Four-Vector Optical Dirac Equation and Spin-Orbital-Hall Effect of Structured Lights

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Abstract

The spin-orbital interaction of light is a crucial concept for understanding electromagnetic properties of material and designing the spin-controlled manipulation of optical fields. Achieving these goals requires a complete description of spin-dependent optical phenomena in the context of vector-wave mechanics. We present an approach to convert the Maxwell equation in generic media into a 4-vector optical Dirac equation, which was found to take almost the same form as the non-Hermitian $\gamma_5$-extension of the massive Dirac equation for fermions with anomalous magnetic momentum moving in an external pseudo-magnetic field. This similarity allows us to investigate optical behaviors of material by effective field theory methods and can find wide applications in metamaterial, photonic topological insulators, etc. By the analogy with non-rotating gravitational fields versus spin-degenerate materials, we demonstrate this method by studying the spin-orbital-Hall effect of structured lights in a gravitational wave. Essentially, our approach could bridge our understanding of the spin-orbit interaction and topological insulators between electronic and photonic systems.

Keywords: optical Dirac equation, structured light, spin-orbit interaction, non-Hermitian photonics
In classical physics, an electromagnetic wave can be described by its amplitude and phase over space-time. In some particular situations where the geometric optics approximation is valid, the correspondence principle allows us to make a simple analysis in term of particle interpretation. The traditional approach is based on the Hamilton-Jacobi theory in which the ray system associated with waves is determined by a Hamilton structure induced from the dispersion relation. Thereby, amplitude and phase of waves are converted into canonical coordinates and momentums associated with the degrees of freedom of particles. In this sense, the variations of optical fields can be equivalently understood by the transverse displacements of light-beam trajectories and vice versa [1, 2].

Moreover, we have realized the correspondence between the polarization feature of electromagnetic wave and the spin of photon, which exhibits, beyond the correspondence principle, another essential aspect of the wave-particle duality in quantum world. Besides the spin angular momentum of photon (SAM), the structured light can carry two types of orbital angular momentum (OAM) - the intrinsic orbital AM (IOAM) and extrinsic orbital AM (EOAM), associated with helical wavefront and helical optical path respectively. Theoretically, with the spin properties encoded in the classical Maxwell theory, incorporating with the non-integral topological phase, we have appreciated a spin-orbital interaction (SOI) of light. The SOI offers a universal concept underlying in a variety of spin-dependent optical phenomena [3–6], and manifests itself by the interplay and mutual conversion between these three types of optical AM. The coupling of SAM and EOAM leads to a family of the spin-Hall effects (SHE) of photon - spin-dependent displacement of light-beam trajectory due to a Lorentz-like force from the Berry potential [7]. It is noted that the global topological phase does not yield a classical force, but only makes transverse displacements on the intensity pattern of light fields [8]. In addition, the SAM-IOAM coupling produces the helicity dependent optical vortex via the spin-to-orbital AM conversion [9–11], and the IOAM-EOAM coupling results in the orbital-Hall effect [12, 13].

There have been various approaches proposed for studying the spin-Hall effect of lights, among which the main method is based on the modified geometrodynamics of polarized light beyond geometric optics approximation. However, the SOI phenomena is only significant at sub-wavelength scales, moreover, if structured lights with the IOAMs are taken into account, a full treatment of vector-wave mechanics becomes more essential. Motivated by this consideration, this work is to develop a four-vector optical Dirac equation from the classical source-free Maxwell equation in generic media, which will enable us to deal with spin-dependent spatial-temporal variations of optical fields within a unified theoretical framework.

**Optical Dirac equation in generic media**

In a general linear medium characterized by the real-valued tensors of permittivity $\epsilon(\mathbf{r})$ and permeability $\mu(\mathbf{r})$ (without including bi-anisotropic property), the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ are related to the auxiliary
fields $\mathbf{D}$ and $\mathbf{H}$ by the local constitutive equation $\mathbf{D} = \varepsilon \cdot \mathbf{E}$, $\mathbf{B} = \mu \cdot \mathbf{H}$. In terms of $\mathbf{D}$ and $\mathbf{B}$, the Maxwell equation takes the form,

$$\begin{align*}
\frac{\partial \mathbf{D}}{\partial t} = \nabla \times (\mu^{-1} \cdot \mathbf{B}), \quad \nabla \cdot \mathbf{D} = 0 \\
\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\varepsilon^{-1} \cdot \mathbf{D}), \quad \nabla \cdot \mathbf{B} = 0
\end{align*}$$

