Amenability, extreme amenability, model-theoretic stability, and dependence property in integral logic

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Abstract

This paper has three parts. First, we study and characterize amenable and extremely amenable topological semigroups in terms of invariant measures using integral logic. We prove definability of some properties of a topological semigroup such as amenability and the fixed point on compacta property. Second, using these results we define types and develop local stability in the framework of integral logic. For a stable formula $\phi$, we prove definability of all complete $\phi$-types over models and deduce from this the fundamental theorem of stability. Third, we study a well-known concept in measure theory, Talagrand’s stability. We show that this concept is a version of non independence property (NIP) and prove a form of definability of types for NIP theories. We also show that the approach of this paper can be used to generalize some results in [8] and [24].

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1 Introduction

Probability logics are logics of probabilistic reasoning. A model theoretic approach aiming
to study probability structures by logical tools was started by Keisler and Hoover (see
[17, 18] for a survey). Among several variants of this logic, they introduced integral
logic $L_\int$ as an equivalent ‘Daniell integral’ presentation for $L_{\omega_1^P}$. Integral logic uses
the language of measure theory, i.e., that of measurable functions and integration. The
resulting framework is close to the usual language of probability theory and encompasses
many usual activities in it. In [3] Bagheri and Pourmahdian developed a finitary version
of integration logic and proved appropriate versions of the compactness theorem and
elementary JEP/AP. The intended models are graded probability structures introduced
by Hoover in [17] and in addition to random variables over probability spaces, they include
dynamical systems and other interesting structures from real analysis. In [21] the authors
showed that many interesting notions such as independence, martingale property, and
some special cases the notion of conditional expectation (as in martingales) are expressible.
Also, the Kolmogorov’s extension theorem was deduced from the compactness property of
model theory. In [20] the authors further used the logical tools to study invariant measures
on compact Hausdorff spaces. Consequently, they gave two proofs of the existence of Haar
measure on compact groups. One might therefore hope to obtain other applications of
the compactness theorem.

Historically one of the great successes of model theory has been Shelah’s stability
theory. Essentially the success of the program may be measured by the fact that the
original set-theoretic criteria are now largely passed over in favor of definitions which
mention ranks or combinatorial properties of a particular formula. On the other hand, a
general trend in model theory is to generalise these model-theoretic notions and tools to
frameworks that go beyond that of first order logic and elementary classes.

In the present paper, on one hand, we study some analytic concepts, amenability and
extremely amenability, using integral logic. On the other hand, we study types and local
stability in this logic. Moreover, we show that the approach of this paper can be used
to generalize some results in [8] and [24]. This approach has two advantages. First, we
underline the strengths of application of logical methods to the other fields of mathematics.
Second, the results obtained by these methods provide a new view on the related subjects
in Analysis and Logic, and open some fruitful areas of research on the similar questions.

To summarize the results of this paper, in the first part (Section 4), we consider an
arbitrary topological semigroup \( S \) and any compact Hausdorff space \( X \) such that \( S \) acts
continuously on \( X \) from the left. Let \( \text{Inv}_X(S) \) be the set of all Radon probability measures
on \( X \) which are left invariant under elements of \( S \). It is shown that the nonemptiness of
\( \text{Inv}_X(S) \) is expressible by a theory \( T_{S,X} \) in integral logic. We then give a characterization
of amenable topological semigroups in terms of invariant measures (Theorem 4.5). Using
the compactness theorem, we give a proof of the fundamental result goes back to N.N. Bo-
golioubov and N.M. Krylov (Theorem 4.11). The interesting fact is that for a topological
semigroup \( S \) the amenability of \( S \) is expressible by a theory \( T_S \) in the framework of inte-
gral logic. Some other new results and different proofs of some known results are given
for extremely amenable topological semigroups (Theorem 4.20, and Propositions 4.22).

Despite the most of results in the first part of the paper are standard, the study of
amenable and extremely amenable semigroups is necessary because it leads us to the
“suitable” and “correct” notation of a type in integral logic. In fact, types are known
mathematical objects, Riesz homomorphisms. Thus, for a complete theory \( T \), the space
of complete types \( S(T) \) can be represented by the spectrum of \( T \). Also, this approach
generalizes the notation of types in classical model theory, continuous logic [8], and op-
erator logics [24]. Thereby, in the second part of the paper (section 5), we define types
and develop local stability. For a stable formula \( \phi \), we prove that all complete \( \phi \)-types
over models are definable, and we deduce from this the fundamental theorem of stability
(Theorem 5.8). We show that a formula \( \phi \) is stable if and only if its Cantor-Bendixson
rank is finite.

In the third part of the paper, we study a form of the dependence propery which is
a well-known measure theoretic property, Talagrand’s stability. Then we prove that for a
weakly dependent formula \( \phi \), all \( \phi \)-types are almost definable (Theorem 6.5). We then
study the Cantor-Bendixson rank and the Vapnik-Chervonenkis dimension in dependence
theories.

It is worth recalling another line of research arisen from ideas of Chang and Keisler [9],
name continuous logic. The idea is recently refined and developed in [8] and [7] by Ben
Yaacov et al. for the class of metric structures which include such important classes of
structures as Banach spaces and measure algebras. Although some results in the present
paper (cf. section 5) are similar to those in [8], but in some senses they are different;
(i) Our approach can be used to generalize the results in [8] and [24] (see Remark 5.12);
(ii) In [5] and [4], Ben Yaacov proved that the theory atomless random variables and the
category of probability algebras are \( \aleph_0 \)-stable. Note that in this paper we do not study
probability measure algebras or \( L^1 \)-spaces, but we study measurable functions in \( L^0 \). On
the other hand, in contrast to [5] and [4], the theory of a probability structure is not
necessarily stable. This leads us to the dichotomy between stable probability structures and unstable probability structures; (iii) Some analytic properties such as probability independence, amenability, extreme amenability and the existence of invariant measures on compact spaces are expressible in the framework of integral logic.

The organization of the paper is as follows. In the next section we review some basic notions from measure theory. In section 3 a summary of results from [3] are given. In section 4, we study amenable and extremely amenable topological semigroups, and give a characterization of (extreme) amenability in terms of (multiplicative) invariant measures. A proof of the Bogolioubov-Krylov theorem is given in section 4. It is shown that the (extreme) amenability of a topological semigroup $S$ is expressible by a theory $T_S$ within in integral logic. In section 5, we conclude with the development of local stability, and we prove the fundamental theory of stability. In section 6, we study NIP theories and give some results.

2 Preliminaries from topological measures theory

In this section we review some basic notions from measure theory. Further details can be found in [11, 12, 14]. Let $X$ be a compact Hausdorff space. The space $C(X, \mathbb{R})$ of continuous real-valued functions on $X$ is denoted here by $C(X)$. Since $X$ is a compact space, every $f \in C(X)$ is bounded and $C(X)$ is a normed vector space with the uniform norm.

The class of Baire sets is defined to be the smallest $\sigma$-algebra $B$ of subsets of $X$ such that each function in $C(X)$ is measurable with respect to $B$. The smallest $\sigma$-algebra containing the open sets is called the class of Borel sets. Clearly, every Baire set is a Borel set, but there are compact spaces where the class of Borel sets is larger than the class of Baire sets. By a Baire (Borel) measure on $X$ we mean a finite measure defined for all Baire (Borel) sets. A Radon measure on $X$ is a Borel measure which is regular. It is known that every Baire measure on a compact space is regular and has a unique extension to a Radon measure.

A topological semigroup is a semigroup $S$ endowed with a Hausdorff topology such that the operation $(x, y) \mapsto xy$ is continuous from $S \times S$ to $S$. By a topological group we mean a group $G$ endowed with a Hausdorff topology such that the group operations $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous from $G \times G$ and $G$ to $G$. A topological group whose topology is (locally) compact and Hausdorff is called a (locally) compact group.

A topological semigroup $S$ is said to act on a topological space $X$ from the left if there is a map $S \times X \to X$ (denoted by $(s, x) \mapsto s \cdot x$ for each $(s, x) \in S \times X$) such that (a) the map $x \mapsto s \cdot x$ is continuous for each $s \in S$, (b) for $s, s' \in S$, $(ss') \cdot x = s \cdot (s' \cdot x)$ for each $x \in X$, and (c) if $S$ has the identity $e$, then $e \cdot x = x$ for each $x \in X$. In addition, the left action is said continuous if $(s, x) \mapsto s \cdot x$ is a continuous map from $S \times X$ to $X$. Similarly one can define a right (continuous) action. If $S$ acts on topological space $X$ from the left (right) and $E \subseteq X$ and $s \in S$ we define

$$s \cdot E = \{s \cdot x : x \in E\} \quad (E \cdot s = \{x \cdot s : x \in E\}).$$
If $f$ is a continuous real-valued function on a topological space $X$ and $s \in S$, we define the left (right) translate of $f$ by $s$, as follows:

$$(f \cdot s)(x) = f(s \cdot x)$$

$$(s \cdot f)(x) = f(x \cdot s).$$

The point of the above definition is to make $f \cdot (ss') = (f \cdot s) \cdot s'$ ($(ss') \cdot f = s \cdot (s' \cdot f)$).

If a topological semigroup $S$ acts on a space $X$ from the left (right), a measure $\mu$ on $X$ is left (right) $S$-invariant if $\mu(s \cdot E) (\mu(E \cdot s))$ is defined and equal to $\mu(E)$ whenever $s \in S$ and $\mu$ measures $E$. If $X$ be a compact Hausdorff space, then a linear functional $I$ on $C(X)$ is called left (right) $S$-invariant if $I(f \cdot s) = I(f)$ ($I(s \cdot f) = I(f)$) for all $s$ in $S$ and $f$ in $C(X)$.

A left (right) Haar measure on a compact group $G$ is a nonzero left (right) $G$-invariant Radon measure $\mu$ on $G$.

**Proposition 2.1** ([11 Proposition 441L]) Let $X$ be a Hausdorff compact space and $S$ a topological semigroup which acts on $X$. A nonzero Radon measure $\mu$ on $X$ is a left (right) $S$-invariant measure iff $\int fd\mu = \int (f \cdot s) d\mu$ ($\int fd\mu = \int (s \cdot f) d\mu$) for all $f \in C(X)$ and $s \in S$.

If $G$ is compact group, then a left Haar measure on $G$ is also a right Haar measure. Also, the Haar measure is unique up to a positive scalar multiple, i.e. if $\mu$ and $\nu$ are Haar measures on a compact group $G$, there exists $c > 0$ such that $\mu = c\nu$.

**Proposition 2.2** Let $X$ be a locally compact Hausdorff space and $C_c(X)$ the space of continuous real-valued functions on $X$ with compact support.

(a) ([11] p. 212) If $I$ is a positive linear functional on $C_c(X)$, there is a unique Radon measure $\mu$ on $X$ such that $I(f) = \int fd\mu$ for all $f \in C_c(X)$.

(b) ([29] p. 358) If $X$ is compact, then the dual of $C(X)$ is (isometrically isomorphic to) the space of all finite signed Baire measures on $X$ with norm defined by $\|\mu\| = |\mu|(X)$.

**Proposition 2.3** ([11] p. 159) Let $N$ be a normed vector space. If $M$ is a closed subspace of $N$ and $x \in N \setminus M$, there exists a bounded linear functional $I$ on $N$ such that $I|_M = 0$, $\|I\| = 1$ and $I(x) = \inf_{y \in M} \|x - y\|$.

Let $(M, B, \mu)$ be a measure space and $\mu^*$ its associated outer measure defined by

$$\mu^*(X) = \inf\{\mu(A) | X \subseteq A \in B\}.$$ 

If $N \subseteq M$, then $B_N = \{A \cap N | A \in B\}$ is a $\sigma$-algebra and $\mu_N = \mu^* | B_N$ is a measure on $N$. $\mu_N$ is called the subspace measure on $N$. A measurable envelope for $N$ is a measurable set $E \in B$ such that $N \subseteq E$ and $\mu(E \cap A) = \mu^*(N \cap A)$ for any $A \in B$. Every $N \subseteq M$ of finite outer measure has an envelope (e.g. take $E \in B$ containing $N$ with $\mu(E) = \mu^*(N)$). If $f : M \to \mathbb{R}$ is measurable, $\int_N f$ abbreviates $\int_N (f \upharpoonright N) d\mu_N$. 

