DERIVED INVARIANTS AND MOTIVES, PART II INTEGRAL DERIVED INVARIANTS AND SOME APPLICATIONS

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Abstract. In this paper we construct new derived invariants with integral coefficients using the theory of motives, and give several applications. Specifically, we obtain the following results: For complex algebraic surfaces, we prove that certain torsion in the abelianized fundamental group is a derived invariant. We prove that the collection of Hodge-Witt cohomology groups is a derived invariant. In particular, Hodge-Witt reduction and ordinary reduction are preserved by derived equivalence when the characteristic is sufficiently large. Finally, using the techniques of non-commutative algebraic geometry, we prove that Serre’s ordinary density conjecture is true for cubic 4-folds which contain a $\mathbb{P}^2$.

Keywords: motives, derived category, non-commutative algebraic geometry, number theory

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1. Introduction

Given an invariant $F$ of varieties, particularly when $F$ comes from a cohomology theory: When is $F$ preserved by equivalences of derived categories of coherent sheaves? Even for the Hodge numbers $h^{i,j}$, for example, this question is a complex problem. If $\text{char}(k) = 0$, then Hodge numbers are derived invariants when the dimension of varieties is less than 3 [PS11, Corollary C], and Kontsevich predicted that this continues in higher dimensions. On the other hand, if $\text{char}(k) = p > 0$, Antieau and Bragg showed that Hodge numbers are derived invariants for smooth proper surfaces [AB19, Thm. 1.3(1)], but Addington and Bragg showed that this does not continues for smooth proper 3-folds [Add17, Thm. 1.1]. For the Abelianization of the Fundamental Group $\pi_1$, Schnell described an example of smooth complex projective 3-folds $X$ and $Y$, such that $X$ and $Y$ are derived equivalent but $\pi_1$ is not preserved by equivalences of derived categories [Sch12], this implies that the torsion in the singular cohomology group $H^2_{\text{Sing}}(-,\mathbb{Z})$ with $\mathbb{Z}$-coefficients is not derived invariant. On the other hand, the even and odd degree singular cohomology groups with $\mathbb{Q}$-coefficients are derived invariants. Our goal is to understand the problem by using the motive theory and Grothendieck-Riemann-Roch theory [BS58] and to construct some integral derived invariants by using integral Grothendieck-Riemann-Roch theory [Pap07], and [Mat], and we give some applications in some areas.

1.1. Manin invariants. Let $k$ be a field and $R$ a commutative ring. We call a graded invariant $F = \bigoplus F^j$ of smooth projective varieties over $k$ with $R$-coefficients which is functorial in Chow groups (details are given in sec. 2) a “Manin invariant with $R$-coefficients”. We list some Manin invariants with $R$-coefficients

- If $k = \mathbb{C}$, the even and odd degree singular cohomology groups $H_{\text{Sing}}^{\text{even}}(X,R)$, $H_{\text{Sing}}^{\text{odd}}(X,R)$.
- If $(l,\text{char}(k)) = 1$ for a prime $l$ and $R = \mathbb{Z}_l$, the Galois representation of the even and odd degree $l$-adic cohomology group $\bigoplus_i H^i_{\text{et}}(X_{\overline{k}},\mathbb{Z}_l(i))$.
- If $k = R$ admits the resolution of singularities, the collection of Hodge cohomology groups $\bigoplus_{j-i=r} H^j_{\text{Zar}}(X,\Omega^i)$ for any $r$.
- If $\text{char}(k) = p > 0$ and let $R$ be the Witt ring $W$ of $k$, the collection of Hodge-Witt cohomology groups $\bigoplus_{j-i=r} H^i(X,W\Omega^j)$ for any $r$.

By the same discussion as in [O105], we prove the following by using the Grothendieck Riemann-Roch theory.

Theorem 1.1 (see Sec.2.2). For a $\mathbb{Q}$-algebra $R$, any Manin invariant with $R$-coefficients $F = F^\bullet$ is a derived invariant for smooth projective varieties.

A conjecture of Orlov [O105, Conj. 1] implies the following conjecture.
Conjecture 1.2. For a \( \mathbb{Q} \)-algebra \( R \), each graded piece \( F^i \) of a Manin invariant \( F^\bullet \) with \( R \)-coefficients is a derived invariant for smooth projective varieties.

This Theorem/Conjecture could explain why Hodge numbers \( h^{i,j} \) of characteristic 0 should be derived invariants and the singular cohomology group with \( \mathbb{Q} \)-coefficients should be derived invariants. However, Hodge numbers \( h^{i,j} \) of characteristic \( p > 0 \) are not derived invariants, and the torsion subgroup of singular cohomology group with \( \mathbb{Z} \)-coefficients are not derived invariants. Since the former are graded pieces of Manin invariants with \( \mathbb{Q} \)-coefficients, the above Theorem/Conjecture would imply that they are derived invariants. On the other hand, since the latter do not have \( \mathbb{Q} \)-coefficients; we cannot apply the Theorem/Conjecture. We study an integral analogue of this story by using the integral Grothendieck Riemann-Roch theorem [Pap07], and [Mat]. We show that Pappas’s result [Pap07] would imply the following.

Theorem 1.3 (Theorem 2.12). If \( \text{char}(k) = 0 \), for a \( \mathbb{Z}[\frac{1}{(3d+1)}] \)-algebra \( R \), any \( d \)-Manin invariant with \( R \)-coefficients \( F \) is a derived invariant for smooth projective varieties whose dimension are less than \( d \).

We show that our result from [Mat] implies the following.

Theorem 1.4 (Theorem 2.13). If \( \text{char}(k) = p > 0 \), for a \( \mathbb{Z}[\frac{1}{(3d+e+1)}] \)-algebra (resp. \( \mathbb{Z}[\frac{1}{(2d+e+1)}] \)-algebra) \( R \), any graded piece of a Manin invariant with \( R \)-coefficients \( F^i \) is a derived invariant for smooth projective varieties whose dimension are less than \( d \) (resp. and which can be embedded into \( \mathbb{P}^k \)).

Remark/Conjecture 1. As with the Conjecture 1.2, we also predict that if \( \text{char}(k) = 0 \) (resp. \( \text{char}(k) = p > 0 \)), for a \( \mathbb{Z}[\frac{1}{(3d+1)}] \)-algebra (resp. \( \mathbb{Z}[\frac{1}{(2d+e+1)}] \)-algebra) \( R \), any graded piece of a Manin invariant with \( R \)-coefficients \( F^i \) is a derived invariant for smooth projective varieties whose dimension are less than \( d \) (resp. and which can be embedded into \( \mathbb{P}^k \)).

These theorems imply that the following Manin invariants are preserved by equivalence of derived categories of smooth projective varieties over \( k \) whose dimension are less than \( d \) and which can be embedded into \( \mathbb{P}^k \).

- If \( k = \mathbb{C} \), the the even and odd degree singular cohomology groups \( H^\text{even}_{\text{sing}}(-, R), H^\text{odd}_{\text{sing}}(-, R) \)
- If \( \text{char}(k) = 0 \), for a prime \( l > 3d + 1 \), the Galois representations of the even and odd degree \( l \)-adic cohomology groups \( \bigoplus_i H^i_{\text{et}}(-, \mathbb{Z}_l)(i), \bigoplus_i H^{2i-1}_{\text{et}}(-, \mathbb{Z}_l)(i) \).
- If \( \text{char}(k) = p > 0 \) and \( p > 2d + e + 1 \), under the assumption of resolution of singularities, the collection of Hodge cohomology groups \( \bigoplus_{j-i=r} H^r_{\text{Zar}}(X, \Omega^j) \) for any \( r \). (Remark. This has been already known for \( p \geq \dim X \) [AV20]).

There are certain applications of these results to mirror symmetry, algebraic geometry of positive characteristic and number theory. We shall explain these applications.

1.2. The application to mirror symmetry. Batyrev and Kreuzer conjectured that the torsion subgroup in Betti cohomology \( \text{Tors}(H^2_{\text{sing}}(-, \mathbb{Z})) \) and \( \text{Tors}(H^3_{\text{sing}}(-, \mathbb{Z})) \) are exchanged by homological mirror symmetry [BK06]. In [GP01], Gross and Popescu gave a counterexample to this conjecture (see [Sch12]). Gross and Popescu constructed a pair of smooth projective varieties \( X \) and \( Y \) such that \( Y \) is to be a (homological) mirror manifold of a (homological) mirror manifold of \( X \) [GP01], Schnell showed that \( X \) and \( Y \) are derived equivalent [Sch12]. If the conjecture of Batyrev and Kreuzer holds, then \( X \) and \( Y \) have the same torsion subgroup of the singular cohomology group \( \text{Tor}H^2_{\text{sing}}(X, \mathbb{Z}) \simeq \text{Tor}H^3_{\text{sing}}(Y, \mathbb{Z}) \), but this isomorphism does not hold [GP01]. The problem with the conjecture of Batyrev and Kreuzer is that the torsion subgroup of the singular cohomology group is not a derived invariant. Inspired by Gross and Popescu’s counter-example and Theorem 1.6 below, we propose the following modified version of the conjecture of Batyrev and Kreuzer.

Conjecture 1.5. Let \( X \) be a complex projective Calabi-Yau 3-fold, and \( M \) be a homological mirror manifold of \( X \). Then the \( m \)-torsion subgroup in Betti cohomology \( (H^2_{\text{sing}}(-, \mathbb{Z}))[m] \) and \( (H^3_{\text{sing}}(-, \mathbb{Z}))[m] \) are exchanged by \( X \) and \( M \) for any number \( m \) such that \( (m,p) = 1 \) for \( p = 2, 3, 5, 7 \).
For a derived equivalence $D^b(X) \simeq D^b(Y)$ of smooth projective varieties, $Y$ is a homological mirror of a mirror manifold of $X$. Thus if the conjecture holds, then $H^2(-, \mathbb{Z})[m]$ and $H^3(-, \mathbb{Z})[m]$ should be derived invariants, and Remark/Conjecture 1 would imply that these two invariants are derived invariants. We can prove that these two invariants are derived invariants for smooth projective surfaces.

**Theorem 1.6 (Theorem 3.1).** Let $X$ and $Y$ be smooth projective surfaces over $\mathbb{C}$. We assume that there is a $\mathbb{C}$-linear equivalence $F : D^b(X) \simeq D^b(Y)$. For a natural number $m$ such that $(m, p) = 1$ for $p = 2, 3, 5, 7$ there are isomorphisms of the $m$-torsion parts of cohomology groups

$$H^i(X, \mathbb{Z})[m] \simeq H^i(Y, \mathbb{Z})[m]$$

for any $i$. In particular, there is an isomorphism of the $m$-torsion parts of the abelianization of the fundamental groups

$$\pi_1^{ab}(X)[m] \simeq \pi_1^{ab}(Y)[m].$$

1.3. **The application to algebraic geometry of positive characteristic.** For a perfect field $k$, we write $W$ for the Witt ring of $k$ and $K$ for the fraction field of $W$. Antieau-Bragg showed that the collection of Hodge-Witt cohomology groups tensored by $K$ for the Witt ring of $k$ and $W$ for the fraction field of $W$. Moreover, the density of the set of finite primes at which $X$ has a good ordinary reduction at is also ordinary.

**Theorem 1.7.** (Theorem 4.3) Let $X$ and $Y$ be smooth projective varieties over a perfect field $k$ with characteristic $p$. Suppose $X, Y \in \text{SmProj}^{d \times d}_e(k)$ for some $e$ and $d$, and $p > 2d + e + 1$. We assume that there is a $k$-linear fully faithful triangulated functor $F : D^b(X) \to D^b(Y)$. Then the followings holds.

1. For any integer $r$, there is a split injective morphism of $W$-modules

$$\bigoplus_{j-i=r} H^i(X, W\Omega^j_X) \rightarrow \bigoplus_{j-i=r} H^i(Y, W\Omega^j_Y).$$

2. For any integer $r$, if $F$ is an equivalence, then there is an isomorphism of $W$-modules

$$\bigoplus_{j-i=r} H^i(X, W\Omega^j_X) \simeq \bigoplus_{j-i=r} H^i(Y, W\Omega^j_Y).$$

3. If $Y$ is Hodge-Witt, then $X$ is also Hodge-Witt.
4. If $Y$ is ordinary, then $X$ is also ordinary.

1.4. **The application to number theory.** We consider the following conjecture.

**Conjecture 1.8.** (Serre conjecture for ordinary density) Let $X/K$ be a smooth projective variety over a number field $K$. Then there is finite extension $L/K$ such that there exists a positive density of primes $v$ of $L$ for which $X_L$ has a good ordinary reduction at $v$.

The conjecture is known for elliptic curves by Serre [Ser89], Abelian surfaces by Ogus [DMOS82, Chapter 6], K3 surfaces by Bogomolov and Zarhin [BZ09], Abelian varieties with complex multiplication and Fermat varieties [Jos16, Theorem 5.1.10]. We can give a non-commutative approach to this conjecture and prove the following.

**Theorem 1.9 (Theorem 6.4).** Let $X$ be a cubic 4-fold which contains $\mathbb{P}^2$ over a number field $K$. Then $X$ satisfies Conjecture 1.8. Moreover, the density of the set of finite primes at which $X_L$ has a good ordinary reduction is one.
1.5. Notation and conventions. We consider the following categories, and ring associated to a field $k$, natural numbers $d, e, r$, and a commutative ring $R$. Where ever possible, we have used notation already existing in the literature.

- **$\text{Sm Proj}_{(e)}^d(k)$**: the full subcategory of $\text{Sm Proj}(S)$ whose objects can be embedded in $\mathbb{P}^e_k$ and dimension of objects is less than or equal to $d$.
- **$\text{KM}_{(e)}^d(k)$**: the smallest full subcategory of $\text{KM}(k)$ which contains the image of the functor $\text{Sm Proj}_{(e)}^d(k) \to \text{KM}(k)$ and is closed under finite coproducts.

**Chow** is given by the class of the diagonal $\Delta_X \in \text{CH}^d(X \times X)_R$. Obviously, given a Manin invariant with $R$-coefficients, we have $\text{Manin} = \text{Manin inv}$.

### Definition 2.1

**Given an $R$-linear category** $A$ which contains any infinite coproducts and covariant functors $F^i : \text{Sm Proj}(k) \to A$ for $i \in \mathbb{Z}$, we call $F = \bigoplus_i F^i$ a Manin invariant if $F = \bigoplus_i F^i$ satisfies following conditions:

1. **(c-1)** For an algebraic cycle $\alpha \in \text{CH}^j(X \times Y)_R$, there is a morphism $F^j(\alpha) : F^j(Y) \to F^{j+i-\text{dim}Y}(X)$ in $A$ for any $j$. In particular, for a morphism of algebraic varieties $f : X \to Y$, the morphism $F^j(Y) \to F^j(X)$ comes form the graph morphism $[f(X)] \in \text{CH}^{\text{dim}Y}(X \times Y)$ is equal to the morphism $F^j(f) : F^j(Y) \to F^j(X)$ for any $j$.

