Counter Terms for Low Momentum Nucleon-Nucleon Interactions

Jason D. Holt 1, T.T.S. Kuo1, G.E. Brown 1, and Scott K. Bogner 2

1Department of Physics, State University of New York at Stony Brook Stony Brook, New York 11794
2Institute for Nuclear Theory, Univ. of Washington, Seattle, WA 98195

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There is much current interest in treating low energy nuclear physics using the renormalization group (RG) and effective field theory (EFT). Inspired by this RG-EFT approach, we study a low-momentum nucleon-nucleon (NN) interaction, \( V_{\text{low}-k} \), obtained by integrating out the fast modes down to the scale \( \Lambda \sim 2 \text{fm}^{-1} \). Since NN experiments can only determine the effective interaction in this low momentum region, our chief purpose is to find such an interaction for complex nuclei whose typical momenta lie below this scale. In this paper we find that \( V_{\text{low}-k} \) can be highly satisfactorily accounted for by the counter terms corresponding to a short range effective interaction. The coefficients \( C_n \) of the power series expansion \( \sum C_n q^n \) for the counter terms have been accurately determined, and results derived from several meson-exchange NN interaction models are compared. The counter terms are found to be important only for the S, P and D partial waves. Scaling behavior of the counter terms is studied. Finally we discuss the use of these methods for computing shell model matrix elements.

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I. INTRODUCTION

Since the pioneering work of Weinberg 1, there has been much progress and interest in treating low-energy nuclear physics using the renormalization group (RG) and effective field theory (EFT) approach 2 3 4 5 7 2 3 3 11. A central idea here is that physics in the infra red region must be insensitive to the details of the short range (high momentum) dynamics. In low-energy nuclear physics, we are probing nuclear systems with low-energy probes of wave length \( \chi \); such probes cannot reveal the short range details at distances much smaller than \( \chi \). Furthermore, our understanding about the short range dynamics is still preliminary, and model dependent. Because of these considerations, a central step in the RG-EFT approach is to divide the fields into two categories: slow fields and fast fields, separated by a chiral symmetry breaking scale \( \Lambda \sim 1 \text{GeV} \). Then by integrating out the fast fields, one obtains an effective field theory for the slow fields only. This approach has been very successful in treating low-energy nuclear systems, as discussed in the references cited above.

In order to have an effective interaction appropriate for complex nuclei in which typical nucleon momenta are \( < k_F \), we separate fast and slow fields by a scale much smaller than \( \Lambda \), namely \( \Lambda \sim 2 \text{fm}^{-1} \). This corresponds to the pion production threshold, a scale often not considered in traditional EFT methods, even though it represents the limit of available 2-nucleon data; experiments give a unique effective interaction only up to this scale. This cutoff has been employed recently by several authors 12 13 14 15 16 17 18 19 20 in a new development for the nucleon-nucleon (NN) interaction: the derivation of a low momentum NN potential, \( V_{\text{low}-k} \). Despite some general similarities, we would like to stress that \( V_{\text{low}-k} \) is not a traditional EFT construct, and would like to now attempt to clarify this issue.

One drawback to the commonly used RG-EFT, which adheres to order-by-order consistency in the power counting, is it’s lack of predictability. The number of unknown parameters increases rapidly as the number of nucleons involved in the process increases. Even if such a method were available for complex nuclei, the parameters would be far from first principle QCD. The key ingredient in our approach is the highly refined standard nuclear physics approach (SNPA) and the objective is to marry the SNPA to an EFT 20.

As an illustration of our strategy, we briefly discuss how this marriage can be effectuated. As summarized recently 27 28, a thesis now developed for some time posits that by combining the SNPA, based on potentials fit to experiments, with modern effective field theory, one can achieve more predictive power than the purist’s EFT alone. This combination of SNPA and effective field theory, called EFT by Kubodera 24, but more properly called more effective effective field theory (MEEFT) as noted by Kubodera, was rediscovered in 12 13 14 15 16 17 18 19 20, and consists of an RG-type approach which limits \( \Lambda \) to 2.1fm\(^{-1} \), because this is equal to the center of mass momentum up to which experiments measuring the nucleon-nucleon scattering phase shifts have been carried out.

