An algorithm for calculating the set of superhedging portfolios and strategies in markets with transaction costs

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Abstract

We study the explicit calculation of the set of superhedging portfolios of contingent claims in a discrete-time market model for \( d \) assets with proportional transaction costs when the underlying probability space is finite. The set of superhedging portfolios can be obtained by a recursive construction involving set operations, going backward in the event tree. We reformulate the problem as a sequence of linear vector optimization problems and solve it by adapting known algorithms. A corresponding superhedging strategy can be obtained going forward in the tree. We discuss the selection of a trading strategy from the set of all superhedging trading strategies. Examples are given involving multiple correlated assets and basket options. Furthermore, we relate existing algorithms for the calculation of the scalar superhedging price to the set-valued algorithm by a recent duality theory for vector optimization problems.

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1 Introduction

In this paper, a method is provided for an explicit calculation of the set of superhedging portfolios of contingent claims in a discrete-time market model for \( d \) assets with proportional transaction costs when the underlying probability space is finite.

The set of superhedging portfolios in markets with transaction costs has been characterized under appropriate no arbitrage conditions using consistent price systems, the pendant to the density process of equivalent martingale measures in markets with transaction costs (\cite{22,23,35}). But until now, an algorithmic approach to calculate the set of superhedging portfolios was only provided for \( d = 2 \) assets in Roux, Zastawniak \cite{33}. In a recent paper Roux, Zastawniak \cite{34} present an extension of the recursive representation in \cite{33} to \( d \) assets, but do not provide

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an efficient method to implement it and thus lack examples beyond one-period markets. The method presented in this paper will overcome this problem.

Most algorithm so far concerned the calculation of the scalar superhedging price in markets with two assets and transaction costs. The scalar superhedging price is the smallest superhedging price if a specific asset (usually the domestic currency or a bond) is chosen as the numéraire. Often, strong assumptions on the size of the transaction costs or the type of contingent claims had to be imposed to provide an algorithm to calculate the scalar replication price and to ensure that this price coincides with the smallest superhedging price (see e.g. [3, 5, 21, 28]). Roux, Tokarz, Zastawniak [32], Roux [31] were able to drop those assumptions and develop an algorithm to calculate the scalar superhedging price in the two asset case that is based on the dual representation of the scalar superhedging price deduced in Jouini, Kallal [21]. The scalar superhedging price is important to deduce price bounds. But only the set of superhedging portfolios provides full information if one wants to carry out a superhedging strategy starting from an initial portfolio that might contain other assets besides the numéraire asset.

This paper concerns three goals. First, an algorithm is presented to calculate the set of all superhedging portfolios in the $d$ asset case, as well as an algorithm to calculate the superhedging strategy when starting from an initial portfolio vector in the set of superhedging portfolios. The $d$ asset case also allows to consider basket options. Secondly, we will show that the superhedging problem in markets with transaction costs leads to a sequence of linear vector optimization problems. This generalizes the well known fact that in frictionless markets, superhedging leads to a sequence of linear optimization problems. Thirdly, we will show how the above mentioned scalar algorithm of [32, 31] can be related to the algorithm presented here by a recent duality theory for vector optimization problems.

To be more precise, we follow the numéraire-free approach of Kabanov [22], Schachermayer [35] to develop an algorithm to calculate the set of all superhedging portfolios. We will show in theorem 3.1 that the set of superhedging portfolios $\text{SHP}_t(X)$ at time $t = 0, 1, \ldots, T$ and node $\omega \in \Omega_t$ of a claim $X$ can be calculated going backwards in the tree by

$$\forall \omega \in \Omega_T \quad \text{SHP}_T(X)(\omega) = X(\omega) + K_T(\omega)$$

$$\forall t \in \{T - 1, \ldots, 1, 0\}, \forall \omega \in \Omega_t \quad \text{SHP}_t(X)(\omega) = \bigcap_{\bar{\omega} \in \text{succ}(\omega)} \text{SHP}_{t+1}(X)(\bar{\omega}) + K_t(\omega),$$

where $K_t$ denotes the solvency cone of the market at time $t$. The corresponding superhedging strategy can be obtained by going forward in the tree. The superhedging strategy might not be uniquely determined and different approaches will be discussed on how to choose a strategy.

The calculation of both, the set of superhedging portfolios and the superhedging strategy for a given initial portfolio vector involves the calculation of an intersection and the calculation of a sum of unbounded polyhedral sets, which could be realized by methods from computational geometry, which are essentially based on the vertex enumeration problem. Instead we reformulate the problems as linear vector optimization problems and solve them with Benson’s algorithm (which also involves vertex enumeration). This reformulation leads to interesting insights into the set (or vector) optimization nature of the problem as in each iteration step a solution of the dual vector optimization problem is used, where we use a geometric duality approach developed in [18].

The first one to connect coherent set-valued risk measures with linear vector optimization problems was Hamel [13], where in section 8 the set-valued average value at risk was formulated.
as a linear vector optimization problem. We pick up on that idea, as the set of superhedging portfolios can be seen as a set-valued coherent risk measure (see [15]), but adopt to the fact that in our dynamic setting and since superhedging strategies are path dependent, one rather formulates the problem as a sequence of linear vector optimization problems in the spirit of dynamic programming (see also section [7]). We obtain an algorithm that grows only polynomial as the trading frequency increases when path-independent payoffs are considered.

It is notable, that even for the determination of the scalar superhedging price the calculation of the set of superhedging portfolios at intermediate nodes is necessary. Thus, the calculation of the set of superhedging portfolios is not more involved than the calculation of the scalar superhedging price. It will turn out that the scalar algorithm based on the dual representation of the scalar superhedging price by Jouini, Kallal [21] is related via geometric duality to the set-valued algorithm. We will show that the set \( SHP_0(X) \) can be recovered from the scalar algorithm if a certain duality map is applied to a function appearing in the penultimate step of the scalar algorithm.

The paper is structured as follows. Section [2] reviews basic definitions and results about market models with proportional transaction costs and superhedging in those markets. In section [3] the above mentioned recursive construction of the set of superhedging portfolios is deduced. An algorithm to calculate a superhedging strategy is presented in section [3.1] as well as a discussion on how to choose an optimal strategy based on different criteria. Section [3] presents examples ranging from multi-period binomial models to correlated trees for multiple assets and includes European call options as well as basket options. Section [5] reformulates the superhedging problem in terms of linear vector optimization. Basic definitions and results from linear vector optimization are reviewed in section [6]. Section [7] presents the dual representation of the scalar superhedging price of Jouini, Kallal [21] generalized to the \( d \) asset case, and provides an algorithm to calculate the scalar superhedging price for finite probability spaces. This generalizes the algorithm of Roux [31], Roux, Tokarz, Zastawniak [32] to \( d \) dimensions, which allows the consideration of basket options. The relationship between the scalar and the set-valued algorithm is deduced in section [7.3] via geometric duality for the corresponding linear vector optimization problems. It shows that the scalar algorithm is in fact a set-valued algorithm.

As a conclusion, in markets with transaction costs, the set of superhedging portfolio vectors as introduced in the numéraire-free approach of Kabanov [22], Schachermayer [35] has to be considered at all intermediate steps, even if one is only interested in the scalar superhedging price for a chosen numéraire at time zero. In the past, mainly scalar algorithms, i.e., algorithms to calculate the smallest superhedging price expressed in a given numéraire, have been studied in the literature. In [33], a set-valued algorithm has been obtained in the two-asset case for American options and was extended to \( d \) assets in [34]. But a lack an efficient implementation for \( d > 2 \) and thus a lack of examples beyond one-period markets in [34] show that our approach, which establishes a connection to linear vector optimization and thus allows the use of well established algorithm to solve the problem, is an advantage. In this paper, we present an algorithm to calculate the set of all superhedging portfolios that works for any fixed number of assets in a discrete time model on a finite probability space.

The method presented will also prove useful for the calculation of other (dynamic) coherent risk measures in markets with transaction costs, price bounds like good deal bounds, or when calculating solutions of portfolio optimization problems in markets with transaction costs. These problems will be discussed in a separate paper.
2 Preliminaries

Consider a financial market where \( d \) assets can be traded over finite discrete time \( t = 0, 1, \ldots, T \). As stochastic base we fix a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)\). We assume without loss of generality that \( \mathcal{F}_0 \) is trivial.

A portfolio vector at time \( t \) is an \( \mathcal{F}_t \)-measurable random vector \( V_t : \Omega \to \mathbb{R}^d \). The values \( V_t(\omega) \) of portfolio vectors are given in physical units, i.e., the \( i \)-th component of the vector \( V_t(\omega) \) is the number of units of the \( i \)-th asset in the portfolio at time \( t \) for \( i = 1, \ldots, d \). Thus, we do not fix a reference asset, like a currency or some other numéraire, and treat all assets symmetrically as it was initiated by Kabanov [22], see also [35, 23]. To be more precise, let the terms of trade at time \( t \) be modeled via a \( \mathcal{F}_t \)-measurable \( d \times d \) bid-ask-matrix \( \Pi_t \), as in definition 1.1 in [35]. That is, \( \Pi_t \) is a matrix-valued map \( \omega \mapsto \Pi_t(\omega) \), denoting the bid and ask prices for the exchange between the \( d \) assets. The entry \( \pi_{ij} \) of \( \Pi_t \) denotes the number of units of asset \( i \) for which an agent can buy one unit of asset \( j \) at time \( t \), i.e., the pair \( (\frac{1}{\pi_{ij}}, \pi_{ij}) \) denotes the bid- and ask-prices of the asset \( j \) in terms of the asset \( i \). Furthermore, \( \Pi_t \) satisfies

\[
\pi_{ij} > 0, \quad 1 \leq i, j \leq d, \\
\pi_{ii} = 1, \quad 1 \leq i \leq d, \\
\pi_{ij} \leq \pi_{ik} \pi_{kj}, \quad 1 \leq i, j, k \leq d.
\]

The solvency cone \( K_t(\omega) \) is spanned by the vectors \( \pi_{ij} e^j - e^i, 1 \leq i, j \leq d \), and the unit vectors \( e^i, 1 \leq i \leq d \). Some of those vectors might be redundant. Non-redundant generating vectors of solvency cones as well as their positive dual cones in different types of market situations are given in the forthcoming paper [27]. Throughout the paper we assume

\[
\mathbb{R}_+^d \setminus \{0\} \subseteq \text{int} K_t(\omega), \tag{2.1}
\]

which is satisfied if the \( d \) assets are liquid in the sense that the exchange rates between any two assets are positive and finite. The cone \( K_t \) is called solvency cone since it includes precisely those portfolios at time \( t \) which can be exchanged into portfolios with only non-negative components (by trading at the prevailing bid-ask prices at time \( t \)). Thus, \( K_t \) contains all the information about exchange rates and proportional transaction costs at time \( t, t \in \{0, 1, \ldots, T\} \).

By \( L^0_d(\mathcal{F}_t, \mathbb{R}^d) = L^0_d(\Omega, \mathcal{F}_t, P; \mathbb{R}^d) \) we denote the linear space of all \( \mathcal{F}_t \)-measurable random vectors \( X : \Omega \to \mathbb{R}^d \) for \( t = 0, 1, \ldots, T \). We denote by \( L^0_d(\mathcal{F}_t, D_t) = L^0_d(\Omega, \mathcal{F}_t, P; D_t) \) those \( \mathcal{F}_t \)-measurable random vectors that take \( P \)-a.s. values in \( D_t \).

An \( \mathbb{R}^d \)-valued adapted process \( (V_t)_{t=0}^T \) is called a self-financing portfolio process for the market given by \( (K_t)_{t=0}^T \) if

\[
\forall t \in \{0, \ldots, T\} : \quad V_t - V_{t-1} \in -K_t \quad P\text{-a.s.}
\]

with the convention \( V_{-1} = 0 \).

We denote by \( A_T \subseteq L^0_d(\mathcal{F}_T, \mathbb{R}^d) \) the set of random vectors \( V_T : \Omega \to \mathbb{R}^d \), each being the value of a self-financing portfolio process at time \( T \), i.e. \( A_T \) is the set of superhedgable claims.
starting from initial endowment $0 \in \mathbb{R}^d$ at time zero. As it easily follows from the definition of self-financing portfolio processes, $A_T$ is a convex cone and

$$A_T = -L_0^d(\mathcal{F}_0, K_0) - L_0^d(\mathcal{F}_1, K_1) - \ldots - L_0^d(\mathcal{F}_T, K_T).$$

Note that $x_0 + A_T$ is the set of self-financing portfolio processes starting from the initial portfolio vector $V_{-1} = x_0 \in \mathbb{R}^d$ at time $t = 0$.

The fundamental theorem of asset pricing states that a market given by $(K_t)_{t=0}^T$ satisfies the robust no arbitrage condition if and only if there exists a strictly consistent price system $(Z_t)_{t=0}^T$ (see [35, theorem 1.7]). For most of the paper we will assume a finite probability space, then the fundamental theorem of asset pricing simplifies: a market given by $(K_t)_{t=0}^T$ satisfies the no arbitrage condition if and only if there exists a consistent price system $(Z_t)_{t=0}^T$ (see [24, 35]).

The definitions are as follows. The market given by $(K_t)_{t=0}^T$ is said to satisfy the no arbitrage property $(NA)$ if $A_T \cap L_0^d(\mathcal{F}_T, \mathbb{R}_+^d) = \{0\}$. The market is said to satisfy the robust no arbitrage property $(NA^+)$ if there exists a market process $(\tilde{K}_t)_{t=0}^T$ satisfying

$$K_t \subseteq \tilde{K}_t \quad \text{and} \quad K_t \setminus K_t \subseteq \text{int} \tilde{K}_t \quad \text{P-a.s.} \quad (2.2)$$

for all $t \in \{0, 1, \ldots, T\}$ such that

$$\tilde{A}_T \cap L_0^d(\mathcal{F}_T, \mathbb{R}_+^d) = \{0\},$$

where $\tilde{A}_T$ is generated by the self-financing portfolio processes with

$$\forall t \in \{0, \ldots, T\}: \quad V_t - V_{t-1} \in -\tilde{K}_t \quad \text{P-a.s.}$$

By $K_t^+$ we denote the set-valued mapping $\omega \mapsto K_t^+(\omega)$, where $K_t^+(\omega)$ denotes the positive dual cone of the cone $K_t(\omega)$ for $\omega \in \Omega$ and $t \in \{0, 1, \ldots, T\}$. Thus,

$$K_t^+(\omega) = \left\{ v \in \mathbb{R}^d : \forall u \in K_t(\omega) : v^T u \geq 0 \right\}.$$ 

Let $\text{ri} K_t^+$ denote the map $\omega \mapsto \text{ri} K_t^+(\omega)$ where $\text{ri} K_t^+(\omega)$ is the relative interior of $K_t^+(\omega) \subseteq \mathbb{R}^d$, see e.g. [30]. An $\mathbb{R}_+^d$-valued adapted process $Z = (Z_t)_{t=0}^T$ is called a (strictly) consistent price system for the market model $(K_t)_{t=0}^T$ if $Z$ is martingale under $P$ and

$$\forall t \in \{0, 1, \ldots, T\}: \quad Z_t \in K_t^+ \setminus \{0\} \quad (\in \text{ri} K_t^+) \quad \text{P-a.s.}$$

Let us denote the set of all consistent price systems by $Z$.

