Infinite Horizon Reach-Avoid Zero-Sum Games via Deep Reinforcement Learning

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Abstract—In this paper, we consider the infinite-horizon reach-avoid zero-sum game problem, where the goal is to find a set in the state space, referred to as the reach-avoid set, such that the system starting at a state therein could be controlled to reach a given target set without violating constraints under the worst-case disturbance. We address this problem by designing a new Lipschitz continuous value function with a contracting Bellman backup, where the super-zero level set recovers the reach-avoid set. Building upon this, we prove that the proposed method can be adapted to compute the viability kernel, or the set of states which could be controlled to satisfy given constraints, and the backward reachable set, or the set of states that could be driven towards a given target set. Finally, we propose to alleviate the curse of dimensionality issue in high-dimensional problems by extending Conservative Q-Learning, a deep reinforcement learning technique, to learn a value function such that the super-zero level set of the learned value function serves as a (conservative) approximation to the reach-avoid set.

I. INTRODUCTION

Ensuring the safety and performance of autonomous systems is essential for safety-critical systems, such as autonomous driving [1], surgical robots [2], and air traffic control [3]. Some safe control tasks could be modeled as driving the system’s state to a target set in the state space while satisfying certain safety constraints [4]. This is referred to as the reach-avoid problem [5]. However, it is challenging to solve this problem under the presence of uncertainty [6], such as modeling errors and environmental disturbances. One way to accommodate this is to consider the reach-avoid problem under the worst-case scenario, which could be formulated as a zero-sum game between the control inputs and an adversary, accounting for the uncertainty or disturbance, with an objective to compromise the efforts of control inputs [7].

In the general case, this problem is challenging because it requires solving a zero-sum game with nonlinear dynamics, where the objective involves the worst-case performance instead of an average or cumulative performance over time, as in most constrained reinforcement learning (RL) papers [8]–[10]. Moreover, no future constraint violation is considered once the state trajectory enters the target set in the reach-avoid zero-sum game. Since constrained RL aims to learn a policy satisfying constraints at all times, it can only learn a conservative sub-optimal policy for the reach-avoid zero-sum game.

A well-known approach to solving the reach-avoid zero-sum game is the Hamilton-Jacobi (HJ) method [11], [12], in which one can design a value function such that the sign of the value function evaluated at a state encodes the safety and performance information of that state. In addition, reach-avoid games with multiple agents [13] and stochastic systems [14] have also been studied. These methods provide a closed-loop control policy for the continuous-time and finite-horizon zero-sum game setting, where an agent is required to reach a target set safely within a given finite-time horizon.

However, there are two major limitations of existing work on HJ-based methods for finite-horizon reach-avoid games. Firstly, for complex or uncertain dynamical systems, it is difficult to predict a sufficient time horizon to guarantee the feasibility of the finite-time reach-avoid zero-sum game problem. This motivates the formulation of the infinite-horizon reach-avoid zero-sum game, where we remove the requirement of reaching the target set within a pre-specified finite horizon. The infinite-horizon reach-avoid game has been studied in only a few prior works. Among these is [15], in which the reach-avoid problem without an adversary player is considered. The authors introduce the time-discount factor, a parameter discounting the impact of future reward and constraints, to the design of a contractive Bellman backup such that by annealing the time-discount factor to 1 they can obtain the reach-avoid set. Empirical results in [15] suggest the potential of this method in solving the infinite-horizon reach-avoid zero-sum games.

A second limitation is the computational complexity of classical HJ-based methods, which are exponential in the dimension of the state space [16]. Several approaches have been proposed in the literature to alleviate this. A line of work [17]–[19] formulate the reach-avoid zero-sum games as polynomial optimization problems for which there is no need to grid the entire state space. Another line of work on system decomposition, that is, decomposing a high-dimensional control problem into low-dimensional problems, is also a promising direction [20], [21]. Underapproximation of the reach-avoid set is proposed in [22] and an efficient open-loop policy method is developed in [23]. Nevertheless, these approaches presume certain structural assumptions about the dynamical systems and costs, which restrict their applications.

With the power of neural networks, deep RL has been proven to be a promising technique for high-dimensional optimal control tasks [24]–[26], where a policy is derived to maximize the accumulated reward at each time step. It has been shown in [27] that neural networks could be leveraged to approximate the
value function in high-dimensional optimal control problems and therefore alleviate the curse of dimensionality of Hamilton-Jacobi reachability analysis. For example, sinusoidal neural networks is proven to be a good functional approximator for learning the value function of backward-reachable-tube problem for high-dimensional dynamical systems [28]. A deep reinforcement learning-based method is proposed in [29] to learn a neural network value function for viability kernel, the set of initial states from which a state trajectory could be maintained to satisfy pre-specified constraints. In particular, the paper [29] designs a contractive Bellman backup and learns a conservative approximation to the viability kernel. This method is further extended to solve infinite-horizon reach-avoid problem in [15].

In this paper, we propose a new HJ-based method for the infinite-horizon reach-avoid zero-sum game, and we develop a deep reinforcement learning algorithm to alleviate the curse of dimensionality in solving it. Our contributions are threefold. We first propose a new value function for the reach-avoid zero-sum game, where the induced Bellman backup is a contraction mapping and ensures the convergence of value iteration to the unique fixed point. Subsequently, we analyze the designed value function by first proving its Lipschitz continuity, and then show that the new value function could be adapted to compute the viability kernel and the backward reachable set, that is, the set of states from which a state trajectory could be controlled to reach the given target set. Finally, we alleviate the curse of dimensionality issue by proposing a deep RL algorithm, where we extend conservative Q-learning [30] to learn the value function of the infinite-horizon reach-avoid zero-sum game. We obtain upper and lower bounds for the learned value function. Our empirical results suggest that a (conservative) approximation of the reach-avoid set can be learned, even with neural network approximation.

Unlike the previous work [15] where the value function is discontinuous in general and the reach-avoid set can only be obtained when the time-discount factor is annealing to 1, the proposed value function is Lipschitz continuous under certain conditions and can exactly recover the reach-avoid set without annealing the time-discount factor. The obtained reach-avoid set is agnostic to any time-discount factors in the interval [0, 1). In addition, this new value function could be adapted to compute the viability kernel and backward reachable set, which constitutes a unified theoretical perspective on reachability analysis concepts such as reach-avoid set, viability kernel, and backward reachable set.

