Particle physics and cosmology of the string derived no-scale flipped $SU(5)$

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Abstract

In a recent paper, we identified a cosmological sector of a flipped $SU(5)$ model derived in the free fermionic formulation of the heterotic superstring, containing the inflaton and the goldstino superfields with a superpotential leading to Starobinsky type inflation, while $SU(5) \times U(1)$ is still unbroken. Here, we study the properties and phenomenology of the vacuum after the end of inflation, where the gauge group is broken to the Standard Model. We identify a set of vacuum expectation values, triggered by the breaking of an anomalous $U(1)_A$ gauge symmetry at roughly an order of magnitude below the string scale, that solve the $F$ and $D$-flatness supersymmetric conditions up to 6th order in the superpotential which is explicitly computed, leading to a successful particle phenomenology. In particular, all extra colour triplets become superheavy guaranteeing observable proton stability, while the Higgs doublet mass matrix has a massless pair eigenstate with realistic hierarchical Yukawa couplings to quarks and leptons. The supersymmetry breaking scale is constrained to be high, consistent with the non observation of supersymmetric signals at the LHC.
1 Introduction

In this work, we make an important step further on the avenue we started recently [1], towards a string derived microscopic model that provides a simultaneous description of fundamental particle physics and cosmology. The model was constructed in 1989 [2] within the framework of free fermionic formulation of four-dimensional (4d) heterotic superstring [3] and has an observable sector based on the flipped $SU(5) \times U(1)$ gauge group with three chiral families of quarks and leptons [4,5]. The basis vectors of boundary conditions defining the model, as well as its full massless spectrum are given again for self consistency in Appendix A.

In [1], we identified the inflaton among the gauge singlet massless states of the model with the superpartner of a fermion mixed with the Right-handed neutrinos [6]. It acquires a superpotential together with the goldstino at 6th and 8th order in the string slope $\alpha'$-expansion, via vacuum expectation values (VEVs) of fields generated by the breaking of an anomalous $U(1)_A$ gauge symmetry (characteristic in heterotic models [7]) at a calculable scale an order of magnitude below the string mass, related to the anomaly. This scale in string units introduces therefore a small parameter allowing perturbative computations around the free-fermionic point where all fields and moduli are fixed at zero VEVs. As a result, the inflation scale turns out to be about five orders of magnitude lower, in the range of $10^{13}$ GeV, while the superpotential leads to a Starobinsky-type inflation [8] due to the no-scale structure of the low energy effective supergravity [9] which is calculable in our model to all orders in $\alpha'$ [1,10,11]. Note that during inflation, $SU(5) \times U(1)$ remains unbroken because its breaking occurs via a first order phase transition at a critical temperature which is lower that the scale of inflation [12,13].

In this work, we extend the previous analysis to the study of the vacuum of the theory, after the end of inflation, where $SU(5) \times U(1)$ is broken to the Standard Model. More precisely, we find a consistent set of VEVs that solve the F and D-flatness equations up to 6th order in the $\alpha'$-expansion of the superpotential. This vacuum obviously preserves supersymmetry, whose breaking we don’t discuss here. An important result to emphasise is that the requirement of gauge symmetry breaking by a pair of $10 + 10$ leads to a slight reorganisation of the choice of VEVs for the $SU(5) \times U(1)$ gauge singlet states, compared to the set we had during the inflationary phase, in order to satisfy the flatness conditions.

Our analysis for finding the choice of VEVs requires in some cases the knowledge of the exact coefficients of higher dimensional operators, or precise relations among them. Thus, besides applying selection rules to find which of those are non-vanishing, we need to perform some explicit computations of superpotential terms to 5th or 6th order which is highly non-trivial. The challenging part involves correlation functions of several primary operators in the Ising model for which we
concentrate a dedicated section of this paper.

Obviously, one of our goals is to identify the quarks and leptons among the three chiral generations and one vector-like pair, as well as a pair of Higgs doublets with the required Yukawa couplings. On the other hand, one should also ensure that all colour triplet states acquire masses at high scale so that there is no dangerous proton decay. For these reasons, we impose in our choice of VEVs that the colour triplet mass matrix has a non-vanishing determinant, while the weak doublet matrix has exactly one massless eigenstate with components along the doublets that provide a realistic hierarchical Yukawa matrix, taking into account the successful phenomenological analysis of the model that has been done in the past [2, 15–19]. Moreover, the constraints from proton decay impose that the supersymmetry breaking scale $m_{\text{susy}}$ should be at least of order of tens of TeV, compatible with an independent analysis of reheating and nucleosynthesis requiring $m_{\text{susy}}$ to lie in this energy region [13], possibly within the reach of the next generation of high energy hadron colliders.

The outline of the paper is the following. In Section 2, we review briefly our previous results [1], such as the identification of the inflaton and the goldstino, as well as the choice of VEVs giving rise to the inflationary superpotential. In Section 3, we impose the phenomenological constraints realising the triplet-doublet splitting at the string level, making all colour triplets superheavy while leaving massless one pair of massless doublets. In Section 4, we compute and solve the D and F-flatness conditions up the 6th order, taking into account the above constraints. Section 5 is devoted to the explicit computation of some superpotential coefficients involving higher point functions of primary operators in the Ising model. In Section 6, we perform the phenomenological analysis by identifying the electroweak Higgs doublets and the quarks and leptons, and by computing in particular the structure of fermion masses. We also study proton decay by computing the relevant dimension-five operators induced by the Higgs triplets exchange, as well as those emerging directly at the string level. Finally, Section 7 contains some concluding remarks. The paper has also three Appendices. Appendix A contains a brief summary of the ‘revamped’ flipped $SU(5)$ string model, Appendix B contains the list and details on the flatness conditions up to 5th order in the superpotential which we use in our analysis in Section 4, while Appendix C contains operator product expansions and various correlators of the Ising model that we use in Section 5.

2 The revamped flipped model and its cosmology sector

For convenience of the reader, we recall briefly the massless spectrum of the ‘revamped’ flipped $SU(5)$ model (with the original notation of [2]), which consist of:
1. A gauge group $[SU(5) \times U(1)] \times U(1)^4 \times [SO(10) \times SO(6)]$, where besides the observable and hidden sectors in the first and last bracket, there are four $U(1)$s, one combination of which is anomalous and becomes massive.

2. Four generations and one anti-generation of chiral matter in the representations $10 + \bar{5} + 1$ (and $10 + \bar{5} + 1$) of $SU(5) \times U(1)$, denoted as $F_i, F_i, \ell_i^c$ (and $F_i, f_i, \ell_i^c$) where $i$ labels the corresponding vector of boundary conditions.

3. Four pairs of $5 + \bar{5}$ containing the electroweak Higgs doublets, denoted by $(h_i, \bar{h}_i)_{i=1,2,3}$ for those coming from the Neveu-Schwarz (NS) sector and $(h_{45} + \bar{h}_{45})$ for the pair coming from the $b_4 + b_5$ sector.

4. Five vectors $(10, 1) + (1, 6)$ of the hidden gauge group, denoted as $T_i$ and $D_i$, respectively.

5. Six pairs $(4 + \bar{4})$ of $SO(6)$ with fractional electric charges $\pm 1/2$, denoted as $(Z_1 + Z_1)$, $(Z_2 + Z_2)$, $(Y_1 + \bar{Y}_1)$, $(X_1 + \bar{X}_1)$, $Y_2, Y_2', X_2, X_2'$.

6. Ten pairs of non-abelian gauge singlets but charged under $U(1)^4$, denoted by $\Phi_{12}, \Phi_{23}, \Phi_{31}$ (with their conjugates) for those coming from the NS sector, and $\phi_1, \ldots, \phi_4, \phi_5, \phi_{45}$ (with their conjugates) for those coming from the $b_4 + b_5$ sector.

7. Five gauge singlets from the NS sector, denoted as $\Phi_1, \ldots, \Phi_5$.

The full tree-level (trilinear) superpotential reads

$$W_3 = g_s \sqrt{2} \left[ F_1 F_1 h_1 + F_2 F_2 h_2 + F_4 F_4 h_1 + \bar{F}_5 \bar{F}_5 h_2 + F_4 \bar{F}_5 \bar{h}_{45} + F_3 \bar{F}_3 \bar{h}_3 \\
+ 1 \sqrt{2} F_4 \bar{F}_5 \phi_3 + 1 \sqrt{2} f_4 \bar{f}_5 \phi_2 + 1 \sqrt{2} f_4 \bar{f}_5 \phi_2 \\
+ h_1 \bar{h}_2 \Phi_{12} + h_1 \bar{h}_2 \Phi_{12} + h_3 \bar{h}_3 \Phi_{23} + h_3 \bar{h}_3 \Phi_{23} + h_3 \bar{h}_3 \Phi_{31} \\
+ \bar{F}_3 h_1 \Phi_{31} + h_3 \bar{F}_4 \Phi_{45} + \bar{F}_3 h_4 \Phi_{45} + \frac{1}{2} \Phi_{45} \Phi_{45} \\
+ \phi_1 \bar{\phi}_2 \Phi_4 + \bar{\phi}_1 \phi_2 \Phi_4 + \phi_3 \bar{\phi}_4 \Phi_5 + \bar{\phi}_3 \phi_4 \Phi_5 + \frac{1}{2} \phi_{45} \phi_{45} \Phi_3 \\
+ \frac{1}{2} \phi_+ \phi_+ \Phi_3 + \frac{1}{2} \phi_- \phi_- \Phi_3 \right]$$

(2.1)
\[ W_I = M_I z(y - \lambda y^2), \quad (2.2) \]

with \( M_I \) the scale of inflation and \( \lambda \) a parameter close to \( 1 \). Following a detailed analysis of the various possibilities and a study of the relevant non-renormalisable (NR) superpotential interactions up to 10th order in the \( \alpha' \)-expansion, we identified the goldstino with the gauge singlet superfield \( \Phi_4 \) from the NS sector, while the inflaton is a linear combination of \( \phi_3 \) and \( \bar{\phi}_3 \):

\[ y = \sin \omega \phi_3 - \cos \omega \bar{\phi}_3 \, ; \quad \tan \omega = \langle \phi_4 \rangle / \langle \bar{\phi}_4 \rangle, \quad (2.3) \]
\[ z = \Phi_4. \quad (2.4) \]

The orthogonal linear combination to \( y \) is massive, while phenomenological constraints on appropriate reheating through the inflaton decay into neutrinos, baryogenesis and light neutrino masses require \( \langle \phi_4 \rangle / \langle \bar{\phi}_4 \rangle \sim 10^{-3} \) \[ 12-14 \].

The superpotential \((2.2)\) is generated at the 6th and 8th order and reads:

\[ W_I = g_s C_6 \left( g_s \sqrt{2\alpha'} \right)^3 \bar{\phi}_3 \Phi_4 \langle D_1 \rangle \cdot \langle D_5 \rangle \langle T_1 \rangle \cdot \langle T_4 \rangle \quad (2.5) \]
\[ + g_s C_8 \left( g_s \sqrt{2\alpha'} \right)^5 \bar{\phi}_3 \Phi_4 \langle D_1 \rangle \cdot \langle D_4 \rangle \langle T_1 \rangle \cdot \langle T_4 \rangle \langle \Phi_{31} \rangle, \]

where \( C_6 \) and \( C_8 \) stand for the numerical values of the correlators associated to the \( N = 6 \) and \( N = 8 \) NR couplings, respectively. Note the presence of several

\[ ^3\text{The value } \lambda = 1 \text{ corresponds to the scalaron with the properties of an } R^2 \text{ term in the effective action.} \]
hidden sector fields. The reason is that the trilinear superpotential involving only
gauge singlets under the non-abelian gauge group is exact and does not receive $\alpha'$
corrections [20]. It turns out that a typical VEV satisfying the D-term conditions
is of order $\xi$:

$$\xi \sim \frac{M_s}{2\pi} ; \quad M_s \equiv \frac{1}{\sqrt{2\alpha'}}.$$  \hfill (2.6)

Thus, defining $\alpha_s \equiv g_s/2\pi$, one gets the inflation scale $M_I \simeq C_6 \alpha_s^5 M_s \sim 10^{13}$ GeV
and $\lambda$ an order one tuneable parameter. The choice of VEVs that solve the F and
D-flatness conditions, giving rise to the above superpotential are given in the left
panel of Table 1.

### 3 Triplet-doublet splitting in $SU(5) \times U(1)$ string model

In many grand unified theories (GUTs), including $SU(5)$ and $SO(10)$, Standard
Model (SM) Higgs doublets reside in the lowest dimensional gauge group representation(s) together with colour triplets. Nevertheless, a successful phenomenological model has to differentiate between Higgs doublets and additional triplets. The former have to stay light down to the electroweak scale in order to realise the electroweak symmetry breaking and provide fermion masses. The latter mediate nucleon decay and, unless sufficiently heavy, they could lead to proton lifetimes incompatible with existing data. This is often referred to as the doublet-triplet splitting problem. Among possible solutions one singles out the so-called missing partner mechanism that involves the breaking of the GUT gauge symmetry via Higgs fields that include unequal numbers of SM colour triplets and electroweak doublets. The missing partner scenario is elegantly realised in the flipped $SU(5)$ model [5]. In the minimal case, the GUT breaking Higgs fields entail a pair of 10, 10 representations that include one pair of d-type colour triplets 3, 3 but no SM Higgs doublets. Interaction terms of these Higgs fields with a single pair of 5, 5 representations, 10 $\times$ 10 $\times$ 5 + $\overline{10}$ $\times$ $\overline{10}$ $\times$ 5, provide masses for the triplet components leaving the associated doublets in the 5, 5 unaffected, see e.g. [21].

A more complicated situation arises in the string implementation of the flipped $SU(5)$ model [2]. Here, we have four pairs of 5, 5 fields, namely $h_i, \overline{h}_i, i = 1, 2, 3, 4$ and one pair of GUT breaking 10, 10 Higgs fields, referred as $F, \overline{F}$, where $F$ stands for a linear combination of $F_\alpha, \alpha = 1, 2, 3, 4$ and $F \equiv \overline{F}_5$. SM Higgs doublets and additional triplets are assigned to $SU(5) \times U(1)$ representations as follows

$$h_i (5, -1) = H_i \left( 1, 2, -\frac{1}{2} \right) + D_i \left( 3, 1, -\frac{1}{3} \right), \quad (3.1)$$

$$F \left( 10, +\frac{1}{2} \right) = Q_H \left( 3, 2, +\frac{1}{6} \right) + d_H \left( 3, 1, +\frac{1}{3} \right) + \nu_H \left( 1, 1, 0 \right), \quad (3.2)$$
| Field | Assignments in Ref. [1] | Assignments in Section 4 |
|-------|-------------------------|--------------------------|
| $\Phi_{12}$ | 0 | 0 |
| $\Phi_{12}$ | 0 | $\xi^3$ |
| $\Phi_{23}$ | 0 | $\xi^3$ |
| $\Phi_{23}$ | 0 | $\xi^3$ |
| $\Phi_{31}$ | $\xi$ | $\xi$ |
| $\Phi_{31}$ | 0 | $\xi$ |
| $\Phi_1$ | 0 | 0 |
| $\phi_{45}$ | $\xi$ | $\xi$ |
| $\phi_{45}$ | $\xi^3$ | $\xi^3$ |
| $\phi_1$ | 0 | 0 |
| $\phi_2$ | 0 | 0 |
| $\phi_3$ | 0 | 0 |
| $\phi_4$ | $\xi^2$ | $\xi$ |
| $\phi_{1}$ | 0 | $\xi$ |
| $\phi_2$ | 0 | 0 |
| $\phi_3$ | 0 | 0 |
| $\phi_4$ | $\xi^2$ | $\xi^2$ |
| $\phi_{1}$ | $\xi$ | $\xi$ |
| $\phi_2$ | $\xi^3$ | $\xi$ |
| $\phi_{1}$ | $\xi^3$ | $\xi$ |
| $\phi_1$ | $\xi^2$ | $\xi$ |
| $F_1$ | 0 | $\xi^{3/2}$ |
| $F_5$ | 0 | $\xi^{3/2}$ |
| $D_1$ | $\xi$ | $\xi$ |
| $D_2$ | 0 | 0 |
| $D_3$ | 0 | $\xi^2$ |
| $D_4$ | $\xi$ | $\xi$ |
| $D_5$ | $\xi$ | 0 |
| $T_1$ | $\xi$ | $\xi$ |
| $T_2$ | 0 | 0 |
| $T_3$ | $\xi$ | $\xi^2$ |
| $T_4$ | $\xi$ | $\xi^2$ |
| $T_5$ | 0 | 0 |

$D_1^2 = D_4^2 = D_5^2 = 0 \quad D_4 \cdot D_5 \sim \xi^4 \quad D_1^2 + D_2^2 + T_4^2 = 0 \quad T_1^2 = T_3^2 = T_4^2 = T_3 T_4 = 0 \quad T_4^2 = T_1 \cdot T_4 = 0$

Table 1: Field VEV assignments in the F/D-flatness solution of Ref. [1] (column 2) and the solution derived here, in Section 4 (column 3).
and similarly for the conjugate fields $\overline{h}_i, \overline{F}$. Taking into account the tree-level superpotential (2.1), the Higgs doublet mass matrix reads

$$M_H^{(3)} = \begin{pmatrix}
H_1 & H_2 & H_3 & H_{45} \\
0 & \Phi_{12} & \Phi_{31} & 0 \\
\Phi_{12} & 0 & \Phi_{23} & 0 \\
\Phi_{31} & \Phi_{23} & 0 & \phi_{45} \\
0 & 0 & \phi_{45} & \Phi_3
\end{pmatrix},$$

(3.3)

where $\Phi_3, \Phi_{12}, \overline{\Phi}_{12}, \Phi_{23}, \Phi_{31}, \overline{\Phi}_{31}$ stand for flipped SU(5) singlets that can acquire VEVs.

