Enhancing synchronization by directionality in complex networks

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We proposed a method called residual edge-betweenness gradient (REBG) to enhance synchronizability of networks by assignment of link direction while keeping network topology and link weight unchanged. Direction assignment has been shown to improve the synchronizability of undirected networks in general, but we find that in some cases incommunicable components emerge and networks fail to synchronize. We show that the REBG method can effectively avoid the synchronization failure \( R = \lambda_2/\lambda_N = 0 \) which occurs in the residual degree gradient (RDG) method proposed in Phys. Rev. Lett. 103, 228702 (2009). Further experiments show that REBG method enhance synchronizability in networks with community structure as compared with the RDG method.

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Synchronization is an important phenomenon in various fields including biology, physics, engineering, and even sociology [1]. In particular, synchronization in complex networks has been intensively studied in the past decade [2–10]. One important objective in these studies is to enhance the synchronizability [7–10], i.e., the ability to coordinate oscillators in synchronization. Many related methods based on node properties and link weight have been proposed. For example, some researchers took into account the degree centrality [7, 8], the betweenness [9] and the node age [11, 12], and assigned weight on links to enhance synchronizability. Nishikawa and Motter proposed to assign zero weight to particular links which lead to an oriented tree with normalized input strength and no directed loop [12, 13]. Moreover, they proved that tree is the optimal structure on which synchronizability is highest. Recently, the shortest oriented spanning tree is the optimal structure on which synchronizability can be enhanced [12, 13]. However, these methods are all based on assigning weights to links, where the influences of link direction on synchronization have not been intensively considered [1].

How to improve synchronization in directed network is still an unsolved problem. Though previous works suggest that hierarchical structure and the absence of feed-back loop can enhance synchronizability, the underlying mechanism is not clear. On the other hand, the directionality plays an important role in the dynamic of networks [10, 12]. With the understanding of relations between link direction and synchronization, a lot of applications can be made [1]. For example, simply regulating the direction of the phase signal of alternating current can facilitate phase match in power grids without additional construction cost in the topology. With this in mind, the authors in Ref. [20] proposed the residual degree gradient (RDG) method to enhance directed networks’ synchronizability by assigning only the direction of links without changing the entire topology and link weights. They also claimed that RDG method can enhance the network synchronization, contrary to the randomly assigned directional method.

However, instead of enhancing synchronizability, we find that the RDG method results in incommunicable components in some particular cases, which lead to graphs incapable of complete synchronization. Incommunicable components of a network correspond to the components on which information cannot be transmitted from one to the other in either direction. Under these circumstance, the networks can never reach complete synchronized state. In this paper, we propose the so-called residual edge-betweenness gradient (REBG) method to resolve the problem of incommunicable components in the RDG method. By evaluating the betweenness on all edges, we devise an algorithm to assign link direction. The effectiveness of the algorithm lies in the use of edge betweenness, which embeds global information, as compared to the node degree which reflect only local information. We find that REBG is effective in enhancing synchronizability without leading to incommunicable components in most network topologies.

To begin our analysis, we make use of the Master Stability Function which allows us to use the eigenratio \( R = \lambda_{\min}/\lambda_{\max} \) of the Laplacian matrix to represent the synchronizability of a network, with \( \lambda_{\min} \) and \( \lambda_{\max} \) denote respectively the smallest nonzero eigenvalue and the largest eigenvalue [21, 22]. Specifically, \( 0 \leq R \leq 1 \), the larger the value of \( R \), the stronger the synchronizability of networks. According to Ref. [10, 22], we use the real part of eigenvalues from the Laplacian matrix to investigate the propensity for synchronization in directed networks. However, instead of setting \( R = \lambda'_{\min}/\lambda'_{\max} \) (superscript \( r \) denotes the real part of complex number),
we set $R = \frac{\lambda_2}{\lambda_N}$, with $\lambda_2$ to be the second smallest eigenvalue. It is because $\lambda_2 = 0$ in cases when incommunicable components emerge, which is not reflected by $\lambda_{\text{min}}$. In the subsequent analysis, we use $R$ to characterize synchronizability.