An initial portfolio vector $x_0 \in \mathbb{R}^d$ allows to superhedge a claim $X \in L_0^d(\mathcal{F}_T, \mathbb{R}^d)$ if there exists a self-financing portfolio process $V_T \in A_T$ such that $x_0 + V_T = X$ P-a.s., i.e. if $X \in x_0 + A_T$.

Note that the term ‘self-financing’ also allows the agents to ‘throw away’ non-negative quantities of the assets, thus the above condition $x_0 + V_T = X$ P-a.s. really means superhedging and not necessarily perfect replication. Let us denote by $SHP_0(X)$ the set of all those portfolio vectors $x_0 \in \mathbb{R}^d$ at time $t = 0$ that allow to superhedge the claim $X \in L_0^d(\mathcal{F}_T, \mathbb{R}_+^d)$, i.e.,

$$SHP_0(X) = \left\{ x_0 \in \mathbb{R}^d : X \in x_0 + A_T \right\}. \quad (2.3)$$

The set of superhedging portfolios of a claim $X \in L_0^d(\mathcal{F}_T, \mathbb{R}^d)$ can be characterized as follows.
Theorem 2.1 ([35] theorem 4.1, [15] corollary 5.4). Under the robust no arbitrage condition (NA*), for a claim $X \in L^0_d(\mathcal{F}_T, \mathbb{R}^d)$ one has

$$SHP_0(X) = \left\{ x_0 \in \mathbb{R}^d : \forall Z \in \mathbb{Z} \text{ with } E[(X^T Z_T)^-] < \infty : E[X^T Z_T] \leq x_0^T Z_0 \right\}$$ (2.4)

$$= \bigcap_{(Q, w) \in \mathcal{W}^1} (E^Q [X] + G(w)).$$ (2.5)

The function $X \mapsto SHP_0(-X)$ defines a closed set-valued coherent market-compatible risk measure as introduced in [15]. Equation (2.3) is the primal representation of this risk measure and corresponds to the very definition of elements $x_0 \in \mathbb{R}^d$ allowing to superhedge the claim $X \in L^0_d(\mathcal{F}_T, \mathbb{R}^d)$. Equation (2.4) refers to the representation of superhedging portfolios with help of consistent price systems as in [22, 23, 35]. Equation (2.5) corresponds to the dual representation of the set of superhedging portfolios as a coherent risk measure (see [15]), where we denote $G(w) = \{ x \in \mathbb{R}^d : 0 \leq w^T x \}$. The dual elements $(Q, w)$ are defined by

$$\mathcal{W}^1 = \left\{ (Q, w) \in \mathcal{M}_{1,d}^P \times \mathbb{R}^d \setminus \{0\} : \forall t \in \{0, 1, \ldots, T\} : E \left[ \text{diag}(w) \frac{dQ}{dP}\big| \mathcal{F}_t \right] \in L^1_0(\mathcal{F}_t, K_t^+) \right\},$$ (2.6)

and are one-to-one with the set of consistent price systems $\mathbb{Z}$ as it was shown in [15].

In (2.6), $\mathcal{M}_{1,d}^P = \mathcal{M}_{1,d}^P(\Omega, \mathcal{F}_T)$ denotes the set of all vector probability measures with components being absolutely continuous with respect to $P$, i.e. $Q_i : \mathcal{F}_T \rightarrow [0, 1]$ is a probability measure on $(\Omega, \mathcal{F}_T)$ such that $\frac{dQ}{dP} \in L^1(\mathcal{F}_T, \mathbb{R})$ for $i = 1, \ldots, d$.

In the following sections we will show that if the underlying probability space is finite, the set of superhedging portfolios $SHP_0(X)$ as defined in (2.3) can be obtained by a recursive construction, going backward in the event tree, while the corresponding superhedging strategy can be obtained going forward in the tree. Based on this recursive structure, an algorithm to calculate superhedging portfolios and strategies for finite probability spaces will be developed.

3 Recursive representation of the set of superhedging portfolios

Let us assume a finite filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$ with $P(\omega) > 0$ for all $\omega \in \Omega$ and the usual assumptions regarding the filtration, in particular, $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_T = \mathcal{F}$. Let $\Omega_t$ denote the set of atoms of $\mathcal{F}_t$ for $t \in \{0, 1, \ldots, T\}$.

Let us consider a tree that represents the income of information, i.e., the nodes of the tree correspond to the atoms $\omega \in \Omega_t$ of $\mathcal{F}_t$. In the examples in section 4 we will consider recombining trees such that the number of nodes does not grow exponentially with the number of time steps to ensure that the algorithm is computationally manageable. However, we do not need this assumption now.

A node $\bar{\omega} \in \Omega_{t+1}$ is called a successor node of $\omega \in \Omega_t$ ($t \in \{0, \ldots, T-1\}$) if $\bar{\omega} \subseteq \omega$. The set of successor nodes of $\omega \in \Omega_t$ is denoted by

$$\text{succ}(\omega) = \{ \bar{\omega} \in \Omega_{t+1} : \bar{\omega} \subseteq \omega \}.$$ 

Let the market model be described by an adapted stochastic process for the solvency cones $(K_t)_{t=0}^T$. On a finite probability space, that is if $|\Omega| = N \in \mathbb{N}$, the superhedging portfolios of a random payoff $X \in L^0_d(\mathcal{F}_T, \mathbb{R}^d)$ can be written in recursive form. In this setting, we can identify $L^0_d(\mathcal{F}_T, \mathbb{R}^d)$ with $\mathbb{R}^{dN}$.
Theorem 3.1. If the probability space is finite and the no arbitrage condition (NA) holds true, the set of superhedging portfolios \( \text{SHP}_0(X) \subseteq \mathbb{R}^d \), defined in (2.3), of a claim \( X \in L^1_0(\mathcal{F}_T, \mathbb{R}^d) \) satisfies
\[
\emptyset \neq \text{SHP}_0(X) \neq \mathbb{R}^d
\]
and can be obtained recursively via
\[
\forall \omega \in \Omega_T : \text{SHP}_T(X)(\omega) = X(\omega) + K_T(\omega)
\]
\[
\forall t \in \{T - 1, \ldots, 1, 0\}, \forall \omega \in \Omega_t : \text{SHP}_t(X)(\omega) = \bigcap_{\bar{\omega} \in \text{succ}(\omega)} \text{SHP}_{t+1}(X)(\bar{\omega}) + K_t(\omega).
\]

Of course the above theorem is also valid for more general probability spaces if one replaces the (NA) condition with (NA')

Proof. Let \( A_\infty \) denote the recession cone of a convex set \( A \subseteq \mathbb{R}^d \), i.e. \( A_\infty = \{ x \in \mathbb{R}^d : \forall t > 0 \ A + tx \subseteq A \} \), see e.g. [30]. The intersection of finitely many non-empty polyhedral convex sets \( A_1, \ldots, A_m \) satisfying \( (A_i)_\infty \supseteq \mathbb{R}^d_+ \) for \( i = 1, \ldots, m \) is non-empty (e.g. the component-wise maximum over a selection \( a^i \in A_i \) \( i = 1, \ldots, m \)) belongs to the intersection. Since \( K_t(\omega) \supseteq \mathbb{R}^d_+ \) for all \( t \in \{0, \ldots, T\} \) and all \( \omega \in \Omega_t \), the sets \( \text{SHP}_t(X)(\omega) \) defined by (3.2), (3.3) are non-empty. Let \( x_0 \in \text{SHP}_0(X) \) as defined by the recursive construction (3.2), (3.3). Equation (3.3) for \( t = 0 \) is equivalent to the existence of a \( V_0 \in x_0 - K_0 \) such that \( V_0 \in \bigcap_{\omega \in \Omega_0} \text{SHP}_1(X)(\omega) \). In particular, \( V_0 \in \text{SHP}_1(X)(\omega) \) for all \( \omega \in \Omega_1 \). Using the definition of \( \text{SHP}_1(X)(\omega) \), i.e. (3.3) for \( t = 1 \), there exists for each \( \bar{\omega} \in \Omega_1 \) a \( V_1(\bar{\omega}) \in V_0 - K_1(\omega) \) such that \( V_1(\omega) \in \text{SHP}_2(X)(\bar{\omega}) \) for all \( \bar{\omega} \in \text{succ}(\omega) \). Going forward in the tree in the same manner and using (3.2), one can conclude that there exists for each \( \bar{\omega} \in \Omega_T \) with \( \bar{\omega} \in \text{succ}(\omega) \) for some \( \omega \in \Omega_{T-1} \) a \( V_T(\bar{\omega}) \in V_{T-1}(\omega) - K_T(\omega) \) such that \( V_T(\bar{\omega}) = X(\bar{\omega}) \). That means, there exists a self-financing trading strategy \( V_T \in x_0 + A_T \) with \( V_T(\bar{\omega}) = X(\bar{\omega}) \), i.e. \( X \in x_0 + A_T \), which means \( x_0 \) is a superhedging portfolio of \( X \) by (2.3). For the reverse direction, take \( x_0 \in \text{SHP}_0(X) \) as defined in (2.3). For a given path \( (\omega_0, \ldots, \omega_{T-1}) \) with \( \omega_t \in \Omega_t \) and \( \omega_t \in \text{succ}(\omega_{t-1}) \) \( t = 1, \ldots, T - 1 \), there exists \( k_t \in K_t(\omega_t) \) such that \( x_0 - k_0 - \ldots - k_{T-1} \in X(\omega) + K_T(\omega) \) for all \( \omega \in \Omega_T \) with \( \omega \in \text{succ}(\omega_{T-1}) \). Thus,
\[
x_0 - k_0 - \ldots - k_{T-2} \in \bigcap_{\bar{\omega} \in \text{succ}(\omega_{T-1})} \text{SHP}_T(X)(\bar{\omega}) + k_{T-1},
\]
where \( \text{SHP}_T(X) \) is defined as in (3.2). Since this holds for any \( \omega_{T-1} \in \Omega_{T-1} \) with \( \omega_{T-1} \in \text{succ}(\omega_{T-2}) \), we have
\[
x_0 - k_0 - \ldots - k_{T-2} \in \bigcap_{\bar{\omega} \in \text{succ}(\omega)} \text{SHP}_T(X)(\bar{\omega}) + K_{T-1}(\omega) = \text{SHP}_{T-1}(X)(\omega)
\]
via (3.3) for all \( \omega \in \Omega_{T-1} \) with \( \omega \in \text{succ}(\omega_{T-2}) \). Succeeding like this reveals \( x_0 \in \text{SHP}_0(X) \) as defined by the recursive construction (3.2), (3.3). This proves the equivalence of (2.3) and the recursive definition of \( \text{SHP}_0(X) \) in (3.2), (3.3). By the fundamental theorem of asset pricing, no arbitrage implies the existence of a consistent price system \( Z \in Z \) (see [24], [35]). For a finite probability space, \( E[\langle X^T Z_T \rangle] < \infty \) is always satisfied and from theorem 3.1, \( \text{SHP}_0(X) \neq \mathbb{R}^d \) follows.
Clearly, $SHP_t(X)(\omega)$ is the set of superhedging portfolios of $X \in L^0_0(\mathcal{F}_T, \mathbb{R}^d)$ at time $t$ at node $\omega \in \Omega_t$, $t \in \{0, 1, \ldots, T\}$ and (3.1) is satisfied likewise. As a consequence of theorem 3.1 for all $t \in \{0, 1, \ldots, T\}$ and all $\omega \in \Omega_t$, $SHP_t(X)(\omega)$ is a non-empty polyhedral convex set satisfying

$$\text{int } (SHP_t(X)(\omega))_\infty \supseteq \mathbb{R}^d_+ \setminus \{0\}.$$  

The recursive structure (3.2), (3.3) provides a geometric intuition for designing an algorithm to calculate the superhedging portfolios. The set of all possible superhedging portfolios can be calculated going backwards in the tree. The operations involved are the calculation of intersections of unbounded polyhedral sets and the sum of such a polyhedral set with a convex cone. These operations could be realized by methods from computational geometry, which are essentially based on the vertex enumeration problem, see e.g. [1, 6]. For several reasons, which will be discussed in section 5, we proceed in a different way. We reformulate in section 5 the problem as a sequence of linear vector optimization problems and solve them with Benson’s algorithm, i.e. in each iteration step a linear vector optimization problem is solved. Benson’s algorithm also involves vertex enumeration but additionally several linear programs. Note that for path independent payoffs, the recursive pricing procedure described in theorem 3.1 grows only polynomial as the trading frequency increases.

It is well known that in the frictionless case, the superhedging problem is a sequence of linear (scalar) optimization problems (see e.g. chapter 7.1 in [10]). Thus, the interpretation of the superhedging problem in markets with frictions as a sequence of linear vector optimization problems makes very clear how the structure of the problem changes if frictions are allowed and how the frictionless case fits into this framework.

Note that the same geometric intuition as in (3.2), (3.3) appears if one is only interested in calculating the scalar superhedging price (i.e. the smallest superhedging price in a given currency or numéraire) when transaction costs are present. Even in the scalar case, it cannot be avoided to use the set-valued operations (intersection and sum of polyhedral sets). As a consequence of (3.4), the polyhedral convex sets $SHP_t(X)(\omega)$ can be expressed as the epigraphs of piecewise linear functions $f : \mathbb{R}^{d-1} \to \mathbb{R}$. For example, in the two asset binomial model, this recursive structure can be rediscovered in the sequential problem $Q_t$, p. 71 in [3].

On the other hand, for the purpose of comparing our method with algorithms developed for calculating the scalar superhedging price based on a dual description as in [32, 33, 31] it is quite helpful to reformulate (3.2), (3.3) as a sequence of linear vector optimization problems. As we will see in detail in section 5 in each iteration step a solution of the dual vector optimization problem is used, where we use the geometric duality approach developed in [18]. As it will turn out, this geometric duality provides a link between (3.2), (3.3) and above mentioned scalar algorithms and allows to recover the whole set of superhedging portfolios from intermediate results of the scalar algorithm. It is a somewhat surprising insight that the calculation of the set of superhedging portfolios is not more difficult than calculating the scalar superhedging price in markets with transaction costs. The scalar problem will be made precise in section 7 and its relation to theorem 3.1 will be exploit in detail in section 7.3.

**Remark 3.2.** Note that it is sufficient to develop an algorithm for superhedging portfolios and superhedging strategies since the set of subhedging portfolios $SubHP_0(X)$ of a claim $X \in L^0_0(\mathcal{F}_T, \mathbb{R}^d)$ is just the negative of the set of superhedging portfolios of $-X$

$$SubHP_0(X) = -SHP_0(-X).$$

The corresponding strategy for the buyer of a claim $X$ is to superhedge $-X$. 

8
Example 3.3. Let us consider a simple introductory example: a one period binomial model with non-constant proportional transaction costs, where the set of superhedging portfolios has multiple vertices. We will use this example to illustrate the algorithm and to explain differences between the scalar and the set-valued approach. Note that the transaction costs are chosen to be quite large, just for the purpose of obtaining illustrative pictures.

Let asset 0 be a riskless cash account and let us assume for simplicity that interest rates are zero. Asset 1 is a risky stock, whose bid-ask prices \((S^b_t, S^a_t)\) at time \(t = 0\) and \(t = T\) are modeled as follows:

\[(18, 25)\quad (20, 26)\quad (16, 23).\]

We consider a digital option, more specifically an asset or nothing call option with physical delivery and strike \(K = 24\). The payoff is given by

\[X(\omega) = (X_1(\omega), X_2(\omega)) = (0, I_{\{S^a_T \geq K\}}(\omega))T.\]

Thus, the payoff in the up-node is \(X(\omega_1) = (0, 1)^T\) and in the down node \(X(\omega_2) = (0, 0)^T\).