The associated triplet mass matrix depends on the choice of $F$, which is the linear combination of the $F_\alpha$ that acquires VEV. Notice, that there are two candidate up-quark couplings at the tree-level superpotential (2.1), namely $F_4 \overline{F}_5 h_{45}$ and $F_3 \overline{F}_3 h_{3}$. Taking into account the absence of associated down-quark couplings of the form $F_i F_j h_i, i = 1, 2, 3, 45$, both at tree-level and at higher order $N = 4, 5$ superpotential terms [15, 17, 22], we are led to identify the heaviest generation of quarks and leptons with $F_4, \overline{F}_5$. Moreover, in (2.1) there exist three terms providing mass to the down quarks namely $F_1 F_1 h_1, F_4 F_4 h_1$ and $F_2 F_2 h_2$. Bearing in mind that the first two involve the same doublet $H_1$ (residing in $h_1$) and $F_4$ is assigned to the third generation we deduce that $F_1$ has to be associated with the GUT breaking Higgs field $F$. In this case the additional triplet mass matrix can be recast in the form

$$M_D^{(3)} = \begin{pmatrix}
D_1 & D_2 & D_3 & D_{45} & \overline{d}_H \\
0 & \Phi_{12} & \Phi_{31} & 0 & 0 \\
\Phi_{12} & 0 & \Phi_{23} & 0 & 2\overline{F}_5 \\
\Phi_{31} & \Phi_{23} & 0 & \phi_{45} & 0 \\
0 & 0 & \phi_{45} & \Phi_3 & 0 \\
2F_1 & 0 & 0 & 0 & 0
\end{pmatrix},$$

(3.4)

where $d_H, \overline{d}_H$ are the triplets residing in $F_1, \overline{F}_5$ respectively.

The doublet-triplet splitting conditions can now be expressed as

$$\det M_H^{(3)} = 0, \quad \det M_D^{(3)} \neq 0,$$

(3.5)

where

$$\det M_H^{(3)} = (\Phi_{12} \Phi_{23} \Phi_{31} + \overline{\Phi}_{12} \overline{\Phi}_{23} \overline{\Phi}_{31}) \Phi_3 + \Phi_{12} \overline{\Phi}_{12} \phi_{45} \phi_{45},$$

(3.6)

$$\det M_D^{(3)} = -4F_1 \overline{F}_5 (\Phi_{23} \Phi_{31} \Phi_3 + \phi_{45} \phi_{45} \Phi_{12}),$$

(3.7)
are the determinants of relevant mass matrices. Additional constraints arise from low energy phenomenology and more particularly from the requirement to have a top quark mass coupling at the tree-level superpotential and in particular from the aforementioned coupling $F_4 F_5 H_{45}$. Expressing the massless Higgs doublet pair as

$$H = c_1 H_1 + c_2 H_2 + c_3 H_3 + c_{45} H_{45},$$  
$$\tilde{H} = \tilde{c}_1 \tilde{H}_1 + \tilde{c}_2 \tilde{H}_2 + \tilde{c}_3 \tilde{H}_3 + \tilde{c}_{45} \tilde{H}_{45},$$

where $c_i, \tilde{c}_i$ depend on the $SU(5) \times U(1)$ singlet VEVs, this requirement is translated to

$$\tilde{c}_{45} \neq 0.$$  

(3.10)

The constraints (3.5), (3.10) depend on $F_1, F_5$ and nine additional parameters, namely $\Phi_{12}, \Phi_{12}, \Phi_{23}, \Phi_{23}, \Phi_{31}, \Phi_{31}, \phi_{45}, \phi_{45}, \Phi_3$. As these VEVs are subject to non-trivial F/D flatness conditions we seek for solutions of (3.5), (3.10) with the minimum number of vanishing VEVs and no fine-tunings. After some algebra we find no solution with a single vanishing VEV and only two solutions with two vanishing VEVs. These are

(a) $\Phi_{12} = \Phi_3 = 0$, with $\det M_{D}^{(3)} = -4 F_1 F_5 \phi_{45} \phi_{45} \Phi_{12}$ and massless doublets

$$H = \phi_{45} H_1 - \Phi_{31} H_{45},$$  
$$\tilde{H} = \phi_{45} \tilde{H}_2 - \Phi_{23} \tilde{H}_{45}.$$  

(3.11)

(3.12)

(b) $\Phi_{12} = \Phi_{31} = 0$, with $\det M_{D}^{(3)} = -4 F_1 F_5 (\Phi_{23} \Phi_{31} \Phi_3 + \phi_{45} \phi_{45} \Phi_{12})$ and massless doublets

$$H = H_1,$$  
$$
\overline{H} = \Phi_{23} \Phi_{23} \Phi_3 \overline{H}_1 - (\Phi_{31} \Phi_{23} \Phi_3 + \Phi_{12} \phi_{45} \phi_{45}) \overline{H}_2 - \Phi_{12} \Phi_{23} \Phi_3 \overline{H}_3 + \Phi_{12} \Phi_{23} \phi_{45} \overline{H}_{45}
$$  

(3.13)

(3.14)

4 Solution of the F/D-flatness conditions

In supersymmetric string models build in the framework of the free fermionic formulation, many features of low energy phenomenology are specified by the VEVs of Standard Model singlet fields. These in turn, are subject to non-trivial F/D flatness constraints dictated by $N = 1$ supersymmetry. For the flipped string model [2] the solution and phenomenological consequences of F/D flatness conditions have been thoroughly studied in the past [2, 15, 17, 19, 20]. In this section we
re-examine solutions of flatness equations taking into account: (i) compatibility with the recent results concerning string cosmology \(1\) as explained in Section \(2\) and (ii) the results of Section \(3\) regarding doublet-triplet splitting and in particular the requirement to generate triplet masses at tree-level in order to efficiently suppress proton decay.

For the flipped string model under consideration the full gauge group is \(SU(5) \times U(1) \times U(1)^4 \times SU(4) \times SO(10)\) The D-flatness conditions associated with the \(U(1)^4\) gauge group factor are of the form

\[
D_I = \sum_i q_i^I |\varphi_i|^2 + \frac{1}{192 \pi^2 \alpha'} \text{Tr} Q_I = 0 , \quad I = 1, 2, 3, A ,
\]

(4.1)

where \(\varphi_i\) denotes a field with charges \(q_i^I\) under \(U(1)^I\) and \(\text{Tr} Q_1 = \text{Tr} Q_2 = \text{Tr} Q_A = 0, \text{Tr} Q_A = 180\) are the traces of the associated abelian group generators \(2, 23\). After some algebra Eqs. (4.1) can be unravelled by taking appropriate linear combinations \(17\)

\[
|\varphi_{45}|^2 - |\bar{\varphi}_{15}|^2 - \frac{1}{2} (|D_3|^2 + |T_3|^2) + \frac{1}{2} |F_3|^2 = \xi^2 ,
\]

(4.2)

\[
|\varphi_4|^2 - |\varphi_-|^2 - |\bar{\varphi}_+|^2 + |\bar{\varphi}_-|^2 + \frac{1}{2} (|D_3|^2 - |T_3|^2) - \frac{1}{2} |F_3|^2 = \xi^2 ,
\]

(4.3)

\[
|\Phi_{31}|^2 - |\bar{\Phi}_{31}|^2 - |\Phi_{23}|^2 + |\bar{\Phi}_{23}|^2 - \frac{1}{2} (|D_1|^2 + |D_2|^2 + |D_3|^2 + |D_4|^2 - |D_5|^2)
\]

\[- \frac{1}{2} (|T_1|^2 + |T_2|^2 + |T_3|^2 - |T_4|^2 + |T_5|^2) = 3\xi^2 ,
\]

(4.4)

\[
|\Phi_{23}|^2 - |\bar{\Phi}_{23}|^2 - |\Phi_{12}|^2 + |\bar{\Phi}_{12}|^2 + \frac{1}{2} \sum_{i=1}^4 \left( |\phi_i|^2 - |\bar{\phi}_i|^2 \right) + |\phi_-|^2 - |\bar{\phi}_+|^2
\]

\[+ \frac{1}{2} (|D_1|^2 + |D_3|^2 + |D_4|^2) + \frac{1}{2} (|T_1|^2 - |T_4|^2) - \frac{1}{2} (|F_2|^2 - |\bar{F}_5|^2) = 0 ,
\]

(4.5)

where

\[
\xi^2 = \frac{1}{16 \pi^2 \alpha'} .
\]

(4.6)

The F-flatness equations are derived from the superpotential \(W\)

\[
F_i = \frac{\partial W}{\partial \varphi_i} = 0 .
\]

(4.7)

At tree-level \(W = W_3\) given in Eq. (2.14). However, the superpotential receives additional contributions from non-renormalisable (NR) terms at higher orders.

These come from terms of the form \(\varphi_1 \varphi_2 \varphi_3 \varphi^{N-3}, N > 3\), where \(\varphi^{N-3}\) denotes a product of \(N - 3\) field VEVs. Apart from gauge invariance these terms are subject
to intricate string selection rules ascribed to world-sheet superalgebra [20, 24].
Using a computer program that successively applies all selection criteria we find
15 candidate NR superpotential couplings for $N = 4$ and 256 couplings for $N = 5$. However, for the sake of simplicity we start our analysis from the tree-level superpotential (2.1) and we will take into account higher order non-renormalisable contributions at a later stage. The tree-level F-flatness equations give

$$\frac{\partial W_3}{\partial \Phi_{12}} = \sum_{i=1}^{4} \phi_i^2 + \Phi_{23} \Phi_{31} + \phi_+ \phi_- = 0,$$

(4.8)

$$\frac{\partial W_3}{\partial \Phi_{12}} = \sum_{i=1}^{4} \bar{\phi}_i^2 + \bar{\Phi}_{23} \bar{\Phi}_{31} + \bar{\phi}_+ \bar{\phi}_- = 0,$$

(4.9)

$$\frac{\partial W_3}{\partial \Phi_{31}} = \Phi_{12} \Phi_{23} + D_2^2 + T_2^2 + T_5^2 = 0,$$

(4.10)

$$\frac{\partial W_3}{\partial \Phi_{31}} = \bar{\Phi}_{12} \bar{\Phi}_{23} + D_2^5 = 0,$$

(4.11)

$$\frac{\partial W_3}{\partial \Phi_{23}} = \Phi_{12} \Phi_{31} + T_4^2 = 0,$$

(4.12)

$$\frac{\partial W_3}{\partial \Phi_{23}} = \bar{\Phi}_{12} \bar{\Phi}_{31} + D_1^2 + D_4^2 + T_1^2 = 0,$$

(4.13)

$$\frac{\partial W_3}{\partial \Phi_3} = \sum_{i=1}^{4} \phi_4 \phi_i + \phi_4 \phi_{45} + \phi_- \bar{\phi}_- + \phi_+ \phi_+ = 0,$$

(4.14)

$$\frac{\partial W_3}{\partial \Phi_4} = \phi_2 \Phi_1 + \phi_1 \phi_2 = 0,$$

(4.15)

$$\frac{\partial W_3}{\partial \Phi_5} = \phi_4 \Phi_3 + \phi_3 \phi_4 = 0,$$

(4.16)

$$\frac{\partial W_3}{\partial \phi_1} = 2\phi_1 \Phi_{12} + \Phi_3 \phi_4 + \Phi_4 \phi_2 = 0,$$

(4.17)

$$\frac{\partial W_3}{\partial \phi_2} = 2\phi_2 \Phi_{12} + \Phi_4 \phi_1 + \Phi_3 \phi_2 + T_4 \cdot T_5 = 0,$$

(4.18)

$$\frac{\partial W_3}{\partial \phi_3} = F_4 \Phi_5 + 2\phi_3 \Phi_{12} + \Phi_3 \phi_3 + \Phi_5 \phi_4 = 0,$$

(4.19)

$$\frac{\partial W_3}{\partial \phi_4} = 2\phi_4 \Phi_{12} + \Phi_5 \phi_3 + \Phi_3 \phi_4 = 0,$$

(4.20)

$$\frac{\partial W_3}{\partial \phi_1} = 2\bar{\phi}_1 \bar{\Phi}_{12} + \Phi_3 \phi_1 + \Phi_4 \phi_2 = 0,$$

(4.21)

$$\frac{\partial W_3}{\partial \phi_2} = 2\bar{\phi}_2 \bar{\Phi}_{12} + \Phi_4 \phi_1 + \Phi_3 \phi_2 = 0,$$

(4.22)
\[
\frac{\partial W_3}{\partial \phi_3} = 2\Phi_3 \Phi_{12} + \Phi_3 \phi_3 + \Phi_5 \phi_4 + D_4 \cdot D_5 = 0, \quad (4.23)
\]
\[
\frac{\partial W_3}{\partial \phi_4} = 2\Phi_4 \Phi_{12} + \Phi_5 \phi_3 + \Phi_3 \phi_4 = 0, \quad (4.24)
\]
\[
\frac{\partial W_3}{\partial \phi_+} = \Phi_3 \phi_+ + \Phi_{12} \phi_- = 0, \quad (4.25)
\]
\[
\frac{\partial W_3}{\partial \phi_-} = \Phi_3 \phi_- + \Phi_{12} \phi_+ = 0, \quad (4.26)
\]
\[
\frac{\partial W_3}{\partial \phi_{45}} = \Phi_3 \phi_{45} = 0, \quad (4.27)
\]
\[
\frac{\partial W_3}{\partial F_1} = \Phi_3 \phi_3 = 0, \quad (4.28)
\]
\[
\frac{\partial W_3}{\partial F_5} = F_4 \phi_3 = 0, \quad (4.29)
\]
\[
\frac{\partial W_3}{\partial T_1} = 2\Phi_{23} T_1 = 0, \quad (4.30)
\]
\[
\frac{\partial W_3}{\partial T_2} = 2\Phi_{31} T_2 = 0, \quad (4.31)
\]
\[
\frac{\partial W_3}{\partial T_4} = 2\Phi_{23} T_4 + \phi_2 T_5 = 0, \quad (4.32)
\]
\[
\frac{\partial W_3}{\partial T_5} = \phi_2 T_4 + 2\Phi_{31} T_5 = 0, \quad (4.33)
\]
\[
\frac{\partial W_3}{\partial D_1} = 2D_1 \Phi_{23} = 0, \quad (4.34)
\]
\[
\frac{\partial W_3}{\partial D_2} = 2D_2 \Phi_{31} = 0, \quad (4.35)
\]
\[
\frac{\partial W_3}{\partial D_4} = 2D_4 \Phi_{23} + D_5 \Phi_{3} / \sqrt{2} = 0, \quad (4.36)
\]
\[
\frac{\partial W_3}{\partial D_5} = 2D_5 \Phi_{31} + D_4 \Phi_{3} / \sqrt{2} = 0. \quad (4.37)
\]

The massless spectrum of the string derived flipped SU(5) model comprises four fermion generations \(F_\alpha, f_\beta, \ell_\gamma, \alpha = 1, 2, 3, 4, \beta = 1, 2, 3, 5\) and one anti-generation.
The solution of the F/D-flatness equations depends on the choice of the flipped SU(5) breaking Higgs fields $F, \bar{F}$ and the assignment of the fermion generations. Following the discussion in Section 3 we choose $F = A_1 F_1 + A_3 F_3, \bar{F} = F_5$ with $A_1^2 + A_3^2 = 1$ and set

$$F_2 = F_4 = 0.$$  (4.41)

