The Low Energy Spectrum of Trapped Bosons in the Gross-Pitaevskii Regime

Christian Brennecke

Institute for Applied Mathematics, University of Bonn, Endenicher Allee 60, 53115 Bonn, Germany

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Abstract

Bogoliubov Theory\cite{9} provides important predictions for the low energy properties of the weakly interacting Bose gas. Recently, Bogoliubov’s predictions could be justified rigorously in\cite{7} for translation invariant systems in the Gross-Pitaevskii regime, where \(N\) bosons in \(\Lambda = [0; 1]^3 \subset \mathbb{R}^3\) interact through a potential whose scattering length is of size \(N^{-1}\). In this note, we review recent results from\cite{14}, a joint work with B. Schlein and S. Schraven, which extends the analysis of\cite{7} to systems of bosons in \(\mathbb{R}^3\) that are trapped by an external potential.

1 Introduction and Main Results

The rigorous understanding of low energy properties of dilute Bose gases has been an active field of research in the last decades after the first experimental realization of Bose-Einstein condensates in trapped atomic gases\cite{2,15}. As a mathematical model for such experimental setups, we consider systems of \(N\) trapped bosons moving in \(\mathbb{R}^3\) in the Gross-Pitaevskii regime. The energy of the system is described by

\[
H_N = \sum_{j=1}^{N} (-\Delta_{x_j} + V_{\text{ext}}(x_j)) + \sum_{1 \leq i < j \leq N} N^2 V(N(x_i - x_j)),
\]

acting in a dense subspace of \(L^2_{\text{loc}}(\mathbb{R}^3)\), the subspace of \(L^2(\mathbb{R}^{3N})\)-functions which are invariant under permutations of the \(N\) particle coordinates. We assume that the trapping potential \(V_{\text{ext}} \in L^\infty_{\text{loc}}(\mathbb{R}^3)\) is such that \(V_{\text{ext}}(x) \to \infty\) for \(|x| \to \infty\) and that \(V \in L^3(\mathbb{R}^3)\) is pointwise non-negative, radial and of compact support.

We are interested in the computation of the low energy spectrum of \(H_N\) in the limit \(N \to \infty\). Notice that, due to the presence of a trapping potential, the particles typically move in a region of volume \(O(1)\). By a simple scaling argument, the system is equivalent
to a system of \( N \) particles interacting through the unscaled interaction \( V \) and trapped in a region of volume \( \mathcal{O}(N^3) \). That is, the Gross-Pitaevskii limit can be thought of as a joint thermodynamic and low density limit (with the particle density of size \( N^{-2} \)). In particular, in view of Bogoliubov theory \([9]\), one may expect that the low energy spectrum depends on the interaction to leading order only through its scattering length.

Recall that the scattering length \( a_0 \) of \( V \) is defined through the solution \( f \) of the zero energy scattering equation

\[
\left(-\Delta + \frac{1}{2} V \right)f = 0 \quad \text{with} \quad \lim_{|x| \to \infty} f(x) = 1. \tag{1.2}
\]

For \( x \in \mathbb{R}^3 \) outside the support of \( V \), \( f \) is explicitly given by

\[
f(x) = 1 - \frac{a_0}{|x|}
\]

for some constant \( a_0 \), the \textit{scattering length} of \( V \). A simple computation shows that

\[
a_0 = \frac{1}{8\pi} \int_{\mathbb{R}^3} dx \frac{V(x)}{f(x)}
\]

and that the solution of \((1.2)\) with the rescaled potential \( N^2 V(N.) \) is equal to \( f(N.) \), i.e.

\[
\left(-\Delta + \frac{1}{2} N^2 V(N.) \right)f(N.) = 0.
\]

This implies that the scattering length of the interaction \( N^2 V(N.) \) is equal to \( a_0 N^{-1} \). The size \( N^{-1} \) of the scattering length characterizes the Gross-Pitaevskii regime.

The leading order contribution to the ground state energy \( E_N \) of \( H_N \) is well-known and has been derived in \([29, 28, 31]\), showing that

\[
\lim_{N \to \infty} \frac{E_N}{N} = \inf_{\psi \in L^2(\mathbb{R}^3): \|\psi\|_2 = 1} \mathcal{E}_{GP}(\psi).
\]

Here, \( \mathcal{E}_{GP} \) denotes the Gross-Pitaevskii functional which is defined by

\[
\mathcal{E}_{GP}(\psi) = \int_{\mathbb{R}^3} \left( |\nabla \psi(x)|^2 + V_{\text{ext}}(x)|\psi(x)|^2 + 4\pi a_0 |\psi(x)|^4 \right) dx.
\]

By standard arguments, \( \mathcal{E}_{GP} \) admits a unique normalized, strictly positive minimizer which we denote from now on by \( \varphi_0 \in L^2(\mathbb{R}^3) \). It solves the Euler-Lagrange equation

\[
-\Delta \varphi_0 + V_{\text{ext}} \varphi_0 + 8\pi a_0 |\varphi_0|^2 \varphi_0 = \varepsilon_{GP} \varphi_0, \quad \text{where} \quad \varepsilon_{GP} = \mathcal{E}_{GP}(\varphi_0) + 4\pi a_0 \|\varphi_0\|_4^4.
\]

In order to determine the next to leading order contribution to the ground state energy as well as the excitation energies of \( H_N \), we need to use the fact that approximate ground states exhibit complete Bose-Einstein condensation (BEC). Indeed, it follows from \([27, 28, 31]\) that every sequence of normalized wave functions \( \psi_N \in L^2(\mathbb{R}^{3N}) \) with

\[
\lim_{N \to \infty} \frac{1}{N} \langle \psi_N, H_N \psi_N \rangle = \mathcal{E}_{GP}(\varphi_0),
\]
that is, every sequence of approximate ground states, exhibits complete Bose-Einstein condensation into the minimizer $\varphi_0$ of $\mathcal{E}_{GP}$. Mathematically, this means that the one-particle reduced density $\gamma_N^{(1)} = \text{tr}_{2,\ldots,N} |\psi_N\rangle\langle \psi_N|$ associated with $\psi_N$ satisfies

$$\lim_{N \to \infty} \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle = 1. \tag{1.3}$$

Notice that this is equivalent to $\lim_{N \to \infty} \gamma_N^{(1)} = |\varphi_0\rangle\langle \varphi_0|$ in the trace class topology.