The calculation of the set of superhedging portfolios by the recursive procedure described in theorem 3.1 and illustrated in figure 1 reveals that \(SHP_0(X)\) has two vertices, one at \((0, 1)^T\) and one at \((-80, 5)^T\) and a recession cone equal to the solvency cone \(K_0\) at initial time which is generated by \((-18, 1)^T\) and \((25, -1)^T\). Figure 2 shows the set of sub- and superhedging portfolios.

The scalar superhedging price is given by 25$ and corresponds to the buy and hold strategy that superreplicates the claim. The strategy is to transfer the initial position \((25, 0)^T\) into the vertex \((0, 1)^T\) of \(SHP_0(X)\) at initial time and hold this portfolio until terminal time.

However, the knowledge of the scalar superhedging price and the corresponding strategy does not give information about optimal strategies if one already owns some shares of the stock. For example, if one owns 5 stocks and is short 80 units cash at initial time (the portfolio corresponding to the second vertex of \(SHP_0(X)\)), one cannot reach the scalar superhedging price, that is the portfolio \((25, 0)^T\), by selling the stock at initial time. This is illustrated in figure 3. Therefore, the portfolio \((-80, 5)^T\) does not allow to superreplicate if initial positions are "cash only" positions.

Consequently, if initial positions in several eligible assets (here stock and cash) are allowed instead of only one (like cash) the cost of superreplication can be reduced: The portfolio \((-80, 5)^T \in SHP_0(X)\) clearly allows to superreplicate (in this example even to replicate) the claim. The scalar superhedging price gives information about price bounds, but for the purpose of actually carrying out a superhedging strategy, only the set of all superhedging portfolios gives full information on optimal strategies.

3.1 Calculation of the superhedging strategy

From the proof of theorem 3.1 one can see that the superhedging strategy, when starting from a particular element in the set of superhedging portfolios, can be calculated going forward in the tree. In the presence of transaction costs optimal superhedging strategies are, in general, path-dependent even for path-independent payoffs. But one only needs to compute the superhedging strategy along the realized path, one step at a time, in real time.
Figure 1: Example 3.3: Illustration of the recursive algorithm (3.2), (3.3) of theorem 3.1.

Figure 2: Example 3.3: The set $-\text{SHP}_0(-X)$ of subhedging portfolios and the set $\text{SHP}_0(X)$ of superhedging portfolios.

Figure 3: Example 3.3: $P_1$ can be exchanged into $P_4 = (10, 0)^T$, but not into the scalar superhedging price $P_3 = (25, 0)^T$. 


Proof. The assertions follow from the recursion (3.2), (3.3) and the proof of theorem 3.1. In particular, it follows that the sets in (3.5), (3.6) are non-empty and thus the existence of a superhedging strategy follows. Let the probability space be finite. For a claim \( X \in L^0_0(\mathcal{F}, \mathbb{R}^d) \) and a given initial portfolio vector \( x_0 \in \mathbb{R}^d \) with \( x_0 \in \text{SHP}_0(X) \), there exists a superhedging strategy for a path \( (\omega_0, \ldots, \omega_T) \) with \( \omega_t \in \Omega_t \) and \( \omega_t \in \text{succ}(\omega_{t-1}) \) (\( t = 1, \ldots, T \)). Such a strategy is given by a self-financing portfolio process \( V_0, V_1, ..., V_T \) defined by
\[
V_0 = x_0 - k_0 \quad \text{and} \quad \forall t \in \{1, \ldots, T\} \colon \quad V_t = V_{t-1} - k_t
\]
as it can be seen from the definition of a self-financing portfolio process.

**Theorem 3.5.** Let the probability space be finite. For a claim \( X \in L^0_0(\mathcal{F}, \mathbb{R}^d) \) and a given initial portfolio vector \( x_0 \in \mathbb{R}^d \) with \( x_0 \in \text{SHP}_0(X) \), there exists a superhedging strategy for a path \( (\omega_0, \ldots, \omega_T) \) with \( \omega_t \in \Omega_t \) and \( \omega_t \in \text{succ}(\omega_{t-1}) \) (\( t = 1, \ldots, T \)). Such a strategy is given by a self-financing portfolio process \( V_0, V_1, ..., V_T \) defined by
\[
V_0 \in (\{x_0\} - K_0) \cap \bigcap_{\omega \in \text{succ}(\omega_0)} \text{SHP}_1(X)(\bar{\omega}), \quad \text{(3.5)}
\]
\[
V_t \in (\{V_{t-1}\} - K_t(\omega_t)) \cap \bigcap_{\omega \in \text{succ}(\omega_t)} \text{SHP}_{t+1}(X)(\bar{\omega}), \quad \text{(3.6)}
\]
for all \( t \in \{1, ..., T\} \).

Proof. The assertions follow from the recursion (3.2), (3.3) and the proof of theorem 3.1. In particular, it follows that the sets in (3.5), (3.6) are non-empty and thus the existence of a superhedging strategy follows.

Note that a superhedging strategy is typically not uniquely determined. In the following we will discuss how to choose an optimal superhedging strategy based on specific criteria. Since the strategies are in general superhedging and not replication strategies, it might be possible in certain scenarios to withdraw cash or assets without endangering the superhedging criteria. Thus, one possible criterion to choose a strategy might be to withdraw as much as possible of certain assets at intermediate points in time. A different criterion would be to avoid trading if possible. It is also possible to explore the tradeoffs between multiple objectives in a multi-criteria strategy selection process using vector optimization.

At each time \( t = 0, 1, ..., T \), choosing a superhedging strategy at a node \( \omega \in \Omega_t \) with endowment \( x = V_t(\omega) \in \text{SHP}_t(X)(\omega) \) means choosing a triplet \((v, y, k)\), where \( v = V_{t+1} \in \text{SHP}_{t+1} \) is the portfolio after any trades \( k \in K_t(\omega) \) and withdrawals of a portfolio \( y \in \mathbb{R}^d \) at time \( t \). That means, \( V_{t+1} = V_t - y - k \) and the portfolio \( V_{t+1} \) hold from time \( t \) to \( t + 1 \) and presents the initial endowment (before trades are made) in the next iteration step.

**Max-withdrawal superhedging strategy.**

Let us assume an investor following a superhedging strategy wants to withdraw as much of a certain portfolio \( y \in \mathbb{R}^d \setminus \{0\} \) as possible at each intermediate point in time.

Let \( t \in \{1, \ldots, T\}, \omega \in \Omega_t, x \in \text{SHP}_t(X)(\omega) \) and let \( K_t(\omega) \in \mathbb{R}^{d \times s} \) be a matrix containing the \( s \) generating vectors of \( K_t(\omega) \). We assume that inequality representations of the polyhedral sets \( \text{SHP}_{t+1}(X)(\omega) \) are known. An algorithm to compute them will be given in section 5. Solving the following LP with variables \((v, \alpha, z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^s\), we obtain a portfolio \( \bar{v} \) according to
Theorem 3.5 which has the property that a maximal amount of the portfolio \( y \) is withdrawn at time \( t \). If \( t < T \), we take

\[
\max \alpha \quad \text{s.t.} \quad v \in \bigcap_{\omega \in \text{succ}(\omega)} \text{SHP}_{t+1}(X)(\omega), \quad v + \alpha y + \tilde{K}_t(\omega)z = x, \quad \alpha \geq 0, \quad z \geq 0. \tag{3.7}
\]

At time \( T \), we solve

\[
\max \alpha \quad \text{s.t.} \quad v \geq X(\omega), \quad v + \alpha y + \tilde{K}_t(\omega)z = x, \quad \alpha \geq 0, \quad z \geq 0. \tag{3.8}
\]

From a solution \((\bar{v}, \bar{\alpha}, \bar{z})\), we get the new portfolio \( V_{t+1} = \bar{v} \). The portfolio \( \bar{y} = \bar{\alpha}y \in \mathbb{R}_+^d \) describes the withdrawal and \( k = \tilde{K}_t(\omega)\bar{z} \) is the corresponding trade.

**Theorem 3.6.** If the market satisfies the no arbitrage property (NA), then there exists a solution \((\bar{v}, \bar{\alpha}, \bar{z})\) for (3.7) and (3.8).

**Proof.** The feasible set is non-empty by Theorem 3.5 and the fact that any nonnegative withdrawal \( y \) belongs to \( \mathcal{K}_t(\omega) \). Assume there does not exist a solution \((\bar{v}, \bar{\alpha}, \bar{z})\) for (3.7) or (3.8), i.e. the value of the problem is \(+\infty\). Denoting by \( C = (\text{SHP}_t(X)(\omega))_\infty \) the recession cone of \( \text{SHP}_t(X)(\omega) \), we obtain \( y \in -C \). But \( y \in \text{int} C \) by (3.4). Hence \( 0 \in \text{int} C \) and thus \( C = \mathbb{R}^d \). It follows that \( \text{SHP}_t(X)(\omega) = \mathbb{R}^d \), which contradicts Theorem 3.1.

An important special case is the max-cash superhedging strategy, which is obtained by setting \( y = (1, 0, \ldots, 0)^T \), where the first component is assumed to correspond to the cash account of interest.

**Min-trading superhedging strategy.**

Another possible strategy to choose a superhedging portfolio is avoid trading whenever possible. The reason to minimize trading activities is that trading in markets with transaction costs is an irreversible process. The natural question how to measure trading activities meaningful leads to the problem to weight trading in case of multiple assets with different bid-ask spreads. For the case the generating vectors of the solvency cone are given by

\[
\tilde{K}_t = \begin{pmatrix}
(S_t^a)^1 & -(S_t^b)^1 & (S_t^a)^2 & -(S_t^b)^2 & \ldots & (S_t^a)^{d-1} & -(S_t^b)^{d-1} \\
-1 & 1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & -1 & 1 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & -1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & -1 & 1
\end{pmatrix}, \tag{3.9}
\]

where \((S_t^a)^i, (S_t^b)^i\) are the bid and ask prices \( \frac{1}{\pi^i} \) and \( \pi^i \) of asset \( i, i = 2, \ldots, d \) in units of asset 1, we will assume that the weights are chosen proportional to the corresponding bid-ask spreads \( \gamma^i_t := (S_t^a)^i - (S_t^b)^i \). Solvency cones of the above form appear for example is all assets are denoted in the same currency and the riskless asset is the currency itself or a bond with zero bid-ask spread (see [27] for more details). We obtain a weight vector

\[
\gamma = \left\{ \gamma^1_t, \gamma^2_t, \gamma^2_t, \ldots, \gamma^{d-1}_t, \gamma^{d-1}_t \right\}^T. \tag{3.10}
\]
For \( t \in \{1, \ldots, T\} \) and \( \omega \in \Omega_t \) and a portfolio \( x \in \text{SHP}_t(X)(\omega) \) we solve one of the following LPs. If \( t < T \), we take
\[
\min \gamma^T z \quad \text{s.t.} \quad v \in \bigcap_{\omega \in \text{succ}(\omega)} \text{SHP}_{t+1}(X)(\omega), \quad v + \widetilde{K}_t(\omega)z = x, \quad z \geq 0,
\]
where we denote by \( \widetilde{K}_t(\omega) \) the generating vectors of \( K_t(\omega) \). At time \( T \) we solve
\[
\min \gamma^T z \quad \text{s.t.} \quad v \geq X(\omega), \quad v + \widetilde{K}_t(\omega)z = x, \quad z \geq 0.
\]
Since the feasible sets are non-empty by theorem 3.5 and the objective function is bounded below by zero, there always exist solutions to (3.10) and (3.11).

**Bi-criteria strategy selection.**

Naturally there is a tradeoff between a max-withdrawal and a min-trading strategy, which can be displayed to the agents by solving a vector optimization problem with two objectives.

Using the above notation we solve in case of \( t < T \) the following linear vector optimization problem:
\[
\min(-\alpha, \gamma^T z)^T \quad \text{s.t.} \quad v \in \bigcap_{\omega \in \text{succ}(\omega)} \text{SHP}_{t+1}(X)(\omega), \quad v + \alpha y + \widetilde{K}_t(\omega)z = x, \quad \alpha \geq 0, \quad z \geq 0.
\]
(3.12)

At time \( T \) we solve
\[
\min(-\alpha, \gamma^T z)^T \quad \text{s.t.} \quad v \geq X(\omega), \quad v + \alpha y + \widetilde{K}_t(\omega)z = x, \quad \alpha \geq 0, \quad z \geq 0.
\]
(3.13)

If the market satisfies the no arbitrage property (NA), the problems (3.12) and (3.13) are feasible and bounded in the sense that there exists \( l \in \mathbb{R}^2 \) such that \( (-\alpha, \gamma^T z)^T \in \{l\} + \mathbb{R}_+^2 \) for all feasible points \( (v, \alpha, z) \) (compare theorem 3.6). From [26] theorem 5.20 and the considerations in section 5.1 in [26] we conclude that a (finitely generated) solution always exists. Solutions to (3.12) and (3.13) can be obtained by Benson’s algorithm, compare [26] chapters 4 and 5. The solution concept is also outlined below in section 6.

An example involving the max-cash, min-trade and the bi-criteria strategy selection is given in example 4.4 in the next section.

**4 Examples**

If one is interested in the superhedging portfolios for initial positions in only a few of the \( d \) assets, for example in just a few currencies, or even in just one currency, one would calculate
\[
\text{SHP}^M_0(X) = \text{SHP}_0(X) \cap M,
\]
(4.1)
for \( M = \{\sum_{i \in I} s_i e^i, s_i \in \mathbb{R}\} \), where the assets \( i \in I \) with \( I \subseteq \{1, \ldots, d\} \) are the ones of interest. This simply involves one more operation (the intersection) in the algorithm. The introduction of \( M \) is quite useful if the number of assets \( d \) is very high, one owns just a few of the \( d \) assets and thus is interested in superhedging portfolios just starting with those assets, or if \( \text{SHP}_0(X) \) is too complex to be visualized. In particular, the \( i \)-th component of the vertex of \( \text{SHP}^M_0(X) \)
for $M = \{se^i, s \in \mathbb{R}\}$ coincides with the smallest superhedging prices if asset $i \in \{1, \ldots, d\}$ is chosen as the numéraire, thus coinciding with the scalar superhedging price $\pi_0^a(X)$.

A second possibility to calculate the scalar superhedging price $\pi_0^a(X)$ w.r.t. a numéraire asset $i$ is by normalizing the inequality representation $\text{SHP}_0(X) = \{x \in \mathbb{R}^d | Bx \geq b\}$ obtained by the SHP-Algorithm in section 3 below in the following way: Transform $Bx \geq b$ into $\tilde{B}x \geq \tilde{b}$ such that each element in the $i$th row of $\tilde{B}$ is equal to 1. This is always possible by (3.4) and $\text{SHP}_0(X) \neq \mathbb{R}^d$. Then, the largest component of $\tilde{b}$ is $\pi_0^a(X)$ w.r.t. the numéraire asset $i$. This relation will also play a role when the connection between the set-valued approach and scalar algorithms is established and will be discussed in more detail in section 7.3.

Both methods provide a way to compare our results with the algorithm for the $d = 2$ case of [32, 31] for calculating the scalar superhedging price when the numéraire asset is the riskless asset (see example 4.2).

