ARX Model Identification using Generalized Spectral Decomposition

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Abstract: This article is concerned with the identification of autoregressive with exogenous inputs (ARX) models. Most of the existing approaches like prediction error minimization and state-space framework are widely accepted and utilized for the estimation of ARX models but are known to deliver unbiased and consistent parameter estimates for a correctly supplied guess of input-output orders and delay.

In this paper, we propose a novel automated framework which recovers orders, delay, output noise distribution along with parameter estimates. The primary tool utilized in the proposed framework is generalized spectral decomposition. The proposed algorithm systematically estimates all the parameters in two steps. The first step utilizes estimates of the order by examining the generalized eigenvalues, and the second step estimates the parameter from the generalized eigenvectors. Simulation studies are presented to demonstrate the efficacy of the proposed method and are observed to deliver consistent estimates even at low signal to noise ratio (SNR).

Keywords: System identification, eigenvalue problem, ARX model, Linear systems, Order determination

1. INTRODUCTION

System identification is a broad field that primarily deals with the study of parameter estimation of a formulated model (Ljung, 1999). It has been an active area of research due to its numerous applications in various domains like signal processing (Oppenheim, 1999), control and fault detection (Sepulchre et al., 2012; Maleki et al., 2014) tasks.

An ideal system identification algorithm is expected to deliver efficient and consistent parameter estimates from noisy signals in an automated manner requiring minimal user intervention. Accurate knowledge of these model parameters improves the quality of several tasks like spectral analysis, filtering (Haykin, 2008) and controller design (Elliott and Nelson, 1993). These basic functions can be used in various specific applications like room acoustics (Xue et al., 2017), speaker localization (Doclo and Moonen, 2003), room geometry estimation (Moore et al., 2013).

In this work, we consider the problem of model estimation of a general ARX model for single input and single output (SISO) systems as shown in Figure 1.

As this is a fundamental problem, abundant literature exists in the field system identification (Ljung, 1999; Georgiou et al., 1992). Most of the existing algorithms like prediction error minimization (PEM), state-space frame-
Bode plots (Tangirala, 2014) can be utilized for the estimation of model structure.

(2) During-Estimation: This class of methods tries to estimate the model structure along with model parameters. The core idea is to utilize regularization or sparsity constraints in compressed sensing techniques for model structure determination (Perepu and Tangirala, 2015).

(3) Post-Estimation: In these approaches, a certain model structure is assumed, and its correctness is checked after estimation. This class of methods includes approaches like recursively estimating ARX models, checking under parameterization, or over parameterization of the model through residual analysis or hypothesis testing of estimated parameters (Tangirala, 2014).

The user is compelled to try most of these approaches practically in an iterative manner. For example, if the model structure is found to be unsatisfactory at the post-estimation stage, the user is bound to try other combinations of input-output order and delay from the beginning of the model estimation stage. Some of these approaches are heuristic, which operates under restrictive assumptions and lacks theoretical rigor.

The approaches mentioned above can also be classified into parametric and non-parametric methods for model structure determination. The prime issue with these approaches is that they operate for a given model structure and parameter estimation in an uncoupled manner. This turns out to be the prime reason for forcing the user to try multiple combinations of input-output order and delay.

In this work, we consider the challenging problem of jointly estimating order and model parameters without any availability of a priori system information. The proposed framework provides the freedom to estimate order and model parameters jointly. With a slight abuse of terminology, this can be seen as an attempt to couple the parametric and non-parametric methods for model structure determination. The novelty of this work lies in the proposal of an automated algorithm that estimates model order, delay, and parameters without any prior knowledge on the system.

The proposed algorithm utilizes the generalized spectral decomposition framework and is heavily inspired from Maurya et al. (2016). Spectral decomposition framework has been gaining recent attention for system identification. Some of the closely related works on slightly different problems are by Vermoersch and De Moor (2019); De Moor (2019); Zheng and Ohta (2018).