(1)

We firstly introduce some necessary notations and relevant rules. According to the Maxwell equation Eq.(1), we need to deal with the following kind of operation

$$\hat{H} \circ = \nabla \times (\Pi \circ)$$

(2)

where $\Pi = \{\Pi_{ij}, i, j = 1, 2, 3\}$ is a tensor of rank 2, which can be either $\varepsilon^{-1}$ or $\mu^{-1}$ for a linear electromagnetic media. For simplicity, $\Pi$ is also assumed to be a constant symmetric tensor. In the global Cartesian coordinate $\mathbf{r} = (x, y, z)$ associated with the laboratory reference frame, the operation $\hat{H}$ can be converted to a matrix multiplication, $\hat{H} = (\hat{k} \cdot \mathbf{s}) \cdot \Pi$, where $\hat{k} = -i \nabla$ is the momentum operator, and $\mathbf{s}$ is the spin-1 operator given by the adjoint representation of $SO(3)$, i.e., $\{s_i\}_{ik} = -i \epsilon_{ijk}$. Taking account of the transversality condition for electromagnetic waves, we are currently used to make decomposition of an electromagnetic vector by its transverse components (denoted by the subscript $\perp$) with respect to a given propagating direction, e.g. the $z$ axis with the basic vector $\mathbf{e}_z$. The transverse components can be further projected to the complex helicity-basis $\{\mathbf{e}_\pm\}$ satisfying $(\mathbf{e}_\pm \cdot \mathbf{s})\mathbf{e}_\pm = \pm \mathbf{e}_\pm$, which is related to the ‘Cartesian frame’ explicitly by $\mathbf{e}_\pm = (\mathbf{e}_x \pm i \mathbf{e}_y) / \sqrt{2}$.

Accordingly, for a given vector $\mathbf{V}$, its components in the Cartesian frame are related to those in the helicity space by a unitary transformation $\hat{U}$, $(V_x, V_y, V_z)^T = \hat{U}(V_+, V_-, V_\perp)^T$. Transforming to the helicity space, the operator becomes $\hat{H} \rightarrow \hat{U}^{-1} \hat{H} \hat{U} = \hat{U}^{-1}(\hat{k} \cdot \mathbf{s})\hat{U} \hat{U}^{-1} \Pi \hat{U}$. Under the action of $\hat{H}$, the transverse components of $\mathbf{V}$ are given by

$$(\hat{H} \mathbf{V})_\perp = \sigma_3[(Q_0 \mathbf{I} + \mathbf{Q} \cdot \sigma_\perp)\hat{k}_z \perp - \hat{k}_\perp \mathbf{q}^\dagger \mathbf{V} \perp + (\hat{k}_z \mathbf{q} - \hat{k}_\perp q_0) V_z]$$

(3)

where $\mathbf{q} = \{Q_+, Q_-\}^T$ is the dipole momentum with the components $Q_{\pm 1} = (\Pi_{13} \mp i \Pi_{23}) / \sqrt{2}$, $Q_0 \equiv \frac{1}{2} (\Pi_{11} + \Pi_{22})$ is the monopole, $q_0 = \Pi_{33}$, $\mathbf{I}$ is the $2 \times 2$ unity matrix, $\mathbf{Q} \cdot \sigma_\perp = Q_{+2} \sigma_+ + Q_{-2} \sigma_-$ is the surface term with the quadrupoles $Q_{\pm 2} = \frac{1}{2} (\Pi_{11} - \Pi_{22}) \mp i \Pi_{12}$, $\sigma_\perp$ denotes for the transverse Pauli matrices $\{\sigma_i, i = 1, 2\}$, and $\sigma_\pm = \frac{1}{2} (\sigma_1 \pm i \sigma_2)$ by convention.

