SYMPLECTIC RESOLUTIONS FOR MULTIPLICATIVE QUIVER VARIETIES AND CHARACTER VARIETIES FOR PUNCTURED SURFACES

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Abstract. We study the algebraic symplectic geometry of multiplicative quiver varieties, which are moduli spaces of representations of certain quiver algebras, introduced by Crawley-Boevey and Shaw [CBS06], called multiplicative preprojective algebras. They are multiplicative analogues of Nakajima quiver varieties. They include character varieties of (open) Riemann surfaces fixing conjugacy class closures of the monodromies around punctures, when the quiver is “crab-shaped”. We prove that, under suitable hypotheses on the dimension vector of the representations, or the conjugacy classes of monodromies in the character variety case, the normalisations of such moduli spaces are symplectic singularities and the existence of a symplectic resolution depends on a combinatorial condition on the quiver and the dimension vector. These results are analogous to those obtained by Bellamy and the first author in the ordinary quiver variety case, and for character varieties of closed Riemann surfaces. At the end of the paper, we outline some conjectural generalisations to moduli spaces of objects in 2-Calabi–Yau categories.

1. Introduction

1.1. Motivation. This paper is devoted to the study of the symplectic algebraic geometry of coarse moduli spaces of (semistable) representations of multiplicative preprojective algebras, which can be thought of as multiplicative analogues of Nakajima quiver varieties [Nak94], which includes character varieties of (open) Riemann surfaces. In particular, our attention is focused on tackling two main problems: the first is to understand whether these multiplicative quiver varieties are symplectic singularities, as defined by Beauville in [Bea00]; the second is to classify all the possible cases in which they admit symplectic resolutions. Multiplicative preprojective algebras were first defined by Crawley-Boevey and Shaw in [CBS06], with the aim of better understanding Katz’s middle convolution operation for rigid local systems, [Kat96]. Another important application contained in the seminal paper [CBS06] is the solution of (one direction of) the multiplicative Deligne-Simpson problem in terms of the root data of a certain star-shaped quiver. Moduli spaces of representations, in the sense of King [Kin94], of these algebras give rise to the so-called multiplicative quiver varieties. Ordinary (Nakajima) quiver varieties have appeared in numerous places in representation theory, algebraic geometry, and mathematical physics; their homology theories are closely related to the representation theory of Kac–Moody Lie algebras [Nak94], and their quantum cohomology is closely related to quantum R-matrices and the Casimir connection [MO12]. A number of authors have studied multiplicative quiver varieties since their definition: among others, Jordan [Jor14] considered quantisations of such varieties from a representation theoretic point of view by constructing flat q-deformations of the algebra of differential operators on certain affine spaces; a more geometric approach was used by Yamakawa in [Yam08], where a symplectic structure on these moduli spaces was defined and studied. Some of these results will be recalled in the next sections of the present paper. More recently, (derived) multiplicative preprojective algebras appeared in the study of wrapped Fukaya categories of certain Weinstein 4-manifolds constructed by plumbing cotangent bundles of Riemann surfaces: see [EL17]. In the recent work of Chalykh and Fairow [CF17], multiplicative quiver varieties are used to construct a new integrable system generalising the Ruijsenaars–Schneider system, which plays a central role in supersymmetric gauge theory and cyclotomic DAHAs. Moreover, in work of McBreen and Webster [MW18], and in their ongoing work together with Gammage, related to [BK16 §7], mirror symmetry is studied for multiplicative hypertoric varieties, which include multiplicative quiver varieties when the dimensions are one. Given their appearance in so many different contexts, it seems natural to perform a careful analysis on multiplicative quiver varieties from the point of view of symplectic algebraic geometry.
The subject of symplectic resolutions and the more general symplectic singularities (the latter dating to Beauville [Bea00]) has recently gained importance in many areas of mathematics and physics. Their quantisations subsume many of the important examples of algebras appearing in representation theory (Cherednik and symplectic reflection algebras, D-modules on flag varieties and representations of Lie algebras, quantised hypertoric and quiver algebras, etc.). There is a growing theory of symplectic duality, or three-dimensional physical mirror symmetry ([BN] [Nak16] [CHZ14] [BLPW16] and many others), between pairs of these varieties. Pioneering work of Braverman, Maulik and Okounkov [BMO11] (continued in the aforementioned [MO12] and in many other places) show that their quantum cohomology is also deeply tied to connections arising in representation theory, related to derived autoequivalences of duals in the sense of homological mirror symmetry (two-dimensional field theories). Since, as mentioned before, quiver varieties play such an important role here, it is expected that multiplicative quiver varieties will as well. Moreover, the varieties in question are instances of moduli spaces parametrising geometric objects. The study of such spaces and their singularities is, in general, important in algebraic geometry.

For all of these reasons, it is natural to ask when multiplicative quiver varieties have symplectic singularities and admit symplectic resolutions. We largely answer these questions, leaving a couple cases (the so-called "(2,2)"-cases related to O’Grady’s examples [OG99], and the so-called isotropic cases, which are multiplicative analogues of symmetric powers of du Val singularities, that appear to require local structure theory. Our methods generalise those of [BS16a], which we largely follow. They build on Crawley-Boevey and Shaw’s pioneering work on multiplicative quiver varieties (and extensions by Yamakawa [Yam08]), and apply (as in [BS16a]) Drezet’s factoriality criteria [Dre91] and Flenner’s theorem [Fle88] on extendability of differential forms beyond codimension four.

1.2. Summary of results on character varieties. Since they are the easiest to state and perhaps of the broadest interest, we first explain the results on character varieties that follow from our considerations on multiplicative quiver varieties. Fix a connected compact Riemann surface $X$ of genus $g \geq 0$, let $S = \{p_1, \ldots, p_k\} \subset X$ be a subset of $k \geq 0$ points, and fix a tuple $C = (C_1, \ldots, C_k)$ of conjugacy classes $C_i \subset GL_n(\mathbb{C})$, $i = 1, \ldots, k$. Let $X^\circ := X \setminus \{p_1, \ldots, p_k\}$ be the corresponding punctured surface, and let $\gamma_i$ be the homotopy class in $\pi_1(X^\circ)$ of some choice of loop around the puncture $p_i$ (having the same free homotopy class as a small counterclockwise loop around $p_i$). We define the character variety of $X^\circ$ with monodromies in $\mathbb{C}$ as follows:

\begin{equation}
\mathcal{X}(g, k, \mathbb{C}) := \{ \chi : \pi_1(X^\circ) \to GL_n \mid \chi(\gamma_i) \in \mathbb{C}\}.
\end{equation}

As recalled in Section 5 below, this has the structure of an affine algebraic variety. Note that $X$ (or $X^\circ$) does not appear in the notation on the left-hand side, since the result does not depend on the choice of $X$ up to isomorphism (only the identification of $\pi_1(X^\circ)$ is relevant).

Observe that, in order for this character variety to be nonempty, we must have $\prod_{i=1}^k \det(C_i) = 1$, where we let $\det(C_i)$ be defined as the determinant of any element of $C_i$. Let us assume this from now on. Given $m \geq 1$ we let $m \cdot C = (C_1^{m}, \ldots, C_k^{m})$. We call $C$ $q$-divisible if $C = m \cdot C'$ for $m \geq 2$ and $\prod_{i=1}^k \det(C_i') = 1$. Call it $q$-indivisible if it is not $q$-divisible. Below, $q$-indis四个意识 will be the most important criterion for the existence of symplectic resolutions for $\mathcal{X}(g, k, \mathbb{C})$.

For each $C_i$, let the minimal polynomial of any $A \in C_i$ be $(x - \xi_{i,1}) \cdots (x - \xi_{i,w_i})$, ordered so that the sequence $\alpha_{i,j} := \text{rank}(A - \xi_{i,j}) \cdots (A - \xi_{i,j})$ has the property that $\alpha_{i,j} - \alpha_{i,j+1}$ is non-increasing in $j$ (for $0 \leq j \leq w_i - 1$, setting $\alpha_{i,0} = n$). This is possible since the non-increasing property obviously holds when all the $\xi_{i,j}$ are equal. The following quantities will have importance for us:

\begin{equation}
\ell := \sum_{i=1}^k w_i - 1, \quad p(\alpha) := 1 + n^2(g - 1) + n\ell + \sum_{i=1}^k \sum_{j=1}^{w_i - 1} \alpha_{i,j} \alpha_{i,j+1} - \sum_{i=1}^k \sum_{j=1}^{w_i} \alpha_{i,j}^2.
\end{equation}

The quantity $2p(\alpha)$ is the “expected dimension” of the character variety, which is its actual dimension in many cases, as explained below.

The sequences $\alpha_{i,j}$, together with the rank $n$, can naturally be arranged on a star-shaped undirected graph, call it $G_C$, with node labeled by the rank, $n$, and the vertices of the $i$-th branch labeled $\alpha_{i,1}, \ldots, \alpha_{i,w_i-1}$, in order going away from the node. (The crab-shaped quiver associated to the variety, which will be important later on, is obtained by orienting the edges of $G_C$ and then adding $g$ loops at the node.)
Our main results on character varieties can be summarised as follows. We divide separately into the genus 0 and the positive genus cases.

Recall here that a symplectic singularity is a normal variety \( X \) equipped with a symplectic structure \( \omega_{\text{reg}} \) on the smooth locus \( X_{\text{reg}} \) such that, for any (or equivalently every) resolution of singularities \( \rho : \tilde{X} \to X \), \( \rho^* \omega_{\text{reg}} \) extends to a regular two-form \( \tilde{\omega} \in \Omega^2(\tilde{X}) \). The map \( \rho \) is furthermore a symplectic resolution if \( \tilde{\omega} \) is non-degenerate.

**Theorem 1.1.** Let \( g = 0 \) and fix \( n \) and conjugacy classes \( C_1, \ldots, C_k \subseteq \text{GL}_n(\mathbb{C}) \) as above.

- If \( \ell < 2n \), then one of the following exclusive possibilities occur, and can be computed by an explicit algorithm:
  - \( X(0, k, \overline{C}) \) is empty;
  - \( X(0, k, \overline{C}) \) is a point;
  - There is a canonical datum \((n', k', C_1', \ldots, C_k', \iota)\) of \( n' < n \), \( k' \leq k \), and conjugacy classes \( C_1', \ldots, C_k' \subseteq \text{GL}_{n'}(\mathbb{C}) \) such that \( \ell' \) (defined as above) satisfies \( \ell' \geq 2n' \), and an isomorphism
    \[ \iota : X(0, k', \overline{C}) \to X(0, k, \overline{C}). \]

  Suppose, therefore, that \( \ell \geq 2n \).

- If \( C \) is \( q \)-indivisible, then \( X(0, k, \overline{C}) \) admits a projective symplectic resolution (via geometric invariant theory). Therefore, its normalisation is a symplectic singularity. Moreover, \( \dim X(0, k, \overline{C}) = 2p(\alpha) \).

- Suppose that \( C \) is \( q \)-divisible. Then, unless one of the following conditions is satisfied, the normalisation of \( X(0, k, \overline{C}) \) is a symplectic singularity which does not admit a symplectic resolution (in fact, it contains a singular terminal factorial open subset):
  
  (a) the graph and labeling \( \alpha \) are twice one of those listed in Theorem 7.7 below (note in this case \( k \in \{3, 4, 5\} \) and \( n \leq 24 \));
  
  (b) the graph \( C_G \) is affine Dynkin and the labeling \( \alpha = m\delta \) is an imaginary root for this graph (here \( k \in \{3, 4\} \));
  
  (c) The pair \((C_G, \alpha)\) is obtained from an affine Dynkin one, \((C'_G, m\delta)\), appearing in the previous case, by attaching single vertex, at which \( \alpha \) has a prime value \( p \) dividing the collection \( C'_G \), which is attached to an extending vertex of \( C'_G \) (i.e., one with \( \delta \) valued at one). Again, here \( k \in \{3, 4\} \).

  Moreover, in all cases except case (b) above, \( \dim X(0, k, \overline{C}) = 2p(\alpha) \).

As mentioned in the theorem, the technique used to show non-existence of symplectic resolutions is by identifying an open singular factorial terminal subset. It is well-known that singular factorial terminal varieties cannot admit crepant resolutions (and hence not symplectic resolutions) by Van der Waerden purity (by factoriality, any crepant resolution would have to have exceptional locus which is a divisor, but by terminality, any crepant resolution has exceptional locus of codimension at least two).

**Remark 1.2.** Regarding the excluded cases of the theorem, we have some expectations: In case (a), conjecturally a symplectic resolution can be obtained by first applying geometric invariant theory and then blowing up the singular locus; this would follow from Conjecture 7.16. We don’t make conjectures for cases (b) and (c), but in the analogous ordinary (Nakajima) quiver varieties, case (b) does admit a symplectic resolution (via framing the quiver and then applying geometric invariant theory), whereas case (c) does not admit one (by [BS16a]).

In case (c) above, it actually follows from the proof of the theorem that the normalisation of \( X(0, k, C) \) is a symplectic singularity, since applying geometric invariant theory results in a partial crepant resolution by a singular factorial terminal variety. (Note that this partial crepant resolution itself cannot admit a symplectic resolution, but we cannot deduce the same property for \( X(0, k, C) \) itself, as such an implication is only known for conical varieties, by [BS16b] Theorem 2.2)).

**Remark 1.3.** Note that, when \( k \leq 2 \), the character variety is always a point (or empty).

**Theorem 1.4.** Suppose that \( g \geq 1 \). Then the following holds:

- If \( C \) is \( q \)-indivisible, then \( X(g, k, \overline{C}) \) admits a projective symplectic resolution (via geometric invariant theory). Therefore, its normalisation is a symplectic singularity. Moreover, it has dimension \( 2p(\alpha) \).
If $C$ is $q$-divisible, then unless one of the following conditions is satisfied, then the normalisation of $\mathcal{X}(g, k, C)$ is a symplectic singularity which does not admit a symplectic resolution (in fact, it contains a singular terminal factorial open subset):

(a) $g = 2$, $k = 0$, and $n = 2$;
(b) $g = 1$, $k = 0$;
(c) $g = 1$, $k = 1$, $w_1 = 2$, and $\alpha_{1,1} = p$, with $p$ prime.

Moreover, in all cases except case (b), $\dim \mathcal{X}(g, k, C) = 2p(\alpha)$.

Remark 1.5. The cases (a), (b), and (c) are analogous to those in Theorem 1.1. However, here, in cases (a) and (b), a symplectic resolution is known to be obtained by the procedures described in Remark 1.2, see, e.g., [BS16a, Section 8]. In case (c), on the other hand, by analogy with the case of Nakajima quiver varieties (see [BS16a]), we expect a symplectic resolution not to exist, but we leave the case open. As in Remark 1.2, we know that the normalisation of the variety in case (c) is a symplectic singularity (it follows from the proof of the theorem).

The proof of these theorems is given in Section 6.7, they are consequences of the main results of the paper on multiplicative quiver varieties (particularly Corollary 6.27).

Remark 1.6. Actually, the results above (slightly modified) should apply not merely to character varieties, but to twisted character varieties, where we replace $\pi_1(X^*)$ by a finite central extension, corresponding to setting the relation $\prod_{i=1}^k [A_i, B_i] \prod_{j=1}^k M_j = 1$ to be a root of unity times the identity matrix. To prove such a statement would require a straightforward generalisation of [CB13] and of Section 3 below. With this in hand, these results would follow from Corollary 6.27 just as before. For some more details, see Section 1.6 of the introduction, where we describe roughly how to translate this corollary into the setting of twisted character varieties.

1.3. Multiplicative quiver varieties with special dimension vectors. Recall that a quiver $Q$ is a directed graph. We let $Q_0$ denote the set of vertices and $Q_1$ the set of arrows (=edges). Given $Q$ together with a tuple of non-zero complex numbers $q \in (\mathbb{C}^\times)^{Q_0}$, one can define the multiplicative preprojective algebra $\Lambda_q(Q)$, over the semisimple ring $\mathbb{C}^{Q_0}$ (see Section 2.2 below). To a representation, we associate a dimension vector in $\mathbb{N}^{Q_0}$. Given furthermore a stability parameter $\theta \in \mathbb{Z}^{Q_0}$, one can define a variety, denoted $\mathcal{M}_{q,\theta}(Q, \alpha)$, which is a coarse moduli space of $\theta$-semistable representations of $\Lambda_q(Q)$ of dimension vector $\alpha$. It is natural to ask what the dimension vectors of $\theta$-stable representations are. Towards this end, one considers a combinatorially-defined subset $\Sigma_{q,\theta} \subseteq \mathbb{N}^{Q_0}$ of the set of all possible dimension vectors (defined in Section 2.2 below). It has the property that, for $\alpha \in \Sigma_{q,\theta}$, the $\theta$-stable locus is dense in $\mathcal{M}_{q,\theta}(Q, \alpha)$ (and it is always open). However, it is unknown in general if $\mathcal{M}_{q,\theta}(Q, \alpha)$ is non-empty. It is expected, but not known, that these conversely describe all dimension vectors of stable representations, i.e.:

(*) If there is a $\theta$-stable representation of $\Lambda_q(Q)$ of dimension $\alpha \in \mathbb{N}^{Q_0}$, then $\alpha \in \Sigma_{q,\theta}$.

In the case $\theta = 0$, Crawley-Boevey pointed out a work in progress with Hubery towards a proof of (*) (thanks to Crawley-Boevey for pointing this out). We prove a weakened version of (*) below (Corollary 0.18), replacing $\Sigma_{q,\theta}$ by a larger set. Note that, if (*) holds and furthermore $\mathcal{M}_{q,\theta}(Q, \alpha) \neq \emptyset$ for all $\alpha \in \Sigma_{q,\theta}$, then put together we would obtain a characterisation of the set $\Sigma_{q,\theta}$: in this case, $\alpha \in \Sigma_{q,\theta}$ if and only if there exists a $\theta$-stable representation of dimension $\alpha$. However, this is, again, unknown.

To define $\mathcal{M}_{q,\theta}(Q, \alpha)$, we require $\alpha \cdot \theta = 0$, and for it to be non-empty, we require that $q^\alpha := \prod_{i \in Q_0} q_i^{\alpha_i} = 1$. Let $N_{q,\theta} := \{ \alpha \in \mathbb{N}^{Q_0} \mid q^\alpha = 1, \alpha \cdot \theta = 0 \}$. We call a vector $\alpha \in N_{q,\theta}$ $q$-indivisible if $\frac{1}{q} \alpha \notin N_{q,\theta}$ for any $m \geq 2$. Equivalently, writing $\alpha = m \beta$ for $m = \gcd(\alpha_i)$, we have that $q^\beta$ is a primitive $m$-th root of unity.

Note that, if $\alpha \in N_{q,\theta}$ is indivisible, it is clearly $q$-indivisible, although the converse does not hold in general. (Unlike in the case of character varieties, here the $q$ in “$q$-(in)divisible” refers to an actual parameter; see Remark 1.12 for an explanation how the two notions nonetheless coincide.)

We denote by $p$ the following function:

$$p : \mathbb{N}^{Q_0} \to \mathbb{Z}, \quad p(\alpha) = 1 - \frac{1}{2} (\alpha, \alpha) \geq 0,$$

where $(-, -)$ denotes the Cartan–Tits form associated to the quiver $Q$ (see Section 2.1 for more details). Geometrically, $2p(\alpha)$ gives the “expected dimension” of $\mathcal{M}_{q,\theta}(Q, \alpha)$ (which is the actual dimension if $\alpha \in \Sigma_{q,\theta}$.
and $\mathcal{M}_{q,\theta}(Q, \alpha) \neq \emptyset$: see Remark 2.23 below). If $p(\alpha) = 1$, i.e., $(\alpha, \alpha) = 0$, then $\alpha$ is called isotropic. Otherwise it is called anisotropic.

One of the main results of this paper, proved in Section 4, is the following:

**Theorem 1.7.** Let $\alpha \in \Sigma_{q,\theta}$ and assume that $\alpha \neq 2\beta$ for $\beta \in N_{q,\theta}$ and $p(\beta) = 2$. Then, assuming it is non-empty, $\mathcal{M}_{q,\theta}(Q, \alpha)$ satisfies the following:

- its normalisation is a symplectic singularity;
- if $\alpha$ is $q$-indivisible, then for suitable generic $\theta'$, it admits a symplectic resolution of the form $\mathcal{M}_{q,\theta'}(Q, \alpha) \to \mathcal{M}_{q,\theta}(Q, \alpha)$;
- if $\alpha = m\beta$ for $\beta \in \Sigma_{q,\theta}$ and $m \geq 2$, and $\mathcal{M}_{q,\theta}(Q, \beta) \neq \emptyset$, then $\mathcal{M}_{q,\theta}(Q, \alpha)$ does not admit a symplectic resolution. Moreover, for suitable generic $\theta'$, $\mathcal{M}_{q,\theta'}(Q, \alpha)$ is a singular factorial terminalisation. In fact, $\mathcal{M}_{q,\theta}(Q, \alpha)$ itself contains a singular, factorial, terminal open subset.

Implicit in Theorem 1.7 is the fact (see Lemma 2.16 and Corollary 2.24 below) that, for all $\alpha \in \Sigma_{q,\theta}$, $\mathcal{M}_{q,\theta}(Q, \alpha) \to \mathcal{M}_{q,\theta}(Q, \alpha)$ is a projective birational Poisson morphism for suitable $\theta'$. This implies, by definition, that it is a symplectic resolution if the source is smooth symplectic. In the last part of the theorem, by singular factorial terminalisation, we mean a projective birational Poisson morphism with source a singular factorial terminal variety.

In the case of generic $\theta$, the theorem can be simplified as follows, avoiding the need to check if a vector is in $\Sigma_{q,\theta}$. First, note that $\Sigma_{q,\theta}$, by definition, is a subset of the set of roots for the quiver (which equals the set of roots of the associated Kac-Moody Lie algebra in suitable cases). The real roots are those vectors obtained from elementary vectors $e_i, i \in Q_0$ by simple reflections $\alpha \mapsto \alpha - (\alpha, e_i)e_i$; the imaginary roots are those obtained by such reflections from nonnegative (or non-positive) vectors with connected support and nonpositive Cartan pairing with each $e_i$.

**Corollary 1.8.** Fix an imaginary root $\alpha$ for $Q$. Let $q$ be such that $q^\alpha = 1$. Let $\theta$ be generic (inside the hyperplane $\{\theta \cdot \alpha = 0\}$). Then:

(i) We have $\alpha \in \Sigma_{q,\theta}$ if and only if $\alpha$ is $q$-indivisible or anisotropic. If $\alpha$ is $q$-indivisible, $\mathcal{M}_{q,\theta}(Q, \alpha)$ is smooth symplectic.

(ii) Assume $\alpha$ is $q$-divisible and anisotropic. Moreover, assume that $\alpha \neq 2\beta$ for $p(\beta) = 2$ and $q^\theta = 1$. Then $\mathcal{M}_{q,\theta}(Q, \alpha)$ is a (normal) symplectic singularity.

(iii) Under the assumptions of (ii), we have the following:

- If there exists a $\theta$-stable representation of $\Lambda^r(Q)$ of dimension $\frac{1}{m}\alpha$, for some $m \geq 2$, then $\mathcal{M}_{q,\theta}(Q, \alpha)$ is singular, factorial, and terminal, and hence does not admit a symplectic resolution.

- If, on the other hand, there are no $\theta$-stable representations of $\Lambda^r(Q)$ of dimension $r\alpha$ for all rational $r < 1$, then $\mathcal{M}_{q,\theta}(Q, \alpha)$ is smooth.

Note that, in the general case with $\alpha \in \Sigma_{q,\theta}$, the above corollary always describes the source of projective birational Poisson morphisms obtained by suitably varying $\theta$.

**Remark 1.9.** Note that every $r\alpha, r \in \mathbb{Q}_{<1}$ appearing in the theorem is also in $\Sigma_{q,\theta}$, by part (i). Thus, if there exists a $\theta$-stable representation of every dimension in $\Sigma_{q,\theta}$, then in part (iii) we are necessarily in the first case. This condition holds in the additive case (with $\lambda \in \mathbb{R}^{Q_0}$), by [BS16a], but we don’t have any other evidence that this holds here. Also, note that the two cases are not exhaustive, so it could happen that there are some stable representations of dimension $r\alpha$ but not when $r = \frac{1}{m}$. In this (unexpected) situation, it would require more detailed analysis to determine whether a symplectic resolution exists.

**Remark 1.10.** In the case left out of the theorem, where $\alpha = 2\beta$ for some $\beta \in N_{q,\theta}$ satisfying $p(\beta) = 2$ (we call this the “$(2,2)$-case”), we conjecture, as in the special case of character varieties of rank two local systems on genus two surfaces handled in [BS16a], that $\mathcal{M}_{q,\theta}(Q, \alpha)$ has a symplectic resolution obtained by blowing up the singular locus of $\mathcal{M}_{q,\theta'}(Q, \alpha)$, for suitably generic $\theta'$. However, in order to prove this, it is necessary to understand the étale local structure of $\mathcal{M}_{q,\theta'}(Q, \alpha)$, while at the moment all of our techniques are global in nature. In Section 4 we discuss an approach to understand the local structure of the multiplicative quiver varieties, based on the conjectural 2-Calabi–Yau property of the multiplicative preprojective algebra for non-Dynkin quivers.
Remark 1.11. Note that, as part of Corollary 1.8 when \( \theta \) is generic (and \( \alpha \in \Sigma_{q, \theta} \)), we prove normality of the variety \( \mathcal{M}_{q, \theta}(Q, \alpha) \), see Proposition 1.10. Moreover, we conjecture normality for all \( \theta \) (as well as for the \((2, 2)\)-case). Such a result requires a local understanding of the varieties \( \mathcal{M}_{q, \theta}(Q, \alpha) \), which would again follow from the conjectural 2-Calabi–Yau property for \( \Lambda^q(Q) \) when \( Q \) is not Dynkin: see Section 7.

