2-iterated Sheffer polynomials

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Abstract: In this article, the 2-iterated Sheffer polynomials are introduced by means of generating function and operational representation. Using the theory of Riordan arrays and relations between the Sheffer sequences and Riordan arrays, a determinantal definition for these polynomials is established. The quasi-monomial and other properties of these polynomials are derived. The generating function, determinantal definition, quasi-monomial and other properties for some new members belonging to this family are also considered.

Keywords: 2-iterated Sheffer polynomials; Differential equation; Determinantal definition.

1. Introduction and preliminaries

Sequences of polynomials play a fundamental role in mathematics. One of the most famous classes of polynomial sequences is the class of Sheffer sequences, which contains many important sequences such as those formed by Bernoulli polynomials, Euler polynomials, Abel polynomials, Hermite polynomials, Laguerre polynomials, etc. and contains the classes of associated sequences and Appell sequences as two subclasses. Roman et. al. in [27,28] studied the Sheffer sequences systematically by the theory of modern umbral calculus.

Roman [26] further developed the theory of umbral calculus and generalized the concept of Sheffer sequences so that more special polynomial sequences are included, such as the sequences related to Gegenbauer polynomials, Chebyshev polynomials and Jacobi polynomials.

Let $K$ be a field of characteristic zero. Let $F$ be the set of all formal power series in the variable $t$ over $K$. Thus an element of $F$ has the form

$$f(t) = \sum_{k=0}^{\infty} a_k t^k,$$

where $a_k \in K$ for all $k \in \mathbb{N} := \{0,1,2,\ldots\}$. The order $O(f(t))$ of a power series $f(t)$ is the smallest integer $k$ for which the coefficient of $t^k$ does not vanish. The series $f(t)$ has a multiplicative inverse, denoted by $f(t)^{-1}$ or $\frac{1}{f(t)}$, if and only if $O(f(t)) = 0$. Then $f(t)$ is called an invertible series. The series $f(t)$ has a compositional inverse, denoted by $\tilde{f}(t)$ and satisfying $f(\tilde{f}(t)) = \tilde{f}(f(t)) = t$, if and only if $O(f(t)) = 1$. Then $f(t)$ is called a delta series.

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Let $f(t)$ be a delta series and $g(t)$ be an invertible series of the following forms:

$$f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!}, \quad f_0 = 0, \quad f_1 \neq 0 \quad (1.2a)$$

and

$$g(t) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!}, \quad g_0 \neq 0 \quad (1.2b)$$

According to Roman [27, p.18 (Theorem 2.3.4)], the polynomial sequence $s_n(x)$ is uniquely determined by two (formal) power series given by equations (1.2a) and (1.2b). The exponential generating function of $s_n(x)$ is then given by

$$\frac{1}{g(f(t))} e^{x(f(t))} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!} \quad (1.3)$$

for all $x \in \mathbb{C}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$. The sequence $s_n(x)$ in equation (1.3) is the Sheffer sequence for the pair $(g(t), f(t))$.

It has been given in [18] that the Sheffer polynomials $s_n(x)$ are ‘quasi-monomial’, for this see [11, 29]. The associated raising and lowering operators are given by

$$\hat{M}_s = \left( x - \frac{g'(\partial_x)}{g(\partial_x)} \right) \frac{1}{f'(\partial_x)} \quad (1.4a)$$

and

$$\hat{P}_s = f(\partial_x), \quad (1.4b)$$

respectively, where $\partial_x := \frac{\partial}{\partial x}$.

Also, for an invertible series $g(t)$ and a delta series $f(t)$, the sequence $(s_n(x))_{n \in \mathbb{N}}$ is Sheffer for the pair $(g(t), f(t))$ if and only if

$$\langle g(t)f(t)^k|s_n(x) \rangle = c_n \delta_{n,k}, \quad (1.5)$$

for all $n, k \geq 0$, where $\delta_{n,k}$ is the kronecker delta. Particularly, the Sheffer sequence for $(1, f(t))$ is called the associated sequence for $f(t)$ defined by the generating function of the form:

$$e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} \tilde{s}_n(x) \frac{t^n}{n!} \quad (1.6)$$

and the Sheffer sequence for $(g(t), t)$ is called the Appell sequence [3] for $g(t)$ defined by the generating function of the form [27]:

$$\frac{1}{g(t)} e^{xt} = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!} \quad (1.7)$$

Now, we recall some preliminaries from [32].
Let $P$ be the algebra of polynomials in the variable $x$ over $K$ and $P^*$ be the dual vector space of all functionals on $P$. Let $(c_n)_{n \in \mathbb{N}}$ be a fixed sequence of nonzero constants. Then for each $f(t)$ in $\mathcal{F}$, we define a linear functional $f(t)$ in $P^*$ is defined by

$$\langle f(t) | x^n \rangle = c_n a_n$$

and define a linear operator $f(t)$ on $P$ is defined by

$$f(t)x^n = \sum_{k=0}^{n} c_n a_k x^{n-k}.$$  \hspace{1cm}(1.9)

Now, we give the concept of Riordan arrays, which was introduced by Shapiro et. al. \cite{23} and further studied by many authors, for example see \cite{3,8,15-17,24,25}.

For an invertible series $g(t)$ and a delta series $f(t)$, a generalized Riordan array with respect to the sequence $(c_n)_{n \in \mathbb{N}}$ is a pair $(g(t), f(t))$, which is an infinite, lower triangular array $(a_{n,k})_{0 \leq k \leq n < \infty}$ according to the following rule:

$$a_{n,k} = \left[ t^n \over c_n \right] g(t) (f(t))^k / c_k,$$ \hspace{1cm}(1.10)

where the functions $g(t)(f(t))^k / c_k$ are called the column generating functions of the Riordan array. Particularly, the classical Riordan arrays corresponds to the case $c_n = 1$ and the exponential Riordan arrays corresponds to the case of $c_n = n!$. One of the most important applications of the theory of the Riordan arrays is to deal with summations of the form $\sum_{k=0}^{n} a_{n,k} h_k$ \cite{24,25}.

Also, for any fixed sequence $(c_n)_{n \in \mathbb{N}}$, the set of all Riordan arrays $(g(t), f(t))$ is a group under matrix multiplication and is called a Riordan group with respect to $(c_n)_{n \in \mathbb{N}}$. The identity of this group is $(1, t)$ and the inverse of the array $(g(t), f(t))$ is $(1/g(\bar{f}(t)), \bar{f}(t))$.

Costabile et. al. in \cite{9} proposed a determinantal definition for the classical Bernoulli polynomials. Further, Costabile and Longo in \cite{10} introduced the determinantal definition for the Appell sequences. Later on, by using the theory of Riordan arrays and relation between the Sheffer sequences and Riordan arrays, the determinantal definition of Appell sequences is extended to Sheffer sequences \cite{32}. The relations between the Sheffer sequences and Riordan arrays for the case of classical Sheffer sequences and classical Riordan arrays are given in \cite{16} and for the case of generalized Sheffer sequences and generalized Riordan arrays are given in \cite{15,31}.

Let $(s_n(x))_{n \in \mathbb{N}}$ be Sheffer for $(g(t), f(t))$ and suppose

$$x^n = \sum_{k=0}^{n} a_{n,k} s_k(x),$$

then $a_{n,k}$ is the $(n, k)$ entry of the Riordan array $(g(t), f(t))$ \cite{31}. 

\hspace{1cm}3
In view of above fact, the following determinantal definition for the Sheffer sequences holds true \[32\]:

Let \((s_n(x))_{n \in \mathbb{N}}\) be Sheffer for \((g(t), f(t))\), then we have

\[
\begin{align*}
  s_0(x) &= \frac{1}{a_{0,0}}, \\
  s_n(x) &= \frac{(-1)^n}{a_{0,0}a_{1,1}...a_{n,n}} \det \begin{pmatrix}
  1 & x & x^2 & \cdots & x^{n-1} & x^n \\
  a_{0,0} & a_{1,0} & a_{2,0} & \cdots & a_{n-1,0} & a_{n,0} \\
  0 & a_{1,1} & a_{2,1} & \cdots & a_{n-1,1} & a_{n,1} \\
  0 & 0 & a_{2,2} & \cdots & a_{n-1,2} & a_{n,2} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n,n-1}
\end{pmatrix},
\end{align*}
\]

\[
= \frac{(-1)^n}{a_{0,0}a_{1,1}...a_{n,n}} \det \left( X_{n+1} \right)_{S_{n \times (n+1)}},
\]

where \(X_{n+1} = (1, x, x^2, \ldots, x^n)\), \(S_{n \times (n+1)} = (a_{j-1,i-1})_{1 \leq i \leq n, 1 \leq j \leq n+1}\) and \(a_{n,k}\) is the \((n, k)\) entry of the Riordan array \((g(t), f(t))\).

