Photon Green’s function and the Casimir energy in a medium

Israel Klich

Departments of Applied Mathematics and Physics,
Technion - Israel Institute of Technology, Haifa 32000 Israel

Abstract

A new expansion is established for the Green’s function of the electromagnetic field in a medium with arbitrary $\epsilon$ and $\mu$. The obtained Born series are shown to consist of two types of interactions - the usual terms (denoted $P$) that appear in the Lifshitz theory combined with a new kind of terms (which we denote by $Q$) associated with the changes in the permeability of the medium. Within this framework the case of uniform velocity of light ($\epsilon\mu = \text{const}$) is studied.

We obtain expressions for the Casimir energy density and the first non-vanishing contribution is manipulated to a simplified form. For (arbitrary) spherically symmetric $\mu$ we obtain a simple expression for the electromagnetic energy density, and as an example we obtain from it the Casimir energy of a dielectric-diamagnetic ball. It seems that the technique presented can be applied to a variety of problems directly, without expanding the eigenmodes of the problem and using boundary condition considerations.

1 Introduction

A theory describing the fluctuations of electromagnetic fields in dielectric medium was developed long ago by Lifshitz, Pitaevskii and others, and is now classical textbook material [1, 2, 3]. Among other things the theory permits efficient and general treatment of the Van der Waals interaction and
calculation of Casimir energies. This theory was developed mainly for cases where the medium has arbitrary unhomogenous permittivity $\epsilon(r, \omega)$ while the material considered is not magnetoactive (with uniform $\mu$). In another classical work, Balian and Duplantier [4] derived an expansion for the Green’s function of the electromagnetic field in vacuum, but in the presence of a conducting boundary surface. The terms in this expansion are related to the geometry of the surface, and are given as explicit integrals, which differ from terms encountered in [1]. This expansion was then applied to the study of the statistical properties of the electromagnetic eigenmodes and led to an expansion for the Casimir energy under arbitrary conducting boundary conditions at zero and finite temperatures [5].

Although having no direct relation to the dielectric material problem, for the special case where $\epsilon \mu = \text{const}$ (this condition will be referred to as the uniform velocity of light - UVL in short) there is such a connection. This was first apparent in the spherical problem where the UVL scenario was first introduced in the present context by Brevik and Kolbendsvent [6]. The problem of a dielectric-diamagnetic ball in a medium under the UVL condition is evaluated by expanding either the Green’s function or the appropriate characteristic functions for the modes in terms of the diluteness parameter $\xi = (\mu - \mu')/(\mu + \mu')$ (where $\mu$ and $\mu'$ are the permeabilities inside and outside the ball respectively). It was shown that one can calculate the term proportional to $\xi^2$ exactly by means of an integral over the free Green’s function of the Helmholtz equation. While introduced as a formal trick, it coincides exactly with the two scattering result in [5] up to multiplication by $\xi^2$. This leads us to develop a general expansion for the Green’s function in dielectric-diamagnetic media.

The expansion that will be presented coincides with the approach of Lifshitz when $\mu$ is constant, and bears strong formal similarity to the multiple scattering expansion when considering dielectric bodies immersed in a medium in which UVL holds. It is interesting to point out that while the parameter encountered in many calculations in the literature of the Green’s function in the latter case is $\xi$, the natural parameter appearing in the present expansion is $\log[(1 - \xi)/(1 + \xi)]$.

We start by deriving a Born type series for the Green’s function in a medium with arbitrary $\epsilon$ and $\mu$. The series are shown to consist of two types of interactions - the usual terms (which will be denoted $P$) that appear in the Lifshitz theory combined with a new kind of terms (which we denote by $Q$) associated with the changes in the permeability of the medium. The $P$ terms are proportional to $(1 - \epsilon \mu)$ and thus, when UVL holds only $Q$ terms contribute to the Green’s function. Next we investigate the properties of the $Q$ terms are investigated and it is shown that in the UVL case the
magnetic and electric Green’s functions are closely related - a feature which is not present in general settings of dielectric medium. We proceed to use the Green’s function for the calculation of the Casimir energy of the medium. We study in detail the contribution of the second term in the expansion to the energy density of the electromagnetic field, and obtain an integral formula. For the case of an arbitrary radially dependent $\mu$ this contribution can be considerably simplified. As an example, we use the radial formula to calculate the Casimir energy of a dielectric-diamagnetic ball and show that the exact energy to order $\xi^2$ is easily recovered.

