Two-loop test of the $\mathcal{N} = 6$ Chern-Simons theory S-matrix

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Abstract

Starting from the integrable two-loop spin-chain Hamiltonian describing the anomalous dimensions of scalar operators in the planar $\mathcal{N} = 6$ superconformal Chern-Simons theory of ABJM, we perform a direct coordinate Bethe ansatz computation of the corresponding two-loop S-matrix. The result matches with the weak-coupling limit of the scalar sector of the all-loop S-matrix which we have recently proposed. In particular, we confirm that the scattering of $\mathcal{A}$ and $\mathcal{B}$ particles is reflectionless. As a warm up, we first review the analogous computation of the one-loop S-matrix from the one-loop dilatation operator for the scalar sector of planar $\mathcal{N} = 4$ superconformal Yang-Mills theory, and compare the result with the all-loop $SU(2|2)^2$ S-matrix.

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1 Introduction

Exact factorized $S$-matrices [1] play a key role in the understanding of integrable models. Planar four-dimensional $\mathcal{N} = 4$ superconformal Yang-Mills (YM) theory (and therefore, according to the $AdS_5/CFT_4$ correspondence [2], a certain type IIB superstring theory on $AdS_5 \times S^5$) is believed to be integrable (see [3]-[6] and references therein). A corresponding exact factorized $S$-matrix with $SU(2|2)^2$ symmetry has been proposed (see [7]-[14] and references therein), which leads [8, 15, 16] to the all-loop Bethe ansatz equations (BAEs) [17].

Aharony, Bergman, Jafferis and Maldacena (ABJM) [18] recently proposed an analogous $AdS_4/CFT_3$ correspondence relating planar three-dimensional $\mathcal{N} = 6$ superconformal Chern-Simons (CS) theory to type IIA superstring theory on $AdS_4 \times CP^3$. Minahan and Zarembo [19] subsequently found that the scalar sector of $\mathcal{N} = 6$ CS is integrable at the leading two-loop order, and proposed two-loop BAEs for the full theory (see also [20]). Moreover, evidence for classical integrability of the dual string sigma model (large-coupling limit) was discovered in [21, 22, 23]. On the basis of these results, and assuming integrability to all orders, Gromov and Vieira then conjectured all-loop BAEs [24].

Based on the symmetries and the spectrum of elementary excitations [19, 25, 26], we proposed an exact factorized $AdS_4/CFT_3$ $S$-matrix [28]. As a check, we verified that this $S$-matrix leads to the all-loop BAEs in [24]. An unusual feature of this $S$-matrix is that the scattering of $\mathcal{A}$ and $\mathcal{B}$ particles is reflectionless. (A similar $S$-matrix which is not reflectionless is not consistent with the known two-loop BAEs [29].) For further related developments of the $AdS_4/CFT_3$ correspondence, see [30] and references therein.

Considerable guesswork has entered into the above-mentioned all-loop results. While there is substantial evidence for the all-loop BAEs and $S$-matrix in the well-studied $AdS_5/CFT_4$ case, the same cannot be said for the rapidly-evolving $AdS_4/CFT_3$ case.

In an effort to further check our proposed $S$-matrix, we perform here a direct coordinate Bethe ansatz computation of the two-loop $S$-matrix, starting from the integrable two-loop spin-chain Hamiltonian describing the anomalous dimensions of scalar operators in planar $\mathcal{N} = 6$ CS [19]. The result matches with the weak-coupling limit of the scalar sector of our all-loop $S$-matrix [28]. In particular, we confirm that the scattering of $\mathcal{A}$ and $\mathcal{B}$ particles is reflectionless. As a warm up, we first review the analogous computation by Berenstein and Vázquez [5] of the one-loop $S$-matrix from the one-loop dilatation operator for the scalar sector of planar $\mathcal{N} = 4$ YM [23], and compare the result with the all-loop $SU(2|2)^2$ $S$-matrix.

The outline of this paper is as follows. In Sec. 2 we review the simpler case of $\mathcal{N} = 4$ YM. In Sec. 3 we analyze the $\mathcal{N} = 6$ CS case, relegating most of the details of $\mathcal{A} − \mathcal{B}$ scattering
to an appendix. We briefly discuss our results in Sec. 4.

2 One-loop $S$-matrix in the scalar sector of $\mathcal{N} = 4$ YM

As is well known, $\mathcal{N} = 4$ YM has six scalar fields $\Phi_i(x)$ ($i = 1, \ldots, 6$) in the adjoint representation of $SU(N)$. It is convenient to associate single-trace gauge-invariant scalar operators with states of an $SO(6)$ quantum spin chain with $L$ sites,

$$\text{tr} \Phi_{i_1}(x) \cdots \Phi_{i_L}(x) \leftrightarrow |\Phi_{i_1} \cdots \Phi_{i_L}\rangle,$$

where $\Phi_i$ on the RHS are 6-dimensional elementary vectors with components $(\Phi_i)_j = \delta_{i,j}$.

The one-loop anomalous dimensions of these operators are described by the integrable $SO(6)$ quantum spin-chain Hamiltonian

$$\Gamma = \frac{\lambda}{8\pi^2} H, \quad H = \sum_{l=1}^{L} \left( 1 - P_{l,l+1} + \frac{1}{2} K_{l,l+1} \right),$$

where $\lambda = g_{YM}^2 N$ is the 't Hooft coupling, $P$ is the permutation operator,

$$P \Phi_i \otimes \Phi_j = \Phi_j \otimes \Phi_i,$$

and the projector $K$ acts as

$$K \Phi_i \otimes \Phi_j = \delta_{ij} \left( \sum_{k=1}^{6} \Phi_k \otimes \Phi_k \right).$$

It is convenient to define the complex combinations

$$X = \Phi_1 + i\Phi_2, \quad Y = \Phi_3 + i\Phi_4, \quad Z = \Phi_5 + i\Phi_6,$$

and to denote the corresponding complex conjugates with a bar, $\bar{X} = \Phi_1 - i\Phi_2$, etc. For $\phi_1, \phi_2 \in \{X, \bar{X}, Y, \bar{Y}, Z, \bar{Z}\}$,

$$P \phi_1 \otimes \phi_2 = \phi_2 \otimes \phi_1,$$

and

$$K \phi_1 \otimes \phi_2 = \begin{cases} 0 & \text{if } \phi_1 \neq \bar{\phi}_2 \\ X \otimes \bar{X} + \bar{X} \otimes X + Y \otimes \bar{Y} + \bar{Y} \otimes Y + Z \otimes \bar{Z} + \bar{Z} \otimes Z & \text{if } \phi_1 = \bar{\phi}_2 \end{cases}.$$
2.1 Coordinate Bethe ansatz

We take $|Z^L\rangle$ as the vacuum state, which evidently is an eigenstate of $H$ with zero energy. One-particle excited states ("magnons") with momentum $p$ are given by

$$|\psi(p)\rangle_\phi = \sum_{x=1}^L e^{ipx} |x\rangle_\phi,$$

where

$$|x\rangle_\phi = |\frac{1}{1} \frac{x}{Z} \cdots \frac{x}{1} \phi \cdots \frac{L}{Z}\rangle$$

is the state obtained from the vacuum by replacing a single $Z$ at site $x$ with an "impurity" $\phi$, which can be either $X, \bar{X}, Y, \bar{Y}$ (but not $\bar{Z}$, which can be regarded as a two-particle bound state). Indeed, one can easily check that (2.8) is an eigenstate of $H$ with eigenvalue $E = \epsilon(p)$, where

$$\epsilon(p) = 4 \sin^2(p/2).$$

In order to compute the two-particle $S$-matrix, we must construct all possible two-particle eigenstates. Let

$$|x_1, x_2\rangle_{\phi_1\phi_2} = |\frac{1}{1} \frac{x_1}{Z} \cdots \frac{x_1}{\phi_1} \cdots \frac{x_2}{\phi_2} \cdots \frac{L}{Z}\rangle$$

denote the state obtained from the vacuum by replacing the $Z$'s at sites $x_1$ and $x_2$ with impurities $\phi_1$ and $\phi_2$, respectively, where $x_1 < x_2$. Following Berenstein and Vázquez [5], we distinguish the following three cases:

$\phi_1 = \phi_2$:

The case of two particles of the same type (i.e., $\phi_1 = \phi_2 \equiv \phi \in \{X, \bar{X}, Y, \bar{Y}\}$) is equivalent to the well-known case originally considered by Bethe in his seminal investigation of the Heisenberg model. (See, e.g., the review by Plefka in [6].) The two-particle eigenstates are given by

$$|\psi\rangle = \sum_{x_1 < x_2} f(x_1, x_2) |x_1, x_2\rangle_{\phi\phi}$$

where

$$f(x_1, x_2) = e^{i(p_1x_1 + p_2x_2)} + S(p_2, p_1) e^{i(p_2x_1 + p_1x_2)}.$$

$$\epsilon(p) = 4 \sin^2(p/2).$$

In order to compute the two-particle $S$-matrix, we must construct all possible two-particle eigenstates. Let

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denote the state obtained from the vacuum by replacing the $Z$'s at sites $x_1$ and $x_2$ with impurities $\phi_1$ and $\phi_2$, respectively, where $x_1 < x_2$. Following Berenstein and Vázquez [5], we distinguish the following three cases:

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where

$$f(x_1, x_2) = e^{i(p_1x_1 + p_2x_2)} + S(p_2, p_1) e^{i(p_2x_1 + p_1x_2)}.$$
Indeed, these states satisfy
\[ H|\psi\rangle = E|\psi\rangle \] (2.14)
with
\[ E = \epsilon(p_1) + \epsilon(p_2), \] (2.15)
where \( \epsilon(p) \) is given by (2.10). It also follows from (2.14) that the S-matrix for \( \phi - \phi \) scattering is given by
\[ S(p_2, p_1) = \frac{u_2 - u_1 + i}{u_2 - u_1 - i}, \] (2.16)
where \( u_j = u(p_j) \) and
\[ u(p) = \frac{1}{2} \text{cot}(p/2). \] (2.17)

\( \phi_1 \neq \bar{\phi_2} \):

If the two particles are not of the same type, but \( \phi_1 \neq \bar{\phi_2} \), then the two-particle eigenstates are of the form
\[ |\psi\rangle = \sum_{x_1 < x_2} \left\{ f_{\phi_1 \phi_2}(x_1, x_2) |x_1, x_2\rangle_{\phi_1 \phi_2} + f_{\phi_2 \phi_1}(x_1, x_2) |x_1, x_2\rangle_{\phi_2 \phi_1} \right\}, \] (2.18)
where
\[ f_{\phi_1 \phi_2}(x_1, x_2) = A_{\phi_1 \phi_2}(12) e^{i(p_1 x_1 + p_2 x_2)} + A_{\phi_2 \phi_1}(21) e^{i(p_2 x_1 + p_1 x_2)}. \] (2.19)

One finds [5]
\[ \left( \begin{array}{c} A_{\phi_1 \phi_2}(21) \\ A_{\phi_2 \phi_1}(21) \end{array} \right) = \left( \begin{array}{cc} R(p_2, p_1) & T(p_2, p_1) \\ T(p_2, p_1) & R(p_2, p_1) \end{array} \right) \left( \begin{array}{c} A_{\phi_1 \phi_2}(12) \\ A_{\phi_2 \phi_1}(12) \end{array} \right), \] (2.20)
where the transmission and reflection amplitudes are given by
\[ T(p_2, p_1) = \frac{u_2 - u_1}{u_2 - u_1 - i}, \quad R(p_2, p_1) = \frac{i}{u_2 - u_1 - i}, \] (2.21)
respectively.

\( \phi_1 = \bar{\phi_2} \):

In the case \( \phi_1 = \bar{\phi_2} \in \{X, \bar{X}, Y, \bar{Y}\} \), the two-particle eigenstates are given by
\[ |\psi\rangle = \sum_{x_1 < x_2} \sum_{\phi = X, \bar{X}, Y, \bar{Y}} \left\{ f_{\phi \phi}(x_1, x_2) |x_1, x_2\rangle_{\phi \phi} + f_{\bar{\phi} \phi}(x_1, x_2) |x_1, x_2\rangle_{\bar{\phi} \phi} \right\} + \sum_{x_1} f_{\bar{Z}}(x_1) |x_1\rangle_{\bar{Z}}, \] (2.22)
where \( f_{\phi, \phi_i}(x_1, x_2) \) are again given by (2.19), and

\[
f_\tilde{Z}(x_1) = A_\tilde{Z} e^{i(p_1 + p_2)x_1}.
\] (2.23)

One finds [5]

\[
\begin{pmatrix}
A_{X\tilde{X}}(21) \\
A_{\tilde{X}X}(21) \\
A_{YY}(21) \\
A_{\tilde{Y}Y}(21)
\end{pmatrix} =
\begin{pmatrix}
R(p_2, p_1) & T(p_2, p_1) & S(p_2, p_1) & S(p_2, p_1) \\
T(p_2, p_1) & R(p_2, p_1) & S(p_2, p_1) & S(p_2, p_1) \\
S(p_2, p_1) & S(p_2, p_1) & R(p_2, p_1) & T(p_2, p_1) \\
S(p_2, p_1) & S(p_2, p_1) & T(p_2, p_1) & R(p_2, p_1)
\end{pmatrix}
\begin{pmatrix}
A_{X\tilde{X}}(12) \\
A_{\tilde{X}X}(12) \\
A_{YY}(12) \\
A_{\tilde{Y}Y}(12)
\end{pmatrix},
\] (2.24)

where

\[
T(p_2, p_1) = \frac{(u_2 - u_1)^2}{(u_2 - u_1 - i)(u_2 - u_1 + i)},
\]

\[
R(p_2, p_1) = \frac{-1}{(u_2 - u_1 - i)(u_2 - u_1 + i)},
\]

\[
S(p_2, p_1) = \frac{-i(u_2 - u_1)}{(u_2 - u_1 - i)(u_2 - u_1 + i)}.
\] (2.25)

### 2.2 Comparison with the all-loop S-matrix

We now wish to compare the above scattering amplitudes with the weak-coupling limit of the all-loop \( SU(2|2) \otimes SU(2|2) \) S-matrix [8]-[14]. This check has not (to our knowledge) been presented elsewhere, and will serve as a useful guide for the \( \mathcal{N} = 6 \) CS case. It is convenient to express the latter in terms of two mutually commuting sets of Zamolodchikov-Faddeev operators \( A_i^\dagger(p), \tilde{A}_i^\dagger(p) \) \((i = 1, \ldots, 4)\),

\[
A_i^\dagger(p_1) A_j^\dagger(p_2) = \sum_{i', j'} S_0(p_1, p_2) \tilde{S}_{i,j}^{i', j'}(p_1, p_2) A_{i'}^\dagger(p_2) A_{j'}^\dagger(p_1),
\]

\[
\tilde{A}_i^\dagger(p_1) \tilde{A}_j^\dagger(p_2) = \sum_{i', j'} S_0(p_1, p_2) \tilde{S}_{i,j}^{i', j'}(p_1, p_2) \tilde{A}_{i'}^\dagger(p_2) \tilde{A}_{j'}^\dagger(p_1),
\]

\[
A_i^\dagger(p_1) \tilde{A}_j^\dagger(p_2) = \tilde{A}_j^\dagger(p_2) A_i^\dagger(p_1).
\] (2.26)

We identify the scalar one-particle states as follows,

\[
X(p) = A_1^\dagger(p) \tilde{A}_2^\dagger(p), \quad \tilde{X}(p) = A_2^\dagger(p) \tilde{A}_1^\dagger(p),
\]

\[
Y(p) = A_2^\dagger(p) \tilde{A}_1^\dagger(p), \quad \tilde{Y}(p) = A_1^\dagger(p) \tilde{A}_2^\dagger(p).
\] (2.27)

The only non-vanishing amplitudes in the scalar sector are

\[
\tilde{S}_{aa}^a(p_1, p_2) = A, \quad \tilde{S}_{ab}^a(p_1, p_2) = \frac{1}{2}(A - B), \quad \tilde{S}_{ab}^b(p_1, p_2) = \frac{1}{2}(A + B),
\] (2.28)
where \( a, b \in \{1, 2\} \) with \( a \neq b \). Here
\[
A = \frac{x_2^- - x_1^+}{x_2^+ - x_1^-},
\]
\[
B = -\left[ \frac{x_2^- - x_1^+}{x_2^+ - x_1^-} + 2 \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^- x_2^+ - x_1^+ x_2^-)} \right],
\]
(2.29)
where \( x_i^\pm = x(p_i)^\pm \) with
\[
\frac{x^+}{x^-} = e^{iu}, \quad x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g},
\]
(2.30)
and \( g = \sqrt{x}/(4\pi) \). Moreover, the scalar factor is given by
\[
S_0(p_1, p_2)^2 = \frac{x_1^- - x_2^+}{x_1^+ - x_2^-} \left( 1 - \frac{1}{x_1^+ x_2^-} \right) \sigma(p_1, p_2)^2,
\]
where \( \sigma(p_1, p_2) \) is the BES dressing factor [12, 14]. In the weak-coupling \((g \to 0)\) limit,
\[
x^\pm \to \frac{1}{g} \left( u \pm \frac{i}{2} \right),
\]
(2.32)
Therefore
\[
A \to \frac{u_1 - u_2 + i}{u_1 - u_2 - i}, \quad B \to -1,
\]
(2.33)
and
\[
S_0^2 \to \frac{u_1 - u_2 - i}{u_1 - u_2 + i},
\]
(2.34)
since \( \sigma(p_1, p_2) \to 1 \).

