On the deformed oscillator and the deformed derivative associated with the Tsallis $q$-exponential

Ramaswamy Jagannathan$^{1,†}$ and Sameen Ahmed Khan$^{2,‡}$

$^1$Retired Faculty (Physics), The Institute of Mathematical Sciences
4th Cross Street, Central Institutes of Technology (CIT) Campus
Tharamani, Chennai 600113, India

$^2$Department of Mathematics and Sciences
College of Arts and Applied Sciences (CAAS), Dhofar University
Post Box No. 2509, Postal Code: 211, Salalah, Oman

Emails: †jagan@imsc.res.in, ‡rohelakhan@yahoo.com

Abstract

The Tsallis $q$-exponential function $e_q(x) = (1 + (1 - q)x)^{1/(1-q)}$ is found to be associated with the deformed oscillator defined by the relations $[N, a^\dagger] = a^\dagger, [N, a] = -a,$ and $[a, a^\dagger] = \phi_T(N + 1) - \phi_T(N),$ with $\phi_T(N) = N/(1 + (1 - q)(1 - N)).$ In a Bargmann-like representation of this deformed oscillator the annihilation operator $a$ corresponds to a deformed derivative with the Tsallis $q$-exponential functions as its eigenfunctions, and the Tsallis $q$-exponential functions become the coherent states of the deformed oscillator.

Keywords: Nonextensive statistical mechanics, Tsallis $q$-exponential, deformed exponentials, deformed oscillators, deformed numbers, deformed derivatives, coherent states.

1 Introduction

The canonical boson oscillator is defined by the commutation relations

$$[N, b^\dagger] = b^\dagger, \quad [N, b] = -b, \quad [b, b^\dagger] = 1,$$

where $b^\dagger, b$ and $N$ are the creation, annihilation and number operators, respectively. These relations imply that $b^\dagger b = N$ and $bb^\dagger = N + 1.$ The Fock representation of the boson algebra (1) is given by

$$b^\dagger |n\rangle = \sqrt{n + 1} |n + 1\rangle, \quad b |n\rangle = \sqrt{n} |n - 1\rangle, \quad N |n\rangle = n |n\rangle,$$

$$n = 0, 1, 2, \ldots ,$$

where $b^\dagger |n\rangle = \sqrt{n + 1} |n + 1\rangle, b |n\rangle = \sqrt{n} |n - 1\rangle, N |n\rangle = n |n\rangle.$
with \( \{ |n\rangle | n = 0, 1, 2, \ldots \} \) being an orthonormal basis. Starting with the ground state \( |0\rangle \) one can build the higher states as

\[
|n\rangle = \frac{b^n}{\sqrt{n!}} |0\rangle.
\] (3)

Coherent states are eigenfunctions of \( b \):

\[
b |\alpha\rangle = \alpha |\alpha\rangle,
\] (4)

where \( \alpha \) is any complex number. The solution for \( |\alpha\rangle \) is

\[
|\alpha\rangle = e^{-\frac{1}{2} |\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,
\] (5)

with the normalization \( \langle \alpha |\alpha\rangle = 1 \). We can write

\[
|\alpha\rangle = e^{-\frac{1}{2} |\alpha|^2} e^{\alpha b^\dagger} |0\rangle.
\] (6)

In the Bargmann-like representation

\[
b^\dagger = x, \quad b = D = \frac{d}{dx}, \quad N = xD,
\] (7)

the monomials

\[
\xi_n(x) = \frac{x^n}{\sqrt{n!}}, \quad n = 0, 1, 2, \ldots ,
\] (8)

carry the Fock representation (2) as given by

\[
b\xi_n = \sqrt{n+1} \xi_{n+1}, \quad b\xi_n = \sqrt{n} \xi_{n-1}, \quad N\xi_n = n\xi_n,
\] (9)

\[n = 0, 1, 2, \ldots .
\]

The monomials \( \{ \xi_n(x) | n = 0, 1, 2, \ldots \} \) are seen to form an orthonormal basis with respect to the inner product

\[
\langle f |g \rangle = [f^*(D)g(x)]|_{x=0}.
\] (10)

Note that in this representation \( e^{\alpha x} \) is an eigenfunction of \( b \) for any complex number \( \alpha \). Under the inner product (10) it is seen that \( \langle e^{\alpha x} | e^{\beta x} \rangle = e^{\alpha \beta} \). Thus, in this representation, the coherent states are given by

\[
\psi_\alpha(x) = e^{-\frac{1}{2} |\alpha|^2} e^{\alpha x} = e^{-\frac{1}{2} |\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \xi_n(x),
\] (11)
which are normalized as $\langle \psi_\alpha | \psi_\alpha \rangle = 1$.