Substituting the longitudinal $z$ component into Eq.(3) by the transversality condition, $V_z = -\hat{k}_z^{-1} \hat{k}_\perp \mathbf{V} \perp$, we can find the effective operator $\hat{H}_\perp \mathbf{V} \perp = (\hat{H} \mathbf{V})_\perp$, a $2 \times 2$ matrix operator acting on transverse 2-vectors,

$$\hat{H}_\perp = \sigma_3 [H_0 + \mathbf{H} \cdot \sigma_\perp]$$

(4)
where

\[ H_0 = \hat{k}_z Q_0 + q_0 \frac{1}{k_z} \hat{k}_+ \hat{k}_- - \mathbf{q} \cdot \hat{k} \]

\[ \mathbf{H} = \hat{k}_z Q + q_0 \frac{1}{k_z} \hat{k}_Q - \hat{k}_q \]

with

\[ \hat{k}_q \cdot \sigma_{\perp} = 2 [Q_{\perp} \hat{k}_+ \sigma_+ + Q_{\perp} \hat{k}_- \sigma_-] \]

\[ \hat{k}_Q \cdot \sigma_{\perp} = \hat{k}_+^2 \sigma_+ + \hat{k}_-^2 \sigma_- \]

Now, we are embarking on applying the above results to the Maxwell equations Eq.(1). Taking the same notation rule as described above, we can write the Maxwell equation for the transverse components as

\[ i \frac{\partial \mathbf{D}_{\perp}}{\partial t} = (H_0^B - \mathbf{H}^B \cdot \sigma_{\perp}) i \sigma_3 \mathbf{B}_{\perp} \]

\[ i \frac{\partial i \sigma_3 \mathbf{B}_{\perp}}{\partial t} = (H_0^D + \mathbf{H}^D \cdot \sigma_{\perp}) \mathbf{D}_{\perp} \]

where the superscripts \( D \) and \( B \) denote for the corresponding quantities obtained from the dielectric matrices \( \epsilon^{-1} \) and \( \mu^{-1} \) respectively. Alternative to the current definitions of wavefunction function of photons using the Riemann-Silberstein vectors \([14–19]\), we define 2-vector wavefunctions of photons in the helicity space,

\[ \Psi_\pm = \mathbf{D}_{\perp} \pm i \sigma_3 \mathbf{B}_{\perp} \]

It is easy to see that \( \Psi_\pm \) correspond to the positive and negative energy states respectively. Furthermore, combining them to form a 4-vector wavefunction \( \Psi = (\Psi_+, \Psi_-)^T \) and introducing the following effective mass and momentum operators,

\[ \hat{m}_\pm = \frac{1}{2} (H_0^D \pm H_0^B), \quad \hat{p}_\pm = \frac{1}{2} (\mathbf{H}^D \pm \mathbf{H}^B) \]

eventually, we can arrive at a compact Dirac form of the Maxwell equation,

\[ i \frac{\partial \Psi_{\perp}}{\partial t} = \left[ \gamma_0 (\hat{m}_+ + \gamma_5 \hat{m}_-) + \gamma_\perp \cdot \hat{p}_+ \right] \Psi_{\perp} + (\sigma_3 \otimes \sigma_{\perp}) \cdot \hat{p}_- \Psi_{\perp} \]

Remarkably, Eq.(9) is akin to the algebraic non-Hermitian \( \gamma_5 \)-extension of the massive Dirac equation for fermions with anomalous magnetic momentum moving in an external magnetic field, which was introduced firstly by Bender et.al \([20, 21]\). The difference with the Dirac equation comes from the additional \( \beta \) operation in the effective momentum term by \( \gamma_i = \beta \alpha_i \), which makes the optical field a non-Hermitian system because of the anti-Hermicity \( \gamma_i^\dagger = -\gamma_i \). It has been realized that the \( \gamma_5 \) extension is a critical generalization to the Dirac model, and has a wide range of applications in physics. Actually, it constitutes the low energy model for the bulk states in topological insulators \([22–24]\). This similarity will help us to understand photonic topological insulators from
electronic systems. The other applications could be found in chiral magnetic effect of non-Hermitian fermionic systems [25] and neutrino physics [26, 27].

