Regular black holes via the Kerr-Schild construction in DHOST theories

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Abstract. We extend the standard Kerr-Schild solution generating method to higher order scalar tensor theories that are shift-invariant for the scalar field. Certain degeneracy conditions, crucial for the absence of Ostrogradski ghosts, are found to be required for the validity of the Kerr-Schild ansatz while on the other hand, theories with no parity symmetry are excluded from the solution generating method. The extended Kerr-Schild symmetry turns out to be a very useful tool to easily construct black hole solutions from simple seed configurations. In particular, the generating method developed is adapted to construct generic black holes but also regular black hole solutions within Degenerate Higher Order Scalar Tensor (DHOST) theories. As a particular example we show how to construct explicitly the Hayward metric as a solution to a specific DHOST theory.

Keywords: GR black holes, modified gravity

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1 Introduction

Black holes are one of the most intriguing objects in theories of gravitation, and their interest has been considerably reinforced with the recent detection of gravitational waves coming from the coalescence of rotating black holes. In this context, a good understanding of their physical properties and construction would be crucial for the comprehension and interpretation of recent and future observations. This remark applies particularly to the prototype Kerr metric [1] which is known to be the unique stationary and axisymmetric black hole solution of the vacuum Einstein equations. It is well-known that the derivation of the Kerr solution is a highly nontrivial task, due to the nonlinear character of Einstein’s partial differential equations. One of the ways to face this problem is to consider an “exact perturbation” about the flat Minkowski metric $\eta_{\mu\nu}$ as

$$g_{\mu\nu} = \eta_{\mu\nu} - 2H(x)l_\mu l_\nu,$$  \hspace{1cm} (1.1)

where $l_\mu$ is the tangent vector to a shear-free and geodesic null congruence and $H$ is a scalar function [2, 3]. This metric ansatz commonly known as the original Kerr-Schild ansatz presents the advantage of linearizing the Ricci tensor, and hence the vacuum Einstein equations reduce to a linear system of equations. Nevertheless, it is important to note that only a very specific selection of the shear-free and geodesic null congruences of the Kerr-Schild ansatz (1.1) will reproduce the Kerr solution.

The original Kerr-Schild ansatz can also be extended for a not necessarily flat metric as follows,

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} - 2H(x)l_\mu l_\nu,$$ \hspace{1cm} (1.2)

where $g^{(0)}_{\mu\nu}$ is called the seed metric and $l$ is a null and geodesic vector field with respect to both metrics, i.e.

$$g^{\mu\nu}l_\mu l_\nu = g^{(0)\mu\nu}l_\mu l_\nu = 0, \hspace{1cm} (\nabla_\mu l_\nu)l^\mu = (\nabla^{(0)}_\mu l_\nu)l^\mu = 0.$$

As significative examples of Kerr-Schild metrics, i.e., metrics that are represented through the Kerr-Schild ansatz (1.2), we can mention the pp (resp. AdS) wave background which describes exact gravitational waves propagating in the Minkowski (resp. AdS) space, with a
flat (resp. AdS) seed metric. As already mentioned, another interesting class of solutions that may fit within the Kerr-Schild ansatz is provided by static or spinning black hole solutions. Indeed, it is appealing that in vacuum, most of the metrics describing black holes are of the Kerr-Schild form in the sense that there exists a coordinate system where the metric $g$ can be written as eq. (1.2). A remarkable exception is given by the five-dimensional black ring solution which has a rather distinctive topology, [4].

In the case of black holes, the Kerr-Schild ansatz (1.2) can also be seen as a geometrical way of introducing the mass of the black hole through the scalar function $H$ while the seed metric corresponds to the asymptotic spacetime region. The other parameters of the black hole must be encoded in a nontrivial way in the seed metric. For example, in the case of the higher-dimensional Kerr-(A)dS metrics [5, 6], the seed metric is described by the (A)dS spacetime written in ellipsoidal coordinates in the presence of some parameters $J_i$ whose interpretation as angular velocities will be effective only for the resulting Kerr-Schild metric. In addition to providing a geometrical and physical interpretation of the metric, the Kerr-Schild ansatz has also been proven to be a powerful tool in deriving stationary black hole solutions of vacuum Einstein equations with or without cosmological constant [1, 2, 5, 6]. These examples highlight the importance of the Kerr-Schild ansatz in vacuum. This is in contrast to the case involving matter sources, where now the picture becomes rather more complicated. In fact, the difficulty with the presence of source is to find an appropriate ansatz for the extra dynamical fields to be fully compatible with the equations of motion. For example, it is simple to show that for the Einstein-Maxwell equations, demanding the gauge potential to be proportional to the null and geodesic vector field will be consistent only in four dimensions [7, 8]. In higher dimension, $D > 4$, such an ansatz for the potential field is incompatible with the equations of motion, and this is clearly one of the reasons behind the lack of an explicit higher-dimensional Kerr-Newman solution.

In spite of the difficulties of implementing a Kerr-Schild ansatz with matter, we would like to investigate its feasibility in the somewhat analogous case of scalar tensor theories. Here the gravitational dynamical fields are given by the metric $g$ and a scalar field denoted by $\phi$. Scalar tensor theories are one of the simplest modified gravity theories which extend General Relativity with one (or more) scalar degrees of freedom. Quite a few years back, Horndeski presented the most general (single) scalar-tensor theory with second order equations of motion [9]. The requirement not to have more than two derivatives in the equations of motion is connected to an extension of Ostrogradski’s no go theorem [10], which states that, under some assumptions, higher-order derivative theories are unstable acquiring (an Ostrogradski) ghost degree of freedom. More recently however, it has been shown that some particular degenerate versions of higher-order scalar-tensor theories can propagate healthy degrees of freedom. The most general such Lagrangian depending quadratically on second-order derivatives of a scalar field was constructed in refs. [11, 13], and dubbed Degenerate Higher Order Scalar Tensor (DHOST) theory (or extended scalar tensor theory (EST)). Such DHOST theories have very interesting and rich phenomenology, as we will also see here. In particular, there exists a subclass of DHOST theories where gravitational waves propagate at the speed of light in perfect agreement with the observed results [16–18] (see also the critical analysis of [19]). This DHOST sector has recently attracted a lot of attention in particular for the search of spherically symmetric as well as rotating stealth black hole solutions, see refs. [20, 26] with asymptotic dark energy properties.

\[1\]

