GLOBAL WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR
THE JORDAN–MOORE–GIBSON–THOMPSON EQUATION WITH
ARBITRARILY LARGE HIGHER-ORDER SOBOLEV NORMS

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Abstract. In this paper, we consider the 3D Jordan–Moore–Gibson–Thompson equation arising in nonlinear acoustics. First, we prove that the solution exists globally in time provided that the lower order Sobolev norms of the initial data are considered to be small, while the higher-order norms can be arbitrarily large. This improves some available results in the literature. Second, we prove a new decay estimate for the linearized model and removing the $L^1$-assumption on the initial data. The proof of this decay estimate is based on the high-frequency and low-frequency decomposition of the solution together with an interpolation inequality related to Sobolev spaces with negative order.

1. Introduction

In this paper, we consider the nonlinear Jordan–Moore–Gibson–Thompson (JMGT) equation:

\begin{equation}
\tau u_{ttt} + u_{tt} - c^2 \Delta u - \beta \Delta u_t = \frac{\partial}{\partial t} \left( \frac{1}{c^2} \frac{B}{2A} (u_t)^2 + |\nabla u|^2 \right),
\end{equation}

where $x \in \mathbb{R}^3$ (Cauchy problem in 3D) and $t > 0$ and $u = u(x,t)$ denotes the acoustic velocity potential. We consider the initial conditions

\begin{equation}
u(t = 0) = u_0, \quad u_t(t = 0) = u_1, \quad u_{tt}(t = 0) = u_2.
\end{equation}

The JMGT equation with different types of damping mechanisms has received a substantial amount of attention in recent years and this because of its wide applications in medicine and industry [10, 13, 22, 23].

Equation (1.1a) is an alternative model to the classical Kuznetsov equation

\begin{equation}
u_{tt} - c^2 \Delta u - \beta \Delta u_t = \frac{\partial}{\partial t} \left( \frac{1}{c^2} \frac{B}{2A} (u_t)^2 + |\nabla u|^2 \right),
\end{equation}

where $u = u(x,t)$ represents the acoustic velocity potential for $x \in \mathbb{R}^3$ and $t > 0$; see [18]. The equation (1.2) can be obtained as an approximation of the governing equations of fluid mechanics by means of asymptotic expansions in powers of small parameters; see [4, 8, 17, 18]. The constants $c > 0$ and $\beta > 0$ are the speed and the diffusivity of sound, respectively. The parameter of nonlinearity $B/A$ arises in the Taylor expansion of the variations of pressure in a medium in terms of the variations of density; cf. [1]. The extra term $\tau u_{ttt}$ appearing in (1.1a) is due to the replacement of the Fourier law of heat conduction in the equation of the conservation of energy by the Cattaneo (or Maxwell–Cattaneo) law which accounts for finite speed of propagation of

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the heat transfer and eliminates the paradox of infinite speed of propagation for pure heat conduction associated with the Fourier law.

The starting point of the nonlinear analysis lies in the results for the linearization (1.3)
\[ \tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - \beta \Delta u_t = 0. \]
This equation is known as the Moore–Gibson–Thompson equation (although, as mentioned in [3], this model originally appears in the work of Stokes [30]). Interestingly, equation (1.3) also arises in viscoelasticity theory under the name of standard linear model of viscoelasticity; see [11] and references given therein.

Equation (1.3) has been extensively studied lately; see, for example [3–6, 14, 20, 21, 27, 28], and the references therein. In particular in [14] (see also [15]), the authors considered the linear equation in bounded domains
\[ \tau u_{ttt} + \alpha u_{tt} + c^2 A u + \beta A u_t = 0, \]
where \( A \) is a positive self-adjoint operator. They proved that when the diffusivity of the sound is strictly positive (\( \beta > 0 \)), the linear dynamics is described by a strongly continuous semigroup, which is exponentially stable provided the dissipativity condition \( \gamma := \alpha - \tau c^2 / \beta > 0 \) is fulfilled. The study of the controllability properties of the MGT type equations can be found for instance in [3, 20]. The MGT equation in \( \mathbb{R}^N \) with a power source nonlinearity of the form \( |u|^p \) has been considered in [5] where some blow up results have been shown for the critical case \( \tau c^2 = \alpha \beta \).

The MGT and JMGT equations with a memory term have been also investigated recently. For the MGT with memory, the reader is referred to [2, 9, 31] and to [19, 24, 25] for the JMGT with memory. The singular limit problem when \( \tau \to 0 \) has been rigorously justified in [16]. The authors in [16] showed that in bounded domain, the limit of (1.1a) as \( \tau \to 0 \) leads to the Kuznetsov equation (i.e., Eq (1.1a) with \( \tau = 0 \)). Concerning the large time asymptotic stability, the author and Pellicer showed in [27] the following decay estimate of the solution of the Cauchy problem associated to (1.3):
\[ (1.4) \quad \| V(t) \|_{L^2(\mathbb{R}^N)} \lesssim (1 + t)^{-N/4} (\| V_0 \|_{L^1(\mathbb{R}^N)} + \| V_0 \|_{L^2(\mathbb{R}^N)}). \]
with \( V = (u_t + \tau u_{tt}, \nabla (u + \tau u_t), \nabla u_t) \). The method used to prove (1.4) is based on a pointwise energy estimates in the Fourier space together with suitable asymptotic integral estimates. The decay rate in (1.4) under the \( L^1 \) assumption on the initial data seems sharp since it matches the decay rate of the heat kernel.

The global well posedness and large time behaviour of the solution to the Cauchy problem associated to the nonlinear 3D model (1.1) has been recently investigated in [29]. More precisely, under the assumption \( 0 < \tau c^2 < \beta \) and by using the contraction mapping theorem in appropriately chosen spaces, the authors showed a local existence result in some appropriate functional spaces. In addition using a bootstrap argument, a global existence result and decay estimates for the solution with small initial data were proved. The decay estimate obtained in [29] agrees with the one of the linearized models given in [27].

Our main goal in this paper is first to improve the global existence result in [29] by removing the smallness assumption on the higher-order Sobolev norms. More precisely, we only assume the lower-order Sobolev norms of initial data to be small, while the higher-order norms can be arbitrarily large. To achieve this, and inspired by [12] we use
different estimates than those in [29] in order to control the nonlinearity in more precise way. Second, as in (1.4), to prove the decay rate of the solution, it is common to take initial data to be in $L^1(\mathbb{R}^n)$ and combine this with energy estimates in $H^s(\mathbb{R}^n)$, $s \geq 0$. However, this may create some difficulties, especially for the nonlinear problems since in some situations it is important to propagate this assumption of the $L^1$-initial data over time, which is not always the case. Hence, it is very important to replace this $L^1$ space by $H^{-\gamma}$, $\gamma > 0$ which is an $L^2$-type space. In fact, we prove (see Theorem 3.2) instead of (1.3), the following decay estimate:

$$ \|V(t)\|_{L^2} \lesssim (1 + t)^{-\gamma}, \quad \gamma > 0, $$

provided that the initial data are in $H^{-\gamma}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. The proof of the decay estimate (1.5) is based on the high-frequency and low-frequency decomposition of the solution together with an interpolation inequality related to Sobolev spaces with negative order (see Lemma A.3 below). In fact, we prove that the low-frequency part of the solution behaves similarly to the solution of the heat equation

$$ \partial_t \psi - \Delta \psi = 0, \quad \text{in} \quad \mathbb{R}^3 $$

and hence, we recover the decay rate in [12] for equation (1.6) in the Sobolev space of a negative order. For the high-frequency part, we show that it follows the decay rate of the “toy” model

$$ \partial_t \psi + \psi = 0, \quad \text{in} \quad \mathbb{R}^3, $$

which is an exponential decay rate.

The rest of this paper is organized as follows: Section 2 contains the necessary theoretical preliminaries, which allow us to rewrite the equation with the corresponding initial data as a first-order Cauchy problem and define the main energy norm with the associated dissipative norm. We also recall a local well-posedness result from [29]. In Section 3 we state and discuss our main result. Section 4 is devoted to the proof of the global existence result. Section 5 is dedicated to the proof of the decay estimates of the linearized problem. In Appendix A, we present the Gagliardo–Nirenberg inequality together with some Sobolev interpolation inequalities that we used in the proofs.

1.1. Notation. Throughout the paper, the constant $C$ denotes a generic positive constant that does not depend on time, and can have different values on different occasions. We often write $f \lesssim g$ where there exists a constant $C > 0$, independent of parameters of interest such that $f \leq C g$ and we analogously define $f \gtrsim g$. We sometimes use the notation $f \lesssim_{\alpha} g$ if we want to emphasize that the implicit constant depends on some parameter $\alpha$. The notation $f \approx g$ is used when there exists a constant $C > 0$ such that $C^{-1} g \leq f \leq C g$.

