On the geometry and regularity of largest subsolutions for a free boundary problem in $\mathbb{R}^2$: elliptic case

Betul Orcan-Ekmekci

Abstract We consider a one-phase exterior Bernoulli type free boundary problem in $\mathbb{R}^2$ with a non constant, only continuous, free boundary condition. In this article, we present geometry and regularity properties of the largest viscosity subsolution of this problem. Moreover, we provide density bounds for the positivity set and its complement near the free boundary.

Keywords Free boundary value problems · Viscosity solutions · Bernoulli problem · Non-constant free boundary condition · The largest viscosity subsolution

Mathematics Subject Classification (2000) 35R35 · 35R60 · 35D40 · 35B40 · 35B65

1 Introduction

In this paper, we consider a one-phase exterior Bernoulli type free boundary problem (FBP) in $\mathbb{R}^2$ with a non constant, only continuous, free boundary condition; and study the geometry and regularity of the largest viscosity subsolution of this FBP. The set up of the problem is the following: for a given bounded open domain $D \subseteq \mathbb{R}^2$, we look for a continuous nonnegative function $u : \mathbb{R}^2 \setminus D \mapsto [0, +\infty)$ that satisfies:

$$\begin{align*}
\Delta u &= 0, & \text{in } \Omega(u) \setminus D, \\
 u &= g(x), & \text{on } \partial D, \\
u &= 0, |\nabla u|^2 &= f(x), & \text{on } \partial \Omega(u),
\end{align*}$$

where $\Omega(u) = \{x \in \mathbb{R}^2 | u(x) > 0\}$; $g(x)$ and $f(x)$ are positive continuous functions. For the existence of such a function $u$, we need an ellipticity condition on $f(x)$: assume that there exist $\Lambda, \lambda > 0$ such that $0 < \lambda < f(x) < \Lambda$, for all $x \in \mathbb{R}^2$. 

Communicated by O. Savin.

B. Orcan-Ekmekci (✉)
Mathematics Department, Rice University, Houston, TX, USA
e-mail: orcan@rice.edu
This problem could be regarded as a modeling of diverse range of physical phenomena, encompassing problems such as in evolutionary case flame propagation, G-equations, and phase transitions; and in nonlinear case capillary drops on a flat or inclined surface. If we consider the evolution problem corresponding to (1.1), then some literature samples among many are: for the heat equation, Caffarelli and Vázquez in [12]; and for the front propagation problem in terms of pulsating wave solutions Berestycki and Hamel, [3]. The problem (1.1) could be viewed as a linearized version of the capillary drop problem; in the variational case Caffarelli and Friedman have geometry and regularity results in [10]; inhomogeneous surface and inclined surface cases were studied by Caffarelli and Mellet, [7,6].

Under more smooth free boundary condition $f(x)$, the variational and weak version of (1.1) were studied; the analysis of the variational problem was presented by Alt and Caffarelli in [1]; for the two-phase problem, Alt et al. in [2]; for the three-dimensional case, Caffarelli et al. in [11]; the geometry of the free boundary in terms of the weak solution, Kenig and Toro in [14]. For more references, see the book of Caffarelli and Salsa [8]. Most of these results require that free boundary condition $f(x)$ is at least Lipschitz and the media is periodic.

When we consider viscosity solutions, the least supersolution and the largest subsolution of (1.1) do not need to be the same but they do need to trap all the other viscosity solutions. In this context, an example would be a linearized version of a drop sliding through an inclined plane with random parallel grooves. In that case, we expect the leading edge of the drop to be steeper, the drop getting stuck on the grooves or “the least supersolution of the free boundary problem”, while the back edge getting hang to the grooves and is flatter, “the largest subsolution”. The least viscosity supersolution of (1.1) has been studied extensively under smooth and periodic data, [4,5]. Among the reasons for its popularity, one is that because it has much better non degeneracy properties and is much simpler. For some models, the largest viscosity subsolution is the proper object of study. On the other hand, there were no prior regularity results for the largest subsolution and this was the starting point for us. In this study, we show that the known results about the least supersolution have analogues which hold for the largest subsolution. We also show in this work the largest subsolution has very nice geometric characterizations around the free boundary that are still open problems for the least supersolution.

Another objective of this study is to research, in an optimal possible way, whether the known results about the “deterministic” cases of these type of problems hold in the “random” case. Depending on the structure of the physical problem, randomization can be found as the dependency of probabilistic variable either on the differential equation or on the given data. In any of these cases, the given regularity of randomized part(s) should be lowered up to continuity or even measurability in order to analyze the problem. The main challenges of the analysis of these cases start at this point.

Motivated by the study of random media, we allow for the data to be highly oscillatory. Thus, we only require $f(x)$ to be positive, bounded, and measurable function. Unlike previous studies, only regularity on $f$ is the continuity and the continuity of $f(x)$ is only necessary to be able to obtain the continuous viscosity solution. Some of our results are restricted by the dimension; for instance, we cannot generalize some results to $\mathbb{R}^n$ because of the lost of linear growth near free boundary, instead, the order of the lower bound of the growth rate near free boundary point $x_0$ becomes $r^{n-1}$ in $B_r(x_0)$.

The paper is organized as follows: Sect. 2 presents the construction of the largest subsolution of (1.1). Section 3 presents the results of Lipschitz and Non-Degeneracy properties. In Sect. 4, we will show the geometric properties of the free boundary. Because of weak assumptions on the free boundary condition, one would expect to have a very unstable free boundary (highly oscillatory) on the contrary Sect. 4 guarantees us that, locally, the normal-
ized neighborhood of the free boundary has two components with positive densities with one of them is the positivity set and the other one is the zero level set, i.e. locally, free boundary does not have high and irregular oscillation.

Our results in this paper are the following:

**Theorem 1.1** Let $u$ be the largest subsolution of (1.1), then $u$ has the following properties:

(i) $u$ is a viscosity solution of (1.1),
(ii) $u$ is Lipschitz,
(iii) $u$ is Non-Degenerate,
(iv) $u$ has a nontrivial growth near free boundary,
(v) locally, $\Omega(u)$ has a single component $\Gamma$ with a positively dense complement.

**Remark 1.2** Parts (i-v) are valid for dimension 2.

Proof of Theorem 1.1 will be given separately in next sections starting with Sect. 3.

### 2 On definitions of viscosity solutions

In this section, we review the definitions of viscosity subsolution, supersolution, and solution of (1.1) as in [4].

**Definition 2.1** $u$ is a viscosity subsolution of (1.1) if $u$ is a continuous function in $\mathbb{R}^2 \setminus D$ and satisfies the following conditions:

1. $\Delta u \geq 0$, in $\Omega(u) \setminus D$,
2. $u \leq g(x)$, on $\partial D$,
3. *Free Boundary Condition (FBC)*: If $x_0 \in \partial \Omega(u)$ has a tangent ball from outside of $\Omega(u)$, then $|\nabla u(x_0)|^2 \geq f(x_0)$; that is, for $v$ is the normal unit vector inward to $\Omega(u)$ at $x_0$, if $u(x) \leq \alpha \langle x - x_0, v \rangle + o(|x - x_0|)$ in a neighborhood of $x_0$, then $\alpha \geq \sqrt{f(x_0)}$.

**Definition 2.2** $u$ is a viscosity supersolution of (1.1) if $u$ is a continuous function in $\mathbb{R}^2 \setminus D$ and satisfies the following conditions:

1. $\Delta u \leq 0$, in $\Omega(u)$,
2. $u \geq g(x)$, on $\partial D$,
3. *Free Boundary Condition (FBC)*: If $x_0 \in \partial \Omega(u)$ has a tangent ball from inside of $\Omega(u)$, then $|\nabla u(x_0)|^2 \leq f(x_0)$; that is, for $v$ is the normal unit vector inward to $\Omega(u)$ at $x_0$, if $u(x) \geq \alpha \langle x - x_0, v \rangle + o(|x - x_0|)$ in a neighborhood of $x_0$, then $\alpha \leq \sqrt{f(x_0)}$.

Moreover, $u$ is a viscosity solution of (1.1) if it is both a viscosity sub- and supersolution of (1.1).

Heuristically, Definitions 2.1 and 2.2 imply the following facts for a viscosity solution, $v$, of (1.1):

1. By Definition 2.1: If $x_0 \in \partial \Omega(v)$ has a tangent ball from outside of $\Omega(v)$, then whenever $v(x)$ is touched by a plane from above at $x_0$, then the slope of the plane should be at least $\sqrt{f(x_0)}$.
2. By Definition 2.2: If $x_0 \in \partial \Omega(v)$ has a tangent ball from outside of $\Omega(v)$, then whenever $v(x)$ is touched by a plane from below at $x_0$, then the slope of the plane should be at most $\sqrt{f(x_0)}$. 

\[ \text{Springer} \]
Remark 2.3 We cannot have a comparison principle in between viscosity sub- and supersolutions. On the other hand, there is a comparison principle between viscosity super (or subsolution) and classical sub (or supersolution). In other words, when the classical sub (or supersolution) touches the viscosity super (or subsolution) from below (or above) on the free boundary, they should be either equal to each other or create a contradiction. It is one of reasons why we define the free boundary condition under the condition of a tangent ball to the free boundary from inside (outside) of $\Omega(u)$ in the definitions.

From now on, sub- or supersolution of an equation mean being a sub- or supersolution in the viscosity sense unless otherwise stated. Here is an example for a simple case in order to see the heuristic picture:

Example 2.4 If we consider \((1.1)\) with conditions as see the heuristic picture: the free boundary from inside (outside) of $\Omega(u)$, they should be either equal to each other or create a contradiction. It is one of reasons why we define the free boundary condition under the condition of a tangent ball to the free boundary from inside (outside) of $\Omega(u)$ in the definitions.

In Example 2.4, variational and viscosity solutions are equal. Reader can consult to [1] for the definition and analysis of the variational case of the problem \((1.1)\). In [5], an example is given for the variational solution and the least supersolution are not equal. The following example shows that the variational solution and the largest subsolution for an adjusted problem do not need to be the same, either.

Example 2.5 Let us consider the following free boundary problem in $B_{8R}(0)$ for some $R > 1$,

\[
\begin{align*}
\triangle u &= 0 & \text{in } \Omega(u) \setminus B_1(0), \\
u &= 1, & \text{on } \partial B_{8R}(0), \\
u &= 0, & \text{on } \partial B_1(0), \\
u &= 0, |\nabla u|^2 = f(x), & \text{on } \partial \Omega(u),
\end{align*}
\]

(2.1)

where $f(x)$ is a radially symmetric smooth function such that

\[
f(x) \equiv \begin{cases} 
4, & \text{if } x \in B_{8R}(0) \setminus B_{6R}(0) \\
1, & \text{if } x \in B_{8R}(0). 
\end{cases}
\]

In order to construct a solution, we can use a truncated harmonic function that needs to satisfy a different free boundary condition for different solutions. Variational solution has a free boundary condition that will average the function $f(x)$ on the other hand the largest subsolution takes the smallest possible value of $f(x)$ as the free boundary condition. Let us denote $\langle f \rangle = \frac{1}{|B_{8R}(0)|} \int_{B_{8R}(0)} f(x)dx$, then the variational solution is of the form as the truncated harmonic function in $B_{8R}(0) \setminus B_1(0)$ $u(x) = \langle \frac{1}{\ln(R)} \ln(\frac{|x|}{R}) \rangle_+$ where $R > r > 1$ satisfies $|\nabla w(x)|^2 = \langle f \rangle$ on $\partial B_r(0)$. On the other hand, the largest subsolution is again of the form as the truncated harmonic function in $B_{8R}(0) \setminus B_1(0)$ but $u(x) = \langle \frac{1}{\ln(R/r)} \ln(\frac{|x|}{r}) \rangle_+$, where $R > r > r_1 > 1$, and satisfies $|\nabla u(x)|^2 = 1$ on $\partial B_{r_1}(0)$.
In order to obtain the largest subsolution of (1.1), we adapt PERRON’s method for viscosity solutions (See [13]). Intuitively, for a fixed classical supersolution \( v \), if we take the largest viscosity subsolution \( u \) which is smaller than \( v \), then we obtain a viscosity solution as in Perron’s method. Thus, let us construct a classical supersolution of (1.1) by taking a suitable harmonic function. Without loss of generality, we assume that \( 0 \in D \). Then, let us take two balls \( B_{R_0}(0) \) and \( B_r(0) \) both containing \( D \). Let \( h(x) = (\frac{1}{\ln(\frac{|x|}{R_0})} \ln(\frac{|x|}{R_0}))^+ \) be the subharmonic function in \( \mathbb{R}^2 \setminus D \), then choose \( R_0 \) as

\[
R_0 = \inf \left\{ R | R > r \quad \text{and} \quad \frac{1}{R^2 \ln^2(r/R_0)} < \lambda \right\}.
\]

