State-feedback anisotropy-based robust control of linear systems with polytopic uncertainties

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Abstract. In this paper, linear discrete-time systems with polytopic uncertainties affected by random external disturbances are under consideration. The input disturbance is supposed to be a stochastic signal with known mean anisotropy, which stands for a spectral color of the signal. The anisotropic norm of the system indicates its stochastic gain from the input disturbance with the same mean anisotropy level to the output of the system. The problem is to find state-feedback control law that robustly stabilizes the uncertain system and guarantees desired performance index subject to input disturbance for all possible uncertainties. In order to solve this problem, Lyapunov functions technique and matrix inequalities approach are used to obtain the numerically effective procedure. To illustrate the efficiency of the proposed conditions, a numerical example is considered.

Introduction
In many practical applications, the real control plant does not match its mathematical model due to several reasons. Among them unmodeled dynamics and technological tolerance to some components of the system. Such kind of mismatch may lead to substantial performance degradation or even loss of stability of the plant, closed by controller that is developed for mathematical model with precisely known parameters. To overcome this problem, a lot of approaches to robust analysis and robust control of uncertain plant were established [1–3]. The aim of the robust control is to provide closed-loop stability and guaranteed performance for all possible values of uncertainties. The solution being obtained for the uncertain system depends on the uncertainty description. The attractive way is to represent parametric uncertainty as a convex polytope.

Linear systems with polytopic uncertainties have been extensively studied last three decades. A lot of fruitfull results dedicated to robust stability analysis and robust control can be found in [4–9]. To provide closed-loop linear system performance against external disturbances the most popular criteria are $\mathcal{H}_2$ and $\mathcal{H}_\infty$. $\mathcal{H}_2$ norm describes dispersion of the system output when the input disturbance is white Gaussian noise while $\mathcal{H}_\infty$ norm indicates system’s gain of the input disturbance at the whole frequency range. Both aforementioned performance indices have substantial drawbacks. $\mathcal{H}_2$ controllers lack of robustness while $\mathcal{H}_\infty$ controllers lead to unnecessary conservatism and control energy.

Additional restrictions on the spectral density of the input disturbance allow overcoming the aforementioned disadvantages of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ approaches in some sense. This allows us to adjust the controllers more precisely depending on the spectral density of the input disturbance. Such a theory is known as “stochastic $\mathcal{H}_\infty$ control theory” or “anisotropy-based control theory” [10–12]. The main
concepts of the anisotropy-based theory are the anisotropy of a random vector, the mean anisotropy of a random sequence, and the anisotropic norm of the system. The anisotropy of a random vector is the minimum value of the relative entropy (Kulbak-Leibler information divergence) between the probability density functions of the random vector and the Gaussian signal with a zero mean and a scalar covariance matrix. The mean anisotropy is the limit of the anisotropy of a vector consisting of \( n \) random vectors divided by the number \( n \) when \( n \) tends to infinity. The mean anisotropy characterizes "spectral color" of the input sequence or the difference between the sequence and the Gaussian white noise.

The induced \( \mathcal{H}_2 \) norm of the system, the input of which is a random signal with bounded mean anisotropy, is called the anisotropic norm of the stationary system. The anisotropic norm of the system lies between the scaled \( \mathcal{H}_2 \) norm and the \( \mathcal{H}_\infty \) norm. In terms of the anisotropy-based control theory, the \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) norms are limiting cases of the anisotropic norm (when the mean anisotropy tends to zero and to infinity, respectively). Thus, anisotropy-based control approaches generalize \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) approaches with respect to the description of input disturbances. Convex optimization technique applied to anisotropy-based control theory made it possible to develop numerically effective methods of performance analysis and control design for linear systems with precisely known parameters [13–14] and with norm-bounded parametric uncertainties [14–15].

In this paper a problem of robust anisotropy-based control of discrete-time linear systems with polytopic uncertainties are under consideration. Numerical effectiveness of the proposed methods is provided by using matrix inequalities and convex optimization. The rest part of the paper is organized as follows. Section Preliminaries presents basic definitions of anisotropy-based control theory. Section Problem Statement introduces the problem to be solved. In Section Main Result conditions of robust anisotropy-based state-feedback control are derived. Section Numerical Example illustrates effectiveness of the proposed method. Finally, conclusive remarks are given in section Conclusion.

