Quantum Hall effect on Riemann surfaces

Carlos Tejero Prieto
Departamento de Matemáticas and
Instituto Universitario de Física Fundamental y Matemáticas
Universidad de Salamanca
Spain
E-mail: carlost@usal.es

Abstract. We study the family of Landau Hamiltonians compatible with a magnetic field on a Riemann surface $S$ by means of Fourier-Mukai and Nahm transforms. Starting from the geometric formulation of adiabatic charge transport on Riemann surfaces, we prove that Hall conductivity is proportional to the intersection product on the first homology group of $S$ and therefore it is quantized. Finally, by using the theory of determinant bundles developed by Bismut, Gillet and Soul, we compute the adiabatic curvature of the spectral bundles defined by the holomorphic Landau levels. We prove that it is given by the polarization of the jacobian variety of the Riemann surface, plus a term depending on the relative analytic torsion.

1. Introduction
The classical Hall effect, giving rise to the so-called Hall conductivity, was observed in 1879 by E. Hall. Almost exactly a century later, in 1980, K. von Klitzing [1] discovered the integer quantum Hall effect which implies the quantization of Hall conductivity. Soon afterwards, in 1981, R. Laughlin [2] gave the first physical explanation based on a gauge argument on a cylinder. The remarkable exactness of this effect triggered a worldwide interest on it which eventually lead to the discovery of the the fractional quantum Hall effect by D. Tsui, H Stormer, A. Gossard in 1982 [3]. The same year, D. Thouless, M. Kohmoto, P. Nightingale and M. den Nijs [4] gave the first topological explanation for the quantization of the Hall conductivity by means of the perturbative Kubo’s formula applied to a Flat torus configuration space. With the same geometry, J. Avron-R. Seiler [5], Q. Niu-D. J. Thouless [6], M. Kohmoto [7], and Q. Niu-D. J. Thouless-Y. S Wu [8] showed in 1985 that Kubo’s formula is given by the Chern number of a line bundle over the torus of magnetic fluxes. The next conceptual step was carried out by J. Avron, R. Seiler and L. Yaffe in 1987 [9], when they interpreted Kubo’s formula in the framework of adiabatic transport theory and proved its validity to second order. This approach allowed them to show that Hall conductivity was given by the adiabatic curvature of spectral projectors associated with a family of Schrödinger operators parametrized by the Aharonov-Bohm potentials. Later, in 1990, M. Klein, R. Seiler [10] proved the validity of Kubo’s formula to any perturbation order. Finally, J. Avron, R. Seiler and P. G. Zograf [11] outlined in 1994 the computation of the adiabatic conductance for the first Landau level for strong magnetic fields.

In this paper we give a geometric description of the spectral bundles defined by the family of Landau Hamiltonians on a Riemann surface $S$ by means of holomorphic spectral geometry techniques. This will allow us to determine the Hall conductivity for holomorphic Landau levels in terms of Chern classes of spectral bundles.
In order to achieve this goal we first use a particular instance of Nahm’s transform to describe the spectral bundles associated with the holomorphic Landau levels. Then we show that this Nahm transform is equivalent to an integral functor associated with the jacobian of the Riemann surface $S$. This identification allows us to determine the topological invariants of the spectral bundles and hence to prove the quantization of the Hall conductivity of holomorphic Landau levels on a Riemann surface.

After this we go one step further and compute the adiabatic conductance for holomorphic Landau levels. We show that this is equivalent to determining the adiabatic curvature of the spectral bundles and in turn this is the same as computing the curvature of the determinant bundles of the spectral bundles with respect to the natural Hermitian metrics induced on them. These metrics are conformal to the Quillen metrics and we can compute the curvature of the latter by means of the techniques developed by Bismut, Gillet and Soulé [12, 13, 14]. As a final result we prove that the adiabatic conductance of the holomorphic Landau levels is determined by the polarization of the jacobian of the Riemann surface plus a term which depends on the relative analytic torsion.

The results expounded here are based in the papers [15, 16], the reader is referred to them for further details.

2. Geometric quantization of the Landau-Hall problem

Let $(S, g)$ be an oriented Riemannian surface and $B \in \Omega^2(S)$ a covariantly constant magnetic field, thus $B = \hat{B} \Omega_2$, where $\hat{B} \in \mathbb{R}$ and $\Omega_2$ is the Riemannian area element.

The Landau-Hall problem for particles of charge $e$ and mass $m$ is the Hamiltonian dynamical system $(T^* S, \omega_2, H)$, where $\omega_2 = d\theta + eB$, and $H = \frac{||\omega||^2}{2m}$ is the Landau Hamiltonian.

It can be shown that the symplectic manifold $(T^* S, \omega_2)$ is quantizable if and only if $(S, eB)$ is quantizable, that is, if and only if $[\frac{\omega^B}{2\pi}] \in H^2(X, \mathbb{Z})$. Moreover, the Hamiltonian $H$ is quantizable in the BKS scheme and if $\mathcal{L} = (L, \langle , \rangle, \nabla) \to S$ is a prequantization bundle with Hermitian metric $\langle , \rangle$ and unitary connection $\nabla$, then the Schrödinger operator associated with $H$ is

$$\hat{H} = \frac{\hbar^2}{2m} (\nabla^* \nabla + \frac{R}{6}),$$

where $\nabla^* \nabla$ is the Bochner Laplacian of the connection $\nabla$ and $R$ is the scalar curvature of the riemannian metric $g$.

The different Schrödinger operators associated with the magnetic field $B$ are parametrized by the set of equivalence classes of prequantization line bundles of $(S, eB)$. This set is given by $H^1(S, U(1))$, the space of Aharanov-Bohm potentials.

If we consider a prequantization bundle $\mathcal{L} = (L, \langle , \rangle, \nabla) \to S$ then any other prequantization bundle $\mathcal{L}' = (L', \langle , \rangle', \nabla')$ can be obtained by twisting $\mathcal{L}$ by a flat Hermitian line bundle $L_0$.

