Schlesinger transformations
for elliptic isomonodromic deformations

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Abstract

Schlesinger transformations are discrete monodromy preserving symmetry transformations of the classical Schlesinger system. Generalizing well-known results from the Riemann sphere we construct these transformations for isomonodromic deformations on genus one Riemann surfaces. Their action on the system’s tau-function is computed and we obtain an explicit expression for the ratio of the old and the transformed tau-function.

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1 Introduction

The theory of isomonodromic deformations of ordinary matrix differential equations of the type

\[ \frac{d\Psi}{d\lambda} = A(\lambda) \Psi, \]

where \( A(\lambda) \) is a matrix-valued meromorphic function on \( \mathbb{C} \), is a classical area intimately related to the matrix Riemann-Hilbert problem on the Riemann sphere. Over the last 20 years this has become a powerful tool in areas like soliton theory, statistical mechanics, theory of random matrices, quantum field theory etc. The main object associated with the isomonodromic deformation equations is the so-called \( \tau \)-function which turns out to be closely related to the Fredholm determinant of certain integral operators associated

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to the Riemann-Hilbert problem. After the classical work of Schlesinger the major contributions to the development of the subject were made in the papers of Jimbo, Miwa and their collaborators in the early 80’s.

There are only a few cases where the matrix Riemann-Hilbert problem may be solved explicitly in terms of known special functions (in addition to the mentioned papers of the Kyoto school see also the recent work where certain classes of solutions were obtained using the theory of theta-functions). However, as was already discovered by Schlesinger himself, there exists a large class of transformations which allow to get an infinite chain of new solutions starting from the known ones. They share the characteristic feature that they shift the eigenvalues of the residues of the connection $A(\lambda)$ by integer or half-integer values, thus changing the associated monodromies by sign only. These transformations – nowadays called Schlesinger transformations – were systematically studied in. In particular, it turns out that being written in terms of the $\tau$-functions the superposition laws of these transformations provide a big supply of discrete integrable systems.

The natural question of generalizing the theory of isomonodromic deformations on the sphere to higher genus surfaces was addressed by several authors. Here, we mention the contributions of Okamoto and Iwasaki.

For the case of the torus, recently two different explicit forms of isomonodromic deformations were proposed. In, two of the present authors studied isomonodromic deformations of non-singlevalued meromorphic connection on the torus whose “twists” (which determine the transformation of the connection $A(\lambda)$ with respect to tracing along basic cycles of the torus) vary with respect to the deformation parameters. The isomonodromic deformation equations for these connections hence contain transcendental dependence on the dynamical variables, which makes it difficult to analyse this system in a way analogous to the Schlesinger system on the sphere. On the other hand, Takasaki considered connections on the torus whose twists remain invariant with respect to the parameters of deformation. In Takasaki’s form, the equations of isomonodromic deformations have already the same degree of non-linearity as the ordinary Schlesinger system.

The purpose of the present paper is the extension of the notion of Schlesinger transformations from the Riemann sphere to the isomonodromy deformation equations on the torus with constant twists (we call this the elliptic Schlesinger system). In particular, in analogy to the ordinary Schlesinger system, it turns out to be possible to derive the action of elliptic Schlesinger transformation on the $\tau$-function. Thereby, we realize the first steps of the program to extend the results to the elliptic case. Throughout, we restrict to the case $A(\lambda) \in \mathfrak{sl}(2, \mathbb{C})$.

The paper is organized as follows. In section 2 we remind the framework of isomonodromic transformations on the sphere and reproduce some facts about the Schlesinger transformations on the Riemann sphere in a form convenient for generalization to the elliptic case. In section 3 we describe the elliptic Schlesinger system with constant twists. In particular, we find the simple formula

$$H_\mu = -\frac{1}{2\pi i} \oint \text{tr} A^2(\lambda) d\lambda ,$$

for the Hamiltonian which generates the isomonodromic evolution with respect to the module $\mu$ of the torus. This evolution is hence of the same type as the isomonodromic evolution with respect to the position of the singularities $\lambda_j$ of $A(\lambda)$, which is generated
by the contour integrals
\[ H_j = \frac{1}{4\pi i} \oint_{\gamma_j} \text{tr} A^2(\lambda) d\lambda . \] (1.3)

Generalizing the construction from the Riemann sphere we subsequently implement the elliptic Schlesinger transformations.

Finally, in section 4 we determine the action of the elliptic Schlesinger transformations on the \( \tau \)-function of the system. The transformed \( \tau \)-function \( \hat{\tau} \) differs from the old one by a factor which may be explicitly integrated in terms of the characteristic parameters of the solution \( \Psi \) of (1.1). For elementary elliptic Schlesinger transformations which shift the eigenvalues of the residues of \( A(\lambda) \) in two singularities \( \lambda_k \) and \( \lambda_l \) by 1/2, the main result is given by Theorem 4.1 below:

\[ \hat{\tau} (\{\lambda_j\}, \mu) = \left\{ \left[ w_1 w_2 w_3 \left( \frac{\lambda_k - \lambda_l}{2} \right) \right]^{1/2} \det \left[ G J^{1/2} \right] \right\} \cdot \tau (\{\lambda_j\}, \mu) , \] (1.4)

with certain elliptic functions \( w_\lambda \), and where \( J \) and \( G \) are parameters in the local expansion of the solution \( \Psi \) to (1.1) around the singularities \( \lambda_k \) and \( \lambda_l \), to be defined explicitly below.

2 Isomonodromic deformations on the Riemann sphere and Schlesinger transformations

2.1 Schlesinger system on the Riemann sphere

Consider the following ordinary linear differential equation for a matrix-valued function \( \Psi(\lambda) \in SL(2, \mathbb{C}) \):

\[ \frac{d\Psi}{d\lambda} = A(\lambda) \Psi \equiv \sum_{j=1}^{N} \frac{A_j}{\lambda - \lambda_j} \Psi , \] (2.1)

where the residues \( A_j \in \mathfrak{sl}(2, \mathbb{C}) \) are independent of \( \lambda \). Regularity at \( \lambda = \infty \) requires

\[ \sum_{j=1}^{N} A_j = 0 , \] (2.2)

and allows to further impose the initial condition \( \Psi(\lambda = \infty) = I \). The matrix \( \Psi(\lambda) \) defined in this way lives on the universal covering \( \mathbb{C} P^1 \setminus \{\lambda_1, \ldots, \lambda_N\} \). Its asymptotical expansion near the singularities \( \lambda_j \) is given by

\[ \Psi(\lambda) = G_j \Psi_j \cdot (\lambda - \lambda_j)^{T_j} C_j , \] (2.3)

with \( G_j, C_j \in SL(2, \mathbb{C}) \) constant, \( \Psi_j = I + O(\lambda - \lambda_j) \in SL(2, \mathbb{C}) \) holomorphic around \( \lambda = \lambda_j \), and where \( T_j \) is a traceless diagonal matrix with eigenvalues \( \pm t_j \). The residues \( A_j \) of (2.1) are encoded in the local expansion as

\[ A_j = G_j T_j G_j^{-1} . \] (2.4)

Upon analytical continuation around \( \lambda = \lambda_j \), the function \( \Psi(\lambda) \) in \( \mathbb{C} P^1 \setminus \{\lambda_1, \ldots, \lambda_N\} \) changes by right multiplication with some monodromy matrices \( M_j \)

\[ \Psi(\lambda) \rightarrow \Psi(\lambda) M_j , \] (2.5)

\[ M_j = C_j^{-1} e^{2\pi iT_j} C_j . \]
In the sequel we shall consider the generic case when none of \( t_j \) is integer or half-integer.

