WEIGHTED COMPOSITION OPERATORS ON WEAK HOLOMORPHIC SPACES AND APPLICATION TO WEAK BLOCH-TYPE SPACES ON THE UNIT BALL OF A HILBERT SPACE

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Abstract. Let $E$ be a space of holomorphic functions on the unit ball $B_X$ of a Banach space $X$. In this work, we introduce a Banach structure associated to $E$ on the linear space $W(E)$ containing $Y$-valued holomorphic functions on $B_X$ such that $w \circ f \in E$ for every $w \in W$, a separating subspace of the dual $Y'$ of a Banach $Y$. We establish the relation between the boundedness, the (weak) compactness of the weighted composition operators $W_{\psi,\varphi} : f \mapsto \psi \cdot (f \circ \varphi)$ on $E$ and $\tilde{W}_{\psi,\varphi} : g \mapsto \psi \cdot (g \circ \varphi)$ on $W(E)$ via some characterizations of the separating subspace $W$. As an application, via the estimates for the restrictions of $\psi$ and $\varphi$ to a $m$-dimensional subspace of $X$ for some $m \geq 2$, we characterize the properties mentioned above of $W_{\psi,\varphi}$ on Bloch-type spaces $B_\mu(B_X)$ of holomorphic functions on the unit ball $B_X$ of an infinite-dimensional Hilbert space as well as their the associated spaces $WB_\mu(B_X,Y)$, where $\mu$ is a normal weight on $B_X$.

1. Introduction

Let $X,Y$ be complex Banach spaces and $W \subset Y'$ be a separating subspace of the dual $Y'$ of $Y$. We consider Banach spaces $E_1, E_2$ of holomorphic functions on the unit ball $B_X$ of $X$ and $W$-associated spaces $WE_1(Y), WE_2(Y)$ of $Y$-valued holomorphic functions on $B_X$ in the following sense:

$$WE_i(Y) := \{ f : B_X \to Y : f \text{ is locally bounded and } w \circ f \in E_i, \forall w \in W \}$$

equipped with the norm

$$\|f\|_{WE_i(Y)} := \sup_{w \in W, ||w|| \leq 1} \|w \circ f\|_{E_i}, \quad i = 1, 2.$$ 

For a holomorphic self-map $\varphi$ of $B_X$ and a holomorphic function $\psi$ on $B_X$, the weighted composition operators $W_{\psi,\varphi} : E_1 \to E_2$ and $\tilde{W}_{\psi,\varphi} : WE_1(Y) \to WE_2(Y)$ are defined by

$$W_{\psi,\varphi}(f) := \psi \cdot (f \circ \varphi) \quad \forall f \in E,$$

$$\tilde{W}_{\psi,\varphi}(g) := \psi \cdot (g \circ \varphi) \quad \forall g \in WE_1(Y).$$

When the function $\psi$ is identically 1, the operators reduce to the composition operators $C_\varphi$ and $\tilde{C}_\varphi$. A main problem in the investigation of such operators is to relate function theoretic properties of $\psi$ and $\varphi$ to operator theoretic properties of $W_{\psi,\varphi}$ and $\tilde{W}_{\psi,\varphi}$.

The problem of studying of weighted composition operators on various Banach spaces of holomorphic functions on the unit disk or the unit ball (in finite and infinite dimensional
spaces), such as Hardy and Bergman spaces, the space $H^\infty$ of all bounded holomorphic functions, the disk algebra and weighted Banach spaces with sup-norm, etc. received a special attention of many authors during the past several decades. They appeared in some works with different applications. There is a great number of topics on operators of such a type: boundedness and compactness, compact differences, topological structure, dynamical and ergodic properties.

The study of weighted composition operators between vector-valued function spaces involves some important basic principles which hold for large classes of function spaces. In this paper we are interested in the dependence of the boundedness and the (weak) compactness of $\tilde{W}_{\psi, \varphi}$ on the respective properties of $W_{\psi, \varphi}$ as well as their restrictions of $W_{\psi, \varphi}$ to some finite-dimensional ones. A positive answer to this problem is very important because in certain situations, the study of the above properties of $\tilde{W}_{\psi, \varphi}$ can be reduced to studying $W_{\psi, \varphi}$, or even, its restriction to the unit ball $B_m \subset \mathbb{C}^m$ for some $m \in \mathbb{N}$.

First, in Section 2, we introduce some fundamental properties of the Banach space of Banach-valued holomorphic functions $W$-associated to $E$. The important result in this section is the linearization theorem for spaces $WE(Y)$ (Theorem 2.2) that allows us to identify $WE(Y)$ with the space of $Y$-valued continuous linear operators on the predual space $E'$ of $E$. This is useful in our investigation on the boundedness and the (weak) compactness of operators $\tilde{W}_{\psi, \varphi}$ in the next section.

The main theorem in Section 3 is the answer of the problem posed above. We introduce some assumptions on the subspace $W$ of the dual $Y'$ of $Y$ to give necessary as well as sufficient conditions for the boundedness, the (weak) compactness of $\tilde{W}_{\psi, \varphi}$ via the respective properties of $W_{\psi, \varphi}$ (Theorem 3.1).

The remaining part of the paper contains some applications of the above results to the Bloch-type space of Banach-valued holomorphic functions on the unit ball of an infinite-dimensional Hilbert space.

The notion of classical Bloch space $B$ of holomorphic functions on the unit disk $B_1$ of $\mathbb{C}$ was extended by R. M. Timoney \cite{Timoney1, Timoney2} by considering bounded homogeneous domains in $\mathbb{C}^n$, for example, the unit ball $B_n$ and the polydisk $D_n$. Recently, O. Blasco, P. Galindo, A. Miralles \cite{BGM} introduce and investigate Bloch functions on the unit ball $B_X$ of an infinite-dimensional Hilbert space. It is well known that several results and characterizations of Bloch space on $B_n$ can be extended and still hold in this infinite dimensional setting.

Modifying the well-known definitions and results in \cite{BGM}, in Section 4 we introduce Bloch-type space $B_\mu(B_X, Y)$ of $Y$-valued holomorphic functions on the unit ball $B_X$ of an infinite-dimensional Hilbert space where $\mu$ is a radial, normal weight on $B_X$. It is shown that if the restrictions of a function on $B_X$ to finite-dimensional subspaces have their Bloch norms uniformly bounded then this function belongs to $B(B_X)$ and conversely. Motivated by this fact, we introduce the notions of gradient norm, radial derivate norm, affine norm on $B_\mu(B_X, Y)$ and prove the equivalence between them. Another equivalent norm for $B(B_X, Y)$ (in the case $\mu(z) = 1 - ||z||^2$) which is invariant-modulo the constant functions under the action of the automorphism of the $B_X$ is also presented in this section.

In Section 5, via the estimates for the restrictions of the functions $\psi$ and $\varphi$ to $B_m$ for some $m \geq 2$, we characterize the boundedness and the compactness of the operators $W_{\psi, \varphi}$ between the (little) Bloch-type spaces $B_\mu(B_X)$, $B_{\mu, 0}(B_X)$ as well as the equivalent relationships between them (Theorems 6.1 and 6.3). It should be noted that a necessary condition (but not sufficient) and a sufficient condition (but not necessary) for the compactness of $W_{\psi, \varphi}$ are also obtained after any necessary minor modifications for the
holomorphic self-map \( \varphi \) (Remark 6.1 and Theorem 6.4). Finally, we finish the paper with the presentation the necessary as well as sufficient conditions for the boundedness and the (weak) compactness of \( \overline{W}_{0,\varphi} \) which are immediate consequences of Theorem 3.1.

Throughout this paper, we use the notions \( X \lesssim Y \) and \( X \asymp Y \) for non negative quantities \( X \) and \( Y \) to mean \( X \leq CY \) and, respectively, \( Y/C \leq X \leq CY \) for some inessential constant \( C > 0 \).

2. Weak holomorphic spaces and Linearization Theorem

Let \( X, Y \) be complex Banach spaces. Denote by \( B_X \) the closed unit ball of \( X \) (we write \( B_n \) instead of \( B_{C^n} \)).

By \( H(B_X, Y) \) we denote the vector space of \( Y \)-valued holomorphic functions on \( B_X \). A holomorphic function \( f \in H(B_X, Y) \) is called locally bounded holomorphic on \( B_X \) if for every \( z \in B_X \) there exists a neighbourhood \( U_z \) of \( 0 \in X \) such that \( f(U_z) \) is bounded. Put

\[
H_{LB}(B_X, Y) = \{ f \in H(B_X, Y) : f \text{ is locally bounded on } B_X \}.
\]

Suppose that \( E \) is a Banach space of holomorphic functions \( B_X \to \mathbb{C} \) such that

(\text{e1}) \( E \) contains the constant functions,

(\text{e2}) the closed unit ball \( B_E \) is compact in the compact open topology \( \tau_{co} \) of \( B_X \).

It is easy to check that the properties (\text{e1}), (\text{e2}) are satisfied by a large number of well-known function spaces, such as classical Hardy, Bergman, BMOA, and Bloch spaces.

Let \( W \subset Y' \) be a separating subspace of the dual \( Y' \) of \( Y \). We say that the space \( \mathcal{W}E(Y) := \{ f : B_X \to Y : f \text{ is locally bounded and } w \circ f \in E, \forall w \in W \} \)
equipped with the norm

\[
\|f\|_{\mathcal{W}E(Y)} := \sup_{w \in W, \|w\| \leq 1} \|w \circ f\|_E.
\]
is the Banach space \( \mathcal{W} \)-associated to \( E \) of \( Y \)-valued functions.

**Proposition 2.1.** Let \( X, Y \) be complex Banach spaces and \( W \subset Y' \) be a separating subspace. Let \( E \) be a Banach space of holomorphic functions \( B_X \to \mathbb{C} \) satisfying (\text{e1})-(\text{e2}) and \( \mathcal{W}E(Y) \) be the Banach space \( \mathcal{W} \)-associated to \( E \). Then, the following assertions hold:

(\text{we1}) \( f \mapsto f \otimes y \) defines a bounded linear operator \( P_y : E \to \mathcal{W}E(Y) \) for any \( y \in Y \), where \( (f \otimes y)(z) = f(z)y \) for \( z \in B_X \),

(\text{we2}) \( g \mapsto w \circ g \) defines a bounded linear operator \( Q_w : \mathcal{W}E(Y) \to E \) for any \( w \in W \),

(\text{we3}) For all \( z \in B_X \) the point evaluations \( \tilde{\delta}_z : \mathcal{W}E(Y) \to \langle Y, \sigma(Y, W) \rangle \), where \( \tilde{\delta}_z(g) = g(z) \), are continuous.

In the case the hypothesis “separating” of \( W \) is replaced by a stronger one that \( W \) is “almost norming”, we obtain the assertion (\text{we3}') below instead of (\text{we3}):

(\text{we3}') For all \( z \in B_X \) the point evaluations \( \delta_z : \mathcal{W}E(Y) \to Y \) are bounded.

Here, the subspace \( W \) of \( Y' \) is called almost norming if

\[
g_W(x) := \sup_{w \in W, \|w\| \leq 1} |w(x)|
\]
defines an equivalent norm on \( Y \).
Proof. (i) Fix $y \in Y$. In fact, for every $f \in E$ we have $w \circ (f \otimes y) = w(y)f$. Then
\[
\|P_f(y)\|_{W(E)} = \sup_{\|w\| \leq 1} \|w \circ (f \otimes y)\|_E = \sup_{\|w\| \leq 1} \|w(y)f\|_E \\
\leq \|w\| \cdot \|y\| \cdot \|f\|_E \leq \|y\| \cdot \|f\|_E.
\]
Thus (we1) holds.

(ii) Fix $w \in W$; for every $g \in W(E)$ we have
\[
\|Q_w(g)\|_E = \|w \circ g\|_E = \|w\| \left\| \frac{w}{\|w\|} \circ g \right\|_E \\
\leq \|w\| \sup_{\|u\| \leq 1} \|u \circ g\|_E = \|w\| \cdot \|g\|_{W(E)}. \\
\]
Thus (we2) is true.

(iii) Fix $z \in B_X$. Note first that since $E$ satisfies (e1) and (e2), then the evaluation maps $\delta_z : E' \to \mathbb{C}$ for $z \in B_X$ where $\delta_z(f) = f(z)$ for $f \in E$.

It is obvious that $w(\delta_z(g)) = \delta_z(w \circ g)$ for every $g \in W(E)$ and for every $w \in W$. Let $V$ be a $\sigma(Y,W)$-neighbourhood of 0 in $Y$. Without loss of generality we may assume $V = \{y \in Y : \|w(y)\| < 1\}$ for some $w \in W$. Then $\delta_z(\|\delta_z\|^{-1}\|w\|^{-1}B_{W(E)}(Y)) \subset V$, where $B_{W(E)}(Y) = \{g \in W(E) : \|g\|_{W(E)} < 1\}$ is the unit ball of $W(E)$. Indeed, for every $g \in B_{W(E)}(Y)$ we have
\[
|w(\delta_z(\|\delta_z\|^{-1}\|w\|^{-1}g))| = |\|\delta_z\|^{-1}\|\delta_z\|^{-1}w \circ g| \\
\leq \|\delta_z\|^{-1}\|\delta_z\|^{-1}w \circ g| \\
\leq \sup_{u \in W, \|u\| \leq 1} \|u \circ g\|_E = \|g\|_{W(E)} < 1.
\]
Thus, (we3) holds for $(E, W(E))$.

In the case where $W$ is almost norming, since $q_w$ defines an equivalent norm, there exists $C > 0$ such that
\[
\|\delta_z(g)\| = \|g(z)\| \leq C q_w(g(z)) = C \sup_{w \in W, \|w\| \leq 1} |w(g(z))| \\
\leq C \sup_{w \in W, \|w\| \leq 1} \|w \circ g\| = C \|g\|_{W(E)} \quad \forall g \in W(E).
\]
The assertion (we3') is proved. \hfill \qed

Finally, we discuss the linearization theorem for spaces $W(E)$ which is useful in our investigation on the boundedness and the (weak) compactness of operators $\tilde{W}_{\varphi,\varphi}$ in the next section.

**Theorem 2.2** (Linearization). Let $X, Y$ be complex Banach spaces and $W \subset Y'$ be a separating subspace. Let $E$ be a Banach space of holomorphic functions $B_X \to \mathbb{C}$ satisfying (e1)-(e2). Then there exist a Banach space $^*E$ and a mapping $\delta_X \in H(B_X, ^*E)$ with the following universal property: A function $f \in W(E)$ if and only if there is a unique mapping $T_f \in L(^*E, Y)$ such that $T_f \circ \delta_X = f$. This property characterize $^*E$ uniquely up to an isometric isomorphism.

Moreover, the mapping
\[
\Phi: f \in W(E) \mapsto T_f \in L(^*E, Y)
\]

is a topological isomorphism.
Proof. Let us denote by \( \ast E \) the closed subspace of all linear functionals \( u \in E' \) such that \( u|_{B_E} \) is \( \tau_{co} \)-continuous. By the Ng Theorem \([Ng \, Theorem \, 1]\) the evaluation mapping 

\[ J : E \to (\ast E)' \]

given by 

\[ (Jf)(u) = u(f) \quad \forall u \in \ast E, \]

is a topological isomorphism.

Let \( \delta_X : B_X \to \ast E \) be the evaluation mapping given by

\[ \delta_X(x) = \delta_x \]

with \( \delta_x(g) := g(x) \) for all \( g \in E \).

Since

\[ (Jg) \circ \delta_X(x) = \delta_x(g) = g(x) \]

for all \( g \in E, x \in B_X, \) and \( J \) is surjective, we can check that the map \( \delta_X : B_X \to \ast E \) is weak holomorphic, and hence holomorphic because \( E \) is Banach.

Next we prove that

\[ \delta_X(x) = \delta_x \]

for all \( x \in B_X \), and hence \( \delta \) is separating, according to \([QLD, \, Lemma \, 4.2]\) we have \( f \in H_{LB}(B_X, Y) \). Next, it follows from \( (\ast E)' = E \) that \( u \circ f \in E \) for each \( u \in W \), and then \( f \in WE(Y) \).

Now we will prove the converse of the statement. Fix \( f \in WE(Y) \).

