Quantum simulator for the Schwinger effect with atoms in bi-chromatic optical lattices

Nikodem Szpak* and Ralf Schützhold†
Fakultät für Physik, Universität Duisburg-Essen, Duisburg, Germany
(Dated: January 13, 2013)

Ultra-cold atoms in specifically designed optical lattices can be used to mimic the many-particle Hamiltonian describing electrons and positrons in an external electric field. This facilitates the experimental simulation of (so far unobserved) fundamental quantum phenomena such as the Schwinger effect, i.e., spontaneous electron-positron pair creation out of the vacuum by a strong electric field.

PACS numbers: 67.85.-d, 03.65.Pm, 12.20.-m.

Introduction There are several fundamental predictions of relativistic quantum field theory which have so far eluded a direct experimental verification. One prominent example is the Schwinger [1] effect (historically more accurate would be the name Sauter-Schwinger effect, see [2] and [3]), i.e., the spontaneous creation of electron/positron-pairs out of the vacuum by a strong electric field. For a constant electric field $E$, the leading-order $e^+e^-$ pair creation probability scales as [1–3]

$$P_{e^+e^-} \sim \exp \left\{ - \frac{e^3}{\hbar} \frac{M^2}{qE} \right\} = \exp \left\{ - \frac{E_S}{E} \right\}, \tag{1}$$

where $E_S = M^2e^3/(\hbar q)$ is the critical field strength determined by the elementary charge $q$ and the mass $M$ of an electron (or positron). The above expression (1) for $P_{e^+e^-}$ does not permit a Taylor expansion in $q$, i.e., it is inherently non-perturbative and thus cannot be represented by any finite set of Feynman diagrams.

Unfortunately, our theoretical understanding of this non-perturbative QED effect is still very incomplete. Apart from the constant field case, only very simple field configurations where the electric field either depends on time $E(t)$ or on one spatial coordinate such as $E(x)$ are fully solved. For example, recently it has been found that the occurrence of two different frequency scales in a time-dependent field $E(t)$ can induce drastic changes in the (momentum dependent) pair creation probability [4, 5]. Moreover, the impact of interactions between the electron and the positron of the created pair, as well as between them and other electrons/positrons is not understood. This ignorance is unsatisfactory not only from a theoretical point of view but also in view of planned experiments which envisage field strengths not too far below the critical field strength $E_S$ and thus could be able to probe this effect experimentally [6].

These considerations motivate the investigation of the Schwinger effect via a different line of approach. By suitably designing a laboratory system, we could reproduce the quantum many-particle Hamiltonian describing electrons and positrons in an electric field and thereby obtain a quantum simulator for the Schwinger effect. This would facilitate the investigation of space-time dependent electric fields such as $E(t, x)$ and should also provide some insight into the role of interactions. It should be stressed here that our proposal goes beyond the simulation of the (classical or first-quantized) Dirac equation on the single-particle level, see, e.g., [8–14], but aims at the full quantum many-particle Hamiltonian. A correct description of many-body effects such as particle-hole creation (including the impact of interactions) requires creation and annihilation operators in second quantization. There are some proposals for the second-quantized Dirac Hamiltonian [15–20] but they consider scenarios which are more involved than the set-up discussed here and aim at different models and effects. Similarly, the recent observation of Klein tunnelling in graphene [21, 22] deals with massless Dirac particles – but the mass gap is crucial for the non-perturbative Schwinger effect, cf. Eq. (1).

The model We start with the Dirac equation [7] describing electrons and positrons propagating in an electromagnetic vector potential $A_\mu$ which are described by the spinor wave-function $\Psi$ ($\hbar = c = 1$)

$$\gamma^\mu(i\partial_\mu - qA_\mu)\Psi - M\Psi = 0. \tag{2}$$

For simplicity, we consider 1+1 dimensions ($\mu = 0, 1$) where the Dirac matrices $\gamma^\mu$ satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ can be represented by the Pauli matrices $\gamma^0 = \sigma_3$ and $\gamma^1 = -i\sigma_1$. Since in one spatial dimension there is no magnetic field we can choose the gauge $qA_0 = \Phi$ and $A_1 = 0$. As a result, the Dirac equation simplifies to

$$i\partial_t \Psi(t, x) = (-i\sigma_2 \partial_x + M\sigma_3 + \Phi)\Psi(t, x). \tag{3}$$

