Hecke algebras are usually defined algebraically, via generators and relations. We give a new algebro-geometric construction of affine and double-affine Hecke algebras (the former is known as the Iwahori-Hecke algebra, and the latter was introduced by Cherednik [Ch1]) in terms of residues. More generally, to any generalized Cartan matrix \( A \) and a point \( q \) in a 1-dimensional complex algebraic group \( C \) we associate an associative algebra \( H_q \). If \( A \) is of finite type and \( C = C^\times \), the algebra \( H_q \) is the affine Hecke algebra of the corresponding finite root system. If \( A \) is of affine type and \( C = C^\times \) then \( H_q \) is, essentially, the Cherednik algebra. The case \( C = C^\times + \) corresponds to ‘degenerate’ counterparts of the above objects considered by Drinfeld [Dr] and Lusztig [L2]. Finally, taking \( C \) to be an elliptic curve one gets some new elliptic analogues of the affine Hecke algebra.

Let \( W \) be the Weyl group and \( T \) the maximal torus of a Kac-Moody group associated to the Cartan matrix \( A \). Write \( X_*(T) \) for the lattice of one-parameter subgroups in \( T \) and \( C(T) \) for the field of rational functions on \( T \) with the natural \( W \)-action. There is a well-known description of affine Hecke algebra as the subalgebra of \( \text{End}_{C} C(T) \) generated by the so-called Demazure-Lusztig operators, cf. [Lu]. We observe that all the operators obtained in this way have poles of a very special kind. Thus, for any 1-dimensional algebraic group \( C \), we can give an ‘external’ definition of an algebra \( H_q \) as the subalgebra of operators on the vector space of rational functions on \( C \otimes_\mathbb{Q} X_*(T) \) consisting of the elements with a certain specific singularity type. All standard properties of Hecke algebras (e.g. the normal form of elements of the Cherednik algebra) follow readily from this definition. Our construction completes and somewhat clarifies the results of Kostant-Kumar [KK1] on the equivariant cohomology of the affine flag manifold.

1. Main construction.
1.1. Let \( A = \|a_{ij}\|_{i,j=1}^n \) be a symmetrizable generalized Cartan matrix.
To simplify notation, we will write $f$ is assumed to be affine, with the exception of §4 devoted to the elliptic case.

Throughout the paper we fix a root data, see [Sp], associated to $A$, that is:

- A free abelian group $X$ such that $rk(X) + rk(A) = 2n$;
- $\mathbb{Z}$-independent elements $\alpha_1, ..., \alpha_n \in X$, called simple roots;
- $\mathbb{Z}$-independent elements $\alpha_1^\vee, ..., \alpha_n^\vee \in X^\vee$, such that $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$, where $X^\vee := Hom(X, \mathbb{Z})$ and $\langle, \rangle : X^\vee \times X \to \mathbb{Z}$ is the natural pairing. The $\alpha_i^\vee$ are called simple coroots.

The vector space $\mathbb{C} \otimes_{\mathbb{Z}} X$ together with the collections of simple roots and coroots, as above, is called a realization of $A$, see [Ka]. Given a realization of $A$, one associates (see [Ka]) the Kac-Moody Lie algebra $g(A)$ with Cartan subalgebra $\mathbb{C} \otimes_{\mathbb{Z}} X^\vee$. Similarly, given a root data, one associates the Kac-Moody group $G(A)$ with maximal torus $\mathbb{C}^* \otimes_{\mathbb{Z}} X^\vee$ (the group $G(A)$ will not be used in this paper).

Write $R \subset X$ for the set of roots and $W$ for the Weyl group of $G(A)$. The group $W$ is generated by the simple reflections $s_i, \; i = 1, ..., n$, corresponding to the $\alpha_i$. Let $R^\text{re} \subset R$ be the set of real roots, i.e., roots of the form $w\alpha_i$, $w \in W$, $i = 1, ..., n$. For any $\alpha \in R^\text{re}$ the corresponding reflection, $s_\alpha$, belongs to $W$, in particular, $s_i = s_{\alpha_i}$. One has a decomposition $R = R_+ \sqcup R_-$ into positive and negative roots, and we put $R^\text{re}_+ := R^\text{re} \cap R_+$.

From now on, we fix $C$, a 1-dimensional complex algebraic group written multiplicatively. We form the abelian algebraic group $T = C \otimes_{\mathbb{Z}} X^\vee$, and write $C(T)$ for the field of rational functions on $T$. Associated to any $\lambda \in X$ is a group homomorphism $C \otimes_{\mathbb{Z}} X^\vee \to C$ denoted $t \mapsto t^\lambda$. Explicitly, if $t = c \otimes x$, where $c \in C$, $x \in X^\vee$, we have $t^\lambda = c^{(\lambda, x)}$. Given a point $q \in C$ we put $T_{\lambda, q} := \{ t^\lambda = q \}$, a divisor in $T$. This is a reduced but not necessarily connected subvariety of $T$. If $q = 1$ we simply write $T_{\lambda} = T_{\lambda, 1}$ for the kernel divisor.

1.2. There is a $W$-action on $T = C \otimes_{\mathbb{Z}} X^\vee$ induced by the natural $W$-action on the second factor. This gives a left $W$-action $\upsilon : f \mapsto \upsilon f$ on the field $C(T)$ by the formula $\upsilon f(t) = f(w^{-1} \cdot t)$. Introduce a twisted group algebra, $C(T)[W]$, as the complex vector space $C(T) \otimes_{\mathbb{C}} C[W]$ with multiplication:

$$(f \otimes w) \cdot (g \otimes y) = (f \cdot \upsilon y) \otimes (w \cdot y) \quad f, g \in C(T), \; w, y \in W$$

To simplify notation, we will write $f[w]$ instead of $f \otimes w$ in the future.

Throughout the paper we fix an element $q \in C$, $q \neq \pm 1$. Also, the curve $C$ is assumed to be affine, with the exception of §4 devoted to the elliptic case.
Definition 1.3. Let $\tilde{\mathcal{H}}$ be the C-linear subspace in $\mathbb{C}(T)[W]$ consisting of the elements $\sum_{w \in W} f_w[w]$ such that:

1. Each function $f_w$ has no other singularities but first order poles at the divisors $T^\alpha$, for a finite number of roots $\alpha \in R^e_+$. 
2. For every $w \in W$ and $\alpha \in R^e_+$, we have $\text{Res}_{T^\alpha}(f_w) + \text{Res}_{T^\alpha}(f_{s\alpha w}) = 0$.

Let $\mathcal{H}_q$ be the subspace of $\tilde{\mathcal{H}}$ consisting of elements as above satisfying the following additional condition:

3. The function $f_w$ vanishes on $T^\alpha,q−2$ whenever $\alpha \in R^e_+$ and $w^{−1}(\alpha) \in R$.

Theorem 1.4. Both $\tilde{\mathcal{H}}$ and $\mathcal{H}_q$ are subalgebras in $\mathbb{C}(T)[W]$.

