The Laplacian Flow of Locally Conformal Calibrated $G_2$-Structures

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Abstract: We consider the Laplacian flow of locally conformal calibrated $G_2$-structures as a natural extension to these structures of the well-known Laplacian flow of calibrated $G_2$-structures. We study the Laplacian flow for two explicit examples of locally conformal calibrated $G_2$ manifolds and, in both cases, we obtain a flow of locally conformal calibrated $G_2$-structures, which are ancient solutions, that is they are defined on a time interval of the form $(-\infty, T)$, where $T > 0$ is a real number. Moreover, for each of these examples, we prove that the underlying metrics $g(t)$ of the solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric as $t$ goes to $-\infty$, and they blow-up at a finite-time singularity.

Keywords: locally conformal calibrated $G_2$-structures; Laplacian flow; solvable Lie algebras

1. Introduction

A $G_2$-structure on a 7-manifold $M$ can be characterized by the existence of a globally defined 3-form $\varphi$ (the $G_2$ form) on $M$, which can be written at each point as

$$
\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},
$$

with respect to some local coframe $\{e^1, \ldots, e^7\}$ on $M$. Here, $e^{127}$ stands for $e^1 \wedge e^2 \wedge e^7$, and so on. A $G_2$-structure $\varphi$ induces a Riemannian metric $g_\varphi$ and a volume form $dV_{g_\varphi}$ on $M$ given by

$$
g_\varphi(X, Y) dV_{g_\varphi} = \frac{1}{6} i_X \varphi \wedge i_Y \varphi \wedge \varphi,
$$

for any pair of vector fields $X, Y$ on $M$, where $i_X$ denotes the contraction by $X$.

The classes of $G_2$-structures can be described in terms of the exterior derivatives of the 3-form $\varphi$ and the 4-form $* \varphi$ [1,2], where $* \varphi$ is the Hodge operator defined from $g_\varphi$ and $dV_{g_\varphi}$. If the 3-form $\varphi$ is closed and coclosed, then the holonomy group of $g_\varphi$ is a subgroup of the exceptional Lie group $G_2$ [2], and the metric $g_\varphi$ is Ricci-flat [3]. When this happens, the $G_2$-structure is said to be torsion-free [4]. This condition has a variational formulation, due to Hitchin [5,6]. The first compact examples of Riemannian manifolds with holonomy $G_2$ were constructed first by Joyce [7,8], and then by Kovalev [9]. Recently, other examples of compact manifolds with holonomy $G_2$ were obtained in [10,11]. Explicit examples on solvable Lie groups were also constructed in [12]. A $G_2$-structure $\varphi$ is called locally conformal parallel if $\varphi$ satisfies the two following conditions

$$
d\varphi = \theta \wedge \varphi, \quad d(*\varphi) = \frac{4}{3} \theta \wedge *\varphi,
$$

Keywords: locally conformal calibrated $G_2$-structures; Laplacian flow; solvable Lie algebras
for some closed non-vanishing 1-form $\theta$, which is known as the Lee form of the $G_2$-structure. Such a $G_2$-structure is locally conformal to one which is torsion-free. Ivanov, Parton and Piccinni in [13] prove that a compact locally conformal parallel $G_2$ manifold is a mapping torus bundle over the circle $S^1$ with fibre a simply connected nearly Kähler manifold of dimension six and finite structure group.

We remind that a $G_2$-structure $\varphi$ is called closed (or calibrated according to [14]) if $d\varphi = 0$. In this paper we will focus our attention on the class of locally conformal calibrated $G_2$-structures, which are characterized by the condition

$$d\varphi = \theta \wedge \varphi,$$

where $\theta$ is a closed non-vanishing 1-form, which is also known as the Lee form of the $G_2$-structure. We will refer to a manifold equipped with such a structure as a locally conformal calibrated $G_2$ manifold. Each point of such a manifold has an open neighborhood $U$ where $\theta = df$, for some $f \in \mathcal{F}(U)$ with $\mathcal{F}(U)$ being the algebra of the real differentiable functions on $U$, and the 3-form $e^{-f} \varphi$ defines a calibrated $G_2$-structure on $U$. Hence, locally conformal calibrated $G_2$-structures are locally conformal equivalent to calibrated $G_2$-structures, and they can be considered analogous in dimension 7 to the locally conformal symplectic manifolds, which have been studied in [15–21] and the references therein. Some results of locally conformal calibrated $G_2$ manifolds were given in [22–25]. In fact, in [24] the first author and Ugarte introduced a differential complex for locally conformal calibrated $G_2$ manifolds, and such manifolds were characterized as the ones endowed with a $G_2$-structure $\varphi$ for which the space of differential forms annihilated by $\varphi$ becomes a differential subcomplex of the de Rham’s complex. Moreover, in [23] it is proved that a similar result to that of Ivanov, Parton and Piccinni holds for compact 7-manifolds with a suitable locally conformal calibrated $G_2$-structure. More recently, a structure result for Lie algebras with an exact locally conformal calibrated $G_2$-structure was proved by Bazzoni and Raffero in [22], where it is also shown that none of the non-Abelian nilpotent Lie algebras with closed $G_2$-structures admits locally conformal calibrated $G_2$-structures.

Compact $G_2$-calibrated manifolds have interesting curvature properties. As we mentioned before, a $G_2$ holonomy manifold is Ricci-flat or, equivalently, both Einstein and scalar-flat. But on a compact calibrated $G_2$ manifold, both the Einstein condition [26] and scalar-flatness [27] are equivalent to the holonomy being contained in $G_2$. In fact, Bryant in [27] shows that the scalar curvature is always non-positive.

Locally conformal calibrated $G_2$-structures $\varphi$ whose underlying Riemannian metric $g_\varphi$ is Einstein have been studied in [25], where it was shown that in the compact case the scalar curvature of $g_\varphi$ can not be positive. Then, Fino and Raffero in [25] show that a compact homogeneous 7-manifold cannot admit an invariant Einstein locally conformal calibrated $G_2$-structure $\varphi$ unless the underlying metric $g_\varphi$ is flat. However, in contrast to the compact homogeneous case, a non-compact example of homogeneous manifold $S$ endowed with a locally conformal calibrated $G_2$-structure whose associated Riemannian metric is Einstein and non Ricci-flat was given in [25]. The manifold $S$ is a simply connected solvable Lie group which is not unimodular (see Section 4.2 for details).

On the other hand, in [23] it is given an example of a compact manifold $N$ with a locally conformal calibrated $G_2$-structure. The manifold $N$ is a compact solvmannifold, that is $N$ is a compact quotient of a simply connected solvable Lie group $K$ by a lattice, endowed with an invariant locally conformal calibrated $G_2$-structure.