Also, let \((s_n(x))_{n \in \mathbb{N}}\) be the sequence of polynomials defined by equations (1.12) and (1.13), where \(a_{n,k}\) is the \((n, k)\) entry of the Riordan array \((g(t), f(t))\), then

\[
s_n(x) = \sum_{k=0}^{n} b_{n,k} x^k,
\]

where \(b_{n,k}\) is the \((n, k)\) entry of the Riordan array \((1/g(f(t)), f(t))\) and \((s_n(x))_{n \in \mathbb{N}}\) be Sheffer for \((g(t), f(t))\) \[31\].

Motivated by the theory of Riordan arrays and relation between the Sheffer sequences and Riordan array, in this paper, a new family of the 2-iterated Sheffer polynomials is introduced by means of generating function. The determinantal definition of the 2-iterated Sheffer polynomials is established by using the relation between Sheffer sequences and Riordan array. The quasi-monomial properties of these polynomials are derived. The 2-iterated associated Sheffer polynomials are deduced and their properties are considered. Examples of some members belonging to these families are given.

2. 2-iterated Sheffer polynomials

The 2-iterated Sheffer polynomials (2ISP in the following) are introduced by means of generating function. Further, a determinantal definition of the 2ISP is given.
In order to obtain the generating function of the 2ISP, we prove the following result:

**Theorem 2.1.** The 2ISP are defined by the following generating function:

\[
\frac{1}{g_1(f_1(t))} \frac{1}{g_2(f_2(f_1(t)))} \exp(x \tilde{f}_2(\tilde{f}_1(t))) = \sum_{n=0}^{\infty} s_n^{[2]}(x) \frac{t^n}{n!}.
\]  

(2.1)

**Proof.** Let \( s_n^{(1)}(x) \) and \( s_n^{(2)}(x) \) be Sheffer for \((g_1(t), f_1(t))\) and \((g_2(t), f_2(t))\), respectively be two different polynomials defined by the generating functions of the forms:

\[
\frac{1}{g_1(f_1(t))} \exp(x \tilde{f}_1(t))) = \sum_{n=0}^{\infty} s_n^{(1)}(x) \frac{t^n}{n!},
\]

(2.2a)

and

\[
\frac{1}{g_2(f_2(t))} \exp(x \tilde{f}_2(t))) = \sum_{n=0}^{\infty} s_n^{(2)}(x) \frac{t^n}{n!},
\]

(2.2b)

respectively.

Expanding the exponential function and then replacing the powers \( x^0, x^1, \ldots, x^n \) by the polynomials \( s_0^{(2)}(x), s_1^{(2)}(x), \ldots, s_n^{(2)}(x) \), respectively in both sides of equation (2.2a), so that we have

\[
\frac{1}{g_1(f_1(t))} \left[ 1 + s_1^{(2)}(x) \frac{\tilde{f}_1(t)}{1!} + \ldots + s_n^{(2)}(x) \frac{\tilde{f}_1(t)^n}{n!} \right] = \sum_{n=0}^{\infty} s_n^{(1)}(s_1^{(2)}(x)) \frac{t^n}{n!}.
\]

(2.3)

Using equation (2.2b) with \( t \) replaced by \( \tilde{f}_1(t) \) in the l.h.s. and denoting the resultant 2ISP in the r.h.s. of equation (2.3) by

\[
s_n^{[2]}(x) = s_n^{(1)}(s_1^{(2)}(x)),
\]

(2.4)

we get assertion (2.1).

**Remark 2.1.** We remark that equation (2.4) is the operational correspondence between the 2ISP \( s_n^{[2]}(x) \) and Sheffer polynomials \( s_n^{(1)}(x) \).

**Remark 2.2.** We know that the Sheffer sequence for \((g(t), t)\) becomes the Appell sequence \( A_n(x) \). Therefore, taking \( f_1(t) = f_2(t) = t \) which gives \( \frac{1}{g_1(f_1(t))} = \frac{1}{g_1(t)} \) and \( \frac{1}{g_2(f_2(f_1(t)))} = \frac{1}{g_2(t)} \) in equation (2.1) yields the generating function for the 2-iterated Appell polynomials (2IAP) \( A_n^{[2]}(x) \) [19].

**Remark 2.3.** We know that the Sheffer sequence for \((1, f(t))\) becomes the associated Sheffer sequence \( \tilde{s}_n(x) \). Therefore, taking \( g_1(t) = g_2(t) = 1 \) which gives \( \frac{1}{g_1(f_1(t))} = \frac{1}{g_2(f_2(f_1(t)))} = 1 \) in equation (2.1) yields the following consequence of Theorem 2.1:
Corollary 2.1. The 2-iterated associated Sheffer polynomials \( (2\text{IAS}) \) \( \tilde{s}_{n}^{[2]}(x) \) are defined by the following generating function:

\[
\exp(x\tilde{f}_2(\tilde{f}_1(t)))) = \sum_{n=0}^{\infty} \tilde{s}_{n}^{[2]}(x) \frac{t^n}{n!}. \tag{2.5}
\]

Remark 2.4. The following operational correspondence between the 2IAS \( \tilde{s}_{n}^{[2]}(x) \) and associated Sheffer sequences \( \tilde{s}_{n}(x) \) holds:

\[
\tilde{s}_{n}^{[2]}(x) = \tilde{s}_{n}^{(1)}(\tilde{s}_{n}^{(2)}(x)). \tag{2.6}
\]

It is shown in [32] that for \( s_{n}^{(1)}(x) \) and \( s_{n}^{(2)}(x) \) be Sheffer for \( (g_1(t), f_1(t)) \) and \( (g_2(t), f_2(t)) \), respectively be two different Sheffer sequences defined by

\[
s_{n}^{(1)}(x) = \sum_{k=0}^{n} b_{n,k}x^k \tag{2.7a}
\]

and

\[
s_{n}^{(2)}(x) = \sum_{k=0}^{n} d_{n,k}x^k, \tag{2.7b}
\]

where \( b_{n,k} \) is the \((n, k)\) entry of the Riordan array \( (1/g_1(\tilde{f}_1(t)), \tilde{f}_1(t)) \) and \( d_{n,k} \) is the \((n, k)\) entry of the Riordan array \( (1/g_2(\tilde{f}_2(t)), \tilde{f}_2(t)) \), then the umbral composition of \( s_{n}^{(1)}(x) \) and \( s_{n}^{(2)}(x) \) is the sequence \( (s_{n}^{(2)}(s_{n}^{(1)}(x)))_{n \in \mathbb{N}} \) defined by

\[
s_{n}^{(2)}(s_{n}^{(1)}(x)) = \sum_{k=0}^{n} d_{n,k}s_{k}(x) = \sum_{k=0}^{n} d_{n,k}\sum_{j=0}^{k} b_{k,j}x^j = \sum_{j=0}^{n} \left( \sum_{k=j}^{n} d_{n,k}b_{k,j} \right)x^j = \sum_{j=0}^{n} p_{n,j}x^j, \tag{2.8}
\]

where \( p_{n,j} \) is the \((n, j)\) entry of the Riordan array \( \left( \frac{1}{g_2(f_2(t))g_1(f_1(f_2(t))), \tilde{f}_1(\tilde{f}_2(t))} \right) \) and \( s_{n}^{(2)}(s_{n}^{(1)}(x)) \) is Sheffer for \( (g_1(t)g_2(f_1(t)), f_2(f_1(t))) \).

Remark 2.5. We remark that, the 2ISP \( s_{n}^{[2]}(x) \) are actually the composition of \( s_{n}^{(1)}(x) \) and \( s_{n}^{(2)}(x) \) and are defined by

\[
s_{n}^{[2]}(x) = \sum_{k=0}^{n} d_{n,k}s_{k}(x), \tag{2.9}
\]

where \( d_{n,k} \) is the \((n, k)\) entry of the Riordan array \( \left( \frac{1}{g_2(f_2(t))g_1(f_1(f_2(t))), \tilde{f}_1(\tilde{f}_2(t))} \right) \).

The series definition (2.9) can also be obtained by replacing the powers \( x^1 \) and \( x^k \) by the polynomials \( s_{n}^{(1)}(x) \) and \( s_{n}^{(2)}(x) \), respectively, in equation (2.7b) and then using equation (2.4) in the l.h.s. of resultant equation.