1.5. Proof: We first show that $\tilde{\mathcal{H}}$ is a subalgebra.

Fix a root $\alpha \in R^e_+$. Let $P = \sum P_w[w]$ and $Q = \sum Q_w[w]$ be two elements of $\tilde{\mathcal{H}}$. Then

$$P \cdot Q = \sum_{u \in W} F_u[u], \quad \text{where} \quad F_u = \sum_{wy = u} P_w \cdot w^u Q_y \quad (1.5.1)$$

To check (1.3.1) we must show that each coefficient $F_u$ has at most first order pole at generic points of the hypersurface $T^\alpha$. To that end, observe that, for fixed $u \in W$, the set of pairs $(w, y)$ such that $wy = u$ has a fixpoint free involution $(w, y) \mapsto (s\alpha w, w^{−1} s\alpha wy)$. Split the sum in (1.5.1) into partial sums, each taken over an orbit of this involution. A partial sum has the form

$$P_w \cdot u^w Q_y + P_{s\alpha w} \cdot s\alpha w Q_{w^{−1} s\alpha wy} \quad (1.5.2)$$

Adding and subtracting the term $P_{s\alpha w} \cdot u^w Q_y$ and using that $w^{−1} s\alpha w = s_w^{−1}(\alpha)$, one rewrites (1.5.2) as follows

$$(P_w + P_{s\alpha w}) \cdot u^w Q_y - P_{s\alpha w} \cdot \left(u^w Q_y - s_w^w Q_{s_w^{−1}(\alpha)} y\right) \quad (1.5.3)$$

Observe now that the function $P_w + P_{s\alpha w}$ is regular at $T^\alpha$ by (1.3.2), while the function $u^w Q_y$ has at most simple pole at $T^\alpha$. Hence, the first summand in (1.5.3) has at most simple pole at $T^\alpha$. Similarly, condition (1.3.2) applied to $Q_y$ and the root $w^{−1}(\alpha)$ says that the function $Q_y + Q_{s_w^{−1}(\alpha)} y$ is regular at the divisor $T^{w−1}(\alpha) = w^{−1}(T^\alpha)$. Hence, the function $u^w Q_y + u^w Q_{s_w^{−1}(\alpha)} y$ which takes an element $t \in T$ into $Q_y(w^{−1} t) + Q_{s_w^{−1}(\alpha)} y(w^{−1} t)$, is regular for $t \in T^\alpha$. 

3
Observe finally that, for any function \( f \), we have \( \text{Res}_{T_\alpha}(f) = -\text{Res}_{T_\alpha}(\sigma^w f) \). Therefore the expression \( uQ_y - s_{\alpha u}Q_{s_{w^{-1}(\alpha)}y} \) in the second term of (1.5.3) is regular at \( T_\alpha \). Thus, (1.5.3) has at most simple pole at \( T_\alpha \), and condition (1.3.1) holds for \( P \cdot Q \).

We now check that condition (1.3.2) holds for \( P \cdot Q \) and any root \( \alpha \). Given \( u \in W \), we must show that the sum (cf. (1.5.1)):
\[
F_u + F_{s_\alpha u} = \sum_{w y = u} (P_w \cdot uQ_y + P_{s_\alpha w} \cdot s_{\alpha w}Q_y)
\]  
(1.5.4)
is regular at \( T_\alpha \). To that end, we split the sum in (1.5.4) into partial sums corresponding to \( w \) running over an individual orbit of the involution \( w \mapsto s_\alpha w \) and \( y \) running over an individual orbit of the involution \( y \mapsto w^{-1}s_\alpha wy \) respectively. Such a partial sum has the form
\[
P_w \cdot uQ_y + P_{s_\alpha w} \cdot s_{\alpha w}Q_{s_{w^{-1}(\alpha)}y} + P_{s_\alpha w} \cdot s_{\alpha w}Q_y + P_w \cdot s_{\alpha w}Q_{s_{w^{-1}(\alpha)}y}
\]
Regrouping terms, we obtain
\[
P_w \cdot (uQ_y + s_{\alpha w}Q_{s_{w^{-1}(\alpha)}y}) + P_{s_\alpha w} \cdot s_{\alpha w}(Q_{s_{w^{-1}(\alpha)}y} + uQ_y)
\]  
(1.5.5)
Applying condition (1.3.2) we see as above that in (1.5.5) the two expressions in the brackets are both regular at \( T_\alpha \). Furthermore, observe that the restrictions of \( uQ_y + s_{\alpha w}Q_{s_{w^{-1}(\alpha)}y} \) and \( s_{\alpha w}(Q_{s_{w^{-1}(\alpha)}y} + uQ_y) \) to \( T_\alpha \) are the same. Writing \( r \) for this restriction we find that the residue of (1.5.5) at \( T_\alpha \) equals
\[
r \cdot \text{Res}_{T_\alpha}(P_w + P_{s_\alpha w})
\]
The second factor vanishes by (1.3.2). Thus, we have proved that \( \mathcal{H} \) is a subalgebra.

(1.5.6) **Notation**: given \( w \in W \), write \( D(w) \) for the set of \( \alpha \in R_+^e \) such that \( w^{-1}(\alpha) \in R_- \).

We now prove (1.3.3) holds for \( P \cdot Q \) provided it holds for both \( P \) and \( Q \). Let \( u \in W \). Then \( \alpha \in D(u) \) implies \( \alpha \in R_+^e \) and \( u^{-1}(\alpha) \in R_- \). We claim that every summand, \( F_u \), in (1.5.1) vanishes at generic points of \( T_{\alpha,q^{-2}} \). Indeed, let \( wy = u \). Then \( y^{-1}w^{-1}(\alpha) \in R_- \), and there are two alternatives: either \( w^{-1}(\alpha) \in R_+ \), or \( w^{-1}(\alpha) \in R_- \). If \( w^{-1}(\alpha) \in R_+ \), then \( w^{-1}(\alpha) \in D(y) \), hence \( Q_y \) vanishes on \( T_{w^{-1}(\alpha),q^{-2}} \). This means that the factor \( uQ_y \) in (1.5.1) vanishes along \( T_{\alpha,q^{-2}} \), Similarly, if \( w^{-1}(\alpha) \in R_- \), then the factor \( P_w \) in (1.5.1) vanishes on \( T_{\alpha,q^{-2}} \).
Thus in both cases the summand in $F_u$ corresponding to any given $(w, y)$ vanishes on $T_{\alpha, q^{-2}}$, and the theorem follows. □

**Remarks.**

(a) We can, if we want, introduce as many “quantization parameters” $q_\nu$ as there are $W$-orbits on $R^{re}$, by modifying (1.3.3) in an obvious way. Proof of theorem 1.4 still goes through.

(b) One can modify definition 1.3 letting the point $q$ vary. This way one gets an algebra $H_C$ over $C[C]$, the ring of regular functions on $C$ (if $C = C^\times$ we have $C[C] = C[q, q^{-1}]$, in accordance with standard definition).

(c) If the Cartan matrix $A$ is of affine type the Lie algebra $\mathfrak{g}(A)$ differs from its derived algebra $\mathfrak{g}_{der}(A) = [\mathfrak{g}(A), \mathfrak{g}(A)]$. It is then possible to take $X^\vee$ as a lattice in $\mathfrak{h}_{der}$, the Cartan subalgebra of $g_{der}(A)$ rather than $g(A)$. One then regards the roots as elements of the $\mathfrak{h}^*_{der}$ (see §6) and writes $\Phi_{der}$ for the resulting root data.

The following results will be proved in sections 5 and 6 respectively.

**Theorem 1.7.** If $A$ is of finite type, then $H_q$ is isomorphic to the affine Iwahori-Hecke algebra associated to $A^t$, the transpose of $A$.

**Theorem 1.8.** If $A$ is of affine type, then the algebra $H_q$ associated to the root data $\Phi_{der}$, is isomorphic to the double affine Hecke algebra (cf. §6).

2. Geometric interpretation.

We have found somewhat ‘strange’ conditions (1.3.1-3) as a result of computation of the equivariant K-theory of the Steinberg variety. The latter is related to Hecke algebras due to the work of Kazhdan-Lusztig [KL] (in the finite case); see Garland-Grojnowski announcement [GG] in the general case. An alternative geometric interpretation of the algebra $H_q$ will be given in theorem 2.2 below.