The rest of the paper is organized as follows. In Section II, we formulate the infinite-horizon reach-avoid problem. We present our main theoretical results in Sections III – IV with proofs provided in the Appendix. In Section VI, we illustrate our algorithm in multiple experiments. Finally, we conclude the paper in Section VII.

II. PROBLEM FORMULATION

We denote the state of a system by $x \in \mathbb{R}^n$. Define $u := \{u_0, u_1, \ldots\}$ and $d := \{d_0, d_1, \ldots\}$, where $u_t \in \mathcal{U} \subseteq \mathbb{R}^m$ and $d_t \in \mathcal{D} \subseteq \mathbb{R}^d$ are a sequence of control inputs and a sequence of disturbances, respectively. We assume $\mathcal{U}$ and $\mathcal{D}$ to be compact sets. Let $\xi_{x,d}^u(t)$ be the state trajectory evaluated at time $t$, which evolves according to the update rule

$$\xi_{x,d}^u(t) = \begin{cases} x, & t = 0, \\ f(\xi_{x,d}^u(t), u_t, d_t), & t \in \mathbb{Z}_+. \end{cases}$$

where $\mathbb{Z}_+$ is the set of all non-negative integers and the system dynamics $f(\cdot, \cdot) : \mathbb{R}^n \times \mathcal{U} \times \mathcal{D} \to \mathbb{R}^n$ is assumed to be Lipschitz continuous in the state.

Many control problems involving safety-critical systems can be interpreted as driving the system’s state to a specific region while satisfying certain safety constraints. This intuition can be formalized by introducing the concept of target set as the set of desirable states to reach as well as the concept of constraint set as the set of all the states complying with the given constraints. The task is to design control inputs such that even under the worst-case disturbance the system state trajectory hits the target set while staying in the constraint set all times. We represent the target set and constraint set by $T = \{x \in \mathbb{R}^n : r(x) > 0\}$ and $C = \{x \in \mathbb{R}^n : c(x) > 0\}$, respectively, for some Lipschitz continuous and bounded functions $r(\cdot) : \mathbb{R}^n \to \mathbb{R}$ and $c(\cdot) : \mathbb{R}^n \to \mathbb{R}$.

At each time instance $t \in \mathbb{Z}_+$, the control input $u_t$ aims to move the state towards the target set while staying inside the constraint set, whereas the disturbance $d_t$ attempts to drive the state away from the target set and the constraint set. In this work, we consider a conservative setting, where at each time instance the control input plays before the disturbance.

We propose to account for this playing order by adopting the notion of non-anticipative strategy:

**Definition 1** (Non-anticipative strategy). A map $\phi : \{u_t\}_{t=0}^\infty \to \{d_t\}_{t=0}^\infty$, where $u_t \in \mathcal{U}$ and $d_t \in \mathcal{D}$, $\forall t \in \mathbb{Z}_+$, is a non-anticipative strategy if it satisfies the following condition: Let $\{u_t\}_{t=0}^\infty$ and $\{\bar{u}_t\}_{t=0}^\infty$ be two sequences of control inputs. Let $\{d_t\}_{t=0}^\infty = \phi(\{u_t\}_{t=0}^\infty)$ and $\{d_t\}_{t=0}^\infty = \phi(\{\bar{u}_t\}_{t=0}^\infty)$. For all $T \geq 0$, if $u_t = u_t$, $\forall t \in \{0, 1, \ldots, T\}$, then $d_t = d_t$, $\forall t \in \{0, 1, \ldots, T\}$.

The intuition behind this non-anticipative strategy is that, given two sequences of control inputs sharing the same values for the first $T$ entries, the corresponding entries of the two sequences of disturbance must also be the same, meaning the disturbance cannot use any information about the future control inputs when taking its own action. We denote by $\Phi$ the set of all non-anticipative strategies. Building upon the previous discussion on reach-avoid zero-sum games, we are now ready to define the meaning reach-avoid set:

$$\mathcal{R}(\mathcal{T}, \mathcal{C}) := \{x \in \mathbb{R}^n : \forall \phi \in \Phi, \exists \{u_t\}_{t=0}^\infty \text{ and } T \geq 0, \text{ s.t., } \forall t \in [0, T], \xi_{x,\phi(\{u_t\})}^u(t) \in \mathcal{T} \land \xi_{x,\phi(\{u_t\})}^u(t) \in \mathcal{C}\},$$

where 's.t.' is the abbreviation of the phrase 'such that'. As such, we formalize the problem to be studied in this paper below.

**Problem 1** (Infinite-horizon reach-avoid game). Let $\mathcal{T} := \{x \in \mathbb{R}^n : r(x) > 0\}$ and $\mathcal{C} := \{x \in \mathbb{R}^n : c(x) > 0\}$ be the target and the constraint set, respectively. Find the reach-avoid set $\mathcal{R}(\mathcal{T}, \mathcal{C})$ and an optimal sequence of control inputs $\{u_t\}_{t=0}^\infty$.\]
The sign of the value function encodes some crucial safety information about each state in $\mathcal{RA}(\mathcal{T}, \mathcal{C})$ such that under the worst-case disturbance $\{d_t\}_{t=0}^\infty$, the state trajectory will reach the target set without violating the constraints.

## III. A New Value Function for Infinite-horizon Reach-Avoid Games

In this section, we address Problem 1 by introducing a new Lipschitz continuous value function such that its sign evaluated at a state indicates whether that state can be driven to the target set without violating constraints. To be more specific, we will show that the super-zero level set of the designed value function recovers the reach-avoid set in Problem 1. Subsequently, we derive a Bellman backup equation. The fixed-point iteration based on this Bellman backup equation induces a unique fixed point solution, which will be proven to be the designed value function $V$. Finally, we will show that the designed value function is Lipschitz continuous, which is a favorable property for HJ analysis [31].