Now, we derive the determinantal definition for the 2ISP \( s_{n}^{[2]}(x) \). For this we prove the following result:
Theorem 2.2. The 2ISP $s_n^{[2]}(x)$ of degree $n$ are defined by

$$ s_0^{[2]}(x) = \frac{1}{a_{0,0}}, \quad s_n^{[2]}(x) = \frac{(-1)^n}{a_{0,0}a_{1,1} \ldots a_{n,n}} \det \begin{pmatrix} 1 & s_1^{(2)}(x) & s_2^{(2)}(x) & \cdots & s_{n-1}^{(2)}(x) & s_n^{(2)}(x) \\ a_{0,0} & a_{1,0} & a_{2,0} & \cdots & a_{n-1,0} & a_{n,0} \\ 0 & a_{1,1} & a_{2,1} & \cdots & a_{n-1,1} & a_{n,1} \\ 0 & 0 & a_{2,2} & \cdots & a_{n-1,2} & a_{n,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \end{pmatrix}, $$

$$ = \frac{(-1)^n}{a_{0,0}a_{1,1} \ldots a_{n,n}} \det \left( \begin{array}{c} S_{n+1}(x) \\ M_{n \times (n+1)} \end{array} \right), \tag{2.11}$$

where $S_{n+1}(x) = (1, s_1^{(2)}(x), \ldots, s_n^{(2)}(x))$, $M_{n \times (n+1)} = (a_{j-1,i-1})_{1 \leq i \leq n, 1 \leq j \leq n+1}$ and $a_{n,k}$ is the $(n,k)$ entry of the Riordan array $(g_1(t)g_2(f_1(t)), f_2(f_1(t)))$.

Proof. Replacing the power $x^n$ by the polynomial $s_n^{(2)}(x)$ in the l.h.s. and $x$ by $s_1^{(2)}(x)$ in r.h.s. of equation (1.11) and then using operational correspondence (2.4) in the r.h.s. of resultant equation, we find

$$ s_n^{(2)}(x) = \sum_{k=0}^{n} a_{n,k} s_k^{[2]}(x), \tag{2.12} $$

where $a_{n,k}$ is the $(n,k)$ entry of the Riordan array $(g_1(t)g_2(f_1(t)), f_2(f_1(t)))$.

The above identity leads to the following system of infinite equations in the unknown $s_n^{[2]}(x)$ for $n = 0, 1, \ldots$:

$$ \begin{cases} a_{0,0} s_0^{[2]}(x) = 1, \\ a_{1,0} s_0^{[2]}(x) + a_{1,1} s_1^{[2]}(x) = s_1^{(2)}(x), \\ a_{2,0} s_0^{[2]}(x) + a_{2,1} s_1^{[2]}(x) + a_{2,2} s_2^{[2]}(x) = s_2^{(2)}(x), \\ \vdots \\ a_{n,0} s_0^{[2]}(x) + a_{n,1} s_1^{[2]}(x) + a_{n,2} s_2^{[2]}(x) + \ldots + a_{n,n} s_n^{[2]}(x) = s_n^{(2)}(x). \end{cases} \tag{2.13} $$

From first equation of system (2.13), we get assertion (2.10). Also, the special form of system (2.13) (lower triangular) allows us to work out the unknown $s_n^{[2]}(x)$. Operating with the first $n + 1$ equations simply by applying the Cramer’s rule, we have
polynomials obtained are of the form:

\[ n, k \] is then the \( \tilde{\text{(2.6).}} \) assertion \( \tilde{\text{(2.11).}} \)

noting that the determinant of a square matrix is the same as that of its transpose, we obtain

Then, bringing \((n + 1)\)-th column to the first place by \( n \) transpositions of adjacent column and noting that the determinant of a square matrix is the same as that of its transpose, we obtain assertion \( \tilde{\text{(2.11).}} \)

**Remark 2.6.** We know that for \((1, f(t))\), the Sheffer sequences \( s_n(x) \) become the associated sequences \( \tilde{s}_n(x) \), i.e., for \( c_n = n! \) in equation \( \tilde{\text{(1.9),}} \) \( (s_n(x))_{n \in \mathbb{N}} \) is associated to \( f(t) \) and \( a_{n,k} \) is then the \((n, k)\) entry of the Riordan array \((1, f(t))\) and coefficients of the associated Sheffer polynomials obtained are of the form:

\[
\begin{align*}
  a_{n,0} &= \left[ \frac{\tilde{t}^n}{c_n} \right] \left( \frac{f(t)}{c_0} \right)^{0} = \frac{c_n}{c_0} \tilde{t}^n \| 1 = \delta_{n,0}, \\
  \text{In view of equation } \tilde{\text{(2.15),}} \text{ the following determinantal definition of the associated Sheffer sequences } \tilde{s}_n(x) \text{ holds true } \tilde{\text{(2.16):}}
\end{align*}
\]

\[
\tilde{s}_n(x) = \frac{1}{a_{0,0}} \begin{vmatrix} 
  x & x^2 & \cdots & x^{n-1} & x^n \\
  a_{1,1} & a_{2,1} & \cdots & a_{n-1,1} & a_{n,1} \\
  0 & a_{2,2} & \cdots & a_{n-1,2} & a_{n,2} \\
  \vdots & \vdots & \cdots & \vdots & \vdots \\
  0 & \cdots & \cdots & \cdots & a_{n-1,n-1} & a_{n,n-1} \\
\end{vmatrix},
\]

\[
\tilde{s}_n(x) = \frac{(-1)^n}{a_{0,0}a_{1,1} \cdots a_{n,n}} \det \left( \frac{\tilde{S}_n(x)}{\tilde{M}_{(n-1) \times n}} \right),
\]

where \( \tilde{S}_n(x) = (x, \ldots, x^n), \tilde{M}_{(n-1) \times n} = (a_{j,i})_{1 \leq i \leq n-1, 1 \leq j \leq n} \) and \( a_{n,k} \) is the \((n, k)\) entry of the Riordan array \((1, f(t))\).
Using equation (2.15) and replacing $s_n^{(2)}(x)$ by $\tilde{s}_n^{(2)}(x)$ in the r.h.s. of the determinantal definition (2.10) and (2.11) of the 2ISP $s_n^{[2]}(x)$, we obtain the following consequence of Theorem 2.2:

**Corollary 2.2.** The 2IASP $\tilde{s}_n^{[2]}(x)$ of degree $n$ are defined by

$$\tilde{s}_0^{[2]}(x) = \frac{1}{a_{0,0}},$$

$$\tilde{s}_n^{[2]}(x) = \frac{(-1)^n}{a_{0,0}a_{1,1} \cdots a_{n,n}} \begin{vmatrix} s_0^{(2)}(x) & s_1^{(2)}(x) & \cdots & s_{n-1}^{(2)}(x) & \tilde{s}_n^{(2)}(x) \\ a_{1,1} & a_{2,1} & \cdots & a_{n-1,1} & a_{n,1} \\ 0 & a_{2,2} & \cdots & a_{n-1,2} & a_{n,2} \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \end{vmatrix},$$

(2.18)

$$\tilde{s}_1^{[2]}(x) \cdots \tilde{s}_n^{[2]}(x) = \frac{(-1)^n}{a_{0,0}a_{1,1} \cdots a_{n,n}} \det \left( \begin{array}{c} \tilde{S}_n(x) \\ \tilde{M}_{(n-1)\times n} \end{array} \right),$$

(2.19)

where $\tilde{S}_n(x) = (\tilde{s}_1^{(2)}(x), \ldots, \tilde{s}_n^{(2)}(x))$, $\tilde{M}_{(n-1)\times n} = (a_{j,i})_{1 \leq i \leq n-1, 1 \leq j \leq n}$ and $a_{n,k}$ is the $(n, k)$ entry of the Riordan array $(1, f_2(f_1(t)))$.

**Remark 2.7.** We know that for $(g(t), t)$, the Sheffer sequences $s_n(x)$ become Appell sequences $A_n(x)$. The coefficients $a_{n,k}$ in equation (1.13) are then the $(n, k)$ entry of the exponential Riordan array $(g(t), t)$ as follows:

$$a_{n,k} = \left[ \frac{t^n}{n!} \right] g(t) \frac{t^k}{k!} = \frac{n!}{k!} [t^{n-k}] g(t) = \binom{n}{k} g_{n-k}. \quad (2.20)$$

Therefore, using equation (2.20) in the determinantal definition (2.10) and (2.11) of the 2ISP $s_n^{[2]}(x)$ yields the determinantal definition of the 2IAP $A_n^{[2]}(x)$, which is given in [20].

3. Quasi-monomial and other properties

In this section, we frame the 2ISP $s_n^{[2]}(x)$ within the context of monomiality principle and derive certain other properties of the 2ISP $s_n^{[2]}(x)$.

