X-RAY TRANSFORM IN ASYMPTOTICALLY CONIC SPACES

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Abstract. In this article, we study the properties of the geodesic X-ray transform for
asymptotically Euclidean or conic Riemannian metrics and show injectivity under non-
trapping and no conjugate point assumptions. We also define a notion of lens data for
such metrics and study the associated inverse problem.

1. Introduction

On Euclidean space \((\mathbb{R}^n, g_0)\), the linear operator \(I_0\) mapping a function \(f\) to the set of
its integrals

\[
I_0 f(\gamma) = \int_{\mathbb{R}} f(\gamma(r)) \, dr
\]

along lines \(\gamma\) is called X-ray transform, or Radon transform in dimension 2. It is known
to be invertible using Radon inversion formula and is the basis of X-ray tomography. This
operator, when acting on functions in a fixed convex compact support, say for example the
unit ball \(B\), is also the linearisation of the following natural geometric inverse problem: is
there a metric \(g = e^{2f} g_0\) conformal to the Euclidean metric \(g_0\) (with \(f \in C^\infty_c(B)\)) so that
there is a conjugation \(\psi : S_{g_0} \mathbb{R}^n \to S_g \mathbb{R}^n\) between the geodesic flow of \(g_0\) and \(g\) on their
respective unit tangent bundles, which is equal to the Identity outside \(S_{g_0} B\), i.e.

\[
\varphi_t^g(\psi(x, v)) = \psi(\varphi_t^{g_0}(x, v)), \quad \psi(x, v) = (x, v) \text{ if } |x| \geq 1
\]

This conjugation property can also be written in terms of the equality between two func-
tions called scattering map and rescaled lengths, that we shall introduce below. More
generally, one can define an X-ray transform on symmetric tensors of order \(m \in \mathbb{N}\) by the
formula

\[
I_m f(\gamma) = \int f_\gamma(t) (\otimes^m \dot{\gamma}(t)) \, dt = \int_{\mathbb{R}^n} \sum_{i_1, i_2, \ldots, i_m=1}^n f_{i_1i_2\ldots i_m}(\gamma(t)) \dot{\gamma}_1(t) \dot{\gamma}_2(t) \ldots \dot{\gamma}_m(t) \, dt.
\]

The case \(m = 2\) corresponds to the same linearised problem as above but replacing con-
formal metrics \(g = e^{2f} g_0\) by any possible metric \(g\) on \(\mathbb{R}^n\) which is a compact perturba-
tion of \(g_0\). This rigidity problem was solved by Gromov [Gr], with an alternate proof by Croke
[Cr], for the class of metrics \(g\) with no conjugate points. This property is also called bound-
ary rigidity of the Euclidean metric on \(B\).
In this paper, we investigate a similar problem but for non-compact perturbations of $\mathbb{R}^n$, and more generally non-compact perturbations of metric cones. A metric cone is a warped product $(0, \infty)_r \times N$ with metric

$$g_0(r, y) = dr^2 + r^2h_0(y, dy)$$

where $(N, h_0)$ is a closed Riemannian manifold of dimension $n - 1$. Here we work with smooth metrics and we can take our model to be any smoothing of the metric cone at the cone tip $r = 0$, and more generally any Riemannian metric which is asymptotic near infinity to the region $r \geq 1$ of the cone. To be precise, our Riemannian manifold $(M, g)$ metric will be called asymptotically conic if $M$ is the interior of a smooth compact manifold with boundary $\partial M$ and there is a smooth boundary defining function of $\partial M$ such that, in a product decomposition $[0, \epsilon)_\rho \times \partial M$ near the boundary,

$$g(\rho, y) = \frac{d\rho^2}{\rho^2} + \frac{h_\rho(y)}{\rho^2}$$

where $h_\rho$ is a smooth 1-parameter of metrics on $\partial M$, $y$ being coordinates on $\partial M$; see subsection 2.1. Here $\partial M$ plays the role of the section $N$ of the cone described above and $\rho$ plays the role of $1/r$. Using the $r$ variable, this means that $g$ has an asymptotic expansion in powers of $1/r$ near infinity, and with leading term the exact conic metric. We say that $g$ is non-trapping if each complete geodesic $\gamma(t)$ of $g$ tends to $\partial M$ as $t \to \pm \infty$; more generally such geodesics $\gamma$ is said to be non-trapped. For example small perturbations of the Euclidean metric on $\mathbb{R}^n$ are non-trapping. These types of metrics have been studied intensively in scattering theory for the wave equation [Me1, MeZw, HaVa, JoSB1, JoSB2, SBWu]. For asymptotically conic metrics we define the X-ray transform $I_m$ on symmetric tensors by

$$I_m : C_c^\infty(M; S^m(T^*M)) \to L^\infty_{\text{loc}}(\mathcal{G}), \quad I_m f(\gamma) := \int_{-\infty}^{\infty} f_\gamma(t)(\otimes^m \dot{\gamma}(t))dt$$

where $\mathcal{G}$ is the set of complete non-trapped geodesics of $g$. The kernel of $I_m$ contains the space of exact $(m - 1)$-tensors $\{Dh; h \in C_c^\infty(M; S^{m-1}(T^*M))\}$ where $D$ is the symmetrized covariant derivative, that is, $Dh = S(\nabla h)$ where $S : (T^*M)^{\otimes m} \to (T^*M)^{\otimes m}$ is the symmetrization operator,

$$S(\sum_{i_1, \ldots, i_m} h_{i_1i_2\ldots i_m}dx^{i_1} \otimes \cdots \otimes dx^{i_m}) = \sum_{i_1, \ldots, i_m} \frac{1}{m!} (\sum_{\sigma} h_{i_{\sigma(1)}i_{\sigma(2)}\ldots i_{\sigma(m)}})dx^{i_1} \otimes \cdots \otimes dx^{i_m},$$

where $\sigma$ runs over all permutations of the set $\{1, 2, \ldots, m\}$. We say that $I_m$ is solenoidal injective if this is an equality.
A smooth symmetric scattering tensor of order \(m\) is a smooth symmetric tensor \(h \in C^\infty(M; S^mT^*M)\) on \(M\) which can be written near \(\rho = 0\) under the form

\[
h = \sum_{j=0}^{m} S \left( \frac{h_j}{\rho^j} \otimes \left( \frac{d\rho}{\rho^2} \right)^{\otimes(m-j)} \right)
\]

where \(h_j \in C^\infty(M; S^j(T^*\partial M))\) and \(S\) is the symmetrization operator on tensors. Note that they have bounded pointwise norm with respect to \(g\). The space of smooth scattering symmetric tensors will be denoted \(C^\infty(M; S^m_{\text{sc}}T^*M))\).

**Theorem 1.1.** Let \((M, g)\) be either a non-trapping asymptotically conic manifold without conjugate points or a negatively curved asymptotically conic manifold, possibly with trapping, and let \(k > n/2 + 1\).

i) If \(f \in \rho^k C^\infty(M)\) and \(I_0 f = 0\), then \(f = 0\).

ii) If \(f \in \rho^k C^\infty(M, scT^*M)\) and \(I_1 f = 0\) then there exists \(u \in \rho^{k-1} C^\infty(M)\) such that \(f = du\).

iii) If in addition \((M, g)\) has non-positive curvature, then for each \(f \in \rho^k C^\infty(M; S^m_{\text{sc}}T^*M)\) satisfying \(I_m f = 0\) with \(m > 1\), there is \(u \in C^\infty(M; S^{m-1}T^*M) \cap \rho^{k-1} L^\infty\) such that \(Du = f\).

Assuming that the boundary has additional geometric properties, we obtain similar results with weaker assumptions on the function \(f\). Indeed, the geodesics that stay very far near infinity (i.e. close to \(\partial M\)) approach in the compactification \(\overline{M}\) the geodesics traveling for time \(\pi\) in the boundary \((\partial M, h_0)\). This leads to some kind of time-\(\pi\) ray transform on the boundary, which turns out to be injective if for example the radius of injectivity of \((\partial M, h_0)\) is larger than \(\pi\); see Proposition 3.4.

We emphasize that the geometric assumption on \((M, g)\) in Theorem 1.1 does not include non-trivial decaying perturbations of the Euclidean metrics, as is shown in the companion paper \([GMT]\): indeed, asymptotically Euclidean metrics on \(\mathbb{R}^n\) with no conjugate points must be Euclidean. However there are plenty of examples satisfying the assumptions of Theorem 1.1, for example some with non-positive curvature. The boundary \(\partial \overline{M}\) could typically be a round sphere of curvature less than 1; see section 2.3.

In the case of Cartan-Hadamard surfaces with curvature decaying at order \(O(d^{-\kappa})\) for some \(\kappa > 2\) \((d\) being to distance to a fixed point), Lehtonen-Railo-Salo \([LRS]\) proved injectivity of X-ray for tensors with the same decay assumptions on \(f\). Their result is more general in terms of the behaviour at infinity than ours, but it is weaker in the sense that it requires non-positive curvature for functions and 1-forms.

We remark that one possible application of that work could be to apply our injectivity results for getting stability in the inverse scattering problem for the wave-equation on such manifolds, since stability estimates for the wave equation often can be reduced to X-ray
stability estimates, and the proof of injectivity easily brings stability estimates.

In a second part of the article, we define the notion of lens data in that setting, namely the scattering map $S_g$ and the renormalized length $L_g$ of geodesics, in a way similar to the work [GGSU] for asymptotically hyperbolic manifolds. The renormalized length of a non-trapped complete geodesic $\gamma$ is

$$L_g(\gamma) := \lim_{\epsilon \to 0} \ell_g(\gamma \cap \{ \rho \geq \epsilon \}) - 2/\epsilon$$

and the scattering map is a symplectic map $S_g : T^*\partial\overline{M} \to T^*\partial\overline{M}$ which encodes the asymptotics at $t \to \pm \infty$ for $(\gamma(t), \dot{\gamma}(t))$ on $S^*M$; it is defined using a time rescaling of the geodesic flow of $g$, see Section 4.1. We show that being in the kernel of the linearisation of this pair $(L_g, S_g)$ at a given metric $g_0$ means that the X-ray transform $I_2$ (for $g_0$) of the variation of metrics must vanish, implying a deformation rigidity result. A non-trivial aspect in this analysis is the determination, in certain geometric cases, of the jets of the metrics at $\partial\overline{M}$ from the scattering map: we show for example that the scattering map determines the full metric asymptotics if the boundary metric has ergodic geodesic flow at time $\pi$ or more generally if it has negative curvature. This analysis is also strongly used in our companion paper [GMT] on the non-existence of non-trivial asymptotically Euclidean metrics on $\mathbb{R}^n$.

We conclude with a deformation rigidity result in negative curvature:

**Theorem 1.2.** Let $(M, g(s))$, $s \in (-1, 1)$, be a one-parameter family of negatively curved asymptotically conic manifolds which metric $g(s)$ depends smoothly on $s$ such that near $\partial\overline{M}$ the metric has the form $g(s) = \frac{d\rho^2}{\rho^2} + \frac{h(s)}{\rho^2}$, where $h(s)|_{\partial\overline{M}} = h(0)|_{\partial\overline{M}}$ and $(\partial\overline{M}, h(0))$ has negative curvature. Assume the family has constant lens data, that is,

$$S_{g(s)} = S_{g(0)}, \quad L_{g(s)} = L_{g(0)}.$$

Then there exists a family of diffeomorphisms $\psi(s) : M \to M$ with $\psi|_{\partial\overline{M}} = \text{Id}$ which satisfies $g(0) = \psi(s)^*g(s)$.

A similar result holds with the non-trapping and no conjugate point assumptions, provided the family agrees to high enough order at the boundary (depending on the dimension so that we can apply Theorem 1.1)

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2. Geometric Preliminaries

2.1. Asymptotically conic metrics and normal form. Asymptotically conic or scattering manifolds are complete Riemannian manifolds \((M, g)\) where \(M\) is the interior of a smooth compact manifold with boundary \(\overline{M}\) and \(g\) is a smooth metric on \(M\) satisfying a certain asymptotic structure at \(\partial \overline{M}\) that we now describe. Let \(\rho_0 \in C^\infty(\overline{M}; \mathbb{R}^+)\) be a smooth boundary defining function, i.e. \(\partial \overline{M} = \{ \rho_0 = 0 \}\) and \(d\rho_0|_{\partial \overline{M}}(y) \neq 0\) for all \(y \in \partial \overline{M}\). First, following Melrose [Me1], we recall that there is a smooth bundle, called the scattering tangent bundle and denoted by \(\text{sc}T\overline{M} \to \overline{M}\), whose space of smooth sections can be identified with the space of smooth vector fields of the form \(\rho V\), where \(V\) are smooth vector fields on \(\overline{M}\) that are tangent to the boundary \(\partial \overline{M}\). The vector bundle dual to \(\text{sc}T\overline{M}\) will be denoted by \(\text{sc}^*T\overline{M}\). Near \(\partial \overline{M}\), \(d\rho_0/\rho_0^2, d\eta_1/\rho_0, \ldots, d\eta_{n-1}/\rho_0\) is a local frame.

**Definition 2.1.** A Riemannian metric \(g\) on \(M\) is called asymptotically conic if \(g \in C^\infty(\overline{M}; S^2(\text{sc}^*T^*\overline{M}))\) and there exists a boundary defining function \(\rho_0\) such that \(\rho_0^{-2} |d\rho_0|_g = 1 + \mathcal{O}(\rho_0^0)\). We say that it is asymptotically conic to order \(m \geq 1\) if moreover \(g - g_0 \in \rho_0^m C^\infty(\overline{M}; S^2(\text{sc}^*T^*\overline{M}))\) for some smooth metric \(g_0\) on \(M\) that is equal to an exact conic metric \(g_0 = d\rho^2/\rho^4 + h_0/\rho^2\) near \(\partial \overline{M}\), \(h_0\) being a smooth Riemannian metric on \(\partial \overline{M}\).

In particular, near \(\partial \overline{M}\), one has

\[
g - g_0 = \rho_0^m \left( a \frac{d\rho_0^2}{\rho_0^2} + \sum_j b_j \frac{d\rho_0}{\rho_0^2} dy_j + \sum_{i,j=1}^{n-1} \ell_{ij} dy_i dy_j \right) \tag{2.1}
\]

where \(a, b_j, \ell_{ij} \in C^\infty(\overline{M})\) with \(\rho_0^m a = \mathcal{O}(\rho_0^{\max(2,m)})\). A metric cone is \((0, \infty)_r \times N\) with metric \(dr^2 + r^2 h_0\) if \((N, h_0)\) is a compact Riemannian manifold. Setting \(\rho = 1/r\), the metric becomes \(d\rho^2/\rho^4 + h_0/\rho^2\) outside \(r = 0\), thus smoothing the tip \(r = 0\) of the cone indeed gives an asymptotically conic metric. In [JoSB2], Joshi-Sa Barreto proved that an asymptotically conic metric admits an approximate normal form near the boundary \(\partial \overline{M}\).

An exact normal form can in fact easily be obtained by reducing to a non-characteristic first order PDE, this suggestion appears for example in Graham-Kantor [GrKa]: we give here a short self-contained proof based on this argument (we also need to compare the exact cone case to the perturbed one).

**Lemma 2.2.** Let \(g\) be asymptotically conic to order \(m \geq 1\). Then there is a boundary defining function \(\rho \in C^\infty(\overline{M})\) satisfying

\[
\frac{\nabla g \rho|_g}{\rho^2} = 1, \quad \rho = \rho_0(1 + \mathcal{O}(\rho_0^m)). \tag{2.2}
\]

If \(m \geq 2\), the function \(\rho\) is uniquely determined near the boundary by the equation (2.2), while if \(m = 1\), such a \(\rho\) is not unique: for each \(\omega_0 \in C^\infty(\partial \overline{M})\) there is a function \(\rho = \rho_0 + \omega_0 \rho_0^2 + \mathcal{O}(\rho_0^3)\) such that \(\rho^{-2} |\nabla g \rho|_g = 1\), uniquely defined near \(\partial \overline{M}\). For each such
\(\rho\), there is a smooth diffeomorphism
\[
\psi : [0, \epsilon)_s \times \partial M \rightarrow U \subset M
\]
ono onto a collar neighborhood \(U\) of \(\partial M\) such that \(\psi(0, \cdot)|_{\partial M} = \text{Id}\), \(\psi^* \rho = s\) and
\[
\psi^* g = \frac{ds^2}{s^4} + \frac{h_s}{s^2},
\]
where \(h_s\) is a smooth family of Riemannian metrics on \(\partial M\) such that
\[
h_s - h_0 \in s^m C^\infty([0, \epsilon) \times \partial M; S^2(T^*\partial M)).
\]
The diffeomorphism \(\psi\) is given by the expression \(\psi(s, y) = e^{sZ_\rho}(y)\), where \(Z_\rho := -4\nabla^g \rho \in C^\infty(M; TM)\). When \(m = 1\), if \(h_s\) and \(h_s\) are the smooth family of Riemannian metrics on \(\partial M\) associated with two boundary defining function \(\rho, \rho\) with \(\partial \rho = \rho + \omega_0 \rho^2 + \mathcal{O}(\rho^3)\), then
\[
\hat{h}_s = h_s + s(L_{\xi_s} h_0 + 2\omega_0 h_0) + \mathcal{O}(s^2).
\]
Proof. We search for \(\rho = \rho_0 e^{\rho\rho\omega}\) with \(\omega \in C^\infty(M)\) such that \(|d\rho/\rho^2|_g = 1\) near \(\partial M\). This is equivalent to the equation
\[
2\rho_0^{-3}(\nabla^g \rho_0)(\rho_0 \omega) = e^{2\rho_0 \omega} - 1 - \rho_0^2 \frac{d\omega}{\rho_0^2} - \rho_0^2 \omega^2 - 2\rho_0^2 g\left(\frac{d\omega}{\rho_0^2}, \frac{d\rho_0}{\rho_0^2}\right) + \rho_0^2 F
\]
with \(F := \rho_0^{-2}(1 - |d\rho_0/\rho_0^2|_g) \in \rho^{\max(m-2,0)} C^\infty(M)\). A direct computation gives
\[
\nabla^g \rho_0 = \rho_0 \partial_{\rho_0} + \rho_0^{m+3} V
\]
for some smooth vector field on \(M\) that has the form \(V = a \rho_0 \partial_{\rho_0} + \sum_j b_j \partial_{y_j}\) near \(\partial M\) with \(a, b_j \in C^\infty(M)\). Then (2.4) can be rewritten as
\[
2\partial_{\rho_0} \omega = \rho_0^{-2} \left(e^{2\rho_0 \omega} - 1 - 2\rho_0 \omega\right) + G_g(\rho_0, y, \omega, \partial_{\rho_0}, \omega, \partial_{\rho_0}, \omega).
\]
where \(G_g\) is \(C^\infty\) in the variables \(\rho_0, y\) and polynomial of degree 2 in the last 3 variables, and \(G_g(0, y, \omega, U, V)\) is independent of \(U\). Note that \((e^x - 1 - x) = x^2 \sum_j x^j/(j + 2)!\) is smooth, then we see that the equation (2.5) with boundary condition \(\omega|_{\rho_0=0} = \omega_0\) at \(\partial M\) is a non-characteristic first order non-linear PDE with \(C^\infty\) coefficients. It thus has a unique solution \(\omega\) near \(\partial M\) that is \(C^\infty(M)\). If we choose \(\omega|_{\partial M} = 0\), we obtain a particular solution of (2.4). Notice that \(\omega = 0\) is the solution of (2.4) with \(g\) is replaced by \(\tilde{g}_0\) and \(\omega|_{\partial M} = 0\), and that
\[
G_g(\rho_0, y, \omega, \partial_{\rho_0}, \omega, \partial_{\rho_0}, \omega) = G_{g_0}(\rho_0, y, \omega, \partial_{\rho_0}, \omega, \partial_{\rho_0}, \omega) + F + \rho_0^{m-1} \tilde{G}(\rho_0, y, \omega, \rho_0 \partial_{\rho_0}, \omega, \partial_{\rho_0}, \omega)
\]
for some \(\tilde{G}\) that is \(C^\infty\) in the variables \((\rho_0, y)\), polynomial of order 2 in the last 3 variables. To simplify notations, we write \(G_g(\omega)\) instead of \(G_g(\rho_0, y, \omega, \partial_{\rho_0}, \omega, \partial_{\rho_0}, \omega)\) using the
expression of $G_{g_0}(\omega)$ obtained from (2.4) with $g_0$ instead of $g$,  
\[ 2\partial_\rho \omega = \rho_0^{m-1} \tilde{G}(\omega) + F + \rho_0^{-2} \left( e^{2\rho_0 \omega} - 1 - 2\rho_0 \omega \right) + G_{g_0}(\omega) \]
\[ = -|\partial_\omega|_{h_0}^2 + \omega^2 + \mathcal{O}(|\rho_0 \partial_\rho \omega|(|\omega| + |\rho_0 \partial_\rho \omega|)) + \rho_0^{m-1} \tilde{G}(\omega) + F. \]  
(2.6)

Since $\omega \in C^\infty$, we can write its Taylor expansion, and assuming that
\[ \omega = \mathcal{O}(\rho_0^\ell), \quad \partial_\rho \omega = \mathcal{O}(\rho_0^\ell), \quad \partial_\rho \omega = \mathcal{O}(\rho_0^{\ell-1}) \]
for some $\ell \geq 1$ and plugging this into (2.6), we deduce that $\partial_\rho \omega = \mathcal{O}(\rho_0^{m-2}) + \mathcal{O}(\rho_0^{2\ell})$, which shows that $\omega = \mathcal{O}(\rho_0^{m-1}) + \mathcal{O}(\rho_0^{2\ell+1})$. By induction, this implies that $\omega = \mathcal{O}(\rho_0^{m-1})$ near $\rho_0$. Finally, let $Z_\rho := \nabla^g \rho / \rho^4 = \nabla^g \rho / |\nabla^g \rho|_g^2$. This is a $C^\infty$ vector field near $\partial M$ that is transverse to $\partial M$. We consider its flow $\phi_s$ and the diffeomorphism $\psi : [0, \epsilon) \times \partial M \to M$ defined by $(s, y) \mapsto \phi_s(y)$ is $C^\infty$ and it is direct to check that it satisfies the desired properties by using that $\rho = \rho_0(1 + \mathcal{O}(\rho_0^m))$. Moreover one has uniqueness of such a defining function if $m \geq 2$ since $\rho$ is determined by $\omega_0$ (which needs to be 0 in order to get $\rho - \rho_0 = \mathcal{O}(\rho_0^2)$).

