Symmetry and symplectic reduction

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Abstract

This encyclopedia article briefly reviews without proofs some of the main results in symplectic reduction. The article recalls most the necessary prerequisites to understand the main results, namely, group actions, momentum maps, and coadjoint orbits, among others.

1 Introduction

The use of symmetries in the quantitative and qualitative study of dynamical systems has a long history that goes back to the founders of mechanics. In most cases, the symmetries of a system are used to implement a procedure generically known under the name of reduction that restricts the study of its dynamics to a system of smaller dimension. This procedure is also used in a purely geometric context to construct new nontrivial manifolds having various additional structures.

Most of the reduction methods can be seen as constructions that systematize the techniques of elimination of variables found in classical mechanics. These procedures consist basically of two steps. First, one restricts the dynamics to flow invariant submanifolds of the system in question and secondly one projects the restricted dynamics onto the symmetry orbit quotients of the spaces constructed in the first step. Sometimes, the flow invariant manifolds appear as the level sets of a momentum map induced by the symmetry of the system.

2 Symmetry reduction

The symmetries of a system. The standard mathematical fashion to describe the symmetries of a dynamical system $X \in \mathfrak{X}(M)$ defined on a manifold $M$ ($\mathfrak{X}(M)$ denotes the Lie algebra of smooth vector fields on $M$ endowed with the Jacobi-Lie bracket $[,\cdot]$) consists in studying its invariance properties with respect to a smooth Lie group $\Phi : G \times M \to M$ (continuous symmetries) or Lie algebra $\phi : \mathfrak{g} \to \mathfrak{X}(M)$ (infinitesimal symmetry) action. Recall that $\Phi$ is a (left) action if the map $g \in G \mapsto \Phi(g, \cdot) \in \text{Diff}(M)$ is a group homomorphism, where $\text{Diff}(M)$ denotes the group of smooth diffeomorphisms of the manifold $M$. The map $\phi$ is a (left) Lie algebra action if the map $\xi \in \mathfrak{g} \mapsto \phi(\xi) \in \mathfrak{X}(M)$ is a Lie algebra anti-homomorphism and the map $(m, \xi) \in M \times \mathfrak{g} \mapsto \phi(\xi)(m) \in TM$ is smooth. The vector field $X$ is said to be $G$-symmetric whenever it is equivariant with respect to the $G$-action $\Phi$, that is, $X \circ \Phi_g = T\Phi_g \circ X$, for any $g \in G$. The space of $G$-symmetric vector fields on $M$ is denoted by $\mathfrak{X}(M)^G$. The flow $F_t$ of a

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$G$-symmetric vector field $X \in \mathfrak{X}(M)^G$ is $G$-equivariant, that is, $F_t \circ \Phi_g = \Phi_g \circ F_t$, for any $g \in G$. The vector field $X$ is said to be $g$-symmetric if $[\phi(\xi), X] = 0$, for any $\xi \in \mathfrak{g}$.

If $\mathfrak{g}$ is the Lie algebra of the Lie group $G$, then the infinitesimal generators $\xi_M \in \mathfrak{X}(M)$ of a smooth $G$-group action defined by $\xi_M(m) := \frac{d}{dt}|_{t=0} \Phi(\exp t \xi, m)$, $\xi \in \mathfrak{g}$, $m \in M$, constitute a smooth Lie algebra $\mathfrak{g}$-action and we denote in this case $\phi(\xi) = \xi_M$.

If $m \in M$, the closed Lie subgroup $G_m := \{ g \in G \mid \Phi(g, m) = m \}$ is called the isotropy or symmetry subgroup of $m$. Similarly, the Lie subalgebra $\mathfrak{g}_m := \{ \xi \in \mathfrak{g} \mid \phi(\xi)(m) = 0 \}$ is called the isotropy or symmetry subalgebra of $m$. If $\mathfrak{g}$ is the Lie algebra of $G$ and the Lie algebra action is given by the infinitesimal generators, then $\mathfrak{g}_m$ is the Lie algebra of $G_m$. The action is called free if $G_m = \{e\}$ for every $m \in M$ and locally free if $\mathfrak{g}_m = \{0\}$ for every $m \in M$. We will write interchangeably $\Phi(g, m) = \Phi_g(m) = \Phi^m(g) = g \cdot m$, for $m \in M$ and $g \in G$.

In this article we will focus mainly on continuous symmetries induced by proper Lie group actions. The action $\Phi$ is called proper whenever for any two convergent sequences $\{m_n\}_{n \in \mathbb{N}}$ and $\{g_n \cdot m_n := \Phi(g_n, m_n)\}_{n \in \mathbb{N}}$ in $M$, there exists a convergent subsequence $\{g_{n_k}\}_{k \in \mathbb{N}}$ in $G$. Compact group actions are obviously proper.

**Symmetry reduction of vector fields.** Let $M$ be a smooth manifold and $G$ a Lie group acting properly on $M$. Let $X \in \mathfrak{X}(M)^G$ and $F_t$ be the its (necessarily equivariant) flow. For any isotropy subgroup $H$ of the $G$-action on $M$, the $H$-isotropy type submanifold $M_H := \{ m \in M \mid G_m = H \}$ is preserved by the flow $F_t$. This property is known as the law of conservation of isotropy. The properness of the action guarantees that $G_m$ is compact and that the (connected components of) $M_H$ are embedded submanifolds of $M$ for any closed subgroup $H$ of $G$. The manifolds $M_H$ are, in general, not closed in $M$. Moreover, the quotient group $N(H)/H$ (where $N(H)$ denotes the normalizer of $H$ in $G$) acts freely and properly on $M_H$. Hence, if $\pi_H : M_H \to M_H/(N(H)/H)$ denotes the projection onto orbit space and $i_H : M_H \hookrightarrow M$ is the injection, the vector field $X$ induces a unique vector field $X^H$ on the quotient $M_H/(N(H)/H)$ defined by $X^H \cdot \pi_H = T \pi_H \circ X \circ i_H$, whose flow $F^H_t$ is given by $F^H_t \circ \pi_H = \pi_H \circ F_t \circ i_H$. We will refer to $X^H \in \mathfrak{X}(M_H/(N(H)/H))$ as the $H$-isotropy type reduced vector field induced by $X$.

