VARIATIONS ON VAN KAMPEN’S METHOD

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ABSTRACT. We give a detailed account of the classical Van Kampen method for computing presentations of fundamental groups of complements of complex algebraic curves, and of a variant of this method, working with arbitrary projections (even with vertical asymptotes).

Introduction In the 1930’s, Van Kampen described a general technique for computing presentations of fundamental groups of complements of complex algebraic curves. Though Van Kampen’s original approach was essentially valid, some technical details were not entirely clear and were later reformulated in more modern and rigorous terms (see for example the account by Chéniot, [C]).

It is possible to transform Van Kampen’s “method” into an entirely constructive algorithm. To my knowledge, two implementations have been realized, one by Jorge Carmona, the other by Jean Michel and myself (GAP package VKCURVE, [VK]).

The goal of the present note is to clarify some aspects which are usually neglected but must be addressed to obtain an efficient implementation. Also, the “Van Kampen’s method” explained here differs from the classical one, which assumes the choice of a “generic” projection: our variant method works with an arbitrary projection. The reason for what may appear to be a superfluous sophistication (since “generic” projections always exists and are easy to find) is that working with a non-generic projection may be computationally more efficient. The variant method explained here is implemented in VKCURVE, and has already been used to find previously unknown presentations.

Let \( P \in \mathbb{C}[X,Y] \). The equation \( P(X,Y) = 0 \) defines an algebraic curve \( C \in \mathbb{C}^2 \). Our goal is to find a presentation for the fundamental group of \( \mathbb{C}^2 - C \) (the method can be adapted to work with projective curves, as it is briefly mentioned at the end of section 2). Without loss of generality, we may (and will) assume that \( P \) is quadratfrei. View \( P \) as a polynomial in \( X \) depending on the parameter \( Y \):

\[
P = \alpha_0(Y)X^d + \alpha_1(Y)X^{d-1} + \cdots + \alpha_d(Y),
\]

with \( \alpha_0(Y) \neq 0 \). To study \( \mathbb{C}^2 - C \), we decompose it according to the fibers of the projection \( \mathbb{C}^2 \to \mathbb{C}, (x,y) \mapsto y \). Up to changing the variables, one could assume that \( d \) equals the total degree of \( P \) (the projection is then said to be “generic”); however, for reasons detailed
below, we do not make this assumption that we have a generic projection. For all but a finite number of exceptional values for \( y_0 \), the equation \( P(X, y_0) = 0 \) has exactly \( d \) distinct solutions in \( X \). The main idea in Van Kampen’s method is that to compute a presentation, it essentially suffices to be able to track these \( d \) solutions when the parameter \( Y \) varies along certain loops (around the exceptional values). These \( d \) solutions form certain braids with \( d \) strings, called monodromy braids.

In complexity terms, the most expensive part of the algorithm is the computation of the monodromy braids. Since the computation time increases with the number of strings, it is tempting to keep it reduced by working with non-generic projections, with a smaller number of strings. However, these projections frequently involve vertical asymptotes (to us, vertical lines are lines with equations of the form \( Y = y_0 \), thus our \( X \)-axis is vertical, and our \( Y \)-axis is horizontal; vertical asymptotes can only appear when \( \alpha_0 \) is not a scalar). Classical Van Kampen method is not adapted to deal with vertical asymptotes but, as we explain in sections 3 and 4, some corrections can be introduced to make it work.

The structure of this note is as follows: after some preliminaries in section 1, we describe the main steps of the algorithm in section 2. Sections 3 and 4 are devoted to two technical points – they contain the only original material of this note.

This note does not cover all aspects of the effective implementation of Van Kampen’s method. The most serious gap is that we do not explain how to perform step \( a \) of Procedure 12; this, with other features of VKCURVE, will be described in a forthcoming joint paper with Jean Michel.

Note. A modification of Van Kampen’s method dealing with vertical asymptotes is also proposed in [ACCLM]. Our approach is probably more or less equivalent to theirs, but formulated in a way which is closer to a fully automated procedure. The author thanks Jorge Carmona for useful electronic discussions.

1. Topological preliminaries

Though we will use them only in dimension 2, we formulate the results of this section in arbitrary dimension, since it doesn’t cost more; for the same reason, we work in the projective space.

Let \( n \) be a positive integer, let \( \mathcal{H} \) be an algebraic hypersurface in the complex projective space \( \mathbb{P}^n \).

Let \( x_0 \) be a basepoint in \( \mathbb{P}^n - \mathcal{H} \). We are interested in generating \( \pi_1(\mathbb{P}^n - \mathcal{H}, x_0) \). There is a natural class of elements of this groups, the meridians (also called generators-of-the-monodromy), in which to pick the generators. To construct a meridian, one needs to choose:

1) a smooth point \( x \in \mathcal{H} \);
2) a path from \( x_0 \) to \( x \), intersecting \( \mathcal{H} \) only at the endpoint \( x \).

