Mean field linear–quadratic control: Uniform stabilization and social optimality

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Abstract

This paper is concerned with uniform stabilization and social optimality for general mean field linear–quadratic control systems, where subsystems are coupled via individual dynamics and costs, and the state weight is not assumed with the definiteness condition. For the finite-horizon problem, we first obtain a set of forward–backward stochastic differential equations (FBSDEs) from variational analysis, and construct a feedback-type control by decoupling the FBSDEs. For the infinite-horizon problem, by using solutions to two Riccati equations, we design a set of decentralized control laws, which is further proved to be asymptotically social optimal. Some equivalent conditions are given for uniform stabilization of the systems in different cases, respectively. Finally, the proposed decentralized controls are compared to the asymptotic optimal strategies in previous works.

1. Introduction

Mean field games have drawn increasing attention in many fields including system control, applied mathematics and economics (Bensoussan, Frehse, & Yam, 2013; Caines, Huang, & Malhamé, 2017; Gomes & Saude, 2014). The mean field game involves a very large population of small interacting players with the feature that while the influence of each one is negligible, the impact of the overall population is significant. By combining mean field approximations and individual's best response, the dimensionality difficulty is overcome. Mean field games and control have found wide applications, including smart grids (Ren, Busic, Busic, & Meyn, 2017; Li, Ma, Li, Chen, & Gu, 2019; Ma, Callaway, & Hiskens, 2013), finance, economics (Chan & Sircar, 2015), and social sciences (Bauso, Tentmbe, & Basar, 2016), etc.

By now, mean field games have been intensively studied in the LQ (linear–quadratic) framework (Bensoussan, Sung, Yam, & Yung, 2016; Elliott, Li, & Ni, 2013; Huang, Caines, & Malhamé, 2007; Li & Zhang, 2008; Moon & Basar, 2017; Wang & Zhang, 2012b). Huang et al. developed the Nash certainty equivalence (NCE) based on the fixed-point method and designed an $\epsilon$-Nash equilibrium for mean field LQ games with discount costs by the NCE approach (Huang et al., 2007). The NCE approach was then applied to the cases with long run average costs (Li & Zhang, 2008) and with Markov jump parameters (Wang & Zhang, 2012b), respectively. The works (Bensoussan et al., 2016; Carmona & Delarue, 2013) employed the adjoint equation approach and the fixed-point theorem to obtain sufficient conditions for the existence of the equilibrium strategy over a finite horizon. For other aspects of mean field games, readers are referred to Carmona and Delarue (2013), Huang, Malhamé, and Caines (2006), Lasry and Lions (2007) and Yin, Mehta, Meyn, and Shanbhag (2012) for nonlinear mean field games, Weintraub, Benkard, and Van Roy (2008) for oblivious equilibrium in dynamic games, Huang (2010) and Wang and Zhang (2012a) for mean field games with major players, Huang and Huang (2017) and Moon and Basar (2017) for robust mean field games.

Besides noncooperative games, social optima in mean field models have also attracted much interest. The social optimum control refers to that all the players cooperate to optimize the common social cost—the sum of individual costs, which is a type of team decision problem (Ho, 1980). Huang et al. considered social optima in mean field LQ control, and provided an asymptotic...
the organization of the paper is as follows. In Section 2, the socially optimal control problem is formulated. In Section 3, we construct asymptotically optimal decentralized control laws by tackling FBSDEs for the finite-horizon case. In Section 4, for the infinite-horizon case, the asymptotically optimal controls are designed and analyzed, and some equivalent conditions are further given for uniform stabilization in different cases. In Section 5, some numerical examples are given to show the effectiveness of the proposed control laws. Section 6 concludes the paper.

The main contributions of the paper are summarized as follows.

- We first obtain necessary and sufficient existence conditions of finite-horizon centralized optimal control by variational analysis, and then design a feedback-type decentralized control by tackling FBSDEs with mean field approximations.
- In the case $Q \geq 0$, the necessary and sufficient conditions are given for uniform stabilization of the systems with the help of the system’s observability and detectability.
- In the case that $Q$ is indefinite, the necessary and sufficient conditions are given for uniform stabilization of the systems using the Hamiltonian matrices.
- The asymptotically optimal decentralized controls are obtained under very basic assumptions (without verifying the fixed-point condition). The corresponding social costs are explicitly given by virtue of the solutions to two Riccati equations.

This paper investigates uniform stabilization and social optimality for linear–quadratic mean-field control systems, where subsystems (agents) are coupled via dynamics and individual costs. The state weight $Q$ is not limited to positive semi-definite. This model can be taken as a generation of robust mean-field subsystems (agents) are coupled via dynamics and individual costs. The decentralized control laws are further shown to have asymptotic social optimality. For the infinite-horizon case, we design a set of decentralized control laws by using solutions of two Riccati equations, and the proposed decentralized control laws are compared to the feedback strategies in previous works. Finally, some numerical examples are given to illustrate the effectiveness of the proposed control laws.
convenience of the statement, we assume \( W_i \) is scalar and \( \sigma \in C_{\text{loc}}([0, \infty), \mathbb{R}^p) \). For the finite-horizon problem, our results hold for the case that the matrices \( A, B, G, \ldots \) depend on \( t \).

Assume

(A1) The initial states of agents \( x_i(0), i = 1, \ldots, N \) are mutually independent and have the same mathematical expectation. \( x_i(0) = x_0, \mathbb{E}[x_i(0)] = \bar{x}_0, i = 1, \ldots, N \). There exists a constant \( C_0 \) (independent of \( N \)) such that \( \max_{1 \leq i \leq N} \mathbb{E}[\|x_i(0)\|^2] < C_0 \).

3. The finite-horizon problem

For the convenience of design, we first consider the following finite-horizon problem:

\[
(P1) \quad \inf_{u \in L^2_{\text{loc}}(0, T; \mathbb{R}^m)} J_{\text{loc}}^f(u),
\]

where \( J_{\text{loc}}^f(u) = \sum_{i=1}^N J_i^f(u) \) and \( F_i = \sigma \left( \bigcup_{j=1}^N \mathcal{F}_j \right) \). Here

\[
J_i^f(u) = \mathbb{E} \left[ \int_0^T e^{-\rho t} \left( \|x_i(t) - \Gamma x_i^N(t) - \eta(t)\|^2_Q + \|u_i(t)\|^2_R \right) dt \right].
\]

We first give equivalent conditions for the convexity of (P1).

**Proposition 3.1.** (i) Problem (P1) is convex in \( u \) if and only if for any \( u_i \in L^2_{\text{loc}}(0, T; \mathbb{R}^m), i = 1, \ldots, N \),

\[
\sum_{i=1}^N \int_0^T e^{-\rho t} \left( \|y_i(t) - \Gamma y_i^N(t)\|^2_Q + \|u_i(t)\|^2_R \right) dt \geq 0,
\]

where \( y_i^N = \sum_{j=1}^N y_j/N \) and \( y_i \) satisfies

\[
dy_i(t) = [Ay_i(t) + Gy_i^N(t) + Bu_i(t)] dt,
\]

\( y_i(0) = 0, \ i = 1, 2, \ldots, N \).

(ii) Problem (P1) is uniformly convex in \( u \) if and only if for any \( u_i \in L^2_{\text{loc}}(0, T; \mathbb{R}^m), i = 1, \ldots, N \), there exists \( \gamma > 0 \) such that

\[
\sum_{i=1}^N \int_0^T e^{-\rho t} \left( \|y_i(t) - \Gamma y_i^N(t)\|^2_Q + \|u_i(t)\|^2_R \right) dt \geq \gamma \sum_{i=1}^N \int_0^T e^{-\rho t} \|u_i(t)\|^2 dt.
\]

**Proof.** Let \( x_i \) and \( \bar{x}_i \) be the state processes of agent \( i \) with the control \( v \) and \( \bar{v} \), respectively. Take any \( \lambda_1 \in [0, 1] \) and let \( \lambda_2 = 1 - \lambda_1 \). Then

\[
\lambda_1 J_{\text{loc}}^f(v) + \lambda_2 J_{\text{loc}}^f(\bar{v}) - J_{\text{loc}}^f(\lambda_1 v + \lambda_2 \bar{v}) = \lambda_1 \lambda_2 \sum_{i=1}^N \int_0^T \left( \|x_i(t) - \bar{x}_i(t)\|^2_Q + \|\bar{v}(t) - \bar{v}(t)\|^2_R \right) dt.
\]

Denote \( u = v - \bar{v}, y_1 = x_1 - \bar{x}_1 \). Thus, \( y_1 \) satisfies (4). By the definition of (uniform) convexity, the lemma follows. \( \square \)

By examining the variation of \( J_{\text{loc}}^f \), we obtain the necessary and sufficient conditions for the existence of centralized optimal control of (P1). To simplify the presentation later, we denote

\[
\begin{align*}
\delta &\triangleq \Gamma Q + Q \Gamma - \Gamma Q \Gamma, \\
\hat{\eta} &\triangleq Q \eta - \Gamma Q \eta.
\end{align*}
\]

**Theorem 3.1.** Suppose \( R > 0 \). Then (P1) has a set of optimal control laws if and only if Problem (P1) is convex in \( u \) and the following equation system admits a set of solutions \( (x_i, p_i, \beta_i, i = 1, \ldots, N) \):

\[
\begin{align*}
dx_i(t) = & (Ax_i(t) - BR^{-1}B^T p_i(t) + Gx_i^N(t)) dt + (\sigma(t) dW(t), \\
dp_i(t) = & - \left[ (A - \rho I) p_i(t) + G^T p_i^N(t) + Qx_i(t) \right] dt \\
& + \left[ \mathbb{E}[x_i^N(t)] + \hat{\eta}(t) \right] dt + \sum_{j=1}^N p_j(t) dt, \\
x_i(0) = & x_0, \quad p_i(T) = 0, \quad i = 1, \ldots, N,
\end{align*}
\]

where \( p_i^N(t) = \frac{1}{N} \sum_{j=1}^N p_j(t) \), and furthermore the optimal control is given by \( \tilde{u}_i(t) = -R^{-1}B^T p_i(t) \).

