Unveiling regions in multi-scale Feynman integrals using singularities and power geometry

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Motivation

1. The determination of quantum field theoretical observables in perturbative domain relies on the evaluation of Feynman diagrams.

2. With the increasing accuracy in the collider experiments, it is necessary to calculate higher order corrections in order for obtaining better prediction from theory side.

3. This leads to the topic of evaluation of multi-scale, multi-loop Feynman diagrams, which are very difficult to solve analytically.

4. When complete analytic calculations are not available, methods for systematic approximation are necessary.

5. The Method of Regions is one of the useful ways to systematically perform the asymptotic analysis of a given Feynman diagram in a given limit.
The Method of Regions (MoR)

1. For a given Feynman diagram, identify all the scales.
2. Divide the loop momentum domain into several regions based on the scaling prescriptions. In each of the regions, construct a suitable small expansion parameter which is basically the ratio of small and large scales.
3. Taylor expand the integrand in each of the regions.
4. After the expansion, integrate the expanded terms over the whole integration domain.
5. All the scaleless integrals are set to zero.
6. Sum of the contributions of all of the regions gives the result for the original Feynman integral in an expanded form.
Consider the following integral,

$$F = \int Dk \frac{1}{((k + p)^2)^{n_1}(k^2 - m^2)^{n_2}}$$

- The considered limit is $|p^2| \gg m^2$.
- Two regions are there, hard region ($k \sim p$) and soft region ($k \sim m$).

The integral is given by,

$$F = F^h + F^s - F^{h,s}$$

$F^{h,s}$ produces scaleless contribution and hence vanish in DR.

$$F = F^h + F^s$$

Jantzen, J. High Energ. Phys. 2011, 76 (2011)
$F^h = \mu^{2\epsilon} e^{\epsilon \gamma E} e^{-i\pi n_{12}} (-p^2 - i0)^{2-n_{12}-\epsilon} \frac{\Gamma(n_{12} - 2 + \epsilon)\Gamma(2 - n_1 - \epsilon)\Gamma(2 - n_2 - \epsilon)}{\Gamma(n_1)\Gamma(n_2)\Gamma(4 - n_{12} - 2\epsilon)}$
$\times \, _2F_1(n_{12} - 2 + \epsilon, n_{12} - 3 + 2\epsilon; n_2 - 1 + \epsilon; \frac{m^2}{p^2})$

$F^s = \mu^{2\epsilon} e^{\epsilon \gamma E} e^{-i\pi n_{12}} (m^2)^{2-n_2-\epsilon} (-p^2 - i0)^{-n_1} \frac{\Gamma(n_{2} - 2 + \epsilon)}{\Gamma(n_2)}$
$\times \, _2F_1(n_1, n_1 - 1 + \epsilon : 3 - n_2 - \epsilon; \frac{m^2}{p^2})$

$F = \mu^{2\epsilon} e^{\epsilon \gamma E} e^{-i\pi n_{12}} (m^2)^{2-n_{12}-\epsilon} \frac{\Gamma(2 - n_1 - \epsilon)\Gamma(n_{12} - 2 + \epsilon)}{\Gamma(n_2)\Gamma(2 - \epsilon)}$
$\times \, _2F_1(n_1, n_{12} - 2 + \epsilon; 2 - \epsilon; \frac{p^2 + i0}{m^2})$
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Geometric approach for the identification of regions

1. There is one geometric approach ASY\textsuperscript{1} based on the analysis in Feynman parameter space which finds the regions.

2. We present our own development for unveiling the regions by looking at Landau singularities and the analysis of power geometry.

3. We call this algorithm ASPIRE (Algebro-geometric analysis of Singular Polynomials for Identification of REgions).

\textsuperscript{1}Pak, Smirnov, Eur.Phys.J.C 71 (2011) Jantzen, Smirnov, Smirnov, Eur.Phys.J.C 72 (2012) 2139
A Feynman graph having $l$-loops, $m$-denominators, and $r$-external momenta $(p_1, \ldots, p_r)$ in $d$-dimension has the form,

\[
I(n) = \int \prod_{i=1}^{l} \frac{d^dl_i}{\pi^{\frac{d}{2}}} \prod_{j=1}^{m} \frac{1}{(A_{ik}l_i \cdot l_k + 2B_{ik}l_i \cdot p_k + C_j)^{n_j}}
\]

where $A, B$ are respectively $l \times l, l \times r$ matrices and $C_j$ are constants. In alpha-parametric form, $I(n)$ can be written as,

\[
I(n) = \frac{\Gamma(|n| - \frac{ld}{2})}{\prod_{j=1}^{m} \Gamma(n_j)} \int \prod_{j=1}^{m} d\alpha_j \alpha_j^{n_j-1} \delta \left(1 - \sum_{j=1}^{m} \alpha_j \right) \frac{\mathcal{U}|n| - \frac{(l+1)d}{2}}{\mathcal{F}|n| - \frac{ld}{2}}
\]