2 Born type expansion

The statistical properties of the electromagnetic field in a medium are described by the appropriate photonic Green’s function. The electromagnetic fields are derived from the electromagnetic potentials $A^\alpha, \alpha = 0, \ldots, 3$. (It is convenient to work in the gauge $A^0 = 0$.) The Green’s function is now defined by:

$$D_{ik}(X_1, X_2) = i < TA_i(X_1)A_k(X_2) >,$$

(1)

where $X_1, X_2$ are 4-vectors $(X^0_1, X^3_1)$ and $i, k = 1, \ldots, 3$. The angular brackets denote averaging with respect to the Gibbs distribution and $T$ denotes the time ordering operator. In general we will be interested in the retarded Green’s function, $D_{ik}$, which is defined by:

$$D_{ik}(X_1, X_2) = \begin{cases} 
    i < A_i(X_1)A_k(X_2) - A_k(X_2)A_i(X_1) > & t_1 < t_2 \\
    0 & \text{otherwise}
\end{cases}$$

(2)

This function, when Fourier transformed with respect to the time difference is actually the generalized susceptibility of the system \[2\]. It is known that in a medium with a given permittivity tensor $\epsilon_{ij}$ and permeability tensor $\mu_{ij}$, $D$ satisfies the equation:

$$\nabla \times \mu^{-1} \nabla \times D + \frac{\omega^2}{c^2} \epsilon D = -4\pi \hbar I \delta(r - r')$$

(3)

where $I$ is the 3-dimensional unit matrix. In what follows we shall work in units where $c = \hbar = 1$.

It will be convenient to introduce the following notation, which is suitable for some of the calculations ahead: Let $a$ be a 3-vector; we define the cross product operator with $a$ as $[a]_{ijk} = \epsilon_{ijk}a_j$. This operator acts on vector valued
functions by $[a] \vec{r} = \vec{a} \times \vec{r}$. Thus, for example, the curl operator is denoted $[\nabla]$. Furthermore we adopt the convention that the same notation is used for the operators and their kernels. With this notation the equation for the Green’s function (multiplied by $4\pi$) of dielectric media has the following form:

$$([\nabla] \mu^{-1} [\nabla] - \omega^2 \epsilon) D = -\mathbb{I}$$

(4)

In the following we assume that $\mu(r)$ is a scalar function (of course, in the general case both $\mu$ and $\epsilon$ are tensors).

To write this equation in a manner convenient for the following we multiply by $\mu$ and use the fact that $\mu[\nabla] \mu^{-1} = -[\nabla \log \mu] + [\nabla]$. Thus obtaining:

$$([\nabla]^2 - [\nabla \log \mu][\nabla] - \omega^2 \epsilon \mu) D = -\mu \mathbb{I}$$

(5)

The inverse of the operator $([\nabla]^2 - \omega^2)$ is well known and will be denoted $D_0$. The kernel of $D_0$, which is the Green’s function in vacuum, is given by the formula:

$$D_0(\omega; r, r') = -\left(\mathbb{I} + \frac{1}{\omega^2} \nabla \otimes \nabla\right) g_0(r - r'),$$

(6)

where $g_0$ is the Green’s function for the Helmholtz equation:

$$\triangle g_0 + \omega^2 g_0 = -\delta(r - r'),$$

(7)

given by:

$$g_0(r - r') = \frac{1}{4\pi|r - r'|} \exp^{(i\omega|r - r'|)}.$$  

(8)

Now we wish to use the known $D_0$ to express $D$ via a Born type series. Denote $P = \mathcal{P} + \mathcal{Q}$ where:

$$\mathcal{P} = \omega^2 \mathbb{I}(1 - \mu \epsilon)$$

(9)

and

$$\mathcal{Q} = -[\nabla \log(\mu)][\nabla]$$

(10)

\footnote{The convenience of this notation will arise from the necessity to evaluate expressions such as $\text{Tr}(a \times b \times c \times)$. Written in this form one has to keep track of the order of the multiplication because of the nonassociative nature of vector products. Working simply with matrices such as $[a]$ will prove to be very transparent.}
Then:
\[ D = ((I - PD_0)D_0^{-1})^{-1} \mu = D_0(I - PD_0)^{-1} \mu \]  \hspace{1cm} (11)

Thus we get the following formal series for \( D \):
\[ D = D_0\mu + D_0(\mathcal{P} + Q)D_0\mu + D_0(\mathcal{P} + Q)D_0(\mathcal{P} + Q)D_0\mu + \ldots \]  \hspace{1cm} (12)

Thus the series for \( D \) consist of two types of interaction. The \( \mathcal{P} \) terms are well known and can be found in \([1]\). However, to our knowledge the magnetoactive terms \( \mathcal{Q} \) are new in this context and are related to the changes in the permeability of the medium.