1.4. Character varieties as (open subsets of) multiplicative quiver varieties. In Section 3, extending results of CBS00 and Yam08, we explain how character varieties identify as natural open subsets of the multiplicative quiver varieties for crab-shaped quivers (Theorem 8.6). Namely, the character variety identifies as an open subset of a multiplicative quiver variety for the crab-shaped quiver described in Section 1.2 with appropriate parameter \( q \in (\mathbb{C}^\times)^{Q_0} \). The open subset is defined by requiring the loops to act invertibly (which means that, in the genus zero case, the character variety equals the multiplicative character variety).

Remark 1.12. By the above correspondence, a collection of conjugacy classes \( C \subseteq GL_n(\mathbb{C}^\times) \) is \( q \)-divisible in the sense of Section 1.2 (where \( q \) is not yet a parameter) if and only if, for the associated quiver \( Q \), dimension vector \( \alpha \in \mathbb{N}^{Q_0} \), and parameter \( q \in (\mathbb{C}^\times)^{Q_0} \), the vector \( \alpha \) is \( q \)-divisible in the sense of Section 1.3. We hope that this abuse of notation aids understanding.

It is then an interesting question which character varieties exhibit the different properties discussed above, in particular, which ones are the \((2, 2)\)-cases where an “O’Grady” type resolution is expected? (We already mentioned these cases in Theorems 1.1 and 1.4 under (a)). The classification is achieved in Theorems 5.1 and 5.3: all of these are in the genus zero case (i.e., they are star-shaped quivers), with three to five punctures and particular monodromy, as classified in Theorem 5.1, except for two cases in Theorem 5.3. The latter cases correspond to once-punctured tori and to closed genus two surfaces (with even rank and rank two local systems, respectively, the former having particular monodromy about the puncture).

1.5. General dimension vectors. Although it is difficult to study directly quiver varieties of dimensions \( \alpha \notin \Sigma_{q, \theta} \), in the additive setting this issue is alleviated by Crawley-Boevey’s canonical decomposition, expressing an arbitrary variety as a product of varieties for dimension vectors in \( \Sigma_{q, \theta} \) (CBS00, Theorem 1.1) (extended to \( \theta \neq 0 \) in BS16a, Proposition 2.1). In Theorem 6.17 below, we provide a version of this decomposition in the multiplicative setting using reflection functors, following the proof of CBS00, which is weaker in the sense that the dimension vectors of the factors need not be in \( \Sigma_{q, \theta} \), and hence the factors could further decompose (although it is not known in general if they do). One of the reasons why we must give the weaker statement is the unavailability of (*) for a certain multiplicative moment map is flat; \( \Sigma_{q, \theta} \subseteq \Sigma_{q, \theta} \) is the subset of isotropic roots. See Sections 2 and 6 for details on these definitions.

Theorem 1.13. Assume that \( \mathcal{M}_{q, \theta}(Q, \alpha) \) is non-empty.

(1) There is a unique decomposition \( \alpha = \beta^{(1)} + \cdots + \beta^{(k)} \) with \( \beta^{(i)} \in \Sigma_{q, \theta} \cup \mathbb{N}_{\geq 2} \cdot \Sigma_{q, \theta}^{iso} \), satisfying the properties:
- (a) When \( p(\beta^{(i)}) = 0 \) then \( \beta^{(i)} \in \Sigma_{q, \theta} \);
- (b) Every other element \( \beta^{(i)} \) has the property \( \beta^{(i)} \cdot \gamma \leq 0 \) for every real root \( \gamma \in \mathbb{Q}_{q, \theta} \);
- (c) Any other decomposition into \( \Sigma_{q, \theta} \cup \mathbb{N}_{\geq 2} \cdot \Sigma_{q, \theta}^{iso} \) satisfying (a) and (b) is a refinement of this one.

(2) The direct sum map produces an isomorphism (of reduced varieties), using the decomposition in (1),

\[
\prod_{i=1}^{k} \mathcal{M}_{q, \theta}(Q, \beta^{(i)}) \cong \mathcal{M}_{q, \theta}(Q, \alpha).
\]

(3) Assume that this decomposition has neither elements \( \beta^{(i)} \in \mathbb{N}_{\geq 2} \cdot \Sigma_{q, \theta}^{iso}(Q, \alpha) \) nor \( \beta^{(i)} = 2\alpha \) for \( \alpha \in \mathbb{N}_{q, \theta} \) and \( p(\alpha) = 2 \). Then:
- The normalisation of \( \mathcal{M}_{q, \theta}(Q, \alpha) \) is a symplectic singularity;
- Each factor \( \mathcal{M}_{q, \theta}(Q, \beta^{(i)}) \) with \( \beta^{(i)} \notin \Sigma_{q, \theta} \) admits a symplectic resolution.
If for any factor $\beta(i)$ there exists a $\theta$-stable representation of dimension $\gamma(i) = \frac{1}{m}\beta(i)$ with $m \geq 2$, then $M_{q,\theta}(Q, \alpha)$ does not admit a symplectic resolution. In fact, it has an open, singular, terminal, factorial subset.

Putting everything together, in Corollary 6.27 we are able to give a classification of crab-shaped settings whose multiplicative quiver varieties admit symplectic resolutions. By Theorem 3.6 we also deduce the corresponding statement for character varieties (Theorems 1.1 and 1.4), which are open subsets of these varieties, for $\theta = 0$ and for certain values of the parameter $q$. To state the result, first recall that the Jordan quiver is the quiver with one vertex and one arrow (a loop). The fundamental region $F(Q)$ consists of those nonzero vectors $\alpha \in \mathbb{N}^{Q_0}$ with connected support and with $(\alpha, e_i) \leq 0$ for all $i$. As we explain below, by applying certain reflection functors, we can reduce to this case. We give a simplified version of the statement of Corollary 6.27 below; see the full statement for precise details.

**Corollary 1.14.** Let $Q$ be a crab-shaped quiver and $\alpha \in N_{q,\theta}$ a vector in the fundamental region with $\alpha_i > 0$ for all $i \in Q_0$. Further assume that $(Q, \alpha)$ is not one of the following cases:

(a) $\beta := \frac{1}{m}\alpha$ is integral, $q^3 = 1$, and $(Q, \beta)$ is one of the quivers in Theorem 5.1 and Theorem 5.3

(b) $Q$ is affine Dynkin of type $A_0$ (i.e., the Jordan quiver with one vertex and one arrow), $D_4$ or $E_6, E_7, E_8$ and $\alpha$ is a $q$-divisible multiple of the indivisible imaginary root $\delta$ of $Q$.

Then:

- The normalisation of $M_{q,\theta}(Q, \alpha)$ is a symplectic singularity;
- If $\alpha$ is $q$-indivisible, $M_{q,\theta}(Q, \alpha)$ admits a symplectic resolution;
- If $\alpha$ is $q$-divisible, and $\alpha$ is not: (c) a prime multiple of one of the quivers listed in Theorem 6.10 (b2) below (a framed affine Dynkin quiver with dimension vector $(1, m\delta)$ with $m\delta$ $q$-divisible) with $\theta \cdot \delta = 0$, then $M_{q,\theta}(Q, \alpha)$ does not admit a symplectic resolution (it contains an open singular factorial terminal subset).

Thus, after reducing to the fundamental region, a symplectic resolution exists if and only if the dimension vector is $q$-indivisible, unless we are in one of the following three open cases:

(a) twice one of the dimension vectors appearing in Theorems 5.1 and 5.3 below, which correspond to one of certain (twisted) character varieties of a sphere with 3 to 5 punctures (with rank at most 24), of a once-punctured torus (of rank 4) or a closed genus two surface (of rank 2);

(b) a $q$-divisible imaginary root on an extended Dynkin quiver (of type $A_0, D_4, E_6, E_7, E_8$), which corresponds to either a (twisted) character variety of a closed torus (type $A$), or a (twisted) character variety of a sphere with 3 or 4 punctures (type $E$ or $D$, respectively) with particular rank and monodromy conditions;

(c) a prime multiple of the vector $(1, \delta)$ on a framed affine Dynkin quiver (again of type $A_0, D_4, E_6, E_7, E_8$) with $\theta \cdot \delta = 0$, which corresponds again to a certain character variety of a once-punctured torus or a sphere with 3 or 4 punctures.

Here when we say “correspond”, we mean precisely that, for $\theta = 0$, the multiplicative quiver varieties equal the given (twisted) character varieties, whereas for the genus $\geq 1$ case, the latter is the open subset of the former where the transformations corresponding to loops are invertible. Setting $\theta \neq 0$ gives a partial resolution (which may be an actual one, as in case of $\theta$ generic and $\alpha$ $q$-indivisible).

**Remark 1.15.** For the ordinary (untwisted) character varieties of closed genus one or two surfaces appearing in the lists for cases (b) and (a) above, a symplectic resolution exists, as we already noted in Section 1.2. The proof makes use of Poincaré–Verdier duality for closed surfaces; perhaps a suitable generalisation of this for orbifolds would allow us to extend those results to the orbifold case. If so, we could remove $A_0$ from case (b) and the quiver with one vertex and two loops from (a).

**1.6. Outline of the paper.** The outline of the paper is as follows: in Section 2 we recall some basic facts about quivers and root systems and establish the notation that shall be used throughout the paper. We then recall the definition of multiplicative preprojective algebras and outline some of their algebraic properties. These are needed in the construction, via Geometric Invariant Theory (GIT), of their moduli spaces of semistable representations, following [Kin94]. In Definition 2.5 we introduce the subset $\Sigma_{q,\theta}$ appearing in the theorem above. There are a number of properties of multiplicative quiver varieties with dimension vector
in $\Sigma_{q,\theta}$ that were originally formulated and proved in [CBS06] in the case of a trivial stability condition, i.e., $\theta = 0$, and that have been extended in the present paper in the more general case of $\theta$-semistable representations, i.e., for $\theta \neq 0$.

In Section 3 we prove that, for quivers of special type, namely those which are crab-shaped (see Figure 1), there is an isomorphism between (an open subset of) the corresponding multiplicative quiver variety and a character variety arising from considering representations of the fundamental group of a punctured Riemann surface where the monodromies of loops around the punctures are assumed to lie in the closure of certain conjugacy classes. In order to build such a correspondence we exploit [CBS06] Lemma 8.2 and Theorem 1.1. Note that an instance of the correspondence between multiplicative quiver varieties and local systems on punctured surfaces already appeared in [Yam08], where a proof is given for the case of the punctured projective line. Our result applies to all genera. Thanks to this correspondence, and to the results proved in Section 4, we are able to extend the work of Bellamy and the first author from closed Riemann surfaces to open ones. Another interesting aspect of this correspondence is that it could be conjecturally combined with the Non-abelian Hodge Theorem to extend the main results of [Tir17], proved by the second author, in the context of moduli spaces of parabolic Higgs bundles. More details on this topic are provided in Section 7 where possible future research directions of the present work are discussed.

Section 4 contains the proof of Theorem 1.7. A careful study of the singularities of multiplicative quiver varieties is carried out: first, we detect which points in these moduli spaces are indeed singular, proving the following

**Proposition 1.16** (Proposition 4.3 below). Let $\alpha \in \Sigma_{q,\theta}$. Then, the singular locus of $M_{q,\theta}(Q,\alpha)$ is given by the strictly $\theta$-semistable points.

The remaining part of the section is devoted to the study of the nature of the singular locus. To this end, we use techniques from the work [BS16a] of Bellamy and the first author to prove that, under suitable hypotheses, the singularities are symplectic. We also prove that, under certain conditions, the moduli space $M_{q,\theta}(Q,\alpha)$ contains an open subset which is singular, factorial, and terminal. As a consequence, $M_{q,\theta}(Q,\alpha)$ does not admit a symplectic resolution. Moreover, for generic $\theta$, we see that the open subset is the entire variety. The only case left out by Theorem 1.7, when $\alpha = 2\beta$ for $\beta \in N_{q,\theta}$ and $p(\beta) = 2$, is more subtle than the others. The corresponding result in the context of ordinary quiver varieties, treated in [BS16a], is based on the study of the local structure of such varieties. In our case, such a tool is still not available, but will hopefully be the object of future research.

In Section 5, namely in Theorems 5.1 and 5.3, we combinatorially classify all the the pairs $(Q,\alpha)$, formed by a crab-shaped quiver and a corresponding dimension vector in the fundamental region such that $(p(\gcd(\alpha)\ominus\alpha), \gcd(\alpha)) = (2,2)$. This is relevant as these are the cases expected, for generic $\theta$, to admit “O’Grady”-type resolutions (i.e., by blowing up the singular locus). This is also important since, by Theorem 1.13, it allows us to recognise whether a dimension vector in the fundamental region is expected to admit a symplectic resolution or not.

In Section 6 we face the problem of existence of symplectic resolutions of multiplicative quiver varieties for general dimension vectors. In order to do so, we follow the approach of Bellamy and the first author. In particular, we prove that a multiplicative quiver variety has a canonical decomposition into natural factors, see Theorem 6.17. This can be viewed as a multiplicative analogue of Crawley-Boevey’s decomposition [CB02], and we follow his proof, obtaining some more factors due to the unavailability of (*) and some local structure results. Our result makes it possible to solve the problem by understanding it only at the level of such indecomposable factors, which are multiplicative quiver varieties with particular dimension vectors. We then extend the GIT construction of symplectic resolutions by varying $\theta$ to dimension vectors not in $\Sigma$ (Theorem 6.22), this includes multiplicative analogues of framed quiver varieties such as Hilbert schemes of $\mathbb{C}^2$ and of hyperkähler almost locally Euclidean spaces. In Theorem 6.26 we make use of our canonical decomposition and, modulo some cases for which the question remains still open, we classify all multiplicative quiver varieties with arbitrary dimension vector that admit a symplectic resolution. As an application of this result, by restricting to crab-shaped quivers, we give an explicit classification of the character varieties of punctured surfaces admitting symplectic resolutions (Corollary 6.27), combining Theorem 6.26 and the results of Section 5.
Last, Section 7 contains some open questions which naturally arise from the study carried out in the present paper. One open question which would be interesting to tackle regards the possibility to extend the main results of [1517] starting from the correspondence outlined in Section 3. In Section 7 we provide some details on this topic by describing which moduli space one would need to consider, via the non-abelian Hodge Theorem in the non-compact case [Sim90], and we conjecture a generalisation of the Isosingularity Theorem for such moduli spaces. To conclude, we outline a general setting and pose a number of questions which should generalise the work of the present and many other papers, e.g., [ASL15, BS16a, KL07, Tri14]; it seems that many of the techniques exploited in the mentioned works are particular instances of theorems which conjecturally hold in the context of moduli spaces of semistable objects in 2-Calabi–Yau categories, under suitable hypotheses. This assertion is motivated also by the work of Bocklandt, Galluzzi and Vaccarino, [BGV16], who studied moduli spaces of representations of 2-Calabi–Yau algebras and proved that such varieties locally look like representations of (ordinary) preprojective algebras. This seems to be a singular, local, undervived version of the phenomenon that representation varieties of Calabi–Yau algebras are (shifted) symplectic (as announced by Brav and Dyckerhoff; see a similar result in [Yeu]).

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2. Multiplicative quiver varieties

In this section we give the definition of multiplicative quiver varieties following [CBS06] and recall some basic properties of such moduli spaces which will be useful in the arguments of the proof of our main theorems. In addition to these known results, we prove a new one, concerning the normality of the aforementioned varieties.

Throughout the paper, we work over the field $\mathbb{C}$ of complex numbers.

2.1. Preliminaries on quivers and root systems. We recall the basic definitions and fix the notations from the theory of quiver representations. For a finite quiver $Q$, let $Q_0$ and $Q_1$ denote the set of vertices and the set of arrows of $Q$, respectively. Moreover, for an arrow $a \in Q_1$, let $h(a)$ and $t(a)$ denote the head and the tail of $a$, respectively. For a dimension vector $\alpha \in \mathbb{N}^{Q_0}$, we will denoe by $\text{Rep}(Q, \alpha)$ the space of representations of $Q$ of dimension $\alpha$, which is naturally acted upon by the group $GL(\alpha) := \prod_{i \in Q_0} GL(\alpha_i)$.

The coordinate vector at vertex $i$ is denoted $e_i$. The set $\mathbb{N}^{Q_0}$ of dimension vectors is partially ordered by $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ for all $i$ and we say that $\alpha > \beta$ if $\alpha \geq \beta$ with $\alpha \neq \beta$. The vector $\alpha$ is called sincere if $\alpha_i > 0$ for all $i$. The Euler (or Ringel) form on $\mathbb{Z}^{Q_0}$ is defined by

$$\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_{t(a)} \beta_{h(a)}.$$

Let $\langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$ denote the corresponding Cartan (or Tits) form and set $p(\alpha) = 1 - \langle \alpha, \alpha \rangle$. The fundamental region $F(Q)$ is the set of nonzero $\alpha \in \mathbb{N}^{Q_0}$ with connected support and with $\langle \alpha, e_i \rangle \leq 0$ for all $i$. For $q \in (\mathbb{C}^\times)^{Q_0}$ and $\alpha \in \mathbb{N}^{Q_0}$, let $q^\alpha := \prod_{i \in Q_0} q_i^{\alpha_i}$.

If $i$ is a loopfree vertex, so $p(e_i) = 0$, there is a reflection $s_i : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}$ defined by $s_i(\alpha) = \alpha - \langle \alpha, e_i \rangle e_i$. The real roots (respectively imaginary roots) are the elements of $\mathbb{Z}^{Q_0}$ which can be obtained from the coordinate vector at a loopfree vertex (respectively $\pm$ an element of the fundamental region) by applying some sequence of reflections at loopfree vertices. Let $R^+$ denote the set of positive roots. Recall that a root $\beta$ is isotropic imaginary if $p(\beta) = 1$ and anisotropic imaginary if $p(\beta) > 1$. We say that a dimension vector $\alpha$ is indivisible if the greatest common divisor of the $\alpha_i$ is one.
2.2. Multiplicative preprojective algebras. We now define the quiver algebras whose moduli of representations are the varieties of interest in the present paper. To this purpose, let $Q$ be a quiver, fixed once and for all in this section. First, recall that, for a vector $\lambda \in \mathbb{C}^{Q_0}$, the deformed preprojective algebra $\Pi^\lambda(Q)$ is the quotient of the path algebra $\mathbb{C}Q$ of the doubled quiver $\overline{Q}$ by the relation

$$\sum_{x \in Q_1} [x, x^*] = \sum_{i \in Q_0} \lambda_i e_i,$$

where $x^*$ denotes the dual loop to $x$ in $\overline{Q}_1$; it is well-known that Nakajima quiver varieties can be interpreted as moduli spaces of $(\theta$-semistable) representations of such algebras. As one might expect, the defining relation for multiplicative preprojective algebras is a multiplicative analogue of the above equation: choose $q \in (\mathbb{C}^\times)^{Q_0}$ and define $A(Q)$ to be the universal localisation of the path algebra $\mathbb{C}Q$ such that $1 + xx^*$ and $1 + x^*x$ are invertible, for $x \in Q_1$. Then, following [CBS06 Definition 1.2], the multiplicative preprojective algebra $\Lambda^q(Q)$ is defined as the quotient of $A(Q)$ by the relation

$$\prod_{x \in Q_1} (1 + xx^*) \varepsilon(x) = \sum_{i \in Q_0} q_i e_i,$$

where $\varepsilon(x)$ equals 1 if $x \in Q_1$ and $-1$ otherwise and the product is ordered by an arbitrary choice of ordering “$<$” on $Q_1$. It is known, by [CBS06 Theorem 1.4] that, up to isomorphism, $\Lambda^q(Q)$ does not depend on the orientation of the quiver or the chosen ordering on $\overline{Q}_1$.

**Remark 2.1.** When the quiver $Q$ is clear from the context, we will use the shortened notation $\Lambda^q$ in place of $\Lambda^q(Q)$.

Analogously to the additive case mentioned above, representations of $\Lambda^q(Q)$ are representations of the underlying quiver $\overline{Q}$, $\{ (V_i)_{i \in Q_0}, (\phi_j)_{j \in Q_1} \}$, satisfying the additional relations:

$$\text{Id}_{V_{h(j)}} + \phi_j \phi_j^* \text{ is an invertible endomorphism of } V_{h(j)} \text{ for all } j \in \overline{Q}_1$$

$$\prod_{j \in \overline{Q}_1, h(j) = i} (\text{Id}_{V_{h(j)}} + \phi_j \phi_j^*)^{\varepsilon(j)} = q_i \text{Id}_{V_i} \text{ for all } i \in Q_0,$$

where, for an edge $j \in \overline{Q}_1$, $\phi_j^*$ denotes the linear map $\phi_j^*$, $j^*$ being the dual edge of $j$.

For a positive vector $\alpha \in \mathbb{N}^{Q_0}$, we denote by $\text{Rep}(\Lambda^q, \alpha)$ the set of representations of $\Lambda^q$ with $V_i = \mathbb{C}^{\alpha_i}$ for all $i$. This can be given an obvious affine scheme structure via the subset of matrices satisfying the obvious polynomial equations. We will work below with the reduced subvariety of this affine scheme.

**Remark 2.2.** By taking determinants of the defining relation for the multiplicative preprojective algebra, one can easily see that if $\Lambda^q$ has a representation of dimension vector $\alpha$, then $q^\alpha = 1$, which, thus, is a necessary condition to be satisfied in order to have non-empty moduli spaces.

The following results, which will be used in the next sections, are proved in [CBS06]. It is worth pointing out that, even though we work over $\mathbb{C}$, these statements hold true over an arbitrary field $K$.

**Proposition 2.3.** If $X$ and $Y$ are finite-dimensional representations of $\Lambda^q$, then

$$\dim \text{Ext}^1_{\Lambda^q}(X, Y) = \dim \text{Hom}_{\Lambda^q}(X, Y) + \dim \text{Hom}_{\Lambda^q}(Y, X) - (\dim X, \dim Y).$$

The following result concerns the geometry of the space $\text{Rep}(\Lambda^q, \alpha)$ of representations of the algebra $\Lambda^q$, when a dimension vector $\alpha \in \mathbb{N}^{Q_0}$ is fixed. Define $g_\alpha$ as $g_\alpha := -1 + \sum_{i \in Q_0} \alpha_i^2$.

**Proposition 2.4.** $\text{Rep}(\Lambda^q, \alpha)$ is an affine variety, and every irreducible component has dimension at least $g_\alpha + 2p(\alpha)$. The subset $T \subset \text{Rep}(\Lambda^q, \alpha)$ of representations $X$ with trivial endomorphism algebra, $\text{End}(X) = \mathbb{C}$, is open and, if non-empty, smooth of dimension $g_\alpha + 2p(\alpha)$.

In order to state the next result, we introduce a subset of the set of all possible dimension vectors for representations of $\Lambda^q$, which is the multiplicative analogue of the set $\Sigma_\lambda$ introduced by Crawley-Boevey in [CB02] and extensively used in [BST16a] (generalised to $\Sigma_{\lambda, \theta}$) to prove the existence of a canonical decomposition for quiver varieties with general stability conditions. In the next definition, one should keep in mind that $\theta \in \mathbb{Z}^{Q_0}$ will play the role of a stability condition for a GIT quotient.
Definition 2.5. Fix $q \in (\mathbb{C}^\times)^{Q_0}$ and $\theta \in \mathbb{Z}^{Q_0}$ and set $N_{q, \theta} := \{ \alpha \in \mathbb{N}^{Q_0} \mid q^\alpha = 1, \alpha \cdot \theta = 0 \}$. Define $R^+_{q, \theta} := R^+ \cap N_{q, \theta}$. Then,

$$\Sigma_{q, \theta} := \left\{ \alpha \in R^+_{q, \theta} \ \bigg| \ p(\alpha) > \sum_{i=1}^{r} p\left(\beta^{(i)}\right) \text{ for any decomposition } \alpha = \beta^{(1)} + \cdots + \beta^{(r)} \text{ with } r \geq 2, \ \beta^{(i)} \in R^+_{q, \theta} \right\}.$$  

When $\theta = 0$, we shall use the shortened notation $\Sigma_q$ in place of $\Sigma_{q,0}$.

Definition 2.6. A dimension vector $\alpha \in N_{q, \theta}$ is said to be $q$-indivisible if $\frac{1}{m} \alpha \notin N_{q, \theta}$ for all $m \geq 2$. Equivalently, for $\alpha = m\beta$ and $\beta$ indivisible, then $q^\beta$ is a primitive $m$-th root of unity.

The following property is part of the motivation for the definition of $\Sigma_q$ (which we generalise in Proposition 2.22 to the $\theta \neq 0$ case):

Proposition 2.7. [CBS06 Theorem 1.11] If $\alpha \in \Sigma_q$, then, if non-empty, $\text{Rep}(\Lambda^q, \alpha)$ is a complete intersection, equidimensional of dimension $q_\alpha + 2p(\alpha)$, and the set of simple representations is open and dense.