Studies on $q$-deformed oscillator commutation relations started in 1970s from different points of view (see [1, 2, 3]; see [4] for an account of the early history of $q$-deformed commutation relations). With the advent of quantum groups in 1980s, representation theory of quantum algebras led to $q$-oscillators [5, 6, 7, 8] and initiated extensive studies on them and their applications. In general, a deformed oscillator algebra can be prescribed by the relations

\[ [N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad [a, a^\dagger] = \phi(N + 1) - \phi(N). \tag{12} \]

where $\phi(n)$, sometimes called the deformation, or structure, function, characterizes the deformed oscillator. For the boson oscillator $\phi(N) = N$. The Fock representation can be constructed easily as follows:

\[ a^\dagger|n\rangle = \sqrt{\phi(n + 1)}|n + 1\rangle, \quad a|n\rangle = \sqrt{\phi(n)}|n - 1\rangle, \quad N|n\rangle = n|n\rangle, \quad n = 0, 1, 2, \ldots \tag{13}\]

Note that $a^\dagger a = \phi(N)$. Now, defining

\[
\begin{align*}
\phi(0)! &= 1, \\
\phi(n)! &= \phi(n)\phi(n-1)\phi(n-2)\ldots\phi(2)\phi(1), \quad n = 1, 2, \ldots 
\end{align*} \tag{14}
\]

we can write

\[ |n\rangle = \frac{a^{\dagger n}}{\sqrt{\phi(n)!}}|0\rangle. \tag{15} \]

Defining the deformed exponential function

\[ e_x^\phi = \sum_{n=0}^{\infty} \frac{x^n}{\phi(n)!}, \tag{16} \]

it can be verified that the normalized coherent states of the deformed oscillator (12), eigenstates of $a$, are given by

\[ |\alpha\rangle = \left( e_{\phi}^{|\alpha|^2} \right)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{\phi(n)!}} |n\rangle = \left( e_{\phi}^{|\alpha|^2} \right)^{-\frac{1}{2}} e_\phi^{\alpha a^\dagger} |0\rangle. \tag{17} \]

Let

\[ D_\phi f(x) = \left[ \frac{1}{x} \phi(xD) \right] f(x). \tag{18} \]
Note that analogous to the relation
\[ D x^n = n x^{n-1}. \]  
we have
\[ D_\phi x^n = \phi(n) x^{n-1}. \]  
The Bargmann-like representation is
\[ a^\dagger = x, \quad a = D_\phi, \quad N = x D. \]  
so that the monomials
\[ \xi_n(x; \phi) = \frac{x^n}{\sqrt{\phi(n)!}}, \quad n = 0, 1, 2, \ldots, \]  
carry the Fock representation as given by
\[ a^\dagger \xi_n = \sqrt{\phi(n+1)} \xi_{n+1}, \quad a \xi_n = \sqrt{\phi(n)} \xi_{n-1}, \quad N \xi_n = n \xi_n, \]  
\[ n = 0, 1, 2, \ldots. \]  
The monomials \( \{ \xi_n(x; \phi) | n = 0, 1, 2, \ldots \} \) form an orthonormal basis with respect to the inner product
\[ \langle f | g \rangle = \left[ f^* (D_\phi) g(x) \right]_x=0. \]  
Note that in this representation \( e^\alpha x_\phi \) is an eigenfunction of \( a \) for any complex number \( \alpha \). Under the inner product
\[ \langle e^\alpha x_\phi | e^\alpha x_\phi \rangle = e^{\alpha^2}. \]  
Thus, in this representation, the coherent states are given by
\[ \psi_\alpha(x; \phi) = \left( e^{|\alpha|^2} \right)^{-\frac{1}{2}} e^\alpha x_\phi = \left( e^{|\alpha|^2} \right)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{\phi(n)!}} \xi_n(x; \phi), \]  
which are normalized as \( \langle \psi_\alpha | \psi_\alpha \rangle = 1. \) This leads to the result that the eigenfunctions of the deformed derivative \( D_\phi \), or \( a \), are the deformed exponential functions \( e^\alpha x_\phi \) representing the corresponding coherent states of the deformed oscillator in the Bargmann-like representation. We review a few
well known examples in the next section (see also [9]). The deformed derivative $D_\phi$ has been used to develop a very general theory of deformation of classical hypergeometric functions [10].

It is known since the early days of deformed oscillators that the creation and annihilation operators of the deformed oscillator (12) can be written in terms of the boson creation and annihilation operators as

$$a^\dagger = \sqrt{\phi(N)/N} b^\dagger, \quad a = b \sqrt{\phi(N)/N}, \quad N = b^\dagger b.$$  \hspace{1cm} (27)

Writing $\sqrt{\phi(N)/N} = f(N)$, the deformed oscillator (12) has been presented as

$$a^\dagger = f(N) b^\dagger, \quad a = bf(N), \quad N = b^\dagger b,$$  \hspace{1cm} (28)

called an $f$-oscillator, and a general theory of $f$-oscillators has been developed [11] with many applications to nonlinear physics including nonlinear coherent states, or $f$-coherent states, relevant for quantum optics (see, e.g., [12]). In terms of $f(n)$ we can rewrite the deformed exponential function (16) as

$$e^x(f) = \sum_{n=0}^{\infty} \frac{1}{f^2(n)!} \frac{x^n}{n!},$$  \hspace{1cm} (29)

where $f^2(n)! = f^2(n)f^2(n-1)\ldots f^2(2)f^2(1)$. Then, the normalized coherent states of the $f$-oscillator (28), eigenstates of $a$, are given by

$$|\alpha\rangle = (e^{|\alpha|^2})^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{1}{f(n)!} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,$$  \hspace{1cm} (30)

with $f(n)! = f(n)f(n-1)\ldots f(2)f(1)$. These states (30), eigenstates of $a = bf(N)$, are called nonlinear coherent states characterized by $f(n)$, or $f$-coherent states, of the boson oscillator. They correspond to nonlinear coherent states of photons in the context of quantum optics.

