THE ALGEBRAIC STRUCTURE OF THE ONSAGER ALGEBRA

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Abstract

We study the Lie algebra structure of the Onsager algebra from the ideal theoretic point of view. A structure theorem of ideals in the Onsager algebra is obtained with the connection to the finite-dimensional representations. We also discuss the solvable algebra aspect of the Onsager algebra through the formal Lie algebra theory.

1 Introduction

In this note, we are going to discuss the algebraic structure of a certain infinite-dimensional Lie algebra which appeared in the seminal work of Onsager of 1944 on the solution of 2D Ising model \([1]\). Due to some other simpler and powerful methods introduced later in the study of Ising model, the algebra in the Onsager’s original work, now called the Onsager algebra, had not received enough attention in the substantial years until the 1980s when its new-found role appeared in the superintegrable chiral Potts model, [2-12] (for a review of the subject, see [3] and references therein). Since the work of Onsager it was known that there exists an intimate relationship between the Onsager algebra and \(sl_2\), a connection now clarified in [8, 14]. Namely, the Onsager algebra is isomorphic to the fixed subalgebra of \(sl_2\)-loop algebra (or alternatively, its central extension \(A^{(1)}_1\), ) by a certain involution. A generalization of the Onsager algebra to other Kac-Moody algebras was later introduced in [13, 15], with some interesting relations with integrable motions. Due to the close relationship of the Onsager algebra with 2D integrable models in statistical mechanics, especially in the chiral Potts model, a thorough mathematical understanding of this infinite-dimensional algebra is desirable to warrant the further investigation. In this note, we summarize our recent results on the algebraic study of the structure of the Onsager algebra. Detailed derivations, as well as extended references to the literature, may be found in [16]. We have established the structure theorem of a certain class of (Lie)-ideals of the Onsager algebra, originated from the study of (reducible or irreducible) finite dimensional representations of the Onsager algebra, through the theory of the

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The DG-condition on \( A \) with \( \{ A, Z \} = 2 \) forms the Onsager algebra, which means they satisfy the Onsager relations, and the chiral angles, which was studied by Howes, Kadanoff and M. den Nijs [11]. Note that the Potts model is given by reciprocal polynomials. In the process, we have found a profound structure of the Onsager algebra with some interesting applications to solvable and nilpotent Lie algebras. The mathematical results obtained would be expected to have some feedback to the original physical theory. Such a program is now under progress and partial results are promising.

### 2 Hamiltonian of Superintegrable Chiral Potts Model

The superintegrable chiral Potts \( N \)-state spin chain Hamiltonian has the following form of a parameter \( k' \),

\[
H(k') = H_0 + k'H_1,
\]

with \( H_0, H_1 \) the Hermitian operators acting on the vector space of \( L \)-tensor of \( \mathbb{C}^N \), defined by

\[
H_0 = -2 \sum_{l=1}^{L} \sum_{n=1}^{N-1} (1 - \omega^{-n})X_{l}^{n}, \quad H_1 = -2 \sum_{l=1}^{L} \sum_{n=1}^{N-1} (1 - \omega^{-n})Z_{l}^{n}Z_{l+1}^{N-n},
\]

where \( \omega = e^{i\pi/3}, X_l = I \otimes \ldots \otimes X \otimes \ldots \otimes I, Z_l = I \otimes \ldots \otimes Z \otimes \ldots \otimes I, (Z_{L+1} = Z_1) \). Here \( I \) is the identity operator, and \( X, Z \) are the operators of \( \mathbb{C}^N \) with the Weyl commutation relation, \( ZX = \omega XZ \), which are defined by \( X|m > = |m+1 >, Z|m > = \omega^m|m >, m \in \mathbb{Z}_N \). The operator \( H(k') \) is Hermitian for real \( k' \), hence with the real eigenvalues. When \( N = 2 \), \( X, Z \), become the Pauli matrices, \( \sigma^1, \sigma^3 \), then it is the Ising quantum chain [17]:

\[
-H(k') = \sum_{l=1}^{L} \sigma_{l}^{1} + k' \sum_{l=1}^{L} \sigma_{l}^{3} \sigma_{l+1}^{3}.
\]

