ON THE ASYMPTOTIC ACCURACY OF THE UNION BOUND

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ABSTRACT. A new lower bound on the error probability of maximum likelihood decoding of a binary code on a binary symmetric channel (BSC) was proved in Barg and McGregor (2004, cs.IT/0407011). It was observed in that paper that this bound leads to a new region of code rates in which the random coding exponent is asymptotically tight, giving a new region in which the reliability of the BSC is known exactly. The present paper explains a relation of these results to the union bound on the error probability.

1. INTRODUCTION

This is a companion paper to [6]. Suppose that a code $C$ is used on a BSC($p$) and decoded according to the maximum likelihood procedure. The error probability of decoding $P_e(C, p)$ can be estimated from above using the distance distribution of $C$ together with the union bound. As a general rule of thumb, this bound gives a good estimate of the error probability for low channel noise and is loose for high noise. Quantifying this heuristic is a difficult problem related not just to the distance distribution but also to structural properties of the code. Rigorous results are attainable only in the asymptotic setting when the code length $n$ tends to infinity (therefore in effect we will study families of codes rather than individual codes without always saying so). The inaccuracy of the union bound is related to the fact that intersections of half-spaces related to codewords other than the transmitted one, are counted more than once. It turns out that under certain conditions adding the measure of these intersections does not change the exponential asymptotics of the actual value of the error probability. The first result of this type was obtained by Gallager [13] who proved that for the ensemble of random codes and for rate $R < R_{\text{crit}}$, where $R_{\text{crit}}$ is the so-called critical rate of the channel (see below), the union bound gives the correct exponent of the average error probability for this ensemble (this quantity is different from the error probability of a typical random code, and both are different from the error probability of decoding for a typical linear code, see [5]). The proof in [13] is based on the fact that the error probability of decoding into a list of size two decreases exponentially faster than the estimate of $P_e(C, p)$ given by the union bound. A similar result can be proved for the ensemble of random linear codes using the ensemble-average coset weight distribution.

Subsequent results of this type are substantially more involved. They are related to universal bounds on the distance distribution of codes [16] and rely upon various methods of proving lower bounds on $P_e$ given the distance distribution. One such method, due to [15], was used in [16] to prove new estimates of the reliability function of the BSC [16] and the power-constrained AWGN channel [2]. Other methods known are due to [8,9] and [10]. The main question addressed by this analysis is the value of the code rate $R_*$ such that for rates $R \leq R_*$ the union bound can be claimed to be exponentially tight.

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The paper is organized as follows. In Sect. 2 we discuss the problem statement. Sect. 3 is devoted to general lower estimates of the error probability $P_e(C, p)$ given the distance distribution of the code $C$. In Sect. 4 we study the relation between the random coding exponent (the exponent of the error probability for a typical linear code) and the union bounds on this probability. Our context is that of geometry of decoding of random linear codes. We explain how different bounds on codes are related to the union bound on the error probability. Then in Sect. 5 we put everything together and show that a part of the random coding exponent just below the critical rate of the channel gives the actual value of the channel reliability. Some concluding remarks are presented in the final Section 6.

2. STATEMENT OF THE PROBLEM

We consider transmission with binary codes of length $n$ over a BSC with crossover probability $p$. Let $X = \{0, 1\}^n$ be the $n$-dimensional Hamming space. Let $C(n, M = 2^Rn) \subset X$ be a code of rate $R$ and let $x_i \in C$ be the transmitted vector. Under this condition the probability that a vector $y$ is received equals $P(y|x_i) = p^{n - |y + x_i|} (1 - p)^{|y + x_i|}$, where $|\cdot|$ is the Hamming weight.