1. Preliminaries

Now we give basic definitions of anisotropy-based theory, such as anisotropy of a random vector, mean anisotropy of the sequence, and anisotropic norm of the system.

Let \( W = \{w(k)\}_{k \in \mathbb{Z}} \) be a stationary Gaussian sequence of random \( m \)-dimensional vectors. Assembling the elements of \( W \), associated with the interval \([0, N-1]\), into a random vector

\[
W_{0:N-1} = \begin{bmatrix} w(0) \\ \vdots \\ w(N-1) \end{bmatrix},
\]

we assume that \( W_{0:N-1} \) is absolutely continuously distributed for every \( N > 0 \).

1.1. Definition 1. Anisotropy \( A(W_{0:N-1}) \) is defined as the minimal value of relative entropy with respect to the Gaussian distributions in \( \mathbb{R}^{mN} \) with zero mean and scalar covariance matrix described by

\[
A(W_{0:N-1}) = \frac{m}{2} \ln \left( \frac{2m \pi e}{m} \mathcal{E}(|W_{0:N-1}|^2) \right) - h(W_{0:N-1})
\]

where

\[
h(W_{0:N-1}) = -\mathbb{E} \ln f(W_{0:N-1}) = -\int_{\mathbb{R}^{mN}} f_N(w) \ln f_N(w) dw
\]
is a differential entropy, \( f_X : \mathbb{R}^m \rightarrow \mathbb{R}_+ \) is the probability density function of the vector \( W_{0:N-1} \), \( E \) denotes mathematical expectation.

### 1.2. Definition 2

Mean anisotropy of the sequence \( W \) is defined by the expression

\[
\overline{A}(W) = \lim_{N \to \infty} \frac{A(W_{0:N-1})}{N}.
\]

Consider a stable, linear, discrete-time system \( F \) written in a state-space representation

\[
x(k+1) = Ax(k) + Bw(k),
\]

\[
y(k) = Cx(k) + Dw(k).
\]

\( W = \{w(k)\}_{k \in \mathbb{Z}} \) is a stationary Gaussian sequence of \( m \)-dimensional random vectors with a bounded mean anisotropy level \( \overline{A}(W) \leq a \) \((a \geq 0)\) and zero mean, \( x(k) \in \mathbb{R}^n \), \( y(k) \in \mathbb{R}^p \).

For the given system \( F \) with the input signal \( W = \{w(k)\}_{k \in \mathbb{Z}} \) the root mean-square gain can be defined as

\[
Q(F, W) = \frac{\|Y\|_p}{\|W\|_p}
\]

where

\[
\|Y\|_p = \left( \lim_{N \to \infty} \frac{1}{2N+1} \sum_{k=-N}^{N} E |y(k)|^p \right)^{1/p}
\]

is a power norm of the Gaussian stationary sequence \( Y = \{y(k)\}_{k \in \mathbb{Z}} \). Anisotropic norm of system \( F \) can be defined in time domain in the following way.

### 1.3. Definition 3

For a given parameter \( a \geq 0 \) anisotropic norm of the system \( F \) (2)–(3) is defined by

\[
\left\| F \right\|_a = \sup_{A(W) \leq a} Q(F, W).
\]

The mean anisotropy functional \( \overline{A}(W) \) of the stationary Gaussian random sequence \( W \) can be presented in terms of its spectral density \( S(\omega) \) as

\[
\overline{A}(W) = -\frac{1}{4\pi} \int_{-\pi}^{\pi} \ln \det \left( \frac{mS(\omega)}{\|W\|_p^2} \right) d\omega.
\]

Anisotropic norm \( \left\| F \right\|_a \) describes the stochastic gain of system \( F \) with respect to the input sequence \( W \) with a known mean anisotropy level. It's necessary to mention that the anisotropic norm lies between \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) norms of the same system. If \( \overline{A}(W) = 0 \), then \( \left\| F \right\|_0 = \frac{\|F\|_m}{\sqrt{m}} \). Moreover,

\[
\lim_{a \to \infty} \left\| F \right\|_a = \|F\|_\infty.
\]

For more information about basic concepts of anisotropy-based control theory, see also [10–12,17].

