Abstract. Strongly coupled gravitational systems describe Einstein gravity and matter in the limit that Newton’s constant $G$ is assumed to be very large. The nonlinear evolution of these systems may be solved analytically in the classical and semiclassical limits by employing a Green function analysis. Using functional methods in a Hamilton-Jacobi setting, one may compute the generating functional (‘the phase of the wavefunctional’) which satisfies both the energy constraint and the momentum constraint. Previous results are extended to encompass the imposition of an arbitrary initial hypersurface. A Lagrange multiplier in the generating functional restricts the initial fields, and also allows one to formulate the energy constraint on the initial hypersurface. Classical evolution follows as a result of minimizing the generating functional with respect to the initial fields. Examples are given describing Einstein gravity interacting with either a dust field and/or a scalar field. Green functions are explicitly determined for (1) gravity, dust, a scalar field and a cosmological constant and (2) gravity and a scalar field interacting with an exponential potential. This formalism is useful in solving problems of cosmology and of gravitational collapse.

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In the limit that Newton’s constant $G$ approaches infinity, the mathematical equations describing Einstein gravity coupled to matter simplify dramatically. One may safely drop second order spatial gradients while first order spatial gradients are retained. Using Hamilton-Jacobi (HJ) theory and the semiclassical Green function method, a preceding paper (Salopek 1998) demonstrated how to construct a general class
of solutions for matter and gravity in the strong coupling limit. Here, those powerful methods will be extended to formulate the initial hypersurface problem.

Strongly coupled gravitational systems are useful in studies of cosmology and of gravitational collapse. In cosmology, strongly coupled gravity has been used to describe the evolution of ‘long-wavelength fields’ arising, for example, from inflation which then serve as the primordial initial conditions for structure formation. The wavelength of these fields exceeds the local value of the Hubble radius, and different spatial points are no longer in causal contact (Salopek 1991, Salopek and Bond 1990 and Salopek 1998). A strongly coupled expansion also appears in string theory formulations of cosmology (Veneziano 1997). In problems of gravitational collapse, strongly coupled gravity describes ‘velocity-dominated’ evolution where the gravitational potential terms which contain spatial gradients may be neglected as space collapses rapidly into a singularity. Numerical studies (Berger 1998, Berger and Moncrief 1993, Hern and Stewart 1998) indicate that many singularities that appear in general relativity are velocity-dominated. In addition, Hern and Stewart (1998) have shown numerically that the evolution of a certain class of Gowdy models may mimic that of velocity-dominated models even when spatial derivative terms are not negligible.

The two essential ideas behind generating general solutions to strongly coupled gravity and matter are: (1) computing the semiclassical Green function solution of the energy constraint and (2) convolving the Green function with an arbitrary, gauge-invariant, initial state. In this paper, I will generalize the Hamilton-Jacobi formalism for strongly coupled gravity and matter to encompass the situation where one specifies the fields on some arbitrary initial hypersurface in superspace.

Hamilton-Jacobi theory for general relativity is useful for two very important reasons:

(1) It can be used to formulate a primitive theory of quantum gravity. At present, there are many problems in solving the Wheeler-DeWitt equation (DeWitt 1967) which is the canonical equation for quantum gravity. The worst is perhaps the problem of infinities. This issue may be temporarily postponed by expanding the wavefunctional in powers of \( \hbar \). At the lowest order, one must solve the Hamilton-Jacobi equation for general relativity which is mathematically self-consistent and does not require renormalization. This paper develops tools that allow one to solve the HJ equation in the strongly coupled limit. At higher order in \( \hbar \), infinities would certainly appear and they would have to treated at some later date presumably using some new ideas. However, an expansion in powers of \( \hbar \) has been very successful in field theory, and one strongly suspects that it will be useful in the gravitational context although much work remains to be done. (Future prospects for a quantum description of cosmology and gravity have been discussed by Hartle 1997.)

(2) It yields a covariant formulation of the gravitational field. Since its conception in
the 1960's, it has been known that the Wheeler-DeWitt equation refers to the 3-metric $\gamma_{ab}$, but that it does not depend on the lapse $N$ nor the shift $N_i$. Quite often, this leads to problems in interpreting various approximate solutions. In a semiclassical setting, the Hamilton-Jacobi equation also does not depend on $N$ and $N_i$. This is actually a blessing because a solution of the HJ equation is valid for all choices of the lapse and shift. In this sense, it provides a covariant description of gravity.

HJ methods have been particularly useful in solving the long-wavelength problem of cosmology. In fact, they yield a transparent and elegant solution for the nonlinear evolution of long-wavelength fluctuations in cosmological models (Salopek 1991, Salopek and Bond 1990, Salopek 1998). (They are also being applied to string cosmology by Saygili 1998). In the long-wavelength limit, I feel that attempts to include nonlinear effects using higher order perturbation methods by Abramo et al (1997) are actually more difficult to apply and interpret, and they tend to obscure the fundamental simplicity of the long-wavelength problem: different spatial points evolve as homogeneous and independent universes. (Unruh 1998 has discussed the results of Abramo et al 1997 in more detail.) HJ methods are very powerful because they allow one to glue these independent spatial points using the momentum constraint to form one universe. This feature is crucial if one wishes to compute higher order terms in the strong coupling expansion (Parry, Salopek and Stewart 1994). Nonlinear long-wavelength fields also figure prominently in stochastic inflation (Vilenkin 1983, Starobinski 1986, Salopek and Bond 1991, Linde et al 1994, Vilenkin 1998). There, the probability distribution for long-wavelength fields is governed by a diffusion-type equation.

Other applications of HJ theory to cosmology include: (1) the construction of inflation models which produce non-Gaussian fluctuations (Salopek 1992) —these models are still of observational interest (Fan and Bardeen 1992, Moscardini et al 1993); (2) a relativistic formulation of the Zel'dovich approximation which describes how nonlinear pancake structures form in the distribution of galaxies (Croudace et al 1994, Salopek et al 1994); (3) a covariant computation of density perturbations from inflation with one or two scalar fields (Salopek and Stewart 1995, Salopek 1995).

In classic textbooks (Lanczos 1970, Goldstein 1981), one usually discusses Hamilton-Jacobi theory for unconstrained systems in the context of canonical transformations. However, for general relativity, the presence of constraints prevents a straightforward application of these standard methods. For example, one may have expected that one could apply the method of canonical transformations to gravity after a suitable phase space-reduction. However, since gravity is a nonlinear theory, this has not been achieved in practice except in some very special cases (linear perturbation theory, minisuperspace models, etc.). Rather, one should apply the powerful Dirac formulation of constrained systems to the constraints of general relativity (interacting with a scalar field $\phi$ just to
be specific),

\[ \mathcal{H}[\pi^{ab}(x), \pi^{\phi}(x), \gamma_{ab}(x), \phi(x)] = 0, \quad \text{energy constraint} \quad (1a) \]

\[ \mathcal{H}_d[\pi^{ab}(x), \pi^{a}(x), \gamma_{ab}(x), \phi(x)] = 0, \quad \text{momentum constraint} \quad (1b) \]

where one replaces the momenta by functional derivatives of the generating functional, \( S[\gamma_{ab}(x), \phi(x)] \):

\[ \pi^{ab}(x) = \frac{\delta S}{\delta \gamma_{ab}(x)}, \quad \pi^{\phi}(x) = \frac{\delta S}{\delta \phi(x)}. \quad (2) \]

The two constraint equations \((1a-b)\) are self-contained equations for \( S \), and they may be taken as the starting point for a HJ formulation of gravity. If \( S \) is real, one recovers classical general relativity by integrating the definition of the momenta:

\[ \left( \dot{\gamma}_{ij} - N_{ij} - N_{ji} \right) / N = 2\kappa \gamma^{-1/2} \left( 2\gamma_{ik}\gamma_{jl} - \gamma_{ij}\gamma_{kl} \right) \frac{\delta S}{\delta \gamma_{kl}}, \quad (3a) \]

\[ \left( \dot{\phi} - N^i\phi_{,i} \right) / N = \kappa \gamma^{-1/2} \frac{\delta S}{\delta \phi}, \quad (3b) \]

where

\[ \kappa \equiv 8\pi G \equiv \frac{8\pi}{m_P^2} \quad (4) \]

denotes the gravitational coupling constant. Actually, HJ methods allow one to integrate the classical evolution equations \((3a-b)\) in an elegant way.

In section 1, I give a brief review of the semiclassical Green function method of solving strongly coupled gravitational systems as advanced by Salopek (1998). In section 2, I describe how to generalize the method by introducing a Lagrange multiplier in the generating functional to include the case of specifying the fields on an arbitrary initial hypersurface in superspace. In earlier work, it was found that one of the initial fields in the Green function must be set to zero. A specific choice was made previously, but here I would like to examine other possibilities.

In section 3, I show how to reduce the number of degrees of freedom appearing in the energy constraint by four. In sections 4 and 5, I will construct nontrivial Green functions for the following situations:

1. Gravity, dust, a massless scalar field and a cosmological constant,
2. Gravity and a scalar field with exponential potential.

Section 6 uses specific examples to illustrate the generalized Green function method as applied to gravity, dust and a scalar field (without cosmological constant). Section 7 adds a cosmological constant. In section 8, I consider an advanced example involving gravity and dust. Here the initial hypersurface is defined in terms of the initial dust field and the Ricci scalar of the initial 3-metric. Since this system is not analytically
solvable, I expand the generating functional in a series. This example illustrates quite clearly how the energy constraint restricts the initial fields. I also show how to construct the 4-metric using the HJ formalism. A summary and conclusions are given in section 9.

(In this paper, the matter fields have been rescaled by factors of the gravitational coupling constant $\kappa$, and consequently $\kappa$ disappears from most expressions. Consult Salopek (1998) for definitions.)