Assume first that $A$ is a Cartan matrix of finite type. Then we have $rk(A) = n = \dim T$, and $W$ is a finite reflection group. The algebra $C[T]$ of regular functions on $T$ is, by the Pittie-Steinberg theorem, a free module over $C[T]^W$, the subalgebra of $W$-invariants. Observe next that there is a natural $C[T]^W$-linear action of the algebra $C(T)[W]$ on $C(T)$. The action of $P = \sum P_w[w]$ is given by the formula

$$f \mapsto \hat{P}(f) := \sum_{w \in W} P_w \cdot w^\vee f$$

(2.1)
Let $J_q \subset \mathbb{C}[T]$ denote the principal ideal generated by the element

$$\Delta = \prod_{\alpha \in R_+} (q^{-1}t^{\alpha/2} - q^{-1}t^{-\alpha/2}) \in \mathbb{C}[T]$$

**Theorem 2.2.** If $A$ is a Cartan matrix of finite type, then the assignment $P \mapsto \hat{P}$, see (2.1), yields a natural algebra isomorphism:

$$H_q \simeq \{u \in \text{End}_{\mathbb{C}[T]} \mathbb{C}[T] \mid u(J_q) \subset J_q\}.$$

We will prove Theorem 2.2 in the next section. It will be deduced from the following

**Proposition 2.3.** If $A$ is of finite type, then the assignment $P \mapsto \hat{P}$ yields an algebra isomorphism

$$\tilde{H} = \text{End}_{\mathbb{C}[T]} \mathbb{C}[T].$$

The isomorphism shows that $\tilde{H}$ is a matrix algebra over $\mathbb{C}[T]^W$. Observe further that Theorem 2.2 reads: $H_q \simeq \text{End}_{\mathbb{C}[T]} \mathbb{C}[T] \cap \text{End}_{\mathbb{C}[T]} J_q$. Using that, $J_q = \Delta \cdot \mathbb{C}[T]$, we obtain the following

**Corollary 2.4.** $H_q = \tilde{H} \cap \Delta \cdot \tilde{H} \cdot \Delta^{-1} \square$.

In the rest of this section we assume $A$ to be an arbitrary Cartan matrix and study the structure of the algebra $H_q$ in more detail.

Observe first that the algebra $\mathbb{C}[T]$ may be identified with a (non-central) subalgebra of $H_q$ via the assignment $f \mapsto f[1]$. Further, for each $i = 1, 2, \ldots, n$, introduce the following element:

$$\sigma_i = \left(\frac{qt^{\alpha_i} - q^{-1}}{t^{\alpha_i} - 1}\right)[s_i] - \frac{q - q^{-1}}{t^{\alpha_i} - 1} [1] \in \mathbb{C}(T)[W].$$

(2.4)

These elements belong to $H_q$ and are called the Demazure-Lusztig operators, cf. [L1]. Moreover, it is well known (see, e.g., [KL]) that the elements $\sigma_i$, $i = 1, 2, \ldots, n$ satisfy the braid relations: if $w \in W$ is any element, then for any reduced decomposition $w = s_{i_1} \cdots s_{i_k}$ the product $\sigma_w = \sigma_{i_1} \cdots \sigma_{i_k}$ depends only on $w$ but not on the choice of a reduced decomposition.

**Theorem 2.5.** Let $A$ be an arbitrary Cartan matrix. Then $H_q$ is a free left $\mathbb{C}[T]$-module with the base $\{\sigma_w, w \in W\}$. 
Let $H^\sigma$ denote the left $\mathbb{C}[T]$-submodule in $\mathbb{C}(T)[W]$ generated by $\{\sigma_w, w \in W\}$. We will prove theorem 2.5 by showing that $H_q = H^\sigma$. We proceed by induction on `$\preceq$', the Bruhat (partial) order on the Weyl group $W$. For each $w \in W$, introduce the subspace

$$\mathbb{C}(T)[W]_{\leq w} = \{ \sum_{y \in W} f_y[y] \in \mathbb{C}(T)[W] | f_y = 0 \text{ unless } y \preceq w \}$$

and similarly define $\mathbb{C}(T)[W]_{< w}$. This gives an increasing filtration on $\mathbb{C}(T)[W]$ labeled by the partially ordered set $W$. We write $H_{\leq w}$ and $H_{\sigma \leq w}$ for the corresponding induced filtrations on the subspaces $H_q$ and $H^\sigma$ respectively.

Further, given $w \in W$, write $\mathbb{C}[T] \theta_w$ for the rank one $\mathbb{C}[T]$-submodule in $\mathbb{C}(T)[W]$ with generator $\theta_w = \prod_{\alpha \in D(w)} \left( q^{\alpha \cdot w - 1} \right)$ (notation 1.5.6).

**Lemma 2.8.** (i) Let $f = \sum_{y \leq w} f_y[y] \in H_{\leq w}$. Then $f_w \in \mathbb{C}(T) \theta_w$.

(ii) $\sigma_w - \theta_w[w] \in \mathbb{C}(T)[W]_{< w}$.

**Proof:** (i) Observe that $s_\alpha \cdot w > w$ whenever $\alpha \in D(w)$, cf. [Bou]. Hence, for $f \in H_{\leq w}$ and $\alpha \not\in R_+ \setminus D(w)$, we have $f_{s_\alpha w} = 0$, and conditions (1.3.1-3) say that

1. The only possible singularities of $f_w$ are first order poles at the divisors $T_\alpha$ for $\alpha \in D(w)$;

2. $f_w$ vanishes at $T_{\alpha, q^{-2}}$ for $\alpha \in D(w)$.

The space of rational functions subject to these conditions is precisely the free $\mathbb{C}[T]$-submodule generated by $\theta_w$. This proves part (i).

To prove claim (ii), write a reduced expression $w = s_{i_1} \cdots s_{i_k}$. Observe that the collection of roots

$$\{ \alpha_{i_k}, s_{i_k-1}(\alpha_{i_k}), s_{i_k-2}s_{i_{k-1}}(\alpha_{i_k}), \ldots, s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \} = D(w) \quad (2.9)$$

gives a particular enumeration of all the elements of the set $D(w)$, cf. [Bou]. Write $\theta_\alpha := \frac{q^{\alpha \cdot w} - q^{-1}}{q^{\alpha \cdot s} - 1}$, and recall that $\sigma_i = \theta_{\alpha_i}[s_i] + \frac{q^\alpha - q^{-1}}{q^{\alpha s} - 1}[1]$. Thus, by
definition of \( \sigma_w \), we find
\[
\sigma_w = \sigma_{i_1} \cdots \sigma_{i_k} \\
= \sigma_{i_1} \cdots \sigma_{i_{k-1}} \cdot (\theta_{\alpha_{i_k}}[s_{i_k}]) + \text{ lower terms} \\
= \sigma_{i_1} \cdots \sigma_{i_{k-2}} \cdot (s_{i_{k-1}} \theta_{\alpha_{i_k}}[s_{k-1}s_{i_k}]) + \text{ lower terms} = \\
\ldots \\
= \theta_{\alpha_{i_1}} s_{i_1} \theta_{\alpha_{i_2}} s_{i_1} s_{i_2} \theta_{\alpha_{i_3}} \ldots s_{i_{k-1}} s_{i_k-1} \theta_{\alpha_{i_k}} \cdot [s_{i_1} \cdots s_{i_k}] + \text{ lower terms} \\
= \prod_{\alpha \in D(w)} \left( \frac{qt^\alpha - q^{-1}}{t^\alpha - 1} \right) [w] + \text{ lower terms} = \theta_{w}[w] + \text{ lower terms},
\]
where ‘lower terms’ stands for a linear combination \( \sum_{y<w} P_y[y] \). The lemma follows. \( \square \)

**Proof of Theorem 2.5:** Theorem 1.4 implies that, for any \( y \in W \), we have \( \sigma_y \in H_q \). Clearly, \( H_{\leq}^{\sigma} \) is a free left \( \mathbb{C}[T] \)-module with base \( \{ \sigma_y, y \leq w \} \). Whence, \( H_{\leq}^{\sigma} \subseteq H_{\leq}^{\sigma} \). We prove that this inclusion is actually an equality by induction on \( w \) (relative to the Bruhat order).