This reduction technique has been widely exploited in handling specific dynamical systems. When the symmetry group $G$ is compact and we are dealing with a linear action the construction of the quotient $M_H/(N(H)/H)$ can be implemented in a very explicit and convenient manner by using the invariant polynomials of the action and the theorems of Hilbert and Schwarz-Mather.

### 3 Symplectic reduction

**Symplectic or Marsden-Weinstein reduction** is a procedure that implements symmetry reduction for the symmetric Hamiltonian systems defined on a symplectic manifold $(M, \omega)$. The particular case in which the symplectic manifold is a cotangent bundle is dealt with separately in \cite{Marsden1974}. We recall that the Hamiltonian vector field $X_h \in \mathfrak{X}(M)$ associated to the Hamiltonian function $h \in C^\infty(M)$ is uniquely determined by the equality $\omega(X_h, \cdot) = dh$. In this context, the symmetries $\Phi : G \times M \to M$ of interest are given by symplectic or canonical transformations, that is, $\Phi^* \omega = \omega$, for any $g \in G$. For canonical actions each $G$-invariant function $h \in C^\infty(M)^G$ has an associated $G$-symmetric Hamiltonian vector field $X_h$. A Lie algebra action $\varphi$ is called symplectic or canonical, if $\mathcal{L}_{\varphi(\xi)} \omega = 0$ for all $\xi \in \mathfrak{g}$, where $\mathcal{L}$ denotes the Lie derivative operator. If the Lie algebra action is induced from a canonical Lie group action by taking its infinitesimal generators, then it is also canonical.
Momentum maps. The symmetry reduction described in the previous section for general vector fields does not produce a well-adapted answer for symplectic manifolds \((M, \omega)\) in the sense that the reduced spaces \(M_H/(N(H)/H)\) are, in general, not symplectic. To solve this problem one has to use the conservation laws associated to the canonical action, that often appear as **momentum maps**.

Let \(G\) be a Lie group acting canonically on the symplectic manifold \((M, \omega)\). Suppose that for any \(\xi \in \mathfrak{g}\), the vector field \(\xi_M\) is Hamiltonian, with Hamiltonian function \(J^\xi \in C^\infty(M)\) and that \(\xi \in \mathfrak{g} \mapsto J^\xi \in C^\infty(M)\) is linear. The map \(J : M \to \mathfrak{g}^*\) defined by the relation \((J(z), \xi) = J^\xi(z)\), for all \(\xi \in \mathfrak{g}\) and \(z \in M\), is called a **momentum map** of the \(G\)-action. Momentum maps, if they exist, are determined up to a constant in \(\mathfrak{g}^*\) for any connected component of \(M\).

**Examples 3.1**

(i) **Linear momentum.** The phase space of an \(N\)-particle system is the cotangent space \(T^*\mathbb{R}^{3N}\) endowed with its canonical symplectic structure. The additive group \(\mathbb{R}^3\), whose Lie algebra is Abelian and is also equal to \(\mathbb{R}^3\), acts canonically on it by spatial translation on each factor: 
\[
v \cdot (q_i, p^i) = (q_i + v, p^i),\]
with \(i = 1, \ldots, N\). This action has an associated momentum map \(J : T^*\mathbb{R}^{3N} \to \mathbb{R}^3\), where we identified the dual of \(\mathbb{R}^3\) with itself using the Euclidean inner product, that coincides with the classical **linear momentum** \(J(q_i, p^i) = \sum_{i=1}^N p_i\).

(ii) **Angular momentum.** Let \(SO(3)\) act on \(\mathbb{R}^3\) and then, by lift, on \(T^*\mathbb{R}^3\), that is, \(A \cdot (q, p) = (Aq, Ap)\). This action is canonical and has as associated momentum map \(J : T^*\mathbb{R}^3 \to \mathfrak{so}(3)^* \cong \mathbb{R}^3\) the classical **angular momentum** \(J(q, p) = q \times p\).

(iii) **Lifted actions on cotangent bundles.** The previous two examples are particular cases of the following situation. Let \(\Phi : G \times M \to M\) be a smooth Lie group action. The (left) **cotangent lifted action** of \(G\) on \(T^*Q\) is given by \(g \cdot \alpha_q := T_{\rho_g}^*\Phi_{-1}(\alpha_q)\) for \(g \in G\) and \(\alpha_q \in T^*Q\). Cotangent lifted actions preserve the canonical one-form on \(T^*Q\) and hence are canonical. They admit an associated momentum map \(J : T^*Q \to \mathfrak{g}^*\) given by \((J(\alpha_q), \xi) = \alpha_q(\xi_Q(q))\), for any \(\alpha_q \in T^*Q\) and any \(\xi \in \mathfrak{g}\).

(iv) **Symplectic linear actions.** Let \((V, \omega)\) be a symplectic linear space and let \(G\) be a subgroup of the linear symplectic group, acting naturally on \(V\). By the choice of \(G\) this action is canonical and has a momentum map given by \((J(v), \xi) = \frac{1}{2}\omega(\xi_V(v), v)\), for \(\xi \in \mathfrak{g}\) and \(v \in V\) arbitrary.

**Properties of the momentum map.** The main feature of the momentum map that makes it of interest for use in reduction is that it encodes conservation laws for \(G\)-symmetric Hamiltonian systems. **Noether’s Theorem** states that the momentum map is a constant of the motion for the Hamiltonian vector field \(X_h\) associated to any \(G\)-invariant function \(h \in C^\infty(M)^G\).

