We have found the algebraic structure of the two-qubit quantum Rabi model behind the possibility of its novel exceptional eigenstates with finite photon numbers by analyzing the Hamiltonian in the photon number space. The exceptional solutions with at most 1 photon exist in the whole qubit–photon coupling regime with constant eigenenergy equal to single photon energy $\omega_\eta$, which can be clearly demonstrated from the Hamiltonian structure. With a similar method, we find that these special dark-state-like eigenstates (the eigenenergy is coupling-independent, but the wave function is coupling-dependent) commonly exist for the two-qubit Jaynes–Cummings model, with $E = N\omega_\eta$ ($N = -1, 0, 1, ...$), and one of them is also the eigenstate of the
two-qubit quantum Rabi model, which is interesting for application in a simpler way. Besides, using Bogoliubov operators, we analytically retrieve the solution of the general two-qubit quantum Rabi model. In this concise and physical way, we clearly see how the eigenvalues of the infinite-dimensional two-qubit quantum Rabi Hamiltonian are determined by a convergent power series, so that the solution can reach arbitrary accuracy conveniently because of the convergence property.

Keywords: two-qubit quantum Rabi model, algebraic structure, bogoliubov operators, solvability, dark-state-like eigenstates

1. Introduction

The quantum Rabi model [1] describes the interaction between a bosonic mode and a two-level system—probably the simplest interaction between light and matter. Its semiclassical form was first introduced by Rabi in nuclear magnetic resonance [2]. In 1963, Jaynes and Cummings [1] found its application in describing the interaction between a two-level molecular and a single mode photon field. With the development of experiments, many systems can be described by this model in quantum optics [3], condensed matter [4], cavity quantum electrodynamics (QED) [5], circuit QED [6], quantum dots [7], trapped ions [8] and so on. Although this model takes a very simple form, its analytical solution was not so easy to obtain, so various approximations were made, one of which is the famous rotating-wave approximation [1]. In 2011, its solution was analytically found by Braak [9] in the Bargmann space [10]. It can describe the ultrastrong qubit–photon coupling regime, which has been reached in recent circuit QED experiments [11], where the rotating wave approximation breaks down. After that, much research was conducted on the full Rabi Hamiltonian, including recovering the solution of the Rabi model [12–14], real-time dynamics [15], the solution of the two-qubit Rabi Hamiltonian [16–22], dynamical correlation functions [23], and so on [24–34].

The two-qubit system is fundamental to the construction of the universal quantum gate. Various qubit–qubit interactions are applied to generate qubit–qubit entanglement and realize quantum computation [35, 36], one of which is mediated by a resonant cavity, described by the two-qubit quantum Rabi model [20]. In this case, the ultrafast two-qubit quantum gate can be constructed in the ultrastrong qubit–photon coupling regime [37]. Besides, the distant qubits can be coupled through a resonant cavity and the coherent quantum state storage and transfer can be realized [38]. Working for the whole qubit–photon coupling regime, the two-qubit quantum Rabi model can be applied in many systems in quantum optics [39] and quantum information [40]. Its analytical solution was obtained in [20] by means of the Bargmann space approach, and also in [22] with extended coherent states representation. One interesting result is that coupling-dependent eigenstates exist in the whole coupling regime with constant eigenenergy, reminiscent of ‘dark states’ or ‘trapping states’ [19], but they are coupling-dependent and the photon number is bounded from above at 1, which is novel and interesting. Besides, there are exceptional solutions with finite photon numbers N, which are not present in the one-qubit Rabi model. These special solutions are very interesting and may have potential application, however, the algebraic structure behind the possibility of these special solutions needs to be clarified.
In this paper, we have clarified the algebraic structure of the two-qubit quantum Rabi model for its special exceptional solutions with finite photon numbers found in [20]. By analyzing the Hamiltonian in the photon number space, we find the condition for a closed subspace, i.e. the algebraic structure is related to the permutation symmetry of the qubit–photon coupling terms for the two qubits. Even more interestingly, the exceptional solution with at most 1 photon exists in the whole coupling regime with constant eigenenergy equal to single photon energy $\hbar \omega$, which can be clearly found from the algebraic structure. These eigenstates are partly like ‘dark states’, but are coupling dependent and the photon number is bounded from above, so they are interesting for application. According to the algebraic structure of the two-qubit quantum Rabi model, we may conjecture that there are similar dark-state-like solutions to those models with homogenous qubit–photon coupling terms. For example, we consider the two-qubit Jaynes–Cummings model [38], which is commonly applied for simplicity in the coupling regime $g \ll \omega$ [41]. Very interestingly, under a similar condition, we find many dark-state-like eigenstates, existing in the whole coupling regime with constant eigenenergy $E = N\hbar \omega$ $(N = -1, 0, 1, \ldots)$, one of which is also the eigenstate of the two-qubit Rabi model. Since the Jaynes–Cummings model is simpler than the Rabi model, we may find out the possible application of these dark-state-like eigenstates more easily. On the other hand, we analytically retrieve the solution of the two-qubit quantum Rabi model using Bogoliubov operators. With this physical and straightforward method, we find a way to obtain its solution by a convergent power series, so that we can choose a reasonable cutoff in practical calculation and the solution can reach arbitrary accuracy.