Since $L_0$ is flat, its is trivializable, hence

$$L' = L \otimes L_0 \simeq L,$$

$$\nabla' = \nabla - \frac{i}{\hbar} A \quad A \in \Omega^1(S) \text{ closed}.$$

3. Families of Landau operators

On a compact surface $S$ there is an isomorphism

$$H^1(S, U(1)) \simeq H^1(S, \mathbb{R})/H^1(S, \mathbb{Z})$$
and $J(S) = H^1(S, \mathbb{R})/H^1(S, \mathbb{Z})$ is the Lazzeri model of the jacobian variety of $S$.

Any flat line bundle on the Riemann surface $(S, g)$ is endowed with a natural holomorphic structure. Thus, $H^1(S, U(1))$ gets identified with the Picard group $\text{Pic}^0(S)$ of degree zero holomorphic line bundles.

Therefore there are isomorphisms

$$J(S) \simeq H^1(S, U(1)) \simeq \text{Pic}^0(S),$$

that correspond to the identifications

$$\begin{align*}
\{ \text{Jacobian variety of } S \} & \leftrightarrow \{ \text{Aharanov-Bohm potentials} \} \\
& \leftrightarrow \{ \text{Degree zero holomorphic line bundles} \}
\end{align*}$$

To each presentation of the space of Aharanov-Bohm potentials either as $J(S)$ or $\text{Pic}^0(S)$ we can associate a family of Schrödinger operators parametrized by these spaces.

### 3.1. Family of Landau operators parametrized by $J(S)$

In this case we keep the line bundle fixed and parametrize the connections.

Let $\mathcal{L} = (L, \langle \cdot, \cdot \rangle, \nabla) \to S$ be a prequantization bundle on $(S, eB)$. We have seen that the Landau Hamiltonian $\hat{H}$ is quantizable and its Schrödinger operator, acting on $L^2(S, L)$, is:

$$\hat{H} = \frac{\hbar^2}{2m}(\nabla^* \nabla + \frac{R}{6}) \quad \text{(Landau operator)}.$$

The spaces of eigensections of $\hat{H}$ are called Landau levels. We have also seen that any other prequantization bundle is of the form $(L, \langle \cdot, \cdot \rangle, \nabla' = \nabla - \frac{i}{\hbar}A)$ where $A \in \Omega^1(S)$ is closed. We denote by

$$\hat{H}(A) = \frac{\hbar^2}{2m}(\nabla'^* \nabla' + \frac{R}{6}),$$

the corresponding Landau operator acting on $L^2(S, L)$.

We get this way a family of operators $\hat{H}(A)$ parametrized by $J(S)$, whose elements act on the Hilbert space $L^2(S, L)$. We consider the trivial Hilbertian bundle

$$\mathcal{H} = J(S) \times L^2(S, L) \to J(S).$$

$\mathcal{H}$ is endowed with the flat connection $\tilde{\nabla}$ associated with the trivialization. The family of operators $\hat{H}(A)$ can be thought as a section $\hat{H} \in \Gamma(J(S), \text{End}(\mathcal{H}))$.

### 3.2. Family of operators defined by a relative operator

From a geometrical point of view, a $C^\infty$ family of differential operators parametrized by a differentiable manifold $T$ is represented by a relative differential operator.

**Definition 1** Let $\pi : X \to T$ be a submersion and let $E \to X$, $F \to X$ be two vector bundles. A $k$-th order relative differential operator is a $C^\infty(T)$-linear map

$$D : \Gamma(X, E) \to \Gamma(X, F),$$

that factorizes through the $k$-th order relative jet extension

$$j^k_{X/T} : \Gamma(X, E) \to \Gamma(X, j^k_{X/T}(E)),$$
by means of a $C^\infty(X)$-linear mapping $\overline{D} : \Gamma(X, J^k_{X/T}(E)) \to \Gamma(X, F)$. That is, there is a commutative diagram

$$
\begin{array}{ccc}
\Gamma(X, E) & \xrightarrow{D} & \Gamma(X, F) \\
\downarrow_j & & \downarrow_{\overline{D}} \\
\Gamma(X, J^k_{X/T}(E)) & & \end{array}
$$

For any $t \in T$, set $X_t = \pi^{-1}(t)$ and let $F_t \to X_t$, $E_t \to X_t$ be the restrictions of the vector bundles $E, F$ to the fiber $X_t$.

A relative differential operator $D : \Gamma(X, E) \to \Gamma(X, F)$, induces in a natural way differential operators

$$
D_t : \Gamma(X_t, E_t) \to \Gamma(X_t, F_t)
$$
in this way we obtain the family of differential operators defined by $D$ and parametrized by $T$.

3.3. Family of Landau operators parametrized by $\text{Pic}^0(S)$

Now we deform the line bundle.

It is well known that $\text{Pic}^0(S)$ parametrizes the gauge equivalence classes of flat Hermitian line bundles on $S$.

Let $\mathcal{P}_S \to S \times \text{Pic}^0(S)$ be a Poincaré line bundle. $\mathcal{P}_S$ can be endowed with a unitary connection $\nabla_p$, such that the restriction of $(\mathcal{P}_S, \nabla_p)$ to every fiber $S_x \equiv S \times \{x\}$ of the natural projection $\pi_{\text{Pic}^0(S)} : S \times \text{Pic}^0(S) \to \text{Pic}^0(S)$ is a flat bundle in the gauge equivalence class defined by $\xi \in \text{Pic}^0(S)$.

Let $\mathcal{L} = (L, \langle \cdot, \cdot \rangle, \nabla) \to S$ be a prequantization bundle for $(S, eB)$ and let $\pi_S : S \times \text{Pic}^0(S) \to S$ be the natural projection.

We endow $\pi_S^*\mathcal{L} \otimes \mathcal{P}_S \to S \times \text{Pic}^0(S)$ with the product connection $\nabla = \pi_S^*\nabla \otimes 1 + 1 \otimes \nabla_p$, that can be considered as a relative differential operator for the projection $\pi_{\text{Pic}^0(S)} : S \times \text{Pic}^0(S) \to \text{Pic}^0(S)$.