Combined with (4.2), this implies $\bar{\phi}_{45} \neq 0$, which, in conjunction with (4.29), leads to $\Phi_3 = 0$. This rejects the second solution of Section 3 leaving us with

$$\Phi_{12} = \Phi_3 = 0$$  (4.42)

and $\Phi_{13}, \Phi_{31}, \Phi_{21}, \Phi_{23}, \phi_{45}, \bar{\phi}_{45} \neq 0$ in order solve the doublet-triplet problem at tree-level. Eq. (4.12) then implies

$$T_2^2 = 0,$$  (4.43)

and (4.31) yields

$$\phi_3 = 0$$  (4.44)

as $F_5 \neq 0$. To protect $\bar{F}_5$, associated in Section 3 with the 3rd fermion generation, from receiving tree-level mass via the superpotential term $f_4 \bar{F}_5 \bar{\phi}_2$ we have to impose also

$$\bar{\phi}_2 = 0.$$  (4.45)

Next we solve Eqs. (4.15), (4.16) by choosing

$$\phi_2 = \bar{\phi}_3 = 0$$  (4.46)

and Eqs. (4.10), (4.22) using

$$\Phi_4 = \Phi_5 = 0.$$  (4.47)

Keeping in mind that $\Phi_{31}, \bar{\Phi}_{31} \neq 0$ Eqs. (4.34), (4.36), (4.38), (4.40) lead to

$$T_2 = T_5 = D_2 = D_5 = 0.$$  (4.48)

The remaining equations can be solved after considering the aforementioned non-renormalisable corrections to the superpotential in (2.1). Taking into account the full superpotential $W_5$ incorporating NR contributions up to order $N = 5$ and utilising the tree-level partial solution (4.41)-(4.48) the F-flatness equations yield

$$\frac{\partial W_5}{\partial \Phi_{12}} = \phi_1^2 + \phi_4^2 + \phi_+ \phi_- + \Phi_{23} \Phi_{31} = 0,$$  (4.49)
\[
\frac{\partial W_5}{\partial \Phi_{12}} = \overline{\phi}_1^2 + \phi_4^2 + \phi_+ \phi_- + \overline{\Phi}_{23} \overline{\Phi}_{31} + \left\{ F_1^2 \Phi_{12} \right\} = 0 , \\
\frac{\partial W_5}{\partial \Phi_{31}} = \left\{ \left( \overline{\phi}_1^2 + \phi_4^2 + \phi_+ \phi_- \right) \left( D_1^2 + D_4^2 + T_1^2 \right) \right\} = 0 , \\
\frac{\partial W_5}{\partial \Phi_{23}} = \Phi_{12} \Phi_{23} = 0 , \\
\frac{\partial W_5}{\partial \Phi_{31}} = \phi_1 \phi_1 + \phi_3 \phi_4 + \phi_- \phi_- + \phi_+ \phi_+ + \phi_{45} \phi_{45} = 0 , \\
\frac{\partial W_5}{\partial F_1} = \left\{ F_1 F_5 \overline{\Phi}_{12} \right\} = 0 , \\
\frac{\partial W_5}{\partial F_2} = \left\{ F_5 F_1 \overline{\Phi}_{12} \right\} = 0 , \\
\frac{\partial W_5}{\partial \phi_3} = \left\{ (F_1 F_5) (T_1 \cdot T_4) \right\} = 0 , \\
\frac{\partial W_5}{\partial \phi_1} = 2 \overline{\phi}_1 \left[ \Phi_{12} + \left\{ \Phi_{31} \left( D_1^2 + D_4^2 + T_1^2 \right) \right\} \right] = 0 , \\
\frac{\partial W_5}{\partial \phi_4} = 2 \overline{\phi}_4 \left[ \Phi_{12} + \left\{ \Phi_{31} \left( D_1^2 + D_4^2 + T_1^2 \right) \right\} \right] = 0 , \\
\frac{\partial W_5}{\partial \phi_-} = \phi_- \left[ \Phi_{12} + \left\{ \Phi_{31} \left( D_1^2 + D_4^2 + T_1^2 \right) \right\} \right] = 0 , \\
\frac{\partial W_5}{\partial \phi_+} = \phi_+ \left[ \Phi_{12} + \left\{ \Phi_{31} \left( D_1^2 + D_4^2 + T_1^2 \right) \right\} \right] = 0 , \\
\frac{\partial W_5}{\partial T_1} = 2 T_1 \left[ \Phi_{23} + \left\{ \Phi_{31} \left( \overline{\phi}_1^2 + \phi_4^2 + \phi_+ \phi_- \right) \right\} \right] = 0 , \\
\frac{\partial W_5}{\partial T_4} = 2 T_4 \left[ \Phi_{23} + \left\{ \Phi_{31} \left( \phi_1^2 + \phi_4^2 + \phi_+ \phi_- \right) \right\} \right] = 0 , \\
\frac{\partial W_5}{\partial D_1} = 2 D_1 \left[ \Phi_{23} + \left\{ \Phi_{31} \left( \overline{\phi}_1^2 + \phi_4^2 + \phi_+ \phi_- \right) \right\} \right] = 0 , \\
\frac{\partial W_5}{\partial D_4} = 2 D_4 \left[ \Phi_{23} + \left\{ \Phi_{31} \left( \phi_1^2 + \phi_4^2 + \phi_+ \phi_- \right) \right\} \right] = 0 , \\
\frac{\partial W_5}{\partial D_5} = \left\{ D_3 \left( F_3 F_5 \right) \right\} = 0 ,
\]

where we have omitted order one numerical coefficients of the NR terms (terms in curly brackets).

Higher order superpotential terms lead to additional constraints on the SM singlet VEVs in order to assure compatibility with low energy phenomenology.
First, superpotential terms of the form $F_1 \overline{f}_j h_k \varphi^a, j = 1, 2, 3, 5, k = 1, 2, 3, 45$ could induce mixings of leptons with Higgs doublets. The effective superpotential $W_5$ comprises three such terms

$$ F_1 \overline{f}_4 h_{45} \phi_1 (1 + \Phi_2) + F_1 \overline{f}_5 h_{45} (T_1 \cdot T_4) , \quad (4.67) $$

where we dropped curly brackets (denoting omission of numerical coefficients), for simplicity. To eliminate them we set

$$ \phi_1 = 0 , \ T_1 \cdot T_4 = 0 . \quad (4.68) $$

Second, the fifth order superpotential $W_5$ includes two terms that provide mass for the surplus pairs of fermions $f, \overline{f}$ and $\ell^c, \overline{\ell}^c$

$$ \left( f_3 \overline{f}_3 + \overline{\ell}_3 \ell_3 \right) (T_3 \cdot T_4) , \quad (4.69) $$
as long as

$$ T_3 \cdot T_4 \neq 0 . \quad (4.70) $$

Summarizing the VEV assignments above, we have

$$ F_2 = F_4 = 0 , \Phi_{12} = \Phi_3 = \Phi_4 = \Phi_5 = \phi_1 = \phi_2 = \phi_3 = \phi_2 = \phi_3 = 0 , \quad (4.71) $$
together with

$$ \overline{\Phi}_{12}, \Phi_{31}, \overline{\Phi}_{31}, \Phi_{23}, \overline{\Phi}_{23}, \phi_{45}, T_3 \cdot T_4 \neq 0 . \quad (4.72) $$

Let us now proceed with the solution of the remaining equations. Notice that the system of D/F-flatness constraints contains a single fixed parameter $\xi$ (4.6), which is about 0.1 in string units. We can thus attempt to find a perturbative solution expressing all remaining VEVs as power series in $\xi$. Assuming that (to leading order) $\Phi_{31, \overline{\Phi}_{31}, \phi_{45}, \phi_4, \phi_+, \phi_-}, \overline{\phi}_1, \overline{\phi}_- \sim \xi$, $F_1 \overline{F}_5 \sim \xi^3$, Eq. (4.63) can be solved with respect to $\Phi_{23}$

$$ \Phi_{23} = - \left\{ \Phi_{31} \left( \phi_4^2 + \phi_+ \phi_- \right) \right\} \sim \xi^3 , \quad (4.73) $$

Eqs. (4.62), (4.64), (4.65) with respect to $\overline{\Phi}_{23}$

$$ \overline{\Phi}_{23} = - \left\{ \Phi_{31} \left( \overline{\phi}_1^2 + \overline{\phi}_4^2 + \overline{\phi}_+ \overline{\phi}_- \right) \right\} \sim \xi^3 , \quad (4.74) $$

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and Eqs. (4.58)-(4.61) with respect to $\Phi_{12}$

$$\overline{\Phi}_{12} = - \left\{ \Phi_{31} \left( D_1^2 + D_4^2 + T_1^2 \right) \right\} \sim \xi^3 . \quad (4.75)$$

The solutions (4.74), (4.75) need further clarification, since both involve three different equations that should be compatible ((4.60) and (4.61) come from the same superpotential term). Actually, their validity depends on the numerical factors of the fifth order terms that we have omitted (terms in curly brackets). A detailed analysis of Eqs. (4.50)-(4.53), (4.58)-(4.62) and (4.64)-(4.65) taking into account all numerical coefficients is presented in the next section and Appendix B where we compute the F-flatness solutions of a generic superpotential of the form

$$w = \overline{\Phi}_{23}\overline{\Phi}_{31}\overline{\Phi}_{12} + \overline{\Phi}_{12} \left( \overline{\phi}_1^2 + \overline{\phi}_2^2 + \overline{\phi}_4^2 + \overline{\phi}_+ \overline{\phi}_- \right) + \left( D_1^2 + D_4^2 + T_1^2 \right) \overline{\Phi}_{23} + \left( \alpha_1 \overline{\phi}_1^2 + \alpha_2 \overline{\phi}_2^2 + \alpha_3 \overline{\phi}_4^2 + \alpha_4 \overline{\phi}_+ \overline{\phi}_- \right) D_2^2 \Phi_{31} \quad (4.76)$$

As shown in Section 5 an explicit calculation of the related fifth order superpotential couplings supports the following coupling relations

$$-\frac{\alpha_1}{3} = -\frac{\gamma_1}{3} = -\frac{\beta_1}{2} = \frac{\beta_4}{2} = \frac{\beta_4}{2} = \alpha_4 , \quad (4.77)$$

$$\alpha_2 = \alpha_3 = \gamma_2 = \gamma_3 = \gamma_4 = \alpha_4 , \quad \beta_2 = 0 . \quad (4.78)$$

These in turn, following the analysis of Appendix B lead to the subsequent solution of the associated F-flatness equations to order $\xi^4$

$$\overline{\Phi}_{12} = -\alpha_4 D_4^2 \Phi_{31} \quad (4.79)$$

$$T_1^2 = -D_1^2 - D_4^2 - \overline{T}_{31} \overline{\Phi}_{12} , \quad (4.80)$$

$$\overline{\Phi}_{23} = - \left\{ \overline{\Phi}_{31} \left( \phi_4^2 + \phi_+ \phi_- \right) \right\} \quad (4.81)$$

$$\overline{\Phi}_{23} = -2 \Phi_{31} \alpha_4 \left( -\overline{\phi}_1^2 + \overline{\phi}_4^2 + \overline{\phi}_+ \overline{\phi}_- \right) \quad (4.82)$$

$$\overline{\phi}_1^2 = -\phi_4^2 - \phi_+ \phi_- - \overline{T}_{23} \overline{\Phi}_{31} \, \quad (4.83)$$

where in (4.82) we used the exact coefficients from the computation.

The remaining F and D-flatness equations to order $\xi^4$ give

$$\phi_4^2 + \phi_+ \phi_- = 0 , \quad (4.84)$$

$$\phi_4 \overline{\phi}_4 + \phi_- \overline{\phi}_- + \phi_+ \overline{\phi}_+ + \phi_{45} \overline{\phi}_{45} = 0 , \quad (4.85)$$

$$\left\{ D_3 \left( F_3 \overline{F}_5 \right) \right\} = 0 . \quad (4.86)$$
and
\[ |\phi_{45}|^2 - |\phi_{45}^*|^2 + \frac{1}{2} \left( |F_3|^2 - |D_3|^2 - |T_3|^2 \right) = \xi^2, \]
(4.87)
\[ |\phi_+|^2 - |\phi_-|^2 - |\phi_+^*|^2 - |\phi_-^*|^2 + \frac{1}{2} \left( |D_4|^2 - |T_3|^2 - |F_3|^2 \right) = \xi^2, \]
(4.88)
\[ |\Phi_{31}|^2 - |\overline{\Phi}_{31}|^2 - \frac{1}{2} \left( |D_1|^2 + |D_3|^2 + |D_4|^2 \right) - \frac{1}{2} \left( |T_1|^2 + |T_3|^2 - |T_4|^2 \right) = 3\xi^2, \]
(4.89)
\[ \frac{1}{2} \left( |\phi_4|^2 - |\phi_1|^2 - |\phi_4|^2 \right) + |\phi_+|^2 - |\phi_+^*|^2 + \frac{1}{2} \left( |D_1|^2 + |D_3|^2 + |D_4|^2 \right) + \frac{1}{2} \left( |T_1|^2 - |T_4|^2 - |\overline{F}_5|^2 \right) = 0, \]
(4.90)
\[ |A_1 F_1|^2 + |A_3 F_3|^2 = |\overline{F}_5|^2. \]
(4.91)

In addition, we have the \( SU(4) \simeq SO(6) \) and \( SO(10) \) D-flatness which can be cast in the form
\[ \sum_{i=1,4}^n D_i^* \tau^a D_i = 0, \quad a = 1, \ldots, 15, \]
(4.92)
\[ \sum_{i=1,3,4}^n T_i^* \lambda^A T_i = 0, \quad A = 1, \ldots, 45, \]
(4.93)

where \( \tau^a, \lambda^A \) are the \( SO(6), SO(10) \) generators respectively. These can be solved by utilising an antisymmetric representation of the \( SO(2n) \) group generators
\[ (M_{ab})_{ij} = -i \left( \delta_{ai} \delta_{bj} - \delta_{aj} \delta_{bi} \right); \quad b > a = 1, \ldots, 2n; \quad i, j = 1, \ldots, 2n. \]
(4.94)

In this way, Eqs. (4.92) can be easily solved by choosing the VEVs \( D_1, D_3, D_4 \)
real. Although this is not applicable for Eqs. (4.93) because of the constraints
\( T_4^2 = T_1 \cdot T_4 = 0 \), we can find other explicit solutions. For example, using the ansatz
\[ T_1 = \left( re^{-i\theta}, ire^{-i\theta}, b, 0, \ldots, 0 \right), \]
(4.95)
\[ T_4 = \left( a, ia, 0, 0, \ldots, 0 \right), \]
(4.96)
\[ T_3 = \left( ic, c + ie, 0, \ldots, 0 \right), \]
(4.97)

Eqs. (4.93) reduce to three independent constraints solved by
\[ r^2 = c^2 - a^2, \quad e^2 = b^2 r^2 / c^2 - c^2, \quad \tan \theta = \frac{c}{e}, \]
(4.98)

with \( a, b, c \) free real parameters. Moreover, this solution guarantees \( T_4^2 = T_1 \cdot T_4 = 0 \)
and gives \( T_1^2 = b^2 \) and \( T_3 \cdot T_4 = 2iac \), depending on the free parameters.

A natural question to ask at this point is whether higher order terms in the superpotential could destabilise our flatness solution. A definite answer to this
would require the calculation of the full superpotential to a rather high order, e.g. $N = 10$, and the solution of the associated flatness equations, which is a very difficult task from the technical point of view. Here, we have shown that our perturbative solution is valid to order $\xi^4$ when taking into account superpotential terms up to and including $N = 5$. Moreover, we have checked the $N = 6$ contributions and it turns out that the aforementioned flatness solution holds to order $\xi^5$ provided $\Phi_2 = 0$, $D_3(F_3 F_5) \lesssim \xi^6$, $T_3 T_1 \lesssim \xi^3$ and an additional condition relating the VEVs of $D_1$, $T_1$. This supports the perturbative validity of our solution at higher orders. The above VEV assignments that solve the F/D-flatness equations are summarised in the right panel of Table 1.