1. Review of HJ solutions for strongly coupled gravity and matter

In a HJ formulation, the energy constraint and the momentum constraint describing the 3-metric $\gamma_{ab}(x)$ interacting with a dust field $\chi(x)$ are, respectively, (see, e.g., Salopek 1998),

$$\mathcal{H}(x)/\kappa = \gamma^{-1/2} \left( 2\gamma_{ac}\gamma_{bd} - \gamma_{ab}\gamma_{cd} \right) \frac{\delta S}{\delta \gamma_{ab}} \frac{\delta S}{\delta \gamma_{cd}} + \frac{\delta S}{\delta \chi} = 0,$$

$$\mathcal{H}_i(x) = -2 \left( \gamma_{ik} \frac{\delta S}{\delta \gamma_{kj}} \right)_{,j} + \frac{\delta S}{\delta \gamma_{kl}} \gamma_{kl,i} + \frac{\delta S}{\delta \chi} \chi_{,i} = 0. \tag{5b}$$

I will refer to $(\gamma_{ab}(x), \chi(x))$ as the ‘original fields’. The semiclassical Green function method of solving these two equations is described below.

1.1. Semiclassical Green function method

A complete solution of the energy constraint eq.(5a) is one where the number of parameter fields equals the number of original fields. The Green function solution given below is a complete solution to the energy constraint with the additional property that the parameter fields, $[\gamma_{ab}^{(0)}(x), \chi^{(0)}(x)]$, may be interpreted as ‘initial fields’ on some initial hypersurface in superspace:

$$\mathcal{G}[(\gamma_{ab}(x), \chi(x)|\gamma_{ab}^{(0)}(x), \chi^{(0)}(x)) =$$

$$\frac{4}{3} \int d^3x \frac{1}{\left( \chi(x) - \chi^{(0)}(x) \right)} \left[ 2\gamma^{1/4} \gamma^{1/4}_{(0)} \cosh\left( \frac{3}{8} z \right) - \gamma^{1/2} - \gamma^{1/2}_{(0)} \right], \tag{6a}$$

where

$$z = -\frac{1}{2} \sqrt{\text{Tr} \left\{ \ln \left( [h][h^{-1}] \right) \ln \left( [h][h^{-1}] \right) \right\}}, \tag{6b}$$

and $[h]$ and $[h^{(0)}]$ are matrices, each with unit determinant, whose components are given by

$$[h]_{ab} = \gamma^{-1/3}_{ab}, \quad [h^{(0)}]_{ab} = \gamma^{-1/3}_{(0)} \gamma^{(0)}_{ab}. \tag{6c}$$
The integrand of the Green function, eq. (6a), has the familiar HJ form,
\[
\frac{1}{2} \frac{D^2}{(t - t(0))},
\]  
(7a)
describing a free particle; the time \( t \) may be identified with the proper time \( \chi \) of the dust particle, and \( D^2 \) is the distance-squared between ‘two points’, \((\gamma_{ab}, \gamma_{ab}^{(0)})\), in the space of 3-metrics:
\[
D^2 = \frac{8}{3} \left[ 2\gamma^{1/4} \gamma_{(0)}^{1/4} \cosh(\sqrt{\frac{3}{8}} z) - \gamma^{1/2} - \gamma_{(0)}^{1/2} \right].
\]  
(7b)
This distance function, eq. (7b), was first derived by DeWitt (1967).

In order to ensure consistency of the approach, it was necessary to set one of the parameter fields to zero. (This point will be discussed in further detail in section 2). For example, in Salopek (1998) the initial dust field was set to zero:
\[
\chi_{(0)}(x) = 0.
\]  
(8)
One then constructs a general solution to the energy constraint and the momentum constraint through a superposition over an arbitrary ‘initial state’ \( \mathcal{I} \):
\[
\mathcal{S}[\gamma_{ab}(x), \chi(x)] = \mathcal{G}[\gamma_{ab}(x), \chi(x) | \gamma_{ab}^{(0)}(x), \chi_{(0)}(x) = 0] + \mathcal{I}[\gamma_{ab}^{(0)}],
\]  
(9a)
where the initial 3-metric \( \gamma_{ab}^{(0)}(x) \) has been chosen to minimize \( \mathcal{S} \),
\[
0 = \frac{\delta \mathcal{G}}{\delta \gamma_{ab}^{(0)}} + \frac{\delta \mathcal{I}}{\delta \gamma_{ab}^{(0)}}.
\]  
(9b)
Here \( \mathcal{I} \) is an arbitrary “gauge-invariant” functional of the initial 3-metric:
\[
0 = -2 \left( \gamma_{ik}^{(0)} \frac{\delta \mathcal{I}}{\delta \gamma_{kj}^{(0)}} \right)_{,j} + \frac{\delta \mathcal{I}}{\delta \gamma_{kl}^{(0)}} \gamma_{(0)}^{(0)} g_{kl,i}.
\]  
(9c)
That is, \( \mathcal{I}[\gamma_{ab}^{(0)}(x)] \), is invariant under reparametrizations of the spatial coordinates. After solving the minimization equation (9b), one solves for the initial 3-metric as a function of the original fields,
\[
\gamma_{ab}^{(0)}(x) \equiv \gamma_{ab}^{(0)} [\gamma_{ab}(x), \chi(x)],
\]  
(10)
One then substitutes this result into eq. (9d) to determine the generating functional \( \mathcal{S}[\gamma_{ab}(x), \chi(x)] \) which depends solely on the original fields.

The solution (9c) is an Ansatz. It was determined by trial and error. One can verify that it satisfies the energy constraint (5a) by computing the functional derivatives of \( \mathcal{S} \) with respect to \( \gamma_{ab}(x) \) and \( \chi(x) \). It also satisfies the momentum constraint (5b) because the initial state \( \mathcal{I} \) was chosen to satisfy the momentum constraint initially, eq. (9c), and the momentum constraint is preserved upon evolution because its Poisson bracket with the Hamiltonian vanishes. Many examples given in Salopek (1998) demonstrate how to exploit the Green function method and the superposition principle in practice.
1.2. Initial hypersurface prescription

The simplest interpretation of setting the parameter field \( \chi(0)(x) \) to zero, eq. (8), is that one is choosing an ‘initial hypersurface’ where the original dust field vanishes. However, this choice is not forced upon us. In fact, in defining an initial hypersurface, one may set any one scalar function of the parameter fields to zero. I will refer to this process as the initial hypersurface prescription. For example, if, in addition to a dust field, there is a scalar field \( \phi(x) \) and if \( \phi(0)(x) \) denotes the corresponding initial field, one can instead set

\[
\phi(0)(x) = 0. \tag{11a}
\]

This choice corresponds to defining an initial hypersurface where the scalar field vanishes. In addition, one now assumes that \( \mathcal{I} \) is a gauge-invariant functional,

\[
\mathcal{I} \equiv \mathcal{I}[\gamma(0)_{ab}(x), \chi(0)(x)], \tag{11b}
\]

of the remaining parameter fields, \( \gamma(0)_{ab}(x) \) and \( \chi(0)(x) \). The initial hypersurface condition can presumably contain spatial derivatives such as,

\[
\phi^2(0) + \gamma(0)_{ab}(x) \phi(0)_{,a} \phi(0)_{,b} = 0, \tag{11c}
\]

or even the Ricci scalar \( R(0) \) associated with the initial 3-metric such as,

\[
\chi(0) - BR(0) = 0, \quad (B \text{ is a constant}). \tag{11d}
\]

This last example will be treated in detail in section 8.

In general one should be able to impose any one constraint on the parameter field such as eqs. (11a-c) by employing a Lagrange multiplier. I will develop this idea in the next section.

2. The Lagrange multiplier method

Although a Green function solution such as eq. (6a-c) to the energy constraint (1d) is definitely a useful device, it leads to an inconsistency if one tries to construct solutions to the energy constraint and the momentum constraint by superimposing over all of the initial fields. This mathematical reason for this inconsistency is that the initial fields are not all independent since they obey the energy constraint on the initial hypersurface. Apparently, the initial energy constraint reduces the number of degrees of freedom by one field per spatial point. In order to avoid this problem, it was suggested in Salopek (1998) that one arbitrarily set one of the parameter fields to zero before superimposing over the remaining parameter fields. As was argued on physical grounds in section 1.2, this choice defines the initial hypersurface. In the present section, I will expand on this theme by introducing a Lagrange multiplier \( L(x) \) to impose this reduction in the number
of parameter fields. This simple extension leads to a much wider field of applications for the HJ formalism. The role of the initial energy constraint will be discussed using an example in section 8.

2.1. Formal HJ solution for gravity, dust and a scalar field

In this section, I will construct a formal HJ solution to a strongly coupled system consisting of gravity, a dust field and a scalar field $\phi(x)$ with potential $V(\phi)$. This system will be described by the following constraint equations:

$$
\mathcal{H}(x)/\kappa = \gamma^{-1/2} \left( 2\gamma_{ac}\gamma_{bd} - \gamma_{ab}\gamma_{cd} \right) \frac{\delta S}{\delta \gamma_{ab}} \frac{\delta S}{\delta \gamma_{cd}} + \frac{\delta S}{\delta \chi} + \frac{\gamma^{-1/2}}{2} \left( \frac{\delta S}{\delta \phi} \right)^2 + \gamma^{1/2} V(\phi) = 0,
$$

$$
\mathcal{H}_i(x) = -2 \left( \gamma_{ik} \frac{\delta S}{\delta \gamma_{kj}} \right)_{,j} + \frac{\delta S}{\delta \gamma_{kl}} \gamma_{kl,i} + \frac{\delta S}{\delta \phi} \phi,i + \frac{\delta S}{\delta \chi} \chi,i = 0.
$$

(12a)

(12b)

For this case, there are eight degrees of freedom per spatial point associated with the original fields, $\gamma_{ab}(x)$, $\phi(x)$ and $\chi(x)$.