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Proposition 2.4 ([12, p. 38]) Let \((M, \mathcal{B}, \mu)\) be a measure space, \(N \subseteq M\), and \(f\) be an integrable function defined on \(M\).

(a) If \(f\) is nonnegative then \(f \upharpoonright N\) is \(\mu_N\)-integrable and \(\int_N f \leq \int f\).

(b) If either \(N\) is of full outer measure in \(M\) or \(f\) is zero almost everywhere on \(M - N\), then \(\int_N f = \int_M f\).

3 Integral logic

In this section we give a brief review of integral logic from [3, 20]. Results from [3, 20] are stated without proof. All languages are assumed to contain of unary relation and constant symbols. Let \(L\) be a language. To each relation symbol \(R \in L\) we assign a nonnegative real number \(\flat R \geq 0\) called the universal bound of \(R\).

Definition 3.1 The family of \(L\)-formulas and their universal bounds is defined as follows:

1. If \(R\) is a relation symbol, then \(R(\vec{x})\) is an atomic formula with bound \(\flat R\).

2. If \(\phi\) and \(\psi\) are formulas and \(r, s \in \mathbb{R}\), then so are \(r\phi + s\psi\) and \(\phi \times \psi\) with bounds \(|r|\flat\phi + |s|\flat\psi\) and \(\flat\phi \flat \psi\), respectively.

3. If \(\phi\) is a formula and \(x\) is a variable, then \(|\phi|\) is a formula with bound \(\flat\phi\).

4. If \(\phi\) is a formula and \(x\) is a variable, then \(\int \phi \, dx\) is a formula with bound \(\flat \phi\).

Note that \(\phi^+ = \frac{1}{2}(\phi + |\phi|)\) and \(\max(\phi, \psi) = (\phi - \psi)^+ + \psi\) and similarly \(\phi^-\) and \(\min(\phi, \psi)\) are formulas.

Definition 3.2 An \(L\)-structure is a probability measure space \(M = (M, \mathcal{B}, \mu)\) equipped with:

- for each constant symbol \(c \in L\), an element \(c^M \in M\);
- for each relation symbol \(R \in L\), a measurable map \(R^M : M \to [-\flat R, \flat R]\).

\(L\)-structures are denoted by \(M, N\) etc. The notion of free variable is defined as usual and one writes \(\phi(\vec{x})\) to display them. If \(M\) is an \(L\)-structure, for each formula \(\phi(\vec{x})\) and \(\vec{a} \in M\), \(\phi^M(\vec{a})\) is defined inductively starting from atomic formulas. In particular,

\[
\left( \int \phi(\vec{x}, \vec{y}) \, dy \right)^M(\vec{a}) = \int \phi^M(\vec{a}, \vec{y}) \, dy.
\]

An easy induction shows that every \(\phi^M(\vec{x})\) is a well-defined measurable function and \(|\phi^M(\vec{x})| \leq \flat \phi\). Indeed, for every \(\phi(\vec{x}, \vec{y})\) and \(\vec{a}, \phi^M(\vec{a}, \vec{y})\) is measurable. Moreover, we have \(\int \int \phi \, dx \, dy = \int \int \phi \, dy \, dx\).
A formula is closed if no free variable occurs in it. A statement is an expression of the form \( \phi(\bar{x}) \geq r \) or \( \phi(\bar{x}) = r \). Closed statements are defined similarly. Any set of closed statements is called a theory. The theory of a structure \( M \) is the collection of closed statements satisfied in it. Such theories are called complete. \( M, N \) are elementarily equivalent (written \( M \equiv N \)) if they have the same theory. The notion \( M \models \Gamma \) is defined in the obvious way. If \( T \) is an \( L \)-theory, two formulas \( \phi(\bar{x}), \psi(\bar{x}) \) are said to be \( T \)-equivalent if the statement \( \phi = \psi \) a.e. is satisfied in every model of \( T \). We say \( T \) has quantifier-elimination if every formula is \( T \)-equivalent to a quantifier-free formula (i.e. without \( \exists \)).

Ultaprodut of a family \( M_i, i \in I \) of structures over an ultrafilter \( D \) is an \( L \)-structure and denoted by \( M = \prod_D M_i \) (cf. \([3, 20]\)).

**Theorem 3.3** (Fundamental theorem) For each \( \phi(\bar{x}) \) and \([a_i^1], \ldots, [a_i^n] \in M\)
\[
\phi^M([a_i^1], \ldots, [a_i^n]) = \lim_D \phi^{M_i}(a_i^1, \ldots, a_i^n).
\]

An immediate consequence of the fundamental theorem is the following whose proof is just a modification of its analog in the usual first order logic.

**Theorem 3.4** (Compactness theorem) Any finitely satisfiable set of closed statements is satisfiable.

**Definition 3.5** (i) If \( M \subseteq N \), \( M \) is a substructure of \( N \), denoted by \( M \subseteq N \), if \( M \) has the subspace measures and for each \( R \in L \) and \( \bar{a} \in M \), \( R^M(\bar{a}) = R^N(\bar{a}) \). If these equalities hold for almost all \( \bar{a} \), \( M \) is called an almost substructure of \( N \) and is denoted by \( M \subseteq_a N \).

(ii) An injection \( f : M \rightarrow N \) is called an elementary embedding if for each \( \phi \) and \( \bar{a} \in M \), \( \phi^M(\bar{a}) = \phi^N(f(\bar{a})) \). It is an almost elementary embedding if for each \( \phi \) this holds almost surely for \( \bar{a} \in M \). If \( f \) is the inclusion, these are respectively denoted by \( M \preceq N \), and \( M \preceq_a N \). \( f \) is said to be almost surjective if its range has full measure. One also defines isomorphism (resp. almost isomorphism) as a surjective (resp. almost surjective) elementary (resp. almost elementary) embedding.

The fact that \( \preceq \) (resp. \( \preceq_a \)) is stronger than \( \subseteq \) (resp. \( \subseteq_a \)) is a consequence of Tarski-Vaught test (cf. \([3]\)). Among the two notions of isomorphism, the notion of almost isomorphism is more useful, however, the exact isomorphism appears naturally in some cases. In ergodic theory, a map which is an (exact) isomorphism after removing some negligible sets from its domain and codomain is called an isomorphism. This notion is equivalent to the notion of almost isomorphism.

A structure is called minimal if it has no redundant measurable sets, i.e., for any substructure \( M' = (M, A, \mu \upharpoonright A) \) where \( A \subseteq B \), one has \( A = B \). In fact, every structure is isomorphic to a minimal structure, which can be explicitly described.

**Proposition 3.6** Let \( M = (M, \mathcal{B}, \mu) \) be an \( L \)-structure and \( \mathcal{A} \) be the \( \sigma \)-algebra generated by the sets of the form \( \{x \in M : \phi^M(x) > 0\} \) where \( \phi \) is any formula with parameters in \( M \). Then, \( M' = (M, \mathcal{A}, \mu \upharpoonright A) \) is a minimal measure \( L \)-structure isomorphic to \( M \).
Next we are going to prove a key result, which plays an important role in the rest of this paper. Assume $X$ is a compact Hausdorff space. Let $L_X$ be the language consisting of a unary relation symbol $R_f$ for each $f \in C(X)$ and a constant symbol $c_a$ for each $a \in X$. Let $M$ be an $L_X$-structure with the following properties:

- $X \subseteq M$;
- the restriction of $R_f^M$ to $X$ is $f$, particularly $R_1^M = 1$;
- $R_{f+g}^M = R_f^M + R_g^M$ and $R_{r \cdot f}^M = r \cdot R_f^M$ for each $f, g \in C(X)$ and real number $r$;
- $R_{f \cdot g}^M = R_f^M \times R_g^M$ for each $f, g \in C(X)$;
- $R_{\text{max}(f,g)}^M = \max(R_f^M, R_g^M)$ for each $f, g \in C(X)$.

The next proposition shows that the subspace measure $\mu_X$ on $X$ behaves like the measure $\mu$ on $M$. In fact, $(X, B_X, \mu_X)$ with the natural interpretation of relation and constant symbols is an elementary substructure of $M$.

**Proposition 3.7** Assume $X$ and $M$ are as above.

(a) The subspace measure $\mu_X$ on $X$ is a regular Baire measure such that $\int f d\mu_X = \int R_f d\mu$ for each $f \in C(X)$.

(b) There exists a Radon measure $\bar{\mu}_X$ on $X$ such that $\int f d\bar{\mu}_X = \int R_f d\mu$ for each $f \in C(X)$.

**Proof.** (a). By Proposition 2.1, it suffices to show that $X$ is of full outer measure in $M$. We assume $M$ is minimal. By Proposition 3.6,

$$\mu_X(X) = \inf \left\{ \sum_{1}^{\infty} \mu(A_k) : X \subseteq \bigcup_{1}^{\infty} A_k \right\}$$

where $A_k = (R_{f_k}^M)^{-1}(0, \infty)$ for a $f_k \in C(X)$ because every formula $\phi$ is equal to a relation symbol $R_f$. We show that $\mu(\bigcup_k A_k) = 1$ for every sequence $(A_k)_{k \in \mathbb{N}}$ such that $X \subseteq \bigcup_{1}^{\infty} A_k$. If $X \subseteq \bigcup_k f_k^{-1}(0, \infty)$, then there exist $f_1, \ldots, f_n$ such that $X = \bigcup_1^n f_k^{-1}(0, \infty)$ because $X$ is compact. If $f = \max(f_1, \ldots, f_n)$, then $X = f^{-1}(0, \infty)$. Thus, $X \subseteq (R_{f}^M)^{-1}(0, \infty)$ because $R_{f}^M = \max(R_{f_1}^M, \ldots, R_{f_n}^M)$. Since $X$ is compact and $f$ is continuous, there exist real numbers $s \geq r > 0$ such that $X = f^{-1}[r, s]$. Also, we can easily check that $M = (R_{f}^M)^{-1}(0, \infty)$ since $R_{f}^M \geq r$. Thus, $\mu(\bigcup_k A_k) \geq \mu((R_{f}^M)^{-1}(0, \infty)) = 1$, i.e. $\mu_X(X) = 1$. We may assume $\mu_X$ is a Baire measure. Also, we know that every Baire measure on a compact space is regular.

(b). It is known that every Baire regular measure on a compact space has a unique extension to a Radon measure (cf. [29, p. 341]). Let $\bar{\mu}_X$ be the unique extension of $\mu_X$ to a Radon measure on $X$. Since only the values of $\bar{\mu}_X$ on Baire sets matter for $\int f d\bar{\mu}_X$, we have $\int f d\bar{\mu}_X = \int f d\mu_X$ for each $f \in C(X)$.

\[\square\]
4 Amenability and extreme amenability

In this section we study and characterize amenable and extremely amenable topological semigroups in terms of invariant measures using integral logic. First, we give two conditions equivalent to the existence of measures on a compact Hausdorff space $X$ invariant under a semigroup $S$ which acts on it from the left. We then characterize (extremely) amenable topological semigroups in terms of (multiplicative) invariant measures. It is shown that all compact groups, abelian topological semigroups, and all locally finite topological groups are amenable. An interesting fact is that for a topological semigroup $S$ the (extreme) amenability of $S$ is expressible by a theory $T_{S}$ ($T_{x}$) in the framework of integral logic. Therefore, it is shown that a locally compact group $G$ has no Borel paradoxical decomposition iff the theory $T_{G}$ is finitely satisfiable.

Let $X$ be a compact Hausdorff space and $S$ be a semigroup which acts on $X$ from the left. Let $L_{X}$ be the language consisting of a unary relation symbol $R_{f}$ for each $f \in C(X)$ and a constant symbol $c_{a}$ for each $a \in X$ and $T_{S,X}$ be the theory with the following axiom:

1. $R_{1} = 1$,
2. $\int R_{1}dx = 1$,
3. $R_{f}(c_{a}) = f(a)$ for each $R_{f}, c_{a} \in L_{X}$,
4. $R_{f+g} = R_{f} + R_{g}$ for each $R_{f}, R_{g} \in L_{X}$,
5. $R_{r \times f} = r \times R_{f}$ for each $R_{f} \in L_{X}$ and $r \in \mathbb{R}$,
6. $R_{f \times g} = R_{f} \times R_{g}$ for each $R_{f}, R_{g} \in L_{X}$,
7. $R_{\text{max}(f,g)} = \text{max}(R_{f}, R_{g})$ for each $R_{f}, R_{g} \in L_{X}$,
8. $\int R_{f}(x)dx = \int R_{(f,s)}(x)dx$ for each $R_{f} \in L_{X}$ and $s \in S$,

where $(f \cdot s)(x) = f(s \cdot x)$.