Now, for the case where $m = 1$, we see that the choices of $\rho$ are determined by the boundary value $\omega_0$: assume $\hat{\rho} = \rho + \omega_0 \rho^2 + \mathcal{O}(\rho^3)$ are two normal forms and $g = d\hat{\rho}^2 / \rho^4 + h_0 / \rho^2$ and $h_0 = h_0 + \rho h_0 + \mathcal{O}(\rho^2)$. Let $\hat{\phi}_s = e^{s\hat{\rho}}$; to check (2.3), we compute that $\partial_s (s^2 \phi_s^* g)(\partial_{j_1}, \partial_{j_2})|_{s=0} = (h_1 + \mathcal{L}_{\hat{\rho}} h_0 + 2 \omega_0 h_0)(\partial_{j_1}, \partial_{j_2})$. Now an easy computation shows that $Z_{\rho_0}|_{\partial M} = \partial_\rho + \nabla^0 \omega_0$ and this gives (2.3). \[ \square \]

In what follows, we will study $g$ in normal form, i.e. $g = d\rho^2 / \rho^4 + h_\rho / \rho^2$ near $\partial M$, for some smooth family $h_\rho$ of metrics on $\partial M$.

### 2.2. Curvature tensor

Let us first compute the decay of the curvature tensor near infinity.

**Proposition 2.3.** Let $V, W \in \ker d\rho$ of length $g_\rho(V, V) = g_\rho(W, W) = 1$ and $g_\rho(V, W) = 0$ and let $Z = \rho^2 \partial_\rho$. For $p = (\rho, y) \in M$ sufficiently close to $\partial M$, the sectional curvature $K_p$ and Riemann tensor $R_p$ satisfy:

1) $|K_p(V, W)| = \mathcal{O}(\rho^2)$.

2) $|K_p(V, Z)| = \mathcal{O}(\rho^4)$.

3) $\langle R_p(V, W)W, Z \rangle = \mathcal{O}(\rho^2)$.

If in addition $(\partial M, h_0)$ has sectional curvature $+1$, then $K_p(V, W) = \mathcal{O}(\rho^3)$.

**Proof.** Let us define the subbundle $E = \ker d\rho \subset T\overline{M}$ and $\mathcal{E} = C^\infty(\overline{M}; E)$. We have $V = \rho \overline{V}$ and $W = \rho \overline{W}$ for some smooth $\overline{V}, \overline{W} \in \mathcal{E}$, and let $Z = \rho^2 \partial_\rho = \rho^2 Z$ for $Z \in C^\infty(\overline{M}; T\overline{M})$. A direct computation gives for all $V, W \in \rho \mathcal{E}$

\[ [Z, V] - \rho V \in \rho^3 \mathcal{E}, \quad [V, W] \in \rho^2 \mathcal{E} \]

Using Koszul formula, the Levi-Civita connection satisfies for all $V, W \in \rho \mathcal{E}$

\[ \nabla_V W = \rho^2 \nabla_V \overline{W} + a(V, W)Z, \quad \nabla_Z Z = 0, \quad g(\nabla_Z V, Z) = 0, \]
where $\nabla^h$ is the connection of $h(x)$ on the hypersurfaces given by level sets of $x$ and $\alpha(V, W)$ some smooth function on $M$.

We claim that the curvature in the direction $V, W \in \mathcal{E}$ (with $\langle V, W \rangle = 0$ and $|V|_g = 1 = |W|_g$) is given by

$$K(V, W) = \rho^2(K^h(\nabla^h V, \nabla^h W) - 1) + \frac{\rho^3}{2}((\partial_\rho h)(\nabla^h V, \nabla^h V) + (\partial_\rho h)(\nabla^h W, \nabla^h W))$$

$$- \frac{\rho^4}{4}((\partial_\rho h)(\nabla^h V, \nabla^h V)(\partial_\rho h)(\nabla^h W, \nabla^h W) - ((\partial_\rho h)(\nabla^h V, \nabla^h W))^2)$$

(2.7)

Indeed, we compute for an orthonormal basis $(Y_j)_{j=1}^{n-1} \in \rho \mathcal{E}$ of $\ker d\rho$ for the metric $g$

$$\langle \nabla_W \nabla_V W, V \rangle = \langle \nabla_W \langle \nabla_V W, Z \rangle Z, V \rangle + \sum_{j=1}^{n-1} \langle \nabla_W \langle \nabla_V W, Y_j \rangle Y_j, V \rangle$$

$$= \langle \nabla_V W, Z \rangle \langle \nabla_W Z, V \rangle + \rho^2 h(\nabla^h_W \nabla^h_V \nabla^h - \nabla^h \nabla^h V)$$

$$= -\langle \nabla_V W, Z \rangle \langle Z, \nabla_W V \rangle + \rho^2 h(\nabla^h_W \nabla^h_V, \nabla^h V)$$

$$= -((\nabla_V W, Z)\rangle^2 + \rho^2 h(\nabla^h_W \nabla^h_V, \nabla^h V)$$

$$= -\frac{1}{4}\rho^4(\partial_\rho h(\nabla^h V, \nabla^h W))^2 + \rho^2 h(\nabla^h_W \nabla^h_V, \nabla^h V)$$

The last equality comes from using Koszul formula. Also

$$\langle \nabla_V \nabla_W W, V \rangle = \langle \nabla_V \sum_j \langle \nabla_W W, Y_j \rangle Y_j, V \rangle + \langle \nabla_V \langle \nabla_W W, Z \rangle Z, V \rangle$$

$$= \rho^2 \langle \nabla^h_W \nabla^h_V \nabla^h - \nabla^h \nabla^h V \rangle + \langle W, \nabla_W Z \rangle \langle [V, Z], V \rangle$$

$$= \rho^2 \langle \nabla^h_W \nabla^h_V, \nabla^h V \rangle - \langle W, [Z, W] \rangle \langle [V, Z], V \rangle$$

$$= \rho^2 \langle \nabla^h_W \nabla^h_V, \nabla^h V \rangle - W, \rho W + \rho^3 (\partial_\rho, \nabla^h V) \rangle \langle V, \rho V + \rho^3 (\partial_\rho, \nabla^h V) \rangle$$

Now using the fact that $\partial_\rho h(\nabla^h W, \nabla^h V)) = 0 = \partial_\rho h(\nabla^h V, \nabla^h V))$, we see that $2h([\partial_\rho, \nabla^h V], \nabla^h V) = -\partial_\rho h(\nabla^h W, \nabla^h W)$ and the same with $\nabla^h V$ replacing $\nabla^h W$. We thus obtain

$$\langle \nabla_V \nabla_W W, V \rangle = \rho^2 \langle \nabla^h_W \nabla^h_V, \nabla^h V \rangle - \rho^2 - \frac{\rho^4}{4} \partial_\rho h(\nabla^h V, \nabla^h V) \partial_\rho h(\nabla^h W, \nabla^h W)$$

$$+ \frac{1}{2} \rho^3 (\partial_\rho h(\nabla^h V, \nabla^h V) + \partial_\rho h(\nabla^h W, \nabla^h W))$$

Since also $\langle \nabla_W W, V \rangle = \rho^2 h(\nabla^h_W \nabla^h_V, \nabla^h V)$, we can combining these computations to deduce (2.7).

Next, we compute the curvature in mixed direction $K(Z, V)$.

$$\langle \nabla_Z \nabla_V V, Z \rangle = Z \langle \nabla_V V, Z \rangle = -Z \langle V, \nabla_V Z \rangle = Z \langle \rho h(\nabla^h V, \partial_\rho, \rho V) \rangle$$

$$= \rho^2 - \rho^3 \partial_\rho h(\nabla^h V, \nabla^h V) - \frac{1}{2} \rho^4 \partial_\rho (\partial_\rho h(\nabla^h V, \nabla^h V))$$

(2.8)
We can choose $V$ to be parallel with respect to $Z$ in the collar $\rho \in (0, \epsilon)$, so that
$$\langle \nabla_V \nabla_Z V, Z \rangle = 0.$$  

The equation $\nabla_Z V = 0$ can be written as (here $S = h^{-1} \partial_\rho h$ denotes the shape operator for the hypersurfaces $\rho = \text{const}$)
$$[\partial_\rho, V] = -\frac{1}{2} S V.$$

And finally, we have
$$-\langle \nabla_{[Z, V]} Z, Z \rangle = \langle V, \nabla_{[Z, V]} Z \rangle = \langle V, \nabla_Z [Z, V] - [Z, [Z, V]] \rangle$$
$$= h(V, \nabla_{\rho \partial_\rho} [Z, V] - 2 \rho^2 [\partial_\rho, V] - \rho^3 [\partial_\rho^2, V])$$
$$= \rho^3 h(V, \nabla_{\rho \partial_\rho} [Z, V] - [\partial_\rho^2, V])$$
$$= \rho^3 h(V, \nabla_{\rho \partial_\rho} V - [\partial_\rho, V] + \rho \nabla_{\partial_\rho} [\partial_\rho, V] - \rho [\partial_\rho^2, V])$$
$$= -\rho^2 + \frac{\rho^3}{2} \partial_\rho h(V, \nabla) + \rho^4 h(V, \nabla_{\rho \partial_\rho} V - [\partial_\rho^2, V]).$$

But we also have, using $\nabla_{\partial_\rho} V = 0$,
$$\rho^4 h(V, \nabla_{\partial_\rho} [\partial_\rho, V]) = \rho^6 (\partial_\rho (g(V, [\partial_\rho, V])) + \rho^3 h(V, [\partial_\rho, V])$$
$$= -\rho^3 h(V, [\partial_\rho, V]) + \rho^4 \partial_\rho (h(V, [\partial_\rho, V]))$$
$$= \frac{1}{2} \rho^3 \partial_\rho h(V, V) - \frac{\rho^4}{2} \partial_\rho h(V, V)$$
and, by differentiating twice $h(V, V) = 1$ with respect to $\partial_\rho$,
$$-h([\partial_\rho^2, V], V) - \frac{1}{2} \partial_\rho^2 h(V, V) = h([\partial_\rho, V], [\partial_\rho, V]) + 2 \partial_\rho h([\partial_\rho, V], V).$$

Thus, combining with (2.8) we obtain that
$$K(Z, V) = -\frac{\rho^4}{2} \partial_\rho^2 h(V, V) + \rho^4 h([\partial_\rho, V], [\partial_\rho, V])$$
$$= -\frac{\rho^4}{2} \partial_\rho^2 h(V, V) + \frac{\rho^4}{4} h(SV, SV).$$

This implies that $K(Z, V) = O(\rho^4)$ and $K(V, W) = O(\rho^2)$. Note that if $h$ has curvature $+1$, then $K(V, W) = O(\rho^3)$.

For the last statement, taking $V, W$ such that $\nabla_Z W = \nabla_Z V = 0$, then
$$R(Z, W, V, W) = \langle \nabla_Z \nabla_W V - \nabla_{[Z, W]} V, W \rangle = Z \langle \nabla_W V, W \rangle - \rho^2 h(\nabla_{[\partial_\rho, W]} V, W)$$
$$= Z \langle [W, V], W \rangle - \rho^2 h(\nabla_{[\partial_\rho, W]} V, W)$$
$$= -\rho^2 h([W, V], W) - \rho^2 h(\nabla_{[\partial_\rho, W]} V, W) + O(\rho^3) = O(\rho^3)$$

concluding the proof. \qed
2.3. **Examples with no conjugate points.** Let us also give an example of a simply connected asymptotically conic manifold with no conjugate points that is not the Euclidian metric. Take $\mathbb{R}^n$ and use the radial coordinates $(r, \theta)$ so that the Euclidean metric is $g_0 = dr^2 + r^2 d\theta^2$ with $d\theta^2$ the metric of curvature $+1$ on the round sphere $S^{n-1}$. Take now a new metric on $\mathbb{R}^n$ given by in polar coordinates $$g = dr^2 + f(r)^2 d\theta^2, \quad f''(r) \geq 0, \quad f(r) = r \text{ in } [0, 1], \quad f(r) = ar \text{ in } [4, \infty)$$ where $a > 1$ and $f \in C^\infty([0, \infty))$. Then the metric is smooth on $\mathbb{R}^n$ and is asymptotically conic, as can be seen by setting $\rho = 1/r$ outside the ball $\{r \leq 1\}$. Moreover, a standard computation gives that the sectional curvature of $g$ is for any $V, W$ of unit length and tangent to the level sets $r = \text{const}$ $$K(\partial_r, V) = -\frac{f''(r)}{f(r)} \leq 0, \quad K(V, W) = \frac{1}{f^2(r)}(1 - (f'(r))^2) \leq 0.$$ In particular this metric has no conjugate points and is a Cartan-Hadamard manifold.

In fact, we can choose $f$ so that $f(r) = \sinh(r)$ in $r \in [2, 3]$ in which case the curvature is $-1$ in $r \in [2, 3]$ and by a result of Gulliver [Gu, Theorem 3], there is a new metric $g'$ on $\mathbb{R}^n$ such that $g' = g$ outside a compact subregion $\Omega$ of $\{r \in (2, 3)\}$, the curvature of $g'$ is positive in an open subset $U \subset \Omega$ and $g'$ has no conjugate points. This shows that there are asymptotically conic metrics on $\mathbb{R}^n$ that have positive curvature and no conjugate points, they are therefore non-trapping but not Cartan-Hadamard manifolds and do not enter in the class studied by Lehtonen-Railo-Salo [LRS].

In fact, for asymptotically conic metrics $g$ on $\mathbb{R}^2$, it can be shown that if the boundary metric $h(0)$ is such that $\partial M = S^1$ has length less than $2\pi$, then $g$ must have conjugate points, see [GMT].

2.4. **The manifold $\overline{S^* M}$, geodesic vector field, Liouville measure.** The unit cotangent bundle $S^* M = \{(x, \xi) \in T^* M; \|\xi\|_g = 1\}$ is non-compact. There is a natural compactification by taking the scattering unit cotangent bundle $\overline{sc S^* M} = \{(x, \xi) \in \overline{sc T^* M}; \|\xi\|_g = 1\}$ but the geodesic flow does not behave well near the boundary of that compactification. It is more convenient to work on a non-compact manifold with boundary on which the geodesic vector field is transversal to the boundary. This manifold is denoted by $\overline{S^* M}$, its interior is $S^* \mathbb{R}^n$ and its boundary consists of two connected components diffeomorphic to $T^* \partial \overline{M}$. The manifold $\overline{S^* M}$ is defined using the following procedure. The unit scattering cotangent bundle is $$\overline{sc S^* M} := \{(x, \xi) \in \overline{sc T^* M}; \|\xi\|_g = 1\}.$$ It is a compact smooth manifold with boundary. The local coordinates $(\rho, y)$ on $\overline{M}$ near $\partial \overline{M}$ induce local coordinates $(\rho, y, \zeta_0, \eta)$ on $\overline{sc T^* M}$ by writing each $\xi \in \overline{sc T^* M}$ under the
form

\[ \xi = \xi_0 \frac{d\rho}{\rho^2} + \sum_{i=1}^{n-1} \eta_i \frac{dy_i}{\rho}, \]  

(2.10)

and \( \xi \in \text{sc}S^*M \) means that \( \xi_0^2 + |\eta|^2_{h_{\rho}} = 1 \). A quick study of the integral curves of the geodesic vector field on \( S^*M \) shows that geodesics going to infinity have their \((\xi_0, \eta)\) components tending to \((\pm 1, 0)\), which suggest to use polar coordinates around the incoming/outgoing sets \( \{\rho = 0, \xi_0 = \pm 1, \eta = 0\} \) in \( \text{sc}S^*\overline{M} \). We thus consider the manifold

\[ [\text{sc}S^*\overline{M}; L_{\pm}] \]

obtained by performing a radial blow-up of \( \text{sc}S^*\overline{M} \) at the incoming/outgoing submanifolds \( L_{\pm} := \{\xi_0 = \mp 1, \rho = 0\} \subset \text{sc}S^*\overline{M} \). This is obtained by replacing \( L_{\pm} \) by the inward pointing spherical normal bundle \( NL_{\pm}/\mathbb{R}^+ \) of the submanifold \( L_{\pm} \) of \( \text{sc}S^*\overline{M} \). The blow-down map \( \beta : [\text{sc}S^*\overline{M}; L_{\pm}] \to \text{sc}S^*\overline{M} \) is the identity outside \( NL_{\pm}/\mathbb{R}^+ \) and is the natural projection \( NL_{\pm}/\mathbb{R}^+ \to L_{\pm} \) when restricted to \( NL_{\pm}/\mathbb{R}^+ \). A function on the blow-up is smooth if outside \( NL_{\pm}/\mathbb{R}^+ \) it is the pull back by \( \beta \) of a smooth function, and if near \( NL_{\pm}/\mathbb{R}^+ \) it can be written as a smooth function in polar coordinates around \( L_{\pm} \), i.e. it is a smooth function of the variables

\[ y, R := \sqrt{|\eta|^2 + \rho^2}, Y := \frac{\eta}{\sqrt{|\eta|^2 + \rho^2}}, Z := \frac{\rho}{\sqrt{|\eta|^2 + \rho^2}}. \]  

(2.11)

Note that in these coordinates, \( \beta \) can be described near \( NL_{\pm}/\mathbb{R}^+ \) as

\[ \beta \left( y, \sqrt{|\eta|^2 + \rho^2}, \frac{\eta}{\sqrt{|\eta|^2 + \rho^2}}, \frac{\rho}{\sqrt{|\eta|^2 + \rho^2}} \right) = (\rho, y, \xi_0 = \mp \sqrt{1 - \rho^2 |\eta|^2_{h_{\rho}}}, \eta). \]

We refer to the lecture notes [Me2, Chapter 5] for details about blow-ups. The manifold \([\text{sc}S^*\overline{M}; L_{\pm}]\) is a smooth manifold with codimension 2 corners. The boundary hypersurface of \([\text{sc}S^*\overline{M}; L_{\pm}]\) corresponding to the pull-back of \( \{\rho = 0, \eta \neq 0\} \) to \([\text{sc}S^*\overline{M}; L_{\pm}]\) by the blow-down map \( \beta : [\text{sc}S^*\overline{M}; L_{\pm}] \to \text{sc}S^*\overline{M} \) is denoted by \( \text{bf} \). In other words, \( \text{bf} = \text{cl}(\beta^{-1}(\{\rho = 0, \eta \neq 0\})) \subset [\text{sc}S^*\overline{M}; L_{\pm}] \). Then the boundary of \([\text{sc}S^*\overline{M}; L_{\pm}] \setminus \text{bf} \) is the union of two boundary faces obtained from the blow-up, they are the half-sphere bundles \( NL_{\pm}/\mathbb{R}^+ \) and are defined by \( R = 0 \) in the smooth coordinates (2.11) on \([\text{sc}S^*\overline{M}; L_{\pm}]\), with interior denoted by \( \partial_{\pm}S^*\overline{M} \). These two new boundary faces are isomorphic to \( T^*\partial \overline{M} \): using that in the region \( \pm \xi_0 > 0 \), one has \( L_{\mp} = \{\rho = 0, \eta = 0\} \), the projective coordinates

\[ \rho, y, \eta := \eta/\rho \]  

(2.12)

are smooth coordinates in a neighborhood of the interior of these new boundary faces. Moreover \( \partial_{\pm}S^*M = \{\rho = 0\} \) in this neighborhood (in the projective coordinates): then
\((y, \eta)\) restricted to \(\partial_\pm S^*M\) provide a diffeomorphism with \(T^*\partial M\). The variable \(\bar{\xi}_0\) is determined by \((\rho, y, \eta)\) near \(\partial_\pm S^*M\) by the equation

\[ \bar{\xi}_0^2 + \rho^2|\eta|_{h_\rho}^2 = 1. \]  

(2.13)

We then define the non-compact manifold with boundary

\[ \overline{S^*M} := \frac{\partial_\pm S^*M; L_\pm}{bf}. \]

The coordinates \((\rho, y, \xi_0, \eta)\) satisfying the condition (2.13) provide well-defined smooth coordinates on \(\overline{S^*M}\). We also note that \(\rho\) is a smooth boundary defining function of \(\partial_\pm S^*M\) in \(\overline{S^*M}\). More informally, the space \(\overline{S^*M}\) corresponds to \(S^*M\) with two copies of \(T^*\partial M\) glued at \(\rho = 0, \bar{\xi}_0 = \pm 1\) in a way that smooth functions on \(\overline{S^*M}\) correspond to smooth functions \(f \in C^\infty(S^*M)\) which can be written under the form \(f(x) = F(\rho, y, \bar{\xi}_0, \eta)\) near \(\rho = 0\) by using the coordinates (2.12), with \(F \in C^\infty([0, \epsilon) \times \partial M \times [-1, 1] \times T^*\partial M\).

Recall that two normal forms with functions \(\rho\) and \(\hat{\rho}\) are related by \(\hat{\rho} = \rho + \omega_0 \rho^2 + \mathcal{O}(\rho^3)\) for some \(\omega_0 \in C^\infty(\partial M)\) in Lemma 2.2. Thus the induced coordinates \((\hat{\rho}, \hat{y}, \hat{\xi}_0, \hat{\eta})\) are related to \((\rho, y, \xi_0, \eta)\) by

\[ \hat{y} = y + \mathcal{O}(\rho), \quad \hat{\xi}_0 = \xi_0 + \mathcal{O}(\rho), \quad \hat{\eta} = \eta + d\omega_0 + \mathcal{O}(\rho^2). \]

(2.14)

The canonical Liouville 1-form on \(T^*M\) is denoted \(\alpha\), in local coordinates near \(\partial M\) it is given by

\[ \alpha = \xi_0 d\rho + \eta dy, \]

and \(d\alpha\) is the canonical Liouville symplectic form of \(T^*M\). The geodesic vector field \(X\) is the Hamilton vector field of the energy functional

\[ H(x, \xi) = \frac{1}{2}|\xi|_{g_\rho} = \frac{1}{2}(\bar{\xi}_0^2 + \rho^2|\eta|_{h_\rho}^2) \]

As usual, we can restrict \(\alpha\) to \(S^*M\) as a contact form satisfying \(\alpha(X) = 1\) and \(i_X d\alpha = 0\), so that \(X\) is the Reeb vector field of \(\alpha\). A direct computation yields

\[ X = \rho^2 \bar{\xi}_0 \partial_\rho + \rho^2 \sum_{i,j} h^{ij} \eta_i \partial_{y_j} - (\rho^2|\eta|^2 + \frac{1}{2} \rho^2 \partial_\rho |\eta|_{h_\rho}^2) \partial_\xi_0 - \frac{1}{2} \rho^2 \sum_j \partial_{y_j}(|\eta|_{h_\rho}^2) \partial_{y_j}. \]

(2.15)

In particular,

\[ \overline{X} := \rho^{-2} X \]

(2.16)

extends smoothly down to \(\partial \overline{S^*M} = \{\rho = 0\}\) in \(\overline{S^*M}\) and is equal at \(\{\rho = 0\}\) to

\[ \overline{X}|_{\partial \overline{S^*M}} = \bar{\xi}_0 \partial_\rho + \sum_{i,j} h^{ij} \eta_i \partial_{y_j} - \frac{1}{2} \sum_{i,j} \partial_{y_j}(|\eta|_{h_0}^2) \partial_{y_j} = \bar{\xi}_0 \partial_\rho + Y, \]

(2.17)

where \(Y\) is the Hamilton field of \(\frac{1}{2} |\eta|_{h_0}^2\) on \((\partial M, h_0)\). We deduce
Lemma 2.4. The vector field $X$ extends smoothly to $\overline{S^*M}$ and can be factorized under the form $X = \rho^2 \overline{X}$ with $\overline{X} \in C^\infty(S^*M; TS^*\overline{M})$ a smooth vector field transverse to the boundary $\partial \overline{S^*M}$.