3 The case of uniform velocity of light

Uniform velocity of light in the medium implies \( \varepsilon \mu \equiv I \). This considerably simplifies the expansion (12), since it just leaves the \( \mathcal{Q} \) scattering terms. We are then left with the expansion:
\[ D = D_0\mu + D_0QD_0\mu + D_0QD_0QD_0\mu + \ldots \]  \hspace{1cm} (13)

In the following we are going to apply \( \mathcal{Q} \) to the free Green’s function \( D_0 \). For this purpose it is convenient to introduce
\[ G_0 \equiv [\nabla]D_0 = [\nabla g_0] \]  \hspace{1cm} (14)

Thus the action of \( \mathcal{Q} \) on \( D_0 \) is given by
\[ \mathcal{Q}D_0 = -[\nabla \log \mu]G_0 \]  \hspace{1cm} (15)

So that (13) can be written in the form:
\[ D = D_0\mu - D_0[\nabla \log \mu]G_0\mu + \ldots (-1)^{n-1}D_0([\nabla \log \mu]G_0)^n\mu - \ldots \]  \hspace{1cm} (16)

The terms in (16) can be written as explicit integrals using relation (A.3):
\[ D(r, r') = \mu(r')D_0 - \mu(r') \int D_0(r, r_1)g_0(r_1 - r')[\nabla \log \mu(r_1)][u_{1r'}] + \mu(r') \int D_0(r, r_1)g_0(r_1 - r_2)g_0(r_2 - r')[\nabla \log \mu(r_1)][u_{1r_2}][\nabla \log \mu(r_2)][u_{2r'}] - \ldots \]  \hspace{1cm} (17)

Convergence of the series relates of course on the nature of \( \mu(r) \). For specific cases, such as of a body immersed in dielectric-diamagnetic medium the expansion will have a natural parameter, as will be explained in section 4.

\(^3\)Taking the rotor of \( D_0 \) simplifies this function considerably since we are only left with first order derivatives of the scalar Helmholtz propagator: indeed \( \epsilon_{ijl}\partial_j\partial_l\equiv0 \)
3.1 The magnetic Green’s function and its relation to the electric Green’s function

Another quantity of interest is the magnetic Green’s function, given by:\[ \nabla \times D \times -\nabla \]. In this section we show that the expansion for the magnetic Green’s function is similar to that of the electric Green’s function, the main difference being alternating signs of the terms. This property is important for calculation of the Casimir energy, since it cancels odd terms in the expansion.

Indeed let us write\[ D = (I + D_0[\nabla \log \mu][\nabla])^{-1}D_0\mu = D_0(I + [\nabla \log \mu]G_0)^{-1}\mu \] (18)

Thus,\[ \nabla \times D \times -\nabla = \nabla(\nabla[D^t]^t = [\nabla][[\nabla]D_0(I + (D_0[\nabla \log \mu][\nabla])^t)^{-1})^t = [\nabla][I + D_0[\nabla \log \mu][\nabla])^{-1}(-D_0[\nabla \log \mu]\mu + [\nabla]D_0\mu) = (I + G_0[\nabla \log \mu])^{-1}(-G_0[\nabla \log \mu]\mu + \omega^2D_0\mu - \mu) = \omega^2(I + G_0[\nabla \log \mu])^{-1}D_0\mu - \mu\] (19)

(Here we used the elementary property \((1 + AB)^{-1}A = A(1 + BA)^{-1}\).)

We can now write the expansion of the magnetic Green’s function in the form:\[ \nabla \times D \times -\nabla = \] (20)

\[-\mu + \omega^2D_0\mu + \omega^2(D_0[\nabla \log \mu]G_0)^t\mu ....\omega^2(D_0([\nabla \log \mu]G_0)^n)^t\mu ....\]

Comparing this expansion with the one for \(D\) (Eq(16)), we see that the magnetic Green’s function has the same type of expansion as \(D^t\), but with constant signs, owing to \([\nabla \log \mu]\) being the only antisymmetric operator in the expansion. A similar symmetry was obtained in a different way for the Green’s function in vacuum with conducting boundary conditions in [4] and its implication on the distribution of eigenmodes is discussed. The implication of this symmetry to the calculation of the Casimir energy, is that in the series for the energy the main term one has to consider is the second order term in the expansion. This follows since the zeroth term is usually subtracted as the free problem solution while the first term cancels between the electric contribution and the magnetic contribution.

