Is the Dirichlet Space a Quotient of $DA_n$?

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To Björn Jawerth, for many good memories.

Abstract. We show that the Dirichlet space is not a quotient of the Drury-Arveson space on the $n$-ball for any finite $n$. The proof is based on a quantitative comparison of the metrics induced by the Hilbert spaces.

1. Statement of the Result

We will consider reproducing kernel Hilbert spaces, RKHS’s, on the balls $B^n \subset \mathbb{C}^n$, $n = 1, 2, \ldots, \infty$. We interpret $\mathbb{C}^\infty$ as the space $\ell^2(\mathbb{Z}_+)$ and $B^\infty$ as its unit ball.

We are interested in $D$, the Dirichlet space, which consists of holomorphic functions $f$ defined on the unit disk $B^1 = D$, $f(z) = \sum a_n z^n$, normed using $\|f\|^2_D = \sum (n+1) |a_n|^2$. The space has a reproducing kernel, $h_z$, for evaluating functions at $z$, given by

$$h_z(w) = \frac{1}{\bar{z}w} \log \frac{1}{1 - \bar{z}w}, \quad zw \neq 0,$$

$$= 1, \quad zw = 0.$$

We denote the normalized kernels by $\hat{h}_z$.

The other spaces we consider are the Drury-Arveson spaces, $DA_n, 1, 2, \ldots, \infty$. The space $DA_n$ is the Hilbert space of holomorphic functions on $B^n$ which is defined by the reproducing kernel $j_z^{(n)}$; for $z, w \in B^n$,

$$j_z^{(n)}(w) = \frac{1}{1 - \langle w, z \rangle}.$$

Here $\langle w, z \rangle$ is the standard inner product on $\mathbb{C}^n$. We will denote the normalized kernels by $j_z^{(n)}$ and generally omit "$(n)$".

We are interested in the following

**Question 1:** Is there, for some finite $n$, a map $\Phi : B^1 \to B^n$, and a positive function $\lambda$ defined on $B^1$ so that for all $z, w \in B^1$

$$h(w, z) = \lambda(z) \lambda(w) j_z^{(n)}(\Phi(w), \Phi(z)) \quad ?$$

The main result in this paper is that Question 1 has a negative answer.

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2. Background

The spaces $\mathcal{D}$ and $DA_n$ have irreducible complete Pick kernels, CPK. An introduction to such spaces is in [AgMc] and we will make free use of the results there. More recent information is in [Sha].

When the general theory of RKHS with CPK is applied to the space $\mathcal{D}$ it insures that the variation of Question 1 in which $n$ is not required to be finite has a positive answer. That result holds in general; if $K$ is a RKHS of functions on a space $X$, and if $K$ has a CPK, then there is a map $\Phi_X : X \to B^{n(K)}$ so that the analog of (1.1) holds. Again, $n(K) = \infty$ may be required.

Given such a $\Phi_X$ we define its range, $\text{Ran}(\Phi_X)$, to be the image of $X$ in $B^{n(K)}$. Let $\text{Span}(\Phi_X(X))$ be the closed linear span of the $DA_n^{n(K)}$-kernel functions for $x \in \text{Ran}(\Phi_X)$, and let $\text{Van}(\Phi_X(X))$ be the space of functions in $DA_n^{n(K)}$ which vanish on $\text{Ran}(\Phi_X)$. That is:

$$\text{Ran}(\Phi_X) = \{ \Phi_X(x) : x \in X \} \subset B^{n(K)},$$

$$\text{Span}(\Phi_X) = \text{closed span of } \{ j^{n(K)}_\xi : \xi \in \text{Ran}(\Phi_X) \} \subset DA_n^{n(K)},$$

$$\text{Van}(\Phi_X(X)) = \{ f \in DA_n^{n(K)} : f(\xi) = 0 \ \forall \xi \in \text{Ran}(\Phi_X) \} \subset DA_n^{n(K)}.$$

The map which takes kernel functions for $K$ to kernel functions in $DA_n^{n(K)}$, $k_x \to j^{n(K)}_{\Phi_X(x)}$, extends by linearity and continuity to a surjective isometry from $K$ to $\text{Span}(\Phi_X)$. Also, considering the definitions we see that $\text{Span}(\Phi_X)^\perp = \text{Van}(\Phi_X(X))$.