In order to derive the multiplicative and derivative operators for the 2ISP $s_n^{[2]}(x)$, we prove the following result:

**Theorem 3.1.** The 2ISP $s_n^{[2]}(x)$ are quasi-monomial with respect to the following multiplicative and derivative operators:

$$\tilde{M}_s^{[2]} = \left( x - \frac{g_1'(f_2(\partial_x))}{g_1(f_2(\partial_x))} f_1'(f_2(\partial_x)) - \frac{g_2'(\partial_x)}{g_2(\partial_x)} \right) \frac{1}{f_1'(f_2(\partial_x)) f_2'(\partial_x)}$$

(3.1a)
and

\[ \hat{P}_x^{[2]} = f_1(f_2(\partial_x)), \]  

(3.1b)

respectively, where \( \partial_x := \frac{\partial}{\partial x} \).

**Proof.** Differentiating equation (2.1) partially with respect to \( t \), we find

\[
\left( x f_2'(f_2(f_1(t))) f_1'(f_1(t)) - \frac{g'_2(f_2(f_1(t)))}{g_2(f_2(f_1(t)))} f_2'(f_2(f_1(t))) f_1'(f_1(t)) - \frac{g'_1(f_1(t))}{g_1(f_1(t))} f_1'(f_1(t)) \right)
\]

\[
\frac{1}{g_1(f_1(t))} \frac{1}{g_2(f_2(f_1(t)))} \exp(x f_2(f_1(t))) = \sum_{n=0}^{\infty} s_{n+1}^{[2]}(x) \frac{t^n}{n!},
\]

(3.2)

which can also be simplified as

\[
\left( x - \frac{g'_2(f_2(f_1(t)))}{g_2(f_2(f_1(t)))} f_2'(f_2(f_1(t))) f_1'(f_1(t)) \right)
\]

\[
\frac{1}{g_1(f_1(t))} \frac{1}{g_2(f_2(f_1(t)))} \exp(x f_2(f_1(t))) = \sum_{n=0}^{\infty} s_{n+1}^{[2]}(x) \frac{t^n}{n!}.
\]

(3.3)

Since \( g_1(t) \) and \( g_2(t) \) are invertible series of \( t \), therefore \( \frac{g'_1(t)}{g_1(t)} \) and \( \frac{g'_2(t)}{g_2(t)} \) possess power series expansions of \( t \). Thus, in view of the following identity for the 2ISP \( s_n^{[2]}(x) \):

\[
\partial_x \left\{ \frac{1}{g_1(f_1(t))} \frac{1}{g_2(f_2(f_1(t)))} \exp(x f_2(f_1(t))) \right\} = f_2(f_1(t)) \left( \frac{1}{g_1(f_1(t))} \frac{1}{g_2(f_2(f_1(t)))} \exp(x f_2(f_1(t))) \right),
\]

(3.4a)

or, equivalently

\[
f_1(f_2(\partial_x)) \left\{ \frac{1}{g_1(f_1(t))} \frac{1}{g_2(f_2(f_1(t)))} \exp(x f_2(f_1(t))) \right\} = t \left( \frac{1}{g_1(f_1(t))} \frac{1}{g_2(f_2(f_1(t)))} \exp(x f_2(f_1(t))) \right),
\]

(3.4b)

equation (3.3) becomes

\[
\left( x - \frac{g'_1(f_2(\partial_x))}{g_1(f_2(\partial_x))} f_1'(f_2(\partial_x)) - \frac{g'_2(\partial_x)}{g_2(\partial_x)} \right)
\]

\[
\frac{1}{g_1(f_1(t))} \frac{1}{g_2(f_2(f_1(t)))} \exp(x f_2(f_1(t))) = \sum_{n=0}^{\infty} s_{n+1}^{[2]}(x) \frac{t^n}{n!}.
\]

(3.5)

Again, using generating function (2.1) in the l.h.s. of equation (3.5) and then rearranging the summation yields

\[
\sum_{n=0}^{\infty} \left( \left( x - \frac{g'_1(f_2(\partial_x))}{g_1(f_2(\partial_x))} f_1'(f_2(\partial_x)) - \frac{g'_2(\partial_x)}{g_2(\partial_x)} \right) \frac{1}{g_1(f_1(t))} \frac{1}{g_2(f_2(f_1(t)))} \right) \left\{ s_{n+1}^{[2]}(x) \frac{t^n}{n!} \right\} = \sum_{n=0}^{\infty} s_{n+1}^{[2]}(x) \frac{t^n}{n!}.
\]

(3.6)
Equating the coefficients of the same powers of \( t \) in both sides of the above equation, we find

\[
\left( x - \frac{g_1'(f_2(\partial_x))}{f_1'(f_2(\partial_x))} f_1'(f_2(\partial_x)) - \frac{g_2'(\partial_x)}{g_2(\partial_x)} \right) \left( \frac{1}{f_1'(f_2(\partial_x)) f_2'(\partial_x)} \right) \left\{ s_n^{[2]}(x) \right\} = s_{n+1}^{[2]}(x), \tag{3.7}
\]

which in view of monomiality principle equation \( \hat{M}\{s_n(x)\} = s_{n+1}(x) \) for \( s_n^{[2]}(x) \) yields assertion (3.1a).

In order to prove assertion (3.1b), we use generating function (2.1) in both sides of the identity (3.4b), so that we have

\[
f_1(f_2(\partial_x)) \left\{ \sum_{n=0}^{\infty} s_n^{[2]}(x) \frac{t^n}{n!} \right\} = \sum_{n=0}^{\infty} s_n^{[2]}(x) \frac{t^n}{(n-1)!}.
\tag{3.8}
\]

Rearranging the summation in the l.h.s. of equation (3.8) and then equating the coefficients of the same powers of \( t \) in both sides of the resultant equation, we find

\[
f_1(f_2(\partial_x)) \left\{ s_n^{[2]}(x) \right\} = n \ s_n^{[2]}(x), \ n \geq 1,
\tag{3.9}
\]

which in view of monomiality principle equation \( \hat{P}\{s_n(x)\} = n s_{n-1}(x) \) for \( s_n^{[2]}(x) \) yields assertion (3.1b).

**Theorem 3.2.** The 2ISP \( s_n^{[2]}(x) \) satisfy the following differential equation:

\[
\left( \left( x - \frac{g_1'(f_2(\partial_x))}{f_1'(f_2(\partial_x))} f_1'(f_2(\partial_x)) - \frac{g_2'(\partial_x)}{g_2(\partial_x)} \right) \left( \frac{1}{f_1'(f_2(\partial_x)) f_2'(\partial_x)} \right) - n \right) \left\{ s_n^{[2]}(x) \right\} = 0. \tag{3.10}
\]

**Proof.** Using equations (3.1a) and (3.1b) in monomiality principle equation \( \hat{M}\hat{P}\{s_n(x)\} = ns_n(x) \) for \( s_n^{[2]}(x) \), we get assertion (3.10).

**Remark 3.1.** We know that the Sheffer sequence for \((1, f(t))\) becomes the associated Sheffer sequence \( s_n(x) \). Therefore, taking \( g_1(t) = g_2(t) = 1 \) which implies \( g_1'(t) = g_2'(t) = 0 \) in equations (3.1a), (3.1b) and (3.10), we get the following consequence of Theorem 3.1 and Theorem 3.2:

**Corollary 3.1.** The 2IASP \( s_n^{[2]}(x) \) are quasi-monomial with respect to the following multiplicative and derivative operators:

\[
\hat{M}_{s_n^{[2]}} = \frac{x}{f_1'(f_2(\partial_x)) f_2'(\partial_x)}, \tag{3.11a}
\]

and

\[
\hat{P}_{s_n^{[2]}} = f_1(f_2(\partial_x)), \tag{3.11b}
\]

respectively.

**Corollary 3.2.** The 2IASP \( s_n^{[2]}(x) \) satisfy the following differential equation:

\[
\left( \frac{x \ f_1(f_2(\partial_x))}{f_1'(f_2(\partial_x)) f_2'(\partial_x)} - n \right) \overline{s}_n^{[2]}(x) = 0. \tag{3.12}
\]
Remark 3.2. We know that for \((g(t), t)\), the Sheffer sequences \(s_n(x)\) reduce to the Appell sequences \(A_n(x)\). Therefore, taking \(f_1(t) = f_2(t) = t\) which implies \(f'_1(t) = f'_2(t) = 1\) in equations (3.1a), (3.1b) and (3.10) yields the multiplicative and derivative operators and differential equation for the 21AP \(A^{[2]}_n(x)\), which are given in [19].