**Proof.** Suppose that \( \tilde{u}_i = -R^{-1}B^T p_i \), where \( (p_i, \beta_i, i = 1, \ldots, N) \) is a set of solutions to the second equation in (5). Denote by \( \bar{x}_i \) the state of agent \( i \) under the control \( \tilde{u}_i \). For any \( u_i \in L^2_{\text{loc}}(0, T; \mathbb{R}^m) \) and \( \theta \in \mathbb{R} (0 \neq 0) \), let \( u_i^\theta = u_i + \theta u_i \). Denote by \( \tilde{x}_i^\theta \) the solution of the following perturbed state equation

\[
dx_i^\theta(t) = \left[ Ax_i^\theta(t) + B(\tilde{u}_i(t) + \theta u_i(t)) + f(t) \right] dt + (\sigma(t) dW(t), \\
\tilde{x}_i^\theta(0) = x_0, \quad i = 1, 2, \ldots, N.
\]

Let \( y_i = (x_i^\theta - \bar{x}_i)/\theta \). It can be verified that \( y_i \) satisfies (4). Then by Itô’s formula, for any \( i = 1, \ldots, N \),

\[
0 = \mathbb{E} [e^{-\rho T} p_i(T, y_i(T)) - \langle p_i(0), y_i(0) \rangle] = \mathbb{E} \int_0^T e^{-\rho t} \left[ \left( -(A - \rho I)^T p_i(t) + G^T p_i^N(t) + Qx_i(t) \right) \right] \\
+ \mathbb{E} \left[ x_i^N(t) + \hat{\eta}(t), y_i(t) \right] (p_i(t), (A - \rho I)y_i(t) + G^T p_i(t), y_i(t)) dt,
\]

which implies

\[
0 = \sum_{i=1}^N \mathbb{E} \int_0^T e^{-\rho t} \left[ \left( -G^T p_i^N(t) + Qx_i(t) \right) \right] \\
+ \mathbb{E} \left[ x_i^N(t) + \hat{\eta}(t), y_i(t) \right] (p_i(t) + G^T p_i(t), y_i(t)) dt.
\]

From (3), we have

\[
J_{\text{loc}}^f(\tilde{u} + \theta u) - J_{\text{loc}}^f(\tilde{u}) = 2\theta I_1 + \theta^2 I_2
\]

where \( \tilde{u} = (\bar{u}_1, \ldots, \bar{u}_N), \)

\[
I_1 = \sum_{i=1}^N \int_0^T e^{-\rho t} \left[ Q(\bar{x}_i(t) - (\Gamma \bar{x}_i^N(t)) + \eta(t)) \right] y_i^2 dt, \\
I_2 = \sum_{i=1}^N \int_0^T e^{-\rho t} \left[ \|y_i(t) - \Gamma y_i^N(t)\|^2 + \|u_i(t)\|^2_R \right] dt.
\]

Note that (suppressing the time \( t \))

\[
\sum_{i=1}^N \int_0^T e^{-\rho t} \left[ Q(\bar{x}_i - (\Gamma \bar{x}_i^N) + \eta) \right] \gamma y_i^N dt
\]

\[
= \mathbb{E} \int_0^T e^{-\rho t} \left[ I_1 + \frac{1}{N} \sum_{i=1}^N y_i^2 \right] dt
\]
\begin{align*}
= & \sum_{j=1}^{N} \mathbb{E} \int_{0}^{T} e^{-\rho t} \left( \frac{\Gamma^2}{N} \sum_{i=1}^{N} (\tilde{x}_i - (\Gamma \tilde{\chi}^{(N)} + \eta)) , y_j \right) dt \\
= & \sum_{j=1}^{N} \mathbb{E} \int_{0}^{T} e^{-\rho t} \left[ Q \left( \tilde{x}_i - (\Gamma \tilde{\chi}^{(N)} + \eta) \right) , y_j \right] dt.
\end{align*}

From (6), one can obtain that
\begin{align*}
I_1 = & \sum_{i=1}^{N} \int_{0}^{T} e^{-\rho t} \left[ \left( Q(\tilde{x}_i - (\Gamma \tilde{\chi}^{(N)} + \eta)) , y_i \right) \right] dt \\
& + \sum_{i=1}^{N} \int_{0}^{T} e^{-\rho t} \left[ \left( -G \tilde{p}^{(N)}(t) + Qx_i(t) \right) \right] dt \\
& + \mathbb{E} \int_{0}^{T} e^{-\rho t} \left[ \left( \sum_{i=1}^{N} (\tilde{\gamma}^{(N)} + \tilde{\eta})(t) + (\tilde{u}_i + B\tilde{p}_i, u_i) \right) \right] dt \\
= & \sum_{i=1}^{N} \int_{0}^{T} e^{-\rho t} \left[ N \tilde{u}_i + B\tilde{p}_i, u_i \right] dt.
\end{align*}

From (7), \( \tilde{u} \) is a minimizer to Problem (P1) if and only if \( I_2 = 0 \) and \( I_1 = 0 \). By Proposition 3.1, \( I_2 = 0 \) if and only if (P1) is convex. \( I_1 = 0 \) is equivalent to
\[ \tilde{u}_i = -BR^{-1}B\tilde{p}_i. \]

Thus, we have the optimality system (5). This implies that (5) admits a solution \((\tilde{x}_i, \tilde{p}_i, \tilde{\gamma}_i, \tilde{\eta}_i, i, j = 1, \ldots, N)\).

On other hand, if the equation system (5) admits a solution \((\tilde{x}_i, \tilde{p}_i, \tilde{\gamma}_i, \tilde{\eta}_i, i, j = 1, \ldots, N)\). Let \( \tilde{u}_i = -BR^{-1}B\tilde{p}_i. \) If (P1) is convex, then \( \tilde{u} \) is a minimizer to Problem (P1).

It follows from (5) that
\begin{align*}
\begin{cases}
\begin{aligned}
dx^{(N)}(t) = & \left( A + G \tilde{x}^{(N)}(t) - BR^{-1}B\tilde{p}^{(N)}(t) + f(t) \right) dt \\
& + \frac{1}{N} \sum_{i=1}^{N} \sigma(t) dW_i(t),
\end{aligned} \\
\begin{aligned}
dp^{(N)}(t) = & \left( A + G - \rho I \right) \tilde{p}^{(N)}(t) \\
& + (I - G) \tilde{x}^{(N)}(t) - \tilde{\eta}(t) \right) dt \\
& + \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \beta_j \tilde{u}_j(t) dW_j(t),
\end{aligned}
\end{cases}
\end{align*}

\begin{align*}
\tilde{x}^{(N)}(0) = & \frac{1}{N} \sum_{i=1}^{N} x_{i0}, \quad \tilde{p}^{(N)}(T) = 0.
\end{align*}

Let \( p(t) = P(t)\tilde{x}(t) + K(t)\tilde{x}^{(N)}(t) + s(t) , t \geq 0 \). Then by (5), (9), and Itô’s formula (suppressing the time \( t \)),
\begin{align*}
\begin{cases}
\begin{aligned}
dp_1 = & \dot{P}(t)\tilde{x}(t) + \dot{K}(t)\tilde{x}^{(N)}(t) + s(t), \quad t \geq 0.
\end{aligned} \\
\begin{aligned}
& + G \tilde{x}^{(N)} + f \right) dt + \sigma dW(t) \\
& + K \left( \left( A + G \right) \tilde{x}^{(N)} - BR^{-1}B\tilde{p}^{(N)} \right) dt \\
& + s) \right) dt \\
& \left( A - \rho \right) \tilde{x}^{(N)} \\
& + G \left( \tilde{p}^{(N)} + \tilde{\eta} \right) dt + \frac{1}{N} \sum_{i=1}^{N} \sigma dW_i(t)
\end{cases}
\end{align*}

This implies \( \beta_j = \frac{1}{2}K\sigma + \rho \sigma \), \( \beta_j = \frac{1}{2}K\sigma \), \( j \neq i \),
\begin{align*}
\rho P(t) = & \dot{P}(t) + A^2P(t) + P(t)A + Q \\
& - \dot{P}(t)BR^{-1}B^T P(t) + P(t)BR^{-1}B^T P(t) + K(t)A + G + G^T P(t) + P(t) \left( - \dot{P}(t) + K(t)BR^{-1}B^T (P(t) + K(t)) \right) \\
& + P(t)BR^{-1}B^T P(t) - \Xi, \quad K(t) = 0.
\end{align*}

\begin{align*}
\rho s(t) = & \dot{s}(t) + [A - BR^{-1}B^T (P + K)^T] s(t) \\
& + (P + K) [f(t) - \tilde{\eta}(t)], \quad s(T) = 0.
\end{align*}

Remark 3.1. Note that (11) is not a standard Riccati equation. Its solvability may be referred to Abou-Kandil, Freiling, Ionescu, and Jank (2003). In particular, by Theorem 4.3 in Ma and Yong (1999, Chapter 2), if \( \det \left[ \begin{bmatrix} 0 & I \end{bmatrix} \right] > 0 \) with
\begin{align*}
A = & \begin{bmatrix} A - BR^{-1}B^T & -BR^{-1}B^T \end{bmatrix} - A^2 + \frac{1}{2} I, \quad \text{then we have}
\end{align*}
\begin{align*}
P(t) = & \begin{bmatrix} 0 & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & I \end{bmatrix}.
\end{align*}

Remark 3.2. Denote \( \Pi = P + K \). Then from (10) and (11), \( \Pi \) satisfies
\begin{align*}
\rho \Pi(t) = & \dot{\Pi}(t) + \left( A + G \right)^T \Pi(t) + \Pi(t) \left( A + G \right) \\
& - \Pi(t)BR^{-1}B^T \Pi(t) + \left( I - G \right)^T Q \left( I - G \right),
\end{align*}
with \( \Pi(T) = 0 \). By Sun et al. (2016, Theorem 4.5), the solvability of (10) and (11) is equivalent to the uniform convexity of two optimal control problems. Particularly, if \( Q \geq 0 \), then (10) and (11) admit a unique solution, respectively.