(i) The case of \( Y = C \): We define \( T_f : \ast E \to W' \) by

\[ T_f u(\varphi) = T_{\varphi \circ f}(u) = u(\varphi \circ f) \quad \forall u \in \ast E, \forall \varphi \in W, \]

i.e. \( T_{\varphi \circ f} \) is as defined in the case (i).

It is easy to check that \( T_f \in L(\ast E, W') \) and \( \|T_f\| = \|f\|_{WE(Y)} \), hence, \( \Phi \) is an isometric isomorphism.

Furthermore,

\[ (T_f \delta_x)(\varphi) = (\varphi \circ f)(x) \]

for every \( x \in B_X \) and \( \varphi \in W \) and, therefore, since \( W \) is separating we get \( T_f \delta_x = f(x) \in Y \) for every \( x \in B_X \). Then, by \([QLD, \, Lemma \, 4.2]\) \( T_f \in L(\ast E, Y) \). The uniqueness of \( T_f \) follows also from the fact \([QLD, \, Lemma \, 4.2]\) that \( \delta_X(B_X) \) generates a dense subspace of \( \ast E \).

Finally, the uniqueness of \( \ast E \) up to an isometric isomorphism follows from the universal property, together with the isometry \( \|T_f\| = \|f\|_{WE(Y)} \). This completes the proof. \( \square \)
3. The weighted composition operators

Let $E_1$ and $E_2$ be Banach spaces of holomorphic functions $B_X \to \mathbb{C}$ satisfying the conditions (e1) and (e2). Let $\psi \in H(B_X)$ and $\varphi \in S(B_X)$, the set of holomorphic self-maps of $B_X$. Consider the operators $W_{\psi,\varphi} : E_1 \to E_2$ and $\tilde{W}_{\psi,\varphi} : WE_1(Y) \to WE_2(Y)$ given by

$$W_{\psi,\varphi}(f) := \psi \cdot (f \circ \varphi), \quad \tilde{W}_{\psi,\varphi}(g) := \psi \cdot (g \circ \varphi) \quad \forall f \in E_1, \forall g \in WE_1(Y).$$

**Theorem 3.1.** Let $X, Y$ be complex Banach spaces and $W \subset Y'$ be a subspace. Let $E_1$ and $E_2$ be Banach spaces of holomorphic functions $B_X \to \mathbb{C}$ satisfying (e1)-(e2). Let $\psi \in H(B_X)$ and $\varphi \in S(B_X)$.

1. If $W$ is separating then $W_{\psi,\varphi}$ is bounded if and only if $\tilde{W}_{\psi,\varphi}$ is bounded;
2. If $W$ is almost norming and $W_{\psi,\varphi}$ compact then:
   a. $\tilde{W}_{\psi,\varphi}$ is compact if and only if the identity map $I_Y : Y \to Y$ is compact, i.e. $\dim Y < \infty$;
   b. $\tilde{W}_{\psi,\varphi}$ is weakly compact if and only if the identity map $I_Y : Y \to Y$ is weakly compact.

**Proof.** (1) Let $y \in Y$, $w \in W$ such that $\|y\| = \|w\| = 1$ and $w(y) = 1$. Consider the maps $P_y$ and $Q_w$ as in Proposition 2.1. It is easy to check that

$$W_{\psi,\varphi} = Q_w \circ \tilde{W}_{\psi,\varphi} \circ P_y.$$  

By Proposition 2.1 the operators $P_y, Q_w$ are bounded, hence we have $W_{\psi,\varphi}$ is bounded if $\tilde{W}_{\psi,\varphi}$ is bounded.

In the converse direction, note that for every $g \in WE_1(Y)$ and $w \in W$ we have

$$\|w \circ (\tilde{W}_{\psi,\varphi}(g))\|_{E_2} = \|W_{\psi,\varphi}(w \circ g)\|_{E_2} \leq \|W_{\psi,\varphi}\| \cdot \|w \circ g\|_{E_1}.$$ 

Consequently, $\|\tilde{W}_{\psi,\varphi}\| \leq \|W_{\psi,\varphi}\|$.

(2) Assume that $W$ is almost norming and $W_{\psi,\varphi}$ is compact. Let $z_0 \in B_X$ such that $\psi(z_0) \neq 0$. Put

$$R : Y \to WE_1(Y) \quad \text{and} \quad S : WE_2(Y) \to Y$$

$$y \mapsto f_y, \quad h \mapsto \psi(z_0)^{-1}h(z_0),$$

where $f_y(z) := 1 \odot y \equiv y$ for every $z \in B_X$. Here, the function $1 \in E_1$ by the condition (e1). It is easy to see that

$$I_Y = S \circ \tilde{W}_{\psi,\varphi} \circ R.$$  

Note that $S = \psi(z_0)^{-1}\delta_{z_0}$. It follows from (we1) and (we3') that $R$ and $S$ are bounded operators.

Therefore, by Proposition 2.2 we have $(W_{\psi,\varphi})^*(E_2) \subset E_1$. Put $T_{\psi,\varphi} : L(E_1, Y) \to L(E_2, Y)$ given by

$$A \mapsto I_Y \circ A \circ (W_{\psi,\varphi})^*|_{E_2}.$$ 

By the compactness of $W_{\psi,\varphi}$ we have $(W_{\psi,\varphi})^*|_{E_2} : E_2 \to E_1$ is compact. Next, we will show that

$$\tilde{W}_{\psi,\varphi} = \Phi_2 \circ T_{\psi,\varphi} \circ \Phi_1^{-1}.$$
where $\Phi_1 : L(^*E_1, Y) \to WE_1(Y)$ and $\Phi_2 : L(^*E_2, Y) \to WE_2(Y)$ are isometric isomorphisms from Theorem 2.2. Indeed, we have

$$((\Phi_2 \circ T_{\psi,\varphi} \circ \Phi_1^{-1})(f))(z) = \left(\Phi_2(\Phi_1^{-1}(f) \circ (W_{\psi,\varphi})^*|_{E_1})\right)(z) = \Phi_1^{-1}(f)((W_{\psi,\varphi})^*(\delta_z))$$

$$= \Phi_1^{-1}(f)(\psi(z)\delta_{\varphi(z)}) = \psi(z)\Phi_1^{-1}(f)(\delta_{\varphi(z)} = \psi(z)(f \circ \varphi)(z)$$

for every $f \in E_1$ and $z \in B_X$.

(a) First assume that $\tilde{W}_{\psi,\varphi}$ is compact. By the boundedness of the operators $R, S$, 3.1 implies the compactness of $I_Y$.

Now, suppose $I_Y$ is compact. By the compactness of $(W_{\psi,\varphi})^*|_{E_1}$, it follows from [ST, Theorem 2.1 and Remark 2.4] that $T_{\psi,\varphi}$ is compact. Hence, by (3.2), $\tilde{W}_{\psi,\varphi}$ is compact.

(b) We assume that $\tilde{W}_{\psi,\varphi}$ is weakly compact. Then, (3.1) implies that $I_Y$ is weakly compact.

In the contrary case, suppose $I_Y$ is weakly compact. By an argument analogous to the case (a) but using [ST, Proposition 2.3 and Remark 2.4] instead of [ST, Theorem 2.1 and Remark 2.4] we get $\tilde{W}_{\psi,\varphi}$ is weakly compact.

Remark 3.1. In the case $W$ is separating and $Y$ is separable, we have $W$ is almost norming and the identity operator $I_Y$ is weakly compact. Then as in (b) we get $\tilde{W}_{\psi,\varphi}$ is weakly compact if $W_{\psi,\varphi}$ is compact.

4. The Bloch-type spaces on the unit ball of a Hilbert space

Throughout the forthcoming, unless otherwise specified, we shall denote by $X$ a complex Hilbert space with the open unit ball $B_X$ and $Y$ a Banach space. Denote

$$H^\infty(B_X, Y) = \left\{f \in H(B_X, Y) : \sup_{z \in B_X} \|f(z)\| < \infty\right\}.$$ 

It is easy to check that $H^\infty(B_X, Y)$ is Banach under the sup-norm

$$\|f\|_\infty := \sup_{z \in B_X} \|f(z)\|.$$

Let $(e_k)_{k \in \Gamma}$ be an orthonormal basis of $X$ that we fix at once. Then every $z \in X$ can be written as

$$z = \sum_{k \in \Gamma} z_k e_k, \quad \overline{z} = \sum_{k \in \Gamma} \overline{z_k} e_k.$$

Given $f \in H(B_X, Y)$ and $z \in B_X$. Then, for each $u \in Y'$, we denote by $\nabla(u \circ f)(z)$ the gradient of $u \circ f$ at $z$. It is the unique element in $X$ representing the linear functional $u \circ f'(z) = (u \circ f)'(z) \in Y'$. We can write

$$\nabla(u \circ f)(z) = \left(\frac{\partial(u \circ f)}{\partial z_k}(z)\right)_{k \in \Gamma},$$

hence, for every $x \in X$

$$(u \circ f')(z)(x) = (u \circ f)'(z)(x) = \sum_{k \in \Gamma} \frac{\partial(u \circ f)}{\partial z_k}(z)x_k = \langle x, \nabla(u \circ f)(z)\rangle.$$

Now for every $z \in B_X$ we define

$$\|\nabla f(z)\| := \sup_{u \in Y', \|u\| = 1} \|\nabla(u \circ f)(z)\|,$$

$$\|R f(z)\| := \sup_{u \in Y', \|u\| = 1} |R(u \circ f)(z)|,$$
where
\[ R(u \circ f)(z) = \langle \nabla (u \circ f)(z), \overline{z} \rangle \]
is the radial derivative of \( u \circ f \) at \( z \). It is obvious that \( \| Rf(z) \| \leq \| \nabla f(z) \| \| z \| < \| \nabla f(z) \| \)
for every \( z \in B_X \).

For \( \varphi \in S(B_X) \), the set of holomorphic self-maps on \( B_X \) we write \( \varphi(z) = \sum_{k \in \Gamma} \varphi_k(z) \) and \( \varphi'(z) : X \to X \) its derivative at \( z \), and \( R\varphi(z) = \langle \varphi'(z), \overline{z} \rangle \) its radial derivative at \( z \).

**Definition 4.1.** A positive, continuous function \( \mu \) on the interval \([0, 1]\) is called normal if there are three constants \( 0 \leq \delta < 1 \) and \( 0 < a < b < \infty \) such that
\[
(W_1) \quad \frac{\mu(t)}{(1-t)^\alpha} \text{ is decreasing on } [\delta, 1), \quad \lim_{t \to 1} \frac{\mu(t)}{(1-t)^\alpha} = 0,
\]
\[
(W_2) \quad \frac{\mu(t)}{(1-t)^\beta} \text{ is increasing on } [\delta, 1), \quad \lim_{t \to 1} \frac{\mu(t)}{(1-t)^\beta} = \infty.
\]

If we say that a function \( \mu : B_X \to [0, \infty) \) is normal, we also assume that it is radial, that is, \( \mu(z) = \mu(\|z\|) \) for every \( z \in B_X \).

Then, it follows from \((W_1)\) that a normal function \( \mu \) is strictly decreasing on \([\delta, 1)\) and \( \mu(t) \to 0 \) as \( t \to 1 \).

On the other hand, by \((W_2)\) we easily obtain that
\[
(1.1) \quad S_{\mu} := \sup_{t \in [0,1)} \frac{(1-t)^b}{\mu(t)} < \infty.
\]

Throughout this paper, the weight \( \mu \) always is assumed to be normal. In the sequel, when no confusion can arise, we will use the symbol \( \hat{\circ} \) to denote either \( \nabla \) or \( R \).

We define Bloch-type spaces on the unit ball \( B_X \) as follows:
\[
\mathcal{B}^\phi_{\mu}(B_X, Y) := \{ f \in H(B_X, Y) : \| f \|_{s\mathcal{B}^\phi_{\mu}(B_X, Y)} := \sup_{z \in B_X} \mu(z) \| \hat{\nabla} f(z) \| < \infty \}.
\]

It is easy to check \( \| \cdot \|_{s\mathcal{B}^\phi_{\mu}(B_X, Y)} \) is a semi-norm on \( \mathcal{B}^\phi_{\mu}(B_X, Y) \) and this space is Banach under the sup-norm
\[
\| f \|_{\mathcal{B}^\phi_{\mu}(B_X, Y)} := \| f(0) \| + \| f \|_{s\mathcal{B}^\phi_{\mu}(B_X, Y)}.
\]

We also define little Bloch-type spaces on the unit ball \( B_X \) as follows:
\[
\mathcal{B}^\phi_{\mu,0}(B_X, Y) := \{ f \in \mathcal{B}^\phi_{\mu}(B_X, Y) : \lim_{\| z \| \to 1} \mu(z) \| \hat{\nabla} f(z) \| = 0 \}
\]
endowed with the norm induced by \( \mathcal{B}^\phi_{\mu}(B_X, Y) \).

In the case \( Y = \mathbb{C} \) we write \( \mathcal{B}^\phi_{\mu}(B_X) \), \( \mathcal{B}^\phi_{\mu,0}(B_X) \) instead of the respective notations.

It is clear that for every separating subspace \( W \) of \( Y \) we have
\[
\mathcal{B}^\phi_{\mu}(B_X, Y) \subset WB^\phi_{\mu}(B_X) \subset WB^\phi_{\mu,0}(B_X)(Y), \quad \mathcal{B}^\phi_{\mu,0}(B_X, Y) \subset WB^\phi_{\mu,0}(B_X)(Y).
\]

For \( \mu(z) = 1 - \| z \|^2 \) we write \( \mathcal{B}^\phi(B_X, Y) \) instead of \( \mathcal{B}^\phi_{\mu}(B_X, Y) \) and when \( \dim X = m \), \( Y = \mathbb{C} \) we obtain correspondingly the classical Bloch-type space \( \mathcal{B}^\phi(\mathbb{B}_m) \).

We will show below that the study of Bloch-type spaces on the unit ball can be reduced to studying functions defined on finite dimensional subspaces.

For each \( m \in \mathbb{N} \) we denote
\[
z_{[m]} := (z_1, \ldots, z_m) \in \mathbb{B}_m
\]
where \( \mathbb{B}_m \) is the open unit ball in \( \mathbb{C}^m \). For \( m \geq 2 \) by
\[
\text{OS}_m := \{ x = (x_1, \ldots, x_m), \ x_k \in X, \langle x_k, x_j \rangle = \delta_{kj} \}
\]
we denote the family of orthonormal systems of order $m$.

It is clear that $OS_1$ is the unit sphere of $X$.

For every $x \in OS_m$, $f \in H(B_X, Y)$ we define

$$f_x(z_{[m]}) = f\left(\sum_{k=1}^{m} z_k x_k\right).$$

Then

$$\nabla(u \circ f_x)(z_{[m]}) = \left(\frac{\partial (u \circ f_x)}{\partial z_j} \left(\sum_{k=1}^{m} z_k x_k\right)\right)_{j \in \Gamma}$$

for every $u \in Y'$, and hence

$$(\text{4.2}) \quad \left\|\nabla f_x(z_{[m]})\right\| = \left\|\nabla f\left(\sum_{k=1}^{m} z_k x_k\right)\right\|.$$ 

Now, for each finite subset $F \subset \Gamma$, in symbol $|F| < \infty$, we denote by $B_{[F]}$ the unit ball of span$\{e_k, k \in F\}$ and $f_F = f_x$ where $x = \{e_k, k \in F\}$. For each $z \in B_X$ and each $F \subset \Gamma$ finite we write

$$z_F = \sum_{k \in F} z_k e_k \in B_{[F]}.$$

**Definition 4.2.** Let $B_1$ be the open unit ball in $C$ and $f \in H(B_X, Y)$. We define an affine semi-norm as follows

$$\|f\|_{sB^a_{\mu}(B_X, Y)} := \sup_{\|x\|=1} \|f(\cdot x)\|_{sB^a_{\mu}(B_1, Y)}$$

where $f(\cdot x) : B_1 \to Y$ given by $f(\cdot x)(\lambda) = f(\lambda x)$ for every $\lambda \in B_1$, and

$$\|f(\cdot x)\|_{sB^a_{\mu}(B_1, Y)} = \sup_{\lambda \in B_1} \mu(\lambda x)\|f(\cdot x)(\lambda)\|.$$

It is easy to see that $\|\cdot\|_{sB^a_{\mu}(B_X, Y)}$ is a semi-norm on $B_{\mu}(B_X, Y)$. We denote

$$B^a_{\mu}(B_X, Y) := \{f \in B_{\mu}(B_X, Y) : \|f\|_{sB^a_{\mu}(B_X, Y)} < \infty\}.$$ 

It is also easy to check that $B^a_{\mu}(B_X, Y)$ is Banach under the norm

$$\|f\|_{B^a_{\mu}(B_X, Y)} := \|f(0)\| + \|f\|_{sB^a_{\mu}(B_X, Y)}.$$ 

We also define little affine Bloch-type spaces on the unit ball $B_X$ as follows:

$$B^a_{\mu,0}(B_X, Y) := \{f \in B^a_{\mu}(B_X, Y) : \lim_{|\lambda| \to 1} \sup_{\|x\|=1} \mu(\lambda x)\|f(\cdot x)(\lambda)\| = 0\}.$$ 

As the above, for $\mu(z) = 1 - \|z\|^2$ we use notation $B$ instead of $B_{\mu}$.