The assumption of independence of all monodromy matrices \( M_i \) of the positions of the singularities \( \lambda_j \): \( \partial M_i / \partial \lambda_j = 0 \) is called the isomonodromy condition; it implies the following dependence of \( \Psi(\lambda) \) on \( \lambda_j \)

\[
\frac{\partial \Psi}{\partial \lambda_j} = - \frac{A_j}{\lambda - \lambda_j} \Psi ,
\] (2.6)
as follows from (2.3) and normalization of \( \Psi(\lambda) \) at \( \infty \). Compatibility of (2.1) and (2.6) then is equivalent to the classical Schlesinger system [1]

\[
\frac{\partial A_j}{\partial \lambda_i} = \frac{[A_j, A_i]}{\lambda_j - \lambda_i} , \quad i \neq j , \quad \frac{\partial A_j}{\partial \lambda_j} = - \sum_{i \neq j} \frac{[A_j, A_i]}{\lambda_j - \lambda_i} .
\] (2.7)
describing the dependence of the residues \( A_j \) on the \( \lambda_i \). Obviously, the eigenvalues \( t_i \) of the \( A_j \) are integrals of motion of the Schlesinger system. The functions \( G_j \) accordingly have the following dependance [3]:

\[
\frac{\partial G_j}{\partial \lambda_i} = \frac{A_i G_j}{\lambda_i - \lambda_j} , \quad i \neq j , \quad \frac{\partial G_j}{\partial \lambda_j} = - \sum_{i \neq j} \frac{A_i G_j}{\lambda_i - \lambda_j} ,
\] (2.8)
which obviously implies (2.7).

To introduce the notion of the \( \tau \)-function for the Schlesinger system, one notes that (2.7) is a multi-time Hamiltonian system [2] with respect to the Poisson structure on the residues \( A_j \)

\[
\{ A^A_j , A^B_j \} = \delta_{ij} \epsilon^{ABC} A^C_j ,
\] (2.9)
\((\alpha, \beta, \gamma \) denoting \( \mathfrak{sl}(2) \) algebra indices with the completely antisymmetric structure constants \( \epsilon^{ABC} \)) and Hamiltonians

\[
H_i = \frac{1}{4 \pi i} \oint_{\lambda_i} \text{tr} A^2(\lambda) d\lambda = \frac{1}{2} \sum_{j \neq i} \frac{\text{tr} A_i A_j}{\lambda_j - \lambda_i} .
\] (2.10)
Explicitly, (2.7) takes the form

\[
\frac{\partial A_j}{\partial \lambda_i} = \{ H_i , A_j \} ,
\] (2.11)
and all the Hamiltonians \( H_j \) Poisson-commute.

The \( \tau \)-function \( \tau(\{\lambda_j\}) \) of the Schlesinger system then is defined as the generating functions of the Hamiltonians

\[
\frac{\partial \ln \tau}{\partial \lambda_j} = H_j ,
\] (2.12)
where compatibility of these equations follows from (2.7). This \( \tau \)-function is closely related to the Fredholm determinant of a certain integral operator associated to the Riemann-Hilbert problem (see [13] for details).
2.2 Schlesinger transformations

Schlesinger transformations are symmetry transformations of the Schlesinger system (2.7) which map a given solution \( \{A_j(\{\lambda_i\})\} \) to another solution \( \{\hat{A}_j(\{\lambda_i\})\} \) with the same number and positions of poles \( \lambda_j \) such that the related eigenvalues \( t_j \) are shifted by integer or half-integer values \( t_j \to t_j + n_j/2, \quad n_j \in \mathbb{Z} \). The monodromy matrices \( M_j \) hence remain invariant or change sign under this transformation. To be brief, we do not consider Schlesinger transformations involving the point \( \lambda = \infty \). Moreover, we shall restrict ourselves to elementary Schlesinger transformations, which change only two \( t_j \)'s, say, \( t_k \) and \( t_l \) for \( k \neq l \) by \( \pm 1/2 \). The transformed variables will be denoted by \( \hat{\Psi}, \hat{A}_j, \hat{t}_j \), etc. Without loss of generality we consider the case

\[
\hat{t}_j = \begin{cases} 
  t_j + \frac{1}{2} & \text{for } j = k, l \\
  t_j & \text{else}
\end{cases}.
\] (2.13)

Our presentation here mainly follows [14]. For the transformed function \( \hat{\Psi} \) we make the ansatz

\[
\hat{\Psi}(\lambda) = F(\lambda) \Psi(\lambda),
\] (2.14)

with

\[
F(\lambda) = \sqrt{\frac{\lambda - \lambda_k}{\lambda - \lambda_l}} S_+ + \sqrt{\frac{\lambda - \lambda_l}{\lambda - \lambda_k}} S_-,
\] (2.15)

where the matrices \( S_\pm \) do not depend on \( \lambda \) and are uniquely determined by [14]:

\[
S_\pm^2 = S_\pm, \quad S_+ + S_- = I, \quad S_+ G_i^1 = S_- G_k^1 = 0.
\] (2.16)

By \( G_j^\alpha \) here we denote the \( \alpha \)-th column of the matrix \( G_j \) (\( \alpha = 1, 2 \)). Combining the columns \( G_k^1 \) and \( G_l^1 \) into a \( 2 \times 2 \) matrix

\[
G = (G_k^1, G_l^1),
\] (2.17)

we can deduce from (2.16) the following simple formula for \( S_\pm \):

\[
S_\pm = GP_\pm G^{-1},
\] (2.18)

with projection matrices

\[
P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

It is easy to check using the local expansion of \( \Psi \) at the singularities \( \lambda_j \) (2.3) and the defining relations for \( S_\pm \) (2.18) that the transformed function \( \hat{\Psi} \) at \( \lambda_j \) has a local expansion of the form (2.3) with the same matrices \( C_j \) and the desired transformation (2.13) of the \( t_j \). The matrices \( G_j \) change to new matrices \( \hat{G}_j \). Thus, \( \hat{\Psi} \) satisfies the system

\[
\frac{\partial \hat{\Psi}}{\partial \lambda} = \sum_{j=1}^N \frac{\hat{A}_j}{\lambda - \lambda_j} \hat{\Psi}, \quad \frac{\partial \hat{\Psi}}{\partial \lambda_j} = -\frac{\hat{A}_j}{\lambda - \lambda_j} \hat{\Psi},
\] (2.19)

where the functions \( \hat{A}_j(\{\lambda_i\}) \) build a new solution of the Schlesinger system (2.7).
On the level of the residues $A_j$, the form of the Schlesinger transformation is not very transparent; however, it turns out that the associated $\tau$-function transforms in a rather simple way. Namely, for $\hat{\Psi}$ we find
\[
\text{tr} \, \hat{A}^2 = \text{tr} \, A^2 + 2 \text{tr} \left[ F^{-1} \frac{dF}{d\lambda} \right] A + \text{tr} \left[ F^{-1} \frac{dF}{d\lambda} \right]^2 .
\]
The Hamiltonians $H_j$ for $j \neq k, l$ hence transform as
\[
\hat{H}_j - H_j = \left( \frac{1}{\lambda_j - \lambda_k} - \frac{1}{\lambda_j - \lambda_l} \right) \text{tr} \, [A_j S_+] + \text{tr} \left[ A_j G P_+ G^{-1} \right] - \frac{1}{2(\lambda_k - \lambda_l)} .
\]
(2.20)
(2.21)

