Abstract—Boolean networks (BNs) are discrete-time systems where gene nodes are inter-connected (here we call such connection rule among nodes as network structure), and the dynamics of each gene node is determined by logical functions. In this paper, we propose a new framework on pinning control design for global stabilization of BNs based on BNs’ network structure (named as NS-based distributed pinning control). By deleting the minimum number of edges, the network structure becomes acyclic. Then, a NS-based distributed pinning control is designed to achieve global stabilization. Compared with existing literature, the design of NS-based distributed pinning control is not based on the state transition matrix of BNs. Hence, the computational complexity in this paper is reduced from $O(2^n \times 2^n)$ to $O(2^n \times 2^K)$, where $n$ is the number of nodes and $K < n$ is the largest number of in-neighbors of nodes in a BN. In addition, without using state transition matrix, global state information is no longer needed, and the design of pinning control is just based on neighbors’ local information, which is easier to be implemented. The proposed method is well demonstrated by several biological networks with different network sizes. The results are shown to be simple and concise, while the traditional pinning control can not be implemented for BNs with such a large dimension.

Index Terms—Boolean networks, Global stabilization, Distributed pinning control, Network structure, Acyclic, Semi-tensor product of matrices.

I. INTRODUCTION

In genetic regulatory networks, the behavior of genes or cells generates a logical phenomenon of activities, like active/inactive, expressed/unexpressed. As a typical logical system, Boolean network (BN) was first proposed by Kauffman to discover the internal behaviors of genes in genetic regulatory networks [1]. Since then, investigations on Boolean networks (BNs) have been a hot-spot to explore evolution patterns and structures [2], [3]. For example, BNs can be successfully used to model Drosophila segment polarity network [4], T helper cells regulatory network [5], lac operon [6] and neurotransmitter signaling pathway [7]. In a BN each node representing a cell or a gene takes a value from two logical variables 1 and 0. In addition, the state evolution for each node is determined by a logical function described by logical operators and its neighboring nodes. This implies that the dynamical behavior of a BN is determined by both network structure describing the neighbor relationship between gene nodes and also by the value update logical functions. Since then, BNs have attracted great attention such as in the study of the oscillations of signal transduction networks [8] and identifying BNs’ structure [9].

In addition, many fundamental results have been obtained to disclose the relationships between network structure of a BN and its dynamics such as the number of steady states and cyclic attractors; see [10], [11].

Recently, a new matrix product called semi-tensor product (STP) approach was proposed in [12], which brings a new direction for further study of BNs. The STP of matrices defines a new matrix product, which breaks the traditional dimension matching condition of matrix product. It was firstly proposed by Cheng and his colleagues [12] as a convenient tool to analyze logical functions, BNs and other finite-valued systems. Based on a bijective equivalence between logical variables and vectors, any logical function can be expressed by a standard algebraic form. Under the framework of algebraic forms, a great deal of results have been established in the past few decades [13]–[22]. Some fundamental problems in control fields concerning Boolean (control) networks have been widely addressed, including probabilistic BNs [23], controllability [24], [25], stability and stabilization [26]–[28], and disturbance decoupling [29], [30]. In addition, partially-observed Boolean systems have been paid much attention, which are used to model the stochasticity in both states and measurement processes [31]–[34].

In biological systems or genetic networks, it is important to design therapeutic interventions that steer patients to desirable states, such as healthy one, and maintain this state afterward [35]. For example, some kinds of control actions (called node and edge deletions control actions) have been proposed to identify control targets in some biological networks, such as p53-mdm2 network and T cell lymphocyte granular leukemia survival signaling network [36]. In [36], such node deletion can model the control action of knockout of a node, while edge deletion can model the action of a drug that inactivates the corresponding interaction among two gene products.

Over the past years, stability and stabilization of BNs have attracted much attention. For example, stabilization of Boolean control networks (BCNs) based on two kinds of controls (open-loop control and state feedback control) has been addressed in [26]. Later, the results are extended to the issue of stabilization to some periodic cycles [37]. In [27], Li et al. firstly proposed an approach to design state feedback stabilization for BNs, and further Li et al. presented a new way to design all possible feedback stabilizers in [38]. However, in these related papers, controllers are imposed to all nodes or
applied randomly to some nodes of BNs. Due to the introduction of pinning control technique [39], [40], investigations on stabilization of BCNs under pinning controllers have been a hot-spot [41]–[43]. For example, in [39], stabilization under pinning controllers has been studied, and an algorithm to design pinning controllers is firstly presented based on the state transition matrix of BNs.

Given a BN with \( n \) nodes, using the STP method, one can obtain its algebraic form: \( x(t + 1) = Lx(t) \) (matrix \( L \) is called the state transition matrix with dimension \( 2^n \times 2^n \)). According to the traditional pinning control method based on the state transition matrix \( L \) (named as \( L \)-based pinning control) proposed in [39], \( L \)-based pinning control can be designed to achieve stabilization by changing columns of matrix \( L \). This implies that the information of matrix \( L \) is needed in [39], and one cannot design pinning controllers without matrix \( L \). Recently, most of the results concerning the design of pinning controllers are based on this method; see [39]–[46]. However, the dimension of matrix \( L \) grows dramatically with the number of nodes. Thus, traditional \( L \)-based pinning control method in [39]–[46] seems to be hard for implementation on some large-dimension genetic networks, which is one of the main drawbacks. In addition, since \( L \)-based pinning control method is based on the information of matrix \( L \), the pinning controllers are determined by global state information of all the nodes. That traditional \( L \)-based pinning control design is in the form of \( u(t) = g(x_1(t), \ldots, x_n(t)) \), where \( x_1(t), \ldots, x_n(t) \) are global states of a BN. This is another drawback of traditional \( L \)-based pinning control design, which will lead to a high dimensional controller design. To the best of our knowledge, there is no result available for studying stabilization of BNs based on the network structure of BNs and using neighbors’ local information.

In our previous paper [40], the goal was to study the controllability problem of BCNs where some specific nodes are selected to be controlled. However, [40] does not present the method on how to determine the pinning controlled nodes and design the corresponding controllers. In addition, the obtained controllability criteria in [40] are based on the state transition matrix \( L \), and this would lead to an extremely high computational complexity and a very complex form of controllers. Moreover, in [40], the controllers are free control sequences which can be arbitrary values, but not state feedback controllers. In this paper, a new framework to design state feedback pinning control is proposed based on network structure of BNs without using the traditional state transition matrix \( L \). In some real-world systems, it is quite difficult to obtain system’s global information, but possible to get information from its local neighbors. Thus, a more efficient and lower dimension controller strategy needs to be proposed based on local neighbors of controlled nodes.

Motivated by the above discussions, this paper makes the following fundamental contributions:

(I) In this paper, a new concept named NS-based distributed pinning control design is firstly proposed to achieve global stabilization of BNs. The design of NS-based distributed pinning control is based on the network structure of BNs describing the coupling connections among nodes, and the neighbors’ local information. Compared with existing references on traditional \( L \)-based pinning control [39]–[40], the information of state transition matrix \( L \) is no longer needed in this paper;

(II) The computational complexity will be dramatically reduced from \( O(2^n \times 2^n) \) to \( O(2 \times 2^k) \), where \( K \) is the largest number of in-neighbors of nodes. The method of designing NS-based distributed pinning control in this paper can be implemented for some large-dimension BNs, which is well illustrated in several biological networks with different network sizes like 90 nodes;

(III) By deleting the minimum number of edges such that the network structure is acyclic, global stability will be guaranteed. Then, a NS-based distributed pinning control can be designed based on the neighbors’ local information but not on global state information. This NS-based distributed pinning method leads to a lower dimensional controller design and easier for implementation than those traditional methods of designing controllers;

(IV) In addition, if a system needs to achieve global stabilization with respect to (w.r.t) a prescribed steady state, a second step is presented to find the minimum number of controlled nodes. The whole process can be rewritten as an integer linear programming problem.

The remainder of this paper is presented as follows: In Section II some preliminaries about the STP of matrices and digraph are introduced. In Section III BNs, problem formulation and global stability are introduced. In Section IV the main results on NS-based distributed pinning control design for global stabilization are presented. In Section V two T-LGL survival signaling biological networks, respectively, with 6 and 29 nodes, are presented to demonstrate the validity of the results. Section VI is given to well illustrate some comparisons and the main contributions of this paper, a brief conclusion is given in Section VII.

Some basic notations are given as follows:

- \( \mathbb{I}_n = (1,1,\ldots,1) \); 
- \( \mathbb{D} = \{0,1\} \): the logical domain; 
- \( \mathbb{N}^+ \): the set of positive integers; 
- \( \mathbb{R}_{m \times n} \): the set of \( m \times n \) real matrices; 
- \( |S| \) is the cardinal number of a given set \( S \); 
- \( [a,b] = \{a,a+1,\ldots,b\} \), where \( a < b \) and \( a,b \in \mathbb{N}^+ \); 
- \( \Delta_n = \{\delta_n^1,\ldots,\delta_n^n\} \) denotes the columns’ set of \( I_n \); 
- \( \text{Col}(I)(A) \): the \( i \)-th column of matrix \( A \). \( \text{Col}(A) \) denotes the set of columns; 
- \( L = [\delta_1^1,\ldots,\delta_1^n] \) is called a logical matrix, and simply denote it by \( L = [\delta_1,\ldots,i_1] \); 
- \( \mathcal{L}_{m \times n} \): the set of logical matrices with dimension \( m \times n \); 
- \( [a]_S \) be the set of positive integers, \( [a]_S = \{j: 0 < j < a, j \in S\} \), where \( S \) is a set of positive integers; 
- Given numbers \( x_1,\ldots,x_n \) and an index set \( I = \{i_1,\ldots,i_m\} \subseteq [1,n] \), define \( [x]_I = [x_i]_{i \in I} \); 
- Swap matrices \( W_{m,n} = [I_m \otimes \delta_1^1, I_m \otimes \delta_2^1,\ldots, I_m \otimes \delta_n^1] \); 
- Power-reducing matrix \( \Phi_{2^n} = [I_2 \otimes 1, 2^n + 2, (3 - 1)2^n + 3, (4 - 1)2^n + 4,\ldots,(2^n - 2)2^n + 2^n - 1, 2^{2n}] \).
II. Some Preliminaries

A. Semi-tensor product (STP) of matrices

STP of matrices is a generalization of conventional matrix product, which deals with the case that the dimension-matching condition of matrix product is not satisfied \[ 12 \].