Note that (1) says that the interpretation of $R_{1}$ is the constant function 1, (2) means that we have a probability measure, (3) says that $f$ is a subset of the interpretation of $R_{f}$, (4)–(7) that the family of the interpretations of relation symbols is a vector lattice, and (8) means that the measure is left $S$-invariant. $T_{S,X}$ is called the theory of left $S$-invariant measures on $X$.

As a consequence of the compactness theorem we give conditions equivalent to the existence of a left $S$-invariant Radon measure on $X$. Later, we give results based on these conditions. Let Inv$_{X}(S)$ be the set of all regular Borel probability measures on $X$ which are left $S$-invariant.

**Proposition 4.1** Assume $S, X$ and $T_{S,X}$ are as above. Then the following are equivalent:

(i) $\text{Inv}_{X}(S) \neq \emptyset$. 


(ii) $T_{S,X}$ is finitely satisfiable.

Proof. (i)$\Rightarrow$(ii) is obvious. For the converse, by the compactness theorem, there exists a model $M$ of $T_{S,X}$. By Urysohn’s lemma, one can easily verify that $X \subseteq M$. By Proposition 3.7(b), there exists a Radon measure $\bar{\mu}_X$ on $X$ such that $\int f \, d\bar{\mu}_X = \int R^M_X d\mu$ for each $f \in C(X)$. Therefore, $\bar{\mu}_X$ is a nonzero regular Borel left $S$-invariant measure on $X$. \hfill \Box

We give another condition equivalent to the existence of a left $S$-invariant Radon measure on $X$.

**Proposition 4.2** Let $S$ be a semigroup with identity. If $S$ acts from the left on a compact Hausdorff space $X$, then the following are equivalent:

(i) $\text{Inv}_X(S) \neq \emptyset$.

(ii) For every elements $s_1, \ldots, s_n$ of $S$ and elements $f_1, \ldots, f_n$ of $C(X)$ we have

$$\left\| 1 - \sum_{i=1}^n (f_i \cdot s_i - f_i) \right\| \geq 1.$$

Proof. (i)$\Rightarrow$(ii): Let $h = \sum_{i=1}^n (f_i \cdot s_i - f_i)$. If $\sup_{x \in X} |1 - h(x)| = 1 - \epsilon$ where $\epsilon$ is a positive real number, then $\epsilon < h(x) < 2$ for all $x \in X$, thereby $\int h \, d\mu > \epsilon$ for every probability measure $\mu$ on $X$, i.e., $\text{Inv}_X(S) = \emptyset$. For the converse, let $L_X$ be the language consisting of a unary relation symbol $R_f$ for each $f \in C(X)$ and a constant symbol $c_a$ for each $a \in X$ and $T_{S,X}$ be the theory of left $S$-invariant measures on $X$. By Proposition 4.1 it suffices to show that $T_{S,X}$ is finitely satisfiable. Assume $\Gamma$ is a finite subset of $T_{S,X}$ such that for each $i \leq n$ and $j \leq m$ the statement $\int R_{f_i} \, dx = \int R_{f_i \cdot s_j} \, dx$ is in $\Gamma$. Thus, $f_1, \ldots, f_n$ are in $C(X)$ and $s_1, \ldots, s_m$ are in $S$. Let $M$ be the closure of the subspace generated by $f_i - f_j \cdot s_j$ for each $i \leq n$ and $j \leq m$. Since $S$ has an identity, clearly $\inf_{h \in M} \|1 - h\| = 1$. Let $K$ be a subspace of $C(X)$ such that $M + K = C(X)$ and $M \cap K = 0$. By Proposition 2.3 define $I$ to be 0 on $M$ and a nonzero bounded linear functional on $K$ such that $I(1) = \|I\| = 1$. By Proposition 2.2(b), there exists a Baire signed measure $\mu$ on $X$ such that $\int (f_i - f_j \cdot s_j) \, d\mu = 0$ for each $i \leq n$ and $j \leq m$. Also, $\mu$ is a nonzero positive measure because $\mu(X) = \int 1 \, d\mu = I(1) = \|I\| = |\mu|(X)$. Hence $(X, \mu)$ with the natural interpretation of relation and constant symbols is a model of $\Gamma$. \hfill \Box

### 4.1 Amenability

In this subsection we define amenable topological semigroups and characterize them in terms of invariant measures. Also, we show that all compact groups and locally finite topological groups are amenable. Let $S$ be a topological semigroup, and $C_b(S)$ the Banach space of all bounded real-valued continuous functions on $S$ with the usual supremum norm. For $s \in S$ and $f \in C_b(S)$, let $f \cdot s$ and $s \cdot f$ be the elements in $C_b(S)$ defined by
\((f \cdot s)(t) = f(st)\) and \((s \cdot f)(t) = f(ts)\), \(t \in S\), respectively. A subspace \(E\) of \(C_b(S)\) is left (right) invariant if \(f \cdot s \in E\) \((s \cdot f \in E)\) for all \(s \in S, f \in E\). If \(E\) is both left and right invariant, then \(E\) is called invariant.

Let \(E\) be a left invariant closed subspace of \(C_b(S)\) that contains \(1\), the constant 1 function on \(S\). A mean on \(E\) is a linear functional \(I\) on \(E\) such that

1. \(I(1) = 1\),
2. \(I(f) \geq 0\) if \(f \geq 0\).

A mean \(I\) on a left (right) invariant closed subspace \(E\) of \(C_b(S)\) that contains 1 is said to be left (right) invariant if \(I(f \cdot s) = I(f)\) \((I(s \cdot f) = I(f))\) for all \(f \in E\) and \(s \in S\).

We define the subspace \(LUC(S)\) of all left uniformly continuous functions in \(C_b(S)\) which plays an important role in the rest of this paper. For a topological semigroup \(S\) set

\[LUC(S) = \{ f \in C_b(S) : \text{the map } s \mapsto f \cdot s \text{ is (norm) continuous from } S \text{ to } C_b(S) \}.\]

Similarly one can define the subspace \(RUC(S)\) of all right uniformly continuous functions in \(C_b(S)\). It is known that \(LUC(S)\) and \(RUC(S)\) are closed and invariant subalgebras of \(C_b(S)\). They are also closed under the lattice operations (cf. \cite{25} Lemmas 1.1 and 1.2). Therefore, \(LUC(S)\) and \(RUC(S)\) are \(M\)-spaces with the unit 1.

**Definition 4.3** A topological semigroup \(S\) is said to be left (right) amenable if \(LUC(S)\) \((RUC(S))\) admits a left (right) invariant mean. A topological semigroup \(S\) is called amenable if it is both left and right amenable.

We now characterize amenable topological semigroups in terms of invariant measures, for which we need the following lemma.

**Lemma 4.4** Let \(S\) be a topological semigroup.

(i) If \(X\) is a closed and invariant subset of \(\{ I \in LUC(S)^* : \| I \| = 1 \}\), then the natural action of \(S\) on \(X\) is continuous.

(ii) If \(X\) is a compact Hausdorff space and \(\cdot\) is a continuous action of \(S\) on \(X\) (by the left side), then, for each \(f \in C(X)\), the map \(s \mapsto f \cdot s\) from \(S\) to \(C(X)\) is (norm) continuous.

**Proof.** (i): Assume that \(s, s' \in S\) and \(I, I' \in X\). Then for each \(f \in LUC(S)\) we have

\[
|((s' \cdot I')(f) - (s \cdot I)(f))| = |I'(f \cdot s') - I(f \cdot s)|
\leq |I'(f \cdot s') - I'(f \cdot s)| + |I'(f \cdot s) - I(f \cdot s)|
= |I'(f \cdot s' - f \cdot s)| + |I'(f \cdot s) - I(f \cdot s)|
\leq \| I' \| \times \| f \cdot s' - f \cdot s \| + \| I'(f \cdot s) - I(f \cdot s) \|
= \| f \cdot s' - f \cdot s \| + \| I'(f \cdot s) - I(f \cdot s) \|.
\]
Therefore the continuity of \((s, I) \mapsto I \cdot s\) follows from the continuity \(s \mapsto f \cdot s\).

(ii): Let \(f \in C(X), s_0 \in S\) and \(\epsilon > 0\), and let \(U\) be the subset of \(S \times X\) given by
\[
U = \{(s, x) : |f(s_0 \cdot x) - f(s \cdot x)| < \epsilon\}.
\]
Then \(U\) is open and \(\{s_0\} \times X \subseteq U\). Hence there is a neighborhood \(V\) of \(s_0\) such that \(V \times X \subseteq U\), and it follows that \(\|f \cdot s_0 - f \cdot s\| < \epsilon\) whenever \(s \in V\).

We now come to the main theorem of this subsection (see [14, Corollary 449E] for the group case).

**Theorem 4.5** Let \(S\) be a topological semigroup with identity. Then the following are equivalent:

(i) \(S\) is left amenable.

(ii) Whenever \(X\) is a non-empty compact Hausdorff space and \(\cdot\) is a continuous action of \(S\) on \(X\) (by the left side), then \(\text{Inv}_X(S) \neq \emptyset\).

**Proof.** (i)⇒(ii): By Proposition [14.2] it suffices to show that \(\sup_{x \in X} |1 - h(x)| \geq 1\), which \(h\) is of the form \(\sum_{i=1}^n (f_i \cdot s_i - f_i)\) where \(s_1, \ldots, s_n\) are elements of \(S\) and \(f_1, \ldots, f_n\) are in \(C(X)\). If not, then \(\sup_{x \in X} h(x) < 0\). Let \(I\) be a left invariant mean on \(LUC(S)\). Fix a positive linear functional \(\Lambda\) on \(C(X)\). Define \(\tilde{f} : S \to \mathbb{R}\) by \(\tilde{f}(s) = \Lambda(f \cdot s)\) for each \(f \in C(X)\). We claim that \(\tilde{f} \in LUC(S)\). By Lemma [4.4(ii)], the map \(s \mapsto f \cdot s\) is norm continuous from \(S\) to \(C(X)\). It is easy to verify that the continuity of \(s \mapsto \tilde{f} \cdot s\) follows from the continuity of \(s \mapsto f \cdot s\). Define \(J : C(X) \to \mathbb{R}\) by \(J(f) = I(\tilde{f})\). Obviously \(J\) is a left invariant positive functional on \(C(X)\). Therefore, \(J(h) = 0\) since \(J\) is invariant. But \(J(h) < 0\) since \(J\) is positive and \(h < 0\).

(ii)⇒(i): It is easy to check that the set \(M_U(S)\) of all means on \(LUC(S)\) is a weak* compact subset of \(LUC(S)^*\). Note that by Lemma [4.4(i)], the natural action of \(S\) from the left on \(M_U(S)\) is continuous. Let \(\mu\) be a left \(S\)-invariant Radon probability measure on \(M_U(S)\). Define \(I_\mu : LUC(S) \to \mathbb{R}\) by \(I_\mu(g) = \int \tilde{g} \, d\mu\), where \(\tilde{g} : M_U(S) \to \mathbb{R}\) is defined by \(\tilde{g}(J) = J(g)\). Clearly, \(I_\mu\) is a left invariant mean on \(LUC(S)\).

**Remark 4.6** If \(S = G\) be locally compact group, then an invariant mean on \(LUC(G)\) extends to an invariant mean on the space \(C_b(G)\) of all bounded real-valued continuous functions on \(G\) (cf. [30, Theorem 1.1.9, p. 21]).

A topological semigroup can be left, but not right, amenable (e.g., consider the semigroup \(S = \{a, b\}\) with the following operation: \(a \cdot a = b \cdot a = a, a \cdot b = b \cdot b = b\). It is easy to check that this semigroup is left, but not right, amenable). Of course, if \(S\) be a topological group, then \(S\) is amenable if and only if it is left (or right) amenable. Basically it depends on the fact that the operation \(g \mapsto g^{-1}\) transposes the order of products, and therefore interchanges left and right. Also, we will show that any abelian topological semigroup is (both left and right) amenable (Corollary [4.12]).