An Ansatz solution to the energy and momentum constraints is

$$
S[\gamma_{ab}(x), \phi(x), \chi(x)] = \mathcal{G} + \mathcal{I} + \mathcal{L}.
$$

(13)

I will assume that the Green function $\mathcal{G}$ satisfies the energy constraint eq.(12a) with $S$ replaced by $\mathcal{G}$:

$$
\mathcal{H}(x)/\kappa = \gamma^{-1/2} \left( 2\gamma_{ac}\gamma_{bd} - \gamma_{ab}\gamma_{cd} \right) \frac{\delta \mathcal{G}}{\delta \gamma_{ab}} \frac{\delta \mathcal{G}}{\delta \gamma_{cd}} + \frac{\delta \mathcal{G}}{\delta \chi} + \frac{\gamma^{-1/2}}{2} \left( \frac{\delta \mathcal{G}}{\delta \phi} \right)^2 + \gamma^{1/2} V(\phi) = 0.
$$

(14a)

It is a complete solution in that it depends on eight inhomogeneous parameter fields (initial fields), $\gamma_{ab}^{(0)}(x)$, $\phi^{(0)}(x)$ and $\chi^{(0)}(x)$:

$$
\mathcal{G} \equiv \mathcal{G}[\gamma_{ab}(x), \phi(x), \chi(x)|\gamma_{ab}^{(0)}(x), \phi^{(0)}(x), \chi^{(0)}(x)].
$$

(14b)

\mathcal{I}, the ‘generalized initial state’, is a functional of all the parameter fields,

$$
\mathcal{I} \equiv \mathcal{I}[\gamma_{ab}^{(0)}(x), \phi^{(0)}(x), \chi^{(0)}(x)].
$$

(14c)

The new ingredient, $\mathcal{L}$, is a functional of the parameter fields which is linear in the Lagrange multiplier $L(x)$,

$$
\mathcal{L} = \int d^3x \gamma_{ab}^{1/2} L(x) f[\gamma_{ab}^{(0)}(x), \phi^{(0)}(x), \chi^{(0)}(x)].
$$

(14d)

Here $f$ is some scalar function of the parameter fields, $\gamma_{ab}^{(0)}(x)$, $\phi^{(0)}(x)$ and $\chi^{(0)}(x)$ which will, in general, contain spatial derivatives. I will demand that $\mathcal{I}$ and $\mathcal{L}$ be invariant
under spatial coordinate reparameterizations of the parameter fields and the Lagrange multiplier $L(x)$:

\[ 0 = -2 \left( \frac{\gamma^{(0)}_{ik} \delta L}{\delta \gamma^{(0)}_{kl}} \right)_{,j} + \frac{\delta L}{\delta \gamma^{(0)}_{kl}} \gamma^{(0)}_{kl,i} + \frac{\delta L}{\delta \phi^{(0)}(x)} \phi^{(0)}_{,i} + \frac{\delta L}{\delta \chi^{(0)}(x)} \chi^{(0)}_{,i} \]  

\[ 0 = -2 \left( \gamma^{(0)}_{ik} \frac{\delta L}{\delta \gamma^{(0)}_{kl}} \right)_{,j} + \frac{\delta L}{\delta \gamma^{(0)}_{kl}} \gamma^{(0)}_{kl,i} + \frac{\delta L}{\delta \phi^{(0)}(x)} \phi^{(0)}_{,i} + \frac{\delta L}{\delta \chi^{(0)}(x)} \chi^{(0)}_{,i} + \frac{\delta L}{\delta L} L_{,i} \]  

The parameter fields and the Lagrange multiplier are chosen to minimize the generating functional, eq.(13):

\[ 0 = \frac{\delta S}{\delta \gamma^{(0)}_{ab}(x)} , \]  

\[ 0 = \frac{\delta S}{\delta \phi^{(0)}(x)} , \]  

\[ 0 = \frac{\delta S}{\delta \chi^{(0)}(x)} , \]  

\[ 0 = \frac{\delta S}{\delta L(x)} . \]  

The minimization process leads to a solution of the classical Einstein equations in the strongly coupled limit.

Variation with respect to the Lagrange multiplier $L(x)$, eq.(16d), implies that the parameter fields are constrained according to,

\[ 0 = f \left[ \gamma^{(0)}_{ab}(x), \phi^{(0)}(x), \chi^{(0)}(x) \right] . \]  

This effectively reduces the number of parameter fields by one in accordance with the initial hypersurface prescription of section 1.2. The minimization method will be successful if one can solve eqs.(16a-d) for the parameter fields and the Lagrange multiplier in terms of the original fields:

\[ \gamma^{(0)}_{ab}(x) \equiv \gamma^{(0)}_{ab}[\gamma_{ab}(x), \phi(x), \chi(x)] , \]  

\[ \phi^{(0)}(x) \equiv \phi^{(0)}[\gamma_{ab}(x), \phi(x), \chi(x)] , \]  

\[ \chi^{(0)}(x) \equiv \chi^{(0)}[\gamma_{ab}(x), \phi(x), \chi(x)] , \]  

\[ L(x) \equiv L[\gamma_{ab}(x), \phi(x), \chi(x)] . \]  

If, for example, there are too many parameter fields, something will go wrong at this stage. Substitution of the above into eq.(13) leads to the ultimate goal, the generating functional $S$, which depends only on the original fields.
2.2. Justification of Ansatz

In order to justify the Ansatz (13), it is necessary to compute the functional derivatives of the generating functional with respect to the original fields.

Before one applies the minimization prescription, $S$ is a functional of the original fields, the parameter fields and the Lagrange multiplier:

$$S \equiv S \left[ \gamma_{ab}(x), \phi(x), \chi(x)|\gamma^{(0)}_{ab}(x), \phi(0)(x), \chi(0)(x), L(x) \right].$$

After applying the minimization prescription, the parameter fields and the Lagrange multiplier become functions of the original fields through eqs.(18a-d). Hence a functional derivative of $S$ with the respect to one of the original fields, say $\phi(x)$, can be computed by the chain rule:

$$\left. \frac{\delta S}{\delta \phi(x)} \right|_{\gamma_{ab}, \chi} = \left. \frac{\delta G}{\delta \phi(x)} \right|_{\gamma_{ab}, \chi, \gamma^{(0)}_{ab}, \phi(0), \chi(0)} + \int d^3y \left[ \frac{\delta S}{\delta \gamma^{(0)}_{ab}(y)} \frac{\delta \gamma^{(0)}_{ab}(y)}{\delta \phi(x)} + \frac{\delta S}{\delta \phi(0)(y)} \frac{\delta \phi(0)(y)}{\delta \phi(x)} + \frac{\delta S}{\delta \chi^{(0)}(y)} \frac{\delta \chi^{(0)}(y)}{\delta \phi(x)} + \frac{\delta S}{\delta L(y)} \frac{\delta L(y)}{\delta \phi(x)} \right].$$

(20a)

All of the terms in eq.(20a) which are integrated over $y$ vanish by virtue of the minimization conditions, eqs.(16a-d). Hence a functional derivative of $S$ with respect to $\phi(x)$ coincides with a functional derivative of $G$ with respect to $\phi(x)$:

$$\left. \frac{\delta S}{\delta \phi(x)} \right|_{\gamma_{ab}, \chi} = \left. \frac{\delta G}{\delta \phi(x)} \right|_{\gamma_{ab}, \chi, \gamma^{(0)}_{ab}, \phi(0), \chi(0)}.$$

(20b)

This argument also holds for functional derivatives with respect to the other original fields, $\gamma_{ab}(x)$ and $\chi(x)$.

Hence, $S$ given by eq.(13) satisfies the energy constraint (12a) because:

(1) $G$ satisfies the energy constraint (14a), and

(2) functional derivatives of $G$ and $S$ are identical, which was illustrated in the previous paragraph.

Moreover, $S$ also satisfies the momentum constraint (12b) since I demanded invariance under reparameterization of the spatial coordinates of both $I$ and $L$ through eqs.(15a-b). Since reparametrization invariance is valid initially, it is guaranteed to be valid at other times.

Thus, $S$ given by eq.(13) is a HJ solution to both the energy constraint and the momentum constraint provided one can solve for the initial fields and the Lagrange multiplier in terms of the original fields, eqs.(18a-d).
2.2.1. Discussion of Lagrange multiplier method  I will now demonstrate the physical significance of the Lagrange multiplier method. The functionals, $\mathcal{I}$ and $\mathcal{L}$, appearing in eq.(13) effectively define the ‘initial setting’ of the parameter fields.

As was mentioned briefly in section 1.2, different choices of the $f$ function, eq.(17), specify the initial hypersurface. For example, if

$$ f = \chi(0) , $$

then variation of the Lagrange multiplier implies that $\chi(0)(x)$ vanishes. After setting $\chi(0)(x) = 0$ in the generating functional, eq.(13), one recovers eq.(9a-c) which is the result given in Salopek (1998). There, it was shown by explicit construction that if the dust field were zero, $\chi(x) = 0$, then the generating functional $\mathcal{S}$ and the initial state $\mathcal{I}$ coincide:

$$ \mathcal{S}[\gamma_{ab}(x), \chi(x) = 0] = \mathcal{I}[\gamma_{ab}^{(0)}(x) = \gamma_{ab}(x)] . $$

(21b)

In this sense, $\chi(x) = 0$ denotes the initial hypersurface which was chosen among all possible choices by setting the parameter field $\chi(0)(x) = 0$.

If on the other hand,

$$ f = \phi(0) , $$

(22a)

then the scalar field is uniformly zero on the initial hypersurface: $\phi(x) = 0$. As was shown in Salopek (1998) for the case of a single scalar field interacting with gravity, if $\phi(x) = 0$, the generating functional $\mathcal{S}$ and initial state $\mathcal{I}$ coincide:

$$ \mathcal{S}[\gamma_{ab}(x), \phi(x) = 0] = \mathcal{I}[\gamma_{ab}^{(0)}(x) = \gamma_{ab}(x)] . $$

(22b)

Once again, setting a parameter field to zero effectively determines the initial hypersurface in superspace.

