \left\{ (l_A, l_I); \ l_A \in [0, 1], \ l_I \in [0, 1], \ 0 \leq L(t) \leq \hat{L} \right\} \)
subject to (1.1)-(1.6), (1.7), where $\hat{L}$ is a fixed constant, $\hat{L} > L_0$. The upper bound $\hat{L}$ for $L$ is justified by the fact that we try to catch in the class $L$ as much as possible individuals from the classes $A$ and $I$, but not the total population. The aim is to detect, by enforcing the testing, at least a part of the population $A_0 + I_0$ in order to isolate it. It should be said that the lower bound $L \geq 0$ is not a constraint because this follows from a property which will be proved for the solution to the state system.

We note that $(P)$ is an optimal control problem with the state constraint $L(t) \in [0, \hat{L}]$ which will require a more elaborated treatment. In fact, for such a problem, the maximum principle (the first order conditions of optimality) lead to a singular dual backward system. Problem $(P_0)$ is much simpler, as we shall see, and this entitles us to begin our study by approaching first problem $(P)$ and dealing after then with problem $(P_3)$. Thus, we start to solve $(P)$ by assuming that the values of $A$, $I$, $S$ at the time $T$ are calculated after solving $(P_0)$ and they become the known $A_0$, $I_0$, $S_0$ by resetting the time at 0.

We approach problem $(P)$ by an optimal control technique. In Section 2, after proving the existence and uniqueness of the solution to the state system (1.1)-(1.6), we show that there exists at least a solution $(l_A^*, l_I^*)$ to problem $(P)$ in Proposition 2.2. For this problem with state restrictions, the optimality conditions cannot be directly calculated, but via an approximating problem $(P_\varepsilon)$ indexed along a positive parameter $\varepsilon$, which contains appropriate penalized terms replacing the state constraint. This is introduced in Section 2.1. The convergence of a sequence of solutions to $(P_\varepsilon)$ precisely to a certain chosen solution to $(P)$ is proved in Proposition 2.4 and the approximating optimality conditions are provided in Proposition 2.5. Relying on appropriate estimates for the solution to the dual system proved in Proposition 2.6, local conditions of optimality for problem $(P)$ are obtained by passing to the limit in the approximating ones, in Theorem 2.8. Problem $(P_0)$ is solved in Section 3 and the optimality conditions are given in Proposition 3.2. An investigation of the asymptotic stability of the system done in Section 4 finds the conditions under which the disease can evolve towards an asymptotic equilibrium state and allows the definition of the reproduction rate. Some final interpretations in Section 5 complete the paper.

2 Problem $(P)$

We begin with the proof of the well-posedness of the state system. Its solution will be sometimes denoted by $X := (S, A, I, L, R)$.

By a solution to (1.1)-(1.6) on $[0, T]$ we mean an $\mathbb{R}^5$-valued absolutely continuous function $X = (S, A, I, L, R)$ on $[0, T]$ which satisfies (1.1)-(1.6) a.e. on $(0, T)$.

**Proposition 2.1.** Let $T > 0$ and $(S_0, A_0, I_0, L_0, R_0) \geq 0$. The state system (1.1)-(1.6) has a unique global solution $(S, A, I, L, R) \in (W^{1, \infty}(0, T))^5$. The solution is continuous with respect to the data $(l_A, l_I)$.

**Proof.** In system (1.1)-(1.5) we apply the Banach fixed point theorem, using
the set
\[ \mathcal{M} = \{ A \in C([0, T]; \ 0 \leq A(t) \leq N \text{ for all } t \in [0, T]) \}. \]
Let us pick \( a \in \mathcal{M} \) and fix it in the system
\[
\begin{align*}
S'(t) &= -\beta_I(t)S(t)I(t) - \beta_A(t)S(t)a(t) + \xi(t)R(t), \quad (2.1) \\
A'(t) &= \beta_I(t)S(t)I(t) + \beta_A(t)S(t)a(t) - (\sigma + \mu_A + l_A)A(t), \quad (2.2) \\
I'(t) &= \sigma a(t) - (\mu_I + l_I)I(t), \quad (2.3) \\
L'(t) &= l_A a(t) + l_I I(t) - \mu_L L(t), \quad (2.4) \\
R'(t) &= \mu A a(t) + \mu_I I(t) + \mu_L L(t) - \xi(t)R(t), \quad (2.5)
\end{align*}
\]
with the initial condition \( (\ref{eq:initial}) \). We define \( \Psi : \mathcal{M} \to C([0, T]) \) by \( \Psi(a) = A \) where \((S, A, I, L, R)\) is the solution to \((\ref{eq:system}), (\ref{eq:initial}), (\ref{eq:boundary})\) and show that \( \Psi(\mathcal{M}) \subset \mathcal{M} \) and that \( \Psi \) is a contraction on \( \mathcal{M} \). By \((\ref{eq:contraction})\) we deduce, using the formula of variation of constants, that
\[
I(t) = I_0 e^{-\kappa_2 t} + \sigma \int_0^t e^{-\kappa_2 (t-s)} a(s) ds, \ t \in [0, T],
\]
and we have \( I \in C([0, T]) \cap W^{1, \infty}(0, T) \). Moreover, \( I(t) \geq 0 \) for all \( t \geq 0 \). Applying successively the same formula in \((\ref{eq:system}), (\ref{eq:initial}), (\ref{eq:boundary})\) we obtain that \( L, R, S, A \in C([0, T]) \cap W^{1, \infty}(0, T) \) and each of them is nonnegative. By \((\ref{eq:contractivity})\) each component is less or equal to \( N \). Thus, \( \Psi(\mathcal{M}) \subset \mathcal{M} \). It remains to prove that \( \Psi \) is a contraction. Let us take two solutions to \((\ref{eq:system}), (\ref{eq:initial}), (\ref{eq:boundary})\) \((S, A, I, L, R)\) and \((\bar{S}, \bar{A}, \bar{I}, \bar{L}, \bar{R})\) corresponding to \( a \) and \( \bar{a} \) respectively, with the same initial condition. By calculating \((A - \bar{A})(t), (S - \bar{S})(t), ..., (R - \bar{R})(t)\) by each corresponding equation we obtain
\[
\sup_{t \in [0, T]} |(A - \bar{A})(t)| \leq C_1(T) \sup_{t \in [0, T]} |(a - \bar{a})(t)|
\]
where \( C_1(T) \) is a polynomial in \( T \) with coefficients consisting in sums of the constant systems parameters and the \( L^\infty \)-norms of the time dependent system parameters. This shows that \( \Psi \) is a contraction for small \( T \), that is we obtain a local solution. Since all solution components are bounded by \( N \) it follows that the solution is global (see e.g., \cite{3}, p. 41, Theorem 2.15).

Let \((l_A^n, l_I^n)\) be a sequence such that \( l_A^n \to l_A, l_I^n \to l_I \) as \( n \to \infty \) and let \( X^n \) and \( X \) be the solutions to \((\ref{eq:system}), (\ref{eq:initial})\) corresponding to these data, respectively. Since the solution \( X^n \in (W^{1, \infty}(0, T))^5 \) it follows that \( X^n \to X \) uniformly in \([0, T]\) and weak* in \((W^{1, \infty}(0, T))^5 \) and by passing to the limit in \((\ref{eq:system}), (\ref{eq:boundary})\) we deduce that \( \bar{X}'(t) = \lim_{n \to \infty} (X^n)'(t) \), whence it follows that \( \bar{X} \) is the solution to \((\ref{eq:system}), (\ref{eq:boundary})\).

\begin{proposition}
Problem \((P)\) has at least one solution \((l_A^*, l_I^*)\).
\end{proposition}

\begin{proof}
It is obvious that an admissible pair exists. For example, for \( l_A = l_I = 0 \) we get \( L(t) \leq L_0 < L \) and \( J(l_A, l_I) < \infty \). Let \( d := \inf J(l_A, l_I) \geq 0 \). Let us
consider a minimizing sequence \((l^n_A, l^n_I)\), satisfying the restrictions in \((P)\). The minimizing sequence also satisfies
\[
d \leq J(l^n_A, l^n_I) \leq d + \frac{1}{n}, \text{ for } n \geq 1.
\]
This implies that \(l^n_A \to l^*_A\) and \(l^n_I \to l^*_I\) as \(n \to \infty\) and \(l^*_A, l^*_I \in [0, 1]\). We denote by \(X^n\) the solution to the state system corresponding to the minimizing sequence. By Proposition 2.1 this solution exists and belong to \((W^{1,\infty}(0,T))^5\) and each component belongs to \([0, N]\). Then, on a subsequence, \(X^n \to X^*, (X^n)' \to (X^*)'\) weak* in \((L^{\infty}(0,T))^5\). Therefore, by Arzelà theorem it follows that \(X^n \to X^*\) uniformly in \([0, T]\), and so \(X^n(0) \to X^*(0) = X_0\) and \(X^n(T) \to X^*(T)\). According to the last part of Proposition 2.1, it follows that \(X^*\) is the solution to the state system corresponding to \((l^*_A, l^*_I)\). Finally, by the weakly lower semicontinuity of the norms we get \(\lim_{n \to \infty} J(l^n_A, l^n_I) = J(l^*_A, l^*_I)\) and so \((l^*_A, l^*_I)\) turns out to be optimal in \((P)\).