Since Hamilton introduced the Ricci flow in 1982 [28], geometric flows have been an important tool in studying geometric structures on manifolds. In $G_2$ geometry, geometric flows for different $G_2$-structures have been proposed. Let $M$ be a 7-manifold endowed with a calibrated $G_2$-structure $\varphi$. The Laplacian flow starting from $\varphi$ is the initial value problem

$$\frac{d}{dt}\varphi(t) = \Delta_t \varphi(t),$$

$$d\varphi(t) = 0,$$

$$\varphi(0) = \varphi,$$

where $\varphi(t)$ is a closed $G_2$ form on $M$, and $\Delta_t = d d^* + d^* d$ is the Hodge Laplacian operator associated with the metric $g(t) = g_{\varphi(t)}$ induced by the 3-form $\varphi(t)$. This flow was introduced by Bryant in [27] as
a tool to find torsion-free $G_2$-structures on compact manifolds. Short-time existence and uniqueness of the solution when $M$ is compact were proved in [29]. The analytic and geometric properties of the Laplacian flow have been deeply investigated in the series of papers [30–32]. Non-compact examples where the flow converges to a flat $G_2$-structure have been given in [33].

In [34], a flow evolving the 4-form $\psi = *_\varphi \varphi$ in the direction of minus its Hodge Laplacian was introduced, and it is called Laplacian coflow of $\varphi$. This flow preserves the condition of the $G_2$-structure $\varphi$ being coclosed, that is $\psi(t)$ is closed for any $t$, and it was studied in [34] for two explicit examples of coclosed $G_2$-structures. But no general result is known about the short time existence of the coflow. A modified Laplacian coflow was introduced by Grigorian in [35] (see also [36]). There it was proved that for compact manifolds, the modified Laplacian coflow has a unique solution $\psi(t)$ for the short time period $t \in [0, \epsilon)$, for some $\epsilon > 0$. Geometric properties of both coflows on the 7-dimensional Heisenberg group and on 7-dimensional almost-abelian Lie groups were proved in [37,38], respectively.

Some work has also been done on other related flows of $G_2$-structures—such as the Laplacian flow and the Laplacian coflow, for locally conformal parallel $G_2$-structures. These flows has been originally proposed by the second author with Otal and Villacampa in [39], and the first examples of long time solutions of the flows are given in [39].

In this note, for any locally conformal calibrated $G_2$-structure $\varphi$ on a manifold $M$, we consider the Laplacian flow of $\varphi$ given by

$$
\frac{d}{dt} \varphi(t) = \Delta_\varphi(t),
$$

$$
d_\varphi(t) = \theta(t) \wedge \varphi(t),
$$

$$
\varphi(0) = \varphi.
$$

We do not known any general result on the short time existence of solution for this flow. Nevertheless, in Section 4 (Theorems 1 and 2), for each of the aforementioned examples of solvable Lie groups $K$ and $S$ with a locally conformal calibrated $G_2$-structure, we show that the solution of the before Laplacian flow is ancient, that is it is defined on a time interval of the form $(-\infty, T)$, where $T > 0$ is a real number. Moreover, for each of the two examples $K$ and $S$, we show that the underlying metrics $g(t) = g(t)\varphi$ of the solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric as $t$ goes to $-\infty$, and they blow-up in finite-time. As we mentioned before, the Lie group $S$ has a locally conformal calibrated $G_2$-structure inducing an Einstein metric. We prove that the solution $\varphi(t)$ of the flow on $S$ induces an Einstein metric for all time $t$ where $\varphi(t)$ is defined.

2. $G_2$-Structures

Let $M$ be a 7-dimensional manifold with a $G_2$-structure defined by a 3-form $\varphi$. Denote by $\varphi$ the 4-form $\psi = *_\varphi \varphi$, where $*_\varphi$ is the Hodge star operator of the metric $\varphi$ induced by $\varphi$. Let $(\Omega^*(M), d)$ be the de Rham complex of differential forms on $M$. Then, Bryant in [27] proved that the forms $d\varphi$ and $d\psi$ are such that

$$
\begin{align*}
    d\varphi &= \tau_0 \psi + 3 \tau_1 \wedge \varphi + *_\varphi \tau_3, \\
    d\psi &= 4 \tau_1 \wedge \varphi - *_\varphi \tau_2,
\end{align*}
$$

where $\tau_0 \in \Omega^0(M), \tau_1 \in \Omega^1(M), \tau_2 \in \Omega^2(M) \text{ and } \tau_3 \in \Omega^3(M)$. Here $\Omega^i(M)$ and $\Omega^i_2(M)$ are the spaces

$$
\Omega^i_2(M) = \{ \alpha \in \Omega^i(M) \mid \alpha \wedge \varphi = -*_\varphi \alpha \},
$$

$$
\Omega^3_2(M) = \{ \beta \in \Omega^3(M) \mid \beta \wedge \varphi = 0 = \beta \wedge *_\varphi \varphi \}.
$$

The differential forms $\tau_i$ ($i = 0, 1, 2, 3$) that appear in (3), are called the intrinsic torsion forms of $\varphi$. In terms of the torsion forms, some classes of $G_2$-structures with the defining conditions are recalled in the Table 1.

Note that if a manifold $M$ has a locally conformal calibrated $G_2$-structure $\varphi$, then

$$
d\varphi = \theta \wedge \varphi,$$
with θ the Lee form of ϕ. Thus, taking into account (3), the torsion form τ1 of the G2 form ϕ can be expressed in terms of the Lee form θ as \( τ_1 = \frac{1}{2} θ \). Moreover (see [24]), the torsion forms \( τ_1 \) and \( τ_2 \) of ϕ can be obtained as follows:

\[
\begin{align*}
τ_1 &= -\frac{1}{12} \ast ϕ (\ast ϕ dϕ ∧ ϕ), \\
τ_2 &= \ast ϕ (4τ_1 ∧ (\ast ϕ) − d \ast ϕ).
\end{align*}
\]

3. The Laplacian Flow of Locally Conformal Calibrated G2-Structures

In this section, we introduce the Laplacian flow of a locally conformal calibrated G2-structure on a manifold \( M \) and, for its equations, we show some properties that help us solve the flow when \( M \) is a Lie group.

**Definition 1.** Let \( M \) be a 7-manifold with a locally conformal calibrated G2-structure \( ϕ \). We say that a time-dependent G2-structure \( ϕ(t) \) on \( M \), defined for \( t \) in some real open interval, satisfies the Laplacian flow system of \( ϕ \) if, for all times \( t \) for which \( ϕ(t) \) is defined, we have

\[
\begin{align*}
\frac{d}{dt} ϕ(t) &= Δ_t ϕ(t), \\
d ϕ(t) &= θ(t) ∧ ϕ(t), \\
ϕ(0) &= ϕ,
\end{align*}
\]

where \( θ(t) \) is the Lee form of \( ϕ(t) \), and \( Δ_t = d d^* + d^* d \) is the Hodge Laplacian operator associated with the metric \( g(t) = g_{ϕ(t)} \) induced by the 3-form \( ϕ(t) \).