The choices for $f$ are limitless and they really depend on the problem at hand. The reader should be warned, however, that if $f$ depends on spatial derivatives of the parameter fields, then it may be difficult in practice to solve for the Lagrange multiplier and the parameter fields as required by eqs.(18a-d). Such a case is discussed in section 8.

The general formalism is now complete. The basic result is encapsulated in the simple expression for the generating functional, eq.(13): $\mathcal{S} = \mathcal{G} + \mathcal{I} + \mathcal{L}$. The Green function $\mathcal{G}$ effectively describes how the system described by the original fields $[\gamma_{ab}(x), \phi(x), \chi(x)]$ evolves from initial fields $[\gamma_{ab}^{(0)}(x), \phi^{(0)}(x), \chi^{(0)}(x)]$. $\mathcal{I}$ and $\mathcal{L}$ describe the initial setting. $\mathcal{I}$ is the initial state; $\mathcal{L}$, the Lagrange multiplier term, prescribes the initial hypersurface. Classical evolution follows from the minimization of the generating functional with respect to the initial fields and the Lagrange multiplier.
3. Reduced energy constraint

In sections 4 and 5, I will expand the list of known Green functions to include (1) gravity, dust, a massless scalar field and a cosmological constant and (2) gravity and a scalar field with an exponential potential. Before constructing these solutions, I will describe how to reduce the energy constraint to a more manageable form.

In searching for the Green function solution to the energy constraint eq.\((12a)\) describing gravity, dust, and a scalar field with potential, one must deal with eight degrees of freedom: two matter fields \(\phi\) and \(\chi\), and the six components of the symmetric matrix, \(\gamma_{ab}\). One may think that it is hopeless to solve this equation, but the presence of symmetries makes the task manageable. One attempts a solution of the form,

\[ G \equiv G[\phi(x), \chi(x), \alpha(x), z(x)] , \tag{23a} \]

where the 3-metric degrees of freedom are parameterized by \(\alpha\) and \(z\),

\[ \alpha = \frac{1}{6} \ln \gamma , \quad z = \frac{1}{2} \sqrt{\text{Tr} \left\{ \ln \left[ [h]^{-1}_{(0)} \right] \ln \left( [h]^{-1}_{(0)} \right) \right\}} , \tag{23b} \]

and \([h]\) and \([h^{(0)}]\) were defined in eq.\((6a)\). The ‘reduced energy constraint’ involves only four degrees of freedom:

\[ \frac{\delta G}{\delta \chi} - \frac{1}{12} e^{-3\alpha} \left( \frac{\delta G}{\delta \alpha} \right)^2 + \frac{1}{2} e^{-3\alpha} \left( \frac{\delta G}{\delta z} \right)^2 + \frac{1}{2} e^{-3\alpha} \left( \frac{\delta G}{\delta \phi} \right)^2 + e^{3\alpha} V(\phi) = 0 , \tag{23c} \]

(reduced energy constraint).

Not every solution to the energy constraint has the form \((23a)\) but the Green function does. In fact, the Green function solutions given in sections 4 and 5 are derived from the reduced energy constraint \((23c)\). It is much easier to work with the reduced equation than the full energy constraint \((12a)\) because the former does not contain tensor indices.

4. Green function solution: gravity, dust, massless scalar field with cosmological constant

If the potential \(V(\phi) = V_0\) describes a cosmological constant, then the Green function solution of the energy constraint \((12a)\) for gravity, dust and a massless scalar field is:

\[ G[\gamma_{ab}(x), \chi(x), \phi(x)|\gamma_{ab}^{(0)}(x), \chi(0)(x), \phi(0)(x)] = \int d^3x \frac{2H_0}{\sinh \left[ \frac{3H_0}{2} \left( \chi - \chi(0) \right) \right]} \]

\[ \left\{ 2\gamma^{1/4} \gamma^{1/4} (0) \cosh \left( \sqrt{\frac{3}{8}} \sqrt{z^2 + (\phi - \phi(0))^2} \right) \right. \]

\[ - \left( \gamma^{1/2} + \gamma^{1/2} (0) \right) \cosh \left[ \frac{3H_0}{2} \left( \chi - \chi(0) \right) \right] \} , \tag{24a} \]
where
\[ H_0 = \sqrt{\frac{V_0}{3}}. \] (24b)

This solution depends on eight parameter fields, \( (\gamma_{ab}^{(0)}(x), \chi^{(0)}(x), \phi^{(0)}(x)) \). I will now discuss special cases of this rich solution.

4.1. Gravity, dust and cosmological constant

The Green function solution for gravity, dust and a cosmological constant (i.e., no scalar field) is found by setting
\[ \phi - \phi^{(0)} = 0 \] (25)
in eq.(24a). A complete solution depending on seven parameters, \( (\gamma_{ab}^{(0)}(x), \chi^{(0)}(x)) \), is:
\[
G[\gamma_{ab}(x), \chi(x) | \gamma_{ab}^{(0)}(x), \chi^{(0)}(x)] = \int d^3x \frac{2H_0}{\sinh \left[ \frac{3H_0}{2} (\chi - \chi^{(0)}) \right]} \left\{ 2 \gamma^{1/4} \gamma^{1/4} \frac{1}{8} \sqrt{3} \cosh \left( \sqrt{\frac{3}{8}} z \right) - \left( \gamma^{1/2} + \gamma^{1/2}^{(0)} \right) \cosh \left[ \frac{3H_0}{2} (\chi - \chi^{(0)}) \right] \right\}. \] (26)

One can readily verify that the above solution reduces to the case of gravity and dust, eq.(6a), in the limit that the cosmological constant vanishes, \( H_0 \to 0 \).

In general, it is not permissible to arbitrarily specify original fields in the Green function (24a), and then expect that they will satisfy the energy constraint. However, the solution given in eq.(26) may be derived from eq.(24a) using the following argument. If one assumes that the functionals, \( \mathcal{I} \) and \( \mathcal{L} \), in eq.(13) that define the initial setting are independent of \( \phi^{(0)}(x) \), then eq.(25) follows from minimization with respect to \( \phi^{(0)}(x) \). As a result, the scalar field dependence in the Green function (24a) drops out, and one loses this degree of freedom.

4.2. Gravity, massless scalar field and cosmological constant

The Green function,
\[
G[\gamma_{ab}(x), \phi(x) | \gamma_{ab}^{(0)}(x), \phi^{(0)}(x)] = -\sqrt{\frac{4V_0}{3}} \int d^3x \left[ \gamma + \gamma^{(0)} - 2\gamma^{1/2} \gamma^{1/2} \cosh \left( \sqrt{\frac{3}{2}} \sqrt{z^2 + (\phi - \phi^{(0)})^2} \right) \right]^{1/2}, \] (27)
for gravity, a massless scalar field and a cosmological constant (i.e., no dust field) was given in Salopek (1998). The sign preceding the Green function is arbitrary and I have chosen it to correspond to an expanding universe. One may derive it from the Green
function eq. (24) in the following way: if the functionals, \( \mathcal{I} \) and \( \mathcal{L} \), in eq. (13) that define the initial setting are independent of \( \chi(0)(x) \), then minimization of \( \mathcal{S} \) with respect to \( \chi(0)(x) \) implies
\[
\frac{\delta \mathcal{G}}{\delta \chi(0)(x)} = 0
\]
which yields eq. (27). Thus the dust degree of freedom drops out.

4.3. Gravity, dust and a massless scalar field

The Green function for dust, gravity and a massless scalar field (\textit{i.e.}, no cosmological constant) is found from eq. (24) by considering the limit that the cosmological constant vanishes, \( H_0 \to 0 \):
\[
\mathcal{G}^{\gamma_{ab}(x), \phi(x), \chi(x)|\gamma_{ab}(0), \phi(0), \chi(0)} = \frac{4}{3} \int d^3x \frac{1}{(\chi(x) - \chi(0)(x))} \left[ 2\gamma^{1/4} \gamma_{(0)}^{1/4} \cosh \left( \sqrt{\frac{3}{8}} \sqrt{z^2 + (\phi - \phi(0))^2} \right) - \gamma^{1/2} - \gamma_{(0)}^{1/2} \right],
\]

4.4. Special Case: \( \gamma(0) \to 0 \) limit

Note that in the limit that the determinant of the initial 3-metric approaches zero, \( \gamma(0) \to 0 \), the Green function for gravity, dust, scalar field and a cosmological constant, eq. (24), takes on the simple form
\[
\mathcal{G} \to -2 \int d^3x \gamma^{1/2} H \left( \chi | \chi(0) \right),
\]
which describes the integral over the original 3-geometry of the the Hubble function
\[
H \left( \chi | \chi(0) \right) = H_0 \coth \left( \frac{3H_0}{2} (\chi - \chi(0)) \right).
\]
The Hubble function describes gravity, dust and a cosmological constant (the scalar field drops out), and it was first computed by Salopek and Stewart (1992).

