Single-field consistency relation and $\delta N$-formalism

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Abstract. According to the equivalence principal, the long wavelength perturbations must not have any dynamical effect on the short scale physics up to $\mathcal{O}(k_s^2/k_L^2)$. Their effect can be always absorbed to a coordinate transformation locally. So any physical effect of such a perturbation appears only on scales larger than the scale of the perturbation. The bispectrum in the squeezed limit of the curvature perturbation in single-field slow-roll inflation is a good example, where the long wavelength effect is encoded in the spectral index through Maldacena’s consistency relation. This implies that one should be able to derive the bispectrum in the squeezed limit without resorting to the in-in formalism in which one computes perturbative corrections field-theoretically. In this short paper, we show that the $\delta N$ formalism as it is, or more generically the separate universe approach, when applied carefully can indeed lead to the correct result for the bispectrum in the squeezed limit. Hence despite the common belief that the $\delta N$ formalism is incapable of recovering the consistency relation within itself, it is in fact self-contained and consistent.

Keywords: cosmological perturbation theory, inflation

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1 Introduction

The $\delta N$ formalism is a powerful tool for calculating the curvature perturbation from inflation without resorting to the standard cosmological perturbation theory. The main advantage of this method is that the superhorizon curvature perturbation can be obtained solely by solving background homogeneous equations. The $\delta N$ formalism in essence is based on the spatial gradient expansion at its leading order, which is called the separate universe approach. It reveals that any smoothed patch of a perturbed universe on the Hubble horizon scale evolves locally as a homogeneous and isotropic universe. Thus a long wavelength perturbation on scales much greater than the Hubble horizon size can be treated as a homogeneous perturbation to a fiducial background universe on each Hubble horizon scale. An important consequence of this fact is that superhorizon curvature perturbations are equivalent to variations of total number of $e$-folds of expansion of background geometry [1–5]. There has been an attempt to extend the above idea to anisotropic backgrounds [6, 7].

Nevertheless, a seemingly drawback of the this formalism is when one tries to calculate the bispectrum of the curvature perturbation in the simplest inflationary model, i.e., single-field slow-roll inflation. In this case, the non-Gaussianity is local in the sense that the bispectrum is dominated by its squeezed limit configuration, and is conveniently characterized by the non-Gaussianity parameter $f_{NL}$. In the $\delta N$ formalism, $R_c = N_{\phi} \delta \phi_s + N_{\phi \phi} \delta \phi_s \delta \phi_s + \ldots$ to the second order, conventionally it is computed as

$$\frac{6}{5} f_{NL}(p)_{p=p_1=p_2=p_3} = \left[ \frac{N_{\delta \phi}}{N_{\phi}} + \frac{\langle \delta \phi_s(p) \delta \phi_s(p - p_3) \delta \phi_s(p_3) \rangle}{2 N_{\phi} P_{\delta \phi_s}(p_3) P_{\delta \phi_s}(p)} \right] = \eta/2 + \epsilon, \quad (1.1)$$

where the first term in the square brackets (or the $\eta/2$ term) is the contribution captured by the conventional $\delta N$ formalism, while the second term (or the $\epsilon$ term) is regarded as an intrinsic non-Gaussianity of the inflaton fluctuations that the $\delta N$ formalism seemingly fails to address [8]. However, an alternative $\delta N$ approach proposed in [9].

However, according to Einstein’s equivalence principle, the leading order physical effect of a large-scale inhomogeneity on the small-scale perturbations is of the order of $(k_l/k_s)^2$, where $k_l$ and $k_s$ are the characteristic wavenumbers of the large-scale and small-scale perturbations, respectively. Equivalently, one can say that long wavelength perturbations have no “dynamical” effect on the short modes and their only effect is to modify the homogeneous
background from the fiducial background universe. This strongly suggests that the above intrinsic non-Gaussianity is not genuinely intrinsic.

The main purpose of this paper is to show that, contrary to general belief, the delta N formalism, as it is, can in fact give the correct result for the non-Gaussianity parameter $f_{NL}$. In particular, we explicitly show that the seemingly “intrinsic” contribution can be captured via careful application of the standard $\delta N$ formalism.

2 Gradient expansion and separate universe approach

It was shown in [4] that the uniform-$N$ foliation of the spacetime has the unique property that the scalar field equations as well as the Einstein equations take the background form for superhorizon perturbations. The flat slicing, in which the spatial metric takes the spatially flat form, belongs to this family of foliations on superhorizon scales. Thus in terms of the local number of e-folds $N$ (where local means on the scale of the Hubble horizon) as time variable,

$$N = \int \tilde{H} d\tau$$

(2.1)

where $\tilde{H} = \theta(x,t)/3$ and $d\tau = (1 + A)dt$, the Friedmann and the scalar field equations are expressed as [4, 12]

$$\tilde{H} \frac{d}{dN} \left( \tilde{H} \frac{d}{dN} \phi(x,N) \right) + 3\tilde{H}^2 \frac{d}{dN} \phi(x,N) + V_{,\phi}(\phi(x,N)) = 0,$$

(2.2)

$$3M_P^2\tilde{H}^2 = \rho(x,N) \approx V(\phi(x,N)),$$

(2.3)

where $M_P$ is the Planck mass. This shows that a large-scale scalar field perturbation on every smoothed patch of the Hubble horizon scale evolves independently as in a homogeneous FLRW universe. This justifies the separate universe approach for this model.