Now, we prove the conjugate representation of the 2ISP \(s^{[2]}_n(x)\). For this, we prove the following result:

Theorem 3.2. Let \(s^{[2]}_n(x)\) be Sheffer for \((g_1(t)g_2(f_1(t)), f_2(f_1(t)))\), then the following conjugate representation for the 2ISP \(s^{[2]}_n(x)\) holds true:

\[
s^{[2]}_n(x) = \sum_{k=0}^{n} \frac{(g_2(f_2(t))g_1(f_1(f_2(t))))^{-1}(f_1(f_2(t)))^k | s_n(x))}{c_k} s_k(x). \tag{3.13}
\]

Proof. Since \(s^{[2]}_n(x)\) are defined by equation (2.9), where \(d_{n,k}\) is the \((n,k)\) entry of the Riordan array \(\left(\frac{1}{g_2(f_2(t))g_1(f_1(f_2(t)))}, f_1(f_2(t))\right)\). Then, according to the definition of Riordan arrays, we have

\[
d_{n,k} = \left[\frac{t^n}{c_n}\right] \frac{1}{g_2(f_2(t))g_1(f_1(f_2(t)))} \frac{(f_1(f_2(t)))^k}{c_k}. \tag{3.14}
\]

Simplifying the above equation yields

\[
d_{n,k} = \frac{1}{c_k} \left[\frac{t^n}{c_n}\right] (g_2(f_2(t))g_1(f_1(f_2(t))))^{-1}((f_1(f_2(t)))^k | s_n(x)), \tag{3.15}
\]

Now, using equation (3.15) in equation (2.9), we are led to assertion (3.13).

In the next section, examples of some members belonging to the 2ISP and 2IASP are considered.

4. Examples

We recall that the Laguerre polynomials \(L^{(\alpha)}_n(x)\) of order \(\alpha\) form the Sheffer sequence for the pair \(\left(\frac{1}{(1-t)^{\alpha+1}}, \frac{t}{1-t} \right)\) [2][22][27] are defined by the generating function:

\[
\frac{1}{(1-t)^{\alpha+1}} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L^{(\alpha)}_n(x) t^n, \tag{4.1}
\]

which for \(\alpha = 0\) gives the generating function of the Laguerre polynomials \(L_n(x)\) as [2]:

\[
\frac{1}{(1-t)} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L^{(0)}_n(x) t^n. \tag{4.2}
\]
Also, the polynomials of the Gegenbauer case denoted by \((-\lambda_n)^s_n(x)\) considered in [32] form the Sheffer sequence for the pair \(\left(\frac{1}{1+\sqrt{1-t^2}}, \frac{t}{1+\sqrt{1-t^2}}\right)\) and are defined by the generating function of the form:

\[
(1 + t^2)^{\lambda - \lambda_0}(1 - 2xt + t^2)^{-\lambda} = \sum_{n=0}^{\infty} \binom{-\lambda}{n} s_n(x)t^n,
\]

which for \(\lambda_0 = \lambda\) gives the generating function of the Gegenbauer polynomials \(C_n^{(\lambda)}(x)\) as [1, 22, 26]:

\[
(1 - 2xt + t^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x)t^n.
\]

We note that corresponding to each member belonging to the Sheffer family, there exists a new special polynomial belonging to the 2ISP family. Thus, by making suitable choice for the functions \(g(t)\) and \(f(t)\) in equation (2.1), we get the generating function for the corresponding member belonging to the 2ISP family. The other properties of these special polynomials can be obtained from the results derived in previous sections.

We consider the following examples:

**Example 4.1.** Taking \(g_1(t) = g_2(t) = \frac{1}{(1-t)^{\alpha+1}}\) and \(f_1(t) = f_2(t) = \frac{t}{1-t}\) in the l.h.s. of generating function (2.1), we find that the resultant 2-iterated Laguerre polynomials (2ILP) of order \(\alpha\), which may be denoted by \(L^{(\alpha)[2]}_n(x)\) in the r.h.s. are defined by the following generating function:

\[
\frac{1}{(1-t)^{\alpha+1}(1-t)^{-\alpha-1}} \exp(xt) = \sum_{n=0}^{\infty} L^{(\alpha)[2]}_n(x)t^n.
\]

The operational correspondence between the 2ILP of order \(\alpha\) \(L^{(\alpha)[2]}_n(x)\) and Laguerre polynomials of order, \(\alpha\) \(L^{(\alpha)}_n(x)\) is given by:

\[
L^{(\alpha)[2]}(x) = L^{(\alpha,1)}(L^{(\alpha,2)}_1(x)).
\]

The 2ILP of order \(\alpha\), \(L^{(\alpha)[2]}_n(x)\) are quasi-monomial with respect to the following multiplicative and derivative operators:

\[
\hat{M}_{L^{(\alpha)[2]}} = \left( x - (\alpha + 1)(\partial_x - 1)^3 - \frac{(\alpha + 1)}{\partial_x - 1} \right),
\]

\[
\hat{P}_{L^{(\alpha)[2]}} = \partial_x.
\]

The 2ILP of order \(\alpha\), \(L^{(\alpha)[2]}_n(x)\) satisfy the following differential equation:

\[
\left( x\partial_x - (\alpha + 1)(\partial_x - 1)^3 \partial_x - \frac{(\alpha + 1)\partial_x}{\partial_x - 1} - n \right) L^{(\alpha)[2]}_n(x) = 0.
\]
Consider the following series definition for the Laguerre polynomials of order \( \alpha \), \( L_n^{(\alpha)}(x) \) \cite{27}:

\[
L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left( \frac{n + \alpha}{n - k} \right) x^k.
\]  

(4.9)

Replacing the power \( x^k \) by the polynomial \( L_k^{(\alpha)}(x) \) in the r.h.s. and \( x \) by \( L_1^{(\alpha)}(x) \) in the l.h.s. of equation (4.9) and then using equation (4.6) in the l.h.s. of resultant equation yields the following series definition for the 2ILP of order \( l \) of equation (4.9) and then using equation (4.6) in the l.h.s. of resultant equation yields the following series definition for the 2ILP of order \( \alpha \), \( L_n^{(\alpha)[2]}(x) \):

\[
L_n^{(\alpha)[2]}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left( \frac{n + \alpha}{n - k} \right) L_1^{(\alpha)}(x).
\]  

(4.10)

It has been shown in \cite{32} that for \( a_{n,k} = (-1)^k \frac{n!}{k!} (\alpha + n - 1) \) the determinantal definition of the Sheffer polynomials given by equations (1.12) and (1.13) reduces to the determinantal definition of the Laguerre polynomials of order \( \alpha \), \( L_n^{(\alpha)}(x) \).

Therefore, taking \( s_n(x) = L_n^{(\alpha)}(x) \) and \( a_{n,k} = (-1)^k \frac{n!}{k!} (\alpha + n - 1) \) in equations (2.10) and (2.11), we find that the following determinantal definition of the 2ILP of order \( \alpha \), \( L_n^{(\alpha)[2]}(x) \) holds true:

\[
L_0^{(\alpha)[2]}(x) = 1,
\]  

(4.11)

\[
L_n^{(\alpha)[2]}(x) = (-1)^{\frac{n(n+3)}{2}} \det \begin{pmatrix}
1 & L_1^{(\alpha)}(x) & L_2^{(\alpha)}(x) & \cdots & L_{n-1}^{(\alpha)}(x) & L_n^{(\alpha)}(x) \\
1 & \alpha + 1 & (\alpha + 2) & \cdots & (\alpha + n - 1) & (\alpha + n) \\
0 & -1 & -2(\alpha + 2) & \cdots & -(n-1)(\alpha + n-1) & -n(\alpha + n) \\
0 & 0 & 1 & \cdots & \frac{(n-1)(n-2)}{2}(\alpha + n-1) & \frac{n(n-1)}{2}(\alpha + n) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & (-1)^{n-1} & (-1)^{n-1}n(\alpha + n)
\end{pmatrix},
\]

\[
= (-1)^{\frac{n(n+3)}{2}} \det \begin{pmatrix}
S_{n+1}(x) \\
M_{n \times (n+1)}
\end{pmatrix},
\]

(4.12)

where \( S_{n+1}(x) = (1, L_1^{(\alpha)}(x), \ldots, L_n^{(\alpha)}(x)) \), \( M_{n \times (n+1)} = (a_{j-i,1} \leq i \leq n, 1 \leq j \leq n+1) \) and \( L_n^{(\alpha)}(x) (n = 0, 1, \ldots) \) are the Laguerre polynomials of order \( \alpha \).