where $\mathcal{U}, \mathcal{F}$ are the Symanzik polynomials (of degree $l$ and $(l+1)$ respectively) and $|n| = n_1 + n_2 + \cdots + n_m$. 
The singularities of Feynman integrals in the parametric form are given by,

\[ F = 0 \]
\[ \frac{\partial F}{\partial \alpha_i} = 0 \]
Consider a generic polynomial in $n$-variables,

$$g(X) = \sum g_Q X^Q, \quad Q \in S(g)$$

where $X = \{x_1, x_2, \ldots, x_n\}$ and $Q = \{Q_1, Q_2, \ldots, Q_n\}$, where $Q_i$ are the exponents of the variables $x_i$ for each monomial, i.e., of the form $x_1^{Q_1} x_2^{Q_2} \cdots$ and the $Q_i$s are a set of natural numbers. Let $\chi = \{X^0\}$ be a set of points such that $g(X^0) = 0$. 
Support: The support $S(g)$ is defined as the set of all vector exponents. For example, given a polynomial in two variables $(x, y)$

$$g(x, y) = xy + x^2 + x^2y + xy^3 + x^3y$$

$X = \{x, y\}$, and $S(g) = \{(1, 1), (1, 3), (3, 1), (2, 0), (2, 1)\}$.

Newton Polytope: The Newton Polytope or Newton Polyhedron is the convex hull of the support $S(g)$.

Generalized faces: The boundary subsets $\{S'\}$ of the Newton Polytope are its faces $\Gamma_{d,j}^d$, where $d$ is the dimension and $j$ labels the face.
Normal cone: For a vector $P \in S(g)$ belonging to $\mathbb{R}^n$ and $Q$ belonging to the dual space $\mathbb{R}^*_n$, the scalar product $c = \langle P, Q \rangle$ is defined. The set of all points for which $c$ achieves maximum value for the points $P \in S(g)$ on the generalized facets $\Gamma^d_j$ of the Newton polytope is called the normal cone of the facets.

Cone of the problem: The Cone of the problem is a set of vectors $K = (s_1, \ldots, s_n)$ such that curves of the form

$$x_1 = a_1 t^{s_1} \quad x_2 = a_2 t^{s_2} \quad \ldots \quad x_n = a_n t^{s_n},$$

where $t$ parametrizes the polynomial, fill those regions of the $X$-variables space that we are interested in.

Truncated polynomial: The truncation of the sum on the boundary subset is defined as

$$\hat{g}_j^{(d)} = \sum g_Q X^Q \quad Q \in S'_j.$$

Such a truncated polynomial should be quasi-homogenous.
**Figure:** Newton polytope of support

| Face | Boundary Subset | Truncated Polynomial |
|------|-----------------|---------------------|
| 1    | $S_1''\,(1,1),(1,3)$ | $xy + xy^3$ |
| 2    | $S_2''\,(1,3),(3,1)$ | $xy^3 + x^3y$ |
| 3    | $S_3''\,(3,1),(2,0)$ | $x^3y + x^2$ |
| 4    | $S_4''\,(2,0),(1,1)$ | $x^2 + xy$ |

**Table:** A Table showing the Boundary subsets and the associated normals of Newton Polytope
If for $t \to \infty$, the curve

$$x = at^{p_1} (1 + \mathcal{O}(1)), \ y = bt^{p_2} (1 + \mathcal{O}(1)), \ z = ct^{p_3} (1 + \mathcal{O}(1)),$$

where $a, b, c$ and $p_i$ are constants, belongs to the set $g = \{X : g(X) = 0\}$ and the vector $P = (p_1, p_2, p_3)$ belongs to $U^d_j$, then the first approximation $x = at^{p_1}, y = bt^{p_2}, z = ct^{p_3}$ of the curve satisfies the truncated equation $\hat{g}^d_j(X) = 0$.

Bruno, Batkhin, Program Comput Soft 38, 5772 (2012)
The algorithm ASPIRE

- Construct the polynomial $G = F + U$.
- Find the Gröbner Basis for the set of Landau equations.
- For every neighborhood in the alpha-parameter space, perform linear transformations to map the nearest solution curves of the Gröbner basis elements to the origin, coordinate axes, coordinate planes.
- Using the definition of the small threshold expansion parameter $x$, in terms of the kinematic invariants, re-express all the constant coefficients like mass and external momenta appearing in the above equations.
For every transformation applied to the alpha-parameters, find the support of $G$ which has the structure, $S(G) = (Q_0, Q)$, where $Q_0$ is the vector exponent of the small expansion parameter $x$ and $Q = (q_1, q_2, ..., q_n)$ are vector exponents of the alpha-parameters.