2. **(c-2)** For an algebraic cycle $A = \sum \alpha_i \in \text{CH}^*(X \times Y)$ where $\alpha_i$ is a $i$-codimensional cycle in $X \times Y$, we let $F(A)$ denote the morphism $F(Y) = \bigoplus_j F^j(Y) \to F(X) = \bigoplus_j F^j(X)$ in $A$ whose $(j, l)$-component is given by $F(\alpha_{i+j-\text{dim}Y-l}) : F^j(Y) \to F^l(X)$. If given an algebraic cycle $B$ in $\text{CH}^*(Y \times Z)$ then $F = \bigoplus_i F^i$ satisfies the following equation of morphism in $A$:

$$F(A) \circ F(B) = F(r_*(p^*Aq^*B))$$

as morphisms from $F(Z)$ to $F(X)$ where $p$ (resp. $r, q$) is the projection from $X \times Y \times Z$ to $X \times Y$ (resp. to $Y \times Z$, to $X \times Z$).

We call $F = \bigoplus_i F^i$ a d-Manin invariant if $F = \bigoplus_i F^i$ satisfies conditions (c-1) and (c-2) for smooth projective varieties $X, Y$ and $Z$ whose dimension are less than $d$.

To relate Manin invariants with motives theory, we now recall the definition of the Manin’s category of motives $\text{C}_{k,R}$ from [Man68] for a base field $k$ with $R$-coefficients. We let $\text{C}_{k,R}$ denote the $R$-linear category whose objects are smooth projective varieties over $k$ and for smooth projective varieties $X$ and $Y$, maps from $X$ to $Y$ are given by the Chow group $\text{CH}^*(X \times Y)_R$ with $R$-coefficients. The composition of two morphisms is defined using intersection product. The identity $\text{id}_X$ is given by the class of the diagonal $\Delta_X \in \text{CH}^{\text{dim}X}(X \times X)_R$. Set the full subcategory $\text{C}_{k,R}^{\leq d}$ whose objects are varieties whose dimension are less than $d$. Obviously, given a Manin invariant with $R$-coefficients $F = \bigoplus_j F^j$, then there is the functor $\mathcal{F} : \text{C}_{k,R} \to A$ which sends a smooth projective variety $X$ to $F(X) = \bigoplus_j F^j(X)$ and $A \in \text{CH}^*(X \times Y)$ to $F(A)$ and also given a $d$-Manin invariant with $R$-coefficients $F = \bigoplus_j F^j$ then there is the functor $\mathcal{F} : \text{C}_{k,R}^{\leq d} \to A$. Conversely, an object $G \in \text{DM}(k, R)$ induces Manin invariants $G_{even}$ and $G_{odd}$ for any $m \in \mathbb{Z}$.
Proposition 2.2. Given an object \( G \in \text{DM}(k,R) \). For a smooth projective variety \( X \), we denote \( \text{Hom}_{\text{DM}}(M(X)(j)[2j],G) \) and \( \text{Hom}_{\text{DM}}(M(X)(j)[2j+1],G) \) by \( G^j_{\text{even}}(X) \) and \( G^j_{\text{odd}}(X) \), respectively. Then \( G_{\text{even}} = \bigoplus_{j \in \mathbb{Z}} G^j_{\text{even}} \) and \( G_{\text{odd}} = \bigoplus_{j \in \mathbb{Z}} G^j_{\text{odd}} \) are Manin invariants.

Proof. In [Voe00], Voevodsky has proved that there is a fully faithful functor (see [Voe00, Corollary 4.2.5 and Theorem 4.3.7] or [Kel17, Remark 5.3.21. (iii)]):

\[
\text{Chow}^{\text{eff}}(k)_R \hookrightarrow \text{DM}^{\text{eff}}(k,R).
\]

Thus an algebraic cycle \( A = \Sigma_{i \in \mathbb{Z}} a_i \in \text{CH}^*(X \times_k Y) \) induces a morphism

\[
\bigoplus_{j \in \mathbb{Z}} M(X)(j)[2j] \to \bigoplus_{i \in \mathbb{Z}} M(Y)(i)[2i]
\]

whose \((j,i)\)-component is given by \( A_{i-j+\dim Y} \). Also \( A \) induces a morphism

\[
\bigoplus_{j \in \mathbb{Z}} M(X)(j)[2j+1] \to \bigoplus_{i \in \mathbb{Z}} M(Y)(i)[2i+1]
\]

whose \((j,i)\)-component is given by \( A_{i-j+\dim Y} \). Morphisms (2.2) and (2.3) are functorial in Chow groups, thus functors

\[
G^j_{\text{even}} : \text{SmProj}(k) \to (R\text{-mod}); \quad X \mapsto \text{Hom}_{\text{DM}}(M(X)(i)[2i],G)
\]

for \( i \in \mathbb{Z} \) satisfies conditions (c-1) and (c-2), and also functors

\[
G^j_{\text{even}} : \text{SmProj}(k) \to (R\text{-mod}); \quad X \mapsto \text{Hom}_{\text{DM}}(M(X)(i)[2i+1],G)
\]

for \( i \in \mathbb{Z} \) satisfies conditions (c-1) and (c-2). \( \square \)

We shall extend this story to an object \( G \in \text{DM}^{\text{eff}}(k,R) \). By Voevodsky’s cancellation theorem [Voe10], we have the fully faithful \( R \)-linear functor \( \text{DM}^{\text{eff}}(k,R) \hookrightarrow \text{DM}(k,R) \). Define \( G^j_{\text{even}}(X) \) and \( G^j_{\text{odd}}(X) \) as

\[
\text{Hom}_{\text{DM}}(M(X)(j)[2j],G) \quad \text{and} \quad \text{Hom}_{\text{DM}}(M(X)(j)[2j+1],G),
\]

respectively, for any smooth projective variety \( X \) and \( j \in \mathbb{Z} \). Then \( G_{\text{even}} = \bigoplus G^j_{\text{even}} \) and \( G_{\text{odd}} = \bigoplus G^j_{\text{odd}} \) are Manin invariants. But we don’t wish to do this story, because it is hard to calculate \( \text{Hom}_{\text{DM}}(M(X)(j)[2j],G) \) in the case \( j < 0 \). For example, there is an object \( \Omega^n \in \text{DM}^{\text{eff}}(k,k) \) which represents Hodge cohomology for smooth proper varieties, moreover for \( j \geq 0 \) we have an isomorphism

\[
\text{Hom}_{\text{DM}}(M(X)(j)[2j],\Omega^n[m]) \simeq H^{n-j}(X,\Omega^{n-j}_X).\]

On the other hand, for \( j < 0 \), it is very hard to calculate \( \text{Hom}_{\text{DM}}(M(X)(j)[2j],\Omega^n[m]) \), and actually we don’t know it yet:

\[
\text{Hom}_{\text{DM}}(M(X)(j)[2j],\Omega^n[m]) = ???.
\]

To avoid this, we will assume the object \( G \) satisfies a condition on the finiteness of the cohomological dimension.

Definition 2.3. For an object \( G \in \text{DM}^{\text{eff}}(k,R) \), we say \( G \) satisfies finiteness of cohomological dimension if the following condition holds:

- There is an increasing sequence of natural numbers \( \{a_d\}_{d \in \mathbb{Z}} \) satisfying the following: for a natural number \( d \), given a smooth projective variety \( X \) whose dimension is less than \( d \), then we have \( \text{Hom}_{\text{DM}^{\text{eff}}}(M(X),G[m]) = 0 \) for any \( m > a_d \).

We call such a sequence \( \{a_n\} \) cohomological dimension of \( G \).

Example 2.4. \( \{0\}_n \) is a cohomological dimension of \( \mathbb{Z} \in \text{DM}^{\text{eff}}(k,\mathbb{Z}) \) since we know

\[
\text{Hom}_{\text{DM}^{\text{eff}}}(M(X),\mathbb{Z}[m]) \simeq H^{m,0}(X) \simeq \begin{cases} \mathbb{Z} & m = 0 \\ 0 & m > 0 \end{cases}
\]

for any variety \( X \). \( \{d\}_d \) is a cohomological dimension of \( \Omega^m \in \text{DM}^{\text{eff}}(k,k) \).
For an $R$-linear category $A$ containing all products and an $R$-linear triangulated functor $G : \text{DM}^{\text{eff}}(k, R) \to D(A)$ which sends coproducts to products, by Brown representability theorem [Nee96, Theorem 3.1] there is an object $G \in \text{DM}^{\text{eff}}(k, R)$ such that following isomorphisms holds for any motive $M \in \text{DM}^{\text{eff}}(k, R)$ and $m \in \mathbb{Z}$:

$$\text{Hom}_{\text{DM}^{\text{eff}}}(M, G[m]) \simeq H^m(G(M)).$$

We shall say that the object $G \in \text{DM}^{\text{eff}}$ represents $G$.

**Proposition 2.5.** We assume $A$ has all products. Given an object $G \in \text{DM}^{\text{eff}}(k, R)$ satisfying finiteness of cohomological dimension and which represents an $R$-linear triangulated functor $\text{DM}^{\text{eff}}(k, R) \to D(A)$, and choose a cohomological dimension $\{a_n\}$ of $G$. Choose natural numbers $d$ and $m > a_{3d}$. We set

$$G^i_m(X) = \begin{cases} \text{Hom}_{\text{DM}^{\text{eff}}}(M(X)(j)[2j], G[m]) & j > 2d \\ 0 & j \leq 2d \end{cases}$$

for any smooth projective variety $X$ whose dimension is less than $d$. Then

$$G_m = \bigoplus G^i_m : \text{Sm Proj}(k) \to A$$

is a $d$-Manin invariant with $R$-coefficients.

Before we prove the proposition, we shall give an example. Binda-Park-Ostvær constructed the Hodge object $\underline{\Omega}^n$ in the triangulated category of log motives $\log\text{DM}(k, k)$ for any $n \in \mathbb{N}$ [BPO20, section 9] which represents Hodge cohomology group for log smooth pairs. They also showed that there is an adjunction [BPO20, (5.2.1)]

$$\log\text{DM}^{\text{eff}}(k, R) \xrightarrow{\perp} \text{DM}^{\text{eff}}(k, R).$$

Under the assumption of resolution of singularities, they also showed that for smooth proper variety $X$ over $k$, there is an isomorphism $R\omega^* M(X) \simeq M(X, \text{triv}_X)$, where $\text{triv}_X$ is the trivial log structure on $X$, and there are adjunctions [BPO20, proposition 8.2.12]

$$\log\text{DM}^{\text{eff}}(k, R) \xrightarrow{\perp} \text{DM}^{\text{eff}}(k, R).$$

In this setting, there is an object $\Omega^{n}_{\log} \in \log\text{DM}^{\text{eff}}(k, k)$ which represents Hodge cohomology group [BPO20, section 9]. Let us denote $\underline{\Omega}^n = R\omega^* \Omega^n_{\log}$. For a smooth variety $X$, we choose a good compactification $(\overline{X}, X^\infty)$ of $X$, then there is an isomorphism of $k$-vector spaces:

$$\text{Hom}_{\text{DM}^{\text{eff}}(k, k)}(M(X), \underline{\Omega}^n[m]) \simeq H^m(\overline{X}, \Omega^n_{\overline{X}/k}(\log |X^\infty|)).$$

This induces that $\{n\}$ is a cohomological dimension of $\Omega^n$. Let us compute the Hodge cohomology group of the Tate twist. Firstly we recall the residue exact sequence. For an integer $i \geq 1$, we fix a hyperplane $H = X \times \mathbb{P}^{i-1} \hookrightarrow X \times \mathbb{P}^i$. This regular embedding induces the residue exact sequence:

$$0 \to \Omega^n_{X \times \mathbb{P}^i} \xrightarrow{a} \Omega^n_{X \times \mathbb{P}^i}(\log H) \to \Omega^{n-1}_{H} \to 0$$

where $a$ is the natural inclusion. This exact sequence induces the following long exact sequence:

$$(2.4) \quad H^n_{\text{Zar}}(H, \Omega^n_H) \to H^n_{\text{Zar}}(X \times \mathbb{P}^i, \Omega^n_{X \times \mathbb{P}^i}) \xrightarrow{H^n_{\text{Zar}}(X \times \mathbb{P}^i, a)} H^n_{\text{Zar}}(X \times \mathbb{P}^i, \Omega^n_{X \times \mathbb{P}^i}(\log H)) \to H^n_{\text{Zar}}(H, \Omega^n_{H}) \to \cdots.$$ 

**Lemma 2.6.** The map $H^n(X \times \mathbb{P}^i, a)$ is a split surjective for all $n$. 
Proof. We take a section \( s : X \hookrightarrow X \times \mathbb{P}^1 \) which intersects properly with \( H \). Since \( s \) intersects properly with \( H \), there is the natural map \( s^* : H^n_{Zar}(X \times \mathbb{P}^1, \Omega^m_{X \times \mathbb{P}^1}(\log H)) \to H^n_{Zar}(X, \Omega^m_X) \) induced by \( s \), and it is an isomorphism [BP20, corollary 9.2.2]. We call this isomorphism \( \square \)-invariance. There is the following sequence of varieties

\[
\begin{array}{c}
X \\
\downarrow s \\
X \times \mathbb{P}^1 \\
\downarrow f \\
X
\end{array}
\]

where \( f \) is the projection. This induces the following sequence of \( k \)-vector spaces

\[
H^n(X, \Omega^m_X) \xrightarrow{s^*} H^n_{Zar}(X \times \mathbb{P}^1, \Omega^m_{X \times \mathbb{P}^1}(\log H)) \xrightarrow{\id} H^n(X, \Omega^m_X)
\]

where \( H^n(\Omega^m) \) (resp. \( H^n(f, \Omega^m) \)) is the map induced by the natural map \( \Omega^m_{X \times \mathbb{P}^1} \to s_* \Omega^m_X \) (resp. \( \Omega^m_X \to f_* \Omega^m_{X \times \mathbb{P}^1} \)). Since the map \( H^n(s, \Omega^m) \) is the composition of \( H^n(X \times \mathbb{P}^1, a) \), and \( s^* \), by the \( \square \)-invariance of \( s^* \) we have the following sequence:

\[
H^n(X, \Omega^m_X) \xrightarrow{s^*} H^n_{Zar}(X \times \mathbb{P}^1, \Omega^m_{X \times \mathbb{P}^1}(\log H)) \xrightarrow{\id} H^n(X, \Omega^m_X).
\]

Thanks to this sequence, we obtain the claim. \( \square \)

We keep the notation. By the diagram (2.5) we obtain the equality \( \ker(H^n(s, \Omega^m)) = \text{Coker}(H^n(f, \Omega^m)) \), and by the diagram (2.6) we have the equality

\[
\text{Ker}(H^n(X \times \mathbb{P}^1, a)) \simeq \text{Ker}(H^n(s, \Omega^m)) = \text{Coker}(H^n(f, \Omega^m)).
\]

Besides by lemma 2.6 and the long exact sequence (2.4) we obtain the following:

\[
H^n_{Zar}(H, \Omega^m_H) \simeq \text{Ker}(H^n(X \times \mathbb{P}^1, a)) \simeq \text{Coker}(H^n(f, \Omega^m)).
\]

