KURATOWSKI MONOIDS OF $n$-TOPOLOGICAL SPACES

T. BANAKH, O. CHERVAK, T. MARTynyuk, M. PylyPovycH, A. RAVSKy, M. SIMKIV

Dedicated to the 120-th birthday of K. Kuratowski (1896-1980)

Abstract. Generalizing the famous 14-set closure-complement Theorem of Kuratowski from 1922, we prove that for a set $X$ endowed with $n$ pairwise comparable topologies $\tau_1 \subset \cdots \subset \tau_n$, by repeated application of the operations of complement and closure in the topologies $\tau_1, \ldots, \tau_n$ to a subset $A \subset X$ we can obtain at most $2K(n) = 2\sum_{i,j=0}^n \binom{i+j}{i}\binom{i+j}{j}$ distinct sets.

1. Introduction

This paper was motivated by the famous Kuratowski 14-set closure-complement Theorem \(^1\), which says that for any topological space $(X, \tau)$ the operators of complement $c : \mathcal{P}(X) \to \mathcal{P}(X)$, $c : A \to X \setminus A$ and closure $\overline{\tau} : \mathcal{P}(X) \to \mathcal{P}(X)$, $\overline{\tau} : A \mapsto \overline{A}$, generate a submonoid $\langle c, \overline{\tau} \rangle$ of cardinality $\leq 14$ in the monoid $\mathcal{P}(X)^{\mathcal{P}(X)}$ of all self-maps of the power-set $\mathcal{P}(X)$ of $X$.

In \(^2\) Shallitt and Willard constructed two commuting closure operators $p, q : \mathcal{P}(X) \to \mathcal{P}(X)$ on the power-set $\mathcal{P}(X)$ of a countable set $X$ such that the submonoid $\langle p, q, c \rangle \subset \mathcal{P}(X)^{\mathcal{P}(X)}$ generated by these closure operators and the operator of complement is infinite. In Example 3.1 below we shall define two metrizable topologies $\tau_1$ and $\tau_2$ on a countable set $X$ such that the closure operators $\overline{\tau}_1$ and $\overline{\tau}_2$ in the topologies $\tau_1$ and $\tau_2$ generate an infinite submonoid $\langle \overline{\tau}_1, \overline{\tau}_2 \rangle$ in the monoid $\mathcal{P}(X)^{\mathcal{P}(X)}$ of all self-maps of $\mathcal{P}(X)$. Moreover, for some set $A \subset X$ the set $\{ f(A) : f \in \{ \overline{\tau}_1, \overline{\tau}_2 \} \}$ is infinite. This shows that Kuratowski’s 14-set theorem does not generalize to spaces endowed with two or more topologies.

The situation changes dramatically if two topologies $\tau_1$ and $\tau_2$ on a set $X$ are comparable, i.e., one of these topologies is contained in the other. In this case we shall prove that the closure operators $\overline{\tau}_1, \overline{\tau}_2 : \mathcal{P}(X) \to \mathcal{P}(X)$ induced by these topologies together with the operator $c$ of complement generate a submonoid $\langle \overline{\tau}_1, \overline{\tau}_2, c \rangle \subset \mathcal{P}(X)$ of cardinality $\leq 126$. In fact, we shall consider this problem in a more general context of multitopological spaces and polytopological spaces.

By a multitopological space we understand a set $X$ endowed with a family $\mathcal{T}$ of topologies on $X$. A multitopological space $(X, \mathcal{T})$ is called polytopological if the family of its topologies $\mathcal{T}$ is linearly ordered by the inclusion relation. A typical example of a polytopological space is the real line endowed with the Euclidean and Sorgenfrey topologies. Another natural example of a polytopological space is any Banach space, carrying the norm and weak topologies. A dual Banach space is an example of a polytopological space carrying three topologies: the norm topology, the weak topology and the $*$-weak topology. A topological space $(X, \tau)$ can be thought as a polytopological space $(X, \{ \tau \})$ endowed with the family $\{ \tau \}$ consisting of a single topology $\tau$.

For a topology $\tau$ on a set $X$ by $\overline{\tau} : \mathcal{P}(X) \to \mathcal{P}(X)$ and $\overline{\tau} : \mathcal{P}(X) \to \mathcal{P}(X)$ we shall denote the operators of taking the interior and closure with respect to the topology $\tau$. These operators assign to each subset $A \subset X$ its interior $\overline{\tau}(A)$ and closure $\overline{\tau}(A)$, respectively. Since $\tau = \{ \overline{\tau}(A) : A \subset X \} = \{ X \setminus \overline{\tau}(A) : A \subset X \}$ the topology $\tau$ can be recovered from the operators $\overline{\tau}$ and $\overline{\tau}$.

For a multitopological space $X = (X, \mathcal{T})$ the submonoid $K(X) = \langle \overline{\tau}, \overline{\tau} : \tau \in \mathcal{T} \rangle$

---

\(^1\)A complete bibliography related to the Kuratowski 14-set closure-complement Theorem is collected on the web-site [http://www.mathtransit.com/cornucopia.php](http://www.mathtransit.com/cornucopia.php) created by Mark Bowron.
in \( \mathcal{P}(X)^{\mathcal{P}(X)} \) generated by the interior and closure operators \( \bar{\tau}, \bar{\tau} \) for \( \tau \in \mathcal{T} \), will be called the Kuratowski monoid of the multitopological space \( X \). A somewhat larger submonoid

\[
K_2(X) = \langle c, \bar{\tau} : \tau \in \mathcal{T} \rangle
\]

in \( \mathcal{P}(X)^{\mathcal{P}(X)} \) generated by the operator of complement \( c \) and the closure operators \( \bar{\tau}, \tau \in \mathcal{T} \), will be called the full Kuratowski monoid of the multitopological space \( X \).

Taking into account that \( \bar{\tau} = c \circ \bar{\tau} \circ c \) and \( \bar{\tau} = c \circ \bar{\tau} \circ c \), we see that \( K(X) \subseteq K_2(X) \) and moreover,

\[
K_2(X) = K(X) \cup (c \circ K(X)),
\]

which implies that \( |K_2(X)| \leq 2 \cdot |K(X)| \).

The notion of a multitopological space has one disadvantage: multitopological spaces do not form a category (it is not clear what to understand under a morphism of multitopological spaces). This problem with multitopological spaces can be easily fixed by introducing their parametric version called \( L \)-topological spaces where \((L, \leq)\) is a partially ordered set.

Given a subset \( X \) we denote by \( \text{Top}(X) \) the family of all possible topologies on \( X \), partially ordered by the inclusion relation. The family \( \text{Top}(X) \) is a lattice whose smallest element is the anti-discrete topology \( \tau_o \) and the largest element is the discrete topology \( \tau_d \) on \( X \). Observe that for the discrete topology the operators \( \bar{\tau}_d \) and \( \bar{\tau}_d \) coincide with the identity operator \( 1_X \) on \( \mathcal{P}(X) \).

Let \((L, \leq)\) be a partially ordered set. By definition, an \( L \)-topology on a set \( X \) is any monotone map \( \tau : L \to \text{Top}(X) \). The monotonicity of \( \tau \) means that for any elements \( i \leq j \) in \( L \) we get \( \tau(i) \subseteq \tau(j) \). In the sequel for an element \( i \in L \) it will be convenient to denote the topology \( \tau(i) \) by \( \tau_i \). By an \( L \)-topological space we shall understand a pair \((X, \tau)\) consisting of a set \( X \) and an \( L \)-topology \( \tau : L \to \text{Top}(X) \) on \( X \).

By a morphism between two \( L \)-topological spaces \((X, \tau)\) and \((Y, \sigma)\) we understand a map \( f : X \to Y \) which is continuous as a map between topological spaces \((X, \tau_i)\) and \((Y, \sigma_i)\) for every \( i \in L \). \( L \)-Topological spaces and their morphisms form a category called the category of \( L \)-topological spaces. Each \( L \)-topological space \( X = (X, \tau) \) can be thought as a multitopological space endowed with the family of topologies \( \{\tau_i\}_{i \in L} \). If the set \( L \) is linearly ordered, then the multitopological space \((X, \{\tau_i\}_{i \in L})\) is polytopological.

So we can speak about the Kuratowski monoid \( K(X) \) and the full Kuratowski monoid \( K_2(X) \) of an \( L \)-topological space \( X \).

We shall be especially interested in (full) Kuratowski monoids of \( n \)-topological spaces where \( n = \{0, \ldots, n-1\} \) is a finite non-zero ordinal (or a natural number). Observe that \( n \)-topological spaces can be thought as sets endowed with \( n \)-topologies \( \tau_0 \subseteq \tau_1 \subseteq \cdots \subseteq \tau_{n-1} \).

We shall prove that the upper bound for the cardinality of the Kuratowski monoid \( K(X) \) of an \( n \)-topological space \( X \) is given by the number

\[
K(n) = \sum_{i,j=0}^{n} \binom{n}{i+j} \cdot \binom{i+j}{j}
\]

where \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \) is the binomial coefficient.

The main result of this paper is the following theorem.

**Theorem 1.1.** For any \( n \)-topological space \( X = (X, \mathcal{T}) \) its Kuratowski monoid \( K(X) \) has cardinality \( |K(X)| \leq K(n) \) and its full Kuratowski monoid \( K_2(X) \) has cardinality \( |K_2(X)| \leq 2 \cdot K(n) \).

The upper bounds \( |K(X)| \leq K(n) \) and \( |K_2(X)| \leq 2 \cdot K(n) \) given in Theorem 1.1 are exact as shown in our next theorem.

**Theorem 1.2.** For every \( n \in \omega \) there is an \( n \)-topological space \( X = (X, \mathcal{T}) \) such that \( |K(X)| = K(n) \) and \( |K_2(X)| = 2 \cdot K(n) \).

This theorem will be proved in Section 8 (see Corollary 8.2). The asymptotics of the sequence \( K(n) \) is described in the following theorem, which will be proved in Section 6.

**Theorem 1.3.** There exists \( \lim_{n \to \infty} K(n)/(2n) = \sup_{n \to \infty} K(n)/(2n) = 16/9 \) which implies that \( K(n) = (16/9 + o(1)) \cdot (2n) \) for \( n \leq 9 \).

The values of the sequences \( K(n) \) and \( 2K(n) \) for \( n \leq 9 \), calculated with help of computer are presented in the following table:
KURATOWSKI MONOIDS OF $n$-TOPOLOGICAL SPACES

| n  | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $K(n)$ | 1   | 7   | 63  | 697 | 8549 | 111033 | 1495677 | 20667463 | 291020283 | 4157865643 |
| $2K(n)$ | 2   | 14  | 126 | 1394 | 17098 | 222066 | 2991359 | 41334926 | 582040566 | 8315731286 |

In particular, the Kuratowski monoid $K(X)$ of any topological space $X = (X, \tau)$ consists of 7 elements (some of which can coincide [2]):

1, $\check{\tau}$, $\check{\check{\tau}}$, $\check{\check{\check{\tau}}}$, $\check{\check{\check{\check{\tau}}}}$, $\check{\check{\check{\check{\check{\tau}}}}}$.

For a 2-topological space $X = (X, \tau)$ endowed with two topologies $\tau_0 \subset \tau_1$ the number of elements of the Kuratowski monoid $K(X)$ increases to 63 (see Proposition 9.2):

1, $\check{\tau}_0$, $\check{\check{\tau}}$, $\check{\check{\check{\tau}}}$, $\check{\check{\check{\check{\tau}}}}$, $\check{\check{\check{\check{\check{\tau}}}}}$.

### 2. The Kuratowski Monoid of a Saturated Polytopological Space

In this section we introduce a class of $n$-topological spaces $X = (X, \mathcal{T})$ whose Kuratowski monoids $K(X)$ have cardinality strictly smaller than $K(n)$.

A multitopological space $(X, \mathcal{T})$ is called saturated if for any topologies $\tau_0, \tau_1 \in \mathcal{T}$ each non-empty open subset $U \in \tau_0$ has non-empty interior in the topology $\tau_1$. A typical example of a saturated multitopological space is the real line $\mathbb{R}$ endowed with the Euclidean and Sorgenfrey topologies $\tau_0 \subset \tau_1$.

For a linearly ordered set $L$ an $L$-topological space $(X, \tau)$ is defined to be saturated if the multitopological space $(X, \{\tau_i\}_{i \in L})$ is saturated.

**Theorem 2.1.** For a saturated polytopological space $X = (X, \tau)$ the Kuratowski monoid $K(X)$ coincides with the set

$$\{1\} \cup \{\check{\tau}, \check{\check{\tau}}, \check{\check{\check{\tau}}}, \check{\check{\check{\check{\tau}}}} : \tau \in \mathcal{T}\}$$

and hence has cardinality $|K(X)| \leq 1 + 6 \cdot |\mathcal{T}|$.

**Proof.** The definition of a saturated polytopological space $X = (X, \tau)$ implies that $\check{\tau}_0 \check{\tau}_1 = \check{\tau}_0 \check{\tau}_0$ for any topologies $\tau_0, \tau_1 \in \mathcal{T}$. Applying to this equality the operator $c$ of taking complement, we get

$$\check{\tau}_0 \check{\tau}_1 = c\tau_0 c\tau_1 c = c\tau_0 \tau_0 c = c\tau_0 c\tau_0 c = \check{\tau}_0 \check{\tau}_0.$$

This implies that

$$K(X) = \bigcup_{\tau \in \mathcal{T}} K(X, \tau) = \bigcup_{\tau \in \mathcal{T}} \{1, \check{\tau}, \check{\check{\tau}}, \check{\check{\check{\tau}}}, \check{\check{\check{\check{\tau}}}}\}$$

and hence $|K(X)| \leq 1 + \sum_{\tau \in \mathcal{T}} (|K(X, \tau)| - 1) \leq 1 + 6 \cdot |\mathcal{T}|$. □

**Example 2.2.** Let $X = (\mathbb{R}, \{\tau_0, \tau_1\})$ be the real line $\mathbb{R}$ endowed with the Euclidean topology $\tau_0$ and the Sorgenfrey topology $\tau_1$. The Kuratowski monoid $K(X)$ has cardinality $|K(X)| = 13$. Moreover, for some set $A \subset \mathbb{R}$ the family $K(X)A = \{f(A) : A \in K(X)\}$ has cardinality $|K(X)A| = |K(X)| = 13$.