A classification up to cubic order in the second-order derivative of the field equations also exists [14, 15].
Our guiding example in order to implement the Kerr-Schild procedure within scalar tensor theories will be the Kerr-Newman solution. In this case, because of the presence of the extra gauge field \( A_\mu \), we have to extend the notion of a seed metric to a seed configuration defined as

\[
\begin{align*}
    ds_0^2 &= -dt^2 + \frac{\Sigma(r, \theta)}{r^2 + a^2} dr^2 + \Sigma(r, \theta) d\theta^2 + (r^2 + a^2) \sin^2 \theta d\varphi^2, \\
    \Sigma(r, \theta) &= r^2 + a^2 \cos^2 \theta,
\end{align*}
\]



\[A_\mu^{(0)} dx^\mu = 0.\] (1.3)

Using this seed configuration, the Kerr-Newman solution in the Kerr-Schild representation (1.2) reads

\[
\begin{align*}
    ds^2 &= ds_0^2 - \frac{2r}{\Sigma(r, \theta)} \left( q^2 - \frac{\mu}{2r} \right) l \otimes l, \\
    l &= dt + \frac{\Sigma(r, \theta)}{r^2 + a^2} dr - a \sin^2 \theta d\varphi \\
    A_\mu dx^\mu &= \frac{qr}{\Sigma(r, \theta)} l. \quad (1.4)
\end{align*}
\]

The reason for detailing this example is to observe that, in spite of the changes of the metric and the gauge potential, the norm of this latter remains unchanged through the Kerr-Schild transformation, i.e.

\[g^{(0)\mu\nu} A_\mu^{(0)} A_\nu^{(0)} = g^{\mu\nu} A_\mu A_\nu = 0.\] (1.5)

Now, the idea is to adapt this example to scalar tensor theories with the aim of constructing black holes within the Kerr-Schild transformation. More precisely, we generate nontrivial black hole solutions from a simple seed configuration. In this paper, for simplicity, we will not tackle the case of four dimensional stationary black hole solutions in scalar-tensor theories. Already in \( D = 4 \) dimensions such stationary metrics, unlike in GR or special cases [24], do not even have the circularity property (see for example the discussion in [27]). In \( D = 3 \), where circularity is trivially guaranteed, we will find the relevant solution which turns out to be the BTZ solution [28] with an effective cosmological constant.

We consider DHOST theories invariant under the shift symmetry of the scalar field \( \phi \rightarrow \phi + \) constant, with Lagrangian of the form \( \mathcal{L}_{\text{DHOST}} := \mathcal{L}(g, \partial g, \partial^2 g, \partial \phi, \partial^2 \phi) \), and as said before we will restrict ourselves to the static case with a seed configuration given by

\[
\begin{align*}
    ds_0^2 &= -h_0(r) dt^2 + \frac{dr^2}{f_0(r)} + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right), \\
    \phi^{(0)}(t, r) &= qt + \psi^{(0)}(r). \quad (1.6)
\end{align*}
\]

Because of the shift-invariance of the theory, we eventually allow the scalar field to depend linearly on the time coordinate. For a static metric, the Kerr-Schild transformation (1.2) from the seed metric (1.6) can be more simply re-written as

\[
\begin{align*}
    g_{\mu\nu} &= g_{\mu\nu}^{(0)} + \mu a(r) l_\mu l_\nu, \\
    l &= dt - \frac{dr}{\sqrt{f_0(r) h_0(r)}}, \quad (1.7)
\end{align*}
\]

where \( \mu \) is a constant parameter proportional to the mass of the black hole solution, and \( a(r) \) is the scalar function to be determined. The resulting metric (1.7), after a redefinition of the time coordinate

\[
    dt \rightarrow dt - \mu a(r) \frac{dr}{\sqrt{f_0(r) h_0(r)} (h_0(r) + \mu a(r))}
\]

– 3 –
acquires a diagonal form

\[ ds^2 = -(h_0(r) - \mu a(r)) dt^2 + \frac{h_0(r) dr^2}{f_0(r) (h_0(r) - \mu a(r))} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \]  

(1.8)

and hence, the net effect of a Kerr-Schild transformation in the metric functions is given by

\[ h_0(r) \rightarrow h(r) = h_0(r) - \mu a(r), \quad f_0(r) \rightarrow f(r) = \frac{f_0(r) (h_0(r) - \mu a(r))}{h_0(r)} . \]

(1.9)

Most importantly we need to specify the change of the scalar field through the Kerr-Schild transformation. In analogy with the Kerr-Newman example (1.5), we will demand that the kinetic term of the scalar field denoted by \( X \) remains unchanged \(( but \ not \ necessarily \ constant \) under the Kerr-Schild transformations of the metric, i.e.

\[ X^{(0)} := g^{(0)\mu\nu} \partial_\mu \phi^{(0)} \partial_\nu \phi^{(0)} = X := g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \]

(1.10)

At a first glance, this condition seems to be quite restrictive and not natural. Indeed, it is simple to see that this condition does not hold for the black hole solutions of scalar fields nonminimally coupled to Einstein gravity \([29, 31]\). Nevertheless, as shown below, for general DHOST theories with shift and parity symmetries, \( \phi \rightarrow \phi + \text{constant} \) and \( \phi \rightarrow -\phi \), the set of equations is always invariant under the condition (1.10).

Let us specify concretely the procedure to follow. Our starting point is the static seed configuration (1.6) which is solution of some DHOST theory described by the action

\[ S(g, \phi) = \int d^4x \sqrt{-g} \mathcal{L}_{DHOST} . \]

We will say that this theory is \textit{Kerr-Schild invariant} with the transformations (1.9)–(1.10) if the variation of the action is quasi-invariant\(^2\) for a specific selection \( a(r) = a(r, X(r)) \). Concretely, this will be achieved if

\[ S(g, \phi) - S(g^{(0)}, \phi^{(0)}) = \int dr \mathcal{E} (r, a(r), a'(r), X(r), X'(r)) + b.t., \]

(1.11)

for a function \( a(r) = a(r, X(r)) \) solving the differential equation given by \( \mathcal{E}(r, a(r), a'(r), X(r), X'(r)) = 0 \). We stress that the possible dependence of the scalar function \( a \) on the kinetic term \( X(r) \) can be justified from the fact that this latter is the unique scalar quantity that remains invariant under the Kerr-Schild procedure (1.10). In practice, this can be viewed as a generating method to easily construct black hole configurations from a simple seed solution. Let us illustrate the Kerr-Schild process with the simple example of the shift-symmetric quadratic Horndeski theory,

\[ S[g, \phi] = \int d^4x \sqrt{-g} \left\{ G_2(X) + G_4(X) R - 2 G_{4,X}(X) \left[ (\Box \phi)^2 - (\nabla_\mu \nabla_\nu \phi)(\nabla^\mu \nabla^\nu \phi) \right] \right\} . \]