In the sequel, we make use of a quantitative version of (1.3) with optimal rate of convergence that follows from [13]. More precisely, the main result of [13] is that (under the assumptions (1) and (2) in (1.6) stated below) there exists some $C > 0$ such that

$$H_N \geq N \mathcal{E}_{GP}(\varphi_0) + C^{-1} \sum_{j=1}^{N} (1 - |\varphi_0\rangle\langle \varphi_0|)_{x_j} - C. \tag{1.4}$$

In particular, if $\psi_N \in L^2(R^{3N})$ is a normalized sequence of approximate ground states such that $\langle \psi_N, H_N \psi_N \rangle \leq N \mathcal{E}_{GP}(\varphi_0) + \zeta$ for some $\zeta > 0$, then $\gamma_N^{(1)}$ satisfies

$$1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle \leq C \frac{(C + \zeta)}{N}.$$ 

The lower bound (1.4) generalizes the main results of [5, 8], dealing with translation invariant systems, to systems of trapped particles in $R^3$. For sufficiently small $a_0$, (1.4) has previously been proved in [30] (adapting a complete-the-square argument of [11] to systems of trapped bosons) and in [22] (for translation invariant systems, simplifying the arguments of [5]). The bound (1.4) also follows from [32] (with no additional smallness restriction on the size of $a_0$) on which we comment briefly below. For further recent results on BEC, including regimes beyond the Gross-Pitaevskii scaling, see [1, 18].

To state our main results, we denote by $H_{GP}$ the self-adjoint operator

$$H_{GP} = -\Delta + V_{ext} + 8\pi a_0 \varphi_0^2 - \varepsilon_{GP} \tag{1.5}$$

and we make our assumptions on $V$ and $V_{ext}$ more precise:

(1) $V \in L^3(R^3), V(x) \geq 0$ for a.e. $x \in R^3, V(x) = V(y)$ if $|x| = |y|$, $V$ compactly supported,

(2) $V_{ext} \in C^1(\mathbb{R}^3; [0; \infty)), V_{ext}(x) \to \infty$ as $|x| \to \infty$,

$\exists C > 0 \forall x, y \in R^3 : V_{ext}(x + y) \leq C(V_{ext}(x) + C)(V_{ext}(y) + C)$,

$\nabla V_{ext}$ has at most exponential growth as $|x| \to \infty$,

(3) $(H_{GP} + 1)^{-3/4 - \varepsilon} e^{-\alpha |x|}$ is a Hilbert-Schmidt operator, for all $\varepsilon > 0$ and some $\alpha > 0$.

**Theorem 1.1.** ([14, Theorem 1.1]). Let $V, V_{ext}$ and $H_{GP}$ satisfy (1.6). Then, there exists $\rho > 0$ so that, in the limit $N \to \infty$, the ground state energy $E_N$ of $H_N$ is given by

$$E_N = N \mathcal{E}_{GP}(\varphi_0) - 4\pi a_0 \| \varphi_0 \|^4 + E_{Bag} + O(N^{-\rho}), \tag{1.7}$$
where $E_{\text{Bog}}$ is the finite constant (independent of $N$)

$$E_{\text{Bog}} = \frac{1}{2} \lim_{\delta \to 0} \left\{ \text{tr}_{\perp \Phi_0} \left[ \left( H_{GP}^{1/2} (H_{GP} + 16\pi a_0 \mathbb{1}_\delta \varphi_0^2) H_{GP}^{1/2} \right)^{1/2} - H_{GP} - 8\pi a_0 \mathbb{1}_\delta \varphi_0^2 \right] ight. 

+ \left. \frac{(8\pi a_0)^2}{2} \text{tr} \left[ \varphi_0^2 \mathbb{1}_\delta - \frac{1}{\Delta} \mathbb{1}_\delta \varphi_0^2 \right] \right\}$$

(1.8)

with $\mathbb{1}_\delta$ denoting the approximation of the identity with $\mathbb{1}_\delta (x, y) = (2\pi \delta)^{-3/2} e^{-(x-y)^2/2\delta^2}$ and where $\text{tr}_{\perp \varphi_0}$ denotes the trace over the orthogonal complement $\{ \varphi_0 \} \perp$.

Moreover, the spectrum of $H_N - E_N$ below a threshold $\zeta > 0$ (assuming $\zeta \leq C N^{\rho/5 - \epsilon}$ for some sufficiently small $\epsilon > 0$) consists of finite sums of the form

$$\sum_{i=1}^\infty n_i e_i + \mathcal{O}(N^{-\rho}(1 + \zeta^5 + N^{-5\rho} \zeta^{30})), \quad (1.9)$$

where $(e_j)_{j \in \mathbb{N}}$ denote the eigenvalues of the operator $E$ which is defined by

$$E = \left( H_{GP}^{1/2} (H_{GP} + 16\pi a_0 \varphi_0^2) H_{GP}^{1/2} \right)^{1/2}$$

(1.10)

and where $n_i \in \mathbb{N}$ with $n_i \neq 0$ for finitely many $i \in \mathbb{N}$.

Remarks:

1) The form (1.9) of the excitation spectrum of $H_N$ was conjectured in [21], providing results comparable to Theorem 1.1 in the mean field limit (see also [37, 16, 26] for previous and related results on the derivation of the excitation spectrum for mean field systems of bosons). In the mean field regime, more precise expansions in powers of $N^{-1}$ for the ground state energy and low energy excitation energies have been obtained in [35, 10]. For further background and results on mean field systems, see for instance the recent survey article [36].

2) Theorem 1.1 generalizes the main results of [7] (dealing with translation invariant systems) to systems of trapped bosons. On a formal level, one can recover the results of [7] by setting $V_{\text{ext}} = 0$ in $\Lambda = [0; 1]^3$, $V_{\text{ext}} = \infty$ otherwise and by imposing periodic boundary conditions in $\Lambda$. In this setting, the condensate wave function is described by $\varphi = 1_{\Lambda \in L^2(\Lambda)}$ and $H_{GP} = -\Delta$. In particular, $H_{GP}$ commutes with the multiplication operator $16\pi a_0 \varphi_0^2 = 16\pi a_0$ and $E$, defined in (1.10), is the self-adjoint operator that multiplies in Fourier space by $\sqrt{|p|^4 + 16\pi a_0 |p|^2}$, for $p \in 2\pi \mathbb{Z}^3$. For trapped systems as in [14], on the other hand, $H_{GP}$ and $16\pi a_0 \varphi_0^2$ do not commute anymore which makes the analysis technically more involved, compared to [7].