Suppose that a state trajectory reaches a target set $\mathcal{T}$ at time $t$, i.e., $\xi_x^d(t) \in \mathcal{T}$, while being maintained within the constraint set $\mathcal{C}$ for all $t \in [0,1)$. Under an arbitrary sequence of disturbances $d = \{d_t\}_{t=0}^\infty$, we observe that such a sequence of control inputs $u = \{u_t\}_{t=0}^\infty$ should satisfy

$$\max_{u_0} \min_{d_0} \max_{u_1} \min_{d_1} \cdots \max_{u_{t-1}} \min_{d_{t-1}} \gamma^t r(x, u, d(t)),$$

where

$$\min_{\tau = 0, \ldots, t} \gamma^\tau c(\xi_x^u d(\tau)) > 0.$$

Therefore, to verify the existence of a sequence of control inputs driving the state trajectory to the target set safely from an initial state $x_0$, it suffices to check the sign of the following term:

$$\max_{u_0} \min_{d_0} \max_{u_1} \min_{d_1} \cdots \sup_{t=0, \ldots, T} \min_{\tau = 0, \ldots, t} \gamma^\tau c(\xi_x^u d(\tau)).$$

By using the notion of the non-anticipative strategy $\phi$, we can simplify (2) to the term

$$\inf_{\phi} \max_{u} \sup_{t=0, \ldots, T} \min_{\tau = 0, \ldots, t} \gamma^\tau c(\xi_x^{u, \phi(u)}(\tau)).$$

This implies that a new value function for Problem 1 can be defined as

$$V(x) := \inf_{\phi} \max_{u} \sup_{t=0, \ldots, T} \min_{\tau = 0, \ldots, t} \gamma^\tau c(\xi_x^{u, \phi(u)}(\tau)), \quad \forall x \in \mathbb{R}^n. \tag{3}$$

The sign of the value function encodes some crucial safety information about each state, that is, whether the state could be driven towards the target set while satisfying the given constraints. More precisely, $V(x) > 0$ if for every disturbance strategy $\phi$, there exist $u = \{u_t\}_{t=0}^\infty$ and $T < \infty$ such that $\xi_x^{u, \phi(u)}(T) \in \mathcal{T}$ and $\xi_x^{u, \phi(u)}(t) \in \mathcal{C}$ for all $t \in [0, T]$. Based on this intuition, we characterize the relationship between the super-zero level set of the value function $V$ and the reach-avoid set next.

### Theorem 1.

Consider the value function $V(x)$ defined in (3). For every $\gamma \in [0,1)$, it holds that $\{x \in \mathbb{R}^n : V(x) > 0\} = \mathcal{RA}(\mathcal{T}, \mathcal{C})$.

We propose to compute the value function $V$ using dynamic programming. To this end, a contractive Bellman backup will be derived below.

### Theorem 2.

Let $\gamma \in [0,1)$ be the time-discount factor. Suppose $U : \mathbb{R}^n \to \mathbb{R}$ is a bounded function. Consider the Bellman backup $B[\cdot]$ defined as,

$$B[U](x) := \min \{c(x), \max \{r(x, \gamma \min_{u} U(f(x, u, d))\}\}.$$

Then, (3) is the unique solution to the Bellman backup equation,

$$V^{k+1}(x) := B[V^k](x), \quad \forall k \in \mathbb{Z}_+.$$

Since the fixed-point iteration based method using a contraction mapping $B$ will guarantee the convergence to a fixed point within that bounded function space, the Bellman backup ensures the convergence of value iteration to (3), i.e., $\lim_{k \to \infty} V^k = V$.

Once we obtain the value function $V$, we can calculate the following Q function:

$$Q(x, u, d) := \min \{c(x), \max \{r(x, \gamma V(f(x, u, d))\}\}, \quad \forall x \in \mathbb{R}^n. \tag{6}$$

where $V(x) = \max_u \min_d Q(x, u, d)$. Given a state $x \in \mathbb{R}^n$, we can find an optimal control input and disturbance by $u^* = \arg \max_u \min_d Q(x, u, d)$ and $d^* = \arg \min_d Q(x, u, d)$, respectively. This provides a way to extract the optimal control and worst-case disturbance for Problem 1. Therefore, the proposed value function $V$ constitutes a solution to Problem 1 in the sense that one can find the reach-avoid set, the optimal sequence of control inputs, and the worst-case sequence of disturbances.

In what follows, we characterize the Lipschitz continuity of the proposed value function $V$. We show that the Lipschitz continuity of the value function $V$ could be ensured under certain conditions on the dynamics and the time-discount factor.

### Theorem 3 (Lipschitz continuity).

Suppose that the bounded functions $r(\cdot)$ and $c(\cdot)$ are $L_r$- and $L_c$-Lipschitz continuous, respectively. Assume also that the dynamics $f(x, u, d)$ is $L_f$-Lipschitz continuous in $x$, for all $u \in \mathcal{U}$ and $d \in \mathcal{D}$. Let $L := \max(L_r, L_c)$. Then, $V(x)$ is $L$-Lipschitz continuous if $L_f \gamma < 1$.

### Remark 1.

It is known that the sample complexity of neural network approximation could be improved if the function to be approximated is continuous [32]. Therefore, the Lipschitz continuity of the value function $V$ is a beneficial property yielding reliable empirical performance when we deploy neural networks approximation to handle curse of dimensionality as highlighted in Section VI-B.
IV. COMPUTING THE VIABILITY KERNEL AND THE BACKWARD REACHABLE SET

In this section, we show that the designed value function \( \lambda \) can be adapted to compute the viability kernel \( \mathcal{C} \) and the backward reachable set \( \mathcal{B} \). We first present the formal definitions of these notions below.

We adopt the same definition of viability kernel as in \([29]\), where the viability kernel is defined with respect to a closed set \( \mathcal{C} = \{ x \in \mathbb{R}^n : c(x) \geq 0 \} \), i.e., the closure of the constraint set \( C \), as
\[
\Omega(\mathcal{C}) := \{ x \in \mathbb{R}^n : \forall \phi \in \Phi, \exists u, s.t., \xi_x^{u,\phi(u)}(t) \in \mathcal{C}, \forall t \geq 0 \},
\]
which contains all states \( x \) from which a state trajectory \( \xi_x^{u,\phi(u)}(t) \) can be drawn such that \( \xi_x^{u,\phi(u)}(t) \in \mathcal{C}, \forall t > 0 \). This set is of importance when the control task is to make the system satisfy some safety-critical constraints in real-world applications \([29]\).

The backward-reachable set \( \mathcal{B} \) of a target set \( T = \{ x \in \mathbb{R}^n : r(x) > 0 \} \) is defined as
\[
\mathcal{R}(T) := \{ x \in \mathbb{R}^n : \forall \phi \in \Phi, \exists u, \exists t \geq 0, s.t., \xi_x^{u,\phi(u)}(t) \in T \},
\]
which includes all states that can be driven towards the target set \( T \) in finite time. The backward reachable set is a useful concept in reachability analysis \([28]\) because it specifies the set of states that can be controlled to reach a given target set. For instance, if an airplane is in the backward reachable set of another one, then it indicates that the two planes could collide with each other if they are not properly controlled.