Therefore, the transformed $\tau$-function $\hat{\tau}$ is given by
\[
\hat{\tau} = f(\lambda_k, \lambda_l) \det G \cdot \tau
\]
with some function $f(\lambda_k, \lambda_l)$ to be determined from the transformation of $H_k, H_l$. In analogy to above we find that
\[
\hat{H}_k - H_k = \sum_{j \neq k} \text{tr} \left[ A_j S_+ \right] - \frac{1}{2(\lambda_k - \lambda_l)} .
\]
(2.22)

Remark 2.1 The other elementary Schlesinger transformations like
\[
\hat{t}_j = \begin{cases} 
  t_j + \frac{1}{2} & \text{for } j = k \\
  t_j - \frac{1}{2} & \text{for } j = l \\
  t_j & \text{else}
\end{cases}
\]
(2.23)

etc., may be obtained in a similar way by building the matrix $G$ from $G^1_k$ and $G^2_l$ instead of (2.17), etc., Moreover, all such transformations with different $k$ and $l$ may be superposed to get the general Schlesinger transformation which simultaneously shifts an arbitrary number of the $t_j$ by some integer or half-integer constants. These general transformations were in detail studied in [3, 4, 5]. In their framework, $\det G$ from (2.22) is introduced as particular matrix element $(G^{-1}_k G_l)_{12}$ (an identity which holds for $2 \times 2$ matrices).
Remark 2.2 Carefully comparing (2.22) to the result of [4], we find an additional factor of \((\lambda_k - \lambda_l)^{-1/2}\). This is due to the fact, that we treat the \(sl(2, \mathbb{C})\) case rather than the \(gl(2, \mathbb{C})\) case which is done in [4]. Indeed, this amounts to a simple renormalization of the \(\Psi\)-function by e.g. \(\sqrt{\frac{\lambda - \lambda_k}{\lambda - \lambda_l}}\) after the Schlesinger transformation (2.13), which generates precisely this additional factor.

3 Schlesinger transformations for isomonodromic deformations on the torus

In this section, we generalize the construction of Schlesinger transformations described above to the case of genus one Riemann surfaces. To this end we first review some basic elliptic functions and subsequently the isomonodromic deformations on the torus in the setting of [12].

3.1 Some elliptic functions

The elliptic theta-function with characteristic \([p, q] (p, q \in \mathbb{C})\) on a torus \(E\) is defined by the series

\[
\vartheta[p, q](\lambda|\mu) = \sum_{m \in \mathbb{Z}} e^{\pi i m(p+\mu) + \pi i (m+p)(\lambda+q)} .
\]  

(3.1)

Let us introduce on the torus \(E\) the standard Jacobi theta-functions:

\[
\begin{align*}
\vartheta_1(\lambda) &\equiv -\vartheta \left[ \frac{1}{2}, \frac{1}{2} \right] (\lambda|\mu) , \\
\vartheta_2(\lambda) &\equiv \vartheta \left[ \frac{1}{2}, 0 \right] (\lambda|\mu) , \\
\vartheta_3(\lambda) &\equiv \vartheta(\lambda) \equiv \vartheta[0,0](\lambda|\mu) , \\
\vartheta_4(\lambda) &\equiv \vartheta \left[ 0, \frac{1}{2} \right] (\lambda|\mu) ,
\end{align*}
\]  

(3.2)

and corresponding theta-constants

\[
\vartheta_j \equiv \vartheta_j(0) , \quad j = 2, 3, 4 .
\]

We define the following three combinations of Jacobi theta-functions:

\[
\begin{align*}
w_1(\lambda) &= \pi \vartheta_2 \vartheta_3 \frac{\partial \vartheta_1(\lambda)}{\partial \vartheta_1(\lambda)} , \\
w_2(\lambda) &= \pi \vartheta_2 \vartheta_4 \frac{\partial \vartheta_3(\lambda)}{\partial \vartheta_1(\lambda)} , \\
w_3(\lambda) &= \pi \vartheta_3 \vartheta_4 \frac{\partial \vartheta_2(\lambda)}{\partial \vartheta_1(\lambda)} .
\end{align*}
\]  

(3.3)

All these functions have simple poles at \(\lambda = 0\) with residue 1. Moreover, they possess the following periodicity properties:

\[
\begin{align*}
w_1(\lambda + 1) &= -w_1(\lambda) & w_1(\lambda + \mu) &= w_1(\lambda) , \\
w_2(\lambda + 1) &= -w_2(\lambda) & w_2(\lambda + \mu) &= -w_2(\lambda) , \\
w_3(\lambda + 1) &= w_3(\lambda) & w_3(\lambda + \mu) &= -w_3(\lambda) .
\end{align*}
\]  

(3.4)

In the sequel we shall also need the following functions \(Z_\lambda\):

\[
\begin{align*}
Z_1 &= \frac{w_1}{2\pi i} \frac{\partial \vartheta_1(\lambda)}{\partial \vartheta_1(\lambda)} , \\
Z_2 &= \frac{w_2}{2\pi i} \frac{\partial \vartheta_2(\lambda)}{\partial \vartheta_2(\lambda)} , \\
Z_3 &= \frac{w_3}{2\pi i} \frac{\partial \vartheta_3(\lambda)}{\partial \vartheta_3(\lambda)} .
\end{align*}
\]  

(3.5)
with the following periodicity properties:

\[
Z_1(\lambda + 1) = -Z_1(\lambda) \quad Z_1(\lambda + \mu) = Z_1(\lambda) - w_1, \quad (3.6)
\]
\[
Z_2(\lambda + 1) = -Z_2(\lambda) \quad Z_2(\lambda + \mu) = -Z_2(\lambda) + w_2, \quad (3.6)
\]
\[
Z_3(\lambda + 1) = Z_3(\lambda) \quad Z_3(\lambda + \mu) = -Z_3(\lambda) + w_3. \quad (3.6)
\]

It is easy to verify the identity

\[
\frac{dw_A}{d\mu}(\lambda) = \frac{dZ_A}{d\lambda}(\lambda). \quad (3.7)
\]

Let us check this for example, for \( A = 1 \). Both sides of (3.7) are holomorphic in \( E \).