2. Preliminaries

We rewrite the right-hand side of equation (1.1a) in the form

$$ \frac{\partial}{\partial t} \left( \frac{1}{c^2 A^2} (ut)^2 + |\nabla u|^2 \right) = \frac{B}{c^2 A} u_t u_{tt} + 2 \nabla u \cdot \nabla u_t, $$

and introduce the new variables

$$ v = u_t \quad \text{and} \quad w = u_{tt}, $$
and without loss of generality, we assume from now on that $c = 1$. Then equation (1.1a) can be rewritten as the following first-order system

$$(2.1a) \begin{cases} u_t = v, \\ v_t = w, \\ \tau w_t = \Delta u + \beta \Delta v - w + \frac{B}{A}v w + 2\nabla u \cdot \nabla v, \end{cases}$$

with the initial data (1.1b) rewritten as

$$(2.1b) u(t = 0) = u_0, \quad v(t = 0) = v_0, \quad w(t = 0) = w_0.$$ 

Let $U = (u, v, w)$ be the solution of (2.1). In order to state our main result, for $k \geq 0$, we introduce the energy $E_k[U](t)$ of order $k$ and the corresponding dissipation $D_k[U](t)$ as follows:

$$E_k[U](t) = \sup_{0 \leq \sigma \leq t} \left( \|\nabla^k (v + \tau w)(\sigma)\|_{L^2}^2 + \|\Delta \nabla^k v(\sigma)\|_{L^2}^2 + \|\nabla^{k+1} v(\sigma)\|_{L^2}^2 + \|\Delta \nabla^k (u + \tau v)(\sigma)\|_{L^2}^2 + \|\nabla^{k+1} (u + \tau v)(\sigma)\|_{L^2}^2 + \|\nabla^k w(\sigma)\|_{L^2}^2 \right),$$

and

$$(2.2) D_k[U](t) = \int_0^t D_k(\sigma) d\sigma$$

with

$$D_k[U](t) = \left( \|\nabla^{k+1} v(t)\|_{L^2}^2 + \|\Delta \nabla^k v(t)\|_{L^2}^2 + \|\nabla^k w(t)\|_{L^2}^2 + \|\Delta \nabla^k (u + \tau v)(t)\|_{L^2}^2 + \|\nabla^{k+1} (u + \tau v)(t)\|_{L^2}^2 + \|\nabla^k w(t)\|_{L^2}^2 \right).$$

Let $V = (v + \tau w, \nabla (u + \tau v), \nabla v)$. It is clear that for all $t \geq 0$,

$$E_k[U](t) \approx \|\nabla^k V(t)\|_{L^2}^2 + \|\nabla^{k+1} V(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2.$$ 

For a positive integer $s \geq 1$ that will be fixed later on, we define

$$(2.3) E_s^2[U](t) = \sum_{k=0}^{s} E_k^2[U](t) \quad \text{and} \quad D_s^2[U](t) = \sum_{k=0}^{s} D_k^2[U](t).$$

To introduce energies with negative indices, we first define the operator $\Lambda^\gamma$ for $\gamma \in \mathbb{R}$ by

$$\Lambda^\gamma f(x) = \int_{\mathbb{R}^3} |\xi|^\gamma \hat{f}(\gamma) 2^{2i\pi x \cdot \xi} d\xi,$$

where $\hat{f}$ is the Fourier transform of $f$. The homogenous Sobolev spaces $H^\gamma$ consist of all $f$ for which

$$\|f\|_{H^\gamma} = \|\Lambda^\gamma f\|_{L^2} = |||\gamma \hat{f}\|_{L^2}$$

is finite. We can then define the energy functional associated to the negative Sobolev spaces as

$$E_{-\gamma}^2[U](t) = \sup_{0 \leq \sigma \leq t} \left( \|\Lambda^{-\gamma} (v + \tau w)(\sigma)\|_{H^1}^2 + \|\Lambda^{-\gamma} \Delta v(\sigma)\|_{L^2}^2 + \|\Lambda^{-\gamma} \nabla v(\sigma)\|_{L^2}^2 + \|\Lambda^{-\gamma} \Delta (u + \tau v)(\sigma)\|_{L^2}^2 + \|\Lambda^{-\gamma} \nabla (u + \tau v)(\sigma)\|_{L^2}^2 + \|\Lambda^{-\gamma} w(\sigma)\|_{L^2}^2 \right).$$
The associated dissipative term is given by
\[
\mathcal{D}_{\gamma}^2[U](t) = \int_0^t \left( \| \Lambda^{-\gamma} \nabla v(\sigma) \|_{L^2}^2 + \| \Lambda^{-\gamma} \Delta v(\sigma) \|_{L^2}^2 + \| \Lambda^{-\gamma} w(\sigma) \|_{L^2}^2 \right) d\sigma.
\]
(2.4)

Furthermore, the following decay estimate of the linearized problem holds:
\[
\| u(t) \|_{H^m} \leq C_0 \| u_0 \|_{H^m} + \| u_0 \|_{H^m}^{\gamma} + \| u_0 \|_{H^m}^{\gamma+1},
\]
for some $C_0 > 0$.

In the following theorem, we recall a local well-posedness result obtained in [29].

**Theorem 2.1** (see Theorem 1.2 in [29]). Assume that $0 < \tau < \beta$ and let $s > \frac{5}{2}$. Let $U_0 = (u_0, v_0, w_0)^T$ be such that
\[
E_s^2[U](0) = \| v_0 + \tau u_0 \|_{H^{s+1}}^2 + \| \Delta v_0 \|_{H^s}^2 + \| \nabla v_0 \|_{H^s}^2
\]
(2.5)

and
\[
\| \Delta(u_0 + \tau v_0) \|_{H^s}^2 + \| \nabla(u_0 + \tau v_0) \|_{H^s}^2 \leq \tilde{\delta}_0
\]
for some $\tilde{\delta}_0 > 0$. Then, there exists a small time $T = T(E_s(0)) > 0$ such that problem (1.1) has a unique solution $u$ on $[0, T] \times \mathbb{R}^3$ satisfying
\[
E_s^2[U](T) + \mathcal{D}_s^2[U](T) \leq C_{\tilde{\delta}_0},
\]
where $E_s^2[U](T)$ and $\mathcal{D}_s^2[U](T)$ are given in (2.3), determining the regularity of $u$, and $C_{\tilde{\delta}_0}$ is a positive constant depending on $\tilde{\delta}_0$.

3. **Main results**

In this section, we state and discuss our main results. The global existence result is stated in Theorem 3.1, while the decay estimate for the linearized problem is given in Theorem 3.2.

**Theorem 3.1.** Assume that $0 < \tau < \beta$ and let $s \geq 3$ be an integer. Let $s_0 = \max\{2s/3 + 1, [s/2] + 2\} \leq m \leq s$. Assume that $u_0, v_0, w_0$ are such that $E_s[U](0) < \infty$. Then there exists a small positive constant $\delta$, such that if
\[
E_{s_0}^2[U](0) = \| v_0 + \tau u_0 \|_{H^{s_0+1}}^2 + \| \Delta v_0 \|_{H^{s_0}}^2 + \| \nabla v_0 \|_{H^{s_0}}^2
\]
(3.1)

and
\[
\| \Delta(u_0 + \tau v_0) \|_{H^{s_0}}^2 + \| \nabla(u_0 + \tau v_0) \|_{H^{s_0}}^2 + \| w_0 \|_{H^{s_0}}^2 \leq \delta,
\]
then problem (1.1) admits a unique global-in-time solution satisfying
\[
E_m^2[U](t) + \mathcal{D}_m^2[U](t) \leq E_m^2[U](0), \quad t \geq 0,
\]
where $s_0 \leq m \leq s$.

In the following theorem, we state a decay estimate of the solution of the linearized problem associated to (2.1).

**Theorem 3.2.** Let $U$ be the solution of the linearized problem associated to (2.1). Assume that $0 < \tau < \beta$. Let $\gamma > 0$ and let $U(0)$ be such that $E_\gamma^2[U](0) < \infty$. Then, it holds that
\[
E_\gamma^2[U](t) + \mathcal{D}_\gamma^2[U](t) \leq E_\gamma^2[U](0),
\]
(3.3)

In addition, the following decay estimate of the linearized problem hold:
\[
\| V(t) \|_{L^2} \lesssim C_0 (1 + t)^{-\gamma}.
\]
(3.4)

Here $C_0$ is a positive constant that depends on the initial data, but is independent of $t$. 
3.1. **Discussion of the main result.** Before moving onto the proof, we briefly discuss the statements made above in Theorems 3.1 and 3.2.