Theorem 3.2. Assume (A1) holds, and (10)–(11) admit a solution, respectively. Then (P1) has an optimal control
\begin{align*}
\tilde{u}(t) = -BR^{-1}B^T P(t) \tilde{x}(t) + K(t)\tilde{x}^{(N)}(t) + s(t),
\end{align*}
where \( P, K \) and \( s \) are determined by (10)–(12).

To prove Theorem 3.2, we first provide a lemma, which plays a key role in the later analysis.

Lemma 3.1. If (10) and (11) admit a solution, respectively, then Problem (P1) is uniformly convex.

Proof. By (10), (13), and direct calculations, we have
\begin{align*}
& \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} e^{-\rho t} \left( \|y(t) - \Gamma \tilde{y}^{(N)}(t)\|_Q^2 + \|u(t)\|_R^2 \right) dt \\
& = \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} e^{-\rho t} \left( \|y(t)\|_Q^2 - \|\tilde{y}^{(N)}(t)\|_Q^2 + \|u(t)\|_R^2 \right) dt \\
& = \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} e^{-\rho t} \left( \|y(t) - \tilde{y}^{(N)}(t)\|_Q^2 + \|\tilde{y}^{(N)}(t)\|_{Q-2}^2 \\
& + \|u(t) - u^{(N)}(t)\|_R^2 + \|u^{(N)}(t)\|_R^2 \right) dt \\
& \leq \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} e^{-\rho t} \left( \|y(t) - \tilde{y}^{(N)}(t)\|_Q^2 + \|\tilde{y}^{(N)}(t)\|_{Q-2}^2 \\
& + \|u(t)\|_R^2 + \|u^{(N)}(t)\|_R^2 \right) dt
\end{align*}
where the last line follows by Sun et al. (2016, Lemma 2.3). From Theorem 3.1, the lemma follows. □

Proof of Theorem 3.2. Since (10) and (11) have a solution, respectively, then by Ma and Yong (1999, Chapter 2, §4), (9) admits a unique solution. Thus, the FBSDE (5) is decoupled and the existence of a solution follows. From Lemma 3.1, (P1) is uniformly convex. By Theorem 3.1, (P1) has an optimal control admitting a unique solution. Thus, the FBSDE (5) is decoupled and no fixed-point equation is needed.

Here, we first obtain the centralized open-loop control laws are designed. Note that in this case and are fully decoupled and no fixed-point equation is needed.

Theorem 3.3. Assume that (A1) holds, and (10)–(11) admit a solution, respectively. The set of decentralized control laws , , in (15) has asymptotic social optimality, i.e.,

\[
\frac{1}{N} J^{\text{foc}}(\tilde{u}) - \frac{1}{N} \inf_{u \in \mathcal{U}(0,T;\mathbb{R}^n)} J^{\text{foc}}(u) \to 0 \quad (N \to \infty),
\]

and the corresponding social cost is given by

\[
J^{\text{foc}}(\tilde{u}) = \sum_{i=1}^{N} \mathbb{E} \left[ \frac{1}{2} \left\| x_i(t) - x_i^{(N)}(t) \right\|_{\mathcal{F}^t}^2 + \left\| x_i(t) \right\|_{\mathcal{F}^t}^2 \right] + Nq_T + Ne_T,
\]

where

\[
q_T = \int_{0}^{T} e^{-\rho t} \left[ \| \sigma(t) \|_{L_2^T} + \| \sigma(t) \|_{\mathcal{F}^t} \right] dt,
\]

\[
e_T = \mathbb{E} \int_{0}^{T} e^{-\rho t} \| B^* (x(t) - x^*(t)) \|_{\mathcal{F}^t}^2 dt.
\]

Proof. See Appendix A. □

4. The infinite-horizon problem

Based on the analysis in Section 3, we may design the following decentralized control laws for Problem (P):

\[
\hat{u}_i(t) = -R^{-1}B^*[P\hat{x}_i(t) + (\Pi - P)\tilde{x}(t) + s(t)],
\]

\[
t \geq 0, \quad i = 1, \ldots, N,
\]

where and are maximal solutions\(^1\) to the equations

\[
\rho \Pi = A^T \Pi + P - \Pi BR^{-1}B^*P + Q, \quad (21)
\]

\[
\rho \Pi = (A + G)^T \Pi + \Pi (A + G) - \Pi BR^{-1}B^* \Pi + Q - \Xi, \quad (22)
\]

and \( s, \tilde{x}, \Xi \in C_{\rho/2}(\mathcal{L}^0, \mathbb{R}^n) \) are determined by

\[
\rho \Pi(t) = (A + G)^T \Pi(t) + \Pi(t) (A + G) - \Pi BR^{-1}B^* \Pi(t) + Q - \Xi(t),
\]

\[
\hat{x}(t) = (A + G)\tilde{x}(t) - BR^{-1}B^*[\Pi(t)\tilde{x}(t) + s(t)] + f(t), \quad \hat{x}(0) = \tilde{x}_0.
\]

Here \( s(0) \) is to be determined, and the existence conditions of \( P, \Pi, s, \tilde{x} \) need to be investigated further.

4.1. Uniform stabilization of subsystems

We now list some basic assumptions for reference:

(A2) The system \((A - \frac{\rho}{2} I, B)\) is stabilizable, and \((A + G - \frac{\rho}{2} I, B)\) is Hurwitz, where \( A = A - BR^{-1}B^*P \).

(A3) \( Q \geq 0, (A - \frac{\rho}{2} I, \sqrt{Q}) \) is detectable, and \((A+G - \frac{\rho}{2} I, \sqrt{Q}(I-I'))\) is detectable.

Assumptions (A2) and (A3) are basic in the study of the LQ optimal control problem. We will show that under some conditions, (A2) is also necessary for uniform stabilization of multiagent systems. In many cases, (A3) may be weakened to the following assumption.

(A3') \( Q \geq 0, (A - \frac{\rho}{2} I, \sqrt{Q}) \) is detectable, and \((A+G - \frac{\rho}{2} I, \sqrt{Q}(I-I'))\) is detectable.

Lemma 4.1. Under (A2)–(A3), (21) and (22) admit unique solutions \( P > 0, \Pi > 0 \), respectively, and (23)–(24) admits a set of unique solutions \( s, \tilde{x} \in C_{\rho/2}(\mathcal{L}^0, \mathbb{R}^n) \).

Proof. From (A2)–(A3) and (Anderson & Moore, 1990), (21) and (22) admit unique solutions \( P > 0, \Pi > 0 \) such that \( A - BR^{-1}B^*P - \frac{\rho}{2} I \) and \( A + G - BR^{-1}B^* \Pi - \frac{\rho}{2} I \) are Hurwitz, respectively. From an argument in Wang and Zhang (2012a, Appendix A), we obtain \( s, \tilde{x} \in C_{\rho/2}(\mathcal{L}^0, \mathbb{R}^n) \) if and only if

\[
s(t) = \int_{t}^{\infty} e^{-(A+G-\frac{\rho}{2} I)(t-s)} (\Pi (f(t) - \tilde{n}(t))) dt.
\]

Lemma 4.2. Let (A1)–(A3) hold. Then for Problem (P),

\[
\mathbb{E} \int_{0}^{\infty} e^{-\rho t} \| \hat{x}(N)(t) - \tilde{x}(t) \|_{\mathcal{F}^t}^2 dt = \mathcal{O}(1/N),
\]

where \( \hat{x}(N) = \sum_{i=1}^{N} \hat{x}_i \), and \( \hat{x} \) satisfies (24).

Proof. See Appendix B. □

It is shown that the decentralized control laws (15) uniformly stabilize the systems (1).

---

\(^1\) For a Riccati equation (e.g., (21)), \( P \) is called a maximal solution if for any solutions \( P', P - P' \geq 0 \).
Theorem 4.1. Let (A1)–(A3) hold. Then for any $N$,

$$\sum_{i=1}^{N} \mathbb{E} \int_0^\infty e^{-\rho t} \left( \|\dot{x}_i(t)\|^2 + \|\dot{u}_i(t)\|^2 \right) dt < \infty. \quad (26)$$

Proof. See Appendix B. \Box

We now give two equivalent conditions for uniform stabilization of multiagent systems.

Theorem 4.2. Let (A3) hold. Assume that (21)–(22) admit symmetric solutions. Then for Problem (P) the following statements are equivalent:

(i) For any initial condition $(\dot{x}_1(0), \ldots, \dot{x}_N(0))$ satisfying (A1),

$$\sum_{i=1}^{N} \mathbb{E} \int_0^\infty e^{-\rho t} \left( \|\dot{x}_i(t)\|^2 + \|\dot{u}_i(t)\|^2 \right) dt < \infty. \quad (27)$$

(ii) Eqs. (21) and (22) admit unique maximal solutions such that $P > 0$, $\Pi > 0$, and $\dot{A} + G - \frac{\rho}{2}I$ is Hurwitz.

