An invariant of Legendrian and transverse links from open book decompositions of contact 3-manifolds

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Abstract

We introduce a generalization of the Lisca-Ozsváth-Stipsicz-Szabó Legendrian invariant $\mathcal{L}$ to links in every rational homology sphere, using the collapsed version of link Floer homology.

We represent a Legendrian link $L$ in a contact 3-manifold $(M,\xi)$ with a diagram $D$, given by an open book decomposition of $(M,\xi)$ adapted to $L$, and we construct a chain complex $cCFL^{-}(D)$ with a special cycle in it denoted by $L(D)$. Then, given two diagrams $D_1$ and $D_2$ which represent Legendrian isotopic links, we prove that there is a map between the corresponding chain complexes, that induces an isomorphism in homology and sends $L(D_1)$ into $L(D_2)$.

Moreover, a connected sum formula is also proved and we use it to give some applications about non-loose Legendrian links; that are links such that the restriction of $\xi$ on their complement is tight.

1 Introduction

The Legendrian invariant $\mathcal{L}$ was first introduced in [12]. In this paper we generalize $\mathcal{L}$ to Legendrian links in rational homology contact 3-spheres. For sake of simplicity, we consider only null-homologous links, which in this settings are links whose all components represent trivial classes in homology.

A contact structure $\xi$ on a differentiable 3-manifold $M$ is a smooth 2-plane field that is given as the kernel of a 1-form $\alpha$ such that $\alpha \wedge d\alpha > 0$. A Legendrian $n$-component link $L \hookrightarrow (M,\xi)$ is a link such that the tangent space $T_pL$ is contained in $\xi_p$ for every point $p$. A link $T \hookrightarrow (M,\xi)$ is transverse if $T_pT \oplus \xi_p = T_pM$ for every $p \in T$ and $\alpha > 0$ on $T$.

A Legendrian isotopy between $L$ and $L'$ is a smooth isotopy $F_t$ of $M$, where $t \in [0,1]$, such that $F_0(L) = L$, $F_1(L) = L'$ and $F_t(L)$ is Legendrian for every $t$.

A 3-manifold $M$ can be represented with an open book decomposition $(B,\pi)$. Suppose $L$ is a Legendrian $n$-component link in $M$, equipped with a contact structure $\xi$. When some compatibility conditions with $\xi$ and $L$, which are explained in Section 2, are satisfied $(B,\pi)$, together with an appropriate set $A$ of embedded arcs in the page $S_1$, is said to be adapted to the Legendrian link $L$. In particular, we prove the following theorem.

**Theorem 1.1.** Every Legendrian link $L$ in a contact 3-manifold $(M,\xi)$ admits an adapted open book decomposition $(B,\pi,A)$ which is compatible with the triple $(L,M,\xi)$. Moreover, the contact framing of $L$ coincides with the framing induced on $L$ by the page $S_1$. 
Suppose from now on that $M$ is a rational homology sphere. A \textit{multi-pointed Heegaard diagram} is defined in [17] as two collections of $g+n-1$ curves and two sets of $n$ basepoints in a genus $g$ surface $\Sigma$ which describe a link in $M$. As shown in Section 3, one can associate, up to isotopy, a diagram of this kind to every adapted open book decomposition, compatible with the triple $(L, M, \xi)$. This diagram is called a Legendrian Heegaard diagram and it is denoted with $D_{(B,\pi,A)}$. The surface $\Sigma$ in $D_{(B,\pi,A)}$ is obtained by gluing the pages $S_1$ and $S_\infty$ together, moreover all the basepoints are contained in $S_1$. In Heegaard Floer theory the diagram $D_{(B,\pi,A)}$ gives a bigraded chain complex $(cCFL^-(D_{(B,\pi,A)}), \partial^-)$, generated by some discrete subsets of points in $\Sigma$ [19] and whose homology is an $\mathbb{F}[U]$-module, where $\mathbb{F}$ is the field with two elements, called \textit{collapsed link Floer homology}. Such group it is a smooth link invariant and it is denoted by $cHFL^-(M,L)$, see [14, 17]. Furthermore, there is only one cycle in $cCFL^-(D_{(B,\pi,A)})$, see [14, 22], that lies on the page $S_1$. This cycle is denoted by $L_{(D_{(B,\pi,A)})}$. We show that the homology class of the cycle $L_{(D_{(B,\pi,A)})}$ in $cHFL^-(M,L)$ is a Legendrian link invariant.

**Theorem 1.2.** Let us consider a Legendrian Heegaard diagram $D$, given by an open book compatible with a triple $(L, M, \xi)$, where $M$ is a rational homology 3-sphere and $L$ is a null-homologous Legendrian $n$-component link. Let us take the cycle $\Sigma(D) \in cCFL^-(D, t_{\Sigma(D)})$. Then, the isomorphism class of $[\Sigma(D)]$ in $cHFL^-(M, L, \xi)$ is a Legendrian invariant of $(L, M, \xi)$ and we denote it with $\Sigma(L,M,\xi)$. Furthermore, the Spin$^c$ structure $t_{\Sigma(D)}$ coincides with $t_{\xi}$.

In [1] Baldwin, Vela-Vick and Vértesi, using a different construction, introduced another invariant of Legendrian links in contact 3-manifolds which generalizes $\Sigma$ in the case of knots in the standard 3-sphere. The same argument in [1] implies that this invariant coincides with our $\Sigma$ for every Legendrian link in $(\mathbb{S}^3, \xi_{st})$.

The smooth link type is clearly a Legendrian invariant. Together with the Thurston-Bennequin and rotation numbers are called classical invariant of a Legendrian link. The \textit{Thurston-Bennequin and rotation numbers} of a null-homologous link $L$ in a rational homology contact 3-sphere $(M, \xi)$ are defined as follows. The first one, denoted with $tb(L)$, is the linking number of the contact framing $\xi_L$ of $L$ respect to $\xi$ and a Seifert surface for $L$. While, the second one, denoted with $rot(L)$, is defined as the numerical obstruction to extending a non-zero vector field, everywhere tangent to the knot, to a Seifert surface of $K$ (see [6]).

If in $(M, \xi)$ there is an embedded disk $E$ such that $tb(\partial E) = 0$ then the structure $\xi$ is overtwisted; otherwise it is called tight. Furthermore, in overtwisted structures there are two types of Legendrian and transverse links: \textit{non-loose} if their complement is tight and \textit{loose} if it is overtwisted. We have the following proposition.

**Proposition 1.3.** Let $L$ be a loose Legendrian link in an overtwisted contact 3-manifold $(M, \xi)$. Then we have that $\Sigma(L,M,\xi) = [0]$.

We use the invariant $\Sigma$ to prove the existence of non-loose Legendrian $n$-component links, with loose components, in many overtwisted contact 3-manifolds.
Theorem 1.4. Suppose that \((M, \xi)\) is an overtwisted 3-manifold such that there exists a contact structure \(\zeta\) with \(\xi = \xi_\zeta\) and \(\widehat{c}(M, \zeta) \neq [0]\). Then there is a non-split Legendrian \(n\)-component link \(L\), for every \(n \geq 1\), such that \(\Sigma(L, M, \xi)\) is non-zero and all of its components are loose. In particular, \(L\) is non-loose and stays non-loose after a sequence of negative stabilizations.

Furthermore, the transverse link \(T_L\), obtained as transverse push-off of \(L\), is again non-split and \(\Sigma(T_L, M, \xi) := \Sigma(L, M, \xi)\) is non-zero, which means that \(T_L\) is also non-loose.

Furthermore, we give examples of non-loose, non-simple Legendrian and transverse link types. Here non-simple means that there exists a pair of Legendrian (transverse) links that have the same classical invariants and their components are Legendrian (transverse) isotopic knots, but they are not Legendrian (transverse) isotopic (as links). Moreover, with non-loose we mean that such a pair consists of two non-loose Legendrian (transverse) links.

Theorem 1.5. Suppose that \((M, \xi)\) is an overtwisted 3-manifold as in Theorem 1.4. Then in \((M, \xi)\) there is a pair of non-loose, non-split Legendrian (transverse) \(n\)-component links, with the same classical invariants and Legendrian (transversely) isotopic components, but that are not Legendrian (transversely) isotopic.

This paper is organized as follows. In Section 2 we define open book decompositions and we describe the compatibility condition with Legendrian links. Moreover, we show that such adapted open books always exist. In Section 3 we give some remarks on link Floer homology. In Section 4 we define \(\Sigma\) and we prove its invariance under Legendrian isotopy. In Section 5 we show some properties of the invariant \(\Sigma\), including the relations with the contact invariant \(\widehat{c}\) and the classical invariants of Legendrian links. In Section 6 we generalize the transverse invariant \(\Sigma\) to links and we describe some of its properties. Finally, in Section 7 we give some applications of our invariants.

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2 Open book decompositions adapted to a Legendrian link

2.1 Definition

Let us start with a contact 3-manifold \((M, \xi)\). We say that an open book decomposition for \(M\) is a pair \((B, \pi)\) where

- the binding \(B\) is a smooth link in \(M\);
- the map \(\pi : M \setminus B \to S^1\) is a locally trivial fibration such that \(\pi^{-1}(\theta) = S_\theta\) is a compact surface with \(\partial S_\theta = B\) for every \(\theta \in S^1\). The surfaces \(S_\theta\) are called pages of the open book.
Moreover, the pair \((B, \pi)\) supports \(\xi\) if, up to isotopy, we can find a 1-form \(\alpha\) for \(\xi\) such that

- the 2-form \(d\alpha\) is a positive area form on each page \(S_0\);
- we have \(\alpha > 0\) on the binding \(B\).

Let us take the page \(S_1 = \pi^{-1}(1)\), which is an oriented, connected, compact surface of genus \(g\) and with \(l\) boundary components. Suppose that an \(n\)-component Legendrian link \(L\) in \((M, \xi)\) sits inside \(S_1\) and its components represent \(n\) independent elements in \(H_1(S_1; \mathbb{F})\). We say that a collection of disjoint, simple arcs \(A = \{a_1, ..., a_{2g+l+n-2}\}\) is a system of generators for \(S_1\) adapted to \(L\) if the following conditions hold:

1. the subset \(A^1 \sqcup A^2 \subset A\), where \(A^1 = \{a_1, ..., a_n\}\) and \(A^2 = \{a_{n+1}, ..., a_{2g+l-1}\}\), is a basis of \(H_1(S_1, \partial S_1; \mathbb{F}) \cong \mathbb{F}^{2g+l-1}\);
2. we have that \(L \cap a_i = \{1\ \text{pt}\}\) for every \(a_i \in A^1\) and \(L \cap a_i = \emptyset\) for \(i > n\). The arcs in the subset \(A^1\) are called distinguished arcs;
3. the surface \(S_1 \setminus (a_1 \cup ... \cup a_n)\) has \(n\) connected components; each one containing exactly one component of \(L\). The arcs in \(A \setminus A^1\) that appear twice on the same component of the boundary of \(S_1 \setminus (a_1 \cup ... \cup a_n)\) are called dead arcs; the others separating arcs;
4. the elements in the subset \(A^3 = \{a_{2g+l}, ..., a_{2g+l+n-2}\}\) are separating arcs such that the disk \(D = S_1 \setminus (A^1 \sqcup A^2)\) is disconnected into \(n\) disks and every arc in \(A^3\) separates a unique pair of components of \(D\).

With this definition in place we say that the triple \((B, \pi, A)\) is an open book decomposition adapted to the Legendrian link \(L\) if

- the pair \((B, \pi)\) is compatible with \((M, \xi)\);
- the link \(L\) is contained in the page \(S_1\);
- the \(n\) components of \(L\) are independent in \(S_1\);
- the set \(A\) is a system of generators for \(S_1\) adapted to \(L\).

In this case we also say that the adapted open book decomposition \((B, \pi, A)\) is compatible with the triple \((L, M, \xi)\). It is important to observe that, since the components of \(L\) are required to be independent in homology, we only consider open book decompositions with pages not diffeomorphic to a disk.

2.2 Existence

We need to prove that open book decompositions adapted to Legendrian links always exist. In order to do this we recall the definition of contact cell decomposition (of a contact 3-manifold) and ribbon of a Legendrian graph.

A contact cell decomposition of \((M, \xi)\) is a finite CW-decomposition of \(M\) such that

1. the 1-skeleton is a connected Legendrian graph;
2. each 2-cell \(E\) satisfies \(\text{tb}(\partial E) = -1\);
3. the contact structure $\xi$ is tight when restricted to each 3-cell.

Moreover, if we have a Legendrian link $L \hookrightarrow (M, \xi)$ then we also suppose that

4. the 1-skeleton contains $L$.