The paper is organized as follows. In section 2, we clarify the algebraic structure behind the possibility of exceptional solutions with finite photon numbers obtained in [20] and also find the special dark-state-like solutions of the two-qubit Jaynes–Cummings model. In section 3, we analytically retrieve the solution of the two-qubit quantum Rabi model using Bogoliubov operators. Finally, we make some conclusions in section 4.

### 2. Algebraic structure for exceptional solutions with finite photon numbers

The Hamiltonian of the two-qubit quantum Rabi model reads ($\hbar = 1$) [17, 20]

$$H_{\text{q}} = \omega a^\dagger a + g_1 \sigma_1 \left( a + a^\dagger \right) + g_2 \sigma_2 \left( a + a^\dagger \right) + \Delta_1 \sigma_1 + \Delta_2 \sigma_2,$$

where $a^\dagger$ and $a$ are the single mode photon creation and annihilation operators with frequency $\omega$, respectively. $\sigma_i$ ($i = x, y, z$) are the Pauli matrices. $2\Delta_1, 2\Delta_2$ are the energy level splittings of the two qubits. $g_1$ and $g_2$ are the qubit–photon coupling constants for the two qubits respectively. There are exceptional solutions with finite photon numbers $N$ obtained by analyzing the recurrence relation of the coefficients in [20]. However, the algebraic structure behind the possibility of these novel exceptional solutions needs to be clarified.

Exceptional solutions with finite photon numbers $N$ imply that there are closed subspaces in the photon number representation, i.e. the algebraic structure. Here we demonstrate that the closed subspaces are related to the permutation symmetry of the qubit–photon coupling terms by analyzing the structure of the Hamiltonian in the photon number space. $H_{\text{q}}$ (1) possess a $\mathbb{Z}_2$ symmetry with the transformation $R = \exp(i\alpha a^\dagger a) \otimes \sigma_z \otimes \sigma_2$. Taking odd parity for example, supposing the initial state $|\psi\rangle$ is in a subspace formed by $\{|M, e, g\}, |M, g, e\}, |M + 1, g, g\}, |M + 1, e, e\}$, ..., $|N - 1, g, g\}, |N - 1, e, e\}$, $|N, e, g\}, |N, g, e\}, |N, e, e\}$, with the coefficient $\{c_1, c_2, c_3, c_4, c_{1+1}, \ldots, c_{N+1}, c_2\}$, where $M$ and $N$ are even photon numbers, then the Hamiltonian reads ($\omega$ is set to 1)
If for $|\psi\rangle = H |\psi\rangle$, the coefficients of $\{ |N + 1, g, g\rangle, |N + 1, e, e\rangle \}$ and $\{ |M - 1, g, g\rangle, |M - 1, e, e\rangle \}$ equal to 0, then this subspace is closed. For the first case, we obtain

$$\sqrt{N + 1} g_1 c_{1,N} + \sqrt{N + 1} g_2 c_{2,N} = 0,$$

(3)

$$\sqrt{N + 1} g_2 c_{1,N} + \sqrt{N + 1} g_1 c_{2,N} = 0,$$

(4)

where $c_{1,N}$ and $c_{2,N}$ are the coefficients of $|N, g, e\rangle$ and $|N, g, e\rangle$ respectively. From (3) and (4) and $g_1, g_2 > 0$, we obtain $g_1 = g_2$ and $c_{1,N} = -c_{2,N}$. By using the time-independent Schrödinger equation, we obtain

$$\sqrt{N} g_1 c_{1,N-1} + \sqrt{N} g_2 c_{2,N-1} + (N + \Delta_1 - \Delta_2) c_{1,N} = E c_{1,N},$$

(5)

$$\sqrt{N} g_2 c_{1,N-1} + \sqrt{N} g_1 c_{2,N-1} + (N + \Delta_2 - \Delta_1) c_{2,N} = E c_{2,N},$$

(6)

so that

$$E = N,$$

(7)

$$\Delta_2 - \Delta_1 c_{1,N} \equiv (\sqrt{N} g_1 c_{1,N-1} + \sqrt{N} g_2 c_{2,N-1}).$$

(8)

For the special case $\Delta_1 = \Delta_2$ and $c_{1,N-1} = c_{2,N-1} = 0$, there is an invariant subspace formed by $\{ |N, e, g\rangle, |N, g, e\rangle \}$, and the eigenstate is

$$|\psi\rangle_N = \frac{1}{\sqrt{2}} (|N, g, e\rangle - |N, e, g\rangle),$$

(9)

which is the famous ‘dark state’ [19, 20], where the spin singlet is decoupled from the photon field.