There is a natural volume form on $S^*M$, namely the Liouville measure given by

$$\mu = \alpha \wedge (d\alpha)^{n-1}.$$  

The boundary $\partial \overline{S^*M}$ identifies to two copies of $T^*\overline{M}$ and is thus a natural symplectic manifold with the canonical Liouville form $\sum_j d\eta_j \wedge dy_j$.

Lemma 2.5. The forms $\rho^2 \alpha$ and $d\alpha$ extend smoothly to $\overline{S^*M}$ and $i_\partial^* d\alpha$ restricts to the canonical Liouville symplectic form of $\partial_\pm S^*M \simeq T^*\overline{M}$ if $i_\partial : \partial \overline{S^*M} \to S^*M$ is the inclusion map. The volume form $\mu$ is such that $\rho^2 \mu$ extends smoothly on $\overline{S^*M}$ and $i_\partial^* i_X \mu$ is the canonical Liouville symplectic volume form on $\partial \overline{S^*M}$.

Proof. We can write $\alpha = \xi_0 d\rho + \sum_j \eta_j dy_j$ so $\rho^2 \alpha$ extends smoothly to $\overline{S^*M}$. Now $d\alpha = \rho^{-2} d\xi_0 \wedge d\rho + \sum_j d\eta_j \wedge dy_j$ and differentiating $1 = \xi_0^2 + \rho^2 |\eta|[^2\rho]$ gives $d\xi_0 \wedge d\rho = -\rho |\eta|[^2\rho] + O(\rho^2)$ on $T(S^*M)$ thus $d\alpha$ extends smoothly to $\partial \overline{S^*M}$ and $i_\partial^* d\alpha$ restricts to the canonical Liouville symplectic form of $\partial_\pm S^*M$. From the discussion above, $\rho^2 \mu$ extends smoothly, and $i_X \mu = (d\alpha)^{n-1}$, thus pulls-back to the symplectic measure on $T^*(\partial M) \simeq \partial_\pm S^*M$. \hfill $\square$

Since also $\rho^2 \mu = \xi_0 d\rho \wedge (\sum_j \eta_j dy_j)^{n-1} + O(\rho)$, we see that the orientation induced by $\rho^2 \mu$ and $(\sum_j \eta_j dy_j)^{n-1}$ agree (resp. are opposite) on $\partial_+ S^*M$ (resp. on $\partial_- S^*M$).

2.5. Connection map and Sasaki metric on $S^*M$. The manifold $T^*M$ has a natural metric structure called the Sasaki metric defined so that the horizontal space, defined through the Levi-Civita connection, and the vertical space are orthogonal; we refer to [Pa, Chapter 1.3] for details. The Sasaki metric will be denoted by $G$. The projection on the base $\pi : T^*M \to M$ allows to define the vertical bundle $V := \ker d\pi \subset T(T^*M)$ and there is a map called connection map $K : T(T^*M) \to T^*M$ and $H := \ker K$ is called the horizontal space if $z \in T^*M$. The maps

$$K : V \to T^*M, \quad d\pi : H \to TM$$

are isomorphisms and we will call horizontal (resp. vertical) lift $\xi^h \in H_{(x,\xi)}$ (resp. $\xi^v \in V_x$) of a point $z = (x,\xi) \in T^*M$ the element $\xi^h \in H_z$ (resp. $\xi^v \in V_z$) so that $K(\xi^v) = \xi$. We have a similar decomposition $T(S^*M) = H \oplus V$ over $S^*M$ but $V$ becomes $(n-1)$-dimensional. We let $Z \to S^*M$ be the bundle with fibers $Z_{(x,\xi)} = \{v \in T_xM : \xi(v) = 0\}$; then the maps $d\pi|_H : H \cap \ker \alpha \to Z$ and $K|_V : V \to Z$ are isomorphisms (see [PSU]). We will call horizontal lift $\xi^h \in H_{(x,\xi)}$ of a point $(x,\xi) \in S^*M$ the element $\xi^h \in H_{(x,\xi)}$ so that $d\pi(\xi^h) = \xi$. The vector field $X$ also acts on sections of $Z$ by using parallel transport along
geodesics of \( g \): for \( v \in \Gamma(\mathcal{Z}) \), \( v(\varphi_t(x, \xi)) \) is a vector field along the geodesic \( \pi(\varphi_t(x, \xi)) \), and we define

\[
Xv(x, \xi) := \nabla_{\partial_t} v(\varphi_t(x, \xi))|_{t=0}
\]

where \( \nabla \) is the Levi-Civita connection pulled-back to \( \mathcal{Z} \) over \( S^*M \).

If an element \( z = (x, \xi) \in T^*M \) is expressed as \( x = (\rho, y) \) and \( \xi = \xi_0 \frac{d\rho}{\rho} + \sum_j \eta_j dy_j = \xi_0 d\rho + \sum_j \eta_j dy_j \), and if we use the coordinate on \( T^*M \) given by \( (\rho, y, \xi_0, \eta) \) and the coordinate frame for \( T(T^*M) \) is \( \{\partial_\rho, \partial_y, \partial_{\xi_0}, \partial_\eta\} \), the vertical lift of \( \xi \) is given by

\[
\xi^v = \xi_0 \partial_\rho + \sum_j \eta_j \partial_{\eta_j} \in \mathcal{V}.
\]

Using the coordinates \( (\rho, y, \xi_0, \eta) \) we have \( \xi^v = \xi_0 \partial_{\xi_0} + \sum_j \eta_j \partial_{\eta_j} \). The horizontal lift of a vector \( Z = Z_0 \partial_\rho + \sum_{j=1}^{n-1} Z_j \partial_{y_j} \in T_x M \) to a vector over the element \( z = (x, \xi) \in T^*M \) is given by

\[
Z^h = Z + \sum_{i,j,k} \xi_i \Gamma^{k}_{ij} Z_j \partial_{\xi_i}, \in \mathcal{H}
\]

where \( \xi_i = \eta_i \) for \( i \geq 1 \) and \( \Gamma^{k}_{ij} = dx_k(\nabla_{\partial_{x_i}} \partial_{x_j}) \). In particular, some computations give

\[
\partial_\rho = \partial_\rho - 2\rho^{-1} \xi_0 \partial_\xi_0 + \rho^{-1} L_x^0(\eta, \partial_\eta),
\]

\[
\partial^h_{y_j} = \partial_{y_j} + \rho^{-1} \xi_0 J_x^j(\partial_\eta) + \rho K_x^j(\eta) \partial_{\xi_0} + L^j(\eta, \partial_\eta)
\]

where \( L_x^j(a, b) \) are smooth in \( x = (\rho, y) \) (down to \( \rho = 0 \)) and bilinear in \( (a, b) \), \( J_x^j(b), K_x^j(a) \) are smooth in \( (\rho, y) \) (down to \( \rho = 0 \)) and linear in \( a \) and \( b \). The Sasaki metric satisfies

\[
G(\xi^v, \xi^v) = g^{-1}(\xi, \xi) = g^h(\xi_0, \xi_0) + \rho^2 |\eta|^2_{h_\rho} = \xi_0^2 + \rho^2 |\eta|^2_{h_\rho},
\]

\[
G(Z^h, Z^h) = g(Z, Z) = \rho^{-4} Z_0^2 + \rho^{-2} \sum_{j=1}^{n-1} Z_j \partial_{y_j}|_{h_\rho}^2 = \rho^{-4} Z_0^2 + \rho^{-2} \sum_{ij=1}^{n-1} h_{ij} Z_j Z_i,
\]

\[
G(Z^h, \xi^v) = 0,
\]

in particular \( \mathcal{H} \oplus \mathcal{V} \) is an orthogonal decomposition for the metric \( G \). Using the local frame \( \{\partial^h_\rho, \partial^h_{y_j}, \partial_{\xi_0}, \partial_\eta\} \) of \( T(T^*M) \), the Sasaki metric can be expressed as

\[
G = \rho^{-4} \delta^h_\rho + \rho^{-2} \sum_{jk} h_{kj} \delta_{y_j} \delta_{y_k} + d\xi_0^2 + \rho^2 \sum_{jk} h^{jk} d\eta_j d\eta_k
\]

where \( \{\delta_\rho, \delta_{y_j}, d\xi_0, d\eta_j\} \) denotes the dual frame (1-forms on \( T^*M \)).

To compute the gradient \( \nabla^G u \) of \( u : S^*M \rightarrow \mathbb{R} \) with respect to \( G \), it suffices to compute \( \tilde{\nabla}^G (u \circ p)|_{S^*M} \) where \( p : T^*M \rightarrow S^*M \) is defined by \( p(x, \xi) = (x, \xi/|\xi|) \) and \( \tilde{\nabla}^G \) denotes the gradient with respect to \( G \) in \( T^*M \). We get, writing \( \tilde{u} = u \circ p \),

\[
\tilde{\nabla}^G \tilde{u} = \rho^4 \partial^h_\rho \tilde{u} \partial^h_\rho + \rho^2 \sum_{jk} h^{jk} \delta^h_{y_j} \tilde{u} \delta^h_{y_k} + \partial_{\xi_0} \tilde{u} \partial_{\xi_0} + \rho^{-2} \sum_{k,j} h_{jk} \partial_{\eta_k} \tilde{u} \partial_{\eta_j}.
\]
with the condition $\bar{\xi}_0 \partial_{\bar{\xi}_0} \bar{u} + \sum_j \eta_j \partial_{\eta_j} \bar{u} = 0$. We deduce that

$$\|\nabla^G u\|_G^2 = \rho^4 (\partial^h \rho u)^2 + \rho^2 \sum_{k,j} h_{kj} \partial^h \eta_j u \partial^h \eta_k u + (\partial_{\xi_0} u)^2 + \rho^{-2} \sum_{k,j} h_{kj} \partial_{\eta_j} u \partial_{\eta_k} u \quad (2.20)$$

We will also write $\nabla^h u$ and $\nabla^v u$ for the horizontal and vertical gradient, which are the orthogonal projections of $\nabla^G u$ onto $\mathcal{H}$ and $\mathcal{V}$ with respect to $G$.

2.6. Lift of tensors. Denote by $C^\infty(M; S^m(\pi^* T^* M))$ the smooth symmetric scattering tensors, i.e. smooth sections of the bundle of symmetric tensors in $\pi^* T^* M$. There exists a natural lift

$$\pi^*_m : C^\infty(M; S^m T^* M) \to C^\infty(S^* M), \quad \pi^*_m f(x,\xi) := f(x)(\otimes^m \xi^\sharp) \quad (2.21)$$

where $\xi^\sharp \in T_x M$ is the dual to $\xi \in T^*_x M$ with respect to $g$. Let us consider the action of $\pi^*_m$ on $C^\infty(M; S^m(\pi^* T^* M))$. When viewed as a function on the smooth compact manifold with boundary $\pi^* M := \{(x,\xi) \in \pi^* T^* M; |\xi|_g = 1\}$, it is direct to see that $\pi^*_m f \in C^\infty(\pi^* M)$. Now we also need to view $\pi^*_m$ as a function on $\pi^* M$. This can be computed explicitly in the coordinate given by $\xi = \bar{\xi}_0 d\rho + \sum_j \eta_j dy^j$ where $\bar{\xi}_0^2 + \rho^2 |\eta|^2_h = 1$; since $\xi = \bar{\xi}_0 d\rho + \sum_j \rho \eta_j dy^j$ and $\xi^\sharp = \bar{\xi}_0 \rho^2 \partial_\rho + \rho^2 \sum_i h_{ij} \eta_i \partial_{y^j}$, we see that if $f \in C^\infty(M; S^m(\pi^* T^* M))$ then

$$\pi^*_m f \in C^\infty(S^* M), \quad \pi^*_m f \in \rho^{-m-\ell} C^\infty(\pi^* M)$$

where $i_{\rho^2 \partial_\rho}$ denotes interior product. On smooth sections of $S^m(\pi^* T^* M) \cap \ker(i_{\rho^2 \partial_\rho})^{\ell+1}$ we get

$$|\pi^*_m f(x,\xi)| \leq C \|f\|_{C^0(\pi^* M; S^m(\pi^* T^* M))} \rho^{m-\ell} |\eta|_h^{m-\ell} \leq C \|f\|_{C^0(\pi^* M; S^m(\pi^* T^* M))} \quad (2.22)$$

for some uniform $C > 0$, and so the same estimate thus holds on $C^0(\pi^* M; S^m(\pi^* T^* M))$. Similarly, by direct computation using (2.20) and (2.18),

$$\|\nabla^G \pi^*_m f\|_G^2 \leq C \|f\|_{C^1(\pi^* M; S^m(\pi^* T^* M))}^2 \rho |\eta|_h^{2m-2-2\ell} \leq C \|f\|_{C^1(\pi^* M; S^m(\pi^* T^* M))}^2 \quad (2.23)$$

on smooth sections of $S^m(\pi^* T^* M) \cap \ker(i_{\rho^2 \partial_\rho})^{\ell+1}$ for some uniform $C > 0$.

It is well known that on a manifold with smooth metric, trace-free elements of $S^m T^* M$ lifts to fiberwise homogeneous harmonic polynomials on $T^* M$ with respect to the vertical Laplacian $\Delta^v := (\nabla^v)^* \nabla^v$ which then restricts to spherical harmonics on $S^* M$ (see [Sh]) which we denote by $\Omega_m$.

2.7. Integral curves of $X$ pointing towards infinity. Recall that the rescaled vector field $\bar{X}$ was defined as $\rho^{-2} X$ where $X$ is the geodesic vector field on $S^* M$ given by the expression (near $\rho = 0$) (2.15). We first prove a lemma about the flow $\varphi_t$ of $X$, which in turn is a rescaling of the flow of $\bar{X}$. In what follows, we let

$$W^\pm := \{z = (\rho, \eta, \bar{\xi}_0, \eta) \in \pi^* M; \rho \leq \epsilon, \pm \bar{\xi}_0 \leq 0\}. $$
Lemma 2.6. For $\epsilon > 0$ small enough, there is $C > 0$ such that for all $z \in W^+_\epsilon$ and $t \geq 0$

$$\frac{\rho(z)}{1 + \rho(z)t} \leq \rho(\phi_t(z)) \leq \frac{C\rho(z)}{1 + \rho(z)t}, \quad |\eta(z)| e^{-C\rho(z)} \leq |\eta(\phi_t(z))| \leq |\eta(z)| e^{C\rho(z)}.$$  

Furthermore,

$$0 \leq 1 - \xi_0(\phi_t(z))^2 \leq C(1 - \xi_0(z)^2) \left(\frac{1}{1 + \rho(z)t}\right)^2 e^{C\rho(z)}.$$  

The same holds for the reverse time flow $\phi_{-t}(z)$ for $z \in W^-_\epsilon$.

Proof. W e rewrite $\phi_t(z) = (\rho(t), y(t), \xi_0(t), \eta(t))$. We first note that $\partial_t(\frac{1}{\rho(t)}) \leq -\xi_0 \leq 1$, which gives that

$$\rho(t) \geq \frac{\rho(0)}{1 + \rho(0)t}. \quad (2.25)$$

Furthermore we get from

$$\dot{\rho} = \rho^2 \xi_0, \quad \dot{\xi}_0 = -\rho^3(|\eta|^2 + \frac{\rho}{2} \partial_\rho h(\eta, \eta))$$

that for $\rho(0) \leq \epsilon$ small enough and $\xi_0(0) \leq 0$, one has that $\dot{\rho} \leq 0$ and $\xi_0(t) \leq 0$ for all $t > 0$. We now establish the upper bound on $|\eta(t)|$. We also get from the flow equation

$$-C\rho^2|\eta|^2 \leq \partial_t|\eta|^2 = \dot{\rho}(t)\partial_\rho h(\eta, \eta) \leq C\rho^2|\eta|^2$$

(here notice that the $\dot{y}\partial_y|\eta|^2$ and the $2h(\dot{\eta}, \eta)$ terms cancel out each other). By Grönwall’s inequality one gets

$$|\eta(0)| e^{-C\int_0^t \rho^2(s)ds} \leq |\eta(t)| \leq |\eta(0)| e^{C\int_0^t \rho^2(s)ds}. \quad (2.26)$$

We see here that it is natural to consider an estimate for the $L^2$ norm of $\rho(t)$. To this end we first observe that since $\rho(t)$ is decreasing and $\rho(0) \leq \epsilon$, thus for sufficiently small $\epsilon > 0$

$$\rho(t)^2|\eta(t)|^2 + \rho(t)^3\partial_\rho h(\eta(t), \eta(t)) \geq \frac{\rho(t)^2}{2} |\eta(t)|^2 = \frac{1}{2} (1 - |\xi_0(t)|^2).$$

Combining this with (2.15) and (2.25), we have

$$\xi_0 \leq \frac{-1}{2} \frac{\rho(0)}{1 + \rho(0)t} (1 - |\xi_0(t)|^2) = \frac{-1}{2} \frac{\rho(0)}{1 + \rho(0)t} (1 + \xi_0)(1 - \xi_0) \quad (2.27)$$

$$\leq \frac{-1}{2} \frac{\rho(0)}{1 + \rho(0)t} (1 + \xi_0).$$

The last inequality uses the fact that $\xi_0(t) \leq 0$ for all $t > 0$. Applying Grönwall again we see that $1 + \xi_0(t) \leq \frac{(1 + \xi_0(0))}{\sqrt{1 + \rho(0)t}}$ or

$$-1 \leq \xi_0(t) \leq -1 + \frac{1 + \xi_0(0)}{\sqrt{1 + \rho(0)t}}. \quad (2.28)$$
Inserting this into $\partial_t(\frac{1}{\rho}) = -\xi_0$ we see that

$$\frac{1}{\rho(t)} \geq \frac{1}{\rho(0)} + t - 2 \frac{1 + \xi_0(0)}{\rho(0)} \sqrt{1 + \rho(0)t} + 2 \frac{1 + \xi_0(0)}{\rho(0)}$$

or

$$\rho(t) \leq \frac{\rho(0)}{1 + \rho(0)t(1 - 2 \frac{(1 + \xi_0(0))(\sqrt{1 + \rho(0)t} - 1)}{\rho(0)t})}. \tag{2.29}$$

This gives the integrability estimate

$$\int_0^\infty \rho(t)^2 dt \leq \rho(0) \int_0^\infty \frac{1}{1 + s(1 - 2 \frac{(1 + \xi_0(0))(\sqrt{1 + s} - 1)}{s})^2} ds.$$ 

Using that $\xi_0(0) \leq 0$ and $0 \leq 1 - 2 \frac{\sqrt{1 + s - 1}}{s} \leq 1$ is strictly increasing we see that $\int_0^\infty \rho(t)^2 dt \leq C \rho(0)$ where $C$ is independent of the choice of $z \in W_c^+$. Going back to (2.26) we deduce that

$$|\eta(t)| \leq |\eta(0)| e^{C \rho(0)} \quad \forall t > 0.$$ 

Furthermore if we let $c_0 > 0$ be the number such that $2 \frac{(\sqrt{1 + s - 1})}{s} < 1/2$ for all $s > c_0$ we have from (2.29) that if $\rho(0)t \geq c_0$ then $\rho(t) \leq \frac{2\rho(0)}{1 + \rho(0)t}$. If $\rho(0)t \leq c_0$ then

$$(1 + c_0) \frac{\rho(0)}{1 + \rho(0)t} \geq \rho(0) \geq \rho(t)$$

since $\dot{\rho}(t) \leq 0$. This estimate along with (2.25) gives the estimates for $\rho(t)$. The estimate for $1 - \xi_0(t)^2 = \rho(t)^2 |\eta(t)|^2$ follows immediately.

2.8. Rescaled Dynamic. If $\phi_t$ is the flow generated by $X$, one can see that for $z_0 \in S^*M$, $\varphi_{\tau}(z_0) := \phi_{t(\tau)}(z_0)$ is the flow for the rescaled vector field $\overline{X}$ by setting

$$\tau(t) := \int_0^t \rho^2(\phi_s(z_0))ds, \quad t(\tau) := \int_0^\tau \rho^{-2}(\overline{\phi}_s(z_0))ds. \tag{2.30}$$

In what follows, for a fixed $z_0 \in S^*M$ we will denote by $z(\tau) := \varphi_{\tau}(z_0)$ whereas $z(t) := \phi_t(z_0)$ will denote the unscaled dynamic with $\tau$ and $t$ related by the change of variable given by (2.30). We define the rescaled arrival times for $z \in S^*M$

$$\tau_+(z) = \sup\{\tau \geq 0; \forall s < \tau, \varphi_s(z) \notin \partial_+ S^*M\},$$

$$\tau_-(z) = \inf\{\tau \leq 0; \forall s > \tau, \varphi_s(z) \notin \partial_- S^*M\}. \tag{2.31}$$

We notice that this quantity depends on the choice of $\rho$. 

2.9. Geodesics arbitrarily close to infinity. We first discuss the exact conic case near infinity, i.e. when \( h = h_0 \) is constant in \( \rho \). In this case the vector field \( \overline{X} \) is given by the formula (2.16). We then obtain
\[
\dot{\rho}(\tau) = -\rho|\eta|^2_{h_0},
\]
and a direct computation shows that \( |\eta(\tau)|_{h_0} = |\eta_0|_{h_0} \) is constant with \( \dot{\rho}(0) = \overline{\xi}_0(0) = 1 \), so we obtain
\[
\rho(\tau) = \frac{1}{|\eta_0|_{h_0}} \sin(\tau|\eta_0|_{h_0}).
\]
We see that for large \( |\eta_0| \), the geodesic stays close to infinity. We would like to assert that for large initial speed \( |\eta_0| \) the rescaled dynamics approximate the one given by a warped product metric. To this end we first prove a lemma about the rescaled dynamic for short time.

**Lemma 2.7.** Let \( \epsilon > 0 \) be sufficiently small and suppose that \( \overline{\varphi}_c(z_0) \) is a flow line contained in \( W^+_\epsilon \cup W^-_\epsilon \) for \( z_0 \in \partial_\ast S^*M \). There exist positive constants \( c \) and \( C \) such that \( c\tau \leq \rho(\tau) \leq C\tau \) for all \( \tau \) such that \( \overline{\varphi}_c(z_0) \) is contained in \( W^-_\epsilon \) and \( c(\tau_c(z_0) - \tau) \leq \rho(\tau) \leq C(\tau_c(z_0) - \tau) \) for all \( \tau \) such that \( \overline{\varphi}_c(z_0) \) is contained in \( W^+_\epsilon \).