### 4.1 Two asset case

**Example 4.1.** Let us consider a digital option similar to example 3.3 but in a multi-period framework and smaller transaction costs. Let asset 1 be a riskless bond $B$ with an annual interest rate of 3%, face value $B_T = 1$, maturity one year, frequent compounding with $n = 100$ time intervals, i.e. $B_0 = (1 + \frac{r}{n})^{-n}$ and no transaction costs for the bond, i.e. $B_t^b = B_t^a = B_t$ for all $t$. Let the mid-market stock price $S$ follow a Cox-Ross-Rubinstein binomial model,

$$S_t = \varepsilon_t S_{t-1},$$

for $t = 1, \ldots, T$, where $\varepsilon_1, \varepsilon_2, \ldots$ is a sequence of independent identically distributed random variables taking two possible values $e^{\sigma \Delta t}$ or $e^{-\sigma \Delta t}$, where $\Delta t$ is the length of one time step. The initial stock price is $S_0 = 18$, volatility $\sigma = 0.2$, and transaction costs are constant $\lambda = 0.04$. Let the bid and the ask price at time $t$, respectively, be given by

$$S_t^b = S_t(1 - \lambda), \quad S_t^a = S_t(1 + \lambda). \quad (4.2)$$

An asset or nothing call option with physical delivery, maturity 1 year, strike $K = 19$ and payoff

$$X(\omega) = (X_1(\omega), X_2(\omega))^T = \left(0, I_{\{S_T \geq K\}}(\omega)\right)^T$$

is considered. The set $\text{SHP}_0(X)$ has two vertices, one at $(0, 1)^T$ and one at $(-24.92, 2.39)^T$ and a recession cone equal to the solvency cone $K_0$ at initial time which is generated by $(-S_0^b/B_0, 1)^T$ and $(S_0^a/B_0, -1)^T$.

The scalar superhedging price in the numéraire asset (the bond) is $\pi_0^a(X) = 19.29$ units of the bond, which corresponds to a scalar superhedging price in the domestic currency of $\pi^a(X) = 18.72 = S_0^a$ and the corresponding strategy is the buy and hold strategy. Note that in contrast to the trivial strategy one obtains for the scalar superhedging, the superhedging strategy can be more involved, when the initial portfolio vector is not cash-only.

**Example 4.2.** Let asset 1 be a riskless bond $B$ with an effective interest rate of $r_e = 10\%$, frequent compounding, face value $B_T = 1$, maturity 1 year, i.e. $B_0 = (1 + r_e)^{-1}$, and no transaction costs for the bond, i.e. $B_t^b = B_t^a = B_t$ for all $t$. Let the stock price $S$ follow a Cox-Ross-Rubinstein binomial model as in example 4.1. The initial stock price is $S_0 = 100$,
volatility $\sigma = 0.2$, maturity $T = 1$ year and transaction costs are constant $\lambda = 0.00125$. Let the bid and ask prices at time $t$ be given as in \cite{42}. Consider a call option with maturity 1 year, physical delivery and strike $K = 80$ whose payoff is a function on the mid-market price, i.e.

$$X(\omega) = (X_1(\omega), X_2(\omega))^T = (-KI_{\{S_T > K\}}(\omega), I_{\{S_T > K\}}(\omega))^T.$$ 

The set of superhedging and subhedging portfolios is given by its vertices and recession cones. For different values of $n$, the vertices (in units of bond and stock) are recorded in table 1. The recession cone of $SHP_0(X)$ is always $K_0$, generated by $(-S^b_0/B_0, 1)^T$ and $(S^a_0/B_0, -1)^T$, whereas the recession cone of $-SHP_0(-X)$ is $-K_0$. The scalar price bounds $\pi^b(X), \pi^a(X)$ in the domestic currency are also recorded in table 1.

For comparison purpose, we also give the scalar price bounds $\pi^b(X), \pi^a(X)$ if there are no transaction costs at $t = 0$ as considered in \cite{32, 31, 5, 28}. We are able to replicate the scalar results as given in table 1 of \cite{32} and table 3.1 and 3.2 of \cite{31}, where the different values of the parameters $K$ and $\lambda$ are $K \in \{80, 90, 100, 110, 120\}$ and $\lambda \in \{0\% , 0.125\% , 0.5\% , 0.75\% , 2\% \}$. Minor deviations (all less than 0.001) from table 3.2 of \cite{31} appear in a few instances for the bid prices in the $n = 1000$ case and one deviation of 0.014 for $n = 1000$ that is recorded in table 1. The case $n = 1800$ was not considered in \cite{32, 31}. Here, we just present the results for $K = 80$ and $\lambda = 0.125\%$.

| $n$  | 6   | 13  | 52  | 250 | 1000 | 1800 |
|------|-----|-----|-----|-----|------|------|
| vertex of $-SHP_0(-X)$ | $(-74.434, 0.953)$ | $(-74.699, 0.956)$ | $(-75.477, 0.962)$ | $(-76.348, 0.969)$ | $(-78.049, 0.983)$ | $(-79.049, 0.992)$ |
| lower price bound $\pi^b(X)$ | 27.552 | 27.537 | 27.462 | 27.381 | 27.249 | 27.191 |
| vertex of $SHP_0(X)$ | $(-73.814, 0.948)$ | $(-73.857, 0.949)$ | $(-73.857, 0.949)$ | $(-72.856, 0.941)$ | $(-71.244, 0.929)$ | $(-70.209, 0.921)$ |
| upper price bound $\pi^a(X)$ | 27.854 | 27.866 | 27.872 | 27.994 | 28.213 | 28.370 |

| $n$  | 6   | 13  | 52  | 250 | 1000 | 1800 |
|------|-----|-----|-----|-----|------|------|
| lower price bound $\pi^b(X)$ | 27.671 | 27.656 | 27.582 | 27.502 | 27.372$^a$ | 27.315$^b$ |
| upper price bound $\pi^a(X)$ | 27.735 | 27.747 | 27.753 | 27.876 | 28.097 | 28.255$^b$ |

$^a$ differs from value 27.386 in \cite{31}

$^b$ not considered in \cite{32, 31}

Table 1: set-valued and scalar sub- and superhedging portfolios of European call options

Note that, if the bond is chosen as the numéraire asset, the scalar superhedging price $\pi^b_1(X)$ is given in units of the bond, and one needs to multiply it by $B_0$ to obtain the scalar superhedging price $\pi^a_1(X)$ in the domestic currency that is recorded in table 1. It is worth pointing out that there are parameter constellations that lead to multiple vertices for the set of superhedging or subhedging portfolios. For example, $-SHP_0(-X)$ for $\lambda = 2\%$, $K = 110$ and $n = 52$ has 8 vertices given by the columns of the following matrix

$$
\begin{pmatrix}
-34.743 & -48.097 & -79.757 & -88.323 & -91.778 & -84.331 & -54.520 & -41.461 \\
0.322 & 0.445 & 0.732 & 0.809 & 0.840 & 0.774 & 0.504 & 0.384
\end{pmatrix}
$$
with a scalar subhedging price of \( \pi^b(X) = -0.023 \) (in the domestic currency) if transaction costs are considered at all time points, and \( \pi^b(X) = 0.865 \) if no transaction costs are considered at \( t = 0 \). The set \(-SHP_0(-X)\) for \( \lambda = 2\% \), \( K = 110 \) and \( n = 250 \) has 3 vertices given by

\[
\begin{pmatrix}
2.370 & -107.125 & -110.107 \\
-0.036 & 0.974 & 1.001
\end{pmatrix}
\]

with a scalar subhedging price of \( \pi^b(X) = -1.546 \) if transaction costs are considered at all time points, and \( \pi^b(X) = -0.038 \) if no transaction costs are considered at \( t = 0 \). Note that negative bid prices might occur when physical delivery is considered in markets with transaction costs. This issue was discussed and resolved in [29], see also remark 3.30 in [31].

### 4.2 Multiple correlated assets and basket options

We are interested in the set of superhedging portfolios of options involving multiple correlated assets. We will use a multi-dimensional tree that approximates a \( d - 1 \)-dimensional Black Scholes model for \( d - 1 \) risky assets, where the stock price dynamics under the risk neutral measure \( Q \) are given by

\[
dS^i_t = S^i_t(r dt + \sigma_i dW^i_t), \quad i = 1, ..., d - 1
\]

for Brownian motions \( W^i \) and \( W^j \) with correlation \( \rho_{i,j} \in [-1, 1] \) for \( i \neq j \). We will follow the method in [25] to set up a tree for the correlated risky assets by transforming the stock price process \( S \) into a process \( Y \) with independent components. This tree will have \( 2^{d-1} \) branches in each node and will be recombining with \( (t + 1)^{d-1} \) nodes at time \( t \) with \( t \in \{0, 1, ..., T\} \). Thus, a node can be identified by an index \((t, j_1, ..., j_{d-1})\) for \( t \in \{0, 1, ..., T\} \) and \( 1 \leq j_i \leq t + 1 \) for all \( i \in \{1, ..., d - 1\} \). For \( d = 3 \) the nodes at time \( t \) can be described by the indices \((j_1, j_2)\) of the elements of a matrix \( M_t \in \mathbb{R}^{(t+1) \times (t+1)} \). The values of the process \( Y \) at such a node can be obtained as follows.

Let \( \Sigma \) be the covariance matrix of the log asset prices and \( GG^T = \Sigma \) be the Cholesky decomposition of \( \Sigma \). Let \( n \) be the number of time intervals and \( \Delta t \) the length of one time interval. Let us denote \( \alpha = G^{-1}(r - \frac{1}{2} \sigma^2) \). The initial value of the process \( Y \) is given by \( Y_0 = G^{-1}(X_0) \) with \( X_0 = (\log(S^1_0), ..., \log(S^{d-1}_0)) \). The value of the process \( Y \) at node \((t, j_1, ..., j_{d-1})\) is given by

\[
Y^i_t = Y^{i}_0 + t\alpha_i \Delta t + (2j_i - t - 2)\sqrt{\Delta t}, \quad i = 1, ..., d - 1.
\]  

(4.3)

for \( t \in \{0, 1, ..., T\} \) and \( 1 \leq j_i \leq t + 1 \). We omit the index \((j_1, ..., j_{d-1})\) for \( Y_t(j_1, ..., j_{d-1}) \) and hope not to cause confusion. The value of the original stock price vector \( S \) at this node \((t, j_1, ..., j_{d-1})\) is

\[
S^i_t = \exp(GY^i_t), \quad i = 1, ..., d - 1.
\]

(4.4)

Now, let us assume for simplicity that the proportional transaction costs are constant for each of the risky assets and are given by \( \lambda = (\lambda^1, ..., \lambda^{d-1}) \). Thus the bid and ask prices at node \((t, j_1, ..., j_{d-1})\) are given by

\[
(S^b)^i_t = S^i_t(1 - \lambda^i), \quad (S^a)^i_t = S^i_t(1 + \lambda^i), \quad i = 1, ..., d - 1.
\]

(4.5)
Furthermore, let us assume there is a riskless asset with dynamics \((B_t)_{t=0}^T\). Transaction costs in the riskless asset can be incorporated by considering bid-ask prices \(B^B_t \leq B^B_t\) for \(t = 0, 1, \ldots, T\). For \(d = 3\), if both risky assets are denoted in the domestic currency (the currency of the riskless asset), if \(\lambda_1, \lambda_2 > 0\), and if we assume an exchange between the two risky assets cannot be made directly, only via cash by selling one asset and buying the other, we obtained the following tree model for the solvency cone process \(K_t\). At node \((t, j_1, j_2)\) the generating vectors of the solvency cone are given by \(\pi^i e^i - e^j\), \(0 \leq i, j \leq 2\) (see section 4), i.e., by the columns of the following matrix

\[
\begin{pmatrix}
\frac{(S^1_0)^1}{B^B_0} & \frac{(S^1_0)^2}{B^B_0} & -\frac{(S^1_0)^2}{B^B_0} & 0 & 0 \\
-1 & 1 & 0 & 0 & 1 \\
0 & 0 & -1 & 1 & \frac{(S^2_0)^1}{(S^1_0)^2} - \frac{(S^2_0)^1}{(S^1_0)^2} \\
\end{pmatrix}
\]  

(4.6)

If there are no transaction costs for the riskless asset, i.e. \(B^B_t = B^B_t\), the last two generating vectors in (4.6) are redundant and can be omitted. Non-redundant generating vectors of positive dual cones of solvency cones \(K_t\) are given in [27].

Note, if there is an assets denoted in a currency different from the domestic currency, one would rather use a discrete approximation of a mean-reverting process than a geometric Brownian motion for this asset. In this case the model for the process \(Y\) with independent components would be similar to above, but the transformation (4.4) needs to be adapted to the new setting, see for example [19].

Random proportional transaction costs for an asset can be modeled analogously, by treating them as another (correlated) risky asset. In (4.5) one would just replace the constant \(\lambda\) by the value of the stochastic process \(\lambda\) at node \((t, j_1, \ldots, j_{d-1})\).

**Example 4.3.** (exchange option) Let us consider a European option in which at expiration, the holder can exchange one unit of asset 2 and receive one unit of asset 1. Let asset 0 be a riskless bond \(B\) with annual interest rate \(r\) under frequent compounding and face value \(B_T = 1\). We assume constant transaction costs \(\lambda_0\) for the bond with bid and ask prices as in (4.5). Assets 1 and 2 are two correlated stocks \(S^1\) and \(S^2\), denoted in the same currency as the bond, with initial stock price for the first stock \(S^1_0 = 45\), volatility \(\sigma_1 = 0.15\), constant transaction costs \(\lambda_1\) and for the second stock \(S^2_0 = 50\), \(\sigma_2 = 0.2\), \(\lambda_2\) and correlation \(\rho = 0.2\) between both stocks. The tree is modeled as described in section 4.2. The bid and ask prices are given as in (4.5).

Consider an exchange option with physical delivery. The payoff is given by

\[
X(\omega) = (X_1(\omega), X_2(\omega), X_3(\omega))^T = \left(0, I_{\{S^2_0 \geq S^2_0\}}(\omega), -I_{\{S^2_0 \geq S^2_0\}}(\omega)\right)^T.
\]

The maturity is one year. Table 2 gives the vertices of \(SHP_0(X)\) in units of (bond, asset 1, asset 2)\(^T\) and the scalar superhedging prices \(\pi^a(X)\) in units of bond and \(\pi^a(X)\) in the domestic currency for different values for \(r\) and \(\lambda = (\lambda_1, \lambda_2, \lambda_3)^T\). The recession cone of \(SHP_0(X)\) is equal to \(K_0\) generated by the vectors given in (4.6). Note that \(\pi^a(X)\) in the domestic currency can be calculated straight forward if \(\lambda_0 = 0\) by \(\pi^a(X) = \pi^a_0(X)B_0\). If \(\lambda_0 > 0\), the scalar
superhedging price $\pi^a(X)$ in the domestic currency can be calculated by adding to $SHP_0(X)$ (as a 3 dimensional object in a four dimensional space, where the cash axis was added) the four dimensional cone $\tilde{K}_0$, which is generated by the bid and ask prices of the bond and the stocks as in (3.9), and calculating the vertex of the intersection with the cash axis. The reason is that the transaction costs for the bond might lead to the effect that there might be cheaper ways to trade cash into the set $SHP_0(X)$ than to trade the pure cash position $(\pi_0^a(X))^+ B_0^a - (\pi_0^a(X))^- B_0^b$ into the pure bond position $\pi_0^a(X)$, see the fifth example in table 2 where $\pi_0^a(X) = 6.988 < 7.011 = (\pi_0^a(X))^+ B_0^a - (\pi_0^a(X))^- B_0^b$.