If $\Delta_1 \neq \Delta_2$, considering the coefficient of $\{ |M - 1, g, g\rangle, |M - 1, e, e\rangle \}$ must be 0, it is required that $E = M$, which contradicts with $E = N$, so that the only possible choice is $M = 0$. Now we have obtained a closed subspace (algebraic structure) formed by $\{ |0, e, g\rangle, |0, g, e\rangle, |1, g, g\rangle, |1, e, e\rangle, \cdots, |N - 1, g, g\rangle, |N - 1, e, e\rangle, |N, e, g\rangle, |N, g, e\rangle \}$, with the condition

$$g_1 = g_2,$$

(10)

$$E = N.$$

(11)

Then by using the time-independent Schrödinger equation, we can obtain exceptional solutions with finite photon number $N$ for a certain choice of parameters $\Delta_1, \Delta_2,$ and $g = g_1 + g_2$. For example, if $N = 2$, the determinant of the matrix
must equal to 0, which gives

\[
\left( \Delta_1^2 - \Delta_2^2 \right) \left( 4 - (\Delta_1 - \Delta_2)^2 \right) \left( 1 - (\Delta_1 + \Delta_2)^2 \right) - 2g^2 = 0. \tag{13}
\]

This is the condition for an odd parity solution with the photon number bounded from above at \( N = 2 \), coinciding with [20], which depends on \( \Delta_1, \Delta_2 \) and \( g \). So now, we have found that the algebraic structure and exceptional solutions with finite photon numbers \( N \). Furthermore, it is very interesting for the solution with \( N = 1 \), whose existing condition is independent of \( g \). The closed subspace is formed by \( \{|0, e, g\}, |0, g, e\}, |1, g, g\}, |1, e, e\} \), and the condition is

\[
\det \begin{vmatrix}
\Delta_1 - \Delta_2 - 1 & 0 & g/2 & g/2 \\
0 & \Delta_2 - \Delta_1 - 1 & g/2 & g/2 \\
g/2 & g/2 & -\Delta_1 - \Delta_2 & 0 \\
g/2 & g/2 & 0 & \Delta_2 + \Delta_1
\end{vmatrix} = 0, \tag{14}
\]

which gives

\[
(\Delta_1 + \Delta_2)^2 \left( (\Delta_1 - \Delta_2)^2 - 1 \right) = 0, \tag{15}
\]

which is independent of \( g \), coinciding with [20]. So for \( \Delta_1 - \Delta_2 = 1 = \hbar \omega \) and \( \Delta_1 - \Delta_2 = -1 = -\hbar \omega \), we obtain two exceptional solutions

\[
|\psi_{k1}\rangle = \frac{1}{\mathcal{N}} \left( \frac{2(\Delta_1 + \Delta_2)}{g} \{0, e, g\} + \{1, g, g\} - \{1, e, e\} \right), \tag{16}
\]

\[
|\psi_{k2}\rangle = \frac{1}{\mathcal{N}} \left( \frac{2(\Delta_1 + \Delta_2)}{g} \{0, g, e\} + \{1, g, g\} - \{1, e, e\} \right), \tag{17}
\]

respectively, where \( \mathcal{N} = \sqrt{4(\Delta_1 + \Delta_2)^2 + 2g^2/g} \). For example, choosing \( \Delta_1 = 1.4, \Delta_2 = 0.4, g_1 = g_2 \), the numerical spectrum of the two-qubit quantum Rabi model is shown in figure 1. The horizontal line at \( E = 1 = \hbar \omega \) corresponds to the special eigenstate \(|\psi_{k1}\rangle\) (16). This eigenstate exists in the whole coupling regime with constant eigenenergy, like ‘dark states’, but is coupling dependent, and with at most 1 photon.

For even parity, similarly, we obtain one such special eigenstate

\[
|\psi_e\rangle = \frac{1}{\mathcal{N}} \left( \frac{2(\Delta_1 - \Delta_2)}{g} \{0, e, e\} - \{1, e, e\} + \{1, g, g\} \right), \tag{18}
\]

with the condition \( \Delta_1 + \Delta_2 = 1 = \omega \), \( g_1 = g_2 \) and \( E = 1 = \hbar \omega \), consistent with [20]. Now, we have demonstrated all the exceptional eigenstates of the two-qubit quantum Rabi model with finite photon numbers presented in [20] by finding its algebraic structure in the photon number space.