- Similarly to the result in [12], we only assume the lower-order Sobolev norms of initial data to be small, while the higher-order norms can be arbitrarily large. This improves the recent result of [29, Theorem 1.1] where all the norms up to order $s$ are assumed to be small. To do this, and inspired by [12], we employ different techniques to tackle nonlinear terms rather than the usual commutator estimates. More precisely, we use Sobolev interpolation of the Gagliardo–Nirenberg inequality between higher-order and lower-order spatial derivatives to tackle the nonlinear terms.

- The decay rate for the linearized equation obtained in [27] holds under the assumption that the initial data $V_0 \in L^1(\mathbb{R}^3)$. Theorem 3.1 does not require the initial data to be in $L^1(\mathbb{R}^3)$. Instead, we take the initial data to be in $H^{-\gamma}$, which is obvious due to (3.3), that this norm is preserved. This can be shown (under some restrictions on $\gamma$) to hold also for the nonlinear problem. However, it seems difficult to extend the decay estimate (3.4) to the nonlinear problem since the cut-off operators defined in (5.4) induce some commutators that are difficult to control by the lower frequency dissipative terms. The decay estimates of the nonlinear problem provided in [12] are mainly based on an estimate of the form

$$\frac{d}{dt} \| \nabla^\ell V(t) \|_{L^2} + \| \nabla^{\ell+1} V(t) \|_{L^2} \leq 0.$$ 

Such estimate seems difficult to obtain in our situation due to the nature of our equation (1.1).

- Theorem 3.2 holds for all $\gamma > 0$. The decay rate obtained in [12] is restricted to the case $\gamma \in [0, 3/2)$; this restriction is needed to control the nonlinear terms.

4. **Energy estimates**

The main goal of this section is to use the energy method to derive the main estimates of the solution, which will be used to prove Theorem 3.1. In fact, we prove by a continuity argument that the energy $E_m[U](t)$ is uniformly bounded for all time if $\delta$ is sufficiently small. The main idea in the proof is to bound the nonlinear terms by $E_{s_0}[U](t)D_m^2[U](t)$ and get the estimate (4.7). As a result, if we prove that $E_{s_0}[U](t) \leq \varepsilon$ provided that $\delta$ is sufficiently small, then we can absorb the last term in (4.7) by the left-hand side. To control the nonlinear terms, we do not use the commutator estimates as in [29], instead and inspired by [12], we use Sobolev interpolation of the Gagliardo–Nirenberg inequality between higher-order and lower-order spatial derivatives.

Let $s_0$ be as in Theorem 3.1. We now use a bootstrap argument to show that $E_{s_0}[U](t)$ is uniformly bounded. We recall that

$$E_{s_0}^2[U](t) = \sum_{k=0}^{s_0} E_k^2[U](t).$$

We derive our estimates under the a priori assumption

$$E_{s_0}^2[U](t) \leq \varepsilon^2$$ (4.1)
and show that
\[ E^2_{s_0} [U](t) \leq \frac{1}{2} \varepsilon^2. \]
Hence, we deduce that \( E^2_{s_0} [U](t) \leq \varepsilon^2 \) provided that the initial energy \( E^2_{s_0} [U](0) \) is small enough. First, we have the following estimate

**Proposition 4.1** (First-order energy estimate). Let \( E^2_{s_0} [U](t) \leq \varepsilon^2 \) for some \( \varepsilon > 0 \) and a fixed integer \( 5/2 < s_0 < s \). Then

\[ \mathcal{E}^2_0 [U](t) + \mathcal{D}^2_0 [U](t) \lesssim \mathcal{E}^2_0 [U](0) + \varepsilon \mathcal{D}^2_0 [U](t). \]  

**Proof.** According to [29, Estimate (2.39)], the following energy estimate holds:

\[ \mathcal{E}^2_0 [U](t) + \mathcal{D}^2_0 [U](t) \lesssim \mathcal{E}^2_0 [U](0) + \mathcal{E}_0 [U](t) \mathcal{D}^2_0 [U](t) + M_0 [U](t) \mathcal{D}^2_0 [U](t), \]  

where

\[ M_0 [U](t) = \sup_{0 \leq s \leq t} \left( \|v(s)\|_{L^\infty} + \|(v + \tau w)(s)\|_{L^\infty} + \|\nabla (v + \tau w)(s)\|_{L^\infty} + \|\nabla u(s)\|_{L^\infty} + \|\nabla v(s)\|_{L^\infty} \right). \]

Using the Sobolev embedding theorem (recall that \( s_0 > 5/2 \)) together with the assumption on \( E^2_{s_0} [U](t) \) yields

\[ M_0 [U](t) + \mathcal{E}_0 [U](t) \lesssim E_{s_0} [U](t) \lesssim \varepsilon. \]

Plugging this inequality into (4.3) further yields the desired bound. \( \square \)

To prove a higher-order version of this energy estimate, we apply the operator \( \nabla^k \), \( k \geq 1 \) to (2.1a). We obtain for \( U := \nabla^k u, V := \nabla^k v \) and \( W := \nabla^k w \)

\[
\begin{align*}
\partial_t U &= V, \\
\partial_t V &= W, \\
\tau \partial_t W &= \Delta U + \beta \Delta V - W + \nabla^k \left( \frac{B}{A} vw + 2 \nabla u \cdot \nabla v \right).
\end{align*}
\]

Let us also define the right-hand side functionals as

\[ R^{(k)} = \nabla^k \left( \frac{B}{A} vw + 2 \nabla u \cdot \nabla v \right). \]

The following estimate holds; cf. [29, Estimate (2.50)].

**Proposition 4.2** (Higher-order energy estimate, [29]). Under the assumptions of Proposition 4.1, for all \( 1 \leq k \leq s \), it holds

\[ \mathcal{E}^2_k [U](t) + \mathcal{D}^2_k [U](t) \lesssim \mathcal{E}^2_k [U](0) + \sum_{i=1}^5 \int_0^t 1^{(k)}_i (\sigma) d\sigma \]
with

\[
I_1^{(k)} = \left| \int_{\mathbb{R}^3} R^{(k)}(t) (V + \tau W) \, dx \right|, \quad I_2^{(k)} = \left| \int_{\mathbb{R}^3} \nabla R^{(k)} \nabla (V + \tau W) \, dx \right|, \\
I_3^{(k)} = \left| \int_{\mathbb{R}^3} R^{(k)} \Delta (U + \tau V) \, dx \right|, \quad I_4^{(k)} = \left| \int_{\mathbb{R}^3} \nabla R^{(k)} \nabla V \, dx \right|, \\
I_5^{(k)} = \left| \int_{\mathbb{R}^3} R^{(k)} W \, dx \right|.
\]

(4.6)

Thus our proof reduces to estimating the right-hand side terms \(I_1^{(k)}, \ldots, I_5^{(k)}\). This will be done through several lemmas (see Lemmas 4.1–4.5 below). Inspired by \[12\], we use a different method to handle the nonlinearities compared to \[29\]. In particular, we will make extensive use of the Gagliardo–Nirenberg inequality (A.2), which will allow us to interpolate between higher-order and lower-order Sobolev norms and “close” the nonlinear estimates.

We thus wish to show an estimate of the form

\[
E_2^2[U](t) + D_2^2[U](t) \leq E_2^2[U](0) + E_{s_0}[U](t) D_2^2[U](t),
\]

which improves the one stated in \[29\], where \(E_s(t)\) replaces \(E_{s_0}(t)\) in (4.7).

4.1. Estimates of the terms \(I_i^{(k)}\), \(1 \leq i \leq 5\). The goal of this section is to provide the appropriate estimates of the last term on the right-hand side of the estimate (4.5).

