Regular continuum systems of point particles. I: systems without interaction

Chubarikov V. N., Lykov A. A., Malyshev V. A.*

Abstract

Normally, in mathematics and physics, only point particle systems, which are either finite or countable, are studied. We introduce new formal mathematical object called regular continuum system of point particles (with continuum number of particles). Initially, each particle is characterized by the pair: (initial coordinate, initial velocity) in $R^d$. Moreover, all initial coordinates are different and fill up some domain in $R^d$. Each particle moves via normal Newtonian dynamics under influence of some external force, but there is no interaction between particles. If the external force is bounded then trajectories of any two particles in the phase space do not intersect. More exactly, at any time moment any two particles have either different coordinates or different velocities. The system is called regular if there are no particle collisions in the coordinate space.

The regularity condition is necessary for the velocity of the particle, situated at a given time at a given space point, were uniquely defined. Then the classical Euler equation for the field of velocities has rigorous meaning. Though the continuum of particles is in fact a continuum medium, the crucial notion of regularity was not studied in mathematical literature.

It appeared that the seeming simplicity of the object (absence of interaction) is delusive. Even for simple external forces we could not find simple necessary and sufficient regularity conditions. However, we found a rich list of examples, one-dimensional and multi-dimensional, where we could get regularity conditions on different time intervals. In conclusion we formulate many unsolved problems for regular systems with interaction.

Key Words

point particle dynamics, continuum media, Euler equation, absence of collisions

1 Introduction

Now we give the exact definition of the central object we will study.

Regular continuum system $M_T$ of point particles is the set of subsets $\Lambda_t \in R^d$ enumerated by the time moments $t \in [0, T), 0 < T \leq \infty$. Moreover, $\Lambda_0$ is assumed to be the closure of some open connected subset of $R^d$ with piece-wise smooth boundary $\partial \Lambda_0$. Each point of this domain is considered as a «material particle» of infinitely small mass. The dynamics is defined by the system of one-to-one mappings (diffeomorphisms) $U_t = U_{0,t} : \Lambda_0 \to \Lambda_t, t \in [0, T)$. All these mappings are assumed to be sufficiently smooth in $x$ and piece-wise smooth in $t$, and $U_0(x)$ is the identity map. Thus, each point (particle) $x \in \Lambda_0$ has its own trajectory in $R^d$: $y(t, x) = U_t(x)$, where $y(0, x) = x$ is the initial coordinate of this particle. It follows from the definition, that the particles never collide, that is $y(t, x) \neq y(t, x')$ for any $t$ and $x \neq x'$.

$M_T$ is called a system without interaction, if $y(t, x)$ are the solutions of the following equations

$$\frac{d^2 y(t, x)}{dt^2} = F_x(y(t, x))$$

for some given functions: initial velocity $v(x)$ and external forces $F_x(y)$, possibly different for different particles. Further on we assume that either $F_x(y) = F(y)$ does not depend on $x$ or $F_x(y) = \frac{F(y)}{m(x)}$ for some functions $F(y)$ and $m(x) > 0$, see section 4 below. It is always assumed that $v(x)$ and $m(x)$ are sufficiently smooth in $x \in \Lambda_0$, and $F(y)$ is smooth or piece-wise smooth in $y$. Moreover, it is always assumed that any equation (1) has a unique solution on all considered interval $[0, T)$. Unless otherwise stated, we put $m(x) = 1$.

Obviously, the conception of continuum media as consisting of the continuum number of particles of infinitely small mass, is well known in mathematics, see for example [7], p. 56. The goal of this paper is to stress the importance of the notion of regularity, and give examples of such systems. If the smoothness of $y(t, x)$ follows

* Lomonosov Moscow State University, Faculty of Mechanics and Mathematics, Vorobyevy Gory 1, Moscow, 119991, Russia
from general theorems of the theory of ordinary differential equations, then the main difficulty is to prove the absence of collisions. We remind that we consider the trajectories not in the phase space $R^d \times R^d$, but their projections on the coordinate space $R^d$. The term «regular» hints that more general definitions are possible.

2 Main results

2.1 One-dimensional systems

Smooth force Firstly, note that if $v(x)$ and $F(y)$ are positive and non-decreasing functions then there will be no collisions, because a particle cannot catch up the particles, which are initially (at $t = 0$) to the right of it.

Now we put $A_0 = [0,1]$, and $F(y), y \in [-\infty, \infty)$, is assumed to be smooth. Define the potential energy at any point $y$ and the full energy of the particle at time $t$, initially at point $x$, correspondingly to the equation (1),

$$U(y) = -\int_0^y F(z)dz, H_r(x) = \frac{u^2(t, y(t, x))}{2} + U(y(t, x))$$

where 

$$u(t, y(t, x)) = \frac{dy(t, x)}{dt} = v(t, x)$$

is the velocity of the particle being at time $t$ at the point $y$.

We shall prove first the simpler, but more intuitive statement, and later we shall discuss technically more difficult one. Let $T(x, y)$ the first time moment when the point $x \in [0,1]$ will be at the point $y$.