The derivative \(TJ\) of the momentum map satisfies the following two properties: range \((T_mJ) = (\mathfrak{g}_m)^\circ\) and ker \(T_mJ = (\mathfrak{g} \cdot m)^\omega\), for any \(m \in M\), where \((\mathfrak{g}_m)^\circ\) denotes the annihilator in \(\mathfrak{g}^*\) of the isotropy subalgebra \(\mathfrak{g}_m\) of \(m\), \(\mathfrak{g} \cdot m := T_m(G \cdot m) = \{\xi_M(m) \mid \xi \in \mathfrak{g}\}\) is the tangent space at \(m\) to the \(G\)-orbit that contains this point, and \((\mathfrak{g} \cdot m)^\omega\) is the symplectic orthogonal space to \(\mathfrak{g} \cdot m\) in the symplectic vector space \((T_mM, \omega(m))\). The first relation is sometimes called the **bifurcation lemma** since it establishes a link between the symmetry of a point and the rank of the momentum map at that point.

The existence of the momentum map for a given canonical action is not guaranteed. A momentum map exists if and only if the linear map \(\rho : [\xi] \in \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \mapsto [\omega(\xi_M, \cdot)] \in H^1(M, \mathbb{R})\) is identically zero. Thus if \(H^1(M, \mathbb{R}) = 0\) or \(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] = H^1(\mathfrak{g}, \mathbb{R}) = 0\) then \(\rho \equiv 0\). In particular, if \(\mathfrak{g}\) is semisimple, the First Whitehead Lemma states that \(H^1(\mathfrak{g}, \mathbb{R}) = 0\) and therefore a momentum map always exists for canonical semisimple Lie algebra actions.

A natural question to ask is when the map \(\xi, \eta \mapsto \{\xi, \eta\} = \{J^\xi, J^\eta\}\), \(\xi, \eta \in \mathfrak{g}\), is a Lie algebra homomorphism, that is, \(\{\xi, \eta\} = \{J^\xi, J^\eta\}\), \(\xi, \eta \in \mathfrak{g}\). Here \(\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to \mathbb{R}\) is the Poisson bracket of \(C^\infty(M)\) and \(\{J^\xi, J^\eta\}\) is the symplectic form \(\omega(\xi_M, \cdot)\) evaluated at \(m\) using the relevant representation of \(G\).
$C^\infty(M)$ denotes the Poisson bracket associated to the symplectic form $\omega$ of $M$ defined by $\{f, h\} := \omega(X_f, X_h)$, $f, h \in C^\infty(M)$. This is the case if and only if $T_\ast J(z)(\xi_M(z)) = -\text{ad}_z^* J(z)$, for any $\xi \in g, \eta \in M$, where $\text{ad}^*$ is the dual of the adjoint representation $\text{ad} : (\xi, \eta) \in g \times g \mapsto [\xi, \eta] \in g$ of $g$ on itself. A momentum map that satisfies this relation is called \textit{infinitesimally equivariant}. The reason behind this terminology is that this is the infinitesimal version of \textit{global} or \textit{coadjoint equivariance}. $J$ is \textit{equivariant} if $\text{Ad}^*_{g^-1} \circ J = J \circ \Phi_g$ or, equivalently, $J^\text{Ad}_{\xi}(g \cdot z) = J(\xi)(z)$, for all $g \in G, \xi \in \mathfrak{g}$, and $z \in M$; $\text{Ad}^*$ denotes the dual of the adjoint representation $\text{Ad}$ of $G$ on $\mathfrak{g}$. Actions admitting infinitesimally equivariant momentum maps are called \textit{Hamiltonian actions} and Lie group actions with coadjoint equivariant momentum maps are called \textit{globally Hamiltonian actions}. If the symmetry group $G$ is connected then global and infinitesimal equivariance of the momentum map are equivalent concepts. If $g$ acts canonically on $(M, \omega)$ and $H^1(g, \mathbb{R}) = \{0\}$ then this action admits at most one infinitesimally equivariant momentum map.

Since momentum maps are not uniquely defined, one may ask whether one can choose them to be equivariant. It turns out that if the momentum map is associated to the action of a compact Lie group this can always be done. Momentum maps of cotangent lifted actions are also equivariant as are momentum maps defined by symplectic linear actions. Canonical actions of semisimple Lie algebras on symplectic manifolds admit infinitesimally equivariant momentum maps, since the Second Whitehead Lemma states that $H^2(g, \mathbb{R}) = 0$ if $g$ is semisimple. We shall identify below a specific element of $H^2(g, \mathbb{R})$ which is the obstruction to the equivariance of a momentum map (assuming it exists).

Even though, in general, it is not possible to choose a coadjoint equivariant momentum map, it turns out that when the symplectic manifold is connected there is an affine action on the dual of the Lie algebra with respect to which the momentum map is equivariant. Define the \textit{non-equivariance one-cocycle} associated to $J$ as the map $\sigma : G \longrightarrow \mathfrak{g}^\ast$ given by $g \mapsto J(\Phi_g)(z) - \text{Ad}_{g^-1}^* J(z))$. The connectivity of $M$ implies that the right hand side of this equality is independent of the point $z \in M$. In addition, $\sigma$ is a (left) $\mathfrak{g}^\ast$-valued one-cocycle on $G$ with respect to the coadjoint representation of $G$ on $\mathfrak{g}^\ast$, that is, $\sigma(gh) = \sigma(g) + \text{Ad}_{g^-1}^* \sigma(h)$ for all $g, h \in G$. Relative to the affine action $\Theta : G \times g^\ast \longrightarrow g^\ast$ given by $(g, \mu) \mapsto \text{Ad}_{g^-1}^* \mu + \sigma(g)$, the momentum map $J$ is equivariant. The Reduction Lemma, the main technical ingredient in the proof of the reduction theorem, states that for any $m \in M$ we have

$$\mathfrak{g}_J(m) \cdot m = g \cdot m \cap \ker T_m J = g \cdot m \cap (g \cdot m)^\omega,$$

where $\mathfrak{g}_J(m)$ is the Lie algebra of the isotropy group $G_J(m)$ of $J(m) \in \mathfrak{g}^\ast$ with respect to the affine action of $G$ on $\mathfrak{g}^\ast$ induced by the non-equivariance one-cocycle of $J$.

