Classification of (2+1)–Dimensional Growing Surfaces Using Schramm–Loewner Evolution

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Statistical behavior and scaling properties of iso-height lines in three different saturated two-dimensional grown surfaces with controversial universality classes are investigated using ideas from Schramm-Loewner evolution (SLEα). We present some evidence that the iso-height lines in the ballistic deposition (BD), Eden and restricted solid-on-solid (RSOS) models have conformally invariant properties all in the same universality class as the self-avoiding random walk (SAW), equivalently SLEκ/4. This leads to the conclusion that all these discrete growth models fall into the same universality class as the Kardar-Parisi-Zhang (KPZ) equation in two dimensions.

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Nonequilibrium growth processes exhibit nontrivial scaling behavior which are often characterized and classified by three exponents, the dynamical exponent z and the growth exponent β [1–6]. Analytic results for the values of these exponents are scarce and one has to depend on numerical analysis. In some cases, namely the two-dimensional (2D) Kardar-Parisi-Zhang (KPZ) equation, numerical results are not definitive and ambiguities remain. On the other hand it is by no means clear that this set of exponents is exhaustive and other characterizing exponents may exist. In this paper, we propose a new method for analysis of two dimensional rough surfaces based on Schramm-Loewner evolution [7]. In this method iso-height lines in the saturated regime are analyzed as random simple paths in which no self-crossing occurs. This leads to an extra characteristic for a grown surface, namely κ, the diffusivity coefficient of SLE. Our approach arrives at a sharp conclusion on the universality class of growing surfaces in two dimensions.

The discrete ballistic deposition (BD) [8], Eden, and restricted solid-on-solid (RSOS) models [9] (for a review and definitions of these models see [2]), are believed to be in the same universality class as the KPZ equation [3] which describes a non-conserved growth.

The evidence that two models belong to the same universality class can be given in many ways. One of the most direct approaches, as pointed out in [10], is to show that the two models correspond to the same fixed point system. In this direction and based on master-equation approach, the KPZ equation has been exactly derived for the RSOS model [11], which indicates that these two models belong to the same universality class.

However, the story for the BD model is more controversial. A continuum equation is derived from the BD microscopic rules in [10] which deviates from the KPZ equation (the model which is considered in [10] is the next-nearest-neighbor (NNN) BD model, slightly different from the nearest-neighbor (NN) BD model in our present paper). Despite this deviation, the symmetry arguments suggest that the 1D BD system is in the same universality class of the KPZ equation while for the 2D case, the absence of the rotational symmetry in the derived continuum equation violates the a priori reason for them belonging to the same universality class. An exact lattice Langevin equation for the BD model has been derived in [12], whose continuum limit is shown to be dominated by the KPZ equation. Although for a 1D substrate the solution of the exact lattice Langevin equation yields the KPZ scaling exponents, but for a 2D substrate its scaling exponents are again different from those obtained from simulations [12].

Another way to determine the universality class of a rough surface is to compute its exponents α, β and κ. Numerical results are consistent with the proposition that RSOS and KPZ models belong to the same universality class, but the situation is more controversial when considering the BD and Eden models.

For the KPZ equation in d = 1, the exact values α = 1/2 and β = 1/3 are known [3]. The estimated values obtained by various numerical works on BD in d = 1 for roughness exponent and growth exponent range from α = 0.42 to 0.506 and β = 0.3 to 0.339 [13–17]. Among the results, those obtained by Reis [17] are close enough to the exact KPZ values. In d = 2, there is no exact computation of the exponents for the KPZ system, nevertheless, various numerical and theoretical approaches have been applied to measure the exponents. The simulations based on direct numerical integration of the KPZ equation in 2D give α = 0.37 to 0.4 [18, 19], and the values obtained by various theoretical methods range from α = 0.29 to 0.4 [20–23]. Among the theoretical approaches, application of the mode-coupling approximation for the KPZ equation in 2D [22] yielded α ≃ 0.38, in good agreement with the values found from simulations. However, the result α ≃ 0.29 obtained by the self-consistent expansion for 2D KPZ equation [20] displays a discrepancy with the results of simulations.

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TABLE I: Scaling exponents for BD, Eden, RSOS and KPZ [30] models in two dimensions obtained by our simulations. With the exception of KPZ, the averages were taken over $10^3$ independent simulation runs for different square substrates of size $50 \leq L \leq 700$.

| Model | $\alpha$  | $z$     | $\beta$  |
|-------|-----------|---------|----------|
| BD    | 0.28(2)   | 1.70(5) | 0.15(1)  |
| Eden  | 0.36(2)   | 1.65(5) | 0.205(15)|
| RSOS  | 0.393(10) | 1.58(3) | 0.240(5) |
| KPZ   | 0.37(1)   | 1.61(3) | 0.23(1)  |

The diversity of the obtained values of various simulations for 2D BD model ranging from 0.26 to 0.38 for $\alpha$ and 0.21 to 0.24 for $\beta$ [14–17, 24, 25], does not indeed provide a convincing evidence that it belongs to the same universality class of the KPZ model. The same story holds for the Eden model with scattered reported results

for $\alpha$ ranging from 0.20 to 0.39 [2] in (2+1) dimensions.