3) In (1.8), we cannot take directly the limit $\delta \to 0$, replacing $\mathbb{1}_\delta$ with the identity, because the resulting operator is not a trace class operator. However, using the
assumptions (1.6), we can compute the limit defining $E_{\text{Bog}}$ explicitly and find that

$$E_{\text{Bog}} = \frac{\kappa^2}{4}(8\pi a_0)^2 \text{tr} \left[ \varphi_0^2 (-\Delta + \kappa^2)^{-1}(-\Delta)^{-1}\varphi_0^2 \right]$$

$$+ \frac{1}{4}(8\pi a_0)^2 \left\| \varphi_0 (-\Delta + \kappa^2)^{-1}[\varphi_0, -\Delta](-\Delta + \kappa^2)^{-1/2} \right\|_{\text{HS}}^2$$

$$+ \frac{1}{4}(8\pi a_0)^2 \left\| \varphi_0 (-\Delta + \kappa^2)^{-1}\nabla \varphi_0 \right\|_{\text{HS}}^2$$

$$+ \left( \frac{8\pi a_0}{4\kappa^2} \right)^2 \| \varphi_0^3 \|^2 + \frac{(8\pi a_0)^2}{4} \| (H_{\text{GP}} + \kappa^2)^{-1/2} Q \varphi_0 \|^2$$

$$- \left( \frac{8\pi a_0}{4\kappa^2} \right)^2 \kappa^2 \text{tr}_{\perp \varphi_0} \left[ \varphi_0^2 Q \frac{1}{H_{\text{GP}}(H_{\text{GP}} + \kappa^2)^{-1/2}} Q \varphi_0^2 \right]$$

$$- \left( \frac{8\pi a_0}{4\kappa^2} \right)^2 \frac{1}{\pi} \int_0^\infty ds \sqrt{s} \text{tr}_{\perp \varphi_0} \left[ \frac{H_{\text{GP}}^2}{s + H_{\text{GP}}^2} \left[ \frac{\varphi_0^2}{s + H_{\text{GP}}^2}, \frac{H_{\text{GP}}^2}{s + H_{\text{GP}}^2} \right] \frac{H_{\text{GP}}^2}{s + H_{\text{GP}}^2} \right]$$

$$- \left( \frac{8\pi a_0}{4\kappa^2} \right)^2 \frac{1}{\pi} \int_0^\infty ds \sqrt{s} \text{tr}_{\perp \varphi_0} \left[ \frac{1}{s + H_{\text{GP}}^2} H_{\text{GP}}^2 \varphi_0^2 H_{\text{GP}}^2 \right]^3$$

$$\times \frac{1}{s + H_{\text{GP}}^2 (H_{\text{GP}} + 16\pi a_0 \varphi_0^2) H_{\text{GP}}^2}$$

for any $\kappa > 0$ large enough. In particular, all contributions on the right hand side of (1.11) are finite.

4) The form of the excitation spectrum (1.9) has been derived independently in [32], valid under slightly more general assumptions on $V$ and $V_{\text{ext}}$, compared to (1.6). We refer to [32, 14] for detailed comparisons of the assumptions and the proofs.

5) The formulas (1.7) and (1.11) show that $E_N$ depends on the interaction $V$ only through its scattering length $a_0$, up to errors vanishing in the limit $N \to \infty$. In particular, (1.7) is the analogue of the Lee-Huang-Yang formula [23, 24, 25]

$$E_{N,L} = 4\pi \rho a_0 \left( 1 + \frac{128}{15\sqrt{\pi}}(\rho a_0^3)^{1/2} + o((\rho a_0^3)^{1/2}) \right)$$

(1.12)

for the ground state energy $E_{N,L}$ of $N$ bosons trapped in $\Lambda_L = [0; L]^3$, in the thermodynamic limit where $N, L \to \infty$ with the density $\rho = NL^{-3}$ fixed. (1.12) has been rigorously established in [38] (upper bound) and [19, 20] (lower bound).

In the following sections, we outline the main steps that lead to the proof of Theorem 1.1. Our proof is based on a rigorous implementation of Bogoliubov’s method [9], previously established in [7] for translation invariant systems in the Gross-Pitaevskii regime. For the details of the implementation, we refer to [14].
2 Excitation Hamiltonians

For our analysis, it is convenient to switch to a Fock space setting, factoring out the Bose-Einstein condensate. This allows us to focus on the analysis of the orthogonal excitations around the condensate. We follow here [26] and consider the unitary map $U_N : L^2_{\varphi_0} (\mathbb{R}^3)^N \rightarrow \mathcal{F}_{\varphi_0}^{\leq N}$ defined by $U_N \psi_N = (\alpha_0, \alpha_1, \ldots, \alpha_N)$ if

$$\psi_N = \alpha_0 \varphi_0^{\otimes N} + \alpha_1 \otimes_s \varphi_0^{\otimes (N-1)} + \ldots + \alpha_N$$

with $\alpha_j \in L^2_{\varphi_0} (\mathbb{R}^3)^{\otimes j}$, for $j = 0, 1, \ldots, N$. Here, $\otimes_s$ denotes the symmetric tensor product and $\mathcal{F}_{\varphi_0}^{\leq N}$ denotes the truncated excitation Fock space

$$\mathcal{F}_{\varphi_0}^{\leq N} = \bigoplus_{n=0}^{N} L^2_{\varphi_0} (\mathbb{R}^3)^{\otimes n}$$

that is built over the orthogonal complement $L^2_{\perp \varphi_0} (\mathbb{R}^3) = \{ \varphi_0 \}^\perp$. The property [165] of BEC translates in the Fock space setting to the statement that

$$\lim_{N \rightarrow \infty} \left( 1 - (\varphi_0, \gamma_N^{(1)} \varphi_0) \right) = \lim_{N \rightarrow \infty} N^{-1} (U_N \psi_N, N U_N \psi_N) = 0,$$

where $N$ denotes the number of particles operator in $\mathcal{F}_{\varphi_0}^{\leq N}$, defined by $(N \xi)^{(n)} = n \xi^{(n)}$ for every $\xi = (\xi^{(1)}, \ldots, \xi^{(N)}) \in \mathcal{F}_{\varphi_0}^{\leq N}$. In other words, condensation into $\varphi_0$ means that the average number of excitations is small compared to $N$, the total number of particles.