(iii) (A2) holds.

Proof. See Appendix C. \Box

Corollary 1. Assume that (A3) holds and $G = 0$. Assume that (21)–(22) admit symmetric solutions. Then the following statements are equivalent:

(i) For any $(\dot{x}_1(0), \ldots, \dot{x}_N(0))$ satisfying (A1),

$$\sum_{i=1}^{N} \mathbb{E} \int_0^\infty e^{-\rho t} \left( \|\dot{x}_i(t)\|^2 + \|\dot{u}_i(t)\|^2 \right) dt < \infty. \quad (26)$$

(ii) Eqs. (21) and (22) admit unique maximal solutions such that $P > 0$, $\Pi > 0$, respectively.

(iii) The system $(A - \frac{\rho}{2}I, B)$ is stabilizable.

When (A3) is weakened to (A3'), we have the following equivalent conditions of uniform stabilization.

Theorem 4.3. Let (A3') hold. Assume that (21)–(22) admit solutions. Then the following are equivalent:

(i) For any initial $(\dot{x}_1(0), \ldots, \dot{x}_N(0))$ satisfying (A1),

$$\sum_{i=1}^{N} \mathbb{E} \int_0^\infty e^{-\rho t} \left( \|\dot{x}_i(t)\|^2 + \|\dot{u}_i(t)\|^2 \right) dt < \infty. \quad (27)$$

(ii) Eqs. (21) and (22) admit unique maximal solutions $P \geq 0$, $\Pi \geq 0$, and $\dot{A} + G - \frac{\rho}{2}I$ is Hurwitz.

(iii) (A2) holds.

Proof. See Appendix C. \Box

For the more general case that $Q$ are indefinite, we have the following equivalent conditions for uniform stabilization of all the subsystems. Assume (A3') both $M_1$ and $M_2$ have no eigenvalues on the imaginary axis, where

$$M_1 = \begin{bmatrix} A - \frac{\rho}{2}I & B R^{-1}B^T \\ Q & -A^T + \frac{\rho}{2}I \end{bmatrix},$$

$$M_2 = \begin{bmatrix} A + G - \frac{\rho}{2}I & B R^{-1}B^T \\ Q - \Sigma & -(A + G)^T + \frac{\rho}{2}I \end{bmatrix}. $$

Theorem 4.4. Assume that (A3') holds, and (21)–(22) admit solutions. Then the following are equivalent:

(i) For any $(\dot{x}_1(0), \ldots, \dot{x}_N(0))$ satisfying (A1),

$$\sum_{i=1}^{N} \mathbb{E} \int_0^\infty e^{-\rho t} \left( \|\dot{x}_i(t)\|^2 + \|\dot{u}_i(t)\|^2 \right) dt < \infty. \quad (26)$$

(ii) Eqs. (21) and (22) admit unique $\rho$-stabilizing solutions$^2$ (which are also the maximal solutions), and $\dot{A} + G - \frac{\rho}{2}I$ is Hurwitz.

(iii) (A2) holds.

Remark 4.1. $M_1$ and $M_2$ are Hamiltonian matrices. The Hamiltonian matrix plays a significant role in studying general algebraic Riccati equations. See more details of the property of Hamiltonian matrices in Abou-Kandil et al. (2003) and Molinari (1977).

Remark 4.2. For the case $Q = 0$ and $G = 0$, the Hamiltonian matrices reduce to

$$M_1 = M_2 = \begin{bmatrix} A - \frac{\rho}{2}I & -A R^{-1}B^T \\ 0 & 0 \end{bmatrix}. $$

Then it follows from Theorem 4.4 that if $A - \frac{\rho}{2}I$ have no eigenvalues on the imaginary axis, the decentralized controls (15) uniformly stabilize the systems (1) if and only if $(A - \frac{\rho}{2}I, B)$ is stabilizable. Since $Q = 0$ and $A - \frac{\rho}{2}I$ is not Hurwitz necessarily, the system $(A - \frac{\rho}{2}I, \sqrt{Q})$ is not detectable, which implies that the assumptions of Theorem 4.3 in Huang et al. (2012) do not hold.

To show Theorem 4.4, we need two lemmas. The first lemma is copied from Molinari (1977, Theorem 6).

Lemma 4.3. Eqs. (21) and (22) admit unique $\rho$-stabilizing solutions (which are also the maximal solutions) if and only if (A2) and (A3') hold.

Lemma 4.4. Let (A1) hold. Assume that (21) and (22) admit $\rho$-stabilizing solutions, respectively, and $\dot{A} + G - \frac{\rho}{2}I$ is Hurwitz. Then

$$\sum_{i=1}^{N} \mathbb{E} \int_0^\infty e^{-\rho t} \left( \|\dot{x}_i(t)\|^2 + \|\dot{u}_i(t)\|^2 \right) dt < \infty. \quad (26)$$

Proof. From the definition of $\rho$-stabilizing solutions, $A - BR^{-1}B^T P - \frac{\rho}{2}I$ and $A + G - BR^{-1}B^T \Pi - \frac{\rho}{2}I$ are Hurwitz. By the argument in the proof of Theorem 4.1, the lemma follows. \Box

Proof of Theorem 4.4. By using Lemmas 4.3 and 4.4 together with a similar argument in the proof of Theorem 4.1, the theorem follows. \Box

Example 1. Consider a scalar system with $A = a$, $B = b$, $G = g$, $Q = q$, $\Gamma = \gamma$, $R = r > 0$. Then

$$M_1 = \begin{bmatrix} a - \frac{\rho}{2}b^2/r & b^2/rq \\ q/a + \rho/2 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} a + g - \rho/2 & b^2/rq(1 - \gamma)^2 \\ q(1 - \gamma)^2/a + \rho/2 \end{bmatrix}. $$

By direct computations, neither $M_1$ nor $M_2$ has eigenvalues in the imaginary axis if and only if

$$a - \frac{\rho}{2}b^2 + \frac{b^2}{r}q > 0, \quad (28)$$

$$a + g - \frac{\rho}{2}b^2 + \frac{b^2}{r}(1 - \gamma)^2q > 0. \quad (29)$$

$^2$ For a Riccati equation (21), $P$ is called a $\rho$-stabilizing solution if $P$ satisfies (21) and all the eigenvalues of $A - BR^{-1}B^T P - \frac{\rho}{2}I$ are in left half-plane.
Note that if \( q > 0 \) (or \( a - \rho/2 < 0 \), \( q = 0 \), i.e., \((a - \rho/2, \sqrt{q})\) is observable (detectable), then (28) holds, and if \( (1 - \gamma)^2 q > 0 \) \((a+g-\rho/2 < 0, q = 0)\), i.e., \((a+g-\rho/2, \sqrt{(1-\gamma)})\) is observable (detectable), then (29) holds.

For this model, the Riccati equation (21) is written as

\[
b^2 p^2 - 2(a-\rho)p - q = 0. \tag{30}
\]

Let \( \Delta = 4[(a - \rho/2)^2 + b^2q/r] \). If (28) holds then \( \Delta > 0 \), which implies (30) admits two solutions. If \( q > 0 \) then (30) has a unique positive solution such that \( a-b^2p/r-\rho/2 = -\sqrt{\Delta}/2 < 0 \). If \( q = 0 \) and \( a - \rho/2 < 0 \) then (30) has a unique non-negative solution \( p = 0 \) such that \( a-b^2p/r-\rho/2 = a - \rho/2 \leq 0 \).

Assume that (28) and (29) hold. By Theorem 4.4, the system is uniformly stable if and only if \((a-\rho/2, b)\) is stabilizable (i.e., \( b \neq 0 \) or \( a - \rho/2 < 0 \)) and \( a-b^2p/r-\rho/2 + g < 0 \). Note that \( a-b^2p/r-\rho/2 < 0 \). When \( q \leq 0 \), we have \( a-b^2p/r-\rho/2 + g < 0 \).

**Example 2.** We further consider the model in Example 1 for the case that \( a + g = \rho/2 \) and \( \gamma = 1 \) (i.e., (29) does not hold). In this case, the Riccati equation (22) admits a unique solution \( \Pi = 0 \). (23) becomes \( \rho \dot{\beta}(t) = \dot{s}(t) + \frac{2}{\rho} \ddot{s}(t) \) and has a unique solution \( \beta(t) \equiv 0 \) in \( C_{p/o}([0, \infty), \mathbb{R}) \). Thus, \( \dddot{x} \) satisfies

\[
\begin{align*}
\dddot{x} &= \frac{\rho}{2} \dddot{x}(t) + f(t). \tag{31}
\end{align*}
\]

Assume that \( f \) is a constant. Then (31) does not admit a solution in \( C_{p/o}([0, \infty), \mathbb{R}) \) unless \( \dddot{x}(0) = -2f/\rho \).

4.2. Asymptotic social optimality

Now we are in a position to state the asymptotic optimality of the decentralized control.