2.1 The approximating problem \((P_\varepsilon)\)

Let \(\varepsilon > 0\) and let \((l^*_A, l^*_I)\) be optimal in \((P)\). We introduce the adapted approximating cost functional
\[
J_\varepsilon(l_A, l_I) = \frac{\alpha_0}{2} \int_0^T (A^2(t) + I^2(t)) dt + \frac{\alpha_1}{2} (l^2_A + l^2_I) + \frac{\alpha_2}{2\varepsilon} \int_0^T \left((L(t) - \hat{L})^+\right)^2 dt + \frac{1}{2} (l_A - l^*_A)^2 + \frac{1}{2} (l_I - l^*_I)^2
\]
and study the following approximating problem \((P_\varepsilon)\),

Minimize \(\{J_\varepsilon(l_A, l_I); l_A \in [0, 1], l_I \in [0, 1]\}\)

subject to \((1.1)-(1.6), (1.7)\).

We observe that the state constraint in \((P)\) is replaced here by the penalization of the \(L^2\)-norm of the positive part of \((L(t) - \hat{L})\), where \(\alpha_2 > 0\). The last two penalization terms in \((2.6)\) ensure the convergence of the approximating solution to the chosen optimal controller \((l^*_A, l^*_I)\) in \((P)\).

Proposition 2.3. \(P_\varepsilon\) has at least one solution, \((l^*_{A,\varepsilon}, l^*_{I,\varepsilon})\) with the corresponding state \(X^*_{\varepsilon} \in (W^{1,\infty}(0,T))^5\), having the components in \([0, N]\).

Proof. First of all we see that there is at least an admissible triplet, let it be \((l^*_{A,\varepsilon}, l^*_{I,\varepsilon})\) an optimal one in \((P)\) with the corresponding global state \(X^{**}\), belonging to \((W^{1,\infty}(0,T))^5\) and \(L^{**} \leq \hat{L}\). Then, \(J_\varepsilon(l^*_{A,\varepsilon}, l^*_{I,\varepsilon}) < \infty\). Hence, the admissible set in \((P_\varepsilon)\) is not empty and since \(J_\varepsilon(l_A, l_I) \geq 0\), there exists \(d_\varepsilon := \inf J_\varepsilon(l_A, l_I) \geq 0\). We take a minimizing sequence \((l^n_{A,\varepsilon}, l^n_{I,\varepsilon})\) with the corresponding solution \(X^n_{\varepsilon}\) to the state system in the class of global solutions, with the components in the interval \([0, N]\). Recall that the initial condition
$X_0^n = X_0$ is nonnegative. We have $d_x \leq J_x(l^n_{A,x}, l^n_{I,x} \geq d + \frac{1}{n}$, for $n \geq 1$. Then, $l^n_{A,x} \to l^\ast_{A,x}, l^n_{I,x} \to l^\ast_{I,x}$ and $l^n_{A,x}, l^n_{I,x} \in [0, 1]$. By (2.6) it follows that $(L^n_\infty - \tilde{L})^+, (A^n_\infty)_n, (P^n_\infty)_n$ are bounded in $L^{2}(0, T)$ and so by (1.4), (1.5), (1.1)-(1.3) we get that $(L^n_\infty)_n, (R^n_\infty)_n, (S^n_\infty)_n, (A^n_\infty)_n, (P^n_\infty)_n$ are bounded in $W^{1, \infty}(0, T)$. We infer that $X^n_\infty \to X^\ast_\infty$ uniformly in $[0, T]$. The limit is bounded and satisfies the state system corresponding to $(l^n_{A,x}, l^n_{I,x})$. In addition we note that since $L \to L$ is continuous, we have $(L^n_\infty - \tilde{L})^+, (L^\ast_\infty - \tilde{L})^+$ uniformly in $[0, T]$ and so, \lim_{n \to \infty} J_x(l^n_{A,x}, l^n_{I,x}) = J_x(l^\ast_{A,x}, l^\ast_{I,x})$. Moreover, all components are nonnegative and less or equal to $N$. All these prove that $(l^\ast_{A,x}, l^\ast_{I,x})$ is optimal in $(P_\infty)$. \hfill \Box

**Proposition 2.4.** Let $\{(l^n_{A,x}, l^n_{I,x}), X^n_\infty\}$ and $\{(l^\ast_{A,x}, l^\ast_{I,x}), X^\ast_\infty\}$ be optimal in $(P)$ and in $(P_\infty)$, respectively. Then,

$$l^\ast_{A,x} \to l^\ast_{A}, l^\ast_{I,x} \to l^\ast_{I}, \text{ as } \varepsilon \to 0,$$

$$X_\infty \to X^\ast \text{ weak* in } (W^{1, \infty}(0, T))^5 \text{ and uniformly in } [0, T], \text{ as } \varepsilon \to 0.$$

**Proof.** If $(l^\ast_{A,x}, l^\ast_{I,x})$ is optimal, then $J_x(l^\ast_{A,x}, l^\ast_{I,x}) \leq J_x(l_{A}, l_{I})$, for all $(l_{A}, l_{I})$ satisfying the constraints in $(P)$. In particular, we can set $(l_{A}, l_{I}) = (l^n_{A,x}, l^n_{I,x})$ which is the optimal triplet chosen in $(P)$ and then the previous inequality becomes

$$J_x(l^n_{A,x}, l^n_{I,x}) = \frac{\alpha_0}{2} \int_0^T ((A^n_\infty(t))^2 + (I^n_\infty(t))^2)dt + \frac{\alpha_1}{2} (l^n_{A,x}^2 + l^n_{I,x}^2) (2.9)$$

$$+ \frac{\alpha_2}{2\varepsilon} \int_0^T ((L^n_\infty(t) - \tilde{L})^+)^2(t)dt + \frac{1}{2} (l^n_{A,x} - l^n_{A})^2 + \frac{1}{2} (l^n_{I,x} - l^n_{I})^2$$

$$\leq J_x(l^n_{A,x}, l^n_{I,x}) = \frac{\alpha_0}{2} \int_0^T ((A^n_\infty(t))^2 + (I^n_\infty(t))^2)dt + \frac{\alpha_1}{2} (l^n_{A,x}^2 + l^n_{I,x}^2),$$

because $L^\ast \leq \tilde{L}$ in $(P)$. Thus, the left-hand side is bounded and we have $l^n_{A,x} \to l^\ast_{A}, l^n_{I,x} \to l^\ast_{I}, \text{ as } \varepsilon \to 0$. By Proposition 2.1, we infer that $X_\infty \to X^\ast$ weak* in $(W^{1, \infty}(0, T))^5$ and uniformly in $[0, T]$. By (2.9) we can write

$$\frac{\alpha_2}{2\varepsilon} \int_0^T ((L^n_\infty(t) - \tilde{L})^+)^2(t)dt \leq \text{ constant},$$

which implies that $\int_0^T ((L^n_\infty(t) - \tilde{L})^+)^2(t)dt \to 0$ and so, $(L^\ast_\infty(t) - \tilde{L})^+ \to 0$ strongly in $L^2(0, T)$. On the other hand, $(L^\ast(t) - \tilde{L})^+(t) \to (L^\ast - \tilde{L})^+(t)$ for all $t \in [0, T]$, so that $L^\ast(t) \leq \tilde{L}$, which ends the proof. \hfill \Box

### 2.2 The approximating optimality conditions

Let $\lambda > 0$ and set the variations

$$l^n_{A,x} = l^n_{A,x} + \lambda \omega_A, \omega_A = \tilde{l}_A - l^n_{A,x}, \tilde{l}_A \in [0, 1],$$

$$l^n_{I,x} = l^n_{I,x} + \lambda \omega_I, \omega_I = \tilde{l}_I - l^n_{I,x}, \tilde{l}_I \in [0, 1].$$
Let us denote \( x^\epsilon := X^\epsilon_{\lambda, x}i^\epsilon_{\lambda, x} - X^* \), where \( X^\epsilon_{\lambda, x}i^\epsilon_{\lambda, x} \) is the solution to \((1.1)-(1.6)\) corresponding to \((l^\lambda_{A,e}, l^\epsilon_{l, e})\), satisfying \((1.7)\) and \( X^*_e \) is the optimal state corresponding to \((l^\lambda_{A,e}, l^\epsilon_{l, e})\).

We introduce the linearized system for problem \((P)\)

\[
\begin{align*}
  s' &= -k_{0,\epsilon} s - \beta_1 S^*_e i - \beta_A S^*_e a + \xi r, \\
  a' &= k_{0,\epsilon} s + \beta_1^* S^*_e i + k_{3,\epsilon} a - \omega_A A^*_e, \\
  i' &= \sigma a - k_{2,\epsilon} i - \omega_I e, \\
  l' &= L_{A,e} a + l_{l, e} i - \mu_l l + \omega_A A^*_e + \omega_I I^*_e, \\
  r' &= \mu_A a + \mu_I i + \mu_l l - \xi r,
\end{align*}
\] (2.10)

for a.a. \( t > 0 \), with the initial condition

\[
s(0) = 0, \quad a(0) = 0, \quad i(0) = 0, \quad l(0) = 0, \quad r(0) = 0, \quad (2.11)
\]

where

\[
k_{0,\epsilon} = \beta_A A^*_e + \beta_I I^*_e, \quad k_{1,\epsilon} = \sigma + \mu_A + l^*_{A,e}, \quad k_{2,\epsilon} = \mu_I + l^*_{l, e}, \quad k_{3,\epsilon} = \beta_A S^*_e - k_{1,\epsilon}.
\] (2.12)

First, by the known results for linear systems we infer that (2.10) has a unique global solution \( x^\epsilon = (s, a, i, l, r) \) \( (W^{1,\infty}(0, T)) \). By a direct calculation, using the continuity with respect to the data of the solution to the state system it can be easily proved that \( x^\epsilon \rightarrow x^\epsilon := (s, a, i, l, r) \) strongly in \( C([0, T]) \), as \( \lambda \rightarrow 0 \), so that (2.10) stands for the system in variations.