Thus, the “more effective” in MEEFT arises from the guarantee that all experimental data will be reproduced by \( V_{\text{low}-k} \), which will contain no Fourier components higher than those at which experiments have been carried out and analyzed. Therefore, \( V_{\text{low}-k} \) is reliable up
to \( k = \Lambda \). This seems to be adequate for a description of shell model properties of complex nuclei.

In practice, the construction of \( V_{\text{low}-k} \) begins with one of a number of available realistic models \cite{21,22,23,24,25} for the nucleon-nucleon (NN) potential \( V_{NN} \). While these models agree well in the low momentum (long range) region, where they are just given by the one pion exchange interaction, in the high momentum (short range) region they are rather uncertain and, in fact, differ significantly from one another. Naturally, it would be desirable to remove these uncertainties and model dependence from the high momentum components of the various modern NN potentials. Following our RG-type approach to achieve this end, it would seem to be appropriate to integrate out the high momentum modes of the various \( V_{NN} \) models. This is how the \( V_{\text{low}-k} \) was derived, and in section II, we provide a brief outline of the derivation based on T-matrix equivalence.

The low-momentum NN potential, \( V_{\text{low}-k} \), reproduces the deuteron binding energy, low-energy NN phase shifts and the low-momentum half-on-shell T-matrix. Furthermore, it is a smooth potential and can be used directly in nuclear many body calculations, avoiding the calculation of the Brueckner G-matrix \cite{12,13,14,18,19,20}. Shell model nuclear structure calculations using \( V_{\text{low}-k} \) have indeed yielded very encouraging results \cite{13,19,20}. These calculations apply the same \( V_{\text{low}-k} \) to a wide range of nuclei, including those in the sd-shell, tin region, and heavy nuclei in the lead region. At the present, no algebraic form exists for \( V_{\text{low}-k} \), and it should be useful to have one. In the present work we study the feasibility of finding such an expression.

A central result of modern renormalization theory is that a general RG decimation generates an infinite series of counter terms \cite{8,9,10,11} consistent with the input interaction. When we derive our low momentum interaction, the high momentum modes of the input interaction are integrated out. Does this decimation also generate a series of counter terms? If so, then what are the properties of the counter terms so generated? We study these questions in section III where we carry out an accurate determination of the counter terms and show that this approach reproduces not only \( V_{\text{low}-k} \), but the deuteron binding energy and low-energy phase shifts as well. We shall discuss that the counter terms represent generally a short range effective interaction and are important only for partial waves with angular momentum \( l \leq 2 \). The scaling behavior of the counter terms with respect to the decimation momentum will be studied in section IV. Finally in section V, we examine the prospect for using \( V_{\text{low}-k} \) and the counter term method in shell model calculations.

### II. T-Matrix Equivalence

Since our method for deriving the low-momentum interaction \( V_{\text{low}-k} \) has been described elsewhere \cite{12,13,14}, in the following we only outline the derivation. We obtain \( V_{\text{low}-k} \) from a realistic \( V_{NN} \) model, such as the CD-Bonn model, by integrating out the high-momentum components. This integration is carried out with the requirement that the deuteron binding energy and low-energy phase shifts of \( V_{NN} \) are preserved by \( V_{\text{low}-k} \). This preservation may be satisfied by the following T-matrix equivalence approach \cite{12,13,14}. We start from the half-on-shell T-matrix

\[
T(k', k, k^2) = V_{NN}(k', k) + \int_0^\infty q^2 dq V_{NN}(k', q) \times \frac{1}{k^2 - q^2 + i0^+} T(q, k, k^2),
\]

noting that the intermediate state momentum \( q \) is integrated from 0 to \( \infty \). We then define an effective low-momentum T-matrix by

\[
T_{\text{low}-k}(p', p, p^2) = V_{\text{low}-k}(p', p) + \int_0^\Lambda q^2 dq V_{\text{low}-k}(p', q) \times \frac{1}{p'^2 - q^2 + i0^+} T_{\text{low}-k}(q, p, p^2)
\]

where \( \Lambda \) denotes a momentum space cut-off and \( (p', p) \leq \Lambda \). We choose \( \Lambda \sim 2\text{fm}^{-1} \), essentially the momentum up to which the experiments give us information in the phase shift analysis. Note that in Eq.(2) the intermediate state momentum is integrated from 0 to \( \Lambda \). We require the above T-matrices satisfying the condition

\[
T(p', p, p^2) = T_{\text{low}-k}(p', p, p^2); \quad (p', p) \leq \Lambda.
\]