**Proposition 4.1.** Let $f \in H(B_X, Y)$. The following are equivalent:

1. $f \in B^\circ_{\mu}(B_X, Y)$;
2. $\sup\{\|f_F\|_{B^\circ_{\mu}(B_{[F]}, Y)} : F \subset \Gamma, |F| < \infty\} < \infty$;
3. $\sup_{x \in OS_m} \|f_x\|_{B^\circ_{\mu}(B_{[m]}, Y)} < \infty$ for every $m \geq 2$;
4. There exists $m \geq 2$ such that $\sup_{x \in OS_m} \|f_x\|_{B^\circ_{\mu}(B_{[m]}, Y)} < \infty$.

Moreover, for each $m \geq 2$

$$(\text{4.3}) \quad \|f\|_{B^\circ_{\mu}(B_X, Y)} = \sup_{|F| < \infty} \|f_F\|_{sB^\circ_{\mu}(B_{[F]}, Y)} = \sup_{x \in OS_m} \|f_x\|_{sB^\circ_{\mu}(B_{[m]}, Y)}.$$
Thus fix $f$, then it follows from the assumption (2) and
\[ \left\| \nabla f_F(z_{[F]}) \right\| = \left\| \nabla f \left( \sum_{j \in F} z_j e_j \right) \right\|. \]

Denote $\mu^{[F]} = \mu \big|_{\text{span}\{e_k, k \in F\}}$. Since $\left\| \sum_{j \in F} z_j e_j \right\| = \left\| z_{[F]} \right\|$ we get
\[
\left\| f_F \right\|_{s_{B_Y^\infty(B_{[F]}, Y)}} = \sup_{z_{[F]} \in B_{[F]}} \mu^{[F]}(z_{[F]}) \left\| \nabla f_F(z_{[F]}) \right\|
\leq \sup_{z \in B_X} \mu^{[F]}(z_{[F]}) \left\| \nabla f \left( \sum_{j \in F} z_j e_j \right) \right\|
\leq \left\| f \right\|_{s_{B_Y^\infty(B_X, Y)}}.
\]

In particular, we obtain (2).

(2) $\Rightarrow$ (1): Let $z = \sum_{k \in \Gamma} z_k e_k$. We denote the partial sums of this series by $s_n$. Because $f$ is holomorphic, $\frac{\partial f}{\partial z_j}$ are continuous. Then
\[
\left\| \nabla f(z) \right\| = \sup_{u \in Y', \|u\| = 1} \left\| \nabla (u \circ f)(z) \right\|
= \sup_{u \in Y', \|u\| = 1} \lim_{n \to \infty} \left\| \nabla (u \circ f)(s_n) \right\|
\leq \sup_{u \in Y', \|u\| = 1} \sup_{F \subset \Gamma, |F| < \infty} \left\| \nabla (u \circ f)(z_{[F]}) \right\|
= \sup_{F \subset \Gamma, |F| < \infty} \left\| \nabla f_F(z_{[F]}) \right\|.
\]

Then, it follows from the assumption (2) and $\left\| z_{[F]} \right\| \leq \| z \|$, that
\[
\mu^{[F]}(z_{[F]}) \left\| \nabla f(z) \right\| \leq \mu^{[F]}(z_{[F]}) \left\| \nabla f(z_{[F]}) \right\|
\leq \sup_{F \subset \Gamma, |F| < \infty} \mu^{[F]}(z_{[F]}) \left\| \nabla f_F(z_{[F]}) \right\| < \infty.
\]

Thus $f \in B_{\mu}^\infty(B_X, Y)$.

(1) $\Rightarrow$ (3): It is analogous to (1) $\Rightarrow$ (2).

(3) $\Rightarrow$ (4): It is obvious.

(4) $\Rightarrow$ (1): Assume there exists $m \geq 2$ such that $\sup_{x \in \text{OS}_m} \left\| f_x \right\|_{B(B_X, Y)} < \infty$. We fix $z \in B_X$, $z \neq 0$. Consider $x = (\frac{z}{\|z\|}, x_2, \ldots, x_m) \in \text{OS}_m$ and put $z_{[m]} := (\|z\|, 0, \ldots, 0) \in B_m$. Then $\|z_{[m]}\| = \|z\|$ and
\[
\left\| \nabla f_x(z_{[m]}) \right\| = \left\| \nabla f \left( \sum_{k=1}^m z_k x_k \right) \right\| = \left\| \nabla f(z) \right\|.
\]

This implies that
\[
\left\| f \right\|_{s_{B_Y^\infty(B_X, Y)}} = \sup_{z \in B_X} \mu(z) \left\| \nabla f(z) \right\|
\leq \sup_{z \in B_X} \mu(z_{[m]}) \left\| \nabla f_x(z_{[m]}) \right\|
\leq \sup_{x \in \text{OS}_m} \left\| f_x \right\|_{B(B_m, Y)} < \infty.
\]

Thus $f \in B_{\mu}^\infty(B_X, Y)$.

On the other hand, it is obvious that
\[
\sup_{x \in \text{OS}_m} \left\| f_x \right\|_{B_Y^\infty(B_m, Y)} \leq \left\| f \right\|_{s_{B_Y^\infty(B_X, Y)}} \quad \forall m \geq 2.
\]
Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (1): The proof is straight-forward by putting $x \in OS_m$ and $z_{[m]} \in B_m$ as in the proof of (4) $\Rightarrow$ (1) in Proposition 4.1 for each $z \in B_X$ with $\|z\| > \varrho$. □

In the next proofs below we need the following lemma.

**Lemma 4.3.** For every $f \in B^\Lambda_\mu (B_X, Y)$ and $x \in X$ with $\|x\| = 1$ we have

\begin{equation}
R_f(\lambda x) = \lambda f'(\cdot\,x)(\lambda) \quad \forall \lambda \in B_1
\end{equation}

and

\begin{equation}
f'(\cdot\,x)(\lambda)(\mu) = f'(\lambda x)(\mu x) \quad \forall \lambda, \mu \in B_1.
\end{equation}
Proof. (i) First, it follows from the Bessel inequality that every \( x \in X \) has only a countable number of non-zero Fourier coefficients \( \langle x, e_j \rangle \). Indeed, for every \( \varepsilon > 0 \) the set \( \{ j \in \Gamma : |\langle x, e_j \rangle| > \varepsilon \} \) is finite. Then we still have \( x = \sum_{j \in \Gamma} \langle x, e_j \rangle e_j = \sum_{j \in \Gamma} x_j e_j \) where the sum is in fact a countable one, and it is independent of the particular enumeration of the countable number of non-zero summands. Hence, we can write \( x = \sum_{j=1}^{\infty} x_j e_j \). Then, by the definitions of \( f(\cdot) \) and \( f'(\cdot)(\lambda) \) we have

\[
\left\| \frac{1}{t} \sum_{k=1}^{\infty} \left( f\left( \sum_{\lambda=1}^{k} \lambda x_{j} e_{j} + t \lambda x_{k} e_{k} + \sum_{j=k+1}^{\infty} (\lambda + t \lambda) x_{j} e_{j} \right) \right) - \frac{f\left( \sum_{\lambda=1}^{k} \lambda x_{j} e_{j} + \sum_{j=k+1}^{\infty} (\lambda + t \lambda) x_{j} e_{j} \right) - \lambda f'(\cdot)(\lambda)}{t} \right\| = \frac{\| f((\lambda + t \lambda) x) - f(\lambda x) \|}{t} - \lambda f'(\cdot)(\lambda) \rightarrow 0 \text{ as } t \rightarrow 0.
\]

Hence (4.11) is proved.

(ii) For \( \lambda, \eta \in B_1 \) we have

\[
\| \eta f'(\cdot)(\lambda) - f'(\lambda x)(\eta x) \| = \left\| \frac{f(\lambda x + t \eta x) - f(\lambda x) - \eta f'(\cdot)(\lambda)}{t} \right\| = \| f(\lambda x + t \eta x) - f(\lambda x) \| + \| \eta f'(\cdot)(\lambda) \| \rightarrow 0 \text{ as } t \rightarrow 0.
\]

Then \( f'(\cdot)(\lambda)(\eta) = \eta f'(\cdot)(\lambda) = f'(\lambda x)(\eta x) \), and (4.10) is proved. \( \square \)

**Proposition 4.4.** (1) The spaces \( B^{R}_{\mu}(B_{X,Y}) \) and \( B^{aff}_{\mu}(B_{X,Y}) \) coincide. Moreover,

\[
\| f \|_{sB^{R}_{\mu}(B_{X,Y})} \leq \| f \|_{sB^{aff}_{\mu}(B_{X,Y})} \lesssim \| f \|_{sB^{R}_{\mu}(B_{X,Y})} \forall f \in B^{R}_{\mu}(B_{X,Y}).
\]

(2) The spaces \( B^{R}_{\mu,0}(B_{X,Y}) \) and \( B^{aff}_{\mu,0}(B_{X,Y}) \) coincide.

**Proof.** (1)(i) Let \( f \in B^{aff}_{\mu}(B_{X,Y}) \). In order to prove \( f \in B^{R}_{\mu}(B_{X,Y}) \) it suffices to show that

\[
\text{(4.11)} \quad Rf(z) = \| f \| \int_{\mathbb{B}} \left( \frac{z}{\| z \|} \right)^{\| z \|} \forall z \in B_{X} \setminus \{ 0 \}.
\]

It is easy to see that (4.11) follows immediately from (4.9) for \( y = \frac{z}{\| z \|} \) and \( \lambda = \| z \| \) for every \( z \in B_{X} \setminus \{ 0 \} \). Moreover, it follows from (4.11) that

\[
\| f \|_{sB^{R}_{\mu}(B_{X,Y})} \leq \| f \|_{sB^{aff}_{\mu}(B_{X,Y})}.
\]

(ii) Let \( f \in B^{R}_{\mu}(B_{X,Y}) \) and \( x \in X \) be such that \( \| x \| = 1 \). Since \( f \) is holomorphic at \( 0 \in B_{X} \), its derivative \( f' : B_{X} \rightarrow L(X,Y) \) is also holomorphic, and thus there are \( r \in (0,1) \) and \( M > 0 \) such that

\[
\| f'(z) \|_{L(X,Y)} \leq M \quad \forall z \in \overline{B}(0,r) := \{ u \in X : \| u \| \leq r \}.
\]
Then, by (4.11) we have
\[
\sup_{|\lambda| \leq r} \mu(\lambda x)\|f'(\cdot)(\lambda)\| = \sup_{|\lambda| \leq r} \mu(\lambda x) \sup_{|\eta| \leq 1} \|f'(\cdot)(\lambda)(\eta)\|
\]
\[
= \sup_{|\lambda| \leq r} \mu(\lambda x) \sup_{|\eta| \leq 1} \|f'(\cdot)(\lambda)(\eta)\|
\]
\[
\leq \sup_{|\lambda| \leq r} \mu(\lambda x)\|f'(\cdot)(\lambda)\| \leq M.
\]

For the case where \(\|z\| > r\), by (4.9), (4.10) and the monotony increases of the function \(\frac{1}{\lambda}\) we have
\[
\mu(\lambda x)\|f'(\cdot)(\lambda)\| = \mu(\lambda x)((1 - |\lambda|)\|f'(\cdot)(\lambda)\| + \mu(\lambda x)\|f'(\cdot)(\lambda)\|)
\]
\[
\leq \mu(\lambda x)((1 - |\lambda|)\frac{1 - r}{r}\|f'(\cdot)(\lambda)\| + \mu(\lambda x)\|Rf(\lambda x)\|)
\]
\[
\leq \left(\mu(\lambda x)\frac{1 - r}{r} + \mu(\lambda x)\right)\|Rf(\lambda x)\|.
\]
This implies that
\[
\sup_{|\lambda| > r} \mu(\lambda x)\|f'(\cdot)(\lambda)\| \leq \frac{1}{r} \sup_{z \in B_X} \mu(z)\|Rf(z)\|.
\]

Therefore, \(f \in B^B_{\mu}(B_X, Y)\), and we also obtain \(\|f\|_{sB^B_{\mu}(B_X, Y)} \leq \frac{1}{r} \|f\|_{sB^B_{\mu}(B_X, Y)}\).

(2) Let \(f \in B^B_{\mu,0}(B_X, Y)\). Then, using (4.11) it is easy to see that \(f \in B^B_{\mu,0}(B_X, Y)\). In the converse direction, it follows from (4.12) that \(f \in B^B_{\mu,0}(B_X, Y)\) if \(f \in B^B_{\mu,0}(B_X, Y)\). \(\Box\)

Next, we will compare the spaces \(B^B_{\mu}(B_X, Y)\) and \(B^B_{\mu,0}(B_X, Y)\).

We need a vector-valued version of Lemma 4.11 in \([11]\). First we note that
\[
(4.13) \quad f \in B_{\mu}(B_1, Y) \quad \text{if and only if} \quad u \circ f \in B_{\mu}(B_1) \text{ for all } u \in Y'
\]
and, interchanging the suprema, we have that
\[
(4.14) \quad \|f\|_{B^B_{\mu}(B_1, Y)} \simeq \sup_{\|u\|=1} \|u \circ f\|_{B^B_{\mu}(B_1)}.
\]

**Lemma 4.5.** Let \(f \in B^B_{\mu}(B_2, Y)\). If there exists \(M > 0\) such that for any \(x = (x_1, x_2) \in B_2\), the function \(f(\cdot)(x)(\lambda)\) satisfies \(\|f(\cdot)(x)\|_{sB^s_{\mu}(B_1, Y)} \leq M\), then
\[
\mu((x_1, 0))\|\nabla f(x_1, 0)\| \leq 2\sqrt{2}MR_{\mu} \quad \forall x_1 \in \mathbb{C}, |x_1| < 1
\]
where \(R_{\mu} := 1 + \max_{t \in [0, \delta]} \mu(t) \int_0^t \frac{dt}{\mu(t)}\).

**Proof.** Fix \(u \in Y'\) with \(\|u\| = 1\). By the hypothesis, \(f(\cdot) \in B(B_1, Y)\). Then it follows from (4.13) that \(u \circ f(\cdot) \in B(B_1)\).
\[
\|u \circ f(\cdot)\|_{sB^s_{\mu,0}} \leq \|u\|\|f(\cdot)\|_{sB^s_{\mu,0}} \leq M.
\]
First of all, the hypotheses imply that
\[
\mu((x_1, 0))\left|\frac{\partial(u \circ f)}{\partial x_1}(x_1, 0)\right| \leq M.
\]
and so it is sufficient to show that
\[
\mu((x_1, 0))\left|\frac{\partial(u \circ f)}{\partial x_2}(x_1, 0)\right| \leq 2\sqrt{2}M.
\]
Indeed, from the hypotheses, we have

\[ |f(z) - f(0)| = \left| \int_0^1 (\nabla f(tz), \tau) dt \right| \leq M \int_0^1 \frac{||t| dt}{\mu(t)}. \]

Then, using the Cauchy integral formula and a simple estimate, we obtain

\[
\mu((x_1, 0)) \left| \frac{\partial (u \circ f)}{\partial x_2} (x_1, 0) \right|
\leq \mu((x_1, 0)) \frac{1}{2\pi} \int_{|w|=1/\sqrt{2}} \frac{\|u\| |f(x_1, w) - f(0) + f(0) - f(x_1, 0)|}{|w|^2} dw
\leq \mu((x_1, 0)) \frac{2M}{2\pi} \int_{|w|=1/\sqrt{2}} \frac{dw}{w^2} \leq 2\sqrt{2} M R_{\mu}
\]

as required. \(\square\)

**Theorem 4.6.** (1) The spaces \(B_{\mu}^{X}(B_X, Y)\) and \(B_{\mu}^{R}(B_X, Y)\) coincide. Moreover,

\[ \|f\|_{B_{\mu}^{R}(B_X, Y)} \asymp \|f\|_{B_{\mu}^{X}(B_X, Y)}. \]

(2) The spaces \(B_{\mu,0}^{X}(B_X, Y)\) and \(B_{\mu,0}^{R}(B_X, Y)\) coincide.