To these choices, one associates a loop as follows: start from \( x_0 \), follow \( \gamma \); just before reaching \( x \), make a positive full turn around \( \mathcal{H} \) (by the local inversion theorem, in the neighbourhood of a smooth point, the complement of an hypersurface “looks like” the complement of an hyperplane, and the local fundamental group is isomorphic to \( \mathbb{Z} \) – the standard orientation of \( \mathbb{C} \) telling which generator is the positive one); return to \( x_0 \) following \( \gamma \) backwards.

The reader should convince himself that this makes sense and that we have defined an element \( s_\gamma \in \pi_1(\mathbb{P}^n - \mathcal{H}, x_0) \). Of course, different choices may yield the same element. However, although the element \( x \in \mathcal{H} \) smooth is not uniquely determined by \( s_\gamma \), it should be noted that it belongs to exactly one of irreducible components of \( \mathcal{H} \) (since intersections of components belong to the singular locus), and that this component \( D \) is uniquely determined by \( s_\gamma \) (to see it, integrate over \( s_\gamma \) the inverses of defining polynomials). We will say that \( s_\gamma \) is a meridian of \( \mathcal{H} \) around \( D \). It is important to note this notion depends not only on \( D \), but also on the remaining components of \( \mathcal{H} \), since these have to be avoided when choosing the path from \( x_0 \) to a point of \( D \).

We will get rid of all topological technicalities by admitting without proof the following folk-lemma:

**Lemma 1.** Let \( D \) be an irreducible component of \( \mathcal{H} \). We have \( \mathcal{H} = \mathcal{H}' \cup D \), with \( D \not\subset \mathcal{H}' \), where \( \mathcal{H}' \) is the union of the remaining components.

(i) The meridians of \( \mathcal{H} \) around \( D \) form a single conjugacy class.

(ii) Consider the embedding \( \mathbb{P}^n - \mathcal{H} \hookrightarrow \mathbb{P}^n - \mathcal{H}' \), and the associated morphism \( \phi \) between fundamental groups. Then \( \phi \) is surjective, and its kernel is generated by the meridians of \( \mathcal{H} \) around \( D \).

One may already observe that this lemma has a meaning in terms of generators and relations. First, an induction from (ii) proves that \( \pi_1(\mathbb{P}^n - \mathcal{H}) \) is generated by meridians. Secondly, assume that we already know a presentation of \( \pi_1(\mathbb{P}^n - \mathcal{H}) \), with generators corresponding to meridians; then, by simply forgetting (as generators, and in the relations) those generators which are meridians around \( D \), we obtain a presentation of \( \pi_1(\mathbb{P}^n - \mathcal{H}') \). We will use this later.

One may also note that the complement of \( m \) points in \( \mathbb{C} \) is relevant to the above discussion (with \( n = 1 \)). Since we will need it the next sections, let us fix the following *ad hoc* terminology:

**Definition 2.** Let \( x_1, \ldots, x_m \) be \( m \) distinct points in \( \mathbb{C} \). Let \( x_0 \in \mathbb{C} - \{x_1, \ldots, x_m\} \). We define a planar tree connecting \( x_0 \) to \( \{x_1, \ldots, x_m\} \) to be a subset of \( \mathbb{C} \) homeomorphic to a tree, containing \( \{x_0, \ldots, x_m\} \) and such that \( x_1, \ldots, x_m \) are leaves.

Assume we have fixed such a planar tree \( T \). For each \( i \in \{1, \ldots, m\} \), there is (up to reparametrization) a unique path in \( T \) connecting \( x_0 \) to
by definition, \( \{s_1, \ldots, s_m\} \).

It is clear, in the above definition, that the \( s_i \)'s realize an explicit isomorphism between \( \pi_1(\mathbb{C} - \{x_1, \ldots, x_m\}, x_0) \) and the free group on \( m \) generators.

2. THE MAIN IDEA: A FIBRATION ARGUMENT

After these preliminaries, we move to the central matter. Let

\[
P = \alpha_0(Y)X^d + \alpha_1(Y)X^{d-1} + \cdots + \alpha_d(Y)
\]

be as in the introduction. For a "generic" choice of \( y_0 \), the equation \( P(X, y_0) = 0 \) has \( d \) solutions in \( X \). Here "generic" means that \( y_0 \) should not be a zero of the discriminant \( \Delta \in \mathbb{C}[Y] \) of \( P \).

Let \( y_1, \ldots, y_r \) be the distinct roots of \( \Delta \). Let \( B := \mathbb{C} - \{y_1, \ldots, y_r\} \). Let \( E := \{(x, y) \in \mathbb{C} \times B | P(x, y) \neq 0\} \). Using the classical fact that the roots of a polynomial are continuous functions of its coefficients, we see that the map \( p : E \to B, (x, y) \mapsto y \) is a locally trivial fibration, with fibers homeomorphic to the complement of \( d \) points in \( \mathbb{C} \).