3.2 The symmetry when relaxing the UVL condition

What can be said when keeping, in addition, the \(P\) terms? Can we still have cancellation or similarity between the electric and magnetic Green’s
function? Cancellation in the spirit of the preceding section can be achieved by expanding as follows: Let $D_\epsilon$ be the solution of

$$([\nabla][\nabla] - \omega^2\epsilon\mu)D_\epsilon = -I$$

(21)

And accordingly $G_\epsilon = [\nabla]D_\epsilon$. Then it is possible to expand $D$ in terms of $D_\epsilon$. Using the fact that $D_\epsilon$ is a solution for a non magnetoactive problem and therefore has the symmetry property $D_{ik}(r, r') = D_{ki}(r', r)$. One may show that there is again a symmetry between the expansions of the magnetic and electric Green’s functions, similarly to the one discussed in the previous section.

4 Dielectric-diamagnetic body immersed in a medium

In this section we consider a body $B$ with constant dielectric properties $\epsilon, \mu$ immersed in a medium with different dielectric properties $\epsilon', \mu'$. Let $\partial B$ denote the boundary of the body, and let $s$ be coordinates on this boundary. In this case the $Q$ tensor, as a function of its first variable, will be supported on the boundary of the body (more generally, on $\partial B \times \mathbb{R}^3$) and has the following structure:

$$Q(s, r) = \log\left(\frac{\mu}{\mu'}\right)[n(s)]G_0(s, r)$$

(22)

where $n(s)$ is the unit normal to $\partial B$ at $s$.

When UVL holds, the expansion for $D$ is:

$$D(r, r') = \mu(r')D_0 - \log\left(\frac{\mu}{\mu'}\right)\int_{\partial B} ds_1 D_0(r, s_1)[n(s_1)]G_0(s_1, r') +$$

(23)

$$\log\left(\frac{\mu}{\mu'}\right)^2\int_{\partial B} ds_1 ds_2 D_0(r, s_1)[n(s_1)]G_0(s_1, s_2)[n(s_2)]G_0(s_2, r') - ....$$

This can be further simplified using (17):

$$D(r, r') = \mu(r')D_0 - \log\left(\frac{\mu}{\mu'}\right)\mu(r') \int_{\partial B} ds_1 D_0(r, s_1)g_0(s - r')[n_1][u_{sr'}] +$$

(24)

$$\mu(r') \log\left(\frac{\mu}{\mu'}\right)^2\int_{\partial B} ds_1 ds_2 D_0(r, s_1)g_0(s_1 - s_2)g_0(s_2 - r')[n_{s_1}][u_{srs}][n_{s_2}][u_{s2r'}] - ....$$

Thus the Green’s function is expanded in terms of the parameter: $\log\left(\frac{\mu}{\mu'}\right)$.

It is interesting to relate this parameter to the parameter $\xi$, which appears in the literature [6] when considering the case of dielectric-diamagnetic ball or cylinder. To see the relation we expand:

$$\log\left(\frac{\mu}{\mu'}\right) = \log\left(\frac{1 - \xi}{1 + \xi}\right) = -2\xi - 2\xi^3/3 + ...$$

(25)
Hence the two expansions can now be related by assuming $\xi$ small. In particular the second term, proportional to $(\log \mu/\mu')^2$ is exactly one forth of the coefficient of $\xi^2$ in the common expansions of the Casimir energy.

5 Energy density from Green’s function

Fluctuation dissipation theory says that (with our definition of $D$), at temperature $T = 1/\beta$ the correlation functions of the fields $A_i$ are related to the generalized susceptibility $D$ by \[ \tag{26} \]

\[ (A_i(r)A_j(r'))_\omega = -i4\pi \coth\left(\frac{\beta\omega}{2}\right)(D_{ij}(r, r') - D_{ji}(r', r)) \]

Now one can check that \[ (E_iE_k)_\omega = \omega^2 (A_iA_k)_\omega \]

and that \[ (B_iB_k)_\omega = -\nabla \times (A_iA_k)_\omega \times \nabla. \]

Note that there is no minus sign before the correlator of the electric fields, this can be verified by returning to the time parameter and using $E = -\dot{A}$. Thus the energy density per $d\omega$ of the electromagnetic field at temperature $T$ is given by \[ \rho_{T}(r, \omega) = -\coth\left(\frac{\beta\omega}{2}\right)\rho(r, \omega) \]

where \[ \rho(r, \omega) = \frac{1}{4\pi}\lim_{r'\to r}\Im\text{Tr}[\omega^2 \epsilon(r)D(r, r') + \frac{1}{\mu(r)}\nabla \times D \times \nabla] \]

This expression for $\rho$ is usually divergent and should be regularized. The natural regularization scheme in this problem would be to subtract the energy density of the spatially homogeneous problem for this result.