Moreover, from the periodicity properties of \( w_1 \) and \( Z_1 \) we have:

\[
\frac{dZ_1}{d\lambda}(\lambda + 1) = -\frac{dZ_1}{d\lambda}(\lambda) \quad \frac{dZ_1}{d\lambda}(\lambda + \mu) = \frac{dZ_1}{d\lambda}(\lambda) - \frac{dw_1}{d\lambda}(\lambda). \quad (3.7)
\]

Therefore, the difference \( dw_1/d\mu - dZ_1/d\lambda \) is holomorphic in \( E \), invariant with respect to the \( \mu \)-shifts of \( \lambda \) and changes sign with respect to unit shifts of \( \lambda \). It hence vanishes and (3.7) is fulfilled.

Let us list some further useful properties of the functions \( w_A \) and \( Z_A \):

- The functions \( w_A \) may be represented as ratios of Jacobi’s elliptic functions as follows:
  \[
  w_1(\lambda) = \frac{1}{\text{sn}(\lambda)}, \quad w_2(\lambda) = \frac{\text{dn}(\lambda)}{\text{sn}(\lambda)}, \quad w_3(\lambda) = \frac{\text{cn}(\lambda)}{\text{sn}(\lambda)}. \quad (3.8)
  \]

- The functions \( w_A \) satisfy the following differential equation:
  \[
  \frac{dw_1(\lambda)}{d\lambda} = -w_2(\lambda)w_3(\lambda), \quad (3.9)
  \]
  and cyclic permutations thereof. This relation may be easily proved comparing the behaviour at \( \lambda = 0 \) and the twist properties of both sides of (3.9).

- The functions \( w_A \) satisfy the following summation theorem which easily follows from the summation theorem for Jacobi functions:
  \[
  w_1(\lambda + \lambda') - w_1(\lambda - \lambda') = \frac{2w_1(\lambda')w_2(\lambda)w_3(\lambda)}{w_1^2(\lambda) - w_1^2(\lambda')}, \quad (3.10)
  \]
  and cyclic permutations thereof. At \( \lambda = \lambda' \) this relation implies
  \[
  2w_1(2\lambda) = \frac{\partial}{\partial \lambda} \ln \left( \frac{w_1}{w_2w_3} \right). \quad (3.11)
  \]

- For any values of \( A, B \), the difference of squares \( w_A^2(\lambda) - w_B^2(\lambda) \) is independent of \( \lambda \), as follows from its single-valuedness and holomorphy on \( E \). From the well-known relations between the squares of Jacobi elliptic functions we find more precisely that \( w_1^2(\lambda) - w_2^2(\lambda) = x; \ w_2^2(\lambda) - w_3^2(\lambda) = 1 \) where \( x = x(\mu) \) is the module parameter of the torus \( E \) which arises from realizing the torus as two-sheet covering of the complex plane with branch points \( 0, 1, x, \infty \).
• The previous property in particular implies that the expression $w^2(\lambda) - w^2(\lambda')$ does not depend on $a$ for any values of $\lambda$ and $\lambda'$.

• We have the following formula for integration of the product of two functions $w_a$ along the basic $a$-cycle of the torus $E$

$$\oint_a w_a(\lambda - \lambda_1) w_a(\lambda - \lambda_2) d\lambda = 2\pi i Z_a(\lambda_1 - \lambda_2).$$ (3.12)

This formula can be verified by checking the analyticity and periodicity properties of both sides in, say, the $\lambda_1$-plane.

### 3.2 Isomonodromic deformations on the torus

Consider the elliptic curve $E$ with periods 1 and $\mu$ together with the canonical basis of cycles $(a, b)$. A (naive) straightforward generalization of the idea of isomonodromic deformations from the complex plane to the torus $E$ runs into difficulties related to the absence of meromorphic functions on the torus with just one simple pole. An independent variation of the simple poles of a meromorphic connection $A$ on the torus preserving the monodromies around the singularities and basic cycles is impossible for the following simple reason. Existence of such a deformation would imply a version of (2.6) with the function $j A(\lambda - \lambda_j)$ on the r.h.s. being substituted by a meromorphic function with only one simple pole on the torus, which gives rise to the contradiction. Therefore, one of the underlying assumptions has to be relaxed.

E.g. one may consider the case where not all the poles of the connection $A$ are varied independently. Another possibility is the assumption that some of the poles of $A$ are of order higher than one [9]. A third alternative which we shall consider here, is to relax the condition of single-valuedness of the connection $A$ on $E$ and assume that $A$ has “twists” with respect to analytical continuation along the basic cycles $a$ and $b$, i.e.

$$A(\lambda + 1) = QA(\lambda)Q^{-1}, \quad A(\lambda + \mu) = RA(\lambda)R^{-1},$$

where the matrices $Q, R$ do not depend on $\lambda$. By a gauge transformation of the form $A \rightarrow SAS^{-1} + dSS^{-1}$ with $S$ holomorphic but possibly multi-valued, one may bring the connection into a form where $Q = I$ and $R = e^{\kappa \sigma_3}$, where $\sigma_i$ denote the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

The equations of isomonodromic deformations with this choice of the twist were considered in [11] where the multi-valuedness of $A$ had a natural origin in the holomorphic gauge fixing of Chern-Simons theory on the punctured torus. The resulting equations however are rather complicated in comparison with the Schlesinger system on the sphere. This is due to the fact that the twist $\kappa$ itself becomes a dynamical variable – i.e. changes under isomonodromic deformations – and in generic situation has a highly non-trivial $\lambda_j$-dependence. Therefore, instead of being bilinear with respect to the dynamical variables, this Schlesinger system on the torus becomes highly transcendental.

An alternative form of the elliptic Schlesinger system was proposed by Takasaki [12] who considered the restriction $Q = \sigma_3, R = \sigma_1$, related to the classical limit of Etingof’s elliptic version of the Knizhnik-Zamolodchikov-Bernard system on the torus [13, 14, 15].
This choice of fixing the twists turns out to be compatible with the isomonodromic deformations equations, therefore essentially simplifying the dynamics as compared to [11]. It results into studying isomonodromic deformations of the system

$$\frac{d\Psi}{d\lambda} = A(\lambda) \Psi, \quad (3.13)$$

$$A(\lambda) \equiv \sum_{j=1}^{N} \sum_{\lambda=1}^{3} A^j_{\lambda} w_{\lambda}(\lambda - \lambda_j),$$

with $\lambda \in \mathbb{C}$ and functions $w_{\lambda}$ from (3.3). The connection $A(\lambda)$ obviously has only simple poles on $E$ and the following twist properties, cf. (3.4)

$$A(\lambda + 1) = \sigma_3 A(\lambda) \sigma_3, \quad A(\lambda + \mu) = \sigma_1 A(\lambda) \sigma_1. \quad (3.14)$$

Since the residues of all $w_{\lambda}$ at $\lambda = 0$ coincide, the residue of $A(\lambda)$ at $\lambda_j$ is

$$A_j \equiv \sum_{\lambda} A^j_{\lambda} \sigma_\lambda.$$

As in the case of the Riemann sphere, the function $\Psi$ has regular singularities at $\lambda = \lambda_j$ with the same local properties (2.3)–(2.5). The twist properties of $\Psi$ take the form

$$\Psi(\lambda + 1) = \sigma_3 \Psi(\lambda) M_a \quad \Psi(\lambda + \mu) = \sigma_1 \Psi(\lambda) M_b, \quad (3.15)$$

with monodromy matrices $M_a$, $M_b$ along the basic cycles of the torus. Moreover, as in the case of Riemann sphere, $\Psi(\lambda)$ has monodromies $M_j$ around the singularities $\lambda_j$.

