SHARP CONVERGENCE RATES FOR EMPIRICAL OPTIMAL TRANSPORT WITH SMOOTH COSTS

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We revisit the question of characterizing the convergence rate of plug-in estimators of optimal transport costs. It is well known that an empirical measure comprising independent samples from an absolutely continuous distribution on $\mathbb{R}^d$ converges to that distribution at the rate $n^{-1/d}$ in Wasserstein distance, which can be used to prove that plug-in estimators of many optimal transport costs converge at this same rate. However, we show that when the cost is smooth, this analysis is loose: plug-in estimators based on empirical measures converge quadratically faster, at the rate $n^{-2/d}$. As a corollary, we show that the Wasserstein distance between two distributions is significantly easier to estimate when the measures are well-separated. We also prove lower bounds, showing not only that our analysis of the plug-in estimator is tight, but also that no other estimator can enjoy significantly faster rates of convergence uniformly over all pairs of measures. Our proofs rely on empirical process theory arguments based on tight control of $L^2$ covering numbers for locally Lipschitz and semi-concave functions. As a byproduct of our proofs, we derive $L^\infty$ estimates on the displacement induced by the optimal coupling between any two measures satisfying suitable concentration and anticoncentration conditions, for a wide range of cost functions.

1. Introduction. Optimal transport costs have received a recent surge of interest in applied probability and statistics. Arising from the classical optimal transport problem (Villani, 2003), this family of divergences measures the work required to couple two probability distributions in terms of a cost function over the space upon which they are defined. This fact makes them a powerful tool for comparing measures in a manner which is sensitive to the geometry of the underlying space, and has motivated their use in areas such as computer vision (Rubner, Tomasi and Guibas, 2000), generative modeling (Arjovsky, Chintala and Bottou, 2017), and computational biology (Orlova et al., 2016), among many others. We refer the reader to the monographs of Panaretos and Zemel (2019), Santambrogio (2015), and Peyré and Cuturi (2019) for surveys of their respective applications in statistics, applied mathematics, and machine learning.

In many of these applications, it is necessary to estimate the optimal transport cost between two measures on the basis of independent observations. This raises the fundamental question of characterizing the expected convergence rate of empirical estimators of these costs. Though this question has been studied in great generality in the literature, the goal of this paper is to highlight some unexpected phenomena that arise when the cost function is smooth.

For concreteness, we focus throughout on the optimal transport problem over the Euclidean space $\mathbb{R}^d$ for some integer $d \geq 1$. Let $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^d$, and let $\mathcal{P}(\mathcal{X})$ denote the set of Borel probability measures with support contained in $\mathcal{X}$. Let $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$. Given

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a nonnegative cost function \( c : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \), the optimal transport cost based on \( c \) is defined by
\[
\mathcal{T}_c(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, y) d\pi(x, y).
\]
Here, \( \Pi(\mu, \nu) \) denotes the set of couplings, that is, joint Borel probability measures over \( \mathcal{X} \times \mathcal{Y} \) with respective marginals \( \mu \) and \( \nu \).

In statistical contexts, the measures \( \mu \) and \( \nu \) are typically unknown, and it is necessary to estimate the optimal transport cost between them on the basis of i.i.d. observations \( X_1, \ldots, X_n \sim \mu \) and \( Y_1, \ldots, Y_n \sim \nu \). A canonical choice is the plug-in estimator \( \mathcal{T}_c(\mu_n, \nu_n) \), obtained by replacing \( \mu \) and \( \nu \) by their corresponding empirical measures:
\[
\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}, \quad \nu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{Y_i}.
\]
We call this quantity the empirical optimal transport cost, and we seek sharp upper and lower bounds on the expected gap between the empirical optimal transport cost and its population counterpart:
\[
(1) \quad \Delta_n(c) = \mathbb{E}[\mathcal{T}_c(\mu_n, \nu_n) - \mathcal{T}_c(\mu, \nu)].
\]
We highlight the dependence of \( \Delta_n \) on \( c \) because a key finding of our work is that the rate of decay of \( \Delta_n \) is driven by properties of the cost, and can improve significantly when \( c \) is smooth.

To illustrate the phenomena we have in mind, we turn to perhaps the most widely-used cost functions: those of the form \( c_p(x, y) = \|x - y\|^p \), \( p \geq 1 \), where \( \| \cdot \| \) denotes the Euclidean metric on \( \mathbb{R}^d \). These costs give rise to the \( p \)-Wasserstein distances, defined by \( W_p = \mathcal{T}_{c_p}^{1/p} \).

The convergence rate of the empirical \( p \)-Wasserstein distance \( W_p(\mu_n, \nu_n) \) to its population counterpart \( W_p(\mu, \nu) \) is a well-studied problem; for instance, assuming for simplicity of exposition that \( \mathcal{X} = \mathcal{Y} \) is a compact set, Fournier and Guillin (2015) prove that there exists a constant \( C_d > 0 \), depending only on \( d \) and \( \mathcal{X} \), such that
\[
(2) \quad \mathbb{E}W_p(\mu_n, \mu) \leq \left[ \mathbb{EW}_p^p(\mu_n, \mu) \right]^{1/p} \leq C_d n^{-1/d},
\]
whenever \( d > 2p \). Since \( W_p \) is a metric, it follows that
\[
(3) \quad \mathbb{E}|W_p(\mu_n, \nu_n) - W_p(\mu, \nu)| \leq \mathbb{E}W_p(\mu_n, \mu) + \mathbb{E}W_p(\nu_n, \nu) \leq 2C_d n^{-1/d}.
\]
The \( n^{-1/d} \) rate in equation (3) is well known to be inherent to statistical optimal transport problems. In particular, it was shown by Niles-Weed and Rigollet (2022) that, up to polynomial factors, no estimator of \( W_p(\mu, \nu) \) improves on the rate in equation (3) uniformly over all pairs of measures \( (\mu, \nu) \). Nevertheless, one of the main contributions of this paper is to show that this bound is only tight when \( \mu = \nu \), and can otherwise be improved up to quadratically. Indeed, our results imply the bound
\[
(4) \quad \Delta_n(c_p) = \mathbb{E}[W_p^p(\mu_n, \nu_n) - W_p^p(\mu, \nu)] \lesssim \left\{ \begin{array}{ll}
\frac{n^{-p/d}}{2}, & 1 \leq p \leq 2 \\
\frac{n^{-2/d}}{2}, & 2 \leq p < \infty,
\end{array} \right.
\]
which, as we shall see, entails
\[
(5) \quad \mathbb{E}|W_p(\mu_n, \nu_n) - W_p(\mu, \nu)| \lesssim \delta_0^{1-p} \left\{ \begin{array}{ll}
\frac{n^{-p/d}}{2}, & 1 \leq p \leq 2 \\
\frac{n^{-2/d}}{2}, & 2 \leq p < \infty,
\end{array} \right. \quad \text{if } W_p(\mu, \nu) \geq \delta_0 > 0.
\]
Whenever \( p > 1 \), equations (4) and (5) provide a significant sharpening of the naive estimate in equation (3). We show that such improvements arise due to the Hölder smoothness of the
cost $c_p$, and in fact, similar rates of convergence for $\Delta_n(c)$ are enjoyed by a much broader collection of smooth cost functions. Beyond smoothness assumptions on $c$, we establish our main results under the following broad structural condition, which is presumed throughout the sequel,

**(H0)** The cost function $c : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is nonnegative, and takes the form $c(x, y) = h(x - y)$ where $h : \mathbb{R}^d \to \mathbb{R}_+$ is convex, even, and lower semi-continuous.

Before summarizing our main results and comparing them to further existing literature, we begin with an idealized example which illustrates the role of these conditions.

1.1. Example: Location Families. For simplicity, we limit this example to upper bounding the following one-sample analogue of $\Delta_n(c)$,

$$\mathbb{E}|\mathcal{T}_c(\mu_n, \nu) - \mathcal{T}_c(\mu, \nu)|.$$  

We also continue to assume for simplicity that $\mathcal{X} = \mathcal{Y}$ is a convex and compact set. Let $c$ be any cost function such that condition (H0) holds, and assume there exists $\alpha \in (1, 2]$ such that $h \in C^\alpha(\mathcal{X})$. Here, $C^\alpha(\mathcal{X})$ denotes the Hölder space over $\mathcal{X}$ with regularity $\alpha$, which is defined in Section 1.4 together with all other notational conventions used in the sequel.

Let $\mu, \nu \in \mathcal{P}(\mathcal{X})$ be any two measures differing only by a location transformation with respect to a fixed vector $z_0 \in \mathbb{R}^d$, in the sense that $\nu = T_{0,\#} \mu := \mu(T_{0}^{-1}(\cdot))$, where $T_{0}(z) = z + z_0$ and $\#$ denotes the pushforward operator. In this example, it is simple to find an optimal coupling between $\mu$ and $\nu$. Indeed, recall that $h$ is convex and even under (H0), thus for all $\pi \in \Pi(\mu, \nu)$, Jensen’s inequality implies

$$\int h(x - y) d\pi(x, y) \geq h \left( \int xd\mu(x) - \int yd\nu(y) \right) = h(z_0).$$

Thus, $\mathcal{T}_c(\mu, \nu) \geq h(z_0)$, and the lower bound is achieved by the coupling $\pi = (Id, T_0)_\# \mu$, implying that $T_0$ is an optimal transport map from $\mu$ to $\nu$. On the other hand, for all couplings $\pi_n \in \Pi(\mu_n, \mu)$, $\gamma_n = (Id, T_0)_\# \pi_n$ is a (typically suboptimal) coupling between $\mu_n$ and $\nu$, whence

$$\mathcal{T}_c(\mu_n, \nu) \leq \int c(z, y) d\gamma_n(z, y) = \int c(z, T_0(x)) d\pi_n(z, x).$$

Since we assumed that the Hölder norm $\Lambda := \|h\|_{C^\alpha(\mathcal{X})}$ is finite, $h$ is close to its first-order Taylor expansion. Specifically, we obtain from the above display,

$$\mathcal{T}_c(\mu_n, \nu) \leq \int \left[ h(x - T_0(x)) + \langle \nabla h(x - T_0(x)), x - z \rangle + \Lambda \|x - z\|^{\alpha} \right] d\pi_n(z, x).$$

Due to the marginal constraints in the definition of $\pi_n$, equation (6) is tantamount to

$$\mathcal{T}_c(\mu_n, \nu) \leq \mathcal{T}_c(\mu, \nu) + \int \langle \nabla h(z_0), \cdot \rangle d(\mu_n - \mu) + \Lambda \int \|x - z\|^{\alpha} d\pi_n(x, z).$$

The final term of the above display is manifestly the $\|\cdot\|^{\alpha}$-transport cost between $\mu_n$ and $\mu$, with respect to a possibly suboptimal coupling $\pi_n \in \Pi(\mu_n, \mu)$. Since it holds for any choice of $\pi_n$, taking the infimum over such couplings leads to

$$\mathcal{T}_c(\mu_n, \nu) \leq \mathcal{T}_c(\mu, \nu) + \int \langle \nabla h(z_0), \cdot \rangle d(\mu_n - \mu) + \Lambda W_\alpha^\alpha(\mu_n, \mu).$$

The second term on the right-hand side of the above display is a mean-zero sample average, and hence typically decays at the rate $n^{-1/2}$ in probability. Equation (8) thus provides an upper bound on $\mathcal{T}_c(\mu_n, \nu) - \mathcal{T}_c(\mu, \nu)$ which is primarily driven by the rate of convergence of
the empirical measure under the optimal transport cost with respect to $\|\cdot\|^\alpha$, which we refer to as the $\alpha$-transport cost in the sequel. By equation (2), we arrive at the following one-sided estimate whenever $d \geq 5$,

$$
\mathbb{E}\left[T_\alpha(\mu_n, \nu) - T_\alpha(\mu, \nu)\right] \leq \Lambda \mathbb{E}\left[W_\alpha^\alpha(\mu_n, \mu)\right] \leq \Lambda C_d n^{-\alpha/d}.
$$

Although equation (9) does not imply an upper bound in expected absolute value, it captures the main features of our problem; as we shall see, a simple extension of the above derivations leads to the bound

$$
\mathbb{E}\left[|T_\alpha(\mu_n, \nu) - T_\alpha(\mu, \nu)|\right] \leq C_0 n^{-\alpha/d}, \quad \text{for all } d \geq 5,
$$

for a large enough constant $C_0 > 0$ depending on $d, \mathcal{X}$ and $\Lambda$. Equation (10) shows that, for the class of cost functions under consideration, the rate of convergence of the empirical optimal transport cost is largely driven by the smoothness of $c$ when $\mu$ and $\nu$ differ merely in mean. In particular, notice that the cost $h(x) = \|x\|^p$ satisfies $h \in C^p$ for all $p \geq 1$, which implies the previously announced result (4) in the special case of one-sample location families.

This fast rate of convergence in equation (10) arose in the present example because the first-order term in the Taylor expansion (6) is negligible, leading to a rate driven only by its remainder. While this argument cannot easily be extended to general measures $\mu$ and $\nu$, its conclusion turns out to be generic, as we now describe.

1.2. Our Contributions. The primary contribution of this paper is to provide sharp upper and lower bounds on $\Delta_n(\mathcal{C})$ for smooth costs satisfying condition (H0). In this setting, our main result informally states that whenever $h \in C^\alpha$ for some $\alpha > 0$,

$$
\Delta_n(\mathcal{C}) \lesssim \begin{cases} 
n^{-\alpha/d}, & 0 \leq \alpha \leq 2 \\
n^{-2/\alpha}, & 2 \leq \alpha < \infty
\end{cases}, \quad \text{for all } d \geq 5.
$$

This upper bound is stated formally in Theorem 2 under the assumption that $\mu$ and $\nu$ admit bounded support. Under additional conditions on $c$, we extend this result to measures $\mu$ and $\nu$ with unbounded support, satisfying appropriate tail assumptions, in Theorem 12 and Corollary 13. As in equation (5), our results have natural implications for the convergence rate of empirical Wasserstein distances, which we discuss in Corollary 4. In view of Section 1.1, the convergence rate (11) admits a natural interpretation: the first order term in a formal expansion of the empirical optimal transport cost is typically negligible when $d \geq 5$, leading to a rate that improves with the smoothness parameter $\alpha \in (0, 2)$. When $\alpha \geq 2$, the quadratic term in this expansion is not negligible, thus faster rates do not occur without stronger conditions.

At the heart of our proofs is the Kantorovich dual formulation of the optimal transport problem—summarized in Section 1.4—which allows us to reduce the problem of bounding $\Delta_n(\mathcal{C})$ to that of bounding the expected suprema of empirical processes indexed by collections of sufficiently regular Kantorovich potentials. While characterizing the regularity of these potentials is routine when $\mu$ and $\nu$ are compactly supported (Gangbo and McCann (1996), Appendix C), the bulk of our efforts lies in the case where they admit unbounded support. In this setting, one of our key technical contributions is to provide quantitative $L^\infty$ estimates on the displacement induced by the optimal coupling between any two measures satisfying appropriate tail conditions (Theorem 11). For instance, the following is a special case of our result for the $p$-transport cost.

**Theorem (Informal).** Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ and $p > 1$. Let $\nu$ be a $\sigma^2$-sub-Gaussian measure (Boucheron, Lugosi and Massart, 2013) and $\mu$ have finite $p$-th moment, and assume...
there exist constants $c_1, c_2 > 0$ such that $\mu(B_{2,1}) \geq c_1 \exp(-c_2 \|x\|^2)$ for all $x \in \mathbb{R}^d$. Then, for any optimal coupling $\pi$ between $\mu$ and $\nu$ with respect to the cost $c_p(x,y) = \|x - y\|^p$, 

$$
\|y\| \lesssim \sigma(\|x\| + 1), \quad \text{for } \pi\text{-a.e. } (x,y).
$$

(12)

In particular, if there exists an optimal transport map $T$ from $\mu$ to $\nu$ with respect to $c_p$, then 

$$
\|T(x)\| \lesssim \sigma(\|x\| + 1), \quad \text{for } \mu\text{-a.e. } x.
$$

Analogues of equation (12) have previously been derived by Colombo and Fathi (2021) in the special case where $\mu$ is a Gaussian measure and $p = 2$, and we further discuss these results below the statement of Theorem 11. As we shall see, equation (12) leads to estimates on the local Lipschitz constants of Kantorovich potentials between any two, possibly atomic probability measures, and forms the basis of our main results when $\mu$ and $\nu$ have unbounded support. These results are quantitative analogues of the fact, proved by Gangbo and McCann (1996), that Kantorovich potentials are locally Lipschitz under mild smoothness conditions on $c$.

In Section 4.1 we explicitly construct measures $\mu$ and $\nu$ for which inequality (11) is achieved up to universal constants, inspired by the example in Section 1.1. While this result proves that our upper bounds cannot generally be improved, it does not preclude the possibility that there exists another estimator $\tilde{T}_n$, i.e. a measurable function of $X_1, Y_1, \ldots, X_n, Y_n$, for which the quantity $\mathbb{E}[\tilde{T}_n - T(\mu, \nu)]$ scales at a faster rate than that of equation (11), uniformly over pairs of measures $\mu, \nu$. We prove in Section 4.2 that, in an information theoretic sense, such an improvement is not possible up to polylogarithmic factors.

Though we prove inequality (11) for all $d \geq 5$, notice that it does not generally hold for all $d \geq 1$. Indeed, it is a simple observation that the empirical optimal transport cost cannot generally achieve a faster rate of convergence than $n^{-1/2}$ (Niles-Weed and Rigollet, 2022). The probabilistic behavior of the empirical costs is therefore qualitatively different in low dimension. While our proof techniques for bounded measures can be extended to the case $d \leq 4$, they do not appear to yield tight results for certain values of $\alpha > 0$; see Remark 1 below. Similarly, our techniques for unbounded measures do not generally appear to be tight in the low-dimensional case. Since our goal in this paper is to obtain sharp convergence rates, we assume in what follows that $d \geq 5$, where we are able to establish exact results.

Outline of the remainder of the paper. In Sections 1.3 and 1.4, we review prior work and recall some important preliminary results on the duality theory of transport costs. Section 2 contains our main results for compactly supported measures. In Section 3, we extend these results to the unbounded case. Lower bounds appear in Section 4. The proofs of certain intermediary results from Sections 2–4 are respectively deferred to Appendices A–C.

1.3. Related Work. Upper bounds on the expected deviation $\Delta_n(c)$ are available in the literature for several special cases. The closest to our setting is the quadratic cost $c_2(x,y) = \|x - y\|^2$, for which Chizat et al. (2020) prove that $\Delta_n(c_2) \leq n^{-2/d}$ when $\mu$ and $\nu$ are compactly supported. Their proof hinges upon the Knott-Smith optimality criterion, which allows them to relate $\Delta_n(c_2)$ to suprema of empirical processes indexed by convex potentials, which are in fact globally Lipschitz since $\mu$ and $\nu$ are assumed compact. Empirical processes indexed by globally Lipschitz convex functions are well-studied (Bronstein, 1976; Guntuboyina and Sen, 2012), and lead to their result. Our results extend theirs in two directions: we replace $c_2$ by any smooth cost, and we remove the condition that the measures be compactly supported. When $\mu$ and $\nu$ are compactly supported and $\alpha = 2$, our proof strategy mirrors that of Chizat et al. (2020): though the potentials arising for other costs are not necessarily convex, it is still possible to use existing empirical process theory bounds
to obtain sharp rates. On the other hand, when $\mu$ and $\nu$ have unbounded support, the relevant potentials may not even be globally Lipschitz, and our proof requires significant new techniques.

Faster rates of convergence for estimating optimal transport costs are achievable under strong conditions on $X$ and $Y$. For instance, when $c$ is a metric raised to a power $p \geq 1$, the bound $\Delta_n(c) \lesssim n^{-1/2}$ is known to hold when $X$ and $Y$ are one-dimensional (Munk and Czado, 1998; Freitag and Munk, 2005; Bobkov and Ledoux, 2019; del Barrio, Gordaliza and Loubes, 2019; Manole, Balakrishnan and Wasserman, 2022) or countable (Sommerfeld and Munk, 2018; Tameling, Sommerfeld and Munk, 2019). In both of these cases, the corresponding empirical $p$-Wasserstein distance is known to exhibit distinct convergence rates depending on whether $\mu$ and $\nu$ are vanishingly close or not, similar to our findings in equation (5). While these two examples form important special cases, their underlying proof techniques are closely tied to characterizations of the optimal transport problem which are only available for discrete and one-dimensional measures, and do not shed more general light on the behaviour of $\mathcal{T}_c(\mu_n, \nu_n)$.