\textbf{The Symplectic Reduction Theorem.} The symplectic reduction procedure that we now present consists of constructing a new symplectic manifold out of a given symmetric one in which the conservation laws encoded in the form of a momentum map and the degeneracies associated to the symmetry have been eliminated. This strategy allows the reduction of a symmetric Hamiltonian dynamical system to a dimensionally smaller one. This reduction procedure preserves the symplectic category, that is, if we start with a Hamiltonian system on a symplectic manifold, the reduced system is also a Hamiltonian system on a symplectic manifold. The reduced symplectic manifold is usually referred to as the \textit{symplectic or Marsden-Weinstein reduced space}.

\textbf{Theorem 3.2} Let $\Phi : G \times M \longrightarrow M$ be a free proper canonical action of the Lie group $G$ on the connected symplectic manifold $(M, \omega)$. Suppose that this action has an associated momentum map $J : M \longrightarrow \mathfrak{g}^\ast$, with non equivariance one-cocycle $\sigma : G \longrightarrow \mathfrak{g}^\ast$. Let $\mu \in \mathfrak{g}^\ast$ be a value of $J$ and denote by $G_\mu$ the isotropy of $\mu$ under the affine action of $G$ on $\mathfrak{g}^\ast$. Then:
(i) The space $M_{\mu} := J^{-1}(\mu)/G_{\mu}$ is a regular quotient manifold and, moreover, it is a symplectic manifold with symplectic form $\omega_{\mu}$ uniquely characterized by the relation

$$\pi_{\mu}^\ast \omega = i_{\mu}^\ast \omega.$$ 

The maps $i_{\mu} : J^{-1}(\mu) \hookrightarrow M$ and $\pi_{\mu} : J^{-1}(\mu) \rightarrow J^{-1}(\mu)/G_{\mu}$ denote the inclusion and the projection, respectively. The pair $(M_{\mu}, \omega_{\mu})$ is called the symplectic point reduced space.

(ii) Let $h \in C^\infty(M)^G$ be a $G$-invariant Hamiltonian. The flow $F_t$ of the Hamiltonian vector field $X_h$ leaves the connected components of $J^{-1}(\mu)$ invariant and commutes with the $G$-action, so it induces a flow $F_t^\mu$ on $M_{\mu}$ defined by $\pi_{\mu} \circ F_t \circ i_{\mu} = F_t^\mu \circ \pi_{\mu}$.

(iii) The vector field generated by the flow $F_t^\mu$ on $(M_{\mu}, \omega_{\mu})$ is Hamiltonian with associated reduced Hamiltonian function $h_{\mu} \in C^\infty(M_{\mu})$ defined by $h_{\mu} \circ \pi_{\mu} = h \circ i_{\mu}$. The vector fields $X_h$ and $X_{h_{\mu}}$ are $\pi_{\mu}$-related. The triple $(M_{\mu}, \omega_{\mu}, h_{\mu})$ is called the reduced Hamiltonian system.

(iv) Let $k \in C^\infty(M)^G$ be another $G$-invariant function. Then $\{h, k\}$ is also $G$-invariant and $\{h, k\}_\mu = \{h_{\mu}, k_{\mu}\}_\mu$, where $\{\cdot, \cdot\}_\mu$ denotes the Poisson bracket associated to the symplectic form $\omega_{\mu}$ on $M_{\mu}$.

Reconstruction of dynamics. We pose now the question converse to the reduction of a Hamiltonian system. Assume that an integral curve $c_{\mu}(t)$ of the reduced Hamiltonian system $X_{h_{\mu}}$ on $(M_{\mu}, \omega_{\mu})$ is known. Let $m_0 \in J^{-1}(\mu)$ be given. One can determine from this data the integral curve of the Hamiltonian system $X_h$ with initial condition $m_0$. In other words, one can reconstruct the solution of the given system knowing the corresponding reduced solution. The general method of reconstruction is the following. Pick a smooth curve $d(t)$ in $J^{-1}(\mu)$ such that $d(0) = m_0$ and $\pi_{\mu}(d(t)) = c_{\mu}(t)$. Then, if $c(t)$ denotes the integral curve of $X_h$ with $c(0) = m_0$, we can write $c(t) = g(t) \cdot d(t)$ for some smooth curve $g(t)$ in $G_{\mu}$ that is obtained in two steps. First, one finds a smooth curve $\xi(t)$ in $g_{\mu}$ such that $\xi(t)_M(d(t)) = X_h(d(t)) - d(t)$. With the $\xi(t) \in g_{\mu}$ just obtained, one solves the non-autonomous differential equation $\dot{g}(t) = T_{g(t)} L_{\xi(t)} \xi(t)$ on $G_{\mu}$ with $g(0) = e$.

The orbit formulation of the Symplectic Reduction Theorem. There is an alternative approach to the reduction theorem which consists of choosing as numerator of the symplectic reduced space the group invariant saturation of the level sets of the momentum map. This option produces as a result a space that is symplectomorphic to the Marsden-Weinstein quotient but presents the advantage of being more appropriate in the context of quantization problems. Additionally, this approach makes easier the comparison of the symplectic reduced spaces corresponding to different values of the momentum map which is important in the context of Poisson reduction [17]. In carrying out this construction one needs to use the natural symplectic structures that one can define on the orbits of the affine action of a group on the dual of its Lie algebra and that we now quickly review.

