SMOOTH VOLUME RIGIDITY FOR MANIFOLDS WITH NEGATIVELY CURVED TARGETS

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Abstract. We establish conditions for a continuous map of nonzero degree between a smooth closed manifold and a negatively curved manifold of dimension greater than four to be homotopic to a smooth cover, and in particular a diffeomorphism when the degree is one. The conditions hold when the volumes or entropy-volumes of the two manifolds differ by less than a uniform constant after an appropriate normalization of the metrics. The results are qualitatively sharp in the sense that all dependencies are necessary. We present a number of corollaries including a corresponding finiteness result. Notably, the method of proof does not rely on a $C^\alpha$ or Gromov-Hausdorff precompactness result nor on surgery technology.

1. Introduction

A basic topological question asks when a continuous map of degree one between two smooth manifolds is homotopic to a diffeomorphism. In a series of papers (see [FJ89], [FJ90], [FJ93]), Farrell and Jones established their celebrated topological rigidity result stating that any homotopy equivalence between any closed manifold and a closed nonpositively curved manifold of dimension at least 5 is homotopic to a homeomorphism. However, they also showed in [FJ89a] that smooth rigidity fails; there are closed negatively curved Riemannian manifolds $(M, g)$ and $(N, g_o)$ which are homeomorphic but not diffeomorphic. Moreover, for any $\delta > 0$ they have examples where the sectional curvatures of $N$ satisfy $K_{g_0} \equiv -1$ and those of $M$ satisfy $-1 - \delta \leq K_g \leq -1$. In a separate paper, [FJ94b], they also gave a set of four criteria, in terms of an ideal boundary conjugacy, for when a homotopy equivalence between two nonpositively curved manifold may be realized by a diffeomorphism (see Section 5 for details).

The main purposes of this paper is to establish a volumetric condition for the smooth rigidity of continuous maps with negatively curved targets. We will also present some generalizations and corollaries.

Theorem 1.1 (Volume Gap). Let $f : M \to N$ be any continuous map between two smooth closed manifolds of dimension $n > 4$. Choose any Riemannian metric $g$ on $M$ normalized to have sectional curvature bound $K_g \geq -1$, and suppose $N$ admits a negatively curved metric $g_o$ normalized to have $-\rho^2 \leq K_{g_o} \leq -1$. There is a constant $C > 0$ such that if

$$\text{Vol}_g(M) \leq |\deg(f)| \text{Vol}_{g_o}(N) + C,$$

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then $f$ is homotopic to a smooth covering map of degree $|\deg(f)|$. Moreover, $C$ depends only on $n$, the injectivity radius of $(M, g)$, the pinching constant $\rho \geq 1$ and $|\deg(f)||N|$, where $|N|$ is the simplicial volume of $N$.

**Remarks 1.2.**

- The constant $C$ always satisfies $C < \text{Vol}_g(M)$ so that the volume constraint is never satisfied when $f$ has degree zero. When the degree is not zero, the resulting local diffeomorphism is given by an explicit construction from the original continuous map $f$. We will show with some examples (see Section 5.7) that the dependence of the constant $C$ on both the injectivity radius of $(M, g)$ and $\rho$ is necessary. Also, the injectivity radius dependency can be exchanged for a lower bound on the normalized volume of sufficiently small balls such as $\inf_{p \in M} \inf_{0 < r \leq 1} \frac{1}{r} \text{Vol}_g(B(p, r))$.

- Note that $|N|$ only depends on $\pi_1(N)$ and in even dimensions we may replace the dependence of $C$ on $\|M\|$ by $|\chi(N)|$. Also, we will see that the dependence of $C$ on $|\deg(f)||N|$ can also be exchanged for a dependence on $|\nabla Rm(g_o)|$, where $Rm$ is the curvature operator on $\Lambda^2 T N$. Thus one may remove the dependence of $C$ on $|\deg(f)|$ if needed.

- Under the hypotheses of Theorem 1.1, Besson, Courtois and Gallot proved in [BCG98] that $\text{Vol}_g(M) \geq |\deg(f)| \text{Vol}_{g_o}(N)$ with equality being achieved if and only if $N$ and $M$ both have constant curvature $-1$ and $f$ is homotopic to a Riemannian cover. From this point of view, Theorem 1.1 can be viewed as a coarse (topological) version of their result.

- If $(M, g)$ satisfies the hypotheses of Theorem 1.1 for some $(N, g_o)$ and fixed value of the constant $C > 0$, then $(M, g)$ has Ricci curvatures bounded below, and injectivity radius and volume bounded above. Hence by Theorem 0.2 of [AC92], it was already known that there are at most a finite number of possible diffeomorphism types for such $(M, g)$.

- Bessières ([Bes98]) first established the special case of Theorem 1.1 when $C = 0$ and $(N, g_o)$ is hyperbolic. Specifically, he extends the main result of [BCG95] to show that if $f : M \to N$ is a map of nonzero degree with $N$ hyperbolic and

$$\text{Minvol}(M) = |\deg(f)| \text{Minvol}(N) = |\deg(f)| \text{Vol}_{g_o}(N),$$

then $M$ admits a hyperbolic metric and $f$ is homotopic to a smooth cover of degree $\deg(f)$. Moreover in [Bes00], he produces an example of a non-compact finite volume hyperbolic manifold $N$ and a manifold $M$, not homeomorphic to $N$, together with a degree one map $f : M \to N$ such that $\text{Minvol}(M) \leq \text{Minvol}(N)$ and their simplicial volumes coincide, $\|M\| = |N|$. Hence, a finite volume version of Theorem 1.1 must address more than just a suitable replacement for the dependence of $C$ on the injectivity radius.

We will derive Theorem 1.1 as a special case of two other progressively more general results. For any finite volume Riemannian manifold $(M, g)$, define the
volume growth entropy of the metric $g$ to be,

$$h(g) = \limsup_{R \to \infty} \frac{\log \text{Vol}_g(B(x, R))}{R}$$

where $B(x, R)$ is the ball of radius $R$ in the Riemannian universal cover $\tilde{M}$ about $x \in \tilde{M}$. The definition is independent of $x$. Moreover Manning showed that for $M$ closed and nonpositively curved, the limit always exists and equals the topological entropy of the geodesic flow on $M$ ([Man79]).

For a negatively curved Riemannian manifold $(N, g_o)$ we can consider the quantity

$$u(g_o) = \inf_{x \in N} \inf_{\lambda \in P(S_x N)} \sqrt{n} \left( \frac{\det \lambda (\text{Hess}_x(B_v))}{\sqrt{\det \lambda (v \otimes v)}} \right)^{1/n}$$

where $P(S_x N)$ is the space of probability measures on the unit tangent sphere $S_x N$, $v \in S_x N$ is the variable of integration by $\lambda$ and $B_v$ is the Busemann function associated to $v$. There are four important properties of the quantity $u(g_o)$ which can be easily derived from the work in [BCG99]: it scales the same way the entropy does, namely $u(c \cdot g_o) = \sqrt{c} u(g_o)$, it satisfies $u(g_o) \geq a(n-1)$ (resp. $u(g_o) \leq b(n-1)$) whenever $K_{g_o} \leq -a^2$ (resp. $K_{g_o} \geq -b^2$), $h(g_o) \geq u(g_o)$ and if $g_0$ is a locally symmetric metric, then $u(g_0) = h(g_0)$.

We now establish some notation for what follows. For any Riemannian manifold $(M, g)$, we denote its injectivity radius by $\text{injrad}(g)$ and its universal cover by $(\tilde{M}, \tilde{g})$. Set $\kappa(g) = \sqrt{\inf_{P \in Gr_2(TM)} K_g(P)}$. Whenever $\kappa(g) = 0$, we have $\text{Ric}(g) \geq 0$ and so $h(g) = 0$. Therefore by the aforementioned result of [BCG99], if $N$ admits a negatively curved metric and there is a map $f : M \to N$ of nonzero degree, then $\kappa(g) > 0$. For any negatively curved Riemannian manifold $(N, g_o)$ we define the pinching constant to be $\rho(g_o) = \sqrt{\inf_{P \in Gr_2(TN)} K_{g_o}(P)}$.

We now state the normalization free version of Theorem 1.4.

**Theorem 1.3 (Smooth Entropy-Volume Rigidity).** Let $f : M \to N$ be a continuous map of nonzero degree between any closed Riemannian manifold $(M, g)$ and a closed negatively curved manifold $(N, g_o)$ of dimension $n > 4$. There is a constant $C$ depending only on $\frac{h(g)}{\kappa(g)}$, $\kappa(g) \cdot \text{injrad}(g)$, $|\text{deg}(f)| \|N\|$ and $\rho(g_o)$ such that if

$$h(g)^n \text{Vol}_g(M) \leq |\text{deg}(f)| u(g_o)^n \text{Vol}_{g_o}(N) + C,$$

then $f$ is homotopic to a smooth covering map of degree $|\text{deg}(f)|$.

**Remark 1.4.** Here the quantities $C$, $\frac{h(g)}{\kappa(g)}$, $\kappa(g) \cdot \text{injrad}(g)$, $\rho(g_o)$, $h(g)^n \text{Vol}_g(M)$ and $u(g_o)^n \text{Vol}_{g_o}(N)$ are all invariant under scaling either of the metrics $g$ or $g_o$. Moreover, $C$ necessarily tends to 0 if either $\kappa(g) \cdot \text{injrad}(g)$ or $\frac{h(g)}{\kappa(g)}$ tends to zero or if $\rho(g_o)$ tends to infinity. (see [7.3]).

For any closed orientable topological $n$-manifold $N$ admitting a metric of negative curvature, we let $\mathcal{M}_{\delta, k}(N)$ be the family of Riemannian $n$-manifolds $(M, g)$ with $\kappa(g) \text{injrad}(g) > \delta$ and admitting a degree $k$ continuous map to a fixed topological manifold $N$. Similarly we define $\mathcal{N}_{n, \rho}$ to be the family of closed $n$-manifolds $(N, g_o)$ with $-a^2 \rho^2 \leq K_{g_o} \leq -a^2$ for any $a > 0$. 

We can optimize each side of the inequality in Theorem 1.3 over any smooth equivalence class of metrics as follows. Suppose $M_\phi$ and $N_{\phi_o}$ represent topological $n$-manifolds $M$ and $N$ for $n > 4$ equipped with two specific smooth structures $\phi$ and $\phi_o$ respectively. By passing to subsequences, we may always choose a sequence $\{g_i\}$ of metrics achieving the infimum,

$$\inf_{(M_\phi, g) \in M_{\phi,k}} h(g)^n \text{Vol}_g(M)$$

such that the limits $\text{Vol}_k(M_\phi) = \lim_i \text{Vol}_{g_i}(M)$ and $h = \lim_i h(g_i)$ both exist. Similarly define the supremum of the volumes of $N$ by metrics $g_o$ rescaled so that $u(g_o) = h$ to be

$$\text{Vol}_\rho(N_{\phi_o}) = \sup \{ \text{Vol}_{g_o}(N) \mid (N_{\phi_o}, g_o) \in N_{\phi_o, \rho} \text{ and } u(g_o) = h\}.$$

The following is an immediate corollary of Theorem 1.3 and Remarks 1.2.

**Corollary 1.5.** For given smooth topological manifolds $M_\phi$ and $N_{\phi_o}$ of dimension $n > 4$ as above, there is a constant $C > 0$ depending only on $\delta$ and $\rho$ such that $M_\phi$ is diffeomorphic to a degree $k$ cover of $N_{\phi_o}$ if

$$\text{Vol}_k(M_\phi) \leq k \text{Vol}_\rho(N_{\phi_o}) + C.$$

The following theorem of Gromov (1.7 of [Gro78]) shows that the additive curvature pinching constant, $\epsilon$, in the Farrell and Jones examples must depend on the volume of $N$.

**Theorem 1.6.** For $(M, g)$ closed of dimension $n \geq 4$, there is a $\epsilon > 0$ depending only on an upper bound for $\text{Vol}_g(M)$ such that if $-1 - \epsilon \leq K g \leq -1$, then $M$ is diffeomorphic to a hyperbolic manifold.