**Definition 2** The family of Landau operators parametrized by $\text{Pic}^0(S)$ is the family defined by the relative Schrödinger operator

$$
\hat{H} = \frac{\hbar^2}{2m}(\nabla^*\nabla + \frac{R}{6}), \quad (\text{Universal Landau operator})
$$
defined on the line bundle $\pi_S^*\mathcal{L} \otimes \mathcal{P}_S \to S \times \text{Pic}^0(S)$.

Comparing it with the family parametrized by $J(S)$, in this case we get a Hilbertian bundle

$$
\hat{\mathcal{H}} \to \text{Pic}^0(S),
$$
whose fiber at $[L_0] \in \text{Pic}^0(S)$ is $L^2(S, L \otimes L_0)$. Moreover, $\hat{\mathcal{H}}$ is endowed with a flat connection $\hat{\nabla}$ introduced by Bismut [12].

3.4. Equivalence of the families of Landau operators parametrized by $J(S)$ and $\text{Pic}^0(S)$

It can be proved that the Abel-Jacobi isomorphism $J(S) \simeq \text{Pic}^0(S)$ establishes an isomorphism between the families of Landau operators parametrized by $J(S)$ and $\text{Pic}^0(S)$.

We call any of them the family of Landau operators on the Riemann surface $S$ corresponding to the magnetic field $B$. Bearing in mind the isomorphism induced by the Abel-Jacobi map we can use these families in an equivalent way.
4. Adiabatic charge transport on Riemann surfaces

By means of an extension of Kato’s Adiabatic Theorem, Avron, Seiler and Jaffe [9] interpreted Hall conductivity as the adiabatic curvature of a family of spectral projectors parametrized by the Aharanov-Bohm potentials.

A time dependent quantum system is described by a selfadjoint operator $\hat{H}(t)$ defined on a Hilbert space. Let $\tau$ be a parameter defining the temporal scale of the system. The dynamics of the system is determined by the time dependent Schrödinger equation

$$i\hbar \frac{\partial \psi_\tau(t)}{\partial t} = \hat{H}(t/\tau)\psi_\tau(t).$$

The adiabatic limit of the system corresponds to the limit $\tau \to \infty$.

To formulate the Adiabatic Theorem one has to make several hypothesis on the uniparametric family of operators $\hat{H}(\sigma)$ with $\sigma = \frac{t}{\tau}$.

A1. The family $\hat{H}(\sigma)$ is smooth.

A2. The spectrum of $\hat{H}(\sigma)$ has a gap whose size is uniformly bound. Therefore, we can define a projector $P(\sigma)$ onto the part of the spectrum separated by the gap.

A3. The rank of $P(\sigma)$ is finite.

Let $\psi_\tau(0) \in \text{Im} P(0)$, the Adiabatic Theorem states that the evolved state $\psi_\tau(\sigma)$, where $\sigma \sim 1$, is in $\text{Im} P(\sigma)$ up to correction terms.

**Theorem 1** If $\hat{H}(\sigma)$ satisfies conditions A1-A3 and $U_\tau(\sigma)$ is the evolution operator of the Schrödinger equation, then one has

$$U_\tau(\sigma)P(0)U_\tau(\sigma)^{-1} = P(\sigma) + O(\tau^{-1})$$

The magnitude of the error term depends on $\tau$ and the size of the spectral gap.

4.1. Charge transport on a Riemann surface

Let $\{\alpha_1, \ldots, \alpha_{2p}\}$ be a symplectic basis of $H_1(S, \mathbb{Z})$ and let $\{\omega_1, \ldots, \omega_{2p}\}$ be the dual basis of $H^1(S, \mathbb{Z})$ formed by harmonic 1-forms, that is

$$\int_{\alpha_i} \omega_j = \delta_{ij}.$$  

We say that $\omega_j$ is the Aharanov-Bohm potential associated with $\alpha_j$.

If $A(t)$ is a curve in $J(S) = H^1(S, \mathbb{R})/H^1(S, \mathbb{Z})$ then $D = \frac{\partial A}{\partial t} \in \mathfrak{X}(J(S))$ is a vector field that by Faraday’s law gets identified with an electromotive force.

Let $\psi$ be a solution to the Schrödinger equation $i\hbar \frac{\partial \psi}{\partial t} = \hat{H}(A(t))\psi$. The intrinsic expression of the current 1-form associated with the state of the system described by $\psi$ and induced by the variation of the Aharanov-Bohm potentials given by $A(t)$, is

$$I(\psi) = \frac{e}{\hbar} d\langle \psi, \hat{H}\psi \rangle + i\frac{e^2}{\hbar} i_D \langle \tilde{\nabla} \psi, \tilde{\nabla} \psi \rangle$$

where $\tilde{\nabla}$ is the connection on the Hilbertian bundle $\mathcal{H} \to J(S)$.

The 2-form $C = i\frac{e^2}{\hbar} \langle \tilde{\nabla} \psi, \tilde{\nabla} \psi \rangle$ relating currents with electromotive forces is called the conductance 2-form. The current induced along $\alpha_j \in H_1(S, \mathbb{Z})$ by the variation $A(t)$ is given by:

$$I(\alpha_j, \psi) = i_D I(\psi)$$

where $D_j$ is the vector field on $J(S)$ associated with $\omega_j$.

The charge transport along the homology class $\alpha_j$ due to the variation of the Aharanov-Bohm potentials $A(t)$ is given by

$$Q(\alpha_j, \psi) = \int_0^t I(\alpha_j, \psi) dt$$

Finally, the charge transport along $\alpha_j$ while the Aharanov-Bohm potential associated with $\alpha_k$ experiments a unity increment, that is $D = D_k$, is denoted $Q(\alpha_k, \alpha_j)$.  

4.2. Adiabatic charge transport
The idea behind adiabatic transport is to replace $\psi$ by its adiabatic evolution and estimate the difference by means of the Adiabatic Theorem.