**Proof.** Let \( \tau_0 > 0 \) be such that \( z_1 := \overline{\varphi}_{-\tau}(z_0) \in W^-_\epsilon \). Consider the unscaled backward flow \( \varphi_{-\tau}(z_1) \) along the same trajectory. By (2.30) one has that \( \tau_0 = \int_0^\infty \rho^2(\varphi_{-\tau}(z_1))dt \), which by Lemma 2.6 can be estimated above and below by \( c\rho(z_1) \leq \tau_0 \leq C\rho(z_1) \) and this completes the proof for the \( W^+_\epsilon \) case. The \( W^-_\epsilon \) case can be dealt with the same way. \( \square \)

**Lemma 2.8.** There is \( C > 0 \) such that each \( z_0 = (y_0, \eta_0) \in \partial_\ast S^*M \), if \( |\eta_0| \) is sufficiently large we have \( \rho(\tau) = \tau + O(\tau^3) \) for all \( \tau \in [0, \tau_+(y_0, \eta_0)] \) where \( |\eta_0|\tau_+(y_0, \eta_0) = \pi + O(|\eta_0|^{-1}) \), and
\[
\sup_{\tau \in [0, \tau_+(y_0, \eta_0)]} |\rho(\tau)| \leq C|\eta_0|, \quad \sup_{\tau \in [0, \tau_+(y_0, \eta_0)]} \frac{|\eta(\tau)|}{|\eta_0|} - 1 \leq \frac{C}{|\eta_0|}.
\]

**Proof.** Let \( z_0 := (y_0, \eta_0) \in \partial_\ast S^*M \). Applying Lemma 2.6 backwards one sees that \( |\eta(\tau)| \leq C|\eta_0| \) for all \( \tau > 0 \) such that \( \overline{\varphi}_c(z_0) \) contains \( W^-_\epsilon \). Since \( \overline{\xi}^2_0 + \rho^2|\eta|^2_h = 1 \) along the trajectory, taking \( |\eta_0| < \epsilon/C \), we see that the first time \( \tau_{\text{max}} \) so that \( \overline{\varphi}_c(z_0) \notin W^-_\epsilon \) for \( \tau > \tau_{\text{max}} \) needs to satisfy \( \overline{\xi}_0(\tau_{\text{max}}) = 0 \) and \( \varphi_{\text{max}}(z_0) \in W^+_\epsilon \). By Lemma 2.6 (and its proof), \( \overline{\varphi}_c(z_0) \) is contained in \( W^+_\epsilon \) for all \( \tau > \tau_{\text{max}} \) and \( \rho(\tau) = O(1) \leq C|\eta_0| \) for some \( C \). We also see that for \( |\eta_0| \) sufficiently large, \( c|\eta_0|^{-1} \leq \rho_{\text{max}} \leq C|\eta_0|^{-1} \) where \( \rho_{\text{max}} = \rho(\tau_{\text{max}}) \) is the maximum value of \( \rho(\tau) \) for \( \tau \in [0, \tau_+(y_0, \eta_0)] \) and \( c, C \) independent of \( \eta_0 \). By Lemma 2.7 one has that
\[
c''|\eta_0|^{-1} \leq \rho_{\text{max}} \leq C'\rho_{\text{max}} \leq C''|\eta_0|^{-1}.
\]
for some positive \( c', c'', C', C'' \) independent of \( \eta_0 \). The same argument yields that
\[
\frac{c}{|\eta_0|} \leq \tau_+(y_0, \eta_0) \leq \frac{C}{|\eta_0|}.
\]
for $|\eta_0|$ sufficiently large. We introduce polar coordinates on $S^*M$ given by

$$\eta_0 = \cos \theta, \quad \rho|\eta|_h = \sin \theta. \quad (2.34)$$

Differentiating the cosine term we have (using the expression of $\nabla$ from (2.15))

$$-\left(\rho|\eta|_h^2 + \frac{\rho^2}{2} \partial_{\rho h}(\eta, \eta)\right) = \frac{d}{d\tau} \eta_0 = -\dot{\theta} \sin \theta = -\dot{\theta} \rho|\eta|$$

which becomes

$$\dot{\theta}(\tau) = |\eta|_h + \frac{\rho}{2} \partial_{\rho h}(\eta, \eta) = |\eta_0| + \int_0^\tau \partial_s|\eta(|\varphi_s(z_0)|)|_h ds + \frac{\rho}{2} \partial_{\rho h}(\eta, \eta).$$

Again from (2.15) and $|\eta|_h^2 = (1 - \eta_0^2)\rho^{-2}$, it is direct to check that $\partial_s|\eta(|\varphi_s(z_0)|)|_h = \frac{1}{2} \xi_0 \partial_{\rho h}(\eta, \eta)$. Inserting this expression into the integral we obtain the double-sided bound on $\dot{\theta}$:

$$|\eta_0| - C|\eta_0|\tau^+ \leq \dot{\theta} \leq |\eta_0| + C|\eta_0|\tau^+$$

where we have simplified the notation $\tau^+ := \tau^+(y_0, \eta_0)$. Going back to (2.34) one sees that $z(\tau) \in \partial S^*M$ precisely when $\xi_0 = -1$ or $\theta = \pi$. Integrating the two-sided differential inequality and evaluating at $\tau = \tau^+$ we arrive at

$$|\eta_0|\tau^+ - C|\eta_0|(\tau^+)^2 \leq \theta(\tau^+) = \pi \leq |\eta_0|\tau^+ + C|\eta_0|(\tau^+)^2.$$

Dividing through by $|\eta_0|$ and applying (2.33) we see that

$$\tau^+ \in \left[\frac{\pi}{|\eta_0|} - \frac{C}{|\eta_0|^2}, \frac{\pi}{|\eta_0|} + \frac{C}{|\eta_0|^2}\right].$$

The estimate for $\left|\frac{|\eta_0(\tau)|}{|\eta_0|} - 1\right|$ follows from these estimates and Lemma 2.6.

We would like to get more detailed description of the asymptotics of the dynamics as $|\eta_0| \to \infty$. To this end, fix $(y_0, \eta_0) \in \partial S^*M$ with $|\eta_0|_h = 1$ and consider the rescaled dynamics $\varphi_s(z_0) := (\rho(\tau), y(\tau), \xi_0(\tau), \eta(\tau))$ with initial condition $z_0 := (y_0, \epsilon^{-1}\eta_0) \in \partial S^*M$. It is convenient to rescale again and introduce variables

$$\tilde{\rho}(s) := \epsilon^{-1}\rho(\epsilon s), \quad \tilde{\xi}_0(s) := \xi_0(\epsilon s), \quad \tilde{\eta}(s) := \epsilon \eta(\epsilon s), \quad \tilde{y}(s) = y(\epsilon s).$$

In coordinates, we get equations

$$\dot{\tilde{\rho}} = \tilde{\xi}_0, \quad \dot{\tilde{\xi}}_0 = -\tilde{\rho}(|\tilde{\eta}|^2 + \frac{\tilde{\rho}}{2} (\partial_{\rho h}(\tilde{\eta}, \tilde{\eta})),
\dot{\tilde{\eta}}_j = \sum_k h_{\epsilon \rho}^{jk} \tilde{\eta}_k, \quad \dot{\tilde{\eta}}_j = -\frac{1}{2} \partial_{y_j}|\tilde{\eta}|_{\epsilon \rho}^2. \quad (2.35)$$
Note that using these relations we can derive a convenient representation for $|\eta|^2$. Indeed, differentiating the relation $|\eta|^2 = \frac{1 - \tilde{\xi}_0^2}{\rho^2}$ we have using (2.35) that $\partial_\rho|\eta|^2 = \tilde{\xi}_0 \partial_\rho h(\eta, \tilde{\eta})$. Therefore,

$$|\eta(s)|^2 = 1 + \epsilon \int_0^s \tilde{\xi}_0 \partial_\rho h(\eta, \tilde{\eta}) dt$$

(2.36)

is a way to keep track of the evolution of $|\eta|^2$ (and the same holds for $|\tilde{\eta}|$). This expression also allows us to obtain a convenient representation for $\epsilon^{-1} \tau_g^+(y, \epsilon^{-1}\eta_0)$. First, we consider the variable $\theta(\cdot) : [0, \epsilon^{-1} \tau_g^+(y, \epsilon^{-1}\eta_0)] \to [0, \pi]$ defined by

$$\tilde{\xi}_0(s) = \cos \theta(s), \quad \tilde{\rho}(s)|\eta(s)|_h = \sin \theta(s).$$

Observe that $\theta(0) = 0$ and $\theta(\epsilon^{-1} \tau_g^+(y, \epsilon^{-1}\eta_0)) = \pi$. Differentiating the second equation and using (2.35) we obtain the integral relation

$$\theta(s) = \int_0^s |\eta| + \frac{\epsilon \tilde{\rho}}{2} \partial_\rho h(\eta, \tilde{\eta}) dt = s + \epsilon \int_0^s \left( \int_0^t \tilde{\xi}_0 \partial_\rho h(\eta, \tilde{\eta}) du + \frac{\tilde{\rho}}{2} \partial_\rho h(\eta, \tilde{\eta}) \right) dt.$$

where we used (2.36) for the second identity. Setting $s = \epsilon^{-1} \tau_g^+(y, \epsilon^{-1}\eta_0)$ we obtain

$$\pi = \epsilon^{-1} \tau_g^+(y, \epsilon^{-1}\eta_0) + \epsilon \int_0^{\epsilon^{-1} \tau_g^+(y, \epsilon^{-1}\eta_0)} \left( \int_0^t \tilde{\xi}_0 \partial_\rho h(\eta, \tilde{\eta}) du + \frac{\tilde{\rho}}{2} \partial_\rho h(\eta, \tilde{\eta}) \right) ds$$

(2.37)

which implies

$$\tau_g^+(y, \epsilon^{-1}\eta_0) = \epsilon \pi + O(\epsilon^2).$$

(2.38)

We then obtain a description of the integral curves as $\epsilon \to 0$:

**Lemma 2.9.** Fix $(y_0, \eta_0) \in \partial_- S^* M$ with $|\eta_0| h_0 = 1$ and set $\tau_\epsilon = \epsilon^{-1} \tau_g^+(y_0, \epsilon^{-1}\eta_0)$. For $\epsilon > 0$ sufficiently small, the solutions to (2.35) for $s \in [0, \tau_\epsilon]$ are of the form

$$\tilde{\rho}(s) = \frac{\tau_\epsilon}{\epsilon \pi} \sin \left( \frac{\epsilon \pi s}{\tau_\epsilon} \right) + r_\epsilon(s), \quad \tilde{\xi}_0(s) = \cos \left( \frac{\epsilon \pi s}{\tau_\epsilon} \right) + q_\epsilon(s)$$

where $0 = r_\epsilon(0) = \dot{r}_\epsilon(0) = \dot{\dot{r}}_\epsilon(0) = r_\epsilon(\epsilon^{-1} \tau_\epsilon) = \dot{r}_\epsilon(\epsilon^{-1} \tau_\epsilon) = \dot{\dot{r}}_\epsilon(\epsilon^{-1} \tau_\epsilon)$ and

$$\sup_{s \in (0, \epsilon^{-1} \tau_\epsilon)} |q_\epsilon(s)| \leq C \epsilon, \quad \sup_{s \in (0, \epsilon^{-1} \tau_\epsilon)} \frac{|r_\epsilon(s)|}{\sin^3 \left( \frac{\epsilon \pi s}{\tau_\epsilon} \right)} \leq C \epsilon.$$

(2.39)

**Proof.** We define $r_\epsilon$ and $q_\epsilon$ by the expression in the Lemma, with boundary condition $r_\epsilon(0) = r_\epsilon(\epsilon^{-1} \tau_\epsilon) = q_\epsilon(0) = q_\epsilon(\epsilon^{-1} \tau_\epsilon) = 0$. Using (2.35) and (2.36), we see that $r_\epsilon$ and $q_\epsilon$ must solve

$$\dot{r}_\epsilon = q_\epsilon, \quad \dot{q}_\epsilon = \frac{c_\epsilon}{\epsilon} \int_0^s \tilde{\xi}_0 \partial_\rho h(\eta, \tilde{\eta}) dt - \frac{\rho^2}{2} \partial_\rho h(\eta, \tilde{\eta}) + \left( \frac{\pi}{\tau_\epsilon} - \frac{\epsilon}{\pi} \right) \sin \left( \frac{\pi \tau_\epsilon}{\tau_\epsilon} \right) - r_\epsilon.$$

Note that due to (2.37) we have that $(\frac{\pi}{\tau_\epsilon} - \frac{\epsilon}{\pi}) = c_\epsilon \epsilon$ where

$$c_\epsilon = \frac{\pi + \epsilon^{-1} \tau_\epsilon}{\pi \epsilon^{-1} \tau_\epsilon} \int_0^{\epsilon^{-1} \tau_\epsilon} \left( \int_0^t \tilde{\xi}_0 \partial_\rho h(\eta, \tilde{\eta}) du + \frac{\tilde{\rho}}{2} \partial_\rho h(\eta, \tilde{\eta}) \right) dt = O(1)$$. 


as $\epsilon \to 0$. So the system can be simplified to

$$
\dot{r}_\epsilon = q_\epsilon, \quad \dot{q}_\epsilon = -r_\epsilon + \epsilon k_\epsilon
$$

with $k_\epsilon$ given by

$$
k_\epsilon(s) = -\tilde{\rho} \int_0^s \tilde{\xi}_0 \partial_\rho h(\tilde{\eta}, \tilde{\eta}) dt - \frac{\partial^2}{2} \partial_\rho h(\tilde{\eta}, \tilde{\eta}) + c_\epsilon \sin\left(\frac{\pi s \epsilon}{\tau_\epsilon}\right).
$$

Solving the ODE systems with vanishing initial conditions one then gets the following representation for $r_\epsilon(s)$:

$$
r_\epsilon(s) = \epsilon \left( - \cos(s) \int_0^s \sin(t) k_\epsilon(t) dt + \sin(s) \int_0^s k_\epsilon(t) \cos(t) dt \right).
$$

Similarly solving the ODE with vanishing end conditions gives that

$$
r_\epsilon(s) = \epsilon \left( - \cos(s) \int_s^{s+1} \sin(t) k_\epsilon(t) dt + \sin(s) \int_s^{s+1} k_\epsilon(t) \cos(t) dt \right).
$$

Note that $k_\epsilon(s) = O(s)$ as $s \to 0$ and $k_\epsilon(s) = O(|s - \epsilon^{-1}\tau_\epsilon|)$ as $s \to \epsilon^{-1}\tau_\epsilon$. From this we see that $r_\epsilon(s) = O(\epsilon^3)$ as $s \to 0$ and a similar estimate as $s \to \epsilon^{-1}\tau_\epsilon$, thus (2.39) is satisfied.

Note that due to the smoothness of the tensor $h$ up to the boundary $\partial M$, as $\epsilon \to 0$ the equation in (2.35) for the $(\tilde{y}, \tilde{\eta})$ dynamic converges to the equation

$$
\dot{y}_j = \sum_k h_0^{jk} \eta_k, \quad \dot{\eta}_j = -\frac{1}{2} \partial_{y_j} |\eta|^2_{h_0}
$$

where $h_0$ is the metric on $\partial \overline{M}$. Note that this is nothing but the Hamilton flow equation of the Hamiltonian vector field $H_0$ of the Hamiltonian $\frac{1}{2} |\eta|^2_{h_0}$ on $T^*\partial \overline{M}$. In particular, if

$$(\dot{y}(s), \dot{\eta}(s)) := e^{sH_0}(y_0, \eta_0) \in T^*\partial \overline{M}$$

is the integral curve solution of this equation with initial condition $(\dot{y}(0), \dot{\eta}(0)) = (y_0, \eta_0)$, we get from Duhamel formula

$$
d_{h_0}((\tilde{y}(s), \tilde{\eta}(s)), (\dot{y}(s), \dot{\eta}(s))) = O(\epsilon)
$$

(2.40)

where we view $(\tilde{y}(s), \tilde{\eta}(s))$ as a curve in $T^*\partial \overline{M}$ and $d_{h_0}$ is the distance for the Sasaki metric associated to $h_0$ in $T^*\partial \overline{M}$ (or equivalently any other Riemannian metric).
3. X-Ray transform

We first define the X-ray transform on functions on $S^*M$, and this can be seen as some boundary value of a natural boundary value problem on $\overline{S^*M}$. Except in Subsection 3.4.3, we shall always assume that the metric $g$ is non-trapping in the sense that $\tau_+(z) < \infty$ for all $z \in S^*M$, where $\tau_+$ is defined in (2.31). Observe that due to (2.30) and the estimates for $\rho(t)$ in Lemma 2.6, the condition $|\tau_+(z)| < \infty$ is equivalent to saying that $\rho(\varphi_t(z)) \to 0$ as $t \to \pm \infty$.

3.1. Boundary value problem and X-ray. Let us consider the boundary value problem

$$- Xu = f, \quad u|_{\partial S^*M} = 0$$

where $f \in C^\infty_c(S^*M)$. Assuming that the metric $g$ is non-trapping, there is a unique smooth solution $u \in C^\infty(S^*M)$ given by

$$u(z) := \int_0^\infty f(\varphi_t(z))dt.$$

This defines an operator $R_+: C^\infty_c(S^*M) \to C^\infty(S^*M)$ by setting $R_+ f = \int_0^\infty f \circ \varphi_t dt$. We would like to extend this to functions which are not compactly supported and to points in $z \in \overline{S^*M}$ (i.e. all the way up to the boundary). To this end, observe that by making a change of variable $t = t(\tau)$ of (2.30) one can define the forward/backward resolvent

$$R_+ f(z) := \int_0^{\tau_+(z)} \frac{f(\varphi_\tau(z))}{\rho^2(\tau)} d\tau, \quad R_- f(z) = - \int_0^{\tau_-(z)} \frac{f(\varphi_\tau(z))}{\rho^2(\tau)} d\tau. \quad (3.1)$$

Note that this definition extends to functions $f \in \rho^2 C^\infty(S^*M)$. Writing $f = \rho^2 \overline{f}$ one gets $Rf(z) := \int_0^{\tau_-(z)} \overline{f}(\varphi_\tau(z)) d\tau$. Again, due to the non-trapping assumption this is a smooth function in $S^*M$. Furthermore, since $\overline{f}$ and $\varphi_\tau$ are both smooth all the way up to $\partial S^*M$, this extends to be a smooth function on $\overline{S^*M}$.

**Definition 3.1.** The X-ray transform of a function $f \in \rho^2 C^\infty(S^*M)$ is defined by $I f := R_+ f|_{\partial S^*M} = \int_0^{\tau_+(z)} \rho^{-2}(\tau) f(\varphi_\tau(z)) d\tau$.

First, observe that

$$f \in \ker I \cap \rho^2 C^\infty(\overline{S^*M}) \iff R_+ f = R_- f. \quad (3.2)$$

By (2.22), for symmetric tensors $f \in \rho^k C^\infty(\overline{M}; S^m(\kappa T^*\overline{M})) \cap \ker(\iota_{\mu^2 \partial_\nu})^{\ell+1}$, one has $\pi^*_m f \in \rho^{k+m-\ell} C^\infty(\overline{S^*M})$ and thus, for $k + m - \ell \geq 2$

$$I_m := I \pi^*_m : \rho^k C^\infty(\overline{M}; S^m(\kappa T^*\overline{M})) \cap \ker(\iota_{\mu^2 \partial_\nu})^{\ell+1} \to C^\infty(\partial_+ S^*M) \quad (3.3)$$

is well-defined and continuous. We call it the X-ray transform on $m$-tensors. In the next subsections we will study the kernel of $I_m$. 
3.2. Choice of gauge. Let $B_\pm : \overline{S^*M} \to \partial^\pm S^*M$ be the endpoint map defined by

$$B_\pm(z) := \varphi_{\tau_\pm}(z),$$

where $\tau_\pm$ are defined in (2.31). Since $R_+ Xf = -f + f \circ B_+$ for $f \in C^\infty(\overline{S^*M})$, we see that

$$IXf = f|_{\partial^+ S^*M} \circ B_+ - f|_{\partial^- S^*M}$$

and in particular $IXf = 0$ if $f \in \rho C^\infty(\overline{S^*M})$. Now it is also a direct computation to check that $X\pi_m^*f = \pi_m^*Df$ if $f \in C^\infty(M; S^m T^*M)$ and where $D := S\nabla$ is the symmetrised covariant derivative. As a consequence, if $f \in \rho C^\infty(M; S^{m-1}(sc T^*M))$, then $I_m Df = 0$. Since $I_m$ has a natural kernel, we can always put a function $f$ in a certain gauge using that kernel.

**Lemma 3.2.** Let $f \in \rho^k C^\infty(M; S^m(sc T^*M))$ with $k \geq 2$. There exists a tensor $u \in \rho^{k-1} C^\infty(M; S^{m-1}(sc T^*M))$ such that $f - Du \in \rho^k C^\infty(M; S^m(sc T^*M))$ and $\iota_{\rho^2 \partial_\rho}(f - Du) = 0$ near $\partial M$.

**Proof.** We proceed as in [GGSU] and consider a collar neighbourhood $[0, \epsilon)_\rho \times \partial M \subset M$ and express $f$ in this neighbourhood as $f = S(f_j \odot \left(\frac{d\rho}{\rho^2}\right)^{m-j})$ where $f_j \in \rho^k C^\infty(M; S^j(sc T^*M))$ satisfies $\iota_{\rho^2 \partial_\rho} f_j = 0$. We will proceed by induction on $j$. By a direct computation

$$\nabla \frac{d\rho}{\rho^2} = \frac{1}{2} \partial_\rho h_\rho - \frac{h_\rho}{\rho} \in \rho C^\infty(M; S^2(sc T^*M)) \cap \ker \iota_{\partial_\rho}. \quad (3.4)$$

Similarly, if $\omega \in \rho^k C^\infty(M; sc T^*M)$ is tangential to $\partial M$ near $\rho = 0$ (i.e. in $\ker \iota_{\rho^2 \partial_\rho}$) then

$$\nabla \omega = d\rho \otimes \partial_\omega + 2S(\omega \otimes \frac{d\rho}{\rho}) + S(d\rho \otimes A\omega) + \nabla^h \omega \quad (3.5)$$

for some smooth endomorphism $A \in C^\infty(M; \text{End}(T^*\partial M))$ and $\nabla^h$ denotes the Levi-Civita connection of $h_\rho$ on $\partial M$. We begin by setting $q_0(\rho, y) := \int_0^\rho s^{-2} f_0(s, y) ds \in \rho^{k-1} C^\infty(M)$ so that, using (3.4),

$$D(q_0(\frac{d\rho}{\rho^2})^{m-1})(\rho^2 \partial_\rho, \ldots, \rho^2 \partial_\rho) = f_0.$$

This means that $f - D(q_0(\frac{d\rho}{\rho^2})^{m-1}) = \sum_{j=1}^m S(f_j \odot (\frac{d\rho}{\rho^2})^{m-j})$ where $f_j \in \rho^k C^\infty(M; S^j(sc T^*M))$ satisfies $\iota_{\rho^2 \partial_\rho} f_j = 0$ near $\partial M$, using again (3.4).