**Definition 2.1:** \[ 12 \] Given two matrices \( A \in \mathcal{R}_{n \times m} \) and \( B \in \mathcal{R}_{p \times q} \), the STP of \( A \) and \( B \), denoted by \( A \times B \), is defined as:

\[ A \times B = (A \otimes I_{m/p})(B \otimes I_{p/l}), \tag{1} \]

where \( l \) is the least common multiple of \( m \) and \( p \), \( \otimes \) is the Kronecker product of matrices.

In order to facilitate use of the STP method in BNs, identify “1” and “0” with vectors, \( 1 \sim \delta_{1}^{1}, 0 \sim \delta_{2}^{2} \), respectively. Under this framework, the equivalent algebraic form for any logical function can be obtained.

**Proposition 2.1:** \[ 12 \] Let \( f(a_{1}, \ldots, a_{n}) : (\mathcal{D})^{n} \rightarrow \mathcal{D} \) be a logical function. Then for every \( (a_{1}, \ldots, a_{n}) \in (\Delta_{2})^{n} \), there exists a unique matrix \( F \in \mathcal{L}_{2 \times 2^{n}} \) such that

\[ f(a_{1}, \ldots, a_{n}) = F \times a_{1} \times \cdots \times a_{n}. \tag{2} \]

Here, \( F \) is called the structure matrix of logical function \( f \).

Then, some basic structure matrices of logical operators are obtained, such as negation “\( \neg \)”, conjunction “\( \land \)”, disjunction “\( \lor \)”, conditional “\( \Rightarrow \)”, and bi-conditional “\( \Leftrightarrow \)”, whose structure matrices are denoted by \( M_{\neg} = \delta_{2}[1, 2], M_{\land} = \delta_{2}[1, 2, 2], M_{\lor} = \delta_{2}[1, 1, 2], M_{\Rightarrow} = \delta_{2}[1, 2, 1] \) and \( M_{\Leftrightarrow} = \delta_{2}[1, 2, 2] \).

Some detailed introductions for STP method and algebraic representations of logical functions can be found in \[ 12, 40 \].

B. Digraphs

Let \( G = (V, E) \) be a digraph, which consists of a non-empty finite set \( V \) of elements called vertices and a finite set of ordered pairs \( E \) called edges. Here, we call \( V \) the vertex set and \( E \) the edge set of digraph \( G \). Let the ordered pair \( v_i \rightarrow v_j \) denote an edge from \( v_i \in V \) to \( v_j \in V \) in \( G \). In addition, we call \( v_i \) an in-vertex of \( v_j \) and \( v_j \) an out-vertex of \( v_i \), respectively. Let \( \mathcal{N}_{in}(v_i) \) and \( \mathcal{N}_{out}(v_i) \) denote the sets of in-neighbors and out-neighbors of vertex \( v_i \), respectively.

Given two vertices of digraph \( G \), \( v_i \) and \( v_j \). Let \( w_{ij} \) denote a walk from \( v_i \) to \( v_j \), that is a sequence \( v_{i0} \rightarrow v_{i1} \rightarrow \cdots \rightarrow v_{in} \) (with \( v_{i0} = v_i \) and \( v_{in} = v_j \)) in which \( v_{ik} \rightarrow v_{ik+1} \) is an edge of \( E \) for all \( k \in [0, m-1] \). If all the vertices in the walk are pairwise distinct, a walk is also called a path. A closed walk is a walk \( w_{ij} \) when the starting vertex and the ending vertex are the same, i.e. \( v_i = v_j \). In addition, if there is no repetition of vertices in the walk other than the starting vertex and the ending vertex, a walk is called a cycle.

III. Global Stability of Boolean Networks

A Boolean function \( f : \mathcal{D}^{n} \rightarrow \mathcal{D} \) consisting of \( n \) Boolean variables \( x_1, \ldots, x_n \in \mathcal{D} \) is a discrete-time finite state dynamical system. The value update rule associated with a BN can be described by a set of Boolean functions \( f_1, \ldots, f_n : x(t+1) = f_i(x_1(t), \ldots, x_n(t)), i \in [1, n] \). Let \( x(t) = (x_1(t), \ldots, x_n(t)) \in \mathcal{D}^n \) be the state at time \( t \).

Note that using Boolean algebra, some variables may be nonfunctional in a logical function. For example, if \( f(x_1, x_2) = (x_1 \land x_2) \lor (x_1 \land \neg x_2) \), then function \( f \) is not dependent on variable \( x_2 \), because \( (x_1 \land x_2) \lor (x_1 \land \neg x_2) = x_1 \); while if \( f(x_1, x_2) = x_1 \lor x_2 \), then function \( f \) is dependent on variables \( x_1 \) and \( x_2 \). Then, we introduce the dependency of logical functions on variables.

**Definition 3.1:** A logical function \( f(x_1, \ldots, x_i, \ldots, x_n) : \mathcal{D}^n \rightarrow \mathcal{D} \) is said to be dependent on variable \( x_i \) if there exists a tuple \( \bar{x} \in \mathcal{D}^{n-1} \) such that \( f(\bar{x}, x_i) \neq f(\bar{x}, \neg x_i) \), where \( \bar{x} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \). Here, variable \( x_i \) is called a functional variable in function \( f(\ldots, x_i, \ldots) \), otherwise, it is called a nonfunctional variable. In addition, if there is no nonfunctional variables, then it is called minimally represented.

Based on the value update rules \( f_1, \ldots, f_n \), we can introduce the interaction digraph describing the coupling connections among nodes of a BN which here is called network structure.

**Definition 3.2:** Let \( f = (f_1, \ldots, f_n) \) be the value update rules associated with a BN. The interaction digraph of a BN is a digraph denoted by \( G_{id} = (V, E) \) of \( n \) vertices, the vertices set is \( V = \{1, \ldots, n\} \). An edge \( i \rightarrow j \) (simplified by \( e_{ij} \in E \)) exists in \( G_{id} = (V, E) \) if and only if \( f_j \) is dependent on \( x_i \).

Since a Boolean function \( f : \mathcal{D}^n \rightarrow \mathcal{D} \) can be minimally represented, in this paper the following minimally represented BN is considered:

\[ x_i(t+1) = f_i([x_j(t)])_{j \in k}, \quad i \in [1, n]. \tag{3} \]

Here, \( x_i \in \mathcal{D}, i \in [1, n] \), are the \( i \)-th node of system \( \{ \mathcal{N}_k, i \in [1, n] \} \), are the index sets of in-neighbors of node \( x_i \) expressing the adjacency relation of nodes, \( f_i : \mathcal{D}^{\mathcal{N}_k} \rightarrow \mathcal{D} \), \( i \in [1, n] \), are Boolean functions. For example, given a function \( f(x_1, x_2) = x_1 \lor x_2, \mathcal{N}_2 = \{1, 2\} \). Let \( x(t) = (x_1(t), \ldots, x_n(t)) \in \mathcal{D}^n \) be the state of system \( \{ \mathcal{N}_k \} \) at time step \( t \).

Then, the definitions for steady states and cyclic attractors of system \( \{ \mathcal{N}_k \} \) are introduced as follows:

**Definition 3.3:** The state \( x(t) \in \mathcal{D}^n \) is called a steady state at time \( x(t+1) = x(t), t = 0, 1, 2, \ldots \) The sequence of states \( \{x(t), x(t+1), \ldots, x(t+l-1)\} \) is called a cyclic attractor with length \( l \) if \( x(t+l) = x(t), x(t+i) \neq x(t), i \in [1, l-1] \).

According to Proposition 2.1, suppose that the structure matrices of \( f_i \) are \( A_i \in \mathcal{L}_{2 \times 2^{\mathcal{N}_i}}, i \in [1, n] \), then one can obtain the following algebraic form:

\[ x_i(t+1) = A_i \times x_i(t), \quad i \in [1, n]. \tag{4} \]

For example, given for the general function \( x(t+1) := f(x_1(t), x_2(t)) = x_1(t) \lor x_2(t) \), one has its algebraic form:

\[ x(t+1) := A \times x_1(t) \times x_2(t) = \delta_{2}[1, 1, 1, 2] \times x_1(t) \times x_2(t). \]

**Remark 3.1:** Suppose that for the set of in-neighbors of node \( x_i, \mathcal{N}_i = \{b_{j1}', b_{j2}', \ldots, b_{j5}', \ldots, b_{j5}', \ldots, b_{j5}', \ldots, n \} \). For the indexes \( j_i \in [1, n], j \in [1, |\mathcal{N}_i|] \), we assume that \( 1 \leq b_{j1}' < b_{j2}' < \cdots < b_{j5}' \leq n \). For simplicity, we always assume that \( \mathcal{N}_{j_i} \subseteq \mathcal{N}_i \). For example, let \( \mathcal{N}_1 = \{3, 5, 6\}, \) under the ordering assumption, \( \mathcal{N}_{j_i} \subseteq \mathcal{N}_i \), one can obtain the equivalent representation:

\[ x(t+1) = Lx(t), \tag{5} \]
where $L \in \mathcal{L}_{2^n \times 2^n}$ is called the state transition matrix of system $[5]$. The detailed calculations for the state transition matrix $L$ can be obtained by using the STP method $[12, 40]$. For example, consider a BN with 2 nodes below:

$x_1(t+1) = x_3(t), x_2(t+1) = \neg x_1(t)$, one has $x_1(t+1) = x_2(t), x_2(t+1) = \delta_2[2,1]x_1(t)$, which leads to $x(t+1) = x_1(t+1)x_2(t+1) = x_3(t)\delta_2[2,1]x_1(t) = (I_2 \otimes \delta_2[2,1])x_1(t) = (I_2 \otimes \delta_2[2,1])W_{2,2}x_1(t)x_2(t) := Lx(t) = \delta_2[2,4,1,3]x(t)$.

