Local normality of infravacua and relative normalizers for relativistic systems

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Abstract

We revisit the problem of local normality of Kraus-Polley-Reents infravacuum representations and provide a straightforward proof based on the Araki-Yamagami criterion. We apply this result to the theory of superselection sectors. Namely, we extend the novel formalism of second conjugate classes and relative normalizers to the local relativistic setting.

Keywords: Infrared problems, quasifree states, Shale-Stinespring theory.

1 Introduction

The infrared problem in QFT is a maze of difficulties caused by massless particles. In the algebraic approach one aspect of this problems is a multitude of superselection sectors differing by soft photon clouds which escape detection. It is therefore natural to group these sectors into equivalence classes and several definitions of such charge classes are available in the literature [Bu82, BR14, CD19]. The most recent approach from [CD19], based on a novel concept of the relative normalizer (see formula (2.3) below), concerns the structure of the group of automorphisms Aut(Ω) of the C*-algebra Ω of observables. A restrictive aspect of this group theoretic approach is that all the relevant representations of Ω must be expressible as automorphisms in the defining (‘vacuum’) representation. The conventional Kraus-Polley-Reents (KPR) infravacuum representation [Re74, KPR77, Kr82], which describes a background radiation blurring the soft photon clouds, is immediately expressible by an automorphism in a non-relativistic case considered in [CD19]. However, in the relativistic setting it is not obvious if an associated automorphism exists and one may wonder if the
group theoretic formalism of \[\text{CD19}\] generalizes to this context. As we show in Section 2, using a result of Takesaki \[\text{Ta70}\], this is actually the case, provided that the infravacuum representation is locally normal.

For us this by itself is sufficient motivation to revisit the problem of local normality of KPR representations of the massless scalar free field. This property is actually claimed by Kunhardt in \[\text{Ku98, Proposition 3.4}\], but only some hints for the proof are given with a reference for details to an unpublished work of F. Hars. However, we are not going to reconstruct the strategy indicated in this reference as it is based on the phase space condition \(C \# \text{BP90}\). Firstly, to our knowledge, this condition has only been verified for free scalar fields but not e.g. for free electromagnetism. This would suffice for the present paper, but not for planned generalizations. Secondly, and more importantly, condition \(C \#\) is not expected to hold for unbounded regions such us, e.g., future lightcones. Yet we consider lightcone normality of infravacuum representations an important question for future research. It is relevant, in particular, for exemplifying the abstract constructions of Buchholz and Roberts from \[\text{BR14}\]. We remark that a much simpler question of lightcone normality of coherent states has been settled only recently in \[\text{CD20}\].

In this paper we aim for a more optimal strategy for proving the local normality of KPR representations. This question is related to the well-known Shale-Stinespring problem of unitary implementation of symplectic transformations on Fock space \[\text{Sh62, SS65, Ru78}\], which is nowadays textbook material \[\text{Ar, DG1, HSSS12}\]. However, since we aim for local and not global normality (the latter is actually in conflict with the infravacuum property (2.4)), the symplectic form is effectively degenerate, which excludes the above formulations. A Shale-Stinespring type theorem valid in the degenerate case was proven by Araki and Yamagami \[\text{AY82}\] and we will rely on this result here. Actually, the same route was taken in several other investigations of local normality in scalar free field theory on flat and curved spacetime, e.g. for Hadamard states \[\text{Ve94}\] and for certain infravacua in the two-dimensional massless case \[\text{BFR21}\].

Our paper is organized as follows: In Section 2 we demonstrate that local normality allows to generalize the formalism of relative normalizers and second conjugate classes from \[\text{CD19}\] to the relativistic framework. In Section 3 we describe a class of quasi-free representations of the massless scalar field, which are given by symplectic transformations \(T\). We list conditions on \(T\) which imply the irreducibility and infravacuum property of these representations. In Section 4 we use a result of Araki and Yamagami \[\text{AY82}\] to formulate conditions on \(T\) which guarantee local normality of the resulting representation. These conditions are verified in Section 5 in the case of the KPR infravacuum map. Given observations from Section 2 this concludes a construction of a non-trivial relative normalizer in the local relativistic case.

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2 Relative normalizers for relativistic systems

We focus here on the most recent approach to building equivalence classes of sectors \[\text{[CD19]},\] which can be explained in very general terms: Let \( \mathfrak{A} \) be a C\(^\ast\)-algebra and \( G := \text{Aut}(\mathfrak{A}) \) its automorphism group. \( G \) acts on the set of sectors \( X \), i.e., orbits of pure states under the action of the group of inner automorphisms. Given a distinguished vacuum sector \( x_0 \in X \), the second conjugate class of \( x = x_0 \cdot g_x \), \( g_x \in G \), w.r.t. a background \( a \in G \) is given by

\[
[\bar{x}]^a := [x]_{a^{-1} \cdot G_{x_0} \cdot a},
\]

where \( G_{x_0} \) is the stabilizer group of \( x_0 \) and the r.h.s. of (2.1) denotes the orbit of \( x \) under \( a^{-1} \cdot G_{x_0} \cdot a \). As discussed in \[\text{[CD19]},\] this definition is motivated by conventional superselection theory, where the conjugation is involutive.

