GLOBAL EXISTENCE OF NON-NEWTONIAN INCOMPRESSIBLE FLUID IN HALF SPACE
WITH NONHOMOGENEOUS INITIAL-BOUNDARY DATA

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ABSTRACT. In this study, we investigate the global existence of weak solutions of non-Newtonian incompressible fluids governed by (1.1). When $u_0 \in B^{\alpha \frac{d}{2}}_{p,q}(\mathbb{R}^n) \cap B^{1-\frac{p}{2}}_{p,q}(\mathbb{R}^n) \cap B^{1+\frac{p}{2}}_{p,q}(\mathbb{R}^n)$ is given, we will find the weak solutions for the equation (1.1) in the function space $C_t(0, \infty; B^{\alpha \frac{d}{2}}_{p,q}(\mathbb{R}^n)) \cap C_t(0, \infty; B^{\frac{p}{2}}_{p,q}(\mathbb{R}^n)) \cap L^{\infty}(0, \infty; W^{1}_\infty(\mathbb{R}^n)) \cap L^{2n/(n+4)} \cap L^{2n/(n+4)}$ $n+2 < p < \infty$, $1 \leq q \leq \infty$, $1 + \frac{4n}{p} < \alpha < 2$. We show the existence of weak solutions in the anisotropic Besov spaces $\dot{B}^{\alpha \frac{d}{2}}_{p,q}(\mathbb{R}^n \times (0, \infty))$ (see Theorem 1.2) and we show the embedding $\dot{B}^{\frac{p}{2}}_{p,q}(\mathbb{R}^n \times (0, \infty)) \subset C_t(0, \infty; B^{\alpha \frac{d}{2}}_{p,q}(\mathbb{R}^n))$ (see Lemma 2.3).

For the global existence of solutions, we assume that the extra stress tensor $S$ is represented by $S(\alpha) = \mathbb{D}(\alpha)\alpha$, where $\mathbb{D}(0)$ is a uniformly elliptic matrix and $\mathbb{D} \in C^3(B(0,1))$, where $B(0,1)$ is open ball in $\mathbb{R}^{2n}$ whose center is origin and radius is 1. Note that $S_1$, $S_2$ and $S_3$ introduced in (1.2) satisfy our assumptions.

Keywords and phrases: Non-Newtonian equations, initial-boundary value, half space.

1. Introduction

In this study, we investigate the global existence of solutions for a non-Newtonian incompressible fluid governed by the following system:

\[ u_t - \text{div}(S(Du)) + \nabla p = -\text{div}(u \otimes u), \quad \text{div} u = 0 \text{ in } \mathbb{R}^n_+ \times (0, \infty), \]

\[ u|_{t=0} = u_0, \quad u|_{x_n=0} = 0, \tag{1.1} \]

where $u_0$ is the given initial velocity and $u$ and $p$ are the unknown velocity and pressure, respectively. Here, $(Du)_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, $i, j = 1, 2, \ldots, n$ and $S : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ is the extra stress tensor. When $S(Du) = 2Du$, the equations (1.1) become usual Navier-Stokes equations.

The standard models of $S(Du)$ are as follows:

\[ S_1(Du) = (\mu_0 + \mu_1 |Du|)^{d-2}Du, \]
\[ S_2(Du) = (\mu_0 + \mu_1 |Du|^{2})^{\frac{d}{2}}Du, \]
\[ S_3(Du) = (\mu_0 + \mu_1 |Du|^{d-2})Du, \tag{1.2} \]

where $\mu_0, \mu_1 > 0$ are positive real numbers. Here, $|\mathbb{A}| = \left( \sum_{1 \leq i,j \leq n} |a_{ij}|^2 \right)^{\frac{1}{2}}$ for matrix $\mathbb{A} = (a_{ij})_{1 \leq i,j \leq n}$. We call the fluid shear-thickening if $d > 2$ and shear-thinning if $1 < d < 2$.  

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We review the prior results regarding for the existence which are relevant to our result. Ladyženskaja ([38, 39]) was the first to study the existence of weak solutions of (1.1) in bounded domains with no-slip boundary condition \( u|_{x_n} = 0 \), and Nečas et al. also studied this problem intensively. After them, the non-Newtonian incompressible fluids were studied by many mathematicians (see [11, 12, 23, 37, 46] pertaining to the whole space and see [5, 8, 13, 30, 36, 45, 55, 56, 57, 59] for bounded domains with no-slip boundary condition).

In the engineering literatures, the non-Newtonian fluids under the no-slip condition are no longer well accepted, as shown by the extensive and thorough discussion in [43].

In smooth bounded domains, the pure slip boundary condition case was studied by Bulíček, Málek and Rajagopal [10] (also see [44]) and the pure Neumann boundary condition case was studied by Bothe and Prüss [11] (also see [51]).

In this study, we investigate the nonhomogeneous boundary problem for non-Newtonian incompressible fluid \( u|_{x_n} = 0 \). Over the past decade many mathematicians have studied the Newtonian incompressible fluids based on nonhomogeneous boundary data (See [2, 3, 4, 15, 16, 17, 26, 27, 28, 29, 32, 33, 34, 35, 41, 47, 58] and the references therein).

Recently, Bae and Kang [12] showed the existence of a low regular solutions of (1.1) in \( \mathbb{R}^n \) with assumptions \( S(Du) = F(Du)Du, F(A) = F(|A|^2), F(0) = I \) and \( F \in C^2([0, \infty)) \), where \( I \) is identity matrix in \( \mathbb{R}^n \).

Motivated by Bae and Kang’s results, in this study we show the existence of a weak solutions with minimal regularity and decay conditions of \( S(Du) \).

We introduce our assumptions for \( S(Du) \). We assume that \( S(Du) \) is represented by \( S(Du) = F(Du)Du \), where \( F: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n} \) is satisfied that

\[
(A) \quad F \in C^2(B(0,1)), \quad \text{where} \ B(0,1) \text{ is open unit ball in } \mathbb{R}^{n \times n} \text{ and } F(0) \text{ is uniformly elliptic matrix.}
\]

Note that \( S_1, S_2 \) and \( S_3 \) introduced in (1.2) satisfy our assumptions with \( F_1(A) = (\mu_0 + \mu_1 |A|)^{d-2}I, F_2(A) = (\mu_0 + \mu_1 |A|^2)^{d-2}I \) and \( F_3(A) = (\mu_0 + \mu_1 |A|^{d-2})I \).

Our main results are follows:

**Theorem 1.1.** Let \( S(Du) = F(Du)Du \) and \( F: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n} \) satisfy the assumptions (A). Let \( n + 2 < p < \infty \) and \( 1 \leq q \leq \infty \) such that \( 1 + \frac{2}{p} < \alpha < 2 \). Let \( u_0 \in B_{p,q}^{\alpha-\frac{2}{p}-1} (\mathbb{R}^n) \cap B_{p,\frac{n+2}{p},\frac{n+2}{p}}^{1-\frac{2}{p}} (\mathbb{R}^n) \cap B_{p,1}^{\frac{1+\frac{2}{p}}{p}} (\mathbb{R}^n) \)
such that \( \text{div} u_0 = 0 \) and \( u_0|_{\mathbb{R}^{n-1} \times (0, \infty)} = 0 \). There is \( \delta > 0 \) such that if

\[
\|u_0\|_{\dot{B}^{\frac{n}{p}-\frac{n}{q}}_{\frac{p}{2}, \frac{q}{2}}(\mathbb{R}^n_+)} + \|u_0\|_{\dot{B}^{\frac{n}{p}+\frac{n}{q}}_{p,1}(\mathbb{R}^n_+)} + \|u_0\|_{\dot{B}^{\frac{n}{p}-\frac{n}{q}}_{p,q}(\mathbb{R}^n_+)} < \delta,
\]

then, (1.1) has a solution \( u \) satisfying

\[
u \in C_{b}([0, \infty); \dot{B}^{\alpha-\frac{2}{p}}_{p,q}(\mathbb{R}^n_+)) \cap C_{b}([0, \infty); \dot{W}^{1-\frac{n}{p}}_{\frac{2}{p}, \frac{p}{2}}(\mathbb{R}^n_+)) \cap L^\infty(0, \infty, \dot{W}^{1}_{\frac{p}{2}, \frac{p}{2}}(\mathbb{R}^n_+)).
\]

To prove Theorem 1.1 we show the existence of the solutions of the equation (1.1) in anisotropic Besov spaces \( \dot{B}^{\alpha-\frac{2}{p}}_{p,q}(\mathbb{R}^n_+ \times (0, \infty)) \) (see Lemma 2.3). In showing the existence of the solutions, the anisotropic Besov spaces have several merits compared to the function spaces \( C_{b}((0, \infty); \dot{B}^{\alpha-\frac{2}{p}}_{p,q}(\mathbb{R}^n_+)) \). The parabolic type singular integral operators are bounded in \( \dot{B}^{\alpha-\frac{2}{p}}_{p,q}(\mathbb{R}^n_+ \times (0, \infty)) \) but not in \( C_{b}((0, \infty); \dot{B}^{\alpha-\frac{2}{p}}_{p,q}(\mathbb{R}^n_+)) \). This means that we can represent the solutions with the integral forms and using it we can obtain \( \dot{B}^{\alpha-\frac{2}{p}}_{p,q}(\mathbb{R}^n_+ \times (0, \infty)) \)-norm estimates of solutions (see Section 4).

We consider equations (1.1) with non-zero boundary condition, \( u|_{t=0} \neq 0 \). Theorem 1.1 is special case of the following results.

**Theorem 1.2.** Let \( S(Du) = \mathbb{F}(Du)Du \) and \( \mathbb{F} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n} \) satisfy the assumptions (A). Let \( n + 2 < p < \infty \) and \( 1 \leq q \leq \infty \) such that \( 1 + \frac{n+2}{p} < \alpha < 2 \). Let

\[
\begin{align*}
u_0 &\in \dot{B}^{\alpha-\frac{2}{p}}_{p,q}(\mathbb{R}^n_+) \cap \dot{B}^{1-\frac{n}{p}}_{\frac{2}{p}, \frac{p}{2}}(\mathbb{R}^n_+) \cap \dot{B}^{1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+), \\
g &\in \dot{B}^{\alpha-\frac{2}{p}+\frac{n}{q}}_{p,q}(\mathbb{R}^{n-1} \times (0, \infty)) \cap \dot{B}^{1-\frac{n}{p}+\frac{1}{2}}_{\frac{2}{p}, \frac{p}{2}}(\mathbb{R}^{n-1} \times (0, \infty)) \cap \dot{B}^{1+\frac{n}{p}+\frac{1}{2}}_{p,1}(\mathbb{R}^{n-1} \times (0, \infty)), \\
g_n &\in A^{\alpha-\frac{2}{p}+\frac{n}{q}}_{p,q}(\mathbb{R}^{n-1} \times (0, \infty)) \cap A^{1-\frac{n}{p}+\frac{1}{2}}_{\frac{2}{p}, \frac{p}{2}}(\mathbb{R}^{n-1} \times (0, \infty)) \cap A^{1+\frac{n}{p}+\frac{1}{2}}_{p,1}(\mathbb{R}^{n-1} \times (0, \infty))
\end{align*}
\]

such that \( u_0 \) and \( g \) satisfy the following conditions;

\[
div u_0 = 0, \quad \text{and} \quad g|_{t=0} = u_0|_{x_0=0}.
\]

If

\[
\begin{align*}
\|u_0\|_{\dot{B}^{\frac{n}{p}-\frac{n}{q}}_{\frac{p}{2}, \frac{q}{2}}(\mathbb{R}^n_+)} + \|u_0\|_{\dot{B}^{\frac{n}{p}+\frac{n}{q}}_{p,1}(\mathbb{R}^n_+)} + \|u_0\|_{\dot{B}^{\frac{n}{p}-\frac{n}{q}}_{p,q}(\mathbb{R}^n_+)} + \|g\|_{\dot{B}^{1-\frac{n}{p}+\frac{1}{2}}_{\frac{2}{p}, \frac{p}{2}}(\mathbb{R}^{n-1} \times (0, \infty))} + \|g\|_{\dot{B}^{1+\frac{n}{p}+\frac{1}{2}}_{p,1}(\mathbb{R}^{n-1} \times (0, \infty))} + \|g\|_{A^{\alpha-\frac{2}{p}+\frac{n}{q}}_{p,q}(\mathbb{R}^{n-1} \times (0, \infty))} + \|g_n\|_{A^{1-\frac{n}{p}+\frac{1}{2}}_{\frac{2}{p}, \frac{p}{2}}(\mathbb{R}^{n-1} \times (0, \infty))} + \|g_n\|_{A^{1+\frac{n}{p}+\frac{1}{2}}_{p,1}(\mathbb{R}^{n-1} \times (0, \infty))} + \|g_n\|_{A^{\alpha-\frac{2}{p}+\frac{n}{q}}_{p,q}(\mathbb{R}^{n-1} \times (0, \infty))} < \delta,
\end{align*}
\]
for small $\delta > 0$, then the equations (1.1) with the boundary condition $u|_{x_n=0} = g$ has a solution $u$ satisfying

$$
u \in \dot{B}^{\alpha, q}_{p, q}(\mathbb{R}^n_+ \times (0, \infty)) \cap \dot{W}^{1+\frac{\alpha}{p}, q}_{p, q}(\mathbb{R}^n_+ \times (0, \infty)) \cap L^\infty(0, \infty; \dot{W}^{1, q}_{p, q}(\mathbb{R}^n_+)).$$

(1.4)

Moreover, there is $\delta_0 > 0$ such that if $u_1$ is another solution of (1.1) satisfying $u_1 \in \dot{W}^{1+\frac{\alpha}{p}, q}_{p, q}(\mathbb{R}^n_+ \times (0, \infty)) \cap L^\infty(0, \infty; \dot{W}^{1, q}_{p, q}(\mathbb{R}^n_+))$ with $\|u_1\|_{\dot{W}^{1+\frac{\alpha}{p}, q}_{p, q}(\mathbb{R}^n_+ \times (0, \infty))} + \|Du_1\|_{L^\infty(\mathbb{R}^n_+ \times (0, \infty))} < \delta_0$, then $u = u_1$.

The function spaces are introduced in Section 2.

We note that the homogeneous Besov spaces $\dot{B}^{\alpha, q}_{p, q}(\mathbb{R}^n_+ \times (0, \infty))$ are not complete if $\alpha > \frac{n+2}{p}$. Because we use the iteration method to find solutions, it looks like the function space $\dot{B}^{\alpha, q}_{p, q}(\mathbb{R}^n_+ \times (0, \infty))$, $\alpha > 1 + \frac{n+2}{p}$ are not reasonable for the solution spaces. To overcome it, we add the complete function space $\dot{W}^{1+\frac{\alpha}{p}, q}_{p, q}(\mathbb{R}^n_+ \times (0, \infty))$ (see Theorem B.2). Our approximate solutions $u^m$ are uniformly bounded and Cauchy sequence in $\dot{B}^{\alpha, q}_{p, q}(\mathbb{R}^n_+ \times (0, \infty)) \cap \dot{W}^{1+\frac{\alpha}{p}, q}_{p, q}(\mathbb{R}^n_+ \times (0, \infty))$. Then, by completeness of $\dot{B}^{\alpha, q}_{p, q}(\mathbb{R}^n_+ \times (0, \infty)) \cap \dot{W}^{1+\frac{\alpha}{p}, q}_{p, q}(\mathbb{R}^n_+ \times (0, \infty))$, $u^m$ converges to $u$ in $\dot{B}^{\alpha, q}_{p, q}(\mathbb{R}^n_+ \times (0, \infty)) \cap \dot{W}^{1+\frac{\alpha}{p}, q}_{p, q}(\mathbb{R}^n_+ \times (0, \infty))$.

We note that the restriction $\alpha < 2$ (regularity of solutions) is based on the assumptions of $S(Du)$. With an improvement in the regularity of $\mathbb{F}$, the regularity of the solutions in Theorem 1.2 will be also improved (see [36]).

For simplicity, we assume that $\mathbb{F}(0) = \mathbb{I}$ is the identity matrix. Let

$$\sigma(A) := \mathbb{F}(A) - \mathbb{F}(0).$$

(1.5)

Accordingly, the first equations in (1.1) are denoted by

$$u_t - \Delta u + \nabla p = \text{div} (\sigma(Du) Du - u \otimes u).$$

(1.6)

Thus, for the proof of Theorem 1.2, the following initial-boundary value problem of the Stokes equations in $\mathbb{R}^n_+ \times (0, \infty)$ is necessary:

$$w_t - \Delta w + \nabla p = f, \quad \text{div} w = 0 \quad \text{in} \quad \mathbb{R}^n_+ \times (0, \infty),$$

$$w|_{t=0} = u_0, \quad w|_{x_n=0} = g,$$

(1.6)

where $f = \text{div} \mathbb{F}(\text{div} (\sigma(Du) Du - u \otimes u)).$

The following theorem states our results on the unique solvability of the Stokes equations (1.6).