Denote the 1-skeleton of a contact cell decomposition of $(M, \xi)$ with $G$. Then $G$ is a Legendrian graph and its ribbon is a compact surface $S_G$ satisfying:

- $S_G$ retracts onto $G$;
- $T_p S_G = \xi_p$ for every $p \in G$;
- $T_p S_G \neq \xi_p$ for every $p \in S \setminus G$.

We say that an adapted open book decomposition $(B, \pi, A)$, compatible with the triple $(L, M, \xi)$, comes from a contact cell decomposition if $S = \pi^{-1}(1)$ is a ribbon of the 1-skeleton of $(M, \xi)$.

**Proof of Theorem 1.1.** Corollary 4.23 in [7] assures us that we can always find an open book decomposition $(B, \pi)$ which comes from a contact cell decomposition of $(M, \xi)$ and the link $L$

![Figure 1](image.png)

Figure 1: We add a new separating arc which is parallel to $L_i$ except near the distinguished arc.

$L$ is contained in $S_1$; where the page $S_1$ is precisely a ribbon of the 1-skeleton of $(M, \xi)$. The proof of this corollary also gives that the two framings of $L$ agree.

The components of $L$ are independent because it is easy to see from the construction that there is a collection of disjoint, properly embedded arcs $\{a_1, \ldots, a_n\}$ in $S_1$ such that

$$L_i \cap a_i = \{1 \text{ pt}\} \quad \text{and} \quad L_i \cap \left( \bigcup_{j \neq i} a_j \right) = \emptyset$$

for every $i$. To conclude we only need to show that there exists a system of generators $A = \{a_1, \ldots, a_{2g+l+n-2}\}$ for $S_1$ which is adapted to $L$.

The arcs $a_1, \ldots, a_n$ are taken as before. If we complete $L$ to a basis of $H_1(S; \mathbb{F})$ then Alexander duality gives a basis $\{a_1, \ldots, a_{2g+l-1}\}$ with the same property. We define $a_{2g+l}, \ldots, a_{2g+l+n-2}$ in the following way. Each new separating arc is parallel to $L_i$, extended by following the distinguished arc until the boundary of $S_1$ as in Figure 1. Clearly, it disconnects the surface, because the first $2g + l - 1$ arcs are already a basis of $H_1(S, \partial S; \mathbb{F})$. If one of the components contains no distinguished arcs, like in Figure 2 then we choose the other endpoint of $a_i$ to extend the arc.
Figure 2: The picture appears similar to Figure 1 but this time the new arc follows the distinguished arc in the opposite direction.

In the following paper we use adapted open book decompositions to present Legendrian links in contact 3-manifolds. Moreover, we study how to relate two different open book decompositions representing isotopic Legendrian links.

2.3 Abstract open books

An abstract open book is a quintuple \((S, \Phi, \mathcal{A}, z, w)\) defined as follows. We start with the pair \((S, \Phi)\). We have that \(S = S_{g,l}\) is an oriented, connected, compact surface of genus \(g\) and with \(l\) boundary components, not diffeomorphic to a disk. While \(\Phi\) is the isotopy class of a diffeomorphism of \(S\) into itself which is the identity on \(\partial S\). The class \(\Phi\) is called monodromy.

The pair \((S, \Phi)\) determines a contact 3-manifold up to contactomorphism. The construction is described in [7] Definition 2.3, Lemma 2.4 and Theorem 3.13.

The set \(\mathcal{A}\) consists of two collections of properly embedded arcs, \(B = \{b_1, ..., b_{2g+l+n-2}\}\) and \(C = \{c_1, ..., c_{2g+l+n-2}\}\) in \(S\) with \(n \geq 1\), such that all the arcs in \(B\) are disjoint, all the arcs in \(C\) are disjoint and each pair \(b_i, c_i\) appears as in Figure 3. We suppose that each strip, the grey area between \(b_i\) and \(c_i\), is disjoint from the others. We also want \(B\) and \(C\) to represent two system of generators for the relative homology group \(H_1(S, \partial S; \mathbb{F})\). In this way, if we name the strips \(\mathcal{A}_i\), we have that \(S \setminus \bigcup b_i, S \setminus \bigcup c_i\) and \(S \setminus \bigcup \mathcal{A}_i\) have exactly \(n\) connected components.

Finally, \(z\) and \(w\) are two sets of basepoints: \(w = \{w_1, ..., w_n\}\) and \(z = \{z_1, ..., z_n\}\). We require these sets to have the following properties:

- there is a \(z_i\) in each component of \(S \setminus \bigcup \mathcal{A}_i\), with the condition that every component contains exactly one element of \(z\);
- each \(w_i\) is inside one of the strips \(\mathcal{A}_i\), between the arcs \(b_i\) and \(c_i\), with the property that every strip contains at most one element of \(w\). See Figure 4.
we choose $z$ and $w$ in a way that each component of $S \setminus B$ and $S \setminus C$ contains exactly one element of $z$ and one element of $w$.

We can draw an $n$-component link inside $S$ using the following procedure: we go from the $z$’s to the $w$’s by crossing $B$ and from $w$ to $z$ by crossing $C$, as shown in Figure 5. Moreover, we observe that the components of the link are independent in $S$.

Using the Legendrian realization theorem (Theorem 2.7) in [12] and the procedure we described we can prove that every abstract open book determines a Legendrian link in a contact 3-manifold up to contactomorphism.

We are now interested in proving that an adapted open book decomposition $(B, \pi, A)$ always determines an abstract open book.

Proposition 2.1. We can associate to an adapted open book decomposition $(B, \pi, A)$, compatible with the triple $(L, M, \xi)$, an abstract open book $(S, \Phi, A, z, w)$ up to isotopy.

Proof. The surface $S$ is obviously the page $\pi^{-1}(1)$. Now consider the subsets of unit complex numbers $I_{\pm} \subset S^1 \subset \mathbb{C}$ with non-negative and non-positive imaginary part. Since they are contractible, we have that $\pi|_{\pi^{-1}(I_{\pm})}$ is a trivial bundle. This gives two diffeomorphisms between the pages $S_1$ and $S_{-1}$. The monodromy $\Phi$ is precisely the isotopy class of the composition of these diffeomorphisms.

At this point, we want to define the strips $A$. Hence, we need the collections of arcs $B$ and $C$: starting from the system of generators $A$, which is adapted to $L \subset S$, we take them to be both isotopic to $A$, in “strip position” like in Figure 3 and such that $L$ does not cross the intersections of the arcs in $B$ with the ones in $C$. We only have an ambiguity on, following the orientation of $L$, which is the first arc intersected by $L$. To solve this problem we have to follow the rule that we fixed in Figure 5.

Now we need to fix the basepoints. We put the $z$’s on $L$; exactly one on each component of $L \setminus (L \cap A)$. The points in $z$ on different components of $L$ stay in different domains because of Condition 3 in the definition of adapted system of generators. Then $S \setminus A$ has $n$ connected components, since the components of $L$ are independent, and each of these contains exactly one element of $z$. Since the $z$’s are outside of the strips then we have that each component of $S \setminus B$ and $S \setminus C$ also contains only one $z_i$. 

Figure 4: On the left $S_i$ and $S_j$ are different components of $S \setminus \bigcup A_i$.

Figure 5: The link is oriented accordingly to the basepoints.
The $w$’s are still put on $L$, but inside the strips containing the $n$ distinguished arcs. The points $w_1, ..., w_n$ correspond to $A_1, ..., A_n$.

\section{Heegaard Floer homology}

\subsection{Legendrian Heegaard diagrams}

Heegaard Floer homology has been introduced by Ozsváth and Szabó in [16]; later it was generalized to knots and links in [17] and independently by Rasmussen, in his PhD thesis [21]. In its original formulation links and 3-manifolds were presented with Heegaard diagrams, but in this work we only use a specific type of these diagrams, obtained from adapted open book decompositions, that we call Legendrian Heegaard diagrams.

From now on a 3-manifold $M$ is always supposed to be a rational homology sphere. Given an adapted open book decomposition $(B, \pi, A)$, compatible with the triple $(L, M, \xi)$, a Legendrian Heegaard diagram consists of a quintuple $(\Sigma, \alpha, \beta, w, z)$ where $\Sigma$ is a closed, oriented surface, $\alpha$ and $\beta$ are two collections of curves in $\Sigma$ and $w$ and $z$ are two sets of $n$ basepoints in $\Sigma$.

The surface $\Sigma$ is $S_1 \cup S_{-1}$, where $S_{\pm 1} = \pi^{-1}(\pm 1)$; since $\pi$ is a locally trivial fibration the pages $S_1$ and $S_{-1}$ are diffeomorphic, but we reverse the orientation of the second one when we glue them together.

We have that $A = \{a_1, ..., a_{2g+l+n-2}\}$ and we choose $B = \{b_1, ..., b_{2g+l+n-2}\}$ in a way that $A$ and $B$ are like in Figure 3. We recall that $g$ is the genus of $S_1$, $l$ is the number of boundary components of $S_1$ and $n$ is the number of components of $L$. Then we define $\alpha_i = b_i \cup \overline{b_i}$ and $\beta_i = a_i \cup \overline{a_i}$ for every $i = 1, ..., 2g + l + n - 2$ as follows: the arcs $\overline{a_i}$ and $\overline{b_i}$ are defined as $h(a)$ and $(h \circ \Phi)(b)$ respectively, where $h : S_1 \to S_{-1}$ is the Identity, and $\Phi$ is the monodromy, which is fixed by the open book as seen in Proposition 2.1. Finally, the $z$’s and the $w$’s are the set of baspoints that we introduced in Subsection 2.3.

In the settings of [16] and [17] the Legendrian Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ is a diagram for the (smooth) link $L$ in the 3-manifold $\overline{M}$ (19 Section 3), that is $M$ considered with the opposite orientation. We remark that here $\alpha$ and $\beta$ are swapped respect to the papers of Ozsváth and Szabó.

We observe that, given $(B, \pi, A)$, the only freedom in the choice of the Legendrian Heegaard diagram is in the arcs $\overline{b_1}, ..., \overline{b_{2g+l+n-2}}$ inside $S_{-1}$, which can be taken in the isotopy class of $\Phi(b_1), ..., \Phi(b_{2g+l+n-2})$.  

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{A diagram for the standard Legendrian 2-unlink in $(S^3, \xi_{st})$.}
\end{figure}
3.2 Basics of Heegaard Floer theory

Let us consider a Legendrian Heegaard diagram $D = (\Sigma, \alpha, \beta, w, z)$, coming from an adapted open book decomposition compatible with $(L, M, \xi)$. Consider the $(2g + l + n - 2)$-dimensional tori

$$T_{\alpha} = \alpha_1 \times \ldots \times \alpha_{2g+l+n-2}$$

and

$$T_{\beta} = \beta_1 \times \ldots \times \beta_{2g+l+n-2}$$

in the symmetric power $\text{Sym}^{2g+l+n-2}(\Sigma)$ of $\Sigma$ and define $\widehat{CF}(D)$ and $cCFL^{-}(D)$ respectively as the $F$-vector space and the free $F[U]$-module generated by the elements of the transverse intersection $T_{\alpha} \cap T_{\beta}$.

Fix an appropriate symplectic and compatible almost-complex structure $(\omega, J)$ on $\text{Sym}^{2g+l+n-2}(\Sigma)$. For every relative homology class $\phi \in \pi_2(x, y)$ we define $M(\phi)$ as the moduli space of $J$-holomorphic maps from the unit disk $D \subset \mathbb{C}$ to $(\text{Sym}^{2g+l+n-2}(\Sigma), J)$ with the appropriate boundary conditions, see [16]. The formal dimension of $M(\phi)$, denoted by $\mu(\phi)$, is the Maslov index; moreover, we call $\widehat{M}(\phi)$ the quotient $M(\phi)/\mathbb{R}$ given by translation.

Since we are working on 3-manifolds, we use the definition of Spin$^c$ structure given by Turaev in [22]: an isotopy class, away from a point, of nowhere vanishing vector fields on the manifold. As described in [19] Section 3.3, we have two well-defined maps

$$s_w, s_z : T_{\alpha} \cap T_{\beta} \longrightarrow \text{Spin}^c(M) \cong H^2(M; \mathbb{Z})$$

which send an intersection point $x \in T_{\alpha} \cap T_{\beta}$ into the Spin$^c$ structures $s_w(x)$ and $s_z(x)$ on $M$. These maps are obtained by associating, to every $x \in T_{\alpha} \cap T_{\beta}$, two global 2-plane fields $\pi_w(x)$ and $\pi_z(x)$ on $M$, whose restrictions are the corresponding Spin$^c$ structures.