5. Green function solution: gravity and scalar field with exponential potential

Consider a scalar field with exponential potential
\[
V(\phi) = V_0 \exp \left( -\sqrt{\frac{2}{p}} \phi \right).
\]
The reduced HJ equation (23) becomes:
\[
-\frac{1}{12} e^{-3\alpha} \left( \frac{\delta \mathcal{G}}{\delta \alpha} \right)^2 + \frac{1}{2} e^{-3\alpha} \left( \frac{\delta \mathcal{G}}{\delta z} \right)^2 + \frac{1}{2} e^{-3\alpha} \left( \frac{\delta \mathcal{G}}{\delta \phi} \right)^2 + e^{3\alpha} V_0 \exp \left( -\sqrt{\frac{2}{p}} \phi \right) = 0.
\]
The Green function solution which depends on seven parameter fields \([\gamma^{(0)}_{ab}(x), \phi^{(0)}(x)]\) is

\[
\mathcal{G} = -\sqrt{\frac{4V_0}{3(1 - 1/(3p))}} \int d^3x \sqrt{f} ,
\]

(33a)

where \(f\) is an abbreviation for

\[
f = \gamma e^{-\sqrt{2p} \phi} + \gamma^{(0)} e^{-\sqrt{2p} \phi^{(0)}} - 2 \gamma^{1/2} \gamma^{1/2} \exp \left[ -\frac{1}{\sqrt{2p}} (\phi + \phi^{(0)}) \right] \cosh \theta ,
\]

(33b)

and

\[
\theta = \sqrt{\frac{3}{2}} \left[ \phi - \phi^{(0)} - \frac{1}{\sqrt{18p}} \ln(\gamma/\gamma^{(0)}) \right]^2 + \left( 1 - \frac{1}{3p} \right) z^2 .
\]

(33c)

The sign in front of the Green function is arbitrary and I have chosen a negative sign to describe an expanding universe.

5.1. \( p \to \infty \)

As \( p \to \infty \), the potential for the scalar field approaches a constant, and one recovers the case of gravity, a massless scalar field and a cosmological constant described by eq.(27).

5.2. Special Case: \( \gamma^{(0)} \to 0 \) limit

Note that as \( \gamma^{(0)} \to 0 \), the Green function approaches

\[
\mathcal{G} \to -2 \int d^3x \gamma^{1/2} H(\phi) ,
\]

(34a)

where the Hubble function was computed by Salopek and Bond (1990):

\[
H(\phi) = \sqrt{\frac{V_0}{3(1 - 1/(3p))}} \exp \left( -\frac{\phi}{\sqrt{2p}} \right) .
\]

(34b)

6. Examples of Lagrange multiplier method: gravity, dust and scalar field

I will illustrate the Lagrange multiplier method by considering various cases of gravity, dust, and a massless scalar field whose evolution is described by the Green function eq.(29). The case with cosmological constant is also interesting but the algebra is more complicated and I will devote section 7 to it.

The Lagrange multiplier method for solving the HJ equation had been foreshadowed in an earlier paper (Salopek 1991). I will show how to recover these earlier results from the generalized formalism described in section 2.

In the present section, I will consider the special case where the functionals \( \mathcal{I} \) and \( \mathcal{L} \) which specify the initial setting in eq.(13) do not contain any spatial gradients. Many examples are given.
6.1. **Initial setting contains no spatial gradients**

I will choose the function $f$ appearing in the Lagrange multiplier term (14d) to be an arbitrary function of $\phi(0)$ and $\chi(0)$:

$$f \equiv f(\phi(0), \chi(0)),$$

(35)

and I will choose the initial state

$$I = \int d^3x \gamma_{1/2} (\phi(0), \chi(0)),$$

(36)

to be an integral over the initial volume of some arbitrary function $g$ of $\phi(0)$ and $\chi(0)$. This system will be tractable because no spatial gradients appear.

Minimization of the generating functional eq.(13) with respect to the initial 3-metric $\gamma_{ab}(0)$ implies

$$z = 0,$$

(37a)

$$\gamma_{1/4} = \gamma^{1/4} \cosh \left[ \sqrt{\frac{3}{8}} (\phi - \phi(0)) \right] \left/ \left[ 1 - \frac{3}{4} (\chi - \chi(0)) (g + Lf) \right] \right. .$$

(37b)

Substitution into eq.(13) yields the reduced generating functional

$$S = -\frac{4}{3} \int d^3x \gamma^{1/2} \frac{1}{\chi(0)} + \frac{4}{3} \int d^3x \gamma^{1/2} \cosh^2 \left[ \sqrt{\frac{3}{8}} (\phi - \phi(0)) \right] \frac{\cosh^2 \left[ \sqrt{\frac{3}{8}} (\phi - \phi(0)) \right]}{\chi(0)} \left[ 1 - \frac{3}{4} (\chi - \chi(0)) (g + Lf) \right]$$

(38)

which is still subject to minimization with respect to $\phi(0)(x)$, $\chi(0)(x)$ and $L(x)$. Since the purpose of $L$ is solely to impose the constraint $f = 0$, the above generating function is equivalent to:

$$S = -\frac{4}{3} \int d^3x \gamma^{1/2} \frac{1}{\chi(0)} + \frac{4}{3} \int d^3x \gamma^{1/2} \frac{\cosh^2 \left[ \sqrt{\frac{3}{8}} (\phi - \phi(0)) \right]}{\chi(0)} \left[ 1 - \frac{3}{4} (\chi - \chi(0)) (g + Lf) \right]$$

$$+ \int d^3x \gamma^{1/2} L(x) f (\phi(0)(x), \chi(0)(x))$$

(minimized with respect to $\phi(0)(x)$, $\chi(0)(x)$ and $L(x)$).

(39)

Here, I have set $f = 0$ in eq.(38), and then appended a Lagrange multiplier term to it to recover the equivalent condition.

6.1.1. **Recovery of previous results**  The application of the Lagrange multiplier method to gravitational systems was suggested in an earlier paper by Salopek (1991). It will be
shown how these special results may be derived using the generalized methods presented in section 2.

For two or more scalar fields denoted by $\phi_a$, $a = 1, ..., n$, $a \geq 2$, it was demonstrated in Salopek (1991) that one could obtain a special class of solutions to the long-wavelength problem by attempting an Ansatz which was an integral over the volume element of the Hubble function, $H(\phi_a)$:

$$S[\gamma_{ab}(x), \phi_a(x)] = -2 \int d^3 x \gamma^{1/2} H[\phi_a(x)] ,$$  \hspace{1cm} (40a)

provided that the Hubble function $H(\phi_a)$ satisfies the separated Hamilton-Jacobi (SHJE):

$$H^2 = \frac{2}{3} \sum_{a=1}^{n} \left( \frac{\partial H}{\partial \phi_a} \right)^2 + \frac{V(\phi)}{3} .$$  \hspace{1cm} (40b)

If one is fortunate to find a solution, $\tilde{H}(\phi_a|\tilde{\phi}_a)$, of the SHJE \((40b)\) which depends on $n$ parameters, $\tilde{\phi}_a$, one may construct another solution $H(\phi_a)$,

$$H(\phi_a) = \tilde{H}(\phi_a|\tilde{\phi}_a) ,$$  \hspace{1cm} (42a)

by choosing the parameters to minimize the n-parameter solution with respect $\tilde{\phi}_a$ assuming that the parameters are subject to a constraint:

$$f(\tilde{\phi}_a) = 0 .$$  \hspace{1cm} (42b)

By introducing a Lagrange multiplier $\lambda$, one can write this new solution as

$$H = \tilde{H}(\phi_a|\tilde{\phi}_a) + \lambda f(\tilde{\phi}_a) ,$$  \hspace{1cm} (43a)

where $H$ is minimized with respect to $\lambda$ and $\tilde{\phi}_a$:

$$\frac{\partial H}{\partial \lambda} = 0 , \quad \frac{\partial H}{\partial \tilde{\phi}_a} = 0 .$$  \hspace{1cm} (43b)

Now in this paper, I have focussed on a single scalar field and a single dust field, but all past experience has show that a dust field can basically be treated as scalar field with a different term appearing in the energy constraint. In this subsection, for the purpose of simplicity, I will set $g(\phi(0), \chi(0)) = 0$ in eq.\((39)\), and then obtain the following generating functional,

$$S = \frac{4}{3} \int d^3 x \gamma^{1/2} \sinh^2 \left[ \sqrt{\frac{8}{3}} \left( \phi - \phi(0) \right) \right] \frac{\left( \chi - \chi(0) \right)}{\left( \chi - \chi(0) \right)}$$

$$+ \int d^3 x \gamma^{1/2} L(x) f(\phi(0)(x), \chi(0)(x)) ,$$

(minimized with respect to $\phi(0)(x)$, $\chi(0)(x)$ and $L(x)$).  \hspace{1cm} (44)
However, this generating functional is of the form of eq. (40a) and eq. (43a) where
\[ H = \tilde{H} [\phi, \chi | \phi(0), \chi(0)] + \lambda f (\phi(0)(x), \chi(0)(x)) , \quad (\lambda = -L/2), \quad (45a) \]
is minimized with respect to \( \lambda, \phi(0) \) and \( \chi(0) \). Here, \( \tilde{H} \),
\[ \tilde{H} [\phi, \chi | \phi(0), \chi(0)] = -\frac{2}{3} \sinh^2 \left( \sqrt{\frac{3}{8}} (\phi - \phi(0)) \right) \left( \chi - \chi(0) \right), \quad (45b) \]
satisfies the following SHJE:
\[ \tilde{H}^2 = -\frac{2}{3} \frac{\partial \tilde{H}}{\partial \chi} + 2 \left( \frac{\partial \tilde{H}}{\partial \phi} \right)^2. \quad (45c) \]
Hence the generalized Lagrange multiplier method recovers the special case considered in Salopek (1991) which was described for \( n \) scalar fields in eqs. (43a-b).

I will now consider two special cases for \( f \) corresponding to initial hypersurfaces where firstly the dust field is uniform and where secondly the scalar field is uniform.