Now, an interesting question is whether there exists a deformed oscillator with which the Tsallis $q$-exponential function [13] [14],

$$e_q(x) = (1 + (1-q)x)^{\frac{1}{1-q}},$$  \hspace{1cm} (31)

is associated such that corresponding coherent states are given by $\{e_q(aa^\dagger)|0\rangle | \alpha \in \mathbb{C} \}$. Or, in other words, is there a deformed oscillator for
which in the Bargmann-like representation the annihilation operator $a$ will be a deformed derivative with the Tsallis $q$-exponential function as its eigenfunction. In the following we present the deformed oscillator and the deformed derivative associated with the Tsallis $q$-exponential function, after reviewing briefly the general structure of deformed oscillators and the basic properties of the Tsallis $q$-exponential function.

2 Deformed oscillators

The $q$-oscillator with

\[
\begin{align*}
[N, a^\dagger] &= a^\dagger, \quad [N, a] = -a, \quad [a, a^\dagger] = \phi(N + 1) - \phi(N), \\
\phi(N) &= [N]_q = \frac{1 - q^N}{1 - q},
\end{align*}
\]

was studied much before the advent of quantum groups (see [1, 2, 3, 4]). Since

\[
[N + 1]_q - q[N]_q = 1,
\]

the relation between $a$ and $a^\dagger$ is written as

\[
aa^\dagger - qa^\dagger a = 1.
\]

The Fock representation is

\[
\begin{align*}
a^\dagger |n\rangle &= \sqrt{[n + 1]_q}|n + 1\rangle, \quad a|n\rangle = \sqrt{[n]_q}|n - 1\rangle, \quad N|n\rangle = n|n\rangle, \\
n &= 0, 1, 2, \ldots,
\end{align*}
\]

where

\[
[n]_q = \frac{1 - q^n}{1 - q}
\]

is Heine’s basic number, or $q$-number (see, e.g., [15, 16]). Note that

\[
\lim_{q \to 1}[n]_q = n.
\]

Now, with the definitions

\[
[0]_q! = 1, \quad [n]_q! = [n]_q[n - 1]_q[n - 2]_q \cdots [2]_q [1]_q, \quad n = 1, 2, \ldots,
\]

we have

\[
|n\rangle = \frac{a^{+n}}{\sqrt{[n]_q!}}|0\rangle.
\]
Defining $q$-exponential function

$$e^x_q = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!},$$

(39)

the normalized coherent states of the $q$-oscillator (32) are given by

$$|\alpha\rangle = \left(e^{|\alpha|^2}_q\right)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]_q!}} |n\rangle = \left(e^{|\alpha|^2}_q\right)^{-\frac{1}{2}} e^{\alpha a^\dagger} |0\rangle.$$  

(40)

In the Bargmann-like representation

$$a^\dagger = x, \quad a = D_q = \frac{1}{x} [xD_q] = \frac{1}{x} \left(\frac{1 - q x D}{1 - q}\right), \quad N = x D,$$

(41)

where $D_q$ is the Jackson derivative operator (see e.g., [15, 16]) such that

$$D_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x}.$$  

(42)

Note that

$$D_q x^n = [n]_q x^{n-1}.$$  

(43)

In the Bargmann-like representation the monomials

$$\xi_n(x; q) = \frac{x^n}{\sqrt{[n]_q!}}, \quad n = 0, 1, 2, \ldots,$$

(44)

carry the Fock representation of the $q$-oscillator algebra (32) as given by

$$a^\dagger \xi_n = \sqrt{[n + 1]_q} \xi_{n+1}, \quad a \xi_n = \sqrt{[n]_q} \xi_{n-1}, \quad N \xi_n = n \xi_n, \quad n = 0, 1, 2, \ldots.$$  

(45)

The monomials $\{\xi_n(x; q) | n = 0, 1, 2, \ldots\}$ form an orthonormal basis with respect to the inner product

$$\langle f | g \rangle = [f^* (D_q) g(x)]|_{x=0}.$$  

(46)

It follows from (43) that the $q$-exponential function

$$e^{\alpha x}_q = \sum_{n=0}^{\infty} \frac{\alpha^n x^n}{[n]_q!},$$

(47)
for any complex number $\alpha$, becomes an eigenfunction of $a$, and the normalized coherent states of the $q$-oscillator are given by

$$\left( e_q^{\alpha^2} \right)^{-\frac{1}{2}} e_q^{\alpha x} = \left( e_q^{\alpha^2} \right)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \xi_n(x; q).$$

(48)

The development of quantum groups and the representation theory of associated quantum algebras led first to the $(q^{-1}, q)$-oscillator [5, 6] (see also [7, 8]):