Thanks to compactness of integral logic we have the following known result.
Proposition 4.7 Let $S$ be a topological semigroup with identity. Suppose that there is a family $\{S_\alpha\}_{\alpha \in I}$ of subsemigroups of $S$ such that

(i) $\bigcup_{\alpha \in I} S_\alpha$ is dense in $S$;
(ii) $S_\alpha$ is an amenable subsemigroup with identity for all $\alpha \in I$;
(iii) For any $\alpha_1, \alpha_2 \in I$, there exists $\alpha_3 \in I$ such that $S_{\alpha_1} \cup S_{\alpha_2} \subseteq S_{\alpha_3}$.

Then $S$ is also amenable.

Proof. Let $S' = \bigcup_{\alpha \in I} S_\alpha$ and $X$ be a compact Hausdorff space and $\cdot$ a left continuous action of $S'$ on $X$. By assumptions, the theory $T_{S',X}$ of left $S'$-invariant measures on $X$ is finitely satisfiable. By Proposition 4.1, as $X$ and $\cdot$ are arbitrary, $S'$ is amenable. Assume that $I$ is an $S'$-invariant mean on $LUC(S')$. Define $J : LUC(S) \to \mathbb{R}$ by $J(f) = I(f \restriction S')$ for each $f \in LUC(S)$. We can easily check that $J$ is an left invariant mean on $LUC(S)$ because $S'$ is dense. Similarly, one can show that $S$ is right amenable. \qed

Corollary 4.8 If every finitely generated subsemigroup (with identity) of a topological semigroup $S$ is amenable, then $S$ is also amenable.

Note that the converse may fail. As an example let $S'$ be any finitely generated non-amenable semigroup (e.g., the free group on two generators), and let $S$ contain $S'$ and one new element $s_0$ such that $s_0 s = s = s_0 s_0 = s_0$, and $S'$ is a subsemigroup of $S$. Then $S$ has an invariant mean $I(f) = f(s_0)$. The subsemigroup $S'$ has not.

It is known that every locally compact group possesses a Haar measure (cf. [1]), but not every locally compact group is amenable. The free group on two generators, with the discrete topology is a non-amenable locally compact group (cf. [14] Example 49G, p. 399). Of course, every compact group is amenable. Indeed, assume that $G$ acts continuously from the left on a compact Hausdorff space $X$. Fix $x_0 \in X$ and set $\phi(a) = a \cdot x_0$ for $a \in G$; then $\phi$ is continuous. Let $\mu$ be the Haar probability measure on $G$, and $\nu$ the Radon probability measure $\mu \phi^{-1}$ on $X$. Clearly $\nu$ is $G$-invariant. As $X$ and $\cdot$ are arbitrary, we have the following result.

Proposition 4.9 Every compact group is amenable.

A group $G$ is called locally finite if every finite subset of $G$ generates a finite subgroup of $G$. An immediate consequence of the above results is the following.

Proposition 4.10 Let $G$ be a topological group such that the union of the finite subsets of $G$ that generate a compact subgroup is dense. Then $G$ is amenable. In particular, every locally finite topological group is amenable.
4.2 Commutativity

The usual proof of the Bogolioubov-Krylov theorem uses the Markov-Kakutani fixed point theorem. Now, we give a proof of this theorem by using the compactness theorem and induction.

**Theorem 4.11 (Bogolioubov-Krylov)** Assume that $S$ be an abelian semigroup which acts from the left on a compact Hausdorff space $X$. Then $\text{Inv}_X(S) \neq \emptyset$.

**Proof.** By Proposition 4.1, it suffices to consider the case when $S$ is finite. We prove the theorem by induction on the number of elements of $S$. Let $D$ be a non-principal ultrafilter on $\mathbb{N}$ and $x_0$ any point of $X$. If $S = \{s\}$, then define $\mu_1$ by $\int f d\mu_1 = \lim_{n \to D} \frac{1}{n+1} \sum_{k=0}^{n} (f \cdot s^k)(x_0)$ for every $f \in C(X)$. It is easy to check that $\mu_1$ is invariant with respect to $s$. By induction hypothesis, there exists a measure $\nu$ on $X$ which is invariant with respect to $s_1, \ldots, s_{n-1}$. By Proposition 2.2(a), define the measure $\mu$ by $\int f d\mu = \lim_{n \to D} \frac{1}{n+1} \sum_{k=0}^{n} \int (f \cdot s^k_n) d\nu$ for every $f \in C(X)$. We can easily check that $\mu$ is invariant with respect to $s_1, \ldots, s_n$. Indeed, it is easy to verify that $\mu$ is $s_n$-invariant. Also, for each $i \leq n - 1$, we have

$$\int (f \cdot s_i) d\mu = \lim_{n \to D} \frac{1}{n+1} \sum_{k=0}^{n} \int (f \cdot s_i) \cdot s^k_n d\nu$$

$$= \lim_{n \to D} \frac{1}{n+1} \sum_{k=0}^{n} \int (f \cdot s^k_n) \cdot s_i d\nu \quad \text{commutativity}$$

$$= \lim_{n \to D} \frac{1}{n+1} \sum_{k=0}^{n} \int (f \cdot s^k_n) d\nu \quad \nu \text{ is } s_i \text{-invariant}$$

$$= \int f d\mu.$$  

Therefore, $\mu$ is the desired measure, so the theorem follows. \qed

An immediate consequence of the Bogolioubov-Krylov theorem is the following.

**Corollary 4.12** Any abelian topological semigroup is amenable.

The theorem 4.11 gives another proof of the existence of Haar measure on abelian compact groups. By the same method one can also give a functional analytic proof of the existence of Haar measures on abelian locally compact groups. The author will present elsewhere a proof of this theorem using the same method.

**Corollary 4.13 (Mazur-Orlicz)** Let $F$ be a family of commuting mappings of a set $X$ onto itself. Then there exists a mean on $B(X)$ which is $F$-invariant. In particular, every closed linear subspace $E$ of $B(X)$ such that $f \circ h \in E$ whenever $f \in E$ and $h \in F$ has an $F$-invariant mean.

**Proof.** Use Theorem 4.11 \qed
4.3 Paradoxical decompositions

The problematic of amenability has grown out of the famous Banach-Tarski paradox (which essentially amounts to the non-amenability of the free groups on two generators). We continue this paper by looking at the connection between satisfiability and paradoxical decompositions. Let $G$ be a discrete group acting on a nonempty set $X$. Then $E \subseteq X$ is called $G$-paradoxical if there are pairwise disjoint subsets $A_1, \ldots, A_m, B_1, \ldots, B_n$ of $E$ along with $g_1, \ldots, g_m, h_1, \ldots, h_n \in G$ such that $E = \bigcup_{i=1}^{m} g_i \cdot A_i = \bigcup_{i=1}^{n} h_i \cdot B_i$. $X$ is said to be $G$-paradoxical if it has a $G$-paradoxical subset. A group $G$ is called paradoxical if it is $G$-paradoxical. Clearly an amenable group is non-paradoxical. A remarkable fact is that the converse is also true, which follows from the following result of Tarski.

**Theorem 4.14** ([30], p. 7) Assume that $G$ and $X$ are as above. Then there exists a finitely additive, $G$-invariant measure on $X$ defined for all subsets of $X$ if and only if $X$ is not $G$-paradoxical.

A locally compact group $G$ admits a Borel paradoxical decomposition if it has a paradoxical decomposition such that the sets $A_1, \ldots, A_m, B_1, \ldots, B_n$ in the above definition are Borel sets. Paterson [26] proved that a locally compact group $G$ is not amenable if and only if $G$ admits a Borel paradoxical decomposition. The question of whether the non-existence of such suitable paradoxical decompositions characterizes the amenable, topological groups seems to be open (cf. [32]).

Now, we show that the amenability of a topological semigroup is expressible by a theory in integral logic. Note that for a semigroup $S$ the dual of the space $B(S)$ of all bounded real-valued functions on $S$ is the space of all signed charges on all subsets of $S$ (cf. [11], p. 496]). Therefore, a mean $I$ on $B(S)$ is represented by a (positive) charge $\nu_I$. If $\nu_I$ is a charge which is not countably additive, then $(S, \nu_I)$ is not a structure in integral logic. Nevertheless, thanks to the representation theorem for $M$-spaces, the amenability of a topological semigroup is expressible. Indeed, consider a topological semigroup $S$ and let $\sigma(S) = \sigma(LUC(S))$ be the set of Riesz homomorphisms $h : LUC(S) \to \mathbb{R}$ such that $h(1) = 1$ (cf. [13], p. 222]). The set $\sigma(S)$ is sometimes called the spectrum of $LUC(S)$. We shall see later that $\sigma(S)$ is the space of all complete types of a theory. Note that, by Proposition 353P(d) in [13], p. 243], $\sigma(S)$ is the set $M_U(S)$ of all multiplicative means on $LUC(S)$. First, we remark that $\sigma(S)$ is a weak* compact subset of $LUC(S)^*$ and $\|h\| = 1$ for every $h \in \sigma(S)$, and hence by Lemma 4.4(i), the natural action of $S$ on $\sigma(S)$ is continuous. The space $LUC(S)$ can be identified, as normed Riesz space, with $C(\sigma(S))$, because $LUC(S)$ is an $M$-space with standard order unit 1 and $\sigma(S)$ is a compact Hausdorff space (cf. [13] Corollary 354L, p. 249]). The identification is the map $f \mapsto \hat{f}$ where $\hat{f}(h) = h(f)$ for $f \in LUC(S)$ and $h \in \sigma(S)$. By Proposition 2.2(a), the identification of $LUC(S)$ with $C(\sigma(S))$ means that we have a one-to-one correspondence $\mu \leftrightarrow I_\mu$ between Radon probability measures $\mu$ on $\sigma(S)$ and positive linear functionals $I_\mu$ on $LUC(S)$ such that $I_\mu(1) = 1$, given by the formula $I_\mu(f) = \int \hat{f} d\mu$ for $f \in LUC(S)$. 

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Now
\[ I_\mu \text{ is invariant } \iff I_\mu(f \cdot s) = I_\mu(f) \quad \text{for every } f \in \text{LUC}(S) \text{ and } s \in S \]
\[ \iff \int \hat{f} \cdot sd\mu = \int \hat{f}d\mu \quad \text{for every } f \in \text{LUC}(S) \text{ and } s \in S \]
\[ \iff \int (\hat{f} \cdot s)d\mu = \int \hat{f}d\mu \quad \text{for every } f \in \text{LUC}(S) \text{ and } s \in S \]
\[ \iff \mu \text{ is invariant.} \]

So there is a one-to-one correspondence between Radon probability left \( S \)-invariant measures on \( \sigma(S) \) and left \( S \)-invariant means on \( \text{LUC}(S) \). Summarizing, we have the following.

**Proposition 4.15** Assume that \( S \) and \( T_S \) are as above. Then the following are equivalent:

(i) \( S \) is amenable.

(ii) \( T_S \) is (finitely) satisfiable.

If \( S \) is a locally compact group, then (i) and (ii) are equivalent to

(iii) \( S \) is not Borel paradoxical.

In fact we can say more: if \( S \) and \( T_S \) are as above, then the cardinal of the set of all left \( S \)-invariant means on \( \text{LUC}(S) \) is equal to the number of models of \( T_S \) up to almost isomorphism. Indeed, if \( \mu \neq \nu \) are (left) \( S \)-invariant measures on \( \sigma(S) \) then \( (\sigma(S), \mathcal{B}, \mu) \) and \( (\sigma(S), \mathcal{B}, \nu) \) with the natural interpretation of relation and constant symbols are different models of \( T_S \). Conversely, assume that \( M = (\sigma(S), \mathcal{B}, \mu_M) \) is a model of \( T_S \). By Proposition 3.17 the substructure \( M' = (\sigma(S), \mathcal{B}_{\sigma(S)}, \mu_M | \sigma(S)) \) is also a model of \( T_S \) and the inclusion map \( \sigma(S) \to M \) covers a full measure subset of \( M \). Therefore, \( M' \simeq_a M \). Clearly, the unique extension of \( \mu_M | \sigma(S) \) to a Radon measure on \( \sigma(S) \) is left \( S \)-invariant. To summarize:

**Proposition 4.16** Assume that \( S \) and \( T_S \) are as above. Then there is a bijection from the set of all models of \( T_S \) to the set of all left \( S \)-invariant means on \( \text{LUC}(S) \).