Remark 3.2: Note that the dimension of matrix $L \in \mathcal{L}_{2^n \times 2^n}$ grows exponentially with the size of network (depending on the number of nodes). Thus, all the algorithms based on matrix $L$ have the exponential complexity, as one is required to deal with matrix of size $(2^n \times 2^n)$. This is a noteworthy drawback with regard to computational complexity and has already attracted much attention $[47]$. As one can see from the below discussions and analysis, in this paper, the design of NS-based distributed pinning controllers is not based on matrix $L$ but on matrices $A_i \in \mathcal{L}_{2^n \times 2^n}$ in system $[4]$. Thus, the approach proposed in this paper breaks the restriction of dimension and the number of nodes, and also reduces calculations.

Based on algebraic form $[5]$, some results about global stability of BNs are presented below.

Definition 3.4: $[12, 48]$ System (3) is said to be globally stable if there exists a unique steady state as the attractor with no other cyclic attractors. Especially, system $[3]$ is said to be globally stable w.r.t. state $x^* \in \Delta_{2^n} \sim X^* \in \mathcal{P}^n$, if for any initial state $x_0 \in \Delta_{2^n} \sim X_0 \in \mathcal{P}^n$, there exists an integer $T$, such that $x(t,x_0) = x^*, t \geq T$.

Theorem 3.1: $[48]$ System (3) is globally stable if the interaction digraph $G_{id} = (V, E)$ is acyclic.

Theorem 3.2: $[12]$ Assume that stable state is $x^* = \delta_2^\gamma, \gamma \in [1,2^n]$. Then, system (3) is globally stable w.r.t. $x^* = \delta_2^\gamma$ if and only if there exists an integer $T$, such that $\text{Col}(L^T) = \{ \delta_2^\gamma \}$.

Define the set of initial states that can reach state $\delta_2^\gamma$ at the $k$-th step as follows: $R_k(\delta_2^\gamma) = \{ x(0) \in \Delta_{2^n} : x(k, x(0)) = \delta_2^\gamma \}$. Then, let $R(\delta_2^\gamma) = \bigcup_{k=1}^{\infty} R_k(\delta_2^\gamma)$, which is the set of initial states that can reach state $\delta_2^\gamma$.

Theorem 3.3: $[39]$ System (3) is globally stable w.r.t. $x^* = \delta_2^\gamma$ if and only if (1) $\text{Col}_V(L) = \delta_2^\gamma$; and (2) there exists an integer $T$, such that $R_T(\delta_2^\gamma) = \Delta_{2^n}$.

Example 3.1: Consider a BN with 5 nodes as follows:

$$
\begin{align*}
&x_1(t+1) = \neg x_3(t), &x_4(t+1) = x_1(t) \land x_2(t), \\
&x_2(t+1) = x_1(t), &x_5(t+1) = 1, \\
&x_3(t+1) = x_2(t) \lor x_4(t), &x_5(t+1) = 1,
\end{align*}
$$

where $x_1, x_2, x_3, x_4, x_5 \in \mathcal{P}$.

Then, one can obtain its interaction digraph $G_{id} = (V, E)$ as shown in Fig. 1. According to Theorem 3.1 system $[6]$ is globally stable since the interaction digraph $G_{id} = (V, E)$ is acyclic. Using STP method and denoting $x(t) = \chi_{[1,5]} x_j(t)$, one can obtain the algebraic form: $x(t+1) = Lx(t)$, where $L = \delta_2[17, 1, 17, 1, 7, 19, 7, 19, 7, 19, 7, 19, 7, 27, 11, 27, 11, 27, 11, 27, 11, 27, 15, 27, 15, 27, 15, 27, 15, 27, 15]$.

According to Theorem 3.2 one can also obtain that system $[6]$ is globally stable w.r.t. state $\delta_2^v$, such that $\text{Col}(L^3) = \{ \delta_2^v \}$. Fig. 2 shows the state transition graph of system $[6]$. However, compared with the interaction digraph $G_{id} = (V, E)$, the state transition graph with 32 nodes is more complex, and its size will grow exponentially with the number of nodes.

Remark 3.3: It should be noted that if pinning control design is based on the interaction digraph, the computational complexity will be dramatically reduced. Fortunately, Theorem 3.3 presents a sufficient condition for global stability, which will be used to design pinning controllers in the following section. The main idea is to delete some specific edges such that the interaction digraph $G_{id} = (V, E)$ becomes acyclic.

IV. NS-BASED DISTRIBUTED PINNING CONTROL FOR GLOBAL STABILIZATION

Now, in this section, a new approach to design a NS-based distributed pinning control for global stabilization is proposed, by considering whether the interaction digraph $G_{id} = (V, E)$ is acyclic. Two main steps are introduced to design NS-based distributed pinning control to achieve global stabilization:

(1) delete the minimum number of edges in the interaction digraph $G_{id} = (V, E)$, such that $G_{id} = (V, E)$ becomes acyclic;

(2) transform the structure matrices $A_1, \ldots, A_n$, such that state $\delta_2^v$ will be the unique steady state. It should be noted that the control action by deleting edges of interaction digraph $G_{id} = (V, E)$ is plausible in genetic networks. As studied in $[36]$, two types of control actions (one is deletion of edges and another one is deletion of nodes) have been used in studying Boolean molecular networks. For example, the deletion of an edge can be achieved by the use of therapeutic drugs to control some specific gene interactions. In the following subsections, we will use these two steps
to design a NS-based distributed pinning control to achieve global stabilization w.r.t. any given steady state based on the interaction digraph \( G_{id} = (V,E) \).

A. Step 1: delete edges to lead interaction digraph \( G_{id} = (V,E) \) to be acyclic

Now, we consider the first main step, that is to delete the minimum number of edges in the interaction digraph \( G_{id} = (V,E) \), such that the reduced digraph is acyclic. By using the depth-first search algorithm\(^1\), one can obtain the cycles and fixed points in the digraph \( G_{id} = (V,E) \). Firstly, we present some notations. Given an edge \( e \in E \) in \( G_{id} = (V,E) \), let \( O_-(e) \) be the starting vertex and \( O_+(e) \) be the ending vertex of the edge \( e \in E \). For example, consider an edge from node 1 to node 2 (\( e = 1 \rightarrow 2 \)) in the digraph \( G_{id} = (V,E) \) shown in Fig. [1] then one has \( O_-(e) = \{1\}, O_+(e) = \{2\} \). Then, we consider the following problem of deleting the minimum number of edges such that the interaction digraph \( G_{id} = (V,E) \) becomes acyclic. Firstly, the concept of (minimum) feedback arc set for a given digraph is introduced.

Definition 4.1: A feedback arc set is defined as a subset of edges containing at least one edge of every cycle in a feedback arc set. For example the cardinality of the minimal feedback arc set satisfying (8) in practice.

Problem 1: Assume that \( \Omega_1 = \{v_{11}^1, \ldots, v_{1\kappa}^1\}, \ldots, \Omega_\kappa = \{v_{11}^\kappa, \ldots, v_{1\kappa}^\kappa\} \) are the minimal feedback arc sets with cardinality \( c_2 \in [1,|E|] \), which are obtained in the interaction digraph \( G_{id} = (V,E) \). Here, \( v_{11}^1, \ldots, v_{1\kappa}^1, \ldots, v_{11}^\kappa, \ldots, v_{1\kappa}^\kappa \in E, \kappa \in \mathbb{N}^+ \). Then, minimize the cost function

\[
    c_1 = \min \{ |\pi_j| : j = 1, \ldots, \kappa \}, \tag{7}
\]

subject to conditions

\[
    \pi_j \triangleq \bigcup_{i=1}^{c_2} O_+(v_i^j), j \in [1,\kappa]. \tag{8}
\]

Remark 4.2: As one can see from Problem 1, (7) and (8) are presented to find the minimum number of ending vertices among the edges in the minimal feedback arc sets, such that the digraph \( G_{id} = (V,E) \) becomes acyclic. During the past few years, a great number of researches have been focused on finding the relationship between steady states and the minimum feedback arc set in interaction digraph of BNs [11]. In general, many real-world genetic regulatory networks are sparse. Thus it will be feasible to find a minimum feedback arc set satisfying (8) in practice.

Assume that \( \Omega_\zeta = \{v_{11}^\zeta, \ldots, v_{1\kappa}^\zeta\} = \{\Omega_1, \ldots, \Omega_\kappa\} \) is one feasible minimal feedback arc set with cardinality \( c_2 \), which satisfies conditions (7) and (8). Then, we further assume that

\[
    \bigcup_{i=1}^{c_2} O_+(v_i^j) \triangleq \{\omega_1, \ldots, \omega_\kappa\}, \quad \omega_1, \ldots, \omega_\kappa \in V. \tag{9}
\]

Thus, under these assumptions, nodes \( x_i, i \in \{\omega_1, \ldots, \omega_\kappa\} \), should be controlled after considering Problem 1. This implies that vertices \( \omega_1, \ldots, \omega_\kappa \) are the corresponding ending vertices for the edges \( v_{11}^j, \ldots, v_{1\kappa}^j \subset E \). Here, we assume that the edges \( v_{11}^j, \ldots, v_{1\kappa}^j \subset E \) share a same ending vertex \( \omega_1 \), i.e. \( \bigcup_{j=1}^{c_2} O_+(v_i^j) \triangleq \{\omega_1\} \). The edges \( \ldots, v_{11}^j, \ldots, v_{1\kappa}^j \subset E \) share a same ending vertex \( \omega_\kappa \), i.e. \( \bigcup_{j=1}^{c_2} O_+(v_i^j) \triangleq \{\omega_\kappa\} \). In addition, \( \bigcup_{j=1}^{c_2} O_+(v_i^j) \triangleq \{\theta_1^j, \ldots, \theta_\kappa^j\} \). Then, if \( \bigcup_{j=1}^{c_2} O_+(v_i^j) \triangleq \{\theta_1^j, \ldots, \theta_\kappa^j\} \) and \( \theta_1^j < \ldots < \theta_\kappa^j \), \( \theta_1^j < \ldots < \theta_\kappa^j \).