\[ H_{\text{jtjc}} = a'a + g_1 \left( \sigma_1^z a + \sigma_1^- a^\dagger \right) + g_2 \left( \sigma_2^z a + \sigma_2^- a^\dagger \right) + \Delta_1 \sigma_1z + \Delta_2 \sigma_2z. \]  

(19)

It is easy to find \( C = a'a + \frac{1}{2}(\sigma_1z + \sigma_2z + 2) \) commutes with \( H_{\text{jtjc}} \), so there is a conserved quantity \( C = 2 \), and it is easy to testify \( |\psi_e\rangle \) is also an eigenstate of \( H_{\text{jtjc}} \) existing in the whole coupling regime with constant eigenenergy for \( g_1 = g_2 \) and \( \Delta_1 + \Delta_2 = 1 \). To find out all such kinds of eigenstates, we study the eigenproblem of \( H_{\text{jtjc}} \). For \( C = N \ (N > 1) \), the Hamiltonian in the subspace \( \{|N - 2, e, e\}, \{|N - 1, e, g\}, \{|N - 1, g, e\}, \{|N - g, g\}\} \) reads

\[
\begin{pmatrix}
N - 2 + \Delta_1 + \Delta_2 & \sqrt{N - 1} g_2 & 0 \\
0 & N - 1 + \Delta_1 - \Delta_2 & \sqrt{N - 1} g_1 \\
\sqrt{N - 1} g_2 & 0 & N - 1 - \Delta_1 + \Delta_2 \\
\sqrt{N - 1} g_1 & \sqrt{N} g_2 & N - \Delta_1 - \Delta_2
\end{pmatrix}.
\]

(20)

Using the time-independent Schrödinger equation, we find the eigenvalue \( E \) is determined by

\[
\begin{align*}
& (E - N + \Delta_1 + \Delta_2)(E - N + 2 - \Delta_1 - \Delta_2) \left[ (E - N + 1)^2 - (\Delta_1 - \Delta_2)^2 \right] \\
+ & (g_1^2 + g_2^2) \left[ (E - N + 1)(E - N + \Delta_1 + \Delta_2) - 2N(E - N + 1) \right] \\
+ & (g_1^2 - g_2^2) \left[ (E^2 - N^2) + (\Delta_1^2 - \Delta_2^2)(2N - 1) \right] \\
+ & (E + N)(\Delta_2 - \Delta_1) = 0.
\end{align*}
\]

(21)
The condition (21) is generally dependent on $g_1$ and $g_2$, but there are two special cases. The first is the ‘dark state’ $|\psi\rangle = \frac{1}{\sqrt{2}}(|N - 1, e, g\rangle - |N - 1, g, e\rangle)$, with the condition $g_1 = g_2$ and $\Delta_1 = \Delta_2$, just the same as the case of the two-qubit quantum Rabi model [18, 19]. The spin singlet is decoupled from the photon field, so the eigenenergy and eigenstate are coupling-independent. The second case is partly like the ‘dark state’—the eigenenergy is also coupling independent, but the eigenstate is not. For $g_1 = g_2$ and $\Delta_1 + \Delta_2 = 1$, (21) reduces to

$$(E - N + 1)^2 \left[ (E - N + 1)^2 + \left( \frac{1}{2} - N \right) g^2 - (\Delta_1 - \Delta_2)^2 \right] = 0,$$

where $g = g_1 + g_2$. For $E = N - 1$, the condition is $g$-independent. Besides, the eigenenergies are symmetric about $E = N - 1$ and there are two degenerate eigenstates with $E = N - 1$ existing in the whole coupling regime

$$|\psi_{C=N_0}\rangle = \frac{1}{A} \left( \frac{2(\Delta_1 - \Delta_2)}{\sqrt{N - 1} g} |N - 2, e, e\rangle - |N - 1, e, g\rangle + |N - 1, g, e\rangle \right),$$

$$|\psi_{C=0}\rangle = \frac{1}{B} \left( \frac{\sqrt{N - 1} g}{(\Delta_1 - \Delta_2)} |N - 2, e, e\rangle + |N - 1, e, g\rangle - |N - 1, g, e\rangle \right) + \frac{(N - 1) g^2 + 2(\Delta_2 - \Delta_1)^2}{\sqrt{N} g(\Delta_2 - \Delta_1)} |N, g, g\rangle,$$

where $\frac{1}{A}$ and $\frac{1}{B}$ are the normalizing constants. For $N = 2$, $|\psi_{C=2}\rangle$ is just $|\psi\rangle$, (17), which is the eigenstate of the two-qubit quantum Rabi model.

