A Characterization of MDS Symbol-pair Codes over Two Types of Alphabets

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Abstract
Symbol-pair codes are block codes with symbol-pair metrics designed to protect against pair-errors that may occur in high-density data storage systems. MDS symbol-pair codes are optimal in the sense that it can attain the highest pair-error correctability within the same code length and code size. Constructing MDS symbol-pair codes is one of the main topics in symbol-pair codes. In this paper, we characterize the symbol-pair distances of some constacyclic codes of arbitrary lengths over finite fields and a class of finite chain rings. Using the characterization of symbol-pair distance, we present several classes of MDS symbol-pair constacyclic codes and show that there is no other MDS symbol-pair code except for what we present. Moreover, some of these MDS symbol-pair constacyclic codes over the finite chain rings cannot be obtained by previous work.

Keywords: Symbol-pair codes, MDS codes, constacyclic codes

1. Introduction
Modern high-density data storage systems may not read the transmitted information individually as classic information transmission due to physical limitations. Motivated by this fact, Cassuto and Blaum \cite{1} developed symbol-pair code over symbol-pair read channel whose outputs are overlapping pairs.
of symbols. The efficient decoding algorithms for cyclic codes over symbol-pair read channels are shown in [2–4].

Let \( \Xi \) be an alphabet of \( q \) elements with \( q \geq 2 \). A code \( C \) over \( \Xi \) of length \( n \) is a subset of \( \Xi^n \). The elements in \( C \) are called codewords. We use the bold letter to denote a vector in the sequel. Let \( \mathbf{x} = (x_0, x_1, \ldots, x_{n-1}), \mathbf{y} = (y_0, y_1, \ldots, y_{n-1}) \) be vectors in \( \Xi^n \). A vector \( \mathbf{x} \) transmitted in the symbol-pair read channel is read as

\[
\pi(\mathbf{x}) = ((x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_0)).
\]

We call \( \pi(\mathbf{x}) \) as a symbol-pair vector of \( \mathbf{x} \). The symbol-pair distance between \( \mathbf{x} \) and \( \mathbf{y} \) is defined as the Hamming distance between \( \pi(\mathbf{x}) \) and \( \pi(\mathbf{y}) \), i.e.,

\[
d_{sp}(\mathbf{x}, \mathbf{y}) = d_H(\pi(\mathbf{x}), \pi(\mathbf{y})) = |\{i : (x_i, x_{i+1}) \neq (y_i, y_{i+1})\}|.
\]

The (minimum) symbol-pair distance of \( C \) is defined as

\[
d_{sp}(C) = \min\{d_{sp}(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in C \text{ and } \mathbf{x} \neq \mathbf{y}\}.
\]

For a code \( C \) of length \( n \) over \( \Xi \) with symbol-pair distance \( d_{sp} \), the upper bound on the code size of \( C \), called Singleton bound for symbol-pair codes [5], is

\[
|C| \leq q^n - d_{sp} + 2.
\]  \hspace{1cm} (1)

A symbol-pair code whose parameters satisfy (1) with equality is called maximum distance separable (MDS). According to (1), MDS symbol-pair codes possess the largest symbol-pair distance under the same code length and code size, which indicates that MDS symbol-pair codes are a class of optimal symbol-pair codes that can have high pair error-correcting capability since the symbol-pair distance is a tool to measure the pair error-correcting capability of the codes.

Constructing MDS symbol-pair codes is meaningful both in theoretical and practical. The research on constructing MDS symbol-pair codes is active in recent years [6–11]. Many MDS symbol-pair codes are obtained by analyzing the generator polynomials of constacyclic codes. See [6–8, 12] for example. In [9], Dinh et al. characterize the symbol-pair distances of all constacyclic codes of length \( p^s \) over \( \mathbb{F}_{p^m} \) and obtain all the MDS symbol-pair codes of prime power lengths. These results are generalized in two different directions. One of the directions is to construct MDS symbol-pair constacyclic codes over different alphabets such as the finite chain ring \( \mathbb{F}_{p^m} + u \mathbb{F}_{p^m} \).
([10, 13]). The other direction is extending the code length of constacyclic codes to some other special code lengths such as $2p^s$ ([11, 14]).

Let $\lambda$ be a nonzero element in $\mathbb{F}_p^m$ and $n$ be a positive integer coprime to $p$. Due to the complicated irreducible factorization of $x^n - \lambda$ in $\mathbb{F}_p^m[x]$, the algebraic structure of $\lambda$-constacyclic codes of length $np^s$ over $\mathbb{F}_p^m$ are not obtained completely. Therefore it is difficult to analyze the symbol-pair distance of constacyclic codes of length $np^s$. In [15], the authors discussed the structure of a special class of constacyclic codes of length $np^s$ over $\mathbb{F}_p^m$. This work inspires us to analyze the symbol-pair distances of these constacyclic codes. Our motivation is to characterize the MDS symbol-pair codes among the larger class of constacyclic codes and obtain new MDS symbol-pair codes with more flexible parameters.

In this paper, we consider some constacyclic codes of length $np^s$ over two different alphabets, which are finite fields $\mathbb{F}_p^m$ and finite chain rings $\mathbb{F}_p^m + u\mathbb{F}_p^m$, where $p$ is a prime, $m$ is a positive integer and $u^2 = 0$. Let $\alpha_0$ be a nonzero element in $\mathbb{F}_p^m$ such that $x^n - \alpha_0$ is irreducible over $\mathbb{F}_p^m$. Denote $\alpha = \alpha_0^p$. Let $\beta$ be an element in $\mathbb{F}_p^m$. We completely characterize the symbol-pair distances of $\alpha$-constacyclic codes over $\mathbb{F}_p^m$ and $(\alpha + u\beta)$-constacyclic codes over $\mathbb{F}_p^m + u\mathbb{F}_p^m$. We present several classes of MDS symbol-pair constacyclic codes in Table 1 and Table 2. Some of these codes are obtained in previous work and we remark the references in the tables. Some of these codes are obtained in this paper.

| Generator Polynomial | Dimension | Pair Distance | Remark | Ref. |
|----------------------|-----------|---------------|--------|------|
| $x - \alpha_0$       | $p^s - 1$ | 3             |        | [9]  |
| $(x - \alpha_0)^2$   | $p^s - 2$ | 4             |        | [9]  |
| $(x - \alpha_0)^4$   | $5$       | 6             | $p = 3$
|                      |           |               | $s = 2$ | [9]  |
| $(x - \alpha_0)^k$   | $p - k$   | $k + 2$       | $s = 1$
|                      |           |               | $1 \leq k \leq p - 2$ | [9]  |
| $(x - \alpha_0)^{p^s - 2}$ | $2$ | $p^s$ |        | [9]  |
| $x^2 - \alpha_0$     | $2p^s - 2$ | 4              |        | [14] |
| $(x^2 - \alpha_0)^k$ | $2p - 2k$ | $2k + 2$      | $s = 1$
|                      |           |               | $1 \leq k \leq p - 2$ | [14] |
| $(x^2 - \alpha_0)^{p^s - 1}$ | $2$ | $2p^s$ |        | [14] |

Table 1: MDS symbol-pair $\alpha$-constacyclic codes of length $np^s$ over $\mathbb{F}_p^m$. 

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Table 2: MDS symbol-pair $\alpha$-constacyclic codes of length $np^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$, where $b(x)$ is either zero or a unit in $\mathbb{F}_{p^m}[x]/(x^{np^s} - \alpha)$

The codes in Table 1 are MDS symbol-pair $\alpha$-constacyclic codes over $\mathbb{F}_{p^m}$. For any positive integer $n$, we prove that there is no other MDS symbol-pair $\alpha$-constacyclic code of length $np^s$ except for the codes in Table 1.

The codes in Table 2 are MDS symbol-pair $(\alpha + u\beta)$-constacyclic codes over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. Notice that the codes considered in [16] is a subcase of the codes we considered in this paper which confine $n = 1$. In [16], Dinh et al. gave two classes of MDS symbol-pair codes with parameters $(2^s, 2^m(2^s-1)+4, 3, 2^{s-2})$ and $(3^s, 3^m(2\cdot 3^{s-1}+4), 2\cdot 3^{s-1})$, but these two classes are actually not MDS symbol-pair codes. We will give a detailed analysis of these two classes of codes in section 4. Besides, we also obtain three new classes of MDS symbol-pair $(\alpha + u\beta)$-constacyclic codes. Moreover, we prove that there is no other MDS symbol-pair $(\alpha + u\beta)$-constacyclic codes of length $np^s$ except for the
codes we present in Table 2.