For photonic systems, we observe that the emergence of the $\gamma_5$ extension is due to the so called $\epsilon$-$\mu$ mismatch, which breaks the spin-degenerate condition. Apparently, the additional effective mass $\gamma_5 \hat{m}_-$ corresponds to an external pseudoscalar potential. As usual, $\hat{m}_-$ must behave like a scalar under a Poincare transformation, except $\hat{m}_-$ is parity odd because of the negative parity of $\gamma_5$. As indicated by Eq.(5) and Eq.(8), $\hat{m}_-$ is proportional to the linear momentum, and so is mostly parity odd. Moreover, the extra momentum term $\gamma_5 \hat{p}_-$ is equivalent to an anomalous magnetic moment $\delta \mu_a$ in an external magnetic field $B$ by the interaction $\sim \delta \mu_a \sigma_3 \otimes (\sigma \cdot B)$, or in the covariant form $\frac{1}{2} \delta \mu_a \sigma^{\mu\nu} F_{\mu\nu}$ [28]. It is worth emphasizing here that the $\gamma_5$ extension offers an alternative perspective for understanding the electromagnetic properties of material and related optical phenomena such as chirality of materials, spin-orbital Hall effect, spin-to-orbital conversions, etc.

The following remarks is necessary. Above optical Dirac equation is derived based on the assumption of real symmetric dielectric tensors. Indeed, including anti-symmetric components, e.g. associated with gyrotropic materials, will make the dipole and quadrupole operators $\hat{k}_q$ and $\hat{k}_Q$ no longer in-plane transverse vectors, and the resulting longitudinal component would give rise to the chirality of materials and produce the optical activity.

In the same spirit of above derivation, it is straightforward to generalize the optical Dirac equation for gyrotropic, biisotropic and bianisotropic materials, even for inhomogeneous materials. For helical optical paths, the optical Dirac equation can be also extended to general 3-d curvilinear coordinates [5, 29], in which the Berry potential will be naturally introduced through the moving tetrad.

**Example: gravitational fields as spin-degenerate materials**

For light propagation in a gravitational field, the curved space background can be transformed to a linear optical medium, whose optical properties characterized by the effective dielectric tensor are fully specified by the spacetime geometries [30]. By this analogue, the gravitational spin-Hall effect can be easily understood [31–35]. Practically, taking an appropriate 3+1 decomposition of space-time metric, the Maxwell equation can be written in the same form as Eq.(1), where the permittivity and permeability tensors obey a simple relation with the spatial metric $\gamma = \{g_{ij}\}$, $\epsilon^{-1} = \mu^{-1} = \gamma$, satisfying the spin-degenerate condition. In this case, the optical Dirac equation Eq.(9) reduces to

$$i \frac{\partial}{\partial t} \Psi = (\beta \hat{m} + \beta \alpha \cdot \hat{p}) \Psi$$

in which the effective mass term is

$$\hat{m} = Q_0 \hat{k}_z + \frac{q_0}{2k_z} \hat{k}_\perp^2 - q \cdot \hat{k}_\perp$$
and the momentum components

\[ p_\pm = Q_{\pm} 2 \hat{k}_z - 2 Q_{\pm} \hat{k}_\perp + \frac{q_0 k_\perp^2}{k_z}, \quad p_z = 0, \quad \text{(12)} \]

It is easy to verify that Eq.(10) describes a \( \mathcal{P}\mathcal{T} \) symmetric non-Hermitian system\(^{36–38}\), and would have both the positive and negative energy solutions. For a photon with given negative energy, it can be regarded as a mirror photon with the equal positive energy and opposite helicity. Actually, this mirror duality between positive and negative energy states can be understood by the following facts. Firstly, the effective “mass” given by Eq.(11) is a pseudo scalar operator with the odd parity \( \mathcal{P} \) and odd time-reversal symmetry \( \mathcal{T} \). Secondly, the global “chiral” transformation \( \Psi \rightarrow \gamma_5 \Psi \) is equivalent to making an exchange between the positive and negative states. Taking account of \( \gamma_5 \) anti-commuting with \( \gamma_0 = \beta \) and \( \gamma_i = \beta \sigma_i \), we can see that the optical Dirac equation is invariant under the \( \gamma_5 \) transformation combining with \( k \rightarrow -k \) (either \( \mathcal{T} \) or \( \mathcal{P} \)) transformation. Accordingly, the appearance of extra \( \beta \) in the momentum term of the optical Dirac equation can be also understood, otherwise, this invariance will be broken. Furthermore, this symmetry, as such, is associated with the duality of Maxwell equations in vacuum. In fact, a simple analysis indicates that the duality transformation is equivalent to the \( \gamma_5 \mathcal{T} \) transformation, and the equality \( \epsilon = \mu \) of gravitational fields is necessary for the duality symmetry.