Then, we get \( |\nabla h(x)|^2 = \frac{1}{R_0^2 \ln^2(r/R_0)} = \lambda \) on \( \partial B_{R_0}(0) \). Hence, \( h(x) \) is a classical supersolution of (1.1). Therefore, any viscosity subsolution should be smaller than \( h \) by the comparison principle. From now on, let us denote

\[
u = \sup\{ v \in C(\mathbb{R}^2 \setminus D) | v \leq h \quad \text{and} \quad v \text{ is a subsolution of (1.1)} \}.
\]

**Theorem 2.6** \( u \) is a viscosity solution of (1.1) and it is the largest subsolution of (1.1).

**Proof** Since, \( u \) is the supremum of subsolutions, it is also a subsolution. Therefore, it is the largest one. Next, we will show that \( u \) is a supersolution.

Claim 1: \( \triangle u = 0 \) in \( \Omega(u) \) : Assume not, then there exists a point \( y_0 \in \Omega(u) \) such that \( \triangle u(y_0) > 0 \). Let us take a neighborhood of \( y_0 \) in \( \Omega(u) \), say \( B_r(y_0) \), and the harmonic function \( w(x) \) in \( B_{r/2}(y_0) \) with \( w(x) = u(x) \) for \( x \in B_r(y_0) \setminus B_{r/2}(y_0) \). Since, \( u \leq h \) and both \( w \) and \( h \) are harmonic functions in \( B_r(y_0) \), we have \( w \leq h \) in \( B_r(y_0) \). Then, we obtain a larger subsolution \( u \) than \( u \) by defining \( v = w \) in \( B_r(y_0) \) and \( v = u \) in \( \Omega(u) \setminus B_r(y_0) \) which is a contradiction.

Claim 2: If \( x_0 \in \partial \Omega(u) \) has a tangent ball from inside of \( \Omega(u) \), then \( |\nabla u(x_0)|^2 \leq f(x_0) \) : Assume not, then there there exists a point \( x_0 \in \partial \Omega(u) \), having a tangent ball from inside of \( \Omega(u) \), such that for \( u(x) \geq \alpha(x - x_0, v) + o(|x - x_0|) \) in a neighborhood of \( x_0 \) we have \( \alpha > \sqrt{f(x_0)} \). Thus, by the Hopf principle, \( v \) cannot touch \( u \) at \( x_0 \). Therefore, \( x_0 \) has a neighborhood which is in \( \Omega(u) \), say \( B_{r_0}(x_0) \subset \Omega(u) \). We can take a tangent ball, \( B_{r_0}(y_0) \), small enough so that \( B_{r_0}(y_0) \subset \Omega(u) \setminus B_{r_0}(x_0) \). Since, \( f \) is continuous and \( \sqrt{f(x)} < \alpha \), \( x_0 \) has a neighborhood such that \( \sqrt{f(x)} < \alpha \). Let us choose a neighborhood of \( x_0 \), say \( B_{r_0}(x_0) \), small enough that satisfies \( B_{r_0}(x_0) \subset \Omega(v) \), \( \sqrt{f(x)} < \alpha \) in \( B_{r_0}(x_0) \), and \( B_{r/2}(z) \subset B_{r_0}(x_0) \) is an inside tangent ball. Since \( u \) is harmonic in \( \Omega(u) \) and positive, by the Harnack inequality, we have \( \inf_{B_{r/4}(z)} u(x_0) \geq cu(z) \), for some \( c > 0 \). Let us denote \( u(z) = m > 0 \) and define \( w(x) \), for some \( 0 < r_0 < r \),

\[
w(x) = \begin{cases}
\frac{cm}{\ln(\frac{|x - z|}{r_0})}, & x \in \mathbb{R}^2 \setminus B_{r_0}(z) \\
\frac{cm}{\ln(\frac{|x - z|}{r_0})} & x \in B_{r_0}(z)
\end{cases}
\]

then \( w \) is harmonic in \( B_r(z) \setminus B_{r_0}(z) \).

Consider \( h = \max(u, w) \), we claim that it is a subsolution of (1.1) larger than \( u \) if we choose \( r_0 \) small enough, \( h \) is a harmonic function in \( \Omega(h) \setminus D \), \( h = u \leq g \) on \( \partial D \), and \( h \leq v \); so we only need to show that it satisfies the FBC, (iii) in Definition 2.1. Let \( y_0 \in \partial \Omega(h) \) and it has a tangent ball from outside of \( \Omega(h) \). If \( y_0 \in \partial \Omega(u) \), then \( u \) already satisfies the FBC, so does \( h \). If \( y_0 \in \partial \Omega(w) \), then \( |\nabla w|^2(y_0) = \frac{cm}{\ln(\frac{|x - z|}{r_0})}^2 = \alpha \) which implies, again, \( h \) is a subsolution.
of (1.1). Moreover, \( h \) is larger than \( u \) which contradicts to \( u \) being the largest subsolution. Hence, the result follows.

By Claim 1 and 2, \( u \) is a supersolution of (1.1). Thus, \( u \) is a viscosity solution of (1.1). □

3 Lipschitz and non-degeneracy properties

In this section, we focus on the regularity properties of the largest subsolution, \( u \), of (1.1). Lipschitz regularity is valid in any dimension. On the other hand, Non-Degeneracy property is restricted to two-dimensional case and this is one of the main difficulties for the subsolution theory. In higher dimensions, \( \mathbb{R}^n \), one can obtain the growth rate near free boundary as \( r^{n-1} \).

In \( \mathbb{R}^2 \), this corresponds to nontrivial linear growth rate which directly implies local geometric properties near free boundary.

3.1 Lipschitz property

**Theorem 3.1** \( u \) is Lipschitz.

**Proof** Since, this is a one-phase problem, it is enough to show that \( u(x) \leq Cd(x, \partial \Omega(u)) \), for some universal constant \( C > 0 \) and every \( x \in \Omega(u) \). Because of the local estimates on derivatives for harmonic functions, we have

\[
|\nabla u(x_0)| \leq \frac{C_1}{r^\alpha} \|u\|_{L_1(B_r(x_0))}.
\]

If we can show that \( u(x) \leq Cd(x, \partial \Omega(u)) \), then we will obtain \( |\nabla u(x_0)| \leq C_1 C \), for some universal constant \( C > 0 \). Actually, this result implies more than Lipschitz property, instead we obtain a uniform bound on the gradient of \( u \). Thus, by the scaling argument around any free boundary point \( x_0 \), we obtain \( u(x) = u(x_0) + o(x^1 - x_0^1)_+ + o(r_j) \), in some rotated system where \( x = (x^1, x^2) \in \mathbb{R}^2 \). Here \( o(r_j) \) depends on \( x_0 \) and we scale \( u(x) \) at \( x_0 \) with the sequence \( r_j \), where \( r_j \to 0 \), as \( u_{r_j}(x) = u_{r_j}(x_0 - y_0) \).

Let us prove that \( u(x) \leq Cd(x, \partial \Omega(u)) \), for some universal constant \( C > 0 \). By way of contradiction: Assume that there exists a point \( x_0 \in \Omega(u) \) with \( d(x, \partial \Omega(u)) = 1 \) such that \( u(x_0) > M \), for some large \( M \). Since \( u \) is harmonic in \( \Omega(u) \) and positive, by the Harnack inequality, we have inf_{B_{1/2}(x_0)} u(x) \geq cM, for some \( c > 0 \). Let us define \( w(x) \), shown in Fig. 2,

\[
w(x) = \begin{cases} 
cM \left( \frac{\ln(|x - x_0|/(1 + \epsilon))}{\ln(1/2(1 + \epsilon))} \right) +, & x \in \mathbb{R}^2 \setminus B_{1/2}(x_0) 
cM, & x \in B_{1/2}(x_0)
\end{cases}
\]

then \( w \) is harmonic in \( B_{1+\epsilon}(x_0) \setminus B_{1/2}(x_0) \).

Consider \( v = \max(u, w) \), we claim that it is a subsolution of (1.1) larger than \( u \) if we choose \( M \) large enough. \( v \) is a harmonic function in \( \Omega(v) \setminus D \) and \( v = u \leq g \) on \( \partial D \), so we only need to show that it satisfies the FBC, (iii) in Definition 2.1. Let \( y_0 \in \partial \Omega(v) \) and it has a tangent ball from outside of \( \Omega(v) \). If \( y_0 \in \partial \Omega(u) \), then \( u \) already satisfies the FBC, so does \( v \). If \( y_0 \in \partial \Omega(w) \), then \( |\nabla w|^2(y_0) = \frac{cM}{(1+\epsilon)\ln(1/2(1+\epsilon))} \geq \Lambda \) which implies, again, \( v \) is a subsolution of (1.1). Moreover, \( v \) is larger than \( u \) which contradicts to \( u \) being the largest subsolution. Hence, the result follows. □
3.2 Non-degeneracy property

Notational Comment: We write $A \sim B$ to mean that, for some universal constants $m, M > 0$, $m \cdot B \leq A \leq M \cdot B$.

In this subsection, we prove that $u$ is Non-Degenerate, that is: there exists a universal constant $\kappa > 0$ such that $\sup_{B_r(x_0)} u(x) \geq \kappa r$, for every $x_0 \in \partial \Omega(u)$, by going rigorously through heuristic observations:

\[ 0 = u(x_0) = \int_{\partial B_r(x_0)} u \nu ds + \int_{B_r(x_0)} (G \Delta u) dx, \]

where $G(y, x_0) = \frac{1}{4\pi} \ln(\frac{1}{r} |y - x_0|)$ with $G \equiv 0$ on $\partial B_r(x_0)$. Thus

\[ - \int_{\partial B_r(x_0)} u \nu ds = \int_{B_r(x_0)} (G \Delta u) dx. \]

Therefore, we will estimate $u(x)$ in $\partial B_r(x_0)$ by $\int_{B_r(x_0)} \Delta u dx$, i.e. by the total mass of $\Delta u$ in $B_r(x_0)$.

First, we will show that, for the normalized problem:

\[ \text{The total mass of } \Delta u \sim 1 \text{ in } B_1(0) \text{ if we assume that } 0 \in \partial \Omega(u). \]

We need to show that there exist some constants $c, C > 0$ such that $c \leq \int_{B_1(0)} \Delta u dx \leq C$.

- For the upper bound of $\int_{B_1(0)} (\Delta u) dx$:
  Since, $u$ has a uniform gradient bound in $B_1(0)$, by the Divergence theorem, we have
  \[ \int_{B_1(0)} (\Delta u) dx = \int_{\partial B_1(0)} u \nu dx, \]
  we estimate $u \nu$ by the first order incremental quotient for $x_0 \in \partial B_1(0)$ and obtain the upper bound.