**Example 4.4.** (outperformance option: superhedging portfolios and strategies) Let us consider an outperformance option. Let asset 0 be a riskless cash account with zero interest rates. Assets 1 and 2 are two correlated stocks $S^1$ and $S^2$, denoted in the same currency as the cash account, with initial stock price for the first stock $S_0^1 = 50$, volatility $\sigma_1 = 0.15$, constant transaction costs $\lambda_1 = 0.2$ and for the second stock $S_0^2 = 45$, $\sigma_2 = 0.2$, $\lambda_2 = 0.1$ and correlation $\rho = 0.2$ between both stocks. The tree is modeled as described in section 4.2. The bid and ask is given as in (4.5).

We want to calculate the set of superhedging portfolios of an outperformance option with physical delivery and strike $K = 47$. The payoff is given by

$$X(\omega) = (X_1(\omega), X_2(\omega), X_3(\omega))^T$$

$$= \left( -KI_{\{s^a_1, s^a_2 \geq K\}}(\omega) , I_{\{s^a_1 \geq s^a_2 \text{ and } s^a_2 \geq K\}}(\omega) , I_{\{s^a_2 > s^a_1 \text{ and } s^a_2 \geq K\}}(\omega) \right)^T.$$ 

The maturity is one year. We will use only a small number of time intervals $n = 4$ to illustrate different possibilities of choosing optimal superhedging strategies as described in section 3.1. The set of superhedging portfolios $SHP_0(X)$ has two vertices

$$\begin{pmatrix} -27.404 \\ 0.514 \\ 0.388 \end{pmatrix}, \quad \begin{pmatrix} -34.254 \\ 0.567 \\ 0.480 \end{pmatrix}$$

and a recession cone equal to the solvency cone $K_0$. The scalar superhedging price is $\pi^a(X) = 22.624$ in the domestic currency and the scalar subhedging price is $\pi^b(X) = -8.633$. Negative bid prices might occur when physical delivery is considered in markets with transaction costs. This issue was discussed and resolved in [29], see also remark 3.30 in [31].

Let us compute superhedging strategies for a given path using the methods in section 3.1. We fix a path given by the sequence of indices $j_1 = 1, 2, 3, 4$ and $j_2 = 1, 2, 3, 4$ at times $t = 0, 1, ..., 4$ and an initial portfolio vector given by the first vertex $x_0 = (-27.404, 0.514, 0.388)^T$. Instead of visualizing all possible strategies as a subset of $\mathbb{R}^3$, the bi-criteria problems (3.12) and (3.13) are considered. Plotting the upper images, i.e. the values of the $\mathbb{R}^2$-valued objective function and all (component-wise) larger vectors, we can visualize the tradeoffs and select $\mathbb{R}^2_+$-minimal vectors. In particular, all vertices of the upper images are minimal.

Figure 4 illustrates the strategy of following the max-cash approach at all points in time. At $t = 2, 2.882$ can be withdrawn while still guaranteeing superhedging. However, if one follows the min-trading strategy as illustrated in figure 5 and does not take the money out at $t = 2$, one can withdraw even 6.143 at terminal time $t = 4$ for this particular path. Thus, by leaving money inside, a larger payout at a later date is possible. However, one could also loose the money left inside. Figure 6 illustrates this in an example of a mixed strategy: at $t = 2$ an
\[
\begin{align*}
\pi^n_0(X) &= \pi^n(X) \\
\pi^n_0(X) (\text{in bonds}) &= 6.789 \\
\pi^n_0(X) (\text{in cash}) &= 8.158
\end{align*}
\]

\[
\begin{align*}
\pi^n_0(X) &= \pi^n(X) \\
\pi^n_0(X) (\text{in bonds}) &= 7.134 \\
\pi^n_0(X) (\text{in cash}) &= 8.576
\end{align*}
\]

\[
\begin{align*}
\pi^n_0(X) &= \pi^n(X) \\
\pi^n_0(X) (\text{in bonds}) &= 4.240 \\
\pi^n_0(X) (\text{in cash}) &= 4.429
\end{align*}
\]

\[
\begin{align*}
\pi^n_0(X) &= \pi^n(X) \\
\pi^n_0(X) (\text{in bonds}) &= 7.418 \\
\pi^n_0(X) (\text{in cash}) &= 8.167
\end{align*}
\]

\[
\begin{align*}
\pi^n_0(X) &= \pi^n(X) \\
\pi^n_0(X) (\text{in bonds}) &= 4.310 \\
\pi^n_0(X) (\text{in cash}) &= 4.318
\end{align*}
\]

Table 2: The set of superhedging portfolios of an exchange option with and without transaction costs for the bond (see example 4)
efficient point in between max-cash and min-trading is chosen, and at time \( t = 3 \) min-trading is chosen which corresponds to a withdrawal of 3.006\$, which is 1.536\$ less than one could have taken out with max cash. However, this money is lost at \( t = 4 \).

Figure 4: Example 4.4 with max-cash selection. In the first picture we see that no trading (\( y_2 \)-axis) is necessary and no withdrawal (\( y_1 \)-axis) is possible at time \( t = 0 \). At time \( t = 1 \), trading is necessary and no withdrawal is possible. We see that for \( t = 0,1,3,4 \) exactly one minimal point exists, which refers to both a max-cash and a min-trading strategy. At time \( t = 2 \) we choose point \( P_1 \) which refers to a max-cash strategy where 2.882\$ are withdrawn. A strategy that refers to \( P_1 \) entails a high trading activity. As no further withdrawal is possible, the sum of all withdrawals is 2.882\$.

Figure 5: Example 4.4 with min-trading selection. The first difference in comparison to figure 4 is at time \( t = 2 \), where we choose the point \( P_2 \). The possible withdrawal of 2.882\$ is waived and a min-trading strategy is chosen. As a result we have different choices at time \( t = 3 \) and a withdrawal of 5.704\$ would be possible now. But a better result can be obtained if the withdrawal is waived and a min-trading strategy is taken by choosing point \( P_3 \). At time \( t = T = 4 \), we have two choices. To obtain the withdrawal only in cash, we choose point \( P_4 \). The sum of all withdrawals is 6.143\$. This shows that the min-trading strategy may yield higher total withdrawals.

5 Reformulating and solving the problem

There are at least four reasons to relate the recursive representation of the set \( SHP_0(X) \) of superhedging portfolios in (3.2) and (3.3) to linear vector optimization:

1. The problem can be formulated as a sequence of linear vector optimization problems.

2. Vectorial duality theory \[18, 26\] in an extended form \[16\] leads to interesting insights into the problem’s nature, see also section 7.3.

3. Existing algorithms \[31, 32\] for the scalar superhedging price as well as their generalization to the case of more than two assets can be interpreted in the framework of vectorial duality theory, see section 7.
Figure 6: Example 4.4: loss of money by min-trading strategy. The first difference in comparison to figures 4 and 5 is at time $t = 2$, where we choose the minimal point $P_5$. The withdrawal at $t = 2$ is 1.222$. At time $t = 3$ a withdrawal of 3.320$ would be possible connected with trading. But we only take the smaller amount 1.7841$ which can be withdrawn without trading (choice of point $P_6$). We see that no further withdrawal is possible at time $t = T = 4$. This means that the total withdrawal is only 3.006$. This means that the money which was waived to withdraw at $t = 3$ by choosing the min-trading strategy has been lost. The max-cash strategy at $t = 3$ had result the higher withdrawal 4.542$.

4. The set of superhedging portfolios can be computed using existing algorithms from linear vector optimization (in each iteration step). To this end the algorithms from the literature [2, 9, 26] have been extended to more general ordering cones, see [16].

We consider a linear optimization problem with a $q$-dimensional objective function. The image space $\mathbb{R}^q$ is partially ordered by a polyhedral convex cone $C \subseteq \mathbb{R}^q$ that contains no lines and has non-empty interior. For $y, z \in \mathbb{R}^q$ we write $y \leq_C z$, or shortly $y \leq z$, if $z - y \in C$. We consider the problem to minimize $P : \mathbb{R}^d \rightarrow \mathbb{R}^q$ with respect to $\leq_C$ subject to $Bx \geq b$, \hspace{1cm} (P)

where $P$ is linear, i.e. $P \in \mathbb{R}^{q \times d}$, $B \in \mathbb{R}^{m \times d}$, $b \in \mathbb{R}^m$. The feasible set of (P) is denoted $S := \{ x \in \mathbb{R}^d \mid Bx \geq b \}$.

The dual problem to (P) is

maximize $D^* : \mathbb{R}^m \times \mathbb{R}^q \rightarrow \mathbb{R}^q$ with respect to $\leq_K$ over $T$, \hspace{1cm} (D*)

with (linear) objective function

$D^* : \mathbb{R}^m \times \mathbb{R}^q \rightarrow \mathbb{R}^q$, \hspace{1cm} $D^*(u, w) := (w_1, \ldots, w_{q-1}, b^T u)^T$,

ordering cone

$K := \mathbb{R}_+ \cdot (0, 0, \ldots, 0, 1)^T$,

and feasible set

$T := \{ (u, w) \in \mathbb{R}^m \times \mathbb{R}^q \mid u \geq 0, \ B^T u = P^T w, \ c^T w = 1, \ w \in C^+ \}$;

where $c$ is a fixed vector in $\text{int} C$ and $C^+ := \{ w \in \mathbb{R}^q \mid \forall y \in C : \ w^T y \geq 0 \}$ is the dual cone of $C$. This kind of vectorial duality, called geometric duality, has been introduced in [18]. A more detailed discussion is given below in section 6. For the moment we are content with the fact that Benson’s algorithm [2, 9, 26, 16] can be used to compute solutions (as defined below) to both the primal and the dual problem. The set of superhedging portfolios can be computed by the following algorithm.
SHP-Algorithm

Input:
for all $t \in \{0, 1, \ldots, T\}$, for all $\omega \in \Omega_t$: $\widetilde{K}_t(\omega)$ (matrix of generating vectors of $K_t(\omega)$),
for all $\omega \in \Omega_T$: $X(\omega)$,

Output:
for all $t \in \{0, 1, \ldots, T\}$, for all $\omega \in \Omega_t$: $SHP_t(X)(\omega) = \{ x \in \mathbb{R}^d \mid B^\omega x \geq b^\omega \}$.

00: begin
01: for all $\omega \in \Omega_T$
02: begin
03: \( B^\omega = \widetilde{K}_T(\omega)^\dagger \); \( b^\omega = (B^\omega)^T \cdot X(\omega) \);
04: end;
05: for $t=T-1$ downto 0
06: begin
07: for all $\omega \in \Omega_t$
08: begin
09: \( B = \{ B^\bar{\omega} \mid \bar{\omega} \in \text{succ}(\omega) \} \);
10: \( b = \{ b^\bar{\omega} \mid \bar{\omega} \in \text{succ}(\omega) \} \);
11: \( P = \text{LiquidationMap}(K_t(\omega)) \);
12: \( C = P \cdot \widetilde{K}_t(\omega) \);
13: compute a solution \( \{(u^1, w^1), \ldots, (u^k, w^k)\} \) to \( \text{[D']} \);
14: \( B^\omega = (P^T u^1, \ldots, P^T u^k)^T \);
15: \( b^\omega = (b^T u^1, \ldots, b^T u^k)^T \);
16: end;
17: end;
18: end;
19: end.

The SHP-Algorithm yields the set of superhedging portfolios by computing a sequence of linear vector optimization problems together with several elementary operations. To be more precise, the algorithm consists of a reformulation of the recursive description of the set of superhedging portfolios given in (3.2) and (3.3). Solving the linear vector optimization problem \( \text{[D']} \) in line 14 means to compute an inequality representation of \( P[S] + C \), where \( S = \{ x \in \mathbb{R}^d \mid Bx \geq b \} \) with \( P, B, b, C \) in each step as in lines 10-13 of the SHP-Algorithm. Under the assumptions of theorem 3.1 the algorithm computes inequality representations of $SHP_t(X)(\omega)$ for all $t \in \{T, \ldots, 0\}$ and all $\omega \in \Omega_t$.

Let us discuss the steps of the SHP-Algorithm in detail. In line 03 we need to compute the generating vector of the dual cone which is a standard problem of computational geometry. For special classes of solvency cones we can exploit an explicit description of the dual cone, see [27]. In lines 03 and 04 we store an inequality representation \( \{ x \in \mathbb{R}^d \mid B^\omega x \geq b^\omega \} \) of the sets

\[ SHP_T(X)(\omega) = X_T(w) + K_T(\omega) \]

for all $\omega \in \Omega_T$.

In lines 10 and 11, an inequality representation \( S = \{ x \in \mathbb{R}^d \mid Bx \geq b \} \) of

\[ \bigcap_{\bar{\omega} \in \text{succ}(\omega)} SHP_{t+1}(X)(\bar{\omega}), \quad (\omega \in \Omega_t) \]
is obtained. The set $S$ is understood to be the feasible set of the primal linear vector optimization problem $[P]$, where the objective function and the ordering cone are computed in lines 12 and 13. The objective function $P$ in line 12 is the liquidation map, which corresponds to an exchange of those assets with no transaction costs into a single asset. For instance, if there are no transaction costs between assets $i$ and $j$, the matrix

$$P_{ji} = (e^{1}, \ldots, e^{i-1}, e^{i} + \pi_{ij} e^{j}, e^{i+1}, \ldots, e^{j-1}, e^{j+1}, \ldots, e^{d})^T$$

is considered, where $\pi_{ij}$ is the exchange rate, the price of asset $j$ in terms of asset $i$, as defined in section 2. If $x$ is a portfolio with $d$ assets then $P_{ji}x$ is a portfolio with $d - 1$ assets which is obtained from $x$ by exchanging asset $j$ in asset $i$ without transaction costs. The same procedure is applied to the new portfolio if there are further pairs of assets having no transaction costs. This process determines the liquidation map $P$ for solvency cones as introduced in section 2. At the end we have transaction costs between any two assets. As a consequence, the cone $C$ computed in line 13 (the ordering cone of $[P]$) contains no lines. This is important for the usage of Benson’s algorithm.

A solution to the vector optimization problem $[P]$ can be interpreted as a finite set of efficient points (portfolios) and directions that contain complete information to describe the polyhedron $P[S] + C$, which is the set of superhedging portfolios after applying the liquidation map. A solution to the dual problem $[D^*]$ provides an inequality representation of the set $P[S] + C$. An inequality representation of $\text{SHP}(X)_t(\omega)$ is then immediately obtained using the liquidation map (lines 15 and 16), $t \in \{0, \ldots, T - 1\}$.

Note that we can use Benson’s algorithm to compute a solution to $[D^*]$ in line 14. This algorithm yields simultaneously a solution to $[P]$. This means that we also obtain a representation of the superhedging portfolios $\text{SHP}_t(X)(\omega)$ by finitely many efficient portfolios $\{x^1, \ldots, x^s\} \subseteq \mathbb{R}^d$ and efficient directions $\{\hat{x}^1, \ldots, \hat{x}^t\} \subseteq \mathbb{R}^d \setminus \{0\}$, that is, a representation of the form

$$\text{SHP}_t(X)(\omega) = \text{co} \{x^1, \ldots, x^s\} + \text{cone} \{\hat{x}^1, \ldots, \hat{x}^t\} + K_t(\omega).$$

A definition of what is meant by efficient is given in the next section.