The isomonodromy condition on the torus implies that all monodromies $M_j$, $M_a$ and $M_b$ are independent of the positions of singularities $\lambda_j$ and the module $\mu$ of the torus. As on the Riemann sphere this implies that the function $\partial \Psi / \partial \lambda_j \Psi^{-1}$ has the only simple pole at $\lambda = \lambda_j$ with residue $-A_j$. In addition, it has the following twist properties

$$\frac{\partial \Psi}{\partial \lambda_j} \Psi^{-1}(\lambda + 1) = \sigma_3 \frac{\partial \Psi}{\partial \lambda_j} \Psi^{-1}(\lambda) \sigma_3,$$

$$\frac{\partial \Psi}{\partial \lambda_j} \Psi^{-1}(\lambda + \mu) = \sigma_1 \left( \frac{\partial \Psi}{\partial \mu} \Psi^{-1}(\lambda) - \frac{\partial \Psi}{\partial \lambda} \Psi^{-1}(\lambda) \right) \sigma_1.$$

Therefore,

$$\frac{\partial \Psi}{\partial \lambda_j} = - \sum_{\lambda=1}^{3} A^j_{\lambda} w_{\lambda}(\lambda - \lambda_j) \sigma_\lambda \Psi. \quad (3.16)$$

To derive the equation with respect to module $\mu$ we observe that $\partial \Psi / \partial \mu \Psi^{-1}$ is holomorphic at $\lambda = \lambda_j$ (but not at $\lambda = \lambda_j + \mu$) and has twist properties

$$\frac{\partial \Psi}{\partial \mu} \Psi^{-1}(\lambda + 1) = \sigma_3 \frac{\partial \Psi}{\partial \mu} \Psi^{-1}(\lambda) \sigma_3,$$

$$\frac{\partial \Psi}{\partial \mu} \Psi^{-1}(\lambda + \mu) = \sigma_1 \left( \frac{\partial \Psi}{\partial \mu} \Psi^{-1}(\lambda) - \frac{\partial \Psi}{\partial \lambda} \Psi^{-1}(\lambda) \right) \sigma_1.$$

Taking into account the periodicity properties of the functions $Z_\lambda$ (3.6), this hence implies

$$\frac{\partial \Psi}{\partial \mu} = \sum_{j=1}^{N} \sum_{\lambda=1}^{3} A^j_{\lambda} Z_\lambda \sigma_\lambda \Psi. \quad (3.17)$$

The compatibility conditions of the equations (3.13), (3.16) and (3.17) then yield the $\lambda_i$ and $\mu$ dependence of the residues $A_j$. The result is summarized in the following
Theorem 3.1 [12] Isomonodromic deformations of the system (3.13) are described by the following elliptic version of the Schlesinger system:

\[
\frac{dA_j}{d\lambda_i} = \left[ A_j, \sum_{A=1}^{3} A_i^A w_A(\lambda_j - \lambda_i) \sigma_A \right], \quad i \neq j , \tag{3.18}
\]

\[
\frac{dA_j}{d\lambda_j} = -\sum_{i \neq j} \left[ A_j, \sum_{A=1}^{3} A_i^A w_A(\lambda_j - \lambda_i) \sigma_A \right],
\]

\[
\frac{dA_j}{d\mu} = -\sum_{i=1}^{N} \left[ A_j, \sum_{A=1}^{3} A_i^A Z_A(\lambda_j - \lambda_i) \sigma_A \right].
\]

The corresponding equations for the matrices \(G_j\) from (2.3) take a form analogous to the equations (2.8) on the Riemann sphere:

\[
\frac{\partial G_j}{\partial \lambda_i} = \sum_A A_i^A w_A(\lambda_i - \lambda_j) \sigma_A G_j , \quad \frac{\partial G_j}{\partial \lambda_j} = -\sum_{i=1}^{N} \sum_A A_i^A w_A(\lambda_i - \lambda_j) \sigma_A G_j. \tag{3.19}
\]

The system (3.18) admits a multi-time Hamiltonian formulation with respect to the Poisson structure (2.9) on the residues

\[
\{ A_i^A, A_j^B \} = \delta_{ij} \varepsilon^{ABC} A_j^C.
\]

This is summarized as

**Theorem 3.2** The elliptic Schlesinger system (3.18) is a multi-time Hamiltonian system with respect to the Poisson bracket

\[
\left\{ \frac{1}{2} A(\lambda), \frac{2}{2} A(\lambda') \right\} = \left[ r(\lambda - \lambda'), \frac{1}{2} A(\lambda) + \frac{2}{2} A(\lambda') \right], \tag{3.20}
\]

with the elliptic classical r-matrix \(r\) given by

\[
r(\lambda) = \sum_A w_A(\lambda) \sigma_A \otimes \sigma_A .
\]

The Hamiltonians describing deformation with respect to the variables \(\lambda_i\) and to the module \(\mu\) of the torus are respectively given by

\[
H_i = \frac{1}{4\pi i} \oint_{\lambda_i} \text{tr} A^2(\lambda) d\lambda = \sum_{j \neq i} \sum_a A_j^A A_i^A w_A(\lambda_j - \lambda_i) , \tag{3.21}
\]

\[
H_\mu = -\frac{1}{2\pi i} \oint_a \text{tr} A^2(\lambda) d\lambda = -\sum_{i,j} \sum_a A_j^A A_i^A Z_A(\lambda_i - \lambda_j) , \tag{3.22}
\]

and mutually Poisson commute.

This follows from straight-forward calculation. The representation of \(H_\mu\) as contour integral along the basic \(a\)-cycle in (3.22) may be derived using the relations (3.12). The fact, that all Hamiltonians Poisson-commute is a direct consequence of

\[
\{ \text{tr} A^2(\lambda), \text{tr} A^2(\lambda') \} = 0, \tag{3.23}
\]
which in turn follows immediately from (3.20).

Now we are in position to define the \( \tau \)-function of the elliptic Schlesinger system (3.18) as generating function \( \tau (\{ \lambda_j \}, \mu) \) of the Hamiltonians

\[
\frac{\partial \ln \tau}{\partial \lambda_j} = H_j, \quad \frac{\partial \ln \tau}{\partial \mu} = H_\mu, 
\]

(3.24)

whereby it is uniquely determined up to an arbitrary \((\mu, \{ \gamma_j \})\)-independent multiplicative constant. As usual, consistency of this definition is a corollary of the elliptic Schlesinger system.

