A FAMILY OF CYCLIC QUARTIC FIELDS WITH EXPLICIT FUNDAMENTAL UNITS

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Abstract. We construct a family of quartic polynomials with cyclic Galois group and show that the roots of the polynomials are fundamental units or generate a subgroup of index 5.

The goal of this paper is to prove the following.

Theorem 1. Let $s$ be an integer such that $3s^2 - 4s + 4$ is a square. Let $K_s$ be the splitting field of

$$F_s(t) = t^4 + (4s^3 - 4s^2 + 8s - 4)t^3 + (-6s^2 - 6)t^2 + 4t + 1,$$

Then $\text{Gal}(K_s/\mathbb{Q})$ is cyclic of order 4. If $s^2 + 2$ is squarefree and $s \neq 0$, then $\pm 1$ and the roots of $F_s(t)$ generate either the unit group of the ring of algebraic integers in $K_s$ or a subgroup of index 5.

Computational evidence (see the end of Section 5) indicates that the index 5 case does not occur, but we have not yet proved this.

Families of cyclic quartic fields with explicit units have been studied in the past (for example, [4], [8], [9]), but it does not seem that this family has been studied previously. In [1], families of cyclic cubic fields were constructed and the method led naturally to studying integral points on a model of the elliptic modular surface $X(3)$. In the present case, we are led to study integral points on a degree 4 cover of the surface $X(2)$, but the Diophantine properties are not as transparent. The family we study lives above a singular fiber of $X(2)$. It would be interesting to know if there are other families living as curves on the surface.

1. The polynomials

As in [1], we start with the action of the Galois group given by a linear fractional transformation, but this time we take the matrix to have order 4 in $\text{PGL}_2(\mathbb{Z})$. Let $f, g$ be integers and let

$$M = \begin{pmatrix} f & -1 \\ f^2 + g^2 & -g \end{pmatrix}.$$ 

Then

$$M^2 = \frac{f - g}{2} \begin{pmatrix} f + g & -2 \\ f^2 + g^2 & -g - f \end{pmatrix}, \quad M^3 = \frac{(f - g)^2}{2} \begin{pmatrix} g & -1 \\ f^2 + g^2 & -f \end{pmatrix}.$$ 

The action of the Galois group is to be $\theta \mapsto M\theta$. We want $\theta$ to be a unit, so we take it to have norm 1:

$$\theta \cdot \frac{f\theta - 1}{f^2 + g^2\theta - g} \cdot \frac{(f + g)\theta - 2}{(f^2 + g^2)\theta - g - f} \cdot \frac{g\theta - 1}{f^2 + g^2\theta - f} = 1.$$
If \( fg(f + g) \neq 0 \), this relation can be rearranged to say that \( \theta \) is a root of

\[
(2) \quad t^4 \frac{(f^2 + g^2)^3 + 4(f^2 + g^2) + 16fg}{4fg(f + g)} + 3\left(\frac{(f^2 + g^2)^2 + 4}{4fg}\right)t^2 - \frac{(f + g)^4 - 4f^2g^2 + 4}{2fg(f + g)}t + 1 = 0.
\]

Since this is symmetric in \( f \) and \( g \), we make the substitutions \( s = f + g \) and \( p = fg \) to obtain

\[
(3) \quad t^4 - \frac{(s^2 - 2p)^3 + 4s^2 + 8p}{4sp}t^3 + \frac{3((s^2 - 2p)^2 + 4)}{4sp}t^2 - \frac{s^4 - 4p^2 + 4}{2sp} + 1.
\]

Let \( L = -s^4 - 4p^2 + 4 \). The polynomial becomes

\[
(4) \quad t^4 + (2s^3 + Ls^2 - 4ps + 2Lp)t^3 + (-3s^2 - 3Ls + 6p)t^2 + 2Lt + 1.
\]

Therefore, if \( L \in \mathbb{Z} \), or if \( 2L \in \mathbb{Z} \) and \( s \) is even, the roots of the polynomial are units. Of course, since \( s = f + g \) and \( p = fg \), there is the extra condition that \( s^2 - 4p \) is a square.

**Remark.** The divisibility condition \( 4sp \mid s^4 - 4p^2 + 4 \) that makes \( L \) integral is the same type of condition that appears in the cubic case, where the paper [1] has the condition \( fg \mid f^3 + g^3 + 1 \). These both seem to be analogues of the condition \( r \mid 4n \) for the Richaud-Degert real quadratic fields \( \mathbb{Q}(\sqrt{n^2 + r}) \) (see [12]).

Note that changing \( s \) to \(-s\) while holding \( p \) constant corresponds to changing the signs of \( f \) and \( g \). This has the effect of

\[
L \mapsto -L, \quad t \mapsto -t
\]

in (4). Therefore, we can for simplicity assume that \( L > 0 \) (the case \( L = 0 \) cannot occur).