Let $G$ be a Lie group, $\sigma : G \rightarrow g^*$ a coadjoint one-cocycle, and $\mu \in g^*$. Let $O_\mu$ be the orbit through $\mu$ of the affine $G$-action on $g^*$ associated to $\sigma$. If $\Sigma : g \times g \rightarrow \mathbb{R}$ defined by $\Sigma(\xi, \eta) := \frac{d}{dt}_{t=0} \langle \sigma(\exp(t\xi)), \eta \rangle$ is a real valued Lie algebra two-cocycle (which is always the case if $\sigma$ is the derivative of a smooth real valued group two-cocycle or if $\sigma$ is the non-equivariance one-cocycle of a momentum map), then $\Sigma : g \times g \rightarrow \mathbb{R}$ is skew symmetric and $\Sigma([\xi, \eta], \xi) + \Sigma([\eta, \xi], \xi) + \Sigma([\xi, \xi], \eta) = 0$ for all $\xi, \eta, \zeta \in g$, then the affine orbit $O_\mu$ is a symplectic manifold with $G$-invariant symplectic structure $\omega_{O_\mu}^\pm$ given by

$$\omega_{O_\mu}^\pm(\nu)(\xi_{g^*}(\nu), \eta_{g^*}(\nu)) = \pm \langle \nu, [\xi, \eta] \rangle \mp \Sigma(\xi, \eta),$$

(3.1)
for arbitrary $\nu \in \mathcal{O}_\mu$, and $\xi, \eta \in \mathfrak{g}$. The symbol $\xi_\mathfrak{g}^+(\nu) := -\text{ad}_\xi^*\nu + \Sigma(\xi, \cdot)$ denotes the infinitesimal generator of the affine action on $\mathfrak{g}^*$ associated to $\xi \in \mathfrak{g}$.

The symplectic structures $\omega_\mathcal{O}_\mu^+$ on $\mathcal{O}_\mu$ are called the ±-orbit or Kostant-Kirillov-Souriau (KKS) symplectic forms.

This symplectic form can be obtained from Theorem 3.2 by considering the symplectic reduction of the cotangent bundle $T^*G$ endowed with the magnetic symplectic structure $\omega_T := \omega_{\text{can}} - \pi^*B_\Sigma$, where $\omega_{\text{can}}$ is the canonical symplectic form on $T^*G$, $\pi : T^*G \to G$ is the projection onto the base, and $B_\Sigma \in \Omega^2(G)^G$ is a left invariant two-form on $G$ whose value at the identity is the Lie algebra two-cocycle $\Sigma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. Since $\Sigma$ is a cocycle it follows that $B_\Sigma$ is closed and hence $\omega_T$ is a symplectic form.

Moreover, the lifting of the left translations on $G$ provides a canonical $G$-action on $T^*G$ that has a momentum map given by $J(g, \mu) = \Theta(g, \mu)$, $(g, \mu) \in G \times \mathfrak{g}^* \simeq T^*G$, where the trivialization $G \times \mathfrak{g}^* \simeq T^*G$ is obtained via left translations. Symplectic reduction using these ingredients yields symplectic reduced spaces that are naturally symplectically diffeomorphic to the affine orbits $\mathcal{O}_\mu$ with the symplectic form $\omega_\mu^+$.

**Theorem 3.3 (Symplectic orbit reduction)** Let $\Phi : G \times M \to M$ be a free proper canonical action of the Lie group $G$ on the connected symplectic manifold $(M, \omega)$. Suppose that this action has an associated momentum map $J : M \to \mathfrak{g}^*$, with non equivariance one-cocycle $\sigma : G \to \mathfrak{g}^*$. Let $\mathcal{O}_\mu := G \cdot \mu \subset \mathfrak{g}^*$ be the $G$-orbit of the point $\mu \in \mathfrak{g}^*$ with respect to the affine action of $G$ on $\mathfrak{g}^*$ associated to $\sigma$.

Then the set $M_{\mathcal{O}_\mu} := J^{-1}(\mathcal{O}_\mu)/G$ is a regular quotient symplectic manifold with the symplectic form $\omega_{\mathcal{O}_\mu}$ uniquely characterized by the relation $i_{\mathcal{O}_\mu}^* \omega = \pi_1^* \omega_{\mathcal{O}_\mu} + J_{\mathcal{O}_\mu}^* \omega_\mu^+$, where $J_{\mathcal{O}_\mu}$ is the restriction of $J$ to $J^{-1}(\mathcal{O}_\mu)$ and $\omega_{\mathcal{O}_\mu}^+$ is the ±-symplectic structure on the affine orbit $\mathcal{O}_\mu$. The maps $i_{\mathcal{O}_\mu} : J^{-1}(\mathcal{O}_\mu) \to M$ and $\pi_{\mathcal{O}_\mu} : J^{-1}(\mathcal{O}_\mu) \to M_{\mathcal{O}_\mu}$ are natural injection and the projection, respectively. The pair $(M_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ is called the symplectic orbit reduced space. Statements similar to (ii) through (iv) in Theorem 3.2 can be formulated for the orbit reduced spaces $(M_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$.

We emphasize that given a momentum value $\mu \in \mathfrak{g}^*$, the reduced spaces $M_\mu$ and $M_{\mathcal{O}_\mu}$ are symplectically diffeomorphic via the projection to the quotients of the inclusion $J^{-1}(\mu) \to J^{-1}(\mathcal{O}_\mu)$.

Reduction at a general point can be replaced by reduction at zero at the expense of enlarging the manifold by the affine orbit. Consider the canonical diagonal action of $G$ on the symplectic difference $M \oplus \mathcal{O}_\mu$ which is the manifold $M \times \mathcal{O}_\mu$ with the symplectic form $\pi_1^* \omega - \pi_2^* \omega_{\mathcal{O}_\mu}$, where $\pi_1 : M \times \mathcal{O}_\mu \to M$ and $\pi_2 : M \times \mathcal{O}_\mu \to \mathcal{O}_\mu$ are the projections. A momentum map for this action is given by $J \circ \pi_1 - \pi_2 : M \oplus \mathcal{O}_\mu^+ \to \mathfrak{g}^*$. Let $(M \oplus \mathcal{O}_\mu)_0 := ((J \circ \pi_1 - \pi_2)^{-1}(0)/G, (\omega \oplus \omega_{\mathcal{O}_\mu}^+)_0)$ be the symplectic point reduced space at zero.

**Theorem 3.4 (Shifting theorem)** Under the hypotheses of the Symplectic Orbit Reduction Theorem 3.3, the symplectic orbit reduced space $M_{\mathcal{O}_\mu}$, the point reduced spaces $M_\mu$, and $(M \oplus \mathcal{O}_\mu)_0$ are symplectically diffeomorphic.