The soft photon clouds in this setting are sectors of the form \( x_0 \cdot s \), \( s \in S \), where the subgroup \( S \subset G \) is not contained in \( G_{x_0} \), that is, \( x_0 \cdot s \neq x_0 \) for some \( s \in S \). The second conjugate class (2.1) serves its purpose, if the background \( a \) is chosen in such a way, that

\[
[x_0 \cdot s]^a = [x_0]^a.
\]

A convenient sufficient condition is that \( a \) is an element of the relative normalizer \[\text{[CD19]}\]

\[
N_G(R, S) := \{ g \in G | g \cdot S \cdot g^{-1} \subset R \},
\]

where \( R \subset G_{x_0} \) is a subgroup. (We drop here the assumption \( R \subset S \) from \[\text{[CD19]}\] as it is not needed for relation (2.2)).

A search for suitable backgrounds, i.e., elements of \( N_G(R, S) \), naturally leads to infravacuum representations \( \pi : \mathfrak{A} \rightarrow B(\mathcal{H}) \). By definition, they satisfy

\[
\pi \cdot s \simeq \pi, \quad s \in S,
\]

where \( \simeq \) denotes unitary equivalence. In the non-relativistic setting of \[\text{[CD19]}\] the KPR infravacuum representation has the form \( \pi = \pi_{\text{id}} \circ \alpha \), where \( \pi_{\text{id}} \) is the defining vacuum representation and \( \alpha \) is an automorphism of \( \mathfrak{A} \). From this and (2.4) we immediately get \( \alpha \cdot s \cdot \alpha^{-1} = \text{Ad}(U) \) for some unitary \( U \) on the vacuum Hilbert space, hence \( \alpha \) belongs to the relative normalizer (2.3).

As mentioned in the Introduction, in the relativistic setting the KPR infravacuum representations are not immediately expressible by automorphisms. However, it turns out that representations of local nets of von Neumann algebras are closely related to automorphisms provided that they are \textit{locally normal}. This is a content of a theorem by Takesaki \[\text{[Ta70, Theorem 12]},\] which we now recall in a form adapted to our situation: Let \( \mathfrak{A} \) be the global C\(^\ast\)-algebra of a net \( \mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}) \) of infinite dimensional von Neumann algebras, labelled by open, bounded regions \( \mathcal{O} \subset \mathbb{R}^4 \), satisfying isotomy and locality. In addition, we assume the split property, that is for any open, bounded region \( \mathcal{O}_1 \) there is another open, bounded region \( \mathcal{O}_2 \) and a type I factor \( \mathcal{R} \) s.t.

\[
\mathfrak{A}(\mathcal{O}_1) \subset \mathcal{R} \subset \mathfrak{A}(\mathcal{O}_2).
\]
Given such structure, we say that a representation \( \pi \) of \( \mathfrak{A} \) is \textbf{locally normal} if it is \( \sigma \)-weakly continuous on each local subalgebra \( \mathfrak{A}(O) \). There holds the following:

**Theorem 2.1.** [Ta70, Corollary 13] Let \( \mathfrak{A} \) be as above. Suppose its defining representation \( \pi_{\text{id}} \) acts irreducibly on a separable Hilbert space. Let \( \pi \) be an irreducible, locally normal representation of \( \mathfrak{A} \) on a separable Hilbert space. Then there exists an automorphism \( \alpha \) s.t.

\[
\pi_{\text{id}} \cdot \alpha \simeq \pi. \quad (2.6)
\]

We remark that the proper sequential funnel of type \( I_\infty \) factors in \( \mathfrak{A} \), assumed in [Ta70], is readily constructed from the factors \( \mathcal{R} \) in (2.5). The properness condition [Ta70, Definition 6] is verified using isotony and locality as well as the fact that the relative commutant of type I factors is type I. This follows from [Ta, p.300] and the fact that a type \( I_\infty \) factor is quasi-equivalent to \( B(\mathcal{H}) \).

Now suppose that \( \pi \) from Theorem 2.1 satisfies in addition the infravacuum property (2.4). Then we have for some unitaries \( U, U', U'' \) on \( \mathcal{H} \)

\[
\pi_{\text{id}} \cdot \alpha \cdot s = \text{Ad}(U) \cdot \pi_{\text{id}} \cdot s = \text{Ad}(U') \cdot \pi = \text{Ad}(U'') \cdot \pi_{\text{id}} \cdot \alpha. \quad (2.7)
\]

Since \( \pi_{\text{id}} \) is the defining representation, this means \( \alpha \cdot s \cdot \alpha^{-1} = \text{Ad}(U'') \). Thus we obtain:

**Corollary 2.2.** Let \( \pi, \pi_{\text{id}} \) be as in theorem (2.1) and, in addition, \( \pi \) be an infravacuum representation in the sense of (2.4). Then the automorphism \( \alpha \) of (2.6) is an element of the relative normalizer \( N_{\mathcal{G}}(\mathcal{R}, S) \) of (2.3).

