Shadow Sequences of Integers: From Fibonacci to Markov and Back

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When Sophie Morier-Genoud and I took over the editorship of the Gems and Curiosities column from Sergei Tabachnikov a year ago, we understood that there was a long tradition of very large shoes to fill. In particular, in its previous incarnation under the name Mathematical Entertainments, this column enjoyed its golden years under the watch of David Gale. For an account, see the amazing book [4]. Integer sequences was one of the main subjects discussed in the column at that time [3], and the sequences now called Gale–Robinson and Somos sequences first appeared in the column. Gale’s “riddles” about integer sequences strongly influenced combinatorics; see, for instance, the powerful and now classical reference [2], in which Gale’s riddles were solved.

Although any attempt to come up to the high bar set by David Gale on the subject of integer sequences is doomed to fail, I will discuss here a large class of integer sequences given by recurrence relations. The following general idea looks crazy. What if every integer sequence has another integer sequence that follows it like a shadow? I will demonstrate that this is indeed the case, though perhaps not for every integer sequence (I unfortunately don’t know of any exceptions), but for many of them.

Does this mean that the number of known and registered sequences in the Online Encyclopedia of Integer Sequences (OEIS) [6], now more than 350,000, will double? The answer is no for two reasons: the shadow of a known sequence can also be a known sequence, and one sequence can have several shadows.

Dual Numbers

A pair of integers $A = (a, \alpha) \in \mathbb{Z}^2$ can be organized as a linear expression

$$A := a + \alpha \epsilon,$$

where $\epsilon$ is a formal variable. If $\epsilon$ is a square root of $-1$, then $A$ is a complex number, called a Gaussian integer. But if $\epsilon$ satisfies the condition $\epsilon^2 = 0$, then $A$ is called a dual number. Dual numbers were introduced by William Clifford in 1873, and they have applications in geometry and mathematical physics. In geometry, dual numbers are used in working with the space of oriented lines in $\mathbb{R}^3$, a useful device for geometric optics and computer vision.

Let $(a_n)_{n \in \mathbb{N}}$ be an integer sequence whose first $k$ entries, $a_1, \ldots, a_k$, for some fixed integer $k > 0$, are given as initial conditions. Each of the remaining entries $a_{n+k}$, $n \in \mathbb{N}$, is determined by the $k$ entries that precede it:

$$a_{n+k} = R(a_{n+k-1}, \ldots, a_n),$$

for all $n \in \mathbb{N},$
where $R$ is a (generally rational) expression. Assume that a sequence of dual numbers $(A_n)_{n \in \mathbb{N}}$, $A_n = a_n + a_n \varepsilon$, satisfies the same recurrence

$$A_{n+k} = R(A_{n+k-1}, \ldots, A_n).$$

The expression on the right-hand side then has two components: $R = R_0 + R_1 \varepsilon$, and $a_{n+k}$ is then determined by the $\varepsilon$-component $R_1$ and some initial conditions. Moreover, for some “mysterious” reasons, the new sequence $(a_n)_{n \in \mathbb{N}}$ turns out to be an integer sequence!

This idea to construct dual integer sequences was suggested in [7, 8] and applied to the Gale–Robinson and Somos sequences. Today, I will go further and apply it to several other interesting sequences, making the method more universal. Let us call such dual sequences “shadows.” The main example is that of the Markov numbers.

**The Most Honest Sequence**

Perhaps the only way to demonstrate that a general method has a chance to work is to consider examples. The skeptical reader may interrupt me here and say, “OK, then! Let us take the ‘most honest’ sequence of all positive integers:

$$a_n = 1, 2, 3, 4, 5, 6, 7, \ldots,$$

registered in the OEIS under the (somewhat surprising) number A000027. What ought to be the shadow of A000027??

Let’s see .... The sequence satisfies a linear recurrence $a_{n+1} = a_n + 1$, which on substituting into it a sequence of dual numbers $A_n = a_n + a_n \varepsilon$ will produce a, $a$, $a$, $\ldots$ with arbitrary $a$, a sad constant sequence. But I promised shadows of sequences, not shadows of poorly chosen recurrence relations!