Let \( f = \sum_{y \leq w} f_y[y] \in H_{\leq}^{\sigma} \). Lemma 2.8(i) shows that \( f_w = P \cdot \theta_w \), for some \( P \in \mathbb{C}[T] \). Furthermore, part (ii) of the same lemma yields \( f - P \cdot \sigma_w \in H_{<}^{\sigma} \). But the induction hypothesis yields \( H_{<}^{\sigma} \subseteq H_{<}^{\sigma} \). Hence, \( f = (f - P \cdot \sigma_w) + P \cdot \sigma_w \in H_{<}^{\sigma} + H_{\leq}^{\sigma} \), and the theorem follows. \( \square \)

**Remark 2.10.** Theorem 2.2 implies in particular that, for any simple root \( \alpha_i \), one has \( \sigma_i(\Delta) \in J_q \). In fact, the following more precise result holds:
\[
\sigma_i(\Delta) = -q^{-1} \cdot \Delta.
\]

**Proof:** For a simple root \( \alpha_i \), set \( \Delta_i = q^{-1}t^{-\alpha_i/2} - qt^{\alpha_i/2} \). Direct calculation gives \( \Delta_i = (\sigma_i - q)^{e_{\alpha_i}/2} \). Hence, the Hecke relation \( (\sigma_i + q^{-1})(\sigma_i - q) = 0 \) yields
\[
(\sigma_i + q^{-1})\Delta_i = (\sigma_i + q^{-1})(\sigma_i - q)t^{\alpha_i/2} = 0 \tag{2.11}
\]
This proves the claim in the \( sl_2 \)-case.

In general, fix a simple root \( \alpha_i \), and write \( \Delta = \Delta' \cdot \Delta_i \) where
\[
\Delta' = \prod_{\alpha \in R_+, \alpha \neq \alpha_i} (q^{-1}t^{\alpha/2} - qt^{-\alpha/2})
\]
Observe, that the simple reflection \( s_i \) acts by permutation of the set \( R_+ \setminus \{\alpha_i\} \). Therefore, we get \( s^i \Delta' = \Delta' \). Thus, from formula (2.1) we calculate
\[
\sigma_i(\Delta) = \left( \frac{qt^\alpha_i - q^{-1}}{t^\alpha_i - 1} \right) \left( \frac{q - q^{-1}}{t^\alpha_i - 1} \right) \left( \Delta' \cdot \Delta_i \right)
= \frac{qt^\alpha_i - q^{-1}}{t^\alpha_i - 1} \cdot s_i \Delta' \cdot s_i \Delta_i - \frac{q - q^{-1}}{t^\alpha_i - 1} \Delta' \cdot \Delta_i
= \Delta' \cdot \frac{qt^\alpha_i - q^{-1}}{t^\alpha_i - 1} \cdot s_i \Delta_i - \Delta' \cdot \frac{q - q^{-1}}{t^\alpha_i - 1} \cdot \Delta_i
= \Delta' \cdot \sigma_i(\Delta_i) \quad \text{by (2.11)}
= \Delta' \cdot (-q^{-1}) \cdot \Delta_i = -q^{-1} \cdot \Delta \quad \blacksquare.
\]

3. Polynomial representation.

In this section we will study in more detail the action of the algebras \(H_q\) and \(\tilde{H}\) on the vector space \(\mathbb{C}(T)\) given by formula (2.1) and give proofs of Proposition 2.3 and Theorem 2.2. The curve \(C\) is supposed to be affine, and the matrix \(A\) is supposed to be of finite type.

**Lemma 3.1.**
(i) Let \(P \in \tilde{H}\). Then, for any regular function \(f \in \mathbb{C}[T]\), we have \(\hat{P}(f) \in \mathbb{C}[T]\).

(ii) Assume \(f \in \mathbb{C}[T]\) satisfies the following condition: for each positive root \(\alpha \in R_+\), the function \(f\) vanishes at \(T_{\alpha,q^{-\alpha}}\). Then, for any \(P \in H_q\), the function \(\hat{P}(f)\) satisfies similar condition.

**Proof:**
The argument is very similar to the one used in the proof of theorem 1.4. To prove (i), fix a root \(\alpha \in R_+\) and write \(P = \sum P_w[w]\). Split the sum \(\hat{P}(f) = \sum w P_w \cdot w f\) into partial sums of the form
\[
P_w \cdot w f + P_{s_w} \cdot s_w f
\]
Setting \(g = w f\) and regrouping terms, one rewrites this as follows
\[
(P_w + P_{s_w}) \cdot g - P_{s_w} \cdot (g - s_w g)
\]
Observe now that the function \(P_w + P_{s_w}\) is regular at \(T_\alpha\) by (1.3.2), while the function \(g\) is regular everywhere. Hence the the first summand in (3.2) has no singularity at \(T_\alpha\). Similarly, \(P_{s_w}\) has at most single pole at \(T_\alpha\), while the function \(g - s_w g\) vanishes at \(T_\alpha\). Hence, the second summand in (3.2) has no singularity at \(T_\alpha\). Thus, we have shown that, for any root \(\alpha\), the function \(\hat{P}(f)\) is regular at generic points of \(T_\alpha\). Therefore, it is regular everywhere, and part (i) follows. Part (ii) is proved in a similar way,
repeating the verification of condition (1.3.3) in the proof of theorem 1.4. □.

Recall next that the ring $\mathbb{C}[T]^W$ is the coordinate ring of the orbi-space $T/W$, an affine algebraic variety. Write $I_a \subset \mathbb{C}[T]^W$ for the maximal ideal corresponding to a point $a \in T/W$. Given a $\mathbb{C}[T]^W$-module $M$, set $M_a := M/I_a \cdot M$, the geometric fiber of $M$ at $a$.

Observe that by lemma 3.1(i) we have a well-defined map
\[
\Phi : \tilde{H} \to \text{End}_{\mathbb{C}[T]^W} \mathbb{C}[T], \tag{3.3}
\]
which is, moreover, a morphism of $\mathbb{C}[T]^W$-modules. Given a point $a \in T/W$, write
\[
\Phi_a : \tilde{H}_a \to \text{End}_{\mathbb{C}[T]^W} (\mathbb{C}[T]_a) \tag{3.4}
\]
for the induced map of geometric fibers at $a$.

Let $\pi : T \to T/W$ be the natural projection. We have the following

**Lemma 3.5.** Let $t \in T$ be such that one of the following two conditions holds:

1. For any $\alpha \in R_+$, we have $t \not\in T_\alpha$;
2. $t \in T_\alpha$ for exactly one root $\alpha \in R_+$.

Then, for $a = \pi(t)$, the map (3.4) is surjective.

**Proof:** Recall that $T/W$ is a smooth algebraic variety (e.g., if $W$ is the symmetric group, $T/W$ is the symmetric power of the curve $C$).