3.3 Schlesinger transformations for elliptic isomonodromy deformations

The natural generalization of the notion of Schlesinger transformations on the Riemann sphere as introduced above to the elliptic case is the following. Starting from any solution of the elliptic Schlesinger system (3.18) with associated function \( \Psi \) satisfying (3.13) and (3.15) we construct a new solution \( \hat{A}_j, \hat{\Psi} \) with eigenvalues \( \hat{t}_j \) which differ from the \( t_j \) by integer or half-integer values. In particular, we will consider the elliptic analog of the elementary Schlesinger transformation (2.13) on the Riemann sphere. The following construction is inspired by the papers [18], [19].

As a natural elliptic analog of the function \( F(\lambda) \) from (2.15) we shall choose the following ansatz

\[
F(\lambda) = \frac{f(\lambda)}{\sqrt{\det f(\lambda)}}, 
\]

(3.25)

\[
f(\lambda) = \frac{1}{2} + \sum_{\lambda=1}^{3} J_\lambda \, w_\lambda (\lambda - \frac{1}{2} (\lambda_k + \lambda_l)) \, \sigma_\lambda, 
\]

where the functions \( J_\lambda(\lambda_j, \mu) \) depend on \( G_k \) and \( G_l \) and will be defined below. We formulate the result of this section in the following

**Theorem 3.3** Let the functions \( \{ A_j(\{ \lambda_i \}) \} \) satisfy the elliptic Schlesinger system (3.18) with twist properties (3.14) and let the function \( \Psi \) satisfy the associated linear system (3.13). For two arbitrary non-coinciding poles \( \lambda_k \) and \( \lambda_l \), define the new function

\[
\hat{\Psi}(\lambda) \equiv F(\lambda) \Psi(\lambda), 
\]

(3.26)

with \( F(\lambda) \) from formula (3.25) with \( \lambda \)-independent coefficients \( J_\lambda \) defined by

\[
\sum_{\lambda} J_\lambda \, w_\lambda (\frac{1}{2} (\lambda_k - \lambda_l)) \, \sigma_\lambda \equiv -\frac{1}{2} G \sigma_3 G^{-1}, 
\]

(3.27)

where as above we denote by \( G \) the matrix (2.17) containing the first columns of the matrices \( G_k \) and \( G_l \).

The function \( \hat{\Psi}(\lambda) \) then satisfies the equations (3.13), (3.16), (3.17) and the twist conditions (3.15) with the transformed functions

\[
\hat{A}_j(\{ \lambda_i \}) \equiv \text{res}_{\lambda=\lambda_j} \left\{ \frac{d\hat{\Psi}}{d\lambda} \hat{\Psi}^{-1} \right\}. 
\]

(3.28)
In turn, the functions $\hat{A}_j$ satisfy the elliptic Schlesinger system (3.18). For the eigenvalues $t_j$ we have

$$
\hat{t}_j = \begin{cases} 
t_j + \frac{1}{2} & \text{for } j = k, l \\
t_j & \text{else}
\end{cases}.
$$

The monodromy matrices $\hat{M}_j$, $\hat{M}_a$ and $\hat{M}_b$ of the function $\hat{\Psi}$ coincide with the monodromies of $\Psi$, except for $\hat{M}_k = -M_k$ and $\hat{M}_l = -M_l$.

**Proof.** The proof consists of two parts. The first part is to check that locally in the neighbourhoods of the singularities $\lambda_j$ the situation looks exactly like the situation on the Riemann sphere. The second (global) part is to check that no new singularities arise apart from the $\lambda_j$ and that the new function $\hat{\Psi}$ satisfies the required twist conditions (3.15).

The proper local behaviour of function $\hat{\Psi}$ is ensured by the relations

$$
S_\pm^2 = S_\pm, \quad S_+ + S_- = I, \quad S_+ G_l^1 = S_- G_k^1 = 0;
$$

for

$$
S_\pm = \frac{1}{2} + \sum_{\lambda} J_{\lambda} w_{\lambda}(\frac{1}{2}(\lambda_k - \lambda_l)) \sigma_{\lambda} = GP_{\pm} G^{-1},
$$

which in complete analogy to (2.11) describe annihilation of the vectors $G_k^1$ and $G_l^1$ by the matrices $f(\lambda_k)$ and $f(\lambda_l)$, respectively. Obviously, equations (3.29) are a consequence of (3.27). Similarly to the case of the sphere, it is then easy to verify that (3.29) provide the required asymptotical expansions (2.3) for the function $\hat{\Psi}$ with parameters $\hat{G}_j$, $C_j$ and $\hat{t}_j$.

Concerning the global behavior of $\hat{\Psi}$ we note that the prefactor $(\det f(\lambda))^{-1/2}$ in (3.23) provides the condition $\det \hat{\Psi} = 1$ and kills the simple pole of $f(\lambda)$ at $\lambda = (\lambda_k + \lambda_l)/2$. Therefore, the only singularities of $F(\lambda)$ on $E$ are the zeros of $\det f(\lambda)$. Since $\det f(\lambda)$ has only one pole – this is the second order pole at $\lambda = (\lambda_k + \lambda_l)/2$ – it must have also two zeros on $E$ whose sum according to Abel's theorem equals $\lambda_k + \lambda_l$. According to (3.24) these are precisely $\lambda_k$ and $\lambda_l$. It remains to check that $\hat{\Psi}$ satisfies conditions (3.15) with the same matrices $M_a$ and $M_b$. This follows from the twist properties

$$
f(\lambda + 1) = \sigma_3 f(\lambda) \sigma_3, \quad f(\lambda + \mu) = \sigma_1 f(\lambda) \sigma_1,
$$

which in turn is a consequence of (3.25) and the periodicity properties (3.4) of the functions $w_j(\lambda)$.

\[\square\]

4 The action of elliptic Schlesinger transformations on the $\tau$-function

In this section we shall present and prove the elliptic analog of formula (2.22) describing the transformation of the $\tau$-function under the action of elliptic Schlesinger transformations.
Theorem 4.1 The \( \tilde{\tau} \)-function corresponding to the Schlesinger-transformed solution \( \tilde{A}_j \) (3.28) of the elliptic Schlesinger system is related to the \( \tau \)-function corresponding to the solution \( A_j \) as follows

\[
\tilde{\tau}(\{\lambda_j\}, \mu) = \left\{ \left[ w_1 w_2 w_3 \left( \frac{\lambda_k - \lambda_l}{2} \right) \right]^{1/2} \det [G J^{1/2}] \right\} \cdot \tau(\{\lambda_j\}, \mu),
\]

where \( G \) is the matrix (2.17) containing the first columns of the matrices \( G_k, G_l \); \( J \) is the matrix

\[
J = \sum_{A=1}^{3} J_A \sigma_A
\]

and the functions \( J_A \) are defined in terms of \( G \) via (3.27).

Proof. We start noting that

\[
\sum_A [J_A w_A (\frac{1}{2} (\lambda_k - \lambda_l))]^2 = \frac{1}{4},
\]

as follows from taking the determinant of (3.27). In particular, this shows that upon replacing \( w_A(\lambda) \rightarrow 1/\lambda \), formula (4.1) indeed reproduces the result for the Riemann sphere (2.22).