The proposed algorithm can be seen as comprising two major steps. The first step jointly estimates the equation order and noise distribution properties by analyzing the generalized eigenvalues of the covariance matrix for the lagged input-output data. In this paper, we refer autocovariance function (ACVF) of the noise \(v[k]\) in Figure 1 as noise distribution properties. This step requires an initial guess for the ACVF of noise, which is assumed to be not available a priori. So, we estimate the ACVF iteratively.

The second step involves estimating the parameters of the linear difference equation relating to input and output data. In the end, the proposed algorithm systematically recovers input-output orders, delays, parameters of a difference equation, and ACVF of noise in an automated manner with minimal user intervention.

The rest of the paper is organized as follows. In Section 2, we describe the full estimation problem formally and manifests the use of principal component analysis (PCA) and its variants in system identification. Section 3 starts by presenting the equivalence of generalized spectral decomposition and existing algorithms working in the PCA framework. It also consists of the proposed algorithm under various assumptions. Most of the key ideas are demonstrated using a simple second-order system case study. Section 4 contains simulation studies that elucidate the merits of working with the proposed algorithm. Concluding remarks and directions for future work are addressed in Section 5.

2. FOUNDATIONS

A general parametric deterministic linear time-invariant (LTI) dynamic model among input \((u^*)\) - output \((y^*)\) variable is described by

\[
y^*[k] + \sum_{i=1}^{n_u} a_i y^*[k-i] = \sum_{j=D}^{n_y} b_j u^*[k-j] \tag{1}
\]

where \(n_u\) and \(n_y\) are output and input order, respectively and \(D\) is the input-output delay. We define the term equation order, \(\eta = \max(n_y, n_u)\). The problem statement is to estimate the coefficients \(\{a_i\}_{i=1}^{n_u}\; \{b_j\}_{j=1}^{n_y}\) along with \(n_y, n_u\) and \(D\) from \(N\) noisy samples of output \(y^*[k]\) denoted by \(y[k]\) and noise-free input \(u^*[k]\) shown below

\[
y[k] = y^*[k] + v[k] \tag{2a}
\]

\[
v[k] + \sum_{i=1}^{n_v} a_i v[k-i] = e[k], \quad e[k] \sim \mathcal{N}(0, \sigma_e^2) \tag{2b}
\]

The measured output signal, denoted by \(y[k]\) has ARX model structure with noise-free input \(u^*[k]\) as shown in Figure 1, where \(A(z^{-1}) = 1 + \sum_{i=1}^{n_u} a_i z^{-i}\) and \(B(z^{-1}) = \sum_{j=1}^{D} b_j z^{-j}\).

2.1 PCA & Dynamic PCA

In this section, we briefly discuss the application of principal component analysis (PCA) and its variants in the context of system identification. The model in Eqn. (1) can viewed as

\[
\theta^T x[k] = 0 \tag{3}
\]

\[
\theta = [1 \ a_1 \ a_2 \ldots \ a_{n_u} \ -b_{D+1} \ -b_{D+2} \ldots \ -b_{n_v}]^T \tag{4a}
\]

\[
x[k] = [y^*[k] \ldots y^*[k-n_y] \ u^*[k-D] \ldots u^*[k-n_u]]^T \tag{4b}
\]

Equation 3 can be interpreted as a constrained model where the parameter vector \(\theta\) is orthogonal to \(x[k]\) for \(k = 1, 2, \ldots, N\). It is well established that basis for an orthogonal space can be obtained from singular vectors using singular value decomposition (SVD) or equivalently PCA (Joliffe, 2002; Rao, 1964) of \(X\), where \(X\) is defined as follows:

\[
X = [x[k] \ x[k+1] \ldots x[N]]^T \tag{5}
\]

The problem of estimating parameter vector \(\theta\) from noise-free measurements can be solved using multiple
frameworks such as ordinary least squares (Xu et al., 1995) and PCA which minimizes the total least squares (TLS) error function (Jolliffe, 2002). To illustrate the above idea consider a second order LTI system as shown below:

\[ y[k] - 0.4y[k-1] + 0.6u[k-2] = 2u[k-1] \]  