**Theorem 4.5.** Let (A1)-(A3) hold. For Problem (P), the set of decentralized control laws \( \{\hat{u}_1, \ldots, \hat{u}_N\} \) given by (20) has asymptotic social optimality, i.e.,

\[
\frac{1}{N} J_{soc}(\hat{u}) - \frac{1}{N} \inf_{u \in \mathcal{U}_N} J_{soc}(u) = O(1/\sqrt{N}).
\]

**Proof.** We first prove that for \( u \in \mathcal{U}_N, J_{soc}(u) < NC_1 \) implies that

\[
\sum_{i=1}^{N} \mathbb{E} \int_0^\infty e^{-pt} \left( |\dot{x}_i(t)|^2 + |u_i(t)|^2 \right) dt < NC_2, \tag{32}
\]

for all \( i = 1, \ldots, N \). From \( J_{soc}(u) < NC_1 \), we have \( \sum_{i=1}^{N} \mathbb{E} \int_0^\infty e^{-pt} |u_i(t)|^2 dt < NC \) and

\[
\sum_{i=1}^{N} \mathbb{E} \int_0^\infty e^{-pt} \left( |\dot{x}_i(t)| - \Gamma x_i(t) \right)^2 dt < NC, \tag{33}
\]

which further implies that

\[
\mathbb{E} \int_0^\infty e^{-pt} \left( |(I-\Gamma)x_i(t)|^2 \right) dt \leq \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \int_0^\infty e^{-pt} \left( |\dot{x}_i(t)| - \Gamma x_i(t) \right)^2 dt < C. \tag{34}
\]

By (1) we have

\[
\begin{align*}
\dddot{x}_i(t) &= [(A + G)x_i(t) + Bu_i(t) + f(t)] dt \\
&\quad + \frac{1}{N} \sum_{i=1}^{N} \sigma(t) dW_i(t),
\end{align*}
\]

which leads to for any \( r \in [0, 1] \),

\[
\chi_i(t) = e^{(A+G)r}x_i(t) + \int_{t}^{t-r} e^{(A+G)(t-\tau)} \left[ Bu_i(\tau) + f(\tau) \right] d\tau + \frac{1}{N} \sum_{i=1}^{N} \int_{t-r}^{t} e^{(A+G)(t-\tau)} \sigma(\tau) dW_i(\tau). \tag{35}
\]

By \( J_{soc}(u) < C_1 \) and basic SDE estimates, we can find a constant \( C \) such that

\[
\mathbb{E} \int_0^\infty e^{-pt} \left( \int_{t-r}^{t} e^{(A+G)(t-\tau)} \sigma(\tau) dW_i(\tau) \right)^2 dt \leq C. \tag{36}
\]

From (34) and (35) we obtain

\[
\begin{align*}
\mathbb{E} \int_0^\infty e^{-pt} \left[ \int_{t-r}^{t} e^{(A+G)^2(\tau)} \left( x_i(\tau) + \int_0^\tau e^{(A+G)(\tau-\tau')} \sigma(\tau') dW_i(\tau') \right) d\tau \right] dt &\leq C,
\end{align*}
\]

which implies that for any \( r \in [0, 1] \),

\[
\begin{align*}
\mathbb{E} \int_0^\infty e^{-pt} \left( \int_{t-r}^{t} e^{(A+G)^2(\tau)} \left( x_i(\tau) + \int_0^\tau e^{(A+G)(\tau-\tau')} \sigma(\tau') dW_i(\tau') \right) d\tau \right) dt &\leq C.
\end{align*}
\]

By taking integration with respect to \( r \in [0, 1] \), we obtain

\[
\begin{align*}
\mathbb{E} \int_0^\infty e^{-pt} \left( \int_{t-r}^{t} e^{(A+G)^2(\tau)} \left( x_i(\tau) + \int_0^\tau e^{(A+G)(\tau-\tau')} \sigma(\tau') dW_i(\tau') \right) d\tau \right) dt &\leq C.
\end{align*}
\]

This together with (A3) leads to

\[
\mathbb{E} \int_0^\infty e^{-pt} \left| x_i(\tau) \right|^2 dt < C, \tag{36}
\]

which with (33) further gives

\[
\sum_{i=1}^{N} \mathbb{E} \int_0^\infty e^{-pt} \left| x_i(\tau) \right|^2 dt < NC. \tag{37}
\]

By (1), we have

\[
\dot{x}_i(t) = e^{\rho t} \dot{x}_i(t - r) + \int_{t-r}^{t} e^{(A-\frac{\rho}{2})\tau} \left[ Bu_i(\tau) + f(\tau) + Gx_i(\tau) \right] d\tau + \frac{1}{N} \sum_{i=1}^{N} \int_{t-r}^{t} e^{(A-\frac{\rho}{2})\tau} \sigma(\tau) dW_i(\tau). \tag{38}
\]

It follows from (36) that

\[
\mathbb{E} \int_0^\infty e^{-pt} \left( \int_{t-r}^{t} e^{(A-\frac{\rho}{2})\tau} Gx_i(\tau) d\tau \right)^2 dt \leq \mathbb{E} \int_0^\infty e^{-pt} \left( Gx_i(\tau) \right)^2 \int_{t-r}^{t} \left| e^{(A-\frac{\rho}{2})\tau} \right|^2 d\tau dt \leq C.
\]

From (37) and (38), we obtain that

\[
\sum_{i=1}^{N} \mathbb{E} \int_0^\infty e^{-pt} \left| x_i(\tau) \right|^2 dt < NC.
\]

This together with (A3) implies that

\[
\sum_{i=1}^{N} \mathbb{E} \int_0^\infty e^{-pt} \left| \dot{x}_i(\tau) \right|^2 + \left| \ddot{x}_i(\tau) \right|^2 dt < NC,
\]

which gives (32). From this with Theorem 4.1,
By a similar argument to the proof of Theorem 3.3 combined with Lemma 4.2, the conclusion follows. □

If (A3) is replaced by (A3'), the decentralized control (20) still has asymptotic social optimality.

**Corollary 2.** Assume that (A1)–(A2), (A3') hold. The decentralized control (20) is asymptotically social optimal.

**Proof.** Without loss of generality, we simply assume $A + G = \text{diag}(A_1, A_2)$, where $A_1 - (\rho/2)I$ is Hurwitz, and $- (A_2 - (\rho/2)I)$ is Hurwitz (if necessary, we may apply a nonsingular linear transformation as in the proof of Theorem 4.3). Write $x(t) = [x_1^T, z_2^T]^T$ and $Q^{1/2}(I - \Gamma) = [S_1, S_2]$ such that $\| (I - \Gamma) x(n)(t) \|^2_Q = \| S_1 x_1(t) + S_2 x_2(t) \|^2$, and $(A_2 - (\rho/2)I, S_2)$ is observable. By the proof of Theorem 4.1 or (Huang, 2010), $E \int_0^\infty e^{-\rho t} \|u(n)(t)\|^2 dt < \infty$ implies $E \int_0^\infty e^{-\rho t} \|z_2(t)\|^2 dt < \infty$, which together with (34) gives $E \int_0^\infty e^{-\rho t} \|S_2 x_2(t)\|^2 dt < \infty$. This and the observability of $(A_2 - (\rho/2)I, S_2)$ leads to $E \int_0^\infty e^{-\rho t} \|2x_2(t)\|^2 dt < \infty$. Thus, $E \int_0^\infty e^{-\rho t} \|x(n)(t)\|^2 dt < \infty$. The other parts of the proof are similar to that of Theorem 4.5. □

For the case that $Q$ are indefinite, we have the following result of asymptotic optimality.

**Theorem 4.6.** Let (A1)–(A2), (A3') hold. Assume (21)–(22) admit negative definite solutions $P^* < 0$ and $\Gamma^* < 0$, respectively. Then, the set of decentralized control (20) is asymptotically socially optimal. Furthermore, if $|x_0|$ have the same variance, then the asymptotic average social optimality is given by

$$
\lim_{N \to \infty} \frac{1}{N} J_{soc}(\bar{u}) = E \sum_{i=1}^{N} \int_0^\infty e^{-\rho t} \left( \|\sigma(t)\|^2_{P_i} + \|\bar{u}(t)\|^2_{\Pi_i} \right) dt
$$

$$
= \frac{1}{N} \sum_{i=1}^{N} \int_0^\infty e^{-\rho t} \left( \|\sigma(t)\|^2_{P_i} + \|\bar{u}(t)\|^2_{\Pi_i} \right) dt.
$$

**Proof.** From the above assumptions and Theorem 4.4, the Riccati equation (21) admits a $\rho$-stabilizing solution $P$ and a negative definite solution $\Gamma^*$. By a similar argument in the proof of Lemma 3.1, we obtain for any $u \in U_n$,

$$
J_{soc}(u) = \frac{1}{N} \sum_{i=1}^{N} \int_0^\infty e^{-\rho t} \left( \|x_i - x(n)(0)\|^2_{P_i} + \|x(n)(0)\|^2_{\Pi_i} \right.
$$

$$
\left. + 2\bar{u}_i(T) x_i(T) + \|x_i(T)\|^2_{\Pi_i} + \|u_i^n\|^2_{\Pi_i} \right) dt
$$

$$
= \frac{1}{N} \sum_{i=1}^{N} \int_0^\infty e^{-\rho t} \left( \|x_i - x(n)(0)\|^2_{P_i} + \|x(n)(0)\|^2_{\Pi_i} \right.
$$

$$
+ 2\bar{u}_i(T) x_i(T) + \|x_i(T)\|^2_{\Pi_i} + \|u_i^n\|^2_{\Pi_i} \right) dt
$$

$$
+ \frac{1}{N} \sum_{i=1}^{N} \int_0^\infty e^{-\rho t} \left( \|u_i^n + R^{-1} B^T \Gamma x_i(n)\|^2_{\Pi_i} \right.
$$

$$
\left. + \|u_i^n + R^{-1} B^T \Gamma x_i(n)\|^2_{\Pi_i} \right) dt + q_{\infty}.
$$

By Willems (1971, Theorem 8), the centralized optimal control exists and the optimal state is $\rho$-stable. Thus, we only need to consider the following control set

$$
U_c = \left\{ (u_1, \ldots, u_N) | u_i(t) \text{ is adapted to } \mathcal{F}_i, \quad E \int_0^\infty e^{-\rho t} \|x_i(t)\|^2 dt < \infty, \forall i \right\}.
$$