The proof of (4.1) according to the definition of the \( \tau \)-function (3.24) now consists of three parts; it has to be checked that

\[
\tilde{H}_j - H_j = \frac{\partial}{\partial \lambda_j} \ln \left\{ \left[ \sum_{A=1}^{3} J_A^2 \right]^{1/2} \det G \right\} \quad \text{for} \quad j \neq k, l,
\]

\[
\tilde{H}_k - H_k = \frac{\partial}{\partial \lambda_k} \ln \left\{ \left[ w_1 w_2 w_3 \left( \frac{1}{2} (\lambda_k - \lambda_l) \right) \sum_{A=1}^{3} J_A^2 \right]^{1/2} \det G \right\},
\]

\[
\tilde{H}_\mu - H_\mu = \frac{\partial}{\partial \mu} \ln \left\{ \left[ w_1 w_2 w_3 \left( \frac{1}{2} (\lambda_k - \lambda_l) \right) \sum_{A=1}^{3} J_A^2 \right]^{1/2} \det G \right\}.
\]

To obtain the l.h.s. of these equations we make use of the representation of the Hamiltonians as contour integrals (3.21), (3.22) and compute the difference

\[
\text{tr} \tilde{A}^2 - \text{tr} A^2 = 2 \text{tr} \left[ F^{-1} \frac{dF}{d\lambda} A \right] + \text{tr} \left[ F^{-1} \frac{dF}{d\lambda} \right]^2.
\]

It is

\[
F^{-1} \frac{dF}{d\lambda} = - \frac{2}{1 - 4 \sum_A (J_A w_A)^2} \times
\]

\[
\times \left( \sum_a J_A w_A' \sigma_A - i \sum_{A,B,C} \varepsilon^{ABC} J_B J_C w_A (w_B^2 - w_C^2) \sigma_A \right) \left( \lambda - \frac{1}{2} (\lambda_k + \lambda_l) \right),
\]

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where all the elliptic functions \(w_A\) on the r.h.s. are taken at the argument \(\lambda - \frac{1}{2}(\lambda_k + \lambda_l)\).

Making use of (3.9), (3.10), (4.2) and the fact that the combination \(w^2_A(\lambda) - w^2_A(\lambda')\) does not depend on \(\lambda\), this expression simplifies to

\[
F^{-1} \frac{dF}{d\lambda} = - \frac{1}{4} \sum_A |J^2_A| \sum_A J^A w_A(\lambda - \lambda_k) - w_A(\lambda - \lambda_l) \frac{w_A(\lambda(\lambda_k - \lambda_l))}{\sigma_A} \tag{4.5}
\]

\[- \frac{i}{4} \sum_{A,B,C} \varepsilon^{ABC} J^A J^B J^C (w^2_B - w^2_C) \frac{w_A(\lambda - \lambda_k) + w_A(\lambda - \lambda_l)}{w_B w_C(\frac{1}{2}(\lambda_k - \lambda_l))} \sigma_A.
\]

The argument in the combinations \(w^2_B - w^2_C\) has been skipped since they are \(\lambda\)-independent.

Let us start by proving (4.3a). According to (4.4) and (4.5) we find that

\[
\hat{H}_j - H_j = - \frac{1}{2} \sum_A |J^2_A| \sum_A J^A w_A(\lambda - \lambda_k) - w_A(\lambda - \lambda_l) \frac{w_A(\lambda(\lambda_k - \lambda_l))}{\sigma_A} \tag{4.6}
\]

\[- \frac{i}{2} \sum_{A,B,C} \varepsilon^{ABC} J^A J^B J^C (w^2_B - w^2_C) \frac{w_A(\lambda - \lambda_k) + w_A(\lambda - \lambda_l)}{w_B w_C(\frac{1}{2}(\lambda_k - \lambda_l))} .
\]

This should be compared to the r.h.s. of (4.3a). From (3.19) we find that

\[
\frac{\partial G}{\partial \lambda_j} = \sum_A A^A j^A \sigma_A \{ w_A(\lambda_j - \lambda_k) GP_+ + w_A(\lambda_j - \lambda_l) GP_- \} .
\]

Thus,

\[
\frac{\partial}{\partial \lambda_j} \{ \ln \det G \} = \text{tr} \left[ \frac{\partial G}{\partial \lambda_j} G^{-1} \right] \tag{4.7}
\]

\[= -2 \sum_A A^A j^A w_A(\lambda(\lambda_k - \lambda_l)) (w_A(\lambda_j - \lambda_k) - w_A(\lambda_j - \lambda_l)) .
\]

It is slightly more complicated to calculate the first term in the r.h.s. of (4.3a). Equations (3.27) and (3.19) imply that

\[
2 \sum_A \frac{\partial J^A j^A}{\partial \lambda_j} w_A(\lambda(\lambda_k - \lambda_l)) \sigma_A = - \frac{\partial}{\partial \lambda_j} \left\{ G \sigma_3 G^{-1} \right\} \tag{4.8}
\]

\[= - \sum_A A^A j^A (w_A(\lambda_j - \lambda_k) S_- - w_A(\lambda_j - \lambda_l) S_+ \) \sigma_A
\]

\[+ G \sigma_3 G^{-1} \sum_A A^A j^A (w_A(\lambda_j - \lambda_k) S_- - w_A(\lambda_j - \lambda_l) S_+) \sigma_A .
\]

Reexpressing \(G \sigma_3 G^{-1}\) and \(S_\pm\) in terms of \(J^A\) according to (3.27) and (3.29), we get after some calculation

\[
\frac{\partial}{\partial \lambda_j} \sum_A J^2_A = - \sum_A J^A j^A \frac{w_A(\lambda - \lambda_k) - w_A(\lambda - \lambda_l)}{w_A(\lambda(\lambda_k - \lambda_l))} \left( 1 - 4w_A^2(\frac{1}{2}(\lambda_k - \lambda_l)) \sum_b J^2_B \right)
\]

\[- \frac{i}{2} \sum_{A,B,C} \varepsilon^{ABC} J^A J^B J^C (w^2_B - w^2_C) \frac{w_A(\lambda_j - \lambda_k) + w_A(\lambda_j - \lambda_l)}{w_B w_C(\frac{1}{2}(\lambda_k - \lambda_l))} .
\]
such that combining this with (4.6) and (4.7) we indeed recover (4.3a). This determines the function $\tilde{\tau}$ already up to some factor depending on $\lambda_k$, $\lambda_l$ and $\mu$ in accordance with formula (4.4).