Assume the following:

1) $v(x) > 0$,

2) for all $x \in [0,1]$ and all $y \geq x$ the functions $H_0(x) - U(y) > 0$. In particular, this is the case when $F(y)$ is positive for all $y \geq 0$ (in this case all particles move to the right).

Theorem 1 Under these assumptions the following conditions are equivalent:

1) on the interval $[0, \infty)$ there will not be collisions of particles;

2) for all $y > 0$ the functions $T(x, y)$ is strictly decreasing in $x$ on the interval $x \in [0, \min\{y, 1\}]$;

3) for any $y > 0$ and for $x \in [0, \min\{y, 1\}]$

$$\frac{1}{v(x)} + \frac{v(x)v'_x(x) - F(x)}{2\sqrt{2}}\int_x^y \frac{dz}{(H_0(x) - U(z))^\frac{3}{2}} \geq 0$$

where the equality is possible only on the discrete subset of points.

Theorem 2 Let now $v(x) \geq 0, m(x), F(y) > 0 \in C^1(\mathbb{R})$. Then there will not be collisions iff for all $x \in [0, \min\{y, 1\}]$ and $y > x$ the following inequality holds:

$$H_0(x)\left(\frac{1}{\sqrt{2(H_0(x) - U(y))}} F(y) + \int_x^y \frac{1}{\sqrt{2(H_0(x) - U(z))}} \frac{F'(z)}{F''(z)}dz\right) < \frac{v'(x)\sqrt{m(x)}}{F(x)} + \frac{v(x)m'(x)}{2F(x)\sqrt{m(x)}}$$

In particular, when $v(x) = 0, m(x) = 1$ for all $x \in [0,1]$, the inequality (2) is equivalent to the following:

$$\frac{1}{\sqrt{U(x) - U(y)}} + F(y)\int_x^y \frac{1}{\sqrt{U(x) - U(z)}} \frac{F'(z)}{F''(z)}dz > 0$$

Note that the similar assertion holds if the functions $F(y), v(x), m(x)$ are piece-wise smooth.

Piece-wise constant force

Theorem 3 1) (one gap) Let for some $F_1 > 0, F_2 \geq 0$ and $A > 1$

$$F(x) = F_1, 0 \leq x < A, \ F(x) = F_2, x \geq A$$

If $v(x) = 0$ for all $x \in [0,1]$, then there are no collisions iff $F_2 \geq F_1$. 
If \( v(x) \geq 0 \) for all \( x \in [0,1] \), then there will not be collisions iff for all \( x \in [0,1] \) both of following inequalities hold:

\[
- 2(A - x)v'(x) < v(x) + \sqrt{D(x)},
\]

\[
v'(x)((F_1 - F_2)v(x) + F_2\sqrt{D(x)}) \geq F_1(F_1 - F_2),
\]

where

\[
D(x) = v^2(x) + 2F_1(A - x)
\]

2) (two gaps) assume that \( v(x) = 0 \) for all \( x \in [0,1] \), and also that for some \( 0 < F_2 < F_1, F_2 < F_3 \) and \( 1 < A < B \)

\[
F(x) = F_1, 0 \leq x < A, \quad F(x) = F_2, x \in [A, B), \quad F(x) = F_3, x \geq B
\]

Then there will not be collisions iff the following inequality holds:

\[
B - A \leq \alpha(A - 1), \quad \alpha = \frac{F_1(F_3 - F_1)(F_3(F_1 - F_2) + F_1(F_3 - F_2))}{(F_1 - F_2)^2F_3^2},
\]

From this statement the following necessary condition for the absence of collisions follows: \( F_3 > F_1 \). Notice also that the set of all \( B > A > 1 \), for which there will not be collisions, is not empty under the condition that \( F_3 > F_1 \).

Unexpected corollary of the second statement of point 1) of the theorem for the case \( F_2 = 0 \) is the following simple sufficient (but not necessary) condition for the absence of collisions:

\[
v'(x) \geq \frac{F_1}{\sqrt{F_1x + v^2(0)}}
\]

2.2 Multi-dimensional systems

Multi-dimensional analog of monotonicity of the force  Remind that if the external force does not decrease and initial velocities also do not decrease then there will not be collisions. We will prove the following multi-dimensional generalization of this fact.

**Theorem 4** Assume that the force \( F(y) \) is such that for all \( x, y \in \mathbb{R}^d \) the following inequality holds:

\[
(F(y) - F(x), y - x) \geq 0.
\]

Assume also that for all \( x_1, x_2 \in \Lambda \)

\[
(v(x_2) - v(x_1), x_2 - x_1) \geq 0.
\]

Then there will not be collisions on the time interval \([0, \infty)\).

**Linear force**

Assume that the force \( F \) is linear, that is

\[
F(y) = Ay + b,
\]

for some \((d \times d)\)-matrix \( A \) and \( b \in \mathbb{R}^d \).

Further on we will assume that all eigenvalues \( \lambda_1, \ldots, \lambda_d \) of \( A \) are real, and there exists basis \( u_i \) of the space \( \mathbb{R}^d \), consisting of the eigenvectors of \( A \), so that \( Au_i = \lambda_i u_i, \ i = 1, \ldots, d \).