On the other hand, quantum field theory (QFT) tells us the amplitude of quantum vacuum fluctuations. The flat slicing is a privileged foliation of spacetime for QFT computations as well because it gives a closed form of the action in terms of the scalar field perturbation. In particular, the amplitude of the $\delta \phi$ on a flat slice is easily computed and whose Fourier amplitude is given by $\delta \phi_k = H/\sqrt{2k^3}$ at $k \lesssim H_0$. In real space this can be regarded as a random walk of the inflaton with $\delta \phi = \pm H/(2\pi)$ in one e-folding time on each Hubble horizon patch [10, 11]. Namely,

$$<\delta \phi_0^2(x,t)> = \left( \frac{H(t)}{2\pi} \right)^2,$$

(2.4)

where the subscript 0 on $\delta \phi$ is attached since we have assumed that the Hubble parameter is space-independent. However, on scales much larger than the Hubble scale, $H$ may spatially vary $H = H(x,t)$. This implies

$$<\delta \phi^2(x,t)> = \left( \frac{H(x,t)}{2\pi} \right)^2.$$

(2.5)

3 Large scale modulation in $\delta \phi$

What we have discussed above has a simple outcome: a short wavelength perturbation (of $\delta \phi$ on the Hubble horizon scale) is modulated by a long wavelength perturbation (of the Hubble
parameter $H$). In the language of the separate universe approach, or the gradient expansion at leading order, the amplitude of $\delta \phi$ on the flat slicing at the time when the scale of interest has just exceeded the horizon size is proportional to the local Hubble parameter $H(x)$ at that moment. Thus denoting the fiducial value of $H(x, t)$ by $\bar{H}(t)$, and the corresponding inflaton perturbation by $\delta\bar{\phi}(x, t)$, the actual inflaton perturbation $\delta\phi(x, t)$ is modulated as

$$\delta\phi(x) = \delta\bar{\phi}(x) \frac{H(x)}{\bar{H}},$$  \hspace{1cm} (3.1)

where and in what follows we omit the time $t$ from the arguments for notational simplicity. The idea is shown in figure 1.

Now let us evaluate $H(x)$. At leading order in the slow-roll approximation, we have $3M_p^2H^2 = V(\phi)$. Thus we have

$$\frac{H^2(x)}{H^2} = \frac{V(\phi + \delta\phi_L)}{V(\phi)} = 1 + \frac{V'}{V} \delta\phi_L(x) + \cdots,$$  \hspace{1cm} (3.2)

or

$$\frac{H(x)}{H} = 1 + \frac{1}{2} \frac{V'}{V} \delta\phi_L(x) + \cdots,$$  \hspace{1cm} (3.3)

where $\delta\phi_L$ indicates that it is the long-wavelength part of $\delta\phi$. Inserting the above to eq. (3.1) gives

$$\delta\phi(x) = \delta\bar{\phi}(x) \left( 1 + \frac{1}{2} \frac{V'}{V} \delta\phi_L(x) \right) + \cdots.$$  \hspace{1cm} (3.4)

In passing, it may be worth mentioning that the above estimate of $\delta H$ is perfectly consistent with the result of linear perturbation analysis at long wavelength limit. If we recall the fact that the curvature perturbation on the flat slicing $R_c$ is conserved on superhorizon scales, and that the gauge transformation from the comoving slicing to the flat slicing on which $R = 0$ is given by

$$0 = R_c - HT,$$  \hspace{1cm} (3.5)
hence the lapse function perturbation on the flat slicing $A_f$ on large scales is given by

$$A_f = 0 - \dot{T} = -\frac{d}{dt} \left( \frac{R_c}{H} \right) = -\epsilon R_c,$$

(3.6)

where $\epsilon \equiv -\dot{H}/H^2$ and the subscript $f$ is for flat slicing. Thus one finds

$$\frac{\delta H_f}{H} = -A_f = \epsilon R_c = -\frac{H}{\dot{\phi}} \delta \phi_f.$$

(3.7)

We can easily show that this coincides with eq. (3.3) in the slow-roll limit.

