A NOTE ON CARLITZ’S TYPE \( q \)-CHANGHEE NUMBERS AND POLYNOMIALS

DMITRY V. DOLGY, GWAN-WOO JANG, HYUCK-IN KWON, AND TAEKYUN KIM

Abstract. In this paper, we consider the Carlitz’s type \( q \)-analogue of Changhee numbers and polynomials and we give some explicit formulae for these numbers and polynomials.

1. Introduction

Let \( p \) be an odd prime number. Throughout this paper, \( \mathbb{Z}_p \), \( \mathbb{Q}_p \) and \( \mathbb{C}_p \) will denote the ring of \( p \)-adic integers, the field of \( p \)-adic numbers and the completion of the algebraic closure of \( \mathbb{Q}_p \). The \( p \)-adic norm is normalized as \( |p|_p = \frac{1}{p} \). Let \( q \) be an indeterminate in \( \mathbb{C}_p \) such that \( |1 - q|_p < p^{-1} \). The \( q \)-analogue of number \( x \) is defined as \( [x]_q = q^x - 1 - q^{x-1} \). As is well known, the Euler polynomials are defined by the generating function to be

\[
\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad \text{ (see [1 - 14]).} \tag{1.1}
\]

When \( x = 0 \), \( E_n = E_n(0), (n \geq 0) \), are called the Euler numbers. In [1,2,3] L. Carlitz considered the \( q \)-analogue of Euler numbers which are given by the recurrence relation as follows:

\[
\mathcal{E}_{0,q} = 1, \quad q(q\mathcal{E}_q + 1)^n + \mathcal{E}_{n,q} = \begin{cases} 
[2]_{q^n}, & \text{if } n = 0, \\
0, & \text{if } n > 1.
\end{cases}
\]

with the usual convention about replacing \( \mathcal{E}_q^n \) by \( \mathcal{E}_{n,q} \).

He also considered \( q \)-Euler polynomials which are defined by

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\[ E_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_{q}^{n-l} q^l x_{l,q}, \quad \text{(see [2, 3]).} \quad (1.2) \]

In [8, 9, 10], Kim defined the fermionic \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) as follows:

\[ I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} p^{N-1} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \quad (1.3) \]

where \( f(x) \) is a continuous function on \( \mathbb{Z}_p \) and \([x]_{-q} = \frac{1+q}{1(-q)^x}\).

From (1.3), he derived the following formula for the Carlitz’s \( q \)-Euler polynomials:

\[ \int_{\mathbb{Z}_p} [x+y]^n_q d\mu_{-q}(y) = E_{n,q}(x), \quad (n \geq 0), \quad \text{(see [7, 10]).} \quad (1.4) \]

When \( x = 0 \), \( E_{n,q} = \int_{\mathbb{Z}_p} [x]^n_q d\mu_{-q}(x) \) are Carlitz’s \( q \)-Euler numbers.

The Changhee polynomials are defined by the generating function to be

\[ \frac{2}{2+t(1+t)^x} = \sum_{n=0}^{\infty} Ch_n(x) \frac{t^n}{n!}, \quad \text{(see [5, 6]).} \quad (1.5) \]

Thus, by (1.5), we get

\[ E_n(x) = \sum_{k=0}^{n} S_2(n,k) Ch_k(x), \quad Ch_n(x) = \sum_{k=0}^{n} S_1(n,k) E_k(x), \quad (n \geq 0), \quad (1.6) \]

Where \( S_2(n,k) \) is Stirling number of the second kind and \( S_1(n,k) \) is the Stirling number of the first kind. In [10], the higher-order Carlitz’s \( q \)-Euler polynomials are written by the fermionic \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) as follows:

\[ \sum_{n=0}^{\infty} e_{n,q}^{(r)}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{[x_1+\cdots+x_r+x]} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r), \quad (n \geq 0). \quad (1.7) \]

In this paper, we consider the Carlitz’s type \( q \)-Changhee polynomials and numbers and we give explicit formulas for these numbers and polynomials.
2. Carlitz’s type $q$-Changhee polynomials

In this section, we assume that $t \in \mathbb{C}_p$ with $|t|_p < p^{-1}$. From (1.3) and (1.5), we note that

$$
\int_{\mathbb{Z}_p} (1 + t)^{x+y} d\mu_{-1}(y) = \frac{2}{2 + t} (1 + t)^z = \sum_{n=0}^{\infty} Ch_n(x) \frac{t^n}{n!}, \quad (n \geq 0),
$$

(2.1)

Thus, by (2.1), we get

$$
\int_{\mathbb{Z}_p} (x + y)^n d\mu_{-1}(y) = Ch_n(x), \quad (n \geq 0),
$$

(2.2)

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$.