**Lemma 4.1** (Estimate of \(I_1^{(k)}\)). For any \(1 \leq k \leq s\), it holds that

\[
I_1^{(k)} \lesssim \varepsilon \left( \| \nabla^k v \|_{L^2}^2 + \| \nabla^{k+1} v \|_{L^2}^2 + \| \nabla^{k+2} u \|_{L^2}^2 + \| \nabla^k w \|_{L^2}^2 + \| \nabla^{k+1} (v + \tau w) \|_{L^2}^2 \right)
\]

\[
\lesssim \varepsilon \left( \mathcal{D}_{k-1}^2[U](t) + \mathcal{D}_{k}^2[U](t) \right).
\]

**Proof.** Recall the definition of \(R^{(k)}\) (4.3). We have

\[
I_1^{(k)} = \int_{\mathbb{R}^3} \left| \nabla^{k-1} \left( \frac{B}{A} v w + 2 \nabla u \cdot \nabla v \right) \nabla^{k+1} (v + \tau w) \right| \, dx
\]

\[
= \frac{B}{A} \int_{\mathbb{R}^3} \left| \sum_{0 \leq \ell \leq k-1} \mathcal{C}_{k-1}^\ell \nabla^{k-1-\ell} v \nabla^\ell u \nabla^{k+1} (v + \tau w) \right| \, dx
\]

\[
+ 2 \int_{\mathbb{R}^3} \left| \sum_{0 \leq \ell \leq k-1} \mathcal{C}_{k-1}^\ell \nabla^{k-1-\ell} v \nabla^\ell u \nabla v \nabla^{k+1} (v + \tau w) \right| \, dx =: I_{1;1}^{(k)} + I_{1;2}^{(k)}.
\]

The term \(I_{1;1}^{(k)}\) can be estimates as follows:

\[
I_{1;1}^{(k)} \lesssim \sum_{0 \leq \ell \leq k-1} \| \nabla^{k-1-\ell} v \|_{L^3} \| \nabla^\ell u \|_{L^6} \| \nabla^{k+1} (v + \tau w) \|_{L^2}.
\]

Employing the Gagliardo–Nirenberg inequality (A.1) yields

\[
\| \nabla^\ell u \|_{L^6} \lesssim \| \nabla^k w \|_{L^2}^{\frac{12}{k-2}} \| \nabla^\ell w \|_{L^2}^{\frac{12}{k-2}} \| \nabla^{k+1} (v + \tau w) \|_{L^2}, \quad 0 \leq \ell \leq k-1.
\]

(4.9)
Now, by applying the Sobolev–Gagliardo–Nirenberg inequality \((A.2)\), we obtain

\[
\|\nabla^{k-\ell} v\|_{L^3} \lesssim \|\nabla^{m_0} v\|_{L^2}^{\frac{1+\ell}{k}} \|\nabla^k v\|_{L^2}^{\frac{1-k}{k}}, \quad 0 \leq \ell \leq k - 1
\]

with

\[
\frac{1}{3} = \frac{k - 1 - \ell}{3} + \left(\frac{1}{2} - \frac{m_0}{3}\right) \frac{1+\ell}{k} + \left(\frac{1}{2} - \frac{k}{3}\right) \left(1 - \frac{1+\ell}{k}\right).
\]

This relation implies

\[
m_0 = \frac{k}{2(1+\ell)} \leq \frac{k}{2} \leq \frac{s-1}{2}.
\]

It is clear that for \(s_0 = [(s-1)/2] + 1\) we have

\[
\|\nabla^{m_0} v(t)\|_{L^2} \lesssim \mathcal{E}_{s_0}(t).
\]

Hence, by plugging estimates \((4.10)\) and \((4.11)\) into \((4.9)\), and making use of assumption \((4.1)\) on \(E\), we obtain

\[
I_{1;1}^{(k)} \lesssim \sum_{0 \leq \ell \leq k-1} \|\nabla^{m_0} v\|_{L^2}^{\frac{1+\ell}{k}} \|\nabla^k v\|_{L^2} \|\nabla^{k+1}(v + \tau w)\|_{L^2}
\]

\[
\lesssim \varepsilon \sum_{0 \leq \ell \leq k-1} \|\nabla^k v\|_{L^2} \|\nabla^{k+1}(v + \tau w)\|_{L^2}.
\]

Young’s inequality implies that

\[
I_{1;1}^{(k)} \lesssim \varepsilon \left(\|\nabla^k v\|_{L^2}^2 + \|\nabla^k w\|_{L^2}^2 + \|\nabla^{k+1}(v + \tau w)\|_{L^2}^2\right).
\]

Now, we estimate \(I_{1;2}^{(k)}\). We have

\[
I_{1;2}^{(k)} \lesssim \sum_{0 \leq \ell \leq k-1} \|\nabla^{k-\ell} u\|_{L^6} \|\nabla^{\ell+1} v\|_{L^6} \|\nabla^{k+1}(v + \tau w)\|_{L^2}.
\]

By employing again the Gagliardo–Nirenberg inequality, we infer

\[
\|\nabla^{\ell+1} v\|_{L^6} \lesssim \|v\|_{L^2}^{\frac{\ell+2}{\ell+4}} \|\nabla^{k+1} v\|_{L^2}^{\frac{\ell+2}{\ell+4}}.
\]

We then also have by using the Sobolev–Gagliardo–Nirenberg inequality \((A.2)\),

\[
\|\nabla^{k-\ell} u\|_{L^3} \lesssim \|\nabla^{m_1+1} u\|_{L^2}^{\frac{\ell+2}{1+k}} \|\nabla^{k+2} u\|_{L^2}^{\frac{\ell+2}{1+k}}, \quad 0 \leq \ell \leq k - 1
\]

with

\[
\frac{1}{3} = \frac{k - \ell}{3} + \left(\frac{1}{2} - \frac{k + 2}{3}\right) \left(1 - \frac{\ell + 2}{1+k}\right) + \left(\frac{1}{2} - \frac{m_1 + 1}{3}\right) \frac{\ell + 2}{1+k}.
\]

This yields

\[
m_1 = \frac{k + 1}{2(2 + \ell)} \leq \frac{k + 1}{4} \leq \frac{s}{4}.
\]

Thus for \(s_0 = [\frac{s}{4}] + 1\), it holds that

\[
\|\nabla^{m_1+1} u(t)\|_{L^2} \lesssim \mathcal{E}_{s_0}[U](t).
\]
Consequently, by making use of the assumption (4.1) and the fact that $\|v\|_{L^2} \lesssim \mathcal{E}_0[U](t)$, we have

$$I_{1;2}^{(k)} \lesssim \sum_{0 \leq \ell \leq k-1} \|\nabla^{k-\ell}u\|_{L^2} \|\nabla^{\ell+1}v\|_{L^2} \|\nabla^{\ell+1}(v + \tau w)\|_{L^2} \|\nabla^{\ell+1}(v + \tau w)\|_{L^2}$$

$$\lesssim \sum_{0 \leq \ell \leq k-1} \|v\|^\ell \|\nabla^{k+1}v\|_{L^2} \|\nabla^{m_1+1}u\|_{L^2} \|\nabla^{k+1}(v + \tau w)\|_{L^2}$$

$$\lesssim \varepsilon \sum_{0 \leq \ell \leq k-1} \|\nabla^{k+1}v\|_{L^2} \|\nabla^{k+2}u\|_{L^2} \|\nabla^{k+1}(v + \tau w)\|_{L^2}.$$

Applying Young’s inequality yields

(4.13) $$I_{1;2}^{(k)} \lesssim \varepsilon \left(\|\nabla^{k+1}v\|_{L^2}^2 + \|\nabla^{k+2}u\|_{L^2}^2 + \|\nabla^{k+1}(v + \tau w)\|_{L^2}^2\right).$$

Hence, (4.8) holds on account of (4.12) and (4.13). □

Next we wish to estimate $I_2^{(k)}$, defined in (4.6).

**Lemma 4.2 (Estimate of $I_2^{(k)}$).** For any $1 \leq k \leq s$, it holds that

(4.14) $$I_2^{(k)} \lesssim \varepsilon \left(\|\nabla^{k+2}u\|_{L^2}^2 + \|\nabla^{k+1}w\|_{L^2}^2 + \|\nabla^{k+1}(v + \tau w)\|_{L^2}^2 + \|\nabla^{k+2}v\|_{L^2}^2\right)$$

$$\lesssim \varepsilon \mathcal{E}_k^2[U](t).$$

**Proof.** Recall that

$$\nabla R^{(k)} = \nabla^{k+1} \left(\frac{B}{A} vw + 2\nabla u \nabla v\right).$$

Thus, we have

$$I_2^{(k)} = \int_{\mathbb{R}^3} \left|\nabla^{k+1} \left(\frac{B}{A} vw + 2\nabla u \cdot \nabla v\right) \nabla^{k+1}(v + \tau w)\right| \, dx$$

$$= I_{2;1}^{(k)} + I_{2;2}^{(k)}.$$
We estimate $I_{2;1}^{(k)}$ as follows:

$$
I_{2;1}^{(k)} \lesssim \int_{\mathbb{R}^3} \left| \nabla^{k+1}(vw) \nabla^{k+1}(v + \tau w) \right| \, dx
$$

$$
= \int_{\mathbb{R}^3} \left| \sum_{0 \leq \ell \leq k+1} \nabla^{k+1-\ell} v \nabla^\ell w \nabla^{k+1}(v + \tau w) \right| \, dx
$$

$$
\lesssim \int_{\mathbb{R}^3} \left| \sum_{0 \leq \ell \leq k+1} \nabla^{k+1-\ell} v \nabla^\ell w \nabla^{k+1}(v + \tau w) \right| \, dx
$$

$$
= C \int_{\mathbb{R}^3} \left| \sum_{0 \leq \ell \leq k} \nabla^{k+1-\ell} v \nabla^\ell w \nabla^{k+1}(v + \tau w) \right| \, dx
$$

$$
+ C \int_{\mathbb{R}^3} \left| v \nabla^{k+1} w \nabla^{k+1}(v + \tau w) \right| \, dx
$$

$$
\lesssim \sum_{0 \leq \ell \leq k+1} \| \nabla^{k+1-\ell} v \|_{L^3} \| \nabla^\ell w\|_{L^6} \| \nabla^{k+1}(v + \tau w)\|_{L^2}.
$$

Using Hölder’s inequality, the term on the right-hand side of (4.15) corresponding to $\ell = k+1$ can be estimated in the following manner:

$$
\int_{\mathbb{R}^3} \left| \nabla^{k+1-\ell} v \nabla^\ell w \nabla^{k+1}(v + \tau w) \right| \, dx \lesssim \| v \|_{L^\infty} \| \nabla^{k+1} w \|_{L^2} \| \nabla^{k+1}(v + \tau w)\|_{L^2}
$$

$$
\lesssim \varepsilon \left( \| \nabla^{k+1} w \|_{L^2}^2 + \| \nabla^{k+1}(v + \tau w)\|_{L^2}^2 \right),
$$

where we have used the Sobolev embedding theorem.