Though the naive bound in equation (3) is loose for $p > 1$ when $W_p(\mu, \nu)$ is bounded away from zero, Liang (2019) and Niles-Weed and Rigollet (2022) show that it cannot generally be improved by more than a polylogarithmic factor when no separation conditions are placed on $\mu$ and $\nu$. Recall that this upper bound arose from the convergence rate of $\mu_n$ under the $p$-Wasserstein distance in equation (2). The study of such convergence rates was initiated by Dudley (1969) in the special case $p = 1$, who also used arguments from empirical process theory, due to the dual characterization of $W_1$ as a supremum over Lipschitz functions (Villani, 2003). For $p > 1$, distinct techniques have been used to study this problem in great generality by Boissard and Le Gouic (2014); Fournier and Guillin (2015); Bobkov and Ledoux (2019); Weed and Bach (2019); Singh and Póczos (2019); Lei (2020), and references therein. Dudley (1969) also derived deterministic lower bounds on the quality of approximating $\mu$ by any discrete measure supported on $n$ points under $W_1$—we build upon these results to obtain our lower bounds on $\Delta_n(c)$ in Section 4.1.

Another line of work has sought to understand optimal rates of estimation for Wasserstein distances when the densities—rather than the cost—are smooth. These works (Liang, 2021; Singh et al., 2018; Niles-Weed and Berthet, 2022) show that the plug-in empirical estimator $W_p(\mu_n, \nu_n)$ for $W_p(\mu, \nu)$ is suboptimal if $\mu$ and $\nu$ have smooth densities, but that replacing $\mu_n$ and $\nu_n$ by appropriate nonparametric density estimators suffices to obtain optimal rates of estimation. Under similar conditions on $\mu$ and $\nu$, it is also possible to construct appropriate smooth estimators of the optimal map between $\mu$ and $\nu$ (Hütter and Rigollet, 2021). Our work takes a quite different perspective: rather than adding additional conditions on $\mu$ and $\nu$, we show that the rates of convergence of empirical estimators improve under additional smoothness conditions on the cost.

### 1.4. Notation and Further Background on the Optimal Transport Problem

Our proofs make repeated use of the Kantorovich dual formulation of the optimal transport problem (Villani (2008), Theorem 5.10), which we now describe. Define for all $\varphi \in L^1(\mu)$ and $\psi \in L^1(\nu)$ the functional

$$J_{\mu, \nu}(\varphi, \psi) = \int \varphi d\mu + \int \psi d\nu.$$

The regularity condition (H0) is sufficient to imply

$$\mathcal{T}_c(\mu, \nu) = \sup_{(\varphi, \psi) \in \Phi_c(\mu, \nu)} J_{\mu, \nu}(\varphi, \psi), \tag{13}$$
where $\Phi_c(\mu, \nu)$ denotes the set of pairs $(\varphi, \psi) \in L^1(\mu) \times L^1(\nu)$ such that $\varphi(x) + \psi(y) \leq c(x, y)$ for all $x \in X$ and $y \in Y$. If we further assume $c(x, y) \leq c_1(x) + c_2(y)$ for some $c_1 \in L^1(\mu)$ and $c_2 \in L^1(\nu)$, then the supremum in equation (13) is achieved. Any pair $(\varphi, \psi)$ achieving the supremum is called a pair of optimal (Kantorovich) potentials.

We shall say that a function $g : \mathbb{R}^d \to \mathbb{R}$ taking values in the extended real line $\mathbb{R} = \mathbb{R} \cup \{ -\infty \}$ is $c$-concave (Gangbo and McCann, 1996) if it is not identically $-\infty$, and if there exists a nonempty set $A \subseteq \mathbb{R}^d \times \mathbb{R}$ such that

$$g(y) = \inf_{(x, \lambda) \in A} \{ c(x, y) - \lambda \}.$$ 

The canonical example of $c$-concave functions are $c$-conjugates, for which $A$ is the graph of a map $f$. Specifically, given $f : X \to \mathbb{R}$ not identically $-\infty$, the $c$-conjugate of $f$ is given by

$$f^c : Y \to \mathbb{R}, \quad f^c(y) = \inf_{x \in X} \{ c(x, y) - f(x) \}. \quad (14)$$

Whenever a map $f$ is merely defined over a nonempty subset $\mathcal{X}_0 \subseteq \mathcal{X}$ in the sequel, we extend its definition to $\mathcal{X}$ by setting $f(\mathcal{X} \setminus \mathcal{X}_0) = \{ -\infty \}$. In this case, it is clear that the infimum in the above display can be restricted to $\mathcal{X}_0$.

When the supremum in the Kantorovich duality (13) is achieved by a pair $(\varphi, \psi) \in \Phi_c(\mu, \nu)$, it is easy to see that $(\varphi, \varphi^c)$ also lies in $\Phi_c(\mu, \nu)$ and can only increase the value of $J_{\mu, \nu}(\varphi, \psi)$. It follows that $(\varphi, \varphi^c)$ is itself a pair of optimal Kantorovich potentials, and one obtains (Villani, 2008),

$$\mathcal{T}_c(\mu, \nu) = \sup_{\varphi \in L^1(\mu)} \int \varphi d\mu + \int \varphi^c d\nu. \quad (15)$$

Notice that if $c(x, y) = -x^\top y$, then the definition (14) of $c$-conjugate reduces to $-(-f)^*$, where for any convex function $h$ on $\mathbb{R}^d$, $h^*(y) = \sup_{x \in \mathbb{R}^d} \{ x^\top y - h(x) \}$ denotes its Legendre-Fenchel transform. It is well known that if $h$ is also lower semi-continuous and not identically infinite, the supremum in the definition of $h^*(y)$ is achieved by a point in its subdifferential; specifically, one has the relation

$$y \in \partial h(x) \iff x \in \partial h^*(y) \iff x^\top y = h(x) + h^*(y).$$

To derive analogous notions for $c$-concave functions, define the $c$-superdifferential of a $c$-concave function $f : \mathcal{X} \to \mathbb{R}$ by

$$\partial^c f = \{(x, y) \in \mathcal{X} \times Y : c(v, y) - f(v) \geq c(x, y) - f(x), \forall v \in \mathcal{X} \}.$$ 

Furthermore, let $\partial^c f(x) = \{ y \in \mathbb{R}^d : (x, y) \in \partial^c f \}$ and $\partial^c f(B) = \bigcup_{x \in B} \partial^c f(x)$, for all $B \subseteq \mathcal{X}$. The following Lemma summarizes the main properties of $c$-concave functions which we shall require. Some of the statements which follow are weaker than necessary, but sufficient for our purposes.

**Lemma 1.** Let $f : \mathcal{X} \to \mathbb{R}$ be $c$-concave, and assume condition (H0).

(i) (Villani (2008), Proposition 5.8) We have, $f^{cc} = f$.

(ii) (Villani (2003), Remark 1.13) Assume the cost $c$ is bounded. Then, the supremum in equation (15) is achieved by a $c$-concave function $\varphi \in L^1(\mu)$ such that $0 \leq \varphi \leq \| c \|_\infty$ and $-\| c \|_\infty \leq \varphi^c \leq 0$.

(iii) (Gangbo and McCann (1996), Theorem 2.7) $\partial^c f$ is $c$-cyclically monotone, in the sense that for any permutation $\sigma$ on $k \geq 1$ letters and any $(x_1, y_1), \ldots, (x_k, y_k) \in \partial^c f$,

$$\sum_{j=1}^k c(x_j, y_j) \leq \sum_{j=1}^k c(x_{\sigma(j)}, y_j).$$
(iv) (Gangbo and McCann (1996), Proposition C.4) Assume further that \( h \) is superlinear. For any given \( x \in \mathbb{R}^d \), assume there exists a neighborhood of \( x \) over which \( f \) is bounded. Then, the \( c \)-superdifferential \( \partial^c f(x) \) is nonempty. Therefore, if \( f \) is locally bounded over \( \mathbb{R}^d \), it holds that for all \( x, y \in \mathbb{R}^d \),

\[
y \in \partial^c f(x) \iff x \in \partial^c f^c(y) \iff c(x, y) = f(x) + f^c(y).
\]

In particular,

\[
f(x) = \inf_{y \in \partial^c f(x)} \{ c(x, y) - f^c(y) \}, \quad f^c(y) = \inf_{x \in \partial^c f^c(y)} \{ c(x, y) - f(x) \}.
\]

Furthermore, if \( f \) is in fact an optimal Kantorovich potential for the optimal transport problem (15) from \( \mu \) to \( \nu \), and if \( T_\epsilon(\mu, \nu) < \infty \), then for any optimal coupling \( \pi \in \Pi(\mu, \nu) \), \( \supp(\pi) \subseteq \partial^c f \).

We close this section with a summary of notational conventions used in the sequel.

**Notation.** Given \( a, b \in \mathbb{R} \), we write \( a \vee b = \max\{a, b\} \) and \( a \wedge b = \min\{a, b\} \). Given a set \( \Omega \subseteq \mathbb{R}^d \) and a function \( f : \Omega \to \mathbb{R} \) which is differentiable to order \( k \geq 1 \), and a multi-index \( \beta \in \mathbb{N}_0^d \), we write \( |\beta| = \sum_{i=1}^d \beta_i \), and for all \( |\beta| \leq k \), \( D^\beta f = \partial^{|\beta|} f/\partial x_1^{\beta_1} \ldots \partial x_d^{\beta_d} \).

Given \( \alpha > 0 \), let \( \lfloor \alpha \rfloor \) denote the largest integer strictly less than \( \alpha \) (for instance, \( \lfloor \alpha \rfloor = 0 \) when \( \alpha = 1 \)). The Hölder space \( C^\beta(\Omega) \) is defined as the set of functions \( f : \Omega \to \mathbb{R} \) admitting at least \( \lfloor \alpha \rfloor \) continuous derivatives over the interior \( \Omega^o \) of \( \Omega \), which extend continuously up to the boundary of \( \Omega \), and are such that the Hölder norm

\[
\|f\|_{C^\beta(\Omega)} = \sum_{|\beta| = 0}^{\lfloor \alpha \rfloor} \sup_{|\beta| = j} \|D^\beta f\|_{L^\infty(\Omega)} + \sum_{|\beta| = \lfloor \alpha \rfloor} \sup_{x, y \in \Omega, x \neq y} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x - y|^{|\beta| - \lfloor \alpha \rfloor}}
\]

is finite. Notice that when \( k \geq 1 \) is an integer, \( C^k(\Omega) \) is also commonly denoted \( C^{k-1,1}(\Omega) \).

Given a measure space \( (\Omega, \mathcal{F}, \nu) \), we interchangeably use the symbols \( \|f\|_{L^p(\Omega)} \) and \( \|f\|_{L^p(\nu)} \) to denote the \( L^p \) norm \( (\int_\Omega |f(x)|^p d\nu(x))^{1/p} \) for any \( 1 \leq p \leq \infty \) and any Borel-measurable function \( f : \Omega \to \mathbb{R} \). We drop the suffix \( \Omega \) and simply write \( C^\beta \) and \( L^p \) when \( \Omega \) is clear from context. Given a map \( T : \Omega \to \Omega \), the pushforward of \( \nu \) under \( T \) is denoted by \( T_\# \nu = \nu(T^{-1}(\cdot)) \). Given a vector \( x \in \mathbb{R}^d \) and \( 1 \leq p \leq \infty \), we write \( \|x\|_{L^p} = (\sum_{i=1}^d |x_i|^p)^{1/p} \).

When \( p = 2 \), we drop the subscript \( \ell_2 \) and simply write \( \|\cdot\| \). Given a square matrix \( A = (a_{ij}) \in \mathbb{R}^{d \times d} \), \( \|A\|_{op} = \sup\{\|Ax\| : x \in \mathbb{R}^d, \|x\| = 1\} \) denotes its operator norm, and \( \|A\|_\infty = \max_{1 \leq i, j \leq d} |a_{ij}| \) its entrywise \( \ell_\infty \) norm. The convolution of two functions \( f, g : \mathbb{R}^d \to \mathbb{R} \) is denoted \( (f * g)(x) = \int f(y)g(x - y)dy \), and that of \( f \) and a Borel probability measure \( \mu \) on \( \mathbb{R}^d \) is given by \( (f * \mu)(x) = \int f(x - y) d\mu(y) \), for all \( x \in \mathbb{R}^d \).

The closed ball centered at \( x \in \mathbb{R}^d \) of radius \( r \geq 0 \) is denoted \( B_{x,r} = \{x \in \mathbb{R}^d : \|x\| \leq r\} \), and we drop the subscript \( r \) when it equals 1: \( B_x = B_{x,1} \). \( \mathcal{L} \) denotes the Lebesgue measure on \( \mathbb{R}^d \).

A **universal constant** is any constant \( C > 0 \) which is tacitly permitted to depend on \( d, \mathcal{X}, \mathcal{Y} \), and may also depend on additional data when specified. Given sequences of nonnegative real numbers \( (a_n)_{n=1}^{\infty} \) and \( (b_n)_{n=1}^{\infty} \), we write \( a_n \lesssim b_n \) if there exists a universal constant \( C > 0 \) such that \( a_n \leq Cb_n \) for all \( n \geq 1 \). We also write \( a_n \asymp b_n \) if \( b_n \lesssim a_n \lesssim b_n \).

Finally, as discussed in Section 1.2, we assume \( d \geq 5 \) throughout the remainder of this manuscript, unless otherwise stated.

### 2. Upper Bounds for Compactly Supported Measures.

We begin by bounding the rate of convergence of the empirical optimal transport cost in the special case where \( \mathcal{X}, \mathcal{Y} \) and \( \mathcal{Z} = \mathcal{X} - \mathcal{Y} = \{x - y : x \in \mathcal{X}, y \in \mathcal{Y}\} \) satisfy the following condition.
(S1) $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^d$ are convex and compact sets with nonempty interior. Furthermore, we have $\mathcal{X}, \mathcal{Y}, Z \subseteq B_{0,1}$.

The assumption of compactness of $\mathcal{X}$ and $\mathcal{Y}$ will be relaxed in the following section, under concentration and anticoncentration conditions on the measures. Once $\mathcal{X}$ and $\mathcal{Y}$ are assumed compact, notice that the final assumption of (S1) can always be satisfied up to rescaling and recentering. Furthermore, we recall that the supports of $\mu$ and $\nu$ are merely assumed to be contained in $\mathcal{X}$ and $\mathcal{Y}$, and thus need not be convex themselves.

We shall also assume throughout this section that the cost $c$ satisfies condition (H0) and the following smoothness condition.

(H1) There exists $\alpha \in (0, 2]$ and a convex open set $Z_1$ such that $Z \subseteq Z_1 \subseteq B_{0,2}$, and $h \in \mathcal{C}^\alpha(Z_1)$. Furthermore, we have $0 \leq h \leq 1$ on $Z_1$. We write $\Lambda := 1 \vee \|h\|_{\mathcal{C}^\alpha(Z_1)} < \infty$.

For any measures $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$, recall that $X_1, \ldots, X_n \sim \mu$ and $Y_1, \ldots, Y_n \sim \nu$ denote i.i.d. samples, with corresponding empirical measures $\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$ and $\nu_n = \frac{1}{n} \sum_{i=1}^n \delta_{Y_i}$. The main result of this section is now stated as follows.

**Theorem 2.** Assume conditions (S1), (H0), and (H1). Then, there exists a constant $C > 0$ depending only on $d, \alpha, \mathcal{X}, \mathcal{Y}, Z_1$ such that

$$\sup_{\mu \in \mathcal{P}(\mathcal{X}), \nu \in \mathcal{P}(\mathcal{Y})} \mathbb{E}_{\mu, \nu} |\mathcal{T}_c(\mu_n, \nu_n) - \mathcal{T}_c(\mu, \nu)| \leq C \Lambda n^{-\alpha/d}.$$ 

Theorem 2 proves that the convergence rates anticipated in Section 1.1, for measures differing only in mean, in fact hold for all compactly supported measures. In particular, the $n^{-\alpha/d}$ rate of convergence is achievable as soon as $h \in \mathcal{C}^\alpha$, for $\alpha \in (0, 2]$, though is not generally claimed to improve further when $\alpha > 2$. For instance, the quadratic cost $||\cdot||^2$ lies in $\mathcal{C}^\infty$, but one cannot hope for a faster convergence rate than $n^{-2/d}$. Indeed, we derive matching lower bounds in Section 4 under closely related assumptions on the cost function $c$, which imply that the upper bound of Theorem 2 is generally unimprovable.

A careful investigation of our proof reveals that Theorem 2 in fact continues to hold for nonconvex costs $h$. We nevertheless prefer to retain the assumption of convexity in condition (H0) since it is required for the remainder of our main results; in particular, we do not claim that the convergence rate in Theorem 2 is sharp when $h$ is not convex.

By letting $\alpha$ vanish, Theorem 2 suggests that the empirical optimal transport cost does not generally converge at any polynomial rate for cost functions which fail to be uniformly Hölder continuous. Indeed, absent any smoothness assumptions on $c$, $\mathcal{T}_c(\mu_n, \nu_n)$ may not even converge in $L^1(\mathbb{P})$, as can be seen by taking $c$ to be the Hamming metric. In this case, $\mathcal{T}_c(\mu_n, \nu_n)$ is simply the Total Variation distance between $\mu_n$ and $\nu_n$, which almost surely equals unity when $\mu$ and $\nu$ are absolutely continuous with respect to the Lebesgue measure.

As discussed in Section 1, perhaps the most widely-used cost functions satisfying conditions (H0) and (H1) are norms over $\mathbb{R}^d$ raised to a power greater than one. We illustrate the conclusion of Theorem 2 for such an example.

**Corollary 3 (Powers of $\ell_r$ Norms).** Let $\mathcal{X}, \mathcal{Y}$ satisfy condition (S1), and define the cost $c_{p,r}(x, y) = ||x - y||^p_{\ell_r}$ for all $x \in \mathcal{X}, y \in \mathcal{Y}$ and $p, r \geq 1$. Let $\mu \in \mathcal{P}(\mathcal{X}), \nu \in \mathcal{P}(\mathcal{Y})$.

(i) We have for all $p, r \geq 1$, $\mathbb{E} |\mathcal{T}_{c_{p,r}}(\mu_n, \nu_n) - \mathcal{T}_{c_{p,r}}(\mu, \nu)| \lesssim n^{-(2\wedge p/r)/d}$. In particular, specializing to $r = 2$,

$$\mathbb{E} |W_p(\mu_n, \nu_n) - W_p(\mu, \nu)| \lesssim \begin{cases} n^{-p/d}, & 1 \leq p < 2 \\ n^{-2/d}, & 2 \leq p < \infty. \end{cases}$$
(ii) If \( \mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^d \) are disjoint, then for all \( p \geq 1 \) and \( r \geq 2 \),
\[
\mathbb{E} \left| T_{p,r} (\mu_n, \nu_n) - T_{p,r} (\mu, \nu) \right| \lesssim n^{-2/d}.
\]

The proof is deferred to Appendix A.1. Corollary 3(i) follows from the fact that \( \| \cdot \|_r^{p} \in C^{2\wedge p \wedge r}(\mathcal{Z}_1) \) for any bounded open set \( \mathcal{Z}_1 \), for all \( p, r \geq 1 \). When \( r \geq 2 \), notice that \( \| \cdot \|_r^{p} \) is smooth away from the origin, so that the condition \( \| \cdot \|_r^{p} \in C^{2}(\mathcal{Z}_1) \) can be satisfied for any \( p \geq 1 \) whenever the closed set \( \mathcal{Z} \subseteq \mathcal{Z}_1 \) does not contain the point zero. This observation leads to Corollary 3(ii). This last point implies the rather surprising fact that for all measures \( \mu \) and \( \nu \) admitting disjoint and compact support, one has
\[
\mathbb{E} \left| W_1 (\mu_n, \nu_n) - W_1 (\mu, \nu) \right| \lesssim n^{-2/d}.
\]
When the measures \( \mu \) and \( \nu \) are not vanishingly close, Corollary 3 also translates into convergence rates for empirical Wasserstein distances.

**Corollary 4 (Wasserstein Distances).** Let \( p \geq 1 \). Let \( \mathcal{X}, \mathcal{Y} \) satisfy condition (S1), and let \( \mu \in \mathcal{P}(\mathcal{X}), \nu \in \mathcal{P}(\mathcal{Y}) \). Assume \( W_p (\mu, \nu) \geq \delta_0 \), for some constant \( \delta_0 > 0 \). Then,
\[
\mathbb{E} \left| W_p (\mu_n, \nu_n) - W_p (\mu, \nu) \right| \lesssim \delta_0^{1-p} \left\{ \begin{array}{ll}
n^{-p/d}, & 1 \leq p < 2 \\
n^{-2/d}, & 2 \leq p < \infty.
\end{array} \right.
\]

**Proof.** By the numerical inequality \( |x - y| \leq y^{1-p} |x^p - y^p| \) for all \( x, y \geq 0 \), \( p \geq 1 \), one has
\[
\mathbb{E} \left| W_p (\mu_n, \nu_n) - W_p (\mu, \nu) \right| \leq \delta_0^{1-p} \mathbb{E} \left| W_p (\mu_n, \nu_n) - W_p (\mu, \nu) \right|.
\]
The claim thus follows from Corollary 3. \( \square \)

2.1. **Proof of Theorem 2.** We divide our argument into three cases.