For $C = 1$, the subspace is formed by $\{|0, e, c\}, |0, g, e\}, |1, g, g\rangle\}$, and the eigenvalues satisfy

$$E = \left[ (\Delta_1 - \Delta_2)^2 + E(1 - \Delta_1 - \Delta_2) - E^2 + g_1^2 + g_2^2 \right] + (\Delta_1 + \Delta_2 - 1)(\Delta_1 - \Delta_2)^2 + (g_1^2 - g_2^2)(\Delta_1 - \Delta_2) = 0.$$

For $g_1 = g_2$ and $\Delta_1 + \Delta_2 = 1$, (25) reduces to

$$E \left[ E^2 - \frac{1}{2} g^2 - (\Delta_1 - \Delta_2)^2 \right] = 0.$$

So an eigenstate exists in the whole coupling regime with constant eigenenergy $E = 0$

$$|\psi_{C=0}\rangle = \frac{1}{N} \left( \frac{2(\Delta_1 - \Delta_2)}{g} |1, g, g\rangle - |0, e, g\rangle + |0, g, e\rangle \right).$$

For $C = 0$, the eigenstate is $|0, g, g\rangle$, with constant eigenenergy $E = -1$.

To conclude, for identical-coupling $g_1 = g_2$ and quasi-resonant condition $\Delta_1 + \Delta_2 = 1 = \omega$, the spectrum of the two-qubit Jaynes–Cummings Hamiltonian $H_{JC}$ is very regular and interesting: there are horizontal lines at $E = N_0 (N = -1, 0, 1, \ldots)$, and the energy curve with the same $C = N$ are symmetric about the line $E = N - 1$. For $C = 0, 1$, one kind of eigenstate exists in the whole coupling regime with constant eigenenergy, while for other cases, there are two such kinds of degenerate eigenstates, one of which for $C = 2$ is also the eigenstate of the two-qubit quantum Rabi model. With constant eigenenergy, these eigenstates are partly like the ‘dark state’, but they are coupling dependent. Choosing $\Delta_1 = 0.7,$
$\Delta_2 = 0.3$ and $g_1 = g_2 = g/2$, the spectra of the two-qubit Jaynes–Cummings model and Rabi model are compared in Figure 2.

### 3. Solvability of the two-qubit quantum Rabi model using Bogoliubov operators

First for convenience, we make unitary transformations $S_1 = \frac{1}{\sqrt{2}}(\sigma_{1z} + \sigma_{1z})$ and $S_2 = \frac{1}{\sqrt{2}}(\sigma_{2z} + \sigma_{2z})$ to the two-qubit Rabi Hamiltonian (1) to obtain ($\omega$ is set to 1)

$$H_{tq}' = a^\dagger a + g_1 \sigma_{1z} \left( a + a^\dagger \right) + g_2 \sigma_{2z} \left( a + a^\dagger \right) + \Delta_1 \sigma_{1z} + \Delta_2 \sigma_{2z}. \quad (28)$$

$H_{tq}'$ has a conserved parity with the $\mathbb{Z}_2$ transformation $R = T \otimes \sigma_{1z} \otimes \sigma_{2z}$, where $T = \exp(i \alpha a^\dagger a)$, giving us a way to diagonalize the Hamiltonian in the basis of $\{|e\rangle_1, |g\rangle_1\}$, which is the eigenvector of $\sigma_{1z}$. Applying the Fulton–Gouterman transformation [16, 42]

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ T \otimes \sigma_{2z} & -T \otimes \sigma_{2z} \end{pmatrix} \quad (29)$$

in the basis of $\{|e\rangle_1, |g\rangle_1\}$, we obtain

$$U^\dagger H_{tq}' U = \begin{pmatrix} H_t & 0 \\ 0 & H_- \end{pmatrix} \quad (30)$$

where

$$H_\pm = a^\dagger a + g_1 \left( a + a^\dagger \right) + g_2 \left( a + a^\dagger \right) \sigma_{2z} + \Delta_1 \sigma_{2z} \pm \Delta_1 T \sigma_{2z}, \quad (31)$$

acting on the subspace of $U^\dagger R U$ with eigenvalues $\pm 1$. First we consider $H_+$. For $H_-$, we just need to substitute $-\Delta_1$ for $\Delta_1$. In the basis of $\{|e\rangle_2 \otimes |\phi_1\rangle, |g\rangle_2 \otimes |\phi_2\rangle\}$, where $|\phi_1\rangle$ and $|\phi_2\rangle$ are photon field states, $H_+$ is expanded as
\[
\begin{pmatrix}
a^\dagger a + g(a + a^\dagger) & \Delta_2 + \Delta_1 T \\
\Delta_2 + \Delta_1 T & a^\dagger a + g'(a + a^\dagger)
\end{pmatrix},
\]

(32)

where \( g = g_1 + g_2, \ g' = g_1 - g_2 \). To remove the linear terms of \( a^\dagger \) and \( a \), we use the following Bogoliubov operators

\[ A = a + g, \ B = a + g'. \]

(33)