Consider case (1) first. Then $t$ is regular and the map $\pi : T \to T/W$ is unramified over $a$. Hence, the fiber $\pi^{-1}(a) = \{w(t), w \in W\}$ is the $W$-orbit consisting of $\#W$ elements. This yields a natural $W$-equivariant isomorphism of vector spaces $\mathbb{C}[T]_a \simeq \mathbb{C}[W]$, where the line $\mathbb{C} \cdot [w] \subset \mathbb{C}[W]$ is identified with the geometric fiber of $\mathbb{C}[T]$ at the point $w(t)$. On the other hand, theorem 2.5 shows that $\tilde{H}_a$ is a $(\#W)^2$-dimensional $\mathbb{C}$-algebra generated by the elements $\{[y], y \in W\}$ and $\{f_a : f \in \mathbb{C}[T]\}$. Furthermore, it follows from definitions that the map $\Phi_a : \tilde{H}_a \to \text{End}_C \mathbb{C}[W]$ arising from (3.4) sends $[y]$ to the operator of left translation by $w^{-1}$, and $f_a$ to the diagonal operator acting on the line $\mathbb{C} \cdot [w] \subset \mathbb{C}[W]$ via multiplication by $f(w^{-1}(t))$, the value of $f$ at the point $w(t)$. It is clear further, that the operators of those two types generate the whole algebra $\text{End}_C \mathbb{C}[W]$. That proves the lemma in case (1).
Assume now $t \in T_{\alpha}$ as in case (2). Then the isotropy group of $t$ in $W$ is \( \langle s_{\alpha} \rangle = \{1, s_{\alpha}\} \), the two-element subgroup in $W$. Locally near $t$, the projection $\pi : T \to T/W$ is a two-branch ramified covering of the form $(z_1, z_2, \ldots, z_r) \mapsto (z_1^2, z_2, \ldots, z_r)$. Hence, the induced linear map of the cotangent spaces $d\pi^* : T^*_a(T/W) \to T^*_t(T)$ has 1-codimensional image. Furthermore, the root $\alpha \in \text{Lie}(T)^* \simeq T^*_t(T)$ does not belong to the image of $d\pi^*$. Thus, there is a canonical direct sum decomposition

\[
T^*_t(T) = d\pi^*(T^*_a(T/W)) \oplus \mathbb{C} \cdot \alpha
\]  

(3.6)

Write $pr_{\alpha} : T^*_t(T) \to \mathbb{C} \cdot \alpha$ for the second projection.

To any regular function $f$ on a neighborhood of $t$ we assign a vector in the two-dimensional vector space $\mathbb{C} \oplus \mathbb{C}\alpha$ with base 1 and $\alpha$. The assignment is given by the formula

\[
f \mapsto \tau(f) = f(t) \cdot 1 + pr_{\alpha}(df_t),
\]

where $df_t$ stands for the differential of $f$ at $t$. Further, make $\mathbb{C} \oplus \mathbb{C}\alpha$ into $\langle s_{\alpha}\rangle$-module setting $s_{\alpha}(1) = 1$ and $s_{\alpha}(\alpha) = -\alpha$. Then, $\tau$ becomes $\langle s_{\alpha}\rangle$-equivariant map. Moreover, decomposition (3.6) shows that $\tau(f)$ depends only on $f_a$, the image of $f$ in $\mathbb{C}[T]_a = \mathbb{C}[T]/I_a \cdot \mathbb{C}[T]$.

It is clear that the algebra $\text{End}_\mathbb{C}(\mathbb{C} \oplus \mathbb{C}\alpha)$ is generated by the following operators:

\[
A = \begin{pmatrix} 1 & \alpha \\ \alpha & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ \alpha & 2 \end{pmatrix}, \quad C = \text{Id}
\]

On the other hand, consider the $\mathbb{C}(T)[W]$-action on $\mathbb{C}(T)$. It is straightforward to verify that, for any $f \in \mathbb{C}(T)$ regular at $t$, one has

\[
\tau \left( (t^\alpha - 1) \cdot f \right) = A(\tau(f)), \quad \tau \left( \left( \frac{1}{t^\alpha - 1} - \frac{1}{t^\alpha - 1} [s_{\alpha}] \cdot f \right) \right) = B(\tau(f))
\]  

(3.7)

Observe next that we have a $W$-equivariant isomorphism of sets $\pi^{-1}(a) \simeq W/\langle s_{\alpha}\rangle$. It follows the the map $\tau$ defined above can be uniquely extended to a natural $W$-module isomorphism

\[
\mathbb{C}[T]_a \cong Ind^W_{\langle s_{\alpha}\rangle} (\mathbb{C} \oplus \mathbb{C}\alpha) = \mathbb{C}[W] \otimes_{\mathbb{C}[\langle s_{\alpha}\rangle]} (\mathbb{C} \oplus \mathbb{C}\alpha)
\]

Thus, we may view (3.4) as a homomorphism

\[
\Phi_a : \tilde{H}_a \to \text{End}_\mathbb{C} \left( Ind^W_{\langle s_{\alpha}\rangle} (\mathbb{C} \oplus \mathbb{C}\alpha) \right)
\]  

(3.8)
To prove that $\Phi_a$ is surjective, we observe, as in proof of case (1), that the image of $\Phi_a$ contains left translations by $W$ as well as all 'diagonal' operators, given by multiplication by a scalar-valued function on $W/\langle s_\alpha \rangle$. In addition, $\tilde{H}_a$ contains the element which is equal to $t^\alpha - 1$, resp. $\frac{1}{t^\alpha - 1} - \frac{1}{t^\alpha - 1} [s_\alpha]$, at $t$, and equal to 0 at all other points of the fiber $\pi^{-1}(a)$. Equation (3.7) then shows that the image of $\Phi_a$ contains all operators that coomute with left translation and act as the operators $A$, resp. $B$, fiberwise. Altogether, all the listed operators generate the RHS of (3.8). That completes the proof of case (2) of the lemma. □

Proof of proposition 2.3: By the Pittie-Steinberg theorem, the object on the RHS of (3.3) is a free $C[T]^W$-module of rank $(\#W)^2$. Similarly, theorem 2.5 implies that the object on the LHS of (3.3) is a free $C[T]^W$-module of the same rank. Hence, proving the proposition suffices it to show that the map $\Phi$ in (3.3) is surjective. Furthermore, the modules being free, surjectivity of $\Phi$ follows from the surjectivity of the corresponding maps $\Phi_a$, for all points $a$ in the complement of a codimension two subset of $T/W$. But the points $\pi(t)$ such that $t$ satisfies neither of the conditions of lemma 3.5 form a codimension two subset. Thus, proposition 2.3 follows from the lemma. □.

