Torsion of anisotropic and composite cylindrical rod

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Abstract. In work the limit state of cylindrical and prismatic rods from anisotropic ideal rigid-plastic material is investigated under torsion for arbitrary condition of plasticity; and the torsion of anisotropic cylindrical rod with elliptic section and rod with piecewise smooth cross-section contour under the condition of Mises-Hill plasticity is considered; the limit state of a composite cylindrical rod with elliptical cross-section is studied under torsion.

1. Introduction
Torsion is a type of deformation of solids characterized by mutual rotation of its cross sections under the influence of moments operating in these sections. Torsion of rods is quite common in engineering practice, especially in mechanical engineering. Shafts of engines and machines, axles of wagons and locomotives, helical springs, etc. work by torsion. The theory of torsion of isotropic and anisotropic rods belongs to the number of developed sections of the theory of an ideal rigid-plastic solid. At the same time, researches on the theory of torsion of non-uniform rods are not enough. In particular, researches on the theory of torsion of isotropic and anisotropic ideal rigid-plastic rods are presented in works [1]-[5]. Torsion of non-uniform and compound rods is considered in works [6], [7], [8], [9]. The results obtained in the work can be applied in mechanical engineering in determining of ultimate loadings on the rod as a result of torsion, in solving of new problems of the theory of limit state, in calculations of the bearing capacity of various solids and structures.

2. Materials and methods
The torsion of rods made of an ideal rigid-plastic material is considered in this work. It is assumed that the plastic properties of the rod material depend on the direction or coordinates of the point, that is, the rod material has anisotropy or heterogeneity. In the work the anisotropy of the material of the rod is determined by the condition of the plasticity of Mises-Hill [10]. The rod of heterogeneous material is represented by a composite rod, when different plasticity conditions are true in different parts of the rod.

The semi-inverse Saint-Venant’s method was used in the construction of the solution of the basic relations. Integrals of basic relations are determined by the method of characteristics.
3 Results and discussion

3.1 Torsion of an anisotropic rod under arbitrary plasticity condition
Consider a cylindrical or prismatic rigid-plastic rod oriented in a rectangular coordinate system $xyz$. Generatrices of the rod are parallel to the axis $z$. Assume that the rod consists of anisotropic ideal-plastic material. The rod is twisted around its axis, the lateral surface of the rod is free from loadings.

Let us assume that the stress state arising in the rod is characterized by following values of the tension components:

$$
\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0, \\
\tau_{xz} = \tau_{xz}(x, y), \tau_{yz} = \tau_{yz}(x, y).
$$

(1)

In general case, the plasticity condition has the form

$$
f(\tau_{xz}, \tau_{yz}) = 0,
$$

(2)

and the only equation of equilibrium takes the form

$$
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0.
$$

(3)

As it follows from our assumptions, the equality is carried out on the contour of cross section of the rod

$$
\frac{dy}{dx} = \frac{\tau_{yz}}{\tau_{xz}}.
$$

(4)

Taking into account (2), the characteristics of equation (3) are determined by the ratio

$$
\frac{dy}{dx} = \frac{\partial f}{\partial \tau_{xz}}
$$

and have the form

$$
px + qy = \text{const},
$$

(5)

and along the characteristics we have the ratios

$$
\tau_{xz} = c_1 = \text{const}, \tau_{yz} = c_2 = \text{const},
$$

(6)

where $p = \frac{\partial f}{\partial \tau_{xz}}(c_1, c_2), q = \frac{\partial f}{\partial \tau_{yz}}(c_1, c_2), f(c_1, c_2) = 0$.

Thus, according to (5), (6) and (7), characteristics of the ratio (3) are lines orthogonal to the gradient vector to the yield curve (2) at the corresponding point.

3.2 Torsion of an anisotropic rod with elliptical section
Consider an anisotropic cylindrical rod, which contour $L$ of the cross section is an ellipse (figure 1a)

$$
L: \frac{x^2}{a_0^2} + \frac{y^2}{b_0^2} = 1.
$$

(8)

Assume that the plasticity condition (2) is given in the form (figure 1b)

$$
\frac{\tau^2_{xz}}{a^2} + \frac{\tau^2_{yz}}{b^2} = 1.
$$

(9)
Figure 1. Cross section of an anisotropic rod with characteristic and yield curve.