**Theorem 5** Assume that all eigenvalues of the matrix \( A \) are non negative, and that for all \( x_1, x_2 \in \Lambda \)

\[
(v(x_2) - v(x_1), x_2 - x_1) \geq 0,
\]

Then there will not be collisions.
Piece-wise constant force

**Lemma 1** Let $F(y) = F$ for all $y \in \mathbb{R}^d$ and for some constant vector $F \in \mathbb{R}^d$. The particle $x_1, x_2$ collide iff the vectors $R(x_1, x_2) = x_2 - x_1$ and $V(x_1, x_2) = v(x_2) - v(x_1)$ are parallel and the following inequality holds:

$$ (R(x_1, x_2), V(x_1, x_2)) < 0, $$

where $(\cdot, \cdot)$ is the standard euclidean product in $\mathbb{R}^d$.

Assume that the force $F$ is defined as:

$$ F(y) = \begin{cases} F_1, & y \in \Pi_1 = \{y = (y^1, \ldots, y^d) \in \mathbb{R}^d : y^d < A\} \\ F_2, & y \in \Pi_2 = \{y = (y^1, \ldots, y^d) \in \mathbb{R}^d : y^d \geq A\} \end{cases}, $$

where $F_k = (F_k^1, F_k^2, \ldots, F_k^d) \in \mathbb{R}^d$, $k = 1, 2$ are constant vectors, and the parameter $A > 0$. We shall assume also that $F_1^d > 0$ for definiteness. Assume also that $\Lambda \subset \Pi_1$.

The following statement is both natural and somewhat unexpected generalization of (3).

**Theorem 6** Assume that $v(x) = 0$ for all $x \in \Lambda$ and $F_2^d \geq 0$. Then there is no collisions iff $F_1^d \leq F_2^d$.

Note that the condition $F_2^d \geq 0$ is necessary for the particle could not return to the set $\Pi_1$ after hitting the set $\Pi_2$. Without this condition particle could oscillate between the sets $\Pi_1, \Pi_2$. Then the analysis becomes more complicated.

Central field on the plane

Let $d = 2$ and, besides euclidean $x = (x^1, x^2)$ coordinate we shall use also polar coordinates $(r, \phi)$ on the plane:

$$ x^1 = r \cos \phi, \quad x^2 = r \sin \phi. $$

Let $\Lambda_0$ be bounded and does not contain the origin. Then it is contained in some annulus

$$ O(R_1, R_2) = \{x : 0 < R_1 < r < R_2 < \infty\} $$

The force is assumed to be central, that is directed along the radius vector $r$ of the point $x$, and equal to

$$ F(x) = -\frac{\partial U(r)}{\partial r} \frac{r}{r}, \quad y \in \mathbb{R}^2, $$

where the potential energy $U$ is a smooth scalar function on $(0, \infty)$, $|\cdot|$ is the euclidean norm.

Denote $r(t, x), \phi(t, x)$ the norm and the angle of the point $y(t, x)$ at time moment $t$. Note that the trajectory

$$ y(t, x) = (r(t, x), \phi(t, x)). $$

is uniquely defined by the initial velocities field $v(0, x), \quad x \in \Lambda$, or the functions

$$ \frac{dr(0, x)}{dt}, \quad \frac{d\phi(0, x)}{dt}, \quad x \in \Lambda $$

We need also the following assumptions:

1. For all points $x \in \Lambda$, the functions

$$ \frac{dr(0, x)}{dt} = g(|x|) > 0, \quad \frac{d\phi(0, x)}{dt} = h(|x|) $$

depend only on $r$, and the first one is positive.

2. For all $r_2 \geq r_1 > R_1$

$$ -\frac{dU(r_2)}{dr_2} + \frac{M^2(r_1)}{r_2^2} \geq 0, $$

where $M(r) = r^2 h(r)$ is the kinetic momentum.

As we will see later, these conditions garanty that $r(t, x)$ monotonically increases to infinity with $t$. 

**Theorem 7** Under the formulated assumptions, for the absence of collisions it is sufficient that for any $R_1 < r_1 < r_2, u_2 > r_1$

\[
\int_{r_1}^{r_2} \frac{d}{dr_1} \frac{1}{\sqrt{2(E_0(r_1) - V(z, r_1))}} dz < \frac{1}{g(r_1)},
\]

where

\[
E_0(r) = \frac{1}{2}g^2(r) + U(r) + \frac{1}{2}r^2h^2(r), \quad V(z, r) = U(z) + \frac{r^4h^2(r)}{2z^2}.
\]

Note that the dynamics of $\Lambda_0$ can be described as follows. All intersection points of $\Lambda_0$ with the circle $\gamma_r$ of radius $r > 0$ will become at time $t$ on the circle of some radius $R(t, r)$, and simultaneously will be rotated around the origin on the same angle $\phi(t, r)$. Moreover, $R(t, r)$ and $\phi(t, r)$ depend only of $r$ and $t$.