The following corollary of Theorem 1.1 is an equivalent statement of the above theorem in the $n > 4$ case, but by an alternate proof which we will provide in Section 5.

**Corollary 1.7.** For $(M, g)$ closed of dimension $n > 4$ there is a $\epsilon > 0$ depending only on $\|M\|$ such that if $M$ has pinched curvatures $-1 - \epsilon \leq K g \leq -1$, then $M$ is diffeomorphic to a hyperbolic manifold.

This result implies the previous one since since by 1.4 of [Gro78], there are only a finite number of possible diffeomorphism types under the assumptions of Theorem 1.1. In fact, 1.6 implies 1.7 since, under the assumptions of the Corollary, a theorem of Thurston’s (see Sections 0.3 and 1.2 of [Gro82b]) implies $\text{Vol}_g(M)$ is bounded by a uniform constant times the simplicial norm of $M$ which is a homotopy invariant. In particular, the corollary is known to hold in dimension 4 as well. These results are false in dimension 3, as shown by the examples of homotopy inequivalent manifolds with bounded volumes and curvatures tending to $-1$ found in [Gro78]. In fact, these examples can be chosen to be hyperbolic by the work of Thurston [Thu77].

Both the examples of Gromov and Thurston [GT87] in dimension $n \geq 4$ and the counterexamples of Farrell and Jones ([FJ94a] and Farrell, Jones and Ontaneda ([FJO98]) in dimension $n > 4$ mentioned earlier show that $\delta$ in the above corollary must depend on $\pi_1(M)$. For the Gromov-Thurston examples in dimension $n > 4$, we see this dependence explicitly since $\delta < C \log i$ where $i$ is the degree of the ramified covers over a fixed manifold which they use as their examples. The same statement
in dimension \( n \geq 4 \) can be derived directly from Theorem 1.6 in conjunction with Wang’s finiteness theorem [Wan72].

Another principal feature of Theorems 1.3 and 1.8 is that they do not rely on a Cheeger-Gromov type compactness theorem. In fact, even the family of closed Riemannian manifolds \( M \) with fixed \( \pi_1(M) \), curvatures and injectivity radius bounded below, and admitting a map of nonzero degree onto a fixed negatively curved manifold is not precompact in the Gromov-Hausdorff topology, since one may metrically connect sum any such \( M \) with a sufficiently large dilation of an arbitrary simply connected closed nonnegatively curved manifold and stay within this family. As such, we will indicate how we can sometimes use Theorem 1.1 to replace Anderson and Cheeger-Gromov type compactness arguments (e.g. [Che69, AC91, AC92]) to obtain smooth topological finiteness results. For instance, if we fix the topology, then we have the following smooth finiteness theorem of Belegradek (see also Fukaya [Fuk84]).

**Theorem 1.8** ([Bel02]). For \( n \geq 3 \) and constants \( b \geq a > 0 \), there are only a finite number of diffeomorphism types of finite volume manifolds with fixed \( \pi_1 \) and \(-b^2 \leq K_{\text{sec}} \leq -a^2 \).

In Section 5 we will show that this theorem, in the case of closed manifolds of dimension \( n \geq 4 \), follows from Theorem 1.3. We will also prove there a generalization of this to the following finiteness theorem which arises as a corollary of Theorem 1.8.

**Corollary 1.9.** Fix any topological manifold \( N \) of dimension \( n \geq 4 \) admitting a negatively curved metric, and let

\[
V(\delta) = \sup \{ u(g_o)^n \text{Vol}_{g_o}(N) + C(n, \delta, \rho) : \rho \geq 1 \text{ and } (N, g_o) \in \mathcal{N}_{n, \rho} \},
\]

where \( C(n, \delta, \rho) \) is the constant from Theorem 1.3 for \( |\text{deg}(f)| = 1 \). If \( M \) represents the class of all Riemannian manifolds \((M, g)\) admitting a degree one map to \( N \) and satisfying the entropy-volume bound \( h(g)^n \text{Vol}_g(M) < V(\delta) \), then \( M \) has only a finite number of diffeomorphism types.

Now we state our most general theorem. For a \( C^1 \) map \( F : M \rightarrow N \) between two manifolds let \( T_r(F) \subset M \) be the \( r \)-tubular neighborhood of the critical points of \( F \), i.e.

\[
T_r(F) = \bigcup_{\{x \mid \text{Jac}_r(F) = 0\}} B(x, r).
\]

Theorems 1.1 and 1.8 are special cases of the following more general theorem.

**Theorem 1.10.** Let \( f : M \rightarrow N \) be a continuous map of nonzero degree between any closed Riemannian manifold \((M, g)\) and a closed negatively curved manifold \((N, g_o)\) of dimension \( n \geq 4 \). There exists a \( C^1 \) map \( F : M \rightarrow N \) homotopic to \( f \) and a number \( r \) depending only on \( n, \rho, |\text{deg}(f)| \|N\| \) and \( n(g) \cdot \text{injrad}(\bar{g}) \) such that

\[
h(g)^n \text{Vol}_g(M) \geq u(g_o)^n |\text{deg}(f)| \text{Vol}_{g_o}(N) + \frac{1}{2} h(g)^n \text{Vol}_g\left(T_{r(F)}\right).
\]

Theorem 1.10 implies that adding “smooth topology” to \( M \) uniformly increases its volume. For instance, starting with \( M = N \) and adding \( k \) \( i \)-handles, for any \( i = 1, \ldots, n - 1 \), with bounded normalized injectivity radius to \( M \) increases the entropy-volume of \( M \) by at least \( kC \) for a fixed constant \( C \). This follows from the injectivity radius bound, since for the resulting degree 1 map \( F \) there must be at
least \( k \) critical points, one on each handle, separated by a distance of at least the injectivity radius. This bound could be made sharper by taking better account of the entire critical locus for handles. It is generally easy to detect the topological change resulting from adding handles. However, there are many more subtle ways of changing smooth topology. A less intuitive example would be to keep the topology fixed, and allow changes to the smooth structure. If \((M_1, g_1)\) and \((M_2, g_2)\) are two homeomorphic, but nondiffeomorphic, negatively curved manifolds, then we cannot have \( u(g_i) = h(g_i) \) for both \( i = 1, 2 \). Otherwise either the above inequality holds, or we could reverse the roles of \( M_1 \) and \( M_2 \) so that it holds. There are some other general situations where we automatically have a degree one map. The following corollary gives one such example.

**Corollary 1.11.** Let \((N, g_o)\) be closed with \(-\rho^2 \leq K_{g_o} \leq -1\). For any smooth manifold \( Q \) and metric \( g \) on \( N \# Q \) rescaled so that \( K_g \geq -1 \), there is a constant \( C = C(n, \text{injrad}(g), \rho) \) such that if

\[
\text{Vol}_g(N \# Q) \leq \text{Vol}_{g_o}(N) + C,
\]

then \( N \# Q \) is diffeomorphic to \( N \).

In Section 2 we recall the construction of the generalized natural maps \( F_s \) due to Besson, Courtois and Gallot. There we also reduce the proof of Theorems 1.1, 1.3 and 1.10 to a key estimate. In Section 3 we derive the main components of our main estimate, and in Section 4 we put these together. Finally, in Section 5 we prove the remaining corollaries and some additional related results.

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2. Preliminaries

Let \((M, g)\) and \((N, g_o)\) be closed, orientable manifolds and let \( f : M \rightarrow N \) be a degree \( d \) map. Since the quantities in the inequality of Theorem 1.10 are scale invariant in both \( g \) and \( g_o \), we will from now on, unless otherwise stated, assume that we have scaled the metrics so that the sectional curvatures of \( N, g_o \) are bounded from above by \(-1\) and that those of \((M, g)\) from below by \(-1\). This normalization removes two extra parameters that we would otherwise have to drag around.

We begin by recalling the construction of the natural maps \( F_s : M \rightarrow N \) due to Besson, Courtois and Gallot in its present form. Let \( \tilde{f} : \tilde{M} \rightarrow \tilde{N} \) denote the lift of \( f \) to the universal covers. For each \( s > 0 \) and \( x \in M \) consider the measure \( \mu_x^s \) on \( \tilde{M} \) in the Lebesgue class with density

\[
d\mu_x^s(z) = e^{-sd(x, z)}
\]

where \( d \) is the distance function of \( \tilde{M} \). Recall the definition of the volume growth entropy \( h(g) \). For all \( s > h(g) \) and all \( x \in \tilde{M} \) the total measure \( \| \mu_x^s \| \) of \( \mu_x^s \) is finite.

Consider the push-forward measure \( \bar{f}_* \mu_x^s \) on \( \tilde{N} \), and define a measure \( \sigma_x^s \) on \( \partial \tilde{N} \) in the following way. For \( z \in \tilde{N} \), let \( \nu_z \) be the “visual” or Patterson-Sullivan
measures normalized to be probability measures on \( \partial \tilde{N} \) (see [BCG95]), and for \( U \subset \partial \tilde{N} \) measurable define

\[
\sigma_x^s(U) = \int_{\tilde{N}} \nu_z(U) d(\tilde{f}^* \mu_x^s)(z).
\]

That is, we take \( \sigma_x^s \) to be convolution of the push-forward measure \( \tilde{f}^* \mu_x^s \) with the visual measures \( \nu_z \). Notice that for all \( s, x, \| \mu_x^s \| = \| \sigma_x^s \| \), so the measure \( \sigma_x^s \) is finite for \( s > h(g) \).

For \( \theta \in \partial \tilde{N} \) denote by \( B_\theta(y) \) the Busemann function of \( N \) (normalized so that \( B_\theta(O) = 0 \) for some fixed origin \( O \in \tilde{N} \)) and consider the function on \( \tilde{N} \) defined by

\[
\sigma_x^s(y) = \int_{\partial \tilde{N}} B_\theta(y) d\sigma_x^s(\theta).
\]

This is a proper strictly convex function, hence it has a unique minimum [BCG95], which we call the barycenter of the measure \( \sigma_x^s \) and denote by \( Bar(\sigma_x^s) \).

This construction is much more general: Given any finite measure \( \lambda \) on \( \tilde{N} \) we can define as in (1) a measure \( \sigma_\lambda \) as the convolution of \( \lambda \) with the family of visual measures. For example the convolution of the Dirac-measure \( \delta_z \) with support \( z \in \tilde{N} \) is the visual measure \( \nu_z \). Similarly, we can define for every finite measure \( m \) of \( \partial \tilde{N} \) the function \( B_m \) as in (3). The function \( B_m \) is proper and convex if \( m \) has no atoms. If this is the case, we define \( Bar(m) \), the barycenter of \( m \), to be the unique minimum of \( B_m \).

For all \( s > h(g) \), the map \( \tilde{F}_s : \tilde{M} \to \tilde{N} \) defined by \( x \mapsto Bar(\sigma_x^s) \) is equivariant under the action of \( \pi_1(M) \) and \( \pi_1(N) \) and so descends to the natural map \( F_s : M \to N \). The following is a collection and restatement of some of the important properties of the natural map due to Besson, Courtois and Gallot [BCG95, BCG98]. In the statements found there, the authors used \( h(g_o) \) instead of \( u(g_o) \) in the case the target is a locally rank one symmetric space or else \( n - 1 \) for the case when the target is negatively curved with upper curvature bound \(-1\). However, their method of proof was to show the following more general version, and then show separately that \( h(g_o) = u(g_o) \) when \( g_o \) is locally symmetric and that \( n - 1 \geq u(g_o) \) when \( K_{g_o} \leq -1 \).

**Theorem 2.1.** Let \((M, g)\) and \((N, g_o)\) be closed orientable manifolds, let \( f : M \to N \) be a map of nonzero degree and assume that the sectional curvature of \( N \) is pinched and bounded from above by \(-1\). For all \( s > h(g) \) and all \( x \in M \),

1. The natural maps \( F_s \) are at least \( C^1 \).
2. The map \( \tilde{\Psi}_s : [0, 1] \times \tilde{M} \to \tilde{N} \) defined by \( \tilde{\Psi}_s(t, x) = Bar \left( t f^*(x) + (1 - t) \sigma_x^s \right) \)
   is equivariant and induces a continuous homotopy between \( f \) and \( F_s \).
3. \( |Jac(F_s)(x)| \leq (\frac{1}{n(g_o)})^{n} \).