The above equations define the effective low momentum interaction \( V_{\text{low}-k} \), and are satisfied by the solution \cite{12,13,14}

\[
V_{\text{low}-k} = \hat{Q} - \hat{Q} \int \hat{Q} + \hat{Q} \int \hat{Q} \int \hat{Q} - \hat{Q} \int \hat{Q} \int \hat{Q} \int \hat{Q} + \cdots
\]

which is just the Kuo-Lee-Ratcliff folded-diagram effective interaction \cite{30,31}. Here \( \hat{Q} \) represents the irreducible vertex function whose intermediate states are all beyond \( \Lambda \); \( \hat{Q} \) is the same as \( \hat{Q} \) except with its terms first order in the interaction removed.

For any decimation momentum \( \Lambda \), the above \( V_{\text{low}-k} \) can be calculated highly accurately (essentially exactly) using either the Andreozzi-Lee-Suzuki \cite{32,33} (ALS) or the Krenciogowa-Kuo \cite{34} iteration methods. These procedures preserve the deuteron binding energy in addition
to the half-on-shell T matrix, which implies preservation of the phase shifts. After obtaining the above \( V_{low-k} \), which is not Hermitian, we further perform an Okubo transformation \( \Sigma \) to make it Hermitian. This is done using the method given in Refs. \[36, 37\]. Here we first calculate the eigenvalues and eigenvectors of the operator \( \omega \) where \( \omega \) is the wave operator obtained with the ALS method. Then the Hermitian \( V_{low-k} \) is calculated in terms of these quantities in a convenient way (see Eq.(23) of Ref. \[37\]).

III. COUNTER TERMS

Here we study whether the low-momentum interaction \( V_{low-k} \) can be well represented by the low-momentum part of the original NN interaction supplemented by certain simple counter terms. Specifically, we consider

\[
V_{low-k}(q, q') \simeq V_{bare}(q, q') + V_{counter}(q, q') \quad (q, q') \leq \Lambda
\]

where \( V_{bare} \) is some bare NN potential \( V_{NN} \). Since \( V_{low-k} \) and \( V_{bare} \) are both known, there is no question about the existence of \( V_{counter} \) in general. Our aim is, however, to investigate if \( V_{counter} \) can be well represented by a short ranged effective interaction, such as a smeared out delta function. We shall check this conjecture by assuming a suitable momentum expansion form for \( V_{counter} \) and investigating how well it can satisfy the above equality. Since \( V_{bare} \) is generally given according to partial waves as is \( V_{low-k} \), we shall proceed to determine \( V_{counter} \) separately for each partial wave. At very small radial distance, the Bessel function \( j_l(qr) \) behaves like \( (qr)^l \). We assume that \( V_{counter} \) is a very short ranged interaction.

Hence the leading term in a momentum expansion of the partial wave matrix element \( \langle q|V_{counter}|q'\rangle \) is proportional to \( q'q'' \). Thus we consider the following expansion for the partial wave counter term potential, namely

\[
\langle q|V_{counter}|q'\rangle = q'q''(C_0 + C_2(q^2 + q'^2) + C_4(q^4 + q'^4) + C_6(q^6 + q'^6) + C_8(q^8 + q'^8) + \cdots)
\]

(6)

The counter term coefficients will be determined by standard \( \chi \)-square fitting procedure so that the difference between \( V_{low-k} \) and \( V_{bare+V_{counter}} \) is minimized.