Combining these observations we have that $K$ is the Hilbert space quotient of $DA_n^{n(K)}$ by $\text{Van}(\Phi_X(X))$:

$$K \approx DA_n^{n(K)} \ominus \text{Van}(\Phi_X(X))$$

$$\approx \text{Van}(\Phi_X(X))^\perp$$

$$\approx \text{Span}(\Phi_X)$$

This representation of $K$ as a quotient of $DA_n^{n(K)}$ is the source of the title of this paper.

In this situation it is natural to wonder about the optimal value of $n(K)$ for given $K$. The author learned of this question a few years ago in discussions with Ken Davidson and Orr Shalit, and the results here have their origin in those conversations. These and similar questions have also been considered by John McCarthy and Orr Shalit [McSh].

3. A Reformulation Using the Metric $\delta$

We will recast this Hilbert space question as one about isometric mappings between metric spaces.

Suppose $K$ is a RKHS of functions on $X$ with reproducing kernels $\{k_x : x \in X\}$ and normalized reproducing kernels $\{\tilde{k}_x\}$. Define, for all $x, w \in X$

$$\delta(x, w) = \delta_K(x, w) = \sqrt{1 - \left| \left\langle \tilde{k}_x, \tilde{k}_w \right\rangle \right|^2}.$$

For any $x \in X$ let $P_x$ be the Hilbert space projection of $K$ onto the span of $k_x$.

PROPOSITION 3.1 (Coburn [Cob]). $\delta(z, w) = \|P_z - P_w\|$. In particular $\delta$ is a metric on $X$. 

Proof. See [Cob] or [ARSW]. □

This metric will be our main tool. If \( K = D \) the formula for \( \delta_D \) does not simplify algebraically. On the other hand, there are informative algebraic rewrites of the formula for \( \delta_{DA_n} \).

We begin with the case \( n = 1 \). Note that \( DA_1 \) is the classical Hardy space of the disk. Using the definitions we find that for \( z, w \in B^1 = D \),

\[
\delta_{DA_1}(z, w) = \sqrt{1 - \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{z}w|^2}}.
\]

When we use the fundamental identity, for points \( x, w \in B^1 = D \)

\[
1 - \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{z}w|^2} = \frac{|z - w|}{1 - \bar{z}w},
\]

we find that \( \delta_{H^2} = \rho_1 \), the pseudohyperbolic metric on the disk;

\[
\rho_1(z, w) = \frac{|z - w|}{1 - \bar{z}w}.
\]

That metric is characterized by the fact that for any \( z \in D \), \( \rho_1(0, z) = |z| \) together with the fact that \( \rho_1 \) is invariant under holomorphic automorphisms of the disk.

For general \( n \) we have something very similar. From the definitions we see that for \( z, w \in B^n \),

\[
\delta_{DA_n}(z, w) = \sqrt{1 - \frac{\langle \hat{z}j_z, \hat{w}j_w \rangle^2}{|1 - \bar{z}w|^2}} = \sqrt{1 - \frac{(1 - \|z\|^2)(1 - \|w\|^2)}{|1 - \bar{z}w|^2}}.
\]

Although it is less well known, there is also a pseudohyperbolic metric \( \rho_n \) on \( B^n \). For our purposes a good reference on that metric is [DuWe]. There is an identity for simplifying the expression for \( \delta_{DA_n} \), similar to but more complicated than (3.2). With it one finds that \( \delta_{DA_n} = \rho_n \). In analogy with \( n = 1 \), \( \rho_n \) is characterized by knowing that for any \( z \in B^n \)

\[
\rho_n(0, z) = \|z\|
\]

(3.3) and that \( \rho_n \) is invariant under holomorphic automorphisms of the ball.

Although we will not use this fact, we note in passing that the metric space \((B^n, \delta_{DA_n}) = (B^n, \rho_n)\) is a standard model for complex hyperbolic geometry, [Gol].

Suppose now that Question 1 has a positive answer, and let \( \Phi, \lambda \) be the objects guaranteed by that answer. Using (1.1) and the previous discussion of the \( \delta \)'s and \( \rho \)'s, we would have, for all \( z, w \in D \),

\[
\delta_D(z, w) = \delta_{DA_n}(\Phi(z), \Phi(w)) = \rho_n(\Phi(z), \Phi(w))
\]

(The factors of \( \lambda \) all cancel.) Hence, to get a negative answer to Question 1 it suffices to get a negative answer to the following question:

**Question 2:** Is there a finite \( n \) for which there is an isometric mapping \( \Phi \) of the metric space \((D, \delta_D)\) into the metric space \((B^n, \delta_{DA_n})\)?

We will now give a negative answer to that question.
Theorem 3.2. Question 2, and hence also Question 1, have negative answers.