Let $D(x)$ be the decision region of max–likelihood decoding for a codevector $x$. Given that $x_i$ is transmitted, the error probability of maximum likelihood decoding equals $P_e(x_i) = P_i(X \setminus D(x_i))$. The (average) error probability of decoding for the code $C$ equals

$$P_e(C, p) = \frac{1}{M} \sum_{i=1}^{M} P_e(x_i).$$

Computing this probability directly is prohibitively difficult in most nontrivial examples, therefore, there has been much interest in bounding it from both sides. As in [6], we focus on lower bounds on $P_e(C, p)$. For a given code sequence, define its error exponent as

$$E(p) = \lim_{n \to \infty} \frac{1}{n} \log \frac{1}{P_e(C, p)}.$$

We will also apply the results of the paper to the largest attainable exponent of the error probability of decoding defined as

$$E(R, p) = \lim_{n \to \infty} \frac{1}{n} \log \max_{C \subseteq X, R(C) = R} \frac{1}{P_e(C, p)}.$$

This quantity is also called the reliability function of the BSC.

Let us fix an arbitrary ordering of the codewords. Define the local distance distribution of the code $C$ with respect to the codeword $x_i$. This is a set of $n + 1$ numbers $B_{i0}, \ldots, B_{iw}, \ldots, B_{in}$, where $B_{iw}$ is the number of neighbors of $x_i$ in the code at distance $w$. Below we will mostly concentrate on lower bounds on the probability $P_e(x_i)$ given the local distance distribution. We will consider codes of exponentially growing size for which the error probability $P_e(C, p)$ declines exponentially fast. In this situation, given the average distance distribution of the code $C$, we can isolate a subcode of the same exponential order in which the local distance distribution for every codeword is asymptotically the same as the average distribution. Therefore, the bound $P_e(x_i)$ can be used to obtain a bound on $P_e(C, p)$ with the same exponent. This argument is presented in detail in [2, 9], so we will rely on it here without further discussion.
Notation. Let $C = \{x_1, \ldots, x_M\}$ be a code. For a subset $Y \subset X$ let

$$P_i(Y) = \sum_{y \in Y} P(y|x_i).$$

Let $\pi(w)$ be the error probability for two codewords at distance $w$, i.e., the probability of transmitting $x_i$ and decoding $x_j$, where $d(x_i, x_j) = w$ and $d(\cdot, \cdot)$ denotes the Hamming distance. By the union bound,

$$P_e(x_i) \leq \sum_{w=1}^{n} B^i_w \pi(w). \tag{1}$$

Letting $\pi(\omega n) = 2^{n A(\omega) + o(n)}$, we have $A(\omega) = \frac{1}{n} \log 2 \sqrt{p(1-p)}$. Then

$$\frac{1}{n} \log \frac{1}{P_e(x_i)} \geq -A(\omega) - \mu(\omega), \tag{2}$$

where $\mu(\omega) = \frac{1}{n} \log B^i_w$.

By $h(x)$ we denote the binary entropy function. We also use the divergence $D(x\|y) = h(x) + x \log y + (1-x) \log(1-y)$ (the logarithms are binary).

Bounds on codes. Define

$$\delta(R) = \limsup_{n \to \infty} \max_{C:|C|=2^{nR}} \frac{d(C)}{n}.$$

There exist code sequences (for instance, typical codes from the ensemble of random linear codes) whose relative distance approaches the quantity $\delta_{GV}(R) = h^{-1}(1-R)$ which is called the Gilbert-Varshamov (GV) distance. Thus,

$$\delta(R) \geq \delta_{GV}(R).$$

On the other hand, by the Elias bound,

$$\delta(R) \leq \delta_E(R) := 2 \delta_{GV}(R)(1 - \delta_{GV}(R)),$$

where the quantity $\delta_E(R)$ is sometimes called the Elias distance. A better upper estimate of $\delta(R)$ is provided by the JPL bound [17]:

$$\delta(R) \leq \bar{\delta} := \min_{0 \leq \alpha \leq \frac{1}{2}} G(\alpha, \tau)$$

where $G(\alpha, \tau) = 2^{\alpha(1-\alpha)-\tau(1-\tau)}$, and where $\tau$ satisfies $h(\tau) = 1-R - h(\alpha)$. For $0 \leq R \leq 0.305$ this bound takes a simpler form: $\bar{\delta} = \phi(h^{-1}(R))$, where $\phi(x) = \frac{1}{2} - \sqrt{x(1-x)}$. Denote by $\bar{R}(\bar{\delta})$ the inverse function of $\bar{\delta}(R)$ which is well defined because $\bar{\delta}$ is a monotone decreasing function of $R$.