We note that for \( l \) and \( l' \) being both S-wave, the leading term for \( V_{counter} \) is \( C_0 \) which is momentum independent and corresponds to a delta interaction. For other channels, however, the leading term for the counter potential is momentum dependent. For example, for \( l \) and \( l' \) being both P-wave, the leading term is \( qq' C_0 \).

If we take the projection operator \( P \) to project onto states with momentum less than \( \Lambda \), namely \( P = \sum_{k < \Lambda} |k\rangle\langle k| \), we can rewrite \( V_{bare} \) of Eq.(5) as \( PV_{bare}P \). From now on we shall also abbreviate \( V_{counter} \) as \( V_{CT} \). The above counter term approach for \( V_{low-k} \) is similar to that used in RG-EFT \[4, 5\], as both methods impose some cutoff, integrate out the modes above that cutoff, and then attempt to recover the information contained in those states in a counter term series. Ultimately, the main difference is the the cutoff scale used and the resulting potential to which \( V_{CT} \) is added. As already noted, in the representation of \( V_{low-k} \) given above, we use a momentum cutoff above which no constraining data exists,
and thus our “slow” potential is a bare NN potential projected onto the low momentum states. In the EFT scheme, the “slow” potential is obtained by integrating out heavy dynamical degrees of freedom above the chiral symmetry breaking scale. Hence, counter terms are added to \( V_{\text{light}} \), which is given by light mesons below the cutoff scale, and usually taken to be \( V_{\text{one-pion}} \) with unrestricted pion momentum.

We have also tried to fit \( V_{\text{low-k}} \) solely to a low order momentum expansion, and found the results to be less than satisfactory; the number of terms needed to produce an accurate representation is much higher than the order used here. Attempts at fitting \( (V_{\text{one-pion}} + V_{\text{CT}}) \) also generated results far less satisfactory than fitting it by \( (V_{\text{bare}} + V_{\text{CT}}) \). We shall discuss these points later.

As mentioned earlier, the counter term coefficients are determined using standard fitting techniques. We perform this fitting over all partial wave channels, and find consistently very good agreement. In Fig. 1 we compare some \( ^1S_0 \) and \( ^3S_1 \) matrix elements of \( (PV_{\text{bare}} + V_{\text{CT}}) \) with those of \( V_{\text{low-k}} \) for momenta below the cutoff \( \Lambda \). A similar comparison for the \( ^3S_1 - ^3D_1 \) and \( ^3D_1 \) channels is displayed in Fig. 2. We have also obtained very good agreement for the \( P \)-waves, as displayed in Fig. 3. It may be mentioned that here the momentum factor \( qq' \) of Eq. (6) is essential in achieving the good fit as shown. As demonstrated by these figures, the two methods yield nearly identical effective interactions, lending strong support that \( V_{\text{low-k}} \) can be very accurately represented by \( (PV_{\text{bare}} + V_{\text{CT}}) \). Furthermore \( V_{\text{CT}} \) is a very short ranged effective interaction.

To further check the accuracy of our counter term approach, we have calculated the phase shifts given by \( V_{\text{low-k}} \) and compared them with those given by \( (V_{\text{bare}} + V_{\text{CT}}) \), as seen in Fig 4, where we have found that the phase shifts are almost exactly preserved by the counter term approach. The phase shifts are plotted up to a lab energy of 325 MeV, and only at this high momentum, do they begin to differ slightly from those obtained from \( V_{\text{low-k}} \). In passing, we mention that the deuteron binding energy is also accurately reproduced by our counter term approach (e.g., for the CD Bonn potential the deuteron binding energy given by the full potential versus the counter term approach are respectively 2.225 and 2.224 MeV). Thus, we can accurately reproduce \( V_{\text{low-k}} \) and all its results by using \( (PV_{\text{bare}} + V_{\text{CT}}) \).