As the known fact, when there is isolated node or community in an undirected network, the network can never reach complete synchronization since there is no information flow between the isolated components. Under this circumstances, the synchronizability index $R = \frac{\lambda_2}{\lambda_N} = 0$. We call this phenomenon ($R = 0$) synchronization failure. Generally speaking, synchronization failure is more likely in directed networks as isolated components are not necessary. We first denote the cut-vertex as the only node through which two or more components communicate with each other. In directed networks, the synchronization failure happens whenever cut-vertex have only incoming links. In this case, there is absolutely no communication between those components and the cut-vertex becomes an information sink. Hence, synchronization cannot be achieved among the incommunicable components.

In order to examine synchronization failure, we first describe briefly the RDG method \[20\]. The RDG method is proposed to enhance the synchronizability of undirected networks by simply assigning the link direction. They refer links without yet assignment of direction as residual edges, and the number of connected residual edges as the residual degree of a node. In each step, they select the node with the smallest residual degree and set a maximum of $\lceil (k/2) \rceil$ of its residual edges pointing to it \[24\], where $\langle k \rangle$ is the average degree in the original network. Once a node has been selected, it will not be chosen again. Nodes have not yet been selected are called residual nodes. In addition, there is a directionality $\alpha$ to control the final fraction of links with direction assignment. When $\alpha = 0$, all links remain undirected. When $\alpha = 1$, all links are assigned direction. The RDG assignments are finished when there is no residual node left in the network.

The RDG method may lead to directed graph with synchronization failure, i.e., cut-vertex with only incoming links. A simple example is given in Fig. 1 (a). According to the rule of RDG, node 1 ($k = 2$) will be selected first and the two remaining communities will be left incommunicable. As we have discussed, this RDG network can never reach complete synchronized state.

According to \[20\], RDG was supposed to provide a practical way for the synchronization improvement in power grid networks, which have exponential degree distribution \[25\]. So we first study the RDG method in random exponential networks \[24\]. Random exponential networks in our context refer to random networks with exponential degree distribution. We also tested the RDG method in random scale-free networks since power-law degree distributions are widely observed in empirical data \[20\]. The degree distribution are respectively given by $P(k) \sim e^{-\beta k}$ and $P(k) \sim k^{-\gamma}$. For each $\beta$ and $\gamma$, we tested the RDG method on 100 network realizations. To examine synchronization failure, we make sure that there is no isolated nodes or communities in the original networks which masks the effect of incommunicable components. We find that many resultant RDG networks are with $R = 0$. The failure rate, i.e. the fraction of realization incapable of complete synchronization, is reported in Fig. 2. As shown in Fig. 2 (a) and (b), when $\alpha$ increase (i.e., more links are assigned direction), failure rate increases for both exponential and power-law networks. These results imply that synchronization failure is common in both power-grid-like and scale-free networks, given RDG is used as the method for direction assignment. We further tested the case of $\alpha = 1$, which leads to the largest synchronization improvement as reported in \[20\]. By varying $\beta$ and $\gamma$, we find that failure is high when $\beta$ and $\gamma$ are large. We also remark that failure rate decreases with $k_{\text{min}}$ but increases with $\theta$.
All these suggest that synchronization failure is not a rare phenomenon and therefore a new method of direction assignment is required to prevent failure.

We thus introduce the residual edge-betweenness gradient (REBG) method to solve the problem of synchronization failure. Instead of the node degree, we take the edge-betweenness into account. First of all, we define \( s_i \) for node \( i \) as

\[
s_i = \sum_{j=1}^{N} a_{ij} l_{ij}^\theta, \tag{1}
\]

where \( a_{ij} = 1 \) when there exists an undirected link between \( i \) and \( j \) and otherwise 0. \( l_{ij} \) is the betweenness of the link between \( i \) and \( j \) evaluated on the original undirected networks, subjected to a power \( \theta \) with \( 0 \leq \theta \leq 1 \).

To assign link direction, we select the node with the smallest residual \( s_i \) in each step and assign an incoming direction for a maximum of \( \lfloor (k)/2 \rfloor \) of its residual links. As more directed links are assigned, the residual \( s_i \) has to be updated at every step. If there are multiple nodes of the smallest \( s_i \), we choose the node with the smallest initial \( s_i \) first. The REBG method stops when there is no residual node left in the network. We remark that when the parameter \( \theta = 0 \), \( s_i = k_i \) and the REBG reduces to the RDG method. When \( \theta > 0 \), the REBG method contains the global information delivered from the edge betweenness.