Since our manifold $M$ already comes with a contact structure $\xi$, we have that both $M$ and $\overline{M}$ are equipped with a specific Spin$^c$ structure, induced by $\xi$, that we denote with $t_\xi$. The elements of $T_{\alpha} \cap T_{\beta}$ can be partitioned according to the Spin$^c$ structures on $\overline{M}$, resulting in decompositions

$$\widehat{CF}(D) = \bigoplus_{t \in \text{Spin}^c(\overline{M})} \widehat{CF}(D, t) \quad \text{and} \quad cCFL^{-}(D) = \bigoplus_{t \in \text{Spin}^c(\overline{M})} cCFL^{-}(D, t);$$

where $\widehat{CF}(D, t)$ is generated by the intersection points $x \in T_{\alpha} \cap T_{\beta}$ such that $s_z(x) = t$, while $cCFL^{-}(D, t)$ by the ones such that $s_w(x) = t$. Note that the Spin$^c$ structures in the two splittings may be different.

For every $\phi \in \pi_2(x, y)$ we call

$$n_z(\phi) = \# \phi \cap \{z_i\} \times \text{Sym}^{2g+l+n-3}(\Sigma)$$

and

$$n_w(\phi) = \# \phi \cap \{w_i\} \times \text{Sym}^{2g+l+n-3}(\Sigma);$$

where here we mean algebraic intersection. Moreover, we have

$$n_z(\phi) = \sum_{i=1}^{n} n_{z_i}(\phi) \quad \text{and} \quad n_w(\phi) = \sum_{i=1}^{n} n_{w_i}(\phi).$$
We define the differential $\hat{\partial}_z : \widehat{C\!F}(D, t) \to \widehat{C\!F}(D, t)$ as follows:

$$\hat{\partial}_z x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x, y), \mu(\phi) = 1, n_z(\phi) = 0} \# |\mathcal{M}(\phi)| \cdot y$$

for every $x \in T_\alpha \cap T_\beta$. We note that, since we are interested in $\phi$’s that are image of some $J$-holomorphic disks, we have that $n_z(\phi), n_w(\phi) \geq 0$ (Lemma 3.2 in [16]). This means that $n_z(\phi) = n_w(\phi) = 0$ if and only if $n_z(\phi) = n_w(\phi) = 0$ for every $i = 1, \ldots, n$.

The map $\hat{\partial}_z$ is well-defined if the diagram $D$ is admissible, which means that every $\phi \in \pi_2(x, x)$ with $n_w(\phi) = 0$, representing a non-trivial homology class, has both positive and negative local multiplicities. Usually we have a distinction between weak and strong admissibility, but if $M$ is a rational homology 3-sphere then the weakly and strongly admissible conditions coincide; see Remark 4.11 in [16] and Definition 3.5 and Subsection 4.1 in [19]. Given a diagram we can always achieve admissibility with isotopies.

**Proposition 3.1.** Suppose $D = (\Sigma, \alpha, \beta, w, z)$ is a Legendrian Heegaard diagram given by an adapted open book decomposition compatible with $(L, M, \xi)$, where $M$ is a rational homology 3-sphere. Then $D$ is always admissible up to isotope the arcs in $B$.

*Proof.* It follows from Proposition 3.6 in [19] and Theorem 2.1 in [20].

The fact that $\hat{\partial}_z \circ \hat{\partial}_z = 0$ is proved in [16]. This gives that the pair $\left(\widehat{C\!F}(D, t), \hat{\partial}_z\right)$ is a chain complex.

In the definition of $\left(\widehat{C\!F}(D, t), \hat{\partial}_z\right)$ we never use the basepoints in $w$, in fact the complex does not depend on the link $L$, but only on the number of its components. Moreover, in [16] is proved that the homology $\widehat{HF}(D, t)$ of $\left(\widehat{C\!F}(D, t), \hat{\partial}_z\right)$ is an invariant of the Spin$^c$ 3-manifold $(\overline{M}, t)$ if the number of basepoints in $D$ is fixed. When $n = 1$ the homology group is usually denoted with $\widehat{HF}(\overline{M}, t)$.

In addition, since $M$ is a rational homology sphere, $\widehat{C\!F}(D, t)$ comes with an additional $\mathbb{Q}$-grading, called Maslov grading [15], given by $M_z(x) = d_3(\pi_z(x))$; where $d_3$ is the Hopf invariant of a nowhere zero vector field. The differential $\hat{\partial}_z$ drops the Maslov grading by one and then we have that

$$\widehat{HF}(D, t) = \bigoplus_{d \in \mathbb{Q}} \widehat{HF}_d(D, t) ;$$

where each $\widehat{HF}_d(D, t)$ is a finite dimensional $\mathbb{F}$-vector space.

### 3.3 Link Floer homology

We consider the free $\mathbb{F}[U]$-module $cCFL^-(D, t)$ and we define a new differential

$$\partial^- : cCFL^-(D, t) \to cCFL^-(D, t)$$

in the following way:

$$\partial^- x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x, y), \mu(\phi) = 1, n_z(\phi) = 0} \# |\mathcal{M}(\phi)| \cdot U^{n_w(\phi)} y$$
and
\[ \partial^- (Ux) = U \cdot \partial^- x \]
for every \( x \in T_\alpha \cap T_\beta \). If \( D \) is admissible then \( \partial^- \) is also well-defined. Moreover, the fact that \( \partial^- \circ \partial^- = 0 \) is proved in Lemma 4.3 in [19]. Hence, the pair \((cCFL^-(D,t), \partial^-)\) is a chain complex.

From [19] we know that the homology of \((cCFL^-(D,t), \partial^-)\), that is denoted with \(cHFL^-(\overline{M}, L, t)\), is a smooth isotopy invariant of \( L \) in \( \overline{M} \).

As before we can define the Maslov grading as \( M(x) = M_w(x) = d_3(\pi_w(x)) \) for every intersection point and we extend it by taking \( M(Ux) = M(x) - 2 \). Note that this definition of Maslov grading is different from the one used in the previous subsection; in fact, now the set \( w \) appears in place of \( z \). In order to avoid confusion, we denote the Maslov grading of \( x \) with \( M(x) \) only in the case of links; otherwise we specify which set of basepoints is used in the definition. Again the Maslov grading gives an \( \mathbb{F} \)-splitting
\[ cHFL^-(\overline{M}, L, t) = \bigoplus_{d \in \mathbb{Q}} cHFL^d_-(\overline{M}, L, t) ; \]
where \( cHFL^-(\overline{M}, L, t) \) is a finitely generated \( \mathbb{F}[U] \)-module and each \( cHFL^d_-(\overline{M}, L, t) \) is a finite dimensional \( \mathbb{F} \)-vector space.

In the case of null-homologous links we can also define a \( \mathbb{Z}_2 \)-grading on \( cCFL^-(D,t) \) called Alexander grading and denoted with \( A \). Let us call \( M_L \) the 3-manifold with boundary \( M \setminus \nu(L) \). Since \( L \) has \( n \) components, we have that \( \partial M_L \) consists of \( n \) disjoint tori. On this kind of 3-manifold we define a relative Spin\(^c\) structure as in [19]: the isotopy class, away from a point, of a nowhere vanishing vector field such that the restriction on each boundary torus coincides with the canonical one (see [22]). We denote the set of the relative Spin\(^c\) structures on \( M_L \) by \( \text{Spin}^c(M,L) \); then we have an identification of \( \text{Spin}^c(M,L) \) with the relative cohomology group \( H^2(M_L, \partial \nu(L); \mathbb{Z}) \). Moreover, from [19] we have the following map
\[ s_{w,z} : T_\alpha \cap T_\beta \rightarrow \text{Spin}^c(M,L) . \]
Clearly, the relative Spin\(^c\) structure \( s_{w,z}(x) \) extends to the actual Spin\(^c\) structure \( s_w(x) \). Poincaré duality gives that
\[ H^2(M_L, \partial \nu(L); \mathbb{Z}) \cong H_1(M_L; \mathbb{Z}) \cong H_1(L; \mathbb{Z}) \oplus H_1(M; \mathbb{Z}) \cong \mathbb{Z}^n \oplus H^2(M; \mathbb{Z}) ; \]
where we recall that \( H^2(M; \mathbb{Z}) \) is a finite group. A basis of the \( \mathbb{Z}^n \) summand is given by the cohomology classes \( \{PD[\mu_i]\}_{i=1,\ldots,n} \); where \( \mu_i \) is a meridian of the \( i \)-th component of \( L \), oriented coherently. Then we have that
\[ s_{w,z}(x) = \sum_{i=1}^n 2s_i \cdot PD[\mu_i] + s_w(x) , \]
where each \( s_i \) is an integer. Since \( L \) admits a Seifert surface \( F \), we define the Alexander absolute grading as follows:
\[ A(x) = \sum_{i=1}^n s_i = \frac{s_{w,z}(x)[F]}{2} , \]
extended on the whole \( cCFL^-(D,t) \) by saying that \( A(Ux) = A(x) - 1 \).
We have that $\partial^-$ preserves the Alexander grading and then there is another $\mathbb{F}$-splitting
\[
cHFL^- (\mathcal{M}, L, t) = \bigoplus_{d,s \in \mathbb{Q}} cHFL^-_{d,s} (\mathcal{M}, L, t);
\]
where each $cHFL^-_{d,s} (\mathcal{M}, L, t)$ is a finite dimensional $\mathbb{F}$-vector space.

4 The invariant

4.1 Special intersection points in Legendrian Heegaard diagrams

In this subsection we define a cycle in the link Floer chain complex that we previously introduced. The corresponding homology class will be our Legendrian invariant. Let us consider the only intersection point of $D = (\Sigma, \alpha, \beta, w, z)$ which lies in the page $S_1$. We recall that, in general, the intersection points live in the space $T_{\alpha} \cap T_{\beta}$, but they can be represented inside $\Sigma$. We denote this element with $L(\mathcal{D})$.

Proposition 4.1. The intersection point $L(\mathcal{D})$ is such that $\partial^- L(\mathcal{D}) = 0$ and then $L(\mathcal{D})$ is a cycle in $cCFL^- (\mathcal{D}, t_{L(\mathcal{D})})$; where $t_{L(\mathcal{D})}$ is the Spin$^c$ structure that it induces on $M$.

Proof. Every $\phi \in \pi_2(L(\mathcal{D}), y)$ such that $\mu(\phi) = 1$, where $y \in T_{\alpha} \cap T_{\beta}$, has the property that $n_z(\phi) > 0$. The claim follows easily from the definition of the differential. \qed

In Subsection 4.4 we show that more can be said on the Spin$^c$ structure $t_{L(\mathcal{D})}$. Now we spend a few words about the Ozsváth-Szabó contact invariant $c(\xi)$, introduced in [18].

Given a Legendrian Heegaard diagram $D$. Let us call $c(\mathcal{D})$ the only intersection point which lies on the page $S_1$ as before, but now considered as an element in $(\widehat{CF}(D, t_{c(\mathcal{D})}), \widehat{\partial}_z)$. The proof of Proposition 4.1 says that $c(\mathcal{D})$ is also a cycle. Moreover, we have the following theorem.

Theorem 4.2 (Ozsváth and Szabó). Let us consider a Legendrian Heegaard diagram $D$ with a single basepoint $z$, given by an open book compatible with a pair $(M, \xi)$, where $M$ is a rational homology 3-sphere. Let us take the cycle $c(\mathcal{D}) \in \widehat{CF}(D, t_{c(\mathcal{D})})$.

Then the isomorphism class of $[c(\mathcal{D})] \in \widehat{HF}(\mathcal{M}, t_{\xi})$ is a contact invariant of $(M, \xi)$ and we denote it with $\widehat{c}(M, \xi)$.

Furthermore, we have the following properties:

- the Spin$^c$ structure $t_{c(\mathcal{D})}$ coincides with $t_{\xi}$;
- the Maslov grading of $c(\mathcal{D})$ is given by $M_z(c(\mathcal{D})) = -d_3(M, \xi)$.

The proof of this theorem comes from [18], where Ozsváth and Szabó first define the invariant $\widehat{c}(M, \xi)$, and [11], where Honda, Kazez and Matić give the reformulation using open book decompositions that we use in this paper.

We want to prove a result similar to Theorem 4.2, but for Legendrian links. In other words, we show that the isomorphism class of the element $[\mathcal{L}(\mathcal{D})]$ inside the homology group $cHFL^- (\mathcal{M}, L, t_{\mathcal{L}(\mathcal{D})})$ is a Legendrian invariant of the triple $(L, M, \xi)$. Let us be more specific: we consider a Legendrian link $L \hookrightarrow (M, \xi)$ in a rational homology contact 3-sphere; we associate two open book decompositions compatible with $(L, M, \xi)$, say $(B_1, \pi_1, A_1)$ and $(B_2, \pi_2, A_2)$, and these determine (up to isotopy) two Legendrian Heegaard diagrams, that...
we call $D_1 = (\Sigma_1, \alpha_1, \beta_1, w_1, z_1)$ and $D_2 = (\Sigma_2, \alpha_2, \beta_2, w_2, z_2)$ respectively. Then we want to find a chain map

$$\Psi_{(D_1, D_2)} : cCFL^-(D_1, t) \rightarrow cCFL^-(D_2, t),$$

that induces an isomorphism in homology, preserves the bigrading and it is such that $\Psi_{(D_1, D_2)}(L(D_1)) = L(D_2)$; where $t = t_{L(D_1)} = t_{L(D_2)} \in \text{Spin}^c(M)$.