**Proof.** The upper bound $|K(X)| \leq 13$ follows from Theorem 2.1. To prove the lower bound $|K(X)| \geq 13$, consider the subset

$$A = \bigcup_{n=0}^{\infty} (3^{-2n-1}, 3^{-2n}) \cup (1, 2) \cup \{3\} \cup ([4, 5) \cap \mathbb{Q})$$

and observe that the following 13 subsets of $\mathbb{R}$ are pairwise distinct, witnessing that $|K(X)| \geq |\{f(A) : f \in K(X)\}| \geq 13$. 

Observe that the family \( \{ f(A) : f \in K(X) \} \) is infinite and hence the Kuratowski monoid \( K(X) \) of the multitopological space \( X = (X, \{ \tau_0, \tau_1 \}) \) is infinite too.

\[ \square \]

Theorem 2.2 gives a partial answer to the following general problem.

**Problem 2.3.** Which properties of a polytopological space \( X \) are reflected in the algebraic structure of its Kuratowski monoid \( K(X) \)?

### 3. An example of a multitopological space with infinite Kuratowski monoid

In this section we shall construct the following example announced in the introduction.

**Example 3.1.** There is a countable space \( X \) endowed with two (incomparable) metrizable topologies \( \tau_0, \tau_1 \) such that the Kuratowski monoid \( K(X) \) of the multitopological space \( X = (X, \{ \tau_0, \tau_1 \}) \) is infinite. Moreover, for some set \( A \subset X \) the family \( \{ f(A) : f \in K(X) \} \) is infinite.

**Proof.** Take any countable metrizable topological space \( X \) containing a decreasing sequence of non-empty subsets \( (X_n)_{n \in \omega} \) such that \( X_0 = X, \bigcap_{n \in \omega} X_n = \emptyset \) and \( X_{n+1} \) is nowhere dense in \( X_n \) for all \( n \in \omega \).

To find such a space \( X \), take the convergent sequence \( S_0 = \{0\} \cup \{2^{-n} : n \in \omega \} \) and consider the subspace

\[
X = \{(x_k)_{k \in \omega} \in S_0^\omega : \exists n \in \omega \ \forall k \geq n \ x_k = 0\} \setminus \{0\}^\omega
\]

of the countable power \( S_0^\omega \) endowed with the Tychonoff product topology. It is easy to see that the subsets

\[
X_n = \{(x_k)_{k \in \omega} \in X : \forall k < n \ x_k = 0\}, \quad n \in \omega,
\]

have the required properties: \( X_0 = X, \bigcap_{n \in \omega} X_n = \emptyset \) and \( X_{n+1} \) is nowhere dense in \( X_n \).

On the space \( X \) consider two topologies

\[
\tau_0 = \{U \subset X : \forall n \in \omega \ U \cap (X_{2n+1} \setminus X_{2n+2}) \text{ is open in } X_{2n} \setminus X_{2n+2}\}
\]

and

\[
\tau_1 = \{U \subset X : \forall n \in \omega \ U \cap (X_{2n+1} \setminus X_{2n+3}) \text{ is open in } X_{2n+1} \setminus X_{2n+3}\}.
\]

Observe that \( \tau_0 \) coincides with the topology of the topological sum \( \bigoplus_{n \in \omega} X_{2n} \setminus X_{2n+2} \) while \( \tau_2 \) coincides with the topology of the topological sum \( \bigoplus_{n \in \omega} X_{2n+1} \setminus X_{2n+3} \).

We claim that the multitopological space \( X = (X, \{ \tau_0, \tau_1 \}) \) has the required properties.

By \( \tilde{\tau}_1, \tilde{\tau}_2 : P(X) \to P(X) \) we denote the closure operators in the topologies \( \tau_1 \) and \( \tau_2 \), respectively. The nowhere density of \( X_{n+1} \) in \( X_n \) for all \( n \in \omega \) and the definition of the topologies \( \tau_0, \tau_1 \) imply that for every \( n \in \omega \)

1. \( \tilde{\tau}_0(X \setminus X_{2n+1}) = X \setminus X_{2n+2}; \)
2. \( \tilde{\tau}_1(X \setminus X_{2n+2}) = X \setminus X_{2n+3}; \)
3. \( \tilde{\tau}_1(X \setminus X_{2n+1}) = X \setminus X_{2n+3}; \)
4. \( \tilde{\tau}_1^2(X \setminus X_1) = X \setminus X_{2n+1}. \)

Therefore, for the set \( A = X \setminus X_1 \) the sets \( (\tilde{\tau}_1 \tilde{\tau}_0)(A), n \in \omega \), are pairwise distinct, which implies that the family \( \{ f(A) : f \in K(X) \} \) is infinite and hence the Kuratowski monoid \( K(X) \) of the multitopological space \( X = (X, \{ \tau_0, \tau_1 \}) \) is infinite too.

\[ \square \]
4. Kuratowski monoids

To prove Theorem 1.1 we shall use the natural structure of partial order on the monoid $P(X)^P(X)$. For two maps $f, g \in P(X)^P(X)$ we write $f \leq g$ if $f(A) \subseteq g(A)$ for every subset $A \subset X$. This partial order turns $P(X)$ into a partially ordered monoid.

By a partially ordered monoid we understand a monoid $M$ endowed with a partial order $\leq$ which is compatible with the semigroup operation of $M$ in the sense that for any points $x, y, z \in M$ the inequality $x \leq y$ implies $xz \leq yz$ and $zx \leq zy$. Recall that a monoid is a semigroup $S$ possessing a two-sided unit $1 \in S$.

Observe that for two comparable topologies $\tau_1 \subseteq \tau_2$ on a set $X$ we get

$$\hat{\tau}_1 \leq \hat{\tau}_2 \leq 1_X \leq \hat{\tau}_2 \leq \hat{\tau}_1$$

where $1_X : P(X) \to P(X)$ is the identity transformation of $P(X)$. Now we see that for a polytopological space $X = (X, T)$ its Kuratowski monoid $K(X) = \langle \hat{\tau}, \tau : \tau \in T \rangle$ is generated by the linearly ordered set

$$L(X) = \{ \hat{\tau} : \tau \in T \} \cup \{ 1_X \} \cup \{ \hat{\tau} : \tau \in T \}.$$

This leads to the following

**Definition 4.1.** A Kuratowski monoid is a partially ordered monoid $K$ generated by a finite linearly ordered set $L$ containing the unit $1$ of $K$ and consisting of idempotents.

The set $L$ will be called a linear generating set of the Kuratowski monoid $K$. This set can be written as the union $L = L_- \cup \{ 1 \} \cup L_+$ where $L_- = \{ x \in L : x < 1 \}$ and $L_+ = \{ x \in L : x > 1 \}$ are the sets of negative and positive generating elements of $K$.

A Kuratowski monoid $K$ is called a Kuratowski monoid of type $(n, p)$ if $K$ has a linear generating set $L$ such that $|L_-| = n$ and $|L_+| = p$.

For two numbers $n, p \in \omega$ consider the number

$$K(n, p) = \sum_{i=0}^{n} \sum_{j=0}^{p} \binom{i+j}{i} \cdot \binom{i+j}{j}$$

and observe that $K(n) = K(n, n)$ for every $n \in \omega$.

It is easy to see that for each polytopological space $X = (X, T)$ endowed with $n = |T|$ topologies, its Kuratowski monoid $K(X)$ is a Kuratowski monoid of type $(n, n)$ or $(n-1, n-1)$. The latter case happens if the polytopology $T$ of $X$ contains the discrete topology $\tau_d$ on $X$. In this case $\hat{\tau}_d = 1_X = \hat{\tau}_d$.

Now we see that Theorem 1.1 is a partial case of the following more general theorem, which will be proved in Section 7 (more precisely, in Theorem 7.1).

**Theorem 4.2.** Each Kuratowski monoid $K$ of type $(n, p)$ has cardinality $|K| \leq K(n, p)$.

The values of the double sequence $K(n, p)$ for $n, p \leq 9$ were calculated by computer:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----|---|---|---|---|---|---|---|---|---|---|
| 0   | 1 | 2 | 3 | 4 | 6 | 7 | 8 | 9 | 10|
| 1   | 2 | 7 | 17 | 34 | 60 | 97 | 147 | 212 | 294 | 395|
| 2   | 3 | 17 | 63 | 180 | 431 | 909 | 1743 | 3104 | 5211 | 8337|
| 3   | 4 | 34 | 180 | 697 | 2173 | 5787 | 13677 | 29438 | 58770 | 110296|
| 4   | 5 | 60 | 431 | 2173 | 8549 | 28039 | 80029 | 204690 | 479047 | 1041798|
| 5   | 6 | 97 | 909 | 5787 | 28039 | 11033 | 376467 | 1128392 | 3059118 | 7629873|
| 6   | 7 | 147 | 1743 | 13677 | 80029 | 376467 | 1495677 | 5192258 | 16140993 | 45761773|
| 7   | 8 | 212 | 3104 | 29438 | 204690 | 1128392 | 5192258 | 20667463 | 73025423 | 233519803|
| 8   | 9 | 294 | 5211 | 58770 | 479047 | 3059118 | 7629873 | 291020283 | 1042490763 |
| 9   | 10| 395 | 8337 | 110296 | 1041798 | 7629873 | 45761773 | 233519803 | 1042490763 | 4157865643|

5. Kuratowski words

Theorem 4.2 will be proved by showing that each element of a Kuratowski monoid $K$ with linear generating set $L$ can be represented by a Kuratowski word in the alphabet $L$. Kuratowski words are defined as follows.

By a pointed linearly ordered set we understand a linearly ordered set $L$ with a distinguished element $1 \in L$ called the unit of $L$. This element divides the set $L \setminus \{ 1 \}$ into negative and positive parts $L_- = \{ x \in L : x < 1 \}$ and $L_+ = \{ x \in L : x > 1 \}$, respectively. By $FS_L = \bigcup_{n=1}^{\infty} L^n$ we denote the free semigroup over $L$. It consists
of non-empty words in the alphabet \( L \). The semigroup operation on \( FS_L \) is defined as the concatenation of words. The set \( L \) is identified with the set \( L^1 \) of words of length 1 in the alphabet \( L \).

A word \( w = x_1 \ldots x_n \in FS_L \) of length \( n \) is called alternating if for each natural number \( i \) with \( 1 \leq i < n \) the doubleton \( \{x_i, x_{i+1}\} \) intersects both sets \( L_- \) and \( L_+ \). According to this definition, words of length 1 also are alternating. On the other hand, an alternating word of length \( \geq 2 \) does not contain a letter equal to 1.

An alternating word \( x_0 \cdots x_n \in FS_L \) of length \( n + 1 \geq 2 \) is defined to be

- a \( v_x \)-word if there is an integer number \( m \in \{0, \ldots, n - 1\} \) such that the sequences \( (x_{m+2i})_{0 \leq i \leq \frac{n-m}{2}} \) are strictly increasing in \( L_- \) and the sequences \( (x_{m+2i+1})_{0 \leq i \leq \frac{n-m}{2}} \) are strictly decreasing in \( L_+ \);
- a \( v^-_x \)-word if there is a number \( m \in \{0, \ldots, n - 1\} \) such that the sequences \( (x_{m+2i})_{0 \leq i \leq \frac{n-m}{2}} \) are strictly decreasing in \( L_+ \) and the sequences \( (x_{m+2i+1})_{0 \leq i \leq \frac{n-m}{2}} \) are strictly increasing in \( L_- \);
- a \( w_x \)-word if there is a number \( m \in \{1, \ldots, n - 1\} \) such that \( x_{m-1} = x_{m+1} \in L_- \), the sequences \( (x_{m+1+2i})_{0 \leq i \leq \frac{n-m}{2}} \), \((x_{m-1+2i})_{0 \leq i \leq \frac{n-m}{2}} \) are strictly increasing in \( L_- \) and the sequences \((x_{m+2i})_{0 \leq i \leq \frac{n-m}{2}} \), \((x_{m+2i+1})_{0 \leq i \leq \frac{n-m}{2}} \) are strictly decreasing in \( L_+ \);
- a \( w^-_x \)-word if there is a number \( m \in \{1, \ldots, n - 1\} \) such that \( x_{m+1} = x_{m-1} \in L_+ \), the sequences \((x_{m+3+2i})_{0 \leq i \leq \frac{n-m+2}{2}} \), \((x_{m-1+2i})_{0 \leq i \leq \frac{n-m-1}{2}} \) are strictly decreasing in \( L_- \) and the sequences \((x_{m+2i})_{0 \leq i \leq \frac{n-m}{2}} \), \((x_{m+2i+2})_{0 \leq i \leq \frac{n-m}{2}} \) are strictly increasing in \( L_+ \).

By \( V_x \) (resp. \( V^+_x \), \( V^-_x \), \( W_x \), \( W^-_x \)) we denote the family of all \( v_x \)-words (resp. \( v^+_x \)-words, \( v^-_x \)-word, \( w_x \)-words) in the alphabet \( L \). It is easy to see that the families of words \( V_x \), \( V^+_x \), \( W_x \), \( W^-_x \) are pairwise disjoint. Words that belong to the set \( K_L = L^1 \cup V_x \cup V^+_x \cup W_x \cup W^-_x \) are called Kuratowski words in the alphabet \( L \).

Let us calculate the cardinality of the set \( K_L \) depending on the cardinalities \( n = |L_-| \) and \( p = |L_+| \) of the negative and positive parts of \( L \).

For non-negative integers \( n, r \) by \( \binom{n}{r} \) we denote the cardinality of the set of \( r \)-element subsets of an \( n \)-element set. It is clear that

\[
\binom{n}{r} = \begin{cases} \frac{n!}{r!(n-r)!} & \text{if } 0 \leq r \leq n; \\ 0 & \text{otherwise.}
\end{cases}
\]

The numbers \( \binom{n}{r} \) will be called binomial coefficients. The following properties of binomial coefficients are well-known (see, e.g. [3, §5.1]).

**Lemma 5.1.** For any non-negative integer numbers \( m, n, k \) we get

1. \( \binom{n}{k} = \binom{n}{n-k} \),
2. \( \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \), and
3. \( \binom{n+m}{k} = \sum_{l=0}^{n} \binom{n}{l} \binom{m}{k-l} \).

In the following theorem we calculate the cardinality of the set \( K_L \) of Kuratowski words.