We introduce the backward dual system for the variables \( (p, q, d, e, f) \) as

\[
\begin{align*}
p'_e - k_{0,\epsilon} p_e + k_{0,\epsilon} q_e &= 0, \\
q'_e - \beta_A A^*_e p_e + k_{3,\epsilon} q_e + \sigma d_e + l^*_{A,e} e_e + \mu_A f_e &= -\alpha_0 A^*_e, \\
d'_e - \beta_1 S^*_e q_e - k_{2,\epsilon} d_e + l^*_{l, e} e_e + \mu_l f_e &= -\alpha_0 I^*_e, \\
\theta'_e - \mu_A e_e + \mu_l f_e + \frac{\alpha_2}{\epsilon}(L^*_e - \hat{L})^+ &= 0, \\
f'_e + \xi p_e - \xi f_e &= 0.
\end{align*}
\] (2.13)

for a.a. \( t > 0 \), with the final conditions

\[
p_e(T) = 0, \quad q_e(T) = 0, \quad d_e(T) = 0, \quad e_e(T) = 0, \quad f_e(T) = 0.
\] (2.14)

The linear system (2.13)-(2.18) has, for each \( \epsilon > 0 \), a unique global solution \( (p, q, d, e, f) \) \( (W^{1,\infty}(0, T)) \).

Let \( N_{[0,1]}(z) \) be the normal cone to the set \([0, 1] \),

\[
N_{[0,1]}(z) = \begin{cases} 
  \mathbb{R}^-, & \text{if } z = 0 \\
  0, & \text{if } z \in (0, 1), \\
  \mathbb{R}^+, & \text{if } z = 1.
\end{cases}
\]

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We recall that the projection of a point \( z \) on a set \( K \subset \mathbb{R}^d, \ d \in \mathbb{N} \), is defined by \( P_K(z) := (I_d + \kappa \partial I_K)^{-1}(z) \), for all \( \kappa > 0, \ I_d \) being the identity operator.

**Proposition 2.5.** Let \( (l_{A,\varepsilon}^*, l_{I,\varepsilon}^*) \) be optimal in \( (P_\varepsilon) \) with the state \( X_{\varepsilon}^* \). Then,

\[
\begin{align*}
l_{A,\varepsilon}^* &= P_{[0,1]} \left( \frac{1}{\alpha_1 + 1} \left( \int_0^T A_{\varepsilon}^*(q_{\varepsilon} - e_{\varepsilon}) dt + l_A^* \right) \right), \\
l_{I,\varepsilon}^* &= P_{[0,1]} \left( \frac{1}{\alpha_1 + 1} \left( \int_0^T I_{\varepsilon}^*(d_{\varepsilon} - e_{\varepsilon}) dt + l_I^* \right) \right),
\end{align*}
\]

where \((p_{\varepsilon}, q_{\varepsilon}, d_{\varepsilon}, e_{\varepsilon}, f_{\varepsilon}) \) is the solution to the backward dual system \((2.13)-(2.18)\).

**Proof.** Let us multiply the equations for \( s, a, i, l, r \) in \((2.10)\) by \( p_{\varepsilon}, q_{\varepsilon}, d_{\varepsilon}, e_{\varepsilon}, f_{\varepsilon} \), respectively and integrate over \((0, T)\). By integrating by parts and taking into account the equations in the dual system and the initial conditions in the system in variations, we obtain

\[
\begin{align*}
\alpha_0 \int_0^T (A_{\varepsilon}^* a + I_{\varepsilon}^* i) dt + \frac{\alpha_2}{\varepsilon} \int_0^T (L_{\varepsilon}^* - \hat{L})^+ ld t &= \int_0^T (\omega_A A_{\varepsilon}^*(e_{\varepsilon} - q_{\varepsilon})) + \omega_I I_{\varepsilon}^*(e_{\varepsilon} - d_{\varepsilon})) dt. \\
\end{align*}
\]

On the other hand, for \((l_{A,\varepsilon}^*, l_{I,\varepsilon}^*) \) optimal in \((P_\varepsilon)\) we can write

\[
J_{\varepsilon}(l_{A,\varepsilon}^*, l_{I,\varepsilon}^*) \geq J_{\varepsilon}(l_{A,\varepsilon}^*, l_{I,\varepsilon}^*).
\]

By replacing the expression of the cost functional \( J_{\varepsilon} \), performing some algebra, dividing by \( \lambda \) and passing to the limit as \( \lambda \to 0 \) we obtain

\[
\begin{align*}
\alpha_0 \int_0^T (A_{\varepsilon}^* a + I_{\varepsilon}^* i) dt + \alpha_1 (l_{A,\varepsilon}^* \omega_A + l_{I,\varepsilon}^* \omega_I) \\
+ \frac{\alpha_2}{\varepsilon} \int_0^T (L_{\varepsilon}^* - \hat{L})^+ ld t + (l_{A,\varepsilon}^* - l_{A}^*) \omega_A + (l_{I,\varepsilon}^* - l_{I}^*) \omega_I &\geq 0.
\end{align*}
\]

By comparison with \((2.20)\) we deduce

\[
\begin{align*}
\int_0^T \omega_A A_{\varepsilon}^*(e_{\varepsilon} - q_{\varepsilon}) dt + ((\alpha_1 + 1)l_{A,\varepsilon}^* - l_{A}^*) \omega_A \\
+ \int_0^T \omega_I I_{\varepsilon}^*(e_{\varepsilon} - d_{\varepsilon}) dt + ((\alpha_1 + 1)l_{I,\varepsilon}^* - l_{I}^*) \omega_I &\geq 0.
\end{align*}
\]

Recall the setting of \( \omega_A \) and \( \omega_I \) and choose, in particular, \( \bar{l}_A = l_{A,\varepsilon}^* \), meaning that we keep \( l_{A,\varepsilon}^* \) fixed and give a variation only to \( l_{I,\varepsilon}^* \). Then, \((2.22)\) yields

\[
\begin{align*}
\left( - \int_0^T I_{\varepsilon}^*(e_{\varepsilon} - d_{\varepsilon}) dt - (\alpha_1 + 1)l_{I,\varepsilon}^* + l_I^* \right) (l_{I,\varepsilon}^* - \bar{l}_I) &\geq 0
\end{align*}
\]
for all \( t_1 \in [0, 1] \), which implies

\[
- \int_0^T I^*_e(e - d_e)dt - (\alpha_1 + 1) I_{e, \epsilon}^* + t_1^* \in N_{[0,1]}(t^*_1), \tag{2.23}
\]

whence we have the second relation in (2.19).

Then, let us set \( t_1 = t_{I, \epsilon}^* \). By (2.22) we obtain that

\[
- \int_0^T A^*_e(e - q_e)dt - (\alpha_1 + 1) I_{e, \epsilon}^* + t_1^* \in N_{[0,1]}(t^*_1), \tag{2.24}
\]

which yields the first relation in (2.19). \( \square \)

2.3 The optimality conditions for problem \((P)\)

We begin by proving the boundedness of the solution to the dual system.

**Proposition 2.6.** There exists \( T_{loc} \), such that for \( T < T_{loc} \) we have

\[
\int_0^T |e'(t)| dt \leq C, \tag{2.25}
\]

\[
|p_z(t)| + |q_z(t)| + |l_z(t)| + |f_z(t)| \leq C, \text{ for } t \in (0, T), \tag{2.26}
\]

independently of \( \epsilon \).

