Numerically stable algorithm for identification of linear dynamical systems by extended instrumental variables

D V Ivanov¹, A I Zhdanov²

¹Samara State University of Transport, Svobody street 2 B, Samara, Russia, 443066
²Samara State Technical University, Molodogvardejskaya street 244, Samara, Russia, 443100

Abstract. Instrumental variables are widely used to identify linear dynamical systems. The advantages of instrumental variables include low computational complexity, as well as the possibility of identification for different models of color noise. Often the method of instrumental variables leads to ill-conditioned problems, which significantly limits the application of this method. The paper proposes a solution to the problem of extended instrumental variables based on augmented normal equations. Test examples showed high accuracy of the proposed approach.

1. Introduction
The method of instrumental variables is widely used to solve problems of the identification of dynamic systems [1-3]. A different choice of instrumental variables allows solving identification problems for output error models, ARMAX models, ARARX models, BJ models [1-3], as well as for models with errors in variables [4,5]. Instrumental variables are used to identify dynamic systems described by equations with derivatives and fractional-order differences [6, 7].

The disadvantages of the instrumental variable method include the conflicting requirements for the choice of an instrumental variable: it must be correlated with a regression vector and not correlated with interference. Often the method of instrumental variables leads to ill-conditioned tasks, which significantly limits the scope of its application. The paper proposes a solution to the problem of extended instrumental variables based on an augmented system of equations. Test cases showed high accuracy of the proposed approach.

2. Problem statement
A stable dynamic system is described by the equation:

\[ z_i - \sum_{m=1}^{n} b^{(m)}_{i-m} z_{i-m} = \sum_{m=0}^{n} a^{(m)}_{i-m} x_{i-m}. \] (1)

Depending on the parameterization and application points of the noise, different classes of models can be obtained.

1. The model with an error in the equation

\[ z_i - \sum_{m=1}^{n} b^{(m)}_{i-m} z_{i-m} = \sum_{m=0}^{n} a^{(m)}_{i-m} x_{i-m} + \xi_i. \] (2)
as well as its generalizations ARMAX, ARARX, ARARMAX, etc.

2. The model with an output error

\[ z_i - \sum_{m=1}^{n} b^{(m)} z_{i-m} = \sum_{m=0}^{n} a^{(m)} x_{i-m}, \quad y_i = z_i + \xi_i, \]  

(3)
as well as its generalization model BJ (Box-Jenkins).

3. The model with errors in all variables

\[ z_i - \sum_{m=1}^{n} b^{(m)} z_{i-m} = \sum_{m=0}^{n} a^{(m)} x_{i-m}, \quad y_i = z_i + \xi_i, \quad w_i = x_i + \zeta_i. \]  

(4)

Various combinations of these models are also possible or noise can be fractional [8]. We represent models (2) - (4) in the form of

\[ y_i = \phi_i^T \theta + e_i, \]  

(5)

where \( \theta = (b^{(1)}, \ldots, b^{(n)}, a^{(0)}, \ldots, a^{(n+1)})^T \in \mathbb{R}^{n+1}; \) for the model (2) there is the vector \( \phi_i = (z_{i-1}, \ldots, z_{i-n}, x_{i-1}, \ldots, x_{i-n})^T \in \mathbb{R}^{n+1}; \) for the model (3) there is the vector \( \phi_i = (y_{i-1}, \ldots, y_{i-n}, x_{i-1}, \ldots, x_{i-n})^T \in \mathbb{R}^{n+1}; \) for the model (4) there is the vector \( \phi_i = (z_{i-1}, \ldots, z_{i-n}, x_{i-1}, \ldots, x_{i-n})^T \in \mathbb{R}^{n+1}; \) \( \{e_i\} \) is a sequence of random variables with \( \mathbb{E}(e_i) = 0, \mathbb{E}(e_i^2) = \sigma_i^2 > 0, \) where \( \mathbb{E} \) is the mathematical expectation operator.

