Behavior of solutions to a Petrovsky equation with damping and variable-exponent sources

Menglan Liao & Zhong Tan*

School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
Email: liaoml14@mails.jlu.edu.cn, tan85@xmu.edu.cn

Received June 29, 2021; accepted October 8, 2021; published online March 24, 2022

Abstract This paper deals with the following Petrovsky equation with damping and nonlinear sources:

\[ u_{tt} + \Delta^2 u - M(\|\nabla u\|_2^2)\Delta u - \Delta u_t + |u_t|^{m(x)-2}u_t = |u|^{p(x)-2}u \]

under initial-boundary value conditions, where \( M(s) = a + bs^\gamma \) is a positive \( C^1 \) function with the parameters \( a > 0, b > 0, \gamma \geq 1, \) and \( m(x) \) and \( p(x) \) are given measurable functions. The upper bound of the blow-up time is derived for low initial energy by the differential inequality technique. For \( m(x) \equiv 2, \) in particular, the upper bound of the blow-up time is obtained by the combination of Levine’s concavity method and some differential inequalities under high initial energy. In addition, we discuss the lower bound of the blow-up time by making full use of the strong damping. Moreover, we present the global existence of solutions and an energy decay estimate by establishing some energy estimates.

Keywords Petrovsky equation, damping, variable-exponent source, blow-up, energy decay rate

MSC(2020) 35L35, 35B44, 35A01, 35B40

Citation: Liao M L, Tan Z. Behavior of solutions to a Petrovsky equation with damping and variable-exponent sources. Sci China Math, 2023, 66: 285–302, https://doi.org/10.1007/s11425-021-1926-x

1 Introduction

It is well known that nonlinear wave equations can be used to describe a variety of problems in physics, engineering, chemistry, material science and other sciences. The study of nonlinear wave equations also has great significance in mathematical analysis. Guesmia [7] considered the fourth-order wave equation

\[ u_{tt} + \Delta^2 u + q(x)u + g(u_t) = 0 \]

for a continuous and increasing function \( g \) with \( g(0) = 0 \) and a bounded function \( q, \) and proved a global existence result and a regularity result. At the same time, decay results for weak and strong solutions were established as well under suitable growth conditions on \( g. \) Messaoudi [20] studied the nonlinearly damped semilinear Petrovsky equation

\[ u_{tt} + \Delta^2 u + a|u_t|^{m-2}u_t = b|u|^{p-2}u, \]

*Corresponding author
where \(a\) and \(b\) are positive constants. He proved the existence of a local weak solution and showed that this solution blows up in finite time if \(p > m\) with negative initial energy and the solution is global if \(m \geq p\). Wu and Tsai [28] extended the results in [20], and proved that the solution is global in time under some conditions without \(m \geq p\). They also proved that the local solution blows up in finite time if \(p > m\) and the initial energy is nonnegative. The decay estimates of the energy function and the estimates of the lifespan of solutions were given. The blow-up result had been further improved by Chen and Zhou [3], where they proved that the solution blows up in finite time if the positive initial energy satisfies a suitable condition. Li et al. [14] studied the following strongly damped Petrovsky system with nonlinear damping:

\[
 u_{tt} + \Delta^2 u - \Delta u_t + |u_t|^{m-1}u_t = b|u|^{p-1}u,
\]

where they proved the global existence of the solution under conditions without any relation between \(m\) and \(p\), and established an exponential decay rate. They also showed that the solution blows up in finite time if \(p > m\) and the initial energy is less than the potential well depth. Guo and Li [8] discussed the boundary problem:

\[
 u_{tt} + \Delta^2 u - \Delta u + \omega \Delta u_t + \alpha(t)u_t = b|u|^{p-2}u.
\]

In fact, Guo and Li [8] extended and improved the results in [26]. Readers can refer to [9, 10, 13, 19, 25] and the references therein for more other results.

To describe the nonlinear vibrations of an elastic string, Kirchhoff [11] first introduced the following equation:

\[
 \rho u_{tt} + \delta u_t = \left\{ p_0 + \frac{Eh}{2L} \int_0^L u_x^2 \, dx \right\} u_{xx} + f, \quad 0 \leq x \leq L, \quad t \geq 0,
\]

where \(u = u(x, t)\) is the lateral deflection, \(E\) is the Young’s modulus, \(\rho\) is the mass density, \(h\) is the cross-section area, \(L\) is the length, \(p_0\) is the initial axial tension, \(\delta\) is the resistance modulus, and \(f\) is the external force. In recent years, the problem of Kirchhoff type has been further developed (see [27, 29–31]). Zhou [31], in particular, considered a Kirchhoff type plate equation

\[
 u_{tt} + \alpha \Delta^2 u - a \Delta u - b \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{\gamma} \Delta u + \lambda u_t = \mu |u|^{p-2}u,
\]

where he showed the blow-up of solutions and the lifespan estimates for three different ranges of initial energy. The global existence of solutions was proved by the potential well theory, and decay estimates of the energy function were established by Nakao’s inequality.

With the rapid development of the mathematical theory, much attention has been paid to the study of mathematical nonlinear models of hyperbolic, parabolic and elliptic equations with variable exponents. For example, Messaoudi et al. [22] considered the following nonlinear wave equation with variable exponents:

\[
 u_{tt} - \Delta u + a|u_t|^{m(x)-2}u_t = b|u|^{p(x)-2}u.
\]

They established the existence of a unique weak solution by using the Faedo-Galerkin method under suitable assumptions, and also proved the finite time blow-up of solutions.

Inspired by the works mentioned above, in this paper, we are concerned with the following initial-boundary problem:

\[
 \begin{cases}
 u_{tt} + \Delta^2 u - M(\|\nabla u\|_2^2) \Delta u - \Delta u_t + |u_t|^{m(x)-2}u_t = |u|^{p(x)-2}u & \text{in } \Omega \times (0, T), \\
 u(x, t) = \frac{\partial u(x, t)}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0, T), \\
 u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) & \text{in } \Omega,
\end{cases}
\]

where \(\Omega\) is a bounded smooth domain in \(\mathbb{R}^N(N \geq 1)\), \(\nu\) is the unit outward normal vector on \(\partial \Omega\), \(u_0(x) \in H_0^2(\Omega)\) and \(u_1(x) \in L^2(\Omega)\). \(M(s) = a + bs^\gamma\) is a positive \(C^1\) function with the parameters \(a > 0, \)
b > 0 and γ ≥ 1. The exponents \(m(x)\) and \(p(x)\) are given measurable functions on \(\Omega\) satisfying
\[
\begin{align*}
m(x) & \in [m^-, m^+] \subset (1, \infty), & p(x) & \in [p^-, p^+] \subset (1, \infty), & \forall x & \in \Omega
\end{align*}
\]
and the log-Hölder continuity condition, i.e., for some \(A > 0\) and any \(0 < \delta < 1\),
\[
|q(x) - q(y)| \leq -\frac{A}{\log ||x - y||} \quad \text{for all } x, y \in \Omega \quad \text{with } |x - y| < \delta.
\]
Here,
\[
m^- := \text{ess inf}_{x \in \Omega} m(x), \quad m^+ := \text{ess sup}_{x \in \Omega} m(x),
\]
\[
p^- := \text{ess inf}_{x \in \Omega} p(x), \quad p^+ := \text{ess sup}_{x \in \Omega} p(x).
\]

To the best of our knowledge, there is no known general theory concerned with the existence and the nonexistence of solutions to the problems like (1.1). Antontsev et al. [1] obtained the existence of local solutions by using the Banach contraction mapping principle under suitable assumptions, and gave a blow-up result with negative initial energy when \(-M(||\nabla u||^2_2)\Delta u\) in the problem (1.1) is absent, but they did not discuss the blow-up phenomena with positive initial energy and other properties of solutions. Subsequently, they studied the local existence of the solution under suitable conditions, and investigated the nonexistence of solutions with negative initial energy in the absence of the strong damping \(\Delta u_t\) in [2]. It seems that one cannot directly apply the classical potential well method to construct some invariant sets and to analyze the behavior of solutions as in [14,31] because for all \(f \in L^{p(x)}(\Omega)\),
\[
\min\{\|f\|_{p^-}^{p^-}, \|f\|_{p^+}^{p^+}\} \leq \int_\Omega |f|^{p(x)}dx \leq \max\{\|f\|_{p^-}^{p^-}, \|f\|_{p^+}^{p^+}\},
\]
which is different from \(\|f\|_p = \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}}\) for all \(f \in L^p(\Omega)\).

In this paper, we develop a new technique to discuss bounds of the blow-up time and decay rates. The rest of this paper is organized as follows. In Section 2, we introduce the Banach spaces that will be suitable for studying the problem (1.1), some notations and useful lemmas for later use. Section 3 will be devoted to discussing the lifespan of solutions, i.e., the upper and lower bounds of the blow-up time. In Section 4, some energy estimates are used to prove a uniform decay rate of the solution.