2.3. Reflection functors for $\Lambda^q(Q)$. As in the additive case, one can define reflection functors for the multiplicative preprojective algebra $\Lambda^q(Q)$: let $v$ a loopfree vertex in $Q$ and define

$$u_v : (\mathbb{C}^\times)^{Q_0} \to (\mathbb{C}^\times)^{Q_0}, \quad u_v(q) = q_v^{-e_v} q_w.$$  

It is easy to see that the map $u_v$ satisfies the following identity:

$$\left(u_v(q)\right)^\alpha = q^{s_v(\alpha)},$$  

where $s_v$ is the reflection map defined in Subsection 2.1. The main result concerning such maps is analogous to the properties of reflection functors for $\Pi^\lambda(Q)$.

Proposition 2.8. [CBS06 Theorem 1.7] If $v$ is a loopfree vertex and $q_v \neq 1$, then there is an equivalence of categories $F_q$ from the category of representations of $\Lambda^q$ to the category of representations of $\Lambda^{u_v(q)}$, acting on dimension vectors through the reflection $s_v$. The inverse equivalence is given by $F_{u_v(q)}$.

We will need also reflections on $\theta$. Define

$$r_v : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}, \quad r_v(\theta) = \theta - (e_v, e_w)\theta_v.$$  

Definition 2.9. The map $(q, \theta, \alpha) \mapsto (u_v(q), r_v(\theta), s_v(\alpha))$, is called a reflection. If $\theta_v \neq 0$ or $q_v \neq 1$, it is called an admissible reflection.

We will explain below isomorphisms of multiplicative quiver varieties, due to Yamakawa, which are closely related to the above equivalence.

2.4. Moduli of representations of $\Lambda^q(Q)$. We shall now outline the construction of the varieties of interest for the present work. As mentioned above, the general definition involves a stability condition $\theta \in \mathbb{Z}^{Q_0}$, which we fix for the rest of this section.

Given a dimension vector $\alpha$, from the seminal work of King, [Kin94], one can introduce the following definition of $\theta$-(semi)stability on $\text{Rep}(Q, \alpha)$.

Definition 2.10. Let $M$ be a finite-dimensional representation of $\Lambda^q$ such that $\dim M \cdot \theta = 0$. $M$ is said to be $\theta$-semistable if, for any sub-module $N \subset M$

$$\theta \cdot \dim N \leq 0.$$  

$M$ is said to be $\theta$-stable if the strict inequality holds. Finally, $M$ is said to be $\theta$-polystable if it is a direct sum of $\theta$-stable representations. Given a set (or scheme) $X$ of representations, let $X^{\theta-s}$ and $X^{\theta-ss}$ denote the $\theta$-stable and $\theta$-semistable loci, respectively.

We will also need the corresponding partial ordering:

Definition 2.11. We say that $\theta' \geq \theta$ if every $\theta'$-semistable representation is also $\theta$-semistable.
Remark 2.12. By [Kin94, Proposition 3.1], one has that the above definition of stability coincides with the usual one coming from GIT: indeed, consider the character
\[ \chi_\theta : GL(\alpha) \rightarrow \mathbb{C}^\times, \quad (g_i)_{i \in Q_0} \mapsto \prod_{i \in Q_0} (\det g_i)^{-\theta_i}. \]
It defines a linearisation on the trivial line bundle \( \text{Rep}(Q, \alpha) \times \mathbb{C} \) of the action of \( GL(\alpha) \) on \( \text{Rep}(Q, \alpha) \); thus, one can define the notion of \( \chi_\theta \)-stability à la Mumford, [MPK02]. The aforementioned result of King proves that \( M \) is \( \theta \)-(semi)stable if and only if it is \( \chi_\theta \)-(semi)stable.

Using the notion above one can construct the moduli space of (semistable) representations of \( \Lambda \) as follows (see [Yam08, §2], for the details): define \( \text{Rep}^{\theta-ss}(Q, \alpha) \subset \text{Rep}(Q, \alpha) \) to be the subset of \( \theta \)-semistable representations and set \( \text{Rep}^{\circ}(Q, \alpha) \) to be
\[ \text{Rep}^{\circ}(Q, \alpha) = \{(\phi_j)_{j \in Q_1} \in \text{Rep}(Q, \alpha) \mid \det(1 + \phi_j \phi_j^*) \neq 0, j \in Q_1 \}. \]
One can then consider the map
\[ \Phi : \text{Rep}^{\circ}(Q, \alpha) \rightarrow GL(\alpha), \]
defined by the formula
\[ \Phi((\phi_j)_{j \in Q_1}) = \prod_{j \in Q_1} (1 + \phi_j \phi_j^*)^{\epsilon(j)}. \]
Let us identify \( \mathbb{C}^\times \) also with the scalar matrices in \( GL(\alpha_i) \), and hence \( (\mathbb{C}^\times)^{Q_0} \) also with a subset of \( GL(\alpha) \). Fixing \( q \in (\mathbb{C}^\times)^{Q_0} \), one has that \( \text{Rep}(\Lambda^s(Q), \alpha) \) is the set-theoretic preimage \( \Phi^{-1}(q) \). Thus, one can give the following

**Definition 2.13.** The multiplicative quiver variety \( M_{q, \theta}(Q, \alpha) \) is the GIT quotient
\[ M_{q, \theta}(Q, \alpha) := (\text{Rep}^{\theta-ss}(Q, \alpha) \cap \Phi^{-1}(q)) \sslash GL(\alpha). \]

**Remark 2.14.** The reason for the terminology in the previous definition is apparent: indeed, the equations defining the multiplicative preprojective relation are modifications of the ones used to define the usual deformed preprojective algebras, whose moduli of (semistable) representations are indeed Nakajima quiver varieties.

It is worth recalling a fundamental result of King, which gives a moduli-theoretic interpretation—in the sense of (representable) moduli functors—to \( M_{q, \theta}(Q, \alpha) \).

**Theorem 2.15.** [Kin94, Propositions 3.1 and 3.2] Assume \( \theta \in \mathbb{Z}^{Q_0} \). Then, \( M_{q, \theta}(Q, \alpha) \) is a coarse moduli space for families of \( \theta \)-semistable representations up to \( S \)-equivalence.

Here two \( \theta \)-semistable representations are \( S \)-equivalent if and only if they have the same composition factors into \( \theta \)-stable representations (i.e., they have filtrations whose subquotients are isomorphic \( \theta \)-stable representations). This means that every point in \( M_{q, \theta}(Q, \alpha) \) has a unique representative which is \( \theta \)-polystable, up to isomorphism.

Precisely as in [BS16a, Lemma 2.4], we have the following instance of the well-known principle of GIT:

**Lemma 2.16.** [BS16a, Lemma 2.4] Let \( \alpha \in N_{q, \theta} \) be such that \( M_{q, \theta}(Q, \alpha) \neq \emptyset \). Take \( \theta' \geq \theta \). Then we have a projective Poisson morphism \( M_{q, \theta'}(Q, \alpha) \rightarrow M_{q, \theta}(Q, \alpha) \) induced by the inclusion \( \Phi^{-1}(q)^{\theta'-ss} \subseteq \Phi^{-1}(q)^{\theta-ss} \).

We caution that this morphism need not be surjective (and indeed the source could be empty when the target is not). However, in many cases, as we will see, it produces a symplectic resolution.

**2.5. Reflection isomorphisms.** There is a multiplicative analogue of the Lusztig-Maffei-Nakajima reflection isomorphisms of quiver varieties (see in particular [Ma12, Theorem 26]), due to Yamakawa, which makes use of the reflection functors \( F_q \). Let us extend the definition of \( M_{q, \theta}(Q, \alpha) \) to \( \alpha \in \mathbb{Z}^{Q_0} \) by setting it to be empty in the case that \( \alpha_i < 0 \) for some \( i \).

**Theorem 2.17.** [Yam08, Theorem 5.1] An admissible reflection \( (q, \theta, \alpha) \rightarrow (u_v(q), r_v(\theta), s_v(\alpha)) \) induces an isomorphism of multiplicative quiver varieties, \( M_{q, \theta}(Q, \alpha) \cong M_{u_v(q), r_v(\theta)}(Q, s_v(\alpha)) \).
2.6. Poisson structure on $\mathcal{M}_{q,0}(Q,\alpha)$. In order to construct a Poisson structure on $\mathcal{M}_{q,0}(Q,\alpha)$, we shall use the theory of quasi-Hamiltonian reductions, first developed in [AMM98] for the case of real manifolds, and then treated by Boalch, [Boa07], and Van den Bergh [Van08a, Van08b] in the holomorphic and algebraic settings. To this end, note that the map $\Phi$ defined above is a group valued moment map for the quasi-Hamiltonian action of GL($\alpha$) on Rep($\overline{Q},\alpha$). Thus, the variety $\mathcal{M}_{q,0}(Q,\alpha)$ can be considered as the quasi-Hamiltonian reduction of Rep($\overline{Q},\alpha$) modulo the action of GL($\alpha$). From the properties of such a reduction, we obtain that $\mathcal{M}_{q,0}(Q,\alpha)$ is a Poisson variety. Moreover, defining

$$\mathcal{M}^s_{q,0}(Q,\alpha) := (\text{Rep}^{s-ss}(\overline{Q},\alpha) \cap \Phi^{-1}(q))/\text{GL}(\alpha),$$

where Rep$^{s-ss}(\overline{Q},\alpha)$ denotes the $\theta$-stable locus, one has the following result, which will be crucial in proving that $\mathcal{M}_{q,0}(Q,\alpha)$ is a symplectic singularity. Note that, in the above definition the quotient is the usual orbit space, if we replace GL($\alpha$) by PGL($\alpha$) = GL($\alpha$)/$\mathbb{C}^*$, as a point in the stable locus has trivial stabiliser group under PGL($\alpha$).

**Proposition 2.18.** [Yam08, Theorem 3.4] $\mathcal{M}^s_{q,0}(Q,\alpha)$, if non-empty, is an equidimensional algebraic symplectic manifold and its dimension is $2p(\alpha)$.

2.7. Stratification by representation type. An important result proved in [CBS06, §7] concerns a natural stratification of the affine quotient $\mathcal{M}_{q,0}(Q,\alpha) = \text{Rep}(\Lambda^q,\alpha)/\text{GL}(\alpha)$, which parametrises semisimple representations of the algebra $\Lambda^q$. This stratification and its generalisation, proved below, to the case of $\theta$-semistable representations are important in order to understand the singular locus of $\mathcal{M}_{q,0}(Q,\alpha)$.

Consider $M \in \text{Rep}^{\theta-ss}(\Lambda^q,\alpha)$. Replace it by the unique $\theta$-polystable representation which is $S$-equivalent to it (see the discussion after Theorem 2.15). $M$ is then said to be of representation type $\tau = (k_1,\beta^{(1)}; \ldots; k_r,\beta^{(r)})$ if it can be decomposed into the direct sum $M \cong M_1^{k_1} \oplus \cdots \oplus M_r^{k_r}$, where $M_i$ is a $\theta$-stable representation of $\Lambda^q$ of dimension vector $\beta_i$, $i = 1, \ldots, r$, and $M_i \neq M_j$ for $i \neq j$.

**Proposition 2.19.** If $\tau$ is a representation type for $\Lambda^q$, then the set $C^r_{q,\theta}(Q,\alpha)$ of $\theta$-semistable representations of type $\tau$ is a locally closed subset of $\mathcal{M}_{q,0}(Q,\alpha)$, which, if non-empty, has dimension $\sum_{i=1}^r 2p(\beta^{(i)})$. $\mathcal{M}_{q,0}(Q,\alpha)$ is the disjoint union of the strata $C_{q,\theta}^r(Q,\alpha)$, where $\tau$ runs over the set of representation types that can occur for $\Lambda^q$.

**Proof.** First, note that the case when $\theta = 0$ is treated in [CBS06] and proved in Lemma 7.1 therein. For the case when $\theta \neq 0$ we use the same arguments. Indeed, the fact that $\mathcal{M}_{q,0}(Q,\alpha)$ is a disjoint union of subsets of a fixed representation type is immediate from the fact that the decomposition of a $\theta$-polystable module into $\theta$-stable modules is unique. This, in turn, holds because, for $\theta$-stable modules $M$ and $N$, we have dim Hom($M,N) \leq 1$, with equality if and only if $M$ and $N$ are isomorphic. Moreover, to prove that each $C_{q,\theta}^r(Q,\alpha)$ is locally closed and of the dimension prescribed by the lemma, one can adapt the proof [CB01, Theorem 1.3]: indeed, those arguments can be repeated in this case as well, replacing Rep($\overline{Q},\alpha$) with Rep$^{s-ss}(\overline{Q},\alpha)$, $\mu^{-1}_{\alpha}(\lambda)$ with $\Phi^{-1}(q)$, the word ‘(semi-)simple’ with ‘(semi-)stable’ in the proof, and noting that everything goes through in the same way because Rep$^{s-ss}(\overline{Q},\alpha)$ is open in Rep($\overline{Q},\alpha$). The only difference is that, in this case, we do not claim irreducibility, since Rep$^{s-ss}(\overline{Q},\beta)$ is not known to be irreducible, even when $\beta \in \Sigma_{q,\theta}$.

We will need also the following property of Rep$^{s-ss}(\Lambda^q(\alpha))$:

**Lemma 2.20.** Every irreducible component of Rep$^{s-ss}(\Lambda^q(\alpha))$ has dimension at least $g_\alpha + 2p(\alpha)$ and the set of $\theta$-stable representations form an open subset of Rep($\Lambda^q(\alpha)$) which, if non-empty, is smooth of dimension $g_\alpha + 2p(\alpha)$.

**Proof.** For the first part, Lemma 6.2 in [CB03] proves the statement in the case when $\theta = 0$, of which the above result is a consequence since Rep$^{s-ss}(\Lambda^q(\alpha))$ is an open subset of Rep($\Lambda^q(\alpha)$): indeed, every irreducible component of the former variety is contained in only one irreducible component of the latter and, hence, the dimension estimate holds. For the second part, one just needs to note that, if $X$ is a $\theta$-stable representation, then End($X$) = $\mathbb{C}$ and, hence, by [CBS06, Theorem 1.10] defines a smooth point of Rep($\Lambda^q(\alpha)$), which implies that it is a smooth point of Rep$^{s-ss}(\Lambda^q(\alpha))$. 

\[ \text{\square} \]
Proposition 2.22. Let \( M \) be dense inside \( \text{Rep} \) holds:

\[ \text{Corollary 2.26.} \]

On the other hand, by Proposition 2.19, one has that

\[ \text{Corollary 2.24.} \]

Proof. By Definition 2.11, \( \text{Rep} \) component of \( \theta \) locus is dense in the \( \theta \) ensured by the fact that the \( \theta \) mapped isomorphically to the \( \theta \) with the word ‘\( \theta \)-stable’ and use Proposition 2.19 instead of [CBS06, Lemma 7.1] and Lemma 2.20 instead of [CBS06, Theorem 1.1].

Proposition 2.21. The inverse image in \( \text{Rep} \) of the stratum of representations of type \( \tau = (k_1, \beta^{(1)}; \ldots; k_r, \beta^{(r)}) \) has dimension at most \( g_\alpha + p(\alpha) + \sum_{i=1}^r p(\beta^{(i)}) \).

Proposition 2.22. Let \( \alpha \in \Sigma_{q,\theta}. \) Then, if non-empty, \( \text{Rep} \) is a complete intersection in \( \text{Rep} \), equidimensional of dimension \( g_\alpha + 2p(\alpha) \). The locus of \( \theta \)-stable representations \( \text{Rep} \) is dense inside \( \text{Rep} \).

Remark 2.23. Note that a consequence of the above proposition is that, if \( \pi : \text{Rep} \to \mathcal{M}_{q,\theta}(Q, \beta) \) is the projection map, the image \( \mathcal{M}^\theta_{q,\theta}(Q, \alpha) := \pi\left(\text{Rep}^{\theta-\text{ss}}(\Lambda^q(Q), \alpha)\right) \) is dense in the moduli space \( \mathcal{M}_{q,\theta}(Q, \alpha) \). As a corollary of this and Proposition 2.18 (or Proposition 2.19), one has that every component of \( \mathcal{M}_{q,\theta}(Q, \alpha) \) has dimension \( 2p(\alpha) \).

A useful corollary of the proposition is the following criterion for birationality of the maps \( \mathcal{M}_{q,\theta}(Q, \beta) \to \mathcal{M}_{q,\theta}(Q, \beta) \). Together with Lemma 2.16 this explains that these maps will be resolutions of singularities when the source is smooth.

Corollary 2.24. Let \( \alpha \in \Sigma_{q,\theta} \) be such that \( \mathcal{M}_{q,\theta}(Q, \alpha) \neq \emptyset \). Take \( \theta' \geq \theta \). Then, the morphism \( \mathcal{M}_{q,\theta'}(Q, \beta) \to \mathcal{M}_{q,\theta}(Q, \beta) \) is birational.

Proof. By Definition 2.11 \( \text{Rep}^{\theta-\text{ss}}(\Lambda^q, \alpha) \) is a subset of \( \text{Rep}^{\theta-\text{ss}}(\Lambda^q, \alpha) \), and it is open. Since the \( \theta \)-stable locus is dense in the \( \theta \)-semistable locus, this implies that the \( \theta \)-stable, \( \theta' \)-semistable locus is dense in the \( \theta' \)-semistable locus. So the \( \theta \)-stable locus is dense in \( \mathcal{M}_{q,\theta'}(Q, \beta) \). Thus, birationality of the desired map is ensured by the fact that the \( \theta \)-stable locus \( \mathcal{M}_{q,\theta}(Q, \beta) \) is open and dense inside \( \mathcal{M}_{q,\theta'}(Q, \beta) \), is mapped isomorphically to the \( \theta \)-stable locus of \( \mathcal{M}_{q,\theta}(Q, \beta) \).

Using the above results, one can derive an important geometric property of the moduli space \( \mathcal{M}_{q,\theta}(Q, \alpha) \). For reasons which are clear in the proof of the proposition, we assume that a certain codimension estimate holds. As usual, let \( \pi : \text{Rep}^{\theta-\text{ss}}(\Lambda^q, \alpha) \to \mathcal{M}_{q,\theta}(Q, \alpha) \) denote the quotient map.

Lemma 2.25. Assume \( \alpha \in \Sigma_{q,\theta} \) and let \( \tau \) be a stratum. The following inequality holds true:

\[ \text{codim}_{\text{Rep}^{\theta-\text{ss}}(\Lambda^q, \alpha)}(C^\tau_{q,\theta}(Q, \alpha)) \geq \frac{1}{2} \text{codim}_{\mathcal{M}_{q,\theta}(Q, \alpha)}(C^\tau_{q,\theta}(Q, \alpha)). \]

Proof. By Proposition 2.22 one has that

\[ \text{codim}(C^\tau_{q,\theta}(Q, \alpha)) = g_\alpha + 2p(\alpha) - \dim C^\tau_{q,\theta}(Q, \alpha). \]

Moreover, from Proposition 2.21 it follows that

\[ g_\alpha + 2p(\alpha) - \dim C^\tau_{q,\theta}(Q, \alpha) \geq g_\alpha + 2p(\alpha) - g_\alpha - p(\alpha) - \sum_{l=1}^r p(\beta^{(l)}) = p(\alpha) - \sum_{l=1}^r p(\beta^{(l)}). \]

On the other hand, by Proposition 2.19 one has that

\[ p(\alpha) - \sum_{l=1}^r p(\beta^{(l)}) = \frac{1}{2} \left( \dim \mathcal{M}_{q,\theta}(Q, \alpha) - \dim C^\tau_{q,\theta}(Q, \alpha) \right), \]

which, combined with the above inequality, leads to the desired statement.

By taking the minimum of these codimensions, we immediately conclude:

Corollary 2.26. Let \( Z \) denote the complement inside \( \mathcal{M}_{q,\theta}(Q, \alpha) \) of the set of \( \theta \)-stable representations \( \mathcal{M}^\theta_{q,\theta}(Q, \alpha) \), i.e., \( Z \) is the union of all the non-open strata of \( \mathcal{M}_{q,\theta}(Q, \alpha) \). Then, the following inequality holds:

\[ \text{codim} \pi^{-1}(Z) \geq \frac{1}{2} \min_{\tau \neq (1,\alpha)} \text{codim} C^\tau_{q,\theta}(Q, \alpha). \]
Proposition 2.27. Consider $\alpha \in \Sigma_{q,\theta}$ and assume that all strata in the non-empty multiplicative quiver variety $M_{q,\theta}(Q,\alpha)$ have codimension at least 4, i.e., assume that
\[
\min_{\tau \neq (1,\alpha)} (\dim M_{q,\theta}(Q,\alpha) - \dim C_{q,\theta}^\tau(Q,\alpha)) \geq 4.
\]
Then, the variety $M_{q,\theta}(Q,\alpha)$ is normal.

Proof. The arguments to prove the above statement are analogous to the ones used in [BS16a, Proposition 8.3]. In particular, we shall use a criterion proved by Crawley-Boevey, [CB03, Corollary 7.2]. We first deal with the case when $\theta = 0$ and then explain how to adapt the arguments for general $\theta$. When $\theta = 0$, $M_{q,0}(Q,\alpha)$ is the categorical quotient $\text{Rep}(\Lambda^q,\alpha)/\text{GL}(\alpha)$ of an affine variety modulo a reductive group. Thus, we only need to show that $\text{Rep}(\Lambda^q,\alpha)$ satisfies Serre’s condition $(S_2)$ and that certain codimension estimates hold true. The first condition is ensured by the fact that, by Proposition 2.27, $\text{Rep}(\Lambda^q,\alpha)$ is a complete intersection and, hence, Cohen-Macaulay, which indeed implies condition $(S_2)$. Now, denote by $S$ the open subset $S \subset M_{q,0}(Q,\alpha)$ of simple representations, which is non-empty by our assumption. $S$ is contained in the smooth locus and hence is normal. Moreover, let $Z$ denote its complement in $M_{q,0}(Q,\alpha)$ and denote with $\pi: \text{Rep}(\Lambda^q,\alpha) \to M_{q,0}(Q,\alpha)$ the quotient map; then, by Corollary 2.26, one has
\[
\dim \text{Rep}(\Lambda^q,\alpha) - \dim \pi^{-1}(Z) \geq \frac{1}{2} \min_{\tau \neq (1,\alpha)} (\dim M_{q,0}(Q,\alpha) - \dim C_{q,0}^\tau(Q,\alpha)),
\]
and the right hand side is greater or equal than two by assumption. Thus, all the hypotheses of [CB03, Corollary 7.2] are satisfied and we can conclude that $M_{q,0}(Q,\alpha)$ is normal. For the case when $\theta \neq 0$, keeping in mind that normality is a local property, we fix a point $x \in M_{q,\theta}(Q,\alpha)$ and aim at proving normality at $x$. This is achieved by choosing an open neighbourhood $V$ of $x$ such that the restriction to $\pi^{-1}(V)$ of the projection morphism $\pi^{-1}(V) \to V$ is an affine quotient (note that this can be done thanks to the properties of the GIT construction). One can now repeat the same arguments as for the $\theta = 0$ case, noting that, by Proposition 2.19, the estimates above hold true also in this more general setting: being Cohen-Macaulay is a local statement and thus the previous part of the proof ensures that $\pi^{-1}(V)$, which is open in $\text{Rep}(\Lambda^q,\alpha)$, satisfies such a property. Moreover, defining $S_\theta$ to be the subset of $V$ of $\theta$-stable representations, then one may proceed as in the first part of the proof to obtain the desired conclusion. \hfill $\Box$

Remark 2.28. In the next sections, we will examine some cases in which the technical assumption in the previous result is satisfied, thus giving explicit examples of when $M_{q,\theta}(Q,\alpha)$ is normal.

3. Punctured character varieties as multiplicative quiver varieties

In this subsection, we explain how it is possible to realise certain character varieties as particular examples of multiplicative quiver varieties by considering quivers of special type, the so-called crab-shaped quivers. Such character varieties parametrise representations of the fundamental group of a compact Riemann surface with a finite number of punctures, where the monodromies at closed loops around such punctures are fixed to lie in (the closure of) certain conjugacy classes. As mentioned in the introduction, these varieties are central in the ground-breaking work of Hausel, Lettelier and Rodriguez-Villegas, who studied ways to compute their Betti numbers using arithmetic methods, see [HLRV11, HLRV13]. We use the language of quiver Riemann surfaces introduced by Crawley-Boevey in [CB13]. Moreover, in what follows, we shall adopt the term punctured character variety to refer to the character variety of a Riemann surface with punctures.