In the case \( i = 1 \), the hyper plain \( H = X \), the projection \( f : X \times \mathbb{P}^1 \to X \) induces an isomorphism in \( \text{DM}^{\text{eff}}(k, k) \):

\[
M(X \times \mathbb{P}^1)_k \simeq M(X)_k \oplus M(X)_k(1)[2]
\]

where the projection to the first factor \( M(X \times \mathbb{P}^1)_k \to M(X)_k \) is the natural map \( M(f) \). This decomposition induces the following isomorphism

\[
\text{Coker}(\text{DM}^{\text{eff}}(k, k)(M(f), \Omega^m[n])) \simeq \text{DM}^{\text{eff}}(k, k)(M(X)(1)[2], \Omega^m[n]).
\]

We know that the object \( \Omega^m \) represents the Hodge cohomology, thus we have the following:

\[
\text{Coker}(\text{DM}^{\text{eff}}(k, k)(M(f), \Omega^m[n])) \simeq \text{Coker}(H^n(f, \Omega^m)),
\]

thanks to this isomorphism, (2.7) and (2.8), we obtain the following

\[
H^n_{Zar}(X, \Omega^m_X) \simeq \text{DM}^{\text{eff}}(k, k)(M(X)(1)[2], \Omega^m[n]).
\]

Now we start to compute the Hodge realization of the Tate twist \( M(X)_k(i)[2i] \). At first, we prove the following:

**Proposition 2.7.** For a proper smooth variety \( X \), there is an isomorphism of \( k \)-vector spaces:

\[
\text{DM}^{\text{eff}}(k, k)(M(X)_k(i)[2i], \Omega^m[n]) \simeq \text{DM}^{\text{eff}}(k, k)(M(X)_k(i-1)[2i-2], \Omega^m[n-1]).
\]
Proof. We will use a projection formula and \((\mathbb{P}^n, \mathbb{P}^n-1)\)-invariance of the Hodge cohomology. Consider projections \(f_i : X \times \mathbb{P}^i \to X\) and \(f_{i-1} : X \times \mathbb{P}^{i-1} \to X\) and fix a hyper plain \(j : X \times \mathbb{P}^{i-1} \to X \times \mathbb{P}^i\), take another hyper plain \(H \simeq X \times \mathbb{P}^{i-1}\) of \(X \times \mathbb{P}^i\) intersects properly with \(j\).

\[
\begin{array}{c}
\text{j}\ast H & \xrightarrow{j'} & H \\
\downarrow & & \downarrow \text{j} \\
X \times \mathbb{P}^{i-1} & \xrightarrow{j} & X \times \mathbb{P}^i \\
f_{i-1} & \searrow & f_i \\
& & X
\end{array}
\]

Thanks to the projection formula, the cone of \(M(j) : M(X \times \mathbb{P}^{i-1}) \to M(X \times \mathbb{P}^i)\) is equal to \(M(X)_{k(i)}[2i]\) and also the cone of \(M(j') : M(j\ast H) \to M(H)\) is equal to \(M(X)_{k(i-1)}[2i-2]\). Let us study a morphism between the residue sequences induced by \(j\):

\[
\begin{array}{c}
0 \to \Omega^m_{X \times \mathbb{P}^i} \to \Omega^m_{X \times \mathbb{P}^i}(\log H) \to \Omega^{m-1}_H \to 0 \\
\downarrow \jmath_* \Omega^m_{X \times \mathbb{P}^{i-1}} \downarrow \jmath_* \Omega^m_{X \times \mathbb{P}^{i-1}}(\log j\ast H) \downarrow \jmath_* \Omega^{m-1}_H \downarrow 0
\end{array}
\]

Now we know \(f_{i-1}\) and \(f_i\) induce an isomorphism \(H^n(X \times \mathbb{P}^{i-1}, \Omega^m_{X \times \mathbb{P}^{i-1}}(\log j\ast H)) \simeq H^n(X, \Omega^m_X)\) and an isomorphism \(H^n(X \times \mathbb{P}^i, \Omega^m_{X \times \mathbb{P}^i}(\log H)) \simeq H^n(X, \Omega^m_X)\). By Lemma 2.6 there is a morphism between split exact sequences:

\[
0 \to H^{n-1}(H, \Omega^{m-1}_H) \to H^n(X \times \mathbb{P}^i, \Omega^m_{X \times \mathbb{P}^i}) \to H^n(X \times \mathbb{P}^i, \Omega^m_{X \times \mathbb{P}^i}(\log H)) \to 0
\]

Since morphisms \(M(j') : M(j\ast H) \to M(H)\) and \(M(j) : M(X \times \mathbb{P}^{i-1}) \to M(X \times \mathbb{P}^i)\) are split injective, we know that the map \(H^{n-1}(j', \Omega^{m-1}) \simeq \text{DM}^{\text{eff}}(k, k)(M(j'), \Omega^{m-1}[n-1])\) and the map \(H^n(j, \Omega^m) \simeq \text{DM}^{\text{eff}}(k, k)(M(j), \Omega^m[n])\) are split surjective, Now we apply the nine lemma to the diagram (2.10) we obtain that

\[
\text{Ker}(H^{n-1}(j', \Omega^{m-1})) \simeq \text{Ker}(H^n(j, \Omega^m)),
\]

thanks to this isomorphism we have the following

\[
\text{DM}^{\text{eff}}(k, k)(M(X)_{k(i-1)}[2i-2], \Omega^{m-1}[n-1]) \simeq \text{Ker}(\text{DM}^{\text{eff}}(k, k)(M(j'), \Omega^{m-1}[n-1]))
\]

\[
\simeq \text{Ker}(H^{n-1}(j', \Omega^{m-1}))
\]

\[
\simeq \text{Ker}(H^n(j, \Omega^m))
\]

\[
\simeq \text{Ker}(\text{DM}^{\text{eff}}(k, k)(M(j), \Omega^m[n]))
\]

\[
\simeq \text{DM}^{\text{eff}}(k, k)(M(X)_{k(i)}[2i], \Omega^m[n]).
\]

This is what we want. \(\square\)

As a corollary of this proposition, we obtain the following:

**Corollary 2.8.** Take an integer \(m \geq 1\). For an integer \(m \geq i \geq 0\) and any \(n \in \mathbb{Z}\) and any smooth proper variety \(X\), there is an isomorphism of \(k\)-vector space

\[
\text{DM}^{\text{eff}}(k, k)(M(X)_{k(i)}[2i], \Omega^m[n]) \simeq H^m_{\text{Zar}}(X, \Omega^m_X),
\]

and for an integer \(i \geq m\) and an integer \(n \in \mathbb{Z}\) there is a vanishing

\[
\text{DM}^{\text{eff}}(k, k)(M(X)_{k(i)}[2i], \Omega^m[n]) = 0.
\]
Proof. In the case $m \geq i \geq 0$, thanks to proposition 2.7 we have the following:

\[
\text{DM}^{\text{eff}}(k, k)(M(X)_k(i)[2i], \Omega^n[n]) \simeq \text{DM}^{\text{eff}}(k, k)(M(X)_k(i-1)[2i-2], \Omega^{m-1}[n-1]) \\
\simeq \text{DM}^{\text{eff}}(k, k)(M(X)_k(i-2)[2i-4], \Omega^{m-2}[n-2]) \\
\simeq \ldots \\
\simeq \text{DM}^{\text{eff}}(k, k)(M(X)_k, \Omega^{m-i}[n-i]) \\
\simeq H^{n-i}_Z(X, \Omega^{m-i}_X).
\]

Thus the statement (2.11) holds. In the case $i > m$, similarly thanks to proposition 2.7 we have the following:

\[
\text{DM}^{\text{eff}}(k, k)(M(X)_k(i)[2i], \Omega^n[n]) \simeq \text{DM}^{\text{eff}}(k, k)(M(X)_k(i-1)[2i-2], \Omega^{m-1}[n-1]) \\
\simeq \text{DM}^{\text{eff}}(k, k)(M(X)_k(i-2)[2i-4], \Omega^{m-2}[n-2]) \\
\simeq \ldots \\
\simeq \text{DM}^{\text{eff}}(k, k)(M(X)_k(i-m)[2i-2m], \Omega^0[n-m]).
\]

Consider the projection $f : X \times \mathbb{P}^{i-m} \to X$. The morphism $f$ induces the isomorphism $H^l(X, \mathcal{O}_X) \simeq H^l(X \times \mathbb{P}^{i-m}, \mathcal{O}_{X \times \mathbb{P}^{i-m}})$ for all $l$, hence the map $\text{DM}^{\text{eff}}(k, k)(M(f), \Omega^n[l])$ is an isomorphism for all $l$. On the other hand, thanks to the projection formula, we know

\[\text{Coker}(\text{DM}^{\text{eff}}(k, k)(M(f), \Omega^0[l])) \simeq \text{DM}^{\text{eff}}(k, k)(M(\Omega)^1[2] \oplus \ldots \oplus M(X)(i-m)[2i-2m], \Omega^0[l]),\]

thus we obtain the vanishing:

\[\text{DM}^{\text{eff}}(k, k)(M(X)(1)[2] \oplus \ldots \oplus M(X)(i-m)[2i-2m], \Omega^0[l]) = 0\]

for all $l$. This implies the vanishing $\text{DM}^{\text{eff}}(k, k)(M(X)_k(i-m)[2i-2m], \Omega^0[n-m]) = 0$. We can finish the proof. \qed

Let us apply Proposition 2.5 for $G = \Omega^n$. For $m > 3d$ and a smooth projective variety $X$ whose dimension is less than $d$, by Corollary 2.8 we have

\[G^d(X) = \begin{cases} \text{Hom}_{\text{DM}^m}(M(X)(j)[2j], \Omega^n[n]) \simeq H^{m-j}(X, \Omega^{n-j}_{X/k}) & j > 2d \\ 0 & j \leq 2d \end{cases}\]

Thus we know $G_m = \bigoplus_j G^j_m = \bigoplus_{j > d} H^{m-j}(X, \Omega^{n-j}_{X/k}) \oplus \bigoplus_{j \leq d} 0 \simeq \bigoplus_{j \in \mathbb{Z}} H^{m-j}(X, \Omega^{n-j}_{X/k})$ is a $d$-Manin invariant with $k$-coefficients, where we use $H^{m-j}(X, \Omega^{n-j}_{X/k}) = 0$ for $j \leq d$ since $m-j > d$.

Proof of Proposition 2.5. Given smooth projective varieties $X$, $Y$ and $Z$ over $k$. For algebraic cycles $A = \Sigma_i \alpha_i \in \text{CH}^*(X \times_k Y)$ and $B = \Sigma_i \beta_i \in \text{CH}^*(Y \times_k Z)$, there are morphisms in $\text{DM}(k, R)$:

\[\bigoplus_{i \in \mathbb{Z}} M(X)(i)[2i] \xrightarrow{A} \bigoplus_{l \in \mathbb{Z}} M(Y)(l)[2l] \quad \text{and} \quad \bigoplus_{l \in \mathbb{Z}} M(Y)(l)[2l] \xrightarrow{B} \bigoplus_{e \in \mathbb{Z}} M(Z)(e)[2e].\]

Let us restrict these morphisms. Voevodsky proved the following isomorphism [Voe00]

\[\text{Hom}_{\text{DM}(k, R)}(M(X)(i)[2i], M(Y)(l)[2l]) \simeq \text{CH}^{\dim Y + l-i}(X \times_k Y)_R.\]

Since we assume $\max\{\dim X, \dim Y\} \leq d$, if $|l-i| > d$ then $\text{CH}^{\dim Y + l-i}(X \times_k Y)_R = 0$, thus we have

\[\text{Hom}_{\text{DM}(k, R)}(M(X)(i)[2i], M(Y)(l)[2l]) = 0 \quad \text{when} \quad |l-i| > d,\]

and we have a commutative diagram

\[
\begin{array}{ccc}
\bigoplus_{i \in \mathbb{Z}} M(X)(i)[2i] & \overset{A}{\longrightarrow} & \bigoplus_{l \in \mathbb{Z}} M(Y)(l)[2l] \\
\downarrow & & \downarrow \\
\bigoplus_{i \geq 2d} M(X)(i)[2i] & \overset{\tilde{A}}{\longrightarrow} & \bigoplus_{l \geq d} M(Y)(l)[2l] \\
\downarrow & & \downarrow \\
\bigoplus_{i \geq 2d} M(X)(i)[2i] & \overset{\tilde{B}}{\longrightarrow} & \bigoplus_{e \geq 0} M(Z)(e)[2e]
\end{array}
\]
For any smooth projective variety \( Z \) whose dimension is less than \( d \), since we know \( \dim \mathbb{P}^d \times_k W \leq 3d \) and \( m > a_{3d} \) thus we have
\[
\text{Hom}_{\mathcal{DM}^{eff}}(\mathbb{P}^d \times_k W, G[m]) = 0.
\]
For \( 2d \geq l \geq 0 \), \( M(W)(l)[2l] \) is a direct summand of \( \mathbb{P}^{2d} \times_k W \), thus we have an equality
\[
\text{Hom}_{\mathcal{DM}^{eff}}(M(W)(l)[2l], G[m]) = 0 \quad \text{for any} \quad 0 \leq l \leq 2d.
\]
We shall apply the functor \( \text{Hom}_{\mathcal{DM}}(\_, G[m]) \) to the above diagram.
\[
\begin{array}{c}
\bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{DM}^{eff}}(M(X)(i)[2i], G[m]) \\
\bigoplus_{l \in \mathbb{Z}} \text{Hom}_{\mathcal{DM}^{eff}}(M(Y)(l)[2l], G[m]) \\
\bigoplus_{e \in \mathbb{Z}} \text{Hom}_{\mathcal{DM}^{eff}}(M(Z)(e)[2e], G[m])
\end{array}
\]
where vertical maps are split injective. Since the vertical maps preserve the graded structure, we obtain the claim. \( \square \)

We shall extend this story to an \( R \)-linear functor \( \mathcal{DM}^{eff}_{gm}(k, R) \rightarrow D(A) \).

**Definition 2.9.** For an \( R \)-linear functor \( \Gamma : \mathcal{DM}^{eff}_{gm}(k, R) \rightarrow D(A) \), we say \( \Gamma \) satisfies finiteness of cohomological dimension if the following condition holds:

- There is an increasing sequence of natural numbers \( \{a_d\}_{d \in \mathbb{Z}} \) satisfying the following: for a natural number \( d \), given a smooth projective variety \( X \) whose dimension is less than \( d \), then \( H^m(\Gamma(M(X))) = 0 \) for any \( m > a_d \).

We call such a sequence \( \{a_n\} \) cohomological dimension of \( \Gamma \).

By the same discussion, we can prove the following.

**Proposition 2.10.** We assume \( A \) has all coproducts. Given an \( R \)-linear triangulated functor \( \Gamma : \mathcal{DM}^{eff}_{gm}(k, R) \rightarrow D(A) \) which satisfies finiteness of cohomological dimension, and choose a cohomological dimension \( \{a_n\} \) of \( \Gamma \). Choose a natural numbers \( d \) and \( m > a_{3d} \). We set
\[
\Gamma^j_m(X) = \begin{cases} 
H^m(\Gamma(M(X)(j)[2j])) & j > 2d \\
0 & j \leq 2d
\end{cases}
\]
for any smooth projective variety \( X \) whose dimension is less than \( d \). Then
\[
\Gamma_m = \bigoplus \Gamma^j_m : \text{Sm Proj}(k) \rightarrow A
\]
is a \( d \)-Manin invariant with \( R \)-coefficients.