3. Lower bounds on $P_e(C, p)$

In this section we review the known lower estimates of the probability $P_e(x_i)$ given the local distance distribution of the code. Let $C(i) = \{x \in C : d(x, x_i) = w\}$ for some fixed value of $w$. Given two different vectors $x_i, x_j \in C$, let

$$X_{ij} \subset \hat{X}_{ij} := \{y \in X : d(x_j, y) \leq d(x_i, y)\}$$

be an arbitrary subset.
3.1. **Kounias’ bound** [15]. This (obvious) bound states that
\[
P_e(x_i) \geq \sum_{j \in C(i)} \left\{ P_i(X_{ij}) - \sum_{k \in C(i) \setminus \{x_j\}} P_i(X_{ij} \cap X_{ik}) \right\}.
\]

In principle, here and hereafter \(C(i)\) can be an arbitrary subcode of \(C\) that does not contain \(x_i\).

3.2. **Burnashev’s method** [8] [7] [6]. This method was originally suggested for the AWGN channel and was adapted to the BSC in [6]. The error probability of decoding is estimated by carefully taking account of the probability of the subsets \(X_{ij} \cap X_{ik} \neq j\) for \(x_j, x_k \in C(i)\) and for some suitable definition of the subsets \(X_{ij}\). Let \(x_i, x_j, x_k \in C(i), d(x_i, x_j) = d(x_i, x_k) = \omega n, d(x_j, x_k) = \lambda n\). Let
\[
X_{ij} = \{ y \in X : d(x_i, y) = d(x_j, y) = \frac{\omega n}{2} + pn(1 - \omega) \}.
\]

Denote by \(B(\omega, \lambda)\) the negative exponent of the probability \(P_i(X_{ik} | X_{ij})\),

\[
B(\omega, \lambda) = -\omega - (1 - \omega) h(p) + \max_{\eta \in [\frac{\lambda p}{\omega}, \min(\frac{\lambda p}{\omega}, 1 - \omega)]} \left( \lambda h\left( \frac{2\eta}{\lambda} \right) + (\omega - \lambda/2) h\left( \frac{\omega - 2\eta}{2\omega - \lambda} \right) + (1 - \omega - \lambda/2) h\left( \frac{p(1 - \omega) - \eta}{1 - \omega - \lambda/2} \right) \right).
\]

The main result of [6] is given by

**Theorem 1.** Let \((C_i)_{i \geq 1}\) be a sequence of codes with rate \(R\), relative distance \(\delta\) and distance distribution satisfying \(B_{\omega n} \geq 2^{n\beta(\omega) - o(n)}\), where \(\beta(\omega) > 0\) for all \(\delta \leq \omega \leq 1\). The error probability of max-likelihood decoding of these codes satisfies \(P_e(C, p) \geq 2^{-En + o(n)}\), where

\[
P_e = \min_{\delta \leq \omega \leq 1} \max_{0 \leq \lambda \leq \omega} \left[ \text{max}\left( -\beta(\omega) - A(\omega), B(\omega, \lambda) - A(\lambda) \right) \right].
\]

As it turns out, for sufficiently low code rates \(R\), the first term under the maximum in (5) dominates the estimate. This shows that for code rates \(R \leq R_e\) the union bound is exponentially tight, where \(R_e\) is some value of the rate than depends on the distance distribution of the code and on the noise level in the channel. We will study the values of \(R_e\) in Sect.5 for the problem of bounding the channel reliability function.

3.3. **The method of Cohen and Merhav: de Caen’s inequality and its generalizations.** D. de Caen [11] suggested a new lower bound on the probability of a finite union of events. While an elementary result (essentially, Cauchy-Schwarz), this bound is sometimes the best among the inequalities of this type. De Caen’s inequality was used to compute lower bounds on the error probability via the distance distribution in [18] [14]. Cohen and Merhav [10] generalized de Caen’s inequality by introducing a weighting function that depends on the weight of the error vector and derived a lower bound on \(P_e(C, p)\) by optimizing on this function. Their result can be stated as follows.