**Theorem 5.2.** For any finite pointed linearly ordered set \( L \) with \( n = |L_-| \) and \( p = |L_+| \) we get

1. \( |V_x| = \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n}{l} \binom{p}{k-l} \binom{k+1}{l+1} \binom{n-l}{r} \),
2. \( |V^+_x| = \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n}{l} \binom{p}{k-l} \binom{k+1}{l} \binom{n-l}{r} \),
3. \( |W_x| = \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n}{l} \binom{p}{k-l} \binom{k+1}{l} \binom{n-l}{r} \),
4. \( |W^-_x| = \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n}{l} \binom{p}{k-l} \binom{k+1}{l} \binom{n-l}{r} \),
5. \( |K_L| = \sum_{k=0}^{n} \binom{n}{k} \binom{p}{k} = K(n, p) \).

**Proof.**

1. To calculate the number of \( v_x \)-words, fix any \( v_x \)-word \( v \in V_x \) and write it as an alternating word \( v = x_k \ldots x_{2m} \ldots x_q \) such that the sequences \((x_{2m+2i})_{0 \leq i \leq \frac{2m}{2}} \) and \((x_{2m+2i-1})_{0 \leq i \leq \frac{2m-1}{2}} \) are strictly increasing in \( L_- \) and the sequences \((x_{2m+1+2i})_{0 \leq i \leq \frac{2m}{2}} \) and \((x_{2m+1+2i-1})_{0 \leq i \leq \frac{2m-1}{2}} \) are strictly decreasing in \( L_+ \). It follows that the sequence \((x_{2m+2i})_{1 \leq i \leq \frac{2m}{2}} \) is a strictly increasing sequence of length \( r = \left\lfloor \frac{2m}{2} \right\rfloor \) in the
linearly ordered set $A = \{x \in L : x > x_{2m}\} \subset L_+$. The number of such sequences is equal to \( \binom{q}{a} \) where the cardinality $a = |A|$ can vary from 0 (if $x_{2m}$ is the largest element of the set $L_+$) till $n - 1$ (if $x_{2m}$ is the smallest element of $L_+$). By analogy, $(x_{2m-2i})_{1 \leq i \leq \frac{2m-k}{2}}$ is a strictly increasing sequence of length $l = \left\lfloor \frac{2m-k}{2} \right\rfloor$ in the linearly ordered set $A$ and the number of such sequences is equal to \( \binom{\left\lfloor \frac{2m-k}{2} \right\rfloor}{i} \).

If $l = \left\lfloor \frac{2m-k}{2} \right\rfloor$, then $2m = k + 2l$ and $(x_{2m+1-2i})_{1 \leq i \leq \frac{2m-k}{2}}$ is a strictly decreasing sequence of length $\left\lfloor \frac{2m+1-k}{2} \right\rfloor = l$ in the linearly ordered set $B = \{x \in L : x < x_{2m+1}\} \subset L_+$. The number of such sequences is equal to \( \binom{\left\lfloor \frac{2m+1-k}{2} \right\rfloor}{i} \) where $b = |B| < p$. If $l = \frac{2m-k-2}{2}$, then $2m = k + 1 + 2l$ and $(x_{2m+1-2i})_{1 \leq i \leq \frac{2m+1-k}{2}}$ is a strictly increasing sequence of length $\left\lfloor \frac{2m+1-k}{2} \right\rfloor = l + 1$ in the set $B$. The number of such sequences is equal to \( \binom{\left\lfloor \frac{2m+1-k}{2} \right\rfloor}{i+1} \).

If $r = \frac{q-2m}{2}$, then $2m = q - 2r$ and $(x_{2m+1+2i})_{1 \leq i \leq \frac{q-2m}{2}}$ is a strictly decreasing sequence of length $\left\lfloor \frac{q-2m-1}{2} \right\rfloor = r - 1$ in the linearly ordered set $B$. The number of such sequences is equal to \( \binom{\left\lfloor \frac{q-2m-1}{2} \right\rfloor}{i} \). If $r = \frac{q-2m-1}{2}$, then $2m = q - 1 - 2r$ and $(x_{2m+1+2i})_{1 \leq i \leq \frac{q-2m-1}{2}}$ is a strictly decreasing sequences of length $\left\lfloor \frac{q-2m-1}{2} \right\rfloor = r$ in the linearly ordered set $B$. The number of such sequences is equal to \( \binom{\left\lfloor \frac{q-2m-1}{2} \right\rfloor}{i} \). Summing up and applying Lemma 5.1(2), we conclude that the family $V_\ast$ of all $V_\ast$-words has cardinality

$$|V_\ast| = \sum_{a=0}^{n-1} \sum_{l=0}^{p-1} \sum_{a=0}^{a} \sum_{b=0}^{b} \binom{\left\lfloor \frac{2m-k}{2} \right\rfloor}{i} \binom{\left\lfloor \frac{2m+1-k}{2} \right\rfloor}{i+1} \binom{\left\lfloor \frac{q-2m-1}{2} \right\rfloor}{i} \binom{\left\lfloor \frac{q-2m-1}{2} \right\rfloor}{i+1}.$$

2. By analogy we can prove that

$$|W_\ast| = \sum_{a=0}^{n-1} \sum_{l=0}^{p-1} \sum_{a=0}^{a} \sum_{b=0}^{b} \binom{\left\lfloor \frac{2m-k}{2} \right\rfloor}{i} \binom{\left\lfloor \frac{2m+1-k}{2} \right\rfloor}{i+1} \binom{\left\lfloor \frac{q-2m-1}{2} \right\rfloor}{i} \binom{\left\lfloor \frac{q-2m-1}{2} \right\rfloor}{i+1}.$$

3. To calculate the number of $W_\ast$-words, fix any $W_\ast$-word $w \in W_\ast$ and write it as an alternating word $w = x_k \ldots x_{2m} \ldots x_q$ such that $k < 2m < q$, $x_{2m} \in L_-$, $x_{2m-1} = x_{2m+1} \in L_+$, the sequences $(x_{2m+2i})_{0 \leq i \leq \frac{2m-k}{2}}$ are strictly increasing in $L_-$ whereas the sequences $(x_{2m+1+2i})_{0 \leq i \leq \frac{q-2m-1}{2}}$ and $(x_{2m-1-2i})_{0 \leq i \leq \frac{q-2m-1}{2}}$ are strictly decreasing in $L_+$. It follows that $(x_{2m-2i})_{1 \leq i \leq \frac{2m-k}{2}}$ is a strictly increasing sequence of length $l = \left\lfloor \frac{2m-k}{2} \right\rfloor$ in the linearly ordered set $A = \{x \in L : x > x_{2m}\} \subset L_+$. The number of such sequences is equal to \( \binom{\left\lfloor \frac{2m-k}{2} \right\rfloor}{i} \) where $a = |A| < n$. By analogy, $(x_{2m+1+2i})_{0 \leq i \leq \frac{q-2m-1}{2}}$ is a strictly increasing sequence of length $r = \left\lfloor \frac{q-2m}{2} \right\rfloor$ in the linearly ordered set $A$ and the number of such sequences is equal to \( \binom{\left\lfloor \frac{q-2m}{2} \right\rfloor}{i} \).

If $l = \frac{2m-k}{2}$, then $2m = 2l + k$ and $(x_{2m-1-2i})_{1 \leq i \leq \frac{2m-k}{2}}$ is a strictly decreasing sequence of length $\left\lfloor \frac{2m-k}{2} \right\rfloor = l - 1$ in the linearly ordered set $B = \{x \in L : x < x_{2m-1} = x_{2m+1}\} \subset L_+$. The number of such sequences is equal to \( \binom{\left\lfloor \frac{2m-k}{2} \right\rfloor}{i} \) where $b = |B| < p$. If $l = \frac{2m-k-2}{2}$, then $(x_{2m-1-2i})_{1 \leq i \leq \frac{2m-k}{2}}$ is a strictly decreasing sequence of length $\left\lfloor \frac{2m-k-1}{2} \right\rfloor = l$ in the set $B$. The number of such sequences is equal to \( \binom{\left\lfloor \frac{2m-k-1}{2} \right\rfloor}{i} \).

If $r = \frac{q-2m}{2}$, then $2m = q - 2r$ and $(x_{2m+1+2i})_{0 \leq i \leq \frac{q-2m}{2}}$ is a strictly decreasing sequence of length $\left\lfloor \frac{q-2m}{2} \right\rfloor = r - 1$ in the linearly ordered set $B$. The number of such sequences is equal to \( \binom{\left\lfloor \frac{q-2m}{2} \right\rfloor}{i} \). If $r = \frac{q-2m-1}{2}$, then $2m = q - 1 - 2r$ and $(x_{2m+1+2i})_{1 \leq i \leq \frac{q-2m-1}{2}}$ is a strictly decreasing sequence of length $\left\lfloor \frac{q-2m-1}{2} \right\rfloor = r$ in the linearly ordered set $B$. The number of such sequences is equal to \( \binom{\left\lfloor \frac{q-2m-1}{2} \right\rfloor}{i} \). Summing up, we conclude that the family $W_\ast$ of all $W_\ast$-words has cardinality

$$|W_\ast| = \sum_{a=0}^{n-1} \sum_{l=0}^{p-1} \sum_{a=0}^{a} \sum_{b=0}^{b} \binom{\left\lfloor \frac{2m-k}{2} \right\rfloor}{i} \binom{\left\lfloor \frac{2m+1-k}{2} \right\rfloor}{i+1} \binom{\left\lfloor \frac{q-2m-1}{2} \right\rfloor}{i} \binom{\left\lfloor \frac{q-2m-1}{2} \right\rfloor}{i+1}.$$
4. By analogy we can prove that
\[
|\mathcal{W}_-| = \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \sum_{l=0}^{a} \sum_{r=0}^{a} \binom{a}{l} \binom{r}{p} \binom{b+1}{a} \binom{b}{a}.
\]

5. By the preceding items
\[
|\mathcal{K}_L| = |L| + |\mathcal{V}_x| + |\mathcal{V}_z| + |\mathcal{W}_-| + |\mathcal{W}_+| = 1 + n + p + |\mathcal{V}_x| + |\mathcal{W}_-| + |\mathcal{V}_z| + |\mathcal{W}_+| =
\]
\[
= 1 + n + p + \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \sum_{l=0}^{a} \sum_{r=0}^{a} \binom{a}{l} \binom{r}{p} \left( \binom{b+1}{a} \binom{b}{a} \binom{b}{a} \binom{b}{a} \binom{b}{a} \right)
\]
\[
= 1 + n + p + \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \sum_{l=0}^{a} \sum_{r=0}^{a} \binom{a}{l} \binom{r}{p} \left( \binom{b+2}{a} \binom{b+2}{a} \binom{b+2}{a} \binom{b+2}{a} \binom{b+2}{a} \right)
\]
\[
= 1 + n + p + \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \sum_{l=0}^{a} \sum_{r=0}^{a} \binom{a}{l} \binom{r}{p} \left( \binom{b+2}{a} \binom{b+2}{a} \binom{b+2}{a} \binom{b+2}{a} \binom{b+2}{a} \right)
\]
\[
= 1 + n + p + \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \sum_{l=0}^{a} \sum_{r=0}^{a} \binom{a}{l} \binom{r}{p} \left( \binom{b+1}{a} \binom{b}{a} \binom{a+b+2}{a} \right)
\]
\[
= 1 + n + p + \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \sum_{l=0}^{a} \sum_{r=0}^{a} \binom{a}{l} \binom{r}{p} \left( \binom{b+1}{a} \binom{b}{a} \binom{a+b+1}{a} \right) = K(n, p).
\]

6. An asymptotics of the sequence \(K(n)\)

In this section we study the asymptotical growth of the sequence \(K(n) = K(n, n)\) and prove Theorem 6.3 announced in the Introduction as a corollary of the following results.

For every integers \(0 \leq a, b \leq n\) put
\[
c_{a,b}(n) = \frac{(2n-a-b)}{(2n)} = \frac{n(n-1) \cdots (n-a+1) \cdot n(n-1) \cdots (n-b+1)}{2n(2n-1) \cdots (2n-a-b+1)}
\]
and observe that
\[
\lim_{n \to \infty} c_{a,b}(n) = 2^{-(a+b)}.
\]

For every \(n \geq 0\) put \(k(n) = K(n)/(2n)^2\) and observe that
\[
k(n) = \frac{K(n)}{(2n)^2} = \sum_{i,j=0}^{n} \frac{(i+j)}{(2n)^2} = \sum_{a,b=0}^{n} \frac{(n-a-b)^2}{(2n)^2} = \sum_{a,b=0}^{n} c_{a,b}(n)^2.
\]

**Proposition 6.1.** For every \(n \geq 0\) we have \(k(n) \leq \frac{16}{9}\).

**Proof.** \(k(0) = 1, k(1) = k(2) = \frac{7}{4} < \frac{16}{9}, k(3) = \frac{969}{709} < \frac{16}{9}, k(4) = \frac{9549}{4909} < \frac{16}{9}\). Suppose now that for some \(n \geq 5\) we have proved that \(k(n-1) \leq \frac{16}{9}\).

We shall use the following two lemmas.

**Lemma 6.2.** For each \(0 < a \leq n\) we have \(c_{a,0}(n) < 2^{-a}\).

**Proof.** This lemma follows from the equality \(c_{a,0}(n) = \frac{n(n-1) \cdots (n-a+1)}{2n(2n-1) \cdots (2n-a+1)}\) and the inequality \((2n-l) > 2(2n-l)\) holding for all \(0 < l \leq n\).

**Lemma 6.3.** \(\frac{16}{9}c_{1,1}(n)^2 + 2(c_{1,0}(n)^2 + c_{2,0}(n)^2) < \frac{1}{4} + 2\left( \frac{1}{4} + \frac{1}{16} \right)\) for \(n \geq 5\).

**Proof.** Observe that \(c_{1,1}(n) = \frac{n}{2(2n-1)}\), \(c_{1,0}(n) = 1/2\), and \(c_{2,0}(n) = \frac{n-1}{2(2n-1)}\). Routine transformations show that the inequality in the lemma are equivalent to \(19 < 4n\), which holds for \(n \geq 5\). \(\square\)
Now we have that
\[ k(n) = c_{0,0}(n)^2 + 2 \sum_{a=1}^{n} c_{a,0}(n)^2 + \sum_{a,b=0}^{n-1} c_{a,b}(n)^2 = 1 + 2 \sum_{a=1}^{n} c_{a,0}(n)^2 + \sum_{a,b=0}^{n-1} c_{a+1,b+1}(n)^2 = \]
\[ = 1 + 2 \sum_{a=1}^{n} c_{a,0}(n)^2 + \sum_{a,b=0}^{n-1} \frac{n^2}{(2n-1)^2} c_{a,b}(n-1)^2 = \]
\[ = 1 + 2 \sum_{a=1}^{n} c_{a,0}(n)^2 + 2 \sum_{a=1}^{n-2} c_{a,0}(n)^2 + c_{1,1}(n)^2 k(n-1) < \]
\[ < 1 + 2 \sum_{a=1}^{n} \frac{1}{4^a} + 2(c_{1,0}(n)^2 + c_{2,0}(n)^2) + c_{1,1}(n)^2 \cdot \frac{16}{9} < \]
\[ < 1 + 2 \sum_{a=3}^{\infty} \frac{1}{4^a} + 2\left(\frac{1}{4} + \frac{1}{4^2}\right) + \frac{1}{9} = \frac{16}{9} \]
according to Lemma 6.3.