## 2 Preliminaries

Throughout this paper, we denote by \(\| \cdot \|_p\) the \(L^p(\Omega)\) norm for \(1 \leq p \leq \infty\). We equip \(H_0^1(\Omega)\) with the norm \(\|u\|_{H_0^1(\Omega)} = \|\Delta u\|_2\) for \(u \in H_0^1(\Omega)\), which is equivalent to the standard one due to the Poincaré inequality. Firstly, let us introduce the space \(L^{p(x)}(\Omega)\) in [4,5]. Set
\[
p : \Omega \to (1, \infty)
\]
(2.1)
to be a measurable function. Define
\[
A_{p(x)}(f) = \int_\Omega |f|^{p(x)}dx < \infty
\]
and
\[
L^{p(x)}(\Omega) = \{ f \text{ is measurable on } \Omega : A_{p(x)}(f) < \infty \}
\]
equipped with the Luxemburg norm
\[
\|f\|_{p(x)} = \inf\left\{ \lambda > 0 : A_{p(x)}\left(\frac{f}{\lambda}\right) \leq 1 \right\}.
\]
Let us assume that
\[
p(x) \in [p^-, p^+] \subset (1, \infty) \quad \text{a.e. } x \in \Omega.
\]
(2.2)
Lemma 2.1. Let (2.1) and (2.2) be fulfilled. Then for every $f \in L^{p(x)}(\Omega)$,

$$
\min\{\|f\|_{p_+}^{p_+}, \|f\|_{p_-}^{p_-}\} \leq A_{p(x)}(f) \leq \max\{\|f\|_{p_+}^{p_+}, \|f\|_{p_-}^{p_-}\}.
$$

Lemma 2.2. Let $p(x)$ and $q(x)$ satisfy (2.1) and (2.2). If $p(x) \geq q(x)$ a.e. in $\Omega$, then there is a continuous embedding $L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ and the embedding constant is less than or equal to $1 + |\Omega|$. Let us follow the proof of [1, Theorem 3.2] or [22, Theorem 3.3]. The local existence of solutions to the problem (1.1) without the proof is as follows.

Theorem 2.3. Suppose that $u_0(x) \in H_0^2(\Omega)$, $u_1(x) \in L^2(\Omega)$ and

$$
2 \leq m^- \leq m(x) \leq m^+ < \begin{cases} 
\infty & \text{for } N \leq 4, \\
2N/(N-4) & \text{for } N \geq 5,
\end{cases}
$$

$$
2 < p^- \leq p(x) \leq p^+ < \begin{cases} 
\infty & \text{for } N \leq 4, \\
(2N-2)/(N-4) & \text{for } N \geq 5.
\end{cases}
$$

Then there exists a unique local weak solution $u := u(x, t) \in L^\infty(0, T; H_0^2(\Omega))$ for the problem (1.1) with $u_t \in L^\infty(0, T; L^2(\Omega)) \cap L^{m^-}(0, T; L^{m(x)}(\Omega)) \cap L^2(0, T; H_0^2(\Omega))$.

Define

$$
\alpha_1 := (B_1^2)^{\frac{2}{p-2}}, \quad E_1 := \left(\frac{1}{2} - \frac{1}{p}\right)\frac{\alpha_1}{2},
$$

with $B_1 = \max\{1, B\}$. Here, $B$ is the embedding constant of the embedding $H_0^2(\Omega) \hookrightarrow L^{p(x)}(\Omega)$, i.e., there exists an optimal constant $B$ such that

$$
\|u\|_{p(x)} \leq B\|\Delta u\|_2, \quad \forall u \in H_0^2(\Omega).
$$

Define the energy functional

$$
E(t) = \frac{1}{2}\|u_t\|^2_2 + \frac{1}{2}\|\Delta u\|^2_2 + \frac{a}{2}\|\nabla u\|^2_2 + \frac{b}{2(\gamma + 1)}\|\nabla u\|^{2(\gamma+1)} - \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx.
$$

Direct computation implies

$$
E'(t) = -\int_\Omega |u_t|^{m(x)}dx - \|\nabla u_t\|^2_2 \leq 0.
$$

Lemma 2.4. Let $E(0) < E_1$ and

$$
2 < p^- \leq p(x) \leq p^+ < \begin{cases} 
\infty & \text{for } N \leq 4, \\
2N/(N-4) & \text{for } N \geq 5.
\end{cases}
$$

Assume that $u$ is a solution with $u \in C([0, T]; H_0^2(\Omega))$ for the problem (1.1). Then

1. for $B_1^2\|\Delta u_0\|^2_2 > \alpha_1$, there exists a positive constant $\alpha_2 > \alpha_1$ such that

$$
B_1^2\|\Delta u\|^2_2 \geq \alpha_2, \quad \forall t \geq 0;
$$

2. for $B_1^2\|\Delta u_0\|^2_2 < \alpha_1$, there exists a positive constant $0 < \alpha_2 < \alpha_1$ such that

$$
B_1^2\|\Delta u\|^2_2 \leq \alpha_2, \quad \forall t \geq 0.
$$

Proof. Using (2.6), Lemma 2.1 and (2.5) yields that

$$
E(t) \geq \frac{1}{2}\|\Delta u\|^2_2 \left(\frac{1}{p} - \frac{1}{p^+} \max\{\|u\|_{p_+}^{p_+}, \|u\|_{p_-}^{p_-}\}\right)
$$

$$
\geq \frac{1}{2}\|\Delta u\|^2_2 \left(\frac{1}{p} - \frac{1}{p} \max\{B^{p_+}\|\Delta u\|^2_2, B^{p_-}\|\Delta u\|^2_2\}\right)
$$
where \( \alpha := \alpha(t) = B_1^2 \| \Delta u \|_2 \). By direct computation, \( G(\alpha) \) satisfies the following:

\[
G'(\alpha) = \begin{cases} 
\frac{1}{2B_1^2} - \frac{p^+}{2p^-} \alpha^{\frac{p^+ - 2}{2}} & < 0, \quad \alpha > 1, \\
\frac{1}{2B_1^2} - \frac{1}{2} & < 0, \quad 0 < \alpha < 1,
\end{cases}
\]

\[
G'(1) = \frac{1}{2B_1^2} - \frac{p^+}{2p^-} < 0, \quad G'(1) = \frac{1}{2B_1^2} - \frac{1}{2} < 0,
\]

Thus, \( G(\alpha) \) is strictly increasing for \( 0 < \alpha < \alpha_1 \), strictly decreasing for \( \alpha_1 < \alpha \), \( G(\alpha) \to -\infty \) as \( \alpha \to +\infty \), and \( G(\alpha_1) = E_1 \). Since \( E(0) < E_1 \), there exist \( \alpha_2 \) and \( \tilde{\alpha}_2 \) with \( \alpha_2 > \alpha_1 > \tilde{\alpha}_2 \) such that

\[
G(\alpha_2) = G(\tilde{\alpha}_2) = E(0). \quad \text{Set} \quad \alpha_0 := B_1^2 \| \Delta u_0 \|_2^2. \quad \text{Then}
\]

\[
G(\alpha_0) \leq E(0) = G(\alpha_2) = G(\tilde{\alpha}_2). \quad (2.12)
\]

(1) If \( B_1^2 \| \Delta u_0 \|_2^2 > \alpha_1 \), then (2.12) implies \( \alpha_0 > \alpha_2 \). To prove (2.9), we suppose by contradiction that for some \( t_0 > 0 \), \( \alpha(t_0) < \alpha_2 \). The continuity of \( \alpha(t) \) illustrates that we could choose \( t_0 \) such that \( \alpha_1 < \alpha(t_0) < \alpha_2 \), and we have \( E(0) = G(\alpha_2) < G(\alpha(t_0)) \leq E(t_0) \), which contradicts (2.7).

(2) If \( B_1^2 \| \Delta u_0 \|_2^2 < \alpha_1 \), then (2.12) implies \( \alpha_0 \leq \alpha_2 \). Similar to (1), we suppose by contradiction that for some \( t^0 > 0 \), \( \alpha(t^0) > \alpha_2 \). The continuity of \( \alpha(t) \) illustrates that we could choose \( t^0 \) such that \( \tilde{\alpha}_2 < \alpha(t^0) < \alpha_1 \), and we have \( E(0) = G(\tilde{\alpha}_2) < G(\alpha(t^0)) \leq E(t^0) \), which contradicts (2.7).

**Lemma 2.5.** Set \( H(t) = E_2 - E(t) \) for \( t \geq 0 \), where \( E_2 \in \mathcal{E}(\Omega) \) is sufficiently close to \( E(0) \). If all the conditions of Lemma 2.4 hold, then for all \( t \geq 0 \),

\[
0 < H(0) \leq H(t) \leq \int_{\Omega} \frac{1}{p(x)} |u(p)dx| \leq \frac{1}{p} \int_{\Omega} |u| |p(x)| dx. \quad (2.13)
\]

**Proof.** (2.7) implies that \( H(t) \) is nondecreasing with respect to \( t \), and thus for \( t \geq 0 \), \( H(t) \geq H(0) = E_2 - E(0) > 0 \). (2.6) and (2.9) illustrate

\[
H(t) \leq E_2 - \frac{1}{2} \| \Delta u \|_2^2 + \int_{\Omega} \frac{1}{p(x)} |u| |p(x)| dx \leq E_2 - \frac{\alpha_2}{2B_1^2} + \int_{\Omega} \frac{1}{p(x)} |u| |p(x)| dx \leq E_1 - \frac{\alpha_1}{2B_1^2} + \int_{\Omega} \frac{1}{p(x)} |u| |p(x)| dx.
\]

This completes the proof.

**Remark 2.6.** In Lemma 2.5, \( H(t) = E_2 - E(t) \), where \( E_2 \in \mathcal{F}(\Omega) \) is sufficiently close to \( E(0) \), is necessary. This necessity can be seen in the proof of Theorem 3.1.