### 3 Symplectic maps and quasi-free representations

Let us introduce a vector space \( L := D(\mathbb{R}^3; \mathbb{C}) \oplus D(\mathbb{R}^3; \mathbb{C}) \), whose elements are pairs of functions \( G = (G_1, G_2) \). We equip it with the symplectic form

\[
\sigma(G, G') = \int_{\mathbb{R}^3} (\overline{G_1}G'_2 - \overline{G_2}G'_1) dx. \quad (3.1)
\]

Now let \( \mu(k) := |k| \) and consider the vector spaces \( \mathcal{L}_1 := \mu^{-1/2}\hat{D}(\mathbb{R}^3; \mathbb{C}), \mathcal{L}_2 := \mu^{1/2}\hat{D}(\mathbb{R}^3; \mathbb{C}) \), where hat denotes the Fourier transform. We denote elements of \( \mathcal{L} := \mathcal{L}_1 \oplus \mathcal{L}_2 \) by \( F = (F_1, F_2) \) and define a symplectic form on \( \mathcal{L} \) by extending \( \sigma \) to \( L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \). We note that \( \sigma(F, F') = \sigma(G, G') \) thus the mapping

\[
L \ni (G_1, G_2) \mapsto (\mu^{-1/2}G_1, \mu^{1/2}G_2) \in \mathcal{L}, \quad (3.2)
\]

preserves the symplectic form. The subspaces of \( L \) and \( \mathcal{L} \), determined by \( G_1, G_2 \) supported in a ball \( O_r \) of radius \( r > 0 \), centered at zero, will be denoted by \( L_r, \mathcal{L}_r \).

Now define two complex-linear maps

\[
T_1 : \mathcal{L}_1 \to L^2(\mathbb{R}^3), \quad T_2 : \mathcal{L}_2 \to L^2(\mathbb{R}^3) \quad \text{s.t.} \quad \langle T_1F_1, T_2F_2 \rangle = \langle F_1, F_2 \rangle, \quad (3.3)
\]
where \( \langle \cdot, \cdot \rangle \) is the scalar product in \( L^2(\mathbb{R}^3) \). Consequently, \( T : \mathcal{L} \mapsto L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \) given by \( T(F_1, F_2) = (T_1 F_1, T_2 F_2) \) is a symplectic map. We also require, that \( T_1, T_2 \) commute with complex conjugation in configuration space, as this will be needed in (3.15) below.

Now we impose the infravacuum property on this map. We introduce the following subspace of the algebraic dual \( \mathcal{L}^* \) of \( \mathcal{L} \):

\[
\mathcal{L}^*_S := \mu^{-3/2} \chi(\mu) C^\infty_{\text{sym}}(S^2),
\]

(3.4)

where \( \chi \) is the sharp characteristic function of some fixed interval containing zero and \( C^\infty_{\text{sym}}(S^2) \) denotes smooth, real-valued functions on the sphere, symmetric under \( k \mapsto -k \). (Due to this later property, these functions are invariant under complex conjugation in configuration space). We say that the map \( T \) has the **infravacuum property w.r.t.** \( \mathcal{L}^*_S \) if for any \( v \in \mathcal{L}^*_S \) there exists an element of \( L^2(\mathbb{R}^3) \), which we denote \( T_1 v \), s.t.

\[
\langle v, F_2 \rangle = \langle T_1 v, T_2 F_2 \rangle \text{ for all } F_2 \in \mathcal{L}_2.
\]

(3.5)

We note that the \( L^2 \)-pairing on the l.h.s. of (3.5) is well defined and (3.5) extends relation (3.3).

Now let \( \mathcal{F} \) be the symmetric Fock space and denote the usual creation and annihilation operators by \( a^*, a \) and the Fock space vacuum by \( \Omega \). For any \( G = (G_1, G_2) \) consider the scalar quantum field and canonical momentum in a representation specified by \( T \):

\[
\phi_T(G_1) := \frac{1}{\sqrt{2}} \left( a^* (T_1 \mu^{-1/2} \widehat{G}_1) + a (T_1 \mu^{-1/2} \widehat{G}_1) \right),
\]

(3.6)

\[
\pi_T(G_2) := \frac{1}{\sqrt{2}} \left( a^* (i T_2 \mu^{1/2} \widehat{G}_2) + a (i T_2 \mu^{1/2} \widehat{G}_2) \right),
\]

(3.7)

\[
\Phi_T(G) := \phi_T(G_1) + \pi_T(G_2).
\]

(3.8)

The case \( T = \text{id} \), which reproduces the usual (vacuum) representation will be indicated by dropping the index \( T \). We introduce the local von Neumann algebra, corresponding to a double cone \( \mathcal{O}_r \), whose base is the ball \( O_r \),

\[
\mathfrak{A}(\mathcal{O}_r) := \{ e^{i \Phi(G)} \mid G \text{ real-valued, } \text{supp}(G) \subset O_r \}'',
\]

(3.9)

and the global \( C^* \)-algebra \( \mathfrak{A} := \bigcup_{r > 0} \mathfrak{A}(\mathcal{O}_r) \). The algebras \( \mathfrak{A}(\mathcal{O}) \), corresponding to arbitrary open bounded regions \( \mathcal{O} \subset \mathbb{R}^4 \) are now obtained in a standard manner [Bo00]. It is well known that this net of algebras satisfies properties listed above Theorem 2.1, in particular the split property [BW86][BJ87].

