GRADIENT ESTIMATES OF VERY WEAK SOLUTIONS TO GENERAL QUASILINEAR ELLIPTIC EQUATIONS

SUN-SIG BYUN AND MINKYU LIM

Abstract. We establish a gradient estimate for a very weak solution to a quasilinear elliptic equation with a nonstandard growth condition, which is a natural generalization of the $p$-Laplace equation. We investigate the maximum extent for the gradient estimate to hold without imposing any regularity assumption on the nonlinearity other than basic structure assumptions. Our results also include a higher integrability result of the gradient and the existence for the very weak solutions to such nonlinear problems.

1. Introduction

In this paper we study the existence and regularity issues regarding nonlinear elliptic equations with nonstandard growth of the form

$$\text{div } A(x, Du) = \text{div } \left( \frac{\varphi(|f|)}{|f|} f \right) \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is a bounded open set and $\varphi \in C^1([0, \infty)) \cap C^2((0, \infty))$ is a given Young function which is convex and increasing. A Carathéodory map $A = A(x, \xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is the nonlinearity which satisfies the following growth and monotonicity conditions:

$$\begin{cases}
|A(x, \xi)| \leq L \varphi'(|\xi|) \\
(A(x, \xi) - A(x, \zeta), \xi - \zeta) \geq \nu \varphi''(|\xi| + |\zeta|)|\xi - \zeta|^2
\end{cases}$$

for a.e $x \in \Omega$, $\xi, \zeta \in \mathbb{R}^n \setminus \{0\}$ with some constants $0 < \nu \leq L < +\infty$. We assume that there exists a constant $s_\varphi \geq 1$ such that

$$\varphi(0) = 0, \quad \frac{1}{s_\varphi} \leq \frac{t \varphi''(t)}{\varphi'(t)} \leq s_\varphi \quad (t > 0).$$

We are concerned with gradient estimates of a solution to the problem (1.1) with only basic structural assumptions (1.2)-(1.3) but without any regularity assumption on the nonlinearity $x \mapsto A(x, z)$. More precisely, we want to prove that there exists a positive constant $\delta_0$ depending on $n, s_\varphi, \nu$ and $L$ such that the following implication

$$\varphi(|f|) \in L^q_{\text{loc}}(\Omega) \quad \Rightarrow \quad \varphi(|Du|) \in L^q_{\text{loc}}(\Omega)$$

holds for every $q \in [1 - \delta_0, 1 + \delta_0]$. For $q \in [1, 1 + \delta_0]$ with some small $\delta_0 > 0$, this follows from the classical theory and a higher integrability result for a weak solution.
as in [14, 17, 27]. In this regard, we mainly consider the case of \( q \in [1 - \delta_0, 1) \) with some small \( \delta_0 > 0 \) and there is indeed no such an existence and regularity theory as far as we are concerned in the literature.

The condition (1.3) includes the classical case of \( \phi(t) = t^p \) with \( p > 1 \). Then our problem (1.1) is reduced to the elliptic problems of the \( p \)-Laplacian type and implication (1.4) of interest to many researchers becomes

\[
|f|^p \in L^q_{\text{loc}}(\Omega) \implies |Du|^p \in L^q_{\text{loc}}(\Omega)
\]

for some positive number \( q \) lying in the range of \( \left( \frac{p-1}{p}, \infty \right) \) depending on further regularity assumptions on the nonlinearity \( A = A(x, z) \). This kind of estimate was first proved by Calderón and Zygmund in [9] for the linear case that \( p = 2 \) in the range of \( q \in \left( \frac{1}{2}, \infty \right) \). For the nonlinear case that \( p \neq 2 \) and \( q \in (1, \infty) \), there have been extensive regularity results according to given analytic and geometric settings as in [7, 8, 22, 25]. On the other hand, the implication (1.3) with \( q \in \left( \frac{p-1}{p}, 1 \right) \) for the \( p \)-Laplace problem still remains a wide open problem, partly due to the fact that the duality argument cannot be applicable to the nonlinear problem, as pointed out by Iwaniec [22].

There have been notable works with the gradient estimates below the natural exponent, that is when \( q < 1 \), when the right hand side of (1.1) is a bounded Radon measure, see for instance [4, 5, 28, 30]. Recently, these kinds of estimates have been extended to the generalized \( p \)-Laplacian equations in [11, 12, 15]. We refer to [10, 29] for a further discussion in this direction. On the other hand, when we have the divergence data of \( \text{div} \left( |f|^{p-2} f \right) \) in the right hand side, it was shown in [23] that there exists a small positive constant \( \delta_0 \) independent of \( f \) and \( u \) such that (1.5) holds in the range of \( q \in [1 - \delta_0, 1] \), see also [3, 23, 26]. However, such results have not yet been extended to the nonlinear equations with nonstandard growth including (1.1). The reason is that a generalization of Hodge decomposition or Lipschitz truncation method to Orlicz spaces should be properly formalized in advance to deal with very weak solutions. In this paper we exploit the Lipschitz truncation method recently developed in [18, 19] in order to find an appropriate form of the sub-natural gradient estimates. In this regard, the present paper provides a window to obtain the sub-natural estimates for other highly nonlinear problems.