6.2. Uniform dust field initially

As was discussed before in section 2, if one takes
\[ f (\phi(0)(x), \chi(0)(x)) = \chi(0) \quad (46a) \]
then variation of the Lagrange multiplier implies that
\[ \chi(0) = 0 , \quad (46b) \]
and that the initial hypersurface is one where the dust field is uniformly zero. In this case, I will further assume that \( g \) is an arbitrary function of \( \phi(0) \):
\[ g \equiv g(\phi(0)). \quad (46c) \]
One finds that the generating functional eq. (39) is:
\[ S[\gamma_{ab}(x), \phi(x)] = \frac{4}{3} \int d^3 x \sqrt{\gamma}^{1/2} \frac{1}{\chi} \left\{ \cosh^2 \left( \sqrt{\frac{3}{8}} (\phi - \phi(0)) \right) \left( 1 - \frac{3}{4} \chi^2 g \right) \right\} - 1 \], \quad (47a) \]
\[ \tanh \left( \sqrt{\frac{3}{8}} (\phi - \phi(0)) \right) = \sqrt{\frac{3}{8}} \chi \frac{\partial \phi}{\partial \phi(0)} \left( 1 - \frac{3}{4} \chi^2 g \right). \quad (47b) \]
The second equation is the minimization condition of the first with respect to \( \phi(0) \).

For various choices of \( g \), I have computed the resulting generating functional.

For \( g \) a constant:
\[ g = C \quad (a \ constant), \quad (48a) \]
\[ S[\gamma_{ab}(x), \chi(x)] = -\frac{4}{3} \int d^3x \gamma^{1/2} \frac{1}{\left[\chi - 4/(3C)\right]} \cdot \] (48b)

For \( g \) proportional to an exponential:

\[ g = C \exp \left( \sqrt{\frac{3}{2}} \phi_0 \right) \,, \quad \] (49a)

\[ S[\gamma_{ab}(x), \chi(x)] = C \int d^3x \gamma^{1/2} \exp \left( \sqrt{\frac{3}{2}} \phi \right) \,. \] (49b)

For \( g \) proportional to \( \cosh^2(\sqrt{3/8} \phi_0) \):

\[ g = C \cosh^2 \left( \sqrt{\frac{3}{8}} \phi_0 \right) \,, \quad \] (50a)

\[ S[\gamma_{ab}(x), \chi(x)] = -\frac{4}{3} \int d^3x \gamma^{1/2} \cosh^2 \left( \sqrt{\frac{3}{8}} \phi \right) \frac{\left[\chi - \chi_0\right]}{\left[\chi - 4/(3C)\right]} \] (50b)

Thus the Green function method leads to many solutions of strongly coupled gravitational systems.

6.3. Uniform scalar field initially

Setting \( f = \phi_0 \) implies that the scalar field is uniformly zero on the initial hypersurface. In this case, I now assume that \( g \) is an arbitrary function of \( \chi_0 \):

\[ g \equiv g \left( \chi_0 \right) \,. \] (51)

In this case, the generating functional eq.(39) becomes:

\[ S = -\frac{4}{3} \int d^3x \gamma^{1/2} \frac{1}{\left(\chi - \chi_0\right)} \]

\[ + \frac{4}{3} \int d^3x \gamma^{1/2} \cosh^2 \left( \sqrt{\frac{3}{8}} \phi \right) \frac{\left[\chi - \chi_0\right]}{\left[1 - \frac{3}{4} \left(\chi - \chi_0\right) g \right]} \,, \] (52a)

\[ \tanh^2 \left( \sqrt{\frac{3}{8}} \phi \right) = \frac{\left(\chi - \chi_0\right)^2}{\left[1 - \frac{3}{4} \left(\chi - \chi_0\right) g \right]^2} \left( \frac{9}{16} g^2 - \frac{3}{4} \partial g \right) \] (52b)

Equation (52b) is the minimization condition of eq.(52a) with respect to \( \chi_0 \).

For various choices of \( g \), I have computed the resulting generating functional. For \( g \) a constant:

\[ g = C \quad \text{(a constant)} \,, \] (53a)
\[ S[\gamma_{ab}(x), \chi(x)] = C \int d^3 x \gamma^{1/2} \exp \left( \sqrt{\frac{3}{2}} \phi \right). \] (53b)

which is the same result found in eq.(49b) using a different route.

For \( g = -4/(3\chi(0)) \), one obtains

\[
g = -\frac{4}{3} \frac{1}{\chi(0)}, \quad (54a)
\]

\[
S = -\frac{4}{3} \int d^3 x \gamma^{1/2} \frac{1}{\chi}. \quad (54b)
\]

For \( g = C/\chi(0) \),

\[
g = \frac{C}{\chi(0)}, \quad (55a)
\]

\[
S = \frac{4}{3} \int d^3 x \gamma^{1/2} \frac{\sinh^2 \left( \sqrt{\frac{3}{8}} \phi + \beta \right)}{\chi}. \quad (55b)
\]

where

\[
\sinh^2 \beta = \frac{3C}{4}. \quad (55c)
\]

What is rather unusual is that for

\[
C = -\frac{4}{3} \quad (56a)
\]

one finds \( \beta = i\pi/2 \), and the resulting generating functional is

\[
S = -\frac{4}{3} \int d^3 x \gamma^{1/2} \frac{\cosh^2 \left( \sqrt{\frac{3}{8}} \phi \right)}{\chi}. \quad (56b)
\]

which is certainly a solution to the energy and momentum constraints but it is not the solution given in eq.(54a,b)! However, note that when \( \phi = 0 \), they do agree —- they share the same initial functional, \( I \). These two results reflect the multivalued solutions of the minimization equation (52b).

7. Lagrange multiplier method: gravity, dust, massless scalar field with cosmological constant

I will repeat the analysis of section 6 but I will now add a cosmological constant to the system of gravity, dust and a massless scalar field. The Green function \( G \) was given in eq.(24a).
When the functionals, $\mathcal{I}$ and $\mathcal{L}$,
\begin{align}
\mathcal{I} &= \int d^3x \gamma^{1/2} g \left( \phi(0)(x), \chi(0)(x) \right), \\
\mathcal{L} &= \int d^3x \gamma^{1/2} L(x) f \left( \phi(0)(x), \chi(0)(x) \right),
\end{align}
(57a, 57b)
which determine the initial setting do not contain spatial gradients, then the
minimization of $S$, eq.(13), with respect to the initial 3-metric is straightforward:
\begin{align}
z &= 0, \\
\left( \frac{\gamma(0)}{\gamma} \right)^{1/4} &= \frac{\cosh \left[ \sqrt{\frac{3}{8}} \left( \phi - \phi(0) \right) \right]}{\cosh \left[ \sqrt{\frac{3}{8}} \left( \chi - \chi(0) \right) \right] - \frac{(g+L)}{2H_0} \sinh \left[ \frac{3H_0}{2} \left( \chi - \chi(0) \right) \right]},
\end{align}
(58a, 58b)
giving
\begin{align}
\frac{S}{(2H_0)} &= -\int d^3x \gamma^{1/2} \cotanh \left[ \frac{3H_0}{2} \left( \chi - \chi(0) \right) \right] + \\
&\int d^3x \gamma^{1/2} \cosh \left[ \sqrt{\frac{3}{8}} \left( \phi - \phi(0) \right) \right] \sinh \left[ \frac{3H_0}{2} \left( \chi - \chi(0) \right) \right] \frac{1}{\cosh \left[ \frac{3H_0}{2} \left( \chi - \chi(0) \right) \right] - \frac{g}{2H_0} \sinh \left[ \frac{3H_0}{2} \left( \chi - \chi(0) \right) \right]} \\
&+ \frac{1}{(2H_0)} \int d^3x \gamma^{1/2} L f.
\end{align}
(59)
In the above equation, I followed the analysis of section 6 by setting $f = 0$ in the
generating functional and then recovering this same condition by adding a Lagrange
multiplier term which is a linear functional in $L(x)$.

7.1. Uniform dust field initially

I will assume that initial state is given on an initial hypersurface where the dust field is
zero which is characterized by a vanishing value of $\chi(0)(x)$:
\begin{align}
f &= \chi(0)(x) = 0.
\end{align}
(60)
The Lagrange multiplier term will thus be removed. Moreover, I will assume that $g$,
which characterizes the initial state, is a function solely of the initial scalar field $\phi(0)$:
\begin{align}
g &\equiv g \left( \phi(0) \right).
\end{align}
(61)
After minimizing with respect to $\phi(0)$, the generating functional is:
\begin{align}
S[\gamma_{ab}(x), \phi(x), \chi(x)]/(2H_0) &= -\int d^3x \gamma^{1/2} \cotanh \theta + \\
&\int d^3x \gamma^{1/2} \frac{1}{\sinh^2 \theta} \left[ \cotanh \theta - h \left( \phi(0) \right) \right] ^2 - \frac{2}{3} \left( \frac{dh}{d\phi(0)} \right)^2,
\end{align}
(62a)
where $\theta$ and $h$ are dimensionless representations of $\chi$ and $g$,

$$\theta \equiv \frac{3H_0\chi}{2}, \quad h \equiv g\left(\phi_{(0)}\right)/(2H_0).$$

(62b)

The parameter field $\phi_{(0)}(x)$ is determined implicitly through the algebraic equation,

$$\tanh\left[\sqrt{\frac{3}{8}}\left(\phi - \phi_{(0)}(x)\right)\right] = \frac{\sqrt{\frac{3}{8}} \frac{dh}{d\phi_{(0)}}}{\cotanh\theta - h\left(\phi_{(0)}\right)}.$$  \hspace{1cm} (62c)

7.1.1. Exact solution \hspace{1cm} One obtains a non-trivial exact solution if one assumes that $g$ has the following form:

$$g\left(\phi_{(0)}\right) \equiv h = C \exp\left(-\sqrt{\frac{3}{2}}\phi_{(0)}\right) + D + E \exp\left(\sqrt{\frac{3}{2}}\phi_{(0)}\right).$$  \hspace{1cm} (63a)

One can then solve $\phi_{(0)}$ explicitly using eq.(62c), and the generating functional is

$$S[\gamma_{ab}(x), \phi(x), \chi(x)] = -2 \int d^3x \, \gamma^{1/2} H(\phi, \chi),$$

(63b)

where the Hubble function $H(\phi, \chi)$ is given by

$$H(\phi, \chi)/H_0 = \cotanh\theta - \frac{1}{\sinh^2\theta} \left[\frac{Eb + C/b - D + \cotanh\theta}{(\cotanh\theta - D)^2 - 4CE}\right],$$

(63c)

and $b$ denotes

$$b = \exp\left(\sqrt{\frac{3}{2}}\phi\right).$$  \hspace{1cm} (63d)

As a check of this method, one may verify that $H$ is a solution of the SHJE,

$$H^2 = -\frac{2}{3} \frac{\partial H}{\partial \chi} + \frac{2}{3} \left(\frac{\partial H}{\partial \phi}\right)^2 + \frac{V_0}{3}.\quad (64)$$

Some special cases illuminate the solution.