2.1.1. **Case 1: \( \alpha = 2 \).** Under conditions (S1), (H0) and (H1), it follows from Lemma 1(ii) that there exist Kantorovitch potentials \( \varphi_n : \mathcal{X} \to \mathbb{R} \) and \( \psi_n : \mathcal{Y} \to \mathbb{R} \) such that \( T_c (\mu_n, \nu_n) = J_{\mu_n, \nu_n} (\varphi_n, \psi_n) \) and \( |\varphi_n|, |\psi_n| \leq 1 \). Furthermore, since \( \mu \) and \( \nu \) are compactly supported, it follows immediately from the definition of the \( c \)-conjugate that \( (\varphi_n, \psi_n) \in \Phi_c (\mu, \nu) \), whence
\[
T_c (\mu, \nu) = \sup_{(\varphi, \psi) \in \Phi_c (\mu, \nu)} J_{\mu, \nu} (\varphi, \psi)
\]
\[
\geq J_{\mu, \nu} (\varphi_n, \psi_n) = J_{\mu_n, \nu_n} (\varphi_n, \psi_n) + \int \varphi_n d(\mu - \mu_n) + \int \psi_n d(\nu - \nu_n).
\]
On the other hand, recalling that \( T_c (\mu_n, \nu_n) = J_{\mu_n, \nu_n} (\varphi_n, \psi_n) \), we derive
\[
T_c (\mu_n, \nu_n) - T_c (\mu, \nu) \leq \int \varphi_n d(\mu_n - \mu) + \int \psi_n d(\nu_n - \nu).
\]
Our goal is now to bound the empirical processes arising on the right-hand side of the above display. Due to the compactness of \( \mathcal{X} \) and \( \mathcal{Y} \), it can be deduced from Gangbo and McCann (1996) that \( \varphi_n \) and \( \psi_n \) are Lipschitz and semi-concave. The following Lemma is an analogue of their results with explicit constants, whose proof is included in Appendix A.2 for completeness.

**Lemma 5.** Assume conditions (S1) and (H0), and that condition (H1) holds with \( \alpha = 2 \). Then the maps
\[
\varphi_n : x \in \mathcal{X} \mapsto \varphi_n (x) - \frac{\Lambda}{2} \| x \|^2, \quad \tilde{\psi}_n : y \in \mathcal{Y} \mapsto \psi_n (y) - \frac{\Lambda}{2} \| y \|^2
\]
are concave and \((2\Lambda)\)-Lipschitz. Furthermore, \( |\varphi_n|, |\tilde{\psi}_n| \leq 2\Lambda \).
For any $L, U > 0$, let $\mathcal{F}_{L,U}(K)$ denote the set of $L$-Lipschitz convex functions $f : K \to \mathbb{R}$ over a convex set $K \subseteq \mathbb{R}^d$, such that $|f| \leq U$. Recalling the convexity of $\mathcal{X}$ and $\mathcal{Y}$ under condition (81), define

$$\Delta_n = \sup_{f \in \mathcal{F}_{1,1}(\mathcal{X})} \int f d(\mu_n - \mu) + \sup_{g \in \mathcal{F}_{1,1}(\mathcal{Y})} \int g d(\nu_n - \nu).$$

By Lemma 5, we have $(-\tilde{\varphi}_n/2 \Lambda) \in \mathcal{F}_{1,1}(\mathcal{X})$ and $(-\tilde{\psi}_n/2 \Lambda) \in \mathcal{F}_{1,1}(\mathcal{Y})$, thus together with equation (19) we obtain

$$\mathcal{T}_c(\mu_n, \nu_n) - \mathcal{T}_c(\mu, \nu) \leq 2 \Delta_n + \frac{\Lambda}{2} \int \|\| d((\mu_n - \mu) + (\nu_n - \nu)).$$

(20)

On the other hand, lower bounds on $\mathcal{T}_c(\mu_n, \nu_n) - \mathcal{T}_c(\mu, \nu)$ are simple to obtain. As before, there exists a pair of optimal Kantorovich potentials $(\varphi_0, \psi_0) \in \Phi_c(\mu, \nu)$ such that $|\varphi_0| \vee |\psi_0| \leq 1$ and $\mathcal{T}_c(\mu, \nu) = J_{\mu,\nu}(\varphi_0, \psi_0)$. Therefore,

$$\mathcal{T}_c(\mu_n, \nu_n) \geq J_{\mu_n,\nu_n}(\varphi_0, \psi_0) = \mathcal{T}_c(\mu, \nu) + \int \varphi_0 d(\mu_n - \mu) + \int \psi_0 d(\nu_n - \nu).$$

Combining the previous two displays, we deduce

$$\mathbb{E}[\mathcal{T}_c(\mu_n, \nu_n) - \mathcal{T}_c(\mu, \nu)] \leq 2 \Delta_n + \frac{\Lambda}{2} \int \|\| d((\mu_n - \mu) + (\nu_n - \nu))$$

$$+ \mathbb{E} \left| \int \varphi_0 d(\mu_n - \mu) \right| + \mathbb{E} \left| \int \psi_0 d(\nu_n - \nu) \right| \lesssim \Lambda \left( \mathbb{E}[\Delta_n] + n^{-1/2} \right),$$

(21)

where the final bound is a straightforward consequence of Chebyshev’s inequality, and where we have used the fact that $|\varphi_0|, |\psi_0| \leq 1$. It thus remains to bound $\mathbb{E}[\Delta_n]$. This last is a sum of expected suprema of empirical processes indexed by convex Lipschitz functions, upper bounds for which can be obtained via Dudley’s chaining technique (Dudley, 2014) in terms of the metric entropy of the class $\mathcal{F}_{L,U}(K)$. Specifically, recall that for all $\epsilon > 0$, the $\epsilon$-metric entropy of a set $A$ contained in a metric space $(\mathcal{X}, \eta)$ is the logarithm of the $\epsilon$-covering number $N(\epsilon, A, \eta)$ of $(A, \eta)$, defined by

$$N(\epsilon, A, \eta) = \inf \{ N \geq 1 : \exists \{x_1, \ldots, x_N\} \subseteq \mathcal{X}, \forall x \in A, \exists 1 \leq i \leq N : \eta(x, x_i) \leq \epsilon \}.$$

The following version of Dudley’s bound will be sufficient for our purposes, and can be deduced for instance from Lemma 16 of von Luxburg and Bousquet (2004) (see also Lemma 3.2 of van de Geer (2000)).

**Lemma 6 (von Luxburg and Bousquet (2004)).** Let $\mathcal{G}$ be a set of real-valued measurable functions on $\mathbb{R}^d$. Then,

$$\mathbb{E} \left[ \sup_{g \in \mathcal{G}} \int g d(\mu_n - \mu) \right] \lesssim \mathbb{E} \left[ \inf_{\tau > 0} \left\{ \tau + \frac{1}{\sqrt{n}} \int_{\tau}^{\infty} \sqrt{\log N(\epsilon, \mathcal{G}, L^2(\mu_n))} \right\} \right].$$

(22)

Tight bounds on the metric entropy of the class $\mathcal{F}_{L,U}(K)$ are well known, and were first obtained in general dimension $d$ by Bronshtein (1976) (see also Dudley (2014)). The following is a version of Bronshtein’s result stated in Theorem 1 of Guntuboyina and Sen (2012) with explicit dependence on the constants $L$ and $U$, which we shall also use in Section 3.

**Lemma 7 (Bronshtein (1976)).** There exist universal constants $C, \epsilon_0 > 0$ such that for every $L, U > 0$ and $b > a$, we have for all $\epsilon \leq \epsilon_0(U + L(b - a))$,

$$\log N(\epsilon, \mathcal{F}_{L,U}([a, b]^d), L^\infty) \leq C \left( \frac{U + L(b - a)}{\epsilon} \right)^\frac{d}{2}.$$
Notice that, by condition (S1),
\[ N(\cdot, \mathcal{F}_{1,1}(X), L^2(\mu)) \leq N(\cdot, \mathcal{F}_{1,1}(X), L^\infty) \leq N(\cdot, \mathcal{F}_{1,1}([-1,1]^d), L^\infty), \]
and the same upper bound holds for the covering number \( N(\cdot, \mathcal{F}_{1,1}(Y), L^2(\nu)) \). Combine these facts with equation (21) and with Lemmas 6–7 to deduce that for any \( \tau > 0 \),
\[ (23) \]
\[ \mathbb{E}\left| \mathcal{T}_c(\mu, \nu_n) - \mathcal{T}_c(\mu, \nu) \right| \lesssim \Lambda \left[ n^{-1/2} + \tau + \frac{1}{\sqrt{n}} \int_{\tau}^{\infty} e^{-\frac{4}{9}d} \, dx \right] \lesssim \Lambda \left[ n^{-1/2} + \tau + \frac{1/4}{\sqrt{n}} \right], \]
where we have used the assumption \( d \geq 5 \). Choosing \( \tau \asymp n^{-2/d} \) leads to the claimed bound,
\[ \mathbb{E}\left| \mathcal{T}_c(\mu, \nu_n) - \mathcal{T}_c(\mu, \nu) \right| \lesssim \Lambda n^{-2/d}. \]

2.1.2. Case 2: \( 1 < \alpha < 2 \). We prove the claim using a smooth approximation of the cost \( h \), thereby appealing to the result of Case 1. Let \( K: \mathbb{R}^d \to \mathbb{R}_+ \) be an even, smooth mollifier with support lying in \( B_{0,1} \). For \( \sigma > 0 \), write \( K_\sigma(x) = \sigma^{-d} K(x/\sigma) \). Define the modified cost function \( c_\sigma(x, y) = h_\sigma(x-y) \), where \( h_\sigma = h \ast K_\sigma \).

**Lemma 8.** Assume conditions (S1), (H0)–(H1) hold for some \( \alpha \in (1, 2) \). Then, there exist universal constants \( C > 0 \) and \( \epsilon \in (0, 1) \) such that for all \( \sigma \in (0, \epsilon) \), the following statements hold.

(i) We have, \( \|h - h_\sigma\|_{L^\infty(Z)} \leq \Lambda \sigma^\alpha \).

(ii) The cost function \( c_\sigma \) itself satisfies condition (H0), and satisfies condition (H1) in the sense that there exists an open set \( \tilde{Z}_1 \supseteq Z \) contained in \( B_{0,2} \) such that \( h_\sigma \leq 1 \) on \( \tilde{Z}_1 \) and
\[ \|h_\sigma\|_{C^2(\tilde{Z}_1)} \leq \Lambda_\sigma := C\Lambda \sigma^{\alpha - 2}. \]

The proof of Lemma 8 appears in Appendix A.3. Notice that
\[ \sup_{\tilde{\mu} \in \mathcal{P}(X)} \sup_{\tilde{\nu} \in \mathcal{P}(Y)} |\mathcal{T}_c(\tilde{\mu}, \tilde{\nu}) - \mathcal{T}_{c_\sigma}(\tilde{\mu}, \tilde{\nu})| \leq \|h - h_\sigma\|_{L^\infty(Z)}, \]
thus Lemma 8(i) implies
\[ \mathbb{E}\left| \mathcal{T}_c(\mu, \nu_n) - \mathcal{T}_c(\mu, \nu) \right| \]
\[ \leq \mathbb{E}\left| \mathcal{T}_c(\mu, \nu_n) - \mathcal{T}_{c_\sigma}(\mu, \nu_n) \right| + \mathbb{E}\left| \mathcal{T}_{c_\sigma}(\mu, \nu_n) - \mathcal{T}_{c_\sigma}(\mu, \nu) \right| + \mathbb{E}\left| \mathcal{T}_{c_\sigma}(\mu, \nu) - \mathcal{T}_c(\mu, \nu) \right| \]
\[ \leq 2\Lambda \sigma^\alpha + \mathbb{E}\left| \mathcal{T}_{c_\sigma}(\mu, \nu_n) - \mathcal{T}_{c_\sigma}(\mu, \nu) \right|. \]

On the other hand, by Lemma 8(ii), we may apply the result of Case 1 to obtain,
\[ \mathbb{E}\left| \mathcal{T}_{c_\sigma}(\mu, \nu_n) - \mathcal{T}_{c_\sigma}(\mu, \nu) \right| \lesssim \Lambda \sigma^{\alpha - 2} n^{-2/d}. \]

Altogether, we deduce that for any \( \sigma \in (0, \epsilon) \),
\[ \mathbb{E}\left| \mathcal{T}_c(\mu, \nu_n) - \mathcal{T}_c(\mu, \nu) \right| \lesssim \Lambda \left[ \sigma^\alpha + \sigma^{\alpha - 2} n^{-2} \right]. \]
Choosing \( \sigma \asymp n^{-1/d} \) leads to an upper bound scaling at the rate \( \Lambda n^{-\alpha/d} \) on the right-hand side of the above display, as claimed.
2.1.3. Case 3: $0 < \alpha \leq 1$. When $\alpha \in (0, 1]$, the claim follows by a simpler argument than that of Case 1. For any bounded set $K \subseteq \mathbb{R}^d$ and $L > 0$, define the $\alpha$-Hölder ball $C^\alpha(K; L) = \{ f \in C^\alpha(K) : \| f \|_{C^\alpha(K)} \leq L \}$, and let $\varphi_n$ and $\psi_n$ be defined as in Case 1. When $\alpha < 2$, Lemma 5 no longer guarantees that these potentials are semi-concave, however the following Hölder estimate is easily derived, and stated without proof.

**Lemma 9.** Assume conditions (S1) and (H0), and that condition (H1) holds with $\alpha \in (0, 1]$. Then, $\varphi_n \in C^\alpha(X; \Lambda)$ and $\psi_n \in C^\alpha(Y; \Lambda)$ for all $n \geq 1$.

By an analogous reduction as in Case 1, we therefore have for any $\tau > 0$,

$$\mathbb{E} | T_c(\mu_n, \nu_n) - T_c(\mu, \nu) | \lesssim \Lambda \left[ n^{-1/2} + \tau + \frac{1}{\sqrt{n}} \int_\tau^{\infty} \sqrt{\log N(\varepsilon, C^\alpha([-1,1]^d; 1), L^\infty)} d\varepsilon \right].$$

By Theorem 2.7.1 of van der Vaart and Wellner (1996), one has

$$\log N(\varepsilon, C^\alpha([-1,1]^d; 1), L^\infty) \lesssim \varepsilon^{-d/\alpha}, \quad \varepsilon > 0,$$

implying that the right-hand side of the penultimate display is of order $\Lambda n^{-\alpha/d}$ if $\tau \asymp n^{-\alpha/d}$ and $d \geq 5 > 2\alpha$. The claim follows.

**Remark 1.** Though Theorem 12 is only stated when $d \geq 5$, a simple extension of our proof yields the rate $n^{-1/2}$ whenever $d \leq 4$ and $\alpha = 2$, up to a logarithmic factor when $d = 4$; this follows by taking $\tau = 0$ in equation (23). A similar extension can be made when $\alpha \in (0, 1]$. On the other hand, our mollification step for the case $\alpha \in (1, 2)$ does not appear to yield a sharp convergence rate when $d < 2\alpha < 5$. After a preprint of our paper was made available, the work of Hundrieser, Staudt and Munk (2022) has extended Theorem 2, by showing that in all dimensions $d \geq 1$, and for all $\alpha \in (0, 2]$, the upper bound

$$\mathbb{E} | T_c(\mu_n, \nu_n) - T_c(\mu, \nu) | \lesssim \begin{cases} n^{-\alpha/d}, & 2\alpha < d \\ n^{-\alpha/d} \log n, & 2\alpha = d \\ n^{-1/2}, & 2\alpha > d \end{cases} \quad (24)$$

holds under the same assumptions as those of Theorem 2. In particular, this result recovers our Theorem 2 when $d \geq 5$.

3. Upper Bounds for Unbounded Measures under Tail Conditions. We now turn to upper bounding the rate of convergence of the empirical optimal transport cost for measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ with unbounded support, under suitable tail conditions. We shall assume that the cost function $c$ satisfies the following smoothness assumption, which is a suitable generalization of condition (H1) to the present setting.

**(H2)** $h \in C^2_{\text{loc}}(\mathbb{R}^d)$, and there exist $p, \Lambda \geq 1$ such that $\| h \|_{C^2(B_{\rho,r})} \leq \Lambda r^p$ for all $r > 1$.

Notice that unlike in Section 2, we limit our exposition to costs lying in $C^2_{\text{loc}}$ rather than $C^\alpha_{\text{loc}}$ for all $\alpha \in (0, 2]$. As we shall see, our main result is nevertheless sufficiently general to cover the costs $h(x) = \| x \|_p$ for all $p > 1$, via an approximation argument.

Our upper bounds in Section 2 hinged upon Lemma 5, which provided quantitative estimates on the Lipschitz and semi-concavity constants of optimal Kantorovich potentials, for any sufficiently smooth cost function over a compact set. In contrast, we now only assume a local Hölder estimate on $c$ in assumption (H2), thus the optimal Kantorovich potentials between two measures on $\mathbb{R}^d$ will generally not be *globally* Lipschitz or semi-concave. While these properties nevertheless hold *locally* under rather general conditions (Gangbo and McCann, 1996), we are not aware of existing quantitative estimates under the conditions required...
for our development. One of our key technical contributions in this section is to obtain such quantitative bounds, which we now describe before stating our main result. We begin with the following straightforward generalization of Lemma 5, whose proof appears in Appendix B.1.

**Lemma 10.** Let $c$ be a cost function satisfying conditions (H0) and (H2) with $h$ superlinear. Given two measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, let $\pi \in \Pi(\mu, \nu)$ be an optimal coupling with respect to $c$, and assume there exists a locally bounded $c$-concave function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that $\text{supp}(\pi) \subseteq \partial^c \varphi$. Let $r \geq 1$, and let

$$
\Lambda_r = \Lambda \sup \left\{ \|x - y\|^p : x \in B_{0,r}, \ y \in \partial^c \varphi(B_{0,r}) \right\}.
$$

Then, there exist universal constants $c_1, c_2 > 0$ such that the map

$$
\phi : B_{0,r} \to \mathbb{R}, \ \phi(x) = \varphi(x) - c_1 \Lambda_r \|x\|^2,
$$

is concave, and Lipschitz with parameter $c_2 r \Lambda_r$.

Lemma 10 shows that the local Lipschitz and semi-concavity constants for a Kantorovich potential $\varphi$ are largely driven by the maximal displacement induced by the coupling $\pi$ over points lying in $B_{0,r}$. We will show how $L^\infty$ estimates on these displacements can be obtained under the following conditions on $\mu, \nu$.

(i) **Super-Gaussian Anticoncentration.** We will say a measure $\mu$ is $(\gamma, b)$-super-Gaussian for some $\gamma, b > 0$ if for any $x \in \mathbb{R}^d$,

$$
\mu(B_x) \geq b \cdot \mathbb{P}(Z \in B_x), \quad \text{where } Z \sim N(0, \gamma^2).
$$

(ii) **Sub-Weibull Concentration.** A measure $\nu$ is said to be $(\sigma, \beta)$-sub-Weibull (Kuchibhotla and Chakrabortty, 2022; Vladimirova et al., 2020) for some $\sigma > 0$ and $0 < \beta \leq 2$ if

$$
\int \exp \left[ \frac{1}{2} \left( \frac{\|y\|}{\sigma} \right) ^\beta \right] d\nu(y) \leq 2.
$$

The assumption of super-Gaussianity implies an anticoncentration bound for the underlying measure, in the sense that it cannot place significantly less probability mass than a Gaussian distribution in any unit-radius ball. In Appendix B.9, we show that a measure is super-Gaussian whenever it admits a regular Lebesgue density in the sense of Polyanskiy and Wu (2016). For instance, Polyanskiy and Wu show that for any probability measure $\mu$ with finite first moment, the mixture distribution $K_r \ast \mu$ admits a regular density, where $K_r$ is the $N(0, \tau^2 I_d)$ density for some $\tau > 0$. Any such measure is thus also super-Gaussian. We also note that absolute continuity is not necessary for super-Gaussianity; for example, given any (possibly atomic) measure $\rho \in \mathcal{P}(\mathbb{R}^d)$, any super-Gaussian measure $\mu$, and any $\lambda \in (0, 1)$, the measure $\lambda \rho + (1 - \lambda) \mu$ is also super-Gaussian.

On the other hand, the sub-Weibull condition is a concentration assumption which generalizes the well-known sub-Gaussian and sub-exponential conditions, which respectively correspond to the cases $\beta = 2$ and $\beta = 1$ up to rescaling of the constant $\sigma$ (Boucheron, Lugosi and Massart, 2013). Indeed, it is a straightforward consequence of Markov’s inequality that if $Y$ has $(\sigma, \beta)$-sub-Weibull distribution for some $\beta \in (0, 2]$ and $\sigma > 0$, then for all $u > 0$,

$$
\mathbb{P}(\|Y\| \geq u) \leq 2 \exp \left\{ -\frac{1}{2} \left( \frac{u}{\sigma} \right) ^\beta \right\}.
$$

Finally, we shall require the following condition on the cost function $c$. 