**Proof.** Let us choose \( y_1 < \hat{L}, \rho > 0 \) and \( \theta \in \mathbb{R} \), with \( |\theta| = 1 \) such that \( y_1 + \rho \theta \leq \hat{L} \), where \( L_0 < \hat{L} \). Then, we have \( y_1 + \rho \theta \leq \hat{L} \) and this holds if \( \rho \leq \hat{L} - y_1 \). The value \( y_1 \) is at our free choice and we choose \( 0 < y_1 < L_0 \). We multiply (2.16) by \( (L_z^* - (y_1 + \rho \theta)) \) and integrate over \((0, T)\). We have

\[
\int_0^T (L_z^* - y_1)e_z' dt - \int_0^T \rho \theta e_z' dt + \frac{\alpha_2}{\epsilon} \int_0^T (L_z^* - \hat{L})^+(L_z^* - y_1 - \rho \theta) dt = \int_0^T (\mu L e_z - \mu L f_z)(L_z^* - y_1 - \rho \theta) dt. \tag{2.27}
\]

Let us define \( \varphi(y) = \frac{1}{2} (y + \theta)^2, \varphi : [0, \infty) \to [0, \infty] \). Its subdifferential \( \partial \varphi(y) = y^+ \) and \( \varphi(y) - \varphi(z) \leq \partial \varphi(y)(y - z) \) for all \( z \leq \hat{L} \). In the second term on the left-hand side of (2.27) we set

\[
\theta := -\frac{e_z'}{e_z'}1_{\{t : e_z'(t) \neq 0\}},
\]

where \( 1_{\{t : e_z'(t) \neq 0\}} \) is the characteristic function of the set indicated as subscript. Using the relation for the subdifferential of \( \varphi \) and integrating by parts the first term on the left-hand side of (2.27) we have

\[
(L_z^*(T) - y_1)e_z(T) - (L_z^*(0) - y_1)e_z(0) - \int_0^T (L_z^*)' e_z dt + \rho \int_0^T |e_z'(t)| dt + \frac{\alpha_2}{\epsilon} \int_0^T ((L_z^* - \hat{L})^+)^2 dt - \frac{\alpha_2}{\epsilon} \int_0^T ((y_1 + \rho \theta) - \hat{L})^+)^2 dt \leq \int_0^T (\mu L e_z - \mu L f_z) dt - \int_0^T (\mu L e_z - \mu L f_z)(y_1 + \rho \theta) dt.
\]
Since \( y_1 + \rho \theta \leq \hat{L}, e_\varepsilon(T) = 0 \), using eq. \([1.4]\) and making some rearrangements we are led to the relation

\[
\rho \int_0^T |e_\varepsilon'(t)| \, dt \leq |L_0 - y_1| |e_\varepsilon(0)| + \int_0^T (e_\varepsilon(l_{A,\varepsilon}^* A_\varepsilon^* + l_{I,\varepsilon}^* I_\varepsilon^*) - \mu_L f_\varepsilon L_\varepsilon^*) dt - \int_0^T (\mu_L e_\varepsilon - \mu_L f_\varepsilon)(y_1 + \rho \theta) dt
\]

\[
\leq \int_0^T ((|e_\varepsilon| + |f_\varepsilon|)) (l_{A,\varepsilon}^* |A_\varepsilon^*| + l_{I,\varepsilon}^* |I_\varepsilon^*| + \mu_L |L_\varepsilon^*| + \mu_L y_1 + \mu_L \rho) \, dt.
\]

Denoting

\[
E_\varepsilon := \int_0^T |e_\varepsilon'(t)| \, dt,
\]

\[
F_0 = |L_0 - y_1|, F_{1,\varepsilon} = l_{A,\varepsilon}^* \|A_\varepsilon^*\|_\infty + l_{I,\varepsilon}^* \|I_\varepsilon^*\|_\infty + \mu_L \|L_\varepsilon^*\|_\infty + \mu_L y_1
\]

where \( \|\|_\infty = \|\|_{L^\infty(0,T)} \), we have

\[
\rho \int_0^T |e_\varepsilon'(t)| \, dt \leq F_0(e_\varepsilon(0)) + \int_0^T (|e_\varepsilon| + |f_\varepsilon|))(F_{1,\varepsilon} + \mu_L \rho) \, dt.
\] (2.29)

Now, since \( e_\varepsilon(T) = 0 \), we note that

\[
|e_\varepsilon(0)| = \left| \int_0^T e_\varepsilon'(t) \, dt \right| \leq E_\varepsilon, \quad |e_\varepsilon(t)| \leq 2E_\varepsilon \text{ for all } t \in [0,T].
\] (2.30)

We multiply eqs. \([2.13],[2.15],[2.17]\) by \( p_\varepsilon, q_\varepsilon, d_\varepsilon, f_\varepsilon \) respectively, integrate over \((0,t)\) and sum up. We get

\[
\frac{1}{2} S_\varepsilon^2(t) = \frac{1}{2} \alpha_0^2 \left( \|A_\varepsilon^*\|_\infty^2 + \|I_\varepsilon^*\|_\infty^2 \right) T + \int_0^t F_\varepsilon S_\varepsilon^2(\tau) d\tau
\]

\[
+ l_{A,\varepsilon}^* \int_0^t |e_\varepsilon| |q_\varepsilon| d\tau + l_{I,\varepsilon}^* \int_0^t |e_\varepsilon| |d_\varepsilon| d\tau,
\]

whence

\[
S_\varepsilon^2(t) \leq \alpha_0^2 \left( \|A_\varepsilon^*\|_\infty^2 + \|I_\varepsilon^*\|_\infty^2 \right) T + \int_0^t 2G_\varepsilon S_\varepsilon^2(\tau) d\tau + 8E_\varepsilon^2 T,
\]

where \( S_\varepsilon^2(t) = |p_\varepsilon(t)|^2 + |q_\varepsilon(t)|^2 + |d_\varepsilon(t)|^2 + |f_\varepsilon(t)|^2 \). Here, \( F_\varepsilon \) and \( G_\varepsilon \) consist in sums of the constant coefficients of the equations in the dual system plus the \( L^\infty \)-norms of the time dependent coefficients. These sums also include \( l_{A,\varepsilon}^2 \) and \( l_{I,\varepsilon}^2 \). By the Gronwall’s lemma, the previous inequality yields the estimate

\[
|p_\varepsilon(t)| + |q_\varepsilon(t)| + |d_\varepsilon(t)| + |f_\varepsilon(t)| \leq \alpha_0(\|A_\varepsilon^*\|_\infty + \|I_\varepsilon^*\|_\infty + 4E_\varepsilon)\sqrt{T}e^{G_\varepsilon T},
\] (2.31)

the right-hand side being bounded independently of \( \varepsilon \), since \( A_\varepsilon^*, I_\varepsilon^* \) tend uniformly to \( A^*, I^* \), \( l_{A,\varepsilon}^* \to l_{A}^* \), \( l_{I,\varepsilon}^* \to l_{I}^* \), \( G_\varepsilon \to G \) which is constant.
We go back to (2.29) and using (2.31) we write

$$
\rho E_{\varepsilon} \leq F_0 E_{\varepsilon} + (|e_{\varepsilon}| + |f_{\varepsilon}|)F_{1,\varepsilon}T + \mu_L \rho(|e_{\varepsilon}| + |f_{\varepsilon}|)T
\leq (F_0 + 4\alpha_0 F_{1,\varepsilon}T\sqrt{T}e^{G_{\varepsilon}T})\varepsilon_{\varepsilon} + 4\mu_L \rho_0 T\sqrt{T}e^{G_{\varepsilon}T} + \alpha_0 F_{2,\varepsilon}(F_{1,\varepsilon} + \mu_L \rho)T\sqrt{T}e^{G_{\varepsilon}T},
$$

where $F_{2,\varepsilon} := \|A^*_{\varepsilon}\|_{\infty} + \|I^*_{\varepsilon}\|_{\infty}$. This implies

$$
E_{\varepsilon}\left(\rho(1 - 4\mu_L \alpha_0 e^{G_{\varepsilon}T}\sqrt{T}) - (F_0 + 4\alpha_0 F_{1,\varepsilon}T\sqrt{T}e^{G_{\varepsilon}T})\right) \leq \alpha_0 F_{2,\varepsilon}(F_{1,\varepsilon} + \mu_L \rho)T\sqrt{T}e^{G_{\varepsilon}T}. \tag{2.32}
$$

We have to prove that the coefficients of $\rho$ and $E_{\varepsilon}$ are positive, at least on a small interval. Since the approximating optimal solution tends uniformly to the optimal solution in $(P)$, we infer that

$$
\mu_L \Rightarrow \mu_{L,0} = 0, \quad i = 1, 2,
$$

$$
F_1 = I_{A^*} \|A^*\|_{\infty} + I_{\varepsilon} \|I^*\|_{\infty} + \mu_L \|I^*\|_{\infty} + \mu_L y_1, \quad F_2 = \|A^*\|_{\infty} + \|I^*\|_{\infty},
$$

$$
G = C(\|\beta_A\|_{\infty} + \|\beta_I\|_{\infty} + \sigma + \mu_A + I_{A^*} + \mu_I + I_{\varepsilon} + \|\xi\|_{\infty} + I_{A}^2 + I_{I}^2),
$$

with $C$ a constant. We note that the function $t \rightarrow 4\mu_L \alpha_0 e^{G_{\varepsilon}t}\sqrt{T}$ is positive, strictly increasing for $t > 0$ and vanish at 0. We have

$$
4\mu_L \alpha_0 e^{G_{\varepsilon}t}\sqrt{T} \leq 4\mu_L \alpha_0 e^{G_{\varepsilon}t}\sqrt{T} + O(\varepsilon) < 1 \text{ as } \varepsilon \to 0. \tag{2.33}
$$

Then, there exists $T_1 > 0$ such that (2.33) takes place on $(0, T_1)$.