To discuss the regularity of solutions to the equation (1.1) with \( f \) having a low degree of integrability, a few concepts of solutions have been introduced in the literature. Among them, the very weak solution presented therein might not have a finite natural energy, and so it requires a test function to be at least Lipschitz continuous in the weak formulation of the equation (1.1). Indeed, various kind of Lipschitz truncation methods have been developed as in [3, 24, 26]. Here we introduce a suitable Lipschitz truncation method keeping zero boundary condition for the generalized \( p \)-Laplacian equation.

Our paper is organized as follows. In the next section we introduce the notion of very weak solution to the problem and an Orlicz-Sobolev space to state our main theorem. In Section 3 we provide a variety of analytic tools including a Lipschitz truncation method which will be commonly used in the later sections. Existence and higher integrability issues for a very weak solution to the homogeneous equation will be investigated in Section 4. The last section is devoted to dealing with the comparison estimates in the balls under consideration for proving Theorem 2.1.
2. Notations and main results

We start this section with the concept of very weak solution. \( u \in W^{1,1}(\Omega) \) is said to be a very weak solution to (1.1) if \( \varphi'(|Du|) \in L^1(\Omega) \), \( \varphi'(|f|) \in L^1(\Omega) \) and

\[
\int_{\Omega} \langle A(x,Dw),D\varphi \rangle \, dx = \int_{\Omega} \left( \frac{\varphi'(|f|)}{|f|} f, D\eta \right) \, dx
\]

holds for every \( \eta \in C_0^\infty(\Omega) \), where we have denoted by \( C_0^\infty(\Omega) \) to mean that the set of smooth functions with compact support in \( \Omega \).

We next introduce standard notations. \( B_y(r) \) is the open ball with center \( y \in \mathbb{R}^n \) and radius \( r > 0 \). For an integrable function \( v \) defined on a bounded measurable subset \( E \subset \mathbb{R}^n \), we briefly denote the integral average over \( E \) as

\[
\mathcal{F}_E := \frac{1}{|E|} \int_{E} v(x) \, dx
\]

where \( |E| \) is the Lebesgue measure of \( E \). We now introduce auxiliary vector fields defined by

\[
V(z) := \left[ \frac{\varphi'(|z|)}{|z|} \right]^\frac{1}{q} \quad (z \in \mathbb{R}^n \setminus \{0\}).
\]

Note that if (1.3) is satisfied, then there holds

\[
|V(z_1) - V(z_2)|^2 \approx \varphi''(|z_1| + |z_2|)|z_1 - z_2|^2 \quad (z_1, z_2 \in \mathbb{R}^n \setminus \{0\}),
\]

where the implied constant depends on \( n \) and \( s_\varphi \). Then (1.2) yields

\[
\langle A(x,z_1) - A(x,z_2), z_1 - z_2 \rangle \geq cV(z_1) - V(z_2)^2 \quad (z_1, z_2 \in \mathbb{R}^n \setminus \{0\})
\]

for some positive constant \( c \) depending on \( n, \nu \) and \( s_\varphi \). Moreover, for any \( \varepsilon > 0 \), we have

\[
\varphi(|z_1 - z_2|) \leq c_{\varepsilon} |V(z_1) - V(z_2)|^2 + \varepsilon \varphi(|z_2|) \quad (z_1, z_2 \in \mathbb{R}^n \setminus \{0\}),
\]

where the constant \( c_{\varepsilon} \) depends on \( s_\varphi \) and \( \varepsilon \). For a further discussion regarding (2.2) through (2.5), we refer to [17, 19, 34].

As mentioned earlier, this paper aims at obtaining gradient estimates (2.6) below the natural exponent for very weak solutions to (1.1) in an appropriate setting of Orlicz spaces. Since the associated Young function \( \varphi \) satisfies the so-called \( \Delta_2 \)-condition and \( \nabla_2 \)-condition when (1.3) holds (see [31]), the set \( L^{\varphi}(\Omega) \) consisting of all measurable functions \( v \) on \( \Omega \) with

\[
\int_{\Omega} \varphi(|v(x)|) \, dx < \infty
\]

becomes a separable reflexive Banach space, where the norm is given by

\[
\|v\|_{L^{\varphi}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \varphi\left( \frac{|v(x)|}{\lambda} \right) \, dx \leq 1 \right\}.
\]

We remark that with (1.3) and \( q \) being sufficiently close to 1, \( L^{\varphi^q}(\Omega) \) also becomes a separable reflexive Banach space, where we have defined \( \varphi^q(t) := [\varphi(t)]^q \) for \( t \geq 0 \). See Lemma 3.2. The Orlicz-Sobolev space \( W^{1,\varphi} (\Omega) \) is a function space consisting of all measurable functions \( v \in L^{\varphi}(\Omega) \) whose weak derivatives \( Dv \) also belong to \( L^{\varphi}(\Omega; \mathbb{R}^n) \). The norm of \( W^{1,\varphi} (\Omega) \) is given by

\[
\|v\|_{W^{1,\varphi}(\Omega)} = \|v\|_{L^{\varphi}(\Omega)} + \|Dv\|_{L^{\varphi}(\Omega)}.
\]
Note that $C^\infty(B_\rho) = W^{1,\varphi}(B_\rho)$ for any ball $B_\rho \subset \mathbb{R}^n$, where the completion is taken with respect to the $W^{1,\varphi}(B_\rho)$ norm. $W^{1,\varphi}_0(\Omega)$ is defined as the closure of $C^\infty_0(\Omega)$ in $W^{1,\varphi}(\Omega)$. We refer to [20] for a further discussion regarding this Orlicz-Sobolev space.