Proof of theorem 2.2: In view of proposition 2.3, the theorem is clearly equivalent to the equality, cf. corollary 2.4: $H_q = \tilde{H} \cap \Delta \cdot \tilde{H} \cdot \Delta^{-1}$. Theorem 2.5 and remark 2.10 imply that the LHS of the equality is contained in the RHS. To prove the opposite inclusion, let $P = \sum_w P_w[w] \in \tilde{H}$. Then we calculate

$$
\Delta^{-1} \cdot P \cdot \Delta = \sum_w \Delta^{-1} \cdot P_w \cdot w \cdot \Delta
= \sum_w \prod_{\alpha \in R_+} (q^{-1} t^{\alpha/2} - qt^{-\alpha/2})^{-1} \cdot P_w \cdot \prod_{\alpha \in R_+} (q^{-1} t^{w(\alpha)/2} - qt^{-w(\alpha)/2})
= \sum_w \prod_{\alpha \in R_+} (q^{-1} t^{\alpha/2} - qt^{-\alpha/2})^{-1} \prod_{w^{-1}(\alpha) \in R_+} (q^{-1} t^{\alpha/2} - qt^{-\alpha/2}) \cdot P_w
= \sum_w \prod_{\alpha \in D(w)} \frac{q^{-1} t^{w(\alpha)/2} - qt^{-w(\alpha)/2}}{q^{-1} t^{\alpha/2} - qt^{-\alpha/2}} \cdot P_w
$$

We see that the assumption $\Delta^{-1} \cdot P \cdot \Delta \in \tilde{H}$ implies that, for any $w \in W$,
the function \( \prod_{\alpha \in D(w)} \frac{q^{-1}t^{w(\alpha)/2}-qt^{-w(\alpha)/2}}{q^{-1}t^{\alpha/2}-qt^{-\alpha/2}} \cdot P_w \) has no singularities. Whence, \( P_w \) is divisible by \( \prod_{\alpha \in D(w)} (q^{-1}t^{\alpha/2}-qt^{-\alpha/2}) \), and condition (1.3.3) follows.

\[ \square \]

4. Elliptic case.

In this section we keep the assumption that the Cartan matrix \( A \) is of finite type but allow the 1-dimensional algebraic group \( C \) to be arbitrary, an elliptic curve in particular. It is convenient, in this more general setting, to use the language of sheaves. Write \( \mathcal{O}_X \) for the sheaf of regular functions on an algebraic variety \( X \).

As in section 3, let \( \pi : T \to T/W \) be the natural projection. Observe that the group \( W \) acts freely on the Zariski-open subset \( T^{reg} := T \setminus (\cup_{\alpha \in RT} T_\alpha) \) so that \( \pi \) is a finite flat morphism unramified over \( \pi(T^{reg}) \). It follows from the flatness that for any locally free sheaf \( \mathcal{O}_T \)-sheaf \( \mathcal{L} \) its direct image, \( \pi_* \mathcal{L} \), is a locally free sheaf on \( T/W \).

Let \( j : T^{reg} \hookrightarrow T \) be the embedding. It will be useful to embed \( \pi_* \mathcal{O}_T \) into the larger sheaf, \( \pi_* (j_* \mathcal{O}_T \otimes_{\mathcal{O}_T} \nabla_{\mathcal{L}}) \). This is clearly a sheaf of \( \mathcal{O}_T/W \)-algebras with a compatible \( W \)-action along the fibers.

We introduce a sheaf-theoretic analogue of the algebra \( \mathbb{C}(T)[W] \). It is a sheaf \( \mathcal{O}[W] \) on \( T/W \) (not on \( T^{reg}/W \)) defined as the skew tensor product of \( \pi_* (j_* \mathcal{O}_T \otimes_{\mathcal{O}_T} \nabla_{\mathcal{L}}) \) and \( \mathbb{C}[W] \) over \( \mathbb{C} \). Thus, \( \mathcal{O}[W] \) is a sheaf of \( \mathcal{O}_{T/W} \)-algebras, quasi-coherent (but not coherent) as a sheaf of \( \mathcal{O}_{T/W} \)-modules. Clearly, there is a natural action-morphism of sheaves on \( T/W \):

\[
\mathcal{O}[W] \otimes \pi_* (j_* \mathcal{O}_T \otimes_{\mathcal{O}_T} \nabla_{\mathcal{L}}) \to \pi_* (\mathcal{O}_T \otimes\mathcal{L})
\]

Definition 4.2. Let \( \mathcal{H}, \text{ resp. } \mathcal{H}_{\Pi} \), be the subsheaf of \( \mathcal{O}[W] \) whose local sections satisfy conditions (1.3.1-2), resp. (1.3.1-3).

Proposition 4.3. Both \( \mathcal{H} \) and \( \mathcal{H}_{\Pi} \) are subsheaves of algebras in \( \mathcal{O}[W] \). The action (2.1) of \( \mathcal{H} \) on \( \pi_* (j_* \mathcal{O}_T \otimes_{\mathcal{O}_T} \nabla_{\mathcal{L}}) \) preserves \( \pi_* \mathcal{O}_T \).

Proof is entirely similar to the proof of Theorem 1.4 and is omitted.

Let \( \mathcal{J}_{\Pi} \subset \mathcal{O}_T \) be the subsheaf of ideals corresponding to the divisor \( \sum_{\alpha \in R^+_e} T_{\alpha,q-2} \), and \( \pi_* \mathcal{J}_{\Pi} \subset \pi_* \mathcal{O}_T \) its direct image. Our proof of theorem 2.4 yields, in fact, the following local result.
Theorem 4.4. (a) The $\hat{\mathcal{H}}$-action of $\hat{\mathcal{H}}$ on $\pi_*\mathcal{O}_T$, cf. prop. 4.3., gives an isomorphism $\hat{\mathcal{H}} \simeq \text{End}(\pi_*\mathcal{O}_T)$ of sheaves of algebras on $T/W$. Moreover, we have

$$\mathcal{H}_\Pi = \{ u \in \hat{\mathcal{H}} \mid u(\pi_*\mathcal{J}_\Pi) \subset \pi_*\mathcal{J}_\Pi \} \quad \square.$$ 

Now let $\mathcal{C}$ be an elliptic curve. The variety $T/W$ in this case is known to be a weighted projective space. By the above, the sheaf $\mathcal{H}_\Pi$ is a locally free rank $\#W^2$ sheaf of algebras on $T/W$. We will consider global sections of $\mathcal{H}_\Pi$ over an appropriate open subset.

Let $\xi \in \mathcal{C}$, $\xi \neq 1$ be a point of order 2. Observe that the set $\bigcup_{\alpha \in \mathbb{R}_+^2} T_{\alpha, \xi}$ is $W$-stable. Its complement is a Zariski-open subset in $T/W$:

$$U = (T \setminus \bigcup_{\alpha \in \mathbb{R}_+^2} T_{\alpha, \xi})/W$$

Choose a complex uniformization $\mathcal{C} = \mathbb{C}/\Lambda$, where $\Lambda$ is a lattice in $\mathbb{C}$. Let $sn(z)$ be the Jacobi sine function on $\mathbb{C}/\Lambda$ associated with the choice of the second order point $\xi$. It is uniquely characterized by the following conditions: it has simple zeroes at 0 and $\xi$, simple poles at the two other points of order 2 and its derivative at 0 is equal to 1. For each simple root $\alpha_i$ we have

$$\sigma_i := \frac{sn(q^{-2})}{sn(t^{\alpha_i})}[1] + \left( 1 - \frac{sn(q^{-2})}{sn(t^{\alpha_i})} \right)[s] \in \Gamma(U, \mathcal{H}_\Pi) \quad (4.5)$$

One calculates that, for any $i$, we have $\sigma_i^2 = [1]$, but these elements do not satisfy the braid relations [BE] [Gu].

Assume, for instance, we are in the $SL_2$-case. Then, $T = \mathcal{C}$ and $W = \{1, s\}$. Therefore, we find $T/W = \mathcal{C}/\{\pm 1\}$ and $U = (\mathcal{C} \setminus \{\xi\})/\{\pm 1\}$. Let $\wp^{(m)}$ denote the $m$-th derivative of the Weierstrass $\wp$-function, and write $\sigma$ for the (only) element in $\Gamma(U, \mathcal{H}_\Pi)$ given by formula (4.5). We have the following result.

Proposition 4.6. In the $SL_2$-case the algebra $\Gamma(U, \mathcal{H}_\Pi)$ has $\mathbb{C}$-basis formed by the following elements:

$$[1], \sigma, \wp^{(m)}(t - \xi)[1], (\wp^{(m)}(t - \xi) - \wp^{(m)}(q^{-2} - \xi))[s],$$

where $t \in \mathbb{C}/(\{\pm 1\} \ltimes \Lambda)$, and $m \geq 0$. 