One can do the condition 2) weaker, assuming that $g(|x|) \geq 0$. Then the proof should be changed along the plan similar to that in the theorem 2.

## 3 Proofs

### 3.1 One dimensional systems

**Smooth force - Theorem 1** The equivalence of 1) and 2) is obvious - this means that no particle will catch up another particle, situated at time $t = 0$ to the right of it. To prove 3) note that from the energy conservation $H_1(x) = H_0(x)$ the following formula follows

\[
T(x, y) = \int_x^y \frac{dz}{\sqrt{2(H_0(x) - U(z))}} \tag{6}
\]

Note that under our conditions the function $U$ is non-increasing in $x$, thus the expressions under square root in \([6]\) are always non-negative. One has now only to calculate the derivative

\[
\frac{dT(x, y)}{dx} = -\frac{1}{\sqrt{2(H_0(x) - U(x))}} + \frac{1}{2} \int_x^y \frac{d}{dx} \left(\frac{1}{(H_0(x) - U(z))^2}\right) dz =
\]

\[
= -\frac{1}{v(x)} - \frac{1}{2\sqrt{2}} \int_x^y (v_v - F(x)) dz \leq 0 \tag{7}
\]

**Smooth force - Theorem 2** One can integrate by parts in the formula for $T(x, y)$:

\[
\sqrt{2}T(x, y) = -\int_x^y \frac{2}{U'(z)} d\sqrt{H_0(x) - U(z)} =
\]

\[
= 2 \left(-\sqrt{\frac{H_0(x) - U(y)}{U'(y)}} + \sqrt{\frac{H_0(x) - U(x)}{U'(x)}}\right) + 2 \int_x^y \sqrt{\frac{H_0(x) - U(z)}{U'(z)}} \left(\frac{1}{U'(z)}\right)' dz =
\]

\[
= 2h(x, y) + 2g(x, y),
\]

where

\[
h(x, y) = -\sqrt{\frac{H_0(x) - U(y)}{U'(y)}} + \sqrt{\frac{H_0(x) - U(x)}{U'(x)}} = \frac{\sqrt{H_0(x) - U(y)}}{F(y)} - \frac{\sqrt{m(x)}v(x)}{\sqrt{2F(x)}}
\]

\[
g(x, y) = \int_x^y \sqrt{\frac{H_0(x) - U(z)}{U'(z)}} \left(\frac{1}{U'(z)}\right)' dz = \int_x^y \sqrt{\frac{H_0(x) - U(z)}{U'(z)}} \frac{F'(z)}{F^2(z)} dz
\]

The derivative of the first summand is:

\[
\frac{d}{dx} h(x, y) = H'_0(x) \frac{1}{2\sqrt{H_0(x) - U(y)}} \frac{1}{F(y)} - \frac{v(x)\sqrt{m(x)}}{2\sqrt{2F(x)}} + \frac{v(x)F'(x)\sqrt{m(x)}}{\sqrt{2F^2(x)}} - \frac{v(x)m'(x)}{2\sqrt{2F(x)\sqrt{m(x)}}}
\]

Using the known formula:

\[
\int_x^y \frac{d}{dz} f(x, z) dz = -f(x, x) + \int_x^y \frac{d}{dx} f(x, z) dz.
\]

we get

\[
\frac{d}{dx} g(x, y) = -\sqrt{\frac{H_0(x) - U(x)}{F^2(x)}} F'(x) + \frac{1}{2} H'_0(x) \int_x^y \frac{1}{\sqrt{\frac{H_0(x) - U(z)}{F^2(z)}}} dz.
\]
This gives the proof
\[
\frac{d}{dx}T(x, y) = \sqrt{2} \frac{d}{dx}h(x, y) + \sqrt{2} \frac{d}{dx}g(x, y) = \\
= H_0'(x) \frac{1}{\sqrt{2(H_0(x) - U(y))}} \frac{1}{F(y)} \left( - \frac{v'(x) \sqrt{m(x)}}{F(x)} - \frac{v(x)m'(x)}{2F(x)\sqrt{m(x)}} + \right) \\
+ H_0'(x) \int_x^y \frac{1}{\sqrt{2(H_0(z) - U(z))}} \frac{F'(z)}{F^2(z)} dz
\]

**Piece-wise constant force - Theorem**

This theorem can be proved using the Theorem 2 (more exactly, on its analog for piece-wise smooth force \(F(x)\)). But the following simpler proof is more useful.

Let us prove the first assertion of the theorem. Obviously, on the time interval \([0, \infty)\) there will not be collisions iff \(v(T(0, A), x)\) is non-decreasing in \(x \in [0, 1]\). Evidently

\[
v(T(0, A), x) = v(T(x, A), x) + F_2(T(0, A) - T(x, A)), \quad v(T(x, A), x) = F_1T(x, A).
\]

\[
\frac{dv(T(0, A), x)}{dx} = F_1 \frac{dT(x, A)}{dx} - F_2 \frac{dT(x, A)}{dx} = (F_1 - F_2) \frac{dT(x, A)}{dx}.
\]

It is clear that \(\frac{dT(x, A)}{dx} < 0\) for all \(x \in [0, 1]\), that gives the assertion.