**Remark 2.2.** The appropriate version of the above theorem also holds when \( M \) or \( N \) are not orientable, assuming that \( f \) induces an orientation true homomorphism between the fundamental groups.

The above theorem shows that the maps \( F_s \) have a calibration property which will be crucial to our result. However, essentially all of the difficulties in the proof of the main theorems are encountered in proving the following key result whose proof we will postpone.
Theorem 2.3. The gradient of the Jacobian of the natural map exists off of the critical locus of $F_\varepsilon$ and admits the following bound,
\[ \|\nabla \text{Jac}(F_\varepsilon)\| \leq s^n \left( C(n)(1 + s)(1 + \eta^{-n}) + 5s\beta \right), \]
where $C(n)$ is a constant depending only on $n$, $\eta$ is the injectivity radius of $(\tilde{M}, \tilde{g})$ and $\beta = \max \{ \rho^3, \|\nabla \text{Rm}(g_0)\| \}$.

We will prove this theorem in Section 4. Using this we can easily prove Theorem 1.10.

Proof of Theorem 1.10. We begin by replacing the metric $g_0$ on $N$ with a nearby one with nearly the same volume and curvature pinching, but with better derivatives of curvature. The main theorem of BMORS employs a Ricci flow theorem of Hamilton’s to show that on the space of all closed Riemannian manifolds $(N, g)$ with $-1 \leq K(g) \leq 1$, the metrics can be smoothed. Namely, there are uniform constants $T(n), c(n)$ and $c(n, m)$ and metrics $g_\varepsilon$ with Riemannian connection $\nabla_\varepsilon$ such that
\[ e^{-c(n)\varepsilon} g \leq g_\varepsilon \leq e^{c(n)\varepsilon} g, \quad |\nabla - \nabla_\varepsilon| \leq c(n)\varepsilon, \quad |\nabla_\varepsilon^m \text{Rm}(g_\varepsilon)| \leq \frac{c(n, m)}{\varepsilon}. \]

Moreover, this was extended in Proposition 2.5 of Ron90 (whose proof Ron attributes to T. Ilmanen and W.-X. Shi) to show there is a constant $c(n)$ such that
\[ \inf K_g - c(n)\varepsilon \leq K_{g_\varepsilon} \leq \sup K_g + c(n)\varepsilon. \]
(See also Shi92 and Kap07.)

Applying this to the metric $\tilde{g}_0 = \rho \cdot g_0$, with lower curvature bound $-1$, we obtain $C^1$ close metrics $\tilde{g}_\varepsilon$ with the listed properties.

We now renormalize $\tilde{g}_\varepsilon$ to the metric $g_\varepsilon = \left( \frac{1}{\rho} - c(n)\varepsilon \right) \tilde{g}_\varepsilon$ so that $g_\varepsilon$ has curvatures at most $-1$. The above controls imply, $|u(g_\varepsilon) - u(g_0)| \leq c(n, \rho, \varepsilon)$ and $|\text{Vol}_{g_\varepsilon}(N) - \text{Vol}_{g_0}(N)| < c(n, \rho, \varepsilon)$ for some constant $c(n, \rho, \varepsilon)$ tending to 0 as $\varepsilon \to 0$. In particular, since $u(g_0)$ is bounded in terms of $\rho$ under the curvature assumption, we have $u(g_0)|\text{deg}(f)|\text{Vol}_{g_0}(N) \leq u(g_\varepsilon)|\text{deg}(f)|\text{Vol}_{g_\varepsilon}(N) + c(n, \rho, \varepsilon)|\text{deg}(f)|\text{Vol}_{g_0}(N)$.

Hence for any $\delta > 0$ there is a sufficiently small $\varepsilon$ depending only on $n, \rho, |\text{deg}(f)|\text{Vol}_{g_0}(N)$ and $\delta$ such that $u(g_\varepsilon)|\text{deg}(f)|\text{Vol}_{g_\varepsilon}(N) \leq u(g_\varepsilon)|\text{deg}(f)|\text{Vol}_{g_\varepsilon}(N) + \delta$. In other words, after decreasing the size of the constant $C$ in the statement by a small uniform amount, we may assume the metric $g_\varepsilon$ is such that $\beta = \max \{ \rho^3, \|\nabla \text{Rm}(g_\varepsilon)\| \}$ is bounded by a constant involving only $n, \rho, |\text{deg}(f)|\text{Vol}_{g_\varepsilon}(N)$.

If $\text{deg}(f) = 0$, then the inequality is trivially true. Hence, we will assume $\text{deg}(f) \neq 0$. By the gradient estimate of Theorem 2.3, we have that the Jacobian is at most $\|\nabla \text{Jac}(F_\varepsilon)\| / r$ on $T_r(F_\varepsilon)$. We want an $r$ such $\|\nabla \text{Jac}(F_\varepsilon)\| / r \leq s^n C r \leq d_n(g_0, g_\varepsilon)^n$. So we take $r = \frac{1}{2Cn(g_\varepsilon)}$ where $C = C(n, \eta, \beta)$. Note that $C$ may be treated as independent of $s$ since $0 \leq h(g) \leq n - 1$ and we will choose $s$ sufficiently close to $h(g)$. Since $n - 1 \leq u(g_\varepsilon) \leq \rho(n - 1) \leq \beta^+(n - 1)$, $r$ also depends only on $n, \eta$ and the constant $\beta$.

We are assuming the metric $g_0$ has been smoothed, so that $\beta \leq C_2$ where $C_2 = C_2(n, \rho, |\text{deg}(f)|\text{Vol}_{g_0}(N))$. Hence the radius $r$ depends only on $n, \rho, \eta$ and $|\text{deg}(f)|\text{Vol}_{g_0}(N)$. Finally, in Gro82a it is shown that under our curvature assumptions $\text{Vol}_{g_\varepsilon}(N)$ is bounded above and below by constants depending only on $n$ and $\rho$. (The generalized Gauss-Bonnet formula shows that the proportionality of $\text{Vol}_{g_\varepsilon}(N)$ to $|\chi(M)|$ is bounded above and below in the even dimensional case.)
On \( T_r(F_s) \) we have the estimate,

\[
|\text{Jac } F_s(x)| < \frac{1}{2} \left( \frac{s}{u(g_o)} \right)^n.
\]

Integrating, we have

\[
|\deg(f)| \text{Vol}_{g_o}(N) = \left| \int_N \deg(f) dg_o \right|
\leq \int_M |f^*dg_o|
\leq \int_M |\text{Jac } F_s(y)| dg(y)
\leq \int_{M-T_r(F_s)} |\text{Jac } F_s(y)| dg(y) + \int_{T_r(F_s)} |\text{Jac } F_s(y)| dg(y)
< \left( \frac{s}{u(g_o)} \right)^n \left( \text{Vol}_g(M) - \frac{1}{2} \text{Vol}_g(T_r(F_s)) \right).
\]

Finally, take \( s \to h(g) \) and multiply through by \( u(g_o)^n \). Recall that we have scaled \( g \) so that \( \kappa(g) = 1 \). If we scale a metric by a constant \( \frac{1}{k} > 0 \) then \( \kappa \left( \frac{1}{k} \right) = c \kappa(g) \), \( h \left( \frac{1}{k} \right) = c h(g) \) and \( \text{Vol}_g \left( B_\frac{1}{k} \left( p, \frac{1}{c} \right) \right) = c^{-n} \text{Vol}_g \left( B_g(p,r) \right) \). Therefore scaling the metric back we obtain the given expression.

\[\square\]

Now we show that Theorems 1.3 and 1.1 easily follow.

**Proof of Theorem 1.3.** We use the previous theorem to obtain a condition under which \( F_s \) can have no critical points. Suppose the critical locus of some \( F_s \) for \( s \) very close to \( h(g) \), is not empty. Then it contains at least one point \( p \). Hence if \( C' \leq \frac{1}{2} \text{Vol}_g(B(p,r)) \), then the inequality could not be satisfied. Hence \( F_s \) would be a local diffeomorphism, and in particular, a smooth \( C^1 \) cover. Again, recall that we have scaled \( g \) so that \( \kappa(g) = 1 \).

Lastly, we recall a couple of standard results of differential topology. Any \( C^1 \) structure on \( M \) is \( C^1 \) equivalent to a \( C^k \) structure for \( k \in [1, \infty) \cup \{\infty, \omega\} \). Similarly, any \( C^1 \) (local) diffeomorphism is homotopic to a \( C^k \) (local) diffeomorphism for any \( k \in [1, \infty) \cup \{\infty, \omega\} \) (e.g see Chapter 2 and Theorem 2.10 of [Hir76]). Hence, we obtain the \( C^\infty \) covering map, call it \( F \), stated in the conclusion which is homotopic to the original \( C^0 \) map \( f \). Using mollifiers, we can construct \( F \) explicitly from \( F_s \) and hence, explicitly from \( f \).

\[\square\]

**Remark 2.4.** If one were interested in the minimum regularity possible, then in order to state the theorem note that we only need a \( C^2 \) structure on \( M \) and a \( C^3 \) structure on \( N \). In this case, we obtain a \( C^2 \) covering map. Also, in the case when \( N \) is negatively curved then we obtain a \( C^1 \) limit map \( \lim_{s \to h(g)} F_s \) [BCG96], however it is unlikely if this exists when \( M \) has mixed curvatures.

**Proof of Theorem 1.1.** For this we note that under the curvature assumptions, \( K_g \geq -1 \) and \( K_{g_o} \leq -1 \), we have \( h(g) \leq n - 1 \) and \( u(g_o) \geq n - 1 \). If \( F_s \) has a critical point at \( p \), then by the proof of Theorem 1.3 we have \( h(g)^n \left( \text{Vol}_g(M) - \frac{1}{2} \text{Vol}_g(B(p,r)) \right) \leq
u(g_s)^n |\text{deg}(f)| \text{Vol}_{g_s}(N). Therefore we obtain,

\[(n - 1)^n \left( \text{Vol}_g(M) - \frac{1}{2} \text{Vol}_g(B(p, r)) \right) \geq h(g)^n \left( \text{Vol}_g(M) - \frac{1}{2} \text{Vol}_g(B(p, r)) \right) \geq u(g_s)^n |\text{deg}(f)| \text{Vol}_{g_s}(N) \geq (n - 1)^n |\text{deg}(f)| \text{Vol}_{g_s}(N).\]

To finish, we note that by a classical result of Berger (see [Cro8]) for an improved constant), \( \text{Vol}_g(B(p, r)) \geq C(n)r^n \) for all \( r \leq \delta \) where \( C(n) \) only depends on \( n \) and \( \delta \) is the injectivity radius of \( M \).

\[\square\]

3. Jacobian Estimates

The barycenter of \( \sigma_\lambda \) is defined to be the minimum of the \( C^1 \)-function \( B_{\sigma_\lambda}(\cdot) \). In particular, \( \text{Bar}(\sigma_\lambda) = x \) if and only if the gradient of \( B_{\sigma_\lambda} \) vanishes at \( x \). This gradient can be computed as follows

\[
\nabla_x B_{\sigma_\lambda} = \int_{\partial N} \nabla_x B_\theta \, d\sigma_\lambda(\theta) = \int_{\partial N} \int_{\partial \tilde{N}} \nabla_x B_\theta \, d\nu_z(\theta) \, d\lambda(z),
\]

where \( \nabla_x B_\theta \) is the unit vector in \( T_x \tilde{N} \) pointing to \( \theta \in \partial \tilde{N} \). Applying this to \( \lambda = \mu_s^* \), we have \( \sigma_\lambda = \sigma_y^* \) and the gradient vanishes at \( x = \tilde{F}_s(y) \). We denote by \( r_z \) the function \( r_z(x) = d(x, z) \). Taking the covariant derivative of the gradient with respect to \( y \), i.e. directions \( v \in T_y \tilde{M} \), yields

\[
0 = D_y \nabla_{F_s(y)} B_{\sigma_y^*} = \int_{\partial \tilde{N}} D_{y_{F_s(y)}} \nabla B_\theta \, d\sigma_y^*(\theta) - s \int_{\tilde{M}} \int_{\partial \tilde{N}} \nabla_{F_s(y)} B_\theta \, d\nu_{f(z)}(\theta) \, d\mu_y^*(z).
\]

Therefore we have,

\[
d_y F_s = s \left( \int_{\partial \tilde{N}} D_{y_{F_s(y)}} B_\theta \, d\sigma_y^* \right)^{-1} \int_{\tilde{M}} \left( \int_{\partial \tilde{N}} \nabla_{F_s(y)} B_\theta \, d\nu_{f(z)}(\theta) \right) \otimes d_y r_z(y) \, d\mu_y^*(z),
\]

where \( D_{y_{F_s(y)}} B_\theta \) is the \((1,1)\)-tensor associated to the Hessian of \( B_\theta \) at the point \( F_s(y) \). More specifically, it is the self adjoint linear map from \( T_{F_s(y)} \tilde{N} \rightarrow T_{F_s(y)} \tilde{N} \) such that \( D_{y_{F_s(y)}} B_\theta (\nabla_{F_s(y)} B_\theta) = 0 \) and \( D_{y_{F_s(y)}} B_\theta \) restricted to \( (\nabla_{F_s(y)} B_\theta)^{\perp} \) is the second fundamental form of the horosphere through \( F_s(y) \) and tangent to \( \theta \).