The remaining of this paper is organized as follows. In Section 2, we present some preliminaries and notations. In Section 3, we give the symbol-pair distances of all $\alpha$-constacyclic codes of length $np^s$ over $\mathbb{F}_{p^m}$ and show all the MDS symbol-pair $\alpha$-constacyclic codes of length $np^s$ among these codes. In Section 4, we determine the symbol-pair distances of some of $(\alpha + u\beta)$-constacyclic codes of length $np^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ and exhibit all the MDS symbol-pair codes among these codes.

2. Preliminaries

In this section, we give some notations and results that will be used in the sequel.

Let $R$ be a finite commutative ring with identity. A code $C$ over $R$ is called linear if $C$ is a submodule of $R^n$. The symbol-pair weight of a vector $x$ in $R^n$ is the symbol-pair distance between $x$ and the all-zero vector $0$ of $R^n$, denoted by $wt_{sp}(x)$. The symbol-pair distance of a linear code is equal to the minimum symbol-pair weight of nonzero codewords of the linear code.

For a unit $\lambda$ of $R$, the $\lambda$-constacyclic shift $\tau_{\lambda}$ on $R^n$ is defined by:

$$\tau_{\lambda}(x_0, x_1, \ldots, x_{n-1}) = (\lambda x_{n-1}, x_0, x_1, \ldots, x_{n-2}).$$

A linear code $C$ is said to be $\lambda$-constacyclic if $\tau_{\lambda}(C) = C$. Each codeword $c = (c_0, c_1, \ldots, c_{n-1})$ in $C$ is customarily identified with its polynomial representation $c(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$ in $R[x]/\langle x^n - \lambda \rangle$. In the ring $R[x]/\langle x^n - \lambda \rangle$, $xc(x)$ corresponds to performing a $\lambda$-constacyclic shift on $c$. The following theorem shows the algebraic property of constacyclic codes.

Proposition 2.1. [17] A linear code $C$ of length $n$ over $R$ is a $\lambda$-constacyclic code if and only if $C$ is an ideal of the quotient ring $R[x]/\langle x^n - \lambda \rangle$.

The ideal of $R[x]/\langle x^n - \lambda \rangle$ is generated by a factor of $x^n - \lambda$. Let $\lambda$ be a unit of $R$. In this paper, we mainly consider the constacyclic codes of length $np^s$, i.e., an ideal of the residue ring $R[x]/\langle x^{np^s} - \lambda \rangle$. If $R$ is a Frobenius ring, one can find a unit $\lambda_0$ such that $\lambda_0^{p^s} = \lambda$. Therefore, we have $x^{np^s} - \lambda = (x^n - \lambda_0)^{p^s}$. In this paper, we make an assumption that $x^n - \lambda_0$ is irreducible in $R[x]$. The following result shows the irreducibility of binomials over finite fields.
Proposition 2.2. [18] Let \( n \geq 2 \) be an integer and \( \lambda \in \mathbb{F}_q^* \). Then the binomial \( x^n - \lambda \) is irreducible in \( \mathbb{F}_q[x] \) if and only if the following two conditions are satisfied:

(i) each prime factor of \( n \) divides the order \( e \) of \( \lambda \) in \( \mathbb{F}_q^* \), but not \( 2^{-1} \);  
(ii) \( q \equiv 1(\mod 4) \) if \( n \equiv 0(\mod 4) \).

According to Proposition 2.2, if \( n \) satisfy the condition (ii) of Proposition 2.2 and all the prime factors of \( n \) divide \( q - 1 \), there exists \( \lambda \in \mathbb{F}_q^* \) such that \( x^n - \lambda \) is irreducible over \( \mathbb{F}_q \).

2.1. Constacyclic Codes over \( \mathbb{F}_{p^m} \)

Let \( \alpha \) be a nonzero element in \( \mathbb{F}_{p^m} \). We present some results of \( \alpha \)-constacyclic codes of length \( np^s \) over \( \mathbb{F}_{p^m} \) in this subsection. Denote by \( \mathbb{F} \) the quotient ring \( \mathbb{F}_{p^m}[x]/\langle x^{np^s} - \alpha \rangle \). The structures and the minimum Hamming distances of \( \alpha \)-constacyclic codes are given in the following theorem.

Theorem 2.3. [15, Theorem 3.6] Let \( \mathbb{F}_{p^m} \) be a finite field and \( n \) be a positive integer with \( \gcd(n, p) = 1 \). Suppose that \( x^n - \alpha_0 \) is irreducible over \( \mathbb{F}_{p^m} \) for \( \alpha_0 \in \mathbb{F}_{p^m}^* \) and \( \alpha = \alpha_0^{p^s} \). Then the \( \alpha \)-constacyclic codes of length \( np^s \) over \( \mathbb{F}_{p^m} \) are of the form \( \mathcal{C}_i = \langle (x^n - \alpha_0)^i \rangle \), where \( 0 \leq i \leq p^s \). And the minimum Hamming distance of \( \mathcal{C}_i \) is given by

\[
d_H(\mathcal{C}_i) = \begin{cases} 
1, & \text{if } i = 0, \\
(\theta + 2)p^k, & \text{if } p^s - p^s - k + \theta p^s - k - 1 + 1 \leq i \leq p^s - p^s - k + (\theta + 1)p^s - k - 1, \\
0 & \text{if } i = p^s.
\end{cases}
\]

where \( 0 \leq \theta \leq p - 2 \) and \( 0 \leq k \leq s - 1 \).

For simplicity, we use the notation \( \mathcal{C}_i \) to denote the \( \alpha \)-constacyclic codes of length \( np^s \) with generator polynomial \( (x^n - \alpha_0)^i \), where \( 0 \leq i \leq p^s \). The following lemma shows a formula to compute the Hamming weight of the codeword \( (x^n - \alpha_0)^i \) in \( \mathcal{C}_i \).

Lemma 2.4. [19, Lemma 1] For any nonnegative integer \( i < p^s \), let \( i = i_{s-1}p^{s-1} + \cdots + i_1 p + i_0 \), where \( 0 \leq i_0, i_1, \ldots, i_{s-1} \leq p - 1 \), which means that \( (i_{s-1}, \ldots, i_0) \) is the \( p \)-adic expansion of \( i \). Then

\[
\text{wt}_H((x^n - \alpha_0)^i) = \prod_{j=0}^{s-1} (i_j + 1).
\]
The following lemma shows the relationship between the symbol-pair distance and the Hamming distance.

**Lemma 2.5.** [1, Theorem 2] For two codewords \( x, y \) in a code \( C \) of length \( n \) with \( 0 < d_H(x, y) < n \), define the set \( S_H = \{ j \mid x_j \neq y_j \} \). Let \( S_H = \bigcup_{i=1}^L B_i \) be a minimal partition of the set \( S_H \) to subsets of consecutive indices (indices may wrap around modulo \( n \)). Then

\[
d_{sp}(x, y) = d_H(x, y) + L.
\]

To calculate the symbol-pair distances, we will use the concept of the coefficient weight of polynomials, which was first proposed in [20]. For a polynomial \( f(x) = a_n x^n + \cdots + a_1 x + a_0 \) of degree \( n \), the coefficient weight of \( f \), which denoted by \( \text{cw}(f) \), is

\[
\text{cw}(f) = \begin{cases} 0, & \text{if } f \text{ is a monomial} \\
\min \{|i-j| : a_i \neq 0, a_j \neq 0, i \neq j\}, & \text{otherwise.}
\end{cases}
\]

Intuitively, \( \text{cw}(f) \) is the smallest distance among exponents of nonzero terms of \( f(x) \). It is shown in [9] that if \( 0 \leq \deg(g(x)) \leq \text{cw}(f(x)) - 2 \) and \( \deg(f(x)) + \deg(g(x)) \leq n - 2 \), then

\[
\text{wt}_{sp}(f(x)g(x)) = \text{wt}_H(f(x)) \cdot \text{wt}_{sp}(g(x)).
\]

### 2.2. Constacyclic Codes over \( \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} \)

Let \( \alpha + u\beta \) be a unit in \( \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} \), i.e., \( \alpha \neq 0 \). This subsection gives the structures of \((\alpha + u\beta)\)-constacyclic codes of length \( np^s \) over \( \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} \). Denote by \( \mathcal{R} \) the quotient ring \( (\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x]/\langle x^{np^s} - \alpha - u\beta \rangle \).

Note that the structures of the ideals of \( \mathcal{R} \) are quite different when the value of \( \beta \) is equal to zero or not. The following lemma shows that all the ideals of \( \mathcal{R} \) are principal ideals in the case of \( \beta \neq 0 \).

