Effect of Modified Dispersion Relations on Immirzi Parameter

Ô. Acık and Ü. Ertem

Department of Physics, Ankara University, Faculty of Sciences, 06100, Tandoğan-Ankara, Turkey

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Black hole entropy calculations which are based on counting of microstates and based on modified dispersion relations in the framework of loop quantum gravity are considered. We suggest that the inconsistency of two approaches can be explained by different ways. This inconsistency can affect the definition and constancy of the Immirzi parameter or order of the modification constants of dispersion relations. Possible results of these effects are discussed.

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I. INTRODUCTION

In Loop Quantum Gravity (LQG) [1], there is a free parameter $\gamma$, called Immirzi parameter that has no effect in the classical theory but has an effect in the quantum theory. So, it reflects the quantization ambiguity of the theory. LQG is based on a connection formulation of general relativity whose phase space variables are an $SU(2)$ connection and a densitized triad. However, there is a one parameter family of canonical transformations which lead to the same Hamiltonian formulation of general relativity. The parameter that labels the family of canonical transformations is the Immirzi parameter. Different values of $\gamma$ are reflected in the different forms of the Hamiltonian constraint. If $\gamma$ is selected as a complex number (more specifically the complex number $i$), then the Hamiltonian constraint is simpler than in the ADM formulation and quantization can be easier, but in this case, one has some extra reality constraints. The pure complexity of $\gamma$ is not a necessity and if it is selected as a real number, then the Hamiltonian constraint is more complicated, but it is still manageable for quantization and there is no reality constraints in this case [1, 2]. However, in quantum theory $\gamma$ does not vanish and still present in the spectrum calculations of operators. In LQG, geometric operators such as area and volume have discrete eigenvalues and $\gamma$ appears in the spectrum of these operators. So, finding the value of $\gamma$ means determining the area and volume quantums. Hence, there is a quantization ambiguity for quantum gravity and choosing a value of the Immirzi parameter needs an explanation, namely it must be fixed by theoretical or experimental ways.

Immirzi parameter appears also in the computation of black hole entropy in the framework of LQG, because of the relation between the area and entropy of a black hole. So, one can determine the value of $\gamma$ by comparing black hole entropy-area relation found in LQG to the semiclassical Bekenstein-Hawking (BH) entropy-area relation [3]. Black hole entropy-area relation have been found by counting of microscopic states for a fixed area value, and besides the linear area term, there is also an $\ln$ correction term in the found relation [4, 5].

On the other hand, in LQG, some theoretical calculations reveal the presence of corrections to energy-momentum relations, so it implies some modifications of dispersion relations [6, 7, 8, 9]. This kind of modification effects may be observed at gamma ray and ultra high energy cosmic ray threshold anomalies [10]. Modified dispersion relations can be understood in the framework of Deformed Special Relativity (DSR) which refers that there is an observer independent invariant energy scale, Planck energy, besides the invariant speed of light [10, 11, 12, 13, 14, 15, 16]. But modifications of dispersion relations implies some modifications to particle localization limit and this results some corrections to black hole entropy-area relation [17]. In this case also, there is an $\ln$ correction term, but this time a term proportional to square root of area also exists. So, for consistency one must consider the entropy-area relation with correction terms in determining the Immirzi parameter $\gamma$. This indicates some possibilities like restrictions on modifications to dispersion relations, or different values of $\gamma$ for different scales.

In this paper, by comparing the two different approaches of finding black hole entropy in the framework of LQG, we discuss the explanations for the inconsistencies between the two approaches and find some possibilities about non-constancy of Immirzi parameter and orders of modification constants of dispersion relations. Organization of the paper is as follows. In section 2, we summarized the procedure of finding Immirzi parameter with counting microscopic states of a black hole in the framework of LQG. Section 3 discusses how the entropy-area relation can be modified with modification of dispersion relations. In section 4, we find an equation for $\gamma$ with considering modified entropy-area relation. This section also includes some limits to coefficients of modification terms of dispersion relations for the consistency with fixed $\gamma$. Some all order modifications of dispersion relations are also discussed for comparison with coefficient limits. In section 5, possible effects of scale dependence of $\gamma$ are ar-
wed and section 6 concludes the paper.