Choose a basepoint \( y_0 \in B \). Let \( F \) be the fiber over \( y_0 \). In \( F \), choose a basepoint \( x_0 \) (this choice is not innocent – we will return to this in the next section). Since \( F \) is connected and \( \pi_2(B) = 1 \), the fibration exact sequence basically amounts to

\[
1 \to \pi_1(F, x_0) \to \pi_1(E, (x_0, y_0)) \to \pi_1(B, y_0) \to 1.
\]

Both \( \pi_1(F, x_0) \) and \( \pi_1(B, y_0) \) are isomorphic to free groups, of ranks respectively \( d \) and \( r \). Any short exact sequence landing on a free group is split, so \( \pi_1(E, (x_0, y_0)) \) must be a semi-direct product \( F_d \ltimes F_r \).

Let us be more specific. Choose a planar tree connecting \( x_0 \) to the roots of \( P(X, y_0) \). This spider defines free generators \( f_1, \ldots, f_d \) of \( \pi_1(F, x_0) \), each of them being a meridian around one of the roots.

Similarly, choose a planar tree connecting \( y_0 \) to the roots of \( \Delta \). We obtain \( r \) meridians \( g_1, \ldots, g_r \) generating freely \( \pi_1(B, y_0) \).

Whichever way we choose to lift \( g_1, \ldots, g_r \) to elements \( \tilde{g}_1, \ldots, \tilde{g}_r \in \pi_1(E, (x_0, y_0)) \), we have a semi-direct product structure. The conjugacy action \( f_i \mapsto \tilde{g}_j^{-1} f_i \tilde{g}_j \) defines an monodromy automorphism \( \phi_j \in \text{Aut}(F_d) \) (we will explain in section 4 how to compute these automorphisms).

We obviously have the presentation

\[
\pi_1(E, (x_0, y_0)) \simeq \langle f_1, \ldots, f_d, \tilde{g}_1, \ldots, \tilde{g}_r | \tilde{g}_j^{-1} f_i \tilde{g}_j = \phi_j(f_i) \rangle
\]

(where by \( \phi_j(f_i) \) we mean the corresponding word in the \( f_1, \ldots, f_d \) and their inverses).

For each root \( y_j \) of \( \Delta \), define \( L_j \) to be the line in \( \mathbb{C}^2 \) of equation \( Y = y_j \). Clearly,

\[
E = \mathbb{C}^2 - \mathcal{C} \cup L_1 \cup \cdots \cup L_r.
\]
Nothing prevents some of the $L_j$'s to be included in $C$. The space we are interested in, $C^2 - C$, is obtained from $E$ by adding (or, more exactly, by forgetting to remove) the $L_j$'s which are not included in $C$. What we are tempted to do is to use Lemma 1 (ii), and to forget the corresponding $\tilde{g}_j$'s in the above presentation. If we had chosen the $\tilde{g}_j$'s to be meridians, we would obtain a presentation for $\pi_1(C^2 - C)$. The following Lemma proves that this strategy works:

**Lemma 3.** Let $g_j \in \pi_1(B,y_0)$ be a meridian around $y_j$. There exists in $\pi_1(E,(x_0,y_0))$ a meridian $\tilde{g}_j$ around $L_j$ such that $\pi_1(E)\tilde{g}_j = g_j$.

**Proof.** This is a particular case of Lemma 2.4. (whose proof relies on an easy general position argument). For a more constructive argument, see next section. \(\square\)

Van Kampen’s method is the following procedure, which summarizes the above discussion:

**Procedure 4** (‘‘Van Kampen’s method’’). Start with a (quadrafrei) polynomial $P \in \mathbb{C}[X,Y]$.

1) Compute $\Delta \in \mathbb{C}[Y]$ and its roots $y_1, \ldots, y_r$.
2) Choose a basepoint $y_0 \in \mathbb{C}'y_1, \ldots, y_r \}$. Construct generating meridians $g_1, \ldots, g_r$ in $\pi_1(\mathbb{C}'y_1, \ldots, y_r, y_0)$. Choose $x_0 \in \mathbb{C}$ such that $P(x_0) \neq 0$. Lift $g_1, \ldots, g_r$ to elements $\tilde{g}_1, \ldots, \tilde{g}_r \in \pi_1(E,(x_0,y_0))$ which are meridians around the $L_j$'s.
3) Choose generating meridians $f_1, \ldots, f_d$ for the fundamental group of the fiber over $y_0$. Compute the monodromy automorphisms $\phi_1, \ldots, \phi_r$.
4) Let $J := \{j \in \{1, \ldots, r\} | L_j \not\subset C\}$. We obtain the following presentation for $\pi_1(C^2 - C, (x_0,y_0))$:

$$\langle f_1, \ldots, f_d, \tilde{g}_1, \ldots, \tilde{g}_r \mid \forall i \in \{1, \ldots, d\}, \forall j \in \{1, \ldots, r\}, \tilde{g}_j^{-1}f_i\tilde{g}_j = \phi_j(f_i) \rangle$$