- The proof of the lower bound is very technical but the idea is the following:
  \[ u \text{ is harmonic in } B_1(0) \cap \Omega(u) \text{ so } \int_{B_1(0)} (\Delta u) dx \text{ will be nonzero only on } B_1(0) \cap \partial \Omega(u). \]
– If we can show that there exists a partition of the interval \((0, 1)\) with some mutually disjoint intervals of the form \(\left(0, |y_0| - r_y, |y_0| + r_y\right)\) where \(\int_{B_{r_y}(y_0)} (\Delta u) dx \geq c r_y\), then we get

\[
\int_{B_1(0)} (\Delta u) dx \geq \sum_{\text{the partition of (0,1)}} \int_{B_{r_y}(y_0)} (\Delta u) dx \geq \sum_{\text{the partition of (0,1)}} c r_y = c/2.
\]

– Thus, in order to obtain the above partition, firstly, we show that

* Lemma: If \(x_0 \in \bar{\Omega}(u)\) with \(d = d(x_0, \partial \Omega(u))\), then for any \(d < r < 1\), we have \(\partial B_r(x_0) \cap \{u = 0\}^o \neq \emptyset\). Since, \(0 \in \partial \Omega(u)\), this lemma implies that for any \(r \in (0, 1)\), we have \(\partial B_r(0) \cap \{u = 0\}^o \neq \emptyset\). Thus, we can obtain a tangent ball from outside to \(\Omega(u)\) for any \(r \in (0, 1)\). These tangent ball radii will be our candidate partition elements.

* Theorem: If \(x_0 \in \partial \Omega(u)\) has a tangent ball from outside of \(\Omega(u)\), say \(B_r(y) \subseteq \Omega^c(u)\), then \(u\) grows linearly in \(B_r(x_0)\). This theorem will imply the lower bound of \(\Delta u\) in \(B_r(x_0)\).

We shall show these in detail in a series of lemmas.

**Lemma 3.2** Let \(x_0 \in \bar{\Omega}(u)\) with \(d = d(x_0, \partial \Omega(u))\), then we have \(\partial B_r(x_0) \cap \{u = 0\}^o \neq \emptyset\), for any \(d < r < 1\).

**Proof** (By way of contradiction) Assume that there exists \(r_0\) such that \(\partial B_{r_0}(x_0) \cap \{u = 0\}^o = \emptyset\). Let us define the harmonic function \(h(x)\) as

\[
\begin{align*}
\Delta h &= 0 \quad \text{in } B_{r_0}(x_0), \\
h &= u \quad \text{in } \mathbb{R}^2 \setminus B_{r_0}(x_0).
\end{align*}
\]

By the Maximum principle, \(h(x) > 0\) in \(B_{r_0}(x_0)\) and it is actually a subsolution: If \(h(y) = 0\), then \(u(y) = 0\). If \(y \in \partial \Omega(h)\) and has a tangent ball from outside of \(\Omega(h)\), then \(y \in \partial \Omega(u)\) and since \(\partial B_{r_0}(x_0) \cap \{u = 0\}^o = \emptyset\) it has a tangent ball from outside of \(\Omega(u)\). If \(h(x) \leq \alpha (x - y, v)^+ + o(\|x - y\|)\) in a neighborhood of \(y\), then we have \(u(x) \leq h(x) \leq \alpha (x - y, v)^+ + o(\|x - y\|)\) so \(\alpha \leq f(y)\) by the FBC, \((iii)\) in Definition 2.1, satisfied by \(u\). This implies \(w = \max\{u, h\}\) is a subsolution of \((1.1)\) which is larger than \(u\). Contradiction for \(u\) being the largest subsolution. Hence, the result follows. \(\square\)

**Theorem 3.3** Let \(x_0 \in \partial \Omega(u)\) with a tangent ball from outside of \(\Omega(u)\), say \(B_r(y) \subseteq \Omega^c(u)\), then \(u\) grows linearly in \(B_r(x_0)\); that is, there exist universal constants \(C_1, C_2 > 0\) such that

\[
C_1 r \leq \sup_{B_r(x_0)} u \leq C_2 r.
\]

**Proof** (By way of contradiction) Assume that \(x_0 \in \partial \Omega(u)\) with a tangent ball \(B_r(y) \subseteq \Omega^c(u)\) from outside of \(\Omega(u)\) and \(u\) does not grow linearly in \(B_r(x_0)\). Since \(u\) is Lipschitz, we already have \(u \leq C_2 r\) in \(B_r(x_0)\), for some \(C_2 > 0\). Hence, suppose that the first inequality is not true, then there exists \(\delta > 0\) sufficiently small such that \(\sup_{B_{r_0}(x_0)} u \leq \delta r\). Let us define the harmonic function

\[
h(x) = \frac{2\delta r}{\ln 2} \ln \left\| \frac{x - y}{r} \right\| \quad \text{in } B_r(y)^c.
\]

Shown in Fig. 3. Then, \(h(x) \geq u(x)\) in \(B_{r_0}(x_0) \cap \Omega(u)\). We can choose \(\delta > 0\) small enough so that \(\nabla h^2(x) = \left[\frac{2\delta}{\ln 2}\right]^2 < \lambda\) on \(\partial B_r(y)\). Then, \(u(x) \leq \frac{2\delta}{\ln 2} \|x - x_0, v\| + o(\|x - x_0\|)\) in a
Fig. 3 $h(x) \geq u(x)$ in $B_r(x_0) \cap \Omega(u)$ and $|\nabla h|^2(x) = \left[\frac{2\delta}{\ln 2}\right]^2$.

Fig. 4 Project ball $B_{r_r}(y_r)$ onto $R_1$.

neighborhood of $x_0$. Hence, $\left[\frac{2\delta}{\ln 2}\right]^2 \geq f(x_0)$ so that we get a contradiction:

$$\lambda > \left[\frac{2\delta}{\ln 2}\right]^2 \geq f(x_0) > \lambda.$$ 

Hence, there exists $C_1 > 0$, universal, such that $C_1 r \leq \sup_{B_r(x_0)} u$. \hfill \Box

**Lemma 3.4** The total mass of $\Delta u \sim 1$ in $B_1(0)$.

**Proof** First of all, the total mass of $\Delta u$ in $B_1(0)$ is bounded by above since $u$ is Lipschitz in $B_1(0)$, by the Divergence Theorem, we have

$$\int_{B_1(0)} (\Delta u)dx = \int_{\partial B_1(0)} u_v ds,$$

we estimate $u_v$ by the first order incremental quotient for $x_0 \in \partial B_1(0)$ as

$$\frac{u(x_0 + sv) - u(x_0)}{s} \leq \frac{C|sv|}{s} \leq C,$$

where $C > 0$ is the universal Lipschitz constant of $u$. Therefore,

$$\int_{B_1(0)} (\Delta u)dx = \int_{\partial B_1(0)} u_v ds \leq 2\pi C.$$

**Remark 3.5** We apply the Divergence Theorem without any further detail that goes through a smoothing argument. $u_v$ is well-defined, at least, in terms of $L^1_{loc}$ which guarantees the existence of a smooth sequence converging to $u_v$ in $L^1_{loc}$. Thus, we can apply the Divergence Theorem by going through the limiting argument that we skip the details.

Next, we shall determine the lower bound for the total mass of $\Delta u$ in $B_1(0)$. Since $0 \in \partial \Omega(u)$, by Lemma 3.2, we have $\partial B_r(0) \cap \{x|u(x) = 0\}^c \neq \emptyset$ for any $r \in (0, 1)$. Hence,
for each radius \( r < 1 \), there exists a ball \( B_{\varepsilon_r}(x_r) \subseteq \{x | u(x) = 0\}^c \) and it is tangent to \( \partial \Omega(u) \) at some point \( y_r \in \partial \Omega(u) \). Pick a ray in \( B_1(0) \), say \( R_1 = \{r\eta | r \in [0, 1], \eta \in S^1\} \). Let us use \( R_1 \) in order to pick a partition of \((0, 1)\) in terms of \( \varepsilon_r \). Consider \( \bigcup_{r \in [0, 1]} B_{\varepsilon_r}(y_r) \) and rotate all these balls until their centers intersects with \( R_1 \). Hence, we obtain a covering of another ball whose center is on \( R \) (a disjoint subfamily \( \bigcup \) their original places, i.e. consider only the subset of \( \Omega_1 \) at some point mutually disjoint partition of \((0, 1)\). Let us use \( \varepsilon_r \) such that \( 0 \leq \varepsilon_r \leq 2 \), and \( \Delta \Omega \) covers \( K \) in order to pick a partition of \( \Delta \Omega \) such that 2\( \varepsilon_r \) covers \( K \). Now, resend back this subfamily onto their original places, i.e. consider only the subset of \( \bigcup_{r \in [0, 1]} B_{\varepsilon_r}(y_r) \) whose translations are in the disjoint subfamily \( \{S_r\} \). With this subfamily and by using their radii we obtain the mutually disjoint partition of \((0, 1)\) mentioned at the beginning of this Sect. 3.2. Next, we show that the total mass of \( \Delta u \) in \( B_{\varepsilon_r}(y_r) \) is at least \( c\varepsilon_r \) so that when we add them up we get \( 1/2 \) (it is because \( 2\varepsilon_r \) covers \( K \)). In this part of the proof, we will use the linear growth property of \( u \) in \( B_{\varepsilon_r}(y_r) \) which is true by Theorem 3.3. Let \( w \) be the harmonic function such that

\[
\begin{align*}
\Delta w &= 0 \quad \text{in} \; B_{\varepsilon_r}(y_r), \\
w &= u \quad \text{on} \; \partial B_{\varepsilon_r}(y_r).
\end{align*}
\]

By the Divergence theorem, we have

\[
0 = \int_{\partial B_{\varepsilon_r}(y_r)} (u - w)(u - w) + \int_{B_{\varepsilon_r}(y_r)} \nabla^2(u - w)dx \\
\geq \int_{B_{\varepsilon_r}(y_r)} \Delta(u - w)(u - w)dx + \int_{B_{\varepsilon_r/2}(z_r)} \nabla^2(u - w)dx \\
\geq \int_{B_{\varepsilon_r}(y_r)} \Delta(u - w)(u - w)dx + \frac{C_2}{\varepsilon_r^2} \int_{B_{\varepsilon_r/2}(z_r)} \left( \frac{\varepsilon_r}{2} \right)^2 dx \\
\geq \int_{B_{\varepsilon_r}(y_r)} \Delta(u - w)(u - w)dx + C_3 \varepsilon_r^2,
\]

where \( z_r \in \Omega^c(u) \) such that \( u \equiv 0 \) and \( w \) grows linearly in \( B_{\varepsilon_r/2}(z_r) \). Since, \( 0 < w - u \leq C_1 \varepsilon_r \) and \( w \) is harmonic in \( B_{\varepsilon_r}(y_r) \), we get

\[
C_4 \int_{B_{\varepsilon_r}(y_r)} \varepsilon_r \Delta u dx \geq - \int_{B_{\varepsilon_r}(y_r)} (\Delta(u - w)(u - w))dx \geq C_3 \varepsilon_r^2 \\
\int_{B_{\varepsilon_r}(y_r)} (\Delta u)dx \geq C_5 \varepsilon_r.
\]
Thus, by adding these $\Delta u$ masses in these balls, $B_{r_j}(y_j)$, over $r_j$, we get the total mass of $\Delta u$ in $B_1(0)$ is at least $C_5 > 0$, i.e.

$$C \geq \int_{B_1(0)} (\Delta u) dx \geq \sum_{r_j} \int_{B_{r_j}(y_j)} (\Delta u) dx \geq C_5.$$ 

\[ \square \]

**Theorem 3.6** $u$ is Non-Degenerate, i.e. there exists a universal constant $\kappa > 0$ such that $\sup_{B_r(x_0)} u(x) \geq \kappa r$, for every $x_0 \in \partial \Omega(u)$.