6 Expansion for energy density and density of modes in a medium with UVL

As was previously shown, in the UVL case there is a symmetry between the electric and magnetic Green’s functions. As a result the calculations are
considerably simplified. In the expression for the energy density the zeroth order terms are cancelled by the vacuum $D_0$ function, and moreover we are left solely with the even terms of the expansion. All the higher order terms are given by explicit integrals in which the limit $r' \to r$ can be taken at the outset. Thus, for example, the first term to be considered in the energy density is:

$$\rho_T^{(2)}(r, \omega) = -\coth\left(\frac{\beta\omega}{2}\right)\rho^{(2)}(r, \omega)$$ \tag{31}

Where

$$\rho^{(2)}(r, \omega) = \frac{1}{2\pi}\omega^2 \Im \int g'_0(r_1 - r_2)g'_0(r_2 - r)$$

$$\times \text{Tr}[(\nabla \log \mu(r_1)][u_{r_1r_2}][\nabla \log \mu(r_2)]G_0(r_2, r)D_0(r, r_1)]$$ \tag{32}

To find the density of the Casimir energy with respect to $\omega$ this expression is to be integrated over the volume considered. It is convenient to integrate first over $r$ (see [4]). In our notation:

$$\int \, dr \, G_0(r_2, r)D_0(r, r_1) = [\nabla r_2](I + \frac{1}{\omega^2} \nabla \otimes \nabla r_1) \int \, dr \, g_0(r_2 - r)g_0(r - r_1)$$ \tag{33}

Using the identity:

$$\int \, dr \, g_0(r_2 - r)g_0(r - r_1) = \frac{1}{2\omega}\partial_\omega g_0(r_2 - r_1)$$ \tag{34}

We obtain the result:

$$\int \, dr \, G_0(r_2, r)D_0(r, r_1) = \frac{1}{2\omega}\partial_\omega G_0(r_2 - r_1)$$ \tag{35}

Namely we have the operator identity, $G_0D_0 = \frac{1}{2\omega}\partial_\omega G_0$. This can be done at any order in the perturbation series so that we obtain the following general expression:

$$\rho^{(2n)}(\omega) = \frac{1}{2\pi}\omega^2 \Im \int \text{Tr}[(\nabla \log \mu(r_1)][u_{r_1r_2}][\nabla \log \mu(r_2)][u_{r_2n,r_1}]$$

$$\times g'_0(r_1 - r_2) \cdots g'_0(r_{2n-1} - r_{2n})\frac{1}{2\omega}\partial_\omega g'_0(r_{2n} - r_1)$$ \tag{36}

In principle the Casimir energy can be obtained from these expression to any order and for any $\mu$. However, in most cases this is impossible to carry out, and one has to be content with a first few orders.
7 Casimir energy: second order results

In this section we focus on the second term in the expansion for the density (36). We will obtain a simplified expression for this term, however, one must bear in mind that for any specific problem it is necessary to check whether this term is indeed larger than the other terms in the expansion. To cast the second term in a more convenient form, it is useful to introduce the function:

$$g_2(r_1 - r_2) = g_0^2(r_1 - r_2) \frac{1}{2\omega} \partial_\omega g_0^2(r_2 - r_1) = \frac{e^{2i\omega|r_2 - r_1|}}{32\pi^2|r_2 - r_1|} \left( \frac{1}{|r_2 - r_1|} - i\omega \right),$$

(37)

so that the second order contribution to the energy is given by (36):

$$\rho^{(2)}(\omega) = \frac{1}{2\pi} \omega^2 \text{Im} \int \text{Tr}([\nabla \log \mu(r_1)][u_{r_1r_2}][\nabla \log \mu(r_2)][u_{r_2r_1}]) g_2(r_1 - r_2).$$

(38)

Using the relation (A.10)

$$Tr([a][b][c][d]) = 2(a \cdot b)(b \cdot c)$$

(39)