For two particles of the same type, the scattering amplitude is evidently given by
\[
S(p_1, p_2) \equiv \left( S_0(p_1, p_2) \, \hat{\sigma}_{a\bar{a}}(p_1, p_2) \right)^2 = S_0^2 \, A^2 \to \frac{u_1 - u_2 + i}{u_1 - u_2 - i},
\]
(2.35)
in agreement with (2.16).

We now consider the case \( \phi_1 \neq \bar{\phi}_2 \), e.g.,
\[
X(p_1) \, Y(p_2) = T(p_1, p_2) \, Y(p_2) \, X(p_1) + R(p_1, p_2) \, X(p_2) \, Y(p_1).
\]
(2.36)
It follows from (2.26)-(2.28) and (2.33), (2.34) that
\[
T(p_1, p_2) = \frac{1}{2} S_0^2 A(A - B) \to \frac{u_1 - u_2}{u_1 - u_2 - i},
\]
\[
R(p_1, p_2) = \frac{1}{2} S_0^2 A(A + B) \to \frac{i}{u_1 - u_2 - i},
\]
(2.37)
in agreement with (2.21).

Finally, we consider the case \( \phi_1 = \bar{\phi}_2 \), e.g.,

\[
X(p_1) \bar{X}(p_2) = T(p_1, p_2) \bar{X}(p_2) X(p_1) + R(p_1, p_2) X(p_2) \bar{X}(p_1) + S(p_1, p_2) Y(p_2) \bar{Y}(p_1) + S(p_1, p_2) \bar{Y}(p_2) Y(p_1) \, .
\]

(2.38)

It follows from (2.26)-(2.28) and (2.33), (2.34) that

\[
T(p_1, p_2) = \frac{1}{4} S_0^2 (A - B)^2 \to \frac{(u_1 - u_2)^2}{(u_1 - u_2 - i)(u_1 - u_2 + i)},
\]

\[
R(p_1, p_2) = \frac{1}{4} S_0^2 (A + B)^2 \to \frac{-1}{(u_1 - u_2 - i)(u_1 - u_2 + i)},
\]

\[
S(p_1, p_2) = \frac{1}{4} S_0^2 (A - B)(A + B) \to \frac{i(u_1 - u_2)}{(u_1 - u_2 - i)(u_1 - u_2 + i)},
\]

(2.39)

in agreement with (2.25).\(^1\)

In short, the all-loop \( AdS_5/CFT_4 \) \( S \)-matrix correctly reproduces the \( \mathcal{N} = 4 \) YM one-loop scalar-sector scattering amplitudes, as expected. In the next section, we perform a similar check of the \( AdS_4/CFT_3 \) \( S \)-matrix.

### 3 Two-loop \( S \)-matrix in the scalar sector of \( \mathcal{N} = 6 \) CS

The \( \mathcal{N} = 6 \) CS theory \(^{18}\) has a pair of scalar fields \( A_i(x) \) \( (i = 1, 2) \) in the bifundamental representation \( (\mathbf{N}, \bar{\mathbf{N}}) \) of the \( SU(N) \times SU(N) \) gauge group, and another pair of scalar fields \( B_i(x) \) \( (i = 1, 2) \) in the conjugate representation \( (\bar{\mathbf{N}}, \mathbf{N}) \). These fields can be grouped into \( SU(4) \) multiplets \( Y^A(x) \),

\[
Y^A = (A_1, A_2, B_1^\dagger, B_2^\dagger), \quad Y_A^\dagger = (A_1^\dagger, A_2^\dagger, B_1, B_2) \, .
\]

(3.1)

Following \(^{19}\), we associate single-trace gauge-invariant scalar operators with states of an alternating \( SU(4) \) quantum spin chain with \( 2L \) sites,

\[
\text{tr} \, Y^{A_1}(x) Y_{B_1}(x) \cdots Y^{A_L}(x) Y_{B_L}(x) \Leftrightarrow |Y^{A_1} Y_{B_1}^\dagger \cdots Y^{A_L} Y_{B_L}^\dagger\rangle, \quad (3.2)
\]

where \( Y^A \) on the RHS are 4-dimensional elementary vectors with components \( (Y^A)_j = \delta_{A,j} \). The two-loop anomalous dimensions of these operators are described by the integrable

\(^1\)There is a sign discrepancy in \( S(p_1, p_2) \) which perhaps can be reconciled by a gauge transformation in (2.38), e.g., \( Y \to -Y \) while leaving others unchanged.
alternating $SU(4)$ quantum spin-chain Hamiltonian \[19\]

\[
\Gamma = \lambda^2 H, \quad H = \sum_{i=1}^{2L} \left( 1 - P_{l,l+2} + \frac{1}{2} \{ K_{l,l+1}, P_{l,l+2} \} \right),
\]

where $\lambda = N/k$ is the 't Hooft coupling\[2\] $P$ is the permutation operator, and the projector $K$ acts as

\[
K Y^A \otimes Y_B^\dagger = \delta_B^A \sum_{C=1}^4 Y_C^\dagger \otimes Y_C^C, \quad K Y_B^\dagger \otimes Y^A = \delta_B^A \sum_{C=1}^4 Y_C^\dagger \otimes Y^C.
\]

That is,

\[
KA_i \otimes A_j^\dagger = KB_i^\dagger \otimes B_j = \delta_{ij} \sum_{k=1}^2 \left( A_k \otimes A_k^\dagger + B_k^\dagger \otimes B_k \right),
\]

\[
KA_i^\dagger \otimes A_j = KB_i \otimes B_j^\dagger = \delta_{ij} \sum_{k=1}^2 \left( A_k^\dagger \otimes A_k + B_k \otimes B_k^\dagger \right),
\]

\[
KA_i \otimes B_j = KB_i \otimes A_j = KA_i^\dagger \otimes B_j^\dagger = KB_i^\dagger \otimes A_j^\dagger = 0.
\]

### 3.1 Coordinate Bethe ansatz

Following \[25, 26\], we take the state with $L$ pairs of $(A_1 B_1)$, i.e.,

\[
\left| (A_1 B_1)^L \right>
\]

as the vacuum state, which evidently is an eigenstate of $H$ with zero energy. It is convenient to label the $(A_1 B_1)$ pairs by $x \in \{1, \ldots, L\}$. There are two types of one-particle excited states with momentum $p$, called “$A$-particles” and “$B$-particles.” The former are given by

\[
|\psi(p)\rangle^A_\phi = \sum_{x=1}^L e^{ipx} |x\rangle^A_\phi,
\]

where

\[
|x\rangle^A_\phi = | (A_1 B_1) \cdots \frac{1}{\sqrt{2}} (\phi B_1) \cdots (A_1 B_1) \rangle
\]

is the state obtained from the vacuum by replacing the $A_1$ from pair $x$ with an “impurity” $\phi$, which can be either $A_2$ or $B_2^\dagger$ (but not $B_1^\dagger$, which can be regarded as a two-particle bound state). Similarly, the “$B$-particles” are given by

\[
|\psi(p)\rangle^B_\phi = \sum_{x=1}^L e^{ipx} |x\rangle^B_\phi,
\]

\[2\] The action has two $SU(N)$ Chern-Simons terms with integer levels $k$ and $-k$, respectively.
where
\[ |x\rangle_\phi^B = | (A_1 B_1) \cdots (A_1 \phi) \cdots (A_1 B_1) \rangle \] (3.10)
is the state obtained from the vacuum by replacing the \( B_1 \) from pair \( x \) with an “impurity” \( \phi \), which can be either \( A_2^\dagger \) or \( B_2 \) (but not \( A_1^\dagger \), which can be regarded as a two-particle bound state). Indeed, both (3.7) and (3.9) are eigenstates of \( H \) with eigenvalue \( E = \epsilon(p) \), where \( \epsilon(p) \) is given by (2.10).