In order to solve the first equation of the flow (5) for our examples, we follow the approach of [39]. Let \( G \) be a simply connected solvable Lie group of dimension 7 with Lie algebra \( g \). Let \( \{ e^1, \ldots, e^7 \} \) be a basis of the dual space \( g^* \) of \( g \), and let \( f_i = f_i(t) \) (\( i = 1, \ldots, 7 \)) be some differentiable real functions depending on a parameter \( t \in I \subset \mathbb{R} \) such that \( f_i(0) = 1 \) and \( f_i(t) \neq 0 \), for any \( t \in I \), where \( I \) is a real open interval. For each \( t \in I \), we consider the basis \( \{ x^1, \ldots, x^7 \} \) of \( g^* \) defined by

\[
x^i = x^i(t) = f_i(t) e^i, \quad 1 ≤ i ≤ 7.
\]

We consider the one-parameter family of left invariant G2-structures \( ϕ(t) \) on \( G \) given by

\[
ϕ(t) = x^{127} + x^{347} + x^{567} + x^{135} − x^{146} − x^{236} − x^{245},
\]

\[
f_{127} e^{127} + f_{347} e^{347} + f_{567} e^{567} + f_{135} e^{135} − f_{146} e^{146} − f_{236} e^{236} − f_{245} e^{245},
\]

where \( f_{ijk} = f_{ijk}(t) \) stands for the product \( f_i(t)f_j(t)f_k(t) \).

Now, we introduce the function \( ε(i, j, k) \) on ordered indices \( (i, j, k) \) as follows:

\[
ε(i, j, k) = \begin{cases} 
1 & \text{if } (i, j, k) \in A = \{(1, 2, 7), (1, 3, 5), (3, 4, 7), (5, 6, 7)\}; \\
-1 & \text{if } (i, j, k) \in B = \{(1, 4, 6), (2, 3, 6), (2, 4, 5)\}; \\
0 & \text{otherwise}.
\end{cases}
\]
Thus, the $G_2$ form $\varphi$ defined in (1), can be reexpressed as $\varphi = \sum_{(i,j,k) \in A \cup B} \epsilon(i,j,k) x^{ijk}$, and the $G_2$ form $\varphi(t)$ given by (6) becomes

$$\varphi(t) = \sum_{(i,j,k) \in A \cup B} \epsilon(i,j,k) x^{ijk}.$$ 

Therefore,

$$\frac{d}{dt} \varphi(t) = \sum_{(i,j,k) \in A \cup B} \epsilon(i,j,k) \frac{d f_{ijk}}{dt} e^{ijk} = \sum_{(i,j,k) \in A \cup B} \epsilon(i,j,k) \frac{\partial f_{ijk}}{\partial t} x^{ijk} = \sum_{(i,j,k) \in A \cup B} \epsilon(i,j,k) \frac{d}{dt} \left(\ln f_{ijk}\right) x^{ijk}.$$

Moreover, we have

$$\Delta_t \varphi(t) = \sum_{(i,j,k) \in A \cup B} \epsilon(i,j,k) \Delta_{ijk} x^{ijk} + \sum_{1 \leq l < m < n \leq 7, (l,m,n) \not\in A \cup B} \Delta_{lmn} x^{lmn},$$

where $\epsilon(i,j,k) \Delta_{ijk}$ is the coefficient in $x^{ijk}$ of $\Delta_t \varphi(t)$ if $(i,j,k) \in A \cup B$ (i.e., if $\epsilon(i,j,k) \neq 0$), and $\Delta_{lmn}$ is the coefficient in $x^{lmn}$ of $\Delta_t \varphi(t)$ if $1 \leq l < m < n \leq 7$ and $\epsilon(l,m,n) = 0$. Consequently, the first equation of the flow (5) is equivalent to the system of differential equations

$$\begin{cases} 
\Delta_{ijk} = \frac{(f_{ijk})'}{f_{ijk}} & \text{if } (i,j,k) \in A \cup B, \\
\Delta_{lmn} = 0 & \text{if } 1 \leq l < m < n \leq 7 \text{ and } (l,m,n) \not\in A \cup B, 
\end{cases}$$

that is,

$$\begin{cases} 
\Delta_{ijk} = \frac{d}{dt} \ln(f_{ijk}) & \text{if } (i,j,k) \in A \cup B, \\
\Delta_{lmn} = 0 & \text{if } 1 \leq l < m < n \leq 7 \text{ and } (l,m,n) \not\in A \cup B.
\end{cases}$$

We will also use the following properties of $\Delta_{ijk}$.

**Lemma 1.** Let $\varphi(t)$ be a family of left invariant $G_2$-structures on the Lie group $G$ solving the system (7), and such that $\varphi(t)$ can be expressed as (6), for some functions $f_i = f_i(t)$. For ordered indices $(i,j,k)$ and $(p,q,r) \in A \cup B$ (that is, $\epsilon(i,j,k)$ and $\epsilon(p,q,r)$ are both non-zero) we have

i) if $\Delta_{ijk} = \Delta_{pqr}$, then $f_{ijk} = f_{pqr}$;

ii) if $f_{ijk} \Delta_{ijk} = f_{pqr} \Delta_{pqr}$, then $f_{ijk} = f_{pqr}$;

iii) if $\Delta_{ijk} + \Delta_{pqr} = 0$, then $f_{ijk} f_{pqr} = 1$;

iv) if $f_{ijk} \Delta_{ijk} + f_{pqr} \Delta_{pqr} = 0$, then $f_{ijk} + f_{pqr} = 2$.

**Proof.** The first statement of this Lemma was proved in [39]. Nevertheless, we point out how to prove it. Since $\Delta_{ijk} = \Delta_{pqr}$, the system (8) implies that $\frac{d}{dt} \ln f_{ijk} = \frac{d}{dt} \ln f_{pqr}$. Hence, $\ln f_{ijk} = \ln f_{pqr} + C$, for some constant $C$. Now, using that $f_i(0) = 1$, for $i = 1, \ldots, 7$, we have that $C = 0$. So, $f_{ijk} = f_{pqr}$, which proves i).

Now, let us suppose that $f_{ijk} \Delta_{ijk} = f_{pqr} \Delta_{pqr}$, for some $i,j,k,p,q,r$ with $1 \leq i < j < k \leq 7$ and $1 \leq p < q < r \leq 7$. From (7), we get

$$(f_{ijk})' = (f_{pqr})'.$$
Integrating this equation, we obtain $f_{ijk} = f_{pqr} + C$, for some constant $C$. Since $f_i(0) = 1$, for all $i = 1, \ldots, 7$, we have $C = 0$, and so $f_{ijk} = f_{pqr}$. This proves $ii$.