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad [a, a^\dagger] = \phi(N + 1) - \phi(N),$$

$$\phi(N) = [N]_{(q^{-1}, q)} = \frac{q^{-N} - q^N}{q^{-1} - q}. \quad (49)$$

With the relation

$$[N + 1]_{(q^{-1}, q)} - q^{-1} [N]_{(q^{-1}, q)} = q^N, \quad (50)$$

the relation between $a$ and $a^\dagger$ becomes

$$aa^\dagger - q^{-1} a^\dagger a = q^N. \quad (51)$$

The Fock representation is

$$a^\dagger |n\rangle = \sqrt{[n + 1]_{(q^{-1}, q)}} |n + 1\rangle, \quad a |n\rangle = \sqrt{[n]_{(q^{-1}, q)}} |n - 1\rangle,$$

$$N |n\rangle = n |n\rangle, \quad n = 0, 1, 2, \ldots, \quad (52)$$

where

$$[n]_{(q^{-1}, q)} = \frac{q^{-n} - q^n}{q^{-1} - q} \quad (53)$$

defines the $(q^{-1}, q)$-basic number. Note that $\lim_{q \to 1} [n]_{(q^{-1}, q)} = n$. Now, with the definitions

$$[0]_{(q^{-1}, q)}! = 1,$$

$$[n]_{(q^{-1}, q)}! = [n]_{(q^{-1}, q)}[n - 1]_{(q^{-1}, q)}[n - 2]_{(q^{-1}, q)} \ldots [2]_{(q^{-1}, q)}[1]_{(q^{-1}, q)}, \quad n = 1, 2, \ldots, \quad (54)$$

we have

$$|n\rangle = \frac{a_+^n}{\sqrt{[n]_{(q^{-1}, q)}!}} |0\rangle. \quad (55)$$
Defining the \((q^{-1}, q)\)-exponential function as
\[
e^{x(q^{-1}, q)} = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{(q^{-1}, q)}!},
\]
the normalized coherent states of the \((q^{-1}, q)\)-oscillator \([49]\) are given by
\[
|\alpha\rangle = \left(e^{\left|\alpha\right|^2}_{(q^{-1}, q)}\right)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]_{(q^{-1}, q)}!}} |n\rangle = \left(e^{\left|\alpha\right|^2}_{(q^{-1}, q)}\right)^{-\frac{1}{2}} e^{\alpha a^\dagger}_{(q^{-1}, q)} |0\rangle.
\]
The Bargmann-like representation is given by
\[
a^\dagger = x, \quad a = D_{(q^{-1}, q)} = \frac{1}{x} \left(\frac{q^{-xD} - q^{xD}}{q^{-1} - q}\right), \quad N = xD.
\]
Note that
\[
D_{(q^{-1}, q)} f(x) = \frac{f(q^{-1}x) - f(qx)}{(q^{-1} - q)x}, \quad D_{(q^{-1}, q)} x^n = [n]_{(q^{-1}, q)} x^{n-1}.
\]
Then, the monomials
\[
\xi_n(x; q^{-1}, q) = \frac{x^n}{\sqrt{[n]_{(q^{-1}, q)}!}}, \quad n = 0, 1, 2, \ldots
\]
carry the Fock representation of the \((q^{-1}, q)\)-oscillator algebra \([49]\) as given by
\[
a^\dagger \xi_n = \sqrt{[n + 1]_{(q^{-1}, q)}} \xi_{n+1}, \quad a \xi_n = \sqrt{[n]_{(q^{-1}, q)}} \xi_{n-1},
\]
\[
N \xi_n = n \xi_n, \quad n = 0, 1, 2, \ldots
\]
The monomials \(\{\xi_n(x; q^{-1}, q) | n = 0, 1, 2, \ldots\}\) form an orthonormal basis with respect to the inner product
\[
\langle f | g \rangle = \left[ f^* \left(D_{(q^{-1}, q)} \right) g(x) \right]_{x=0}.
\]
In this Bargmann-like representation, the \((q^{-1}, q)\)-exponential function
\[
e^{\alpha x}_{(q^{-1}, q)} = \sum_{n=0}^{\infty} \frac{\alpha^n x^n}{[n]_{(q^{-1}, q)}!},
\]
for any complex number $\alpha$, is an eigenfunction of $a$, and

$$
(e^{\frac{|a|^2}{(q^{-1},q)}})^{-\frac{1}{2}} e^{a x}_{(q^{-1},q)} = \left(e^{\frac{|a|^2}{(q^{-1},q)}}\right)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{|n|_{(q^{-1},q)}!}} \xi_n(x; q^{-1}, q)
$$

(64)
is a normalized coherent state of the $(q^{-1},q)$-oscillator [13].

Representation theory of two-parameter quantum algebras led to the generalization of the $(q^{-1},q)$-oscillator to the $(p,q)$-oscillator [17] (see also [18, 19]):

$$
[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad [a, a^\dagger] = \phi(N + 1) - \phi(N),
$$

$$
\phi(N) = [N]_{(p,q)} = \frac{p^N - q^N}{p - q}.
$$

(65)