In order to compute the two-particle \( S \)-matrix, we must construct all possible two-particle eigenstates.

### 3.1.1 \( A-A \) scattering

Let
\[ |x_1, x_2\rangle_{\phi_1, \phi_2}^{AA} = | (A_1 B_1) \cdots (A_1 \phi_1) \cdots (A_1 \phi_2) \cdots (A_1 B_1) \rangle \] (3.11)
denote the state obtained from the vacuum by replacing the \( A_1 \)'s from pairs \( x_1 \) and \( x_2 \) with impurities \( \phi_1 \) and \( \phi_2 \), respectively, where \( x_1 < x_2 \) and \( \phi_i \in \{ A_2, B_2^\dagger \} \). We distinguish two cases:

**\( \phi_1 = \phi_2 \):**

The case of two \( A \)-particles of the same type (i.e., \( \phi_1 = \phi_2 \equiv \phi \in \{ A_2, B_2^\dagger \} \)) is again the same as in the Heisenberg model. The two-particle eigenstates are given by
\[ |\psi\rangle = \sum_{x_1 < x_2} f(x_1, x_2) |x_1, x_2\rangle_{\phi_1, \phi_2}^{AA} \] (3.12)
where \( f(x_1, x_2) \) is given by (2.13). These states have energy (2.15), and the \( S \)-matrix is again given by (2.16),
\[ S(p_2, p_1) = \frac{u_2 - u_1 + i}{u_2 - u_1 - i}. \] (3.13)

**\( \phi_1 \neq \phi_2 \):**

If the two \( A \)-particles are not of the same type (e.g., \( \phi_1 = A_2, \phi_2 = B_2^\dagger \)), then the two-particle eigenstates are of the form
\[ |\psi\rangle = \sum_{x_1 < x_2} \left\{ f_{\phi_1, \phi_2}(x_1, x_2) |x_1, x_2\rangle_{\phi_1, \phi_2}^{AA} + f_{\phi_2, \phi_1}(x_1, x_2) |x_1, x_2\rangle_{\phi_2, \phi_1}^{AA} \right\}, \] (3.14)
where \( f_{\phi, \phi'}(x_1, x_2) \) is again given by (2.19). Since \( K \) on these states is zero, the \( S \)-matrix is again given by (2.21),
\[ T(p_2, p_1) = \frac{u_2 - u_1}{u_2 - u_1 - i}, \quad R(p_2, p_1) = \frac{i}{u_2 - u_1 - i}. \] (3.15)
3.1.2 \( \mathcal{B} - \mathcal{B} \) scattering

Let

\[
|x_1, x_2 \rangle^{\mathcal{B}\mathcal{B}}_{\phi_1, \phi_2} = | (A_1 B_1 \cdots (A_1 \phi_1 \cdots (A_1 \phi_2 \cdots (A_1 B_1) \rangle \quad (3.16)
\]

denote the state obtained from the vacuum by replacing the \( B_1 \)'s from pairs \( x_1 \) and \( x_2 \) with impurities \( \phi_1 \) and \( \phi_2 \), respectively, where \( x_1 < x_2 \) and \( \phi_i \in \{A^\dagger_2, B_2\} \). The eigenstates with two \( \mathcal{B} \)-particle are given by expressions similar to those with two \( \mathcal{A} \)-particles (namely, (3.12) and (3.14) with \( |x_1, x_2 \rangle^{\mathcal{A}\mathcal{A}}_{\phi_i, \phi_j} \leftrightarrow |x_1, x_2 \rangle^{\mathcal{B}\mathcal{B}}_{\phi_i, \phi_j} \), and we obtain the same results (3.13), (3.15) for the scattering amplitudes.

3.1.3 \( \mathcal{A} - \mathcal{B} \) scattering

In order to analyze \( \mathcal{A} - \mathcal{B} \) scattering, we define the states

\[
|x_1, x_2 \rangle^{\mathcal{A}\mathcal{B}}_{\phi_1, \phi_2} = | (A_1 B_1 \cdots (A_1 \phi_1 \cdots (A_1 \phi_2 \cdots (A_1 B_1) \rangle , \quad (3.17)
\]

where \( x_1 < x_2 \) and \( \phi_1 \in \{A_2, B_2^\dagger\}, \phi_2 \in \{A_2^\dagger, B_2\} \). We distinguish two cases:

\( \phi_1 \neq \phi_2^\dagger \):

If \( \phi_1 \neq \phi_2^\dagger \) (e.g., \( \phi_1 = A_2, \phi_2 = B_2 \)), then \( K \) on the states (3.17) is zero. As noted in [19], we are left with two decoupled \( SU(2) \) chains on the even and odd sites. Hence, there is trivial scattering between \( \mathcal{A} \) and \( \mathcal{B} \) particles.

\( \phi_1 = \phi_2^\dagger \):

If \( \phi_1 = \phi_2^\dagger \) (e.g., \( \phi_1 = A_2, \phi_2 = A_2^\dagger \)), then the eigenstates are given by

\[
|\psi\rangle = \sum_{x_1 < x_2} \sum_{\phi = A_2, B_2^\dagger} \left\{ f_{\phi \phi^\dagger}(x_1, x_2) |x_1, x_2 \rangle^{\mathcal{A}\mathcal{B}}_{\phi_1, \phi_2} + f_{\phi^\dagger \phi}(x_1, x_2) |x_1, x_2 \rangle^{\mathcal{A}\mathcal{B}}_{\phi_2, \phi_1} \right\}
\]

\[
+ \sum_{x_1} \sum_{k=1}^2 \left\{ f_{A_k A_k^\dagger}(x_1) |A_k A_k^\dagger \rangle + f_{B_k^\dagger B_k}(x_1) |B_k^\dagger B_k \rangle \right\} , \quad (3.18)
\]

where

\[
|x \rangle_{\phi_i, \phi_j} = | (A_1 B_1 \cdots (\phi_i \phi_j) \cdots (A_1 B_1) \rangle \quad (3.19)
\]
is the state obtained from the vacuum by replacing the \((A_1 B_1)\) pair at \(x\) with \((\phi_i \phi_j)\).
We assume that \(f_{\phi_i \phi_j}(x_1, x_2)\) are again given by (2.19), and
\[
f_{\phi_i \phi_j}(x_1) = A_{\phi_i \phi_j} e^{i(p_1 + p_2)x_1}.
\] (3.20)

After a lengthy computation (see the Appendix for further details), we find
\[
\begin{pmatrix}
A_{A_2 A_2}^{(21)}(12) \\
A_{A_2 A_2}^{(21)} \\
A_{B_2 B_2}^{(21)} \\
A_{B_2 B_2}^{(21)}(12)
\end{pmatrix} =
\begin{pmatrix}
0 & T(p_2, p_1) & 0 & S(p_2, p_1) \\
T(p_2, p_1) & 0 & S(p_2, p_1) & 0 \\
0 & S(p_2, p_1) & 0 & T(p_2, p_1) \\
S(p_2, p_1) & 0 & T(p_2, p_1) & 0
\end{pmatrix}
\begin{pmatrix}
A_{A_2 A_2}^{(12)}(12) \\
A_{A_2 A_2}^{(12)} \\
A_{B_2 B_2}^{(12)} \\
A_{B_2 B_2}^{(12)}(12)
\end{pmatrix}
\] (3.21)
where
\[
T(p_2, p_1) = \frac{u_1 - u_2}{u_1 - u_2 - i}, \quad S(p_2, p_1) = \frac{i}{u_1 - u_2 - i}.
\] (3.22)
Note that the scattering is reflectionless.

Similar results can be obtained for \(B - A\) scattering.