(H3) We have $h(0) = 0$. Furthermore, there exist constants $p > 1$, $\kappa \geq 1$, and a convex differentiable function $\omega : (1, \infty) \to (1, \infty)$ such that

$$h(z) = \omega(\|z\|), \quad \text{and} \quad \frac{1}{\kappa} \|z\|^{p-1} \leq \omega'(\|z\|) \leq \kappa \|z\|^{p-1} \quad \text{for all} \quad \|z\| > 1.$$  

Condition (H3) implies that the function $h$ is superlinear, with order of growth comparable to that of $\|\cdot\|^p$ for some $p > 1$. Aside from the assumption $h(0) = 0$, which can always be satisfied by translation, we emphasize that condition (H3) does not constrain the behavior of $h$ near zero, but is nevertheless stronger than the conditions assumed in Section 2. Therefore, we provide several examples of cost functions satisfying these two conditions before turning to our main results.

The most important example of a cost satisfying our assumptions is, of course, $\|\cdot\|^p$ for $p \geq 2$. However, when $p \in (1, 2)$, the cost $\|\cdot\|^p$ does not satisfy condition (H2); to study this case, we will employ an approximation argument with the cost function $h_\epsilon(x) = (\|x\|^2 + \epsilon^{2/p})^{p/2} - \epsilon$, which satisfies (H2)–(H3) for any $\epsilon > 0$ and $p > 1$. This cost function has been of interest in its own right in the optimal transport literature, as it forms an approximation of $\|\cdot\|^p$ which satisfies the celebrated Ma-Trudinger-Wang regularity conditions even when $p \neq 2$ (Ma, Trudinger and Wang, 2005; Li, Santambrogio and Wang, 2014).

Conditions (H2)–(H3) also hold for costs that have different power-type behaviors at the origin and infinity, such as $h(z) = \lambda_p \|z\|^p + \lambda_q \|z\|^q$ for $p > q \geq 2$, which arise in the study of modified transport problems with congestion costs (Brasco, Carlier and Santambrogio, 2010; Carlier, Jimenez and Santambrogio, 2008).

More generally, conditions (H2)–(H3) are satisfied by any twice continuously differentiable convex cost function of the form

$$h(x) \propto \begin{cases} \|x\|^p, & \|x\| > 1 \\ h_0(x), & \|x\| \leq 1, \end{cases}$$

where $h_0(0) = 0$ and $p \geq 2$. This family includes, for instance, smooth approximations of the truncated cost $h(x) = \|x\|^p I(x \in B_0^c)$, and $\ell_p$ analogues of Huber’s loss function.

While the smoothness condition (H2) will be needed in order to appeal to Lemma 10, assumption (H3) is sufficient to obtain the following result, which plays a central role in our development.

**Theorem 11.** Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ and assume $\nu$ is $(\sigma, \beta)$-sub-Weibull. Let $c$ be a cost function satisfying conditions (H0) and (H3), and let $\pi \in \Pi(\mu, \nu)$ have $c$-cyclically monotone support. Then, there exists a constant $C > 0$ depending on $d, p, \beta, \kappa$ such that for any $c$-concave function $\varphi$ satisfying $\text{supp}(\pi) \subseteq \partial^c \varphi$,

$$\sup_{y \in \partial^c \varphi(x)} \|y\| \leq C \sigma \left( (\|x\| + 1) \vee \sup_{\|x-y\| \leq 2} \left[ \log \left( \frac{1}{\mu(B_y)} \right) \right]^{1/2} \right), \quad x \in \mathbb{R}^d. \quad (26)$$

In particular, if $\mu$ is $(\gamma, b)$-super-Gaussian, $h$ is strictly convex, and $T_c(\mu, \nu) < \infty$, then the unique optimal transport map $T$ pushing $\mu$ forward onto $\nu$ satisfies for $\mu$-a.e. $x \in \mathbb{R}^d$,

$$\|T(x)\| \leq C' \sigma (\|x\| + 1)^{1/2}, \quad (27)$$

for a constant $C' > 0$ depending on $d, \kappa, p, \beta, \gamma, b$.

We defer the proof to Section 3.2. Theorem 11 implies that any optimal transport plan between $\mu$ and $\nu$ does not move probability mass from any point $x \in \mathbb{R}^d$ by more than a polynomial of $\|x\|$. To obtain this result, we required an anticoncentration assumption on
the source measure $\mu$ and a concentration assumption on the target measure $\nu$, ensuring that their tails are sufficiently comparable to avoid large transports of mass. It is easy to see that assumptions of this nature are necessary: for instance, if $\mu$ were compactly supported and $\nu$ were supported over $\mathbb{R}^d$, any transport plan in $\Pi(\mu, \nu)$ would couple a nonzero amount of mass from the bounded support of $\mu$ with points lying at an arbitrarily far distance.

In the special case where $\nu$ is sub-Gaussian, its tails are no heavier than those of a super-Gaussian measure $\mu$. Equation (27) shows that the optimal transport map from $\mu$ to $\nu$ grows at most linearly in this regime, irrespective of the order of growth $p$ of the cost function. This bound is clearly unimprovable in general, as can be seen by taking $\mu = \nu$.

Our proof of Theorem 11 is inspired by its non-quantitative analogues proven by Gangbo and McCann (1996), and by Colombo and Fathi (2021) who derived analogous quantitative bounds for the special case where $\mu$ is a Gaussian measure and $h(x) = \|x\|^2$. Unlike Colombo and Fathi (2021), our result holds for any cost function satisfying conditions (H0) and (H3), and for general measures $\mu, \nu$ which are not presumed to be absolutely continuous with respect to the Lebesgue measure. We shall require this level of generality in the sequel, when Theorem 11 will be invoked for $\mu$ and $\nu$ replaced by their empirical counterparts.

Equipped with Theorem 11, we are ready to state the main result of this section.

**Theorem 12.** Assume conditions (H0), (H2) and (H3) hold. Assume further that $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ are both $(\sigma, \beta)$-sub-Weibull, and $(\gamma, b)$-super-Gaussian. Then, there exists a constant $C > 0$ depending on $\sigma, \beta, \gamma, b, \kappa, p, d$ such that

$$\mathbb{E}\left| T_c(\mu_n, \nu_n) - T_c(\mu, \nu) \right| \leq C n^{-\frac{\sigma}{4}}.$$

Theorem 12 shows that, for $C^\alpha_{\text{loc}}$ convex costs with polynomial rate of growth, the $n^{-2/d}$ rate of convergence obtained for compactly supported measures in Section 2 carries over to unboundedly supported measures, with tails satisfying suitable concentration and anticoncentration conditions. While Theorem 12 does not provide upper bounds for $C^\alpha_{\text{loc}}$ costs when $\alpha < 2$, it is sufficiently general to deduce the following special case.

**Corollary 13.** Assume $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ satisfy the same conditions as Theorem 12. Then, for all $p > 1$,

$$\mathbb{E}\left| W_p^p(\mu_n, \nu_n) - W_p^p(\mu, \nu) \right| \leq \begin{cases} n^{-p/d}, & 1 < p \leq 2 \\ n^{-2/d}, & 2 \leq p < \infty. \end{cases}$$

For the regime $p \geq 2$, this result follows as a direct consequence of Theorem 12, while for the regime $1 < p < 2$, we achieve the claim by using a smooth uniform approximation of $\|\cdot\|^p$ which satisfies the conditions of Theorem 12. The proof is deferred to Appendix B.7.

By reasoning identically as in Corollary 4, one can also deduce a convergence rate for empirical Wasserstein distances between measures with unbounded support. In particular, equation (17) continues to hold for all $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ satisfying the conditions of Theorem 12, and satisfying $W_p(\mu, \nu) \geq \delta_0 > 0$.

3.1. **Proof of Theorem 12.** As in the proof of Theorem 2, we shall reduce the problem of bounding the $L^1(\mathbb{P})$ convergence rate of $T_c(\mu_n, \nu_n)$ to that of bounding the supremum of an empirical process. Unlike in Theorem 2, the relevant empirical process in this section will be indexed by locally semi-concave Lipschitz functions, with quantitative local Lipschitz and semi-concavity moduli obtained in part by appealing to Theorem 11. Our proof proceeds with eight steps, the first five of which carry out this reduction, and the last three of which
bound the resulting empirical process. Throughout the proof, \( C, C', C_i, c_i > 0, i \geq 0 \), denote constants possibly depending on \( \sigma, \beta, \gamma, b, \kappa, p, d \), which do not depend on \( \Lambda \) or otherwise on \( c, \mu \) and \( \nu \), and whose value may change from line to line. Likewise, the symbols \( \lesssim \) and \( \asymp \) hide constants possibly depending on the former quantities. All intermediary results appearing in the sequel are proven in Appendix B.

**Step 0: Setup.** Let \( L_j = [-3^j, 3^j]^d \) for all \( j \geq 0 \). For all \( j \geq 1 \), let \( I_{jk} \) denote the \( m_d := 3^d - 1 \) cubes of side-length \( 2 \cdot 3^{j-1} \) forming the natural partition of \( L_j \setminus L_{j-1} \). For notational convenience, set \( I_0 = I_{0k} = L_0 \) for all \( k = 1, \ldots, m_d \), and define

\[
I_j := L_j \setminus L_{j-1} = \bigcup_{k=1}^{m_d} I_{jk}, \quad j \geq 1.
\]

Note that \( \{I_j : j \geq 0\} \) and \( \{I_{jk} : j \geq 0, 1 \leq k \leq m_d\} \) are partitions of \( \mathbb{R}^d \), up to measure-zero intersections. We also write \( \ell_j = \sup_{x \in I_j} \|x\| = \sqrt{d}3^j \) for all \( j \geq 0 \).

Let \( \mathcal{F} \) denote the set of convex functions over \( \mathbb{R}^d \). Recall that \( \mathcal{F}_{m,u}(I) \) denotes the set of \( m \)-Lipschitz convex functions over a convex set \( I \subseteq \mathbb{R}^d \), which are uniformly bounded over \( I \) by \( u > 0 \). Let \( M = (M_j : j \geq 0) \) and \( U = (U_j : j \geq 0) \) denote sequences of nonnegative real numbers, and let

\[
K_{M,U} = \left\{ f : \mathbb{R}^d \to \mathbb{R} : (-f) |_{I_{jk}} \in \mathcal{F}_{M,U_j}(I_{jk}), \, j \geq 0, 1 \leq k \leq m_d \right\},
\]

**Step 1: Extension of Kantorovich Potentials.** Let

\[
R_n = \max_{1 \leq i \leq n} \|X_i\| \vee \|Y_i\|.
\]

Conditions (H0) and (H3) imply that \( h(z) \leq \kappa \|z\|^p \) for all \( z \in \mathbb{R}^d \), thus \( h \) is bounded above by \( \bar{R}_n := \kappa (2R_n)^p \) over \( B_{0,2R_n} \). It can then be deduced from Lemma 1(ii) that there exist potentials \( f_n : \text{supp}(\mu_n) \to [-\bar{R}_n, 0] \) and \( g_n : \text{supp}(\nu_n) \to [0, \bar{R}_n] \) such that \( (f_n, g_n) \in \Phi_c(\mu_n, \nu_n) \) and \( (f_n, g_n) \) is optimal for the optimal transport problem from \( \mu_n \) to \( \nu_n \). We extend the domain of \( f_n \) and \( g_n \) to \( \mathbb{R}^d \) using the following construction. Define for all \( y \in \mathbb{R}^d \),

\[
\eta_n(y) = \inf_{x \in \text{supp}(\mu_n)} \left\{ c(x, y) - f_n(x) \right\} \wedge \bar{R}_n,
\]

and for all \( x, y \in \mathbb{R}^d \),

\[
\varphi_n(x) = \eta_n(x) = \inf_{y \in \mathbb{R}^d} \left\{ c(x, y) - \eta_n(y) \right\}, \quad \psi_n(y) = \eta_n(y) = \inf_{x \in \mathbb{R}^d} \left\{ c(x, y) - \eta_n(x) \right\}.
\]

**Lemma 14.** Given an optimal coupling \( \pi_n \) between \( \mu_n \) and \( \nu_n \), the following hold.

(i) For all \( x, y \in \mathbb{R}^d \), \( \varphi_n(x) + \psi_n(y) \leq c(x, y) \).
(ii) \( \varphi_n(x) = f_n(x) \) for all \( x \in \text{supp}(\mu_n) \), and \( \psi_n(y) = g_n(y) \) for all \( y \in \text{supp}(\nu_n) \). In particular,

\[
\mathcal{T}_c(\mu_n, \nu_n) = \int \varphi_n d\mu_n + \int \psi_n d\nu_n.
\]

(iii) For all \( x \in \mathbb{R}^d \), \( |\varphi_n(x)| \vee |\psi_n(x)| \leq \bar{R}_n \).
(iv) For all \( (x, y) \in \text{supp}(\pi_n), (x, y) \in \partial^c \varphi_n \) and \( (y, x) \in \partial^c \psi_n \).

By Lemma 14(iii), \( \varphi_n \) and \( \psi_n \) are bounded, so \( \varphi_n \in L^1(\mu) \) and \( \psi_n \in L^1(\nu) \). This fact combined with Lemma 14(i) guarantees that \( (\varphi_n, \psi_n) \in \Phi_c(\mu, \nu) \), whence

\[
\mathcal{T}(\mu, \nu) = \sup_{(\varphi, \psi) \in \Phi_c(\mu, \nu)} \int \varphi d\mu + \int \psi d\nu.
\]
It remains to bound the last two terms on the right-hand side of the above display. We shall do so by first proving that \( \varphi_n, \psi_n \in \mathcal{K}_{M,U} \) with high probability, for suitable sequences \( M \) and \( U \). We focus on \( \varphi_n \), and a symmetric argument can be used for \( \psi_n \).

By Lemma 10, recall that the Lipschitz and semi-concavity moduli of \( \varphi_n|_{I_{j,k}} \) are largely driven by the magnitude of the \( c \)-superdifferential \( \partial^c \varphi_n(I_{j,k}) \). The bulk of our effort will go into bounding this quantity. In fact, it will suffice to bound that of \( \partial^c \varphi_n(L_j) \), for all \( j \geq 0 \). To do so, we proceed with the following step, in view of invoking Theorem 11 with the measures \( \mu_n \) and \( \nu_n \).

**Step 2: Global Concentration and Local Anticoncentration of \( \mu_n, \nu_n \).** Fix \( \rho = \frac{2p}{\beta} \lor \frac{2}{4} \), and set

\[
V_{1,n} = \int \exp \left( \frac{\|x\|^\beta}{2\rho \sigma^\beta} \right) d\mu_n(x), \quad V_{2,n} = \int \exp \left( \frac{\|y\|^\beta}{2\rho \sigma^\beta} \right) d\nu_n(y).
\]

By Jensen’s inequality, notice that

\[
\int \exp \left( \frac{\|x\|^\beta}{2\rho V_{1,n} \sigma^\beta} \right) d\mu_n(x) \leq V_{1,n}^{1/2V_{1,n}} \leq 2,
\]

implying that \( \mu_n \) is \((\sigma(\rho V_{1,n})^{1/\beta}, \beta)\)-sub-Weibull. Similarly, \( \nu_n \) is \((\sigma(\rho V_{2,n})^{1/\beta}, \beta)\)-sub-Weibull, implying that both are \((\sigma(\rho V_n)^{1/\beta}, \beta)\)-sub-Weibull when \( V_n = V_{1,n} + V_{2,n} \).

We further show that \( \mu_n \) satisfies a high-probability anticoncentration bound in a sufficiently small region about the origin. Specifically, define the integer

\[
J_n = \left\lfloor \frac{1}{2} \log_3 \left( \frac{\gamma^2}{4d} \log n \right) \right\rfloor,
\]

and the event

\[
A_n = \bigcap_{j=0}^{J_n} \left\{ \inf_{x \in L_j} \inf_{\|x-y\| \leq 2} \mu_n(B_y) \geq \frac{C_1}{2} \exp(-\ell_j^2 / \gamma^2) \right\},
\]

where we recall that the parameter \( \gamma \) arises from the super-Gaussianity assumption on \( \mu \) and \( \nu \). The following result is proven using elementary tools from empirical process theory.

**Lemma 15.** There exists a sufficiently large choice of the constant \( C_1 > 0 \), depending only on \( \gamma \), such that \( \mathbb{P}(A_n^C) \lesssim 1/n \).

**Step 3: Bounding \( \partial^c \varphi_n \).** Step 2 will allow us to bound \( \partial^c \varphi_n(L_j) \) whenever \( 0 \leq j \leq J_n \) by invoking Theorem 11. On the other hand, \( \mu_n \) may place insufficient mass outside the box \( L_{J_n} \) to appeal to Theorem 11 when \( j > J_n \), thus we treat this case separately below.

- **Regime 1:** \( 0 \leq j \leq J_n \). By Step 2, \( \nu_n \) is \((\sigma(\rho V_n)^{1/\beta}, \beta)\)-sub-Weibull, and \( \text{supp}(\pi_n) \subseteq \partial^c \varphi_n \) by Lemma 14(iv). Therefore, by Theorem 11, we have for all \( 0 \leq j \leq J_n \),

\[
\sup_{y \in \partial^c \varphi_n(L_j)} \|y\| \lesssim V_n^{1/2} \left\{ (\ell_j + 1) \lor \sup_{\|x-y\| \leq 2} \left[ \log \left( \frac{1}{\mu_n(B_y)} \right) \right]^{\frac{1}{\beta}} \right\}.
\]

Over the event \( A_n \), we therefore have uniformly in \( 0 \leq j \leq J_n \),

\[
\sup_{y \in \partial^c \varphi_n(L_j)} \|y\| \lesssim V_n^{\frac{1}{2}} \left[ (\ell_j + 1 + \ell_j^2 / \gamma^2) \right]^{\frac{1}{\beta + 1}} \lesssim V_n^{\frac{1}{2}} \ell_j^{\frac{2}{\beta + 1}} \lesssim V_n^{\frac{1}{2}} 3^{q_1},
\]

for a large enough exponent \( q_1 \geq 1 \).
• **Regime 2:** $J_n < j < \infty$. In this regime, it will suffice to provide a crude bound on \( \partial^c \varphi_n(L_j) \). We begin with the following result, which is a quantitative generalization of Proposition C.4 of Gangbo and McCann (1996).

**Proposition 16.** Let $R, \tau \geq 4$. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a c-concave function such that $|\varphi| \leq R$ over $B_{0,\tau}$. Then, under conditions (H0) and (H3), we have

$$\sup_{y \in \partial^c \varphi(B_{0,\tau/2})} \| y \|^{p-1} \leq C_{p,\kappa}(r^{p-1} + R),$$

for a constant $C_{p,\kappa} > 0$ depending only on $p$ and $\kappa$.

By Proposition 16, it will suffice to bound $|\varphi_n(x)|$ uniformly over $x \in L_j$, for all $j \geq J_n$. Recall from Lemma 14 that $|\varphi_n(x)| \leq \overline{p}_n$, thus it suffices to bound $\overline{p}_n$. Define $D_n = \sigma(4 \log n)^{1/\beta}$ and the event $A'_n = \{ R_n \leq D_n \}$. Apply a union bound together with the sub-Weibull assumption on $\mu$ and $\nu$ to deduce that

$$\mathbb{P}((A'_n)^c) \lesssim n \exp \left\{ -\frac{1}{2} \left( \frac{D_n}{\sigma} \right)^\beta \right\} \lesssim \frac{1}{n}. \tag{32}$$

Over the event $A'_n$, we therefore have $|\varphi_n(x)| \leq \overline{p}_n \lesssim D_n^p$ for all $x \in \mathbb{R}^d$. Combined with Proposition 16, we deduce

$$\sup_{y \in \partial^c \varphi_n(L_j)} \| y \| \lesssim \left[ \ell_j + D_n^{p-1} \right] \lesssim (\log n)^{\frac{1}{2(\alpha-1)}} 2^j \lesssim (3^j \log n)^{q_2},$$

for a large enough exponent $q_2 > 0$.

In summary, the following holds over the event $A_n \cap A'_n$, uniformly in $j \geq 0$,

$$\sup_{y \in \partial^c \varphi_n(L_j)} \| y \| \leq H_j := C_2 \begin{cases} 3^{j \eta_1} V_n^{\frac{1}{2}}, & 0 \leq j \leq J_n \\ (3^j \log n)^{q_2}, & j > J_n. \end{cases} \tag{33}$$

**Step 4: Bounding the Lipschitz and Semi-Concavity Moduli of $\varphi_n$.** Define for a large enough constant $C_3 > 0$,

$$\xi_n(x) = C_3 \| x \|^2 \sum_{j=0}^\infty H_j^p I(x \in I_j),$$

and set $\tilde{\varphi}_n(x) = \frac{1}{2} \varphi_n(x) - \xi_n(x)$. Under condition (H3), it follows from Lemma 10 that $\tilde{\varphi}_n|_{I_{jk}}$ is $C_4 H_j^p \ell_j^2$-Lipschitz and concave for all $j \geq 0$ and $k = 1, \ldots, m_d$. Furthermore, the map $\tilde{\varphi}_n - \tilde{\varphi}_n(0)$ is bounded over $L_j$, and hence also over $I_{jk}$, by $C_5 H_j^p \ell_j^2$. Thus, there exist sufficiently large exponents $r_i \geq 1, 1 \leq i \leq 4$, such that if $M = (M_j)_{j=0}^\infty$, then

$$M_j = C_4 \begin{cases} 3^{j r_1}, & 0 \leq j \leq J_n \\ (3^j \log n)^{r_2}, & j > J_n, \end{cases}, \quad U_j = C_4 \begin{cases} 3^{j r_3}, & 0 \leq j \leq J_n \\ (3^j \log n)^{r_4}, & j > J_n. \end{cases} \tag{34}$$

then $V_n^{-\frac{1}{2}} (\tilde{\varphi}_n - \tilde{\varphi}_n(0)) \in K_{M, U}$, over the event $A_n \cap A'_n$.