14
Proof: An element of $\Gamma(U, \mathcal{H}_U)$ has the form $f_1(t)[1] + f_s(t)[s]$ where $f_1$ and $f_s$ are meromorphic functions on $\mathbb{C}$. By definition of the sheaf $\mathcal{H}_U$, these functions are regular for $z \neq 0, \xi$ (where $z$ denotes the coordinate on $\mathbb{C}$), and have at most a simple pole at $z = 0$ with $\text{Res}_0 f_1 + \text{Res}_0 f_s = 0$. Set $r = \text{Res}_0 f_1$. Then each of the functions $f_1(z) - r/sn(z)$ and $f_s(z) + r/sn(z)$ is regular off the point $\xi$ and is therefore a linear combination of 1 and $\wp^m(z - \xi)$. Hence, we have

$$f_1(t) = c_0 + \frac{c_1}{sn(t)} + \sum_{i \geq 2} c_i \wp^{i-2}(t - \xi), \quad f_s(t) = d_0 - \frac{c_1}{sn(t)} + \sum_{i \geq 2} d_i \wp^{i-2}(t - \xi)$$

Subtracting linear combinations of 1 and $\sigma$ we can kill the first two terms in $f_1$ and the second term in $f_s$. After that the residue condition becomes void, for $\wp^{(i)}$ have no residues. Thus, the coefficients $c_i, i \geq 2$ may be arbitrary. Further, the only remaining condition on $f_s$ is that $f_s(t) = 0$ for $t = q^{-2}$. Clearly, functions $\wp^m(t - \xi) - \wp^m(q^{-2} - \xi)$ form a basis in the space of all functions satisfying this condition and regular off the point $\{t = \xi\}$. Proposition follows. \(\square\)

5. Affine Hecke algebra.

Let $\Phi = (A, X)$ be a root data with Cartan matrix $A$ of finite type. Then, there is a unique (up to isomorphism) reductive algebraic group $G = G(\Phi)$ over $\mathbb{C}$ with Lie algebra $\mathfrak{g}(A)$ and Cartan subalgebra $\mathfrak{h} = \mathbb{C} \otimes_\mathbb{Z} X^\vee$.

Write $Q \subset \mathfrak{h}^*$ and $Q^\vee \subset \mathfrak{h}$ for the root and coroot lattices, respectively, and let $P = \text{Hom}_\mathbb{Z}(Q^\vee, \mathbb{Z})$ and $P^\vee = \text{Hom}_\mathbb{Z}(Q, \mathbb{Z})$ be the corresponding weight and coweight lattices. We have

$$Q \subset X \subset P \quad \text{and} \quad Q^\vee \subset X^\vee \subset P^\vee$$

The group $\Pi_\Phi = X^\vee/Q^\vee$ is known to be isomorphic to the center of $G$.

Let $\theta \in R_+$ be the maximal root. The affine Weyl group $W^a = W^a(\Phi)$ of $\Phi$ is, by definition, the group of affine transformations of $\mathfrak{h}$ generated by $W$ and $s_0$, additional reflection with respect to the affine hyperplane $\{h \in \mathfrak{h} : (\theta, h) + 1 = 0\}$. Thus $W^a$ is a Coxeter group with generators $s_i, i = 0, \ldots, n$. It is known that $W^a$ is a semidirect product of $W$ and the co-root lattice $Q^\vee$.

Let $\widetilde{W} = \widetilde{W}(\Phi)$ be the semidirect product of $W$ and $X^\vee \supset Q^\vee$. It is called the extended Weyl group of $\Phi$ and is not, in general, a Coxeter group.
It is clear that $W^a \subset \tilde{W}$ is a normal subgroup and we have $\tilde{W}/W^a = \Pi_\Phi$. Denote $X = X \oplus \mathbb{Z}$, and $R = R \times \mathbb{Z} \subset \tilde{X}$, the affine root system associated to $R$. Its positive part is

$$\tilde{R}_+ = \{ (\alpha, k) \in X \oplus \mathbb{Z} : \alpha \in R, k > 0 \text{ or } \alpha \in R_+, k \geq 0 \}.$$ 

The group $\tilde{W}$ acts on $\tilde{X}$ by group automorphisms preserving $\tilde{R}$. Let $S$ be the system of simple roots of $\tilde{R}_+$. It consists of $(\alpha_i, 0)$, $i = 1, \ldots, n$, which will be still denoted by $\alpha_i$, and $\alpha_0 = (\theta, 1)$.

Write $l(w) := \# D(w)$ for the length of $w \in W^a \subset \tilde{W}$, cf. sect. 1.5.1. We extend the length function to $\tilde{W}$ by the same formula $l(w) := \# D(w)$. With this definition, the set $\{ w \in \tilde{W} : l(w) = 0 \}$ is a subgroup which is identified with $\Pi_\Phi$. The group $\Pi_\Phi$ acts on $S$ by permutations. Thus $\tilde{W} = \Pi_\Phi \ltimes W^a$, the semidirect product with the commutation relations $\pi s_\alpha \pi^{-1} = s_{\pi(\alpha)}$, $\alpha \in S$.

Let $q \in \mathbb{C}^*$ be a non-zero complex number. The affine Hecke algebra $\hat{H}_q = \hat{H}_q(\Phi)$ can be defined in several different ways.

**5.2 First definition:** $\hat{H}_q$ is the $\mathbb{C}$-algebra with basis $T_w, w \in \tilde{W}$ and multiplication given by the rules:

$$(T_w + q^{-1})(T_w - q) = 0, \quad w \in \{ s_0, \ldots, s_n \}, \quad (5.2.1)$$

$$T_w T_y = T_{wy}, \quad \text{if } l(wy) = l(w) + l(y). \quad (5.2.2)$$

Denote by $\hat{H}_q^a \subset \hat{H}_q$ the subspace spanned by the $T_w, w \in W^a$ only. This is clearly a subalgebra and $\hat{H}_q \simeq \hat{H}_q^a[\Pi_\Phi]$, is the twisted group algebra for the $\Pi_\Phi$-action on $\hat{H}_q^a$. In other words, $\hat{H}_q$ is generated by the sets $\{ T_w, w \in W^a \}$ and $\{ T_\pi, \pi \in \Pi_\Phi \}$ with the relations (5.2.1-2) for the $T_u$’s, and the relations

$$T_\pi T_{\pi'} = T_{\pi + \pi'}, \quad T_\pi T_{s_\alpha} T_\pi = T_{s_{\pi(\alpha)}}, \alpha \in \tilde{R}_+. \quad (5.2.3)$$

**5.3 Second definition:** We have $\tilde{W} \simeq W \ltimes X^\vee$. Accordingly, one has a presentation of $\hat{H}_q$ by generators $\{ T_w, w \in W \}$ and $\{ Y_\lambda, \lambda \in X^\vee \}$ subject to the relations:

(5.3.1) The $T_w, w \in W$, satisfy (4.2.1-2).

(5.3.2) $Y_\lambda Y_\mu = Y_{\lambda + \mu}$.

(5.3.3) $T_{s_i} Y_\lambda = Y_\lambda T_{s_i}$, if $(\alpha_i^\vee, \lambda) = 0$.

(5.3.4) $T_{s_i} Y_{s_i(\lambda)} T_{s_i} = q Y_\lambda$, if $(\alpha_i^\vee, \lambda) = 1$. 