We obtain

$$\rho^{(2)}(\omega) = \frac{1}{2\pi} \omega^2 \text{Im} \int (\nabla \log \mu(r_1) \cdot u_{r_1r_2})(\nabla \log \mu(r_2) \cdot u_{r_2r_1}) g_2(r_1 - r_2)$$

(40)

This is our final result for the first contribution to the Casimir energy for arbitrary \(\mu\). For simple geometries this can be still simplified. For example, if we assume that \(\mu\) is a function of one coordinate only, namely \(\mu = \mu(z)\), Then \([\nabla \log \mu] = (\log \mu(z))'\hat{z}\) and the second term is

$$\rho^{(2)}_T(\omega) = -\frac{1}{2\pi} \omega^2 \text{coth} \left( \frac{\beta \omega}{2} \right) \text{Im} \int 2(\hat{z} \cdot u_{r_1r_2})^2 g_2(r_2 - r_1)(\log \mu(z_1))'(\log \mu(z_2))'$$

(41)

The energy density per unit area for this configuration can be now obtained by integrating this expression over \(r_2\) and the \(z\) coordinate of \(r_1\).

7.1 Second order term for spherically symmetric \(\mu\)

For cases with high symmetry the equation (40) for the energy density can be further simplified. Indeed, let us assume that we are dealing with a medium with spherical symmetry. This type of mediums can be encountered in a variety of problems from QCD bag models to cosmology. In our formalism
this amounts to consider a magnetic permeability $\mu$ which is a radial function. Thus $[\nabla \log \mu] = (\log \mu)'[u_0]$. The second term (40) is then given by

$$
\rho^{(2)}(\omega) = -\frac{e^2}{2\pi} \text{Im} \int \text{Tr}[[u_{0r_1}] [u_{r_1 r_2}] [u_{0r_2}] [u_{r_2 r_1}]] \times g_2(r_1 - r_2)(\log \mu(|r_1|))'(\log \mu(|r_2|))'
$$

To carry the integration further, we change to bipolar coordinates $u = |r_2 - r_1|$ and $v = |r_2|$ and $r = |r_1|$. Thus:

$$(u_{0r_1} \cdot u_{r_1 r_2})(u_{0r_2} \cdot u_{r_2 r_1}) = \frac{(v^2 - u^2 - r^2)}{2ru} \frac{(r^2 - v^2 - u^2)}{2vu} = \frac{(u^4 - (v^2 - r^2)^2)}{4u^2 v^2 r}$$

and the integration becomes:

$$\text{Im} \int_0^\infty 4\pi r^2 dr (\log \mu(r))' \int \frac{2\pi u}{r} dv (u^2 - v^2 - r^2)^2 \frac{g_2(u)(\log \mu(v))'}{2u^2 v^2}$$

Where the integration domain is such that $u$, $v$ and $r$ can form a triangle. More specifically, the integrations are given by:

$$I(r) \equiv \left( \int_0^r du \int_{r-u}^{r+u} dv + \int_r^\infty du \int_{u-r}^{u+r} dv \right) \left( \frac{(u^4 - (v^2 - r^2)^2)}{2u} g_2(u)(\log \mu(v))' \right)$$

Finally, the second order contribution for the Casimir energy in this case is given by the formula:

$$\rho_T^{(2)}(\omega) = -4\pi \omega^2 \coth\left(\frac{\beta \omega}{2}\right) \text{Im} \int_0^\infty dr I(r)(\log \mu(r))'$$

This equation is our final formula for the Casimir energy of a UVL configuration with arbitrary radially symmetric $\mu$. As an example, in the next subsection we implement it for a one-line calculation of the Casimir energy of a dielectric diamagnetic ball.

### 7.2 Example: The dielectric-diamagnetic ball

It was first shown by Boyer [7], that, contrary to intuition, the Casimir force on a conducting spherical shell is repulsive. A large number of subsequent calculations by various techniques [9, 5] verified Boyer’s result. In the following, we show that the calculation of the Casimir energy of a ball of radius $a$ immersed in a medium, under the UVL condition is immediate using the
technique developed above. In this case \( \mu(r) = \mu \) if \( r < a \) and \( \mu(r) = \mu' \) if \( r > a \). Let \( \kappa = \log \frac{\mu}{\mu'} \). Then from (16) it is immediate that the density of states is of the form

\[
\rho_T^{(2)}(\omega) = -4\pi \omega^2 \coth\left(\frac{\beta \omega}{2}\right) \text{Im}(\kappa I(a))
\]  