Our strategy is to study how Legendrian isotopic triples are related to one another, which means to define a finite set of moves between two adapted open book decompositions. Then find the maps $\Psi_{(D_1, D_2)}$ in these particular cases.

### 4.2 Open books adapted to Legendrian isotopic links

We want to show that, given two Legendrian isotopic links $L_1, L_2 \hookrightarrow (M, \xi)$, two open book decompositions $(B_i, \pi_i, A_i)$, compatible with the triples $(L_i, M, \xi)$, are related by a finite sequence of moves.

#### 4.2.1 Global isotopies

The first lemma follows easily.

**Lemma 4.3.** Let us consider an adapted open book decomposition $(B_1, \pi_1, A_1)$, compatible with the triple $(L_1, M, \xi)$, and suppose that there is a contact isotopy of $(M, \xi)$, sending $L_1$ into $L_2$.

Then the time-1 map of the isotopy is a diffeomorphism $F : M \rightarrow M$ such that $(F(B_1), \pi_1 \circ F^{-1}, F(A_1))$ is an adapted open book decomposition, compatible with $(L_2, M, \xi)$.

This lemma says that, up to global contact isotopies, we can consider $(B_i, \pi_i, A_i)$ to be both compatible with a triple $(L, M, \xi)$, where the link $L$ is Legendrian isotopic to $L_i$ for $i = 1, 2$. In other words, we can just study the relation between two open book decompositions compatible with a single triple $(L, M, \xi)$.

#### 4.2.2 Positive stabilizations

Let us start with a pair $(S, \Phi)$. A positive stabilization of $(S, \Phi)$ is the pair $\left(\tilde{S}, \tilde{\Phi}\right)$ obtained in the following way:

- the surface $\tilde{S}$ is given by adding a 1-handle $H$ to $S$;
- the monodromy $\tilde{\Phi}$ is isotopic to $\Phi' \circ D_\gamma$. The map $\Phi'$ concides with $\Phi$ on $S$ and it is the Identity on $H$. While $D_\gamma$ is the right-handed Dehn twist along a curve $\gamma$; which intersects $S \cap H$ transversely precisely in the attaching sphere of $H$. See Figure 7.

We say that $(B', \pi', A')$ is a positive stabilization of $(B, \pi, A)$ if

- the pair $(S', \Phi')$, obtained from $(B', \pi')$, is a positive stabilization of $(S, \Phi)$, the one coming from $(B, \pi)$;
- the system of generators $A'$ is isotopic to $A \cup \{a\}$, where $a$ is the cocore of $H$ as in Figure 7.
Figure 7: The arc $\gamma \cap \tilde{S}$ is a generic arc in the interior of $S$.

We recall the following theorem, proved by Giroux [10]. More details can be found in [7] Section 3 and 4.

**Theorem 4.4.** Two open book decompositions $(B_i, \pi_i, A_i)$ are compatible with contact isotopic triples $(L_i, M, \xi_i)$ if and only if they admit isotopic positive stabilizations.

In other words, we may need to stabilize both open books many times and eventually we obtain other two open books $(B, \pi, \tilde{A}_i)$, both compatible with $L \hookrightarrow (M, \xi)$; which is contact isotopic to $(L_i, M, \xi_i)$ for $i = 1, 2$.

### 4.2.3 Admissible arc slides

Take an adapted system of generators $A$ for an $n$-component link $L$, lying inside a surface $S$. We define admissible arc slide, a move that change $A = \{a_1, ..., a_i, ..., a_j, a_{2g+l+k-2}\}$ into $A' = \{a_1, ..., a_i + a_j, ..., a_j, ..., a_{2g+l+k-2}\}$; where $a_j$ is not a distinguished arc and one of the endpoints of $a_i$ and $a_j$ are adjacent, like in Figure 8. We can prove the following proposition.

**Proposition 4.5.** Let us consider two open book decompositions $(B, \pi, A_i)$, compatible with the Legendrian link $L \hookrightarrow (M, \xi)$. Then, after a finite number of admissible arc slides and isotopies on $A_i$, the open books coincide.

We need two preliminary lemmas.

**Lemma 4.6.** An admissible arc slide, from $A$ to $A'$, can be inverted. In the sense that we can perform another admissible arc slide, now from $A'$ to $A''$, such that $A''$ is isotopic to $A$.

**Proof.** If the arc slide changes the arc $a_i$ into $a_i + a_j$ then it is easy to see that we can just slide $a_i + a_j$ over an arc $a'_j$, isotopic to $a_j$; in a way that $a_i + a_j + a'_j$ is isotopic to $a_i$. \[\square\]

**Lemma 4.7.** Suppose that we perform an admissible arc slide that changes $a_i$ into $a_i + a_j$. Then we have the following facts:

a) the set $\{a_1, ..., a_{2g+l-1}\}$ is a basis of $H_1(S, \partial S; \mathbb{F})$ and $i \leq 2g + l - 1$ if and only if $\{a_1, ..., a_i + a_j, ..., a_{2g+l-1}\}$ is a basis of $H_1(S, \partial S; \mathbb{F})$;
b) The arc \( a_i \) is distinguished if and only if \( a_i + a_j \) is a distinguished arc;

c) The arc \( a_i \) is separating if and only if \( a_i + a_j \) is a separating arc;

d) The arc \( a_i \) is dead if and only if \( a_i + a_j \) is a dead arc.

**Proof.**
a) Obviously, the arc \( a_i + a_j \) represents the sum of the relative homology classes of \( a_i \) and \( a_j \). At this point we just need basic linear algebra.

b) This claim is trivial.

c) We take \( \{a_1, \ldots, a_{2g+l-1}\} \) as a basis of the first relative homology group of \( S \). We can suppose that \( i \geq 2g+l \). This is because if \( a_i \) is the only separating arc for a component of \( S \setminus \{a_1, \ldots, a_{2g+l-1}\} \) then \( a_i \) is homologically trivial; otherwise, we can switch \( a_i \) with another separating arc for the same component.

Now, if \( a_i + a_j \) is not a separating arc then \( S \setminus \{a_1, \ldots, a_{2g+l-1}, a_i + a_j\} \) should remain connected, but this is impossible because \( \{a_1, \ldots, a_{2g+l-1}\} \) is a basis.

The other implication follows easily from Lemma 4.6.

d) It follows from b) and c).

The strategy of the proof of Proposition 4.5 is, say \( S_1, \ldots, S_n \) and \( S'_1, \ldots, S'_n \) are the components of \( S \) minus the separating arcs of \( A_1 \) and \( A_2 \) respectively, we modify all the separating arcs, in both \( A_1 \) and \( A_2 \), with admissible arc slides; in order for \( S_i \) to coincide with \( S'_i \) for every \( i = 1, \ldots, n \). Moreover, we also want that each separating arc in \( A_1 \) becomes isotopic to a separating arc in \( A_2 \).

At the end, the components of \( S \) minus the separating arcs will be \( n \) surfaces, each one containing a component of \( L \), a distinguished arc and the same number of dead arcs. We conclude applying \( n \) times Proposition 3.2 in [12]; which is the knot case of this proposition.

**Proof of Proposition 4.5.** We want to prove that there is an adapted system of generators \( A \) for \( L \) in \( S_1 = \pi^{-1}(1) \) such that \( A \) is obtained, from \( A_1 \) and \( A_2 \), by a sequence of admissible arc slides (and isotopies).

We start from the component \( S_1 \). We can suppose that \( \partial S_1 \) contains the separating arc \( a_{2g+l} \subset A_1^3 \subset A_1 \) and the distinguished arc \( a_1 \subset A_1^1 \subset A_1 \), with almost adjacent endpoints. We can have two cases: in the first one there are no other separating arcs in the portion of \( \partial S_1 \) on the opposite side of \( a_{2g+l} \). In the second case they appear; possibly more than one,

![Figure 9](image_url)

Figure 9: Case 1 is on the left. The figure actually shows a portion of \( S \setminus A_1 \).

but we can suppose that there is exactly one of them. See Figure [9]
Figure 10: There are no other separating arcs, except for \( a_{2g+l} \) in \( S_1 \).

Case 1. When we consider \( S'_1 \), which contains the same component of \( L \) that is in \( S_1 \), after some arc slides we have that it appears as in Figure 9 (left). This is because in the same figure we see that \( S_1 \) is split in two pieces by \( L_1 \) and the innermost one is not connected in any way to other components of \( S \); in fact there are no separating arcs on that side. This means that the same holds for \( S'_1 \) too. At this point it is easy to see that \( S_1 \) can be modified to be like in Figure 10 more explicitly, the separating arcs are parallel and the distinguished arcs lie in the region where \( L_1 \) is.

Case 2. As before we have that also \( S'_1 \) appears like in the right part of Figure 9 (always after some arc slides). The reason is the same of previous case. Hence, now we can modify \( S_1 \) to be like in Figure 11 just in the same way as we did in Case 1.

We have obtained that the separating arcs are fixed on \( S_1 \) and \( S'_1 \) and then the surfaces now have isotopic boundaries. Hence, we can move to another component \( S_2 \), whose boundary contains a separating arc that has not yet been fixed, and we repeat the same process described before. We may need to slide some separating arcs over the ones that we already fixed in the previous step, but this is not a problem. We just iterate this procedure until all the \( S_i \) coincides with \( S'_i \) and this completes the proof.
The results of this subsection prove the following theorem.

**Theorem 4.8.** If \( L_1, L_2 \hookrightarrow (M, \xi) \) are Legendrian isotopic links then the open book decompositions \((B_i, \pi_i, A_i)\), which are compatible with the corresponding triples, are related by a finite sequence of global contact isotopies, positive stabilizations and admissible arc slides.

Though they are easy to deal with, we do not have to forget that, when we define the corresponding Legendrian Heegaard diagrams, we need to consider the choices of the monodromy \( \Phi \) and the families of arcs \( a, b \) and basepoints \( z, w \) inside their isotopy classes.

### 4.3 Invariance

#### 4.3.1 Definition of the diagrams and global isotopies

If two open book decompositions are related by a global isotopy then it easy to see that the induced abstract open book coincide, up to conjugation of the monodromy and isotopy of the curves and the basepoints in the diagrams.

Hence, let us consider an abstract open book \((S, \Phi, A, z, w)\) and recall that in \( S \) we have another set of arcs \( B \), as explained in Subsections 2.3 and 3.1. The first check is easy: in fact if we perturb the basepoints inside the corresponding components of \( S \setminus A \cup B \) then even the complex \((cCFL^-(D, \nu), \partial^-)\) does not change; where \( D \) is a Legendrian Heegaard diagram obtained from \((S, \Phi, A, z, w)\). The same is true for an isotopy of \( S \).

Now for what it concerns \( S \) we are done, but when we define the chain complex we also consider the closed surface \( \Sigma \), obtained by gluing together \( S \) and \( \overline{S} \). We still have some choices on \( \overline{S} \), in fact by Proposition 3.1 we may need to modify the arcs \( \overline{B} \) (see Subsection 3.1) in their isotopy classes to achieve admissibility for \( D = (\Sigma, \alpha, \beta, w, z) \). Then the proof rests on the following proposition.

**Proposition 4.9.** Suppose that two curves in a Heegaard diagram are related by the move shown in Figure 12. Then we can find a map \( \Psi_{(D_1, D_2)} \) as in Equation (1).

**Proof.** The map \( \Psi_{(D_1, D_2)} \) is constructed using a Hamiltonian diffeomorphism of the surface, as described in [16] Subsection 7.3. Since the new disks appear in \( \overline{S} \), we have that \( \Sigma(D_1) \) is sent to \( \Sigma(D_2) \) which lie both in \( S \).

#### 4.3.2 Admissible arc slides

An arc slide \( \{..., a_i, ..., a_j, ...\} \to \{..., a_i + a_j, ..., a_j, ...\} \) in \( S \) corresponds to a handleslide \( \{..., \alpha_i, ..., \alpha_j, ...\} \to \{..., \alpha'_i, ..., \alpha_j, ...\} \) in \( \Sigma \), where \( \alpha'_i = \alpha_i + a_j \cup a_i + a_j \subset \Sigma \). Thus, a chain map \( \Psi_{(D_1, D_2)} \), which induces an isomorphism in homology, is obtained by counting holomorphic triangles, as explained in [19] Subsection 6.3 and Section 7. The admissibility of the arc slide is required only to avoid crossing a basepoint in \( w \).
Remember that for every arc slide we actually have two handleslides. In fact we need to slide both the $\alpha$ and the $\beta$ curves.