II. IMMIRZI PARAMETER FROM BLACK HOLE ENTROPY

Black holes in general relativity obey some laws that resemble the thermodynamical principles. In this sense, the area of the event horizon is related to entropy. The Bekenstein-Hawking formula gives the entropy of a black hole which is proportional to horizon area of the black hole $A$;

$$ S = \frac{A}{4L_p^2} $$

where $L_p$ is the Planck length. A quantum theory of gravity must provide a mechanism for microscopic states of a black hole which explains this entropy relation. In LQG framework, the fixed horizon area of a black hole $A$ can be obtained from different intersections of edges of a spin network with the horizon. Spin networks are the basis for kinematical Hilbert space of LQG and they are eigenstates of the geometric operators. That different possibilities of intersections constitute the microscopic states of a black hole. Edges of a spin network are labeled by $SU(2)$ representations $j = 1/2, 1, 3/2,...$. So, different microscopic states represent the different intersections with different spin labels which result the same area value. Area operator has discrete eigenvalues and the area spectrum includes the Immirzi parameter $\gamma$;

$$ A = 8\pi\gamma L_p^2 \sum_i \sqrt{j_i(j_i+1)} $$

where the sum is over intersections. Thus $\gamma$ will appear in entropy-area relation and can be fixed by comparing with BH entropy for large area values.

Calculations about counting of microscopic states of a black hole has been achieved by several people [4, 5]. The result is that entropy-area relation includes an $ln$ correction term;

$$ S = \frac{\gamma_0}{\gamma} \frac{A}{4L_p^2} - \frac{1}{2} \ln(\frac{A}{L_p^2}) + O(\frac{L_p^2}{A}) $$

where $\gamma_0$ satisfies the equation;

$$ \sum_i (2j_i + 1) \exp(-2\pi\gamma_0 \sqrt{j_i(j_i+1)}) = 1 $$

The solution of this equation can be found approximately as $\gamma_0 = 0.27398...$. By comparing with BH entropy for large $A/L_p^2$ values, it can be seen that $\gamma$ must be equal to $\gamma_0$. Fixing $\gamma$ means fixing the quantum of area, and one can find from (2) that minimum possible area value of a surface. But this determination of $\gamma$ is valid only for large area values. If it is also valid for small area values then there can not be correction terms rather than the $ln$ term to entropy in calculations for finding entropy-area relations by using different methods. But in the next section we will see that if modified dispersion relations are considered there is a correction term to entropy which is proportional to square root of area and this will effect the fixing of $\gamma$.

III. BLACK HOLE ENTROPY FROM MODIFIED DISPERSION RELATIONS

Several theoretical calculations about light propagation and neutrino propagation in LQG [6, 7, 8, 9] predict that the usual relation between energy and momentum which comes from special relativity, may be modified at Planck scales in the form of

$$ E^2 = p^2 + m^2 + \alpha_1 L_p E^3 + \alpha_2 L_p^2 E^4 + O(L_p^3 E^5) $$

where $\alpha_1$ and $\alpha_2$ are constants of order one. This kind of modification of dispersion relations can be explained by alternative possibilities [10]. Some of them are: (i) No effect of Planck scale phenomena can be observed in low energies and hence modification of dispersion relations has no results for observable phenomena, (ii) Lorentz invariance breaks down and there is a preferred frame at the Planck scale, (iii) Relativity of inertial frames maintained but Planck length or Planck energy becomes an observer independent quantity. This possibility is called Deformed Special Relativity (DSR). Experimentally, modification effects of dispersion relations may be observed by gamma ray and ultra high energy cosmic ray thresholds [10].