**Lemma 2.6.** [21, Theorem 3.3] Let \( x^n - \alpha_0 \) be an irreducible polynomial in \( \mathbb{F}_{p^m}[x] \), \( \alpha = \alpha_0^{p^s} \), and \( \beta \) be a nonzero element in \( \mathbb{F}_{p^m} \). Then the ring \( \mathcal{R} = (\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x]/\langle x^{np^s} - \alpha - u\beta \rangle \) is a chain ring whose ideal chain is as follows

\[
\mathcal{R} = \langle 1 \rangle \supseteq \langle x^n - \alpha_0 \rangle \supseteq \cdots \supseteq \langle (x^n - \alpha_0)^{2p^s-1} \rangle \supseteq \langle (x^n - \alpha_0)^{2p^s} \rangle = \langle 0 \rangle.
\]

In other words, \((\alpha + u\beta)\)-constacyclic codes of length \( np^s \) over \( \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} \) are precisely the ideals \( \mathcal{D}_i = \langle (x^n - \alpha_0)^i \rangle \) of \( \mathcal{R} \), where \( 0 \leq i \leq 2p^s \). The number of codewords of \((\alpha + u\beta)\)-constacyclic code \( \mathcal{D}_i \) is \( p^{mn(2p^s-i)} \). In particular, \( \langle (x^n - \alpha_0)^p \rangle = \langle u \rangle \).
Observing that the symbol-pair distances of $C_{sp}$ follow that $w_t$ where $0 \leq t \leq 3$. MDS Symbol-pair codes over $\mathbb{F}_{p^n} + u\mathbb{F}_{p^n}$ of length $np^s$, i.e. all ideals of the ring $\mathcal{F} + u\mathcal{F}$, are given by the following three types:

(I) $\mathcal{D} = \langle (x^n - \alpha_0)^k \rangle$, where $0 \leq k \leq p^s$, with $|\mathcal{D}| = p^{2mn(p^s-k)}$

(II) $\mathcal{D} = \langle (x^n - \alpha_0)^j b(x) + u(x^n - \alpha_0)^k \rangle$, where $0 \leq k \leq p^s - 1$, $\left\lceil \frac{p^s+k}{2} \right\rceil \leq j \leq p^s - 1$ and either $b(x)$ is 0 or $b(x)$ is a unit in $\mathcal{F}$, with $|\mathcal{D}| = p^{mn(p^s-k)}$.

(III) $\mathcal{D} \cong \langle (x^n - \alpha_0)^j b(x) + u(x^n - \alpha_0)^k, (x^n - \alpha_0)^{k+t} \rangle$, where $0 \leq k \leq p^s - 2$, $1 \leq t \leq p^s - k - 1$, $k + \left\lceil \frac{t}{2} \right\rceil \leq j \leq k + t$, and either $b(x)$ is 0 or $b(x)$ is a unit in $\mathcal{F}$, with $|\mathcal{D}| = p^{mn(2p^s-2k-t)}$.

Note that $\mathbb{F}_{p^m}$ is a subfield of $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. We define the subfield subcode of codes over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ as the set of codewords whose components are in $\mathbb{F}_{p^m}$. We use the notation $\mathcal{D}|_{\mathcal{F}}$ to denote the subfield subcode of $\mathcal{D}$ and $d_{sp}(\mathcal{D}|_{\mathcal{F}})$ to denote the symbol-pair distance of $\mathcal{D}|_{\mathcal{F}}$. We can represent the polynomial $c(x)$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ as $c(x) = a(x) + ub(x)$, where $a(x), b(x) \in \mathbb{F}_{p^m}[x]$. Observing that $c_i = a_i + ub_i = 0$ if and only if $a_i = b_i = 0$, where $c_i, a_i$ and $b_i$ are coefficients of $x^i$ in polynomials $c(x), a(x)$ and $b(x)$, respectively. It follows that $w_{t_{sp}}(c(x)) \geq \max \{w_{t_{sp}}(a(x)), w_{t_{sp}}(b(x))\}$.

3. MDS Symbol-pair codes over $\mathbb{F}_{p^m}$

3.1. Symbol-pair distances of constacyclic codes

Denote $\mathcal{C}_i = \langle (x^n - \alpha)^i \rangle$ as the $\alpha$-constacyclic codes of length $np^s$ over $\mathbb{F}_{p^m}$, where $0 \leq i \leq p^s$. In this subsection, we give a complete characterization of the symbol-pair distances of $\mathcal{C}_i$. The symbol-pair distances for the cases that $n = 1$ and $n \geq 2$ are different, and we only give the analysis of the case for $n \geq 2$. For the related results of the case of $n = 1$, we refer the readers to [9].

We discuss the symbol-pair distance of $\mathcal{C}_i$ depends on the value of $i$. For the trivial cases that $i = 0$ and $i = p^s$, we have

$$d_{sp}(\mathcal{C}_0) = d_{sp}(\mathcal{F}) = 2$$

and

$$d_{sp}(\mathcal{C}_{p^s}) = d_{sp}(\langle 0 \rangle) = 0.$$
In order to consider the symbol-pair distances of $\mathcal{C}_i$ for $1 \leq i \leq p^s - 1$, we divide the set \( \{ i \in \mathbb{N}, 1 \leq i \leq p^s - 1 \} \) into \( s(p - 1) \) parts, i.e.,

\[
\bigcup_{0 \leq \theta \leq p - 2} \{ i \in \mathbb{N}, p^s - p^{s-k} + \theta p^{s-k-1} + 1 \leq i \leq p^s - p^{s-k} + (\theta + 1)p^{s-k-1} \}
= \{ i \in \mathbb{N}, 1 \leq i \leq p^s - 1 \}.
\]

Note that \( d_{sp}(\mathcal{C}_i) \leq d_{sp}(\mathcal{C}_j) \) if \( i \leq j \) since \( \mathcal{C}_i \supseteq \mathcal{C}_j \). In order to determine the symbol-pair distances of \( \mathcal{C}_i \) for \( p^s - p^{s-k} + \theta p^{s-k-1} + 1 \leq i \leq p^s - p^{s-k} + (\theta + 1)p^{s-k-1} \), where \( 0 \leq k \leq s - 1 \) and \( 0 \leq \theta \leq p - 2 \), we consider an upper bound \( U \) on the symbol-pair distance of \( \mathcal{C}_{p^s - p^{s-k} + (\theta + 1)p^{s-k-1}} \) and a lower bound \( L \) on the symbol-pair distance of \( \mathcal{C}_{p^s - p^{s-k} + \theta p^{s-k-1} + 1} \). Observing that \( \mathcal{C}_{p^s - p^{s-k} + \theta p^{s-k-1} + 1} \supseteq \mathcal{C}_{p^s - p^{s-k} + (\theta + 1)p^{s-k-1}}, \) therefore,

\[
L \leq d_{sp}(\mathcal{C}_{p^s - p^{s-k} + \theta p^{s-k-1} + 1}) \leq d_{sp}(\mathcal{C}_{p^s - p^{s-k} + (\theta + 1)p^{s-k-1}}) \leq U.
\]

If \( L = U \), then we obtain the symbol-pair distances of \( \mathcal{C}_i \) for each \( i \) belongs to the interval \( [p^s - p^{s-k} + \theta p^{s-k-1} + 1, p^s - p^{s-k} + (\theta + 1)p^{s-k-1}] \). The following lemma shows the corresponding upper bound.

**Lemma 3.1.** Let \( n, k, \theta \) be integers such that \( n \geq 2 \), \( 0 \leq k \leq s - 1 \) and \( 0 \leq \theta \leq p - 2 \). Then \( d_{sp}(\mathcal{C}_{p^s - p^{s-k} + (\theta + 1)p^{s-k-1}}) \leq 2(\theta + 2)p^k. \)

**Proof.** By Lemma 2.4, we have

\[
\text{wt}_{H}( (x^n - \alpha_0)^{p^s - p^{s-k} + (\theta + 1)p^{s-k-1}} ) = (\theta + 2)p^k.
\]

Since

\[
\text{cw}( (x^n - \alpha_0)^{p^s - p^{s-k} + (\theta + 1)p^{s-k-1}} ) \geq n \geq 2,
\]

it follows that

\[
\text{wt}_{sp}( (x^n - \alpha_0)^{p^s - p^{s-k} + (\theta + 1)p^{s-k-1}} ) = 2(\theta + 2)p^k.
\]

Thus

\[
d_{sp}(\mathcal{C}_{p^s - p^{s-k} + (\theta + 1)p^{s-k-1}}) \leq 2(\theta + 2)p^k.
\]

\square

The lower bounds of symbol-pair distances of \( \mathcal{C}_{p^s - p^{s-k} + \theta p^{s-k-1} + 1} \) is more complicated and we consider it in four subcases:

(i) \( \theta = 0, k = 0; \)
\( \theta = 0, 1 \leq k \leq s - 2; \)
\( 1 \leq \theta \leq p - 2, 0 \leq k \leq s - 2; \)
\( 0 \leq \theta \leq p - 2, k = s - 1. \)

We start with the first case of \( k = 0 \) and \( \theta = 0. \)

**Lemma 3.2.** The pair distance \( d_{sp}(C_1) \) of \( C_1 = \langle x^n - \lambda_0 \rangle \subseteq \mathbb{F}_{p^m}[x]/\langle x^{np^s} - \lambda \rangle \) is greater than or equal to 4.