16
It is known that the elements $T_wY_\lambda, \ w \in W, \ \lambda \in X,$ form a $\mathbb{C}$-basis of $\hat{H}_q$.

Recall the Demazure-Lusztig elements $\sigma_i \in H_q(\Phi)$ given, for $i = 1, \ldots, n,$ by formula (2.4). Theorem 1.7 is a consequence of the following more precise result.

**Theorem 5.4.** The assignment $T_w \mapsto \sigma_w, \ Y_\lambda \mapsto t^\lambda$ extends to an algebra isomorphism $f: \hat{H}_q \rightarrow H_q(\Phi^\vee)$, where $\Phi^\vee$ is the root data dual to $\Phi$.

**Proof:** The existence of $f$ is proved by verifying the relations, which is well known. The fact that $f$ is an isomorphism follows since $\{T_wY_\lambda\}$ is a $\mathbb{C}$-basis in $\hat{H}$, while $\{\sigma_w t^\lambda\}$ is a $\mathbb{C}$-basis in $H_q$, by theorem 2.5.

6. Cherednik algebra.

**6.1 Affine root systems reviewed.** Let $A$ be a generalized Cartan matrix of untwisted affine type. We write it as $A = \|a_{ij}\|_{i,j=0}^n$, so that:

$\overline{A} = \|a_{ij}\|_{i,j=1}^n$ is a (finite type) Cartan matrix from which $A$ is obtained by affinization.

$\overline{g} = g(\overline{A})$ is the corresponding finite-dimensional Lie algebra.

$\overline{G}$ is the simply connected reductive group over $\mathbb{C}$ with Lie algebra $\overline{g}$.

$\overline{R}, \overline{W}, \overline{T}$ etc., are the root system, the Weyl group, the maximal torus, etc. for $\overline{G}$. Thus, $R = \{\langle \alpha, k \rangle, \alpha \in \overline{R}, k \in \mathbb{Z}\}$. The simple root $\alpha_0$ is $(-\theta, 1)$ where $\theta$ is the maximal root of $\overline{R}$.

Notice that the lattice of characters of $\overline{T}$ is $\overline{Q}$, the dual of the coroot lattice of $\overline{g}$.

$\Pi = \overline{T}^\vee / \overline{Q}^\vee$ is the center of the simply connected group associated to $\overline{g}$.

We have the identification of vector spaces:

$g = g(A) = \mathbb{C} \cdot d \oplus \mathbb{C} \cdot c \oplus \overline{g}[z, z^{-1}]$, where $c$ is the central element, $d$ is the infinitesimal rotation, i.e., $[d, g(z)] = z \frac{dg}{dz}$ for $g(z) \in \overline{g}[z, z^{-1}]$.

$g_{der} = g_{der}(A) = \mathbb{C} \cdot c \oplus \overline{g}[z, z^{-1}]$ is the derived algebra of $g$, considered for the purpose of comparing our construction with that of Cherednik.

$G_{der} = \overline{G}((z))$ is the Kac-Moody group associated with $g_{der}$. Its center will be denoted $\mathbb{C}^\ast_\zeta$. It is isomorphic to $\mathbb{C}^\ast$ and we think of $\zeta$ as a coordinate therein.

$\mathfrak{h}_{der} = \mathbb{C} \cdot c \oplus \overline{g}$ is the Cartan subalgebra of $g_{der}$, and $T_{der} = \mathbb{C}^\ast_\zeta \times \overline{T}$ the maximal torus of $G_{der}$.
The \( W \)-action on \( \mathbb{C}[T_{\text{der}}] \), the ring of regular functions on \( T_{\text{der}} \), is given by the formulas:

\[
\begin{align*}
    s_i(t^\lambda \zeta^l) &= t^{s_i(\lambda)} \zeta^l, \quad i = 1, ..., n, \lambda \in \overline{P}, l \in \mathbb{Z}, \\
    s_0(t^\lambda \zeta^l) &= t^{\lambda - \langle \lambda, \theta^\vee \rangle \theta} \zeta^{l+1}.
\end{align*}
\]

(6.1) The double affine Hecke algebra \( \mathcal{H}_q \) associated to \( A \) was defined by Cherednik [Ch2] to be the \( \mathbb{C} \)-algebra (depending on a parameter \( q \in \mathbb{C}^*, q \neq \pm 1 \)) on generators \( \delta, T_i, i = 0, ..., n, Y_\lambda, \lambda \in \overline{P}, T_\pi, \pi \in \Pi \), subject to the following relations:

(6.2.0) The element \( \delta \) is central.

(6.2.1) The \( T_i, T_\pi \) generate the affine Hecke algebra \( \mathcal{H}_q \) corresponding to the finite root system \( (\overline{A}, \overline{P}) \).

(6.2.2) The \( T_i, i = 1, ..., n \) and the \( Y_\lambda \) generate the affine Hecke algebra corresponding to the dual finite root system \( (\overline{A}, \overline{P}^\vee) \).

(6.2.3) \( T_iY_\lambda = Y_\lambda T_i \) for any \( i = 0, ..., n, \lambda \in \overline{P} \) such that \( \langle \lambda, \alpha_i^\vee \rangle > 1 \). Here we set \( \alpha_0^\vee = -\theta^\vee \).

(6.2.4) \( T_iY_\lambda = Y_\lambda T_i \) for any \( i = 0, ..., n, \lambda \in \overline{P} \) such that \( \langle \lambda, \alpha_i^\vee \rangle > 1 \).

(6.2.5) \( T_iY_\lambda = Y_\lambda T_i \) for any \( i = 0, ..., n, \lambda \in \overline{P} \) such that \( \langle \lambda, \theta^\vee \rangle > 1 \).

(6.2.6) \( T_\pi Y_\lambda T_\pi^{-1} = Y_{\pi(\lambda)} \).

We are going to compare the algebra \( \mathcal{H}_q \) with \( \mathbb{H}_q(\Phi_{\text{der}}) \), where \( \Phi_{\text{der}} = (A, X) \) is the root data such that \( X \subset h_{\text{der}}^* \) is the lattice dual to \( X^\vee = \bigoplus_{i=0}^n \alpha_i^\vee \subset h_{\text{der}} \). The subalgebra in \( \mathcal{H}_q \) generated by \( \delta, Y_\lambda, \lambda \in \overline{P} \) is naturally identified with \( \mathbb{C}[T_{\text{der}}] \), where \( \delta \) corresponds to the coordinate function, \( \zeta \), on the center of the derived Kac-Moody group. Recall also the elements \( \sigma_i, i = 0, ..., n \) of \( \mathbb{H}_q(\Phi_{\text{der}}) \) from (2.4).

**Theorem 6.3.1.** The assignment \( Y_\lambda \delta^l \mapsto t^\lambda \zeta^l, T_i \mapsto \sigma_i, i = 0, ..., n \) gives an algebra isomorphism \( \mathcal{H}_q \rightarrow \mathbb{H}_q(\Phi_{\text{der}}) \).

**Proof.** We have first to verify that the relations of \( \mathcal{H}_q \) hold in \( \mathbb{H}_q(\Phi_{\text{der}}) \), which is known, see, e.g., [Ki]. This means that we have a homomorphism, denote it \( f \). To show that it is an isomorphism we use the fact, proved by Cherednik, that the elements \( Y_\lambda \delta^l T_w, w \in W \), form a \( \mathbb{C} \)-basis in \( \mathcal{H}_q \). A similar statement holds for \( \mathbb{H}_q(\Phi_{\text{der}}) \), by Theorem 2.5. This implies that \( f \)
gives isomorphisms on the factors of the natural filtrations in $H_q(\Phi_{der})$ and $\hat{H}_q$, labeled by $W$, and thus is an isomorphism.

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