The family considered in [4] (the “simplest quartic fields”) corresponds to the matrix \( M \) with \( f = 1 \) and \( g = -1 \). However, the relation [4] becomes trivial in this case and does not yield the polynomial defining the family.

### 2. The Surface

When the analogous construction was done for the cubic case in [1], the expression for one of the coefficients yielded an equation for \( X(3) \). In the present case, we are looking for integral points \((f, g, L)\) on the surface

\[
X : (f + g)^4 - 4f^2g^2 + 4 + 2Lfg(f + g) = 0.
\]

This is a double cover of the surface

\[
s^4 - 4p^2 + 4 + 2Lps = 0,
\]

which can be transformed (see, for example, [3]) to

\[
y^2 = (x + 4)(x - 4)(x + L^2).
\]

Therefore, \( X \) is a degree 4 cover of the elliptic surface \( X(2) \), whose fibers are Legendre elliptic curves. The bad fibers \( L = \pm 2 \) are the ones that play a role in what follows.

A computer search produced several pairs \((f, g)\) for which \( L \) is integral, and almost all of them had \( L = \pm 2 \).
As mentioned above, a simple transformation changes the examples with $L = -2$ into examples with $L = 2$.

### 3. A family of fields

From now on, we make the assumption:

$$L = 2.$$

Since $s^4 - 4p^2 + 4Lps + 4 = 0$ is a quadratic in $p$, the quadratic formula yields

$$p = \frac{Ls}{2} \pm \frac{1}{2} (s^2 + 2).$$

If $p = s + \frac{1}{2} (s^2 + 2)$, the polynomial in Equation (4) becomes $(t + 1)^4$, so we always take

$$p = s - \frac{1}{2} (s^2 + 2).$$

We now have the polynomial

$$F_s(t) = t^4 + (4s^3 - 4s^2 + 8s - 4)t^3 + (-6s^2 - 6)t^2 + 4t + 1,$$

with the side condition that

$$3s^2 - 4s + 4$$

is a square (this is a rewriting of $s^2 - 4p$). This condition immediately implies the following:

$$s \text{ is even}, \quad s^2 + 2 \equiv 2 \pmod{4}, \quad p \text{ is odd},$$

where the last follows from (5). Since $p$ is odd, we must have $f$ and $g$ are odd.

The original parameters $f$ and $g$ are given by

$$f, g = \frac{s \pm \sqrt{3s^2 - 4s + 4}}{2}.$$
Suppose now that Equation (7) holds. Then
\[ v^2 = 3s^2 - 4s + 4, \]
for some \( v \), which forces \( v = 2w \) and \( s = 2u \) for some \( u, w \). This yields
\[ (3u - 1)^2 - 3w^2 = -2. \]
The solutions are given by
\[ (3u - 1) \pm w\sqrt{3} = (-1)^{n+1}(1 + \sqrt{3})(2 + \sqrt{3})^n, \]
with \( n \in \mathbb{Z} \) (the first choice of signs is arbitrary; the second sign is chosen in order to make the right side congruent to \(-1 \mod \sqrt{3})\).

This gives the following values of \( s \):
\[ 4, -12, 48, -176, 660, -2460, 9184, -34272. \]
Every third value \((4, -176, 9184, \ldots)\) has \( s^2 + 2 \) divisible by \( 3^2 \). The other values listed yield squarefree values of \( s^2 + 2 \), although it is not known whether the sequence yields infinitely many values of \( s \) such that \( s^2 + 2 \) is squarefree. Questions of this type seem similar to questions about squarefree Mersenne numbers, most of which are unsolved. For the first 100 values of \( s \) (that is, for \( s \) arising from \( 1 \leq n \leq 100 \) in Equation (11)), all values of \( s^2 + 2 \) or \((s^2 + 2)/9\) are squarefree.

The discriminant of the polynomial \( F_s(t) \) in Equation (11) is
\[ 256(3s^2 - 4s + 4)^3(s^2 + 2)^3. \]
Since \( 3s^2 - 4s + 4 \) is a square, the discriminant is a square times \( s^2 + 2 \). But \( s^2 + 2 \) is never a square, so \( k_s = \mathbb{Q}(\sqrt{s^2 + 2}) \subseteq K_s \). Therefore, once we show that the Galois group is cyclic, we know that \( k_s \) is the unique quadratic subfield.

We first show that \( F_s \) is irreducible, then identify the Galois action.