Now, we solve the eigenvalue problem of Eq.(10) under the geometric approximation. On the leading order, only the first terms in both Eq.(11) and Eq.(12), \( \sim O(k_z) \) are kept. Let \( \omega = \lambda k_z \), the energy-eigenvalue equation is found to satisfy

\[ \lambda^2 + |Q_+|^2 = Q_0^2 \]

The real solutions require \( |Q_0| > |Q_+| \), which is always valid for weak gravitational fields. As expected, the eigenvalue equation has two solutions with the positive and negative energies. For the positive energy solution with \( \lambda_+ \), we have

\[ |R\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -\eta Q_- \end{pmatrix} ; \quad |L\rangle = \begin{pmatrix} 0 \\ 1 \\ -\eta Q_+ \\ 0 \end{pmatrix} \quad \text{(13)} \]

where \( \eta = (\lambda_+ + Q_0)^{-1} \). For general positive-energy solutions to Eq.(10), we expand by a linear superposition of the two helicity eigenstates Eq.(13),

\[ \Psi = (\varphi_R |R\rangle + \varphi_L |L\rangle)e^{-i\omega t + ik_z z} \quad \text{(14)} \]

Defining the 2-vector \( \Phi = \left( \varphi_R, \varphi_L \right)^T \) and substituting Eq.(14) into Eq.(10), we are led to an optical Schrödinger equation

\[ i \frac{\partial \Phi}{\partial z} = -\frac{q_0}{2k_z} \nabla_\perp^2 \Phi + V(r_\perp)\Phi \quad \text{(15)} \]
with the complex potential

\[ V(r_\perp) = -q \cdot k_\perp + \sigma_3 \frac{1}{2k_z} \left[ Q_{+2}k^2_\perp - Q_{-2}k^2_+ \right] \quad (16) \]

in which only the terms of linear order of \( O(Q) \) are kept at the level of linearized gravity theory. Clearly, the above equation is similar to the simplest optical model describing the paraxial propagation of light in an inhomogeneous dielectric medium with a complex refractive index \( V = n_r + i n_i \)[39, 40], while we are now discussing a two-component system. The potential Eq.(16) consists of two parts - the real part gives a dipole interaction, and the imaginary part is a non-Hermitian quadrupole interaction responsible for the optical gain or loss arising from the emission and absorption of gravitons. On the other hand, the Hamiltonian Eq.(15) indicates that the opposite helicity states have decoupled, implying no helicity transition would occur.

In any metric gravity theory, GWs can have, at most, six distinct polarization modes, including two tensor-types (spin-2), two vector types (spin-1) and two scalar-types (spin-0). The dipole momentum \( q \) in Eq.(16), \( H_D = -q \cdot k_\perp \) is associated with the vector-type. Einstein theory only allows for the existence of the tensor plus (+) and cross(\( \times \)) polarizations. Accordingly, the dipole moment vector can be written as

\[ q = h^+(e_g \cdot e_k)[(e_g \times e_k) \times e_k] + h^x(e_g \times e_k) \quad (17) \]

where \( e_g \) and \( e_k \) denote for the unit vectors in the propagating directions of GWs and lights respectively. If the incident GW propagates parallel to the light beam, the dipole is identically zero, \( q = 0 \), as expected in the TT gauge. This dipole interaction is physically due to the coupling of GW polarization and transverse momentum of photons. This helicity-helicity coupling consists of the two contributions, one from \( \propto h^+ e_g \cdot k_\perp \), the other from \( \propto h^x e_g \cdot (e_k \times k_\perp) \). Formally, the former is due to the GW polarization coupled with the OAM density flow of photons \( S_o \propto \text{Im}[E^* \cdot (\nabla E)] \), and the latter with the spin density flow \( S_c \propto \text{Im}[\nabla \times (E^* \times E)] \)[41].