2.2. In this section, we shall show Manin invariants relates to non-commutative algebraic geometry. Let us recall the category \( \mathcal{KM}(k) \) of Gillet-Soulé’s \( K \)-motives from [GS09, Def.5.1, 5.4, 5.6] for a field \( k \) and a commutative ring \( R \). The category \( \mathcal{KM}(k)_R \) is the idempotent completion of the category whose objects are the regular projective \( k \)-varieties over \( k \) and whose morphisms, for regular projective \( k \)-varieties \( X, Y \), are given by
\[
\mathcal{KM}(k)_R(X, Y) := K_0(X \times Y)_R.
\]
Composition is defined as follows:
\[
\mathcal{KM}(k)_R(X, Y) \times \mathcal{KM}(k)_R(Y, Z) \rightarrow \mathcal{KM}(k)_R(X \times Z)
\]
\[
[F] \times [G] \mapsto p_{13*}[p_{12}^*F \otimes p_{23}^*G].
\]
The identity of \( X \) is given by the object in \( \mathcal{KM}(k)_R(X, X) \) corresponding to \( [\Delta_X^*O_X] \in K_0(X \times X)_R \) where \( \Delta_X \) is the diagonal morphism of \( X \). By Grothendieck Riemann-Roch theory [BS58], if \( R \) is a \( \mathbb{Q} \)-algebra there is an \( R \)-linear functor (see [Tab14] for the details)
\[
\omega_{k,R} : \mathcal{KM}(k)_R \rightarrow C_{k,R}.
\]
For a derived equivalence of smooth projective varieties $F : D^b(X) \xrightarrow{\sim} D^b(Y)$ over $k$, there is the object $P \in D^b(X \times Y)$ which represents $F$ and induces an isomorphism $X \xrightarrow{P} Y$ in $\mathbf{KM}(k)_R$ (see [Orl03] for the details). Given a Manin invariant $F = \bigoplus F^* : \mathbf{SmProj}(k) \to \mathcal{A}$ with $R$-coefficients, then there is a functor $\mathcal{F} : C_{k,R} \to \mathcal{A}$. Thus if $R$ is a $\mathbb{Q}$-algebra then we have a sequence of functors

$$\mathbf{KM}(k)_R \xrightarrow{\omega_{k,R}} C_{k,R} \xrightarrow{\mathcal{F}} \mathcal{A}.$$ 

Furthermore, we obtain the following result.

**Theorem 2.11.** For a $\mathbb{Q}$-algebra $R$, any Manin invariant with $R$-coefficients $\mathcal{F} = F^*$ is a derived invariant for smooth projective varieties.

**Proof.** For a derived equivalence of smooth projective varieties $D^b(X) \simeq D^b(Y)$, by [Mat, Lemma 3.2] we have an isomorphism $X_R \simeq Y_R$ in $\mathbf{KM}(k)_R$. Thus we obtain an isomorphism $F(X) = F \circ \omega_{k,R}(X) \simeq \mathcal{F} \circ \omega_{k,R}(Y) = F(Y)$ in $\mathcal{A}$. \qed

To continue this story when $R$ is not a $\mathbb{Q}$-algebra we have studied integral Grothendieck Riemann-Roch theory [Mat, Theorem 1.1]. Consider the full subcategory $\mathbf{SmProj}^{\leq d}(\mathcal{S})$ of $\mathbf{SmProj}(\mathcal{S})$ such that dimension of objects are less than or equal to $d$, and the full subcategory $\mathbf{SmProj}^{(e)}_{\underline{d}}(\mathcal{S})$ of $\mathbf{SmProj}^{\leq d}(\mathcal{S})$ whose objects can be embedded in $\mathbb{P}^e_k$. Now we define $\mathbf{KM}^{\leq d}(k)$ (resp. $\mathbf{KM}^{(e)}_{\underline{d}}(k)$) as the smallest full subcategory of $\mathbf{KM}(k)$ which contains the image of the functor $\mathbf{SmProj}^{\leq d}(k) \to \mathbf{KM}(k)$ (resp. $\mathbf{SmProj}^{(e)}_{\underline{d}}(k) \to \mathbf{KM}(k)$) and is closed under finite coproducts. We recall the comparison between non-commutative motives and Chow motives with integral coefficients which is proved in [Mat, Theorem 3.1]. For natural numbers $d$ and $e$, if $\text{ch}(k) = 0$ let $R$ be a $\mathbb{Z}[\frac{1}{(2d+1)!}]$-algebra (resp. if $\text{ch}(k) = p$ let $R$ be a $\mathbb{Z}[\frac{1}{(2d+e+1)!}]$-algebra), we have constructed an $R$-linear functor (see [Mat, Corollary A.3] (resp. [Mat, Theorem 3.1]));

$$\Phi_R : \mathbf{KM}^{\leq d}(k)_R \to \text{Chow}(k)_R/ - \otimes T_R \quad \text{(resp. } \Phi_R : \mathbf{KM}^{(e)}_{\underline{d}}(k)_R \to \text{Chow}(k)_R/ - \otimes T_R).$$

Via this functor (2.13), the fully faithful functor $\theta : C_{k,R} \to \text{Chow}(k)_R/ - \otimes T_R$ induces the following $R$-linear functor

$$\omega_{k,R} : \mathbf{KM}^{\leq d}(k)_R \to C^{\leq d}_{k,R} \quad \text{(resp. } \omega_{k,R} : \mathbf{KM}^{(e)}_{\underline{d}}(k)_R \to C^{\leq d}_{k,R});$$

For a derived equivalent smooth projective varieties $F : D^b(X) \xrightarrow{\sim} D^b(Y)$ over $k$, there is an isomorphism $X \xrightarrow{P} Y$ in $\mathbf{KM}(k)_R$. Given a $d$-Manin invariant $F = \bigoplus F^* : \mathbf{SmProj}(k) \to \mathcal{A}$ with $R$-coefficients, then there is a functor $\mathcal{F} : C^{\leq d}_{k,R} \to \mathcal{A}$. Thus we have a sequence of functors

$$\mathbf{KM}^{\leq d}(k)_R \xrightarrow{\omega_{k,R}} C^{\leq d}_{k,R} \xrightarrow{\mathcal{F}} \mathcal{A} \quad \text{(resp. } \mathbf{KM}^{(e)}_{\underline{d}}(k)_R \xrightarrow{\omega_{k,R}} C^{\leq d}_{k,R} \xrightarrow{\mathcal{F}} \mathcal{A}).$$

Thus we obtain the following result.

**Theorem 2.12.** If $\text{char}(k) = 0$, for a $\mathbb{Z}[[\frac{1}{(2d+1)!}]]$-algebra $R$, any $d$-Manin invariant with $R$-coefficient $\mathcal{F}$ is a derived invariant for smooth projective varieties whose dimension are less than $d$.

**Theorem 2.13.** If $\text{char}(k) = p > 0$, for a $\mathbb{Z}[[\frac{1}{(2d+e+1)!}]]$-algebra $R$, any $d$-Manin invariant with $R$-coefficient $\mathcal{F}$ is a derived invariant for smooth projective varieties whose dimension are less than $d$ and which can be embedded into $\mathbb{P}^e_k$.

3. The Application to Mirror Symmetry

This section assumes the base field $k$ is of characteristic 0. For an embedding of fields $\sigma : k \hookrightarrow \mathbb{C}$, there is the Betti realization with $R$-coefficients of Chow motives:

$$\Gamma^R_{\text{sing}} : \mathbf{DM}_{\text{eff}}^+(k, R) \xrightarrow{\sigma^*} \mathbf{DM}_{\text{eff}}^+(\mathbb{C}, R) \xrightarrow{\text{RHom}} \mathbf{DA}^\text{eff,\acute{e}t}(\mathbb{C}, R) \xrightarrow{\text{Bli}^0(\mathbb{C})} D(\text{R-mod}),$$
where $\sigma^*$ is the natural base change functor along $\sigma$, and $\text{Bti}(\mathbb{C})$ is the Betti realization of motives constructed in [Ayo14c, Définition 1.7] and the equivalence $\text{Ro}_k$ is proved by Ayoub [Ayo14a, Theorem 4.4]. For a smooth projective variety $X$ over $k$, the functor $\Gamma^R_{\text{sing}}$ induces an isomorphism of $R$-modules

$$H^i(\Gamma^R_{\text{sing}}(M(X)_R)) \simeq H^i_{\text{sing}}(X^\sigma_{\text{an}}, R),$$

and for any positive integer $j \in \mathbb{Z}_{\geq 0}$ there is an isomorphism of $R$-modules

$$(3.1) \hspace{1cm} H^i(\Gamma^R_{\text{sing}}(M(X)_R(j)[2j])) \simeq H^i_{\text{sing}}(X^\sigma_{\text{an}}, R).$$

Let us denote $R_{\text{Bti}} \in \text{DM}^{\text{eff}}(k, R)$ the object which represents $\Gamma^R_{\text{sing}}$. We know $\{2n\}_n$ is a cohomological dimension of $R_{\text{Bti}}$. Choose a natural number $d$. For a natural number $m > 6d$, we set

$$R^i_{\text{Bti}, m}(X) = \bigoplus_{j > 2d} R^i_{\text{Bti}, m}(X) \simeq \bigoplus_{j > 2d} H^{m-2j}_{\text{sing}}(X^\sigma_{\text{an}}, R) \begin{cases} \oplus_{i: \text{even}} H^i_{\text{sing}}(X^\sigma_{\text{an}}, R) & \text{if } m: \text{even} \\ \oplus_{i: \text{odd}} H^i_{\text{sing}}(X^\sigma_{\text{an}}, R) & \text{if } m: \text{odd} \end{cases}$$

since $H^j(X^\sigma_{\text{an}}, R) = 0$ for any $j > 2d$, and by Proposition 2.5 we know $R^i_{\text{Bti}, m}$ is a $d$-Manin invariant. If $R$ is a $\mathbb{Z}_{[(3d+1)!]}$-algebra then by the sequence of functors (2.14) there is a functor

$$(3.2) \hspace{1cm} R^i_{\text{Bti}, m} : \text{KM}^{\leq d}(k)_R \to (R\text{-mod})$$

**Theorem 3.1.** We fix an embedding of fields $\sigma : k \hookrightarrow \mathbb{C}$. Let $X$ and $Y$ be smooth projective varieties over $k$. We denote $d = \dim Y$. We assume that there is a $k$-linear fully faithful triangulated functor $F : D^b(X) \to D^b(Y)$. Then the following holds.

1. There are split injective morphism of $\mathbb{Z}_{[(3d+1)!]}$-module

$$\Phi^{\text{even}}_{\text{sing}}(F_\sigma) : H^\text{even}_{\text{sing}}(X_\sigma, \mathbb{Z}_{[(3d+1)!]}) \hookrightarrow H^\text{even}_{\text{sing}}(Y_\sigma, \mathbb{Z}_{[(3d+1)!]}),$$

$$\Phi^{\text{odd}}_{\text{sing}}(F_\sigma) : H^\text{odd}_{\text{sing}}(X_\sigma, \mathbb{Z}_{[(3d+1)!]}) \hookrightarrow H^\text{odd}_{\text{sing}}(Y_\sigma, \mathbb{Z}_{[(3d+1)!]}).$$

2. If $F$ is an equivalence then $\Phi^{\text{even}}_{\text{sing}}(F_\sigma)$ and $\Phi^{\text{odd}}_{\text{sing}}(F_\sigma)$ are isomorphisms.

**Proof.** We denote $R = \mathbb{Z}_{[(3d+1)!]}$. By the assumption, we know $\dim X \leq d$. Thus $X_R, Y_R \in \text{KM}^{\leq d}(k)_R$ for some $e \in \mathbb{N}$. For a fully faithful functor (resp. equivalent functor) $F : D^b(X) \to D^b(Y)$, there is a split injective map (resp. isomorphism)

$$X_R \hookrightarrow Y_R \text{ (resp. } X_R \simeq Y_R \text{)}$$

in $\text{KM}^{\leq d}(k)_R$ (see [Mat, Lemma 3.2]). The claim follows from the functor (3.2). \hfill \Box

**Theorem 3.2.** Let $X$ and $Y$ be smooth projective surfaces over $\mathbb{C}$. We assume that there is a $\mathbb{C}$-linear equivalence $F : D^b(X) \simeq D^b(Y)$. For a natural number $m$ such that $(m, p) = 1$ for $p = 2, 3, 5, 7$ then there are isomorphisms of the $m$-torsion parts of cohomology groups

$$H^i(X, \mathbb{Z})[m] \simeq H^i(Y, \mathbb{Z})[m]$$

and equalities

$$H^i(X, \mathbb{Z}) \simeq H^i(Y, \mathbb{Z})[m]$$

for any $i$. In particular, there is an isomorphism of the $m$-torsion parts of the abelianization of the fundamental groups

$$\pi^a_1(X)[m] \simeq \pi^a_1(Y)[m].$$
Proof. For simplicity of notation we write $R$ instead of $\mathbb{Z}[\frac{1}{n}]$. For a natural number $m$ such that $(m, p) = 1$ for $p = 2, 3, 5, 7$, by Theorem 3.1 there is an isomorphism of $m$-torsion groups

$$H^+_\text{sing}(X_{\sigma}, R)[m] \simeq H^+_\text{sing}(Y_{\sigma}, R)[m]$$

for $+ = \text{even,odd}$. Since $H^i(X_{\sigma}, R)$ and $H^i(Y_{\sigma}, R)$ are torsion free groups for $i = 0, 1, 4$, and vanish for $i < 0, i > 4$, this produces the isomorphisms claimed in the statement. The second statement follows from the universal coefficients theorem.

Corollary 3.3. Let $G$ be a finite group, and $X$ be a complex projective surface with faithful $G$-action $G \curvearrowright X$. If the quotient $X/G$ is also a complex projective manifold and the canonical map $X \to X/G$ induces an equivalence of bounded derived categories, then the vanishing

$$G_{ab}^G[p] = 0$$

holds for any prime $p > 7$.

Conjecture 3.4. Let $G$ be a finite group, and $X$ be a complex projective d-fold with faithful $G$-action $G \curvearrowright X$. If the quotient $X/G$ is also a complex projective manifold and the canonical map $X \to X/G$ induces an equivalence of bounded derived categories, then the vanishing

$$G_{ab}^G[p] = 0$$

holds for any prime $p > 3d + 1$.

Example 3.5. In [GP01], Gross and Popescu constructed a simply-connected complex projective manifold $Y$ which is an abelian surface fibration to $\mathbb{P}^1$ with exactly 64 sections with a group action $H = (\mathbb{Z}/8\mathbb{Z})^2 \curvearrowright Y$, and conjectured that the quotient $Y/H$ should be the mirror of mirror of $Y$. Homological mirror symmetry would therefore predict that $D^b(Y) \simeq D^b(Y/H)$, and actually the equivalence is proved by Schnell [Sch12]. Since $Y$ is simply-connected, we know isomorphisms $\pi(Y) = 0$ and $\pi(Y/H) = H$, by universal coefficients theorem we know $\text{Tors}(H^2(Y, \mathbb{Z})) \neq \text{Tors}(H^2(Y/H, \mathbb{Z}))$, but for any $(m, 2) = 1$, we have $\text{Tors}(H^2(Y, \mathbb{Z})[m] \simeq \text{Tors}(H^2(Y/H, \mathbb{Z}))[m] = 0$.