**Lemma 2.7.** Let \( p^- > 2(\gamma + 1) \) and \( m(x) = 2 \) hold. If the initial datum \( u_0 \in H^1_0(\Omega) \) and \( u_1 \in L^2(\Omega) \) such that

\[
0 < E(0) < \frac{C}{p} \int_{\Omega} u_0 u_1 dx, \quad (2.14)
\]

then the weak solution \( u \) to the problem (1.1) satisfies

\[
\int_{\Omega} u u_t dx - \frac{p^-}{C} E(t) \geq \left( \int_{\Omega} u_0 u_1 dx - \frac{p^-}{C} E(0) \right) e^{C t} > 0 \quad (2.15)
\]

for any \( t \in [0, T) \), where

\[
C = \min \left\{ 2 + p^-, \frac{2p^-(p^- - 2)a}{1 + (2p^- + 1)S^2} \right\} \quad (2.16)
\]

with \( S \) being the optimal embedding constant of \( H^1_0(\Omega) \hookrightarrow L^2(\Omega) \).
Proof. The idea of this proof comes from [9, 16, 24]. It is direct that
\[
\frac{d}{dt} \int_{\Omega} uu_t dx = \|u_t\|_2^2 + \int_{\Omega} uu_{tt} dx \\
= \|u_t\|_2^2 - \|\Delta u\|_2^2 - a\|\nabla u\|_2^2 - b\|\nabla u\|_2^{2(\gamma + 1)} \\
- \int_{\Omega} \nabla u \nabla u_t dx - \int_{\Omega} uu_t dx + \int_{\Omega} |u|^p(x) dx \\
\geq \frac{p^+ - 2}{2} \|u_t\|_2^2 + \frac{p^- - 2}{2} \|\Delta u\|_2^2 + \frac{p^- - 2}{2} a\|\nabla u\|_2^2 \\
+ \frac{p^- - 2(\gamma + 1)}{2(\gamma + 1)} b\|\nabla u\|_2^{2(\gamma + 1)} - \int_{\Omega} \nabla u \nabla u_t dx - \int_{\Omega} uu_t dx - p^- E(t),
\]  
where the first equality of (1.1) and (2.6) are used. Taking full advantage of Young’s inequality, we have
\[
\int_{\Omega} \nabla u \nabla u_t dx \leq C \frac{4p}{\gamma} \|\nabla u\|_2^2 + \frac{p^-}{C} \|u_t\|_2^2,
\]
\[
\int_{\Omega} uu_t dx \leq C \frac{4p}{\gamma} \|u\|_4^2 + \frac{p^-}{C} \|u_t\|_2^2.
\]
By combining (2.17)–(2.19), and using the embedding theorem \(H_0^1(\Omega) \hookrightarrow L^2(\Omega)\), we know that
\[
\frac{d}{dt} \int_{\Omega} uu_t dx \geq \frac{p^+ + 2}{2} \|u_t\|_2^2 + \left[ \frac{p^- - 2}{2} a - \frac{C}{4p^-} \right] \|\nabla u\|_2^2 - C \frac{4p}{\gamma} \|u\|_4^2 \\
+ \frac{p^-}{C} (\|\nabla u_t\|_2^2 + \|u_t\|_2^2) - p^- E(t) \\
\geq \frac{p^+ + 2}{2} \|u_t\|_2^2 + \left[ \frac{p^- - 2}{2} a - \frac{C}{4p^-} \right] \|u\|_4^2 - C \frac{4p}{\gamma} \left\{ \|u\|_2^2 - p^- E(t) \right\} \\
+ \frac{p^-}{C} (\|\nabla u_t\|_2^2 + \|u_t\|_2^2) - p^- E(t). \tag{2.20}
\]
Define
\[
\Psi(t) = \int_{\Omega} uu_t dx - \frac{p^-}{C} E(t).
\]
Recalling (2.16), and then combining (2.20) and (2.7), one has
\[
\frac{d}{dt} \Psi(t) \geq \frac{p^+ + 2}{2} \|u_t\|_2^2 + \left[ \frac{p^- - 2}{2} a - \frac{C}{4p^-} \right] \|u\|_4^2 - C \frac{4p}{\gamma} \left\{ \|u\|_2^2 - p^- E(t) \right\} \\
\geq C \left\{ \frac{1}{2} \|u\|_2^2 + \frac{1}{2} \|u_t\|_2^2 - \frac{p^-}{C} E(t) \right\} \geq C \Psi(t). \tag{2.21}
\]
Noticing that \(\Psi(0) = \int_{\Omega} u_0 u_1 dx - \frac{p^-}{C} E(0) > 0\), by Gronwall’s inequality, we have
\[
\Psi(t) \geq e^{Ct} \Psi(0) > 0.
\]
This proof is completed. \(\square\)

Lemma 2.8 (See [12, 17]). Suppose that a positive, twice-differentiable function \(\psi(t)\) satisfies the inequality
\[
\psi''(t)\psi(t) - (1 + \theta)(\psi'(t))^2 \geq 0,
\]
where \(\theta > 0\). If \(\psi(0) > 0\) and \(\psi'(0) > 0\), then \(\psi(t) \to \infty\) as \(t \to t_1 \leq t_2 = \frac{\psi(0)}{\psi'(0)}\).

3 Blow-up results

In this section, the blow-up phenomenon will be discussed. Moreover, the lifespan of solutions will be derived as well.
3.1 Blow-up for low initial energy

By constructing an ordinary differential inequality, we present the blow-up result for low initial energy as follows.

**Theorem 3.1.** Let (2.8) and \( \max\{m^+, 2(\gamma + 1)\} < p^- \) hold. Provided that all the conditions of Lemma 2.4(1) hold, the solution \( u \) of the problem (1.1) blows up at some finite time \( T^* \) in the sense of \( \lim_{t \to T^-} u(t) = +\infty \) and the blow-up time \( T^* \) can be estimated from above as follows:

\[
T^* \leq F^- \frac{\sigma}{\sigma - 1}(0) \frac{M_1}{M_2} - \frac{\sigma}{\sigma - 1},
\]

where

\[
0 < \sigma \leq \min \left\{ \frac{p^- - m^+}{p^- (m^+ - 1)}, \frac{p^- - 2}{2p^-}, \frac{\gamma}{\gamma + 1} \right\} < \frac{1}{2},
\]

and \( M_1 \) and \( M_2 \) will be presented in (3.7) and (3.15), respectively.

**Proof.** This proof goes back to our previous paper [18]. Let us recall \( H(t) = E_2 - E(t) \) for \( t \geq 0 \). Define an auxiliary function

\[
F(t) = H^{1-\sigma}(t) + \varepsilon \left( \int_\Omega u_t u dx + \frac{1}{2} \|\nabla u\|^2 \right).
\]

The proof is divided into three steps:

**Step 1.** Estimate for \( F'(t) \). Directly differentiating \( F(t) \), recalling (1.1), and using \( \varepsilon p^-(1 - \varepsilon_1)H(t) \) with \( 0 < \varepsilon_1 < 1 \), we get

\[
F'(t) = (1 - \sigma)H^{-\sigma}(t)H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \|\Delta u\|^2 - \varepsilon a\|\nabla u\|^2
\]

\[
- \varepsilon b\|\nabla u\|^2(2\gamma + 1) - \varepsilon \int_\Omega |u_t|^{m(x) - 2}u_t u dx + \varepsilon \int_\Omega |u|^{p(x)} dx
\]

\[
\geq (1 - \sigma)H^{-\sigma}(t)H'(t) + \varepsilon \left[ 1 + \frac{p^- (1 - \varepsilon_1)}{2} \right] \|u_t\|^2
\]

\[
+ \varepsilon\left[ \frac{p^- (1 - \varepsilon_1)}{2} - 1 \right] \|\Delta u\|^2 + \varepsilon a\left[ \frac{p^- (1 - \varepsilon_1)}{2} - 1 \right] \|\nabla u\|^2
\]

\[
+ \varepsilon b\left[ \frac{p^- (1 - \varepsilon_1)}{2r + 1} - 1 \right] \|\nabla u\|^2(2\gamma + 1) - \varepsilon \int_\Omega |u_t|^{m(x) - 2}u_t u dx
\]

\[
+ \varepsilon p^-(1 - \varepsilon_1)H(t) - \varepsilon p^-(1 - \varepsilon_1)E_2 + \varepsilon_1 \int_\Omega |u|^{p(x)} dx.
\]

Applying Young’s inequality with \( \varepsilon_2 > 1 \) and \( H'(t) = -E'(t) \), Lemma 2.1 and the embedding \( L^{p(x)}(\Omega) \hookrightarrow L^{m(x)}(\Omega) \), we easily have

\[
\int_\Omega |u_t|^{m(x) - 2}u_t u dx \leq \varepsilon_2 H^{-\sigma}(t) \int_\Omega |u_t|^{m(x)} dx + \frac{1}{\varepsilon_2^{m-1}} \int_\Omega |u|^{m(x)} H\sigma(m(x) - 1) dx
\]

\[
\leq \varepsilon_2 H^{-\sigma}(t) \int_\Omega |u_t|^{m(x)} dx + \frac{C_1}{\varepsilon_2^{m-1}} H\sigma(m(x) - 1) \int_\Omega |u|^{m(x)} dx
\]

\[
\leq \varepsilon_2 H^{-\sigma}(t) \int_\Omega |u_t|^{m(x)} dx + \frac{C_2 H\sigma(m(x) - 1)}{\varepsilon_2^{m-1}} \max \{\|u\|_{p(x)}^{m(x)}, \|u\|_{p(x)}^{-m(x)}\}
\]

\[
\leq \varepsilon_2 H^{-\sigma}(t) H'(t) + \frac{C_2 H\sigma(m(x) - 1)}{\varepsilon_2^{m-1}} \max \{\|u\|_{p(x)}^{m(x)}, \|u\|_{p(x)}^{-m(x)}\},
\]