We have carried out extensive simulations of the RSOS, NN-BD, and Eden models to estimate the values of the three exponents $\alpha$, $z$ and $\beta$ in (2+1) dimensions. In some cases our results take values out of the above mentioned ranges (see table I). Although the two well-known scaling relations $\beta = \alpha/z$ and $\alpha + z = 2$ [26] are obeyed, within the statistical errors. Nevertheless, due to the significant difference between the exponent values of these three models, it is not possible to conclude that they belong to the same universality class.

A new tool for study of domain walls in critical systems is the theory of Schramm-Loewner Evolution (or SLE$_\kappa$) [7] (for a review see [27]). The diffusivity constant $\kappa$ determines the critical exponents hence the universality class of the system in question. Some authors have argued that SLE may be applied to turbulence [28] and surface growth phenomena [29–31] as well. Recently, we have reported some evidence of conformal invariance in the statistical properties of iso-height lines in the saturated growth models including an experimentally grown $WO_3$ surface [29], and numerical study of the KPZ equation done by direct integration of the discretized KPZ equation in (2+1) dimensions [30, 31]. The mere observation of scale invariance in a 2D physical system does not necessarily imply conformal invariance [32, 33]. However arguments pointing to conformal invariance of 2D growing surfaces have been attempted [34]. This suggests the need for more stringent tests of conformal invariance in such systems.

In this paper, we report the results of extensive simulations on three different discrete growth models i.e., RSOS, NN-BD and Eden models. We present evidence that iso-height lines on saturated surfaces are SLE curves of the same diffusivity $\kappa = 8/3$ indicating that in this spirit and within statistical errors, all these models belong to the KPZ universality class.

The growth simulations were undertaken on an anisotropic geometry i.e., strips of size $L_x \times L_y$ with $L_y = L$, $L_x = 3L$ and $50 \leq L \leq 10^3$ (for Eden model, due to the long CPU time, the simulations were done up to size $L = 600$). Periodic boundary conditions were applied in both directions (Strip geometry was chosen for congruence with dipolar SLE). We gathered a number of $5 \times 10^3$ of saturated samples of each size for each of the three RSOS, BD and Eden models. The samples were obtained from $10^2$ independent simulation runs. During each simulation run, each of the 50 samples was selected after $10^3$ time steps (in units of the number of lattice sites) after the saturation time.

![FIG. 1](image1.png)

**FIG. 1:** (Color online) The positive-height clusters shown in different colors, and the corresponding spanning iso-height lines (solid lines).

For each height configuration $\{h_i\}$, a level cut is made at the mean height, say $h := 0$. Then the cluster heights were defined as individual sets of connected sites with positive height which were identified by using the Hoshen-Kopelman algorithm [35]. All spanning clusters along $y$–direction were marked. For each spanning cluster, a walker algorithm was applied to determine each of its perimeters which connects the lower boundary to the upper one i.e., the spanning iso-height lines (Fig. 1). The iso-height lines were identified uniquely by using the tie-breaking rule on the square lattice, described in [36]. Thereby, an ensemble of contour lines of fixed linear size

![FIG. 2](image2.png)

**FIG. 2:** (Color online) The average length $l$ of a spanning iso-height line of the saturated growth models versus the system size $L$, $l \sim L^{4/3}$. The solid line shows the expected result for SAW. The error bars are almost the same size as the symbols.
$L$ was obtained for each surface ensemble of different size.

We base our arguments on four different tests acertaining that the iso-height lines are SLE: the fractal dimension, winding angle statistics, left passage probability and direct SLE test.

i) Fractal dimension. For conformally invariant curves the fractal dimension is related to the diffusivity $\kappa$ by the relation $d_f = 1 + \kappa/8$. The fractal dimension and the conjectured value of diffusivity for SAW are known to be $d_f = 4/3 = 1.33$ and $\kappa = 8/3 = 2.66$.

In Fig. 2 we show our computed results for the fractal dimension of the iso-height lines for different models. The best fit to our data in the whole range of the lattice sizes yields $d_f = 1.345(7), 1.330(5)$ and $1.335(4)$ for BD, Eden and RSOS models, respectively. In all of our measurements for BD model, our analysis on slightly larger system sizes $150 \leq L \leq 10^3$ corresponds to the same results as the Eden and RSOS models. For example we find $d_f = 1.337(5)$ for the BD model in this size range. As shown in Fig. 2, our results are well compatible with the fractal dimension of SAW and SLE$_{8/3}$.

ii) Winding angle statistics. The winding angle between two end points of a finite SAW in two dimensions, is studied in [37] using Coulomb gas methods. They found that the winding angle is Gaussian distributed with a variance of $\sim (8/g) \ln L$, where $L$ is the distance between the end points and $g$ is Coulomb gas coupling parameter which is related to $\kappa$ by $g = 4/\kappa$. They have also shown that the winding angle at a single end point relative to the global average direction of the curve is a Gaussian with variance of $(4/g) \ln L$.