**Proof.** (1) Let us show that \(\|f\|_{sB_{\mu}^{X}(B_X, Y)} \leq 2\sqrt{2} R_{\mu} \|f\|_{sB_{\mu}^{R}(B_X, Y)}\) and the result follows using Proposition [4.3].

Fix \(u \in Y'\) with \(\|u\| = 1\). Let \(z \in B_X\) and \(v \in X\) with \(\|v\| = 1\) be fixed. We may assume that \(\dim X \geq 2\). Then there exist orthonormal unit vectors \(e_1, e_2 \in X\) and \(s, r_1, r_2 \in \mathbb{C}\) with \(|s| < 1\) and \(|t_1|^2 + |t_2|^2 = 1\) such that \(z = se_1, v = t_1e_1 + t_2e_2\). For \(f \in B_{\mu}^{R}(B_X, Y)\) put

\[ F(z_1, z_2) = (u \circ f)(z_1e_1 + z_2e_2), \quad (z_2, z_2) \in B_2. \]

Then \(F \in H(B_X)\) and it is easy to check that \(F\) satisfies the assumptions of Lemma [4.3]. Then

\[ \mu(z) |\nabla (u \circ f)(z)| = \mu(s) |\nabla (u \circ f)(se_1)| = \mu(s, 0) |\nabla F(s, 0)| \leq 2\sqrt{2} M R_{\mu}, \]

hence, \(\|f\|_{sB_{\mu}^{X}(B_X, Y)} \leq 2\sqrt{2} R_{\mu} \|f\|_{sB_{\mu}^{R}(B_X, Y)}\) as required.

(2) Because \(\|R(f(z))\| < \|\nabla f(z)\|\) for every \(z \in B_X\), it suffices to show that \(B_{\mu,0}^{R}(B_X, Y) \subset B_{\mu,0}^{X}(B_X, Y)\). Let \(f \in B_{\mu,0}^{R}(B_X, Y)\) and consider the function \(F(z_1, z_2)\) defined in the proof of the part (1). It is clear that \(RF(z_1, z_2) = R(u \circ f)(z_1e_2 + z_2e_2), \quad \frac{\partial F}{\partial z_1}(z_1, 0) = \langle \nabla(u \circ f)(z_1e_1), e_1 \rangle\) \(\) and \(\frac{\partial F}{\partial z_2}(z_1, 0) = \langle \nabla(u \circ f)(z_1e_1), e_2 \rangle\).

Fix \(r_0 \in (\delta, 1)\) and assume that \(|z_1| =: r \geq r_0\). Since \(\delta < r_0 < \sqrt{t^2 + r_0^2(1 - (t/r)^2)} \leq r\) for \(t \in [0, r]\) and \(\mu\) is strictly decreasing on \([\delta, 1]\), we have

\[ \mu(\sqrt{t^2 + r_0^2(1 - (t/r)^2)}) \geq \mu(r), \quad t \in [0, r]. \]
Then, for \( t \in [0, r] \) by Cauchy’s integral formula, we have

\[
\left| \frac{\partial (RF)}{\partial z_2}(t, 0) \right| = \frac{1}{2\pi i} \int_{|z_2|=r_0} \frac{RF(t, z_2)dz_2}{z_2^2} = \frac{1}{2\pi} \int_{|z_2|=r_0} \frac{Rf(te_1 + z_2e_2)dz_2}{z_2^2} \\
\leq \max_{|z_2|=r_0} \frac{|Rf(te_1 + z_2e_2)|}{r_0 \sqrt{1 - (r/t)^2}} \\
\leq \sup_{|z| \leq |z| < 1} \mu(|z|) |Rf(z)| \\
\frac{\mu(r)}{r_0 \sqrt{1 - (t/r)^2}}.
\]

Then, by this estimate and Lemma 6.4.5, for \( |z_1| = r \geq r_0 \) we have

\[
|z_1| \frac{\partial F}{\partial z_2}(z_1, 0) = \left| \int_0^r \frac{\partial (RF)}{\partial z_2}(t, 0) dt \right| \\
\leq \int_0^r \sup_{|z| \leq |z| < 1} \mu(|z|) |Rf(z)| \\
\frac{\mu(r)}{r_0 \sqrt{1 - (t/r)^2}} \\
\left| \frac{\pi |z_1|}{2\mu(|z_1|)r_0} \sup_{|z| < 1} \mu(|z|) |Rf(z)| \right.
\]

It implies that

\[
|z_1| \frac{\partial F}{\partial z_2}(z_1, 0) \leq \pi |z_1| \frac{1}{2\mu(|z_1|)r_0 \sqrt{1 - (t/r)^2}} \sup_{|z| \leq |z| < 1} \mu(|z|) |Rf(z)| \quad \text{for} \ |z_1| \geq r_0.
\]

On the other hand, we also have

\[
|z_1| \frac{\partial F}{\partial z_1}(z_1, 0) \leq \frac{1}{\mu(|z_1|)r_0 \sqrt{1 - (t/r)^2}} \sup_{|z| \leq |z| < 1} \mu(|z|) |Rf(z)| \quad \text{for} \ |z_1| \geq r_0.
\]

From (4.16) and (4.17) we obtain

\[
\mu(|z|) |\nabla f(z), v| = \mu(s) |(\nabla f(s_1), t_1e_1 + t_2e_2)| \\
\leq \mu(s) \left| \frac{\partial F}{\partial z_1}(s, 0) + t_2 \frac{\partial F}{\partial z_2}(s, 0) \right| \\
\leq \mu(s) \left( \left| \frac{\partial F}{\partial z_1}(s, 0) \right|^2 + \left| \frac{\partial F}{\partial z_2}(s, 0) \right|^2 \right)^{1/2} \\
\leq \frac{\pi}{\sqrt{2\delta}} \sup_{|z| \leq |z| < 1} \mu(|z|) |Rf(z)|, \quad \|z\| \geq r_0, \|v\| = 1.
\]

Now, by the hypothesis, for every \( \varepsilon > 0 \) we can find \( r_0 \in (\delta, 1) \) such that \( \mu(|z|) |Rf(z)| < \varepsilon \) for \( \|z\| > r_0 \). Therefore, it follows from (4.18) that \( \lim_{\|z\| \to 1} \mu(|z|) |\nabla f(z)| = 0 \), that means \( f \in B_{\mu, 0}^\nabla (B_X, Y) \).

We can now combine the results of Proposition 4.4 and Lemma 4.5 with an argument analogous to the Theorem 2.6 in [BGM] and obtain the following theorem:
We recall the following result of Blasco and his colleagues in [BGM]:

Let \( \text{Definition 4.3.} \)

Theorem 4.9.

The spaces \( B^g_{µ}(B_X,Y), B^R_{µ}(B_X,Y) \) and \( B^\text{aff}_{µ,0}(B_X,Y) \) coincide. Moreover,

\[
\|f\|_{B^g_{µ}(B_X,Y)} \leq \|f\|_{B^v_{µ}(B_X,Y)} \leq 2\sqrt{2}R_µ \|f\|_{B^\text{aff}_{µ}(B_X,Y)}.
\]

Next, we present a Möbius invariant norm for the Bloch-type space \( WB(B_X,Y) \).

Möbius transformations on a Hilbert space \( X \) are the mappings \( φ_a, a \in B_X \), defined as follows:

\[
φ_a(z) = \frac{a - P_µ(z) - saQ_µ(z)}{1 - \langle z, a \rangle}, \quad z \in B_X
\]

where \( s_a = \sqrt{1 - \|a\|^2} \), \( P_µ \) is the orthogonal projection from \( X \) onto the one dimensional subspace \( [a] \) generated by \( a \), and \( Q_µ \) is the orthogonal projection from \( X \) onto \( X \odot [a] \). It is clear that

\[
P_µ(z) = \frac{\langle z, a \rangle}{\|a\|^2}a, \quad (z \in X) \quad \text{and} \quad Q_µ(z) = z - \frac{\langle z, a \rangle}{\|a\|^2}a, \quad (z \in B_X).
\]

When \( a = 0 \), we simply define \( φ_a(z) = -z \). It is obvious that each \( φ_a \) is a holomorphic mapping from \( B_X \) into \( X \).

We will also need the following facts about the pseudohyperbolic distance in \( B_X \). It is given by

\[
g_X(x,y) := \|φ_−y(x)\| \quad \text{for any } x,y \in B_X.
\]

For details concerning Möbius transformations and the pseudohyperbolic distance we refer to the book of K. Zhu [Zhu].

It is well known that, in the case \( n \geq 2 \), the equality \( \|f \circ φ\|_{B^v(B_n,Y)} = \|f\|_{B^v(B_n,Y)} \) is false. Our goal is to find a semi-norm on \( B(B_X,Y) \) which is invariant under the automorphisms of the ball \( B_X \).

Definition 4.3. Let \( X \) be a complex Hilbert space, \( Y \) be a Banach space and \( f \in H(B_X,Y) \). Consider the invariant gradient norm

\[
\|∇f(z)\| := \|∇(f \circ φ_z)(0)\| \quad \text{for any } z \in B_X.
\]

We recall the following result of Blasco and his colleagues in [BGM]:

Lemma 4.8 (Lemma 3.5, [BGM]). Let \( f \in H(B_X) \). Then

\[
\|∇f(z)\| = \sup_{w \neq 0} \frac{|⟨∇f(z), w⟩|}{\sqrt{(1 - \|z\|^2)||w||^2 + ⟨w, z⟩^2}}.
\]

We define invariant semi-norm as follows

\[
\|f\|_{B^\text{inv}(B_X,Y)} := \sup_{z \in B_X} \|∇f(z)\| = \sup_{z \in B_X} \|∇(u \circ f)(z)\|.
\]

We denote

\[
B^\text{inv}(B_X,Y) := \{ f \in B(B_X,Y) : \|f\|_{B^\text{inv}(B_X,Y)} < \infty \}.
\]

It is also easy to check that \( B^\text{inv}(B_X,Y) \) is Banach under the norm

\[
\|f\|_{B^\text{inv}(B_X,Y)} := \|f(0)\| + \|f\|_{B^\text{inv}(B_X,Y)}.
\]

Now, applying Theorem 3.8 in [BGM] to the functions \( u \circ f \) for every \( u \in Y' \) we obtain the following:

Theorem 4.9. The spaces \( B^v(B_X,Y), \) and \( B^\text{inv}(B_X,Y) \) coincide. Moreover,

\[
\|f\|_{B^v(B_X,Y)} \leq \|f\|_{B^\text{inv}(B_X,Y)} \leq \|f\|_{B^v(B_X,Y)}.
\]
Now let $W \subset Y'$ be a separating subspace of the dual $Y'$. Applying Proposition 4.4, Theorems 4.7 and 4.9 to functions $w \circ f$ for each $f \in H(B_X, Y)$ and $w \in W$ we obtain the equivalence of the norms in associated Bloch-type spaces:

$$\|wB^w_w(B_X)\| \cong \|wB^w(B_X)\| \cong \|wB^w(B_X)\| \cong \|wB_\mu(B_X)\| \cong \|wB_\mu(B_X)\| \cong \|wB_\mu(B_X)\|.$$ 

Hence, for the sake of simplicity, from now on we write $B_\mu$ instead of $B^R_\mu$.

Now, we show that $WB_\mu(B_X)(Y)$, $WB_{\mu,0}(B_X)(Y)$ satisfy (we1)-(we3).

We need the following lemma whose proof parallels that of Lemma 13 in [SW] and will be omitted.

**Lemma 4.10.** Let $\mu$ be a normal weight on $B_X$. Then there exists $C_\mu > 0$ such that

$$C_\mu \leq \frac{\mu(r)}{\mu(r^2)} \leq 1 \ \forall r \in [0, 1].$$

**Proposition 4.11.** Let $W \subset Y'$ be a separating subspace. Let $\mu$ be a normal weight on $B_X$. Then $B_\mu(B_X)$, $B_{\mu,0}(B_X)$ satisfy (e1) and (e2), and hence, $WB_\mu(B_X)(Y)$ and $WB_{\mu,0}(B_X)(Y)$ satisfy (we1)-(we3).

**Proof.** It is obvious that $B_\mu(B_X)$, $B_{\mu,0}(B_X)$ satisfy (e1).

Because $B_{\mu,0}(B_X)$ is the subspace of $B_\mu(B_X)$ it suffices to check (e2) for the space $B_\mu(B_X)$.

In order to prove (e2) holds for $B_\mu(B_X)$ we will show that the closed unit ball $U$ of $B_\mu(B_X)$ is pointwise bounded and equicontinuous.

(i) First we prove that $U$ is pointwise bounded. It suffices to prove that

$$|f(z)| \leq \max \left\{1, \int_0^\infty \frac{|f|_{B_\mu(B_X)}}{\mu(t)} \right\} \|f\|_{B_\mu(B_X)} \forall f \in B_\mu(B_X), \forall z \in B_X.$$

Fix $f \in B_\mu(B_X)$ and put $g(z) = f(z) - f(0)$ for every $z \in B_X$. Note that $g(0) = 0$ and $\|g\|_{B_\mu(B_X)} = \|f\|_{B_\mu(B_X)}$. As in Lemma 4.5 by Cauchy-Schwarz inequality we have

$$|g(z)| \leq \int_0^1 \frac{|f|_{B_\mu(B_X)}|z|}{\mu(tz)} dt = \frac{\|f\|_{B_\mu(B_X)}}{\mu(t)} \int_0^1 \frac{|z|}{\mu(t)} dt = \frac{\|g\|_{B_\mu(B_X)}}{\mu(t)} \int_0^1 \frac{|z|}{\mu(t)} dt.$$

Consequently,

$$|f(z)| \leq |f(0)| + |g(z)| \leq |f(0)| + \|g\|_{B_\mu(B_X)} \int_0^1 \frac{|z|}{\mu(t)} dt = \|f\|_{B_\mu(B_X)} \int_0^1 \frac{|z|}{\mu(t)} dt = \|f\|_{B_\mu(B_X)} + \left(\int_0^1 \frac{|z|}{\mu(t)} - 1\right) \|f\|_{B_\mu(B_X)}$$

$$\leq \max \left\{1, \int_0^1 \frac{|z|}{\mu(t)} dt \right\} \|f\|_{B_\mu(B_X)}.$$

(ii) Next, we show that $U$ is equicontinuous. For each $f \in U$, by Proposition 4.1 we can find $m \geq 2$ such that

$$\|f\|_{B_\mu(B_X)} = \sup_{y \in OS_m} \|f_y\|_{B_\mu(B_m)}.$$
Remark

Consequently,

\[ \| \Delta \| z_{[m]} \| - f_{e_{[m]}}(w_{[m]}) \| \leq \beta(z_{[m]}, w_{[m]}) \sup_{x_{[m]} \in \mathbb{B}_m} \| \bar{\nabla} f_{e_{[m]}}(x_{[m]}) \| \]

\[ \leq \beta(z_{[m]}, w_{[m]}) \sup_{x_{[m]} \in \mathbb{B}_m} \sup_{y \in \mathbb{B}_m \setminus \{0\}} \frac{|\langle \nabla f_{e_{[m]}}, y \rangle (1 - \|x_{[m]}\|^2)}{\sqrt{(1 - \|x_{[m]}\|^2)\|y\|^2 + \langle y, x_{[m]} \rangle^2}} \]

\[ \leq \beta(z_{[m]}, w_{[m]}) C_\mu^{-1} \sup_{x_{[m]} \in \mathbb{B}_m} \frac{\mu^{[m]}(\|x_{[m]}\|)}{\mu^{[m]}(\|x_{[m]}\|)} \| f_{e_{[m]}} \|_{B_\mu(\mathbb{B}_m)} \sqrt{1 - \|x_{[m]}\|^2}}{\mu^{[m]}(\|x_{[m]}\|^2)} \]

where \( \beta \) is the Bergman metric on \( \mathbb{B}_m \) given by

\[ \beta(s, t) = \frac{1}{2} \log \frac{1 + |(\varphi_m)_s(t)|}{1 - |(\varphi_m)_s(t)|} \]

with \( (\varphi_m)_s \) is the involutive automorphism of \( \mathbb{B}_m \) that interchanges 0 and \( s \).