In one spatial dimension, there is also no spin, hence the wave function has only two components $\Psi = (\Psi^1, \Psi^2)$. The Hamiltonian for the classical Dirac field reads then

$$H = \int dx \bar{\Psi} i(-i\sigma_2 \partial_x + M\sigma_3 + \Phi)\Psi. \tag{4}$$

As the next step, we discretize the space dimension and introduce a regular grid (lattice) $x_n = na$ with a positive grid (lattice) constant $a$ and integers $n \in \mathbb{Z}$. The discretization of the wave function $\Psi_n(t) := \sqrt{a}\Psi(t, x_n)$, defined now at the grid points $x_n$, gives rise to a discretized derivative $\sqrt{a}\partial_x \Psi(t, x_n) \to [\Psi_{n+1}(t) - \Psi_{n-1}(t)]/(2a)$. Finally, replacing the $x$-integral by a sum, we obtain

$$H \to \sum_n \Psi_n^\dagger \left[ \frac{-i\sigma_2}{2a}(\Psi_{n+1} - \Psi_{n-1}) + M\sigma_3\Psi_n + \Phi_n\Psi_n \right]. \tag{5}$$
In order to obtain the quantum many-body Hamiltonian, we quantize the discretized Dirac field operators via the fermionic anti-commutation relations

\[ \{ \hat{\Psi}^\alpha_n(t), \hat{\Psi}^\beta_m(t) \} = 0, \quad \{ \hat{\Psi}^\alpha_n(t), [\hat{\Psi}^\beta_n(t)]^\dagger \} = \delta_{nm}\delta^{\alpha\beta}. \quad (6) \]

Using \( \hat{\Psi}_n = (\hat{a}_n, \hat{b}_n) \), i.e., \( \hat{\Psi}_n^\dagger = \hat{a}_n \) and \( \hat{\Psi}_n = \hat{b}_n \), the discretized many-particle Hamiltonian reads

\[
\hat{H} = \frac{1}{2a} \sum_n \left[ \hat{b}_{n+1}^\dagger \hat{a}_n - \hat{b}_n^\dagger \hat{a}_{n+1} + \text{h.c.} \right] + \sum_n \left[ (\Phi_n + M) \hat{a}_n^\dagger \hat{a}_n + (\Phi_n - M) \hat{b}_n^\dagger \hat{b}_n \right]. \quad (7)
\]

The first term describes jumping between the neighboring grid points while the remaining two terms can be treated as a combination of external potentials. Due to the specific form of the jumping, the lattice splits into two disconnected sub-lattices: (A) containing \( \hat{a}_{2n} \) and \( \hat{b}_{2n+1} \) and (B) containing \( \hat{a}_{2n+1} \) and \( \hat{b}_{2n} \) with integers \( n \). Since the two sub-lattices behave basically in the same way, it is sufficient to consider only one of them, say A. Identifying \( \hat{c}_{2n} = (-1)^n \hat{a}_{2n} \) and \( \hat{c}_{2n+1} = (-1)^{n+1} \hat{b}_{2n+1} \), we obtain the form of the well-known Fermi-Hubbard Hamiltonian for a one-dimensional lattice

\[
\hat{H} = -J \sum_n \left[ \hat{c}_{n+1}^\dagger \hat{c}_n + \hat{c}_n^\dagger \hat{c}_{n+1} \right] + \sum_n V_n \hat{c}_n^\dagger \hat{c}_n, \quad (8)
\]

with hopping rate \( J = 1/(2a) \) and on-site potentials \( V_{2n} = \Phi_{2n} + M \) and \( V_{2n+1} = \Phi_{2n+1} - M \). This Hamiltonian will be the starting point for the design of the optical lattice analogy. But before we proceed, we note that the free part \( \hat{H}_0 \) of this Hamiltonian, i.e., with \( \Phi_n = 0 \), can be explicitly diagonalized. Performing a discrete Fourier transform on the lattice

\[
\hat{a}_p := \sum_n e^{-i 2 \pi n a p} \hat{a}_{2n}, \quad \hat{b}_p := \sum_n e^{-i (2n+1) a p} \hat{b}_{2n+1}, \quad (9)
\]