(47)

Now explicitly calculating \( I(a) \) we have:

\[
I(a) = \kappa \int_0^{2a} \frac{u^3}{2} q_2(u) \]  

(48)

Performing the integral in (48) and substituting in (17), we have:

\[
\rho^{(2)}(\omega) = -\text{Im}\left[ \frac{-1 + e^{4in\omega} - 4ie^{4in\omega}a\omega - 4e^{4in\omega}\omega^2 a^2}{8\pi} \right] \xi^2 + O(\xi^4)
\]  

(49)

(Where we used (25) to write the result in terms of \( \xi \).) Taking the imaginary part of the expression on the right gives one a simple way of eliminating the divergence in this problem, which arises upon integrating over frequencies. This simply eliminates the term proportional to \( 1/8\pi \) in the energy density. Alternatively, this term is independent of radius and thus can be eliminated by subtracting the unconstrained system where \( a \to \infty \). In any case we recover the result obtained in [8] by mode summation.

At zero temperature, the integration over frequencies is conveniently carried out on the imaginary axis to yield the known result to order \( \xi^2 \):

\[
E = \int_{-\infty}^{\infty} d\omega \frac{\xi^2}{8\pi} e^{-4a|\omega|}(1 + 4a|\omega| + 4a|\omega|^2) = \frac{\xi^2}{2a} \frac{5}{16\pi}
\]  

(50)

Since the energy is a positive quantity decreasing with radius the Casimir pressure on the boundary will be outward. At finite temperatures, one has to take into account the \( \coth\left(\frac{\beta \omega}{2}\right) \) factors that appear in (29). This is conveniently done by changing the integration to summation over the imaginary Matsubara frequencies. In [10] this was done using the expression (49), with the result:

\[
E_C(a, T) = \frac{T\xi^2}{4} + \frac{T\xi^2}{2(e^{8a\pi T}-1)} + \frac{4a\pi^2 e^{8a\pi T}}{(e^{8a\pi T}-1)^2} + \frac{8a^2 e^{2T}\xi^2 e^{8a\pi T}}{(e^{8a\pi T}-1)}
\]  

(51)

This result is exact (to order \( \xi^2 \)) for all temperatures. It is easy to check that the pressure on the boundary remains outward at all temperatures.
8 Concluding remarks

In this paper we presented a technique appropriate for evaluation of the Green’s function of dielectric media, especially in cases where uniform velocity of light (UVL) holds. The perturbative technique developed allows one to deal with the Casimir effect for geometries not studied in other approaches. It was shown that for the UVL case the magnetic and electric Green’s functions are closely related - a feature which is not present in general settings of dielectric medium. Due to this property the first contribution to the energy density of the electromagnetic field comes from the second term in the expansion. We have shown that equation (40) is an explicit integral representation of this contribution. For cases of spherical symmetry we derived a simplified formula, namely (46), for the energy density of a configuration with an arbitrary radial \( \mu \). From this formula the energy of a dielectric-diamagnetic ball follows easily, without recourse to expansion of the modes themselves or boundary condition considerations, other problems can now be also accessed in the same manner.

Acknowledgments

I wish to thank L.P. Pitaevskii for helpful discussions. I also wish to thank A. Elgart, A. Mann, and M. Revzen for many useful remarks.

Appendix: Identities

For the convenience of the reader we list here a few simple relations:

1. 

\[
[b][a] = a \otimes b - a \cdot b I \quad (A.1)
\]

This is easy to see by working with indices:

\[
([b][a])_{im} = \epsilon_{ijk}b_j\epsilon_{klm}a_l = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})a_l b_j = a_i b_m - \delta_{im}a_l b_l \quad (A.2)
\]

2. 

\[
G_0(r, r') = g_0(r - r')[u_{rr'}] \quad (A.3)
\]

where \( u_{rr'} \) is a unit vector in the direction from \( r \) to \( r' \). Indeed:

\[
G_0(r, r') = [\nabla]D_0 = -[\nabla]g_0(r - r')I = -[\nabla_{(r-r')}g_0(r - r')] = g_0(r - r')[u_{rr'}] \quad (A.4)
\]
3. From these two we conclude that:
\[ QD_0 = [\nabla \log \mu(\mathbf{r})]G_0(\mathbf{r}, \mathbf{r}') = g_0'(\mathbf{r} - \mathbf{r'})[\nabla \log \mu(\mathbf{r})][\mathbf{u}_{\mathbf{r}_1\mathbf{r}_2}] \]  
(A.5)