**Proof** (By way of contradiction) Assume that there exists $x_0 \in \partial \Omega(u)$ with $r > 0$ such that $\sup_{B_r(x_0)} u(x) < \delta r$, for some sufficiently small $\delta > 0$. By Green’s representation theorem, we have

$$0 = u(x_0) = \int_{\partial B_r(x_0)} u G_v ds + \int_{B_r(x_0)} (G \Delta u) dx,$$

where $G(y, x_0) = \frac{1}{2\pi} \ln \left( \frac{1}{r} |y - x_0| \right)$ with $G \equiv 0$ on $\partial B_r(x_0)$. Hence,

$$\int_{\partial B_r(x_0)} u G_v ds = - \int_{B_r(x_0)} (G \Delta u) dx \geq - \int_{B_{r/2}(x_0)} G \Delta u dx.$$

Then, by Lemma 3.4 and $G(y, x_0) \leq -C_1$ in $B_{r/2}(x_0)$, we get

$$\int_{\partial B_r(x_0)} u G_v ds \geq - \int_{B_{r/2}(x_0)} (G \Delta u) dx \geq C_2 r,$$

for some universal constant $C_2 > 0$.

$G_v \sim \frac{1}{r}$ on $\partial B_r(x_0)$ so we obtain by the assumption

$$C_3 \delta r \geq \int_{\partial B_r(x_0)} u G_v ds \geq C_2 r.$$

Contradiction; we can choose $\delta > 0$ small enough so that the above inequality fails. Hence, we get the result. \[ \square \]

### 4 Locally, $\Omega(u)$ has a single component with a positively dense complement

In this section, we give a geometric characterization around the free boundary. We localize the problem around the free boundary. Let us consider a point $x_0 \in \partial \Omega(u)$ and a neighborhood of $x_0$, $B_r(x_0)$, such that all the components of $\Omega(u)$ reach up to at least $B_{r/2}(x_0)$ and all the components of $\{x | u(x) = 0\}$ reach 0 with a connected subset of $\partial \Omega(u)$. Then, we normalize this neighborhood to $B_1(0)$ by taking $\tilde{u}(x) = \frac{u(x_0 - r)}{r}$ for $x \in B_1(0)$. For the sake of simplicity, we denote the normalized function $\tilde{u}(x)$ as $u(x)$. We will show that $\Omega(u)$ has a single component with a positively dense complement in $B_1(0)$ with the following steps:

1. Let $\Upsilon$ be any connected component of $\{x | u(x) = 0\}$, then $\Upsilon$ has some nice geometric properties.
(2) Let $\Gamma$ be any connected component of $\Omega(u)$ in $B_1(0)$, then $u$ has a nontrivial linear growth in $\Gamma$, i.e. $u$ has Non-Degeneracy component by component.

(3) $\Omega(u)$ has at most two connected components.

(4) There is only one component with a positively dense complement.

4.1 On some properties of the open components of zero-level set

Main results of this section are the following:

- There is no open component of $\{x|u(x) = 0\}^o$ which is strictly contained in $B_1(0)$
- Let $\Upsilon$ be any open component of $\{x|u(x) = 0\}^o$ with $r_0 = d(\Upsilon, 0)$, then the contribution of the mass of $\Delta u$ in $\Upsilon \sim (1-r_0)$ in $B_1(0)$,
- Let $\Upsilon$ be connected to $0$ with the connected subset, $C_\Upsilon$, of $\partial \Omega(u)$, then $\partial \Upsilon \cap \partial B_1(0) \neq \emptyset$ with one of the following two conclusions: either
  
  (1) for any $\eta, \varepsilon > 0$ and $x \in C_\Upsilon \cap B_{1-\eta}(0)$ we have $B_\varepsilon(x) \cap \{x|u(x) = 0\}^o \neq \emptyset$ and the set
  \[
  \bigcup_{x \in C_\Upsilon \cap B_{1-\eta}(0)} \{O| O \text{ is a component of } \{x|u(x) = 0\}^o \text{ such that } B_\varepsilon(x) \cap O \neq \emptyset
  \]
  has finitely many elements, or
  
  (2) $0 \in \partial \Upsilon$.

**Remark 4.1** When we talk about the mass of $\Delta u$ in a set $A$, we mean the mass of $\Delta u$ in the closure of $A$.

Next two lemmas provide us some lower bound estimates on the growth rate of some harmonic functions:

**Lemma 4.2** Let $h(x) = \langle x, e_2 \rangle_+ = x_2^+$ in $\mathbb{R}^2$ and $w$ be a harmonic function such that $w \geq h$ in $B_1(0)$ and $w = 0$ on $\partial B_1(0) \cap \{x|\langle x, e_2 \rangle < -\delta\}$. Then, for $v$ is the inner normal vector to $\Omega(w)$ at $x_0 \in \partial \Omega(w) \cap \partial B_1(0) \cap \{x| - \delta < \langle x, e_2 \rangle < 0\}$ we have $w_v(x_0) \geq \ln(1/\delta)$, for some $C > 0$.

**Proof** We can write down the Poisson formula for $w$ and estimate $w_v(x_0)$ by the first order incremental quotient, so we have $x \in B_1(0)$,

\[
w(x) = \frac{1 - |x|^2}{2} \int_{\partial B_1(0)} \frac{w(y)}{|x - y|^2} dS(y).
\]

Let us denote $\langle x_0 + sv, e_1 \rangle = a$ and $\langle x_0 + sv, e_2 \rangle = b \leq 0$, then we have

\[
w(x_0 + sv) - w(x_0) = \frac{1 - |x_0 + sv|^2}{2s} \int_{\partial B_1(0)} \frac{w(y)}{|x_0 + sv - y|^2} dS(y)
\]

\[
\geq \frac{[1 - |x_0 + sv|^2]}{2s} \int_{\partial B_1(0)} \frac{h(y)}{|x_0 + sv - y|^2} dS(y)
\]

\[
= \frac{[1 - |x_0 + sv|^2]}{2s} \int_{\partial B_1(0) \cap \{y_2 > 0\}} \frac{y_2}{|x_0 + sv - y|^2} dS(y)
\]
Let us take the limit as \( s \to 0 \) on both sides, then for \( x_0 \in \partial \Omega(w) \cap \partial B_1(0) \cap \{ x | - \delta < \langle x, e_2 \rangle < 0 \} \) we have

\[
w_v(x_0) \geq C \int_{\partial B_1(0) \cap \{ y_2 > 0 \}} \frac{y_2}{|x_0 - y|^2} dS(y)
\]

\[
= C \int_{\partial B_1(0) \cap \{ y_2 > 0 \}} \frac{y_2}{2 - 2(x_0, y)} dS(y)
\]

\[
\geq C \int_{\partial B_1(0) \cap \{ y_2 > 0 \}} \frac{y_2}{2 - 2y_1 x_1} dS(y)
\]

\[
= -C \ln [1 - x_1] \geq C \ln (1/\delta),
\]

since \( x_1 \geq \sqrt{1 - \delta^2} \) where \( x_1 = (x_0, e_1) \). Hence, we get the result. \( \square \)

**Lemma 4.3** Let \( h(x) = (\langle x, \pm e_2 \rangle \pm \delta)_+ \in \mathbb{R}^2 \) and \( w \) be a harmonic function such that \( w \geq h \) in \( B_1(0) \). Then, for \( v \) is the inner normal vector to \( \Omega(w) \) at \( x_0 \in \partial \Omega(w) \) we have \( w_v(x_0) \geq \frac{C}{\delta} \) for some \( C > 0 \).

**Proof** Notice that \( \{ x | w(x) = 0 \} \subseteq \partial B_1(0) \) and we can write down the Poisson formula for \( w \) and estimate \( w_v(x_0) \) by the first order incremental quotient, as we did in the proof of Lemma 4.2, so we have

\[
\frac{w(x_0 + sv) - w(x_0)}{s} = 1 - \frac{|x_0 + sv|^2}{2\pi s} \int_{\partial B_1(0)} \frac{w(y)}{|x_0 + sv - y|^2} dS(y)
\]

\[
\geq \frac{[1 - |x_0 + sv|^2]}{2\pi s} \int_{\partial B_1(0)} \frac{h(y)}{|x_0 + sv - y|^2} dS(y)
\]

\[
\geq \frac{[1 - |x_0 + sv|^2]}{2\pi s} \int_{\partial B_1(0) \cap |x_0 - y| < \delta} \frac{h(y)}{\delta^2} dS(y)
\]

Let us take the limit as \( s \to 0 \) on both sides, we get

\[
w_v(x_0) \geq C \int_{\partial B_1(0) \cap |x_0 - y| < \delta} \frac{1}{\delta^2} dS(y) = \frac{C}{\delta}.
\]

\( \square \)

**Lemma 4.4** There is no open component of \( \{ x | u(x) = 0 \}^o \) which is strictly contained in \( B_1(0) \).

**Proof** (By way of contradiction) Assume there is an open component of \( \{ x | u(x) = 0 \}^o \), say \( \Gamma \) which is strictly contained in \( B_1(0) \), then there is a tangent ball \( B_r(y) \subseteq \{ x | u(x) = 0 \}^o \) to \( \Omega(u) \), let \( x_0 \in \partial \Omega(u) \cap \partial B_r(y) \). Then, by Lemma 3.3, \( x_0 \) has a ball \( B_r(x_0) \) such that \( u \) has a linear growth in \( B_r(x_0) \). Consider the domain \( \Sigma = B_r(x_0) \cup \Gamma \), shown in Fig. 5, and \( h(x) \) be the harmonic function such that

\[
\begin{aligned}
\Delta h &= 0 & & \text{in } \Sigma, \\
h &= u & & \text{in } B_1(0) \setminus \Sigma.
\end{aligned}
\]
Then, \( h \geq u \) in \( \Sigma \). We claim that \( v = \max\{u, h\} \) is a larger subsolution than \( u \), we know that \( h \) is a harmonic function with \( h \neq u \), so \( v \) is a subharmonic function in \( \Omega(v) \) so it is enough to show that \( v \) satisfies the FBC, \((iii)\) in Definition 2.1: Let \( y_0 \in \partial \Omega(v) \) such that \( y_0 \) has a tangent ball from outside of \( \Omega(v) \) and \( \eta \) be the inner normal vector into \( \Omega(v) \), then \( y_0 \in \partial \Sigma \) by the Maximum principle. Moreover, the only possible region for \( y_0 \) is either in a neighborhood of the intersection points \( \partial B_r(x_0) \cap \partial \Gamma \) (since outside of these neighborhoods zero-level set of \( h(x) \) can be only a curve) or \( y_0 \in \partial \Omega(u) \) such that \( y_0 \) has a tangent ball from outside of \( \Omega(u) \).

In the first case, we estimate \( v_\eta(y_0) \) by Lemma 4.2. \( v \) is a harmonic function which is bigger than \( l(x) = \alpha(x - x_0, v)_+ \) in \( B_r(x_0) \) for some direction \( v \in \mathbb{S}^1 \) and \( \alpha > 0 \) since \( u \) grows linearly in \( B_r(x_0) \). Then, we have

\[
v_\eta(y_0) \geq C \ln(1/|y_0 - x_0|) > 0,
\]

Hence, \( h_\eta(y_0) \geq C \ln(1/r) \). Therefore, by choosing \( r \) small enough we guarantee that \( v \) is a subsolution of \((1.1)\).

In the second case, if we have \( v(x) \leq \alpha(x - y_0, v)_+ + o(|x - y_0|) \) for some \( \alpha > 0 \), then we have \( \alpha \geq \sqrt{f(y_0)} \) since \( y_0 \in \partial \Omega(u) \) with a tangent ball from outside of \( \Omega(u) \),

\[
u(x) \leq v(x) \leq \alpha(x - y_0, v)_+ + o(|x - y_0|),
\]

and \( u \) satisfies the FBC, \((iii)\) in Definition 2.1, i.e. \( \alpha \geq \sqrt{f(y_0)} \). Hence, \( v \) is a subsolution of \((1.1)\).