**Lemma 3.** Let \(|s| \geq 3\). The roots of \( F_s(t) \) satisfy
\[ r_1 = -4s^3 + 4s^2 - 8s + 4 - (3/2)s^{-1} - (3/2)s^{-2} + \theta_1 s^{-4}, \text{ with } 1 \leq \theta_1 \leq 2, \]
\[ r_2 = \frac{1 + \sqrt{3}}{2} s^{-1} + \frac{3 + \sqrt{3}}{6} s^{-2} - \frac{1}{3 \sqrt{3}} s^{-3} + \theta_2 s^{-4}, \text{ with } -3/2 \leq \theta_2 \leq -1/2, \]
\[ r_3 = (1/2)s^{-1} + (1/2)s^{-2} - \theta_3 s^{-4}, \text{ with } 0 \leq \theta_3 \leq 1, \]
\[ r_4 = \frac{1 - \sqrt{3}}{2} s^{-1} + \frac{3 - \sqrt{3}}{6} s^{-2} + \frac{1}{3 \sqrt{3}} s^{-3} + \theta_2 s^{-4}, \text{ with } -1/2 \leq \theta_2 \leq 1/2. \]

**Proof.** Let \( \tau_1 = -4s^3 + 4s^2 - 8s + 4 - (3/2)s^{-1} - (3/2)s^{-2} \). Substitute \( \tau_1 + s^{-4} \) into \( F_s(t) \). The result is a degree 21 polynomial \( P_1(s) \) divided by \( s^{16} \). The real roots of \( P_1(s) \) all have absolute value less than 2, and \( P_1(s) \) has a positive top coefficient. Since \( P_1(s) \) is positive for large \( s \) and does not change sign in the interval \((2.1, \infty)\), we see that \( P_1(s) > 0 \) for \( s \geq 3 \). Similarly, \( P_1(s) < 0 \) when \( s \leq -3 \).

Now substitute \( \tau + 2s^{-4} \) into \( F_s(t) \). The result is a degree 21 polynomial \( Q_1(s) \) divided by \( s^{16} \). The real roots of \( Q_1(s) \) all have absolute value less than 1, and \( Q_1(s) \) has a negative top coefficient. It follows that \( Q_1(s) < 0 \) for \( s \geq 1 \) and \( Q_1(s) > 0 \) when \( s \leq -1 \).

Fix \( s \) with \( s \geq 3 \). Then \( F_s(\tau_1 + s^{-4}) > 0 > F_s(\tau_1 + 2s^{-4}) \). Therefore, there is a zero \( r_1 \) of \( F_s(t) \) that satisfies the stated conditions. The case where \( s \leq -3 \) is similar.

The proofs for \( r_2, r_3, r_4 \) are similar. \( \square \)
These expansions of \( r_1 \) and \( r_3 \) were found by letting \( s = 10^{100} \) and finding the roots of \( F_s(t) \) numerically. The coefficients of the expansions were then easy to deduce from the decimal expansions of the roots. For \( r_2 \) and \( r_4 \), the expansions of \( r_2 + r_4 \) and \((r_2 - r_4)/\sqrt{3}\) had simple forms, and the above were obtained from these.

If we take \( f = (s + \sqrt{3s^2 - 4s + 4})/2 \) and \( g = (s - \sqrt{3s^2 - 4s + 4})/2 \), then the linear fractional transformation \( M \) maps \( r_j \) to \( r_{j+1} \). Given the approximation to \( r_1 \), we could obtain the other approximations from the action of \( M \), but estimating the error terms would be harder.

**Lemma 4.** Let \( s \in \mathbb{Z} \). Then \( F_s(t) \) is irreducible in \( \mathbb{Q}[t] \).

**Proof.** The only possible rational roots of \( F_s \) are \( \pm 1 \). But \( F_s(\pm 1) \neq 0 \) when \( s \in \mathbb{Z} \). Therefore, \( F_s \) does not have a linear factor, so, if it factors, it must have two quadratic factors, and they must have integer coefficients. This means that \( r_2 + r_4 \) is an integer. Therefore, \( F_s \) cannot factor into quadratic factors.

**Lemma 5.** Let \( K_s \) be the splitting field of \( F_s \) and assume \( 3s^2 - 4s + 4 \) is a square. Then Gal\( (K_s/\mathbb{Q}) \) is cyclic, and the linear fractional transformation \( M \) gives the Galois action on the roots of \( F_s(t) \).

**Proof.** Since \( 3s^2 - 4s + 4 \) is a square, the parameters \( f \) and \( g \) exist (see Equation (8)). Let \( \theta \) be a root of \( F_s \). Then \( \theta \) satisfies Equation (11). Let \( \theta' = M\theta \), the result of applying the linear fractional transformation \( M \) to \( \theta \). Since \( M \) cyclically permutes the factors in Equation (11), we see that \( \theta' \) also satisfies this equation, and therefore \( F_s(\theta') = 0 \). Therefore, \( \mathbb{Q}(r_1) \) contains all the roots of \( F_s \). Since \( F_s \) is irreducible, it follows that the Galois group of \( F_s \) is cyclic of order 4 and is generated by \( M \).

4. **The discriminant**

**Proposition 6.** Suppose that \( s^2 + 2 \) is squarefree and \( 3s^2 - 4s + 4 \) is a square. Then the discriminant of \( K_s \) is \( 2^8(s^2 + 2)^3 \).