Next, we show that $\rho(1 - 4\mu_L \alpha_0 e^{G_{\varepsilon}T}\sqrt{T}) - (F_0 + 4\alpha_0 F_{1,\varepsilon}T\sqrt{T}e^{G_{\varepsilon}T}) > 0$ for $t \in (0, T_2)$. Indeed, this comes to

$$
F_0 + 4\alpha_0(F_{1,\varepsilon} + \rho\mu_L)t\sqrt{T}e^{G_{\varepsilon}t} \leq F_0 + 4\alpha_0(F_{1,\varepsilon} + \rho\mu_L)t\sqrt{T}e^{G_{\varepsilon}t} + O(\varepsilon) < \rho. \tag{2.34}
$$

The function $t \rightarrow F_0 + 4\alpha_0(F_{1,\varepsilon} + \rho\mu_L)t\sqrt{T}e^{G_{\varepsilon}t}$ is positive, strictly increasing and equal with $F_0$ at 0 and we note that

$$
F_0 = |L_0 - y_1| < \rho. \tag{2.35}
$$

This implies that there exists $T_2 > 0$ such that (2.34) is satisfied on $(0, T_2)$. It remains to check (2.35).

We recall that we let $y_1 \in (L_0, L_0)$ and if we choose $\rho \in (L_0 - y_1, \hat{L} - y_1)$ it follows that (2.35), $y_1 + \rho \leq \hat{L}$ and $y_1 - \rho \leq \hat{L}$ hold. The last one is true because by the choice of $\rho$ we have $y_1 - \rho \leq 2y_1 - L_0 \leq L_0 < \hat{L}$. By (2.32)-(2.34) we can write that

$$
\int_0^T |e'(\xi)| \, dt \leq \frac{\alpha_0 F_{2,\varepsilon}(F_{1,\varepsilon} + \mu_L \rho)T\sqrt{T}e^{G_{\varepsilon}T}}{\rho(1 - 4\mu_L \alpha_0 e^{G_{\varepsilon}T}\sqrt{T}) - (F_0 + 4\alpha_0 F_{1,\varepsilon}T\sqrt{T}e^{G_{\varepsilon}T})},
$$

for $T \in (0, T_{loc})$, $T_{loc} := \min\{T_1, T_2\}. \tag{2.36}$
More exactly, $T_1$ and $T_2$ are the solutions to
\[ 4\mu_L\alpha_0 e^{Gt}t^{3/2} = 1 \text{ and } F_0 + 4\alpha_0 (F_{1,e} + \rho \mu_L)te^{Gt} = \rho, \] (2.37)
respectively. Since the right-hand side in (2.36) is bounded we conclude with (2.25), while (2.26) is implied by (2.31).

**Remark 2.7.** Now, we recall a few definitions and results necessary in the proof of the next theorem. We denote by $BV([0,T])$ the space of functions $v : [0,T] \to \mathbb{R}$ with bounded variation, that is
\[ \|v\|_{BV([0,T])} = \sup \left\{ \sum_{i=0}^{M-1} |v(t_{i+1}) - v(t_i)| ; 0 = t_0 < t_1 < \ldots < t_M = T \right\} < \infty, \]
and by $\mathcal{M}([0,T])$ the dual of the separable space $C([0,T])$. The space $\mathcal{M}([0,T])$ contains the bounded Radon measures defined on $[0,T]$. We also recall by the Lebesgue decomposition theorem (see e.g. [14]), that every $\mu \in (L^\infty(0,T))^*$ can be uniquely written as
\[ \mu = \mu_a + \mu_s, \] (2.38)
where $\mu_a \in L^1(0,T)$ and $\mu_s$ is a singular measure. This means that for each $\varepsilon > 0$ there exists a Lebesgue measurable set $S \subset [0,T]$ with $\operatorname{meas}([0,T]\setminus S) \leq \varepsilon$ and $\mu_s(\varphi) = 0$ for all $\varphi \in L^\infty(S)$.

Next, we recall that every $v \in BV([0,T])$ has a unique decomposition,
\[ v = v^a + v^s, \] (2.39)
where $v^a \in AC[0,T]$ and $v^s \in BV([0,T])$. Here, $AC[0,T]$ is the space of absolutely continuous functions on $[0,T]$ and $v^s$ is a singular part (for instance it can be a jump function with bounded variation or a function with bounded variation with a.e. zero derivative).

We note that if $v \in BV([0,T])$, then its distributional derivative $\frac{dv}{dt} := \mu$ belongs to $(L^\infty(0,T))^*$, and in virtue of the Lebesgue decomposition, it is represented by the sum of the absolutely continuous part and the singular part
\[ \frac{dv}{dt} = \mu_a + \mu_s = \frac{dv^a}{dt} + \frac{dv^s}{dt} \in \mathcal{D}'(0,T), \] (2.40)
where $\mathcal{D}'(0,T)$ is the space of Schwartz distributions on $(0,T)$.

In the next theorem we shall pass to the limit in the approximating optimality conditions. To this end, we introduce the system
\[ p' - k_0^*p + k_0^*q = 0, \text{ a.e. } t \in (0,T), \] (2.41)
\[ q' - \beta_A S^* + k_3^*q + \sigma d + l_3^*e + \mu_A f = -\alpha_0 A^*, \text{ a.e. } t \in (0,T), \] (2.42)
\[ d' - \beta_1 S^* + \beta_1 S^* - k_2^*d + l_1^*e + \mu_I f = -\alpha_0 I^*, \text{ a.e. } t \in (0,T), \] (2.43)
where

\[ e' - \mu_L e + \mu_L f + \nu = 0, \text{ in } \mathcal{D}'(0, T), \tag{2.44} \]

\[ f' + \xi p - \xi f = 0, \text{ a.e. } t \in (0, T), \tag{2.45} \]

\[ p(T) = 0, \ q(T) = 0, \ d(T) = 0, \ e(T) = 0, \ f(T) = 0, \tag{2.46} \]

\[ \nu = \alpha_2 \lim_{\varepsilon \to 0} \left( \frac{1}{\varepsilon} (L^* - \tilde{L})^+ \right) \text{ weak* in } (L^\infty(0, T))^*, \tag{2.47} \]

We denote by \( n \) the normal cone to \( K \) at \( z \),

\[ n = \begin{cases} 0, & \text{if } z < L \\ \mathbb{R}^+, & \text{if } z = L \end{cases} \tag{2.49} \]

We also define

\[ \tilde{K} = \{ y \in \mathbb{R}; -\infty < y \leq \tilde{L}, \tilde{K} = \{ v \in C([0, T]); \nu(t) \leq \tilde{K} \text{ for all } t \in [0, T] \}. \]

We denote by \( N_{\tilde{K}}(\zeta) \) the normal cone to \( \tilde{K} \) at \( z \),

\[ N_{\tilde{K}}(\zeta) = \left\{ \eta \in \mathcal{M}([0, T]); \eta(\zeta - z) \geq 0, \forall z \in \tilde{K} \right\} \tag{2.50} \]

We also define

\[ \hat{K} = \{ y \in \mathbb{R}; -\infty < y \leq \hat{L}, \hat{K} = \{ v \in C([0, T]); v(t) \leq \hat{K} \text{ for all } t \in [0, T] \}. \]

Theorem 2.8. Let \( (l^*_A, l^*_I) \) be optimal in \((P)\) with the corresponding state \( X^* \). Then, if \( T < T_{\text{loc}} \) defined in \([2.36],[2.37]\), the optimality conditions for problem \((P)\) read:

\[ l^*_A = P_{[0,1]} \left( \frac{1}{\alpha} \int_0^T A^*(q - e)dt \right), \ l^*_I = P_{[0,1]} \left( \frac{1}{\alpha} \int_0^T I^*(d - e)dt \right) \tag{2.51} \]

where \((p, q, d, e, f)\) is the solution of the dual system \([2.41],[2.48]\) with

\[ (p, q, d, f) \in (W^{1,\infty}(0, T))^4, \ e \in BV([0, T], (e')_s \in \mathcal{M}([0, T])) \tag{2.52} \]

Moreover, we have \( e' = (e')_a + (e')_s, \ (e')_a \in L^1(0, T), \ (e')_s \in \mathcal{M}([0, T]), \)

\[ e'_a(t) + v_a(t) = (\mu_L e - \mu_L f)(t), \text{ a.e. } t \in (0, T), \tag{2.53} \]

\[ (e'_s)' + v_s = 0, \text{ in } \mathcal{D}'(0, T), \tag{2.54} \]

where

\[ v_a(t) \in N_{\tilde{K}}(L^*(t)), \text{ a.e. } t \in [0, T], \tag{2.55} \]
\[ \nu_s(\varphi) \geq 0, \ \text{supp} \ \nu_s \subset \{ t \in [0, T]; \ L^*(t) = \hat{L} \}. \]  

(2.56)

**Proof.** We shall establish some estimates in order to pass to the limit in the approximating optimality conditions determined in Proposition 2.5. Let \( T < T_{10}^\infty \), let \((l^*_A, l^*_I)\) be optimal in \((P)\) with the corresponding state \( X^* \) and let us consider \( \{(l^*_A, l^*_I), X^*_\varepsilon\} \) optimal in \((P)_\varepsilon\). Recalling Proposition 2.4 and the continuity property from Proposition 2.1, we have \( l^*_A, l^*_I, X^*_\varepsilon \to l^*_A, l^*_I, X^* \) and

\[ X^*_\varepsilon \to X^* \text{ weak* in } W^{1,\infty}(0, T) \text{ and uniformly on } [0, T]. \]

By (2.26) each component of the solution to the dual system, but \( e_\varepsilon \), follows to be bounded in \( L^\infty(0, T) \). Moreover, by (2.13)-(2.15), (2.17) it follows that \( (p_\varepsilon^0), (q_\varepsilon^0), (d_\varepsilon^0), (f_\varepsilon^0) \) are bounded in \( L^\infty(0, T) \), so that, on a subsequence,

\[ p_\varepsilon \to p, \ q_\varepsilon \to q, \ d_\varepsilon \to d, \ f_\varepsilon \to f \text{ weak* in } W^{1,\infty}(0, T) \]  

(2.57)

and uniformly in \([0, T]\), as \( \varepsilon \to 0 \).