Proposition 6.4. There exists a limit \( \lim_{n \to \infty} k(n) = 16/9 \).

Proof. The equality (1) implies that \( \lim_{n \to \infty} k(n) = \sum_{a,b=0}^{\infty} 2^{-2(a+b)} = \sum_{i=0}^{\infty} (i + 1)4^{-i} = 16/9 \). By Proposition 6.1 \( \lim_{n \to \infty} k(n) = 16/9 \).

By Stirling’s approximation, \( \lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n} (n/e)^n} = 1 \), which yields the second equality in Theorem 1.3.

7. Representing elements of Kuratowski monoids by Kuratowski words

Let \( K \) be a Kuratowski monoid with linear generating set \( L \) and let \( L_{\leq} = \{ x \in L : x < 1 \} \) and \( L_+ = \{ x \in L : x > 1 \} \) be the negative and positive parts of \( L \), respectively. Let \( FS_L = \bigcup_{n=1}^{\infty} L^n \) be the free semigroup over \( L \) and \( \pi : FS_L \to K \) be the homomorphism assigning to each word \( x_1 \ldots x_n \in FS_L \) the product \( x_1 \cdots x_n \) of its letters in \( K \). The homomorphism \( \pi : FS_L \to K \) induces a congruence \( \sim \) on \( FS_L \) which identifies two words \( u, v \in FS_L \) iff \( \pi(u) = \pi(v) \).

A word \( w \in FS_L \) is called irreducible if \( w \) has the smallest possible length in its equivalence class \( [w]_{\sim} = \{ u \in FS_L : u \sim w \} \). Since the set of natural numbers is well-ordered, for each element \( x \in K \) there is an irreducible word \( w \in FS_L \) such that \( x = \pi(w) \). Consequently, the cardinality of \( K \) does not exceed the cardinality of the set of irreducible words in \( FS_L \).

Theorem 7.1. Each irreducible word in \( FS_L \) is a Kuratowski word. Consequently, \( \pi(K_L) = K \) and \( |K| \leq |K_L| \).

If the set \( L \) is finite, then \( |K| = |K_L| = K(n,p) \) where \( n = |L_{\leq}| \) and \( p = |L_+| \).

Proof. We divide the proof of Theorem 7.1 into a series of lemmas.

Lemma 7.2. For any elements \( x, y \in L_{\leq} \cup \{1\} \) we get \( xy = \min\{x, y\} \).

Proof. Since \( L \) is linearly ordered, either \( x \leq y \) or \( y \leq x \).

If \( x \leq y \), then multiplying this inequality by \( x \), we get \( x = xx \leq xy \). On the other hand, multiplying the inequality \( y \leq 1 \) by \( x \), we get the reverse inequality \( xy \leq x = x \). Taking into account that \( x \leq xy \leq x \), we conclude that \( x = xy \).

If \( y \leq x \), then multiplying this inequality by \( y \), we get \( y = yy \leq xy \). On the other hand, multiplying the inequality \( x \leq 1 \) by \( y \), we get \( xy \leq 1y = y \). Taking into account that \( y \leq xy \leq y \), we conclude that \( xy = y = \min\{x, y\} \).

By analogy we can prove:

Lemma 7.3. For any elements \( x, y \in L_+ \cup \{1\} \) we get \( xy = \max\{x, y\} \).

Proof. If \( x \leq y \), then multiplying this inequality by \( y \), we obtain \( xy \leq yy = y \). On the other hand, multiplying the inequality \( 1 \leq x \) by \( y \), we get \( y = 1y \leq xy \). So, \( xy = y = \max\{x, y\} \).

If \( y \leq x \), then after multiplication by \( x \), we obtain \( xy \leq xx = x \). On the other hand, multiplying the inequality \( 1 \leq y \) by \( x \), we get \( x \leq xy \) and hence \( xy = x = \max\{x, y\} \).
Recall that a word $x_1 \ldots x_n \in FS_L$ is alternating if for each natural number $i$ with $1 \leq i < n$ the doubleton $\{x_i, x_{i+1}\}$ intersects both sets $L_-$ and $L_+$. According to this definition, one-letter words also are alternating. Lemmas 7.2 and 7.3 imply:

**Lemma 7.4.** Each irreducible word $w \in FS_L$ is alternating.

The following lemma will help us to reduce certain alternating words of length 4.

**Lemma 7.5.** If $x_1x_2x_3x_4 \in FS_L$ is an alternating word in the alphabet $L$ such that $x_1x_3 = x_1$ and $x_2x_4 = x_4$ in $K$, then $x_1x_2x_3x_4 \sim x_1x_4$ in $K$ and hence $x_1x_2x_3x_4 \sim x_1x_4$.

**Proof.** Two cases are possible.

1) $x_1, x_3 \in L_-$ and $x_2, x_4 \in L_+$. In this case the equalities $x_1x_3 = x_1$ and $x_2x_4 = x_4$ imply that $x_1 \leq x_3$ and $x_2 \leq x_4$ (see Lemmas 7.2 and 7.3). To see that $x_1x_2x_3x_4 = x_1x_4$, observe that $x_1x_2x_3x_4 \leq x_1x_2 \cdot 1 \cdot x_4 = x_1x_2x_4 = x_1x_4$.

On the other hand,

$$x_1x_4 = (x_1x_3)x_4 = x_1 \cdot 1 \cdot x_3x_4 \leq x_1x_2x_3x_4.$$  

These two inequalities imply the desired equality $x_1x_2x_3x_4 = x_1x_4$.

2) $x_1, x_3 \in L_+$ and $x_2, x_4 \in L_-$. In this case the equalities $x_1x_3 = x_1$ and $x_2x_4 = x_4$ imply that $x_1 \geq x_3$ and $x_2 \geq x_4$ (see Lemmas 7.2 and 7.3). To see that $x_1x_2x_3x_4 = x_1x_4$, observe that $x_1x_2x_3x_4 \leq x_1 \cdot 1 \cdot x_3x_4 = x_1x_3x_4 = x_1x_4$.

On the other hand,

$$x_1x_4 = x_1(x_2x_4) = x_1x_2 \cdot 1 \cdot x_4 \leq x_1x_2x_3x_4.$$  

These two inequalities imply the desired equality $x_1x_2x_3x_4 = x_1x_4$.

Now we are able to prove that each irreducible word $w \in FS_L$ is a Kuratowski word. If $w$ consists of a single letter, then it is trivially Kuratowski and we are done. So, we assume that $w$ has length $\geq 2$. By Lemma 7.4 the word $w$ is alternating and hence can be written as the product $w = x_k \cdots x_n$ for some $k \in \{0, 1\}$ and $n > k$ such that $x_2 \in L_-$ for all integer numbers $i$ with $k \leq 2i \leq n$ and $x_{2i-1} \in L_+$ for all integer numbers $i$ with $k < 2i - 1 \leq n$.

Let $m$ be the smallest number such that $k \leq 2m \leq n$ and $x_{2m} = \min\{x_{2i} : k \leq 2i \leq n\}$ in $L_-$. First we shall analyze the structure of the subword $x_k \cdots x_{2m}$ of the word $w = x_k \ldots x_n$.

**Lemma 7.6.** The sequence $(x_{2m-2i})_{0 \leq i \leq \frac{m-1}{2}}$ is strictly increasing in $L_-$.

**Proof.** Assume conversely that $x_{2i} \geq x_{2i-2}$ for some number $i \leq m$ and assume that $i$ is the largest possible number with this property. The definition of the number $m$ guarantees that $i < m$. Consequently, $x_{2i-2} \leq x_{2i}$ and $x_{2i} > x_{2i+2}$. Taking into account that $x_{2i-2}, x_{2i}, x_{2i+2} \in L_-$ and applying Lemma 7.2 we get $x_{2i-2}x_{2i} = \min\{x_{2i-2}, x_{2i}\} = x_{2i-2}$ and $x_{2i+2}x_{2i+2} = \min\{x_{2i+2}, x_{2i+2}\} = x_{2i+2}$.

Now consider the elements $x_{2i-1}, x_{2i+1} \in L_+$. If $x_{2i-1} \leq x_{2i+1}$, then $x_{2i-1}x_{2i+1} = x_{2i+1}$ by Lemma 7.3 and by Lemma 7.5 the alternating word $x_{2i-2}x_{2i-1}x_{2i+1} = x_{2i-2}x_{2i+1}$ is reducible as $x_{2i-2}x_{2i} = x_{2i-2}$ and $x_{2i-1}x_{2i+1} = x_{2i+1}$.

If $x_{2i-1} > x_{2i+1}$, then $x_{2i-1}x_{2i+1} = x_{2i-1}$ by Lemma 7.5. Lemma 7.5 guarantees that the alternating word $x_{2i-1}x_{2i-1}x_{2i+1}x_{2i+2} = x_{2i-1}x_{2i+2}$ is reducible as $x_{2i-1}x_{2i+1} = x_{2i-1}$ and $x_{2i+1}x_{2i+2} = x_{2i+2}$.

Therefore, the word $x_k \cdots x_n$ contains a reducible subword and hence is reducible, which contradicts the choice of this word. This contradiction shows that $x_{2i-2} < x_{2i}$ and hence $x_{2i-2}x_{2i} = \min\{x_{2i-2}, x_{2i}\} = x_{2i-2} \neq x_{2i}$ according to Lemma 7.2.

**Lemma 7.7.** The sequence $(x_{2m-2i})_{0 \leq i \leq \frac{m-1}{2}}$ is strictly decreasing in $L_+$.

**Proof.** Assume conversely that $x_{2i-1} \geq x_{2i+1}$ for some number $i < m$ with $k \leq 2i - 1$. Since $x_{2i-1}, x_{2i+1} \in L_+$, Lemma 7.3 implies that $x_{2i-1}x_{2i+1} = \max\{x_{2i-1}, x_{2i+1}\} = x_{2i-1}$. By Lemma 7.3 $x_{2i}x_{2i+2} = x_{2i+2}$. Then by Lemma 7.5 the alternating word $x_{2i-1}x_{2i}x_{2i+1}x_{2i+2}$ is equal to $x_{2i-1}x_{2i+2}$. This implies that the word $x_k \cdots x_n$ is reducible, which contradicts the choice of this word. This contradiction shows that $x_{2i-1} < x_{2i+1}$ and hence $x_{2i-1}x_{2i+1} = \max\{x_{2i-1}, x_{2i+1}\} = x_{2i+1} \neq x_{2i-1}$ according to Lemma 7.3.

Next, we consider the subword $x_{2m-1} \cdots x_n$ of the word $w = x_k \cdots x_n$.

**Lemma 7.8.** The sequence $(x_{2i+2})_{1 \leq i \leq \frac{m-2}{2}}$ is strictly increasing in $L_-$. 

**Proof.** Assume conversely that $x_{2i} \geq x_{2i+2}$ for some number $i \leq m - 1$ with $k \leq 2i$. Since $x_{2i-1}, x_{2i+1} \in L_-$, Lemma 7.2 implies that $x_{2i-1}x_{2i+1} = x_{2i-1}$ and $x_{2i}x_{2i+2} = x_{2i+2}$. Then by Lemma 7.5 the alternating word $x_{2i}x_{2i+1}x_{2i+2}$ is equal to $x_{2i}x_{2i+2}$. This implies that the word $x_k \cdots x_n$ is reducible, which contradicts the choice of this word. This contradiction shows that $x_{2i} < x_{2i+2}$ and hence $x_{2i}x_{2i+1} = x_{2i+1} \neq x_{2i}$ according to Lemma 7.2.
Proof. Assume conversely that \( x_{2i} \geq x_{2i+2} \) for some number \( i > m \) with \( 2i + 2 \leq n \). We can assume that \( i \) is the smallest possible number with this property. Then either \( i = m + 1 \) or else \( x_{2i+2} < x_{2i} \). If \( i = m + 1 \), then \( x_{2m} = x_{2i-2} \leq x_{2i} \) by the choice of \( m \). In both cases we get \( x_{2i-2} \leq x_{2i} \), which implies \( x_{2i-2}x_{2i} = \min\{x_{2i-2}, x_{2i}\} = x_{2i-2} \) according to Lemma 7.2. The same Lemma 7.2 implies that \( x_{2i}x_{2i+2} = \min\{x_{2i}, x_{2i+2}\} = x_{2i+2} \).

Now consider the elements \( x_{2i-1}, x_{2i+1} \in L_+ \). If \( x_{2i-1} \geq x_{2i+1} \), then \( x_{2i-1}x_{2i+1} = \max\{x_{2i-1}, x_{2i+1}\} = x_{2i-1} \) and the alternating word \( x_{2i-1}x_{2i+1}x_{2i+1}x_{2i+2} = x_{2i-1}x_{2i+2} \) is reducible according to Lemma 7.5. If \( x_{2i-1} < x_{2i+1} \), then \( x_{2i-1}x_{2i+1} = \max\{x_{2i-1}, x_{2i+1}\} = x_{2i+1} \) and the alternating word \( x_{2i-1}x_{2i+1}x_{2i+1} = x_{2i-1}x_{2i+2} \) is reducible by Lemma 7.5. But this contradicts the irreducibility of the word \( x_k \cdots x_n \). So, \( x_{2i} < x_{2i+2} \) and \( x_{2i}x_{2i+2} = \min\{x_{2i}, x_{2i+2}\} = x_{2i+2} \) according to Lemma 7.2.

Lemma 7.9. The sequence \( (x_{2m+1+2})_{0 \leq i \leq \frac{n-2}{2}} \) is strictly decreasing in \( L_+ \).