We now state our main result in this paper.

**Theorem 2.1.** Assume [12] and [13]. Then there exists a small positive constant $\delta_0 = \delta_0(n, s_\varphi, \nu, L)$ such that for all $f \in L^{q_0}(\Omega; \mathbb{R}^n)$ with $q \in [1 - \delta_0, 1 + \delta_0]$, any very weak solution $u \in W^{1,\varphi^1-\delta_0}(\Omega)$ of [12] satisfies that

$$u \in W^{1,\varphi^q}_{\text{loc}}(\Omega)$$

with the estimate

$$\int_{B_{R/2}} |\varphi(|Du|)|^q \, dx \leq c \left[ \left( \int_{B_R} |\varphi(|Du|)|^{1-\delta_0} \, dx \right)^{\frac{s}{s-\delta_0}} + \int_{B_R} |\varphi(|f|)|^q \, dx \right]$$

for some positive constant $c$ depending only on $n, s_\varphi, \nu, L$, whenever $B_R \subset \Omega$.

We would like to point out that the estimate (2.6) becomes the standard Calderón-Zygmund type estimate if we choose $q$ in the range of $(1, \infty)$, which holds under a suitable regularity condition on the nonlinearity $A$ in [14] with respect to $x$, as follows from [6] [13] [34]. Therefore, the above theorem implies that we can extend the exponent range to $q \in [1 - \delta_0, \infty)$ for the gradient estimate, which includes the estimates below the natural energy level. On the other hand, the gradient estimate still holds in the range of $q \in [1 - \delta_0, 1 + \delta_0]$ without any such a condition imposed on the nonlinearity.

### 3. Preliminaries

In this section we provide some analytic tools to prove our main estimate (2.6).

Let $\mathcal{M}$ be the Hardy-Littlewood maximal function defined by

$$\mathcal{M}(f)(y) = \sup_{\rho > 0} \int_{B_{\rho}(y)} |f(x)| \, dx \quad (x \in \mathbb{R}^n)$$

for $f \in L^{1}_{\text{loc}}(\mathbb{R}^n)$. We recall that an $A_s$-weight ($s > 1$) is a non-negative function $w \in L^{1}_{\text{loc}}(\mathbb{R}^n)$ satisfying

$$[w]_{A_s} := \sup_{B \subset \mathcal{R}^n} \left( \frac{1}{\int_{B} w(x) \, dx} \right) \left( \int_{B} |w(x)|^{\frac{s}{s-1}} \, dx \right)^{s-1} < \infty.$$ (3.2)

Then we say that $w$ belongs to the Muckenhoupt class $A_s$ and the quantity $[w]_{A_s}$ is referred to the $A_s$-constant of $w$. It is well known that if $w$ is an $A_s$-weight, then there exists a constant $c$ depending only on $s$ and $[w]_{A_s}$ such that

$$\int_{B_p} [\mathcal{M}(f)(x)]^s w(x) \, dx \leq c \int_{B_p} |f(x)|^s w(x) \, dx \quad (B_p \subset \mathbb{R}^n)$$

for every measurable function $f \in L^{1}_{\text{loc}}(\mathbb{R}^n)$ with the right hand side of [32] being finite [32] Chapter 5]. The following lemma includes standard classical theory regarding maximal functions and $A_s$ weights.
**Lemma 3.1.** Let $0 < \tau < 1$. Then for every nonnegative measurable function $f$ such that $f^{1-\delta} \in L^1(\mathbb{R}^n)$ for some $\delta \in (0, \frac{1}{1-\tau}]$, there holds
\[
\int_{B_\rho} [M(f^\tau)(x)]^{\frac{1}{1-\tau}} dx \leq c \int_{B_\rho} [f(x)]^{1-\delta} dx \tag{3.4}
\]
for some positive constant $c$ depending on $n$ and $\tau$, whenever $B_\rho \subset \mathbb{R}^n$. Moreover, $[M(f^\tau)]^{-\frac{\delta}{\tau}}$ is in the Muckenhoupt class $A_\tau$ with
\[
[[M(f^\tau)(x)]^{-\frac{\delta}{\tau}}]_{A_\tau} \leq c, \tag{3.5}
\]
where the constant $c$ depends on $n$ and $\tau$.

**Proof.** Since $\frac{1-\delta}{1-\tau} > \frac{1}{2}$, (3.4) follows from the boundedness of the maximal function, where the constant $c$ appearing in (3.4) only depends on $n$ and $\tau$, not on $\delta$, see [32] for more details about maximal function. Now, note that if $\epsilon \leq \frac{1}{2}$, then for any ball $B_s(z) \subset \mathbb{R}^n$, we have
\[
\int_{B_s(z)} [M(f^\tau)]^\epsilon dx \leq c[M(f^\tau)(z)]^\epsilon \text{ a.e. } z \in \mathbb{R}^n, \tag{3.6}
\]
where the constant $c$ depends only on $n$ (see for instance [32] p.214). Then since $\frac{\delta}{1-\tau} \leq \frac{1}{2}$, we observe that for any ball $B_\rho(y) \subset \mathbb{R}^n$,
\[
\int_{B_\rho(y)} [M(f^\tau)]^{-\frac{\delta}{\tau}} dx \left( \int_{B_\rho(y)} [M(f^\tau)]^{\frac{1}{1-\tau}} dx \right)^{\frac{1-\delta}{\tau}} \\
\leq \left( \int_{B_\rho(y)} [M(f^\tau)]^{-\frac{\delta}{\tau}} dx \right)^{\frac{1-\delta}{\tau}} \left( \inf_{z \in B_\rho(y)} 2^n \int_{B_{2s}(z)} [M(f^\tau)]^{\frac{1}{1-\tau}} dx \right)^{\frac{\delta}{\tau}} \\
\leq c \left\{ \inf_{z \in B_\rho(y)} M(f^\tau)(z) \right\}^{-\frac{\delta}{\tau}} \left\{ \inf_{z \in B_\rho(y)} M(f^\tau)(z) \right\}^{\frac{\delta}{\tau}} \leq c \tag{3.7}
\]
for some positive constant $c$ depending only on $n$ and $\tau$. Recalling (3.2), we conclude the proof of (3.5).