To complete the induction, we show that if $j \in [1, m-1]$ and $f_j \in \rho^k C^\infty(M; S^j(sc T^*M))$ satisfies $\iota_{\rho^2 \partial_\rho} f_j = 0$ near $\partial M$, we can construct $q_j \in \rho^{k-1} C^\infty(M; S^j(sc T^*M))$ which satisfies $\iota_{\rho^2 \partial_\rho} q_j = 0$ near $\partial M$ and solves

$$DS(q_j \odot (\frac{d\rho}{\rho^2})^{m-j}) = S(f_j \odot (\frac{d\rho}{\rho^2})^{m-j}) + T \quad (3.6)$$
where \( T = S(T_0 \otimes \frac{d\rho}{\rho})^{m-j-1} + S(T_1 \otimes \frac{d\rho}{\rho})^{m-j-2} \) with \( T_0 \in \rho^k \mathcal{C}^\infty(M; S^{j+1}(sc^* T^* M)) \) and \( T_1 \in \rho^k \mathcal{C}^\infty(M; S^{j+2}(sc^* T^* M)) \) satisfying \( \iota_{\rho^2 \partial_{\rho}} T_0 = \iota_{\rho^2 \partial_{\rho}} T_1 = 0 \) (when \( j = m - 1 \) the term involving \( T_1 \) vanishes). To this end, using (3.5) and (3.4), we compute

\[
DS(q_j \otimes \left( \frac{d\rho}{\rho} \right)^{m-j-1}) = S((\rho^2 \partial_{\rho} q_j + 2\rho q_j + \rho^2 B q_j) \otimes \left( \frac{d\rho}{\rho} \right)^{m-j}) + T
\]

with \( T \) as above if \( q_j \in \rho^{k-1} \mathcal{C}^\infty(M; S^j(sc^* T^* M)) \cap \ker \iota_{\rho^2 \partial_{\rho}} \), for some smooth endomorphism \( B \in \mathcal{C}^\infty(M; \text{End}(S^j T^* \partial M)) \). It then suffices to solve the ODE

\[
(\partial_{\rho} + B)(\rho^2 q_j) = \bar{f}_j
\]

which has a solution \( q_j \in \rho^{k-1} \mathcal{C}^\infty(M; S^j(sc^* T^* M)) \).

Thus, in what follows, for studying the kernel of \( I_m \), we will always assume that we are working with tensors in \( \ker \iota_{\rho^2 \partial_{\rho}} \).

3.3. **Boundary determinations.** The first property that we will take advantage of is that an element in the kernel of \( I_m \) must have its leading behaviour at \( \partial M \) satisfying some property on the boundary \( (\partial M, h_0) \). A consequence of (2.40) and Lemma 2.9 is the following

**Lemma 3.3.** Let \( f \in \mathcal{C}^\infty(M; S^m(sc^* T^* M)) \), then for all \( k \geq 2 \) and \( z_0 = (y_0, \eta_0) \in T^* \partial M \) with \( |\eta_0|_{h_0} = 1 \), we have for \( \gamma(\tau) = \pi_0(\bar{\tau}_\tau(z_0)) \)

\[
\lim_{\epsilon \to 0} \epsilon^{-k} \int_0^{\tau^+ (y_0, \epsilon^{-1} \eta_0)} \rho^{k-2}(\tau) f(\rho^2 \dot{\gamma}(\tau), \ldots, \rho^2 \ddot{\gamma}(\tau)) d\tau
\]

\[
= \sum_{\ell=0}^{m} \int_0^\pi \sin^{k+m-\ell-2}(s) \cos^{\ell}(s) \iota_{\partial_{h_0}} (t_{\ell \rho^2 \partial_{\rho}} \ldots t_{\ell \rho^2 \partial_{\rho}} f)(\hat{\alpha}(s), \ldots, \hat{\alpha}(s))
\]

where \( \alpha : [0, \pi] \to \partial M \) is the unit speed geodesic for the metric \( h_0 \) with initial condition \((\alpha(0), \hat{\alpha}(0)) = (y_0, \eta_0^* )\) with \( \eta_0^* \in T_{y_0} \partial M \) the dual of \( \eta_0 \) by \( h_0 \).

**Proof.** In coordinates we can write \( f = \sum_{\ell=0}^{m} f_{\ell,i_1,\ldots,i_{m-\ell}} \left( \frac{d\rho}{\rho} \right)^{\ell} \frac{d\eta_{i_1}}{\rho} \ldots \frac{d\eta_{i_{m-\ell}}}{\rho} \) and since

\[
\dot{\gamma}(\tau) = \bar{\tau}_0(\tau) \partial_{\rho} + \sum_{k} h_{jk}^i(\tau) \eta_j(\tau) \partial_{y_k}
\]

we get

\[
\epsilon^{-k} \int_0^{\tau^+ (y_0, \epsilon^{-1} \eta_0)} \rho^{k-2}(\tau) f(\rho^2 \dot{\gamma}(\tau), \ldots, \rho^2 \ddot{\gamma}(\tau)) d\tau
\]

\[
= \epsilon^{-k} \sum_{\ell=0}^{m} \int_0^{\tau^+ (y_0, \epsilon^{-1} \eta_0)} \rho^{k+m-\ell-2}(\tau) f_{\ell,i_1,\ldots,i_{m-\ell}} (\tau) \bar{\tau}_0(\tau) \eta^*_{i_1}(\tau) \ldots \eta^*_{i_{m-\ell}}(\tau) d\tau,
\]
with \( \eta^* \) the dual of \( \eta \) by \( h_\rho \). Make the change of variable \( s = \varepsilon^{-1} \tau \) and take the limit as \( \varepsilon \to 0 \) using (2.40), Lemma 2.9 and (2.38), we obtain the desired result. \( \square \)

This leading behaviour can be viewed as a sort of ray-transform on the boundary but for geodesic segments of length \( \pi \). We recall that for a closed manifold \((N, h_0)\), the X-ray transform on \( m \)-symmetric tensors is called \( s \)-injective if for each \( u \in C^\infty(N; S^mT^*N) \) satisfying \( \int_\gamma \pi^*_m u = 0 \) for all closed geodesic \( \gamma \) on \( N \), then \( u = Df \) for some \( f \in C^\infty(N; S^{m-1}T^*N) \) (and for \( m = 0 \) we ask that \( u = 0 \)). From [JoSB1] and Lemma 3.3, we obtain directly

**Proposition 3.4.** Suppose \((\partial \overline{M}, h_0)\) is a Riemannian manifold of dimension \( n \geq 2 \) which either has injectivity radius is strictly bigger than \( \pi \), or its X-ray transform on functions is injective or it is a sphere of radius \( r \notin \{1/k; k \in \mathbb{N}, k \geq 2 \} \). If \( f \in \rho^2C^\infty(\overline{M}) \) satisfy \( I_0f = 0 \), then \( f \in \rho^\infty C^\infty(\overline{M}) \).

**Proof.** We write \( f = \rho^2 \overline{f} \) where \( \overline{f} \in C^\infty(\overline{M}) \) and we have that \( \overline{f} = \sum_{k=0}^N \overline{f}_k \rho^k + O(\rho^{N+1}) \) with \( \overline{f}_k \in C^\infty(\partial \overline{M}) \). We proceed by induction: if \( \overline{f}_j = 0 \) for all \( j \leq k - 1 \), we have by

\[
0 = |\eta_0|^{k-1} \int_0^{\tau^+ (y_0, \eta_0)} \overline{f}(\pi_0(\varphi_\tau(y_0, \eta_0))) d\tau \to \int_0^{\pi} \sin^k(s) \overline{f}_k(\alpha_{y_0, v}(s)) ds
\]

as \( |\eta_0| \to \infty \), with \( v = \eta_0/|\eta_0| \in S^*_y \partial \overline{M} \) and \( \alpha_{y, v}(s) \) the geodesic for \( h_0 \) with initial condition \( \alpha_{y, v}(0) = y, \dot{\alpha}_{y, v}(0) = v \). The proof of [JoSB1, Theorem 4.2] implies \( \overline{f}_k = 0 \). \( \square \)

For 1-forms, it is possible to show, using similar arguments as in [JoSB1] and Lemma 3.2 that if \( I_1 \omega = 0 \) for \( \omega \in \rho^2C^\infty(\overline{M}; scT^*\overline{M}) \) then there is \( u \in \rho C^\infty(\overline{M}) \) such that \( \omega - du \in \rho^\infty C^\infty(\overline{M}; scT^*\overline{M}) \) under the assumption that \((\partial \overline{M}, h_0)\) is a closed Riemannian manifold with injectivity radius \( \text{inj}(h_0) > 2\pi \) and such that the X-ray transform on 1-forms is \( s \)-injective.

### 3.4. Resolvent Estimates and Pestov Identity.

#### 3.4.1. Jacobi Fields Near Infinity.

The curvature estimates in Proposition 2.3 allows us to deduce estimates on the Jacobi fields.

**Lemma 3.5.** There is \( C > 0 \) such that for each \( z \in W^+ \), if \( \gamma(t) := \pi_0(\varphi_t(z)) \) and \( J(t) \) is a smooth vector field along \( \gamma \), we have the following estimate

\[
|R_{\gamma(t)}(\dot{\gamma}(t), J(t))\dot{\gamma}(t)|_g \leq C \rho^4(t)|J(t)|_g
\]

for all \( \pm t \geq 0 \).

\(^\text{1}\)Although it is strangely not written in the statement of Theorem 4.2 of [JoSB1], the authors considered the case of the sphere with radius 1 in the proof.
Proof. We write $z = (x, \eta) \in SM$ with $\rho(x) \leq \epsilon$ and $\dot{\gamma}(t) = \bar{\xi}_0(t)\rho^2(t)\partial_\rho + \rho(t)^2\eta(t)$, with $|\eta(t)|_g \leq C$ and $d\rho(\eta(t)) = 0$ for some uniform $C > 0$ using Lemma 2.6. By Proposition 2.3 we have

$$\langle R_{\gamma(t)}(\gamma(t), J(t))\dot{\gamma}(t), \rho^2\partial_\rho \rangle \leq C(|J_0(t)|_g^6 + |J_1(t)|_g^5(t))$$

where $J_0(t) = g(J(t), \rho^2\partial_\rho)$ and $J_1(t) = J(t) - J_0(t)\rho^2\partial_\rho$, while for each $Y(t) \in \ker d\rho$ with $|Y(t)|_g = 1$

$$\langle R_{\gamma(t)}(\gamma(t), J(t))\dot{\gamma}(t), Y(t) \rangle \leq C(|J_0(t)|_g^5(t) + |J_1(t)|_g^4(t))$$

which ends the proof. \qed

Lemma 3.6. There is $C > 0$ such that for all $z \in W^+_t$ and $J(t)$ a Jacobi field along $\gamma(t) := \pi_0(\varphi_t(z))$, we have the following estimates:

$$|\dot{J}(t)|_g \leq C|J(0)|_g\rho^2(0) + C|\dot{J}(0)|_g,$$

$$|J(t)|_g \leq |J(0)|_g + C|J(0)|_g\rho^3(0)t + C|\dot{J}(0)|_g t.$$ (3.7)

Proof. Consider the function given by $F(t) := \langle R_{\gamma(t)}(\dot{\gamma}(t), J(t))\dot{\gamma}(t), \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|_g} \rangle$. We have $\frac{d}{dt}|\dot{J}(t)|_g = F(t)$ since $J$ is a Jacobi field. Define the barrier function $B(t)$ by

$$\dot{B}(t) := |F(t)|, \quad B(0) := |\dot{J}(0)|_g.$$  

One then can easily see that

$$B(t) \geq |\dot{J}(0)|_g + \int_0^t \partial_s|\dot{J}(s)|_g ds = |\dot{J}(t)|_g$$  

and is non-decreasing. Furthermore, due to the asymptotics of Lemma 2.6 and the estimates of Lemma 3.5 we have

$$|F(t)| \leq C\left(\frac{\rho(0)}{1 + \rho(0)t}\right)^4 |J(t)|_g.$$  

Therefore one can write for $t > 0$

$$\dot{B}(t) \leq C\left(\frac{\rho(0)}{1 + \rho(0)t}\right)^4 |J(t)|_g = C\left(\frac{\rho(0)}{1 + \rho(0)t}\right)^4 \left(|J(0)|_g + \int_0^t \partial_s|J(s)|_g ds\right).$$

Differentiating $|J(s)|_g$ and using (3.8) we obtain

$$\dot{B}(t) \leq C\left(\frac{\rho(0)}{1 + \rho(0)t}\right)^4 \left(|J(0)|_g + \int_0^t B(s) ds\right).$$

And now use the fact that $B(s)$ is increasing we arrive at

$$\dot{B}(t) \leq C\left(\frac{\rho(0)}{1 + \rho(0)t}\right)^4 \left(|J(0)|_g + tB(t)\right).$$
If \( B(0) = |\dot{J}(0)| = 0 \) we have
\[
B(t) \leq C|J(0)|_g \int_0^t \left( \frac{\rho(0)}{1 + \rho(0)s} \right)^4 ds + C \int_0^t \left( \frac{\rho(0)}{1 + \rho(0)s} \right)^4 sB(s)ds.
\]
Then by Grönwall’s lemma,
\[
B(t) \leq |J(0)|_g \rho(0)^3 \int_0^{\rho(0)t} \left( \frac{1}{1 + s} \right)^4 ds \exp \left( C\rho(0)^2 \int_0^{\rho(0)t} \left( \frac{1}{1 + r} \right)^4 rdr \right) \leq C\rho(0)^3|J(0)|_g.
\]
So we obtain
\[
|\dot{J}(t)|_g \leq C|J(0)|_g \rho(0)^3, \quad |J(t)| \leq |J(0)|_g + C\rho(0)^3|J(0)|_g t
\] (3.10)
if \( \dot{J}(0) = 0 \). Suppose now \( \dot{J}(0) = 0 \). Then \( \dot{B}(t) \leq C\left( \frac{\rho(0)}{1 + \rho(0)t} \right)^4 tB(t) \) and we obtain
\[
|\dot{J}(t)|_g \leq B(t) \leq |\dot{J}(0)|_g \exp \left( C\rho^2(0) \int_0^{\rho(0)t} \left( \frac{1}{1 + s} \right)^4 ds \right) \leq C|\dot{J}(0)|_g,
\]
\[
|J(t)|_g \leq |\dot{J}(0)|_g \int_0^t \exp \left( C\rho^2(0) \int_0^{\rho(0)s} \left( \frac{1}{1 + u} \right)^4 udu \right) ds \leq C|\dot{J}(0)|_g t
\] (3.11)
Consequently combining (3.10) and (3.11) we have the desired estimates. \( \Box \)

3.4.2. Resolvent mapping properties. We next describe the solution of \( Xu = f \) with boundary conditions \( u|_{\partial_\pm S^*M} = 0 \) when \( f \) are symmetric tensors.

**Lemma 3.7.** Let \( f \in \rho^kC^\infty(\overline{M}; S^{m(scT^*\overline{M})}) \) satisfy \( \iota_{\partial_\rho}f = 0 \) near \( \partial M \). If \( m \geq 1 \), let \( k + m > 3 \) and define \( u_\pm := R_\pm \pi^*_Mf \) as in (3.1). There is \( C > 0 \) such that for all \( z = (\rho, y, \xi_0, \eta) \in W_c^\pm \),
\[
|u_\pm(z)| \leq C\rho^{k-1}|\rho\eta|_{h_\rho}^m,
\]
\[
\|\nabla^v u_\pm(z)\|_G \leq C\rho^{k-2}|\rho\eta|_{h_\rho}^{m-1}, \quad \|\nabla^h u_\pm(z)\|_G \leq C\rho^{k-1}|\rho\eta|_{h_\rho}^{m-1}.
\] (3.12)
The resolvent thus satisfies the estimate
\[
|u_\pm(z)| + \|\nabla^v u_\pm(z)\|_G + \|\nabla^h u_\pm(z)\|_G \leq C\rho^{k-2}.
\]
If \( m = 0 \), one gets the same estimates but with \( \rho^{k-1} \) on the right hand side. Furthermore, if \( f \in \rho^\infty C^\infty(\overline{M}; S^{m(scT^*\overline{M})}) \) then \( u_\pm \) vanishes to infinite order at \( \partial_\pm S^*M \).

**Proof.** We only do this for \( u = u_+ \) as the \( u_- \) case is exactly the same. By definition one writes in using the decomposition \( \xi = \xi_0 d\rho/\rho^2 + \eta \) of cotangent vectors near \( \partial S^*\overline{M} \) (with \( \iota_\partial, \eta = 0 \))
\[
u(\rho, y, \xi_0, \eta) = \int_0^\infty f_\eta(t)(\eta^2(t), \ldots, \eta^2(t))dt
\] (3.13)
where \( (\gamma(t), \xi_0(t), \eta(t)) \in S^*\overline{M} \) is the geodesic with initial condition \( (\rho, y, \xi_0, \eta) \in S^*\overline{M} \) and with \( h_\rho(\eta^2(t), \cdot)/\rho^2(t) = \eta(t) \). The first inequality in (3.12) then follows from Lemma 2.6 and the definition of \( scT^*\overline{M} \).
For the estimates on the derivatives, we see from computing using the chain rule that
\[
\|\nabla^h u(z)\|_G \leq \sup_{\|V\|=1} \int_0^\infty \|d(\pi_m^* f)_{\varphi_t(z)}\|_G \|J_{(V,0)}(t), \dot{J}_{(V,0)}(t))\|_C dt
\]
where \(J_{V,W}(t)\) is the Jacobi field with initial condition \(J_{V,W}(0) = V\) and \(\dot{J}_{V,W}(0) = W\). Using (2.24), we have
\[
\|\nabla^h u(z)\|_G \leq C \sup_{\|V\|=1} \int_0^\infty \rho(t)^k |\rho(t)\eta(t)|_h^{(m-1)} \|J_{(V,0)}(t), \dot{J}_{(V,0)}(t))\|_C dt
\]
and applying the estimates of Lemma 2.6 and (3.7) we obtain
\[
\|\nabla^h u(z)\|_G \leq C \rho^{k+m-1} \sup_{\|V\|=1} \int_0^\infty \left( \frac{1}{1 + \rho(0)t} \right)^{k+m-1} |\eta(0)|_h^{(m-1)} \|J_{(V,0)}(t), \dot{J}_{(V,0)}(t))\|_C dt
\]
\[
\leq C \rho^{k-1} |\rho\eta|_h^{m-1}.
\]

For obtaining the analogous estimate for \(\|\nabla^v u(z)\|_G\), we repeat the same process and get
\[
\|\nabla^v u(z)\|_G \leq C \rho^k |\rho\eta|_h^{m-1} \sup_{\|V\|=1} \int_0^\infty \left( \frac{1}{1 + \rho(0)t} \right)^{k+m-1} \|J_{(V,0)}(t), \dot{J}_{(0,V)}(t))\|_C dt
\]
\[
\leq C \rho^{k-2} |\rho\eta|_h^{m-1}.
\]

The case \(m = 0\) is similar with an improvement of one power of \(\rho\). The last statement is a direct consequence of the expression \(u = R_+ f\) in terms of the smooth flows \(\varphi_t\) on \(S^* M\).

We then derive the following

**Corollary 3.8.** Let \(f \in \rho^k C^\infty(M; S^m(\infty T^* M))\) satisfy \(\iota_{\rho^2 \partial_t} f = 0\) near \(\partial M\). If \(m \geq 1\), we also assume \(k + m > 3\). If \(I_m f = 0\), there exists \(u \in C^\infty(S^* M)\) satisfying \(X u = f\) and the bounds near \(\rho = 0\)
\[
|u(z)| + \|\nabla^v u(z)\|_G + \|\nabla^h u(z)\|_G \leq C \rho(z)^{k-1}.
\]

**Proof.** The condition \(I_m f = 0\) is equivalent to \(R_+ \pi_m^* f = R_- \pi_m^* f\) by (3.2) and thus it suffices to set \(u = R_+ \pi_m^* f\) and apply Lemma 3.7. \(\square\)

### 3.4.3. Case with trapping

We briefly discuss the case where the curvature of \(g\) is negative, in which case the trapped set is a hyperbolic set in the sense of Anosov. The dynamics and resolvent are constructed in [DyGu] and the application to X-ray is done for the compact setting in [Gui]. In that case the trapped set has measure 0 and the set of \(z \in \partial_- S^* M\) for which \(\tau_+(z) < \infty\) is an open set of full measure and we will say that \(I_m f = 0\) if \(I_m f(z) = 0\) for all such \(z \in \partial_- S^* M\). Using that the trapped set is contained in a strictly convex compact region \(\{\rho \geq \epsilon\} \subset S^* M\) for some \(\epsilon > 0\) small, it is straightforward to apply the analysis of [DyGu, Gui] in our setting. The argument are mutatis mutandis the same as
in the asymptotically hyperbolic case discussed in [GGSU, Proposition 3.11 and Lemma 3.12]: combined with the analysis near $\partial \overline{M}$ done to prove Lemma 3.7, we obtain

**Lemma 3.9.** Assume that $g$ is asymptotically conic with negative curvature, but with a non-trivial trapped set. Then the conclusion of Corollary 3.8 hold true exactly as in the non-trapping case.

3.4.4. Pestov identities. We next give the Pestov identity for tensors. Let $M_\epsilon := \{ z \in M; \rho(z) \geq \epsilon \}$, and notice that $M_\epsilon$ is strictly convex for small $\epsilon > 0$.