**Remark 4.1.** We remark that by taking \( \alpha = 0 \) in the results derived above for the 2-iterated Laguerre polynomials (2ILP) of order \( \alpha \), we get the corresponding results for the 2-iterated Laguerre polynomials (2ILP), which may be denoted by \( L_n^{[2]}(x) \).
Example 4.2. Taking \( g_1(t) = g_2(t) = \left( \frac{2}{1 + \sqrt{1 - \tau^2}} \right)^{\lambda_0} \) and \( f_1(t) = f_2(t) = \frac{-t}{1 + \sqrt{1 - \tau^2}} \) in the l.h.s. of generating function (2.1), we find that the resultant 2-iterated polynomials of the Gegenbauer case (2IPoGc), which may be denoted by \((-\lambda_n) s_n^{[2]}(x)\) in the r.h.s. are defined by the following generating function:

\[
\left( \frac{1 + t^2}{1 + 6t^2 + t^4} \right)^{\lambda_0} \exp \left( \frac{4xt(1 + t^2)}{1 + 6t^2 + t^4} \right) = \sum_{n=0}^{\infty} \left( -\lambda_n \right) s_n^{[2]}(x) t^n.
\] (4.13)

The operational correspondence between the 2IPoGc \( s_n^{[2]}(x) \) and polynomials of the Gegenbauer case \( s_n(x) \) is given by:

\[
 s_n^{[2]}(x) = s_n^{(1)}(s_1^{(2)}(x)).
\] (4.14)

The 2IPoGc \((-\lambda_n) s_n^{[2]}(x)\) are quasi-monomial with respect to the following multiplicative and derivative operators:

\[
\hat{M}^{(\alpha)[2]}_s = \left( x - \lambda_0 2^{\lambda_0 - 1} \frac{\partial_x (1 + \sqrt{1 - \partial_x^2})^{\lambda_0 + 2}}{\left( 2(1 - \partial_x^2 + \sqrt{1 - \partial_x^2})^2 \right)} \frac{\lambda_0 \partial_x}{2(1 - \partial_x^2 + \sqrt{1 - \partial_x^2})} \right) \frac{\partial_x (1 + \sqrt{1 - \partial_x^2})^{\lambda_0 + 1}}{\partial_x (1 + \sqrt{1 - \partial_x^2})^{\lambda_0 + 1}}
\]

\[
\times \frac{1}{\sqrt{1 - \partial_x^2}}
\] (4.15)

\[
\hat{P}^{(\alpha)[2]}_s = \frac{-\partial_x}{1 + \sqrt{1 - \partial_x^2} + \sqrt{2(1 - \partial_x^2 + \sqrt{1 - \partial_x^2})}}.
\] (4.16)

The 2IPoGc \((-\lambda_n) s_n^{[2]}(x)\) satisfy the following differential equation:

\[
\left( x + \frac{\lambda_0 2^{\lambda_0 - 1} \partial_x (1 + \sqrt{1 - \partial_x^2})^{\lambda_0 + 2}}{\left( 2(1 - \partial_x^2 + \sqrt{1 - \partial_x^2})^2 \right)} \partial_x (1 + \sqrt{1 - \partial_x^2})^{\lambda_0 + 1} \right) \frac{\lambda_0 \partial_x}{\sqrt{1 - \partial_x^2} (1 + \sqrt{1 - \partial_x^2})} \partial_x (1 + \sqrt{1 - \partial_x^2})^{\lambda_0 + 1}
\]

\[
+ \left( x + \lambda_0 2^{\lambda_0 - 1} \partial_x (1 + \sqrt{1 - \partial_x^2})^{\lambda_0 + 2} \right) \frac{\lambda_0 \partial_x}{\sqrt{1 - \partial_x^2} (1 + \sqrt{1 - \partial_x^2})} \partial_x (1 + \sqrt{1 - \partial_x^2})^{\lambda_0 + 1}
\]

\[
\times \frac{1}{\sqrt{1 - \partial_x^2} - n} s_n^{[2]}(x) = 0.
\] (4.17)

It has been shown in [32] that by taking

\[
 a_{n,k} = \begin{cases} 
 0 & \text{if } n - k \text{ is odd}, \\
 \frac{c_n (-1)^k \lambda_0 + k}{c_n^{(2^k)} \lambda_0 + n} & \text{if } n - k \text{ is even},
 \end{cases}
\] (4.18)

where \( c_n = 1/(\lambda_n) \), the determinantal definition of the Sheffer polynomials given by equations (1.12) and (1.13) reduces to the determinantal definition of the polynomials of Gegenbauer case.
\((-\lambda)^n s_n(x)\).

Therefore, taking \(s_n^{(2)}(x) = s_n x\) and using equation (4.18) in equations (2.10) and (2.11), we find that the following determinantal definition of the 2IPoGc \((-\lambda)^n s_n^{(2)}(x)\) holds true:

\[
\begin{align*}
\frac{\lambda_0}{2\lambda(x+1)} & = 0 \\
0 & = \frac{1}{2} \cdot 0 \\
\vdots & = \vdots \\
0 & = 0 \\
\end{align*}
\]

\[
\frac{\lambda_0(\lambda_0+n-1)!}{2^n (\frac{\lambda}{2})!(\lambda+1)\ldots(\lambda+n-1)(\lambda+\frac{n}{2})!}
\]

\[
\frac{n(\lambda_0+n-1)!}{2^n (\frac{\lambda}{2})!(\lambda+2)\ldots(\lambda+n-1)(\lambda+\frac{n+2}{2})!}
\]

\[
\frac{\lambda_0}{2\lambda(x+1)} = (-1)^{\frac{n(n+3)}{2} - \frac{n(n+1)}{2}} \det \left( \begin{array}{c} \lambda_n \lambda_{n-1} \lambda_n \lambda_{n-1} \lambda_n \lambda_{n-1} \\ 1 \ 0 \ \lambda_{n+1} \ 0 \\ 1 \ 0 \ \lambda_{n+1} \ 0 \\ 0 \ 0 \ \lambda_{n+1} \ 0 \\ . \ . \ . \ . \ \\ 0 \ 0 \ 0 \ \lambda_{n+1} \ 0 \end{array} \right),
\]

where \(S_{n+1}(x) = (1, s_1(x), \ldots, s_n(x))\), \(M_{n\times(n+1)} = (a_{j-1,i-1})_{1\leq i\leq n, 1\leq j\leq n+1}\) and \(s_n(x)\) \((n = 0, 1, \ldots)\) are the polynomials of Gegenbauer case.

**Remark 4.2.** We remark that, by taking \(\lambda_0 = \lambda\) in the results derived above for the 2IPoGc \((-\lambda)^n s_n^{(2)}(x)\) we get the corresponding results for the 2-iterated Gegenbauer polynomials (2IGP) which may be denoted by \(C_n^{(\lambda)[2]}(x)\).

**Remark 4.3.** It is given in [1] that, for \(\lambda = 1\), the Gegenbauer polynomials becomes the Chebyshev polynomials of the second kind \(U_n(x)\), i.e., \(U_n(x) = C_n^{(1)}(x)\) and for \(\lambda = 1/2\), the Gegenbauer polynomials becomes the Legendre polynomials \(P_n(x)\), i.e., \(P_n(x) = C_n^{(1/2)}(x)\). Therefore, taking \(\lambda = 1\) and \(\lambda = 1/2\) in the results of the 2-iterated Gegenbauer polynomials (2IGP) \(C_n^{(\lambda)[2]}(x)\), we obtained the corresponding results of the 2-iterated Chebyshev polynomials of the second kind \(U_n^{(2)}(x)\) and 2-iterated Legendre polynomials \(P_n^{(2)}(x)\).

Now, we proceed to introduce certain members belonging to the 2IASP. We recall that the associated Sheffer family contains the falling factorials \(\left(\frac{z}{n}\right)\) and exponential polynomials \(\phi_n(x)\) as the important members.

The falling factorial \(\left(\frac{z}{n}\right)\), associated to \(f(t) = e^{\alpha t} - 1\) [27], is defined by the generating
function:
\[
\exp(xa^{-1}\log(1 + t)) = \sum_{n=0}^{\infty} \frac{(x^a)}{n} \frac{t^n}{n!}
\]  
(4.21)

and the exponential polynomials \(\phi_n(x)\) associated to \(f(t) = \log(1 + t)\) \([5, 27]\) defined by the generating function:
\[
\exp(x(e^t - 1)) = \sum_{n=0}^{\infty} \phi_n(x) \frac{t^n}{n!}.
\]  
(4.22)

Also, for each member belonging to the associated Sheffer family, there exists a new special polynomial belonging to the 2IASP family. Thus, by making suitable choice for the functions \(f_1(t)\) and \(f_2(t)\) in equation (2.5), we get the generating function for the corresponding member belonging to the 2IASP family. The other properties of these special polynomials can be obtained from the results derived in previous sections.