In what follows, we show how the value function \( \lambda \) can be adapted to compute the viability kernel and backward-reachable set.

**Proposition 1.** Assume that \( r(x) = -1 \) for all \( x \in \mathbb{R}^n \). It holds that \( V(x) \leq 0 \) for all \( x \in \mathbb{R}^n \). In addition, \( V(x) = 0 \) if and only if \( x \in \Omega(\mathcal{C}) \).

**Remark 2.** The value function in Proposition 1 could serve as a control barrier function \([33]\), from which we can derive a policy keeping the system states within a given set of the state space.

Similarly, we can compute the backward-reachable set by substituting particular constraint functions into \( \lambda \).

**Proposition 2.** Assume that \( c(x) = 1 \) for all \( x \in \mathbb{R}^n \). It holds that \( V(x) \geq 0 \) for all \( x \in \mathbb{R}^n \). In addition, \( V(x) > 0 \) if and only if \( x \in \mathcal{R}(T) \).

Given the above results, our method provides a new theoretical angle to computing the reach-avoid set, viability kernel, and backward reachable set for infinite-horizon games, where a common value function \( \lambda \) can be adaptively used to compute multiple important sets for safety-critical analysis.

V. A DEEP REINFORCEMENT LEARNING ALGORITHM

In this section, we develop a deep RL algorithm, which alleviates the curse of dimensionality issue for high dimensional problems. In particular, since the satisfaction of given safety constraints is vital for safety-critical systems, we propose to extend conservative Q-learning (CQL) \([30]\) to Problem 1 and develop a deep RL algorithm for solving it, where the conservatism is favorable due to the neural network approximation error. More precisely, we learn a value function parameterized by a neural network which in theory is a lower bound of the value function \( \tilde{\lambda} \). The super-zero level set of the converged value function in CQL is a subset of \( \mathcal{R}A(T, C) \), i.e., a conservative approximation of \( \mathcal{R}A(T, C) \).

The reason why CQL manages to learn a value function lower bound \( \tilde{\lambda} \) is that CQL minimizes not only the Bellman backup error but also the value function itself at each iteration \([30]\). Since the super-zero level set of the value function \( \tilde{\lambda} \) is the reach-avoid set \( \mathcal{R}A(T, C) \), the super-zero level set of a function that is a lower bound of \( \tilde{\lambda} \) is a subset of \( \mathcal{R}A(T, C) \).

Similar to prior works on Deep RL \([24]\), \([25]\), \([30]\), we approximate the Q function \( \tilde{\lambda} \) by a neural network \( Q_\theta(x, u, d) : \mathbb{R}^n \times U \times D \to \mathbb{R} \), where \( \theta \in \mathbb{R}^M \) is the vector of parameters of the neural network and \( M \) is the total number of parameters. We define the neural network value function as \( V_\theta(x) := \max_u \min_d Q_\theta(x, u, d) \), \( \forall x \in \mathbb{R}^n \). We adopt the CQL framework \([30]\) to learn the value function \( \tilde{\lambda} \) by replacing the Bellman backup therein with \( \tilde{\lambda} \). With \( \lambda \geq 0 \), we extend the loss function in \([30]\) to the reach-avoid zero-sum game setting and propose the following loss function
\[
L(\theta) := \theta \in \mathbb{R}^M \to \mathbb{R} \text{ for the neural network parameters:}
\]
\[
L(\theta) := \mathbb{E}_{x \sim \mu} [||V_\theta(x) - B[V_\theta(x)]||_2^2 + \lambda V_\theta(x)],
\]
where \( \mu \) is the uniform distribution over a compact set \( \mathcal{X} \subseteq \mathbb{R}^n \) and the set \( \mathcal{X} \) is subject to user’s choice.

As suggested in \([30]\), given a Bellman backup \( \mathcal{B} \), its corresponding CQL Bellman backup \( B_{CQL}[\cdot] \) can be derived as
\[
B_{CQL}[U](x) := B[U](x) - \lambda,
\]
where \( U : \mathbb{R}^n \to \mathbb{R} \) is a bounded function and \( B[\cdot] \) is the Bellman backup \( \mathcal{B} \). One can show that this Bellman backup is a contraction mapping by a similar reasoning as in the proof of Theorem 2.

Given a bounded functional space, the fixed-point theorem \([34]\) ensures that the value iteration based on \( \tilde{\lambda} \) will converge to a unique value function, which we refer to as \( V_{CQL}(x) \).

We characterize the relationship between the nominal value function \( V(x) \) in \([3]\) and its conservative counterpart \( V_{CQL}(x) \) below.

**Theorem 4.** Let \( V(x) \) and \( V_{CQL}(x) \) be the nominal value function defined in \([3]\) and CQL value function, respectively. We have
\[
V(x) - \frac{\lambda}{1 - \gamma} \leq V_{CQL}(x) \leq V(x) - \lambda, \quad \forall x \in \mathbb{R}^n.
\]

Theorem 4 implies that the value function learned by CQL would be a lower bound of the true value function \( \tilde{\lambda} \). However, the learned value function from CQL would still be bounded, and it is not too pessimistic, i.e., no state will have a value going to negative infinity. We remark here that the bounds in Theorem 4 are specialized for the reach-avoid zero-sum game.

Building on the above results, we propose Algorithm 1, where we substitute the Bellman backup in the algorithm of CQL \([30]\) with \( \lambda \). As in DQN \([24]\), we adopt experience
Algorithm 1: Conservative Reach-Avoid Deep Q-learning

1 Initialize the action-value function $Q_{\theta_0}(x, u, d)$ with random weights $\theta_0$. Select a stepsize $\alpha > 0$, the total number of iterations $K$, the batch size $J$, and the rollout horizon $T$;