We now provide several properties of the Young function $\varphi$.

**Lemma 3.2.** Suppose $\varphi$ satisfies (1.3). Then the following holds:

(a) For any $t > 0$, we have
\[
\frac{1}{s_\varphi} + 1 \leq \frac{t \varphi'(t)}{\varphi(t)} \leq 1 + s_\varphi. \tag{3.8}
\]

(b) For any $0 < \lambda \leq 1$ and $1 \leq \Lambda < \infty$, we have
\[
\begin{cases}
\lambda^{1+s_\varphi} \varphi(t) \leq \varphi(\lambda t) \leq \lambda^{(1/s_\varphi)+1} \varphi(t) & (t > 0) \\
\Lambda^{(1/s_\varphi)+1} \varphi(t) \leq \varphi(\Lambda t) \leq \Lambda^{1+s_\varphi} \varphi(t) & (t > 0).
\end{cases} \tag{3.9}
\]

(c) There exists a constant $\delta = \delta(s_\varphi) > 0$ such that for all $q \in [1 - \delta, 1 + \delta]$,
\[
\frac{1}{2s_\varphi} \leq \frac{t[\varphi^q(t)]''}{[\varphi^q(t)]'} \leq 2s_\varphi \quad (t > 0), \tag{3.10}
\]
that (b) implies that \(\lim_{t \to 0} t[\varphi(t)]^{-\delta} (t \geq 0)\).

**Proof.** We first refer to [27] for the finding of (a) and (b). A direct computation gives

\[
\frac{t[q\varphi(t)]''}{[q\varphi(t)]'} = t\varphi''(t)\varphi(t) + (q - 1)\frac{t\varphi'(t)}{\varphi(t)}.
\]

(3.12)

Therefore, taking \(\hat{\delta} = \frac{1}{2s_\varphi}\), we have

\[
\frac{t[q\varphi(t)]''}{[q\varphi(t)]'} - \frac{t\varphi''(t)}{\varphi(t)} \leq (q - 1)\frac{t\varphi'(t)}{\varphi(t)} \leq 2\hat{\delta}s_\varphi \leq \frac{1}{2s_\varphi}.
\]

(3.13)

Combining (3.12) with (3.13), we get (3.10). Differentiating \(\varphi^q(t)\), we have

\[
[q\varphi(t)]' = q\varphi'(t)[\varphi(t)]^{q-1} \geq 0,
\]

(3.14)

\[
[q\varphi(t)]'' = q[q\varphi(t)]^{q-1}\left[\frac{\varphi''(t) + (q - 1)[\varphi'(t)]^2}{\varphi(t)}\right] \geq q[q\varphi(t)]^{q-1}\left[\frac{\varphi''(t)}{\varphi(t)} - \hat{\delta}\frac{[\varphi'(t)]^2}{\varphi(t)}\right] \geq 0.
\]

(3.15)

Then we conclude that \(\varphi^q(t)\) is increasing and convex. To show (3.11), we first note that (b) implies that \(\lim_{t \to 0} t[\varphi(t)]^{-\delta} = 0\). Therefore, we obtain

\[
t[\varphi(t)]^{-\delta} = \int_0^t \left\{s[\varphi(s)]^{-\delta}\right\}' ds
\]

\[
= \int_0^t [\varphi(s)]^{-\delta} - \delta \left\{s\varphi'(s)[\varphi(s)]^{-1-\delta}\right\} ds
\]

\[
\geq \int_0^t [\varphi(s)]^{-\delta} - \frac{1}{2s_\varphi}[\varphi(s)]^{-\delta} ds \geq \frac{1}{2} \int_0^t [\varphi(s)]^{-\delta} ds.
\]

(3.16)

Lemma 3.2 shows that for any \(q \in [1 - \hat{\delta}, 1]\), the spaces \(L^{q\varphi}(\Omega)\) and \(W^{1,q\varphi}(\Omega)\) are well-defined and every function \(f \in L^{q\varphi}(\Omega, \mathbb{R}^n)\) satisfies \(\varphi'(|f|) \in L^1(\Omega)\). Now we introduce some variation of Young’s inequalities related to the Young function \(\varphi\), which will be frequently used later in Section 4.