After considering Problem 1 based on the minimal feedback arc set, the following problem is further considered to find feasible structure matrices for designing NS-based distributed pinning control.

Problem 2: Consider the nodes \( x_i, i \in \{\omega_1, \ldots, \omega_\kappa\} \) that should be controlled, find matrices \( \hat{A}_{\omega_1} \in \mathcal{L}_{2\times 2^{|\omega_1|}}, \ldots, \hat{A}_{\omega_\kappa} \in \mathcal{L}_{2\times 2^{|\omega_\kappa|}}, \ldots, \hat{A}_{\omega_1} \in \mathcal{L}_{2\times 2^{|\omega_1|}}, \ldots, \hat{A}_{\omega_\kappa} \in \mathcal{L}_{2\times 2^{|\omega_\kappa|}} \), such that

\[
    \left\{ \begin{array}{l}
    \hat{A}_{\omega_1} = \hat{A}_{\omega_1}(I_{2^{|\omega_1|}} \otimes I_{2^{|\omega_1|}}), \\
    \hat{A}_{\omega_\kappa} = \hat{A}_{\omega_\kappa}(I_{2^{|\omega_\kappa|}} \otimes I_{2^{|\omega_\kappa|}}), \\
    \ldots
    \end{array} \right. \tag{10}
\]

Here, we assume that \( \mathcal{N}_{\omega_1} \triangleq \mathcal{N}_{\omega_1}(\theta_1^j, \ldots, \theta_\kappa^j) \), \( \mathcal{N}_{\omega_\kappa} \triangleq \mathcal{N}_{\omega_\kappa}(\theta_1^j, \ldots, \theta_\kappa^j) \), and so on. Using the property of swap matrix \( W_{m,n} \), that is \( W_{m,n} \sigma_1 \otimes \sigma_2 = \sigma_2 \otimes \sigma_1 \), one has the following lemma.

Lemma 4.1: For \( x_{\omega_1}(t+1) = A_{\omega_1} x_{\omega_1}(t) \in [1,|\omega_1|], \) one can swap the positions of neighbors in the following form:

\[
    x_{\omega_1}(t+1) = A_{\omega_1} W_{\mathcal{N}_{\omega_1}} x_{\omega_1}(t) \times x_{\omega_\kappa}(t), \quad j \in [1,|\omega_1|], \tag{11}
\]

where \( W_{\mathcal{N}_{\omega_1}} \triangleq \bigotimes_{i=1}^{c_2} \bigotimes_{i=1}^{c_2} \mathcal{N}_{\omega_1}(I_{2^{|\omega_1|}} \otimes I_{2^{|\omega_1|}}) \), and \( A_{\omega_\kappa} \triangleq A_{\omega_1} W_{\mathcal{N}_{\omega_1}} W_{\mathcal{N}_{\omega_\kappa}} \).

Then, based on the above Problems 1 and 2, one can design a NS-based distributed pinning control for nodes \( x_j, j \in \{\omega_1, \ldots, \omega_\kappa\} \), as follows:

\[
    \left\{ \begin{array}{l}
    x_j(t+1) = u_j(t) \oplus f_j(x_j(t)), \quad j \in \{\omega_1, \ldots, \omega_\kappa\}, \\\n    u_j(t) = g_j(x_j(t)), \quad j \in \{\omega_1, \ldots, \omega_\kappa\}, \\\n    x_i(t+1) = f_i(x_i(t)), \quad i \in [1,|\omega_1| \setminus \{\omega_1, \ldots, \omega_\kappa\}], \tag{12}
    \end{array} \right.
\]

\(^1\)Depth-first search algorithm is a fundamental recursive algorithm following the edges of a digraph to find vertices connected to the source [49].
Here, \( u_j \in \mathcal{D}, j \in \{\omega_1, \ldots, \omega_k\} \), are NS-based distributed pinning controllers for nodes \( x_j, j \in \{\omega_1, \ldots, \omega_k\} \), binary logical functions to be determined in the following sequel. Functions \( g_j : \mathcal{P}_j \rightarrow \mathcal{D}, j \in \{\omega_1, \ldots, \omega_k\} \), are Boolean functions of controllers \( u_j \) depending on the corresponding neighbors of nodes \( x_j (j \in \{\omega_1, \ldots, \omega_k\}) \), which will be determined later.

Since any logical function can be equivalently expressed by corresponding structure matrix, let \( M_{\omega_j} \in \mathbb{Z}^{2 \times 4} \) and \( K_j \in \mathbb{L}_{2 \times 2}^{x_1/n_j} \). For nodes \( x_j, j \in \{\omega_1, \ldots, \omega_k\} \), using STP, Eq. (12) can be expressed in the following algebraic forms,

\[
\begin{align*}
x_j(t+1) &= M_{\omega_j} u_j(t) A_j \forall i \in \{\omega_1, \ldots, \omega_k\} x_i(t) \forall i \in \{\omega_1, \ldots, \omega_k\} x_j(t), \\
&= M_{\omega_j} K_{j}(I_{x_1/n_j} \circ \tilde{A}_j) \Phi_{2 \times 2} \forall i \in \{\omega_1, \ldots, \omega_k\} x_i(t), \\
&= K_j \forall i \in \{\omega_1, \ldots, \omega_k\} x_i(t) \forall i \in \{\omega_1, \ldots, \omega_k\} x_j(t).
\end{align*}
\]  

(13)

According to Problems 1 and 2, if \( j \in \{\omega_1, \ldots, \omega_k\} \), system \( x_j(t+1) = M_{\omega_j} K_j(I_{x_1/n_j} \circ \tilde{A}_j) \Phi_{2 \times 2} \forall i \in \{\omega_1, \ldots, \omega_k\} x_i(t) \forall i \in \{\omega_1, \ldots, \omega_k\} x_j(t), \) becomes \( x_j(t+1) = A_j \forall i \in \{\omega_1, \ldots, \omega_k\} x_i(t) \forall i \in \{\omega_1, \ldots, \omega_k\} x_j(t) \), then one can obtain \( x_j(t+1) = A_j \forall i \in \{\omega_1, \ldots, \omega_k\} x_i(t) \). Thus, according to \( x_j(t+1) = A_j \forall i \in \{\omega_1, \ldots, \omega_k\} x_i(t) \), in the interaction digraph \( G_{id} = (V, E) \), the edges from nodes \( v_{1}, \ldots, v_{1} \) to node \( x_{0}, \ldots, \), and the edges from the nodes \( v_{1}', \ldots, v_{x_{1}} \) to node \( x_{0}, \ldots, \) will be deleted.

In order to design NS-based distributed pinning controllers, one needs to solve matrices \( M_{\omega_j} \in \mathbb{Z}^{2 \times 4} \) and \( K_j \in \mathbb{L}_{2 \times 2}^{x_1/n_j} \), \( j \in \{\omega_1, \ldots, \omega_k\} \) from the following equations,

\[
\begin{align*}
\tilde{A}_{\omega_1} &= M_{\omega_1} K_{\omega_1}(I_{x_1/n_1} \circ \tilde{A}_1) \Phi_{2 \times 2} \forall i \in \{\omega_1, \ldots, \omega_k\} x_i(t), \\
&= \cdots \\
\tilde{A}_{\omega_k} &= M_{\omega_k} K_{\omega_k}(I_{x_1/n_k} \circ \tilde{A}_k) \Phi_{2 \times 2} \forall i \in \{\omega_1, \ldots, \omega_k\} x_i(t).
\end{align*}
\]  

(14)

Fortunately, Li has proved [44] that Eq. (14) is solvable. Lemma 4.2: [44] Eq. (14) is solvable.

Theorem 4.1: By solving matrices \( M_{\omega_j} \in \mathbb{Z}^{2 \times 4} \) and \( K_j \in \mathbb{L}_{2 \times 2}^{x_1/n_j} \), \( j \in \{\omega_1, \ldots, \omega_k\} \) from Eq. (14), BN under a NS-based distributed pinning control in the form of Eq. (12) will be globally stabilized.

Proof. If BN (3) is under the NS-based distributed pinning control in the form of (12), then one can obtain its algebraic form for the controlled nodes \( x_j, j \in \{\omega_1, \ldots, \omega_k\} \),

\[
x_j(t+1) = M_{\omega_j} u_j(t) A_j \forall i \in \{\omega_1, \ldots, \omega_k\} x_i(t) \forall i \in \{\omega_1, \ldots, \omega_k\} x_j(t),
\]

(15)

This implies that in the interaction digraph \( G_{id} = (V, E) \), the NS-based distributed pinning controllers (12) will delete the edges from nodes \( \theta_{1}^{1}, \ldots, \theta_{1}^{n} \) to node \( x_{0}^{1}, \ldots, \), and the edges from nodes \( \theta_{1}^{1}, \ldots, \theta_{1}^{n} \) to node \( x_{0}^{1} \). According to Problem 1, these deleted edges will lead the interaction digraph \( G_{id} = (V, E) \) to be acyclic. Then, based on Theorem 3.1, system (12) will be globally stabilized, which completes the proof.