Proof. Assume conversely that \( x_{2i-1} \leq x_{2i+1} \) for some \( i > m \) with \( 2i + 1 \leq n \). Then \( x_{2i-1}x_{2i+1} = \max\{x_{2i-1}, x_{2i+1}\} = x_{2i+1} \) according to Lemma 7.3. If \( i = m + 1 \), then \( x_{2i-2}x_{2i+1} = x_{2m}x_{2i} = \min\{x_{2i}, x_{2i+1}\} = x_{2m} \) by the choice of the number \( m \). If \( i > m + 1 \), then \( x_{2i-2}x_{2i+1} \) is reducible by Lemma 7.5. By Lemma 7.3 the alternating word \( x_{2i-2}x_{2i+1}x_{2i+1}x_{2i+2} = x_{2i-2}x_{2i+2} \) is reducible, which contradicts the irreducibility of the word \( x_k \cdots x_n \). This contradiction shows that \( x_{2i-1} > x_{2i+1} \) and hence \( x_{2i-1}x_{2i+1} = \max\{x_{2i-1}, x_{2i+1}\} = x_{2i-1} \) \( \neq x_{2i+1} \) according to Lemma 7.3.

Now we are ready to complete the proof of Theorem 7.1. Five cases are possible.

1) \( 2m = n \). In this case \( x_k \ldots x_n = x_k \ldots x_2m \) is a \( V_\exists \)-word by Lemmas 7.6 and 7.9.

2) \( 2m + 1 = n \) and \( k = 2m \). In this case \( x_k \ldots x_n = x_{2m}x_{2m+1} \) is a \( V_\exists \)-word.

3) \( 2m + 1 = n \) and \( k \leq 2m - 1 \). This case has three subcases.

a) If \( x_{2m-1} < x_{2m+1} \), then \( x_k \ldots x_n = x_k \ldots x_{2m-1}x_{2m}x_{2m+1} \) is a \( V_\exists \)-word by Lemmas 7.6 and 7.7.

b) If \( x_{2m-1} > x_{2m+1} \), then \( x_k \ldots x_n = x_k \ldots x_{2m-1}x_{2m}x_{2m+1} \) is a \( V_\exists \)-word by Lemmas 7.6 and 7.4.

3c) If \( x_{2m-1} = x_{2m+1} \), then \( x_k \ldots x_n = x_k \ldots x_{2m-1}x_{2m}x_{2m+1} \) is a \( W_\exists \)-word by Lemmas 7.6 and 7.7.

4) \( 2m + 2 \leq n \) and \( k = 2m \). Since \( x_{2m} \leq x_{2m+2} \), this case has two subcases.

a) If \( x_{2m} < x_{2m+2} \), then \( x_k \ldots x_n = x_k \ldots x_{2m}x_{2m+1}x_{2m+2} \ldots x_n \) is a \( V_\exists \)-word by Lemmas 7.8 and 7.9.

b) If \( x_{2m} = x_{2m+2} \), then \( x_k \ldots x_n = x_k \ldots x_{2m}x_{2m+1}x_{2m+2} \ldots x_n \) is a \( W_\exists \)-word by Lemmas 7.8 and 7.9.

5) \( 2m + 2 \leq n \) and \( k \leq 2m - 1 \). This case has four subcases.

a) \( x_{2m} < x_{2m+2} \) and \( x_{2m-1} < x_{2m+1} \). In this case \( x_k \ldots x_n = x_k \ldots x_{2m}x_{2m+1} \ldots x_n \) is a \( V_\exists \)-word by Lemmas 7.6, 7.9.

b) \( x_{2m} < x_{2m+2} \) and \( x_{2m-1} > x_{2m+1} \). In this case \( x_k \ldots x_n = x_k \ldots x_{2m}x_{2m-1}x_{2m} \ldots x_n \) is a \( V_\exists \)-word by Lemmas 7.6, 7.9.

b) \( x_{2m} < x_{2m+2} \) and \( x_{2m-1} = x_{2m+1} \). In this case \( x_k \ldots x_n = x_k \ldots x_{2m}x_{2m-1}x_{2m} \ldots x_n \) is a \( W_\exists \)-word by Lemmas 7.6, 7.9.

5c) \( x_{2m} = x_{2m+2} \). In this case we shall prove that \( x_{2m-1} > x_{2m+1} \). Assuming that \( x_{2m-1} \leq x_{2m+1} \) we can apply Lemma 7.3 to conclude that \( x_{2m-1}x_{2m+1} = \max\{x_{2m-1}, x_{2m+1}\} = x_{2m-1} \). It follows from \( x_{2m} = x_{2m+2} \) that \( x_{2m}x_{2m+2} = x_{2m+2} \). By Lemma 7.5 the alternating word \( x_{2m-1}x_{2m}x_{2m+1}x_{2m+2} \) is reducible, which is a contradiction. So, \( x_{2m-1} > x_{2m+1} \) and hence \( x_k \ldots x_n = x_k \ldots x_{2m}x_{2m+1}x_{2m+2} \ldots x_n \) is a \( W_\exists \)-word by Lemmas 7.6, 7.9.

Therefore each irreducible word in \( FS_L \) is a Kuratowski word, which implies that \( |K| \leq |K_L| \). If the set \( L \) is finite, then the set \( K_L \) of Kuratowski words over \( L \) has cardinality \( |K_L| = K(|L_-|, |L_+|) \), see Theorem 5.2.

8. Separation of Kuratowski words by homomorphisms

In the preceding section we proved that any element of a Kuratowski monoid \( K \) with a linear generating set \( L \) can be represented by a Kuratowski word \( w \in K_L \). In this section we shall prove that Kuratowski words can be separated by homomorphisms into the Kuratowski monoids of suitable 2-topological spaces.

Given an \( n \)-topological space \( X = (\{X_i\}_{i \in \Bbb N}, \tau) \), observe that the linear generating set

\[
L(X) = \{t_i\}_{i \in \Bbb N} \cup \{1_X\} \cup \{\bar t_i\}_{i \in \Bbb N}
\]

of its Kuratowski monoid \( K(X) \) is symmetric.

This observation motivates the following definition. A *linearly ordered set* is a linearly ordered set \( L \) endowed with an involutive bijection \( * : L \rightarrow L \), \( * : \ell \mapsto \ell^* \), that has a unique fixed point \( 1 \in L \) and is
decreasing in the sense that for any elements \( x < y \) in \( L \) we get \( x^* > y^* \). Each \( * \)-linearly ordered set \( L \) is pointed — the unit of \( L \) is the unique fixed point of the involution \( * : L \to L \). Observe that the structure of a \( * \)-linearly ordered set \( L \) is determined by the structure of its negative part \( L_- \).

A map \( f : L \to \Lambda \) between two \( * \)-linearly ordered sets \( L, \Lambda \) will be called a \( * \)-morphism if \( f \) is monotone (in the sense that for any elements \( x \leq y \) in \( L \) we get \( f(x) \leq f(y) \)) and preserves the involution (in the sense that \( f(x^*) = f(x)^* \) for every \( x \) in \( L \). Since \( f(1) = f(1^*) = f(1)^* \), the image \( f(1) \) of the unit of \( L \) coincides with the unit of \( \Lambda \). Observe that each \( * \)-morphism \( f : L \to \Lambda \) is uniquely determined by its restriction \( f|_{L_-} \).

For a \( * \)-linearly ordered set \( L \), the involution \( * : L \to L \) of \( L \) has a unique extension to an involutive semigroup isomorphism \( * : FS_L \to FS_L \) of the free semigroup over \( L \). The image of a word \( w \in FS_L \) under this involutive isomorphism will be denoted by \( w^* \).

Let \( X = (X, T) \) be a polytopological space and

\[
L(X) = \{ \tau : \tau \in T \} \cup \{ 1_X \} \cup \{ \tau : \tau \in T \}
\]

be the linear generating set of the Kuratowski monoid \( K(X) \) of \( X \). Observe that each topology \( \tau \) is determined by its interior operator \( \bar{\tau} \) (since \( \tau = \{ \bar{\tau}(A) : A \subset X \} \)). This implies that the interior operators \( \bar{\tau}, \bar{\tau} \in T \), are pairwise distinct. The same is true for the closure operators \( \bar{\tau}, \bar{\tau} \in T \). This allows us to define a bijective involution \( * : L(X) \to L(X) \) letting \( \bar{\tau}^* = \bar{\tau} \) and \( \bar{\tau}^* = \bar{\tau} \) for every \( \tau \in T \). This involution turns \( L(X) \) into a \( * \)-linearly ordered set.

Let \( L \) be a \( * \)-linearly ordered set. Choose any point \( c \notin L \) and consider the free semigroup \( FS_{L \cup \{ c \}} \) over the set \( L \cup \{ c \} \). This semigroup consists of words in the alphabet \( L \cup \{ c \} \). Let \( X = (X, T) \) be a polytopological space and \( L(X) \) be the linear generating set of the Kuratowski monoid \( K(X) \) of \( X \). Let \( c_X : \mathcal{P}(X) \to \mathcal{P}(X), c_X : A \mapsto X \setminus A \), denote the operator of taking complement.

Given any \( * \)-morphism \( f : L \to L(X) \) let \( \bar{f} : FS_{L \cup \{ c \}} \to K_2(X) \) be a (unique) semigroup homomorphism such that \( \bar{f}(1) = 1_X, \bar{f}(c) = c_X \), and \( \bar{f}(\ell) = f(\ell) \) for \( \ell \in L \). The homomorphism \( \bar{f} \) will be called the Kuratowski extension of \( f \).

Observe that \( \bar{f}(K_L) \subset K(X) \). In the semigroup \( FS_{L \cup \{ c \}} \) consider the subset

\[
\widetilde{K}_L = K_L \cup \{ cw : w \in K_L \} \subset FS_{L \cup \{ c \}}
\]

whose elements will be called full Kuratowski words.

**Theorem 8.1.** For any \( * \)-linearly ordered set \( L \) and any two distinct words \( u, v \in \widetilde{K}_L \) there is a 2-topological space \( X \), and a \( * \)-morphism \( f : L \to L(X) \) whose Kuratowski extension \( \bar{f} : FS_{L \cup \{ c \}} \to K_2(X) \) separates the words \( u, v \) in the sense that \( \bar{f}(u) \neq \bar{f}(v) \).

**Proof.** In most of the cases the underlying set of the 2-topological space \( X \) will be a set \( X = \{ x, y \} \) containing two pairwise distinct points \( x, y \) and the topologies of \( X \) are equal to one of four possible topologies on \( X \):

- \( \tau_d = \{ \emptyset, \{ x \}, \{ y \}, X \} \), the discrete topology on \( X \);
- \( \tau_a = \{ \emptyset, X \} \), the anti-discrete topology on \( X \);
- \( \tau_s = \{ \emptyset, \{ x \}, X \} \);
- \( \tau_y = \{ \emptyset, \{ y \}, X \} \).

Fix any two distinct words \( u, v \in \widetilde{K}_L \) and consider four cases.

1) \( u \in K_L \) and \( v \notin K_L \). In this case consider the 2-topological space \( X = (X, (\tau_a, \tau_d)) \). Then for the \( * \)-morphism \( f : L \to \{ 1_X \} \subset L(X) \) we get \( \bar{f}(u) = 1_X \neq c_X = \bar{f}(v) \).

2) \( v \in K_L \) and \( u \notin K_L \). In this case take the 2-topological space \( X \) from the preceding case and observe that for the \( * \)-morphism \( f : L \to \{ 1_X \} \subset L(X) \) we get \( \bar{f}(u) = c_X \neq 1_X = \bar{f}(v) \).

3) \( u, v \in K_L \). Denote by \( u_0, v_0 \in L \) the last letters of the words \( u, v \), respectively. Consider two cases.

3a) \( u_0 \neq v_0 \). We lose no generality assuming that \( u_0 < v_0 \) (in the linearly ordered set \( L \)).

Five subcases are possible:

3aa) \( u_0 = 1 \) and \( v_0 \in L_+ \). In this case consider the 2-topological space \( X = (\{ x, y \}, (\tau_a, \tau_d)) \) and the \( * \)-morphism \( f : L \to L(X) \) assigning to each \( \ell \in L_- \) the operator \( \tau_\ell_a \). Then for the subset \( A = \{ x \} \) of \( X \) and the operators \( \bar{u} = \bar{f}(u) \) and \( \bar{v} = \bar{f}(v) \), we get \( \bar{u}(A) = A \neq X = \tau_\ell_a(A) = \bar{v}(A) \), which implies that \( \bar{f}(u) \neq \bar{f}(v) \).

3ab) \( u_0 \in L_- \) and \( v_0 = 1 \). In this case, take the 2-topological space \( X \), the subset \( A = \{ x \} \), and the \( * \)-morphism \( f : L \to L(X) \) from the preceding case. Then \( \bar{u}(A) = \emptyset \neq A = \bar{v}(A) \), which implies that \( \bar{f}(u) \neq \bar{f}(v) \).
3ac) \( u_0 \in L_- \) and \( v_0 \in L_+ \). In this case, take the 2-topological space \( X \), the subset \( A = \{x\} \) and the \(*\)-morphism \( f : L \to L(X) \) from case (3aa). Then \( \hat{u}(A) = \emptyset \neq X = \hat{v}(A) \), which implies that \( \hat{f}(u) \neq \hat{f}(v) \).

3ad) \( u_0, v_0 \in L_- \). Consider the 2-topological space \( X = (\{x, y\}, (\tau_a, \tau_x)) \) and the \(*\)-morphism \( f : L \to L(X) \) assigning to each \( \ell \in L_- \) the operator

\[
\hat{f}(\ell) = \begin{cases} 
\hat{\tau}_a & \text{if } \ell \leq u_0; \\
\hat{\tau}_x & \text{if } \ell > u_0.
\end{cases}
\]

Put \( \hat{u} = \hat{f}(u) \) and \( \hat{v} = \hat{f}(v) \). Observe that for the subset \( A = \{x\} \) we get \( \hat{u}_0(A) = \hat{\tau}_a(A) = \emptyset \) and hence \( \hat{u}(A) = \emptyset \). Next, we evaluate \( \hat{v}(A) \). Write the Kuratowski word \( v = v_0 \ldots v_q \) where \( v_0, \ldots, v_q \in L \{1\} \). If \( q = 0 \), then \( \hat{v}(A) = \hat{v}_0(A) = \hat{\tau}_x(A) = A \neq \emptyset = \hat{u}(A) \). If \( q > 0 \), then \( \hat{v}_1 = \hat{v}_0(A) = \hat{v}_1(A) = \{\hat{\tau}_a(A), \hat{\tau}_x(A)\} = \{X\} \) and hence \( \hat{v}(A) = X \neq \emptyset = \hat{u}(A) \). This yields the desired inequality \( \hat{f}(u) \neq \hat{f}(v) \).