Fix a connected compact Riemann surface $X$ of genus $g \geq 0$, let $S = \{p_1,\ldots,p_k\} \subset X$ be the set of punctures and fix a tuple $\mathcal{C} = (C_1,\ldots,C_k)$ of conjugacy classes $C_i \subset \text{GL}_n(\mathbb{C})$, $i = 1,\ldots,k$. Recall that the fundamental group $\pi_1(X \setminus S)$ of the punctured surface $X \setminus S$ admits the following presentation:
\[
\pi_1(X \setminus S) = \langle a_1,\ldots,a_g,b_1,\ldots,b_g,c_1,\ldots,c_k \mid [a_1,b_1] \cdots [a_g,b_g]c_1 \cdots c_k = 1 \rangle,
\]
where $[a,b] = aba^{-1}b^{-1}$ denotes the commutator. Note that the generators $c_1,\ldots,c_k$ represent homotopy classes of closed loops around the punctures, in the same free homotopy classes as small counter-clockwise loops around the punctures. Thus, a representation of $\pi_1(X \setminus S)$ whose monodromies about the punctures are in the conjugacy classes $C_i$ is given by a tuple of matrices $(A_1,\ldots,A_g,B_1,\ldots,B_g,C_1,\ldots,C_k) \in \text{GL}_n(\mathbb{C})^{2g} \times $
\(C_1 \times \cdots \times C_k\), satisfying the relation
\[
\prod_{i=1}^{g}[A_i, B_i] \prod_{j=1}^{k} C_j = I.
\]
Given the above, from the fact that isomorphic representations correspond to conjugate matrices, one has that the character variety \(X(g, k, \mathbf{C})\) associated to the pair \((X, S)\) and monodromies lying in the conjugacy classes fixed above is isomorphic to the affine quotient
\[
\left\{ (A_1, \ldots, A_g, B_1, \ldots, B_g, C_1, \ldots, C_k) \in \text{GL}_n(\mathbf{C})^{2g} \times \mathbf{C}^1 \times \cdots \times \mathbf{C}^k \mid \prod_{i=1}^{g}[A_i, B_i] \prod_{j=1}^{k} C_j = I \right\} / \text{GL}_n(\mathbf{C}).
\]

Remark 3.1. Note that the closures \(\overline{C}_i\) are affine varieties and hence the quotient is indeed that of an affine variety by an algebraic group.

We shall now explain how to realise the variety \(X(g, k, \mathbf{C})\) as an open subset of a multiplicative quiver variety, using an equivalence of categories proved in [CB13]. As mentioned above, such a correspondence holds when one considers the so-called crab-shaped quivers, i.e., quivers such that there exists a vertex \(v\) satisfying the following condition: the set of arrows is formed by loops at \(v\) and a finite number of legs ending at \(v\). See Figure 3. A star-shaped quiver is a crab-shaped quiver with no loops.

![Figure 1. A crab-shaped quiver with 2 loops and 3 legs, of length 2, 3 and 1 respectively.](image)

For the remainder of this subsection, the following notation will be used: \(g\), for the number of loops around the central vertex; \(k\) for the number of legs and \(l_i\), for \(i = 1, \ldots, k\), for the length of the \(i\)-th leg. As we shall see, \(g\) contains the information regarding the genus of the surface, while the integers \(k\) and \(l_i\) encode information about the (prescribed) conjugacy classes of the monodromies of the loops around the punctures.

Definition 3.2. [CB13 §2] A Riemann surface quiver \(\Gamma\) is a quiver whose set of vertices has the structure of a Riemann surface \(X\) with finitely many connected components. \(\Gamma\) is said to be compact if \(X\) is compact. A point \(p \in X\) is called marked if it is a head or a tail of an arrow of \(\Gamma\).

Definition 3.3. Given a Riemann surface quiver \(\Gamma\), the component quiver \([\Gamma]\) of \(\Gamma\), is the quiver whose set of vertices is the set of connected components of \(\Gamma\) and arrows given by \([a] : [p] \to [q]\) for any arrow \(a : p \to q\), where \(p\) and \(q\) are points of \(X\) and \([p]\) denotes the connected component of \(X\) containing \(p\).

Remark 3.4. Although, by definition, there are in general infinitely many vertices, we will consider (Riemann surface) quivers with finitely many arrows.

Following closely [CB13 §5, §8], starting from a Riemann surface quiver \(\Gamma\), it is possible to define two categories of representations, \(\text{Rep}_\sigma(\pi(\Gamma))\) and \(\text{Rep}^{\Lambda^3([\Gamma])}\), whose equivalence is the key point to proving the correspondence between multiplicative quiver varieties and punctured character varieties. Fix a quiver Riemann surface \(\Gamma\) and let \(\{X_i\}_{i \in I}\) the set of connected components of the underlying Riemann surface \(X\). For each \(i \in I\) let \(D_i\) be the set of marked points of \(\Gamma\) contained in \(X_i\). Moreover, let \(D = \cup_i D_i\): fix \(\sigma \in (\mathbb{C}^*)^{D}, b_i \in X_i \setminus D_i\) and, for each \(p \in D_i\) fix a loop \(l_p \in \pi_1(X_i \setminus D_i, b_i)\) around \(p\).
Rep$_{\sigma}(\pi(\Gamma))$ is defined to be the category whose objects are given by collections $(V_i, \rho_i, \rho_a, \rho_a^*)$ consisting of representations $\rho_i : \pi_1(X_i \setminus D_i, b_i) \to \text{GL}(V_i)$ for $i \in I$ and linear maps $\rho_a : V_i \to V_j$ and $\rho_a^* : V_j \to V_i$ for each arrow $a : p \to q$ in $\Gamma$, where $X_i = [p]$ and $X_j = [q]$, satisfying

$$\sigma_p^{-1} \rho_i(\ell_p)^{-1} = 1_{V_i} + \rho_a^* \rho_a \quad \text{and} \quad \sigma_q \rho_j(\ell_q) = 1_{V_j} + \rho_a \rho_a^*$$

and whose morphisms are the natural ones. Consider the component quiver $[\Gamma]$ and define $Q$ to be the quiver obtained from $[\Gamma]$ by adjoining $g_i$ loops at each vertex $i$, where $g_i$ is the genus of $X_i$. Moreover, define $q \in (\mathbb{C}^\times)^I$ by $q_i = \prod_{p \in D_i} \sigma_p$. We define $\text{Rep} \Lambda^q([\Gamma])'$ to be the category of representations of the multiplicative preprojective algebra $\Lambda^q(Q)$ in which the linear maps representing the added loops are invertible.

**Lemma 3.5.** [CB13 Proposition 2] There is an equivalence of categories

$$\text{Rep}_\sigma(\pi(\Gamma)) \simeq \text{Rep} \Lambda^q([\Gamma])'.$$

This induces a $\text{GL}(\alpha)$-equivariant isomorphism of affine algebraic varieties,

$$\text{Rep}_\sigma(\pi(\Gamma), \alpha) \overset{\sim}{\to} \text{Rep}(\Lambda^q([\Gamma])', \alpha),$$

defined as the collections of representations with $V_i = \mathbb{C}^{\alpha_i}$ for all $i$.

**Proof.** The first statement is precisely [CB13 Proposition 2]. For the second, both $\text{Rep}_\sigma(\pi(\Gamma), \alpha)$ and $\text{Rep}(\Lambda^q([\Gamma])', \alpha)$ are acted upon by the group $\text{GL}(\alpha)$ and the above equivalence of categories implies that there is a $\text{GL}(\alpha)$-equivariant bijection as desired. Moreover, $\text{Rep}_\sigma(\pi(\Gamma), \alpha)$ and $\text{Rep}(\Lambda^q([\Gamma])', \alpha)$ are easily seen to be affine algebraic varieties, defined as tuples of matrices satisfying certain polynomial relations, with certain polynomials inverted. To see that the above map is a $\text{GL}(\alpha)$-equivariant algebra isomorphism, observe that the proof of [CB13 Proposition 2] uses explicit invertible polynomial formulae. Therefore, the map descends to an isomorphism between the affine quotients for the action of $\text{GL}(\alpha)$. □

In order to explain how the above equivalence of categories implies the correspondence between character varieties and preprojective algebras, we shall explain how it is possible to encode the datum of a number of conjugacy classes into a star-shaped quiver. We follow [CBS06 §8] and [CB04 §2]: fix $k$ conjugacy classes $C_1, \ldots, C_k$ in $GL_n(\mathbb{C})$, for $k \geq 1$. We can encode the datum of such conjugacy classes in a combinatorial object as follows: take $A_i \in C_i$ and let $w_i \geq 1$ be the degree of its minimal polynomial, for $i = 1, \ldots, k$; choose elements $\xi_{ij} \in \mathbb{C}^\times$, $1 \leq i \leq k$, $1 \leq j \leq w_i$, such that

$$(A_i - \xi_{i1} I) \cdots (A_i - \xi_{iw_i} I) = 0.$$ 

By construction, one may take $\xi_{ij}$ as the roots of the minimal polynomial of $A_i$, for $i = 1, \ldots, k$. The closure of the conjugacy class $C_i$ is then determined by the ranks of the partial products

$$\alpha_{ij} = \text{rank}(A_i - \xi_{i1} I) \cdots (A_i - \xi_{ij} I),$$

for $A_i \in C_i$ and $1 \leq j \leq w_i - 1$. In addition, if we set $\alpha_0 = n$, we get a dimension vector $\alpha$ for the following quiver $Q_w$

$$[1, 1] \quad [1, 2] \quad \cdots \quad [1, w_1 - 1]$$

$$[2, 1] \quad [2, 2] \quad \cdots \quad [2, w_2 - 1]$$

$$\vdots \quad \vdots \quad \quad \quad \vdots$$

$$[k, 1] \quad [k, 2] \quad \cdots \quad [k, w_k - 1]$$

Now, for every $i \in \{1, \ldots, k\}$, let $t_i$ be an integer $\geq 2$ and define a Riemann surface quiver $\Gamma$ as follows: its underlying Riemann surface $X$ is given by the disjoint union

$$X = X_0 \sqcup \bigsqcup_{i \in \{1, \ldots, k\}, j \in \{1, \ldots, t_i\}} \mathbb{P}^1_{i,j},$$
where $X_0$ is an arbitrary closed Riemann surface of genus $g$ (the choice does not matter), and $\mathbb{P}^1_{i,j}$ is simply a copy of $\mathbb{P}^1$ for the index $(i, j)$. For each pair of indices $(i, j)$ fix a point $p_{i,j} \in \mathbb{P}^1_{i,j}$, and, for $i = 1, \ldots, k$, $p_i \in X_0$. Define $D$ as before to be 
\[ D = \{p_{i,j}\} \cup \{p_i\}. \]
The arrows of $\Gamma$ are listed as follows:
\[ a_{i,0} : p_{i,1} \to p_i, \text{ for } i = 1, \ldots, k; \]
\[ a_{i,j} : p_{i,j+1} \to p_{i,j}, \text{ for } i = 1, \ldots, k, j = 1, \ldots, t_i - 1. \]

**Figure 2.** An example of a Riemann surface quiver associated with a tuple of conjugacy classes.

Unfolding the definition for the objects of the category $\text{Rep}_\sigma \pi(\Gamma)$, one has that for such a Riemann surface quiver $\Gamma$ these are representations 
\[ \rho : \pi_1(X_0 \setminus \{p_1, \ldots, p_k\}) \to \text{GL}(V), \]
and linear maps 
\[ \rho_{i,0} : V_{i,1} \to V, \quad \rho_{i,j} : V_{i,j+1} \to V_{i,j} \]
and 
\[ \rho_{i,j}^* : V \to V_{i,1}, \quad \rho_{i,j}^* : V_{i,j} \to V_{i,j+1}, \]
for $i = 1 \ldots, k$ and $j = 1, \ldots, t_i - 1$, such that, if $l_i \in \pi_1(X_0 \setminus \{p_1, \ldots, p_k\})$ is the loop around $p_i$, $i = 1, \ldots, k$, 
the linear automorphism $\rho(l_i)$ satisfies the condition
\[ \sigma_i \rho(l_i) = 1_V + \rho_{i,0} \rho_{i,0}^* \]
and the linear maps $\rho_{i,j}$ and $\rho_{i,j}^*$ satisfy the equations:
\[ \sigma_{i,j+1} \rho_{i,j+1} = 1_{V_{i,j+1}} + \rho_{i,j}^* \rho_{i,j}, \]
\[ \sigma_{i,j} \rho_{i,j} = 1_{V_{i,j}} + \rho_{i,j+1} \rho_{i,j+1}^*, \]
for $i = 1, \ldots, k$, $j = 1, \ldots, t_i - 1$ and $l = 1, \ldots, t_i$, which, setting $j = l - 1$ and summing the equations involving operators on the same space $V_{i,l}$, can be rewritten as 
\[ \rho(l_i) = \sigma_i^{-1} 1_V + \sigma_i^{-1} \rho_{i,0} \rho_{i,0}^*, \]
\[ \rho_{i,j-1} \rho_{i,j-1} - \rho_{i,j} \rho_{i,j}^* = (\sigma_{i,j} - \sigma_{i,j}) 1_{V_{i,j}}, \]
\[ \sigma_{i,l} \rho_{i,l} = 1_{V_{i,l}} + \rho_{i,l} \rho_{i,l}^*, \]
for $i = 1, \ldots, k$ and $j = 1, \ldots, t_i - 1$. Now, we specialise to the case $t_i = w_i - 1$ and assume that $\dim V_{i,j} = \alpha_{i,j}$ and $\dim V = n$, where $w_i$ and $\alpha_{i,j}$ are defined as before. Through some simple algebraic computations, it is possible to see that, given $\xi_{i,j}$ as before, it is possible to find corresponding $\sigma_{i,j}$, defined as 
\[ \sigma_0 = \frac{1}{\prod_{i=1}^k \xi_{i,j}}, \quad \sigma_{i,j} = \frac{\xi_{i,j}}{\xi_{i,j+1}}, \]
such that the above sets of equations can be rewritten in terms of linear operators $\phi_{i,j}$ and $\psi_{i,j}$, for $i = 1, \ldots, k$ and $j = 1, \ldots, w_i - 1$, 
\[ V \xrightarrow{\phi_{i,1}} V_{i,1} \xrightarrow{\phi_{i,2}} V_{i,2} \xrightarrow{\phi_{i,w_i-1}} \cdots \xrightarrow{\phi_{i,w_i-1}} V_{i,w_i-1} \]
satisfying
\[
\begin{align*}
\rho(l_i) - \psi_{i1} \phi_{i1} &= \xi_{i1} 1_V \\
\phi_{i,j}\psi_{i,j} - \psi_{i,j+1}\phi_{i,j+1} &= (\xi_{i,j+1} - \xi_{i,j}) 1_{V_{ij}} \\
\phi_{i,w_1-1}\psi_{i,w_1-1} &= (\xi_{i,w_1} - \xi_{i,w_1-1}) 1_{V_{i,w_1-1}},
\end{align*}
\]
which, by [CB04 Theorem 2.1], implies that \(\rho(l_i)\) lies in the closure of the conjugacy class \(C_i, i = 1, \ldots, k\). In fact, this theorem says that this is a necessary and sufficient condition; thus, given a representation \(\rho : \pi_1(X_0 \setminus \{p_1, \ldots, p_k\}) \to GL(V)\), where \(dim V = n\) and \(\rho(l_i) \in \overline{C_i}\) for prescribed conjugacy classes \(C_1, \ldots, C_k\) in \(GL_n(\mathbb{C})\), we can find linear maps \(\rho_i, j\) and \(\rho^*_i, j\) and vector spaces \(V_{i,j}\) of dimension \(\alpha_{i,j}\) as above, such that the tuple \((V, \rho, V_{i,j}, \rho_i, j, \rho^*_i, j)\) is an object of the category \(\text{Rep}_\pi\). Then, combining this with Lemma 3.15, one has the following result.

**Theorem 3.6.** There is an isomorphism between the character variety \(\mathcal{X}(g, k, \mathbb{C})\) and the affine quotient \(\mathcal{M}_{q,0}(\Gamma, \alpha) := \text{Rep} \Lambda^q(\Gamma, \alpha)' // GL(\alpha)\).

**Remark 3.7.** From its definition, one can see that \(\text{Rep} \Lambda^q(\Gamma, \alpha)'\) is an open affine \(GL(\alpha)\)-invariant subset of \(\text{Rep}(\Lambda^q(\Gamma, \alpha))\) which is obtained by inverting certain \(GL(\alpha)\)-invariant functions (the determinants of the linear transformations corresponding to loops at the node). Since the quotient in the affine case is obtained by passing to \(G\)-invariant functions, i.e., \(\text{Spec} B//G = \text{Spec} B^G\), we deduce that the affine quotient \(\mathcal{M}_{q,0}(\Gamma, \alpha)\) can be identified with an open subset of the multiplicative quiver variety \(\mathcal{M}_{q,0}(\Gamma, \alpha)\). This is important because, as outlined in the following section, in order to show the non-existence of symplectic resolutions we prove that certain such varieties contain an open subset which is factorial and terminal.

**Remark 3.8.** We note that, in the star-shaped case, this result follows from [CBS06 Section 8]. Moreover, in the general case, Yamakawa proves a similar result to the one obtained in this section in the language of local systems on punctured surfaces, see [Yam08 Theorem 4.14] for more details.

4. Singularities of multiplicative quiver varieties

Throughout this section, which is devoted to the study of the singularities \(\mathcal{M}_{q,\theta}(Q, \alpha)\) and to the proof of Theorem 1.7, we use the notation introduced in Section 2.

In order to carry out this analysis, in Section 4.4 we describe the singular locus of the varieties in question. As one might expect, for \(\alpha \in \Sigma_{q,\theta}\), this is given by the locus of strictly semistable representations. This follows because these varieties are Poisson, the stable locus is symplectic and smooth, and its complement has codimension at least two. Since a generically non-degenerate Poisson structure on a smooth variety can only degenerate along a divisor (the vanishing locus of the Pfaffian of the Poisson bivector), we conclude that the entire smooth locus is non-degenerate. Since the strictly semistable locus is degenerate, it must therefore be singular. Moreover, in the case where \(\alpha\) is \(q\)-indivisible, a symplectic resolution can be obtained by varying \(\theta\), by Lemma 2.16 and Corollary 2.24. This proves the second statement of Theorem 1.7.

In Section 4.4 we complete the proof of Theorem 1.7 by considering strata of representation type \(\nu\beta\), where \(\alpha = n\beta\) and \(\nu\) is a partition of \(n\). We compute their codimension. As a consequence, taking \(\beta\) to be \(q\)-indivisible, for suitable \(\theta' \geq \theta\), \(\mathcal{M}_{q,\theta'}(Q, \alpha)\) has singularities in codimension \(\geq 4\). Hence by Flenner’s theorem [Fle88], its normalisation is a symplectic singularity, which proves the first statement of Theorem 1.7. Finally, we show, using Drezet’s criterion of factoriality, that the singularities along most strata \(\nu\beta\) are factorial and terminal. This proves the final statement of Theorem 1.7. Note that Section 4.4 closely follows [BS16a], where the analogous strata are considered for ordinary quiver varieties.

4.1. Singular locus of \(\mathcal{M}_{q,\theta}(Q, \alpha)\) for \(\alpha \in \Sigma_{q,\theta}\). Before proving the main statement, we need an auxiliary result which is valid for any variety endowed with a Poisson structure.

**Lemma 4.1.** Let \(X\) be a smooth variety and \(\pi \in \wedge^2 TX\) a generically non-degenerate Poisson bivector. Let \(D\) be the degenerate locus of \(\pi\). Then, if nonempty, \(D\) is a divisor.

**Proof.** By generic non-degeneracy, \(\dim X\) has to be even, therefore \(\dim X = 2d\). Define the top polyvector field \(\gamma = \wedge^d \pi\). Then, \(D\) coincides with the zero locus of \(\gamma\). On the other hand, \(\gamma\) is a section of a line bundle and, therefore, its zero locus is a divisor (if nonempty). \(\square\)
This implies the following criterion for the singular locus of a Poisson variety:

**Corollary 4.2.** Let $X$ be a Poisson variety which is smooth and symplectic in the complement of a closed Poisson subvariety $Z \subseteq X$ which has codimension at least two everywhere. Then $Z$ equals the set-theoretic singular locus of $X$.

**Proof.** Suppose for a contradiction that $X$ is smooth at a point $z \in Z$. Since $Z$ is a closed Poisson subvariety, the Poisson structure of $X$ is degenerate at $z$. It follows from Lemma 4.1 that the degeneracy locus of $X$ has codimension 1 at $z$. However, this locus is contained in $Z$, which has codimension at least two at $z$. This is a contradiction. □

**Proposition 4.3.** Let $\alpha \in \Sigma_{\theta, \rho}$. The smooth locus of $M_{\theta, \rho}(Q, \alpha)$ is $M_{\theta, \rho}^{s}(Q, \alpha)$.

**Proof.** By Proposition 2.18, $M_{\theta, \rho}^{s}(Q, \alpha)$ is smooth and symplectic. Let $Z$ be the complement. It is the union of all the non-open strata of $M_{\theta, \rho}(Q, \alpha)$. There are finitely many and these all have purely even dimension; hence $Z$ has codimension at least two everywhere (as $M_{\theta, \rho}^{s}$ is dense and it has purely even dimension, $2p(\alpha)$). Furthermore, we claim that $Z$ is a Poisson subvariety, i.e., all Hamiltonian vector fields are tangent to it. Indeed, Hamiltonian vector fields descend from GL($\alpha$)-invariant Hamiltonian vector fields on representation varieties. These integrate to formal automorphisms which commute with the $G$-action, which hence preserve the stratification by conjugacy classes of stabiliser. Therefore, the hypotheses of Corollary 4.2 are satisfied, and the statement follows. □

**Remark 4.4.** It is reasonable to ask if a stronger statement is true, which makes sense for general $\alpha$: are the connected components of the representation type strata the symplectic leaves? Equivalently, do the Hamiltonian vector fields span the tangent spaces to the representation type strata? If so, then (a) $M_{\theta, \rho}(Q, \alpha)$ has finitely many symplectic leaves, and (b) the representation type strata are all smooth. The converse statement also holds: if a stratum is smooth and it is a union of finitely many symplectic leaves, its Poisson structure must be non-degenerate outside a locus of codimension at least two. So Lemma 4.1 implies that it is actually non-degenerate.

Let us comment briefly on conditions (a) and (b). First, if $M_{\theta, \rho}(Q, \alpha)$ is a symplectic singularity, it has finitely many symplectic leaves, by [Kal06, Theorem 2.5]. Next, for a representation type $\tau = (k_{1}, \beta^{(1)}, \ldots; k_{r}, \beta^{(r)})$, there is an obvious surjection $(M_{\theta, \rho}(Q, \beta^{(1)}) \times \cdots \times M_{\theta, \rho}(Q, \beta^{(r)}))^{\text{dist}} \to C_{\theta, \rho}(Q, \alpha)$ with smooth source, where the dist refers to the open subset where the elements of the $i$-th and $j$-th factors are unequal for all $i$ and $j$. It seems reasonable to expect this to be a covering, in which case the stratum is smooth.

**4.2. Generalities on symplectic singularities.** In order to prove the first statement of Theorem 1.7 we need a criterion for the normalisation of a variety to have symplectic singularities. This is an extension of [BS16a, Lemma 6.12], using [Kal09, Theorem 1.5]:

**Proposition 4.5.** Let $X$ be a Poisson variety and assume that $\pi : Y \to X$ is a proper birational Poisson morphism from a variety $Y$ with symplectic singularities. Then the normalisation $X'$ of $X$ has symplectic singularities. Moreover, the induced map $\pi' : Y \to X'$ is Poisson.

**Proof.** In [BS16a, Lemma 6.12], the result is proved under the assumption that $X$ is in fact normal. To conclude the lemma from this result, we may apply [Kal09, Corollary 1.4, Theorem 1.5]. By these results (and their proofs), given a Poisson variety $X$, the normalisation $X'$ has a unique Poisson structure such that the normalisation map $\nu : X' \to X$ is a Poisson morphism. The map $\pi$ factors through $\nu$, and the induced map $\pi' : Y \to X'$ must be Poisson, since the Poisson bracket on $O_{X'}$ is the unique extension of the Poisson bracket on $O_{X}$ to a biderivation $O_{X'} \times O_{X'} \to O_{Y}$. Then the fact that $X'$ has symplectic singularities follows from [BS16a, Lemma 6.12]. □

**Remark 4.6.** For convenience, we will apply this result even in the case where $\pi$ is a symplectic resolution. However, in this case, the statement follows from definitions, without really requiring the results of [BS16a, Kal09], as follows. The map $\pi : Y \to X$ factors through $\pi' : Y \to X'$, which induces on $X'$ a unique Poisson structure such that $\pi'$ is Poisson; as $Y$ is non-degenerate and its symplectic form is pulled back from $X'$, $X'$ must also be non-degenerate on the smooth locus. By definition, $X'$ is then a symplectic singularity. Since
π is dominant, the Poisson structure on X is uniquely determined from the one on Y, and must be the one obtained from X’ via the inclusion O_X → O_X’. This proves the last statement.

Remark 4.7. Actually, in the above proposition, the biconditional holds: X has symplectic singularities if and only if Y does. Moreover, one can generalise to the case where X is a nonreduced Poisson scheme: in this case, the map π factors through the reduced subvariety X^{red}, which is canonically Poisson by [Kal09, Corollary 1.4].

4.3. The q-indivisible case. We now prove the second statement of Theorem [17]. Suppose that β ∈ Σ_{q,θ} is q-indivisible.

First suppose that β is real. In this case, by [CBS06, Theorem 2.1], Λ^0(Q) admits a simple rigid representation X and any other representation Y of the same dimension must be isomorphic to X, which means that the variety M_{q,θ}(Q, β) is a point. So there is nothing to prove.

Next suppose that β is imaginary, in this case one may proceed as follows: by choosing a generic stability parameter θ’ ≥ θ, there is a projective symplectic resolution

\[ \pi : M_{q,θ}(Q, β) \rightarrow M_{q,θ}(Q, β) ; \]

indeed, by [Yam08, Proposition 3.5], for θ’ generic, the stable locus M_{q,θ}(Q, β), which is smooth, coincides with the semistable locus and hence we can find θ’ ≥ θ such that M_{q,θ} is smooth and symplectic. Moreover, the fact that the morphism π exists and is projective and Poisson follows, in the θ = 0 case, from the very definitions of affine and GIT quotient and, for general θ, from Lemma 2.16. Finally, birationality of π is ensured by Corollary 2.24. Thus, we can conclude that M_{q,θ}(Q, β) admits a symplectic resolution, given by the morphism π. By Proposition 4.5 (or Remark 4.6), this implies that the normalisation of M_{q,θ}(Q, β) has symplectic singularities.