**Proof.** The discriminant of the polynomial \( F_s(t) \) is
\[
2^8(s^2 + 2)^3(3s^2 - 4s + 4)^3 = 2^8(s^2 + 2)^3(f - g)^6,
\]
where we have used Lemma 2 to obtain the second expression. We need to show that the factor \( (f - g)^6 \) can be removed. Let \( q \) be an odd prime dividing \( f - g \).

Since \( -4spL = s^2 - 4p^2 + 4 \), we cannot have \( f \equiv g \equiv 0 \pmod{q} \).

If \( fg(f + g) \equiv 0 \pmod{q} \) (equivalently, \( f \equiv 0 \pmod{q} \)), then Equation (2) becomes
\[
F_s(t) \equiv (t - 1/f)^3(t - f^3) \pmod{q}.
\]
Also, Lemma 2 implies that
\[
(11) \quad 0 \equiv (f - g)^2 = 3s^2 - 4s + 4 = 3(f + g)^2 - 4(f + g) + 4 \equiv 12f^2 - 8f + 4 \pmod{q}.
\]
We claim that the roots \( 1/f \) and \( f^3 \) of \( F_s(t) \pmod{q} \) are distinct. Suppose \( 1/f \equiv f^3 \pmod{q} \). The resultant of \( f^4 - 1 \) and \( 12f^2 - 8f + 4 \) is \( 2^{13} \cdot 3 \), so we must have \( q = 3 \).
Lemma 7. Let \( K \) be a prime of \( K \), dividing \( q \) and let \( I \) be the inertia subgroup of \( \operatorname{Gal}(K_{s}/Q) \) for \( q \). If \( q \) divides the discriminant of \( K \), then \( \sigma^{2} \in I \), where \( \sigma \) generates \( \operatorname{Gal}(K_{s}/Q) \). Let \( r_{j} \equiv f^{3} \pmod{q} \). Then the other three roots are congruent to \( 1/f \pmod{q} \). But \( \sigma^{2} \in I \) means that \( f^{3} \equiv r_{j} \equiv \sigma^{2}(r_{j}) \equiv 1/f \pmod{q} \), contradicting the fact that \( f \not\equiv 1/f^{3} \pmod{q} \). Therefore, \( q \) does not divide the discriminant of \( K_{s} \), so \( f - g \) contributes no odd prime factors to the discriminant of \( K_{s} \).

We have proved that the discriminant \( D \) of \( K_{s} \) divides a power of \( 2 \) times \( (s^{2}+2)^{3} \).

The subfield \( k_{s} = Q(\sqrt{s^{2}+2}) \subset K_{s} \) has conductor \( 4(s^{2}+2) \), so \( s^{2}+2 \) is squarefree and congruent to \( 2 \mod 4 \). Let \( \chi \) be a Dirichlet character of order 4 attached to \( k_{s} \). Then \( \chi^{2} \) is the quadratic character attached to \( k_{s} \). Since \( \chi^{2} \) has conductor \( 4(s^{2}+2) \), it follows that \( \chi \) and \( \chi^{-1} \) have conductor divisible by \( 4(s^{2}+2) \). The conductor-discriminant formula implies that \( D \) is divisible by \( 4^{3}(s^{2}+2)^{3} \). We have therefore proved that \( D \) is a power of \( 2 \) times \( (s^{2}+2)/2)^{3} \).

Since \( 2 \) ramifies in \( k_{s}/Q \) and \( K_{s}/Q \) is cyclic, \( 2 \) is totally ramified in \( K_{s}/Q \). This means that \( \chi \) is the product of a character of conductor 16 and a character of odd conductor. The conductor-discriminant formula implies that \( 2^{11} \) is the exact power of \( 2 \) dividing \( D \). Since \( (s^{2}+2)^{3} \) contributes \( 2^{3} \), this completes the proof.

Remark. In order to determine the ramification and discriminant of \( K_{s} \), we could consider the extension \( K_{s}(i)/Q(i) \). Order the roots \( r_{1}, r_{2}, r_{3}, r_{4} \), so that \( Mr_{j} = r_{j+1} \). Let \( \rho = r_{1} + r_{2}i - r_{3} - r_{4}i \).

Computationally, it appears that

\[
\rho^{3} = -2^{6}i(f + gi)^{3} \frac{\pi^{3} \pi}{\bar{\pi}},
\]

where \( \pi = (f + 1)/2 - i(g + 1)/2 \). This would suffice to remove the factor \( (3s^{2} - 4s + 4)^{3} \), since \( \sqrt{p} \) generates the extension \( K(i)/Q(i) \), and since the odd parts of the discriminants of \( K_{s}(i)/Q(i) \) and \( K_{s}/Q \) are equal. However, the verification of Equation (12) seems to be potentially quite involved, which is why we had the incentive to find the above proof.

5. Fundamental units

The purpose of this section is to prove that \( \pm 1 \) and the roots of \( F_{s}(t) \) generate the units of \( K_{s} \). Throughout this section, we assume that \( 3s^{2} - 4s + 4 \) is a square.