It is required to estimate the vector of parameters \( \hat{\theta} \) from equation (5) from the observed data \( \{\phi_i\}, \{y_i\}. \)

3. Identification algorithm

We will present two-step algorithms to estimate parameters for the trend models under study. Suppose that there is a vector such that:

• \( \psi_i \) not correlated with noise \( e_i; \)
• \( \psi_i \) well correlated with the regression vector \( \phi_i, \)

If the dimensions of the regression vector and the vector of instrumental variables are equal

\[ n_{\psi} = \dim \psi_i = \dim \phi_i = n + n_i + 1, \]

then the estimate of the parameter vector \( \hat{\theta}_{\psi} \) by instrumental variables can be found from the solution of the system of equations:

\[ \left( \frac{1}{N} \sum_{i=1}^{N} \psi_i \phi_i^T \right) \hat{\theta} = \left( \frac{1}{N} \sum_{i=1}^{N} \psi_i y_i \right) \Rightarrow \hat{R}_{\psi \psi} \hat{\theta} = \hat{r}_{\psi}, \]  

(6)

where \( \hat{R}_{\psi \psi} = \frac{1}{N} \sum_{i=1}^{N} \psi_i \phi_i^T \) is the matrix \( n_{\psi} \times n_{\psi}, \)

\[ \hat{r}_{\psi} = \frac{1}{N} \sum_{i=1}^{N} \psi_i y_i \in \mathbb{R}^{n_{\psi}}. \]

In [9], a solution to the system of equations (6) was proposed based on the augmented equivalent system of linear algebraic equations:

\[ C \Theta_{\psi \psi} = d, \]  

(7)

where \( C = \begin{pmatrix} \frac{k_1}{\psi^T} & \frac{l_1}{\psi^T} \\ \Phi & 0 \end{pmatrix} \) is the matrix \( (n_{\psi} + N) \times (n_{\psi} + N), \Phi = (\phi_1 \ldots \phi_N)^T \) is the matrix \( N \times n_{\psi} \).
\[ \Psi = \begin{pmatrix} \Psi_1 & \ldots & \Psi_N \end{pmatrix}^T \] is the matrix \( N \times n_y \), \( \Theta_{AEIV} = \begin{pmatrix} k_1 \\ \vdots \\ k_N \end{pmatrix} \in R^{N+n_y} \), \( d = \begin{pmatrix} Y \\ \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix} \end{pmatrix} \in R^{N+n_y} \), \( Y = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \in R^N \),

\[ k_i = \sqrt{\frac{\mu_{\text{min}}}{2}} \], \( I_N \) is the identity matrix of order \( N \), \( \mu_{\text{min}} \) is the minimum eigenvalue of the matrix \( \hat{R}_{yy} \).

The lower boundary of the conditionality of system (7) is determined by the inequality:

\[ \text{cond}_{\text{d}}(C) \geq \frac{1}{2} + \left( \frac{1}{4} + 2\text{cond}_{\text{d}}(R_{yy}) \right)^{1/2}. \tag{8} \]

Despite the decrease in the number of conditionality, the solution of system (8) often gives unsatisfactory estimates. This is because covariance estimates are used as coefficients. In short samples, the method of instrumental variables can give a large error. To improve the accuracy of estimates, the dimension of the vector of instrumental variables is taken greater than the dimension of the regression vector \( n_y = \dim \psi_i > n + n_i + 1 \).

This modification is called the extended method of instrumental variables [1, 2]. The estimate of the parameter vector \( \hat{\theta}_{EV} \) by extended instrumental variables can be found from the expression:

\[ \hat{\theta}_{EV} = \arg \min_{\theta} \left\| \left( \frac{1}{N} \sum_{i=1}^{N} \psi_i \phi_i^T \right) \theta - \left( \frac{1}{N} \sum_{i=1}^{N} \psi_i y_i \right) \right\|_2. \tag{9} \]

The minimum of expression (8) is a solution to the system of normal equations

\[ \hat{\theta}_{EV} = \left( \hat{R}_{yy} \hat{R}_{yy}^T \right)^{-1} \hat{R}_{yy} \hat{\theta}_{ev}. \tag{10} \]

where \( \hat{R}_{yy} \) is the matrix \( n_y \times n + n_i + 1 \).