### 4.4 Extreme amenability

In this subsection we give some other results for extremely amenable topological semigroups. Most of the proofs are straightforward and we omit some unnecessary details. First, we characterize extremely amenable topological semigroups in terms of multiplicative invariant measures (Theorem 4.20). Finally, we prove that the extreme amenability of a topological semigroup is expressible by a theory in integral logic (Proposition 4.22).

A Radon probability measure \( \mu \) on a compact Hausdorff space \( X \) is multiplicative if \( \int fd\mu \times \int gd\mu = \int (f \times g)d\mu \) (the pointwise product) for all \( f, g \in C(X) \).

Let \( S \) be a topological semigroup which acts on a compact hausdorff space \( X \) from the left. Let \( T_{S,X} \) be the theory of left \( S \)-invariant measures on \( X \) with the additional axiom schema

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\[ \int R_{f \times g}(x)dx = \int R_f(x)dx \times \int R_g(x)dx \quad \text{for each } R_f, R_g, R_{f \times g} \in L_X, \]

where \((f \times g)(x) = f(x) \times g(x)\).

Note that (9) says that the measure is multiplicative. \(T_{S,X}\) is called the theory of multiplicative left \(S\)-invariant measures on \(X\).

Let \(\text{MInv}_X(S)\) be the set of all multiplicative, Radon probability measures on \(X\) which are left \(S\)-invariant. A consequence of the compactness theorem is the following.

**Proposition 4.17** Assume \(S, X\) and \(T_{S,X}\) are as above. Then the following are equivalent:

(i) \(\text{MInv}_X(S) \neq \emptyset\).

(ii) \(T_{S,X}\) is finitely satisfiable.

Let \(S\) be a topological semigroup. A mean \(I\) on \(LUC(S)\) is multiplicative if \(I(f) \times I(g) = I(f \times g)\) (the pointwise product) for all \(f, g \in LUC(S)\). We remark that \(LUC(S)\) is a closed and invariant subalgebra of \(C_b(S)\) (cf. \([25, \text{Lemmas 1.1 and 1.2}]\)).

**Definition 4.18** A topological semigroup \(S\) is said to be extremely left (right) amenable if \(LUC(S)\) (\(RUC(S)\)) admits a multiplicative left (right) invariant mean. A topological semigroup \(S\) is called extremely amenable if it is both left and right amenable.

**Remark 4.19** A topological semigroup \(S\) has the left (right) fixed point on compacta property if every continuous action of \(S\) on a compact Hausdorff space by the left (right) side has a fixed point. In \([23]\), Mitchell showed that a topological semigroup \(S\) has a multiplicative left invariant mean on \(LUC(S)\) iff \(S\) has the left fixed point on compacta property. Also, he asked the question: Is there a non trivial extremely amenable group at all? Historically the first example of extremely amenable groups was found in \([16]\). Many further examples of extremely amenable groups may be found in \([27, 28, 14]\).

The next theorem gives a proof of Mitchell’s theorem \([23, \text{Theorem 1}]\) and it also characterizes extremely amenable topological semigroups in terms of multiplicative invariant measures.

**Theorem 4.20** Let \(S\) be a topological semigroup with identity. Then the following are equivalent:

(i) \(S\) is extremely left amenable.

(ii) \(S\) has the left fixed point on compacta property.

(iii) Whenever \(X\) is a non-empty compact Hausdorff space and \(\cdot\) is a continuous action of \(S\) on \(X\) by the left side, then \(\text{MInv}_X(S) \neq \emptyset\).
Proof. (i)⇒(iii): The set \( M_U(S) (= \sigma(S)) \) of all multiplicative means on \( LUC(S) \) is a weak* compact subset of \( LUC(S)* \). By Lemma 4.4(i), the natural action of \( S \) on \( M_U(S) \) (by the left side) is continuous. Also, it is easy to verify that \( \text{MInv}_X(S) \neq \emptyset \) iff for every elements \( s_1, \ldots, s_n \) of \( S \) and elements \( f_1, \ldots, f_n, g_1, \ldots, g_n \) of \( C(X) \) we have \( \|1 - \sum_{i=1}^n g_i \cdot (f_i \cdot s_i - f_i)\| \geq 1 \). (Compare Proposition 4.2.) Now, the proof is a simple adaptation of the proof of Theorem 4.5.

(ii)⇒(iii): Assume that \( x_0 \in X \) is a fixed point, i.e., \( s \cdot x_0 = x_0 \) for every \( s \in S \). Define the measure \( \mu \) by \( \int f \, d\mu = f(x_0) \) for every \( f \in C(X) \). Clearly, \( \mu \) is a multiplicative Radon left \( S \)-invariant measure on \( X \).

(iii)⇒(ii): Assume that \( X \) is a non-empty compact Hausdorff space and \( \cdot \) is a continuous action of \( S \) on \( X \) by the left side. Let \( \mu \) be a multiplicative left \( S \)-invariant Radon probability measure on \( X \). Then the linear functional \( I \) defined by \( I(f) = \int f \, d\mu \) is multiplicative and invariant. Therefore, by Lemma 25 in [10, p. 278], there is a point \( x_0 \) in \( X \) such that \( I(f) = f(x_0) \) for every \( f \in C(X) \). Since \( C(X) \) separates points and \( I \) is invariant, \( x_0 \) is the desired fixed point. \( \square \)

Using the compactness theorem of integral logic, a result similar to Proposition 4.7 can be proved:

**Proposition 4.21** If \( S \) is a topological semigroup with a dense subset \( \bigcup_{\alpha \in I} S_\alpha \) where \( S_\alpha \) are extremely amenable semigroups and for any \( \alpha_1, \alpha_2 \in I \), \( S_{\alpha_1} \cup S_{\alpha_2} \subseteq S_{\alpha_3} \) for some \( \alpha_3 \in I \) then \( S \) is extremely amenable.

At the end of this section we show that the extreme amenability of a topological semigroup is expressible by a theory in integral logic. Let \( S \) be a topological semigroup and \( T_S = T_{S, \sigma(S)} \) be the theory of multiplicative left \( S \)-invariant Radon probability measures on \( \sigma(S) \). In fact, we show that the cardinal of \( \text{MInv}_X(S) \) is equal to the number of models of \( T_S \). By Propositions 4.15 and 4.16 it suffices to show that there is a one-to-one correspondence between multiplicative Radon probability measures on \( \sigma(S) \) and multiplicative means on \( LUC(S) \). Note that the identification of \( LUC(S) \) and \( C(\sigma(S)) \) is algebraic, i.e., \( \widehat{f \times g} = \widehat{f} \times \widehat{g} \) for every \( f, g \in LUC(S) \) (cf. [13] Pro 353P(d), p. 243]. Now

\[
I_\mu \text{ is multiplicative } \iff I_\mu(f \times g) = I_\mu(f) \times I_\mu(g) \text{ for every } f, g \in LUC(S)
\]

\[
\iff \int (\widehat{f \times g}) \, d\mu = \int \widehat{f} \, d\mu \times \int \widehat{g} \, d\mu \text{ for every } f, g \in LUC(S)
\]

\[
\iff \int (\widehat{f} \times \widehat{g}) \, d\mu = \int \widehat{f} \, d\mu \times \int \widehat{g} \, d\mu \text{ for every } f, g \in LUC(S)
\]

\[
\iff \mu \text{ is multiplicative.}
\]

To summarize:

**Proposition 4.22** Assume that \( S \) and \( T_S \) are as above. Then there is a bijection from the set of all models of \( T_S \) to the set of all multiplicative left \( S \)-invariant means on \( LUC(S) \). In particular, \( S \) is extremely left amenable iff \( T_S \) is (finitely) satisfiable.
5 Types and stability

In classical model theory, a complete type is a finitely additive 0-1 valued measure on the formulas. Actually, one can say more, i.e., a complete type is a 0-1 valued Riesz homomorphism on the formulas. Indeed, let $L$ be a first order language, $M$ an $L$-structure, $a \in M$, and $\text{tp}^M(a)$ be the complete type of $a$ in $M$. For each $L$-formula $\phi(x)$, define $f_\phi : M \to \{0, 1\}$ by $f_\phi(b) = 1$ if $M \models \phi(b)$, and $f_\phi(b) = 0$ otherwise. Let $V = \{f_\phi : \phi \in L\}$. One can easily check that $V$ is an (Archimedean) Riesz space. For this we define $f_\phi + f_\psi := f_{\phi \lor \psi}$, $-f_\phi := f_{-\phi}$, and for each $r \in \mathbb{R}$, $r \cdot f_\phi := f_{\phi}$ if $r > 0$, $r \cdot f_\phi := f_{-\phi}$ if $r < 0$, and $r \cdot f_\phi := 0$ if $r = 0$. Also, $f_\phi \leq f_\psi$ if $f_\phi(b) \leq f_\psi(b)$ for each $b \in M$. Clearly, $V$ with this structure is a Riesz space, i.e., it is a partially ordered linear space which is a lattice.

Now, for an $a \in M$, define the Riesz homomorphism $I_a : V \to \{0, 1\}$ by $I_a(f_\phi) = 1$ if $\phi(a) = 1$, and $I_a(f_\phi) = 0$ otherwise, i.e., $I_a(f_\phi) = 1$ iff $\phi \in \text{tp}^M(a)$. In other words, $I_a$ can be interpreted as playing the role of $\text{tp}^M(a)$.

More generally, we consider real valued Riesz homomorphisms. Indeed, consider an arbitrary partially ordered set $L = \{f_\phi : M \to \mathbb{R} : \phi \in L\}$ such that

$$\forall b \in M : f_\phi(b) \leq f_\psi(b) \iff M \models \phi(b) \rightarrow \psi(b)$$
$$f_\phi(b) < f_\psi(b) \iff M \models \neg \phi(b) \land \psi(b).$$

Let $V$ be the linear space generated by $L$. Again, $V$ is an Archimedean Riesz space. Define the Riesz homomorphism $I_a : V \to \mathbb{R}$ by $I_a(f) = f(a)$. It is easy to verify that $\phi \in \text{tp}^M(a)$ iff $I_a(f_{\phi \lor \neg \phi}) \leq I_a(f_\phi)$. Therefore it is natural to conjecture that real valued Riesz homomorphisms on measurable functions should play the role of complete types in the framework of integral logic. Our next goal is to convince the reader that this is indeed the case.

5.1 Types

Let us now return to integral logic. Suppose that $L$ is an arbitrary language, maybe with $n$-ary relation symbols and $n$-ary function symbols. Let $M$ be a graded $L$-structure (cf. [3]), $A \subseteq M$ and $T_A = \text{Th}(M, a)_{a \in A}$. Let $p(x)$ be a set of $L(A)$-statements in free variable $x$. We shall say that $p(x)$ is a type over $A$ if $p(x) \cup T_A$ is satisfiable. A complete type over $A$ is a maximal type over $A$. We let $S^M(A)$ be the set of all complete types over $A$. The type of $a$ in $M$ over $A$, denoted by $\text{tp}^M(a/A)$, is the set of all $L(A)$-statements satisfied in $M$ by $a$. We give a characterization of complete types. First we need some definitions.

Let $\mathcal{L}_A$ be the family of all interpretations $\phi^M$ in $M$ where $\phi$ is an $L(A)$-formula (not a statement) with a free variable $x$. $\mathcal{L}_A$ is an Archimedean Riesz space of measurable functions on $M$. Let $\sigma_A(M)$ be the set of Riesz homomorphisms $I : \mathcal{L}_A \to \mathbb{R}$ such that $I(1) = 1$. The set $\sigma_A(M)$ is called the spectrum of $T_A$. Note that $\sigma_A(M)$ is a weak* compact subset of $\mathcal{L}_A^*$. The next proposition shows that a complete type can be coded by a Riesz homomorphism and gives a characterization of complete types.

Proposition 5.1 Assume that $M$, $A$ and $T_A$ are as above.
(i) There is a bijection from $S^M(A)$ to $\sigma_A(M)$.

(ii) $p \in S^M(A)$ if and only if there is an elementary extension $N$ of $M$ and $a \in N$ such that $p = tp^N(a/A)$.