(6)

The input \( u^* \) is chosen to be a full band pseudo random binary signal (PRBS) of length 1023. The output \( y^* \) is generated according to Eqn. (6). The next step is to apply PCA by using eigenvalue decomposition (EVD) on generated according to Eqn. (6). The next step is to perform spectral decomposition on the sample covariance matrix, \( S_X = \frac{1}{N-\eta}X^T X \) as shown below:

\[ S_X V_0^* = V_0^* \Lambda_0^* \]  

(7)

where \( \Lambda_0^*, V_0^* \) consists of eigenvalues and corresponding eigenvectors respectively. By performing EVD, the minimum eigenvalue is observed to be 0 and the corresponding eigenvector is

\[ V_0^* = [0.4256 -0.1703 0.2554 -0.8513] \]  

(8)

The estimate of parameter vector can be obtained from above by normalizing the coefficient of dependent variable \( y[k] \) to be unity:

\[ \theta = [1 -0.4 0.6 -2]^T \]  

(9)

It can be observed that the estimated parameters are same as used in simulating the data in Eqn. (6).

The entire identification task becomes challenging for noisy output measurements. PCA and its variants deliver unbiased estimates of parameter vector only for homoskedastic errors case, meaning both output (\( y^* \)) and input (\( u^* \)) are contaminated by Gaussian white noise of same variance (Jolliffe, 2002).

In our case, the output noise is colored, and the input is noise-free. In the next subsection, we discuss more generalized variants of PCA which can handle some of these issues.

2.2 Iterative PCA & Dynamic IPCA

In this subsection, we discuss the key idea to handle heteroskedastic errors used in Maurya et al. (2018) and make relaxing assumptions that the order and noise variances are known. It was originally proposed for different settings but we discuss only the few relevant aspects for the proposed work.

Iterative PCA Narasimhan and Shah (2008) was proposed to handle the heteroskedastic errors case. Dynamic iterative PCA referred as DIPCA (Maurya et al., 2018) was proposed recently and can be viewed as nature extension of IPCA for dynamic models. One of the novel contributions of these work is order determination by analysis of singular values.

Dynamic IPCA (Maurya et al., 2018) algorithm assumed the presence of noise in both input and output variables as shown in Figure 2. This is contrary to our settings as the input is assumed to be noise-free. But these are discussed briefly to understand the prolific merits of handling heteroskedasticity and order determination using the EVD framework, as shown later in Section 3.2.

Consider both the input and output measurements are corrupted by Gaussian white noise of different variances as shown below:

\[ y[k] = y^*[k] + e_y[k] \]

\[ u[k] = u^*[k] + e_u[k] \]

where \( e_y[k] \sim N(0, \sigma_y^2) \), \( e_u[k] \sim N(0, \sigma_u^2) \). The collection of \( N \) measurements for this case can be given by:

\[ z[k] = [y[k] \ldots y[k-n_y] \ u[k-D] \ldots u[k-n_u]]^T \]

(10a)

\[ Z = [z^T[n] \ z^T[n+1] \ldots z^T[N]] \]

(10b)

The noise covariance is a diagonal matrix of dimension \((n_a + n_b - D + 1) \times (n_a + n_b - D + 1)\) with its entries as, \( \text{diag}(\Sigma_e) = [\sigma^2_{e_1} 1_{n_a} \ 1_{n_b} \sigma^2_{e_2} 1_{n_a-D+1}] \), where \( 1_m \) denotes a vector containing all the \( m \) elements as unity.