To estimate the second term, observe that the term $\| \nabla^\ell w\|_{L^6}$ can be handled as in (4.10). In other words,

$$
\| \nabla^\ell w\|_{L^6} \lesssim \| \nabla^{k+1} w\|_{L^2}^{\frac{1+\ell}{1+k}} \| w\|_{L^2}^{\frac{1+\ell}{1+k}}, \quad 0 \leq \ell \leq k.
$$

To estimate the term $\| \nabla^{k+1-\ell} v\|_{L^3}$, we apply the Sobolev–Gagliardo–Nirenberg inequality (A.2),

$$
\| \nabla^{k+1-\ell} v\|_{L^3} \lesssim \| \nabla^{m_2+1} v\|_{L^2}^{\frac{1+\ell}{1+k}} \| \nabla^{k+2} v\|_{L^2}^{\frac{1+\ell}{1+k}}, \quad 0 \leq \ell \leq k
$$

with

$$
\frac{1}{3} = \frac{k+1-\ell}{3} + \left( \frac{2}{3} - \frac{k+2}{3} \right) \left( 1 - \frac{\ell+1}{k+1} \right) + \left( \frac{1}{2} - \frac{m_2+1}{3} \right) \frac{\ell+1}{k+1}.
$$

The above equation gives

$$
m_2 = \frac{1+k}{2(1+\ell)} \leq \frac{1+s}{2}.
$$

As before, for $s_0 \geq [(1+s)/2] + 1$, we have

$$
\| \nabla^{m_2+1} v(t)\|_{L^2} \lesssim \varepsilon_{s_0}[U](t).
$$
Hence, by collecting (4.17) and (4.18), we obtain (4.19)
\[\sum_{0 \leq \ell \leq k} \|\nabla^{k+1-\ell} v\|_{L^2} \|\nabla^{\ell} w\|_{L^6} \|\nabla^{k+1}(v + \tau w)\|_{L^2}\]
\[\lesssim \sum_{0 \leq \ell \leq k} \|\nabla^{k+1} w\|_{L^2} \|w\|_{L^6} \|\nabla^{m+1} v\|_{L^2} \|\nabla^{k+2} v\|_{L^2} \|\nabla^{k+1}(v + \tau w)\|_{L^2}\]
\[\lesssim \varepsilon \left(\|\nabla^{k+1} w\|_{L^2}^2 + \|\nabla^{k+1}(v + \tau w)\|_{L^2}^2\right).
\]

Hence, collecting (4.16) and (4.19), we obtain (4.20)
\[I^{(k)}_{2:1} \lesssim \varepsilon \left(\|\nabla^{k+1} w\|_{L^2}^2 + \|\nabla^{k+1}(v + \tau w)\|_{L^2}^2\right).
\]

Next we estimate \(I^{(k)}_{2:2}\). We have
\[I^{(k)}_{2:2} = 2 \int_{\mathbb{R}^3} \left|\nabla^{k+1}(\nabla u \nabla v)\nabla^{k+1}(v + \tau w)\right| dx
\]
\[= C \int_{\mathbb{R}^3} \sum_{0 \leq \ell \leq k+1} \nabla^{k+2-\ell} u \nabla^{\ell+1} v \nabla^{k+1}(v + \tau w) dx.
\]

We split the above sum into three cases: \(\ell = 0, \ell = k+1, \) and \(1 \leq \ell \leq k\). Thus
\[I^{(k)}_{2:2} \lesssim \int_{\mathbb{R}^3} \left|\nabla^{k+2} u \nabla^{k+1}(v + \tau w)\right| dx
\]
\[+ \int_{\mathbb{R}^3} \left|\nabla u \nabla^{k+2} v \nabla^{k+1}(v + \tau w)\right| dx
\]
\[+ \int_{\mathbb{R}^3} \left|\sum_{1 \leq \ell \leq k} \nabla^{k+2-\ell} u \nabla^{\ell+1} v \nabla^{k+1}(v + \tau w)\right| dx.
\]

As before, the first term in (4.21) is estimated by using Hölder’s inequality and the Sobolev embedding theorem,
\[\int_{\mathbb{R}^3} \left|\nabla^{k+2} u \nabla^{k+1}(v + \tau w)\right| dx \lesssim \|\nabla v\|_{L^6} \|\nabla^{k+2} u\|_{L^2} \|\nabla^{k+1}(v + \tau w)\|_{L^2}\]
\[\lesssim \varepsilon \left(\|\nabla^{k+2} u\|_{L^2}^2 + \|\nabla^{k+1}(v + \tau w)\|_{L^2}^2\right).
\]

Similarly, we estimate the second term on the right-hand side of (4.21) as
\[\int_{\mathbb{R}^3} \left|\nabla u \nabla^{k+2} v \nabla^{k+1}(v + \tau w)\right| dx \lesssim \|\nabla u\|_{L^6} \|\nabla^{k+2} v\|_{L^2} \|\nabla^{k+1}(v + \tau w)\|_{L^2}\]
\[\lesssim \varepsilon \left(\|\nabla^{k+2} v\|_{L^2}^2 + \|\nabla^{k+1}(v + \tau w)\|_{L^2}^2\right).
\]

For the third term on the right-hand side of (4.21), we write
\[\int_{\mathbb{R}^3} \left|\sum_{1 \leq \ell \leq k} \nabla^{k+2-\ell} u \nabla^{\ell+1} v \nabla^{k+1}(v + \tau w)\right| dx \lesssim \sum_{1 \leq \ell \leq k} \|\nabla^{k+2-\ell} u\|_{L^3} \|\nabla^{\ell+1} v\|_{L^6} \|\nabla^{k+1}(v + \tau w)\|_{L^2}.
\]
By applying the Gagliardo–Nirenberg inequality (A.1), we obtain
\[ \|\nabla^{\ell+1}v\|_{L^6} \lesssim \|v\|_{L^{\infty}}^{1-\frac{2\ell+1}{2k+1}} \|\nabla^{k+2}v\|_{L^2}^{\frac{2\ell+1}{2k+1}}, \quad 1 \leq \ell \leq k. \]
We also have, by suing (A.2),
\[ \|\nabla^{k+2-\ell}u\|_{L^3} \lesssim \|\nabla^{m_3+1}u\|_{L^2}^2 \|\nabla^{k+2}u\|_{L^2}^{1-\frac{2\ell+1}{2k+1}}, \quad 1 \leq \ell \leq k \]
with
\[ \frac{1}{3} = \frac{k+2-\ell}{3} + \left( \frac{1}{2} - \frac{k+2}{3} \right) \left( 1 - \frac{2\ell+1}{2k+1} \right) + \left( \frac{1}{2} - \frac{m_3+1}{3} \right) \frac{2\ell+1}{2k+1}. \]
This yields
\[ m_3 = \frac{1}{2} + \frac{1+2k}{1+2\ell} \leq \frac{2k}{3} + \frac{5}{6} \leq \frac{2s}{3} + \frac{5}{6} \quad \text{since} \quad \ell \geq 1. \]
Hence, for \( s_0 \geq [2s/3] + 1 \), we have \( \|\nabla^{m_3+1}u(t)\|_{L^2} \lesssim E_{s_0}(t) \). Also, using the Sobolev embedding theorem together with (4.1), we obtain \( \|v\|_{L^{\infty}} \lesssim E_{s_0}[U](t) \lesssim \varepsilon \). Consequently, we obtain
\[ (4.24) \quad \int_{\mathbb{R}^3} \left| \sum_{1 \leq \ell \leq k} \nabla^{k+2-\ell}u \nabla^{\ell+1}v \nabla^{k+1}(v+\tau w) \right| \, dx. \]
\[ \lesssim \sum_{1 \leq \ell \leq k} \|\nabla^{m_3+1}u\|_{L^2}^2 \|v\|_{L^{\infty}}^{1-\frac{2\ell+1}{2k+1}} \|\nabla^{k+2}u\|_{L^2}^{\frac{2\ell+1}{2k+1}} \|\nabla^{k+2}v\|_{L^2}^{\frac{2\ell+1}{2k+1}} \|\nabla^{k+1}(v+\tau w)\|_{L^2} \]
\[ \lesssim \varepsilon \left( \|\nabla^{k+2}u\|_{L^2}^2 + \|\nabla^{k+2}v\|_{L^2}^2 + \|\nabla^{k+1}(v+\tau w)\|_{L^2}^2 \right). \]
Therefore, from (4.22), (4.23) and (4.24), we deduce that
\[ (4.25) \quad I_{2;2}^{(k)} \lesssim \varepsilon \left( \|\nabla^{k+2}u\|_{L^2}^2 + \|\nabla^{k+2}v\|_{L^2}^2 + \|\nabla^{k+1}(v+\tau w)\|_{L^2}^2 \right). \]
Hence, (4.11) holds by collecting (4.20) and (4.25). This finishes the proof of Lemma 4.2. \( \square \)