4. The Application to Algebraic Geometry of Positive Characteristic

Definition 4.1. For a smooth proper variety $X$ over a perfect field $k$ of characteristic $p$, the variety $X$ is said to be ordinary if $H^i(X, d\Omega^1_{X/k}) = 0$ for any $i$ and $j > 0$. $X$ is said to be Hodge-Witt if the $W$-module $H^i(X, W\Omega^j_{X/k})$ is finitely generated over $W$ for any $i$ and $j$.

Proposition 4.2. ([Jos16, Theorem 4.1.3]) For a smooth proper variety $X$, the following are equivalent:

1. $X$ is ordinary,
2. $X \times X$ is ordinary,
3. $X \times X$ is Hodge-Witt.

Proof. As a product of ordinary varieties is ordinary [Ill90], thus (1) $\implies$ (2). It is well known that an ordinary variety is Hodge-Witt variety, thus (2) $\implies$ (3). By [Eke85, Prop III 7.2(ii)] if $X \times X$ is Hodge-Witt then $X$ is ordinary, thus we obtain (3) $\implies$ (1). 

In [CR12], for a perfect field $k$ of characteristic $p$ and $k$-scheme $S$, Rülling-Chatzistamatiou have constructed a $W$-linear functor (see [KSY16, Theorem 1]):

$$C_{S,W} \to (W\mathcal{O}_S\text{-module})$$

$$f : X \to S \mapsto \bigoplus_{i,j \in \mathbb{Z}} R^if_*W\Omega^j_X.$$ 

In the case $S = \text{Spec} k$, the functor induces the following additive functor (see [CR12, Lemma 3.5.5] and [KSY16, Appendix B])

$$\Gamma^r_{HW} : C_{k,W} \to (W\text{-module})$$

$$X \mapsto \bigoplus_{j-i=r} H^i(X, W\Omega^j_X).$$
Theorem 4.3. Let $X$ and $Y$ be smooth projective varieties over a perfect field $k$ with characteristic $p$. Suppose $X, Y \in \text{SmProj}_{(e)}^{\leq d}(k)$ for some $e$ and $d$, and $p > 2d + e + 1$. We assume that there is a $k$-linear fully faithful triangulated functor $F : D^b(X) \to D^b(Y)$. Then the following holds

1. For an integer $r$, there is a split injective morphism of $W$-module
   $$\bigoplus_{j-i=r} H^i(X, W\Omega^j_X) \hookrightarrow \bigoplus_{j-i=r} H^i(Y, W\Omega^j_Y).$$

2. For an integer $r$, if $F$ is an equivalence, then there is an isomorphism of $W$-module
   $$\bigoplus_{j-i=r} H^i(X, W\Omega^j_X) \cong \bigoplus_{j-i=r} H^i(Y, W\Omega^j_Y).$$

3. If $Y$ is Hodge-Witt, then $X$ is also Hodge-Witt.
4. If $Y$ is ordinary, then $X$ is also ordinary.

Proof. (1). By [Mat, Lemma 3.2], there is an split injective
   $$X \hookrightarrow Y$$
   in $\text{KM}(k)$. For simplicity of notation we write $R$ instead of $\mathbb{Z}[\frac{1}{(2d+e+1)!}]$. The $R$-linear functor
   $$\omega_R : \text{KM}_{(e)}^{\leq d}(k)_R \to C_{k,R}$$
induces the split injective
   $$\theta : X_R \hookrightarrow Y_R$$
in $C_{k,R}$. Since $p > 2d + e + 1$, the Witt ring $W$ is $R$-algebra, thus the $R$-linear functor (4.2) induces the split injective map of $W$-module
   $$\bigoplus_{j-i=r} H^i(X, W\Omega^j_X) \hookrightarrow \bigoplus_{j-i=r} H^i(Y, W\Omega^j_Y).$$

(2). Since there is an equivalence of dg-categories
   $$\text{perf}_{dg}(X) \cong \text{perf}_{dg}(Y),$$
we have an isomorphism of non-commutative motives
   $$U(X)_R \cong U(Y)_R.$$ By the $R$-linear functor $\omega_R : \text{KM}_{(e)}^{\leq d}(k)_R \to C_{k,R}$, we have an isomorphism
   $$X_R \cong Y_R$$
in $C_{k,R}$. The functor (4.2) induces an isomorphism of $W$-module
   $$\bigoplus_{j-i=r} H^i(X, W\Omega^j_X) \cong \bigoplus_{j-i=r} H^i(Y, W\Omega^j_Y).$$

(3). By (1), we obtain the claim. (4). By [Mat, Corollary 3.4], there is a split injective morphism
   $$M(X)_R(2dy)[2i] \hookrightarrow \bigoplus_{i=0}^{d_X+dy} M(Y)_R(i)[2i]$$
in $\text{DM}^{\text{eff}}(k, R)$. By the tensor structure in $\text{DM}^{\text{eff}}(k, R)$, we have a split injective map
   $$h(X \times X) \hookrightarrow \bigoplus_{i=0}^{d_X+dy} \bigoplus_{j=0}^{d_X+dy} h(Y \times Y) \otimes T^{i+j}$$
in $\text{Chow}(k)_R$. Thus we have a split injective
   $$(X \times X)_R \hookrightarrow \bigoplus_{i=0}^{d_X+dy} \bigoplus_{j=0}^{d_X+dy} (Y \times Y)_R$$
5. Descent of semi-orthogonal decomposition

In this section, we shall show that the property of the semi-orthogonal decomposition of derived categories is stable with respect to field extensions. Theorem 5.1 follows from Proposition 5.7 and Lemma 5.13 below.

**Theorem 5.1.** Take smooth projective varieties $X$ and $Y_1,...,Y_j$ over $K$ and Azumaya algebras $\alpha_e$ on $Y_e$ for each $e$. If there is a semi-orthogonal decomposition

$$D^b(X_K) \overset{\text{SOD}}{\simeq} \langle D^b(Y_{1,K}, \alpha_{1,K}), ..., D^b(Y_{j,K}, \alpha_{j,K}) \rangle,$$

then there is a finite extension $L/K$ such that there is a semi-orthogonal decomposition

$$D^b(X_L) \overset{\text{SOD}}{\simeq} \langle D^b(Y_{1,L}, \alpha_{1,L}), ..., D^b(Y_{j,L}, \alpha_{j,L}) \rangle.$$

5.1. In this section, we shall recall the twisted sheaves theory and twisted Fourier-Mukai transformation from [Cal00] and [CS07]. For a noetherian scheme $X$, let $R$ be a $O_X$-algebra. Assume that $R$ is finitely generated projective $O_X$-module and that the morphism of $O_X$-modules:

$$R \otimes_{O_X} R^{\text{op}} \rightarrow \text{End}_{O_X}(R)$$

is an isomorphism, where $R^{\text{op}}$ is the opposite algebra of $R$. Then $R$ is called an Azumaya algebra on $X$. We say that two Azumaya algebra $R$ and $P$ on $X$ are Morita equivalent, written $R \sim_M P$, if there are finitely generated projective $O$-modules $F$ and $F'$ such that

$$R \otimes_{O_X} \text{End}_{O_X}(F) \simeq P \otimes_{O_X} \text{End}_{O_X}(F').$$

We consider the class $\{\text{Azumaya algebra on } X\}$ of Azumaya algebras on $X$. The Brauer group $Br(X)$ is the group given by:

$$Br(X) = \{\text{Azumaya algebra on } X\} / \sim_M$$

with a group structure given by the tensor product over $O_X$. Gabber and de Jong showed that if $X$ is quasi-compact and separated then $Br(X)$ is isomorphic to the torsion in the étale cohomology group $H^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{tor}}$, [DJ, Theorem]. We identify an Azumaya algebra $R$ with the corresponding element $\alpha \in H^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{tor}}$. For an Azumaya algebra $\alpha$, there is an étale covering $\{U_i \rightarrow X\}$ such that $\alpha$ is represented by a Čech cocycle $\alpha_{ijk} \in \Gamma(U_i \times_X U_j \times_X U_k, \mathbb{G}_m)$. An $\alpha$-twisted sheaf is given by a system $(M_i, \phi_{ij})$ where each $M_i$ is quasi-coherent $O_{U_i}$-module on $U_i$, and where $\phi_{ij} : M_i \otimes_{O_{U_i}} O_{U_j} \rightarrow M_j \otimes_{O_{U_j}} O_{U_i}$ are isomorphism and satisfies

$$\phi_{jk} \circ \phi_{ij}|_{U_{ijk}} = \alpha_{ijk} \phi_{ik}|_{U_{ijk}}$$

over $U_{ijk}$. Let us denote $\text{qCoh}(X, \alpha)$ the Abelian category of $\alpha$-twisted sheaves and Coh$(X, \alpha)$ the Abelian category of $\alpha$-twisted coherent sheaves, and we write its bounded derived category by $D^b(X, \alpha)$. We shall recall Canonaco-Stellar’s result. For two Azumaya algebra $\alpha$ and $\beta$ on $X$, take an étale covering $\{U_i \rightarrow X\}$ such that $\alpha$ and $\beta$ can be represented by $\alpha_{ijk} \in \Gamma(U_i \times_X U_j \times_X U_k, \mathbb{G}_m)$ and $\beta_{ijk} \in \Gamma(U_i \times_X U_j \times_X U_k, \mathbb{G}_m)$. For an $\alpha$-twisted sheaf $M_a = (M_{a,i}, \phi_{a,ij})$ and a $\beta$-twisted sheaf $M_b = (M_{b,i}, \phi_{b,ij})$, let us denote $M_a \otimes M_b$ the $\alpha\beta$-twisted sheaf given by the system:

$$\{(M_{a,i} \otimes_{O_{U_i}} M_{b,i}, \phi_{a,ij} \otimes_{O_{U_{ij}}} \phi_{b,ij})\}.$$

For an $\alpha$-twisted sheaf $P = (M_{a,i}, \phi_{a,ij})$, we say $P$ is projective if each $M_{a,i}$ is a locally free $O_{U_i}$-module. For a projective bounded complex $P^\bullet$ of $\alpha$-twisted sheaves and a complex $M^\bullet$ of $\beta$-twisted sheaves, let us denote $P^\bullet \otimes^L M^\bullet$ the total complex $\text{Tot}(P^\bullet \otimes M^\bullet)$. There is a functor

$$\text{Comp}^b(\text{Coh}(X, \beta)) \overset{P^\bullet \otimes^L -}{\longrightarrow} \text{Comp}^b(\text{Coh}(X, \alpha \beta)).$$

Moreover, this extends to the derived functor:

$$D^b(X, \beta) \overset{P^\bullet \otimes^L -}{\longrightarrow} D^b(X, \alpha \beta).$$
For a flat morphism \( f : Y \to X \) of noetherian separated schemes and an Azumaya algebra \( \alpha \) on \( X \), an \( \alpha \)-twisted sheaf \( (\mathcal{M}_i, \phi_{ij}) \), we shall study the pull back functor. The pull back \( f^*(\mathcal{N}_i, \phi_{ij}) \) is given by \( (f^*\mathcal{M}_i, f^*\phi_{ij}) \) and it is a \( f^*\alpha \)-twisted sheaf, and there are functors \( f^* : \text{qCoh}(X, \alpha) \to \text{qCoh}(Y, f^*\alpha) \) and \( f^* : \text{Coh}(X, \alpha) \to \text{Coh}(Y, f^*\alpha) \).

For a flat projective morphism \( f : Y \to X \) of noetherian and separated schemes and an Azumaya algebra \( \alpha \) on \( X \), choose an étale covering \( \{U_i \to X\} \) and a Čech cocyle \( \alpha_{ijk} \in \Gamma(U_i \times_X U_j \times_X U_k, \mathbb{G}_m) \) representing \( \alpha \). Then \( f^*\alpha \) is represented by the étale covering \( \{f^{-1}U_i \to Y\} \) and \( f^*\alpha_{ijk} \in \Gamma(f^{-1}U_i \times_Y f^{-1}U_j \times_Y f^{-1}U_k, \mathbb{G}_m) \). For a \( f^*\alpha \)-twisted sheaf \( (\mathcal{M}_i, \phi_{ij}) \), the push forward \( f_*(\mathcal{N}_i, \phi_{ij}) \) is given by \( (f_*\mathcal{N}_i, f_*\phi_{ij}) \) where \( f_*\phi_{ij} : f_*\mathcal{N}_i \otimes_U U_{ij} \simeq f_*\mathcal{N}_j \otimes_U U_{ij} \) is the isomorphism which fits into the following commutative diagram:

\[
\begin{array}{c}
f_*(\mathcal{N}_i \otimes \mathcal{O}_{f^{-1}U_i}) \otimes \mathcal{O}_{U_i} \cong f_*(\mathcal{N}_i \otimes \mathcal{O}_{U_i}) \otimes \mathcal{O}_{U_i} \\
f_*(\mathcal{N}_j \otimes \mathcal{O}_{f^{-1}U_j}) \otimes \mathcal{O}_{U_i} \cong f_*(\mathcal{N}_j \otimes \mathcal{O}_{U_i}) \otimes \mathcal{O}_{U_i}
\end{array}
\]

Lemma 5.2. For a flat projective morphism \( g : V \to U \) and \( w \in \Gamma(U, \mathcal{O}_U^*) \) and coherent sheaves \( F \) and \( G \) on \( V \), we assume there is an isomorphism \( \phi : F \simeq G \). Then \( g^*w \) induces an isomorphism \( g^*w \cdot \phi : F \simeq G \) and satisfies the following

\[
g_*(g^*w \cdot \phi) \simeq w \cdot g_*\phi
\]

as isomorphisms from \( f_*F \) to \( f_*G \).

