Entropy and the Uncertainty Principle

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Abstract. We generalize, improve and unify theorems of Rumin, and Maassen–Uffink about classical entropies associated with quantum density matrices. These theorems refer to the classical entropies of the diagonals of a density matrix in two different bases. Thus, they provide a kind of uncertainty principle. Our inequalities are sharp because they are exact in the high-temperature or semi-classical limit.

1. Introduction

The von Neumann entropy of a quantum state (density matrix) can be calculated either in momentum space or in configuration space and the two are equal and non-negative. They can even be zero. Nevertheless, the corresponding classical entropies, determined by the diagonals of the two representations of the density matrix, can be different, and they can even be negative, but their sum cannot be arbitrarily small. This sum of the classical entropies can thus serve as a measure of the quantum mechanical uncertainty principle.

This point of view was advocated by Deutsch [3], who, among other things, proved a lower bound on this sum, which was later improved by Maassen and Uffink [8], following a conjecture of Kraus [6]. These inequalities were obtained for a general pair of bases, not just momentum and configuration space. In the momentum-configuration bases, an improvement on these previous inequalities was made by Rumin [9], who was able to add a term to the inequality involving the largest eigenvalue of the density matrix. He raised the question whether this additional term could be further improved by using a larger quantity, namely, the von Neumann entropy of the density matrix. In this paper, we prove that this surmise is correct.
We prove even more by combining the Maassen–Uffink investigation with the Rumin surmise. Rumin was concerned with the momentum–configuration space duality, whereas Maassen–Uffink were concerned with arbitrary pairs of bases of the Hilbert space. For this, they introduced a parameter $c$ which somehow quantifies the disparity between the two bases. As one might expect, the $k, x$ pair has the largest $c$ value, i.e., $c = 1$. We show how our theorem applies to any pair with the corresponding $c$-dependent improvement found in [8].

Our theorem and simple proof are supported by a semi-classical intuition, as evidenced by our use of the Golden–Thompson inequality. The only other ingredient in our proof is the Gibbs variational principle. Because our constant in Theorem 2.1 agrees with the semi-classical limit, it is the best possible.

2. Rumin’s Conjecture and Its Generalizations

For any trace class operator $\gamma \geq 0$ on $L^2(\mathbb{R}^d)$, we denote by $\rho_\gamma(x) = \gamma(x, x)$ its density; see (2.2) for a precise definition. Moreover, its Fourier transform is

$\hat{\gamma}(k, k') = \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{2\pi i (k \cdot x - k \cdot x')} \gamma(x, x') \, dx \, dx'$

and

$\rho_{\hat{\gamma}}(k) = \hat{\gamma}(k, k) = \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{2\pi ik (x - x')} \gamma(x, x') \, dx \, dx'$.

We note that if $\text{Tr} \, \gamma = 1$, then

$\int_{\mathbb{R}^d} \rho_\gamma(x) \, dx = \int_{\mathbb{R}^d} \rho_{\hat{\gamma}}(k) \, dk = 1$.

Our main result is

**Theorem 2.1.** For any $\gamma \geq 0$ with $\text{Tr} \, \gamma = 1$ and

$\int_{\mathbb{R}^d} \rho_\gamma(x) \ln_+ \rho_\gamma(x) \, dx < \infty$ and $\int_{\mathbb{R}^d} \rho_{\hat{\gamma}}(k) \ln_+ \rho_{\hat{\gamma}}(k) \, dk < \infty$,

where $\ln_+ \rho = \max\{\ln \rho, 0\}$, one has

$-\int_{\mathbb{R}^d} \rho_\gamma(x) \ln \rho_\gamma(x) \, dx - \int_{\mathbb{R}^d} \rho_{\hat{\gamma}}(k) \ln \rho_{\hat{\gamma}}(k) \, dk \geq -\text{Tr} \, \gamma \ln \gamma$. (2.1)

**Remarks.**

(1) While the entropy on the right side of (2.1) is necessarily non-negative, those on the left side can have either sign.

(2) Inequality (2.1) is saturated in the semi-classical limit. This can be verified by taking $\gamma = Z_{\beta}^{-1} \exp(-\beta(-\Delta + x^2))$ and letting $\beta \to 0$ (see [9]).

(3) For $\gamma$ of rank one, this is Hirschman’s inequality [5]. This was improved by Beckner [1]. Because of (2), however, this improvement is not possible if one allows for mixed states (i.e., $\gamma$ of higher rank).