Lemma 7. Let \( s \neq 0 \) and suppose \( s^{2} + 2 \) is squarefree. Then

\[
\epsilon = -r_{1}r_{3} = s^{2} + 1 + \sqrt{3s^{2} + 2}
\]

is the fundamental unit of the ring of integers of \( Q(\sqrt{s^{2}+2}) \).

Proof. Since \( s^{2} + 2 \not\equiv 1 \pmod{4} \), the fundamental unit is in \( \mathbb{Z}[\sqrt{s^{2}+2}] \) (that is, there is no 2 in the denominator). If \( \epsilon_{0} = a + b\sqrt{s^{2}+2} \) is the fundamental unit, so \( a, b > 0 \), then \( \epsilon_{0}^{2} > s^{2} + 1 + s\sqrt{s^{2}+2} = \epsilon \). Since \( \epsilon \) is a power of \( \epsilon_{0} \), we must have \( \epsilon = \epsilon_{0} \).

From Lemma 3 we see that \( r_{1}r_{3} \approx -2s^{2} \). In particular, \( 1 < -r_{1}r_{3} < \epsilon \), so \( r_{1}r_{3} = -\epsilon \). \( \square \)
Note: We did not need to know the ordering of the $r_j$ under the Galois group to obtain this result, since $-r_1r_3$ is the only combination with the same approximate size as $\epsilon$. In fact, once we know this, if $\sigma$ is a generator of $\text{Gal}(K_k/\mathbb{Q})$ then $\sigma^2$ maps $r_1$ to $r_3$, and hence $\sigma$ or $\sigma^{-1}$ permutes the roots $r_1, r_2, r_3, r_4$ cyclically (that is, $r_j \mapsto r_{j+1}$). Of course, we know that $M = \sigma$ or $\sigma^{-1}$, depending on the choice of signs in Equation (8).

**Proposition 8.** Let $K$ be a totally real number field with $\text{Gal}(K/\mathbb{Q})$ cyclic of order 4. Let $R_K$ and $D_K$ be the regulator and discriminant of $K$. Let $k$ be the quadratic subfield of $K$ and let $\epsilon$ and $d_k$ be the fundamental unit and discriminant of $k$. Then

$$\frac{1}{4} \log^2 \left( \frac{D_K}{16d_k^2} \right) \leq \frac{R_K}{\log \epsilon}.$$ 

If $d_k > 150$, then

$$\frac{1}{4} \log^2 \left( \frac{D_K}{4.84d_k^2} \right) < \frac{R_K}{\log \epsilon}.$$ 

**Proof.** This result is implicit in [2], [4], and [14], for example. However, since it does not seem to be explicit in the literature, we include the proof for the convenience of the reader.

The result is easily verified for $\mathbb{Q}(\zeta_{16})^+$, the maximal real subfield of the 16th cyclotomic field. In all other cases, an odd prime ramifies in $K/\mathbb{Q}$, so we may assume that $K/k$ ramifies at some prime of $k$ that divides an odd prime of $\mathbb{Z}$.

Let $E_K$ and $E_k$ be the units of $K$ and $k$, and let $\sigma$ generate $\text{Gal}(K/\mathbb{Q})$. Then $E_K/E_k$ is a $\mathbb{Z}[\iota]$-module, where $i$ acts as $\sigma$. We claim that this module is torsion-free. If $\alpha \in \mathbb{Z}[\iota]$ maps $u \in E_K$ into $E_k$, then so does $\text{Norm}(\alpha)$, so it suffices to show that the $\mathbb{Z}$-torsion is trivial.

If $u \in E_K$ and $u^n \in E_k$, then $(\sigma^2u)^n = u^n$, so $\sigma^2u = \pm u$, since $k$ is real. Therefore, $\sigma^2(u^2) = u^2$, so $u^2 \in E_k$. Therefore, $k(u)/k$ ramifies at most at the primes above 2, hence is trivial since $K/k$ is assumed to ramify at an odd prime. Therefore, $u \in E_k$, so $E_K/E_k$ is torsion-free.

Since $E_K/E_k$ has $\mathbb{Z}$-rank 2, it has $\mathbb{Z}[\iota]$-rank 1, so there is a unit $\eta$ that generates it as a $\mathbb{Z}[\iota]$-module. Let $\eta^{1+\sigma^2} = \delta \in E_k$, and let $\eta' = \sigma(\eta)$. Then \{±1, $\epsilon$, $\eta$, $\eta'$\} generates $E_K$ as a $\mathbb{Z}$-module.