The conditionality of the system of equations (10) is greater than the system of equations (6). To reduce conditioning, system (10) can be represented as an augmented equivalent system of equations:

\[ \check{C} \Theta_{AEIV} = \check{d}, \tag{11} \]

where \( \check{C} = \begin{pmatrix} k_2 I_{n_y} & \hat{R}_{yy} \\ \vdots & \vdots \\ \hat{R}_{yy} & 0 \end{pmatrix} \) is the matrix \( (n_y + n + n_i + 1) \times (n_y + n + n_i + 1) \), \( \Theta_{AEIV} = \begin{pmatrix} k_2^{-1} \\ \vdots \\ 0 \end{pmatrix} \in R^{n_y+n+n_i+1} \),

\[ \check{d} = \begin{pmatrix} R_{yy} \\ \vdots \\ 0 \end{pmatrix} \in R^{n_y+n+n_i+1}, \quad k_2 = \frac{\mu_{\text{min}}}{\sqrt{2}}. \]

Conditionality (11) and the choice of the optimal factor are considered in [10, 11], and is determined by the expression:

\[ \text{cond}_{\text{d}}(\check{C}) = \frac{1}{2} + \left( \frac{1}{4} + 2\text{cond}_{\text{d}}(R_{yy}^T R_{yy}) \right)^{1/2}. \]

Thus, system (11) has a smaller number of conditionality in comparison with (10), and, as a rule, the errors in the estimates of covariance are less than in (9).

4. Test Example

The dynamic system is described by the equation:

\[ y_i = A_1 e^{-b_1 i} + A_2 e^{-b_2 i} + \xi_i, \tag{12} \]

The parameters \( b_1, b_2 \) included in equation (12) are nonlinear, which makes the identification procedure more difficult. Using the Z-transform [12], we represent equation (12) in the form of a difference equation linear in parameters with an error in the output signal:

\[ z_i = c_1 z_{i-1} + c_2 z_{i-2} + d_1 \delta_{i-1} - d_2 \delta_{i-2}, \tag{13} \]
\[ y_i = z_i + \xi_i, \]

where \[ c_1 = \exp(-b_1 \cdot \Delta t) + \exp(-b_2 \cdot \Delta t), \quad c_2 = \exp(-b_1 \cdot \Delta t) \cdot \exp(-b_2 \cdot \Delta t), \quad d_1 = A_1 + A_2, \]
\[ d_2 = A_1 \exp(-b_1 \cdot \Delta t) + A_2 \exp(-b_2 \cdot \Delta t), \quad \delta_i \text{ is the delta function.} \]

For \( i > 2 \) equation (13) takes the form:
\[ z_i = c_1 z_{i-1} + c_2 z_{i-2}, \quad y_i = z_i + \xi_i. \] (14)

To estimate the coefficients \( c_1, c_2 \) (14), the least squares method cannot be applied, since the generalized error is not white noise:
\[ \varepsilon_i = y_i - c_1 y_{i-1} - c_2 y_{i-2} = \xi_i - c_1 \xi_{i-1} - c_2 \xi_{i-2}. \]

We test the algorithms considered in the article using the example of a dynamic system described by the equation:
\[ y_i = 4e^{-4} + 10e^{-0.9998} + \xi_i, \] (15)

The number of observations is \( N = 40 \), the sampling interval is \( \Delta t = 0.2 \). For \( i > 2 \) the difference equation corresponding to equation (14) takes the form:
\[ z_i = 1.637625268682286 \cdot 10^4 z_{i-1} + 0.670454123452141 \cdot 10^4 z_{i-2}, \quad y_i = z_i + \xi_i. \] (16)

The choice of the vector of instrumental variables:
1. for instrumental variables (6), (7)
\[ \Psi_i = (y_{i-3}, y_{i-4})^T \in \mathbb{R}^2. \]
2. for extended instrumental variables (9), (10)
\[ \Psi_i = (y_{i-3}, ..., y_{i-N})^T \in \mathbb{R}^{N-2}. \]