We can rewrite the previous expression more concisely as

\[
d_y F_s = s \left( \int_{\partial \tilde{N}} D_{y\theta} \, d\sigma_y^* \right)^{-1} \int_{\tilde{M}} \int_{\partial \tilde{N}} \nabla B_\theta \, d\nu_{f(z)}(\theta) \, d\mu_y^*(z).
\]
For any $v \in T_y\tilde{M}$ we have, assuming the directional derivatives exist,

$$\nabla_v \text{Jac}(F) = s^n \nabla_v \frac{\det H}{\det A} = s^n \frac{\det H}{\det A} \left( \text{Tr}(\nabla_v H H^{-1}) - \text{Tr}(\nabla_v A^{-1}) \right)$$

(3)

where the traces and determinants are with respect to the metrics $g_o$ on $T_{F,(y)}\tilde{N}$ and $g$ on $T_y\tilde{M}$.

We can compute the derivative terms as,

$$\nabla_v H = \left( \int_M \int_{\partial\tilde{N}} \left[ D_{d_y F,(v)} dB_\theta \otimes dr_z + \nabla B_\theta \otimes Ddr_z(v) \right. \right.$$

$$\left. \left. - s \nabla B_\theta \otimes dr_z \langle \nabla y r_z, v \rangle \right] d\nu_{\tilde{f}(z)}(\theta) d\mu_y^*(z) \right).$$

and

$$\nabla_v A = \int_M \int_{\partial\tilde{N}} \left[ D_{d_y F,(v)} DdB_\theta - s DdB_\theta \langle \nabla y r_z, v \rangle \right] d\nu_{\tilde{f}(z)}(\theta) d\mu_y^*(z).$$

The existence of $\nabla_v \text{Jac}(F)$ will follow from the continuity of the terms, assuming they can be bounded. Except where otherwise specified, for the remainder of the paper $\|A\|$ will represent the operator norm (largest singular value) on the tensor $A$ induced from the metric norm on tangent vectors and cotangent vectors. For a measure $\nu$, the quantity $\|\nu\|$ is its total mass. We will concentrate on the estimates of $\circled{2}$ and $\circled{3}$ in terms of $\circled{1}$ for the remainder of this section.

Recall that $\eta$ is the injectivity radius of the universal cover of $M$. Note that $\eta = \injrad(\tilde{g}) \geq \injrad g = \delta$.

**Proposition 3.1.** We have $\|\nabla H\| \leq C(1 + s) (1 + \eta^{-n}) \|\mu_y^*\|$ for a constant $C$ depending only on $n$.

Before proving this, we will need a lemma.

**Lemma 3.2.** We have

$$\int_{\tilde{M}} |\Delta_y r_z| d\mu_y^*(z) \leq C(1 + s) (1 + \eta^{-n}) \|\mu_y^*\|,$$

where $C$ is a constant depending only on $n$ and $\eta$ is the injectivity radius of $\tilde{M}$.

**Proof.** Set $r_z(y) = d(y, z)$. Since spherical Jacobi tensors satisfy the Sturm-Liouville equation, we have $\Delta_y d(y, z) = \Delta_z d(y, z)$ (see [EH90]). So we estimate $\int_{\tilde{M}} |\Delta_z r_y| d\mu_y^*(z)$. If $n_\Omega$ is the outward pointing normal to a domain $\Omega \subset \tilde{M}$ with smooth boundary,
then we have by Stokes Theorem,
\[
\int_{\partial \Omega} \langle \nabla e^{-sr_y}, \hat{n}_t \rangle \, dg_{\partial \Omega} = \int_{\Omega} \Delta e^{-sr_y} \, dg \\
= \int_{\Omega} \text{div}(-se^{-sr_y} \nabla y_y) \, dg \\
= \int_{\Omega} -se^{-sr_y} \Delta y_y + \langle \nabla - se^{-sr_y}, \nabla y_y \rangle \, dg \\
= \int_{\Omega} -se^{-sr_y} \Delta y_y + s^2 e^{-sr_y} \langle \nabla y_y, \nabla y_y \rangle \, dg.
\]
Since $|\nabla y_y| = 1$, we obtain
\[
\int_{\Omega} \Delta y_y e^{-sr_y} \, dg = se^{sr_y} \int_{\Omega} e^{-sr_y} \, dg - \frac{1}{s} \int_{\partial \Omega} \langle \nabla e^{-sr_y}, \hat{n}_t \rangle \, dg_{\partial \Omega} \\
= s \int_{\Omega} e^{-sr_y} \, dg + \int_{\partial \Omega} e^{-sr_y} \langle \nabla y_y, \hat{n}_t \rangle \, dg_{\partial \Omega}.
\]
Let $\text{Cut}_y \subset \tilde{M}$ denote the cut locus from $y$, and for any $r > 0$, set $V(r) = \exp_y^{-1}(B(y, r) - \text{Cut}_y)$. The set $V(r) \subset T_y M$ is star convex from 0. By retracting each ray from $y$ to the points in $\partial V(r)$ by an appropriate amount between 0 and $\epsilon$ we can obtain a domain $V_\epsilon(r) \subset V(r)$ with smooth boundary. By construction $V_\epsilon(r)$ is star convex from 0 and since $\exp_y$ is a diffeomorphism on $V(r)$, the image $\Omega_\epsilon = \exp_y V_\epsilon(r)$ will have smooth boundary and is star convex from $y$. If $\hat{n}_t$ is the outward pointing normal field to $\partial \Omega_\epsilon$ then the star convexity of $V_\epsilon(r)$ implies that $\langle \nabla y y, \hat{n}_t(z) \rangle \geq 0$ for all $z \in \partial \Omega_\epsilon$. Applying the previous estimate to $\Omega = \Omega_\epsilon$ we obtain,
\[
\int_{\Omega_\epsilon} \Delta y_y e^{-sr_y} \, dg = s \int_{\Omega_\epsilon} e^{-sr_y} \, dg + \int_{\partial \Omega_\epsilon} e^{-sr_y} \langle \nabla y_y, \hat{n}_t \rangle \, dg_{\partial \Omega_\epsilon} \\
\geq s \int_{\Omega_\epsilon} e^{-sr_y} \, dg.
\]
Taking $\epsilon \to 0$ we obtain that $\int_{B(y, r) - \text{Cut}_y} \Delta y_y e^{-sr_y} \, dg \geq s \int_{B(y, r) - \text{Cut}_y} e^{-sr_y} \, dg$. In fact $\Delta y_y \, dg$ extends to the cut locus as well as a signed measure (or see the bottom of p.257 in [Cha93] for how to define $\Delta y_y$ using support functions). Since the cut locus has 0 measure, and the Laplacian is controlled there, we can ignore it and simply write,
\[
\int_{B(y, r)} \Delta y_y e^{-sr_y} \, dg \geq s \int_{B(y, r)} e^{-sr_y} \, dg.
\]
Finally, taking $r \to \infty$ we obtain,
\[
\int_{\tilde{M}} \Delta y y \, d\mu^y \geq s \|\mu^y\|.
\]
Set $\Delta^+ y_y = \max\{\Delta y_y, 0\}$ and $\Delta^- y_y = \min\{\Delta y_y, 0\}$. These are defined off of the cut locus of $y$ in $\tilde{M}$. Recall that this has measure 0. The lower curvature bound on $\tilde{M}$ implies by a standard Ricci comparison result (see e.g. [Cha93]) that off of the cut locus of $\tilde{M}$ from $y$, the mean curvature of distance spheres, i.e. the trace of
the shape operator, of $S(y, r_y(z))$ is less than \((n - 1) \coth(r_y(z))\). In other words, 
\[ \Delta^+ r_y(z) \leq (n - 1) \coth(r_y(z)). \]
Therefore we have

\[ \int_M |\Delta r_y| d\mu^*_y(z) = \int_M \Delta^+ r_y \ d\mu^*_y(z) - \int_M \Delta^- r_y \ d\mu^*_y(z) \]

\[ = 2 \int_M \Delta^+ r_y \ d\mu^*_y(z) - \int_M \Delta r_y \ d\mu^*_y(z) \]

\[ \leq 2(n - 1) \int_M \coth(r_y) \ d\mu^*_y(z) - s \|\mu^*_y\| \]

On the other hand, for the restricted metric $g'$ on $S(y, t)$ we may write $\text{Vol}_{g'}(S(y, t)) = \int_{S \otimes M} \text{dvol}(v, t) dv$ in terms of the radial spherical volume element $\text{dvol}$. From the curvature assumptions we have $\text{dvol}(v, t) \leq (\sinh(t))^{n-1}$.

Hence we have

\[ \int_{B(y, \frac{e^{st}}{1+s})} \coth(r_y) d\mu^*_y(z) \leq \omega_{n-1} \int_0^{\frac{1}{1+s}} \cosh(t) \sinh^{n-2}(t) e^{-st} \ dt \leq \frac{\omega_{n-1}}{(1+s)^{n-1}}. \]

For the last inequality we used the fact that $\sinh(t) \leq \frac{e^t}{1+t}$ and that $\cosh(t) < 2$ on the interval.

Since $\coth(t) \leq 1 + \frac{1}{t}$ for all $t > 0$, we have

\[ \int_{M \setminus B(y, \frac{e^{st}}{1+s})} \coth(r_y) d\mu^*_y(z) \leq (3 + s) \int_{M \setminus B(y, \frac{e^{st}}{1+s})} d\mu^*_y(z) \leq (3 + s) \|\mu^*_y\|. \]

Hence,

\[ \int_M |\Delta r_y| d\mu^*_y(z) \leq 2(n - 1) \left( (3 + s) \|\mu^*_y\| + \frac{\omega_{n-1}}{(1+s)^{n-1}} \right) - s \|\mu^*_y\|. \]

For later use it will be essential that this bound scale linearly in $\|\mu^*_y\|$ as in the statement of the lemma. Hence we must bound the $\frac{\omega_{n-1}}{(1+s)^{n-1}}$ in terms of the size of this measure.

By a result of Berger (see [Cro80], [Cro88] for a stronger version) there is a constant $C_1(n) \geq \frac{\omega_{n-1}}{n}$ depending only on $n$, such that $\text{Vol}_g(B(x, r)) \geq C_1(n) r^n$ for all $r < \eta$. We then have,
where the first equality is by expressing the volume of the sphere as the derivative of the volume of the ball. The next inequality is just restricting the integral to the finite domain $[0, \eta]$. The next equality is integration by parts. The next inequality follows from the estimate for the volume of balls up to the injectivity radius mentioned above. The integral is then evaluated by noting that the function $(1 + s)\frac{\eta^2}{1 + s} + 1$ is either monotone or unimodal in $s$ depending on $\eta$, and hence it is bounded below by its limits as $s \to 0$ or $s \to \infty$. The last inequality follows from noting that the first term is larger than $\frac{1}{3}C_1(n)\eta^n$ for $s \leq \frac{1}{6}$ and then choosing a sufficiently small constant $C_2(n)$.