The component \((e_\varepsilon)_\varepsilon\) is bounded in \( L^\infty(0, T) \) by (2.30) and its derivative is in \( L^1(0, T) \), by (2.25). These imply that \( e_\varepsilon \in BV([0, T]) \) and by Helly’s theorem (see e.g., [2], p. 47) it follows that

\[ e_\varepsilon(t) \to e(t), \text{ for all } t \in [0, T], \text{ as } \varepsilon \to 0. \]  

(2.58)

Going back to (2.16) we deduce that \( \frac{1}{\varepsilon}(L^*_\varepsilon - \hat{L})^+ \in L^1(0, T) \). Now, we assert that \((e'_\varepsilon)_\varepsilon\) and \( \frac{1}{\varepsilon}(L^*_\varepsilon - \hat{L})^+\) are weak* compact in \( (L^\infty(0, T))^* \), the dual of \( L^\infty(0, T) \). This is pointed out in the proof of Corollary 2B in [13], but this assertion does not follow directly from Alaoglu theorem. An argument can be found in [13], and we resume it below.

Let us consider the linear operator \( \Psi : C([0, T]) \to L^\infty(0, T) \), \( \Psi z = \tilde{\Psi} z \), which maps a continuous function into the corresponding class of equivalence \( \tilde{\Psi} \) (of all functions a.e. equal). Its adjoint \( \Psi^* : (L^\infty(0, T))^* \to M([0, T]) \), is defined by \( \langle \Psi^* \mu \rangle (z) := \mu(\Psi z) \) for any \( z \in C([0, T]) \). If \( (\mu_n)_n \) is bounded in \( (L^\infty(0, T))^* \) and also in \( M([0, T]) \), then \( (\Psi^* \mu_n)_n \) is bounded in \( M([0, T]) \) and using the Alaoglu theorem it follows that \( (\Psi^* \mu_n)_n \) is weak* sequentially compact in \( M([0, T]) \).

Therefore, it follows that \( (\mu_n)_n \) is weak* sequentially compact in \( M([0, T]) \). Passing to the limit in \( \mu_n(\Psi z) = (\Psi^* \mu_n)(z) \) we get \( \mu(\Psi z) = (\Psi^* \mu)(z) \) for any \( \Psi \in L^\infty(0, T) \) which is of the form \( \Psi z \) with \( z \in C([0, T]) \). Then, due to the Hahn-Banach theorem, \( \mu \) can be extended to all \( L^\infty(0, T) \) and so we conclude that \( (\mu_n)_n \) is weak* sequentially compact in \((L^\infty(0, T))^*\).

Therefore, one can extract a subsequence such that

\[ e'_\varepsilon \to q', \ \frac{\alpha_\varepsilon}{\varepsilon}(L^*_\varepsilon - \hat{L})^+ \to \nu \text{ weak* in } (L^\infty(0, T))^* \subset M([0, T]). \]  

(2.59)

Thus, relying on (2.57)-(2.59), we can pass to the limit in (2.13)-(2.15) and obtain (2.41)-(2.43).
Now, we move to (2.19), or more exactly in (2.23) and (2.24) and pass to the limit. The left-hand side of (2.23) converges and since the normal cone is maximal monotone, hence strongly-strongly closed it follows that

$$- \int_0^T I^*(e - d)dt - \alpha_1 l_i^* \in N_{[0,1]}(l_i^*),$$

which implies the second relation in (2.51). Similarly, we proceed in (2.24) and obtain

$$- \int_0^T A^*(e - q)dt - \alpha_1 l_i^* \in N_{[0,1]}(l_i^*),$$

whence we get the first relation in (2.51).

Finally, we detail equation (2.44). Since $\frac{1}{\varepsilon} z +$ is the subdifferential of the function $\frac{1}{2} (z^+)^2$, we can write

$$\frac{\alpha_2}{\varepsilon} \int_0^T (L^*_z - \hat{L})^+ ((L^*_z - \hat{L}) - (z(t) - \hat{L}))dt \geq \frac{\alpha_2}{2\varepsilon} ((L^*_z - \hat{L})^+) - \frac{1}{2} ((z(t) - \hat{L})^+) \geq 0,$$

for all $z \in \hat{K}$. At limit we obtain

$$\nu(L^* - z) \geq 0, \text{ for all } z \in \hat{K}, \quad (2.60)$$

whence, $\nu \in N_{\hat{K}}(L^*)$.

Since $\nu \in (L^\infty(0, T))^* \subset M([0, T])$, recalling (2.38) and (2.40), we can represent $\nu = \nu_a + \nu_s$, where $\nu_a$ is the absolutely continuous part (in the sense of measure) and $\nu_s$ is the singular part of $\nu$. Also, $e' \in M([0, T])$ and $e' = (e')_a + (e')_s$ where $(e')_a = e'_a \in L^1(0, T)$. Then, (2.44) can be rewritten as in (2.53)-(2.54). Relation (2.60) implies that $\nu_a \in N_{\hat{K}}(L^*)$,

$$\nu_a(t) \geq 0, \; \nu_a(t) = 0 \text{ on } \{ t \in [0, T]; \; L^*(t) < \hat{L} \}, \quad (2.61)$$

$$\nu_s(\varphi) \geq 0, \; \nu_s(\varphi) = 0 \; \text{if} \; \varphi \in \overset{\circ}{K}, \quad (2.62)$$

where $\overset{\circ}{K} = \{ \varphi \in C([0, T]); \; \varphi(t) < \hat{L} \text{ for } t \in [0, T] \}$ is the interior of $\hat{K}$, while $\nu_s$ has the support on the boundary of $\hat{K}$. Recalling that

$$\text{supp } \nu_s = \{ \Sigma \subset [0, T]; \; \nu_s \neq 0 \text{ on } \Sigma \}$$

it follows that $\text{supp } \nu_s \subset \{ t \in (0, T); \; z(t) = \hat{L} \}$. Thus, we actually get (2.56), as claimed.

3 Problem \((P_0)\)

In this section we treat problem \((P_0)\) associated to the cost functional (1.11).

**Proposition 3.1.** Problem \((P_0)\) has at least one solution \((\beta^*_i, A^*_0, I^*_0)\).
The proof is led on the basis of similar arguments as in Proposition 2.2, using the result of existence and uniqueness of the solution to the state system given in Proposition 2.1.

The optimality conditions can be directly determined, after writing the system in variations and the adjoint system. Let us define $K_0 = \{(y, z) \in \mathbb{R}^2; \ y \geq 0, \ z \geq 0, \ y + z \leq N_0\}, \ K_+ = \{y \in \mathbb{R}; \ y \geq 0\}.

Let $\lambda > 0$ and set the variations

$$\beta^\lambda_I = \beta^+_I + \lambda u, \ u = \tilde{\beta}^+_I - \beta^*_I, \ \text{where} \ \tilde{\beta}^+_I(t) \in K_+, \ \text{a.e.} \ t \geq 0,$$

$$A^\lambda_0 = A^*_0 + \lambda w, \ w = A_0 - A^*_0, \ I^\lambda_0 = I^*_0 + \lambda v, \ v = I_0 - I^*_0,$$

where $(\tilde{A}_0, I_0) \in K_0.$ (3.1)

Let us denote $x^\lambda = \frac{X_{\beta^+_I} - X^*}{\beta^+_I - \beta^*_I} - X^*$, where $X_{\beta^+_I} = A^*_0, I^*_0$ is the solution to (1.1)-(1.6) corresponding to $(\beta^+_I, \ A^*_0, I^*_0)$, satisfying (1.7) and $X^*$ is the optimal state corresponding to $(\beta^*_I, A^*_0, I^*_0)$.