(1.12)

After some manipulations, the variation of this action through the Kerr-Schild transformations (1.9)–(1.10) yields (up to a boundary term)

\[ S(g, \phi) - S(g^{(0)}, \phi^{(0)}) = -8\pi \mu \int dr \sqrt{\frac{f_0(r)}{h_0(r)}} \left[ a(r) + ra'(r) \right] \left[ -2G_{4,X} X + G_4 \right] . \]

(1.13)

\(^2\)By quasi-invariant, we mean an action invariant up to boundary terms denoted by b.t..
Hence, we conclude that starting from any seed configuration (1.6) solution of the quadratic Horndeski theory (1.12), the transformed metric and scalar field defined respectively by (1.8) and (1.10) will be a black hole solution of the same theory provided that the metric function $a$ satisfies $a(r) + ra'(r) = 0$, in other words for a Coulomb fall off, $a(r) = \frac{1}{r}$. It is also interesting to stress that this Coulomb behavior for the mass term is independent of the seed configuration. This is in complete accordance with the existence of black hole solutions for this quadratic Horndeski theory which are asymptotically flat and (A)dS, see refs. [20, 32–36] but also for solutions with rather exotic asymptotics, [37].

What is more interesting with this generating method is that we can go further than GR and construct exotic solutions such as regular black holes in scalar tensor theories (see for example [38]). Such solutions were thought to exist in higher order theories (see for example chapter 8 of [39]) but the vast functional space of the coupling functions made the theories and solutions hard to find. We will achieve this here, precisely due to the Kerr-Schild ansatz, as the seed solution and the mass function $a$ can be found in two independent steps. We will see that we can first adequately fix the mass function and then reverse engineer the seed theory in order to find a full regular black hole solution. It is very probable that our method can be extended to different asymptotic spacetimes as well as other exotic solutions such as wormholes or solitons.

The paper is organized as follows: in section 2 we shall acquire the validity of the Kerr-Schild ansatz in general higher order scalar-tensor theories. In particular, we will see that certain degeneracy conditions, necessary for healthy theories, will actually be required for the validity of Kerr-Schild symmetries while on the other hand, certain healthy scalar tensor theories such as the DGP Galileon, or the Gauss-Bonnet, will not adhere to the Kerr-Schild symmetries. We will then find new static black hole solutions, with differing mass fall-off, with relative ease within very general DHOST theories. With hindsight from section 2 we will then move on to construct black holes without a central curvature singularity. As a direct application of this procedure, we will present a special DHOST theory that sources the regular Hayward metric black hole [40]. In the last section, we will summarize our results and discuss some generalizations of our work (in particular for generalized Proca theories). Finally, an appendix will be devoted to the three-dimensional rotating case of the quadratic Horndeski theory.

## 2 Kerr-Schild symmetry for generalized scalar tensor theories

Before treating the physical case of DHOST theories, we first consider a more general class of scalar tensor theories which contains up to second order covariant derivatives of the scalar field and whose action is,

\[
S[g, \phi] = \int d^4x \sqrt{-g} \left[ K(X) + G(X)R + F_1(X)\Box \phi + F_2(X)G^{\mu \nu} \phi_{\mu \nu} + A_1(X)\phi_{\mu \nu} \phi^{\mu \nu} \right. \\
\left. + A_2(X)(\Box \phi)^2 + A_3(X)\Box \phi \phi^{\mu \nu} \phi_{\mu \nu} + A_4(X)\phi^{\mu \nu} \phi_{\mu \nu} \phi_{\nu \rho} + A_5(X) (\phi^{\mu \nu} \phi_{\mu \nu} \phi^{\rho})^2 \right].
\]

(2.1)

This theory includes shift-symmetric Horndeski and DHOST theories but also theories which can have Ostrogradski ghosts. Our aim is to check when the Kerr-Schild ansatz is valid and if certain conditions of degeneracy are required by the ansatz itself. For simplicity we have defined $\phi_{\mu \nu} = \nabla_\mu \nabla_\nu \phi$ and $K, G, F_1, F_2$ and the $A_i$ are a priori arbitrary functions of the
kinetic term \( X = \partial_{\mu} \phi \partial^{\mu} \phi \) which in turn ensures the shift invariance \( \phi \to \phi + \text{cst} \) of the action. For latter convenience, we also define

\[
\mathcal{H}(X) = A_1(X) X - G(X), \quad \mathcal{B}(X) = A_3(X) X + 4G_X(X) - 2A_1(X), \\
\mathcal{Z}(X) = A_3(X) + A_4(X) + X A_5(X).
\] (2.2)

As previously, we are interested under which conditions the action (2.1) is (quasi)invariant under a Kerr-Schild transformation (1.9) leaving invariant the kinetic term (1.10). It is interesting to note that a necessary condition for the action to be (quasi)invariant is to impose \( A_2(X) = -A_1(X) \) and \( F_1 = F_2 = 0 \). While the former is the Horndeski degeneracy condition the latter conditions tell us that theories without parity symmetry \( (\phi \leftrightarrow -\phi) \) are excluded from the Kerr-Schild solution generating method. This includes therefore the \( G_3 \) and \( G_5 \) Galileons (see for example [34]). Only under these conditions, the variation of the action can be of the form of eq. (1.11), i.e.

\[
S[\bar{g}, \bar{\phi}] - S[g, \phi] = -\mu \pi \int dr \sqrt{h_0(r)} \left[ a(r)P(r, X) + a'(r)Q(r, X) \right] + b.t.,
\] (2.3)

where the expressions of \( P \) and \( Q \) read

\[
P(r, X) = r^2(X')^2 \mathcal{Z}(X) + 4RX'B(X) - 8\mathcal{H}(X), \quad Q(r, X) = r^2X'B(X) - 8r\mathcal{Z}(X).
\] (2.4)

Hence, in order for the transformations (1.9)–(1.10) to be a symmetry of the action, the function \( a \) must be given by

\[
a(r) = e^{-\int \frac{P(r, X)}{Q(r, X)} dr},
\] (2.5)

provided \( Q(r, X) \neq 0 \), and where of course the integration constant can be re-absorbed into the mass parameter \( \mu \). Note that \( K(X) \) is not involved in the above mass integral in accord with our requirement for \( X \) (1.10). Also, in accordance to classical no hair theorems [46] the vanilla case of a minimally coupled scalar field \( (K = -\frac{1}{2}X, G = 1) \), i.e. \( \mathcal{H}(X) = -1 \) and \( \mathcal{B}(X) = 0 \), is clearly excluded. It is also interesting to note from the equation (2.5) that the standard Coulomb fall off for the mass \( a(r) \propto \frac{1}{r} \) will be ensured only if \( Q(r, X) = rP(r, X) \).