Let the point \((x_0, y_0) \in L\), and the characteristic \(l\) of the relation (2) passes through it. According to (6), (7), (8) and (9) the equation of the family of characteristics \(l\) has the form

\[ a^2b^2x_0(y - y_0) - a_0^2b^2y_0(x - x_0) = 0, \]  

(10)

Figure 2. Field of characteristics and envelope.
and stress components are given along the characteristics in the form

$$
\tau_{xz} = - \frac{a_0^2 ab y_0}{\sqrt{a_0^2 b^2 y_0^2 + b_0 a^2 x_0^2}}, \quad \tau_{yz} = \frac{b_0^2 ab x_0}{\sqrt{a_0^2 b^2 y_0^2 + b_0 a^2 x_0^2}}.
$$

(11)

The family of characteristics (10) has an envelope (Fig. 2)

$$(a_0 b^2 x)^2 + (b_0 a^2 y)^2 = (a_0^2 b^2 - a^2 b_0^2)^2.$$

(12)

In particular, if the equality $\frac{a}{b} = \frac{a_0}{b_0}$ is satisfied, then the ratios (10) and (11) take the form

$$
y = \frac{y_0}{x_0} x,
$$

(13)

$$
\tau_{xz} = - \frac{a y_0}{b_0}, \quad \tau_{yz} = \frac{b x_0}{a_0}.
$$

(14)

The envelope (12) degenerates to the origin. For figure 2 characteristics are drawn in solid lines, and the envelope of characteristics is dotted. Bold line – break line voltages.

3.3 Torsion of an anisotropic rod with a piecewise smooth cross-section contour

Consider an anisotropic rigid-plastic rod with a piecewise smooth cross-section contour ABCD (figure 3). The cross-section contour of the rod consists of two straight sections BC, DA, determined by the equations: $y = \pm c$, and two arcs AB, CD of the ellipse, the equation of which has the form (8).

![Figure 3](image-url)

**Figure 3.** Cross section of the rod with a piecewise smooth contour.

The rod is twisted around the z axis by equal and opposite pairs of forces. The lateral surface of the rod is considered free from loads.

The stress state of the rod is determined by the relations (1), the equilibrium equation (3), and the condition of plasticity (9). Assume that $\frac{a}{b} = \frac{a_0}{b_0}$. Taking into account the assumptions made and according to (6), (7), (8) and (9) we have two families of characteristics.

The first family of characteristics has the equation

$$
x = \text{const}.
$$

(15)
Along this family of characteristics, stress components have the form
\[ \tau_{xz} = \pm a, \quad \tau_{yz} = 0. \] (16)

The second family of characteristics is determined by the ratio (13), along which the stress components have the form (14).
In this connection, there is uncertainty in the determination of the stress state of the rod, which leads to the lines of stress rupture, emerging from the points A, B, C and D (figure 4). The lines of stress rupture BM and CN are determined by the ratio
\[ b_0^2 x^2 = a_0^2 (b_0 - d)(2y + b_0 - d), \] (17)
and the lines of stress rupture AM and DN - from the equation
\[ b_0^2 x^2 = a_0^2 (b_0 - d)(-2y + b_0 - d). \] (18)

![Figure 4. Field of characteristics and line of stress rupture.](image)

3.4 Torsion of a composite rod with elliptical section
Consider a composite cylindrical rigid-plastic rod. Section contour of the rod by the plane \( z = \text{const} \) is determined by the ratio (8).
Assume that section of the rod consists of two anisotropic areas separated by AOB broken line (figure 5a, 5b).
Figure 5. Cross section of a composite rod.

The rod is twisted around the z axis by equal and opposite pairs of forces. Lateral surface of the rod is considered free from loadings.

Stress state of the rod is determined by the ratios (1) and the plasticity conditions

\[ \frac{\tau_{xx}}{a_1^2} + \frac{\tau_{yz}}{b_1^2} = 1 \]  

in area I,

\[ \frac{\tau_{xx}}{a_2^2} + \frac{\tau_{yz}}{b_2^2} = 1 \]  

in area II, by equilibrium equation (3), where \( b_1 \leq b_2, a_1 \leq a_2 \).

According to (5), characteristics of the ratio (3) in each area are straight lines orthogonal to the gradient vector to the curves (19) and (20), respectively. The vector of tangent stress:

\[ \vec{\tau} = \tau_{xx} \vec{i} + \tau_{yz} \vec{j}, \]  

does not change along the characteristics and is directed along the tangent to the contour (8) of cross-section of the rod.

Let

\[ \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_0}{a_0}. \]  

Then the equation of characteristics in both areas has the form (13).