In $\mathcal{F}_{\varphi_0}^{\leq N}$, we then consider $L_N = U_N H_N U_N^\dagger$ which reads in second quantized form

$$L_N = \langle \varphi_0 | \left( -\Delta + V_{\text{ext}} + \frac{1}{2} (N^3 V(N \cdot) * |\varphi_0|^2) \right) \varphi_0 \rangle (N - N)$$

$$- \frac{1}{2} \langle \varphi_0, (N^3 V(N \cdot) * |\varphi_0|^2) \varphi_0 \rangle (N + 1)(1 - N/N)$$

$$+ \left( \sqrt{N} b \left( (N^3 V(N \cdot) * |\varphi_0|^2 - 8\pi a_0 |\varphi_0|^2) \varphi_0 \right) + \frac{N + 1}{\sqrt{N}} b \left( (N^3 V(N \cdot) * |\varphi_0|^2) \varphi_0 \right) + \text{h.c.} \right)$$

$$+ \int dx \ a_x^*(\Delta_x) a_x + \int dx \ V_{\text{ext}}(x) a_x^* a_x$$

(2.1)

$$+ \int dx dy N^3 V(N(x - y)) \varphi_0^2(y) \left( b_x^* b_x - \frac{1}{N} a_x^* a_x \right)$$

$$+ \int dx dy N^3 V(N(x - y)) \varphi_0(0) \varphi_0(y) \left( b_x^* b_y - \frac{1}{N} a_x^* a_y \right)$$

$$+ \int dx dy N^3 V(N(x - y)) \varphi_0(0) \varphi_0(x) \left( b_x^* b_y + b_x b_y \right)$$

$$+ \int dx dy N^{5/2} V(N(x - y)) \varphi_0(y) \left( b_x^* a_y a_x + a_x^* a_y b_x \right)$$

$$+ \int dx dy N^2 V(N(x - y)) a_x^* a_y^* a_x a_x.$$
Here, $a_x, a_y^*$, for $x, y \in \mathbb{R}^3$, denote the usual creation and annihilation operators, satisfying the canonical commutation relations $[a_x, a_y^*] = \delta(x - y)$ and $[a_x, a_y] = [a_x^*, a_y^*] = 0$, and the operators $b_x, b_y^*$ denote modified creation and annihilation operators, defined by

$$b_x = \sqrt{1 - N/N} a_x, \quad b_x^* = a_x^* \sqrt{1 - N/N}.$$  

The operators $b_x, b_y^*$ preserve the number of particles truncation in $\mathcal{F}_{± \varphi}^N$ and we have

$$[b_x, b_y^*] = (1 - N^{-1}N)\delta(x - y) - N^{-1}a_x^*a_x, \quad [b_x, b_y] = [b_x^*, b_y^*] = 0.$$  

Hence, on states with a low number of excitations $N \ll N$, the modified field operators satisfy the usual commutation relations up to errors that vanish in the limit $N \to \infty$.

Extracting the low energy spectrum of $H_N$ by analyzing $\mathcal{L}_N$ directly seems difficult, because we face a number of serious problems. First, the constant contribution in (2.1), the right hand side in (2.1) for the moment as an operator in $\mathcal{F}_\varphi$ in the dynamical context [4, 12]. To motivate the approach heuristically, let’s consider

$$\varphi the state

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Extracting the low energy spectrum of $H_N$ by analyzing $\mathcal{L}_N$ directly seems difficult, because we face a number of serious problems. First, the constant contribution in (2.1),

$$N(\varphi_0, (−\Delta + V_{\text{ext}} + \frac{1}{2}(N^3V(\varphi) * |\varphi_0|^2))\varphi_0) = N\mathcal{E}_{GP}(\varphi_0) + O(N),$$

is off by a quantity of order $O(N)$ from the correct leading order energy $N\mathcal{E}_{GP}(\varphi_0)$.

Second, the terms that are linear in the creation and annihilation operators,

$$\sqrt{N}b((N^3V(\varphi) * |\varphi_0|^2 - 8\pi a_0|\varphi_0|^2) \varphi_0) + \text{h.c.},$$

are a priori of size $O(N^{1/2})$ and, similarly, the simple form bounds

$$\pm \frac{1}{2} \int dxdy N^3V(N(x - y))\varphi_0(y)\varphi_0(x)(b_x^*b_y^* + b_xb_y)$$

$$\leq \frac{1}{2} \int dxdy N^2V(N(x - y))a_x^*a_y a_x + \frac{N}{2} ||V||_1 ||\varphi_0||_\infty^2 ||\varphi_0||^2_2$$

indicate that the off-diagonal pairing terms contribute to order $O(N)$. The reason for these difficulties is that through the map $U_N$, we expand $H_N$ around the energy of the state $\varphi_0^\otimes N$, which neglects the correlations among the particles. The correlations, however, are crucial even for extracting the leading order term $N\mathcal{E}_{GP}(\varphi_0)$.

In order to resolve these difficulties, the first step of our analysis consists of renormalizing $\mathcal{L}_N$ through conjugation by a suitable unitary map that extracts the missing correlation energies of order $O(N)$. Here, we make use of ideas that were first implemented in the dynamical context [4, 12]. To motivate the approach heuristically, let’s consider

$$\varphi the right hand side in (2.1) for the moment as an operator in $\mathcal{F}_\varphi = \bigoplus_{n=0}^{N} L^2(\mathbb{R}^{3n})$, ignoring the orthogonality constraints in $\mathcal{F}_{± \varphi}^N$. Setting

$$\mathcal{H}_N = \int dx a_x^*(-\Delta_x)a_x + \int dx V_{\text{ext}}(x)a_x^*a_x + \frac{1}{2} \int dxdy N^2V(N(x - y))a_x^*a_y^*a_y a_x$$

and $\tilde{\eta}(x; y) = (−N)(1 - f)(N(x - y))\varphi_0(x)\varphi_0(y)$, recalling that $f$ denotes the solution of the zero energy scattering equation (1.2), as well as

$$B(\tilde{\eta}) = \frac{1}{2} \int dxdy \tilde{\eta}(x; y)b_x^*b_y^* - \text{h.c.},$$