5 Computation of higher order superpotential terms

In the context of the free fermionic formulation of the heterotic superstring the effective $N = 1$ superpotential is fully calculable. Actually, the trilinear coupling constants are fixed by conformal symmetry alone [25]; the non-vanishing ones are of the form $kg_s$, where $g_s$ is the string coupling and $k \in \{1/\sqrt{2}, 1, \sqrt{2}\}$. The computation of the coupling constants of higher order NR terms, $N > 3$, is in general more intricate as it entails the calculation of $N$-point correlation functions [28]. Though suppressed by inverse powers of the string scale, these terms turn out to play an important role in low energy phenomenology. For example, they can account for fermion mass hierarchies and provide intermediate scale masses for exotic states. In our analysis they are also important in ensuring the existence of a particular solution of the F-flatness constraints as explained in Section 4 and Appendix B. For this purpose, we compute in this section, the coupling constants $\alpha_i, \beta_i, \gamma_i, i = 1, 4$ of the fifth order NR terms appearing in Eq. (4.76)

\[
(\alpha_1 \phi_1^2 + \alpha_2 \phi_2^2 + \alpha_3 \phi_4^2 + \alpha_4 \phi_+ \phi_-) D_1^2 \Phi_{31} + (\beta_1 \phi_1^2 + \beta_2 \phi_2^2 + \beta_3 \phi_3^2 + \beta_4 \phi_+ \phi_-) D_1^2 \Phi_{31} + (\gamma_1 \phi_1^2 + \gamma_2 \phi_2^2 + \gamma_3 \phi_3^2 + \gamma_4 \phi_+ \phi_-) T_1^2 \Phi_{31}.
\] (5.1)

Following [28] the coupling constant of a NR superpotential term of the form

\[
\int d^2\theta X_1 \ldots X_N, \ N \geq 3
\] (5.2)

is proportional to the correlation function

\[
\langle \Psi_1 \Psi_2 \Phi_3 \ldots \Phi_N \rangle,
\] (5.3)

where $\Psi_1, \Psi_2$ are the fermionic components of the superfields $X_1, X_2$ respectively and $\Phi_i$ stand for the bosonic components of $X_i, i = 3, \ldots, N$. Evaluation of the
correlator requires knowing the vertex operators of the associated fields which (in
the 0-ghost picture) are world-sheet operators of conformal dimensions \((1, 1)\). The
vertex operators are expressed in terms of the world-sheet degrees of freedom.
These include 22 real left-moving fermions \(\{\psi^\mu, \chi^I, y^I, \omega^I, I = 1, \ldots, 6\}\) together with 12 real \(\{\overline{y}^I, \overline{\omega}^I, I = 1, \ldots, 6\}\) and 16 complex \(\{\overline{\psi}^{1,\ldots,5}, \overline{\eta}^{1,2,3}, \overline{\phi}^{1,\ldots,8}\}\) right-moving fermions. In the case at hand, the world-sheet supercurrent takes the form

\[
T_F = \psi^\mu \partial X_\mu + i \sum_{I=1}^{6} \chi^I y^I \omega^I. \tag{5.4}
\]

Furthermore, the particular choice of basis vectors allows for the bosonisation of
the fermionic fields \(\chi^1, \ldots, \chi^6\) as follows

\[
e^{\pm i S_{k,k+1}} = \frac{1}{\sqrt{2}} (\chi^k \pm i \chi^{k+1}) ; k = 1, 3, 5. \tag{5.5}
\]

In terms of the bosonised fields, the supercurrent takes the form

\[
T_F = T_F^0 + T_F^- + T_F^+, \tag{5.6}
\]

where

\[
T_F^0 = \psi^\mu \partial X_\mu, \tag{5.7}
\]

\[
T_F^- = \sum_{k=1,3,5} e^{-i S_{k,k+1}} \tau_{k,k+1}, \tag{5.8}
\]

\[
T_F^+ = -(T_F^-)^*, \tag{5.9}
\]

and

\[
\tau_{k,\ell} = \frac{i}{\sqrt{2}} (y^k \omega^\ell + iy^\ell \omega^k). \tag{5.10}
\]

From the remaining real fermions only two pairs can be complexified, namely
\(\omega^2, \omega^3\) and \(\overline{\omega}^2, \overline{\omega}^3\). The rest of them can be grouped as nine left-right moving fermion pairs \(\{y^I, \overline{y}^I\}, I = 1, \ldots, 6\) and \(\{\omega^I, \overline{\omega}^I\}, I = 1, 4, 6\), ascribed to critical Ising models.

In this framework, a general vertex operator of the bosonic component of a chiral
superfield in the canonical picture (ghost charge -1) and vanishing momentum
is of the form

\[
V_{-1}^B(z) = e^{-c(z)} e^{i \alpha_1 S_{12}(z)} e^{i \alpha_2 S_{34}(z)} e^{i \alpha_3 S_{56}(z)} G(z, \overline{z}), \tag{5.11}
\]

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where $c$ stands for the ghost field. Here, $\alpha_i \in \{0, \pm \frac{1}{2}, \pm 1\}$ and $G(z, \bar{z})$ stands for a conformal field with dimensions $\left(\frac{1}{2} - \sum_{i=1}^{3} \frac{\alpha_i^2}{2}, 1\right)$ comprised of exponentials of the remaining bosonised fields and of primary Ising fields $\sigma_+(z, \bar{z}), \sigma_-(z, \bar{z}), f(z), \bar{f}(\bar{z})$, corresponding to the order, disorder and the left/right fermion operators respectively. The fermionic partner of (5.11) in the canonical -1/2 picture takes the form

$$V_{-1/2}^F(z) = e^{-c/2(z)S_a e^{i(\alpha_1-1/2)S_{12}}(z) e^{i(\alpha_2-1/2)S_{34}}(z) e^{i(\alpha_3-1/2)S_{56}}(z) G(z, \bar{z})},$$

where $S_a$ represents a space-time spinor field. In the computation of the correlator (5.3) the bosonic fields $\Phi_1, \ldots, \Phi_N$ have to be picture changed to the 0-ghost picture. This procedure can be carried out using the standard picture-changing formula

$$V_0^B(z, \bar{z}) = \lim_{w \to z} e^c(w) T_F(w) V_{-1}^B(z, \bar{z})$$

and the relevant operator product expansions.

Let us now proceed with the computation of the correlator

$$\langle \psi_{1(-1/2)}^F \psi_{2(-1/2)}^F \Phi_{31(-1)}^B \bar{\psi}_{4(0)} \bar{\psi}_{2(0)} \rangle,$$

where $\psi_{1}^F \psi_{2}^F \in \{\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_4, \bar{\phi}_5\}$ and $\varphi_{1}^B \varphi_{2}^B \in \{D_1^2, D_3^2, T_1^2\}$. The vertex operators of the fields involved are (in the canonical fermionic/bosonic picture)

$$\bar{\phi}_{1(-1/2)}^F = e^{-c/2} S_6 e^{-i \frac{2}{5} S_{56} \sigma_{y_2}^y \sigma_{y_1}^y \sigma_{y_4}^y \sigma_{y_5}^y} e^{-\frac{i}{2} \bar{\Pi}_{1} e^{i \frac{1}{2} \bar{\Pi}_{2}}},$$

$$\bar{\phi}_{2(-1/2)}^F = e^{-c/2} S_6 e^{-i \frac{2}{5} S_{56} \sigma_{y_2}^y \sigma_{y_3}^y \sigma_{y_4}^y \sigma_{y_5}^y} e^{-\frac{i}{2} \bar{\Pi}_{1} e^{i \frac{1}{2} \bar{\Pi}_{2}}},$$

$$\bar{\phi}_{4(-1/2)}^F = e^{-c/2} S_6 e^{-i \frac{2}{5} S_{56} \sigma_{y_2}^y \sigma_{y_3}^y \sigma_{y_5}^y \sigma_{y_4}^y} e^{-\frac{i}{2} \bar{\Pi}_{1} e^{i \frac{1}{2} \bar{\Pi}_{2}}},$$

$$\bar{\phi}_{5(-1/2)}^F = e^{-c/2} S_6 e^{-i \frac{2}{5} S_{56} \sigma_{y_2}^y \sigma_{y_3}^y \sigma_{y_4}^y \sigma_{y_5}^y} e^{-\frac{i}{2} \bar{\Pi}_{1} e^{i \frac{1}{2} \bar{\Pi}_{2}}},$$

$$\bar{\phi}_{7(-1/2)}^F = e^{-c/2} S_6 e^{-i \frac{2}{5} S_{56} \sigma_{y_2}^y \sigma_{y_3}^y \sigma_{y_5}^y \sigma_{y_4}^y} e^{-\frac{i}{2} \bar{\Pi}_{1} e^{i \frac{1}{2} \bar{\Pi}_{2}}},$$

$$D_{1(-1)}^B = e^{c} e^{i \frac{2}{5} S_{56} \sigma_{y_2}^y \sigma_{y_3}^y \sigma_{y_4}^y \sigma_{y_5}^y \sigma_{y_6}^y} e^{i \frac{1}{2} \Pi_{3} e^{i \frac{1}{2} \Pi_{2}}} e^{i J_6 W_6},$$

$$D_{3(-1)}^B = e^{c} e^{i \frac{2}{5} S_{56} \sigma_{y_2}^y \sigma_{y_3}^y \sigma_{y_5}^y \sigma_{y_4}^y \sigma_{y_6}^y} e^{i \frac{1}{2} \Pi_{3} e^{i \frac{1}{2} \Pi_{2}}} e^{i J_6 W_6},$$

$$T_{1(-1)}^B = e^{c} e^{i \frac{2}{5} S_{56} \sigma_{y_2}^y \sigma_{y_3}^y \sigma_{y_5}^y \sigma_{y_4}^y \sigma_{y_6}^y} e^{i \frac{1}{2} \Pi_{3} e^{i \frac{1}{2} \Pi_{2}}} e^{i J_10 W_{10}},$$

where $\sigma_\pm$ denotes the order/disorder operators of the Ising pair $\{f, \bar{f}\}$, $e^{i \Pi_i}, i = 1, 2, 3$ and $e^{i \Pi_4}$ stand for the bosonised fermions $\bar{\sigma}_i, i = 1, 2, 3$ and $\bar{\sigma}_5, \bar{\sigma}_6$ respectively; $e^{i J_6 W_6}$ and $e^{i J_{10} W_{10}}$ represent the bosonised fermions ascribed to the vectorial representations of the hidden gauge group $6$ of $SO(6) \simeq SU(4)$ and $10$ of
SO(10), respectively, where $J_6, J_{10}$ the associated bosonic fields and $W_6, W_{10}$ the corresponding charges. Using (5.13), we can then derive the expressions for the picture-changed bosonic fields that contribute to the correlator (5.14)

$$D_{1(0)}^B = \frac{i}{2} e^{-\frac{1}{2} S_{44}} e^{\frac{1}{4} S_{66}} \left[ \sigma_+ \sigma_+ \sigma_+ \sigma_+ \omega^3 + i \sigma_+ \sigma_+ \sigma_+ \sigma_+ \omega^4 \right] e^{-\frac{1}{2} \mathcal{P}_2} e^{\frac{1}{2} \mathcal{P}_3} e^{i J_{6} W_6}$$

$$D_{4(0)}^B = \frac{i}{2} e^{-\frac{1}{2} S_{44}} e^{\frac{1}{4} S_{66}} \left[ \sigma_y \sigma_y \sigma_+ \sigma_+ \omega^3 + i \sigma_+ \sigma_+ \sigma_+ \sigma_+ \omega^4 \right] e^{-\frac{1}{2} \mathcal{P}_2} e^{\frac{1}{2} \mathcal{P}_3} e^{i J_{6} W_6}$$

$$T_{1(0)}^B = \frac{i}{2} e^{-\frac{1}{2} S_{44}} e^{\frac{1}{4} S_{66}} \left[ \sigma_+ \sigma_+ \sigma_+ \sigma_+ \omega^3 + i \sigma_+ \sigma_+ \sigma_+ \sigma_+ \omega^4 \right] e^{-\frac{1}{2} \mathcal{P}_2} e^{\frac{1}{2} \mathcal{P}_3} e^{i J_{10} W_{10}}$$

(5.26)

Putting everything together, we get

$$\alpha_1 = \left( \bar{\phi}^F_{1(-1/2)} \bar{\phi}^F_{1(-1/2)} \Phi^B_{3(-1)} D_{1(0)} D_{1(0)} \right)$$

$$= \frac{g_5^3 \sqrt{2}}{(2\pi)^2} \prod_{i=1}^{5} d^2 z_i \left( e^{-c/2(1)} e^{-c/2(2)} e^{-c(3)} \right) \langle S_a(1) S_b(2) \rangle$$

$$\times \left( e^{i S_{34}(3)} e^{-\frac{i}{2} S_{34}(4)} e^{-\frac{i}{2} S_{34}(5)} \right) \left( e^{-\frac{i}{2} S_{66}(1)} e^{-\frac{i}{2} S_{66}(2)} e^{\frac{i}{2} S_{66}(3)} \right) \left( e^{i \mathcal{P}_2} e^{i \mathcal{P}_3} \right) \left( e^{i J_{6} W_6} \right)$$

$$\times \left( e^{i \mathcal{P}_2} e^{i \mathcal{P}_3} \right) \left( e^{i J_{10} W_{10}} \right)$$

$$\langle e^{i J_{10} W_{10}}(5) e^{i J_{10} W_{10}}(6) \rangle,$$

(5.27)

where

$$\Sigma(1, 2, 5, 6) = -\frac{1}{4} (\Sigma_1 - \Sigma_2),$$

(5.28)

with

$$\Sigma_1 = \langle \sigma_{y+}(1) \sigma_{y+}(2) \rangle \langle \sigma_{y+}(1) \sigma_{y+}(2) \rangle \langle \sigma_{y+}(5) \sigma_{y+}(6) \rangle \langle \sigma_{y+}(5) \sigma_{y+}(6) \rangle,$$

$$\times \langle \sigma_{y+}(1) \sigma_{y+}(2) \sigma_{y+}(5) \sigma_{y+}(6) \rangle \langle \mathcal{F}(1) \mathcal{F}(2) \sigma_{y+}(5) \sigma_{y+}(6) \rangle \langle \omega^3(5) \omega^3(6) \rangle$$

(5.29)

$$\Sigma_2 = \langle \sigma_{y+}(1) \sigma_{y+}(2) \rangle \langle \sigma_{y+}(1) \sigma_{y+}(2) \rangle \langle \sigma_{y+}(5) \sigma_{y+}(6) \rangle \langle \sigma_{y+}(5) \sigma_{y+}(6) \rangle,$$

$$\times \langle \sigma_{y+}(1) \sigma_{y+}(2) \sigma_{y+}(5) \sigma_{y+}(6) \rangle \langle \mathcal{F}(1) \mathcal{F}(2) \sigma_{y+}(5) \sigma_{y+}(6) \rangle \langle \omega^4(5) \omega^4(6) \rangle.$$n

(5.30)

Substituting the conformal correlators from Appendix C we get

$$\alpha_1 = -\frac{g_5^3 \sqrt{2}}{4 (2\pi)^2} I_{\alpha_1},$$

(5.31)
with

\[ I_{\alpha_1} = \int d^2w \int d^2z \left[ K_1^{\alpha_1}(w, z) - K_2^{\alpha_1}(w, z) \right]. \quad (5.32) \]

Here

\[ K_1^{\alpha_1}(w, z) = \frac{1}{2\sqrt{2}} |w|^{-1} |1 - w|^{-1} |w - z|^{-1} |z|^{-3/4} \times (1 - |z| + |1 - z|)^{1/2}, \quad (5.33) \]

\[ K_2^{\alpha_1}(w, z) = \frac{1}{4\sqrt{2}} |w|^{-1} |1 - w|^{-1} |w - z|^{-1} |z|^{-2} (1 - |z| + |1 - z|)^{1/2}, \quad (5.34) \]

where we have utilised conformal invariance to set \( z_1 = \infty, z_2 = 1, z_3 = w, z_4 = z, z_5 = 0. \) It should be emphasised that both \( K_1^{\alpha_1} \) and \( K_2^{\alpha_1} \) exhibit a pole at \( z = 0, \) however, the poles cancel exactly in the expression of Eq. (5.32) ensuring the convergence of the integral. Next, we change variables \( z \to 1 - z, w \to 1 - w \) in \( K_1^{\alpha_1}(w, z) \)

\[ K_1^{\alpha_1}(w, z) = \frac{1}{2\sqrt{2}} |w|^{-1} |1 - w|^{-1} |w - z|^{-1} |z|^{-3/4} (1 - z)^{-2} (1 - |z| + |1 - z|)^{1/2} \times (\sqrt{z} + \sqrt{\bar{z}} |z|) \quad (5.35) \]

and then replace \( K_1^{\alpha_1}(w, z) \) with \( (K_1^{\alpha_1}(w, z) + K_1^{\alpha_1}(\bar{w}, \bar{z})) / 2 \) in (5.32) that enables us to recast \( K_1^{\alpha_1} \) in a purely real form:

\[ K_1^{\alpha_1}(w, z) = \frac{1}{2\sqrt{2}} |w|^{-1} |1 - w|^{-1} |w - z|^{-1} |z|^{-3/4} (1 - z)^{-2} (1 - |z| + |1 - z|)^{1/2} \times (1 + |z|) \text{Re} \sqrt{z}. \quad (5.36) \]

