On the new nonlinear properties of the nonlinear heat conductivity problem in nondivergence form

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ON THE NEW NONLINEAR PROPERTIES OF THE NONLINEAR HEAT CONDUCTIVITY PROBLEM IN NONDIVERGENCE FORM

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Abstract

In this article, we discuss one problem of nonlinear thermal conductivity with double nonlinearity; an exact analytical solution has been found for it, the analysis of which allows revealing a number of characteristic features of thermal processes in nonlinear media. The following nonlinear effects have been established: the inertial effect, the finite propagation velocity of thermal disturbances, the spatial localization of heat, and the effect of the finite time of the existence of a thermal structure in an absorption medium.

Keywords: degenerate nonlinear, parabolic equation, not divergent, exact solution, new effects, localization, estimate.

Mathematics Subject Classification (2010): 35C05.

Introduction

When we study the processes of energy transfer in high-temperature media, a number of their special properties should be considered. For example, the temperature dependence of the heat capacity and thermal conductivity coefficient of a medium, it is necessary to take into account the contribution to the energy balance of volume radiation, exothermic and endothermic processes of ionization, chemical reactions, combustion, etc. Consideration of these factors determines the nonlinearity of the energy transfer equation. Along with this, one can also take into account convective heat transfer and its influence on the evolution of the process under study.

1 Results and findings

The intensive development of the nonlinear theory of transfer was stimulated by studies in plasma physics [1, 2]. Recently, fundamental results have been obtained here and a number of nonlinear effects determining the properties of inertia and localization of thermal processes have been discovered [2]-[6].

We discuss the following problem on the influence of an instantaneous concentrated heat source in an incompressible nonlinear medium with a coefficient with double nonlinearity of thermal conductivity of temperature and volumetric absorption of thermal energy, whose power depends on temperature and obviously on time according to a power law. Such an unsteady heat conduction process is described by the following Cauchy problem for a quasilinear parabolic equation

\[
\frac{\partial u}{\partial t} = \nabla \cdot \left( \kappa(T) \nabla u \right) + f(x,t),
\]

subject to the initial condition

\[
u(x,0) = v_0(x),
\]

and the boundary conditions

\[
\frac{\partial u}{\partial n} = g(x,t),
\]

where \(\kappa(T)\) is the thermal conductivity coefficient as a function of temperature, \(v_0(x)\) is the initial temperature distribution, \(g(x,t)\) is the heat source distribution, and \(n\) is the outward normal to the boundary of the domain.

The solution to this problem is given by the Green's function method, which involves finding the Green's function \(G(x,t|\xi,\eta)\) for the heat equation

\[
\frac{\partial G}{\partial t} = \nabla \cdot \left( \kappa(T) \nabla G \right),
\]

subject to the initial condition

\[
G(x,t|\xi,\eta) = \delta(x-\xi) \delta(t-\eta),
\]

and the boundary conditions

\[
\frac{\partial G}{\partial n} = 0.
\]

The exact solution to the problem is then given by

\[
u(x,t) = \int_{\Omega} \int_0^t \nabla \cdot \left( \kappa(T(x,t-\eta)) \nabla G(x,t|\xi,\eta) \right) \delta(x-\xi) \delta(t-\eta) d\xi d\eta,
\]

subject to the initial condition

\[
u(x,0) = v_0(x),
\]

and the boundary conditions

\[
\frac{\partial \nu}{\partial n} = 0.
\]

This solution provides a valuable insight into the behavior of heat transfer in nonlinear media, particularly in situations where the temperature and thermal conductivity are not constant.
\[ \frac{\partial u}{\partial t} = u^n \nabla (u^{m-1} |\nabla u|^{p-2} \nabla u) + \text{div}(v(t)u) - b(t)u^q, \]  
\[ u(0, x) = Q_0 \delta(x), \quad (t > 0, x \in \mathbb{R}^N) \]

Here \( u(x, t) \) – temperature, \( b > 0, b(t)u^q \) - volumetric heat absorption power; \( Q_0 \) - value that determines the energy of a heat source at the initial time; \( \delta(x) \) – delta-shaped function characterizing the initial temperature distribution of a concentrated heat source placed at the origin.