In implementations of step 1, one only computes approximations of $y_1, \ldots, y_r$; it is not difficult to make sure they are good enough for our purposes (which is to find loops circling the actual roots). In the next two sections, we will describe explicit constructive versions of steps 2 and 3. To be really useful, step 4 should be followed by a procedure “simplifying” the initial presentation, which is usually highly redundant and unpleasant. This problem has no general solution, though some heuristics, implemented in VKCURVE, happen to be quite effective on many practical examples; these heuristics will not be described here.

**Projective Van Kampen method** The above procedure explains how to compute the fundamental group of the complement of an affine complex algebraic curve. To deal with a projective curve $C \subset \mathbb{C}P^2$ proceed as follows: decompose $\mathbb{C}P^2$ as $\mathbb{C}^2 \cup \mathbb{C}P^1$; let $C' := C \cap \mathbb{C}^2$ with the above method, compute a presentation for $\mathbb{C}^2 - C = \mathbb{C}P^2 - C \cup \mathbb{C}P^1$. 
let \( s_\infty \in \pi_1(\mathbb{CP}^2 - \mathcal{C} \cup \mathbb{P}^1) \) be a meridian of \( \mathcal{C} \cup \mathbb{CP}^1 \) around \( \mathbb{P}^1 \) (in most situations, if \( f_1, \ldots, f_d \) are as in section 4, \( s_\infty := (f_d f_{d-1} \cdots f_1)^{-1} \) is suitable; see [ACCLM], Prop. 2.10); by Lemma 1, by adding the relation \( s_\infty = 1 \), one obtains a presentation for \( \pi_1(\mathbb{CP}^2 - \mathcal{C}) \).

3. LIFTING MERIDIANS

In this section, we discuss the problem of lifting generating meridians \( g_1, \ldots, g_r \) of \( \pi_1(\mathcal{B}, y_0) \) to elements \( \tilde{g}_1, \ldots, \tilde{g}_r \) of \( \pi_1(\mathcal{E}, (x_0, y_0)) \). As explained in the previous section, we would like these \( \tilde{g}_j \)'s to be meridians around the \( L_j \)'s. Thanks to Lemma 3, we know that this is possible. However, we would like to do this in a constructive manner – something we could easily instruct a computer to do.

Practically, to encode the \( g_j \)'s, we choose representants \( \gamma_1, \ldots, \gamma_r \) in the loop space \( \Omega(\mathcal{B}, y_0) \) (in VKCURVE, we actually work with piecewise-linear loops, with endpoints in \( \mathbb{Q}[i] \)).

Thus the most natural way of lifting the \( g_j \)'s is to take the elements represented by the loops \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_r \), defined as follows: for all \( t \in [0, 1] \) and all \( j \), we set \( \tilde{\gamma}_j(t) := (x_0, \gamma_j(t)) \). Two possible problems arise from this idea:

- The first problem is that \( \tilde{\gamma}_j \) may not be a loop in \( \mathcal{E} \): nothing prevents it from intersecting \( \mathcal{C} \) (although, by general position arguments, it should only happen for a finite number of unlucky choices for \( x_0 \)).
- The second problem is a more serious one: the \( \tilde{\gamma}_j \)'s may not represent meridians.

The classical Van Kampen method assumes that \( \mathcal{C} \) does not have vertical asymptotes. With this assumption, a compactness argument can be used to eliminate both problems: the supports of the \( \gamma_j \)'s can be assumed to be all included in a disk \( D \), and, if we choose \( x_0 \) large enough (in module), we can guarantee that \( \mathcal{C} \) does not intersect the lifted disk \( (x_0, D) \). With such a choice, it follows easily that the \( \tilde{\gamma}_j \)'s indeed represent meridians around the \( L_j \)'s.

However, as explained in the introduction, we do not want to assume the absence of vertical asymptotes. To ensure that none of the above two problems occurs, we rely on the following criterion (as in Procedure 4, we set \( J := \{ j \in \{1, \ldots, r\} | L_j \not\subset \mathcal{C} \} \)):

**Lemma 5** (Explicit Step 2). Let \( Q := P \prod_{j \in J} (Y - y_j) \). Let \( \nabla \in \mathbb{C}[X] \) be the discriminant of \( Q \), viewed as polynomial in \( Y \) with coefficients in \( \mathbb{C}[X] \). Choose \( x_0 \in \mathbb{C} \) which is not a root of \( \nabla \). Denote by \( S \) the set of solutions in \( Y \) of \( Q(x_0, Y) = 0 \). For all \( j \in \{1, \ldots, r\} \), one has \( y_j \in S \).