This condition is achieved for constant kinetic term \( X = \text{constant} \) or for

\[
r\mathcal{Z}(X) X' + 3B(X) = 0.
\] (2.6)

In particular, this last condition is satisfied for the quadratic Horndeski case (1.12) in accordance with the results presented in the introduction. Going beyond the quadratic Horndeski case therefore already hints on a much richer behavior with differing fall-off mass functions.

### 2.1 Degenerate Higher Order Scalar Tensor theories (DHOST)

We now move on to consider the physical subclass of the scalar tensor theories (2.1) invariant under the reflection \( \phi \to -\phi \) and free of Ostrogradski ghosts [11–13]. The first condition imposes that \( F_1 = F_2 = 0 \) while the absence of Ostrogradski ghosts may be ensured through
the following restrictions,

\[
A_1 = -A_2 \neq \frac{G}{X}, \\
A_4 = \frac{1}{8(G - X A_1)^2} \left\{ 4G \left[ 3(-A_1 + 2G X)^2 - 2A_3 G \right] - A_3 X^2 (16A_1 G X + A_3 G) \\
+ 4X \left[ -3A_2 A_3 G + 16A_1^2 G X - 16A_1 G X^2 - 4A_3^2 + 2A_3 G X \right] \right\}, \\
A_5 = \frac{1}{8(G - X A_1)^2} (2A_1 - X A_3 - 4G X) (A_1 (2A_1 + 3X A_3 - 4G X) - 4A_3 G). \tag{2.7}
\]

Note that while \(A_1 = -A_2\) was also necessary for the validity of the Kerr-Schild ansatz the latter two conditions are not.

### 2.1.1 DHOST solutions with a time independent scalar field \(q = 0\)

For simplicity, let us consider the case where the scalar field does not depend on time, i.e. \(q = 0\). This case has been considered recently in [26] and some solutions with \(X\) non constant were given. Using the notations (2.2), the field equations can be written in a tractable way as

\[
X [2(A_1 G) X + G A_3] + r^2 \left[ (KH)_X + \frac{3}{4}KB \right] = 0, \tag{2.8a}
\]

\[
-3(3r f + 8(B r X') f \mathcal{H}) \left( \frac{r f'}{f} + 4 \right) - 32 f \mathcal{H} \left[ \frac{K r^2 + 2G}{f} + 2 \mathcal{H} \left( \frac{r f'}{f} + 1 \right) \right] = 0, \tag{2.8b}
\]

\[
r^2 (16 B X X'' + 3 B^2 X'^2 + 8 H X' r (3r f' - 16 \mathcal{H} X) + 16 r^2 \mathcal{H} B X'' - 64 \mathcal{H}^2 \left[ \frac{r f'}{f} + 1 \right] + \frac{2G + r^2 K}{2f \mathcal{H}} \right] = 0, \tag{2.8c}
\]

where we have assumed that \(\mathcal{H} \neq 0\) and \(K r^2 + 2G \neq 0\). The equations (2.8a) and (2.8b) are identical to those given in [26] while the latter is the \(tt\)-equation which simplifies considerably in this notation. It is interesting to note that (2.8a) fixes \(X\) independently of the spacetime metric [26], condition that is in adequation with the invariance of the kinetic term (1.10) through a Kerr-Schild transformation.

Using these variables it is straightforward to verify that the mass function for the Kerr-Schild transformation (2.5) is given by,

\[
a(r) = \frac{1}{r} e^{\frac{3}{2}} \int dX \frac{\mathcal{H}}{\mathcal{H}}. \tag{2.9}
\]

The same conclusion can be obtained from the field equations (2.8). As already mentioned, for invariant \(X\) as defined by (1.10), the equation (2.8a) is trivially invariant under the Kerr-Schild transformation of the metric (1.9) while the invariance of the eq. (2.8b) will be ensured only if \(a(r)\) is given by (2.9). Finally, the remaining equation (2.8c) under the Kerr-Schild transformations (1.10)–(1.9) with \(a\) given by (2.9) will be mapped to itself provided eq. (2.8b) is satisfied.

Rather than solving the curved field equations given above it suffices to find a seed solution and use the Kerr-Schild transformation to evaluate the mass function (2.9). We will choose to do this in the case of flat spacetime although numerous other cases are possible such as dS, adS, or other more exotic vacua. One needs only to put the relevant metric...
ansatz for the seed metric solution in order to proceed. For a flat seed metric $f_0 = h_0 = 1$, the field equations read,

$$X[2(A_1 G)_X + G A_3] + r^2 \left[(K \mathcal{H})_X + \frac{3}{4} K B\right] = 0,$$

$$- 3(B r X')^2 + 32(B r X') \mathcal{H} - 32 \mathcal{H}(K r^2 + 2 A_1 X) = 0,$$

$$r^2(16 B_X H + 3 B X')^2 - 128 r H X X' + 16 r^2 B X' - 32 \mathcal{H}(2 A_1 X + r^2 K) = 0.$$

Given a particular theory, equation (2.10a) will provide an algebraic expression for $X$ whereas the remaining equations (2.10b) and (2.10c) will impose compatibility constraints onto the coupling constants and parameters of the model under consideration. In fact, from (2.10b) we obtain

$$B r X' = \frac{16 H}{3} \left(1 \pm \sqrt{1 - \frac{3(K r^2 + 2 A_1 X)}{8 \mathcal{H}}} \right)$$

with $\mathcal{H} \neq 0$, whereas from (2.10c) and (2.10b) we have

$$-(B r^2 X')' + 8 r \mathcal{H} X X' + 4(2 A_1 X + r^2 K) = 0$$

Due to the structure of the field equations, one approach is to keep $B$ and $\mathcal{H}$ of a given order, hence we consider

$$G = \gamma X^n, \quad A_1 = \alpha X^{n-1}, \quad A_3 = \beta X^{n-2}, \quad K = -\Lambda X^m$$