Since

$$
[N + 1]_{(p,q)} - p[N]_{(p,q)} = q^N,
$$

(66)
we can write

$$
a a^\dagger - p a^\dagger a = q^N.
$$

(67)
The Fock representation is

$$
a^\dagger |n\rangle = \sqrt{[n + 1]_{(p,q)} |n + 1\rangle}, \quad a |n\rangle = \sqrt{[n]_{(p,q)} |n - 1\rangle},
$$

$$
N |n\rangle = n |n\rangle, \quad n = 0, 1, 2, \ldots ,
$$

(68)
where

$$
[n]_{(p,q)} = \frac{p^n - q^n}{p - q}
$$

(69)
defines the $(p,q)$-basic number, or the twin-basic number [20]. Note that $\lim_{p,q \rightarrow 1} [n]_{(p,q)} = n$. Now, with the definitions

$$
[0]_{(p,q)}! = 1, \quad [n]_{(p,q)}! = [n]_{(p,q)} [n - 1]_{(p,q)} [n - 2]_{(p,q)} \cdots [2]_{(p,q)} [1]_{(p,q)}, \quad n = 1, 2, \ldots ,
$$

(70)
we have

$$
|n\rangle = \frac{a^{n}}{\sqrt{[n]_{(p,q)}!}} |0\rangle.
$$

(71)
Defining the $(p,q)$-exponential function as

$$
e^{x}_{(p,q)} = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{(p,q)}!},
$$

(72)
the normalized coherent states of the \((p, q)\)-oscillator \([65]\) are given by

\[
|\alpha\rangle = \left( e^{\langle |\alpha|^{2}\rangle} \right)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \left( e^{\langle |\alpha|^{2}\rangle} \right)^{-\frac{1}{2}} e^{\alpha a^\dagger} |0\rangle.
\]

(73)

The Bargmann-like representation of the \((p, q)\)-oscillator algebra \([65]\) is given by

\[
a^\dagger = x, \quad a = D_{(p, q)} = \frac{1}{x} \left( \frac{p^x D - q^x D}{p - q} \right), \quad N = xD.
\]

(74)

The \((p, q)\)-derivative \(D_{(p, q)}\) is such that

\[
D_{(p, q)} f(x) = \frac{f(px) - f(qx)}{(p - q)x}, \quad D_{(p, q)} x^n = [n]_{(p, q)} x^{n-1}.
\]

(75)

The monomials

\[
\xi_n(x; p, q) = \frac{x^n}{\sqrt{[n]_{(p, q)}!}}, \quad n = 0, 1, 2, \ldots ,
\]

(76)

carry the Fock representation:

\[
a^\dagger \xi_n = \sqrt{[n + 1]_{(p, q)}} \xi_{n+1}, \quad a \xi_n = \sqrt{[n]_{(p, q)}} \xi_{n-1},
\]

\[
N \xi_n = n \xi_n, \quad n = 0, 1, 2, \ldots .
\]

(77)

The monomials \(\{\xi_n(x; p, q) | n = 0, 1, 2, \ldots \}\) form an orthonormal basis with respect to the inner product

\[
\langle f | g \rangle = \left[ f^* \left( D_{(p, q)} g(x) \right) \right]_{x=0}.
\]

(78)

Note that, in this Bargmann-like representation, the \((p, q)\)-exponential function

\[
e^{\alpha x}_{(p, q)} = \sum_{n=0}^{\infty} \frac{\alpha^n x^n}{[n]_{(p, q)}!},
\]

(79)

for any complex number \(\alpha\), is an eigenfunction of \(a\) and a normalized coherent state of the \((p, q)\)-oscillator is given by

\[
\left( e^{\langle |\alpha|^{2}\rangle} \right)^{-\frac{1}{2}} e^{\alpha x}_{(p, q)} = \left( e^{\langle |\alpha|^{2}\rangle} \right)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]_{(p, q)}!}} \xi_n(x; p, q).
\]

(80)
Note that the canonical boson oscillator \((1)\), the \(q\)-oscillator \((32)\), and the \((q^{-1}, q)\)-oscillator \((49)\), are special cases of the \((p, q)\)-oscillator \((65)\) corresponding to \((p = 1, q = 1)\), \(p = 1\), and \(p = q^{-1}\), respectively. The canonical fermion oscillator corresponds to \((p = −1, q = 1)\). The \(q\)-fermion oscillator \([21, 22]\) and the Tamm-Dancoff oscillator \([23, 24]\) are also special cases of the \((p, q)\)-oscillator corresponding to \(p = −q^{-1}\) and \(p = q\), respectively. There exist several other deformed oscillator structures (see, e.g., \([25, 26, 27, 28, 29]\)) with single and multiple deformation parameters.

### 3 Tsallis \(q\)-exponential function

The Tsallis \(q\)-exponential function \([13, 14]\) is a natural generalization of the exponential function and it is not surprising that its applications are ubiquitous in natural sciences (see the bibliography in \([30]\); see also \([31]\)). In the definition of the exponential function

\[
e^x = \lim_{N \to \infty} \left(1 + \frac{x}{N}\right)^N, \tag{81}
\]

if we replace the infinitesimally small \(1/N\) by \(1−q\), with \(0 < 1−q \approx 0\), then we get the Tsallis \(q\)-exponential function,

\[
e_q(x) = (1 + (1−q)x)^\frac{1}{1−q}, \tag{82}
\]

such that

\[
\lim_{q \to 1} e_q(x) = e^x. \tag{83}
\]

The corresponding \(q\)-logarithm is defined by

\[
\log_q x = \frac{x^{1−q} − 1}{1−q}, \tag{84}
\]

such that

\[
\log_q (e_q(x)) = x. \tag{85}
\]