**Proof.** Verify that a codeword with symbol-pair weight two must be of form \( ux^j \), which is invertible in \( \mathbb{F}_p \). Hence there is no codeword in \( C_1 \) with symbol-pair weight two. Note that a codeword with symbol-pair weight three has the form \( u_0x^j + u_1x^{j+1} \), where \( 0 \leq j \leq np^s - 1 \). It follows that \( x^n - \alpha_0 \) divides \( u_0x^j + u_1x^{j+1} = (u_0 + u_1x)x^j \), hence \( x^n - \alpha_0 \) divides \( u_0 + u_1x \), which is impossible since the degree of \( x^n - \alpha_0 \) is greater than that of \( u_0 + u_1x \). Hence there is no codeword in \( C_1 \) with symbol-pair weight three. Therefore, we obtain \( d_{sp}(C_1) \geq 4. \)

The following lemma shows the lower bound of the minimum pair-distance of \( C_{ps^{p^s-k+1}} \) in the case that \( \theta = 0 \) and \( 1 \leq k \leq s - 2. \)

**Lemma 3.3.** Let \( n, k \) be integers such that \( n \geq 2 \) and \( 1 \leq k \leq s - 2. \) Then \( d_{sp}(C_{ps^{p^s-k+1}}) \geq 4p^k. \)

**Proof.** Let \( c(x) \) be any nonzero codeword in \( C_{ps^{p^s-k+1}} \). Then there is a nonzero element \( f(x) \) in \( \mathcal{F} \) such that \( c(x) = (x^n - \alpha_0)^{p^s - p^s - k + 1} f(x) \) with \( \deg(f) < np^s - n(p^s - p^s - k + 1) = n(p^s - k - 1) \). Let \( g(x) = (x^n - \alpha_0)f(x) \). Then \( \deg(g) < np^{s-k}, \) wt_H(\( g(x) \)) \( \geq 2, \) and

\[
c(x) = (x^n - \alpha_0)^{p^s - p^s - k} g(x)
= \left[ \sum_{j=0}^{p^k-1} \binom{p^k-1}{j} (-\alpha_0)^{p^s-k(p^k-j-1)} x^{np^s-jk} \right] g(x).
\]

We discuss the symbol-pair weight of \( c(x) \) in the following three cases.

Case 1: If \( \deg(g) \leq np^{s-k} - 2, \) then

\[
\text{cw}((x^n - \alpha_0)^{p^s - p^s - k}) = np^{s-k} \geq \deg(g) + 2
\]
and

\[
\deg((x^n - \alpha_0)^{p^s - p^s - k}) + \deg(g) \leq np^s - 2.
\]
By equation (2), we have
\[
\text{wt}_\text{sp}(c(x)) = \text{wt}_H((x^n - \alpha_0)^{p^s - p^{s-k}}) \cdot \text{wt}_\text{sp}(g(x)) = p^k \text{wt}_\text{sp}(g(x)).
\]

According to Lemma 3.2, \(\text{wt}_\text{sp}(g(x)) \geq d_{sp}(C_1) \geq 4\), which deduces that \(\text{wt}_\text{sp}(c(x)) \geq 4p^k\).

Case 2: If \(\deg(g) = np^{s-k} - 1\) and \(g(0) = 0\), then there is an integer \(l > 0\) such that \(g(x) = x^l g'(x)\), where \(\deg(g') \leq np^{s-k} - 2\). Clearly,
\[
\text{wt}_\text{sp}(c(x)) = \text{wt}_\text{sp}((x^n - \alpha_0)^{p^s - p^{s-k}} g(x)) = \text{wt}_\text{sp}((x^n - \alpha_0)^{p^s - p^{s-k}} x^l g'(x)) = \text{wt}_\text{sp}((x^n - \alpha_0)^{p^s - p^{s-k}} g'(x)).
\]

Similar to the proof in Case 1, we have \(\text{wt}_\text{sp}(c(x)) \geq 4p^k\).

Case 3: If \(\deg(g) = np^{s-k} - 1\) and \(g(0) \neq 0\), then \(g(x) = (x^n - \alpha_0) f(x)\) is an element in \((x^n - \alpha_0)\) of the ring \(\mathbb{F}_{pm}[x]/(x^{np^{s-k}} - \alpha_0^{np^{s-k}})\), i.e., a code-word of an \(\alpha_0^{np^{s-k}}\)-constacyclic code of length \(np^{s-k}\) over \(\mathbb{F}_{pm}\). According to Lemma 3.2, \(\text{wt}_\text{sp}(g(x)) \geq 4\), which implies that \(g(x)\) cannot be the form \(r_0 + r_1 x^{np^{s-k} - 1}\), where \(r_0, r_1 \neq 0\). Hence \(\text{wt}_H(g(x)) \geq 3\). When \(\text{wt}_H(g(x)) \geq 4\), we have
\[
\text{wt}_\text{sp}(c(x)) \geq \text{wt}_H(c(x)) = \text{wt}_H((x^n - \alpha_0)^{p^s - p^{s-k}}) \cdot \text{wt}_H(g(x)) \geq 4p^k.
\]

When \(\text{wt}_H(g(x)) = 3\), we assume that
\[
g(x) = r_0 + r_1 x^l + r_2 x^{np^{s-k} - 1},
\]
where \(0 < l < np^{s-k} - 1\) and \(r_0, r_1, r_2 \neq 0\). Let \(S_H\) be a set of the exponents of nonzero terms of \(c(x)\). Then the minimal partition of the set \(S_H\) to subsets of consecutive indices may be the following three cases:

if \(l = 1\),
\[
S_H = \cup_{1 \leq j \leq p^k - 1} \{np^{s-k} j - 1, np^{s-k} j, np^{s-k} j + 1\} \cup \{0, 1, np^s - 1\};
\]

if \(l = np^{s-k} - 2\),
\[
S_H = \cup_{1 \leq j \leq p^k - 1} \{np^{s-k} j - 2, np^{s-k} j - 1, np^{s-k} j\} \cup \{0, np^s - 1, np^s - 2\};
\]
if $1 < l < np^{s-k} - 2$,

$$S_H = \bigcup_{1 \leq j \leq p^k-1} \{np^{s-k}j - 1, np^{s-k}j \} \cup \{np^{s-k}j + l \} \cup \{0, np^{s} - 1 \} \cup \{l \}.$$  

According to the above three cases, we have $d_{sp}(C_{p^s-p^{s-k}+1}) \geq 4p^k$. This completes the proof. □

The following lemma is considering the case of $0 \leq k \leq s - 2$ and $1 \leq \theta \leq p - 2$.

**Lemma 3.4.** Let $n, k, \theta$ be integers such that $n \geq 2$, $0 \leq k \leq s - 2$, and $1 \leq \theta \leq p - 2$. Then $d_{sp}(C_{p^s-p^{s-k}+\theta p^{s-k-1}+1}) \geq 2(\theta + 2)p^k$.

**Proof.** Let $c(x)$ be any nonzero codeword in $C_{p^s-p^{s-k}+\theta p^{s-k-1}+1}$. Then there is a nonzero element $f(x)$ in $\mathcal{F}$ such that $c(x) = (x^n - \alpha_0)p^{s-p^{s-k}+\theta p^{s-k-1}+1} f(x)$ with $\deg(f) < n[(p-\theta)p^{s-k-1} - 1]$. Let $g(x) = (x^n - \alpha_0) f(x)$. Then $\deg(g) < n(p-\theta)p^{s-k-1}$, $wt_H(g(x)) \geq 2$, and

$$c(x) = (x^n - \alpha_0)p^{s-p^{s-k}+\theta p^{s-k-1}+1} g(x) = (x^{np^{s-k-1}} - \alpha_0^{p^{s-k-1}})p^{k+1+\theta} g(x).$$

Suppose that $\mathcal{T} = \{i_1, \cdots, i_r\}$ is a set of the exponents of nonzero terms of $g(x)$. For an integer $i$, let $S_i$ be a set of integers congruent to $i$ modulo $np^{s-k-1}$, i.e., $S_i = \{j \mid j \equiv i \pmod{np^{s-k-1}}\}$. We consider two cases that $\mathcal{T} \subset S_{i_1}$ and $\mathcal{T} \not\subset S_{i_1}$.