This choice directly allows us to evaluate the mass term,

$$\mathcal{H} = (\alpha - \gamma) X^n = \mathcal{H}_0 X^n$$

$$B = (\beta + 4 \gamma n - 2 \alpha) X^{n-1} = B_0 X^{n-1}$$

and therefore

$$a(r) = \frac{1}{r} X \frac{3 \mathcal{H}_0}{8}$$

Second, from (2.8a) we can algebraically solve the kinetic term $X(r)$ explicitly

$$X^{n-m} = \delta r^2$$

where

$$\delta = \Lambda \frac{4 m \alpha - 4 m \gamma + 3 \beta + 4 \alpha n - 6 \alpha + 8 \gamma n}{4 \gamma (-2 \alpha + \beta + 4 \alpha n)} = \Lambda \frac{4 (m + n) \mathcal{H}_0 + 3 B_0}{4 \gamma (B_0 + 4 n \mathcal{H}_0)}.$$  

Finally, we note that (2.8b) and (2.10c) reduce into two constraints for the theory,

$$B_0 = (n - m) \frac{8 \mathcal{H}_0}{3} \left(1 \pm \sqrt{1 - \frac{3(2 \delta \alpha - \Lambda)}{8 \mathcal{H}_0 \delta}} \right)$$

$$B_0 \frac{3 n - m}{n - m} = \mathcal{H}_0 n + \frac{2(2 \alpha \delta - \Lambda)(n - m)}{\delta}$$

Straightforwardly, equations (2.19) and (2.20) can be re-written in the following manner

$$\frac{3 B_0}{8 \mathcal{H}_0} = (n - m)(1 \pm \sqrt{1 - \epsilon}),$$

$$\frac{3 B_0}{8 \mathcal{H}_0} = \frac{n - m}{3 n - m}(3 n + 2(n - m) \epsilon),$$
where we have defined the auxiliary quantity, $\epsilon = \frac{3}{8H_0} \frac{2\delta \alpha - \Lambda}{s}$, which is required to be smaller than one. This form of the constraints gives direct access to the form of the mass function (2.16). On the other hand, compatibility of (2.21) implies that

$$1 \pm \sqrt{1 - \epsilon} = \frac{3n}{3n - m} + 2\epsilon \frac{n - m}{3n - m}$$

(2.22)

must be satisfied. The later condition defines two branches of solutions:

**Branch 1.** For this branch we have that

$$1 + \sqrt{1 - \epsilon} = \frac{3n}{3n - m} + 2\epsilon \frac{n - m}{3n - m}.$$  

(2.23)

Solving for $\epsilon$ implies $\epsilon = 3/4$ which gives the following constraint on the parameters,

$$4m\alpha - 2\alpha + \beta - 4m\gamma - 4n\alpha + 8\gamma n = 0.$$  

(2.24)

It so happens that the above condition satisfies both equations (2.21). By plugging this result into the mass function (2.16) we observe that there is no dependence on the parameters, giving a mass function of the form

$$a(r) = r^2$$

(2.25)

where in this case the integration constant $\mu$ as defined through $\mu a(r)$ in the metric (1.8) plays the role of a cosmological constant changing the asymptotic behavior of the solution relative to the size and magnitude of $\mu$.

However, the fact that both of the equations (2.21) are satisfied with one condition (2.24) signals degeneracy of the equations. This happens because the choice (2.13) with (2.24) gives $2G + r^2K = 0$ for which equations (2.10a)–(2.10c) are not satisfied. At the end a careful resolution of the initial Euler-Lagrange equations give us an additional condition,

$$\alpha (m - 3n) + 2n\gamma = 0.$$  

(2.26)

To summarize, when both conditions (2.24) and (2.26) are satisfied, eq. (2.13) and (2.25) is solution. This solution is seemingly quite special as the ‘mass’ integration constant behaves as a cosmological constant completely independently of the bulk cosmological constant term obtained in $K$ (for $m = 0$ for example).

**Branch 2.** For the negative branch we get

$$1 - \sqrt{1 - \epsilon} = \frac{3n}{3n - m} + 2\epsilon \frac{n - m}{3n - m}$$

(2.27)

As in the previous case we solve for $\epsilon$, which now reads

$$\epsilon = \frac{n(2m - 3n)}{(m - n)^2} \leq 1.$$  

(2.28)

Using the above relation, from (2.21) we obtain,

$$\frac{3\rho_0}{8H_0} = -n.$$  

(2.29)
Solving the above two constraints gives us two relations between the coefficients of the Lagrangian,
\[ 2n\gamma = \alpha(3n - m), \]
\[ 3\beta = 2(3 + m - 7n)\alpha. \]  
Thus for this branch the mass function depends on \( n \) and \( m \),
\[ a(r) = r^{n-m\over n-m}, \]  
with the integration constant is given by \( \mu \) in our representation (1.8). This branch offers a far richer structure for the solutions. We note that we get a Coulomb mass term if \( n = 0 \). The resulting metric after the Kerr-Schild procedure is given by,
\[ f = h = 1 - \mu r^{n-m\over n-m}, \]  
and describes an asymptotically flat black hole spacetime for \( m > 3n \) and \( m > n \). Note that \( X \) is then singular at \( r = 0 \) while it goes to zero as \( r \to \infty \). The family of black hole solutions includes the case where we have a canonical kinetic term \( m = 1 \). We also note the generality of the solution obtained with relative ease, as integrating the equations with Kerr-Schild reduces to finding a flat spacetime solution. Note for completeness that the solution given here, agrees with the static asymptotically flat solution found in [26], by solving the curved field equations (2.8a)–(2.8c) with \( m = \alpha = 0 \) upon which we have \( a(r) = r^{-3} \).

3 Regular black hole solutions

In the previous section we constructed a large class of black hole solutions in DHOST theories. Starting from a seed configuration, represented by \((h_0, f_0, X_0)\) of some DHOST theory, we constructed a black hole solution with the same scalar kinetic term \( X = X_0 \) (where \( X = X_0 \) is not necessarily constant) and a mass term given by (2.5). There is an important technical advantage within the method, which distinguishes two independent steps: finding the seed solution and seeking the mass fall-off term. We now apply this technique in order to construct a regular black hole solution within DHOST theory. By regular black hole we mean a black hole solution with at least one regular event horizon and no central curvature singularity. The first such regular black hole geometries were proposed in [41] but no source of this solution was given. Later on, the authors of refs. [42, 43] were the first to present an exact regular black hole solution of General Relativity coupled to a specific nonlinear electrodynamic source.