### 3.2 Comparison with the all-loop S-matrix

We now wish to compare the above scattering amplitudes with the weak-coupling limit of the all-loop \(SU(2|2)\) S-matrix \[28\]. It is convenient to express the latter in terms of two sets of Zamolodchikov-Faddeev operators \(A_i^\dagger(p), B_i^\dagger(p) (i = 1, \ldots, 4)\) corresponding to the \(A, B\) particles, respectively,
\[
A_i^\dagger(p_1) A_j^\dagger(p_2) = \sum_{i',j'} S_0(p_1, p_2) \tilde{S}_{ij}^{i'j'}(p_1, p_2) A_{i'}^\dagger(p_2) A_{j'}^\dagger(p_1),
\] (3.23)
\[
B_i^\dagger(p_1) B_j^\dagger(p_2) = \sum_{i',j'} S_0(p_1, p_2) \tilde{S}_{ij}^{i'j'}(p_1, p_2) B_{i'}^\dagger(p_2) B_{j'}^\dagger(p_1),
\] (3.24)
\[
A_i^\dagger(p_1) B_j^\dagger(p_2) = \sum_{i',j'} \tilde{S}_0(p_1, p_2) \tilde{S}_{ij}^{i'j'}(p_1, p_2) B_{i'}^\dagger(p_2) A_{j'}^\dagger(p_1),
\] (3.25)
\[
B_i^\dagger(p_1) A_j^\dagger(p_2) = \sum_{i',j'} \tilde{S}_0(p_1, p_2) \tilde{S}_{ij}^{i'j'}(p_1, p_2) A_{i'}^\dagger(p_2) B_{j'}^\dagger(p_1).
\] (3.26)

The absence of \(A_i^\dagger(p_2) B_i^\dagger(p_1)\) terms on the RHS of (3.25) (and similarly, of \(B_i^\dagger(p_2) A_i^\dagger(p_1)\) terms on the RHS of (3.26)) means that the scattering is reflectionless.

We identify the scalar one-particle states as follows,
\[
A_1^\dagger(p)|0\rangle = \sum_x e^{ipx} |x\rangle_{A_2}^A, \quad A_2^\dagger(p)|0\rangle = \sum_x e^{ipx} |x\rangle_{B_2}^A,
\] (3.27)
\[
B_1^\dagger(p)|0\rangle = \sum_x e^{ipx} |x\rangle_{B_2}^B, \quad B_2^\dagger(p)|0\rangle = \sum_x e^{ipx} |x\rangle_{A_2}^B.
\]
The $SU(2|2)$ $S$-matrix elements $\tilde{S}_{i}^{ij}(p_1, p_2)$ are the same as before (2.28), (2.29), where $x^{\pm}$ satisfy (2.30) and [25 [26 27]

$$g = h(\lambda),$$

(3.28)

with $h(\lambda) \sim \lambda$ for small $\lambda$, and $h(\lambda) \sim \sqrt{\lambda/2}$ for large $\lambda$. The scalar factors are given by (cf. (2.31))

$$S_0(p_1, p_2) = \frac{1 - \frac{1}{x_1 x_2}}{1 - \frac{1}{x_1 x_2}} \sigma(p_1, p_2), \quad \tilde{S}_0(p_1, p_2) = \frac{x_1 - x_2}{x_1 + x_2} \sigma(p_1, p_2).$$

(3.29)

In the weak-coupling ($g \to 0$) limit,

$$S_0 \to 1, \quad \tilde{S}_0 \to \frac{u_1 - u_2 - i}{u_1 - u_2 + i}.$$  

(3.30)

### 3.2.1 $A-A$ scattering

For two $A$ particles of the same type (i.e., both $A_a$ with $a \in \{1, 2\}$), the scattering amplitude is evidently given by

$$S(p_1, p_2) \equiv S_0(p_1, p_2) \tilde{S}_0^{aa}(p_1, p_2) = S_0 A \to \frac{u_1 - u_2 + i}{u_1 - u_2 - i},$$

(3.31)

in agreement with (3.13). Although the same expression also appears in the $\mathcal{N} = 4$ YM case (2.35), note that the latter follows from the all-loop $S$-matrix (2.26) in a rather different way.

For two $A$ particles of different type (i.e., $A_a$ and $A_b$ with $a, b \in \{1, 2\}$ and $a \neq b$), it follows from (3.23), (2.28), (2.33) that

$$A_a(p_1) A_b(p_2) = T(p_1, p_2) A_b(p_2) A_a(p_1) + R(p_1, p_2) A_a(p_2) A_b(p_1),$$

(3.32)

where

$$T(p_1, p_2) = \frac{1}{2} S_0(A - B) \to \frac{u_1 - u_2}{u_1 - u_2 - i},$$

$$R(p_1, p_2) = \frac{1}{2} S_0(A + B) \to \frac{i}{u_1 - u_2 - i},$$

(3.33)

in agreement with (3.15). Again, the same expressions arise in the $\mathcal{N} = 4$ YM case (2.37) in a different way.

### 3.2.2 $B-B$ scattering

According to (3.24), the $B-B$ and $A-A$ scattering amplitudes are equal, in agreement with the results from Sec. 3.1.2.
3.2.3 $A - B$ scattering

According to (3.25), the $A - B$ scattering amplitude is

$$\tilde{S}_0(p_1, p_2) \tilde{S}_{aa}^{a}(p_1, p_2) = \tilde{S}_0 A \rightarrow 1,$$

(3.34)
in agreement with the results from Sec.3.1.3 for the case $\phi_1 \neq \phi_2$. Note that the scalar factor $\tilde{S}_0$ (3.29) is essential for obtaining this result.

For $A_a - B_b$ scattering (with $a, b \in \{1, 2\}$ and $a \neq b$), it follows from (3.25) that

$$A^\dagger_a(p_1) B^\dagger_b(p_2) = T(p_1, p_2) B^\dagger_b(p_2) A^\dagger_a(p_1) + S(p_1, p_2) B^\dagger_b(p_2) A^\dagger_a(p_1),$$

(3.35)
where

$$T(p_1, p_2) = \frac{1}{2} \tilde{S}_0(A - B) \rightarrow \frac{u_1 - u_2}{u_1 - u_2 + i},$$

$$S(p_1, p_2) = \frac{1}{2} \tilde{S}_0(A + B) \rightarrow \frac{i}{u_1 - u_2 + i},$$

(3.36)
which agrees with (3.22).³

4 Discussion

We have found that the all-loop $AdS_4/CFT_3$ $S$-matrix (3.23) - (3.26) correctly reproduces the $\mathcal{N} = 6$ CS two-loop scalar-sector scattering amplitudes. The scalar factors (3.29), which differ from the $AdS_5/CFT_4$ scalar factor (2.31), play a crucial role. In particular, we have confirmed that the scattering of $A$ and $B$ particles is reflectionless. This gives greater confidence in the correctness of the all-loop $S$-matrix, and in the corresponding all-loop BAEs [24].

We have restricted our analysis to the scalar sector of $\mathcal{N} = 6$ CS, since this is the only sector for which an explicit Hamiltonian has been available [19]. Very recently, the Hamiltonian for the full two-loop $OSp(6|4)$ spin chain has been found [31, 32]. Hence, it should now be possible to extend the present analysis to other sectors, and thereby further check the all-loop $S$-matrix.

It would also be interesting to extend the present analysis beyond two loops. This could provide further information about the important function $h(\lambda)$ (3.28) and the dressing phase in the $S$-matrix. However, such an analysis must wait until the higher-loop Hamiltonian becomes available.