4. Preliminary Estimates

We want to estimate $\delta_D(0,z)$ for $z$ near the boundary. The situation is obviously rotation invariant so without loss of generality we suppose $z$ is real and positive. For convenience we write $z = 1 - \sigma$ where $\sigma$ is defined by $2\sigma - \sigma^2 = e^{-K}$ for some positive $K$. Thus $1 - z^2 = e^{-K}$. We then have, with $K$ large

$$1 - \delta^2_D(0,z) = 1 - \log(1 - z^2) = 1 - \frac{1}{K},$$

(4.1)

$$\delta_D(0,z) = \sqrt{1 - \frac{1}{K}}.$$ 

Now we estimate the $\delta_D$ distance that $z$ is moved by a rotation through the small angle $\sigma$; that is, we want to estimate $\delta_D(z, ze^{i\sigma})$. From the definitions we have

$$1 - \delta^2_D(z, ze^{i\sigma}) = \left| \frac{\log(1 - z^2 e^{i\sigma})}{\log(1 - z^2)} \right|^2.$$

We have $z^2 = 1 - 2\sigma + \sigma^2$. Using the Taylor series for $\log(1 - x)$ and for $\exp x$ we find

$$1 - \delta^2_D(z, ze^{i\sigma}) = \frac{\log[(2 - i)\sigma + (2i - \frac{1}{2})\sigma^2 + O(\sigma^3)]}{\log e^{-K}}.$$

$$= \frac{\log(2 - i) + \log(1 + O(\sigma))}{-K}.$$

We now use the estimate $\log(1 + x) = A + iB$ with $A, B$ real and $A > 0$. Hence, with $O(1)$ and $O(\sigma)$ denoting real quantities, we have

$$1 - \delta^2_D(z, ze^{i\sigma}) = \left| \frac{-K + \log(1 - i/2) + O(\sigma)}{-K} \right|^2.$$

Write $\log(1 - i/2) = A + iB$ with $A, B$ real and $A > 0$. Hence, with $O(1)$ and $O(\sigma)$ denoting real quantities, we have

$$1 - \delta^2_D(z, ze^{i\sigma}) = \left| 1 - \frac{A + O(\sigma)}{K} + i\frac{O(1)}{K} \right|^2.$$

(4.2)

$$= 1 - \frac{2A + O(1)}{K} + \frac{O(1)}{K^2}.$$

The invariant (Poincare-Bergman, hyperbolic) volume of a pseuohyperbolic ball of radius $r$ is a function of the radius only, not the center. If the center is selected to be the origin then one can compute the volume explicitly. We record the formula from [DuWe] (??), if the radius is $r$ then the volume is

$$V(r) = \frac{r^{2n}}{(1 - r^2)^n}. $$

(4.3)

In particular

$$V(r) = r^{2n} + O(r^{2n+1}) \quad \text{as} \quad r \to 0$$

$$= 2^n(1 - r)^{-n} + O((1 - r)^{-n+1}) \quad \text{as} \quad r \to 1.$$
5. Proof of the Theorem

Proof. Suppose the answer to Question 2 is positive. Select and fix a finite $n$ and a map $\Phi$ whose existences are insured by that answer.

Fix a large positive $K$ with the property that, with $\sigma$ defined as above, $N = 2\pi/\sigma$ is an integer.

Consider the circle centered at the origin and with radius $1 - \sigma$. On that circle select $N$ equally spaced points $\{z_i\}$. Now consider the image points $\{\Phi(z_i)\}$. Because $\Phi$ is an isometry, and noting (4.1), we have

$$\delta_{DA}(0, \Phi(z_i)) = \delta_D(0, z_i) = \sqrt{1 - \frac{1}{K}}$$

Hence all of the $\{\Phi(z_i)\}$ lie on the sphere $S$ in $\mathbb{B}^n$ centered at the origin and with $\delta_{DA}$ radius $\sqrt{1 - 1/K}$. Also, again using the isometry property, and now noting (4.2), we have, for $z_i \neq z_j$, and for some $B > 0$

$$\delta_{DA}(\Phi(z_i), \Phi(z_j)) = \delta_D(z_i, z_j) > \sqrt{B/K}.$$ 

Hence if we pick and fix a small $C > 0$, then $\delta_{DA}$ balls $\{B_i\}$ centered at the points $\{\Phi(z_i)\}$ and having radius $\sqrt{C/K}$ will be disjoint. These balls have centers on $S$, the boundary of the ball $B_{\delta_{DA}}(0, \sqrt{1 - 1/K})$, and hence certainly do not lie inside that ball. However they do lie inside a slightly large concentric ball whose radius we now estimate. The metric $\delta_{DA}$ satisfies a strengthened version of the triangle inequality. [DuWe] equ. (**)]. Hence if $\zeta$ is inside one of the $B_i$ then, recalling that we are interested in large $K$, we can make the following estimates; the first line is the strengthened triangle inequality for $\delta_{DA}$.