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we are going to conjugate the terms on the right hand side in (2.1) by the unitary operator exponential $e^{B(\tilde{\eta})} : \mathcal{F}^\leq N \to \mathcal{F}^\leq N$, using the second order Taylor expansion

$$e^{-B(\tilde{\eta})}Oe^{B(\tilde{\eta})} \approx O + [O, B(\tilde{\eta})] + \frac{1}{2}[[O, B(\tilde{\eta})], B(\tilde{\eta})]$$

for observables $O$ in $\mathcal{F}^\leq N$. The main observation that follows from [4, 12] is that conjugation by $e^{-B(\tilde{\eta})}(\cdot)e^{B(\tilde{\eta})}$ leads to the renormalizations

$$e^{-B(\tilde{\eta})}\left(\sqrt{N}b \left(\langle N^3 V(N\cdot) * \langle |\varphi_0|^2 \rangle \varphi_0 \right) + \text{h.c.}\right)e^{B(\tilde{\eta})}$$

$$+ \int dxdy N^{5/2}V(N(x - y))\varphi_0(y)e^{-B(\tilde{\eta})}\left(b_x^* a_y^* a_x + a_y^* a_y b_x \right)e^{B(\tilde{\eta})}$$

$$\approx \int dxdy N^{5/2}V(N(x - y))\varphi_0(y)\left(b_x^* a_y^* a_x + a_y^* a_y b_x \right)$$

and

$$N\langle \varphi_0, (\Delta + V_{\text{ext}} + \frac{1}{2}N^3 V(N\cdot) * |\varphi_0|^2 \rangle \varphi_0 \rangle + e^{-B(\tilde{\eta})}\mathcal{H}_N e^{B(\tilde{\eta})}$$

$$+ \frac{1}{2} \int dxdy N^3 V(N(x - y))\varphi_0(y)\varphi_0(x)e^{-B(\tilde{\eta})}\left(b_x^* b_y + b_x b_y \right)$$

$$\approx NE_{\text{GP}}(\varphi_0) + \mathcal{H}_N,$$

while the remaining parts on the r.h.s. in (2.1) are essentially left unchanged, up to errors of order $O(1)$ which are well under control. In other words, conjugation by $e^{-B(\tilde{\eta})}(\cdot)e^{B(\tilde{\eta})}$ leads to the cancellation of the large linear and quadratic pairing terms in (2.1), and it renormalizes the constant $O(N)$ contribution to the correct energy $NE_{\text{GP}}(\varphi_0)$.

To make the above heuristics rigorous, we need to make a few technical adjustments. First of all, instead of conjugating with the exponential of $B(\tilde{\eta})$, we choose a related operator $B = B(\eta)$ with kernel $\eta \in (Q \otimes Q)L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, for $Q = 1 - |\varphi_0\rangle\langle \varphi_0|$. This ensures that $e^B$ is a unitary map from $\mathcal{F}^\leq N$ to itself. Moreover, if $\|\eta\|_2$ is sufficiently small, we know from [12, 7, 8, 13] that one has the exact expansion

$$e^{-B}b(g)e^B = b(\cosh_\eta(g)) + b^*(\sinh_\eta(g)) + d_\eta(g),$$

for every $g \in L^2_{\perp \varphi_0}(\mathbb{R}^3)$, where $b(g) = \int_{\mathbb{R}^3} dx \overline{g(x)b_x}$ and where

$$\cosh_\eta = \sum_{n=0}^{\infty} \frac{\eta^{2n}}{(2n)!}, \quad \sinh_\eta = \sum_{n=0}^{\infty} \frac{\eta^{2n+1}}{(2n + 1)!}.$$  

Moreover, the remainder $d_\eta(g)$ in (2.4) is small on states with a small number $N \ll N$ of excitations, in the sense that $\|d_\eta(g)\xi\| \leq CN^{-1}\|g\||(N + 1)^{3/2}\|\xi\|$ for all $\xi \in \mathcal{F}^\leq N_{\perp \varphi_0}$.

To choose a kernel $\eta \in (Q \otimes Q)L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ with small $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ norm that still leads to the cancellations (2.2) and (2.3) (up to lower order corrections), one could directly modify $\tilde{\eta}$ from above, for example by imposing a suitable cutoff in position space.
Proposition 2.1. Assume (1.6), let $\mathcal{G}_N = e^{-B} \mathcal{L}_N e^B$ and let $E_N$ denote the ground state energy of $H_N$, defined in (1.1). Then, the following holds true:

a) We have that $|E_N - N\mathcal{E}_{GP}(\varphi_0)| \leq C$ and

$$\mathcal{G}_N - E_N = \mathcal{H}_N + \Delta_N,$$

where the error term $\Delta_N$ is such that for every $\delta > 0$, there exists $C > 0$ with

$$\pm \Delta_N \leq \delta \mathcal{H}_N + C(N + 1).$$

Furthermore, for every $k \in \mathbb{N}$ there exists a $C > 0$ such that

$$\pm \text{ad}^{(k)}_{i\mathcal{N}}(\mathcal{G}_N) = \pm \text{ad}^{(k)}_{i\mathcal{N}}(\Delta_N) = \pm [i\mathcal{N}, \ldots [i\mathcal{N}, \Delta_N] \ldots] \leq C(\mathcal{H}_N + 1).$$

b) Let $\sigma = \sinh_\eta$ and $\gamma = \cosh_\eta$ be defined as in (2.5) and let $\kappa_{\mathcal{G}_N}$ denote the constant

$$\kappa_{\mathcal{G}_N} = N\langle \varphi_0, (-\Delta + V_{\text{ext}} + \overline{V}(0)\varphi_0^2/2)\varphi_0 \rangle - 4\pi a_0\|\varphi_0\|^4$$

$$+ \text{tr} \left( \sigma (-\Delta + V_{\text{ext}} - \varepsilon_{GP}) \sigma \right)$$

$$+ \text{tr} \left( \gamma [N^3 V(N(x-y))\varphi_0(x)\varphi_0(y)] \sigma \right)$$

$$+ \text{tr} \left( \sigma [N^3 V(N(x-y))\varphi_0^2 + N^3 V(N(x-y))\varphi_0(x)\varphi_0(y)] \sigma \right)$$