**Proposition 1** A shadow of A000027 is the sequence A0000292,

$$a_n = 0, 1, 4, 10, 20, 35, 56, \ldots,$$

called the tetrahedral numbers and given explicitly by

$$a_n = \frac{(n-1)n(n+1)}{6}.$$

To explain this, notice that besides the above linear recurrence, A000027 satisfies another, more interesting, nonlinear recurrence:

$$a_n a_{n+2} = a_{n+1}^2 - 1. \quad (1)$$

Substituting $A_n = a_n + a_n \varepsilon$ instead of $a_n$ and collecting the $\varepsilon$-terms gives the following linear recurrence for $a_n$:

$$a_n a_{n+2} = 2a_{n+1} a_{n+1} - a_{n+2} a_n. \quad (2)$$

More precisely,

$$a_{n+2} = \frac{2(n+1)}{n} a_{n+1} - \frac{(n+2)}{n} a_n. \quad (3)$$

At first glance, it is not clear that $a_n$ will always be an integer as $n$ grows. But it is an easy exercise to check that (3) is satisfied for the tetrahedral sequence. Therefore, choosing the initial conditions $(a_1, a_2) = (0, 1)$ indeed gives sequence A0000292.

Interestingly, the alternative choice $(a_1, a_2) = (1, 0)$ leads to the sequence

$$a_n = 1, 0, -3, -9, -19, -34, -55, \ldots,$$

which is (up to sign) the same sequence A0000292 decreased by 1 (yet registered as A062748). An arbitrary solution of (3) is a linear combination of A0000292 and A062748.

Strictly speaking, every such linear combination should be called a shadow of A000027. However, A0000292, which corresponds to the simplest choice of initial conditions that does not produce negative entries, seems to be the most natural choice.

Another interesting observation is that the sequence of tetrahedral numbers A0000292 is actually the convolution of A000027 with itself.

To end our warmup example of A000027, let me add that besides (1), the sequence satisfies many other recurrences, for instance, $a_n a_{n+1} = a_{n+1} a_{n+2} - 2$. However, the shadowing procedure described here does not lead to any additional integer sequences $(a_n)$. I cannot see any good candidate for a shadow of A000027 other than A0000292.

**The Shadows of Fibonacci and Catalan**

The Fibonacci numbers (sequence A000045)

$$F_n = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

satisfy the linear recurrence $F_{n+2} = F_{n+1} + F_n$, but once again, this linear recurrence is not interesting for purposes of shadowing.

**Proposition 2** A shadow of the Fibonacci sequence is the sequence A001629:

$$\varphi_n = 0, 1, 2, 5, 10, 20, 38, 71, 130, 235, 420, 744, 1308, \ldots,$$

which is the convolution of the Fibonacci sequence with itself.

Indeed, consider the Cassini identity

$$F_n F_{n+2} = F_{n+1}^2 - (-1)^n,$$

so called after the first director of the Paris Observatory, Giovanni Domenico Cassini, and substitute into it $F_n = F_n + \varphi_n \varepsilon$. The recurrence for $\varphi_n$ is then similar to (2):

$$\varphi_{n+2} = \frac{2F_{n+1} \varphi_{n+1} - F_n \varphi_n}{F_n}.$$
is to combinatorics what the Fibonacci sequence is to nature, for they appear in hundreds of combinatorial problems. The Catalan numbers obey the following recurrence, which was known already to Euler:

\[ C_{n+1} = C_n C_0 + C_{n-1} + \cdots + C_0 C_0. \]

On taking the sequence of dual numbers

\[ C_n = C_0 C_n + C_1 C_{n-1} + \cdots + C_n C_0, \]

and substituting it into this recurrence, we see that the \( C_0 \)-part is determined by

\[ \gamma_{n+1} = 2(C_0 \gamma_n + C_1 \gamma_{n-1} + \cdots + C_n \gamma_0). \]

I leave it as an exercise to check that choosing the initial conditions \( (\gamma_0, \gamma_1) = (0, 1) \) leads to the sequence A000984 of central binomial coefficients \( \binom{2n}{n} \):

\[ \gamma_n = 0, 1, 2, 6, 20, 70, 252, 924, 3432, 12870, 48620, 184756, \ldots \]

Choosing the initial conditions \( (\gamma_0, \gamma_1) = (1, 0) \) leads to A162551, the sequence of double binomials \( \frac{2^n}{n} \):

\[ \gamma_n = 1, 0, 2, 8, 30, 112, 420, 1584, 6006, 22880, 87516, 335920, \ldots \]

Both sequences and their linear combinations are good candidates for a shadow of the Catalan numbers.

Let me recall some elements of Markov theory. Obviously, \((1, 1, 1)\) is a solution of \((4)\), and a theorem of which Russian mathematicians will always be proud is the following.