We consider the following examples:

**Example 4.3.** Taking \(f_1(t) = f_2(t) = e^{at} - 1\) in the l.h.s. of generating function (2.5), we find that the resultant 2-iterated falling factorial (2IFF), denoted by \((\frac{x}{a})^{[2]}_n\) in the r.h.s. are defined by the following generating function:
\[
\exp(xa^{-1}\log(1 + a^{-1}\log(1 + t))) = \sum_{n=0}^{\infty} \frac{(x^a)}{n} \frac{t^n}{n!},
\]  
(4.23)

The operational correspondence between the 2IFF \((\frac{x}{a})^{[2]}_n\) and falling factorial \((\frac{x}{a})_n\) is given by:
\[
(\frac{x}{a})^{[2]}_n = (\frac{x}{a})^{(2)}_n.
\]  
(4.24)

The 2IFF \((\frac{x}{a})^{[2]}_n\) are quasi-monomial with respect to the following multiplicative and derivative operators:
\[
\hat{M}(\frac{x}{a})^{[2]}_n = \frac{x}{ae^{ae^{a\partial_x}}},
\]  
(4.25a)

\[
\hat{P}(\frac{x}{a})^{[2]}_n = e^{a(e^{a\partial_x} - 1)} - 1.
\]  
(4.25b)

The 2IFF \((\frac{x}{a})^{[2]}_n\) satisfy the following differential equation:
\[
\left(\frac{x}{ae^{ae^{a\partial_x}}ae^{a\partial_x}} - n\right)(\frac{x}{a})^{[2]}_n = 0.
\]  
(4.26)

Consider the following series definition for the falling factorial \((\frac{x}{a})^{[2]}_n\) \([27]\):
\[
(\frac{x}{a})_n = \sum_{k=0}^{n} a^n s(n, k)x^k,
\]  
(4.27)
where $s(n, k)$ are the Stirling numbers of the first kind defined by $s(n, k) = \binom{n}{k}$, $k, n \in \mathbb{N}$, $1 \leq k < n$.

Replacing the power $x^k$ by the polynomial $\left( \frac{x}{a} \right)_k$ in the r.h.s. and $x$ by $\left( \frac{x}{a} \right)_1$ in the l.h.s. of equation (4.27) and then using equation (4.24) in the l.h.s. of resultant equation yields the following series definition for the 2IFF $\left( \frac{x}{a} \right)_n^{[2]}$:

$$\left( \frac{x}{a} \right)_n^{[2]} = \sum_{k=0}^{n} a^n s(n, k) \left( \frac{x}{a} \right)_k.$$ (4.28)

It has been shown in [32], that for $a_{n,k} = a^n S(n, k)$, where $S(n, k)$ are the Stirling numbers of the second kind in equations (2.16) and (2.17), the determinantal definition of the associated Sheffer polynomials reduces to the determinantal definition of falling factorials $\left( \frac{x}{a} \right)_n^{[2]}$.

Therefore, taking $\widetilde{s}_n(x) = \left( \frac{x}{a} \right)_n$ and $a_{n,k} = a^n S(n, k)$ in the r.h.s. of equations (2.18) and (2.19), we find that the 2IFF $\left( \frac{x}{a} \right)_n^{[2]}$ are defined by the following determinantal definition:

$$\left( \frac{x}{a} \right)_n^{[2]} = 1,$$ (4.29)

$$\left( \frac{x}{a} \right)_n^{[2]} = \frac{(-1)^{n+1}}{a^{\binom{n+1}{2}}} \det \begin{pmatrix} 0 & a^2 S(2, 1) & \cdots & a^{n-1} S(n-1, 2) & a^n S(n, 2) \\ aS(1, 1) & a^2 S(2, 1) & \cdots & a^{n-1} S(n-1, 1) & a^n S(n, 1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a^{n-1} S(n-1, n-1) & a^n S(n, n-1) \end{pmatrix},$$ (4.30)

where $\widetilde{S}_n(x) = \left( \frac{x}{a} \right)_1, \ldots, \left( \frac{x}{a} \right)_n$, $\widetilde{M}_{(n-1) \times n} = (a_{j,i})_{1 \leq i \leq n-1, 1 \leq j \leq n}$ and $\left( \frac{x}{a} \right)_n (n = 0, 1, \ldots)$ are the falling factorials.

**Example 4.4.** Taking $f_1(t) = f_2(t) = \log(1 + t)$ in the l.h.s. of generating function (2.5), we find that the resultant 2-iterated exponential polynomials (2IEP) denoted by $\phi_n^{[2]}(x)$ in the r.h.s. are defined by the following generating function:

$$e^{(e^t-1)} - 1 = \sum_{n=0}^{\infty} \phi_n^{[2]}(x) \frac{t^n}{n!}.$$ (4.31)
The operational correspondence between the 2IEP $\phi_n^{[2]}(x)$ and exponential polynomials $\phi_n(x)$ is given by:

$$\phi_n^{[2]}(x) = \phi_n^{(1)}(\phi_1^{(2)}(x)).$$ (4.32)

The 2IEP $\phi_n^{[2]}(x)$ are quasi-monomial with respect to the following multiplicative and derivative operators:

$$\hat{M}_{\phi^{[2]}} = \frac{x}{(1 + \log(1 + t))(1 + t)},$$ (4.33a)

$$\hat{P}_{\phi^{[2]}} = \log(1 + \log(1 + t)).$$ (4.33b)

The 2IEP $\phi_n^{[2]}(x)$ satisfy the following differential equation:

$$\left(\frac{x \log(1 + \log(1 + t))}{(1 + \log(1 + t))(1 + t)} - n\right) \phi_n^{[2]}(x) = 0.$$ (4.34)

Consider the following series definition for the exponential polynomials $\phi_n(x)$ [27]:

$$\phi_n(x) = \sum_{k=0}^{n} S(n,k)x^k,$$ (4.35)

where $S(n,k)$ are the Stirling numbers of the second kind defined by

$$S(n,k) = \left\{ \begin{array}{c} n \\ k \end{array} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n.$$ (4.36)

Replacing the power $x^k$ by the polynomial $\phi_k(x)$ in the r.h.s. and $x$ by $\phi_1(x)$ in the l.h.s. of equation (4.30) and then using equation (4.35) in the l.h.s. of resultant equation yields the following series definition for the 2IEP $\phi_n^{[2]}(x)$:

$$\phi_n^{[2]}(x) = \sum_{k=0}^{n} S(n,k)\phi_k(x),$$ (4.37)

It has been shown in [32], that for $a_{n,k} = s(n,k)$, where $s(n,k)$ are the Stirling numbers of the first kind in equations (2.16) and (2.17), the determinantal definition of the associated Sheffer polynomials reduces to the determinantal definition of exponential polynomials $\phi_n(x)$.

Therefore, taking $\tilde{s}_n(x) = \phi_n(x)$ and $a_{n,k} = s(n,k)$ in equations (2.18) and (2.19), we find that the 2-iterated exponential polynomials (2IEP) $\phi_n^{[2]}(x)$ are defined by the following determinantal definition:
\[ \phi_0^{[2]}(x) = 1, \quad \phi_1(x) \quad \phi_2(x) \quad \cdots \quad \phi_{n-1}(x) \quad \phi_n(x) \]
\[ s(1, 1) \quad s(2, 1) \quad \cdots \quad s(n - 1, 1) \quad s(n, 1) \]
\[ 0 \quad s(2, 2) \quad \cdots \quad s(n - 1, 2) \quad s(n, 2) \]
\[ \vdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \]
\[ 0 \quad 0 \quad \cdots \quad s(n - 1, n - 1) \quad s(n, n - 1) \]
\[ = (-1)^{n+1} \det \left( \frac{\mathcal{S}_n(x)}{\mathcal{M}_{(n-1)\times n}} \right), \quad (437) \]

where \( \mathcal{S}_n(x) = (\phi_1(x), \ldots, \phi_n(x)) \), \( \mathcal{M}_{(n-1)\times n} = (a_{j,i})_{1\leq i\leq n-1, 1\leq j\leq n} \) and \( \phi_n(x) \ (n = 0, 1, \ldots) \) are the exponential polynomials.