$$+ \frac{1}{2} \int dxdy N^2 V(N(x-y))|\langle \sigma_x, \gamma_y \rangle|^2.$$
Here, \((N^3V(N_\ast)\varphi_0^2)\) acts as multiplication operator and we identify kernels like \(N^3V(N_\ast - y)\varphi_0(x)\varphi_0(y)\) with their associated Hilbert Schmidt operators. Let

\[
\Phi = \gamma(-\Delta + V_{\text{ext}} - \varepsilon_{\text{GP}})\gamma + \sigma(-\Delta + V_{\text{ext}} - \varepsilon_{\text{GP}})\sigma + \gamma(N^3V(N_\ast)\varphi_0^2 + N^3V(N_\ast - y)\varphi_0(x)\varphi_0(y))\gamma + \sigma(N^3V(N_\ast)\varphi_0^2 + N^3V(N_\ast - y)\varphi_0(x)\varphi_0(y))\sigma + \left(\gamma(N^3(V_f)(N_\ast - y)\varphi_0(x)\varphi_0(y))\sigma + \text{h.c.}\right),
\]

and

\[
\Gamma = \gamma(N^3(V_f)(N_\ast - y)\varphi_0(x)\varphi_0(y))\gamma + \sigma(N^3(V_f)(N_\ast - y)\varphi_0(x)\varphi_0(y))\sigma + \left[\sigma(-\Delta + V_{\text{ext}} - \varepsilon_{\text{GP}})\gamma + \text{h.c.}\right] + \left[\sigma(N^3V(N_\ast)\varphi_0^2 + N^3V(N_\ast - y)\varphi_0(x)\varphi_0(y))\gamma + \text{h.c.}\right).
\]

With \(\Phi\) and \(\Gamma\), we define the quadratic Fock space Hamiltonian

\[
Q_{G_N} = \int dxdy \Phi(x;y)b_x^*b_y + \frac{1}{2}\int dxdy \Gamma(x;y)(b_x^*b_y^* + b_xb_y),
\]

and we denote by \(C_{G_N}\) the cubic Fock space operator

\[
C_{G_N} = \int dxdy N^{5/2}V(N_\ast - y)\varphi_0(y)b_x^*b_y^*(b(\gamma_\ast) + b^*(\sigma_\ast)) + \text{h.c.},
\]

where \(\gamma_\ast(x) = \gamma(x;x)\) and \(\sigma_\ast(x) = \sigma(x;x)\) for \(x \in \mathbb{R}^3\). Then, \(G_N\) is equal to

\[
G_N = \kappa_{G_N} + Q_{G_N} + C_{G_N} + V_N + \mathcal{E}_{G_N}\tag{2.11}
\]

for some self-adjoint operator \(\mathcal{E}_{G_N}\) which is bounded in \(\mathcal{F}_{\perp\varphi_0}^{\leq N}\) by

\[
\pm \mathcal{E}_{G_N} \leq CN^{-1/2}(H_N + N^2 + 1)(N + 1).\tag{2.12}
\]

Proposition 2.1 implements the heuristic renormalizations in (2.2) and (2.3) rigorously, noticing that \(\kappa_{G_N} = N\mathcal{E}_{G_N}(\varphi_0) + O(1)\) and \(\|\Gamma\|_2 \leq C\), uniformly in \(N \in N\), for \(\kappa_{G_N}\) and \(\Gamma\) defined in (2.7) and (2.8), respectively (see [13] for the details). Moreover, the error \(\mathcal{E}_{G_N}\) in (2.11) satisfying (2.12), is negligible on low energy eigenstates. Indeed, this is a direct consequence of the following proposition which was proved in [13] Theorem 2.6] by combining the main result (1.4) of [13] with the commutator estimates (2.6).

**Proposition 2.2.** Assume (1.6) and let \(E_N\) denote the ground state energy of \(H_N\), defined in (1.1). For \(\zeta > 0\), let \(\psi_N \in L^2(\mathbb{R}^{2N})\) with \(\|\psi_N\| = 1\) be such that

\[
\psi_N = 1_{(-\infty,E_N + \zeta]}(H_N)\psi_N
\]

and let \(\xi_N = e^{-BU_N}\psi_N \in \mathcal{F}_{\perp\varphi_0}^{\leq N}\) denote the renormalized excitation vector related to \(\psi_N\). Then, for every \(j \in N\) there is a constant \(C > 0\) such that

\[
\langle \xi_N, (H_N + 1)(N + 1)^j \xi_N \rangle \leq C(1 + \zeta^{2j+1}).
\]
With Bogoliubov’s arguments [9] in mind, we may be tempted to neglect the cubic and quartic contributions $C_{GN}$ and $V_N$ on the r.h.s. in (2.10) and diagonalize the remaining quadratic operator $Q_{GN}$ in (2.9) to determine the spectrum of $G_N$ up to order $O(1)$. Proceeding this way does, however, not yield the right spectrum, because the terms $C_{GN}$ and $V_N$ still contain important energy contributions of order $O(1)$. This has been a key observation for the analysis in [7] (see also [38] and [33, 34] related to this point) so that, in order to conclude Theorem 1.1, we first need to extract the missing $O(1)$ energies.

To this end, we proceed similarly as in the first step, now having the goal to renormalize the cubic contribution $C_{GN}$ in (2.10). To understand the main idea on a heuristic level, consider for simplicity first the cubic operator

$$\int dxdy N^{5/2}V(N(x-y))\varphi_0(y)b_x^*b_y^*b_z + \text{h.c.} \quad (2.13)$$

that is obtained from $C_{GN}$ by replacing $\gamma_x \approx \delta_x$ and $\sigma_x \approx 0$ (to leading order in $\eta$). By analogy to the quadratic renormalization through $e^{B}$, a straightforward computation involving the scattering equation (1.2) implies that

$$\int dxdy (-N(1-f)(N(x-y))\varphi_0(y)[K_N + V_N, b_x^*b_y^*b_z + \text{h.c.}]
\approx -\int dxdy N^{5/2}V(N(x-y))\varphi_0(y)b_x^*b_y^*b_z + \text{h.c.},$$

up to errors of order $O(1)$. What this indicates is that we can cancel the cubic term (2.13) by conjugating $G_N$ through a unitary operator exponential that is now cubic in the (modified) creation and annihilation operators, having the form

$$\tilde{A} = \frac{1}{\sqrt{N}} \int dxdy (-N(1-f)(N(x-y))\varphi_0(y)b_x^*b_y^*b_z + \text{h.c.} \quad (2.14)$$