where \( C_1 = \min\{H(0), 1\} \) and \( C_2 = (1 + |\Omega|) m^+ C_1^{\sigma(m(x) - m^+)} \). On the other hand, Lemmas 2.1 and 2.5
imply
\[
\|u\|_{p(x)}^{m^+} \leq \max \left\{ \left( \int_\Omega |u|^{p(x)} \, dx \right)^{\frac{m^+}{p^+}}, \left( \int_\Omega |u|^{p(x)} \, dx \right)^{\frac{m^+}{p^+}} \right\}
\]
\[
\leq \max \{ (p^- H(t))^{\frac{m^+}{p^+} - \frac{m^+}{p^+} \epsilon}, 1 \} \left( \int_\Omega |u|^{p(x)} \, dx \right)^{\frac{m^+}{p^+}},
\]
\[
\|u\|_{p(x)}^{m^-} \leq \max \{ (p^- H(t))^{\frac{m^-}{p^-} - \frac{m^-}{p^-} \epsilon}, (p^- H(t))^{\frac{m^- - m^+}{p^-}} \} \left( \int_\Omega |u|^{p(x)} \, dx \right)^{\frac{m^+}{p^+}},
\]
which illustrate
\[
\max \{\|u\|_{p(x)}^{m^+}, \|u\|_{p(x)}^{m^-} \} \leq C_3 \left( \int_\Omega |u|^{p(x)} \, dx \right)^{\frac{m^+}{p^+}},
\]
where \(C_3 = 2(\min\{p^- H(0), 1\})^{\frac{m^-}{p^-} - \frac{m^+}{p^+}}\). Recalling \(0 < \sigma \leq \frac{p^- - m^+}{p^- (m^+ - 1)}\) and Lemma 2.5, apparently, we have
\[
H^{\sigma(m^+ - 1)}(t) \max \{\|u\|_{p(x)}^{m^+}, \|u\|_{p(x)}^{m^-} \} \leq C_3 H^{\sigma(m^+ - 1)}(t) \left( \int_\Omega |u|^{p(x)} \, dx \right)^{\frac{m^+}{p^+}}
\]
\[
= C_3 H^{\sigma(m^+ - 1) + \frac{m^-}{p^-} - 1}(t) H^{1 - \frac{m^+}{p^+}}(t) H^{\sigma(m^+ - 1) + \frac{m^-}{p^-} - 1}(0) \left( \int_\Omega |u|^{p(x)} \, dx \right)^{\frac{m^+}{p^+}}
\]
\[
\leq C_3 \left( \frac{1}{p^+} \right)^{1 - \frac{m^+}{p^+}} \left( \int_\Omega |u|^{p(x)} \, dx \right)^{1 - \frac{m^+}{p^+}} H^{\sigma(m^+ - 1) + \frac{m^-}{p^-} - 1}(0) \left( \int_\Omega |u|^{p(x)} \, dx \right)^{\frac{m^+}{p^+}}
\]
\[
\leq C_3 \left( \frac{1}{p^+} \right)^{1 - \frac{m^+}{p^+}} C_1^{\sigma(m^+ - 1) + \frac{m^-}{p^-} - 1} \int_\Omega |u|^{p(x)} \, dx.
\]
It follows from (3.1), (3.2) and (3.4) that
\[
F'(t) \geq (1 - \sigma - \varepsilon \varepsilon_2) H^{-\sigma}(t) H'(t) + \varepsilon \left[ 1 + \frac{p^- (1 - \varepsilon_1)}{2} \right] \|u_t\|_2^2
\]
\[
+ \varepsilon \left[ \frac{p^- (1 - \varepsilon_1)}{2} - 1 \right] \|\Delta u\|_2^2 + \varepsilon p^- (1 - \varepsilon_1) H(t) - \varepsilon p^- (1 - \varepsilon_1) E_2
\]
\[
+ \varepsilon a \left[ \frac{p^- (1 - \varepsilon_1)}{2} - 1 \right] \|\nabla u\|_2^2 + \varepsilon b \left[ \frac{p^- (1 - \varepsilon_1)}{2(\gamma + 1)} - 1 \right] \|\nabla u\|_2^{(\gamma + 1)}
\]
\[
+ \varepsilon \left[ \frac{p^- (1 - \varepsilon_1)}{2} - 1 \right] \|\Delta u\|_2^2 - \varepsilon p^- (1 - \varepsilon_1) E_2 \geq \varepsilon \left[ \frac{p^- (1 - \varepsilon_1)}{2} - 1 \right] \|\nabla u\|_2^2 + \varepsilon p^- (1 - \varepsilon_1) E_2
\]
\[
\geq \varepsilon \left[ \frac{p^- (1 - \varepsilon_1)}{2} - 1 \right] \|\Delta u\|_2^2 - \varepsilon p^- (1 - \varepsilon_1) E_2 \geq 0.
\]
Let us fix the constant $\varepsilon_2$ such that

$$\varepsilon_1 > \frac{C_1^{\sigma(m+1)+\frac{m^+}{p} - 1} C_2 C_3 (\frac{1}{p})^{1-\frac{m^+}{p}}}{\varepsilon_2^{m^--1}},$$

and then choose $\varepsilon$ so small that $1 - \sigma > \varepsilon \varepsilon_2$. Therefore, (3.5) can be written as

$$F'(t) \geq M_1 \left( \|u_t\|_2^2 + H(t) + \|\nabla u\|_2^{2(\gamma+1)} + \|\nabla u\|_2^2 + \int_{\Omega} |u|^p(x) \, dx \right).$$

(3.6)

where

$$M_1 = \varepsilon \min \left\{ 1 + \frac{p^{-1}(1 - \varepsilon_1)}{2}, (1 - \varepsilon_1)p^{-1}, a \left[ \frac{p^{-1}(1 - \varepsilon_1)}{2} - 1 \right], b \left[ \frac{p^{-1}(1 - \varepsilon_1)}{2(2r + 1)} - 1 \right], \frac{\varepsilon - C_1^{\sigma(m+1)+\frac{m^+}{p} - 1} C_2 C_3 (\frac{1}{p})^{1-\frac{m^+}{p}}}{\varepsilon_2^{m^--1}} \right\}.$$

(3.7)

**Step 2.** Estimate for $F^{\frac{1}{1-p}}(t)$. We are now in a position to consider

$$F^{\frac{1}{1-p}}(t) = \left[ H^{1-\sigma}(t) + \varepsilon \left( \int_{\Omega} u_t u dx + \frac{1}{2} \|\nabla u\|_2^2 \right) \right]^{\frac{1}{1-p}}.$$

(3.8)

On the one hand, applying Hölder’s inequality, the embedding $L^p(x)(\Omega) \hookrightarrow L^2(\Omega)$ and Young’s inequality shows

$$\left| \int_{\Omega} u_t u dx \right|^{\frac{1}{1-p}} \leq \left( \|u_t\|_2 \|u\|_2 \right)^{\frac{1}{1-p}} \leq (1 + |\Omega|)^{\frac{1}{1-p}} \|u_t\|_2^{\frac{1}{1-p}} \|u\|_p^{\frac{1}{1-p}} \leq C_4 \|u_t\|_2^2 + C_5 \|u\|_{p(x)}^{\frac{2}{2(1-\sigma)-1}},$$

where

$$C_4 = \frac{(1 + |\Omega|)^{\frac{1}{1-p}}}{2(1-\sigma)}, \quad C_5 = \frac{(1 + |\Omega|)^{\frac{1}{1-p}} [2(1-\sigma) - 1]}{2(1-\sigma)}.$$

Recalling $0 < \sigma \leq \frac{p^{-2}}{2p}$, Lemmas 2.1 and 2.5, one obtains

$$\|u\|_{p(x)}^{\frac{1}{1-p}} \leq \max \left\{ \left( \int_{\Omega} |u|^{p(x)} \, dx \right)^{\frac{p^{-2}[2(1-\sigma)-1]}{p^{-2}[2(1-\sigma)-1]}} \right\} \leq \max \left\{ (p^{-} H(t))^{\frac{2^{-}[2(1-\sigma)-1]}{p^{-}[2(1-\sigma)-1]}}, (p^{-} H(t))^{\frac{2-\sigma}{p^{-}[2(1-\sigma)-1]}} \right\} \int_{\Omega} |u|^{p(x)} \, dx \leq C_6 \int_{\Omega} |u|^{p(x)} \, dx$$

(3.10)

with $C_6 = \left( \min \{ p^{-} H(0), 1 \} \right)^{\frac{2^{-}[2(1-\sigma)-1]}{p^{-}[2(1-\sigma)-1]}}$. Inserting (3.10) into (3.9) yields

$$\left| \int_{\Omega} u_t u dx \right|^{\frac{1}{1-p}} \leq C_4 \|u_t\|_2^2 + C_5 C_6 \int_{\Omega} |u|^{p(x)} \, dx.$$

(3.11)

On the other hand, there exists a positive constant $C_7$ such that

$$\|\nabla u\|_2^{\frac{2}{\gamma+1}} \leq C_7 (\|\nabla u\|_2^2 + \|\nabla u\|_2^{2(\gamma+1)}).$$

(3.12)

