Energy-momentum tensor for a scalar Casimir apparatus in a weak gravitational field: Neumann conditions

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We consider a Casimir apparatus consisting of two perfectly conducting parallel plates, subject to the weak gravitational field of the Earth. The aim of this paper is the calculation of the energy-momentum tensor of this system for a free, real massless scalar field satisfying Neumann boundary conditions on the plates. The small gravity acceleration (here considered as not varying between the two plates) allows us to perform all calculations to first order in this parameter. Some interesting results are found: a correction, depending on the gravity acceleration, to the well-known Casimir energy and pressure on the plates. Moreover, this scheme predicts a tiny force in the upwards direction acting on the apparatus. These results are supported by two consistency checks: the covariant conservation of the energy-momentum tensor and the vanishing of its regularized trace, when the scalar field is conformally coupled to gravity.

I. INTRODUCTION

Quantum field theory in curved spacetime, although far from being a definitive theory unifying the quantum theories with gravitation, offers nevertheless some intriguing results, such as the well-known Hawking radiation [1] and the closely related Unruh effect [2]. Moreover, in the recent literature, a number of papers studying the influence of a gravitational field on the energy stored in a Casimir cavity appeared [3, 4, 5, 6], in particular nowadays the theoretical prediction that the vacuum fluctuations follow the equivalence principle seems to be demonstrated [4, 5, 7, 8].

The main result is that there seems to be full agreement on the fact that Casimir energy gravitates, i.e. a Casimir cavity storing an energy $E_0$ experiences a force of magnitude $F = \frac{\kappa}{c^2} |E_0|$, $g$ being the gravity acceleration. The present work is the natural development of our previous article [6], where we computed the vacuum expectation value of the renormalized energy-momentum tensor of a massless scalar field in a Casimir cavity. The scalar field was there assumed to satisfy Dirichlet boundary conditions on the parallel plates constituting the cavity. The analogy with the electromagnetic case [9], where the components of the potential satisfy a mixture of Dirichlet and Neumann conditions, motivated the present analysis. Here we show that the Neumann boundary conditions yield equivalent results. Interestingly, combining these results with those obtained in [6], i.e. considering a two-component field satisfying mixed boundary condition, the electromagnetic case [5, 9] is exactly reproduced.

As expected, the conformal and minimal coupling of the scalar field with gravity yield different results. We find that the known divergences on the boundaries [10] (the plates of the apparatus) lead to finite physical quantities only in the conformal coupling case. Quite different is the mixed boundary conditions case considered in the second part of Sec. III, where we find that the energy stored and the pressures are independent of the coupling constant $\xi$, to first order in the gravity acceleration.

II. ENERGY-MOMENTUM TENSOR

Since we rely heavily on our work in Ref. [6], we refer the reader to it for all technical details. It is enough to say that, starting from the basic formalism for scalar fields in curved spacetime [11, 12], we use the covariant geodesic point separation method of Ref. [13] to expand the Green functions to first order in the parameter $\epsilon \equiv \frac{2g}{a^2}$, $a$ being the distance between the plates. By virtue of translation invariance, one can perform a Fourier analysis of the Green functions, with the associated reduced Green functions, which obey Neumann boundary conditions on parallel plates, i.e.

$$\left. \frac{\partial \gamma^{(i)}}{\partial z} \right|_{z=0} = \left. \frac{\partial \gamma^{(i)}}{\partial z} \right|_{z=a} = 0, \quad i = 0, 1. \quad (2.1)$$

We therefore obtain, to zeroth order in $\epsilon$,

$$\gamma^{(0)}(z, z') = -\frac{\cos(\lambda z_0) \cos(\lambda(a - z_0))}{\lambda \sin(\lambda a)}, \quad (2.2)$$
we can evaluate the energy-momentum tensor up to first order in $\epsilon$,

$$
\gamma^{(1)}(z, z') = \frac{1}{4a\lambda^2} \left\{ \left[ k_0^2 - \lambda^2 \right](z + z') - k_0^2 \left( \frac{z}{\lambda^2} + z' \frac{\partial}{\partial z} \right) \right. \\
+ \left( \frac{k_0^2}{\lambda^2} - 1 \right) \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial z'} \right) \right. \gamma^{(0)}(z, z') \\
- a^2 k_0^2 \cos(\lambda z) \cos(\lambda z') \right\}.
$$

(2.3)

