Conformally Invariant Off-shell Strings

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ABSTRACT
Recent advances in non-critical string theory allow a unique continuation of critical
Polyakov string amplitudes to off-shell momenta, while preserving conformal invar-
ance. These continuations possess unusual, apparently stringy, characteristics, as
we illustrate with our results for three-point functions.

1. The Polyakov functional integral

The Polyakov path integral\(^1\) appears only a recipe for perturbative com-
putations of scattering amplitudes, yet it serves as the foundation for our entire
understanding of string theory. It is natural to ask whether this framework might
yield any insight into the off-shell properties of strings. We report on how the recent
progress in understanding non-critical strings provides a new technique to calculate
off-shell Polyakov amplitudes\(^2\). This approach preserves conformal invariance, and
so these amplitudes lend themselves to analysis by essentially the standard tech-
niques of conformal field theory.

Space-time scattering amplitudes of string excitations are calculated as cor-
relation functions of vertex operators in a functional integral over the metric on the
string world-sheet and the space-time string configurations:

\[
\left\langle \prod_i \int d^2 z_i \sqrt{g} V_i(z_i) \right\rangle \equiv \int \frac{Dg \, DX}{\text{vol.(Diff} \times \text{Weyl)}} \, e^{-S[g,X]} \prod_i \int d^2 z_i \sqrt{g} V_i(z_i). \tag{1}
\]

The measure is divided by the ‘volume’ of the symmetries of the classical action
\(S \equiv 1/(8\pi) \int d^2 z \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X_\mu\), with \(\mu = 1, \ldots, D\) — namely, diffeomorphisms
and local Weyl rescalings on the world-sheet. Choosing conformal gauge, \(g_{ab} \equiv

\[\text{Talk by R.C.M. at Strings ’93, Lawrence Berkeley Laboratory}\]
\[ e^{2\phi} g_{ab}(m), \text{ and fixing diffeomorphisms à la Faddeev-Popov, one finds that these functional integrals reduce to} \]

\[
\int \! dm \frac{D\phi}{\text{vol.}(\text{Weyl})} \frac{D X \ Det'_{\text{FP}}}{\text{vol.}(\text{c.k.v.})} e^{-S[g, X]} \prod_i \int \! d^2 z_i \sqrt{\hat{g}(m)} V_i(z_i). \tag{2}
\]

Here, c.k.v. stands for the conformal Killing vectors which arise if the world-sheet is a sphere or a torus, and \( dm \) denotes the measure for integrating over the moduli of surfaces with one or more handles. In Eq. (2), the Weyl factor should decouple from the theory, and the integration \( D\phi \) would cancel against the volume of the group of Weyl rescalings in the denominator. This decoupling is only actually achieved if Weyl rescaling survives as a symmetry of the quantum path integral. This requires that \( D = 26 \) in order to cancel the anomalous dependences on the Weyl field in the measure factor, \( DX \ Det'_{\text{FP}}/\text{vol.}(\text{c.k.v.}) \). Also, one must impose various spacetime mass-shell and polarization/gauge conditions on the external string states to avoid any anomalous Weyl dependences from normal-ordering the vertex operators. Combined, these restrictions ensure that \( \phi \) is decoupled from on-shell correlation functions in critical string theory, and the Weyl factor simply disappears from the functional integral.

Therefore the mass-shell conditions are obtained from requiring Weyl invariance. It follows, in the Polyakov approach, that the calculation of amplitudes for off-shell string states requires the ability to compute correlation functions of vertex operators with an anomalous Weyl dependence, in the normalized measure \( D\phi/\text{vol.}(\text{Weyl}) \). Why are such computations difficult? The problem resides in the non-linearity of the Riemannian metric that defines \( D\phi \). The norm on infinitesimal changes of the conformal factor is constructed with the full world-sheet metric \( g_{ab} \)

\[
(\delta\phi, \delta\phi) = \int \! d^2 x \sqrt{\hat{g}}(\delta\phi)^2 = \int \! d^2 x \sqrt{\hat{g}} e^{2\phi}(\delta\phi)^2, \tag{3}
\]

which then explicitly depends on \( \phi \). The functional integral over \( \phi \) would be a standard quantum field theory with the measure, \( D_0\phi \), defined by the translation-invariant norm, \( (\delta\phi, \delta\phi)_0 = \int \! d^2 x \sqrt{\hat{g}}(\delta\phi)^2 \). The relation between these two measures is remarkably simple,

\[
D\phi = D_0\phi \exp \left( S_L - \frac{\mu}{\pi} \int \! d^2 z e^{\alpha\phi} \right), \tag{4}
\]

where \( S_L \equiv \int \! \frac{d^2 z}{6\pi} \left[ \partial_\phi \bar{\partial} \phi + \frac{1}{4} \sqrt{\hat{g}} \hat{R} \phi \right] \). The ‘cosmological constant’ \( \mu \) is the coefficient of a local counterterm, and remains undetermined in this computation. The constant \( \alpha \) in this interaction is explicitly fixed (see below). This relation (4) originally arose in the study of two-dimensional gravity coupled to conformal matter\(^3\). It is important to note that the derivation of Eq. (4) is entirely independent of the rest
of the functional integrals involved. Thus it remains valid for our studies of off-shell amplitudes.