4 **Singular reduction**

In the previous section we carried out symplectic reduction for free and proper actions. The freeness guarantees via the bifurcation lemma that the momentum map $J$ is a submersion and hence the level sets $J^{-1}(\mu)$ are smooth manifolds. Freeness and properness ensures that the orbit spaces $M_\mu := J^{-1}(\mu)/G_\mu$ are regular quotient manifolds. The theory of singular reduction studies the properties of the orbit space $M_\mu$ when the hypothesis on the freeness of the action is dropped. The main result in this situation shows
that these quotients are symplectic Whitney stratified spaces in the sense that the strata are symplectic manifolds in a very natural way; moreover, the local properties of this Whitney stratification make it into what is called a cone space. This statement is referred to as the Symplectic Stratification Theorem and adapts to the symplectic symmetric context the stratification theorem of the orbit space of a proper Lie group action by using its orbit type manifolds. In order to present this result we review the necessary definitions and results on stratified spaces.

**Stratified spaces.** Let \( Z \) be a locally finite partition of the topological space \( P \) into smooth manifolds \( S_i \subset P, i \in I \). We assume that the manifolds \( S_i \subset P, i \in I \), with their manifold topology are locally closed topological subspaces of \( P \). The pair \((P, Z)\) is a decomposition of \( P \) with pieces in \( Z \) when the following condition is satisfied:

\[(DS)\] If \( R, S \in Z \) are such that \( R \cap \bar{S} \neq \emptyset \), then \( R \subset \bar{S} \). In this case we write \( R \preceq S \). If, in addition, \( R \neq S \) we say that \( R \) is incident to \( S \) or that it is a boundary piece of \( S \) and write \( R \prec S \).

Condition \((DS)\) is called the frontier condition and the pair \((P, Z)\) is called a decomposed space. The dimension of \( P \) is defined as \( \dim P = \sup \{ \dim S_i \mid S_i \in Z \} \). If \( k \in \mathbb{N} \), the \( k \)-skeleton \( P^k \) of \( P \) is the union of all the pieces of dimension smaller than or equal to \( k \); its topology is the relative topology induced by \( P \). The depth \( dp(z) \) of any \( z \in (P, Z) \) is defined as

\[
dp(z) := \sup \{ k \in \mathbb{N} \mid \exists S_0, S_1, \ldots, S_k \in Z \text{ with } z \in S_0 \prec S_1 \prec \ldots \prec S_k \}.
\]

Since for any two elements \( x, y \in S \) in the same piece \( S \in P \) we have \( dp(x) = dp(y) \), the depth \( dp(S) \) of the piece \( S \) is well defined by \( dp(S) := dp(x), x \in S \). Finally, the depth \( dp(P) \) of \((P, Z)\) is defined by \( dp(P) := \sup \{ dp(S) \mid S \in Z \} \).

A continuous mapping \( f : P \rightarrow Q \) between the decomposed spaces \((P, Z)\) and \((Q, \mathcal{Y})\) is a morphism of decomposed spaces if for every piece \( S \in Z \), there is a piece \( T \in \mathcal{Y} \) such that \( f(S) \subset T \) and the restriction \( f|_S : S \rightarrow T \) is smooth. If \((P, Z)\) and \((P, T)\) are two decompositions of the same topological space we say that \( Z \) is coarser than \( T \) or that \( T \) is finer than \( Z \) if the identity mapping \((P, T) \rightarrow (P, Z)\) is a morphism of decomposed spaces. A topological subspace \( Q \subset P \) is a decomposed subspace of \((P, Z)\) if for all pieces \( S \in Z \), the intersection \( S \cap Q \) is a submanifold of \( S \) and the corresponding partition \( Z \cap Q \) forms a decomposition of \( Q \).

Let \( P \) be a topological space and \( z \in P \). Two subsets \( A \) and \( B \) of \( P \) are said to be equivalent at \( z \) if there is an open neighborhood \( U \) of \( z \) such that \( A \cap U = B \cap U \). This relation constitutes an equivalence relation on the power set of \( P \). The class of all sets equivalent to a given subset \( A \) at \( z \) will be denoted by \([A]_z\) and called the set germ of \( A \) at \( z \). If \( A \subset B \subset P \) we say that \([A]_z\) is a subgerm of \([B]_z\), and denote \([A]_z \subset [B]_z\).

A stratification of the topological space \( P \) is a map \( S \) that associates to any \( z \in P \) the set germ \( S(z) \) of a closed subset of \( P \) such that the following condition is satisfied:

\[(ST)\] For every \( z \in P \) there is a neighborhood \( U \) of \( z \) and a decomposition \( Z \) of \( U \) such that for all \( y \in U \) the germ \( S(y) \) coincides with the set germ of the piece of \( Z \) that contains \( y \).

The pair \((P, S)\) is called a stratified space. Any decomposition of \( P \) defines a stratification of \( P \) by associating to each of its points the set germ of the piece in which it is contained. The converse is, by definition, locally true.

**The strata.** Two decompositions \( Z_1 \) and \( Z_2 \) of \( P \) are said to be equivalent if they induce the same stratification of \( P \). If \( Z_1 \) and \( Z_2 \) are equivalent decompositions of \( P \) then, for all \( z \in P \), we have that
\[ dp_z^i(z) = dp_{z_j}(z). \] Any stratified space \((P, S)\) has a unique decomposition \(Z_S\) associated with the following maximality property: for any open subset \(U \subset P\) and any decomposition \(Z\) of \(P\) inducing \(S\) over \(U\), the restriction of \(Z_S\) to \(U\) is coarser than the restriction of \(Z\) to \(U\). The decomposition \(Z_S\) is called the canonical decomposition associated to the stratification \((P, S)\). It is often denoted by \(S\) and its pieces are called the strata of \(P\). The local finiteness of the decomposition \(Z_S\) implies that for any stratum \(S\) of \((P, S)\) there are only finitely many strata \(R\) with \(S \prec R\). In the sequel the symbol \(S\) in the stratification \((P, S)\) will denote both the map that associates to each point a set germ and the set of pieces associated to the canonical decomposition induced by the stratification of \(P\).