**Theorem 1.3.** Let $1 < p < \infty$, $1 \leq q \leq \infty$ and $1 < \alpha < 2$. 

(1) Let

\[ u_0 \in \dot{B}^{a-\frac{d}{2}}_{p,q}(\mathbb{R}^n_+), \quad \mathcal{F} \in L^p(0, \infty; \dot{B}^{a-1}_{p,q}(\mathbb{R}^n_+)), \]

\[ g \in \dot{B}^{a-\frac{d}{2}+\frac{1}{2}}_{p,q}(\mathbb{R}^{n-1} \times (0, \infty)), \quad g_n \in \dot{A}^{a-\frac{d}{2}+\frac{1}{2}}_{p,q}(\mathbb{R}^{n-1} \times (0, \infty)) \]

such that \( u_0 \) and \( g \) satisfy (1.3). Accordingly, equation (1.6) has a solution \( w \in \dot{B}^{a-\frac{d}{2}}_{p,q}(\mathbb{R}^n_+ \times (0, \infty)) \) with the following inequalities

\[
\|w\|_{\dot{B}^{a-\frac{d}{2}}_{p,q}(\mathbb{R}^n_+ \times (0, \infty))} \leq c \left( \|u_0\|_{\dot{B}^{a-\frac{d}{2}}_{p,q}(\mathbb{R}^n_+)} + \|g\|_{\dot{B}^{a-\frac{d}{2}+\frac{1}{2}}_{p,q}(\mathbb{R}^{n-1} \times (0, \infty))} 
+ \|g_n\|_{\dot{A}^{a-\frac{d}{2}+\frac{1}{2}}_{p,q}(\mathbb{R}^{n-1} \times (0, \infty))} + \|\mathcal{F}\|_{L^p(\mathbb{R}^n_+ \times (0, \infty))} \right). \tag{1.7}
\]

(2) Let

\[ u_0 \in \dot{B}^{1-\frac{d}{2}}_{p,p}(\mathbb{R}^n_+), \quad \mathcal{F} \in L^p(\mathbb{R}^n_+ \times (0, \infty)), \]

\[ g \in \dot{B}^{1-\frac{d}{2}+\frac{1}{2}}_{p,p}(\mathbb{R}^{n-1} \times (0, \infty)), \quad g_n \in \dot{A}^{1-\frac{d}{2}+\frac{1}{2}}_{p,p}(\mathbb{R}^{n-1} \times (0, \infty)) \]

such that \( u_0 \) and \( g \) satisfy (1.3). Then, equation (1.6) has a solution \( w \in W^{1,p}_p(\mathbb{R}^n_+ \times (0, \infty)) \) with the following inequalities

\[
\|w\|_{W^{1,p}_p(\mathbb{R}^n_+ \times (0, \infty))} \leq c \left( \|u_0\|_{\dot{B}^{1-\frac{d}{2}}_{p,p}(\mathbb{R}^n_+)} + \|g\|_{\dot{B}^{1-\frac{d}{2}+\frac{1}{2}}_{p,p}(\mathbb{R}^{n-1} \times (0, \infty))} 
+ \|g_n\|_{\dot{A}^{1-\frac{d}{2}+\frac{1}{2}}_{p,p}(\mathbb{R}^{n-1} \times (0, \infty))} + \|\mathcal{F}\|_{L^p(\mathbb{R}^n_+ \times (0, \infty))} \right). \tag{1.8}
\]

In particular, if \( p < n + 2 \), then the solution \( w \) of (1.6) is a unique in \( W^{1,p}_p(\mathbb{R}^n_+ \times (0, \infty)) \).

We organize this paper as follows. In Section 2, we introduce the function spaces, and the definitions of the weak solutions of Stokes equations and non-Newtonian Navier-Stokes equations. In Section 3, the various estimates of operators related to Newtonian and Gaussian kernels are provided. In Section 4, we complete the proof of Theorem 1.3. In Section 5, we provide the proof of Theorem 1.2 by applying the estimates in Theorem 1.3 to the approximate solutions.

2. Notations and Definitions

The points of spaces \( \mathbb{R}^{n-1} \) and \( \mathbb{R}^n \) are denoted by \( x' \) and \( x = (x', x_n) \), respectively. In addition, multiple derivatives are denoted by \( D_t^\beta D_x^m = \frac{\partial^m}{\partial x^m} \frac{\partial^{\beta t}}{\partial t} \) for multi-index \( \beta \) and nonnegative integer \( m \). Throughout this paper, we denote various generic constants by using \( c \). Let \( \mathbb{R}^n_+ = \{ x = (x', x_n) : x_n > 0 \} \) and \( \overline{\mathbb{R}^n_+} = \{ x = (x', x_n) : x_n \geq 0 \} \).

For the Banach space \( X \), we denote the usual Bochner space by \( L^p(0, \infty; X), 1 \leq p \leq \infty \). For \( 0 < \theta < 1 \) and \( 1 \leq p \leq \infty \), we denote the complex interpolation and the real interpolation space of the normed space \( X \) and \( Y \) as \( [X, Y]_\theta \) and \( (X, Y)_{\theta, p} \), respectively. For \( 1 \leq p \leq \infty \), \( p' = \frac{p}{p-1} \).
Let $\Omega$ be either $\mathbb{R}^n$, $\mathbb{R}^{n-1}$, or $\mathbb{R}^n_+$. Let $1 \leq p \leq \infty$ and $k$ be a nonnegative integer. The norms of usual Lebesgue space $L^p(\Omega)$, the usual homogeneous Sobolev space $W^k_p(\Omega)$ and the usual homogeneous Besov space $B^s_{p,q}(\Omega)$, $1 \leq p \leq \infty$ are written as $\| \cdot \|_{L^p(\Omega)}$, $\| \cdot \|_{W^k_p(\Omega)}$ and $\| \cdot \|_{B^s_{p,q}(\Omega)}$ respectively.

Now, we introduce anisotropic Besov spaces and their properties (See Chapter 5 of \cite{54}, and Chapter 3 of \cite{6} for the definition of spaces and their properties, although different notations are used in the books).

Let $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. We define an anisotropic homogeneous Sobolev space $\dot{W}^s_p(\mathbb{R}^{n+1})$, $n \geq 1$ by

$$\dot{W}^s_p(\mathbb{R}^{n+1}) = \{ f \in \mathcal{S}'(\mathbb{R}^{n+1}) \mid f = h_s * g, \text{ for some } g \in L^p(\mathbb{R}^{n+1}) \}$$

with norm $\| f \|_{\dot{W}^s_p(\mathbb{R}^{n+1})} := \| g \|_{L^p(\mathbb{R}^{n+1})} = \| h \ast \cdot f \|_{L^p(\mathbb{R}^{n+1})}$, where $\ast$ is a convolution in $\mathbb{R}^{n+1}$ and $\mathcal{S}'(\mathbb{R}^{n+1})$ is the dual space of the Schwartz space $\mathcal{S}(\mathbb{R}^{n+1})$. Here, $h_s$ is a distribution in $\mathbb{R}^{n+1}$ whose Fourier transform in $\mathbb{R}^{n+1}$ is defined by

$$\mathcal{F}_{x,t}(h_s)(\xi, \tau) = c_s(4\pi^2|\xi|^2 + 2\pi i \tau)^{-\frac{s}{2}}, \quad (\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}.$$ 

where $\mathcal{F}_{x,t} = \mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ is the inverse Fourier transform in $\mathbb{R}^{n+1}$, respectively. We also denote $\mathcal{F}^{-1}$ as the Fourier transform in $\mathbb{R}^{n+1}$. Here, the function $(4\pi^2|\xi|^2 + i\tau)^{-\frac{s}{2}}$ is defined as $e^{-\frac{s}{2} \ln(4\pi^2|\xi|^2 + i\tau)}$, where the branch line of the logarithm that is a negative real line for real arguments is used.

Let $D^\frac{1}{2}_t f(t) = \frac{1}{\sqrt{\pi}} D_t \int_{-\infty}^{t} \frac{f(s)}{(t-s)^{\frac{1}{2}}} ds$ and $D^\frac{k}{2}_t f = D^\frac{1}{2}_t D^\frac{k-1}{2}_t f$. Note that $D^{-\frac{1}{2}}_t f(\tau) = (a_0 + ib_0 \text{sign}(\tau))|\tau|^{-\frac{1}{2}} \hat{f}(\tau)$ for certain complex numbers $a_0$ and $b_0$ (see Section 2 in \cite{20}). Using the multiplier theorem for $L^p(\mathbb{R}^{n+1})$ (see Section 2 in \cite{20} or Section 6 in \cite{9}), in the case $\alpha = k \in \mathbb{N} \cup \{0\}$,

$$\| f \|_{\dot{W}^s_p(\mathbb{R}^{n+1})} \approx \sum_{|\beta|+l = k} \| D_{\alpha}^{\beta} D^\frac{1}{2}_t f \|_{L^p(\mathbb{R}^{n+1})}.$$ 

(See Theorem \ref{thm:multiplier}) In particular, $\| f \|_{\dot{W}^s_p(\mathbb{R}^{n+1})} = \| f \|_{L^p(\mathbb{R}^{n+1})}$ and

$$\| f \|_{\dot{W}^s_p(\mathbb{R}^{n+1})} = \| D^\frac{1}{2}_t f \|_{L^p(\mathbb{R}^{n+1})} + \| D_{\alpha} f \|_{L^p(\mathbb{R}^{n+1})},$$

$$\| f \|_{\dot{W}^s_p(\mathbb{R}^{n+1})} = \| D_t f \|_{L^p(\mathbb{R}^{n+1})} + \| D^2_{\alpha} f \|_{L^p(\mathbb{R}^{n+1})}.$$ 

See \cite{20}.

We fix a Schwartz function $\phi \in \mathcal{S}(\mathbb{R}^{n+1})$ satisfying $\hat{\phi}(\xi, \tau) > 0$ on $\frac{1}{2} < |\xi| + |\tau|^\frac{1}{2} < 2$, $\hat{\phi}(\xi, \tau) = 0$ elsewhere, and $\sum_{j=-\infty}^{\infty} \hat{\phi}(2^{-j} \xi, 2^{-2j} \tau) = 1$ for $(\xi, \tau) \neq 0$. Let

$$\hat{\phi}_j(\xi, \tau) := \hat{\phi}(2^{-j} \xi, 2^{-2j} \tau) \quad (j = 0, \pm 1, \pm 2, \cdots).$$
We define the homogeneous anisotropic Besov space $\dot{B}^{s, \frac{r}{r}}_{p, q}(\mathbb{R}^{n+1})$ as

$$
\dot{B}^{s, \frac{r}{r}}_{p, q}(\mathbb{R}^{n+1}) = \{ f \mid \| f \|_{\dot{B}^{s, \frac{r}{r}}_{p, q}(\mathbb{R}^{n+1})} < \infty \},
$$

where

$$
\| f \|_{\dot{B}^{s, \frac{r}{r}}_{p, q}(\mathbb{R}^{n+1})} = \left( \sum_{-\infty < j < \infty} 2^{qsj} \| f \ast \phi_j \|_{L^p(\mathbb{R}^{n+1})}^{q} \right)^{\frac{1}{q}},
$$

where $\ast$ is the $n + 1$ dimensional convolution.

**Remark 2.1.**

1. We consider $W^{s, \frac{r}{r}}_{p}(\mathbb{R}^{n+1})$ and $\dot{B}^{s, \frac{r}{r}}_{p, q}(\mathbb{R}^{n+1})$ as the quotient spaces with polynomial space with respect to $(x, t)$ and so $H^s(\mathbb{R}^{n+1})$ and $B^{s, \frac{r}{r}}_{p, q}(\mathbb{R}^{n+1})$ are normed spaces.
2. For $s \in \mathbb{R}$, $1 < p < \infty$, the anisotropic homogeneous Sobolev space $W^{s, \frac{r}{r}}_{p}(\mathbb{R}^{n+1})$ norm is equivalent to

$$
\| \mathcal{F}^{-1} \left( \sum_{-\infty < k < \infty} (4\pi^2|\xi|^2 + 2\pi \tau) \hat{\phi}(2^{-k\xi}, 2^{-2k\tau}) \hat{f} \right) \|_{L^p(\mathbb{R}^{n+1})}.
$$

(2.1)

(See Appendix A)

3. Note that $W^{s, \frac{r}{r}}_{p}(\mathbb{R}^{n+1})$ and $\dot{B}^{s, \frac{r}{r}}_{p, q}(\mathbb{R}^{n+1})$ are complete spaces for $0 < s < \frac{n+2}{p}$ (see Theorem B.1).

The properties of the anisotropic homogeneous spaces are comparable with the properties of usual homogeneous spaces. In particular, the following properties will be used later.

**Proposition 2.2.**

1. For $1 \leq p$, $q_0$, $q_1$, $r \leq \infty$ and $s_1$, $s_2 \in \mathbb{R}$,

$$(W^{s_0, \frac{r}{r}}_{p_0}(\mathbb{R}^{n+1}), W^{s_1, \frac{r}{r}}_{p_1}(\mathbb{R}^{n+1}))_{\theta, r} = \dot{B}^{s, \frac{r}{r}}_{p, q}(\mathbb{R}^{n+1}),$$

$$(B^{s_0, \frac{r}{r}}_{p_0, q_0}(\mathbb{R}^{n+1}), B^{s_1, \frac{r}{r}}_{p_1, q_1}(\mathbb{R}^{n+1}))_{\theta, r} = \dot{B}^{s, \frac{r}{r}}_{p, q}(\mathbb{R}^{n+1}),$$

$$[W^{s_0, \frac{r}{r}}_{p_0}(\mathbb{R}^{n+1}), W^{s_1, \frac{r}{r}}_{p_1}(\mathbb{R}^{n+1})]_{\theta} = \dot{W}^{s, \frac{r}{r}}_{p}(\mathbb{R}^{n+1}),$$

where $0 < \theta < 1$ and $s = s_0(1 - \theta) + s_1 \theta$.

2. Suppose that $1 \leq p_0 \leq p_1 < \infty$, $1 \leq r_0 \leq r_1 \leq \infty$, and $s_0 \geq s_1$ with $s_0 - \frac{n+2}{p_0} = s_1 - \frac{n+2}{p_1}$.

Accordingly, the following inclusions hold

$$(W^{s_0, \frac{r}{r}}_{p_0}(\mathbb{R}^{n+1}) \subset W^{s_1, \frac{r}{r}}_{p_1}(\mathbb{R}^{n+1}), \quad 1 < p, p_1 < \infty, \quad [W^{s_0, \frac{r}{r}}_{p_0}(\mathbb{R}^{n+1}), W^{s_1, \frac{r}{r}}_{p_1}(\mathbb{R}^{n+1})]_{\theta} = \dot{W}^{s, \frac{r}{r}}_{p}(\mathbb{R}^{n+1}),$$

where $0 < \theta < 1$ and $s = s_0(1 - \theta) + s_1 \theta$.

3. For $0 < s$ and $1 < p < \infty$,

$$W^{s, \frac{r}{r}}_{p}(\mathbb{R}^{n+1}) = L^p(\mathbb{R}; W^{s}(\mathbb{R}^n)) \cap L^p(\mathbb{R}^n; \dot{W}^{s, \frac{r}{r}}_{p}(\mathbb{R})).$$
(4) Let $1 < p, q < \infty$ and $0 < s < 1$. The $\dot{B}^{s, \frac{q}{p}}_{p,q}(\mathbb{R}^{n+1})$-norm of $f$ is equivalent to
\[
\|f\|_{\dot{B}^{s, \frac{q}{p}}_{p,q}(\mathbb{R}^{n+1})} = \left( \int_{\mathbb{R}^{n+1}} \left( \int_{\mathbb{R}^{n+1}} |f(x + y, t + \tau) - f(x, t)|^q \, dy \, d\tau \right)^{\frac{p}{q}} \, dx \right)^{\frac{1}{p}}.
\]

(5) If $f \in \dot{B}^{s, \frac{q}{p}}_{p,q}(\mathbb{R}^{n+1})$ or $f \in \dot{W}^{s, \frac{q}{p}}_{p,q}(\mathbb{R}^{n+1})$, $s > \frac{2}{p}$, then $f|_{t=0} \in \dot{B}^{s, \frac{q}{p}}_{p,q}(\mathbb{R}^n)$ with
\[
\|f|_{t=0}\|_{\dot{B}^{s, \frac{q}{p}}_{p,q}(\mathbb{R}^n)} \leq C \|f\|_{\dot{B}^{s, \frac{q}{p}}_{p,q}(\mathbb{R}^{n+1})}, \quad \|f|_{t=0}\|_{\dot{B}^{s, \frac{q}{p}}_{p,q}(\mathbb{R}^n)} \leq C \|f\|_{\dot{W}^{s, \frac{q}{p}}_{p,q}(\mathbb{R}^{n+1})}.
\]

Remark 2.3. We refer to (2) of Remark 2.1 and Theorem 3.7.1 in [6] and Theorem 6.4.5 in [9] for the proof of (1), Theorem 6.5.1 in [9] for (2), Theorem 3.7.3 in [6] for (3), Theorem 3.6.1 in [6] for (4) and Theorem 4.5.2 in [6] and Theorem 6.6.1 in [9] for (5).

$\dot{B}^{s, \frac{q}{p}}_{p,q}(\mathbb{R}^n \times (0, \infty))$ refers to the restriction of $\dot{B}^{s, \frac{q}{p}}_{p,q}(\mathbb{R}^n \times \mathbb{R})$ with norm
\[
\|f\|_{\dot{B}^{s, \frac{q}{p}}_{p,q}(\mathbb{R}^n \times (0, \infty))} = \inf \{\|F\|_{\dot{B}^{s, \frac{q}{p}}_{p,q}(\mathbb{R}^n \times \mathbb{R})} : F \in \dot{B}^{s, \frac{q}{p}}_{p,q}(\mathbb{R}^n \times \mathbb{R}) \text{ with } F|_{\mathbb{R}_+^n \times (0, \infty)} = f \}.
\]

Similarly, we define $\dot{W}^{s, \frac{q}{p}}_{p,q}(\mathbb{R}^n \times (0, \infty))$.

Now, we explain interpolation spaces defined in $\mathbb{R}^n_+ \times (0, \infty)$.