**Step 5: Empirical Process Reduction.** We deduce from Step 4 that, over $A_n \cap A'_n$,

$$\left| \int \varphi_n d(\mu_n - \mu) \right| = \Lambda \left| \int \tilde{\varphi}_n d(\mu_n - \mu) + \int \xi_n d(\mu_n - \mu) \right| \leq \Lambda \left| \int (\tilde{\varphi}_n - \tilde{\varphi}_n(0)) d(\mu_n - \mu) \right| + \Lambda \left| \int \xi_n d(\mu_n - \mu) \right|.$$
\[ \leq \Lambda V_n^{\frac{2}{p}} \sup_{f \in \mathcal{K}_{M,U}} \int f d(\mu_n - \mu) + \Lambda \left| \int \xi_n d(\mu_n - \mu) \right| \text{.} \]

Apply the same argument over the event
\[ E_n = A_n \cap A'_n \cap \bigcup_{j=0}^{J_n} \left\{ \inf_{x \in I_j} \inf_{y \in \mathcal{B}_j} \nu_n(B_y) \geq \frac{C_1}{2} \exp\left(-\ell_j^2/\gamma^2\right) \right\} \]

so that, \( V_n^{-\beta/\rho} \left( \psi_n - \tilde{\psi}_n(0) \right) \in \mathcal{K}_{M,U} \), where \( \tilde{\psi}_n(y) = \frac{1}{\Lambda} \psi_n(y) - \xi_n(y) \), up to increasing the constants \( C, C', C_3 > 0 \), and that \( P(E_n^c) \lesssim 1/n \). We thus have, over \( E_n \),
\[ \int \psi_n d(\nu_n - \nu) \leq \Lambda V_n^{\frac{2}{p}} \sup_{g \in \mathcal{K}_{M,U}} \int g d(\nu_n - \nu) + \Lambda \left| \int \xi_n d(\nu_n - \nu) \right| \text{.} \]

In the sequel, we write
\[ \Delta_n = \sup_{f \in \mathcal{K}_{M,U}} \int f d(\mu_n - \mu) + \sup_{g \in \mathcal{K}_{M,U}} \int g d(\nu_n - \nu), \quad \mathcal{X}_n = \left| \int \xi_n d(\mu_n - \mu) \right| + \left| \int \xi_n d(\nu_n - \nu) \right| \text{,} \]

so that,
\[ \mathbb{E}\left\{ I_{E_n} \left[ \left| \int \varphi_n d(\mu_n - \mu) \right| + \left| \int \psi_n d(\nu_n - \nu) \right| \right] \right\} \leq \Lambda \mathbb{E}[V_n^{\frac{2}{p}} \Delta_n] + \Lambda \mathbb{E}[\mathcal{X}_n] \]
\[ \leq \Lambda \left( \mathbb{E}\left[ V_n^{\frac{2p}{3}} \right] \mathbb{E}\left[ \Delta_n^2 \right] \right)^{1/2} + \Lambda \mathbb{E}[\mathcal{X}_n] \]

Since \( \mu \) and \( \nu \) are \((\sigma, \beta)\)-sub-Weibull, and since \( \rho \geq 2p/\beta \), it readily follows from Jensen’s inequality that \( \mathbb{E}V_n^{2p/\beta} \leq 2 \). Deduce that
\[ \mathbb{E}\left\{ I_{E_n} \left[ \left| \int \varphi_n d(\mu_n - \mu) \right| + \left| \int \psi_n d(\nu_n - \nu) \right| \right] \right\} \leq \Lambda \sqrt{\mathbb{E}[\Delta_n^2]} + \Lambda \mathbb{E}[\mathcal{X}_n] \text{.} \]

**Step 6: Metric Entropy Bound.** Key to bounding \( \mathbb{E}[\Delta_n^2] \) is the following upper bound on the \( L^2(\mu_n) \) metric entropy of the class \( \mathcal{K}_{M,U} \).

**Proposition 17.** There exists \( C_5 > 0 \) such that for all \( \epsilon > 0 \),
\[ \log N(\epsilon, \mathcal{K}_{M,U}, L^2(\mu_n)) \leq C_5 \cdot V_n^{\frac{2}{p}} \epsilon^{-\frac{2}{3}} \text{.} \]

**Proof.** Using Lemma 7, we prove the following result in Appendix B.5, inspired by Corollary 2.7.4 of van der Vaart and Wellner (1996).

**Lemma 18.** For all \( \epsilon > 0 \),
\[ \log N(\epsilon, \mathcal{K}_{M,U}, L^2(\mu_n)) \lesssim \left( \frac{1}{\epsilon} \right)^{\frac{2}{3}} \left( \sum_{j=0}^{m_j} (U_j + \text{diam}(I_{jk}) M_j)^{\frac{2d}{p+1}} \mu_n(I_{jk})^{\frac{d}{p+1}} \right)^{\frac{p+1}{d}} \text{.} \]

In particular, Lemma 18 implies,
\[ \log N(\epsilon, \mathcal{K}_{M,U}, L^2(\mu_n)) \lesssim \left( \frac{1}{\epsilon} \right)^{\frac{2}{3}} \left( \sum_{j=0}^{\infty} (U_j + 3^j M_j)^{\frac{2d}{p+1}} \mu_n(I_j)^{\frac{d}{p+1}} \right)^{\frac{p+1}{d}} \text{.} \]
By Markov’s inequality, notice that for all $j \geq 0$,
\[
\mu_n(I_j) \leq \mu_n(B_{3j-1}^\epsilon) \lesssim \frac{\int \exp \left( \frac{\|x\|^4}{2\rho^2} \right) d\mu_n(x)}{\exp \left( \frac{3(j-1)^\beta}{2\rho^2} \right)} \leq V_n \exp(-c_13^{j\beta}),
\]
so that
\[
\log N(\epsilon, \mathcal{K}_{M,U}, L^2(\mu_n)) \lesssim \left( \frac{1}{\epsilon} \right)^{\frac{d}{2}} V_n^\frac{d}{2} \left( \sum_{j=0}^{J_n} (U_j + 3^jM_j)^\frac{2d}{3d+4} \exp(-c_13^{j\beta}) \right)^{\frac{4+4d}{4d+4}} \leq \left( \frac{1}{\epsilon} \right)^{\frac{d}{2}} V_n^\frac{d}{2} \left( \sum_{j=0}^{J_n} (3^{j+1}M_j)^\frac{2d}{3d+4} \exp(-c_13^{j\beta}) \right)^{\frac{4+4d}{4d+4}}.
\]

We deduce that there exist constants $c_2, c_3 > 0$ such that
\[
\log N(\epsilon, \mathcal{K}_{M,U}, L^2(\mu_n)) \lesssim \left( \frac{1}{\epsilon} \right)^{\frac{d}{2}} V_n^\frac{d}{2} \left[ 1 + (\log n)^{c_2} \sum_{j=J_n+1}^{\infty} 3^{j\beta} \exp(-c_33^{j\beta}) \right]^{\frac{4+4d}{4d+4}} \lesssim \left( \frac{1}{\epsilon} \right)^{\frac{d}{2}} V_n^\frac{d}{2}.
\]

Notice that $\sum_{j=0}^{\infty} 3^{j\beta} \exp(-c_33^{j\beta}) < \infty$, thus the first summation on the right-hand side of the above display is finite. For the second summation, notice that there exists $J_0 > 0$ such that for all $j \geq J_0$, $3^{j\beta} \leq \exp(c_43^{j\beta})$ where $c_4 = c_3/2$. Thus, since $J_n \asymp \log \log n$, we obtain
\[
\log N(\epsilon, \mathcal{K}_{M,U}, L^2(\mu_n)) \lesssim \left( \frac{1}{\epsilon} \right)^{\frac{d}{2}} V_n^\frac{d}{2} \left[ 1 + (\log n)^{c_2} \sum_{j=J_n+1}^{\infty} \exp(-c_43^{j\beta}) \right]^{\frac{4+4d}{4d+4}} \lesssim \left( \frac{1}{\epsilon} \right)^{\frac{d}{2}} V_n^\frac{d}{2}.
\]

This proves Proposition 17.

**Step 7: Chaining.** Equipped with Proposition 17, we are now in a position to bound the expected (squared) supremum of the empirical process indexed by $\mathcal{K}_{M,U}$. We begin by noting the following.

**Lemma 19.** It holds that
\[
\mathbb{E} \left[ \left( \sup_{f \in \mathcal{K}_{M,U}} \int f d(\mu_n - \mu) \right)^2 \right] \lesssim \frac{(\log n)^{2\tau_4}}{n} + \mathbb{E} \left[ \sup_{f \in \mathcal{K}_{M,U}} \int f d(\mu_n - \mu) \right]^2.
\]

By combining Lemma 19 with Lemma 6 and Proposition 17, we deduce that for all $\tau > 0$,
\[
\mathbb{E} \left[ \left( \sup_{f \in \mathcal{K}_{M,U}} \int f d(\mu_n - \mu) \right)^2 \right] \lesssim \frac{(\log n)^{2\tau_4}}{n} + \left( \tau + \mathbb{E} V_n^\frac{d}{2} \int_0^\infty \left( \frac{1}{\epsilon} \right)^{\frac{d}{2}} d\epsilon \right)^2.
\]
Since $\mu$ and $\nu$ are $(\sigma, \beta)$-sub-Weibull, and since $\rho \geq d/4$, we again have by Jensen’s inequality that $\mathbb{E}[V_{i,n}^{d/4}] \leq 2$ for both $i = 1, 2$, implying that $\mathbb{E}[V_{n}^{d/4}] \leq c_5$. Choosing $\tau \asymp n^{-2/d}$ in the above display thus leads to a bound scaling at the rate $n^{-4/d}$. Upon repeating the same argument for $\nu_n$, we obtain

$$\sqrt{\mathbb{E}[\Delta_n^2]} \lesssim n^{-2/d}. \quad (35)$$

**Step 8: Conclusion.** Let $(\varphi_0, \psi_0) \in \Phi_c(\mu, \nu)$ be a pair of optimal Kantorovich potentials between $\mu$ and $\nu$. It follows similarly as in the proof of Theorem 2 that

$$\mathcal{T}_c(\mu_n, \nu_n) - \mathcal{T}_c(\mu, \nu) \geq \int \varphi_0 d(\mu_n - \mu) + \int \psi_0 d(\nu_n - \nu) =: \Gamma_n.$$ 

Combine the above display with equations (29), (34) and (35) to deduce

$$\mathbb{E}\left| \mathcal{T}_c(\mu_n, \nu_n) - \mathcal{T}_c(\mu, \nu) \right|$$

$$\leq \mathbb{E}|\Gamma_n| + \mathbb{E}\left[ \int \varphi_n d(\mu_n - \mu) \right] + \mathbb{E}\left[ \int \psi_n d(\nu_n - \nu) \right]$$

$$\lesssim \mathbb{E}|\Gamma_n| + \Lambda\left\{ \mathbb{E}[\mathcal{X}_n] + \sqrt{\mathbb{E}[\Delta_n^2]} \right\} + \mathbb{E}\left[ I_{E_n} \int \varphi_n d(\mu_n - \mu) \right] + \mathbb{E}\left[ I_{E_n} \int \psi_n d(\nu_n - \nu) \right]$$

$$\lesssim \mathbb{E}|\Gamma_n| + \Lambda\left\{ \mathbb{E}[\mathcal{X}_n] + \sqrt{\mathbb{E}[\Delta_n^2]} \right\} + \sqrt{\mathbb{P}(E_n^c)\mathbb{E}[R_n^2]}$$

$$\lesssim \mathbb{E}|\Gamma_n| + \Lambda\left\{ \mathbb{E}[\mathcal{X}_n] + n^{-2/d} \right\} + n^{-1/2}(\log n)^{c_1}. \quad (36)$$

The quantities $\Gamma_n$ and $\mathcal{X}_n$ are simple to bound in expectation, as we now show.

**Lemma 20.** For any $\epsilon > 0$, $\mathbb{E}|\Gamma_n| \vee \mathbb{E}[\mathcal{X}_n] \lesssim n^{\epsilon - \frac{1}{2}}$.

Since $d \geq 5$, combining equation (36) with Lemma 20 leads to the claim.

**3.2. Proof of Theorem 11.** Fix $x \in \mathbb{R}^d$. If $\partial^{c}\varphi(x)$ is empty, then there is nothing to show, thus suppose otherwise. Choose $y_x \in \partial^{c}\varphi(x)$. Let $K_p = 2(3x^2)^{1/(p-1)}$, and notice that we may assume

$$\|y_x\| \geq 4(K_p + 1)(\|x\| + 1), \quad (37)$$

as otherwise we are done. In particular, this assumption implies $\|y_x - x\| \geq 4$, thus the following ball is non-empty

$$U = \left\{ u \in \mathbb{R}^d : \|u - y_x\| \leq \|x - y_x\| - 1 \right\}.$$ 

Furthermore, define $\xi = (y_x - x)/\|y_x - x\|$, and note that the ball $S := B_{x + 2\xi}$ of radius 1 centered at $x + 2\xi$ is contained in $U$.

If $\partial^{c}\varphi(S) = \emptyset$, then the condition $\text{supp}(\pi) \subseteq \partial^{c}\varphi$ implies $\mu(S) = 0$, in which case the right-hand side of equation (26) is infinite and the claim is trivial. Thus, assume otherwise, and pick $u \in S$ for which $\partial^{c}\varphi(u)$ is nonempty. Notice that $\|x - u\| \leq 3$. Furthermore, let $y_u \in \partial^{c}\varphi(u)$ be arbitrary. Since $\varphi$ is $c$-concave, the set $\partial^{c}\varphi$ is $c$-cyclically monotone by Lemma 1(iii). In particular,

$$c(x, y_x) - c(u, y_x) \leq c(x, y_u) - c(u, y_u).$$
Thus, using condition (H3),
\[ c(x, y_u) - c(u, y_u) \geq \omega(\|x - y_u\|) - \omega(\|u - y_u\|) \]  
(Since \( \|x - y_u\| \wedge \|u - y_u\| \geq 1 \))
\[ \geq \omega((\|x - y_u\|) - \omega((\|x - y_u\| - 1) \) \) (Since \( u \in U \) and \( \omega \) is increasing)
\[ \geq \omega'(\|x - y_u\| - 1) \]  
(By convexity of \( \omega \))
\[ \geq \frac{1}{\kappa}(\|x - y_u\| - 1)^{p-1} \]  
(By condition (H3))
\[ \geq \frac{1}{\kappa 2^{p-1}} \|x - y_u\|^{p-1}. \]  
(Since \( \|x - y_u\| \geq 4 \))

Now, conditions (H0) and (H3) imply that \( h(z) \leq \kappa \) for all \( z \in B_{0,1} \). Furthermore, the preceding display combined with equation (37) implies that \( c(x, y_u) > \kappa \), whence \( \|x - y_u\| \geq 1 \). We may thus again apply conditions (H0) and (H3) to obtain,
\[ c(x, y_u) - c(u, y_u) \leq \langle \nabla h(x - y_u), x - u \rangle \leq \omega'(\|x - y_u\|) \|x - u\| \leq 3\kappa \|x - y_u\|^{p-1}. \]

We deduce that \( \|x - y_u\| \leq K_p \|x - y_u\| \), whence,
\[ \|y_u\| \leq K_p \|x - y_u\| + \|x\| \leq K_p \|y_u\| + \|x\| (K_p + 1) \leq K_p \|y_u\| + \frac{1}{4} \|y_u\|, \]
where the last inequality is due to equation (37). We thus have \( \|y_u\| \geq C \|y_x\| \) for a constant \( C > 0 \) depending only on \( d, p, \kappa \). It follows that,
\[ \partial_c \varphi(S) \subseteq \left\{ v \in \mathbb{R}^d : \|v\| \geq C \|y_x\| \right\}. \]

Given \( Y \sim \nu \), we deduce from the sub-Weibull condition on \( \nu \) that
\[ \nu(\partial_c \varphi(S)) \leq \mathbb{P}(\|Y\| \geq C \|y_x\|) \leq \exp \left( -\frac{C^2 \|y_x\|^2}{2\sigma^2} \right). \]

On the other hand, using the fact that \( \text{supp}(\pi) \subseteq \partial_c \varphi \), one has
\[ \nu(\partial_c \varphi(S)) = \pi(\mathbb{R}^d \times \partial_c \varphi(S)) \geq \pi(S \times \partial_c \varphi(S)) = \mu(S), \]
so that,
\[ \exp \left( -\frac{C^2 \|y_x\|^2}{2\sigma^2} \right) \geq \nu(\partial_c \varphi(S)) \geq \mu(S) \geq \inf_{y : \|x - y\| \leq 2} \mu(B_y). \]  

The first claim follows. To prove the second claim, recall from the definition of \((\gamma, b)\)-super-Gaussianity that for all \( y \in \mathbb{R}^d \) such that \( \|x - y\| \leq 2 \),
\[
\mu(B_y) \geq \frac{b}{\sqrt{2\pi \gamma^2}} \mathcal{L}(B_y) \inf_{z \in B_y} \exp(-\|z\|^2/2\gamma^2) \geq C_1 \exp(-\|x\|^2/C_1),
\]
for a constant \( C_1 > 0 \) depending on \( d, b, \gamma \). By Lemma 1(iv), since \( T_c(\mu, \nu) < \infty \), any optimal coupling \( \pi \) between \( \mu \) and \( \nu \) lies in the support of a \( c \)-concave potential \( \varphi \). We may therefore apply equation (38) to deduce that, for some constant \( C' > 0 \), any such coupling satisfies
\[ \|y\| \leq C' \sigma(\|x\| + 1)^{\frac{3}{2}}, \quad \pi\text{-a.e.} \ (x, y). \]

Furthermore, since \( \mu \) is absolutely continuous with respect to the Lebesgue measure, notice that the conditions of Gangbo and McCann (1996), Theorem 1.2, are satisfied under conditions (H0) and (H3) and the strict convexity of \( h \). Therefore, there exists a unique optimal transport map \( T \) from \( \mu \) to \( \nu \), so that the measure \( \pi \) in the above display may be taken to be \((Id, T)\#\mu \). The claim follows.
4. Lower Bounds. In this section, we derive two lower bounds which imply that the rates of convergence derived in Sections 2 and 3 are typically unimprovable. In Section 4.1, we obtain lower bounds on the rate of convergence of the empirical optimal transport cost, while in Section 4.2, we derive a minimax lower bound which implies that, up to polylogarithmic factors, no estimator of $\mathcal{T}_c(\mu, \nu)$ can achieve a faster rate of convergence than the empirical estimator uniformly over all pairs of measures $\mu, \nu$.

In order to state our lower bounds, we require an assumption on the maximal Hölder exponent $\alpha \in (0, 2]$ achievable by the cost $h$. To state such an assumption, recall that our upper bounds were based, for instance, on the condition $\Lambda = 1 \vee \|h\|_{C^{\alpha}(Z)} < \infty$, for some $\alpha \in (0, 2]$, which in particular implies that for all $z, z_0 \in Z$,

$$h(z) - h(z_0) \leq \begin{cases} \Lambda \|z - z_0\|^\alpha, & \alpha \leq 1 \\ \langle \nabla h(z_0), z - z_0 \rangle + \Lambda \|z - z_0\|^\alpha, & \alpha > 1 \end{cases}.$$  

We shall assume the following dual condition throughout this section.

(H4) $\mathcal{X}$ and $\mathcal{Y}$ are convex sets with nonempty interior, and are such that $h$ is differentiable over $Z = \mathcal{X} - \mathcal{Y}$. Furthermore, there exist $\lambda > 0$, $\alpha \in (0, 2]$, and $z_0 = x_0 - y_0 \in Z$ such that $x_0 \in \text{int}(\mathcal{X})$, $y_0 \in \text{int}(\mathcal{Y})$, and for all $z \in Z$,

$$h(z) - h(z_0) \geq \begin{cases} \lambda \|z - z_0\|^\alpha, & \alpha \leq 1 \\ \langle \nabla h(z_0), z - z_0 \rangle + \lambda \|z - z_0\|^\alpha, & \alpha > 1 \end{cases}.$$  

Notice that condition (H4) implies that $\mathcal{X}$ and $\mathcal{Y}$ have positive Lebesgue measure over $\mathbb{R}^d$. The presence of this assumption can be anticipated from the fact that empirical optimal transport costs may achieve improved rates of convergence when $\mathcal{X}$ and $\mathcal{Y}$ have intrinsic dimension less than $d$ (Weed and Bach, 2019). It is straightforward to verify that conditions (H0) and (H4) are satisfied by the cost $h(x) = \|x\|^p$ when $1 \leq p \leq 2$ with $\alpha = p$ and $z_0 = 0$. These conditions are also satisfied for $2 < p < \infty$ and $\alpha = 2$, whenever there exists a neighborhood of zero which is not contained in $Z$. We also note that condition (H4) is satisfied with $\alpha = 2$ by any differentiable and $(2\lambda)$-strongly convex function $h$ over $Z$.