In the viewpoint of (1.4), we consider the Carlitz’s type $q$-Changhee polynomials which are derived from the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ as follows:

$$
\int_{\mathbb{Z}_p} (1 + t)^{x+y} q d\mu_{-1}(y) = \sum_{n=0}^{\infty} Ch_{n,q}(x) \frac{t^n}{n!}.
$$

(2.3)

Thus, by (2.3), we get

$$
\sum_{n=0}^{\infty} Ch_{n,q}(x) \frac{t^n}{n!} = \sum_{k=0}^{\infty} \int_{\mathbb{Z}_p} [x + y]^k q d\mu_{-1}(y) \frac{1}{k!} \left( \log(1 + t) \right)^k
$$

$$
= \sum_{k=0}^{\infty} \int_{\mathbb{Z}_p} [x + y]^k q d\mu_{-1}(y) \sum_{n=k}^{\infty} S_1(n, k) \frac{t^n}{n!}
$$

(2.4)

$$
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} E_{k,q}(x) S_1(n, k) \right) \frac{t^n}{n!}
$$

Indeed,

$$
\sum_{k=0}^{n} S_1(n, k) \int_{\mathbb{Z}_p} [x + y]^k q d\mu_{-1}(y) = \sum_{k=0}^{n} S_1(n, k) \frac{1}{(1-q)^k} \sum_{l=0}^{k} \binom{k}{l} q^l (-1)^l \frac{[2]_q}{1 + q^{l+1}}
$$

$$
= [2]_q \sum_{k=0}^{n} \sum_{l=0}^{k} \frac{1}{(1-q)^k} \binom{k}{l} q^l (-1)^l S_1(n, k) \frac{1}{1 + q^{l+1}}.
$$

Therefore, by (2.4), we obtain the following theorem.
Theorem 2.1. For \( n \geq 0 \), we have

\[
Ch_{n,q}(x) = [2]^n \sum_{k=0}^{n} \frac{1}{(1-q)^k} \binom{k}{l} q^lx(-1)^k \frac{S_1(n,k)}{1+q^{l+1}}
\]

\[
= \sum_{k=0}^{n} S_1(n,k) E_{n,q}(x).
\]

From (1.4), we note that

\[
\sum_{n=0}^{\infty} \varepsilon_{n,q}(x) \frac{t^n}{n!} = \int_{Z_p} e^{[x+y]q^t d\mu_{-q}(y)}.
\] (2.5)

By (2.5), we get

\[
\sum_{k=0}^{\infty} Ch_{k,q}(x) \frac{1}{k!} (e^t - 1)^k = \int_{Z_p} e^{[x+y]q^t d\mu_{-q}(y)} = \sum_{n=0}^{\infty} \varepsilon_{n,q}(x) \frac{t^n}{n!}.
\] (2.6)

On the other hand,

\[
\sum_{k=0}^{\infty} Ch_{k,q}(x) \frac{1}{k!} (e^t - 1)^k = \sum_{k=0}^{\infty} Ch_{k,q}(x) \sum_{n=k}^{\infty} S_2(n,k) \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} Ch_{k,q}(x) S_2(n,k) \right) \frac{t^n}{n!}.
\] (2.7)

Thus, by (2.6) and (2.7), we get the following theorem.

Theorem 2.2. For \( n \geq 0 \), we have

\[
\varepsilon_{n,q}(x) = \sum_{k=0}^{n} Ch_{k,q}(x) S_2(n,k).
\]
From Theorem 1, we note that
\[
\sum_{n=0}^{\infty} Ch_{n,q}(x) \frac{t^n}{n!} = [2]q \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} S_1(n,k) \frac{1}{(1-q)^k} \sum_{l=0}^{k} \binom{k}{l} q^l x (-1)^l \sum_{m=0}^{\infty} (-q^l+1)^m \right) \frac{t^n}{n!},
\]
(2.8)

Therefore, by (2.8), we obtain the following theorem.