An shown in Fig. 1 (b), the REBG method can effectively avoid the problem of synchronization failure. In this simple network, whenever node 1 is chosen before node 3 and 7, synchronization failure occurs. We note that node 1 is the cut-vertex connecting the two communities, its edges are of high betweenness. To see how failure is avoided, we evaluate the initial \( s_1 \) before any direction assignment, as given by \( s_1 = 20^\theta + 20^\theta \), \( s_x = 1^\theta + 1^\theta + 6^\theta \) for \( x = 2, 4, 5, 6, 8, 9 \) and \( s_y = 6^\theta + 6^\theta + 6^\theta + 20^\theta \) for \( y = 3, 7 \). By tracing all the possibilities of subsequent direction assignment, we find that synchronization failure does not occur provided that node 1 is not selected at the first assignment. In this case, \( s_1 > s_x \) which implies \( \theta \geq 0.18 \). We denote this value as \( \theta_c \), which marks the value of \( \theta \) from which failure ceases. We remark that the value of \( \theta_c \) is different for different topology.

We then examine the failure rate in the random exponential networks and the random scale-free networks. As shown in Fig. 2(c) and (d), failure rate vanishes for both networks when \( \theta = 0.01, 0.2 \) and 1. These results suggest \( \theta_c \approx 0 \), i.e., failure ceases when edge betweenness is considered in direction assignment. It implies that \( \theta \) in Eq. (1) should be positive to make complete synchronized state possible.

Finally, we compare the synchronizability index \( R \) between REBG and RDG methods. Both random exponential networks and random scale-free networks are examined. For better illustration, we report the difference \( D = R_{REBG} - R_{RDG} \) as a function of \( \theta \) and \( \beta \) in Fig. 3(a), and \( \theta \) and \( \gamma \) in Fig. 3(b). The positive \( D \) shows that the REBG method results in higher synchronizability as compared to the RDG method. This enhancement is mainly resulted from the prevention of the synchronization failure. One may question the validity of indicating synchronizability by \( R \) with only real part of eigenvalues. We show in Fig. 4 that for the regime when \( \beta \) and \( \gamma \) is large, the resultant networks are free of directed loops, which support the validity of positive \( D \) (as obtained by \( R \)) in this regime. These results also explain the negative \( D \) observed with intermediate value of \( \beta \) and \( \gamma \). From the lines of \( \theta = 0 \) and \( \theta = 1 \), we see that the number of networks with directed loops is higher when \( \theta = 1 \), which hinders synchronization and lead to unfavorable results from REBG. We further show that the REBG method with \( \theta = 0.2 \) are similar to the RDG in terms of the fraction of loopy realizations, suggesting \( \theta = 0.2 \) is an effective value for direction assignment, and at the same time avoids synchronization failure. Hence, one can use the REBG method with small \( \theta \) to enhance synchronizability effectively.

We further compare the RDG and the REBG methods in graphs where failure does not occur. Here, we study the two methods in Girvan Newman benchmark (GN-benchmark) network which consists of 128 nodes and is divided into 4 communities [27]. In GN-benchmark, \( k_{\text{inter}} + k_{\text{outer}} = 16 \), where \( k_{\text{inter}} \) is the average node degree in each community and \( k_{\text{outer}} \) is the average node degree among different communities. We show in Fig. 5 the results of \( k_{\text{outer}} = 1 \) from which the GN-benchmark
networks are highly clustered. We can see from Fig. 5(a) that the synchronization enhancement comes mainly from $\lambda_2$. Moreover, we tested the Kuramoto model on the resultant RDG networks and REBG networks. The oscillator on node $i$ of the networks is described by $\dot{\theta}_i = \omega_i + \sigma \sum_{j} A_{ij} \sin(\theta_j - \theta_i)$ in which $A$ is the adjacency matrix and $A_{ij}$ denote the information flow from $j$ to $i$, and the collective phase synchronization can be investigated by the order parameter defined as $r(t) = \langle |(\sum_{j=1}^{N} e^{i \theta_j(t)/N})\rangle$. From Fig. 5(d), it is obvious that $r(t)$ of the REBG networks converges faster than that of the RDG networks. These results show that the REBG method lead to greater improvement in synchronizability than the RDG method in highly clustered networks, despite the absence of failure in RDG networks.