**Stratified spaces with smooth structure.** Let \((P, S)\) be a stratified space. A singular or stratified chart of \(P\) is a homeomorphism \(\phi : U \rightarrow \phi(U) \subset \mathbb{R}^n\) from an open set \(U \subset P\) to a subset of \(\mathbb{R}^n\) such that for every stratum \(S \in S\) the image \(\phi(U \cap S)\) is a submanifold of \(\mathbb{R}^n\) and the restriction \(\phi|_{U \cap S} : U \cap S \rightarrow \phi(U \cap S)\) is a diffeomorphism. Two singular charts \(\phi : U \rightarrow \phi(U) \subset \mathbb{R}^n\) and \(\varphi : V \rightarrow \varphi(V) \subset \mathbb{R}^m\) are compatible if for any \(z \in U \cap V\) there exist an open neighborhood \(W \subset U \cap V\) of \(z\), a natural number \(N \geq \max\{n, m\}\), open neighborhoods \(O, O' \subset \mathbb{R}^N\) of \(\phi(U) \times \{0\} \) and \(\varphi(V) \times \{0\}\), respectively, and a diffeomorphism \(\psi : O \rightarrow O'\) such that \(i_m \circ \varphi|_W = \psi \circ i_n \circ \phi|_W\), where \(i_n\) and \(i_m\) denote the natural embeddings of \(\mathbb{R}^n\) and \(\mathbb{R}^m\) into \(\mathbb{R}^N\) by using the first \(n\) and \(m\) coordinates, respectively. The notion of singular or stratified atlas is the natural generalization for stratifications of the concept of atlas existing for smooth manifolds. Analogously, we can talk of compatible and maximal stratified atlases. If the stratified space \((P, S)\) has a well defined maximal atlas, then we say that this atlas determines a smooth or differentiable structure on \(P\). We will refer to \((P, S)\) as a smooth stratified space.

**The Whitney conditions.** Let \(M\) be a manifold and \(R, S \subset M\) two submanifolds. We say that the pair \((R, S)\) satisfies the Whitney condition (A) at the point \(z \in R\) if the following condition is satisfied:

(A) For any sequence of points \(\{z_n\}_{n \in \mathbb{N}}\) in \(S\) converging to \(z \in R\) for which the sequence of tangent spaces \(\{T_{z_n}S\}_{n \in \mathbb{N}}\) converges in the Grassmann bundle of \(\dim S\)-dimensional subspaces of \(TM\) to \(\tau \subset T_zM\), we have that \(T_zR \subset \tau\).

Let \(\phi : U \rightarrow \mathbb{R}^n\) be a smooth chart of \(M\) around the point \(z\). The Whitney condition (B) at the point \(z \in R\) with respect to the chart \((U, \phi)\) is given by the following statement:

(B) Let \(\{x_n\}_{n \in \mathbb{N}} \subset R \cap U\) and \(\{y_n\}_{n \in \mathbb{N}} \subset S \cap U\) be two sequences with the same limit \(z = \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n\) and such that \(x_n \neq y_n\), for all \(n \in \mathbb{N}\). Suppose that the set of connecting lines \(\phi(x_n) \phi(y_n) \subset \mathbb{R}^n\) converges in projective space to a line \(L\) and that the sequence of tangent spaces \(\{T_{y_n}S\}_{n \in \mathbb{N}}\) converges in the Grassmann bundle of \(\dim S\)-dimensional subspaces of \(TM\) to \(\tau \subset T_zM\). Then, \((T_z\phi)^{-1}(L) \subset \tau\).

If the condition (A) (respectively (B)) is verified for every point \(z \in R\), the pair \((R, S)\) is said to satisfy the Whitney condition (A) (respectively (B)). It can be verified that Whitney’s condition (B) does not depend on the chart used to formulate it. A stratified space with smooth structure such that for every pair of strata Whitney’s condition (B) is satisfied is called a Whitney space.

**Cone spaces and local triviality.** Let \(P\) be a topological space. Consider the equivalence relation \(\sim\) in the product \(P \times [0, \infty)\) given by \((z, a) \sim (z', a')\) if and only if \(a = a' = 0\). We define the cone \(CP\) on \(P\) as the quotient topological space \(P \times [0, \infty) / \sim\). If \(P\) is a smooth manifold then the cone \(CP\) is a
decomposed space with two pieces, namely, $P \times (0, \infty)$ and the \textit{vertex} which is the class corresponding to any element of the form $(z, 0)$, $z \in P$, that is, $P \times \{0\}$. Analogously, if $(P, \mathcal{Z})$ is a decomposed (stratified) space then the associated cone $CP$ is also a decomposed (stratified) space whose pieces (strata) are the vertex and the sets of the form $S \times (0, \infty)$, with $S \in \mathcal{Z}$. This implies, in particular, that $\dim CP = \dim P + 1$ and $dp(CP) = dp(P) + 1$.

A stratified space $(P, \mathcal{S})$ is said to be \textit{locally trivial} if for any $z \in P$ there exist a neighborhood $U$ of $z$, a stratified space $(F, S^F)$, a distinguished point $0 \in F$, and an isomorphism of stratified spaces

$$\psi : U \to (S \cap U) \times F,$$

where $S$ is the stratum that contains $z$ and $\psi$ satisfies $\psi^{-1}(y, 0) = y$, for all $y \in S \cap U$. When $F$ is given by a cone $CL$ over a compact stratified space $L$ then $L$ is called the \textit{link} of $z$.

An important corollary of \textit{Thom’s First Isotopy Lemma} guarantees that every Whitney stratified space is locally trivial. A converse to this implication needs the introduction of cone spaces. Their definition is given by recursion on the depth of the space.