Such a modification causes an effect to the Planck scale particle localization limit [17]. An absolute limit on the localization of a particle of energy is given by $E \geq \frac{1}{\delta x}$. But, if one considers (5), then particle localization limit can be found as follows;

$$ E \geq \frac{1}{\delta x} - \alpha_1 \frac{L_p}{(\delta x)^2} + (\frac{11}{8} \alpha_1^2 - \frac{3}{2} \alpha_2) \frac{L_p^2}{(\delta x)^3} + O(\frac{L_p^3}{(\delta x)^4}) $$

The particle localization limit must be considered to derive the BH entropy-area relation. So if (6) is valid then black hole entropy relation will change because of modification terms. This has been calculated in [17] and found that modified entropy is

$$ S \approx \frac{A}{4L_p^2} + \sqrt{\frac{A}{L_p}} + \frac{\alpha_1}{\sqrt{2}} + \frac{3}{2} \alpha_2 - \frac{11}{8} \alpha_1^2 \pi \ln \frac{A}{L_p} $$

If both $\alpha_1$ and $\alpha_2$ are vanish then entropy is equal to BH entropy. If only $\alpha_1$ vanish then there is only the $ln$ correction term and that is consistent with the entropy corrections which are found from counting of microscopic states (which is mentioned in [17, 18]), but this correspondence fixes the value of $\alpha_2$. Generally if $\alpha_1$ and $\alpha_2$ are different from zero then there is a correction term which is proportional to square root of area. From these discussions one can conclude that the $\alpha_1$ coefficient of
modified dispersion relations must be zero, but we will see in the next section that this is not the only possibility. On the other hand, the existence of the square root area term will restrict the order of $\alpha_1$ because of the constancy of the Immirzi parameter.

IV. IMMIRZI PARAMETER FROM MODIFIED BLACK HOLE ENTROPY

We have seen that there are two manifestations of black hole entropy in the framework of LQG. One from counting of microstates and one from modification of dispersion relations. Both includes a leading term corresponding to BH entropy and an $\ln$ correction term. But, while there is a square root of area term in the modification of dispersion relations approach, there is no such a term in the counting of microstates approach. For the consistency of the two approaches, these two entropy relations must be equal.

The second term of equation (3) and the third term of equation (7) are $\ln$ correction terms, so we must equalize the coefficients of these terms and we find a relation between $\alpha_1$ and $\alpha_2$ modification constants of dispersion relations

$$12\alpha_2 - 11\alpha_1^2 = -\frac{4}{\pi}. \quad (8)$$

If there is no Planck order modification to dispersion relations namely $\alpha_1 = 0$, so two entropy relations are consistent, then $\alpha_2$ must be equal to $-\frac{1}{12\pi}$. On the other hand, if $\alpha_1 \neq 0$ then we must equalize the first term of (3) and the first two terms of (7),

$$\frac{\gamma_0}{\alpha} = \frac{A}{4L_p^2} + \alpha_1 \sqrt{\frac{\alpha}{L_p^2}}. \quad (9)$$

In this case we can not fix the value of Immirzi parameter to $\gamma_0$. By using (2) and the definition $\sum \sqrt{j_i(j_i+1)} = J$, one can find that $\gamma$ is equal to

$$\gamma = \left[\frac{1}{2J\gamma_0}(\alpha_1 \pm \sqrt{\alpha_1^2 + 2J\gamma_0})\right]^{-2}. \quad (10)$$

So, $\gamma$ is dependent to $\alpha_1$, and it is also dependent to $J$, but this means that the value of the Immirzi parameter changes with the number of intersections of edges, hence with the scale determined by the area of the corresponding surface. However, if $J^{1/2} \gg \alpha_1$ then (10) transforms to $\gamma \simeq \gamma_0$, namely for the large area values $\gamma$ goes to $\gamma_0$. This is expected from the counting of microstates approach. But for the small values of $J$, $\gamma$ is changing, this is a contradiction with the constancy of $\gamma$. If it is constant, then it must be equal to same quantity for all area values. The smallest value of $J$ is comes from an edge with $j = 1/2$ and it is $J_{min} = \sqrt{3}/2$. So, if $\gamma$ is a constant and is equal to $\gamma_0$, then $\alpha_1 \ll J_{min}$, and this means that

$$\alpha_1 \ll 1. \quad (11)$$

So, for the constancy of $\gamma$ there is no need to $\alpha_1 = 0$, but it must be much smaller than 1.