We consider a representation \( \pi_T : \mathfrak{A} \to \mathcal{B}(\mathcal{F}) \) defined by

\[
\pi_T(e^{i \Phi(G)}) = e^{i \Phi_T(G)}, \quad G \in D(\mathbb{R}^3; \mathbb{R}) \oplus D(\mathbb{R}^3; \mathbb{R}).
\]

(3.10)

We recall that \( \pi_T \) is **irreducible** if

\[
\{ T_1 F(G)_1 + i T_2 F(G)_2 \mid G \in D(\mathbb{R}^3; \mathbb{R}) \oplus D(\mathbb{R}^3; \mathbb{R}) \} = L^2(\mathbb{R}^3),
\]

(3.11)
where $F$ is defined in (3.2) [Ku98, Section 3.1]. To state the infravacuum property for these representations, we introduce the coherent automorphisms of $\mathfrak{A}$ by extending the relation

$$\alpha_v(e^{i\Phi(G)}) = e^{-i\sigma((v,0),F(G))}e^{i\Phi(G)}, \quad G \in D(\mathbb{R}^3;\mathbb{R}) \oplus D(\mathbb{R}^3;\mathbb{R}),$$

(3.12)

for $v \in L^*_S$. (Here we could write $\sigma((v,0),F(G)) = \langle v, F(G) \rangle$, by analogy with (3.1), since the $L^2$-pairing between elements of $L^*_S$ and $L_2$ is well defined). We note a simple lemma which is implicit in [Ku98]:

**Lemma 3.1.** Suppose that $T$ has the infravacuum property w.r.t. $L^*_S$. Then $\pi_T$ has the infravacuum property w.r.t. $S := \{ \alpha_v | v \in L^*_S \}$, i.e.,

$$\pi_T \cdot s \simeq \pi_T, \quad s \in S,$$

(3.13)

where $\simeq$ denotes the unitary equivalence. Furthermore, $\pi_T$ is not unitarily equivalent to the defining representation $\pi_{id}$.

**Proof.** For $v \in L^*_S$ the automorphism $\alpha_v$ is defined as in (3.12). We have, by the infravacuum property of $T$,

$$\pi_T \circ \alpha_v = \text{Ad} U_v \circ \pi_T,$$

(3.14)

where $U_v := e^{\frac{1}{\sqrt{2}}(a^*(T_1v) + a(T_1v))}$ is a unitary on $\mathcal{F}$. This follows from the computation

$$\pi_T \circ \alpha_v(e^{i\Phi(G)}) = e^{-i\sigma(T(v,0),TF(G))}e^{i\Phi_T(G)} = U_v \pi_T(e^{i\Phi(G)})U_v^*, $$

(3.15)

which uses the CCR and the infravacuum property of $T$ defined in (3.5). Now suppose that $\pi_{id} = \text{Ad} U \circ \pi_T$ for some unitary $U$. Then, by (3.14),

$$\pi_{id} \circ \alpha_v = \text{Ad} U \circ \pi_T \circ \alpha_v = \text{Ad}(U U_v) \circ \pi_T = \text{Ad}(UU_v U^*) \circ \pi_{id} $$

(3.16)

This is a contradiction, since $\pi_{id} \circ \alpha_v$, is disjoint from $\pi_{id}$ for some non-zero $v$ (cf. e.g. [Ku98,CD19]). □

### 4 Local normality of quasi-free representations

The map $T$ in this section satisfies relation (3.3) and commutes with complex conjugation in configuration space. We do not require here the infravacuum (3.5) or the irreducibility property (3.11). We will justify the following criterion for local normality:

**Theorem 4.1.** Fix $r > 0$ and let $\chi_r \in D(\mathbb{R}^3;\mathbb{R})$ be an approximate characteristic function\footnote{$\chi_r$ should be equal to one on $O_r$ and vanish outside of a slightly larger set.} of $O_r$. Define operators $\chi_{1,r} := \mu^{-1/2}\chi_r \mu^{1/2}$ and $\chi_{2,r} := \mu^{1/2}\chi_r \mu^{-1/2}$, which are bounded by Lemma A.2. Suppose that the following conditions hold:
1. $T_j \chi_{j,r}$ extend from $\mathcal{L}_j$ to bounded operators on $L^2(\mathbb{R}^3)$ and there exists $c_r > 0$ s.t.
\[
c_r \chi_{j,r}^* \chi_{j,r} \leq \chi_{j,r}^* (T_j^* T_j) \chi_{j,r} \leq c_r^{-1} \chi_{j,r}^* \chi_{j,r}, \quad j = 1, 2. \tag{4.1}
\]

2. The following operators are trace class on $L^2(\mathbb{R}^3)$
\[
K_{1,r} := \chi_r \mu^{-1/2} (T_1^* T_1 - 1) \mu^{-1/2} \chi_r, \quad K_{2,r} := \chi_r \mu^{1/2} (T_2^* T_2 - 1) \mu^{1/2} \chi_r. \tag{4.2}
\]

Then $\pi_r$ is $\sigma$-weakly continuous on $\mathfrak{A}(\mathcal{O}_r)$.