Choose \( y_0 \in \mathbb{C} - S \), and for all \( j \in \{1, \ldots, r\} \) choose a loop \( \gamma_j \in \Omega(\mathbb{C} - S, y_0) \) representing a meridian of \( S \) around \( y_j \). Then, for all \( j \),
the path $\tilde{\gamma}_j := (x_0, \gamma_j)$ represents a meridian of $C \cup L_1 \cup \cdots \cup L_r$ around $L_j$.

Proof. For all $j \in \{1, \ldots, r\}$, either $j \in J$, or $L_j \subseteq C$; in both case, it is clear that $y_j \in S$. The path $\tilde{\gamma}_j$ avoids $C \cup L_1 \cup \cdots \cup L_r$, since the intersection of this curve with the line $X = x_0$ is precisely described by $S$. Proving that $\tilde{\gamma}_j$ represents a meridian of $C \cup L_1 \cup \cdots \cup L_r$ around $L_j$ essentially amounts to checking that $(x_0, y_j)$ is a smooth point of $C \cup L_1 \cup \cdots \cup L_r$ or, by an immediate reformulation, that $L_j$ is the only component of $C \cup L_1 \cup \cdots \cup L_r$ in which $(x_0, y_j)$ lies. If this was not satisfied, the polynomial $Q(x_0, Y)$ would have multiple roots in $Y$, which contradicts the assumption on $x_0$. \qed

4. Computing the monodromy automorphisms

At the end of the second step of Procedure 4 as detailed in Lemma 5 we are provided with:

- A basepoint $(x_0, y_0) \in E$.
- Loops $\gamma_1, \ldots, \gamma_r \in \Omega(B, y_0)$ such that each horizontally lifted loop $\tilde{\gamma}_j = (x_0, \gamma_j)$ is in $\Omega(E, (x_0, y_0))$ and represents a meridian of $C \cup L_1 \cup \cdots \cup L_r$ around $L_j$.

As it was explained in the previous section, in the classical Van Kampen method where one assumes the absence of vertical asymptotes, one can get rid of many problems by choosing $x_0$ far enough; then, to compute the monodromy automorphism $\phi_j$ corresponding to $\tilde{\gamma}_j$, it suffices to track the solutions in $X$ of $P(X, Y)$ when $Y$ moves along $\gamma_j$: this defines a monodromy braid $b_j$ on $d$ strings, from which the automorphism $\phi_j$ can be deduced using the standard Hurwitz action of the braid group on the free group (see Definition 9 and Lemma 10).

However, since we decided to work in a situation allowing vertical asymptotes, we may no longer assume that $x_0$ is “far enough”; in particular, it may occur that the strings of the monodromy braid turn around $x_0$, in which case the monodromy automorphism cannot be computed by Hurwitz formulas (the example given at the end of this section should convince the reader that there is a serious obstruction – this is not just a matter of being smart when choosing $x_0$). However, by adding to the monodromy braid an additional string fixed at $x_0$, one obtain extra information which can be used to modify Hurwitz formulas in a suitable way. This is what we detail in the present section.

To simplify notation, we fix some $j \in \{1, \ldots, r\}$, and write $\gamma$ and $\tilde{\gamma}$ instead of $\gamma_j$ and $\tilde{\gamma}_j$; our goal is to compute the corresponding monodromy automorphism $\phi$.

Let $\{x_1, \ldots, x_{d+1}\}$ be the set of solutions in $X$ of $(X - x_0)P(X, y_0) = 0$. For the sake of simplicity, we assume that the $x_i$’s have distinct real parts (this is always true up to rescaling), and that $\Re(x_1) < \Re(x_2) <$
\[ \cdots < \Re(x_{d+1}) \text{ (this is always true up to reordering). Among the } x_i \text{'s is } x_0, \text{ say } x_0 = x_{i_0}. \]

Let \( X_{t+1} \) be the configuration space of \( d+1 \) points in the complex line. Tracking the solutions of \((X - x_0)P(X, \gamma(t)) = 0 \) for \( t \in [0,1] \), we obtain an element of \( \Omega(X_{d+1}; \{x_1, \ldots, x_{d+1}\}) \), which represents an element \( b_\gamma \) of the braid group \( \pi_1(X_{d+1}; \{x_1, \ldots, x_{d+1}\}) \).

For all \( i \in \{1, \ldots, d+1\} \), we denote by \( x_i(t) \) the string of the monodromy braid starting at \( x_i \), i.e., the unique continuous path \([0,1] \to \mathbb{C} \) such that \( \forall t \in [0,1], P(x_i(t), \gamma(t)) = 0 \) and \( x_i(0) = x_i \).

