A LIOUVILLE PROPERTY FOR GRADIENT GRAPHS AND A BERNSTEIN PROBLEM FOR HAMILTONIAN STATIONARY EQUATIONS.

Abstract. Using an rotation of Yuan, we observe that the gradient graph of any semiconvex function is a Liouville manifold, that is, does not admit bounded harmonic functions. As a corollary, we find that any solution of the fourth order Hamiltonian stationary equation satisfying

$$\theta \geq (n - 2) \frac{\pi}{2} + \delta$$

for some $\delta > 0$ must be a quadratic.

In this short note, we record the following.

**Theorem 1.** Suppose that $u$ is semi-convex. Then the gradient graph

$$\Gamma = \{(x, Du(x)) : x \in \mathbb{R}^n\}$$

with the induced submanifold metric, is a Liouville manifold.

Recall that a manifold has the Liouville property if all bounded harmonic functions are constant. A rotation of Yuan [Yua02] allows us to write the Laplace operator as a uniformly elliptic divergence operator. The result then follows readily from the De-Giorgi-Nash-Moser theory.

We are interested in studying a fourth order special Lagrangian type equation. Let $\lambda_i$ be the eigenvalues of the Hessian $D^2 u$. The lagrangian phase is given by

$$\theta = \sum_{i=1}^{n} \arctan \lambda_i.$$

We extend the following generalization of theorems of Yuan [Yua02][Yua06].

**Theorem 2.** Let $g$ be the metric induced on $\Gamma = (x, Du(x))$. Suppose that $u$ is an entire solution of the fourth order Hamiltonian stationary equation

(0.1) $$\Delta_g \theta = 0$$

with

(0.2) $$|\theta(u)| > (n - 2) \frac{\pi}{2} + \delta$$

or

$$D^2 u \geq 0.$$

Then $u$ is a quadratic function, that is, $\Gamma$ is a plane.

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For a fixed bounded domain $\Omega \subset \mathbb{R}^n$ consider the volume functional
\[
F_\Omega(u) = \int_\Omega \sqrt{\det \left(I + (D^2 u)^T D^2 u\right)}\,dx.
\]
A function $u$ is critical for $F_\Omega(u)$ under compactly supported variations of the scalar function if and only if $u$ satisfies the equation (0.1) c.f [SW03 Proposition 2.2]. In other words, the gradient graph of $u$ has smallest volume compared with other gradient graphs. Recall that if $u$ satisfies the special Lagrangian equation [HL82]
\[
D\theta = 0
\]
then $u$ is critical under all variations of the surface.

The Liouville property, together with (0.2) will force $\theta$ to satisfy (0.3), so $\Gamma$ is a minimal surface. It then follows immediatly from a result of Yuan that $\Gamma$ is a plane. For Bernstein results for (0.1) with a volume growth constraint, and more discussion of the problem, see [Mes01].

1. Proof

1.1. Proof of Theorem 1. If $u$ is semiconvex, then there exists a value $M$ such that
\[
D^2 u + MI_n \geq 0.
\]
It follows that
\[
\arctan \lambda_i \geq - \arctan M
\]
for all $\lambda_i$.

Letting
\[
\delta = \frac{\pi}{2} - \arctan M > 0,
\]
and
\[
D^2 u \geq \tan \left(\delta - \frac{\pi}{2}\right).
\]
The Yuan rotation from [Yua06 section 2] is as follows.

Consider the map
\[
T(x) = \cos \left(\frac{\delta}{n}\right) x + \sin \left(\frac{\delta}{n}\right) Du(x).
\]
Differentiating
\[
(1.1) \quad DT = \cos \left(\frac{\delta}{n}\right) I + \sin \left(\frac{\delta}{n}\right) D^2 u(x)
\]
\[
\geq \cos \left(\frac{\delta}{n}\right) I + \sin \left(\frac{\delta}{n}\right) \tan \left(\delta - \frac{\pi}{2}\right) I
\]
\[
= \cos \left(\frac{\delta}{n}\right) \left(1 + \tan \left(\frac{\delta}{n}\right) \tan \left(\delta - \frac{\pi}{2}\right)\right) I.
\]
Recalling the formula
\[ \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} \]
we see
\[ 1 + \tan \left( \frac{\delta}{n} \right) \tan \left( \frac{\pi}{2} - \frac{\delta}{n} \right) = \frac{\tan \left( \frac{\delta}{n} \right) - \tan \left( \frac{\pi}{2} - \frac{\delta}{n} \right)}{\tan \left( \frac{\delta}{n} - \left( \frac{\pi}{2} - \frac{\delta}{n} \right) \right)} = \frac{\tan \left( \frac{\delta}{n} \right) + \tan \left( \frac{\pi}{2} - \delta \right)}{\tan \left( \frac{\pi}{2} - \delta \frac{n-1}{n} \right)}. \]
It follows that
\[ DT \geq \cos \left( \frac{\delta}{n} \right) \tan \left( \frac{\delta}{n} \right) + \tan \left( \frac{\pi}{2} - \delta \right) I > 0, \]
and the map
\[ T : \mathbb{R}^n \to \mathbb{R}^n \]
is a diffeomorphism.