3ae) \( u_0, v_0 \in L_+ \). In this case we can consider the conjugated words \( u^* \) and \( v^* \) and observe that their last letters are distinct and belong to the set \( L_- \). By the preceding item, there are a 2-topological space \( X \) and a \(*\)-morphism \( f : L \to L(X) \) such that \( \hat{f}(u^*) \neq \hat{f}(v^*) \). Then \( \hat{f}(u)^* = \hat{f}(u^*) \neq \hat{f}(v^*) = \hat{f}(v)^* \) and hence \( \hat{f}(u) \neq \hat{f}(v) \).

Next, consider the case:

3b) \( u_0 = v_0 \). It follows that \( u_0 = v_0 \in L \{1\} \). Write the Kuratowski words \( u, v \) as \( u = u_p \ldots u_0 \) and \( v = v_q \ldots v_0 \) where \( u_i, v_i \in L \{1\} \). For the sake of consistency it will be convenient to assume that \( u_i = 1 \) for every integer \( i \notin \{0, \ldots, p\} \) and \( v_j = 1 \) for every integer \( j \notin \{0, \ldots, q\} \). Un particular, \( u_{-1} = v_{-1} = 1 \).

Since \( u \neq v \), the number \( k = \min\{i \geq 0 : u_i \neq v_i\} \) is well-defined and does not exceed \( \max\{p, q\} \). It follows from \( u_0 = v_0 \) that \( k \geq 1 \). By the definition of \( k \) we also get the equality \( u_{k-1} \ldots u_0 = v_{k-1} \ldots v_0 \). First we consider the case \( u_{k-1} = v_{k-1} \in L_+ \). Since \( u_k \neq v_k \), we lose no generality assuming that \( u_k < v_k \). Since \( u, v \) are alternating words, the inclusion \( u_{k-1} = v_{k-1} \in L_+ \) implies \( u_k, v_k \in L_- \cup \{1\} \) and \( u_k \in L_- \).

Two cases are possible:

3ba) \( u_k < u_{k-2} \) (this case includes also the case of \( k = 1 \) in which \( u_1 = 1 = u_{-1} \)). Since \( u \in K_L \), the inequality \( u_k < u_{k-2} \) implies that the sequence \( (u_{k-2i})_{0 \leq i \leq \frac{k}{2}} \) is strictly increasing in \( L_- \) and the sequence \( (u_{k-1-2i})_{0 \leq i \leq \frac{k}{2}} = (v_{k-1-2i})_{0 \leq i \leq \frac{k}{2}} \) is strictly decreasing in \( L_+ \).

Now consider two subcases:

3baa) \( v_{k+1} > v_{k-1} \). In this case \( (v^*_{k+1-2i})_{0 \leq i \leq \frac{k}{2}} \) is a strictly increasing sequence in \( L_- \). Depending on the relation between the elements \( u_k \) and \( v^*_{k+1} \) of \( L_- \) we shall distinguish two subcases.

3baaa) \( v^*_{k+1} \leq u_k \). In this case consider the 2-topological space \( X = (X, (\tau_a, \tau_y)) \) where \( X = \{x, y\} \) and define the \(*\)-morphism \( f : L \to L(X) \) assigning to each \( \ell \in L_- \) the operator

\[
\hat{f}(\ell) = \begin{cases} 
\hat{\tau}_a & \text{if } \ell \leq v^*_{k+1}; \\
\hat{\tau}_x & \text{if } v^*_{k+1} < \ell \leq u_k; \\
1_X & \text{if } u_k \leq \ell.
\end{cases}
\]

Consider the subset \( A = \{x\} \subset X \). Since the sequence \( (u_{k-2i})_{0 \leq i \leq \frac{k}{2}} \) is strictly increasing in \( L_- \), for every positive number \( i \leq \frac{k}{2} \) we get \( v_{k-2i} = u_{k-2i} > u_k \) and hence \( \hat{u}_{k-2i} = \hat{v}_{k-2i} = 1_X \), which implies that \( \hat{u}_k = \hat{v}_k = \hat{v}_{k-2i} = \hat{v}_{k-2i-2i} = 1_X \), and hence \( \hat{u}_k = \hat{v}_k = \hat{v}_{k-2i} = \hat{v}_{k-2i-2i} = 1_X \), and hence \( \hat{u}_k = \hat{v}_k = \hat{v}_{k-2i} = \hat{v}_{k-2i-2i} = 1_X \). On the other hand, for every positive \( i \leq \frac{k}{2} \) we get \( u^*_{k+1-2i} = v^*_{k+1-2i} = v_{k+1} \), and hence \( \hat{u}_{k+1-2i} = \hat{v}_{k+1-2i} \in \{\hat{\tau}_y, 1_X\} \), which implies \( \hat{u}_{k+1-2i} = \hat{v}_{k+1-2i} \in \{\hat{\tau}_y(A), 1_X(A)\} = \{A\} \). So, \( \hat{u}_k = \hat{v}_k \).

Observe that \( \hat{u}_k \neq \hat{u}_k(A) = \hat{\tau}_y(A) = \emptyset \) and hence \( \hat{u}(A) = \emptyset \). On the other hand, the inequality \( v_k > u_k \) implies \( \hat{v}_k = 1_X \) and then \( \hat{v}_k \neq \hat{u}_k \). This yields the desired inequality \( \hat{f}(u) \neq \hat{f}(v) \).

3baab) \( v^*_{k+1} > u_k \). In this case consider the 2-topological space \( X = (X, (\tau_a, \tau_x)) \) where \( X = \{x, y\} \) and define a \(*\)-morphism \( f : L \to L(X) \) assigning to each \( \ell \in L_- \) the operator

\[
\hat{f}(\ell) = \begin{cases} 
\hat{\tau}_a & \text{if } \ell \leq u_k; \\
\hat{\tau}_x & \text{if } u_k < \ell \leq v^*_{k+1}; \\
1_X & \text{if } u_k < \ell.
\end{cases}
\]

Consider the subset \( A = \{x\} \subset X \). Since the sequence \( (u_{k-2i})_{0 \leq i \leq \frac{k}{2}} \) is strictly increasing in \( L_- \), for every positive \( i \leq \frac{k}{2} \) we get \( v_{k-2i} = u_{k-2i} > u_k \) and hence \( \hat{u}_{k-2i} = \hat{v}_{k-2i} = 1_X \), which implies that \( \hat{u}_{k-2i}(A) = \hat{v}_{k-2i}(A) = A \).
\( \hat{u}_{k-2i}(A) \in \{ \tilde{\tau}_x(A), 1_X(A) \} = \{ A \} \). Since the sequence \( (v_{k+1-2i})_{0 \leq i \leq \frac{k+1}{2}} \) is strictly increasing in \( L_+ \), for every positive \( i \leq \frac{k+1}{2} \) we get \( u^*_{k+1-2i} = v^*_{k+1-2i} \) and hence \( \hat{u}_{k+1-2i} = \hat{v}_{k+1-2i} = 1_X \), which implies
\[ \hat{u}_{k+1-2i}(A) = \tilde{\tau}_{k+1-2i}(A) = A. \]
So, \( \hat{u}_{k-1} \cdots \hat{u}_0(A) = \hat{u}_{k-1} \cdots \hat{u}_0(A) = A \).

Observe that \( \hat{u}_k \cdots \hat{u}_1(A) = \tilde{\tau}_x(A) = \emptyset \) and hence \( \hat{u}(A) = \emptyset \). Next, we evaluate \( \hat{v}(A) \). If \( v_k = 1 \), then \( \hat{v}(A) = \hat{v}_k \cdots \hat{v}_1(A) = A \neq \emptyset = \hat{u}(A) \). If \( v_k \neq 1 \) but \( v_{k+1} = 1 \), then \( u_k < v_k \) implies that \( \hat{v}_k \in \{ \tilde{\tau}_x, 1_X \} \) and hence \( \hat{v}(A) = \hat{v}_k \cdots \hat{v}_0(A) = \tilde{\tau}_x(A), 1_X(A) = \{ A \} \) and hence \( \hat{v}(A) = A \neq \emptyset = \hat{u}(A) \). If \( v_{k+1} \neq 1 \), then \( \hat{v}_{k+1} \cdots \hat{v}_1(A) = \hat{v}_{k+1}(A) = \tilde{\tau}_x(A) = X \) and then \( \hat{v}(A) = X \neq \emptyset = \hat{u}(A) \). This yields the desired inequality \( \hat{f}(u) \neq \hat{f}(v) \).

Next, consider the subcase:

3bab) \( v_{k+1} = v_{k-1} \). In this case \( v \in W_x \) and hence the sequences \( (v_{k-2i})_{0 \leq i \leq \frac{k-1}{2}} \) and \( (v_{k+1+2i})_{0 \leq i \leq \frac{k-1}{2}} \) are strictly increasing in \( L_- \) whereas the sequences \( (v_{k-1-2i})_{0 \leq i \leq \frac{k-1}{2}} \) and \( (v_{k+1+2i})_{0 \leq i \leq \frac{k-1}{2}} \) are strictly decreasing in \( L_+ \). Consider the 2-topological space \( X = \{(x, y), (\tau_x, \tau_y)\} \) and define a \(*\)-morphism \( f : L \to L(X) \) assigning to each element \( \ell \in L_- \) the operator
\[
f(\ell) = \begin{cases} 
\tilde{\tau}_y & \text{if } \ell \leq u_k; \\
1_X & \text{if } \ell > u_k.
\end{cases}
\]
Also consider the subset \( A = \{ x \} \) in the 2-topological space \( X \).

Taking into account that \( u_k < u_k \leq u_{k+2i} \) for any \( i \in \mathbb{Z} \) with \( 0 \leq k + 2i \leq q \), we conclude that \( \hat{v}_{k+2i} = 1_X \).

On the other hand, for every \( i \in \mathbb{Z} \) with \( 1 \leq k + 1 + 2i \leq q \) we get \( \hat{v}_{k+1+2i} \in \{ \tilde{\tau}_y, 1_X \} \) and hence \( \hat{v}_{k+1+2i}(A) \in \{ \tilde{\tau}_y(A), 1_X(A) \} = \{ A \} \). This implies that \( \hat{v}(A) = A \).

On the other hand, \( \hat{u}_k \cdots \hat{u}_0(A) = \hat{u}_k \hat{v}_{k-1} \cdots \hat{v}_0(A) = \hat{u}_k(A) = \tilde{\tau}_y(A) = \emptyset \) and then \( \hat{u}(A) = \emptyset \neq A = \hat{v}(A) \). This implies that \( \hat{f}(u) \neq \hat{f}(v) \).

Finally, consider the subcase:

3bac) \( v_{k+1} < v_{k-1} \). Taking into account that \( v \in K_L \) and the sequence \( (v_{k-1-2i})_{0 \leq i \leq \frac{k-1}{2}} \) is strictly decreasing in \( L_+ \), we conclude that \( v_{k-1} > v_{k-1+2i} \) for any non-zero integer number \( i \) with \( 0 \leq k - 1 + 2i \leq q \) and \( u_k < \min(u_{k-2}, u_k) = \min(v_{k-2}, v_k) \leq v_{k-2+2i} \) for any integer number \( i \) with \( k + 2i \in \{ 0, \ldots, q \} \). Take the 2-topological space \( X \), the subset \( A = \{ x \} \), and the \(*\)-morphism \( f : L \to L(X) \) from the case (3bab). By analogy with the preceding case it can be shown that \( \hat{v}(A) = A \neq \emptyset = \hat{u}(A) \) and hence \( \hat{f}(u) \neq \hat{f}(v) \). This completes the proof of case (3bab).

So, now we consider the case:

3bb) \( u_k \geq u_{k-2} \). This case is more difficult and requires to consider a set \( X = \{ x, y, z \} \) of cardinality \( |X| = 3 \) endowed with the topologies:
\[
\begin{align*}
\tau_{x,z} &= \{ \emptyset, \{ x \}, \{ z \}, \{ x, z \}, X \}, \\
\tau_{x,z,y} &= \{ \emptyset, \{ x \}, \{ y \}, \{ z \}, \{ x, y \}, \{ x, z \}, X \}, \\
\tau_{x,z,y} &= \{ \emptyset, \{ x \}, \{ y \}, \{ x, y \}, \{ x, z \}, X \}.
\end{align*}
\]
In \( X \) consider the subset \( A = \{ x \} \).

The inequality \( u_{k-2} \leq u_k \in L_- \) and the inclusion \( u \in K_L \) imply that the sequence \( (u_{k-1+2i})_{0 \leq i \leq \frac{k-1}{2}} \) is strictly decreasing in \( L_+ \). By analogy, the (strict) inequality \( v_{k-2} = v_{k-2} \leq u_k < v_k \) implies that sequence \( (v_{k-1+2i})_{0 \leq i \leq \frac{k-1}{2}} \) is strictly increasing in \( L_- \) and \( (v_{k-1+2i})_{0 \leq i \leq \frac{k-1}{2}} \) is strictly decreasing in \( L_+ \). Let \( u_{k-1}^* \) be the letter in \( L_+ \), symmetric to the letter \( u_{k-1} \) with respect to \( 1 \). Depending on the relation between \( u_{k-1}^* \) and \( u_k \) we consider two subcases:

3ba) \( u_{k-1}^* \leq u_k \). In this case consider the 2-topological space \( X = (X, (\tau_{x,z}, \tau_{x,z,y})) \) where \( X = \{ x, y, z \} \) and define the \(*\)-morphism \( f : L \to L(X) \) assigning to every element \( \ell \in L_- \) the operator
\[
f(\ell) = \begin{cases} 
\tilde{\tau}_x & \text{if } \ell \leq u_{k-1}^*; \\
\tilde{\tau}_{x,z} & \text{if } u_{k-1}^* < \ell \leq u_k; \\
1_X & \text{if } u_k < \ell.
\end{cases}
\]
Consider the subset \( A = \{ x \} \) of \( X \). Observe that for every \( \ell \in L \) we get \( \{ x \} \subset \hat{\ell}(\{ x \}) \subset \hat{\ell}(\{ x, y \}) \subset \{ x, y \} \). Then
\[
\{ x \} \subset \hat{u}_{k-2} \hat{u}_{k-3} \cdots \hat{u}_0(A) \subset \{ x, y \}
\]
and
\[
\{ x, y \} = \tilde{\tau}_{x,z}(\{ x \}) \subset \tilde{\tau}_{x,z}(\{ x, y \}) \subset \hat{u}_{k-1} \hat{u}_{k-2} \cdots \hat{u}_0(A) \subset \hat{u}_{k-1}(\{ x, y \}) = \tilde{\tau}_{x,z}(\{ x, y \}) = \{ x, y \}.
\]
So, \( \hat{u}_{k-1} \cdots \hat{u}_0(A) = \hat{u}_{k-1} \cdots \hat{u}_0(A) = \{x, y\} \) and \( \hat{u}_k \cdots \hat{u}_1(A) = \hat{u}_k(\{x, y\}) = \hat{r}_{x, y}\). Taking into account that \( u_k < v_k \) and the sequence \( (u_k, v_k)_{0 \leq k \leq \infty} \) is strictly increasing in \( L^- \), we conclude that \( \{u_k, v_k\}_{0 \leq k \leq \infty} \subset \{\hat{r}_{x, y}, 1\} \), which implies that \( \hat{v}(A) = \hat{v}_k \cdots \hat{v}_0(A) \subset \{x, y\} \neq \{x\} \cup \hat{u}(A) \). This yields the desired inequality \( \hat{f}(u) \neq \hat{f}(v) \).