In summary, we introduced the residual edge betweenness gradient (REBG) method for direction assignment, which overcomes the problem of emergence of incommunicable components in the residual degree gradient (RDG) method. The effectiveness of our method lies in the use of edge betweenness, which reflects global network information when compared to the node degree in RDG. Further tests of REBG and RDG in highly clustered networks show that REBG can lead to greater synchronizability improvement in networks despite the absence of failure problem. For daily applications, such incommunicable components brings huge loss to the electrical systems when power grids are unable to reach complete synchronization. Hence, the REBG method is effective in improving synchronizability which may lead to wide applications.

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![FIG. 5: (Color online) The synchronizability of the REBG method and the RDG method in GN-benchmark($k_{outer} = 1$) as measured by (a) $\lambda_2^R$, (b) $\lambda_2^N$, and (c) $R = \lambda_2^R/\lambda_2^N$ as a function of $\theta$. (d) The order parameter $r(t)$ of Kuramoto model ($\sigma = 10$) on the RDG networks and REBG networks($\theta = 1$). The results are obtained by averaging 100 independent realizations.](image)

[1] A. Arenas, A. Diaz-Guilera, J. Kurths, Y. Moreno, and C.-S. Zhou, Phys. Rep. 469, 93 (2008).
[2] T. Nishikawa, A. E. Motter, Y.-C. Lai, and F. C. Hoppensteadt, Phys. Rev. Lett., 91, 014101 (2003).
[3] H. Hong, B. J. Kim, M. Y. Choi, and H. Park, Phys. Rev. E, 69, 067105 (2004).
[4] P. N. McGraw and M. Menzinger, Phys. Rev. E, 72, 014101 (2005).
[5] J. Gomez-Gardenes, Y. Moreno, and A. Arenas, Phys. Rev. Lett., 98, 034101 (2007).
[6] T. Nishikawa, and A. E. Motter, Proc. Natl. Acad. Sci. U.S.A. 107, 10342 (2010).
[7] C.-S. Zhou, A. E. Motter, and J. Kurths, Phys. Rev. Lett., 96, 034101 (2006).
[8] A. E. Motter, C.-S. Zhou, and J. Kurths, Phys. Rev. E, 71, 016116 (2005).
[9] M. Chavez, D.-U. Hwang, A. Amann, H. G. E. Hentschel, and S. Boccaletti, Phys. Rev. Lett., 94, 218701 (2005).
[10] D.-U. Hwang, M. Chavez, A. Amann, and S. Boccaletti, Phys. Rev. Lett., 94, 138701 (2005).
[11] Y.-F. Lu, M. Zhao, T. Zhou, and B.-H. Wang, Phys. Rev. E, 76, 057103 (2007).
[12] T. Nishikawa and A. E. Motter, Phys. Rev. E, 73, 065106 (2006).
[13] T. Nishikawa and A. E. Motter, Physica D, 224, 77 (2006).
[14] A. Zeng, Y. Hu, and Z. Di, Europhys. Lett., 87, 48002 (2009).
[15] T. Zhou, M. Zhao, and C.-S Zhou, New J. Phys., 12, 043030 (2010).
[16] G. Bianconi, N. Gulbahce, and A. E. Motter, Phys. Rev. Lett. 100, 118701 (2008).
[17] A. Zeng, Y.-Q. Hu, and Z. Di, Phys. Rev. E, 81, 046211 (2010).
[18] G. Zamora-Lopez, V. Zlatic, C.-S. Zhou, H. Stefancic, and J. Kurths, Phys. Rev. E 77, 016106 (2008).
[19] S. M. Park and B. J. Kim, Phys. Rev. E 74, 026114 (2006).
[20] S.-W. Son, B. J. Kim, H. Hong, and H. Jeong, Phys. Rev. Lett. 103, 228702 (2009).
[21] M. Barahona and L. M. Pecora, Phys. Rev. Lett., 89, 054101 (2002).
[22] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett., 80, 2109 (1998).
[23] K. S. Fink, G. Johnson, T. Carroll, D. Mar, and L. Pecora, Phys. Rev. E 61, 5080 (2000).
[24] Different from [20], please note that the edge direction here is the direction of information flow.
[25] R. Albert, I. Albert, and G. L. Nakarado, Phys. Rev. E 69, 025103(R) (2004).
[26] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. E 64, 026118 (2001).
[27] M. E. J. Newman and M. Girvan, phys. Rev. E 69, 026113 (2004).