For \( N = 3 \), one obtains the \( \mathbb{Z}_3 \)-symmetrical self-dual chiral clock model with the chiral angles \( \varphi = \phi = \frac{\pi}{3}, \)

\[
-\frac{\sqrt{3}}{2}H(k') = \sum_{l=1}^{L} (e^{\frac{i\pi}{3}}X_{l} + e^{\frac{i\pi}{3}}X_{l}^{2}) + k' \sum_{l=1}^{L} (e^{\frac{2i\pi}{3}}Z_{l}Z_{l+1}^{2} + e^{\frac{2i\pi}{3}}Z_{l}^{2}Z_{l+1})
\]

which was studied by Howes, Kadanoff and M. den Nijs [11]. Note that the Potts model is given by the chiral angles, \( \varphi = \phi = 0 \). For a general \( N \), \( H(k') \) was constructed in a paper of G. von Gehlen and R. Rittenberg [10], in which the models were shown to be ”superintegrable” in the sense that the Dolan-Grady (DG) condition is satisfied for \( A_0 = -2N^{-1}H_0, A_1 = -2N^{-1}H_0, \)

\[
[A_1, [A_1, [A_1, A_0]]] = 16[A_1, A_0], \quad [A_0, [A_0, [A_0, A_1]]] = 16[A_0, A_1].
\]

For a pair of operators, \( A_0, A_1 \), we denote, \( 4G_1 = [A_1, A_0] \), and define an infinite sequence of operators, \( A_m, G_m, (m \in \mathbb{Z}) \), by relations,

\[
A_{m-1} - A_{m+1} = \frac{1}{2}[A_{m}, G_{1}], \quad G_{m} = \frac{1}{4}[A_{m}, A_{0}].
\]

The DG-condition on \( A_1, A_0 \), are equivalent to the statement that the collection of \( A_m, G_m \) forms the Onsager algebra, which means they satisfy the Onsager relations,

\[
[A_m, A_l] = 4G_{m-l}, \quad [A_m, G_l] = 2(A_{m-l} - A_{m+l}), \quad [G_m, G_l] = 0.
\]
The above relations ensure that the \( k' \)-dependence eigenvalues of \( H(k') \) have the following special form as in the Ising model,

\[
a + bk' + 2N \sum_{j=1}^{n} m_j \sqrt{1+k'^2-2k'\cos(\theta_j)} ,
\]

where \( a, b, \theta_j \), are reals, and \( m_j \)'s take the values, \( m_j = -s_j, (-s_j+1), \ldots, s_j \) with \( s_j \) a positive half-integer. Mathematically, this result follows from the classification of the finite-dimensional unitary representations of the Onsager algebra, by the relationship of the algebra with the loop algebra of \( sl_2 \), \( L(sl_2) := \mathbb{C}[t, t^{-1}] \otimes sl_2 \), which was identified in [14] as follows.

**Definition.** The Onsager algebra, denoted by \( OA \), is defined as one of the following equivalent conditions:

(\( i \)) \( OA \) = the universal Lie algebra generated by two elements, \( A_0, A_1 \), with the DG-condition.

(ii) \( OA \) = the Lie-subalgebra of \( L(sl_2) \) fixed by the involution \( \hat{\theta} \),

\[
\hat{\theta} : p(t) \mapsto p(t^{-1}) , \quad e \mapsto f , \quad f \mapsto e , \quad h \mapsto -h 
\]

where \( p(t) \in \mathbb{C}[t, t^{-1}] \), \( e, f, h \) are the standard basis of \( sl_2 \) with \( [e, f] = h \), \( [h, e] = 2e \), \( [h, f] = -2f \).