Firstly we use the Bogoliubov operator \( A \). The time-independent Schrödinger equation reads

\[
\left( A^\dagger A - g^2 - E \right) \left\{ \phi_1 \right\} + \Delta_2 \left\{ \phi_2 \right\} + \Delta_1 \left\{ \phi_4 \right\} = 0,
\]

(34)

\[
\left[ A^\dagger A + (g' - g)(A + A^\dagger) + g^2 - 2gg' - E \right] \left\{ \phi_2 \right\} + \Delta_2 \left\{ \phi_1 \right\} + \Delta_1 \left\{ \phi_3 \right\} = 0,
\]

(35)

where \( \left\{ \phi_1 \right\} = T \left\{ \phi_1 \right\}, \ \left\{ \phi_2 \right\} = T \left\{ \phi_2 \right\} \). To apply the reflection symmetry, we make the transformation \( T \) to (34) and (35) to obtain

\[
\left( A^\dagger A - 2g(A + A^\dagger) + 3g^2 - E \right) \left\{ \phi_3 \right\} + \Delta_2 \left\{ \phi_4 \right\} + \Delta_1 \left\{ \phi_5 \right\} = 0,
\]

(36)

\[
\left[ A^\dagger A - (g' + g)(A + A^\dagger) + g^2 + 2gg' - E \right] \left\{ \phi_4 \right\} + \Delta_2 \left\{ \phi_3 \right\} + \Delta_1 \left\{ \phi_6 \right\} = 0.
\]

(37)

We expand the photon field states \( \left\{ \phi_j \right\}, \ j = 1, \ldots, 4 \) in terms of the normalized orthogonal extended coherent state [12]

\[
|n, g\rangle = \frac{e^{-\frac{n^2}{2} - g^2}}{\sqrt{n!}} \left( a^\dagger + g \right)^n,
\]

(38)

which is the eigenstate of \( A^\dagger A \), and obtain

\[
\left\{ \phi_j \right\} = e^{\frac{g^2}{2}} \sum_{n=0}^{\infty} \sqrt{n!} a_{j,n} |n, g\rangle, \ j = 1, \ldots, 4.
\]

(39)

Substituting (39) into (34)–(37), and left multiply \( \langle m, g | \), we obtain the recurrence relations for \( a_{j,m} \)

\[
(E - m + g^2) a_{1,m} = \Delta_1 a_{4,m} + \Delta_2 a_{2,m},
\]

(40)

\[
(g' - g)(m + 1) a_{2,m+1} = (E - m + 2gg' - g^2) a_{2,m} - (g' - g) a_{2,m-1} - \Delta_1 a_{3,m} - \Delta_2 a_{1,m},
\]

(41)

\[
2g(m + 1) a_{3,m+1} = (m - E + 3g^2) a_{3,m} - 2ga_{3,m-1} + \Delta_1 a_{2,m} + \Delta_2 a_{4,m},
\]

(42)

\[
(g + g')(m + 1) a_{4,m+1} = (m - E + 2gg' + g^2) a_{4,m} - (g + g') a_{4,m-1} + \Delta_1 a_{1,m} + \Delta_2 a_{3,m}.
\]

(43)

It is seen that the coefficients \( a_{j,m} \) depend on three initial conditions, which can be chosen as \( \{a_{1,0}, a_{2,0}, a_{3,0}\} \).
Then we consider the Bogoliubov operator $B = a + g'$. Now $H_a$ is given as
\[
\left(\begin{array}{c}
B'B + (g - g')(B + B') + (g')^2 - 2gg' - E
\end{array}\right)
\Delta_2 + \Delta_1 + \Delta_2 + \Delta_1 = 0.
\]

Applying transformation $T$ to the time-independent Schrödinger equation, we obtain four equations similar to (34)–(37)
\[
\left(\begin{array}{c}
B'B - (g')^2 - E
\end{array}\right)
\Delta_2 + \Delta_1 = 0.
\]
\[
\left(\begin{array}{c}
B'B + (g' + g)(B + B') + (g')^2 + 2gg' - E
\end{array}\right)
\Delta_2 + \Delta_1 = 0.
\]
\[
\left(\begin{array}{c}
B'B + 2g'(B + B') + 3(g')^2 - E
\end{array}\right)
\Delta_2 + \Delta_1 = 0.
\]