Let us pause to prove this inequality. If $\|\nabla u\|_2 < 1$, it directly follows from the fact $\frac{2}{\gamma+1} > 2$ that $\|\nabla u\|_2^{\frac{2}{\gamma+1}} \leq \|\nabla u\|_2^2$. If $\|\nabla u\|_2 \geq 1$, recalling $\sigma < \frac{\gamma}{\gamma+1}$, we have $\frac{2}{\gamma+1} < 2(\gamma+1)$, which implies $\|\nabla u\|_2^{\frac{2}{\gamma+1}} \leq \|\nabla u\|_2^{2(\gamma+1)}$. Therefore, combining (3.10), (3.12), (3.8) and

$$(a_1 + a_2 + \cdots + a_m)^l \leq 2^{(m-1)(l-1)}(a_1^l + a_2^l + \cdots + a_m^l),$$

(3.13)
which implies by Gronwall’s inequality,

\begin{align*}
\int_0^t \frac{1}{\epsilon} |u_t| & \leq \frac{2^{\frac{1}{\epsilon}}}{\epsilon} \left( \int_0^t u_t \, dx \right)^{\frac{1}{\epsilon}} + \left( \frac{1}{2} \right)^{\frac{1}{\epsilon}} \| \nabla u \|^2_{\epsilon} \\
& \leq 2^{\frac{1}{\epsilon}} \left( H(t) + \epsilon \frac{1}{\epsilon} C_4 \| u_t \|^2 + \epsilon \frac{1}{\epsilon} C_5 C_6 \int_\Omega |u|^{\epsilon} \, dx \\
& \quad + \left( \frac{1}{2} \right)^{\frac{1}{\epsilon}} C_7 \| \nabla u \|^2_{\epsilon} + \left( \frac{1}{2} \right)^{\frac{1}{\epsilon}} C_7 \| \nabla u \|^{2(\gamma + 1)}_{\epsilon} \right) \\
& \leq M_2 \left( H(t) + \| u_t \|^2 + \| \nabla u \|^2 + \| \nabla u \|^2 \right) + \int_\Omega |u|^{\epsilon} \, dx,
\end{align*}

where

\begin{equation}
M_2 = 2^{\frac{1}{\epsilon}} \max \left\{ 1, \frac{\epsilon}{\epsilon} C_4, \frac{\epsilon}{\epsilon} C_5 C_6, \left( \frac{1}{2} \right)^{\frac{1}{\epsilon}} C_7 \right\}.
\end{equation}

\textbf{Step 3.} The blow-up result. Combining (3.6) and (3.14), obviously, we have \( F^{\frac{1}{\epsilon}}(t) \leq \frac{M_1}{M_2} F^*(t) \), which implies by Gronwall’s inequality,

\begin{equation}
F^{\frac{1}{\epsilon}}(t) \leq \frac{1}{F^{\frac{1}{\epsilon}}(0) - \frac{M_1}{M_2} \frac{1}{1 - \sigma} t},
\end{equation}

which yields \( F(t) \rightarrow +\infty \) in finite time \( T^* \) and

\[ T^* \leq F^{\frac{1}{\epsilon}}(0) \frac{M_1}{M_2} \frac{1 - \sigma}{\sigma}. \]

Here, we fix some \( \epsilon > 0 \) such that

\[ F(0) = H^{1-\sigma}(0) + \epsilon \left( \int_\Omega u_t u_0 \, dx + \frac{1}{2} \| \nabla u_0 \|^2 \right) > 0. \]

In what follows, we prove

\[ \lim_{t \rightarrow T^*} F(t) \rightarrow +\infty \Rightarrow \lim_{t \rightarrow T^*} \| u \|_{p(x)} = +\infty. \]

Let us consider the following three cases based on the definition of \( F(t) \):

\textbf{Case 1.} \( H(t) \rightarrow +\infty \). In this case, Lemma 2.5 yields \( \int_\Omega |u|^{p(x)} \, dx \rightarrow +\infty \). It easily follows from Lemma 2.1 that \( \lim_{t \rightarrow T^*} \| u \|_{p(x)} = +\infty \).

\textbf{Case 2.} \( \int_\Omega u_t \, dx \rightarrow +\infty \). Cauchy’s inequality and the embedding \( H^1_0(\Omega) \hookrightarrow L^2(\Omega) \) with the optimal constant \( S > 0 \) illustrate

\[ \int_\Omega u_t \, dx \leq \frac{1}{2} \| u_t \|^2 + \frac{1}{2} \| u \|^2 \leq \frac{1}{2} \| u_t \|^2 + \frac{1}{2} \| u \|^2 + \frac{1}{2} S^2 \| \nabla u \|^2. \]

Recalling (2.7) and \( E(t) \leq E(0) < E_1 \), we have

\[ \frac{1}{2} \| u_t \|^2 + \frac{a}{2} \| \nabla u \|^2 \leq \frac{1}{2} \| u_t \|^2 + \frac{1}{2} \| u \|^2 + \frac{a}{2} \| \nabla u \|^2 + \frac{a}{2} \| \Delta u \|^2 + \frac{b}{2(\gamma + 1)} \| \nabla u \|^2 \]

\[ \leq E(t) + \int_\Omega \frac{1}{p(x)} |u|^{p(x)} \, dx \leq E(0) + \frac{1}{p} \int_\Omega |u|^p \, dx.
\]

Combining (3.16), (3.17) and Lemma 2.1, we directly find that if there exists \( \int_\Omega u_t \, dx \rightarrow +\infty \), then \( \lim_{t \rightarrow T^*} \| u \|_{p(x)} = +\infty \).

\textbf{Case 3.} \( \| \nabla u \|^2 \rightarrow +\infty \). Here, \( \lim_{t \rightarrow T^*} \| u \|_{p(x)} = +\infty \) is clear due to (3.17). \( \square \)
3.2 Blow-up for high initial energy

In this subsection, we are committed to proving the finite time blow-up for high initial energy and to estimating the upper bound of the blow-up time when the exponent \( m(x) \equiv 2 \).

**Theorem 3.2.** Let all the assumptions in Lemma 2.7 be fulfilled. Then the solution \( u \) for the problem (1.1) blows up in finite time.

**Proof.** Let us prove this theorem by contradiction, i.e., assume that the solution \( u \) for the problem (1.1) is a global solution of the problem (1.1), we have \( E(t) \geq 0 \) for all \( t \in [0, \infty) \). Otherwise, there exists \( t_0 \in [0, \infty) \) such that \( E(t_0) < 0 \). Choose \( u(x, t_0) \) as the initial data, and then Theorem 3.1 indicates that \( u \) blows up in finite time, which leads to a contradiction. Thus, (2.7) implies \( 0 \leq E(t) \leq E(0) \). Furthermore, (3.18) can be rewritten as

\[
\| u \|_2 \leq \| u_0 \|_2 + \sqrt{t} (E(0))^\frac{1}{2} 
\text{ for } t \in [0, \infty).
\]

On the other hand, (2.15) indicates

\[
\frac{d}{dt} \| u \|_2^2 = 2 \int_\Omega uu_x dx \geq 2\Psi(0)e^{Ct} + 2p^- E(t) \geq 2\Psi(0)e^{Ct} > 0.
\]

Integrating (3.20) from 0 to \( t \) yields

\[
\| u \|_2^2 = \| u_0 \|_2^2 + 2 \int_0^t \int_\Omega uu_x dx d\tau \geq \| u_0 \|_2^2 + 2 \int_0^t e^{C\tau} \Psi(0) d\tau
= \| u_0 \|_2^2 + \frac{2}{C}(e^{Ct} - 1)\Psi(0),
\]

which contradicts (3.19) for \( t \) sufficiently large. Thus, the solution \( u \) for the problem (1.1) blows up in finite time.

**Theorem 3.3.** Let all the assumptions in Lemma 2.7 be fulfilled. In addition, if

\[
E(0) \leq \frac{C}{2p^-} \| u_0 \|_2^2,
\]

then the solution \( u \) for the problem (1.1) blows up at some finite time \( T^* \) in the sense of

\[
\lim_{t \to T^*^-} \left( \| u \|_2^2 + \int_0^t (\| u \|_2^2 + \| \nabla u \|_2^2) ds \right) = \infty,
\]

and the upper bound of the blow-up time is given by

\[
T^* \leq \frac{2(\| u_0 \|_2^2 + \rho \omega^2)}{(p^- - 2)\int_\Omega u_0 u_1 dx + \rho \omega - 2\| \nabla u_0 \|_2^2},
\]

where \( \rho = \frac{-2p^- E(0) + C \| u_0 \|_2^2}{xp^-} \) and \( \omega > 0 \) is sufficiently large such that

\[
(p^- - 2)\left[ \int_\Omega u_0 u_1 dx + \rho \omega \right] - 2\| \nabla u_0 \|_2^2 > 0.
\]

(3.23)
Proof. Obviously, Theorem 3.2 implies that the solution \( u \) for the problem (1.1) blows up in finite time. Denote by \( T^* \) the blow-up time. Now, we need to estimate the upper bound of \( T^* \).