4.4. The q-divisible case. In this subsection, we prove the first and third statements of Theorem [17]. We may assume that α is q-divisible: this is automatic in the third part, whereas in the first part, the result follows from the second part (proved in the preceding subsection) in the q-indivisible case. This means that α is anisotropic, by the following result:

Lemma 4.8. Let α ∈ N_{q,θ} be q-divisible. Then α ∈ Σ_{q,θ} only if α is anisotropic. Conversely, if α = mβ and β ∈ Σ_{q,θ} is anisotropic, then α ∈ Σ_{q,θ}.

Proof. This is a generalisation of [CB02, Proposition 1.2] (in view of Remarks 0.7 and 0.8, with the same proof. For details, see Corollary 6.20(ii) below (whose proof is independent of any of the results of this section)).

Recall that a weighted partition of n is a sequence ν = (l_1, ν_1;...; l_k, ν_k) such that ν_1 ≥ ⋯ ≥ ν_k and ∑_{i=1}^k l_iν_i = n. If ν is a partition of n, we shall denote by νβ the representation type (l_1, ν_1β;...; l_k, ν_kβ).

Lemma 4.9. (1) The set Σ_{q,θ} contains \{mβ | m ≥ 1\};
(2) dim C_{q,θ}^{\nu β}(Q, nβ) = 2 \left( k + (p(\beta) - 1) \sum_{i=1}^k ν_i^2 \right);
(3) for (p(β), n) \neq (2, 2), dim M_{q,θ}(Q, nβ) = dim C_{q,θ}^{\nu β}(Q, nβ) ≥ 4 for all ν \neq (1, n).
(4) for (p(β), n) \neq (2, 2) and ν \neq (1, n), one has dim M_{q,θ}(Q, nβ) = dim C_{q,θ}^{\nu β}(Q, nβ) ≥ 8 unless one of the following holds: (i) (p(β), n) = (2, 3) and ν = (1, 2; 1, 1); (ii) (p(β), n) = (3, 2) and ν = (1, 1; 1, 1).

Proof. The arguments are completely analogous to those of [BS16a, Lemma 6.1], except here that we use the dimension estimates given by Proposition 2.19. The first statement is a consequence of Lemma 4.8.

Note that the above result has the following interesting consequence.

Proposition 4.10. Assume that all θ-stable representations of dimension γ < nβ have γ = mβ for some m. For example, this is true for β q-indivisible and θ generic. Moreover, assume that (p(β), n) \neq (2, 2). Then, M_{q,θ}(Q, nβ) is normal.

Proof. This is an immediate consequence of Proposition 2.27 and point (3) of Lemma 4.9 given that, by assumption on θ, all strata except for the open one have codimension greater than 4.
Now, let \( \alpha \in \Sigma_{q,\theta} \) be \( q \)-divisible. Write \( \alpha = n\beta \) for \( \beta \) \( q \)-indivisible and \( n \geq 2 \). For generic \( \theta' \geq \theta \), the only strata of \( M_{q,\theta'}(\alpha) \) are those given by multipartitions of \( n \) times \( \beta \). If \( (p(\beta), n) \neq (2, 2) \), then, taking into account Remark 2.23, all non-open strata have codimension at least four by Lemma 4.9.(3). Therefore \( M_{q,\theta'}(Q, \alpha) \) is a symplectic singularity by Flenner’s Theorem [Fle88]. Now, the map \( M_{q,\theta'}(Q, \alpha) \to M_{q,\theta}(Q, \alpha) \) is birational, projective, and Poisson by Corollary 2.24 and Lemma 2.16. Therefore, the normalisation of \( M_{q,\theta}(Q, \alpha) \) is itself a symplectic singularity by Proposition 4.5. This proves the first statement of Theorem 1.7.

It remains to prove the final statement of Theorem 1.7. For this purpose, assume that \( \alpha = n\beta \) for \( n \geq 2 \) and that \( \beta \in N_{q,\theta} \) (not necessarily \( q \)-indivisible or in \( \Sigma_{q,\theta} \)), such that there exists a \( \theta \)-stable representation of dimension \( \beta \). Let \( U \) be the union of all the strata indexed by \( \nu\beta \) for \( \nu \) a weighted partitions of \( n \),

\[
U := \bigcup_{\nu} C_{q,\theta}^{\nu\beta}(Q, \alpha).
\]

As well as for the previous lemma, to prove the following result one can repeat verbatim the arguments used in [BS16a, Lemma 6.2].

**Lemma 4.11.** The subset \( U \) is open in \( M_{q,\theta}(Q, \alpha) \). If \( \theta \) is generic and \( \beta \) is \( q \)-indivisible, this subset is the entire variety.

In order to prove that \( U \) is factorial, we shall follow the approach of [BS16a], which was itself inspired by some results of Drezet, [Dre91], on factoriality of points in moduli spaces of semistable sheaves on rational surfaces. Assuming the notation above, with \( \pi : \text{Rep}^{\theta-ss}(\Lambda^q, \alpha) \to M_{q,\theta}(Q, \alpha) \) denoting the quotient map, define \( V := \pi^{-1}(U) \). We aim at proving that \( V \) is a local complete intersection and that it is factorial and normal. We shall then descend the factoriality property to the subvariety \( U \).

**Proposition 4.12.** \( V \) is a local complete intersection, factorial and normal.

The proof of this proposition follows closely the arguments used in [BS16a, Proposition 6.5].

**Proof of Proposition 4.12.** Since \( V \) is open inside \( \text{Rep}^{\theta-ss}(\Lambda^q, \alpha) \), Proposition 2.22 implies that it is a local complete intersection. To prove normality and factoriality, recall that a local complete intersection satisfies Serre’s \( S_2 \) property, so Serre’s criterion implies that it is normal if it is smooth outside a locus of codimension at least 2. Moreover, by a result of Grothendieck ([KLS06, Theorem 3.12]), a local complete intersection which is smooth outside a locus of codimension at least 4 is factorial. Put together, to show that \( V \) is normal and factorial, it suffices to show that it is smooth outside of a locus of codimension at least 4. For this, one can repeat verbatim the arguments used in [BS16a, Proposition 6.5], replacing Corollary 6.4 and Lemma 6.1 with Lemma 2.23 and Lemma 4.9 respectively. \( \square \)

In order to descend factoriality from \( V \) to \( U \) we use Drezet’s method. In particular, [BS16a, Theorem 6.7] holds true in this context as well with no change in the proof of the result, as we have already made sure that all of the tools used there are still applicable here, Proposition 4.12 being the most important one. Thus, the corresponding statement of [BS16a, Corollary 6.9] is the following:

**Theorem 4.13.** \( U \) is a factorial variety.

We omit the proof, as it is exactly the same as in [BS16a, Corollary 6.9].

Using the previous theorem and the estimates on the codimension of the singular locus, one can conclude that \( M_{q,\theta}(Q, \alpha) \) does not admit a symplectic resolution. We state this formally below, where we also recall our running hypotheses for the reader’s convenience.

**Theorem 4.14.** Let \( \alpha = n\beta \in \Sigma_{q,\theta} \) be anisotropic imaginary, for \( n \geq 2 \), such that there exists a \( \theta \)-stable representation of \( \Lambda^q \) of dimension \( \beta \), and \( (p(\beta), n) \neq (2, 2) \). Then \( M_{q,\theta}(Q, \alpha) \) has an open subset which is factorial, terminal, and singular. Hence it does not admit a symplectic resolution. Moreover, if \( \theta \) is generic and \( \alpha \) is \( q \)-indivisible, then this open subset is the entire variety.

**Proof.** The subset \( U \) is singular, since it contains the non-open stratum \( (n, \beta) \). It is factorial by Theorem 4.13. Under the assumptions, the singular strata in \( U \) all have codimension at least four, hence also the singular locus. Thus, \( U \) is terminal by [Nam01], since it has symplectic singularities and the singular locus has codimension at least four. \( \square \)
This completes the proof of the third and final statement of Theorem 1.7. Its proof is complete.

4.5. **Proof of Corollary 1.8** Write \( \alpha = m \beta \) for \( \beta \) \( q \)-indivisible. Note that for \( \theta \) generic, then the only possible decompositions of \( \alpha \) are into multiples of \( \beta \). If \( \alpha \) is \( q \)-indivisible, it therefore follows trivially that \( \alpha \in \Sigma_{q, \theta} \); since the only stratum in \( M_{q, \theta}(Q, \alpha) \) is the open one of stable representations, it also follows from Proposition 2.18 that \( M_{q, \theta} \) is smooth symplectic. Suppose that \( \alpha \) is \( q \)-divisible. It then follows from Corollary 6.20(ii) below that \( \alpha \) is in \( \Sigma_{q, \theta} \) if and only if it is anisotropic. This completes the proof of part (i).

Part (ii) follows from Proposition 4.10 and Theorem 1.7. The first statement of part (iii) follows from Theorem 1.7. Finally, the last statement follows from 2.18 because, in this case, there is only one stratum in \( M_{q, \theta}(Q, \alpha) \), consisting of \( \theta \)-stable representations.

4.6. **The anisotropic imaginary** \((p(\alpha), n) = (2, 2)\) case. The only case left out in this analysis is that of \( 2 \alpha \in \Sigma_{q, \theta} \) for \( \alpha \in N_{q, \theta} \) satisfying \( p(\alpha) = 2 \). The analogous question of existence of a symplectic resolution in the setting of Nakajima quiver varieties is settled in [BS16a, Theorem 1.6], where it is shown that, for generic \( \theta \), blowing up the ideal sheaf defining the singular locus gives a symplectic resolution of singularities. This is achieved by showing that, étale locally, the variety is isomorphic to the product of \( \mathbb{C}^4 \) with the closure of the cone given by the six dimensional adjoint orbit closure to \( Sp(V) \), for \( V \) a 4-dimensional symplectic vector space, see [BS16a, Theorem 5.1] and the references therein. Given this, one might conjecture that an analogous result holds for multiplicative quiver varieties, and such a result should be proved by studying the étale local structure of the variety; in fact, by Artin’s approximation theorem [Art69], it would be sufficient to give a description of the formal neighbourhood of a point. This will be discussed in a future work. For more details, see Section 6.

5. **Combinatorics of multiplicative quiver varieties**

In this section we study some combinatorial problems which are related to the geometry of multiplicative quiver varieties. Indeed, an interesting problem is to classify all the possible \( (2, 2) \)-cases: these are the main \( q \)-divisible cases for which we conjecture that there exists a symplectic resolution. In the next subsection we carry out these computations in the case of crab-shaped quivers, i.e., we classify all punctured character varieties that can, conjecturally, be symplectically resolved. We shall see how most of the \( (2, 2) \)-cases occur in the case of star-shaped quivers, i.e., when the surface has genus \( 0 \). It is also important to point out that the classification below yields a classification of the crab-shaped quivers for which the factoriality and terminality results apply: Suppose \( Q \) is crab-shaped. As we will explain in Section 6, we can reduce to the case that \( \alpha \in \mathcal{F}(Q) \). Moreover, if \( \alpha \) is \( q \)-indivisible, it is actually in \( \Sigma_{q, \theta} \) (by Theorem 6.16 below), hence Theorem 1.7 implies that \( M_{q, \theta}(Q, \alpha) \) admits a projective symplectic resolution. Suppose, on the contrary, that \( \alpha = n \beta \) for \( \beta \in \Sigma_{q, \theta} \) \( q \)-indivisible and \( n \geq 2 \). Then one of the following holds:

(a) \( \beta \) is one of the cases described below \( (p(\beta) = 2) \); then for \( n = 2 \), a symplectic resolution of \( M_{q, \theta}(Q, \alpha) \) is conjecturally obtained by varying \( \theta \) and blowing up the reduced singular locus; while for \( n > 2 \), \( M_{q, \theta}(Q, \alpha) \) is factorial, terminal, and singular for generic \( \theta \);

(b) The support of \( \alpha \) is affine Dynkin, and \( \beta \) is an imaginary root \( (\beta = m \delta \text{ for } m \geq 1) \);

(c) We are in another case. As we will explain in the proof of Corollary 6.27, \( M_{q, \theta}(Q, \alpha) \neq \emptyset \). Thus, by Theorem 1.7, \( M_{q, \theta}(Q, \alpha) \) is factorial, terminal, and singular.

Finally, it is possible to have \( \alpha \in \mathcal{F}(Q) \) \( q \)-divisible but not a multiple of an element of \( \Sigma_{q, \theta} \). In this case, as explained in Theorem 6.16 in the next section, \( \alpha \) must have a very special form. Namely, \( \alpha = p \beta \) for \( p \geq 2 \) prime, and \( Q \) is a quiver obtained from an affine Dynkin one (of types \( \tilde{A}_6, \tilde{D}_4, \tilde{E}_6, \tilde{E}_7 \), or \( \tilde{E}_8 \)) by adding an additional vertex and arrow connecting it to an extending vertex of the affine Dynkin quiver. Furthermore, the dimension vector \( \beta = (m \delta, 1) \), with notation meaning putting \( m \) times the primitive imaginary root of the affine Dynkin quiver and then attaching a new vertex to an extending vertex of the quiver. The extending vertex of the former quiver has dimension \( m \), whereas the new vertex has dimension one. In this case, by expectation (*), we don’t anticipate a stable representation of dimension \( \alpha \) to exist, so the reasoning above likely does not apply. We therefore leave the question of existence of symplectic resolutions open in this case.

Thus, modulo some general results we will explain in Section 6, the combinatorial work in this section completes the classification of dimension vectors in the crab-shaped case which either admit symplectic
resolutions, do not admit them, or for which the question is left open (with an expectation or a conjecture given).

5.1. (2,2) cases for crab-shaped quivers. The analysis is based on some standard numerical arguments and the constraints on the dimension vector $\alpha$ for it to satisfy the conditions of Section 3, i.e., it has to represent the multiplicities of the eigenvalues in the prescribed conjugacy class.

**Theorem 5.1.** There are exactly 13 pairs $(Q, \alpha)$, where $Q$ is a star-shaped quiver as in Section 3 and $\alpha \in F(Q)$ is in the fundamental region, such that $p(\alpha) = 2$. Such pairs are depicted as follows, where a vertex is substituted by the corresponding entry of the dimension vector:

(5.1.1)

\[
\begin{array}{cccc}
1 & 1 & 1 \\
1 & 2 & 1 \\
\end{array}
\]

(5.1.2)

\[
\begin{array}{cccc}
1 & 2 & 1 \\
1 & 3 & 2 & 1 \\
\end{array}
\]

(5.1.3)

\[
\begin{array}{cccc}
2 & 2 \\
1 & 4 & 3 & 2 & 1 \\
\end{array}
\]

(5.1.4)

\[
\begin{array}{cccc}
3 & 2 & 1 \\
1 & 2 & 4 & 3 & 3 & 1 \\
\end{array}
\]

(5.1.5)

\[
\begin{array}{cccc}
2 & 2 \\
2 & 4 & 2 & 1 \\
\end{array}
\]

(5.1.6)

\[
\begin{array}{cccc}
4 & 3 & 2 & 1 \\
2 & 5 & 4 & 3 & 2 & 1 \\
\end{array}
\]
Remark 5.2. It is important to highlight that quivers (5.1.5), (5.1.9), (5.1.11), (5.1.13) are the affine Dynkin quivers $\tilde{D}_4$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, respectively, with dimension vector given by $(2\delta, 1)$, where $\delta$ is the minimal isotropic imaginary root of the corresponding quiver. See Remark 5.5 for the significance of this.
Proof of Theorem 5.1. Note that \( p(\alpha) = 2 \) if and only if \( \langle \alpha, \alpha \rangle = -1 \). Let us calculate the value of \( \langle \alpha, \alpha \rangle \) explicitly, for \( \alpha \) a general dimension vector. The general star-shaped quiver has \( g \) loops and \( k \) legs, each of which has \( l_i \) arrows, \( i = 1, \ldots, k \). We have

\[
\langle \alpha, \alpha \rangle = (1 - g)n^2 + \sum_{i,j} \alpha_{i,j}^2 - n \sum_{i=1}^{k} \alpha_i - 1 - \sum_{i=1}^{k} \sum_{j=1}^{l_i} \alpha_i \alpha_{i,j+1}.
\]

Assume now that \( \alpha \in \mathcal{F}(Q) \) and that \( \langle \alpha, \alpha \rangle = -1 \); then, given that \( \langle \alpha, \alpha \rangle = \sum_{i \in Q_0} \alpha_i \langle \alpha, e_i \rangle = \sum_{i \in Q_0} \alpha_i \langle e_i, \alpha \rangle \), this implies that there can only be two possibilities:

1. There exists a unique vertex \( i \in Q_0 \) such that either \( \alpha_i = 1 \) and \( \langle \alpha, e_i \rangle = -2 \), or \( \alpha_i = 2 \) and \( \langle \alpha, e_i \rangle = -1 \), with \( \langle \alpha, e_j \rangle = 0 \) for \( j \neq i \); this implies that, denoted by \( \text{Adj}(i) \) the set of vertices which are adjacent to \( i \), one has \( \sum_{j \in \text{Adj}(i)} \alpha_j = 5 \) for \( \alpha_i = 2 \) and \( \sum_{j \in \text{Adj}(i)} \alpha_j = 4 \) for \( \alpha_i = 1 \);

2. There are two distinct vertices \( i \) and \( i' \) such that \( \langle \alpha, e_i \rangle = \langle \alpha, e_{i'} \rangle = -1 \) and \( \alpha_i = \alpha_{i'} = 1 \), with \( \langle \alpha, e_j \rangle = 0 \) for \( j \neq i, i' \). In this case one has \( \sum_{j \in \text{Adj}(k)} \alpha_j = 3 \) for \( k = i, i' \).

In this case, if \( i \) or \( i' \) are the central vertex, then the only possibility is given by the quiver (1) in the statement of the theorem. Otherwise, we have that, if \( v \) is the central vertex, then \( \langle \alpha, e_v \rangle = 0 \), which implies that \( \sum_{j \in \text{Adj}(v)} \alpha_j = 2n \), where \( \alpha_v = n \); indeed,

\[
0 = \langle \alpha, e_v \rangle = \langle \alpha, e_v \rangle + \langle e_v, \alpha \rangle = n - \sum_{k \to v} \alpha_k + n - \sum_{v \to l} \alpha_l = 2n - \sum_{j \in \text{Adj}(v)} \alpha_j.
\]

Now, fix a branch along which none of the special vertices \( i \) and \( i' \) appear, let \( l \) be its length and let \( \beta_0 = n, \beta_1, \ldots, \beta_l \) the components of the vector \( \alpha \) along the branch.

Then, using that \( \langle \alpha, e_j \rangle = 0 \) for \( j \neq i, i' \), we get the recursive formula

\[
2\beta_j = \beta_{j-1} + \beta_{j+1},
\]

for \( j = 1, \ldots, l - 1 \), and also \( \beta_{l-1} = 2\beta_l \), which implies that

\[
\beta_j = (l + 1 - j)\beta_l.
\]

Therefore, the branch has the form

\[
n \rightarrow n - c \rightarrow n - 2c \rightarrow \cdots \rightarrow c,
\]

where \( c \) is a positive integer such that \( c | n \). Moreover, in order for condition a) to be satisfied there has to be one branch ending with one of the following

\[
5 \rightarrow 2, \quad 4 \rightarrow 2 \rightarrow 1, \quad 4 \rightarrow 1,
\]

and, thus, having the form

\[
(\S) \quad n - 3 \rightarrow \ldots, \rightarrow 5 \rightarrow 2,
\]

\[
n - 2 \rightarrow \ldots, \rightarrow 4 \rightarrow 2 \rightarrow 1,
\]

\[
n - 3 \rightarrow \ldots, \rightarrow 4 \rightarrow 1
\]

respectively; for condition b), there have to be two branches ending as

\[
3 \rightarrow 1,
\]

having the form

\[
(\SS) \quad n - 2 \rightarrow \ldots \rightarrow 3 \rightarrow 1.
\]
Therefore, we are left to consider a star-shaped quiver where all but one or two branches are as follows:

Moreover, if the quiver satisfies condition a), then \( l = k - 1 \) and there is an additional branch having one of the forms in (§); on the other hand, if the quiver is as in case b), then \( l = k - 2 \) and there are two additional legs of the form described by (§§).

We shall now use some numerical arguments to prove that, among all such possibilities, only the ones listed in the statement of the theorem can actually occur. First, let us spell out how the equality \( 0 = (\alpha, e_v) \) can be rephrased: one has that

\[
0 = (\alpha, e_v) \iff 2n = \sum_{i=1}^{k} (n - a_i) \iff \sum_{i=1}^{k} a_i = (k - 2)n.
\]

Therefore, one has the following possibilities:

a) in the cases of a branch ending with \( 5 \to 2 \) or \( 4 \to 1 \) the equality \( 0 = (\alpha, e_v) \) reads as

\[
\frac{3}{n} + \sum_{i=1}^{k-1} \frac{1}{n/a_i} = k - 2,
\]

where \( n \equiv 2 \pmod{3} \) and \( n \equiv 1 \pmod{3} \), respectively, and \( n > a_i \geq 2, \ a_i | n \) for every \( i \); these shall be mentioned in the following as cases a.1 and a.2. On the other hand for a branch ending with \( 4 \to 2 \to 1 \) we have

\[
\frac{2}{n} + \sum_{i=1}^{k-2} \frac{1}{n/a_i} = k - 2,
\]

where \( n \) has to be even and \( a_i | n \); this is renamed as case a.3.

b) there are two branches \( 3 \to 1 \) and \( 0 = (\alpha, e_v) \) is equivalent to

\[
\frac{4}{n} + \sum_{i=1}^{k-2} \frac{1}{n/a_i} = k - 2,
\]

and \( a_i | n \) for every \( i \) and \( n \) has to be odd, and \( a_i < n \), for every \( i \).

In cases a.1 and a.2 one has that \( n \geq 4 \) which forces \( k \leq 4 \): indeed, one has that for \( n \geq 4 \ n/a_i \geq 2 \) and, therefore,

\[
\frac{3}{n} + \sum_{i=1}^{k-1} \frac{1}{n/a_i} \leq \frac{3}{4} + \frac{k - 1}{2} < \frac{k + 1}{2},
\]

which implies that

\[
k - 2 < \frac{k + 1}{2},
\]

and thus \( k \leq 4 \). For \( k = 4 \) and \( n = 4 \) it is easily checked that quiver (5.1.3) in the statement of the result is the only possibility. If \( k = 3 \), then the following inequality holds:

\[
\left( \frac{1}{2} + \frac{1}{4} \right) n \geq \left( 1 - \frac{3}{n} \right) n = n - 3,
\]
which forces $n \leq 12$; one can check that case a) cannot be realised for $n = 4, 7, 11$ and that the cases $n = 5, 8, 10$ give quivers \((5.1.6), (5.1.10)\) and \((5.1.12)\) respectively. Next, for case a.3 one has: $n$ even and $k \leq 4$: indeed, since $n \geq 4$, from equation \((5.1.14)\), one has that

\[
\frac{2}{4} + \frac{k - 1}{2} \geq k - 2,
\]

which implies that

\[
\frac{k}{2} \leq 2.
\]

If $k = 4$ and $n = 4$ then one gets quiver \((5.1.5)\). If $k = 3$, then $n \leq 12$: indeed, from equation \((5.1.14)\), we have

\[
\frac{2}{n} + \frac{1}{2} + \frac{1}{3} \geq 1
\]

This leads to: quiver \((5.1.3)\) for $n = 4$, quivers \((5.1.8)\) and \((5.1.9)\) for $n = 6$, quiver \((5.1.11)\) for $n = 8$, and quiver \((5.1.13)\) for $n = 12$.

We turn now to case b): $n$ is odd and $\geq 3$; this implies that $k \leq 4$: indeed, in the same way as the previous case, equation \((5.1.14)\) gives

\[
\frac{4}{3} + \frac{k - 2}{3} \geq k - 2 \implies \frac{8}{3} \geq \frac{2k}{3}.
\]

Therefore, setting $k = 4$ forces $n = 3$, which leads to quiver \((5.1.2)\). When $k = 3$ one has that $n \leq 6$, which implies that $n = 3$ or $n = 5$. One checks that $n = 3$ is impossible, whereas $n = 5$ gives quiver \((5.1.7)\). Since we have dealt with all the possible cases, the proof is complete.

**Theorem 5.3.** Assume that $g \geq 1$. Then, the only pairs $(Q, \alpha)$, where $Q$ is a crab-shaped quiver and $\alpha \in F(Q)$ is such that $\langle \alpha, \alpha \rangle = -1$ are the following:

\[
(5.1.15)
\]

\[
(5.1.16)
\]

**Remark 5.4.** Parallel to Remark 5.2, in the second case above, the quiver and dimension vector are also of the form $(2\delta, 1)$ where $\delta = (1)$ is the primitive imaginary root of affine type $\tilde{A}_0$ (the Jordan quiver with one vertex and one arrow).