Having defined \(e_q(x)\) for \(1−q \approx 0\) one may extend it to other values of \(q\) and explore its uses. Note that \(e_q(x)\) satisfies the nonlinear differential equation

\[
\frac{df(x)}{dx} = f(x)^q. \tag{86}
\]
It has been shown [32] that expansion in Taylor series leads to the expression

\[
e_q(x) = 1 + \sum_{n=1}^{\infty} \frac{Q_{n-1}}{n!} x^n, \quad Q_0 = 1, \quad Q_n = q(2q-1)(3q-2)\ldots(nq-(n-1)), \quad n = 1, 2, \ldots.
\]  

(87)

Writing the expression for \(e^{x_f}\) in (29) as

\[
e^{x_f} = 1 + \sum_{n=1}^{\infty} \frac{1}{f^n(n)!} \frac{x^n}{n!},
\]  

(88)

and comparing this expression with (87) one identifies \(e_q(x)\) with \(e^{x_f}\) corresponding to

\[
f(0)! = 1, \quad f(n)! = \frac{1}{\sqrt{Q_{n-1}}}, \quad n = 1, 2, \ldots.
\]  

(89)

In [33] this choice of \(f(n)!\) has been substituted in (30) to get the normalized states

\[
|\alpha\rangle = \mathcal{N}(\alpha) \left\{ |0\rangle + \sum_{n=1}^{\infty} \sqrt{Q_{n-1}} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \right\},
\]

\[
\mathcal{N}(\alpha) = \left( 1 + \sum_{n=1}^{\infty} Q_{n-1} \frac{|\alpha|^2}{n!} \right)^{-\frac{1}{2}},
\]  

(90)

called the \(f\)-coherent states attached to the Tsallis \(q\)-exponential function. This \(f\)-coherent state has been studied in detail in [33].

Noting that in (87)

\[
Q_{n-1} = (1 + (1-q)(1-n))!, \quad n = 1, 2, \ldots,
\]  

(91)

the Taylor series expression for \(e_q(x)\) has been rewritten in [34] to read

\[
e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{(1-q)}!},
\]  

(92)
with 
\[
[n]_{(1-q)} = \frac{n}{1 + (1 - q)(1 - n)}, \quad n = 0, 1, 2, \ldots ,
\]
\[
[n]_{(1-q)}! = [n]_{(1-q)}[n-1]_{(1-q)}[n-2]_{(1-q)} \cdots [2]_{(1-q)}[1]_{(1-q)}. \tag{93}
\]

Note that
\[
[0]_{(1-q)} = 0, \quad [1]_{(1-q)} = 1, \quad \lim_{q \to 1} [n]_{(1-q)} = n. \tag{94}
\]

Regarding \([n]_{(1-q)}\) as a deformed number, a scheme of \(q\)-deformation of non-linear maps has been proposed \cite{34} and this proposal has been found to have many applications (see, e.g., \cite{35, 36, 37, 38, 39, 40}). Now, note that the expression in \((92)\) for \(e^q(x)\) is of the same form as in \((16)\) for \(e^x\) with
\[
\phi(n) = [n]_{(1-q)}. \tag{95}
\]

Let us close this section with an interesting observation. Note that we can write
\[
e^q(x) = e^{\log\left(\frac{1+(1-q)x}{1-q}\right)} = e^{\sum_{n=1}^{\infty} \frac{(1-q)^n-1}{1-q} x^n}
\]
\[
= e^{x-(1-q)\frac{x^2}{2}+(1-q)^2\frac{x^3}{3}-(1-q)^3\frac{x^4}{4}+-\ldots}, \tag{96}
\]
showing explicitly \(\lim_{q \to 1} e^q(x) = e^x\). In \cite{41} (see also \cite{42}) it has been shown that if
\[
h(x) = \sum_{n=1}^{\infty} a_n x^n, \tag{97}
\]
then 
\[
e^{h(x)} = \sum_{n=0}^{\infty} c_n x^n, \tag{98}
\]
where \(c_0 = 1, \quad c_1 = a_1, \) and
\[
c_n = a_n + \frac{1}{n} \sum_{j=1}^{n-1} j c_{n-j} a_j, \quad n = 2, 3, \ldots . \tag{99}
\]

Using this result the \(q\)-exponential function \(e^x_q\) in \((39)\) has been expressed as the exponential of an infinite series in \cite{43}. For the Tsallis \(q\)-exponential
function we have

\[ h(x) = \sum_{n=1}^{\infty} \frac{(-1-q)^{n-1}}{n} x^n, \]

\[ e^{h(x)} = e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!(1-q)!} = 1 + x + \sum_{n=2}^{\infty} \frac{(1+(1-q)(1-n))!}{n!} x^n. \]  

(100)

We find \( c_0 = 1 \) and \( c_1 = 1 = a_1 \) as should be. Now, substituting

\[ a_n = \frac{(-1-q)^{n-1}}{n}, \quad c_n = \frac{(1+(1-q)(1-n))!}{n!}, \quad \text{for } n \geq 2, \]  

