ON BERNSTEIN–TYPE INEQUALITIES FOR POLYNOMIALS INVOLVING THE POLAR DERIVATIVE

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Abstract. In this paper, we establish some upper bound estimates for the polar derivative of a polynomial not vanishing in a disk $|z| < k$, $k \geq 1$ with a zero of multiplicity $s$, $0 \leq s \leq n - 1$ at the origin. The obtained results enable us to derive polar derivative analogues of some well known Bernstein-type inequalities as special cases.

1. Introduction

By $\mathbb{P}_n$ we denote the space of all complex polynomials $P(z) := \sum_{v=0}^{n} a_v z^n$ of degree $n$. If $P \in \mathbb{P}_n$, then by the famous Bernstein inequality [3], we have

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|.$$  \hspace{1cm} (1)

Equality holds in (1) if and only if $P(z)$ has all its zeros at origin. If we restrict ourselves to the class of polynomials $P(z)$ having no zero in $|z| < 1$, then (1) can be sharpened. In fact, Erdős conjectured and later Lax [6] proved that if $P(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|.$$ \hspace{1cm} (2)

It was shown by Frappier et al. [4] that if $P \in \mathbb{P}_n$, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{1 \leq l \leq 2n} |P(e^{\frac{i\pi}{n}})|.$$ \hspace{1cm} (3)

Clearly (3) is a refinement of (1), since the maximum of $|P(z)|$ on $|z| = 1$ may be larger than maximum of $|P(z)|$ taken over $2n^{th}$ roots of unity as one can show by taking a simple example $P(z) = z^n + ia$, $a > 0$.

The inequality (3) was improved by Aziz [1] by showing that

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} (M_\alpha + M_{\alpha+\pi}),$$ \hspace{1cm} (4)
where

$$M_\alpha = \max_{1 \leq l \leq n} |P(e^{i(\alpha+2\pi l)/n})|$$

(5)

for all real \(\alpha\).

In the same paper, Aziz [1] also improved inequality (4) for a restricted class of polynomials not vanishing in the unit disk \(|z| < 1\). In fact, he proved that if \(P \in \mathbb{P}_n\) and \(P(z) \neq 0\) in \(|z| < 1\), then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \left(M_\alpha^2 + M_\alpha^2 + 2m^2\right)^{1/2},$$

(6)

where \(M_\alpha\) is as defined in (5).

As a refinement of (6), Rather and Shah [7] in the same paper proved that if \(P \in \mathbb{P}_n\) and \(P(z) \neq 0\) in \(|z| < 1\), then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \left(M_\alpha^2 + M_\alpha^2 - 2m^2\right)^{1/2}.$$  

(7)

where \(m = \min_{|z|=1} |P(z)|\) and \(M_\alpha\) is as defined in (5).

As a generalization of (7), Rather and Shah [7] in the same paper proved that if \(P \in \mathbb{P}_n\) and \(P(z) \neq 0\) in \(|z| < k\), \(k \geq 1\), then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{\sqrt{2(1+k^2)}} \left(M_\alpha^2 + M_\alpha^2 - 2m_k^2\right)^{1/2},$$

(8)

where \(m_k = \min_{|z|=k} |P(z)|\) and \(M_\alpha\) is as defined in (5).

Recently Sunil et al. [8], extended inequality (8) to the class of polynomials \(P(z) = z^s \left(a_0 + \sum_{v=\mu}^{n-s} a_v z^v\right)\), \(1 \leq \mu \leq n-s\) having \(s\)-fold zero at \(z = 0\) and the remaining \(n-s\) zeros in \(|z| \geq k\), \(k \geq 1\) and obtained the following result.

**Theorem A.** If \(P \in \mathbb{P}_n\) and \(P(z) = z^s \left(a_0 + \sum_{v=\mu}^{n-s} a_v z^v\right)\), \(1 \leq \mu \leq n-s, 0 \leq s \leq n-1\) such that \(P(z)\) has \(s\)-fold zero at \(z = 0\) and the remaining \(n-s\) zeros in \(|z| \geq k\), \(k \geq 1\), then

$$\max_{|z|=1} |P'(z)| \leq s \max_{|z|=1} |P(z)| + \frac{n-s}{\sqrt{2(1+k^2)}} \left(M_\alpha^2 + M_\alpha^2 - 2m_k^2\right)^{1/2},$$

(9)

where \(m_k = \min_{|z|=k} |P(z)|\) and

$$M_\alpha^* = \max_{1 \leq l \leq n-s} \left|P(e^{i(\alpha+2\pi l)/n-s})\right|.$$  

(10)

Over the last few decades different authors produced a large number of different versions and generalizations of the above inequalities by introducing restrictions on
the multiplicity of zero at \( z = 0 \), the modulus of largest root of \( P(z) \), restrictions on coefficients, using higher order derivatives etc. Before proceeding to our main results, let us introduce the concept of polar derivative involved.