In a similar way, we find

\[ \alpha_4 = \langle \Phi_{(+1/2)\Phi_{(-1/2)}}^{B} D_{1(0)} D_{1(0)} \rangle = \frac{g_3^3 \sqrt{2}}{2 \pi^2} I_{\alpha_4}, \quad (5.37) \]

with

\[ I_{\alpha_4} = \int d^2w \int d^2z \left[ K_1^{\alpha_4}(w, z) - K_2^{\alpha_4}(w, z) \right], \quad (5.38) \]

and

\[ K_1^{\alpha_4}(w, z) = \frac{1}{\sqrt{2}} |w|^{-1} |1 - w|^{-1} |w - z|^{-1} |z|^{-2} (1 - |z| + |1 - z|)^{1/2}, \quad (5.39) \]
\[ K_2^{α_4}(w, z) = \frac{1}{2\sqrt{2}} |w|^{-1} |1 - w|^{-1} |w - z|^{-1} |z|^{-1} |1 - z|^{-3/4} (1 - |z| + |1 - z|)^{1/2} \times (1 + |z|) \text{Re} \sqrt{z}. \]  

(5.40)

Repeating the calculation for

\[ α_2 = \langle \bar{φ}_2(-1/2) \bar{φ}_2(-1/2) \Phi_{31(-1)}^B D_{1(0)} D_{1(0)} \rangle, \]  

(5.41)

\[ α_3 = \langle \bar{φ}_4(-1/2) \bar{φ}_4(-1/2) \Phi_{31(-1)}^B D_{1(0)} D_{1(0)} \rangle, \]  

(5.42)

we find

\[ α_2 = α_3 = α_4. \]  

(5.43)

The correlator functions \( β_1, β_2 \) and \( β_4 \) can be computed similarly

\[ β_1 = \langle \bar{φ}_1(-1/2) \bar{φ}_1(-1/2) \Phi_{31(-1)}^B D_{4(0)} D_{4(0)} \rangle = \frac{g_3^3 \sqrt{2}}{(2\pi)^2} I_{β_1}, \]  

(5.44)

\[ β_4 = \langle \bar{φ}_4(-1/2) \bar{φ}_4(-1/2) \Phi_{31(-1)}^B D_{4(0)} D_{4(0)} \rangle = \frac{g_3^3 \sqrt{2}}{(2\pi)^2} I_{β_4}, \]  

(5.45)

where

\[ I_{β_1} = -I_{β_4} = \int d^2 w \int d^2 z \left[ K_1^{β_1}(w, z) - K_2^{β_1}(w, z) \right], \]  

(5.46)

and

\[ K_1^{β_1}(w, z) = \frac{1}{2} |w|^{-1} |1 - w|^{-1} |w - z|^{-1} |z|^{-2} |1 - z|^{-1} (|1 - z| + |1 - z| + 1), \]  

(5.47)

\[ K_2^{β_1}(w, z) = \frac{1}{2} |w|^{-1} |w - z|^{-1} |1 - w|^{-1} |z|^{-2} |1 - z|^{-1} (|1 - z| + |z| + 1). \]  

(5.48)

The two terms in the integrand are here nicely combined, cancelling explicitly the pole at \( z = 0 \) and yielding

\[ I_{β_1} = -I_{β_4} = -\int d^2 w \int d^2 z \ |w|^{-1} |1 - w|^{-1} |w - z|^{-1} |z|^{-1} |1 - z|^{-1}. \]  

(5.49)

Repeating for

\[ β_2 = \langle \bar{φ}_2(-1/2) \bar{φ}_2(-1/2) \Phi_{31(-1)}^B D_{4(0)} D_{4(0)} \rangle, \]  

(5.50)

\[ β_3 = \langle \bar{φ}_4(-1/2) \bar{φ}_4(-1/2) \Phi_{31(-1)}^B D_{4(0)} D_{4(0)} \rangle, \]  

(5.51)
we obtain
\[ \beta_2 = 0, \ \beta_3 = \beta_4. \] (5.52)

Finally, an analysis of the \( \gamma_1, \gamma_4 \) correlators gives
\[ \gamma_1 = \langle \bar{\phi}_1 (-1/2) \phi_1 (-1/2) \Phi_{31}^B (-1) T_1 (0) D_4 (0) \rangle = \frac{g_1^2 \sqrt{2}}{(2\pi)^2} I_{\beta_1} = \frac{g_4^2 \sqrt{2}}{(2\pi)^2} I_{\alpha_1} = \alpha_1, \] (5.53)
\[ \gamma_4 = \langle \bar{\phi}_1 (-1/2) \phi_1 (-1/2) \Phi_{31}^B (-1) T_1 (0) D_4 (0) \rangle = \frac{g_4^3 \sqrt{2}}{(2\pi)^2} I_{\beta_4} = \frac{g_4^3 \sqrt{2}}{(2\pi)^2} I_{\alpha_4} = \alpha_4. \] (5.54)

In the same way we get
\[ \gamma_2 = \langle \bar{\phi}_2 (-1/2) \phi_2 (-1/2) \Phi_{31}^B (-1) T_1 (0) D_4 (0) \rangle = \gamma_4 = \alpha_4, \] (5.55)
\[ \gamma_3 = \langle \bar{\phi}_4 (-1/2) \phi_4 (-1/2) \Phi_{31}^B (-1) T_1 (0) D_4 (0) \rangle = \gamma_4 = \alpha_4. \] (5.56)

Summarising, the couplings under consideration involve three independent integrals \( I_{\alpha_1}, I_{\alpha_4}, I_{\beta_1} \) defined in (5.32), (5.38) and (5.46), respectively. We can calculate them numerically after changing to polar coordinates \( w = |w| e^{iu}, z = |z| e^{iv} \).

Using Mathematica we get
\[ I_{\alpha_1} = -567.20, \ I_{\alpha_4} = 189.07, \ I_{\beta_1} = -378.14 \] (5.57)

with an error of the order of 0.1%. This supports the relations
\[ I_{\alpha_1} = -3I_{\alpha_4}, \ I_{\beta_1} = -2I_{\alpha_4}, \] (5.58)
yielding
\[ \alpha_1 = -3\alpha_4, \ \beta_1 = 3\beta_4. \] (5.59)

When combined with Eqs. (5.53), (5.52), (5.53), (5.54), (5.56) the last equation gives a set of coupling relations
\[ -\frac{\alpha_1}{3} = -\frac{\gamma_1}{3} = -\frac{\beta_1}{2} = \frac{\beta_3}{2} = \frac{\beta_4}{2} = \gamma_3 = \gamma_4 = \alpha_3 = \alpha_4, \] (5.60)
\[ \alpha_2 = \gamma_2 = \alpha_4, \ \beta_2 = 0. \] (5.61)

6 Fermion masses and proton decay

In this section we study the phenomenological consequences of the flatness solution of Section 4. Of key importance to low energy phenomenology are the surviving SM
Higgs doublets. At tree-level these are given in Eqs. (3.11), (3.12). However, both the doublet and the triplet mass matrices receive additional contributions from NR terms of the form \( h_i h_j \phi^N \), \( i, j = 1, 2, 3, 45 \), where \( \phi^N \) is a combination of VEVs, arising from a NR term of order \( N > 3 \). A comprehensive computer search up to and including \( N = 7 \) order yields numerous NR terms of this form. However, after applying the flatness solution of Section 4 with the additional simplifying assumptions

\[
\Phi_1 = \Phi_2 = 0, \quad \bar{\phi}_{45} \lesssim \xi^2, \tag{6.1}
\]

the doublet mass matrix reduces to

\[
M^{(7)}_H = \begin{pmatrix}
H_1 & H_2 & H_3 & H_{45} \\
0 & \Phi_{12} & \Phi_{31} & 0 \\
0 & A(7) & \Phi_{23} & 0 \\
0 & B(5) & \bar{\phi}_{45} & 0
\end{pmatrix} + \mathcal{O}(\xi^6), \tag{6.2}
\]

where

\[
B(5) = \{ \bar{\phi}_{45} (D_1^2 + D_4^2 + T_1^2) \} \sim \xi^2 \bar{\phi}_{45}, \tag{6.3}
\]

\[
A(7) = \{ \Phi_{31} (\bar{\phi}_1^2 + \bar{\phi}_4^2 + \bar{\phi}_+ \bar{\phi}_-) (D_1^2 + D_4^2 + T_1^2) \} \sim \xi^5. \tag{6.4}
\]

Diagonalising we get a massless doublet pair

\[
H \propto \phi_{45} H_1 - \bar{\Phi}_{31} H_{45}, \tag{6.5}
\]

\[
\bar{H} \propto \{ \bar{B}(5) \bar{\Phi}_{23} - A(7) \bar{\phi}_{45} \} \bar{H}_1 + \{ \bar{\Phi}_{12} \bar{\phi}_{45} - \bar{B}(5) \bar{\Phi}_{31} \} \bar{H}_2 + \{ A(7) \bar{\Phi}_{31} - \bar{\Phi}_{12} \bar{\Phi}_{23} \} \bar{H}_{45} \tag{6.6}
\]

In the last equation the coefficient of \( \bar{H}_1 \) is of order \( \mathcal{O}(\xi^5 \bar{\phi}_{45}) \) while those of the components \( \bar{H}_2 \) and \( \bar{H}_{45} \) are of orders \( \mathcal{O}(\xi^3 \bar{\phi}_{45}) \) and \( \mathcal{O}(\xi^6) \) respectively. We therefore impose

\[
\bar{\Phi}_{12} \phi_{45} - \bar{B}(5) \Phi_{31} \lesssim \xi^6, \tag{6.7}
\]

so that \( \bar{H}_{45} \) is the dominant component in the \( \bar{H} \) Higgs doublet. This can be done for instance by choosing \( \bar{\phi}_{45} \sim \xi^3 \) and tuning slightly the combination \( (\bar{\Phi}_{12} - \{ D_1^2 + D_4^2 + T_1^2 \}) \). As a result, we obtain:

\[
H = \cos \theta H_1 - \sin \theta H_{45}; \quad \tan \theta = \langle \bar{\Phi}_{31} \rangle / \langle \phi_{45} \rangle \tag{6.8}
\]
\[ H = \cos \bar{\theta} H_{45} + \sin \bar{\theta} \cos \bar{\vartheta} H_2 + \sin \bar{\theta} \sin \bar{\vartheta} H_1 \quad ; \] (6.9)

\[ \tan \bar{\theta} \cos \bar{\vartheta} = \frac{\langle \Phi_{12} \rangle \langle \phi_{45} \rangle - B(5) \langle \Phi_{31} \rangle}{A(7) \langle \Phi_{31} \rangle - \langle \Phi_{12} \rangle \langle \Phi_{23} \rangle} ; \quad \tan \bar{\vartheta} = \frac{B(5) \langle \Phi_{23} \rangle - A(7) \langle \phi_{45} \rangle}{\langle \Phi_{12} \rangle \langle \phi_{45} \rangle - B(5) \langle \Phi_{31} \rangle}. \]

It follows that

\[ H_1 = \cos \theta H + \ldots ; \quad H_{45} = \sin \theta H + \ldots \]

\[ H_{45} = \cos \bar{\theta} H + \ldots ; \quad H_2 = \sin \bar{\theta} \cos \bar{\vartheta} H + \ldots ; \quad H_1 = \sin \bar{\theta} \sin \bar{\vartheta} H + \ldots \] (6.10)

where the dots stand for (superheavy) massive doublets. To simplify the analysis, in the following we will take \( \cos \theta \sim \cos \bar{\theta} \sim 1 \).

### 6.1 Fermion masses

We notice that the physical Higgs which provides masses to the charged \( Q = 2/3 \) up quarks is a mixture of \( H_{45} \) (as a necessary leading component), \( H_2 \) and \( H_1 \), while for \( Q = -1/3 \) down quarks and charged leptons, the physical Higgs is a mixture of \( H_1 \) (as a necessary leading component) and \( H_{45} \). Indeed, the leading electroweak Higgs components are exactly what the trilinear super potential (2.1) suggests in order to provide masses to the heaviest third generation: \( t, b \) and \( \tau \):

\[ W_3 \supset g_s \sqrt{2} \{ F_4 \bar{t}_5 H_{45} + (F_4 F_4 + f_1 \ell_1 c_1) H_1 \} . \] (6.11)

Actually, more than thirty years ago, this model predicted that the mass of the top quark is around \( \sim 170-180 \) GeV [2], as was observed in 1995 in Fermilab. This is a consequence of the fact that the top Yukawa coupling evolves at low energies towards a fixed point for its ratio to the QCD gauge coupling. Furthermore, we get the relation at the GUT scale \( m_b = m_\tau \), following the equality of the corresponding Yukawa couplings, which is apparent from the above expression of the trilinear superpotential. This is a successful mass relation assuming a suitable supersymmetric spectrum [26, 27]. Hence, we have the particle identification for the third generation:

\[ F_4 = \{(t, b), b', \nu_\tau \}_L ; \quad \bar{t}_5 \supset t^c_L ; \quad \bar{f}_1 \supset (\nu_\tau, \tau)_L ; \quad l^c_1 = \tau^c_L \] (6.12)

We next look for possible fermion mass terms at higher NR orders in the superpotential involving the massless Higgs doublets (6.5) and (6.6). Omitting corrections to the Yukawa couplings of the 3rd generation, we find the following list up to 5th order:

- Up quarks:

\[ F_2 \bar{t}_5 H_{45} \phi_4 \] (6.13)
• Down quarks:

\[ F_2 F_3 h_1 \left( \phi_1^2 - \phi_+ \phi_- + \lambda \phi_4^2 \right) \quad (6.14) \]

• Charged leptons:

\[ \overline{\mathcal{F}_2} \ell_5 h_1 \left( \phi_1^2 - \phi_+ \phi_- + \lambda \phi_4^2 \right) + \overline{\mathcal{F}_5} \ell_4 h_1 \left( \phi_1^2 + \phi_+ \phi_- - \phi_4^2 \right) \quad (6.15) \]

where we displayed only the non-vanishing dominant contributions for the choice of VEVs that solved the flatness conditions in Table I and we made a convenient choice for the co-cycle factors ambiguity in the 5-point amplitudes fixing the relative signs, while \( \lambda \) is an (irrelevant) order one constant (see below).

Using the flatness condition (4.50):

\[ \phi_2^1 + \phi^+ - \phi^+ - \phi_4^2 = -\Phi_3 \Phi_2 = O(\xi^4) \quad (6.16) \]

one can identify all members of the 2nd generation:

\[ F_2 = \{(c, s), s^c, \nu_L^c\} \quad ; \quad \overline{\mathcal{F}_2} = \{c^e, (\nu_\mu, \mu)\}_L \quad ; \quad \ell_5^e = \mu_L^c \quad (6.17) \]

with Yukawa couplings suppressed by two orders of magnitude compared to those of the 3rd generation. Indeed, (6.14) provides a successful mass to the charm quark when \( \phi_4 \sim O(\xi^2) \), while (6.14) and (6.15) provide successful masses to the strange quark and muon. Moreover, a direct computation of the corresponding coefficients of the above 5th order operators shows that those of \( F_2 F_2 h_1 \) are equal to those of \( \overline{\mathcal{F}_2} \ell_2 h_1 \) [16]. One thus obtains the mass relation \( m_s = m_\mu \).

It remains the identification of the first generation. Taking into account the mass term (4.69), the identification (6.12), (6.17) and the leftover operator for the charged leptons Yukawa couplings, one gets:

\[ F_3 = \{(u, d), d^c, \nu_L^e\}_L \quad ; \quad \overline{\mathcal{F}_3} \supset \{(\nu_e, e)\}_L \quad ; \quad \ell_1 = u_L^e \quad (6.18) \]

Combining (6.12) and (6.18), we thus obtain:

\[ \overline{\mathcal{F}_5} = \{l^e, (\nu_e, e)\}_L \quad ; \quad \overline{\mathcal{F}_1} = \{u^c, (\nu_\tau, \tau)\}_L \quad (6.19) \]

It is amazing how in this string model the three generations are distributed with rather interesting consequences for inter-generation mixing, proton decay modes and possibly flavour changing lepton number. Up to this order, there are no Yukawa coupling for the up and down quarks, which is compatible with the order of our approximation. On the other hand, the second operator in (6.15) leads to a Yukawa coupling for the electron. Using the flatness relation (6.16) and the
one finds that the electron mass is two orders of magnitude lower than the muon mass, which is a remarkable successful relation [16].

Focussing now to possible mass terms for the quarks of the first generation, we extracted all relevant NR terms up to 7th order (included) for our choice of VEVs, that we display below, omitting higher order corrections to existing Yukawa couplings:

- Up quarks:
  \[ F_1 f_1 h_2 \Phi_{31} \phi_{45} \bar{\phi}_1 + F_3 f_5 h_{45} (D_3 D_4) \bar{\Phi}_{23} \]
  \[ + F_3 f_1 h_{45} \left[ \bar{\phi}_- (T_1 T_3) \Phi_{31} + \bar{\phi}_1 (D_1 D_3) \Phi_{31} \right] \]  
  (6.20)

- Down quarks:
  \[ F_1 F_3 h_1 (D_1 D_3) \bar{\Phi}_{23} + F_3 F_4 h_1 (D_3 D_4) \bar{\Phi}_{23} \]  
  (6.21)

- Charged leptons:
  \[ f_5 \xi \left( h_1 \bar{\Phi}_{31} + h_{45} \phi_{45} \right) \left( \phi^2_1 + \phi^2_4 + \phi_+ \phi_- \right) \Phi_{31} \]  
  (6.22)

Eq. (6.22) gives a correction to the electron Yukawa coupling of order \( O(\xi^6) \). The term in the second line of (6.20) leads to an up quark Yukawa coupling of the right order of magnitude \( O(\xi^5) \), provided \( T_3 \) and \( D_3 \) are of order \( O(\xi^2) \), consistently with the 6th order flatness conditions discussed at the end of section 4. Finally, the second term in the first line of (6.20), as well as the second term of (6.21) lead to quark mixing between the first and third generation of order \( O(\xi^6) \).