3.bbb) \( u_k < u_k^* \). In this case consider the 2-topological space \( X = (X, (\tau_{x, z}, \tau_{x, z, xy})) \) where \( X = \{x, y, z\} \). Define a \( * \)-morphism \( f : L \to L(X) \) assigning to every element \( \ell \in L^- \) the operator

\[
\hat{f}(\ell) = \begin{cases} 
\hat{r}_{x, z} & \text{if } \ell \leq u_k; \\
\hat{r}_{x, z, xy} & \text{if } u_k \leq \ell \leq u_{k-1}^*; \\
1_X & \text{if } u_{k-1}^* < \ell.
\end{cases}
\]

Consider the subset \( A = \{x\} \) of \( X \). It follows that

\[
\hat{u}_{k-1} \cdots \hat{u}_0(\{x\}) = \hat{u}_{k-1}(\{x\}) = \{\hat{r}_{x, z}(\{x\}), \hat{r}_{x, z, xy}(\{x, y\})\} = \{\{x\}\}
\]

and hence \( \hat{u}_k \cdots \hat{u}_0(A) = \hat{u}_k(\{x, y\}) = \hat{r}_{x, z}(\{x, y\}) = \{x\} \). Taking into account that the sequence \( (u_k, v_k)_{0 \leq k \leq \infty} \) is strictly increasing in \( L^- \), we conclude that \( \hat{u}_{k-1}^* = 1_X \) for all positive \( i \leq \frac{v_k - u_k}{2} \).

This implies that \( \hat{v}(A) = \hat{v}_k \cdots \hat{v}_0(A) \subset A = \{x\} \).

On the other hand, \( \hat{u}_k \cdots \hat{u}_0(A) = \hat{v}_k(\{x, y\}) \subset \{\hat{r}_{x, z}(\{x, y\}), 1_X(\{x, y\})\} = \{\{x, y\}\} \). Taking into account that \( u_k < v_k \) and the sequence \( (u_k, v_k)_{0 \leq k \leq \infty} \) is strictly increasing, we conclude that \( \hat{v}_k(\{x, y\}) = \{x, y\} \subset \{\hat{r}_{x, z}(\{x, y\}), 1_X(\{x, y\})\} \), which implies that \( \hat{v}(A) = \hat{v}_k \cdots \hat{v}_0(A) \not\subset \{x, y\} \). So, \( \hat{u}(A) \neq \hat{v}(A) \), which implies that \( \hat{f}(u) \neq \hat{f}(v) \).

This completes the proof of case (3) under the assumption \( u_{k-1} = v_{k-1} \in L^+ \). If \( u_{k-1} = v_{k-1} \in L^- \), then we can consider the dual words \( u^* = u_k \cdots u_0^* \) and \( v^* = v_k \cdots v_0^* \). For these dual words we get \( u_i^* = v_i^* \) for \( i < k \) and \( v_{k-1}^* = u_{k-1}^* \in L^+ \). In this case the preceding proof yields an 2-topological space \( X \) and a \( * \)-morphism \( f : L \to L(X) \) such that \( \hat{f}(u^*) \neq \hat{f}(v^*) \neq \hat{f}(v)^* \), which implies that \( \hat{f}(u) \neq \hat{f}(v) \).

Finally, consider the case:

4) \( u, v \in \tilde{K}_L \setminus K_L \). Then \( u = cu' \) and \( v = cv' \) for some distinct words \( u', v' \in K_L \). By the case (3), we can find a 2-topological space \( X \) and a \( * \)-morphism \( f : L \to L(X) \) such that \( \hat{f}(u') \neq \hat{f}(v') \). This means that \( \hat{f}(u')(A) \neq \hat{f}(v')(A) \) for some subset \( A \subset X \). Then

\[
\hat{f}(u)(A) = \hat{f}(u')(A) = \hat{f}(c\hat{f}(u')(A)) = c_X(\hat{f}(u')(A)) \neq c_X(\hat{f}(v')(A)) = \hat{f}(c\hat{f}(v')(A)) = \hat{f}(v)(A),
\]

which means that \( \hat{f}(u) \neq \hat{f}(v) \). \( \square \)

The following corollary of Theorem 8.1 implies Theorem 10.2 and shows that the upper bound in Theorem 10.1 is exact.

**Corollary 8.2.** For any \( * \)-linearly ordered set \( L \) there is an \( L^- \)-topological space \( X = (X, (\tau_\ell)_{\ell \in L^-}) \) such that the unique semigroup homomorphism \( \pi : FS\ L_{L^+} \to K_2(X) \) such that \( \pi(1) = 1_X \), \( \pi(c) = c_X \), \( \pi(\ell) = \hat{r}_\ell \), \( \pi(\ell') = \hat{r}_\ell \) for \( \ell \in L^- \) maps bijectively the set \( K_2(X) \) onto \( K(X) \) and the set \( \tilde{K}_L \) onto \( K_2(X) \). If the set \( L^- \) has finite cardinality \( n \), then the Kuratowski monoid \( K(X) \) of \( X \) has cardinality \( |K(X)| = K(n, n) = K(n) \) and the full Kuratowski monoid \( K_2(X) \) of \( X \) has cardinality \( |K_2(X)| = 2 \cdot K(n) \).

**Proof.** Let \( \tilde{K}_L \setminus \{u, v\} = \{u, v\} \in \tilde{K}_L \times \tilde{K}_L : u \neq v \} \). By Theorem 8.1, for any distinct words \( u, v \in \tilde{K}_L \) there exist 2-topological space \( X_{u, v} = (X_{u, v}, (\tau_{u, v}, \hat{r}_{u, v})^*) \) and a \( * \)-morphism \( f_{u, v} : L \to L(X) \) whose Kuratowski extension \( \hat{f}_{u, v} : FS\ L_{L^+} \to K_2(X_{u, v}) \) separates the words \( u, v \) in the sense that \( \hat{f}(u) \neq \hat{f}(v) \). This means that \( \hat{f}_{u, v}(u)(A_{u, v}) \neq \hat{f}_{u, v}(v)(A_{u, v}) \) for some subset \( A_{u, v} \subset X_{u, v} \). Let \( \delta_{u, v} \) denote the discrete topology on the set \( X_{u, v} \).

Define the \( L^- \)-topology \( \tau_{u, v} : L^- \to Top(X_{u, v}) \) on \( X_{u, v} \) by the formula

\[
\tau_{u, v}(\ell) = \begin{cases} 
\tau_{u, v} & \text{if } \ell = \hat{r}_{u, v}; \\
\tau_{u, v}^* & \text{if } \ell = \hat{r}_{u, v}^*; \\
\delta_{u, v} & \text{otherwise}.
\end{cases}
\]
Consider the $L_*$-topological space $X^{-}_{u,v} = (X_{u,v}, \tau_{u,v})$ and observe that its full Kuratowski monoid coincides with its full Kuratowski monoid of the 2-topological space $X_{u,v}$. Moreover, the Kuratowski extension $\hat{f}_{u,v} : F_{S_{L_*}} \to K_2(X_{u,v}) = K_2(\tau_{u,v})$ of the morphism $f_{u,v}$ has the properties $\hat{f}_{u,v}(1) = 1_{X_{u,v}}, f_{u,v}(c) = c_{X_{u,v}}, f_{u,v}(\ell) = \tau_{u,v}(\ell)$, and $f(\ell^*) = \tau_{u,v}(\ell^*)$ for $\ell \in L_*$.

We lose no generality assuming that for any distinct pairs $(u, v), (u', v') \in \bar{K}_L^2 \setminus \Delta$ the sets $X_{u,v}$ and $X_{u',v'}$ are disjoint. This allows us to consider the disjoint union $X = \bigcup \{X_{u,v} : (u, v) \in \bar{K}_L^2 \setminus \Delta\}$ and the subset $A = \bigcup \{A_{u,v} : (u, v) \in \bar{K}_L^2 \setminus \Delta\}$ in $X$. For every $\ell \in L_*$ consider the topology $\tau_\ell$ on the set $X$ generated by the base $\bigcup \{\tau_{u,v}(\ell) : (u, v) \in \bar{K}_L^2 \setminus \Delta\}$. We claim that the $L_*$-topological space $X = (X, \tau)$ (which is the direct sum of $L_*$-topological spaces $X_{u,v}$) and the subset $A \subset X$ have the desired property: for any two distinct words $u, v \in \bar{K}_L$, we get $\hat{u}(A) \neq \hat{v}(A)$, where $\hat{u}$ and $\hat{v}$ are the images of $u$ and $v$ under the (unique) semigroup homomorphism $\pi : F_{S_{L_*}} \to K_2(X)$ such that $\pi(1) = 1_X, \pi(c) = c_X, \pi(\ell) = \tau_\ell, \pi(\ell^*) = \tau_\ell^*$ for $\ell \in L_*$. This follows from the fact that $\hat{u}(A) \cap X_{u,v} = f_{u,v}(u)(A_{u,v}) \neq f_{u,v}(v)(A_{u,v}) = \hat{v}(A) \cap X_{u,v}$.

This means that the restriction $\pi : \bar{K}_L \to K_2(X)$ is injective. By Theorem 7.1 $\pi(\bar{K}_L) = K(X)$. Since $K_2(X) = K(X) \cup \{c_X \circ w : w \in K(X)\}$, we get also that $\pi(\bar{K}_L) = K_2(X)$. This means that the homomorphism $\pi$ maps bijectively the set $\bar{K}_L$ onto $K(X)$ and the set $\bar{K}_L$ onto $K_2(X)$.

If the set $L_*$ has finite cardinality $n$, then the set $\bar{K}_L$ has cardinality $|\bar{K}_L| = K(n, n) = K(n)$ (according to Theorem 5.2) and hence $|K(X)| = |\bar{K}_L| = K(n, n)$ and $|K_2(X)| = |\bar{K}_L| = 2 \cdot K(n, n) = 2 \cdot K(n)$. \hfill \Box

9. Free Kuratowski monoids

In this section we shall discuss free Kuratowski monoids over pointed linearly ordered sets. By a pointed linearly ordered set we understand a linearly ordered set $(L, \leq)$ with a distinguished point $1 \in L$ called the unit of $L$. The subsets $L^- = \{x \in L : x \leq 1\}$ and $L^+ = \{x \in L : 1 < x\}$ are called the negative and positive parts of $L$, respectively. A function $f : L \to \Lambda$ between two pointed linearly ordered sets is called a morphism if $f(1) = 1$ and $f$ is monotone in the sense that $f(x) \leq f(y)$ for any elements $x \leq y$ of $L$.

Each pointed linearly ordered set $L$ determines the free Kuratowski monoid $F_{K L}$ defined as follows. On the free semigroup $F_{S_L} = \bigcup_{n=1}^{\infty} L^n$ consider the smallest compatible partial preorder $\preceq$ extending the linear order $\leq$ of the set $L = L^1 \subset F_{S_L}$ containing the pairs $(x, x), (x, 1x), (1x, x), (x^2, x)$, and $(x^2, x)$ for $x \in L$. The compatible partial preorder $\preceq$ generates the congruence $\rho_\preceq = \{(v, w) \in F_{S_L} : v \preceq w, w \preceq v\}$ on $F_{S_L}$ identifying the words $x, x1, 1x, x^2$ for any $x \in L$. The quotient semigroup $F_{K L} = F_{S_L}/\rho_\preceq$ endowed with the quotient partial order is called the free Kuratowski monoid generated by the pointed linearly ordered set $L$. By $q_L : F_{S_L} \to F_{K L}$ we shall denote the (monotone) quotient homomorphism. The restriction $\eta_L = q_L|L : L \to F_{K L}$ is called the canonical embedding of the pointed linearly ordered set $L$ into its free Kuratowski monoid. In Proposition 5.2 we shall see that $\eta$ is indeed injective.

First we show that the free Kuratowski monoid $F_{K L}$ is free in the categorical sense.

\textbf{Proposition 9.1.} For any pointed linearly ordered set $L$ and any Kuratowski monoid $K$ with a linear generating set $\Lambda$ any morphism $f : L \to \Lambda$ determines a unique monotone semigroup homomorphism $\hat{f} : F_{K L} \to K$ such that $\hat{f} \circ \eta_L = f$.

\textbf{Proof.} Let $\hat{f} : F_{S_L} \to K$ be the unique semigroup homomorphism extending the morphism $f : L \to \Lambda$ into $K \subset L$. The partial order $\preceq$ of the Kuratowski monoid $K$ induces the compatible partial preorder $\preceq$ on $F_{S_L}$ defined by $u \preceq v$ iff $\hat{f}(u) \preceq \hat{f}(v)$. The monotonicity of $f$ implies that the partial preorder $\preceq$ contains the linear order of the set $L$. Then the minimality of the partial preorder $\preceq$ implies that $\preceq \leq \preceq$. This allows us to find a unique monotone semigroup homomorphism $\hat{f} : F_{K L} \to K$ such that $\hat{f} \circ \eta_L = f$ and hence $\hat{f} \circ \eta_L = \hat{f} \circ q_L|L = f|L = f$. \hfill \Box

\textbf{Proposition 9.2.} For any pointed linearly ordered set $L$ the quotient homomorphism $q_L : F_{S_L} \to F_{K L}$ maps bijectively the set $\bar{K}_L$ of Kuratowski words onto $F_{K L}$.