**Lemma 3.3.** Suppose \(\varphi\) satisfies (1.3) and let \(\hat{\delta}\) be given in Lemma 3.2. Then for any \(\varepsilon \in (0, 1]\) and any \(\delta \in (0, \hat{\delta}]\), there hold

\[
t[\varphi(s)]^{-\delta}\varphi'(s) \leq \varepsilon[\varphi(t)]^{1-\delta} + c_\varepsilon[\varphi(s)]^{1-\delta} (t, s \geq 0)
\]

(3.17)

and

\[
t[\varphi(t)]^{-\delta}\varphi'(s) \leq \varepsilon[\varphi(t)]^{1-\delta} + c_\varepsilon[\varphi(s)]^{1-\delta} (t, s \geq 0),
\]

(3.18)

where the constant \(c_\varepsilon\) depends only on \(s_\varphi\) and \(\varepsilon\).
Proof. First observe that for any \( \tilde{s} > 0 \),
\[
\left[ \frac{\varphi(\tilde{s})}{\tilde{s}} \right]^{1-\delta} = \left[ (1 - \delta) \frac{\varphi'(\tilde{s})}{\tilde{s}} - \frac{\varphi(\tilde{s})}{\tilde{s}^2} \right] \left[ \varphi(\tilde{s}) \right]^{-\delta} \geq 0.
\]
Then since \( \left[ \frac{\varphi(\tilde{s})}{\tilde{s}} \right]^{1-\delta} \) is increasing, we have
\[
\tilde{t} \left[ \frac{\varphi(\tilde{s})}{\tilde{s}} \right]^{1-\delta} \leq \tilde{t} \left[ \frac{\varphi(\tilde{t})}{\tilde{t}} \right]^{1-\delta} + \tilde{s} \left[ \frac{\varphi(s)}{s} \right]^{1-\delta} = \left[ \varphi(\tilde{t}) \right]^{1-\delta} + \left[ \varphi(s) \right]^{1-\delta}
\]
for any \( \tilde{t}, \tilde{s} > 0 \). Putting \( \tilde{t} = \varepsilon t \) and \( \tilde{s} = \varepsilon^{-2s_\varphi} s \) for \( \varepsilon \in (0, 1) \), we obtain
\[
t \left[ \frac{\varphi(s)}{s} \right]^{1-\delta} = \varepsilon^{-2s_\varphi - 1} \left[ \frac{\varphi(\varepsilon t)}{\varepsilon t} \right]^{1-\delta} \leq \varepsilon t \left[ \frac{\varepsilon^{-2s_\varphi} \varphi(s)}{\varepsilon^{-2s_\varphi} s} \right]^{1-\delta} \leq \varepsilon t \left[ \varphi(\varepsilon^{-2s_\varphi} s) \right]^{1-\delta} \leq \varepsilon [\varphi(t)]^{1-\delta} + \varepsilon^{-4s_\varphi} [\varphi(s)]^{1-\delta}.
\]
Taking (3.18) into account, we get the desired inequality (3.17). Similarly, (3.11) implies that \( t [\varphi(t)]^{-\delta} \) is an increasing function. Therefore, we have
\[
\tilde{t} [\varphi(\tilde{t})]^{-\delta} \frac{\varphi(\tilde{s})}{\tilde{s}} \leq [\varphi(\tilde{t})]^{1-\delta} + \left[ \varphi(s) \right]^{1-\delta}
\]
for any \( \tilde{t}, \tilde{s} > 0 \). Putting \( \tilde{t} = \varepsilon t \) and \( \tilde{s} = \varepsilon^{-2s_\varphi} s \) for \( \varepsilon \in (0, 1) \), we have
\[
t [\varphi(t)]^{-\delta} \frac{\varphi(s)}{s} = \frac{\varepsilon t}{\varepsilon \varphi(t)^{\delta}} \times \frac{\varepsilon^{-s_\varphi + 1} \varphi(s)}{\varepsilon^{-s_\varphi} s} \leq \frac{\varepsilon t}{\left[ (1/s_\varphi) + 1 \right] \varphi(t)^{\delta}} \times \frac{\varepsilon^{-s_\varphi + 1} \varphi(s)}{\varepsilon^{-s_\varphi} s} \leq \varepsilon t \left[ \frac{\varepsilon^{-s_\varphi} \varphi(\varepsilon t)}{\varepsilon^{-s_\varphi} s} \right]^{1-\delta} \leq \varepsilon [\varphi(t)]^{1-\delta} + \varepsilon^{-4s_\varphi} [\varphi(s)]^{1-\delta},
\]
which implies the inequality (3.18). \( \square \)

We next introduce an Orlicz-Sobolev-Poincaré type inequality. The proof can be found in [17] Theorem 7.

Lemma 3.4. Suppose \( \varphi \) satisfies (1.3) and let \( \hat{\delta} \) be the number given in Lemma 3.2. Then there exists a constant \( \theta = \theta(n, s_\varphi) \in [1 - \hat{\delta}, 1] \) such that for any \( w \in W^{1, \varphi}(B_\rho) \), there holds
\[
\int_{B_\rho} \varphi \left( \frac{|w - w_p|}{\rho} \right) \, dx \leq c \left( \int_{B_\rho} [\varphi(|Dw|)]^\theta \, dx \right)^{\frac{1}{\theta}} \tag{3.24}
\]
for some positive constant \( c \) depending only on \( n \) and \( s_\varphi \).