Similar results can also be proved for certain other known members belonging to the Sheffer family. These members are listed in Table 1:

**Table 1. Some known Sheffer polynomials.**

| S.No. | \( A(t); H(t) \) | \( g(t); f(t) \) | Generating Functions | Polynomials |
|-------|-------------------|------------------|----------------------|-------------|
| I.    | \( e^{-t^2}; 2t \) | \( e^{\frac{1}{2}t^2} ; \frac{1}{2}t \) | \( e^{2xt - t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \) | Hermite polynomials \( H_n(x) \) \([2]\) |
| II.   | \( e^{-\nu t}; \nu t \) | \( e^{\frac{1}{\nu}t^2} ; \frac{1}{\nu}t \) | \( \exp(\nu xt - t^2) = \sum_{n=0}^{\infty} H_{n,\nu}(x) \frac{t^n}{n!} \) | Generalized Hermite polynomials \( H_{n,\nu}(x) \) \([13]\) |
| III.  | \( (1-t)^{-1}; 1 \) | \( (1-t)^{-1} \) | \( \frac{1}{1 - t} \exp \left( \frac{t}{1-t} \right) = \sum_{n=0}^{\infty} L_n(x) \frac{t^n}{n!} \) | Laguerre polynomials \( L_n(x) \) \([2]\) |
| IV.   | \( \frac{1}{1 - \ln \left( \frac{e^{x} - 1}{x} \right)} \) | \( \frac{1}{1 - \ln \left( \frac{e^{x} - 1}{x} \right)} \) | \( \frac{1}{1 - \ln \left( \frac{e^{x} - 1}{x} \right)} \exp \left( \frac{a(x^2 - 1)}{a(x^2 - 1)} \right) \) | Poisson-Charlier polynomials \( a_n^{(0)}(x) \) \([2, 14]\) |
| V.    | \( e^{x^2}; 1 - e^x \) | \( \frac{1}{1 - t} \exp \left( \frac{t}{1-\ln(1-t)} \right) = \sum_{n=0}^{\infty} \frac{a(x)}{n!} \frac{x^n}{n!} \) | \( \frac{1}{1 - \ln \left( \frac{e^{x} - 1}{x} \right)} \exp \left( \frac{a(x^2 - 1)}{a(x^2 - 1)} \right) \) | Poisson-Charlier polynomials \( a_n^{(0)}(x) \) \([2, 14]\) |
| VI.   | \( \frac{1}{1 - \ln(1-t)} \exp \left( \frac{a(x^2 - 1)}{a(x^2 - 1)} \right) \) | \( \frac{1}{1 - \ln(1-t)} \exp \left( \frac{a(x^2 - 1)}{a(x^2 - 1)} \right) \) | \( \frac{1}{1 - \ln \left( \frac{e^{x} - 1}{x} \right)} \exp \left( \frac{a(x^2 - 1)}{a(x^2 - 1)} \right) \) | Poisson-Charlier polynomials \( a_n^{(0)}(x) \) \([2, 14]\) |
| VII.  | \( \frac{1}{1 - \ln(1-t)} \exp \left( \frac{a(x^2 - 1)}{a(x^2 - 1)} \right) \) | \( \frac{1}{1 - \ln(1-t)} \exp \left( \frac{a(x^2 - 1)}{a(x^2 - 1)} \right) \) | \( \frac{1}{1 - \ln \left( \frac{e^{x} - 1}{x} \right)} \exp \left( \frac{a(x^2 - 1)}{a(x^2 - 1)} \right) \) | Poisson-Charlier polynomials \( a_n^{(0)}(x) \) \([2, 14]\) |
| VIII. | \( \frac{1}{1 - \ln(1-t)} \exp \left( \frac{a(x^2 - 1)}{a(x^2 - 1)} \right) \) | \( \frac{1}{1 - \ln(1-t)} \exp \left( \frac{a(x^2 - 1)}{a(x^2 - 1)} \right) \) | \( \frac{1}{1 - \ln \left( \frac{e^{x} - 1}{x} \right)} \exp \left( \frac{a(x^2 - 1)}{a(x^2 - 1)} \right) \) | Poisson-Charlier polynomials \( a_n^{(0)}(x) \) \([2, 14]\) |
| IX.   | \( \frac{1}{1 - \ln \left( \frac{e^{x} - 1}{x} \right)} \exp \left( \frac{a(x^2 - 1)}{a(x^2 - 1)} \right) \) | \( \frac{1}{1 - \ln \left( \frac{e^{x} - 1}{x} \right)} \exp \left( \frac{a(x^2 - 1)}{a(x^2 - 1)} \right) \) | \( \frac{1}{1 - \ln \left( \frac{e^{x} - 1}{x} \right)} \exp \left( \frac{a(x^2 - 1)}{a(x^2 - 1)} \right) \) | Poisson-Charlier polynomials \( a_n^{(0)}(x) \) \([2, 14]\) |
| X.    | \( \frac{1}{1 - \sqrt{1 + x^2}} \arctan(t) \) | \( \frac{1}{1 - \sqrt{1 + x^2}} \arctan(t) \) | \( \frac{1}{1 - \sqrt{1 + x^2}} \arctan(t) \) | Shively's pseudo-Laguerre polynomials \( R_n(a, x) \) \([23]\) |
| XI.   | \( \frac{1}{1 - \sqrt{1 + x^2}} \arctan(t) \) | \( \frac{1}{1 - \sqrt{1 + x^2}} \arctan(t) \) | \( \frac{1}{1 - \sqrt{1 + x^2}} \arctan(t) \) | Shively's pseudo-Laguerre polynomials \( R_n(a, x) \) \([23]\) |
Now, we proceed to plot the graphs related to the 2ILP of order, \( \alpha L_n^{(3)[2]}(x) \), 2ILP \( L_n^{(2)}(x) \), 2IFF \( (x)_{n}^{[2]} \) and 2IEP \( \phi_n^{[2]}(x) \) for \( n = 4 \). For this, we need the first few values of \( L_n^{(\alpha)}(x) \), \( L_n(x) \), \( (x)_n \) and \( \phi_n(x) \). We give the list of first five \( L_n^{(\alpha)}(x) \), \( L_n(x) \), \( (x)_n \) and \( \phi_n(x) \) in Table 2.

**Table 2. First five \( L_n^{(\alpha)}(x) \), \( L_n(x) \), \( (x)_n \) and \( \phi_n(x) \).**

| \( n \) | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| \( L_n^{(\alpha)(x)} \) | 1 | \( 1 + \alpha - x \) | \( (1 + \alpha)(2 + \alpha) \) | \( -(2 + \alpha)x + \frac{1}{2}x^2 \) | \( \frac{(1 + \alpha)(2 + \alpha)(3 + \alpha)}{2} \) | \( \frac{(1 + \alpha)(2 + \alpha)(3 + \alpha)(3 - \alpha - x)}{2} \) |
| \( L_n(x) \) | 1 | \( -x \) | \( x^2 - 4x + 2 \) | \( -x^2 + 9x^3 - 18x + 6 \) | \( \frac{x^2 - 16x^3 + 72x^2 - 96x + 24}{2} \) |
| \( (x)_n \) | 0 | \( x \) | \( x^2 - x \) | \( x^3 - 6x^2 + 12x - 6 \) |
| \( \phi_n(x) \) | 1 | \( x \) | \( x^2 + x \) | \( x^3 + 3x^2 + x \) |

In view of equations (4.10), (4.28), (4.37) and Table 2, we find the first few values of \( L_n^{(3)[2]}(x) \), \( L_n^{[2]}(x) \), \( (x)_{n}^{[2]} \) and \( \phi_n^{[2]}(x) \). We present the values of first five \( L_n^{(3)[2]}(x) \), \( L_n^{[2]}(x) \), \( (x)_{n}^{[2]} \) and \( \phi_n^{[2]}(x) \) in Table 3.

**Table 3. First five \( L_n^{(3)[2]}(x) \), \( L_n^{[2]}(x) \), \( (x)_{n}^{[2]} \) and \( \phi_n^{[2]}(x) \).**

| \( n \) | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| \( L_n^{(3)[2]}(x) \) | 1 | \( x \) | \( x^2 + x - \frac{5}{2} \) | \( x^3 - 6x^2 + 7x \) | \( x^4 - 12x^3 + 40x^2 - 35x \) |
| \( L_n^{[2]}(x) \) | 1 | \( x \) | \( x^2 - 2x \) | \( x^3 + 6x^2 + 5x \) | \( x^4 + 12x^3 + 32x^2 + 15x \) |
| \( (x)_{n}^{[2]} \) | 0 | \( x \) | \( x^2 - 2x \) | \( x^3 - 6x^2 + 7x \) | \( x^4 - 12x^3 + 40x^2 - 35x \) |
| \( \phi_n^{[2]}(x) \) | 1 | \( x \) | \( x^2 + 2x \) | \( x^3 + 6x^2 + 5x \) | \( x^4 + 12x^3 + 32x^2 + 15x \) |

From Table 2, we get the following graphs:
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