### 2. Problem statement

Consider the following discrete-time time-invariant linear system
\[ x(k+1) = A(\Theta)x(k) + B_x(\Theta)u(k) + B_w(\Theta)w(k), \]  
\[ z(k) = C(\Theta)x(k) + D_x(\Theta)w(k) \]  

where \( x(k) \in \mathbb{R}^n \) is a state vector, \( w(k) \in \mathbb{R}^m \) is a random stationary sequence with bounded mean anisotropy level \( \overline{\| W \|} \leq a \ (a \geq 0) \), \( u(k) \in \mathbb{R}^p \) is a control input, \( z(k) \in \mathbb{R}^p \) is a controllable output. Matrices \( A(\Theta), B_x(\Theta), B_w(\Theta), C(\Theta), D_x(\Theta) \) are defined as

\[ A(\Theta) = \sum_{i=1}^{r} \Theta_i A_i, \quad B_x(\Theta) = \sum_{i=1}^{r} \Theta_i B_{x,i}, \quad B_w(\Theta) = \sum_{i=1}^{r} \Theta_i B_{w,i}, \]  
\[ C(\Theta) = \sum_{i=1}^{r} \Theta_i C_i, \quad D_x(\Theta) = \sum_{i=1}^{r} \Theta_i D_{x,i}. \]  

Parameters \( \Theta \) are limited by

\[ \sum_{i=1}^{r} \Theta_i = 1, \quad \Theta_i \geq 0, \quad \Theta_i \in \mathbb{R}, \quad \forall i = 1, r. \]  

3. Problem statement.
For a known mean anisotropy level \( a \geq 0 \) and a given scalar value \( \gamma > 0 \) the aim is to find a state-feedback control law \( u(k) = Kx(k) \) that robustly stabilizes closed-loop system (7)–(8) and guarantees disturbance attenuation level less than \( \gamma \) (i.e. \( \| F_z \|_0 < \gamma \)) for all possible uncertainties, satisfying (9)–(10).

4. Main Result
Before we solve the problem, it’s necessary to find conditions of robust stability and anisotropic norm boundedness for unforced system (7)–(8). These conditions are listed below.

**Theorem 1.**
Unforced system (7)–(8) is robustly stable and meets anisotropy-based performance \( \gamma > 0 \) for a given mean anisotropy level \( a \geq 0 \) and all possible uncertainties defined by (9)–(10) if there exist matrices \( P(\Theta) > 0, \Psi > 0 \), and a scalar value \( \eta: \eta > \gamma^2 \) such that the following matrix inequalities hold:

\[ \eta - \left( e^{-2a} \det \Psi \right)^{1/n} < \gamma^2, \]  
\[ \begin{bmatrix} \Psi - \eta I_m & * & * \\ B_w(\Theta) & -P^{-1}(\Theta) & * \\ D_x(\Theta) & 0 & -I_p \end{bmatrix} < 0, \]  
\[ \begin{bmatrix} -P(\Theta) & * & * & * \\ 0 & -\eta I_m & * & * \\ A(\Theta) & B_w(\Theta) & -P^{-1}(\Theta) & * \\ C(\Theta) & D_x(\Theta) & 0 & -I_p \end{bmatrix} < 0. \]  

Notation \( * \) denotes transposed blocks from lower part of the matrix.
Result from Theorem 1 is a half-way result of Theorem 1 in [18] and is omitted here. Theorem 1 defines parameter-dependent anisotropy-based bounded real lemma for unforced system (7)–(8). Based on these conditions, control design procedure can be implemented as follows.