- Find the convex hull of the support.
- Find the boundary subset for every facet of every Newton polytope.
- Find the normal cone for each of the facets. This amounts to finding the normal vector to the surfaces.
- Using the theorem we conclude that the above truncated polynomials are satisfied by the following expressions for the alpha-parameters

$$\alpha_1 = a_1 x^{p_1}, \alpha_2 = a_2 x^{p_2}, ..., \alpha_n = a_n x^{p_n}.$$ 

Here $a_i \in \mathbb{C}$ and the set of $P = \{p_i\}$ defines the region. The normal to the surface corresponding to the truncated polynomial gives us $P$. 
Consider the integral,

\[ I = \int \frac{d^d k}{(k^2 - m^2)((k - q)^2 - m^2)}. \]

Here \( q \) is the external momentum, \( k \), the loop momentum and the threshold expansion parameter is defined as \( y = m^2 - \frac{q^2}{4} \).
We find the $\mathcal{U}, \mathcal{F}$ function using the code UF.m,

$$\text{UF} \left[ \{k\}, \left\{ -(k^2 - m^2), -((k - q)^2 - m^2) \right\}, \left\{ q^2 \rightarrow qq, m^2 \rightarrow \frac{qq}{4} + y \right\} \right]$$

yielding an output

$$\left\{ x_1 + x_2, \frac{1}{4} qq x_1^2 - \frac{1}{2} qq x_1 x_2 + \frac{1}{4} qq x_2^2 + x_1^2 y + 2 x_1 x_2 y + x_2^2 y, 1 \right\}$$

The first and second elements are the $\mathcal{U}$ and $\mathcal{F}$ polynomials respectively, while the third element of the output is the number of loops, which is 1 in this example.
We wish to find the locations of the singularities in the Alpha-parameter space with the help of Landau equations,

\[ \mathcal{F} = 0, \]

\[ \frac{\partial \mathcal{F}}{\partial x_1} = \frac{\partial \mathcal{F}}{\partial x_2} = 0. \]

Next we find the Gröbner basis of the set of Landau Equations for which, we use the Mathematica function `GroebnerBasis` via the command

\[
\text{GroebnerBasis}\left[\left\{\mathcal{F}, \frac{\partial \mathcal{F}}{\partial x_1}, \frac{\partial \mathcal{F}}{\partial x_2}\right\}, \{x_1, x_2\}\right]
\]

which gives the elements,

\[ \mathcal{G} = \{q^2yx_2, (x_1 + x_2)y, q^2(x_1 - x_2)\}. \]
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Figure: Partitioning of alpha parameter space by solution curves of the Gröbner basis elements.
List of all distinct transformations:

- Identity transformation:
  \[ T_1 \equiv \{ x_1 \rightarrow x_1, x_2 \rightarrow x_2 \} \]

- Non-trivial transformations:
  \[ T_2 \equiv \{ x_1 \rightarrow \frac{x_1}{2}, x_2 \rightarrow x_2 + \frac{x_1}{2} \} \]
  \[ T_3 \equiv \{ x_1 \rightarrow x_1 + \frac{x_2}{2}, x_2 \rightarrow \frac{x_2}{2} \} \]
we go on to compute $G$ with all the above transformations $T = \{T_1, T_2, T_3\}$, where the $G = \{G_1, G_2, G_3\}$ corresponding to the three transformations.

$$G_1 \equiv \frac{1}{4} q^2 x_1^2 - \frac{1}{2} q^2 x_1 x_2 + \frac{1}{4} q^2 x_2^2 + x x_1^2 + 2 x x_1 x_2 + x_1 + x x_2^2 + x_2,$$

$$G_2 \equiv \frac{1}{4} q^2 x_1^2 + x x_1^2 + 2 x x_1 x_2 + x_1 + x x_2^2 + x_2,$$

$$G_3 \equiv \frac{1}{4} q^2 x_2^2 + x x_1^2 + 2 x x_1 x_2 + x_1 + x x_2^2 + x_2$$

Here, we have substituted $y \rightarrow x$. 
The support $S_i$ of $G_i$, where $i$ enumerates the three polynomials coming from the three transformations are,

$$S_1 = \begin{pmatrix}
0 & 1 & 0 \\
0 & 2 & 0 \\
1 & 2 & 0 \\
0 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 1 \\
0 & 0 & 2 \\
1 & 0 & 2 \\
\end{pmatrix}, \quad S_2 = \begin{pmatrix}
0 & 1 & 0 \\
0 & 2 & 0 \\
1 & 2 & 0 \\
0 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 1 \\
0 & 0 & 2 \\
1 & 0 & 2 \\
\end{pmatrix}, \quad S_3 = \begin{pmatrix}
0 & 1 & 0 \\
1 & 2 & 0 \\
0 & 0 & 1 \\
0 & 1 & 1 \\
1 & 0 & 2 \\
1 & 0 & 2 \\
\end{pmatrix}$$
The regions

| Normal      | Facet : top/bottom(1/−1) |
|-------------|--------------------------|
| \{0, 0\}   | \{-1\}                  |
| \{-1, -1\} | \{1\}                   |
| \{-1/2, -1\}| \{-1\}                  |
| \{-1, -1/2\}| \{-1\}                  |

\[^2\text{For the bottom facets, } \vec{r}.\vec{v} = c, \text{ for points on the facets and } \vec{r}.\vec{v} > c \text{ for the rest of the points.}\]
Conclusion

- ASPIRE provides an useful method for unveiling the regions associated with a given Feynman diagram in a given limit.
- The regions are basically the scalings coming from the bottom facets.
Thank You!