In particular,

$$\frac{\omega_{n-1}}{(1 + s)^{n-1}} \leq (1 + s)C_3(n)\eta^{-n}\|\mu_y^n\|$$

for some constant $C_3(n)$ depending only on $n$. Putting this together with the estimate  and choosing $C(n) = \max\{6n - 6, (2n - 2)C_3(n)\}$ gives the lemma.

\[ \square \]

**Remark 3.3.** We could have shortened the end of the previous lemma slightly if the lemma just asked for an unspecified bound in terms of $s$ and the injectivity radius of $M$. In fact, the linear dependence on $s$ and $\eta$ is optimal up to constants. To see this, we can take a manifold with $h(g) = 1$ and with $y$ at the tip of a long spike with injectivity radius $\eta$.

This estimate is the only term which does not have $s$ as a factor. This is in fact necessary since if $M$ has constant curvature $-1$, then $\|\int_M \omega_x d\mu_x^n\| > \|\mu_y^n\|$. 

We now establish a couple of nontrivial properties of the operator norm which we will need later.

**Lemma 3.4.** For any square matrix $C \in M_{n \times n}$ and symmetric matrices $A \in M_{n \times n}$ and $B \in M_{n \times n}$ satisfying $\|A(v)\| \leq \|B(v)\|$ for all $v \in \mathbb{R}^n$ we have

$$\|AC\| \leq \|BC\| \quad \text{and} \quad \|CA\| \leq \|CB\|. $$

**Proof.** We may assume $B$ is invertible, otherwise we take limits in $\text{GL}_n$. Since $\langle Av, Av \rangle \leq (Bv, Bv)$ by replacing $v$ with $B^{-1}v$ we have $\langle AB^{-1}v, AB^{-1}v \rangle \leq \langle v, v \rangle$. Then $\|v\| \leq \|AB^{-1}v\| = \|B^{-1}A^*v\| = \|B^{-1}Av\|$. Setting $w$ to be the unit vector
\[ w = \frac{B^{-1}Av}{\|B^{-1}Av\|} \] for any unit vector \( v \) we obtain
\[ \|Av\| \leq \frac{\|Av\|}{\|B^{-1}Av\|} = \frac{\|BB^{-1}Av\|}{\|B^{-1}Av\|} = \|Bw\| . \]

On the other hand \( Av = Bw \frac{\|Av\|}{\|Bw\|} \). So taking \( v \) to be a unit vector with \( \|CAv\| = \|CA\| \), we obtain
\[ \|CA\| = \|CAv\| = \|CBw\| \frac{\|Av\|}{\|Bw\|} \leq \|CBw\| \leq \|CB\| . \]

Since \( \|CA\| = \|A^*C^*\| = \|AC\| \), we obtain \( \|AC^*\| \leq \|BC^*\| \) which gives the second result since \( C \) was arbitrary. \( \square \)

**Lemma 3.5.** For any matrix \( A \), positive definite matrices \( B_i \), and numbers \( \rho_i \) with \( |\rho_i| \leq C \) for \( i = 1, \ldots, k \), we have
\[ \left\| A \left( \sum_{i=1}^k \rho_i B_i \right) \right\| \leq 3C \left\| A \left( \sum_{i=1}^k B_i \right) \right\| \quad \text{and} \quad \left\| \left( \sum_{i=1}^k \rho_i B_i \right) A \right\| \leq 3C \left\| \left( \sum_{i=1}^k B_i \right) A \right\| . \]

**Proof.** Set \( B_p = \sum_{i=1}^k \rho_i B_i \) and \( B = \sum_{i=1}^k B_i \). First assume that each \( \rho_i > 0 \). For any vector \( v \), we have \( v, B_i v \geq 0 \) and so
\[ \langle v, B_p(v) \rangle = \sum_{i=1}^k \rho_i \langle v, B_i(v) \rangle \leq C \sum_{i=1}^k \langle v, B_i(v) \rangle = C \langle v, B(v) \rangle . \]

In particular, this is true for all eigenvectors, so \( \langle B_p(v), B_p(v) \rangle \leq C^2 \langle B(v), B(v) \rangle \) for all vectors. Hence for some choice of unit \( v \) we have
\[ \|B_p A\|^2 = \langle B_p A(v), B_p A(v) \rangle \leq C^2 \langle B A(v), B A(v) \rangle = C^2 \|BA(v)\|^2 \leq C^2 \|BA\|^2 . \]

For the case when \( \rho_i \) may be negative we have \( \|B_p A\| = \|B_p + C A + CBA\| \leq \|B_p + C A\| + C \|BA\| \leq 2C \|BA\| + \|BA\| \) by the previous case. Transpose invariance yields the result with \( A \) and \( B \) reversed. \( \square \)

**Proof of Proposition 3.4.** We treat each of the three terms separately. First, since \( |dr_z(v)| \leq 1 \) for any \( v \) and \( DdF_{s}(y) B_\theta \) is positive semi-definite and symmetric, by Lemma 3.3 we have,
\[
\left\| \int_{\tilde{M}} \int_{\partial \tilde{N}} DdF_{s}(y) B_\theta \circ dF_s(y) \otimes dr_z(v) \, dv_f(z) (\theta) \, d\mu_y(z) \right\|
= \left\| \left( \int_{\tilde{M}} dr_z(v) \int_{\partial \tilde{N}} DdF_{s}(y) B_\theta \, dv_f(z) (\theta) \, d\mu_y(z) \right) \, dF_s(y) \right\|
\leq 3 \left\| \left( \int_{\partial \tilde{N}} Dd\theta B_\theta \, d\sigma^* (\theta) \right) \, dF_s(y) \right\|
= 3 \left\| \mu_s \delta_{A^{-1}H} \right\| = 3s \|H\| .
\]

The singular values of this quantity are all bounded by \( 3s \|\mu_s^*\| \).

By the previous lemma we have a bound on \( \int_{\tilde{M}} |\text{Tr} Ddr_z| \, d\mu_y^* \). However, the lower curvature bound of \( -1 \) on \( \tilde{M} \) implies that all eigenvalues of \( Ddr_z \) are less than \( \coth(r_z) \). So as in the proof of the lemma, we have that the eigenvalues of \( \int_{\tilde{M}} Ddr_z \, d\mu_y^* \) are bounded from above by \( [(3 + s) + (1 + s)C_\eta^{-\eta}] \|\mu_y^*\| \). Combining this with the bound from the lemma of the integral of the absolute value of the
In other words, we obtain a bound for how negative these eigenvalues can be. Putting these together, we obtain that

$$\int_M \|Ddr_z\|\,d\mu_y(z) < 2C(1+s)\left(1 + \eta^{-n}\right)\|\mu_y^*\|$$

where $C$ is a constant depending only on $n$.

For the final term we have $\|\nabla B_\theta\| = 1 = \|dr_z\|$. Hence the final term has singular values bounded by $\|\mu_y^*\|$.

The singular values of the sum of tensors is bounded by the sum of their singular values. Therefore after absorbing an extra factor of 6 into our constant $C$, we have the given universal bound on the tensor $\frac{1}{\|\mu_y\|} \|\nabla H\|$.

We now proceed to the second main estimate. Recall that we set $\beta = \max\{\rho^3, \|\nabla Rm\|\}$ to be the bound on the derivatives up to first order of the curvature tensor of $N$.

**Proposition 3.6.** We have $\|A^{-1}\nabla A\|_\infty \leq 4\beta s\|\mu_y^*\|\|A^{-1}\| + s$.

Again we will prove this proposition by dealing with each of the terms separately. However, first we must deal with the regularity of a fixed horosphere.

The strong unstable foliation $W^{su}$ for the geodesic flow on $SN$ is in general only Hölder continuous whenever $\rho > 2$. On the other hand, it follows from a version of the Hadamard-Perron Theorem (or see Theorem 8 of [Ana69]) that the leaves of this foliation are individually $C^\infty$. In particular, on a closed manifold of negative curvature, the horospheres $B_\theta^{-1}(0)$ are $C^\infty$ submanifolds for fixed $\theta$ (see also [HIH77]). It is a fairly well-known result, e.g Remark 3.3 in Chapter 4 of [Bal95], that for fixed $\theta$, $\nabla_v DdB_\theta$ depends on $\|\nabla Rm\|$ and $\rho$. However, we are not aware of any explicit estimates to this effect. Therefore, the next lemma makes this dependency precise.

**Lemma 3.7.** For any $v \in SN$ and $\theta \in \partial N$, we have $\|\nabla_v DdB_\theta\| \leq 2\beta$.

**Proof.** Recall that for fixed $\theta$ the symmetric tensor $DdB_\theta$ is 0 in the direction $\nabla B_\theta$. In particular, for all $u \in SN$, $(\nabla_u DdB_\theta)(\nabla B_\theta) = \nabla_u(DdB_\theta(\nabla B_\theta)) - DdB_\theta(\nabla_u \nabla B_\theta) = -DdB_\theta^2(u)$.

Let $w$ represent the geodesic vector field defined at each point $z \in \tilde{N}$ by $w(z) = \nabla_z B_\theta$. In other words, $w(z)$ is the unique unit vector at $z$ pointing toward $\theta$. We indicate the bounds in terms of the curvature tensor. The Ricatti equation gives,

$$\nabla_w DdB_\theta + (DdB_\theta)^2 + R(w, \cdot, w) = 0.$$ 

Observe that the vector field $w$ can be viewed as a stable submanifold of $\tilde{SN}$. Choose an extension of $v$ which is an unstable Jacobi field along the flow lines of $w$. Then $\{w, v\} = 0$ and so

$$R(w, v, \cdot) = \nabla_{[w, v]} - \nabla_{w, \nabla v} = \nabla_v \nabla w - \nabla_w \nabla v.$$ 

Covariantly differentiating this with respect to $v$, and applying the Ricatti identity we obtain,

$$0 = \nabla_v \nabla_w DdB_\theta + \nabla_v(DdB_\theta)^2 + \nabla_v R(w, \cdot, w)$$

$$= \nabla_w \nabla_v DdB_\theta + R(w, v, \cdot) DdB_\theta + \nabla_v(DdB_\theta) DdB_\theta + DdB_\theta \nabla_v(DdB_\theta) + \nabla_v R(w, \cdot, w).$$

In other words,

$$\nabla_w \nabla_v DdB_\theta = -DdB_\theta \nabla_v(DdB_\theta) - \nabla_v(DdB_\theta) DdB_\theta - R(w, v, \cdot) DdB_\theta - \nabla_v R(w, \cdot, w).$$
Recall that for any symmetric 2-tensor $U$, $\nabla_v U$ is symmetric since $(\nabla_v U)(X, Y) = \nabla_v (U(X, Y)) - U(\nabla_v X, Y) - U(X, \nabla_v Y)$. Also observe that $R(w, v, \cdot)$ is skew symmetric, but that $R(w, v, \cdot)\nabla^2_b$ is symmetric since $\nabla^2_b \nabla (D\nabla^2_b) + \nabla (D\nabla^2_b) \nabla^2_b$ is symmetric along with the remaining terms of equation (4). By assumption the eigenvalues of $D\nabla^2_b$ are between 1 and $\rho$, and the eigenvalues of $\nabla_v R(w, \cdot, v)$ are between $-\beta$ and $\beta$. Similarly the eigenvalues of $R(w, v, \cdot)\nabla^2_b$ are between $-\rho^3$ and $\rho^3$. Hence, the differential equation dictates that if an eigenvalue of $\nabla_v (D\nabla^2_b)$ is larger than $\beta$ then the corresponding unit eigenvector $u$ must be perpendicular to $w$ and satisfies $(u, \nabla_v \nabla^2_b (D\nabla^2_b)(u)) < 0$.