**Theorem 2.** Let \(x_j, x_k \in C(i)\) be arbitrary vectors, \(j \neq k\). Then

\[
P_e(x_i) \geq \sum_{y \in X_{ij}} P(y | x_i) \eta(|y|)^2 \left( \frac{B_w^i}{\sum_{y \in X_{ij}} P(y | x_i) \eta(|y|)^2 + (B_w^i - 1) \sum_{y \in X_{ij} \cap X_{ik}} P(y | x_i) \eta^2(|y|)} \right)^2.
\]

where \(\eta(\cdot)\) is an arbitrary weight function.
Taking $C(i)$ to be the set of neighbors of $x_i$ at the minimum distance $d$, paper \cite{10} obtains a bound on $P_e(x_i)$ formed of two pieces. Similarly to Theorem \cite{11}, Theorem \cite{12} implies that for low rates the exponent of $P_e(x_i)$ asymptotically coincides with the exponent of the union bound. The condition on the code rate for the union bound on $P_e(x_i)$ to be (exponentially) tight proved in \cite{10} Prop. 5.3 can be written as follows:

\begin{equation}
B_d(i) \mathbb{P}_i(X_{ij} \cap X_{ik}) \lesssim \mathbb{P}_i(X_{ij}),
\end{equation}

where $x_j, x_k \in C(i)$ are arbitrary (different) codewords and $\lesssim$ refers to an inequality for the exponents.\footnote{Note that (7) relies on $X_{ij}$ instead of $\hat{X}_{ij}$. The reason for this is explained in the end of Sect.5 below.}

4. DECODING GEOMETRY OF RANDOM LINEAR CODES AND THE UNION BOUND

4.1. Decoding of random linear codes. Consider the ensemble of linear codes defined by $(n-k) \times n$ parity-check matrices with independent random components chosen with equal probability from \{0, 1\}. Let $R = k/n$. The ensemble-average weight distribution has the form $A_{\omega n} \approx 2^{n(R+1-h(\omega))}, \omega = 0, (1/n), \ldots, (n-1)/n, 1$. The minimum relative distance $\delta$ of a typical code from the ensemble approaches the Gilbert-Varshamov bound $\delta_{GV}(R) = h^{-1}(1-R)$. Computing the error probability $P_e(C)$ for such a code, we obtain an upper bound on the BSC reliability of the form $E(R, p) \geq E_0(R, p)$, where $E_0(R, p)$ is the “random coding exponent,”

\begin{equation}
E_0(R, p) = \begin{cases} 
-\delta_{GV}(R) \log_2 2\sqrt{p(1-p)} & 0 \leq R \leq R_x, \\
D(\rho_0\|p) + R_{\text{crit}} - R & R_x \leq R \leq R_{\text{crit}}, \\
D(\delta_{GV}(R)\|p) & R_{\text{crit}} \leq R \leq 1 - h(p),
\end{cases}
\end{equation}

where

\begin{align}
R_x &= 1 - h_2(\omega_0) \\
R_{\text{crit}} &= 1 - h_2(\rho_0)
\end{align}

\begin{equation}
\rho_0 = \frac{\sqrt{p}}{\sqrt{p} + \sqrt{1-p}}, \quad \omega_0 := 2\rho_0(1 - \rho_0) = \frac{2\sqrt{p(1-p)}}{1 + 2\sqrt{p(1-p)}}.
\end{equation}

This is a classical result of coding theory due to P. Elias and R. Gallager. Concise, self-contained proofs that are suitable for our context appear in \cite{5,4}.

A part of this result that is used below is related to the typical weight $\omega_{\text{typ}}$ of the incorrectly decoded codeword in the case of decoding error\footnote{The expression for $P_e(C)$ is a finite sum of binomial-type probabilities. Asymptotically for large $n$ it is dominated by weights of incorrectly decoded codewords in a small segment around some value, which is called a typical weight of incorrect codewords.}. For the cases (a)-(c) of (8) the values of $\omega_{\text{typ}}$ are as follows \cite{5}:

(a) $\omega_{\text{typ}} = \delta_{GV}(R)$

(b) $\omega_{\text{typ}} = \omega_0$

(c) $\omega_{\text{typ}} = \delta_{E}(R)$.

