BCS Diquark Condensation in the 3 + 1d Lattice NJL Model

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We present preliminary evidence of BCS diquark condensation in the 3 + 1 dimensional Nambu–Jona-Lasinio (NJL) model at non-zero chemical potential ($\mu$) on the lattice. Large $N$ results are used to match the model’s parameters to low energy, zero density phenomenology. A diquark source $j$ is added in a partially quenched approximation to enable the measurement of lattice diquark observables. In particular measurements are made of the diquark condensate and susceptibilities as functions of $j$ which support the existence of a BCS phase at high $\mu$.

1. Introduction

Despite recent advances in the study of QCD at small but non-zero density on the lattice [1] the sign problem still persists in the study of dense, low temperature, exotic matter phases. One approach is to study four-fermi theories which, unlike SU(2)$^C$ QCD (in which the $qq$ baryons are bosonic), can display Fermi-surface effects such as BCS superfluidity. Although these models exhibit no confinement, one variant, the SU(2)$^L \times$SU(2)$^R$ NJL model, possesses the same global symmetries as 2 flavour QCD, which should allow us to investigate the phase structure of this theory.

2. Lattice Model and Parameter Choice

The model studied here, which uses staggered lattice fermions, is the 3+1 dimensional version of that studied in [2] and references therein. In particular the model has the action

$$S = \Psi^\dagger A \Psi + \frac{2}{g^2} \sum_{\vec{x}} \left( \sigma^2 + \vec{\pi} \cdot \vec{\pi} \right),$$

where the bispinor $\Psi$ is written in terms of independent isospinor fermionic fields via $\Psi^\dagger = (\chi, \chi^\dagger)$, and the auxiliary bosonic fields $\sigma$ and $\vec{\pi}$ are introduced in the standard way. Written in the Gor’kov basis the fermion matrix is

$$A = \frac{1}{2} \begin{pmatrix} j\tau_2 & M \\ -M^\dagger & j\tau_2 \end{pmatrix},$$

where the matrix $M$ is identical to the 3+1d form of that given in [2]. The diquark sources $j$ and $\vec{j}$ differ from those in [2] by a factor of 2 which allows us to identify them with Majorana masses.

Being a theory of point-like interactions the NJL model has no continuum limit for $d \geq 4$ leaving the physics sensitive to simulation parameters. In order to ensure we simulate in a physical regime these parameters are fitted to physical observables calculated in the large $N$ limit at $\mu = 0$ [3]. In particular we evaluate dimensionless ratios between the pion decay rate $f_\pi$, the constituent quark mass $m^*$ and the pion mass $m_{\pi}$, and fit to the phenomenological values of 93MeV, 350MeV and 138MeV respectively. Using these fits we determine the bare quark mass $a m_0 = 0.002$ and the inverse coupling $a^2/g^2 = 0.565$. We also extract the large $N$ lattice spacing $a^{-1} = 1076$MeV, which allows us to present results in physical units.

3. Preliminary Results

The above model was simulated with a Hybrid Monte Carlo algorithm on $L_s^3 \times L_t$ lattices with $L_s = L_t = 12, 16$ and $20$. A partially quenched approximation was used in which the diquark sources are set to zero during the update of the auxiliary fields but are non-zero during the measurement of diquark observables.

In order to study chiral symmetry breaking we...
measure the chiral condensate $\langle \chi \chi \rangle$ and baryon number density $n_B$, which are defined by

$$\langle \chi \chi \rangle = \frac{1}{V} \frac{\partial \ln Z}{\partial m_0}$$

(3)

and

$$n_B = \frac{1}{2V} \frac{\partial \ln Z}{\partial \mu}.$$ 

(4)

Plots of $\langle \chi \chi \rangle$ and $n_B$ as functions of chemical potential are presented in Figure 1. Both the large $N$ solutions and the lattice data have been extrapolated to the limit $V^{-1} \to 0$. Chiral symmetry is broken in the low $\mu$ phase, as signalled by a non-zero chiral condensate, and is approximately restored as $\mu$ is increased through what appears to be a crossover. The order of the chiral phase transition in the NJL model is strongly dependent on the parameters used [3].

In the diquark sector, the operators

$$qq_\pm(x) = \chi^{tr} \frac{\tau_2}{4} \chi(x) \pm \chi \frac{\tau_2}{4} \chi^{tr}(x)$$

(5)

allow one to define the diquark condensate as

$$\langle qq_+ \rangle = \frac{1}{V} \frac{\partial \ln Z}{\partial j_+},$$

(6)

where $j_\pm = j \pm 7$.

Figure 2 shows the condensate plotted as a function of $j$ extrapolated to $L_t^{-1} \to 0$ (we simulate with $j = 7$ throughout). For $\mu = 0$ a quadratic fit through the data is consistent with the condensate vanishing as $j \to 0$. For high $\mu$ the fit is consistent with a non-zero diquark condensate which is similar in magnitude to the chiral condensate in the vacuum. We believe that finite volume effects cause the points with $j \leq 0.2$ to fall below the curve. In particular, in the high $\mu$ phase as $j \to 0$ we expect an exact Goldstone mode which leads the correlation length to diverge. For this reason these points have been ignored during the fits.

Finally, if we define susceptibilities

$$\chi_\pm = \sum_x \langle qq_+(0)qq_\pm(x) \rangle,$$

(7)

we can clearly distinguish between the two phases.
by studying the ratio
\[ R = \lim_{j_+ \to 0} \frac{\chi_+}{\chi_-}. \]  
(8)

If a U(1) baryon number symmetry is manifest, the two susceptibilities should be identical up to a sign factor and the ratio should equal 1. If the symmetry is broken, the Ward identity
\[ \chi_-|_{j_+ = 0} = \frac{\langle qq_+ \rangle}{j_+} \]  
(9)

predicts that \( \chi_- \) should diverge and the ratio should vanish.

Figure 3 shows the susceptibility ratio \( R \) vs. \( j \) for high and low \( \mu \).

Figure 3 shows the susceptibility ratio as a function of \( j \), again extrapolated to \( L_t^{-1} \to 0 \). A linear fit through the \( \mu = 0 \) data is consistent with a manifest U(1)\(_B\) symmetry. For \( \mu a = 1.0 \) we see markedly different behaviour, with a fit (again discarding \( j \leq 0.2 \)) suggesting an intercept close to zero and therefore a broken baryon number symmetry.

4. Summary and Outlook

We have provided preliminary evidence of a BCS superfluid phase in the 3+1d NJL model at high \( \mu \), with an order parameter approximately equal in magnitude to the chiral order parameter at \( \mu = 0 \). The behaviour of the susceptibility ratio defined in (8) further supports a broken U(1)\(_B\) symmetry in the high \( \mu \) phase analogous to the broken U(1)\(_{EM}\) in BCS superconductors. This study provides the first evidence of a BCS phase in lattice field theory, which we interpret as evidence of a colour-superconducting phase at high \( \mu \) in QCD.

Clearly, more data are required for intermediate values of \( \mu \) to understand the nature of the diquark transition. To have more control over the finite volume effects shown in Figures 2 and 3 it may also be necessary to study lattices with \( L_s \neq L_t \). Finally, in the future we wish to study the fermion dispersion relation. This will allow us to present a comparison between the chiral mass gap \( \Sigma \) and the BCS superfluid gap \( \Delta \) which can, in principle, be experimentally determined.

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