It should be noted that line 14 could be replaced by methods from computational geometry. Essentially one can use vertex enumeration methods for (unbounded) polyhedra. Benson’s algorithm also involves a vertex enumeration method and additionally several LPs to be solved. Beyond the fact that Benson’s algorithm can be easily applied, we obtain interesting insights into the nature of set-valued superhedging portfolios by relating them to vector optimization. Therefore, we continue this discussion after introducing some basic notions and results from linear vector optimization.

Note that the output of the SHP-Algorithm can be used to compute superhedging strategies as discussed in section 3.1 where we assumed that inequality representations of $\text{SHP}_t(X)$ are known.

6 Some basics on linear vector optimization

It should be noted first that there are different approaches to solution concepts and duality for linear vector optimization problems in the literature, see e.g. [20, 8, 4, 26] and the references...
therein. Here we only present geometric duality \cite{18,26,16} and solution concepts as introduced in \cite{26}.

Let us first recall some facts on polyhedra. A polyhedron (or polyhedral set) \( A \) in \( \mathbb{R}^q \) is defined to be the intersection of finitely many halfspaces, that is

\[
A = \bigcap_{i=1}^r \{ y \in \mathbb{R}^q | (z^i)^T y \geq \gamma_i \} \quad (z^i \neq 0).
\]

Every non-empty polyhedron \( A \) in \( \mathbb{R}^q \) can be expressed by finitely many points \( x^1, \ldots, x^s \in \mathbb{R}^q \) and directions \( k^1, \ldots, k^t \in \mathbb{R}^q \setminus \{0\} \):

\[
A = \left\{ \sum_{i=1}^s \lambda_i x^i + \sum_{j=1}^t \mu_j k^j \bigg| \lambda_i \geq 0 \ (i = 1, \ldots, s), \sum_{i=1}^s \lambda_i = 1, \ \mu_j \geq 0 \ (j = 1, \ldots, t) \right\}.
\]

This can be also written as

\[
A = \text{co} \left\{ x^1, \ldots, x^s \right\} + \text{cone} \left\{ k^1, \ldots, k^t \right\}.
\]

Note that we set \( \text{cone} \emptyset = \{0\} \), which applies in the bounded case where no directions occur. The set \( A_\infty := \text{cone} \left\{ k^1, \ldots, k^t \right\} \) is the recession cone of \( A \). As usual in computational geometry, a finite set of halfspaces defining a polyhedron \( A \) is called \( H \)-representation (or inequality representation) of \( A \), whereas a finite set of points and directions defining \( A \) is called \( V \)-representation of \( A \). A bounded polyhedron is called polytope.

Let us consider problem (\( P \)) as introduced in section 5 with objective matrix \( P \in \mathbb{R}^{q \times d} \), polyhedral ordering cone \( C \subseteq \mathbb{R}^q \) that contains no lines and has non-empty interior, and feasible set \( S = \{ x \in \mathbb{R}^d | Bx \geq b \} \). A solution to (\( P \)) (as defined in \cite{26}) is a feasible point \( \bar{u}, \bar{w} \) with \( P\bar{z} \leq_C P\bar{x} \) and \( P\bar{z} \neq P\bar{x} \). The set of efficient points of (\( P \)) is denoted by \( \text{Eff}(P) \). Efficient directions can be introduced by the homogeneous problem

minimize \( P : \mathbb{R}^d \to \mathbb{R}^q \) with respect to \( \leq_C \) subject to \( Bx \geq 0 \). \hspace{1cm} (\( P^h \))

Nonzero efficient points of (\( P^h \)) are called efficient directions of (\( P \)).

A solution to (\( P^h \)) is a non-empty finite subset \( S \subseteq \text{Eff}(P) \) together with a (possibly empty) subset \( S^h \subseteq \text{Eff}(P^h) \) such that

\[
\text{co} P[S] + \text{cone} P[S^h] + C = P[S] + C.
\]

The latter condition means that a \( V \)-representation of the set \( P : \mathbb{R}^d \to \mathbb{R}^q \) given by a solution. The set \( P \) is called upper image of (\( P \)).

Likewise a feasible point \( (\bar{u}, \bar{w}) \in T \) is said to be efficient for (\( D^* \)) (as introduced in section 5) if there is no \( (u, w) \in T \) such that \( D^*(u, w) \geq_K D^*(\bar{u}, \bar{w}) \). Efficient directions do not occur in the dual problem. The set of efficient points for (\( D^* \)) is denoted by \( \text{Eff}(D^*) \). A solution to (\( D^* \)) is a non-empty finite subset \( \bar{T} \subseteq \text{Eff}(D^*) \) such that

\[
\text{co} D^*[\bar{T}] - K = \text{co} D^*[T] - K.
\]
This means that a solution can be used to get a $V$-representation of the set $D^*: = D^*[T] - K$, which is called the lower image of $(D^*)$.

To give a first flavor of the duality relation between $(P)$ and $(D^*)$, let us mention here that a solution to $(P)$ also provides an $H$-representation of $D^*: = D^*[T] - K$ and a solution to $(D^*)$ provides an $H$-representation of $P = P[S] + C$, which is exactly what we use in the above algorithm.

The duality relation between $P$ and $D^*$ is similar to the classical idea of duality of polytopes. Two polytopes $A$ and $A^*$ in $\mathbb{R}^q$ are said to be dual to each other if there exists a one-to-one mapping $\Psi$ between the set of all faces of $A$ and the set of all faces of $A^*$ such that $\Psi$ is inclusion reversing, i.e. faces $F^1$ and $F^2$ of $A$ satisfy $F^1 \subseteq F^2$ if and only if the faces $\Psi(F^1)$ and $\Psi(F^2)$ of $A^*$ satisfy $\Psi(F^1) \supseteq \Psi(F^2)$, see e.g. [12].

A duality mapping $\Psi$ for the vector optimization problems $(P)$ and $(D^*)$ is now introduced in several steps. We assume throughout that there exists a vector $c \in \text{int} C$ such that $c_q = 1$. (6.1)

For the parameter $c$, we consider the following bi-affine function

$$\varphi : \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}, \quad \varphi(y, y^*) := \sum_{i=1}^{q-1} y_i y^*_i + y_q \left(1 - \sum_{i=1}^{q-1} c_i y^*_i \right) - y^*_q.$$  

This coupling function has been introduced in [18] for the choice $c = (1, \ldots, 1)^T$.

The coupling function $\varphi$ is used to define the following two injective hyperplane-valued maps

$$H : \mathbb{R}^q \Rightarrow \mathbb{R}^q, \quad H(y^*) := \{ y \in \mathbb{R}^q | \varphi(y, y^*) = 0 \},$$

$$H^* : \mathbb{R}^q \Rightarrow \mathbb{R}^q, \quad H^*(y) := \{ y^* \in \mathbb{R}^q | \varphi(y, y^*) = 0 \}.$$  

The mapping $H$ is used to define the duality function

$$\Psi : 2^{\mathbb{R}^q} \to 2^{\mathbb{R}^q}, \quad \Psi(F^*) := \bigcap_{y^* \in F^*} H(y^*) \cap P,$$

where $2^{\mathbb{R}^q}$ denotes the power set of $\mathbb{R}^q$.

Weak duality reads as follows.

**Theorem 6.1** ([18], [26]). The following implication is true:

$$[y \in P \land y^* \in D^*] \implies \varphi(y, y^*) \geq 0.$$  

A motivation can be given by the following geometric interpretation. Weak duality implies the inclusions

$$D^* \subseteq \{ y^* \in \mathbb{R}^q | \forall y \in P : \varphi(y, y^*) \geq 0 \} \quad \text{and} \quad P \subseteq \{ y \in \mathbb{R}^q | \forall y^* \in D^* : \varphi(y, y^*) \geq 0 \},$$

whereas the following strong duality theorem yields even equality.

**Theorem 6.2** ([18], [26]). Let the feasible sets $S$ of $(P)$ and $T$ of $(D^*)$ be non-empty. Then

$$[\forall y^* \in D^* : \varphi(y, y^*) \geq 0] \implies y \in P,$$

$$[\forall y \in P : \varphi(y, y^*) \geq 0] \implies y^* \in D^*.$$
Geometric duality is a further duality statement that takes into account the facial structure of the sets $\mathcal{P}$ and $\mathcal{D}^*$. 

**Theorem 6.3** ([18, 17]). $\Psi$ is an inclusion reversing one-to-one map between the set of all $K$-maximal proper faces of $\mathcal{D}^*$ and the set of all proper faces of $\mathcal{P}$. The inverse map is given by

$$
\Psi^{-1}(F) = \bigcap_{y \in F} H^*(y) \cap \mathcal{D}^*.
$$

Moreover, if $F^*$ is a $K$-maximal proper face of $\mathcal{D}^*$, then

$$
\dim F^* + \dim \Psi(F^*) = q - 1.
$$

As shown in [26], the non-$K$-maximal proper facets (i.e. faces of dimension $q - 1$) of $\mathcal{D}^*$ correspond to the extreme directions of $\mathcal{P}$ by a similar relation, where the coupling function has to be replaced by $\hat{\varphi} : \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}$, $\hat{\varphi}(y, y^*) := \varphi(y, y^*) + y^*_q$. Let us illustrate the main ideas by two examples.

**Example 6.4.** Consider problem (P) with the data

$$
P = \begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}, \quad B = \begin{pmatrix}
2 & 1 \\
1 & 2 \\
1 & 0 \\
0 & 1
\end{pmatrix}, \quad b = \begin{pmatrix}
6 \\
6 \\
0 \\
0
\end{pmatrix},
$$

and the ordering cone

$$
C = \text{cone} \left\{ \begin{pmatrix}
-3 \\
1
\end{pmatrix}, \begin{pmatrix}
1 \\
2
\end{pmatrix} \right\}.
$$

The pair $(\tilde{S}, \tilde{S}^h)$ with $\tilde{S} = \{(2,2)^T, (6,0)^T\}$ and $\tilde{S}^h = \{(1,0)^T\}$ is a solution to (P). It contains the information to construct the upper image $\mathcal{P}$ via $\mathcal{P} = \text{co} P[\tilde{S}] + \text{cone} P[\tilde{S}^h] + C$, see figure 7 for an illustration.
Choosing \( c = (0, 1)^T \in \text{int} \, C \), we obtain the geometric dual problem \( \mathcal{D}^* \) with the objective function
\[
D^*(u_1, u_2, u_3, u_4, w_1, w_2) = (w_1, 6u_1 + 6w_2)^T
\]
and the feasible set \( T \subseteq \mathbb{R}^4 \times \mathbb{R}^2 \) which is given by the equalities
\[
2u_1 + u_2 + u_3 - w_1 - w_2 = 0 \\
u_1 + 2u_2 + u_4 + w_1 - w_2 = 0 \\
w_2 = 1
\]
and the inequalities
\[
\begin{align*}
  u_1 &\geq 0 \\
u_2 &\geq 0 \\
u_3 &\geq 0 \\
u_4 &\geq 0
\end{align*}
\]
The set
\[
\bar{T} = \left\{(0, 0, 0, 2, -1, 1)^T, \left(0, \frac{2}{3}, 0, 0, -\frac{1}{3}, 1\right)^T, \left(\frac{2}{3}, 0, 0, 0, \frac{1}{3}, 1\right)^T\right\}
\]
provides a solution to \( \mathcal{D}^* \). It contains the required information to construct the lower image \( \mathcal{D}^*: = D^*[T] - K \) via \( \mathcal{D}^* = D^*[\bar{T}] - K \). The vertices of \( \mathcal{D}^* \) are
\[
D^*[\bar{T}] = \left\{(-1, 0)^T, \left(-\frac{1}{3}, 4\right)^T, \left(\frac{1}{3}, 4\right)^T\right\},
\]
and together with the extreme direction \( (0, 1)^T \) of the cone \( K \) we have a \( V \)-representation of \( \mathcal{D}^* \). Using the coupling function \( \varphi \) we immediately obtain an \( H \)-representation of the upper image \( \mathcal{P} \). For example, taking \( y^1 = (-1, 0)^T \in D^*[\bar{T}] \), the inequality \( \varphi(y, y^1) \geq 0 \) yields \( -y_1 + y_2 \geq 0 \). An \( H \)-representation of \( \mathcal{P} = P[S] + C \) is obtained in this way as
\[
\mathcal{P} = \left\{y \in \mathbb{R}^2 \mid -y_1 + y_2 \geq 0, -\frac{1}{3}y_1 + y_2 \geq 4, \frac{1}{3}y_1 + y_2 \geq 4\right\}.
\]
Likewise we obtain an \( H \)-representation of \( \mathcal{D}^* \) from a solution to \( (P) \). For \( x^1 := (2, 2)^T \in \bar{S} \) and \( y^1 := Px^1 = (0, 4)^T \) the inequality \( \varphi(y^1, y^*) \geq 0 \) yields \( y_2^* \leq 4 \) and for \( x^2 := (6, 0)^T \in \bar{S} \) and \( y^2 := Px^2 = (6, 6)^T \) we get \( 6y_1^* - y_2^* \geq -6 \). Further inequalities are obtained from \( \bar{S}^h \) and from the extreme directions of the ordering cone \( C \). To this end, \( \varphi \) has to be replaced by \( \hat{\varphi} \), which means that the \( y_2^* \)-component has to be deleted. Taking \( \hat{x}^1 := (1, 0)^T \in \bar{S}^h \) and \( \hat{y}^1 := P\hat{x}^1 = (1, 1)^T \), we get \( y_1^* \geq -1 \). Taking \( \hat{y}^2 = (-3, 1)^T \) and \( \hat{y}^3 = (1, 2)^T \) we get \( y_1^* \leq \frac{1}{3} \) and \( y_1^* \geq -2 \), where the latter inequality is redundant. Thus
\[
\mathcal{D}^* = \left\{y^* \in \mathbb{R}^2 \mid y_2^* \leq 4, 6y_1^* - y_2^* \geq -6, -1 \leq y_1^* \leq \frac{1}{3}\right\},
\]
see figure [7].
Figure 8: The upper and lower images of Example 6.5. The dark gray facets, the plain edges and all vertices of $D^*$ are $K$-maximal and the light gray facets and the dashed edges of $D^*$ are not.

Example 6.5. Consider problem (P) with the data

$$P = \begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
2 & 4 & 4 \\
4 & 2 & 4 \\
4 & 4 & 2 \\
1 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad b = \begin{pmatrix}
3 \\
3 \\
3 \\
1 \\
0 \\
0 \\
0
\end{pmatrix}, \quad C = \mathbb{R}^3_+.$$

and the dual problem (D*) for the parameter $c = (1,1,1)^T$. The upper image $P$ has 6 vertices, 18 edges and 13 facets. The lower image $D^*$ has 13 vertices, 18 $K$-maximal edges and 6 $K$-maximal facets, see figure 8 for an illustration.

Note that in [18, 26] only the case $c = (1,1,...,1)^T$ and $C = \mathbb{R}^q_+$ is considered. The proofs of the general case are straightforward and will be published in a forthcoming paper [16]. Moreover, Theorem 6.3 (for $c$ as in (6.1)) is a special case of geometric duality for convex vector optimization problems, see example 3 in [17].

7 Scalar superhedging price

The set of superhedging portfolios plays an important role when one is actually interested in carrying out a strategy starting from an initial portfolio vector that can contain more assets than just cash. Clearly, in this case, it is not enough to know the scalar superhedging price.