Next consider the map
\[ \tilde{D} = Du \circ T^{-1}. \]
By \( (1.1) \),
\[ D\tilde{D}(y) = D^2u(T^{-1}(y)) \left[ \cos \left( \frac{\delta}{n} \right) I + \sin \left( \frac{\delta}{n} \right) D^2u(T^{-1}(y)) \right]^{-1}. \]
Diagonalizing \( D^2u \) at \( T^{-1}(y) \) we see
\[ D\tilde{D}|_y \leq \max_i \frac{\lambda_i}{\cos \left( \frac{\delta}{n} \right) + \sin \left( \frac{\delta}{n} \right) \lambda_i} \leq \frac{1}{\sin \left( \frac{\delta}{n} \right)} = M_0 < \infty. \]
So the map
\[ G : \mathbb{R}^n \to \Gamma \subset \mathbb{R}^n \times \mathbb{R}^n \]
given by
\[ G(x) = (T^{-1}(x), Du \circ T^{-1}) \]
is a diffeomorphism onto the gradient graph \( \Gamma \). Thus the pulled-back metric is given by
\[ g = I_n + G^* \bar{g} \]
and satisfies
\[ I_n \leq g \leq (1 + M_0^2)I_n. \]
It follows that the Laplacian, given by
\[ \Delta_g f = \frac{\partial_j (\sqrt{\det gg^{ij}} \partial_i f)}{\sqrt{\det g}}, \]
is a uniformly elliptic divergence type operator.

The remaining proof is standard, but we include it for completeness.

We recall the Harnack inequality of De-Giorgi-Nash-Moser (cf \cite[Theorem 8.20]{GT01})
Theorem 3. Let $u \geq 0$ be a solution of
\[ \partial_j \left( a^{ij}(x) \partial_i f(x) \right) = 0 \]
on $B_4(0)$ with
\[ 0 < \varepsilon I_n \leq a^{ij} \leq \frac{1}{\varepsilon} I_n. \]
There exists a constant $C$ such that
\[ \sup_{B_1(0)} f \leq C \inf_{B_1(0)} f. \]

We may assume that either $f$ or $(-f)$ is bounded below, and we may add a constant and assume $\inf f = 0$. Notice that
\[ f_R(x) = f \left( \frac{x}{R} \right) \]
is a solution of the equation
\[ \partial_j \left( a^{ij} \left( \frac{x}{R} \right) \partial_i f_R(x) \right) = 0 \]
so satisfies the hypothesis of the Harnack inequality. In particular
\[ \sup_{B_R(0)} f \leq C \inf_{B_R(0)} f \]
for every ball $B_R(0)$ with a fixed constant $C$. Taking $R \to \infty$ gives $\sup f = 0$.

In fact, we can state a slightly more general theorem.

Theorem 4. Suppose that $F(D^2u)$ is an elliptic functional, and let $g$ be the induced metric on the gradient graph. If $u$ is semi-convex and
\[ \Delta_g F(D^2u) = 0 \]
then
\[ F(D^2u) = \text{const.} \]

Proof. If $u$ is semiconvex, then there exists a value $M$ such that
\[ D^2u - MI_n \geq 0. \]
It follows by ellipticity that
\[ F(D^2u) \geq F(MI_n) > -\infty. \]
The result follows immediately from our main theorem. \hfill \qed
1.2. **Proof of Theorem 2.** The function $\theta$ is odd in $u$, so we need only show that

\[
\theta(u) > (n - 2) \frac{\pi}{2} + \delta
\]

implies that $u$ is semiconvex. If

\[
\lambda_i < \arctan(\delta - \frac{\pi}{2})
\]

then we must have

\[
\sum_{i \neq j} \arctan \lambda_j > (n - 2) \frac{\pi}{2} + \delta - (\delta - \frac{\pi}{2}) = (n - 1) \frac{\pi}{2}
\]

which is clearly a contradiction as

\[
\arctan \lambda_j \leq \frac{\pi}{2}
\]

We conclude that

\[
D^2 u \geq \arctan(\delta - \frac{\pi}{2})
\]

and $u$ is semiconvex. Thus $\theta(u) = \text{const}$, by Theorem 1. The result follows by the main results in [Yua02][Yua06].

**References**

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