A down quark Yukawa coupling can appear when \( F_3 \) gets a small VEV which is consistent with the flatness conditions when \( \langle F_3 \rangle / \langle F_1 \rangle < \xi \), as explained in the end of section 4. In this case, the GUT Higgs \( F \) and the first generation \( F'_3 \) are given by the linear combinations:

\[ F = A_1 F_1 + A_3 F_3 \simeq F_1 + \varepsilon F_3 \]
\[ F'_3 = -A_3 F_1 + A_1 F_3 \simeq F_3 - \varepsilon F_1 \]  
(6.23)

where the constants \( A_i \) satisfy \( \sum_i |A_i|^2 = 1 \) and enter in the flatness condition \( (4.91) \), while \( \varepsilon \sim O(\xi^{3/2}) \). Thus, the tree-level superpotential term \( F_1 F_1 h_1 \) in (2.11) generate a Yukawa coupling for the down quark of the right order of magnitude \( O(\xi^3) \).

Our results on the masses of quarks and leptons and their relations (at the string scale) are summarised below:

\[ m_t = g v_{45} \quad m_b = g v_1 \quad m_\tau = m_b \]
\[ m_e \sim \xi^2 m_t \quad m_\tau \sim \xi^2 m_b \quad m_\mu = m_s \]
\[ m_u \sim \xi^3 m_t \quad m_d \sim \xi^3 m_b \quad m_e \sim \xi^4 m_\tau \]  
(6.24)
where $g = g_s \sqrt{2}$ is the GUT gauge coupling and $v_{45}, v_1$ denote the VEVs of $H_{45}, H_1$ Higgs doublets, respectively (see Eqs. (6.10), (6.8)). It follows that $\tan \beta \sim m_t/m_b \simeq 40$.

The relation $m_\mu = m_s$ is apparently problematic. Moreover, we have not obtained a $d-s$ quark mixing. Both issues can be in principle addressed by allowing an appropriate non-zero VEV for $F_2$ generalising the flipped $SU(5)$ breaking mixing VEVs in (6.23). This introduces a mixing between $F_2$ and $F_3$ which could account for the Cabibo angle and correct the relation $m_\mu = m_s$. However, a separate analysis is needed that could also include neutrino masses and mixings which goes beyond the scope of this paper.

We consider that equation (6.24) belongs to the highlights of this work, and as such, we need to pause and reflect on its importance. We have achieved, for the first time ever to our knowledge, to compute explicitly the mass spectrum of quarks and (charged) leptons of the Standard Model in String Theory. By calculating the superpotential at the $N = 3$ (tree) level, we identified the content of the third generation, i.e. the particles that get Yukawa couplings proportional to the string coupling constant $g_s$ at this level. We discussed above about the top quark Yukawa coupling and its consequences of predicting in 1989 the top quark mass in the 170-180 GeV range, as observed in 1995 at FNAL to be around 173 GeV. This particular top-quark Yukawa coupling triggers the radiative electroweak breaking of $SU(2) \times U(1)$ at low energies, thus explaining the gauge hierarchy $M_W/M_{GUT} \sim \mathcal{O}(10^{-16})$ a natural way. Actually, because our string model is of no-scale type, it leads to a determination of the SUSY breaking scale in the $\mathcal{O}(\text{TeV})$ region. Concerning the masses of the bottom quark and $\tau$-lepton, we get the relation $m_b = m_\tau$ at the string scale, as well as the equality of the top and bottom Yukawa couplings that leads to the determination of $\tan \beta \sim m_t/m_b \simeq 40$, which eventually would be determined dynamically through the no-scale mechanism.

For the next two generations, we need to calculate non renormalisable corrections in $\alpha'$ in the superpotential, corresponding to $N = 4, 5, 6, \ldots$, that we have done using the general method of Ref. [28]. The $N$-th order NR terms contain $(N - 3)$ fields that will need to get VEVs. The way that all these fields get their VEVs is through the endemic, in the string models we are considering, existence of an ‘anomalous’ abelian gauge symmetry $U(1)_A$ that enforces non-trivial VEVs for some ‘charged’ fields. Eventually, in order to satisfy the F- and D-type flatness conditions, a set of fields get dynamically VEVs of the order $\xi \approx 1/10 M_s$, and thus we have a perturbative expansion parameter! Thus, all the masses of the second and first generations are found to be determined, involving powers of $\xi^n$, $n = 2, 3, 4, 5$, multiplying, for normalisation, the corresponding masses of the third generation. In other words, the masses of the second and first generations are dynamically determined as $m_t, m_b, m_\tau$ and $\xi$ and are dynamically fixed!
Now, we can really appreciate the structure of Eq. (6.24), as it provides a very successful mass estimation for all quarks and charged leptons.

6.2 Proton decay

Let us now turn to the problem of proton decay. Proton decay has been for more than forty years a real headache for theorists. Basically, it is a main prediction of GUTs that has not been vindicated experimentally. The present lower limits on proton decay are of the order of $10^{34} - 10^{35}$ years depending on the particular decay mode. Unlike the Standard Model where one can show that, because of its particle content, contains no baryon (B) and lepton (L) number violating interactions [29] in Grand Unified Theories, these interactions are endemic. Furthermore, SUSY GUTs contain dangerous $d = 5$ B and L violating interactions that may lead to very rapid proton decay. With the advent of superstring theory, generally, the proton decay problem became more acute. The reason being that the low energy spectrum contains a plethora of particles that may provide B, L violating interactions leading to a rather rapid proton decay.

In the case of the string derived flipped $SU(5)$ model under consideration there are two sources of $d = 5$ baryon number violating operators. The first consists of the usual dimension five operator $QQQL$ ascribed to the exchange of additional triplets in the massless string spectrum. The second comprises effective $QQQL$ operators generated from non-renormalisable string couplings arising from the exchange of massive string modes.

Let us start with the triplet exchange induced dimension-five operators. As explained in Section 3 we have five pairs of additional triplets accommodated in the fields $h_i, \overline{h}_i, i = 1, 2, 3, 45$ and the flipped $SU(5)$ breaking Higgs multiplets $F, \overline{F}$ defined in this section. At tree-level the triplets mass matrix is given by (3.4) where $d_H$ now stands for the additional triplet combination $d_1^c + \varepsilon d_3^c$ and $F_1$ is replaced by $F$. For our flatness solution of Table 1 and taking into account non-renormalisable interactions up to $N = 7$, the extra triplet mass matrix to order $\xi^5$ reads

\[
M_D^{(7)} = \begin{pmatrix}
D_1 & D_2 & D_3 & D_{45} & \overline{d}_H \\
0 & \Phi_{12} & \Phi_{31} & 0 & \overline{s}_1^{(5)} \\
0 & A^{(7)} & \Phi_{23} & 0 & 2\overline{F}_5 \\
\Phi_{31} & \Phi_{23} & 0 & \phi_{45} & \overline{s}_3^{(5)} \\
0 & \overline{B}^{(5)} & \phi_{45} & 0 & 0 \\
2F & s_2^{(5)} & 0 & s_4^{(5)} & s \\
\end{pmatrix} + O(\xi)^6, \quad (6.25)
\]
where
\[
\begin{align*}
\bar{s}_1^{(5)} &= \{ \mathcal{F}_5 \left( \phi_4^2 + \phi_+ \phi_- \right) \}, \\
\bar{s}_3^{(5)} &= \{ \mathcal{F}_5 \left( D_4^2 + D_2^4 + T_4^2 \right) \}, \\
\bar{s}_2^{(5)} &= \{ F \left( \phi_4^2 + \phi_+ \phi_- \right) \}, \\
\bar{s}_4^{(5)} &= \{ F \Phi_{31} \phi_{45} \}, \\
s &= \{ F \bar{F}_5 \mathcal{T}_{12} \}.
\end{align*}
\]

Assuming \( F \bar{F}_5 \sim \xi^3 \) the determinant is given by
\[
\det \left( M_d^{(7)} \right) \sim \left( F \phi_{45} - \Phi_{31} \bar{s}_4^{(5)} \right) \left( \bar{B}^{(5)} \mathcal{F}_5 \Phi_{31} - \mathcal{F}_5 \Phi_{12} \phi_{45} - \bar{B}^{(5)} \bar{s}_1^{(5)} \Phi_{23} + A_2^{(7)} \bar{s}_1^{(5)} \phi_{45} \right) \\
\sim F \bar{F}_5 \phi_{45} \left( \bar{B}^{(5)} \Phi_{31} - \Phi_{12} \phi_{45} \right) + \cdots \sim \xi^{10}
\]
ensuring that all triplets are massive. A detailed calculation shows that the orders of magnitude of the triplets mass eigenstates are: \( \xi, \xi, \xi, \xi, \xi^5 \). This is consistent with our approximation utilised in (6.25), as the lightest eigenvalue is of order \( \mathcal{O}(\xi^5) \) rendering higher order contributions in (6.25) irrelevant.

Following the analysis of [30], triplet exchange \( QQQL \) type dimension-five operators for a general superpotential of the form
\[
f_{ij} a f_i f_j h_a + \bar{y}_{ij} F_i f_j \bar{h}_a + \mathcal{K}_a (M_D)_{ab} h_b
\]

involving \( n \) additional fiveplets are proportional to
\[
\mathcal{O}_{ijkl}^{QQQL} \sim \frac{1}{\det \left( M_D^{(7)} \right)} \sum_{a,b=1}^n y_{kl}^a \text{cof} \left( M_D^{(7)} \right)_{ab} f_{ij}^b.
\]

In our case, a direct computation yields
\[
\text{cof} \left( M_D^{(7)} \right)_{1,1} = F_5 \phi_{45} \phi_{45} \bar{s}_2^{(5)} + \cdots \sim \xi^9, \\
\text{cof} \left( M_D^{(7)} \right)_{1,2} = -\phi_{45} \phi_{45} \bar{s}_1^{(5)} + \cdots \sim \xi^{11}, \\
\text{cof} \left( M_D^{(7)} \right)_{1,3} = F_5 \Phi_{31} \phi_{45} \bar{s}_2^{(5)} + \cdots \sim \xi^{11}, \\
\text{cof} \left( M_D^{(7)} \right)_{1,45} = -F_5 \Phi_{31} \phi_{45} \bar{s}_2^{(5)} + \cdots \sim \xi^7, \\
\text{cof} \left( M_D^{(7)} \right)_{2,1} = -F_1 F_5 \phi_{45} \phi_{45} \bar{s}_2^{(5)} + \cdots \sim \xi^7, \\
\text{cof} \left( M_D^{(7)} \right)_{2,2} = F_1 \phi_{45} \phi_{45} \bar{s}_1^{(5)} + \cdots \sim \xi^9,
\]
\[
\text{cof} \left( M_D^{(7)} \right)_{2,3} = 0, \quad (6.40)
\]
\[
\text{cof} \left( M_D^{(7)} \right)_{2,45} = F_1 \overline{\Phi}_{31} \phi_{45} + \cdots \sim \xi^5, \quad (6.41)
\]
\[
\text{cof} \left( M_D^{(7)} \right)_{3,1} = F_1 \overline{\Phi}_{45} B^{(5)} + \cdots \sim \xi^9, \quad (6.42)
\]
\[
\text{cof} \left( M_D^{(7)} \right)_{3,2} = -F_1 \phi_{45} B^{(5)} \overline{s}_1 + \cdots \sim \xi^{11}, \quad (6.43)
\]
\[
\text{cof} \left( M_D^{(7)} \right)_{3,3} = 0, \quad (6.44)
\]
\[
\text{cof} \left( M_D^{(7)} \right)_{3,45} = -F_1 \overline{\Phi}_{45} \Phi_{12} + \cdots \sim \xi^7 \quad (6.45)
\]
\[
\text{cof} \left( M_D^{(7)} \right)_{45,1} = \overline{\Phi}_{45} \left( \Phi_{31} s_2^{(5)} - F_1 \Phi_{23} \right) + \cdots \sim \xi^9, \quad (6.46)
\]
\[
\text{cof} \left( M_D^{(7)} \right)_{45,2} = \left( \overline{\Phi}_{31} s_2^{(5)} - F_1 \Phi_{23} \right) \overline{s}_1 + \left( \overline{\Phi}_{12} \phi_{45} - \Phi_{31} B^{(5)} \right) F_1 \overline{s}_3^{(5)} + \cdots \sim \xi^{11}, \quad (6.47)
\]
\[
\text{cof} \left( M_D^{(7)} \right)_{45,3} = -F_1 \overline{\Phi}_5 \left( \overline{\Phi}_{12} \phi_{45} - B^{(5)} \Phi_{31} \right) + \cdots \sim \xi^9, \quad (6.48)
\]
\[
\text{cof} \left( M_D^{(7)} \right)_{45,45} = \overline{\Phi}_{31} \left( \overline{\Phi}_{31} s_2^{(5)} - F_1 \Phi_{23} \right) + \cdots \sim \xi^7. \quad (6.49)
\]

Triplet-exchange dimension-five operators can be generated via couplings of the type \( \Phi F h N_3 \) and \( \overline{\Phi} \overline{h} \Phi M_3 \), arising at orders \( N \) and \( M \) respectively, where \( \phi_{N_3}, \phi_{M_3} \) stand for combinations of field VEVs. At tree-level \( (N = M = 3) \) we have a single pair of couplings of this type, namely \( F_2 F_2 h_2, F_4 \overline{h}_5 \overline{h}_{45} \), that can give rise to an effective \( F_2 F_2 h_2 \) operator. Actually, this operator corresponds to the dominant contribution in the \( \xi \) expansion, among (6.44)-(6.49), arising from \( \text{cof} \left( M_D^{(7)} \right)_{2,45} \) which is proportional to

\[
\frac{\text{cof} \left( M_D^{(7)} \right)_{2,45}}{\det \left( M_D^{(7)} \right)} \sim \frac{1}{\xi^5}. \quad (6.50)
\]

However, the associated \( d = 5 \) operator is further suppressed by at least a factor \( \xi^3 \), since \( F_1, F_2 \) accommodate third and second generation quarks respectively. In fact the associated \( QQQL \) operator is suppressed by an effective triplet mass of order \( \xi^2 M_P \sim 10^{16} \) GeV which leads to proton lifetime exceeding current experimental limits for a SUSY breaking scale of the order of \( m_{\text{susy}} \gtrsim 10^2 \) TeV [31]. Additional tree-level operators could be generated via the \( F_1 - F_3 \) mixing introduced in Section 6.1 to generate down quark mass (see (6.23)). The term \( F_1 h_1 \) induces an effective coupling of the form \( F_3 h_3 h_1 \) that combined with the \( \overline{F} \overline{F}_4 h_45 \) leads to an effective
The $d = 5$ operator of the form $F_3 F_3 F_4 \mathcal{F}_5$. However, this operator gets an extra suppression $\xi^3$ due to the mixing and $\xi^7$ due to Eq. (6.45), leading to an effective triplet scale of the order of $M_P$ and thus becomes subdominant.

At higher order ($N = 3, M = 4$) one could use the terms $F_2 F_2 h_2$ and $F_2 \mathcal{F}_2 h_{45} \phi_4$ to form an effective dimension five operator of the type $F_2 F_2 \mathcal{F}_2$. In this case we have smaller family mixing, of the order of $\xi^2$, however, we get an additional suppression of order $\xi^2$ from the additional VEV $\phi_4$. Furthermore, higher order contributions ($N, M > 3$) are relatively suppressed by a factor of $\xi^2$ in the worst case scenario. The same is true for all other operators in Eqs. (6.34)-(6.50).