To prove $iii$, we use (8), and we obtain

$$\ln(f_{ijk} \cdot f_{pqr}) = C,$$

for some constant $C$. But $f_i(0) = 1$, for all $i = 1, \ldots, 7$, imply that $C = 0$, that is

$$f_{ijk} \cdot f_{pqr} = 1.$$

Finally, let us suppose that $f_{ijk} \Delta_{ijk} + f_{pqr} \Delta_{pqr} = 0$, for some $i, j, k, p, q, r$ with $1 \leq i < j < k \leq 7$ and $1 \leq p < q < r \leq 7$. Then, using (7), we get $(f_{ijk})' = -(f_{pqr})'$. Integrating this equation, we obtain $f_{ijk} = -f_{pqr} + C$, for some constant $C$. But $C = 2$ since $f_i(0) = 1$, for all $i = 1, \ldots, 7$. Thus, $f_{ijk} + f_{pqr} = 2$, which completes the proof. \(\square\)

4. Solutions of the Laplacian Flow on Locally Conformal Calibrated $G_2$ Solvmanifolds

Lie groups admitting left invariant locally conformal calibrated $G_2$-structures constitute a convenient setting where it is possible to investigate the behaviour of the Laplacian flow (5) in the non-compact case.

In this section, we consider two examples of solvable Lie groups $K$ and $S$, each of them with a left invariant locally conformal calibrated $G_2$-structure, and we show that in both cases the solution is ancient (i.e. it is defined in some interval $(-\infty, T)$, with $0 < T < +\infty$) and the induced metrics blow-up at a finite-time singularity.

4.1. The Laplacian Flow on $K$

Let $K$ be the simply connected and solvable Lie group of dimension 7 whose Lie algebra $k$ is defined by

$$k = \left( e^{37}, e^{47}, -e^{17}, -e^{27}, e^{14} + e^{23}, e^{13} - e^{24}, 0 \right).$$

Here, $e^{37}$ stands for $e^3 \wedge e^7$, and so on; and $(e^{37}, e^{47}, -e^{17}, -e^{27}, e^{14} + e^{23}, e^{13} - e^{24}, 0)$ means that there is a basis $\{e^1, \ldots, e^7\}$ of the dual space $k^*$ of $k$, satisfying

\begin{align*}
de^1 &= e^{37}, & de^2 &= e^{47}, & de^3 &= -e^{17}, & de^4 &= -e^{27}, \\
de^5 &= e^{14} + e^{23}, & de^6 &= e^{13} - e^{24}, & de^7 &= 0, \end{align*}

where $d$ denotes the Chevalley-Eilenberg differential on $k^*$.

The 3-form $\varphi$ on $K$ given by

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{256} - e^{245}$$

(10)

defines a left invariant locally conformal calibrated $G_2$-structure on the Lie group $K$, with Lee form $\theta = e^7$, and so with torsion form $\tau_1 = \frac{1}{2}e^7$. In fact,

$$d\varphi = -e^{1357} + e^{1467} + e^{2367} + e^{2457} = e^7 \wedge \varphi.$$

In [23] it is proved that there exists a lattice $\Gamma$ in $K$, so that the quotient space of right cosets $\Gamma \backslash K$ is a compact solvmanifold endowed with an invariant locally conformal calibrated $G_2$-structure $\varphi$, with Lee form $\theta = e^7$.

However, we should note that in the following Theorem, we will show a solution of the Laplacian flow (5) of the $G_2$ form $\varphi$ (defined by (10)) on the Lie group $K$. Such a solution does not solve the Laplacian flow of $\varphi$ on the compact quotient $\Gamma \backslash K$ since we will consider the Hodge Laplacian operator $\Delta_t$ on the Lie algebra $k$ of $K$ and we cannot check the Hodge Laplacian operator on the compact space $\Gamma \backslash K$. 

**Theorem 1.** The family of locally conformal calibrated $G_2$-structures $\varphi(t)$ on $K$ given by

$$
\varphi(t) = e^{127} + e^{347} + (1 - \frac{8}{3} t)^{-3/2} \left( e^{567} + e^{135} - e^{146} - e^{236} - e^{245} \right)
$$

(11)

is the solution for the Laplacian flow (5) of the $G_2$ form $\varphi$ given by (10), where $t \in \left( -\infty, \frac{3}{8} \right)$. The Lee form $\theta(t)$ of $\varphi(t)$ is $e^7$. Moreover, the underlying metrics $g(t)$ of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in $K$, as $t$ goes to $-\infty$, and they blow-up as $t$ goes to $\frac{3}{8}$.

**Proof.** As in Section 2, let $f_i = f_i(t)$ ($i = 1, \ldots, 7$) be some differentiable real functions depending on a parameter $t \in I \subset \mathbb{R}$ such that $f_i(0) = 1$ and $f_i(t) \neq 0$, for any $t \in I$, where $I$ is a real open interval. For each $t \in I$, we consider the basis $\{x^1, \ldots, x^7\}$ of local invariant 1-forms on $K$ defined by

$$
x^i = x^i(t) = f_i(t) \, dt, \quad 1 \leq i \leq 7.
$$

Taking into account (9), the structure equations of $K$ with respect to the basis $\{x^1, \ldots, x^7\}$ are

$$
dx^1 = \frac{f_1}{f_7} x^{37}, \quad dx^2 = \frac{f_2}{f_4} x^{47}, \quad dx^3 = -\frac{f_3}{f_7} x^{17}, \quad dx^4 = -\frac{f_4}{f_7} x^{27},
$$

$$
dx^5 = \frac{f_5}{f_{14}} x^{14} + \frac{f_5}{f_{23}} x^{23}, \quad dx^6 = \frac{f_6}{f_{13}} x^{13} - \frac{f_6}{f_{24}} x^{24}, \quad dx^7 = 0.
$$

(12)

From now on, we write $f_{ij} = f_{ij}(t) = f_i(t) f_j(t), f_{ijk} = f_{ijk}(t) = f_i(t) f_j(t) f_k(t)$, and so forth. Then, for any $t \in I$, we consider the $G_2$-structure $\varphi(t)$ on $K$ given by

$$
\varphi(t) = x^{127} + x^{347} + x^{567} + x^{135} - x^{146} - x^{236} - x^{245}
$$

$$
= f_{127} e^{127} + f_{347} e^{347} + f_{567} e^{567} + f_{135} e^{135} - f_{146} e^{146} - f_{236} e^{236} - f_{245} e^{245}.
$$

(13)

Note that the 3-form $\varphi(t)$ defined by (13) is such that $\varphi(0) = \varphi$ and, for any $t$, $\varphi(t)$ determines the metric $g(t)$ on $K$ such that the basis $\{x_i = \frac{1}{f_i} e_i; i = 1, \ldots, 7\}$ of left invariant vector fields on $K$ dual to $\{x^1, \ldots, x^7\}$ is orthonormal. So, $g(t)(e_i, e_j) = f_i f_j$, and hence $f_i = f_i(t) > 0$.