For any $u \in U_c$ satisfying $J_{soc}(u) \leq NC$, we have

$$
J_{soc}(u) = \sum_{i=1}^{N} \int_0^\infty e^{-\rho t} \left( \|x_i - x(n)(0)\|^2_{P_i} + \|x(n)(0)\|^2_{\Pi_i} + 2\bar{u}_i(T) x_i(T) \right.
$$

$$
\left. + \|u_i^n + R^{-1} B^T \Gamma x_i(n)\|^2_{\Pi_i} \right) dt + q_{\infty}.
$$

Denote $v(N) = u(N) + R^{-1} B^T \Gamma x(N)$, then from (1),

$$
\frac{dx(n)(t)}{dt} = (A + G - BR^{-1} B^T \Pi) x(n)(t) dt + B v(N)(t) dt + \frac{1}{N} \sum_{i=1}^{N} \sigma_i(t) dw_i(t).
$$

By Huang (2010), there exist $C_1, C_2 > 0$ such that

$$
E \int_0^\infty e^{-\rho t} \|x(n)(t)\|^2 dt \leq C_1 E \int_0^\infty e^{-\rho t} \|u(n)\|^2 dt + C_2.
$$

This together with (40) gives

$$
\sum_{i=1}^{N} \int_0^\infty e^{-\rho t} \left( \|x_i - x(n)(0)\|^2_{P_i} + \|x(n)(0)\|^2_{\Pi_i} \right) dt
$$

$$
= \frac{1}{N} \sum_{i=1}^{N} \int_0^\infty e^{-\rho t} \left( \|x_i - x(n)(0)\|^2_{P_i} + \|x(n)(0)\|^2_{\Pi_i} \right) dt
$$

$$
\leq \frac{NC_1}{N} \int_0^\infty e^{-\rho t} \|u_i^n\|^2 + NC_4 \leq NC.
$$

Similarly, we have

$$
\sum_{i=1}^{N} \int_0^\infty e^{-\rho t} \left( \|x_i - x(n)(0)\|^2 + \|u_i^n - u(n)\|^2 \right) dt \leq NC.
$$

The remainder of the proof can follow by that of Theorem 3.3. For the case that $|x_0|$ have the same variance, from (17), the asymptotic average social optimum (lim$_{N \to \infty} \frac{1}{N} J_{soc}(\bar{u})$) is given by

$$
E \int_0^\infty e^{-\rho t} \left( \|x_i - x(n)(0)\|^2 + \|x(n)(0)\|^2 \right) dt
$$

$$
+ \frac{1}{N} \sum_{i=1}^{N} \int_0^\infty e^{-\rho t} \left( \|u_i^n + R^{-1} B^T \Gamma x_i(n)\|^2_{\Pi_i} \right.
$$

$$
\left. + \|u_i^n + R^{-1} B^T \Gamma x_i(n)\|^2_{\Pi_i} \right) dt + q_{\infty}.
$$

**Remark 4.3.** The work Huang et al. (2012) investigated mean field LQ problem (P) with $Q \geq 0$. To obtain asymptotic social optimality, they need $Q \geq 0$ and $I - \Gamma$ is nonsingular. In Corollary 2, we have loosed the assumption to (A3'), i.e., $(A - (\rho/2)I, \sqrt{Q})$ and $(A - (\rho/2)I, \sqrt{Q}(I - \Gamma))$ are detectable. In Theorem 4.6, we further give the condition for the case of indefinite $Q$. Particularly, for the scalar case, the condition is equivalent to (28)–(29). It can be verified that the assumption $Q > 0$ and $I - \Gamma$ is nonsingular implies (28)–(29), but the converse is not true.

**4.3. Comparison to previous solutions**

In this section, we compare the proposed decentralized control laws with the feedback decentralized strategies in previous works.
We first introduce a definition from Basar and Olshder (1982).

**Definition 4.1.** For a control problem with an admissible control set $\mathcal{U}$, a control law $u \in \mathcal{U}$ is said to be a representation of another control $u^* \in \mathcal{U}$ if
(i) they both generate the same unique state trajectory, and
(ii) they both have the same open-loop value on this trajectory.

For Problem (P), let $f = 0$, and $G = 0$. In Huang et al. (2012, Theorem 4.3), the decentralized control laws are given by
\begin{equation}
\hat{u}_i(t) = -R^{-1}B^T(P\hat{x}_i(t) + \hat{s}(t)), \quad i = 1, \ldots, N,
\end{equation}
where $P$ is the stabilizing solution of (21), and $\hat{s} = \hat{K}\hat{x} + \phi$. Here $\hat{K}$ satisfies
\begin{equation}
\rho \hat{K} = \hat{K}\hat{A} + \hat{A}^T\hat{K} - \hat{K}BR^{-1}B^T\hat{K} - \Sigma, \quad \hat{\phi} \in C_{\rho/2}(0, \infty, \mathbb{R}^n),
\end{equation}
and $\phi$, $\bar{\phi}$ are given by
\begin{align*}
\frac{d\bar{x}_i}{dt} &= \bar{A}\bar{x}_i(t) - BR^{-1}B^T(\bar{K}\bar{x}_i(t) + \phi(t)), \bar{x}_i(0) = \bar{x}_0,
\frac{d\phi}{dt} &= -[A - BR^{-1}B^T(P + \hat{K}) - \rho]\phi(t) + \bar{\eta}(t),
\end{align*}
in which $\hat{A} = A - BR^{-1}B^TP$ and $\phi(0)$ is to be determined by $\phi \in C_{\rho/2}(0, \infty, \mathbb{R}^n)$. By comparing this with (22)–(24), one can obtain that $\hat{K} = \Pi - P$, $\bar{x} = \bar{x}_i$ and $\phi = s$. From the above discussion, we have the equivalence of the two sets of decentralized control laws.

**Proposition 4.1.** The set of decentralized control laws $\{\hat{u}_1, \ldots, \hat{u}_N\}$ in (20) is a representation of $\{\bar{u}_1, \ldots, \bar{u}_N\}$ given by (42).

**Remark 4.4.** The work Huang et al. (2012) studied the problem (P) with $Q \geq 0$ by the fixed-point approach. In Theorem 4.3, they have shown that the fixed-point equation admits a unique solution, when $(A - (\rho/2)L, \sqrt{Q})$ is detectable and $\Sigma = \Gamma^TQ' + Q'\Gamma - \Gamma'Q\Gamma \leq 0$. In fact, the above assumption is merely a sufficient condition to ensure (A3') $(A - (\rho/2)L, \sqrt{Q} - \Sigma)$ is detectable.

**Remark 4.5.** The work Huang and Zhou (2020) investigated asymptotic solvability of mean field Q games by the re-scaling method. They considered (1)–(2) with $Q \geq 0$ and derived a low-dimensional ordinary differential equation system by dynamic programming. Actually, the method proposed in this paper can be viewed as a type of direct approach. Different from Huang and Zhou (2020), we tackle directly high-dimensional FBSDEs along the line of maximum principle.

5. Numerical examples

Now, two numerical examples are given to illustrate the effectiveness of the proposed decentralized control.

We first consider a scalar system with 30 agents in Problem (P). Take $A = 0.8, B = R = 1, Q = -0.1, G = -0.2, f(t) = 1, \sigma(t) = 0.2, \rho = 0.6, \Gamma = 0.2, \eta = 5$ in (1)–(2). The initial states of 50 agents are taken independently from a normal distribution $N(5, 0.3)$. Note that $B \neq 0$, and $\bar{A} + G - \frac{\sigma}{2}I = -0.5873 < 0$. Then (A1)–(A2) hold. Since $M_1 = \begin{bmatrix} 0.5 & 1 \\ -0.1 & -0.5 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0.3 \\ -0.064 \end{bmatrix}$ have no eigenvalues on the imaginary axis, (A3') also holds. Under the control law (20), the trajectories of $\bar{x}$ and $\bar{x}^{(N)}$ in Problem (P) are shown in Fig. 1. It can be seen that $\bar{x}$ and $\bar{x}^{(N)}$ coincide well, which illustrate the consistency of mean field approximations.

Denote $\epsilon = \frac{1}{R \text{soc}(\bar{u})} - \frac{1}{R \text{inf}_{\text{dec}} \text{soc}(u)}$. By Theorems 3.3 and 4.6, we obtain $\epsilon = \int_0^\infty e^{-\mu} \|B^T[K(x^{(N)}(t) - x_i(t))\sigma_i]dt$. The cost gap $\epsilon$ is demonstrated in Fig. 2 where the agent number $N$ grows from 1 to 200.

Finally, we consider the 2-dimensional case of Problem (P). Take parameters as follows: $A = \begin{bmatrix} 0.1 & 0 \\ -1 & 0.2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, G = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Gamma = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \eta = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, f = [1 \ 1]^T$ and $\sigma = [0.5 \ 0.5]^T$. Denote $\hat{x}_i(t) = [\hat{x}_i^1(t) \ \hat{x}_i^2(t)]^T$. Both of $\hat{x}_i^1(0)$ and $\hat{x}_i^2(0)$ are taken independently from a normal distribution $N(5, 0.5)$. Under the control laws (20), the trajectories of $\hat{x}_i^1$ and $\hat{x}_i^2$, $i = 1, \ldots, N$ are shown in Figs. 3 and 4, respectively. The curves of $\hat{x}_i^1, i = 1, \ldots, 30$ soon converge to 0 with some fluctuation. The curves of $\hat{x}_i^2, i = 1, \ldots, 30$ first increase and then grow up before the time 40. After a period of time, they converge to a constant, which verify the $\rho$-stability of agents.