For $k \in \mathbb{N} \cup \{0\}$, we define $\dot{W}^{2k,\frac{q}{p}}_{p}(\mathbb{R}^n_+ \times (0, \infty))$ by
\[
\dot{W}^{2k,\frac{q}{p}}_{p}(\mathbb{R}^n_+ \times (0, \infty)) = \{f \in \dot{W}^{2k,\frac{q}{p}}_{p}(\mathbb{R}^n_+ \times (0, \infty)) : \sum_{2l+|\beta|=2k} \|D_x^l D_x^\beta f\|_{L^p(\mathbb{R}^n_+ \times (0, \infty))} < \infty\}.
\]

We also define the interpolation spaces $\dot{W}^{s, \frac{q}{p}}_{p}(\mathbb{R}^n_+ \times (0, \infty))$ and $\dot{B}^{s, \frac{q}{p}}_{p,q}(\mathbb{R}^n_+ \times (0, \infty))$ by
\[
\dot{W}^{s, \frac{q}{p}}_{p}(\mathbb{R}^n_+ \times (0, \infty)) := [\dot{L}^p(\mathbb{R}^n_+ \times (0, \infty)), \dot{W}^{2k,\frac{q}{p}}_{p}(\mathbb{R}^n_+ \times (0, \infty))]_{\theta},
\]
\[
\dot{B}^{s, \frac{q}{p}}_{p,q}(\mathbb{R}^n_+ \times (0, \infty)) := (\dot{L}^p(\mathbb{R}^n_+ \times (0, \infty)), \dot{W}^{2k,\frac{q}{p}}_{p}(\mathbb{R}^n_+ \times (0, \infty)))_{\theta,q},
\]
where $0 < \theta < 1$ and $s = 2k\theta$.

For $f \in \dot{W}^{2k,\frac{q}{p}}_{p}(\mathbb{R}^n_+ \times (0, \infty))$, we define extension $E f \in \dot{W}^{2k,\frac{q}{p}}_{p}(\mathbb{R}^{n+1})$ of $f$ by the following process. First, for the given function $f : \mathbb{R}^n_+ \times (0, \infty) \to \mathbb{R}$, we define extension of $E_1 f$ over $\mathbb{R}^n_+ \times (0, \infty)$ as follows:
\[
E_1 f(x, t) = \begin{cases} f(x, t) & \text{for } x_n > 0, \\ \sum_{j=1}^{2k+1} \lambda_j f(x', -jx_n, t) & \text{for } x_n < 0, \end{cases}
\]
where $\lambda_1, \lambda_2, \ldots, \lambda_{2k+1}$ satisfy $\sum_{j=1}^{2k+1} (-j)^l \lambda_j = 1$, $0 \leq l \leq 2k$. Next, for $f : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$, we define extension of $E_2 f$ over $\mathbb{R}^{n+1}$ by
\[
E_2 f(x, t) = \begin{cases} f(x, t) & \text{for } t > 0, \\ \sum_{j=1}^{k+1} \eta_j f(x, -jt) & \text{for } t < 0, \end{cases}
\]
Accordingly, \( RF \). This implies the following:

\[ \left. \begin{array}{l}
\text{Similarly, we obtain} \\
\text{From (2.2) and (2.4),}
\end{array} \right\} \]

\[ \left. \begin{array}{l}
\text{Let } s > 0 \text{ be a non-natural number. Based on the property of the complex interpolation,}
\end{array} \right\} \]

\[ E : W_p^{2i,i}(\mathbb{R}_+^n \times (0, \infty)) \to W_p^{2i,2}(\mathbb{R}_+^{n+1}) \text{ is bounded operator. This implies the followings:} \]

\[ \| f \|_{W_p^{2i,2}(\mathbb{R}_+^n \times (0, \infty))} \leq c \| f \|_{W_p^{2i,i}(\mathbb{R}_+^{n+1})} \leq c \| f \|_{W_p^{2i,2}(\mathbb{R}_+^{n+1})}. \]

\[ \text{Accordingly,} \]

\[ W_p^{2i,2}(\mathbb{R}_+^n \times (0, \infty)) \subset W_p^{2i,2}(\mathbb{R}_+^{n+1}). \]  \hspace{1cm} (2.2)

\[ \text{Let } 2k < s < 2k+2 \text{ for } k \in \mathbb{N} \cup \{0\}. \]

\[ \text{Let } R : W_p^{2i,i}(\mathbb{R}_+^{n+1}) \to W_p^{2i,i}(\mathbb{R}_+^n \times (0, \infty)), \]

\[ \| f \|_{W_p^{2i,i}(\mathbb{R}_+^n \times (0, \infty))} \leq c \| f \|_{W_p^{2i,i}(\mathbb{R}_+^{n+1})}. \]

\[ \| RF \|_{W_p^{2i,2}(\mathbb{R}_+^n \times (0, \infty))} \leq c \| f \|_{W_p^{2i,2}(\mathbb{R}_+^{n+1})}. \]  \hspace{1cm} (2.3)

\[ \text{Let } f \in W_p^{2i,2}(\mathbb{R}_+^n \times (0, \infty)). \]

\[ \text{By the definition of } W_p^{2i,2}(\mathbb{R}_+^n \times (0, \infty)), \]

\[ \text{such that } RF = f \text{ and } \| f \|_{W_p^{2i,2}(\mathbb{R}_+^{n+1})} \leq 2 \| f \|_{W_p^{2i,2}(\mathbb{R}_+^n \times (0, \infty))}. \]

\[ \text{Then, from (2.3), we have} \]

\[ \| f \|_{W_p^{2i,2}(\mathbb{R}_+^n \times (0, \infty))} \leq 2c \| f \|_{W_p^{2i,2}(\mathbb{R}_+^{n+1})}. \]

\[ \text{This implies the following:} \]

\[ W_p^{2i,2}(\mathbb{R}_+^n \times (0, \infty)) \subset W_p^{2i,2}(\mathbb{R}_+^n \times (0, \infty)). \]  \hspace{1cm} (2.4)

\[ \text{From (2.2) and (2.4),} \]

\[ W_p^{2i,2}(\mathbb{R}_+^n \times (0, \infty)) = [L^p(\mathbb{R}_+^n \times (0, \infty)), \mathcal{W}^{2k,i}_p(\mathbb{R}_+^n \times (0, \infty))]_{\theta}. \]  \hspace{1cm} (2.5)

\[ \text{Similarly, we obtain} \]

\[ \mathcal{B}_{r,p}^{2i,2}(\mathbb{R}_+^n \times (0, \infty)) = (L^p(\mathbb{R}_+^n \times (0, \infty)), \mathcal{W}^{2k,i}_p(\mathbb{R}_+^n \times (0, \infty)))_{\theta,r} \quad 1 \leq p, r \leq \infty, \quad \forall s > 0. \]  \hspace{1cm} (2.6)

\[ \text{Hereafter, we use the notations } L^p = L^p(\mathbb{R}_+^n \times (0, \infty)), W_p^{2i,2} = W_p^{2i,2}(\mathbb{R}_+^n \times (0, \infty)) \text{ and } \mathcal{B}_{r,p}^{2i,2} = \mathcal{B}_{r,p}^{2i,2}(\mathbb{R}_+^n \times (0, \infty)). \]

\[ \text{From Proposition 2.2, we obtain the following results.} \]

**Proposition 2.4.** \hspace{1cm} (1) For \( 1 \leq p, q_0, q_1, r \leq \infty, \)

\[ (W^{q_0,q_0}_p, W^{q_1,q_1}_p)_{\theta,r} = B^{2i,2}_{p,r}, \quad (B^{q_0,q_0}_{p,q_0}, B^{q_1,q_1}_{p,q_1})_{\theta,r} = B^{2i,2}_{p,r}, \quad [W^{q_0,q_0}_p, W^{q_1,q_1}_p]_{\theta} = W^{2i,2}_p, \]

\[ \text{where } 0 < \theta < 1 \text{ and } s = s_0(1 - \theta) + s_1 \theta. \]
(2) Suppose that $1 \leq p_0 \leq p_1 \leq \infty$, $1 \leq r_0 \leq r_1 \leq \infty$ and $s_0 \geq s_1$ with $s_0 = \frac{n+2}{p_0} = s_1 = \frac{n+2}{p_1}$.

Accordingly, the following inclusions hold

$$W^{s_0, \frac{2}{p_0}}_{p_0} \subset W^{s_1, \frac{2}{p_1}}_{p_1}, \quad \dot{B}^{s_0, \frac{2}{p_0}}_{p_0} \subset \dot{B}^{s_1, \frac{2}{p_1}}_{p_1}.$$ 

In particular,

$$W^{1, \frac{n+2}{2}}_{n+2} \subset W^{0,0}_{n+2} = L^{n+2}. \quad (2.7)$$

(3) For $0 < s$ and $1 < p < \infty$,

$$W^{s, \frac{2}{p}}_{p} = L^p(0, \infty; W^{s, \frac{2}{p}}(\mathbb{R}^n_+)) \cap L^p(\mathbb{R}^n_+, W^{\frac{2}{p}}_{p}(0, \infty)).$$

(4) If $f \in \dot{B}^{s, \frac{2}{p}}_{p, q}$ or $f \in W^{s, \frac{2}{p}}_{p}$, $1 < p < \infty$, $1 \leq q \leq \infty$, $s > \frac{2}{p}$, then $f|_{t=0} \in \dot{B}^{s-\frac{2}{p}}_{p, q}(\mathbb{R}^n)$ or $f|_{t=0} \in \dot{B}^{s-\frac{2}{p}}_{p, q}(\mathbb{R}^n)$, respectively with

$$\|f|_{t=0}\|_{\dot{B}^{s-\frac{2}{p}}_{p, q}(\mathbb{R}^n)} \leq c\|f\|_{W^{s, \frac{2}{p}}_{p}}, \quad \|f|_{t=0}\|_{\dot{B}^{s-\frac{2}{p}}_{p, q}(\mathbb{R}^n)} \leq c\|f\|_{\dot{B}^{s, \frac{2}{p}}_{p, q}}.$$

From the definition of anisotropic Besov space in $\mathbb{R}^{n+1}$, we obtain the following lemma;

**Lemma 2.5.** Let $1 < p < \infty$ and $1 \leq q \leq \infty$ and $s > \frac{1}{p}$. Accordingly,

$$\|f|_{x=0}\|_{\dot{B}^{s, \frac{2}{p}}_{p, q}(\mathbb{R}^n \times (0, \infty))} \leq c\|f\|_{W^{s, \frac{2}{p}}_{p}}, \quad \|f|_{x=0}\|_{\dot{B}^{s, \frac{2}{p}}_{p, q}(\mathbb{R}^n \times (0, \infty))} \leq c\|f\|_{\dot{B}^{s, \frac{2}{p}}_{p, q}}.$$

See Theorem 4.5.2 in [5].

We say that a distribution $f$ in $\mathbb{R}^n_+ \times \mathbb{R}$ is contained in function space $W^{\frac{k}{p}}_{p}(0, \infty; \dot{B}^{\frac{1}{p}}_{\frac{1}{p}+\frac{1}{p}}(\mathbb{R}^{n+1}))$, $k \in \mathbb{N} \cup \{0\}$ if $f$ satisfies

$$\|f\|_{W^{\frac{k}{p}}_{p}(0, \infty; \dot{B}^{\frac{1}{p}}_{\frac{1}{p}+\frac{1}{p}}(\mathbb{R}^{n+1}))} : = \|D_t^{\frac{k}{p}}f\|_{L^p(0, \infty; \dot{B}^{\frac{1}{p}}_{\frac{1}{p}+\frac{1}{p}}(\mathbb{R}^{n+1}))} < \infty,$$

where $\dot{B}^{\frac{1}{p}}_{\frac{1}{p}+\frac{1}{p}}(\mathbb{R}^{n+1})$ is dual space of $\dot{B}^{\frac{1}{p}}_{p, p}(\mathbb{R}^{n+1})$, $\frac{1}{p} + \frac{1}{p} = 1$.

Let $k < s < k + 1$ with $s = \theta k + (1 - \theta)(k + 1)$ for $0 < \theta < 1$ and $1 \leq q \leq \infty$. We define function space $A^{s, \frac{2}{p}}_{p, q}(\mathbb{R}^{n+1} \times (0, \infty))$ by

$$A^{s, \frac{2}{p}}_{p, q}(\mathbb{R}^{n+1} \times (0, \infty)) : = \left(L^p(0, \infty; \dot{B}^{s, \frac{1}{p}}_{p, p}(\mathbb{R}^{n+1})) \cap W^{s, \frac{2}{p}}_{p}(0, \infty; \dot{B}^{s, \frac{1}{p}}_{p, p}(\mathbb{R}^{n+1})) \right)^{\theta,q}.$$ 

The following Hölder type inequality is a well-known result for usual homogeneous Besov space in $\mathbb{R}^n$ (see Lemma 2.2 in [14]).

**Lemma 2.6.** Let $0 < s$, $1 \leq p \leq r_i, p_i \leq \infty$, $1 \leq q \leq \infty$ with $\frac{1}{r_i} + \frac{1}{p_i} = \frac{1}{p}$, $i = 1, 2$. Accordingly,

$$\|f_1 f_2\|_{\dot{B}^{s, \frac{2}{p}}_{p, q}} \leq c\left(\|f_1\|_{\dot{B}^{s, \frac{2}{p}}_{p, q}} \|f_2\|_{L^r_{\mathbb{T}}(\mathbb{T})} + \|f_1\|_{L^p} \|f_2\|_{\dot{B}^{s, \frac{2}{p}}_{p, q}} \right). \quad (2.9)$$
Proof. Let $E$ be an extension operator defined by the above statement. From the proof of Lemma 2.2 in [14], Lemma 2.6 holds for $E f_1$ and $E_2 f$ with homogeneous anisotropic Besov space $\dot{B}^{s,\frac{p}{2}}_{p,q}(\mathbb{R}^{n+1})$. Accordingly, we have

$$
\|f_1 f_2\|_{\dot{B}^{s,\frac{p}{2}}_{p,q}(\mathbb{R}^{n+1})} \leq c \left( \|E f_1\|_{\dot{B}^{s,\frac{p}{2}}_{p,q}(\mathbb{R}^{n+1})} \|E f_2\|_{L^1(\mathbb{R}^{n+1})} + \|E f_1\|_{L^p(\mathbb{R}^{n+1})} \|E f_2\|_{L^2(\mathbb{R}^{n+1})} \right)
$$

Hence, (2.9) holds. \qed

Remark 2.7. The properties (1) and (2) of Proposition 2.4 and Lemma 2.6 hold for Besov space $\dot{B}^{s,\frac{p}{2}}_{p,q}(\mathbb{R}^{n})$ and Sobolev space $W^p_p(\mathbb{R}^{n})$.

The next lemma pertains to the relation between $\dot{B}^{s,\frac{p}{2}}_{p,q}(\mathbb{R}^{n})$ and $C_b([0,\infty); \dot{B}^{s,\frac{p}{2}}_{p,q}(\mathbb{R}^{n}))$.

Lemma 2.8. Let $1 < p < \infty$, $1 \leq q \leq \infty$ and $s > \frac{2}{p}$. Accordingly,

$$
W^p_p \subset C_b([0,\infty); \dot{B}^{s,\frac{p}{2}}_{p,q}(\mathbb{R}^{n})) , \quad \dot{B}^{s,\frac{p}{2}}_{p,q} \subset C_b([0,\infty); \dot{B}^{s,\frac{p}{2}}_{p,q}(\mathbb{R}^{n})) .
$$

We proved Lemma 2.8 in Appendix C.

Lemma 2.9. Let $1 \leq p, q < \infty$ and $0 < s < 1$.

1) For $G \in \dot{B}^{s,\frac{p}{2}}_{p,q} \cap L^\infty$,

$$
\|\sigma (G) G\|_{\dot{B}^{s,\frac{p}{2}}_{p,q}} \leq c \|D \sigma\|_{L^\infty(0,\|G\|_{L^\infty(\mathbb{R}^d)})} \|G\|_{L^\infty(\mathbb{R}^d)} \|G\|_{\dot{B}^{s,\frac{p}{2}}_{p,q}} ,
$$

where $\sigma$ is defined in (1.5).

2) For $G \in \dot{B}^{s,\frac{p}{2}}_{p,q}$,

$$
\|\sigma (G) - \sigma (H)\|_{\dot{B}^{s,\frac{p}{2}}_{p,q}} \leq c \left( \|G - H\|_{\dot{B}^{s,\frac{p}{2}}_{p,q}} + \|H\|_{\dot{B}^{s,\frac{p}{2}}_{p,q}} \right) (\|G - H\|_{L^\infty}) ,
$$

where $c \leq C \left( \|D^2 \sigma\|_{L^\infty(0,\|G\|_{L^\infty},\|H\|_{L^\infty})} + \|D^2 \sigma\|_{L^\infty(0,\|G\|_{L^\infty},\|H\|_{L^\infty})} \right)$.

We proved Lemma 2.9 in Appendix E.

The following is the Gagliardo-Nirenberg type inequality in anisotropic spaces.

Lemma 2.10. Let $p > n + 2$ and $\eta = \frac{n+2}{p+n+2} \frac{p}{2} \frac{p}{2}$ and $\theta = \frac{n+2}{p}$. Accordingly,

$$
\|u\|_{L^p(\mathbb{R}^n \times (0,\infty))} \leq c \|u\|^\theta_{L^{n+2}(\mathbb{R}^n \times (0,\infty))} \|u\|^\eta_{\dot{B}^{s,\frac{p}{2}}_{p,q}(\mathbb{R}^n \times (0,\infty))} , \quad (2.10)
$$

$$
\|u\|_{\dot{B}^{s,\frac{p}{2}}_{p,q}(\mathbb{R}^n \times (0,\infty))} \leq c \|u\|^\theta_{L^{n+2}(\mathbb{R}^n \times (0,\infty))} \|u\|^\eta_{\dot{B}^{s,\frac{p}{2}}_{p,q}(\mathbb{R}^n \times (0,\infty))} . \quad (2.11)
$$
We proved Lemma 2.10 in Appendix F.