**Proof.** We can simultaneously prove (i) and (ii). Assume that $p(x)$ is a complete type over $M$. Define $I_p : \mathcal{L}_A \to \mathbb{R}$ by $I_p(\phi^M) = r$ if $\phi(x) = r$ is in $p(x)$. Clearly, $I_p$ is a Riesz homomorphism on $\mathcal{L}_A$. The map $p \mapsto I_p$ is injective, and we may reasonably prefer to assume that $p = I_p \in \sigma_A(M)$. In particular, for any $a \in M$, $tp^M(a/A) = \{ \phi(x) = \phi^M(a) : \phi \in \mathcal{L}_A \}$ and $I_{tp^M(a/A)}(\phi^M) = \phi^M(a)$. Also, the map $M \mapsto \sigma_A(M)$ defined by $a \mapsto I_{tp^M(a/A)}$ is an inclusion. We can assume that this inclusion is injective. (Indeed, one can assume that the language has a 2-ary relation symbol $e$ with the interpretation $e(a, b) = 1$ if $a = b$, and $e(a, b) = 1$ otherwise (cf. [3, p. 469]). On the other hand, if $\mathcal{L}$ separates $M$, i.e., for each $a \neq b \in M$ there is $\phi^M \in \mathcal{L}$ such that $\phi^M(a) \neq \phi^M(b)$, then the inclusion is again injective.)

Conversely, assume that $I : \mathcal{L}_A \to \mathbb{R}$ is a Riesz homomorphism. We show that $p(x) = \{ \phi(x) = I(\phi^M) : \phi^M \in \mathcal{L}_A \}$ is a complete type over $A$, and $I = I_p$. We know that the space $\mathcal{L}_A$ can be embedded as an order-dense and norm-dense Riesz subspace of $C(\sigma_A(M))$ (cf. [14, Theorem 353M]). Let $N = (\sigma_A(M), \nu)$ be the elementary extension of $M$ with the natural interpretations of symbols and measure. Therefore, the embedding is the map $\phi^M \mapsto \phi^N$ where $\phi^N(I) = I(\phi^M)$ for $\phi^M \in \mathcal{L}_A$ and $I \in \sigma_A(M)$. Clearly, $I$ determines a unique Riesz homomorphism $\hat{I}$ on $C(\sigma_A(M))$. (If $\xi^M = \lim_n \phi^M_n$ where $\phi^M_n \in \mathcal{L}_A$, then $\hat{I}(\xi^M) = \lim_n I(\phi^M_n)$.) Since $\sigma_A(M)$ is compact, there is a point $x_0$ in $\sigma_A(M)$ such that $\hat{I}(\xi^M) = \xi^N(x_0)$ for every $\xi \in C(\sigma_A(M))$. Thus $p(x) = tp^N(x_0/A)$ is a complete type, and $I = I_p$ since $I_p(\phi^M) = I(\phi^M)$ for each $\phi$. Therefore, the map $p \mapsto I_p$ is also surjective. □

We equip $S^M(A) = \sigma_A(M)$ with the related topology induced from $\mathcal{L}_A$. Therefore, $S^M(A)$ is a compact and Hausdorff space. For a complete type $p = tp^M(a/A)$ and a formula $\phi$, we let $\phi(p) = \phi^M(a)$. It is easy to verify that the topology on $S^M(A)$ is the weakest topology in which all the functions $p \mapsto \phi(p)$ are continuous. This topology sometimes called the logic topology.

**Remark 5.2** The elementary extension $N \geq M$ in the proposition 5.1 realizes every type over $M$. Also, it is easy to verify that $M$ is a dense subset of $N = \sigma_A(M)$. Since $\phi^N$'s are continuous, the natural measure on $N$ is Baire and it has a unique extension to a Radon measure $\mu$. From now on we assume $N = (S(M), \mu)$ where $\mu$ is this Radon measure.

**Corollary 5.3** Let $G$ be a topological group. Then $G$ is extremely amenable iff there is a complete type $p$ such that $g \cdot p = p$ for each $g \in G$.

### 5.2 Definable relations

**Definition 5.4** A relation $\xi : M \to [\neg b, b]$ is $\emptyset$-definable if there is a sequence $\phi_k(\bar{x})$ of formulas such that $b_{\phi_k} \leq b$ and $\phi_k \to \xi$ pointwise. A subset is definable if its characteristic function is definable.
This may be defined on the basis of other notions of convergence such as almost uniform convergence, convergence in measure, convergence in the mean etc. However, the corresponding definitions are equivalent. For example if \( \phi_k \) converges in measure to \( \xi \), then it has a subsequence which converges to \( f \) almost everywhere. So, if \( R \) is definable using the first notion of convergence, it is also definable using the second one. In particular, since the measure is finite and \( |\phi_k| \leq b \), \( \phi_k \to \xi \) in measure iff \( \phi_k \to \xi \) in mean iff \( \phi_k \to \xi \) pointwise (see [11]). On the other hand, if \( M \preceq N \) and \( \xi \) is definable in \( M \), then there is a corresponding definable relation \( \xi' \) in \( N \) and it is not hard to see that \( M \preceq_a N \). The set of definable relations is a Banach algebra with the norm defined by \( |\phi| = \sup_x |\phi(x)| \) and this algebra depends only on \( T \). It can be described as the completion of the algebra of formulas with the uniform norm. We denote this completion by \( L(T) \). A relation is \( M \)-definable if it is definable in \( Th(M,a)_{a \in M} \). So, \( L(M) \) is defined in the natural way.

### 5.3 Local stability

Here and in the next section we give two different notations of “stability” of a formula inside a model, a measure theoretic notation and a model theoretic notation. In fact, the measure theoretic notation (Definition 6.1) is a suitable form of the dependent property in classical model theory.

Let \( M \) be a structure and \( \phi(x,y) \) a formula. Assume that \( N \supseteq M \) and \( a \in N \). Let \( p = tp_N^M(a/M) \) be the complete \( \phi \)-type of \( a \) over \( M \), i.e., a function which associates to each instance \( \phi(x,b), b \in M \), the value \( \phi(a,b) \), which will then be denoted by \( \phi(p,b) \). Note that the complete \( \phi \)-type \( p \) uniquely determines a Riesz homomorphism \( I_p : L_\phi \to \mathbb{R} \) where \( L_\phi \) is the Riesz space generated by \( \{\phi(x,b) : b \in M\} \), and \( I_p(\phi(x,b)) = \phi(p,b) \) for each \( b \in M \). We equip \( S_\phi(M) \) with the weakest topology in which all functions \( p \mapsto \phi(p,b), b \in M \) are continuous. Equivalently, if \( \sigma_\phi(M) \) be the spectrum of \( T_\phi = \{\phi \geq r : \phi \geq r \text{ is in } T(M,a)_{a \in M}\} \) (i.e., the set of Riesz homomorphisms \( I : L_\phi \to \mathbb{R} \) such that \( I(1) = 1 \)), then \( S_\phi(M) = \sigma_\phi(M) \) is equipped with the topology induced by the weak* topology on \( L_\phi^* \). Clearly, \( S_\phi(M) \) is compact and Hausdorff. If \( \psi \) is a continuous function on \( S_\phi(M) \) such that \( \psi \) can be expressed as a pointwise limit of algebraic combinations of (at most countably many) functions of the form \( p \mapsto \phi(p,b), b \in M \), then \( \psi \) is called a \( \phi \)-definable relation over \( M \). A definable relation \( \psi(y) \) over \( M \) defines \( p \in S_\phi(M) \) if \( \phi(p,b) = \psi(b) \) for all \( b \in M \).

The next notation is more natural and less technically involved than measure theoretic notation, Definition 6.1 (See Definition 7.1 in [11].)

**Definition 5.5** A formula \( \phi(x,y) \) is called **stable in a structure** \( M \) if there are no \( r > s \) and infinite sequences \( a_n, b_n \in M \) such that for all \( i > j \): \( \phi(a_i,b_j) \geq r \) and \( \phi(a_j,b_i) \leq s \). A formula \( \phi \) is **stable in a theory** \( T \) if it is stable in every model of \( T \).

It is easy to verify that \( \phi(x,y) \) is stable in \( M \) if whenever \( a_n, b_n \in M \) form two sequences we have

\[
\lim_{n} \lim_{m} \phi(a_n, b_m) = \lim_{m} \lim_{n} \phi(a_n, b_m),
\]

provided both limits exist.
Notation 5.6 Similar to the definition of weak dependent property, one can give a weak notation of stability. Indeed, a formula $\phi(x, y)$ is called weakly stable in a structure $\mathcal{M}$ if whenever $E \subseteq \mathcal{M}$, with $\mu(E) > 0$ and $s < r$ in $\mathbb{R}$, there is some $k \geq 1$ such that $(\mu^{2k})S_k(E, s, r) < (\mu E)^{2k}$ where $$S_k(E, s, r) = \{ w \in E^{2^k} : \phi(w_i, w_i+k) \geq r, \phi(w_j, w_i+k) \leq s \text{ for } j < i \leq k \}$$ $$= \{(a_1, \ldots, a_k, b_1, \ldots, b_k) \in E^{2^k} : \phi(a_i, b_j) \geq r, \phi(a_j, b_i) \leq s \text{ for } j < i \}.$$ 

A formula $\phi$ is weakly stable in a theory $T$ if it is weakly stable in every model of $T$. Equivalently, $\phi(x, y)$ is weakly stable in $\mathcal{M}$ if $$\lim_n \lim_m \phi(a_n, b_m) = \lim_m \lim_n \phi(a_n, b_m) \quad \text{for almost every } ((a_n), (b_m)) \in M^\mathbb{N} \times M^\mathbb{N}$$ provided both limits exist. Clearly, a stable formula in a structure (or in a theory) is necessarily weakly stable in the structure (or in the theory).

Fact 5.7 (Grothendieck’s Criterion, [15]) Let $X$ be an arbitrary topological space, $X_0 \subseteq X$ a dense subset. Then the following are equivalent for a subset $A \subseteq C_b(X)$:

(i) The set $A$ is relatively weakly compact in $C_b(X)$.

(ii) The set $A$ is bounded, and whenever $f_n \in A$ and $x_n \in X_0$ form two sequences we have $$\lim_n \lim_m f_n(x_m) = \lim_m \lim_n f_n(x_m),$$ whenever both limits exists.

5.4 Fundamental theorem of stability

In [8] and [4] a continuous version of the definability of types in a stable theory, which is a generalization of the classical one, is proved. Roughly speaking, in continuous logic, for a stable formula $\phi$, the number of $\phi$-types is controlled by the number of continuous functions on the space of $\phi$-types. A similar result holds for a stable formula in integral logic. Also, another result shows that for a weakly dependent formula $\phi$, the number of $\phi$-types (up to an equivalence relation) is controlled by the number of measurable functions on the space of $\phi$-types.

On the other hand, in [6] and [5], Ben Yaacov studied probability algebras and $L^1$-random variables in the frameworks of compact abstract theories (cats) and of continuous logic. Also, Ben Yaacov proved that the theory atomless random variables and the category of probability algebras are $\aleph_0$-stable. Note that in this paper we do not study probability measure algebras or $L^1$-spaces, but we study measurable functions. We shall not identify measurable functions in $L^0$ with their class in $L^1$. On the other hand, in contrast to [6] and [5], the theory of a probability structure is not necessarily stable. This leads us to the dichotomy between stable probability structures and unstable probability structures.

Now, we come quickly to the following theorem. The proof is essentially similar to that in [4], but it works for measure structures.
Theorem 5.8 (Definability of types) Let $\phi(x, y)$ be a formula stable in a structure $M$. Then every $p \in S_\phi(M)$ is definable by a unique $\tilde{\phi}$-definable relation $\psi(y)$ over $M$, where $\tilde{\phi}(y, x) = \phi(x, y)$.