The fact that $\Psi(D_1, D_2)(\mathcal{L}(D_1)) = \mathcal{L}(D_2)$ follows from Lemma 3.5 in [11]; where the arc slides invariance is proved in the open books setting.

4.3.3 Positive stabilizations

At this point, in order to complete the proof of the invariance of $[\mathcal{L}(D)]$, it would be enough to define $\Psi(D, D^+)$, such that $\Psi(D, D^+)(\mathcal{L}(D)) = \mathcal{L}(D^+)$, in the case where $D$ and $D^+$ are obtained from an open book and one of its positive stabilizations. Nevertheless, this is not what we prove. In fact, we define $L$-elementary positive stabilizations as the ones such that the curve $\gamma$, which is the curve in the page $S'$ that we used to perform the stabilization (see Figure 7), intersects the link $L$ (that sits in $S$ and then also in $S'$) in at most one point transversely. Then what we actually prove is the existence of $\Psi(D, D^+)$ for an $L$-elementary stabilization. To do this we only need the following theorem; which is a modification of Giroux’s Theorem 4.4 and whose proof is explained in Section 4 in [7].

Theorem 4.10. If $(B_i, \pi_i)$ are two open book decompositions, compatible with the triple $(L, M, \xi)$, then they admit isotopic $L$-elementary stabilizations. Namely, there is another compatible open book $(B, \pi, A)$ which is isotopic to both $(B_1, \pi_1)$ and $(B_2, \pi_2)$, after an appropriate sequence of $L$-elementary stabilizations.

Since we have already proved invariance under admissible arc slides, we can complete the open books $(B, \pi)$ and $(B^+, \pi^+)$ with every possible adapted system of generators and then eventually define the map $\Psi(D, D^+)$. 

Proposition 4.11. Let us call $S = S_{g,l} = \pi^{-1}(1) \supset L$ the page over 1 of $(B, \pi)$. Then we can always find $A$, an adapted system of generators for $L$ in $S$, with the property that $A$ is disjoint from the arc $\gamma' = \gamma \cap S$; where $\gamma$ is the curve that we used to perform the $L$-elementary stabilization.

Proof. We have to study four cases:

a) the arc $\gamma'$ intersects $L$ (transversely in one point);

b) the intersection $\gamma' \cap L$ is empty and $\gamma'$ does not disconnect $S$;

c) the intersection $\gamma' \cap L$ is empty, the arc $\gamma'$ disconnects $S$ and $L$ is not contained in one of the two resulting connected components of $S$;

d) the intersection $\gamma' \cap L$ is empty, the arc $\gamma'$ disconnects $S$ and the link $L$ lies in one of the resulting connected components of $S$.

Let us start with Case a). We observe that $\gamma'$ does not disconnect $S$; in fact if this was the case then a component of $L$ would be split inside the two resulting components of $S$, thus we would have at least another intersection point between $L$ and $\gamma'$; which is forbidden. We define $A = \{a_1, \ldots, a_{2g+l+n-2}\}$ as follows. Take $a_1$ as a push-off of $\gamma'$; clearly $a_1$ is a distinguished arc, because it intersects $L_1$, a component of $L$, in one point. Now call $a'$ the arc given by taking a push-off of $L_1$ and extend it through $a_1$, on one side of $L_1$; this is the same procedure described in the proof of Theorem 1.1. If $a'$ disconnects $S$ into $S_1, S_2$ and there are components of $L$ that lie in both $S_i$ then we take $a'$ as a separating arc; thus
we extend \( \{a_1, a'\} \) to an adapted system of generators \( A \) by using Theorem 1.1. On the other hand if \( L \) is contained in \( S_2 \) and \( S_1 \) is empty then we consider \( \{a_1, a''\} \); where \( a'' \) is another push-off of \( L_1 \), this time extended through \( a_1 \) on the other side of \( L_1 \). We still have problems when \( a' \) does not disconnect \( S \). We can fix this by taking \( \{a_1, a', a''\} \); which together disconnect \( S \) into two connected components, one of them containing only \( L_1 \) and the other one \( L \setminus L_1 \). Again we extend the set \( \{a_1, a', a''\} \) to \( A \) applying Theorem 1.1.

The other three cases are easier. In Case b) we just denote with \( a_{n+1} \) the push-off of \( \gamma' \) and we complete it to \( A \).

In Case c) we denote with \( a_{2g+1} \) our push-off of \( \gamma' \) and we take it as a separating arc.

Then we can complete it to \( A \).

Finally, Case d) is as follows. Since in this case the push-off is trivial in homology, and it bounds a surface disjoint from \( L \), we can actually ignore it and easily find a set \( A \) which never intersects \( \gamma' \).

Now we have \((S, \Phi, A, z, w)\) the abstract open book obtained from \((B, \pi, A)\). Denote with \((S^+, \Phi^+, A^+, z, w)\) the one coming from \((B^+, \pi^+, A^+)\), where \( S^+ = (\pi^+)^{-1}(1) \) is \( S \) with a 1-handle attached; \( \Phi^+ = \Phi' \circ D \), where \( \Phi' \) coincides with \( \Phi \) on \( S \), extended with the Identity on the new 1-handle; and \( A^+ = A \cup \{a\} \) with \( a \) being the cocore of the new 1-handle (see Figure 7). Then we call \( D \) and \( D^+ \) the corresponding Legendrian Heegaard diagrams.

We define \( \Psi_{(D, D^+)} \) in the following way. For every \( x \in \mathbb{T}_a \cap \mathbb{T}_b \beta_{1\xi(D)} \) it is \( \Psi_{(D, D^+)}(x) = x' \); where \( x' = x \cup \{a \cap b\} \) with \( b \) being the arc in strip position with \( a \), as in Figure 8. It results that \( \Psi_{(D, D^+)} \) is a chain map because the curve \( \alpha = b \cup \overline{b} \) only intersects the curve \( \beta = a \cup \overline{a} \), and moreover it is \( \alpha \cap \beta = \{1 \text{ pt}\} \), since we choose \( A^+ \) in a way that every arc in it is disjoint from \( \gamma' \). We have that \( \Psi_{(D, D^+)} \) induces an isomorphism in homology, because it is an isomorphism also on the level of chain complexes, and sends \( \mathcal{L}(D) \) into \( \mathcal{L}(D^+) \).

### 4.4 Proof of the main theorem

We prove our main result which defines the Legendrian invariant \( \mathcal{L} \).

**Proof of Theorem 4.1** We proved that \( \mathcal{L}(D) \) is a cycle in Proposition 4.1. The invariance follows from the results obtained in this section: in fact, we proved that if \( D_1 \) and \( D_2 \) are Legendrian Heegaard diagrams, representing Legendrian isotopic links, then we have a chain map \( \Psi_{(D_1, D_2)} \) that preserves the bigrading and the Spin\(^c\) structure and sends \( \mathcal{L}(D_1) \) to \( \mathcal{L}(D_2) \).

Now to see which is the corresponding Spin\(^c\) structure we recall that

\[
\mathfrak{s}_\omega(\mathcal{L}(D)) - \mathfrak{s}_\omega(\mathcal{L}(D)) = \text{PD}[L]
\]

from Lemma 2.19 in [1]. Then we have

\[
t_{\mathcal{L}(D)} = \mathfrak{s}_\omega(\mathcal{L}(D)) = \mathfrak{s}_\omega(\mathcal{L}(D)) + \text{PD}[L] = t_{\mathcal{L}(D)} = t_{\xi}.
\]

We note that the invariant can be a \( U \)-torsion class in the group \( cHFL^- (\overline{\mathbb{M}}, L, t_\xi) \), which means that there is a \( k \geq 0 \) such that \( U^k \cdot \mathcal{L}(L, M, \xi) = \{0\} \). Moreover, the cycle \( \mathcal{L}(D) \) possesses a bigrading \((d, s)\); such bigrading is induced on the invariant \( \mathcal{L}(L, M, \xi) \), because all the maps \( \Psi \) defined in this section preserves both the Maslov and the Alexander grading.
5 Properties of $\mathcal{L}$ and connected sums

5.1 Multiplication by $U + 1$ in the link Floer homology group

Take a Legendrian Heegaard diagram $D$, obtained from an adapted open book decomposition compatible with the triple $(L, M, \xi)$; where $M$ is a rational homology 3-sphere and $L$ is a null-homologous Legendrian $n$-component link. We have the following surjective $\mathbb{F}$-linear map:

$$F : cCFL^-(D, t_\xi) \xrightarrow{U=1} \widehat{CF}(D, t_\xi);$$

which is given by setting $U$ equals to 1.

The map $F$ clearly commutes with the differentials and it is such that $F(\mathcal{L}(D)) = c(D)$. It is well-defined and surjective, because every intersection point $x$ is such that $F(x) = x$ and if $s_w(x) = t_\xi$ then $s_\omega(x) = t_\xi$, since $L$ is null-homologous. Moreover, it respects the gradings in the following sense.

**Lemma 5.1.** The map $F$ sends an element with bigrading $(d, s)$ into an element with Maslov grading $d - 2s$.

**Proof.** We have that

$$M_z(F(x)) = d_3(\overline{M}, \pi_z(x)) = d_3(\overline{M}, \pi_w(x)) - s_{w,z}(x)[S] = M(x) - 2A(x),$$

where $S$ is a Seifert surface for $L$. \hfill $\square$

The map $F$ induces $F_*$ in homology:

$$F_* : cHFL^-(\overline{M}, L, t_\xi) \xrightarrow{U=1} \widehat{HF}(\overline{M}, t_\xi) \otimes (\mathbb{F}(-1) \oplus \mathbb{F}(0))^{\otimes(n-1)}.$$  

Then we can prove the following two lemmas.

**Lemma 5.2.** The map $F$ defined above is such that $F_*(\mathcal{L}(L, M, \xi)) = \widehat{c}(M, \xi) \otimes (e_{-1})^{\otimes(n-1)}$, where $e_{-1}$ is the generator of $\mathbb{F}(-1)$.

**Proof.** Denote with $D_1$ a Legendrian Heegaard diagram for the standard Legendrian unknot in $(M, \xi)$. If we perform $n$ consecutive stabilizations on $D_1$ then we easily obtain a diagram $D_n$ for the standard Legendrian $n$-component unlink.

Since $D$ and $D_n$ are Legendrian Heegaard diagrams for the same contact manifold $(M, \xi)$ and they have the same number of basepoints, from \[10\] we know that

$$\widehat{HF}(D, t_\xi) \cong \widehat{HF}(\overline{M}, t_\xi) \otimes (\mathbb{F}(-1) \oplus \mathbb{F}(0))^{\otimes(n-1)}.  \tag{2}$$

Moreover, a little variation of the maps $\Psi$ that we define in Section \[4\] tells us that the isomorphism in Equation (2) also identifies the class $[c(D)]$ with $\widehat{c}(M, \xi) \otimes (e_{-1})^{\otimes(n-1)}$. \hfill $\square$

Before proving the second lemma, we recall that, since the link Floer homology group is an $\mathbb{F}[U]$-module and $\mathbb{F}[U]$ is a principal ideal domain, we have $cHFL^-(\overline{M}, L, t_\xi) \cong \mathbb{F}[U]^r \oplus T$; where $r$ is an integer and $T$ is the torsion $\mathbb{F}[U]$-module.

**Lemma 5.3.** The following two statements hold:

1. $F(x) = 0$ and $x$ is homogeneous with respect to the Alexander grading if and only if $x = 0$;
2. $F_*[x] = [0]$ and $[x_i]$ is homogeneous with respect to the Alexander grading, for every $i = 1, ..., r$, if and only if $[x]$ is torsion. Here the $[x_i]$’s are a decomposition of $[x]$ in the torsion-free quotient of $cHFL^-(M, L, t_\xi)$.

\textbf{Proof.} 1. The if implication is trivial. For the only if, suppose that $F(x) = 0$; this gives that
\[ x = (1 + U)\lambda(U) y_1 + ... + (1 + U)\lambda_t(U) y_t, \]
where $\lambda(U) \in \mathbb{F}[U]$ for every $i = 1, ..., t$ and $y_1, ..., y_t$ are all the intersection points that induce the Spin$^c$ structure $t_\xi$.

Since each $y_i$ is homogeneous and the monomial $U$ drops the Alexander grading by one, we have that $\lambda_i(U) = 0$ for every $i = 1, ..., t$ and then $x = 0$.