To solve the flow (5) of $\varphi$ we determine firstly the functions $f_i$ and the interval $I$ so that $\frac{d}{dt} \varphi(t) = \Delta_t \varphi(t)$, for $t \in I$. We know that

$$
\Delta_t \varphi(t) = (\ast_1 d \ast_1 d - d \ast_1 d \ast_1) \varphi(t).
$$

We calculate separately each of the terms $\ast_1 d \ast_1 d \varphi(t)$ and $-d \ast_1 d \ast_1 \varphi(t)$ of $\Delta_t \varphi(t)$. Taking into account (12) and the fact that the basis $\{x^1(t), \ldots, x^7(t)\}$ is orthonormal, we have

$$
\ast_1 d \ast_1 d \varphi(t) = -\frac{(f_1 f_4 - f_2 f_3) (f_2 f_5 + f_1 f_4) f_5}{f_1 f_2 f_3 f_4 f_7} x^{126} - \frac{(f_1 f_4 - f_2 f_3) (f_2 f_5 + f_1 f_4) f_5}{f_1 f_2 f_3 f_4 f_7} x^{146}
$$

$$
- \frac{(f_2 f_3 - f_1 f_4) (f_1 f_4 + f_2 f_3) f_3}{f_1 f_2 f_3 f_4 f_7} x^{236} + \frac{(f_1 f_4 - f_2 f_3) (f_2 f_3 + f_1 f_4) f_5}{f_1 f_2 f_3 f_4 f_7} x^{346}
$$

$$
+ \frac{(f_2 f_3 f_4 + f_2 f_3 f_4 + f_1 f_3 f_4 + f_1 f_3 f_4)}{f_1 f_2 f_3 f_4} x^{567},
$$

(14)
and, on the other hand, we obtain

\[
\begin{align*}
\Delta_{126} &= \frac{f_2}{f_7} \left( \frac{f_2}{f_1 f_4} - \frac{f_1}{f_2 f_3} \right), \\
\Delta_{346} &= \frac{f_5}{f_7} \left( \frac{f_4}{f_2 f_3} - \frac{f_3}{f_2 f_4} \right).
\end{align*}
\]

Each of these equalities implies that \( f_{14} = f_{23} \), and so

\[
f_{14} = f_{23} \tag{16}
\]

since \( f_i = f_i(t) > 0 \).

Also (14) and (15) imply that the coefficients \( \Delta_{ijk} \), with \( (i, j, k) \in A \cup B \), are given by

\[
\begin{align*}
\Delta_{127} &= -\frac{f_3}{f_1} B_{23} + \frac{f_4}{f_2} B_{14}, \\
\Delta_{135} &= \frac{f_5}{f_3} A, \\
\Delta_{146} &= \frac{f_5}{f_{14}} A - \frac{f_1}{f_3} B_{12} + \frac{f_4}{f_2} B_{34}, \\
\Delta_{567} &= A_2,
\end{align*}
\]

where

\[
\begin{align*}
A &= f_5 \left( \frac{1}{f_{23}} + \frac{1}{f_{14}} \right) + f_6 \left( \frac{1}{f_{13}} + \frac{1}{f_{24}} \right), \\
B_{12} &= \frac{1}{f_7^2} \left( \frac{f_2}{f_4} - \frac{f_1}{f_3} \right), \\
B_{23} &= \frac{1}{f_7^2} \left( \frac{f_2}{f_4} - \frac{f_3}{f_1} \right), \\
B_{14} &= \frac{1}{f_7^2} \left( \frac{f_4}{f_2} - \frac{f_3}{f_1} \right), \\
B_{34} &= \frac{1}{f_7^2} \left( \frac{f_4}{f_2} - \frac{f_5}{f_1} \right), \\
A_2 &= f_5^2 \left( \frac{1}{f_{23}} + \frac{1}{f_{24}} \right) + f_6^2 \left( \frac{1}{f_{13}} + \frac{1}{f_{24}} \right),
\end{align*}
\]

Using (17), one can check that \( f_{135} \Delta_{135} = f_{245} \Delta_{245} \). Thus, \( f_{13} = f_{24} \) by Lemma 1–ii). This equality and (16) imply

\[
f_1 = f_2, \quad f_3 = f_4. \tag{19}
\]

The equalities (19) imply that the functions \( B_{12} \) and \( B_{34} \) defined in (18) are such that \( B_{12} = 0 = B_{34} \).

Hence, \( \Delta_{146} = \frac{f_1}{f_{14}} A \). So, from (17), we have \( f_{146} \Delta_{146} = f_{245} \Delta_{245} \). Now, Lemma 1–ii) and (19) imply

\[
f_5 = f_6. \tag{20}
\]
Moreover, from (18) and (19) we get $B_{14} = -B_{23}$. Then, from (17) we have $f_{127} \Delta_{127} + f_{347} \Delta_{347} = 0$. Now, Lemma 1–iv) implies
\[ f_{12} + f_{34} = 2/f_7. \]

Thus,
\[ f_7 = \frac{2}{(f_1^2 + f_3^2)}. \tag{21} \]

Using the equalities (19) and (21), we obtain that $\Delta_{135} = \Delta_{567}$. Therefore, by Lemma 1–i) we have
\[ f_{13} = f_{67}. \]

From this equality and (21), we obtain
\[ f_6 = \frac{1}{2} f_{13} (f_1^2 + f_3^2). \tag{22} \]

In summary, from (19)–(22), we have
\[ f_1 = f_2, \quad f_3 = f_4, \quad f_5 = f_6 = \frac{1}{2} f_{13} (f_1^2 + f_3^2), \quad f_7 = \frac{2}{f_1^2 + f_3^2}. \tag{23} \]

Now, we can suppose that $f_3 = f_1 = f$ (see below Lemma 2). Then, the previous conditions reduce to
\[ f_1 = f_2 = f_3 = f_4 = f, \quad f_5 = f_6 = f^4, \quad f_7 = f^{-2}. \tag{24} \]

Then, (18), $B_{14} = 0 = B_{23}$ since $f_1 = f_2 = f_3 = f_4$ by (23). So, $\Delta_{127} = 0 = \Delta_{347}$. This implies that the unique non-zero components $\Delta_{ijk}$ of the Laplacian of $\Delta_t \varphi(t)$ are
\[ \Delta_{567} = \Delta_{135} = \Delta_{146} = \Delta_{236} = \Delta_{245} = 4f^4. \]