Let \( F := \mathbb{C} - \{x_1, \ldots, x_{i_0-1}, x_{i_0+1}, \ldots, x_{d+1}\} \). We view \( x_0 \) as a basepoint for \( F \). We also introduce a secondary basepoint \( x_\infty \), which will be used in our argumentation but will not appear in the formulation of the final result. This secondary basepoint is assumed to have a “negative enough” imaginary part, in the sense that it satisfies the following conditions:

\[ \forall t \in [0,1], \forall i \in \{1, \ldots, d+1\}, \Re(x_\infty) < \Re(x_i(t)) \]

and

\[ \forall i \in \{1, \ldots, d\}, \Re \left( \frac{x_{i+1} - x_\infty}{x_i - x_\infty} \right) > 0. \]

It is not difficult to figure out why such \( x_\infty \) exist; let us fix one.

We consider the planar tree \( \bigcup_{i=1}^{d+1} [x_\infty, x_i] \) (where \([x_\infty, x_i] \) denotes the linear segment between \( x_\infty \) and \( x_i \)). We use this tree to describe generators for various fundamental groups (see Definition 2):

- Being a tree connecting \( x_\infty \) to \( \{x_1, \ldots, x_{d+1}\} \), it defines generating meridians
  \[ e_1, \ldots, e_{d+1} \]
  for \( \pi_1(F - \{x_0\}, x_\infty) \).

- Being a tree connecting \( x_\infty \) to \( \{x_1, \ldots, x_{i_0-1}, x_{i_0+1}, \ldots, x_{d+1}\} \), it defines generating meridians for \( \pi_1(F, x_\infty) \). Conveniently abusing notations, we still denote them by
  \[ e_1, \ldots, e_{i_0-1}, e_{i_0+1}, \ldots, e_{d+1}. \]

- Being a tree connecting \( x_0 = x_{i_0} \) to \( \{x_1, \ldots, x_{i_0-1}, x_{i_0+1}, \ldots, x_{d+1}\} \), it defines generating meridians
  \[ f_1, \ldots, f_{i_0-1}, f_{i_0+1}, \ldots, f_{d+1} \]
  for \( \pi_1(F, x_0) \).

Let \( E_\gamma \) be the pullback over \([0,1] \to \gamma \) of the fiber bundle \( E \rightarrow B \); in other words, the fiber \( E_t \) of \( E_\gamma \) over \( t \) is the complement in \( \mathbb{C} \) of \( \{x_1(t), \ldots, x_{i_0-1}(t), x_{i_0+1}(t), \ldots, x_{d+1}(t)\} \). The space \( F \) defined above coincides with the fiber over \( 0 \) (or, equivalently, 1).

**Lemma 6.** There exists a trivialization

\[ \Psi : E_\gamma \simeq F \times [0,1] \]
such that, for all \( t \in [0, 1] \), \( \Psi((x_0, \gamma(t))) = (x_0, t) \) and \( \Psi((x_\infty, \gamma(t))) = (x_\infty, t) \). In particular, the induced homeomorphism \\

\[
\psi : F = E_0 \sim E_1 = F
\]

satisfies \( \psi(x_0) = x_0 \) and \( \psi(x_\infty) = \psi(x_\infty) \).

**Proof.** This is an elementary variation of the standard construction of the map from the braid group to the mapping class group of the punctured plane. Basically, one has to imagine that the plane is a piece of rubber, pinned to a desk at \( x_0 \) and \( x_\infty \), and that we force \( d \) other points to move according to \( b_\gamma \). We leave the details to the reader. \( \Box \)

We choose \( \Psi \) and \( \psi \) as in the lemma. Since \( \psi \) fixes both \( x_0 \) and \( x_\infty \), it induces automorphisms \( \psi_0 \in \text{Aut}(\pi_1(F, x_0)) \) and \( \psi_\infty \in \text{Aut}(\pi_1(F, x_\infty)) \). We will need a third automorphism: since \( \psi(x_0) = x_0 \), \( \psi \) restricts to an homeomorphism \( \tilde{\psi} \) of \( F - \{x_0\} \), and induces an element \( \tilde{\psi}_\infty \in \text{Aut}(\pi_1(F - \{x_0\}, x_\infty)) \).

Let \( \tilde{\gamma} \) be the lifted path \( (x_0, \gamma) \in \pi_1(E, (x_0, y_0)) \). Let \( \phi \in \text{Aut}(\pi(F, x_0)) \) be the associated monodromy automorphism (see section 2).