We now define the weak solution for the Stokes equations (1.6).

**Definition 2.11** (Weak solution of the Stokes equations). Let $1 < p < \infty$ and $1 \leq \alpha \leq 2$. Let $u_0$, $g$ and $\mathcal{F}$ satisfy the same hypotheses as in Theorem 1.3. A vector field $u \in B^{\alpha,q}_{p,q}(\mathbb{R}^n \times (0,\infty))$ is called a weak solution of the Stokes equations (1.6) if the following conditions are satisfied:

$$
\int_0^\infty \int_{\mathbb{R}^n} \left( -u \cdot (\Phi_t + \Delta \Phi) + \mathcal{F} : \nabla \Phi \right) dx \, dt = \int_0^\infty \int_{\mathbb{R}^n} g(x', t) \cdot \frac{\partial \Phi}{\partial n} \, dx' \, dt + \int_{\mathbb{R}^n} u_0(x) \cdot \Phi(x, 0) \, dx
$$

for each $\Phi \in C_\infty_c(\mathbb{R}^n_+ \times [0,\infty))$ with $\text{div}_x \Phi = 0$ and $\Phi|_{x_n=0} = 0$. In addition, for each $\Psi \in C^1_c(\mathbb{R}^n_+)$

$$
\int_{\mathbb{R}^n_+} u(x, t) \cdot \nabla \Psi(x) \, dx = 0 \quad \text{for all} \quad 0 < t < \infty.
$$

(2.12)

Based on the same method, we define the weak solution of the system (1.1).

**Definition 2.12** (Weak solution to the non-Newtonian incompressible fluid). Let $1 < p < \infty$ and $1 \leq \alpha \leq 2$. Let $u_0$ and $g$ satisfy the same hypothesis as in Theorem 1.2. A vector field $u \in B^{\alpha,q}_{p,q}(\mathbb{R}^n \times (0,\infty))$ is called a weak solution of the non-Newtonian fluids (1.1) if the following variational formulations are satisfied:

$$
\int_0^\infty \int_{\mathbb{R}^n} \left( -u \cdot (\Phi_t + S(Du) + (u \otimes u)) : \nabla \Phi \right) dx \, dt = \int_{\mathbb{R}^n} u_0(x) \cdot \Phi(x, 0) \, dx
$$

for each $\Phi \in C_\infty_c(\mathbb{R}^n_+ \times [0,\infty))$ with $\text{div}_x \Phi = 0$ and $\Phi|_{x_n=0} = 0$. In addition, for each $\Psi \in C^1_c(\mathbb{R}^n_+)$, $u$ satisfies (2.12).

(2.13)

3. Preliminaries.

In the sequel, we denote the fundamental solutions of the Laplace equation and the heat equation by $N$ and $\Gamma$, respectively,

$$
N(x) = \begin{cases} 
\frac{1}{(n-2)\omega_n} \frac{1}{|x|^{n-2}}, & n \geq 3, \\
\frac{1}{2\pi} \ln |x|, & n = 2,
\end{cases} \\
\Gamma(x,t) = \begin{cases} 
\frac{1}{\sqrt{4\pi t}} \text{e}^{-\frac{|x|^2}{4t}}, & t > 0, \\
0, & t < 0,
\end{cases}
$$

where $\omega_n$ is the measure of the unit sphere in $\mathbb{R}^n$. 
We define several integral operators:

\[ N \ast f(x,t) = \int_{\mathbb{R}^{n-1}} N(x'-y',x_n)f(y',t)dy', \]

\[ N \ast g(x) = \int_{\mathbb{R}_+^n} N(x-y)g(y)dy, \]

\[ N^\ast g(x) = \int_{\mathbb{R}_+^n} N(x-y^*)g(y)dy \quad y^* = (y',-y_n), \]

\[ \Gamma g(x,t) = \int_{\mathbb{R}_+^n} \Gamma(x-y,t)g(y)dy, \]

\[ \mathcal{U}f(x,t) = \int_0^t \int_{\mathbb{R}^{n-1}} D_x \Gamma(x'-y',x_n,t-s)f(y',s)dy'ds. \]

**Proposition 3.1.** Let \( 0 < \alpha \) and \( k \in \mathbb{N} \cup \{0\} \) and \( 1 < p < \infty \). If \( f \in L^p(0,\infty;B^{\frac{1}{2}-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})) \cap B^\frac{k}{p,\frac{1}{2}}_{p,p}((0,\infty);B^\frac{1}{p,\frac{1}{2}}_{p,p}(\mathbb{R}^{n-1})) \), then,

\[
\|\nabla N \ast f\|_{W^{2k,1}_p} \leq c(\|f\|_{L^p(0,\infty;B^{\frac{1}{2}-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}))} + \|f\|_{W^{1,1}_p((0,\infty);B^{\frac{1}{p,\frac{1}{2}}}_{p,p}(\mathbb{R}^{n-1}))}), \quad k \in \mathbb{N} \cup \{0\},
\]

\[
\|\nabla N \ast f\|_{B^{\frac{1}{2}-\frac{1}{p}}_{p,p}} \leq c\|f\|_{B^{\frac{1}{2}-\frac{1}{p}}_{p,p}}, \quad \alpha > 0.
\]

We proved Proposition 3.1 in Appendix C.

**Proposition 3.2.** Let \( g \in W^{k,\frac{1}{2}}_p \) or \( g \in B^{k,\frac{1}{2}}_{p,\frac{1}{2}} \), \( 1 < p < \infty \), \( 1 \leq q \leq \infty \). Then,

\[
\|\nabla^2 N \ast g\|_{W^{k,\frac{1}{2}}_p}, \quad \|\nabla^2 N^\ast g\|_{W^{k,\frac{1}{2}}_p} \leq c\|g\|_{W^{k,\frac{1}{2}}_p}, \quad k \geq 0,
\]

\[
\|\nabla^2 N \ast g\|_{B^{\frac{1}{2}-\frac{1}{p}}_{p,\frac{1}{2}}}, \quad \|\nabla^2 N^\ast g\|_{B^{\frac{1}{2}-\frac{1}{p}}_{p,\frac{1}{2}}} \leq c\|g\|_{B^{\frac{1}{2}-\frac{1}{p}}_{p,\frac{1}{2}}}, \quad s > 0.
\]

We proved Proposition 3.2 in Appendix D.

**Lemma 3.3.** Let \( 1 < p < \infty \), \( 1 \leq q \leq \infty \) and \( 0 < \alpha < 2 \). Let \( f \in B^{\frac{1}{2}-\frac{1}{2p}}_{p,q}(\mathbb{R}^{n-1} \times (0,\infty)) \) with \( f|_{t=0} = 0 \). Accordingly,

\[
\|\mathcal{U}f\|_{B^{\frac{1}{2}-\frac{1}{p}}_{p,q}} \leq c\|f\|_{B^{\frac{1}{2}-\frac{1}{2p}}_{p,q}(\mathbb{R}^{n-1} \times (0,\infty))}, \quad \|\mathcal{U}f\|_{W^{1,1}_p(\mathbb{R}^{n-1} \times (0,\infty))} \leq c\|f\|_{B^{\frac{1}{2}-\frac{1}{2p}}_{p,q}(\mathbb{R}^{n-1} \times (0,\infty))}.
\]

**Proof.** Let \( \tilde{f} \) be a zero extension of \( f \) with respect to \( t \). Since \( f|_{t=0} = 0 \), \( \|\tilde{f}\|_{B^{\frac{1}{2}-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1} \times \mathbb{R})} \leq c\|f\|_{B^{\frac{1}{2}-\frac{1}{2p}}_{p,q}(\mathbb{R}^{n-1} \times (0,\infty))} \). Let \( \mathcal{U}\tilde{f}(x,t) = \int_{-\infty}^t \int_{\mathbb{R}^{n-1}} D_x \Gamma(x'-y',x_n,t-s)\tilde{f}(y',s)dy'ds \). From [40], we obtain the following estimate

\[
\|\mathcal{U}f\|_{W^{1,1}_p} \leq c\|\mathcal{U}\tilde{f}\|_{W^{1,1}_p(\mathbb{R}^{n-1} \times \mathbb{R})} \leq c\|\tilde{f}\|_{B^{\frac{1}{2}-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1} \times \mathbb{R})} \leq c\|f\|_{B^{\frac{1}{2}-\frac{1}{2p}}_{p,q}(\mathbb{R}^{n-1} \times (0,\infty))}. \quad (3.1)
\]
By the same argument in the proof of Lemma 3.6 in [18],
\[ \| \mathcal{A} f \|_{L^p} \leq c \| \mathcal{A} f \|_{L^p(\mathbb{R}^n \times \mathbb{R})} \leq c \| f \|_{\dot{B}^p_{p,q}(\mathbb{R}^{n-1} \times \mathbb{R})} \leq c \| f \|_{\dot{B}^p_{p,q}(\mathbb{R}^{n-1} \times (0, \infty))}, \tag{3.2} \]
where \( \dot{B}^p_{p,q}(\mathbb{R}^{n-1} \times \mathbb{R}) \) is the dual space of \( \dot{B}^p_{p,q}(\mathbb{R}^{n-1} \times \mathbb{R}) \). Using the properties of complex interpolation and real interpolation between (3.1) and (3.2), we obtain Lemma 3.3. \( \square \)

4. PROOF OF THEOREM 1.3

Because the proofs are similar, we only prove (1) of Theorem 1.3. We will follow the proof of Theorem 1.2 in [18].

Let \( \mathcal{F} \) be an extension of \( \mathcal{F} \) over \( \mathbb{R}^n \times (0, \infty) \) such that \( \| \mathcal{F} \|_{\dot{B}^{1-q}_{p,q}(\mathbb{R}^n \times (0, \infty))} \leq c \| \mathcal{F} \|_{\dot{B}^{1-q}_{p,q}(\mathbb{R}^n \times (0, \infty))} \) and \( \bar{u}_0 \) be an extension of \( u_0 \) with \( \|ar{u}_0\|_{\dot{B}^{1-q}_{p,q}(\mathbb{R}^n \times (0, \infty))} \leq c \| u_0 \|_{\dot{B}^{1-q}_{p,q}(\mathbb{R}^n \times (0, \infty))} \). Let
\[
\begin{align*}
  w^1(x, t) &= \int_0^t \int_{\mathbb{R}^n} \Gamma(x - y, t - s) \mathbb{P} \div \mathcal{F}(y, s) dy ds, \\
  w^2(x, t) &= \int_{\mathbb{R}^n} \Gamma(x - y, t) \bar{u}_0(y) dy,
\end{align*}
\]
where \( (\mathbb{P} f)_t = f_t - \sum_{1 \leq j \leq n} R_i R_j f_j \) is the Helmholtz decomposition operator in \( \mathbb{R}^n \). The following are well-known results
\[ \|w^1\|_{\dot{B}^{k-1}_{p,q}(\mathbb{R}^n \times (0, \infty))} \leq c \| \mathcal{F} \|_{\dot{B}^{k-1}_{p,q}(\mathbb{R}^n \times (0, \infty))} \leq c \| u_0 \|_{\dot{B}^{k-1}_{p,q}(\mathbb{R}^n \times (0, \infty))}, \tag{4.1} \]
\[ \|w^2\|_{\dot{B}^{k-1}_{p,q}(\mathbb{R}^n \times (0, \infty))} \leq c \| \mathcal{F} \|_{\dot{B}^{k-1}_{p,q}(\mathbb{R}^n \times (0, \infty))} \leq c \| u_0 \|_{\dot{B}^{k-1}_{p,q}(\mathbb{R}^n \times (0, \infty))}, \tag{4.2} \]
Using the property of real interpolation, we have
\[ \|w^1\|_{\dot{B}^{k-1}_{p,q}} \leq c \| \mathcal{F} \|_{\dot{B}^{k-1}_{p,q}}, \quad \|w^2\|_{\dot{B}^{k-1}_{p,q}} \leq c \| u_0 \|_{\dot{B}^{k-1}_{p,q}} \quad 1 < \alpha. \tag{4.3} \]

By trace theorems for usual homogeneous Sobolev space \( \dot{W}^k_p(\mathbb{R}^n \times (0, \infty)) \) (see (4) of Proposition 2.4), for \( k \in \mathbb{N} \), we have
\[ \|w^1|_{x_n=0}\|_{L^p(0, \infty; W^k_p(0, \infty))} \leq c \|w^1\|_{L^p(0, \infty; W^k_p(0, \infty))} \leq c \|w^1\|_{\dot{W}^{2k,k}_p}. \tag{4.4} \]
From Section 5.2 in [18], we have
\[ \|w^1|_{x_n=0}\|_{W^k_p(0, \infty; \dot{B}^{1-q}_{p,q}(\mathbb{R}^{n-1} \times (0, \infty)))} \leq c \|w^1\|_{L^p(0, \infty; \dot{B}^{1-q}_{p,q}(\mathbb{R}^{n-1} \times (0, \infty)))} \leq c \|w^1\|_{\dot{W}^{2k,k}_p}. \tag{4.5} \]
From (4.4), (4.5), the definition of \( \dot{A}^{1-q}_{p,q}(\mathbb{R}^{n-1} \times (0, \infty)) \) and the properties of the real interpolations (see (2.8)),
\[ \|w^1|_{x_n=0}\|_{\dot{A}^{1-q}_{p,q}} \leq c \|w^1\|_{\dot{A}^{1-q}_{p,q}}, \quad \alpha > 1. \tag{4.6} \]
Let \( \phi(x, t) = \int_{\mathbb{R}^{n-1}} N(x' - y', x_n) h(y', t) dy', w^3 = \nabla \phi \) and \( \pi = -\phi_t \), where
\[
h := g_{n}(y', t) - \frac{w_1}{n}|x_n=0 - \frac{w_2}{n}|x_n=0.\]
From the compatibility condition (1.3), we have \( h(x', 0) = 0 \).
Accordingly, \((w^3, \pi)\) satisfies
\[
\begin{cases}
  \ w^3_t - \Delta w^3 + \nabla \pi = 0 \quad \text{in} \quad \mathbb{R}^n_+ \times (0, \infty), \\
  \ \text{div} \ w^3 = 0 \quad \text{in} \quad \mathbb{R}^n_+ \times (0, \infty), \\
  \ w^3|_{t=0} = 0, \\
  \ w^3|_{x_n=0} = (R'_i h, \cdots, R'_{n-1} h, h),
\end{cases}
\]
where \( R'_i, i = 1, \cdots, n - 1 \) are \( n - 1 \) dimensional Riesz transform. Moreover, from Proposition 3.1, (4.3) and (4.6), we have
\[
\|w^3\|_{B^a}_{p, q, \alpha} (\mathbb{R}^n_+ \times (0, \infty)) \leq c \|h\|_{A^{a - \frac{\alpha}{p} - \frac{1}{2}} (\mathbb{R}^{n-1} \times (0, \infty))} \leq c \left( \|g_n\|_{A^{a - \frac{\alpha}{p} - \frac{1}{2}} (\mathbb{R}^{n-1} \times (0, \infty))} + \|w^1\|_{B^a}_{p, q, \alpha} (\mathbb{R}^n_+ \times (0, \infty))} + \|w^2\|_{B^a}_{p, q, \alpha} (\mathbb{R}^n_+ \times (0, \infty))} \right).
\]
Let \( G := g - w^1|_{x_n=0} - w^2|_{x_n=0} - w^3|_{x_n=0} \). Note that from the compatibility condition (1.3) \( G|_{t=0} = g|_{t=0} - u_0 |_{x_n=0} = 0 \) and \( G_n = 0 \). We solve the following equations
\[
\begin{cases}
  \ w^4_t - \Delta w^4 + \nabla q = 0 \quad \text{in} \quad \mathbb{R}^n_+ \times (0, \infty), \\
  \ \text{div} \ w^4 = 0 \quad \text{in} \quad \mathbb{R}^n_+ \times (0, \infty), \\
  \ w^4|_{t=0} = 0, \quad w^4|_{x_n=0} = G.
\end{cases}
\]
According to Section 5 in [17], \( w^4 \) can be rewritten in the following form
\[
w^4(x, t) = -\frac{1}{2} \nabla \mathcal{S}(x, t) - 4 \mathcal{S}_{\pi}(x, t) + 4 \frac{\partial}{\partial x_i} \mathcal{S}(x, t), \quad i = 1, \cdots, n,
\]
where \( \mathcal{S} \) is defined by
\[
\mathcal{S}(x, t) = \int_{\mathbb{R}^n_+} \nabla_y (N(x - y) - N(x - y^*)) \cdot F(y, t) dy,
\]
where
\[
F_j := -\frac{1}{2} \nabla \mathcal{G}_j, \quad j = 1, \cdots, n, \quad F_n := \sum_{j=1}^{n-1} \nabla \mathcal{R}_j \mathcal{G}_j(x, t).
\]
Let \( \tilde{G} \) be a zero extension over \( \mathbb{R}^{n-1} \times \mathbb{R} \). Accordingly, from Proposition 3.2 and Lemma 3.3
\[
\|w^4\|_{B^a}_{p, q, \alpha} \leq c \|\nabla \tilde{G}\|_{B^a}_{p, q, \alpha} (\mathbb{R}^n \times \mathbb{R}) \leq c \|G\|_{A^{a - \frac{\alpha}{p} - \frac{1}{2}} (\mathbb{R}^{n-1} \times (0, \infty))} \leq c \left( \|g\|_{A^{a - \frac{\alpha}{p} - \frac{1}{2}} (\mathbb{R}^{n-1} \times (0, \infty))} + \|w^1\|_{B^a}_{p, q, \alpha} (\mathbb{R}^n_+ \times (0, \infty))} + \|w^2\|_{B^a}_{p, q, \alpha} (\mathbb{R}^n_+ \times (0, \infty))} \right).
\]
Note that \( w = w^1 + w^2 + w^3 + w^4 \) is solution of equation (1.6) and summing all estimates, we obtain the estimates (1.7) and (1.8).
Next, we show the uniqueness of solution in $W^{1,\frac{2}{p}}_p$, $p < n + 2$. Let $p^* = \frac{(n+2)p}{n+2-p}$. Note that $W^{1,\frac{2}{p}}_p \subset L^{p^*}$. Suppose that $(u_1,p_1)$ and $(u_2,p_2)$ are weak solutions of the Stokes equations (1.6) in the class $L^{p^*}$ with the same data, then $u_1 - u_2$ satisfies the variational formulation

$$\int_0^\infty \int_{\mathbb{R}^n} (u_1 - u_2) \cdot (-\phi_t - \Delta \phi + \nabla \pi) dx dt = 0$$

for any $\phi \in C_0^\infty(\mathbb{R}^n_+ \times [0, \infty))$ with $\text{div}_x \phi = 0$, $\phi|_{x_n=0} = 0$ for all $t \in (0, \infty)$. Since $\{-\phi_t - \Delta \phi + \nabla \pi : \phi \in C_0^\infty(\mathbb{R}^n_+ \times [0, \infty))$ with $\text{div} \phi(t, \cdot) = 0$, $\phi|_{x_n=0} = 0\}$ is dense in $L^{(q')'}$, we conclude that $u_1 - u_2 = 0$ a.e. in $\mathbb{R}^n_+ \times (0, \infty)$. Therefore, the uniqueness of the solution of the Stokes system (1.6) holds in the class $L^{p^*}$.

5. PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2 by constructing approximate velocities and deriving their uniform convergence in $B^{\alpha, \frac{q}{2}}_{\frac{1}{2}} \cap B^{\frac{n+1}{2}, \frac{n+2}{2}}_{1} \cap W^{1,\frac{2}{p}}$.

5.1. Approximating solutions. Let $u^0 = 0$ and $p^0 = 0$. After obtaining $(u^1, p^1), \cdots, (u^m, p^m)$ construct $(u^{m+1}, p^{m+1})$ which satisfies the following equations

$$u^{m+1}_t - \Delta u^{m+1} + \nabla p^{m+1} = \text{div} f^m, \quad \text{div} u^{m+1} = 0, \quad \text{in} \, \mathbb{R}^n_+ \times (0, \infty),$$

$$u^{m+1}|_{t=0} = u_0, \quad u^{m+1}|_{x_n=0} = g,$$

where $f^m = \sigma(Du^m)Du^m - u^m \otimes u^m$ for $m \geq 1$.

From Theorem 1.3 we have

$$\|u^1\|_{W^{1,\frac{2}{p}}_{\frac{1}{2}}} \leq cM_{01}, \quad \|u^1\|_{B^{\frac{n+1}{2}, \frac{n+2}{2}}_{1}} \leq cM_{02}, \quad \|u^1\|_{B^{\alpha, \frac{q}{2}}_{\frac{1}{2}}} \leq cM_{03},$$

(5.2)

where

$$M_{01} := \|u_0\|_{B^{\frac{1}{p}, \frac{n+1}{2}}_{\frac{2}{p}}(\mathbb{R}^n_+)} + \|g\|_{B^{\frac{1}{p}, \frac{n+1}{2}}_{\frac{2}{p}}(\mathbb{R}^{n-1} \times (0, \infty))} + \|g_n\|_{A^{\frac{1}{p}, \frac{n+1}{2}}_{\frac{2}{p}}(\mathbb{R}^{n-1} \times (0, \infty))},$$

$$M_{02} := \|u_0\|_{B^{\frac{1}{p}, \frac{n+1}{2}}_{\frac{2}{p}}(\mathbb{R}^n_+)} + \|g\|_{B^{\frac{1}{p}, \frac{n+1}{2}}_{\frac{2}{p}}(\mathbb{R}^{n-1} \times (0, \infty))} + \|g_n\|_{A^{\frac{1}{p}, \frac{n+1}{2}}_{\frac{2}{p}}(\mathbb{R}^{n-1} \times (0, \infty))},$$

$$M_{03} := \|u_0\|_{B^{\alpha, \frac{q}{2}}_{\frac{1}{2}}(\mathbb{R}^n_+)} + \|g\|_{B^{\alpha, \frac{q}{2}}_{\frac{1}{2}}(\mathbb{R}^{n-1} \times (0, \infty))} + \|g_n\|_{A^{\alpha, \frac{q}{2}}_{\frac{1}{2}}(\mathbb{R}^{n-1} \times (0, \infty))}.$$

Applying Theorem 1.3 we have

$$\|u^{m+1}\|_{W^{1,\frac{2}{p}}_{\frac{1}{2}}} \leq c(M_{01} + \|u^m \otimes u^m\|_{L^{\frac{n+2}{2}}} + \|\sigma(Du^m)Du^m\|_{\frac{n+2}{2}}),$$

(5.3)

$$\|u^{m+1}\|_{B^{\frac{n+1}{2}, \frac{n+2}{2}}_{1}} \leq c(M_{02} + \|u^m \otimes u^m\|_{B^{\frac{n+1}{2}, \frac{n+2}{2}}_{1}} + \|\sigma(Du^m)Du^m\|_{B^{\frac{n+1}{2}, \frac{n+2}{2}}_{1}}),$$

(5.4)

$$\|u^{m+1}\|_{B^{\alpha, \frac{q}{2}}_{\frac{1}{2}}} \leq c(M_{03} + \|u^m \otimes u^m\|_{B^{\alpha, \frac{q}{2}}_{\frac{1}{2}}} + \|\sigma(Du^m)Du^m\|_{B^{\alpha, \frac{q}{2}}_{\frac{1}{2}}}).$$

(5.5)
From Hölder inequality and \((2)\) in Proposition 2.4,

\[
\|u^m \otimes u^n\|_{L^{\frac{2n}{n-2}}} \leq c \|u^m\|^2_{L^{n+2}} \leq c \|u^m\|^2_{H^{\frac{1}{2}}}. \tag{5.6}
\]

From the definition of anisotropic Besov space, we get \(\|Du^m\|_{L^\infty} \leq c \|u^m\|_{B^{\frac{n+2}{2}, \frac{2}{2}+\frac{n+2}{2p}}} \). Note that \(\sigma(Du^m) = |F(Du^m) - F(0)| \leq c \|Du^m\|\). Then, from Lemma 2.9 if \(\|u^m\|_{L^\infty} \leq 1\), we have

\[
\|\sigma(Du^m)Du^m\|_{L^{\frac{n+2}{2}}} \leq c \|Du^m\|_{L^\infty} \|Du^m\|_{L^{\frac{n+2}{2}}} \leq c \|Du^m\|\tag{5.7}\]

From (5.3), (5.6) and (5.7),

\[
\|u^{m+1}\|_{{\dot{W}}^{\frac{1}{2}}} \leq c \left( M_{01} + \|u^m\|^2 + \|u^m\|_{B^{\frac{n+2}{2}, \frac{2}{2}+\frac{n+2}{2p}}} \right).	ag{5.8}
\]

From Lemma 2.6 and Lemma 2.10, we have

\[
\|u^m \otimes u^n\|_{{\dot{B}}^{\frac{n+2}{2}, \frac{2}{2}+\frac{n+2}{2p}}} \leq c \|u^m\|^2_{{\dot{B}}^{\frac{n+2}{2}, \frac{2}{2}+\frac{n+2}{2p}}} \leq c \|u^m\|^2_{{\dot{B}}^{\frac{n+2}{2}, \frac{2}{2}+\frac{n+2}{2p}}} \leq c \left( \|u^m\|^2_{{\dot{W}}^{\frac{1}{2}}} + \|u^m\|^2_{{\dot{B}}^{\frac{n+2}{2}, \frac{2}{2}+\frac{n+2}{2p}}} \right) \quad \tag{5.9}
\]

From Lemma 2.9 if \(\|Du^m\|_{L^\infty} \leq 1\), then

\[
\|\sigma(Du^m)Du^m\|_{{\dot{B}}^{\frac{n+2}{2}, \frac{2}{2}+\frac{n+2}{2p}}} \leq c \|Du^m\|_{L^\infty} \|Du^m\|_{{\dot{B}}^{\frac{n+2}{2}, \frac{2}{2}+\frac{n+2}{2p}}} \leq c \|u^m\|^2_{{\dot{B}}^{\frac{n+2}{2}, \frac{2}{2}+\frac{n+2}{2p}}} \tag{5.10}
\]

From (5.4), (5.9) and (5.10), we have

\[
\|u^{m+1}\|_{{\dot{B}}^{\frac{n+2}{2}, \frac{2}{2}+\frac{n+2}{2p}}} \leq c \left( M_{02} + \|u^m\|^2_{{\dot{W}}^{\frac{1}{2}}} + \|u^m\|^2_{{\dot{B}}^{\frac{n+2}{2}, \frac{2}{2}+\frac{n+2}{2p}}} \right) \quad \tag{5.11}
\]

From (1) of Proposition 2.2 we have \(B_{\frac{a-1}{a}}^{\frac{a+2}{a}, \frac{2}{2}} \subset B_{\frac{a}{a+2}}^{\frac{a+2}{a}, \frac{2}{2}}\). Then, from Lemma 2.10 we have

\[
\|u^m\|_{{\dot{B}}^{\frac{a-1}{a+2}, \frac{2}{2}}} \leq c \|u^m\|_{{\dot{W}}^{\frac{1}{2}}} \leq c \left( \|u^m\|_{{\dot{W}}^{\frac{1}{2}}} + \|u^m\|_{{\dot{B}}^{\frac{a+2}{a}, \frac{2}{2}}} \right) \quad \tag{5.12}
\]
and since \(|Du^m|_{L^\infty} \leq c|u^m|_{L^{\alpha/2 + 1, \frac{2n}{n+2}}_{B,p,1}}\) from Lemma 2.9 if \(|u^m|_{L^\infty} \leq 1\), then we have

\[
\|\sigma(Du^m)Du^m\|_{a-1, \frac{2n}{n+2}}_{B,p,q} \leq c\|Du^m\|_{L^\infty} \|Du^m\|_{a-1, \frac{2n}{n+2}}_{B,p,q} \leq c\|u^m\|_{a-1, \frac{2n}{n+2}}_{B,p,q} \|u^m\|_{\alpha/2, \frac{n}{2}}_{B,p,q}.
\] (5.13)

From (5.5), (5.12), and (5.13),

\[
\|u^{m+1}\|_{a-1, \frac{2n}{n+2}}_{B,p,q} \leq c(M_{03} + \|u^m\|_{W^{1, \frac{n}{4}}_{B,p,1}} + \|u^m\|_{a-1, \frac{2n}{n+2}}_{B,p,1} + \|u^m\|_{\alpha/2, \frac{n}{2}}_{B,p,1}).
\] (5.14)

We take \(\delta_0\) satisfying \(\delta_0 < \min(\frac{1}{3\delta}, 1)\). Let

\[cM_{01}, \ cM_{02}, \ cM_{03} < \frac{1}{3}\delta_0.\]

From (5.2), we have

\[
\|u^1\|_{W^{1, \frac{n}{4}}_{B,p,1}}, \ |u^1|_{W^{1, \frac{n}{4}}_{B,p,1}}, \ |u^1|_{a-1, \frac{2n}{n+2}}_{B,p,q} < \frac{1}{3}\delta_0 < \delta_0.
\]

Suppose that

\[
\|u^m\|_{W^{1, \frac{n}{4}}_{B,p,1}}, \ |u^m|_{W^{1, \frac{n}{4}}_{B,p,1}}, \ |u^m|_{a-1, \frac{2n}{n+2}}_{B,p,q} < \delta_0.
\]

Accordingly, from (5.8), (5.11), and (5.14),

\[
\|u^{m+1}\|_{W^{1, \frac{n}{4}}_{B,p,1}} \leq c\big(M_{01} + \|u^m\|_{W^{1, \frac{n}{4}}_{B,p,1}} + \|u^m\|_{a-1, \frac{2n}{n+2}}_{B,p,1} \|u^m\|_{W^{1, \frac{n}{4}}_{B,p,1}}\big)
\]

\[
\leq \left(\frac{1}{3}\delta_0 + c\delta_0^2 + c\delta_0^2\right) < \delta_0,
\]

\[
\|u^{m+1}\|_{a-1, \frac{2n}{n+2}}_{B,p,1} \leq c\big(M_{02} + \|u^m\|_{W^{1, \frac{n}{4}}_{B,p,1}} + \|u^m\|_{a-1, \frac{2n}{n+2}}_{B,p,1} \|u^m\|_{W^{1, \frac{n}{4}}_{B,p,1}}\big)
\]

\[
\leq \left(\frac{1}{3}\delta_0 + c\delta_0^2 + c\delta_0^2\right) < \delta_0,
\]

\[
\|u^{m+1}\|_{a-1, \frac{2n}{n+2}}_{B,p,q} \leq c\big(M_{03} + \|u^m\|_{W^{1, \frac{n}{4}}_{B,p,1}} + \|u^m\|_{a-1, \frac{2n}{n+2}}_{B,p,1} \|u^m\|_{W^{1, \frac{n}{4}}_{B,p,1}}\big)
\]

\[
\leq \left(\frac{1}{3}\delta_0 + c\delta_0^2 + c\delta_0^2\right) < \delta_0.
\]

Hence, for all \(m \geq 1\),

\[
\|u^m\|_{W^{1, \frac{n}{4}}_{B,p,1}}, \ |u^m|_{W^{1, \frac{n}{4}}_{B,p,1}}, \ |u^m|_{a-1, \frac{2n}{n+2}}_{B,p,q} < \delta_0.
\] (5.15)
5.2. Uniform convergence. Let $U^m = u^{m+1} - u^m$ and $P^m = p^{m+1} - p^m$. Accordingly, $(U^m, P^m)$ satisfies the following equations

$$
U^m_t - \Delta U^m + \nabla P^m = -\text{div}(u^m \otimes u^m - u^{m-1} \otimes u^{m-1})
- \text{div}(\sigma(Du^m)Du^m - \sigma(Du^{m-1})Du^{m-1}),
\text{div } U^m = 0, \text{ in } \mathbb{R}^n \times (0, \infty),
U^m|_{t=0} = 0, \quad U^m|_{x_n=0} = 0.
$$

Note that

$$
u^m \otimes u^m - u^{m-1} \otimes u^{m-1} = u^m \otimes U^m - U^{m-1} \otimes u^{m-1},
\sigma(Du^m)Du^m - \sigma(Du^{m-1})Du^{m-1} = \sigma(Du^m)DU^m - (\sigma(Du^m) - \sigma(Du^{m-1}))Du^{m-1}.
$$

From Hölder inequality, (2) in Proposition 2.4, we have

$$
||u^m \otimes u^m - u^{m-1} \otimes u^{m-1}||_{L^{\frac{2n^2}{n+2}}} \leq c\left(||u^m||_{L^{n+2}}||U^m||_{L^{n+2}} + ||u^{m-1}||_{L^{n+2}}||U^{m-1}||_{L^{n+2}}\right)
\leq c\left(||u^m||_{W^{1,\frac{2}{n+2}}} + ||u^{m-1}||_{W^{1,\frac{2}{n+2}}}\right)||U^m||_{W^{1,\frac{2}{n+2}}}.
$$

Note that since $\sigma(0) = 0$, by mean-value theorem, we have $|\sigma(Du^m)| \leq ||D\sigma||_{L^\infty(0, ||Du^m||_{L^\infty})}||Du^m||$, and $|\sigma(Du^m) - \sigma(Du^{m-1})| \leq ||D\sigma||_{L^\infty(0, \max(||Du^m||_{L^\infty}, ||Du^{m-1}||_{L^\infty}))}||Du^m - Du^{m-1}||$. Hence, we have

$$
||\sigma(Du^m)Du^m - \sigma(Du^{m-1})Du^{m-1}||_{L^{\frac{2n^2}{n+2}}} \leq c\left(||Du^m||_{L^{\infty}} + ||Du^{m-1}||_{L^{\infty}}\right)||U^m||_{L^{n+2}}
\leq c\left(||u^m||_{B^{1+\frac{n+2}{2p}, \frac{n+2}{2p}}_{p,1}} + ||u^{m-1}||_{B^{1+\frac{n+2}{2p}, \frac{n+2}{2p}}_{p,1}}\right)||U^m||_{W^{1,\frac{2}{n+2}}}
$$

From Lemma 2.10, we have

$$
||u^m||_{B^{p, \frac{n^2+2}{2p}}_{p,1}} \leq c||u^m||_{B^{p, \frac{n^2+2}{2p}}_{p,1}} ||u^m||_{W^{1, \frac{2}{n+2}}} \leq c\left(||u^m||_{W^{1, \frac{2}{n+2}}} + ||u^m||_{B^{1+\frac{n+2}{2p}, \frac{n+2}{2p}}_{p,1}}\right),
$$

(5.16)

$$
||u^m||_{L^{\infty}} \leq c||u^m||_{B^{p, \frac{n^2+2}{2p}}_{p,1}} \leq c\left(||u^m||_{W^{1, \frac{2}{n+2}}} + ||u^m||_{B^{1+\frac{n+2}{2p}, \frac{n+2}{2p}}_{p,1}}\right),
$$

(5.17)

$$
||u^m||_{B^{-1, \frac{n^2+2}{2p}}_{p,q}} \leq c||u^m||_{B^{\frac{2}{p+q}, \frac{n^2+2}{2p+q}}_{p,q}} ||u^m||_{\frac{2}{p+q}} \leq c\left(||u^m||_{W^{\frac{2}{p+q}, \frac{n^2+2}{2p+q}}} + ||u^m||_{B^{\frac{2}{p+q}, \frac{n^2+2}{2p+q}}_{p,q}}\right),
$$

(5.18)

For the first inequality of the third equation, we used the fact $B^{p, \frac{n^2+2}{2p}}_{p,1} \subset B^{p, \frac{n^2+2}{2p}}_{p,q}$. 