**Proposition 3.10.** Let $f \in \rho^k C^\infty(\overline{M}; S^m(T^* M))$ which satisfies $\iota'_{\rho^2 \partial_\rho} f = 0$ near $\partial \overline{M}$ and $I_m f = 0$. Assume $k > \frac{n}{2} + 1$ and set $u := R_+ \pi^*_m f$ defined by (3.1). The following identity holds

$$\| \nabla^v \pi^*_m f \|_{L^2}^2 - \| X \nabla^v u \|_{L^2}^2 = (n-1) \| \pi^*_m f \|_{L^2}^2 - \langle \mathcal{R} \nabla^v u, \nabla^v u \rangle,$$

where $\mathcal{R} : Z \to Z$ is defined by $\mathcal{R}(x, \xi)^{\sharp} := R_x (Z, \xi) \xi^\sharp$ with $R$ the Riemann curvature tensor. In general, for all $u \in \rho^\infty C^\infty(S^* M)$ one has

$$\| \nabla^v X u \|_{L^2}^2 - \| X \nabla^v u \|_{L^2}^2 = (n-1) \| X u \|_{L^2}^2 - \langle \mathcal{R} \nabla^v u, \nabla^v u \rangle_{L^2(S^* M)}.$$

**Proof.** We will only prove the first identity since the same argument applies to the second one. We use the computation of [GGSU, Proof of Theorem 1] and have for $u \in C^\infty(S^* M)$

$$\| \nabla^v X u \|_{L^2(S^* M)}^2 - \| X \nabla^v u \|_{L^2(S^* M)}^2 = (n-1) \| X u \|_{L^2(S^* M)}^2 - \langle \mathcal{R} \nabla^v u, \nabla^v u \rangle_{L^2(S^* M)} + \int_{\partial S^* M} \langle \nabla^v u, \nabla^h u \rangle - (n-1) u X u \rangle_{\mu_\epsilon}$$

where $\mu_\epsilon = \iota'_{\rho = \epsilon} \iota X \mu$ and $\langle \nabla^v u, \nabla^h u \rangle$ is understood as $g(K \nabla^v u, d\pi \nabla^h u)$.

We now argue that when $\epsilon \to 0$ the boundary terms vanish and the terms involving interior integrals converge to the integral on all of $S^* M$. Treating the expression term by term we see first using (2.23) that

$$\| \pi^*_m f \|_{L^2(S^* M)}^2 \leq C \int_{\rho \geq \epsilon} \rho^{2k} d\text{vol}_g \leq C'$$

if $k > n/2$ and $\| \pi^*_m f \|_{L^2(S^* M)}^2 \to \| \pi^*_m f \|_{L^2(S^* M)}^2$ as $\epsilon \to 0$. For the term involving $\langle \mathcal{R} \nabla^v u, \nabla^v u \rangle_{L^2(S^* M)}$, this can be estimated by using $|\langle \mathcal{R} \nabla^v u, \nabla^v u \rangle_g| \leq \| \nabla^v u \|_{L^2}^2 |K(\xi^{\sharp}, \nabla^v u)|$ using Lemma 3.7 and Proposition 2.3, we obtain $|\langle \mathcal{R} \nabla^v u, \nabla^v u \rangle_g| \leq \rho^{2k}$ if $m \geq 1$ and $|\langle \mathcal{R} \nabla^v u, \nabla^v u \rangle_g| \leq \rho^{2k}$ if $m = 0$. If $k > n/2 + 1$, we get as $\epsilon \to 0$ that

$$\langle \mathcal{R} \nabla^v u, \nabla^v u \rangle_{L^2(S^* M)} \to \langle \mathcal{R} \nabla^v u, \nabla^v u \rangle_{L^2(S^* M)}.$$

We now look at $\| \nabla^v X u \|_{L^2(S^* M)}^2$ (recall $X u = \pi^*_m f$): this is equal to $\| \Delta^v \pi^*_m f \|_{L^2(S^* M)}^2$ where $\Delta^v$ is the vertical Laplacian in the fiber, but $f = \sum_{2j \leq m} S(\otimes^j g \otimes f_j)$ for some
trace-free $f_j \in \rho^k C^\infty(\overline{M} ; S^{m-2}(\omega T^* M))$, which then satisfy
\[ \Delta^v \pi_m^* f_j = (m - 2j)(m - 2j + n - 2)\pi_m^* f_j. \]
This implies that $||\nabla^v X u||^2_{L^2(S^* M)} \to ||\nabla^v X u||^2_{L^2(S^* M)}$ as $\epsilon \to 0$ if $k > n/2$. For the $||X \nabla^v u||^2_{L^2}$ term we use the identity $X \nabla^v u = \nabla^v \pi_m^* f - \nabla^h u$ in conjunction with Lemma 3.7 to get $|X \nabla^v u|^2 \leq C\rho^{2k-2}$, thus $||X \nabla^v u||^2_{L^2(S^* M)} \to ||X \nabla^v u||^2_{L^2(S^* M)}$ as $\epsilon \to 0$ if $k > n/2 + 1$.

We conclude with the boundary terms: by Lemma 3.7 we have at $\partial S^* M_e$
\[ |(\nabla^v u, \nabla^h u) g| + |\pi_m^* f|, |u| \leq C\epsilon^{2k-3} + C\epsilon^{2k-1} \]
thus for small $\epsilon$
\[ \int_{\partial S^* M_e} (|(\nabla^v u, \nabla^h u) g| + |\pi_m^* f|, |u|) \mu_e \leq C\epsilon^{2k-2-n} \]
which converges to 0 as $\epsilon \to 0$. This ends the proof for the first statement. The second statement of the Lemma goes the same way, where it suffices to use the fast vanishing of $u$ as $\rho \to 0$.

**Remark:** Note that in the case when $m \geq 1$ and the dimension $n \geq 2$ then the condition $k > 3 - m$ needed in Lemma 3.7 is always valid if we assume as in the Proposition that $k > \frac{n}{2} + 1$.

### 3.5 Injectivity of X-ray transforms
We begin with a Carleman estimate which will be useful in proving injectivity of $I_1$:

**Lemma 3.11.** Let $R \in \langle t \rangle^{-\alpha} L^\infty(\mathbb{R} ; \text{End}(\mathbb{R}^{n-1}))$, then there is $C > 0$ such that for all $U \in \langle t \rangle^{-\alpha} W^{2, \infty}(\mathbb{R} ; \mathbb{R}^{n-1})$ one has the following estimate for $N$ sufficiently large:
\[ \|e^{N \log(1+t^2)} \partial_t e^{-N \log(1+t^2)} U + R(t)U\|^2_{L^2} \geq N\|\dot{U}\|^2_{L^2} + CN^2\|\langle t \rangle^{-1} U\|^2_{L^2}. \]

**Proof.** We first compute
\[ \|e^{N \log(1+t^2)} \partial_t e^{-N \log(1+t^2)} U\|^2 = \int_{\mathbb{R}} \|\dot{U}\|^2 dt + N \int_{\mathbb{R}} \frac{1 + 2(2N - \frac{1}{2})t^2}{(1 + t^2)^2} |U|^2 dt. \tag{3.14} \]
So for $N \geq 2$ we yield that $\|e^{N \log(1+t^2)} \partial_t e^{-N \log(1+t^2)} U\|^2 \geq \int \frac{N}{1 + t^2} |U|^2 dt$. Using this inequality in conjunction with (3.14) we have
\[ \|e^{N \log(1+t^2)} \partial_t^2 e^{-N \log(1+t^2)} U\|^2 \geq N \int |e^{N \log(1+t^2)} \partial_t e^{-N \log(1+t^2)} U|^2 \]
\[ = N \int |\dot{U}|^2 + 2N^2 \int \frac{1 + (2N - \frac{1}{2})t^2}{(1 + t^2)^2} |U|^2 \]
\[ \geq N \int |\dot{U}|^2 + N^2 \int \frac{1}{1 + t^2} |U|^2 dt \]
We see that for $N$ sufficiently large the potential term $RU$ can be absorbed to the right side to obtain the desired inequality. \qed
Lemma 3.12. Let $\gamma$ be a complete geodesic curve on $M$ with $\lim_{t \to \pm \infty} \gamma(t) \in \partial M$. If $J$ is a Jacobi field along $\gamma$ satisfying $\|J(t)\|_g \leq C\langle t \rangle^{-\alpha}$ for some $\alpha > 0$ and $\langle J, \dot{\gamma} \rangle = 0$, then $J = 0$.

Proof. Let $\{\dot{\gamma}, Y_1, \ldots, Y_{n-1}\}$ be a unitary orthonormal frame which is parallel along $\gamma$, and recall that $|\dot{\gamma}(t) \pm \rho^2(\gamma(t))\partial_{\rho} = O(\rho(\gamma(t)))$ as $t \to \pm \infty$ by using Lemmas 2.6, so $(d\rho/\rho^2)(Y_j(t)) = O(\rho(\gamma(t)))$. We write $J = \sum U_t Y_t$, the equation $\ddot{\gamma} = R(\dot{\gamma}, J)\dot{\gamma}$ becomes $\ddot{U}(t) + R(t)U(t) = 0$ with $U(t) \in \langle t \rangle^{-\alpha}L^\infty(\mathbb{R}; \mathbb{R}^{n-1})$ and some $R \in \langle t \rangle^{-4}L^\infty(\mathbb{R}; \text{End}(\mathbb{R}^{n-1}))$ by using Lemma 3.5. We first show that $U(t), \dot{U}(t) \in \langle t \rangle^{-\alpha}L^\infty(\mathbb{R}; \mathbb{R}^{n-1})$. Indeed, we write for $t < T$,

$$\ddot{U}(t) = -\int_t^T \ddot{U}(s)ds + \dot{U}(T) = \int_t^T R(s)U(s)ds + \dot{U}(T).$$

This implies by integrating

$$|\dot{U}(T)|t \leq |U(t)| + |U(0)| + C\int_0^t \int_r^\infty \langle s \rangle^{-(4+\alpha)}dsdr.$$

Taking the lim sup as $T \to \infty$ we obtain

$$\limsup_{T \to \infty} |\dot{U}(T)|t \leq |U(t)| + |U(0)| + C\int_0^t \int_r^\infty \langle s \rangle^{-(3+\alpha)}dsdr.$$

Taking $t \to +\infty$ and using the fact that $U \in L^\infty(\mathbb{R})$ we get that $\limsup_{T \to \infty} |\dot{U}(T)| = 0$. Therefore, we obtain

$$|\dot{U}(t)| = \int_t^\infty |R(s)U(s)|ds \leq C\int_t^\infty \langle s \rangle^{-(4+\alpha)}ds \leq C\langle t \rangle^{-(3+\alpha)}$$

as $t \to +\infty$. Same estimate holds for when $t \to -\infty$ by the analogous argument. Now write

$$|U(t)| \leq \int_t^\infty |\dot{U}(s)|ds \leq C\langle t \rangle^{-(2+\alpha)}$$

as $t \to \infty$ and similarly for $t \to -\infty$. We can repeat the argument and deduce that $U \in \langle t \rangle^{-\alpha}L^\infty$ by induction, which means $\ddot{U}(t) = \int_{-\infty}^t R(s)U(s)ds$ is in $\langle t \rangle^{-\alpha}L^\infty$. Now that we have $U \in \langle t \rangle^{-\alpha}W^{2,\infty}(\mathbb{R}; \mathbb{R}^{n-1})$ we can apply Lemma 3.11 to deduce that $U = 0$. \hfill \Box

We are now in position to prove Theorem 1.1, the injectivity of X-ray transform on tensors.

Proof of Theorem 1.1. We first show i): by Corollary 3.8 (or Lemma 3.9 for the trapping case with negative curvature), we get $O(\rho^{k-1})$ pointwise bounds on the smooth function $u := R_+\pi_0^* = R_-\pi_0^*f \in C^\infty(S^*M)$, and on the derivatives $|\nabla^u u|_G, |\nabla^b u|_G$. We apply Pestov identity from Proposition 3.10 and get

$$0 = \|X\nabla^v u\|_{L^2}^2 + (n-1)\|\pi_0^* f\|_{L^2}^2 - \langle \mathcal{R}\nabla^u u, \nabla^v u \rangle.$$
Then we follow the proof of [GGSU, Theorem 1]: we let \( Z \in C^\infty(S^*M; \mathcal{Z}) \cap \rho^{k-1}L^\infty \) so that \(|XZ|_G \in \rho^{k-1}L^\infty \) and \(|X^2Z|_G \in \rho^{k-1}L^\infty \), and define the quadratic form

\[
A(Z) = \|XZ\|_{L^2(S^*M)}^2 - \langle \mathcal{R}Z, Z \rangle_{L^2(S^*M)}.
\]

By the same argument as in the proof of [GGSU, Theorem 1], we have \( A(Z) \geq 0 \) for such \( Z \): the method is to first replace \( Z \) by \( Z_\epsilon := \chi(\rho/\epsilon)Z \) where \( \chi(s) = 1 \) for \( s \geq 1 \) and \( \chi(s) = 0 \) for \( s \leq 1/2 \), then we get \( A(Z_\epsilon) \geq 0 \) since \( g \) has no conjugate points and \( A(Z_\epsilon) \) is the integrated index form on geodesics, and by letting \( \epsilon \to 0 \) we get \( A(Z) = \lim_{\epsilon \to 0} A(Z_\epsilon) \) using \(|X(\chi(\rho/\epsilon))| \leq C\epsilon \) and the bounds on \( Z \). It then suffices to apply this with \( Z = \nabla^v u \) (we use \([X, \nabla^v]u = -\nabla^h u \) and \([X, \nabla^h]u = \mathcal{R}\nabla^v u \) to get bounds on \( XZ \) and \( X^2Z \) from bounds on \( \nabla^vu \) and \( \nabla^hv \)) and we deduce that \(|X\nabla^v u|_{L^2}^2 - \langle \mathcal{R}\nabla^v u, \nabla^v u \rangle \geq 0 \), so \( f = 0 \).

For ii), we may assume, using Proposition 3.2 that \( \iota_{\rho^2\partial_\rho} f = 0 \) near \( \partial M \). We can then argue as in [GGSU, Proof of Theorem 1] to show that Proposition 3.10 implies that \( X^2\nabla^v u = \mathcal{R}\nabla^v u \) if \( u = R_+\pi^*_m f = R_-\pi^*_m f \) (using \( I_1 f = 0 \)). When restricted to a geodesic curve \( \gamma \subset M \), \( J := \nabla^v u|_{\gamma} \) is then a Jacobi field along \( \gamma \) with (for \( x = \gamma(0) \in M \))

\[
\|J(\gamma(t))\|_g = \|\nabla^v u|_{\gamma(t)}\|_g \leq C\rho(t)^{-2} \leq C(t)^{-k+2}, \quad t \to \pm\infty
\]

by using Lemma 3.7 and Lemma 2.6. Now apply Lemma 3.12 to deduce that \( J(\gamma(t)) = 0 \) identically. Since \( \gamma \) is an arbitrary complete geodesic with end points on \( \partial M \), we have that \( \nabla^v u = 0 \) and therefore \( u = \pi^*_q \) for some \( q \in \rho^{k-1}C^\infty(M) \) and \( Xu = \pi^*_m f \) implies \( dq = f \).

For iii) we follow the idea of [PSU] (see also [GGSU]). First by Proposition 3.2 we may assume \( \iota_{\partial_\rho} f = 0 \) near \( \partial M \). Let \( u = R_+\pi^*_m f = R_-\pi^*_m f \), which is smooth in \( S^*M \) and belongs to \( \rho^kL^\infty(S^*M) \) with similar estimates for its vertical/horizontal derivatives by Proposition 3.8. We write \( u = \sum u_\ell \) where \( u_\ell \) are eigenmodes of the vertical Laplacian \( \Delta^v = \nabla^v*\nabla^v \). Define \( \tilde{u} := u - \sum_{\ell \leq m-1} u_\ell \), then \( X\tilde{u} = \pi^*_m f - \sum_{\ell \leq m-1} Xu_\ell \) has no eigenmodes above \( m \) since \( \pi^*_m f = \sum_{\ell \leq m} f_\ell \). Furthermore, \( X\tilde{u} = \sum_{\ell \geq m} Xu_\ell \) has no eigenmodes below \( m - 1 \). Therefore, \( X\tilde{u} = (X\tilde{u})_m + (X\tilde{u})_{m-1} \). We apply Proposition 3.10 to \( \tilde{u} \) and get

\[
\lambda_m \|(X\tilde{u})_m\|^2 + \lambda_{m-1} \|(X\tilde{u})_{m-1}\|^2 = \|X\nabla^v u\|^2 - \langle \mathcal{R}\nabla^v u, \nabla^v u \rangle + (n - 1)(\|(X\tilde{u})_m\|^2 + \|(X\tilde{u})_{m-1}\|^2).
\]
where \( \lambda_m = m(m + n - 2) \) and \( \lambda_{m-1} = (m - 1)(m + n - 3) \). Now if the curvature is non-positive, one has that \( X \tilde{u} = 0 \): indeed,

\[
\lambda_m \|(X \tilde{u})_m\|^2 + \lambda_{m-1} \|(X \tilde{u})_{m-1}\|^2 \geq \|X \nabla^u \tilde{u}\|^2 + (n - 1)(\|(X \tilde{u})_m\|^2 + \|(X \tilde{u})_{m-1}\|^2) \\
\geq \left( \frac{(m - 1)(m + n - 2)^2}{m + n - 3} + (n - 1) \right) \|(X \tilde{u})_{m-1}\|^2 \\
+ \left( \frac{m(m + n - 1)^2}{m + n - 2} + (n - 1) \right) \|(X \tilde{u})_m\|^2
\]

where we used the computation in [PSU, Lemma 4.3] to deal with \( \|X \nabla^u \tilde{u}\|^2 \). This implies that \( X \tilde{u} = 0 \) with \( \tilde{u} \) decaying to order \( O(\rho^k) \) as \( \rho \to 0 \). Thus necessarily \( \tilde{u} = 0 \) and the proof is complete.

\[\Box\]

4. Renormalized Length and Scattering map

4.1. Scattering map. We define the scattering map using the rescaled flow \( \overline{\varphi}_\tau \).

**Definition 4.1.** For non-trapping asymptotically conic manifolds \((M, g)\), the scattering map \( S_g : T^* \partial \overline{M} \to T^* \partial \overline{M} \) is defined by

\[ S_g(z) := \overline{\varphi}_{\tau_+}(z). \]

In the case of trapping, the scattering map is defined on the subset \( \{ z \in T^* \partial \overline{M}; \tau_+(z) < \infty \} \).

We notice that \( S_g \) is defined using a choice of normal form (in order to identify \( \partial_{\tau} S^* M \) to \( T^* \partial \overline{M} \)), but by \((2.14)\), a change of normal form yields the coordinate change \((y, \eta) \mapsto (y, \eta + d\omega_0)\) on \( T^* \partial \overline{M} \) where \( \omega_0 \in C^\infty(\partial \overline{M}) \), which means that \( S_g \) is simply being conjugated by this transformation of \( T^* \partial \overline{M} \).

**Lemma 4.2.** Let \((y_0, \eta_0) \in T^* \partial \overline{M} \), then

\[
\lim_{\epsilon \to 0} \epsilon.S_g(y_0, \epsilon^{-1} \eta_0) = |\eta_0|_{\eta_0}.\phi_{\pi} \left( y_0, \frac{\eta_0}{|\eta_0|_{\eta_0}} \right) \tag{4.1}
\]

where we defined the fiber dilation action \( c.(y, \eta) := (y, c\eta) \) if \( c > 0 \), \((y, \eta) \in T^* \partial \overline{M} \) and \( \phi_t : T^* \partial \overline{M} \to T^* \partial \overline{M} \) is the Hamilton flow of the Hamiltonian \( \frac{1}{2}|\eta|^2_{\eta_0} \) at time \( t \).

**Proof.** Let us first take \( |\eta_0|_{\eta_0} = 1 \). We use the bound \((2.40)\) at \( s = \epsilon^{-1} \tau_+ = \epsilon^{-1} \tau^+_g(y_0, \epsilon^{-1} \eta_0) \), which by \((2.38)\) is equal to \( \pi + O(\epsilon) \), to deduce that as \( \epsilon \to 0 \),

\[ \epsilon.S_g(y_0, \epsilon^{-1} \eta_0) \to \phi_{\pi}(y_0, \eta_0). \]

For the general case, it suffices to write \( \epsilon.S_g(y_0, \epsilon^{-1} \eta_0) = \epsilon^{\epsilon'} \eta_0 \Rightarrow \epsilon'.S_g(y_0, \epsilon^{\epsilon'} \eta_0) \) with \( \epsilon' := \epsilon |\eta_0|_{\eta_0} \to 0 \).
4.2. Renormalized length. In this section we define the notion of a rescaled length $L_g(\gamma)$ for non-trapped geodesics $\gamma : (-\infty, \infty) \to M$. The first observation we make is that for $\gamma(t) = \pi_0(\varphi_t(z))$ with $z \in S^*M$, we have $\rho(\gamma(t)) \leq C(t)^{-1}$ when $t \to \pm \infty$ by Lemma 2.6. This means that the quantity $L_\lambda^g(\gamma) = \int_{-\infty}^{\infty} \lambda^t(\varphi_t(z))dt$ is finite if $\text{Re}(\lambda) > 1$. The goal is to extend this function holomorphically near $\lambda = 0$:

**Proposition 4.3.** For each complete non-trapped geodesic $\gamma$, the function $L_\lambda^g(\gamma)$ has a meromorphic extension to $\text{Re}(\lambda) > -1$ which is holomorphic at $\lambda = 0$. Furthermore, the renormalized length $L_g(\gamma) := L_0^g(\gamma)$ is also given by

$$L_g(\gamma) = \lim_{\epsilon \to 0} \left( \ell_g(\gamma \cap \{\rho > \epsilon\}) - 2\epsilon^{-1} \right)$$

where $\ell_g$ denotes the length with respect to $g$.