The estimate of \( I_{4}^{(k)} \) can be done as the one of \( I_{2}^{(k)} \), we thus omit the details and just state the result.

**Lemma 4.3** (Estimate of \( I_{4}^{(k)} \)). For any \( 1 \leq k \leq s \), it holds that
\[ (4.26) \quad I_{4}^{(k)} \lesssim \varepsilon \left( \|\nabla^{k+2}u\|_{L^2}^2 + \|\nabla^{k+1}v\|_{L^2}^2 + \|\nabla^{k+1}v\|_{L^2}^2 + \|\nabla^{k+2}v\|_{L^2}^2 \right) \]
\[ \lesssim \varepsilon \mathcal{G}_k^2[U](t). \]

Our goal now is to estimate \( I_{3}^{(k)} \).

**Lemma 4.4** (Estimate of \( I_{3}^{(k)} \)). For any \( 1 \leq k \leq s \), it holds that
\[ (4.27) \quad I_{3}^{(k)} \lesssim \varepsilon \left( \|\nabla^{k+1}v\|_{L^2}^2 + \|\nabla^{k+2}u\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2 \right. \]
\[ + \|\nabla^{k+1}w\|_{L^2}^2 + \|\Delta \nabla^{k}(u+\tau v)\|_{L^2}^2 \]
\[ \lesssim \varepsilon \left( \mathcal{G}_{k-1}^2[U](t) + \mathcal{G}_k^2[U](t) \right). \]
Proof. We have

\[ I_3^{(k)} = \int_{\mathbb{R}^3} |R^{(k)} \Delta (U + \tau V)| \, dx \]

\[ \lesssim \int_{\mathbb{R}^3} \left| \sum_{0 \leq \ell \leq k} \nabla^{k-\ell} v \nabla^\ell w \Delta \nabla^k (u + \tau v) \right| \, dx \]

\[ + \int_{\mathbb{R}^3} \left| \sum_{0 \leq \ell \leq k} \nabla^{k-\ell} \nabla u \nabla^\ell \nabla v \Delta \nabla^k (u + \tau v) \right| \, dx \]

\[ = I_{3:1}^{(k)} + I_{3:2}^{(k)}. \]

First, we estimate \( I_{3:1}^{(k)}. \) We have

\[ I_{3:1}^{(k)} \lesssim \sum_{0 \leq \ell \leq k} \| \nabla^{k-\ell} v \|_{L^3} \| \nabla^\ell w \|_{L^6} \| \Delta \nabla^k (u + \tau v) \|_{L^2}. \]  

Using (A.1), we write

\[ \| \nabla^\ell w \|_{L^6} \lesssim \| \nabla^{k+1} v \|_{L^2}^{1 + \frac{\ell}{k+1}} \| w \|_{L^2}^{1 + \frac{\ell}{k+1}}, \quad 0 \leq \ell \leq k. \]  

Applying (A.2), we obtain

\[ \| \nabla^{k-\ell} v \|_{L^3} \lesssim \| \nabla^{m_4} v \|_{L^2}^{\frac{k+1}{3}} \| \nabla^{k+1} v \|_{L^2}^{1 + \frac{\ell}{k+1}}, \quad 0 \leq \ell \leq k \]

with

\[ \frac{1}{3} = \frac{k - \ell}{3} + \left( \frac{1}{2} - \frac{m_4}{3} \right) \frac{\ell + 1}{k + 1} + \left( \frac{1}{2} - \frac{k + 1}{3} \right) \left( 1 - \frac{\ell + 1}{k + 1} \right). \]

This results in

\[ m_4 = \frac{k + 1}{2(1 + \ell)} \leq \frac{k + 1}{2} \leq \frac{s + 1}{2}, \]

Therefore, for \( s_0 \geq [s/2] + 1, \) we have \( \| \nabla^{m_4} v(t) \|_{L^2} \lesssim \mathcal{E}_{s_0}[U](t). \) Hence, inserting (4.29) and (4.30) into (4.28), we obtain, by making use of (4.1),

\[ I_{3:1}^{(k)} \lesssim \sum_{0 \leq \ell \leq k} \| \nabla^{m_4} v \|_{L^2}^{\frac{k+1}{3}} \| \nabla^{k+1} v \|_{L^2}^{1 + \frac{\ell}{k+1}} \| \nabla^{k+1} w \|_{L^2}^{\frac{1 + \ell}{k+1}} \| w \|_{L^2}^{1 + \frac{\ell}{k+1}} \| \Delta \nabla^k (u + \tau v) \|_{L^2} \]

\[ \lesssim \varepsilon \left( \| \nabla^{k+1} v \|_{L^2}^2 + \| \nabla^{k+1} w \|_{L^2}^2 + \| \Delta \nabla^k (u + \tau v) \|_{L^2}^2 \right). \]
Next, we estimate \( I_{3;2}^{(k)} \). Recall that

\[
I_{3;2}^{(k)} = C \int_{\mathbb{R}^3} \left| \sum_{0 \leq \ell \leq k} \nabla^{k-\ell} \nabla u \nabla^\ell \nabla \Delta \nabla^{k} (u + \tau v) \right| \ dx
\]  
(4.32)

\[
\lesssim \int_{\mathbb{R}^3} \left| \nabla^k \nabla u \nabla \Delta \nabla^{k} (u + \tau v) \right| \ dx
\]
\[
+ \int_{\mathbb{R}^3} \left| \nabla u \nabla^k \nabla \Delta \nabla^{k} (u + \tau v) \right| \ dx
\]
\[
+ \int_{\mathbb{R}^3} \left| \sum_{1 \leq \ell \leq k-1} \nabla^{k-\ell} \nabla u \nabla^\ell \nabla \Delta \nabla^{k} (u + \tau v) \right| \ dx.
\]

We estimate the first term on the right-hand side of (4.35) as

\[
\int_{\mathbb{R}^3} \left| \nabla^k \nabla u \nabla \Delta \nabla^{k} (u + \tau v) \right| \ dx \lesssim \| \nabla v \|_{L^\infty} \| \nabla^{k+1} u \|_{L^2} \| \Delta \nabla^{k} (u + \tau v) \|_{L^2}
\]
\[
\lesssim \varepsilon \left( \| \nabla^{k+1} u \|_{L^2}^2 + \| \Delta \nabla^{k} (u + \tau v) \|_{L^2}^2 \right).
\]  
(4.33)

The second term on the right-hand side of (4.35) is estimated as

\[
\int_{\mathbb{R}^3} \left| \nabla u \nabla^k \nabla \Delta \nabla^{k} (u + \tau v) \right| \ dx \lesssim \| \nabla u \|_{L^\infty} \| \nabla^{k+1} v \|_{L^2} \| \Delta \nabla^{k} (u + \tau v) \|_{L^2}
\]
\[
\lesssim \varepsilon \left( \| \nabla^{k+1} v \|_{L^2}^2 + \| \Delta \nabla^{k} (u + \tau v) \|_{L^2}^2 \right).
\]  
(4.34)

For the last term on the right-hand side of (4.35), we have

\[
\int_{\mathbb{R}^3} \left| \sum_{1 \leq \ell \leq k-1} \nabla^{k-\ell} \nabla u \nabla^\ell \nabla \Delta \nabla^{k} (u + \tau v) \right| \ dx
\]
\[
+ \sum_{1 \leq \ell \leq k-1} \| \nabla^{k+1-\ell} u \|_{L^3} \| \nabla^{\ell+1} v \|_{L^6} \| \Delta \nabla^{k} (u + \tau v) \|_{L^2}.
\]  
(4.35)

