THE DISTANCE RATIO METRIC ON SOME DOMAINS OF \( \mathbb{C} \) PLANE

Slavko Simic

Mathematical Institute SANU, Kneza Mihaila 36, 11000 Belgrade, Serbia

E-mail: ssimic@turing.mi.sanu.ac.rs

2010 Mathematics Subject Classification: Primary 51M10, 30C20.

Keywords: Lipschitz constant, holomorphic functions, Schwarz-Pick lemma.

Abstract We prove a generalization for some \( \mathbb{C} \) domains of Gehring - Palka theorem on Möbius transformations regarding the distance ratio metric. Namely, we show that this theorem is valid for arbitrary holomorphic mappings \( f : \mathbb{H} \rightarrow \mathbb{H} \) or \( f : \mathbb{D} \rightarrow \mathbb{D} \) with the same Lipschitz constant 2.

1. Introduction

For a subdomain \( G \subset \mathbb{R}^n \) and for all \( x, y \in G \) the distance ratio metric \( j_G \) is defined as

\[
j_G(x, y) = \log \left( 1 + \frac{|x - y|}{\min\{d(x, \partial G), d(y, \partial G)\}} \right),
\]

where \( d(x, \partial G) \) denotes the Euclidean distance from \( x \) to \( \partial G \). The distance ratio metric was introduced by F.W. Gehring and B.P. Palka [3] and in the above simplified form by M. Vuorinen [9]. As the "first approximation" of the quasihyperbolic metric, it is frequently used in the study of hyperbolic type metrics ([1],[2],[6],[10]) and geometric theory of functions.

For an open continuous mapping \( f : G \rightarrow G' \) we consider the following condition: there exists a constant \( C \geq 1 \) such that for all \( x, y \in G \) we have

\[
j_{G'}(f(x), f(y)) \leq C j_G(x, y),
\]

or, equivalently, that the mapping

\[
f : (G, j_G) \rightarrow (G', j_{G'})
\]

between metric spaces is Lipschitz continuous with the Lipschitz constant \( C \).

However, unlike other frequently used metrics (hyponetric metric, absolute ratio metric), the distance ratio metric \( j_G \) is not invariant under Möbius transformations. Therefore, it is natural to ask what the Lipschitz constants are for this metric under conformal mappings or Möbius transformations in higher dimension. F. W. Gehring, B. P. Palka and B. G. Osgood proved that \( j \) metric is not changed by more than a factor 2 under Möbius transformations, see [2], [3]:

**Theorem A** \( j \) If \( G \) and \( G' \) are proper subdomains of \( \mathbb{R}^n \) and if \( f \) is a Möbius transformation of \( G \) onto \( G' \), then for all \( x, y \in G \)

\[
j_{G'}(f(x), f(y)) \leq 2 j_G(x, y)
\].
It is an interesting problem to investigate Lipschitz continuity of the distance-ratio metric under some other conformal mappings.

In this paper we show that analogous to the above results are valid for arbitrary holomorphic self-mappings of a half-plane or the open unit disk. Thereby Theorem A is considerably generalized in these cases.

2. Results and proofs

We shall treat firstly the case of upper half-plane $\mathbb{H} := \{ z | \Im z > 0 \}$.

A generalization of Theorem A for this domain is given by the following

**Theorem 1** For any holomorphic mapping $f$, $f : \mathbb{H} \to \mathbb{H}$, and all $z, w \in \mathbb{H}$, we have

$$j_{\mathbb{H}}(f(z), f(w)) \leq 2j_{\mathbb{H}}(z, w). \quad (1.1)$$

The constant 2 is best possible.

**Proof**

Denote $s := \min \{ \Im z, \Im w \}$, $S := \min \{ \Im f(z), \Im f(w) \}$, $T := \max \{ \Im f(z), \Im f(w) \}$. Then

$$j_{\mathbb{H}}(z, w) = \log \left( 1 + \frac{|z - w|}{s} \right); \quad j_{\mathbb{H}}(f(z), f(w)) = \log \left( 1 + \frac{|f(z) - f(w)|}{S} \right).$$

Main tool in the proof will be Julia’s variant of the famous Schwarz-Pick Lemma for the half-plane [cf 2].

**Theorem B** For any holomorphic mapping $f$, $f : \mathbb{H} \to \mathbb{H}$, and all $z, w \in \mathbb{H}$, we have

$$|\frac{f(z) - f(w)}{f(z) - f(w)}| \leq |\frac{z - w}{z - w}|.$$  

Applying this assertion, we get

$$|\frac{f(z) - f(w)}{f(z) - f(w)}|^2 - 1 \geq \frac{z - w}{z - w}^2 - 1,$$

that is

$$|\frac{f(z) - f(w)}{z - w}|^2 \leq \frac{\Im f(z) \Im f(w)}{\Im z \Im w},$$

since for any $x, y \in \mathbb{C}$, the identity

$$|x - y|^2 - |x - y|^2 = 4 \Im x \Im y$$

holds.