Renormalization of the energy-momentum tensor in curved spacetime is often carried out by subtracting the stress tensor constructed by the Schwinger–DeWitt method [13][14]. Here we follow the classical scheme of renormalization of the Casimir effect in flat space, i.e. we subtract, from the energy-momentum tensor of the system, the one associated to a field propagating in free space (without boundaries). Therefore we need also the free Feynman Green functions. After having obtained the full Feynman Green functions, the Hadamard functions are twice the imaginary part of the Feynman, and we can evaluate the energy-momentum tensor up to first order of the expansion

$$
\langle T_{\mu\nu} \rangle \sim \langle T_{\mu\nu}^{(0)} \rangle + \epsilon \langle T_{\mu\nu}^{(1)} \rangle + O(\epsilon^2).
$$

(2.4)

On defining the new variables $s \equiv \frac{\pi}{a\lambda}$ and $s' \equiv \frac{\pi}{a\lambda'}$, the renormalized energy-momentum tensor, to zeroth order in $\epsilon$, is

$$
\langle T_{\mu\nu}^{(0)} \rangle = -\frac{\pi^2}{1440a^4} \text{diag}(1, -1, -1, 3) \\
- \frac{\xi - 1}{6} \frac{2 + \cos 2s}{8a^4 \sin^4 s} \text{diag}(1, -1, -1, 0),
$$

(2.5)

while, to first order, the only nonvanishing components are found to be

$$
\langle T_{00}^{(1)} \rangle = \frac{\pi \csc^2 s}{14400a^4} \left\{ 77\pi + 141s - 20[7 + 2(\pi - s)s]\cot s \\
+ 30[-4\pi + 3s + 4(\pi - s)s \cot s] \csc^2 s + \cos 2s \\
\times (3\pi - s + 150s \csc^2 s) \right\} + \frac{\xi - 1}{6} \frac{\pi \csc^2 s}{48a^4} \\
\times \left\{ 4\pi - 2s - \cot s [1 + 2(\pi - s)s + 6s \cot s] \\
+ 6[-\pi + (\pi - s)s \cot s] \csc^2 s \right\},
$$

(2.6)

$$
\langle T_{11}^{(1)} \rangle = \frac{\pi \csc^2 s}{115200a^4} \left\{ 20[-7 + 22(\pi - s)s] \cos s + 20 \\
\times [7 + 2(\pi - s)s] \cos 3s + (\pi - 2s)(-230 \sin s \\
- 85 \sin 3s + \sin 5s) - \frac{\pi \csc^5 s}{96a^4} \\
\times \left\{ [1 + 11(\pi - s)s] \cos s + [-1 + (\pi - s)s] \\
\times \cos 3s - 2(\pi - 2s)(3 \sin s + \sin 3s) \right\},
$$

(2.7)

$$
\langle T_{22}^{(1)} \rangle = \langle T_{11}^{(1)} \rangle,
$$

(2.8)

$$
\langle T_{33}^{(1)} \rangle = -\frac{\pi^2}{1440a^4} + \frac{\pi s}{720a^4} - \left( \xi - \frac{1}{6} \right) \frac{\pi \cos s}{16a^4 \sin^3 s}.
$$

(2.9)

The consistency of this result is ensured by the following tests. First of all, the computed tensor is found to be covariantly conserved up to first order in the $\epsilon$ parameter, i.e. it satisfies the equation $\nabla_{\mu} \langle T_{\mu\nu} \rangle = 0$. On the other hand we know that, for a conformal scalar field, the following relation between the trace of the tensor and the mass of the field holds: $T_{\mu}^{\mu} = -2m^2 \phi^2$. Hence we expect a vanishing trace when $\xi = \frac{1}{6}$, our scalar field being massless. This is exactly what we have found, because, upon defining $\tau_{\xi} \equiv g_{\mu\nu} \langle T_{\mu\nu} \rangle$, we have

$$
\tau_{\xi} = \left( \xi - \frac{1}{6} \right) \frac{\pi \csc^5 s}{32a^4} \left\{ 6\pi(3 \sin s + \sin 3s) \\
- \epsilon [1 + 11(\pi - s)s] \cos s - (1 - (\pi - s)s) \cos 3s \\
- 2(\pi - 2s)(3 \sin s + \sin 3s) \right\},
$$

(2.10)

that clearly vanishes in the case of conformal coupling.