A calculation shows that the regulator of $K$ is

$$R_K = 2 \log(\epsilon) \left( \log |\eta| - \frac{1}{2} \log |\delta| \right)^2.$$ 

The different of $K/k$ divides $\eta - \sigma^2(\eta) = \eta - \delta/\eta$, so the discriminant of $K/k$ divides

$$\text{Norm}_{K/k}(\eta - \delta/\eta) = -(|\eta - \delta/\eta|)^2.$$ 

Since $D_K = d_k^2 \text{Norm}_{k/\mathbb{Q}}(D_{K/k})$, we find that

$$\frac{D_K}{d_k^2} \text{ divides } (\eta - \delta/\eta)^2 (\eta' \pm 1/(\delta \eta'))^2 \leq (x + 1/x)^2 (y + 1/y)^2;$$

where $x = \text{Max}(|\eta|/|\delta|^{1/2}, |\delta|^{1/2}/|\eta|)$ and $y = \text{Max}(|\eta'|/|\delta|^{1/2}, 1/|\delta|^{1/2} |\eta'|)$.

The conductor-discriminant formula implies that $d_k^2 \mid D_K$. If $1 \leq x, y \leq \sqrt{10}$, then

$$d_k \leq (x + 1/x)^2 (y + 1/y)^2 < 150.$$
Therefore, if \( d_k > 150 \) then at least one of \( x, y \) is larger than \( \sqrt{10} \). If \( x > \sqrt{10} \), then \( x + 1/x < 1.1x \). If \( 1 \leq x \leq \sqrt{10} \), then \( x + 1/x \leq 2x \). Therefore,
\[
(x + 1/x) (y + 1/y) < 2.2xy,
\]
so
\[
\log(D_K/d_k^2) \leq \log (x + 1/x)^2 (y + 1/y)^2 < 2 (\log 2.2 + \log x + \log y)
\]
\[
\leq 2 \left( \log 2.2 + \sqrt{2} (\log^2 x + \log^2 y)^{1/2} \right) \quad \text{(Cauchy-Schwarz)}
\]
\[
= 2 \left( \log 2.2 + (R_K/\log \epsilon)^{1/2} \right).
\]
This yields the last statement of the proposition. Note that by increasing the lower bound for \( d_k \), we could replace 4.84 by any number larger than 4.

If we do not require \( d_k > 150 \), then a slightly simpler argument works (we thank Stéphane Louboutin for pointing this out): We have \( x, y \geq 1 \), so \( x+1/x \leq x+1 \leq 2x \) and similarly for \( y \). Therefore,
\[
(x + 1/x) (y + 1/y) \leq 4xy.
\]
The above argument yields
\[
\frac{1}{4} \log^2 \left( \frac{D_K}{16d_k^4} \right) \leq \frac{R_K}{\log \epsilon}.
\]
This inequality suffices for our purposes. \( \square \)

**Lemma 9.** Assume \( s^2 + 2 \) is squarefree. Let \( E \) be the units of \( K_s \) and let \( U \) be the subgroup generated by \( \pm 1 \) and \( r_1, r_2, r_3, r_4 \). Then \( [E : U] \neq 2, 3, 4, 6, 7, 8 \).

**Proof.** Let \( E_k \) be the units of \( \mathbb{Q}(\sqrt{s^2 + 2}) \). By Lemma 7, \( E_k \subset U \). Let \( E = E/E_k \) and \( U = U/E_k \). Then \( [E : U] = [E : U] \). As in the proof of Proposition 5, \( E \cong \mathbb{Z}[i] \).

The subgroup \( U \) is an ideal of \( \mathbb{Z}[i] \) under this isomorphism.

If \( [E : U] = 2 \), then there exists \( \eta \) that generates \( E \) as a \( \mathbb{Z}[i] \)-module and such that \( (1 + i)\eta \equiv r_1 \mod E_k \). This means that there exists \( \delta \in E_k \) such that \( \eta^{1+\sigma} = r_1 \delta \).

Since \( \sigma^2 \) fixes \( \delta \), we have
\[
\pm 1 = \eta^{1+\sigma+\sigma^2+\sigma^3} = r_1 r_3 \delta^2 = -\epsilon \delta^2.
\]
But \( \epsilon \) is the fundamental unit of \( \mathbb{Q}(\sqrt{s^2 + 2}) \), so this is impossible.

Now suppose \( [E : U] = 4 \). The only ideal of index 4 in \( \mathbb{Z}[i] \) is \( (2) \), so there exists \( \eta \in E \) such that \( \eta^2 = r_1 \delta \), with \( \delta \in E_k \). Taking norms to \( k_s \) yields \( N(\eta)^2 = -\epsilon \delta^2 \).

This is impossible because \( \epsilon \) is the fundamental unit of \( k_s \).

If \( [E : U] = 8 \), then there exists \( \eta \) such that \( \eta^{2(1+\sigma)} = r_1 \delta \), with \( \delta \in E_k \). Taking norms from \( K_s \) to \( k_s \), we find
\[
1 = (\eta^{1+\sigma+\sigma^2+\sigma^3})^2 = r_1^{1+\sigma} \delta^{1+\sigma^2} = -\epsilon \delta^2,
\]
which is impossible.