Let $P$ be the spectral projector associated with $\psi$ and define the spectral bundle $\hat{P} = \text{Im} P \to J(S)$. This endowed with the connection $\hat{\nabla} = P \circ \nabla$ with curvature $\Omega^b = P \circ (\nabla P \wedge \nabla P) \circ P$,

where $\nabla P$ denotes the covariant derivative of $P \in \text{End} (\mathcal{H})$.

**Definition 3** The ordinary 2-form

$$\Omega^P = \text{Tr} \left( P \circ (\nabla P \wedge \nabla P) \circ P \right),$$

is called the **adiabatic curvature** associated with the spectral bundle $\hat{P} \to J(S)$.

We get this way the adiabatic currents $I_{Ad}(\psi)$ and the adiabatic charge transports $Q_{Ad}(\alpha_k, \alpha_j)$. One checks that the adiabatic conductance 2-form is proportional to the adiabatic curvature $C_{Ad} = i\frac{e^2}{\hbar} \Omega^P$.

The average of the adiabatic quantities are given by

$$\langle I_{Ad}(\alpha_j, \psi) \rangle = \frac{e^2}{\hbar} \frac{i}{2\pi} \int_{\alpha_j} i \partial \Omega^P,$$

$$\langle Q_{Ad}(\alpha_k, \alpha_j) \rangle = e \frac{i}{2\pi} \int_{\alpha_k \ast \alpha_j} \Omega^P,$$

where $\alpha_k \ast \alpha_j$ is the Pontryagin product of the homology classes $\alpha_k, \alpha_j \in H_1(J(S), \mathbb{Z})$.

**Theorem 2** Let us suppose that the family of Landau operators on a Riemann surface $S$ satisfies conditions (A1-3) of the Adiabatic Theorem, then one has

(i) $C = C_{Ad} + O(1/\tau)$.

(ii) $\langle Q(\alpha_k, \alpha_j) \rangle = \langle Q_{Ad}(\alpha_k, \alpha_j) \rangle + O(\tau^{-\infty})$ as $\tau \to \infty$.

The formula for the adiabatic transport

$$\langle Q_{Ad}(\alpha_k, \alpha_j) \rangle = \frac{e}{2\pi} \int_{\alpha_k \ast \alpha_j} \Omega^P,$$

is the geometric formulation of Kubo’s formula.

Therefore, on a surface $S$ of genus $p$ there are $p(2p - 1)$ different transport coefficients, and any of them gives rise to a mean Hall conductivity

$$\langle \sigma_H(\alpha_k, \alpha_j) \rangle = \frac{e^2}{\hbar} \frac{i}{2\pi} \int_{\alpha_k \ast \alpha_j} \Omega^P.$$

This is the most common form of Kubo’s formula.

Since $\frac{i}{2\pi} \Omega^P = \frac{i}{2\pi} \text{Tr} (\Omega^P)$ represents the first Chern class $c_1(\hat{P}) \in H^2(J(S), \mathbb{Z})$ of the spectral bundle $\hat{P} \to J(S)$, one has that $\langle Q_{Ad} \rangle$ and $\langle \sigma_H \rangle$ depend only on the cohomology class $c_1(\hat{P})$.

As a consequence, Theorem 2 implies that the charge transport, as well as the Hall conductivity are quantized up to infinitesimal terms in the adiabatic parameter.
5. Nahm transforms on Riemann surfaces
In order to be able to describe the spectral bundles associated with the Landau levels and to calculate their topological invariants, we introduce the Nahm transform associated to the jacobian variety \( J(S) \) of the Riemann surface \( S \). We assume the identification \( J(S) \simeq \text{Pic}^0(S) \).

We have seen that \( J(S) \) parametrizes the gauge equivalence classes of flat line bundles and the restriction of the Poincaré bundle \( \mathcal{P}_S \to S \times J(S) \) endowed with its unitary connection \( \nabla_P \) to any fiber \( S_\xi = S \times \{ \xi \} \) is a flat line bundle \( (\mathcal{P}_\xi, \nabla_{P, \xi}) \to S \) in the equivalence class defined by \( \xi \in J(S) \).

Let \( E \to S \) be a Hermitian vector bundle with a unitary connection \( \nabla \). We endow \( E \otimes \mathcal{P}_\xi \) with the product connection \( \nabla_\xi = \nabla \otimes 1 + 1 \otimes \nabla_{P, \xi} \).

Let us consider the spinor bundle \( \mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^- \) of \( S \) as a spin\(^c\) manifold, with \( \mathcal{S}^+ = \Lambda^{0,0} T^* S \), \( \mathcal{S}^- = \Lambda^{0,1} T^* S \) and let \( \nabla_{\mathcal{S}} \) be the spinorial connection of \( \mathcal{S} \).

We get a family of coupled Dirac operators:

\[
\delta_\xi : \Omega^0(S, \mathcal{S}^+ \otimes E \otimes \mathcal{P}_\xi) \to \Omega^0(S, \mathcal{S}^- \otimes E \otimes \mathcal{P}_\xi).
\]

By the Atiyah-Singer theorem for families, the difference bundle

\[
\{ \ker \delta_\xi \} - \{ \coker \delta_\xi \},
\]

defines an object \( \text{Ind}(D) \) in the \( K \)-theory of \( J(S) \), the index of \( D \).

If either \( \{ \ker \delta_\xi \} \) or \( \{ \coker \delta_\xi \} \) has constant rank, then \( \ker \delta \) and \( \coker \delta \) are vector bundles over \( J(S) \) and moreover

\[
\text{Ind}(D) = [\ker \delta] - [\coker \delta] \in K(J(S)).
\]

5.1. The transformed bundle

**Definition 4** A Hermitian vector bundle with unitary connection \( (E, \nabla) \to S \) is an IT\textsuperscript{0}-pair (Index Theorem) if either \( \coker D = 0 \) or \( \ker D = 0 \). It is an

- IT\textsuperscript{0}-pair if \( \coker D = 0 \).
- IT\textsuperscript{1}-pair if \( \ker D = 0 \).