One of the application of this lemma is the following variant regarding the Lipschitz truncation.
Lemma 3.5. Suppose \( \varphi \) satisfies (1.3). For \( v \in W^{1,\varphi^s}_0(B_\rho) \) with \( B_\rho \subset \mathbb{R}^n \) and for any \( \lambda > 0 \), we write
\[
E_\lambda := \{ x \in B_\rho : \{ M(\varphi(|Dv|)^s)(x) \}^{1/s} > \lambda \}.
\]
Then there exist a Lipschitz function \( v_\lambda \in W^{1,\infty}_0(B_\rho) \) and a positive constant \( c \) depending on \( n \) and \( s_\varphi \) such that
\[
v_\lambda(x) = v(x) \quad \text{and} \quad Dv_\lambda(x) = Dv(x) \quad \text{a.e.} \quad x \in B_\rho \setminus E_\lambda,
\]
where the estimate \( \varphi(|Dv_\lambda|)(x) \leq c\lambda \) holds for a.e. \( x \in B_\rho \).

For any function \( v \) in Orlicz-Sobolev spaces, the above lemma provides an Lipschitz function \( v_\lambda \) identical to \( v \) except on the set \( E_\lambda \) which vanishes as \( \lambda \) goes to \( \infty \). This lemma allows us to choose \( v_\lambda \) as a test function for various divergence types of equations, while the set of difference can be controlled by the original function \( v \) and \( \lambda \). This kind of approximation was first introduced in [1].

The case that \( \varphi(t) = t^p \) of Lemma 3.5 was proved in [16]. The proof is based on a Whitney type decomposition. Since the proof in the literature uses Jensen’s inequality and Poincaré type inequality, the lemma still holds for our general \( \varphi \) under the condition (1.3). See also [18]. For the practical applications of the Lipschitz truncation method to very weak solutions, we refer to [3, 26].

We conclude this section with the following technical lemma which will be used later for deriving the required gradient estimate.

Lemma 3.6. [21 Lemma 6.1] Let \( \phi : [\frac{R}{2}, R] \to \mathbb{R}^n \) be a bounded non-negative function. Suppose that for every choice of \( r_1 \) and \( r_2 \) such that \( \frac{R}{2} \leq r_1 < r_2 \leq R \), we have
\[
\phi(r_1) \leq d\phi(r_2) + \frac{A}{(r_2 - r_1)\beta} + B \tag{3.25}
\]
for some positive numbers \( A, B, \beta > 0 \) and \( d \in (0, 1) \). Then,
\[
\phi(r_1) \leq c \left( \frac{A}{R^\beta} + B \right) \tag{3.26}
\]
for some positive constant \( c \) depending on \( d \) and \( \beta \). Especially, the constant \( c \) continuously depends on \( \beta \).

4. Higher Integrability and Solvability

We investigate several properties of a very weak solution to the equation (1.1). For the sake of convenience, we use the letter \( c \) to mean a generic constants depending on \( n, s_\varphi, \nu \) and \( L \), where the exact values might be different from line to line. Similarly, \( c(\varepsilon) \) represent a generic constant depending on \( n, s_\varphi, \nu, L \) and \( \varepsilon \). We first introduce a higher integrability result of very weak solution to the following homogeneous equation.
\[
\text{div} \ A(x, Dw) = 0 \quad \text{in} \quad \Omega. \tag{4.1}
\]
Lemma 4.1. Assume (1.2) and (1.3). Then there exists a small constant \( \sigma = \sigma(n, s_\varphi, \nu, L) \) such that every very weak solution \( w \in W^{1,\varphi^{1-\sigma}}(\Omega) \) to the problem (4.1) belongs to \( W^{1,\varphi^{-1}}(\Omega) \). Moreover, we have the following estimate
\[
\left( \int_{B_r} [\varphi(|Dw|)]^{1+\sigma} \, dx \right)^{1/\sigma} \leq c \left( \int_{B_{2r}} [\varphi(|Dw|)]^{1-\sigma} \, dx \right)^{1/\sigma}
\]  
for some positive constant \( c > 0 \) depending on \( n, s_\varphi, \nu \) and \( L \), whenever \( B_{2r} \subset \Omega \).

**Proof.** Suppose \( w \in W^{1,\varphi^{1-\delta}}(\Omega) \) is a solution of (4.1) for some \( \delta \in (0, \frac{1}{2\theta}] \), with \( \theta \) given in Lemma 3.4. Fix any ball \( B_{2r}(y) \subset B_{2\rho} \subset \Omega \) and choose a cut-off function \( \eta \in C_0^\infty(B_{2r}(y)) \) such that \( |D\eta| \leq \frac{1}{\rho} \), \( 0 \leq \eta \leq 1 \) and \( \eta \equiv 1 \) on \( B_r(y) \). Write
\[
v := (w - \overline{w})_B \eta \in W^{1,\varphi^{1-\delta}}(B_{2r}(y)).
\]

Then we apply Lemma 3.3 to \( v \) to find that for any \( \lambda > 0 \) one has a Lipschitz function \( v_\lambda \in W^{1,\varphi^{1-\delta}}_0(B_{2r}(y)) \) such that \( v_\lambda = v \) and \( Dv_\lambda = Dv \) a.e. on \( B_{2r}(y) \setminus E_\lambda \) and that \( \varphi(|Dv_\lambda|) \leq c \lambda \) for a.e. on \( B_{2r}(y) \) for some positive constant depending on \( n \) and \( s_\varphi \), where
\[
E_\lambda := \{ x \in B_{2r}(y) \setminus \Omega : \{ M(|\varphi(|Dv_\lambda|)|) \}^{\frac{1}{\varphi}} > \lambda \}.
\]