According to (14), the vector of tangent stress: \( \vec{\tau} \) in areas I and II is respectively determined by the ratios

\[ \vec{\tau} = - \frac{a_1}{b_0} y_0 \vec{i} + \frac{b_1}{a_0} x_0 \vec{j}, \]  

\[ \vec{\tau} = - \frac{a_2}{b_0} y_0 \vec{i} + \frac{b_2}{a_0} x_0 \vec{j}. \]  

A jump of the tangent stresses is inevitable on the line of heterogeneity AOB. Therefore, vector of tangent stress: \( \vec{\tau} \), and accordingly characteristics of the ratio (4) change their direction when passing through the broken line AOB. This leads to additional lines of stress: rupture in area II (figure 6a, 6b).

Let the point A have coordinates \((x_1, y_1)\) and point B has coordinates \((x_2, y_2)\). Then

\[ \tan \varphi_1 = \frac{y_1}{x_1}, \tan \varphi_2 = \frac{y_2}{x_2}. \]
where $\varphi_1$ is an angle formed by the line of heterogeneity OA with axis Ox, $\varphi_2$ is an angle formed by the line of heterogeneity OB with axis Ox. Suppose that:

$$\gamma = \varphi_2 - \varphi_1, \quad 0 < \gamma < 2\pi$$

Figure 6. Field of characteristics and lines of stress rupture.

Then, vector of tangent stress $\vec{\tau}_2$ in the area bounded by a closed curve OACO is determined by the ratio

$$\vec{\tau}_2 = (mx_1 - \frac{a_1}{b_0}y_1)i + \left(\frac{b_1}{a_0}x_1 + my_1\right)j,$$  \hspace{1cm} (26)

where $m = \frac{\sqrt{a_2^2 - a_1^2}}{a_0} = \frac{\sqrt{b_2^2 - b_1^2}}{b_0}$.

Characteristics in this area are given by the equation

$$b_2^2 \left(mx_1 - \frac{a_1}{b_0}y_1\right)x + a_2^2 \left(\frac{b_1}{a_0}x_1 + my_1\right)y = \text{const}.$$  \hspace{1cm} (27)

The line of stress rupture AC is determined from the differential equation

$$\left(\frac{b_1}{b_0^2 x^2 + a_0^2 y^2} - \frac{b_1}{a_0}x_1 + my_1\right)dx + \left(\frac{a_2 a_0 y}{b_0^2 x^2 + a_0^2 y^2} - \left(mx_1 - \frac{a_1}{b_0}y_1\right)\right)dy \equiv 0.$$  \hspace{1cm} (28)

According to (28), the equation of the line of stress rupture AC has the form

$$\frac{b_2}{b_0} \sqrt{b_0^2 x^2 + a_0^2 y^2} - \left(\frac{b_1}{a_0}x_1 + my_1\right)x + \left(mx_1 - \frac{a_1}{b_0} y_1\right)y = a_0(b_2 - b_1).$$  \hspace{1cm} (29)

The vector of tangent stress $\vec{\tau}_2$ in the area bounded by a closed curve OBCO is determined by the ratio

$$\vec{\tau}_2 = (-mx_2 - \frac{a_1}{b_0}y_2)i + \left(\frac{b_1}{a_0}x_2 - my_2\right)j.$$  \hspace{1cm} (30)

Characteristics in this area are given by the equation

$$b_2^2 \left(-mx_2 - \frac{a_1}{b_0}y_2\right)x + a_2^2 \left(\frac{b_1}{a_0}x_2 - my_2\right)y = \text{const}.$$  \hspace{1cm} (31)

The line of stress rupture BC is determined from the differential equation
According to (32), the equation of the line of stress rupture has the form

\[
\frac{b_2 b_0 x}{\sqrt{b_0^2 x^2 + a_0^2 y^2}} + \left( -\frac{b_1}{a_0} x_2 + m y_2 \right) dx + \left( \frac{a_2 a_0 y}{\sqrt{b_0^2 x^2 + a_0^2 y^2}} - \left( m x_2 + \frac{a_1}{b_0} y_2 \right) \right) dy = 0. \quad (32)
\]

Point C is determined from the relations (29) and (33). In the case where the point C lies inside the contour, the segment OC is the stress rupture line. For fig. 6 the characteristics of the basic ratios are drawn in thin lines, and the stress rupture lines are drawn in bold lines.

4 Conclusion

Integrals describing the limit state of an anisotropic rigid-plastic rod at torsion for the arbitrary plasticity condition are obtained.

The stress state at torsion of an anisotropic rod with elliptic section as well as the rod with a piecewise smooth cross-section contour for Mises-Hill plasticity condition is described: the stress components are determined, the characteristics of the basic relations, the envelope of the family of characteristics and the stress rupture line are found.

The limit state of a rigid-plastic composite rod with elliptical section is investigated: the field of characteristics of the basic ratios is constructed, the ratios along the characteristics and the lines of tension rupture are found.

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