For \( P \in \mathbb{P}_n \) the polar derivative \( D_\beta P(z) \) of \( P(z) \) with respect to point \( \beta \) is defined as,

\[
D_\beta P(z) := nP(z) + (\beta - z)P'(z).
\]

Note that \( D_\beta P(z) \) is a polynomial of degree at most \( n - 1 \). This is so called polar derivative of \( P(z) \) with respect to \( \beta \). It generalises the ordinary derivative in the sense that

\[
\lim_{\beta \to \infty} \left\{ \frac{D_\beta P(z)}{\beta} \right\} = P'(z),
\]

(11)

uniformly with respect to \( z \) for \( |z| \leq R \), \( R > 0 \).

In 1998, Aziz and Shah \cite{2} established the polar derivative analogue of (1) by proving that if \( P(z) \) is a polynomial of degree \( n \), then for every complex number \( \beta \) with \( |eta| \geq 1 \),

\[
\max_{|z|=1} |D_\beta P(z)| \leq n|\beta| \max_{|z|=1} |P(z)|.
\]

(12)

Clearly the above inequality generalizes (1) and to obtain (1) from it, simply divide both sides of (12) by \( |\beta| \) and let \( |\beta| \to \infty \).

The main aim of the present paper is to obtain some upper bound estimates for the maximum modulus of polar derivative of a polynomial on a unit disk under the assumption that the polynomial has no zeros in the disk \( |z| < k \), \( k \geq 1 \), but having \( s \)-fold zero at the origin. The obtained results generalizes some already known estimates for the ordinary derivative of polynomial as special cases.

**Theorem 1. (Main)** If \( P(z) = z^s \left( a_0 + \sum_{v=\mu}^{n-s} a_v z^v \right) \), \( 1 \leq \mu \leq n - s \), \( 0 \leq s \leq n - 1 \) is a polynomial of degree \( n \) having \( s \)-fold zero at \( z = 0 \) and remaining \( n - s \) zeros in \( |z| \geq k \), \( k \geq 1 \), then for any complex number \( \beta \) with \( |\beta| \geq 1 \), we have

\[
\max_{|z|=1} |D_\beta P(z)| \leq \left[ n + s(|\beta| - 1) \right] \max_{|z|=1} |P(z)| + \frac{(n-s)(|\beta|-1)}{\sqrt{2(1+k^{2\mu})}} \left( M^*_{\alpha} + M^*_{\alpha + \pi} - \frac{2m_k^2}{k^{2s}} \right)^{\frac{1}{2}},
\]

(13)

\( m_k = \min_{|z|=k} |P(z)| \) and \( M^*_{\alpha} \) is as defined in (10).

**Remark 1.** If we divide both sides of inequality (13) by \( |\beta| \) and let \( |\beta| \to \infty \) and noting (11), we get inequality (9).

If we take \( s = 0 \) and \( \mu = 1 \) in (13), we get the following polar derivative analogue of (8).
COROLLARY 1. If \( P(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) and having no zeros in \(|z| < k, k \geq 1\), then for every complex number \( \beta \) with \(|\beta| \geq 1\), we have

\[
\max_{|z|=1} |D_\beta P(z)| \leq n \max_{|z|=1} |P(z)| + \frac{n(|\beta| - 1)}{\sqrt{2(1 + k^2)}} (M_\alpha^2 + M_\alpha + \pi^2 - 2m_k^2)^{\frac{1}{2}},
\]

\( m_k = \min_{|z|=k} |P(z)| \) and \( M_\alpha \) is as defined in (5).

REMARK 2. Dividing both sides of inequality (14) by \(|\beta|\) and let \(|\beta| \to \infty\) and noting (11), we get inequality (8).

By taking \( s = 0, \mu = 1 \) and \( k = 1 \) in (13), we get the following polar derivative analogue of (7).