If \( \|x_{[m]}\|^2 \leq \delta \) it is clear that

\[ \frac{\sqrt{1 - \|x_{[m]}\|^2}}{\mu^{[m]}(\|x_{[m]}\|)} \leq \frac{1}{m_{\mu, \delta}} < \infty, \]

where \( m_{\mu, \delta} = \min_{t \in [0, \delta]} \mu(t) > 0 \); if \( \|x_{[m]}\|^2 > \delta \) and \( b \geq 1/2 \) we have

\[ \frac{\sqrt{1 - \|x_{[m]}\|^2}}{\mu^{[m]}(\|x_{[m]}\|)} \leq \left(1 - \|x_{[m]}\|^2\right)^b < S_\mu < \infty; \]

if \( \|x_{[m]}\|^2 > \delta \) and \( b < 1/2 \) we get

\[ \frac{\sqrt{1 - \|x_{[m]}\|^2}}{\mu^{[m]}(\|x_{[m]}\|)} = \left(1 - \|x_{[m]}\|^2\right)^b \mu^{[m]}(\|x_{[m]}\|)^{1/2 - b} \leq S_\mu (1 - \delta)^{1/2 - b} < \infty. \]

Consequently,

\[ |f_{e_{[m]}}(z_{[m]}) - f_{e_{[m]}}(w_{[m]})| \leq \beta(z_{[m]}, w_{[m]}) \hat{S}_\mu \| f_{e_{[m]}} \|_{B_\mu(\mathbb{B}_m)} \]

where

\[ \hat{S}_\mu := C_\mu^{-1} \max \{ m_{\mu, 1}, S_\mu (1 - \delta)^{1/2 - b} \}. \]

Since \( \beta(s, t) \) is the infimum of the set consisting of all \( \ell(\gamma) \) where \( \gamma \) is a piecewise smooth curve in \( \mathbb{B}_m \) from \( s \) to \( t \) (see [Z1] p. 25) we have

\[ |f_{e_{[m]}}(z_{[m]}) - f_{e_{[m]}}(w_{[m]})| \leq \hat{S}_\mu \| f_{e_{[m]}} \|_{B_\mu(\mathbb{B}_m)} \leq \hat{S}_\mu \| z - w \|. \]

Consequently,

\[ |f(z) - f(w)| = \lim_{m \to \infty} |f_{e_{[m]}}(z_{[m]}) - f_{e_{[m]}}(w_{[m]})| \leq \hat{S}_\mu \| z - w \|. \]

This yields that \( U \) is equicontinuous.

\[ \square \]

Remark 4.2. (1) In fact, the estimate (4.20) can be written as follows

\[ |f(z)| \leq |f(0)| + \int_0^{\|z\|} dt \mu(t)^b \| f \|_{B_\mu}. \]
It should be noted that, for $\mu(z) = 1 - \|z\|^2$, \( (1 - \|z\|^2) \) will be

\[
(4.21) \quad |f(z)| \leq \max \left\{ 1, \frac{1}{2} \log \frac{1 + \|z\|}{1 - \|z\|} \right\} \|f\|_{B(B_X)} \quad \forall f \in B(B_X), \forall z \in B_X,
\]

and, therefore, by an easy calculation that $\sup_{x \in [0,1)} (1 - x^2) \log \frac{1 + x}{1 - x} < 1$ we have

\[
(4.22) \quad (1 - \|z\|^2)|f(z)| \leq \|f\|_{B(B_X)} \quad \forall f \in B(B_X), \forall z \in B_X.
\]

5. The Test Functions and Auxiliary results

This section provides some preparations to study characterizations the boundedness and compactness of the weighted composition operators between (little) Bloch-type spaces.

In this section we consider $\nu$ is a normal weight on $B_X$ and $\varphi \in S(B_X)$.

We begin this section by constructing test functions that are useful for the proofs of our main results.

First we consider the holomorphic function

\[
(5.1) \quad g(z) := 1 + \sum_{k > k_0} 2^k z^{n_k} \quad \forall z \in B_1
\]

where $k_0 = \left[ \log_2 \frac{1}{\nu(\delta)} \right]$, $n_k = \left[ \frac{1}{1 - r_k} \right]$ with $r_k = \nu^{-1}(1/2^k)$ for every $k \geq 1$. Here the symbol $[x]$ means the greatest integer not more than $x$. By [HW, Theorem 2.3], $g(t)$ is increasing on $[0,1)$ and

\[
(5.2) \quad |g(z)| \leq g(|z|) \in \mathbb{R} \quad \forall z \in B_1,
\]

\[
(5.3) \quad 0 < C_1 := \inf_{t \in [0,1)} \nu(t) g(t) \leq \sup_{t \in [0,1)} \nu(t) g(t) \leq \sup_{z \in B_1} \nu(z) |g(z)| =: C_2 < \infty.
\]

Proposition 5.1. There exists positive constants $C_3$ such that the inequality

\[
(5.4) \quad \int_0^r g(t) dt \leq C_3 \int_0^{r^2} g(t) dt
\]

holds for all $r \in [r_1, 1)$, where $r_1 \in (0, 1)$ is a constant such that $\int_0^{r_1} g(t) dt = 1$.

Proof. Let $\delta > 0$ be the constant in Definition of the normal weight $\nu$. We may assume that $r_1 < \delta^{1/4}$.

In the case $r \in [r_1, \delta^{1/4}]$ we have $\int_0^r g(t) dt$ is bounded above and $\int_0^{r^2} g(t) dt$ is bounded below by a positive constant. So, there exists a constant $C > 0$ such that

\[
\int_0^r g(t) dt \leq C \int_0^{r^2} g(t) dt, \quad r \in [r_1, \delta^{1/4}].
\]
In the case $r \in (\delta^{1/4}, 1)$, by (5.3) and (W2) we have
\[
\int_0^r g(t) dt \leq C_2 \int_0^r \frac{1}{\nu(t)} dt = C_2 \int_0^r \frac{(1-t)^b}{\nu(t)} dt \\
\leq C_2 \frac{(1-r^2)^b}{\nu(r^2)} \frac{r-r^2}{(1-r)^b} \\
= C_2 \frac{(1-r^2)^b (r-r^2)(1+r)^b (1+r^2)^b}{(1-r^4)^b} \\
\leq C_2 \frac{(r-r^2)(1+r)^b (1+r^2)^b}{r^2 - r^4} \int_0^r \frac{(1-t)^b}{\nu(t)} dt \\
\leq \frac{C_2 (1+r)^b (1+r^2)^b}{C_1 (r+r^2)} \int_0^r g(t) dt.
\]
(5.5)

Therefore, there exists a constant $C_3 > 0$ such that
\[
\int_0^r g(t) dt = \int_0^r g(t) dt + \int_r^2 g(t) dt \leq C_3 \int_0^r g(t) dt, \quad r \in (\delta^{1/4}, 1).
\]
This implies that (5.4) is proved. \[\square\]

Now, for $w \in B_X$ fixed such that $\|w\| \geq r$ for some $r > 0$, consider put the test functions
\[
\beta_w(z) := \frac{1}{\|w\|} \int_0^{(z,w)} g(t) dt, \quad z \in B_X,
\]
(5.6)
\[
\gamma_w(z) := \frac{1}{\|w\|^2} g(t) dt \left( \int_0^{(z,w)} g(t) dt \right)^2, \quad z \in B_X,
\]
(5.7)

Let $\{w^n\}_{n \geq 1} \subset B_X$ be a sequence satisfying, for some $r > 0$, $\|w^n\| \geq r$ for every $n \geq 1$ and $\lim_{n \to \infty} w^n = w^0$ with $\|w^0\| = 1$. Under the additional assumption that
\[
\int_0^1 \frac{1}{\nu(t)} < \infty,
\]
for each $n \geq 1$ consider the sequences the functions $\{\gamma_{w^n}\}_{n \geq 1}$ defined by (5.7) and of the functions
\[
\eta_{w^n}(z) := \frac{1}{\|w^n\|^{n+1}} \int_0^{\|w^n\|^n (z,w^n)} g(t) dt, \quad n = 1, 2, \ldots,
\]
(5.8)
\[
\theta_{w^n} := \eta_{w^n} - \beta_{w^n} \quad n = 1, 2, \ldots.
\]

**Proposition 5.2.** We have $\beta_w, \gamma_w, \theta_w \in B_{\nu,0}(B_X)$ and
\[
\|\beta_w\|_{B_{\nu}(B_X)} \leq C_2, \quad \|\gamma_w\|_{B_{\nu}(B_X)} \leq 2C_2C_3, \quad \|\theta_{w^n}\|_{B_{\nu}(B_X)} \leq 2C_2.
\]

**Proof.** It suffices to prove for the function $\beta_w$ because the proofs for other ones are very similar.

(i) Since $\nu(z) \to 0$, from (5.3) we have
\[
\nu(z) \|\nabla \beta_w(z)\| = \frac{1}{\|w\|} \|\nabla \beta_w(z)\|_{B_X} \to 0 \\
\quad \leq \nu(z) g(\|w\|) \to 0 \quad \text{as} \quad \|z\| \to 1.
\]

This implies that $\beta_w \in B_{\nu,0}(B_X)$.
Proposition 5.3. We have
\[ \|\beta_w\|_{\mathcal{B}_\nu(B_X)} = \sup_{z \in B_X} \nu(z)\|R\beta_w(z)\| \leq \nu(z) \frac{1}{\|w\|} \|g(\langle z, w \rangle)\| \leq C_2 < \infty. \]
\[ \square \]

Lemma 5.4. Let \( \psi, f \in H(B_X) \) and \( \varphi \in S(B_X) \). Then, for every \( z \in B_X \) we have
\[ \|\nabla z(\psi \circ \varphi)(z)\| \leq |f(\varphi(z))|\|\nabla_z \psi(z)\| + |\psi(z)|\|\nabla_{\varphi(z)} f(\varphi(z))\|\|\nabla_z \varphi(z)\| \]
and
\[ \|R(\psi \cdot (f \circ \varphi))(z)\| \leq |f(\varphi(z))|\|R\psi(z)\| + |\psi(z)|\|\nabla_{\varphi(z)} f(\varphi(z))\|\|R\varphi(z)\|. \]

In order to study the compactness of the operators \( W_{\psi, \varphi} \), as in [1] we can prove the following.

Lemma 5.5. Let \( E, F \) be two Banach spaces of holomorphic functions on \( B_X \). Suppose that
(1) The point evaluation functionals on \( E \) are continuous;
(2) The closed unit ball of \( E \) is a compact subset of \( E \) in the topology of uniform convergence on compact sets;
Then, $T : E \to F$ is continuous when $E$ and $F$ are given the topology of uniform convergence on compact sets. Then, $T$ is a compact operator if and only if given a bounded sequence $\{f_n\}$ in $E$ such that $f_n \to 0$ uniformly on compact sets, then the sequence $\{Tf_n\}$ converges to zero in the norm of $F$.

We can now combine this result with Montel theorem and (4.20) to obtain the following proposition. The details of the proof are omitted here.

**Proposition 5.6.** Let $\psi \in H(B_X)$ and $\varphi \in S(B_X)$. Then $W_{\psi,\varphi} : \mathcal{B}_\nu(B_X) \to \mathcal{B}_\mu(B_X)$ is compact if and only if $\|W_{\psi,\varphi}(f_n)\|_{\mathcal{B}_\nu(B_X)} \to 0$ for any bounded sequence $\{f_n\}$ in $\mathcal{B}_\nu(B_X)$ converging to 0 uniformly on compact sets in $B_X$.

### 6. Weighted composition operators between Bloch-type spaces

Let $\nu, \mu$ be normal weights on $B_X$. Let $\varphi \in S(B_X)$, the set of holomorphic self-maps on $B_X$ and $\psi \in H(B_X)$. For each $F \subset \Gamma$ finite and $m \in \mathbb{N}$ we write

$$\varphi^{[F]} = \varphi|_{\text{span} \{e_k, k \in F\}}, \quad \varphi^{[m]} = \varphi|_{\text{span} \{e_1, \ldots, e_m\}}$$

and

$$\psi^{[F]} = \psi|_{\text{span} \{e_k, k \in F\}}, \quad \psi^{[m]} = \psi|_{\text{span} \{e_1, \ldots, e_m\}}.$$  

For each $j \in \Gamma$ and $k \geq 1$ we denote

$$\varphi_j(\cdot) := \langle \varphi(\cdot), e_j \rangle, \quad \varphi(k)(\cdot) := \langle \varphi_1(\cdot), \ldots, \varphi_k(\cdot) \rangle.$$  

In this section we investigate the boundedness and the compactness of the operators $W_{\psi,\varphi}$ between the (little) Bloch-type spaces $\mathcal{B}_\nu$ ($\mathcal{B}_{\nu,0}$) and $\mathcal{B}_\mu$, ($\mathcal{B}_{\mu,0}$) via the estimates of $\psi^{[m]}$, $\varphi^{[m]}$ and $\varphi^{(k)}$. Hence, by Theorem 5.1 some characterizations for the boundedness and compactness of the operators $W_{\psi,\varphi}$ between spaces $\mathcal{W}_{\nu}(B_X, Y)$, ($\mathcal{W}_{\nu,0}(B_X, Y)$) and $\mathcal{W}_{\mu}(B_X, Y)$, ($\mathcal{W}_{\mu,0}(B_X, Y)$) will be obtained from these results.

By Proposition 4.3 and Theorem 4.7 in the paper we will only present the results for the spaces $\mathcal{B}_{\mu,0}(B_X)$.

First we investigate the boundedness of weighted composition operators. We use there certain quantities, which will be used in this work. We list them below:

\begin{align}
(6.1a) & \quad \mu^{[m]}(y) \| R\psi^{[m]}(y) \| \max \left\{ 1, \int_0^{\| \varphi^{[m]}(y) \|} \frac{dt}{\nu(t)} \right\}, \\
(6.1b) & \quad \frac{\mu^{[m]}(y) \| \psi^{[m]}(y) \| \| R\varphi^{[m]}(y) \|}{\nu(\varphi^{[m]}(y))}, \\
(6.2a) & \quad \mu^{[F]}(y) \| R\psi^{[F]}(y) \| \max \left\{ 1, \int_0^{\| \varphi^{[F]}(y) \|} \frac{dt}{\nu(t)} \right\}, \\
(6.2b) & \quad \frac{\mu^{[F]}(y) \| \psi^{[F]}(y) \| \| R\varphi^{[F]}(y) \|}{\nu(\varphi^{[F]}(y))}, \\
(6.3a) & \quad \mu(y) \| R\psi(y) \| \max \left\{ 1, \int_0^{\| \varphi^{(k)}(y) \|} \frac{dt}{\nu(t)} \right\}, \\
(6.3b) & \quad \frac{\mu(y) \| \psi(y) \| \| R\varphi^{(k)}(y) \|}{\nu(\varphi^{(k)}(y))},
\end{align}
By (4.20) and (5.9), for every

\[ \|W(6)\| \leq \sup_{y \in B_m} \left( \int_0^1 \|\varphi(t)\| \frac{dt}{\nu(t)} \right)^{\frac{1}{2}}. \]

Moreover, if

\[ \|W(6)\| \leq \sup_{y \in B_m} \left( \int_0^1 \|\varphi(t)\| \frac{dt}{\nu(t)} \right)^{\frac{1}{2}}, \]

Theorem 6.1. Let \( \psi \in H(B_X) \) and \( \varphi \in S(B_X) \). Let \( \mu, \nu \) be normal weights on \( B_X \). Then the following are equivalent:

1. \( M_{\psi, R\varphi}^m \left( \frac{1}{m} \right) < \infty \), \( M_{\psi, R\varphi}^n \left( \frac{1}{n} \right) < \infty \) for some \( m \geq 2 \);
2. \( M_{\psi, R\varphi}^F \left( \frac{1}{F} \right) < \infty \), \( M_{\psi, R\varphi}^F \left( \frac{1}{F} \right) < \infty \) for every \( F \in \Gamma \) finite;
3. \( M_{\psi, R\varphi}^{(k)} < \infty \), \( M_{\psi, R\varphi}^{(k)} < \infty \) for every \( k \geq 1 \);
4. \( M_{\psi, R\varphi}^F < \infty \), \( M_{\psi, R\varphi}^F < \infty \) for every \( F \in B_X \);
5. \( W_{\psi, \varphi} : B_\nu(B_X) \rightarrow B_\nu(B_X) \) is bounded;
6. \( W_{\psi, \varphi} : B_\nu,0(B_X) \rightarrow B_\nu(B_X) \) is bounded.