From our Kerr-Schild procedure, looking for an asymptotically flat regular black hole with a seed metric given by \( h_0 = f_0 = 1 \), will imply that the final metric will be of the form (2.5)
\[ h(r) = f(r) = 1 - \mu X(r) \frac{dX}{dr}. \]
Given the form of the above mass function we can make the following simple hypothesis,
\[ \frac{3E}{5H} = \frac{\lambda}{X} \implies h(r) = f(r) = 1 - \frac{\mu X(r)^{\lambda}}{r}, \]  
where \( \lambda \) is a constant. The idea will be to choose an appropriate kinetic term \( X(r) \) and a parameter \( \lambda \) in order for the metric to be: regular at \( r = 0 \), asymptotically flat and to have
an outer event horizon at some finite $r = r_h$. This would mean that our seed configuration is given by $h_0 = f_0 = 1$ together with $X(r)$, and hence by inverse engineering from the field equations (2.10), one would be able to specify the corresponding DHOST theory, that is to determine the functions $K, G, A_1$ and $A_3$ (as functions of $X$ only). Keeping this in mind we start with hypothesis (3.1) and solve for the underlying theory functions, $K, G, A_3, \ldots$ etc in terms of $X$, its derivatives and $r$. It is important to stress that once we fix $X = X(r)$, at the end we must have that all coupling functions are solely $X$-dependent and not $r$-dependent. This will be achieved if the function $X$ is locally invertible.

Let us proceed by detailing all the different steps of the construction. We first observe that using the hypothesis (3.1), the constraints (2.11) and (2.12) take the form

$$\lambda \frac{r X'}{X} = 2(1 \pm \sqrt{1 - \epsilon})$$

and

$$r \mathcal{H} \left( \lambda \frac{r X'}{3X} - 1 \right)' = \left( \frac{4}{3} \epsilon - 1 \right) \mathcal{H},$$

where for simplicity we have defined

$$\epsilon = \frac{3(K r^2 + 2A_1 X)}{8 \mathcal{H}}.$$  

In addition, from eq. (3.1) and making use of the definition of $\mathcal{B}$, we can trade off $A_3$ and $A_1$

$$A_3 = - \frac{4G X}{X} + \frac{2A_1}{X} + \frac{8\lambda \mathcal{H}}{3X^2}, \quad A_1 = \frac{\mathcal{H} + G}{X}$$

and, hence through the definition of $\epsilon$ one can get either $G$ or $K$ as

$$G + \frac{1}{2} Kr^2 = \left( \frac{4\epsilon}{3} - 1 \right) \mathcal{H}.$$ 

After some algebraic manipulations, eq. (2.8a) is conveniently reduced as

$$2(\mathcal{H} G)_X + r^2 (K \mathcal{H})_X + \frac{2\lambda \mathcal{H}}{X} \left( \frac{4}{3} G + Kr^2 \right) = 0.$$  

Now the picture of the construction is quite clear. From eq. (3.2), the $\epsilon-$ function is obtained, and this allows us to get $\mathcal{H}$ by solving eq. (3.3). Finally, using the last equation (3.7), we will get $K$. In practice, we end up with

$$\epsilon = \left( 2 - \frac{\lambda r X'}{2X} \right) \frac{\lambda r X'}{2X}, \quad \mathcal{H} = \frac{H_0}{X^\lambda (\frac{\lambda r X'}{3X} - 1)}, \quad K = -\frac{A_r}{r X^{\lambda/3}},$$

where $H_0$ is an integration constant, and where

$$A = \frac{H_0 \left( 1 - \frac{\lambda r X'}{X} \right)}{X^{2\lambda/3} \left( \frac{\lambda r X'}{3X} - 1 \right)}.$$  

Having all the functions fixed in terms of $r$ and $X(r)$, we will now implement our second hypothesis. We choose a kinetic term $X(r)$ that is an invertible function in a such way that combined with a judicious choice of $\lambda$, it will provide a regular black hole solution. In order
to ensure the regularity of the solution, the metric should satisfy the following requirements: 
(i) it must be regular at the origin \( r = 0 \) and at infinity (asymptotically flat in our case),  
(ii) it must possess an outer event horizon at some finite \( r = r_h \), and finally (iii) the metric function \( f = h \) must satisfy the Sakharov criterion \([39, 44]\) at the origin, i.e.

\[
f(r) \underset{r\to 0}{\sim} 1 - f_0 r^p, \quad p \geq 2,
\]

which means that the metric function possesses at least a de-Sitter core near the origin. The condition (iii) ensures that any invariant constructed from the contractions of the Riemann curvature tensor will be regular at the origin.

We observe that a relatively simple and invertible function \( X(r) \) that provides a metric function satisfying the conditions (i) and (ii) is given by

\[
X(r) = \frac{\alpha r^2}{\beta r^2 + \gamma}, \quad \Rightarrow \quad r^2 = \frac{\gamma X}{\alpha - \beta X}
\]

where \( \alpha, \beta \) and \( \gamma \) are arbitrary non-zero real constants independent of the mass, while the choice of \( \lambda = 2 \) ensures that condition (iii) is satisfied.\(^3\) Finally, using the relation (3.10) we easily conclude that the DHOST theory parameterized by

\[
\mathcal{H}(X) = \frac{H_0 \alpha^3}{X^2(4\beta X - \alpha)}, \quad G(X) = -\frac{3H_0 \alpha^2 \left(8\beta^2 X^2 - 8\alpha \beta X + \alpha^2\right)}{X^2(4\beta X - \alpha)^2},
\]

\[
K(X) = \frac{8\alpha H_0 \left(16X^4 \beta^4 - 54X^3 \alpha \beta^3 + 63X^2 \beta^2 \alpha^2 - 28\alpha^3 \beta X + 3\alpha^4\right)}{X^3 \gamma (\alpha - 4\beta X)^2}
\]

supports the existence of the following spherically symmetric regular black hole

\[
ds^2 = -\left(1 - \frac{\mu \alpha^2 r^3}{(\beta r^2 + \gamma)^2}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{\mu \alpha^2 r^3}{(\beta r^2 + \gamma)^2}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)
\]

It is easy to observe that solution (3.12) is well behaved at the origin and at infinity, with a Kretschmann invariant

\[
\lim_{r \to 0} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \sim 76 \left(\frac{\alpha}{\gamma}\right)^4 r^2 + \mathcal{O}(r^4)
\]

along with a regular everywhere kinetic term \( X(r) \). Furthermore, for \( \alpha > 0 \) large enough (compared to \( \beta \)) the metric will have an inner and outer event horizon at some positive values of the radial coordinate \( r \). One can of course construct, in a similar fashion, different regular metrics. To end this section, let us instead construct the specific DHOST theory that sources a known regular geometry, the Hayward metric, \([40]\). The interests of this spacetime example are multiple and particularly used to test the formation and evaporation of non-singular black holes, see e.g. \([45]\). For our static ansatz, the regular Hayward black hole metric is given by

\[
ds^2 = -\left(1 - \frac{\mu \alpha r^3}{\beta r^3 + \gamma}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{\mu \alpha r^3}{\beta r^3 + \gamma}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2).
\]