Let us now prove the second statement. Obviously, on the interval \([0, \infty)\) there will not be collisions of \(v(T(0, A), x)\) is non-decreasing in \(x \in [0, 1]\) and \(T(x, A)\) is decreasing in \(x\). Evidently

\[
v(T(0, A), x) = v(T(x, A), x) + F_2(T(0, A) - T(x, A)), \quad v(T(x, A), x) = v(x) + F_1T(x, A).
\]

\[
\frac{dv(T(0, A), x)}{dx} = F_1 \frac{dT(x, A)}{dx} - F_2 \frac{dT(x, A)}{dx} = v'(x) + (F_1 - F_2) \frac{dT(x, A)}{dx}.
\]

Let us see when the function \(T(x, A)\) is decreasing in \(x\). From the equation

\[
x + v(x)t + \frac{F_1}{2}t^2 = A
\]

we get

\[
T(x, A) = \frac{-v(x) + \sqrt{D(x)}}{F_1}, \quad D(x) = v^2(x) + 2F_1(A - x).
\]

Then

\[
\frac{dT(x, A)}{dx} = \frac{-v'(x) + \frac{v(x)v'(x) - F_1}{\sqrt{D(x)}}}{F_1} = v'(x) \frac{v(x) - \sqrt{D(x)}}{F_1 \sqrt{D(x)}} - \frac{1}{\sqrt{D(x)}}.
\]

That is why the condition \(\frac{dT(x, A)}{dx} < 0\) is equivalent to the following inequality:

\[
(v(x) - \sqrt{D(x)})v'(x) < F_1.
\]

Multiplying this inequality on \(v(x) + \sqrt{D(x)}\), from (2) we get equivalent inequality:

\[-2(A - x)v'(x) < v(x) + \sqrt{D(x)}.
\]

Substituting the formula (3) for \(\frac{dT(x, A)}{dx}\) into the formula (5), we get:

\[
\frac{dv(T(0, A), x)}{dx} = v'(x) \left( 1 + (F_1 - F_2) \frac{v(x) - \sqrt{D(x)}}{F_1 \sqrt{D(x)}} \right) \frac{F_1 - F_2}{\sqrt{D(x)}}.
\]

That is why the condition \(\frac{dv(T(0, A), x)}{dx} > 0\) is equivalent to the inequality:

\[v'(x)((F_1 - F_2)v(x) + F_2\sqrt{D(x)}) \geq F_1(F_1 - F_2).
\]

The statement is thus proved.

Let us prove now the third statement. But before the formal proof, we would like to intuitively explain why the situation is possible when two infinitely close particles \(x_1 = x, x_2 = x + dx\) will not collide after the trajectory of the left point \(x_1\) will reach \(A\). Assuming that \(F_2 < F_1\), the distance between the points will

\[
\frac{dx}{dt}(T(x_1, A), x_1) = \frac{dx}{dt}(T(x_2, A), x_2) \Rightarrow x_1(t) = x_1(t) + (F_1 - F_2)v(x_1)\sqrt{D(x_1)}.
\]
decrease linearly in time after the moment \( T_1 = T(x_1, A) \) with velocity \( w = v(T_1, x_1) - v(T_1, x_2) \). As the points are infinitely close, then \( w = a \ dx \) for some constant \( a = a(x_1, x_2) > 0 \). From the other side, the distance \( D = y(T_1, x_2) - y(T_1, x_1) = b \ dx \) between points at time \( T_1 \), for some constant \( b = b(x_1, x_2) > 0 \). That is why the time necessary for the left particle to catch up the right one, equals \( t = t(x_1, x_2) = D/w = \frac{b}{a} \). Thus, this time is already not infinitely small. It appears that this time is separated from zero by some constant \( t^* \) for all \( 0 \leq x_1 < x_2 \leq 1 \), that is why it is sufficient to choose the length of the interval \( B - A \), where the force \( F_2 \) acts, so that any point from \([0, 1]\) pass the interval \([A, B]\) for the time less than \( t^* \). It remains to choose the force \( F_3 \) sufficiently large.

Let \( T = T(0, B) \). It is clear that on the interval \([0, +\infty]\) there will not be collisions iff \( v(T, x) \) is non decreasing in \( x \in [0, 1] \). We have evident equalities:

\[
v(T, x) = v(T(x, B), x) + F_3(T - T(x, B)),
\]

\[
v(T(x, B), x) = v(T(x, A), x) + F_2(T(x, B) - T(x, A)), \quad v(T(x, A), x) = F_1 T(x, A).
\]

This gives

\[
\frac{dv(T, x)}{dx} = F_1 \frac{dT(x, A)}{dx} + F_2 \frac{dT(x, B)}{dx} - F_3 \frac{dT(x, B)}{dx} = \left( F_1 - F_2 \right) \frac{dT(x, A)}{dx} - \left( F_3 - F_2 \right) \frac{dT(x, B)}{dx}.
\]