6. Concluding remarks

In this paper, we have considered uniform stabilization and social optimality for mean field LQ multiagent systems. For finite- and infinite-horizon problems, we design the decentralized control laws by decoupling FBSDEs, respectively, which are further shown to be asymptotically optimal. Some equivalent conditions are further given for uniform stabilization of the systems in
Proof. It follows by (16) that
\[
\dot{x}^{(N)}(t) = \left[ (\tilde{A}(t) + G\tilde{x}^{(N)}(t) - BR^{-1}B^{T}K(t)\bar{x}(t) + s(t) + f(t) \right] dt + \frac{1}{N} \sum_{i=1}^{N} \sigma(t) dW_{i}(t).
\]
which implies \( \max_{0 \leq t \leq T} \mathbb{E}[\|x^{(N)}(t) - \bar{x}(t)\|^2] = O(1/N). \) (A.1)

Proof of Theorem 3.3. Note that \( \inf_{u \in L_{2}^{2}(0,T;\mathbb{R}^{m})} f_{\text{soc}}^{F}(u) \leq f_{\text{soc}}^{F}(\hat{u}). \)
To obtain
\[
\frac{1}{N} f_{\text{soc}}^{F}(\hat{u}) \leq \frac{1}{N} \inf_{u \in L_{2}^{2}(0,T;\mathbb{R}^{m})} f_{\text{soc}}^{F}(u) + O\left(\frac{1}{\sqrt{N}}\right),
\]
we only need to prove for any \( u \in \mathcal{U}^{T} \triangleq \{ u \in L_{2}^{2}(0,T;\mathbb{R}^{m}) : f_{\text{soc}}^{F}(u) \leq f_{\text{soc}}^{F}(\hat{u}) \}, \) the following holds:
\[
\frac{1}{N} f_{\text{soc}}^{F}(\hat{u}) \leq \frac{1}{N} f_{\text{soc}}^{F}(\hat{u}) + O\left(\frac{1}{\sqrt{N}}\right).
\]
We now show that for \( u \in \mathcal{U}^{T} \), \( \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} e^{-\rho t} (\|x_{i}(t)\|^{2} + \|u_{i}(t)\|^{2}) dt < NC_{2} \). By Lemma 3.1, (P1) is uniformly convex which gives there exists \( \delta_{0} > 0 \) such that
\[
\delta_{0} \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} e^{-\rho t} \|u_{i}(t)\|^{2} dt - C \leq f_{\text{soc}}^{F}(u).
\]
Since \( f_{\text{soc}}^{F}(\hat{u}) < NC_{1} \), we have \( f_{\text{soc}}^{F}(u) < NC_{1} \), which implies \( \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} e^{-\rho t} \|u_{i}(t)\|^{2} dt < NC \). This leads to
\[
\mathbb{E} \int_{0}^{T} e^{-\rho t} \|u^{(N)}(t)\|^{2} dt \leq \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} e^{-\rho t} \|u_{i}(t)\|^{2} dt < C,
\]
where \( u^{(N)}(t) = \frac{1}{N} \sum_{i=1}^{N} u_{i}(t) \). By (1),
\[
\dot{x}^{(N)}(t) = \left[ (A + Gx^{(N)}(t) + Bu^{(N)}(t) + f(t) \right] dt + \frac{1}{N} \sum_{i=1}^{N} \sigma(t) dW_{i}(t),
\]
which implies \( \max_{0 \leq t \leq T} \mathbb{E}[\|x^{(N)}(t)\|^2] \leq C \). Note that
\[
x_{i}(t) = e^{At}x_{0} + \int_{0}^{t} e^{(A-t')\sigma(t) dW_{i}(t)} + \int_{0}^{t} e^{(A-t')}Gx^{(N)}(\tau) + Bu_{i}(\tau) + f(\tau) d\tau.
\]
We have
\[
\sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} e^{-\rho t} \|x_i(t)\|^2 dt \\
\leq C \left( \sum_{i=1}^{N} \mathbb{E} \|x_0\|^2 + N \max_{0 \leq t \leq T} \mathbb{E} \|x_i(t)\|^2 \right) + \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} e^{-\rho t} \|u_i(t)\|^2 dt + NC_1 < NC_2.
\] (A.4)

By (14) and (16), we obtain that
\[
\mathbb{E} \int_{0}^{T} e^{-\rho t} \left( \|\hat{x}_i(t)\|^2 + \|\hat{u}_i(t)\|^2 + \|\hat{x}(t)\|^2 \right) dt < C. 
\] (A.5)

Let \( \hat{x}_i = x_i - \hat{x}_i, u_i = u_i - \hat{u}_i \) and \( \hat{x}(N) = \frac{1}{N} \sum_{i=1}^{N} \hat{x}_i \). Then by (1) and (16),
\[
d\hat{x}_i(t) = (A\hat{x}_i(t) + C\hat{x}(N)(t) + B\hat{u}_i(t)) dt, \quad \hat{x}_i(0) = 0. \] (A.6)

From (3), \( J^F_s(u_i) = \sum_{i=1}^{N} (J^F_s(\hat{u}) + \mathcal{I}_i) \), where
\[
J^F_s(\hat{u}) \triangleq \mathbb{E} \int_{0}^{T} e^{-\rho t} \left( \|\hat{x}_i(t) - \Gamma \tilde{x}(N)(t)\|^2_Q + \|\hat{u}_i(t)\|^2_R \right) dt, \\
\mathcal{I}_i = 2\mathbb{E} \int_{0}^{T} e^{-\rho t} \left[ \hat{x}_i(t) - \Gamma \tilde{x}(N)(t) - \eta(t)^T \hat{Q} \times \left( \hat{x}_i(t) - \Gamma \tilde{x}(N)(t) + \hat{u}_i(t) \right) \right] dt. 
\]

By Lemma 3.1 and Proposition 3.1, \( J^F_s(\hat{u}) \geq 0 \). We only need to prove \( \frac{1}{N} \sum_{i=1}^{N} \mathcal{I}_i = O(1/\sqrt{N}) \). By direct computations, one can obtain
\[
\sum_{i=1}^{N} \mathcal{I}_i = 2 \mathbb{E} \int_{0}^{T} e^{-\rho t} \left[ \tilde{x}_i^T \left( \hat{Q} \hat{x}_i - \Gamma^T \tilde{x} - \eta \right) - \Gamma^T Q ((I - \Gamma) \tilde{x} - \eta) \right] dt + \sum_{i=1}^{N} \frac{1}{N} \tilde{R}_i \hat{u}_i \left( I \hat{Q} \right) dt \] (A.7)

By (10)–(12), (A.6) and Itô’s formula,
\[
0 = \mathbb{E} \int_{0}^{T} \sum_{i=1}^{N} e^{-\rho t} \left\{ -\tilde{x}_i^T \left[ \hat{Q} \hat{x}_i - Q (\Gamma \tilde{x} + \eta) \right] - \Gamma^T Q ((I - \Gamma) \tilde{x} - \eta) - \hat{u}_i^T \tilde{R}_i \right\} dt \\
+ \mathbb{E} \int_{0}^{T} e^{-\rho t} \tilde{R}_i \hat{u}_i (I \hat{Q}) dt.
\]

From this and (A.7), we obtain
\[
\frac{1}{N} \sum_{i=1}^{N} \mathcal{I}_i = 2 \mathbb{E} \int_{0}^{T} e^{-\rho t} \left( \tilde{x}_i(t) - \tilde{x}(t) \right)^T \left( I + \Gamma^T \hat{Q} \Gamma \right) \tilde{x}_i(t) dt \\
- \left( \mathbb{E} + \Gamma^T P + PG \right) \tilde{x}(N)(t) dt.
\]

By Lemma A.1, (A.4) and (A.5), we obtain
\[
\frac{1}{N} \sum_{i=1}^{N} \mathcal{I}_i \leq C \mathbb{E} \int_{0}^{T} e^{-\rho t} \left( \|\tilde{x}_i(t)\|^2 + \|\tilde{x}(t)\|^2 \right) dt \\
- \mathbb{E} \int_{0}^{T} e^{-\rho t} \|\tilde{x}(N)(t)\|^2 dt \leq C \mathbb{E} \int_{0}^{T} e^{-\rho t} \|\tilde{x}(N)(t)\|^2 dt \\
- \mathbb{E} \int_{0}^{T} e^{-\rho t} \|\tilde{x}(t)\|^2 dt,
\]

which implies \( \frac{1}{N} \sum_{i=1}^{N} \mathcal{I}_i = O(1/\sqrt{N}) \).