Then, the system of differential Equations (7) reduces to
\[ f^{-5} f' = \frac{2}{3}. \]

Integrating this equation, we obtain
\[ f = \left( C - \frac{8}{3} t \right)^{-\frac{1}{4}}, \quad C = constant. \tag{24} \]

But $f(0) = 1$ implies $C = 1$. Hence,
\[ f = f(t) = \left( 1 - \frac{8}{3} t \right)^{-\frac{1}{4}}. \]

Therefore, the one-parameter family of 3-forms $\varphi(t)$ given by (11) is the solution of the Laplacian flow of $\varphi$ on $K$, and it exists for every $t \in \left( -\infty, \frac{3}{8} \right)$. A simple computation shows that
\[ d\varphi(t) = f^6 \left( -e^{1357} + e^{1467} + e^{2367} + e^{2457} \right) = e^7 \wedge \varphi(t), \]
and so the Lee form $\theta(t)$ of $\varphi(t)$ is $\theta(t) = e^7$.

Now we study the behavior of the underlying metric $g(t)$ of such a solution in the limit for $t \to -\infty$. If we think of the Laplacian flow as a one parameter family of $G_2$ manifolds with a locally conformal calibrated $G_2$-structure, it can be checked that, in the limit, the resulting manifold has
vanishing curvature. For \( t \in (\infty, \frac{3}{8}) \), let us consider the metric \( g(t) \) on \( K \) induced by the \( G_2 \) form \( \varphi(t) \) given by (11). Then,

\[
g(t) = (1 - \frac{8}{3} t)^{-\frac{1}{2}} (e^1)^2 + (1 - \frac{8}{3} t)^{-\frac{1}{2}} (e^2)^2 + (1 - \frac{8}{3} t)^{-\frac{1}{2}} (e^3)^2 + (1 - \frac{8}{3} t)^{-\frac{1}{2}} (e^4)^2 + (1 - \frac{8}{3} t)^{-\frac{1}{2}} (e^5)^2 + (1 - \frac{8}{3} t)^{-\frac{1}{2}} (e^6)^2 + (1 - \frac{8}{3} t)^{-\frac{1}{2}} (e^7)^2.
\]

Then, taking into account the symmetry properties of the Riemannian curvature \( R(t) \) we obtain

\[
R_{1234} = R_{1256} = R_{3456} = -\frac{1}{2(1 - \frac{8}{3} t)},
\]

\[
R_{1313} = R_{1414} = R_{2323} = R_{2424} = \frac{3}{4(1 - \frac{8}{3} t)},
\]

\[
R_{1515} = R_{1616} = R_{2525} = R_{2626} = R_{3535} = R_{3636} = R_{4545} = R_{4646} = R_{1324} = R_{1432} = R_{1526} = R_{1652} = R_{3546} = R_{3654} = -\frac{1}{4(1 - \frac{8}{3} t)},
\]

where \( R_{ijkl} = R(t)(e_i, e_j, e_k, e_l) \). Therefore, \( \lim_{t \to -\infty} R(t) = 0 \).

Furthermore, the curvatures \( R(g(t)) \) of \( g(t) \) blow-up as \( t \) goes to \( \frac{3}{8} \), and the finite-time singularity is of Type I since \( R(g(t)) = O(1 - \frac{8}{3} t)^{-\frac{1}{2}} \) as \( t \to \frac{3}{8} \); in fact,

\[
\lim_{t \to \frac{3}{8}} \frac{|R(g(t))|}{(1 - \frac{8}{3} t)^{-\frac{1}{2}}} < \infty.
\]

\( \square \)

To complete the proof of Theorem 1, we show that under the conditions (19)–(22) the assumption \( f_1 = f_3 \), that we made in its proof, is correct.

**Lemma 2.** If the 3-form \( \varphi(t) \) defined in (13) is the solution for the Laplacian flow (5) of the \( G_2 \) form \( \varphi \) given by (10), then \( f_1(t) = f_3(t) \).

**Proof.** Take \( u = f_1 \) and \( v = f_3 \). We know that if the 3-form \( \varphi(t) \) defined in (13) is the solution for the Laplacian flow (5) of the \( G_2 \) form \( \varphi \), then the equalities (19)–(22) are satisfied. Now, taking into account (17), the equalities (19)–(22) imply that the Hodge Laplacian \( \Delta \varphi(t) \) of \( \varphi(t) \) has the following expression

\[
\Delta \varphi(t) = -\frac{(u^2 - v^2)(u^2 + v^2)^2}{2u^2} x_{127} + \frac{(u^2 - v^2)(u^2 + v^2)^2}{2v^2} x_{347} + (u^2 + v^2)^2 \left( x_{567} + x_{135} - x_{146} - x_{236} - x_{245} \right).
\]

Thus, for \( (i, j, k) \in \{(1, 2, 7), (3, 4, 7)\} \), the equation \( \Delta_{ijk} = \frac{u_{ijk}}{f_{ijk}} \) of the system (7) becomes in both cases

\[
\frac{du}{dt} = -\frac{(u^2 - 2v^2)(u^2 + v^2)^3}{12uv^2},
\]
while for \((i, j, k) \in A \cup B\) with \((1, 2, 7) \neq (i, j, k) \neq (3, 4, 7)\), the equation \(\Delta_{ijk} = \frac{(f_{ik})'}{f_{jk}}\) is expressed as

\[
\frac{dv}{dt} = \frac{(2u^2 - v^2) (u^2 + v^2)^3}{12uv}.
\]

Therefore, the system (7) becomes

\[
\begin{cases}
\frac{du}{dt} = \frac{(u^2 - 2v^2)(u^2 + v^2)^3}{12uv}, \\
\frac{dv}{dt} = \frac{(v^2 - 2v^2)(v^2 + v^2)^3}{12uv}, \\
u(0) = v(0) = 1.
\end{cases}
\] (25)

Thus,

\[
\frac{dv}{du} = \frac{-v(2u^2 - v^2)}{u(u^2 - 2v^2)}.
\] (26)

To solve this differential equation, we consider the change of variable \(w = v/u\). Then, (26) can be expressed as follows:

\[
u \frac{dw}{du} + w = \frac{2 - w^2}{1 - 2w^2}.
\]