Moreover, if \( W_{\psi, \varphi} \) is bounded, the following asymptotic relation holds for some \( m \geq 2 \):

\[ \|W_{\psi, \varphi}^0 \| = |\psi(0)| \max \left\{ 1, \int_0^1 \|\varphi(0)\| \frac{dt}{\mu(t)} \right\} + M_{\psi, R\varphi}^m + M_{\psi, R\varphi}^n. \]

Proof. It is clear that (4) \( \Rightarrow (2) \) \( \Rightarrow (1) \) and (4) \( \Rightarrow (3) \). Since \( B_\nu,0(B_X) \subset B_\nu(B_X) \) the implication (5) \( \Rightarrow (6) \) is obvious.

(1) \( \Rightarrow (5) \): Assume that \( M_{\psi, R\varphi}^m < \infty \) and \( M_{\psi, R\varphi}^{(n)} < \infty \) for some \( m \geq 2 \). For each \( x \in O\), we write \( z_x := \sum_{k=1}^n \lambda_k x_k \). Note that \( \|z_x\| = \|z_{[m]}\| \) and hence \( \mu([m]) \|z_{[m]}\| = \mu([m]) \|z_x\| \).

By (4.20) and (5.9), for every \( f \in B_\nu(B_X) \) and for every \( x \in O\), we have

\[ \|W_{\psi, \varphi}(f)\|_{B_\nu(B_X)} = \sup_{z_x \in \mathcal{B}} \mu([m]) \|R\psi \circ (f \circ \varphi)\|_{B_\nu(B_X)} \]

\[ \leq \sup_{z_x \in \mathcal{B}} \mu([m]) \|R\psi \|_{B_\nu(B_X)} \max \left\{ 1, \int_0^1 \|\varphi(0)\| \frac{dt}{\mu(t)} \right\} \|f\|_{B_\nu(B_X)} \]

\[ + \sup_{z_x \in \mathcal{B}} \mu([m]) \|\varphi(0)\| \|R\psi \|_{B_\nu(B_X)} \max \left\{ 1, \int_0^1 \|\varphi(0)\| \frac{dt}{\mu(t)} \right\} \|f\|_{B_\nu(B_X)} \]

\[ \leq \sup_{z_x \in \mathcal{B}} \mu([m]) \|R\psi \|_{B_\nu(B_X)} \max \left\{ 1, \int_0^1 \|\varphi(0)\| \frac{dt}{\mu(t)} \right\} \|f\|_{B_\nu(B_X)} \]

\[ + \sup_{z_x \in \mathcal{B}} \mu([m]) \|\varphi(0)\| \|R\psi \|_{B_\nu(B_X)} \max \left\{ 1, \int_0^1 \|\varphi(0)\| \frac{dt}{\mu(t)} \right\} \|f\|_{B_\nu(B_X)} \]

\[ \leq (M_{\psi, R\varphi}^m + M_{\psi, R\varphi}^n) \|f\|_{B_\nu(B_X)}. \]
Consequently, by (6.3) we have
\[
\|W_{\psi,\varphi}(f)\|_{\mathcal{B}_\mu(B_X)} = \sup_{x \in \Omega_{\mathcal{B}_m}} \|(W_{\psi,\varphi}(f))_x\|_{\mathcal{B}_\mu^m(\mathcal{B}_m)} \\
(6.6)
\lesssim (M_{\mathcal{R}_{\psi,\varphi}} + M_{\mathcal{R}_{\psi,\varphi}}^m)\|f\|_{\mathcal{B}_\nu(B_X)}.
\]

Thus, (5) is proved.

(3) $\Rightarrow$ (5): Use an argument analogous and an estimate to the previous one.

(6) $\Rightarrow$ (4): Suppose $W_{\psi,\varphi}^0 : \mathcal{B}_{\psi,\varphi}(B_X) \to \mathcal{B}_\mu(B_X)$ is bounded.

First, it is obvious that $\psi \in \mathcal{B}_\mu(B_X)$. Indeed, by considering the function $f = 1 \in \mathcal{B}_{\psi,\varphi}$ it easy to see that
\[
\sup_{z \in B_X} \mu(z)R\psi(z) = \|\psi\|_{\mathcal{B}_\mu(B_X)} = \|W_{\psi,\varphi}^1(1)\|_{\mathcal{B}_\mu(B_X)} \leq \|W_{\psi,\varphi}^0\| < \infty.
\]

Now, we will prove $M_{\psi,\varphi} < \infty$.

For each $k \in \Gamma$ consider the test function $\gamma_k \in \mathcal{B}_\mu(B_X)$ given by $\gamma_k := z_k$ for every $z \in B_X$. Then, since $W_{\psi,\varphi}^0$ is bounded we have
\[
\|W_{\psi,\varphi}^0(\gamma_k)\|_{\mathcal{B}_\nu(B_X)} = \sup_{z \in B_X} \mu(z)|R\psi(z)| = \|\psi\|_{\mathcal{B}_\nu(B_X)} = \|W_{\psi,\varphi}^1(1)\|_{\mathcal{B}_\nu(B_X)} \leq \|W_{\psi,\varphi}^0\| < \infty.
\]

This implies that $\psi \cdot \varphi_k \in \mathcal{B}_\mu(B_X)$.

Assume the contrary, that $M_{\psi,\varphi} = \infty$. Then there exists $\{z^n\}_{n \geq 1} \subset B_X$ such that
\[
\frac{\mu(z^n)|\psi(z^n)||R\varphi(z^n)|}{\nu(\varphi(z^n))} \geq n \quad \forall n \geq 1.
\]

We claim that $\liminf_{n \to \infty} \|\varphi(z^n)\| := r > 0$. Indeed, if this was not the case, since $\nu$ is positive, continuous and $\lim_{\|z\| \to 0} \nu(z) = 0$ we get $\lim \sup_{n \to \infty} \nu(\varphi(z^n)) > 0$. Therefore, from the estimate
\[
\|\mu(z^n)R\psi(z^n)\varphi(z^n) + \mu(z^n)\psi(z^n)R\varphi(z^n)\| \\
\geq \|\mu(z^n)\psi(z^n)R\varphi(z^n)\| - \|\mu(z^n)R\varphi(z^n)\| \to \infty,
\]
we would have $\psi \cdot \varphi_k \notin \mathcal{B}_\mu(B_X)$ for some $k \in \Gamma$. We obtain a contradiction.

Therefore, we may assume that there exists some $\eta > 0$ such that $|\varphi(z^n)| > \eta$ for every $n \geq 1$. Then, putting $w^n = \varphi(z^n)$ we can consider the test function $\beta_w$ given by (5.6).

Then, since $\beta_w(0) = 0$, we have
\[
\nu(z)|R\beta_w(z^n)| = \nu(z)\|w^n\||g(z, w^n)| \leq \nu(z)g(||z||) \leq C_2.
\]

By the boundedness of $W_{\psi,\varphi}^0$, there exists $C_W > 0$ such that
\[
W_{\psi,\varphi}^0(\beta_w) \|\mathcal{B}_\nu(B_X) \leq C_W \|\beta_w\|_{\mathcal{B}_\nu(B_X)} \leq C_W C_2.
\]

By Lemma [4.10] we have
\[
\frac{\nu(\|w^n\|)}{\nu(\|w^n\|^2)} \geq C_\mu \quad \forall n \geq 1.
\]
This implies that

\[
\|W^0_{\psi, \varphi}(\beta_{w^n})\|_{B_w(B_X)}
\geq \|\mu(z^n)R\psi(z^n)\beta_{w^n}(\varphi(z^n))
+ \mu(z^n)\psi(z^n)\nabla\beta_{w^n}(\varphi(z^n))R\varphi(z^n)\|
\geq \|\mu(z^n)\psi(z^n)\nabla\beta_{w^n}(\varphi(z^n))R\varphi(z^n)\|
- \|\mu(z^n)R\psi(z^n)\beta_{w^n}(\varphi(z^n))\|
\geq \frac{\mu(z^n)\psi(z^n)\|R\varphi(z^n)\|}{\nu(\|\varphi(z^n)\|)}\nu(\|w^n\|^2)g(\|w^n\|^2)\frac{\nu(\|w^n\|)}{\nu(\|w^n\|^2)}
- \|\mu(z^n)R\psi(z^n)\beta_{w^n}(\varphi(z^n))\|
\geq nC_1C_\mu - \|\mu(z^n)R\psi(z^n)\beta_{w^n}(\varphi(z^n))\|.
\]

(6.9)

On the other hand,

\[
\|\mu(z^n)R\psi(z^n)\beta_{w^n}(\varphi(z^n))\|
\leq \|W^0_{\psi, \varphi}(\beta_{w^n})\|_{B_w(B_X)} + \|\psi\|_{B_w(B_X)}\nu(\|w^n\|^2)g(\|w^n\|^2)
\leq \|W^0_{\psi, \varphi}(\beta_{w^n})\|_{B_w(B_X)} + \|\psi\|_{B_w(B_X)}C_2
\leq C_2C_W + \|\psi\|_{B_w(B_X)}C_2.
\]

Thus,

\[
2C_2C_W + \|\psi\|_{B_w(B_X)}C_2 \geq nC_1C_\mu \to \infty \quad \text{as } n \to \infty.
\]

This is a contradiction to (6.7).

Finally, we prove \(M_{R\psi, \varphi} < \infty\). Otherwise, we can find \(\{z^n\}_{n \geq 1} \subset B_X\) such that

\[
\mu(z^n)\|R\psi(z^n)\| \int_0^{\|\varphi(z^n)\|} \frac{dt}{\nu(t)} > n.
\]

By \(\psi \in B_{\mu}(B_X)\) this implies that

\[
\int_0^{\|\varphi(z^n)\|} \frac{dt}{\nu(t)} > \frac{n}{\|\psi\|_{B_w(B_X)}}.
\]

Put \(w^n = \varphi(z^n)\). Then, by (6.3), for every \(n \geq 1\) we have

\[
(6.10) \quad \frac{1}{C_1} \int_0^{\|w^n\|} g(t)dt \geq \int_0^{\|w^n\|} \frac{dt}{\nu(t)} > \frac{n}{\|\psi\|_{B_w(B_X)}}.
\]

Therefore, \(\sup_{n \geq 1} \int_0^{\|w^n\|} g(t)dt = \infty\), hence,

\[
\liminf_{n \to \infty} \|\varphi(z^n)\| = \liminf_{n \to \infty} \|w^n\| > 0.
\]
Then, we can consider the sequence \( \{ \gamma_{w^n} \}_{n \geq 1} \) defined by (5.7). By (5.10) and (5.4), for every \( n \geq 1 \), we have

\[
|\gamma_{w^n}(\varphi(z^n))| = \left| \int_0^{(w^n,w^n)} g(t) dt \right| \left| \int_0^{(w^n,w^n)} g(t) dt \right|
\geq \left| \int_0^{(w^n,w^n)} g(t) dt \right| \left| \int_0^{(w^n,w^n)} g(t) dt \right|
\geq \left| \int_0^{(w^n,w^n)} g(t) dt \right| C_3 \int_0^{(w^n,w^n)} \frac{dt}{\nu(t)}
\geq C_3 \int_0^{(w^n,w^n)} \frac{dt}{\nu(t)}.
\]

(6.11)

Then, since \( W_{\psi,\varphi}^0 \) is bounded and \( M_{\psi, \varphi} < \infty \), for every \( n \geq 1 \) we have

\[ n < \mu(z^n) ||R\psi(z^n)|| \int_0^{(w^n,w^n)} \frac{dt}{\nu(t)} \]

\[ \leq \frac{C_4}{C_1} \mu(z^n) ||R\psi(z^n)|| |\gamma_{w^n}(\varphi(z^n))| \]

\[ \leq \frac{C_4}{C_1} \mu(z^n) ||RW_{\psi,\varphi}^{g_n}(z^n)|| + \frac{\mu(z^n)||\psi(z^n)||||R\varphi(z^n)||}{\nu(\varphi(z^n))} ||\gamma_{w^n}||_{B_{\nu}(B_X)} \]

\[ \leq \frac{C_4}{C_1} \left( ||W_{\psi,\varphi}^{g_n}||_{B_{\nu}(X)} + \frac{\mu(z^n)||\psi(z^n)||||R\varphi(z^n)||}{\nu(\varphi(z^n))} ||\gamma_{w^n}||_{B_{\nu}(B_X)} \right) \]

\[ \leq \frac{C_4}{C_1} \left( ||W_{\psi,\varphi}^{g_n}|| \gamma_{w^n}||_{B_{\nu}(B_X)} + \frac{\mu(z^n)||\psi(z^n)||||R\varphi(z^n)||}{\nu(\varphi(z^n))} ||\gamma_{w^n}||_{B_{\nu}(B_X)} \right) < \infty. \]

It is impossible. Therefore, \( M_{R\psi,\varphi} < \infty \).

Finally, combining the above estimates gives (5.3). The proof of Theorem is completed. \( \square \)

Next, we will touch the characterizations for the boundedness of the operators from \( B_{\varphi, 0}(B_X) \) to \( B_{\mu, 0}(B_X) \).

**Theorem 6.2.** Let \( \psi \in H(B_X) \) and \( \varphi \in S(B_X) \). Let \( \mu, \nu \) be normal weights on \( B_X \). Then the following are equivalent:

1. \( \psi[m], \psi[m] \cdot \varphi[m] \in B_{\mu[m], 0}(B_m), M_{\mu[m]}^{\varphi[m]} < \infty, M_{\psi, R\varphi}^{\mu[m]} < \infty \) for some \( m \geq 2 \) and for every \( j \in \Gamma \);
2. \( \psi[F], \psi[F] \cdot \varphi[F] \in B_{\mu[F], 0}(B_F), M_{\mu[F]}^{\varphi[F]} < \infty, M_{\psi, R\varphi}^{\mu[F]} < \infty \) for every \( F \subset \Gamma \) finite and for every \( j \in \Gamma \);
3. \( \psi, \psi \cdot \varphi_j \in B_{\mu, 0}(B_X), M_{R\psi, \varphi}^{(k)} < \infty, M_{\psi, R\varphi}^{(k)} < \infty \) for every \( k \geq 1 \) and for every \( j \in \Gamma \);
4. \( \psi, \psi \cdot \varphi_j \in B_{\mu, 0}(B_X), M_{R\psi, \varphi}^{(k)} < \infty, M_{\psi, R\varphi}^{(k)} < \infty \) for every \( k \geq 1 \) and for every \( j \in \Gamma \);
5. \( W_{\psi, \varphi}^0 : B_{\varphi, 0}(B_X) \to B_{\mu, 0}(B_X) \) is bounded.

In this case, the asymptotic relation for \( \|W_{\psi, \varphi}^0\| \) is as (5.3).