Let us now examine the string induced effective dimension-five operators [18]. These are of the form $10 \times 10 \times 10 \times 5$. An explicit search gives no candidate couplings of this type at the level of $N = 4$ non-renormalisable superpotential, while at $N = 5$ we have the following two terms

$$ F_2^2 F_3 \mathcal{F}_3 \Phi_{23} + F_4^2 F_3 \mathcal{F}_3 \Phi_{31}. \quad (6.51) $$

However, both terms involve $\mathcal{F}_3$ which becomes superheavy in our F/D-flatness solution (see (4.69)). As a result, we have no contributions to proton decay at this level. At $N = 6$ we find two non-vanishing terms

$$ F_3^2 F_4 \mathcal{F}_5 (\Phi_{31} \phi_- + \mathcal{F}_{23} \phi_+). \quad (6.52) $$

Following Table 1, these yield an effective operator $F_3^2 F_4 \mathcal{F}_5$ with a coupling of order $\xi^2$ which is translated to a dimension-five $QQQL$ effective operator with triplet scale of the order of $\xi^{-2} M_P \sim 10^{20}$ GeV, which as explained above is safe for proton decay. Moreover, all operators of the form (6.51) have been shown to vanish explicitly in the case of the flipped $SU(5)$ model as a result of permutation symmetries [18].

Summarising, the leading contribution to dimension-five proton decay operators comes from the $F_2 F_2 F_4 \mathcal{F}_5$ operator arising from $h_2, h_{45}$ triplet pair mediation which is compatible with the experimental bounds for a SUSY breaking scale $m_{\text{SUSY}} \gtrsim \mathcal{O}(10^2)$ TeV. It is interesting to point out that a SUSY breaking scale in the energy region of tens of TeV is also required for cosmological reasons, following an analysis of reheating and nucleosynthesis in the flipped $SU(5) \times U(1)$ model [13]. The model can also accommodate the usual lightest supersymmetric particle (LSP) as a sufficiently stable dark matter candidate [33].

### 7 Concluding remarks

The quest for a Unified Theory of all interactions, including gravity, has been for the last hundred years the ‘holly grail’ of High Energy physics. In our times, it
has been named the Theory of Everything (TOE), and as such it should explain not only all of particle physics but also inflationary cosmology in terms of some fundamental principles. Superstring theory has been heralded as the fundamental framework that has the capacity to provide such a Theory of Everything. There are different formulations of (compactified) superstring theory in four dimensions, and for more than thirty years now, the Free Fermionic Formulation (FFF) has been a very useful tool to perform explicit calculations and construct models that may serve as a TOE.

Recently, we derived from the FFF of superstring theory, a Starobinsky like inflationary model that fits all known cosmological data and connects the inflation scale, calculated dynamically, to the Right-handed neutrino mass, as the inflaton field is a mixture of the heavy sneutrino and some GUT singlet fields provided by superstring theory. The framework is superstring derived no-scale flipped $SU(5)$, that has some unique features, as we discussed in previous works.

Here, we worked out in great detail all the possible physics issues that needed to be resolved. We proved that the F- and D-flatness conditions are satisfied at least to sixth order in the $\alpha'$-expansion of the superpotential, taking into account the fact that in our framework there is always an ‘anomalous’ abelian gauge symmetry. The breaking mechanism of this anomalous $U(1)_A$ entails several fields, mostly singlets, to get VEVs of order $\xi \approx 1/10$ in string units. We use then $\xi$ as an expansion parameter in perturbation theory, and thus we solve the F- and D-flatness conditions and get a specific set of VEVs dynamically. Then, we are using this ‘vacuum’ to determine the triplet-doublet Higgs splitting and getting a pair of ‘massless’ Higgs doublets that provides the radiative electroweak symmetry breaking and the Yukawa couplings for the third generation at the tree level of the superpotential. Actually, about 32 years ago, we predicted the mass of the top-quark in the range of 170-180 GeV [2].

Furthermore, non-renormalisable terms, calculable in our framework, provide a realistic hierarchical fermion mass spectrum with all quark and lepton masses derived dynamically, consistent with the experimental hierarchies. As an example, we mention that for the first time ever, the mass of the electron has been calculated explicitly and in full agreement with its observed value, which is rather remarkable. taking into account the fact of its tiny value, vis a vis the top-quark mass. In addition, we derived some new relations involving quarks and leptons that are experimentally satisfied. Furthermore, the triplet Higgs masses are heavy enough as to provide a possible observable proton decay in very specific modes.

We believe that, all in all, we have for the first time a framework that provides not only a unified picture of particle physics and cosmology, but also a dynamically derived hierarchical mass spectrum, probably observable proton decay and inflationary cosmology in agreement with all cosmological data. We cannot avoid but
close with the same final statement used 32 years ago in the first of reference [18]: “We leave it to the reader to decide how many more miracles she wants to see before abandoning her doubts about flipped $SU(5)$”.

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Appendix A  String construction and spectrum of the flipped
$SU(5) \times U(1)$ model

In this appendix we briefly review the construction and the spectrum of the re-vamped flipped $SU(5) \times U(1)$ string model \cite{2}. The model is defined in the free
fermionic formulation of the heterotic superstring \cite{3} using a set of eight basis
vectors $B = \{\beta_1, \beta_2, \ldots, \beta_8\}$ and a set of phases $c_{ij} = c_{[\beta_i/\beta_j]}$, $i, j = 1, \ldots, 8$ where

\[\beta_1 = S = \{\psi^\mu, \chi^{1, \ldots, 6}\},\]

\[\beta_2 = b_1 = \{\psi^\mu, \chi^{12}, y^{3, \ldots, 6}, \bar{y}^{3, \ldots, 6}, \bar{\psi}^{1, \ldots, 5}, \bar{\eta}\},\]

\[\beta_3 = b_2 = \{\psi^\mu, \chi^{34}, y^{12}, \omega^{56}, \bar{y}^{12}, \bar{\omega}^{56}, \bar{\psi}^{1, \ldots, 5}, \bar{\eta}\},\]

\[\beta_4 = b_3 = \{\psi^\mu, \chi^{56}, \omega^{1, \ldots, 4}, \bar{\omega}^{1, \ldots, 4}, \bar{\psi}^{1, \ldots, 5}, \bar{\eta}\},\]

\[\beta_5 = b_4 = \{\psi^\mu, \chi^{12}, y^{36}, \omega^{45}, \bar{y}^{36}, \bar{\omega}^{45}, \bar{\psi}^{1, \ldots, 5}, \bar{\eta}\},\]

\[\beta_6 = b_5 = \{\psi^\mu, \chi^{34}, y^{26}, \omega^{15}, \bar{y}^{26}, \bar{\omega}^{15}, \bar{\psi}^{1, \ldots, 5}, \bar{\eta}\},\]

\[\beta_7 = \zeta = \{\bar{\phi}^{1, \ldots, 8}\},\]

\[\beta_8 = \alpha = \{y^{46}, \omega^{46}, \bar{y}^{46}, \bar{\omega}^{2346}, \bar{\psi}^{1, \ldots, 5}, \bar{\eta}, \bar{\phi}^{1, \ldots, 4}, \bar{\phi}\},\]  

and $c_{ij} = c_{[\beta_i/\beta_j]} = e^{i\pi\tilde{c}_{ij}}$, with

\[
\tilde{c} = \begin{pmatrix}
S & b_1 & b_2 & b_3 & b_4 & b_5 & \zeta & \alpha \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1/2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1/2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1/2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & -1/2
\end{pmatrix}
\]  

\[\text{(A.2)}\]

In the notation employed in (A.1) included fermions are periodic unless indicated
otherwise (for example, $\frac{1}{2}$ denotes fermions twisted by $-i$). The string model under
consideration possesses $N = 1$ space-time supersymmetry and $SU(5) \times U(1) \times$
$U(1)^4 \times SU(4) \times SO(10)$ gauge symmetry. We use the terms “observable” and “hidden” gauge symmetry to refer to the $SU(5) \times U(1)$ and $SU(4) \times SO(10)$ group factors respectively. The four extra abelian group factors $U(1)^4 = \prod_{i=1}^{4} U(1)_i$ associated with the world-sheet currents $\eta^1 \eta^{1*}$, $\eta^2 \eta^{2*}$, $\eta^3 \eta^{3*}$, $\eta^4 \eta^{4*}$ exhibit anomalies.

As a matter of fact $\text{Tr} U(1)_1 = -36$, $\text{Tr} U(1)_2 = -12$, $\text{Tr} U(1)_3 = -24$, $\text{Tr} U(1)_4 = -12$. However, redefining appropriately we obtain three abelian combinations free of gauge and mixed gravitational anomalies

$$U(1)'_1 = U(1)_3 + 2U(1)_4,$$

$$U(1)'_2 = U(1)_1 - 3U(1)_2,$$

$$U(1)'_3 = 3U(1)_1 + U(1)_2 + 4U(1)_3 - 2U(1)_4,$$

and one anomalous $U(1)$ group factor

$$U(1)_A = -3U(1)_1 - U(1)_2 + 2U(1)_3 - U(1)_4 , \quad \text{Tr} U(1)_A = 180 .$$

The massless matter spectrum consists of: (i) “Observable” sector matter particles listed in Table 2. These are states charged exclusively under the $SU(5) \times U(1)$ and $U(1)^4$ gauge group factor. These comprise three chiral fermion families as well as a family/anti-family pair, residing in $SO(10)$ spinorials/anti-spinorials, coming from the sectors $b_1, b_2, b_3, b_4, b_5$, and three fiveplet/anti-fiveplet pairs, accommodated in $SO(10)$ vectorials, from the sectors $S, S+b_4+b_5$. Bosonic partners come from the sectors $S+b_i, i = 1, \ldots, 4$ and $0, b_4+b_5$ respectively. (ii) “Hidden” sector matter particles, that is, states exclusively charged under $SU(4) \times SO(10) \times U(1)^4$ listed in Table 3. These arise from the sectors $(S) + b_i + 2\alpha(+\zeta), i = 1, \ldots, 4$. (iii) Exotic fractionally charged states coming from the sectors $(S) + b_i \pm \alpha(+\zeta)$, $(S) + b_1 + b_2 + b_3 \pm \alpha(+\zeta)$, $(S) + b_1 + b_2 + b_4 \pm \alpha(+\zeta)$. These are listed in Table 4.

---

2Here we employ a compact notation to denote several sectors contributing to the same field multiplet, for example $(S) + b_i + 2\alpha(+\zeta)$ stands for four sectors: $b_i + 2\alpha, b_i + 2\alpha + \zeta, S + b_i + 2\alpha, S + b_i + 2\alpha + \zeta$. 

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|      | SU(5) | U(1) | U(1) | U(1) | U(1) | SU(4) | SO(10) |
|------|-------|------|------|------|------|-------|--------|
| $S$  |       |      |      |      |      |       |        |
| $h_1$ | 5     | -1   | +1   | 0    | 0    | 0     | 1      |
| $h_{10}$ | 5     | +1   | -1   | 0    | 0    | 0     | 1      |
| $h_2$ | 5     | -1   | 0    | +1   | 0    | 0     | 1      |
| $h_{10}$ | 5     | +1   | 0    | -1   | 0    | 0     | 1      |
| $h_3$ | 5     | -1   | 0    | 0    | +1   | 0     | 1      |
| $h_{10}$ | 5     | +1   | 0    | 0    | -1   | 0     | 1      |
| $\Phi_{12}$ | 1     | 0    | -1   | +1   | 0    | 0     | 1      |
| $\Phi_{12}$ | 1     | 0    | +1   | -1   | 0    | 0     | 1      |
| $\Phi_{31}$ | 1     | 0    | +1   | 0    | +1   | 0     | 1      |
| $\Phi_{31}$ | 1     | 0    | 0    | -1   | +1   | 0     | 1      |
| $\Phi_{23}$ | 1     | 0    | 0    | +1   | -1   | 0     | 1      |
| $\Phi_{23}$ | 1     | 0    | 0    | +1   | -1   | 0     | 1      |
| $\Phi_i, i = 1,\ldots,5$ | 1     | 0    | 0    | 0    | 0    | 0     | 1      |

|      |       |      |      |      |      |       |        |
| $b_1$ | 10    | -1/2 | -1/2 | 0    | 0    | 0     | 1      |
| $f_1$ | 5     | -3/2 | -1/2 | 0    | 0    | 0     | 1      |
| $\ell_1$ | 1     | +5/2 | -1/2 | 0    | 0    | 0     | 1      |
| $b_2$ | 10    | -1/2 | 0    | -1/2 | 0    | 0     | 1      |
| $f_2$ | 5     | -3/2 | 0    | -1/2 | 0    | 0     | 1      |
| $\ell_2$ | 1     | +5/2 | 0    | -1/2 | 0    | 0     | 1      |
| $b_3$ | 10    | -1/2 | 0    | 0    | +1/2 | -1/2 | 1      |
| $f_3$ | 5     | -3/2 | 0    | 0    | +1/2 | +1/2 | 1      |
| $\ell_3$ | 1     | +5/2 | 0    | 0    | +1/2 | +1/2 | 1      |
| $b_4$ | 10    | -1/2 | 0    | -1/2 | 0    | 0     | 1      |
| $f_4$ | 5     | -3/2 | 0    | -1/2 | 0    | 0     | 1      |
| $\ell_4$ | 1     | +5/2 | 0    | -1/2 | 0    | 0     | 1      |
| $b_5$ | 10    | -1/2 | 0    | +1/2 | 0    | 0     | 1      |
| $f_5$ | 5     | -3/2 | 0    | -1/2 | 0    | 0     | 1      |
| $\ell_5$ | 1     | +5/2 | 0    | -1/2 | 0    | 0     | 1      |
| $S + b_4 + b_5$ |       |      |      |      |      |       |        |
| $h_{45}$ | 5     | -1   | -1/2 | -1/2 | 0    | 0     | 1      |
| $h_{45}$ | 5     | +1   | +1/2 | +1/2 | 0    | 0     | 1      |
| $\phi_{45}$ | 1     | 0    | +1/2 | +1/2 | +1   | 0     | 1      |
| $\phi_{45}$ | 1     | 0    | -1/2 | -1/2 | -1   | 0     | 1      |
| $\phi_+$ | 1     | 0    | +1/2 | -1/2 | 0    | +1    | 1      |
| $\phi_-$ | 1     | 0    | -1/2 | +1/2 | 0    | -1    | 1      |
| $\phi_+$ | 1     | 0    | -1/2 | +1/2 | 0    | +1    | 1      |
| $\phi_-$ | 1     | 0    | -1/2 | +1/2 | 0    | -1    | 1      |
| $\phi_{i}, i = 1,\ldots,4$ | 1     | 0    | +1/2 | -1/2 | 0    | 0     | 1      |
| $\phi_{i}, i = 1,\ldots,4$ | 1     | 0    | -1/2 | +1/2 | 0    | 0     | 1      |

Table 2: “Observable” sector massless matter states and their $SU(5) \times U(1) \times U(1)^4 \times SU(4) \times SO(10)$ quantum numbers.
Table 3: “Hidden sector” massless matter states and their \( SU(5) \times U(1) \times U(1)^4 \times SU(4) \times SO(10) \) quantum numbers.

| \( SU(5) \) | \( U(1) \) | \( U(1)_1 \) | \( U(1)_2 \) | \( U(1)_3 \) | \( U(1)_4 \) | \( SU(4) \) | \( SO(10) \) |
|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| \( b_1 + 2\alpha (+\zeta) \) | \( D_1 \) | 1 | 0 | 0 | \(-1/2\) | \(+1/2\) | 0 | 6 | 1 |
| \( D_2 \) | 1 | 0 | 0 | \(-1/2\) | \(+1/2\) | 0 | 1 | 10 |
| \( b_2 + 2\alpha (+\zeta) \) | \( D_3 \) | 1 | 0 | \(-1/2\) | \(-1/2\) | 0 | \(+1/2\) | 6 | 1 |
| \( D_4 \) | 1 | 0 | 0 | \(-1/2\) | \(-1/2\) | 0 | \(-1/2\) | 1 | 10 |
| \( b_3 + 2\alpha (+\zeta) \) | \( D_5 \) | 1 | 0 | \(+1/2\) | 0 | \(-1/2\) | 0 | 6 | 1 |
| \( D_6 \) | 1 | 0 | \(-1/2\) | 0 | \(+1/2\) | 0 | 1 | 10 |

Table 4: Exotic fractionally charged massless matter states and their \( SU(5) \times U(1) \times U(1)^4 \times SU(4) \times SO(10) \) quantum numbers.