We have by exploiting (A.1),

\[
\| \nabla^{\ell+1} v \|_{L^6} \lesssim \| v \|_{L^2}^{1-\frac{4+\ell}{3}} \| \nabla^{k+1} v \|_{L^2}^{\frac{2+\ell}{3}}, \quad 1 \leq \ell \leq k-1.
\]

As before, we apply (A.2) and estimate \( \| \nabla^{k+1-\ell} u \|_{L^3} \) as follows:

\[
\| \nabla^{k+1-\ell} u \|_{L^3} \lesssim \| \nabla^{m_5+1} u \|_{L^2}^{\frac{2+\ell}{1+\ell}} \| \nabla^{k+2} u \|_{L^2}^{1-\frac{2+\ell}{1+\ell}}, \quad 1 \leq \ell \leq k-1,
\]

where

\[
\frac{1}{3} = \frac{k+1-\ell}{3} + \left( \frac{1}{2} - \frac{k+2}{3} \right) \left( 1 - \frac{2+\ell}{1+k} \right) + \left( \frac{1}{2} - \frac{m_5+1}{3} \right) \frac{2+\ell}{1+k},
\]

which implies

\[
m_5 = \frac{3(1+k)}{2(2+\ell)} \leq \frac{k+1}{2} \leq \frac{s+1}{2}, \quad \text{since} \quad \ell \geq 1.
\]
Hence, as before, this implies that for \( s_0 \geq [s/2] + 1 \), we have \( \| \nabla^{m+1} u(t) \|_{L^2} \lesssim \mathcal{E}_{s_0}[U](t) \). Consequently, we obtain from above
\[
(4.36) \quad \sum_{1 \leq \ell \leq k - 1} \| \nabla^{k+1-\ell} u \|_{L^2} \| \nabla^{\ell+1} v \|_{L^2} \| \Delta \nabla^k (u + \tau v) \|_{L^2} \lesssim \sum_{1 \leq \ell \leq k - 1} \| \nabla^{m_5+1} u \|_{L^2} \| \nabla^{k+2} u \|_{L^2} \| \nabla^{\ell+1} v \|_{L^2} \| \nabla^{k+1} v \|_{L^2} \| \Delta \nabla^k (u + \tau v) \|_{L^2} \lesssim \varepsilon \left( \| \nabla^{k+1} v \|_{L^2}^2 + \| \nabla^{k+2} u \|_{L^2}^2 + \| \Delta \nabla^k (u + \tau v) \|_{L^2}^2 \right).
\]
Therefore, from \( (4.33), (4.34) \) and \( (4.36) \), we deduce that
\[
(4.37) \quad I_{3,2}^{(k)} \lesssim \varepsilon \left( \| \nabla^{k+1} v \|_{L^2}^2 + \| \nabla^{k+2} u \|_{L^2}^2 + \| \Delta \nabla^k (u + \tau v) \|_{L^2}^2 \right).
\]
Putting together \( (4.31) \) and \( (4.37) \) yields \( (4.27) \). \( \square \)

Next we derive a bound for \( I_{k}^{(5)} \).

**Lemma 4.5 (Estimate of \( I_{k}^{(5)} \)).** For any \( 1 \leq k \leq s \), it holds that
\[
I_{k}^{(5)} \lesssim \varepsilon \left( \| \nabla^{k+1} v \|_{L^2}^2 + \| \nabla^{k+2} u \|_{L^2}^2 + \| \nabla^{k+1} u \|_{L^2}^2 \right)
\]
\[
(4.38) \quad + \| \nabla^{k+1} w \|_{L^2}^2 + \| \Delta \nabla^k (u + \tau v) \|_{L^2}^2 \) \lesssim \varepsilon \left( \mathcal{E}_{k}^2[U](t) + \mathcal{E}_{k}^2[U](t) \right).
\]

**Proof.** The proof of Lemma 4.5 can be done as the one of Lemma 4.4, where \( \| \Delta \nabla^k (u + \tau v) \|_{L^2} \) is replaced by \( \| \nabla^k w \|_{L^2} \). We omit the details here. \( \square \)

### 4.2. Proof of Theorem 3.1

Let \( s \geq 3 \) and \( s_0 = \max\{[2s/3] + 1, [s/2] + 2\} \leq m \leq s \). By plugging the estimates \( (4.38), (4.14), (4.26), (4.27) \) and \( (4.38) \) into \( (4.5) \), and keeping in mind \( (4.2) \), we obtain
\[
(4.39) \quad \mathcal{E}_{k}^2[U](t) + \mathcal{D}_{k}^2(t) \lesssim \mathcal{E}_{k}^2[U](0) + \varepsilon \left( \mathcal{D}_{k}^2[U](t) + \mathcal{D}_{k-1}^2[U](t) \right), \quad 1 \leq k \leq s.
\]

Summing the above estimate over \( k \) from \( k = 1 \) to \( k = s_0 \) and adding the result to \( (4.2) \), we obtain
\[
(4.40) \quad \mathcal{E}_{s_0}^2[U](t) + \mathcal{D}_{s_0}^2[U](t) \leq \mathcal{E}_{s_0}^2[U](0) + \varepsilon \mathcal{D}_{s_0}^2[U](t).
\]

For \( \varepsilon > 0 \) sufficiently small, this yields
\[
\mathcal{E}_{s_0}^2[U](t) + \mathcal{D}_{s_0}^2[U](t) \leq \mathcal{E}_{s_0}^2[U](0).
\]

By assuming (as in \( (3.1) \)) the initial energy satisfies
\[
\mathcal{E}_{s_0}^2[U](0) \leq \delta < \frac{\varepsilon^2}{2},
\]
we obtain
\[
\mathcal{E}_{s_0}^2[U](t) \leq \frac{\varepsilon^2}{2},
\]
which closes the a priori estimate \( (4.1) \) by a standard continuity argument.
Now, by summing \((4.40)\) over \(1 \leq k \leq m\), adding the result to \((4.2)\), and selecting \(\varepsilon > 0\) small enough, we obtain
\[
\mathcal{E}_m^2[U](t) + \mathcal{D}_m^2[U](t) \leq \mathcal{E}_m^2[U](0), \quad t \geq 0,
\]
which is exactly \((3.2)\). This finishes the proof of Theorem 3.1.

5. The decay estimates—Proof of Theorem 3.2

Our main goal in this section is to prove Theorem 3.2.

We consider the linearized problem:

\[
\begin{cases}
    u_t = v, \\
    v_t = w, \\
    \tau w_t = \Delta u + \beta \Delta v - w,
\end{cases}
\]

with the initial data

\[
\begin{aligned}
    u(t = 0) &= u_0, & v(t = 0) &= v_0, & w(t = 0) &= w_0.
\end{aligned}
\]

Now, we derive an energy estimate for the negative Sobolev norm of the solution of \((5.1)\). We apply \(\Lambda^{-\gamma}\) to \((5.1a)\) and set \(\tilde{u} = \Lambda^{-\gamma}u\), \(\tilde{v} = \Lambda^{-\gamma}v\), and \(\tilde{w} = \Lambda^{-\gamma}w\). This yields

\[
\begin{cases}
    \tilde{u}_t = \tilde{v}, \\
    \tilde{v}_t = \tilde{w}, \\
    \tau \tilde{w}_t = \Delta \tilde{u} + \beta \Delta \tilde{v} - \tilde{w},
\end{cases}
\]

We have the following Proposition.

**Proposition 5.1.** Let \(\gamma > 0\), then it holds that

\[
\mathcal{E}_{-\gamma}^2[U](t) + \mathcal{D}_{-\gamma}^2[U](t) \leq \mathcal{E}_{-\gamma}^2[U](0).
\]

Following similar reasoning as before, and using system \((5.2)\), we obtain \((5.3)\). We omit the details.

Our next goal is to prove the decay bound \((3.4)\). We point out that we cannot apply directly the method in \([12]\) to get the decay estimates due to the restricted use of the interpolation inequality in Sobolev spaces with negative index:

\[
\|\nabla^\ell f\|_{L^2} \leq C \|\nabla^{\ell+1} f\|_{L^2}^{1-\theta} \|\Lambda^{-\gamma} f\|_{L^2}^\theta, \quad \text{where} \quad \theta = \frac{1}{\ell + \gamma + 1};
\]

cf. Lemma A.3. To overcome this difficulty and inspired by \([32]\), the strategy is to split the solution into a low-frequency and a high-frequency part instead.