Expanding the photon field states as $|\psi_j\rangle = e^{(g')^2/2} \sum_{n=0}^{\infty} \sqrt{n!} b_{j,n} |n, g'\rangle$, $j = 1, ..., 4$, where the normalized extended coherent state $|n, g'\rangle$ is the eigenstate of $B$, and left multiplying $|m, g\rangle$, we obtain the recurrence relations for $b_{j,m}$
\[
(g - g')(m + 1)b_{1,m+1} = (E - m + 2gg' - (g')^2)b_{1,m}
\]
\[
+ (g' - g)b_{1,m-1} - \Delta_1 b_{4,m} - \Delta_2 b_{2,m},
\]
\[
(E - m + (g')^2)b_{2,m} = \Delta_1 b_{3,m} + \Delta_2 b_{1,m},
\]
\[
(g + g')(m + 1)b_{3,m+1} = (m - E + 2gg' + (g')^2)b_{3,m}
\]
\[
- (g + g')b_{3,m-1} + \Delta_1 b_{2,m} + \Delta_2 b_{4,m},
\]
\[
2g'(m + 1)b_{4,m+1} = (m - E + 3(g')^2)b_{4,m}
\]
\[
- 2g'b_{4,m-1} + \Delta_1 b_{3,m} + \Delta_2 b_{1,m}.
\]

There are three initial conditions, which can be chosen as $\{b_{1,0}, b_{2,0}, b_{4,0}\}$. To utilize the reflection symmetry $|\phi_j\rangle = T |\phi_j\rangle$, $|\phi_3\rangle = T |\phi_3\rangle$, finally, we expand the photon states in terms of the photon number states as $|\psi_j\rangle = \sum_{n=0}^{\infty} \sqrt{n!} c_{j,n} |n\rangle$, and obtain the recurrence relations for $c_{j,m}$
\[
(m + 1)gc_{1,m+1} = (E - m)c_{1,m} - gc_{1,m-1} - \Delta_2 c_{2,m} - \Delta_1 c_{4,m},
\]
\[
(m + 1)g'c_{2,m+1} = (E - m)c_{2,m} - g'c_{2,m-1} - \Delta_2 c_{1,m} - \Delta_1 c_{3,m},
\]
\[
(m + 1)gc_{3,m+1} = (m - E)c_{3,m} - gc_{3,m-1} + \Delta_2 c_{4,m} + \Delta_1 c_{2,m},
\]
\[
(m + 1)g'c_{4,m+1} = (m - E)c_{4,m} - g'c_{4,m-1} + \Delta_2 c_{3,m} + \Delta_1 c_{1,m}.
\]

Considering $|\psi_1\rangle = T |\psi_1\rangle$, $|\psi_2\rangle = T |\psi_2\rangle$, we obtain $c_{1,m} = (-1)^m c_{1,m}$, $c_{2,m} = (-1)^m c_{2,m}$, so there are only two initial conditions, which can be chosen as $c_{1,0}$ and $c_{2,0}$. For homogenous coupling $g_1 = g_2$ ($g' = 0$), we have
\[ c_{2,m} = \frac{\Delta_2 + (-1)^m \Delta_1}{E - m} c_{1,m}, \]  

and the five-term recurrence relations (53)–(56) reduce to a three-term recurrence relation

\[ (m + 1)g_{1,m+1} = \frac{(E - m)^2 - \left[ (-1)^m \Delta_1 + \Delta_2 \right]^2}{E - m} c_{1,m} - g_{1,m-1}, \]

with initial condition \( c_{1,0} \), just like the case in [17, 19].

States \(|\phi_j\rangle\), \(|\psi_j\rangle\) and \(|\psi_j\rangle\) in different representations should be only different by a constant (here can be chosen as 1) if they are nondegenerate eigenstates with eigenvalue \( E \), so we obtain 8 equations

\[ |\phi_j\rangle = |\psi_j\rangle, \]  

\[ |\psi_j\rangle = |\psi_j\rangle. \]

For practical calculation, we left multiply \( \langle 0| e^\beta a \) where \( \beta \) is chosen arbitrarily, then (59) and (60) are mapped to

\[ \langle 0| e^{\beta_1 a} |\phi_j\rangle = \sum_{m=0}^{\infty} a_{j,m} \exp(-g\beta_1)(\beta_1 + g)^m \]

\[ = \langle 0| e^{\beta_1 a} |\psi_j\rangle = \sum_{m=0}^{\infty} b_{j,m} \exp(-g\beta_1)(\beta_1 + g)^m. \]

\[ \langle 0| e^{\beta_2 a} |\psi_j\rangle = \sum_{m=0}^{\infty} b_{j,m} \exp(-g\beta_2)(\beta_2 + g')^m \]

\[ = \langle 0| e^{\beta_2 a} |\psi_j\rangle = \sum_{m=0}^{\infty} c_{j,m} \beta_2^m. \]