In Fig. 1 the bound $E_0(R, p)$ is shown together with the values $\omega_{\text{typ}}$ as a function of the code rate $R$. As $R$ varies between $R_x$ and $R_{\text{crit}}$, the value of $\omega_{\text{typ}} = \omega_0$ changes its location with respect to $\omega_0$.
the minimum distance of the code, moving from $\delta_{GV}(R)$ to $\delta_{E}(R)$. We note that $\omega_{typ} < \delta_{E}(R)$ as long as $R \leq R_{\text{crit}}$.

4.2. Weight distributions and the union bound on $P_e(x_i)$. It is conjectured that $E_0(R, p)$ gives an exact value of $E(R, p)$ for all $R \in [0, 1 - h(p)]$. In an attempt to prove this, various upper bounds on $E(R, p)$ were established. The tightest known upper bounds are proved by showing that an appropriate version of the union bound in effect is tight (entails no loss of accuracy of the estimate for large $n$).

The weight profile (the exponent of the weight distribution) of a typical random linear code of rate $R$ has the form $R + 1 - h(\omega), \omega \geq \delta_{GV}(R)$. As explained above, only the weights in the region $\delta_{GV}(R) \leq \omega \leq \delta_{E}(R)$ are relevant for the random coding exponent. Let us assume for a moment that

(A) for any code $C$, a given codeword $x_i$ has at least $2^{n(R + 1 - h(\omega))}$ codeword neighbors at relative distance $\omega = g(R)$ were $g$ is some monotone decreasing function;

(B) the union bound gives a tight value of the error exponent in the estimates (5) and/or (6) for some region of low rates, to be specified later.

By (B), we can write an asymptotic estimate of $P_e(x_i)$ using (1) in the reverse direction. Substituting the distance distribution from (A) we would be able to state an upper bound on $E(R, p)$ of the form

\[(12) \quad E(R, p) \leq -(R - 1 + h(g(R))) - A(g(R)).\]
Figure 2. Bounds on the error exponent for the BSC with $p = 0.08$. In the interval $R_1 \leq R \leq R_{\text{crit}}$ the random coding bound $E_0(R, p)$ is tight. A discrepancy between upper and lower bounds on $E(R, p)$ remains for rates in the interval $0 < R < R_1$.

For instance, if (A) were true for $\omega = \delta_{\text{GV}}(R)$ then we would obtain (8a) as an upper bound on $E(R, p)$ (this is a very strong assumption because it implies that the GV bound is tight). In this case $g(R) = \delta_{\text{GV}}(R)$.

We will assume that $g(x)$ is such that the function $-(R - 1 + h(g(R))) - A(g(R))$ is $\cup$-convex (this will be the case in all our examples).

Two important remarks should be made with respect to this argument and Fig. 2. We formulate the first one as

**Lemma 3.** The function on the right-hand side of (12) is tangent to the straight line $D(p_0\|p) + R_{\text{crit}} - R$ at the point $R_1 = g^{-1}(\omega_{\text{typ}})$.

Thus if $g^{-1}(\omega_{\text{typ}}) < R_{\text{crit}}$, the random coding bound $E_0(R, p)$ of (8) gives an exact answer for the channel reliability $E(R, p)$ at the point $R = R_1$. Furthermore, together with the straight-line principle of [19] this implies that $E(R, p) = E_0(R, p)$ for all rates $R_1 \leq R \leq R_{\text{crit}}$. A result of this type will be proved in the next section.

Secondly, if $\omega_{\text{typ}} = \delta_{E}(R)$ then it turns out that almost every error vector from the sphere of typical errors leads to a decoding error (see e.g., [4]). Therefore, for $R \geq R_{\text{crit}}$ instead of (12) we compute a “union bound” of a different type, namely, the probability of an error vector of weight $\delta_{\text{GV}}(R)$ occurring in the channel. This argument is not related to the above assumptions and gives (8c) as an unconditional upper bound on $E(R, p)$ (the sphere-packing bound).