**Proof.** First, observe that, writing $\gamma(t) = (\rho(t), y(t))$ in the product decomposition near $\partial M$, we have by (2.15) and the bounds Lemma 2.6 that there are functions $a, b, c_j$ such that

$$\dot{\rho}(t) = a(t)\rho(t)^2, \quad \dot{a}(t) = b(t)\rho(t)^3, \quad \dot{y}_j(t) = c_j(t)\rho(t)^2,$$

with $\dot{c}_j(t) = O(\rho(t)^2)$, $|b(t)| = O(1)$, $\lim_{t \to \pm \infty} |a(t)| = 1$, $|a(t)| \leq 1$. (4.2)

We use the rescaled variable $\tau(t) := \int_{-\infty}^{t} \rho(\gamma(r))^2dr$ and denote its inverse by $t(\tau)$. Direct computation yields that for curves satisfying (4.2), we get in the rescaled variable that $\bar{\rho}(\tau) := \rho(\gamma(t(\tau)))$ has the following expression near $\tau_- := \lim_{t \to -\infty} t(\tau) = 0$:

$$\bar{\rho}(\tau) = \int_0^{\tau} a(t(r))dr \quad \text{and} \quad a(t(\tau)) = -1 + \int_0^{\tau} b(t(r))\bar{\rho}(r)dr = -1 + O(|\tau|^2)$$

which implies that

$$\bar{\rho}(\tau) = \tau + O(|\tau|^3). \quad (4.3)$$

The analogous asymptotic holds for $\tau$ near $\tau_+ = \lim_{t \to +\infty} t(\tau)$ with $\tau_+ - \tau$ as leading term. We can then write

$$L^\lambda_g(\gamma) := \int_{-\infty}^{\infty} \lambda^\tau(\gamma(s))ds = \int_0^{\tau_+} \bar{\rho}^{\lambda - 2}(\tau)d\tau. \quad (4.4)$$

Using the asymptotic (4.3), it is direct to see that for a fixed $\tau_0 \in (\tau_-, \tau_+)$

$$L^\lambda_g(\gamma) = \frac{\tau_0^{\lambda - 1}}{\lambda - 1} + \frac{(\tau_+ - \tau_0)^{\lambda - 1}}{\lambda - 1} + H_{\lambda, \tau_0}$$

with $H_{\lambda, \tau_0}$ holomorphic in $\text{Re}(\lambda) > -1$. To show that $L_g(\gamma)$ can be obtained as the asymptotic limit of the length, we first note that since $\gamma$ intersects $\{\rho = \epsilon\}$ transversally for $\epsilon > 0$ small enough. By (4.3), the equation $\bar{\rho}(\tau) = \epsilon$ has two solutions $\tau_\pm^\epsilon$ satisfying small such that $\tau_\pm^\epsilon = \tau_\pm \pm \epsilon + O(\epsilon^3)$. We obtain, using in addition (4.3),

$$\ell_g(\gamma \cap \{\rho > \epsilon\}) = \int_{\tau_\pm^\epsilon}^{\tau_+^\epsilon} \bar{\rho}^{-2}(\tau)d\tau = 2\epsilon^{-1} - \tau_0^{-1} - (\tau_+ - \tau_0)^{-1} + H_{0, \tau_0} + o(1)$$
as $\epsilon \to 0$, which proves the claim. □

We would like to investigate how the rescaled length depends on the boundary defining function. Suppose $\rho$ and $\tilde{\rho}$ are the boundary defining function for two coordinate system under which the scattering metric $g$ is in normal form. Then $\tilde{\rho} = \rho + \epsilon^2$ for some function $a \in C^\infty(M)$. Denote by $L_\rho$ and $\tilde{L}_\rho$ to be the rescaled length with respect to $\rho$ and $\tilde{\rho}$. Let $\gamma(t)$ be a unit speed geodesic whose trajectory is the same as the rescaled flow (with respect to $x$) $\varphi_\gamma(y_0, \eta_0)$ for some $(y_0, \eta_0) \in \partial_- S^* M$. By definition the rescaled length with respect to the boundary defining function $\tilde{\rho}$ is given by

$$\tilde{L}_\rho(\gamma) = \int_{-\infty}^\infty \tilde{\rho}^\lambda(\gamma(t))d\tau \mid_{\lambda=0} = \int_{-\infty}^\infty (\rho(\gamma(t)) + a(\gamma(t))\rho^2(\gamma(t)))^\lambda d\tau \mid_{\lambda=0}$$

We make a change of variable $\tau(t) = \int_{-\infty}^t \rho^{-2}(\gamma(t))dt$ as in (4.4), we get that

$$\tilde{L}_\rho(\gamma) = \int_{0}^{\tau(\infty)} \tilde{\rho}^\lambda(\tau)(1 + \tilde{\alpha}(\tau)\tilde{\rho}(\tau))d\tau \mid_{\lambda=0}$$

where $\tilde{\alpha} = a(\varphi_\gamma(y_0, \eta_0))$. Split the integral into three parts we obtain, for any $\tau_0 > 0$,

$$\tilde{L}_\rho(\gamma) = \int_{0}^{\tau_0} + \int_{\tau_0}^{\tau(\infty)-\tau_0} + \int_{\tau(\infty)-\tau_0}^{\tau(\infty)} \tilde{\rho}^\lambda(\tau)(1 + \tilde{\alpha}(\tau)\tilde{\rho}(\tau))d\tau \mid_{\lambda=0}$$

The middle integral extends trivially to $\lambda = 0$ to become $\int_{\tau_0}^{\tau(\infty)-\tau_0} \rho^{-2}(\tau)d\tau$. The integral along $(0, \tau_0)$ can be treated by writing

$$(1 + \tilde{\alpha}(\tau)\tilde{\rho}(\tau))^\lambda = 1 + \lambda\tilde{\alpha}(\tau)\tilde{\rho}(\tau) + \lambda\mathcal{O}(\tilde{\rho}^2(\tau))$$

so that

$$\int_{0}^{\tau_0} \tilde{\rho}^\lambda(\tau)(1 + \tilde{\alpha}(\tau)\tilde{\rho}(\tau))d\tau \mid_{\lambda=0} = \int_{0}^{\tau_0} \tilde{\rho}^{\lambda-2}(\tau) \mid_{\lambda=0} + \lambda \int_{0}^{\tau_0} \tilde{\rho}^{\lambda-1}(\tau)\tilde{\alpha}(\tau) \mid_{\lambda=0}.$$ 

Recall from (4.3) that $\tilde{\alpha}(\tau) = \mathcal{O}(1 + \mathcal{O}(\tau))$ which then gives

$$\int_{0}^{\tau_0} \tilde{\rho}^\lambda(\tau)(1 + \tilde{\alpha}(\tau)\tilde{\rho}(\tau))d\tau \mid_{\lambda=0} = \int_{0}^{\tau_0} \tilde{\rho}^{\lambda-2}(\tau) \mid_{\lambda=0} + \lambda \int_{0}^{\tau_0} \tau^{\lambda-1}(1 + \mathcal{O}(\tau))\tilde{\alpha}(\tau) \mid_{\lambda=0}.$$ 

It is easy to check that the meromorphic extension of the second integral is holomorphic at $\lambda = 0$ and is equal to $\tilde{\alpha}(0)$. Therefore,

$$\int_{0}^{\tau_0} \tilde{\rho}^\lambda(\tau)(1 + \tilde{\alpha}(\tau)\tilde{\rho}(\tau))d\tau \mid_{\lambda=0} = \int_{0}^{\tau_0} \tilde{\rho}^{\lambda-2}(\tau)d\tau \mid_{\lambda=0} + \tilde{\alpha}(0).$$

Similarly the integral on the interval $(\tau(\infty) - \tau_0, \tau(\infty))$ can be treated the same way to give

$$\int_{\tau(\infty)-\tau_0}^{\tau(\infty)} \tilde{\rho}^\lambda(\tau)(1 + \tilde{\alpha}(\tau)\tilde{\rho}(\tau))d\tau \mid_{\lambda=0} = \int_{\tau(\infty)-\tau_0}^{\tau(\infty)} \tilde{\rho}^{\lambda-2}(\tau)d\tau \mid_{\lambda=0} + \tilde{\alpha}(\tau(\infty)).$$
We see therefore that if $\gamma$ is a unit speed geodesic whose trajectory is given by $\varphi_s(p_0, \eta_0)$ with $(p_0, \eta_0) \in \partial_- S^* M$, then

$$\tilde{L}_g(\gamma) - L_g(\gamma) = a(p_0) + a(S_g(p_0, \eta_0)).$$

(4.5)

This allows us to obtain the following:

**Lemma 4.4.** Let $g_1$ and $g_2$ be asymptotically conic metrics with the same scattering maps so that $\rho_0^{-2} |d\rho_0|_{g_j} = 1 + \mathcal{O}(\rho_0^2)$, $j = 1, 2$ for some boundary defining function $\rho_0$. If the rescaled lengths of $g_1$ and $g_2$ agree with respect to a boundary defining function $\rho_0$ satisfying the conditions of Definition 2.1, then they agree for all such boundary defining functions.

### 4.3. Determination of boundary metric from $S_g$.

In this section, we will study some cases where the scattering map determines the metric at the boundary. This part only uses geodesics staying arbitrarily close to $\partial M$ (those corresponding to $|\eta|$ very large), and the arguments thus work whether there is trapping or not, since such geodesics are never trapped. First, we notice that if $c > 0$, we have for each $(y, \eta) \in T^* \partial M$

$$|\eta|_{c\eta_0, \phi^c_{\pi}}(y, \eta) = c|\eta|_{h_0, \phi^{h_0}}(y, \eta) = |\eta|_{h_0, \Phi_{\pi}}^{h_0}(y, \eta)$$

where $\phi^c_{\pi}$ is the Hamilton flow of $|\eta|_{c\eta_0}/2$. This shows that, using only (4.1), we can not determine the boundary metric $h_0$ but at best only multiple of $h_0$ can be recovered.

**Lemma 4.5.** Let $g, g'$ be two asymptotically conic metrics on $M$ in normal form so that, up to pulling-back by a diffeomorphism, $g = d\rho^2/\rho^4 + h_\rho/\rho^2$ and $g' = d\rho^2/\rho^4 + h_\rho'/\rho^2$. If the geodesic flow of the boundary metric $(\partial M, h_0)$ at time $\pi$, i.e. the map $\phi^{h_0}_\pi : S^* \partial M \to S^* \partial M$, is ergodic with respect to the Liouville measure, and if $S_g = S_{g'}$, then $h_0' = c h_0$ for some $c > 0$. This is in particular true if $h_0$ has negative curvature or more generally if it has mixing geodesic flow.

**Proof.** First, we notice that the normal form of Lemma 2.2 for $g$ and $g'$ (with associated boundary defining function $\rho$ and $\rho'$) allows to construct a diffeomorphism $\psi : \overline{M} \to \overline{M}$ so that, near $\partial \overline{M}$, $\psi_* \rho' = \rho$ and $\psi_* g' = \frac{d\rho^2}{\rho^4} + \frac{h'_\rho}{\rho^2}$. We can then use (4.1) to deduce that for all $(y, \eta) \in T^* \partial \overline{M}$ with $|\eta|_{h_0} = 1$,

$$|\eta|_{h_0', \phi^c_{\pi}}(y, \eta) = \phi_{\pi}(y, \eta),$$

where $\phi^c_{\pi}$ is the Hamilton flow of $\frac{1}{2} |\eta|_{h_0'}^2$. This implies that the map $(y, \eta) \mapsto |\eta|_{h_0'}(y)$ is invariant by $\phi_{\pi}$ on the unit tangent bundle $S^* \partial \overline{M}$ of $(\partial \overline{M}, h_0)$, it is thus constant by the ergodicity assumption on $\phi_{\pi}$. This shows that $h_0' = c h_0$ for some $c > 0$. If $\phi_t$ is mixing, one has (assuming Liouville measure $\mu$ on $S^* \partial \overline{M}$ has mass 1 for notational simplicity) for
As before, we use the normal form of Lemma 4.1. Proof.

Let \( y \) be a point for which \( \hat{\pi}(y,e) \) vanishes somewhere. Let \( \hat{f} = f - \frac{1}{\text{vol}_{h_0}(M)} \int_M f \text{dvol}_{h_0} \) so that \( h_0' = e^{2c} e^{2\hat{f}} h_0 \) with \( c = \frac{1}{\text{vol}_{h_0}(M)} \int_M f \text{dvol}_{h_0} \). Note that \( \hat{f} \) vanishes somewhere. Let \( y \in \partial M \) be a point such that \( \hat{f}(y) = 0 \). Then for all \( \eta \in T_y\partial M \) such that \( |\eta|_{h_0} = 1 = e^{-c}|\eta|_{h_0'} \), we have using (4.1)

\[
1 = |\phi_\pi(y,e^{-c}\eta)|_{h_0'} = e^{\hat{f}(\pi_0(\phi_\pi(y,\eta)))} |\phi_\pi(y,e^{-c}\eta)|_{h_0} \\
eq e^{\hat{f}(\pi_0(\phi_\pi(y,\eta)))} |\phi_\pi(y,\eta)|_{h_0} = e^{\hat{f}(\pi_0(\phi_\pi(y,\eta)))}.
\]

Therefore, we have

\[
\hat{f}(y) = 0 \implies \hat{f}(\pi_0(\phi_\pi(y,\eta))) = 0, \quad \forall \eta \in S^*_y\partial M
\] (4.6)

where \( S^*_\partial M \) is the unit tangent bundle for \( h_0 \). Let \( y_1 \in \partial M \) be a point such that \( \hat{f}(y_1) = 0 \) and \( s > 0 \) be small enough such that \( \pi + s \) is smaller than the injectivity radius of \( (\partial M, h_0) \).

If \( y_2 \) is a point for which \( d_{h_0}(y_1, y_2) = s \) then there exists a unit vector \( \eta_1 \in S^*_{y_1} \partial M \) such that \( \pi_0(\phi_s(y_1, \eta_1)) = y_2 \). This means that

\[
d_{h_0}(y_2, \pi_0(\phi_\pi(y_1, \eta_1))) = \pi - s, \quad d_{h_0}(y_2, \phi_{-\pi}(y_1, \eta_1)) = \pi + s.
\]
By continuity of the map \( y \mapsto d_{h_0}(y, y_2) \) on \( S_{y_0}(\pi) \), this means that there exists a point \( y_0 \in \partial M \) such that \( d_{h_0}(y_0, y_1) = d_{h_0}(y_0, y_2) = \pi \). Therefore by (4.6) we have that \( 0 = \tilde{f}(y_1) = \tilde{f}(y_0) = \tilde{f}(y_2) \). Since \( s > 0 \) is an arbitrarily small number this means that \( \tilde{f}(y) \) vanishes in a small neighbourhood of \( y_1 \). The proof is complete by an open-close argument. \( \square \)

4.4. **Determination of the metric jets at \( \partial M \) from \( S_g \).** We consider two asymptotically conic metrics \( g, g' \), and as before the normal form of Lemma 2.2 for \( g \) and \( g' \) (with associated boundary defining function \( \rho \) and \( \rho' \)) allows to construct a diffeomorphism \( \psi : M \to \overline{M} \) so that, near \( \partial M \), \( \psi \circ \rho = \rho \) and \( \psi \circ g' = \frac{d\rho}{d\rho'} + h'_{\rho} \). Up to replacing \( g' \) by \( \psi \circ g' \), we can thus assume that \( g, g' \) are both in normal form with the same boundary defining function \( \rho \), i.e.

\[
g = \frac{d\rho^2}{\rho^4} + h_{\rho}, \quad g' = \frac{d\rho}{\rho^2} + h'_{\rho}
\]

and we will assume that the boundary metrics coincide: \( h_0 = h'_0 \). We consider the Taylor expansion at \( \rho = 0 \) of the dual metrics \( h_{\rho}^{-1} \) and \( h'_{\rho}^{-1} \) to \( h_{\rho} \) and \( h'_{\rho} \):

\[
h_{\rho}^{-1} = \sum_{j=0}^{m} \rho^j h_j + \mathcal{O}(\rho^{m+1}), \quad h'_{\rho}^{-1} = \sum_{j=0}^{m} \rho^j h'_j + \mathcal{O}(\rho^{m+1})
\]

for \( m \in \mathbb{N} \), and we define for each \( j \)

\[
T_j := h_j - h'_j.
\]

Here we view \( h_j, h'_j, T_j \) as homogeneous functions of order 2 on \( T^* \partial M \) and, by abuse of notations, \( h_0 \) denotes both the metric on \( T\partial M \) and \( T^* \partial M \).

We shall apply perturbation theory in the regime \( \epsilon \to 0 \) to the system (2.35). Observe that for each \( \epsilon > 0 \) the ODE system (2.35) is given by a 1-parameter smooth family of vector fields \( \tilde{X}_\epsilon \) given by

\[
\tilde{X}_\epsilon := \tilde{\xi}_0 \partial_{\tilde{\rho}} - \tilde{\rho}(\tilde{\eta}_{\tilde{\rho}}) \partial_{\tilde{\xi}_0} + \frac{\epsilon \tilde{\rho}}{2}(\partial_{\tilde{\rho}}|\tilde{\eta}|_{\tilde{h}_{\tilde{\rho}}}^2)\partial_{\tilde{\xi}_0} + H_{\epsilon \tilde{\rho}}.
\]

The variables \( \tilde{\rho}, \tilde{\xi}_0, (\tilde{y}, \tilde{\eta}) \) belong to \([0,1] \times [-1,1] \times T^* \partial M \), and the vector field \( H_{\epsilon \tilde{\rho}} \) is the Hamilton vector field of \( (\tilde{y}, \tilde{\eta}) \mapsto \frac{1}{2} h_{\epsilon \tilde{\rho}}^{-1}(\tilde{\eta}, \tilde{\eta}) = \frac{1}{2} |\tilde{\eta}|_{\tilde{h}_{\tilde{\rho}}}^2 \) on \( T^* \partial M \) with respect to the Liouville symplectic form. In local coordinates one has

\[
H_{\rho} = \sum_{j,k=1}^{n-1} h_{\rho}^{jk} \tilde{k} \partial_{\tilde{y}_j} - \frac{1}{2} \sum_{j=1}^{n-1} \partial_{\tilde{y}_j} |\tilde{\eta}|_{\tilde{h}_{\tilde{\rho}}}^2 \partial_{\tilde{\xi}_0}.
\]

Remark that a priori the integral curves solving the ODE (2.35) belong to the hypersurface \( \{\tilde{\xi}_0^2 + \tilde{\rho}^2 |\tilde{\eta}|_{\tilde{h}_{\tilde{\rho}}}^2 = 1 \} \) but the expression defining the vector field \( \tilde{X}_\epsilon \) extends smoothly to a neighborhood of that hypersurface.
The vector fields $\tilde{X}_\epsilon$ and $\tilde{X}'_\epsilon$ corresponding to the metrics $g$ and $g'$ have a smooth expansion in powers of $\epsilon$

$$\tilde{X}_\epsilon = \sum_{j=0}^{m} c_j X_j + \mathcal{O}(\epsilon^{m+1}), \quad \tilde{X}'_\epsilon = \sum_{j=0}^{m} c'_j X'_j + \mathcal{O}(\epsilon^{m+1})$$

with $X_j, X'_j$ smooth vector fields. Under the assumption that $h_j = h'_j$ for $j \leq m - 1$, we get in addition that $X_j = X'_j$ for $j \leq m - 1$ and

$$X_m - X'_m = -(\frac{m}{2} + 1) \tilde{\rho}^{m+1} T_m(\tilde{\eta}, \tilde{\eta}) \partial_{\tilde{\xi}_0} + \tilde{\rho}^m (H_m - H'_m) \tag{4.9}$$

where $H_m$ (resp. $H'_m$) is the Hamilton vector field of $\frac{1}{2} h_m(\tilde{\eta}, \tilde{\eta})$ (resp. $\frac{1}{2} h_m(\tilde{\eta}, \tilde{\eta})$) on $T^* \partial M$, and $H_m - H'_m$ is the Hamilton field of $T_m(\tilde{\eta}, \tilde{\eta})$. We introduce on $T^* \partial M$ the coordinate system $(E, \tilde{y}, \tilde{\eta})$ where $E := |\tilde{\eta}|^2_{h_0}$ and $\tilde{\eta} = \tilde{\eta}/\sqrt{E}$.

We set $c_j(s)$ and $c'_j(s)$ to be the trajectories of $\tilde{X}_\epsilon$ and $\tilde{X}'_\epsilon$ respectively with the same initial condition

$$(\tilde{\rho}, \tilde{\xi}_0, E, \tilde{y}, \tilde{\eta}) = (0, 1, 1, y_0, \eta_0) \in [0, 1] \times [-1, 1] \times \mathbb{R}^+ \times S^* \partial M.$$ 

These solutions have a Taylor expansion in powers of $\epsilon$ of the form

$$c_j(s) = \sum_{j=1}^{m} c_j(s) + \mathcal{O}(\epsilon^{m+1}), \quad c'_j(s) = \sum_{j=1}^{m} c'_j(s) + \mathcal{O}(\epsilon^{m+1}).$$

**Lemma 4.7.** Assume that $g, g'$ are two asymptotically conic metrics written in normal form such that their boundary jets $h_j$ and $h'_j$ are equal up to $j \leq m - 1$ for some $m \geq 1$. If the scattering map $S_g$ and $S_{g'}$ agree, then for all $(y_0, \eta_0) \in S^* \partial M$:

$$\int_0^\pi \sin(s)^m H_0 T_m(e^{sH_0}(y_0, \eta_0)) ds = 0 \tag{4.10}$$

if $H_0$ is the Hamilton field of $\frac{1}{2} |\eta|^2_{h_0}$ on $T^* \partial M$. If in addition $H_0 T_m = 0$, then

$$\tilde{\rho}(c_m(\pi)) - \tilde{\rho}(c'_m(\pi)) = -(\frac{m}{2} + 1) \int_0^\pi \sin(s)^{m+2} T_m(e^{sH_0}(y_0, \eta_0)) ds, \tag{4.11}$$

$$\int_0^\pi \cos(s) \sin(s)^{m+1} T_m(e^{sH_0}(y_0, \eta_0)) ds = 0, \tag{4.12}$$

$$\int_0^\pi (\sin(s)^m - (\frac{m}{2} + 1) \sin(s)^{m+2}) T_m(e^{sH_0}(y_0, \eta_0)) ds = 0. \tag{4.13}$$

**Proof.** Observe that when $\epsilon = 0$, the vector field $\tilde{X}_0 = \tilde{X}_\epsilon|_{\epsilon = 0}$ is

$$\tilde{X}_0 = \tilde{\xi}_0 \partial_{\tilde{\rho}} - \tilde{\rho}|\tilde{\eta}|^2_{h_0} \partial_{\tilde{\xi}_0} + H_0 \tag{4.14}$$
and its integral curves with initial condition \((\tilde{\rho}, \tilde{\xi}_0, \tilde{y}, \tilde{\eta})|_{s=0} = (0, 1, y_0, \xi_0)\) when \(|\xi_0| \sim 1\) are given by
\[
c_0(s) = (\sin(s), \cos(s), e^{sH_0}(y_0, \eta_0)).
\]
Note that since \(H_0\) is the geodesic vector field on \(T^*\partial M\) for the metric \(h_0\),
\[
dE(H_0) = 0. \quad (4.15)
\]
In the coordinates \((E, \tilde{y}, \tilde{\eta})\), the first order linearization of \(\tilde{X}_0\) takes the convenient block form:
\[
d\tilde{X}_0 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-E & 0 & -\tilde{\rho} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \partial_E H_0 & d_{\tilde{y},0} H_0
\end{pmatrix}.
\]
(4.16)
The vector fields \(\tilde{X}_\epsilon\) and \(\tilde{X}'_\epsilon\) corresponding to the metrics \(g\) and \(g'\) have a smooth expansion in powers of \(\epsilon\):
\[
\tilde{X}_\epsilon = \sum_{j=0}^{m} \epsilon^j X_j + \mathcal{O}(\epsilon^{m+1}), \quad \tilde{X}'_\epsilon = \sum_{j=0}^{m} \epsilon^j X'_j + \mathcal{O}(\epsilon^{m+1})
\]
with \(X_j, X'_j\) smooth vector fields. Under the assumption that \(h_j = h'_j\) for \(j \leq m - 1\), we get in addition that \(X_j = X'_j\) for \(j \leq m - 1\) and
\[
X_m - X'_m = -(\frac{m}{2} + 1)\tilde{\rho}^{m+1}T_m(\tilde{\eta}, \tilde{\eta})\partial_{\tilde{x}_0} + \tilde{\rho}^{m}(H_m - H'_m) \quad (4.17)
\]
where \(H_m\) (resp. \(H'_m\)) is the Hamilton vector field of \(\frac{1}{2}h_m(\tilde{\eta}, \tilde{\eta})\) (resp. \(\frac{1}{2}h_m(\tilde{\eta}, \tilde{\eta})\)) on \(T^*\partial M\), and \(H_m - H'_m\) is the Hamilton field of \(T_m(\tilde{\eta}, \tilde{\eta})\).