III. CASIMIR ENERGY AND PRESSURE

The energy density $\rho$ stored in our Casimir apparatus can be obtained by projecting the renormalized energy-momentum tensor along a unit timelike vector with components $u^\mu = \left( -\frac{1}{\sqrt{-g_{00}}}, 0, 0, 0 \right)$, so that

$$
\tau_{\xi} = \left( \xi - \frac{1}{6} \right) \frac{\pi \csc^5 s}{32a^4} \left\{ 6\pi(3 \sin s + \sin 3s) \\
- \epsilon [1 + 11(\pi - s)s] \cos s - (1 - (\pi - s)s) \cos 3s \\
- 2(\pi - 2s)(3 \sin s + \sin 3s) \right\},
$$

(2.10)
\[
\rho = \langle T_{\mu\nu} \rangle u^\mu u^\nu
\]
\[
= -\frac{\pi^2}{1440\alpha^4} + \frac{\pi \csc^2 s}{14400\alpha^3} \left\{ 77\pi + 146s - 20(7 + 2(\pi - s)) \cot s + 30 \left[ -4\pi + 3s + 4(\pi - s) \cot s \right] \csc^2 s + 3 \cos 2s(\pi - 2s + 50s \csc^2 s) \right. \\
+ \left. \left( \xi - \frac{1}{6} \right) \left\{ -\frac{\pi^2(2 + \cos 2s)}{8\alpha^4 \sin^4 s} + \frac{\pi \csc^5 s}{192\alpha^4} \left[ -1 + 22(\pi - s) \cos s + (1 + 2(\pi - s)) \cos 3s - 4(\pi - 2s)(3 \sin s + \sin 3s) \right] \right\} \right\}
\] (3.1)

Therefore the energy stored, following Ref. [15], is
\[
E = \frac{A a}{\pi} \lim_{\zeta \to \infty} \int_\zeta^{\infty} ds \sqrt{-g} \rho, 
\]
where \(A\) is the area of the plates. This yields
\[
E_\zeta = -\frac{\pi^2 A}{1440\alpha^3} - \frac{\pi^2 A \epsilon}{5760\alpha^3} - \left( \xi - \frac{1}{6} \right) \frac{\pi A}{4\alpha^3} \\
\times \left( 1 + \frac{\epsilon}{4} \right) \lim_{\zeta \to \infty} \frac{\pi \csc \zeta}{\sin \zeta}.
\]

The conformal coupling case (\(\xi = \frac{1}{2}\)) ensures the finiteness of the above result that, reintroducing the constants \(h, c\) and the explicit expression of \(\epsilon\), reads
\[
E_c = -\frac{\pi^2 hc A}{1440 \alpha^3} \left( 1 + \frac{1}{2} \frac{g a}{c^2} \right).
\]

With analogous arguments we find the pressure on the plates
\[
P_\zeta(z = 0) = \frac{\pi^2 A}{480\alpha^4} + \frac{\pi^2 \epsilon}{1440\alpha^4} + \left( \xi - \frac{1}{6} \right) \frac{\pi \epsilon}{16\alpha^4} \lim_{s \to 0} \frac{\cos s}{\sin^3 s},
\]
and
\[
P_\zeta(z = a) = -\frac{\pi^2 A}{480\alpha^4} + \frac{\pi^2 \epsilon}{1440\alpha^4} - \left( \xi - \frac{1}{6} \right) \frac{\pi \epsilon}{16\alpha^4} \lim_{s \to \pi} \frac{\cos s}{\sin^3 s}.
\]

Once again the divergent terms vanish when \(\xi = \frac{1}{6}\), giving
\[
P_c(z = 0) = -\frac{\pi^2 hc}{480 \alpha^4} \left( 1 + \frac{2}{3} \frac{g a}{c^2} \right),
\]
and
\[
P_c(z = a) = -\frac{\pi^2 hc}{480 \alpha^4} \left( 1 - \frac{2}{3} \frac{g a}{c^2} \right).
\]

The force acting on the system has to be calculated by considering the redshift \(r\) of the point \(\tilde{z}\) where the pressures act, relative to the point \(z_s\) where they are added, i.e. [15]
\[
r(\tilde{z}, z_s) = \sqrt{\frac{|g_{00}(\tilde{z})|}{|g_{00}(z_s)|}} \simeq 1 + \frac{g}{c^2}(\tilde{z} - z_s).
\]

Thus, the net force obtained has magnitude
\[
F = A[P_c(0) r(0, z_s) + P_c(a) r(a, z_s)]
\]
\[
= \frac{\pi^2 A h c}{1440 \alpha^3} = \frac{g}{c^2} |E_0|,
\]
where we have defined \(E_0 \equiv -\frac{\pi^2 h c A}{1440 \alpha^3}\). Therefore, direction (upwards along the \(z\) axis) and magnitude of this force are in full agreement with the equivalence principle.

Note that some interesting effects result from combining the formulas here obtained with those in our previous work [6], where the same problem with Dirichlet boundary conditions was considered. We start by defining the real massless two-component field \(\Phi = \begin{pmatrix} \phi_D \\ \phi_N \end{pmatrix}\), where the subscripts \(D\) and \(N\) indicate that the components satisfy homogeneous Dirichlet and Neumann boundary conditions, respectively. This is more than a toy model, because the electromagnetic Casimir effect with perfect-conductor boundary conditions on parallel plates leads exactly to such a mixture of boundary conditions (see, for example, section 4.5 of Ref. [11]).