COROLLARY 2. If \( P(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) and having no zeros in \(|z| < 1\), then for every complex number \( \beta \) with \(|\beta| \geq 1\), we have

\[
\max_{|z|=1} |D_\beta P(z)| \leq n \max_{|z|=1} |P(z)| + \frac{n(|\beta| - 1)}{2} (M_\alpha^2 + M_\alpha + \pi^2 - 2m^2)^{\frac{1}{2}},
\]

\( m = \min_{|z|=1} |P(z)| \) and \( M_\alpha \) is as defined in (5).

REMARK 3. Dividing both sides of inequality (15) by \(|\beta|\) and let \(|\beta| \to \infty\) and noting (11), we get inequality (7).

From Corollary 2, we easily get the following result.

COROLLARY 3. If \( P(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) and having no zeros in \(|z| < 1\), then for every complex number \( \beta \) with \(|\beta| \geq 1\), we have

\[
\max_{|z|=1} |D_\beta P(z)| \leq n \max_{|z|=1} |P(z)| + \frac{n(|\beta| - 1)}{2} (M_\alpha^2 + M_\alpha + \pi^2)^{\frac{1}{2}},
\]

\( M_\alpha \) is as defined in (5).

REMARK 4. For the class of polynomials having no zeros in the unit disk \( \{ z \in \mathbb{C}; |z| < 1 \} \), the inequality (16) represents a refinement of (12) because, since \(|\beta| \geq 1\) and the maximum of \(|P(z)|\) on \(|z| = 1\) may be larger than the quantity \( \frac{1}{2} (M_\alpha^2 + M_\alpha + \pi^2)^{\frac{1}{2}} \) for every real \( \alpha \).

REMARK 5. Dividing both sides of inequality (16) by \(|\beta|\) and let \(|\beta| \to \infty\) and noting (11), we get inequality (6).
We need the following lemmas to prove the theorem.

The following lemma is due to Sunil et al. [8].

**Lemma 1.** If \( P(z) = a_0 + \sum_{\nu=\mu}^{n} a_{\nu} z^\nu \), \( 1 \leq \mu \leq n \), \( P(z) \neq 0 \) in \( |z| < k \), \( k \geq 1 \) and \( m_k = \min_{|z|=k} |P(z)| \), then for every real \( \alpha \),

\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{\sqrt{2(1+k^{2\mu})}} \left( M_\alpha^2 + M_{\alpha+\pi}^2 - 2m_k^2 \right)^{\frac{1}{2}},
\]

where \( M_\alpha \) is as defined in (5).

The following lemma is a special case of a result due to Govil and Rahman [5].

**Lemma 2.** If \( P(z) \) is a polynomial of degree \( n \), then for \( |z| = 1 \),

\[
|P'(z)| + |Q'(z)| \leq n \max_{|z|=1} |P(z)|,
\]

where \( Q(z) = z^n P\left(\frac{1}{z}\right) \).

Next we prove a lemma in which we generalize lemma 1 to the polar derivative of a polynomial. More precisely, we prove the following.

**Lemma 3.** If \( P \in \mathbb{P}_n \) and \( P(z) = a_0 + \sum_{\nu=\mu}^{n} a_{\nu} z^\nu \), \( 1 \leq \mu \leq n \), \( P(z) \neq 0 \) in \( |z| < k \), \( k \geq 1 \) and \( m_k = \min_{|z|=k} |P(z)| \), then for every real \( \alpha \) and for every complex number \( \beta \) with \( |\beta| \geq 1 \), we have

\[
\max_{|z|=1} |D_\beta P(z)| \leq \frac{n}{\sqrt{2}} \left[ \sqrt{2} |P(z)| + \frac{|\beta| - 1}{\sqrt{1+k^{2\mu}}} \left( M_\alpha^2 + M_{\alpha+\pi}^2 - 2m_k^2 \right)^{\frac{1}{2}} \right],
\]

where \( M_\alpha \) is as defined in (5).