We set \(c_\epsilon(s)\) and \(c'_\epsilon(s)\) to be the trajectories of \(\tilde{X}_\epsilon\) and \(\tilde{X}'_\epsilon\) respectively with the same initial condition
\[
(\tilde{\rho}_0, \tilde{\xi}_0, E, \tilde{y}, \tilde{\eta}) = (0, 1, 1, y_0, \eta_0) \in [0, 1] \times [-1, 1] \times \mathbb{R}^+ \times S^*\partial M.
\]
By Taylor expanding in powers of \(\epsilon\) the equations \(\tilde{X}_\epsilon(c_\epsilon(s)) = \dot{c}_\epsilon(s)\) and \(\tilde{X}'_\epsilon(c'_\epsilon(s)) = \dot{c}'_\epsilon(s)\), we obtain using \(X_j = X'_j\) for \(j \leq m - 1\)
\[
c_j(s) = c'_j(s) \text{ for } j \leq m - 1
\]
and the \(\epsilon^m\) term yields the equation
\[
\dot{c}_m(s) - \dot{c}'_m(s) = X_m(c_0(s)) - X'_m(c_0(s)) + dX_0(c_0(s))(c_m(s) - c'_m(s)).
\]
Writing \(e_m(s) := c_m(s) - c'_m(s)\) and using (4.17), we obtain the linear ODE system
\[
\dot{e}_m(s) = X_m(c_0(s)) - X'_m(c_0(s)) + dX_0(c_0(s)).e_m(s). \quad (4.18)
\]
To solve this equation, we introduce the matrix solution of
\[
\dot{R}(s) = d\tilde{X}_0(c_0(s))R(s), \quad R(0) = Id \quad (4.19)
\]
which can be solved explicitly (in the \((\tilde{\rho}, \tilde{\xi}_0, E, (\tilde{y}, \tilde{\eta}))\) coordinates) as

\[
R(s) = \begin{pmatrix}
\cos(s) & \sin(s) & a_1(s) & 0 \\
-\sin(s) & \cos(s) & a_2(s) & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & K(s) & L(s)
\end{pmatrix}.
\] (4.20)

The function \(L(s)\) solves the ODE \(\dot{L}(s) = dH_0(e^{sH_0}(y, \tilde{\eta}))L(s)\) with \(L(0) = \text{Id}\) on \(\{E = 1\} = S^*\partial M\), and \(a_1, a_2, K\) are smooth functions that do not play any role for later. The function \(e_m(s)\) is then given by

\[
e_m(s) = R(s) \int_0^s R(t)^{-1}(X_m(c_0(t)) - X'_m(c_0(t))) dt.
\] (4.21)

Let \(\tau_\epsilon\) and \(\tau'_\epsilon\) be the positive solutions of \(\tilde{\rho}(c_\epsilon(\tau_\epsilon)) = 0\) and \(\tilde{\rho}(c'_\epsilon(\tau'_\epsilon)) = 0\); we note that \(\tau_0 = \tau'_0 = \pi\). Expanding the equation in powers of \(\epsilon\), we obtain that \(\tau_\epsilon = \tau'_\epsilon + \epsilon^m \tau_m + O(\epsilon^{m+1})\)

and

\[
\tilde{\rho}(c_m(\pi)) - \tilde{\rho}(c'_m(\pi)) + (\tau_j - \tau'_j) d\tilde{\rho}.\tilde{\tilde{X}}_0(c_0(\pi)) = 0.
\]

Since \(\tilde{\xi}_0(c_0(\pi)) = -1\), this gives

\[
(\tau_m - \tau'_m) = \tilde{\rho}(c_m(\pi)) - \tilde{\rho}(c'_m(\pi)).
\] (4.22)

The identity \(S_g = S'_g\) implies that \(c_\epsilon(\tau_\epsilon) = c'_\epsilon(\tau'_\epsilon)\) and taking the \(\epsilon^m\) coefficient of the Taylor expansion of this equation, we deduce that

\[
0 = c_m(\pi) - c'_m(\pi) + \tilde{\tilde{X}}_0(c_0(\pi))(\tau_m - \tau'_m),
\]

which can be rewritten using (4.22) under the form

\[
e_m(\pi) + (\tilde{\rho}(c_m(\pi)) - \tilde{\rho}(c'_m(\pi)))(\tilde{\tilde{X}}_0(c_0(\pi))) = 0.
\] (4.23)

Combining (4.23) and (4.21) we deduce that

\[
R(\pi) \int_0^\pi R(t)^{-1}(X_m(c_0(t)) - X'_m(c_0(t))) dt + (\tilde{\rho}(c_m(\pi)) - \tilde{\rho}(c'_m(\pi)))\tilde{\tilde{X}}_0(c_0(\pi)) = 0.
\] (4.24)

We consider the \(\partial_E\) component of this equation: since \(dE.\tilde{\tilde{X}}_0 = 0\) and \(dE.(\tilde{\tilde{X}}_m - \tilde{\tilde{X}}'_m) = \tilde{\tilde{\rho}}^m dE.(H_m - H'_m) = -2\tilde{\tilde{\rho}}^m H_0 T_m\) by (4.17) (viewing \(T_m\) as a homogeneous function of degree 2 in \(\tilde{\eta}\)), this leads to

\[
0 = \int_0^\pi \sin(s)^m H_0 T_m(e^{sH_0}(y_0, \eta_0)) ds,
\]

which is (4.10).

Assume now that \(T_m\) is a Killing 2-tensor, i.e. that \(H_0 T_m = 0\). This implies

\[(H_m - H'_m) h_0 = -H_0 T_m = 0.\]
In other words, in the coordinate system \((\overline{\rho}, \xi_0, E, \tilde{y}, \tilde{\eta})\), the vector field \(\overline{X}_m - \overline{X}'_m\) given by (4.17) lies in the kernel of \(dE\). Identifying the \(\partial_{\overline{z}_0}\) component of (4.24), we get

\[
\int_0^\pi \left(\frac{m}{2} + 1\right) \sin^{m+1}(s) \cos(s) T_m(e^{sH_0}(y_0, \eta_0)) + b(s) \sin(s)^m H_0 T_m(e^{sH_0}(y_0, \eta_0)) ds = 0
\]

for some function \(b(s)\), implying equation (4.12) if \(H_0 T_m = 0\).

Identifying the \(\partial_\overline{y}\) component of (4.24) and using \(\tilde{X}_0(c_0(\pi)) = -\partial_\overline{y} + H_0\) and (4.17), we obtain (4.11) if \(H_0 T_m = 0\).

To obtain (4.13), we consider the \(H_0\) component of (4.24). Since \(d(e^{sH_0})_{y,\eta} H_0(y, \eta) = H_0(e^{sH_0}(y, \eta))\) for all \(y, \eta \in T^*\partial M\), the direction \(H_0\) is preserved by the linearisation of the geodesic flow \(H_0\) on \(\partial M\), and the same holds for \(\ker \lambda\) if \(\lambda := \sum_j \tilde{\eta}_j dy_j\) is the Liouville 1-form on \(T^*\partial M\). We then have \(L(s)H_0(c_0(0)) = H_0(c_0(s))\) and \(L(s)^{-1}H_0(c_0(s)) = H_0(c_0(0))\). There is a natural projection on this direction by applying \(\lambda\) to (4.24), this gives

\[
\overline{\rho}(c_0(\pi)) - \overline{\rho}(c'_0(\pi)) + \int_0^\pi \sin(s)^m \lambda((H_m - H'_m)(e^{sH_0}(y_0, \eta_0))) ds = 0
\]

But \(\lambda(H_m - H'_m) = T_m\) so we conclude that (4.13) holds. The proof is complete. \(\square\)

**Corollary 4.8.** Assume that \(g, g'\) are asymptotically conic metric with the same boundary metric \(h_0\) and assume that the geodesic flow \(e^{2\pi H_0}\) of \(h_0\) at time \(2\pi\) is ergodic on \(S^*\partial M\) (which is in particular true if \(h_0\) has negative curvature or more generally if it has mixing geodesic flow). If the scattering operator \(S_{g'} = S_g\) agree, then there is a smooth diffeomorphism \(\psi : \overline{M} \to \overline{M}\) fixing the boundary so that \(\psi^* g\) and \(g\) agree to infinite order at the boundary.

**Proof.** Assume \(g\) and \(g'\) agree to order \(m\), i.e. \(h_j = h'_j\) for all \(j < m\) using the notation (4.7). We will show that \(T_m = h_m - h'_m\) must vanish. We apply \(H_0\) to (4.10) and integrate by part (using \(H_0(f \circ e^{sH_0}) = \partial_s (f \circ e^{sH_0})\)) to get for all \((y, \eta) \in S^*\partial M\)

\[
0 = \int_0^\pi \sin(s)^m H_0^2 T_m(e^{sH_0}(y, \eta)) ds = -m \int_0^\pi \cos(s) \sin(s)^{m-1} H_0 T_m(e^{sH_0}(y, \eta)) ds.
\]

Applying again this method and using (4.10), we obtain for \(m > 1\) that

\[
0 = \int_0^\pi \cos(s)^2 \sin(s)^{m-2} H_0 T_m(e^{sH_0}(y, \eta)) ds.
\]

Using (4.10), this gives \(\int_0^\pi \sin(s)^{m-2} H_0 T_m(e^{sH_0}(y, \eta)) ds = 0\). We repeat the operation, and one gets when \(m\) is even that for all \((y, \eta) \in S^*\partial M\)

\[
0 = \int_0^\pi H_0 T_m(e^{sH_0}(y, \eta)) ds = T_m(e^{\pi H_0}(y, \eta)) - T_m(y, \eta).
\]
If $e^\pi H_0$ is ergodic, then necessarily $T_m = c h_0$ for some $c \in \mathbb{R}$ and $H_0 T_m = 0$. If $m$ is odd, we end up with $\int_0^\pi \sin(s) H_0 T_m (e^{s H_0} (y, \eta)) ds = 0$, which gives after another application of $H_0$ and two integrations by parts

$$ (H_0 T_m) \circ e^{s H_0} = -H_0 T_m. \quad (4.25) $$

Here again if $e^{2\pi H_0}$ is ergodic, as $(H_0 T_m) \circ e^{2\pi H_0} = H_0 T_m$, we conclude that $H_0 T_m = c' h_0$ for some $c' \in \mathbb{R}$. But integrating this identity over $S^* \partial M$ shows that $c' = 0$ and thus $T_m = c h_0$ for some $c$ by ergodicity of $e^{2\pi H_0}$. In all case we have $T_m = c h_0$ for some $c \in \mathbb{R}$.

Next, we apply the identity (4.3.13) and obtain

$$ c \int_0^\pi (\sin(s)^m - \frac{m}{2} + 1 \sin(s)^{m+2}) ds = 0. $$

Since $\int_0^\pi \sin(s)^{m+2} ds = \frac{m+1}{m+2} \int_0^\pi \sin(s)^m ds$, we conclude that $c = 0$ if $m \geq 2$. To deal with the case $m = 1$, we can change the normal form by using Lemma 2.2: this amounts to change the boundary defining function $\rho$ in that Lemma to $\hat{\rho} = \rho(1 + c_0 \rho + O(\rho^2))$ for some $c_0 \in \mathbb{R}$ so that, by (2.3), $g$ in this normal form becomes $\frac{ds^2}{s^2} + \frac{h_s}{s^2}$ with

$$ \hat{h}_s - h_s = 2s c_0 h_0 + O(s^3). $$

The change of normal form amounts to pulling-back $g$ by a smooth diffeomorphism $\psi$ on $\overline{M}$, fixing $\partial M$ pointwise and so that $\psi^* \hat{\rho} = \rho$. Since we know that $h_1 - h_1' = c h_0$ for some $c$, we can choose $c_0 = c/2$ so that in a normal form near $\partial M$, the expansion of $\psi^* g$ and $g'$ in normal form agree to order 2, i.e and we are reduced to the case $m = 2$ dealt with above (we also use that $S_{\psi^* g} = S_g$ for such diffeomorphism $\psi$ by the remark following Definition 4.1).

In a companion paper in collaboration with Mazzucchelli [GMT, Theorem 7.5], we prove a boundary determination similar to Corollary 4.8 from the scattering map for the case where the boundary is the canonical sphere with curvature $+1$.

4.5. Deformation Rigidity of Rescaled Lens Map. Assume now that, on $M$, we have a family of non-trapping asymptotically conic metrics of the form $g(s) = \frac{ds^2}{s^2} + \frac{h(s)}{s^2}$ near $\partial M$ for $s \in (-1, 2)$ (we set $g = g(0)$) such that $L_{g(s)} = L_g$ and $S_{g(s)} = S_g$ for each $s$. Furthermore we will assume that $h(s) = h(0) + O(\rho^\infty)$. We will denote by prime the derivative with respect to $s$ at $s = 0$. Observe that if $h(s)$ is a smooth family of tensors which are smooth up to the boundary, then $g'(\cdot, \cdot) \in \rho^\infty C^\infty(M; S^2(\nu T^* M))$ which is annihilated by $\nu_{\partial_0}$ for $\rho > 0$ small. Therefore, by the estimates of Lemma 2.6, for each geodesic $\gamma(t)$ of $g$, $|g'(\gamma, \gamma)| = O(t^{-\infty})$ as $|t| \to \infty$. The goal of this section is to show the

Proposition 4.9. If $S_{g(s)} = S_g$ and $L_{g(s)} = L_g$ for all $s$, then $g' = \partial_s g(s)|_{s=0}$ satisfies $I_2(g') = 0$ if $I_2$ is the X-ray transform associated to $g$. 

Proof. We denote the projection on $\overline{M}$ of the (non-trapped) integral curves of the rescaled geodesic vector fields $\overline{X}_s$ of $g(s)$ by $\overline{\gamma}_s(\tau, z)$, if $z \in \partial_- S^*M$ is the initial value at time $\tau = 0$ (these are simply the geodesics of $g(s)$ with time rescaled). Let $\tau_+(s, z)$ be the time to that $\overline{\gamma}_s^0(z) \in \partial_+ S^*M$ if $\overline{\gamma}_s(z)$ denotes the flow of $\overline{X}_s$ at time $\tau \geq 0$. We let $z' = S_g(z) = S_g(z) = \partial_+ S^*M$. By assumptions, we have $\overline{X}_s = \overline{X}_0 + O(s \rho^\infty)$ when viewed as smooth vector fields on $[0, \epsilon] \times [-1, 1] \times \partial_+ \partial M$, $\partial M$ and $\partial \partial M$, which implies that for each $N > 0$ and $\tau \geq 0$ small
\[
\overline{\gamma}_s(z) = \overline{\gamma}_s^0(z) + O(s \max_{u \leq \tau} \rho(\overline{\gamma}_s(u, z))^N), \quad \overline{\gamma}_s^0(z') = \overline{\gamma}_s^0(z') + O(s \max_{u \leq \tau} \rho(\overline{\gamma}_s(-u, z'))^N).
\]
Let $\overline{\gamma}(\tau, z) : = \partial_s \overline{\gamma}_s(\tau, z)|_{s = 0}$ and dot denotes the $\tau$ derivative. Then we obtain
\[
\overline{\gamma}(\tau, z) = O(\tau^N), \quad \overline{\gamma}_s(\tau, z) = O(\tau^N)
\]
uniformly for $\tau$ small, with the similar bounds for $\overline{\gamma}'(\tau, z')$ and $\overline{\gamma}''(\tau, z')$. We recall that $L_{g(s)}$ is obtain from the formula (4.4) in terms of the curve $\overline{\gamma}_s(\tau, z)$, and we shall vary (4.4) with respect to $s$. Let $\epsilon \in (0, \tau_+(s, z)/4)$ be small. Using that $\rho^4 g(s)(\overline{\gamma}(\tau, z), \overline{\gamma}(\tau, z)) = 1$ we compute for $\text{Re}(\lambda) > 1$ (using $\partial_s (\rho^2 g(s))|_{s = 0} = \rho^2 g' = h'$)
\[
\partial_s \left[ \int_0^\epsilon \rho(\overline{\gamma}_s(\tau, z))^\lambda - 2 \, d\tau \right]|_{s = 0} = \partial_s \left[ \int_0^\epsilon \rho(\overline{\gamma}_s(\tau, z))^\lambda + 2 g(s)(\overline{\gamma}_s(\tau, z), \overline{\gamma}_s(\tau, z)) \, d\tau \right]|_{s = 0}
\]
\[
= \lambda \int_0^\epsilon \rho(\overline{\gamma}(\tau, z))^\lambda - 3 \, d\rho(\overline{\gamma}(\tau, z)) \, d\tau
\]
\[
+ \int_0^\epsilon \rho(\overline{\gamma}(\tau, z))^\lambda \partial_s (\rho^2 g(\overline{\gamma}_s(\tau, z), \overline{\gamma}_s(\tau, z)))|_{s = 0} \, d\tau
\]
\[
+ \int_0^\epsilon \rho(\overline{\gamma}(\tau, z))^\lambda h'(\overline{\gamma}(\tau, z), \overline{\gamma}(\tau, z)) \, d\tau.
\]
Using that $h' \in \rho^\infty C^\infty(\overline{M}; S^2(T^*\overline{M}))$ and the bounds (4.26), we deduce that the integrals above all extend holomorphically to $\lambda \in \mathbb{C}$ since $\rho(\overline{\gamma}(\tau, z)) = O(\tau)$ as $\tau \to 0$, moreover this extension is uniformly $O(\epsilon^N)$ for $\lambda \in \mathbb{C}$ in compact sets. The same argument and estimates also applies to
\[
\partial_s|_{s = 0} \left[ \int_0^\epsilon \rho(\overline{\gamma}_s(\tau, z'))^\lambda - 2 \, d\tau \right].
\]
Next we compute (with $\tau_+(z) : = \tau_+([0, z])$)
\[
\partial_s \left[ \int_{\epsilon}^{\tau_+(s, z) - \epsilon} \rho(\overline{\gamma}_s(\tau, z))^\lambda - 2 \, d\tau \right]|_{s = 0} = \partial_s \left[ \int_{\epsilon}^{\tau_+(s, z) - \epsilon} \rho(\overline{\gamma}(\tau, z))^\lambda + 2 g(\overline{\gamma}_s(\tau, z), \overline{\gamma}_s(\tau, z)) \, d\tau \right]|_{s = 0}
\]
\[
+ \lambda \int_{\epsilon}^{\tau_+(z) - \epsilon} \rho(\overline{\gamma}(\tau, z))^\lambda - 3 \, d\rho(\overline{\gamma}(\tau, z)) \, d\tau
\]
\[
+ \int_{\epsilon}^{\tau_+(z) - \epsilon} \rho(\overline{\gamma}(\tau, z))^\lambda h'(\overline{\gamma}(\tau, z), \overline{\gamma}(\tau, z)) \, d\tau.
\]
where we denoted \( h' = \rho^2 g' \) (defined globally on \( \overline{M} \) and tangential to boundary near \( \partial \overline{M} \)).

We thus obtain

\[
\left( \frac{\partial_s}{\partial s} \left[ \int_0^\gamma \rho(\gamma_s(\tau, z))^{\lambda-2} d\tau \right] \right)_{s=0} = \partial_s \left[ \left. \int_\epsilon^{\tau_+ (s,z) - \epsilon} \rho(\gamma_+(\tau, z))^2 g(\gamma_+(\tau, z), \dot{\gamma}_+(\tau, z)) d\tau \right] \right]_{s=0} + \int_\epsilon^{\tau_+(s,z) - \epsilon} h'(\gamma_+(\tau, z), \dot{\gamma}_+(\tau, z)) d\tau + \mathcal{O}(\epsilon^N).
\]

As \( \epsilon \to 0 \), the second term converges to \( I_2(g') \) by Definition 3.1. Making the change of variable \( \tau \mapsto t(\tau, z) = \int_\epsilon^\tau \rho^{-2}(\gamma_+(\tau, z)) d\tau \) we have

\[
\int_\epsilon^{\tau_+ (s,z) - \epsilon} \rho(\gamma_+(\tau, z))^2 g(\gamma_+(\tau, z), \dot{\gamma}_+(\tau, z)) d\tau = \int_0^{t_s(\epsilon)} g(\gamma_+(t), \dot{\gamma}_+(t)) dt
\]

where \( \gamma_+(t) := \gamma_+(\tau, z) \) and \( t_s(\epsilon) := t(\tau_+(s, z) - \epsilon, z) \). This is the energy functional of the curve \( \gamma_+(t) \) with respect to \( g = g(0) \), and since \( \gamma_0(t) \) is a geodesic for \( g \), we get by the variation formula for the energy

\[
\partial_s \left[ \int_0^{\epsilon} \rho(\gamma_+(\tau, z))^{\lambda-2} d\tau \right] \bigg|_{s=0} = g(\gamma'(\epsilon, z), \dot{\gamma}_0(0) - g(\gamma'(\epsilon, z'), \dot{\gamma}(0)) = \mathcal{O}(\epsilon^N)
\]

for all \( N \). Letting \( \epsilon \to 0 \), we conclude that

\[
\partial_s I_{g(s)}(z) \bigg|_{s=0} = I_2(g')
\]

and that proves the Proposition in the non-trapping case. \( \square \)

**Proof of Theorem 1.2.** We now prove deformation rigidity following the argument of [GGSU]. First by Corollary 4.8 we can assume that \( g(s) = g(0) + \mathcal{O}(\rho^\infty) \). By Proposition 4.9 applied at the point \( s_0 \), we have that at each \( s_0 \), \( I_2(g(s_0))(g'(s_0)) = 0 \). By Proposition 1.1, there exists \( q(s) \in \rho^\infty C^\infty(M; ^{sc}T\ast{M}) \) such that \( g'(s) = Dq(s)/q(s) \), or equivalently

\[
g'(s) = L_{\frac{2}{3}q(s)} g(s), \tag{4.27}
\]

with \( q(s) \) the vector field dual to \( q(s) \) by \( g(s) \). Integrating the vector field produces the desired family of diffeomorphisms. \( \square \)

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