Case 1: When $\mathcal{T} \subset S_{i_1}$. We assume that $g(x) = \sum_{t=1}^{\eta} r_t x^{i_1+np^{s-k-1}u_t}$, where $0 = u_1 < \cdots < u_\eta$. Thus

$$c(x) = (x^{np^{s-k-1}} - \alpha_0^{p^{s-k-1}})p^{k+1+\theta} \sum_{t=1}^{\eta} r_t x^{np^{s-k-1}u_t} x^{i_1}.$$ 

It follows that $cw(c(x)) \geq np^{s-k-1} \geq np \geq 4$. Hence

$$wt_{sp}(c(x)) = 2 \cdot wt_H(c(x)) \geq 2 \cdot d_H(C_{p^s-p^{s-k}+\theta p^{s-k-1}+1}) \geq 2(\theta + 2)p^k.$$ 

Case 2: When $\mathcal{T} \not\subset S_{i_1}$. We only show that when $\mathcal{T} \subset S_{i_1} \cup S_{i_2}$ with $i_1 \neq i_2 \pmod{np^{s-k-1}}$, the rest is similar. Let $g(x) = g_1(x) + g_2(x)$, $g_1(x) = \sum_{t=1}^{\eta_1} r_{t_1} x^{i_1+np^{s-k-1}u_{t_1}}$ and $g_2(x) = \sum_{t=\eta_1+1}^{\eta} r_{t_2} x^{i_2+np^{s-k-1}u_{t_2}}$. Then

$$c(x) = (x^{np^{s-k-1}} - \alpha_0^{p^{s-k-1}})p^{k+1+\theta} \sum_{t=\eta_1+1}^{\eta} r_{t_2} x^{np^{s-k-1}u_{t_2}} x^{i_2}.$$ 

It follows that $cw(c(x)) \geq np^{s-k-1} \geq np \geq 4$. Hence

$$wt_{sp}(c(x)) = 2 \cdot wt_H(c(x)) \geq 2 \cdot d_H(C_{p^s-p^{s-k}+\theta p^{s-k-1}+1}) \geq 2(\theta + 2)p^k.$$ 

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\[ \sum_{t=1}^{n_1} r_t^{(1)} x_{i_t + np^{s-k-1}u_t} \] and \[ g_2(x) = \sum_{t=1}^{n_2} r_t^{(2)} x_{i_t + np^{s-k-1}v_t}, \] where \( 0 = u_1 < \ldots < u_{n_1} \) and \( 0 = v_1 < \ldots < v_{n_2} \). Then

\[
c(x) = \left( x^{np^{s-k-1}} - \alpha_0^{p^{s-k-1}} \right) \theta x^{p^{k+1+p+\theta} (g_1(x) + g_2(x))} + \left( x^{np^{s-k-1}} - \alpha_0^{p^{s-k-1}} \right) x^{p^{k+1+p+\theta} \sum_{t=1}^{n_1} r_t^{(1)} x^{np^{s-k-1}u_t}} x_i^{i_1} + \left( x^{np^{s-k-1}} - \alpha_0^{p^{s-k-1}} \right) x^{p^{k+1+p+\theta} \sum_{t=1}^{n_2} r_t^{(2)} x^{np^{s-k-1}v_t}} x_i^{i_2}.
\]

Let \( S_H \) be a set of the exponents of nonzero terms of \( c(x) \). Then

\[
S_H = \{ i_1 + np^{s-k-1}w_j \mid w_j^{(1)} \in \mathbb{N}, 1 \leq j \leq l_1 \} \\
\cup \{ i_2 + np^{s-k-1}w_j \mid w_j^{(2)} \in \mathbb{N}, 1 \leq j \leq l_2 \},
\]

where \( l_t = \text{wt}_H((x^n - \alpha_0)^{s-p^{s-k-1}} g_t(x)) \) for \( t = 1, 2 \). By Theorem 2.3, \( d_H(C_{p^s-p^{s-k-1}+\theta}) \geq (\theta + 1)p^k \), and hence \( l_1, l_2 \geq (\theta + 1)p^k \). Since \( np^{s-k-1} = 4, S_H \) is at least partitioned into \((\theta + 1)p^k\) subsets of consecutive indices. By Theorem 2.5,

\[
\text{wt}_{sp}(c(x)) \geq 3(\theta + 1)p^k \geq 2(\theta + 2)p^k.
\]

Therefore, we have proved that \( \text{wt}_{sp}(c(x)) \geq 2(\theta + 2)p^k \) holds in all cases, that is, \( d_{sp}(C_{p^s-p^{s-k-1}+\theta}) \geq 2(\theta + 2)p^k \).

The following lemma is about the case of \( k = s-1 \) and \( 0 \leq \theta \leq p-2 \).

**Lemma 3.5.** Let \( n, \theta \) be integers such that \( n \geq 2 \) and \( 1 \leq \theta \leq p-1 \). Then \( d_{sp}(C_{p^s-p+\theta}) \geq 2(\theta + 1)p^{s-1} \).

**Proof.** Let \( c(x) \) be any nonzero codeword in \( C_{p^s-p+\theta} \). Then there is a nonzero element \( f(x) \) in \( F \) such that \( c(x) = (x^n - \alpha_0)^{p^s-p+\theta} f(x) \) with \( \deg(f) < n(p-\theta) \). Suppose that \( T = \{i_1, \ldots, i_\eta\} \) is a set of the exponents of nonzero terms of \( f(x) \). For an integer \( i \), let \( S_i \) be a set of integers congruent to \( i \) modulo \( n \), i.e., \( S_i = \{ j \mid j \equiv i \pmod{n} \} \). We consider the set \( T \) in two cases.

Case 1: When \( T \subset S_{i_1} \). We may assume that \( f(x) = \sum_{t=1}^{\eta} r_t x^{i_t + nu_t} \), where \( 0 = u_1 < \ldots < u_{\eta} \). Then

\[
c(x) = \left( x^n - \alpha_0 \right)^{p^s-p+\theta} \sum_{t=1}^{\eta} r_t x^{nu_t} x_i^{i_1}.
\]
It follows that \( cw(c(x)) \geq n \geq 2 \), and hence,
\[
\text{wt}_{sp}(c(x)) = 2 \cdot \text{wt}_{H}(c(x)) \geq 2 \cdot d_{H}(C_{p^{s}-p+\theta}) \geq 2(\theta + 1)p^{s-1}.
\]

Case 2: When \( T \not\subset S_{i}. \) We may assume that \( T \subset S_{i_{1}} \cup S_{i_{2}} \), where \( i_{1} \not\equiv i_{2} \pmod{n} \). Let \( f(x) = f_{1}(x) + f_{2}(x) \), \( f_{1}(x) = \sum_{t=1}^{\eta_{1}} r_{t}^{(1)} x^{nu_{t}} \), and \( f_{2}(x) = \sum_{t=1}^{\eta_{2}} r_{t}^{(2)} x^{nu_{t}} \), where \( 0 = u_{1} < \ldots < u_{\eta_{1}} \) and \( 0 = v_{1} < \ldots < v_{\eta_{2}} \).

Then
\[
c(x) = \left[(x^{n} - \alpha_{0})^{p^{s}-p+\theta} \sum_{t=1}^{\eta_{1}} r_{t}^{(1)} x^{nu_{t}}\right] x^{i_{1}} + \left[(x^{n} - \alpha_{0})^{p^{s}-p+\theta} \sum_{t=1}^{\eta_{2}} r_{t}^{(2)} x^{nu_{t}}\right] x^{i_{2}}.
\]

Since \( i_{1} \not\equiv i_{2} \pmod{n} \),
\[
\text{wt}_{H}(c(x)) = \text{wt}_{H}\left((x^{n} - \alpha_{0})^{p^{s}-p+\theta} \sum_{t=1}^{\eta_{1}} r_{t}^{(1)} x^{nu_{t}}\right) x^{i_{1}} + \text{wt}_{H}\left((x^{n} - \alpha_{0})^{p^{s}-p+\theta} \sum_{t=1}^{\eta_{2}} r_{t}^{(2)} x^{nu_{t}}\right) x^{i_{2}} \geq 2 \cdot d_{H}(C_{p^{s}-p+\theta}) = 2(\theta + 1)p^{s-1},
\]
which implies that \( \text{wt}_{sp}(c(x)) \geq \text{wt}_{H}(c(x)) \geq 2(\theta + 1)p^{s-1} \). Combining the two cases discussed above, it follows that \( d_{sp}(C_{p^{s}-p+\theta}) \geq 2(\theta + 1)p^{s-1} \). \( \square \)

Combining the upper bound given in Lemma 3.1 and the lower bounds given in Lemma 3.2, 3.3, 3.4, and 3.5, the symbol-pair distances of \( \alpha \)-constacyclic codes of length \( np^{s} \) over \( \mathbb{F}_{p^{m}} \) can be completely determined. In order to maintain the integrity of the theorem, we present the symbol-pair distances for both the cases that \( n = 1 \) and \( n \geq 2 \).