To investigating different qualitative properties of the solutions of the problem Cauchy for particular value of numerical parameters devoted many works [6]-[14]. For instance in the case \( m = k, n = 0, 0 < q < 1 \) by analyzing an exact solution [3] when

\[ q = \frac{p - [m(p - 1) - 1]}{p - 1}, \quad 1 < m < 2, \quad p > m(p - 1) - 1 \]

establish the following properties of solutions: an inertial effect of a finite velocity of propagation of thermal disturbances, spatial heat localization and finite time localization solution effect. Considered the problem of the effect of an instantaneous concentrated heat source in incompressible nonlinear medium with a power reliability of a coefficient of heat conduction on temperature in presence of volume absorption of thermal energy, whose power depends on temperature and explicitly on time by a power law.

In [15], Chen-Wnag discussed the initial-boundary value problem of the equation \( u_t = \text{div}(|u|^p \nabla u^{p-2} \nabla u) + \nabla A(u) \) with \( u(0, x) \in L^q(\Omega) \), \( q \geq 1 \). By modifying the usual Morse iteration, imposing some restrictions on the convection function \( A(s) \), the local \( L^\infty \) – estimates were made and \( u_t \in L^2(\mathbb{R}^N \times (\tau, T)) \) was obtained. In [16], Tsutsumi considered the initial-boundary value problem of the equation \( (u^\beta)_t = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( \frac{\partial u^\beta}{\partial x_i} \right)^{p-2} \frac{\partial u^\beta}{\partial x_i} - \lambda u^\gamma \) with the initial value \( u^\beta_0 \in L^1(\Omega) \). By showing that \( (u^{(\beta+1)/2})_t \in L^2(\mathbb{R}^N \times (\tau, t)) \), Tsutsumi researched the existence, uniqueness, regularity, and behavior of solutions to (8); we will give more information of [14] at the appendix of this article.

It can be seen here

\[ q = \frac{p - [k(p - 2) + n + m - 1]}{p - 1}, \quad 1 < m < 2, \quad p \geq k(p - 2) + n + m - 1 \]  

This task (1) has an accurate analytical solution. In order to show this, we consider the class of radially symmetric solutions of the equation obtained by replacing

\[ u(t, x) = w(t, |\xi| = r), \quad \xi = \int_0^t v(ydy) - x, \quad |\xi| = \left( \sum_{i=1}^N \int_0^t v(ydy) - x_i \right)^{1/2}, \quad x \in \mathbb{R}^N \]

Then the unknown function \( w(t, r) \) satisfy the equation

\[ \frac{\partial w}{\partial t} = w^{n-1} r^{p-2} \left( \frac{\partial w}{\partial r} \right)^{p-2} \frac{\partial w}{\partial r} - b(t)w^q, \quad w(0, |x|) = u_0(x), \]
Further setting

\[ w(t, r) = a(t)(f(t) - r^\gamma)_+^{\gamma_1}, \quad \gamma = \frac{p}{(p-1)}, \quad \gamma_1 = \frac{(p-1)/(k(p-2) + m + n - 2)}{v(1) - \gamma^1} \quad (5) \]

where, \( a(t), f(t) \) - functions to be determined, and through \((n)_+\) marked expression \((n)_+ = \max(0, n)\).