Therefore,

$$\frac{|f(z) - f(w)|^2}{S^2} \leq \frac{|z - w|^2}{s^2} \frac{T}{S} = \frac{|z - w|^2}{s^2} \frac{T - S}{S} \left( 1 + \frac{T - S}{S} \right),$$
i.e.,
\[
\frac{|f(z) - f(w)|}{S} \leq \frac{|z - w|}{s} \sqrt{1 + \frac{|f(z) - f(w)|}{S}},
\]

since \( T - S = |\Im(f(z) - f(w))| \leq |f(z) - f(w)|. \)

Denoting \( \frac{|z - w|}{s} := 2X \in [0, \infty) \), the relation (1.2) gives
\[
1 + \frac{|f(z) - f(w)|}{S} \leq (X + \sqrt{1 + X^2})^2.
\]

Hence,
\[
j_H(f(z), f(w)) = \log\left(1 + \frac{|f(z) - f(w)|}{S}\right) \leq 2\log(X + \sqrt{1 + X^2}) \leq 2\log(1 + 2X) = 2j_H(z, w),
\]

and the proof is done.

To prove that the constant 2 is best possible, choose
\[
f_0(z) = a - \frac{1}{b + z},
\]

where \( a \) and \( b \) are arbitrary real numbers.

Since \( \Im f_0(z) = \frac{\sqrt{z}}{|b + z|^2} \), we conclude that \( f_0 \) maps the upper half-plane into itself.

A calculation of \( j \) values along the line \( \zeta \subset \mathbb{H} \) given by \( \zeta := \{z = i - b + t, \ t \in \mathbb{R}\} \) with \( w := i - b \),
gives
\[
j_\mathbb{H}(z, w) = \log(1 + t); \ j_\mathbb{H}(f_0(z), f_0(w)) = \log(1 + \frac{t}{\sqrt{1 + t^2}}) = \log(1 + t\sqrt{1 + t^2}).
\]

Hence,
\[
\frac{j_\mathbb{H}(f_0(z), f_0(w))}{j_\mathbb{H}(z, w)} = \frac{\log(1 + t\sqrt{1 + t^2})}{\log(1 + t)},
\]

and this expression is tending to 2 as \( t \to \infty \).

It follows that the constant 2 is best possible.

A generalization of Theorem A for the unit disk case is given by the following

\textbf{Theorem 2}  \textit{For any holomorphic mapping} \( f, f : \mathbb{D} \to \mathbb{D} \), \textit{and all} \( z, w \in \mathbb{D} \), \textit{we have}
\[
j_\mathbb{D}(f(z), f(w)) \leq 2j_\mathbb{D}(z, w).
\]  \hfill (2.1)
Proof

Let $\max\{|z|, |w|\} = r$ and suppose that $|f(z)| \geq |f(w)|$. Then

$$j_D(z, w) = \log(1 + \frac{|z - w|}{1 - r}); \quad j_D(f(z), f(w)) = \log(1 + \frac{|f(z) - f(w)|}{1 - |f(z)|}).$$

The proof is based on well-known Schwarz-Pick lemma, stated in the following form.

**Theorem C** Let $f$ be a holomorphic mapping of the open unit disk $\mathbb{D}$ into itself. Then for any $z_1, z_2 \in \mathbb{D}$, we have

$$\frac{|f(z_2) - f(z_1)|}{1 - f(z_1)f(z_2)} \leq \frac{|z_2 - z_1|}{1 - z_1\overline{z_2}}.$$ 

An application of this lemma gives

$$\left| \frac{f(z) - f(w)}{1 - f(z)f(w)} \right|^{-2} - 1 \geq \left| \frac{z - w}{1 - z\overline{w}} \right|^{-2} - 1,$$

that is,

$$\left| \frac{f(z) - f(w)}{z - w} \right| \leq \frac{(1 - |f(z)|^2)(1 - |f(w)|^2)}{(1 - |z|^2)(1 - |w|^2)} \leq \frac{(1 - |f(z)|^2)(1 - |f(w)|^2)}{(1 - r^2)^2},$$

where we used the following identity for complex numbers $x, y$,

$$|1 - x\overline{y}|^2 - |x - y|^2 = (1 - |x|^2)(1 - |y|^2).$$

Therefore,

$$\left| \frac{f(z) - f(w)}{1 - f(z)} \right| \leq \frac{|z - w|}{1 - r} \sqrt{\frac{1 + |f(z)||1 + |f(w)|}{r}} \sqrt{\frac{1 - |f(w)|}{1 - |f(z)|}},$$

i.e.,

$$\left| \frac{f(z) - f(w)}{1 - f(z)} \right| \leq \frac{|z - w|}{1 - r} \frac{1 + |f(z)|}{1 + r} \sqrt{\frac{1 + |f(z) - f(w)|}{1 - |f(z)|}},$$