**Theorem 3.6.** Let \( \alpha_{0} \) be a nonzero element in \( \mathbb{F}_{p^{m}} \) and \( \alpha = \alpha_{0}^{p^{s}} \). Given an \( \alpha \)-constacyclic code of length \( np^{s} \) over \( \mathbb{F}_{p^{m}} \). If it has the form as \( C_{i} = \langle (x^{n} - \alpha_{0})^{i} \rangle \subseteq \mathcal{F} \), for \( i \in \{0, 1, \ldots, p^{s}\} \), then the symbol-pair distance \( d_{sp}(C_{i}) \) is completely determined by:

(i) (Trivial cases) \( d_{sp}(C_{0}) = 2 \) and \( d_{sp}(C_{p^{s}}) = 0 \).
(ii) When $n = 1$,

$$d_{sp}(C_i) = \begin{cases} 
3p^k, & \text{if } i = p^s - p^{s-k} + 1 \text{ and } 0 \leq k \leq s - 2; \\
4p^k, & \text{if } p^s - p^{s-k} + 2 \leq i \leq p^s - p^{s-k} + p^{s-k-1} \\
2(\theta + 2)p^k, & \text{if } p^s - p^{s-k} + \theta p^{s-k-1} + 1 \leq i \leq p^s - p^{s-k} + (\theta + 1)p^{s-k-1}, \\
(\theta + 2)p^{s-1}, & \text{if } i = p^s - p + \theta \text{ and } 1 \leq \theta \leq p - 2; \\
p^s, & \text{if } i = p^s - 1.
\end{cases}$$

(iii) When $n \geq 2$,

$$d_{sp}(C_i) = 2(\theta + 2)p^k,$$

where $p^s - p^{s-k} + \theta p^{s-k-1} + 1 \leq i \leq p^s - p^{s-k} + (\theta + 1)p^{s-k-1}$, $0 \leq k \leq s - 1$ and $0 \leq \theta \leq p - 2$.

3.2. MDS codes

In Table 1, we summarize the MDS symbol-pair codes given in the previous literature. In this subsection, we use the result of symbol-pair distances obtained in the former subsection to prove that when $x^n - \alpha_0$ is irreducible over $\mathbb{F}_{p^m}$, there are no other MDS symbol-pair codes except for these in Table 1.

**Theorem 3.7.** Let $\alpha_0$ be a nonzero element of $\mathbb{F}_{p^m}$ and $\alpha = \alpha_0^{p^s}$. When $x^n - \alpha_0$ is irreducible over $\mathbb{F}_{p^m}$, there are no other nontrivial MDS symbol-pair $\alpha$-constacyclic codes of length $np^s$ over $\mathbb{F}_{p^m}$ except the MDS codes shown in Table 1.

**Proof.** When $x^n - \alpha_0$ is irreducible over $\mathbb{F}_{p^m}$, the $\alpha$-constacyclic codes of length $np^s$ over $\mathbb{F}_{p^m}$ are $C_i = \langle (x^n - \alpha_0)^i \rangle$, where $0 \leq i \leq p^s$. Note that $|C_i| = |\langle (x^n - \alpha_0)^i \rangle| = p^{m(np^s-ni)}$ and the Singleton Bounds for symbol-pair constacyclic codes force $|C_i| \leq p^{m(np^s-d_{sp}(C_i)+2)}$, i.e., $ni \geq d_{sp}(C_i) - 2$ for $i \in \{0, 1, \ldots, p^s - 1\}$. Therefore, $C_i$ is an MDS symbol-pair code if and only if $ni - d_{sp}(C_i) = 2$. If $n$ is equal to 1 or 2, all the MDS codes have been constructed by Dinh et al. [9, 14] and listed in Table 1. If $n \geq 3$, let $i = p^s - p^{s-k} + \theta p^{s-k-1} + \gamma$ with $0 \leq k \leq s - 1$, $0 \leq \theta \leq p - 2$, and
1 \leq \gamma \leq p^{s-k-1}. By Theorem 3.6, we have \( d_{sp}(C_i) = 2(\theta + 2)p^k \), hence

\[
\begin{align*}
ni - d_{sp}(C_i) + 2 &= n(p^s - p^{s-k} + \theta p^{s-k-1} + \gamma) - 2(\theta + 2)p^k + 2 \\
&= [np^{s-k} - 2(\theta + 2)](p^k - 1) + (np^{s-k-1} - 2)\theta + n\gamma - 2 \\
&\geq n - 2 > 0.
\end{align*}
\]

Therefore, there is no other MDS symbol-pair \( \alpha \)-constacyclic code.

\[ \square \]

4. MDS Symbol-pair codes over \( \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} \)

4.1. The pair distances of constacyclic codes over \( \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} \)

Let \( \alpha_0, \beta \in \mathbb{F}_{p^m} \) and \( \alpha_0 \neq 0 \). Denote \( \alpha \) as \( \alpha = \alpha_0^{p^s} \). In this section, we characterize the relationship between the symbol-pair distances of \( \alpha \)-constacyclic codes of length \( np^s \) over \( \mathbb{F}_{p^m} \) and the symbol-pair distances of \( (\alpha + u\beta) \)-constacyclic codes of length \( np^s \) over \( \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} \), where \( n \) is a positive integer coprime to \( p \) and \( x^n - \alpha_0 \) is irreducible over \( \mathbb{F}_{p^m} \). We analyze the symbol-pair distances in the cases that \( \beta \neq 0 \) and \( \beta = 0 \). When \( \beta \neq 0 \), \( \mathcal{R} = (\mathbb{F}_{p^m} + u\mathbb{F}_{p^m})[x]/\langle x^{np^s} - \alpha - u\beta \rangle \) is a chain ring and all the ideals of \( \mathcal{R} \) are \( \mathcal{D}_i = \langle (x^n - \alpha_0)^i \rangle \), where \( 0 \leq i \leq 2p^s \).

**Theorem 4.1.** Let \( \alpha_0 \) be a nonzero element in \( \mathbb{F}_{p^m} \) satisfying that \( x^n - \alpha_0 \) is irreducible over \( \mathbb{F}_{p^m} \). Denote \( \alpha = \alpha_0^{p^s} \). Let \( \beta \) be a nonzero element in \( \mathbb{F}_{p^m} \). The symbol-pair distance of \( \mathcal{D}_i = \langle (x^n - \alpha_0)^i \rangle \) is

\[
d_{sp}(\mathcal{D}_i) = \begin{cases} 
2, & \text{if } 0 \leq i \leq p^s; \\
d_{sp}(\langle (x^n - \alpha_0)^{i-p^s} \rangle_F), & \text{if } p^s + 1 \leq i \leq 2p^s.
\end{cases}
\]

**Proof.** When \( 0 \leq i \leq p^s \), we have \( u(x) = (x^n - \alpha_0)^{p^s} \in \mathcal{D}_i \) and the symbol-pair weight of \( u(x) \) is 2. Combining with \( d_{sp}(\mathcal{D}_i) \geq 2 \), we have the symbol-pair distance of \( \mathcal{D}_i \) is 2.

When \( p^s + 1 \leq i \leq 2p^s \), we have

\[
\langle (x^n - \alpha_0)^i \rangle = \langle u(x^n - \alpha_0)^{i-p^s} \rangle,
\]

which means that the codewords in the code \( \langle (x^n - \alpha_0)^i \rangle \) over \( \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} \) are precisely the codewords in the code \( \langle (x^n - \alpha_0)^{i-p^s} \rangle \) over \( \mathbb{F}_{p^m} \) multiplied with \( u \). Therefore, the symbol-pair distance of \( \mathcal{D}_i \) is equal to that of \( \langle (x^n - \alpha_0)^{i-p^s} \rangle_F \). \[ \square \]
The symbol-pair distance of $D_i$ is more complicated when $\beta = 0$, and we analyze it in three cases according to the three types of $\alpha$-constacyclic codes shown in Theorem 2.7.