On the other hand, if $\alpha_1$ is order one, then $\gamma$ can not be a constant for all area values, and it changes with $J$ and $\alpha_1$. This means that for small area values, the area spectrum must have additional dependence on $j_i$'s and also depends on $\alpha_1$:

$$A = 16\pi L_p^2 \frac{J}{(\alpha_1 \pm \sqrt{\alpha_1^2 + 2J\gamma_0})^2}. \quad (12)$$

For large area values that is $J^{1/2} \gg \alpha_1$, (12) converges to (2) where $\gamma$ equals to $\gamma_0$. Then, in the small area regime, the area spectrum must depends on the different $\gamma$-sectors of the theory. Hence, if $\alpha_1$ is order one, then $\gamma$ will be a scale-dependent parameter, and its values are exactly determined by $\alpha_1$ and the scale of $J$.

A. Comparisons with Some All-Order Dispersion Relations

Some all-order modified dispersion relations have been considered in the frameworks of $\kappa$-Minkowski space-time and Deformed Special Relativity \cite{17}. Various models predict different $\alpha_1$ and $\alpha_2$ coefficients. For consistency, this coefficients in the models must satisfy some requirements mentioned above.

In the framework of $\kappa$-Minkowski space-time, dispersion relations are given by

$$\cosh(E/E_p) - \cosh(m/E_p) - \frac{p^2}{2E_p^2} \exp(E/E_p) = 0.$$ 

In this case $\alpha_1 = -1/2$. So if this theory is true, then in the lack of $\sqrt{\alpha}$ term in black hole entropy from counting of microstates, area spectrum must change with (12), namely depends on the different $\gamma$-sectors of the theory.

Another possibility for dispersion relations is given by

$$\cosh(\sqrt{2}E/E_p) - \cosh(\sqrt{2}m/E_p) - \frac{p^2}{E_p^2} \cosh(\sqrt{2}E/E_p) = 0.$$ 

This case has $\alpha_1 = 0$ and $\alpha_2 = -5/18$. By vanishing of $\alpha_1$, this is consistent with two different entropy calculations, but the value of $\alpha_2$ is inconsistent with predictions from consistency of entropy relations discussed above.

In the case of Deformed Special Relativity, dispersion relations are given by

$$\frac{E^2}{(1 - E/E_p)^2} - \frac{p^2}{(1 - E/E_p)^2} - m^2 = 0.$$ 

This is the case of both $\alpha_1$ and $\alpha_2$ vanish, but still there are some modifications to dispersion relations. Vanishing of $\alpha_2$ is inconsistent with $\ln$ correction terms in entropy relation found from counting of microstates.

The true modification of dispersion relations can only be decided from experiments and observations which are
mentioned in [10]. Then, one can know the exact modification coefficients and compare the results with the consistency conditions mentioned above. On the other hand, if \( \gamma \) is scale-dependent, then this must be observed by future measurements of different scale area values which then must have values of different \( \gamma \)-sectors of the quantum theory.

V. POSSIBLE EFFECTS OF SCALE-DEPENDENT IMMIRZI PARAMETER

In the presence of the first order Planck scale modifications to the dispersion relations, Immirzi parameter \( \gamma \) must satisfy (10) for the consistency of entropy relations that are calculated by two different ways. This means that \( \gamma \) depends on \( J \) and has a scale dependence. Scale dependence of \( \gamma \) affects the area spectrum and area eigenvalues must have an extra \( J \) dependence. So, for small scales area spectrum changes with different \( \gamma \) values, but for \( J^{1/2} \gg \alpha_1 \) area eigenvalues are only affected by multiplication with \( \gamma_0 \). These are also relevant for the spectrum of volume and length operators, since they also depend on \( \gamma \), and are affected similarly by changing of \( \gamma \).