Thus, we construct a larger subsolution than \( u \), contradiction. Hence, the result follows. \( \Box \)

Lemma 4.4 is trivially true for \( \Omega(u) \) by the Maximum principle. Since, the harmonic function which is zero on the boundary is the identically zero function.

**Lemma 4.5** Let \( \Upsilon \) be any open component of \( \{x | u(x) = 0\}^o \) with \( r_0 = d(\Upsilon, 0) \), then the contribution of the mass of \( \Delta u \) in \( \Upsilon \sim (1 - r_0) \) in \( B_1(0) \)

**Proof** The upper bound is trivial, for the lower bound we will use the linear growth in a ball which has a tangent ball from the zero level set. By Lemma 3.3, for any \( r \in (r_0, 1) \), there exists \( y_0 \in \{x | u(x) = 0\}^o \) and \( \varepsilon_{y_0} > 0 \) such that \( B_{\varepsilon_{y_0}}(y_0) \) is tangent to \( \Omega(u) \), say at \( x_0 \) and \( u \) has a linear growth in \( B_{\varepsilon_{y_0}}(x_0) \). Pick a ray in \( B_1(0) \), say \( R_1 \). Now, we use \( R_1 \) in order to pick a partition of \( [r_0, 1] \) in terms of \( \varepsilon_{y_0} \). Consider \( \{B_{\varepsilon_{y_0}}(x_{\varepsilon_{y_0}}) | r \in [r_0, 1]\} \) and project all these balls \( B_{\varepsilon_{y_0}}(x_{\varepsilon_{y_0}}) \) onto \( R_1 \). Hence, we obtain a covering of \( R_1 \) by segments \( S_r = [r - h, r + h] \). Extract a disjoint subfamily \( (S_{r_j}) \) such that \( 2S_{r_j} \) covers \( R_1 \). Now, resend back this subfamily onto their original places. Next, we know that the total mass of \( \Delta u \) in \( B_{\varepsilon_{r_j}}(x_{r_j}) \sim \varepsilon_{r_j} \), by
Lemma 3.4, so that when we add them up we get a lower bound as $c(1 - r_0)$ for some $c > 0$. \hfill $\Box$

Lemma 4.5 is a variation of Lemma 3.4.

Next Lemma gives a geometric characterization of the components of $\{x | u(x) = 0\}^\circ$, either a component can contain 0 on its boundary if not there are only finitely many of those components.

Lemma 4.6 Let $\Upsilon$ be any connected component of $\{x | u(x) = 0\}^\circ$ in $B_1(0)$ which is connected to 0 with the connected subset, $C_\Upsilon$, of $\partial \Omega(u)$. Then $\partial \Upsilon \cap \partial B_1(0) \neq \emptyset$ with one of the following two conclusions: either

1. for any $\eta, \varepsilon > 0$ and $x \in C_\Upsilon \cap B_1 - \eta(0)$ we have $B_\varepsilon(x) \cap \{x | u(x) = 0\}^\circ \neq \emptyset$ and the set
   $$\bigcup_{x \in C_\Upsilon \cap B_1 - \eta(0)} \{O | O \text{ is a component of } \{x | u(x) = 0\}^\circ \text{ such that } B_\varepsilon(x) \cap O \neq \emptyset\}$$
   has finitely many elements, or

2. $0 \in \partial \Upsilon$.

See Fig. 6.

Proof (By way of contradiction) Let $\Upsilon$ is a connected component of $\{x | u(x) = 0\}^\circ$ in $B_1(0)$. By Lemma 4.4, $\Upsilon$ cannot be strictly inside of $B_1(0)$, so the only possibility is that $\partial \Upsilon \cap \partial B_1(0) \neq \emptyset$, $0 \notin \partial \Upsilon$, and either there exists $x_0 \in C_\Upsilon \cap B_1 - \eta(0)$, for some $\eta > 0$, with a ball $B_r(x_0)$ such that $B_r(x_0) \cap \{x | u(x) = 0\}^\circ = \emptyset$ or the set

$$\bigcup_{x \in C_\Upsilon \cap B_1 - \eta(0)} \{O | O \text{ is a component of } \{x | u(x) = 0\}^\circ \text{ such that } B_{\varepsilon_0}(x) \cap O \neq \emptyset\}$$

has infinitely many elements for some $\varepsilon_0 > 0$.

If there exists a ball $B_r(x_0)$ such that $B_r(x_0) \cap \{x | u(x) = 0\}^\circ = \emptyset$, then consider the harmonic function $h(x)$ defined in $B_{r/2}(x_0)$ with $h = u$ on $\partial B_{r/2}(x_0)$. Hence, $h(x) > u(x)$ in $B_{r/2}(x_0)$ moreover $\max\{u, h\}$ is a larger subsolution than $u$. Contradiction.

If the set

$$\bigcup_{x \in C_\Upsilon \cap B_1 - \eta(0)} \{O | O \text{ is a component of } \{x | u(x) = 0\}^\circ \text{ such that } B_{\varepsilon_0}(x) \cap O \neq \emptyset\}$$

has infinitely many elements for some $\varepsilon_0 > 0$, then every component has a contribution to the total mass of $\Delta u$ by Lemma 4.5 which is at least $\min\{\varepsilon_0, \eta\}$. On the other hand, the total mass of $\Delta u$ is finite in $B_1(0)$. Contradiction. Thus, there are at most finitely many distinct connected components of $\{x | u(x) = 0\}^\circ$ in this case. Hence, the result follows. \hfill $\Box$
4.2 \( \text{u} \) has a nontrivial linear growth in \( \Gamma \)

**Lemma 4.7** Let \( \Gamma \) be any connected component of \( \Omega(u) \) in \( B_1(0) \), then \( \mathcal{H}^1(\partial \Gamma) < +\infty \).

**Proof** Let us restrict \( u \) only on \( \Gamma \) and work on the total mass of \( \Delta u \) in \( \Gamma \). Let us denote \( \Delta \equiv \frac{u|_{\partial \Gamma} - u}{|\partial \Gamma|} \) in \( B_1(0) \) with \( \Delta \equiv 0 \) in \( B_1(0) \setminus \Gamma \). By Lemma 3.4, there exists a universal constant \( C > 0 \) such that we have

\[
C \geq \int_{B_1(0)} \Delta u \, dx \geq \int_{\Gamma} \Delta w \, dx.
\]

Since, \( \Gamma \) can be covered by a countable union of all but finitely many disjoint balls of radius \( \varepsilon \) and \( w \) is harmonic in \( \Gamma \), we have

\[
C \geq \int_{\Gamma} \Delta w \, dx = \sum_{j} \int_{B_\varepsilon(x_j)} \Delta w \, dx.
\]

If \( B_\varepsilon(x_j) \subseteq \Gamma \), then \( w \) is harmonic in \( B_\varepsilon(x_j) \); so \( \int_{B_\varepsilon(x_j) \subseteq \Gamma} \Delta w \, dx = 0 \). If \( B_\varepsilon(x_j) \cap \partial \Gamma \neq \emptyset \), then the total mass of \( \Delta w \sim \varepsilon \) in \( B_\varepsilon(x_j) \), by Lemma 3.4; so there exists \( \kappa > 0 \) such that \( \int_{B_\varepsilon(x_j)} \Delta w \, dx \geq \kappa \varepsilon \). Thus,

\[
C \geq \sum_{j} \int_{B_\varepsilon(x_j)} \Delta w \, dx \geq \sum_{\{j|B_\varepsilon(x_j) \cap \partial \Gamma \neq \emptyset\}} \kappa \varepsilon.
\]

Hence, the number of balls with radius \( \varepsilon \) that cover \( \partial \Gamma \) is at most \( \frac{C}{\kappa \varepsilon} \). As a result, we obtain that \( \mathcal{H}^1(\partial \Gamma) < +\infty \).

The following lemma is the Lemma A1 in [9], the result for the domain \( \Omega_2 \). Here, we give an explicit decay rate for the function \( w \).

**Lemma 4.8** Let \( w \) be a harmonic function in \( \Omega \subseteq B_1(0) \) with the following properties:

1. \( B_1(0) \cap \partial \Omega \cap \{x_2 = 0\} = \{0\} \),
2. \( \Omega \) is in the upper half-plane, i.e. \( \Omega \subseteq B_1(0) \cap \{x|x_2 \geq 0\} \),
3. \( w \) is Lipschitz in \( \Omega \).

where \( (x_1, x_2) \) represents the coordinates of the point \( x \in \mathbb{R}^2 \), then for any cone, \( C_0 \), in \( B_1(0) \cap \{x|x_2 \geq 0\} \) and for any \( r \in [0, 1] \), the number of \( n \) that satisfies \( r_n = \frac{r}{2^n} \) with \( \{x \in C_0||x| < r_n/2, w(x) = 0\} \neq \emptyset \) is finite, i.e. \( \Omega \) is tangent to \( \{x|x_2 = 0\} \). Shown in Fig. 7.

Note that \( C_0 \) has the following representation:

\[
C_0 = \{(x_1, x_2) \in B_1(0)|x_2 \geq 0 \text{ and } |x_1| \leq 1-a \} \text{ for some } a > 0.
\]

Let us denote \( B_\Gamma^+(0) = B_r(0) \cap \{x|x_2 \geq 0\} \).

\( \bowtie \) Springer
Proof (By way of contradiction) Assume that there exist a cone $C_0$ and $r > 0$ with a sequence of the form $r_{nk} = \frac{r}{nk}$ such that $\{x \in C_0 : |x| < \frac{r_{nk}}{2}\}$ and $w(x) = 0 \neq \emptyset$. For simplicity, let us represent this sequence by $r_k$, then there exists $y_k$ such that $|y_k| = r_k$ and $w(y_k) = 0$. Let us extend $w$ to $\tilde{w}$ from $\Omega$ to $B_1^+(0)$ by zero. Now $\tilde{w}$ is subharmonic in $B_1^+(0)$ and we define

\[
\varepsilon_0 = \inf \{\varepsilon : \varepsilon x_2 \geq \tilde{w}\} \quad \text{in} \quad B_1^+(0)
\]

\[
\varepsilon_k = \inf \{\varepsilon : \varepsilon x_2 \geq \tilde{w}\} \quad \text{in} \quad B_{r_k}^+(0).
\]

(4.1)

$\{\varepsilon_k\}$ is a nonnegative decreasing sequence bounded above by the Lipschitz constant of $w$, let us denote this constant as $\kappa$. Therefore, $\lim_{k \to \infty} \varepsilon_k$ exists and we denote it as $\alpha$. By the definition of $\alpha$, we have

\[
\tilde{w}(x) \leq \alpha x_2 + (\varepsilon_k - \alpha)x_2 \quad \text{in} \quad B_{r_k}^+(0), \quad \text{for} \quad k = 0, 1, 2, \ldots.
\]

(4.2)

Then, by Lipschitz property, on a fixed portion of the arc $\partial B_{r_k}(0) \cap C_0$, say $\Gamma$, we have

\[
\tilde{w}(x) = \tilde{w}(x) - \tilde{w}(y_k) \leq \kappa \delta_0 r_k
\]

(4.3)

where $\delta_0 > 0$ is sufficiently small and depends on $\Gamma$. Let us write down the Poisson formula for $\tilde{w}$ in $B_{r_k}(0) \setminus B_{r_k/2}(0)$ and get an estimate by using (4.2) and (4.3):

\[
\tilde{w}(x) = \frac{r_k^2 - |x|^2}{2\pi r_k} \int_{\partial B_r(0)} \frac{\tilde{w}(y)}{|x-y|^2} dS(y)
\]

\[
\leq \frac{r_k^2 - |x|^2}{2\pi r_k} \int_{\Gamma} \frac{\tilde{w}(y)}{|x-y|^2} dS(y) + \frac{r_k^2 - |x|^2}{2\pi r_k} \int_{\partial B_{r_k}(0) \cap C_0 \cap \Gamma} \frac{\tilde{w}(y)}{|x-y|^2} dS(y)
\]

\[
\leq \frac{r_k^2 - |x|^2}{2\pi r_k} \int_{\Gamma} \frac{\kappa \delta_0 r_k}{|x-y|^2} dS(y) + \frac{r_k^2 - |x|^2}{2\pi r_k} \int_{\partial B_{r_k}^+(0) \cap C_0 \cap \Gamma} \frac{\varepsilon_k y_2}{|x-y|^2} dS(y).
\]