**Theorem 4.2.** Let $D$ be an $\alpha$-constacyclic code of length $n p^s$ over $\mathbb{F}_{p^m} + u \mathbb{F}_{p^m}$ with type I in Theorem 2.7, i.e., $D = \langle (x^n - \alpha_0)^k \rangle$ for $0 \leq k \leq p^s$. Then $d_{sp}(D) = d_{sp}(\langle (x^n - \alpha_0)^k \rangle_F)$.

**Proof.** Notice that $D \supseteq \langle u(x^n - \alpha_0)^k \rangle$, and hence
\[ d_{sp}(D) \leq d_{sp}(\langle u(x^n - \alpha_0)^k \rangle) = d_{sp}(\langle (x^n - \alpha_0)^k \rangle_F). \tag{3} \]

Next, for any nonzero codeword $c(x)$ in $D$, there are $f_0(x), f_u(x)$ in $\mathbb{F}_{p^m}[x]$ such that
\[ c(x) = [f_0(x) + u f_u(x)] (x^n - \alpha_0)^k = f_0(x) (x^n - \alpha_0)^k + uf_u(x) (x^n - \alpha_0)^k. \]

It follows that
\[ \text{wt}_{sp}(c(x)) \geq \max\{\text{wt}_{sp}(f_0(x) (x^n - \alpha_0)^k), \text{wt}_{sp}(f_u(x) (x^n - \alpha_0)^k)\} \geq d_{sp}(\langle (x^n - \alpha_0)^k \rangle_F). \tag{4} \]

Combining (3) and (4), we have
\[ d_{sp}(D) = d_{sp}(\langle (x^n - \alpha_0)^k \rangle_F). \]

The following theorem shows the symbol-pair distances of the constacyclic codes corresponding to the second type.

**Theorem 4.3.** Let $D$ be an $\alpha$-constacyclic code of length $n p^s$ over $\mathbb{F}_{p^m} + u \mathbb{F}_{p^m}$ with type II in Theorem 2.7, i.e., $D = \langle (x^n - \alpha_0)^j b(x) + u(x^n - \alpha_0)^k \rangle$, where $0 \leq k \leq p^s - 1$, $\left\lceil \frac{p^s+k}{2} \right\rceil \leq j \leq p^s - 1$ and either $b(x)$ is $0$ or $b(x)$ is a unit in $\mathcal{F}$. Then
\[ d_{sp}(D) = \begin{cases} d_{sp}(\langle (x^n - \alpha_0)^k \rangle_F), & \text{if } b(x) = 0; \\ d_{sp}(\langle (x^n - \alpha_0)^{p^s-j+k} \rangle_F), & \text{if } b(x) \text{ is a unit in } \mathcal{F}. \end{cases} \]

**Proof.** If $b(x) = 0$, then $D = \langle u(x^n - \alpha_0)^k \rangle$. Hence
\[ d_{sp}(D) = d_{sp}(\langle u(x^n - \alpha_0)^k \rangle) = d_{sp}(\langle (x^n - \alpha_0)^k \rangle_F). \]

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Assume that $b(x)$ is a unit in $\mathcal{F}$. Since
\[(x^n - \alpha_0)^{p^s - j} \left[(x^n - \alpha_0)^j b(x) + u (x^n - \alpha_0)^k\right] = u (x^n - \alpha_0)^{p^s - j + k},\]
it follows that
\[\{ u (x^n - \alpha_0)^{p^s - j + k}\} \subseteq D,\]
hence
\[d_{sp}(D) \leq d_{sp}(\langle u (x^n - \alpha_0)^{p^s - j + k}\rangle) = d_{sp}(\langle (x^n - \alpha_0)^{p^s - j + k}\rangle F). \tag{5}\]

For any nonzero codeword $c(x)$ in $D$, there are $f_0(x), f_u(x)$ in $\mathbb{F}_{p^m}[x]$ such that
\[c(x) = [f_0(x) + u f_u(x)] \left[(x^n - \alpha_0)^j b(x) + u (x^n - \alpha_0)^k\right]
= f_0(x) (x^n - \alpha_0)^j b(x) + u \left[f_0(x) (x^n - \alpha_0)^k + f_u(x) (x^n - \alpha_0)^j b(x)\right].\]

It follows that
\[\text{wt}_{sp}(c(x)) \geq \max\{\text{wt}_{sp}(f_0(x) (x^n - \alpha_0)^j b(x)), \text{wt}_{sp}(f_0(x) (x^n - \alpha_0)^k + f_u(x) (x^n - \alpha_0)^j b(x))\}.\]

If $(x^n - \alpha_0)^{p^s - j} \mid f_0(x)$, let $f_0(x) = (x^n - \alpha_0)^{p^s - j} f'_0(x)$. Then
\[\text{wt}_{sp}(c(x)) \geq \text{wt}_{sp}(r(x) (x^n - \alpha_0)^{p^s - j + k}) \geq d_{sp}(\langle (x^n - \alpha_0)^{p^s - j + k}\rangle F), \tag{6}\]
where
\[r(x) = f'_0(x) + f_u(x) (x^n - \alpha_0)^{2j - p^s - k} b(x).\]

If $(x^n - \alpha_0)^{p^s - j} \nmid f_0(x)$, then
\[\text{wt}_{sp}(c(x)) \geq \text{wt}_{sp}(f_0(x) (x^n - \alpha_0)^j b(x)).\]

Since $j \geq p^s - j + k$,
\[\text{wt}_{sp}(c(x)) \geq d_{sp}(\langle (x^n - \alpha_0)^{p^s - j + k}\rangle F). \tag{7}\]

According to (5), (6) and (7), we have
\[d_{sp}(D) = d_{sp}(\langle (x^n - \alpha_0)^{p^s-j+k} \rangle_F).\]

**Remark 4.4.** Our results of symbol-pair distances of constacyclic codes over \(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}\) generalize the results of [13], which consider the constacyclic codes under the condition of \(n = 1\).

**Remark 4.5.** According to Theorem 4.3, when \(b(x)\) is a unit, the symbol-pair distance of the constacyclic code \(D = \langle (x - \alpha_0)^j b(x) + u(x - \alpha_0)^k \rangle\) with \(k < 2j - p^s\) is equal to \(d_{sp}(\langle (x - \alpha_0)^{p^s-j+k} \rangle_F)\), but not as [13] claimed that equal to \(d_{sp}(\langle (x - \alpha_0)^j \rangle_F)\). We illustrate an example to show this fact in the following.

**Example 4.6.** Consider a cyclic code \(D = \langle (x - 1)^7 + u(x - 1) \rangle\) of length 9 over the finite ring \(\mathbb{F}_3 + u\mathbb{F}_3\), where \(u^2 = 0\). By Theorem 12 in [13],

\[d_{sp}(D) = d_{sp}(\langle (x - 1)^7 \rangle_F) = 9.\]

However, there is a codeword

\[u(x - 1)^3 = (x - 1)^2[(x - 1)^7 + u(x - 1)] \in D,\]

and

\[\text{wt}_{sp}(u(x - 1)^3) = 4,\]

which means the symbol-pair distance of \(D\) cannot be 9. Actually, according to Theorem 4.3

\[d_{sp}(D) = d_{sp}(\langle (x - 1)^3 \rangle_F) = 4.\]

The following theorem shows the symbol-pair distances of the constacyclic codes corresponding to the type III in Theorem 2.7. The proof is similar to that of the former theorem, and we omit it here.

**Theorem 4.7.** Let \(D\) be an \(\alpha\)-constacyclic code of length \(np^s\) over \(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}\) with type III in Theorem 2.7, i.e., \(D = \langle (x^n - \alpha_0)^j b(x) + u(x^n - \alpha_0)^k (x^n - \alpha_0)^{k+t} \rangle\), where \(0 \leq k \leq p^s - 2\), \(1 \leq t \leq p^s - k - 1\), \(k + \left\lfloor \frac{t}{2} \right\rfloor \leq j \leq k + t\), and either \(b(x) = 0\) or \(b(x)\) is a unit in \(\mathcal{F}\). Then

\[d_{sp}(D) = \begin{cases} 
   d_{sp}(\langle (x^n - \alpha_0)^k \rangle_F), & \text{if } b(x) = 0; \\
   d_{sp}(\langle (x^n - \alpha_0)^{2k+t-j} \rangle_F), & \text{if } b(x) \text{ is a unit in } \mathcal{F}.
\end{cases}\]
4.2. MDS codes

In this subsection, we utilize the symbol-pair distances of \((\alpha + u\beta)\)-constacyclic codes of length \(np^s\) over \(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}\) shown in the former subsection to obtain the MDS symbol-pair codes.