4. Symmetry properties

(i) \[ D_{0ij}(\mathbf{r}, \mathbf{r}') = D_{0ji}(\mathbf{r}', \mathbf{r}) \]
(ii) \[ G_{0ij}(\mathbf{r}, \mathbf{r}') = G_{0ji}(\mathbf{r}', \mathbf{r}) \]
(iii) \[ G_{0ij}(\mathbf{r}, \mathbf{r}') = -G_{0ji}(\mathbf{r}, \mathbf{r}') = -G_{0ij}(\mathbf{r}', \mathbf{r}) \]
(iv) \[ [\mathbf{a}] = -[\mathbf{a}]^t \]

Remark: Property (i) is true in the form \[ D_{ij}(\mathbf{r}, \mathbf{r}') = D_{ji}(\mathbf{r}', \mathbf{r}) \] for any \( D \) which is a susceptibility function for non-magnetoactive medium [I].

5.
\[ A \times \nabla = ([\nabla]A)^t \]
(A.7)

6.
\[ \nabla \times D_0 \times \nabla' = \omega^2 D_0 - \mathbb{I} \]
(A.8)

Indeed:
\[ \nabla \times D_0 \times \nabla' = [\nabla]((\nabla)(D_0)^t) = [\nabla]G_0 = [\nabla][\nabla]D_0 = -\mathbb{I} + \omega^2 D_0 \]
(A.9)

7. Using the relations \((\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d}) = (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \otimes \mathbf{d}\) and \(\text{Tr}(\mathbf{a} \otimes \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})\)
we get
\[ \text{Tr}([\mathbf{u}_{0\mathbf{r}_1}\mathbf{u}_{\mathbf{r}_1\mathbf{r}_2}][\mathbf{u}_{0\mathbf{r}_2}\mathbf{u}_{\mathbf{r}_1\mathbf{r}_2}]) = 2(\mathbf{u}_{0\mathbf{r}_1} \cdot \mathbf{u}_{\mathbf{r}_1\mathbf{r}_2})(\mathbf{u}_{0\mathbf{r}_2} \cdot \mathbf{u}_{\mathbf{r}_1\mathbf{r}_2}) \]  
(A.10)

References

[1] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Mechanics*, Part 2 (Pergamon, Oxford, 1984)

[2] Yu. S. Barash and V. L. Gintzburg, 1972, JETP Lett. 15, 403.

[3] L. V. Keldysh, D. A. Kirzhnitz and A. A. Maradudin, *The dielectric function of condensed systems*, (Elsevier, North-Holland Physics, 1989)

[4] R. Balian and B. Duplantier, Ann. Phys. (N.Y.) 104, 336 (1977).

[5] R. Balian and B. Duplantier, Ann. Phys. (N.Y.) 112, 165 (1978).
[6] I. Brevik and H. Kolbenstvedt, *Phys. Rev.* D25 (1982) 1731.

[7] T. H. Boyer, *Phys. Rev.* 174, 1764 (1968).

[8] I. Klich, *Casimir’s energy of a conducting sphere and of a dilute dielectric ball*, Phys. Rev. D 61 (2000) 025004, [hep-th/9908101](https://arxiv.org/abs/hep-th/9908101).

[9] Here we list but a few of the works on the subject:
B. Davies, *J. Math. Phys.* 13, 1324 (1972);
K. A. Milton, L. L. DeRaad, Jr., and J. Schwinger, Ann. Phys. (N.Y.) 115, 388 (1978);
V. V. Nesterenko and I. G. Pirozhenko, Phys. Rev. D 57, 1284 (1997);
G. Barton, J. Phys. A 32, 525 (1999);
M. Bordag, K. Kirsten, Phys. Rev. D53 (1996) 5753;
I. Brevik and H. Kolbenstvedt Ann. Phys. (N.Y.) 143, 179 (1982).
S. Leseduarte, A. Romeo, Ann. Phys. 250, 448 (1996).
V. Marachevsky, [hep-th/0010214](https://arxiv.org/abs/hep-th/0010214).
G. Lambiase, G. Scarpetta and V. V. Nesterenko, [hep-th/9912176](https://arxiv.org/abs/hep-th/9912176).

[10] I. Klich, J. Feinberg, A. Mann and M. Revzen, *Casimir energy of a dilute dielectric ball with uniform velocity of light at finite temperature*, Phys.Rev. D62 (2000) 045017.