Finally, every ideal in \( \mathbb{Z}[i] \) has index that is a norm from \( \mathbb{Z}[i] \) to \( \mathbb{Z} \). Since 3, 6, 7 are not norms, \( [E : U] \neq 3, 6, 7 \). \( \square \)

We can now show that \( \pm 1 \) and \( r_1, r_2, r_3, r_4 \) generate the units of \( K_s \). We assume that \( |s| \geq 10^5 \). The cases where \( |s| < 10^5 \) can be checked individually.
Let $R$ be the regulator of $K_s$ and let $R'$ be the regulator for the group $U$ in Lemma 5. Then $R'/R = [E : U]$. We have
\[
R' = \pm \det \begin{pmatrix}
\log |r_1| & \log |r_2| & \log |r_3| \\
\log |r_2| & \log |r_3| & \log |r_4| \\
\log |r_3| & \log |r_4| & \log |r_1| \\
\end{pmatrix}.
\]
This equals (see Lemma 5.26(c))
\[
\frac{1}{4} \left( \log |r_1| + i \log |r_2| - \log |r_3| - i \log |r_4| \right) \left( \log |r_1| - \log |r_2| + \log |r_3| - \log |r_4| \right)
\times \left( \log |r_1| - i \log |r_2| - \log |r_3| + i \log |r_4| \right)
= \frac{1}{4} \left( \log^2 |r_1/r_3| + \log^2 |r_2/r_4| \right) (2 \log \epsilon).
\]
Lemma 3 implies that
\[
R'/\epsilon \leq \frac{1}{2} \left( \log^2 (9s^4) + \log^2 (4) \right) < 9 \log^2 |s|
\]
when $|s| \geq 10^5$. Also,
\[
\log^2 \left( \frac{D_{K_s}}{16d_{K_s}^2} \right) = \log^2 (s^2 + 2) \geq 4 \log^2 |s|
\]
when $|s| \geq 1$. Proposition 8 implies that
\[
\log^2 |s| \leq \frac{R'/[E : U]}{\log \epsilon} < \frac{9}{[E : U]} \log^2 |s|.
\]
Therefore,
\[
[E : U] < 9.
\]
By Lemma 4 $[E : U] \neq 2, 3, 4, 6, 7, 8$, so $[E : U] = 1$ or 5. This completes the proof of Theorem 1.

When $s \equiv 4, 5 \pmod{9}$, we have $s^2 + 2 \equiv 0 \pmod{9}$, so $s^2 + 2$ is not squarefree. However, if $(s^2 + 2)/9$ is squarefree, then the effect on $D_{K_s}$ is small enough that the above proof shows that we still obtain either the full group of units or a subgroup of index 5, except when $s = 4$ (where the index $[E_{K_s} : U] = 40$). Computational evidence suggests that the index 5 case does not occur.

Since the units of $K_s$ are fairly small, we expect the class numbers to be large. Here is a table of the class groups for the first few values of $s$ (for the class groups, $a \times b$ means $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z}$). These calculations, and others in this paper, were done using GP/PARI.

For all of the examples in the above table, $-1$ and the roots of the polynomial $F_s(t)$ generate the full group of units for the ring of algebraic integers of the corresponding field, except for $s = 4$, where they generate a subgroup of index 40. In this case, $s^2 + 2 = 18$ is not squarefree, and the unit $17 + 4\sqrt{18} = (s^2 + 1) + s\sqrt{s^2 + 2}$ is the fourth power of the fundamental unit $1 + \sqrt{2}$ of the quadratic subfield.

The growth of the class number can be made explicit. Since $K_s/k_s$ is ramified at 2, the class number of $k_s$ divides the class number of $K_s$. Since the class number of $\mathbb{Q}(\sqrt{s^2 + 2})$ goes to $\infty$ as $s \to \infty$ (this can be made explicit with at most one possible exception; see [12]), the class number of $K_s$ also goes to $\infty$. However, the potential effect of Siegel zeroes for quadratic fields can be overcome since we have a quartic field, so we obtain a result with no exceptions.
Proposition 10. Assume $s^2 + 2$ is squarefree and $3s^2 - 4s + 4$ is a square. Let $h$ be the class number of $K_s$ and let $h_2$ be the class number of $k_s$. If $|s| \geq 10^5$, then
\[
\frac{h}{h_2} \geq \frac{1}{450} \frac{s^2 + 2}{\log^2(|s|) \log^2(8(s^2 + 2)/\pi)}.
\]

Proof. Let $\chi$ be a quartic Dirichlet character associated with $K_s$. Then
\[
\frac{8hR}{\sqrt{D_{K_s}}} = L(1, \chi)L(1, \chi^2)L(1, \chi^3).
\]
We have
\[
L(1, \chi^2) = \frac{2h_2 \log \epsilon}{\sqrt{d_{K_s}}}.
\]
Louboutin [11] has shown that for a non-quadratic primitive Dirichlet character $\chi$ of conductor $c \geq 90000$,
\[
|L(1, \chi)| \geq \frac{1}{10 \log(c/\pi)}.
\]
Therefore,
\[
h \geq \frac{1}{400} \left( \frac{D_{K_s}}{d_{K_s}} \right)^{1/2} \frac{\log \epsilon}{R} \frac{1}{\log^2(c/\pi)},
\]
where $c = 8(s^2 + 2)$. By Equation (13), $R/\log \epsilon \leq R'/\log \epsilon \leq 9 \log^2 |s|$. Putting these together, we obtain the result. \qed

It follows that $h > 1$ when $|s| \geq 10^5$. Since we have computed the class number of $K_s$ for $|s| < 10^5$, we obtain as a corollary that $K_s$ has class number 1 only for $s = 4$.