As an indicator of the quality of the model, the relative standard error of the estimation of the parameters of equation (16) was used:
\[ \delta c = \frac{\| \hat{c} - c \|^2}{\| c \|^2} \cdot 100\%. \]

Table 1 shows the conditions and errors in estimating the parameters according to formulas (6), (7), (10), (11), least-square (LS) estimate and the estimate based on the generalized Rayleigh quotient (GRQ) [13] for Gaussian independent noise with standard deviation \( \sigma_\xi = 10^{-8} \).

| Identification method       | \( \text{cond}_i \)          | \( \delta c, \% \) |
|-----------------------------|-------------------------------|-------------------|
| IV (6)                      | 5.4185e+08                    | 13.08             |
| IV with augment matrix (7)  | 3.2920e+04                    | 13.08             |
| EIV (10)                    | 7.11502e+15                   | 95.61             |
| EIV with augment matrix (11)| 1.1928e+08                    | 0.33              |
| LS                          | 1.2746e+16                    | 95.61             |
| GRQ [13]                    | 1.3271e+16                    | 123.52            |

5. Conclusion
The results of the test example show that the proposed modification of instrumental variables based on the augmented equivalent system (11) can significantly improve the accuracy of estimating system parameters.

6. References
[1] Söderström T and Stoica P 1988 *System Identification* (NJ: Prentice-Hall) 612.
[2] Söderström T and Stoica P 1983 *Instrumental Variable. Methods for System Identification* (Berlin: Springer).
[3] Young P C 2011 *Recursive Estimation and Time-Series Analysis* (Berlin: Springer - Verlag) 504.

[4] Söderström T and Mahata K 2002 On instrumental variable and total least squares approaches for identification of noisy systems *International Journal of Control* 75(6) 381-389 DOI: 10.1080/00207170110112278.

[5] Thil S, Gilson M and Garnier H 2008 On instrumental variable-based methods for errors-in-variables model identification *IFAC Proceedings* 41(2) 426-431 DOI: 10.3182/20080706-5-KR-1001.00072.

[6] Malti R, Victor S, Oustaloup A and Garnier H 2008 An optimal instrumental variable method for continuous-time fractional model identification *IFAC Proceedings* 41(2) 14379-14384 DOI: 10.3182/20080706-5-KR-1001.02436.

[7] Ivanov D V, Sandler I L and Kozlov E V 2018 Identification of fractional linear dynamical systems with autocorrelated errors in variables by generalized instrumental variables *IFAC-PapersOnLine* 51(32) 580-584 DOI: 10.1016/j.ifacol.2018.11.485.

[8] Ivanov D V and Ivanov A V 2017 Identification Fractional Linear Dynamic Systems with fractional errors-in-variables *Journal of Physics: Conf. Series*. 803 012058 DOI: 10.1088/1742-6596/803/1/012058.

[9] Zhdanov A I and Gogoleva S Yu 2001 Solving the ill-conditioned identification problems of the linear dynamic systems by the instrumental variable method *Izvestia of Samara Scientific Center of the Russian Academy of Sciences* 3(1) 128-130.

[10] Zhdanov A I and Sidorov Y V 2020 On a method for calculating generalized normal solutions of underdetermined linear systems *Computer Optics* 44(1) 133-136 DOI: 10.18287/2412-6179-CO-607.

[11] Björck Å 1967 Iterative refinement of linear least squares solutions I *BIT Numerical Mathematics* 7 257-278 DOI: 10.1007/BF01939321.

[12] Refaat E A 2005 *Lecture notes on Z-Transform* (Morrisville NC: Lulu Press).

[13] Ivanov D V 2013 Identification Discrete Fractional Order Linear Dynamic Systems with output-error *Proc. International Siberian Conference on Control and Communications (SIBCON)* DOI: 10.1109/SIBCON.2013.6693623.