We can find \( T(x, B) \). Clearly

\[
y(T(x, A) + s, x) = A + T(x, A)F_1 s + \frac{s^2}{2} F_2,
\]

for \( 0 \leq s \leq T(x, B) - T(x, A) \). Then from the condition \( y(x, T(x, A) + s) = B \) we find that

\[
s = s(x) = \frac{-T(x, A)F_1 + \sqrt{D(x)}}{F_2}, \quad D(x) = T^2(x, A)F_1^2 + 2(B - A)F_2.
\]

This gives

\[
T(x, B) = T(x, A) + s(x) = \frac{-T(x, A)(F_1 - F_2) + \sqrt{D(x)}}{F_2}.
\]

Let us find the derivative:

\[
\frac{dT(x, B)}{dx} = \frac{-\frac{dT(x, A)}{dx}(F_1 - F_2) + \frac{T(x, A)\frac{dT(x, A)}{dx} F_1^2}{\sqrt{D(x)}}}{F_2} = \frac{dT(x, A)}{dx} \frac{(F_1 - F_2) - \frac{T(x, A)F_1^2}{\sqrt{D(x)}}}{F_2}.
\]

Then

\[
\frac{dv(T, x)}{dx} = \frac{dT(x, A)}{dx} \left( (F_1 - F_2) + (F_3 - F_2) \frac{F_1 - F_2 - \frac{T(x, A)F_1^2}{\sqrt{D(x)}}}{F_2} \right) = \frac{dT(x, A)}{dx} \left( \frac{(F_1 - F_2)F_3}{F_2} - \frac{(F_3 - F_2)T(x, A)F_1^2}{F_2 \sqrt{D(x)}} \right)
\]

As \( \frac{dT(x, A)}{dx} < 0 \), then the condition \( \frac{dv(T, x)}{dx} > 0 \) is equivalent to the inequality:

\[
T(x, A) \geq \beta \sqrt{D(x)}, \quad \beta = \frac{(F_1 - F_2)F_3}{(F_3 - F_2)F_1^2}.
\]

Taking the square of the latter inequality, after some transformations

\[
T^2(x, A) \geq \beta^2 (T^2(x, A)F_1^2 + 2(B - A)F_2).
\]

we get the equivalent inequality

\[
T^2(x, A) \geq \frac{2(B - A)F_2 \beta^2}{1 - \beta^2 F_1^2}.
\]
This condition should hold for all \( x \in [0, 1] \). But \( T(x, A) \geq T(1, A) \) for all \( x \in [0, 1] \). Hence, (11) is equivalent to the inequality:

\[
T^2(1, A) \geq \frac{2(B - A)F_2\beta^2}{1 - \beta^2F_1^2}.
\]

Substituting the expression (9) for \( T(1, A) \), with \( v(x) = 0, x = 1 \), to this inequality, we get

\[
(A - 1)\left(1 - \beta^2F_1^2\right) \geq B - A.
\]

Transforming the second factor in the left side of this inequality, we get:

\[
\alpha = \frac{1 - \beta^2F_1^2}{F_1F_2\beta^2} = \frac{1}{F_1F_2} \left(\frac{1}{\beta} - F_1\right) \left(\frac{1}{\beta} + F_1\right) =
\]

\[
= \frac{F_1^2}{F_1F_2(F_1 - F_2)^2F_3^2}((F_3 - F_2)F_1 - (F_1 - F_2)F_3)((F_3 - F_2)F_1 + (F_1 - F_2)F_3) =
\]

\[
= \frac{F_1}{(F_1 - F_2)^2F_3^2}((F_3 - F_2)F_1 + (F_1 - F_2)F_3)
\]

The proof is finished.

### 3.2 Multi-dimensional systems

**Proof of Theorem 4**

For any two unequal points \( x_1, x_2 \in \Lambda \) consider the function:

\[
r(t) = ||y(t, x_2) - y(t, x_1)||^2 = (y(t, x_2) - y(t, x_1), y(t, x_2) - y(t, x_1)).
\]

Differentiation gives

\[
\frac{d^2r(t)}{dt^2} = 2(F(y(t, x_2)) - F(y(t, x_1)), y(t, x_2) - y(t, x_1)) + 2||v(t, x_2) - v(t, x_1)||^2.
\]

Using the conditions on \( F \) we get

\[
\frac{d^2r(t)}{dt^2} \geq 0.
\]

For initial conditions

\[
r(0) = ||x_2 - x_1||^2 > 0, \frac{dr}{dt}(0) = 2(v(x_2) - v(x_1), x_2 - x_1) \geq 0.
\]

Our statement follows from these three inequalities.