Stéphane Louboutin pointed out to us the following: Let $\chi$ be an even Dirichlet character of conductor $f > 1$. Then there is the upper bound (see [10])
\[
|L(1, \chi)| \leq \frac{1}{2} (c + \log f),
\]
where \( c = 2 + \gamma - \log(4\pi) \approx 0.04619 \cdots \) (and where \( \gamma = 0.577 \cdots \)). Let \( K \) be a totally real quartic field of conductor \( f_K \) corresponding to the quartic character \( \chi_K \), and let \( k \) be its quadratic subfield. Then

\[
h_K/h_k = \frac{f_K}{4R/\log \epsilon} |L(1, \chi_K)|^2 \leq \left( \frac{c + \log(f_K)}{2 \log(f_K/16)} \right)^2 f_K,
\]

where we have used Proposition \( \text{[8]} \) to bound \( R/\log \epsilon \). Therefore, \( h_K/h_k < f_K \) when \( f_K > 256\epsilon^c \approx 268.01 \), and a quick search using GP/PARI \( \text{[3]} \) shows that \( h_K < f_K \) also when \( f_K \leq 268 \). Since \( f_K = 8(s^2+2) \) in the situation of Proposition \( \text{[10]} \), we see that the lower bound estimate given there is, up to log factors, the correct order of magnitude.

6. Back to Cubics

The construction of \( F_s(t) \) was inspired by the cubic case. See \( \text{[1]} \), which starts with the element

\[
\begin{pmatrix} f & -h \\ (f^2+g^2-fg)/h & -g \end{pmatrix} \in \text{PGL}_2(\mathbb{Q}),
\]

where \( f, g, h \) can be taken to be distinct integers with \( h \neq 0 \). Assume that the associated linear transformation gives the action of a Galois group on a number \( \theta \), and assume that \( \theta \) has norm 1 to \( \mathbb{Q} \). Then we obtain the equation

\[
\theta \cdot \frac{f\theta - h}{\theta(f^2+g^2-fg)/h} \cdot \frac{g\theta - h}{\theta(f^2+g^2-fg)/h} = 1,
\]

which can be rearranged to say that \( \theta \) is a root of

\[
t^3 + 3(f^2+g^2-fg)h^{-2}t^2 + \lambda t - 1,
\]

where \( \lambda = (f^3+g^3+h^3)/(fg^3) \). If \( h = 1 \) and \( \lambda \in \mathbb{Z} \), we have a polynomial with integral coefficients. Therefore, we want integral points \( (x, y, 1) \) on the elliptic surface \( x^3 + y^3 + 1 = \lambda xy \), which is the elliptic modular surface \( X(3) \).

Following the lead of the quartic case, we look at the singular fiber \( \lambda = 3 \). If \( f \) is an integer, we need \( g \) to be a root of

\[
X^3 - 3fX + f^3 + 1 = (X + f + 1)(X^2 - (f + 1)X + f^2 - f + 1).
\]

When \( f \neq 1 \), the second factor is irreducible over \( \mathbb{Q} \), so we must have \( g = -f - 1 \). This yields the family of polynomials

\[
G_f(t) = t^3 + (9f^2 + 9f + 6)t^2 + 3t - 1,
\]

which could be regarded as the analogue of our family of quartic polynomials. This cubic family appears in \( \text{[7]} \).

Let \( r \) be a root of the polynomial \( x^3 + (3f+3)x^2 + 3fx - 1 \). As pointed out in \( \text{[7]} \), \( -r^3 - r \) is a root of \( G_f(t) \). This can easily be verified by representing \( r \) by the matrix

\[
A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -3f - 3 \\ 0 & 1 & -3f \end{pmatrix},
\]

then computing the characteristic polynomial of \( -A^2 - A \).

The fields obtained from the polynomials \( G_f(t) \) are some of Shanks’s “simplest cubic fields.” However, \(-1\) and the roots of \( G_f(t) \) generate a subgroup of index 3 in the group of units generated by the roots of the polynomial \( x^3 + (3f+3)x^2 + 3fx - 1 \).
In the cubic case, there are also many families of polynomials corresponding to curves on the surface $X(3)$ that do not lie in the singular fiber $\lambda = 3$ (see [1]). It would be interesting to find similar families in the quartic case.

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