**Theorem 2.**

System (7)–(8) is robustly stabilizable by state-feedback control law \( u(k) = K(\Theta)x(k) \) and meets anisotropy-based performance \( \gamma > 0 \) for a given mean anisotropy level \( a \geq 0 \) and all possible uncertainties defined by (9)–(10) if there exist matrices \( \Phi(\Theta) > 0, \Psi > 0, L(\Theta), G(\Theta) \) and a scalar value \( \eta : \eta > \gamma^2 \) such that

\[
\eta - \left( e^{-2a} \det \Psi \right)^{1/m} < \gamma^2, \tag{14}
\]

\[
\begin{bmatrix}
\Psi - \eta I_w & * & * \\
B_w(\Theta) & -\Phi(\Theta) & * \\
D_w(\Theta) & 0 & -I_p
\end{bmatrix} < 0, \tag{15}
\]

\[
\begin{bmatrix}
-G(\Theta) - G^T(\Theta) + \Phi(\Theta) & * & * & * \\
0 & -\eta I_w & * & * \\
A(\Theta)G(\Theta) + B_w(\Theta)L(\Theta) & B_w(\Theta) & -\Phi(\Theta) & * \\
C(\Theta)G(\Theta) & D_w(\Theta) & 0 & -I_p
\end{bmatrix} < 0. \tag{16}
\]

Moreover, feedback gain is defined by

\[
K(\Theta) = L(\Theta)G(\Theta)^{-1}.
\]

**Proof.** Closed-loop system is defined as

\[
x(k + 1) = (A(\Theta) + B_w(\Theta)K(\Theta))x(k) + B_w(\Theta)w(k), \tag{17}
\]

\[
z(k) = C(\Theta)x(k) + D_w(\Theta)w(k). \tag{18}
\]

Introduce notations

\[
\Phi(\Theta) = P^{-1}(\Theta), \quad K(\Theta) = L(\Theta)G^{-1}(\Theta). \tag{19}
\]

Inequalities (14) and (15) follows immediately from (11) and (12) respectively. Inequality (13) now takes the form

\[
\begin{bmatrix}
\Phi^{-1}(\Theta) & * & * & * \\
0 & -\eta I_w & * & * \\
A(\Theta) + B_w(\Theta)K(\Theta) & B_w(\Theta) & -\Phi(\Theta) & * \\
C(\Theta) & D_w(\Theta) & 0 & -I_p
\end{bmatrix} < 0. \tag{20}
\]

Pre- and post-multiplying (20) by a nonsingular matrix

\[
\begin{bmatrix}
G^T(\Theta) & 0 & 0 & 0 \\
0 & I_w & 0 & 0 \\
0 & 0 & I_p & 0 \\
0 & 0 & 0 & I_p
\end{bmatrix}
\]
and its transpose, respectively, we get

\[
\begin{bmatrix}
\Lambda(\Theta) & * & * & * \\
0 & -\eta I_m & * & * \\
A(\Theta)G(\Theta) + B_{ui}(\Theta)L(\Theta) & B_{ui}(\Theta) & -\Phi(\Theta) & * \\
C(\Theta)G(\Theta) & D_{ui}(\Theta) & 0 & -I_p
\end{bmatrix} < 0,
\]

where \( \Lambda(\Theta) = -G^T(\Theta)\Phi^{-1}(\Theta)G(\Theta) \).

Note that for \( \Phi(\Theta) > 0 \) the inequality

\[-(G(\Theta) - \Phi(\Theta))^T\Phi^{-1}(\Theta)(G(\Theta) - \Phi(\Theta)) \leq 0\]

implies \(-G^T(\Theta)\Phi^{-1}(\Theta)G(\Theta) \leq -G(\Theta) - G^T(\Theta) + \Phi(\Theta) \). The last follows that inequality (20) holds if (21) holds. This completes the proof.

Condition of Theorem 2 are also parameter-dependent. These conditions are not suitable for numerical implementation. Hence, we provide parameter-independent control design conditions below.