We will prove this theorem using a criterion for quasi-equivalence of representations of CCR-algebras due to Araki and Yamagami \cite{AY82}. Thus we define a sesquilinear form on $L$
\[
S_T(G, G') := \langle \Omega, \Phi_T(G)^* \Phi_T(G') \Omega \rangle, \tag{4.3}
\]
which for real-valued $G$ satisfies $\langle \Omega, e^{i\Phi(G)} \Omega \rangle = e^{-\frac{1}{2} S_T(G, G)}$, in accordance with \cite{AY82} Proposition 3.4 (iii)]. We observe, by explicit computations, that condition (1.3) of \cite{AY82} holds true\cite{Footnote1}.
\[
S_T(G, G) \geq 0, \quad S_T(G, G') - S_T(G', G) = i \sigma(G, G'). \tag{4.4}
\]

Next, we define the sesquilinear form
\[
(G | G')_T := S_T(G, G') + S_T(G', \overline{G}) = (\langle G_1, \mu^{-1/2} (T_1^* T_1) \mu^{-1/2} G_1' \rangle + \langle G_2, \mu^{1/2} (T_2^* T_2) \mu^{1/2} G_2' \rangle) \tag{4.5}
\]
and note the following fact:

**Lemma 4.2.** The sesquilinear form $(\cdot | \cdot)_T$ is positive definite.

**Proof.** Clearly, if $(G | G)_T = 0$, both terms on the r.h.s. of (4.5) must vanish. Suppose that
\[
\langle G_1, \mu^{-1/2} (T_1^* T_1) \mu^{-1/2} G_1 \rangle = \| T_1 \mu^{-1/2} G_1 \|_2^2 = 0. \tag{4.6}
\]
Then, the property below (3.3) gives
\[
0 = \langle T_2 \mu^{1/2} G_1, T_1 \mu^{-1/2} G_1 \rangle = \langle G_1, G_1 \rangle = 0. \tag{4.7}
\]
The second term on the r.h.s. of (4.5) is treated analogously. $\square$

Of particular importance for us will be the scalar product $(\cdot | \cdot)$ corresponding to $T = \text{id}$. Using it, we can write
\[
S_T(G, G') = (G | \tilde{S}_T G'), \quad \text{where} \quad \tilde{S}_T = \left( \begin{array}{cc} \mu^{1/2} T_1^* T_1 \mu^{-1/2} & i \mu \\ -i \mu^{-1} & \mu^{-1/2} T_2^* T_2 \mu^{1/2} \end{array} \right). \tag{4.8}
\]

Now we state the criterion of Araki-Yamagami in a form adapted to our problem.

\footnote{The origin of the imaginary unit on the r.h.s. can be seen by comparing our Weyl relations $e^{i \Phi_T(G)} e^{i \Phi_T(G')} = e^{-\frac{1}{2} \pi(G, G')} e^{i \Phi_T(G + G')}$ with \cite{AY82} Proposition 3.4 (ii)].}
Theorem 4.3. \cite{AY82} Fix \( r > 0 \). The representation \( \pi_T \) is \( \sigma \)-weakly continuous on \( \mathfrak{A}(O_r) \) if and only if the following two conditions are satisfied:

1. There is \( C_r > 0 \) s.t. \( C_r^{-1}(G|G) \leq (G|G)_T \leq C_r(G|G) \) for all \( G \in L_r \).

2. \( \tilde{S}_T^{1/2} - \tilde{S}^{1/2} \) is a Hilbert-Schmidt operator on the Hilbert space \( (L^cpl_r, (\cdot | \cdot)) \).

It is easy to check that assumptions 1., 2. of Theorem 4.1 imply, respectively, conditions 1., 2., in Theorem 4.3. As the case of condition 1. is obvious, we move on to condition 2. By \cite[Appendix B]{Bu74}, it suffices to show that

\[
\tilde{S}_T - \tilde{S} = \begin{pmatrix}
\mu^{1/2}(T_1^* T_1 - 1) & 0 \\
0 & \mu^{-1/2}(T_2^* T_2 - 1) \mu^{1/2}
\end{pmatrix}
\]

is trace class on \( (L^cpl_r, (\cdot | \cdot)) \). This latter property is implied by the trace class property on \( L^2(\mathbb{R}^3) \) of operators \( K_{1,r}, K_{2,r} \) of (4.2). This concludes the proof of Theorem 4.1.

5 Kraus-Polley-Reents infravacuum maps

In this section we apply Theorem 4.1 to prove local normality of infravacuum representations. To define them, we will use the decomposition \( L^2(\mathbb{R}^3) = L^2(\mathbb{R}_+ \otimes L^2(S^2) \) corresponding to spherical coordinates, where the measure of the second factor is normalized to the area of the sphere \( S^2 \).