Since $v$ is a Jacobi field along the geodesic direction $w$, $[w, v] = 0$ and so $\nabla_w v = \nabla_v w = D\nabla^2_b (v)$. Therefore for any field $X$, $(\nabla_v D\nabla^2_b)(w, X) = \nabla_v (D\nabla^2_b (w, X)) - D\nabla^2_b (\nabla_v w, X) - D\nabla^2_b (w, \nabla_v X) = -D\nabla^2_b (v, X)$ or simply $\nabla_v (D\nabla^2_b)(w) = -D\nabla^2_b (v)$. Similarly, $(\nabla_v R(w, \cdot, v))(w) = -R(w, D\nabla^2_b (v), w)$, which we observe is also orthogonal to $w$. Let $u$ be the unit vector field along the geodesic tangent to $w$ such that $u$ is the eigenvector for the maximal eigenvalue $q(v)$ of $\nabla_v D\nabla^2_b$ at each point of the geodesic. Since $u$ is a unit field, $<\nabla_w u, u> = 0$ and so we have

$$\nabla_w (u, \nabla_v D\nabla^2_b (u)) = (\nabla_w u, \nabla_v D\nabla^2_b (u)) + (u, \nabla_w (\nabla_v D\nabla^2_b (u)))$$

$$= (\nabla_w u, q(0) u) + (u, (\nabla_w \nabla_v D\nabla^2_b (u) + \nabla_v D\nabla^2_b (\nabla_w u))$$

$$= 0 + (u, (\nabla_w \nabla_v D\nabla^2_b (u)) + (\nabla_v D\nabla^2_b (u), \nabla_w u)$$

$$= (u, (\nabla_w \nabla_v D\nabla^2_b (u))).$$

Now we express $u = aX + bw$ for a unit vector $X$ with $\langle X, w \rangle = 0$ and functions $a$ and $b$ along the geodesic tangent to $w$ satisfying $a^2 + b^2 = 1$. Since $\nabla_v D\nabla^2_b (w) = -D\nabla^2_b (v)$, we have

$$q(v)b = q(v)u, w = \langle \nabla_v D\nabla^2_b (u), w \rangle = \langle aX + bw, \nabla_v D\nabla^2_b (w) \rangle = -aD\nabla^2_b (v, X).$$

Since $-D\nabla^2_b (v)$ has eigenvalues with norm at most $\rho^2$, if $q(v) > \rho^2$ then $|b| < |a|$. Evaluating from the earlier formula, we obtain,

$$\nabla_w q(v) = u, (\nabla_w \nabla_v D\nabla^2_b (u))$$

$$= -2(\nabla_v D\nabla^2_b (u), D\nabla^2_b (u)) - (R(v, w, u), D\nabla^2_b (u)) - ((\nabla_v R(w, \cdot, w))(u), u)$$

$$= -2q(v) (u, D\nabla^2_b (u)) - a^2 (R(v, w, X), D\nabla^2_b (X)) - a (\langle \nabla_v R(w, \cdot, w) \rangle (X), u) + b (R(w, D\nabla^2_b (v), w), u)$$

$$= -2q(v)a^2 (X, D\nabla^2_b (X)) - a^2 (R(v, w, X), D\nabla^2_b (X)) - a^2 (\langle \nabla_v R(w, \cdot, w) \rangle (X), u) + 2ab (R(v, w, D\nabla^2_b (v), w), D\nabla^2_b (v))$$

$$\leq -2a^2 q(v) + a^2 \rho^3 + a^2 \beta^2 + 2 |ab| \beta^3 < a^2 (-2q(v) + 4 \beta).$$

Here we have used that for all unit vectors $a, b, c, d$ at any point, $|R(a, b, c, d)| \leq \rho^2$ (e.g. see Lemma 3.7 of [DK78]). Hence if $q(v) > 2 \beta$ then $\nabla_w q(v) < 0$. Now observe that $\nabla_w q\left(\frac{v}{|v|}\right) = \frac{1}{|v|} \nabla_w q(v) + q(v) \nabla_w \frac{v}{|v|}$. Since the vector $v$ was extended as an unstable Jacobi field, $\nabla_w \frac{v}{|v|} < 0$. Hence if $q(v) > 2 \beta$, then $\nabla_w q\left(\frac{v}{|v|}\right) < 0$.

Since the diagonal action of $\pi_1(N)$ on $S\hat{N} \times \partial \hat{N}$ is cocompact and $q(v)$ is continuous, there is a $v \in S\hat{N}$ and a $\theta \in \partial \hat{N}$ where $q(v)$ achieves a maximum. This maximum cannot be larger that $2 \beta$. Otherwise for the extension of $v$ as an unstable unstable Jacobi field along the geodesic through the vector $w$ pointing to $\theta$, $q\left(\frac{v}{|v|}\right)$ is strictly increasing in the $-w$ direction, contradicting that $v, \theta$ was a maximum for $q$. 


Similarly, let $q(v)$ be the function giving the minimal eigenvalue for $\nabla_v DB_\theta$ along $w$, and $u$ is the corresponding eigenvector field. If $q(v) < -2\beta$, then the continuing from the second to last line in the computation of $\nabla_w q(v)\text{ we obtain}
\n\nabla_w q(v) \geq -2a^2 q(v) - a^2 \rho^3 - a^2 \beta - 2|ab|\rho^3 > -a^2 (2q(v) + 4\beta) > 0.

Moreover, $\nabla_w q \left( \frac{w}{\|w\|} \right) = \frac{1}{\|w\|} \nabla_w q(v) + q(v) \nabla_w \frac{w}{\|w\|} > 0$. Analogously to the maximum case, the minimum of $q(v)$ over all $v \in S$ and $\theta \in \partial N$ cannot be less than $-2\beta$ since then it is increasing in the stable direction. \hfill \Box

**Lemma 3.8.** For all $v \in S_\theta \tilde{M}$, we have the uniform bound
\n\begin{align*}
\left\| A^{-1} \int_{\partial N} D_{d_y F_\theta} DB_\theta \sigma_y^* \right\| \leq 4\beta s \| \mu^*_y \| \| A^{-1} \| .
\end{align*}

*Proof.* For each $\theta \in \partial N$, let $w_\theta$ represent the geodesic vector field defined at each point $z \in N$ by $w_\theta(z) = \nabla_z B_\theta$. Fix $u \in S_\theta \tilde{M}$. For all $v \in S_\theta \tilde{M}$, we may write $v = a(\theta) X_\theta + b(\theta) w_\theta$ for a unit vector $X_\theta$ with $\langle X_\theta, w_\theta \rangle = 0$, $b(\theta) = \langle w_\theta, v \rangle$ and $a(\theta) = \sqrt{1 - \langle w_\theta, v \rangle^2}$. Since $\nabla_u DB_\theta(w_\theta) = -DB_\theta^2(u)$ and $a(\theta)^2 = 1 - \langle w_\theta, v \rangle^2$, we have
\n\begin{align*}
\nabla_{d_y F_\theta(z)} DB_\theta(v, v) = (1 - \langle w_\theta, v \rangle^2) \nabla_{d_y F_\theta(z)} DB_\theta(X_\theta, X_\theta) - 2a(\theta)b(\theta) DB_\theta^2(d_y F_\theta(z)).
\end{align*}

Recall that in our notation, $d_y F_\theta = sA^{-1} H$. Since $a(\theta)$ and $b(\theta)$ are at most $1$ and $\| DB_\theta \| \leq \rho$, integrating the above expression gives
\n\begin{align*}
\int_{\partial N} a(\theta)b(\theta) DB_\theta^2(d_y F_\theta(v, v)) d\sigma_y^*(\theta) &\leq \rho \left( sA^{-1} H(u, v) \right) = s \rho \langle AA^{-1} H(u), v \rangle.
\end{align*}

Putting this together with the estimate from Lemma 3.7, we obtain
\n\begin{align*}
\left| \int_{\partial N} D_{d_y F_\theta(z)} DB_\theta(v, v) d\sigma_y^*(\theta) \right| &\leq \| d_y F_\theta(z) \| 2\beta \left( \int_{\partial N} 1 - \langle w_\theta, v \rangle^2 d\sigma_y^*(\theta) \right) + 2 s \rho \langle H(u), v \rangle.
\end{align*}

On the other hand, and because of our normalization $K_{g_y} \leq -1$, we have
\n\begin{align*}
A(v, v) = \int_{\partial N} DB_\theta(v, v) d\sigma_y^*(\theta) &\geq \int_{\partial N} 1 - \langle w_\theta, v \rangle^2 d\sigma_y^*(\theta) = \| \mu_y^* \| (Id - Q)(v, v),
\end{align*}

where $Q = \frac{1}{\| \mu_y^* \|} \int_{\partial N} w_\theta \otimes (w_\theta)^* d\sigma_y^*(\theta)$. Applying Lemma 3.4 twice, using these estimates on each factor, we have
\n\begin{align*}
\left\| A^{-1} \int_{\partial N} D_{d_y F_\theta(z)} DB_\theta d\sigma_y^*(\theta) \right\| &\leq \left\| (I - Q)^{-1} (I - Q) \right\| 2\beta \| d_y F_\theta(z) \| + 2 s \rho \| A^{-1} H(u) \|
\end{align*}

\n\begin{align*}
= (2\beta + 2\rho) \| d_y F_\theta(z) \| \leq 4\beta \| d_y F_\theta(z) \|.
\end{align*}

To complete the proof, we note that $\| H \| \leq \| \mu_y^* \|$. \hfill \Box

*Proof of Proposition 3.4.* The first term of the estimate is controlled by the previous Lemma. Noting that $\nabla r_z$ has unit length where it is defined off of the cut locus of measure 0, the second term is estimated by,
\n\begin{align*}
\left\| A^{-1} \left( \int_{\tilde{M}} \int_{\partial N} sD_{d_y F_\theta(z)} B_\theta \langle \nabla_y r_z, v \rangle d\nu_{\tilde{M}}(\theta) d\mu_y^*(z) \right) \right\| \leq \| sA^{-1} A \| \leq s.
\end{align*}
4. Synthesis of the estimates

Here we treat the second part of the main estimate. In formula $[\boxed{1}]$, the term $[\boxed{3}]$ does not have a uniform upper bound. However, we can bound each of the two terms $[\boxed{2}]$ and $[\boxed{3}]$ in term of the reciprocal of $[\boxed{1}]$.

4.1. Estimate of $[\boxed{1}]$. For any $v \in S_{F_{\bar{y}}}(\bar{N})$ set

$$t_v(\theta) = \angle_{F_{\bar{y}}}(\nabla_{F_{\bar{y}}}B_{\theta},Rv),$$

where $Rv \subset T_{F_{\bar{y}}}(\bar{N})$ denotes the line through $v$. Define $\tau_v$ to be the measure on $[0,\pi/2]$ given by

$$\tau_v(U) = \sigma_y^* \left( \{ \theta \in \partial \bar{N} \mid t_v(\theta) \in U \} \right)$$

for any Borel subset $U \subset [0,\pi/2]$.