Hence, let us consider the unit decomposition

\[
1 = \Psi(\xi) + \Phi(\xi)
\]

where \(\Psi, \Phi \in C_c^\infty(\mathbb{R}^3), 0 \leq \Psi(\xi), \Phi(\xi) \leq 1\) satisfy

\[
\Psi(\xi) = 1, \quad \text{if} \quad |\xi| \leq R, \quad \Psi(\xi) = 0, \quad \text{if} \quad |\xi| \geq 2R
\]

with \(R > 0\). We define \(L_R\) and \(H_R\) as follows:

\[
\hat{L}_R f(\xi) = \Psi(\xi) \hat{f}(\xi) \quad \text{and} \quad \hat{H}_R f(\xi) = \Phi(\xi) \hat{f}(\xi).
\]
Accordingly,\n\begin{equation}
(5.4) \quad f^L = L_R f \quad \text{and} \quad f^H = H_R f.
\end{equation}
We denote by $(\hat{u}, \hat{v}, \hat{w})(\xi, t)$ the Fourier transform of the solution of (5.1a). That is, $(\hat{u}, \hat{v}, \hat{w})(\xi, t) = \mathcal{F}[(u, v, w)(x, t)]$. We define
\begin{equation}
(5.5) \quad \hat{E}(\xi, t) = \frac{1}{2} \left\{ \|\hat{v} + \tau \hat{w}\|^2 + \tau (\beta - \tau)|\xi|^2|\hat{v}|^2 + |\xi|^2|\hat{u} + \tau \hat{v}|^2 \right\}
\end{equation}
with $V = (v + \tau w, \nabla (u + \tau v), \nabla v)$.

We have the following lemma.

**Lemma 5.1.** Assume that $0 < \tau < \beta$. Then, there exists a Lyapunov functional $\hat{L}(\xi, t)$ satisfying for all $t \geq 0$
\begin{equation}
(5.6) \quad \dot{\hat{L}}(\xi, t) \approx \hat{E}(\xi, t) \approx |\hat{V}(\xi, t)|^2
\end{equation}
and
\begin{equation}
(5.7) \quad \frac{d}{dt} \hat{L}(\xi, t) + c \frac{|\xi|^2}{1 + |\xi|^2} \hat{E}(\xi, t) \leq 0.
\end{equation}

The functional $\hat{L}(\xi, t)$ is the same one defined in [27, Eq. (3.20)]. The proof of (5.6) was given in [27, (2.23)], while the proof of (5.7) was in [27, (3.22)].

5.1. **Proof of the estimate** (3.4). In this section, we prove the decay estimate (3.4). We consider system (5.1), write $U = U^L + U^H$ with $U = (u, v, w)$ is the solution of (5.1), $U^L = (u^L, v^L, w^L)$ and $U^H = (u^H, v^H, w^H)$ (see [32] for similar ideas)

**Case 1:** (high frequency)
We multiply the inequality (5.7) by $\Phi^2$, we get
\begin{equation}
\frac{d}{dt}(\Phi^2 \hat{L}(\xi, t)) + c \frac{R^2}{1 + R^2} (\Phi^2 \hat{E}(\xi, t)) \leq 0.
\end{equation}
This implies by using (5.6) together with (5.5) and Plancherel’s identity
\begin{equation}
(5.8) \quad \|V^L(t)\|_{L^2} \lesssim \|V_0\|_{L^2} e^{-c_2 t},
\end{equation}
where the constant $c_2 > 0$ depends on $R$.

**Case 2:** (low frequency)
Now multiplying (5.7) by $\Psi^2$, we get
\begin{equation}
\frac{d}{dt}(\Psi^2 \hat{L}(\xi, t)) + c \frac{|\xi|^2}{1 + |\xi|^2} \hat{E}(\xi, t) \leq 0.
\end{equation}
Hence, using Plancherel’s identity as above, we get
\begin{equation}
(5.9) \quad \frac{d}{dt} \mathcal{L}^L(t) + c_3 \|\nabla V^H(t)\|_{L^2}^2 \leq 0,
\end{equation}
where
\begin{equation}
\mathcal{L}^L(t) = \int_{\mathbb{R}^2} \Psi^2(\xi) L(\xi, t) d\xi,
\end{equation}
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and the constant $c_3 > 0$ depends on $R$.

Applying Lemma A.3, we have

$$\|V\|_{L^2}^{1+\frac{1}{\gamma}} \|\Lambda^{-\gamma} V\|_{L^2}^{-\frac{1}{\gamma}} \lesssim \|\nabla V\|_{L^2}.$$  (5.10)

Using the fact that

$$\|\Lambda^{-\gamma} V(t)\|_{L^2} \lesssim \mathcal{E}_{-\gamma}[U](0),$$

together with (5.10), we obtain from (5.9), that

$$\frac{d}{dt} \mathcal{L}^1(t) + C \|V^1\|_{L^2}^{2(1+1/\gamma)} \left(\mathcal{E}_{-\gamma}[U](0)\right)^{-\frac{2}{\gamma}} \leq 0,$$  (5.11)

where we have used the fact that $\mathcal{E}_{-\gamma}[U^1](0) \leq \mathcal{E}_{-\gamma}[U](0)$.

It is clear that

$$\mathcal{L}^1(t) \approx \|V^1\|_{L^2}^2, \quad \forall t \geq 0.$$  (5.12)

Hence, we get from (5.13),

$$\frac{d}{dt} \mathcal{L}^1(t) + C \left(\mathcal{L}^1(t)\right)^{1+1/\gamma} \left(\mathcal{E}_{-\gamma}[U](0)\right)^{-\frac{2}{\gamma}} \leq 0,$$  (5.13)

Integrating this last inequality, we obtain

$$\mathcal{L}^1(t) \leq C_0 (1 + t)^{-\gamma}$$

where $C_0$ is a positive constant depending on $\mathcal{E}_{-\gamma}[U](0)$. Using (5.12) once again, we obtain

$$\|V^1(t)\|_{L^2} \leq C_0 (1 + t)^{-\gamma/2}.$$  (5.14)

Collecting (5.8) and (5.14), we obtain our decay estimate (3.4).

**Appendix A. Auxiliary inequalities**

In this appendix, we recall some inequalities that have been frequently used in the preceding sections.

**Lemma A.1** (The Gagliardo–Nirenberg interpolation inequality; See [26]). Let $1 \leq p, q, r \leq \infty$, and let $m$ be a positive integer. Then for any integer $j$ with $0 \leq j < m$, we have

$$\|\nabla^j u\|_{L^p} \leq C \|\nabla^m u\|_{L^r}^{\frac{q}{r}} \|u\|_{L^q}^{1-\alpha}$$  (A.1)

where

$$\frac{1}{p} = \frac{j}{n} + \alpha \left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1-\alpha}{q}$$

for $\alpha$ satisfying $j/m \leq \alpha \leq 1$ and $C$ is a positive constant depending only on $n, m, j, q, r$ and $\alpha$. There are the following exceptional cases:

1. If $j = 0$, $rm < n$ and $q = \infty$, then we made the additional assumption that either $u(x) \to 0$ as $|x| \to \infty$ or $u \in L^{q'}$ for some $0 < q' < \infty$.
2. If $1 < r < \infty$ and $m - j - n/r$ is a nonnegative integer, then (A.1) holds only for $j/m \leq \alpha < 1$.

We recall the Sobolev interpolation of the Gagliardo–Nirenberg inequality. See [12, Lemma A.1] for the proof.
Lemma A.2. Let \(2 \leq p \leq +\infty\) and \(0 \leq m, \alpha, \ell\); when \(p = \infty\), we require further that \(m \leq \alpha + 1\) and \(\ell \geq \alpha + 2\). Then, we have that for any \(f \in C_0^\infty(\mathbb{R}^3)\),

\[
\|\nabla^\alpha f\|_{L^p} \leq C\|\nabla^\ell f\|^{\theta}_{L^2}\|\nabla^m f\|^{1-\theta}_{L^2}
\]

where \(0 \leq \theta \leq 1\) and \(\alpha\) satisfy

\[
\frac{1}{p} = \frac{\alpha}{3} + \left(\frac{1}{2} - \frac{\ell}{3}\right)\theta + \left(\frac{1}{2} - \frac{m}{3}\right)(1-\theta).
\]

We also recall the following Sobolev interpolation inequality.

Lemma A.3 (See Lemma A.4 in \[12\]). Let \(\gamma \geq 0\) and \(\ell \geq 0\), then we have

\[
\|\nabla^\ell f\|_{L^2} \leq C\|\nabla^{\ell+1} f\|^{\theta}_{L^2}\|\Lambda^{-\gamma} f\|^{1-\theta}_{L^2}
\]

where \(\theta = \frac{1}{\ell + \gamma + 1}\).

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