Proof. We shall regard \( g^*w \) as an isomorphism \( g^*w : \mathcal{O}_V \simeq \mathcal{O}_V \). By the projection formula, there is a commutative diagram:

\[
\begin{array}{ccc}
g_*(g^*w) & \cong & g_*(F \otimes \mathcal{O}_V) \otimes \mathcal{O}_U \\
g_*(g^*w \cdot \phi) & \cong & g_*(F \otimes \mathcal{O}_V) \otimes \mathcal{O}_U \\
g_*(g^*w \cdot \phi) & \cong & g_*(F) \otimes g^*w \otimes \mathcal{O}_U \\
g_*(g^*w \cdot \phi) & \cong & g_*(g^*w \cdot \phi)
\end{array}
\]

Thus we obtain the claim. \( \square \)

Applying Lemma 5.2 for a flat projective morphism, \( f^{-1}U_i \to V_i \) and an isomorphism \( \phi_{ijk} \circ \phi_{ij}|_{f^{-1}U_{ijk}} : \mathcal{N}_i \otimes \mathcal{O}_{f^{-1}U_i} \otimes \mathcal{O}_{f^{-1}U_{ijk}} \simeq \mathcal{N}_j \otimes \mathcal{O}_{f^{-1}U_j} \otimes \mathcal{O}_{f^{-1}U_{ijk}} \), we know \( f_*(f^*\alpha_{ijk}\phi_{ik}) = \alpha_{ijk}f_\phi_{ik} \) and we obtain

\[
f_\phi_{jk} \circ f_\phi_{ij} = \alpha_{ijk}f_\phi_{ik}.
\]

Thus we know that \( (f_*\mathcal{N}_i, f_*\phi_{ij}) \) is a \( \alpha \)-twisted sheaf and there are functors \( f_* : \text{qCoh}(Y, f^*\alpha) \to \text{qCoh}(X, \alpha) \). If \( (\mathcal{N}_i, \phi_{ij}) \) is a coherent, then \( (f_*\mathcal{N}_i, f_*\phi_{ij}) \) is also coherent since \( f_*\mathcal{N}_i \) is a coherent sheaf on \( U_i \), thus there is a functor \( f_* : \text{Coh}(Y, f^*\alpha) \to \text{Coh}(X, \alpha) \). This functor induces the following

\[
Rf_* : \mathcal{D}^b(Y, f^*\alpha) \to \mathcal{D}^b(X, \alpha),
\]

where we need to check that \( f_* \) is left exact, and each \( R^if_*\mathcal{N}_i, \phi_{ij} \) is coherent for any coherent \( f^*\alpha \)-twisted sheaf. Let us prove this. For a flat morphism \( g : U \to X \), we consider the following Cartesian diagram:

\[
\begin{array}{c}
U \times_X Y \xrightarrow{g} Y \\
\downarrow f \quad \downarrow f \\
U \xrightarrow{g} X
\end{array}
\]
It is easy to check that $g^* \circ f_* \simeq \tilde{f}_* \circ \tilde{g}^*$ as functors $\text{Coh}(Y, f^* \alpha) \to \text{Coh}(U, g^* \alpha)$. We denote the étale map $U_i \to X$ by $q_i$. For any $i$, we consider the following Cartesian diagram:

$$
\begin{array}{c}
  f^{-1}U_i & \xrightarrow{\tilde{q}_i} & Y \\
  f \downarrow & & \downarrow f \\
  U_i & \xrightarrow{q_i} & X \\
\end{array}
$$

Since $q_i$ is an étale morphism, it is enough to prove that $\tilde{f}_*: \text{Coh}(f^{-1}U_i, \tilde{f}^*q_i^*\alpha) \to \text{Coh}(U_i, q_i^*\alpha)$ is left exact, and each $R^i\tilde{f}_*$ preserves coherentness. Since $\alpha$ is trivial on $U_i$ we know that every $f^*q_i^*\alpha$-twisted coherent sheaf is a usual coherent sheaf on $f^{-1}U_i$. Thus we obtain the claim. Thanks to this, we obtain the following functor

$$
Rf_*: D^b(Y, f^*\alpha) \to D^b(X, \alpha).
$$

For a $f^*\alpha$-twisted sheaf $(M_i, \phi_{ij})$, it is easy to see that there is an isomorphism

$$(5.1) \quad R^i f_*(M_i, \phi_{ij}) = (R^i f_* M_i, R^i f_* \phi_{ij}),$$

where $R^i f_* \phi_{ij}$ is the morphism fits into the following diagram

$$
\begin{array}{c}
  R^i f_*(N_i \otimes_{\mathcal{O}_{f^{-1}U_i}} \mathcal{O}_{f^{-1}U_i}) & \xrightarrow{\sim}_{\text{proj.form}} & R^i f_* N_i \otimes_{\mathcal{O}U_i} \mathcal{O}_{U_i} \\
  R^i f_* \phi_{ij} \downarrow & & \downarrow R^i f_* \phi_{ij} \\
  R^i f_*(N_j \otimes_{\mathcal{O}_{f^{-1}U_j}} \mathcal{O}_{f^{-1}U_j}) & \xrightarrow{\sim}_{\text{proj.form}} & R^i f_* N_j \otimes_{\mathcal{O}U_j} \mathcal{O}_{U_i} \\
\end{array}
$$

Here is a big theorem proved by Canonaco-Stellari [CS07].

**Theorem 5.3** (Canonaco-Stellari, Theorem 1.1 [CS07]). Let $X$ and $Y$ be smooth projective varieties over a field $K$, and $\alpha$ and $\beta$ be Azumaya algebra on $X$ and $Y$, respectively. For a triangulated $K$-linear full functor $F : D^b(X, \alpha) \to D^b(Y, \beta)$, there exist $E \in D^b(X \times Y, \alpha^{-1} \boxtimes \beta)$ such that $F$ is equivalent to the following functor

$$(5.2) \quad D^b(X, \alpha) \xrightarrow{q^*} D^b(X \times Y, q^* \alpha) \xrightarrow{E \otimes_{\mathcal{O}_Y} -} D^b(X \times Y, p^* \beta) \xrightarrow{R\rho_*} D^b(Y, \beta)$$

where $p$ is the projection $X \times Y \to Y$ and $q$ is the projection $X \times Y \to X$, and $\alpha^{-1} \boxtimes \beta = q^* \alpha^{-1} \cdot p^* \beta$.

Let us prove that the functor (5.2) compatible with any flat base change.

For flat morphisms $f : Y \to X$ and $g : Z \to X$ of noetherian quasi-compact and separated schemes, Azumaya algebras $\alpha$ on $X$. Let us denote $W$ the Cartesian $Z \times_X Y$.

$$
\begin{array}{c}
  W & \xrightarrow{\tilde{g}} & Y \\
  f \downarrow & & \downarrow f \\
  Z & \xrightarrow{g} & X \\
\end{array}
$$

where $\tilde{g}$ (resp. $\tilde{f}$) is the base of $g$ (resp. $f$) along $f$ (resp. $g$). Consider the following diagram:

$$(5.3) \quad D^b(Y, f^*\alpha) \xrightarrow{\tilde{g}^*} D^b(W, g^* f^*\alpha) \xrightarrow{f^*} D^b(X, \alpha) \xrightarrow{g^*} D^b(Z, g^* \alpha)$$

We shall obtain the following by the definition of flat pullback of twisted sheaves.

**Lemma 5.4.** The diagram 5.3 commutes.
For a flat morphism $f : Y \to X$ of noetherian quasi-compact and separated schemes, Azumaya algebras $\alpha$ and $\beta$ on $X$ and a complex of projective $\alpha$-twisted sheaves $P^\bullet$, we consider the following diagram:

$$
\begin{array}{ccc}
D^b(Y, f^*\beta) & \xrightarrow{f^* P^\bullet \otimes_{\beta} -} & D^b(Y, f^* f^*\beta) \\
\uparrow f^* & & \uparrow f^* \\
D^b(X, \beta) & \xrightarrow{P^\bullet \otimes_{\beta} -} & D^b(X, \alpha\beta)
\end{array}
$$

**Lemma 5.5.** The diagram $(5.4)$ commutes.

**Proof.** Let us denote $P^l = (P^l_1, \psi_{ij}^l)$. For a complex $(\mathcal{M}_i \bullet, \phi_{ij}^\bullet)$ of $\beta$-twisted sheaves on $X$, we know

$$f^* P^\bullet \otimes^L f^* (\mathcal{M}_i \bullet, \phi_{ij}^\bullet) = (\text{Tot}(f^* \mathcal{M}_i \bullet \otimes_{\mathcal{O}_{f^{-1}U_i}} f^* P^l_1), \text{Tot}(f^* \phi_{ij}^\bullet \otimes_{\mathcal{O}_{f^{-1}U_{ij}}} f^* \psi_{ij}^l)).$$

Since $f$ is flat, we have isomorphisms $\text{Tot}(f^* \mathcal{M}_i \bullet \otimes_{\mathcal{O}_{f^{-1}U_i}} f^* P^l_1) \simeq f^* \text{Tot}(\mathcal{M}_i \bullet \otimes_{\mathcal{O}_{U_i}} P^l_1)$ and $\text{Tot}(f^* \phi_{ij}^\bullet \otimes_{\mathcal{O}_{f^{-1}U_{ij}}} f^* \psi_{ij}^l) \simeq f^* \text{Tot}(\phi_{ij}^\bullet \otimes_{\mathcal{O}_{U_{ij}}} \psi_{ij}^l)$. Thus the right hand side of the equation is $f^*(P^\bullet \otimes (\mathcal{M}_i \bullet, \phi_{ij}^\bullet))$. \(\square\)

Given a flat morphism $f : Y \to X$ and projective morphism $g : Z \to X$ of noetherian quasi-compact separated schemes and an Azumaya algebras $\alpha$ on $X$. Let us denote $W$ the Cartesian $Z \times_X Y$.

$$
\begin{array}{ccc}
W & \xrightarrow{\bar{g}} & Y \\
\downarrow \bar{f} & & \downarrow f \\
Z & \xrightarrow{g} & X
\end{array}
$$

Consider the following diagram.

$$
\begin{array}{ccc}
D^b(W, \bar{f}^*g^*\alpha) & \xrightarrow{R\bar{g}^*} & D^b(Y, f^*\alpha) \\
\uparrow \bar{f}^* & & \uparrow f^* \\
D^b(Z, g^*\alpha) & \xrightarrow{Rg^*} & D^b(X, \alpha)
\end{array}
$$

**Lemma 5.6.** The diagram $(5.5)$ commutes.

**Proof.** The claim follows from the isomorphism $(5.1)$. \(\square\)

**Proposition 5.7.** Let $X$ and $Y$ be smooth projective varieties over a field $K$, $\alpha$ and $\beta$ be Azumaya algebras on $X$ and $Y$, respectively. For a triangulated $\overline{K}$-linear fully faithful functor $F : D^b(X_{\overline{K}}, \alpha_{\overline{K}}) \to D^b(Y_{\overline{K}}, \beta_{\overline{K}})$, there is a finite extension $L/K$ such that there exits an $L$-linear fully faithful functor $F_L : D^b(X_L, \alpha_L) \to D^b(Y_L, \beta_L)$ which fits into the following commutative diagram:

$$
\begin{array}{ccc}
D^b(X_{\overline{K}}, \alpha_{\overline{K}}) & \xrightarrow{F} & D^b(Y_{\overline{K}}, \beta_{\overline{K}}) \\
\uparrow F & & \uparrow F \\
D^b(X_L, \alpha_L) & \xrightarrow{F_L} & D^b(Y_L, \beta_L)
\end{array}
$$

where vertical maps are given by the base change along $\overline{K}/L$.

**Proof.** By Proposition 5.3, there is an object $\mathcal{E} \in D^b(X_{\overline{K}} \times_{\overline{K}} Y_{\overline{K}}, \alpha_{\overline{K}}^{-1} \boxtimes \beta_{\overline{K}})$ which represents $F$, i.e. the functor $F$ is isomorphic to the functor

$$D^b(X_{\overline{K}}, \alpha_{\overline{K}}) \xrightarrow{\mathcal{E}} D^b(X_{\overline{K}} \times_{\overline{K}} Y_{\overline{K}}, \alpha_{\overline{K}} \boxtimes \beta_{\overline{K}}) \xrightarrow{\mathcal{E} \otimes_{\beta_{\overline{K}}}^L} D^b(X_{\overline{K}} \times_{\overline{K}} Y_{\overline{K}}, f^* \beta_{\overline{K}}) \xrightarrow{R_{\overline{K}}} D^b(Y_{\overline{K}}, \beta_{\overline{K}}).$$

We can take a finite extension $L/K$ such that there is an object $\mathcal{E}_L \in D^b(X_L \times_L Y_L, \alpha_L^{-1} \boxtimes \beta_L)$ and an isomorphism $f^*\mathcal{E}_L \simeq \mathcal{E}$ where $f$ is a natural map $X_{\overline{K}} \times_{\overline{K}} Y_{\overline{K}} \to X_L \times_L Y_L$. 
Consider the following diagram:

\[
\begin{array}{c}
D^b(X,K) \xrightarrow{\iota} D^b(X \times_K Y, K) \xrightarrow{\varepsilon^\text{L}} D^b(X, Y, \beta_K) \xrightarrow{\beta_K} D^b(Y, \beta_K)
\end{array}
\]

where each of vertical maps are given by flat base change map along \(\text{Spec } K \rightarrow \text{Spec } L\). Be Lemma 5.5 and Lemma 5.6, the above diagram commutes. We denote \(Rq_L^* \circ \varepsilon_L \otimes L \circ q_L^*\) by \(F_L\). Let us show that \(F_L\) is fully faithful. By the base change theorem of coherent sheaves, for objects \(a, b \in D^b(X_L, \alpha)_L\), we have a commutative diagram:

\[
\begin{array}{c}
\text{Hom}_{D^b(X, \alpha)}(a, b_K) \xrightarrow{F} \text{Hom}_{D^b(Y, \beta)}(F(a_K), F(b_K)) \xrightarrow{\sim} \text{Hom}_{D^b(X_L, \alpha_L)}(a, b) \otimes_L K \xrightarrow{\sim} \text{Hom}_{D^b(Y_L, \beta_L)}(F_L(a), F_L(b)) \otimes_L K
\end{array}
\]

Since \(K\) is a flat over \(L\), we obtain the claim. \(\square\)

**Lemma 5.8.** Take smooth projective varieties \(X\) and \(Y_1, \ldots, Y_j\) over \(K\) and Azumaya algebra \(\alpha_e\) on \(Y_e\) for each \(e\). If there is a semi-orthogonal decomposition

\[
D^b(X_K) \simeq D^b(Y_{1,K}, \alpha_{1,K}), \ldots, D^b(Y_{j,K}, \alpha_{j,K}) >,
\]

then there is a finite extension \(L/K\) and a full subcategory \(T \hookrightarrow D^b(X_L)\) such that there is a semi-orthogonal decomposition

\[
T \simeq D^b(Y_{1,L}, \alpha_{1,L}), \ldots, D^b(Y_{j,L}, \alpha_{j,L}) >
\]

**Proof.** By Proposition 5.7, there is a \(L\)-linear fully faithful functor \(F_{e,L} : D^b(Y_{e,L}, \alpha_{e,L}) \hookrightarrow D^b(X_L)\). Let us prove that, for all \(1 \leq i < l \leq e\) and all objects \(a \in D^b(Y_{i,L}, \alpha_{i,L})\) and \(b \in D^b(Y_{l,L}, \alpha_{l,L})\):

\[
\text{Hom}_{D^b(X_L)}(F_{i,L}(a), F_{l,L}(b)) = 0
\]

By the base change theorem of coherent sheaves, this equation follows from the equation

\[
\text{Hom}_{D^b(X_K)}((F_{i,L}(a))_K, (F_{l,L}(b))_K) = \text{Hom}_{D^b(X_K)}(F(a_K), F(b_K)) = 0.
\]

We obtain the claim. \(\square\)

5.2. Let \(Q\) be the inverse limit of the inverse system \((Q_\lambda, u_{\lambda \mu}\}\) in the category of schemes. We assume that the transition \(u_{\lambda \mu} : Q_\lambda \rightarrow Q_\mu\) is finite for any \(\lambda < \mu\). Suppose that \(Q_\lambda\) is quasi-compact and quasi-separated. In this case, for any coherent sheaf \(F\) on \(\varprojlim Q_\lambda\) there is a \(\lambda\) and a coherent sheaf \(F_\lambda\) on \(Q_\lambda\) such that \(F \simeq u_\lambda^* F_\lambda\) where \(u_\lambda\) is the natural map \(Q \rightarrow Q_\lambda\) (see [Gro66, Corollaire (8.5.2.4)]).