Note that for $y = (y_1, y_2) \in \partial B_{r_k}^+(0)$ we have $\frac{r_k^2 - |x|^2}{|x-y|^2} \leq 1$. Moreover, the arc length of $\Gamma$ is sufficiently small by the assumption and it is proportional to $\delta_0$. Thus, we have

\[
\tilde{w}(x) \leq \frac{\kappa \delta_0 r_k}{2} + \frac{\alpha r_k + (\varepsilon_k - \alpha)r_k}{2}.
\]

By taking $k$ sufficiently large enough and $\delta_0$ small enough we can obtain

\[
\frac{\kappa \delta_0}{2} + \frac{(\varepsilon_k - \alpha)}{2} < \alpha/2
\]

so that

\[
\tilde{w}(x) < \alpha r_k \quad \text{in} \quad B_{r_k}^+(0),
\]

which contradicts to the definition of $\alpha$. Hence, the result follows.
So far, we know that for any \( x \in \partial \Omega (u) \) and a given ball \( B_r (x) \), Non-Degeneracy condition will be attained from a connected component of \( \Omega (u) \) in \( B_r (x) \) but not necessarily will be attained from all the components of \( \Omega (u) \). Next, we will show that Non-Degeneracy condition is true for each of the connected component of \( \Omega (u) \) in \( B_r (x) \).

**Lemma 4.9** Let \( \Gamma \) be any connected component of \( \Omega (u) \) in \( B_1 (0) \) and \( h (y) \) is tangent from inside to \( \partial \Gamma \) at \( x_0 \), then there exists a universal constant \( C > 0 \) such that

\[
\sup_{B_h (x_0) \cap \Gamma} u \geq C h, \quad \text{for any } h \sim r.
\]

**Proof** (By way of contradiction) Assume that for sufficiently small \( \delta > 0 \), we can find \( r_n > 0 \) such that

\[
\sup_{B_h (x_0) \cap \Gamma} u \leq \delta r_n.
\]

By Theorem 3.6, \( u \) is Non-Degenerate so there exists \( C > 0 \) such that

\[
\sup_{B_h (x_0)} u \geq C r_n.
\]

Therefore, for \( \delta > 0 \) small enough, \( \Omega (u) \setminus \Gamma \) is nonempty around \( x_0 \) and \( u \) grows linearly in this set. Let us denote this set as \( \Gamma_o \) and we have \( \Gamma_o \subseteq \Gamma^c \). Let \( \eta \) be the inner normal vector of \( \Gamma_o \) at \( x_0 \) then we can normalize \( B_{r_n} (x_0) \) to \( B_1 (0) \) with mapping \( x_0 \mapsto 0 \) and \( \eta \mapsto e_2 \) with \( u(x) = \frac{w(e^{i\theta} (x-x_0))}{r_n} \) where \( \theta \geq 0 \) is the angle in between \( \eta \) and \( e_2 \). By Lemma 4.8, for any cone \( C_0 \subseteq B_1 (0) \cap \{ |x| \geq 0 \} \) and for any \( r \in [0, 1] \), the number of \( k \) that satisfies \( r_k = \frac{r}{2^k} \)

\[
\text{with } \{ x \in C_0 | \langle x \rangle < r_k/2, w(x) = 0 \} \neq \emptyset \text{ is finite. Hence, for } \varepsilon > 0 \text{ there exists } r_n > 0 \text{ small enough such that the arc-length of } \partial B_{r_n} (x_0) \cap \Gamma \text{ is } \pi r_n - \varepsilon \text{ and the arc-length of } \partial B_{r_n} (x_0) \cap \Gamma_o > \pi r_n - \varepsilon. \text{ Now, we can construct a larger subsolution by taking the harmonic function } h(x) \text{ in } B_{r_n} (x_0) \text{ such that } h(x) = u(x) \text{ for } x \in \partial B_{r_n} (x_0). \text{ We can estimate } h_v(x) \text{ for } x \in \partial \Omega (h) \text{ as we did in the proofs of Lemma 4.2 and 4.3. We can write down the Poisson formula for } h \text{ and estimate } h_v(x) \text{ by the first order incremental quotient so we have}
\]

\[
\frac{h(x+sv) - h(x)}{s} = \frac{r_n^2 - |x + sv|^2}{2 \pi r_n s} \int_{\partial B_{r_n} (x_0)} \frac{h(y)}{|x + sv - y|^2} dS(y)
\]

\[
\geq \frac{r_n^2 - |x + sv|^2}{2 \pi r_n s} \int_{\partial B_{r_n} (x_0)} \frac{h(y)}{|x + sv - y|^2} dS(y)
\]

\[
\geq \frac{r_n^2 - |x + sv|^2}{2 \pi r_n s} \int_{\partial B_{r_n} (x_0) \cap \{ |x-y| < 4r \}} \frac{h(y)}{32 \varepsilon^2 + 2 \varepsilon^2} dS(y)
\]

Let us take the limit as \( s \to 0 \) on both sides, we get

\[
h_v(x) \geq C \int_{\partial B_{r_n} (x_0) \cap \{ |x-y| < 4r \}} \frac{1}{32 \varepsilon^2} dS(y) \geq \frac{Cr_n}{4 \varepsilon}.
\]

Hence, for sufficiently small \( \varepsilon > 0 \), we obtain \( |\nabla h(x)|^2 > \Lambda \) which implies that \( \max \{ u, h \} \) is a larger subsolution than \( u \). Contradiction. Hence, the result follows. \( \square \)
**Theorem 4.10** \( u \) has a nontrivial linear growth in \( \Gamma \), i.e. there exist universal constants \( C, c > 0 \) such that for any \( x_0 \in \partial \Gamma \) and any \( r \leq \text{diam}(\Gamma) \).

\[
Cr \geq \sup_{B_r(x_0) \cap \Gamma} u \geq cr. \tag{4.4}
\]

**Proof** First inequality is the direct result of Lipschitz property, so we need to prove the second inequality of (4.4). By Lemma 4.4, we know that \( \Gamma \) is a simple connected domain. Let us just consider \( w = u|_\Gamma \) in \( B_1(0) \) and denote \( d_0 = d(0, \Gamma) \). We will prove this theorem in two steps by combining and adapting the ideas of the proofs of Lemma 3.4 and Theorem 3.6. First, we will show that for any \( r \in [d_0, 1] \), there exists a ball where \( w \) has a nontrivial growth. Second, we will obtain the inequality by way of contradiction with Green’s representation theorem.

Let us start the proof of the first claim: \( \Gamma \) has a curve from \( d_0 \) to \( \partial B_1(0) \) and for any \( r \in [d_0, 1] \), there exists \( x_r \in \Gamma \) and \( B_{\varepsilon_r}(x_r) \in \Gamma \) which is tangent from inside to \( \Gamma \) at some point \( y_r \in \partial \Gamma \). As we did before, pick a ray in \( B_1(0) \), say \( R_1 = \{ r \eta | r \in [0, 1] \}, \eta \in S^1 \}. \) Now, we use \( R_1 \) in order to pick a partition of \([d_0, 1] \) in terms of \( \varepsilon_r \). Consider \( \{ B_{\varepsilon_r}(y_r) | r \in [d_0, 1] \} \) and project all these balls \( B_{\varepsilon_r}(y_r) \) onto \( R_1 \). Hence, we obtain a covering of \( R_1 \cap [d_0, 1] \) by segments \( S_r = \{ r - h, r + h \} \). Extract a disjoint subfamily \( \{ S_{r_j} \} \) s.t. \( 2 S_{r_j} \) covers \( R_1 \cap [d_0, 1] \). Now, resend back this subfamily to their original places. Since, \( B_{\varepsilon_r}(x_r) \) is tangent from inside to \( y_r \), by Lemma 4.9, we have \( \sup_{B_{\varepsilon_r}(y_r) \cap \Gamma} u = \sup_{B_{\varepsilon_r}(y_r)} w \geq c \varepsilon_r \). Now, as following the same steps of the proof of Lemma 3.4, we obtain

\[
\int_{B_{\varepsilon_r}(y_r)} (\Delta w) dx \sim \varepsilon_r. \tag{4.5}
\]

Now, let us prove the second inequality of (4.4) by way of contradiction: Assume that there exists \( x_0 \in \partial \Gamma \) with \( r > 0 \) such that \( \sup_{B_r(x_0)} w(x) < \delta r \), for some sufficiently small \( \delta > 0 \). By Green’s representation theorem, we have

\[
0 = w(x_0) = \int_{\partial B_r(x_0)} wG_{\nu}ds + \int_{B_r(x_0)} (G\Delta w)dx
\]

where \( G(y, x_0) = \frac{1}{2\pi} \ln(\frac{1}{r} |y - x_0|) \) with \( G \equiv 0 \) on \( \partial B_r(x_0) \). Hence,

\[
\int_{\partial B_r(x_0)} wG_{\nu}ds = -\int_{B_r(x_0)} (G\Delta w)dx \geq -\int_{B_{r/2}(x_0)} (G\Delta w)dx.
\]

Then, \( G(y, x_0) \leq -C_1 \) in \( B_{r/2}(x_0) \) and by the first part of the proof we obtain a finite cover of \( B_{r/2}(x_0) \cap \partial \Gamma \) with balls \( \{ B_{\varepsilon_r}(y_r) | r \in [d_0, 1] \} \).

\[
\int_{\partial B_r(x_0)} wG_{\nu}ds \geq -\int_{B_{r/2}(x_0)} (G\Delta w)dx
\]

\[
= -\sum_{B_{r/2}(x_0) \cap B_{\varepsilon_r}(y_r)} (G\Delta w)dx
\]

\[
\geq C_2 r, \quad \text{for some universal constant } C_2 > 0.
\]
The last inequality is true because we obtain $\Delta w \sim \varepsilon_r$ in $B_{\varepsilon r}(y_r)$ by (4.5). $G_v \sim \frac{1}{r}$ on $\partial B_{r}(x_0)$ so we obtain by the assumption

$$C_3 \delta r \geq \int_{\partial B_{r}(x_0)} w G_v ds \geq C_2 r.$$  

Contradiction; we can choose $\delta > 0$ small enough so that the above inequality fails. Hence, we get the result. 

4.3 $\Omega_1(u)$ has a single component with a positively dense complement

So far, we obtain the nontrivial linear growth in every connected component of $\Omega_1(u)$ in $B_1(0)$, Theorem 4.9. Next, we will show that $\Omega_1(u)$ can have at most two components in $B_1(0)$. The intuitive idea is the following: The component needs enough mass in $B_1(0)$ in order to have a nontrivial linear growth in $B_1(0)$. This idea directly connects this fact to the Monotonicity Formula. Because the Monotonicity formula enables us to find that how much mass a positivity set needs in order to have a specific growth-order.