**Theorem 3.**

System (7)–(8) is robustly stabilizable by state-feedback control law \( u(k) = Kx(k) \) and meets anisotropy-based performance \( \gamma > 0 \) for a given mean anisotropy level \( a \geq 0 \) and all possible uncertainties defined by (9)–(10) if there exist matrices \( \Phi_i > 0, \Psi > 0, \bar{L}, \bar{G} \) and a scalar value \( \eta: \eta > \gamma^2 \) such that

\[
\eta - \left( e^{-2a} \det \Psi \right)^{1/m} < \gamma^2,
\]

\[
\begin{bmatrix}
\Psi - \eta I_m & * & * \\
B_{ui} & -\Phi_i & * \\
D_{ui} & 0 & -I_p
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
-\bar{G} - \bar{G}^T + \Phi_i & * & * & * \\
0 & -\eta I_m & * & * \\
A_i\bar{G} + B_{ui}\bar{L} & B_{ui} & -\Phi_i & * \\
C_i\bar{G} & D_{ui} & 0 & -I_p
\end{bmatrix} < 0,
\]

\( i = 1, r \). Moreover, feedback gain is defined by

\[
K = LG^{-1}.
\]

**Proof.** Let matrices \( G(\Theta) = \bar{G}, \ L(\Theta) = \bar{L}, \) and \( P(\Theta) = \sum_{i=1}^{r} \Theta_i P_i \). Taking into account (9)–(10), inequalities (15) and (16) are equivalent to

\[
\sum_{i=1}^{r} \Theta_i \begin{bmatrix}
\Psi - \eta I_m & * & * \\
B_{ui} & -\Phi_i & * \\
D_{ui} & 0 & -I_p
\end{bmatrix} < 0.
\]
and

\[
\sum_{i=1}^{n} \Theta_i \begin{bmatrix}
-G - G^T + \Phi_i & * & * & * \\
0 & -\eta_l & * & * \\
A G + B_w L & B_w & -\Phi_i & * \\
C_i \tilde{G} & D_w & 0 & -I_p \\
\end{bmatrix} < 0
\]

respectively. This lead to the conditions (22)–(24).

5. Numerical Example

Consider the following system

\[
A_1 = \begin{bmatrix}
1.2 & -0.7 \\
0.5 & -0.3 \\
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
1 & 1 \\
-0.5 & -0.7 \\
\end{bmatrix},
\]

\[
B_{w1} = \begin{bmatrix}
-0.5 \\
0 \\
\end{bmatrix}, \quad B_{w2} = \begin{bmatrix}
-0.5 \\
2 \\
\end{bmatrix}, \quad B_{w1} = \begin{bmatrix}
1 \\
1 \\
\end{bmatrix}, \quad B_{w2} = \begin{bmatrix}
1 \\
0 \\
\end{bmatrix}
\]

\[
C_1 = C_2 = \begin{bmatrix}
1 & 0 \\
\end{bmatrix}, \quad D_{w1} = D_{w2} = 0.
\]

System is unstable with \( \Theta_1 = 0.66, \Theta_2 = 0.34 \). Its spectral radius is \( \rho = \sqrt{0.66 A_1 + 0.34 A_2} = 1.1195 \).

We choose \( \alpha = 0.1 \) and minimize anisotropic norm of closed-loop system \( \| F \| \rightarrow \min \gamma \). The minimal value of anisotropic norm is found using Yalmip and given by \( \gamma_{\min} = 2.5136 \) with feedback gain \( K = [-0.9196 \quad -0.2176] \). Actual upper bound of anisotropic norm is computed using step size \( h = 0.001 \) and applying computational algorithm from [13] for \( \Theta_2 = 1 - \Theta_1, 0 \leq \Theta_1 \leq 1 \). The evaluation of anisotropic norm is depicted in Fig. 1. Worst case performance is obtained by this procedure is \( \gamma_{\text{real}} = 2.2727 \) that satisfies Theorem 3.

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Conclusions

In this paper the problem of state-feedback robust anisotropy-based control for discrete-time linear systems with polytopic uncertainties has been considered. The solution of the problem is formulated via matrix inequalities and can be solved using convex optimization techniques. Additional variable \( \tilde{G} \) is used to reduce conservatism of the solution.

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