Some remarks about homeomorphisms, “energy”, and so on

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John Ball keeps asking questions of the following sort. Suppose that one has a homeomorphism from a domain in a Euclidean space onto its image in the same Euclidean space. Assume also that the homeomorphism has “finite energy” with respect to some reasonable functional, which would normally entail something like first distributional derivatives in $L^p$ for some $p$, $1 \leq p < \infty$, and the inverses of the associated differentials lying in some $L^q$, $1 \leq q < \infty$. The latter might be controlled in terms of integrability properties of the reciprocal of the Jacobian of the mapping (the determinant of the differential). Under conditions such as these, what kinds of approximations of the homeomorphism can one make by more regular homeomorphisms, approximations which respect similar integrability conditions for the differentials and their inverses?

Let us restrict ourselves to dimensions 3 and lower, since all sorts of strange things happen in dimensions 4 and larger, and since dimensions less than or equal to 3 are physically relevant (elasticity theory, etc., as Dr. Ball well knows). This type of issue in dimension 1 can be treated in a direct manner, by writing the mapping as the integral of its derivative, and so we focus on dimensions 2 and 3.

There are very famous results about approximating homeomorphisms by piecewise-linear homeomorphisms in dimensions 2 and 3. See [Bin, Moi]. More precisely, these are approximations in $C^0$ senses, which are already quite nontrivial and useful in the study of topology. The results include relative versions, in which a homeomorphism is regularized in some parts while not changing it on other parts where it is already regular.

What about approximations which also respect the differential in some manner?

A basic strategy in making approximations of a function in a Sobolev
space is to first choose a set on which the function behaves nicely, and whose complement has small measure. The nice behavior might involve boundedness of the first derivatives, continuity of the first derivatives, and so forth. Although the complement has small measure, it typically need not have any simple structure, but could be quite scattered and irregular. The second step would be to modify the function on the small set. If there are no additional constraints on the function, then there are relatively simple techniques of extension and regularization. However, if one wants the result to be a homeomorphism, then this second step becomes much more complicated.

For more on approximations of homeomorphisms, if not exactly of this form, see [DonS, Luu3, Sul, TukV2, Väi1].

This type of conundrum seems a bit odd to me, in that for a number of basic situations that would arise in a simple way, there should not be as much trouble. One way to look at this is that often there is something like a one-parameter family of mappings. A related issue is that in topology a basic point is often to construct isotopies between homeomorphisms, i.e., a continuous family of homeomorphisms. This is quite different from homotopies, which are continuous families of mappings which are not required to be injective, even if the mappings that the homotopy goes between are injective.

In particular, there are a number of results to the effect that homeomorphisms which are close in a $C^0$ sense can be connected by an isotopy, as in [Cer, EdwK, Fis, Ham, Kis1, Kis2, Luu1, Sul], and where the isotopy stays close to the original homeomorphisms. This is part of the reason that a $C^0$ approximation can be useful, since otherwise it seems rather weak.

In another direction, let us recall a famous result of Hatcher [Hat1, Hat2, Lau] concerning embedded two-dimensional spheres in $\mathbb{R}^3$. More precisely, one considers smoothly embedded two-dimensional spheres in $\mathbb{R}^3$, and the space of these can be locally identified with the space of smooth real-valued functions on the 2-sphere, because of the tubular neighborhood theorem. Hatcher’s result says that this space is contractable.

For the analogous question in the plane, the Riemann mapping theorem can be used. Paul Schweitzer keeps asking about possible analytic proofs for two-dimensional spheres in $\mathbb{R}^3$.

Note that embedded 2-spheres in $\mathbb{R}^3$ can be viewed as the boundaries of solid 3-dimensional balls, just as Jordan curves in the plane can be viewed as boundaries of 2-dimensional disks.

For another very interesting direction along the lines of analysis and com-
plexity of shapes, see [CanKS, KusS1, KusS2].

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