The key idea in Narasimhan and Shah (2008); Maurya et al. (2018) is to map the data from heteroskedastic errors to homoskedastic space by scaling the data with inverse square root of noise covariance matrix which is assumed to be known. Maurya et al. (2018) describes the method to estimate noise covariance matrix also but we skip that discussion as it is felt to be inessential for this work.

\[ LL^T = \Sigma_e \]

\[ Z_0 = ZL^{-1} \]

(11)

The next step is to perform spectral decomposition on the sample covariance matrix of the scaled data matrix, \( S_{Z_0} = \frac{1}{N-\eta}Z_0^T Z_0 \) as shown below:

\[ S_{Z_0} V_s = V_s \Lambda_s \]

(12)

where \( V_s \) and \( \Lambda_s \) denote the eigenvectors and eigenvalues in scaled domain. It is theoretically proved in Narasimhan and Shah (2008) that the eigenvectors in scaled domain can be mapped to original space using

\[ \Lambda_0 = \Lambda_s^0 + I \]

\[ V_s = L^{-1} V_0^* \]

(13)

(14)

where \( I \) denotes identity matrix. The above equations state the zero eigenvalues of \( S_X \) map to unity in the scaled domain, which helps in order determination. The parameter vector can also be recovered from the eigenvector in the scaled domain.

The above discussion was carried under the assumption that both input-output variables contain Gaussian white noise, but in our case, only one of the variables - output has colored noise and input is noise-free. In the next section, we describe the proposed method which handles this issue.

3. PROPOSED METHOD

From the discussion in the previous section, we have concluded that three main challenges for our case are

(1) Handling of colored noise in output only.
(2) Order and delay determination
(3) Model parameter estimation
To address these challenges one by one, we present the proposed algorithm in three subsections.

3.1 Model identification for known order & ACVF of Noise

In this subsection, we first draw the equivalence of scaling approach in Maurya et al. (2018) to the generalized spectral decomposition. By using some elementary linear algebra properties, it can be shown that Eqn. (11) and Eqn. (12) can be expressed as:

\[ S_ZV = \Sigma_e V L_s \]  

(15)

In this subsection as the ACVF of noise is assumed to be known, we construct appropriate form of \( \Sigma_e \) and utilize generalized spectral decomposition in Eqn. (15). As seen in Eqn. (2b), the noise \( v[k] \) is colored.

To illustrate the key idea reconsider the same system in Eqn. (6). Noisy output measurements (\( y \) in Eqn. (2a)) are generated in accordance to Eqn. (6) and Eqn. (2b) with noise variance of \( \sigma^2_v = 0.4 \). The corresponding value of noise variance was chosen to maintain \( \text{var}(y) = 10 \). For stacking of lagged input-output variables shown in Eqn. (10), the noise covariance is shown below:

\[ \Sigma_e = \begin{bmatrix} \sigma_{vv}[0] & \sigma_{vu}[1] & \sigma_{vu}[2] & 0 \\ \sigma_{vu}[1] & \sigma_{vu}[0] & \sigma_{vu}[1] & 0 \\ \sigma_{vu}[2] & \sigma_{vu}[1] & \sigma_{vu}[0] & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]  

(16)

where \( \sigma_{vu}[l] \) denotes ACVF for \( v[k] \) at lag \( l \). The next step is to perform spectral decomposition of the sample covariance matrix. Equation Eqn. (15) reduces to usual algebra properties, it can be shown that Eqn. (11) and Eqn. (12) can be expressed as:

\[ \text{S}_ZV = \Sigma_e V L_s \]  

(15)

The sample covariance matrix for \( Z_L \) is denoted by \( S_{ZL} \) and the full noise covariance matrix is given by:

\[ S_{ZL} = \frac{1}{N - L} Z_L^T Z_L \]  

(19)

\[ \Sigma_e = \begin{bmatrix} \Sigma_v & 0_p \\ 0_p & 0_p \end{bmatrix} \]  

(20)

where \( 0_p \) denotes a zero matrix of dimension \((L + 1) \times (L + 1) \) and \( \Sigma_v \) is the output noise covariance matrix.

\[ \Sigma_v = \begin{bmatrix} \sigma_{vv}[0] & \sigma_{vu}[1] & \sigma_{vu}[2] & \cdots & \sigma_{vu}[L] \\ \sigma_{vu}[1] & \sigma_{vu}[0] & \sigma_{vu}[1] & \cdots & \cdots \\ \sigma_{vu}[2] & \sigma_{vu}[1] & \sigma_{vu}[0] & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{vy}[L] & \sigma_{vy}[L-1] & \cdots & \cdots & \sigma_{vy}[0] \end{bmatrix} \]  

(21)

where \( \sigma_{vy}[l] \) is a the output noise covariance matrix. It can be easily noticed \( \Sigma_v \) is a symmetric Toeplitz matrix.