After considering Problem 1, under the NS-based distributed pinning control design in the form of (12), with corresponding algebraic form (13), one can obtain the reduced system of (14):

\[
\begin{align*}
x_j(t+1) &= A_j \forall i \in \{\omega_1, \ldots, \omega_k\} x_i(t), j \in \{\omega_1, \ldots, \omega_k\}, \\
x_j(t+1) &= A_j \forall i \in \{\omega_1, \ldots, \omega_k\} x_i(t), j \in [1, n]\setminus\{\omega_1, \ldots, \omega_k\}.
\end{align*}
\]  

(15)

As can be seen from Eq. (15), for nodes \( x_j, j \in \{\omega_1, \ldots, \omega_k\} \), we have imposed state feedback pinning controllers, while for nodes \( x_i, i \in [1, n]\setminus\{\omega_1, \ldots, \omega_k\} \), there is no controller imposed. Thus, the structure matrices for nodes \( x_i, i \in [1, n]\setminus\{\omega_1, \ldots, \omega_k\} \), remain unchanged. However, in order to avoid confusion, here we still denote \( \tilde{A}_i = A_i \) and \( \tilde{N}_i = N_i \), \( i \in [1, n]\setminus\{\omega_1, \ldots, \omega_k\} \), for convenience of the following step.

B. Step 2: further design controllers to achieve global stabilization w.r.t. state \( \delta_{x_j}^{1} \)

Based on Step 1, one needs to design a state feedback pinning control to guarantee that system (15) will achieve global stabilization w.r.t. a given steady state \( \delta_{x_j}^{1} \). In order to further achieve global stabilization w.r.t. the given steady state \( \delta_{x_j}^{1} \), the following problem will be considered to design a state feedback pinning control.

Problem 3: Let \( \delta_{x_j}^{2} = \delta_{x_j}^{1} \). Find matrices \( \tilde{A}_1 \in \mathbb{L}_{2 \times 2}^{x_1/n_1}, \ldots, \tilde{A}_n \in \mathbb{L}_{2 \times 2}^{x_1/n_n} \) and binary variables \( \delta_1, \delta_n \in \mathcal{D} \), minimize the cost function

\[
\begin{align*}
e_3 &= \sum_{i=1}^{n} \delta_i
\end{align*}
\]  

(16)

subject to the following conditions:

\[
\begin{align*}
\delta_1^{1} &= [(1-\delta_1)\tilde{A}_1 + \delta_1\tilde{A}_1] \times \delta_{x_j}^{2}, \\
&\cdots \\
\delta_n^{2} &= [(1-\delta_n)\tilde{A}_n + \delta_n\tilde{A}_n] \times \delta_{x_j}^{2}.
\end{align*}
\]  

(17)

Problem 3 can be rewritten as an integer linear programming (ILP) problem, and can be solved by a suitable solver like Yices SMT Solver [52]. As one can see from (17), if \( \delta_1 = 1, \delta_1^{1} = [(1-\delta_1)\tilde{A}_1 + \delta_1\tilde{A}_1] \times \delta_{x_j}^{2} \) reduces to \( \delta_1^{2} = \tilde{A}_1 \times \delta_{x_j}^{2} \), this implies that node \( x_1 \) should be controlled and its structure matrix \( \tilde{A}_1 \) should be changed to another structure matrix \( \tilde{A}_1 \). Thus, \( \sum_{i=1}^{n} \delta_i \) represents the number of nodes requiring to be further controlled, and by minimizing the cost function \( e_3 = \sum_{i=1}^{n} \delta_i \), one can further determine the minimum number of controlled nodes.

Here, an example is used to illustrate Problem 1 as an ILP problem.

Example 4.1: Consider a BN with 3 nodes: \( x_1(t+1) = x_1(t) \times x_3(t), x_2(t+1) = x_3(t) \) and \( x_3(t+1) = 1, \) where
Using STP method, one can firstly obtain the following algebraic form:

\[
\left\{
\begin{array}{l}
    x_1(t+1) = \hat{A}_1 x_1(t)x_3(t), \\
    x_2(t+1) = \hat{A}_2 x_3(t), \\
    x_3(t+1) = \hat{A}_3 = \delta_2^f.
\end{array}
\right.
\]  \hspace{1cm} (18)

Here, \(\hat{A}_1 = \delta_1[1, 1, 1, 2], \hat{A}_2 = \delta_1[1, 2]\). Let \(x(t) = [x_j(t)]\), one has that \(x(t+1) = Lx(t)\), where \(L = \delta_1[1, 3, 1, 3, 1, 7, 1, 7]\).

According to Theorem 3, system (18) is globally stable, and one can further conclude that it is globally stable w.r.t. state \(\delta_1^f\). Now, we consider the global stabilization of system (18) w.r.t. state \(\delta_1\). Thus, a state feedback pinning control is designed to achieve global stabilization w.r.t. state \(\delta_1^f\) by considering Step 2. Here, we consider Problem 3 and the constraint conditions (19) to find matrices \(\hat{A}_1 \in \mathcal{L}_{2 \times 4}, \hat{A}_2 \in \mathcal{L}_{2 \times 2}, \hat{A}_3 \in \mathcal{L}_{2 \times 1}\) and binary variables \(\delta_1, \delta_2, \delta_3 \in \mathcal{D}\) such that the number of controllers is minimum.

The constraint conditions (19) are given as follows:

\[
\delta_2^f = \begin{bmatrix} 1 - \delta_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} \delta_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 1 - a_{11} & 1 - a_{12} & 1 - a_{13} & 1 - a_{14} \end{bmatrix} \begin{bmatrix} \delta_1 \end{bmatrix}^T, \\
\delta_2^f = \begin{bmatrix} 1 - \delta_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} + \begin{bmatrix} \delta_2 \end{bmatrix} \begin{bmatrix} a_{21} & a_{22} \\ 1 - a_{21} & 1 - a_{22} \end{bmatrix} \begin{bmatrix} \delta_1 \end{bmatrix}^T, \\
\delta_2^f = \begin{bmatrix} 1 - \delta_3 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} + \begin{bmatrix} \delta_3 \end{bmatrix} \begin{bmatrix} a_{31} \\ 1 - a_{31} \end{bmatrix} \begin{bmatrix} \delta_1 \end{bmatrix}^T.
\]  \hspace{1cm} (19)

Here, \(a_{ij} \in \mathcal{D}, i = 1, 2, 3, j = 1, 2, 3, 4, a_{ij} \in \mathcal{D}, \), and binary variables \(\delta_1, \delta_2, \delta_3 \in \mathcal{D}\), minimizing the cost function \(c_3 = \sum_{i=1}^{3} \delta_i\) subject to Eq. (20). Thus, one can find the minimum cost \(c_3 = 2\) under the possible solution \(\delta_1 = \delta_2 = 1, \delta_3 = 0, \) and logical matrices \(\hat{A}_1 \) and \(\hat{A}_2\) satisfying Col_3(\(\hat{A}_1\)) = \(\delta_2^f\) and Col_1(\(\hat{A}_2\)) = \(\delta_2^f\). For example, there exists a feasible solution for \(\hat{A}_1 \) and \(\hat{A}_2\) as \(\hat{A}_1 = \delta_1[1, 1, 2, 2] \) and \(\hat{A}_2 = \delta_2[2, 1]\). In fact, system (18) could be globally stabilizable w.r.t. arbitrary state by considering Problem 3 and using the obtained results. For example, consider global stabilization of system (18) w.r.t. state \(\delta_2^f\) (or \(\delta_1^f\)), by similar calculations of (19), one can find \(\delta_1 = \delta_3 = 0, \delta_2 = 1\) such that \(c_3 = \sum_{i=1}^{3} \delta_i\) equals to 1, and a feasible solution is matrix \(\hat{A}_2 = \delta_2[2, 1]\).

By considering Problem 3, one can find matrices \(\hat{A}_1 \in \mathcal{L}_{2 \times 2}, \ldots, \hat{A}_n \in \mathcal{L}_{2 \times 2}\) and binary variables \(\delta_1, \ldots, \delta_n \in \mathcal{D}\), such that the cost function \(c_3\) is minimized. Here, it is assumed that under the solution \(\delta_1 = \delta_2 = \cdots = \delta_n = 0, \delta_j = 1, j \in \{1, \ldots, n\}\), the cost function \(c_3\) is minimum, that is \(c_3 = l\), with the corresponding matrices \(\hat{A}_j, j \in \{1, \ldots, n\}\).

Then, further design a NS-based distributed pinning control to achieve global stabilization w.r.t. a given steady state \(\delta_2^f\). The controllers are imposed on nodes \(x_p, p \in \{\tau_1, \ldots, \tau_i\}\), in the form of

\[
\left\{
\begin{array}{l}
    x_p(t+1) = \bar{u}_p(t)\bar{x}_p(t), \\
    \bar{u}_p(t) = g_p(x_p(t)),
\end{array}
\right.
\]  \hspace{1cm} (21)

Here, nodes \(x_p, p \in \{\tau_1, \ldots, \tau_i\}\), are further chosen to be controlled, while the rest of the nodes \(x_j, j \in \{1, \ldots, n\}\), remain unchanged, just the same with system obtained after considering Step 1. In addition, the structure matrices of logical functions \(f_p(x_j(t), j \in \mathcal{L})\), \(p \in \{\tau_1, \ldots, \tau_i\}\), are \(\bar{A}_p\), which are found in Problem 2.