³There is a sign discrepancy in $S(p_1, p_2)$. However, the sign of $S(p_1, p_2)$ in (3.35) can be changed by a gauge transformation, e.g. by changing $A_1 \rightarrow -A_1$ and leaving $A_2, B_1, B_2$ unchanged.
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A Details of $\mathcal{A} - \mathcal{B}$ scattering

In order to determine the $\mathcal{A} - \mathcal{B}$ scattering amplitudes, it is necessary to act with the Hamiltonian $H$ on the state $|3.18\rangle$. We catalog here the action of $H$ on the various terms:

$$H|1, x_2\rangle_{A_1 A_2}^{A B} = 4|x_1, x_2\rangle_{A_1 A_2}^{A B} - |x_1 - 1, x_2\rangle_{A_1 A_2}^{A B} - |x_1 + 1, x_2\rangle_{A_1 A_2}^{A B}$$

for $x_1 < x_2 - 1$, \hspace{1cm} (A.1)

$$H|x_1, x_1 + 1\rangle_{A_1 A_2}^{A B} = 4|x_1, x_1 + 1\rangle_{A_1 A_2}^{A B} - |x_1\rangle_{A_1 A_2}^{A B} A_1 - |x_1 + 1\rangle_{A_1 A_2}^{A B} A_2$$

$$- |x_1 - 1, x_1 + 1\rangle_{A_1 A_2}^{A B} A_1 - |x_1, x_1 + 2\rangle_{A_1 A_2}^{A B} A_2,$$ \hspace{1cm} (A.2)

$$H|x_1, x_1 + 1\rangle_{B_1 B_2}^{A B} = 4|x_1, x_1 + 1\rangle_{B_1 B_2}^{A B} - |x_1\rangle_{B_1 B_2}^{A B} B_1 - |x_1 + 1\rangle_{B_1 B_2}^{A B} B_2$$

$$- |x_1 - 1, x_1 + 1\rangle_{B_1 B_2}^{A B} B_1 - |x_1, x_1 + 2\rangle_{B_1 B_2}^{A B} B_2,$$ \hspace{1cm} (A.3)

$$H|x_1, x_1 + 1\rangle_{A_1 A_2}^{A B} = 4|x_1, x_1 + 1\rangle_{A_1 A_2}^{A B} - \frac{1}{2}|x_1\rangle_{A_1 A_2}^{A B} A_2 - \frac{1}{2}|x_1 + 1\rangle_{A_1 A_2}^{A B} A_2$$

$$+ \frac{1}{2}|x_1\rangle_{A_1 A_2}^{A B} A_1 + \frac{1}{2}|x_1 + 1\rangle_{A_1 A_2}^{A B} A_1$$

$$+ \frac{1}{2}|x_1\rangle_{B_1 B_2}^{A B} B_2 + \frac{1}{2}|x_1 + 1\rangle_{B_1 B_2}^{A B} B_2$$

$$- |x_1 - 1, x_1 + 1\rangle_{A_1 A_2}^{A B} A_2 - |x_1, x_1 + 2\rangle_{A_1 A_2}^{A B} A_2,$$ \hspace{1cm} (A.4)

$$H|x_1, x_1 + 1\rangle_{B_1 B_2}^{A B} = 4|x_1, x_1 + 1\rangle_{B_1 B_2}^{A B} - \frac{1}{2}|x_1\rangle_{B_1 B_2}^{A B} B_2 - \frac{1}{2}|x_1 + 1\rangle_{B_1 B_2}^{A B} B_2$$

$$+ \frac{1}{2}|x_1\rangle_{B_1 B_2}^{A B} B_1 + \frac{1}{2}|x_1 + 1\rangle_{B_1 B_2}^{A B} B_1$$

$$+ \frac{1}{2}|x_1\rangle_{A_1 A_2}^{A B} - \frac{1}{2}|x_1 + 1\rangle_{A_1 A_2}^{A B}$$

$$- |x_1 - 1, x_1 + 1\rangle_{B_1 B_2}^{A B} - |x_1, x_1 + 2\rangle_{B_1 B_2}^{A B}.$$ \hspace{1cm} (A.5)
\[ H |x_1\rangle_{A_1A_1^\dagger} = 3 |x_1\rangle_{A_1A_1^\dagger} - \frac{1}{2} |x_1 - 1\rangle_{A_1A_1^\dagger} - \frac{1}{2} |x_1 + 1\rangle_{A_1A_1^\dagger} + \frac{1}{2} |x_1 + 1\rangle_{A_2A_2^\dagger} + |x_1\rangle_{B_1^\dagger B_1} + |x_1 + 1\rangle_{B_1^\dagger B_1}, \]

\[ + \frac{1}{2} |x_1\rangle_{B_2^\dagger B_2} + \frac{1}{2} |x_1 + 1\rangle_{B_2^\dagger B_2} + \frac{1}{2} |x_1 - 1, x_1\rangle_{A_2^\dagger A_2} + \frac{1}{2} |x_1, x_1 + 1\rangle_{A_2^\dagger A_2}, \] (A.6)

\[ H |x_1\rangle_{A_2A_2^\dagger} = 4 |x_1\rangle_{A_2A_2^\dagger} + \frac{1}{2} |x_1 - 1\rangle_{A_1A_1^\dagger} + \frac{1}{2} |x_1\rangle_{A_1A_1^\dagger} + \frac{1}{2} |x_1 - 1, x_1\rangle_{A_2A_2^\dagger} + |x_1 - 1, x_1 + 1\rangle_{A_2A_2^\dagger} + \frac{1}{2} |x_1 - 1, x_1 + 1\rangle_{A_2^\dagger A_2}, \] (A.7)

\[ H |x_1\rangle_{B_1^\dagger B_1} = 3 |x_1\rangle_{B_1^\dagger B_1} - \frac{1}{2} |x_1 - 1\rangle_{B_1^\dagger B_1} - \frac{1}{2} |x_1 + 1\rangle_{B_1^\dagger B_1} + \frac{1}{2} |x_1 - 1\rangle_{A_1A_1^\dagger} + \frac{1}{2} |x_1 - 1, x_1\rangle_{A_2A_2^\dagger} + \frac{1}{2} |x_1 - 1, x_1 + 1\rangle_{A_2A_2^\dagger}, \] (A.8)

\[ H |x_1\rangle_{B_2^\dagger B_2} = 4 |x_1\rangle_{B_2^\dagger B_2} + \frac{1}{2} |x_1\rangle_{B_1^\dagger B_1} + \frac{1}{2} |x_1 - 1\rangle_{B_1^\dagger B_1} + \frac{1}{2} |x_1 + 1\rangle_{B_2^\dagger B_2} + \frac{1}{2} |x_1 - 1, x_1\rangle_{A_2^\dagger A_2} + \frac{1}{2} |x_1 - 1, x_1 + 1\rangle_{A_2^\dagger A_2} + \frac{1}{2} |x_1 - 1, x_1 + 1\rangle_{A_2^\dagger A_2} + \frac{1}{2} |x_1, x_1 + 1\rangle_{A_2^\dagger A_2}, \] (A.9)

The appearance of terms of the form \( |x\rangle_{A_kA_k^\dagger} \) and \( |x\rangle_{B_k^\dagger B_k} \) \( (k = 1, 2) \) on the RHS of (A.2)-(A.5) explains the need for such terms in the eigenstate (3.18).

With the help of the above results, the eigenvalue equation

\[ H |\psi\rangle = E |\psi\rangle \] (A.10)
with \(|\psi\rangle\) and \(E\) given by \((3.18)\) and \((2.15)\), respectively, leads to the following equations for the amplitudes:

\[
0 = \left[ 3 - \frac{1}{2}(e^{i(p_1 + p_2)} + e^{-i(p_1 + p_2)}) - E \right] A_{A_1A_1^*}
\]
\[
+ (1 + e^{i(p_1 + p_2)}) \left( \frac{1}{2} A_{A_2A_1^*} + A_{B_1^*B_1} + \frac{1}{2} A_{B_2^*B_2} \right)
\]
\[
+ \frac{1}{2}(e^{ip_2} + e^{-ip_1}) \left[ A_{A_1^*A_2}^{(12)} + A_{B_2B_2^*}^{(12)} \right]
\]
\[
+ \frac{1}{2}(e^{ip_1} + e^{-ip_2}) \left[ A_{A_1^*A_2}^{(21)} + A_{B_2B_2^*}^{(21)} \right],
\]

\(0 = \frac{1}{2}(1 + e^{-i(p_1 + p_2)}) A_{A_1A_1^*} + (4 - E) A_{A_2A_2^*} + \frac{1}{2}(1 + e^{i(p_1 + p_2)}) A_{B_1^*B_1}
\]
\[
+ (e^{ip_2} + e^{-ip_1}) \left[ -A_{A_2A_1^*}^{(12)} - \frac{1}{2} A_{A_1^*A_2}^{(12)} + \frac{1}{2} A_{B_2B_2^*}^{(12)} \right]
\]
\[
+ (e^{ip_1} + e^{-ip_2}) \left[ -A_{A_2A_1^*}^{(21)} - \frac{1}{2} A_{A_1^*A_2}^{(21)} + \frac{1}{2} A_{B_2B_2^*}^{(21)} \right],
\]