Finally, we assume throughout this section that $X_i$ is independent of $Y_j$ for all $1 \leq i, j \leq n$. Though this condition is not needed to derive our upper bounds, we do not preclude the possibility that they may be sharpened under particular dependence structures between the samples from $\mu$ and $\nu$.

4.1. Lower Bounds for the Empirical Optimal Transport Cost. We begin with the following lower bound on the rate of convergence of the empirical optimal transport cost.

**Proposition 21.** Assume conditions (H0) and (H4). Then,

$$\sup_{\mu \in \mathcal{P}(\mathcal{X})} \, \mathbb{E}_{\mu, \nu}[\mathcal{T}_c(\mu_n, \nu_n) - \mathcal{T}_c(\mu, \nu)] \gtrsim \lambda n^{-\alpha/d}.$$  

Proposition 21 implies that the upper bounds in Theorems 2, 12, and Corollaries thereafter cannot generally be improved, provided that the Hölder exponent $\alpha$ therein is chosen maximally in the sense of condition (H4). As we now show, our lower bound is constructive, and is typically achieved by absolutely continuous measures differing by a location translation, as in Example 1.1.
PROOF. We prove the claim assuming $\alpha \in (1, 2]$, and an analogous argument may be used when $\alpha \in (0, 1]$. Since $x_0 \in \text{int}(\mathcal{X})$ and $y_0 \in \text{int}(\mathcal{Y})$, there exists $\varepsilon > 0$ such that $\mathcal{X}_0 := B_{x_0, \varepsilon} \subseteq \mathcal{X}$ and $\mathcal{Y}_0 := B_{y_0, \varepsilon} \subseteq \mathcal{Y}$. Define the measures

$$
\mu = \frac{\mathcal{L}|_{\mathcal{X}_0}}{\mathcal{L}(\mathcal{X}_0)}, \quad \nu = \frac{\mathcal{L}|_{\mathcal{Y}_0}}{\mathcal{L}(\mathcal{Y}_0)},
$$

where recall that $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}^d$. By construction, $\nu = T_0 \# \mu$ where $T_0(x) = x + z_0$. Since $h$ is convex, it follows by the same argument as in Example 1.1 that $T_0$ is an optimal transport map from $\mu$ to $\nu$.

Let $\gamma_n$ denote an optimal coupling between $\mu_n$ and $\nu$ with respect to the cost $c$. Then, by condition (H4),

$$
T_c(\mu_n, \nu) - T_c(\mu, \nu) = \int \left[ c(x, y) - c(x, T_0(x)) \right] d\gamma_n(x, y)
$$

$$
= \int \left[ h(y - x) - h(z_0) \right] d\gamma_n(x, y)
$$

$$
\geq \int \left[ \langle \nabla h(z_0), y - x - z_0 \rangle + \lambda \| y - x - z_0 \|^{\alpha} \right] d\gamma_n(x, y)
$$

$$
= \int \left[ \langle \nabla h(z_0), y - x \rangle + \lambda \| y - x \|^{\alpha} \right] d\pi_n(x, y),
$$

where $\pi_n = (Id, T_0^{-1}) \# \gamma_n \in \Pi(\mu_n, \mu)$. It follows that

$$
T_c(\mu_n, \nu) - T_c(\mu, \nu) \geq \int \langle \nabla h(z_0), y - x \rangle d\pi_n(x, y) + \lambda W_1^\alpha(\mu_n, \mu)
$$

$$
\geq \int \langle \nabla h(z_0), \cdot \rangle \mathbb{E} \mathbb{E}[w_0] \mu + \lambda W_1^\alpha(\mu_n, \mu).
$$

The first order term on the final line of the above display clearly has mean zero, whence

$$
\mathbb{E} \left[ T_c(\mu_n, \nu) - T_c(\mu, \nu) \right] \geq \lambda \mathbb{E} W_1^\alpha(\mu_n, \mu) \geq \lambda \mathbb{E} W_1^\alpha(\mu, \mu) \geq \lambda n^{-\alpha/d},
$$

where the final inequality follows from Proposition 2.1 of Dudley (1969), due to the absolute continuity of $\mu$ with respect to the Lebesgue measure on $\mathbb{R}^d$. Finally, since $h$ is continuous and $\mathcal{Z} = \mathcal{X}_0 \cap \mathcal{Y}_0$ is compact, $h$ is bounded over $\mathcal{Z}$. Thus, by Lemma 1(ii), there exists a pair of Kantorovich potentials $(\phi_n, \psi_n)$ such that $T_c(\mu_n, \nu) = J_{\mu_n, \nu}(\phi_n, \psi_n)$, whence

$$
T_c(\mu_n, \nu_n) \geq J_{\mu_n, \nu_n}(\phi_n, \psi_n) = T_c(\mu_n, \nu) + \int \psi_n d(\nu_n - \nu).
$$

Since the random variables $X_1, \ldots, X_n$ are independent of $Y_1, \ldots, Y_n$, $\psi_n$ is also independent of $Y_1, \ldots, Y_n$, whence

$$
\mathbb{E} \left[ \int \psi_n d(\nu_n - \nu) \bigg| X_1, \ldots, X_n \right] = 0.
$$

It readily follows that $\mathbb{E} T_c(\mu_n, \nu_n) \geq \mathbb{E} T_c(\mu_n, \nu)$, so that

$$
\mathbb{E} \left| T_c(\mu_n, \nu_n) - T_c(\mu, \nu) \right| \geq \mathbb{E} \left[ T_c(\mu_n, \nu_n) - T_c(\mu, \nu) \right] \geq \mathbb{E} \left[ T_c(\mu_n, \nu) - T_c(\mu, \nu) \right] \geq \lambda n^{-\alpha/d}.
$$

The claim follows. 

\[ \Box \]
4.2. Minimax Lower Bounds. We next turn to deriving a minimax lower bound on the rate of estimating the optimal transport cost between two probability measures. Unlike Proposition 21, our next result will require both condition (H4) and the smoothness condition (H1) from Section 2.

Theorem 22. Assume conditions (H0), (H1) and (H4). Then, there exists a constant $C > 0$ depending on $\lambda, \Lambda, d, X, Y, \alpha$ such that

\[
\inf_{\hat{T}_n} \sup_{(\mu, \nu) \in \mathcal{P}(X) \times \mathcal{P}(Y)} \mathbb{E}_{\mu, \nu}[\hat{T}_n - T_c(\mu, \nu)] \geq C (n \log n)^{-\alpha/d},
\]

where the infimum is over all Borel-measurable functions $\hat{T}_n$ of $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$.

Theorem 22 shows that the convergence rates exhibited throughout this paper for the empirical optimal transport cost estimator cannot be improved by any other estimator uniformly over all pairs of measures in $\mathcal{P}(X) \times \mathcal{P}(Y)$, up to a polylogarithmic factor. Minimax lower bounds scaling at the rate $(n \log n)^{-1/d}$ have previously been established for the problem of estimating $p$-Wasserstein distances by Niles-Weed and Rigollet (2022) (Theorem 11), and we build upon their techniques to prove Theorem 22. The proof is deferred to Appendix C.

APPENDIX A: OMITTED PROOFS FROM SECTION 2

A.1. Proof of Corollary 3. Throughout the proof, $C > 0$ denotes a constant depending only on $d, p, r$, whose value may change from line to line.

The proof is elementary, but tedious. To prove the first claim, it suffices to show that $||-||_{\ell_r}^p \in C^{2 \wedge r \wedge p} (B_{0,2})$. It is clear that $||-||_{\ell_r}$ and $||-||_{\ell_r}^p$ are Lipschitz for any $r, p \geq 1$, thus it suffices to assume $p, r > 1$. In this case, $||-||_{\ell_r}^p$ is differentiable, and for all $l = 1, \ldots, d$,

\[
\frac{\partial ||x||_{\ell_r}^p}{\partial x_l} = px_l |x_l|^{r-2} ||x||_{\ell_r}^{p-r}.
\]

Next, we show that $\frac{\partial ||y||_{\ell_r}^p}{\partial x_l}$ is Hölder continuous over $B_{0,2}$ with suitable exponent, uniformly in $l$. Let $x, y \in B_{0,2}$, and assume without loss of generality that $||x||_{\ell_r} \leq ||y||_{\ell_r}$. Then,

\[
|x_l| |x_l|^{r-2} ||x||_{\ell_r}^{p-r} - y_l |y_l|^{r-2} ||y||_{\ell_r}^{p-r}.
\]

\[
\leq |x_l| |x_l|^{r-2} ||x||_{\ell_r}^{p-r} - x_l |x_l|^{r-2} ||y||_{\ell_r}^{p-r} + |x_l| |x_l|^{r-2} ||y||_{\ell_r}^{p-r} - y_l |y_l|^{r-2} ||y||_{\ell_r}^{p-r}.
\]

\[
= |x_l|^{r-1} \left( ||x||_{\ell_r}^{p-r} - ||y||_{\ell_r}^{p-r} \right) + ||y||_{\ell_r}^{p-r} |x_l| |x_l|^{r-2} - y_l |y_l|^{r-2}.
\]

For the first term, notice that

\[
|x_l|^{r-1} \left( ||x||_{\ell_r}^{p-r} - ||y||_{\ell_r}^{p-r} \right) \leq ||x||_{\ell_r}^{r-1} \left( ||y||_{\ell_r}^{p-r} - ||x||_{\ell_r}^{p-r} \right)
\]

\[
\leq ||y||_{\ell_r}^{p-r} - ||x||_{\ell_r}^{p-r} \leq C ||y - x||^{1 \wedge (p-1)}.
\]

Furthermore, letting $\epsilon_x = \text{sgn} (x_l)$ and $\epsilon_y = \text{sgn} (y_l)$, we have,

\[
||y||_{\ell_r}^{p-r} |x_l| |x_l|^{r-2} - y_l |y_l|^{r-2}.
\]

\[
= ||y||_{\ell_r}^{p-r} |x_l|^{r-1} - \epsilon_x |x_l|^{r-1} - \epsilon_y |y_l|^{r-1} \leq ||y||_{\ell_r}^{p-r} \left( |\epsilon_x| |x_l|^{r-1} - |\epsilon_x| |y_l|^{r-1} \right) + |\epsilon_x| |y_l|^{r-1} - |\epsilon_y| |y_l|^{r-1} \right).
\]
\[
\|y\|^2_{\ell_r} |x| |x|^2 - y |y|^2 \|y\|^2_{\ell_r} \|x| |x|^2 - y |y|^2 \leq C |x| |x|^{1+\lambda(r-1)} + 2|x| |y|^2 \leq C |x| |y|^{1+\lambda(r-1)}.
\]

We now consider several cases. If \( p \geq r \), then we readily obtain
\[
\|y\|^2_{\ell_r} |x| |x|^2 - y |y|^2 \leq C \left( |x| |y|^{1+\lambda(r-1)} + 2|x| |y|^2 \right) \leq C |x| |y|^{1+\lambda(r-1)}.
\]

If instead \( p < r \leq 2 \), then from equation (40), we obtain
\[
\|y\|^2_{\ell_r} |x| |x|^2 - y |y|^2 \leq 3 \|y\|^2_{\ell_r} |x| |y|^2 \leq 3 \|y\|^2_{\ell_r} |x| |y|^2 \leq C |x| |y|^{1+\lambda(r-1)}.
\]

Finally, if \( p < r < 2 \), we have from equation (40),
\[
\|y\|^2_{\ell_r} |x| |x|^2 - y |y|^2 \leq \|y\|^2_{\ell_r} \left( (r-1)(|x| |y|^2 + 2|x| |y|^2) \right) \leq C \|y\|^2_{\ell_r} |x| |y|^2 \leq C \|y\|^2_{\ell_r} |x| |y|^2 \leq C |x| |y|^{1+\lambda(r-1)}.
\]

The preceding displays readily imply that for all \( l, \partial \|y\|^2_{\ell_r} / \partial x_l \in C^{1+\lambda(r-1)}(B_{0,2}) \), implying that \( \|y\|^2_{\ell_r} \in C^{2+\lambda(r-1)}(B_{0,2}) \). The first claim now follows by applying Theorem 2. To prove the second claim, notice that when \( r \geq 2 \), the map in equation (39) is differentiable with respect to \( x_l \) over any open subset of \( B_{0,2} \) which does not contain the origin, with derivative
\[
\frac{\partial^2 \|y\|^2_{\ell_r}}{\partial x_l^2} = p \frac{\partial}{\partial x_l} \|x\|^2_{\ell_r} |x|^2 = p \|x\|^2_{\ell_r} |x|^2 |x|^2 \left( (p-r) |x|^2 + (r-1) \|x\|^2_{\ell_r} \right).
\]

Recall that for all positive semidefinite matrices \( A \in \mathbb{R}^{d \times d} \), the 1-Schatten norm of \( A \) is equal to its trace, so that \( \|A\|_\infty \leq \text{tr}(A) \). Thus, for any \( \varepsilon > 0 \),
\[
\sup_{x \in B_{0,2} \setminus B_{0,\varepsilon}} \|\nabla^2 \|y\|^2_{\ell_r} \|_{\infty} \leq \sup_{x \in B_{0,2} \setminus B_{0,\varepsilon}} \sum_{i=1}^{d} \left( \frac{\partial^2 \|y\|^2_{\ell_r}}{\partial x_i^2} \right) \leq \sup_{x \in B_{0,2} \setminus B_{0,\varepsilon}} \sum_{i=1}^{d} \|x\|^2_{\ell_r} |x|^2 \|_{\infty} < \infty.
\]

It readily follows that \( \|y\|^2_{\ell_r} \in C^{2}(B_{0,2} \setminus B_{0,\varepsilon}) \), with Hölder norm depending only on \( d, r, p, \varepsilon \). Now, since \( \mathcal{X}, \mathcal{Y} \) are convex by condition (S1), so is the set \( Z = \mathcal{X} \setminus \mathcal{Y} \), and since \( \mathcal{X} \) and \( \mathcal{Y} \) are closed and disjoint, there must exist \( \varepsilon > 0 \) such that \( B_{0,\varepsilon} \cap Z = \emptyset \). Choose any convex open set \( Z_1 \) containing \( Z \), and contained in \( B_{0,2} \setminus B_{0,\varepsilon} \), to deduce that condition (H1) holds with \( \alpha = 2 \), and with \( \Lambda \) depending only on \( d, p, r, \mathcal{X}, \mathcal{Y} \). The claim follows.

**A.2. Proof of Lemma 5.** The proof is analogous to that of Proposition C.2 of (Gangbo and McCann, 1996), and is included for completeness. We prove the claim for \( \bar{\varphi}_n \), noting that a symmetric argument can be used for the map \( \bar{\psi}_n \). Define the modified cost function
\[
h_\Lambda : z \in Z \mapsto h(z) - \frac{\Lambda}{2} \|z\|^2.
\]

By condition (H1) with \( \alpha = 2 \), \( \nabla h \) is \( \Lambda \)-Lipschitz over \( Z \), implying that for all \( z_1, z_2 \in Z \),
\[
(\nabla h_\Lambda(z_1) - \nabla h_\Lambda(z_2), z_1 - z_2) = (\nabla h(z_1) - \nabla h(z_2), z_1 - z_2) - \Lambda \|z_1 - z_2\|^2 \leq 0.
\]

It follows that \( -\nabla h_\Lambda \) is monotone, whence \( h_\Lambda \) is concave (Hiriart-Urruty and Lemaréchal (2004), Theorem 4.1.4). Now, notice that for all \( x \in \mathcal{X} \),
\[
\bar{\varphi}_n(x) = \inf_{y \in \mathcal{Y}} \{c(x, y) - \psi_n(y)\} - \frac{\Lambda}{2} \|x\|^2.
\]
implies that \( h \) be a sequence such that 

\[
Z \subseteq \text{open set such that } \epsilon > 0 \text{ of } Z.
\]

Since \( t \) \( \epsilon \) \( \text{semi-continuous by assumption on } \phi \in (41) \), since the support of \( \phi \), we have by a first-order Taylor expansion that for all \( \epsilon \leq 1 \), \( \phi \in (42) \), with \( \epsilon \) arbitrary, the Lipschitz property follows upon repeating a symmetric argument to upper bound \( \tilde{\varphi}_n(x) - \tilde{\varphi}_n(x') \). Finally, since \( |\varphi_n| \leq 1 \), \( |\tilde{\varphi}_n| \leq 2\Lambda \) as \( \Lambda \geq 1 \).

\[= \inf_{y \in Y} \left\{ h(x - y) - \frac{\Lambda}{2} \left( \| y \|^2 - 2(x, y) \right) - \psi_n(y) \right\} \]

By concavity of \( h \), the last line of the above display is an infimum of concave functions of \( x \). It follows that \( \tilde{\varphi}_n \) is Lipschitz, let \( x \in X \) and let \( (y_k) \subseteq Y \) be a sequence such that 

\[\tilde{\varphi}_n(x) \geq c(x, y_k) - \psi_n(y_k) - \frac{\Lambda}{2} \| x \|^2 - k^{-1}. \]

Then, for all \( x' \in X \) and \( k \geq 1 \), 

\[\tilde{\varphi}_n(x') - \tilde{\varphi}_n(x) \leq \left[ c(x', y_k) - \psi_n(y_k) - \frac{\Lambda}{2} \| x' \|^2 \right] - \left[ c(x, y_k) - \psi_n(y_k) - \frac{\Lambda}{2} \| x \|^2 \right] + k^{-1} \]

\[= h(x' - y_k) - h(x - y_k) - \frac{\Lambda}{2} \left( \| x' \|^2 - \| x \|^2 \right) + k^{-1} \]

\[\leq \left( \sup_{z \in Z} \| \nabla h(z) \| \right) \| x' - x \| - \frac{\Lambda}{2} \left( \| x' \| - \| x \| \right) \left( \| x' \| + \| x \| \right) + k^{-1} \]

\[\leq \left( \sup_{z \in Z} \| \nabla h(z) \| + \Lambda \right) \| x' - x \| + k^{-1} \leq 2\Lambda \| x' - x \| + k^{-1}. \]

Since \( k \) is arbitrary, the Lipschitz property follows upon repeating a symmetric argument to upper bound \( \tilde{\varphi}_n(x) - \tilde{\varphi}_n(x') \). Finally, since \( |\varphi_n| \leq 1 \), \( |\tilde{\varphi}_n| \leq 2\Lambda \) as \( \Lambda \geq 1 \).

\[A.3. \text{Proof of Lemma 8.} \] To prove the first part, recall that condition (H1) implies \( h \in \mathcal{C}_\alpha(\mathbb{Z}_1) \) with \( 1 < \alpha < 2 \), and \( \Lambda \geq \| h \|_{\mathcal{C}_\alpha(\mathbb{Z}_1)}. \) For any \( z \in Z \), let \( A_z := \{ u \in \mathbb{R}^d : z - u \in \mathbb{Z}_1 \}. \) Since \( Z \) is compact and \( \mathbb{Z}_1 \) is open, there exists \( \epsilon > 0 \) such that for all \( z \in Z \), \( B_{0, \epsilon} \subseteq A_z \). In particular, if \( \sigma < \epsilon \), then \( u \in A_z \) for all \( z \in Z \) and \( u \) in the support of \( K_\sigma \).

Moreover, we have by a first-order Taylor expansion that for all \( z \in Z \) and \( u \in A_z \), 

\[h(z - u) - h(z) = -\langle \nabla h(z - tu), u \rangle, \]

for some \( t \in (0, 1) \). By convexity of \( Z_1 \), we have \( z - tu \in Z_1 \). It follows that 

\[|h(z - u) - h(z) + \langle \nabla h(z), u \rangle| \leq |\langle \nabla h(z) - \nabla h(z - tu), u \rangle| \leq \Lambda \| u \|^\alpha. \]

Finally, the fact that \( K \) is even implies that \( \int u K_\sigma(u) du = 0 \). Combining these facts, we obtain

\[|h_\sigma(z) - h(z)| = \left| \int \left[ h(z - u) - h(z) \right] K_\sigma(u) du \right| \]

\[\leq \left| \int \left[ h(z - u) - h(z) + \langle \nabla h(z), u \rangle \right] K_\sigma(u) du \right| + \left| \int \langle \nabla h(z), u \rangle K_\sigma(u) du \right| \]

\[\leq \Lambda \int \| u \|^\alpha K_\sigma(u) du \leq \Lambda \sigma^\alpha, \]

since the support of \( K_\sigma \) lies in \( B_{0, \sigma} \). This proves the first claim.