If $D = C = E = 0$, the Hubble function is

$$H = H_0 \tanh\left(\frac{3H_0\chi}{2}\right), \quad (D = C = E = 0).\quad (65)$$

If $D = 0, C = E = -\frac{1}{2}$, the Hubble function is

$$H = H_0 \cosh\left(\sqrt{\frac{3}{2}}\phi\right), \quad (D = 0, C = E = -\frac{1}{2}).\quad (66)$$

If $D = 0$ and $C = E$, the Hubble function is

$$\frac{H}{H_0} = \frac{(1 - 4E^2) \cosh\theta \sinh\theta - 2E \cosh\left(\sqrt{\frac{3}{2}}\phi\right)}{\cosh^2\theta - 4E^2 \sinh^2\theta}, \quad (D = 0, C = E).\quad (67)$$
8. Advanced example: initial hypersurface condition containing spatial gradients

I will now consider an advanced example where the Lagrange multiplier term contains spatial gradients. This example is particularly important because it illustrates how the energy constraint applies to the initial fields.

The system under examination will contain only gravity and dust, and the Green function is given by eq. (54c). The function \( f \) which defines the initial hypersurface will be chosen to be,

\[
f = \chi(0) - BR(0),
\]

where \( B \) is a constant and \( R(0) \) is the Ricci scalar associated with the initial 3-metric. The functional \( I \) is taken to be the simplest non-trivial example, namely the volume of the initial 3-geometry:

\[
I = A \int d^3x \gamma^{1/2}(0).
\]

The full generating functional is the sum of \( G, I \) and \( L \):

\[
S[\gamma_{ab}(x), \chi(x)] =
4 \int d^3x \frac{1}{(\chi(x) - \chi(0)(x))} \left[ 2\gamma^{1/4}(0) \gamma^{1/4}(x) \cosh\left(\frac{3}{8}z\right) - \gamma^{1/2} - \gamma^{1/2}(0) \right],
\]

\[
+ A \int d^3x \gamma^{1/2}(0) + \int d^3x \gamma^{1/2}(x) L(x) \left[ \chi(0)(x) - BR(0)(x) \right] \text{ (minimized with respect to } \gamma^{(0)}_{ab}(x), \chi(0)(x) \text{ and } L(x) \text{ ).}
\]

This example will not be exactly solvable unless \( B \) vanishes. One obtains approximate results by expanding in powers of \( B \).

8.1. Minimization conditions

Variation with respect to \( L(x) \) and \( \chi(0) \) lead immediately to:

\[
0 = \chi(0) - BR(0),
\]

\[
L = -4 \int d^3x \frac{2\left(\gamma/\gamma(0)\right)^{1/4} \cosh\left(\sqrt{\frac{3}{8}}z\right) - \left(\gamma/\gamma(0)\right)^{1/2} - 1}{\left(\chi - \chi(0)\right)^2}.
\]

After minimizing eq. (69) with respect to \( \gamma^{(0)}_{ab} \), the trace and traceless parts are (the trace is defined using the initial 3-metric):

\[
0 = 2\left(\frac{\gamma^{1/4}(0) \gamma^{1/4}(x) \cosh\left(\sqrt{\frac{3}{8}}z\right) - 1}{\chi - \chi(0)} \right) + \frac{3A}{2} + B \left( R(0) L + 2L^c_{\text{c}} \right),
\]
\[
0 = 6^{-1/2} \left( \frac{\gamma}{\gamma(0)} \right)^{1/4} \sinh \left( \sqrt{\frac{2}{3}} \frac{z}{\chi - \chi(0)} \right) \ln \left( [h^{-1}[h(0)]] \right)^{a \ b} 
\]

\[
B \left( [R]^a_{(0) b} - L^a_{:\ :b} + \frac{1}{3} \delta^a_b L^c_{:c} \right) .
\]

8.2. Initial energy constraint

After one minimizes \( S \), the initial fields, \( \gamma^{(0)}_{ab}(x), \chi(0)(x) \) and \( L(x) \), are related in a subtle way. By construction, the Green function \( G \) satisfies the energy constraint:

\[
0 = \frac{\delta G}{\delta \chi(x)} + \gamma^{1/2} \left( 2 \gamma_{ac} \gamma_{bd} - \gamma_{ab} \gamma_{cd} \right) \frac{\delta G}{\delta \gamma_{ab}(x)} \frac{\delta G}{\delta \gamma_{cd}(x)}.
\]

We interpret this equation as stating that the original fields satisfy the energy constraint. However, because of a symmetry transformation that relates the original fields and the initial fields,

\[
\left( \gamma_{ab}(x), \chi(x) \right) \longleftrightarrow \left( \gamma^{(0)}_{ab}(x), -\chi(0)(x) \right),
\]

the initial fields, \( \gamma^{(0)}_{ab}(x) \) and \( \chi(0)(x) \), also satisfy the energy constraint:

\[
0 = -\frac{\delta G}{\delta \chi(0)(x)} + \gamma^{(0)}_{ab} \left( 2 \gamma^{(0)}_{ac} \gamma^{(0)}_{bd} - \gamma^{(0)}_{ab} \gamma^{(0)}_{cd} \right) \frac{\delta G}{\delta \gamma^{(0)}_{ab}(x)} \frac{\delta G}{\delta \gamma^{(0)}_{cd}(x)}.
\]

By virtue of the minimization conditions (see, e.g., eq.(16a-d)), one may thus replace functional derivatives of \( G \) with functional derivatives of \( (I + L) \),

\[
-\frac{\delta G}{\delta \chi(0)(x)} = \frac{\delta (I + L)}{\delta \chi(0)(x)},
\]

\[
-\frac{\delta G}{\delta \gamma^{(0)}_{ab}(x)} = \frac{\delta (I + L)}{\delta \gamma^{(0)}_{ab}(x)},
\]

in eq.(73) which leads to the initial energy constraint,

\[
0 = \frac{\delta (I + L)}{\delta \chi(0)(x)} + \gamma^{1/2} \left( 2 \gamma^{(0)}_{ac} \gamma^{(0)}_{bd} - \gamma^{(0)}_{ab} \gamma^{(0)}_{cd} \right) \frac{\delta (I + L)}{\delta \gamma^{(0)}_{ab}(x)} \frac{\delta (I + L)}{\delta \gamma^{(0)}_{cd}(x)}
\]

(initial energy constraint),

which is independent of the original fields, \( \gamma_{ab}(x) \) and \( \chi(x) \). Explicit evaluation of the functional derivatives in eq.(73) yields a relationship amongst the Lagrange multiplier and the initial fields:

\[
0 = L + 2B^2 m^{ab} m_{ab} - \frac{1}{3} \left( B m + \frac{3}{2} A \right)^2,
\]

(initial energy constraint) (76a)

where the tensor \( m^{ab} \) is given by

\[
m^{ab} = R^{ab}_{(0)} L + \gamma^{ab}_{(0)} L^{ic}_{:c} - L^{ab},
\]

(76b)
and \( m^{ab} \) denotes its traceless component:
\[
\overline{m}^{ab} = m^{ab} - \frac{1}{3} \gamma^{ab}_{(0)} m, \quad \text{with} \quad m = \gamma^{(0)}_{ab} m^{ab}.
\]  
(76c)

In the above, a semicolon (;) denotes a derivative with respect to the initial 3-metric and all indices are raised and lowered using the initial 3-metric. In principle, one may solve for the Lagrange multiplier in terms of the initial 3-metric, although this is difficult to achieve in practice since this is a fourth order, nonlinear partial differential equation.

### 8.3. Classical evolution

Classical evolution is found by solving eqs.\((70a-d)\) for \( \gamma_{ab}(x) \) in terms of the proper time \( \chi(x) \), the initial fields, \( \gamma^{(0)}(x) \) and \( \chi^{(0)}(x) \), and the Lagrange multiplier, \( L(x) \):

\[
\left( \frac{\gamma}{\gamma^{(0)}} \right)^{1/2} = \left[ 1 - \frac{(\chi - \chi^{(0)})}{(\chi^{(1)} - \chi^{(0)})} \right] \left[ 1 - \frac{(\chi - \chi^{(0)})}{(\chi^{(2)} - \chi^{(0)})} \right],
\]  
(77a)

\[
[h] = [h^{(0)}] \exp \left( 2z \frac{\overline{m}[\gamma^{(0)}]}{\sqrt{m^{ab} m_{ab}}} \right),
\]  
(77b)

\[
z = \sqrt{\frac{2}{3}} \ln \left[ 1 - \frac{(\chi^{(1)} - \chi^{(0)})}{(\chi^{(2)} - \chi^{(0)})} \right].
\]  
(77c)

\( \left( \gamma/\gamma^{(0)} \right)^{1/2} \) is a quadratic function of \( (\chi - \chi^{(0)}) \). The terms, \( (\chi^{(1)} - \chi^{(0)}) \) and \( (\chi^{(2)} - \chi^{(0)}) \), denote the two roots of \( \left( \gamma/\gamma^{(0)} \right)^{1/2} \):

\[
(\chi^{(1)} - \chi^{(0)}) = \frac{A}{L} + \frac{2B}{3} \left( R_{(0)} + \frac{2}{L} L^{c;e} \right) - \sqrt{\frac{8B}{3L}} \sqrt{\overline{m}^{ab} \overline{m}_{ab}};
\]  
(77d)

\[
(\chi^{(2)} - \chi^{(0)}) = \frac{A}{L} + \frac{2B}{3} \left( R_{(0)} + \frac{2}{L} L^{c;e} \right) + \sqrt{\frac{8B}{3L}} \sqrt{\overline{m}^{ab} \overline{m}_{ab}}.
\]  
(77e)