for \( p \in (-\pi/2a, \pi/2a) \) and introducing new operators

\[
\left( \begin{array}{c} \hat{A}_p \\ \hat{B}_p \end{array} \right) = \frac{1}{\sqrt{2E_p}} \left( \begin{array}{cc} \sqrt{E_p + M} & \sqrt{E_p - M} \\ -\sqrt{E_p - M} & \sqrt{E_p + M} \end{array} \right) \left( \begin{array}{c} \hat{a}_p \\ \hat{b}_p \end{array} \right), \quad (10)
\]

we obtain the diagonalized form

\[
\hat{H}_0 = \int dp \ E_p \left[ \hat{A}_p^\dagger \hat{A}_p - \hat{B}_p^\dagger \hat{B}_p \right] \quad (11)
\]

with the energy spectrum

\[
E_p = \sqrt{M^2 + \frac{1}{4a^2} \cos^2(2p)} . \quad (12)
\]

It approximates the relativistic energy-momentum relation at the edge of the Brillouin zone, for \( p \approx \pm \pi/(2a) \). The spectrum of \( \hat{H}_0 \) consists thus of two symmetric intervals separated by a gap of \( 2M \). In order to obtain a positive Hamiltonian, we perform the usual re-definition \( B_p^\dagger \leftrightarrow B_p \) which corresponds to changing the vacuum state by filling all \( B_p \) states by a fermion. This is completely analogous to the Dirac sea picture in quantum electrodynamics. In terms of this analogy, \( \hat{A}_p^\dagger \) or \( \hat{A}_p \) create or annihilate an electron whereas \( \hat{B}_p \) or \( \hat{B}_p^\dagger \) create or annihilate a hole in the Dirac sea – which is then a positron. An additional potential \( \Phi_n \), if sufficiently localized in space, will not modify this spectrum but may introduce isolated eigenvalues with eigenstates corresponding to bound states localized in space.

Experimental set-up The Fermi-Hubbard Hamiltonian (8) can be realized with ultra-cold fermionic atoms in a one-dimensional optical lattice with the potential

\[
W(x) = W_0 \sin^2(2kx) + \Delta W \sin^2(kx) , \quad (13)
\]

where \( k = \pi/(2a) \), by taking \( W_0 \gg \Delta W \). Similar settings have already been obtained experimentally [25].

![Fig. 1: Sketch of the dispersion relation (in units of \( J \) and \( 1/a \)) for \( \Delta W = 0 \) (dashed blue curve) and small \( \Delta W > 0 \) (solid and dotted red curves).](image-url)
This nearest-neighbor approximation reproduces the relation (12) for \( J = 1/(2a) \) and holds uniformly for all values of the quasi-momentum \( p \) as long as \( \Delta W \) and \( J \) are small [26]. In the vicinity of the minimum of the upper band and the maximum of the lower band, separated by a gap \( 2M \approx \Delta W \), it reproduces the relativistic energy-momentum relation. The corresponding Hamiltonian has the same form as the one for the discretized Dirac equation (11), thus completing the analogy.

Using again the WKB approximation, we find that the hopping rate \( J \) is mainly determined by the potential strength \( W_0 \), the laser wave-number \( k \) and the mass \( m_{\text{atom}} \) of the atoms (not to be confused with the effective mass \( M \) of the Dirac particles to be simulated)

\[
J \approx \frac{4}{\pi} \sqrt{W_0 E_R} \exp \left\{ \frac{\pi}{4} \sqrt{\frac{W_0}{E_R}} \right\}, \tag{14}
\]

where \( E_R = k^2/(2m_{\text{atom}}) = \pi^2/(8m_{\text{atom}}a^2) \) is the recoil energy. The correction \( \Delta W \) generates the mass gap \( M \approx \Delta W/2 \). The analogue of the \( e^+ e^- \) pair creation can be simulated if the involved scales obey the hierarchy

\[
\omega_{\text{osc}} \gg J \gg M \gg T. \tag{15}
\]

First, the local oscillator frequency \( \omega_{\text{osc}} \) in the potential minima must be larger than \( J \) to ensure the applicability of the single-band Fermi-Hubbard Hamiltonian (8). Second, \( J \gg M \) is required for the continuum limit, i.e., that the discretized expression (5) provides a good approximation. Similarly, the change \( \Delta \Phi_n = \Phi_{n+1} - \Phi_n \) of the analogue of the electrostatic potential \( \Phi_n \) from one site to the next should be much smaller than \( M \). Over many sites, however, this change can well exceed the mass gap \( 2M \), which is basically one of the conditions for the Schwinger effect to occur. Finally, the effective temperature \( T \) should be well below the mass gap \( 2M \) in order to avoid thermal excitations.