**Proof.** Since \( P(z) \) is a polynomial of degree \( n \) which does not vanish in \( |z| < k \), \( k \geq 1 \). Applying the inequality (17), we have

\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{\sqrt{2(1+k^{2\mu})}} \left( M_\alpha^2 + M_{\alpha+\pi}^2 - 2m_k^2 \right)^{\frac{1}{2}}.
\]

Now for \( \beta \in \mathbb{C} \), with \( |\beta| \geq 1 \)

\[
|D_\beta P(z)| = |nP(z) + (\beta - z)P'(z)| = |nP(z) + \beta P'(z) - zP'(z)| \leq |nP(z) - zP'(z)| + |\beta||P'(z)|.
\]
It can be easily seen that
\[ |nP(z) - zP'(z)| = |Q'(z)| \quad \text{for } |z| = 1. \]
Therefore we have for $|z| = 1$
\[ |D_\beta P(z)| \leq |Q'(z)| + |\beta||P'(z)| = |Q'(z)| + |P'(z)| - |P'(z)| + |\beta||P'(z)|. \]
Using lemma 2, we get for $|z| = 1$
\[ |D_\beta P(z)| \leq n|P(z)| + (|\beta| - 1)|P'(z)|, \]
which on using (20), we get
\[ |D_\beta P(z)| \leq n|P(z)| + \frac{n(|\beta| - 1)}{\sqrt{2(1 + k^2\mu)}} (M^2_\alpha + M^2_{\alpha+\pi} - 2m^2_k)^{\frac{1}{2}}. \]
Equivalently,
\[ \max_{|z|=1} |D_\beta P(z)| \leq \frac{n}{\sqrt{2}} \left\{ \sqrt{2}|P(z)| + \frac{(|\beta| - 1)}{\sqrt{1 + k^2\mu}} (M^2_\alpha + M^2_{\alpha+\pi} - 2m^2_k)^{\frac{1}{2}} \right\}, \]
which proves lemma 3.

3. Proof of the theorem

Proof. Let $P(z) = z^s \phi(z)$ where $\phi(z) = a_0 + \sum_{v=\mu}^{n-s} a_v z^v$, $1 \leq \mu \leq n - s$ and $0 \leq s \leq n - 1$ is the polynomial of degree $n - s$ having no zeros in $|z| < k$, $k \geq 1$. Applying lemma 3 to polynomial $\phi(z)$ of degree $n - s$, we get for every $\beta \in \mathbb{C}$, with $|\beta| \geq 1$,
\[ \max_{|z|=1} |D_\beta \phi(z)| \leq \frac{(n - s)}{\sqrt{2}} \left\{ \sqrt{2} \max_{|z|=1} |\phi(z)| + \frac{(|\beta| - 1)}{\sqrt{1 + k^2\mu}} (M^2_\alpha + M^2_{\alpha+\pi} - 2m^2_k)^{\frac{1}{2}} \right\}, \]
where $m'_k = \min_{|z|=k} |\phi(z)|$.
Now for $\beta \in \mathbb{C}$ with $|\beta| \geq 1$, we have
\[ D_\beta P(z) = nP(z) + (\beta - z)P'(z) = nz^s \phi(z) + (\beta - z) \left[ z^s \phi'(z) + \phi(z)z^{s-1} \right] \]
\[ = z^s D_\beta \phi(z) + s\beta z^{s-1} \phi(z), \]
which implies
\[ zD_\beta P(z) = z^{s+1} D_\beta \phi(z) + s\beta P(z). \]
Hence for $|z| = 1$, we get from the above inequality that
\[ |D_\beta P(z)| \leq |D_\beta \phi(z)| + s|\beta||P(z)|, \]
which in particular implies,
\[
\max_{|z|=1} |D_\beta P(z)| \leq \max_{|z|=1} |D_\beta \phi(z)| + s|\beta| \max_{|z|=1} |P(z)|.
\]

This gives by using inequality (21),
\[
\max_{|z|=1} |D_\beta P(z)| \leq \left[ \frac{(n-s)}{\sqrt{2}} \left\{ \sqrt{2} \max_{|z|=1} |\phi(z)| + \frac{(\beta - 1)}{\sqrt{(1 + k^{2}\mu)}} \left( M_{\alpha}^{+2} + M_{\alpha+\pi^{2}}^{+} - 2m_{k}^{2} \right) \right\} \right]^{1/2} + s|\beta| \max_{|z|=1} |P(z)|.
\]
Now the relation between $P(z)$ and $\phi(z)$ is $P(z) = z^k \phi(z)$.
This implies $|P(z)| = |\phi(z)|$ for $|z|=1$ and $m_{k}^{+} = \min_{|z|=k} |\phi(z)| = \frac{1}{k} \min_{|z|=k} |P(z)| = \frac{1}{k} m_{k}$, where $m_{k} = \min_{|z|=k} |P(z)|$.
Using these observations in (22), we get (13).
This completes the observations in (22), we get (13).

This completes the proof of Theorem 1.

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