We shall always make the above identification in what follows. For an element \( X \) in \( L(sl_2) \), the criterion of \( X \) in \( OA \) is now given by

\[
X \in OA \iff X = p(t)e + p(t^{-1})f + q(t)h , \quad \text{with} \quad q(t) + q(t^{-1}) = 0
\]

where \( p(t), q(t) \in \mathbb{C}[t, t^{-1}] \). In fact, \( q(t) \) can always be written in the form, \( q(t) = q_+(t) - q_+(t^{-1}) \) with \( q_+(t) \in \mathbb{C}[t] \).

### 3 Closed Ideals of the Onsager Algebra

A (non-trivial Lie) ideal \( l \) of a Lie algebra \( L \) (over \( \mathbb{C} \)) is called a closed ideal if it satisfies the following condition,

\[
l = \{ x \in L \mid [x, L] \subset l \},
\]

or equivalently, \( L/l \) has the trivial center. By Schur’s lemma, the kernel ideal of an irreducible representation of \( L \) in \( sl_n(\mathbb{C}) \) is always closed, which constitutes an important class of closed ideals. In this section, we shall describe the classification of closed ideals of the Onsager algebra, \( OA \).

**Definition.** (i) Let \( P(t) \) be a non-trivial monic polynomial in \( \mathbb{C}[t] \). We call \( P(t) \) a reciprocal polynomial if \( P(t) = \pm t^dP(t^{-1}) \), where \( d \) is the degree of \( P(t) \).

(ii) For a reciprocal polynomial \( P(t) \), \( l_{P(t)} \) is the ideal of \( OA \) defined by

\[
l_{P(t)} := \{ X = p(t)e + p(t^{-1})f + q(t)h \in OA \mid p(t), q(t) \in P(t)\mathbb{C}[t, t^{-1}] \}.
\]

We shall call \( P(t) \) the generating polynomial of the ideal \( l_{P(t)} \). \( \Box \)
It is easy to see that zeros of a reciprocal polynomial $P(t)$ not equal to $\pm 1$ must occur in the reciprocal pairs, and $I_{P(t)}$ is invariant under the involution of $\mathcal{O}A$, $A_m \mapsto A_{-m}, G_m \mapsto G_{-m}$. Through the Chinese remainder theorem, one can establish the a canonical (Lie-)isomorphism of the quotient algebras,

$$\mathcal{O}A/I_{P(t)} \sim \prod_{j=1}^{J} \mathcal{O}A/l_{P_j(t)}, \quad P(t) := \prod_{j=1}^{J} P_j(t)$$

where $P_j(t)$ are pairwise relatively prime reciprocal polynomials. The role of reciprocal polynomials in the study of the Onsager algebra is given by the following structure theorem of closed ideals of $\mathcal{O}A$.

**Theorem 1.** An ideal $I$ of $\mathcal{O}A$ is closed if and only if $l = I_{P(t)}$ for a reciprocal polynomial $P(t)$ whose zero at $t = \pm 1$ are of the even multiplicity. The ideal $I_{P(t)}$ is characterized as the minimal closed ideal of $\mathcal{O}A$ containing $P(t)e + P(t^{-1})f$. □

For an irreducible special representation of $\mathcal{O}A$ on a finite dimensional vector space $V$, $\rho : \mathcal{O}A \rightarrow sl(V)$, the generating polynomial of the kernel $\text{Ker}(\rho)$ is given by $P(t) = \prod_{j=1}^{n} U_a(t)$, where $a_j \in \mathbb{C}^* - \{\pm 1\}, a_j \neq \pm 1$ for $j \neq i$, and $U_a(t) := (t - a)(t - a^{-1})$. In this situation, the evaluation morphism of $\mathcal{O}A$, induced from $L(sl_2)$, into the sum of $n$ copies of $sl_2$,

$$e_{a_1,...,a_n} : \mathcal{O}A \rightarrow \bigoplus_{i=1}^{n} sl_2, \quad X \mapsto (e_{a_1}(X),...,e_{a_n}(X)),$$

gives rise the isomorphism between $\mathcal{O}A/\text{Ker}(\rho)$ and $\bigoplus_{i=1}^{n} sl_2$. Hence irreducible representations of the $sl_2$-factors determine an irreducible representation of $\mathcal{O}A$.