**Linear force - proof of Theorem 5**

We will show that for any \( x \) the quadratic form

\[
Q(x) = (Ax, x) \geq 0
\]

As any \( x \in \mathbb{R}^d \) can be uniquely written as

\[
x = \sum_{i=1}^{d} x_i u_i, \ x_i \in \mathbb{R}.
\]

then we can define the symmetric matrix \( S = (s_{i,j}) \) by

\[
s_{i,j} = \frac{1}{2}(\lambda_i(u_i, u_j) + \lambda_j(u_j, u_i)) = \frac{1}{2}(\lambda_i + \lambda_j)(u_i, u_j).
\]

As

\[
Q(x) = \sum_{i,j} \lambda_i(u_i, u_j)x_ix_j = (Sx, x),
\]

we can write

\[
S = \Lambda U + U\Lambda,
\]

where \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_d) \) is the diagonal matrix and \( U = ((u_i, u_j)) \). That is why the matrix \( S \) is non negative definite, whence (12) follows. Now, using Theorem 4 we get the final statement.
Starting from the moment existence of collisions. For or it follows that the vectors \( R, V \) collide. That gives the proof.

**Proof of Theorem 7** We remind some known facts concerning particle motion in the central field. In this case the kinetic (angular) momentum

\[
M(t, x) = M(x) = M(|x|) = r^2(t, x) \frac{d\phi(t, x)}{dt},
\]
does not depend on time and equals

\[
M(x) = |x|^2 h(|x|). \tag{13}
\]

Dynamics of the radius vector of \( x \) is defined by the equation

\[
\frac{d^2 r(t, x)}{dt^2} = -\frac{\partial E}{\partial r}, \quad r(0, x) = |x|, \quad \frac{dr(0, x)}{dt} = g(|x|). \tag{14}
\]
It is well known (one line proof) that it satisfies the famous conservation law (Liouville equation)
\[ V(\rho) \] Density in such systems and Euler equation. Here we do several comments about further problems (a lot of them) concerning the density, possible interactions.

**4 Conclusion**

Here we do several comments about further problems (a lot of them) concerning the density, possible interactions in such systems and Euler equation.

**Density** Consider the case when \( F_x(y) = F(y) \) does not depend on \( x \). The density at time \( t = 0 \) is defined as arbitrary smooth positive function \( \rho(0, x) \) on \( \Lambda_0 \), and the density at time \( t \) on \( \Lambda_t \) as

\[ \rho(t, y) = \rho(0, U_t^{-1}y) \]

It is well known (one line proof) that it satisfies the famous conservation law (Liouville equation)

\[ \rho_t + (u\rho)_x = \rho_t + u\rho_x + \rho u_x = 0 \] (15)

Note that in all our examples the density monotonically tends to zero. Let us give examples when it tends to infinity.

Consider a smooth curve \( z(t), t \in [0, \infty) \) such that: \( z(0) = 1, z(t) > 0 \) for any \( t \in [0, \infty) \), \( z(t) \rightarrow 0 \) as \( t \rightarrow \infty \) and \( z'(t) < 0 \), that is \( z(t) \) is strictly decreasing.

Then consider the system with \( \Lambda_0 = (0, 1] \), putting

\[ v(x) = z'(t), F(y) = z''(t) \]

where \( t \) is uniquely defined from the condition \( z(t) = x \). Otherwise speaking, positive force decreases the velocities to zero. Moreover, the particles never get the point \( x = 0 \).
About more general regular systems  Any function \( m(x) \) on \( \Lambda_0 \) can be considered as the mass or charge density, giving some links to real physical forces - gravitational and electrostatic. More general forces \( F_x(y) \), different for different particles, does not seem interesting due to the following simple theorem. Assume that the following non-recurrence conditions holds: for any pair of points \( x, z \) the trajectory \( y(t, x) \) passes the point \( z \) not more than once.

1  Then the regular continuum system can be presented as the system without interaction with external forces \( F_x(y) \).

In fact, consider any trajectory \( y(t, x) = U_t x \) with initial conditions \([\text{1}])\). Then it is sufficient to define the force, acting on the particle \( x \), at point \( y = y(t, x) \), as

\[
F_x(y) = \frac{d^2 y(t, x)}{dt^2}
\]

so that the system defined by the diffeomorphisms \( U_t \) were no interaction system.

Now we say very shortly what could be continuum systems with interaction. Interaction in such systems can local, when the force acting on the particle \( y \) at time moment \( t \), looks like

\[
F(y) = f(\rho(y), \nabla \rho(y))
\]

and non-local with the force

\[
F(y) = \int g(|z|, \rho(y), \rho(y + z))dz
\]

for some functions \( f \) and \( g \).

Of course, the introduced systems are approximations for the corresponding finite \( N \) particle (with very large \( N \)) systems. Intuitively one say that in case \([\text{11}]\) these systems can be approximations for \( N \)-particle systems with two particle interaction decaying at infinity, that is where only interaction with bounded (not depending on \( N \)) number of particles is essential. But the case \([\text{17}]\) can be approximation of systems with so called mean field interaction, where each particle interacts with the number of particles of the order \( N \). Some concrete examples are in progress.