Let us insert some example parameters. For \(^6\text{Li} \) atoms in an optical lattice made of light with a wavelength of 500 nm, the recoil energy \( E_R \) would be around 7 \( \mu \)K. If we choose the potential strength as \( W_0 = 10 \mu \)K, the hopping rate \( J \) would be around 5 \( \mu \)K which is still sufficiently below the local oscillator frequency \( \omega_{\text{osc}} \) of around 34 \( \mu \)K. With \( \Delta W = 1 \mu \)K we would get a mass \( M \) of 500 nK and the effective temperature should be below that value.

**Bose-Fermi mapping** Since it is typically easier to cool down bosonic than fermionic atoms, let us discuss an alternative realization based on bosons in an optical lattice. To this end, we start with the Bose-Hubbard Hamiltonian which has the same form as the Fermi-Hubbard Hamiltonian (8) after replacing the fermionic \( \hat{c}_n \) by bosonic \( \hat{d}_n \) operators, but with an additional on-site repulsion term \( U (\hat{d}_{n}^\dagger \hat{d}_n - 1) \hat{d}_{n}^\dagger \hat{d}_n \). For large \( U \gg J \) (which can be controlled by an external magnetic field via a Feshbach resonance, for example), we obtain the bosonic analogue of “Pauli blocking”, i.e., at most one particle can occupy each site. Neglecting all states with double or higher occupancy, we can map these bosons exactly onto fermions in one spatial dimension via

\[
\hat{c}_n = \hat{d}_n \prod_{m<n} \exp \left( -i \pi \hat{d}_m^\dagger \hat{d}_m \right). \tag{16}
\]

As a result, we obtain the same physics as described by the Fermi-Hubbard Hamiltonian (8).

**Simulation procedure** The above established analogy between the (discretized) quantum many-particle Hamiltonian of electrons and positrons in an electric field, on the one hand, and the (Bose or Fermi) Hubbard model describing ultra-cold atoms in optical lattices, on the other hand, enables laboratory simulations of relativistic phenomena of strong-field QED. Probably the most
prominent of them is the spontaneous pair creation in strong electric fields known also as the Schwinger effect [1–3] which has to date not been confirmed experimentally. The original Schwinger effect [1] deals with a constant electric field \( E \) and would correspond to a static tilted optical lattice with \( \Phi(x) = Ex \). However, in view of the planned experiments [6], an electric field which is localized in space and time is more realistic. In order to clearly distinguish non-perturbative spontaneous pair creation (via tunneling) from other perturbative effects such as dynamical pair creation, the electric field should be slowly varying (compared to the frequency scale of the gap \( 2M \)).

As a result, we envisage an experimental realization sketched in Fig. 2. In order to prepare the initial state, we start with \( \Delta W \gg W_0 \) where the two bands are separated by a large gap. The Dirac sea then corresponds to the state where all lower minima are filled with atoms while all upper minima are empty (half filling of the lattice, see first picture in Fig. 2). If we then decrease \( \Delta W \) adiabatically until \( \Delta W \ll W_0 \) and thus \( J \gg M \), the atoms become delocalized – but still the lower band is filled while the upper band remains empty (second picture in Fig. 2). As the next stage, we slowly switch on an additional potential \( \Phi \) to facilitate tunneling from the lower band to the upper band – the analogue of the Schwinger effect (third picture in Fig. 2). Finally, after slowly switching off the potential \( \Phi \) again (fourth picture in Fig. 2), we increase \( \Delta W \) adiabatically until \( \Delta W \gg W_0 \). By this energetic separation, a particle-hole pair created via tunneling is transformed to an atom in one of the upper minima and, consequently, a missing atom (i.e., hole) in one of the lower minima (fifth picture in Fig. 2). This could be detected by site-resolved imaging [23], for example.