From (5.16), (5.17) and (5.18), we have
\[ \|u^m \otimes U^{m-1}\|_{B_{p,q}^{a-1, \frac{d-1}{2}}} \leq c\left(\|u^m\|_{L^\infty} \|U^{m-1}\|_{L^\infty} + \|U^m\|_{L^\infty} \right) \leq cA_mB_{m-1}, \tag{5.19} \]
where
\[ A_m = \|u^m\|_{W^{1, \frac{1}{2}}_{p,\frac{p}{2}}} + \|U^m\|_{B_{p, \frac{p}{2}}^{1, \frac{1}{2} + \frac{d+2}{2p}}} + \|U^m\|_{B_{p, \frac{p}{2}}^{1, \frac{1}{2} + \frac{d+2}{2p}}} \]
\[ B_m = \|U^m\|_{W^{1, \frac{1}{2}}_{p,\frac{p}{2}}} + \|U^m\|_{B_{p, \frac{p}{2}}^{1, \frac{1}{2} + \frac{d+2}{2p}}} + \|U^m\|_{B_{p, \frac{p}{2}}^{1, \frac{1}{2} + \frac{d+2}{2p}}} \]
\[ \|u^m \otimes U^{m-1}\|_{B_{p,q}^{a-1, \frac{d-1}{2}}} \leq c\left(\|u^m\|_{L^\infty} \|U^{m-1}\|_{a-1, \frac{d-1}{2}} + \|U^m\|_{L^\infty} \right) \leq cA_mB_{m-1}. \tag{5.20} \]
\[ \|\sigma(Du^m)DU^{m-1}\|_{B_{p,q}^{a-1, \frac{d-1}{2}}} \leq c\left(\|\sigma(Du^m)\|_{L^\infty} \|DU^{m-1}\|_{a-1, \frac{d-1}{2}} + \|\sigma(Du^m)\|_{L^\infty} \right) \leq cA_mB_{m-1}. \tag{5.21} \]
\[ \|\sigma(Du^m)DU^{m-1}\|_{B_{p,q}^{a-1, \frac{d-1}{2}}} \leq c\left(\|\sigma(Du^m)\|_{L^\infty} \|DU^{m-1}\|_{a-1, \frac{d-1}{2}} + \|\sigma(Du^m)\|_{L^\infty} \right) \leq cA_mB_{m-1}. \tag{5.22} \]
Finally,
\[ \|(\sigma(Du^m) - \sigma(Du^{m-1}))Du^m\|_{B_{p,q}^{a-1, \frac{d-1}{2}}} \leq c\left\|(\sigma(Du^m) - \sigma(Du^{m-1}))\|_{L^\infty} \|DU^{m-1}\|_{a-1, \frac{d-1}{2}} + \|(\sigma(Du^m)\|_{L^\infty} \right) \leq cA_mB_{m-1}. \tag{5.23} \]
The first term is dominated by
\[ \|DU^{m-1}\|_{L^\infty} \|u^m\|_{B_{p,q}^{a-\frac{1}{2}, \frac{d+2}{2}}} \leq c\|U^{m-1}\|_{B_{p,q}^{1, \frac{1}{2} + \frac{d+2}{2p}}} \|u^m\|_{B_{p,q}^{a-\frac{1}{2}, \frac{d+2}{2}}} \leq cA_mB_{m-1}. \tag{5.24} \]
The second term is dominated by
\[ \left(\|DU^{m-1}\|_{a-1, \frac{d-1}{2}} + \|DU^m\|_{a-1, \frac{d-1}{2}} + \|DU^{m-1}\|_{a-1, \frac{d-1}{2}}\right) \|U^m\|_{B_{p,q}^{1, \frac{1}{2} + \frac{d+2}{2p}}} \leq cA_m\left(1 + (A_m + A_{m-1})\right)B_{m-1}. \tag{5.25} \]
Hence, we have
\[
\| (\sigma(Du^m) - \sigma(Du^{m-1}))Du^m \|_{B_{p,q}^{a-\frac{1}{2}} \cap \tilde{B}_{p,q}^{a+\frac{1}{2}}} \leq c A_m \left( 1 + (A_m + A_{m-1}) \right) B_{m-1}. \tag{5.26}
\]
Similarly, we have
\[
\| (\sigma(Du^m) - \sigma(Du^{m-1}))Du^m \|_{B_{p,1}^{a+\frac{1}{2}} \cap \tilde{B}_{p,1}^{a-\frac{1}{2}}} \leq c A_m \left( 1 + (A_m + A_{m-1}) \right) B_{m-1}. \tag{5.27}
\]
Summing all estimates, from Theorem 1.3, we have
\[
\| U^m \|_{W^{1,\frac{1}{2}}_{a+\frac{1}{2}}, \frac{n+2p}{2}} \leq c \left( \| u^m \otimes u^m - u^{m-1} \otimes u^{m-1} \|_{W^{1,\frac{1}{2}}_{a+\frac{1}{2}}, \frac{n+2p}{2}} + \| \sigma(Du^m)Du^m - \sigma(Du^{m-1})Du^{m-1} \|_{L^{q+\frac{2}{n}}} \right),
\]
\[
\leq c \left( \| u^m \|_{W^{1,\frac{1}{2}}_{a+\frac{1}{2}}, \frac{n+2p}{2}} + \| Du^m \|_{L^{q+\frac{2}{n}}} \right), \tag{5.28}
\]
\[
\| U^m \|_{B_{p,1}^{a+\frac{1}{2}} \cap \tilde{B}_{p,1}^{a-\frac{1}{2}}} \leq c \left( \| u^m \otimes u^m - u^{m-1} \otimes u^{m-1} \|_{B_{p,1}^{a+\frac{1}{2}} \cap \tilde{B}_{p,1}^{a-\frac{1}{2}}} + \| \sigma(Du^m)Du^m - \sigma(Du^{m-1})Du^{m-1} \|_{B_{p,1}^{a+\frac{1}{2}} \cap \tilde{B}_{p,1}^{a-\frac{1}{2}}} \right), \tag{5.29}
\]
\[
\| U^m \|_{B_{p,q}^{a+\frac{1}{2}}} \leq c \left( \| u^m \otimes u^m - u^{m-1} \otimes u^{m-1} \|_{B_{p,q}^{a+\frac{1}{2}}} + \| \sigma(Du^m)Du^m - \sigma(Du^{m-1})Du^{m-1} \|_{B_{p,q}^{a+\frac{1}{2}}} \right), \tag{5.30}
\]
Taking \( c \delta_0 < \frac{1}{2} \), we obtain
\[
B_m < \frac{1}{2} B_{m-1}. \tag{5.31}
\]
This implies that \( \{u_m\} \) is Cauchy sequence in \( W^{1,\frac{1}{2}}_{a+\frac{1}{2}}, \frac{n+2p}{2} \cap B_{p,1}^{a+\frac{1}{2}} \cap \tilde{B}_{p,1}^{a-\frac{1}{2}} \). In particular, \( \{u_m\} \) is Cauchy sequence in \( W^{1,\frac{1}{2}}_{a+\frac{1}{2}}, \frac{n+2p}{2} \cap \tilde{B}_{p,q}^{a+\frac{1}{2}} \). Since \( W^{1,\frac{1}{2}}_{a+\frac{1}{2}}, \frac{n+2p}{2} \cap \tilde{B}_{p,q}^{a+\frac{1}{2}} \) is complete (see Theorem B.2), there is \( u \in W^{1,\frac{1}{2}}_{a+\frac{1}{2}}, \frac{n+2p}{2} \cap \tilde{B}_{p,q}^{a+\frac{1}{2}} \) such that \( u_m \) converges to \( u \) in \( W^{1,\frac{1}{2}}_{a+\frac{1}{2}}, \frac{n+2p}{2} \cap \tilde{B}_{p,q}^{a+\frac{1}{2}} \).

5.3. Existence. Let \( u \) be the same one constructed in the previous section. Since \( u^m \rightarrow u \) in \( W^{1,\frac{1}{2}}_{a+\frac{1}{2}}, \frac{n+2p}{2} \cap \tilde{B}_{p,q}^{a+\frac{1}{2}} \), and by (5.15), we have that \( u|_{x_n=0} = g \) and
\[
\| u \|_{W^{1,\frac{1}{2}}_{a+\frac{1}{2}}} < \delta_0.
\]
Moreover, since \( Du^m \rightarrow Du \) in \( L^{q+\frac{2}{n}} \) and \( \| Du^m \|_{L^{q+\frac{2}{n}}} \leq c \| u^m \|_{B_{p,1}^{a+\frac{1}{2}}} \leq c \delta_0 \) for all \( m \) by (5.15), we have \( \| Du \|_{L^{q+\frac{2}{n}}} \leq c \delta_0 \).
In this section, we will show that \( u \) satisfies weak formulation \((2.13)\), that is, \( u \) is a weak solution of \((1.1)\) with appropriate distribution \( p \). Let \( \Phi \in C^0_0(\mathbb{R}^n_+ \times [0, \infty)) \) with \( \text{div} \Phi = 0 \).

Observe that
\[
\int_{\mathbb{R}^n_+} u_0(x) \cdot \Phi(x) dx = \int_0^\infty \int_{\mathbb{R}^n_+} -u^{m+1} \cdot \Phi_t + ((u^m \otimes u^m) : \nabla \Phi + S(Du^m)) : \nabla \Phi dx dt.
\]

Note that \( u^m \rightarrow u \) in \( W^{1, \frac{n}{2}} \) and so \( u^m \rightarrow u \) in \( L^{n+2} \).

Let \( K \) be a compact subset of \( \mathbb{R}^n_+ \times (0, \infty) \). Note that
\[
\|u^m \otimes u^m - u \otimes u\|_{L^1(K)} \leq \|u^m \otimes (u^m - u) + (u^m - u) \otimes u\|_{L^1(K)} \\
\leq c(\|u^m\|_{L^2(K)} + \|u\|_{L^2(K)})\|u^m - u\|_{L^2(K)} \\
\leq c(K)(\|u^m\|_{L^{n+2}} + \|u\|_{L^{n+2}})\|u^m - u\|_{L^{n+2}} \\
\leq c(K)\delta_0\|u^m - u\|_{L^{n+2}},
\]
\[
\|
\sigma(Du^m)Du^m - \sigma(Du)Du\|_{L^1(K)} \leq c(\|Du\|_{L^{\infty}} + \|Du^m\|_{L^{\infty}})\|Du^m - Du\|_{L^1(K)} \\
\leq c(K)\delta_0\|Du^m - Du\|_{L^{n+2}}.
\]

This implies \( u^m \otimes u^m \) and \( S(Du^m) \) converge to \( u \otimes u \) and \( S(Du) \) in \( L^1(K) \) for all bounded subset \( K \subset \mathbb{R}^n_+ \times (0, \infty) \), respectively. With \( m \) sending to infinity, the identity is
\[
\int_{\mathbb{R}^n} u_0(x) \cdot \Phi(x) dx = \int_0^\infty \int_{\mathbb{R}^n_+} -u \cdot \Phi_t + ((u \otimes u) : \nabla \Phi + S(Du)) : \nabla \Phi dx dt.
\]

Therefore \( u \) is a weak solution of \((1.1)\). This completes the proof of the existence part of Theorem 1.2.

5.4. Uniqueness. Let \( u_1 \in W^{1, \frac{n}{2}}_0 \cap u \in L^\infty(0, \infty, W^{1, \frac{n}{2}}_0(\mathbb{R}^n_+)) \) be another weak solution of the system \((1.1)\) with pressure \( p_1 \) satisfying \( \|u_1\|_{\delta_{p_1}(1, \frac{n}{2}, \frac{n}{2})} + \|Du_1\|_{L^{\infty}} < \delta_0 \), where \( \epsilon \) is chosen in Section 5.2. Let \( U = u - u_1 \) and \( P = p - p_1 \). According, \((U, P)\) satisfies the equations
\[
U_t - \Delta U + \nabla P = -\text{div}(u \otimes u - u_1 \otimes u_1) - \text{div}(\sigma(Du)Du - \sigma(Du_1)Du_1),
\]
\[
\text{div} U = 0, \text{ in } \mathbb{R}^n_+ \times (0, \infty),
\]
\[
U|_{t=0} = 0, \quad U|_{x_n=0} = 0.
\]

The estimate \((5.28)\) implies that
\[
\|U\|_{W^{1, \frac{n}{2}}_{\frac{n}{2}}(0, \infty)} \leq c(\|u\|_{W^{1, \frac{n}{2}}_{\frac{n}{2}}(0, \infty)} + \|Du\|_{L^{\infty}} + \|Du_1\|_{L^{\infty}})\|U\|_{W^{1, \frac{n}{2}}_{\frac{n}{2}}(0, \infty)} \\
< c\delta_0\|U\|_{W^{1, \frac{n}{2}}_{\frac{n}{2}}(0, \infty)} < \frac{1}{2}\|U\|_{W^{1, \frac{n}{2}}_{\frac{n}{2}}(0, \infty)}.\]
Remark 2.1. 

In this section, we prove the following theorem.

**APPENDIX A. PROOF OF (2) OF REMARK 2.1**

First, we show that \( \mathcal{S}_0(\mathbb{R}^{n+1}) = \{f \in \mathcal{S}(\mathbb{R}^{n+1}) | 0 \notin \text{supp } \hat{f} \} \) is dense in \( W^{k,\frac{4}{n}}_p(\mathbb{R}^{n+1}) \). If then, \( \sum_{-\infty < k < \infty} (4\pi^2|\xi| + 2\pi i \tau)^{\frac{2}{n}} \hat{f}(2^{-k} \xi, 2^{-2k} \tau) \hat{f} = (4\pi^2|\xi| + 2\pi i \tau)^{\frac{2}{n}} \hat{f} \) and so (2) of Remark 2.1 is proved.

To prove the claim, let us \( f \in W^{k,\frac{4}{n}}_p(\mathbb{R}^{n+1}) \) and \( \varepsilon > 0 \). Let \( g \in L^p(\mathbb{R}^{n+1}) \) be defined by \( \mathcal{F}(g)(\xi, \tau) = \mathcal{F}((4\pi^2|\xi| + 2\pi i \tau)^{\frac{2}{n}} \hat{f})(\xi, \tau) \). Since \( \mathcal{S}_0(\mathbb{R}^{n+1}) \) is dense in \( L^p(\mathbb{R}^{n+1}) \) (see Lemma 3.6 in [25]), there is \( g_e \in \mathcal{S}_0(\mathbb{R}^{n+1}) \) such that \( \|g - g_e\|_{L^p(\mathbb{R}^{n+1})} < \varepsilon \). Since \( 0 \notin \text{supp } \hat{g}_e \), we get \( f_e := \mathcal{F}^{-1}(4\pi^2|\xi| + 2\pi i \tau)^{-\frac{2}{n}} \hat{g}_e \in \mathcal{S}_0(\mathbb{R}^{n+1}) \). Then, we have \( \|f_e - f\|_{W^{k,\frac{4}{n}}_p(\mathbb{R}^{n+1})} = \|g - g_e\|_{L^p(\mathbb{R}^{n+1})} < \varepsilon \). Hence, \( \mathcal{S}_0(\mathbb{R}^{n+1}) \) is dense in \( W^{k,\frac{4}{n}}_p(\mathbb{R}^{n+1}) \). We complete the proof of (2) of Remark 2.1.

**APPENDIX B.**

In this section, we prove the following theorem.

**Theorem B.1.**

1. For \( 1 \leq p < \infty \) and \( k \in \mathbb{N} \), let us

\[
H^{k,\frac{4}{n}}_p(\mathbb{R}^{n+1}) = \{ f \in \mathcal{S}'(\mathbb{R}^{n+1}) | \| f \|_{H^{k,\frac{4}{n}}_p(\mathbb{R}^{n+1})} := \sum_{|\beta| + 2l = k} \| D_\beta^l D^1 f \|_{L^p(\mathbb{R}^{n+1})} < \infty \}. \tag{B.1}
\]

Then, \( \dot{W}^{k,\frac{4}{n}}_p(\mathbb{R}^{n+1}) = H^{k,\frac{4}{n}}_p(\mathbb{R}^{n+1}) \) with \( \| f \|_{\dot{W}^{k,\frac{4}{n}}_p(\mathbb{R}^{n+1})} = \| f \|_{H^{k,\frac{4}{n}}_p(\mathbb{R}^{n+1})} \). Moreover,

\[
\| f \|_{\dot{W}^{2k,k}_p(\mathbb{R}^{n+1})} \approx \sum_{k_1 + k_2 = k} \| \Delta^{k_1} D^{k_2} f \|_{L^p(\mathbb{R}^{n+1})}. \tag{B.2}
\]

2. Let \( 1 \leq p < n + 2 \) and \( 0 < s \) with \( p < \frac{n+2}{s} \). Then, \( \dot{W}^{s,\frac{4}{n}}_p(\mathbb{R}^{n+1}) \subset L^{\frac{p(n+2)}{n+2-mp}}(\mathbb{R}^{n+1}) \) with

\[
\| f \|_{L^{\frac{p(n+2)}{n+2-mp}}(\mathbb{R}^{n+1})} \leq c \| f \|_{\dot{W}^{s,\frac{4}{n}}_p(\mathbb{R}^{n+1})},
\]

3. Let \( 1 \leq p < n + 2 \), \( 1 \leq q < \infty \) and \( 0 < s \) with \( p < \frac{n+2}{s} \). Then, \( \dot{W}^{s,\frac{4}{n}}_p(\mathbb{R}^{n+1}) \) and \( \dot{B}^{s,\frac{4}{n}}_{p,q}(\mathbb{R}^{n+1}) \) are completion of \( C_0^\infty(\mathbb{R}^{n+1}) \). In particular, \( \dot{W}^{s,\frac{4}{n}}_p(\mathbb{R}^{n+1}) \) and \( \dot{B}^{s,\frac{4}{n}}_{p,q}(\mathbb{R}^{n+1}) \) are complete.