2. Again the if implication is trivial. Now we have that
\[ [x] = \sum_{i=1}^r [x_i] + [x]_T, \]
where $[x]_T$ is the projection of $[x]$ on the torsion submodule $T$. Since $F_*[x] = [0]$, it is
\[ [x] = (1 + U)[z] = (1 + U)\lambda(U)[z_1] + ... + (1 + U)\lambda(U)[z_t] + [x]_T; \]
where it is $[x_i] = (1+U)\lambda_i(U)[z_i]$ for every $i = 1, ..., r$ and the $[z_i]$’s are a homogeneous basis of the torsion-free quotient.

The same argument of 1 implies that the polynomials $\lambda_i(U)$ are all zero and then $[x] = [x]_T$. \hfill \Box

Now we use Lemma 5.3 to show that there is a correspondence between the torsion-free quotient of the link Floer homology group and the multi-pointed hat Heegaard Floer homology.

\textbf{Theorem 5.4.} If $L$ is a Legendrian $n$-component link in $(M, \xi)$ then there exists an isomorphism of $\mathbb{F}[U]$-modules
\[ \frac{cHFL^-(M, L, t_\xi)}{T} \to \left( \widehat{HF}(M, t_\xi) \otimes_{\mathbb{F}} \mathbb{F}[U] \right) \otimes_{\mathbb{F}[U]} \left( \mathbb{F}[U]_{(-1)} \oplus \mathbb{F}[U]_{(0)} \right)^{\otimes (n-1)}; \]
which sends a homology class of bigrading $(d, s)$ into one of Maslov grading $d - 2s$.

\textbf{Proof.} We just have to show that $F_*$ sends $\{[z_1], ..., [z_r]\}$, a homogeneous $\mathbb{F}[U]$-basis of the torsion-free quotient of $cHFL^-(M, L, t_\xi)$, into an $\mathbb{F}$-basis of $\mathcal{X} = \widehat{HF}(M, t_\xi) \otimes (\mathbb{F}_{(-1)} \oplus \mathbb{F}_{(0)})^{\otimes (n-1)}$.

Statement 1 in Lemma 5.3 tells us that $F_*$ is surjective. In fact, if $[y] \in \mathcal{X}$ then it is $0 = \partial_d y = F(\partial^- x)$; where $F(x) = y$. We apply Lemma 5.3 to $\partial^- (x)$, since we can suppose that both $x$ and $y$ are homogeneous, and we obtain that $\partial^- (x) = 0$; then $[x]$ is really a homology class. At this point, it is easy to see that the set $\{F_*[z_1], ..., F_*[z_r]\}$ is a system of generators of $\mathcal{X}$. \hfill \Box

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In order to prove that $F_\ast[z_1],...,F_\ast[z_r]$ are also linearly independent in $X$ we suppose that there is a subset $\{F_\ast[z_{i_1}],...,F_\ast[z_{i_k}]\}$ such that

$$F_\ast[z_{i_1}] + ... + F_\ast[z_{i_k}] = F_\ast[z_{i_1} + ... + z_{i_k}] = [0].$$

Then we apply Statement 2 of lemma 5.3 to $[z_{i_1} + ... + z_{i_k}]$ and we obtain that it is a torsion class. This is a contradiction, because $[z_{i_1}],...,[z_{i_k}]$ are part of an $F[U]$-basis of a torsion-free $F[U]$-module.

Lemmas 5.1 and 5.2 and Theorem 5.4 immediately give the following corollary.

**Corollary 5.5.** The Legendrian invariant $\mathfrak{L}(L,M,\xi)$ is non-torsion if and only if the contact invariant $\hat{\gamma}(M,\xi)$ is non zero.

Furthermore, if $D$ is a Legendrian Heegaard diagram compatible with $(L,M,\xi)$ then the Maslov and Alexander grading of the element $\mathfrak{L}(D)$ are related by the following equality:

$$M(\mathfrak{L}(D)) = -d_3(M,\xi) + 2A(\mathfrak{L}(D)) + 1 - n,$$

where $n$ is the number of component of $L$.

In particular, this corollary says that the Legendrian link invariant $\mathfrak{L}$ is always a torsion class if $\xi$ is overtwisted and always non-torsion if $(M,\xi)$ is strongly symplectically fillable. In fact, from [18] we know that $\hat{\gamma}(M,\xi)$ is zero in the first case and non-zero in the second one.

### 5.2 Relation with classical Legendrian invariants

We proved that the isomorphism class $\mathfrak{L}(L,M,\xi)$ is a Legendrian invariant. This is true also for the Alexander (and Maslov) grading of the element $\mathfrak{L}(D) \in cCFL^-(D,\xi)$, that we denote with $A(\mathfrak{L}(D))$. From Corollary 5.5 we know that $\hat{\gamma}(M,\xi) \neq [0]$ implies that $\mathfrak{L}(L,M,\xi)$ is non-torsion, hence it determines $A(\mathfrak{L}(D))$. On the other hand, if $\hat{\gamma}(M,\xi)$ is zero then a priori the gradings of the element $\mathfrak{L}(D)$ could give more information. Starting from these observations, we want to express the value of $A(\mathfrak{L}(D))$ in terms of the other known invariants of the Legendrian link $L$. Note that Corollary 5.5 also tells us that the Maslov grading of $\mathfrak{L}(D)$ is determined, once we know $A(\mathfrak{L}(D))$.

First we recall that, from the definitions of Thurston-Bennequin and rotation number, we have

$$\text{tb}(L) = \sum_{i=1}^{n} \text{tb}_i(L), \quad \text{where } \text{tb}_i(L) = \text{tb}(L_i) + \text{lk}(L_i, L \setminus L_i)$$

and

$$\text{rot}(L) = \sum_{i=1}^{n} \text{rot}_i(L), \quad \text{where } \text{rot}_i(L) = \text{rot}(L_i).$$

**Theorem 5.6.** Consider $L \hookrightarrow (M,\xi)$ a null-homologous Legendrian $n$-component link in a rational homology contact 3-sphere and $D$ a Legendrian Heegaard diagram, that comes from an open book compatible with $(L,M,\xi)$. Then we have that

$$A(\mathfrak{L}(D)) = \frac{\text{tb}(L) - \text{rot}(L) + n}{2} \quad \text{and} \quad M(\mathfrak{L}(D)) = -d_3(M,\xi) + \text{tb}(L) - \text{rot}(L) + 1.$$
Proof. If $L$ is a knot then the claim has been proved by Ozsváth and Stipsicz in [13] (Theorem 1.6). At this point, in order to obtain the claim for links, we need to relate $A(\Sigma(D))$ with the Alexander grading of the Legendrian invariants of the components $L_i$ of $L$.

A Legendrian Heegaard diagram $D_i$ for the knot $L_i$ is easily gotten from $D$ by removing some curves and basepoints. We denote the intersection point representing the Legendrian invariant of $L_i$ with $\Sigma(D_i)$. Then we have

$$A(\Sigma(D)) = \sum_{i=1}^{n} \left( A(\Sigma(D_i)) + \frac{1}{2} \text{lk}(L_i, L \setminus L_i) \right) = \sum_{i=1}^{n} \left( \frac{\text{tb}(L_i) - \text{rot}(L_i) + 1}{2} + \frac{\text{lk}(L_i, L \setminus L_i)}{2} \right) = \sum_{i=1}^{n} \frac{\text{tb}(L_i) - \text{rot}(L_i) + 1}{2} = \frac{\text{tb}(L) - \text{rot}(L) + n}{2}.$$

\[ \square \]

5.3 The link Floer homology group of a Legendrian connected sum

Take two null-homologous Legendrian links $L_1, L_2$, respectively in the connected contact manifolds $(M_1, \xi_1)$ and $(M_2, \xi_2)$. We can define a Legendrian connected sum of the two links [9] and we denote it with $L_1 \# L_2 \rightarrow (M_1 \# M_2, \xi_1 \# \xi_2)$.

While the contact 3-manifold $(M_1 \# M_2, \xi_1 \# \xi_2)$ is uniquely defined, the Legendrian link $L_1 \# L_2$ depends on the choice of the components used to perform the connected sum. Moreover, we have the following properties [7] [9]:

- $d_3(M_1 \# M_2, \xi_1 \# \xi_2) = d_3(M_1, \xi_1) + d_3(M_2, \xi_2)$;
- $\text{tb}(L_1 \# L_2) = \text{tb}(L_1) + \text{tb}(L_2) + 1$;
- $\text{rot}(L_1 \# L_2) = \text{rot}(L_1) + \text{rot}(L_2)$.

Let us consider two adapted open book decompositions $(B_i, \pi_i, A_i)$, compatible with the triples $(L_i, M_i, \xi_i)$. We can define a third open book $(B, \pi, A)$, for the manifold $M_1 \# M_2$, with the property that $\pi^{-1}(1)$ is a Murasugi sum of the pages $\pi^{-1}_1(1)$ and $\pi^{-1}_2(1)$; see [7] for the definition. The resulting open book is compatible with the triple $(L_1 \# L_2, M_1 \# M_2, \xi_1 \# \xi_2)$; where the Murasugi sum is done along the components involved in the connected sum.

Denote with $D_1, D_2$ and $D$ the Legendrian Heegaard diagrams obtained from the open book decompositions that we introduced before. Then, we have the following theorem.

Theorem 5.7. For every Spin$^c$ structure on $M_1$ and $M_2$ there is a chain map

$$cCFL^-(D, t_1 \# t_2) \longrightarrow cCFL^-(D_1, t_1) \otimes_{\mathbb{F}[t]} cCFL^-(D_2, t_2)$$

that preserves the bigrading and the element $\Sigma(D)$ is sent into $\Sigma(D_1) \otimes \Sigma(D_2)$.

Moreover, this map induces an isomorphism in homology; which means that

$$cHFL^-(\overline{M}_1 \# \overline{M}_2, L_1 \# L_2, t_{\xi_1} \# t_{\xi_2}) \cong cHFL^-(\overline{M}_1, L_1, t_{\xi_1}) \otimes_{\mathbb{F}[t]} cHFL^-(\overline{M}_2, L_2, t_{\xi_2})$$

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as bigraded $\mathbb{F}[U]$-modules and
\[
\mathcal{L}(L_1 \# L_2, M_1 \# M_2, \xi_1 \# \xi_2) = \mathcal{L}(L_1, M_1, \xi_1) \otimes \mathcal{L}(L_2, M_2, \xi_2).
\]

Proof. It follows from [12] Section 7, [17] Section 7 and [19] Section 11.

We note that the link Floer homology group of the connected sum does not depend on the choice of the components. In particular, this means that we can compute the link Floer homology group and the Legendrian invariant of a disjoint union $L_1 \sqcup L_2$. See [2] for the definition.

**Proposition 5.8.** If we denote with $\square_2$ a smooth 2-component unlink and with $\mathcal{O}_2$ the Legendrian 2-component unlink in $(S^3, \xi_{st})$ such that $tb(\mathcal{O}_2) = -2$ then we have
\[
chFL^-(\mathcal{M}, L_1 \sqcup L_2, \xi_1) \cong
\mathcal{L}(L_1 \sqcup L_2, M, \xi) = \mathcal{L}(L_1, M_1, \xi_1) \otimes \mathcal{L}(L_2, M_2, \xi_2) \otimes \mathcal{L}(\mathcal{O}_2).
\]
and
\[
chFL^-(\mathcal{M}, L_1, \xi_1) \otimes \mathbb{F}[U] chFL^-(\mathcal{M}, L_2, \xi_2) \otimes \mathbb{F}[U] chFL^-(\square_2)
\]

Proof. We just apply Theorem 5.7 twice, each time on one of the two components of $\mathcal{O}_2$.

The homology group $chFL^-(\square_2)$ is isomorphic, as bigraded $\mathbb{F}[U]$-module, to $\mathbb{F}[U](-1,0) \oplus \mathbb{F}[U](0,0)$ (a proof can be found in [14]). Furthermore, Theorem 5.6 tells us that
\[
\mathcal{L}(\mathcal{O}_2) = e_{-1,0},
\]
that is the generator of $\mathbb{F}[U]$ with bigrading $(-1,0)$.

This means that, if $\hat{\partial}(\mathcal{M}, \xi)$ is non-zero, we have that
\[
M(\mathcal{L}(L_1 \sqcup L_2, M, \xi)) = M(\mathcal{L}(L_1, M_1, \xi_1)) + M(\mathcal{L}(L_2, M_2, \xi_2)) - 1
\]
and
\[
A(\mathcal{L}(L_1 \sqcup L_2, M, \xi)) = A(\mathcal{L}(L_1, M_1, \xi_1)) + A(\mathcal{L}(L_2, M_2, \xi_2)).
\]

We also observe that:
- $tb(L_1 \sqcup L_2) = tb(L_1) + tb(L_2)$;
- $rot(L_1 \sqcup L_2) = rot(L_1) + rot(L_2)$.