**Remark 4.11** Let $\Omega$ be a sector area enclosed by the arc of length $\frac{2\pi}{\alpha}$ in $B_1(0) \subseteq \mathbb{R}^2$, for some $\alpha \geq 1$, then $h(r, \theta) = r^{\alpha/2} \cos(\frac{\alpha}{2}(\theta + \frac{\pi}{\alpha}))$ is the harmonic function in $\Omega$ with

$$\begin{cases} 
\Delta h = 0, & \text{in } \Omega \\
h(s, b) = h(s, \frac{2\pi}{\alpha}) = 0, & 0 \leq s \leq r.
\end{cases}$$

**Theorem 4.12** For any given $\sigma > 0$, there exist a pair of positive constants, $(\varepsilon, \delta)$, with $0 < \varepsilon < \delta < 1$ such that if $\Omega(u) \cap B_1(0) \setminus B_\varepsilon(0)$ has at least two components all of which intersects with $\partial B_\varepsilon(0)$, then, for some direction $e$, we have

$$u(x) \geq C[\langle x, \pm e \rangle - \delta^2 \sigma]_+, \text{ in } B_\delta(0) \cap \{x|\langle x, \pm e \rangle - \delta^2 \sigma > 0\}.$$  

Shown in Fig. 8.

**Proof** (By way of contradiction) Assume that there exists a $\sigma > 0$ such that, for any $(\varepsilon, \delta)$ pair with $\delta > \varepsilon > 0$, we have $\Omega(u) \cap B_1(0) \setminus B_\varepsilon(0)$ has at least two components and

$$u(x) \leq C[\langle x, e \rangle - \delta^2 \sigma]_+ \text{ in } B_\delta(0) \cap \{x|\langle x, e \rangle - \delta^2 \sigma > 0\}$$

or

$$u(x) \leq C[\langle x, -e \rangle - \delta^2 \sigma]_+ \text{ in } B_\delta(0) \cap \{x|\langle x, -e \rangle - \delta^2 \sigma > 0\},$$

$\square$ Springer
for any direction $e$. Let us pick a sequence $\{\delta_k\}$ such that $\delta_k \to 0$, as $k \to \infty$, a direction $e$, and $\varepsilon_k = (1 - \eta)\delta_k$, for some sufficiently small $\eta > 0$, then we have

$$u(x) \leq C \{x, \pm e - \delta_k^2 \sigma\}_{+} \text{ in } B_{\delta_k}(0) \cap \{x| (x, \pm e - \delta_k^2 \sigma > 0\}, \quad (4.6)$$

in at least one of the directions $e$ or $-e$ and $\Omega(u) \cap B(0) \setminus B_{\varepsilon_k}(0)$ has at least two components that each of them intersects with $\partial B_{\varepsilon_k}(0)$, without loss of generality, let us assume that there are two components and denote them as $\Omega_1$, $\Omega_2$. Since, $\Omega_1$ and $\Omega_2$ intersect with $\partial B_{\varepsilon_k}(0)$, their diameter should be at least $\eta \delta_k$, by construction. Moreover, there exist $x_1 \in \partial \Omega_1 \cap B_{\varepsilon_k}(0)$ and $x_2 \in \partial \Omega_2 \cap B_{\varepsilon_k}(0)$. Therefore, by Theorem 4.10,

$$\sup_{\Omega_1 \cap B_r(x_1)} u \geq Cr \quad \text{and} \quad \sup_{\Omega_2 \cap B_r(x_2)} u \geq Cr, \quad (4.7)$$

for any $r \leq \eta \delta_k \leq \min\{\text{diam} \Omega_1, \text{diam} \Omega_2\}$. We can choose $\delta_k, \eta > 0$ small enough to contradict (4.6). Hence, we obtain the result. □

**Corollary 4.13** There exist universal constants $h, \varepsilon_0 > 0$ such that $\Omega(u)$ has only one component from $B_0(0)$ to $B_1(0)$ for any $\varepsilon < \varepsilon_0$. Moreover, the set $\{x|u(x) = 0\}^o \cap B_{1/2}(0)$ contains a ball $B_0(z)$, for some $z \in B_{1/2}$.

**Proof of the first part** (By way of contradiction) Suppose that for $\varepsilon = 2^{-k_0} > 0$ there exists a $\varepsilon_0 < \varepsilon$ s.t. $B_1(0) \setminus B_{\varepsilon_0}(0)$ has two components of $\Omega(u)$, say $\Omega_1$ and $\Omega_2$. These components are also components of $B_2(0) \setminus B_0(0)$. By Theorem 4.9, $u$ has a nontrivial growth in both $\Omega_1$ and $\Omega_2$. Let us consider $u_1(x) = u(x)|_{\Omega_1}$ and $u_2(x) = u(x)|_{\Omega_2}$, and write down the Monotonicity Formula, the Monotonicity Theorem 5.1, for them by denoting the universal Non-Degeneracy and Lipschitz constants as $c$ and $C$, respectively. Then, we have

$$c^4 \pi^2 \leq J(\varepsilon) \leq J(1) \leq C^4 \pi^2.$$  

Note that we can obtain a lower bound for $J(r)$ by adapting the proof of Monotonicity Theorem 5.1 as follows:

$$J'(r) \leq \frac{J(r)}{J(r)} = \frac{\int_{\partial B_{\varepsilon_k}(0)} |\nabla u_1|^2 d\sigma}{\int_{\partial B_{\varepsilon_k}(0)} |\nabla u_1|^2 dx} + \frac{\int_{\partial B_{\varepsilon_k}(0)} |\nabla u_2|^2 d\sigma}{\int_{\partial B_{\varepsilon_k}(0)} |\nabla u_2|^2 dx} - \frac{4}{r},$$

$\geq \left(\frac{\int_{\partial B_{\varepsilon_k}(0)} (u_1)^2 d\sigma}{\left(\int_{\partial B_{\varepsilon_k}(0)} u_1^2 d\sigma\right)^{1/2}}\right)^{1/2} + \left(\frac{\int_{\partial B_{\varepsilon_k}(0)} (u_2)^2 d\sigma}{\left(\int_{\partial B_{\varepsilon_k}(0)} u_2^2 d\sigma\right)^{1/2}}\right)^{1/2} - \frac{2}{r}.$

If we denote the angular traces of the domains $\Omega_1$ and $\Omega_2$ in the circle of radius $r$ as $\pi t_1(r)$ and $\pi t_2(r)$, respectively, then the sum

$$\left(\frac{\int_{\partial B_{\varepsilon_k}(0)} (u_1)^2 d\sigma}{\left(\int_{\partial B_{\varepsilon_k}(0)} u_1^2 d\sigma\right)^{1/2}}\right)^{1/2} + \left(\frac{\int_{\partial B_{\varepsilon_k}(0)} (u_2)^2 d\sigma}{\left(\int_{\partial B_{\varepsilon_k}(0)} u_2^2 d\sigma\right)^{1/2}}\right)^{1/2}$$

Attains its minimum for two adjacent, complementary arcs with length $\alpha 2\pi r$ and $(1 - \alpha)2\pi r$ and the corresponding eigenfunctions are

$$\sin \frac{\theta}{2\alpha r} \text{ and } \sin \frac{\theta}{2(1 - \alpha)r}.$$

Thus, we obtain

$$\frac{J'(r)}{J(r)} \geq \frac{1}{2rt_1(r)} + \frac{1}{2rt_2(r)} - \frac{2}{r}, \quad (4.8)$$

Springer
Consider the right hand side of (4.8) as a function of \((t_1, t_2)\), i.e. let

\[
F(t_1, t_2) = \frac{1}{2rt_1} + \frac{1}{2rt_2} - \frac{2}{r},
\]

then \(F(t_1, t_2)\) has a minimum of zero at \(t_1 = t_2 = 1\) and it is strictly convex at \(t_1 = t_2 = 1\). Therefore,

\[
F(t_1, t_2) \geq \frac{c_0}{r}[(t_1 - 1)^2 + (t_2 - 1)^2], \quad \text{for some } c_0 > 0.
\]

If we consider \(\tilde{u}(x) = \lceil u(x) - c\varepsilon \rceil_+\), then \(\tilde{u}\) has at least two components \(\tilde{\Omega}_1 \subseteq \Omega_1\) and \(\tilde{\Omega}_2 \subseteq \Omega_2\). Moreover, it will start to grow linearly from say \(K\varepsilon\), for some \(K > 0\). If we denote the Monotonicity Formula, in Monotonicity Theorem 5.1, for \(\tilde{u}\) as \(\tilde{J}(r)\) and the angular traces of the domains \(\tilde{\Omega}_1\) and \(\tilde{\Omega}_2\) in the circle of radius \(r\) as \(\pi_{l_1}(r)\) and \(\pi_{l_2}(r)\), respectively, then we have \(\tilde{J}(r) \sim 1\), for any \(r \in [K\varepsilon, 1]\). Therefore, for \(k_1 > k_0 > 0\) and \(2^{-k_1} > K\varepsilon\), we have

\[
C_1 \geq \int_{2^{-k_1}}^{1} \frac{\tilde{J}(r)}{J(r)} dr \geq \int_{2^{-k_1}}^{+\infty} \frac{c_0}{r}[(\tilde{t}_1 - 1)^2 + (\tilde{t}_2 - 1)^2]dr,
\]

for some \(C_1 > 0\). Let us write down the right hand side integral with diadic representation:

\[
C_1 \geq \int_{2^{-k_1}}^{1} \frac{c_0}{r}[(\tilde{t}_1 - 1)^2 + (\tilde{t}_2 - 1)^2]dr
\]

\[
= -1 \sum_{l=k_1}^{+\infty} \int_{2^{-l+1}}^{2^{-l+1}} \frac{c_0}{r}[(\tilde{t}_1 - 1)^2 + (\tilde{t}_2 - 1)^2]dr
\]

\[
\geq \sum_{l=k_1}^{+\infty} 2^l \int_{2^{-l+1}}^{2^{-l+1}} c_0[(\tilde{t}_1 - 1)^2 + (\tilde{t}_2 - 1)^2]dr.
\]

For \(\eta > 0\) sufficiently small, there exists at least one ring such that

\[
C_1 \eta \geq 2^l \int_{2^{-l+1}}^{2^{-l+1}} c_0[(\tilde{t}_1 - 1)^2 + (\tilde{t}_2 - 1)^2]dr.
\]

i.e. \([(\tilde{t}_1 - 1)^2 + (\tilde{t}_2 - 1)^2]\) is close to zero most of the time therefore \(\tilde{t}_1\) and \(\tilde{t}_2\) are close to 1 most of the time for the radii between \(2^{-l}\) and \(2^{-l+1}\). This contradicts to Lipschitz property of a free boundary point \(x_0 \in B_{r^*}(0)\) of \(\tilde{u}\), where \(r^* = \frac{1}{2}[2^{-l} + 2^{-l+1}]\). There exists at least one free boundary point of \(\tilde{u}\) for each radius such that this point has a neighborhood with only one component of \(\Omega(\tilde{u})\), i.e. for each radius \(r \in [2^{-l}, r^*]\), \(\tilde{t}_1\) and \(\tilde{t}_2\) are different than 1. That is because: for \(z_i \in \partial \tilde{\Omega}_1 \cap \partial B_r(0)\) with \(d(\tilde{\Omega}_1 \cap \partial B_r(0), \tilde{\Omega}_2 \cap \partial B_r(0)) = d(z_1, z_2)\), we can determine whether \(\tilde{t}_1\) and \(\tilde{t}_2\) are different than 1 or not. Since, \(u(z_i) = c\varepsilon\) and \(u\) is Lipschitz, Theorem 3.1, we have

\[
d(z_i, \partial \Omega(u)) \geq \frac{c\varepsilon}{C},
\]
and therefore
\[ d(z_1, z_2) \geq d(z_i, \partial \Omega(u)) \geq \frac{c\varepsilon}{C}. \]

Hence, \( t_1 \) and \( t_2 \) are different than 1 for any \( r \in [2^{-l}, r^*] \). Contradiction, thus the result follows.