To prove the second part, it is easy to see that the cost \( h_\sigma \) is convex, even, and lower semi-continuous by assumption on \( h \), thus \( h_\sigma \) satisfies assumption (H0). Now, let \( \tilde{Z}_1 \) be an open set such that \( Z \subseteq \tilde{Z}_1 \) and such that \( \text{cl}(\tilde{Z}_1) \subseteq \mathbb{Z}_1 \). After possibly decreasing the value of \( \epsilon > 0 \), we may again ensure that \( B_{z, \epsilon} \subseteq \mathbb{Z}_1 \) for all \( z \in \tilde{Z}_1 \). We shall now prove that \( h_\sigma \)

\[\text{(41)} \]

\[\text{(42)} \]

\[\text{(43)} \]

\[\text{(44)} \]
satisfies assumption (H1) with the Hölder norm \( \|h_\sigma\|_{C^2(\bar{Z}_1)} \leq C\Lambda \sigma^{\alpha-2} \) as long as \( \sigma < \epsilon \). That \( h_\sigma \leq 1 \) on \( \bar{Z}_1 \) is immediate, so it suffices to show that \( h_\sigma \) has the requisite Hölder norm.

Define for any given \( z \in Z \) and all \( u \in \mathbb{R}^d \),

\[
\bar{h}(u) = h(u) - h(z) - \langle \nabla h(z), u - z \rangle, \quad \tilde{h}_\sigma = \bar{h} \ast K_\sigma.
\]

As before, for any \( z \in \bar{Z}_1 \) and any \( u \in \mathbb{R}^d \) such that \( \|u - z\| \leq \epsilon \), we have \( u \in Z_1 \), whence a first-order Taylor expansion leads to

\[
|\bar{h}(u)| \leq \Lambda \|z - u\|^\alpha.
\]

We thus obtain for all \( z \in \bar{Z}_1 \),

\[
\|\nabla^2 h_\sigma(z)\|_\infty = \|\nabla^2 \tilde{h}_\sigma(z)\|_\infty \\
\leq \int |\bar{h}(u)|\|\nabla^2 K_\sigma(z - u)\|_\infty du \\
= \sigma^{-d-2} \int |\bar{h}(u)|\|\nabla^2 K((z - u)/\sigma)\|_\infty du \\
\leq \Lambda \sigma^{-d-2} \int |z - u|^\alpha \|\nabla^2 K((z - u)/\sigma)\|_\infty du \\
= \Lambda \sigma^{\alpha-2} \int |u|^\alpha \|\nabla^2 K(u)\|_\infty du \leq C\Lambda \sigma^{\alpha-2},
\]

for some constant \( C \) depending only on \( K \). This proves the second claim.

**APPENDIX B: OMITTED PROOFS FROM SECTION 3**

**B.1. Proof of Lemma 10.** Under condition (H2), recall that for all \( R > 0 \), \( \|h\|_{C^2(B_0,R)} \leq \Lambda R^p \). Set

\[
R = \sup\{\|x - y\| : x \in B_0,r, y \in \partial^c \varphi(B_0,r)\},
\]

and let \( \Lambda_r = \Lambda R^p \). It then follows by the same argument as in the proof of Lemma 5 that the map

\[
h_{\Lambda_r} : z \in B_0,R \mapsto h(z) - \frac{\Lambda_r}{2} \|z\|^2
\]

is concave. Now, the assumptions on \( c \) in Lemma 1(iv) are satisfied under conditions (H0), (H2), and under the assumption of superlinearity of \( h \), thus the assumption of local boundedness on \( \varphi \) ensures that \( \partial^c \varphi(x) \) is nonempty for all \( x \in B_0,r \), and that \( \varphi \) admits the representation

\[
\varphi(x) = \inf_{y \in \partial^c \varphi(B_0,r)} \left\{ c(x, y) - \varphi^c(y) \right\}.
\]

It follows that

\[
\phi(x) = \inf_{y \in \partial^c \varphi(B_0,r)} \left\{ c(x, y) - \varphi^c(y) \right\} - \frac{\Lambda_r}{2} \|x\|^2 \\
= \inf_{y \in \partial^c \varphi(B_0,r)} \left\{ h_{\Lambda_r}(x - y) + \frac{\Lambda_r}{2} \left[ \|y\|^2 - 2\langle x, y \rangle \right] - \varphi^c(y) \right\}.
\]

Notice that \( \|x - y\| \leq R \) for all \( x, y \) appearing in the infimum of the final line in the above display, thus \( h_{\Lambda_r} \) is defined and concave therein. Similarly as in Lemma 5, the last line of the
above display is thus an infimum of concave functions of \( x \), implying that \( \phi \) is concave. To prove that \( \phi \) is Lipschitz, let \( x \in B_{0,r} \) and choose a sequence \( (y_k) \subseteq \mathcal{Y} \) such that

\[
\phi(x) \geq c(x, y_k) - \varphi^c(y_k) - \frac{\Lambda_r}{2} \|x\|^2 - k^{-1}.
\]

Then, for all \( x' \in B_{0,r} \) and \( k \geq 1 \),

\[
\phi(x') - \phi(x) \leq \left[ c(x', y_k) - \varphi^c(y_k) - \frac{\Lambda_r}{2} \|x'\|^2 \right] - \left[ c(x, y_k) - \varphi^c(y_k) - \frac{\Lambda_r}{2} \|x\|^2 \right] + k^{-1}
\]

\[
= h(x' - y_k) - h(x - y_k) - \frac{\Lambda_r}{2} \left( \|x'\|^2 - \|x\|^2 \right) + k^{-1}
\]

\[
\leq \left( \sup_{z \in B_{0,r}} \|\nabla h(z)\| \right) \|x' - x\| + \frac{\Lambda_r}{2} \left( \|x'\| - \|x\| \right) (\|x'\| + \|x\|) + k^{-1}
\]

\[
\leq \left( \sup_{z \in B_{0,r}} \|\nabla h(z)\| + r \Lambda_r \right) \|x' - x\| + k^{-1}
\]

\[
\leq 2r \Lambda_r \|x' - x\| + k^{-1}.
\]

The claim readily follows.  

\[ \square \]

**B.2. Proof of Lemma 14.** Part (i) is immediate by definition of \( c \)-conjugate. For part (ii), note that for any \( x \in \text{supp}(\mu_n) \),

\[
\varphi_n(x) = \inf_{y \in \mathbb{R}^d} \left\{ c(x, y) - \eta_n(y) \right\} \geq \inf_{y \in \mathbb{R}^d} \left\{ c(x, y) - \left[ c(x, y) - f_n(x) \right] \right\} = f_n(x).
\]

Similarly, for any \( y \in \text{supp}(\nu_n) \), since \((f_n, g_n) \in \Phi_c(\mu_n, \nu_n)\),

\[
\psi_n(y) = \inf_{x \in \mathbb{R}^d} \left\{ c(x, y) - \varphi_n(x) \right\}
\]

\[
\geq \inf_{x \in \mathbb{R}^d} \left\{ c(x, y) - \left[ c(x, y) - \eta_n(y) \right] \right\}
\]

\[
= \eta_n(y)
\]

\[
= \inf_{x \in \text{supp}(\mu_n)} \left\{ c(x, y) - f_n(x) \right\} \wedge \overline{f_n}
\]

\[
\geq g_n(y) \wedge \overline{f_n} = g_n(y),
\]

where the final equality uses that \( g_n \) maps into \([0, \overline{f_n}]\). Since \( \mu_n \) and \( \nu_n \) are finitely supported, either of the above inequalities is strict if and only if

\[
\int \varphi_n d\mu_n + \int \psi_n d\nu_n > \int f_n d\mu_n + \int g_n d\nu_n = T_c(\mu_n, \nu_n),
\]

in violation of the optimality of \((f_n, g_n)\). Therefore \( \varphi_n \) and \( \psi_n \) agree with \( f_n \) and \( g_n \) on \( \text{supp}(\mu_n) \) and \( \text{supp}(\nu_n) \), and

\[
T_c(\mu_n, \nu_n) = \int \varphi_n d\mu_n + \int \psi_n d\nu_n.
\]

To prove part (iii), note that \( \eta_n \) is nonnegative over \( \mathbb{R}^d \), since \( f_n \) is nonpositive over \( \text{supp}(\mu_n) \). Therefore, for any \( x \in \mathbb{R}^d \),

\[
\varphi_n(x) = \inf_{y \in \mathbb{R}^d} \left\{ c(x, y) - \eta_n(y) \right\} \leq h(0) - \eta_n(x) \leq 0.
\]
Furthermore, since $\eta_n$ is bounded above by $\overline{R}_n$,

\begin{equation}
\varphi_n(x) \geq \inf_{y \in \mathbb{R}^d} \left\{ c(x, y) - \overline{R}_n \right\} \geq -\overline{R}_n,
\end{equation}

Thus, $|\varphi_n(x)| \leq \overline{R}_n$. Similarly, since $\varphi_n$ is nonpositive, $\psi_n$ is nonnegative, and for all $y \in \mathbb{R}^d$,

$$\psi_n(y) = \inf_{x \in \mathbb{R}^d} \left\{ c(x, y) - \varphi_n(x) \right\} \leq h(0) - \varphi_n(y) \leq \overline{R}_n,$$

where we used equation (45). Thus, $|\psi_n(x)| \leq \overline{R}_n$ as well.

Finally, to prove part (iv), equation (44) and the primal definition of $\mathcal{T}_c(\mu_n, \nu_n) < \infty$ imply

$$\int [c(x, y) - \varphi_n(x) - \psi_n(y)] d\pi_n(x, y) = 0.$$ 

By part (i), the integrand of the above display is nonnegative, thus

$$c(x, y) = \varphi_n(x) + \psi_n(y), \quad \text{for all } (x, y) \in \text{supp}(\pi_n).$$

Since $\varphi_n$ and $\psi_n$ are bounded by part (iii), it must then follow from Lemma 1(iv) that for any $(x, y)$ satisfying the above display, $(x, y) \in \partial^c \varphi_n(x)$ and $(y, x) \in \partial^c \psi_n(y).$ \hfill $\square$

**B.3. Proof of Lemma 15.** Let $B$ denote the set of all balls in $\mathbb{R}^d$. Recall that $B$ has Vapnik-Chervonenkis dimension $d + 2$, thus the Vapnik-Chervonenkis inequality (Vapnik and Chervonenkis, 1968) implies that for all $u > 0$,

\begin{equation}
\mathbb{P} \left( \sup_{B \in B} |\mu_n(B) - \mu(B)| \geq u \right) \leq n^{d+2} \exp \left( -\frac{n u^2}{32} \right).
\end{equation}

By the assumption of $(\gamma, b)$-super-Gaussianity, we have for all $\|y - x\| \leq 2$,

$$\mu(B_y) \gtrsim \int_{B_y} \exp \left( -\|u\|^2 / (2\gamma^2) \right) du \gtrsim \exp \left( -\|x\|^2 / \gamma^2 \right),$$

so that, for all $0 \leq j \leq J_n$,

$$\inf_{x \in I_j} \inf_{\|y - x\| \leq 2} \mu(B_y) \geq C_1 \exp(-\ell_j^2 / \gamma^2).$$

Thus setting $u = C_1 \exp(-\ell_j^2 / \gamma^2) / 2$ in equation (46) for all $0 \leq j \leq J_n$, and applying a union bound, leads to

\begin{equation}
\mathbb{P} (A_n) \lesssim n^{d+2} J_n \exp \left\{ -\frac{C_1^2}{128} n \exp(-2\ell_j^2 / \gamma^2) \right\} = n^{d+2} J_n \exp \left\{ -\frac{C_1^2 \sqrt{n}}{128} \right\} \lesssim \frac{1}{n}.
\end{equation}

The claim follows. \hfill $\square$

**B.4. Proof of Proposition 16.** The proof proceeds using a similar argument as that of Proposition C.4 of Gangbo and McCann (1996). Under conditions (H0) and (H3), it follows from Lemma 1(iv) that $\partial^c \varphi(x)$ is nonempty for all $x \in B_r/2$. For any $y \in \partial^c \varphi(x)$, we have

$$\varphi(x) = c(x, y) - \varphi^c(y).$$

Let $v = x - y$. If $\|v\| \leq r$ there is nothing to prove, so assume otherwise, and define $\xi = 1 - \frac{r}{2\|v\|}$. Our assumption implies that $\xi \in [1/2, 1]$. Furthermore, define

$$u = x + (\xi - 1)v = x - \frac{r}{2} \left( \frac{v}{\|v\|} \right).$$
Then, the penultimate display leads to
\[ h(v) - h(\xi v) = c(x, y) - c(u, y) = c(x, y) - \varphi^c(y) - [c(u, y) - \varphi^c(y)] \leq \varphi(x) - \varphi(u) \leq 2R. \]
This fact, together with the convexity and differentiability of \( h \) away from zero, under condition (H3), implies
\[ \frac{r}{2} \langle \nabla h(\xi v), v / \| v \| \rangle \leq 2R. \]
On the other hand, by condition (H3) we have \( h(0) = 0 \), thus by convexity of \( h \),
\[ \frac{h(\xi v)}{\| \xi v \|} \leq \left\langle \nabla h(\xi v), \frac{\xi v}{\| \xi v \|} \right\rangle \leq \frac{4R}{r}. \]
In particular, since \( h(z) \gtrsim \kappa^{-1} \| z \|^p \) for all \( \| z \| \geq 2 \) under condition (H3), we have \( \| \xi v \|^{p-1} \lesssim 4R/r \), thus since \( \xi \geq 1/2 \) and \( r \geq 1 \), \( \| v \|^{p-1} \lesssim R \), and hence
\[ \| y \|^{p-1} \lesssim \| x \|^{p-1} + R. \]
The claim follows. \( \square \)

**B.5. Proof of Lemma 18.** Let \( M_{jk} = M_j \) and \( \bar{U}_{j,k} = U_j \) for all \( j \geq 0 \) and \( k = 1, \ldots, m_d \). Fix an enumeration \( D_1, D_2, \ldots \) (resp. \( M_1, M_2, \ldots \) and \( \bar{U}_1, \bar{U}_2, \ldots \)) of the set \( \{ I_{jk} : j \geq 0, 1 \leq k \leq m_d \} \) (resp. \( (\bar{M}_{jk}), (\bar{U}_{jk}) \)). Given a sequence \( (a_j)_{j=1}^{\infty} \) of positive real numbers, let \( p_j = N(ea_j, \mathcal{F}_{\bar{M}_j, \bar{U}_j}(D_j), L^\infty) \) and let \( f_{j,1}, \ldots, f_{j,p_j} \) be an \( ea_j \)-cover for \( \mathcal{F}_{\bar{M}_j, \bar{U}_j}(D_j) \) in \( L^\infty \). By Lemma 7, we have
\[ \log p_j \lesssim \left( \frac{\bar{U}_j + \operatorname{diam}(D_j)\bar{M}_j}{ea_j} \right)^{\frac{d}{2}}. \]
Now, it can be directly verified that the set
\[ \left\{ \sum_{j=1}^{\infty} f_{j,k_j} : k_j \in \{1, \ldots, p_j\}, j \geq 0 \right\} \]
forms an \( \epsilon \left( \sum_{j=1}^{\infty} a_j^2 \mu_n(D_j) \right)^{1/2} \)-covering of \( K_{M,U} \) in \( L^2(\mu_n) \), which is of size \( \prod_{j=1}^{\infty} p_j \). Thus,
\[ \log N \left( \epsilon \left[ \sum_{j=1}^{\infty} a_j^2 \mu_n(D_j) \right]^{1/2}, K_{M,U}, L^2(\mu_n) \right) \lesssim \sum_{j=1}^{\infty} \log p_j \lesssim \sum_{j=1}^{\infty} \left( \frac{\bar{U}_j + \operatorname{diam}(D_j)\bar{M}_j}{ea_j} \right)^{\frac{d}{2}}. \]
Now, set
\[ a_j = \left( \frac{\bar{U}_j + \operatorname{diam}(D_j)\bar{M}_j}{ea_j} \right)^{\frac{d}{2}} \mu_n(D_j)^{-\frac{2}{d+2}}. \]
Then
\[ \sum_{j=1}^{\infty} \left( \frac{\bar{U}_j + \operatorname{diam}(D_j)\bar{M}_j}{a_j} \right)^{\frac{d}{2}} = \sum_{j=1}^{\infty} \left( \frac{\bar{U}_j + \operatorname{diam}(D_j)\bar{M}_j}{a_j} \right)^{\frac{d}{2}} \mu_n(D_j)^{\frac{2}{d+2}}, \]
and,
\[ \sum_{j=1}^{\infty} a_j^2 \mu_n(D_j) \leq \sum_{j=1}^{\infty} \left( \frac{\bar{U}_j + \operatorname{diam}(D_j)\bar{M}_j}{a_j} \right)^{\frac{d}{2}} \mu_n(D_j)^{\frac{d}{d+2}}. \]
We deduce that for all $\epsilon > 0$,
\[
\log N(\epsilon, K_{M,U}, L^2(\mu_n)) \lesssim \left( \sum_{j=1}^{\infty} a_j^2 \mu_n(D_j) \right)^{\frac{q}{2}} \sum_{j=1}^{\infty} \left( \frac{\bar{U}_j + \text{diam}(D_j) \bar{M}_j}{c a_j} \right)^{\frac{q}{2}} 
\lesssim \left( \frac{1}{\epsilon} \right)^{\frac{q}{2}} \left( \sum_{j=1}^{\infty} \left( \bar{U}_j + \text{diam}(D_j) \bar{M}_j \right)^{\frac{2q}{4+q}} \mu_n(D_j)^{\frac{q}{4+q}} \right).
\]

The claim follows. \qed

**B.6. Proof of Lemma 19.** To prove the claim, it suffices to show that the variance of the supremum of the empirical process is of the order $(\log n)^{2r_4}/n$. By Boucheron, Lugosi and Massart (2013, Theorem 11.1), it holds that
\[
\text{Var} \left[ \sup_{f \in K_{M,U}} \int f d(\mu_n - \mu) \right] \leq \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E} \left[ \sup_{f \in K_{M,U}} \left( f(X_i) - \mathbb{E} f(X_i) \right)^2 \right] 
\leq \frac{1}{n} \mathbb{E} \left[ \sup_{f \in K_{M,U}} f^2(X_1) \right] 
= \frac{1}{n} \sum_{j=0}^{\infty} \int_{L_j} \left( \sup_{f \in K_{M,U}} f^2(x) \right) d\mu(x) 
\leq \frac{1}{n} \sum_{j=0}^{\infty} U_j^2 \mu(L_j) 
\lesssim \frac{1}{n} \sum_{j=0}^{\infty} (3^j \log n)^{2r_4} \exp(-c_1 3^j \beta) 
\lesssim \frac{(\log n)^{2r_4}}{n}.
\]

The claim readily follows. \qed

**B.7. Proof of Lemma 20.** We begin with $\mathbb{E} |\Gamma_n|$. Since the quantity $\int \varphi_0 d(\mu_n - \mu)$ remains unchanged if a constant is added to the map $\varphi_0$, there is no loss of generality in assuming $\varphi_0(0) = 0$. By Theorem 11 and Lemma 10 it must then follow that $\varphi_0(x) \lesssim 1 + \|x\|^q$ for a sufficiently large exponent $q \geq 1$, implying that
\[
\mathbb{E}[\varphi_0(X)^2] \lesssim 1 + \mathbb{E}[\|X\|^{2q}] \leq C,
\]
where the final inequality holds because $\mu$ is $(\sigma, \beta)$-sub-Weibull, and thus admits $(2q)$-th moment bounded above by a constant depending only on $q, \sigma$ and $\beta$, for all $q \geq 1$. Therefore, by Markov’s inequality,
\[
\mathbb{E} \left[ \int \varphi_0 d(\mu_n - \mu) \right] = \int_0^{\infty} \mathbb{P} \left( \left| \int \varphi_0 d(\mu_n - \mu) \right| \geq u \right) du \leq n^{-1/2} + \int_n^{\infty} \frac{C}{nu^2} du \lesssim \frac{1}{\sqrt{n}}.
\]

Applying a similar argument to $\psi_0$ leads to $\mathbb{E} |\Gamma_n| \lesssim n^{-1/2}$.

Turning to $\mathcal{X}_n$, notice that $|\xi_n(x)| \lesssim (\log n \|x\|)^{q'}$ for all $x \in \mathbb{R}^d$, for a sufficiently large constant $q' > 0$, thus it follows similarly as before that $\mathbb{E} |\mathcal{X}_n| \lesssim (\log n)^{q'} n^{-1/2} \lesssim n^{-\epsilon^{-1/2}}$ for any $\epsilon > 0$. \qed
B.8. Proof of Corollary 13. The claim is straightforward when $p \geq 2$. To prove the claim when $p \in (1, 2)$, abbreviate $h_p(x) = \|x\|^p$, and for all $\epsilon \in [0, 1]$ define the cost

$$\tilde{h}_{p, \epsilon}(x) = \left(\|x\|^2 + \epsilon^\frac{2}{p}\right)^\frac{p}{2} - \epsilon.$$

**Lemma 23.** We have for all $p \in (1, 2)$ and all $\epsilon \in [0, 1]$,

1. $\|h_{p, \epsilon} - h_p\|_{L^\infty} \leq 2\epsilon$.
2. $h_{p, \epsilon}$ satisfies condition (H2) with $\|h\|_{C^2(B_0, \epsilon)} \leq \Lambda_\epsilon r^p$ for all $r \geq 1$, where $\Lambda_\epsilon = c_1 \epsilon^{1-\frac{2}{p}}$ for a universal constant $c_1 > 0$. Furthermore, $h_{p, \epsilon}$ satisfies condition (H3) with $\kappa = 2^{\frac{2}{p} - 1}p$.