The tensor \( m^{ab} \) was given earlier but I will write it down here just to have all of the results in one place:

\[
m^{ab} = R^{ab}_{(0)} L + \gamma^{ab}_{(0)} L^{c;e} - L^{ab}. \]  
(78)

\( \chi^{(0)} \) is given by the initial hypersurface condition:

\[
0 = \chi^{(0)} - BR_{(0)},
\]  
(79)

and the Lagrange multiplier may be solved in terms of the initial 3-metric from the initial energy constraint:

\[
0 = L + 2B^{2} \overline{m}^{ab} \overline{m}_{ab} - \frac{1}{3} \left( B m + \frac{3}{2} A \right)^{2},
\]  
(80)
One may solve for $L$, and subsequently all other quantities, by expanding in powers of $B$. To first order in $B$, one finds:

$$L = \frac{3A^2}{4} \left(1 + BAR(0)\right), \quad (81a)$$

$$\chi(0) = BR(0), \quad (81b)$$

$$m^{ab} = \left(\frac{3A^2}{4}\right) \left[R^{ab}_{(0)} + BA \left(R(0) R^{ab}_{(0)} - R^{ab}_{(0)} + \gamma^{ab}_{(0)} \tilde{D}^2 R(0)\right)\right], \quad (81c)$$

$$m = \left(\frac{3A^2}{4}\right) \left[R(0) + BA \left(R^2(0) + 2\tilde{D}^2 R(0)\right)\right], \quad (81d)$$

$$\chi(1) - \chi(0) = \left(\frac{4}{3A}\right) \left(1 - \frac{BA}{2} R(0)\right) - \sqrt{\frac{8}{3}} B \left(\tilde{R}^{ab}_{(0)} \tilde{R}^{0}_{ab}\right)^{1/2}, \quad (81e)$$

$$\chi(2) - \chi(0) = \left(\frac{4}{3A}\right) \left(1 - \frac{BA}{2} R(0)\right) + \sqrt{\frac{8}{3}} B \left(\tilde{R}^{ab}_{(0)} \tilde{R}^{0}_{ab}\right)^{1/2}, \quad (81f)$$

$$z = \frac{3BA^2}{2} \frac{\chi}{\left(1 - \frac{3A\chi}{4}\right)} \left(\tilde{R}^{ab}_{(0)} \tilde{R}^{0}_{ab}\right)^{1/2}, \quad (81g)$$

$$\gamma_{ab} = \left(1 - \frac{3A\chi}{4}\right)^{4/3} \left[\gamma^{(0)}_{ab} + \frac{AB}{\left(1 - \frac{3A\chi}{4}\right)} \left(3A\chi R^{(0)}_{ab} + R(0) \gamma^{(0)}_{ab}\right)\right]. \quad (81h)$$

In the above, the laplacian of $R(0)$ is denoted by

$$\tilde{D}^2 R(0) \equiv \left(R(0)\right)^{;c}_{;c}. \quad (82)$$

**8.4. Determination of the classical 4-metric**

In solving the classical Einstein equations, one ordinarily computes the 4-metric describing time and space. However, in the HJ formalism, one’s attention is primarily focussed on the 3-metric which describes the spatial geometry. How does one recover the 4-metric? In general, given the generating functional $S[\gamma_{ab}(x), \chi(x)]$, the 4-metric is computed by making an arbitrary choice for the time parameter, and then integrating the definition of the momenta,

$$\left(\dot{\gamma}_{ij} - N_{ij} - N_{ji}\right)/N = 2\kappa \gamma^{-1/2} \left(2\gamma_{ik}\gamma_{jl} - \gamma_{ij}\gamma_{kl}\right), \quad (83a)$$

$$\left(\dot{\chi} - N^i \chi_{,i}\right)/N = \kappa, \quad (83b)$$

which is valid in the strongly coupled limit.
For the problems considered in this section, a natural choice for the time hypersurface is one where the dust field \( \chi \) is uniform,

\[ t = \chi / \kappa \tag{84a} \]

which describes a comoving slice. If one assumes that the shift \( N_i \) vanishes, then eq.(83) implies that the lapse \( N \) is equal to one,

\[ N = 1, \tag{84b} \]

which describes a synchronous gauge. The line-element describing the 4-geometry is then

\[ ds^2 = -\frac{d\chi^2}{\kappa^2} + \gamma_{ab} dx^a dx^b, \tag{85} \]

where the classical evolution of the 3-metric \( \gamma_{ab} \) was computed in the previous section using the HJ minimization prescription; see eq.(81) for the first few terms. At each spatial point, the 4-geometry is locally Kasner (see, for example, Salopek 1998).

### 8.5. Computation of generating functional

Using the exact expressions for classical evolution, eqs.(77-b-e), one may express the generating functional \( S \), eq.(69), in an elegant form which depends on the dust field \( \chi(x) \), the initial fields and the Lagrange multiplier:

\[ S = -\int d^3 x \gamma_{(0)}^{1/2} L \left( \chi - \chi(0) \right) + \mathcal{I} + \mathcal{L}. \tag{86} \]

Here I have eliminated all reference to the original 3-metric \( \gamma_{ab}(x) \). As was found in Salopek (1998), the generating functional is basically linear in \( \chi(x) \). If \( L(x) \) is positive, the integrand of \( S \) decreases in \( \chi(x) \), whereas the opposite is true if \( L(x) \) is negative.

Ultimately, one wishes to compute \( S \) solely as a functional of the original fields, \( \gamma_{ab}(x) \) and \( \chi(x) \). Since this cannot be done exactly for the problem at hand, I will be content to approximate it using a Taylor series in \( B \). First note that the initial 3 metric, \( \gamma_{ab}^{(0)}(x) \) may be expressed as a function of \( \gamma_{ab}(x) \) and \( \chi(x) \):

\[ \gamma_{ab}^{(0)} = k_{ab} - \frac{AB}{(1 - 3A\chi)^4} \left[ 3A\chi R^k_{ab} + \left( 1 - \frac{3A\chi}{2} \right) R^k k_{ab} \right], \tag{87a} \]

which is accurate to first order in \( B \). Here the tensor \( k_{ab} \) is conformally related to the original 3-metric, \( \gamma_{ab} \),

\[ k_{ab} = \left( 1 - \frac{3A\chi}{4} \right)^{-4/3} \gamma_{ab}, \tag{87b} \]

and \( R^k, R^k_{ab} \) denote the corresponding Ricci scalar and tensor, respectively. After imposing the initial hypersurface condition, eq.(79), one may safely drop the Lagrange
multiplier term $\mathcal{L}$ in eq. (86) and the generating functional becomes,

$$S[\gamma_{ab}(x),\chi(x)] = A \int d^3x \frac{\gamma^{1/2}(x)}{(1 - \frac{3A\chi}{4})} - \frac{3A^2B}{4} \int d^3x R^k$$

which is accurate to first order in $B$. Conceptually, there is no problem in extending this expression to higher order in $B$.

9. Summary and conclusions

A strong coupling expansion appears in many investigations of gravitational systems including long-wavelength cosmological fluctuations, gravitational collapse and string theory formulations of cosmology. In the present paper, powerful Hamilton-Jacobi methods have been further developed to allow for a very general solution of strongly coupled gravitational systems.

In section 2, one generalizes the semiclassical Green function method of solving the constraint equations to encompass the specification of an arbitrary initial hypersurface. One assumes an Ansatz for the generating functional $S$, eq. (13), which is the sum of three terms:

$$S = \mathcal{G} + \mathcal{I} + \mathcal{L}.$$  \hspace{1cm} (89a)

The Lagrange multiplier term,

$$\mathcal{L} = \int d^3x \gamma^{1/2}(x) L(x) f \left[ \gamma^{(0)}_{ab}(x), \phi^{(0)}(x), \chi^{(0)}(x) \right]$$

specifies the initial hypersurface,

$$f = 0.$$  \hspace{1cm} (89c)

$I$ is the initial state. The Green function $\mathcal{G}$ determines how the systems evolves from the initial setting. The Ansatz (89a) is justified mathematically by computing its functional derivatives. It hence satisfies the energy constraint. Because gauge-invariance is maintained at each step, the Ansatz satisfies the momentum constraint. Classical evolution follows from minimization of the generating functional (89a) with respect to the initial fields and the Lagrange multiplier $L$.

To illustrate the generalized method, I constructed in sections 4 and 5 Green function solutions for (1) gravity, dust, a massless scalar field and a cosmological constant and (2) gravity interacting with a scalar field with exponential potential. One verifies these solutions after deriving a reduced energy constraint in section 3.

If the functionals, $\mathcal{I}$ and $\mathcal{L}$, which define the initial setting do not contain spatial gradients of the initial fields, then the program is straightforward to implement, and one recovers a primitive form of the minimization principle that had been advanced in an
earlier paper (Salopek 1991). In fact, in sections 6 and 7, exact solutions were derived that had not been previously known.

However, in general the initial hypersurface condition contains spatial gradients, and one must typically resort to an approximation method as was illustrated in section 8 for the case of gravity and dust. There it was shown explicitly how to deal with the initial energy constraint.

The *semiclassical Green function method* has reached a high level of generality which should be sufficiently powerful to treat most strongly coupled gravitational systems of physical and numerical interest. Hamilton-Jacobi methods have been useful in the past for solving nonlinear problems in cosmology. In the future, the methods presented in this paper may shed some light on quantum aspects of the gravitational field.

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