On the other hand, \( \gamma \) enters the classical theory by Holst’s modification of Hilbert-Palatini action;

\[
S_H = - \frac{1}{32\pi G} \int (R_{ab} \wedge e^{ab} - \Lambda/6)  \star 1 - \frac{2}{\gamma} R_{ab} \wedge e^{ab} \tag{13}
\]

where \( \star \) is the Hodge star operator, \( e^a \) is 1-form basis, \( R_{ab} \) is curvature 2-form and \( \Lambda \) is the cosmological constant. The last term can be written as a sum of a torsion square term and an exact term which is topological Nieh-Yan class \( R_{ab} \wedge e^{ab} = T^a \wedge T_a - d(e^a \wedge T_a) \). Second term is a boundary term, so \( \gamma \) controls the width of the fluctuations of the torsion. The mean value of torsion is zero (in the non-existence of matter), but it can fluctuate about the mean value. So, scale dependence of \( \gamma \) means scale dependence of the width of fluctuations of torsion at the quantum level.

Coupling of spinors with (13) gives non-zero torsion and Immirzi parameter has an effect on non-minimal fermion interaction term. The coupling constant is dependent on \( \gamma \) [21, 22], but it is shown in [23] that if the inverse of the coupling constant is equal to \( \gamma \) then the last term of (13) and the non-minimal coupling term together turn to a boundary term. So in this case \( \gamma \) has no effect on classical theory. In [24], it is argued that coupling (16) with quadratic spinor Lagrangian indicates that \( \gamma \) is the ratio between scalar and pseudo-scalar contributions in the theory. With scale-dependence of \( \gamma \), this ratio also has scale dependence.

Another appearance of \( \gamma \) is in Loop Quantum Cosmology (LQC) [25]. Because of the volume operator has a dependence on \( \gamma \), operator of the inverse scale factor \( a^{-3} \) has also a dependence on \( \gamma \). The density operator \( d = a^{-3} \) can be constructed from the inverse scale factor [26]:

\[
d_j(a) = a^{-3} p(\frac{3a^2}{\gamma L_{j}^2})^6 \tag{14}
\]

where \( p(q) \) is a function derived in [27]. If \( a^2 \ll \frac{1}{\gamma} L_{j}^2 \) then \( d_j(a) \sim a^{12} \) and if \( a^2 \gg \frac{1}{\gamma} L_{j}^2 \) then \( d_j(a) \sim a^{-3} \). This is a possible explanation for the inflationary phase in the early universe without using scalar fields. But if \( \gamma \) changes with \( j \) like (12) then (17) has an extra \( j \) dependence and this may effect the early evolution of the universe in the framework of LQC.

VI. SUMMARY AND CONCLUSION

Immirzi parameter can be calculated from counting of microstates of a black hole by comparing the found entropy relation with the BH formula. In counting of microstates approach, the entropy has an \( ln \) correction term. In large scales, this correction term is negligible and \( \gamma \) is strictly equal to a number shown as \( \gamma_0 \). On the other hand, black hole entropy is also calculated from dispersion relations and modification of dispersion relations induces some modifications to BH entropy. But, in this case one has an additional correction term which is proportional to square root of the area besides the \( ln \) correction term. For consistency, these two entropy relations found from different approaches must coincide. Comparing the two enotypes indicates some possibilities about the Immirzi parameter and order of the modification constants of the dispersion relations. These possibilities are as follows:

- \( \alpha_1 \) must be zero, and hence no Planck order modifications to the dispersion relations, so two entropy calculations are consistent, but this time \( \alpha_2 \) must be equal to \( -\frac{1}{\gamma} \).
- \( \alpha_1 \) can be different from zero, but must be \( \ll 1 \), then two approaches are consistent, and \( \gamma \sim \gamma_0 \).
- If \( \alpha_1 \sim 1 \), then the calculations for counting of microstates of a black hole must be modified with a square root of area term.
- If \( \alpha_1 \sim 1 \) and counting of microstates approach is right, then \( \gamma \) must be scale-dependent and hence it has different values for small scales and converges to \( \gamma_0 \) for large area values.

Each of these possibilities give rise to the consistency of the entropy relations. The last possibility has some effects. If \( \gamma \) changes with scale, then spectra of area and volume operators have an extra \( j \) dependence. It effects also width of torsional fluctuations. Varying of \( \gamma \) changes the spectrum of \( d_j \) operator in LQC, and effects the early evolution of the universe. The correct case about the consistency must be decided by the near future experiments.
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