## 4 Completion of the Onsager Algebra and Solvable Lie Algebras

By the previous discussion, the study of closed ideals in $\mathcal{O}A$ can be reduced to the case, $l = I_{P(t)}$ with $P(t) = (t \pm 1)^L$ or $U_a(t)^L$ for $L \in \mathbb{Z}_{\geq 0}$, $a \in \mathbb{C}^* - \{\pm 1\}$. In this report, we shall only discuss the case, $P(t) = (t \pm 1)^L$. As the map, $t \mapsto -t$, gives rise an involution of $\mathcal{O}A$, one has the isomorphism,

$$\mathcal{O}A/I_{(t-1)^L} \simeq \mathcal{O}A/I_{(t+1)^L}.$$ 

We may assume, $P(t) = (t - 1)^L, (L \in \mathbb{Z}_{\geq 0})$. Denote $\pi_L, \pi_{KL}$ the canonical projections,

$$\pi_L : \mathcal{O}A \rightarrow \mathcal{O}A/I_{(t-1)^L}, \quad \pi_{KL} : \mathcal{O}A/I_{(t-1)^L} \rightarrow \mathcal{O}A/I_{(t-1)^K}, \quad L \geq K \geq 0,$$

and $\mathcal{O}\mathcal{A}$ the projective limit of the projective system, $(\mathcal{O}A/I_{(t-1)^L}, \pi_{KL})$. Then $\mathcal{O}\mathcal{A}$ is a Lie algebra and denote the canonical morphism, $\psi_L : \mathcal{O}\mathcal{A} \rightarrow \mathcal{O}A/I_{(t-1)^L}, L \in \mathbb{Z}_{\geq 0}$, with the kernel, $\mathcal{O}\mathcal{A}^L := \text{Ker}(\psi_L)$. We have, $\mathcal{O}\mathcal{A}/\mathcal{O}\mathcal{A}^L \simeq \mathcal{O}A/I_{(t-1)^L}$. We have a filtration of ideals in $\mathcal{O}\mathcal{A}$,

$$\mathcal{O}\mathcal{A} = \mathcal{O}\mathcal{A}^0 \supset \mathcal{O}\mathcal{A}^1 \supset \cdots \supset \mathcal{O}\mathcal{A}^L \supset \cdots.$$ 

There exists a morphism, $\pi : \mathcal{O}A \rightarrow \mathcal{O}\mathcal{A}$, with $\psi_L \pi = \pi_L$. The Lie algebra $\mathcal{O}\mathcal{A}$ is regarded as a completion of $\mathcal{O}A$. For the convenience, we shall write the element $\pi(X)$ of $\mathcal{O}\mathcal{A}$ again by $X$ for $X \in \mathcal{O}A$. The
given by fact, one can express \( \hat{\text{identities among Stirling's numbers}} \) [18], in \( \overline{\mathcal{A}} \) such that the following identities hold in \( \overline{\mathcal{A}} \),

\[
A_m(= \pi(A_m)) = \sum_{k \geq 0} \frac{m(k)}{k!} X_k , \quad G_m(= \pi(G_m)) = \sum_{k \geq 0} (-1)^k \frac{m(k)}{k!} Y_k ,
\]