5. RELIABILITY FUNCTION OF THE BSC

In this section we study an application of the above ideas to bounds on the function $E(R, p)$. Recently linear programming was used to derive bounds on the distance distribution of codes [16, 11]. In particular, paper [16] proves the following lower bound on the distance distribution of an arbitrary code family of rate $R$. 

**Theorem 4.** \[16\] For any family of codes of sufficiently large length and rate \(R\) and any \(\alpha \in [0, 1/2]\) there exists a value \(\omega, 0 \leq \omega \leq G(\alpha, \tau)\) such that \(n^{-1} \log B_{\omega n} \geq \mu(R, \alpha, \omega) - o(1)\), where

\[
\mu(R, \alpha, \omega) = R - 1 + h(\tau) + 2h(\alpha) - 2q(\alpha, \tau, \omega/2) - \omega - (1 - \omega)h\left(\frac{\alpha - \omega/2}{1 - \omega}\right),
\]

\(\tau = h^{-1}(h(\alpha) - 1 + R)\), and where

\[
q(\alpha, \tau, \omega) = h(\tau) + \int_{0}^{\omega} dy \log(P + \sqrt{P^2 - 4Qy^2})/2Q,
\]

where \(P = \alpha(1 - \alpha) - \tau(1 - \tau) - y(1 - 2y)\), \(Q = (\alpha - y)(1 - \alpha - y)\), is the exponent of the Hahn polynomial \(H^\omega_{\alpha n}(\omega n)\).

This theorem was used in \[16\] to tighten the upper bound for \(E(R, p)\) for low rates, giving implicitly a condition for the union bound to be tight for low rates. Using this result together with Theorem \[14\] we observe that there exists a value of the rate \(R = R_0\), a function of \(p\), such that for \(0 \leq R \leq R_0\), the first term in the maximum in \(\phi\) is greater than the second one. The following statement was proved in \[16\].

**Theorem 5.** Let \(\bar{R}(2\rho_0(1 - \rho_0)) \leq R_0\), where \(\rho_0\) is defined in \(\[17\]\). Then

\[
E(R, p) \leq -A(\bar{\delta}) - R + 1 - h(\bar{\delta}) \quad 0 \leq R \leq R_0
\]

(13)

\[
E(R, p) \leq \max_{0 \leq \lambda \leq \bar{\delta}} \max_{0 \leq \omega \leq \delta} B(\omega, \lambda) - A(\lambda) \quad R_0 \leq R
\]

(14)

Explicit optimization in \(\[14\]\) is difficult because of the cubic condition on the optimal value of the parameter \(\eta\) in \(\[14\]\) and for other similar reasons; however, the bound can be computed for a given \(p\). Observe that by \(\[13\]\), for \(R < R_*\) the BSC reliability \(E(R, p)\) is estimated from above by the exponent of the union bound. From Lemma \[3\] the bound \(\[13\]\) is tangent on the straight-line part of \(E_0(R, p)\).

It is clear that \(R_1 < R_{\text{crit}}\) simply because \(\bar{\delta}(R) < \delta_E(R)\), i.e., the JPL function is less than the Elias distance. Observe that for \(p \geq 0.04\), the value \(R_1 \leq 0.287\) (and for \(p \geq 0.05\) even \(R_{\text{crit}} \leq 0.305\)). For rates in this region we have \(\bar{\delta} = \phi(h^{-1}(R))\), and then the point of tangency is given by \(R_1 = \phi(h(\omega_{\text{yp}}))\) (since \(\phi = \phi^{-1}\)).

Now to ensure that \(E(R_1, p) = E_0(R, p)\) it remains to show that the union bound exponent can still be claimed an upper bound on \(E(R, p)\) for \(R = R_1\), or that \(R_1 \leq R_*\). This can be verified by computing the bounds \(\[13\]-\[14\]\) and the value of \(R_*\). The computation leads to the following result (see also Fig.\[2\].

**Theorem 6.** Let \(p, 0.046 \leq p < 1/2\) be the channel transition probability. Then the channel reliability \(E(R, p)\) equals the random coding exponent \(E_0(R, p)\) for \(R_1 \leq R \leq R_{\text{crit}}\).