On applying the QZ algorithm for our example, the minimum generalized eigenvalue is observed to be 0.98, and the parameter vector computed from corresponding generalized eigenvector is:

\[ \hat{\theta} = [1 \ -0.4059 \ 0.6036 \ -2.018] \]  

(17)

It can be observed that the estimated parameters are close to the true values used in Eqn. (6) and the minimum eigenvalue is unity. This establishes the estimates of desired parameters can be obtained for known order and noise characteristics.

3.2 Estimation of order for known ACVF of noise

The discussion in previous subsection was conditioned on the availability of order and noise characteristics. In this subsection, we discuss the key idea for estimation of order for given ACVF of noise.

When the order is unknown, the input-output variables should be stacked up to lag order \( L \) as shown below:

\[ z_L[k] = [y[k] \ y[k-1] \cdots y[k-L] \ u^*[k] \cdots u^*[k-L]]^T \]  

(18a)

\[ Z_L = [z_L[L] \ z_L[L+1] \cdots z_L[N]]^T \]  

(18b)

The number of identified linear relations \( \hat{d} \) for chosen stacking order \( L \) could potentially lead to following three cases:

1. \( L < \eta \): If the maximum lag of stacked variables is less than equation order, then there would be no linear relations and hence \( \hat{d} = 0 \). This is the under-parameterization case which could be detected from the absence of linear relations among stacked variables.

2. \( L = \eta \): As discussed in the previous sub-section, there would be only one linear relation. This is the ideally desired choice of stacking order but we have assumed the equation order to be unavailable and hence can not be used as stacking order.

3. \( L > \eta \): If the input-output variables are stacked in excess, multiple linear relations would be detected. This implies \( \hat{d} > 1 \).

The stacking order, \( L \) is chosen to be at least equation order (\( \eta \)). This is done to identify at least one linear relation among the columns of \( Z_L \). The number of excessively identified relations could be potentially used to estimate the equation order, which is described in this subsection.

The next step is to utilize the generalized eigenvalue decomposition for identification of linear relations:

\[ S_{ZL}V = \Sigma_{el} VL_s \]  

(22)

We reconsider the same case study described in Eqn. (6) for \( L = 3 \). As the stacking order for lagged variables is greater than equation order, we should expect excess linear relations, which are as follows:

1. the first linear relation among the variables \( y[k], y[k-1], y[k-2] \) and \( u[k-1] \).
2. the second linear relation among the delayed version of the same variables, which are \( y[k-1], y[k-2], y[k-3] \) and \( u[k-2] \).

It should be noticed that the identified linear relations could also be linear combination of the above two constraints. This is due to the fact that generalized EVD provides a basis for the constraint matrix.
In Section 2.2, it was discussed that the zero eigenvalues map to unity in scaled domain which helps in order determination.

The last two eigenvalues are observed to be in close proximity of unity, which provides clue for the existence of two linear relations among the lagged input-output variables. For a SISO system, as there exists only one unique linear relationship, the equation order can be estimated from

\[ \hat{\eta} = L - \hat{d} + 1 \tag{23} \]

where \( \hat{d} \) denotes the number of unity eigenvalues. Using this the order is estimated to be \( \hat{\eta} = 3 - 2 + 1 = 2 \). The key idea of order determination is inspired from Narasimhan and Shah (2008); Maurya et al. (2018).