**Theorem 2.3.** The generating function of the Carlitz’s type \( q \)-Changhee polynomials is given by
\[
[2]q \sum_{m=0}^{\infty} (-q)^m (1+t)^{[m+x]_q} = \sum_{n=0}^{\infty} Ch_{n,q}(x) \frac{t^n}{n!}.
\]

In particular, \( x = 0 \), we have
\[
[2]q \sum_{m=0}^{\infty} (-q)^m (1+t)^{[m]_q} = \sum_{n=0}^{\infty} Ch_{n,q} \frac{t^n}{n!}.
\]

From (1.3), we easily note that
\[
q I_{-q}(f_1) + I_{-q}(f) = [2]q f(0), \text{ where } f_1(x) = f(x+1).
\]
(2.9)

Thus, by (2.9), we get
\[
q \int_{\mathbb{Z}_p} (1+t)^{[x+1+y]_q} d\mu_{-q}(y) + \int_{\mathbb{Z}_p} (1+t)^{[x+y]_q} d\mu_{-q}(y) = [2]q (1+t)^{[x]_q}. \quad (2.10)
\]

By (2.3) and (2.10), we get
\[
\sum_{n=0}^{\infty} \left( q Ch_{n,q}(x+1) + Ch_{n,q}(x) \right) \frac{t^n}{n!} = [2]q \sum_{n=0}^{\infty} (x)_q \frac{t^n}{n!}, \quad (2.11)
\]
Comparing the coefficients on the both sides of (2.11), we get

\[ qCh_{n,q}(x + 1) + Ch_{n,q}(x) = [2]_q \left( [x]_q \right)_n = [2]_q \sum_{l=0}^{n} S_1(n, l)[x]_q^l, \quad (n \geq 0). \tag{2.12} \]

Therefore, we obtain the following theorem.

**Theorem 2.4.** For \( n \geq 0 \), we have

\[ qCh_{n,q}(x + 1) + Ch_{n,q}(x) = [2]_q \sum_{l=0}^{n} S_1(n, l)[x]_q^l. \]

From (2.12), we have

\[ \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \left( [x + y]_q \right)_n d\mu_{-q}(y) t^n = \sum_{n=0}^{\infty} \frac{Ch_{n,q}(x) t^n}{n!}. \tag{2.13} \]

Thus, by (2.13), we get

\[ \int_{\mathbb{Z}_p} \left( [x + y]_q \right)_n d\mu_{-q}(y) = \frac{Ch_{n,q}(x)}{n!}, \quad (n \geq 0). \]

Now, we observe that

\[ (1 + t)[x + y]_q = (1 + t)[x]_q + q^r[y]_q = (1 + t) [x]_q \cdot (1 + t) q^r[y]_q. \tag{2.14} \]

Thus, by (2.14), we get

\[ \sum_{n=0}^{\infty} Ch_{n,q}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} (1 + t)[x + y]_q d\mu_{-q}(y) \]

\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{n} S_1(n, k) \int_{\mathbb{Z}_p} \left( [x]_q + q^r[y]_q \right)^k d\mu_{-q}(y) \frac{t^n}{n!} \]

\[ = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{k}{l} S_1(n, k) [x]_q^{k-l} q^r \left[ E_{l,q} \right] \right) \frac{t^n}{n!}. \tag{2.15} \]

Therefore, by (2.15), we obtain the following theorem.