We solve this differential equation by applying separation of variables, and we get the following solution

\[
\ln u + C = -\frac{1}{6} \ln \left(1 - w^2\right) + 2 \ln w = \frac{1}{6} \ln \left(\frac{v^2 - u^2}{u^4}\right),
\]

for some constant \(C\). This equation is equivalent to

\[
\check{\mathcal{C}} u^2 = v^2 \left(\frac{v^2 - u^2}{u^4}\right),
\]

for some constant \(\check{\mathcal{C}}\). Thus, \(\check{\mathcal{C}} = 0\) since \(u(0) = v(0) = 1\). Therefore, since \(v(t) = f_3(t) \neq 0\) for all \(t\), for the functions \(u\) and \(v\) we have three possibilities: \(u = v\), \(u = -v\) or \(v = 0\). But \(u(0) = v(0) = 1\), hence the only possibility is \(u(t) = v(t)\), that is, \(f_1(t) = f_3(t)\). (Here, we would like to note that since \(u(t) = v(t)\), the second differential equation of the system (25) reduces to \(\frac{6}{u} \frac{dv}{dt} = 4u^4\), that is the differential Equation (24), which we have solved before.) \(\Box\)

**Remark 1.** Note that proceeding in a similar way as Lauret did in [40] for the Ricci flow, we can evolve the Lie brackets \(\mu(t)\) instead of the 3-form defining the \(G_2\)-structure, and we can show that the corresponding bracket flow has a solution for every \(t\). In fact, if we fix on \(\mathbb{R}^7\) the 3-form \(x^{127} + x^{347} + x^{567} + x^{135} - x^{146} - x^{236} - x^{245}\), the basis \(\{x_1(t), \ldots, x_7(t)\}\) defines, for every real number \(t \in (-\infty, \frac{3}{2})\), a solvable Lie algebra with bracket \(\mu(t)\) such that \(\mu(0)\) is the Lie bracket of the Lie algebra \(k\) of \(K\). Moreover, the solution of the bracket flow converges to the null bracket corresponding to the abelian Lie algebra as \(t\) goes to \(-\infty\), and it blows-up as \(t\) goes to \(\frac{3}{2}\).

**Remark 2.** Taking into account (4) and (11), one can check that the torsion form \(\tau_2(t)\) of \(\varphi(t)\) is given by

\[
\tau_2(t) = \frac{4}{3} (1 - \frac{8}{3} t)^{-1} \left(e^{12} + e^{34}\right) - \frac{8}{3} \left(1 - \frac{8}{3} t\right)^{-5/2} e^{56}.
\]

Thus, \(\lim_{t \to -\infty} \tau_2(t) = 0\). However, the solution \(\varphi(t)\) does not converge to a locally conformal parallel \(G_2\)-structure as \(t\) goes to \(-\infty\) since, by (11), the \(G_2\) forms \(\varphi(t)\) degenerate when \(t \to -\infty\). Moreover, \(\varphi(t)\) blows-up as \(t\) goes to \(\frac{3}{2}\).
4.2. The Laplacian Flow on $S$

Now we consider the simply connected and solvable Lie group $S$ whose Lie algebra $s$ is defined as follows:

$$s = \begin{pmatrix} 1/2 e^1 \, 1/2 e^2 \, 1/2 e^2 \, 1\, e^{27} \, e^{14} + e^{23} + e^{57}, e^{13} - e^{24} + e^{67}, 0 \end{pmatrix}. \quad (27)$$

Then, the 3-form $\varphi$ given by

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245} \quad (28)$$

defines a left invariant locally conformal calibrated $G_2$-structure on the Lie group $S$, with Lee form $\theta = -e^7$, and so with torsion form $\tau_1 = -\frac{1}{3} e^7$. In fact,

$$d\varphi = e^{1357} - e^{1467} - e^{2367} - e^{2457} = -e^7 \wedge \varphi. \quad (29)$$

Since $S$ is a nonunimodular Lie group, $S$ cannot admit a lattice $\Gamma$ such that the quotient space $\Gamma \backslash S$ is a compact solv manifold. In fact, the linear map $s \to \mathbb{R}$, $X \to \text{tr}(\text{ad} X)$ is such that $\text{tr}(\text{ad} e_7)$ is non-zero, where $\{e_1, \ldots, e_7\}$ is the basis of $s$ dual to the basis $\{e^1, \ldots, e^7\}$ of $s^*$.

**Theorem 2.** The family of locally conformal calibrated $G_2$-structures $\varphi(t)$ on $S$ given by

$$\varphi(t) = (1 - 4t)^{3/4} e^{127} + (1 - 4t)^{3/4} e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245} \quad (29)$$

is the solution for the Laplacian flow (5) of the $G_2$ form $\varphi$ given by (28), where $t \in (-\infty, \frac{1}{4})$. The Lee form $\theta(t)$ of $\varphi(t)$ is $\theta(t) = -e^7$. Moreover, the underlying metrics $g(t)$ of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in $S$, as $t$ goes to $-\infty$, and they blow-up as $t$ goes to $\frac{1}{4}$.

**Proof.** To study the flow (5) of the $G_2$ form $\varphi$ defined in (28), we should proceed as in Theorem 1. However, in order to shorten the proof, we will show directly that the one-parameter family of $G_2$-structures given by (29) is the solution for the flow (5). For this, we consider the differentiable real functions $f_i = f_i(t)$ ($i = 1, \ldots, 7$) given by

$$f_1(t) = (1 - 4t)^{1/8}, \quad i = 1, 2, 3, 4,$n
$$f_5(t) = f_6(t) = (1 - 4t)^{-1/4}, \quad (30)$$
$$f_7(t) = (1 - 4t)^{1/2}.$$

These functions are defined for all $t \in (-\infty, \frac{1}{4})$; moreover, $f_i(t) > 0$, for $t \in (-\infty, \frac{1}{4})$.

Now, for each $t \in (-\infty, \frac{1}{4})$, we consider the basis $\{x^1, \ldots, x^7\}$ of left invariant 1-forms on $S$ defined by

$$x^i = x^i(t) = f_i(t) e^i, \quad 1 \leq i \leq 7.$$

Taking into account (30) and (27), the structure equations of $S$ with respect to the basis $\{x^1, \ldots, x^7\}$ are

$$dx^1 = \frac{1}{2} (1 - 4t)^{-1/2} x^{17}, \quad dx^2 = \frac{1}{2} (1 - 4t)^{-1/2} x^{27},$$
$$dx^3 = \frac{1}{2} (1 - 4t)^{-1/2} x^{37}, \quad dx^4 = \frac{1}{2} (1 - 4t)^{-1/2} x^{47},$$
$$dx^5 = (1 - 4t)^{-1/2} (x^{14} + x^{23} + x^{57}), \quad dx^6 = (1 - 4t)^{-1/2} (x^{13} - x^{24} + x^{67}),$$
$$dx^7 = 0. \quad (31)$$
For any \( t \in \left( -\infty, \frac{1}{4} \right) \), we consider the 3-form \( \varphi(t) \) on \( S \) given by
\[
\varphi(t) = x^{127} + x^{347} + x^{567} + x^{135} - x^{146} - x^{236} - x^{245}.
\] (32)

Then, this 3-form \( \varphi(t) \) defines a \( G_2 \)-structure on \( S \), and it is equal to the 3-form \( \varphi(t) \) defined in (29). Note that the 3-form \( \varphi(t) \) is such that \( \varphi(0) = \varphi \) and, for any \( t \), \( \varphi(t) \) determines the metric \( g(t) \) on \( S \) such that the basis \( \{ x_i = \frac{1}{2} e_i ; i = 1, \ldots, 7 \} \) of left invariant vector fields on \( S \) dual to \( \{ x^1, \ldots, x^7 \} \) is orthonormal. So, \( g(t)(e_i, e_i) = f_i^2 \).