Lemma 23(i) implies

$$\mathbb{E}\left|\mathcal{T}_{h_p}(\mu_n, \nu_n) - \mathcal{T}_{h_p}(\mu, \nu)\right| \leq \mathbb{E}\left|\mathcal{T}_{h_{p, \epsilon}}(\mu_n, \nu_n) - \mathcal{T}_{h_{p, \epsilon}}(\mu, \nu)\right| + 4\epsilon,$$

which together with Lemma 23(ii) and Theorem 12 imply

$$\mathbb{E}\left|\mathcal{T}_{h_p}(\mu_n, \nu_n) - \mathcal{T}_{h_p}(\mu, \nu)\right| \lesssim \epsilon^{1-\frac{2}{p}} + 2\epsilon + \epsilon.$$ 

The right-hand side is minimized by choosing $\epsilon \asymp n^{-p/d}$, leading to the claim. It thus remains to prove Lemma 23.

B.8.1. *Proof of Lemma 23.* Notice that for all $x \in \mathbb{R}^d$,

$$|h_{p, \epsilon}(x) - h_p(x)| = \left(\|x\|^2 + \epsilon^\frac{2}{p}\right)^\frac{p}{2} - \epsilon - \|x\|^p \leq \left(\|x\|^2 + \epsilon^\frac{2}{p}\right)^\frac{p}{2} - \|x\|^p + \epsilon \leq 2\epsilon,$$

thus part (i) follows. To prove part (ii), choose the function $\omega(z) = (z^2 + \epsilon^{2/p})^{p/2} - \epsilon$. We have $h(0) = 0$, and for all $z > 0$,

$$\omega'(z) = p(z^2 + \epsilon^{2/p})^{p/2 - 1}z,$$

so that $h_{p, \epsilon}$ satisfies condition (H3) with $\kappa = 2^{1-\frac{2}{p}}p$. It remains to prove the Hölder estimate. Clearly, $h_{p, \epsilon} \in C^2_{\text{loc}}(\mathbb{R}^d)$ for all $\epsilon > 0$, and

$$\nabla^2 h_{p, \epsilon}(x) = p(p - 2)\left(\|x\|^2 + \epsilon^\frac{2}{p}\right)^\frac{p}{2} - \epsilon - \|x\|^p \leq 2\epsilon,$$

Therefore,

$$\|\nabla^2 h_{p, \epsilon}(x)\|_{op} \lesssim \left(\|x\|^2 + \epsilon^\frac{2}{p}\right)^\frac{p}{2} - \epsilon - \|x\|^p + \epsilon \leq \epsilon^{1-\frac{2}{p}}, \quad \|x\| \leq \epsilon^\frac{1}{p},$$

We thus easily deduce that for all $r \geq 1$,

$$\|h_{p, \epsilon}\|_{C^2(B_0, r)} \lesssim r^p + \epsilon^{1-\frac{2}{p}} \leq \epsilon^{1-\frac{2}{p}} r^p,$$

and the claim follows.

B.9. On the Super-Gaussianity Assumption. We close this Appendix with a simple characterization of super-Gaussianity which was stated in Section 3. Recall that we say a measure $\mu$ is $(\gamma, b)$-super-Gaussian if $\mu(B_2) \geq b \cdot \mathbb{P}(Z \in B_2)$ for any $x \in \mathbb{R}^d$, where $Z \sim N(0, \gamma^2)$. Furthermore, we say that $\mu$ admits a $(\gamma_1, \gamma_2)$-regular density (Polyanskiy and Wu, 2016) for some $\gamma_1, \gamma_2 > 0$ if $\mu$ admits a density $f$ with respect to the Lebesgue measure such that $\log f$ is differentiable and satisfies

$$\|\nabla \log f(x)\| \leq \gamma_1 \|x\| + \gamma_2, \quad \text{for all } x \in \mathbb{R}^d.$$ 

**Lemma 24.** Assume $\mu$ admits a $(\gamma_1, \gamma_2)$-regular density. Then, there exist constants $\sigma, b > 0$ such that $\mu$ is $(\sigma, b)$-super-Gaussian.
B.9.1. Proof of Lemma 24. By a first-order Taylor expansion, we have for all \( x \in \mathbb{R}^d \),
\[
| \log f(x) - \log f(0) | = | \nabla \log f(\bar{x})^\top x |,
\]
for some \( \| \bar{x} \| \leq \| x \| \). Therefore,
\[
| \log f(x) - \log f(0) | \leq \| \nabla \log f(\bar{x}) \| \| x \| \leq \gamma_1 \| x \|^2 + \gamma_2 \| x \| ,
\]
which entails
\[
f(x) \geq \exp \left\{ \log f(0) - \gamma_1 \| x \|^2 - \gamma_2 \| x \| \right\} = f(0) \exp \left\{ - \gamma_1 \| x \|^2 - \gamma_2 \| x \| \right\} .
\]
If \( \| x \| \geq \gamma_2 \), the above display is bounded below by \( f(0) \exp(- (1 + \gamma_1) \| x \|^2) \), while if \( \| x \| \leq \gamma_2 \), it is bounded below by \( f(0) \exp(- \gamma_1 \| x \|^2 + \gamma_2^2) \). In either case, \( f \) is bounded below by a constant multiple of the \( N(0, \gamma_1^{-1}) \) density, thus the claim readily follows.

\[
\text{APPENDIX C: OMITTED PROOFS FROM SECTION 4}
\]

C.1. Proof of Theorem 22. Throughout this section, we respectively denote the \( \chi^2 \)-divergence and the Total Variation distance between two probability measures \( P \ll Q \) by
\[
\chi^2(P, Q) = \int \left( \frac{dP}{dQ} - 1 \right)^2 dQ, \quad \text{TV}(P, Q) = \frac{1}{2} \int \left| \frac{dP}{dQ} - 1 \right| dQ.
\]
Furthermore, similarly as in the proof of Proposition 21, we write \( T_0(z) = z + z_0 \), where \( z_0 \) is defined in condition (H4).

Our proof of Theorem 22 follows similarly as that of Niles-Weed and Rigollet (2022), Theorem 11, which establishes a minimax lower bound for estimating \( p \)-Wasserstein distances. Our key extension of their proof technique is contained in the following result.

Proposition 25. Assume the same conditions as Theorem 22. Given an integer \( m \geq 1 \), let \( u \) be the uniform distribution on \( [m] \). Then, there exist universal constants \( C_1, C_2 > 0 \), a constant \( C_{\lambda, \Lambda} > 0 \) depending on \( \lambda, \Lambda, \alpha \), and a random function \( F : [m] \to \mathcal{X} \), such that for any distribution \( q \) on \( [m] \), we have
\[
C_1 m^{-\alpha/d} \text{TV}(q, u) - C_{\lambda, \Lambda} \sqrt{\frac{\chi^2(q, u)}{m}} \leq \mathcal{T}_c(F \# q, (T_0 \circ F) \# u) - h(z_0) \\
\leq C_2 m^{-\alpha/d} \left( \frac{\chi^2(q, u)}{m} \right)^{\frac{2}{d}} \text{TV}(q, u)^{1 - \frac{2}{d}} + C_{\lambda, \Lambda} \sqrt{\frac{\chi^2(q, u)}{m}},
\]
with probability at least \( 9 \).

Proof. We prove the claim for \( \alpha \in (1, 2] \). An analogous argument may be used to prove the claim when \( \alpha \in (0, 1] \). Recall the notation of condition (H4). Similarly as in the proof of Proposition 21, there exists \( \gamma > 0 \) such that \( \mathcal{X}_0 = B_{\gamma z_0, \gamma} \subseteq \mathcal{X} \) and such that \( \mathcal{Y}_0 = T_0(\mathcal{X}_0) \subseteq \mathcal{Y} \), where \( T_0(z) = z + z_0 \). Now, it is a straightforward observation that \( N(\epsilon, \mathcal{X}_0, ||||) \geq c^d \epsilon^{-d} \), for all \( \epsilon \in (0, 1) \) and for a constant \( c > 0 \) depending only on \( d, \gamma \), which implies that the \( \epsilon \)-packing number of \( \mathcal{X} \) under \( |||| \) is also greater than \( c^d \epsilon^{-d} \) (Wainwright (2019), Lemma 5.5). Therefore, there exists a set \( \mathcal{G}_m = \{ x_1, \ldots, x_m \} \subseteq \mathcal{X}_0 \) such that \( ||x_i - x_j|| \geq m^{-1/d} \) for all \( i \neq j \). We let \( F \) be selected uniformly at random from the set of bijections \( S_m \) from \( [m] = \{ 1, \ldots, m \} \) to \( \mathcal{G}_m \).

We begin by proving the lower bound. Let \( \tilde{\pi} \) denote an optimal coupling between \( F \# q \) and \( (T_0 \circ F) \# u \), and let \( \pi = (Id, T_0^{-1}) \# \tilde{\pi} \in \Pi(F \# q, F \# u) \). We then have,
\[
\mathcal{T}_c(F \# q, (T_0 \circ F) \# u) - h(z_0) = \int [h(y - x) - h(z_0)] d\tilde{\pi}(x, y)
\]
Thus, returning to equation (48), we next bound the term \( \Gamma_m = \int (\nabla h(z_0), y - x) d\pi(x, y) \). Notice that

\[
\mathbb{E}_F[\Gamma_m] = \left\langle \nabla h(z_0), \frac{1}{m} \sum_{G \in S_m} \sum_{j=1}^m (u(j) - q(j)) G(j) \right\rangle
= \left\langle \nabla h(z_0), \sum_{j=1}^m (u(j) - q(j)) \left( \frac{1}{m} \sum_{G \in S_m} G(j) \right) \right\rangle.
\]

The quantity \( \frac{1}{m} \sum_{G \in S_m} G(j) \) takes on the same value for all \( j = 1, \ldots, m \), thus we deduce from the above display that \( \mathbb{E}_F[\Gamma_m] = 0 \). Similarly, notice that for any \( 1 \leq j \neq k \leq m \),

\[
\mathbb{E}_F[\langle \nabla h(z_0), F(j) \rangle \langle \nabla h(z_0), F(k) \rangle] = \frac{1}{m!} \sum_{x \in \mathcal{G}} \sum_{y \in \mathcal{G}} \sum_{y \neq x} \sum_{G \in S_m} G(j) = x G(k) = y \langle \nabla h(z_0), x \rangle \langle \nabla h(z_0), y \rangle
= \frac{1}{m(m-1)} \sum_{x \neq y} \langle \nabla h(z_0), x \rangle \langle \nabla h(z_0), y \rangle =: M_{1,1},
\]

which is again constant in \( j, k \). It follows that

\[
\mathbb{E}_F \left[ \sum_{j \neq k} \langle \nabla h(z_0), F(j) \rangle \langle \nabla h(z_0), F(k) \rangle (q(j) - u(j))(q(k) - u(k)) \right]
= M_{1,1} \sum_{j \neq k} (q(j) - u(j))(q(k) - u(k)) = -M_{1,1} \sum_{j=1}^m \left( q(j) - \frac{1}{m} \right)^2 = -M_{1,1} \frac{\chi^2(q, u)}{m},
\]

whence, letting \( M_2 := \mathbb{E}_F \left[ \langle \nabla h(z_0), F(j) \rangle^2 \right] \), which itself is again constant in \( j \), we obtain

\[
\text{Var}_F[\Gamma_n] = \mathbb{E}_F \left[ \left( \sum_{j=1}^m \langle \nabla h(z_0), F(j) \rangle (q(j) - u(j)) \right)^2 \right]
= \sum_{j=1}^m \mathbb{E}_F[\langle \nabla h(z_0), F(j) \rangle^2 (q(j) - u(j))^2] - M_{1,1} \frac{\chi^2(q, u)}{m} = M_2 - M_{1,1} \frac{\chi^2(q, u)}{m}.
\]

Therefore, by Markov’s inequality, there exists a constant \( C_{\lambda, \Lambda} > 0 \) depending only on \( M_2 \), and hence only on \( \lambda, \Lambda, \alpha \), such that

\[
\mathbb{P} \left( |\Gamma_n| \geq C_{\lambda, \Lambda} \sqrt{\frac{\chi^2(q, u)}{m}} \right) \leq .025.
\]

Thus, returning to equation (48), and recalling that for all \( x, y \in \mathcal{G}_m, \| x - y \| \geq m^{-1/d} I(x \neq y) \), we deduce that for some \( C_1 > 0 \), with probability at least .975,

\[
\mathcal{T}_c(F\# q, (T_0 \circ F)\# u) - h(z_0) \geq C_1 \lambda m^{-\frac{\alpha}{2}} \mathbb{P}_\pi(X \neq Y) + \Gamma_m
\geq C_1 \lambda m^{-\frac{\alpha}{2}} \text{TV}(F\# q, F\# u) + \Gamma_m.
\]
\( \geq C_1 \lambda m^{-\frac{\alpha}{d}} TV(q, u) - C_{\lambda, \Lambda} \sqrt{\frac{\chi^2(q, u)}{m}}. \)

We now prove the upper bound of the claim. Unlike before, we now let \( \pi \) denote an optimal coupling between \( F_{\#} q \) and \( F_{\#} u \), and \( \bar{\pi} = (Id, T_0)_{\#} \pi \in \Pi(F_{\#} q, (T_0 \circ F)_{\#} u) \) a possibly suboptimal coupling. By assumption (H1), we then have

\[
T_c(F_{\#} q, (T_0 \circ F)_{\#} u) - h(z_0) \leq \int \left[ h(y - x) - h(z_0) \right] d\bar{\pi}(x, y) \\
= \int \left[ h(y - x + z_0) - h(z_0) \right] d\pi(x, y) \\
\leq \int \langle \nabla h(z_0), y - x \rangle d\pi(x, y) + \frac{\Delta}{2} \int \| x - y \|^{\alpha} d\pi(x, y) \\
= \Gamma_m + \Lambda W^\alpha_{\alpha} (F_{\#} q, F_{\#} u).
\]

Now, by Niles-Weed and Rigollet (2022), Proposition 9, there exists a constant \( C_2 > 0 \) such that

\[
W^\alpha_{\alpha} (F_{\#} u, F_{\#} q) \leq C_2 m^{-\alpha/d} (\chi^2(q, u))^{\alpha/d} TV(q, u)^{1 - \frac{\alpha}{d}}.
\]

After possibly modifying \( C_2 \), it follows from Markov's inequality that

\[
\mathbb{P} \left( W^\alpha_{\alpha} (F_{\#} u, F_{\#} q) \leq C_2 \Lambda m^{-\alpha/d} (\chi^2(q, u))^{\alpha/d} TV(q, u)^{1 - \frac{\alpha}{d}} \right) \geq .975,
\]

so that, together with equations (49) and (51), we have with probability at least .95,

\[
T_c(F_{\#} u, (T_0 \circ F)_{\#} u) - h(z_0) \leq C_{\lambda, \Lambda} \sqrt{\frac{\chi^2(q, u)}{m}} + C_2 \Lambda m^{-\alpha/d} (\chi^2(q, u))^{\alpha/d} TV(q, u)^{1 - \frac{\alpha}{d}}.
\]

Combining this fact with equation (50) and a union bound leads to the claim. 

We now prove the main Theorem. In what follows, let \( D_m \) denote the set of probability distributions \( q \) on \([m]\) satisfying \( \chi^2(q, u) \leq 9 \). Also, given \( \delta > 0 \), let \( D_{m, \delta} \) denote the subset of distributions in \( D_m \) satisfying \( TV(q, u) \leq \delta \), and by \( D^+_m \) the subset of \( D_m \) satisfying \( TV(q, u) \geq 1/4 \). Furthermore, set

\[
\Delta_m = \frac{C_1 \lambda m^{-\alpha/d}}{16},
\]

and \( \delta = \left( \frac{C_1 \lambda}{288 \Lambda C_2} \right)^{\frac{1}{1 - \frac{\alpha}{d}}} \). Since \( d \geq 5 > 2\alpha \), we may assume that \( m \) is large enough to satisfy

\[
\Delta_m \geq 2C_{\lambda, \Lambda} \sqrt{\frac{9}{m}}.
\]

Then, by Proposition 25, for all \( q \in D^+_{m, \delta} \), we have with probability at least .9,

\[
T_c(F_{\#} q, (T_0 \circ F)_{\#} u) - h(z_0) \leq C_2 \Lambda m^{-\alpha/d} (\chi^2(q, u))^{\frac{\alpha}{d}} TV(q, u)^{1 - \frac{\alpha}{d}} + C_{\lambda, \Lambda} \sqrt{\frac{\chi^2(q, u)}{m}} \\
\leq \frac{C_1 \lambda C_2 \Lambda m^{-\frac{\alpha}{d}} 9^{\frac{\alpha}{d}}}{288 \Lambda C_2} + C_{\lambda, \Lambda} \sqrt{\frac{9}{m}} \leq \frac{\Delta_m}{2} + C_{\lambda, \Lambda} \sqrt{\frac{9}{m}} \leq \Delta_m.
\]

Similarly, for all \( q \in D^+_m \), we have with probability at least .9,

\[
T_c(F_{\#} q, (T_0 \circ F)_{\#} u) - h(z_0) \geq C_1 \lambda m^{-\alpha/d} TV(q, u) - C_{\lambda, \Lambda} \sqrt{\frac{\chi^2(q, u)}{m}}.
\]
\[ \frac{C_1}{4} \lambda n^{-\alpha/d} - C_{\lambda,\Lambda} \left( \frac{9}{m} \right)^{\frac{1}{2}} \geq 4 \Delta_m - C_{\lambda,\Lambda} \left( \frac{9}{m} \right)^{\frac{1}{2}} \geq 3 \Delta_m. \]

Now, for any given estimator \( \hat{T}_n \) based on the independent samples \( X_1, \ldots, X_n \) and \( Y_1, \ldots, Y_n \), define the event \( A = \{ |\hat{T}_n - T_c(F_{\# q}, (T_0 \circ F)_{\# u})| \geq \Delta_m \} \). We have by Markov’s inequality,

\[
\sup_{\mu \in \mathcal{P}(\mathcal{X})} \mathbb{E}_{\mu,\nu} \left| \hat{T}_n - T_c(\mu, \nu) \right| \geq \Delta_m \sup_{\mu \in \mathcal{P}(\mathcal{X})} \mathbb{P}_{\mu,\nu} \left( \left| \hat{T}_n - T_c(\mu, \nu) \right| \geq \Delta_m \right)
\]

Similarly, for all \( q \in D_{m, \delta}^- \), we have

\[
\mathbb{E}_F \mathbb{P}_{F \# q, (T_0 \circ F)_{\# u}} [A]
\]

\[ \geq \mathbb{E}_F \mathbb{P}_{F \# q, (T_0 \circ F)_{\# u}} \left( \hat{T}_n \geq h(z_0) + 2 \Delta_m \text{ and } T_c(\mu, \nu) \leq h(z_0) + \Delta_m \right) \]

\[ \geq \mathbb{E}_F \mathbb{P}_{F \# q, (T_0 \circ F)_{\# u}} \left( \hat{T}_n \geq h(z_0) + 2 \Delta_m \right) - \mathbb{P}_F \left( T_c(\mu, \nu) > h(z_0) + \Delta_m \right) \]

\[ \geq \mathbb{E}_F \mathbb{P}_{F \# q, (T_0 \circ F)_{\# u}} \left( \hat{T}_n \geq h(z_0) + 2 \Delta_m \right) - .1. \]

Similarly, for all \( q \in D_{m, \delta}^+ \),

\[
\mathbb{E}_F \mathbb{P}_{F \# q, (T_0 \circ F)_{\# u}} [A]
\]

\[ \geq \mathbb{E}_F \mathbb{P}_{F \# q, (T_0 \circ F)_{\# u}} \left( \hat{T}_n \leq h(z_0) + 2 \Delta_m \text{ and } T_c(\mu, \nu) \geq h(z_0) + 3 \Delta_m \right) \]

\[ \geq \mathbb{E}_F \mathbb{P}_{F \# q, (T_0 \circ F)_{\# u}} \left( \hat{T}_n \leq h(z_0) + 2 \Delta_m \right) - .1. \]

Returning to equation (52), we thus have,

\[
\inf \sup_{\hat{T}_n, \mu \in \mathcal{P}(\mathcal{X})} \mathbb{E}_{\mu,\nu} \left| \hat{T}_n - T_c(\mu, \nu) \right| \geq \Delta_m \inf_{\psi} \left\{ \sup_{q \in D_{m, \delta}^-} \mathbb{P}_q(\psi = 1) + \sup_{q \in D_{m}^+} \mathbb{P}_q(\psi = 0) - 0.2 \right\},
\]

where the infimum is over all tests based on the samples \( X_1, \ldots, X_n \) and \( Y_1, \ldots, Y_n \). By Proposition 10 of Niles-Weed and Rigollet (2022), the infimum on the right-hand side of the above display is bounded below by a constant if \( m \asymp n \log n \). The claim then follows by definition of \( \Delta_m \).

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