It is easy to see that, starting from the action functional \(S = -\frac{1}{2} \int (\Phi^\dagger \phi) \phi - \xi R \Phi \phi - \frac{\epsilon}{2} \phi x \phi\), the vacuum expectation value of the renormalized energy-momentum tensor associated with this action reads
\[
\langle T_{\mu\nu} \rangle = \langle T^{(D)}_{\mu\nu} \rangle + \langle T^{(N)}_{\mu\nu} \rangle,
\]
with obvious notation.

Thus, on combining the results of the Dirichlet and Neumann cases we have
\[
\langle T^{(0)}_{\mu\nu} \rangle = -\frac{\pi^2}{720\alpha^4} \text{diag}(1, -1, -1, 3),
\]
and
\[
\langle T^{(1)}_{00} \rangle = -\frac{\pi^2}{1200\alpha^4} \left( 1 - \frac{s}{3\pi} \right) + \left( \xi - \frac{1}{5} \right) \frac{\pi}{12\alpha^4} \frac{\cos s}{\sin^3 s},
\]
and
\[
\langle T^{(1)}_{11} \rangle = \frac{\pi^2}{3600\alpha^4} \left( 1 - \frac{2s}{\pi} \right) - \left( \xi - \frac{3}{20} \right) \frac{\pi}{12\alpha^4} \frac{\cos s}{\sin^3 s},
\]

\[ \langle T_{22}^{(1)} \rangle = \langle T_{11}^{(1)} \rangle, \quad \text{(3.15)} \]

\[ \langle T_{33}^{(1)} \rangle = -\frac{\pi^2}{720a^4} \left( 1 - \frac{2s}{\pi} \right). \quad \text{(3.16)} \]

Clearly, the covariant conservation holds, being satisfied separately for \( \langle T_{\mu\nu}^{D} \rangle \) and \( \langle T_{\mu\nu}^{N} \rangle \). The trace, defined in (2.10), is found to be

\[ \tau_t = -\left( \xi - \frac{1}{6} \right) \frac{\pi \epsilon \cos s}{a^4 \sin^2 s}, \quad \text{(3.17)} \]

that once again is vanishing for \( \xi = \frac{1}{6} \).

The Casimir energy is twice the value in (3.4), and the pressures on the plates are twice the values in (3.7) and (3.8). Interestingly, in the mixed case, energy and pressure are both finite quantities for any value of \( \xi \), to first order in \( \epsilon \). Moreover, these results coincide perfectly with those found in the electromagnetic case \( [9] \).

IV. CONCLUDING REMARKS

We have evaluated the vacuum expectation value of the renormalized energy-momentum tensor of a massless scalar field, satisfying Neumann boundary conditions on the parallel plates of a Casimir cavity immered in a weak gravitational field. The calculations have been performed up to first order of the expansion in the parameter \( \epsilon \equiv \frac{2a}{\pi} \).

In agreement with the results found in \( [9] \) and with our previous work \( [6] \), we have found a small correction to the Casimir energy not affected by the gravitational field, and the theoretical prediction that the whole cavity experiences a force proportional to the energy stored and with magnitude \( F = \frac{\Delta E_0}{a^2} \) (up to first order in \( a \)), pointing in the upwards direction. This result is in accordance with those found in \( [4, 7, 8] \) and seems to imply that Casimir energy gravitates, i.e. the vacuum energy stored in a Casimir apparatus behaves, on theoretical ground, like a negative mass in a gravitational field.

The finiteness of the physical quantities was ensured by setting the coupling constant \( \xi \) to \( \frac{1}{6} \) i.e. only for conformal coupling between the scalar field and gravity, at least up to the order of our approximation; further investigations are needed to go to higher orders.

A quite different situation appeared on considering a two-component field satisfying mixed boundary conditions. In this case, even though divergent boundary terms affect the energy-momentum tensor, this yields finite energy and pressures, in complete accordance with those found in \( [9] \). Other valuable related work can be found in Refs. \( [17, 18, 19, 20] \), which focus on quantum field theory in Rindler spacetime. This point of view has been later exploited in Refs. \( [14, 15] \), whose first-order results agree with ours as we said before.

At this stage, further studies are required at least in two directions. First, we might try to use the technique here shown for the calculation of \( \langle T_{\mu\nu} \rangle \) to the second (and even higher) order of the expansion in \( \epsilon \), to further check the agreement with the analysis of Ref. \( [5] \), which relies instead on the uniform asymptotics of Bessel functions; moreover, the agreement of results obtained from different approaches persuades us to extend this kind of analysis to Casimir devices in other configurations \( [21] \).

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