**Lemma 7.** We have \( \phi = \psi_0 \).

**Proof.** Consider the loop \( \tilde{\gamma} = (x_0, \gamma) \in \Omega(E, (x_0, y_0)) \). In the pull-back \( E_\gamma \), it corresponds to the horizontal path \( t \mapsto (x_0, t) \). For any loop \( \omega \in \Omega(F, x_0) \), we may use the trivialization of Lemma 6 to construct a homotopy in \( \Omega(E_\gamma, (x_0, 0)) \) between \( \omega \) (viewed as a loop in the fiber of \( E_\gamma \) over \( 0 \)) and \( \tilde{\gamma}\psi(\omega)\tilde{\gamma}^{-1} \) (where \( \psi(\omega) \) is viewed as a loop in the fiber of \( E_\gamma \) over \( 1 \)). Pushed back in \( E \), this homotopy shows that conjugating by \( \tilde{\gamma} \) is the same as applying \( \psi_0 \). \( \Box \)

We may now explain our strategy for computing \( \phi \). First, we compute \( \tilde{\psi}_\infty \). As announced earlier, since \( x_\infty \) if “far enough”, this can be done using Hurwitz formulas. Then we use this intermediate result in two ways: first, we deduce \( \psi_\infty \); then, we compute the discrepancy between \( \psi_\infty \) and \( \psi_0 \) coming from the change of basepoint.

Since this is the convenient setting for implementing the method, we will suppose we are able to write \( b_\gamma \) as a word in the standard generators of the braid group:

**Definition 8** (Abstract braid group). We denote by \( B_{d+1} \) the group given by the abstract presentation

\[
\left\langle \sigma_1, \ldots, \sigma_d \ \middle| \ \begin{array}{c}
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\
\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for all } i, j \text{ with } i - j > 1
\end{array} \right\rangle
\]

It is well-known that \( B_{d+1} \) is isomorphic to the fundamental group of \( X_{d+1} \). Let us be more specific. For each \( i \in \{1, \ldots, d\} \), the positive twist of two consecutive strings along the segment \( [x_i, x_{i+1}] \) defines an element \( \sigma_i \in \pi_1(X_{d+1}, \{x_1, \ldots, x_{d+1}\}) \); these elements satisfy the
relations of the above definition and realize an explicit isomorphism (this is nothing but the usual way of considering braids via their real projection).

**Definition 9 (Hurwitz action).** For all $i \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, d+1\}$, set

$$
H_i(e_j) := \begin{cases} 
  e_{i+1} & \text{if } i = j \\
  e_{i+1}e_i^{-1}e_{i+1} & \text{if } i = j+1 \\
  e_j & \text{otherwise}
\end{cases}
$$

This defines an automorphism $H_i$ of the free group $\langle e_1, \ldots, e_{d+1} \rangle$. The $H_i$’s satisfy the defining relations of $B_{d+1}$ and induce a morphism $H : B_{d+1} \to \text{Aut}(\langle e_1, \ldots, e_{d+1} \rangle)$.

**Note.** For compatibility with usual conventions for multiplication in fundamental groups, we assume in the above definition and throughout this section that groups of automorphisms act on the right.

**Lemma 10.** The automorphism $\tilde{\psi}_*\infty$ induced by $\tilde{\psi}$ on $\langle e_1, \ldots, e_{d+1} \rangle = \pi_1(F - \{x_0\}, x_\infty)$ is $H(b_\gamma)$.

**Proof.** As in Lemma 6, we can construct, for all braid $b \in B_{d+1}$, an homeomorphism of the pointed space $(F - \{x_0\}, x_\infty)$. This obviously induces a morphism $B_{d+1} \to \text{Aut}(\langle e_1, \ldots, e_{d+1} \rangle)$. This morphism coincides with Hurwitz action (it is enough to check this for the standard generators of $B_{d+1}$, which is easy and classical). The result follows as a particular case. \qed

Via the inclusion $F - \{x_0\} \hookrightarrow F$, we have

$$\pi_1(F, x_\infty) = \langle e_1, \ldots, e_{d+1} \rangle / e_{i_0}.$$ 

As mentioned above, we will still denote by $e_i$ the image of $e_i$ in the quotient. The braid $b_\gamma$ is $x_0$-pure (since the $x_0$-strand is constant); therefore, the automorphism $H(b_\gamma)$ sends $e_{i_0}$ to a conjugate of $e_{i_0}$. In particular, $H(b_\gamma)$ induces an endomorphism of $\langle e_1, \ldots, e_{d+1} \rangle / e_{i_0}$, which is nothing but the automorphism of $\pi_1(F, x_\infty)$ induced by $\psi$.

An automorphism of a topological space yields a natural automorphism of the functor from the fundamental groupoid to the category of groups, which associates to each point the fundamental group at this point. We have the following commutative diagram of isomorphisms, where the vertical arrows are isomorphisms associated to paths connecting the two basepoints:

$$
\begin{array}{ccc}
\pi_1(F, x_0) & \xrightarrow{\psi_*0=\phi} & \pi_1(F, x_0) \\
\downarrow h_{[x_0, x_\infty]} & & \downarrow h_{\psi([x_0, x_\infty])} \\
\pi_1(F, x_\infty) & \xrightarrow{\psi_*\infty} & \pi_1(F, x_\infty)
\end{array}
$$
Proof. Left to the reader and e.