2. Correlation functions

To proceed further, we assume that the correlation functions of interest may be calculated with conformal field theory methods. For non-critical strings, this approach has been verified by comparison with results determined by matrix model techniques. The stress tensor deduced from $S_L$ is $T_L = \frac{1}{6} [ (\partial \phi)^2 - \partial^2 \phi ]$, which makes no contribution to the total central charge. Off-shell vertex operators receive exponential Weyl dressings, $e^{\beta \phi} V$ (just as matter operators in non-critical string theory do), where

$$\beta = \frac{1}{6} \left[ \sqrt{1 - 12 \delta} - 1 \right].$$  \hspace{1cm} (5)

Here, $\delta = k^2 + 2n$ (with $n = -1, 0, 1, \ldots$) is the mass-shell operator of the particular external string state. Explicitly, off-shell tachyon vertex operators are $e^{\beta \phi} e^{ik \cdot X}$ with $\delta = k^2 - 2$, and hence $\beta = (\sqrt{25 - 12k^2} - 1)/6$. The dressing exponents are chosen to vanish when the vertex operator goes on-shell (i.e., $\delta = 0$), which insures that the off-shell amplitudes reduce precisely to the usual on-shell amplitudes. Rather puzzling is the non-analyticity in this prescription at $\delta = \frac{1}{12}$. Further insight into quantum Liouville theory is needed to overcome this barrier, but for the present time, we will limit our attention to $\delta \leq \frac{1}{12}$ or $k^2 \leq \frac{25}{12}$.

Explicit calculations may be carried out for tree-level scattering amplitudes. We omit the details here, but as an example present the result for an off-shell tachyon three point amplitude $A = (\prod_{i=1}^{3} \int d^2z_i \ e^{\beta \phi} e^{ik_i \cdot X})$. The final result, valid for arbitrary external momenta subject only to the constraint $\delta_i \leq \frac{1}{12}$ or $k_i^2 \leq \frac{25}{12}$, is

$$A = X \Gamma(\frac{1+3\gamma}{2}) \Gamma(\frac{1-3\gamma}{2}) \prod_{\ell=0}^{3} \lambda(\gamma + \frac{1}{3}, 1 - 3\beta \ell) \lambda(-\gamma - \frac{1}{3}, \frac{3}{2} + 3\beta \ell)$$  \hspace{1cm} (6)

where we have defined $X \equiv [\mu \Gamma(\frac{1}{3})/\Gamma(\frac{2}{3})]^{-(3\gamma+1)/2}(\frac{2}{3})^{\gamma-2/3}, \gamma \equiv \sum_{i=1}^{3} \beta_i$ and $\beta_0 \equiv -\frac{1}{6}$, as well as

$$\lambda(z, b) = (2\pi)^{z/2} \exp[-3z + (\log 3 - E) f(z)] \prod_{\ell=0}^{\infty} \left( \exp \left[ -3z + \frac{f(z)}{\ell + 1} \right] \right.$$  \hspace{1cm} (7)

$$\left. \left[ \prod_{k=0}^{2} \frac{(\frac{3}{2} z + b + 3\ell + k)(\frac{3}{2} z + b + 3\ell + k + \frac{3}{2})}{(b + 3\ell + k)(b + 3\ell + k + \frac{3}{2})} \right]^{\ell+1} \right) .$$

Here $f(z) = \frac{3}{2} z^2 + (b - \frac{5}{6}) z$, and $E = .577 \ldots$ is Euler’s constant. The $\lambda$-function in Eq. (8) satisfies $\lambda(z+1, b) = \Gamma(\frac{3}{2} z + b)\lambda(z, b)$, amongst other properties.
One of the most interesting, and presumably most physically significant features of this amplitude is the infinite set of poles which it contains. Keeping in mind the restriction $\beta_i \geq -\frac{1}{6}$, the $\lambda$-functions introduce two sequences of poles of order $\ell + 1$ at $\beta_i = \ell + \frac{4}{3} + \frac{1}{\ell}$, and $\beta_i = \ell + \frac{4}{3} + \frac{5}{6}$ for $\ell = 0, 1, 2, \ldots$ and $m = 0, 1, 2$. These are leg poles depending on the external momenta of the individual tachyons. From the relation $k_i^2 = 2 - \beta_i - 3\beta_i^2$, one finds that the poles in the first sequence with $m = 0$, and all of the poles in the second sequence, do not correspond to the physical mass shell of external string states. The prefactor $\Gamma\left(\frac{1+3\gamma}{2}\right)\Gamma\left(\frac{1-3\gamma}{2}\right)$ also contains a sequence of simple poles at $\gamma = -1/3$ and $1/3 + 2n$ with $n$, an non-negative integer. In fact, the latter are cancelled by zeroes appearing in the product of $\lambda$-functions, in particular the $\ell = 0$ term. There remains a single simple pole in the amplitude at $\gamma = -1/3$. In contrast to the previous leg poles, this pole depends collectively on all of the external momenta. Note that $\gamma = -1/3$ is precisely the value of $\gamma$ at which the amplitude is independent of the cosmological constant, or equivalently the world-sheet area. This scale independence appears then to be the physical reason for the pole, but its occurrence is an entirely stringy feature.

3. Conclusions and prospects

It has been our aim here to show that the effort expended on the study of non-critical strings has important physical consequences in critical string theories. Any future progress in non-critical string physics, or in quantum Liouville theory, will be of use in understanding off-shell critical string physics. In particular, advances in the conformal field theory treatment of the Liouville correlators are clearly needed. There are no conceptual barriers to the extension of our results to supersymmetric strings, or to open string theories.

A striking feature of the amplitudes is the presence of poles that do not correspond to excitations in the matter sector (even if combined with the ghost sector). They may indicate the presence of excitations that are entirely stringy in nature. In particular, a pole which depends collectively on all of the external momenta, such as that at $\gamma = -1/3$, is entirely unknown in the amplitudes one obtains from a field theory. In field theories, the off-shell character of the amplitude is a function of individual external states. It is difficult then to imagine how this $\gamma$ dependence could be reproduced in a string field theory. Thus our results may indicate that some fundamentally new framework, other than string field theory, will be required to extend our understanding of critical string theory beyond the Polyakov path integral.

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