\textbf{Definition 4.1} Let $m \in \mathbb{N} \cup \{\infty, \omega\}$. A \textit{cone space} of class $C^m$ and depth $0$ is the union of countably many $C^m$ manifolds together with the stratification whose strata are the unions of the connected components of equal dimension. A cone space of class $C^m$ and depth $d + 1$, $d \in \mathbb{N}$, is a stratified space $(P, \mathcal{S})$ with a $C^m$ differentiable structure such that for any $z \in P$ there exists a connected neighborhood $U$ of $z$, a compact cone space $L$ of class $C^m$ and depth $d$ called the \textit{link}, and a stratified isomorphism

$$\psi : U \to (S \cap U) \times CL,$$

where $S$ is the stratum that contains the point $z$, the map $\psi$ satisfies $\psi^{-1}(y, 0) = y$, for all $y \in S \cap U$, and $0$ is the vertex of the cone $CL$.

If $m \neq 0$ then $L$ is required to be embedded into a sphere via a fixed smooth global singular chart $\varphi : L \to S^1$ that determines the smooth structure of $CL$. More specifically, the smooth structure of $CL$ is generated by the global chart $\tau : [z, t] \in CL \mapsto t\varphi(z) \in \mathbb{R}^{d+1}$. The maps $\psi : U \to (S \cap U) \times CL$ and $\varphi : L \to S^1$ are referred to as a \textit{cone chart} and a \textit{link chart}, respectively. Moreover, if $m \neq 0$ then $\psi$ and $\psi^{-1}$ are required to be differentiable of class $C^m$ as maps between stratified spaces with a smooth structure.

\textbf{The Symplectic Stratification Theorem.} Let $(M, \omega)$ be a connected symplectic manifold acted canonically and properly upon by a Lie group $G$. Suppose that this action has an associated momentum map $J : M \to g^*$ with non-equivariance one-cocycle $\sigma : G \to g^*$. Let $\mu \in g^*$ be a value of $J$, $G_\mu$ the isotropy subgroup of $\mu$ with respect to the affine action $\Theta : G \times g^* \to g^*$ determined by $\sigma$, and let $H \subset G$ be an isotropy subgroup of the $G$-action on $M$. Let $M_H^H$ be the connected component of the $H$-isotropy type manifold that contains a given element $z \in M$ such that $J(z) = \mu$ and let $G_\mu M_H^H$ be its $G_\mu$-saturation. Then the following hold:

(i) The set $J^{-1}(\mu) \cap G_\mu M_H^H$ is a submanifold of $M$.

(ii) The set $M_\mu^{(H)} := [J^{-1}(\mu) \cap G_\mu M_H^H]/G_\mu$ has a unique quotient differentiable structure such that the canonical projection $\pi_\mu^{(H)} : J^{-1}(\mu) \cap G_\mu M_H^H \to M_\mu^{(H)}$ is a surjective submersion.

(iii) There is a unique symplectic structure $\omega_\mu^{(H)}$ on $M_\mu^{(H)}$ characterized by

$$j_\mu^{(H)} \omega = \pi_\mu^{(H)} \ast \omega_\mu^{(H)},$$
where \( i^{(H)}_\mu : J^{-1}(\mu) \cap G \mu M^*_H \hookrightarrow M \) is the natural inclusion. The pairs \((M^{(H)}_\mu , \omega^{(H)}_\mu )\) will be called singular symplectic point strata.

(iv) Let \( h \in C^{\infty}(M)^G \) be a \( G \)-invariant Hamiltonian. Then the flow \( F_t \) of \( X_h \) leaves the connected components of \( J^{-1}(\mu) \cap G \mu M^*_H \) invariant and commutes with the \( G \mu \)-action, so it induces a flow \( F^\mu_t \) on \( M^{(H)}_\mu \) that is characterized by \( \pi^{(H)}_\mu \circ F^\mu_t \circ i^{(H)}_\mu = F^\mu_t \circ \pi^{(H)}_\mu \).

(v) The flow \( F^\mu_t \) is Hamiltonian on \( M^{(H)}_\mu \), with reduced Hamiltonian function \( h^{(H)}_\mu : M^{(H)}_\mu \rightarrow \mathbb{R} \) defined by \( h^{(H)}_\mu \circ \pi^{(H)}_\mu = h \circ i^{(H)}_\mu \). The vector fields \( X_h \) and \( X_{h^{(H)}_\mu} \) are \( \pi^{(H)}_\mu \)-related.

(vi) Let \( k : M \rightarrow \mathbb{R} \) be another \( G \)-invariant function. Then \( \{ h, k \}^{(H)}_\mu = \{ h^{(H)}_\mu , k^{(H)}_\mu \}_{M^{(H)}_\mu} \), where \( \{ , \}_{M^{(H)}_\mu} \) denotes the Poisson bracket induced by the symplectic structure on \( M^{(H)}_\mu \).

**Theorem 4.2 (Symplectic Stratification Theorem)** The quotient \( M_\mu := J^{-1}(\mu)/G_\mu \) is a cone space when considered as a stratified space with strata \( M^{(H)}_\mu \).

As was the case for regular reduction, this theorem can be also formulated from the orbit reduction point of view. Using that approach one can conclude that the orbit reduced spaces \( M_{\mathcal{O}_\mu} \) are cone spaces symplectically stratified by the manifolds \( M^{(H)}_{\mathcal{O}_\mu} := G \cdot (J^{-1}(\mu) \cap M^*_H) / G \) that have symplectic structure uniquely determined by the expression

\[
i^{(H)}_{\mathcal{O}_\mu} \ast \omega = \pi^{(H)}_{\mathcal{O}_\mu} \ast \omega^{(H)}_{\mathcal{O}_\mu} + J^{(H)}_{\mathcal{O}_\mu} \ast \omega_{\mathcal{O}_\mu}^+,
\]

where \( i^{(H)}_{\mathcal{O}_\mu} : G \cdot (J^{-1}(\mu) \cap M^*_H) \hookrightarrow M \) is the inclusion, \( J^{(H)}_{\mathcal{O}_\mu} : G \cdot (J^{-1}(\mu) \cap M^*_H) \rightarrow \mathcal{O}_\mu \) is obtained by restriction of the momentum map \( J \), and \( \omega_{\mathcal{O}_\mu}^+ \) is the \( + \)-symplectic form on \( \mathcal{O}_\mu \). Analogous statements to (i) - (vi) above with obvious modifications are valid.

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