Since the element

\[ \phi = h_{\psi,[x_0,\infty)}^{-1} \psi_{x_\infty} h_{[x_0,\infty)} = (h_{\psi,[x_0,\infty)}^{-1} h_{[x_0,\infty)}) (h_{[x_0,\infty)}^{-1} \psi_{x_\infty} h_{[x_0,\infty)}). \]

is conjugate to \( e_i \) in \( \langle e_1, \ldots, e_{d+1} \rangle \).

Let \( a \) be such that \( H(b_\gamma) (e_{i_0}) = ae_{i_0} a^{-1} \). Let \( \bar{\alpha} \) be the image of \( a \) in \( \langle f_1, \ldots, f_{i_0-1}, f_{i_0+1}, \ldots, f_{d+1} \rangle \) by the morphism sending \( e_i \), \( i \neq i_0 \) to \( f_i \) and \( e_{i_0} \) to \( 1 \). Then \( h_{\psi,[x_0,\infty)}^{-1} h_{[x_0,\infty)} \) is the morphism \( f \mapsto \bar{\alpha}^{-1} f \bar{\alpha} \).

Proof. Left to the reader (Hint: if true, this lemma provides a formula for \( \phi \); first, check that this formula indeed defines a morphism \( b_\gamma \mapsto \phi \); then check it on generators of the group of \( x_0 \)-pure braids on \( d+1 \) strings). \( \square \)

The following procedure is a summary of the results of this section. As promised, the exact choice of \( x_\infty \) does not matter; nor does it matter to have distinct notations for the \( e_i \)'s and the \( f_i \)'s.

Procedure 12 (Explicit Step 3). Suppose \( \gamma \) is one the loops \( \gamma_i \) constructed in Lemma 11. To compute the associated monodromy automorphism \( \phi \), one may proceed as follows:

a) Compute (as a word in the standard generators, using real projection) the monodromy braid \( b_\gamma \) with \( d+1 \) strings.

b) Compute Hurwitz action \( H(b_\gamma) \) on the free group \( \langle f_1, \ldots, f_{d+1} \rangle \).

c) Identify the index \( i_0 \) of the \( x_0 \)-string.

d) Find \( a \) such that \( H(b_\gamma) (f_{i_0}) = af_{i_0} a^{-1} \) (this is trivial to do: take \( a \) to be the first half of a reduced word for \( H(b_\gamma) (f_{i_0}) \)).

e) The composition of \( H(b_\gamma) \) with the automorphism \( f \mapsto a^{-1} fa \) is an automorphism of \( \langle f_1, \ldots, f_{d+1} \rangle \) fixing \( f_{i_0} \). It induces an automorphism of the free group \( \langle f_1, \ldots, f_{d+1} \rangle / f_{i_0} \) of rank \( d \): this is the monodromy automorphism \( \phi \).

Steps b, c, d and e are straightforward to implement, as soon as one works with a software where braid groups, free groups and groups automorphisms are available (this is the case with GAP). Finding an efficient implementation of step a is the main issue.

In the classical method, one may assume that \( x_0 \) has a “large enough” real part; this implies that \( i_0 = d+1 \), that \( H(b_\gamma) \) has no factor \( \sigma_d^{-1} \) and that \( H(b_\gamma) (f_{d+1}) = f_{d+1} \): Hurwitz action does not need a corrective term.

Complexity of the modified method. As mentioned in the introduction, we are interested in non-generic projections because they reduce the number of strings (\( d < \deg P \)). To be able to work in this
context, it is necessary to introduce an additional string. The complexity cost from this additional string is usually much smaller than the gain ($d + 1 \leq \deg P$; also, the additional string, which is constant at $x_0$, is handled very efficiently). The cost of steps $c$, $d$ and $e$ (which are not present in classical Van Kampen method) is negligible. Actually, the only serious side-effect of the variant method is not in Procedure 12 itself, but hidden in Lemma 5: one has to construct more complicated loops (for which step 3a will be more costly). However, the overall balance is positive (this is a purely empirical statement, we did not try to assess the theoretical complexity of our implementation).

To conclude, we illustrate by an example (inspired by Example 3.1 in [ACCLM]) how the monodromy automorphism is affected by the choice of the basepoint. It is a good exercise to apply Procedure 12 to check the claims.

**Example.** Consider the example of the monodromy braid with two strings whose positions are $e^{-i\pi t}$ and $-e^{-i\pi t}$, for $t \in [0, 1]$. This braid occurs when studying the monodromy of $(XY - 1)(XY + 1) = 0$ around the singular fiber $Y = 0$. The induced monodromy automorphism depends on the choice of the basepoint in $\mathbb{C} - \{\pm 1\}$:

i) choosing $x_\infty := -2i$ as basepoint for the fiber, the monodromy automorphism has infinite order;

ii) choosing $x_0 := 0$ as basepoint for the fiber, the monodromy automorphism has order 2. In particular, it cannot be described in terms of usual Hurwitz action.

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