From the analysis of the quadratic renormalization and the fact that our goal is to cancel the full cubic term $C_{GN}$ in $G_N$ (and not only the term in (2.13)), it is clear that we need to make a few technical adjustments to $\tilde{A}$ in (2.14). First of all, to stay consistent with the first renormalization step, instead of working with the correlation factor $(-N(1-f))(N)$, we use the function $(-Nw_\ell)(N)$ on which we impose additionally an $N$-dependent low momentum cutoff. More precisely, for some small $\varepsilon > 0$, we set $\chi_H(p) = \chi(|p| > N\varepsilon)$, denote by $\tilde{\chi}_H$ its inverse Fourier transform and define the kernel

$$\tilde{k}_H(x; y) = (-Nw_\ell(N) * \tilde{\chi}_H)(x-y)\varphi_0(y). \quad (2.15)$$

Through the cutoff $\chi_H$, it turns out that it is enough to compute only a few commutators in the expansion of the cubic conjugation, because most of the error terms become small (in orders of $N$). Second, to cancel not only the cubic term in (2.13), but the full term $C_{GN}$ in (2.10), we need to replace the $b_x$-field in (2.14) by $b(\gamma_x) + b^*(\sigma_x)$. Also here, it turns out to simplify the analysis if we impose high momentum cutoffs on the kernels.
Moreover, the self-adjoint operator $E$ and its inverse Fourier transform. With the notation

$$\sigma_L = \sigma \ast_2 \tilde{y}L, \quad \gamma_L = \gamma \ast_2 \tilde{y}L,$$

where $\ast_2$ denotes convolution in the second variable, we then define the cubic operator

$$A = \frac{1}{\sqrt{N}} \int dx dy \tilde{k}_H(x; y)b_x^* b_y^* \left[\tilde{b}(\gamma_L x) + \tilde{b}^*(\sigma_L x)\right] - \text{h.c.}$$

where $\tilde{b}_x = \tilde{b}(Q_x) = \int dz Q(x; z)b_z$, with $Q = 1 - |\varphi_0\rangle\langle \varphi_0|$. Notice that the use of the operators $\tilde{b}_x, \tilde{b}_y^*$ ensures that $A$ is a unitary map from $\mathcal{F}_{\leq N}^\perp \varphi_0$ to itself.

The following proposition summarizes the main properties of the renormalized excitation Hamiltonian $J_N = e^{-\Delta N L} e^A$. For its proof, see [14, Prop. 2.8].

**Proposition 2.3.** Assume (1.0) and let $0 < 6\tau \leq \varepsilon \leq \frac{1}{2}$. Then

$$J_N = \kappa J_N + QJ_N + V_N + EJ_N,$$

where

$$\kappa J_N = \kappa G_N - \text{tr}(\sigma N^3(V w_L)(N) \ast_2 \varphi_0^2 \sigma) - \text{tr}(\sigma N^3(V w_L)(N(x - y))\varphi_0(x)\varphi_0(y)\sigma)$$

with $\kappa G_N$ defined in (2.7) and where the quadratic operator $QJ_N$ is given by

$$QJ_N = \int dx dy \tilde{\Phi}(x; y)b_x^* b_y^* + \frac{1}{2} \int dx dy \tilde{\Gamma}(x; y)(b_x^* b_y^* + b_x b_y)$$

for

$$\tilde{\Phi} = \gamma(-\Delta + V_{ext} - \varepsilon_{GP})\gamma + \sigma(-\Delta + V_{ext} - \varepsilon_{GP})\sigma$$

$$+ \gamma(8\pi a_0 \varphi_0^2 + N^3(V f_L)(N(x - y))\varphi_0(x)\varphi_0(y))\gamma$$

$$+ \sigma(8\pi a_0 \varphi_0^2 + N^3(V f_L)(N(x - y))\varphi_0(x)\varphi_0(y))\sigma$$

$$+ \left(\gamma(N^3(V f_L)(N(x - y))\varphi_0(x)\varphi_0(y))\sigma + \text{h.c.}\right),$$

and

$$\tilde{\Gamma} = \gamma(N^3(V f_L)(N(x - y))\varphi_0(x)\varphi_0(y))\gamma$$

$$+ \sigma(N^3(V f_L)(N(x - y))\varphi_0(x)\varphi_0(y))\sigma$$

$$+ [\sigma(-\Delta + V_{ext} - \varepsilon_{GP})\gamma + \text{h.c.}]$$

$$+ [\sigma(8\pi a_0 \varphi_0^2 + N^3(V f_L)(N(x - y))\varphi_0(x)\varphi_0(y))\gamma + \text{h.c.}].$$

Moreover, the self-adjoint operator $EJ_N$ is bounded by

$$\pm EJ_N \leq CN^{-\min\{\frac{1}{2}, \frac{1}{2} - \varepsilon\}}(H_N + 1)(N + 1)^3.$$
Compared to Prop. 2.1, notice that the decomposition (2.16) of $J_N$ does not contain a cubic contribution anymore. In other words, conjugating $G_N$ by the cubic exponential $e^{-A(x)}e^A$ establishes rigorously the cubic renormalization outlined after (2.13). To control the error term $E_{J_N}$ in (2.16), we make use of the error bound (2.20) and the following analogue of Prop. 2.2 (see [14, Prop. 2.9] for the proof).

**Proposition 2.4.** Assume (1.6) and let $E_N$ denote the ground state energy of $H_N$. For some $\zeta > 0$, let $\psi_N \in L^2(R^3)$ with $\|\psi_N\| = 1$ be such that $\psi_N = 1_{(-\infty; E_N+\zeta]}(H_N)\psi_N$.

Let $\xi_N = e^{-A}e^{-B}U_N^\dagger \psi_N \in \mathcal{F}_{\bot \phi_0}^{\leq N}$ be the renormalized excitation vector related to $\psi_N$. Then, for any $j \in \mathbb{N}$ there exists some $C > 0$ such that

$$\langle \xi_N, (N+1)^j(H_N+1)\xi_N \rangle \leq C(1 + \zeta^{j+3}).$$

### 3 Diagonalization and Proof of Theorem 1.1

Using the decomposition (2.16) in Prop. 2.3 we can now determine the spectrum $\sigma(J_N)$ of $J_N = e^{-A}e^{-B}U_N^\dagger H_N U_N^\dagger e^B e^A$ (and hence of $H_N$, by unitary equivalence), up to errors that vanish as $N \to \infty$. We choose the momentum cutoff parameters (introduced around (2.15)) $\varepsilon = 6/13$ and $\tau = \varepsilon/6$, so that we can apply Propositions 2.1, 2.2, 2.3 and 2.4.