Suppose that the structure matrices for logical functions \(g_p, p \in \{\tau_1, \ldots, \tau_i\}\), are denoted by \(\bar{K}_p \in \mathcal{L}_{2 \times 2}\), and the structure matrices of function \(\bar{g}_p, p \in \{\tau_1, \ldots, \tau_i\}\), are denoted by \(\bar{M}_{\bar{g}_p} \in \mathcal{L}_{2 \times 4}\). Then, for \(p \in \{\tau_1, \ldots, \tau_i\}\), one can firstly obtain the algebraic form of (21) as follows:

\[
\left\{
\begin{array}{l}
    x_p(t+1) = \bar{M}_{\bar{g}_p} \bar{u}_p(t)x_j^p, \\
    \bar{u}_p(t) = \bar{K}_p x_j^p,
\end{array}
\right.
\]  \hspace{1cm} (22)

In order to determine matrices \(\bar{K}_p \in \mathcal{L}_{2 \times 2}\), \(\bar{M}_{\bar{g}_p} \in \mathcal{L}_{2 \times 4}\), \(p \in \{\tau_1, \ldots, \tau_i\}\), one needs to solve the following equations for \(p \in \{\tau_1, \ldots, \tau_i\}\):

\[
\bar{A}_p = \bar{M}_{\bar{g}_p} \bar{K}_p, \bar{x}_j^p = \bar{g}_p(x_j(t), j \in \mathcal{L}).
\]  \hspace{1cm} (23)

Fortunately, Eq. (23) is solvable and thus it is feasible to design state feedback controllers to achieve global stabilization w.r.t. the prescribed steady state \(\delta_2^f\).

In summary, for simplicity, denote sets \(\mathcal{N}_e = \{\omega_1, \ldots, \omega_k\} \cup \{\tau_1, \ldots, \tau_i\}, \mathcal{N}_e = \{\omega_1, \ldots, \omega_k\} \cup \{\tau_1, \ldots, \tau_i\}\), \(\mathcal{N}_o = \{\omega_1, \ldots, \omega_k\} \cup \mathcal{N}_e, \mathcal{N}_o = \{\tau_1, \ldots, \tau_i\} \cup \mathcal{N}_o\). Finally, after considering Steps 1 and 2, a NS-based distributed pinning control for global stabilization w.r.t. state \(\delta_2^f\) is designed as follows:

\[
\left\{
\begin{array}{l}
    x_j(t+1) = \bar{u}_j(t)\bar{x}_j(t), \\
    \bar{u}_j(t) = f_j(x_j(t), j \in \mathcal{L}_j),
\end{array}
\right.
\]  \hspace{1cm} (24)

Here, \(u_j \) and \(\bar{x}_j, j \in \{\omega_1, \ldots, \omega_k\}\), are controllers obtained in Step 1, while \(\bar{u}_j \) and \(\bar{x}_j, j \in \{\tau_1, \ldots, \tau_i\}\), are controllers obtained in Step 2.

Remark 4.3: During the past few decades, many fundamental results using the state transition matrix \(L\) have been established [13]-[22]. Unfortunately, one can observe that there still exists several disadvantages concerning the state transition matrix \(L\). One is that using system \(x(t+1) = Lx(t)\), the information of network structure is missing, such as nodes’ connection, while matrix \(L\) only reflects the state transition digraph. Another one is that the dimension of \(L\) grows drastically with the size of network, which implies that the existing methods are difficult to be implemented for large-dimension BNs. In this paper, a new framework for designing a NS-based distributed pinning control approach is proposed based on network structure of BNs without using the state transition matrix \(L\). This paper aims to solve the problems
of computational complexity (reduced from $O(2^n \times 2^n)$ to $O(2 \times 2^K)$, where $n$ is the number of nodes and $K$ is the largest number of in-neighbors of nodes) and large-scale networks.

V. APPLICATIONS TO T-LGL SURVIVAL SIGNALING NETWORKS

In this section, two biological networks respectively, with 6 and 29 nodes, are presented to demonstrate the validity of the obtained results on reducing the computational complexity.

Example 5.1: Consider a reduced network in the T-LGL survival signaling network, which consists of six nodes, that is S1P, FLIP, Fas, Ceramide, DISC, Apoptosis [53]. Here, $x_1, x_2, x_3, x_4, x_5, x_6$ are used to represent these six nodes, respectively. The interaction digraph $G_{id} = (V, E)$ is shown in Fig. 3 (a). The Boolean rules $f = (f_1, f_2, f_3, f_4, f_5, f_6)$ governing the nodes’ states are given as follows:

$$
\begin{align*}
  x_1(\ast) &= \neg(x_4 \vee x_5), \quad x_2(\ast) = x_3 \neg(x_1 \vee x_6), \\
  x_2(\ast) &= \neg(x_5 \vee x_6), \quad x_3(\ast) = [x_4 \vee (x_3 \neg x_2)] \wedge \neg x_6, \\
  x_3(\ast) &= \neg(x_1 \vee x_6), \quad x_6(\ast) = x_5 \vee x_6.
\end{align*}
$$

The symbol ($\ast$) indicates the state of next step of the marked node to reduce space. For each node $x_i$, $i \in [1,6]$, one has its corresponding in-neighbors, that is $N_1 = \{4,6\}$, $N_2 = \{5,6\}$, $N_3 = \{1,6\}$, $N_4 = \{1,3,6\}$, $N_5 = \{2,3,4,6\}$ and $N_6 = \{5,6\}$. By using STP method, one has the following algebraic form:

$$
\begin{align*}
  x_1(\ast) &= A_1 x_4 \wedge x_5, \\
  x_2(\ast) &= A_2 x_3 \wedge x_5, \\
  x_3(\ast) &= A_3 x_1 \wedge x_5, \\
  x_6(\ast) &= A_6 x_5.
\end{align*}
$$

Here, $A_1 = A_2 = A_3 = \delta_2[2,2,2,1], A_4 = \delta_2[1,1,2,1,1,2,1], A_5 = \delta_2[2,1,2,2,2,1,2,1,2,1,2,2]$ and $A_6 = \delta_2[1,1,1,2]$. Now, global stability w.r.t. state $x^* = (1,0,0,0,0,1) \sim \delta_{64}^1$ is considered. Setting $x(t) = x^* + Lx(t)$, one has the following algebraic form: $x(t^* + 1) = Lx(t)$, where

$$
L = \delta_{64}^1[59,57,59,42,59,27,59,12,63,61,63,46,63,31,63,16,59,57,59,42,59,25,59,10,63,61,63,46,31,63,16,59,49,59,34,59,19,59,9,63,49,63,34,63,19,63,4,59,49,59,34,59,17,59,2,63,49,63,34,63,19,63,4].
$$

Then, one can conclude that there does not exist an integer $T$ such that $\text{Col}(L^T) = \{\delta_{64}^1\}$, which implies that system [25] is not globally stable w.r.t. state $\delta_{64}^1$.

Next, a NS-based distributed pinning control will be considered to achieve global stabilization w.r.t. state $\delta_{64}^1$. Firstly, consider Step 1 by deleting the minimum number of edges such that the interaction digraph becomes acyclic. By using the depth-first search algorithm, there exist one fixed point and ten cycles in the interaction digraph, that is $\{6\}, \{2,5\}, \{1,4\}, \{4,6\}, \{5,6\}, \{1,3,4\}, \{2,5,6\}, \{4,5,6\}, \{1,4,5,6\}, \{1,3,5,6\}$ and $\{1,3,4,5,6\}$. Consider Problem 1. One can find the minimum number of edges that need to be deleted is equivalent to 4, such that the interaction digraph becomes acyclic. Here, the following edges are deleted, $6 \rightarrow 6, 5 \rightarrow 6, 2 \rightarrow 5$ and $4 \rightarrow 1$. Fig. 3 (b) shows the reduced digraph after deleting edges.

![Fig. 3. The left digraph (a) is the original interaction digraph $G_{id} = (V, E)$ of system (25), where the dashed arrows denote edges to be deleted. The right digraph (b) is the reduced digraph where edges $x_6 \rightarrow x_6, x_5 \rightarrow x_6, x_2 \rightarrow x_5$ and $x_4 \rightarrow x_1$ are deleted.]

Now, by considering Problem 2, one can find matrices $\tilde{A}_1 = [1,1] \otimes \delta_2[2,1], \tilde{A}_5 = [1,1] \otimes \delta_2[1,2,2,1,2,1,2,1], \tilde{A}_6 = \delta_2[1,1,1,1]$ satisfying (10), such that

$$
\begin{align*}
  x_1(\ast) &= \tilde{A}_1 x_4 \wedge x_5, \\
  x_5(\ast) &= \tilde{A}_5 x_2 \wedge x_3 \wedge x_4, \\
  x_6(\ast) &= \tilde{A}_6 x_5 \wedge x_6 \sim \delta_1^1.
\end{align*}
$$

Until now, we can design three state feedback pinning controllers $u_1, u_5, u_6$ for nodes $x_1, x_5, x_6$ in the following form,

$$
\begin{align*}
  x_1(\ast) &= u_1 \oplus f_1(x_4, x_5), \\
  x_5(\ast) &= u_5 \oplus f_5(x_2, x_3, x_4, x_6), \\
  x_6(\ast) &= u_6 \oplus f_6(x_5, x_6),
\end{align*}
$$

Here, $K_1 \in \mathcal{L}_{2 \times 4}, K_2, K_5 \in \mathcal{L}_{2 \times 16}, K_6 \in \mathcal{L}_{2 \times 4}$ are the structure matrices of functions $g_1, g_5, g_6$ and $M_{\otimes 1}, M_{\otimes 5}, M_{\otimes 6} \in \mathcal{L}_{2 \times 4}$ are the structure matrices of functions $\otimes 1, \otimes 5, \otimes 6$. To determine the structure matrices $K_1 \in \mathcal{L}_{2 \times 4}, K_5 \in \mathcal{L}_{2 \times 16}, K_6 \in \mathcal{L}_{2 \times 4}, M_{\otimes 1}, M_{\otimes 5}, M_{\otimes 6} \in \mathcal{L}_{2 \times 4}$, one needs to solve the following equations: $\tilde{A}_1 = M_{\otimes 1} K_1 (I_4 \otimes A_1) \Phi_4, \tilde{A}_5 = M_{\otimes 5} K_5 (I_6 \otimes A_5) \Phi_5, \tilde{A}_6 = M_{\otimes 6} K_6 (I_4 \otimes A_6) \Phi_4$. Then, one can find the following feasible solution: $M_{\otimes 1} = \delta_2[2,1,2,1], K_1 = \delta_2[1,2,1,1], M_{\otimes 5} = \delta_2[1,2,2,1], K_5 = \delta_2[1,1,1,2,1,1,1,1,1,1,1], M_{\otimes 6} = \delta_2[1,1,1,1,2,1], K_6 = \delta_2[2,2,2,1]$. Thus, for those controlled nodes $x_1, x_3, x_6$, one can obtain the corresponding logical dynamics as follows:

$$
\begin{align*}
  x_1(\ast) &= u_1 \leftarrow \neg(x_4 \vee x_5), \\
  x_3(\ast) &= u_5 \leftarrow \{x_4 \vee (x_3 \neg x_2)\} \wedge \neg x_6, \\
  x_6(\ast) &= u_6 \vee x_5 \vee x_6.
\end{align*}
$$