Now let us examine the counter terms themselves. In Table I, we list some of the counter term coefficients, using CD-Bonn as our bare potential. It is seen that the counter terms are significant only for the \( S, P \) and \( D \) partial waves. We do not list results beyond the \( ^3P_2 \) partial waves, as the coefficients for them are all zero up to the level of our numerical accuracy (all entries in the table with magnitudes less than \( 10^{-4} \) have been set to zero). It is of interest that except for the above partial waves, \( V_{\text{low-k}} \) is essentially the same as \( PV_{\text{bare}} \) alone. This behavior is clearly a reflection that \( V_{\text{CT}} \) is basically a very short ranged effective interaction. From the table, \( C_0 \) is clearly the dominant term in the expansion. Coefficients beyond \( C_4 \) are generally small and can be ignored. In the last row of the table, we list the rms deviations between \( V_{\text{low-k}} \) and \( PV_{\text{bare}} + V_{\text{CT}} \); the fit is indeed very good.

Comparing counter term coefficients for different \( V_{\text{bare}} \) potentials can illustrate key differences between those potentials. Thus, in Table II, we compare the low order counter terms obtained for the CD-Bonn and Nijmegen potentials.
We have explored other schemes of fitting. We have determined that one of the chief differences between these potentials is the way in which they treat the short range repulsion. For instance, the Paris potential effectively has a very strong short-range repulsion and consequently its 1S0 term is of much larger magnitude than the others.

Our counter term \( V_{CT} \) has been determined by requiring a best fit between \( V_{bare} + V_{CT} \) and \( V_{low-k} \). We have explored other schemes of fitting: We have tried to determine \( V_{CT} \) by requiring a best fit between \( V_{one-pion} + V_{CT} \) and \( V_{low-k} \). But the results are far from satisfactory, the resulting rms deviation at best fit being too large. It may be of interest to determine \( V_{CT} \) by requiring a best fit between \( V_{low-k} \) and \( V_{CT} \) alone.

We have also tried this, with similar unsatisfactory results (large resulting rms deviation). A possible explanation may be the following. A main portion of \( V_{low-k} \) come from \( V_{two-pion} \) and other higher order processes; it appears that such contributions can not be compensated by the \( V_{CT} \) of the simple low-order form of Eq.(6).

## IV. SCALING OF COUNTER TERMS

In Fig. 5 we display the scaling behavior of counter terms \( C_0 \) and \( C_2 \) with respect to the decimation momentum \( \Lambda \). We note that for the \(^1\)S0 channel, \( C_0 \) and \( C_2 \) display a weak \( \Lambda \) dependence. This is a welcome result. We should note that in MEEFT, we are obliged to choose \( \Lambda = 2.1 \text{fm}^{-1} \), because phase shifts from experiments have been carried out up through this momentum and the more effective in MEEFT means that we fit all available experimental data by \( V_{low-k} \). However, the \( C_0 \) for the \(^3\)S1 channel varies significantly with \( \Lambda \). In this channel, there is tensor force which is a mid-range interaction coming from \( \pi \) and \( \rho \) mesons; it has large momentum components in the intermediate momentum region of several \( \text{fm}^{-1} \). As we lower \( \Lambda \) through this region, we are actually integrating out a predominant portion of the tensor force, in addition to the short range repulsion of \( V_{bare} \). As a result, we would expect the \( C_0 \) to change to compensate for this loss of the tensor force. In contrast, the \( C_0 \) for \(^1\)S0 comes mainly from integrating out only the short range repulsion, and little variation is seen. Conversely, as \( \Lambda \) increases, the tensor force is largely retained within the cutoff, and the two \( C_0 \)’s come close to each other as seen in the figure. To justify this line of thinking, we removed the tensor force to see how it would effect the \(^3\)S1 \( C_0 \). We found that the scaling for this term was indeed quite different, displaying a flat behavior over the entire \( \Lambda \) range; \( C_0 \) only changed from 0.05

### TABLE I: Listing of counter terms for all partial waves obtained from the CD-Bonn potential using \( \Lambda = 2 \text{fm}^{-1} \). The units for the combined quantity \( q^2 q^2 \cdot C_o q^2 \) of Eq.(6) is \( \text{fm} \), with momentum \( q \) in \( \text{fm}^{-1} \). All counter terms for higher order partial waves are zero up to our numerical accuracy.