**Proof of Theorem 5.3.** As in the arguments of the previous theorem, we see that, if $v$ is the central vertex and $\alpha \in F$, then there are 3 possibilities for the value of $(\alpha, e_v)$, i.e., $(\alpha, e_v)$ can be either 0, $-1$ or $-2$. In general one has that

\[
(\alpha, e_v) = 2(1 - g)n - \sum_{i=1}^{k} \alpha_{i,1}.
\]

If $k = 0$, then, by the argument leading to cases (a) and (b) in the proof of Theorem 5.1 one must have $n = 1$ and $g = 2$, which gives the first quiver of the statement of the result. Thus, one is left to show that for $k \geq 1$, there are no crab-shaped quivers satisfying the mentioned conditions other than the second quiver in
the statement of the theorem. If $k \geq 1$, $n > 1$, which implies that either $(\alpha, e_v) = 0$ or $-1$. In the first case, we get that

$$2(1 - g)n = \sum_{i=1}^{k} \alpha_{i,1};$$

but $g \geq 1$, which gives $\sum_{i=1}^{k} \alpha_{i,1} \leq 0$, a contradiction. If $(\alpha, e_v) = -1$, then one must have $n = 2$ and $\alpha_{i,1} = 1$ for $i = 1, \ldots, k$ and $g = 1$. This implies that

$$k = 5 - 4g,$$

which forces $k = 1$. Therefore, we get the quiver

$$\begin{array}{ccc}
\circ & \circ & \\
\downarrow & \downarrow & \downarrow \\
2 & 1 & .
\end{array}$$

Since we dealt with all the possible cases the proof is complete. □

Remark 5.5. Note that, to get a list of all the $(2, 2)$-cases one has to take each of the pairs $(Q, \alpha)$ drawn above and consider the pair $(Q, 2\alpha)$. Moreover, given $q \in (\mathbb{C}^\times)^{Q_0}$ and $\theta \in \mathbb{Z}^{Q_0}$, it follows from Theorem 6.16 below that the given $\alpha$ are in $\Sigma_{q,\theta}$ (since they are already in $\mathcal{F}(Q)$) if and only if the following are satisfied:

(a) they are in $N_{q,\theta}$, i.e., $q^\alpha = 1$ and $\theta \cdot \alpha = 0$, and (b) in the $(2\delta, 1)$ cases (mentioned in Remarks 5.2 and 5.4), $\delta \notin N_{q,\theta}$, i.e., $q^\delta \neq 1$ or $\theta \cdot \delta \neq 0$.

6. General dimension vectors and decomposition

One fundamental tool in the classification theorem [BS16a, Theorem 1.4] is the canonical decomposition of a dimension vector of a quiver variety into summands which lie in $\Sigma_{q,\theta}$, which is the additive version of the set $\Sigma_{q,\theta}$ defined in this paper (one just needs to replace the condition $q^\alpha = 1$ with $\lambda \cdot \alpha = 0$). This appears in Crawley-Boevey’s canonical decomposition in the additive case (extended to the case $\theta \neq 0$ in [BS16a]). Combinatorially, it says:

Lemma 6.1. [CB02, Theorem 1.1], [BS16a, Proposition 2.1] Let $\alpha \in \mathbb{N}R^{+}_{\lambda,\theta}$. Then $\alpha$ admits a unique decomposition $\alpha = n_1\sigma^{(1)} + \cdots + n_k\sigma^{(k)}$ as a sum of elements $\sigma^{(i)} \in \Sigma_{\lambda,\theta}$ such that any other decomposition of $\alpha$ as a sum of elements from $\Sigma_{\lambda,\theta}$ is a refinement of this decomposition.

Geometrically, the statement (together with the consequence for symplectic resolutions) is:

Theorem 6.2. [CB02, Theorem 1.1], [BS16a, Theorem 1.4] The symplectic variety $\mathcal{M}_{\lambda,\theta}(Q, \alpha) = \mu^{-1}(\lambda)^{\theta - ss}/\text{GL}(\alpha)$ is isomorphic to the product

$$\mathcal{M}_{\lambda,\theta}(Q, \alpha) \cong \prod_{i=1}^{k} S^{n_i}\mathcal{M}_{\lambda,\theta}(Q, \sigma^{(i)}).$$

Moreover, it admits a symplectic resolution if and only if each $\mathcal{M}_{\lambda,\theta}(Q, \sigma^{(i)})$ admits a symplectic resolution.

For multiplicative quiver varieties, the combinatorial statement still holds, but it is not clear that such a geometric decomposition holds. We instead prove a weaker statement, which gives a decomposition into factors which might not be minimal, but still has all of the needed properties. Moreover, the resulting classification of symplectic resolutions is the same statement as if the canonical decomposition as above held. As a result we are able to generalise Theorem 1.7 to the case of general dimension vectors (Theorem 6.26), and give its specialisation to the crab-shaped case (Corollary 6.27). To complete the proof, we need to establish that $\mathcal{M}_{q,\theta}(Q, \alpha) \to \mathcal{M}_{q,\theta}(Q, \alpha)$ is a symplectic resolution for many $\alpha$ not in $\Sigma_{q,\theta}$ (Theorem 6.22), in order to handle such factors appearing in the decomposition. In the additive case, such resolutions include the Hilbert schemes of points in $\mathbb{C}^2$ and in hyperkähler ALE spaces (i.e., minimal resolutions of du Val singularities).
6.1. Flat roots. In order to write a product decomposition in the multiplicative setting, the dimension vectors for the factors need to be more general than those in $\Sigma_{q,\theta}$. The dimension vectors turn out to include all of the “flat roots”, which are those for which the moment map is flat (this is true for roots in $\Sigma_{q,\theta}$). This condition is also very important in order to have a geometric understanding of the varieties.

**Definition 6.3.** A vector $\alpha \in N_{q,\theta}$ is called flat if, for every decomposition $\alpha = \alpha^{(1)} + \cdots + \alpha^{(m)}$ with $\alpha^{(i)} \in R_{q,\theta}^+$, we have $p(\alpha) \geq p(\alpha^{(1)}) + \cdots + p(\alpha^{(m)})$. Let $\Sigma_{q,\theta}$ be the set of flat roots.

**Remark 6.4.** As in [CB01, Theorem 1.1], we could alternatively have made the definition only requiring $\alpha^{(i)} \in N_{q,\theta}$. Indeed, it follows from the proof of the decomposition theorem (Theorem 6.14) below that if $\alpha \in N_{q,\theta}$, then there is a decomposition $\alpha = \beta^{(1)} + \cdots + \beta^{(k)}$ with $\beta^{(i)} \in \Sigma_{q,\theta} \cup \mathbb{N}_{\geq 2} \cdot \Sigma_{q,\theta}^{iso} \subseteq R_{q,\theta}^+$ such that $p(\alpha) \leq p(\beta^{(1)}) + \cdots + p(\beta^{(k)})$. Hence, if we know the inequality when the $\alpha^{(i)} \in R_{q,\theta}^+$, we also know it when the $\alpha^{(i)} \in N_{q,\theta}$.

The definition has the following interpretation. Let $\text{SL}(\alpha) := \{ g \in \text{GL}(\alpha) \mid \prod_{i \in Q_0} \det(g_i) = 1 \} \subset \text{GL}(\alpha)$. Note that $\Phi_q$ factors through the inclusion $\text{SL}(\alpha) \to \text{GL}(\alpha)$; let $\overline{\Phi}_\alpha : \text{Rep}^q(\mathbb{C}^\alpha)^{\theta-ss} \to \text{SL}(\alpha)$ be the factored map.

**Proposition 6.5.** If $\alpha$ is a flat root, then $\overline{\Phi}_\alpha$ is flat over a neighbourhood of $q$. In particular, $\Phi_q^{-1}(q)^{\theta-ss}$ is a complete intersection and is equidimensional of dimension $g_q + 2p(\alpha)$.

**Proof.** The second statement follows from the argument of Proposition 2.22 (following [CB01, Theorem 1.1]); all arguments go through with the strict inequality replaced by the non-strict one, except that no statement can be deduced about the stable (or simple) representations forming a dense subset. For the first statement, concerning flatness, note that $\dim \overline{\Phi}_\alpha^{-1}(q)^{\theta-ss} = \dim \text{Rep}^q(\mathbb{C}^\alpha)^{\theta-ss} - \dim \text{SL}(\alpha)$. Then the statement follows from the following general considerations. Given a morphism of varieties $f : X \to Y$, by upper semicontinuity of the fibre dimension, the minimum fibre dimension is $\dim Y - \dim X$ and the locus in $Y$ where the fibres have this minimal dimension is open. Next, it is a standard fact that a morphism from a Cohen-Macaulay variety to a smooth one is flat if and only if the dimension of every fibre equals $\dim X - \dim Y$. Therefore, given any $y \in Y$ such that $f^{-1}(y)$ attains this minimum dimension, there exists an open neighbourhood $U$ of $y$ over which $f$ is flat. \hfill $\square$

Putting this together with Proposition 2.22 we conclude the following analogue of the last statement of Proposition 2.22.

**Corollary 6.6.** For $\alpha$ a flat root, a dense subset of $\Phi_q^{-1}(q)^{\theta-ss}$ is given by the union of preimages of strata of types $(1, \beta^{(1)}; \ldots; 1, \beta^{(r)})$ with $p(\alpha) = \sum_{i=1}^r p(\beta^{(i)})$.

Observe that $\Sigma_{q,\theta} \subseteq \overline{\Sigma}_{q,\theta}$. The opposite inclusion does not hold: for instance, with $(q, \theta) = (1, 0)$, one can take the quiver with two vertices and two arrows, one a loop at the first vertex, the other an arrow to the second vertex. Then the dimension vector $(m, 1)$ is flat for all $m$, but only in $\Sigma_{q,\theta}$ for $m = 1$.

**Remark 6.7.** Note that, for every pair $q, \theta$, there always exists $q'$ such that $N_{q',0} = N_{q,\theta}$, and hence $\Sigma_{q,\theta} = \Sigma_{q',0}$ and $\overline{\Sigma}_{q,\theta} = \overline{\Sigma}_{q',0}$. Indeed, let $z \in \mathbb{C}^\times$ be a multiplicatively independent element from the $q_i$ (i.e., $(z,q_i)/q_i$ is infinite cyclic, where $(\cdot)$ denotes the multiplicative group generated by the given elements). Set $q'_i := q_i z^{\theta_i}$. Then $q'$ has the desired properties.

**Remark 6.8.** Similarly, given any parameters in the additive case, $\lambda \in \mathbb{C}Q_0, \theta \in \mathbb{Z}Q_0$, we also can construct $q' \in (\mathbb{C}^\times)^Q$ such that $N^a_{x,\theta} = N_{q',0}$ where now $N^a$ denotes the additive construction; hence $\Sigma_{x,\theta} = \Sigma_{q',0}$ and $\overline{\Sigma}^a_{x,\theta} = \overline{\Sigma}_{q',0}$.

We recall, following [CB01] and [Su06], how to classify flat roots in terms of the fundamental region.

**Definition 6.9.** We say that the transformation $\alpha \mapsto s_v(\alpha)$ is a $(1)$-reflection if $s_v(\alpha) = \alpha_v - e_v$.

We point out a useful geometric consequence of this definition:

**Proposition 6.10.** Suppose that $\alpha \mapsto s_v(\alpha)$ is a $(1)$-reflection and that $q_v = 1$ and $\theta_v = 0$. Then there is a reflection isomorphism $\mathcal{M}_{q,\theta}(s_v \alpha) \cong \mathcal{M}_{q,\theta}(\alpha)$.
Proof. There is an obvious map \( M_{\alpha}(\alpha - e_v) \rightarrow M_{\alpha}(\alpha) \), given by \( \rho \mapsto \rho \oplus \mathbb{C}_v \), where \( \mathbb{C}_v \) is the trivial representation (all arrows act as zero). We claim that it is an isomorphism. In the decomposition of any \( \theta \)-polystable representation of dimension \( \alpha \) into stable representations, at least one factor must have dimension vector which has positive pairing with \( e_v \). By \([\text{CBS06}, \text{Lemma 5.1}]\) (in the case \( \theta = 0 \), which extends to the general case by replacing simple representations by \( \theta \)-stable ones), this summand must be \( \mathbb{C}_v \) itself. Therefore, the obvious map is an isomorphism.

**Definition 6.11.** Given \( \alpha \in R^+_{q,\theta} \), call a sequence \( v_1, \ldots, v_m \in Q_0 \) a reflecting sequence if, setting \((q^{(i)}, \theta^{(i)}, \alpha^{(i)})) := (u_1, \ldots, u_v(q), \rho_1, \ldots, \rho_v(\theta), s_v, \ldots, s_v(q))\), we have (a) \( \alpha^{(m)} \in \mathcal{F}(Q) \cup \{e_v \mid v \in Q_0\} \), and (b) \( \alpha^{(i)} < \alpha^{(i-1)} \) for all \( i \).

**Lemma 6.12.** A reflecting sequence always exists.

Proof. By definition, \( \alpha \) is a root if and only if there exists a sequence of reflections at loopfree vertices taking \( \alpha \) to either the fundamental region or to an elementary root \( e_v \) (it is imaginary in the former case and real in the latter case). Now, given \( \alpha \in \mathbb{N}^Q_0 \), let \( N_\alpha := |\{\beta \in R^+ \text{ real} \mid (\alpha, \beta) > 0\}| \). Then each reflection satisfying (b) decreases \( N_\alpha \) by one, and a non-trivial reflection not satisfying (b) increases \( N_\alpha \) by one. Now assume that \( \alpha \in R^+_{q,\theta} \). Then \( N_\alpha < \infty \). Since \( s_v(R^+ \setminus \{e_v\}) \subseteq R^+ \), an arbitrary sequence of reflections satisfying (b) will remain in \( \mathbb{N}^Q_0 \). Thus, if \( \alpha \) is real, an arbitrary \( N_\alpha - 1 \) reflections satisfying (b) will send \( \alpha \) to \( e_v \) for some \( v \in Q_0 \), and if \( \alpha \) is imaginary, then an arbitrary \( N_\alpha \) reflections satisfying (b) will take \( \alpha \) to \( \mathcal{F}(Q) \). □

As in \([\text{Su06}, \text{Theorem 1.2}]\), we have the following.

**Theorem 6.13.** Let \( \alpha \in R^+_{q,\theta} \). Pick any sequence of vertices \( v_1, \ldots, v_m \in Q_0 \) such that, for \((q^{(i)}, \theta^{(i)}, \alpha^{(i)})) := (u_1, \ldots, u_v(q), \rho_1, \ldots, \rho_v(\theta), s_v, \ldots, s_v(q))\), we have \( \alpha^{(m)} \in \mathcal{F}(Q) \) and \( \alpha^{(i)} < \alpha^{(i-1)} \). Then \( \alpha \) is flat if and only if (a) \( \alpha^{(m)} \in \mathcal{F}(Q) \) is flat, and (b) for every \( i \), either (b1) \((q^{(i)}, \theta^{(i)}, \alpha^{(i)})\) is an admissible reflection (Definition 2.9) of \((q^{(i-1)}, \theta^{(i-1)}, \alpha^{(i-1)})\) (i.e., \( q^{(i-1)} \neq q^{(i)} \) or \( \theta^{(i-1)} \neq \theta^{(i)} \)), or (b2) \( \alpha^{(i)} \) is a \((-1)\)-reflection of \( \alpha^{(i-1)} \).

Analogously to \([\text{Su06}]\), we will show below that \( \alpha^{(m)} \in \mathcal{F}(Q) \) is flat if and only if it is not of the form \( m\delta \) for \( \delta \) the minimal imaginary root of an affine Dynkin subquiver, \( m \geq 2 \), and \( \ell \geq 1 \) is such that \( q^\delta \) is a primitive \( \ell \)-th root of unity.

The theorem actually gives an algorithm to determine if a root is flat, by playing a variant of the numbers game \([\text{Moz90}]\) (with a cutoff in the inadmissible case as in \([\text{CS11}]\)).

**Proof.** Under an admissible reflection the condition of being flat does not change since \( s_{v_i} : R^+_{q^{(i-1)}, \theta^{(i-1)}} \rightarrow R^+_{q^{(i)}, \theta^{(i)}} \) is a bijection (as \( e_v \notin R^+_{q^{(i-1)}, \theta^{(i-1)}} \), cf. \([\text{CBS01}, \text{Lemma 5.2}]\). Here we let \( q^{(0)} := q \) and \( \theta^{(0)} := \theta \).

If we apply a \((-1)\)-reflection, we claim that the condition of being flat does not change. We only have to show that, if \( \alpha^{(i)} \in \tilde{\Sigma}_{q^{(i)}, \theta^{(i)}} \), then also \( \alpha^{(i-1)} \in \tilde{\Sigma}_{q^{(i-1)}, \theta^{(i-1)}} \), since the converse follows immediately from the definition of flat. Suppose on the contrary that \( \alpha^{(i-1)} = \beta^{(1)} + \cdots + \beta^{(k)} \) is a decomposition with \( \beta^{(j)} \in R^+_{q^{(i-1)}, \theta^{(i)}} \) and \( p(\alpha^{(i-1)}) < p(\beta^{(1)}) + \cdots + p(\beta^{(k)}) \). Since \( (\alpha^{(i-1)}, e_v) = 1 \), there must exist \( j \) with \( \beta^{(j)} \oplus e_v \geq 1 \), and hence also \( \beta^{(j)} \oplus e_v \geq 1 \). Set \( \gamma^{(j)} := \beta^{(j)} - \delta_{\ell} e_v \) for all \( \ell \). Note that \( p(\gamma^{(j)}) \geq p(\beta^{(j)}) \), with equality if and only if \( \beta^{(j)} \oplus e_v = 1 \). Then

\[
p(\alpha^{(i)}) = p(\alpha^{(i-1)}) < p(\beta^{(1)}) + \cdots + p(\beta^{(k)}) \leq p(\gamma^{(1)}) + \cdots + p(\gamma^{(k)}),
\]

so that \( \alpha^{(i)} \notin \tilde{\Sigma}_{q^{(i)}, \theta^{(i)}} \). We have proved the contrapositive.

It remains to show that, if \( (\alpha^{(i-1)}, e_v) > 1 \) and \( (q^{(i)}, \theta^{(i)}) = (1, 0) \), then \( \alpha^{(i-1)} \notin \tilde{\Sigma}_{q^{(i-1)}, \theta^{(i-1)}} \). In this case there can be no loops at \( v_i \), so that \( e_v \) is a real root. Moreover, \( \alpha^{(i-1)} \geq 1 \). Then, \( p(\alpha^{(i-1)} - e_v) = p(\alpha^{(i-1)} - e_v) + p(e_v) - p(\alpha^{(i-1)}) \). So \( \alpha^{(i-1)} \) is not flat. □

**Remark 6.14.** The same theorem as above applies in the additive case, to characterise the analogous set \( \tilde{\Sigma}_{\lambda,\theta} \) of flat roots. Also, note that when \( \theta = 0 \), the above proof simplifies the proof of \([\text{Su06}, \text{Theorem 1.2}]\), since it does not require the classification \([\text{CBS01}, \text{Theorem 8.1}]\) of roots in \( \mathcal{F}(Q) \setminus \Sigma_{\lambda,0} \).
Remark 6.15. Thanks to [GS11] Theorem 3.1, the condition that any (or every) reflecting sequence consists only of admissible and \((-1)\)-reflections is equivalent to the condition that, for every real root \(\beta \in R^+_{q,\theta}\), we have \((\alpha, \beta) \leq 1\).

6.2. Fundamental and flat roots not in \(\Sigma_{q,\theta}\). To complete the characterisation, we need to determine the set \(F(Q) \setminus \Sigma_{q,\theta}\). This follows from [CB01] Theorem 8.1, which computes \(F(Q) \setminus \Sigma_\lambda\) (in the additive case, but which extends to the present setting). We state a sharper and more general version:

**Theorem 6.16.** A root in \(R^+_{q,\theta}\) is not in \(\Sigma_{q,\theta}\) if and only if applying a reflecting sequence as in Theorem 6.13 either one of the reflections is inadmissible, or the resulting element of \(F(Q)\) is one of the following:

- \((a)\) \(m\delta\) with \(\delta\) the indivisible imaginary root for an affine Dynkin subquiver, \(m \geq 2\), and \(\ell\) is such that \(q^\delta\) is a primitive \(\ell\)-th root of unity; or
- \((b)\) The support of \(\alpha\) is \(J \sqcup K\) for \(J, K \subseteq Q_0\) disjoint subsets with exactly one arrow from a vertex \(j \in J\) to a vertex \(k \in K\), \(\alpha_k = 1\), and either:
  - \((b1)\) \(\alpha_i = 1\), and \(\alpha_{|J} \in R^+_{q,\theta}\), or
  - \((b2)\) \(Q_{|J}\) is affine Dynkin and \(|J| = m\delta\) for some \(m \geq 2\), with \(\delta \in R^+_{q,\theta}\).

Moreover, the root is not in \(\Sigma_{q,\theta}\) if and only if one of the reflections is neither admissible nor a \((-1)\)-reflection, or the resulting element of \(F(Q)\) is in case \((a)\).

**Proof.** First, it is clear that if an inadmissible reflection is applied in the sequence, \(p(\alpha) \leq p(s_i\alpha) + p(\alpha - s_i\alpha)\) shows that \(\alpha\) is not in \(\Sigma_{q,\theta}\). So we can assume all reflections are admissible. By [CB01] Theorem 8.1 in the additive case (with \(\theta = 0\)), there is a sequence of admissible reflections taking to one of the given cases, not assuming that the result is in \(F(Q)\). The proof extends verbatim to this case, replacing \(N_\lambda\) by \(N_{q,\theta}\). Although the additive statement has \(\ell = 1\) (or in characteristic \(p\) has \(\ell = p\)), for us we note that if \(q^\delta\) is a primitive \(\ell\)-th root of unity, i.e., \(\ell\delta \in \Sigma_{q,\theta}\), since the only roots with entries dominated by \(\ell\delta\) are real. Note that in [CB01] Theorem 8.1, it is required in case \((b2)\) that \(\delta \in R^+_{q,\theta}\), although this is not in the statement.

We claim that in fact we can take the result to be in \(F(Q)\). Note that, in [CB01], it is not required in conditions \((a), (b)\) that \(\alpha\) be in \(F(Q)\). However, we already know that, in order to have \(\alpha \in \Sigma_{q,\theta}\), there must be an admissible reflection sequence taking \(\alpha\) to \(F(Q)\). Thus we may assume that \(\alpha \in F(Q)\) and moreover that it is sincere. Then, applying [CB01] Theorem 8.1, there is a further sequence of admissible reflections taking \(\alpha\) to one of the forms above. After this we can apply an admissible reflection sequence to get back to an element of \(F(Q)\), necessarily \(\alpha\) again. It is clear that doing so will not change the form as above, since \(\alpha\) was assumed to be sincere, so the reflections cannot shrink the support of \(\alpha\); note that, in cases \((a)\) and \((b2)\), means that no reflections will be applied to the coefficients of the multiple of \(\delta\).

For the converse, it remains to verify that cases \((a)\) and \((b1–2)\) are not in \(\Sigma_{q,\theta}\). In case \((a)\) \(p(n\delta) = 1 < mp(\delta) = m\). In case \((b1)\), \(p(\alpha) = p(\alpha_{|J}) + p(\alpha_{|K})\). Finally, in case \((b1)\), \(p(\alpha) = p(\delta) + p(\alpha - \delta)\).

The statement about flat roots follows from Theorem 6.13 together with the observation that case \((a)\) is not flat (as \(p(n\ell\delta) < mp(\ell\delta)\)), whereas cases \((b1)\) and \((b2)\) are flat: this follows from [Su06] Theorem 1.1, where it is shown that \((b1)\) and \((b2)\) are already flat as elements of \(\Sigma_{1,\theta}\) (which is stronger). □

6.3. Canonical decompositions. Let \(\Sigma^\text{iso}_{q,\theta} \subseteq \Sigma_{q,\theta}\) be the subset of isotropic imaginary roots. Let \(\mathbb{N}_{\geq 2} \cdot \Sigma^\text{iso}_{q,\theta} := \{m\alpha \mid m \geq 2, \alpha \in \Sigma^\text{iso}_{q,\theta}\} \).

**Theorem 6.17.** (i) Given \(\alpha \in N_{q,\theta}\), there exists a unique decomposition \(\alpha = \alpha^{(1)} + \cdots + \alpha^{(m)}\) with \(\alpha^{(i)} \in \Sigma_{q,\theta}\) such that any other such decomposition is a refinement of this one.

(ii) There is also a unique decomposition \(\alpha = \beta^{(1)} + \cdots + \beta^{(k)}\) with \(\beta^{(i)} \in \Sigma_{q,\theta} \cup N_{\geq 2} \cdot \Sigma^\text{iso}_{q,\theta}\), satisfying the properties:

- \((a)\) Every real root \(\beta^{(i)}\) is in \(\Sigma_{q,\theta}\);
- \((b)\) Every other element \(\beta^{(i)}\) has the property \((\beta^{(i)}, \gamma) \leq 0\) for every real \(\gamma \in R^+_{q,\theta}\);
- \((c)\) Any other decomposition into \(\Sigma_{q,\theta} \cup N_{\geq 2} \cdot \Sigma^\text{iso}_{q,\theta}\) satisfying (a) and (b) is a refinement of this one.

Also, \((\beta^{(i)}, \beta^{(j)}) = 0\) if \(\beta^{(i)}, \beta^{(j)}\) are imaginary and \(i \neq j\).

(iii) The decomposition in (i) is a refinement of the one in (ii). The real roots coincide. Writing the refinement as \(\beta^{(i)} = \sum_j \alpha^{(i,j)}\), each pairing \((\alpha^{(i,j)}, \alpha^{(i',j')}) \in \{0, -1\}\), and the intersection graph for the \(\alpha^{(i,j)}\)
(putting in edges for pairings $-1$) is a forest, with connected components precisely the $\beta^{(i)}$. In particular, (ii) is recovered from (i) by iteratively combining together pairs of imaginary roots with pairing $-1$.