The transformed bundle of an IT\textsubscript{k}-pair is the finite rank vector bundle

\[
\tilde{E} = (-1)^k \text{Ind}(D) \to J(S).
\]

\( \mathcal{P}_S \to S \times J(S) \) is a holomorphic line bundle and the connection \( \nabla_P \) is compatible with the holomorphic structure.

The curvature \( F^n \) of \( (E, \nabla) \) has \((1,1)\) type, therefore \( E \) is a holomorphic vector bundle. The Dirac operator \( \delta_\xi \) coincides with the Dolbeault-Dirac operator

\[
\delta_\xi = \sqrt{2} (\bar{\partial}_E \otimes \mathcal{P}_\xi + \partial_{E \otimes \mathcal{P}_\xi})
\]

of \( E \otimes \mathcal{P}_\xi \).

5.2. The transformed Hermitian metric

Let \( \pi_S : S \times J(S) \to S \) be the natural projection and let \( \mathcal{H}_{x_\xi}^\infty \) be the spaces of \( C^\infty \)-sections of the vector bundles \( \pi_\xi^*(\mathcal{S}^+ \otimes E) \otimes \mathcal{P}_S \to S \times J(S) \).

\( \mathcal{H}_{x_\xi}^\infty \) can be thought of as infinite dimensional vector bundles \( \mathcal{H}_{x_\xi}^\infty \to J(S) \) whose fibers \( \mathcal{H}_{x_\xi}^\infty \) at \( \xi \in J(S) \) are the spaces of smooth sections \( \Gamma(S, \mathcal{S}^+ \otimes E \otimes \mathcal{P}_\xi) \). On each fiber we define the \( L^2 \) Hermitian metric

\[
\langle s_1, s_2 \rangle_\xi = \int_S \langle s_1, s_2 \rangle \omega, \quad s_1, s_2 \in \mathcal{H}_{x_\xi}^\infty.
\]
We introduce the Hilbertian bundles $\mathcal{H}_\pm$ whose fibers $\mathcal{H}_{\pm,\xi}$ are the $L^2$-completion of $\mathcal{H}_{\pm,\xi}^\infty$ with respect to the previous metric.

If $(E, \nabla)$ is an IT$_i$-pair, by the Regularity Theorem for elliptic operators $\hat{E}$ is a subbundle of $\mathcal{H}_{\pm,\xi}^\infty$, hence we get an induced Hermitian metric on $\hat{E}$.

5.3. The transformed connection

Following Bismut, we define a connection $\tilde{\nabla}$ on $\mathcal{H}_{\pm,\xi}^\infty$ as follows

$$\tilde{\nabla}Dh = \nabla^1 Dh, \quad \text{for every } D \in \mathfrak{X}(J(S)), \ h \in H_{\pm,\xi}^\infty,$$

$D^H$ is the natural lift of the vector field $D$ on $J(S)$ to $S \times J(S)$ and $\nabla^1$ is the product connection of $\pi_S^*(S^\pm \otimes E) \otimes \mathcal{P}_S$. One checks that $\tilde{\nabla}$ is a flat connection.

We endow $\hat{E}$ with the connection $\hat{\nabla} = \mathcal{P} \circ \tilde{\nabla}$, where $\mathcal{P}$ is the orthogonal projection onto $\hat{E}$.

**Definition 5** Let $(E, \nabla)$ be an IT$_i$-pair. The pair $(\hat{E}, \hat{\nabla})$ is called the Nahm transform of $(E, \nabla)$.

5.4. The curvature of the transformed connection

Let $(\hat{E}, \hat{\nabla})$ be the Nahm transform of an IT$_i$-pair. Taking into account that $\tilde{\nabla}$ is flat, one proves that the curvature of $\hat{\nabla}$ is

$$F_{\hat{\nabla}} = \mathcal{P} \circ (\hat{\nabla}\mathcal{P} \wedge \hat{\nabla}\mathcal{P}) \circ \mathcal{P}.$$ 

6. Integral functors and Fourier-Mukai transforms

Now we introduce an integral functor on the Riemann surface $S$ related to the Fourier-Mukai transform of its jacobian variety $J(S)$. This is the holomorphic analogue of the $C^\infty$ Nahm transform just described.

The use of this integral functor has the following advantages:

(i) Computation of the topological invariants of the spectral bundles.

(ii) Establish geometrical properties of these bundles.

We will see in Section 6.4 that this integral functor is compatible, in a precise sense, with the Nahm transform.

If $X$ is a complex projective variety, $D(X)$ denotes its bounded derived category of complexes of coherent $\mathcal{O}_X$-modules.

Let $X, Y$ be two projective varieties and let $\mathcal{P} \rightarrow X \times Y$ be a vector bundle. We have the natural projections

$$\begin{array}{c}
\pi_X & X \times Y & \pi_Y \\
\downarrow & \downarrow & \downarrow \\
X & \rightarrow & Y
\end{array}$$

We define an integral functor with kernel $\mathcal{P}$ as

$$\Phi^\mathcal{P}_{X \rightarrow Y} : D(X) \rightarrow D(Y)$$

$$\mathcal{E}^* \mapsto \Phi^\mathcal{E}_{X \rightarrow Y}^\mathcal{E}^* = R\pi_{Y*}(\pi_X^*(\mathcal{E}^*) \otimes \mathcal{P}).$$

If $\Phi^\mathcal{P}_{X \rightarrow Y} : D(X) \rightarrow D(Y)$ is an equivalence of categories, we say that it is a Fourier-Mukai transform.
6.1. WIT (Weak Index Theorem) and IT (Index Theorem) conditions.

We write $\Phi = \Phi_{X,S}$ and $\Phi^i(E^*) = H^i(\Phi(E^*))$ is the $i$-th cohomology sheaf of the complex $\Phi(E^*)$.

We denote by $\mathcal{P}_y$ the restriction of $\mathcal{P}$ to $X_y = X \times \{y\}$.

**Definition 6** A sheaf $E$ on $X$ is $\Phi$-WIT$_i$ if $\Phi^i(E) = 0$ for $j \neq i$ and we denote by $\widehat{\Phi}$ the unique non vanishing sheaf $\Phi^i(E)$.