| Wave   | \( C_0 \) | \( C_2 \) | \( C_4 \) | \( C_6 \) | \( C_8 \) | \( \Delta_{\text{rms}} \) |
|--------|-----------|-----------|-----------|-----------|-----------|-----------------|
| \(^1\)S0 | -0.1580   | -1.309E-2 | 3.561E-4  | -8.469E-4 | 0         | -1.191E-4 -1.191E-4 | 0.0002         |
| \(^3\)S1 | -0.4651   | 5.884E-2  | -1.824E-3 | -1.163E-2 | -3.243E-4| 5.181E-4 5.181E-4   | 0.0003         |
| \(^3\)S1 - \(^3\)D1 | 2.879E-2 | -3.581E-3 | 1.969E-3  | -3.676E-3 | -2.049E-4| -2.449E-4 | 0.0025         |
| \(^3\)D1 | -1.943E-3 | -1.918E-4 | 1.799E-4  | 0         | 0         | 0               | \(<0.0001       |
| \(^1\)P1 | -4.311E-2 | -7.305E-4 | 7.999E-4  | -2.049E-4 | -2.594E-4| 0             | \(<0.0001       |
| \(^3\)P0 | -5.566E-2 | 4.821E-4  | 2.874E-4  | 1.098E-4  | 0         | 0               | \(<0.0001       |
| \(^3\)P1 | -5.479E-2 | -5.916E-4 | 8.636E-4  | -2.065E-4 | -2.721E-4| 0             | 0.0002         |
| \(^3\)P2 | -1.294E-2 | 3.075E-4  | 7.469E-4  | -4.289E-4 | -2.179E-4| 0             | 0.0002         |

### TABLE II: Comparison of counter terms for \( V_{low-k} \) obtained from the CD-Bonn, Argonne V18, Nijmegen, and Paris potentials using \( \Lambda = 2 \text{fm}^{-1} \). The units for the counter coefficients \( C_o \)’s are the same as in Table I.

| Wave   | \( C_0 \) | \( C_2 \) | \( C_4 \) | \( C_6 \) | \( C_8 \) | \( \Delta_{\text{rms}} \) |
|--------|-----------|-----------|-----------|-----------|-----------|-----------------|
| \(^1\)S0 | -0.1581   | -0.0131   | 0.0004    | -0.0008   | CDB       | 0.0004         |
| \(^1\)S1 | -0.4651   | 0.0588    | -0.0018   | -0.0116   | V18       | 0.0007         |
| \(^3\)S1 | -1.478E-2 | -0.0822   | 0.0002    | 0.0004    | 0.0002    | 0.0002         |
| \(^3\)S1 - \(^3\)D1 | 0.0288   | -0.0036   | 0.0020    | -0.0037   | 0.0037    | 0.0037         |
| \(^3\)D1 | -1.294E-2 | 0.0242    | 0.0006    | 0.0040    | 0.0004    | 0.0004         |

22. Argonne 23 and Paris 25 NN potentials. The \( C_0 \) coefficients for these potentials are significantly different, indicating that one of the chief differences between these potentials is the way in which they treat the short range repulsion. For instance, the Paris potential effectively has a very strong short-range repulsion and consequently its \( C_0 \) is of much larger magnitude than the others.
at $\Lambda = 1.2\text{fm}$ to 0.07 at $\Lambda = 3.0\text{fm}$.

It should be observed that our Fig. 5 does not cover the small $\Lambda$ region. The reason for this is that we have found that our counter term approach is not applicable to cases with $\Lambda \sim 1\text{fm}^{-1}$ or smaller. In this region, we have not been able to achieve a satisfactory fit between $V_{\text{low}-k}$ and $(V_{\text{bare}} + V_{\text{CT}})$. A likely reason for this is that for small $\Lambda$’s, we are integrating out a major part of the tensor force, which can not be compensated by the simple low order counter terms as indicated by Eq. (6).