We introduce the linearized system for problem $(P_0)$

$$s' = -k^*_0 s - k^*_0 i + \beta^*_I S^* a - k^*_0 a + \xi r - u S^* I^*,
$$
$$a' = k^*_0 s + \beta^*_I S^* i + k^*_0 a + u S^* I^*,
$$
$$i' = \sigma a - k^*_0 i,
$$
$$l' = l_A a + l_I i - \mu L l,
$$
$$r' = \mu A a + \mu I i + \mu L l - \xi r$$

(3.2)

for $t \geq 0$, with the initial condition

$$s(0) = -w - v, \ a(0) = w, \ i(0) = v, \ l(0) = 0, \ r(0) = 0, \ (3.3)$$

where

$$k^*_0 = \beta^*_I I^* + A^*_0, \ k^*_3 = \beta^*_I S^* - k^*_1, \ k_1 = \sigma + \mu_A + l_A, \ k_2 = \mu_I + l_I.$$

(3.4)

We introduce the backward dual system for the variables $(p, q, d, e, f)$ as

$$p' - k^*_0 p + k^*_0 q = 0,$$

$$q' - \beta^*_I S^* p + k^*_3 q + \sigma d + l_A e + \mu_A f = 0,$$

$$d' - \beta^*_I S^* p + \beta^*_I S^* q - k^*_2 d + l_I e + \mu_I f = 0,$$

$$e' - \mu L e + \mu_L f = 0,$$

$$f' + \xi p - \xi f = 0$$

(3.5)

(3.6)

(3.7)

(3.8)

(3.9)

for a.a. $t > 0$, with the final conditions

$$p(T) = 0, \ q(T) = 0, \ d(T) = 0, \ (3.10)$$

$$e(T) = L^*(T) - L_T, \ f(T) = R^*(T) - R_T.$$

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The linear systems (3.2) and (3.5)-(3.10) have unique solutions in \((W^{1,\infty}(0,T))^5\).

Let \(N_{K_0}(\zeta_1, \zeta_2)\) be the normal cone to the set \(K_0 \subset \mathbb{R}^2\). We recall that \(N_{K_0}(\zeta_1, \zeta_2) = \partial I_{K_0}(\zeta_1, \zeta_2)\), where \(\partial I_{K_0}(\zeta_1, \zeta_2)\) is the subdifferential of the indicator set of \(K_0\), that is
\[
\partial I_{K_0}(\zeta_1, \zeta_2) = \{\eta = (\eta_1, \eta_2) \in \mathbb{R}^2; \eta_1(\zeta_1 - z_1) + \eta_2(\zeta_2 - z_2) \geq 0, \text{ for } (z_1, z_2) \in K_0\}.
\]

Let \(N_{K_+}(z)\) be the normal cone to the set \(K_+\), that is
\[
N_{K_+}(z) = \begin{cases} \mathbb{R}^-, & \text{if } z = 0 \\ 0, & \text{if } z > 0. \end{cases}
\]

**Proposition 3.2.** Let \((\beta^*_I, A^*_0, I^*_0)\) be optimal in \((P_0)\) with the state \(X^*\). Then,
\[
\beta^*_I(t) = P_{K_+} \left( \frac{1}{\alpha_1} (p(t) - q(t)) S^*(t) T^*(t) \right), \quad \text{a.e. } t \in (0, T), \quad (3.11)
\]
\[
(A^*_0, I^*_0) = (\Gamma + N_{K_0})^{-1} (\alpha_0^{-1} (p(0) - q(0) + \alpha_0 N_0, p(0) - d(0) + \alpha_0 N_0)), \quad (3.12)
\]
where \(\Gamma = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\) and \((p, q, d, e_x)\) is the solution to the backward dual system (3.5)-(3.10).

**Proof.** Let us multiply the equations for \(s, a, i, l, r\) in \(3.2\) by \(p_x, q_x, d_x, e_x, f_x\), respectively and integrate over \((0, T)\). By integrating by parts we obtain
\[
(L^*(T) - L_T) l(T) + (R^*(T) - R_T) r(T) = \int_0^T S^* I^*(q - p) u dt + p(0)(-w - v) + q(0)w + d(0)v. \quad (3.13)
\]
Since \((\beta^*_I, A^*_0, I^*_0)\) is optimal in \((P_0)\) we have \(J(\beta^*_I, A^*_0, I^*_0) \geq J(\beta^*_I, A^*_0, I^*_0)\) and we deduce
\[
(L^*(T) - L_T) l(T) + (R^*(T) - R_T) r(T) + \alpha_1 \int_0^T \beta^*_I u dt \quad (3.14)
\]
\[
+ \alpha_0 (A^*_0 w + I^*_0 v + (N_0 - A^*_0 - I^*_0)(-w - v)) \geq 0.
\]
By comparison with (3.13) and recalling the setting of \(u, w, v\), we have
\[
\int_0^T [(p - q) S^* I^* - \alpha_1 \beta^*_I (\beta^*_I - \tilde{\beta}_I) dt \quad (3.15)
\]
\[
+ [p(0) - q(0) - 2\alpha_0 A^*_0 + \alpha_0 N_0 - \alpha_0 I^*_0] (A^*_0 - \tilde{A}_0)
\]
\[
+ [p(0) - d(0) - 2\alpha_0 I^*_0 + \alpha_0 N_0 - \alpha_0 A^*_0] (I^*_0 - \tilde{I}_0) \geq 0
\]
for all \(\tilde{\beta}_I(t) \in K_+\), a.e. \(t \in (0, T)\) and \((\tilde{A}_0, \tilde{I}_0) \in K_0\). In particular, by setting \((\tilde{A}_0, \tilde{I}_0) = (A^*_0, I^*_0), \quad (3.15)\) yields
\[
(p(t) - q(t)) S^*(t) I^*(t) - \alpha_1 \beta^*_I(t) \in N_{K_+}(\beta^*_I(t)), \quad \text{a.e. } t \in (0, T), \quad (3.16)
\]
which implies (3.11).

Then, let us set \( \tilde{\beta}_I = \beta^*_I \). By (3.15) we obtain that

\[
(p(0) - q(0) - 2\alpha_0 A^*_0 + \alpha_0 N_0 - \alpha_0 I^*_0, p(0) - d(0) - 2\alpha_0 I^*_0 + \alpha_0 N_0 - \alpha_0 A^*_0) \\
\in N_{K_0}(A^*_0, I^*_0).
\]

This can be still written

\[
\alpha^{-1}_0 (p(0) - q(0) + \alpha_0 N_0, p(0) - d(0) + \alpha_0 N_0) - \Gamma(A^*_0, I^*_0) \in N_{K_0}(A^*_0, I^*_0),
\]

where \( \Gamma = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \). This implies (3.12), because \( N_{K_0} \) is maximal monotone
in \( \mathbb{R}^2 \), and \( \Gamma : \mathbb{R}^2 \to \mathbb{R}^2 \) is positive, so \((\Gamma + N_{K_0})^{-1}\) is Lipschitz. \( \square \)

Remark 3.3. By (3.11) and the fact that the optimal state is nonnegative, it
is clear that

\[
\beta^*_I(t) = \begin{cases} \\
0 & \text{on } \{ t \in [0, T]; \ p(t) \leq q(t) \} \\
\frac{1}{\alpha_1} (p(t) - q(t)) S^*(t) I^*(t) & \text{on } \{ t \in [0, T]; \ p(t) > q(t) \}.
\end{cases}
\]

(3.18)

4 System stability and determination of the reproduction rate

In this section we investigate the system stability, which will help to derive an
expression for the reproduction rate. Once identified \( \beta_I(t) \), one can consider its
average over \((0, T)\) and use it for the stability analysis. An average can be set
for \( \beta_A \), too. The use of the average can be more accurate if \((0, T)\) is short or if
the epidemic has reached a plateau, where the rates do not have large variations.

We discuss the system stability for \( \xi = 0 \), because the situation with \( \xi \)
constant nonzero, meaning that immunity is lost immediately after recovery is
not realistic. We recall that \( N = 1 \).

Theorem 4.1. Let \( \xi = 0 \) and assume that

\[
S_{\infty} := \frac{(\sigma + \mu_A + l_A)(\mu I + l_I)}{\beta} \leq N, \quad \beta := k_2 \beta_A + \sigma \beta_I.
\]

(4.1)

The system \((A, I, L)\) with a positive susceptible population \( S_{\infty} \) is asymptotically
stable if and only if

\[
S_{\infty} < S_{\infty}.
\]

(4.2)

Moreover, all solutions (starting from any nonnegative initial condition) tend to
a stationary state, that is

\[
\lim_{t \to \infty} (S(t), A(t), I(t), L(t), R(t)) = (\tilde{S}_{\infty}, 0, 0, N - \tilde{S}_{\infty})
\]

(4.3)

exists and \( \tilde{S}_{\infty} < S_{\infty} \).
Proof. If $\xi = 0$, the stationary solutions are found as $(S_{\infty}, 0, 0, R_{\infty})$, and we choose $S_{\infty}, R_{\infty} \in [0, N]$. We consider the linearized system, extract the system for the infected compartments $A, I, L$, and define its matrix

$$A_{1,in} = \begin{pmatrix} k_{3,\infty} & \beta I S_{\infty} & 0 \\ \sigma & -k_2 & 0 \\ l_A & l_I & -\mu_L \end{pmatrix},$$

with $k_1, k_2$ given in (1.9) and $k_{3, \infty} = \beta A S_{\infty} - k_1$. The characteristic equation has a negative solution $\lambda = -\mu_L$ and two solutions to the equation

$$P(\lambda) = \lambda^2 + \lambda(k_1 + k_2 - \beta A S_{\infty}) + k_1 k_2 - \beta S_{\infty} = 0.$$  \hfill (4.4)

We prove that the solutions to (4.4) have the real part negative, meaning that $\tilde{S}$ and so it tends to a limit $S_{\infty}$ given in (1.9) and $k_{3, \infty} = \beta A S_{\infty} - k_1$. The condition to be Hurwitz is that $k_1 + k_2 - \beta A S_{\infty} > 0$ and $k_1 k_2 - \beta S_{\infty} > 0$, that is $S_{\infty} \in \left[0, \min \left\{ \frac{k_1 k_2}{\beta A}, \frac{k_1 + k_2}{\beta A} \right\} \right] = \left[0, \frac{k_1 k_2}{\beta A} \right] \subset [0, N]$ by (4.1) and this leads to (4.2).