A sheaf $E$ on $X$ is $\Phi$-IT$_i$ if $H^j(X, E \otimes \mathcal{P}_y) = 0$ for every $j \neq i$ and every $y \in Y$.

By the base change theorem of algebraic geometry one has:

**Theorem 3** $E$ is $\Phi$-IT$_i$ $\iff$ it is $\Phi$-WIT$_i$ and $\widehat{\Phi} = \Phi^i(E)$ is a vector bundle.

6.2. The integral functor associated to the jacobian

The first example of a Fourier-Mukai transform was given by Mukai. Let $X$ be an abelian variety, $\tilde{X}$ its dual abelian variety and let $\mathcal{P} \to X \times \tilde{X}$ be the Poincaré bundle then the Mukai transform is $S \equiv \Phi_{X,\tilde{X}}$.

Let $S$ be a Riemann surface and let $\mathcal{P}_S \to S \times J(S)$ be a Poincaré bundle. We define the integral functor $\Phi_J : D(S) \to D(J(S))$ as $\Phi_J = \Phi_{P_S \to J(S)}$. Since $J(S)$ is an automodual abelian variety, its Mukai transform defines a functor

$$ S : D(J(S)) \to D(J(S)) $$

There is a functorial isomorphism

$$ \Phi_J \simeq S \circ \alpha_* $$

where $\alpha : S \to J(S)$ is the Abel-Jacobi immersion.

6.3. Change of topological invariants under $\Phi_J$

Later, in Section 7, we will see that the spectral bundles associated to the family of Landau operators can be expressed as $\widehat{\mathcal{P}} = \Phi_J(L)$ for a suitable line bundle $L \to S$. Since we have seen that the Hall conductivity is determined by the first Chern class of $\widehat{\mathcal{P}}$, a key problem is the computation of the topological invariants of the transformed sheaves $\Phi_J(L)$.

**Proposition 1** If $E$ is a coherent sheaf on $S$ with Chern character $\text{ch}(E) = (r,d)$, then $\text{ch}(\Phi_J(E)) \in H^*(J(S), \mathbb{Q})$ is given by

$$ \text{ch}(\Phi_J(E)) = (d + r(1 - p), -r[\Theta], 0, \ldots, 0) $$

where $p$ is the genus of $S$ and $[\Theta]$ is the cohomology class defined by the theta divisor $\Theta \subset J(S)$.

There exists a line bundle $L_\Theta \to J(S)$ such that $c_1(L_\Theta) = [\Theta]$ and the Appel-Humbert theorem identifies $c_1(L_\Theta)$ with the cup product

$$ c_1(L_\Theta)(\omega_1, \omega_2) = \int_S \omega_1 \wedge \omega_2. $$

6.4. Compatibility between $\Phi_J$ and the Nahm transform

Let $E \to S$ be a Hermitian bundle with unitary connection $\nabla$.

We have seen that the spin$^c$ Dirac operator $D_\xi$ gets identified with the Dolbeault-Dirac operator of $E \otimes \mathcal{P}_\xi$.

Hodge theory and the Dolbeault isomorphism give isomorphisms

$$ \ker D_\xi \simeq H^0(S_\xi, E \otimes \mathcal{P}_\xi) \quad (1) $$

$$ \text{coker } D_\xi \simeq H^1(S_\xi, E \otimes \mathcal{P}_\xi). \quad (2) $$
Thus, $E$ is IT$_i$ with respect to $\Phi_j \leftrightarrow (E, \nabla)$ is an IT$_i$-pair with respect to the Nahm transform.

There is a natural isomorphism of $C^\infty$ vector bundles induced by Hodge theory

$$\phi_E: \hat{E} \sim \Phi_j^i(E).$$

**Theorem 4** The connection $\hat{\nabla}$ is compatible with the holomorphic structure of $\Phi_j^i(E)$ and $\phi_E: \hat{E} \sim \Phi_j^i(E)$ is a holomorphic isomorphism.

7. Spectral bundles of the family of Landau operators

Let $(S, g)$ be a compact surface of genus $p$ with constant curvature $R$, let $L = (L, \langle \cdot, \cdot \rangle, \nabla) \to S$ be a prequantization bundle and let $\hat{H} = \frac{\hbar^2}{2m}(\nabla^* \nabla + \frac{R}{\hbar})$ be the associated Landau operator.

In order to explicitly describe the spectral bundles we must recall the relationship between the spectral geometry of $\hat{H}$ and the holomorphic structure of $L$. The results we need are contained in [15]. what we need is the following result.

**Theorem 5** Suppose that $|\text{gr}(L)| > \text{gr}(K_S)$, where $K_S$ is the canonical line bundle of $S$.

(i) If $p = 1$, the spectrum of $\hat{H}$ is the set

$$\text{Spec}(\hat{H}) = \{E_q = \hbar|\omega|(q + \frac{1}{2}), \forall q \in \mathbb{Z}, q \geq 0\},$$

where $\omega = \frac{\hat{H}}{m}$.

(ii) If $p > 1$ and $k(L) = \frac{\text{deg}(L)}{\text{deg}(K_S)}, R = -\frac{2}{\pi \hbar}$, then the set

$$\text{Spec}^d(\hat{H}) = \{E_q = \frac{\hbar^2}{2mr^2} \left((k(L))(2q + 1) - q(q + 1) - \frac{1}{3}\right), \forall q \in \mathbb{Z}, 0 \leq q < |k(L)| - 1\}.$$

is contained in the spectrum of $\hat{H}$ and are the lowest eigenvalues.

In both cases the space of eigensections of eigenvalue $E_q$ gets identified either with $H^0(S, K_S^{-q} \otimes L)$ if $\deg L > 0$ or $H^0(S, K_S^{-q} \otimes L^{-1})$ if $\deg L < 0$.

**Definition 7** The subset $\text{Spec}_{\text{hol}}(\hat{H})$ of $\text{Spec}(\hat{H})$ defined in the previous theorem is called the holomorphic spectrum of $\hat{H}$. The eigensections with eigenvalue $E_q \in \text{Spec}_{\text{hol}}(\hat{H})$ form the $q$-th holomorphic Landau level.