On the other hand, if one is interested solely in price bounds for a claim in a certain currency, the scalar superhedging and subhedging prices $\pi^a(X)$ and $\pi^b(X)$ give exactly this information. From robust no arbitrage it follows that the market bid and ask prices $p^b(X)$ and $p^a(X)$ of a claim $X$ have to satisfy

$$p^b(X) \leq \pi^a(X), \quad p^a(X) \geq \pi^b(X) \quad \text{and} \quad p^b(X) \leq p^a(X), \quad (7.1)$$
where \( p^b(X), p^a(X), \pi^b(X), \pi^a(X) \) are all denoted in the same currency. Furthermore, the inequalities
\[
p^b(X) \geq \pi^b(X), \quad \text{and} \quad p^a(X) \leq \pi^a(X)
\]
are reasonable: Obviously, one would rather superreplicate \( X \) with \( \pi^a(X) \) than to buy it at a higher price if \( p^a(X) > \pi^a(X) \), but this last inequality would not create robust arbitrage as long as the inequalities in (7.1) are satisfied. In total, one obtains price bounds
\[
\pi^b(X) \leq p^b(X) \leq p^a(X) \leq \pi^a(X).
\]
In the following, we will discuss how \( \pi^b(X) \) and \( \pi^a(X) \) can be calculated and how the obtained algorithm is related to the previously studied SHP-Algorithm. In section 7.1 theorem 7.1 a dual representation of the scalar superhedging price is given. The result is the \( d \)-dimensional version of Jouini, Kallal [21] and can be obtained by scalarizing the dual representation of the set of superhedging portfolios. The dual representation of the scalar superhedging price allows to deduce dynamic programming equations (corollary 7.2) that allow to implement and efficiently calculate the scalar superhedging price of a claim (algorithm in corollary 7.3 in section 7.2). In section 7.3 the relation between the scalar algorithm and the SHP-Algorithm via geometric duality is discussed. One obtains that the calculation of \( \pi^a(X) \) reveals also the set \( \text{SHP}_0(X) \) if a certain mapping is applied to a function appearing in the penultimate step of the scalar algorithm (lemma 7.5).

### 7.1 Scalarization of the dual representation

In the previous sections, the set \( \text{SHP}_0(X) \) of all initial portfolio vectors that allow to superhedge a claim \( X \in L^0_d(\mathcal{F}_T, \mathbb{R}^d) \) was studied. If one is interested in the calculation of price bounds, it is helpful to study the smallest superhedging prices (and the largest subhedging prices) in the currencies or numéraire of interest. To do so, consider the one dimensional subspace \( M \) (see also (4.11)) given by \( M = \{se^i, s \in \mathbb{R} \} \), where asset number \( i \) is the chosen numéraire. One might repeat this procedure for different currencies/numéraires if one is interested in price bounds of the portfolio \( X \) in different currencies/numéraires. We focus on those superhedging elements that lie in \( M \), i.e. we consider \( \text{SHP}^M_0(X) \). From (2.3) and theorem 2.1 we obtain
\[
\text{SHP}^M_0(X) = \{x_0 \in M : X \in x_0 + A_T\} \tag{7.2}
\]
\[
\cup \{ (Q, w) \in \mathcal{W}^1 : E^Q[X] + G(w) \cap M \} \tag{7.3}
\]
\[
= \{ x_0 \in M : \forall Z \in \mathbb{Z} \text{ with } E[(X^TZ)^-] < \infty : E[X^TZ] \leq x_0^T Z_0 \} \tag{7.4}
\]
To obtain the smallest superhedging price, we apply the scalarization procedure introduced in [13] to the function \( R(X) = \text{SHP}^M_0(X) \). That is, we consider the extended real-valued function \( \varphi_{R,v} : L^0_d(\mathcal{F}_T, \mathbb{R}^d) \to \mathbb{R} \cup \{\pm \infty\} \) given by
\[
\varphi_{R,v}(X) = \inf_{u \in R(X)} v^T u
\]
for \( v \in (K_0^M)^+ \) (see also section 5.1 in [15]). \( (K_0^M)^+ \) denotes the positive dual cone of the cone \( K_0^M = K_0 \cap M \) in \( M \). Thus,
\[
(K_0^M)^+ = \{ v \in M : \forall u \in K_0^M : v^T u \geq 0 \} \subseteq M.
\]
We will apply the scalarization to the dual representation of $SHP_0^M$ given in (7.3), respectively (7.4). Dynamic programming equations can be obtained. For finite probability spaces this leads to an algorithm that goes backwards in the event tree. Recall that in our case $M = \{s e^i, s \in \mathbb{R}\}$, where asset number $i \in \{1,...,d\}$ is the asset of interest, e.g. the USD cash account, or a bond. Thus, we are scalarizing with respect to $v \equiv e^i \in (K_0^M)^+ = \{s e^i, s \in \mathbb{R}_+\}$. The calculation of the smallest superhedging price in the asset of interest, is the calculation of $\pi^a_i(X)$ leads to the a generalization of the well known Jouini, Kallal [21] representation to the $d$ asset case. For simplicity we assume that the solvency cone $K_t$ contains no lines. Then, the solvency cone $K_t(\omega)$ is spanned by the vectors $\pi^ij e^i - e^j$, $1 \leq i,j \leq d$, see section 2. The general case can be derived using the liquidation map in (5.1). The following theorem gives a dual representation of the scalar superhedging price in the spirit of Jouini, Kallal [21] in the $d$ asset case.

**Theorem 7.1.** Under the robust no arbitrage condition (NA'), the scalar superhedging price $\pi^a_i(X)$ in units of asset $i \in \{1,...,d\}$ is given by

$$\pi^a_i(X) = \sup_{(S_t,Q) \in Q^i} E^Q[X^T S_T],$$

(7.5)

where $Q^i$ is the set of all processes $(S_t)_{t=0}^T$ with $S^i_0 \equiv 1$, $S^k_t \leq \pi^{jk} S^i_t$ for all $1 \leq j,k \leq d$ and their equivalent martingale measures $Q$ for which $E^Q[(X^T S_T)^-] < \infty$.

**Proof.** From the scalarization of (7.4) with respect to $v \equiv e^i \in (K_0^M)^+$ one obtains

$$\pi^a_i(X) = \inf_{u \in SHP_0^M(X)} v^T u$$

$$= \min\{t \in \mathbb{R} : \sup_{\{Z \in \mathbb{Z} : E[(X^T Z_T)^-] < \infty\}} E[X^T Z_T] \leq t\}. $$

(7.6)

Note that for every $Z \in \mathbb{Z}$, one can define the corresponding frictionless price of the $d$ assets expressed in asset $i$ as

$$S_t = (Z^1_t / Z^i_t, Z^2_t / Z^i_t, ..., Z^d_t / Z^i_t), $$

(7.7)

i.e., $S^i_t \equiv 1$, and obtains an equivalent martingale measure $Q$ of the process $(S_t)_{t=0}^T$ via

$$\frac{dQ}{dP} = \frac{Z_t^i}{Z_0^i}. $$

Note that $Z_t^i > 0$, $t \in \{0,...,T\}$ is ensured by assumption 2.1. The set $Z$ of all consistent price systems $Z$ is one-to-one (up to a multiplicative factor for $Z$) to the set of processes $(S_t)_{t=0}^T$ with $S^i_0 \equiv 1$, $S^k_t \leq \pi^{jk} S^i_t$ for all $1 \leq j,k \leq d$ and their equivalent martingale measures $Q$. This follows from the observation that $S_t$ is defined by $Z \in \mathbb{Z}$ with $Z \in K^+_t \setminus \{0\}$ via (7.7) and $K_t$ is spanned by the vectors $\pi^{ij} e^i - e^j$, $1 \leq i,j \leq d$. Furthermore, $Z \in \mathbb{Z}$ is a martingale under $P$, which corresponds to $Q$ being a martingale measure for $(S_t)_{t=0}^T$, see [35] p.24/25 for details. Since for every $Z \in \mathbb{Z}$ with $E[(X^T Z_T)^-] < \infty$

$$E[X^T Z_T] = E[X^T Z_T / Z_0^i] = E^Q[X^T S_T], $$

we can rewrite (7.6) as in (7.5). \hfill \Box
The supremum in (7.5) is attained if we replace \( Q_d \) by the enlarged set \( \mathcal{Q}^l \), that contains martingale measures that are not necessarily equivalent to \( P \). We will write dynamic programming equations for problem (7.5) that will allow to implement and efficiently calculate the scalar superhedging price of a claim \( X \in L^0_d(\mathcal{F}_T, \mathbb{R}^d) \).

To do so, let us define the the sets of one-step transition densities as in [7]

\[
\mathcal{D}_t := \{ \xi \in L^1(\mathcal{F}_t, \mathbb{R}_+^T) : E_{t-1}[\xi] = 1 \}, \quad t = 1, ..., T.
\]

Let \( \xi_0 = 1 \). Every sequence \((S_t, \xi_t)\) \( t=0, ..., T \) with \( S_t^i \equiv 1, S_t^k \leq \pi^j S_t^j \) for all \( 1 \leq j, k \leq d \) for \( t = 0, ..., T \) and \( \xi_t \in \mathcal{D}_t, E_{t-1}[\xi_t S_t] = S_{t-1} \) for \( t = 1, ..., T \) defines an element \((S_t, Q)\) in \( \mathcal{Q}^l \) by setting

\[
\frac{dQ}{dP} = \xi_1 \cdots \xi_T.
\]

On the other hand, every element \((S_t, Q)\) \( t=0, ..., T \) induces a sequence with the above properties by setting

\[
N^t := \begin{cases} \frac{E_{t-1}[dQ]}{E_{t-1}[dP]} & \text{on } \{E_{t-1}[dQ] > 0\} \\ 1 & \text{on } \{E_{t-1}[dQ] = 0\}. \end{cases}
\]

Let us denote for \( t = 1, ..., T \)

\[
\mathcal{A}_t^k(S_{t-1}) = \left\{ (S_t, \xi_t) : S_t^i = 1, S_t^k \leq \pi^j S_t^j, 1 \leq j, k \leq d, \xi_t \in \mathcal{D}_t, E_{t-1}[\xi_t S_t] = S_{t-1} \right\}.
\]

From the considerations above, we obtain the following.

**Corollary 7.2.** Assume the robust no arbitrage condition (NA'). The scalar superhedging price \( \pi^a(X) \) given in (7.3) can be written as a sequence of nested optimization problems. Let the value function at time \( T - 1 \) be

\[
V_{T-1}(S_{T-1}) = \max_{(S_T, \xi_T) \in \mathcal{A}_T^k(S_{T-1})} E_{T-1}[\xi_T X^T S_T]. \tag{7.8}
\]

For \( t \in \{T - 2, ..., 0\} \) we define the value function

\[
V_t(S_t) = \max_{(S_{t+1}, \xi_{t+1}) \in \mathcal{A}_{t+1}^k(S_t)} E_t[\xi_{t+1} V_{t+1}(S_{t+1})]. \tag{7.9}
\]

Then

\[
\pi^a_i(X) = \max_{S_0 \in \mathbb{R}^d, S_0^i = 1, S_0^k \leq \pi^j S_0^j, 1 \leq j, k \leq d} V_0(S_0). \tag{7.10}
\]

**Proof.** This follows from the one-to-one correspondence between the set \( \mathcal{Q}^l \) and \((S_t, \xi_t)\) \( t=0, ..., T \) and the tower property. \( \square \)

In the following section an algorithm is described that is based on this recursive equations.
7.2 Algorithm for the scalar superhedging price

Let us assume a finite filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)\) as in section 3. We will split each of the iteration steps in (7.9) into two substeps. In the first steps, one incorporates only the constraint \(S_{t+1}^i = 1, S_{t+1}^k \leq \pi^j k S_{t+1}^j, 1 \leq j, k \leq d\), the second steps coincides with actually solving (7.9). For \(d = 2\), this algorithm coincides with algorithm 4.1 in [32], and algorithm 3.15 in [31].

**Corollary 7.3.** *(Algorithm scalar superhedging price under assumption (NA))*

1. \(V_T(S_T) = \bar{V}_T(S_T) = \left\{ \begin{array}{ll} X^T S_T : & S_T^i = 1, S_T^k \leq \pi^j k S_T^j, 1 \leq j, k \leq d \\
-\infty : & \text{else.} \end{array} \right. \) (7.11)

   This defines a function \(V_T^\omega : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}\) at each node \(\omega \in \Omega_T\).

2. For \(t \in \{T-1, ..., 0\}\) and nodes \(\omega \in \Omega_t\)

   \[
   V_t^\omega(S_t) = \text{cap} \{\bar{V}_{t+1}^\omega(S_{t+1}) : \bar{\omega} \in \text{succ}(\omega)\}. \quad (7.12)
   \]

3. The scalar superhedging price of \(X \in L^0_d(\mathcal{F}_T, \mathbb{R}^d)\) is given by

   \[
   \pi^a_i(X) = \max_{S_0 \in \mathbb{R}^d} \tilde{V}_0(S_0). \quad (7.13)
   \]

   The concave cap function \(\text{cap}\{f_1, ..., f_m\} : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}\) of a finite number of concave functions \(f_i : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}\), \(i = 1, ..., m\) is defined by its hypograph via

   \[
   \text{hypo}(\text{cap}\{f_1, ..., f_m\}) = \overline{\bigcup_{i=1}^m \text{hypo} f_i}, \quad (7.14)
   \]

   compare [31], p.135. The closure can be omitted if \(f_i, i = 1, ..., m\) are polyhedral, i.e., the hypographs of \(f_i\) are polyhedral convex sets.

**Proof.** (proof of corollary 7.3) It follows from the representation of the cap function as an optimization problem as in lemma A.4 in [31] that the two substeps (7.13) (respectively (7.11) for \(t = T\)) and (7.12) coincide with optimization problem (7.9). □

**Remark 7.4.** Obviously, the scalar superhedging price at time \(t\) and node \(\omega \in \Omega_t\) is given by

\[
(\pi^a_i)_t(X)(\omega) = \max_{S_t \in \mathbb{R}^d} \bar{V}_t^\omega(S_t) = \max_{S_t \in \mathbb{R}^d, S_t^i = 1, S_t^k \leq \pi^j k S_t^j, 1 \leq j, k \leq d} V_t^\omega(S_t).
\]

In section 7.3 below, we will discuss how the set of superhedging portfolios \(SHP_t(X)\) at time \(t\) and node \(\omega \in \Omega_t\) can be obtained from \(\bar{V}_t^\omega(S_t)\) via geometric duality.

Equation (7.14) already indicates a relationship between the cap function and set-operations.
7.3 Interpretation in terms of vector optimization and recovering the set of superhedging portfolios

We show in this section that the algorithm for the scalar superhedging price is closely related to the SHP-Algorithm which computes the set of superhedging portfolios. This means that $SHP_t(X)$ is also obtained by the scalar algorithm and, depending on how the scalar algorithm is realized, it practically coincides with a special case of the SHP-Algorithm. This relationship is established using the linear vector optimization reformulation of the recursive representation in theorem 3.1 and geometric duality, see sections 5 and 6.