**Definition 5.9.** Let \(X\) be a quasi-compact and quasi-separated scheme, \(S\) be a countable set. For objects \(E_i \in D^b(X) \ i \in S\) and a set \(\{E_i\}_{i \in S}\), we define the sub category \(< E_i | i \in S >\) for the smallest sub triangulated category containing \(E_i\) and closed under finite coproducts, shift and cone. We say \(\{E_i\}_{i \in S}\) is a generating set of \(D^b(X)\) iff \(D^b(X) = < E_i | i \in S >\).

**Lemma 5.10.** Let \(f : X \rightarrow Y\) be a morphism of projective varieties over a field \(k\). We assume \(Y\) is smooth over \(k\). Consider objects \(E_i \in D^b(Y) \ i \in S\). If an object \(E \in D^b(Y)\) is in \(< E_i | i \in S >\), then the object \(Lf^* E\) is in \(< Lf^* E_i | i \in S >\).

**Proof.** The claim follows from the fact that the functor \(Lf^*\) preserves finite coproducts, shift and cone. \(\square\)

**Lemma 5.11.** For a projective smooth variety \(X\) over a field \(k\), we fix a very ample divisor \(O_X(1)\). Then the set \(\{O_X(i) | 0 \geq i \geq -m\}\) is a generating set of \(D^b(X)\) where \(m = \dim_k H^0(X, O_X(1)) - 1\).
Proof. The set \( \{ O_X(-i) \mid i \in \mathbb{N} \cup \{ 0 \} \} \) is a generating set of \( D^b(X) \) (see [Gro61, Théorème (2.2.1) (iii)]). Consider the closed immersion \( f : X \to \mathbb{P}^m \) corresponding to the very ample divisor \( O_X(1) \). It is well known that the set \( \{ O_{\mathbb{P}^m}, O_{\mathbb{P}^m}(-1), \ldots, O_{\mathbb{P}^m}(-m) \} \) is a generating set of \( D^b(\mathbb{P}^m) \). For any natural number \( i \), we have an isomorphism in \( D^b(X) \):

\[
Lf^*O_{\mathbb{P}^m}(-i) \cong f^*O_{\mathbb{P}^m}(-i) \cong O_X(-i).
\]

Since \( O_{\mathbb{P}^m}, O_{\mathbb{P}^m}(-1), \ldots, O_{\mathbb{P}^m}(-m) \geq D^b(\mathbb{P}^m) \), for any \( l > m, O_{\mathbb{P}^m}(-l) \in O_{\mathbb{P}^m}, O_{\mathbb{P}^m}(-1), \ldots, O_{\mathbb{P}^m}(-m) \geq \). Thus we obtain the following:

\[
O_X(-l) \in O_X, O_X(-1), \ldots, O_X(-m) >.
\]

Thus by Lemma 5.10 we obtain the claim. \( \Box \)

For a projective variety \( X \) over a field \( K \) and a finite extension \( L/K \), we write \( X_L = X \otimes_K L \) and we denote the morphism \( X_{\overline{K}} \to X_L \) by \( f_L \).

Lemma 5.12. For a morphism \( r : F \to \mathcal{G} \in D^b(X_{\overline{K}}) \), there is a finite extension \( M/K \) and a morphism \( r_M : F_M \to \mathcal{G}_M \in D^b(X_M) \) such that \( r \simeq f_M^*r_M \).

Proof. There is a finite extension \( L/K \) and objects \( F_L, \mathcal{G}_L \in D^b(X_L) \) such that \( f_L^*F_L \cong F \) and \( f_L^*\mathcal{G}_L \cong \mathcal{G} \). Suppose that \( \text{Hom}_{D^b(X_L)}(F, \mathcal{G}) \cong K^\otimes m \) for some \( m \in \mathbb{N} \). By the flat base change theorem for coherent cohomology, there is an isomorphism of \( L \)-vector spaces:

\[
\text{Hom}_{D^b(X_L)}(F_L, \mathcal{G}_L) \cong L^\otimes m.
\]

For a vector \( r \in \text{Hom}_{D^b(X_L)}(F, \mathcal{G}) = K^\otimes m \), there is finite extension \( N/K \) such that \( r \in N^\otimes m \). Choose a finite extension \( M/K \) containing \( N \) and \( L \). If we take \( r_M \in \text{Hom}_{D^b(X_M)}(F_M, \mathcal{G}_M) \simeq M^\otimes m \) corresponding to \( r \in M^\otimes m \), we obtain the claim. \( \Box \)

Lemma 5.13. Consider a smooth projective variety \( X \) over a field \( K \) and full subcategories \( E_i : T_i \to D^b(X) \) for \( 1 \leq i \leq j \). We assume that \( D^b(X_{\overline{K}}) \) is generated by \( f_K^*E_i(T_i) \) i.e., it is the smallest subcategory of \( D^b(X_{\overline{K}}) \) which contains all of the objects in \( f_K^*E_i(T_i) \) for any \( 1 \leq i \leq \varepsilon \) and is closed under finite coproducts, cone and shifts, where \( f_K \) is the natural map \( X_{\overline{K}} \to X \). Then there is a finite extension \( L/K \) such that \( D^b(X_L) \) is generated by \( f_L^*E_i(T_i) \), where \( f_L/K \) is the natural map \( X_L \to X \).

Proof. For an object \( G \) in \( D^b(X_{\overline{K}}) \) we consider the following condition (1):

1. There is a finite extension \( M/K \) and an object \( G_M \in D^b(X_M) \) such that \( f_M^*G_M \simeq G \), and \( G_M \in f_M^*E_1(T_1), f_M^*E_2(T_2), \ldots, f_M^*E_n(T_n) > \).

All objects in \( f_K^*E_i(T_i) \) satisfy (1). If an object \( G \in D^b(X_{\overline{K}}) \) satisfies (1), then \( G[l] \) satisfies (1) for any \( l \in \mathbb{Z} \). If objects \( G_1, \ldots, G_l \in D^b(X_{\overline{K}}) \) satisfy (1), then \( \bigoplus G_i \) satisfies (1). For a distinguished triangle in \( D^b(X_{\overline{K}}) \)

\[
G_1 \to G_2 \to G_3 \to G_1[1],
\]

if \( G_1 \) satisfies (1) and \( G_2 \) satisfies (1), then by Lemma 5.12 \( G_3 \) satisfies (1). Thus the condition (1) is closed under any finite coproducts, cone and shift. We fix an immersion \( X \hookrightarrow \mathbb{P}^m_K \). For any finite extension \( M/K \), there is a Cartesian diagram:

\[
\begin{array}{ccc}
X_K & \xrightarrow{i} & \mathbb{P}^m_K \\
\downarrow{f_M} & & \downarrow{f_M} \\
X_M & \xrightarrow{i_M} & \mathbb{P}^m_K \\
\downarrow{f_{M/K}} & & \downarrow{f_{M/K}} \\
X & \xrightarrow{i} & \mathbb{P}^m_K
\end{array}
\]

Let us denote \( O_X(1) \) the very ample divisor given by \( i \) and \( O_{X_M}(1) \) the very ample divisor given by \( i_M \). We note that there is a natural isomorphism \( O_X(i) \simeq f_M^*O_{X_M}(i) \) for any \( i \in \mathbb{Z} \). For any
\[ m \geq l \geq 0, \mathcal{O}(X^\text{gm})(-l) \text{ satisfies the condition (1), there is a finite extension } M_l/K \text{ and an object } H_l \text{ such that } H_l \text{ is in } f_{M_l/K}^* E(T_1), ..., f_{M_l/K}^* E_n(T_n) > \text{ and } \mathcal{O}(X^\text{gm})(-l) \simeq f_{M_l}^* H_l. \]

Since \( f_{M_l} \) is faithfully flat, we know \( H_l \simeq \mathcal{O}_{M_l}(-l). \) Choose a finite extension \( L \) containing all \( M_l \) over \( K \) then \( \mathcal{O}_L(-l) \in f_{L/K}^* E(T_1), ..., f_{L/K}^* E_n(T_n) > \) for any \( m \geq l \geq 0. \) Since we know \( D^b(X_L) = \mathcal{O}_{X_L}, \mathcal{O}_{X_L}(-1), ..., \mathcal{O}_{X_L}(-m) >, \) we have \( D^b(X_L) = f_{L/K}^* E(T_1), ..., f_{L/K}^* E_n(T_n) >. \) \( \square \)

**Theorem 5.1** follows from Proposition 5.7 and Lemma 5.13. We note the following Lemma.

**Lemma 5.14.** For an extension of fields \( L/K \) and a projective variety \( X/K, \) given full subcategories \( E_i : T_j \leftrightarrow D^b(X) \) for \( 1 \leq j \leq i \) such that there is a semi-orthogonal decomposition

\[ D^b(X) \overset{SOD}{\simeq} E_1(T_1), ..., E_i(T_i) > \]

then there is a semi-orthogonal decomposition

\[ D^b(X_L) \overset{SOD}{\simeq} f_{L/K}^* E(T_1), ..., f_{L/K}^* E_i(T_i) >, \]

where \( f_{L/K} \) is a natural morphism \( X_L \to X. \)

**Proof.** There is some \( m \in \mathbb{N} \) such that \( D^b(X) \) is generated by \( \mathcal{O}_X(-l) \) for \( 0 \leq l \leq m \) and also \( D^b(X_L) \) is generated by \( \mathcal{O}_{X_L}(-l) \) for \( 0 \leq l \leq m. \) Since each \( \mathcal{O}_X(-l) \) is in \( < E_1(T_1), ..., E_i(T_i) >, \) the claim follows from Lemma 5.10. \( \square \)

6. The application to number theory

6.1. In this section, we first recall the result of Fontaine-Messing on \( p \)-adic Hodge theory over unramified base, and the result of Achinger on ordinary reduction.

We let \( k \) be a finite field of characteristic \( p > 0 \) and we write \( W \) for the Witt ring of \( k \) and \( K \) for the fraction field of \( W. \) For a smooth proper variety \( X \) over \( K \) of good reduction, we write \( \mathfrak{X} \) for a smooth proper integral model of \( X \) over \( W, \) and \( X_k \) for the special fiber of \( \mathfrak{X}. \) If \( i < p - 1, \) in \([FM87]\) Fontaine-Messing proved that there is an isomorphism of \( W \)-modules (see \([FM87, \text{Theorem A}] \) or \([Min21, \text{Theorem 0.9}] \))

\[ H^i_{\text{cris}}(X_k/W) \simeq H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W. \]

(6.1)

where \( \overline{K} \) is the completion of \( K. \)

We let \( F \) be a complete discrete valuation field of mixed characteristic \( (0, p), \) and we write \( k \) be the residue field of \( \mathcal{O}. \) For a smooth proper variety \( \mathfrak{X} \) over \( \mathcal{O}, \) we write \( X_k \) for the special fiber and \( X_{\overline{K}} \) for the geometric fiber over \( \overline{K} \) which is the completion of \( K \) endowed with its unique absolute value extending the given absolute value \( | \cdot | \) on \( F. \) In \([Ach20]\) Achinger proved the following theorem.

**Theorem 6.1** (Achinger, Proposition 6.7 \([Ach20]\)). Consider the following conditions:

(1) \( X_k \) is ordinary,

(2) \( p \)-adic Galois representations \( H^i(X_{\overline{K}}, \mathbb{Q}_p) \) are ordinary for all \( i. \)

Then (1) \( \implies \) (2). Moreover, (2) \( \implies \) (1) if \( H^*_{\text{cris}}(X_k/W) \) is a free \( W \)-module and \( H^*(\mathfrak{X}, \Omega^i_{\mathfrak{X}/\mathcal{O}}) \) are free \( \mathcal{O} \)-modules for all \( i, j. \)

6.2. For a prime number \( l \in k^*, \) Ivorra, \([Ivo06, \text{(128)}], \) and later Ayoub, \([Ayo14b, \text{functor (55)}] \) constructed the \( l \)-adic realization functor of motives:

\[ \Gamma_{k,l} : DM_{\text{gm}}(k) \to D^\text{ét}(\text{Spec } k, \mathbb{Z}_l), [\pi : X \to k] \mapsto R\pi_*\pi^*\mathbb{Z}_l. \]

Choose an algebraic closure \( \overline{k}/k. \) We set a functor

\[ \Gamma_{\overline{k},l}^{\text{pre}} : DM_{\text{gm}}(k) \to DM_{\text{gm}}(\overline{k}) \overset{\Gamma_{\overline{k},l}}{\to} D^\text{ét}(\text{Spec } \overline{k}, \mathbb{Z}_l), \]

where the first functor is the natural base change along the finite extension \( \overline{k}/k. \) This functor sends \([\pi : X \to \text{Spec } k] \) to \( R\pi_*\overline{\pi}^*\mathbb{Z}_l, \) where \( \overline{\pi} \) is a morphism \( X \times_k \overline{k} \to \text{Spec } \overline{k} \) comes from the base change of \( \pi \)
along the finite extension $\overline{k}/K$. For a motive $N \in \text{DM}_{gm}^{\text{eff}}(k)$ and a morphism $a : N \to M$ in $\text{DM}_{gm}^{\text{eff}}(k)$, $\Gamma_{G_k, l}^{\text{pre}}(N)$ and $\Gamma_{G_k, l}^{\text{pre}}(a)$ are equipped with an action by $G_k$. Thus we know that there is a functor $\Gamma_{G_k, l} : \text{DM}_{gm}^{\text{eff}}(k) \to D(G_k \cap \mathbb{Z}_l\text{-mod})$ such that the functor $\Gamma_{G_k, l}^{\text{pre}} : \text{DM}_{gm}^{\text{eff}}(k) \to \text{DM}_{gm}^{\text{eff}}(\overline{k}) \to D^\text{ét}(\text{Spec } \overline{k}, \mathbb{Z}_l)$ factors though it:

\[
\begin{array}{c}
\text{DM}_{gm}^{\text{eff}}(k) \quad \Gamma_{G_k, l}^{\text{pre}} \quad \Gamma_{G_k, l} \quad D(G_k \cap \mathbb{Z}_l\text{-mod})
\end{array}
\]

For a smooth projective variety $X$ over $k$, we have an isomorphism $H^j(\Gamma_{G_k, l} M(X)) \simeq H^j(X_{\overline{k}}, \mathbb{Z}_l)$. Also we know, for a natural number $j$, there is an isomorphism $H^j(\Gamma_{G_k, l} M(X)(j)[2j]) \simeq H^{i-2j}(X_{\overline{k}}, \mathbb{Z}_l)(-j)$. It is well known that $\{2n\}_n$ is a cohomological dimension of $\Gamma_{G_k, l}$ (see [Mil80, chapter VI, Theorem 1.1]).