According to the analysis in the Bargmann space [9, 14, 20], if (61)–(62) are satisfied for one choice of \( \{\beta_1, \beta_2\} \), then (59)–(60) are satisfied. Since we are dealing with a power series of infinite terms, for convenience, we chose proper \( \beta_1 \) and \( \beta_2 \) to make all the power series convergent. According to the recurrence relations for \( a_{j,m} \) (40)–(43), \( b_{j,m} \) (49)–(52) and \( c_{j,m} \) (53)–(56), we find that the radii of convergence of the corresponding power series are \( |g - g'|, \min\{g - g', 2g'\} \) and \( g' \) respectively. So, for different \( g \) and \( g' \), we can always choose proper \( \beta_1 \) and \( \beta_2 \) to obtain the convergent power series [20], so that finite terms can give reliable results and by choosing proper cutoff, we can obtain the results with arbitrary accuracy. That is the advantage of choosing these three different representations. Because of the linearity of recurrence relations, we can denote

\[ \phi_j(\beta_1) = \langle 0| e^{\beta_1 a} |\phi_j\rangle = \sum_{k=1}^{3} a_{k,0} \phi_{j}^k(\beta_1), \]

\[ \psi_j(\beta_1) = \langle 0| e^{\beta_1 a} |\phi_j\rangle = \sum_{k=1,2,4} b_{k,0} \phi_{j}^k(\beta_1), \]

\[ \psi_j(\beta_2) = \langle 0| e^{\beta_2 a} |\psi_j\rangle = \sum_{k=1,2,4} b_{k,0} \phi_{j}^k(\beta_2). \]
where for example, $\psi_j(\beta_2)$ is obtained by setting $b_{k,0}$ equal to 1 and other initial conditions equal to 0 in (49)–(52), like in [25]. Now we have eight initial conditions for eight equations

$$
\phi_j(\beta_1) = \psi_j(\beta_1), \tag{67}
$$

$$
\phi_j(\beta_2) = \psi_j(\beta_2), \tag{68}
$$

which can be denoted as

$$
M_{jk} \xi_k = 0, \tag{69}
$$

with $\xi = [b_{1,0}, b_{2,0}, a_{3,0}, c_{1,0}, c_{2,0}]^T$. The determinant of $M$, which is just the function of energy $E$ must be equal to 0, so we obtain

$$
G(E) = \text{det}(M) = 0, \tag{70}
$$

which can be used to determine the eigenenergy $E$. Equation (70) takes a similar form to (14) in [20], but is obtained in a more physical way. Choosing $\Delta_1 = 0.4$, $\Delta_2 = 0.3$, $\omega = 1$, $g_i = 3g_2$, $0 \leq g = g_1 + g_2 \leq 3$. $E_{+e}$ and $E_{-e}$ are numerical solutions with even and odd parity respectively, while $E_{+o}$ and $E_{-o}$ are analytical solutions with even and odd parity respectively.

Figure 3. The spectrum of the two-qubit quantum Rabi model with $\Delta_1 = 0.4$, $\Delta_2 = 0.3$, $\omega = 1$, $g_i = 3g_2$, $0 \leq g = g_1 + g_2 \leq 3$. $E_{+e}$ and $E_{-e}$ are numerical solutions with even and odd parity respectively, while $E_{+o}$ and $E_{-o}$ are analytical solutions with even and odd parity respectively.
4. Conclusions

We have clarified the algebraic structure behind the possibility of the exceptional solutions with finite photon numbers found in [20]. By analyzing the Hamiltonian structure in the photon number space, we find that the permutation symmetry of the qubit–photon coupling terms for the two qubits brings about a closed subspace, and hence exceptional solutions for certain parameters. The novel coupling-dependent eigenstates existing in the whole coupling regime with constant eigenenergy $E$ equal to single photon energy $\hbar \omega$ correspond to exceptional solutions with at most 1 photon, with the condition for the qubits energy splittings $\Delta_1 \pm \Delta_2 = \hbar \omega$ or $\Delta_2 - \Delta_1 = \hbar \omega$. We have demonstrated this directly from the Hamiltonian structure. These special eigenstates are partly like ‘dark states’, but are coupling-dependent, so we call them dark-state-like eigenstates here, which may have some potential application.

Furthermore, based on our study on the two-qubit quantum Rabi model, we conjecture that such dark-state-like eigenstates commonly exist in similar models with a permutation symmetry of the qubit–photon coupling terms. For example, for the homogenous coupled two-qubit Jaynes–Cummings model, there are many dark-state-like eigenstates with constant energy $E = N \hbar \omega$ ($N = -1, 0, 1, \ldots$), and one of them is also the eigenstate of the two-qubit quantum Rabi model. Since the Jaynes–Cummings model is simpler than the Rabi model, we may find the application of these special eigenstates more easy.

Besides, using Bogoliubov operators, we have analytically retrieved the solution of the two-qubit quantum Rabi model in a concise way. We find three different representations to expand the Hamiltonian, and the solutions can be determined by a convergent power series. In this way, the eigenproblem of the infinite dimensional Hamiltonian can reasonably reduce to the finite dimension in practical calculation, and the results can reach arbitrary accuracy.

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