Definition 5.1. The Kraus-Polley-Reents infravacuum maps \( T_j : L_j \to L^2(\mathbb{R}^3), j = 1, 2 \), are defined as follows:

- We introduce sequences \( \varepsilon_i := 2^{-(i-1)\kappa} \) and \( b_i := \frac{1}{i} \) for \( i = 1, 2, 3, \ldots \).

- We define functions \( \xi_i(|k|) := \frac{\chi_{[\varepsilon_{i+1}, \varepsilon_i]}(|k|)}{|k|^{3/2}} \in L^2(\mathbb{R}_+) \) and their normalized counterparts \( \tilde{\xi}_i(|k|) := \xi_i(|k|)/\|\xi_i\|_{L^2(\mathbb{R}_+)} \).

- We define the orthogonal projections \( Q_i : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \) and \( \tilde{Q}_i : L^2(S^2) \to L^2(S^2) \) given by

\[
Q_i = \langle \tilde{\xi}_i \rangle \langle \tilde{\xi}_i \rangle \otimes \tilde{Q}_i \quad \text{with} \quad \tilde{Q}_i := \sum_{0 \leq \ell \leq |i|} \sum_{m=-\ell}^{\ell} |Y_{\ell m}\rangle \langle Y_{\ell m}|, \quad (5.1)
\]

where \( Y_{\ell m} \) are the spherical harmonics.

- We introduce the complex-linear maps \( T_j : L_j \to L^2(\mathbb{R}^3), j = 1, 2, \)

\[
T_1 := I + s\text{-lim } \sum_{i=1}^{n} (b_i - 1) Q_i, \quad T_2 := I + s\text{-lim } \sum_{i=1}^{n} (\frac{1}{b_i} - 1) Q_i. \quad (5.2)
\]

These maps are well-defined by Lemma A.7 below. We will denote by \( T_{1,n}, T_{2,n} \) the respective approximants.
This definition is fine-tuned in such a way that \( \pi_T \) is an irreducible infravacuum representation w.r.t. the subgroup \( S \) of coherent automorphisms as in Lemma 3.1, [Kn98, CD19]. Thus we can focus on the problem of local normality.

The assumptions of Theorem 4.1 are formulated in terms of \( T_j^*T_j \), \( j = 1, 2 \). They can be expressed as follows as quadratic forms on \( L_j \):

\[
T_1^2 = I + \sum_{i=1}^{\infty} (b_i^2 - 1)Q_i, \quad T_2^2 = I + \sum_{i=1}^{\infty} (b_i^{-2} - 1)Q_i. \tag{5.3}
\]

We note that

\[
\chi_{j,r}Q_i\chi_{j,r}^* = \chi_{j,r}\chi_{j,r}^*Q_i\chi_{j,r}, \tag{5.4}
\]

where \( \chi_{j,r} \) is an approximate characteristic function of \( O_r \) s.t. \( \chi_{j,r} \leq 1 \). We can write

\[
Q_{i,j,r} := \chi_{j,r}Q_i\chi_{j,r}^* = \sum_{0 \leq \ell \leq 1} \sum_{m=-\ell}^{\ell} \chi_{j,r}^{|\tilde{\xi}_i \otimes Y_{\ell m}|}(\tilde{\xi}_i \otimes Y_{\ell m}|\chi_{j,r}^* \tag{5.5}
\]

where \( \xi_{i,\ell m} = \chi_{j,r}^{|\tilde{\xi}_i \otimes Y_{\ell m}|} \), \( \tilde{\xi}_{i,\ell m} := \frac{\xi_{i,\ell m}}{\|\xi_{i,\ell m}\|} \) and, by Lemma A.1,

\[
\|\xi_{i,\ell m}\|_2^2 \leq C_r \xi_i^2, \quad \|Q_{i,j,r}\| \leq C_r (i + 1)^2 \xi_i^2 \tag{5.6}
\]

Due to these estimates, the following operators

\[
(T_1^2), := I + \sum_{i=1}^{\infty} (b_i^2 - 1)Q_{i,j,r}, \quad (T_2^2), := I + \sum_{i=1}^{\infty} (b_i^{-2} - 1)Q_{i,j,r} \tag{5.7}
\]

are bounded. As they satisfy \( \chi_{j,r}T_j^2\chi_{j,r}^* = \chi_{j,r}(T_j^2)\chi_{j,r} \), we immediately obtain the second inequality in assumption 1. of Theorem 4.1.