2 for $k = 0, \ldots, K$
3 Uniformly sample $x \in X$;
4 Compute $u$ and $d$ such that
\begin{align*}
u_t &\leftarrow \arg \max_u \min_d Q_{\theta_k}(\xi_{x, u, d}(t), u, d) \quad \text{and} \\
d_t &\leftarrow \arg \min_d Q_{\theta_k}(\xi_{x, u, d}(t), u, d);
\end{align*}
5 for $t = 0, \ldots, T$
6 Store $\psi_t = (\xi_{x, u, d}(t), u_t, d_t, \xi_{x, u, d}(t + 1))$ in $\Psi$;
7 end
8 for $j = 1, \ldots, J$
9 Sample a transition $\psi_j$ in $\Psi$;
10 Set $y_j \leftarrow \min\{c(\xi_{x, u, d}(j)), \max\{r(\xi_{x, u, d}(j)), \\
\gamma \max_u \min_d Q_{\theta_k}(\xi_{x, u, d}(j), u, d)\}$;
11 end
12 $\theta_{k+1} \leftarrow \theta_k - \alpha \nabla_{\theta} \left( \sum_{j=1}^{J} ((y_j - \\
Q_{\theta}(\xi_{x, u, d}(j), u_j, d_j))^2 + \lambda Q_{\theta}(\xi_{x, u, d}(j), u_j, d_j)) \right)$;
13 end

replay in Algorithm 1, where we store the state transition $\psi_t = (\xi_{x, u, d}(t), u_t, d_t, \xi_{x, u, d}(t + 1))$ at each time step in a replay memory data-set $\Psi = \{\psi_0, \psi_1, \ldots\}$. Subsequently, we apply Q-learning updates to random samples from $\Psi$ in step 2 of Algorithm 1. It is shown that applying Q-learning to random samples drawn from experience replay could break down the correlation between consecutive transitions and recall rare transitions, which leads to a better convergence performance.

VI. EXPERIMENTS

In this section, we present experiments on Algorithm 1 and show that our method could learn a (conservative) approximation to the reach-avoid set reliably even with neural network approximation.

A. 2D Experiment for Reach-Avoid Game

In this experiment, we compare the reach-avoid set learned by Algorithm 1 with the one learned by tabular Q-learning, where we first grid the continuous state space and then run value iteration over the grid. We treat the reach-avoid set learned by tabular Q-learning as the ground truth solution. We apply Algorithm 1 to learn neural network Q functions with 4 hidden layers, where each hidden layer has 128 neurons with ReLu activation functions. This neural network architecture is chosen because empirically it provides sufficient model capacity for approximating the true value function.

Let the target set and constraint set be $T = \{(x, y) \in \mathbb{R}^2 : 1 - (x^2 + y^2) > 0\}$ and $C = \{(x, y) \in \mathbb{R}^2 : 1 - \frac{(x-2)^2}{1.5^2} - y^2 > 0\}$, respectively. We visualize the target set $T$ and the constraint set $C$ in Figure 1. Consider the following discrete-time double integrator dynamics, with the time constant $\Delta t = 0.02s$:
\begin{equation}
\begin{bmatrix}
x(t + 1)
y(t + 1)
\end{bmatrix} = 
\begin{bmatrix}
x(t) + \Delta t \cdot y(t) 
y(t) + \Delta t \cdot (u(t) + d(t))
\end{bmatrix}
\end{equation}
where $x(t)$ and $y(t)$ are the position and velocity, respectively. The $u(t) \in \{-1, 1\}$ and $d(t) \in \{-0.5, 0.5\}$ are the control action and disturbance at time $t \in \{0, 1, 2, \ldots\}$, respectively.

We see in Figure 2 that the reach-avoid set learned by Algorithm 1 is similar to the one computed by tabular Q-learning. An interesting observation is that, under different time-discount factors, the reach-avoid sets computed by tabular Q-learning are somewhat different. This could potentially be due to the numerical difficulty introduced by small values of $\gamma$. To be more specific, from the definition of the value function, we see that the powers of small constant $\gamma \in [0, 1)$ decays to 0 faster than those for a large parameter $\gamma \in [0, 1)$, and this makes it difficult to distinguish zero from the product of the power of a small constant $\gamma$ and a bounded function term. Despite this numerical difficulty, it can be observed that both tabular Q-learning and Algorithm 1 learn similar reach-avoid sets under different time-discount factors $\gamma \in [0, 1)$.

In addition, as we increase the CQL penalty $\lambda$, the reach-avoid set shrinks to a conservative subset of the true reach-avoid set in both tabular Q-learning and Algorithm 1. For the same $\lambda$, although subject to certain numerical errors, both of these methods yield similar results, which empirically supports Theorem 4.

B. 6D Experiment for Reach-Avoid Game

In this experiment, we consider a three-cart dynamical system, where the dynamics of each cart is modeled as a double integrator. The carts move along different axes. The first cart with the position $x_1(t)$ and velocity $v_1(t)$ moves...
The dynamics is
\[
\begin{align*}
\dot{x}_1(t + 1) &= x_1(t) + \Delta t \cdot v_1(t), \\
\dot{v}_1(t + 1) &= v_1(t) + \Delta t \cdot (u_1 + d_t), \\
\dot{x}_2(t + 1) &= x_2(t) + \Delta t \cdot v_2(t), \\
\dot{v}_2(t + 1) &= v_2(t) + 0.02\Delta t, \\
\dot{x}_3(t + 1) &= x_3(t) + \Delta t \cdot v_3(t), \\
\dot{v}_3(t + 1) &= v_3(t) + 0.02\Delta t
\end{align*}
\]

where \( u_t \in \{-1, 1\} \) and \( d_t \in \{-0.5, 0.5\} \) are the control input and disturbance, respectively. The time-integration constant \( \Delta t \) is equal to 0.02s. The time-discount factor \( \gamma \) is 0.99.

Due to the curse of dimensionality, tabular Q learning explained in the previous experiment suffers numerical difficulties in this 6-dimensional experiment. In this subsection, we apply Algorithm 1 with the same neural network architecture as in Section VI-A. We plot the learned reach-avoid set by projecting it onto a 2-dimensional plane. In Figure 4, we visualize the reach-avoid set as well as the effect of the CQL penalty \( \lambda \) on the learned reach-avoid set. As the penalty \( \lambda \) increases, the volume of the reach-avoid set shrinks while the empirical success rate improves. This suggests that a larger penalty \( \lambda \) induces a more conservative estimation of the reach-avoid set and the policy.

We sample 1000 initial states in each of the three learned reach-avoid set in Figure 4, and then collect the trajectories with maximum length 1000 time steps, under the control and disturbance policies induced by the learned neural network value function. For the three learned value functions, the portion of those 1000 trajectories under each of the three values of \( \lambda \) that successfully reach the target set without violating the constraints are 85.4%, 88.4%, and 91.3%. In addition, we randomly sample 10,000 points in the state space, and observe that the ratios of points lying in the learned reach-avoid sets for \( \lambda = 0.01 \) and \( \lambda = 0.05 \) to those for \( \lambda = 0.0 \) are 99.7% and 82.5%, respectively. However, note that the training epochs taken to learn the three value functions are 1290, 1650, and 1470, respectively.