Assume that the incident vector \( e_g \) of GWs is at a given direction of \( \{\theta_g, \phi_g\} \) in the spherical coordinates, the dipole interaction in the helicity basis can be alternatively expressed by

\[ H_D = i\left[ Q_{-1}\nabla_+ + Q_{+1}\nabla_- \right] \quad (18) \]

where

\[ iQ_\pm = \pm \frac{1}{\sqrt{2}} \sin \theta_g (h^x \pm i \cos \theta_g h^+) e^{\mp i\phi_g} \]

In the cylindrical coordinates, the complex differential operators \( \nabla_\pm \) become

\[ \nabla_\pm = \frac{1}{\sqrt{2}} e^{\mp i\phi} (\partial_\rho \pm \frac{1}{\rho} L_z) \quad \text{where} \quad L_z = -i\partial_\phi \text{ is the orbital angular momentum operator in the z-direction.} \]

Obviously, for the eigenstates \( L_z |l\rangle = l |l\rangle \), \( \nabla_\pm \) are functioning as ladder operators to lowers/raises the OAMs by one unit \( \hbar \).
For instance, we simply consider the $LG_{0}^{±1}$ modes, which carry $\hbar$ of OAM per photon and have a well-known donut-like intensity profile. Due to the dipole interaction, the induced mode involves one by lowering the OAM of one unit, and consequently gives rise to the vortex-free Gaussian-like beams of $|0, 0\rangle_{LG}$ and $|0, 1\rangle_{LG}$ modes, producing a bright spot in the center. There are other two high-order modes with $l = ±2$, each acquiring an extra phase factor but with opposite signs, and thus making a global rotation of the intensity pattern. In Fig.(1), we illustrate the dipole effect by an ideal numerical experiment.

The imaginary part in the complex potential Eq.(16) is a helicity dependent quadrupole interaction. In the Einstein gravity, the quadrupole moment can be expressed explicitly as

$$Q_{±2} = \left[ \frac{1}{2} h^{+} (1 + \cos^{2} \theta_g) \mp i h^{x} \cos \theta_g \right] e^{±i2\phi_g}$$

which includes two tensor polarization modes associated with massless spin-2 graviton. As expected, this quadrupole interaction will cause a transition between two OAM eigenstates subject to the selection rule $\Delta l = ±2$. Due to its helicity dependence, the resulting extra helicity modes would have opposite signs of phases for right and left circularly polarized lights respectively. Intrinsically, this chirality is arising from 'time-reversal symmetry breaking'[42].

As an example, we consider a superposition of two Laguerre-Gaussian modes with opposite OAMs $l = ±2$ and opposite circular polarizations, $|in\rangle = F_{ln}(r_{⊥}, z) \cdot (|↑\rangle e^{i2\phi} + |↓\rangle e^{-i2\phi})/\sqrt{2}$ where $|↑\rangle = (1, 0)^T$, $|↓\rangle = (0, 1)^T$ are two 2-vectors for the spin-up and -down states respectively, and the LG mode has been written as $|n, ±l\rangle_{LG} = F_{ln}(r_{⊥}, z) e^{±il\phi}$. The input beam is a vectorial optical vortex with nonuniform polarization, which can be generated by using liquid-crystal converters[43] or spatially varying dielectric gratings[44]. Applying the linear perturbation analysis, we can find an extra vortex-free optical field produced by the GW's tensor modes, $Q_2 k_z L (k_w/k_z)^2 (F_{00} + 2F_{01} + F_{02}) \times (|↑\rangle e^{i\gamma} + |↓\rangle e^{-i\gamma})$ where $Q_{±2} = Q_2 e^{±i\gamma}$ is used. Obviously, the resulting mode becomes a linear polarization beam with a relative rotation angle $\gamma$, and has a Gaussian-like transverse intensity profile, exhibiting both the central brightening and changes of polarization states. Similar to the dipole effect, the output modes also include high-order modes with $l = ±4$, and the corresponding intensity pattern will be rotated by an angle $\gamma$ that depends on both the GW’s parameters and the direction angle of lights. We demonstrate the quadrupole effect in Fig.(2), where the input light field is assumed to be a single mode $|0, 2\rangle$.