To study the properties of solving the problem (1) by introducing the replacement \( w = v^{\frac{1}{1-n}} \) we transform it into the following divergent equation

\[
\frac{1}{1-n} v^{\frac{n}{1-n}} \frac{\partial v}{\partial t} = v^{\frac{n}{1-n}} \frac{\partial}{\partial x} \left[ v^{\frac{m-1}{1-n}} \left( \frac{\partial v^{\frac{k}{1-n}}}{\partial x} \right)^{p-2} \frac{\partial v^l}{\partial x} \right] - b(t)v^{\frac{q}{1-n}}.
\]

Further

\[
\frac{\partial v}{\partial t} = (1-n) \frac{\partial}{\partial x} \left[ v^{m-1} \left( \frac{\partial v^k}{\partial x} \right)^{p-2} \frac{\partial v^l}{\partial x} \right] - (1-n)b(t)v^q,
\]

where \( \frac{m-1}{1-n} = m_1 - 1, \frac{k}{1-n} = k_1, \frac{1}{1-n} = l, \frac{q}{1-n} = q_1. \)

Computing the derivatives of the function \( w(t, r) \), we have

\[
\frac{\partial w}{\partial t} = \frac{da}{dt} (f(t) - r^\gamma)_+^{\gamma_1} + \gamma_1 a(t) \frac{df}{dt} (f(t) - r^\gamma)^{\gamma_1-1},
\]

\[
(r^{N-1}w^{m-1}) \left| \frac{\partial w^{k_1}}{\partial r} \right|^{p-2} \frac{\partial w^l}{\partial r} = -(k\gamma_1)^{p-2}\gamma_1 a^{k_1(p-2)+m_1+1-l}r^N(f(t) - r^\gamma)^{(k_1\gamma_1-1)(p-2)+(m_1-1+l)}
\]

\[
= -(k\gamma_1)^{p-2}a^{k_1(p-2)+m_1+1-l}r^N \quad w(t, r) \in C(Q)
\]

If \((k_1(p-2) + m_1 + l - 1)\gamma_1 - (p-1) = 0 \quad \gamma_1 = \frac{(p-1)}{(k_1(p-2) + m_1 + l - 1)}\)

Further

\[
(1-n)r^{1-N} \frac{\partial}{\partial r} (r^{N-1}w^{m-1}) \left| \frac{\partial w^{k_1}}{\partial r} \right|^{p-2} \frac{\partial w^l}{\partial r} = -(k\gamma_1)^{p-2}\gamma_1 a^{k_1(p-2)+m_1+1-l}(f(t) - r^\gamma)^{(k_1\gamma_1-1)(p-2)+(m_1-1+l)}
\]

\[
= (1-n)[- (k\gamma_1)^{p-2}\gamma_1 N a^{k_1(p-2)+m_1+1-l}(f(t) - r^\gamma)^{(k_1\gamma_1-1)(p-2)+(m_1-1+l-1)}]
\]

or given that

\[(k_1(p-2) + m_1 + l - 1)\gamma_1 - (p-1) = \gamma_1 \quad (7)\]

expression (6) will be rewritten as

\[
(1-n)r^{1-N} \frac{\partial}{\partial r} (r^{N-1}w^{m-1}) \left| \frac{\partial w^{k_1}}{\partial r} \right|^{p-2} \frac{\partial w^l}{\partial r} = -(k\gamma_1)^{p-2}\gamma_1 a^{k_1(p-2)+l+m_1-1}(f(t) - r^\gamma)^{\gamma_1} + \ldots
\]

\]

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\[
  \begin{align*}
  + & \left[ (k_1\gamma_1)^{p-2} \gamma_1 a^{k_1(p-2)+l+m_1-2} \right] r_1 f(t) - r_1 \gamma_1 - 1]
  \cdot (1 - n)r^{1-N} \frac{\partial}{\partial r} \left( w^{m_1-1} \right) \left[ \frac{\partial w^{k_1}}{\partial r} \right]^{p-2} - \frac{\partial w^l}{\partial r} - b(t)w^{m_1} = \\
  = & \left[ (k_1\gamma_1)^{p-2} \gamma_1 N a^{k_1(p-2)+l+m_1-1} \right] \left( f(t) - r_1 \gamma_1 \right) + \\
  + & \left[ (k_1\gamma_1)^{p-2} \gamma_1 a^{k_1(p-2)+l+m_1-2} r_1 - b(t)a^n \right] \left[ f(t) - r_1 \gamma_1 \right] - 1]
  \end{align*}
\]

Such as \( \gamma_1q = \gamma_1 - 1 \).