**Proof.** It is obvious that (4) \( \Rightarrow (2) \Rightarrow (1) \) and (4) \( \Rightarrow (3) \).
(1) ⇒ (5): First, we note that if \( \psi[m], \psi'[m] \cdot \varphi_j[m] \in B_{\mu[m],0}(B_m) \) it follows from the estimate
\[
\begin{align*}
\mu[m](z_{x})|\psi[m](z_{x})|\| R\varphi_j[m](z_{x}) \| \\
\leq \mu[m](z)|R(\psi \cdot \varphi_j[m])(z_{x})| + \mu[m](z_{x})|\| R\psi(z_{x})\||\| \varphi_j[m](z_{x}) \| \\
\forall z_{x} \in \mathbb{B}_{m}, \forall x \in OS_{m}
\end{align*}
\]
that
\[
\lim_{\|z_{x}\| \to 1} \mu[m](z_{x})|\psi[m](z_{x})|\| R\varphi_j[m](z_{x}) \| = 0,
\]
hence,
\[
(6.13) \quad \lim_{\|z_{x}\| \to 1} \mu[m](z_{x})|\psi[m](z_{x})|\| R\varphi[m](k)(z) \| = 0 \quad \forall k \geq 1,
\]
where \( \varphi[k](\cdot) := (\varphi_1[m](\cdot), \ldots, \varphi_k[m](\cdot)) \).

Assume that \( M_{\mu}[m] < \infty, M_{\psi}[m] < \infty, \psi[m], \psi'[m] \cdot \varphi[m] \in B_{\mu[m],0} \) for some \( m \geq 2 \) and for every \( j \in \Gamma \). By Theorem [6.1] \( W_{\psi,\varphi} : B_{\psi,0}(B_X) \to B_{\mu}(B_X) \) is bounded. It suffices to show that \( W_{\psi,\varphi}^0 f \in B_{\mu,0}(B_X) \) for every \( f \in B_{\psi,0}(B_X) \).

Let \( f \in B_{\psi,0}(B_X) \) be arbitrarily fixed. Let \( \varepsilon > 0 \) and \( k \geq 1 \) be fixed. Then there exists \( r_0 \in (1/2, 1) \) such that
\[
(6.14) \quad \nu(z)|\| Rf(w) \| < \frac{\varepsilon}{6(M_{\mu}[\varphi] + M_{\psi}[\varphi])}, \quad \|w\| \geq r_0.
\]
Put \( \varrho := r_0 + \frac{1-r_0}{2} \). By (4.20) we have
\[
(6.15) \quad K := \sup_{\|w\| \leq \varrho} |f(w)| < \infty,
\]
\[
N := \sup_{\|w\| \leq \varrho} \| Rf(w) \| = \sup_{\|w\| \leq \varrho} \frac{\nu(w)}{\|Rf(w)\|} \leq \frac{\|B_\psi(B_X)\|}{\nu(\varrho)} < \infty \quad w \in B_X.
\]

Since \( \psi[m] \in B_{\mu[m],0}(B_m) \) and (6.13) we can find \( \theta \in (0, 1) \) such that for every \( x \in OS_m \)
\[
(6.16) \quad \mu[m](z_{x})|\psi[m](z_{x})|\| R\varphi[m](k)(z) \| < \frac{\varepsilon}{3(K + N)},
\]
whenever \( \theta < \|z_{x}\| < 1 \).

For \( \theta < \|z_{x}\| < 1 \) we consider two cases:

- The case \( \|w[m]\| := \|\varphi[k](z_{x})\| > r_0 \): Let \( \tilde{w}[m] = r_0 \|w[m]\| \). We have
\[
|f(\varphi[m](z_{x})) - f(\tilde{w}[m])| = |f(w[m]) - f(\tilde{w}[m])| \leq \int_{r_0/\|w[m]\|}^{1} \left| \frac{Rf(tw[m])}{t} \right| dt
\]
\[
\leq \frac{\|w[m]\|}{r_0} \int_{r_0/\|w[m]\|}^{1} \left| Rf(tw[m]) \right| dt
\]
\[
\leq \frac{\varepsilon \|w[m]\|}{6(M_{\mu}[\varphi] + M_{\psi}[\varphi])r_0} \int_{r_0/\|w[m]\|}^{1} \frac{1}{\nu(t\|w[m]\|)} dt
\]
\[
\leq \frac{\varepsilon}{3(M_{\mu}[\varphi] + M_{\psi}[\varphi])} \int_{r_0}^{\|w[m]\|} \frac{1}{\nu(t)} dt.
\]
Then, by (6.14) we have
\[
\mu^{[m]}(z_x) R(\psi^{[m]}(f \circ \varphi_k^{[m]})(z_x))
\]
\[
\leq \mu^{[m]}(z_x) R(\psi^{[m]}(z_x)) |f(\varphi_k^{[m]}(z_x))| + \mu^{[m]}(z_x) |\nabla \varphi_k^{[m]}(z_x) f(\varphi_k^{[m]}(z_x))| R(\varphi_k^{[m]})(z_x)
\]
\[
\leq \mu^{[m]}(z_x) R(\psi^{[m]}(z_x)) |f(\varphi_k^{[m]}(z_x))| - f(\tilde{w}^{[m]}) + \mu^{[m]}(z_x) R(\varphi_k^{[m]}(z_x)) |f(\tilde{w}^{[m]})|
\]
\[
+ \mu^{[m]}(z_x) |\nabla \varphi_k^{[m]}(z_x) f(\varphi_k^{[m]}(z_x))| R(\varphi_k^{[m]})(z_x)
\]
\[
\leq M_{R\psi,\varphi} \frac{\varepsilon}{3(M_{R\psi,\varphi} + M_{\psi, R\varphi})} + K \frac{\varepsilon}{3(K + N)}
\]
\[
+ \sup_{\varphi_k^{[m]}(w) > \nu} \nu(\varphi_k^{[m]}(w)) |\nabla \varphi_k^{[m]}(w) f(\varphi_k^{[m]}(w))| \sup_{w \in B_n} \mu^{[m]}(w) |\psi^{[m]}(w)| R(\varphi_k^{[m]}(w))
\]
\[
= \frac{2\varepsilon}{3} + M_{\psi, R\varphi} \frac{\varepsilon}{6(M_{R\psi,\varphi} + M_{\psi, R\varphi})} < \varepsilon.
\]

• The case \(|\varphi_k^{[m]}(z_x)| \leq r_0\): By (6.15) we have
\[
\mu^{[m]}(z_x) R(\psi^{[m]}(f \circ \varphi_k^{[m]})(z_x))
\]
\[
\leq \mu^{[m]}(z_x) |\psi^{[m]}(z_x)| R(\varphi_k^{[m]}(z_x)) |\nabla \varphi_k^{[m]}(z_x) f(\varphi_k^{[m]}(z_x))| R(\varphi_k^{[m]})(z_x)
\]
\[
< N \frac{\varepsilon}{3(K + N)} + K \frac{\varepsilon}{3(K + N)} < \varepsilon.
\]

Now, since \(\varphi_k^{[m]}(z_x) \rightarrow \varphi^{[m]}(z_x)\) as \(k \rightarrow \infty\), combining two cases, we obtain
\[
\mu^{[m]}(z_x) R(\psi^{[m]}(f \circ \varphi^{[m]}(z_x))) < \varepsilon \quad \forall x \in OS_m, \theta < \|z_x\| < 1.
\]

Finally, fix \(z \in B_X, \theta < \|z\| < 1\). Consider \(x = (\frac{z}{\|z\|}, x_2, \ldots, x_m) \in OS_m\) and put \(z_k := ((z, 0, \ldots, 0) \in \mathbb{C}^m\). Then \(\theta < \|z_k\| = \|z\| < 1\) and by as (6.10) we obtain Consequently,
\[
\mu(z) |R(\psi(f \circ \varphi(z)))| = \mu^{[m]}(z_x) R(\psi^{[m]}(f \circ \varphi^{[m]}(z_x))) < \varepsilon.
\]

hence, \(W_{\psi, \varphi}^{0,0}(f) \in B_{\mu, 0}(B_X)\).

(3) \(\Rightarrow\) (5): Use an argument analogous and an estimate to the previous one.

(5) \(\Rightarrow\) (4): Assume that \(W_{\psi, \varphi}^{0,0}\) is bounded. As in the proof of (6) \(\Rightarrow\) (4) in Theorem 6.1 we obtain that \(\psi \in B_{\mu}(B_X)\) and the estimates \(M_{R\psi, \varphi} < \infty\), \(M_{\psi, R\varphi} < \infty\) because the test function \(\beta_{\psi}\) given by \(\|\psi \circ \varphi\|_{S(B_X)} = \|\psi\|_{S(B_X)}\) belongs to \(B_{\psi, 0}(B_X)\).

It remains to check that \(\psi, \psi \circ \varphi \in B_{\mu, 0}(B_X)\) for every \(j \in \Gamma\). It is clear because it is easy to see that \(\psi = W_{\theta, \psi}^{0,0}(1), \psi \circ \varphi = W_{\theta, \psi}^{0,0}(\varphi_j)\), where \(\varphi_j \in B_{\psi, 0}(B_X)\) given by \(\varphi_j(z) := z_j\) for every \(z \in B_X\).

\[\square\]

Now we investigate the compactness of weighted composition operators \(W_{\psi, \varphi}\).

**Theorem 6.3.** Let \(\psi \in H(B_X), \varphi \in S(B_X)\) and \(\mu, \nu\) be normal weights on \(B_X\) such that \(\int_0^1 \frac{d\mu}{|t_\mu|} = \infty\). Then

(A) The following are equivalent:
Proof. First we prove (B).

Under the additional assumption that there exists $m \geq 2$ such that

\begin{equation}
B[\varphi^m, r] := \{\varphi^m(y) : \|\varphi^m(y)\| < r, y \in \mathbb{B}_m\}
\end{equation}

is relatively compact for every $0 \leq r < 1$.

The assertions (2), (3) and following are equivalent:

(4) $\varphi^m \to 0$, $\psi^m \to 0$ as $\|\varphi^m(y)\| \to 1$;
(5) $\psi^m \to 0$ as $\|\varphi^m(y)\| \to 1$ for every $F \subset \Gamma$ finite;
(6) $\|\varphi^m(y)\| \to 1$.

Now, for every $n \geq n_0$ and $r_0 < \|w^m\| < 1$, with noting that

$$
\hat{w}^m := r_0 \frac{w^m}{\|w^m\|} \in B[\varphi^m, r_0],
$$

by (5.3) and (6.18), as in the proof (1) $\Rightarrow$ (5) in Theorem 6.1 we have
\[
\mu^{[m]}(z_x) \| R(W_{\psi, \varphi} f_n)(z_x) \|
\leq \mu^{[m]}(z_x) \| R\varphi^{[m]}(z_x) \| f_n(\varphi^{[m]}(z_x)) \|
+ \mu^{[m]}(z_x) \| \psi^{[m]}(z_x) \| \nabla \varphi^{[m]}(z_x) f_n(\varphi^{[m]}(z_x)) \| \| R\varphi^{[m]}(z_x) \|
\leq \mu^{[m]}(z_x) \| R\varphi^{[m]}(z_x) \| (f_n(\varphi^{[m]}(z_x)) - f_n(\tilde{w}^{[m]}))
+ \mu^{[m]}(z_x) \| R\varphi^{[m]}(z_x) \| f_n(\tilde{w}^{[m]})
+ \mu^{[m]}(z_x) \| \psi^{[m]}(z_x) \| \nabla \varphi^{[m]}(z_x) f_n(\varphi^{[m]}(z_x)) \| \| R\varphi^{[m]}(z_x) \|
\leq \mu^{[m]}(z_x) \| R\varphi^{[m]}(z_x) \| \| w^{[m]} \| \int_{r_0/\| w^{[m]} \|}^{1} \| Rf(tw^{[m]}) \| dt
+ \| \psi \|_B_{\mu(B_X)} \frac{\varepsilon}{3K} + \frac{\varepsilon}{3K} \| f_n \|_B_{\mu(B_X)}
\leq \mu^{[m]}(z_x) \| R\varphi^{[m]}(z_x) \| \frac{2}{C_2} \int_{r_0/\| w^{[m]} \|}^{1} \frac{1}{\nu(t\| w^{[m]} \|)} dt + \frac{2\varepsilon}{3}
\leq \mu^{[m]}(z_x) \| R\varphi^{[m]}(z_x) \| \frac{2}{C_2} \int_{r_0}^{\| w^{[m]} \|} \frac{dt}{\nu^{[m]}(t)} + \frac{2\varepsilon}{3}
< \frac{\varepsilon}{6/C_2 C_2} + \frac{2\varepsilon}{3} = \varepsilon.
\]

(6.19)

On the other hand, since \( \{ f_n \}_{n \geq 1} \) converges to 0 uniformly on compact subsets of \( B_X \), by Cauchy integral formula and (6.17), it is clear that
\[
\sup_{y \in B(\varphi^{[m]}), r} \| \nabla \varphi f_n(y) \| \to 0 \quad \text{as } n \to \infty.
\]

Then, in the case \( \| w^{[m]} \| \leq r_0 \) with the estimate as above we obtain
\[
\mu^{[m]}(z_x) \| R(W_{\psi, \varphi} f_n)(z_x) \| < \varepsilon \quad \text{for every } n \geq n_0.
\]

(6.20)

Finally, with the note that the estimates (6.19), (6.20) are independent of \( x \in O_{\mu_m} \), we obtain
\[
\| W_{\psi, \varphi} f_n \|_{B_{\mu}(B_X)} = \sup_{x \in O_{\mu_m}} \| W_{\psi, \varphi}^{[m]}((f_n)_x) \|_{B_{\mu}(B_m)} \to 0 \quad \text{as } n \to \infty.
\]

Hence, it implies from Proposition 5.6 that \( W_{\psi, \varphi} \) is compact.

(3) \( \Rightarrow \) (6): Suppose \( W_{\psi, \varphi}^{[0]} \) is compact. Then clearly, \( W_{\psi, \varphi}^{[0]} \) is bounded.

Firstly, assume that (6.14) \( \not\Rightarrow 0 \) as \( \| \varphi(z) \| \to 1, z \in B_X \). Then we can take \( \varepsilon_0 > 0 \) and a sequence \( \{ z^n \}_{n \geq 1} \) in \( B_X \) such that \( \| w^n \| = \| \varphi(z^n) \| \to 1 \) but
\[
\frac{\mu(z^n) |\psi(z^n)| \| R \varphi(z^n) \|}{\nu(\varphi(z^n))} \geq \varepsilon_0 \quad \text{for every } n = 1, 2, \ldots
\]

Consider the sequence \( \{ \gamma_{w^n} \}_{n \geq 1} \) defined by (5.7). By Proposition 5.3, this sequence is bounded in \( B_{\mu}(B_X) \) and converges to 0 uniformly on compact subsets of \( B_X \). By an argument similar to the proof (6) \( \Rightarrow \) (4) of Theorem 6.1, we get the one as the estimate
Theorem 6.4. Let \( \psi \in H(B_X) \) and \( \varphi \in S(B_X) \). Assume that \( W_{\psi,\varphi} : B_\nu(B_X) \to B_\mu(B_X) \) is compact. Then
\[
\varphi(r B_X) \text{ is relatively compact for every } 0 \leq r < 1.
\]
Proof. For every $z \in B_X$, consider the function $\delta_z$ given by $\delta_z(f) = f(z)$ for every $f \in B_{B}(B_X)$. By (4.20), it is clear that $\delta_z \in (B_{B}(B_X))'$. Moreover, we have
\begin{equation}
\frac{1}{2} \|z - w\| \leq \|\delta_z - \delta_w\| \quad \forall z, w \in B_X.
\end{equation}
Indeed, it is easy to check by direct calculation that
\[
\frac{1}{2} \|z - w\| \leq \sqrt{1 - \frac{(1 - \|z\|^2)(1 - \|w\|^2)}{1 - \langle z, w \rangle^2}} = \varrho_X(z, w)
\]
where $\varrho_X$ is the pseudohyperbolic metric in $B_X$ (see [GR, p.99]). On the other hand, we also have
\[
\varrho_X(z, w) = \sup \{\varrho(f(z), f(w)) : f \in H^\infty(B_X) \text{ with } \|f\|_\infty \leq 1\}
\]
(see (3.4) in [BGM]), where $\varrho(x, y) = \left| \frac{x - y}{1 - \langle x, y \rangle} \right|$ is holomorphic from $\mathbb{B}_1$ into $\mathbb{B}_1$ and $f(z) - f(w) \to 0$, it follows from Schwarz’s lemma that $\varrho(f(z), f(w)) \leq |f(z) - f(w)|$ for every $z, w \in B_X$. Consequently,
\[
\varrho_X(z, w) \leq \sup \{|f(z) - f(w)| : f \in H^\infty(B_X) \text{ with } \|f\|_\infty \leq 1\}
\]
\[
\leq \sup \{|\delta_z(f) - \delta_w(f)| : f \in H^\infty(B_X) \text{ with } \|f\|_\infty \leq 1\}
\]
\[
= \|\delta_z - \delta_w\|.
\]
Hence, (6.23) is proved.