[Proof of the second part](By way of contradiction) Assume that, for every \( h_0 > 0 \), there exists \( h < h_0 \), such that we have
\[ \{ x \in B_{1/2}(0) \mid B_h(x) \subseteq \{ x \mid u(x) = 0 \} \} = \emptyset. \]

Let us construct a larger subsolution under the above assumptions. Let \( x \in [B_{1/2}(0) \setminus B_{1/2 - 2h}(0)] \cap \{ x \mid u(x) = 0 \}^o \), then \( B_h(x) \cap \Omega(u) \neq \emptyset \). We claim that there exists a point \( z \in [B_{1/2}(0) \setminus B_{1/2 - 2h}(0)] \cap \{ x \mid u(x) = 0 \}^o \) such that \( B_h(z) \) intersects with \( \Omega(u) \) in two components, as shown in Fig. 9. In order to catch such a ball, we start from the ball \( B_h(x) \), say \( x = (1/2 - h, \theta) \) in polar coordinates for some \( \theta \in [0, 2\pi) \), and rotate this ball in counterclockwise direction. There exists a point \( z \in [B_{1/2}(0) \setminus B_{1/2 - 2h}(0)] \cap \{ x \mid u(x) = 0 \}^o \) such that \( B_h(z) \) is intersected with \( \Omega(u) \) in two components, otherwise we can insert a ball \( B_h(z_0) \) into \( \{ x \mid u(x) = 0 \}^o \) which contradicts to our assumption. Hence, consider \( B_h(z) \) such that it is intersected with \( \Omega(u) \) in two components. Then there exists \( r \in [1/2 - 2h, 1/2] \) and two points \( x_1 \) and \( x_2 \) in \( B_h(z) \) such that \( x_1 = x_2 = r \) and these points are in separate components of \( \Omega(u) \). Now, we can construct a larger subsolution for sufficiently small \( h > 0 \) as follows: Consider the harmonic function \( h(x) \) in \( B_r(0) \) with \( w = u \) in \( B_{1/2}(0) \setminus B_r(0) \). Then consider \( w = \max\{u, h\} \) in \( B_{1/2}(0) \). Thus, \( w \) becomes a larger subsolution than \( u \): as we showed previously, this claim is true if the FBC, (iii) in Definition 2.1, is satisfied by \( w \); so if \( x_0 \in \partial \Omega(w) \) with a tangent ball from outside, \( x_0 \) should be on \( \partial B_r(0) \cap B_h(z) \) and by Lemma 4.3 we have \( w_h(x_0) \geq \frac{C}{h} \) where \( h \) is the distance between two components of \( \Omega(u) \) in \( B_h(z) \) which is sufficiently small this time. Hence, \( w \) is a larger subsolution than \( u \). Contradiction, hence we get the result.

At the beginning of this section, we normalized a neighborhood, \( B_r(x_0) \), of the free boundary which contains components of \( \Omega(u) \) up to the radius \( B_{r/2}(x_0) \). On the other hand, by normalization and Corollary 4.13, this neighborhood can be characterized with only two components as \( \Omega(u) \) and its complement.

Acknowledgements I owe my deepest gratitude to my graduate advisor Luis A. Caffarelli for his supervision and support during my graduate degree at The University of Texas at Austin. His unique perspectives and visualizations for the subjects were the most significant contribution to this paper. I am also thankful to my colleagues Alessio Figalli, Nestor Guillen, and Ray Yang for their constructive feedback and productive exchange of ideas on related subjects.
5 Appendix

Let us remind you the Monotonicity formula for \( \mathbb{R}^2 \), the reader can consult to [8] for detailed theory:

**Theorem 5.1** (Monotonicity Theorem) *Let \( u_1, u_2 \in H^1(B_2(0)) \) be continuous and nonnegative functions in \( B_2(0) \subset \mathbb{R}^2 \), such that they are supported and harmonic in disjoint domains \( \Omega_1, \Omega_2 \), respectively, with \( 0 \in \partial \Omega_i \) and

\[
    u_i = 0 \text{ along } \partial \Omega_i \cap B_1 = \Gamma_i \ (i = 1, 2).
\]

Then the quantity

\[
    J(R) = \frac{1}{R^4} \int_{B_R(0)} |\nabla u_1|^2 dx \cdot \int_{B_R(0)} |\nabla u_2|^2 dx
\]

is monotone increasing in \( R, R \leq 3/2 \).

**Proof** We want to show that \( J'(R) \geq 0 \) a.e. \( R \in (0, 3/2) \). By rescaling, it is enough to prove that \( J'(1) \geq 0 \). Observe that,

\[
    \frac{d}{dr} \int_{B_r(0)} |\nabla u_i|^2 dx = \int_{\partial B_r(0)} |\nabla u_i|^2 d\sigma \in L^1(0, 2)
\]

and

\[
    J'(1) = \int_{\partial B_1(0)} |\nabla u_1|^2 d\sigma \cdot \int_{B_1(0)} |\nabla u_2|^2 dx + \int_{B_1(0)} |\nabla u_1|^2 dx \int_{\partial B_1(0)} |\nabla u_2|^2 d\sigma - 4 \int_{B_1(0)} |\nabla u_1|^2 dx \int_{\partial B_1(0)} |\nabla u_2|^2 d\sigma.
\]

Then, we get

\[
    \frac{J'(1)}{J(1)} = \frac{\int_{\partial B_1(0)} |\nabla u_1|^2 d\sigma}{\int_{B_1(0)} |\nabla u_1|^2 dx} + \frac{\int_{\partial B_1(0)} |\nabla u_2|^2 d\sigma}{\int_{B_1(0)} |\nabla u_2|^2 dx} - 4.
\]

Since \( u_i \) is harmonic and supported in \( \Omega_i \), we have \( \Delta u_i^2 = 2|\nabla u_i|^2 \) which implies

\[
    \int_{B_1(0)} |\nabla u_1|^2 dx = \int_{\partial B_1(0)} u_i(u_i)_r d\sigma
\]

\[
    = \left( \int_{\partial B_1(0)} u_i^2 d\sigma \right)^{1/2} \left( \int_{\partial B_1(0)} (u_i)_r^2 d\sigma \right)^{1/2}
\]

where \( u_r \) denotes the exterior radial derivative of \( u \) along \( \partial B_1(0) \). Let us denote \( u_\theta \) as the tangential derivative of \( u \) along \( \partial B_1(0) \) then we get

\[
    \int_{\partial B_1(0)} |\nabla u_i|^2 d\sigma \geq 2 \left( \int_{\partial B_1(0)} (u_i)_r^2 d\sigma \right)^{1/2} \left( \int_{\partial B_1(0)} (u_i)_\theta^2 d\sigma \right)^{1/2}.
\]
Hence, it is enough to prove that
\[
\frac{J'(1)}{J(1)} \geq \left( \frac{\int_{\partial B_1(0)} (u_1)^2 d\sigma}{\int_{\partial B_1(0)} u_1^2 d\sigma} \right)^{1/2} + \left( \frac{\int_{\partial B_1(0)} (u_2)^2 d\sigma}{\int_{\partial B_1(0)} u_2^2 d\sigma} \right)^{1/2} - 2 \geq 0.
\]
Thus, if we can estimate the minimum of the quotient
\[
\frac{\int_{\Gamma_1} (u_1)^2 d\sigma}{\int_{\Gamma_1} u_1^2 d\sigma},
\]
then we will obtain the result. These quotients are minimized by the first eigenfunction of the domains \( \partial B_1(0) \cap \Omega_i \), respectively. Moreover, since \( \Omega_1 \cap \Omega_2 = \emptyset \), the question is reduced to find the minimizer of the quotient
\[
\inf_{v \in H^1_0(\Gamma)} \frac{\int_{\Gamma} v^2 d\sigma}{\int_{\Gamma} v^2 d\sigma}
\]
for a given domain \( \Gamma \) in \( B_1(0) \) with a measure \( \mu \), i.e. to find \( v \in H^1_0(\Gamma) \) which has the smallest eigenvalue. We obtain, by the symmetrization argument, the optimal domain as a connected arc with the larger the arc the smaller the quotient. Thus, when we consider the domains \( \Gamma_1 \) and \( \Gamma_2 \), and the sum
\[
\left( \frac{\int_{\Gamma_1} (u_1)^2 d\sigma}{\int_{\Gamma_1} u_1^2 d\sigma} \right)^{1/2} + \left( \frac{\int_{\Gamma_2} (u_2)^2 d\sigma}{\int_{\Gamma_2} u_2^2 d\sigma} \right)^{1/2},
\]
then this sum attains its minimum for two adjacent, complementary arcs with \( u_1 \) and \( u_2 \) the corresponding eigenfunctions. If the arcs have length \( \alpha 2\pi \) and \( (1 - \alpha) 2\pi \), then the corresponding eigenfunctions are
\[
\sin \frac{\theta}{2\alpha} \quad \text{and} \quad \sin \frac{\theta}{2(1 - \alpha)}
\]
and the sum
\[
\left( \frac{\int_{\Gamma_1} (u_1)^2 d\sigma}{\int_{\Gamma_1} u_1^2 d\sigma} \right)^{1/2} + \left( \frac{\int_{\Gamma_2} (u_2)^2 d\sigma}{\int_{\Gamma_2} u_2^2 d\sigma} \right)^{1/2} \geq \frac{1}{2\alpha} + \frac{1}{2(1 - \alpha)} \geq 2, \quad \text{for } \alpha \in [0, 1]
\]
which implies the result. \( \square \)

References

1. Alt, H.W., Caffarelli, L.A.: Existence and regularity for a minimum problem with free boundary. J. Reine Angew. Math. 325, 105–144 (1981)
2. Alt, H.W., Caffarelli, L.A., Friedman, A.: Variational problems with two phases and their free boundaries. Trans. Am. Math. Soc. 282(2), 431–461 (1984)
3. Berestycki, H., Hamel, F.: Front propagation in periodic excitable media. Commun. Pure Appl. Math. 55(8), 949–1032 (2002)
4. Caffarelli, L., Lee, K.: Homogenization of oscillating free boundaries: the elliptic case. Commun. Partial Differ. Equ. 32(1–3), 149–162 (2007)
5. Caffarelli, L., Lee, K.-A.: Homogenization of nonvariational viscosity solutions. Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5) 29(1), 89–100 (2005)
6. Caffarelli, L.A., Mellet, A.: Capillary drops: contact angle hysteresis and sticking drops. Calc. Var. 29(2), 141–160 (2007)
7. Caffarelli, L. A., Mellet, A.: Capillary drops on an inhomogeneous surface. In: Perspectives in Nonlinear Partial Differential Equations, Volume 446 of Contemporary Mathematics, pp 175–201. AMS, Providence (2007)
8. Caffarelli, L., Salsa, S.: A Geometric Approach to Free Boundary Problems, Volume 68 of Graduate Studies in Mathematics. American Mathematical Society, Providence (2005)
9. Caffarelli, L.A.: A Harnack inequality approach to the regularity of free boundaries. II. Flat free boundaries are Lipschitz. Commun. Pure Appl. Math. 42(1), 55–78 (1989)
10. Caffarelli, L.A., Friedman, A.: Regularity of the boundary of a capillary drop on an inhomogeneous plane and related variational problems. Rev. Mat. Iberoamericana 1(1), 61–84 (1985)
11. Luis A.C., Jerison, D., Kenig, C.E.: Global energy minimizers for free boundary problems and full regularity in three dimensions. In Noncompact Problems at the Intersection of Geometry, Analysis, and Topology, Volume 350 of Contemporary Mathematics, pp. 83–97. AMS, Providence (2004)
12. Caffarelli, L.A., Vázquez, J.L.: A free-boundary problem for the heat equation arising in flame propagation. Trans. Am. Math. Soc. 347(2), 411–441 (1995)
13. Michael, G.C., Ishii, H., Lions, P.L.: User’s guide to viscosity solutions of second order partial differential equations. Bull. Am. Math. Soc. (N.S.) 27(1), 1–67 (1992)
14. Kenig, C.E., Toro, T.: Free boundary regularity for harmonic measures and Poisson kernels. Ann. Math. (2) 150(2), 369–454 (1999)