According to Theorem 5, $\text{Spec}_{\text{hol}}(\hat{H})$ does not depend on the chosen prequantization bundle. Thus, $\text{Spec}_{\text{hol}}(\hat{H})$ is constant for the family of Landau operators parametrized by $J(S) \simeq \text{Pic}^0(S)$ and we denote it $\sigma_{\text{hol}}$.

In particular the eigensections of the Landau operator $\hat{H}(L_0)$ corresponding to the flat line bundle $L_0$ are given by

- $H^0(S, K_S^{-q} \otimes L \otimes L_0)$ if $\deg L > 0$, and by
- $H^0(S, K_S^{-q} \otimes L^{-1} \otimes L_0)$ if $\deg L < 0$.

We assume that $\text{gr} L > 0$, since if it were not the case it would be enough to replace $L$ by $L^{-1}$ in order to obtain the corresponding results. Hence, for any integer $q \geq 0$, with $\text{gr} (K_S^{-q} \otimes L) > \text{gr} K_S$, there exists an eigenvalue $E_q \in \sigma_{\text{hol}}$ corresponding to a holomorphic Landau level.
Lemma 1 The family $P_q$ of spectral projections associated with an eigenvalue $E_q \in \sigma_{\text{hol}}$ fulfills the hypothesis (A1-A3) of the adiabatic Theorem.

Therefore, the $q$-th holomorphic Landau level defines a spectral bundle $\hat{P}_q \to J(S)$ and we endow it with the induced connection $\hat{\nabla}_q = P_q \circ \nabla$. Let $\nabla_q$ be the connection on $K_S^{-q} \otimes L$ obtained by twisting the connection of $L$ with the connection of $K_S^{-q}$ induced by the Levi-Civita connection.

Theorem 6 Let $\hat{P}_q \to J(S)$ be the spectral bundle defined by the $q$-th holomorphic Landau level, we have:

(i) The pair $(\hat{P}_q, \hat{\nabla}_q)$ is the Nahm transform of $(K_S^{-q} \otimes L, \nabla_q)$.

(ii) There exists a holomorphic isomorphism $\hat{P}_q \sim \Phi_0 J(K_S^{-q} \otimes L)$.

7.1. Determination of adiabatic transport and Hall conductivity

The equivalence of the Nahm transform with the integral functor associated to the jacobian, allows us to study the spectral bundles by means of the machinery of integral functors. In particular by Proposition 1 we have the following result.

Corollary 1 The first Chern class of the spectral bundle $\hat{P}_q \to J(S)$ is $c_1(\hat{P}_q) = c_1(\mathcal{L}_\Theta)$, where $\mathcal{L}_\Theta$ is the principal polarization of $J(S)$ defined by the theta divisor $\Theta$.

Now we can determine the adiabatic transport and Hall conductivity.

Theorem 7 The mean adiabatic transport $\langle Q_{\text{Ad}}(\hat{P}_q) \rangle$ and the mean Hall conductivity $\langle \sigma_H(\hat{P}_q) \rangle$ of the $q$-th holomorphic Landau level are

$$\langle Q_{\text{Ad}}(\hat{P}_q) \rangle(\alpha, \beta) = -e(\alpha, \beta),$$

$$\langle \sigma_H(\hat{P}_q) \rangle(\alpha, \beta) = -\frac{e^2}{h}(\alpha, \beta),$$

where $(\ , \ )$ denotes the intersection product on $H_1(S, \mathbb{Z})$.

7.2. Stability of spectral bundles

The vector bundle $\hat{L}_d = \Phi_0^0(L_d)$, where $L_d \to S$ is a line bundle of degree $d$, with $d > \deg K_S$, is called in the literature the $d$-th Picard sheaf on $J(S)$. These sheaves have been studied by several authors, among them we may cite Ein and Lazarsfeld [17]. Using their results we can prove the following:

Theorem 8 The spectral bundles $\hat{P}_q \to J(S)$ associated with the holomorphic Landau levels are holomorphic vector bundles with respect to the polarization of the jacobian $J(S)$ defined by the theta divisor $\Theta$.

The stability of the spectral bundles $\hat{P}_q \to J(S)$ is a remarkable fact, since in the description of the fractional quantum Hall effect proposed by Varnhagen [18] there also appear stable bundles. These facts seem to suggest a connection between the stability of spectral bundles and the interpretation of quantum Hall effect.
8. Analytic torsion and Quillen metrics on holomorphic determinant bundles
Although we have determined the Hall conductivity as the integral of the adiabatic conductance, our final aim is the computation of the adiabatic conductance that controls the fluctuations of the Hall conductivity and is proportional to the adiabatic curvature.

The adiabatic curvature $\Omega^P$ of the spectral bundle $(\hat{P}_q, \hat{\nabla}_q) \to J(S)$ is the trace of its curvature, that is $\Omega^P = \text{Tr} \Omega^P_q$. Equivalently, $\Omega^P_q$ is the curvature of the connection $\text{det} \hat{\nabla}_q$ induced on the determinant bundle $\text{det} \hat{P}_q \to J(S)$, thus

$$\Omega^P_q = \Omega_{\text{det} \hat{\nabla}_q}.$$

On the other hand, the connection $\hat{\nabla}_q$ on the holomorphic vector bundle $\hat{P}_q \to J(S)$ gets identified with the Chern connection of the Hermitian metric $\langle \cdot, \cdot \rangle_{L^2}$ induced by the $L^2$ metric of the Hilbertian bundle $H_L \to J(S)$.

Hence, the computation of the adiabatic curvature is reduced to finding the curvature of the Chern connection of the determinant bundle $\text{det} \hat{P}_q \to J(S)$ for the Hermitian metric $\langle \cdot, \cdot \rangle_{L^2}$ induced by the $L^2$ metric.

However, our strategy is to compute first the curvature of the Chern connection on the determinant bundle with respect to the Quillen metric, since:

(i) The metric $\langle \cdot, \cdot \rangle_{L^2}$ is conformal to the Quillen metric.