To obtain upper and lower bounds on $\sigma(J_N)$, we apply the min-max principle and compare the eigenvalues of $J_N$ with those of the quadratic Fock space Hamiltonian

$$\tilde{Q}_{J_N} = \kappa_{J_N} + \int dxdy \tilde{\Phi}(x,y) a_x^* a_y + \frac{1}{2} \int dxdy \tilde{\Gamma}(x,y)(a_x^* a_y + \text{h.c.}),$$

defined in the Fock space $\mathcal{F}_{\bot \phi_0} = \bigoplus_{n=0}^\infty L^2_{\bot \phi_0}(\mathbb{R}^3)^{\otimes n}$ (without a restriction on the number of particles). Here, $\kappa_{J_N}$ denotes the constant from (2.17), and the kernels of $\tilde{\Phi}$ and $\tilde{\Gamma}$ were defined in (2.18) and (2.19), respectively.

To see that it is enough to analyze $\tilde{Q}_{J_N}$, notice first that, to get a lower bound on the min-max values of $J_N$, we can drop the non-negative potential energy $V_N$ in (2.16) and we can control the error $E_{J_N}$ in (2.16) through Prop. 2.4. Since $\mathcal{F}_{\bot \phi_0}^{\leq N} \subset \mathcal{F}_{\bot \phi_0}^{\leq N}$, we obtain a lower bound on the spectrum of $J_N$ by computing the spectrum of $\tilde{Q}_{J_N}$ in $\mathcal{F}_{\bot \phi_0}^{\leq N}$. To obtain upper bounds on the min-max values of $J_N$, on the other hand, we construct suitable subspaces in $\mathcal{F}_{\bot \phi_0}^{\leq N}$, built from the eigenvectors of $\tilde{Q}_{J_N}$ truncated to $\mathcal{F}_{\bot \phi_0}^{\leq N}$. It turns out that the potential energy $V_N$ is negligible on such eigenspaces and that the upper and lower bounds on the min-max values of $J_N$ coincide, up to errors vanishing in the limit $N \to \infty$ (see [14, Section 3] for the details).

Hence, let us focus on $\tilde{Q}_{J_N}$ and let us recall how to determine its spectrum. As a quadratic Fock space Hamiltonian, $\tilde{Q}_{J_N}$ is exactly diagonalizable. To diagonalize it, we follow [21] and recall the definitions of $H_{GP}$ and $E$ in (1.5) and (1.10), respectively. We
then identify the operators $H_{GP}, E, \tilde{\Phi}, \tilde{\Gamma}$ with operators mapping $L^2_{\perp \varphi_0}(\mathbb{R}^3)$ back into itself (so that, in particular, $H_{GP}$ and $E$ are invertible in $L^2_{\perp \varphi_0}(\mathbb{R}^3)$). Moreover, we set

$$\tilde{D} = \tilde{\Phi} - \tilde{\Gamma} \quad \text{and} \quad \tilde{E} = (\tilde{D}^{1/2}(\tilde{D} + 2\tilde{\Gamma})\tilde{D}^{1/2})^{1/2}$$

as well as $A = \tilde{D}^{1/2}\tilde{E}^{-1/2}$ and $\alpha = \log(|A^*|)$. Finally, Writing $A = W|A|$ for some partial isometry in $L^2_{\perp \varphi_0}(\mathbb{R}^3)$, by the polar decomposition, we construct a standard Bogoliubov transformation $U$ that diagonalizes $\tilde{Q}_J$. Indeed, denoting by $(\varphi_j)_{j \in \mathbb{N}}$ the eigenbasis of $\tilde{E}$ and setting $a^\sharp_j = a^\sharp(\varphi_j)$ for $\sharp \in \{\cdot, \ast\}$ as well as $\alpha_{ij} = \langle \varphi_i, \alpha \varphi_j \rangle$, we set

$$W = \Gamma(W), \quad X = \frac{1}{2} \sum_{i,j=1}^{\infty} \alpha_{ij} a^*_i a^*_j - \text{h.c.}, \quad U = e^X W.$$

Here, $\Gamma(W)$ denotes the second quantization of the partial isometry $W$, acting in the $n$-particle sector of $F_{\perp \varphi_0}$ as $W \otimes_n$. With $U$ defined above, one verifies that

$$U^* \tilde{Q}_J U = \kappa_J + \frac{1}{2} \text{tr}_{\perp \varphi_0} \left( \frac{1}{2}(\tilde{D}^{1/2}\tilde{E}\tilde{D}^{-1/2} + \tilde{D}^{-1/2}\tilde{E}\tilde{D}^{1/2}) - \tilde{D} - \tilde{\Gamma} \right) + d\Gamma(\tilde{E}),$$

(3.1)

where $\text{tr}_{\perp \varphi_0}$ denotes the trace in $L^2_{\perp \varphi_0}(\mathbb{R}^3)$ and where $d\Gamma(\tilde{E})$ denotes the second quantization of $\tilde{E}$, acting as $\sum_{i=1}^{\infty} \tilde{E}_{\xi_i}$ in the $n$-particle sector of $F_{\perp \varphi_0}$.

Now, using that the eigenvalues of $d\Gamma(\tilde{E})$ are equal to those of $d\Gamma(E)$ (with $E$ defined in (1.10)), up to errors that vanish as $N \to \infty$, we conclude (1.9). Theorem 1.1 then follows by showing that the constant on the right hand side in (3.1), that is

$$\kappa_J + \frac{1}{2} \text{tr}_{\perp \varphi_0} \left( \frac{1}{2}(\tilde{D}^{1/2}\tilde{E}\tilde{D}^{-1/2} + \tilde{D}^{-1/2}\tilde{E}\tilde{D}^{1/2}) - \tilde{D} - \tilde{\Gamma} \right),$$

is equal to the constant on the right hand side in (1.7), up to errors that vanish as $N \to \infty$. For the details of this computation, see [14, Section 3].

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