Note that the creation of a particle-hole pair separated by \( \Delta n \) lattice sites requires the simultaneous tunnelling of \( \Delta n \) particles since two particles are not allowed to occupy the same site. This again emphasizes the many-particle character of our proposal which goes beyond the simulation of the classical Dirac equation. Apart from investigating the pair creation probability for space-time dependent electric fields \( E(t, x) \), this quantum simulator for the Schwinger effect could provide some insight into the impact of interactions. For example, including dipolar interactions of the atoms, we may even switch between attractive and repulsive interactions by aligning the atoms spins parallel or perpendicular to the lattice.

R. S. acknowledges stimulating discussions during the Benasque Workshop on Quantum Simulations in 2011 and financial support from the DFG (SFB-TR12).

---

† e-mail: ralf.schuetzhold@uni-due.de

[1] J. Schwinger, Phys. Rev. 82, 664 (1951).
[2] F. Sauter, Z. Phys. 69, 742 (1931); ibid. 73, 547 (1931).
[3] W. Heisenberg and H. Euler, Z. Phys. 98, 714 (1936).
[4] R. Schützhold, H. Gies, G. Dunne, Phys. Rev. Lett. 101, 130404 (2008); G. V. Dunne, H. Gies, R. Schützhold, Phys. Rev. D80, 111301 (2009).
[5] C. K. Dunhu, G. V. Dunne, Phys. Rev. Lett. 104, 250402 (2010).
[6] See, e.g., the European ELI programme: http://www.extreme-light-infrastructure.eu/
[7] P. A. M. Dirac Proc. Roy. Soc. (London) A 117, 610 (1928); ibid. 118, 351 (1928); ibid. 126, 360 (1930).
[8] D. Witthaut, T. Salger, S. Kling, C. Grossert, and M. Weitz, ArXiv e-prints (2011), 1102.4047.
[9] W. G. Unruh and R. Schützhold, Phys. Rev. D 68, 024008 (2003).
[10] S. Longhi, Phys. Rev. A 81, 022118 (2010).
[11] F. Dreisow, M. Heinrich, R. Keil, A. Tünnermann, S. Nolte, S. Longhi, and A. Szameit, Phys. Rev. Lett. 105, 143902 (2010).
[12] R. Gerritsma, B. P. Lanyon, G. Kirchmair, F. Zähringer, C. Hempel, J. Casanova, J. J. García-Ripoll, E. Solano, R. Blatt, and C. F. Roos, Phys. Rev. Lett. 106, 060503 (2011).
[13] R. Gerritsma, G. Kirchmair, F. Zähringer, E. Solano, R. Blatt, and C. F. Roos, Nature 463, 68 (2010).
[14] L. Lamata, J. León, T. Schütz, and E. Solano, Phys. Rev. Lett. 98, 253005 (2007).
[15] J. I. Cirac, P. Maraner, and J. K. Pachos, Phys. Rev. Lett. 105, 190403 (2010).
[16] S.-L. Zhu, B. Wang, and L.-M. Duan, Phys. Rev. Lett. 98, 260402 (2007).
[17] J.-M. Hou, W.-X. Yang, and X.-J. Liu, Phys. Rev. A 79, 043621 (2009).
[18] L.-K. Lim, C. M. Smith, and A. Hemmerich, Phys. Rev. Lett. 100, 130402 (2008).
[19] O. Boada, A. Celi, J. I. Latorre, and M. Lewenstein, ArXiv e-prints (2010), 1010.1716.
[20] N. Goldman, A. Kubasiak, A. Bermudez, P. Gaspard, M. Lewenstein, and M. A. Martin-Delgado, Phys. Rev. Lett. 103, 035301 (2009).
[21] K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Grigorieva, S. V. Dubonos, and A. A. Firsov, Nature 438, 197 (2005).
[22] M. I. Katsnelson, K. S. Novoselov, and A. K. Geim, Nature Physics 2, 620 (2006).
[23] J. F. Sherson, C. Weitenberg, M. Endres, M. Cheneau, I. Bloch, and S. Kuhr, Nature 467, 68 (2010).
[24] N. L. Balazs, Annals of Physics 53, 421 (1969).
[25] T. Salger, C. Geckeler, S. Kling, and M. Weitz, Phys. Rev. Lett. 99, 190405 (2007).
[26] N. Szpak and R. Schützhold, in preparation.