The following theorem shows that no nontrivial MDS symbol-pair \((\alpha + u\beta)\)-constacyclic code exists when \(\beta \neq 0\).

**Theorem 4.8.** Let \(\alpha, \beta\) be nonzero elements in \(\mathbb{F}_{p^m}\). Denote \(\alpha = \alpha_0^{p^s}\). Suppose that \(x^n - \alpha_0\) is irreducible over \(\mathbb{F}_{p^m}\). Let \(\mathcal{D}_i = \langle (x^n - \alpha_0)^i \rangle \subseteq \mathcal{R}\) be an \((\alpha + u\beta)\)-constacyclic code of length \(np^s\) over \(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}\), where \(0 \leq i \leq 2p^s\). Then \(\mathcal{D}_i\) is an MDS symbol-pair code if and only if \(i = 0\).

**Proof.** By Theorem 2.6,

\[ |\mathcal{D}_i| = |\langle (x^n - \alpha_0)^i \rangle| = p^{mn(2p^s - i)}.\]

The Singleton Bound shows

\[ |\mathcal{D}_i| \leq |\mathcal{R}|^{np^s - d_{sp}(C_i) + 2},\]

which is equivalent to

\[ ni \geq 2d_{sp}(\mathcal{D}_i) - 4.\]

Therefore, \(\mathcal{D}_i\) is an MDS symbol-pair code if and only if

\[ ni = 2d_{sp}(\mathcal{D}_i) - 4. \quad (8)\]

If \(0 \leq i \leq p^s\), we have \(d_{sp}(\mathcal{D}_i) = 2\) by Theorem 4.1. According to (8), we obtain \(i = 0\). If \(p^s + 1 \leq i \leq 2p^s - 1\), then \(d_{sp}(\mathcal{D}_i) = d_{sp}(\langle (x^n - \alpha_0)^{i-p^s} \rangle_F)\).

Applying the Singleton bound on the constacyclic code \(\langle (x^n - \alpha_0)^{i-p^s} \rangle_F\), we obtain

\[ n(i - p^s) \geq d_{sp}(\langle (x^n - \alpha_0)^{i-p^s} \rangle_F) - 2. \quad (9)\]

Reformulate (9) we have

\[ ni \geq d_{sp}(\mathcal{D}_i) - 2 + np^s \geq 2d_{sp}(\mathcal{D}_i) - 2 \]

\[ > 2d_{sp}(\mathcal{D}_i) - 4,\]

which implies no MDS symbol-pair constacyclic code when \(i\) is in the range \(p^s + 1 \leq i \leq 2p^s - 1\).
The following theorem is considering the case of \( \beta = 0 \) and we obtain three new classes of MDS symbol-pair \( \alpha \)-constacyclic codes over \( \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} \).

**Theorem 4.9.** Let \( \alpha_0 \) be a nonzero element in \( \mathbb{F}_{p^m} \) and \( \alpha = \alpha_0^p \). There are three classes of MDS symbol-pair \( \alpha \)-constacyclic codes of length \( 2p^s \) over \( \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} \) as follows:

(i) \( D = \langle (x^2 - \alpha_0) + ub(x) \rangle \), where \( b(x) \) is either zero or a unit in \( \mathcal{F} \);

(ii) \( D = \langle (x^2 - \alpha_0)^{p^s-1} + u(x^2 - \alpha_0)^{p^s-2}b(x) \rangle \), where \( b(x) \) is either zero or a unit in \( \mathcal{F} \);

(iii) \( D = \langle (x^2 - \alpha_0)^j + u(x^2 - \alpha_0)^{k}b(x) \rangle \), where \( s = 1, 1 \leq j \leq p - 1, \ max \{0, 2j - p\} \leq k < j \), and \( b(x) \) is either zero or a unit in \( \mathcal{F} \).

**Proof.**

(i) By Lemma 2.7, the size of \( D \) is \( p^{4m(p^s-1)} \). According to Theorem 4.2 and 4.7, the symbol-pair distance of \( D \) is 4. It achieves the Singleton bound

\[ |D| = p^{4m(p^s-1)} = p^{2m(2p^s-4+2)} \]

with equality. Therefore, \( D \) is an MDS code.

(ii) By Lemma 2.7, the size of \( D \) is \( p^{4m} \). According to Theorem 4.2 and 4.3, the symbol-pair distance of \( D \) is \( 2p^s \). It achieves the Singleton bound

\[ |D| = p^{4m} = p^{2m(2p^s-2p^s+2)} \]

with equality. Therefore, \( D \) is an MDS code.

(iii) By Lemma 2.7, the size of \( D \) is \( p^{4m(p-j)} \). According to Theorem 4.2, 4.3 and 4.7, the symbol-pair distance of \( D \) is \( 2j + 2 \). It achieves the Singleton bound

\[ |D| = p^{4m(p-j)} = p^{2m(2p-2j-2+2)} \]

with equality. Therefore, \( D \) is an MDS code.

**Remark 4.10.** In [10], Dinh et al. gave two more classes of MDS symbol-pair codes with parameters \((2^s, 2^m(2^s-1+4), 3 \cdot 2^{s-2})\) and \((3^s, 3^m(3^s-1+4), 2 \cdot 3^{s-1})\).

The first \( \alpha \)-constacyclic code of length \( 2^s \) over \( \mathbb{F}_{2^m} + u\mathbb{F}_{2^m} \) is

\[ D = \langle (x^n - \alpha_0)^{2^s-3} + u(x^n - \alpha_0)^{2^{s-1}-4} \rangle, \]

where \( s \geq 3 \). By Remark 4.5, the symbol-pair distance of \( C \) is \( d_{sp}(\langle (x - \alpha_0)^{2^{s-1}-1} \rangle) = 4 \), but not \( d_{sp}(\langle (x - \alpha_0)^{2^{s-3}} \rangle) = 3 \cdot 2^{s-2} \). Hence it is not MDS.
The second $\alpha$-constacyclic code of length $3^s$ over $\mathbb{F}_{3^m} + u\mathbb{F}_{3^m}$ is

$$D = \langle (x^n - \alpha_0)^{3^s-5} + u (x^n - \alpha_0)^{3^s-1-4} \rangle,$$

where $s \geq 3$. By Remark 4.5, the symbol-pair distance of $C$ is $d_{sp}(\langle (x - \alpha_0)^{3^s-1+1} \rangle_F) = 6$, but not $d_{sp}(\langle (x - \alpha_0)^{3^s-5} \rangle_F) = 2 \cdot 3^s-1$. Hence it is also not MDS.

Combining Theorem 4.9 and previous work, the MDS symbol-pair $\alpha$-constacyclic codes of length $n p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ are listed in Table 2. When the polynomial $x^n - \alpha_0$ is irreducible over $\mathbb{F}_{p^m}$, we draw the following conclusion. The proof is similar to Theorem 3.7 and omitted here.

**Theorem 4.11.** Let $\alpha_0$ be a nonzero element of $\mathbb{F}_{p^m}$ and $\alpha = \alpha_0^{p^s}$. Let $\beta$ be an element of $\mathbb{F}_{p^m}$. When $x^n - \alpha_0$ is irreducible over $\mathbb{F}_{p^m}$, there are no other nontrivial MDS symbol-pair $\alpha + u\beta$-constacyclic codes of length $n p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ except the MDS codes shown in Table 2.

5. Conclusion

Let $\mathbb{F}_{p^m}$ be a finite field and $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ be a finite ring with $u^2 = 0$. We determine the symbol-pair distances of $\alpha$-constacyclic codes of length $n p^s$ over $\mathbb{F}_{p^m}$ and $(\alpha + u\beta)$-constacyclic codes of length $n p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$, where $n, s$ are positive integers with $\gcd(n, p) = 1$, $\beta \in \mathbb{F}_{p^m}$, and $\alpha = \alpha_0^{p^s} \in \mathbb{F}_{p^m}$ such that $x^n - \alpha_0$ is irreducible over $\mathbb{F}_{p^m}$. Moreover, we show that the non-trivial MDS symbol-pair $\alpha$-constacyclic codes of length $n p^s$ over $\mathbb{F}_{p^m}$ only exist when $n = 1, 2$. Similarly, the non-trivial MDS symbol-pair $(\alpha + u\beta)$-constacyclic codes of length $n p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ only exist when $\beta = 0$ and $n = 1, 2$. Some of these MDS symbol-pair codes we present in this paper are new and have relatively large pair distance. It is an interesting problem to consider the case that $x^n - \alpha_0$ is reducible over $\mathbb{F}_{p^m}$.

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