Moreover, for every \( t \in \left( -\infty, \frac{1}{4} \right) \), \( \varphi(t) \) defines a locally conformal calibrated \( G_2 \)-structure on \( S \). In fact,
\[
d\varphi(t) = e^{1357} - e^{1467} - e^{2367} - e^{2457} = -\theta^7 \wedge \varphi(t),
\]

since on the right-hand side of (29) the terms \( e^{127} \) and \( e^{347} \) are both closed and \( d(e^{567} + e^{135} - e^{146} - e^{236} - e^{245}) = e^{1357} - e^{1467} - e^{2367} - e^{2457} \). So, the Lee form \( \theta(t) \) of \( \varphi(t) \) is \( \theta(t) = -e^7 \).

Next, we show that \( \frac{d}{dt} \varphi(t) = \Delta_t \varphi(t) = (\ast d \ast t - d \ast t \ast t) \varphi(t) \). Using (31) and (32), we obtain
\[
\frac{d}{dt} \varphi(t) = -3(1 - 4t)^{-1} \left( x^{127} + x^{347} \right).
\] (33)

On the other hand, we have
\[
(\ast d \ast t \ast t) \varphi(t) = -4(1 - 4t)^{-1} x^{567} - 2(1 - 4t)^{-1} \left( x^{135} - x^{146} - x^{236} - x^{245} \right),
\] (34)

and
\[
(-d \ast t \ast t) \varphi(t) = -3(1 - 4t)^{-1} \left( x^{127} + x^{347} \right) + 4(1 - 4t)^{-1} x^{567}
+ 2(1 - 4t)^{-1} \left( x^{135} - x^{146} - x^{236} - x^{245} \right).
\] (35)

Therefore, (33), (34) and (35) imply \( \frac{d}{dt} \varphi(t) = \Delta_t \varphi(t) \).

To complete the proof, we study the behavior of the underlying metrics of such a solution in the limit for \( t \to -\infty \). If we think of the Laplacian flow as a one parameter family of \( G_2 \) manifolds with a locally conformal calibrated \( G_2 \)-structure, it can be checked that, in the limit, the resulting manifold has vanishing curvature. Denote by \( g(t), t \in \left( -\infty, \frac{1}{4} \right) \), the metric on \( S \) induced by the \( G_2 \) form \( \varphi(t) \) given by (29). Then, \( g(t) \) has the following expression
\[
g(t) = (1 - 4t)^{\frac{1}{2}} (e^1)^2 + (1 - 4t)^{\frac{1}{2}} (e^2)^2 + (1 - 4t)^{\frac{1}{2}} (e^3)^2 + (1 - 4t)^{\frac{1}{2}} (e^4)^2
+ (1 - 4t)^{-\frac{1}{2}} (e^5)^2 + (1 - 4t)^{-\frac{1}{2}} (e^6)^2 + (1 - 4t)^{-\frac{1}{2}} (e^7)^2.
\]

Now, one can check that every non-vanishing coefficient appearing in the expression of the Riemannian curvature \( R(g(t)) \) of \( g(t) \) is proportional to \( \frac{1}{(1-4t)} \). Therefore, \( \lim_{t \to -\infty} R(0) = 0 \).

Furthermore, the curvatures \( R(g(t)) \) of \( g(t) \) blow-up as \( t \) goes to \( \frac{1}{4} \), and the finite-time singularity is of Type I since \( R(g(t)) = O(1 - 4t)^{-1} \) as \( t \to \frac{1}{4} \); in fact
\[
\lim_{t \to \frac{1}{4}} \frac{|R(g(t))|}{(1 - 4t)^{-1}} < \infty.
\]

\[\square\]

Remark 3. As we have noticed in Remark 1, we can also evolve the Lie brackets \( \nu(t) \) instead of the 3-form defining the left invariant \( G_2 \)-structure on \( S \), and we can show that the corresponding bracket flow has a solution for every \( t \in \left( -\infty, \frac{1}{4} \right) \). In fact, if we fix on \( \mathbb{R}^7 \) the 3-form \( x^{127} + x^{347} + x^{567} + x^{135} - x^{146} - x^{236} - x^{245} \), the basis \( \{ x_1(t), \ldots, x_7(t) \} \) defines, for every real number \( t \in \left( -\infty, \frac{1}{4} \right) \), a solvable Lie algebra with bracket \( \nu(t) \) such that \( \nu(0) \) is the Lie bracket of the Lie algebra \( S \) of \( S \). As for the Lie group \( K \) (see Remark 1), the solution
of the bracket flow converges to the null bracket corresponding to the abelian Lie algebra as $t$ goes to $-\infty$, and it blows-up as $t$ goes to $-\frac{1}{3}$.

**Remark 4.** Taking into account (4) and (29), one can check that the torsion form $\tau_2(t)$ of $\varphi(t)$ is given by

$$\tau_2(t) = \frac{5}{3}(1-4t)^{-1/4}(e^{12} + e^{34}) - \frac{10}{3}(1-4t)^{-1}e^{56}.$$

Thus, $\lim_{t \to -\infty} \tau_2(t) = 0$. However, the solution $\varphi(t)$ does not converge to a locally conformal parallel $G_2$-structure as $t$ goes to $-\infty$ since, by (29), the $G_2$ forms $\varphi(t)$ blow-up when $t \to -\infty$, and $\varphi(t)$ degenerate as $t$ goes to $-\frac{1}{3}$. Note that the metrics behaves differently for $S$ than for $K$. Indeed, the induced metrics by the solution of the Laplacian flow on $S$ blow-up at infinity and at the finite time, while the induced metrics by the solution of the Laplacian flow on $K$ only blow-up as $t$ goes to $\frac{3}{8}$.

**Remark 5.** Note that, for every $t \in \left(-\infty, -\frac{1}{3}\right)$, the metric $g(t)$ is an Einstein metric with negative scalar curvature on the Lie group $S$. In fact, with respect to the orthonormal basis $\{x_1(t), \ldots, x_7(t)\}$, we have

$$\text{Ric}(g(t)) = -\frac{3}{1-4t}g(t) = -\frac{3}{1-4t} \sum_{1 \leq i \leq 7} (x^i)^2.$$
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