Proof. Let $X = S_\phi(M)$ and let $X_0 \subseteq X$ be the collection of those types realized in $M$, which is dense in $X$. Since $X$ is compact, the weak topology on $C(X)$ coincides with pointwise topology. Since every formula is bounded, the set $A = \{\phi^a : p \mapsto \phi(a, p) \mid a \in M\} \subseteq C(X)$ is bounded. (Note that the map $\phi(a, y) \mapsto \phi^a$ is bijective, and $(M, \phi(a, y))_{a \in M} \preceq (X, \phi^a)_{a \in M}$ (see the proof of Proposition 5.1). Therefore a relation $\psi : p \mapsto \psi(p)$ is a $\phi$-definable relation over $M$ iff $\psi$ is continuous and $\psi |_M$ is an $M$-definable relation in the sense of Definition 5.4. By Fact 5.7, since $\phi$ is stable in $M$, $A$ is relatively pointwise compact in $C(X)$. Let $p(x) \in S_\phi(M)$, and let $a_i \in M$ be any net such that $\lim_i \text{tp}_\phi(a_i/M) = p$. Since $A$ is relatively pointwise compact, there is a $\psi \in C(X)$ such that $\lim_i \phi^{a_i}(y) = \psi(y)$. By Lemma 8.19 in [19], $\psi$ is the closure point of a sequence $\phi^{a_i}(y)$ of the family $\{\phi^{a_i}(y)\}_i$. Therefore, there is a subsequence $\phi^{a_{n_k}}(y)$ such that $\lim_k \phi^{a_{n_k}}(y) = \psi(y)$. Clearly, $\psi(y)$ is a $\phi$-definable relation over $M$, and for $b \in M$ we have $\phi(p, b) = \lim_k \phi(a_{n_k}, b) = \psi(b)$. Therefore, $p$ is definable by a $\tilde{\phi}$-definable relation $\psi$ over $M$. If $p$ is definable by $\psi_1, \psi_2$, then $\psi_1(b) = \psi_2(b)$ for all $b \in M$. Since $X_0 \subseteq X$ is dense, $\psi_1 = \psi_2$.

We are now ready to prove the main theorem of this section.

Corollary 5.9 (Fundamental Theorem of Stability) Let $\phi(x, y)$ be a formula and $T$ a theory. Then the following are equivalent.

(i) The formula $\phi$ is stable in $T$.

(ii) For every model $M \models T$, every $\phi$-type over $M$ is definable by a $\tilde{\phi}$-predicate over $M$.

(iii) For each cardinal $\lambda = \kappa^{|N|} \geq |T|$, and model $M \models T$ with $|M| \leq \lambda$, $|S_\phi(M)| \leq \lambda$.

(iv) There exists a cardinal $\lambda = \kappa^{|N|} \geq |T|$ such that for every model $M \models T$, $|M| \leq \lambda$ then $|S_\phi(M)| \leq \lambda$.

Proof. We proved (i) implies (ii) in Theorem 5.8. The implications (ii) $\implies$ (iii) $\implies$ (iv) are clear. For (iv) $\implies$ (i), use many type argument and the downward Löwenheim-Skolem theorem (Proposition 5.13 in [3]).

5.5 Cantor-Bendixson rank

Let $M$ be an structure. By Remark 5.2, $N = (S(M), \mu)$ is an elementary extension of $M$. Moreover, $N$ is a topological measure space, $N$ is compact and $\mu$ is a Radon measure. Similarly, for a formula $\phi(x, y)$, the structure $N_\phi = (S_\phi(M), \mu_\phi)$ also is. In fact, $N_\phi$ has further structures:
Definition 5.10 ([8]) A (compact) topometric space is a triplet \( ⟨X, τ, d⟩ \), where \( τ \) is a (compact) Hausdorff topology and \( d \) a metric on \( X \), satisfying: (i) The metric refines the topology. (ii) For every closed \( F ⊆ X \) and \( ε > 0 \), the closed \( ε \)-neighbourhood of \( F \) is closed in \( X \) as well.

Fact 5.11 \( Nφ \) is a compact topometric space.

Proof. For \( p,q ∈ Sφ(M) \), define \( d(p,q) = sup\{∥φ(p,a) − φ(q,a)∥ : a ∈ M\} \). Clearly, \( d \) is a metric on \( Sφ(M) \), and the topology generated by \( d \) sometimes called the uniform topology. On the other hand, we know that \( p_α → p \) in the logic topology \( τ \) iff \( φ^p_α → φ^p \) in the pointwise topology, or equivalently, iff \( φ^p_α → φ^p \) in the weak topology. Now, it is easy to verify that \( (Sφ(M), τ, d) \) is a compact topometric space. □

Remark 5.12 Let \( U \) be an Archimedean Riesz space with order unit \( e \). Then it can be embedded as an order-dense and norm-dense Riesz subspace of \( C(X) \), where \( X \) is a compact Hausdorff space (cf. 354K Theorem in [13]). For \( a,b ∈ X \), define \( d(a,b) = sup\{∥f(a) − f(b)∥ : f ∈ C(X)\} \). Clearly, \( (X, d) \) is a compact topometric space. Therefore, all results in this paper can be extended to Archimedean Riesz space with order unit, and our approach is appropriate for continuous logic as well as operator logics (cf. [24]).

We have the following continuous version of the Cantor-Bendixson rank.

Definition 5.13 ([8]) Let \( X \) be a compact topometric space. For a fixed \( ε > 0 \), we define a decreasing sequence of closed subsets \( X_{ε,α} \) by induction:

\[
\begin{align*}
X_{ε,0} &= X \\
X_{ε,α} &= \bigcap_{β < α} X_{ε,β} \quad \text{for } α \text{ a limit ordinal} \\
X_{ε,α+1} &= \bigcap\{F ⊆ X_{ε,α} : F \text{ is closed and } diam(X_{ε,α} \setminus F) ≤ ε\} \\
X_{ε,∞} &= \bigcap_{α} X_{ε,α}.
\end{align*}
\]

Where the diameter of a subset \( U ⊆ X \) is defined

\[
diam(U) = sup\{d(x,y) : x,y ∈ U\}.
\]

For any non-empty subset \( U ⊆ X \) we define its \( ε \)-Cantor-Bendixson rank in \( X \) as:

\[
CB_{ε,X}(U) = sup\{α : U ∩ X_{ε,α} ≠ ∅\} ⊆ Ord \cup \{∞\}
\]

The next result characterizes stability in terms of CB ranks. We remark that a structure \( M \) is \( ω \)-saturated if every 1-type over a finite tuple in \( M \) is realised in \( M \).

Proposition 5.14 (cf. [8]) \( φ \) is stable iff for any \( ω \)-saturated model \( M ⊨ T \) where \( |M| = (|T| + κ)^{ℵ₀} \) we have \( CB_{Sφ(M),ε}(Sφ(M)) < ∞ \) for all \( ε \).
Proof. Let $\kappa > |T|$ be any cardinal such that $\kappa = \aleph_0$. Let $\lambda$ be the least cardinal such that $2^\lambda > \kappa$. Assume that $\lambda$ is the least cardinal such that $2^\lambda > \kappa$. Proceed by induction. If $M$ is $2^{<\lambda}$-saturated and $2^{<\lambda} = 2^{<\lambda}$, then we can find a model $M_0 \preceq M$ of cardinality $2^{<\lambda}$ and the types $\{p_\alpha\}_{\alpha < 2^\lambda} \subseteq S_\varphi(M_0)$ such that $d(p_\alpha, p_{\alpha'}) > \epsilon$ for all $\alpha \neq \alpha'$. Therefore, $\|S_\varphi(M_0)\| > |M_0|$, i.e., the density character of $S_\varphi(M_0)$ is bigger than the cardinality of $M_0$. The converse is also standard. □

5.6 Stability and amenability

Now we return to analytic concepts. A topological group is called precompact if it is isomorphic to a subgroup of a compact group. Assume that $G$ acts on a set $X$. A bounded function $f$ on $X$ is called weakly almost periodic if the $G$-orbit of $f$ is weakly relatively compact in the Banach space $l^\infty(X)$ of all bounded real-valued functions on $X$ equipped with the supremum norm. For a topological group $G$, denote by $WAP(G)$ the space of all continuous weakly almost periodic functions on $G$.

Proposition 5.15 Assume that $G$ is a topological group and its theory, $T_G$, is satisfiable. Then the following are equivalent:

(i) $T_G$ is stable.

(ii) $G$ is precompact.

Proof. We know that $T_G$ is stable (i.e., $LUC(G)$ is weakly compact) if and only if $LUC(G) = WAP(G)$. By Theorem 4.5 in [22], $LUC(G) = WAP(G)$ if and only if $G$ is precompact. □

Corollary 5.16 Assume that $G$ and $T_G$ are as above. If $T_G$ is stable, then $G$ is uniquely amenable.

Proof. It is known that for every precompact group $G$, the algebras $LUC(G)$ and $LUC(\hat{G})$ are canonically isomorphic, where $\hat{G}$ denotes the compact completion of $G$. Also, every compact group provides an obvious example of a uniquely amenable group, for which the unique invariant mean comes from the Haar measure. So $G$ is uniquely amenable since $\hat{G}$ is. □

6 NIP

Talagrand [31] gave the first explicit definition of stable set of functions. In fact, the notation of stable set of functions ([14] 465B) is a weak form of a well-known model-theoretic property, the dependent property. The definition is not obvious, but given this the basic properties of stable sets listed in [14] 465C] are natural and easy to check, and we come quickly to the fact that (for complete locally determined spaces) pointwise bounded
stable sets are relatively pointwise compact sets of measurable functions (Fact 6.3). We are now ready for the main definition which is an adapted version of Definition 465B in [14].

**Definition 6.1** A formula $\phi(x, y)$ has the weak dependence property, or is weakly dependent, in a structure $M$ if the set $A = \{\phi(x, b), \phi(a, y) : a, b \in M\}$ is a stable set of functions in the sense of Definition 465B in [14], that is, whenever $E \subseteq M$ is measurable, $\mu(E) > 0$ and $s < r$ in $\mathbb{R}$, there is some $k \geq 1$ such that $(\mu^{2k})^{*}D_{k}(A, E, s, r) < (\mu E)^{2k}$ where

$$D_{k}(A, E, s, r) = \bigcup_{f \in A} \{w \in E^{2k} : f(w_{2i}) \leq s, f(w_{2i+1}) \geq r \text{ for } i < k\}.$$  

A formula $\phi$ has the weak dependence property in a theory $T$ if it has weak dependence property in every model of $T$.

**Notation 6.2** Assume that for each $s < r$ and $k \in \mathbb{N}$ the set $D_{k}(A, E, s, r)$ is measurable in $M$. Then it is easy to verify that $\phi(x, y)$ fails to be weakly dependent, or is strongly independent, in $M$ if and only if there exist $E \subseteq M$, with $\mu(E) > 0$ and $s < r$ in $\mathbb{R}$, such that for each $k \geq 1$, and almost each $w \in E^{k}$, for each $I \subseteq \{1, \ldots, k\}$, there is $f \in A$ with $f(w_{i}) \leq s$ for $i \in I$ and $f(w_{i}) \geq r$ for $i \notin I$ (See Proposition 4 in [31]).

In the above definition if $\mu(E) = \epsilon > 0$ then we say $\phi$ has the strong $\epsilon$-independent property. It is an easy exercise to show that the strong $\epsilon$-independent property is a first order property (in integral logic), or equivalently it is expressible. Clearly, $\phi$ has the weak dependent property if it has not strong $\epsilon$-independent property for all $\epsilon > 0$.

Note that the sets $A_{1} = \{\phi(a, y) : a \in M\}$ and $A_{2} = \{\phi(x, b) : b \in M\}$ are dependent if and only if $A = A_{1} \cup A_{2}$ is dependent (cf. Proposition 465C(a),(d) in [14]). On the other hand, one can easily define the (exact) dependence property. For this, we say $\phi$ fails to be dependent, or is independent, in $M$ iff there exist $s < r$ in $\mathbb{R}$, such that for each $k \geq 1$, there are $w_{1}, \ldots, w_{k} \in M$, such that for each $I \subseteq \{1, \ldots, k\}$, there is $f \in A$ with $f(w_{i}) \leq s$ for $i \in I$ and $f(w_{i}) \geq r$ for $i \notin I$. Clearly, a dependent formula (or theory) is necessarily weakly dependent.

We come quickly to the following fact which is an adapted version of Proposition 465D in [14].

**Fact 6.3** Let $M = (M, \Sigma, \mu)$ be a structure such that $\mu$ is a complete locally determined measure space, and $\phi(x, y)$ a weakly dependent formula. Since every formula is bounded, so $\phi$ is. Therefore, $A = \{\phi(x, b), \phi(a, y) : a, b \in M\}$ is relatively compact in the space of measurable functions for the topology of pointwise convergence.