### 5.4 Stabilizations of a Legendrian link

We know that Legendrian links in the tight $S^3$ can be represented with front projections, see [6] for more details. Then we define positive (negative) stabilization of a Legendrian link $L$ in $(S^3, \xi_{st})$, with front projection $P$, the Legendrian link $L^\pm$ represented by the front projection $P^\pm$; which is obtained by adding two consecutive downward (upward) cusps to $P$. Stabilizations are well-defined up to Legendrian isotopy, in the sense that they do not depend on the choice of the point of $P$ where we add the new cusps.

At this point it is easy for us the define stabilizations in every contact manifold. In fact we say that $L^\pm$ is the positive (negative) stabilization of $L$, a Legendrian link in $(M, \xi)$, if $L^\pm = L \# O^\pm$; where $O^\pm$ is the positive (negative) stabilization of the standard Legendrian unknot $O$ in $(S^3, \xi_{st})$. The Legendrian knots $O^\pm$ are shown as front projections in Figure 23.

The link type remains unchanged after stabilizations. The behaviour of the other classical invariants is given by the following proposition.
Proposition 5.9. For every null-homologous Legendrian link $L$ in a contact 3-manifold $(M, \xi)$ it is
\[ \text{tb}(L^\pm) = \text{tb}(L) - 1 \quad \text{and} \quad \text{rot}(L^\pm) = \text{rot}(L) \pm 1. \]
Furthermore, we have that
\[ \mathcal{L}(L^+, M, \xi) = U \cdot \mathcal{L}(L, M, \xi) \quad \text{and} \quad \mathcal{L}(L^-, M, \xi) = \mathcal{L}(L, M, \xi) \]
in $cHFL^-(\overline{M}, L, \xi)$. 

Proof. The first claim is a standard fact (see [6] for example). The second one follows from Theorem 5.7, which says that we just need to determine $\mathcal{L}(O^\pm)$, the fact that $cHFK^-(\bigcirc) \cong F[U]_{(0,0)}$, which says that $\mathcal{L}(O^\pm)$ is fixed by the classical invariants of $O^\pm$, and the first part of this proposition, which tells us that $\text{tb}(O^\pm) = -2$ and $\text{rot}(O^\pm) = \pm 1$. 

5.5 Loose Legendrian links

Since $\hat{c}(M, \xi)$ is always zero for overtwisted contact manifold, we have that the Legendrian link invariant $\mathcal{L}$ is always torsion in this case. But Proposition 1.3 says more in the case of loose Legendrian links.

Proof of Proposition 1.3. The complement of $L$ in $M$ contains an overtwisted disk $E$. Since $E$ is contractible, we can find a ball $U$ such that $E \subset U \subset M \setminus L$. The restriction of $\xi$ to $U$ is obviously overtwisted; moreover, we can choose $E$ such that $\partial U$ has trivial dividing set. Thus we have that $(M, \xi) = (M, \xi_1) \# (S^3, \xi')$; where $\xi_1$ coincides with $\xi$ near $L$ and $\xi'$ is an overtwisted structure on $S^3$.

We now use the fact that the standard Legendrian unknot $O$ is well-defined, up to Legendrian isotopy, and Theorem 5.7 to say that
\[ \mathcal{L}(L, M, \xi) = \mathcal{L}(L, M, \xi_1) \otimes \mathcal{L}(O, S^3, \xi'). \]
But since $cHFK^-(\bigcirc) \cong F[U]_{(0,0)}$, and then there is no torsion, we know that the Legendrian invariant of an unknot is zero in overtwisted 3-spheres. Then $\mathcal{L}(L, M, \xi)$ is also zero.

This proposition says something about stabilizations. In fact, in principle a stabilization of a non-loose Legendrian link $L \hookrightarrow (M, \xi)$ can be loose, but if $\mathcal{L}(L, M, \xi)$ is non zero then all its negative stabilizations are also non-loose.
6 The transverse case

We recall that there is a way to associate a Legendrian link to a transverse link and vice versa. Given a Legendrian link $L$, we denote with $T_L$ the transverse push-off of $L$, which is transverse. The construction is found in Section 2.9 in [6]. Any two transverse push-offs are transversely isotopic and then $T_L$ is uniquely defined, up to transverse isotopy.

In the same way, if $T$ is a transverse link then we can define a Legendrian approximation $L_T$ of $T$. The procedure is also described in Section 2.9 of [6]. The Legendrian link $L_T$ is not well-defined up to Legendrian isotopy, but only up to negative stabilizations. Then from [8] we have the following theorem.

**Theorem 6.1 (Epstein).** Two transverse links in a contact manifold are transversely isotopic if and only if they admit Legendrian approximations which differ by negative stabilizations.

The only classical invariant of a null-homologous transverse link $T$, other than the smooth link type is the self-linking number $sl(T)$. In every $(M,\xi)$ rational homology contact 3-sphere, we define the number $sl(T)$ as follows:

$$sl(T) = tb(L_T) - rot(L_T).$$

Clearly, from Theorem 6.1 and since negative stabilizations drop both the Thurston-Bennequin and rotation number by one, we have that the self-linking number is a well-defined transverse invariant. Moreover, from the definition of self-linking number, we have the following properties:

- $sl(T_1 \# T_2) = sl(T_1) + sl(T_2) + 1$;
- $sl(T_1 \sqcup T_2) = sl(T_1) + sl(T_2)$.

We can now define a transverse invariant from link Floer homology by taking

$$\Sigma(T, M, \xi) = \mathfrak{L}(L_T, M, \xi),$$

where $L_T$ is a Legendrian approximation of $T$. The invariant $\Sigma$ has the same basic properties of $\mathfrak{L}$ that we recall in the following theorem.

**Theorem 6.2.** The isomorphism class $\Sigma(T, M, \xi)$ in $cHFL^- (\overline{M}, T, \xi)$ is a transverse link invariant. If $n$ is the number of components of $T$ we have that the bigrading of $\Sigma$ is

$$A(\Sigma(T, M, \xi)) = \frac{sl(T) + n}{2} \quad \text{and} \quad M(\Sigma(T, M, \xi)) = -d_3(M, \xi) + sl(T) + 1.$$

Furthermore, it is

$$\Sigma(T_1 \# T_2, M_1 \# M_2, \xi_1 \# \xi_2) = \Sigma(T_1, M_1, \xi_1) \otimes \Sigma(T_2, M_2, \xi_2).$$

**Proof.** Proposition 5.9 and Theorem 6.1 tell us that $\Sigma(T, M, \xi)$ is an invariant. The other properties follow from Theorem 5.6, the definition of self-linking number and the fact that the operations of Legendrian approximation and connected sum commute.

In the case of knots the invariant $\Sigma$ has been introduced first in [12].
7 Applications

7.1 A different version of the Legendrian invariant

Let us consider a Legendrian Heegaard diagram \(D\), given by an open book compatible with a triple \((L, M, \xi)\), where \(M\) is a rational homology 3-sphere and \(L\) is a null-homologous Legendrian \(n\)-component link.

We can define another chain complex by taking the \(F\)-vector space
\[
\hat{C}FL(D, t) = CFL^{-}(D, t) / U = 0
\]
for every \(\text{Spin}^c\) structure \(t\) on \(M\). The corresponding differential is \(\hat{\partial} = \partial^{-}\big|_{U=0}\).

We obtain the hat link Floer homology group
\[
\hat{H}FL(D, t) = \bigoplus_{d, s \in \mathbb{Q}} \hat{H}FL_{d, s}(D, t);
\]
given by
\[
\hat{H}FL_{d, s}(D, t) = \frac{\text{Ker} \hat{\partial}_{d, s}}{\text{Im} \hat{\partial}_{d+1, s}}.
\]

The group \(\hat{H}FL(D, t)\) is a finite dimensional, bigraded \(F\)-vector space and its isomorphism type is invariant under smooth isotopy of the link \(L\) \cite{14}. Hence, we can denote \(\hat{H}FL(D, t)\) with \(\hat{H}FL(M, L, t)\).

The insection point \(L(D)\) is a cycle also in \(\hat{C}FL(D, t)\) and it determines the \(\text{Spin}^c\) structure \(t\). Then we have the following theorem.

**Theorem 7.1.** The isomorphism class of \([L(D)]\) in \(\hat{H}FL(M, L, t)\) is a Legendrian invariant of \((L, M, \xi)\) and we denote it with \(\hat{L}(L, M, \xi)\). Furthermore, if \(\hat{L}(L, M, \xi)\) is non-zero then \(L(M, \xi)\) is also non-zero.

For a null-homologous transverse link \(T \hookrightarrow (M, \xi)\) we have that \(\hat{L}(T, M, \xi) = \hat{L}(T_L, M, \xi)\), where \(T_L\) is a Legendrian push-off of \(T\), is a transverse invariant of \(T\) and it has the same non-vanishing property of \(\hat{L}\).

The proof of this theorem is the same as the one of Theorems \(1.2\) and \(6.2\) except for the non-vanishing property, which follows from the fact that \(CFL(D, t)\) is a quotient of \(cCFL^{-}(D, t)\).

The invariant \(\hat{L}(L, M, \xi)\) can be refined using a naturality property of the link Floer homology group of a connected sum. Suppose that \(L\) is a Legendrian link in a contact 3-sphere \((S^3, \xi)\) such that \(\hat{L}(L, S^3, \xi) \neq \{0\}\). Let \(S\) be a convex, splitting sphere with connected dividing set, which intersects \(L\) transversely in exactly two points. Such a splitting sphere expresses \(L\) as a connected sum of two links \(L_1\) and \(L_2\).

Since \(L = L_1 \#_S L_2\) then its hat Heegaard Floer homology group admits the splitting
\[
\hat{H}FL(L^*) \cong \hat{H}FL(L_1^*) \otimes_F \hat{H}FL(L_2^*),
\]

where the mirror images appear because \(S^3\) has a diffeomorphism that reverses the orientation, then we identify \(\hat{H}FL(-S^3, L)\) with \(\hat{H}FL(S^3, L^*) := \hat{H}FL(L^*)\).
The Alexander grading of $\hat{\mathcal{L}}(L,S^3,\xi)$ is well-defined, because we suppose that the invariant is non-zero. Moreover, we have that

$$A\left(\hat{\mathcal{L}}(L)\right) = A\left(\hat{\mathcal{L}}(L_1)\right) + A\left(\hat{\mathcal{L}}(L_2)\right).$$

The pair $(s_1, s_2)$, where

$$s_i = A\left(\hat{\mathcal{L}}(L_i)\right)$$

for $i = 1, 2$, is called Alexander pair of $\hat{\mathcal{L}}(L,S^3,\xi)$ respect to $S$ and we denote it with $A_S(L, S^3, \xi)$. We have that the Alexander pair is an invariant of $L$ in the sense of the following theorem.

**Theorem 7.2.** Suppose that $L$ is a Legendrian link in $(S^3, \xi)$ such that $\hat{\mathcal{L}}(L,S^3,\xi)$ is non-zero. We also assume that there are two convex, splitting spheres $S_1$ and $S_2$, which decompose $L$ as Legendrian connected sums, such that we can find a smooth isotopy of $M$ that fix $L$ and sends $S_1$ into $S_2$.

Then the two Alexander pairs of $\hat{\mathcal{L}}(L,S^3,\xi)$, respect to $S_1$ and $S_2$, coincide, which means that $A_{S_1}(L,S^3,\xi) = A_{S_2}(L,S^3,\xi)$.

**Proof.** The proof is a link version of the one of Theorems 8.4 and 9.1 in [12].

There is a version of Theorem 7.2 for transverse links.

**Corollary 7.3.** Suppose that $T$ is a transverse link in $(S^3, \xi)$. Assume also that one of its Legendrian approximations $L_T$ respects the hypothesis of Theorem 7.2. Then it is

$$A_{S_1}(T,S^3,\xi) = A_{S_2}(T,S^3,\xi),$$

where the Alexander pair is now defined as $A_{S}(T,S^3,\xi) = A_{S}(L_T,S^3,\xi)$.

**Proof.** It is a consequence of the fact that Legendrian approximations of the same transverse link differ by negative stabilizations. Therefore, the Alexander gradings are the same because negative stabilizations do not change the invariant.

The Alexander pair can be useful in distinguishing Legendrian and transverse links that are not isotopic.

**Proposition 7.4.** Suppose that $L_1$ and $L_2$ are smoothly isotopic Legendrian (transverse) links in $(S^3, \xi)$ which appear as follows.

Say $L_1 \cong L_2$ is a 2-component Legendrian (transverse) link, obtained from three Legendrian (transverse) knots $K$, $H$ and $J$ with prime knot types, defined as follows: take the connected sum of $K\#_S H$ with a (standard) positive Hopf link $H_+$ and $J$, in the way that $K\#_S H$ is summed on the first component of $H_+$ and $J$ on the second one.