The NS-based distributed pinning controllers $u_1, u_5, u_6$ are in the following form,

$$
\begin{align*}
  u_1(t) &= x_4(t) \rightarrow x_6(t), \\
  u_5(t) &= [x_3(t) \vee (x_3(t) \vee x_6(t))] \vee \neg x_2(t), \\
  u_6(t) &= \neg x_5(t) \wedge \neg x_6(t).
\end{align*}
$$

Under the NS-based distributed pinning controllers $u_1, u_5, u_6$
in (30), one can obtain the following algebraic form:

\[
\begin{align*}
&x_1(*) = \hat{A}_1 x_6, \\
&x_2(*) = \hat{A}_2 x_5 \times x_6, \\
&x_3(*) = \hat{A}_3 x_1 \times x_6, \\
&x_4(*) = \hat{A}_4 x_1 \times x_3 \times x_6, \\
&x_5(*) = \hat{A}_5 x_1 \times x_4 \times x_6,
\end{align*}
\]  
(31)

Here, \( \hat{A}_1 = \delta_2[2,1], \hat{A}_2 = \hat{A}_3 = \hat{A}_4, \hat{A}_5 = \delta_2[2,1,2,1,2,1]. \)

Now, Step 2 is considered by designing controllers such that state \( \delta_2^{M_1} \) will be the unique steady state. The following problem is considered: (i) to find matrices \( \hat{M}_{M_1}, \hat{K}_1 \) as state matrices \( \hat{M}_{M_1} \delta_2[2,1,2,1,2,1] \) and \( \hat{K}_1 \delta_2[2,1,2,1,2,1]. \) Thus, matrices \( \hat{M}_{M_1}, \hat{K}_1 \) are chosen. Since \( \delta_1 = 1, \delta_2 = 0, i \in [2,6], \) it implies that we need to further design a state feedback controller \( \hat{u}_1 \) for node \( x_1 \) as follows:

\[
x_1(*) := \hat{u}_1 \hat{M}_{M_1}[u_1] = -(x_4 \times x_6), \quad \hat{u}_1(t) = \hat{g}_1(x_6(t)).
\]  
(32)

Thus, by considering Steps 1 and 2, one can design the following NS-based distributed pinning controllers \( u_1, u_5, u_6, \hat{u}_1 \) in the form of

\[
\begin{align*}
&x_1(*) = \hat{u}_1 \hat{M}_{M_1}[u_1] = -(x_4 \times x_6), \\
&x_2(*) = -(x_3 \times x_6), \\
&x_3(*) = -(x_1 \times x_6), \\
&x_4(*) = x_3 \times -(x_1 \times x_6), \\
&x_5(*) = u_5 \leftrightarrow \{x_4 \times (x_3 \times -x_2)\} \times -x_6), \\
&x_6(*) = u_6 \times x_5 \times x_6.
\end{align*}
\]  
(34)

The corresponding NS-based distributed pinning controllers \( u_1, u_5, u_6, \hat{u}_1 \) are designed as follows:

\[
\begin{align*}
u_1(t) &= x_1(t) \rightarrow x_6(t), \\
u_5(t) &= [x_5(t) \times (x_3(t) \times x_6(t))] \times -x_2(t), \\
u_6(t) &= -x_5(t) \times -x_6(t).
\end{align*}
\]  
(35)

Thus, by controlling nodes \( x_1, x_3, x_5, x_6 \) and four NS-based distributed pinning controllers \( u_1, u_5, u_6, \hat{u}_1 \) in the form of (34) and (35), system (33) is globally stabilized w.r.t. state \( \delta_2^{M_1}. \) Using BoolNet package [54], one can obtain the state graphs for both system (25) and system (34) under pinning control strategies (35), to be shown in Fig. 4 and Fig. 5. From Fig. 4 one can also obtain that system (25) is not globally stable since the state graph has two attractors. From Fig. 5 one can obtain that system (34) under pinning control strategies (35) achieves global stabilization w.r.t. the steady state \((1,0,0,0,0,1).\)

\begin{itemize}
\item **Remark 5.1:** Using the traditional L-based pinning control method [39–40], that is to change certain columns of the state transition matrix \( L, \) one needs at least controlling nodes \( x_1, x_2, x_3, x_6. \) However, using the proposed NS-based distributed pinning control method in this paper, one can achieve global stabilization by controlling nodes \( x_1, x_3, x_5, x_6. \) Without using matrix \( L, \) the computational complexity can be reduced, and the NS-based distributed pinning control method also leads to a lower dimensional controller design than traditional L-based pinning control design.
\end{itemize}

**Example 5.2:** Consider the network model of survival signaling in large granular lymphocyte leukemia with 29 nodes [55], one can obtain its corresponding logical dynamics and its interaction digraph (shown in Fig. 6a)), shown below:

\[
\begin{align*}
&x_1(*) = x_1, \\
x_2(*) = x_2, \\
x_3(*) = x_3, \\
x_4(*) = x_4, \\
x_5(*) = x_5, \\
x_6(*) = x_6, \\
x_7(*) = x_7, \\
x_8(*) = x_8, \\
x_9(*) = x_9, \\
x_{10(*)} = x_{10,} \\
x_{11(*)} = x_{11,} \\
x_{12(*)} = x_{12}, \\
x_{13(*)} = x_{13}, \\
x_{14(*)} = x_{14}, \\
x_{15(*)} = x_{15}, \\
x_{16(*)} = x_{16}, \\
x_{17(*)} = x_{17}, \\
x_{18(*)} = x_{18}, \\
x_{19(*)} = x_{19}, \\
x_{20(*)} = x_{20}, \\
x_{21(*)} = x_{21}, \\
x_{22(*)} = x_{22,} \\
x_{23(*)} = x_{23}, \\
x_{24(*)} = x_{24}, \\
x_{25(*)} = x_{25}, \\
x_{26(*)} = x_{26}, \\
x_{27(*)} = x_{27}, \\
x_{28(*)} = x_{28,} \\
x_{29(*)} = x_{29}.
\end{align*}
\]  
(36)

Here, we use \( x_1, \ldots, x_{29} \) to represent the 29 gene structures, IL15, RAS, ERK, JAK, IL2RBT, STAT3, IFNGT, Fasl, PDGF, PDGFR, PI3K, IL2, Bclxl, TPL2, Sphk, S1p, Sfas, Fas, Disc, Caspase, Apoptosis, LCK, MEK, GZMB, IL2RAT, FasT, RANTES, A20, FLIP. respectively. Here, the symbol * is to denote the next time step to save space. In fact, by resorting to the proposed NS-based distributed pinning control
design approach, one can design the following state feedback pinning control on nodes $x_1, x_9, x_{15}$, in order to achieve global stabilization,

$$x_1(\cdot) = u_1 \land x_1, \quad x_9(\cdot) = u_9 \lor x_9, \quad x_{15}(\cdot) = u_{15} \land (x_{11} \land x_{16}).$$

(37)

The logical dynamics of NS-based distributed pinning controllers $u_1, u_9, u_{15}$, are in the form of

$$u_1 = \neg x_1, \quad u_9 = \neg x_9, \quad u_{15} = x_{11}.$$  

(38)

Then, under pinning control systems (37) and (38), system (35) will achieve global stabilization. Actually, using STP method, one should calculate the state transition matrix $L$ with dimension $536870912 \times 536870912$ for algebraic form $x(t+1) = Lx(t)$. Fig. 6(b) shows part of the state transition graph from 25 initial state nodes among 536870912 global states of system (35) before pinning control, while Fig. 6(c) shows part of the state transition graph from 800 initial state nodes under NS-based distributed pinning control strategies (37) and (38). From Fig. 6(c), one can obtain that system (34) under pinning control strategies (35) achieves global stabilization w.r.t. a unique steady state $\{0,0,\ldots,0,1,1,0,0,0,0,1,1,0,0,1,0,0,0,0\}$. The details of calculation of matrix $L$ are presented in Appendix A, which shows that the traditional L-based pinning control design will be difficult for implementation for networks with more than 10 nodes. Comparing the existing L-based pinning control method [59]–[66], it is almost impossible to analyze the state transition matrix $L$ with dimension $536870912 \times 536870912$ for designing controllers.

Remark 5.2: As shown in Eq. (38), the NS-based distributed pinning controllers are designed by using neighbors’ logical information, but not on global information. For example, the dynamic of controller $u_{15}$ is only determined by one node $x_{11}$, which shows that the dynamic of controller seems quite simple and has a low dimension. However, using the traditional L-based pinning control method, pinning control design is quite difficult for implementation on a network of such size, since one needs to analyze a matrix with dimension $536870912 \times 536870912$, and global information of all the nodes is required to design the controller. Thus, the traditional L-based pinning control method [59]–[66] leads to a quite complicate form of controllers with high dimensions.