In Figure 5, we visualize the control and disturbance policies extracted from the Q function of the neural network corresponding to \( \lambda = 0.0 \) in Figure 4. The two policies are considered as to be reasonable because at most state, the control policy either drives the agent towards the target set or prevent the violation of the constraint. Notice that there are a few states, e.g., some points at the right lower corner of the disturbance law.
in Figure 5 where they do not complement to each other. One possible reason is that we learn a local optimal neural network Q function by minimizing the non-convex loss function [7].

C. 6D Experiments on Learning Viability Kernel and Backward Reachable Set

In this subsection, we apply Algorithm 1 to learn viability kernel and backward reachable set for the 6-dimensional dynamical system in Subsection VI-B. The results empirically confirm Propositions 1 and 2. In the following two experiments, the same neural network architecture as in Section VI-B does not yield satisfactory results and we conjecture that this is due to the limited model capacity. As such, in this subsection, we increase the model capacity by adopting 4-layer neural networks with 256 neurons in each layer. We set the CQL penalty parameter to be $\lambda = 0.0$.

We first consider learning the viability kernel where the constraint set is the same as the one in Subsection VI-B. The reward function is set to be $r(x) = -1$, for all $x \in \mathbb{R}^6$. In the first subplot of Figure 6 we visualize the learned value function. The value function is non-positive, which is predicted by Proposition 2. We visualize the learned viability kernel in the second subplot of Figure 6. We sampled 1000 initial positions in the learned viability kernel and simulated a trajectory for each of them with 600 steps. The portion of those sampled points that can be maintained inside the constraint set is 79.2%.

Subsequently, we consider learning a backward reachable set by leveraging Proposition 2 where the constraint function is set to be a constant function with the value 1. We consider the target set $T = \{x \in \mathbb{R}^6 : \min(2 - |x_1|, 1 - |x_2|, 1 - |x_3|) > 0\}$. We visualize the learned value function and backward reachable set in Figure 7. As shown in Proposition 2 the value function is non-negative. We sampled 1000 points in the obtained backward reachable set. The portion of initial states that can be driven towards the target set within 600 simulation steps is 70.0%.

It is observed that the empirical success rate of viability kernel is higher than backward reachable set. There are multiple reasons behind it. First, the simulation horizon is not long enough for an initial state to be driven towards the backward reachable set. In addition, for backward reachable set, a single failure in control may push the trajectory away from the target set in the future. However, for viability kernel, multiple failures in control are allowed if controls at the boundary of the viability kernel keep the trajectory inside the constraint set.

We remark here that the high-dimensional experiments are challenging because there is a lack of efficient methods to check the sufficiency of neural network model capacity and to solve the non-convex optimization problem [7], which are common problems in deep RL and on-going research directions in the deep RL community [35].

VII. CONCLUSION

In this paper, we investigated the infinite-horizon reach-avoid zero-sum game problem, in which the goal is to learn the reach-avoid set in the state space and an associated policy such that each state in the reach-avoid set could be driven towards a given target set while satisfying constraints. We designed a value function that offers several properties: 1) its super-zero level set coincides with the reach-avoid set and the induced Bellman equation is a contraction mapping; 2) the value function is Lipschitz continuous under certain conditions; and 3) the value function can be adapted to compute the viability kernel and backward reachable set. We proposed to alleviate the curse of dimensionality issue by developing a deep RL algorithm. The provided theoretical and empirical results suggest that our method is able to learn reach-avoid sets reliably even with neural network approximation errors.

APPENDIX

Proof of Theorem 4 We first prove the sufficiency. Suppose that there exist an $\varepsilon > 0$ and a finite $T > 0$ such that for all $\phi \in \Phi$, there is a sequence of control inputs $u = \{u_t\}_{t=0}^\infty$ with the property

$$\min\{\min_{t=0}^{\infty} \gamma^t r(\xi^u_{x_0}(u)(t)), \gamma^T r(\xi^u_{x_0}(u)(T))\} > \varepsilon,$$

which implies that for all $\phi \in \Phi$. We then have $V(x) \geq \varepsilon > 0$.

We show the necessity by a contrapositive statement. Suppose that for all $\varepsilon > 0$, there exists a disturbance strategy $\phi \in \Phi$ such that for all sequences of control inputs $\{u_t\}_{t=0}^\infty$ and all time $T \in \{0, 1, \ldots, \}$,

$$\min\{\min_{t=0}^{T} \gamma^t r(\xi^u_{x_0}(u)(t)), \gamma^T r(\xi^u_{x_0}(u)(T))\} \leq \varepsilon,$$

which is equivalent to that for all $\varepsilon > 0$, there exists a disturbance strategy $\phi$ with the property

$$\max_{\{u_t\}_{t=0}^\infty} \sup_{t=0, \ldots, T} \min\{\gamma^t r(\xi^u_{x_0}(u)(t)), \gamma^T r(\xi^u_{x_0}(u)(T))\} \leq \varepsilon.$$

In another words, for all $\varepsilon > 0$,

$$V(x_0) = \inf_{\phi} \max_{\{u_t\}_{t=0}^\infty} \sup_{t=0, \ldots, T} \min\{\gamma^t r(\xi^u_{x_0}(u)(t)), \gamma^T r(\xi^u_{x_0}(u)(T))\} \geq \varepsilon,$$

which yields that

$$V(x_0) = \inf_{\phi} \max_{\{u_t\}_{t=0}^\infty} \sup_{t=0, \ldots, T} \min\{\gamma^t r(\xi^u_{x_0}(u)(t)), \gamma^T r(\xi^u_{x_0}(u)(T))\} \leq 0.$$

Therefore, $V(x) > 0$ if and only if the problem 3 is finite-time feasible and reachable for the initial state $x$.

Lemma 1. Given a compact set $C \subseteq \mathbb{R}$ and a function $a(\cdot) : \mathbb{R} \to \mathbb{R}$, we have

$$\max_{c \in C} \min(a(c), b) = \min(max_{c \in C} a(c), b), \text{ and } \min_{c \in C} \max(a(c), b) = \min(min_{c \in C} a(c), b).$$

Proof of Theorem 2 We first show that $V(x)$ as defined in 3 satisfies the Bellman backup equation 4, and then show that the proposed Bellman backup 4 is a contraction mapping.