Let us prove (4.3). By setting $\omega = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$ and denoting $Z := (A, I, L)^T$ (T is the transposed) we write by (1.1)-(1.5) the equations

$$Z'(t) = AZ(t) + \omega(\beta A A + \beta I I)S(t)$$

$$= \begin{pmatrix} -k_1 & 0 & 0 \\ \sigma & -k_2 & 0 \\ l_A & l_I & -\mu_L \end{pmatrix} Z(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (\beta A A + \beta I I)S(t),$$

$$S'(t) = -S(t)(\beta A A + \beta I I).$$ \hfill (4.6)

We must show that there exists

$$\lim_{t \to \infty} S(t) = \tilde{S}_{\infty}, \quad \lim_{t \to \infty} Z(t) = (0, 0, 0).$$ \hfill (4.7)

Since by (4.6) we see that $S'(t) \leq 0$ it follows that $S'$ is monotonically decreasing and so it tends to a limit $S_{\infty} \geq 0$. Then, by (4.5), (4.6) we have

$$Z'(t) = AZ(t) - S'(t)\omega$$

and deduce by the formula of variation of constants and integration by parts, that

$$Z(t) = e^{At}Z(0) - \int_0^t e^{A(t-s)}S'(s)\omega ds$$ \hfill (4.8)

$$= \int_0^t e^{At}Z(0) - S(t)\omega + e^{At}S(0)\omega - \int_0^t A e^{A(t-s)}S(s)\omega ds.$$  

We calculate the last term

$$E = \int_0^t A e^{A(t-s)}S(s)\omega ds = E_1 + E_2$$

$$E_1 = \int_0^t A e^{A(t-s)}\tilde{S}_{\infty}\omega ds, \quad E_2 = \int_0^t A e^{A(t-s)}(S(s) - \tilde{S}_{\infty})\omega ds.$$
and get

\[ E_1 = -\int_0^t (e^{A(t-s)})^\dagger S\infty_\omega ds = -(1 - e^{-At})S\infty_\omega = -S\infty_\omega + e^{At}S\infty_\omega. \]

We note that \( A \) is Hurwitz and recall that \( \|e^{At}\| \leq Me^{-\gamma t} \), where \( \gamma < \min_j \{-\text{Re}(\lambda_j)\} \); \( \lambda_j \) are the eigenvalues of \( A \) and \( \|\cdot\| \) is the norm in \( \mathbb{R}^3 \).

Now, for any \( M \), such that \( t > M \) we write

\[
E_2 = A \int_0^t e^{A(t-s)}(S(s) - \tilde{S}_\infty)\omega ds \\
= A \int_0^M e^{A(t-s)}(S(s) - \tilde{S}_\infty)\omega ds + A \int_M^t e^{A(t-s)}(S(s) - \tilde{S}_\infty)\omega ds \\
\leq C \sup_{0 \leq s \leq M} \left| (S(s) - \tilde{S}_\infty)\omega \right| \|A\| \int_0^M e^{-\gamma(t-s)}ds \\
+ C \sup_{M \leq s < \infty} \left| (S(s) - \tilde{S}_\infty)\omega \right| \|A\| \int_M^t e^{-\gamma(t-s)}ds \\
\leq \frac{C}{\gamma} \sup_{0 \leq s \leq M} \left| (S(s) - \tilde{S}_\infty)\omega \right| \|A\|e^{-\gamma M} \leq M \gamma \sup_{M \leq s < \infty} \left| (S(s) - \tilde{S}_\infty)\omega \right| \|A\| [1 - e^{-\gamma(t-M)}],
\]

where \( \|A\| \) is the norm of the matrix \( A \). The first term of the last sum tends to zero as \( t \to \infty \). For the second we give the following argument. Let \( \varepsilon > 0 \) and fix \( M \) such that \( \sup_{M \leq s < \infty} \left| (S(s) - \tilde{S}_\infty)\omega \right| < \varepsilon \). Therefore,

\[
\limsup_{t \to \infty} \sup_{M \leq s < \infty} \left| (S(s) - \tilde{S}_\infty)\omega \right| [1 - e^{-\gamma(t-M)}] < \varepsilon
\]

and since \( \varepsilon \) is arbitrary, it means that this term tends to zero, too. Thus, \( \lim_{t \to \infty} E_2(t) = 0 \). We note that

\[
\lim_{t \to \infty} e^{At}Z(0) = 0, \quad \lim_{t \to \infty} e^{At}S(0)\omega = 0, \quad \lim_{t \to \infty} e^{At}\tilde{S}_\infty\omega = 0,
\]

since \( A \) is Hurwitz. Then, taking into account the first relation in \([4.7]\) and letting \( t \to \infty \) in \([4.8]\) we get

\[
\lim_{t \to \infty} Z(t) = -\tilde{S}_\infty\omega + \tilde{S}_\infty\omega = 0.
\]

Thus, \([4.3]\) follows and \([4.2]\) is a sufficient condition to have the solution \((A, I, L) = (0, 0, 0)\) stable.

It remains to show that \( \tilde{S}_\infty < S_\infty \). Otherwise, if \( S_\infty \geq \tilde{S}_\infty \), we see that the polynomial is no longer Hurwitz and so system \((A, I, L)\) is not asymptotically stable. Moreover, \( \lim_{t \to \infty} R(t) = N - (S(t) + A(t) + I(t) + L(t)) = N - S_\infty := R_\infty \) and we note that \( R_\infty = N - S_\infty > N - S_\infty = N - \frac{k_1k_2}{\beta} \geq 0 \), according to \([4.1]\).
We prove that condition (4.2) is necessary, too. It means that if $S_\infty$ is a steady state and $\lim_{t \to \infty} (A(t), I(t), L(t)) = (0, 0, 0)$, it follows that $S_\infty$ should satisfy (4.2). Let us assume the opposite, that is $S_\infty \geq S_\infty'$. It follows that the above corresponding polynomial $P(\lambda)$ is not Hurwitz and this implies that the system $(A, I, L)$ is not asymptotically stable, meaning that $(A(t), I(t), L(t))$ does no longer tend to 0 as $t \to \infty$. □

The fact that the value $\frac{\beta}{k_1k_2}$ appears as a critical value for the system, gives a justification to define the reproduction rate as

$$R^0 = \frac{k_2\beta_A + \sigma\beta_I}{(\sigma + \mu_A + l_A)(\mu_I + l_I)}.$$  \hspace{1cm} (4.9)

Its epidemiologic interpretation will be given further.

## 5 Conclusions

We solved an inverse and a control problem related to an epidemic model for SARS-CoV-2, with five compartments: susceptible $S$, undetected infected asymptomatic $A$, undetected infected symptomatic $I$, detected by testing and isolated $L$, and recovered $R$. By an optimal control technique we identified the rate of infection $\beta_I$ of the susceptible individuals by the infective $I$ class and the number of undetected individuals in the classes $A$ and $I$ at a time set $t = 0$. Their estimation was relied on the observation of the number of the isolated and recovered people at time $t = 0$ and at another later time $T$. It turned out that the $\beta_I$ is the projection of a point depending on the optimal states and the solution to the dual system, on the set of positive real numbers and the point $(A_0^*, I_0^*)$ is uniquely determined by (3.12). Knowing the evolution of $\beta_I$ during $(0, T)$ and the initial values $A_0^*$ and $I_0^*$ one can estimate the transient evolution of the compartments after the time $T$. Then, the control of the infected classes $A$ and $I$ was done, by means of the coefficients $l_A$ and $l_I$ related to the testing action. The controllers were provided by expressions depending on the optimal states for this problem and the solution to the singular dual system.

The investigation of the system asymptotic stability enhanced the determination of the reproduction rate $R^0$, defined in (4.9), under the assumption of constant coefficients in the state system. Theorem 5.1 characterizes the system behavior under the assumption that the disease induces life immunity. It indicates an asymptotic extinction of the disease, following by the globally asymptotic stability of the solution to a steady state $(\tilde{S}_\infty, 0, 0, 0, N - \tilde{S}_\infty)$, where $\tilde{S}_\infty$ does not exceed the value $S_\infty$ given by (4.2). This is interpreted as the number of individuals that have been never infected (see [8]). Moreover, if $R^0\tilde{S}_\infty < 1$, the epidemic extinguishes, while the case $R^0\tilde{S}_\infty > 1$ shows a massive outbreak, $R^0\tilde{S}_\infty = 1$ being a bifurcation point.

Finally, we underline that, in order to avoid much more calculations in a model with many equations, we used a restraint model with less compartments, including however the most relevant ones. More accurate values for the desired...
parameters to be identified can be obtained developing similar arguments for a
more elaborated model with many compartments supposed to be measurable,
such that the information provided by their observation could be included in the
minimization problem formulation. Also, other parameters, as for example $\beta_A$
can be identified and numerical simulations will be provided in a forthcoming
paper.

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