where \( x^{(n)} \) is the shifted factorial defined by \( x^{(0)} := 1, x^{(n)} := x(x-1) \cdots (x-n+1), n \in \mathbb{Z}_{>0} \). In fact, with the infinite formal sum \( \overline{\mathcal{A}} \) has \( X_k, Y_k \) as the generators of the formal Lie algebra, while \( \overline{\mathcal{A}}^L \) is the ideal generated by \( X_k, Y_k, (k \geq L) \). In \( \mathcal{O} \mathcal{A}/l(1)_L \), one has,

\[
\psi_L(A_m) = \sum_{k=0}^{L-1} \frac{m(k)}{k!} \psi_L(X_k) , \quad \psi_L(G_m) = \sum_{k=0}^{L-1} (-1)^k \frac{m(k)}{k!} \psi_L(Y_k) , \quad m \in \mathbb{Z} .
\]

The relations (1) for \( \mathcal{O} \mathcal{A} \) now become the following relations in \( \overline{\mathcal{A}} \),

\[
\sum_{n,k \geq 0} \frac{a!b!}{n!k!} s_n^a b^k [X_n, X_k] = 4 \sum_{k \geq 0} (-1)^k \frac{(a+b)!}{k!} s_{a+b}^k Y_k ,
\]

\[
\sum_{n,k} (-1)^n \frac{a!b!}{n!k!} s_n^a b^k [Y_n, X_k] = 2(1 - (-1)^a) \sum_k (-1)^k \frac{(a+b)!}{k!} s_{a+b}^k X_k ,
\]

\[
[Y_n, Y_k] = 0 ,
\]

where \( s_n^a, s_n^a, (n, k \in \mathbb{Z}_{\geq 0}) \), are the Stirling numbers, i.e., the integers with the relations, \( x^{(n)} = \sum_{k \geq 0} x^k s_n^k \), \( x^n = \sum_{k \geq 0} x^{(k)} s_n^k \). Note that \( s_n^a = s_n^b = 1 \), and \( s_k^a = 0 \) for \( k > n \). By the identities among Stirling's numbers [18],

\[
\sum_k s_k^a s_k^b = \delta_k^b , \quad \frac{b!}{a!} \sum_k (-1)^k s_n^a s_k^b = (-1)^a \frac{a-1}{b-1} , \quad \frac{a!}{j!} s_j^a = \sum_l \frac{k!(a+l)!}{l!(j+k)!} s_k^j s_{a+l}^j ,
\]

one obtains the commutation relations of \( X_k, Y_k \),

\[
[X_n, X_k] = 4(-1)^n(Y_{n+k} + \sum_{a>0} (a+k-1)Y_{n+k+a}) ,
\]

\[
[Y_n, X_k] = 2((-1)^n - 1)X_{n+k} - \sum_{a>0} (-1)^a (a+n-1)X_{n+k+a} ,
\]

\[
[Y_n, Y_k] = 0 .
\]

By the above relations, \( \overline{\mathcal{A}}^L / \overline{\mathcal{A}}^{L+1} \) is abelian for a positive integer \( L \), hence \( \overline{\mathcal{A}}^{L}/\overline{\mathcal{A}}^{L} \) is a finite-dimensional solvable Lie-algebra. However, there exist certain non-trivial relations among \( Y_k \). In fact, one can express \( Y_{2n} \) in terms of \( Y_{2k+1} \) for \( k \geq n \) in the algebra \( \overline{\mathcal{A}} \); the explicit formula is given by

\[
Y_{2n} = \sum_{k \geq n} (-1)^{k-n+1} \frac{4^{k-n+1} - 1}{k-n+1} B_{k-n+1} (2k-2n+1) Y_{2k+1} ,
\]

where \( B_j (j \geq 1) \) are the Bernoulli numbers, \( B_j = (-1)^j j! 2j+1 \), with \( \frac{x}{e^x-1} = \sum_{j=0}^\infty \frac{b_j}{j!} x^j \). For the study of the Lie algebra structure of \( \overline{\mathcal{A}} \), it is more natural to use a local coordinate system near \( t = 1 \) for expressing elements in \( L(sl_2) \). A convenient variable is, \( \lambda := \frac{t-1}{t} \), near the origin, \( \lambda = 0 \).