Now, substituting the calculated expressions in equation (4) we can have the following

\[
  \frac{da}{dt}(f(t) - r_1 \gamma_1) = \gamma_1 a(t) \frac{df}{dt}(f(t) - r_1 \gamma_1) - 1 = \\
  \gamma_1 \gamma_1 a(t) \frac{df}{dt} - \left[ (k_1\gamma_1)^{p-2} \gamma_1 a^{k_1(p-2)+l+m_1-2} \right] r_1 + \\
  + b(t)a^n \left[ f(t) - r_1 \gamma_1 \right] - 1 = 0.
\] (9)

Hence we have

\[
  \frac{da}{dt} + (k_1\gamma_1)^{p-2} \gamma_1 N a^{k_1(p-2)+l+m_1-1}(f(t) - r_1 \gamma_1) + \\
  + (k_1\gamma_1)^{p-2} \gamma_1 a^{k_1(p-2)+l+m_1-2} r_1 - b(t)a^n \left[ f(t) - r_1 \gamma_1 \right] - 1 = 0.
\] (10)

Now, from here to determine the functions \( a(t), f(t) \) we obtain nonlinear differential equations

\[
  \gamma_1 a(t) \frac{df}{dt} + b(t)a^n = (\gamma_1)^p a^{k_1(p-2)+l+m_1-1} f(t)
\]

\[
  \frac{da}{dt} + (k_1\gamma_1)^{p-2}[(\gamma_1 + N)] a^{k_1(p-2)+l+m_1-1} = 0,
\]

\[
  \frac{da}{dt} + (k_1\gamma_1)^{p-2}[(\gamma_1 + N)] a^{k_1(p-2)+l+m_1-1} = 0,
\]

\[
  \gamma_1 a(t) \frac{df}{dt} - (\gamma_1)^p a^{k_1(p-2)+l+m_1-1} f(t) = b(t)a^n
\]

(11)

equation (9) has the following general solution

\[
  a(t) = [c + (k_1(p-2)+l+m_1-1)(\gamma_1)^p - 1][(\gamma_1 + N)] t^{-\frac{1}{k_1(p-2)+l+m_1-2}} = \\
  = [c + (k_1(p-2)+l+m_1-1)p + (k_1(p-2)+l+m_1-1)N] t^{-\frac{1}{k_1(p-2)+l+m_1-2}}
\]

where with integration constant.

We rewrite equation (11) in the form

\[
  \frac{df}{dt} - b_1(t)f = b_2(t)
\]

(12)
Theorem 1.

Then, taking into account (11) from (12), we have

\[ b_1(t) = [c + \left(\frac{p}{k_1(p - 2) + l + m_1 - 2}\right)^{p-1}(p + (k_1(p - 2) + l + m_1 - 2)Nt)]^{-1}, \]

\[ b_2(t) = -\frac{k_1(p - 2) + l + m_1 - 1}{p}b(t)[a(t)]^{q-1}, \]

\[ b_1(t) = \left[\frac{k_1p}{k_1(p - 2) + l + m_1 - 2}\right]^{p-2}(p + (k_1(p - 2) + l + m_1 - 2)Nt]^{-1}. \]

Hence the resolution tending to \( \infty \) at \( t \to 0 \) has the form

\[ a(t) = \left[\frac{k_1p}{k_1(p - 2) + l + m_1 - 2}\right]^{p-2}(p + (k_1(p - 2) + l + m_1 - 2)Nt]^{-1}\]