By definition, we have

$$V(x_0) = \max\{\min\{r(x_0), c(x_0)\}, \inf_{\phi} \max_{\{u_t\}_{t=0}^\infty} \sup_{t=0, \ldots, T} \min\{\gamma^t r(\xi^u_{x_0}(u)(t)), \min_{\tau=1, \ldots, T} \gamma^\tau c(\xi^u_{x_0}(u)(\tau)), c(x_0)\}\}.$$

By applying Lemma 1 the above equation implies

$$V(x_0) = \min\{c(x_0), \max\{r(x_0), \gamma \max_{u_0, d_0} V(f(x_0, u_0, d_0))\}\}.$$
\(B[|V_2|(x)] \leq \|V_1 - V_2\|_\infty\), where \(V_1\) and \(V_2\) are two arbitrary bounded functions. One can write
\[
|B[V_1(x) - B[V_2(x)]|
\leq |\max\{r(x), \gamma \max_d \min_d V_1(f(x, u, d))\} - \max_d r(x) , \gamma \min_u V_2(f(x, u, d))|]
\leq |\gamma \max_u \min_d V_1(f(x, u, d)) - \gamma \min_u V_2(f(x, u, d))|
\]
where the above inequalities follow from the fact that \(\|\min[a, b] - \min[a, c]\| \leq b - c, \forall a, b, c \in \mathbb{R}\).

Without loss of generality, we suppose that \(\max_u \min_d V_1(f(x, u, d)) \geq \max_u V_0(f(x, u, d))\) and let \(u^* := \arg \max_u \min_d V_1(f(x, u, d))\). In addition, let \(d^* := \arg \min_d V_2(f(x, u^*, d))\). We then have
\[
|\gamma \max_u \min_d V_1(f(x, u, d)) - \gamma \min_u V_2(f(x, u, d))|
\leq |\min_d V_1(f(x, u^*, d)) - \min_d V_2(f(x, u^*, d))|,\]
\[\leq |V_1(x) - V_2(x)| \leq \|V_1 - V_2\|_\infty.
\]
Combining (12) and (13), we have \(|B[V_1(x) - B[V_2(x)]| \leq \gamma \|V_1 - V_2\|_\infty\), for all \(x\). This yields that \(\|B[V_1] - B[V_2]\|_\infty \leq \gamma \|V_1 - V_2\|_\infty\).

**Proof of Theorem 3** Define
\[
P(u, d, t, x) := \min\{\gamma^t r_{-}(x_u, d(t)), \min_{\tau = 0, ..., t} \gamma^\tau c(x_u, d(\tau))\}.
\]

Consider two initial states \(x_1, x_2 \in \mathbb{R}^n\). Given \(\varepsilon > 0\), there exists an non-anticipative strategy \(\phi\) such that
\[
V(x_1) \geq \sup_{u} \inf_{t} P(u, \phi(u), t, x_1) - \varepsilon.
\]
We select a control sequence \(\tilde{u}\) and time \(\tilde{t}\) such that
\[
V(x_2) \leq P(\tilde{u}, \tilde{\phi}(\tilde{u}), \tilde{t}, x_2) + \varepsilon.
\]
Moreover, (15) implies that
\[
V(x_1) \geq P(\tilde{u}, \tilde{\phi}(\tilde{u}), \tilde{t}, x_1) - \varepsilon.
\]
By combining (16) and (17), we obtain
\[
V(x_1) - V(x_2) + 2\varepsilon \geq P(u, \phi(u), t, x_1) - P(u, \phi(u), t, x_2)
\geq \min_{\tau = 0, ..., t} \gamma^\tau c(x_u, d(\tau)) - c(x_u, d(\tau))
\]
\[\geq \min_{\tau = 0, ..., t} \gamma^\tau c(x_u, d(t)) - c(x_u, d(t)).
\]
The last inequality is due to the fact that for any finite values \(a_i, b_i \in \mathbb{R} (i = 1, ..., n)\), we have
\[
\min\{a_1, ..., a_n\} - \min\{b_1, ..., b_n\} \geq \min\{a_i - b_i, ..., a_n - b_n\}.
\]

According to the Lipschitz-continuity assumptions,
\[
\|c(x_u, d(t)) - c(x_u, d(t))\| \leq L^f \|x_1 - x_2\|,
\|c(x_u, d(t)) - c(x_u, d(t))\| \leq L^c \|x_1 - x_2\|,
\|r(x_u, d(t)) - r(x_u, d(t))\| \leq L^r \|x_1 - x_2\|.
\]
Thus, it follows from (18) that
\[
V(x_1) - V(x_2) + 2\varepsilon
\geq -\max_{\tau = 0, ..., t} \gamma^\tau L^f \|x_1 - x_2\|.
\]
The condition \(\gamma L_f < 1\) implies that the coefficient in the right-hand side in (21) is bounded for all \(\tilde{t}\). As a result, \(|V(x_1) - V(x_2)| + 2\varepsilon \geq \frac{2}{\gamma^\tau L_f} \|x_1 - x_2\|\) for some \(C > 0\). Similarly, we can show that \(|V(x_2) - V(x_1)| + 2\varepsilon \geq \frac{2}{\gamma^\tau L_f} \|x_1 - x_2\|\). Combining these two inequalities, we prove Theorem 3.

**Proof of Proposition 2** The non-negativity of \(V(x)\) is due to \(r(x) = -1\), for all \(x\). We then show that \(V(x) = 0\) if and only if \(x \in \Omega(C)\).

(Sufficiency) On hand, suppose that for all \(\phi \in \Phi\), there exists a sequence of control inputs \(u = \{u_t\}_{t=0}^{\infty}\) under which \(c(x_{u, \phi}^u(u(t))) \geq 0, \forall t \in \{0, 1, \ldots\}\). Then, for all \(\phi \in \Phi\),
\[
V(x) \leq \max_{\{u_t\}_{t=0}^{\infty}} \sup_{\phi \in \Phi} \inf_{\{u_t\}_{t=0}^{\infty}} \sup_{\tau = 0, ..., t} \gamma^\tau c(x_{u, \phi}^u(u(\tau)))
\]
\[\leq \max_{\{u_t\}_{t=0}^{\infty}} \sup_{\phi \in \Phi} \inf_{\{u_t\}_{t=0}^{\infty}} 0 = 0.
\]
Thus, \(V(x) = 0\) if for all \(\phi \in \Phi\) there exists a sequence of control inputs \(u = \{u_t\}_{t=0}^{\infty}\) such that \(c(x_{u, \phi}^u(u(t))) \geq 0, \forall t \in \{0, 1, \ldots\}\).