(101)
in (99), and taking \( q = 1 + \tau \), we get an identity

\[ \frac{(1+(n-1)\tau)!}{n!} = \frac{\tau^{n-1}}{n} + \frac{1}{n} \sum_{j=1}^{n-1} \frac{(1+(n-j-1)\tau)!\tau^{j-1}}{(n-j)!}, \]

\[ n = 2, 3, \ldots, \]  

(102)

where \((1+n\tau)! = (1+n\tau)(1+(n-1)\tau)\ldots(1+2\tau)(1+\tau)1.\)

4 The \( q \)-oscillator associated with the Tsallis \( q \)-exponential function

Let us now consider the deformed oscillator with the commutation relations

\[ [N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad [a, a^\dagger] = \phi_T(N+1) - \phi_T(N), \]

\[ \phi_T(N) = |N|_{(1-q)} = \frac{N}{1+(1-q)(1-N)}. \]  

(103)

This deformed oscillator has been studied in [44], in a totally different context, with \((1-q)\) replaced by a different deformation parameter relevant to their study. It is a slightly modified version of the \( \mu \)-oscillator [45] for which \( \phi(N) = N/(1+\mu N) \). As an \( f \)-oscillator it reads

\[ a^\dagger = f(N)b^\dagger, \quad a = bf(N), \quad N = b^\dagger b, \]

\[ f(N) = \sqrt{\frac{\phi_T(N)}{N}} = \sqrt{\frac{|N|_{(1-q)}}{N}} = \frac{1}{\sqrt{(1+(1-q)(1-N))}}, \]  

(104)
The states in (90) are the coherent states of this $f$-oscillator corresponding to the eigenfunctions of $a = bf(N)$. So, let us investigate the relation of this oscillator to the Tsallis $q$-exponential function.

The Fock representation of the deformed oscillator (103) is given by

$$a^\dagger|n\rangle = \sqrt{[n+1](1-q)}|n+1\rangle, \quad a|n\rangle = \sqrt{[n](1-q)}|n-1\rangle, \quad N|n\rangle = n|n\rangle, \quad n = 0, 1, 2, \ldots . \quad (105)$$

It follows that

$$|n\rangle = \frac{a^\dagger^n}{\sqrt{[n](1-q)!}}|0\rangle. \quad (106)$$

With the Tsallis $q$-exponential function expressed as in (92), the normalized coherent states of the deformed oscillator (103) are seen to be given by

$$|\alpha\rangle = (e_q(\alpha^2))^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n](1-q)!}} |n\rangle = (e_q(\alpha^2))^{-\frac{1}{2}} e_q(\alpha a^\dagger)|0\rangle. \quad (107)$$

Noting that

$$|n\rangle = \frac{a^\dagger^n}{\sqrt{[n](1-q)!}}|0\rangle = \frac{(f(N)b^i)^n}{\sqrt{[n](1-q)!}}|0\rangle = \frac{f(n)!b^{in}}{\sqrt{[n](1-q)!}}|0\rangle, \quad (108)$$

it is seen that $|\alpha\rangle$ in (107) is the $f$-coherent state found in (30).

To get the Bargmann-like representation of the deformed oscillator (103) we must have a deformed derivative $D_q^{(T)}$ such that

$$D_q^{(T)}x^n = [n](1-q) x^{n-1}. \quad (109)$$

Following (21), as has been done in (46), one may choose the formal operator $(1/x)\phi(xD) = D/(1 + (1 - q)(1 - xD))$ as the required deformed derivative which satisfies (109) and has $e_q(\alpha x)$ as its eigenfunction for any $\alpha$. However, it is only a formal operator, not suitable to operate on an arbitrary function. Another suggestion for the the required deformed derivative, given in (47), is $(1 + (1 - q)x)D$ for which $e_q(x)$ is an eigenfunction. But, it is also not suitable for the purpose since $e_q(\alpha x)$ is not its eigenfunction for arbitrary $\alpha$ and also it does not satisfy the requirement in (109). To derive the required deformed derivative we follow (48) where the deformed derivative corresponding to the
\(\mu\)-oscillator \([43]\) has been obtained. Slightly modifying the suggestion in \([48]\) we get the desired deformed derivative as

\[ D_q^{(T)} f(x) = \int_0^1 dt \ t^{1-q} D f \left( t^{q-1} x \right). \tag{110} \]

It is obvious that \(\lim_{q \to 1} D_q^{(T)} = D\). It can be verified that

\[ D_q^{(T)} x^n = \int_0^1 dt \ t^{1-q} D \left( t^{(q-1)n} x^n \right) = \int_0^1 dt \ t^{(1-q)(1-n)} D x^n \]

\[ = n x^{n-1} \int_0^1 dt \ t^{(1-q)(1-n)} = \frac{n x^{n-1}}{1 + (1 - q)(1 - n)} \]

\[ = [n]_{(1-q)} x^{n-1}. \tag{111} \]

Then, it follows that

\[ D_q^{(T)} e_q(\alpha x) = D_q^{(T)} \left( \sum_{n=0}^{\infty} \frac{\alpha^n x^n}{[n]_{(1-q)}!} \right) = \alpha e_q(\alpha x). \tag{112} \]

It can also be verified directly that

\[ D_q^{(T)} e_q(\alpha x) = \alpha e_q(\alpha x). \tag{113} \]