It is shown in [38] that the variance in the winding angle at typical points along the curve is $1/4$ as large as the variance in the winding at the end points, i.e.,

$$\langle \theta^2 \rangle = a + (\kappa/4) \ln L. \quad (1)$$

Using the same definition as in [38], we measured $\langle \theta^2 \rangle$ for different models, results are shown in Fig. 3. Data points compare well with Eq. (1) (solid lines), using $\kappa = 8/3$ and a suitable value of the parameter $a$ obtained from the best fit to data for each model. The direct measurement of $\kappa$ also obtained from the best fits to the data shown in the inset of the Fig. 3. We find almost the same value $\kappa/4 = 0.700(20)$ for BD (again for larger sizes), Eden and RSOS models, within the statistical error.

iii) Left passage probability. The probability $P_{\kappa}(\varphi)$ that an SLE$_{\kappa}$ curve, in the upper half-plane, passes to the left of a given point at polar coordinates $(\rho, \varphi)$, is computed by Schramm [39]

$$P_{\kappa}(\varphi) = \frac{1}{2} + \frac{1}{\sqrt{\pi} \Gamma \left(\frac{1}{2}\right)} \frac{\Gamma \left(\frac{1}{\kappa}\right)}{\Gamma \left(\frac{1}{2}\right)} 2F_1 \left(\frac{1}{2}, \frac{3}{2}; \frac{1}{\kappa}; 2 \cot^2(\varphi) \right) \cot(\varphi), \quad (2)$$

where $2F_1$ is the hypergeometric function.

As another check, we measure this quantity (which should also hold for a dipolar SLE near the starting point i.e., $\rho \ll L$) for the contour lines of different growth models.

As shown in Fig. 4, our results for the three models are again in good agreement with the prediction for SLE$_{8/3}$.

iv) Direct SLE test. Using a discrete Loewner evolution and successive appropriate conformal maps, we extracted the Loewner deriving function $\{\xi_t\}$ of each iso-height line represented by the sequences of points $\{z_0, z_1, ..., z_N\}$ in the complex half-plane, with $z_0 = (0, 0)$. We use the function $g_2(z) = \sqrt{(\xi_2 - \xi_1) + 4t + \xi_1}$, with $t = \frac{1}{4} x_1^2$ and $\xi_t = R z_1$, to map all of the points except the first one to a shortened renumbered sequence. After each recursive map the first point in the sequence is swallowed and a sequence of $\{\xi_t\}$ can be obtained for each iso-height line. We have also checked the map appropriate for dipolar SLE [40], and found no significant differences.
We find that the statistics of the deriving function for each curve ensemble of fixed linear size \( L \), converges to a Gaussian process with variance \( \langle \xi_t^2 \rangle = \kappa t \). Finite size effects may be reduced by looking at shorter segments of the curve e.g., when 10% of the total average length of the curves is mapped. An example is shown in the inset of Fig. 5, obtained for the contour ensemble of the RSOS model with \( L = 10^4 \). We observe that \( \kappa \) shows a slight dependence on system size, but reducing with \( L \). We find that the value of the diffusivity \( \kappa \) for larger system sizes approaches the expected value for SAW i.e., \( \kappa = 8/3 \), for all three models. As shown in Fig. 5, for RSOS model this convergence begins in rather smaller sizes but for Eden and BD models larger system sizes are needed.

The more accurate results obtained here for the BD model is slightly different from that reported in [29], this is due to the considerable difference in the number of averaging samples and reduced finite size effects.

Summing up, although the numerical values of three exponents \( \alpha, \beta \) and \( z \) for BD, Eden, RSOS and KPZ models are scattered, the numerical value found for \( \kappa \) is sufficiently sharp to suggest that these models all belong to the same universality class. What remains is the inter-dependence of these exponents and \( \kappa \). The existence of two scaling relations \( \beta = \alpha/z \) and \( \alpha + z = 2 \), guarantees that there is only one independent exponent e.g., the roughness exponent \( \alpha \). On the other hand, this paper introduces a new characteristic value for a rough surface: \( \kappa \). How inter-dependent are these two? There is a powerful scaling argument given in [41] which connects \( d_f \) of a contour line to \( \alpha \) of the same surface, \( d_f = 2 - x_1 - \alpha/2 \), where \( x_1 \) is the loop correlation exponent. Although the exact value of \( x_1 = 1/2 \) is known only for the limiting cases of \( \alpha = 0 \) and 1, but it is conjectured that its value is super universal and is independent of \( \alpha \) for Gaussian surfaces. This leaves the case of 2D KPZ in ambiguity, since it does not follow a Gaussian distribution. A simple minded value \( x_1 = 1/2 \) leads to \( d_f = 4/3 \) giving \( \alpha = 1/3 \) which although elegant [21] but is in conflict with numerical results. Our future efforts will concentrate on revealing the nature of this relationship.

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