In the low energy regime, the canonical quantization of linear perturbations around the Minkowski background yields a massless spin-2 particle, called graviton[45]. Though we believe that the long-range gravity is mediated by the graviton, its existence has never been justified experimentally due to its extraordinary weakness. Measuring graviton is to make sense of its particle nature of GWs in term of wave-particle duality at the quantum level, that is,
to recognize some particle features by direct or indirect detections [46]. One theoretical possibility was suggested to detect a single graviton by the emission and absorption of gravitons [47–49]. Given the extreme weakness of gravity, the momentum or energy transfer is hardly measured by the available technology. In our work, since the SAM of photons keeps unchanged the transition, the changes of OAMs can be attributed to the helicity-2 gravitons. Hopefully, the precision quantum optical measurements may make it possible to detect the angular momentum transfer from gravitons to photons, providing evidence of graviton and unveiling the quantization of gravity.

Concluding Remarks

In this study, on using a modified definition of wavefunction of photons, we establish a 4-vector optical Dirac equation from the Maxwell theory in a general linear medium, which enables us to tackle spin-dependent phenomena in the context of vector-wave mechanics. The optical systems are found to be generally of non-Hermitian, and their $\mathcal{PT}$ symmetry relies on the properties of media. In application to the linearized gravitational wave, we found that, as a spin-degenerate material, it is typically a $\mathcal{PT}$-symmetric non-Hermitian system. Moreover, we demonstrate that there exists the spin-OAM coupling effect between GWs and structured lights, which makes photons undergo dipole and quadrupole transitions between different OAM eigenstates and produces some measurable optical features in the 2-D intensity pattern, typically, the central intensity brightening and macroscopic rotation of the intensity pattern in the transverse plane, indicating an alternative spin-orbital Hall effect of lights.

Our approach can be easily generalized to generic linear media and allows us to apply the conventional field-theory methods to probe the SOI phenomena of lights at sub-wavelength scales. Moreover, the optical Dirac equations may deepen our understanding of the spin Hall effect of light from that in electron-systems, and could have wide applications in metamaterials, photonic topological insulators, non-Hermitian photonics etc.[50–52].

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Appendix A The Ladder Relation and Perturbation Theory

In a flat space, the second term in the right side of Eq.(15) due to the space-time perturbations vanished, and thus Eq.(15) reduces to the familiar paraxial equation, which could have family solutions of structured light with OAMs. In the cylindrical coordinates, the paraxial equation has a set of solutions of the
Laguerre-Gaussian modes

\[
LG^l_n(\rho, z) = \frac{a^l_n}{w(z)} \left( \frac{\sqrt{2\rho}}{w(z)} \right)^{|l|} L^{|l|}_n \left( \frac{2\rho^2}{w(z)^2} \right) \exp \left[ -\frac{\rho^2}{w(z)^2} \right] \\
\cdot \exp \left( i\frac{k\rho^2}{2R} \right) \exp (i\ell\phi) \exp (-i\varphi(z)) \tag{A1}
\]

where \( n \) and \( l \) are the radial and the azimuthal indices, the order of the model is given by \( N = 2n + l \), the constant \( a^l_n = (2^n! / \pi (n + l)!)^{1/2} \), \( w(z) \) is the width of mode, \( R \) the radius of the wavefront curvature, and the Gouy phase factor \( \varphi(z) = (N + 1) \tan^{-1}(z/z_R) \), \( z_R = \frac{1}{2} k w_0^2 \), here \( w_0 = w(z = 0) \) is the beam waist.

For the LG modes, \(|n, l\rangle = LG^l_n(\rho, z)\), it is not difficult to work out the following ladder relations for the lowering and raising operators,

\[
\nabla_\pm |n, \pm l \rangle = k_w \left[ \sqrt{n + l} |n, \pm (l - 1) \rangle + \sqrt{n + l + 1} |n + 1, \pm (l - 1) \rangle \right] \tag{A2}
\]

\[
\nabla_\mp |n, \pm l \rangle = k_w \left[ \sqrt{n + l + 1} |n, \pm (l + 1) \rangle - \sqrt{n} |n - 1, \pm (l + 1) \rangle \right] \tag{A3}
\]

where \( k_w = 1/w_0 \), \( l > 0 \) and \( \pm \) signs are taken meantime on both sides of the equations.