(2.2)

since

$$\frac{1 - |f(w)|}{1 - |f(z)|} = 1 + \frac{|f(z)| - |f(w)|}{1 - |f(z)|} \leq 1 + \frac{|f(z) - f(w)|}{1 - |f(z)|}.$$

To obtain an estimation for $|f(z)|, z \in \mathbb{D}$, suppose that $f(0) = a \in \mathbb{D}$. Applying Theorem C with $z_1 = 0, z_2 = z$, we get

$$|z| \geq \left| \frac{f(z) - a}{1 - af(z)} \right| \geq \frac{|f(z)| - |a|}{1 - |a||f(z)|},$$

that is,

$$|f(z)| \leq \frac{|z| + |a|}{1 + |a||z|} \leq \frac{r + |a|}{1 + |a|r}$$

(2.3)
By (2.3) we get $\frac{1+|f(z)|}{1+|a|} := 2c(a,r) = 2c$ and, denoting $\frac{|z-w|}{1-r} := X \in [0, +\infty)$, the inequality (2.2) gives

$$1 + \frac{|f(z) - f(w)|}{1 - |f(z)|} \leq (cX + \sqrt{1 + c^2X^2})^2.$$ 

Hence,

$$\frac{j_D(f(z), f(w))}{j_D(z, w)} \leq 2 \log(cX + \sqrt{1 + c^2X^2}) \log(1 + X).$$

Now it should be notified that the function $g$,

$$g(X) := cX + \sqrt{1 + c^2X^2} - (1 + X)$$

is negative for $X \in (0,T)$, where $T = \frac{2(1-c)}{2c-1} = \frac{2|a|r+1-|a|}{|a|(1-r)}$.

Since $X = \frac{|z-w|}{1-r} \leq \frac{2c}{1-r}$, it follows that $X \in (0,T)$, i.e., $\log(cX + \sqrt{1 + c^2X^2}) \leq \log(1 + X)$, which provides the proof.

**Remark** We cannot claim in this case that the best possible Lipschitz constant is $C^* = 2$ since we do not know an example of analytic function $f_0 : \mathbb{D} \to \mathbb{D}$ such that

$$\sup_{z,w} \frac{j_D(f_0(z), f_0(w))}{j_D(z, w)} = 2.$$ 

This problem was recently treated in [11] where the following estimation for $C^* = C^*(a)$ is proved

$$1 + |a| \leq C^* \leq \min\{2(1 + |a|), \sqrt{5 + 2|a| + |a|^2}\}; \quad a := f(0).$$

By Theorem 2 this is reduced to

$$1 + |a| \leq C^* \leq 2,$$

but problem of determining best possible Lipschitz constant for the unit disk remains open.

**References**

[1] G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen, *Conformal Invariants, Inequalities and Quasiconformal Maps*, John Wiley & Sons, New York, 1997.

[2] H.P. Boas, *Julius and Julia: Mastering the Art of the Schwarz Lemma*, Amer. Math. Monthly 117, November 2010, 770-785.

[3] F.W. Gehring and B.G. Osgood,*Uniform domains and the quasihyperbolic metric*, J. Analyse Math. 36, 1979, 50-74.

[4] F.W. Gehring and B.P. Palka,*Quasiconformally homogeneous domains*, J. Analyse Math. 30, 1976, 172-199.
[5] P. Hästö, Z. Ibragimov, D. Minda, S. Ponnusamy and S. K. Sahoo, *Isometries of some hyperbolic-type path metrics, and the hyperbolic medial axis*, In the tradition of Ahlfors-Bers, IV, Contemporary Math. 432, 2007, 63–74.

[6] M. Huang, S. Ponnusamy, H. Wang, and X. Wang, *A cosine inequality in the hyperbolic geometry*, Appl. Math. Lett. 23 (2010), no. 8, 887–891.

[7] R. Klén, M. Vuorinen, and X. Zhang, *Quasihyperbolic metric and Möbius transformations*, Proc. Amer. Math. Soc. (to appear) Manuscript 9pp, arXiv: 1108.2967 math.CV.

[8] J. Väisälä, *The free quasiusl**. Freely quasiconformal and related maps in Banach spaces. Quasiconformal geometry and dynamics, (Lublin, 1996), 55–118, Banach Center Publ., 48, Polish Acad. Sci., Warsaw, 1999.

[9] M. Vuorinen, *Conformal invariants and quasiregular mappings*, J. Analyse Math., 45 (1985), 69–115.

[10] M. Vuorinen, *Conformal geometry and quasiregular mappings*, Lecture Notes in Math. 1319, Springer-Verlag, Berlin-Heidelberg, 1988.

[11] S. Simic, *Lipschitz continuity of the distance ratio metric on the unit disk*, Filomat 27/8 (2013).