\(0 = (1 + e^{-i(p_1 + p_2)}) \left[ A_{A_1A_1^*} + \frac{1}{2} A_{A_2A_2^*} + \frac{1}{2} A_{B_1^*B_1} \right]
\]
\[
+ \left[ 3 - \frac{1}{2}(e^{i(p_1 + p_2)} + e^{-i(p_1 + p_2)}) - E \right] A_{B_1^*B_1}
\]
\[
+ \frac{1}{2}(e^{ip_2} + e^{-ip_1}) \left[ A_{A_1^*A_2}^{(12)} + A_{B_2B_2^*}^{(12)} \right]
\]
\[
+ \frac{1}{2}(e^{ip_1} + e^{-ip_2}) \left[ A_{A_1^*A_2}^{(21)} + A_{B_2B_2^*}^{(21)} \right],
\]

\(0 = \frac{1}{2}(1 + e^{-i(p_1 + p_2)}) A_{A_1A_1^*} + \frac{1}{2}(1 + e^{i(p_1 + p_2)}) A_{B_1^*B_1} + (4 - E) A_{B_2B_2^*}
\]
\[
+ (e^{ip_2} + e^{-ip_1}) \left[ -A_{B_1^*B_2}^{(12)} + \frac{1}{2} A_{A_1^*A_2}^{(12)} - \frac{1}{2} A_{B_2B_2^*}^{(12)} \right]
\]
\[
+ (e^{ip_1} + e^{-ip_2}) \left[ -A_{B_1^*B_2}^{(21)} + \frac{1}{2} A_{A_1^*A_2}^{(21)} - \frac{1}{2} A_{B_2B_2^*}^{(21)} \right],
\]

\(0 = e^{ip_2}(4 - e^{-ip_1} - e^{ip_2} - E) A_{A_2A_1^*}^{(12)} + e^{ip_1}(4 - e^{-ip_2} - e^{ip_1} - E) A_{A_2A_2^*}^{(21)}
\]
\[
- (1 + e^{i(p_1 + p_2)}) A_{A_2A_2^*},
\]

\(A.11\)

\(A.12\)

\(A.13\)

\(A.14\)

\(A.15\)
\begin{align}
0 &= e^{ip_2}(4 - e^{-ip_1} - e^{ip_2} - E)A_{B_1^2B_2}(12) + e^{ip_1}(4 - e^{-ip_2} - e^{ip_1} - E)A_{B_2B_2}(21) \\
&+ \frac{1}{2}(1 + e^{i(p_1+p_2)})(A_{A_1A_1} - A_{A_2A_2}^t + A_{B_1^2B_1} + A_{B_2^2B_2}), \\
\end{align}

Eliminating $A_{A_kA_k}^t, A_{B_k^2B_k}$ \((k = 1, 2)\), and then solving for the \((21)\) amplitudes in terms of the \((12)\) amplitudes, we arrive at the results \((3.21), (3.22)\).

**References**

[1] A. B. Zamolodchikov and Al. B. Zamolodchikov, “Factorized $S$ matrices in two-dimensions as the exact solutions of certain relativistic quantum field models,” *Ann. Phys.* **120**, 253 (1979).

[2] J. M. Maldacena, “The large $N$ limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* **2**, 231 (1998) [arXiv:hep-th/9711200].● S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” *Phys. Lett.* **B428**, 105 (1998) [arXiv:hep-th/9802109].● E. Witten, “Anti-de Sitter space and holography,” *Adv. Theor. Math. Phys.* **2**, 253 (1998) [arXiv:hep-th/9802150].

[3] J. A. Minahan and K. Zarembo, “The Bethe-Ansatz for $\mathcal{N} = 4$ Super Yang-Mills,” *JHEP* **0303**, 013 (2003) [arXiv:hep-th/0212208].

[4] N. Beisert and M. Staudacher, “The $\mathcal{N} = 4$ SYM Integrable Super Spin Chain,” *Nucl. Phys.* **B670**, 439 (2003) [arXiv:hep-th/0307042].

[5] D. Berenstein and S. E. Vázquez, “Integrable open spin chains from giant gravitons,” *JHEP* **0506**, 059 (2005) [arXiv:hep-th/0501078].

[6] A. A. Tseytlin, “Spinning strings and AdS/CFT duality,” in Ian Kogan Memorial Volume, *From Fields to Strings: Circumnavigating Theoretical Physics*, M. Shifman, A. Vainshtein, and J. Wheater, eds. (World Scientific, 2004) [arXiv:hep-th/0311139].● N.
Beisert, “The dilatation operator of $\mathcal{N} = 4$ super Yang-Mills theory and integrability,” *Phys. Rept.* **405**, 1 (2005) [arXiv:hep-th/0407277]. • K. Zarembo, “Semiclassical Bethe ansatz and AdS/CFT,” *Comptes Rendus Physique* **5**, 1081 (2004) [Fortsch. Phys. **53**, 647 (2005)] [arXiv:hep-th/0411191]. • J. Plefka, “Spinning strings and integrable spin chains in the AdS/CFT correspondence,” *Living Rev. Rel.* **8**, 9 (2005) [arXiv:hep-th/0507136]. • J. A. Minahan, “A brief introduction to the Bethe ansatz in $\mathcal{N} = 4$ super-Yang-Mills,” *J. Phys.* **A39**, 12657 (2006). • K. Okamura, “Aspects of Integrability in AdS/CFT Duality,” [arXiv:0803.3999].

[7] M. Staudacher, “The factorized $S$-matrix of CFT/AdS,” *JHEP* **0505**, 054 (2005) [arXiv:hep-th/0412188].

[8] N. Beisert, “The $su(2|2)$ dynamic $S$-matrix,” *Adv. Theor. Math. Phys.* **12**, 945 (2008) [arXiv:hep-th/0511082]. • N. Beisert, “The Analytic Bethe Ansatz for a Chain with Centrally Extended $su(2|2)$ Symmetry,” *J. Stat. Mech.* **0701**, P017 (2007) [arXiv:nlin/0610017].

[9] R. A. Janik, “The $AdS_5 \times S^5$ superstring worldsheet $S$-matrix and crossing symmetry,” *Phys. Rev.* **D73**, 086006 (2006) [arXiv:hep-th/0603038].

[10] G. Arutyunov and S. Frolov, “On $AdS_5 \times S^5$ string $S$-matrix,” *Phys. Lett.* **B639**, 378 (2006) [arXiv:hep-th/0604043].

[11] N. Beisert, R. Hernandez and E. Lopez, “A crossing-symmetric phase for $AdS_5 \times S^5$ strings,” *JHEP* **0611**, 070 (2006) [arXiv:hep-th/0609044].

[12] N. Beisert, B. Eden and M. Staudacher, “Transcendentality and crossing,” *J. Stat. Mech.* **0701**, P021 (2007) [arXiv:hep-th/0610251].

[13] G. Arutyunov, S. Frolov and M. Zamaklar, ‘The Zamolodchikov-Faddeev algebra for $AdS_5 \times S^5$ superstring,” *JHEP* **0704**, 002 (2007) [arXiv:hep-th/0612229].

[14] N. Dorey, D.M. Hofman and J.M. Maldacena, “On the singularities of the magnon $S$-matrix,” *Phys. Rev.* **D76**, 025011 (2007) [arXiv:hep-th/0703104].

[15] M. J. Martins and C.S. Melo, “The Bethe ansatz approach for factorizable centrally extended $S$-matrices,” *Nucl. Phys.* **B785**, 246 (2007) [arXiv:hep-th/0703086].

[16] M. de Leeuw, “Coordinate Bethe Ansatz for the String $S$-Matrix,” *J. Phys.* **A40**, 14413 (2007) [arXiv:0705.2369].

[17] N. Beisert and M. Staudacher, “Long-range $PSU(2,2|4)$ Bethe ansaetze for gauge theory and strings,” *Nucl. Phys.* **B727**, 1 (2005) [arXiv:hep-th/0504190].
[18] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, “$\cal N = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,” JHEP 0810, 091 (2008) [arXiv:0806.1218].

[19] J. A. Minahan and K. Zarembo, “The Bethe ansatz for superconformal Chern-Simons,” JHEP 0809, 040 (2008) [arXiv:0806.3951]

[20] D. Bak and S. J. Rey, “Integrable Spin Chain in Superconformal Chern-Simons Theory,” JHEP 0810, 053 (2008) [arXiv:0807.2063]

[21] G. Arutyunov and S. Frolov, “Superstrings on $AdS_4 \times CP^3$ as a Coset Sigma-model,” JHEP 0809, 129 (2008) [arXiv:0806.4940].