**Lemma 4.1.** If for some $0 < \epsilon < 1$, we have $\|A^{-1}\| \geq \frac{1}{\epsilon \|\sigma_y^*\|}$, then for some $v \in S_{F_{\bar{y}}}(\bar{N})$ we have

$$\int_0^{\pi/2} \cos^2(t) d\tau_v(t) \geq (1 - \epsilon) \|\sigma_y^*\|.$$

**Proof.** By our standing assumptions, the eigenvalues of the symmetric tensor $D_x dB_{\theta}$ are all at least 1 except for the eigenvalue in the eigendirection $\nabla_x B_{\theta}$ which is 0. Hence, choosing a basis $\{e_i\}$ for $T_{F_{\bar{y}}}(\bar{N})$, we may write,

$$A = \int_{\partial \bar{N}} O^*_\theta \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} O_{\theta} d\sigma^*_\theta(\theta)$$

for some mapping into the orthogonal group $\theta \mapsto O_{\theta} \in O(T_{F_{\bar{y}}}(\bar{N})$ where $\lambda_i \geq 1$ for $i = 2, \ldots, n$ and $O_{\theta}(\nabla_{F_{\bar{y}}}(B_{\theta}) = e_1$. Suppose for some $v \in S_{F_{\bar{y}}}(\bar{N})$, we have $\|A(v)\| \leq \epsilon \|\sigma_y^*\|$. We note that

$$\angle_{F_{\bar{y}}}(e_1, O_{\theta}(Rv)) = \angle_{F_{\bar{y}}}(\nabla_{F_{\bar{y}}}B_{\theta},Rv) = t(\theta).$$

Now we underestimate each $\lambda_i$ by replacing it with 1. In particular,

$$\langle v, Av \rangle \geq \left< v, \left( \int_{\partial \bar{N}} \text{Id} - \nabla_{F_{\bar{y}}}B_{\theta} \nabla_{F_{\bar{y}}}B_{\theta} \right) d\sigma^*_\theta(\theta) \right>(v)$$

$$= \epsilon \int_{\partial \bar{N}} 1 - \langle v, \nabla_{F_{\bar{y}}}B_{\theta} \rangle^2 d\sigma^*_\theta(\theta).$$

Hence we have

$$\int_0^{\pi/2} 1 - \cos^2(t) d\tau(t) \leq \epsilon \|\sigma_y^*\|.$$

We now consider a $g$-orthonormal basis for $T_{\bar{y}}M$ and a $g_{\sigma}$-orthonormal basis for $T_{F_{\bar{y}}}(\bar{N})$, so that we may discuss the magnitude of $H$ with respect to these two metrics. First we need another lemma.
Lemma 4.2. Suppose $(X, \mu)$ is a probability space and $u, v : X \to S^{n-1} \subset \mathbb{R}^n$ are measurable maps to the unit sphere. Then the singular values $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ of the linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ given by the $(1,1)$-tensor

$$A = \int_X u(x) \otimes v(x)^* d\mu(x)$$

satisfy

$$\sum_{i=1}^n \lambda_i = \sup_{O \in O(n)} \int_X \langle u(x), O(v(x)) \rangle d\mu(x) \leq 1.$$

Proof. Let $A = UDV$ be the singular value decomposition for $A$ where $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and $U, V$ are orthogonal. We have

$$\text{Tr} D = \sup_{O \in O(n)} \text{Tr} [DO] = \sup_{O \in O(n)} \text{Tr} [(U^*AV^*)O] = \sup_{O \in O(n)} \text{Tr} [AV^*OU] = \sup_{O \in O(n)} \text{Tr} [AO]$$

$$= \sup_{O \in O(n)} \int_X \langle u(x) \otimes O^*(v(x)) \rangle d\mu(x) = \sup_{O \in O(n)} \int_X \langle u(x), O(v(x)) \rangle d\mu(x) \leq 1.$$ 

Proposition 4.3. If $\|A^{-1}\| \geq \frac{1}{\epsilon \|\sigma_y^*\|}$ for any $\epsilon \leq 1$, then the singular values of $H$ satisfy

$$0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \|\mu_y\|$$

be the singular values of $H$ given by $\lambda_i = \langle w_i, H(u_i) \rangle$ for a $g$-orthonormal frame $\{u_i\}$ and a $g$-orthonormal frame $\{w_i\}$. We can therefore have

$$\lambda_i = \int_{\partial N} \int_{\partial N} \langle w_i, \nabla_{F_y(y)} B_\theta \rangle \langle \nabla_{g r_z} u_i \rangle d\nu_{f(z)}(\theta) d\mu_y^*(z).$$

Since $\langle \nabla_{g r_z} u_i \rangle \leq 1$ we can estimate

$$\lambda_i \leq \int_{\partial N} \langle w_i, \nabla_{F_y(y)} B_\theta \rangle d\sigma_y^*(\theta).$$

Now let $v$ be the vector provided by Lemma 4.4, and write $\nabla B_\theta^\perp$ for the unit vector along the projection of $\nabla_{F_y(y)} B_\theta$ to $v^\perp$, then

$$\lambda_i \leq \int_{\partial N} \langle w_i, \nabla_{F_y(y)} B_\theta \rangle d\sigma_y^*(\theta)$$

$$= \int_{\partial N} \langle w_i, \cos(t(\theta))v + \sin(t(\theta))\nabla B_\theta^\perp \rangle d\sigma_y^*(\theta)$$

$$\leq \int_{0}^{\frac{\pi}{2}} \langle w_i, v \cos(t) + \|\text{proj}_{v^\perp} (w_i)\| \sin(t) \rangle d\tau(t).$$

Since $\mathbb{R}w_n \oplus \mathbb{R}w_{n-1}$ intersects the subspace $v^\perp$ we have

$$\lambda_{n-1} = \sup_{w \in w_n^\perp} \langle H^*(w), H^*(w) \rangle^{\frac{1}{2}} \leq \sup_{w \in w^\perp} \langle H^*(w), H^*(w) \rangle^{\frac{1}{2}} \leq \sup_{w \in v^\perp} \langle w, H(u) \rangle_{S_y M}$$
where the last inequality holds since we may take \( u = \frac{H^*(w)}{\|H^*(w)\|} \). Hence we have from the previous computation

\[
\frac{\lambda_{n-1}}{\|\sigma^*_y\|} \leq \frac{1}{\|\sigma^*_y\|} \int_0^{\frac{\pi}{2}} \sin(t) \, dt \leq \sqrt{\frac{1}{\|\sigma^*_y\|}} \int_0^{\frac{\pi}{2}} \sin^2(t) \, dt \leq \sqrt{\epsilon}.
\]

Here we have used Hölder’s inequality followed by Lemma 4.1. Therefore, all \( \lambda_i \leq \sqrt{\epsilon \|\sigma^*_y\|} \) for \( i = 1, \ldots, n - 1 \), and so \( \lambda_n \leq \|\sigma^*_y\| - \sum_{i=1}^{n-1} \lambda_i \) by applying Lemma 4.2 to the normalized measure \( \|\sigma^*_y\| \).

\[\square\]

Proof of Theorem 2.3. Recall that we are assuming \( \kappa(g) = 1 \) and we set \( \epsilon = \max \left\{ \frac{\|\nu^*_g\|}{\|A^{-1}\|}, 1 \right\} \). Observe that the matrix associated to \( \langle \det H \rangle H^{-1} \) is the adjoint of the matrix associated to \( H \). By Proposition 1.3, its singular values are at most \( \sqrt{\epsilon^{-2} \|\mu^*_g\|^{-n-1}} \) and one singular value is at most \( \sqrt{\epsilon^{-1} \|\mu^*_g\|^{-n-1}} \).

First we show that the product \( \hat{1} \cdot \hat{2} \) has norm bounded above by \( C(1 + s)(1 + \eta^{-n}) \sqrt{\epsilon^{-5}} \) where \( C \) depends only on \( n \). Given the estimate of \( \nabla \epsilon H \), it just remains to point out that \( \frac{\det H}{\det A} \parallel H^{-1}\parallel \) is bounded above by \( \frac{1}{\epsilon \|\nu^*_g\|} \) whenever \( n \geq 4 \).

On the other hand, \( \hat{1} \cdot \hat{3} \) has norm bounded by \( \frac{\det H}{\det A} \parallel A^{-1} \nabla \epsilon A\parallel \). This is bounded by \( \frac{\det A}{\det A} (4s\beta \parallel A^{-1}\parallel \|\mu^*_g\| + s) \). This is in turn bounded by \( 4s\beta \sqrt{\epsilon^{-5}} + 5s\beta \) whenever \( n > 4 \).  

\[\square\]

5. Applications

In this section we will explore some of the consequences of Theorem 2.10. We first mention a couple of well-known topological conditions for the existence of a map \( f : M \to N \) of nonzero degree. Let \( \overline{N} \) be the cover of \( N \) corresponding to \( f_\ast \pi_1(M) < \pi_1(N) \). Since \( f \) induces a map \( \overline{f} : M \to \overline{N} \), we have the following commutative triangle,

\[
\begin{array}{ccc}
H_n(M) & \overset{k}{\longrightarrow} & H_n(N) \\
\times \text{deg}(f) & \downarrow & \times [f_\ast \pi_1(M) : \pi_1(N)] \\
\downarrow & & \downarrow \\
H_n(N) & & H_n(N)
\end{array}
\]

where the multiplication is with respect to the bases of \( \mathbb{Z} \) determined by the respective fundamental classes and \( k \in \mathbb{Z} \). In particular, the index of \( f_\ast \pi_1(M) \) in \( \pi_1(N) \) divides \( \text{deg}(f) \). Consequently \( \pi_1(M) \) is virtually at least as large as \( \pi_1(N) \).

In the case \( f \) has degree one, one may also deduce that \( f_\ast : H_\ast(M) \to H_\ast(N) \) is a split surjection using the induced map \( f^* : H^{n-*}(N) \to H^{n-*}(M) \) on cohomology together with Poincaré duality. There is a similar statement for general degree as well. Thus one can obtain obstructions from both the homology groups, \( H_\ast(M) \) and \( H_\ast(N) \), and the fundamental groups, \( \pi_1(M) \) and \( \pi_1(N) \), to the existence of a nonzero degree map.

For Theorems 2.3 and 2.10 we would like to obtain some bounds on \( \epsilon(h) \) in terms of other quantities.

For \( \Gamma = \pi_1(M) \) and \( S \) a finite subset of \( \Gamma \), let \( \langle S \rangle < \Gamma \) be the subgroup generated by \( S \). Let \( \phi_S \) be the metric on the Cayley graph of \( \langle S, >, S \rangle \) which is the weighted
simplicial distance where the length of each edge corresponding to a generator $\sigma \in S$ is given by the Riemann distance $d(p, \sigma p)$ in the universal cover $(\tilde{M}, \tilde{g})$. Define
\[
    h_{g,S} = \limsup_{R \to \infty} \frac{\log \# \{ \gamma \in \Gamma : \phi_S(e, \gamma) \leq R \}}{R}.
\]
Manning proved in [Man07] the following formula for the volume growth entropy,
\[
h(g) = \sup \{ h_{g,S} : S \text{ finitely generates } \Gamma \}.
\]
This allows us to obtain a curvature and entropy free restatement of Theorem 5.10. Here the entropy is replaced by a dilatation.

**Corollary 5.1.** For any $(N, g_o) \in \mathcal{N}_{n, \rho}$ with $u(g_o) = h(g_o)$ and $n > 4$ and closed Riemannian manifold $(M, g)$ together with any continuous map $f : M \to N$, there exists a $C^1$ map $F : M \to N$ homotopic to $f$ and an $r > 0$ such that
\[
    \Vol_g(M) \geq H^n |\deg(f)| \Vol_{g_o}(N) + \frac{1}{2} \Vol_g \left( T_{\pi_1(M)}(F) \right),
\]
where
\[
    H = \inf \left\{ \frac{h_{g_o, fS}}{h_{g,S}} : S \text{ finitely generates } \pi_1(M) \right\}.
\]
As before, $r$ depends only on $n, \rho, |\deg(f)||N|$ and $\kappa(g) \injrad \tilde{g}$.

**Proof.** Take a sequence $S_i$ so that $h_{g,S_i} \to h(g)$ from below. Note that $u(g_o) = h(g_o) \geq h_{g_o, S'_i} \geq h(g_o, f_s S_i)$ where $S'_i$ is any generating set for $\pi_1(N)$ containing $f_s S_i = \{ f_s \sigma : \sigma \in S_i \}$. Hence after setting $\epsilon_i = h(g) - h_{g,S_i}$, we have
\[
    (h_{g,S_i} + \epsilon_i)^n \left( \Vol_g(M) - \frac{1}{2} \Vol_g \left( T_{\pi_1(M)}(F) \right) \right) \geq h_{g_o, f, S_i} |\deg(f)| \Vol_{g_o}(N).
\]
Letting $\epsilon_i \to 0$ and noting that $H \leq \lim inf \frac{h_{g_o, fS}}{h_{g,S}}$ finishes the proof. 

Consider a smooth $n$-manifold $(N, g_o)$ with a codimension 0 smooth submanifold $S$, and suppose $M_1 = N \setminus S$ and $M_2$ is another smooth $n$-manifold with boundary admitting a map $f : M_2 \to S$ which is a diffeomorphism from $\partial M_2$ to $\partial S$. Let $\partial f$ denote the restriction of $f$ to $\partial M_2$. There is a degree one map from the adjunction space $M_1 \cup_{\partial f} M_2$ to $N$ formed by crushing $M_2$ to $S$ via $f$.