4 Revisiting bispectrum computation in the $\delta N$ formalism

Let us start by a short review of local non-Gaussianity. The so-called local type non-Gaussianity is defined as

$$R_c(x) = R_{cg}(x) + 3 f_{NL} R_{cg}^2(x).$$

(4.1)

The local type non-Gaussianity predicts a non-vanishing correlation between modes with very different momenta [15]. In particular, the above leads to the bispectrum of the curvature perturbation as

$$\langle R_c(k_1)R_c(k_2)R_c(k_3) \rangle = +6 f_{NL} \mathcal{P}_R \sum_i k_i^3 \prod_i k_i^3,$$

(4.2)

where $\mathcal{P}_R \propto P_R(k)k^3$ is approximately $k$-independent owing to the fact that $P_R(k) \sim 1/k^3$, and hence the bispectrum is peaked at the degenerate triangle of momenta, or the squeezed limit. Conversely, in the single-field slow-roll inflation, although the bispectrum is non-vanishing for non-degenerate triangles, the most important contribution to the bispectrum comes from the squeezed limit which can be well described by the local-type non-Gaussianity, which indicates that the non-linearities are of the superhorizon nature.

According to the above result, the non-linearity of the curvature perturbation in single-field slow-roll inflation is significant when there is a correlation between short and long wavelength modes. In real space this means the most important non-linear term is the product of small and large scale perturbations,

$$R_c(x) \simeq \bar{R}_s(x) + 2 \times 3 f_{NL} \bar{R}_s^2(x) \bar{R}_L^2(x) + \cdots,$$

(4.3)

when superscripts $\bar{R}_s$ and $\bar{R}_L$ refer to “Gaussian” perturbations on small and large length scales, respectively. Thus using the $\delta N$ formalism to second order gives

$$R_c = \delta N = \left[ \frac{\partial N}{\partial \phi} + \frac{\partial^2 N}{\partial \phi^2} \delta \phi_L(x) \right] \delta \phi(x) + \cdots.$$

(4.4)

Rewriting the above in powers of the Gaussian fields by using eq. (3.4) yields

$$R_c = \frac{\partial N}{\partial \bar{\phi}} \delta \bar{\phi}(x) + \left( \frac{\partial N}{\partial \bar{\phi}} \right)^T \frac{V''}{2V} \left( \frac{\partial^2 N}{\partial \bar{\phi}^2} \right) \delta \bar{\phi}(x) \delta \phi_L(x) + \cdots$$

$$= \bar{R}_s(x) + \left( \frac{1}{N_{\bar{\phi}}^2} \frac{V''}{2V} + \frac{N_{\bar{\phi}}}{N_{\bar{\phi}}^2} \right) \bar{R}_s^2(x) \bar{R}_L(x) + \cdots.$$

(4.5)
Comparing the above with eq. (4.3), we find
\[ \frac{6}{5} f_{\text{NL}} = \frac{1}{N_\phi} \frac{V'}{2V} + \frac{N_{\phi\phi}}{N_\phi^2} = \epsilon + \frac{\eta}{2}, \]

(4.6)

where in the last step we used \( N_\phi = V/(M_P^2 V') \) and the definitions of the slow-roll parameters,
\[ \epsilon \equiv -\frac{\dot{H}}{H^2} = M_P^2 \frac{V''}{2V'} = \frac{1}{M_P^2 N_\phi^2}, \]
\[ \eta \equiv \frac{\dot{\epsilon}}{H \epsilon} = 2 \frac{N_{\phi\phi}}{N_\phi^2}. \]

(4.7)

Thus we have recovered the total \( f_{\text{NL}} \) within the \( \delta N \) formalism.

Moreover, using the relations,
\[ \epsilon = \frac{\partial \ln H}{\partial N}, \quad \eta = 2 \frac{\partial \phi}{\partial N} \frac{\partial}{\partial \phi} \ln \frac{\partial N}{\partial \phi}, \]

(4.8)

we obtain
\[ 1 - n_s = \frac{\partial}{\partial N} \ln \left[ \left( \frac{\partial N}{\partial \phi} \right)^2 H^2 \right] = \eta + 2\epsilon = 12 \frac{5}{f_{\text{NL}}}, \]

(4.9)

which is Maldacena’s single-field consistency relation. Thus, the consistency relation may be regarded as another manifestation of the \( \delta N \) formula. As discussed frequently in the literature, the essence of this relation is that the three point correlation function of the curvature perturbation in the squeezed limit is proportional to the linear response of the short wavelength modes to the long wavelength mode, while the dynamical effect of the long wavelength fluctuations is suppressed by a factor \( \mathcal{O}(k_L^2/H^2 a^2) \) as already mentioned. This naturally implies it must be within the scope of the \( \delta N \) formalism, and we have just shown that it is indeed the case.

5 Conclusion

We showed that careful application of the \( \delta N \) formalism recovers the bispectrum of the curvature perturbation in single-field slow-roll inflation as well as Maldacena’s consistency condition. This is in agreement with the fact that the only effect of large length scale perturbations is kinematical. In other words, one can say that Maldacena’s consistency relation is a manifestation of the \( \delta N \) formula.

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