We turn to proving (4.3b). According to (4.4), also the residues of $F^{-1} dF/d\lambda$ at $\lambda_k$ enter the variation of $H_k$. Expanding (4.3) we find

$$F^{-1} \frac{dF}{d\lambda} = -\frac{1}{4(\lambda - \lambda_k)} \sum_A (J_A^2) \left\{ \sum_A \frac{J_A \sigma_A}{w_A(\frac{1}{2}(\lambda_k - \lambda_l))} + \sum_{A,B,C} \frac{i\varepsilon_{ABC} (w_B^2 - w_C^2) J_B J_C \sigma_A}{w_B w_C(\frac{1}{2}(\lambda_k - \lambda_l))} \right\}$$

$$+ \frac{1}{4 \sum_A (J_A^2)} \sum_A J_A \frac{w_A(\lambda_k - \lambda_l)}{w_A(\frac{1}{2}(\lambda_k - \lambda_l))} \sigma_A$$

$$- \frac{i}{4 \sum_A (J_A^2)} \sum_{A,B,C} \varepsilon_{ABC} J_B J_C (w_B^2 - w_C^2) \frac{w_A(\lambda_k - \lambda_l)}{w_B w_C(\frac{1}{2}(\lambda_k - \lambda_l))} \sigma_A$$

$$+ O(\lambda - \lambda_k) \ .$$

As it turns out, in (4.3b) all terms linear in the residues $A_j$ cancel in a way completely analogous to (4.3a) shown above. We hence restrict to the remaining terms. On the l.h.s. we find making repeated use of (3.11) and (4.2):

$$\frac{1}{2} \text{res}_{\lambda = \lambda_k} \text{tr} \left[ F^{-1} \frac{dF}{d\lambda} \right]^2 = -\frac{1}{8 \left[ \sum_A (J_A^2) \right]^2} \sum_A J_A^2 \frac{w_A^2(\lambda_k - \lambda_l)}{w_A(\frac{1}{2}(\lambda_k - \lambda_l))}$$

$$+ \frac{1}{2 \left[ \sum_A (J_A^2) \right]^2} \sum_{(ABC)=(123)} \text{cyclic} J_B^2 J_C^2 \frac{(w_B^2 - w_C^2)^2 w_A(\lambda_k - \lambda_l)}{w_B w_C(\frac{1}{2}(\lambda_k - \lambda_l))}$$

$$= -\frac{1}{2 \left[ \sum_A (J_A^2) \right]^2} \sum_A J_A^2 w_A(\lambda_k - \lambda_l)$$

$$- \frac{1}{2 \left[ \sum_A (J_A^2) \right]^2} \sum_{(ABC)=(123)} \text{cyclic} J_B^2 J_C^2 \frac{w_B w_C(\frac{1}{2}(\lambda_k - \lambda_l))}{w_A(\frac{1}{2}(\lambda_k - \lambda_l))}$$

$$= \frac{1}{2} \frac{\partial}{\partial \lambda_k} \ln (w_1 w_2 w_3(\frac{1}{2}(\lambda_k - \lambda_l)))$$

$$- \frac{1}{\sum_A (J_A^2)} \sum_A J_A^2 \frac{\partial}{\partial \lambda_k} \ln (w_A(\frac{1}{2}(\lambda_k - \lambda_l))) \ . \quad (4.9)$$

The first term in (4.9) obviously cancels against the derivative of the explicit factor $w_1 w_2 w_3$ in (4.3b). To see the origin of the second term we note that instead of (4.8) the $\lambda_k$ derivative of $J_\lambda$ is given by

$$2 \sum_A \frac{\partial J_A}{\partial \lambda_k} \frac{w_A(\frac{1}{2}(\lambda_k - \lambda_l))}{w_A(\frac{1}{2}(\lambda_k - \lambda_l))} \sigma_A = -\frac{\partial}{\partial \lambda_k} \left\{ G \sigma_3 G^{-1} \right\} - 2 \sum_A J_A \frac{\partial}{\partial \lambda_k} \frac{w_A(\frac{1}{2}(\lambda_k - \lambda_l))}{w_A(\frac{1}{2}(\lambda_k - \lambda_l))} \ .$$

The additional term on the r.h.s which has no linear dependence on the residues $A_j$ coincides precisely with the second term in (4.9). Thus, we have shown (4.3b).
To finally prove (4.3c) we first note that

$$\sum_j \text{res}_{\lambda=\lambda_j} \tr \left[ F^{-1} \frac{\partial F}{\partial \mu} F^{-1} \frac{dF}{d\lambda} \right] = -\frac{1}{2\pi i} \oint_{\partial E} \tr \left[ F^{-1} \frac{\partial F}{\partial \mu} F^{-1} \frac{dF}{d\lambda} \right] d\lambda$$

$$= -\frac{1}{2\pi i} \oint_{a} \tr \left[ F^{-1} \frac{dF}{d\lambda} \right]^2 d\lambda , \quad (4.10)$$

where by $\partial E$ we denote a closed path encircling all the singularities. To show the second equality in (4.10), note that the integrand is single-valued with respect to shifts along the $a$-cycle, whereas upon tracing along the $b$-cycle it has the additive twist

$$\tr \left[ F^{-1} \frac{\partial F}{\partial \mu} F^{-1} \frac{dF}{d\lambda} \right] (\lambda + \mu) = \tr \left[ F^{-1} \frac{\partial F}{\partial \mu} F^{-1} \frac{dF}{d\lambda} \right] (\lambda) - \tr \left[ F^{-1} \frac{dF}{d\lambda} \right]^2 (\lambda) .$$

The closed integral along $\partial E$ thus reduces to the integral over the additive twist of the integrand along the $a$-cycle. Equation (4.10) may be used to compute the variation of $H_\mu$ as

$$\hat{H}_\mu - H_\mu = -\frac{1}{2\pi i} \oint_{a} \tr \left[ F^{-1} \frac{dF}{d\lambda} A \right] d\lambda - \frac{1}{4\pi i} \oint_{a} \tr \left[ F^{-1} \frac{dF}{d\lambda} \right]^2 d\lambda$$

$$= \sum_j \text{res}_{\lambda=\lambda_j} \tr \left[ F^{-1} \frac{\partial F}{\partial \mu} A \right] + \frac{1}{2} \sum_j \text{res}_{\lambda=\lambda_j} \tr \left[ F^{-1} \frac{\partial F}{\partial \mu} F^{-1} \frac{dF}{d\lambda} \right]^2 .$$

By some further calculations similar to the one given in the proofs of (4.3a) and (4.3b), this variation can now be shown to coincide with the r.h.s. of (4.3c). We leave the details to the reader. This finishes the proof of Theorem 4.1.

\[\square\]

5 Open problems

In this paper we have extended the construction of elementary Schlesinger transformations for $\mathfrak{sl}(2, \mathbb{C})$-valued meromorphic connections from the Riemann sphere to the torus. The induced transformation of the $\tau$-function of the elliptic Schlesinger system has been explicitly integrated.

There are several ways for a further extension of these results. We hope, that the formula (4.1) will give rise to new integrable chains associated to elliptic curves in a way similar to the construction of integrable chains from isomonodromic deformations on the sphere \([4, 5]\). For a complete generalization of the program of \([2, 4]\) to the elliptic case one should extend the results of this work to higher rank matrices and higher order poles.

An interesting problem would be the generalization of the notion of Schlesinger transformations for isomonodromic deformations on higher genus curves, which seems although rather difficult from the technical point of view, cf. \([20, 21]\). Already on the torus it is a rather nontrivial problem to extend our construction to isomonodromic deformations with variable twist \([1]\) as has been discussed in the text.

In the paper \([22]\) it is explicitly solved a certain class of Riemann-Hilbert problems on the torus which allows to get a class of solutions of the elliptic Schlesinger system in terms of Prym theta-functions.
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