**Proof.** H. Bahouri, J. Y. Chemin and R. Danchin proved Theorem B.1 for \( \dot{H}^s_2(\mathbb{R}^n) \), \( s < \frac{2}{5} \) (see the comment in Proposition 1.32 below, Theorem 1.38 and Proposition 1.34 in [7]).
Since the proofs are similar, we only prove (1) for $k = 1$. Let $f \in \mathcal{S}'(\mathbb{R}^{n+1})$ such that 

$$
\sum_{1 \leq i \leq n} \|D_{x_i}f\|_{L^p(\mathbb{R}^{n+1})} + \|D^\frac{1}{2}_t f\|_{L^p(\mathbb{R}^{n+1})} < \infty.
$$

Then, we have

$$
(4\pi^2|\xi|^2 + 2\pi i \tau)^{\frac{1}{2}} \hat{f} = -\sum_{1 \leq i \leq n} \frac{2\pi i \xi_i}{(4\pi^2|\xi|^2 + 2\pi i \tau)^{\frac{1}{2}}} D_{x_i} \hat{f}
$$

$$
= \frac{2\pi i (a_0 + ib_0 \text{sign}(\tau))^{-1} \text{sign}(\tau)\sqrt{\tau}}{(4\pi^2|\xi|^2 + 2\pi i \tau)^{\frac{1}{2}}} D^\frac{1}{2}_t \hat{f}.
$$

Note that by the Marcinkiewicz multiplier theorem (see Theorem 4.6 in [52]), \( \frac{\xi_i}{(\sqrt{\xi^2 + 2\pi i \tau})^2} \) and \( \frac{(a_0 + ib_0 \text{sign}(\tau))^{-1} \text{sign}(\tau)\sqrt{\tau}}{(\sqrt{\xi^2 + 2\pi i \tau})^2} \) are $L^p$-multipliers. Hence, we have

$$
\|f\|_{W_p^{1,\frac{1}{2}}(\mathbb{R}^{n+1})} = \|\mathcal{F}^{-1}((4\pi^2|\xi|^2 + 2\pi i \tau)^{\frac{1}{2}} \hat{f})\|_{L^p(\mathbb{R}^{n+1})} \leq c \left( \sum_{1 \leq i \leq n} \|D_{x_i}f\|_{L^p(\mathbb{R}^{n+1})} + \|D^\frac{1}{2}_t f\|_{L^p(\mathbb{R}^{n+1})} \right).
$$

Hence, we have $H_p^{1,\frac{1}{2}}(\mathbb{R}^{n+1}) \subset W_p^{1,\frac{1}{2}}(\mathbb{R}^{n+1})$.

Conversely, let $f \in W_p^{1,\frac{1}{2}}(\mathbb{R}^{n+1})$ so that $f \in \mathcal{S}'(\mathbb{R}^{n+1})$ and \( \|\mathcal{F}^{-1}((4\pi^2|\xi|^2 + 2\pi i \tau)^{\frac{1}{2}} \hat{f})\|_{L^p(\mathbb{R}^{n+1})} < \infty \). Then, we have

$$
\widehat{D_{x_i}f} = (2\pi i \xi_i) \hat{f} = \frac{2\pi i \xi_i}{(4\pi^2|\xi|^2 + 2\pi i \tau)^{\frac{1}{2}}} (4\pi^2|\xi|^2 + 2\pi i \tau)^{\frac{1}{2}} \hat{f},
$$

$$
\widehat{D^\frac{1}{2}_t f} = (2\pi i |\tau|^{\frac{1}{2}}) \hat{f} = \frac{2\pi i (a_0 + ib_0 \text{sign}(\tau))^{-1} \text{sign}(\tau)\sqrt{\tau}}{(4\pi^2|\xi|^2 + 2\pi i \tau)^{\frac{1}{2}}} (4\pi^2|\xi|^2 + 2\pi i \tau)^{\frac{1}{2}} \hat{f}.
$$

By the Marcinkie\'wicz multiplier theorem (see Theorem 4.6 in [52]), \( \frac{(a_0 + ib_0 \text{sign}(\tau))^{-1} \text{sign}(\tau)\sqrt{\tau}}{(\sqrt{\xi^2 + 2\pi i \tau})^2} \) is $L^p$-multiplier. Hence, we have

$$
\|D_{x_i}f\|_{L^p(\mathbb{R}^{n+1})}, \|D^\frac{1}{2}_t f\|_{L^p(\mathbb{R}^{n+1})} \leq c \|f\|_{W_p^{1,\frac{1}{2}}(\mathbb{R}^{n+1})}.
$$

Hence, we have $W_p^{1,\frac{1}{2}}(\mathbb{R}^{n+1}) \subset H_p^{1,\frac{1}{2}}(\mathbb{R}^{n+1})$. We complete the proof of (B.1).

Next, \( \widehat{D_{x_i}D_{x_j}f} = (2\pi i \xi_i) \hat{f} = \frac{4\pi^2 \xi_i \xi_j}{(4\pi^2|\xi|^2 + 2\pi i \tau)^{\frac{1}{2}}} (4\pi^2|\xi|^2 + 2\pi i \tau) \hat{f} \). This implies (B.2). We complete the proof of (1).

Now, we prove (2). Let $f \in H_p^{1,\frac{1}{2}}(\mathbb{R}^{n+1})$ so that $f_t - \Delta f \in L^p(\mathbb{R}^{n+1})$. Note that $f = \Gamma \ast (f_t - \Delta f)$, where $\Gamma$ is the fundamental solution of heat equation in $\mathbb{R}^n$. It is well-known that

$$
\|\Gamma \ast (f_t - \Delta f)\|_{L^{\phi'(\mathbb{R}^{n+1})}} \leq c \|f_t - \Delta f\|_{L^p(\mathbb{R}^{n+1})},
$$
Hence, we have
\[
\|f\|_{L^{\frac{(m+2)}{m+2-2p}}(\mathbb{R}^{n+1})} \leq c\|f\|_{\mathcal{H}_p^{2,1}(\mathbb{R}^{n+1})}.
\]  
(B.3)

Let \(m \in \mathbb{N}\) such that \(2m \leq n + 2 < 2(m + 1)\) and \(p < \frac{n+2}{2m}\). Then, we have
\[
\|f\|_{L^{\frac{(n+2)}{n+2-2p}}(\mathbb{R}^{n+1})} \leq c\|f_t - \Delta f\|_{L^{\frac{(n+2)}{n+2-2m-2p}}(\mathbb{R}^{n+1})}
\leq c\|D_t(D_t f - \Delta f) - \Delta(D_t f - \Delta f)\|_{L^{\frac{(n+2)}{n+2-2m-2p}}(\mathbb{R}^{n+1})}
\cdots
\leq c\|\sum_{k_1 + l = m} D_t^k \Delta^l f\|_{L^p(\mathbb{R}^{n+1})}
\leq c\|f\|_{\mathcal{W}_p^{2m,m}(\mathbb{R}^{n+1})}.
\]

Acting complex interpolation between \(L^{p_1}(\mathbb{R}^{n+1})\) and \(L^{\frac{(n+2)p_1}{n+2-2mp_1}}(\mathbb{R}^{n+1})\) for \(1 \leq p_1 < \infty\) and \(1 \leq p < \frac{n+2}{2m}\), we have
\[
\|f\|_{[L^{p_1}(\mathbb{R}^{n+1}), L^{\frac{(n+2)p_1}{n+2-2mp_1}}(\mathbb{R}^{n+1})]} \leq c\|f\|_{[L^{p_1}(\mathbb{R}^{n+1}), \mathcal{W}_p^{2m,m}(\mathbb{R}^{n+1})]}.
\]

Hence, we obtain (2) for \(0 < s \leq 2m\) and \(1 \leq p < \frac{n+2}{2m}\).

For the case \(2m < s < n + 2\), we use the following claim: \(f \in \mathcal{W}_p^{\frac{s}{2}+\epsilon}(\mathbb{R}^{n+1})\) if and only if \(D_x^{2m}f, D_t^m f \in \mathcal{W}_p^{\frac{s}{2}+\epsilon}(\mathbb{R}^{n+1})\).

The claim is direct from the following equation.
\[
(4\pi^2|\xi|^2 + 2\pi i \tau)^{-\frac{s}{2}+\epsilon} \hat{\Delta^m f} = \frac{(-4\pi^2|\xi|^2)^m}{(4\pi^2|\xi|^2 + 2\pi i \tau)^m} (4\pi^2|\xi|^2 + 2\pi i \tau)^{\frac{s}{2}} \hat{f},
\]
\[
(4\pi^2|\xi|^2 + 2\pi i \tau)^{-\frac{s}{2}+\epsilon} \hat{D_t^m f} = \frac{(2\pi i \tau)^m}{(4\pi^2|\xi|^2 + 2\pi i \tau)^m} (4\pi^2|\xi|^2 + 2\pi i \tau)^{\frac{s}{2}} \hat{f}.
\]

Let \(f \in \mathcal{W}_p^{\frac{s}{2}+\epsilon}(\mathbb{R}^{n+1})\) for \(s < n + 2\) and \(1 \leq p < \frac{n+2}{2m}\). From the claim, we have the claim, we have \(\Delta f, D_t f \in \mathcal{W}_p^{\frac{s}{2}-\frac{2}{2m}}(\mathbb{R}^{n+1})\) with
\[
\|\Delta f\|_{\mathcal{W}_p^{\frac{s}{2}-\frac{2}{2m}}(\mathbb{R}^{n+1})} + \|D_t f\|_{\mathcal{W}_p^{\frac{s}{2}-\frac{2}{2m}}(\mathbb{R}^{n+1})} \approx \|f\|_{\mathcal{W}_p^{\frac{s}{2}+\epsilon}(\mathbb{R}^{n+1})}.
\]

From (B.3), (B.1), (2) of Theorem [B.1] for \(0 < s \leq 2m\) and the claim continuously, we have
\[
\|f\|_{L^{\frac{(n+2)p}{n+2-2mp}}(\mathbb{R}^{n+1})} \leq c\|f\|_{\mathcal{W}_p^{\frac{s}{2},\frac{s}{2}}(\mathbb{R}^{n+1})} \leq c\|\Delta f\|_{L^{\frac{(n+2)}{n+2-2mp}}(\mathbb{R}^{n+1})} + \|D_t f\|_{L^{\frac{(n+2)}{n+2-2mp}}(\mathbb{R}^{n+1})}
\leq c\|\Delta f\|_{\mathcal{W}_p^{\frac{s}{2},\frac{s}{2}}(\mathbb{R}^{n+1})} + \|D_t f\|_{\mathcal{W}_p^{\frac{s}{2},\frac{s}{2}}(\mathbb{R}^{n+1})} \approx \|f\|_{\mathcal{W}_p^{\frac{s}{2}+\epsilon}(\mathbb{R}^{n+1})}.
\]

This implies the (2) of Theorem [B.1] for \(H^{\frac{s}{2}}(\mathbb{R}^{n+1})\).

For the proof of (2) of Theorem [B.1] for \(B^{\frac{s}{2}+\epsilon}_{p,\theta}(\mathbb{R}^{n+1})\), we use the following real interpolation
\[
(L^p(\mathbb{R}^{n}), \mathcal{W}_p^{\frac{s}{2},\frac{s}{2}}(\mathbb{R}^{n+1}))_{\theta,\gamma} = B^{\frac{s}{2}+\epsilon}_{p,\theta}(\mathbb{R}^{n+1}),
\]
where \(\theta = 1 - \frac{s}{s_1}\). We complete the proof of (2) of Theorem [B.1].
The completeness of $\dot{W}^\frac{1}{2}_p(\mathbb{R}^{n+1})$ is direct result from (2) of Theorem B.1. The completeness of $\dot{B}^\frac{1}{2}_{r,q}(\mathbb{R}^{n+1})$ is from (B.4) and the property of interpolation spaces. We complete the proof of (3) of Thererem B.1.

Theorem B.2. Let $1 \leq p < n + 2$, $\frac{n+2}{n+\alpha} < r < \infty$, $1 \leq q < \infty$ and $1 < \alpha < 2$. Then, $\dot{W}^\frac{1}{2}_p(\mathbb{R}^{n+1}) \cap \dot{B}^\frac{1}{2}_{r,q}(\mathbb{R}^{n+1})$ are complete.

**Proof.** Note that $f \in \dot{B}^\frac{1}{2}_{r,q}(\mathbb{R}^{n+1})$ if and only if $\mathcal{F}^{-1}((4\pi^2|\xi|^2 + 2\pi i\tau)f) \in \dot{B}^{\alpha-2,\frac{\alpha-2}{2}}_{r,q}(\mathbb{R}^{n+1})$ with $\|f\|_{\dot{B}^\frac{1}{2}_{r,q}(\mathbb{R}^{n+1})} \approx \|\mathcal{F}^{-1}((4\pi^2|\xi|^2 + 2\pi i\tau)f)\|_{\dot{B}^{\alpha-2,\frac{\alpha-2}{2}}_{r,q}(\mathbb{R}^{n+1})}$.

Since $\dot{B}^{\alpha-2,\frac{\alpha-2}{2}}_{r,q}(\mathbb{R}^{n+1})$ is dual space of $\dot{B}^{-\alpha+2,\frac{\alpha+2}{2}}_{r',q'}(\mathbb{R}^{n+1})$ and $\dot{B}^{-\alpha+2,\frac{\alpha+2}{2}}_{r',q'}(\mathbb{R}^{n+1})$ is complete by Theorem B.1, $\dot{B}^\frac{1}{2}_{r,q}(\mathbb{R}^{n+1})$ is complete.

Let $\{f_m\}$ be a Cauchy sequence in $\dot{W}^\frac{1}{2}_p(\mathbb{R}^{n+1}) \cap \dot{B}^\frac{1}{2}_{r,q}(\mathbb{R}^{n+1})$. Then, $\{f_m\}$ be a Cauchy sequence in $\dot{W}^\frac{1}{2}_p(\mathbb{R}^{n+1})$ and $\{\mathcal{F}^{-1}((4\pi^2|\xi|^2 + 2\pi i\tau)f_m)\}$ be a Cauchy sequence in $\dot{B}^{\alpha-2,\frac{\alpha-2}{2}}_{r,q}(\mathbb{R}^{n+1})$. Since $\dot{W}^\frac{1}{2}_p(\mathbb{R}^{n+1})$ and $\dot{B}^{\alpha-2,\frac{\alpha-2}{2}}_{r,q}(\mathbb{R}^{n+1})$ are complete, there are $f \in \dot{W}^\frac{1}{2}_p(\mathbb{R}^{n+1})$ and $g \in \dot{B}^{\alpha-2,\frac{\alpha-2}{2}}_{r,q}(\mathbb{R}^{n+1})$ such that $f_m$ converges to $f$ in $\dot{W}^\frac{1}{2}_p(\mathbb{R}^{n+1})$ and $\{\mathcal{F}^{-1}((4\pi^2|\xi|^2 + 2\pi i\tau)f_m)\}$ converges to $g$ in $\dot{B}^{\alpha-2,\frac{\alpha-2}{2}}_{r,q}(\mathbb{R}^{n+1})$.

Let $\phi \in \mathcal{S}(\mathbb{R}^{n+1})$. Note that $\mathcal{F}^{-1}((4\pi^2|\xi|^2 + 2\pi i\tau)\phi) \in \mathcal{S}(\mathbb{R}^{n+1})$ and $<\mathcal{F}^{-1}((4\pi^2|\xi|^2 + 2\pi i\tau)f_m), \phi> = <f_m, \mathcal{F}^{-1}((4\pi^2|\xi|^2 + 2\pi i\tau)\phi)>, \text{ where } <\cdot, \cdot> \text{ is dual pairing between } \mathcal{S}(\mathbb{R}^{n+1}) \text{ and } \mathcal{S}'(\mathbb{R}^{n+1}).$ Then, we have

$$<\mathcal{F}^{-1}((4\pi^2|\xi|^2 + 2\pi i\tau)f_m), \phi> \rightarrow <g, \phi>,<f_m, \mathcal{F}^{-1}((4\pi^2|\xi|^2 + 2\pi i\tau)\phi)\rightarrow <f, \mathcal{F}^{-1}((4\pi^2|\xi|^2 + 2\pi i\tau)\phi)> \quad \text{(B.5)}$$

This implies $g = \mathcal{F}^{-1}((4\pi^2|\xi|^2 + 2\pi i\tau)\phi)$ and so $f = \mathcal{F}^{-1}((4\pi^2|\xi|^2 + 2\pi i\tau)^{-1}g)$. Hence, we have $f \in \dot{W}^\frac{1}{2}_p(\mathbb{R}^{n+1}) \cap \dot{B}^\frac{1}{2}_{r,q}(\mathbb{R}^{n+1})$. We complete the proof of Theorem B.2.