Moreover, by (10), (13) and direct calculations,
\[
J^F_s(\hat{u}) = \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} e^{-\rho t} \left( \|\hat{x}_i - \hat{x}(N) - \Gamma \tilde{x} - \eta\|^2_Q + \|\hat{u}_i\|^2_R \right) dt \\
= \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} e^{-\rho t} \left( \hat{x}_i(t) \right) \hat{x}_i(t) \hat{x}_i(t) dt - \frac{2\eta^T Q(I - \Gamma) \tilde{x} + \|\hat{u}_i\|^2_R} {dt} \\
= \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} e^{-\rho t} \left( \|\hat{x}_i - \hat{x}(N)\|^2_Q + \|\hat{x}(N)\|^2_Q + \|\hat{u}_i\|^2_R \right) dt \\
- \frac{2\eta^T Q(I - \Gamma) \tilde{x} + \|\hat{u}_i\|^2_R} {dt} \\
= \sum_{i=1}^{N} \mathbb{E} \left[ \|x_0 - x(N)(0)\|^2_Q + \|x(N)(0)\|^2_Q + 2s^T(0)x(N)(0) \right] \\
+ \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} e^{-\rho t} \left( \hat{u}_i - \hat{u}_i \right) + R \hat{u}_i + R^T \hat{u}_i + \hat{u}_i \right)^2 dt + q \tau \\
= \sum_{i=1}^{N} \mathbb{E} \left[ \|x_0 - x(N)(0)\|^2_Q + \|x(N)(0)\|^2_Q + 2s^T(0)x(N)(0) \right] \\
+ Nq \tau + N\epsilon \tau, \\
where q \tau and \epsilon \tau are given by (18)–(19). \quad \square

Appendix B. Proofs of Lemma 4.2 and Theorem 4.1

Proof of Lemma 4.2. From (A.2), we have
\[
\mathbb{E} \int_{0}^{\infty} e^{-\rho t} \|\hat{x}(N)(t) - \tilde{x}(t)\|^2 dt \\
\leq \frac{1}{N} \mathbb{E} \int_{0}^{\infty} e^{-\rho t} \left( \|x_0 - \hat{x}(N)(t)\|^2 + \|x(N)(t) - \tilde{x}(t)\|^2 \right) dt \\
+ \frac{1}{N} \mathbb{E} \int_{0}^{\infty} e^{-\rho t} \left( \|x_0 - \hat{x}(N)(t)\|^2 + \|x(N)(t) - \tilde{x}(t)\|^2 \right) dt \\
\leq \frac{1}{N} \int_{0}^{\infty} e^{-\rho t} \|x(0)\|^2 dt + \frac{1}{N} \int_{0}^{\infty} e^{-\rho t} \|x(N)(t)\|^2 dt \\
\leq \frac{1}{N} \int_{0}^{\infty} e^{-\rho t} \|x(0)\|^2 dt + \frac{1}{N} \int_{0}^{\infty} e^{-\rho t} \|x(N)(t)\|^2 dt \\
\leq O(1/N). \quad \square

Proof of Theorem 4.1. By (A1)–(A3), Lemmas 4.1 and 4.2, we obtain that \( \tilde{x} \in C_{\rho/2}([0, \infty), \mathbb{R}^n) \) and
\[
\mathbb{E} \int_{0}^{\infty} e^{-\rho t} \left( \|\hat{x}(N)(t) - \tilde{x}(t)\|^2 \right) dt = O\left( \frac{1}{\sqrt{N}} \right),
\]

which further gives that \( \mathbb{E} \int_{0}^{\infty} e^{-\rho t} ||\hat{x}(N)(t)||^2 dt < \infty \). Denote \( g = -BR^{-1}B^T((I - P)\tilde{x} + s) + G\hat{x}(N) + f \). Then
\[
\hat{x}(t) = e^{\hat{x}_0} + \int_{0}^{t} e^{(t-s)}g(s) ds + \int_{0}^{t} e^{(t-s)}\sigma dW(s). \quad (B.1)
\]
Note that $\tilde{A} - \frac{\xi}{2} I$ is Hurwitz. By Schwarz’s inequality,
\begin{align*}
E \left[ \int_0^\infty e^{-\rho t} \|\tilde{x}(t)\|^2 dt \right] \\
\leq 3E \left[ \int_0^\infty e^{-\rho t} \int_0^t e^{\xi(t-\tau)} g(\tau)^2 d\tau dt \right] \\
+ 3E \left[ \int_0^\infty e^{-\rho t} \int_0^t \|e^{\xi(t-\tau)} I\|^{\frac{1}{2}} d\tau dt \right] \\
\leq C + 3E \left[ \int_0^\infty e^{-\rho t} \int_0^t \|e^{\xi(t-\tau)} I\|^{\frac{1}{2}} d\tau dt \right] \\
\leq C \sum_{i=1}^N E[\tilde{x}(t)] \\
\leq E \left[ \int_0^\infty e^{-\rho t} \|\tilde{x}(t)\|^2 dt \right] < \infty.
\end{align*}

On the other hand, (A.2) gives
\begin{align*}
E[\tilde{x}^{(N)}(t)-\tilde{x}(t)]^2 = E[\hat{\tilde{x}}^{(N)}(0)-\tilde{x}(0)]^2 \\
+ \frac{1}{R^2} \int_0^t \int_0^t \|e^{\xi(t-\tau)} I\|^{\frac{1}{2}} [\tilde{x}(\tau)] d\tau dt.
\end{align*}

By (B.4) and the arbitrariness of $x_0, i = 1, \ldots, N$, we obtain that $A + G - \frac{\xi}{2} I$ is Hurwitz.
\begin{align*}
(ii) \Rightarrow (iii). \ Let V(t) = e^{-\rho t} \tilde{y}(t) P T \tilde{y}(t), \ where \ \tilde{y}(t) = (A + G) \tilde{y}(t) + \bar{B} \tilde{u}(t), \ \tilde{y}(0) = \bar{y}_0.
\end{align*}

Denote $V$ by $V^*$ when $\tilde{u} = 0 = -R^{-1} B^T T \tilde{y}$. By (22),
\begin{align*}
\frac{dV^*}{dt} &= \frac{\tilde{y}(t)}{[-\rho P + (A + G - BR^{-1} B^T) T]} T \\
&\leq \frac{\tilde{y}(t)}{[-(Q - C) - PBR^{-1} B^T] T}. \leq 0.
\end{align*}

Note that $V^* > 0$. Then $\lim_{t \to \infty} V^*(t)$ exists, which implies
\begin{align*}
\lim_{t \to \infty} V^*(t_0) - V^*(t_0 + T) = 0. \hspace{2cm} (C.5)
\end{align*}

Rewrite $\Pi(t)$ in (13) by $\Pi T$. Then we have $\Pi T + x_0 = \Pi T(0)$. By (13),
\begin{align*}
\int_0^{t_0 + t} e^{-\rho t} \|\tilde{y}(t)\|^2_{Q - \Xi} + \|\tilde{u}(t)\|^2_{R} dt \\
\leq e^{-\rho t} \|\tilde{y}(t)\|^2_{\Pi T(t)} + \int_0^t e^{-\rho t} \|\tilde{u}(t)\|^2_{R} dt \\
\leq e^{-\rho t} \|\tilde{y}(t)\|^2_{\Pi T(t_0 + t)} + \int_0^t e^{-\rho t} \|\tilde{u}(t)\|^2_{R} dt
\end{align*}

This with (C.5) implies
\begin{align*}
\lim_{t_0 \to \infty} e^{-\rho t} \|\tilde{y}(t)\|^2_{\Pi T(0)} \\
\leq \lim_{t_0 \to \infty} \int_0^{t_0 + t} e^{-\rho t} \|\tilde{y}(t)\|^2_{Q - \Xi} + \|\tilde{u}(t)\|^2_{R} dt \\
= \lim_{t_0 \to \infty} |V^*(t_0) - V^*(t_0 + T) | = 0
\end{align*}

By (A.3), one can get that there exists $T > 0$ such that $\Pi T(0) > 0$ (see e.g. Zhang et al. (2019) and Zhang, Zhang, and Chen (2008)). Thus, we have $\lim_{t_0 \to \infty} e^{-\rho t} \|\tilde{y}(t)\|^2_{Q - \Xi} = 0$, which implies that $(A + G - \frac{\xi}{2} I, B)$ is stabilizable. Similarly, we can show $(A - \frac{\xi}{2} I, B)$ is stabilizable.

This part has been proved in Theorem 4.1. \hspace{2cm} (D)
By pre- and post-multiplying by $\xi^T$ and $\xi$ where $\xi = [\xi_1^T, 0]^T$, it follows that

$$0 = \rho e^{-t} U^T P U \xi = \xi^T U^T Q U \xi.$$

From the arbitrariness of $\xi_1$, we obtain $Q_{11} = 0$. Since $Q$ is semi-positive definite, then $Q_{12} = Q_{21} = 0$, and $Q_{22} \geq 0$. By comparing each block matrix of both sides of (C.6), we obtain $A_{21} = 0$. It follows from (C.6) that

$$\rho P_2 = P_2 A_{22}^* + A_{22}^* P_2 + \tilde{Q}_{22}.$$

(7.7)

Let $\xi = [\xi_1^T, \xi_2^T]^T = U^T y^*$, where $y^*$ satisfies $\dot{y}^* = \tilde{A} y^*$. Then we have

$$\dot{\xi}_1 = \tilde{A}_{11} \xi_1 + \tilde{A}_{12} \xi_2,$$

$$\dot{\xi}_2 = \tilde{A}_{22} \xi_2.$$

By Lemma 4.1 of Wonham (1968), the detectability of $(A + G, \{Q - \Sigma\})^{1/2}$ implies the detectability of $(\tilde{A}, \tilde{Q})^{1/2}$. Take $\xi(0) \equiv [\xi_1^T, \xi_2^T]^T$. Then $Q^{1/2} y^* = \tilde{Q}^{1/2} U \xi = 0$, which together with the detectability of $(\tilde{A}, \tilde{Q})^{1/2}$ implies $\xi \rightarrow 0$ and $\tilde{A}_{11}$ is Hurwitz. Denote $S(t) = e^{A_{11} t} \xi_1^T P_2 \xi_2$. By (7.7)

$$S(T) - S(0) = - \int_0^T (\xi_1^T P_2 \xi_2) dt \leq 0$$

which implies $\lim_{t \to \infty} S(t)$ exists. By a similar argument with the proof of Theorem 4.2, we obtain $\lim_{t \to \infty} -e^{-t} \|Q(t)\|_2^2 = 0$ and $P_2(t)(0) > 0$, which gives $\xi \to 0$ and $\tilde{A}_{22}$ is Hurwitz. This with the fact that $\tilde{A}_{11}$ is Hurwitz gives that $\xi$ is stable, which leads to (iii). □

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