[22] B. J. Stefanski, “Green-Schwarz action for Type IIA strings on $AdS_4 \times CP^3$,“ [arXiv:0806.4948].

[23] N. Gromov and P. Vieira, “The AdS4/CFT3 algebraic curve,” [arXiv:0807.0437].

[24] N. Gromov and P. Vieira, “The all loop AdS4/CFT3 Bethe ansatz,” JHEP 0901, 016 (2009) [arXiv:0807.0777].

[25] T. Nishioka and T. Takayanagi, “On Type IIA Penrose Limit and $\cal N = 6$ Chern-Simons Theories,” JHEP 0808, 001 (2008) [arXiv:0806.3391].

[26] D. Gaiotto, S. Giombi and X. Yin, “Spin Chains in $\cal N = 6$ Superconformal Chern-Simons-Matter Theory,” [arXiv:0806.4589].

[27] G. Grignani, T. Harmark and M. Orselli, “The $SU(2) \times SU(2)$ sector in the string dual of $\cal N = 6$ superconformal Chern-Simons theory,” [arXiv:0806.4959].

[28] C. Ahn and R. I. Nepomechie, “$\cal N = 6$ super Chern-Simons theory $S$-matrix and all-loop Bethe ansatz equations,” JHEP 0809, 010 (2008) [arXiv:0807.1924].

[29] C. Ahn and R. I. Nepomechie, “An alternative $S$-matrix for $\cal N = 6$ Chern-Simons theory ?” [arXiv:0810.1915].

[30] M. Benna, I. Klebanov, T. Klose and M. Smedback, “Superconformal Chern-Simons Theories and $AdS_4/CFT_3$ Correspondence,” JHEP 0809, 072 (2008) [arXiv:0806.1519].

• K. Hanaki and H. Lin, “M2-M5 Systems in $\cal N = 6$ Chern-Simons Theory,” JHEP 0809, 067 (2008) [arXiv:0807.2074]. • I. Shenderovich, “Giant magnons in $AdS_4/CFT_3$: dispersion, quantization and finite-size corrections,” [arXiv:0807.2861]. • C. Ahn, P. Bozhilov and R. C. Rashkov, “Neumann-Rosochatius integrable system for strings on $AdS_4 \times CP^3$,“ JHEP 0809, 017 (2008) [arXiv:0807.3134]. • Y. Honma, S. Iso,
Y. Sumitomo, H. Umetsu and S. Zhang, “Generalized Conformal Symmetry and Recovery of $SO(8)$ in Multiple M2 and D2 Branes,” [arXiv:0807.3825]. • T. McLoughlin and R. Roiban, “Spinning strings at one-loop in $AdS_4 \times P^3$,” JHEP 0812, 101 (2008) [arXiv:0807.3965]. • J. Kluson, “D2 to M2 Procedure for D2-Brane DBI Effective Action,” Nucl. Phys. B808, 260 (2009) [arXiv:0807.4054]. • L. F. Alday, G. Arutyunov and D. Bykov, “Semiclassical Quantization of Spinning Strings in $AdS_4 \times CP^3$,” JHEP 0811, 089 (2008) [arXiv:0807.4400]. • C. Krishnan, “AdS4/CFT3 at One Loop,” JHEP 0809, 092 (2008) [arXiv:0807.4561]. • N. Gromov and V. Mikhaylov, “Comment on the Scaling Function in $AdS_4 \times CP^3$,” [arXiv:0807.4897]. • O. Aharony, O. Bergman and D. L. Jafferis, “Fractional M2-branes,” JHEP 0811, 043 (2008) [arXiv:0807.4924]. • G. Bonelli, A. Tanzini and M. Zabzine, “Topological branes, p-algebras and generalized Nahm equations,” [arXiv:0807.5113]. • M. R. Garousi and A. Ghodsi, “Hydrodynamics of $\mathcal{N} = 6$ Superconformal Chern-Simons Theories at Strong Coupling,” [arXiv:0808.0411]. • R. D‘Auria, P. Fre, P. A. Grassi and M. Trigiante, “Superstrings on $AdS_4 \times CP^3$ from Supergravity,” [arXiv:0808.1282]. • D. Fioravanti, P. Grinza and M. Rossi, “The generalised scaling function: a systematic study,” [arXiv:0808.1886]. • D. Berenstein and D. Trancanelli, “Three-dimensional $\mathcal{N} = 6$ SCFT’s and their membrane dynamics,” [arXiv:0808.2503]. • R. C. Rashkov, “A note on the reduction of the $AdS_4 \times CP^3$ string sigma model,” Phys. Rev. D78, 106012 (2008) [arXiv:0808.3057]. • K. Hosomichi, K. M. Lee, S. Lee, J. Park and P. Yi, “A Nonperturbative Test of M2-Brane Theory,” JHEP 0811, 058 (2008) [arXiv:0809.1771]. • J. Kluson and K. L. Panigrahi, “Defects and Wilson Loops in 3d QFT from D-branes in $AdS(4) \times CP^3$,” [arXiv:0809.3355]. • C.-h. Ahn, “Squashing Gravity Dual of $\mathcal{N} = 6$ Superconformal Chern-Simons Gauge Theory,” [arXiv:0809.3684]. • T. McLoughlin, R. Roiban and A. A. Tseytlin, “Quantum spinning strings in $AdS_4 \times CP^3$: testing the Bethe Ansatz proposal,” JHEP 0811, 069 (2008) [arXiv:0809.4038]. • S. Ryang, “Giant Magnon and Spike Solutions with Two Spins in $AdS_4 \times CP^3$,” JHEP 0811, 084 (2008) [arXiv:0809.5106]. • D. Bombardelli and D. Fioravanti, “Finite-Size Corrections of the $CP^3$ Giant Magnons: the Luscher terms,” [arXiv:0810.0704]. • T. Lukowski and O. O. Sax, “Finite size giant magnons in the $SU(2) \times SU(2)$ sector of $AdS_4 \times CP^3$,” JHEP 0812, 073 (2008) [arXiv:0810.1246]. • M. Kreuzer, R. C. Rashkov and M. Schimpf, “Near Flat Space limit of strings on $AdS_4 \times CP^3$,” [arXiv:0810.2008]. • C. Ahn and P. Bozhilov, “Finite-size Effect of the Dyonic Giant Magnons in $\mathcal{N} = 6$ super Chern-Simons Theory,” [arXiv:0810.2079]. • C. Ahn and P. Bozhilov, “M2-brane Perspective on $\mathcal{N} = 6$ Super Chern-Simons Theory at Level k,” JHEP 0812, 049 (2008) [arXiv:0810.2171]. • F. Spill, “Weakly coupled $\mathcal{N} = 4$ Super Yang-Mills and $\mathcal{N} = 6$ Chern-Simons theories from $u(2|2)$ Yangian symmetry,” [arXiv:0810.3897]. • J. Gomis, D. Sorokin and L. Wulff, “The complete $AdS(4) \times CP(3)$
superspace for the type IIA superstring and D-branes,” [arXiv:0811.1566]. • C. Kristjansen, M. Orselli and K. Zoubos, “Non-planar ABJM Theory and Integrability,” [arXiv:0811.2150]. • P. Sundin, “The $AdS(4) \times CP(3)$ string and its Bethe equations in the near plane wave limit,” [arXiv:0811.2775]. • J. H. Baek, S. Hyun, W. Jang and S. H. Yi, “Membrane Dynamics in Three dimensional $\mathcal{N} = 6$ Supersymmetric Chern-Simons Theory,” [arXiv:0812.1772]. • D. Bak, “Zero Modes for the Boundary Giant Magnons,” [arXiv:0812.2645]. • A. Agarwal, N. Beisert and T. McLoughlin, “Scattering in Mass-Deformed $\mathcal{N} \geq 4$ Chern-Simons Models,” [arXiv:0812.3367].

[31] B. I. Zwiebel, “Two-loop Integrability of Planar $\mathcal{N} = 6$ Superconformal Chern-Simons Theory,” [arXiv:0901.0411].

[32] J. A. Minahan, W. Schulgin and K. Zarembo, “Two loop integrability for Chern-Simons theories with $\mathcal{N} = 6$ supersymmetry,” [arXiv:0901.1142].