For a natural number $m > 6d$, we set

\[
\Gamma_{G_k, l, m}^j(X) = \begin{cases}
H^m(\Gamma_{G_k, l} M(X)(j)[2j]) & j > 2d \\
0 & j \leq 2d
\end{cases}
\]

If $\dim X \leq d$, for $m > 6d$, then we have an isomorphism

\[
\Gamma_{G_k, l, m}^j(X) = \bigoplus_{j > 2d} \Gamma_{G_k, l, m}^j(X) \simeq \bigoplus_{j > 2d} H^m_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}_l)(-j)
\]

since $H^j(X_{an}, \mathbb{R}) = 0$ for any $j > 2d$, and by Proposition 2.10 we know $\Gamma_{G_k, l, m}$ is a $d$-Manin invariant. We note that if we take $m = 6d + 1$ then we obtain isomorphism of $\mathbb{Z}_l$-module with $G_k$ action:

\[
\Gamma_{G_k, l, 6d+1}^j(X) \otimes_{\mathbb{Z}_l} \mathbb{Z}_l(3d) \simeq \bigoplus_{i : \text{odd}} H^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}_l)\left(\frac{i-1}{2}\right),
\]

and if we take $m = 6d + 2$ then we obtain isomorphism of $\mathbb{Z}_l$-modules with $G_k$ action:

\[
\Gamma_{G_k, l, 6d+2}^j(X) \otimes_{\mathbb{Z}_l} \mathbb{Z}_l(3d + 1) \simeq \bigoplus_{i : \text{even}} H^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}_l)\left(\frac{i}{2}\right).
\]

There is a $\mathbb{Z}_l$-linear functor $\Gamma_{G_k, l, m} : C_{k, \text{gm}}^{\leq d} \to (G_k \cap \mathbb{Z}_l\text{-mod})$. If $l > \begin{cases} 3d + 1 & \text{if } \text{ch}(k) = 0 \\ 2d + e + 1 & \text{if } \text{ch}(k) > 0 \end{cases}$, then

\[
\mathbb{Z}_l \text{ is a } \begin{cases} \mathbb{Z}[\frac{1}{(3d+1)}]\text{-algebra} & \text{if } \text{ch}(k) = 0 \\ \mathbb{Z}[\frac{1}{(2d+e+1)}]\text{-algebra} & \text{if } \text{ch}(k) > 0 \end{cases}. \]

By the sequence of functors (2.14) there is a functor

\[
\Gamma_{G_k, l, m} : \mathbb{K}M_{\leq d}(k)_R \to (G_k \cap \mathbb{Z}_l\text{-mod})
\]

\[
\text{if } \text{ch}(k) = 0
\]

\[
\text{if } \text{ch}(k) > 0
\]

The following theorem follows from the functor (6.4), and Tabuada-Marcollini’s result (Eq.(6.6) below) of non-commutative motives of an Azumaya algebra [TVdB15, Theorem 2.1].

**Theorem 6.2.** Let $k$ be a field of characteristic 0. Choose natural numbers $d$ and a prime number $l > 3d + 1$, and an algebraic closure $\overline{k}/k$. For smooth projective varieties $X$ and $Y_1, ..., Y_j$ whose dimension are less than $d$ and Azumaya algebras $\alpha_i$ on $Y_i$ such that there is a semi-orthogonal decomposition

\[
D^b(X_{\overline{k}}) \simeq < D^b(Y_{1, \overline{k}}, \alpha_{1, \overline{k}}), ..., D^b(Y_{j, \overline{k}}, \alpha_{j, \overline{k}}) >.
\]

Let us denote rank $\alpha_i = r_i$. If $(l, r_i) = 1$ for any $i$, then there is a finite extension $M/k$ such that for any finite extension $L/M$ there are isomorphisms of $\mathbb{Z}_l$-module with Galois action of $G_L$:

\[
\bigoplus_{i : \text{even}} H^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}_l)(\frac{i}{2}) \simeq \bigoplus_{i = 1}^j \bigoplus_{i : \text{even}} H^i_{\text{ét}}(Y_{i, \overline{k}}, \mathbb{Z}_l)(\frac{i}{2})
\]
\[ \bigoplus_{i: \text{odd}} H^i_{\text{ét}}(X_{\overline{\mathbb{k}}}, \mathbb{Z}_l)(\frac{i-1}{2}) \simeq \bigoplus_{e=1}^j \bigoplus_{i: \text{odd}} H^i_{\text{ét}}(Y_{e, \overline{\mathbb{k}}}, \mathbb{Z}_l)(\frac{i-1}{2}). \]

**Proof.** By Theorem 5.1, there is a finite extension \( M/K \) such that there is a semi-orthogonal decomposition

\[ D^b(X_M) \simeq \text{SOD} < D^b(Y_{1,M}, \alpha_{1,M}), \ldots, D^b(Y_{j,M}, \alpha_{j,M}) >, \]

and for any finite extension \( L/M \), we have a semi-orthogonal decomposition

\[ D^b(X_L) \simeq \text{SOD} < D^b(Y_{1,L}, \alpha_{1,L}), \ldots, D^b(Y_{j,L}, \alpha_{j,L}) >. \]

Thanks to [TVdB15, Theorem 2.1], we know an isomorphism

\[ (6.6) \quad U(X_L)_{\mathbb{Z}_l} \simeq \bigoplus_{i=1}^j U(Y_{i,L})_{\mathbb{Z}_l} \]

in \( \mathbf{KMM}(L, \mathbb{Z}_l) \). In fact, this follows from the above semi-orthogonal decomposition and the fact that \( \mathbb{Z}_l \) is \( \mathbb{Z}[\frac{1}{r_1 \ldots r_j}] \)-algebra. By [Mat, Corollary 3.1], we obtain an isomorphism \( (X_L)_{\mathbb{Z}_l} \simeq \bigoplus_{i=1}^j (Y_{i,L})_{\mathbb{Z}_l} \) in \( \mathbf{KM}^{\leq 0}(k, \mathbb{Z}_l) \). The claim follows from the functors (6.4) and isomorphisms (6.2) and (6.3). \( \square \)

6.3. Let us recall the Serre’s density conjecture for ordinary reduction.

**Conjecture 6.3.** (Serre conjecture for ordinary density) Let \( X/K \) be a smooth projective variety over a number field \( K \). Then there is finite extension \( L/K \) such that a positive density of primes \( v \) of \( L \) for which \( X_L \) has a good ordinary reduction at \( v \).

We prove the following theorem.

**Theorem 6.4.** Let \( X \) be a cubic 4-fold which contains \( \mathbb{P}^2 \) over a number field \( K \). Then \( X \) satisfies the conjecture 6.3. Moreover the density of the set of finite primes in which \( X_L \) has a good ordinary is one.

**Proof.** In [Kuz10], Kuznetsov proved that there is a K3 surface \( S \) over \( \overline{\mathbb{k}} \) and an Azumaya algebra \( \alpha \) of rank 2 on \( S \) satisfying the following semi-orthogonal decomposition (see [Kuz10, Theorem 4.3])

\[ D^b(X_{\overline{\mathbb{k}}}) \simeq \text{SOD} < D^b(S, \alpha), O_{X_{\overline{\mathbb{k}}}}, O_{X_{\overline{\mathbb{k}}}}(1), O_{X_{\overline{\mathbb{k}}}}(2) >. \]

Take a finite extension \( M/K \) such that \( S \) and \( \alpha \) are defined by a K3 surface \( S_M \) and \( \alpha_M \) over \( M \). By Theorem 5.1, there is a finite extension \( L'/M \) such that there is a semi-orthogonal decomposition

\[ D^b(X_{L'}) \simeq \text{SOD} < D^b(S_{L'}, \alpha_{L'}), O_{X_{L'}}, O_{X_{L'}}(1), O_{X_{L'}}(2) >. \]

Since a K3 surface \( S_{L'} \) satisfies the conjecture 6.3 (see [BZ09] or [Jos16]), there is a finite extension \( L/L' \) such that a positive density of primes \( v \) of \( L \) for which \( \tilde{S}_L \) has a good ordinary reduction at \( v \). By Lemma 5.14, there is a semi-orthogonal decomposition

\[ D^b(X_L) \simeq \text{SOD} < D^b(S_L, \alpha_L), O_{X_L}, O_{X_L}(1), O_{X_L}(2) >. \]

For a finite prime \( p \) on \( L \), we consider the following conditions where we write \( p \) for the characteristic of the residue field of \( p \).

(c-1) \( p > 13 \).
(c-2) \( X_L \) and \( \tilde{S}_L \) have good reduction at \( p \).
(c-3) The Hodge cohomology \( H^i(X_p, \Omega^j_{X_p}/\mathcal{O}_{X_p}) \) are free \( \mathcal{O}_{L_p} \)-modules for any \( i, j \) where \( X_p \) is an integral model of \( X_{L_p} \) over \( \mathcal{O}_{L_p} \) and \( L_p \) is the completion of \( L \) at \( p \), and \( \mathcal{O}_{L_p} \) is the integral ring of it.
(c-4) \( \tilde{S}_L \) has ordinary reduction at \( p \)
(c-5) \( L_p \) is unramified over \( \mathbb{Q}_p \).
We first prove that all but finitely many finite primes \( p \) satisfy the condition (c-3). Take a non-empty open subscheme \( \text{Spec} \, R \rightarrow \text{Spec} \, \mathcal{O}_L \) such that \( X_L \) has a projective integral model \( X_R \) over \( R \). Consider the subset \( U \) of \( \text{Spec} \, R \) of the union of supports of Hodge cohomology group \( H^i(X_R, \Omega^j_{X/R}) \) as f.g. \( R \)-modules:
\[
U = \bigcup_{i,j} \text{Supp} \left( \text{Tor}^R_{i,j}(X_R, \Omega^j_{X/R}) \right).
\]
The subset \( U \) is finite subset in \( \text{Spec} \, R \) and doesn't contain the generic point of \( \text{Spec} \, R \). For a finite prime \( p \) on \( L \) such that \( p \in \text{Spec} \, R \), consider the base change diagram of \( X_R \rightarrow \text{Spec} \, R \) along the natural morphism \( R \rightarrow \mathcal{O}_{Lp} \). Since the morphism of rings \( R \rightarrow \mathcal{O}_{Lp} \) is flat, by the base change theorem of coherent cohomology we have an isomorphism of \( \mathcal{O}_{Lp} \)-module:
\[
H^i(X_R, \Omega^j_{X/R}) \otimes_R \mathcal{O}_{Lp} \cong H^i(X_{Lp}, \Omega^j_{X/Lp})
\]
for any \( i, j \). Thus if \( p \) is not in \( U \) then \( H^i(X_{Lp}, \Omega^j_{X/Lp}) \) are f.g. \( \mathcal{O}_{Lp} \)-module for any \( i, j \). We know the subset \( \text{Spec} \, R \setminus U \) is an open subscheme of \( \text{Spec} \, \mathcal{O}_L \), so we obtain that for all but finitely many finite primes, \( p \) satisfies the condition (c-3). It is easy to check that for all but finitely many finite primes, \( p \) satisfies conditions (c-1), (c-2) and (c-5).

By the definition of the finite extension \( L/M \), there is a positive density of primes \( v \) of \( L \) for which \( \widehat{S}_L \) has a good ordinary reduction at \( v \). we obtain that the set of finite primes
\[
S = \{ p: \text{finite prime on } L \mid p \text{ satisfies (c-1) } \sim (c-5) \}
\]
has density one.

Let us prove that for any finite prime \( p \) in \( S \), \( X_L \) has ordinary reduction at \( p \). By the condition (c-1) \( p > 13 = 3 \times 4 + 1 \), the functors (6.4) and isomorphisms (6.2) and (6.3) induces following isomorphisms of \( \mathbb{Z}_p \)-module with \( G_{Lp} \) action:
\[
\bigoplus_{i: \text{even}} H^i_{et}(X_{Lp}, \mathbb{Z}_p)(i/2) \cong \bigoplus_{i: \text{even}} H^i_{et}(\widehat{S}_{Lp}, \mathbb{Z}_p)(i/2) \oplus \mathbb{Z}_p^{\oplus 2}
\]
\[
\bigoplus_{i: \text{odd}} H^i_{et}(X_{Lp}, \mathbb{Z}_p)((i + 1)/2) \cong \bigoplus_{i: \text{odd}} H^i_{et}(\widehat{S}_{Lp}, \mathbb{Z}_p)((i + 1)/2)
\]
Choose an isomorphism of fields \( \sigma : \widehat{T}_p \cong \mathbb{C} \), then we have a non-canonical isomorphism of \( \mathbb{Z}_p \)-modules
\[
H^i_{et}(\widehat{S}_{Lp}, \mathbb{Z}_p) \cong H^i_{Sing}(\widehat{S}^an_{\sigma}, \mathbb{Z}) \otimes \mathbb{Z}_p
\]
where \( \widehat{S}_p \) is the base change of \( \widehat{S}_{Lp} \) along \( \sigma \). Since \( \widehat{S}_p \) is a complex K3 surface, the singular cohomology \( H^i_{Sing}(\widehat{S}^an_{\sigma}, \mathbb{Z}) \) is torsion free for \( i = 0, 2, 4 \), and is 0 for \( i = 1, 3 \). Thus we now know the \( \mathbb{Z}_p \)-module \( H^i_{et}(X_{Lp}, \mathbb{Z}_p) \) is free \( \mathbb{Z}_p \)-module for any \( i \). By conditions (c-1) and (c-5), we can apply Fontaine-Messing theorem for \( X \), we obtain the following isomorphism (see isomorphism (6.1)):
\[
H^i_{crys}(X_{L, \kappa(p)}/W(\kappa(p))) \cong H^i_{et}(X_{Lp}, \mathbb{Z}_p) \otimes \mathbb{Z}_p \ W(\kappa(p))
\]
for any \( i \) (for \( 0 \leq i \leq 2 \dim X_{L, \kappa(p)} - 8 \leq 12 < p - 1 \), the isomorphism is induced by Fontaine-Messing theorem, and for \( i > 8 \) the both of side are zero). Thus we have that \( H^i_{crys}(X_{L, \kappa(p)}/W(\kappa(p))) \) are free \( W(\kappa(p)) \)-modules. Thus we can apply Achinger's result [Ach20, Proposition 6.7] for \( X_p \) where we use the condition (c-3), i.e. to show \( X_{\kappa(p)} \) is ordinary it is enough to show that the \( p \)-adic Galois representation \( G_{Lp} \ni H^i_{et}(X_{Lp}, \mathbb{Q}_p) \) is ordinary representation for any \( i \). By the condition (c-5), we know the \( p \)-adic Galois representation \( G_{Lp} \ni H^i_{et}(\widehat{S}_{Lp}, \mathbb{Q}_p) \) is ordinary representation for any \( i \). Since an ordinary \( p \)-adic representation twisted by the Tate twisted \( \mathbb{Q}_p(i) \) is also ordinary, and a direct summand of an ordinary representation is also ordinary, by the isomorphism
\[
\bigoplus_{i: \text{even}} H^i_{et}(X_{Lp}, \mathbb{Z}_p)(i/2) \cong \bigoplus_{i: \text{even}} H^i_{et}(\widehat{S}_{Lp}, \mathbb{Z}_p)(i/2) \oplus \mathbb{Z}_p^{\oplus 2}
\]
we know $H^i_{\text{ét}}(X_{\mathbb{Z}_p}, \mathbb{Z}_p)$ is an ordinary representation if $i$ is even, and we have $\bigoplus_{i \text{ odd}} H^i_{\text{ét}}(X_{\mathbb{Z}_p}, \mathbb{Z}_p)(i/2) = 0$. □

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