3.3 Estimation of order \& ACVF of noise

In this subsection, the discussion is constructed on realistic assumption of non-availability of any prior information about order and ACVF of noise. The key idea is to use the discussion in Sections 3.1 and 3.2 iteratively. One important aspect for this iterative algorithm is estimation of ACVF of noise for given order. Equation (2) can be interpreted as

\[ \begin{bmatrix} A(z^{-1}) - B(z^{-1}) \end{bmatrix} \begin{bmatrix} y[k] \\ u[i][k] \end{bmatrix} = e[k] \tag{24} \]

For a chosen guess of equation order \( \eta \), we could obtain the estimates of \( A(z^{-1}) - B(z^{-1}) \) using the last generalized eigenvector as described in Section 3.1. Equation Eqn. (24) could also be used to obtain the noise variance estimate \( \sigma_e^2 \).

The next step is to obtain the estimate ACVF of \( v[k] \) for construction of \( \Sigma_e \) in Eqn. (15). This can be done using Wiener-Khinchin theorem (Tangirala, 2014; Ljung, 1999). Any stationary process with ACVF \( \sigma_{vv} \) could also be used to obtain the noise variance estimate \( \sigma_e^2 \).

The simulation results are presented in the next section to demonstrate the functioning of proposed algorithm.
are applied. The parameter estimates and the last 3 eigenvalues obtained at the convergence:

\[
\theta = \begin{bmatrix} 1 & -0.2242 & -0.0012 & -1.96 \end{bmatrix} \quad (26a)
\]
\[
\Lambda_s = \begin{bmatrix} 0.3448 & 0.2026 & 0.1546 \end{bmatrix} \quad (26b)
\]

As none of the eigenvalues are unity, we discard \( \eta_{\text{guess}} = 1 \).

The next step is to increase the guessed order and choose \( \eta_{\text{guess}} = 2 \). The estimated parameter and last 5 eigenvalues for \( L = 5 \) after convergence:

\[
\hat{\theta} = \begin{bmatrix} 1 & -0.409 & 0.611 & -0.004 & -1.969 \end{bmatrix} \quad (27a)
\]
\[
\Lambda_s = \begin{bmatrix} 4.5536 & 1.0688 & 1.0493 & 0.9988 & 0.9689 \end{bmatrix} \quad (27b)
\]

As there are 4 unity eigenvalues, the equation order is estimated to be \( \tilde{\eta} = 5 - 4 + 1 = 2 \). It can also be observed that the estimated parameter vector is in close agreement with Eqn. (6). Bootstrap simulations (Shumway and Stoffer, 2000) are performed to derive the confidence interval of model parameters:

\[
y[k] - 0.399 y[k-1] + 0.601 y[k-2] = 2.003 u[k-1] \quad \pm (0.023) \quad (28)
\]

The noise variance also estimated to be 1.459, which is close to the simulated value. The ACVF of noise is also estimated accurately as the parameter estimates of difference equation are close to simulated theoretical values.

As the \( \tilde{\eta} = \eta_{\text{guess}} = 2 \), the algorithm is converged, but for the sake of completeness, we consider the case of \( \eta_{\text{guess}} = 3 \). The last 3 generalized eigenvalues for \( L = 5 \) are shown below:

\[
\Lambda_s = \begin{bmatrix} 1.09 & 0.29 & 0.0068 \end{bmatrix} \quad (29)
\]

As the number of unity eigenvalues is not 3, we discard \( \eta_{\text{guess}} = 3 \). The above exercise demonstrates that a wrong guess of order can be detected by inspection of generalized eigenvalues. It should be noticed that the proposed framework provides the clues for over-parameterization and under-parameterization, which helps converging to the right order quickly.

In order to demonstrate the performance of proposed algorithm at higher levels of noise, we consider SNR = 3. So the output noise variance is chosen accordingly. We repeat the entire procedure of estimating the model parameters with stacking order of \( L = 5 \) and \( \eta_{\text{guess}} = 2 \).

Fig. 4. ACVF of output noise

We obtain 4 unity eigenvalues and the estimated order is found to be \( \tilde{\eta} = 5 - 4 + 1 = 2 \). The final estimated difference equation is shown below:

\[
y[k] - 0.403 y[k-1] + 0.596 y[k-2] = 2.007 u[k-1] \quad \pm (0.0064) \quad (26a)
\]

It should be noticed that the variance in estimated parameters is higher for this case compared to Eqn. (28) due to low SNR. This shows the appreciable quality model parameter estimates can be obtained even at low SNR.