VI. DISCUSSIONS

Note that in linear time-invariant systems, controlling system $\dot{x} = Ax$, one can consider to change $A$ to a diagonal matrix. However, controlling BNs with discrete-time logical system is quite different with controlling LTI system. Note that given a BN, one can define an interaction digraph, however, the state space of BNs is determined by both the interaction digraph among nodes but also by the logical functions among each nodes. This implies that given two BNs with the same interaction digraph, the state spaces for these two BNs can be quite different. Thus, it brings big difficulties in controlling a BN, since we need the information of interaction digraph and also the information of logical operators among nodes.

In addition, it is practical and reasonable by deleting edges over the interaction digraph of BNs, since gene regulatory networks in most biological systems are sparse shown in [56], [57]. Just as shown in [58], the mean connectivity in Escherichia coli is found between 2 and 3, which shows a rather loosely interconnected structure. Thus, the proposed edge-deleting methodology is practical and reasonable in the majority of gene networks since generally they are sparse. In addition, the proposed edge-deleting methodology can model the action of a drug that inactivates the corresponding interaction among two gene products [56], while deleting nodes of BNs can model the control action blocking of effects of products of genes associated to these nodes.

Consider the reduced networks of T-LGL signaling network with 18 nodes, that is CTLA4, TCR, CREB, IFNG, P2, GPCR, SMAD, Fas, aFas, Ceramide, DISC, Caspase, FLIP, BID, IAP, MCL1, S1P, Apoptosis, which are denoted by $x_1,\ldots,x_{18}$. The detailed interaction digraph (shown in Fig. 7(a)) and logical functions can be found in [59] (see Figure 2 and Table S3). Then, one can design the following NS-based distributed pinning controllers for nodes $x_1, x_5, x_{11}, x_{17}, x_{18}$: $x_1(\cdot) = u_1 \lor x_3, x_5(\cdot) = u_5 \land (x_4 \lor x_5), x_{11}(\cdot) = u_{11} \lor [x_{10} \lor (x_8 \land \neg x_{13})], x_{17}(\cdot) = u_{17} \lor \neg x_{10}, x_{18}(\cdot) = u_{12} \land (x_{12} \lor x_{18})$. In addition, the corresponding NS-based distributed pinning controllers are designed in the following form: $u_1 = \neg x_2, u_5 = \neg (x_4 \lor x_5), u_{11} = x_8 \lor (\neg x_8 \land x_{10}), u_{17} = x_{10}, u_{18} = x_{12}$. Then, under the above designed pinning control, the reduced T-LGL signaling network with 18 nodes will achieve global stabilization w.r.t. a steady state $(1,0,0,0,1,1,1,0,1,0,1,1,0,1,1,1)$. Fig. 7(b) shows part of the state transition digraph randomly from 25 initial states before pinning control, while Fig. 7(c) shows part of the state transition digraph randomly from 500 initial states after pinning control.
In fact, the proposed NS-based distributed pinning control method has been tested in a network model consisting of 90 nodes denoted by $x_1, \ldots, x_{90}$ [60], the detailed logical functions for each node can be found in Table 3 in [60] and the interaction digraph is shown in Fig. 7(a). It is said in [61], “It is, to the best of our knowledge, the largest Boolean model of a cellular network to date”. Due the page limitation, the corresponding logical functions are omitted here. According to the proposed NS-based distributed pinning control, one can design an efficient NS-based distributed pinning control just on 15 nodes $x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10},x_{11},x_{12},x_{13},x_{14},x_{15}$. The corresponding distributed pinning control for these 15 nodes are given below: $x_1(\ast) = u_1 \lor x_1,x_2(\ast) = u_2 \lor x_2,x_3(\ast) = u_3 \lor x_3,x_4(\ast) = u_4 \lor x_4,x_5(\ast) = u_5 \lor x_5 \land \neg x_6 \land \neg x_7,x_6(\ast) = u_6 \lor x_6 \lor x_7,x_7(\ast) = u_7 \lor x_7 \lor x_8 \lor x_9,x_8(\ast) = u_8 \lor x_8 \lor x_9 \lor x_{10},x_9(\ast) = u_9 \lor x_9 \lor x_{10} \lor x_{11},x_{10}(\ast) = u_{10} \lor x_{10} \lor x_{11} \lor x_{12},x_{11}(\ast) = u_{11} \lor x_{11} \lor x_{12} \lor x_{13},x_{12}(\ast) = u_{12} \lor x_{12} \lor x_{13} \lor x_{14},x_{13}(\ast) = u_{13} \lor x_{13} \lor x_{14} \lor x_{15},x_{14}(\ast) = u_{14} \lor x_{14} \lor x_{15}$.

The state transition digraph randomly from 25 initial states before pinning control; (c) Part of state transition digraph randomly from 500 initial states after pinning control. Each dashed arrow represents a state transition, and solid arrows highlight attractors.

To sum up, by comparing the simulations on certain BNs with different sizes, the traditional $L$-based pinning control has limitations on the following certain aspects:

I. Calculations of algebraic system $x(t+1) = Lx(t)$ is always needed, where the state transition matrix $L$ has a dimension $2^n \times 2^n$. Thus, when the number of network nodes $n$ becomes larger, the computational complexity ($O(2^n \times 2^n)$) will be more complicate;

II. Since the traditional $L$-based pinning control [39]–[46] is designed based on the transformation of matrix $L$, then $L$-based pinning controllers depends on all the nodes of BNs and also are in the form of $u(t) = g(x_1(t),x_2(t),\ldots,x_n(t))$. Thus, when network nodes $n$ becomes larger, $L$-based pinning controllers will be more complicate and have high dimensions;

III. $L$-based pinning controlled nodes are determined by matrix $L$, thus when the dimension of $L$ becomes larger,

$L$-based pinning controlled nodes will be more difficult to be found.

Fortunately, in this paper, the above limitations are well solved by the proposed NS-based distributed pinning control, which can be efficiently implemented on any BNs with any network structures:

I. In this paper, during the procedures of designing NS-based distributed pinning control, the calculations of $x(t+1) = Lx(t)$ is avoided, which reduces the high computation complexity $O(2^n \times 2^n)$ from $O(2 \times 2^K)$ ($K$ is the largest number of neighbors among each node);

II. Using the feedback arc set in the network structure of BNs, NS-based distributed pinning controlled nodes can be easily determined under the information of network structure;

III. By using neighbors’ local information not on overall nodes’ information, NS-based distributed pinning controllers only depends on local neighbors and are in the form of $u(t) = g(x_1(t),x_2(t),\ldots,x_k(t))$, where $\{i_1,i_2,\ldots,i_p\} \subseteq \{1,2,\ldots,n\}$ are the neighbors of controlled nodes. Thus, compared with $L$-based pinning controllers [39]–[46], NS-based distributed pinning controllers are much simpler and easier to be designed, and also have lower dimensions.

VII. CONCLUSION

In this paper, a new framework for designing NS-based distributed pinning control has been firstly presented to achieve global stabilization w.r.t. any given steady state based on network structure of BNs, without using state transition matrix. By deleting the minimum number of edges such that the network topology is acyclic means that global stability can be guaranteed. Consequently, based on the acyclic structure, a NS-based distributed pinning control can be designed based on the local neighbors of controlled nodes. In addition, the computational complexity can be reduced using the method proposed in this paper, which can be efficiently applied for large-dimension BNs. A new perspective has been proposed combining the network structure of BNs with NS-based
distributed pinning control design, which aims to solve the problems of computational complexity and large-dimensional networks.

The new framework presented in this paper provides a new way to design pinning control with lower dimensional controllers and less computational load. In addition, the proposed approach may also provide a new direction to find the minimum number of controlled nodes to achieve global stabilization. The summarized comparisons between NS-based distributed pinning control and traditional L-based pinning control can be observed from Fig. 9, which also implies one challenging topic of combining network structure with state transition digraph for designing efficient pinning control. In the near future, the inherent relationship between cycles (especially non-functional cycles) of network structure and global stability will be studied. Further, it would be interesting to design pinning control strategies for partially-known BNs to reduce computational cost.

APPENDIX A: L-BASED PINNING CONTROL DESIGN FOR NETWORK WITH 29 NODES \((L \in \mathcal{L}_{536670912} \times 536670912)\)

Compared with traditional L-based pinning control design \([39]–[46]\), one needs to obtain the state transition matrix \(L\) of a BN via STP method. Then, an algebraic form for each node is obtained as follows:

\[
x_i(t+1) = F_i x(t), \quad i \in [1, 29].
\]  

Here, in order to save space, the detailed calculations for matrices \(F_i, i \in [1, 29]\), are omitted. Then, using the Khatri-Rao product and some basic structure matrices of logical operators, one has the algebraic representation for system \([46]\):

\[
x(t+1) = L x(t),
\]

where \(L = F_1 * F_2 * \cdots * F_{29} \in \mathcal{L}_{536670912} \times 536670912\) is the state transition matrix with high dimension. The detailed calculation of matrix \(L\) can be obtained in \([12], [13]\).

**Definition 7.1:** \([62]\) Given two matrices \(A \in \mathcal{R}_{m \times l}\) and \(B \in \mathcal{R}_{l \times n}\), the Khatri-Rao product of \(A\) and \(B\), is defined as

\[
A \star B = [\text{Col}_1(A) \times \text{Col}_1(B), \ldots, \text{Col}_i(A) \times \text{Col}_i(B)].
\]

Thus, one can obtain the state transition matrix \(L \in \mathcal{L}_{536670912} \times 536670912\). However, due to the high complexity of \(L\), only the indexes of the first 8 columns and the last 8 columns are presented as follows:

\[
L = \delta_{536670912}[199169, 199169, 199169, 199169, 199169, 199169, 199169, \ldots, 536672256, 536672256, 536672256, 536672256, 536672256, 536672256].
\]

The computational time for calculating the state transition matrix \(L\) is about 7 days and 2.3 hours by Matlab 2015b running on 2.6 GHz Intel Core i7, which is relatively slow. Thus, due to the high complexity of the state transition matrix \(L\), it is difficult to design controllers by using the method presented in \([39]–[46]\), to achieve global stabilization.

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