We can apply the perturbation theory to solve Eqn.(15). Since the Laguerre-Gaussian modes form a complete and orthonormal set with respect to the mode indices \( n \) and \( l \) in the polar plane \( \{\rho, \phi\} \), we can make a decomposition such that

\[
\varphi_\pm(z) = \sum_{m,k} \xi_{m,k}^\pm(z) |m, k\rangle \tag{A4}
\]

Substituting this expansion Eq.(A4) into Eq.(15), we have

\[
i \frac{d\xi_{n,l}^\pm(z)}{dz} = \sum_{m,k} \langle n, l | H_\pm | m, k \rangle \xi_{m,k}^\pm \tag{A5}
\]

where

\[
H_\pm = i \left[ Q_{-1} \nabla_+ + Q_{+1} \nabla_- \right] \pm \eta \frac{1 + q_0}{k_z} [Q_{-2} \nabla_+^2 - Q_{+2} \nabla_-^2] \tag{A6}
\]

\( \xi_{n,l}^\pm(z) \) can be obtained by direct integrating over the summation in Eqn.(A5) along the propagating distance \( z \).

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Optical Dirac Equation

Fig. 1 Demonstration of the dipole signal in the GW polarization experiment - We first set up a laboratory coordinate system - the coordinate origin is located at the light source, an array of photodetectors is uniformly placed on the circle centered on the origin, the radius of the circle is L, the xy plane is the plane where the photodetectors are located, and its normal direction points to the zenith, set to the z-axis. In our ideal numerical experiment, we make use of eight detectors, the angle between any two adjacent arms is thus 45°. The split light beams injected from the light source are sent to each photodetectors along the arms. The incident GW is assumed in the direction of $\theta_g = 45^\circ$, $\phi_g = 0^\circ$, and the strains is set to $h^x/h^+ = 1$ fixed at $\theta_g = 90^\circ$, $\phi_g = 0^\circ$ in the TT gauge. Adopting the Hermite-Gauss mode $HG_{10}$, which is a superposition of two LG modes with the opposite OAMs, $l = \pm 1$, $|\text{in}\rangle = HG_{10} = \sqrt{2} (|0, +1\rangle_{LG} + |0, -1\rangle_{LG})$, the additional dipole signal under influence of GWs is composed of the vortex-free mode $\sqrt{2}k_wLQ_1 \cos \gamma (|0, 0\rangle_{LG} + |1, 0\rangle_{LG})$ and the vortex modes $k_wLQ_1 [e^{-i\gamma}|0, +2\rangle_{LG} + e^{i\gamma}|0, -2\rangle_{LG}]$ where the complex dipole momentum has been written as $Q_{\pm 1} = Q_1 e^{\pm i\gamma}$. Given the GW’s strains and incident direction, $Q_1$ and $\gamma$ vary with the azimuthal angle of optical arms. The diagram presents the false color intensity profile of the input (at the center) and output (the surrounding frames) light beams, indicating both the central brightening and global rotation of intensity patterns.
Fig. 2 The quadrupole signal in the GW polarization experiment - The laboratory coordinate system is the same as that described in Fig. (1). The incident GW is assumed in the direction of $\theta_g = 30^\circ, \phi_g = 0^\circ$, and the strains are set to $h^x/h^z = 1$. The input light beam adopts the LG mode $|\ln\rangle = |0, 2\rangle$, whose transverse intensity profile is plotted in the upper right panel. The quadrupole interaction leads to the extra OAM modes $|\text{extra}\rangle = \sqrt{2}kLzQ_2(k_w/k_z)^2[e^{-i\gamma}(|0, 0\rangle + 2|1, 0\rangle + |2, 0\rangle) - \sqrt{6}e^{i\gamma}|0, 4\rangle]$. Given the GW’s properties, $Q_2$ and $\gamma$ are functions of the direction angle $\phi$ of optical arms. In the lower left panel, $\gamma(\phi)$ are plotted for different incident polar angles as indicated by the legends. Besides, the right section (2×3 frames) illustrates the false color intensity profiles of the extra modes (upper) and the interference pattern between the input mode and the extra modes (lower), in different azimuthal angles of optical arms, from left to right corresponding to $45^\circ$, $135^\circ$, $225^\circ$ respectively.