We have that if $A_S(L_1,S^3,\xi) \neq A_S(L_2,S^3,\xi)$ then $L_1$ is not Legendrian (transverse) isotopic to $L_2$.

**Proof.** It follows from Theorem 7.2 and Corollary 7.3 and the fact that, if there is a Legendrian (transverse) isotopy $F$ between $L_1$ and $L_2$, the isotopy $F$ is such that $F(S) = S'$ and we can smoothly isotope $S'$ onto $S$. 


7.2 Non-loose Legendrian links with loose components

It is easy to prove that we can always find loose Legendrian links in every contact 3-manifold; in fact Legendrian links inside a Darboux ball need to be loose. On the other hand, it is not harder to show that the same holds for non-loose Legendrian links. In fact, it is a known [7] that every overtwisted contact 3-manifold is obtained from some $-1$-surgeries and exactly one $+1$-surgery on Legendrian knots in $(S^3,\xi_{st})$; then we just take the Legendrian link given by $n$ parallel contact push-offs of $J^+$, the knot where we perform the $+1$-surgery. Moreover, it is easy to check that this link also has non-loose components.

A more interesting result is to show that, under some hypothesis on $(M,\xi)$, we can also find non-split Legendrian $n$-components links such that $\mathfrak{L}(L,M,\xi) \neq [0]$, which means that they are non-loose from Proposition 1.3 and all of their components are now loose. We start by constructing Legendrian knots with non trivial invariant in all the overtwisted structures on $S^3$.

Consider the family of Legendrian knots $L_j$, where $j \geq 1$, given by the surgery diagram in Figure 14. Using Kirby calculus we easily see that $L_j$ is a positive torus knot $T_{2,2j+1}$ in $S^3$. On the other hand, the Legendrian invariants of $L_j$ and the contact structure where it lives are determined in [12] Chapter 6. Namely, the knots in Figure 14 are Legendrian knots in $(S^3,\xi_{1-2j})$ and their invariants are:

- $\text{tb}(L_j) = 6 + 4(j - 1)$;
- $\text{rot}(L_j) = 7 + 6(j - 1)$.

Furthermore, Proposition 6.2 in [12] tells us that $\hat{\mathfrak{L}}(L_j, S^3, \xi_{1-2j}) \neq [0]$ and then $\mathfrak{L}(L_j, S^3, \xi_{1-2j})$ is a non-zero torsion class in $HFK^{-}(T_{2,2j-1})$. Moreover, both have bigrading $(1, 1-j)$.

Now we want to consider another family of Legendrian knots: the knots $L_{k,l}$, with $k,l \geq 0$, shown in Figure 15. From [12] Chapter 6 we also know that $L_{k,l}$ is a negative torus knot $T_{2,-2k-2l-3}$ in $(S^3,\xi_{2l+2})$ and its invariants are:
Figure 15: Contact surgery presentation for the Legendrian knot $L_{k,l}$.

- $tb(L_{k,l}) = -6 - 4(k + l)$;
- $rot(L_{k,l}) = -7 - 2k - 6l$.

In this case, from Theorems 6.8, 6.9 and 6.10 in [12], we have that the invariant $\hat{\mathcal{L}}$ of the Legendrian knots $L_{0,l}$, $L_{1,1}$ and $L_{1,2}$ is non-zero with bigrading $(-2k, 1 - k + l)$ in the homology group $\hat{HFK}(T_{2,2k+2l+3})$. Obviously, the fact that $\hat{\mathcal{L}}$ is non-zero again implies that the same is true for the invariant $\mathcal{L}$.

At this point, we define the Legendrian knots $K_i$, for every $i \in \mathbb{Z}$, in the following way:

$$K_i = \begin{cases} 
L(j) \# L(1) & \text{if } i = -2j < 0 \\
L(j) & \text{if } i = 1 - 2j < 0 \\
L_{0,j-1} & \text{if } i = 2j > 0 \\
L_{0,j-1} \# L(1) & \text{if } i = 2j - 1 > 0 \\
L_{0,0} \# L(1)^2 & \text{if } i = 0.
\end{cases}$$

Then we have the following result; which is Theorem 7.2 in [12].

**Proposition 7.5** (Lisca, Ozsváth, Stipsicz and Szabó). The Legendrian knot $K_i \hookrightarrow (S^3, \xi_i)$ is such that $\mathcal{L}(K_i, S^3, \xi_i) \neq [0]$ and then it is non-loose for every $i \in \mathbb{Z}$.

**Proof.** It follows easily from the previous computation and the connected sum formula. □

We can now go back to links. Let us take an overtwisted 3-manifold $(M, \xi)$ such that there exists another contact structure $\zeta$ on $M$ with $\partial(M, \zeta) \neq [0]$ and $t_\xi = t_\zeta$; in particular $\zeta$ is tight. Consider $\mathcal{L}$ the standard Legendrian unknot in $(M, \zeta)$. We have that $\mathcal{L}((\mathcal{O}, M, \zeta))$ coincides with $e_{-d_3(M, \zeta),0} \neq [0]$ and the invariant is non-torsion; this is because $\text{HF}^-(\hat{M}, \mathcal{O}, t_\zeta) \cong \mathbb{F}[U]_{(-d_3(M, \zeta),0)}$ and Theorem 5.6.
Fix $B$ a Darboux ball in $(M, \xi)$, which contains $O$. Since $B$ is contact isotopic to $(\mathbb{R}^3, \xi_3)$, we can take the Legendrian link $H_+^{n-1}$ in $B$, that is defined as $n - 1$ connected sums of the standard positive Legendrian Hopf link; see Figure 16. If $n = 1$ we do not do anything and we keep the the knot $O$. The invariant $\mathfrak{S}(H_+^{n-1}, M, \xi)$ is the tensor product of $\mathfrak{S}(O, M, \xi)$, which as we said is non-zero, with $n - 1$ times $\mathfrak{S}(H_+)$ the invariant in the standard $S^3$. An easy computation shows that $\mathfrak{S}(H_+)$ is the only non-torsion element in the group

$$cHFL^{-}(H_-) \cong \mathbb{F}[U]_{(0,0)} \oplus \mathbb{F}[U]_{(1,1)} \oplus \left( \frac{\mathbb{F}[U]}{U \cdot \mathbb{F}[U]} \right)_{(0,0)}$$

with bigrading $(1, 1)$. This means that not only $\mathfrak{S}(H_+^{n-1}, M, \xi)$ is non-torsion, but it is also represented by the top generator of one of the $\mathbb{F}[U]$ towers of $cHFL^{-}(M, H_+^{n-1}, \xi)$. Now we perform a connected sum between $H_+^{n-1}$ and the Legendrian knot $L(1)$, defined before; where we suppose that $L(1)$ is summed to $H_+^{n-1}$ on the rightmost component in Figure 16. We saw that $L(1)$ is a positive trefoil in $(S^3, \xi_{-1})$. Thus $H_+^{n-1} \# L(1)$ lives in $(M, \xi')$; where $\xi'$ is an overtwisted structure such that $\xi' = t_c = t_\xi$ and $d_3(\xi') = d_3(\xi) - 1$.

If we do another connected sum with the Legendrian knot $K_{d_3(\xi)-d_3(\xi)+1}$, this time on the leftmost component of $H_+^{n-1}$, we obtain the Legendrian link

$$L = K_{d_3(\xi)-d_3(\xi)+1} \# H_+^{n-1} \# L(1);$$

which is a link in $M$ equipped with a contact structure that has the same Hopf invariant as $(M, \xi)$ and induces the same Spin$^c$ structure of $\xi$. By the Eliashberg’s classification of overtwisted structures [2], we conclude that $L$ is a Legendrian $n$-component link in $(M, \xi)$. We can now prove Theorem 1.4.

**Proof of Theorem 1.4.** We already saw that the link $L$ exists if the hypothesis of the theorem holds. So first we check that $\mathfrak{S}(L, M, \xi)$ is non-zero. In fact, the invariant is represented by the tensor product of a non-zero torsion element with $\mathfrak{S}(H_+^{n-1}, M, \xi)$. We are working with $\mathbb{F}[U]$-modules and we recall that

$$\mathbb{F}[U] \otimes_{\mathbb{F}[U]} \left( \frac{\mathbb{F}[U]}{U \cdot \mathbb{F}[U]} \right) \cong \frac{\mathbb{F}[U]}{U \cdot \mathbb{F}[U]}$$

and, more precisely, in the $\mathbb{F}[U]$ factor only the generator survives. Then, from what we said before, we have that $\mathfrak{S}(L, M, \xi)$ remains non-zero. Note that this is false if instead we take the negative Hopf link. In this case the invariant $\mathfrak{S}$ does not lie in the top of an $\mathbb{F}[U]$ tower of the homology group and then it vanishes after the tensor product.

This immediately implies that $\mathfrak{S}(T, M, \xi)$ is also non-zero and then the theorem holds for transverse links. Moreover, the fact that the components of $L$ (and $T$) are loose follows easily from the construction of $L$. In fact, we performed connected sums of $(M, \xi)$ with two
overtwisted $S^3$’s and each component lies in one of these three summands. It is only left to prove that $L$ is non-split.

From [3] we know that the connected sum of two tight contact manifolds is still tight. This implies that a non-loose Legendrian link is split if and only if its smooth link type is split. Hence, we just have to show that $L$ is non-split as a smooth $n$-component link. But $L$ is a connected sum of torus links in a 3-ball inside $M$ and we know that $L$ is non-split as a link in $S^3$. Furthermore, if $L$ is split in $M$ then it would be split also in the 3-sphere and this is a contradiction.

7.3 Non-simple link types

In the previous subsection we saw that, under some hypothesis, in an overtwisted 3-manifold $(M, \xi)$ we can find non-loose, non-split Legendrian $n$-components links $L_n$. Consider the links $L'_n$ obtained as the connected sum of $L_n$ with the standard Legendrian unknot in $(S^3, \xi_0)$, where $\xi_0$ is the overtwisted $S^3$ with zero Hopf invariant.

Since $(M, \xi)$ is already overtwisted, we have that $(M, \xi) \# (S^3, \xi_0)$ is contact isotopic to $(M, \xi)$. This means that $L'_n$ is also a non-split Legendrian link in $(M, \xi)$, which is smoothly isotopic to $L_n$ for every $n \geq 1$, but unlike $L_n$ it is clearly loose.

Each component of $L'_n$ has the same classical invariants of a component of $L_n$. Moreover, if $n \geq 2$ then there is an overtwisted disk in their complement. From a result of Dymara in [4] the components of $L'_n$ are Legendrian isotopic to the ones of $L_n$. Hence, we have the following corollary.

Corollary 7.6. The link type of $L_n$ and $L'_n$ in $M$, which is denoted with $L$, is both Legendrian and transverse non-simple.

On the other hand, we can also find non-simple link types where the two Legendrian and transverse representatives are non-loose.

Proposition 7.7. Let us consider the links $L_1 = (L_{0,2} \# L_{1,2}) \# H_+ \# L(1)$ and $L_2 = (L_{1,1} \# L_{0,3}) \# H_+ \# L(1)$ in the contact manifold $(S^3, \xi_{11})$; where in $L_i$ the knots on the left are summed on the first component of $H_+$ and $L(1)$ on the second one. Then $L_1$ and $L_2$ are two non-loose, non-split Legendrian 2-component links, with the same classical invariants and Legendrian isotopic components, but that are not Legendrian isotopic.

In the same way, the transverse push offs of $L_1$ and $L_2$ are two non-loose, non-split transverse 2-component links, with the same classical invariants and transversely isotopic components, but that are not transversely isotopic.

Proof. We apply Theorem 7.2. The Legendrian invariant of $L_1$ and $L_2$ is computed in [12] and it is non-zero; moreover, the Alexander pairs of $\hat{\mathcal{L}}(L_1, S^3, \xi_{11})$ and $\hat{\mathcal{L}}(L_2, S^3, \xi_{11})$ are different. The fact that the components are Legendrian isotopic follows from Dymara’s result [4]. The same argument proves the theorem in the transverse setting.

Using the same construction, the refined version of $\hat{\mathcal{L}}$ and $\hat{\mathcal{T}}$ can be applied to find such examples for links with more than two components in every contact manifold as in Theorem 1.4.

Proof of Theorem 1.5. Let us take a standard Legendrian (transverse) positive Hopf link $H_+$ in $(S^3, \xi_{st})$. On the first component of $H_+$ we perform a connected sum with the knot $L_{0,2} \# L_{1,2}$ in one case and $L_{1,1} \# L_{0,3}$ in the other. While, on the second component of $H_+$,
we sum a non-split Legendrian (transverse) $n$-component link in the overtwisted manifold $(M, \xi')$, where $d_3(\xi') = d_3(\xi) - 12$, with non-zero invariants; those links exist as we know from Theorem 1.4. We conclude by applying the same argument of the proof of Proposition 7.7.

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