Since \( C_0^\infty(B_{2\rho}) \) is weak-* dense in \((L^1(B_{2\rho}))^*\), we can take \( v_\lambda \) as a test function to the equation (4.1). Recalling (1.2), we have
\[
\int_{B_{2r}(y) \setminus E_\lambda} \langle A(x, Dw), Dv_\lambda \rangle \, dx = -\int_{E_\lambda} \langle A(x, Dw), Dv_\lambda \rangle \, dx
\]
\[
\leq c \varphi^{-1}(\lambda) \int_{E_\lambda} \varphi'(|Dw|) \, dx.
\]  
Multiplying both sides of (4.3) by \( \lambda^{-1+\delta} \) and integrating from 0 to \( \infty \) with respect to \( \lambda \), we obtain
\[
I_1 := \int_0^\infty \lambda^{-(1+\delta)} \left( \int_{B_{2r}(y) \setminus E_\lambda} \langle A(x, Dw), Dv_\lambda \rangle \, dx \right) \, d\lambda
\]
\[
\leq c \int_0^\infty \lambda^{-(1+\delta)} \varphi^{-1}(\lambda) \int_{E_\lambda} \varphi'(|Dw|) \, dx \, d\lambda
\]
\[
\leq c \int_{B_{2r}(y)} \left[ \int_0^g(x) \lambda^{-(1+\delta)} \varphi^{-1}(\lambda) \varphi'(|Dw|) \, d\lambda \right] \, dx
\]
\[
= c \int_{B_{2r}(y)} \left[ \int_0^{\varphi^{-1}(g(x))} [\varphi(s)]^{-(1+\delta)} s \varphi'(s) \varphi'(|Dw|) \, ds \right] \, dx
\]
\[
\leq c \int_{B_{2r}(y)} \left[ \int_0^{\varphi^{-1}(g(x))} [\varphi(s)]^{-\delta} \varphi'(|Dw|) \, ds \right] \, dx
\]
\[
\leq c \int_{B_{2r}(y)} \varphi^{-1}(g(x)) [g(x)]^{-\delta} \varphi'(|Dw|) \, dx,
\]  
where we have defined
\[
g(x) := \left\{ M(|\varphi(|Dv|)|) \right\}^{\frac{1}{\varphi}} (x \in B_{2r}(y)).
\]  

(4.5)
Applying Lemma 3.3 to (4.4) with \( t = \varphi^{-1}(g(x)) \) and \( s = |Dw| \), we get

\[
I_1 \leq c \left( \int_{B_{2r}(y)} [g(x)]^{1-\delta} \, dx + \int_{B_{2r}(y)} [\varphi(|Dw|)]^{1-\delta} \, dx \right)
\]

\[
\leq c \left( \int_{B_{2r}(y)} [\varphi(|Dv|)]^{1-\delta} \, dx + \int_{B_{2r}(y)} [\varphi(|Dw|)]^{1-\delta} \, dx \right)
\]

\[
\leq c \left( \int_{B_{2r}(y)} \varphi \left( \frac{|w - \overline{w}_{B_{2r}(y)}|}{r} \right) \right)^{1-\delta} \, dx + \int_{B_{2r}(y)} [\varphi(|Dw|)]^{1-\delta} \, dx
\]

\[
\leq c \int_{B_{2r}(y)} [\varphi(|Dw|)]^{1-\delta} \, dx.
\] (4.6)

Next, we estimate \( I_1 \) from below. We split \( B_{2r}(y) \setminus B_{r}(y) \) as

\[
D_1 = \{ x \in B_{2r}(y) \setminus B_{r}(y) : M([\varphi(|Dv|)]^\theta)(x) \leq \delta^\theta M([\varphi(|Dw|)]^\theta \chi_{B_{2r}(y)})(x) \}
\]

and

\[
D_2 = \{ x \in B_{2r}(y) \setminus B_{r}(y) : M([\varphi(|Dv|)]^\theta)(x) > \delta^\theta M([\varphi(|Dw|)]^\theta \chi_{B_{2r}(y)})(x) \}.
\]

Since \( Dv_\lambda = Dv \) a.e. on \( B_{2r}(y) \setminus E_\lambda \) and \( v = w - \overline{w}_{B_{2r}(y)} \) a.e. on \( B_{r}(y) \), we have

\[
I_1 = \int_{B_{2r}(y)} \int_{0}^{\infty} \lambda^{-\gamma} \langle A(x, Dw), Dv \rangle \, d\lambda \, dx
\]

\[
= \frac{1}{\delta} \int_{B_{2r}(y)} [g(x)]^{-\delta} \langle A(x, Dw), Dv \rangle \, dx
\]

\[
\geq \frac{\nu}{\delta} \int_{B_{r}(y)} g^{-\delta} \varphi''(|Dw|)|Dw|^2 \, dx + \frac{1}{\delta} \int_{D_1} g^{-\delta} \langle A(x, Dw), Dv \rangle \, dx
\]

\[
+ \frac{1}{\delta} \int_{D_2} g^{-\delta} \langle A(x, Dw), wD\eta \rangle \, dx
\]

\[
\geq \frac{1}{\delta} \left( \nu \int_{B_{r}(y)} g^{-\delta} \varphi''(|Dw|)|Dw|^2 \, dx - L \int_{D_1} g^{-\delta} \varphi'(|Dw|)|Dw| \, dx
\]

\[
- 2L \int_{D_2} g^{-\delta} \varphi'(|Dw|) \frac{|w - \overline{w}_{B_{2r}(y)}|}{r} \, dx \right)
\]

\[
=: \frac{1}{\delta} (I_2 - I_3 - I_4).
\] (4.7)