Now we compare our notations.

**Proposition 6.4** Let $\phi(x, y)$ be a stable formula in a theory $T$. Then $\phi$ is weakly dependent in $T$. 

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Proof. Assume that $\phi$ fails to be weakly dependent. Therefore, there is a model $M \models T$, $E \subseteq M$, with $\mu(E) > 0$, and $r > s$ in $\mathbb{R}$ such that $(\mu^{2k})^* D_k (A, E, s, r) = (\mu E)^{2k}$ for each $k$. Then it is easy to verify that for each $k$ there are finite sequences $a_n, b_n \in E$, $n \leq k$ such that for all $j < i \leq k$: $\phi(a_i, b_j) \geq r$ and $\phi(a_j, b_i) \leq s$. Now, by the compactness theorem of model theory, there is an elementary extension $N \succ M$ such that $\phi$ is not stable in $N$. Thus, $\phi$ is not stable in $T$. \hfill \Box

Moreover, one can easily show that a formula is weakly dependent if it is weakly stable.

To summarize:

\[ \begin{array}{c} \phi \text{ is stable} \quad \Downarrow \quad \phi \text{ is dependent} \\ \Downarrow \quad \Downarrow \end{array} \]

\[ \begin{array}{c} \phi \text{ is weakly stable} \quad \Downarrow \quad \phi \text{ is weakly dependent} \end{array} \]

6.1 Almost definability of types

A result similar to the fundamental theorem of stability can be proved for the dependent property. For this, we need some definitions. Let $\psi$ be a measurable function on $(S_\phi(M), \mu_\phi)$ where $\mu_\phi$ is the unique Radon measure induced by $\phi^M(x, b)$ for all $b \in M$. Then $\psi$ is called an almost $\phi$-definable relation over $M$ if there is a sequence $g_n : S_\phi(M) \to \mathbb{R}$, $|g_n| \leq |\phi|$, of continuous functions such that $\lim_n g_n(b) = \psi(b)$ for almost all $b \in S_\phi(M)$. (We note that by the Stone-Weierstrass theorem every continuous function $g_n : S_\phi(M) \to \mathbb{R}$ can be expressed as a uniform limit of algebraic combinations of (at most countably many) functions of the form $p \mapsto \phi(p, b)$, $b \in M$.) An almost definable relation $\psi(y)$ over $M$ defines $p \in S_\phi(M)$ if $\phi(p, b) = \psi(b)$ for almost all $b \in M$, and in this case we say that $p$ is almost definable. Assume that every type $p$ in $S_\phi(M)$ is almost definable by a measurable function $\psi^p$. Then, we say that $p$ is almost equal to $q$, denoted by $p \equiv q$, if $\psi^p(b) = \psi^q(b)$ for almost all $b \in M$. Define $[p] = \{ q \in S_\phi(M) : p \equiv q \}$ and $[S_\phi](M) = \{ [p] : p \in S_\phi(M) \}$.

**Theorem 6.5 (Almost definability of types)** Let $\phi(x, y)$ be a formula weakly dependent in a structure $M$. Then every $p \in S_\phi(M)$ is almost definable by a (unique up to measure) almost $\phi$-definable relation $\psi(y)$ over $M$, where $\phi(y, x) = \phi(x, y)$.

**Proof.** We know that $(M, \mu^M_\phi) \preceq (S_\phi(M), \mu_\phi)$. First, we assume that $(S_\phi(M), \mu_\phi)$ is minimal, i.e., $\mu_\phi$ is Baire and it is not necessarily Radon. (One can easily verify that the subspace measure $\mu_\phi \mid_M$ is the measure $\mu^M_\phi$. Therefore, by Proposition 465C(n) in [13], since the set $\{ \phi(a, y) : a \in M \} \subseteq \mathbb{R}^M$ is weakly dependent with respect to the subspace measure $\mu_\phi \mid_M$, the set $A = \{ \phi(a, y) : a \in M \} \subseteq \mathbb{R}^{S_\phi(M)}$ is also weakly dependent with respect to $\mu_\phi$. By Proposition 465C(i) in [13], the set $A$ is also weakly dependent with respect to the completion $\tilde{\mu}_\phi$ of $\mu_\phi$. Now, let $p(x) \in S_\phi(M)$, and let $a_i \in M$ be any net such that $\lim_i \text{tp}_\phi(a_i/M) = p$. Since $\tilde{\mu}_\phi$ is complete, by Fact 6.3 there is a $\tilde{\mu}_\phi$-measurable function $\psi$ such that $\lim_i \phi^{a_i}(y) = \psi(y)$. Let $\mu_\phi$ be the unique extension of $\mu_\phi$ to a Radon measure. Thereby it is also an extension of $\tilde{\mu}_\phi$. Since $\tilde{\mu}_\phi$ is Radon, by Proposition 7.9 in
Corollary 6.6 Let \( \phi(x, y) \) be a formula and \( T \) a theory. Then (i) \( \implies \) (ii) \( \implies \) (iii).

(i) The formula \( \phi \) is weakly dependent in \( T \).

(ii) For every model \( M \models T \), every \( \bar{\phi} \)-type over \( M \) is almost definable by a \( \bar{\phi} \)-predicate over \( M \).

(iii) For each cardinal \( \lambda = \kappa^{\aleph_0} \geq |T| \), and model \( M \models T \) with \( |M| \leq \lambda \), \( |[S_\phi](M)| \leq \lambda \).

Proof. Clear. \( \square \)

6.2 Almost Cantor-Bendixson rank

A result similar to the Cantor-Bendixson rank for stable formulas holds for the weak dependence property. For this we need some definitions. For a \( \mu_\phi \)-measurable function \( \xi : S_\phi(M) \to [-b_\phi, b_\phi] \) where \( b_\phi \) is the universal bound of \( \phi \), let

\[ \langle \xi \rangle = \{ \chi : S_\phi(M) \to [-b_\phi, b_\phi] \mid \chi \text{ is } \mu_\phi \text{-measurable and } \chi = \xi \text{ a.e.} \} \]

Let \( L^1_\phi = \{ \langle \xi \rangle \mid \xi : S_\phi(M) \to [-b_\phi, b_\phi] \text{ is } \mu_\phi \text{-measurable} \} \). We show that \( L^1_\phi \) has a natural compact topometric structure. Indeed, let \( \delta(\langle \xi \rangle, \langle \xi' \rangle) = \int \mid \xi - \xi' \mid d\mu_\phi \), and \( \langle \xi_\alpha \rangle \to \xi \) \( \implies \) \( \langle \xi_\alpha \rangle \to \xi \) for all \( I \in (L^1)^* \). In fact, the topology generated by the metric \( \delta \) is the norm topology on \( L^1 \) and \( \mathcal{F} \) is the weak topology generated by \( (L^1)^* \). Now, it is easy to verify that \( (L^1_\phi, \delta, \mathcal{F}) \) is a compact topometric space. Indeed, since \( L^1_\phi \) is uniformly integrable, by Theorem 247C in [12], \( L^1_\phi \) is relatively weakly compact. Also, \( L^1_\phi \) is closed in the norm topology. It is well-known that for a convex subset of a locally convex space, the weak closure is equal to the norm closure. Therefore, \( L^1_\phi \) is weakly closed, and hence it is weakly compact. On the other hand, it is well-known that the norm \( L^1 \) is weakly lower semicontinuous (cf. Lemma 6.22 in [1]). To summarize, \( L^1_\phi \) is a compact topometric space.

We remark that if the types \( p, q \) are definable by measurable functions \( \psi^p, \psi^q \), then \( p \equiv q \) iff \( \psi^p(b) = \psi^q(b) \) for almost all \( b \in M \), or equivalently, iff \( \langle \psi^p \rangle = \langle \psi^q \rangle \). (Note that since \( M \preceq (S_\phi(M), \mu_\phi) \) therefore \( \psi^p(b) = \psi^q(b) \) for almost all \( b \in M \) iff \( \psi^p = \psi^q \) \( \mu_\phi \)-almost everywhere.) Therefore, if \( |M| = \kappa^{\aleph_0} \) and \( \phi \) is a formula weakly dependent in the structure \( M \), then \( |L^1_\phi| = |[S_\phi](M)| = \kappa^{\aleph_0} \). (See Theorem 6.6 and the notations before it.) Thereby:

Proposition 6.7 If \( \phi \) is weakly dependent then for any \( \omega \)-saturated model \( M \models T \) where \( |M| = (|T| + \kappa)^{\aleph_0} \) we have \( CB_{L^1_\phi}(L^1_\phi) < \infty \) for all \( \epsilon \).
6.3 Vapnik-Chervonenkis dimension

Let \( \mathcal{F} \) be a class of measurable functions on a set \( X \). We say that the class \( \mathcal{F} \) shatters \( F \subseteq X \) at levels \( s, r \) \((s < r)\) if for each \( G \subseteq F \) there exist \( f \in \mathcal{F} \) such that \( f(x) \leq s \) if \( x \in G \) and \( f(x) \geq r \) if \( x \in F \setminus G \). The \( \epsilon \)-Vapnik-Chervonenkis dimension of \( \mathcal{F} \) on \( X \), denoted by \( VC_\epsilon(\mathcal{F}) \), is

\[
VC_\epsilon(\mathcal{F}) = \sup \{ n : \exists F, |F| = n, \mathcal{F} \text{ shatters } F \text{ at levels } s, s + \epsilon \}.
\]

Assume that \( \phi(x, y) \) is a formula and \( M \) is a structure. Let \( \mathcal{F}_{\phi,M} = \{ \phi(a, y), \phi(x, b) : a, b \in M \} \). By definition, \( \phi \) is dependent in \( M \) if and only if \( VC_\epsilon(\mathcal{F}_{\phi,M}) < \infty \) for each \( \epsilon \). Similarly, \( \phi \) is dependent in theory \( T \) iff for each \( M \models T \), \( VC_\epsilon(\mathcal{F}_{\phi,M}) < \infty \) for each \( \epsilon \).

A class \( \mathcal{F} \) of (uniformly bounded) functions on \( X \) is \( \epsilon \)-uniform Glivenko-Cantelli class if

\[
limit_{\mathcal{F}} \sup_{\mu} \left\{ \left( w_i \right) \in M^\mathbb{N} : \sup_{m \geq n, f \in \mathcal{F}} \left| \frac{1}{m} \sum_{i=0}^{m} f(w_i) - \int f \, d\mu \right| > \epsilon \right\} = 0,
\]

where the supremum is understood with respect to all distributions \( \mu \) over \( X \) (with respect to some suitable \( \sigma \)-algebra of subsets of \( X \) such that \( f \) is \( \mu \)-measurable for all \( f \in \mathcal{F} \)). This show that why we say uniform. We say that \( \mathcal{F} \) is a uniform Glivenko-Cantelli class, if \( \mathcal{F} \) is an \( \epsilon \)-uniform Glivenko-Cantelli class for all \( \epsilon > 0 \).

**Proposition 6.8** ([2]) Let \( \phi(x, y) \) be a formula and \( T \) a theory. Then the following are equivalent:

(i) \( \phi \) is dependent in \( T \).

(ii) For each \( M \models T \) and \( \epsilon > 0 \), \( VC_\epsilon(\mathcal{F}_{\phi,M}) < \infty \).

(iii) For each \( M \models T \), \( \mathcal{F}_{\phi,M} \) is a uniform Glivenko-Cantelli class.

The above proposition says that we can approximate any measure by averages of types. A similar result holds for weak dependent property, Glivenko-Cantelli classes, and a form of VC-dimension which it is more suitable in some sense (cf. [31] or [14]).

7 Conclusion

In the first part of this paper we studied some concrete analytic structures. This study led us to the natural and correct notation of types. The perspective of types in this paper can be used in other logics. For example, this approach seems to be appropriate for continuous logic [8] as well as operator logics [24]. Note that by Remark 5.12 every Archimedean Riesz space with order unit admits a natural compact topometric structure. Therefore, the most of results in this paper can be extended to Archimedean Riesz spaces. Also, the notation of forking and independence, and their connections to measure theory can be studied. One can do much more classifications, the strict order property and others. The
author will study elsewhere them. Finally, all these results suggest that many interesting analytic concepts may be studied by logical methods. Also, these methods provide a new view on the related subjects in Analysis, and open some fruitful areas of research on the similar questions.

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