As we mentioned earlier, the counter terms are all rather small except for the S waves. This is consistent with the RG-EFT approach where the counter term potential is mainly a delta function force $38,39$. When using $\Lambda \sim 2\text{fm}^{-1}$, we see that the $C_0$ coefficients for the $^1S_0$ and $^3S_1$ channels are different. This suggests that the counter term potential is spin dependent and may be written as

$$V_{\text{CT}} \approx (f_0 + f_0' \sigma \cdot \sigma) \delta(\vec{r}),$$

with $f_0 = (3C_0(^3S_1) + C_0(^1S_0))/4$ and $f_0' = (C_0(^3S_1) - C_0(^1S_0))/4$. For the coefficients of Table I, we have $f_0 = -0.382$ fm and $f_0' = -0.077$ fm. As indicated in Fig. 5, this spin-dependent factor $f_0'$ will diminish as $\Lambda$ increases.

### V. COMPARISON OF SHELL MODEL MATRIX ELEMENTS

In shell model calculations, a basic input is the relative matrix elements $\langle nlSJT | V_{\text{eff}} | n'l'SJT \rangle$, where $nl$ and $n'l'$ denote the oscillator wave functions for the relative motion of the two nucleons and $S$ and $T$ are the two-nucleon spin and isospin. The relative momenta $l(l')$ and $S$ couple to total angular momentum $J$. From these matrix elements, the two-particle shell model matrix elements in the laboratory frame can be calculated $38$. Thus if $V_{\text{low}-k}$ and $P V_{\text{bare}} + V_{\text{CT}}$ are to prove useful in a broad sense, they must preserve these matrix elements.

In Table III we provide a comparison between these matrix elements as obtained from several input potentials. To begin, we note that the harmonic oscillator matrix elements for $V_{\text{low}-k}$ are approximately the same regardless of which of the four bare potentials we use. So, as far as providing input for shell model calculations, this is further evidence that the $V_{\text{low}-k}$ is approximately unique. Next, we come to the main result of the table, which is that the matrix elements for $V_{\text{low}-k}$ and $V_{\text{bare}} + V_{\text{CT}}$ are in very good agreement. We have used the CD Bonn for the bare potential, but this holds for the other bare potentials as well. In fact for the $^1S_0$, $^3S_1$, and $^3D_1$ channels, the elements are virtually exact out to four significant figures, and there is only minimal disagreement in the coupled channel.

### VI. CONCLUSION

While there are several models for the nucleon-nucleon potential in use today, they suffer from uncertainty and model dependence. Motivated by RG-EFT ideas, a low momentum NN interaction, $V_{\text{low}-k}$, has been constructed via integrating out the high momentum, model dependent components of these different potentials. The result appears to give an approximately unique representation
of the low momentum NN potential, and the main issue
described in this paper was whether $V_{\text{low} - k}$ could ac-
curately be cast in a form $V_{\text{bare}} + V_{\text{CT}}$, where $V_{\text{CT}}$ is a low
order counter term series, and what the physical signif-
icance of this counter term series was. We have shown
that this was indeed the case as $V_{\text{low} - k}$ is nearly identical
to $V_{NN} + V_{\text{CT}}$ over all partial waves, and that $V_{NN} + V_{\text{CT}}$
reproduces both the deuteron binding energy and the
NN phase shifts in the low momentum region. We have
found that only the leading terms in the counter term
series have much significance, indicating that the counter
term potential is mainly a short range effective inter-
action which can be accurately represented by a simple
low-order momentum expansion. Furthermore, we exam-
ined the scaling properties of the S-wave counter terms
with respect to $\Lambda$ and found that the tensor force is re-
sponsible for the behavioral differences exhibited between the $^3S_1$ channel and the $^1S_0$ channel. Finally, we exam-
ined the potential for using $V_{\text{low} - k}$ and $V_{NN} + V_{\text{CT}}$ in
shell model calculations. Again, it was seen that not only
does $V_{\text{low} - k}$ give approximately identical input matrix el-
ements irrespective of which bare potential was used, but
that the counter term approach provided nearly the same
results. Thus we have shown that the high momentum
information integrated out of each bare NN potential can
be accurately replaced by the counter terms, making the
use of $V_{\text{low} - k}$ in a broad context much more tractable.

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