This is seen as follows. From \((110)\) we have

\[ D_q^{(T)} e_q(\alpha x) = \int_0^1 dt \ t^{1-q} D \left[ (1 + (1 - q)\alpha t^{q-1} x) \right]^{\frac{1}{1-q}} \]

\[ = \alpha \int_0^1 dt \ (1 + (1 - q)\alpha t^{q-1} x) \]

\[ = \alpha \int_0^1 dt \ t^{-q} \left( t^{1-q} + (1 - q)\alpha x \right) \]

\[ = \frac{\alpha}{1 - q} \int_0^1 dt (1 - q) \left( t^{1-q} + (1 - q)\alpha x \right) \]

\[ = \alpha t^{1-q} \left( 1 + (1 - q)\alpha t^{q-1} x \right) \left|_{t=0}^{t=1} \right. \]

\[ = \alpha t^{1-q} e_q \left( \alpha t^{q-1} x \right) \left|_{t=0}^{t=1} \right. \]

\[ = \alpha e_q(\alpha x), \tag{114} \]
proving (113).

Now, the Bargmann-like representation of the deformed $q$-oscillator (103) is given by
\[
a^\dagger = x, \quad a = D_q^{(T)}, \quad N = xD. \tag{115}
\]
Under the inner product
\[
\langle f | g \rangle = \left[ f^* \left( D_q^{(T)} \right) g(x) \right]_{x=0}, \tag{116}
\]
the monomials
\[
\xi_n^{(T)}(x) = \frac{x^n}{\sqrt{[n]_{(1-q)}!}}, \quad n = 0, 1, 2, \ldots , \tag{117}
\]
form an orthonormal basis and carry the Fock representation
\[
a^\dagger \xi_n^{(T)}(x) = \sqrt{[n+1]_{(1-q)}\xi_{n+1}^{(T)}(x)}, \quad a \xi_n^{(T)}(x) = \sqrt{[n]_{(1-q)}\xi_{n-1}^{(T)}(x)},
\]
\[
N \xi_n^{(T)}(x) = n \xi_n^{(T)}(x), \quad n = 0, 1, 2, \ldots . \tag{118}
\]
In the Bargmann-like representation, with
\[
\langle e_q(\alpha x) | e_q(\alpha x) \rangle = e_q \left( \left| \alpha \right|^2 \right), \tag{119}
\]
the normalized coherent states of the deformed oscillator (103), eigenfunctions of $a = D_q^{(T)}$, are given by
\[
\psi_\alpha^{(T)}(x) = \left( e_q \left( \left| \alpha \right|^2 \right) \right)^{-\frac{1}{2}} e_q(\alpha x) = \left( e_q \left( \left| \alpha \right|^2 \right) \right)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n x^n}{[n]_{(1-q)}!} \xi_n^{(T)}(x). \tag{120}
\]

The inverse of the deformed derivative $D_q^{(T)}$, the deformed integral, is seen to be given by
\[
\int d_q^{(T)} x f(x) = \left( \frac{d}{dt} \int dx \ t^{(q-1)} x f(t^{(q-1)} x) \right) \bigg|_0^1. \tag{121}
\]
It can be verified that
\[
\int d_q^{(T)} x x^n = \left( \frac{d}{dt} \int dx \ t^{1+(q-1)n} x^n \right) \bigg|_0^1 = \frac{x^{n+1}}{[n+1]_{(1-q)}}, \tag{122}
\]
as should be, since $D_q^{(T)} x^{n+1} = [n + 1]_{(1-q)} x^n$. Note that
\[
\lim_{q \to 1} \int d_q^{(T)} x f(x) = \int dx \ f(x). \tag{123}
\]
5 Conclusion

In fine, the deformed oscillator with the relations

\[ [N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad [a, a^\dagger] = \phi_T(N + 1) - \phi_T(N), \]  

(123)

with

\[ \phi_T(N) = \frac{N}{1 + (1 - q)(1 - N)}, \]  

(124)

is associated with the Tsallis \( q \)-exponential function

\[ e_q(x) = (1 + (1 - q)x)^{1/(1-q)}. \]  

(125)

In a Bargmann-like representation the annihilation operator \( a \) corresponds to a deformed derivative,

\[ D_q^{(T)} f(x) = \int_0^1 dt \ t^{1-q} D f \left( t^{q-1} x \right), \]  

(126)

with the Tsallis \( q \)-exponential functions as its eigenfunctions, and the Tsallis \( q \)-exponential functions are the coherent states of the deformed oscillator. It may be noted that the expression in (110) for the deformed derivative \( D_q^{(T)} \) for which the Tsallis \( q \)-exponential is an eigenfunction should lead to interesting further studies, particularly in the applications of nonextensive statistical mechanics. A remark in [48], made in the context of \( \mu \)-oscillator, points to the existence of interesting possibilities for extending this deformed derivative and the Tsallis \( q \)-exponential function with more deformation parameters. For example, following the remark in [48], if we substitute the ordinary derivative \( D \) in (110) by the two-parameter derivative \( D_{(r,s)} \) (\( D_{(p,q)} \) in (75) with \( p \) and \( q \) replaced by \( r \) and \( s \)) we would get a \((q, r, s)\)-deformed derivative and its eigenfunction would be the Tsallis \((q, r, s)\)-exponential function.

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