\(^3\)Note that one of the three \( \alpha, \beta \) or \( \gamma \) can be set to 1 without loss of generality.
From our previous derivation, this would correspond to $\lambda = 1$ and

$$X(r) = \frac{\alpha r^3}{\beta r^3 + \gamma},$$

and the following DHOST theory

$$\mathcal{H}(X) = \frac{H_0 \alpha^2}{X^2}, \quad G(X) = \frac{H_0 \alpha (5\alpha - 6\beta X)}{X^2},$$

$$K(X) = \frac{2H_0 (\alpha - \beta X)^2 (3\beta X - 5\alpha)}{\gamma X^3} \left( \frac{\gamma X}{\alpha - \beta X} \right)^{\frac{1}{3}},$$

will source the regular metric (3.14). Note that the scalar field is non-trivial even when the mass parameter $\mu$ is switched off. This may imply that in the ground state of the theory the scalar is not constant, but rather it forms a soliton which does not backreact on the metric.

4 Conclusions

In this paper, we have explored an extension of the Kerr-Schild solution generating method for shift invariant higher order scalar tensor theories. In order to achieve this, we have put forward a Kerr-Schild ansatz for a static configuration including the metric (similarly to GR) and the kinetic term of the scalar field. We have found that quite generically DHOST theories with parity invariance (for the scalar field) obey the Kerr-Schild symmetries with some mass function, that is Coulombian for Horndeski, and possibly non-Coulombian once we go to DHOST theories. It is well known that in GR, the Kerr-Schild method reproduces most known interesting solutions. It is hardly ever the method that is initially used to obtain some solution. This is actually true for most solution generating methods in GR. On the contrary here, for scalar tensor theories, given the complexity of the field equations, and the generality of the coupling functions at hand, we have shown that the method is actually crucial in obtaining new solutions such as black holes or even more exotic vacua like regular black holes. What is important here is the independent two step obtention of the seed metric and the mass fall-off function. It is very probable that our method can be extended to other seed vacua as de-Sitter or even other exotic solutions such as wormholes or solitons. In this sense it is possible that the Kerr-Schild extension may help significantly in obtaining new solutions in scalar tensor theories.

When studying modified gravity theories, it is important to search for novel solutions and to compare them with the solutions in GR. In the case of DHOST theories, for most of the solutions presented in the literature (with a few exceptions) the metric is the same as in GR, such as Schwarzschild-dS or Kerr-dS metrics. In this paper we presented a method for constructing solutions with metric other than known GR metric. Comparing properties of new solutions with their counterparts in GR will help to search for signatures of modified gravity by using modern and future gravity experiments, and, on the other hand, to place constraints on particular modifications of gravity. In particular, the trajectories of stars around supermassive black holes would be clearly modified, giving an opportunity to test the theory. In addition, it would be also interesting to study physical properties of the obtained solutions as compared to GR case. One may expect that the thermodynamics of black holes is considerably modified for the solutions presented in the paper.

Apart from these general observations a natural extension to our work is generating a procedure for stationary and axisymmetric configurations. This task is far from obvious but
a first step in this direction has been made in the three-dimensional quadratic Horndeski theory with a scalar field depending linearly on the time and angular coordinates (see the appendix). Here, the challenges to be met in 4 dimensions are numerous as for example the fact that stationary vacua do not have the circularity property (unlike Ricci flat or Einstein spaces). This already renders difficult the starting ansatz for the seed metric. For certain stealth solutions as the ones reported in [24] our method may be generalized on very similar lines as presented here.

Even if the working examples were for purely radial scalar field $\phi = \phi(r)$, our procedure for constructing solutions also applies for a scalar field depending linearly and separately on the time coordinate, i.e. $\phi(r, t) = qt + \psi(r)$. It will be of interest to look for such configurations in spite of the fact that the presence of a time dependent scalar fields considerably complicates the field equations even for a simple seed configuration. Note that such a solution has been found in ref. [26] (see eqs. (66)–(68–70)) for a subclass of the DHOST theory parameterized by $B = \frac{4}{3}$ and $H = \frac{2}{3} X$, and with a kinetic scalar field given by

$$X(r) = -q^2 \left(1 - 3r^2 \Lambda \right)^{\frac{4}{4 - 3}}.$$

It is interesting to observe that a direct application of our mass term (2.5) yields

$$a(r) \propto \frac{1}{r} \left(1 - \frac{3r^2 \Lambda}{4 \xi} \right)^{\frac{4}{3 - 4}},$$

and perfectly fits with the mass term of the solution reported in ref. [26].

It will be desirable to extend the solution generating method for other matter sources. As a natural candidate, and in a complete analogy with the quadratic Horndeski case, we may consider the generalized Proca action

$$S_{GP} = \int d^4x \sqrt{-g} \left[ G_2(X, F) + G_4(X) R + G_4,X \left((\nabla_\mu A^\mu)^2 - \nabla_\mu A_\nu \nabla^\nu A^\mu \right) \right],$$

where $X = -A_\mu A^\mu / 2$ and $F$ is the standard Maxwell term $F = -F_{\mu\nu} F^{\mu\nu}$. The question is to see under which conditions the generalized Proca action (4.1) can be invariant under the Kerr-Schild transformations. These transformations will be the same for the metric (1.9) while the scalar field condition (1.10) can be naturally replaced by

$$X^{(0)} := -\frac{1}{2} g^{(0)\mu\nu} A_\mu^{(0)} A_\nu^{(0)} = X := -\frac{1}{2} g^{\mu\nu} A_\mu A_\nu, \quad F^{(0)} = F. \quad (4.2)$$

Nevertheless, it is simple to observe that in contrast with the scalar field case, the on-shell action $S_{GP}$ on a purely electric ansatz will explicitly depend on the electric potential $A_t(r)$ and its derivative. As a direct consequence, in order to properly define the notion of Kerr-Schild invariance, this will require that the electric potential $A_t(r)$ does not vary under the Kerr-Schild transformations. On the other hand, since the Proca mass term must remain constant, we are forced to turn on other components of the potential vector $A_\mu$. The most simple option in accordance with some solutions present in the literature [47–50] is to consider a non-zero radial component, and hence consider an ansatz configuration of the form

$$ds^2 = -h(r) dt^2 + \frac{dr^2}{f(r)} + r^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2 \right), \quad A_\mu dx^\mu = A_t(r) dt + A_r(r) dr. \quad (4.3)$$
As anticipated previously, the on-shell action on the ansatz (4.3) depends explicitly on $A_t(r)$, and reads

$$S_{GP} = \int dr r^2 \sqrt{h} \left( \frac{f(A'_t)^2}{2h} \right) X, f(X) = G_2 \left( \frac{r f(A'_t)^2}{2h} \right)$$

$$- \int dr \sqrt{h} \left\{ G_4(X) \left[ r f' + f - 1 \right] + G_4 X \left[ -r (A'_t)^2 + 2hr X' - A^2 + 2rh' X + 2hX \right] \right\}.$$  