**Markov’s theorem.** Every positive integer solution of \((4)\) can be obtained from \((1, 1, 1)\) by a sequence of transformations \( (a, b, c) \mapsto (a', b, c) \), where

\[ a' = \frac{b^2 + c^2}{a}, \]

and permutations of \( a, b, \) and \( c \).

It is clear that transformations of the form \((5)\) will always produce integers. Indeed, it follows directly from \((4)\) that \((5)\) can be rewritten without division as \( a' = 3bc - a \). However, for reasons that I shall not explain, the form \((5)\) is more useful conceptually. It is also easy to check that \((a', b, c)\) remains a solution if \((a, b, c)\) is a solution. The difficult part of the proof is to show that every solution is obtained in this way.

The Markov numbers can be organized with the help of an infinite binary tree. The tree is drawn in the plane, cutting it into infinitely many regions, and every region is labeled by a Markov number. Locally, the picture is this:

![Figure 1](image)

The simplest branches of the Markov tree are those bounded by 1 and 2, the border branches in the figure.
The corresponding triplets of Markov numbers contain the odd Fibonacci numbers \((1, F_{2k-1}, F_{2k+1})\) (A001519) and the odd Pell numbers \((2, P_{2k-1}, P_{2k+1})\) (A001653). Various subsequences and arrangements of the Markov numbers are registered in the OEIS as dozens of entries; see A002559 and related sequences.

**The Shadow of Andrey Andreyevich Markov**

My ultimate goal is to do anything but cast a shadow over the Markov numbers, but it is tempting to apply the general shadowing construction to them. It goes very simply: choose the initial triplet of dual numbers \((A_1, B_1, C_1)\) with

\[
A_1 = 1 + a_1 \varepsilon, \quad B_1 = 1 + b_1 \varepsilon, \quad C_1 = 1 + c_1 \varepsilon,
\]

where \(a_1, b_1, c_1\) are arbitrary integers, and then apply all possible sequences of the transformations \((A, B, C) \mapsto (A', B, C)\), with

\[
A' = \frac{B^2 + C^2}{A},
\]

mixed with permutations of \(A, B,\) and \(C\). On collecting the terms with and without \(\varepsilon\), we obtain

\[
da' = \frac{b^2 + c^2}{a}, \quad A' = \frac{2b \beta + 2c \gamma - da'}{a}.
\]

But this time, it is not at all obvious why \(a'\) should remain an integer.

**Proposition 3.** For an arbitrary choice of the initial conditions (6), the transformations (7), mixed with permutations, produce integer sequences \((a_n)_{n \in \mathbb{N}}, (\beta_n)_{n \in \mathbb{N}},\) and \((\gamma_n)_{n \in \mathbb{N}}\).

This integrality persists thanks to a “miracle,” called the Laurent phenomenon [2], the same miracle that guarantees the integrality of the Somos and Gale–Robinson sequences. The complete proof is too technical to be reproduced here.

There is one choice of initial values \((a_1, \beta_1, \gamma_1)\) that seems to be natural and interesting:

\[
(a_1, \beta_1, \gamma_1) = (0, 1, 1).
\]  

Let me explain. First, it is natural to take the initial values symmetric in the second and third entries, that is, to assume \(\beta_1 = \gamma_1\). Choosing \((a_1, \beta_1, \gamma_1) = (1, 1, 1)\) would produce the Markov numbers again, in the sense that the \(\varepsilon\)-part would coincide with the classical sequence:

\[
a_n = a_n, \quad \beta_n = b_n, \quad \gamma_n = c_n,
\]

which is not very interesting. Choosing \((a_1, \beta_1, \gamma_1) = (1, 0, 0)\) would produce negative numbers. The choice (8) is the only one remaining. Figure 2 shows the tree of Markov numbers together with their shadow.

Let us end with a few observations. The sequence

\[1, 4, 13, 40, 120, 354, 1031, 2972, 8495, \ldots,\]

which appears as a companion of the odd Fibonacci branch, turns out to be a known sequence. It is the delightful sequence A238846, which is the convolution of two bisections of the Fibonacci sequence, \(F_{2n+1}\) and \(P_{2n}\). Other subsequences obtained from the tree by following its branches appear to be new.

What is the role of the shadow Markov tree? Does it mean something? anything? I would be glad if I could solve this riddle, but at this stage, I can only say, along with Winnie the Pooh:

*Cottleston, Cottleston, Cottleston Pie,*

*A fish can’t whistle and neither can I."

*Ask me a riddle and I reply:

"Cottleston, Cottleston, Cottleston Pie."*

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