Equation (12) is a linear equation of the first order. It integrates. Its general solution has the form

\[ f(t) = [c + \left(\frac{p}{k_1(p - 2) + l + m_1 - 2}\right)^{p-1}(p + (k_1(p - 2) + l + m_1 - 1)\cdot Nt)]^{-1}f_0 + \int_0^t b_2(y)e^\int_0^y b_1(y)dy dy \]

at \( c = 0 \) we have

\[ f(t) = [\left(\frac{p}{k_1(p - 2) + l + m_1 - 2}\right)^{p-1}(p + (k_1(p - 2) + l + m_1 - 1)\cdot Nt)]^{-1}f_0 + \int_0^t b_2(y)e^\int_0^y b_1(y)dy dy \]

\[ \sum_{i=1}^{N} (\int_0^t v(y)dy - x_i)^{1/2} = [f(t)]^{(p-1)/p}, \]

\[ \int_0^t v(y)dy < \infty, \ f(t) < \infty, \ \forall t > 0. \]

**Theorem 1.** Let in the equation (1)

\[ q = \frac{p(1 - n) - [k(p - 2) + m]}{(p - 1)(1 - n)}, u_0(x) \leq z(0, x), x \in \mathbb{R}^N \]

where

\[ z(t, r) = a(t)\left(f(t) - r^\gamma\right)_+^{\gamma_1}, \gamma = p/(p - 1), \gamma_1 = (p - 1)(1 - n)/[k(p - 2) + m], \]

\( a(t), f(t) \) - the functions defined above. Then, for the problem (1), there is the phenomenon of KSPV.
Theorem 2. Let in the equation (1)
\[
q = \frac{p(1 - n) - [k(p - 2) + m]}{(p - 1)(1 - n)}, u_0(x) \leq z(0, r), r \in R, f(t) < \infty, t > 0
\]
where
\[
z(t, r) = a(t)(f(t) - r^\gamma) + \gamma_1, \gamma = p/(p - 1), \gamma_1 = (p - 1)(1 - n)/[k(p - 2) + m],
a(t), f(t) - functions defined above. Then for problem (1), the spatial localization of the solution takes place.

2 The case of rapid diffusion \(k_1(p - 2) + m_1 + l - 1 < 0\)

Theorem 3. Let in the equation (1)
\[
q = \frac{p - [k_1(p - 2) + l + m_1 - 1]}{p - 1}, u_0(x) \leq z(0, x), x \in R^N
\]
where
\[
z(t, r) = a(t)(f(t) + r^\gamma) + \gamma_1, \gamma = p/(p - 1), \gamma_1 = (p - 1)/(k_1(p - 2) + l + m_1 - 1),
a(t), f(t) - the functions defined above. Then, to solve problem (1), an estimate takes place. \(u(t, x) \leq z(t, r), r \in R, t > 0\).

The finite lifetime of the thermal pulse is due to the influence of volumetric absorption of thermal energy in the medium under consideration. Indeed, if we consider in it even the initial temperature distribution of the form \(u(x, 0) = const\), then due to volumetric heat absorption, the temperature of the medium will decrease with time.

In the considered problem, the manifestation of the following nonlinear effects is observed: the inertial effect of the finite velocity of propagation of thermal disturbances, the effect of spatial localization of heat, and the impact of the finite time of the existence of a thermal structure in an absorption medium.

3 Conclusions

The results of numerical calculations are given below:

Numerical schemes, algorithms and a set of programs for the tasks in the Python3 environment are developed, the analysis of results on the basis of the received estimates of decisions is carried out.

At \(t > t_m\) volumetric heat absorption becomes the dominant factor in the energy balance, the heating wave is replaced by a cooling wave, and the width of the heat pulse begins to decrease with time. At the moment of time, the heat pulse shrinks to a point and ceases to exist.

Visualization when using a timer:
Figure 1: Parameter values: \( k = 1.1, p = 4, m = 1.1, n = 1.1 \)

Figure 2: Parameter values: \( k = 1.1, p = 4, m = 1.1, n = 1.1 \)
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