As for the first inequality, the case of \( T_2^2 \) is immediate: Since \( (b_i^{-2} - 1) \geq 0 \), we can write

\[
\chi_{2,r}\chi_{2,r}^* \leq \chi_{2,r}(1 + \sum_{i=1}^{\infty} (b_i^{-2} - 1)Q_i)\chi_{2,r}^* = \chi_{2,r}T_2^2 \chi_{2,r}. \tag{5.8}
\]

In the case of \( T_1^2 \) we have \( (b_i^2 - 1) \leq 0 \), thus the above argument does not apply. Instead, we proceed as follows: Fix some \( N \in \mathbb{N} \) and write

\[
\chi_{1,r}T_1^2\chi_{1,r}^* = \chi_{1,r}T_1^2 \chi_{1,r} + \chi_{1,r} \sum_{i=N+1}^{\infty} (b_i^2 - 1)Q_{i,j,r} \chi_{1,r}^*, \tag{5.9}
\]

where \( T_{1,N} \) is the approximant as defined below (5.2). We note that the spectrum of \( T_{1,N}^2 \) can be read off directly from its definition. Thus we can write

\[
T_{1,N}^2 \geq \inf \text{sp}(T_{1,N}^2)I = b_N^2I = N^{-2}I. \tag{5.10}
\]
On the other hand
\[ \left\| \sum_{i=N+1}^{\infty} (b_i^2 - 1)Q_{i,j,r} \right\| \leq \sum_{i=N+1}^{\infty} |b_i^2 - 1| \|Q_{i,j,r}\| \leq C_r \sum_{i=N+1}^{\infty} (i+1)^2 \varepsilon_i^2 \leq C_r' 2^{-N/2}, \tag{5.11} \]

Coming back to (5.9),
\[ \chi_{1,r} T_1^2 \chi_{1,r}^* \geq \left( N^{-2} - C_r' 2^{-N/2} \right) \chi_{1,r} \chi_{1,r}^*. \tag{5.12} \]

Now for any given constant \( C_r' \) we can choose \( N \) s.t. \( N^{-2} - C_r' 2^{-N/2} > 0 \), which concludes our verification of assumption 1. of Theorem 4.1.

To verify assumption 2, we define \( \chi_{1,r}^0 := \chi_r \mu^{-1/2}, \chi_{2,r}^0 := \chi_r \mu^{1/2} \). Analogously as in (5.5), we write
\[ Q_{i,j,r}^0 := \chi_{j,r}^0 Q_i (\chi_{j,r}^0)^* = \sum_{0 \leq \ell \leq \ell_m = -\ell} \| \tilde{\xi}_{i,\ell m}^{j,r,0} \|_2 \langle \tilde{\xi}_{i,\ell m}^{j,r,0} | \tilde{\xi}_{i,\ell m}^{j,r,0} \rangle. \tag{5.13} \]

By items (A.2), (A.4) in Lemma A.1 the estimates of (5.6) hold also in this case, that is,
\[ \| \tilde{\xi}_{i,\ell m}^{j,r,0} \|_2^2 \leq C_r \varepsilon_i^2; \quad \| Q_{i,j,r}^0 \| \leq C_r (i+1)^2 \varepsilon_i^2. \tag{5.14} \]

Thus the operator
\[ K_{1,r} := \chi_{1,r}^0 (T_1^2 - 1)(\chi_{1,r}^0)^* = \sum_{i=1}^{\infty} (b_i^2 - 1)Q_{i,j,r}^0 \tag{5.15} \]
is obviously trace-class on \( L^2(\mathbb{R}^3) \) and the same is true for \( K_{2,r} \).

We summarize our considerations in this paper as follows:

**Theorem 5.2.** Let \( T \) be the KPR map of Definition 5.1. Then the representation \( \pi_T \) is irreducible, locally normal and has the infravacuum property w.r.t. \( S \) defined above (3.15). Thus the automorphism \( \alpha_T \), associated with \( \pi_T \) via (2.6), belongs to the relative normalizer \( N_G(R, S) \), where \( G = \text{Aut}(\mathfrak{A}) \) and \( R = G_{\mathfrak{A}_0} \) is the stabilizer of the vacuum sector.

### A Technical lemmas

**Lemma A.1.** There hold the bounds
\[ \| \mu^{1/2} \chi_r \mu^{-1/2} (\tilde{\xi} \otimes Y_{\ell m}) \|_2 \leq C_r \varepsilon_i, \tag{A.1} \]
\[ \| \chi_r \mu^{-1/2} (\tilde{\xi} \otimes Y_{\ell m}) \|_2 \leq C_r \varepsilon_i, \tag{A.2} \]
\[ \| \mu^{-1/2} \chi_r \mu^{1/2} (\tilde{\xi} \otimes Y_{\ell m}) \|_2 \leq C_r \varepsilon_i^2, \tag{A.3} \]
\[ \| \chi_r \mu^{1/2} (\tilde{\xi} \otimes Y_{\ell m}) \|_2 \leq C_r \varepsilon_i^2. \tag{A.4} \]