(ii) It is the natural $C^\infty$ metric defined on the determinant bundles in terms of the analytic torsion, or equivalently, in terms of the zeta-regularized determinants of certain elliptic operators.

In order to calculate the curvature of Quillen metric we use the techniques developed by Bismut, Gillet and Soulé [12, 13, 14].

Let $\pi : X \to T$ be a holomorphic submersion and let $E \to X$ be a holomorphic vector bundle. The direct image $R\pi_* E$ admits a determinant that is a holomorphic line bundle $\text{det} R\pi_* E \to T$ whose dual $\lambda_{\text{KM}}^{\text{det}}(E) = \langle \text{det} R\pi_* E \rangle^{-1} \to T$ is called the Knudsen-Mumford determinant.

For every $t \in T$, let $X_t = \pi^{-1}(t)$ be the fiber over $t$ and let $H^i(X_t, E_t)$ be the cohomology of the restriction $E_t$ of $E$ to $X_t$. For every fiber $\lambda_{\text{KM}}^{\text{det}}(E)_t$ of the Knudsen-Mumford determinant there is a canonical identification

$$\lambda_{\text{KM}}^{\text{det}}(E)_t = \bigotimes_{i \geq 0} \text{det} H^i(X_t, E_t)^{(-1)^{i+1}}.$$

Let $g^{X/T}$ be a $C^\infty$ relative Kähler metric on $\pi : X \to T$, $H_E$ a Hermitian metric on $E \to X$ and let $D = \sqrt{2(\bar{\partial}_E + \partial^*_E)}$ be the Dolbeault-Dirac operator of $E$.

One defines a family of vector spaces $\lambda(E) \to T$ whose fibers are

$$\lambda(E)_t = (\text{det} \text{Ker} D_t)^{-1} \otimes \text{det} \text{Coker} D_t,$$

$\lambda(E) \to T$ admits a natural structure of holomorphic line bundle, called the Bismut-Gillet-Soulé determinant.

The $L^2$ metric induces a Hermitian metric $\langle \cdot, \cdot \rangle_{L^2}$ on $\lambda(E)$ that in general, due to the changes of dimension in $\text{Ker} D_t$, it is not $C^\infty$. However, if $\text{det}' D^* D$ is the function on the parameter space $T$ whose value at $t \in T$ is the regularized determinant of $\text{det}' D^* D(t) = \text{det}' D_t^* D_t$ of $D_t$ then Quillen proved the following result.

**Theorem 9 (Quillen)** The Quillen metric, defined on the determinant bundle $\lambda(E)$ as

$$\langle \cdot, \cdot \rangle_Q = \text{det}' D^* D \langle \cdot, \cdot \rangle_{L^2},$$

is a $C^\infty$ Hermitian metric.
The value of the function \( T(E, H_E, g^{X/T}) = (\det D^t D)^{\frac{1}{2}} \) at \( t \in T \) coincides with the Ray-Singer analytic torsion \( T(E_t, H_{E_t}, g^{X/t}) \) of the vector bundle \( E_t \rightarrow X_t \). We say that \( T(E) = T(E, H_E, g^{X/T}) \) is the relative analytic torsion of \( E \rightarrow X \).

For every \( t \in T \) there is a canonical isomorphism between the fibers

\[
\lambda(E)_t \simeq \lambda^{\text{KM}}(E)_t,
\]

of the Bismut-Gillet-Soulé and Knudsen-Mumford determinants.

One says that \( \pi : X \rightarrow T \) is locally Kähler if there exists an open cover \( \mathcal{U} \) of \( T \) such that for every \( U \in \mathcal{U} \) there exists a Kähler metric on \( \pi^{-1}(U) \).

Let \( R^{X/T}, \Omega^E \) be the curvatures of \( (\Lambda^{1,0} T^*(X/T), g^{X/T}) \) and \( (E, H_E) \), respectively. We have the following key result.

**Theorem 10 (Bismut-Gillet-Soulé)** Suppose that \( \pi : X \rightarrow T \) is locally Kähler. Then

(i) The canonical identification of the fibers \( \lambda(E)_t \simeq \lambda^{\text{KM}}(E)_t \) defines a holomorphic isomorphism

\[
\lambda(E) \simeq \lambda^{\text{KM}}(E).
\]

(ii) The curvature of the Quillen metric \( \lambda(E) \simeq \lambda^{\text{KM}}(E) \) is the degree 2 component of the differential form on \( T \)

\[
2\pi i \int_{X/T} \text{Td}(\frac{iR^{X/T}}{2\pi}) \wedge \text{Tr}(\exp(\frac{i\Omega^E}{2\pi})),
\]

where \( \int_{X/T} \) denotes integration along the fibers.

**9. Adiabatic conductance of spectral bundles**

The natural projection \( \pi_{J(S)} : S \times J(S) \rightarrow J(S) \) is locally Kähler. Therefore, we can apply the results of Bismut, Gillet and Soulé.

**Theorem 11** The adiabatic curvature \( \Omega^P_q \) of \( \hat{P}_q \rightarrow J(S) \) is

\[
\frac{i\Omega^P_q}{2\pi} = \text{Im} H_\Theta + \frac{i}{\pi} \partial \bar{\partial} \ln(T(\pi_S^* (K_S^{-q} \otimes L) \otimes \mathcal{P}_S)),
\]

where \( H_\Theta \) is the principal polarization of \( J(S) \) and \( T(\pi_S^* (K_S^{-q} \otimes L) \otimes \mathcal{P}_S) \) is the relative analytic torsion.

**Corollary 2** The adiabatic conductance \( C_{\text{Ad}}(\hat{P}_q) = i\frac{e^2}{\hbar} \Omega^P_q \) of the \( q \)-th holomorphic Landau level is

\[
C_{\text{Ad}}(\hat{P}_q) = 2\pi e^2 \hbar \{ \text{Im} H_\Theta + \frac{i}{\pi} \partial \bar{\partial} \ln(T(\pi_S^* (K_S^{-q} \otimes L) \otimes \mathcal{P}_S)) \}.
\]

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