(Necessity) We approach the proof by a contrapositive statement. Suppose that for every \(\phi \in \Phi\) and every sequence of control inputs \(u\), there exists a non-negative integer \(t_{0, u}\) such that \(c(x_{u, \phi}^u(u(t_{0, u}))) < 0\). Then, we have
\[
V(x) \leq \max_{u_{t+1}, \phi_{t+1}, \ldots} \min_{\tau = 0, ..., t} \gamma^\tau c(x_{u, \phi}^u(u(\tau)))
\]
\[= \max_{u_{t+1}} \inf_{\phi_{t+1}} \sup_{\tau = 0, ..., t} \gamma^\tau c(x_{u, \phi}^u(u(\tau))) < 0.
\]
This completes the proof.

**Proof of Proposition 2** The non-negativity of \(V(x)\) is due to \(c(x) = 1\), for all \(x\). Subsequently, it suffices to show that \(V(x) > 0\) if and only if there exists an \(\varepsilon > 0\) such that, for all \(\phi \in \Phi\), there exists a sequence of control inputs that renders \(\gamma^t c(x_{u, \phi}^u(u(t))) > \varepsilon\) and \(\gamma^t > \varepsilon\), for some non-negative integer \(t\).

(Sufficiency) Suppose that there exists \(\varepsilon > 0\) such that for all \(\phi \in \Phi\) there exists a sequence of control inputs \(u = \{u_t\}_{t=0}^{\infty}\), \(\gamma^t c(x_{u, \phi}^u(u(t))) > \varepsilon\), for some \(T_{t, u} \in \mathbb{Z}_+\). Then, we have
\[
V(x) = \inf_{\phi \in \Phi} \max_{\{u_t\}_{t=0}^{\infty}} \sup_{\tau = 0, ..., t} \gamma^\tau c(x_{u, \phi}^u(u(\tau)))
\]
\[\geq \inf_{\phi \in \Phi} \varepsilon > \varepsilon.
\]
(Necessity) We approach the proof by a contrapositive statement. Suppose that for all \(\varepsilon > 0\) there exist \(\phi\) such that for every sequence of control inputs \(u = \{u_t\}_{t=0}^{\infty}\), it holds that \(\gamma^t c(x_{u, \phi}^u(u(t))) \leq \varepsilon\) or \(\gamma^t > \varepsilon\), \(\forall t \in \mathbb{Z}_+\). Then, we have
\[
V(x) \leq \max_{\{u_t\}_{t=0}^{\infty}} \inf_{\phi \in \Phi} \sup_{\tau = 0, ..., t} \gamma^\tau c(x_{u, \phi}^u(u(\tau)))
\]
\[\leq \sup_{\{u_t\}_{t=0}^{\infty}} \varepsilon \leq \varepsilon.
\]
Since the above inequality holds true for all \(\varepsilon > 0\), we have \(V(x) \leq 0\). This completes the proof.

**Lemma 2.** Consider two functions \(g(a, b) : \mathbb{R} \rightarrow \mathbb{R}\) and \(\tilde{g}(a, b) : \mathbb{R} \rightarrow \mathbb{R}\). Suppose that \(g(a, b) \leq \tilde{g}(a, b), \forall a, b\). Then, we have \(\max_a \min_b g(a, b) \leq \max_a \min_b \tilde{g}(a, b)\).
Proof of Theorem 2] Let $V^{(0)}(\cdot) : \mathbb{R}^n \to \mathbb{R}$ be an arbitrary bounded function. Consider the value iteration:

$$V^{(k+1)}(x) = B[V^{(k)}(x)], \quad \forall k \in \mathbb{Z}_+.$$  \tag{22}

Let $V^{(0)}_{\text{CQL}} : \mathbb{R}^n \to \mathbb{R}$, and consider the CQL-based value iteration:

$$V^{(k+1)}_{\text{CQL}}(x) = B_{\text{CQL}}[\gamma^{(k)}_{\text{CQL}}(x)], \quad \forall k \in \mathbb{Z}_+.$$  \tag{23}

Suppose that $V^{(0)}_{\text{CQL}}(x) = V^{(0)}(x), \forall x \in \mathbb{R}^n$. It suffices to show that

$$V^{(k)}(x) - \sum_{i=1}^{k} \gamma^i \lambda \leq V^{(k)}_{\text{CQL}}(x) \leq V^{(k)}(x) - \lambda, \quad \forall x \in \mathbb{R}^n,$$  \tag{24}

because by taking the limit $k \to \infty$, we have

$$\lim_{k \to \infty} V^{(k)}(x) - \frac{\lambda}{1 - \gamma} \leq V^{(k)}_{\text{CQL}}(x) \leq V^{(k)}(x) - \lambda, \forall x \in \mathbb{R}^n.$$  \tag{25}

To this end, we first show the upper bound $V^{(k)}_{\text{CQL}}(x) \leq V^{(k)}(x) - \lambda$ by induction. From (8), we have

$$V^{(k)}_{\text{CQL}}(x) = B[V^{(k)}(x)] \leq V^{(k)}(x) - \lambda, \forall x \in \mathbb{R}^n.$$  \tag{26}

Therefore,

$$V^{(k)}_{\text{CQL}}(x) \leq \min \{c(x), \max_i \{r(x), \gamma \max_u \min_d \{f(x, u, d)\}\} - \lambda \leq V^{(k)}(x) - \lambda,$$

where we substitute (20) into the CQL Bellman backup (8) of the $V^{(k)}_{\text{CQL}}(x)$ and also use the inequality in Lemma 2. By induction, we have $V^{(k)}_{\text{CQL}} \leq V^{(k)}(x) - \lambda, \forall x \in \mathbb{R}^n$. Similarly, we show the lower bound $V^{(k)}_{\text{CQL}}(x) \geq V^{(k)}(x) - \sum_{i=1}^{k} \gamma^i \lambda$ as follows. Since $V^{(1)}_{\text{CQL}}(x) \geq V^{(1)}(x) - \lambda$, we have

$$V^{(2)}_{\text{CQL}}(x) \geq \min \{c(x), \max_i \{r(x), \gamma \max_u \min_d \{f(x, u, d)\} - \lambda \geq \gamma \min \{c(x) - \gamma \lambda, \max_i \{r(x) - \gamma \lambda\}, \gamma \max_u \min_d \{f(x, u, d)\} - \gamma \lambda\} - \lambda = V^{(2)}(x) - \gamma \lambda - \lambda.$$

By induction, we have $V^{(k)}(x) - \sum_{i=1}^{k} \gamma^i \lambda \leq V^{(k)}_{\text{CQL}}(x)$.

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