(iv) The direct sum map produces a Poisson isomorphism (of reduced varieties), using the decomposition in (ii),

$$\prod_{i=1}^{k} M_{q,\theta}(Q, \beta^{(i)}) \cong M_{q,\theta}(Q, \alpha).$$

Notice the following immediate consequence (of the decomposition in (ii)), giving a weakened version of \cite{CB02}:

**Corollary 6.18.** If $\alpha$ is the dimension of a $\theta$-stable representation of $\Lambda^\theta$, then $\alpha \in \Sigma_{q,\theta} \cup \mathbb{N}_{\geq 2} \cdot \Sigma_{iso}^{\ell\delta}$. Moreover, $\alpha$ is related by admissible reflections either to $e_\alpha$ (i.e., $\alpha$ is a real root in $\Sigma_{q,\theta}$), or to an element of the fundamental region.

Note that, in the real case $\alpha \in \Sigma_{q,\theta}$, it is obvious from the properties of admissible reflections that $\alpha$ is the dimension of a $\theta$-stable representation (see \cite{CBS06} Theorem 1.9 for a stronger statement). In the imaginary case, following \cite{CB02}, we only expect a stable representation if $\alpha \in \Sigma_{q,\theta}$, but it is not at all clear how to prove its existence.

In the proof of the theorem, we will produce also an algorithm for constructing the $\beta^{(i)}$, by a sequence of reflections and subtracting roots $e_i$, with the end result an element in $\mathcal{F}(Q)$ whose connected components give the imaginary $\beta^{(i)}$. For the real roots, we obtain the unique decomposition into real roots in $\Sigma_{q,\theta}$ (as a real root which is the sum of multiple real roots cannot be in $\Sigma_{q,\theta}$).

**Remark 6.19.** Observe that the same proof was used in \cite{CB02} in the context of Nakajima quiver varieties, and indeed the result above holds in that setting. However, due to the simplifying properties of that case (such as expectation (*) holding, and $q$-divisibility coinciding with ordinary divisibility), Crawley-Boevey is able to show that the product decomposition in (iv) always refines to one using the decomposition of (i). Hence the statement given in \cite{CB02} is substantially simpler (eliminating parts (ii) and (iii) of the statement and hence parts of the proof below).

**Proof.** We prove the statements by induction on the sum of the entries of $\alpha$. If there is a vertex $v \in Q_0$ at which $(\alpha, e_v) > 0$ and either $q_v \neq 1$ or $\theta_v \neq 0$, then we can apply an admissible reflection. Since admissible reflections preserve the set of flat roots, the statements follows from Theorem 2.17 and the induction hypothesis. So suppose that there is no such vertex. Instead, suppose that $v \in Q_0$ is such that $(\alpha, e_v) > 0$ but $q_v = 1, \theta_v = 0$. Then every decomposition of $\alpha$ into elements of $\Sigma_{q,\theta}$ must have an element having positive Cartan pairing with $e_v$. This cannot happen by definition for the imaginary roots. Since $e_v$ is the only real root in $\Sigma_{q,\theta}$ with positive pairing with $e_v$, this implies that $e_v$ must appear as a summand of every decomposition of types (i) and (ii). Similarly, the argument of the proof of Proposition 6.10 shows that, in this case, the direct sum map yields an isomorphism $M_{q,\theta}(Q, \alpha - e_v) \times M_{q,\theta}(Q, e_v) \cong M_{q,\theta}(Q, \alpha)$. We can apply the induction hypothesis to $\alpha - e_v$.

This reduces the theorem to the case that $(\alpha, e_i) \leq 0$ for all $i$. In this case we can decompose $\alpha$ into its connected components. By Theorem 6.19 all of these are flat roots, except for elements of the form $m \ell \delta$ with $\ell \delta \in \Sigma_{q,\theta}^{iso}$ and $m \geq 2$. This yields the desired decomposition in (ii), as well as the isomorphism in (iv). To further refine into the decomposition of (i), we iteratively apply Theorem 6.19. Eventually this process must terminate since $\alpha$ is finite.

Let us prove that the decomposition is unique. The uniqueness in (ii) is clear, since roots have connected support. For the uniqueness in (i), we can prove this by induction, since every decomposition into roots in $\Sigma_{q,\theta}$ of each of the types (a),(b1),(b2) of components described in Theorem 6.19 must refine a particular one. Namely, in type (a), one can directly apply the argument of \cite{CB02} Lemma 3.2, replacing $\delta$ there by $\ell \delta$. Next, in type (b1), one applies the argument of \cite{CB02} Lemma 5.3: every decomposition into elements of $\Sigma_{q,\theta}$ must refine $\alpha |_J + \alpha |_K$, since otherwise if $\beta \in \Sigma_{q,\theta}$ appears and $\beta = |J| + |K|$ is a non-trivial decomposition, then $p(\beta) = p(\beta |_J) + p(\beta |_K)$ is a contradiction. Finally, in type (b2), again the result follows from the argument of \cite{CB02} Lemma 5.4. We omit the details.

Let us see that the properties in (iii) are satisfied. By construction, the distinct imaginary roots $\beta^{(i)}, \beta^{(j)}$ pair to zero. By the inductive procedure of the preceding paragraph, the refinement into $\alpha^{(i)}$ has the stated
property (the pairings are in \{0, -1\} and the intersection graph is a forest whose connected components are the $\beta^{(i)}$).

As a consequence, we obtain the following description of divisibility criteria for elements of $\Sigma_{q, \theta}$ and $\Sigma_q$: analogous to [BS16a, Theorem 2.2]:

**Corollary 6.20.** Let $\alpha = m\beta$ for $\beta \in R^+_q$ $q$-indivisible and imaginary, and $m \geq 2$.

- (i) $\alpha \in \Sigma_{q, \theta}$ if and only if $\beta \in \Sigma_{q, \theta}$ is anisotropic and the decomposition of Theorem 6.17 (ii) is trivial for $\beta$ (i.e., a reflection sequence involves only admissible reflections).

  In particular, for $\gamma \in \Sigma_{q, \theta}$, every rational multiple $r\gamma \in N_{q, \theta}$ for $r \in \mathbb{Q}_{\leq 1}$ is also in $\Sigma_{q, \theta}$.

- (ii) $\alpha \in \Sigma_{q, \theta}$ if and only if $\beta$ is anisotropic and either $\beta \in \Sigma_{q, \theta}$, or $\beta$ is equivalent via admissible reflections to one of the forms (b1) or (b2) in Theorem 6.16.

  In particular, a rational multiple $r\gamma \in N_{q, \theta}$ for $r \in \mathbb{Q}_{\leq 1}$ is in $\Sigma_{q, \theta}$ unless it is equivalent to one of the forms (b1) or (b2). This can only happen if $r = 1/gcd(\gamma)$ and $q^{\gamma} = 1$.

**Proof.** (i) This follows from the classification of flat roots in Theorems 6.13, 6.16 and 6.17. Observe simply that if a reflection sequence for $\beta$ involves an inadmissible $(-1)$-reflection, then the same sequence for $\alpha$ involves an inadmissible $(-2)$-reflection (which is not allowed). There are no inadmissible reflections if and only if the decomposition of Theorem 6.17 (ii) of $\beta$ has no real roots; then if $\beta$ is itself a root the decomposition is trivial. Finally observe that if $\beta$ has the trivial decomposition, then $m\beta$ also gives a flat root unless $\beta$ is isotropic.

(ii) Assume that $\beta \in \Sigma_{q, \theta}$. After applying admissible reflections, we can assume $\beta \in \mathcal{F}(Q)$. Then $\alpha = m\beta$ will be in $\Sigma$ unless $\beta$ is isotropic, by Theorem 6.16. Conversely, if $\alpha \in \Sigma_{q, \theta}$, again by applying admissible reflections, we can assume $\alpha \in \mathcal{F}(Q)$. Then by Theorem 6.16 we see that $\beta$ is either itself in $\Sigma_{q, \theta}$ or of one of the forms (b1) and (b2).

**Remark 6.21.** Although we are working with reduced varieties throughout the paper, we emphasised this in Theorem 6.17 (iv) because it is not completely clear that the reflection isomorphisms in [Yam08, Theorem 5.1] are defined scheme-theoretically. On the other hand, this is the only obstacle here. That is, if these isomorphisms are defined scheme-theoretically, then the proof would appear to extend to this case, i.e., to not-necessarily-reduced multiplicative quiver schemes.

### 6.4. Symplectic resolutions for $q$-indivisible flat roots.

**Theorem 6.22.** Suppose that $\alpha \in \Sigma_{q, \theta}$ is $q$-indivisible and $M_{q, \theta}(Q, \alpha)$ is non-empty. Then for suitable $\theta' \geq \theta$, $M_{q, \theta'}(Q, \alpha) \to M_{q, \theta}(Q, \alpha)$ is a symplectic resolution.

**Remark 6.23.** Observe that the theorem also holds in the additive setting, where the result is also interesting. Indeed, it explains and generalises the technique of framing used to construct resolutions such as Hilbert schemes of $\mathbb{C}^2$ or of hyperkähler ALE spaces. In the former case, the quiver is again the framed Jordan quiver (with two vertices and two arrows, a loop at the first vertex, and an arrow from the second to the first vertex). The dimension vector is $\alpha = (m, 1)$. The theorem recovers the well-known statement that taking $\theta = 0$ gives a symplectic resolution of the singularity $\text{Sym}^m \mathbb{C}^2$ in the additive case; this identifies with Hilb$^m \mathbb{C}^2$. In the multiplicative case for the same quiver, by Theorem 6.22 and Remark 6.1 after localisation, we obtain a resolution of the character variety of the once-punctured torus in the multiplicative case for rank $m$ local systems with unipotent monodromy $A$ satisfying $\text{rk}(A - I) \leq 1$.

**Proof of Theorem 6.22.** In view of Lemma 2.16 we only have to show that we can find $\theta' \geq \theta$ such that $M_{q, \theta'}(Q, \alpha)$ is smooth and $M_{q, \theta'}(Q, \alpha) \to M_{q, \theta}(Q, \alpha)$ is birational. We will make use of the combinatorial analysis of [CB11, Section 8]. Note that, if $\alpha \in \Sigma_{q, \theta}$, then the result follows from the discussion in Section 4.11 so we can assume that this is not the case.

It will be useful to generalise from integral to rational stability conditions: note that if $\theta' \in Q^{Q_0}$, then the sign of $\theta' \cdot \alpha$ equals the sign of $m\theta \cdot \alpha$ for all $m \geq 1$, and hence this added generality does no harm, as one can always multiply by the greatest common denominator of all the $\theta_i$.

Allowing rational stability conditions, we claim that for $\theta' \in Q^{Q_0}$ in a small enough neighbourhood of $\theta$ (intersected with $\theta' \cdot \alpha = 0$), then $\theta' \geq \theta$ under the ordering of Definition 2.11. Indeed, $\theta' \geq \theta$ follows
if \( \theta \cdot \beta > 0 \Rightarrow \theta' \cdot \beta > 0 \) for all \( \beta < \alpha \), which is an open condition. Moreover, we can guarantee that \( \theta \cdot \beta < 0 \Rightarrow \theta' \cdot \beta < 0 \), i.e., that every \( \theta \)-stable representation is also \( \theta' \)-stable. If, moreover, we choose \( \theta' \) so that \( \theta' \cdot \beta \neq 0 \) with \( \beta < \alpha \), then \( \mathcal{M}_{q,\theta}(Q,\alpha) \) will be smooth, so that we only have to show that \( \mathcal{M}_{q,\theta}(Q,\alpha) \to \mathcal{M}_{q,\theta}(Q,\alpha) \) is birational.

By Corollary 6.10 for each connected component of \( \mathcal{M}_{q,\theta}(Q,\alpha) \), there is a dense stratum of the form 
\[
(1, \beta^{(1)}; \ldots; 1, \beta^{(r)}) \quad \text{with} \quad p(\alpha) = \sum_{i=1}^{r} p(\beta^{(i)}) \quad \text{We need to show that each representation } \rho \text{ in such a stratum in is the boundary of a unique } \text{GL}(\alpha)-\text{orbit in } \text{Rep}_{q^{\theta'}}(\Lambda^q(Q,\alpha)). \quad \text{Equivalently, we must show that there is a unique } \theta'-\text{stable representation } \rho' \text{ up to isomorphism such that } \rho \text{ is the } \theta-\text{polystabilisation of } \rho'.
\]

We will prove the statement by induction on \( \alpha \), with respect to the partial ordering \( \leq \). First, applying admissible and \((-1)\)-reflections, we reduce to the case that \( \alpha \) is in the fundamental region. Indeed, it is clear from Theorem 2.17 and Proposition 6.10 that applying these reflections causes no harm. Note that each \((-1)\)-reflection will modify stratum types by removing a real root from the type; once we are in the fundamental region no real roots will appear.

We first show uniqueness. If \( \rho' \) is as above, then suppose that there is an exact sequence of \( \theta'\)-semistable representations of the form \( 0 \to \psi \to \rho' \to \phi \to 0 \). By our assumptions, the dimension vectors of \( \psi \) and \( \phi \) are symmetric. Let \( C \subseteq \{1, \ldots, r\} \) be a subset of indices such that \( \psi = \sum_{i \in C} \beta^{(i)} \). (This set is unique except in case (b2) with \( \dim \psi = \alpha(2) = \alpha - \delta \)). There exists a unique \( i \in C \) such that \( \langle \beta^{(i)}, \dim \psi \rangle = -1 \) (since \( \alpha(1), \alpha(2) = -1 \)). Now, define \( \theta'(\eta) := \theta'|_{\text{supp } \dim \psi} - (\theta' \cdot \dim \psi)v \), where \( v \) is the unique vertex in \( \text{supp } \dim \psi \) which has non-zero Cartan pairing with \( \text{supp } \dim \phi \) (so \( \dim \psi_{\chi,1} = 1 \)). By construction, \( \theta' \cdot \dim \psi = 0 = \theta'' \cdot \dim \phi \). Moreover, \( \theta' \cdot \beta^{(j)} = \theta' \cdot \beta^{(j)} \) for \( j \in C \setminus \{i\} \). We claim that \( \psi \) is \( \theta''\)-stable. By definition of \( \theta''\)-stability, every nonzero submodule \( \eta \) of \( \psi \) satisfies \( \theta'' \cdot \dim \eta < 0 \). Now, if \( \beta^{(i)} \leq \dim \eta \), then \( \theta'' \cdot \dim \eta = \theta' \cdot \dim \eta < 0 \). On the other hand, if \( \eta \) is a proper submodule of \( \psi \), then \( \psi/\eta \) is a nonzero quotient module with \( \beta^{(i)} \leq \dim (\psi/\eta) \). Then \( \dim (\psi/\eta) \cdot \dim \phi = 0 \). By Proposition 2.23 \( \dim \text{Ext}^{1}(\phi,\psi/\eta) = 0 \). Therefore, we have an exact sequence \( 0 \to \eta \to \rho' \to (\psi/\eta) \oplus \phi \to 0 \). As a consequence, \( \psi/\eta \) itself is a quotient module of \( \rho' \). It follows that \( \theta' \cdot \dim (\psi/\eta) > 0 \). Therefore, \( \theta'' \cdot \dim (\psi/\eta) = \theta' \cdot (\psi/\eta) > 0 \). Therefore, \( \theta'' \cdot \eta < 0 \). We conclude that \( \psi \) is \( \theta''\)-stable, as desired. By induction on \( \alpha \), \( \psi \) is then uniquely determined up to isomorphism.

This completes the proof of uniqueness. We move on to existence, which is similar. Begin with a decomposition given by Theorem 6.10 of type (b1) or (b2). Let us keep the notation \( \alpha(1), \alpha(2) \) defined above. The same construction as above yields modifications \( \theta(1), \theta(2) \) of \( \theta' \) such that \( \theta(1) \cdot \alpha(1) = 0 \). By induction we can take \( \theta(1)\)-stable representations \( \phi, \psi \) of dimension vectors \( \alpha(1) \). Then since \( \langle \alpha(1), \alpha(2) \rangle = -1 \), \( \dim \text{Ext}^{1}(\phi,\psi) = \dim \text{Ext}^{1}(\psi,\phi) = 1 \). Assume that \( \theta' \cdot \phi < 0 \), otherwise swap \( \phi \) and \( \psi \). Then form a non-trivial extension \( 0 \to \phi \to \rho' \to \psi \to 0 \). The same computation as above guarantees that \( \rho' \) is \( \theta''\)-stable.

**Corollary 6.24.** In the situation of the proposition, the normalisation of \( \mathcal{M}_{q,\theta}(Q,\alpha) \) is a symplectic singularity.

**Proof.** This follows since we have constructed a symplectic resolution (see Proposition 4.3 or Remark 4.6). □

**Remark 6.25.** Note that the main step of the proof is to show that \( \mathcal{M}_{q,\theta}(Q,\alpha) \to \mathcal{M}_{q,\theta}(Q,\alpha) \) is birational for suitable \( \theta' \geq \theta \). For this, we did not need the hypothesis that \( \alpha \) is \( q \)-indivisible. On the other hand, by Theorems 6.13 and 6.16 when \( \alpha \in \Sigma_{q,\theta} \setminus \Sigma_{q,\theta} \), \( \alpha \) is actually indivisible (not merely \( q \)-indivisible). For \( \alpha \in \Sigma_{q,\theta} \), the birationality statement is Corollary 2.24 which is easy. (Moreover, the full statement of Theorem 6.22 was established for \( \alpha \in \Sigma_{q,\theta} \) as in Section 4.1.) So it does not really add anything to state the birationality property without the \( q \)-indivisibility hypothesis.
6.5. Symplectic resolutions for general $\alpha$.

**Theorem 6.26.** Assume that $\mathcal{M}_{q,\theta}(Q, \alpha)$ is non-empty and that the decomposition of Theorem 6.17 (ii) has no elements $\beta^{(i)}$ of the forms (a) $\beta^{(i)} = 2\gamma$ for $\gamma \in N_{q,\theta}$ and $p(\gamma) = 2$, or (b) $\beta^{(i)} = m\gamma$ for $m \geq 2$ and $\gamma \in \Sigma^{iso}_{q,\theta}$. Then:

- The normalisation of $\mathcal{M}_{q,\theta}(Q, \alpha)$ is a symplectic singularity;
- Each factor $\mathcal{M}_{q,\theta}(Q, \beta^{(i)})$ with $\beta^{(i)} \notin \Sigma_{q,\theta}$ admits a symplectic resolution;
- If for any factor $\beta^{(i)}$ there exists a $\theta$-stable representation of dimension $\gamma^{(i)} = \frac{1}{m}\beta^{(i)}$ with $m \geq 2$, then $\mathcal{M}_{q,\theta}(Q, \alpha)$ does not admit a symplectic resolution. In fact, it has an open, singular, terminal, factorial subset.

**Proof.** The first statement follows if we show that the normalisation of each factor $\mathcal{M}_{q,\theta}(Q, \beta^{(i)})$ is a symplectic singularity. For the factors such that $\beta^{(i)}$ is in $\Sigma_{q,\theta}$, this is a consequence of Theorem 6.1. For $\beta^{(i)} \notin \Sigma_{q,\theta}$, after applying admissible reflections it follows from Theorem 6.16 that it is indivisible. Hence $\beta^{(i)}$ is itself indivisible. By our assumptions, $\beta^{(i)}$ is flat. The result then follows from Theorem 6.22. This also proves the second statement.

We proceed to the third statement. Under the hypotheses, since we have excluded the isotropic and $(2,2)$-cases, an open subset of $\mathcal{M}_{q,\theta}(Q, \beta^{(i)})$ is factorial terminal singular by Theorem 1.7 (see Theorem 4.11). Hence so is an open subset of $\mathcal{M}_{q,\theta}(Q, \alpha)$, which therefore does not admit a symplectic resolution. □

6.6. Classifications of symplectic resolutions of punctured character varieties. Here, we combine the results of this section and Theorem 6.1 to get a classification of all the character varieties of punctured surfaces which admit a symplectic resolution, modulo the conjectural results of the $(2,2)$-cases. As explained in Section 8, in order to get such a result, it suffices to consider multiplicative quiver varieties of crab-shaped quivers, where the parameter $q$ and the dimension vector $\alpha$ are chosen in an appropriate way; see Theorem 6.6.

Let $Q$ be a crab-shaped quiver, $q \in (\mathbb{C}^\times)^Q_0$ and $\alpha \in N_{q,\theta}$ and consider the corresponding quiver variety $\mathcal{M}_{q,\theta}(Q, \alpha)$. Then, if $\alpha \notin F(Q)$, we can apply the algorithm of Theorem 6.17 and obtain a decomposition where the dimension vectors of the factors are in the fundamental region (such dimension vectors are the connected components of the reflection of $\alpha$). Moreover, note that, in the crab-shaped case, all the vector components not containing the central vertex are Dynkin quivers of type $A$: therefore, the associated multiplicative quiver variety is just a point. This implies that we can assume, without loss of generality, that $\alpha$ be sincere ($\alpha_i > 0$ for all $i$) and in the fundamental region. After having performed this reduction, we can prove the following.

**Corollary 6.27.** Let $Q$ be a crab-shaped quiver and $\alpha \in N_{q,\theta}$ a sincere vector in the fundamental region. Further assume that $(Q, \alpha)$ is not one of the following cases:

(a) $\beta = \frac{1}{2}\alpha \in N_{q,\theta}$ and $(Q, \beta)$ is one of the quivers in Theorem 5.4 and Theorem 5.3;
(b) $Q$ is affine Dynkin (of type $\tilde{A}_0$ (i.e., the Jordan quiver with one vertex and one arrow), $\tilde{D}_4$, or $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$) and $\alpha$ is a $q$-divisible multiple of the indivisible imaginary root $\delta$ of $Q$.

Then:

- The normalisation of $\mathcal{M}_{q,\theta}(Q, \alpha)$ is a symplectic singularity;
- If $\alpha$ is $q$-indivisible, $\mathcal{M}_{q,\theta}(Q, \alpha)$ admits a symplectic resolution;
- If $\alpha = m\beta$ for $m \geq 2$ and there exists a $\theta$-stable representation of $\Lambda^\theta(Q)$ of dimension $\beta$, then $\mathcal{M}_{q,\theta}(Q, \alpha)$ does not admit a symplectic resolution (it contains an open singular factorial terminal subset);
- In the case that $\alpha$ is $q$-divisible, the condition of the preceding part is always satisfied, except possibly in the case: (c) $Q = Q^e \cup \{\ast\}$ is an affine Dynkin quiver $Q^e$ of type $\tilde{A}_0, \tilde{D}_4, \tilde{E}_6, \tilde{E}_7,$ or $\tilde{E}_8$ together with an additional vertex $\{\ast\}$ and an additional arrow from this vertex to one with dimension vector $1$ in $\delta$, and $\alpha$ has the form $(p, p\ell\delta)$ for $p \geq 2$ a prime, $\ell\delta$ $q$-indivisible. Here $\delta$ denotes the indivisible imaginary root of $Q^e$.

**Remark 6.28.** In the final part of the corollary, expectation (*) from the introduction predicts that the exception indeed fails to satisfy the conditions of the preceding part. Nonetheless, we believe that, also in
this case, there should not exist a projective symplectic resolution; this would be implied by Conjecture 7.16 in the appendix together with the consequence that follows it, by [BS16a] Theorem 1.5.

**Proof of Corollary 6.27.** By Theorem 6.17 we know that \( \alpha \) is flat unless it is a positive integral multiple of an isotropic root, excluded in case (b). Then the first and third statements follow from Theorems [1.7] and the second from [6.22]. For the fourth statement, we apply Theorem 6.10 to see that, in the case of the dimension vector can only be in the fundamental region but not in \( \Sigma_{q,\theta} \) if the quiver is a framed affine Dynkin quivers of types \( \tilde{A}_q, \tilde{D}_4, \tilde{E}_6, \tilde{E}_7, \) and \( \tilde{E}_8 \) and the dimension vector is (1, \( \delta \)). Thus only prime multiples of this vector can be \( q \)-divisible but have no factor in \( \Sigma_{q,\theta} \). It then remains only to show that \( \mathcal{M}_{q,\theta}(Q, \beta) \neq \emptyset \) for \( \beta \in \Sigma_{q,\theta} \). In the non-abelian Hodge correspondence. 

The work [Tir17] of the second author is the study of analogous problems in the context of the Higgs bundle moduli spaces, which appear in the picture via the non-abelian Hodge correspondence. 

Open questions and future directions

In this section we pose some questions concerning the cases which are left out from the analysis carried out in the previous sections. This includes the question of to what extent the decomposition in Theorem 5.17 can be refined, as in the addtive case in [CB02].

One interesting direction of research which naturally arises from the results proved in this paper and the work [Tr17] of the second author is the study of analogous problems in the context of the Higgs bundle moduli spaces, which appear in the picture via the non-abelian Hodge correspondence.

We also say a few words on how one might hope to study the local structure of formal moduli spaces of polystable objects in a 2-Calabi–Yau category and prove that, under suitable conditions, formal neighbourhoods of such moduli spaces are quiver varieties associated to a quiver which arises from the deformation
theory of the objects parametrised by the moduli spaces. This is relevant to the present context as it would make it possible to give an alternative and more insightful proof of Proposition 6.3 as explained at the beginning of Section 5.