$$\delta_{DA}(0, \zeta) \leq \frac{\delta_{DA}(0, \Phi(z_i)) + \delta_{DA}(\Phi(z_i), \zeta)}{1 + \delta_{DA}(0, \Phi(z_i))\delta_{DA}(\Phi(z_i), \zeta)}$$

$$\leq \frac{1}{1 + \sqrt{1 - \frac{1}{K}}}$$

$$= 1 - \left(\frac{1}{2} + C^2\right) \frac{1}{K} + O\left(\frac{1}{K^{3/2}}\right)$$

$$\leq 1 - \frac{1}{3K}.$$ 

Thus we have $N = 2\pi/\sigma$ small balls inside $B_{\delta_{DA}}(0, 1 - 1/(3K))$. By our estimates for their radius and for the distance between their centers we see that the small balls are disjoint. Finally, we have an estimate for the number of them; if $K$ is large then $\sigma < 2\pi e^{-K}$ and hence $N > e^K$. However, by comparing volumes, we see that this combination of estimates is impossible.

From (4.4) we find that, with $A$ and $B$ positive constants that are independent of $n$, and also independent of $K$ if $K$ is large, we have:

$$N = \text{number of small balls} \geq e^K$$

$$V_S = \text{volume of each small ball} \geq \left(\frac{A}{K}\right)^n$$

$$V_L = \text{volume of large ball} \leq (BK)^n.$$
We must have \( NV_S \leq V_L \) for all \( K \), no matter how large; but the previous estimates show this is impossible, no matter what the values of \( A, B \).

6. Final Remarks

Although the result just proved and various related results are proved more directly in [McSh], this use of \( \delta \) provides a different insight into what is going on.

It is not clear how general the argument is. It does not seem to apply directly to the Hilbert spaces \( D_\alpha \), \( 0 < \alpha < 1 \), which are defined by the kernel functions \((1 - \bar{z}w)^{-\alpha}\). Perhaps this is not surprising. The space \( D \) is formally a limiting case of these spaces as \( \alpha \to 0 \), However it is known that the metric \( \delta_D \) is fundamentally different from \( \delta_{D_\alpha} \), \( 0 < \alpha < 1 \). That difference is discussed in [Roc]. The theorem is trivially false for \( \alpha = 1 \).

On the other hand the argument seems to give a similar result for the spaces \( H_n \), the HSRKs of functions on the disk defined by the reproducing kernels

\[
h^{(n)}_z(w) = \left( \frac{1}{zw} \log \frac{1}{1 - \bar{z}w} \right)^n, \quad zw \neq 0,
= 1, \quad zw = 0
\]

for \( n = 2, 3, \ldots \). Those spaces are studied in [AMPRS].

It seems plausible that there are local, or even infinitesimal, versions of this argument. Such a result might say that mapping a small \((\mathbb{B}^1, \delta_D)\) neighborhood of \( z \) into \((\mathbb{B}^n, \delta_{D\alpha})\), even approximately isometrically, is increasingly difficult, and eventually impossible, as \( z \) approaches the boundary.

An infinitesimal version might involve a curvature obstacle. One can pass to the Riemannian metric which is the infinitesimal version of the \( \delta_s \). Some discussion of this is in [ARSW] and the references there. Starting from the pseudohyperbolic metric on the ball this produces the classical Bergman-Poincare metric, the sectional curvatures of which are always between two negative constants. There are formulas for the analogous curvatures related to a general metric \( \delta_K \) on the disk but the formulas are daunting. If \( k(z, \bar{z}) \) is the kernel function then the Riemannian metric is \( \alpha \, |dz| \) with \( \alpha^2 = \Delta \log k \). The curvature is then \( \kappa = (-\Delta \log \alpha) / \alpha^2 \).

Finally, it would be interesting to recast these ideas in purely Hilbert space terms. The metric \( \delta \) measures the sine of the angle between reproducing kernels. Hence the analogs of the "approximate isometries" on the metric spaces would be linear maps between spans of sets of kernel functions, where the maps would be subject to an appropriate rigidity constraint.

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