Define the auxiliary function

\[
\Upsilon(t) = \|u\|_2^2 + \int_0^t (\|u\|_2^2 + \|\nabla u\|_2^2)dt + (T^* - t)(\|u_0\|_2^2 + \|\nabla u_0\|_2^2) + \varrho(t + \omega)^2 \quad \text{for} \ t \in [0, T^*).
\]

By direct computation, one has

\[
\Upsilon'(t) = 2 \int_\Omega uu_t dx + 2(\|u\|_2^2 - \|u_0\|_2^2 + \|\nabla u\|_2^2 - \|\nabla u_0\|_2^2) + 2\varrho(t + \omega)
\]

\[
= 2 \int_\Omega uu_t dx + \int_0^t (uu_{\tau} + \nabla u \nabla u_{\tau})dxdt + 2\varrho(t + \omega) \quad \text{for} \ t \in [0, T^*).
\]

From the above equality and the problem (1.1), it is obtained that

\[
\Upsilon''(t) = 2\|u_t\|_2^2 + 2 \int_\Omega uu_t dx + 2 \int_\Omega \nabla u \nabla u_t dx + 2\varrho
\]

\[
= 2\|u_t\|_2^2 + 2\|\Delta u\|_2^2 - 2a\|\nabla u\|_2^2 - 2b\|\nabla u\|_2^{2(\gamma + 1)} + 2 \int_\Omega |u|^{p(x)} dx + 2\varrho
\]

for \( t \in [0, T^*). \) Applying the Cauchy-Schwarz inequality and Young's inequality, one has

\[
\xi(t) = \left[ \int_0^t \left( ||u||_2^2 + ||\nabla u||_2^2 \right) d\tau + \eta(t + \omega)^2 \right] \left[ \|u_t\|_2^2 + \int_0^t (\|u_{\tau}\|_2^2 + \|\nabla u_{\tau}\|_2^2) d\tau + \varrho \right] - \left[ \int_\Omega uu_t dx + \int_0^t (uu_{\tau} + \nabla u \nabla u_{\tau}) dxdt + \varrho(t + \omega) \right]^2 \geq 0
\]

for \( t \in [0, T^*). \) Therefore,

\[
\Upsilon(t) \Upsilon''(t) = \frac{p^* + 2}{4} (\Upsilon'(t))^2
\]

\[
= \Upsilon(t) \Upsilon''(t) - \frac{p^* + 2}{4} \left[ 2 \int_\Omega uu_t dx + \int_0^t (uu_{\tau} + \nabla u \nabla u_{\tau}) dxdt + 2\varrho(t + \omega) \right]^2
\]

\[
= \Upsilon(t) \Upsilon''(t) - (p^* + 2) \Upsilon(t) \left( \|u_t\|_2^2 + \int_0^t (\|u_{\tau}\|_2^2 + \|\nabla u_{\tau}\|_2^2) d\tau + \varrho \right) + (p^* + 2) \xi(t)
\]

\[
+(p^* + 2)(T^* - t)(\|u_0\|_2^2 + \|\nabla u_0\|_2^2) \left( \|u_t\|_2^2 + \int_0^t (\|u_{\tau}\|_2^2 + \|\nabla u_{\tau}\|_2^2) d\tau + \varrho \right)
\]

\[
\geq \Upsilon(t) \eta(t) \quad \text{for} \ t \in [0, T^*),
\]

where

\[
\eta(t) = \Upsilon''(t) - (p^* + 2) \left( \|u_t\|_2^2 + \int_0^t (\|u_{\tau}\|_2^2 + \|\nabla u_{\tau}\|_2^2) d\tau + \varrho \right)
\]

\[
= -p^* \|u_t\|_2^2 - 2\|\Delta u\|_2^2 - 2a\|\nabla u\|_2^2 - 2b\|\nabla u\|_2^{2(\gamma + 1)} + 2 \int_\Omega |u|^{p(x)} dx
\]

\[
- (p^* + 2) \int_0^t (\|u_{\tau}\|_2^2 + \|\nabla u_{\tau}\|_2^2) d\tau - p^* \varrho.
\]

Using (2.6) and (2.7), we obtain

\[
\eta(t) = -2p^* E(t) + (p^* - 2)\|\Delta u\|_2^2 + (p^* - 2)a\|\nabla u\|_2^2
\]

\[
+ (p^* - 2(\gamma + 1))b\|\nabla u\|_2^{2(\gamma + 1)} - (p^* + 2) \int_0^t (\|u_{\tau}\|_2^2 + \|\nabla u_{\tau}\|_2^2) d\tau - p^* \varrho
\]

\[
\geq -2p^* E(0) + (p^* - 2)a\|\nabla u\|_2^2 + (p^* - 2) \int_0^t (\|u_{\tau}\|_2^2 + \|\nabla u_{\tau}\|_2^2) d\tau - p^* \varrho
\]
Note that (3.20) implies
\[ \|u\|^2_t \geq \|u_0\|^2_2 \quad \text{for} \quad t \in [0, T^*). \]  
(3.26)

Thus, by combining (3.24)–(3.26), we see that for \( t \in [0, T^*) \),
\[ \Upsilon(t)\Upsilon''(t) - \frac{p^* + 2}{4} (\Upsilon'(t))^2 \geq \Upsilon(t)\{-2pE(0) + C\|u_0\|^2_2 - \rho \omega \} \geq 0, \]
(3.27)

where we use \( \rho = \frac{-2pE(0) + C\|u_0\|^2_2}{2p} \) and (3.22). Noticing that
\[ \Upsilon(0) = \|u_0\|^2_2 + T^*\|\nabla u_0\|^2_2 + \rho \omega^2 > 0 \quad \text{and} \quad \Upsilon'(0) = 2 \int_{\Omega} u_0 u_1 dx + 2 \rho \omega > 0, \]
and thus making use of Lemma 2.8 yield that \( \Upsilon(t) \rightarrow \infty \) as \( t \rightarrow T^* \) with
\[ T^* \leq \frac{4\Upsilon(0)}{(p^* - 2)\Upsilon'(0)} = \frac{2(\|u_0\|^2_2 + T^*\|\nabla u_0\|^2_2 + \rho \omega^2)}{(p^* - 2)(\int_{\Omega} u_0 u_1 dx + \rho \omega - 2\|\nabla u_0\|^2_2)}. \]

It follows from (3.23) that
\[ T^* \leq \frac{2(\|u_0\|^2_2 + \rho \omega^2)}{(p^* - 2)(\int_{\Omega} u_0 u_1 dx + \rho \omega - 2\|\nabla u_0\|^2_2)}. \]

The proof is completed. \( \square \)

### 3.3 A lower bound of the blow-up time

In what follows, we may make full use of the strong damping \( \Delta u_t \) to give a lower bound of the blow-up time.

**Theorem 3.4.** Let \( N \geq 5 \). If all the conditions of Theorem 3.1 or Theorem 3.3 are satisfied, and
\[ 2 < p^- \leq p(x) \leq p^+ < 2(N - 1)/(N - 4), \]
then the lower bound for the blow-up time \( T^* \) is given by
\[ \int_{R(0)}^{+\infty} \frac{1}{K_1y^{p^+-1} + K_2} dy \leq T^*, \]

where
\[ R(0) = \frac{1}{2}\|u_1\|^2_2 + \frac{1}{2}\|\Delta u_0\|^2_2 + \frac{a}{2}\|\nabla u_0\|^2_2 + \frac{b}{2(\gamma + 1)}\|\nabla u_0\|^2_{2(\gamma + 1)}, \]
(3.28)

and the constants \( K_1 \) and \( K_2 \) are defined in (3.33) and (3.34), respectively.

**Proof.** We define an auxiliary function
\[ R(t) = \frac{1}{2}\|u_t\|^2_2 + \frac{1}{2}\|\Delta u\|^2_2 + \frac{a}{2}\|\nabla u\|^2_2 + \frac{b}{2(\gamma + 1)}\|\nabla u\|^2_{2(\gamma + 1)} \]
(3.29)

by recalling (2.6). Therefore, the conclusion of Theorem 3.1 or Theorem 3.3 and (2.5) indicate \( \lim_{t \to T^-} R(t) = \infty \). It follows from (2.7) that
\[ R'(t) = E'(t) + \int_{\Omega} |u|^{p(x)-2}u_t u_t dx \leq -\|\nabla u_t\|^2_2 + \int_{\Omega} |u|^{p(x)-2}u u_t dx. \]
(3.30)
By using Hölder’s inequality, the embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ with the optimal embedding constant $S_*$ and Young’s inequality, we find that

$$R'(t) \leq \|u|^{p(x)-1}\|u\|_{p(x)}^2 \leq 2S_*\|u|^{p(x)-1}\|u\|_{p(x)}^2 \leq \frac{S_*}{2} \|u\|_{p(x)}^2 \|\nabla u\|_2^2 \leq \frac{S_*}{2} \|\nabla u\|_2^2 \leq \frac{S_*}{2} \left(\int_{\{u>1\}} |u|^{2N(p-\frac{1}{N})} dx \right) \frac{2^{N}}{N+2} + \left(\int_{\{|u|<1\}} |u|^{2N(p-\frac{1}{N})} dx \right) \frac{2^{N}}{N+2} \right)
$$

where we choose $\varepsilon = 2$ in the last inequality. Noting that $\frac{2N(p-1)}{N+2} \leq \frac{2(N-1)}{N}$, and then using the embedding $H^1_0(\Omega) \hookrightarrow L^\frac{2N(p-1)}{N+2}(\Omega)$ and (3.29), one has

$$R'(t) \leq \frac{S_*^2}{4} \|u\|_{(p-1)}^{2(p-1)} \frac{2^{N}}{N+2} + \frac{S_*^2}{4} \frac{\Omega^{\frac{N+2}{N}}}{\Omega} \leq \frac{S_*^2}{4} \frac{B^2(\frac{N}{a})^{\frac{N}{a}}}{\Omega} \frac{2^{N}}{N+2} + \frac{S_*^2}{4} \frac{\Omega^{\frac{N+2}{N}}}{\Omega}
$$

$$\leq K_1(R(t))^{p-1} + K_2,
$$

where

$$K_1 = \frac{S_*^2}{4} \frac{B^2(\frac{N}{a})^{\frac{N}{a}}}{\Omega} \frac{2^{N}}{N+2} \quad \text{and} \quad K_2 = \frac{S_*^2}{4} \frac{\Omega^{\frac{N+2}{N}}}{\Omega} \frac{2^{N}}{N+2}.
$$

Clearly, (3.32) implies

$$\int_{R(0)}^\infty \frac{1}{K_1^{p-1} + K_2} dy \leq T^*.
$$

This completes the proof of this theorem.

\[\square\]

4 The global existence and energy decay estimates

In this section, we are committed to showing the asymptotic stability. Let us first prove the global existence of solutions.