In order to compare the proposed algorithm to existing methods, we estimate the parameters using ordinary least squares (OLS) for this ARX model. The estimates are reported as

\[
y[k] - 0.414 y[k-1] + 0.611 y[k-2] = 2.002 u[k-1] \quad \pm (0.035) \quad (28)
\]

It can be observed that the quality of estimated parameters is equivalent from both the methods. It should be noticed that the proposed framework is capable of deriving the equation order also where the user is forced to supply the right combination of input-output order and delay to OLS algorithm.

4.2 Third Order LTI System

Consider a second order system with third order input dynamics shown below:

\[
y^*[k] - 0.3y^*[k-1] + 0.7y^*[k-2] = 1.2u^*[k-2] + 1.6u^*[k-3] \quad (30)
\]

Through this example, we show the functioning of proposed algorithm for the cases where input order is greater than output order.

Input is chosen to be full length and full band PRBS signal with sample size as 1023. The noise free output is generated according to Eqn. (30) and further it is corrupted with noise to follow ARX model. The noise variance is chosen to be 1.7 to maintain SNR of 6. A snapshot of the input-output data is shown in Figure 5.

As the order is assumed to be unknown, we consider high stacking lag order of \( L = 6 \), which implies there would be \( 2(L + 1) = 14 \) variables. The guessed order is chosen as
The estimated order would be $\eta = L - d + 1 = 6 - 4 + 1 = 3$, which is also equal to the maximum lag in the difference equation shown in Eqn. (30). The estimated noise variance is 1.75 which is also close to the true simulated value.

The convergence of estimated parameters with each iteration is shown in Table 2.

As the estimated order is observed to be equal to guessed order, we terminate the algorithm. It can be clearly observed that the estimated parameters is close to the true values used in simulation from Table 2.

Using the estimated model parameters in Eqn. (28), we evaluate the percentage fit from noise-free output and is observed to be 91%. The above simulations show that model order and parameters can be derived by careful inspection of generalized eigenvalues and eigenvectors. In this section, we considered two case-studies and demonstrated working of each step of the proposed algorithm. The notable takeaways are estimation of equation order and efficacy of the obtained estimates in automated manner without supplying any prior knowledge about the system.

5. CONCLUSION

In this article, we proposed a novel automated framework which derives the equation order and model parameters using generalized spectral decomposition framework. The equation order is determined by analyzing the generalized eigenvalues and subsequently the model parameters of linear difference equation can be obtained from generalized eigenvector. The algorithm can be viewed as a two step iterative procedure. Future work consists of extending the proposed framework to time-varying systems Rapisarda (2018) and from single channel to multichannel with more generalized and complex model structure Box-Jenkins model.

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Appendix A. APPENDIX: QZ ALGORITHM

The QZ algorithm (Moler and Stewart, 1973) is a numerical method to solve the generalized eigenvalue problem, $\mathbf{Ax} = \lambda \mathbf{Bx}$. The eigenvalues and eigenvectors can be easily obtained for invertible $\mathbf{B}$ but QZ algorithm solves the above equation without computation of $\mathbf{B}^{-1}$.

The key idea is to simultaneously transform $\mathbf{A}$ and $\mathbf{B}$ into generalized Schur form by using similarity transformations. This may be stated as we intend to find orthogonal matrices $\mathbf{Q}$ and $\mathbf{Z}$

$$\mathbf{Q}^\top (\mathbf{A} - \lambda \mathbf{B}) \mathbf{Z} = \mathbf{T} - \lambda \mathbf{S} \quad (A.1)$$

where $\mathbf{T}$ is of quasi-upper triangular form and $\mathbf{S}$ is of upper triangular shape. Eigenvalues and eigenvectors can be computed from the diagonals of triangular form.