| \( SU(5) \) | \( U(1) \) | \( U(1)_1 \) | \( U(1)_2 \) | \( U(1)_3 \) | \( U(1)_4 \) | \( SU(4) \) | \( SO(10) \) |
|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| \( b_1 \pm \alpha (+\zeta) \) | \( X_1 \) | 1 | \(-5/4\) | \(-1/4\) | \(+1/4\) | \(+1/4\) | \(+1/2\) | 4 | 1 |
| \( X_2 \) | 1 | \(-5/4\) | \(-1/4\) | \(+1/4\) | \(+1/4\) | \(-1/2\) | 3 | 1 |
| \( b_1 + b_4 + b_5 \pm \alpha (+\zeta) \) | \( Y_1 \) | 1 | \(+5/4\) | \(-1/4\) | \(+1/4\) | \(-1/4\) | \(+1/2\) | 4 | 1 |
| \( Y_2 \) | 1 | \(+5/4\) | \(-1/4\) | \(+1/4\) | \(-1/4\) | \(-1/2\) | 4 | 1 |
| \( b_2 + b_3 + b_5 \pm \alpha (+\zeta) \) | \( Z_2 \) | 1 | \(+5/4\) | \(-1/4\) | \(+3/4\) | \(+1/4\) | 0 | 4 | 1 |
| \( Z_2 \) | 1 | \(-5/4\) | \(-3/4\) | \(+1/4\) | \(-1/4\) | 0 | 4 | 1 |
| \( b_1 + b_2 + b_4 \pm \alpha (+\zeta) \) | \( Y_2' \) | 1 | \(+5/4\) | \(-1/4\) | \(+1/4\) | \(-1/4\) | \(-1/2\) | 4 | 1 |
| \( Y_1' \) | 1 | \(-5/4\) | \(+1/4\) | \(-1/4\) | \(+1/4\) | \(-1/2\) | 4 | 1 |
| \( S + b_2 + b_4 \pm \alpha (+\zeta) \) | \( Z_1 \) | 1 | \(-5/4\) | \(+1/4\) | \(+1/4\) | \(-1/4\) | \(+1/2\) | 4 | 1 |
| \( Z_1 \) | 1 | \(+5/4\) | \(-1/4\) | \(-1/4\) | \(+1/4\) | \(-1/2\) | 4 | 1 |
| \( b_3 \pm \alpha (+\zeta) \) | \( X_1 \) | 1 | \(+5/4\) | \(+1/4\) | \(-1/4\) | \(-1/4\) | \(-1/2\) | 4 | 1 |
| \( X_2 \) | 1 | \(-5/4\) | \(-1/4\) | \(+1/4\) | \(+1/4\) | \(-1/2\) | 3 | 1 |
Appendix B  F-flatness condition analysis

In this appendix we consider solutions of the F-flatness equations of a superpotential of the form

\[ w = \overline{\Phi}_{23} \overline{\Phi}_{31} \Phi_{12} + \Phi_{12} \left( \overline{\phi}_1 + \overline{\phi}_2 + \overline{\phi}_4 + \overline{\phi}_+ \overline{\phi}_- \right) \]

\[ + \left( D_1^2 + D_2^2 + T_1^2 \right) \overline{\Phi}_{23} + \left( \alpha_1 \overline{\phi}_1 + \alpha_2 \overline{\phi}_2 + \alpha_3 \overline{\phi}_4 + \alpha_4 \overline{\phi}_+ \overline{\phi}_- \right) D_2^2 \Phi_{31} \]

\[ + \left( \beta_1 \overline{\phi}_1 + \beta_2 \overline{\phi}_2 + \beta_3 \overline{\phi}_4 + \beta_4 \overline{\phi}_+ \overline{\phi}_- \right) D_3^2 \Phi_{31} + \left( \gamma_1 \overline{\phi}_1 + \gamma_2 \overline{\phi}_2 + \gamma_3 \overline{\phi}_4 + \gamma_4 \overline{\phi}_+ \overline{\phi}_- \right) T_1^2 \Phi_{31}, \]

where \( \alpha_i, \beta_i, \gamma_i, i = 1, 2, 3, 4 \) are the coupling constants of the associated fifth order terms. We focus on perturbative solutions assuming \( \Phi_{31}, \overline{\Phi}_{31} \sim \xi \), where \( \xi \) is a small parameter in appropriate mass units. The F-flatness conditions read

\[ \frac{\partial w}{\partial \overline{\Phi}_{23}} = D_1^2 + D_2^2 + T_1^2 + \overline{\Phi}_{31} \Phi_{12} = 0, \quad (B.2) \]

\[ \frac{\partial w}{\partial \Phi_{23}} = 0, \quad (B.3) \]

\[ \frac{\partial w}{\partial \overline{\Phi}_{12}} = \overline{\phi}_1 + \overline{\phi}_2 + \overline{\phi}_4 + \overline{\phi}_+ \overline{\phi}_- + \overline{\Phi}_{23} \Phi_{31} = 0, \quad (B.4) \]

\[ \frac{\partial w}{\partial \Phi_{31}} = \left( \alpha_1 \overline{\phi}_1 + \alpha_2 \overline{\phi}_2 + \alpha_3 \overline{\phi}_4 + \alpha_4 \overline{\phi}_+ \overline{\phi}_- \right) D_2^2 + \left( \beta_1 \overline{\phi}_1 + \beta_2 \overline{\phi}_2 + \beta_3 \overline{\phi}_4 + \beta_4 \overline{\phi}_+ \overline{\phi}_- \right) D_3^2 + \left( \gamma_1 \overline{\phi}_1 + \gamma_2 \overline{\phi}_2 + \gamma_3 \overline{\phi}_4 + \gamma_4 \overline{\phi}_+ \overline{\phi}_- \right) T_1^2 = 0, \quad (B.5) \]

\[ \frac{\partial w}{\partial \overline{\phi}_1} = 2 \overline{\phi}_1 \left[ \Phi_{12} + \Phi_{31} \left( \alpha_1 D_1^2 + \beta_1 D_2^2 + \gamma_1 T_1^2 \right) \right] = 0, \quad (B.6) \]

\[ \frac{\partial w}{\partial \overline{\phi}_2} = 2 \overline{\phi}_2 \left[ \Phi_{12} + \Phi_{31} \left( \alpha_2 D_1^2 + \beta_2 D_2^2 + \gamma_2 T_1^2 \right) \right] = 0, \quad (B.7) \]

\[ \frac{\partial w}{\partial \overline{\phi}_4} = 2 \overline{\phi}_4 \left[ \Phi_{12} + \Phi_{31} \left( \alpha_3 D_1^2 + \beta_3 D_2^2 + \gamma_3 T_1^2 \right) \right] = 0, \quad (B.8) \]

\[ \frac{\partial w}{\partial \overline{\phi}_+} = \overline{\phi}_+ \left[ \Phi_{12} + \Phi_{31} \left( \alpha_4 D_1^2 + \beta_4 D_2^2 + \gamma_4 T_1^2 \right) \right] = 0, \quad (B.9) \]

\[ \frac{\partial w}{\partial \overline{\phi}_-} = \overline{\phi}_- \left[ \Phi_{12} + \Phi_{31} \left( \alpha_4 D_1^2 + \beta_4 D_2^2 + \gamma_4 T_1^2 \right) \right] = 0, \quad (B.10) \]

\[ \frac{\partial w}{\partial D_1} = 2 D_1 \left[ \Phi_{23} + \Phi_{31} \left( \alpha_1 \overline{\phi}_1 + \alpha_2 \overline{\phi}_2 + \alpha_3 \overline{\phi}_4 + \alpha_4 \overline{\phi}_+ \overline{\phi}_- \right) \right] = 0, \quad (B.11) \]

\[ \frac{\partial w}{\partial D_2} = 2 D_2 \left[ \Phi_{23} + \Phi_{31} \left( \beta_1 \overline{\phi}_1 + \beta_2 \overline{\phi}_2 + \beta_3 \overline{\phi}_4 + \beta_4 \overline{\phi}_+ \overline{\phi}_- \right) \right] = 0, \quad (B.12) \]

\[ \frac{\partial w}{\partial D_3} = 2 T_1 \left[ \Phi_{23} + \Phi_{31} \left( \gamma_1 \overline{\phi}_1 + \gamma_2 \overline{\phi}_2 + \gamma_3 \overline{\phi}_4 + \gamma_4 \overline{\phi}_+ \overline{\phi}_- \right) \right] = 0. \quad (B.13) \]
Using the coupling relations

\[
\begin{align*}
-\frac{\alpha_1}{3} &= -\frac{\gamma_1}{3} = -\frac{\beta_1}{2} = \frac{\beta_3}{2} = \frac{\beta_4}{2} = \alpha_4, \\
\alpha_2 &= \alpha_3 = \gamma_2 = \gamma_3 = \gamma_4 = \alpha_4, \quad \beta_2 = 0,
\end{align*}
\]  
\tag{B.14}

Eqs. \((B.4), (B.11), (B.12)\) and \((B.13)\) take the form

\[
\begin{align*}
\phi_1 + \phi_2 + \phi_4 + \phi_+ \phi_- + \Phi_{23} \Phi_{31} &= 0, \\
D_1 \left[ \Phi_{23} + \Phi_{31} \alpha_4 \left( 3\phi_1^2 + \phi_2^2 + 2\phi_4^2 + 4\phi_+ \phi_- \right) \right] &= 0, \\
D_4 \left[ \Phi_{23} + \Phi_{31} \alpha_4 \left( 2\phi_1^2 + 2\phi_2^2 + 4\phi_4^2 + 4\phi_+ \phi_- \right) \right] &= 0, \\
T_1 \left[ \Phi_{23} + \Phi_{31} \alpha_4 \left( 4\phi_1^2 + 3\phi_2^2 + 4\phi_4^2 + 4\phi_+ \phi_- \right) \right] &= 0.
\end{align*}
\]  
\tag{B.16} - (B.19)

Solving \((B.16)\) with respect to \(\bar{\phi}_1\) and substituting into \((B.17)\) and \((B.18), (B.19)\) we get

\[
\begin{align*}
D_1 \left[ \Phi_{23} + \Phi_{31} \alpha_4 \left( 4\phi_1^2 + 4\phi_2^2 + 4\phi_4^2 + 4\phi_+ \phi_- \right) \right] &= 0, \\
D_4 \left[ \Phi_{23} + \Phi_{31} \alpha_4 \left( 2\phi_1^2 + 2\phi_2^2 + 4\phi_4^2 + 4\phi_+ \phi_- \right) \right] &= 0, \\
T_1 \left[ \Phi_{23} + \Phi_{31} \alpha_4 \left( 4\phi_1^2 + 3\phi_2^2 + 4\phi_4^2 + 4\phi_+ \phi_- \right) \right] &= 0,
\end{align*}
\]  
\tag{B.20} - (B.22)

where we have neglected terms of order \(\xi^5\).

Moreover, Eqs. \((B.2), (B.6), (B.7), (B.8), (B.9)\) yield

\[
\begin{align*}
D_1^2 + D_2^2 + T_1^2 + \Phi_{31} \Phi_{12} &= 0, \\
\bar{\phi}_1 \left[ \Phi_{12} - \alpha_4 \Phi_{31} \left( 3D_1^2 + 2D_2^2 + 3T_1^2 \right) \right] &= 0, \\
\bar{\phi}_2 \left[ \Phi_{12} + \alpha_4 \Phi_{31} \left( \Phi_{12} + D_1^2 + T_1^2 \right) \right] &= 0, \\
\bar{\phi}_4 \left[ \Phi_{12} + \alpha_4 \Phi_{31} \left( \Phi_{12} + D_1^2 + 2D_2^2 + T_1^2 \right) \right] &= 0, \\
\bar{\phi}_+ \left[ \Phi_{12} + \alpha_4 \Phi_{31} \left( \Phi_{12} + D_1^2 + 2D_2^2 + T_1^2 \right) \right] &= 0.
\end{align*}
\]  
\tag{B.23} - (B.27)

Solving \((B.23)\) with respect to \(T_1^2\) and substituting into \((B.24)\) - (B.27) we get

\[
\begin{align*}
\bar{\phi}_1 \left[ \Phi_{12} + \alpha_4 \Phi_{31} D_1^2 \right] &= 0, \\
\bar{\phi}_2 \left[ \Phi_{12} - \alpha_4 \Phi_{31} D_1^2 \right] &= 0, \\
\bar{\phi}_4 \left[ \Phi_{12} + \alpha_4 \Phi_{31} D_1^2 \right] &= 0, \\
\bar{\phi}_+ \left[ \Phi_{12} + \alpha_4 \Phi_{31} D_1^2 \right] &= 0.
\end{align*}
\]  
\tag{B.28} - (B.31)
These are not compatible unless $\bar{\phi}_2 \lesssim \xi^2$. In this case
\begin{align}
\mathbf{G}_{12} &= -\alpha_4 D^2 \Phi_{31}, \\
T^2_1 &= -D^2_1 - D^2_4 - \Phi_{31} \Phi_{12},
\end{align}
and the system (B.20)-(B.22) also yields
\begin{align}
\Phi_{23} &= -2 \Phi_{31} \alpha_4 \left( -\bar{\phi}^2_1 + \bar{\phi}^2_4 + \bar{\phi}_+ \bar{\phi}_- \right), \\
\bar{\phi}^2_1 &= -\bar{\phi}^2_4 - \bar{\phi}_+ \bar{\phi}_- - \bar{\phi}_-^2 - \Phi_{23} \Phi_{31}.
\end{align}
Finally, we consider (B.5). Using, (B.11), (B.12), (B.13) all three factors in parentheses equal to $-\frac{\Phi_{23}}{\Phi_{31}}$. That is
\begin{align}
-\frac{\Phi_{23}}{\Phi_{31}} \left( D^2_1 + D^2_4 + T^2_1 \right) = 0 \Rightarrow \frac{\Phi_{23} \Phi_{12} \Phi_{31}}{\Phi_{31}} = 0
\end{align}
which is satisfied at order $\xi^6$ by the use of (B.2). The same holds for (B.3).

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Appendix C  Operator product expansions, correlators etc

The conformal correlator functions appearing in Section 5 can be calculated using techniques explained in [28], [32]. These include correlators of exponentials which can be easily calculated using

\[ \langle \prod_i e^{i\alpha_i \phi(z_i)} \rangle = \prod_{i<j} (z_{ij})^{\alpha_i \alpha_j}, \quad (C.1) \]

the ghost correlator given by

\[ \langle e^{-c/2}(z_1) e^{-c/2}(z_2) e^{-c}(z_3) \rangle = z_{12}^{-1/4} z_{13}^{-1/2} z_{23}^{-1}, \quad (C.2) \]

and the space-time spinor correlator given by

\[ \langle S_a(z_1) S_b(z_2) \rangle = C_{ab} z_{12}^{-1/2}. \quad (C.3) \]

In addition, we use the following Ising type correlators

\[ \langle \sigma_+^{(1)} \sigma_+^{(2)} \rangle = \langle \sigma_-^{(1)} \sigma_-^{(2)} \rangle = |z_{12}|^{-1/4}, \quad (C.4) \]

\[ \langle f(1) f(2) \rangle = \langle \bar{f}(1) \bar{f}(2) \rangle = \frac{1}{z_{12}}, \quad (C.5) \]

\[ \langle \sigma_+^{(1)} \sigma_-^{(2)} f(3) \rangle = \langle \sigma_-^{(1)} \sigma_+^{(2)} \bar{f}(3) \rangle = \frac{e^{i\pi/4}}{\sqrt{2}} z_{12}^{+3/8} z_{13}^{-1/2} z_{23}^{-1/2} z_{12}^{-1/8}, \quad (C.6) \]

\[ \langle \sigma_+^{(1)} \sigma_-^{(2)} \bar{f}(3) \rangle = \langle \sigma_-^{(1)} \sigma_+^{(2)} f(3) \rangle = \frac{e^{-i\pi/4}}{\sqrt{2}} z_{12}^{-3/8} z_{13}^{-1/2} z_{23}^{-1/2} z_{12}^{1/8}, \quad (C.7) \]

\[ \langle \sigma_+^{(1)} \sigma_+^{(2)} \sigma_+^{(3)} \sigma_+^{(4)} \rangle = \frac{1}{\sqrt{2}} |z_{13} z_{24}|^{1/4} |x|^{1/4} |1-x|^{1/4} (|x| + |1-x| + 1)^{1/2}, \quad (C.8) \]

\[ = \frac{1}{\sqrt{2}} |z_{13} z_{24}|^{-1/4} |x(1-x)|^{1/4} (|x| + |1-x| + 1)^{1/2}, \quad (C.9) \]

\[ \langle \sigma_+^{(1)} \sigma_+^{(2)} \sigma_-^{(3)} \sigma_-^{(4)} \rangle = \frac{1}{\sqrt{2}} |z_{12} z_{34}|^{-1/4} |1-x|^{-1/4} (|1-x| - |x| + 1)^{1/2}, \quad (C.10) \]

\[ \langle \sigma_+^{(1)} \sigma_+^{(2)} f(3) f(4) \rangle = \frac{1}{2} \frac{|z_{12}|^{-1/4}}{z_{34}} \frac{2-x}{\sqrt{1-x}}, \quad (C.11) \]

\[ \langle \sigma_+^{(1)} \sigma_+^{(2)} \bar{f}(3) \bar{f}(4) \rangle = \frac{1}{2} \frac{|z_{12}|^{-1/4}}{z_{34}} \frac{2-\bar{x}}{\sqrt{1-\bar{x}}}, \quad (C.12) \]

where

\[ x = \frac{z_{12} z_{34}}{z_{13} z_{24}}. \quad (C.13) \]
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