It is interesting to note that the terms involving the electric potential $A_t$ are expressions that remain invariant under the Kerr-Schild transformations (1.9) with $A_t$ and $X$ being unchanged. Consequently, the variation under the Kerr-Schild transformations of the generalized Proca action does not involve the electric potential, and is given by (up to a boundary term)

$$S_{GP}(\bar{g}, A_t, X) - S_{GP}(g, A_t, X) = -2\mu \int dr \sqrt{f} \left[ 2G_4 X - G_4 \right] \left[ a + ra' \right].$$  

Finally as in the quartic Horndeski case, the mass term will be of the Coulombian form, i.e. $a(r) = \frac{1}{r}$ in accordance with the existing solutions [47–51]. It will be also interesting to investigate if such invariance can be exported to more general vector tensor theories [51].

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**A Example of rotating solution in three dimensions**

As a first approach to the stationary generalization of the Kerr-Schild method let us present the three-dimensional case for the quadratic Horndeski action,

$$S[g, X] = \int d^3x \sqrt{-g} \left\{ G_2(X) + G_4(X)R - 2G_4 X \left[ (\Box \phi)^2 - \phi_{\mu\nu} \phi^{\mu\nu} \right] \right\}.$$  

Assuming stationarity and since spacetime is circular, the most general metric in 2 + 1 dimensions can be parameterized as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + H^2(r) [d\theta - k(r)dt]^2.$$  

We will consider an ansatz for the scalar field,

$$\phi(t, r, \theta) = qt + \psi(r) + L\theta.$$
where \(q\) and \(L\) are two constants. This ansatz for the scalar is in accord with the Hamilton-Jacobi functional interpretation for stationary spacetime geodesics [24], and allows a mild linear dependence in the \(\theta\) and \(t\) coordinates since \(\partial_{\theta}, \partial_{t}\) are two Killing vectors for (A.2). As always for the Kerr-Schild ansatz we first need to find a seed flat solution of the above spacetime which essentially sums up to finding the general solution given the spacetime symmetry.

For the general \(G_2\) and \(G_4\) Horndeski theory, the \(\epsilon_{rr} = 0\) field equation permits to express the derivative of the function \(f\) as

\[
f' = -\frac{1}{2} H \left[ (2G_{4,X,X}X + G_{4,X}) (k')^2 H^2 - 2G_{2,X} \right]. \tag{A.4}
\]

Injecting this expression into the \(\epsilon_{rr} = 0\) equation, one obtains that the kinetic term \(X\) must satisfy the following functional relation that only depends on \(X\)

\[
2X [G_{2,X}G_{4,X} + G_{2}G_{4,X,X}] - G_{2,X}G_{4} + G_{2}G_{4,X} = 0 \tag{A.5}
\]

which in turn implies that \(X\) must be constant given by one of the zeros of the previous functional equation. On the other hand, the combination

\[
\frac{1}{2} \epsilon_{tt} + k(r) \epsilon_{\theta \theta} + \frac{k(r)^2}{2} \epsilon_{\theta \theta} = 0,
\]

yields

\[
\frac{1}{2} f'^2 [2G_{4,X}X - G_{4}] \frac{H''}{H} = 0, \tag{A.6}
\]

and hence \(H'' = 0\), and the last remaining Einstein equation is equivalent to

\[
\left( H^3 k' \right)' = 0. \tag{A.7}
\]

After some redefinitions of the radial and temporal coordinates, the metric (A.2) for the solution can be written as

\[
ds^2 = -\frac{\beta}{4\alpha} \left( r^2 - b^2 \right) dt^2 + \frac{dr^2}{\frac{4}{\alpha} \left( r^2 - b^2 + \frac{\alpha J^2}{r^2} \right)} + J dt d\theta + r^2 d\theta^2, \tag{A.8}
\]

where \(b\) and \(J\) are two integration constants, and where for simplicity we have defined

\[
\alpha = 2G_{4,X,X}X + G_{4,X}, \quad \beta = 2G_{2,X}. \tag{A.9}
\]

These expressions are evaluated on the constant value \(X\) solution of the equation (A.5). Note that the metric solution is nothing but the BTZ metric with a cosmological constant \(\Lambda = \frac{\beta}{4\alpha}\).

This three-dimensional case has been treated by brute force by solving the field equations. It is then legitimate to wonder about the validity of our Kerr-Schild mechanism in this three-dimensional stationary case. The procedure can be formalized as follows. Firstly, a null geodesic vector field for the seed metric (A.2) is given by \(l = dt - \frac{dr}{f(r)}\), and hence the Kerr-Schild transformation (1.7) after the following redefinitions of the angular and time coordinates as

\[
d\theta \rightarrow d\theta - \frac{\mu a(r) k(r) dr}{f(r)(f(r) - \mu a(r))}, \quad dt \rightarrow dt - \frac{\mu a(r) dr}{f(r)(f(r) - \mu a(r))}
\]
brings the seed metric to the following same form
\[ ds^2 = -(f(r) - \mu a(r)) \, dt^2 + \frac{dr^2}{f(r) - \mu a(r)} + H^2(r) \left[ d\theta - k(r) dt \right]^2. \] (A.10)

In other words, the effect of the Kerr-Schild transformation on the metric functions reads
\[ f(r) \rightarrow f(r) - \mu a(r), \quad H(r) \rightarrow H(r), \quad k(r) \rightarrow k(r). \] (A.11)

Now, it is clear that the equations that fully characterize the solution, namely eqs. (A.4), (A.5), (A.6) and (A.7), are invariant under the transformations (A.11) only for a constant function \( a \) as we have obtained in the metric solution (A.8). The same conclusion can be achieved from the variation of the action (A.1) through the Kerr-Schild changes (A.11) which yields
\[ \delta S = -2\pi \mu \int dr H'(r)a'(r) \left[ -2G_{4,X}X + G_4 \right] + B.T., \] (A.12)

and hence one can again conclude that the mass term must be constant.

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