Previously the bound \(E_0(R, p)\) was known to be tight only for the rates \(R \in [R_{\text{crit}}, 1 - h(p)]\) \(\[12\]\).

Given the rate \(R\) and the distance distribution of the code, the value of \(R_*\) is determined uniquely. Based on the computational evidence, the union bound can be claimed exponentially tight (under the approach of this section) if the code rate satisfies \(\[7\]\). Observe that Theorems \(\[11\]\) lead to the same result because of our particular choice of the subsets \(X_{ij}\). Another possibility is to take \(\tilde{X}_{ij} = \{y \in X : d(x_j, y) \leq d(x_i, y)\}\) in which case these theorems would give a weaker result than \(\[10\]\) (this is the essence of the discussion in \(\[10\]\) p.301)). The region \(\tilde{X}_{ij}\) in Theorem \(\[2\]\) is also suboptimal, but the correction term \(\eta(\cdot)\) performs a transformation to the optimal region \(X_{ij}\).
6. Concluding Remarks, Conjectures

The method of this paper and [6] still stops short of proving that $E(R_0, p)$ is tight for all rates $R_x \leq R \leq R_{\text{crit}}$. The crucial elements of the argument made above are (a) the fact that the JPL bound $\delta(R)$ is better than the Elias bound and (b) the straight-line principle of [19]. Further progress can be related either to an improvement of bounds on codes, which at present looks very difficult, or to new ideas for extending a known bound on $E(R, p)$ for low rates.

We remark that the arguments and results similar to those obtained here for the BSC can be also obtained for a power-constrained AWGN channel. They are briefly discussed in [6]. The geometric picture that describes the relation of the random coding bound and the union bounds in this case is qualitatively the same as that of Sections 4, 5.

If the GV bound is tight, then so is the bound $E_0(R, p)$ on the channel reliability. The converse claim, i.e., the implications of the (putative) tightness of $E_0(R, p)$ for bounds on codes, is not so obvious. To be more precise, the following question seems open.

Open problem 1. Assuming that the bound (8b) gives an exact value of $E(R, p)$ for all $R$ in the interval $(R_x, R_{\text{crit}})$, is it possible (with the current knowledge) that there exists a sequence of codes whose minimum distance asymptotically exceeds the GV distance?

This is certainly not true for code sequences in which the number of codewords of minimum weight grows subexponentially in $n$; however, there exist codes with exponentially many minimum-weight vectors [3]. A weight distribution that might support a positive answer to the above open problem is of the form

\[ B_{\omega n} = \begin{cases} 0 & 0 < \omega < \delta \\ 2^{n\alpha(\omega)} & \delta < \omega, \end{cases} \]

where $\delta \geq \delta_{\text{GV}}$ and $\alpha(\omega) > R + 1 - h(\omega)$. Note that the weight distribution of the code family whose existence in proved in [3] is not of this form and its distance is less that $\delta_{\text{GV}}$. If the answer to this problem is positive, this should not be very difficult.

Given that an upper bound on $E(R, p)$ for some rate $R_0$, the straight-line bound of [19] gives a method of obtaining upper bounds on $E(R, p)$ for rates $R \geq R_0$.

Open problem 2. Given an upper bound on $E(R, p)$ for some rate $R = R_0$ find a way of obtaining upper bounds for $R \leq R_0$.

This problem presently seems difficult.

So far the results for the reliability of the BSC and general discrete memoryless channels (DMCs) have been similar. However, apart from straightforward generalizations, it is not clear how to extend the result of this paper to DMCs. Therefore, let is formulate

Open problem 3. Prove that the random coding bound on the reliability function of a DMC is tight for rates immediately below $R_{\text{crit}}$.

Given the similarity of results for a particular distance distribution of Sect.5 obtained by the methods of [11, 10] and [9, 6], another open question that arises is whether the lower bounds of [9] and [10] are generally related. If this is indeed the case, then the approach of [10] would give a more direct alternative to the successive refinement of the estimate of $P_e(x_i)$ performed in [6]. This would also have consequences in the more general context of hypothesis testing [8].
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