**Corollary 5.2.** Fix $(N, g_o)$ with $-\rho^2 \leq K_{g_o} \leq -1$. For any smooth metric $g$ on $M_1 \cup_{\partial f} M_2$ rescaled so that $K_g \geq -1$, there is a constant $C(n, \injrad(g), \rho)$ such that if
\[
    \Vol_g(M_1 \cup_{\partial f} M_2) \leq \Vol_{g_o}(N) + C,
\]
then $M$ is diffeomorphic to $N$.

A special case of this is Corollary 5.11 where such an $M$ is $M = N \# Q$ for any smooth $n$-manifold $Q$. This is interesting in the context of the following result of Farrell and Jones demonstrating that smooth rigidity fails in the negatively curved category.

**Theorem 5.3 ([FJ89]).** Let $n > 5$ be any dimension for which there exist distinct projectively inequivalent exotic smooth spheres $\Sigma_1, \cdots, \Sigma_k$. for any hyperbolic $n$-manifold $M$ and any $\delta > 0$ there is a finite cover $\tilde{M}$ of $M$ such that
\( \hat{M}, \hat{M} \# \Sigma_1, \hat{M} \# \Sigma_2, \ldots, \hat{M} \# \Sigma_k \) are pairwise homeomorphic but not diffeomorphic. Moreover for \( i = 1, \ldots, k \), there are metrics \( g_i \) on \( M \# \Sigma_i \) such that

\[-1 - \delta \le K_{g_i} \le -1.\]

By another construction, the conclusion of the above theorem also holds in dimensions \( n > 4 \). This theorem has been extended by Farrell separately with Jones (FJ94) and Aravinda (AF04) to show that there are closed manifolds of almost quarter pinched negative curvature which are homeomorphic but not diffeomorphic to complex hyperbolic and quaternionic hyperbolic manifolds. Observe that Theorem 3 implies that the degree of the cover in this theorem must depend on the choice of \( \delta \).

Farrell and Jones (FJ94b) also gave a set of four criteria, in terms of a boundary conjugacy, for when an isomorphism between fundamental groups of two nonpositively curved manifold may be realized by a diffeomorphism.

**Theorem 5.4 (FJ94b).** Given nonpositively closed Riemannian manifolds \((M, g)\) and \((N, g_0)\) and an isomorphism \( \alpha : \pi_1(M) \to \pi_1(N) \), there exists a diffeomorphism \( f : M \to N \) with \( f_* = \alpha \) provided:

1. \( \partial_{\infty} \hat{M} \) and \( \partial_{\infty} \hat{N} \) have a natural \( C^1 \) structure
2. There is a \( C^1 \) conjugacy \( \tilde{h} : \partial_{\infty} \hat{M} \xrightarrow{\sim} \partial_{\infty} \hat{N} \) of the \( C^1 \) actions of \( \pi_1(M) \) on \( \partial_{\infty} \hat{M} \) and \( \pi_1(N) \) on \( \partial_{\infty} \hat{N} \). I.e. \( \tilde{h} \circ \gamma = \alpha(\gamma) \tilde{h} \) for all \( \gamma \in \pi_1(M) \).
3. The conjugacy \( \tilde{h} \) extends to a \( C^0 \) semiconjugacy \( \hat{h} : M \to \hat{N} \), I.e. the lift of a continuous map \( h : M \to N \).
4. \( \chi(M) = 0 \) (e.g. \( n \) is odd).

The principal drawback to applying this theorem is that \( C^1 \) structures on \( \partial \hat{M}, \partial \hat{N} \) are only known to exist when \( M \) and \( N \) either are of higher rank, are locally rank one symmetric spaces, or are quarter pinched and negatively curved. In the negatively curved case a \( C^\infty \) structure would imply that \( M \) and \( N \) are locally symmetric by [FL02]. In light of this, Theorems 3 and 4 can be viewed as an effectively computable gap criterion for smooth equivalence.

Now we turn to proving some of the corollaries mentioned in the introduction.

**Proof of Corollary 1.** If the conclusion does not hold, then there is a sequence of pairwise homotopy equivalent nondiffeomorphic manifolds \((M_i, g_i)\) with \( -1 - \delta_i \le K_{g_i} \le -1 \) for a sequence \( \delta_i \to 0 \). The isomorphisms \((f_{ij})_*: \pi_1(M_i) \to \pi_1(M_j)\) are induced by continuous maps \( f_{ij} \) which can be chosen to be of degree 1 since the \( M_i \) are \( K(\pi_1(M_i), 1) \)'s.

We can rescale the metric \( g_i' = (1 + \delta_i)g_i \) so that \( -1 \le K_{g_i'} \le \frac{1}{1 + \delta_i} \). The real Schwarz lemma of [BCG99] then gives

\[
(1 + \delta_i)^{\frac{n}{2}} \text{Vol}_{g_i}(M_i) = \text{Vol}_{g_i'}(M_i) \ge \text{Vol}_{g_i}(M_j) = \frac{\text{Vol}_{g_i'}(M_j)}{(1 + \delta_j)^n}.
\]

This holds for all \( i, j \), so the volumes form a Cauchy sequence. Moreover, this provides a bound on the injectivity radius as follows. If the injectivity radius of \((M_i, g_i')\) is less than the Margulis constant \( \epsilon \), then any component \( A_{\epsilon, \epsilon} \) of the “thin” set \( A_{\epsilon} := \{ x \in M_j \mid \text{injrad}(x) < \epsilon \} \) consists of a uniform tube neighborhood of a geodesic \( \gamma \) with \( g_i \)-length \( l_i < \epsilon \). The volume of this tube satisfies \( \text{Vol}_{g_i'}(A_{\epsilon, \epsilon}) \ge C \lfloor \log l_i \rfloor \) (see the discussion section in [Rez93]). However \( \text{Vol}_{g_i'}(M_j) \) is bounded above, and therefore the injectivity radius is bounded from below independent of
i. In particular, the $C$ of theorem \[\text{[1.1]}\] depends only on $n$ for this sequence of $M_i$ which contradicts their volumes converging.

From this we can obtain Theorem \[\text{[1.8]}\] in the compact case and $n > 4$.

\textbf{Proof of Theorem [1.8].} We can imitate the proof above, to obtain that the family of pinched negatively curved closed manifolds with fixed $\tau_1(M)$ has uniformly bounded volume from above, and injectivity radius from below. We then can choose a set of manifolds whose volumes are within the constant $C$ of any manifold in this class. This covering number bounds the number of diffeomorphism classes.

\textbf{Proof of Corollary [1.4].} Let $\mathcal{N}$ be the class of all the $(N, g_o)$, that is all smooth structures and metrics on the fixed topological manifold $N$, satisfying the hypotheses. By definition, the members of $\mathcal{N}$ are pairwise homeomorphic. In particular, each element of $\mathcal{M}$ admits a degree one map to all of the elements of $\mathcal{N}$. In particular there is an $(N, g_o) \in \mathcal{N}$ such that $\text{Vol}_g(M) \leq \text{Vol}_{g_o}(N) + C$. Thus we can apply Theorem \[\text{[1.1]}\] to conclude that $M$ and $N$ are diffeomorphic. Since $\mathcal{N}$ is a class with only a finite number of diffeomorphism types, so is $\mathcal{M}$.

\textbf{Remark 5.5.} The above easily generalized to the case where $\mathcal{N}$ is any class of negatively curved manifolds with a finite number of diffeomorphism types and each member of $\mathcal{M}$ maps onto at least element of $\mathcal{N}$, but then the formula for the entropy-volume bound $V(\delta)$ for each $(M, g) \in \mathcal{M}$ must be restricted over those elements of $\mathcal{N}$ admitting a degree one map from $M$.

\textbf{Remark 5.6.} An important point to the above proofs is that we do not need to use Cheeger finiteness (or any other form of Gromov-Hausdorff compactness theorems). Instead we have replaced this step with the direct analytic argument of Theorem \[\text{[1.4]}\].

\textbf{Example 5.7.} We first point out that the constant $C$ in Theorems \[\text{[1.1]}\] and \[\text{[1.3]}\] as well as $r$ in Theorem \[\text{[1.7]}\] must depend on $\delta$. Otherwise, we could take a hyperbolic $(N, g_o)$ and let $M = N \# Q$ the smooth connect sum with a very small diameter Riemannian manifold $(Q, g_Q)$ with arbitrary topology. However, $M$ admits a degree 1 map to $N$ and by scaling $g_Q$, $\text{Vol}_g(M)$ can be made as close to $\text{Vol}_{g_o}(N)$ as desired.

Similarly, the dependence of $C$ and $r$ on $\rho$ is also necessary. Take a fixed hyperbolic manifold $(M, g)$ of finite volume and a sufficiently small injectivity radius $\delta$. By the Margulis Lemma, there is an $\epsilon_o > 0$ such that each component $A_\epsilon$ of the $\epsilon$-thin part $M_\epsilon$ consisting of points with injectivity radius less than $\epsilon$ for $\delta < \epsilon < \epsilon_o$ is topologically an $n-1$ ball bundle over $S^1$, where the 0 section is a short geodesic. The metric is locally $\cosh(r)^2 d\gamma^2 + dr^2 + \sinh(r)^2 d\theta^2$ where $d\gamma$ is the geodesic arc element, $dr$ the radial element and $d\theta^2$ represents the combined spherical metric. For $\epsilon < \delta$ let $g' = g$ on the complement of $A_\epsilon$ and on $A_{\epsilon'}$, $g'$ has constant curvature $-\rho^2$. We can achieve this with a warped product metric of the form $h(r)^2 dr^2 + dh^2 + (h(r)^2 - h(0)^2) d\theta^2$ for a convex function which interpolates between $\cosh(r)$ for $r > \text{diam}(A_\epsilon)$ and $\frac{\cosh(\rho r)}{\rho}$ for $r \leq \text{diam}(A_{\epsilon'})$ with uniformly bounded $\frac{h''}{h}$ and $\frac{h'''}{h}$. For any $\lambda > 0$ we can choose $\epsilon$ sufficiently close to $\delta$ so that $\text{Vol}_g(A_\epsilon) < \lambda$. Now $(M, \rho^2 g')$ has curvature bounded below by $-1$ and constant curvature $-1$ on the tube $A_{\epsilon'}$. If the injectivity radius of this rescaled metric is not sufficiently large on $A_{\epsilon'}$, we can replace $M$ by a cover so that the injectivity radius of $(A_{\epsilon'}, \rho^2 g')$ is sufficiently large to perform the operation described.
Note that since \((A_\nu, \rho^2 g')\) has constant curvature \(-1\), the degree of the cover is independent of \(\rho\). Also assuming that we have chosen an appropriate dimension \(n\), there is an exotic smooth \(n\)-sphere \(\sigma\) such that by Theorem 5.3, \(N = M \# \Sigma\) is not diffeomorphic to \(M\). Observe also that the connect sum and the procedure of controlling the curvature, Proposition 1.3 of \([FJ89a]\), are carried out precisely in a tube of bounded diameter with the metric isometric to the original in a neighborhood of the boundary. Since the \(\rho^2 g'\) diameter of \(A_\nu\) is unbounded as \(\rho \to \infty\), we conclude that there exists a metric, denoted by \(\rho^2 g_o\), on \(N\) which agrees with \(\rho^2 g'\) outside of \(A_\nu \# \Sigma\) and \(-\frac{1}{2} \leq K_{\rho^2 g_o} \leq \frac{1}{2}\) on \(A_\nu\). In particular \(-\frac{1}{2} \rho^2 K_{g_o}\) \(\leq -1\). Summarizing we have \(\text{Vol}_g(M) \leq \text{Vol}(N, g_o) + \lambda\), and \(N\) not diffeomorphic to \(N\). Now we can choose \(\lambda\) to be arbitrarily small which would violate Theorem 5.4 if \(C\) did not depend on \(\rho\).

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