Before getting into these issues, we recall what is known about non-emptiness of varieties, which is unfortunately not much: although we do not have a lot to say about this issue, it is an obvious and relevant open question which we have already mentioned throughout the paper.

7.1. Non-emptiness of multiplicative quiver varieties. As explained in the previous sections, one of the subtleties of in the study of multiplicative quiver varieties is represented by the fact that it is not known in general when they are non-empty (nor how many connected components they have). On the other hand, there are special cases in which non-emptiness can be shown. For example, when \( q = 1 \), then, for any quiver \( Q \) and any vector \( \alpha \in \mathbb{N}^{\partial} \), the zero representation is a suitable element of \( \text{Rep}(\Lambda^0, \alpha) \), since the invertibility condition is automatically satisfied as well as the multiplicative preprojective relation. Thus \( \mathcal{M}_{1,0}(Q, \alpha) \neq \emptyset \). More generally, for every real root \( \beta \in R^+_q, \theta \), then by applying reflection sequences as in Section 6.1 (see also the discussion after Corollary 6.18), we conclude that \( \mathcal{M}_{q,\theta}(Q, \beta) \neq \emptyset \). As a result, if \( \alpha \) can be expressed as a sum of real roots in \( R^+_q, \theta \) (not necessarily coordinate vectors) then \( \mathcal{M}_{q,\theta}(Q, \alpha) \neq \emptyset \).

Another, less trivial, case in which we are guaranteed that \( \mathcal{M}_{q,\theta}(Q, \alpha) \) is non-empty is when \( Q \) is star-shaped and \( \alpha \in N_{q,\theta} \); then, in order to construct a representation of dimension \( \alpha \), one can use the correspondence of between multiplicative quiver varieties and character varieties of punctured surfaces and, further, use parabolic bundles and monodromy, see [CB13] (and [Yam08] for \( q = 0 \)). In the crab-shaped case, we then also know that \( \mathcal{M}_{q,\theta}(Q, \alpha) \neq \emptyset \) for \( \alpha \in N_{q,\theta} \), using Thompson’s thesis [Tho61], as explained in the proof of Corollary 6.27.

We can also ask when the stable locus \( \mathcal{M}^s_{q,\theta}(Q, \alpha) \neq \emptyset \). Note that an answer to this question for all \( \alpha \) also answers the question of non-emptiness of the entire locus, since every point in \( \mathcal{M}_{q,\theta}(Q, \alpha) \) is represented by a polystable representation. Thus, more explicitly, \( \mathcal{M}^s_{q,\theta}(Q, \alpha) \neq \emptyset \) if and only if \( \alpha \) can be represented as a sum of roots \( \alpha^{(i)} \) for which \( \mathcal{M}^s_{q,\theta}(Q, \alpha^{(i)}) \neq \emptyset \). Note that, when \( \alpha \in N_{q,\theta} \), then non-emptiness of \( \mathcal{M}_{q,\theta}(Q, \alpha) \) is equivalent to that of \( \mathcal{M}^s_{q,\theta}(Q, \alpha) \), by Proposition 2.22. Our expectation (*) says that \( \mathcal{M}^s_{q,\theta}(Q, \alpha) \neq \emptyset \) implies \( \alpha \in N_{q,\theta} \).

7.2. Refined decompositions for multiplicative quiver varieties. Recall Crawley-Boevey’s canonical decomposition in the additive case (Theorem 6.2 Lemma 6.1). It is useful to ask to what extent such a decomposition holds in the multiplicative setting, refining the one of Theorem 6.17. Let \( \beta^{(i)}, \alpha^{(i)} \) be as in Theorem 6.17 and group together the \( \alpha^{(i,j)} \) that are equal, yielding distinct \( \gamma^{(i,j)} \) each occurring \( r_{i,j} \) times. Note that, when \( \gamma^{(i,j)} \) is anisotropic, then \( r_{i,j} = 1 \), since \( r_{i,j} \gamma^{(i,j)} \in \Sigma_{q,\theta} \), by the uniqueness of the decomposition in Theorem 6.17 (i).

Question 7.1. Do we have a decomposition as follows:

\[
\mathcal{M}_{q,\theta}(Q, \alpha) \cong \prod_{i,j} S^{r_{i,j}} \mathcal{M}_{q,\theta}(Q, \gamma^{(i,j)})?
\]

The following proposition partly answers this question modulo expectation (*).

Proposition 7.2. If (*), then the decomposition of Theorem 6.17 (iv) refines to one of the form

\[
\mathcal{M}_{q,\theta}(Q, \alpha) \cong \prod_{i,j} \mathcal{M}_{q,\theta}(Q, r_{i,j} \gamma^{(i,j)}).
\]

Moreover, in this case, the direct sum map

\[
S^{r_{i,j}} \mathcal{M}_{q,\theta}(Q, \gamma^{(i,j)}) \to \mathcal{M}_{q,\theta}(Q, r_{i,j} \gamma^{(i,j)})
\]

is surjective.

Proof. It suffices to decompose each of the \( \mathcal{M}_{q,\theta}(Q, \beta^{(i)}) \). By Proposition 6.16, the first statement follows by the arguments of [CB02], Section 5] verbatim, replacing simple representations by \( \theta \)-stable ones. For the second statement, if \( r_{i,j} > 1 \), then \( \gamma^{(i,j)} \) is isotropic. Then, the canonical decomposition of \( r_{i,j} \gamma^{(i,j)} \) appearing in Theorem 6.17 (i) is just as a sum of \( r_{i,j} \) copies of \( \gamma^{(i,j)} \). Thus the statement follows from Theorem 6.17 (i) and expectation (*), since every representation in \( \mathcal{M}_{q,\theta}(Q, r_{i,j} \gamma^{(i,j)}) \) is represented by a polystable one. □
Therefore, modulo (*), Question 7.1 reduces to the following one:

**Question 7.3.** Is the natural map (7.2.3) an isomorphism?

**Example 7.4.** Suppose that $\beta^{(i)}$ is the following dimension vector supported on a framed type $\tilde{E}_6$ quiver:

\[
\begin{pmatrix}
  n & 2n & 3n & 2n & n & 1 \\
\end{pmatrix}
\]

(7.2.4)

Then, by the star-shaped case of Theorem 3.6 (proved in [CBS06, Section 8]), the variety $\mathcal{M}_{1,0}(Q, \beta^{(i)})$ is isomorphic to the character variety of rank 3 local systems on the three-punctured sphere $\Sigma_{0,3} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ with unipotent monodromies: about the first two punctures, there should be $n$ Jordan blocks of size three (or some refinement), and about the third puncture, there should be $n - 2$ Jordan blocks of size four, one Jordan block of size four, and one of size two (or some refinement). On the other hand, $\mathcal{M}_{1,0}(Q, \delta)$ is the character variety of rank 3 local systems on $\Sigma_{0,3}$ with arbitrary unipotent monodromies. Question 7.3 then asks whether the first variety is isomorphic to the $n$-th symmetric power of the second; it does not seem so obvious that this should be the case.

In the additive case, Question 7.3 has a positive answer. One of the difficulties in trying to adapt the proof of this in [CB02, Section 3] is that, in the multiplicative case, it is no longer guaranteed that one of the components of $\gamma^{(i,j)}$ equals one (since $\gamma^{(i,j)}$ need only be $q$-indivisible, not indivisible). However, if (*) holds and Conjecture 7.16 below holds, then it would follow from the étale-local identification of the multiplicative quiver variety with an additive one that (7.2.3) is indeed an isomorphism. Thus we expect a positive answer to Question 7.3 as well.

Note that the proof of Theorem 6.2 ([BS16a, Theorem 1.4]), in the additive case, relied on hyperkähler twistings, for which one needs to assume that the parameter $\lambda$ is real. In fact, some of the issues we face are not yet resolved in the general additive case where both $\lambda \not\in \mathbb{R}$ and $\theta \neq 0$.

### 7.3. Symplectic resolutions and singularities

In view of our results and the flexibility of symplectic singularities, as well as the relationships between multiplicative and additive quiver varieties, we propose the following:

**Conjecture 7.5.** Every multiplicative quiver variety is a symplectic singularity.

Note that a product of Poisson varieties is a symplectic singularity if and only if each of the factors is (because normality and being symplectic on the smooth locus have this property, and in the definition of symplectic singularity it is equivalent to check the extension property for one or all resolutions of singularities). Therefore the conjecture reduces to the case of factors appearing in Theorem 6.17 and if (*) holds, to the case $\alpha \in \Sigma_{q,\theta} \cup \mathbb{N}_{\geq 2} \Sigma_{q,\theta}^{iso}$ by Proposition 7.2. If Question 7.3 additionally has an affirmative answer, (e.g., in the presence of Conjecture 7.16), then we can furthermore reduce to the case $\alpha \in \Sigma_{q,\theta}$.

Next, we ask to what extent the property of having a symplectic resolution is equivalent to the same property for the factors.

**Question 7.6.** (i) Is it true that $\mathcal{M}_{q,\theta}(Q, \alpha)$ admits a symplectic resolution if and only if each of the factors $\mathcal{M}_{q,\theta}(Q, \beta^{(i)})$ does?

(ii) Suppose that (*) holds. Is it true that $\mathcal{M}_{q,\theta}(Q, \alpha)$ admits a symplectic resolution if and only if each of the factors $\mathcal{M}_{q,\theta}(Q, r_{i,j} \gamma^{(i,j)})$, appearing in Proposition 7.2, does?

Note that, when $r_{i,j} > 1$, then $\gamma^{(i,j)} \in \Sigma_{q,\theta}^{iso}$, and hence is $q$-indivisible. Therefore, in this case, $\mathcal{M}_{q,\theta}(Q, \gamma^{(i,j)})$ has a symplectic resolution by varying $\theta$, by Lemma 2.16 and Corollary 2.24 since it is a surface, so does its $r_{i,j}$-th symmetric power, by the corresponding Hilbert scheme. Therefore, if Question 7.3 has a positive answer (e.g., in the presence of Conjecture 7.16), we can ignore these factors in (ii) above,
and only consider the ones with \( r_{i,j} = 1 \). Also notice that all \( q \)-indivisible factors, including those with \( \gamma^{(i,j)} \notin \Sigma_{q,\theta} \), admit symplectic resolutions. Also, if any factor \( \gamma^{(i,j)} \) appears which is a \( \geq 2 \) multiple of the dimension vector of a \( \theta \)-stable representation, there can be no symplectic resolution of \( \mathcal{M}_{q,\theta}(Q,\alpha) \), nor of \( \mathcal{M}_{q,\theta}(Q,\gamma^{(i,j)}) \). So again, for the question (ii), it is enough to consider only the factors \( \gamma^{(i,j)} \) which are anisotropic, \( q \)-divisible, and not a \( \geq 2 \) multiple of the dimension vector of a \( \theta \)-stable representation.

### 7.4. Moduli of parabolic Higgs bundles and the Isosingularity Theorem

We restrict the attention to the case of crab-shaped quivers, the corresponding multiplicative quiver varieties of which, as explained in Section 3, lead to the study of character varieties of (possibly non-compact) Riemann surfaces.

Character varieties of compact Riemann surfaces are important, among many reasons, as they appear as the Betti side of the non-abelian Hodge correspondence, which is a series of results that establish isomorphisms between apparently unrelated moduli spaces, see [Sim94a]. Such a correspondence holds also in the case of non-compact curves, thanks to the work of Simpson, see [Sim90].

For the compact case, in [Tir17], the second author exploited a fundamental result of Simpson, called the Isosingularity Theorem, [Sim94b], to show how the statements proved in [BS16a, §8] could be translated to the Dolbeault side of the non-abelian Hodge correspondence, i.e., to the moduli spaces of semistable Higgs bundles of degree 0.

In the light of the results of this paper and the non-abelian Hodge correspondence in the non-compact setting, it is a natural question to ask whether an analogue of the main theorems of [Tir17] holds for the Dolbeault moduli spaces defined on complex curves with punctures, which turn out to be the moduli spaces of parabolic Higgs bundles. Before stating some conjectural results, we recall the relevant definitions. Motivated by the work of Simpson, [Sim90], we recall filtered local systems, following [Yam08, §4], which gives a slightly different but nonetheless equivalent definition from the one given in Simpson, [Sim90].

#### Definition 7.7. [Yam08, Definition 4.5]

Let \( X \) be a compact Riemann surface and let \( D \subset X \) be a finite subset. Let \( L \) be a local system on \( X \backslash D \). For a collection of non-negative integers \( l = (l_p)_{p \in D} \), a filtered structure on \( L \) of filtration type \( l \) is a collection \((U_p, F_p)_{p \in D}\), where for each \( p \in D \):

1. \( U_p \) is a neighbourhood of \( p \) in \( X \) (we set \( U_p^* := U_p \setminus \{ p \} \)); and
2. \( F_p \) is a filtration

\[
L |_{U_p^*} = F^0_p(L) \supseteq F^1_p(L) \supseteq \cdots \supseteq F^{l_p}_p(L) \supseteq F^{l_p+1}_p(L) = 0
\]

by local subsystems of \( L |_{U_p}^* \).

Two filtered structures \((U_p, F_p)_{p \in D}, (U'_p, F'_p)_{p \in D}\) of the same filtration type are equivalent if for each \( p \in D \), there exists a neighbourhood \( V_p^* \subset U_p \cap U'_p \) of \( p \) such that \( F_p \) and \( F'_p \) coincide on \( V_p^* \). A local system \( L \) together with an equivalence class of filtered structures \( F = [(U_p, F_p)_{p \in D}] \) is called a filtered local system on \((X, D)\) of filtration type \( l \).

From [Yam08] one has also the following definition of (semi)-stability.

#### Definition 7.8. [Yam08, Definition 4.6]

Let \((L, F)\) be a filtered local system on \((X, D)\) of filtration type \( l \). Let \( \beta = (\beta^j_p \mid p \in D, j = 0, \ldots, l_p) \) be a collection of rational numbers satisfying \( \beta^j_p < \beta^j_q \) for any \( p \) and \( i < j \) — such a collection is called a weight. \((L, F)\) on \((X, D)\) is said to be \( \beta \)-semistable if for any non-zero proper local subsystem \( M \subset L \) the following inequality holds:

\[
\sum_{p \in D} \sum_j \beta^j_p \frac{\text{rank} (M \cap F^j_p(L)) / (M \cap F^{j+1}_p(L))}{\text{rank} M} \leq \sum_{p \in D} \sum_j \beta^j_p \frac{\text{rank} (F^j_p(L)/F^{j+1}_p(L))}{\text{rank} L}.
\]

\((L, F)\) is \( \beta \)-stable if the strict inequality always holds.

Yamakawa established a correspondence between semistable filtered local systems and multiplicative quiver varieties of star-shaped quivers. This is, as mentioned, a particular case of the correspondence outlined in Section 3. To shed more light on this correspondence, we spell out the correspondence between the parameters \( q, \alpha, \theta \) defining a multiplicative quiver variety \( \mathcal{M}_{q,\theta}(Q,\alpha) \) of a star shaped quiver and the weights \( \beta \) of a filtered local system \((L, F)\): start with a crab-shaped quiver \( Q \) with vertex set \( Q_0 = \{ (i,j) \mid i \in \{ 1, \ldots, n \}, j \in \{ 1, \ldots, l_i \} \} \) — i.e. \( Q \) has \( n \) legs, each of which has length \( l_i \), for \( i = 1, \ldots, n \). Moreover, let \( \alpha \) be a dimension vector, \( \alpha \in \mathbb{N}^{Q_0}, \xi \in (\mathbb{C}^\times)^{Q_0} \) a collection of non-zero complex numbers and
$\beta \in \mathbb{Q}^{Q_0}$ a collection of rational numbers. Then, on one side, one can consider filtered local systems $(L, F)$ on $(\mathbb{P}^1, \{ p_1, \ldots, p_n \}),$ where $p_i, i = 1, \ldots, n$ are pairwise distinct points in $\mathbb{P}^1$, with stability parameter $\beta$ and such that:

(1) $\text{rank}(L) = \alpha_0,$
(2) $\text{dim } F_{p_i}^j(L) = \alpha_{i,j},$
(3) the local monodromy of $F_{p_i}^j(L)/F_{p_i}^{j+1}(L)$ around $p_i$ is given by the scalar multiplication by $\xi_{p_i}^j$ for all $i, j$.

On the other side, one can consider the multiplicative quiver variety $M_{q, \theta}(Q, \alpha),$ where $Q$ and $\alpha$ are as above and $q$ and $\theta$ are given as

$$q_0 := \prod_i (\xi_{p_i}^0)^{-1}, \quad q_{i,j} = \frac{\xi_{p_i}^{j-1}}{\xi_{p_i}^j},$$
$$\theta_0 := \frac{\sum_{i,j} \theta_{i,j} \alpha_{i,j}}{\alpha_0}, \quad \theta_{i,j} = \frac{\beta_{p_i}^j - \beta_{p_i}^{j-1}}{\alpha_{p_i}}.$$

The other main concept in the non-abelian Hodge correspondence on noncompact curves is that of parabolic Higgs bundle, which we recall below (note that, in [Sim90] the term filtered Higgs bundle is used instead).

**Definition 7.9.** Let $X$ and $D$ be a compact Riemann surface and a reduced divisor on $X$ respectively. Let $E \to X$ be a holomorphic vector bundle on $X$. A **parabolic structure** on $E$ is the datum of weighted flags $(E_{i,p}, \alpha_{i,p})_{p \in D}$

$$E_p = E_{1,p} \supseteq E_{2,p} \supseteq \cdots \supseteq E_{t+1,p} = 0,$$
$$0 \leq \alpha_{1,p} < \cdots < \alpha_{t,p} < 1,$$

for each $p \in D$. A **morphism of parabolic vector bundles** is a morphism of holomorphic vector bundles which preserves the parabolic structure at every point $p \in D$.

**Definition 7.10.** Given a parabolic bundle $(E, (E_{i,p}, \alpha_{i,p})_{p \in D})$, its **parabolic degree** is defined to be

$$\text{pardeg}(E) = \text{deg}(E) + \sum_{p \in D} \sum_i m_i(p) \alpha_{i,p},$$

where $m_i(p) = \text{dim } E_{i,p} - \text{dim } E_{i,p+1}$ is called the **multiplicity** of $\alpha_{i,p}$.

**Remark 7.11.** Given the notion of parabolic degree, stability and semi-stability of a parabolic bundle are defined as in the case of vector bundles, using the parabolic degree in place of the ordinary degree, see [LM10 §2] for more details and an outline on some geometric properties of the corresponding moduli spaces.

**Definition 7.12.** A **parabolic Higgs bundle** on $(X, D)$ is a parabolic bundle $(E, (E_{i,p}, \alpha_{i,p})_{p \in D})$ together with a meromorphic map $\Phi : E \to E \otimes K_X$ with poles of order at most 1 at the points $p \in D$. The residue of $\Phi$ at marked points is assumed to preserve the corresponding filtration.

From the Riemann-Hilbert correspondence, we know that representations of the fundamental group of a punctured surface with fixed monodromies correspond bijectively to filtered local systems. Moreover, in [Sim90], the following theorem is proved.

**Theorem 7.13.** [Sim90, Theorem, p. 718] There is a one-to-one correspondence between (stable) filtered local systems and (stable) parabolic Higgs bundles of degree zero.

Even though we will not go into the details of this correspondence, we shall at least explain how it works at the level of parameters $\alpha$ and $\beta$. To this purpose, let $X$ be a compact Riemann surface and $D \subset X$ a finite subset of distinct points of $X$. Fixing $p \in D$, consider the sets

$$\{ (\lambda, \alpha_p^p) \in \mathbb{C} \times [0, 1) \mid \text{the action of Res}_p \Phi \text{ on } E_{j,p}/E_{j-1,p} \text{ has an eigenvalue } \lambda \},$$
$$\left\{ (\xi, \beta_{p}^k) \in \mathbb{C}^\times \times \mathbb{R} \mid \begin{array}{l}
\text{the monodromy of } F_{p}^{k}(L)/F_{p}^{k+1}(L) \\
\text{along a simple loop around } p \text{ has an eigenvalue } \xi
\end{array} \right\}.$$

Then the correspondence between these two sets is explicitly given by $(\lambda, \alpha) \mapsto (\xi, \beta)$, where

$$\beta := \alpha - \Re \lambda, \quad \xi := \exp(-2\pi \sqrt{-1} \lambda).$$
Another fundamental result of Simpson that is crucially used in [Tir17] is the Isosingularity theorem, which, roughly speaking, states that the moduli spaces of the non-abelian Hodge Theorem in the compact case, i.e., with no punctures, are étale isomorphic at corresponding points. It is still not known whether the same result holds in the non-compact case.

**Conjecture 7.14.** The Isosingularity theorem holds between the moduli space of semistable filtered local systems for fixed parameters and the moduli space of semistable parabolic Higgs bundles of degree zero with corresponding parameters.

From the above conjectural result, in combination with the results proved in Section 4 and Conjecture 7.16 below, one should be able to deduce the following:

**Conjecture 7.15.** The moduli space of semistable parabolic Higgs bundles of degree zero with fixed parameters is a symplectic singularity, admitting a symplectic resolution if and only if the corresponding character variety admits a symplectic resolution.

A possible strategy to prove the results listed above would be to study the local structure of moduli spaces of (semistable) objects in Calabi–Yau categories. More details are provided in the next subsection.

7.5. **Moduli spaces in 2-Calabi–Yau categories.** As mentioned above, the problem of studying the singularities of a variety can be carried out by analysing the local structure of the variety itself around a point.

This method is powerful in certain cases, e.g., when one is able to prove some locally étale isomorphism between the variety of interest and another variety whose singularities are well-known: for example, this is carried out by Kaledin and Lehn in [KL07] and later by Arbarello and Sacca in [AS15], where they prove that, given a strictly semistable bundle in the moduli space of semistable sheaves on a K3 surface with a fixed non-generic polarisation, there exists an étale neighbourhood around that point that is isomorphic to an affine quiver variety, which depends on the point itself. Similar computations, which find their inspiration from [KL07], were also performed by the Bellamy and the first author in [BS16a] in the context of quiver and character varieties.

The fact that such a technique can be used and gives the same results in so many apparently different situations suggests that these are indeed particular cases of a series of theorems which should apply in much greater generality, namely in the context of 2-Calabi–Yau categories. In the work [BGV16] the authors carry out a detailed study of the deformation theory of representation spaces of 2-Calabi–Yau algebras and they show that among all semisimple representations, the ones that correspond to smooth points are precisely the simple ones. It is known that, when $Q$ is a Dynkin quiver and $q = 1$, there is an isomorphism

$$\Lambda^1(Q) \cong \Pi^1(Q),$$

between the multiplicative preprojective algebra and the additive preprojective algebra, as shown in [CB13 Corollary 1]. Moreover, given that the additive preprojective algebras of Dynkin quivers with at least one arrow have infinite homological dimension, the above isomorphism suggests that, in general, multiplicative preprojective algebras are not 2-Calabi–Yau. On the other hand, a conjectural statement can be made for the case of non-Dynkin quivers.

**Conjecture 7.16.** Let $Q$ be a connected non-Dynkin quiver and $q \in (\mathbb{C}^\times)^{Q_0}$. Then $\Lambda^q(Q)$ is a 2-Calabi–Yau algebra.

Assuming the above conjecture, then [BGV16 Theorem 6.3, 6.6] implies the following for $\theta = 0$: given a dimension vector $\alpha \in \Sigma_{q,\theta}$ and a point $x \in \mathcal{M}_{q,\theta}(Q,\alpha)$, then formally locally around $x$, we have an isomorphism

$$\mathcal{M}_{q,\theta}(Q,\alpha) \cong \mathcal{M}_{0,0}(Q',\alpha')$$

for some appropriate quiver $Q'$ and dimension vector $\alpha'$. If we can generalise this to arbitrary $\theta$ and prove the conjecture, there would be many interesting consequences. First, it would make it possible to handle the $(2,2)$-case, where one may construct a symplectic resolution by first performing GIT (replacing $\theta$ by suitably generic $\theta'$) and then performing a blow-up of the singular locus. Moreover, this result would imply normality for $\mathcal{M}_{q,\theta}(Q,\alpha)$, without any assumption on $\alpha$ and $\theta$. 

Secondly, it would be interesting to extend such arguments and results to coarse moduli spaces of objects in 2-Calabi–Yau categories. This would indeed make it possible to reduce the study of singularities of a numerous class of moduli spaces to answering the following question: does the moduli space parametrise objects of a 2-Calabi–Yau category?

Knowing more about the singularities of coarse moduli spaces of objects in Calabi–Yau categories might allow one to prove a generalisation of the Isosingularity theorem in the non-compact case, provided that a suitable version of the 2-Calabi–Yau condition holds for the category of parabolic Higgs bundles.

7.6. Character varieties and Higgs bundles for arbitrary groups. Another interesting problem would be to analyse whether the results of the present paper can extend to character varieties of (punctured) Riemann surfaces with representations in arbitrary groups and, via the non-abelian Hodge correspondence, to the moduli spaces of (parabolic) Higgs principal bundles. Since the results on punctured character varieties contained in the present paper are deduced based on the correspondence described in Section 3 which heavily relies on the fact that one considers representations and conjugacy classes inside GL(n, C), it seems unlikely that the techniques used here could be applied in the more general setting of G-representations. On the other hand, a possible approach could be the one outlined in the previous subsection. To this end, the question to be answered would be the following: given an algebraic group G, are there cases, other than G = GL_n, where the category of morphisms ρ : π_1(X) → G is 2-Calabi–Yau?

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