**Theorem 4.1.** If all the conditions of Lemma 2.4(2) hold, then the local solution $u$ of the problem (1.1) is global.

**Proof.** It directly follows from (2.10), (2.11) and (2.6) that

$$\int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx \leq \frac{1}{p(x)} \max\{B_0^+ \|\Delta u\|^2_2, B_0^- \|\Delta u\|^2_2\}
$$

$$\leq \frac{1}{p(x)} \max\{(B_0^2 \|\Delta u\|^2_2)^{\frac{p-2}{2}}, (B_0^- \|\Delta u\|^2_2)^{\frac{p-2}{2}}\} B_0^2 \|\Delta u\|^2_2
$$

$$\leq \frac{B_0^2}{p(x)} \alpha_2 \frac{p^2}{p-2} \left(2E(t) + 2 \int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx \right),$$

which implies

$$\int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx \leq \frac{2B_0^2 \alpha_2 \frac{p^2}{p-2}}{p(x)-2B_0^2 \alpha_2 \frac{p^2}{p-2}} E(t).
$$

(4.1)
Furthermore, (4.1) and (2.6) illustrate
\[
\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\Delta u\|_2^2 + \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{2(\gamma + 1)}\|\nabla u\|_2^{2(\gamma + 1)} \\
\leq \frac{p^-}{p^- - 2B_1^2\tilde{a}_2^{\frac{\gamma + 2}{\gamma - 2}}}E(t) \leq \frac{p^-}{p^- - 2}E(t) \leq \frac{p^-}{p^- - 2}E(0),
\]
(4.2)
which implies the weak solution \(u\) for the problem (1.1) exists globally. \(\square\)

**Theorem 4.2.** Under all the conditions of Theorem 4.1, suppose that 0 < \(E(0) < \tilde{E}_2\). Then there exists a positive constant \(K\) such that the energy functional satisfies
\[
E(t) \leq E(0)e^{1-Kt},
\]
(4.3)
where \(\tilde{E}_2 = (\frac{p^-}{p^+ - 1})^{\frac{p^- - 2}{p^-}}(\frac{1}{2} - \frac{1}{p^+})\alpha_1^{-\gamma} \in (E(0), E_1)\), and \(K\) will be obtained later.

**Proof.** Obviously, Theorem 4.1 implies that the solution \(u\) is global. We borrow some ideas from [6,15,21]. Multiplying (1.1) by \(u\) and integrating over \(\Omega \times (s, T)\) with \(s < T\) yield
\[
\int_s^T \frac{d}{dt} \left[ \int_\Omega uu_t dx \right] dt + \int_s^T \int_\Omega \nabla u \nabla u_t dx dt + \int_s^T \left[ \|\Delta u\|_2^2 + a\|\nabla u\|_2^2 + b\|\nabla u\|_2^{2(\gamma + 1)} \right] dt \\
= \int_s^T \left[ \int_\Omega |u|^{p(x)} dx + \|u_t\|_2^2 - \int_\Omega |u_t|^{m(x)-2}u_t u dx \right] dt.
\]
By recalling (2.6), and then combining the above equality, one has
\[
\int_s^T \left[ E(t) - \frac{1}{2}\|u_t\|_2^2 + \frac{1}{\Omega p(x)}|u|^{p(x)} dx \right] dt \\
= \int_s^T \left[ \frac{1}{2}\|\Delta u\|_2^2 + \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{2(\gamma + 1)}\|\nabla u\|_2^{2(\gamma + 1)} \right] dt \\
\leq \int_s^T \left[ \|\Delta u\|_2^2 + a\|\nabla u\|_2^2 + b\|\nabla u\|_2^{2(\gamma + 1)} \right] dt \\
= \int_s^T \left[ \int_\Omega |u|^{p(x)} dx + \|u_t\|_2^2 - \int_\Omega |u_t|^{m(x)-2}u_t u dx \right] dt \\
- \int_s^T \frac{d}{dt} \left[ \int_\Omega uu_t dx \right] dt - \int_s^T \int_\Omega \nabla u \nabla u_t dx dt.
\]
By simplifying the above inequality, we get
\[
\int_s^T E(t) dt \leq \int_s^T \int_\Omega \left[ 1 - \frac{1}{p(x)} \right]|u|^{p(x)} dx dt + \frac{3}{2} \int_s^T \|u_t\|_2^2 dt \\
- \int_s^T \int_\Omega |u_t|^{m(x)-2}u_t u dx dt - \int_s^T \frac{d}{dt} \left[ \int_\Omega uu_t dx \right] dt - \int_s^T \int_\Omega \nabla u \nabla u_t dx dt \\
=: J_1 + J_2 + J_3 + J_4 + J_5.
\]
(4.4)

**Step 1.** Estimates for \(J_1\) and \(J_2\). Obviously, (4.1) yields
\[
|J_1| \leq (p^+ - 1) \frac{2B_1^2\tilde{a}_2^{\gamma - 2}}{p^- - 2B_1^2\tilde{a}_2^{\gamma - 2}} \int_s^T E(t) dt.
\]
(4.5)
The embedding \(H^1_0(\Omega) \hookrightarrow L^2(\Omega)\) with the optimal constant \(S > 0\) and (2.7) induce
\[
|J_2| \leq \frac{3S^2}{2} \int_s^T \|\nabla u_t\|_2^2 dt \leq -\frac{3S^2}{2} \int_s^T E'(t) dt \leq \frac{3S^2}{2} E(s).
\]
(4.6)
Step 2. Estimate for $J_3$. On the one hand, by using Young’s inequality with $0 < \varepsilon_4 < 1$ and (2.7), we obtain

$$|J_3| \leq \int_s^T \left[ \int_\Omega \left( \varepsilon_4^{1-m} |u_t(x)|^m + \varepsilon_4 |u(x)|^m \right) dx \right] dt$$

$$\leq -\varepsilon_4^{1-m} \int_s^T E'(t) dt + \varepsilon_4 \int_s^T \int_\Omega |u(x)|^m dxdt$$

$$\leq \int_s^T [E(s) - E(T)] + \varepsilon_4 \int_s^T \int_\Omega |u(x)|^m dxdt$$

$$\leq \varepsilon_4^{1-m} E(s) + \varepsilon_4 \int_s^T \int_{\Omega} |u(x)|^m dxdt.$$  \hspace{1cm} (4.7)

On the other hand, we apply Lemma 2.1 and (4.2) to have

$$\int_\Omega |u(x)|^m dx \leq \max \{ B_{1m}^- \| \Delta u \|_{L^m}^m, B_{1m}^+ \| \Delta u \|_{L^m}^m \}$$

$$\leq B_{1m}^- \| \Delta u \|_{L^m}^m \leq \left( \frac{2p^- E(0)}{p^- - 2} \right) \frac{m-2}{2} \frac{2p^- B_{1m}^-}{p^- - 2} E(t).$$  \hspace{1cm} (4.8)

Therefore, we combine (4.8) and (4.7) to obtain

$$|J_3| \leq \varepsilon_4^{1-m} E(s) + \varepsilon_4 \left( \frac{2p^- E(0)}{p^- - 2} \right) \frac{m-2}{2} \frac{2p^- B_{1m}^-}{p^- - 2} \int_s^T E(t) dt.$$  \hspace{1cm} (4.9)

Step 3. Estimates for $J_4$ and $J_5$. By applying Cauchy’s inequality, (2.7) and (4.2), one shows

$$|J_4| = \left| \int_\Omega u_t(x, s) dx - \int_\Omega u_t(x, T) dx \right|$$

$$\leq \frac{1}{2} \left[ \| u_t(x, s) \|_2^2 + \| u_t(x, s) \|_2^2 + \| u_t(x, T) \|_2^2 + \| u_t(x, T) \|_2^2 \right]$$

$$\leq \frac{B_1^2}{2} \| \Delta u(x, s) \|_2^2 + \| \Delta u(x, T) \|_2^2 + \frac{1}{2} \left[ \| u_t(x, s) \|_2^2 + \| u_t(x, T) \|_2^2 \right]$$

$$\leq \left( \frac{2p^- B_1^2}{p^- - 2} + \frac{2p^-}{p^- - 2} \right) E(s).$$  \hspace{1cm} (4.10)

$$|J_5| = \int_s^T \| \nabla u(t) \|_2 \| \nabla u_t(t) \|_2 dt$$

$$\leq \frac{\varepsilon_5}{2} \int_s^T \| \nabla u(t) \|_2^2 dt + \frac{1}{2\varepsilon_5} \int_s^T \| \nabla u_t(t) \|_2^2 dt$$

$$\leq \frac{p^- \varepsilon_5}{p^- - 2} \int_s^T E(t) dt + \frac{1}{2\varepsilon_5} \int_s^T (-E'(t)) dt$$

$$= \frac{p^- \varepsilon_5}{p^- - 2} \int_s^T E(t) dt + \frac{1}{2\varepsilon_5} E(s).$$  \hspace{1cm} (4.11)

for $\varepsilon_5 > 0$.