**Theorem 2.5.** For \( n \geq 0 \), we have

\[ Ch_{n,q}(x) = \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{k}{l} S_1(n, k) [x]_q^{k-l} q^r \left[ E_{l,q} \right]. \]
From (1.3), we note that

\[
\int_{\mathbb{Z}_p} f(x)d\mu(x) - q^x = \lim_{N \to \infty} \frac{1}{p^N - q} \sum_{x=0}^{p^N-1} f(x)(-q)^x
\]

where \(d \in \mathbb{N}\) with \(d \equiv 1 \pmod{2}\). For \(d \in \mathbb{N}\) with \(d \equiv 1 \pmod{2}\), we have

\[
\int_{\mathbb{Z}_p} f(x)d\mu(y) - q^x = \lim_{N \to \infty} \frac{1}{dp^N - q} \sum_{a=0}^{d-1} \sum_{x=0}^{p^N-1} f(a + dx)(-q)^{a + dx}.
\]

By (2.17), we get

\[
\int_{\mathbb{Z}_p} (1 + t)^{|x+y|} d\mu(y)
\]

\[
= \sum_{a=0}^{d-1} (-q)^a \int_{\mathbb{Z}_p} (1 + t)^{\frac{a+x}{d} + y} q^x d\mu(y)
\]

\[
= \sum_{a=0}^{d-1} (-q)^a \int_{\mathbb{Z}_p} \left( \sum_{k=0}^{\infty} \frac{[d]_{q^k}^k E_{k,q}(\frac{a+x}{d}) S_1(n,k)}{k!} \right) \frac{t^n}{n!}
\]

Therefore, by (2.3) and (2.18), we obtain the following theorem.

**Theorem 2.6.** For \(n \geq 0\), we have

\[
Ch_{n,q}(x) = \sum_{a=0}^{d-1} (-q)^a \sum_{k=0}^{\infty} \frac{[d]_{q^k}^k E_{k,q}(\frac{a+x}{d}) S_1(n,k)}{k!}.
\]

For \(r \in \mathbb{N}\), the higher-order Carlitz’s type \(q\)-Changhee polynomials are also given by the multivariate fermionic \(p\)-adic \(q\)-integral as follows:

\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + t)^{|x_1+x_2+\cdots x_r+x|} d\mu_q(x_1) \cdots d\mu_q(x_r) = \sum_{n=0}^{\infty} \frac{Ch_{n,q}^{(r)}(x)^n}{n!}.
\]
Thus, we note that

\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + t)^{x_1 + x_2 + \cdots + x_r} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r)
\]

\[
= \sum_{k=0}^{\infty} \left( \sum_{n=0}^{\infty} S_1(n, k) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \frac{1}{k!} \left( \log(1 + t) \right)^k \right) t^n
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} S_1(n, k) \mathcal{E}_{k,q}^{(r)}(x) \right) \frac{t^n}{n!}.
\]

(2.20)

By (2.19) and (2.20), we get

\[
Ch^{(r)}_{n,q}(x) = \sum_{k=0}^{n} S_1(n, k) \mathcal{E}_{k,q}^{(r)}(x).
\]

(2.21)

When \( x = 0 \), \( Ch^{(r)}_{n,q} = Ch^{(r)}_{n,q}(0) \) are called the Carlitz’s type \( q \)-Changhee numbers.

By (1.7) and (2.19), we get

\[
\sum_{k=0}^{\infty} Ch^{(r)}_{k,q}(x) \frac{1}{k!} \left( e^t - 1 \right)^k = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{x_1 + x_2 + \cdots + x_r + x} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r)
\]

\[
= \sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(r)}(x).
\]

(2.22)

On the other hand,

\[
\sum_{k=0}^{\infty} Ch^{(r)}_{k,q}(x) \frac{1}{k!} \left( e^t - 1 \right)^k = \sum_{k=0}^{\infty} Ch^{(r)}_{k,q}(x) \sum_{n=k}^{\infty} S_2(n, k) \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} Ch^{(r)}_{k,q}(x) S_2(n, k) \right) \frac{t^n}{n!}.
\]

(2.23)

Comparing the coefficients on the both sides of (2.22) and (2.23), we have

\[
\mathcal{E}_{n,q}^{(r)}(x) = \sum_{k=0}^{n} Ch^{(r)}_{k,q}(x) S_2(n, k).
\]

(2.24)
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Hanrimwon, Kwangwoon University, Seoul 139-701, Republic of Korea
E-mail address: dol@mail.ru

Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
E-mail address: gwjang@kw.ac.kr

Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
E-mail address: sura@kw.ac.kr

Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
E-mail address: tkkim@kw.ac.kr