for some \( C_r \) independent of \( i, \ell, m \).
Proof. Starting with (A.1), we can write
\[
\|\mu^{1/2} \chi_r \mu^{-1/2} (\xi_i \otimes Y_{tm})\|_2 = \|\mu^{1/2} \chi_r \mu^{-1/2} \chi_{[\varepsilon_{i+1}, \varepsilon_i]} (\xi_i \otimes Y_{tm})\|_2 \\
\leq \|\mu^{1/2} \chi_r \mu^{-1/2} \chi_{[\varepsilon_{i+1}, \varepsilon_i]}\|,
\]
where \( \chi_{[\varepsilon_{i+1}, \varepsilon_i]} \) is the operator of multiplication by the sharp characteristic function \(|k| \mapsto \chi_{[\varepsilon_{i+1}, \varepsilon_i]}(|k|)\) of \([\varepsilon_{i+1}, \varepsilon_i]\) and in the last line the operator norm is understood. We will estimate this norm using the Schur lemma [DG, Section B6]: If \( A \) is an operator and \( a \) its kernel, then \( \|A\| \leq (CC')^{1/2} \) provided that
\[
\sup_k \int |a(k, k')|dk \leq C \quad \text{and} \quad \sup_{k'} \int |a(k, k')|dk \leq C'.
\]
In our case \( a(k, k') = (2\pi)^{-3/2} |k|^{1/2} \hat{x}_r(k-k')|k'|^{-1/2} \chi_{[\varepsilon_{i+1}, \varepsilon_i]}(|k'|). \) We have
\[
\int |a(k, k')|dk' \leq (2\pi)^{-3/2} \frac{1}{\varepsilon_{i+1}^{1/2}} \int_{\mathbb{R}^3} (|k-k'|^{1/2} + 1)|\hat{x}_r(k-k')|\chi_{[\varepsilon_{i+1}, \varepsilon_i]}(|k'|)dk' \\
\leq \frac{c}{\varepsilon_{i+1}^{1/2}} \int_{\varepsilon_{i+1}}^{\varepsilon_i} |k|^2 dk' \leq c' \varepsilon_i^{5/2}.
\]
Now the second integral in (A.6) can be estimated as follows
\[
\int |a(k, k')|dk \leq \frac{c}{\varepsilon_{i+1}^{1/2}} \int_{\mathbb{R}^3} (|k-k'|^{1/2} + 1)|\hat{x}_r(k-k')|\chi_{[\varepsilon_{i+1}, \varepsilon_i]}(|k'|)dk \leq c' \varepsilon_i^{1/2}.
\]
Thus we have \( \|\mu^{1/2} \chi_r \mu^{-1/2} \chi_{[\varepsilon_{i+1}, \varepsilon_i]}\| \leq c' \varepsilon_i \) which gives (A.1). Estimate (A.2) is an immediate consequence, since
\[
\|\chi_r \mu^{-1/2} (\xi_i \otimes Y_{tm})\|_2 = \|\chi_r \mu^{-1/2} \mu^{1/2} \chi_r \mu^{-1/2} (\xi_i \otimes Y_{tm})\|_2 \\
\leq \|\chi_r \mu^{-1/2}\| \|C_r \xi_i\|,
\]
and \( \|\chi_r \mu^{-1/2}\| < \infty \) by Lemma A.2 below.

Let us move on to (A.3). In this case we write
\[
\|\mu^{-1/2} \chi_r \mu^{1/2} (\xi_i \otimes Y_{tm})\|_2 \leq \|\mu^{-1/2} \chi_r\| \|\mu \chi_r \mu^{-1/2} \chi_{[\varepsilon_{i+1}, \varepsilon_i]}\|.
\]
Now we estimate the norm of \( \chi_r \mu^{1/2} \chi_{[\varepsilon_{i+1}, \varepsilon_i]} \) using the Schur lemma. The kernel has now the form \( a'(k, k') = (2\pi)^{-3/2} \chi_r(k-k')|k'|^{1/2} \chi_{[\varepsilon_{i+1}, \varepsilon_i]}(k'). \) We immediately see that
\[
\int |a(k, k')|dk' \leq c \varepsilon_i^{7/2}, \quad \int |a(k, k')|dk \leq c \varepsilon_i^{1/2},
\]
which gives (A.3) and (A.4). \( \square \)

**Lemma A.2.** The operators \( \chi_r \mu^{-1/2} \) and \( \mu^{1/2} \chi_r \mu^{-1/2} \) extend from \( D(\mathbb{R}^3; \mathbb{C}) \) to bounded operators on \( L^2(\mathbb{R}^3) \).
Proof. We refer to [Dy08, Lemma 3.2] for boundedness of $\chi_r \mu^{-1/2}$. As for the second operator, its kernel satisfies

$$|a(k, k')| = (2\pi)^{-3/2}|k|^{1/2}|\hat{\chi}_r(k - k')||k'|-^{1/2} \leq (2\pi)^{-3/2}(||k - k'||^{1/2}|\hat{\chi}_r(k - k')||k'|-^{1/2} + |\hat{\chi}_r(k - k')||).$$  \hspace{1cm} (A.12)

Hence, for any $G, G' \in D(\mathbb{R}^3; \mathbb{C})$ we can write

$$|\langle G, \mu^{1/2}\chi_r \mu^{-1/2}G' \rangle| \leq c(\langle |G|, \tilde{\chi}_r \mu^{-1/2}|G'\rangle) + \|G\|_2\|G'\|_2,$$ \hspace{1cm} (A.13)

where $\tilde{\chi}_r$ acts by convolution with the rapidly decaying function $k \mapsto |k|^{1/2}|\hat{\chi}_r(k)|$. Now boundedness of $\tilde{\chi}_r \mu^{-1/2}$ follows by analogous arguments as boundedness of $\chi_r \mu^{-1/2}$. \hspace{1cm} $\square$

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