The structure of \( \overline{\mathcal{A}} \) can be visualized as a formal subalgebra of \( sl_2[[\lambda]](= \mathfrak{C}[[\lambda]] \otimes sl_2) \) as follows.

**Theorem 2.** Define

\[
sl_2 \langle \langle \lambda \rangle \rangle := \mathfrak{C}[[\lambda^2]]h + \lambda \mathfrak{C}[[\lambda^2]]e + \lambda \mathfrak{C}[[\lambda^2]]f \subset sl_2[[\lambda]],
\]

\[
sl_2 \langle \langle \lambda \rangle \rangle^L := \{ sl_2 \langle \langle \lambda \rangle \rangle \} \cap \lambda^L sl_2[[\lambda]] , \quad (L \in \mathbb{Z}_{\geq 0}) .
\]
There is a formal Lie-algebra isomorphism,
\[ \widehat{OA} \cong \mathfrak{sl}_2(\langle \lambda \rangle) , \]
under which \( \widehat{OA}^L \) is corresponding to \( \mathfrak{sl}_2(\langle \lambda \rangle)^L \). As a consequence,
\[ OA/I_{(t-1)^L} \cong \mathfrak{sl}_2(\langle \lambda \rangle)/\mathfrak{sl}_2(\langle \lambda \rangle)^L . \]
\[ \square \]

By the above Theorem, \( OA/I_{(t-1)^L} \) is a solvable Lie algebra of dimension \( L + [t/2] \), which has a basis consisting of \( \psi_L(X_k), \psi_L(Y_l), 0 \leq k, l < L, l \equiv 1 \pmod{2} \). By the structure of \( OA/I_{(t-1)^L} \), one can easily see that the criterion of trivial center for \( OA/I_{(t-1)^L} \), (or equivalently, the closed ideal for \( I_{(t-1)^L} \)), is given by the integer \( L \) to be even. For odd \( L \), the center of \( OA/I_{(t-1)^L} \) is 1-dimensional. Using the relations,
\[ t - 1 = \lambda - 1 + \sqrt{1+\lambda^2} \quad , \quad t^{-1} - 1 = -\lambda - 1 + \sqrt{1+\lambda^2} , \]
one obtains an explicit expression of \( A_m, G_m \), in terms of \( \lambda \) in the above theorem, in particular,
\[ A_0 = 2(e + f) , \quad A_1 = 2(e - f)\lambda + 2(e + f) \sum_{j \geq 0} \left( \frac{1}{2} \right)^{j} \lambda^{2j} . \]

By taking the expression of \( t^2 + 1 \) modulo \( \lambda^L \), one can construct reducible representations of \( OA \) with the kernel ideal generated by \( (t - 1)^L \).

5 Further Remarks

For the understanding of representations of the Onsager algebra, both the reducible and irreducible ones, the structure of \( OA/I_{P(t)} \) with a reciprocal polynomial \( P(t) \) warrants the mathematical investigation. But, just to keep things simple, we restrict our attention in this present report only to the case, \( P(t) = (t \pm 1)^L, L \geq 1 \), or \( U_a(t), a \neq \pm 1 \). With a similar argument, one can also obtain the structure of quotients for closed ideals generated by \( P(t) = U_a(t)^L, L \in \mathbb{Z}_{\geq 0} \), (for the details, see [14]). The results, together with the mixed types, should have some interesting applications and implications in solvable or nilpotent algebras, which we shall leave to future work. Here is the one example which appeared in [14]:

**Example.** \( OA/I_{(t^2+1)^2} \cong \mathfrak{sl}_2[\varepsilon]/\varepsilon^2 \mathfrak{sl}_2[\varepsilon] \). Denote \( e_k, f_k, h_k, (k = 0, 1) \) the basis elements of \( \mathfrak{sl}_2[\varepsilon]/\varepsilon^2 \mathfrak{sl}_2[\varepsilon] \) corresponding to the standard basis of \( \mathfrak{sl}_2 \) with the index \( k \) indicating the grade of \( \varepsilon \). Define the linear map, \( \varphi : OA \rightarrow \mathfrak{sl}_2[\varepsilon]/\varepsilon^2 \mathfrak{sl}_2[\varepsilon] \), by the expression:
\[ \varphi(A_m) = 2(i^m e_0 + i^{-m} f_0 + m i^{-m-1} e_1 + m i^{m+1} f_1) , \]
\[ \varphi(G_m) = (i^m - i^{-m}) h_0 + m (i^{m-1} - i^{-m-1}) h_1 . \]

Then \( \varphi \) is a surjective morphism. Since \( \mathfrak{sl}_2[\varepsilon]/\varepsilon^2 \mathfrak{sl}_2[\varepsilon] \) has only trivial center, \( \text{Ker}(\varphi) \) is a closed ideal with \( (t^2 + 1)^2 \) as the generating polynomial. \( \square \)

Generalizations of the Onsager algebra to other loop algebras or Kac-Moody algebras as in [13, 14] should provide ample examples of solvable algebras of the above type. We hope that the further development of the study of the subject will eventually lead to some interesting results in Lie-theory with possible applications in quantum integrable models.
References

[1] L. Onsager: Phys. Rev. 65 (1944) 117.

[2] G. Albertini, B. M. McCoy, J. H. H. Perk, and S. Tang: Nucl. Phys. B 314 (1989) 741.

[3] G. Albertini, B. M. McCoy, and J. H. H. Perk: Phys. Lett. A 135 (1989) 159; Phys. Lett. A 139 (1989) 204; Adv. Stud. Pure Math., vol. 19, Kinokuniya Academic 1989.

[4] H. Au-Yang, B. M. McCoy, J. H. H. Perk, S. Tang, and M. L. Yan: Phys. Lett. 123 A (1987) 219.

[5] R. J. Baxter: Phys. Letts. A 133 (1988) 185.

[6] R. J. Baxter, J. H. H. Perk and H. Au-Yang: Phys. Letts. A 128 (1988) 138.

[7] R. J. Baxter, V.V. Bazhanov and J. H. H. Perk: Int. J. Mod. Phys. B 4 (1990) 803.

[8] B. Davies: J. Phys. A: Math. Gen. 23 (1990) 2245; J. Math. Phys. 32 (1991) 2945.

[9] L. Dolan and M. Grady: Phys. Rev. D 25 (1982) 1587.

[10] G. von Gehlen and R. Rittenberg, Nucl. Phys. B 257 (1985) 351.

[11] S. Howes, L.P. Kadanoff and M. den Nijs: Nucl. Phys. B 215 (1983) 169.

[12] J. H. H. Perk: in Proc.Symp.inPureMathematics, American Mathematical Society, 1989, Vol. 49, part 1 p. 341.

[13] D. B. Uglov and I. T. Ivanov: J. Stat. Phys. 82 (1996) 87.

[14] S. S. Roan: Onsager’s algebra, loop algebra and chiral Potts model, MPI 91-70, Max-Planck-Institut für Mathematik, Bonn, 1991.

[15] C. Ahn and K. Shigemoto: Modern Phys. Lett. A 6 (1991) 3509.

[16] E. Date and S. S. Roan: The structure of quotients of the Onsager algebra by closed ideals, Preprint math.QA/9911018.

[17] J. B. Kogus: Rev. Mod. Phys. 51 (1979) 659 and references therein.

[18] J. Riordan: An introduction to combinatorial analysis, Wiley publication in statistics 1958; Combinatorial identities, Wiley series in probablity and mathematical statistics 1968.