By combining (4.4)–(4.6) and (4.9)–(4.11), we see that

$$\int_s^T E(t) dt \leq (p^+ - 1) \frac{2B_1^2 \varepsilon_5}{p^- - 2} \int_s^T E(t) dt$$

$$+ \left[ \frac{3S^2}{2} + \varepsilon_4^{1-m} \left( \frac{2p^- B_1^2}{p^- - 2} + \frac{2p^-}{p^- - 2} \right) + \frac{1}{2\varepsilon_5} \right] E(s)$$

$$+ \varepsilon_4 \left( \frac{2p^- E(0)}{p^- - 2} \right) \frac{m-2}{2} \frac{2p^- B_{1m}^-}{p^- - 2} \int_s^T E(t) dt + \frac{p^- \varepsilon_5}{p^- - 2} \int_s^T E(t) dt.$$  \hspace{1cm} (4.12)
The condition
\[ 0 < E(0) = G(\alpha_2) < \tilde{E}_2 = G\left( \left( \frac{p - 1}{p + 2} \right)^{\frac{2}{p - 2}} \alpha_1 \right) \]
and the monotonicity of \( G(\alpha) \) easily imply \( \tilde{\alpha}_2 < (\frac{p + 1}{p + 2})^{\frac{2}{p - 2}} \alpha_1 < \alpha_1 < 1 \), which further illustrates
\[ \delta := (p^+ - 1) \frac{2B_1^2 \tilde{\alpha}_2^{\frac{p^+ - 2}{p^+ - 2}}}{p^- - 2B_1^2 \tilde{\alpha}_2^{\frac{p^- - 2}{p^- - 2}}} < 1. \]
Choose \( 0 < \varepsilon_4 < 1 \) and \( 0 < \varepsilon_5 < 1 \) sufficiently small such that
\[ \epsilon_4 \left( \frac{2p^- E(0)}{p^- - 2} \right)^{\frac{m - 2}{2}} \frac{2p^- B_1^{m^-}}{p^- - 2} = \frac{1 - \delta}{2} \quad \text{and} \quad \frac{p^- \varepsilon_5}{p^- - 2} = \frac{1 - \delta}{2}. \]
Therefore, (4.12) can be rewritten as \( \int_s^T E(t) \, dt \leq \frac{1}{K} E(s) \), where
\[ K = \frac{1}{2^{\frac{1}{m^-}} (\frac{3S^2}{2} + \frac{1 - m^-}{\epsilon_4} + \left( \frac{2p^- B_1^{m^-}}{p^- - 2} + \frac{2p^-}{p^- - 2} \right) + \frac{1}{2\varepsilon_5}) \frac{1}{1 - \delta}}. \]
Let us make \( T \to +\infty \). It follows that
\[ \int_s^{+\infty} E(t) \, dt \leq \frac{1}{K} E(s). \] (4.13)
The classical Gronwall’s inequality illustrates (4.3).

At the end of this paper, we give the result of asymptotic stability of weak solutions. We define asymptotic stability of the problem (1.1) as follows: \( u = 0 \) will be called asymptotically stable (in the mean) if and only if
\[ \lim_{t \to \infty} E(t) = 0 \quad \text{for all solutions } u \text{ to the problem (1.1)}. \]
This notation was first presented by Pucci and Serrin [23].

**Theorem 4.3.** Suppose that all the conditions of Theorem 4.2 are satisfied, and then the rest field \( u = 0 \) is asymptotically stable.

**Proof.** By recalling (4.2), we can easily verify that
\[ E(t) \geq \frac{p^- - 2}{p^-} \left[ \frac{1}{2} \| u_{\tau} \|_2^2 + \frac{1}{2} \| \Delta u \|_2^2 + \frac{a}{2} \| \nabla u \|_2^2 + \frac{b}{2(\gamma + 1)} \| \nabla u \|_2^{2(\gamma + 1)} \right] \geq 0. \] (4.14)
On the other hand, we apply the conclusion of Theorem 4.2 to obtain \( E(t) \leq 0 \) as \( t \to +\infty \). Obviously, this theorem is true.

**Acknowledgements** This work was supported by National Natural Science Foundation of China (Grant No. 12071391). The first author expresses her gratitude to Professor Wenjie Gao and Bin Guo in School of Mathematics, Jilin University for their support and constant encouragement. In particular, the authors thank Professor Baisheng Yan in Department of Mathematics, Michigan State University for improving the quality of this paper. The authors are grateful to the anonymous referees for their careful reading of the manuscript and many helpful suggestions.

**References**
1. Antontsev S, Ferreira J, Pişkin E. Existence and blow up of solutions for a strongly damped Petrovsky equation with variable-exponent nonlinearities. Electron J Differential Equations, 2021, 2021: 1–18
2. Antontsev S, Ferreira J, Pişkin E, et al. Existence and non-existence of solutions for Timoshenko-type equations with variable exponents. Nonlinear Anal Real World Appl, 2021, 61: 103341
3. Chen W Y, Zhou Y. Global nonexistence for a semilinear Petrovsky equation. Nonlinear Anal, 2009, 70: 3203–3208
4 Fan X L, Zhang Q H. Existence of solutions for $p(x)$-Laplacian Dirichlet problem. Nonlinear Anal, 2003, 52: 1843–1852
5 Fan X L, Zhao D. On the spaces $L^p(x)(Ω)$ and $W^{k,p(x)}(Ω)$. J Math Anal Appl, 2001, 263: 424–446
6 Ghegal S, Hamchil I, Messaoudi S A. Global existence and stability of a nonlinear wave equation with variable-exponent nonlinearities. Appl Anal, 2020, 99: 1333–1343
7 Guèsma A. Existence globale et stabilisation interne non linéaire d’un système de Petrovsky. Bull Belg Math Soc Simon Stevin, 1998, 5: 583–594
8 Guo B, Li X L. Bounds for the lifespan of solutions to fourth-order hyperbolic equations with initial data at arbitrary energy level. Taiwanese J Math, 2019, 23: 1461–1477
9 Han Y Z, Li Q. Lifespan of solutions to a damped plate equation with logarithmic nonlinearity. Evol Equ Control Theory, 2022, 11: 25–30
10 Kang J R. Global nonexistence of solutions for von Karman equations with variable exponents. Appl Math Lett, 2018, 86: 249–255
11 Kirchhoff G. Vorlesungen über mathematische Physik. Leipzig: Teubner, 1883
12 Levine H A. Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$. Arch Ration Mech Anal, 1973, 51: 371–386
13 Li F S, Gao Q Y. Blow-up of solution for a nonlinear Petrovsky type equation with memory. Appl Math Comput, 2016, 274: 383–392
14 Li G, Sun Y N, Liu W J. Global existence and blow-up of solutions for a strongly damped Petrovsky system with nonlinear damping. Appl Anal, 2012, 91: 575–586
15 Li X L, Guo B, Liao M L. Asymptotic stability of solutions to quasilinear hyperbolic equations with variable sources. Comput Math Appl, 2020, 79: 1012–1022
16 Liao M L. The lifespan of solutions for a viscoelastic wave equation with a strong damping and logarithmic nonlinearity. Evol Equ Control Theory, 2022, 11: 781–792
17 Liao M L, Gao W J. Blow-up phenomena for a nonlocal $p$-Laplace equation with Neumann boundary conditions. Arch Math (Basel), 2016, 108: 313–324
18 Liao M L, Guo B, Zhu X Y. Bounds for blow-up time to a viscoelastic hyperbolic equation of Kirchhoff type with variable sources. Acta Appl Math, 2020, 170: 755–772
19 Liu L H, Sun F L, Wu Y H. Blow-up of solutions for a nonlinear Petrovsky type equation with initial data at arbitrary high energy level. Bound Value Probl, 2019, 2019: 15
20 Messaoudi S A. Global existence and nonexistence in a system of Petrovsky. J Math Anal Appl, 2002, 265: 296–308
21 Messaoudi S A, Al-Smail J H, Talahmeh A A. Decay for solutions of a nonlinear damped wave equation with variable-exponent nonlinearities. Comput Math Appl, 2018, 76: 1863–1875
22 Messaoudi S A, Talahmeh A A, Al-Smail J H. Nonlinear damped wave equation: Existence and blow-up. Comput Math Appl, 2017, 74: 3024–3041
23 Pucci P, Serrin J. Asymptotic Stability for Nonlinear Parabolic Systems. Dordrecht: Springer, 1996
24 Sun F L, Liu L S, Wu Y H. Global existence and finite time blow-up of solutions for the semilinear pseudo-parabolic equation with a memory term. Appl Anal, 2019, 98: 735–755
25 Tahamtani F, Shahrouzi M. Existence and blow up of solutions to a Petrovsky equation with memory and nonlinear source term. Bound Value Probl, 2012, 2012: 50
26 Wu S T. Lower and upper bounds for the blow-up time of a class of damped fourth-order nonlinear evolution equations. J Dyn Control Syst, 2018, 24: 287–295
27 Wu S T, Tsai L Y. On global existence and blow-up of solutions for an integro-differential equation with strong damping. Taiwanese J Math, 2006, 10: 979–1014
28 Wu S T, Tsai L Y. On global solutions and blow-up of solutions for a nonlinearly damped Petrovsky system. Taiwanese J Math, 2009, 13: 545–558
29 Wu S T, Tsai L Y. Blow-up of positive-initial-energy solutions for an integro-differential equation with nonlinear damping. Taiwanese J Math, 2010, 14: 2043–2058
30 Yang Z F, Gong Z G. Blow-up of solutions for viscoelastic equations of Kirchhoff type with arbitrary positive initial energy. Electron J Differential Equations, 2016, 2016: 1–8
31 Zhou J. Global existence and blow-up of solutions for a Kirchhoff type plate equation with damping. Appl Math Comput, 2015, 265: 807–818