For $0 < r < 1$, the set $V_r := \{z : \|z\| \leq r\} \subset (B_{B}(B_X))'$ is bounded. Then, by the compactness of $W_{\psi, \varphi}$ the set
\[
(W_{\psi, \varphi})^*(V_r) = \{\psi(z)\delta_{\varphi(z)} : \|z\| \leq r\}
\]
is relatively compact in $(H^\infty(B_X))'$.

It should be noted that, for every subset $K$ of the dual of a Banach space $E$ and every bounded subset $D \subset \mathbb{C}$, if the set $\{t\eta : t \in D, \eta \in A\}$ is relatively compact in $E$ then $A \subset E'$ is relatively compact. With this fact in mind, since the set $\{\psi(z) : \|z\| \leq r\}$ is bounded, the set $\{\delta_z, \|z\| \leq r\}$ is relatively compact. Then, it follows from the inequality (6.23) that $\varphi(rB_X)$ is relatively compact.

We introduce below is an example which shows that (6.22) does not imply (6.17).

**Example 6.1.** Let $\{e_j\}_{j \geq 1}$ be an orthonormal sequence in a Hilbert space $X$. Consider the function $\varphi \in S(B_X)$ given by
\[
\varphi(z) := \sum_{n=1}^{\infty} \langle z, e_n \rangle^n e_n \quad \forall z \in B_X.
\]
It is easy to check that $\varphi(rB_X)$ is relatively compact for every $0 < r < 1$.

Now we show that $B[\varphi, \frac{1}{2}]$ is not relatively compact. Consider the sequence $\{z_k\}_{k \geq 1} \subset B_X$ given by
\[
z_k = \frac{1}{\sqrt{4}} e_k \quad \forall k \geq 1.
\]
It is obvious $\|\varphi(z_k)\| < \frac{1}{2}$ for every $k \geq 1$. Then for every $k \geq 1$ and $s > 1$ we have
\[
\|\varphi(z_k) - \varphi(z_{k+s})\| = \frac{\sqrt{2}}{4}.
\]
Thus, we get the desired claim.
Remark 6.2. Under the additional condition that \( \varphi(0) = 0 \), the limits \( 6.1a \to 0 \), \( 6.1b \to 0 \) in Theorem 6.3 hold as \( \|z_m\| \to 1 \). Indeed, in a more general framework, it suffices to show \( \|\varphi(z)\| \leq \|z\| \) for every \( z \in B_X \). That means we have to give an infinite version of Schwarz’s lemma.

For each \( z \in X, z \neq 0 \) and \( w \in \overline{B_X} \), applying classical Schwarz’s lemma to the functions \( \phi_{z,w} : B_1 \to B_1 \) given by

\[
\phi_{z,w}(t) := \langle \varphi(tz/\|z\|), w \rangle \quad \forall t \in B_1,
\]

we have

\[
|\phi_{z,w}(t)| \leq |t|.
\]

Then, choosing \( t = \|z\| \) and \( w = \frac{\varphi(z)}{\|\varphi(z)\|} \) we get the desired inequality.

Corollary 6.5. Assume that \( \sup_{z \in X} \|\varphi(z)\| < \infty \). Then \( W_{\psi,\varphi}, W^0_{\psi,\varphi} \) are compact if and only if \( \varphi(B_X) \) is relatively compact.

Indeed, by the hypotheses, \( 6.17, 6.22 \) and the assumption (6) in Theorem 6.3 always hold.

Theorem 6.6. Let \( \psi \in H(B_X), \varphi \in S(B_X) \) and \( \mu, \nu \) be normal weights on \( B_X \) such that \( \int_0^1 \frac{dt}{\mu(t)} < \infty \). Then

(A) The following are equivalent:

1. \( \psi \in B_{\mu}(B_X) \) and \( 6.3 \to 0 \) as \( \|\varphi_{(k)}(y)\| \to 1 \) for every \( k \geq 1 \);
2. \( W_{\psi,\varphi} : B_{\mu}(B_X) \to B_{\mu}(B_X) \) is compact;
3. \( W^0_{\psi,\varphi} : B_{\nu,\mu}(B_X) \to B_{\mu}(B_X) \) is compact.

(B) Under the additional assumption that there exists \( m \geq 2 \) such that \( 6.17 \) holds, the assertions (2), (3) and following are equivalent:

4. \( \psi^{[m]} \in B_{\mu,\nu}^{[m]}(B_m) \) and \( 6.1a \to 0 \) as \( \|\varphi^{[m]}(y)\| \to 1 \);
5. \( \psi^{[F]} \in B_{\mu,\nu}^{[F]}(B_{|F|}) \) and \( 6.2a \to 0 \) as \( \|\varphi^{[F]}(y)\| \to 1 \) for every \( F \subset \Gamma \) finite;
6. \( \psi \in B_{\mu}(B_X) \) and \( 6.4d \to 0 \) as \( \|\varphi(y)\| \to 1 \).

Proof. As Theorem 6.3 it suffices to prove (4) \( \Rightarrow \) (2) and (3) \( \Rightarrow \) (6).

(4) \( \Rightarrow \) (2): By Theorem 6.3 it suffices to show that \( 6.1a \to 0 \) as \( \|\varphi^{[m]}(y)\| \to 1 \), \( y \in B_m \). It is easy to prove this fact from the assumptions that \( \psi^{[m]} \in B_{\mu,\nu}^{[m]}(B_m) \) and \( \int_0^1 \frac{dt}{\mu(t)} < \infty \).

(3) \( \Rightarrow \) (6): Since \( W^0_{\psi,\varphi} \) is bounded, as in Theorem 6.1 we have \( \psi \in B_{\mu}(B_X) \).

Now, we show that \( 6.4a \to 0 \) as \( \|\varphi(y)\| \to 1 \) for \( y \in B_X \). Otherwise, we would have a sequence \( \{z^n\}_{n \geq 1} \subset B_X \) and some positive constant \( \varepsilon_0 > 0 \) such that such that \( \|w^n\| := \|\varphi(z^n)\| \to 1 \) but

\[
\frac{\mu(z^n)\|\varphi(z^n)\|\|R\varphi(z^n)\|}{\nu(\varphi(z^n))} \geq \varepsilon_0.
\]

We may assume that

\[
6.24 \quad 1 - \frac{1}{n^2} < \|w^n\| < 1 \quad \text{and} \quad \lim_{n \to \infty} \|w^n\| = \|w^n\| = 1.
\]
Consider the sequence \( \{ \theta_w \}_n \) which is bounded in \( \mathcal{B}_o(B_X) \) and converges to 0 uniformly on compact subsets of \( B_X \) (see Proposition 5.3). Then

\[
\| W^0_{\psi, \varphi} \theta_w \|_{\mathcal{B}_o(B_X)} + \sup_{z \in B_n} \mu(z) \| R\psi(z) \| \| \theta_w(\varphi(z)) \| \\
\geq \sup_{z \in B_n} \mu(z) \| \psi(z) \| \| R\psi(z) \| \| \varphi(z) \| \\
\geq \mu(z^n) \| \psi(z^n) \| \| R\varphi(z^n) \| g(||w^n||^{n+2}) - g(||w^n||^2) \\
\geq \mu(z^n) \| \psi(z^n) \| \| R\varphi(z^n) \| \nu(\varphi(z^n)) g(||w^n||^{n+2}) - g(||w^n||^2) \\
\geq \varepsilon_0 \left[ \nu(\|w^n\|^2) g(||w^n||^2) \frac{\nu(||w^n||)}{\nu(||w^n||^{n+2})} - \nu(||w^n||^{n+2}) g(||w^n||^{n+2} \frac{\nu(||w^n||)}{\nu(||w^n||^{n+2})} \right].
\]

On the other hand, it follows from (6.34) that \( \| w^n \|^{n+2} \to 1 \) as \( n \to \infty \). Then

\[
0 \leq \limsup_{n \to \infty} \frac{\nu(\|w^n\|^{n+2})}{\nu(||w^n||^{n+2})} = \limsup_{n \to \infty} \frac{\nu(\|w^n\|^{n+2})}{1} \leq \limsup_{n \to \infty} \frac{1}{(2 + n)^a} = 0.
\]

Therefore, it follows from (6.8) that

\[
\liminf_{n \to \infty} \left[ \| W^0_{\psi, \varphi} \theta_w \|_{\mathcal{B}_o(B_X)} + \mu(z) \| R\psi(z) \| \| \theta_w(\varphi(z)) \| \right] \\
\geq \varepsilon_0 \left[ \liminf_{n \to \infty} \nu(\|w^n\|^2) g(||w^n||^2) \frac{\nu(||w^n||)}{\nu(||w^n||^{n+2})} - \limsup_{n \to \infty} \nu(||w^n||^{n+2}) g(||w^n||^{n+2} \frac{\nu(||w^n||)}{\nu(||w^n||^{n+2})} \right] \\
\geq \varepsilon_0 \inf_{t \in [0,1]} \nu(t) g(t) - 0 > 0.
\]

This contradicts \( \| W^0_{\psi, \varphi} \theta_w \|_{\mathcal{B}_o(B_X)} \to 0 \) and \( \theta_w(\varphi^{[m]}(z^n)) \to 0 \) as \( n \to \infty \).

This concludes the proof. \( \square \)

In the same way as in the proof of Theorems 6.3, 6.6, by using Theorem 6.2 we obtain the following results on the compactness of the operator \( W_{\psi, \varphi} : \mathcal{B}_{\psi, \varphi}(B_X) \to \mathcal{B}_{\psi, \varphi}(B_X) \).

**Theorem 6.7.** Let \( \psi \in H(B_X) \), \( \varphi \in S(B_X) \), and \( \mu, \nu \) be normal weights on \( B_X \) such that \( \int_0^1 \frac{dt}{\nu(t)} = \infty \). Then

(A) The following are equivalent:

1. \( \psi, \varphi \in \mathcal{B}_{\psi, \varphi}(B_X) \), \( \theta_w \to 0 \), \( 6.3a \to 0 \) as \( \| \varphi_{(j)}(y) \| \to 1 \) for every \( j \in \Gamma \) and for every \( k \geq 1 \);
2. \( W_{\psi, \varphi} : \mathcal{B}_{\psi, \varphi}(B_X) \to \mathcal{B}_{\psi, \varphi}(B_X) \) is compact.

(B) Under the additional assumption that there exists \( m \geq 2 \) such that (6.17) holds, the assertion (2) and following are equivalent:

1. \( \psi^{[m]}, \varphi^{[m]} \in \mathcal{B}_{\psi, \varphi}(B_m) \), \( \theta_w \to 0 \), \( 6.1a \to 0 \) as \( \| \varphi^{[m]}(y) \| \to 1 \) for every \( j \in \Gamma \);
(4) \( \psi[F], \psi[F] \cdot \varphi_j^{[F]} \in B_{\psi[F]}(B_{[F]}), \) \( (6.2a) \to 0, \) \( (6.2b) \to 0 \) as \( \|\varphi[F](y)\| \to 1 \) for every \( F \subset \Gamma \) finite and for every \( j \in \Gamma; \)

(5) \( \psi, \psi \cdot \varphi_j \in B_{\mu}(B_X), \) \( (6.3a) \to 0, \) \( (6.3b) \to 0 \) as \( \|\varphi(y)\| \to 1. \)

**Theorem 6.8.** Let \( \psi \in H(B_X), \varphi \in S(B_X), \) and \( \mu, \nu \) be normal weights on \( B_X \) such that \( \int_0^1 \frac{dt}{\nu(t)} < \infty. \) Then

(A) The following are equivalent:

1. \( \psi, \psi \cdot \varphi_j \in B_{\mu,0}(B_X), \) \( (6.3a) \to 0 \) as \( \|\varphi(k)(y)\| \to 1 \) for every \( j \in \Gamma \) and for every \( k \geq 1; \)
2. \( W_{\psi,0}^0 : B_{\nu,0}(B_X) \to B_{\mu,0}(B_X) \) is compact.

(B) Under the additional assumption that there exists \( m \geq 2 \) such that \( (6.17) \) holds, the assertion (2) and following are equivalent:

3. \( \psi[m], \psi[m] \cdot \varphi[m] \in B_{\mu[m]}(B_m), \) \( (6.17) \to 0 \) as \( \|\varphi[m](y)\| \to 1 \) for every \( j \in \Gamma; \)
4. \( \psi[F], \psi[F] \cdot \varphi_j^{[F]} \in B_{\psi[F]}(B_{[F]}), \) \( (6.2a) \to 0 \) as \( \|\varphi[F](y)\| \to 1 \) for every \( F \subset \Gamma \) finite and for every \( j \in \Gamma; \)
5. \( \psi, \psi \cdot \varphi_j \in B_{\mu}(B_X), \) \( (6.3a) \to 0 \) as \( \|\varphi(y)\| \to 1. \)

To finish this paper, combining Theorem 6.1 with the main results in Section 6, we state the characterizations for the boundedness and the compactness of the operators \( W_{\psi,\varphi}, W_{\psi,\varphi}^0, W_{\psi,\varphi}^{0,0}. \)

**Theorem 6.9.** Let \( W \subset Y' \) be a separating subspace. Let \( \psi \in H(B_X) \) and \( \varphi \in S(B_X). \) Let \( \mu, \nu \) be normal weights on \( B_X. \) Then

(1) The following are equivalent:

- \( W_{\psi,\varphi} \) is bounded;
- \( W_{\psi,\varphi}^0 \) is bounded;
- One of the assertions (1)-(4) in Theorem 6.4.

(2) \( W_{\psi,\varphi}^{0,0}, \) is bound if and only if the assumptions (1)-(4) in Theorem 6.2 holds.

**Theorem 6.10.** Let \( W \subset Y' \) be a almost norming subspace. Let \( \psi \in H(B_X), \varphi \in S(B_X) \) and \( \mu, \nu \) be normal weights on \( B_X. \) Assume that one of the following is satisfied:

(a) \( \int_0^1 \frac{dt}{\nu(t)} = \infty \) and the assertion (1) in Theorem 6.3;
(b) \( \int_0^1 \frac{dt}{\nu(t)} = \infty \) and one of the assertions (4)-(6) in Theorem 6.3 with the additional assumption that \( (6.17) \) holds for some \( m \geq 2; \)
(c) \( \int_0^1 \frac{dt}{\nu(t)} < \infty \) and one of the assertions (1)-(3) in Theorem 6.6 with the additional assumption that \( (6.17) \) holds for some \( m \geq 2. \)

Then the following are equivalent:

1. \( W_{\psi,\varphi} \) is (resp. weakly) compact;
2. \( W_{\psi,\varphi}^0 \) is (resp. weakly) compact;
3. The identity map \( I_Y : Y \to Y \) is (resp. weakly) compact.

**Theorem 6.11.** Let \( W \subset Y' \) be a almost norming subspace. Let \( \psi \in H(B_X) \) and \( \varphi \in S(B_X). \) Let \( \mu, \nu \) be normal weights on \( B_X. \) Assume that one of the following is satisfied:

(a) The assertion (2) in Theorem 6.7
(b) One of the assertions (3)-(5) in Theorem 6.7 with the additional assumption that \( (6.17) \) holds for some \( m \geq 2. \)

Then the following are equivalent:
(1) $\widehat{W}^{0,0}_{W,W}$ is (resp. weakly) compact;
(2) The identity map $I_Y : Y \to Y$ is (resp. weakly) compact.

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