\textbf{Proof.} Theorem 7.1 implies that $q_L(\bar{K}_L) = F_{K L}$. To show that $q_L|\bar{K}_L$ is injective, we shall apply Theorem 5.2. Choose any injective morphism $e : L \to L^*$ of the pointed linearly ordered set $L$ into a *-linearly ordered set
$L^*$. Let $\bar{e} : FS_L \rightarrow FS_{L^*}$ be the unique semigroup homomorphism extending the map $e$. The injectivity of $e$ implies the injectivity of the homomorphism $\bar{e}$.

By Proposition 9.1 the morphism $e$ determines a unique monotone semigroup homomorphism $\bar{e} : FK_L \rightarrow FK_{L^*}$ such that $\bar{e} \circ \eta_L = \eta_{L^*} \circ e$. By Theorem 8.2 there exists a polytopological space $X$ and a *-morphism $f : L^* \rightarrow L(X)$ whose Kuratowski extension $f : FS_{L^*} \rightarrow K(X)$ maps bijectively the set $K_{L^*}$ onto $K(X)$.

By Proposition 9.3 the *-morphism $f : L^* \rightarrow L(X) \subset K(X)$ determines a (unique) monotone semigroup homomorphism $\hat{f} : FS_{L^*} \rightarrow K(X)$ such that $\hat{f} \circ \eta_{L^*} = f$. Thus we obtain the commutative diagram

\[
\begin{array}{ccc}
L & \xrightarrow{e} & L^* \\
\downarrow & & \downarrow \\
K_L & \xrightarrow{\bar{e}|K_L} & K_{L^*} \\
\downarrow & & \downarrow \\
FS_L & \xrightarrow{\hat{f}} & FS_{L^*} \\
\downarrow & & \downarrow \\
FK_L & \xrightarrow{\bar{e}} & FK_{L^*} \\
\downarrow & & \downarrow f \\
& K(X) & 
\end{array}
\]

in which the map $\hat{f} \circ \bar{e}|K_L$ is injective. Since $\hat{f} \circ \bar{e} = \hat{f} \circ \bar{e} \circ q_L$, the injectivity of the map $\hat{f} \circ \bar{e}|K_L$ implies the injectivity of the map $q_L|K_L$. Since $q_L(K_L) = FK_L$, the restriction $q_L|K_L : K_L \rightarrow FK_L$ is bijective. \hfill $\square$

Now we prove that the congruence $\rho_\leq$ on $FS_L$ determining the free Kuratowski monoid can be equivalently defined in a more algebraic fashion.

**Proposition 9.3.** For any pointed linearly ordered set $L$ the congruence $\rho_\leq$ on the free semigroup $FS_L$ coincides with the smallest congruence $\rho$ on $FS_L$ containing the pairs $(x,y_1), (x,y_2)$ for any $x \in L$ and the pairs $(x_1y_1,x_2y_2)$, $(y_1y_2,x_1x_2y_1,x_2x_1)$ for any points $x_1, x_2, y_1, y_2 \in L$ with $x_1 \leq x_2 \leq 1 \leq y_1 \leq y_2$.

**Proof.** First we prove that $\rho \subset \rho_\leq$. The definition of the partial preorder $\leq$ implies that $\{(x,x_1), (x,1x), (x,x^2) : x \in L\} \subset \rho_\leq$. Repeating the proofs of Lemma 7.3 we can show that

$$\{(x_1y_1,x_2y_2), (y_1y_2,x_1x_2) : x_1, x_2, y_1, y_2 \in L, x_1 \leq x_2 \leq 1 \leq y_1 \leq y_2\} \subset \rho_\leq.$$

Now the minimality of the congruence $\rho_\leq$ implies that $\rho \subset \rho_\leq$.

Denote by $\rho^\downarrow : FS_L \rightarrow FS_L/\rho$ and $q_L : FS_L \rightarrow FS_L/\rho_\leq = FK_L$ the quotient homomorphisms. Since $\rho \subset \rho_\leq$, there is a unique homomorphism $h : FS_L/\rho \rightarrow FK_L$, making the following diagram commutative:

\[
\begin{array}{ccc}
K_L & \xrightarrow{\rho^\downarrow} & FS_L \\
\downarrow & & \downarrow q_L \\
FS_L/\rho & \xrightarrow{h} & FK_L
\end{array}
\]

In this diagram by $i : K_L \rightarrow FS_L$ we denote the identity inclusion of the set $K_L$ of Kuratowski words into the free semigroup $FS_L$. The proof of Theorem 7.1 implies that $\rho^\downarrow(K_L) = FS_L/\rho$.

On the other hand, Proposition 9.2 guarantees that the restriction $q_L|K_L : K_L \rightarrow FK_L$ is bijective. This implies that the homomorphism $h$ is bijective and hence $\rho_\leq = \rho$. \hfill $\square$

The bijectivity of the restriction $q_L|K_L : K_L \rightarrow FK_L$ and Theorem 5.2 imply:

**Corollary 9.4.** For any finite pointed linearly ordered set $L$ the free Kuratowski monoid $FK_L$ has cardinality

$$|FK_L| = |K_L| = K(n,p) = \sum_{i=0}^{n} \sum_{j=0}^{p} \binom{i+j}{i} \binom{i+j}{j},$$

where $n = |L_-|$ and $p = |L_+|$. 


Given two non-negative natural numbers \( n, p \), fix any pointed linearly ordered set \( L_{n,p} \) with \(|(L_{n,p})_-| = n\) and \(|(L_{n,p})_+| = p\) and denote the free Kuratowski monoid \( FK_{L_{n,p}} \) by \( FK_{n,p} \).

**Proposition 9.5.** For any pointed linearly ordered set \( L \) and any distinct elements \( x, y \in FK_L \) there is a morphism of pointed linearly ordered sets \( f : L \rightarrow L_{2,2} \) such that \( f(x) \neq f(y) \). This implies that \( FK_L \) embeds into some power of the free Kuratowski monoid \( FK_{2,2} \).

**Proof.** Enlarge the pointed linearly ordered set \( L \) to a \(*\)-linearly ordered set \( L^* \) and denote by \( e : L \rightarrow L^* \) the identity embedding. Let \( \tilde{e} : FS_L \rightarrow FS_{L^*} \) be the unique semigroup homomorphism extending \( e \). It is clear that \( \tilde{e} \) is an injective map.

By Proposition 9.2 the restriction \( q_L|K_L : K_L \rightarrow FK_L \) is bijective. So, we can find Kuratowski words \( u, v \in K_L \) such that \( q_L(u) = x \) and \( q_L(v) = y \). By Theorem 8.1 for the Kuratowski words \( u, v \in K_L \subset K_{L^*} \) there exist a 2-topological space \( X \) and a \(*\)-morphism \( g : L^* \rightarrow L(X) \) such that \( \hat{g}(u) \neq \hat{g}(v) \) where \( \hat{g} : FS_{L^*} \rightarrow K(X) \) is the unique semigroup homomorphism extending the \(*\)-morphism \( g \). Moreover, the proof of Theorem 8.1 guarantees that the linear generating set \( L(X) \) of the 2-topological space \( X \) is isomorphic to the \(*\)-linearly ordered set \( L_{2,2} \). So, there exists a (unique) bijective \(*\)-morphism \( \iota : L_{2,2} \rightarrow L(X) \). Let \( f = \iota^{-1} \circ g : L^* \rightarrow L_{2,2} \).

The commutativity of the following diagram

\[
\begin{array}{ccc}
L & \xrightarrow{e} & L^* \\
\downarrow & & \downarrow \quad f \\
\downarrow \quad \tilde{e}|K_L & \quad \tilde{e}'|K_{L^*} & \downarrow \\
K_L & \xrightarrow{g} & K_{L^*} \\
\downarrow & & \downarrow \quad f \\
FS_L & \xrightarrow{q_L} & FK_{L^*} \\
\downarrow & & \downarrow \quad f \\
FK_L & \xrightarrow{\hat{g}} & K(X) \\
\end{array}
\]

and the inequality \( \hat{g} \circ \tilde{e}(u) \neq \hat{g} \circ \tilde{e}(v) \) imply that \( \tilde{f}(x) \neq \tilde{f}(y) \). \( \square \)

In fact, Proposition 9.5 can be improved as follows.

**Proposition 9.6.** For any pointed linearly ordered set \( L \) and any distinct elements \( x, y \in FK_L \) there is a pair \( (n, p) \in \{(1, 2), (2, 1)\} \) and a morphism of pointed linearly ordered sets \( f : L \rightarrow L_{n,p} \) such that \( \tilde{f}(x) \neq \tilde{f}(y) \). This implies that \( FK_L \) embeds into some power of the partially ordered monoid \( FK_{1,2} \times FK_{2,1} \).

**Proof.** Because of Proposition 9.5 it suffices to prove that the points of the free Kuratowski monoid \( FK_{2,2} \) can be separated by monotone homomorphisms into the Kuratowski monoids \( FK_{1,2} \) and \( FK_{2,1} \). Write the \(*\)-linearly ordered set \( L_{2,2} \) as \( L_{2,2} = \{\bar{r}_0, \bar{r}_1, 1, \bar{r}_1, \bar{r}_0\} \) for some elements

\[
\bar{r}_0 < \bar{r}_1 < 1 < \bar{r}_1 < \bar{r}_0.
\]

By analogy the pointed linearly ordered sets \( L_{1,2} \) and \( L_{2,1} \) can be written as \( L_{1,2} = \{\bar{r}, 1, \bar{r}_1, \bar{r}_0\} \) and \( L_{2,1} = \{\bar{r}_0, \bar{r}_1, 1, \bar{r}\} \). Consider the four surjective monotone morphisms

\[
h_{12} : L_{2,2} \rightarrow L_{1,2}, \ h_{23} : L_{2,2} \rightarrow L_{1,2}, \ h_{34} : L_{2,2} \rightarrow L_{2,1} \text{ and } h_{45} : L_{2,2} \rightarrow L_{2,1}
\]

such that

\[
h_{12}(\bar{r}_0) = h_{12}(\bar{r}_1) = \bar{r}, \ h_{23}(\bar{r}_1) = h_{23}(1) = 1, \ h_{34}(1) = h_{34}(\bar{r}_1) = 1, \text{ and } h_{45}(\bar{r}_1) = h_{45}(\bar{r}_0) = \bar{r}.
\]
Kuratowski words in the alphabet determine the structure of any free Kuratowski monoid. The following diagram shows the order structure of

By Proposition 9.2, the free Kuratowski monoid $FK_{2,2}$ can be identified with the 63-element set $\mathcal{K}_{L_{2,2}}$ of Kuratowski words in the alphabet $L_{2,2}$:

In the following two lists we write the pairs $(h_{12}(w), h_{45}(w))$ and $(h_{23}(w), h_{34}(w))$ for the Kuratowski words $w$ from the above list. Analyzing these two lists we can see that the quadruples $(h_{12}(w), h_{45}(w), h_{23}(w), h_{34}(w))$, $w \in \mathcal{K}_{L_{2,2}}$, are pairwise distinct, which means that the elements of the free Kuratowski monoid $FK_{2,2}$ are separated by the homomorphism $(h_{12}, h_{23}, h_{34}, h_{45}) : L_{2,2} \to L_{2,2}^2 \times L_{2,2}^2$.

The homomorphism $(h_{12}, h_{45})$:

The homomorphism $(h_{23}, h_{34})$:

Proposition 9.1 shows that the homomorphisms into free Kuratowski monoids $FK_{n,p}$ for $n+p \leq 3$ completely determine the structure of any free Kuratowski monoid. The following diagram shows the order structure of

[Diagram]

□
the free Kuratowski monoids $FK_{1,1}$ with the linear generating set $L_{1,1} = \{a, 1, x\}$. We identify the elements of $FK_{1,1}$ with the Kuratowski words in the alphabet $L_{1,1}$. For two Kuratowski words $u, v$ an arrow $u \rightarrow v$ indicated that $u \leq v$ in $FK_{1,1}$.

The following diagram shows the order structure of the free Kuratowski monoids $FK_{2,1}$ with the linear generating set $L_{2,1} = \{a, b, 1, x\}$:

The free Kuratowski monoid $K_{1,2}$ is isomorphic to the free Kuratowski monoid $K_{2,1}$ endowed with the reversed partial order.

Proposition 9.7 has an interesting application.

**Proposition 9.7.** All elements of any Kuratowski monoid $K$ are idempotents.

**Proof.** Let $L$ be the linear generating set of the Kuratowski monoid $K$. Since $K$ is a quotient monoid of the free Kuratowski monoid $FK_L$, it suffices to check that all elements of $FK_L$ are idempotents. By Proposition 9.6 the free Kuratowski monoid $FK_L$ embeds into some power of the monoid $FK_{1,2} \times FK_{2,1}$. So, it suffices to check that each element of the free Kuratowski monoids $FK_{1,2}$ and $FK_{2,1}$ is an idempotent. This can be seen by a direct verification of each of 17 elements of $FK_{1,2}$ (or its algebraically isomorphic copy $FK_{2,1}$).

**References**

[1] K. Kuratowski, *Sur l’opération $\overline{A}$ de l’Analysis Situs*, Fund. Math. 3 (1922) 182–199.
[2] B.J. Gardner, M. Jackson, *The Kuratowski Closure-Complement Theorem*, New Zealand J. Math. 38 (2008), 9–44.
[3] R. Graham, D. Knuth, O. Patashnik, *Concrete mathematics. A foundation for computer science*, Addison-Wesley Publishing Company, Reading, MA, 1994.
[4] J. Shallit, R. Willard, *Kuratowski’s Theorem for two closure operators*, preprint (arXiv:1109.1227).

T. Banakh: IVAN FRANKO NATIONAL UNIVERSITY OF LVIV (UKRAINE) AND JAN KOCHANOWSKI UNIVERSITY IN KIELCE (POLAND)
E-mail address: t.o.banakh@gmail.com

O. Chervak, T. Martynyuk, M. Pylypovych, M. Simkiv: IVAN FRANKO NATIONAL UNIVERSITY OF LVIV (UKRAINE)
E-mail address: oschervak@gmail.com, tetyanka.martynyuk@gmail.com, pylypovych@gmail.com, simkiv.markiyan@gmail.com

A. Ravsky: INSTITUTE FOR APPLIED PROBLEMS OF MECHANICS AND MATHEMATICS, LVIV (UKRAINE)
E-mail address: oravsky@mail.ru