**Lemma 7.5.** Let the assumptions of theorem 3.1 be satisfied and let $\tilde{V}^\omega_t(S_t)$ be as in corollary 7.3. For $t=0,\ldots,T$ and $\omega \in \Omega_t$,

$$SHP_t(X)(\omega) = \{ x \in \mathbb{R}^d \mid (S_t)^T x \geq \tilde{V}^\omega_t(S_t), (S_t, \tilde{V}^\omega_t(S_t)) \text{ vertices of hypo } \tilde{V}_t \},$$

(7.15)

**Proof.** We start with a reformulation of the scalar algorithm. As it can be seen from corollary 7.2 and 7.3 using the first three substeps, one obtains for $\omega \in \Omega_{T-1}$

$$\tilde{V}^\omega_{T-1}(S_{T-1}) = \begin{cases} 
\max_{(\xi^\omega, S^\omega)} \sum_{\omega \in \text{succ}(\omega)} \xi^\omega X(\omega)^T S^\omega & \text{if } S^i_{T-1} = 1, S^k_{T-1} \leq \pi^j_{T-1}(\omega) S^j_{T-1}, 1 \leq j, k \leq d \\
-\infty & \text{otherwise}
\end{cases}$$

(7.16)

where the maximum is taken subject to the constraints

$$(S^\omega)^i = 1, (S^\omega)^k \leq \pi^j_{T-1}(\omega) (S^\omega)^j, 1 \leq j, k \leq d, \quad \xi^\omega \geq 0, \quad \sum_{\omega \in \text{succ}(\omega)} \xi^\omega = 1, \quad \sum_{\omega \in \text{succ}(\omega)} \xi^\omega S^\omega = S_{T-1}.$$  

For $t = T-2,\ldots,0$ and $\omega \in \Omega_t$ one has

$$\tilde{V}^\omega_t(S_t) = \begin{cases} 
\max_{(\xi^\omega, S^\omega)} \sum_{\omega \in \text{succ}(\omega)} \xi^\omega \tilde{V}^\omega_{t+1}(S^\omega) & \text{if } S^i_t = 1, S^k_t \leq \pi^j_t(\omega) S^j_t, 1 \leq j, k \leq d \\
-\infty & \text{otherwise},
\end{cases}$$

(7.17)

where the constraints for the maximum are

$$\xi^\omega \geq 0, \quad \sum_{\omega \in \text{succ}(\omega)} \xi^\omega = 1, \quad \sum_{\omega \in \text{succ}(\omega)} \xi^\omega S^\omega = S_t.$$  

Finally, we have

$$\pi^\omega(X) = \max_{S_0 \in \mathbb{R}^d} \tilde{V}_0(S_0).$$

(7.18)

The constraints $S^i_t = 1, S^k_t \leq \pi^j_t(\omega) S^j_t, 1 \leq j, k \leq d$ can be equivalently expressed as $S^i_t = 1, S_t \in K^+_{T-1}$. The constraints in (7.16) concerning $(\xi^\omega, S^\omega)$ can be replaced by $\xi^\omega S^\omega = K^+_{T-1}(\omega) u^\omega$, $u^\omega \geq 0, \sum_{\omega \in \text{succ}(\omega)} K^+_{T-1}(\omega) u^\omega = S_{T-1}$ using the fact that we set $S^i_{T-1} = 1$. From (7.16) we obtain for $\omega \in \Omega_{T-1}$,

$$\tilde{V}^\omega_{T-1}(S_{T-1}) = \begin{cases} 
\max_{u^\omega} \sum_{\omega \in \text{succ}(\omega)} X(\omega)^T K^+_{T-1}(\omega) u^\omega & \text{if } S_{T-1} \in K^+_{T-1}(\omega), S^i_{T-1} = 1 \\
-\infty & \text{otherwise}
\end{cases}$$

(7.19)
Thus (7.17) can be reformulated as

\[ \sum_{\tilde{\omega} \in \text{succ}(\omega)} \tilde{K}^+_T(\tilde{\omega}) u_{\tilde{\omega}} = S_{T-1}. \]

Setting \( B^\omega = (\tilde{K}^+_T(\tilde{\omega}))^T \), \( b^\omega = (\tilde{K}^+_T(\tilde{\omega}))^T X(\tilde{\omega}) \), \( w = (S^1_{T-1}, ..., S^{i-1}_{T-1}, S^{i+1}_{T-1}, ..., S^d_{T-1}, S^i_{T-1}) \) and omitting the \( i \)-th variable \( S^i_{T-1} \) of \( \tilde{V}_{T-1} \) (which does not changes the problem), we see that hypo \( \tilde{V}_{T-1} \) is nothing else than the lower image \( D^* \) of a dual vector optimization problem \( (D^*)^\ast \) where the parameter vector \( c \), which has to be an interior point of the ordering cone, is chosen to be the \( d \)-th unit vector, i.e. \( c = e^d \). Note that the \( i \)-th component of \( S_{T-1} \), corresponds to the \( d \)-th component of \( w \). This means that \( c = e^d \) corresponds to the numéraire-component.

The corresponding primal problem \( (P^i) \) is to minimize

\[ \text{minimize } id : \mathbb{R}^d \to \mathbb{R}^d \text{ w.r.t. } \leq_{K_{T-1}(\omega)} \text{ s.t. } B^\omega x \geq b^\omega, \tilde{\omega} \in \text{succ}(\omega). \] (7.20)

We know by theorem 3.1 that \( \text{SHP}_{T-1}(X)(\omega) = \{ x \in \mathbb{R}^d \mid B^\omega x \geq b^\omega \} \) and the upper image of problem (7.20) is \( P = \text{SHP}_{T-1}(X)(\omega) \). Geometric duality (theorem 6.3) yields that an \( H \)-representation of \( P \) is given by the vertices of \( D^* \), that is,

\[ \text{SHP}_{T-1}(X)(\omega) = \{ x \in \mathbb{R}^d \mid (S_{T-1})^T x \geq \tilde{V}_{T-1}^\omega(S_{T-1}), \] (7.21)

\[ \text{s.t. } (S_{T-1}, \tilde{V}_{T-1}^\omega(S_{T-1})) \text{ vertices of hypo } \tilde{V}_{T-1} \} \]

which can be expressed by a matrix \( B^\omega \) and a vector \( b^\omega \), i.e.

\[ \text{SHP}_{T-1}(X)(\omega) = \{ x \in \mathbb{R}^d \mid B^\omega x \geq b^\omega \}. \] (7.22)

At time \( t \in \{ T - 2, \ldots, 0 \} \) we use elements \( (S_{t+1}, \tilde{V}_{t+1}(S_{t+1})) \) of the graph of \( \tilde{V}_{t+1} \), which can be expressed as a convex combination of vertices of hypo \( \tilde{V}_{t+1} \). The coefficients of the convex combinations can be interpreted as variables \( u \) of the (geometric) dual \( (D^i)^\ast \) of the linear vector optimization problem

\[ \text{minimize } id : \mathbb{R}^d \to \mathbb{R}^d \text{ w.r.t. } \leq_{K_t(\omega)} \text{ s.t. } B^\omega x \geq b^\omega, \tilde{\omega} \in \text{succ}(\omega). \]

Thus (7.17) can be reformulated as

\[ \tilde{V}_{t}^\omega(S_t) = \begin{cases} \max_{u^\omega} \sum_{\tilde{\omega} \in \text{succ}(\omega)} (b^\omega)^T u_{\tilde{\omega}} & \text{if } S_t \in K^+_t(\omega), \quad S^i_t = 1 \\ -\infty & \text{otherwise,} \end{cases} \] (7.23)

where the maximum is taken subject to the constraints

\[ u^\omega \geq 0, \quad \sum_{\tilde{\omega} \in \text{succ}(\omega)} (B^\omega)^T u_{\tilde{\omega}} = S_t. \]

We see that hypo \( \tilde{V}_{t}^\omega \) is again the lower image \( D^* \) of the dual vector optimization problem \( (D^i)^\ast \) for \( c = e^d \). Likewise to above, using theorem 3.1 and geometric duality (theorem 6.3), we obtain (7.15), which completes the proof.
**Remark 7.6.** Note that (7.16) and (7.17) are parametric linear optimization problems. For every choice of the parameter $S_t$, a linear program has to be solved. It is well known that parametric linear problems of the present type correspond to linear vector optimization problems, see e.g. [11].

The above considerations show that the scalar algorithm is closely related to the SHP-Algorithm if the last step (7.18) is omitted. The SHP-Algorithm computes $SHP_0(X)$ by a very similar construction if the parameter $c = e^d$ is chosen and the input data are transformed such that the last component refers to the numéraire asset. In contrast to the SHP-Algorithm, the liquidation map does not occur in lemma 7.5 and its proof. The reason is that geometric duality is not restricted to polyhedral sets that contain no lines, but Benson’s algorithm is. On the other hand it remains open in lemma 7.5 how the vertices of hypo $V_t$ are computed.

**Remark 7.7.** A relation between the function whose epigraph is the set $SHP_t(X)$ and the function $\tilde{V}_t$ could also be deduced via conjugation (Legendre-Fenchel transform) in the spirit of section 4.2 in [33]. A similar construction has been used to prove a geometric duality theorem for convex vector optimization problems [17].

**8 Conclusion**

In this paper, an algorithm in the numéraire-free approach of Kabanov [22], Schachermayer [35] is presented to calculate the set of all superhedging portfolios in a $d$ asset market with proportional transaction costs. Furthermore, an algorithm to calculate the superhedging strategy when starting from an initial portfolio vector in the set of superhedging portfolios is given. The $d$ asset case also allows to consider basket options. Examples are given.

It is shown that the superhedging problem in markets with transaction costs leads to a sequence of linear vector optimization problems. This generalizes the well known fact that in frictionless markets, superhedging leads to a sequence of linear optimization problems.

The above algorithms are the first of its kind. Previous papers, mostly in $d = 2$, focused on the representation and calculation of the scalar superhedging price, which is important to calculate price bounds, but does not give sufficient information if one wants to carry out a superhedging strategy starting from portfolio vectors. We extend those results to the $d$ dimensional case and show how scalar algorithm like [32, 31] or [3] and representation results like [21] are related to the set-valued results presented here. It turns out that the hypograph of a function appearing in in the penultimate step of the scalar algorithm in [32, 31] is the geometric dual (in the sense of vector optimization duality) of the set $SHP_0(X)$. This allows to recover the set $SHP_0(X)$, not from the scalar superhedging price $\pi^a_t(X)$ itself, but from the penultimate result in the scalar algorithm.

In [33] and the recent work [34] set-valued constructions to calculate the set $SHP_0(X)$ are as well presented, but without the connection to vector optimization and corresponding algorithms, which leave the problem open how to implement these set-valued constructions for $d > 2$. The approach taken here, answers this question.

**References**

[1] Barber, C. B., Dobkin, D. P., Huhdanpaa, H.: The quickhull algorithm for convex hulls. ACM Trans. Math. Softw. 22 (4), 469-483 (1996)
[2] Benson, H. P.: An outer approximation algorithm for generating all efficient extreme points in the outcome set of a multiple objective linear programming problem. Journal of Global Optimization 13, 1-24 (1998)

[3] Bensaid, B., Lesne, J.-P., Pagès, H., Scheinkman, J.: Derivative asset pricing with transaction costs. Math. Finance 2 (2), 63-86 (1992)

[4] Bot, R. I., Grad, S. M., Wanka, G.: Duality in vector optimization. Vector Optimization. Dordrecht: Springer (2009)

[5] Boyle, P., Vorst, T.: Option replication in discrete time with transaction costs. J. Finance 47, 271-293 (1992)

[6] Bremner, D., Fukuda, K., Marzetta, A.: Primal-dual methods for vertex and facet enumeration. Discrete Comput. Geom. 20 (3), 333-357 (1998)

[7] Cheridito, P., Kupper, M.: Composition of time-consistent dynamic monetary risk measures in discrete time. International Journal of Theoretical and Applied Finance 14 (1), 137-162 (2011)

[8] Ehrgott, M.: Multicriteria optimization. 2nd edition, Springer-Verlag, Berlin (2005)

[9] Ehrgott, M., Löhne, A., Shao, L.: A dual variant of Benson’s outer approximation algorithm. Journal of Global Optimization, to appear

[10] Föllmer, H., Schied, A.: Stochastic Finance. Walter de Gruyter (2004)

[11] Focke, J.: Vektormaximumprobleme und parametrische Optimierung (German) Math. Operationsforsch. Stat. 4, 365-369 (1973)

[12] Grünbaum, B.: Convex polytopes. prepared by Kaibel,V., Klee, V., Ziegler, G. M., 2nd ed., Graduate texts in mathematics 221. New York, Springer (2003)

[13] Hamel, A. H.: A Fenchel-Rockafellar duality theorem for set-valued optimization. Forthcoming in Optimization (2010), DOI:10.1080/02331934.2010.534794

[14] Hamel, A. H., Heyde, F.: Duality for set-valued measures of risk. SIAM J. on Financial Mathematics 1 (1), 66-95 (2010)

[15] Hamel, A. H., Heyde, F., Rudloff, B.: Set-valued risk measures for conical market models. Mathematics and Financial Economics 5 (1), 1-28 (2011)

[16] Hamel, A. H., Löhne, A., Rudloff, B.: Generalized geometric duality and Benson’s algorithm for applications in finance. Working paper

[17] Heyde, F.: Geometric duality for convex vector optimization problems. Working paper

[18] Heyde, F., Löhne, A.: Geometric duality in multiple objective linear programming. SIAM Journal of Optimization 19 (2), 836-845 (2008)

[19] Hull, J., White, A.: Valuing derivative securities using the explicit finite difference method. The Journal of Financial and Quantitative Analysis 25, 87-100 (1990)
[20] Jahn, J.: Vector optimization. Theory, applications, and extensions. Springer-Verlag, Berlin (2004)

[21] Jouini, E., Kallal, H.: Martingales and arbitrage in securities markets with transaction costs. J. Econom. Theory 66 (1), 178-197 (1995)

[22] Kabanov, Y. M.: Hedging and liquidation under transaction costs in currency markets. Finance and Stochastics 3, 237-248 (1999)

[23] Kabanov, Y. M., Safarian, M.: Markets with transaction costs. Springer (2009)

[24] Kabanov, Y. M., Stricker, Ch.: The Harrison-Pliska arbitrage pricing theorem under transaction costs. Journal of Mathematical Economics 35 (2), 185-196 (2001)

[25] Korn, R., Müller, S.: The decoupling approach to binomial pricing of multi-asset options. Journal of Computational Finance 12 (3), 1-30 (2009)

[26] Löhne, A.: Vector optimization with infimum and supremum. Springer-Verlag, Berlin (2011)

[27] Löhne, A., Rudloff, B.: On solvency cones and their duals in markets with transaction costs. Working paper

[28] Palmer, K.: A note on the Boyle-Vorst discrete-time option pricing model with transaction costs. Math. Finance 11, 357-363 (2001)

[29] Perrakis, S., Lefoll, J.: Derivative asset pricing with transaction costs: an extension. Computational Economics 10, 359-376 (1997)

[30] Rockafellar, R. T.: Convex analysis. Princeton University Press, Princeton (1997)

[31] Roux, A.: Options under transaction costs: Algorithms for pricing and hedging of european and american options under proportional transaction costs and different borrowing and lending rates. VDM Verlag (2008)

[32] Roux, A., Tokarz, K., Zastawniak, T.: Options under proportional transaction costs: An algorithmic approach to pricing and hedging. Acta Applicandae Mathematicae 103, 201-209 (2008)

[33] Roux, A., Zastawniak, T.: American options under proportional transaction costs: Pricing, hedging and stopping algorithms for long and short positions. Acta Applicandae Mathematicae 106, 199-228 (2009)

[34] Roux, A., Zastawniak, T.: American and Bermudan options in currency markets under proportional transaction costs. Working paper, available at [http://arxiv.org/abs/1108.1910](http://arxiv.org/abs/1108.1910)

[35] Schachermayer, W.: The fundamental theorem of asset pricing under proportional transaction costs in finite discrete time. Math. Finance 14 (1), 19-48 (2004)