Euler equations and characteristics  Regularity condition says that for any given \( t, y \in \Lambda_0 \), there is not more than one particle \( x \in \Lambda_0 \) with \( y = y(t, x) \), that is the velocity field is unambiguously defined

\[
u(t, y(t, x)) = \frac{dy(t, x)}{dt}
\]

Then it is easy to prove that this velocity field, for regular system without interaction, satisfies the Euler equation

\[
\frac{\partial u(t, y)}{\partial t} + \sum_{\alpha} \frac{\partial u(t, y)}{\partial y_{\alpha}} u_{\alpha}(t, y) = F(y)
\]

In fact, the acceleration of the particle with trajectory \( y(t, x) \) equals

\[
\frac{du_i(t, y(t, x))}{dt} = \frac{\partial u_i(t, y(t, x))}{\partial t} + \sum \frac{\partial u_i(t, y(t, x))}{\partial y_j} u_j(t, y(t, x)).
\]

thus equal to the force \( F(y(t, x)) \). We repeat once more that the absence of collisions allows such derivation of the Euler equation.

Consider now the Euler equation as the abstract partial differential equation. The Cauchy problem for it with \( t \in [0, \infty), y = (y_1, ..., y_d) \in \mathbb{R}^d \), and initial conditions

\[
u(0, x) = v(x)
\]

was studied in many papers. Part of these results one can find in textbooks and monographs, see \([\text{1}][\text{2}][\text{3}][\text{1}][\text{5}][\text{7}][\text{8}][\text{9}][\text{12}][\text{11}][\text{10}]\).

Let now \( \Lambda_0 \) be the real axis \( \mathbb{R} \), and consider the equation \([\text{11}]\), with similar assumptions on the functions \( F(y), v(x) \) for all \( y, x \in \mathbb{R} \).

It is well known that the characteristics \( y(t, x) \) are parametrized by the points \( x \in \mathbb{R} \) and satisfy the Newton equation, describing thus the movement of particles under the external force \( F(y) \). Moreover, the structure of the set of characteristics (more exactly, the projection of this set on the \( x \)-space) defines existence and uniqueness...
for the Cauchy problem. However, in the general case the conditions for the absence of collisions are quite non trivial, and now only examples of this structure can be well understood.

Consider the following example (see [5]): \( F(y) = 0 \) for all \( y \in \mathbb{R} \) and \( v(x) = -\arctg(x) \). It can be proved that in this case the solution \( u(t,y) \) of equation (18) exists for any \( t \leq 1, \ y \in \mathbb{R} \), but it cannot be continuously prolonged to the domain \( t > 1 \). Correspondingly, one can prove that first collisions in this system appear at time moment \( t = 1 \). This example of \( v(x) \) is a particular case of our remark, concerning the monotonicity of \( v(x) \), in the beginning of the section 2.1.

The following more general result follows from our theorem 2.

**Theorem 8** Let \( v(x) \geq 0, F(y) \in C^2(\mathbb{R}) \) and \( F(y) > 0 \) for any \( y \in \mathbb{R} \). The equation (18) has a smooth solution \( u(t,y) \) for \( t \geq 0, \ y \in \mathbb{R} \), with initial condition \( u(0,x) = v(x) \), iff for all \( x < y \) the following inequality holds:

\[
H_0'(x) \left( \frac{1}{\sqrt{2(H_0(x) - U(y))}} \frac{1}{F(y)} + \int_x^y \frac{1}{\sqrt{2(H_0(z) - U(z))}} \frac{F'(z)}{F^2(z)} dz \right) < \frac{v'(x)}{F(x)}
\]

One can consider also the Cauchy problem in bounded domains with moving boundary, consisting of two points \( L(t) = y(t,0), \ R(t) = y(t,1) \), and some boundary conditions on it, which are the Newton equations

\[
\frac{d^2 L}{dt^2} = F(L), \quad \frac{d^2 R}{dt^2} = F(R)
\]

with initial conditions

\[
L(0) = 0, \ R(0) = 1, \ \frac{dL}{dt}(0) = v(0), \ \frac{dR}{dt}(0) = v(1).
\]

**References**

[1] Arnold V. I. Lectures on partial differential equations. 1995. Moscow.

[2] Arnold V. I. Dopolnitelnye glavy teorii obyknovennyh differentsialnyh uravnenij. 1978. Moscow.

[3] Kruzhkov S. N. Selected papers. 2000. Moscow. Fizmatlit.

[4] Goritskij A. Yu., Kruzhkov S. N., Chechkin G. A. First order partial differential equations (textbook). 1999.

[5] Filippov A. F. Introduction to differential equations. 2007. Moscow.

[6] Bogolyubov N. A. About some statistical methods in mathematical physics. 1945, Kiev Academy of Sciences of Ukrainian SSR.

[7] Rashevsky P. K. Riemann geometry and tensor analysis. 1967. Moscow.

[8] Chorin A., Marsden J.. A mathematical introduction to fluid mechanics. Springer. 1992.

[9] Marchioro C., Pulvirenti M. Mathematical Theory of Incompressible Nonviscous Fluids. Springer. 1993.

[10] Temam R., Miranville A. Mathematical modeling in continuum mechanics. Cambridge Univ. Ppress, 2005.

[11] Talman R. Geometric mechanics. WILEY, Second ed. 2007.

[12] Slezkin N. A. Lectures on molecular hydrodynamics. Moscow State University. 1981.