**APPENDIX C. PROOF OF PROPOSITION 3.1**

Note that $D_{x_i}N \ast f = D_{x_i}N \ast' R_i f$, $i = 1, \cdots, n-1$ and $D_{x_n}N$ is Poisson kernel in $\mathbb{R}^n$, where $R'_i$ are $(n-1)$-dimensinal Riesz transforms. Moreover, it is a well known fact that $D_{x_n}N$ is bounded from $\dot{B}^{\frac{1}{p}}_{p,p}(\mathbb{R}^{n+1})$ to $L^p(\mathbb{R}^n)$ so that $\|D_{x_n}N \ast f\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{\dot{B}^{\frac{1}{p}}_{p,p}(\mathbb{R}^{n+1})}$ and $R'_i$ are bounded in $\dot{B}^{\beta}_{p,p}(\mathbb{R}^{n+1})$ to $B^{\beta}_{p,p}(\mathbb{R}^{n+1})$, $\beta \in \mathbb{R}$ (See [52]). Accordingly, we have

$$\|N \ast f\|_{L^p(0,\infty; \dot{B}^{\frac{1}{p}}_{p,p}(\mathbb{R}^n))} \leq \|R'_i f\|_{L^p(0,\infty; \dot{B}^{\frac{1}{p}}_{p,p}(\mathbb{R}^{n+1}))} \leq \|f\|_{L^p(0,\infty; \dot{B}^{\frac{1}{p}}_{p,p}(\mathbb{R}^{n+1}))}.$$
Since $D^k_t N \ast f = N \ast D^k f$, we have

$$\| N \ast f \|_{L^p(\mathbb{R}^n; \dot{W}^k_p(0, \infty))} \leq c \| N \ast D^k f \|_{L^p(\mathbb{R}^n \times (0, \infty))} \leq c \| D^k_t f \|_{L^p(0, \infty; B^s_{p,p}(\mathbb{R}^{n-1}))} \leq c \| f \|_{\dot{W}^k_p(0, \infty; B^s_{p,p}(\mathbb{R}^{n-1}))}.$$  

From (3) of Proposition 2.4, this implies the first quantity of Proposition 3.1.

The second quantity is from the property of real interpolation and the definition of space $A^\alpha_{p,q}$. We complete the proof of Proposition 3.1.

**Appendix D. Proof of Proposition 3.2**

Since the proofs are exactly same, we only prove in the case of $\nabla^2 N \ast g$. By the Caldron-Zygmund integral theorem and by the property of real interpolation,

$$\| \nabla^2 N \ast g \|_{L^p(0, \infty; \dot{W}^k_p(\mathbb{R}^n))} \leq c \| g \|_{L^p(0, \infty; \dot{W}^k_p(\mathbb{R}^n))} \quad k \geq 0. \quad (D.1)$$

See Proposition 3.4 in [18].

Let $\tilde{g} \in \dot{W}^\frac{k}{p}_p(\mathbb{R})$ be a extension of $g \in \dot{W}^\frac{k}{p}_p(0, \infty)$ such that $\| \tilde{g} \|_{\dot{W}^\frac{k}{p}_p(\mathbb{R})} \leq c \| g \|_{\dot{W}^\frac{k}{p}_p(0, \infty)}$. Accordingly

$$\| \nabla^2 N \ast g \|_{L^p(\mathbb{R}^n; \dot{W}^\frac{k}{p}_p(0, \infty))} \leq c \| \nabla^2 N \ast \tilde{g} \|_{L^p(\mathbb{R}^n; \dot{W}^\frac{k}{p}_p(\mathbb{R}))} = c \| \nabla^2 N \ast D_t \tilde{g} \|_{L^p(\mathbb{R}^n \times \mathbb{R})} \leq c \| \tilde{g} \|_{L^p(\mathbb{R}^n; \dot{W}^\frac{k}{p}_p(\mathbb{R}))} \leq c \| g \|_{L^p(\mathbb{R}^n; \dot{W}^\frac{k}{p}_p(0, \infty))}. \quad (D.2)$$

From (D.1), (D.2) and (3) of Proposition 2.4,

$$\| \nabla^2 N \ast g \|_{\dot{W}^\frac{k}{p}_p} \leq c \| g \|_{\dot{W}^\frac{k}{p}_p} \quad k \geq 0.$$  

Using the property of real interpolation,

$$\| \nabla^2 N \ast g \|_{B^s_{p,q}} \leq c \| g \|_{B^s_{p,q}} \quad s > 0.$$  

We complete the proof of Proposition 3.2.

**Appendix E. Proof of Lemma 2.9**

Let $E : \dot{B}^\frac{s}{p,q}_p \rightarrow \dot{B}^\frac{s}{p,q}_p(\mathbb{R}^{n+1})$ be an extension operator defined in Section 2. Let $\mathcal{G} \in \dot{B}^\frac{s}{p,q}_p \cap L^\infty$ and $\tilde{\mathcal{G}} = E \mathcal{G}$ such that $\| \tilde{\mathcal{G}} \|_{\dot{B}^\frac{s}{p,q}_p(\mathbb{R}^{n+1})} \leq c \| \mathcal{G} \|_{\dot{B}^\frac{s}{p,q}_p}$ and $\| \tilde{\mathcal{G}} \|_{L^\infty(\mathbb{R}^{n+1})} \leq c \| \mathcal{G} \|_{L^\infty}$ since $\sigma(\tilde{\mathcal{G}}) \mathcal{G} \in \mathcal{B}^\infty_{p,q} \cap (0, \infty)$ = $\sigma(\mathcal{G}) \mathcal{G}$, by the definition of definition of homogeneous anisotropic Besov space $B^s_{p,q}$ and from
(4) of Proposition 2.2, we have
\[
\|\sigma(G)G\|_{B^{1/2}_{p,q}} \leq c\|\sigma(G)G\|_{B^{1/2}_{p,q}(\mathbb{R}^{n+1})} \leq c\left(\int_{\mathbb{R}^{n+1}} \frac{1}{\left(|y| + |\tau|^{1/2}\right)^{n+2+ps}} \left(\int_{\mathbb{R}^{n+1}} |\sigma(G(x+y,t+\tau)) - G(x+y,t+\tau)|^p dx dt\right)^{1/p} dy d\tau\right)^{1/2}.
\]

By mean-value theorem, we have
\[
\|\sigma(G(x+y,t+\tau)) - \sigma(G(x+y,t+\tau))\|_{B^{1/2}_{p,q}} \leq \|D\sigma\|_{L^\infty(0,\|G\|_{L^\infty(\mathbb{R}^{n+1})})} \|G(x+y,t+\tau) - G(x+y,t+\tau)\|_{B^{1/2}_{p,q}(\mathbb{R}^{n+1})}
\]

de(\sigma(G(x+y,t+\tau))) \leq \|D\sigma\|_{L^\infty(0,\|G\|_{L^\infty(\mathbb{R}^{n+1})})} \|G(x+y,t+\tau) - G(x+y,t+\tau)\|_{B^{1/2}_{p,q}(\mathbb{R}^{n+1})},
\]

and
\[
\|\sigma(G)G\|_{B^{1/2}_{p,q}} \leq c\|D\sigma\|_{L^\infty(0,\|G\|_{L^\infty(\mathbb{R}^{n+1})})} \|G\|_{L^\infty(\mathbb{R}^{n+1})} \|G(x+y,t+\tau) - G(x+y,t+\tau)\|_{B^{1/2}_{p,q}(\mathbb{R}^{n+1})}.
\]

We complete the proof of (1) of Lemma 2.9.

Next, we prove (2). By direct calculation, we have
\[
\sigma(G(x+y,t+\tau)) - \sigma(\mathbb{H}(x+y,t+\tau)) = \left(\sigma(G(x,t)) - \sigma(\mathbb{H}(x,t))\right)
\]
\[
= \int_0^1 D\sigma(\theta G(x+y,t+\tau) + (1-\theta)\mathbb{H}(x+y,t+\tau)) : \left(G(x+y,t+\tau) - \mathbb{H}(x+y,t+\tau)\right) d\theta
\]
\[
- \int_0^1 D\sigma(\theta G(x,t) + (1-\theta)\mathbb{H}(x,t)) : \left(G(x,t) - \mathbb{H}(x,t)\right) d\theta
\]
\[
= \int_0^1 \left(D\sigma(\theta G(x+y,t+\tau) + (1-\theta)\mathbb{H}(x+y,t+\tau)) - D\sigma(\theta G(x,t) + (1-\theta)\mathbb{H}(x,t))\right)
\]
\[
: \left(G(x+y,t+\tau) - \mathbb{H}(x+y,t+\tau)\right) d\theta
\]
\[
+ \int_0^1 D\sigma(\theta G(x,t) + (1-\theta)\mathbb{H}(x,t)) : \left(G(x+y,t+\tau) - \mathbb{H}(x+y,t+\tau) - G(x,t) + \mathbb{H}(x,t)\right) d\theta
\]
\[
= I_1 + I_2.
\]
Here,

\[ |I_2| \leq \|D\sigma\|_{L^\infty(0, \max(\|\tilde{G}\|_{L^\infty(R^{n+1})}, \|\tilde{H}\|_{L^\infty(R^{n+1})})} \left| \tilde{G}(x + y, t + \tau) - \tilde{H}(x + y, t + \tau) - \tilde{G}(x, t) + \tilde{H}(x, t) \right| \]

For \( I_1 \),

\[
D\sigma(\theta \tilde{G}(x + y, t + \tau) + (1 - \theta) \tilde{H}(x + y, t + \tau)) - D\sigma(\theta \tilde{G}(x, t) + (1 - \theta) \tilde{H}(x, t)) \\
= \int_0^1 \frac{d}{d\gamma} D\sigma\left( \gamma(\theta \tilde{G}(x + y, t + \tau) + (1 - \theta) \tilde{H}(x + y, t + \tau)) + (1 - \gamma)(\theta \tilde{G}(x, t) + (1 - \theta) \tilde{H}(x, t)) \right) d\gamma \\
= \int_0^1 D^2\sigma\left( \gamma(\theta \tilde{G}(x + y, t + \tau) + (1 - \theta) \tilde{H}(x + y, t + \tau)) + (1 - \gamma)(\theta \tilde{G}(x, t) + (1 - \theta) \tilde{H}(x, t)) \right) d\gamma.
\]

Hence, we have

\[
|I_1| \leq \|D^2\sigma\|_{L^\infty(0, \max(\|\tilde{G}\|_{L^\infty(R^{n+1})}, \|\tilde{H}\|_{L^\infty(R^{n+1})})} \left( \|\tilde{G}\|_{L^\infty(R^{n+1})} + \|\tilde{H}\|_{L^\infty(R^{n+1})} \right) \\
\times \left( \left| \tilde{G}(x + y, t + \tau) - \tilde{G}(x, t) \right| + \left| \tilde{H}(x + y, t + \tau) - \tilde{H}(x, t) \right| \right) \|G(x + y, t + \tau) - H(x + y, t + \tau)\|.
\]

Summing all estimate, we have

\[
\|\sigma(G) - \sigma(H)\|_{B^\frac{s}{p,q}} \leq c \|\sigma(\tilde{G}) - \sigma(\tilde{H})\|_{B^\frac{s}{p,q}}(R^{n+1}) \\
\leq c \left( \int_{R^{n+1}} \frac{1}{(|y| + |\tau|^2)^{n+2+ps}} \right) \\
\times \left( \int_{R^{n+1}} |\sigma(\tilde{G}(x + y, t + \tau)) - \sigma(\tilde{H}(x + y, t + \tau)) - \sigma(\tilde{G}(x, t)) + \sigma(\tilde{H}(x, t))| dx dt \right)^{\frac{1}{p}} d\tau \\
\leq c \left( \|\tilde{G} - \tilde{H}\|_{B^\frac{s}{p,q}}(R^{n+1}) + \|\tilde{G}\|_{B^\frac{s}{p,q}}(R^{n+1}) + \|\tilde{H}\|_{B^\frac{s}{p,q}}(R^{n+1}) \right) \left( \|G - H\|_{L^\infty(R^{n+1})} \right) \\
\leq c \left( \|G - H\|_{L^\infty} + \|G\|_{L^\infty} + \|H\|_{L^\infty} \right) \left( \|G - H\|_{L^\infty} \right).
\]

We complete the proof of (2) of Lemma 2.9.

**APPENDIX F. PROOF OF LEMMA 2.10**

By Hölder's inequality and Besov space embedding,

\[
\|u\|_{L^p} \leq \|u\|_{L^\infty}^{1-\theta} \|u\|_{L^{n+2}}^\theta \leq c \|u\|_{B^\frac{s}{p,q}}^{1-\theta} \|u\|_{L^{n+2}}^\theta.
\]

We complete the proof of (2.10).
Let \( \Gamma \) be defined in Section 2. Hence, we prove Lemma 2.8 for the case of \( f \in B_{p,q}^{s,\frac{2}{p}}(\mathbb{R}^n \times (0,\infty)) \).

From (5) in Proposition 2.2, we have

\[
B_{p,q}^{s,\frac{2}{p}} \subset L^{\infty}(0,\infty; B_{p,q}^{s-\frac{2}{p}}(\mathbb{R}^n)).
\]

Let \( \frac{2}{p} < s < 2 \) and \( f \in B_{p,q}^{s,\frac{2}{p}} \) with \( f(x,0) = 0 \) for \( x \in \mathbb{R}^n \). Let \( \tilde{f}(x,t) = f(x,t) \) for \( t > 0 \) and \( \tilde{f}(x,t) = 0 \) for \( t < 0 \). Accordingly, \( \tilde{f} \in B_{p,q}^{s,\frac{2}{p}}(\mathbb{R}^{n+1}) \) with \( \| \tilde{f} \|_{B_{p,q}^{s,\frac{2}{p}}(\mathbb{R}^{n+1})} \leq c \| f \|_{B_{p,q}^{s,\frac{2}{p}}(\mathbb{R}^n \times (0,\infty))} \).

From (5) of Proposition 2.2

\[
\| f(t) \|_{B_{p,q}^{s,\frac{2}{p}}(\mathbb{R}^n)} \leq c \| \tilde{f} \|_{B_{p,q}^{s,\frac{2}{p}}(\mathbb{R}^n \times (0,\infty))} \leq c \| f \|_{B_{p,q}^{s,\frac{2}{p}}(\mathbb{R}^n \times (0,\infty))} \rightarrow 0 \quad \text{as} \quad t \rightarrow 0.
\]

Let \( f_0(x) = f(x,0) \). From (5) of Proposition 2.2 \( f_0 \in B_{p,q}^{s,\frac{2}{p}}(\mathbb{R}^n) \). From well-known result of the Cauchy problem of the heat equation in \( \mathbb{R}^n \times (0,\infty) \), the function defined by \( F(x,t) = \Gamma_t * f_0(x) \) is in \( B_{p,q}^{s,\frac{2}{p}} \) with \( \| F \|_{B_{p,q}^{s,\frac{2}{p}}} \leq c \| f_0 \|_{B_{p,q}^{s,\frac{2}{p}}(\mathbb{R}^n)} \) and \( \| F(t) - f_0 \|_{B_{p,q}^{s,\frac{2}{p}}(\mathbb{R}^n)} \rightarrow 0 \) as \( t \rightarrow 0 \). Since \( F - f \in B_{p,q}^{s,\frac{2}{p}} \) and \( (F - f)|_{t=0} = 0 \), from the above argument, \( \| F(t) - f(t) \|_{B_{p,q}^{s,\frac{2}{p}}(\mathbb{R}^n)} \rightarrow 0 \) as \( t \rightarrow 0 \).

Accordingly

\[
\| f(t) - f_0 \|_{B_{p,q}^{s,\frac{2}{p}}(\mathbb{R}^n)} \leq \| f(t) - F(t) \|_{B_{p,q}^{s,\frac{2}{p}}(\mathbb{R}^n)} + \| F(t) - f_0 \|_{B_{p,q}^{s,\frac{2}{p}}(\mathbb{R}^n)} \rightarrow 0 \quad \text{as} \quad t \rightarrow 0.
\]

This implies

\[
B_{p,q}^{s,\frac{2}{p}} \subset C([0,\infty); B_{p,q}^{s-\frac{2}{p}}(\mathbb{R}^n)).
\]

Hence, we proved Lemma 2.8 for \( \frac{2}{p} < s < 2 \).
Let \(2k + \frac{2}{p} < s < 2k + 2\) for \(k \in \mathbb{N}\). Accordingly \(D_x^{2k} f \in \dot{B}^{s-2k, \frac{s}{2}-k}_{p,q} \) and \(D_x^{2k} f_0 \in \dot{B}^{s-2k-\frac{2}{p}}_{p,q} (\mathbb{R}^n)\). From the above argument,
\[
\|f(t) - f_0\|_{\dot{B}^{s-\frac{2}{p}}_{p,q} (\mathbb{R}^n)} \leq \|D_x^{2k} (f(t) - f_0)\|_{\dot{B}^{s-2k-\frac{2}{p}}_{p,q} (\mathbb{R}^n)} \to 0 \quad \text{as} \quad t \to 0.
\]

Hence, we proved Lemma 2.8 for \(\frac{2}{p} < s\).

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**Conflict of Interest**
The authors have no conflicts to disclose.

**Author Contributions**
Tongkeun Chang: Writing – original draft (equal). Bum Ja Jin: Writing – original draft (equal).

**DATA AVAILABILITY**
The data that support the findings of this study are available from the corresponding author upon reasonable request.

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