We first estimate \( I_2 \) from below. According to Lemma 3.1, \( g^{-\delta} \in A_{1/\theta} \). Then from the boundedness of maximal function(Lemma 3.1), we conclude that

\[
I_2 = \nu \int_{B_{r}(y)} g^{-\delta} \varphi''(|Dw|)|Dw|^2 \, dx \geq c \int_{B_{r}(y)} g^{-\delta} \varphi(|Dw|) \, dx
\]

\[
\geq c \int_{B_{r}(y)} g^{-\delta} \left\{ M([\varphi(|Dw|)]^\theta \chi_{B_{r}(y)}) \right\}^{\frac{1}{\theta}} \, dx.
\] (4.8)

We now find a pointwise upper bound of \( g \). If \( x \in B_{r/2}(y) \), then
where we have used Lemma 3.4 for the last inequality. Comparing the last two terms in (4.9), we have

\[
\{\mathcal{M}([\varphi(|Dw|)]^\theta \chi_{B_r(y)})(x)\}^{1-\delta} \leq 2[g(x)]^{-\delta} \{\mathcal{M}([\varphi(|Dw|)]^\theta \chi_{B_r(y)})(x)\}^{\frac{\delta}{p}} + \left[ c \left( \int_{B_{2r}(y)} [\varphi(|Dw|)]^{\theta} dz \right) \right]^{1-\delta},
\]

(4.10)

for \( x \in B_{r/2}(y) \). This estimate leads to

\[
I_2 \geq c \int_{B_{r/2}(y)} [g(x)]^{-\delta} \{\mathcal{M}([\varphi(|Dw|)]^\theta \chi_{B_r(y)})\}^{\frac{\delta}{p}} dx
\]

\[
\geq c \int_{B_{r/2}(y)} \{\mathcal{M}([\varphi(|Dw|)]^\theta \chi_{B_r(y)})\}^{\frac{\delta}{p}} dx - cr^n \left( \int_{B_{2r}(y)} [\varphi(|Dw|)]^{\theta} dz \right)^{\frac{1-\delta}{\theta}}
\]

\[
\geq c \int_{B_{r/2}(y)} [\varphi(|Dw|)]^{1-\delta} dx - cr^n \left( \int_{B_{2r}(y)} [\varphi(|Dw|)]^{\theta} dx \right)^{\frac{1-\delta}{\theta}}.
\]

(4.11)

Using the definition of \( D_1 \) and Lemma 3.1, \( I_3 \) can be estimated as follows:

\[
I_3 = L \int_{D_1} [g(x)]^{-\delta} \varphi'(|Dw|)|Dw| \, dx
\]

\[
\leq L \int_{B_{2r}(y)} \{\mathcal{M}([\varphi(|Dw|)]^\theta)\}^{-\frac{\delta}{p}} \varphi^{-1} \left( \{\mathcal{M}([\varphi(|Dw|)]^\theta)\}^{\frac{\delta}{p}} \right) |Dw| \, dx
\]

\[
\leq \varepsilon \int_{B_{2r}(y)} [\varphi(|Dw|)]^{1-\delta} dx + c(\varepsilon) \int_{B_{2r}(y)} \{\mathcal{M}([\varphi(|Dw|)]^\theta)\}^{\frac{1-\delta}{\theta}} \, dx
\]

\[
\leq \varepsilon \int_{B_{2r}(y)} [\varphi(|Dw|)]^{1-\delta} dx + c(\varepsilon) \delta^{1-\delta} \int_{B_{2r}(y)} \{\mathcal{M}([\varphi(|Dw|)\chi_{B_{2r}(y)}])^\theta\}^{\frac{1-\delta}{\theta}} \, dx
\]

\[
\leq (\varepsilon + c(\varepsilon)\delta^{1-\delta}) \int_{B_{2r}(y)} [\varphi(|Dw|)]^{1-\delta} dx.
\]

(4.12)

Similarly, we estimate \( I_4 \) using the definition of \( D_2 \). Here we write

\[
h(x) := \{\mathcal{M}([\varphi(|Dw|)\chi_{B_{2r}(y)}])^\theta)(x)\}^{\frac{\delta}{p}} \quad (x \in B_{2r}(y))
\]

(4.13)
and observe that $\varphi(|Dw(x)|) \leq h(x)$ for $x \in B_{2r}(y)$. Then we find that

$$I_4 = 2L \int_{D_2} [g(x)]^{-\delta} \varphi'(|Dw|) \left| \frac{w-w_{B_{2r}(y)}}{r} \right| dx$$

$$= 2L \int_{B_{2r}(y)} \left\{ M\left(|\varphi(|Dv|)|^\theta \right) \right\}^{-\frac{\delta}{\theta}} \varphi'(|Dw|) \left| \frac{w-w_{B_{2r}(y)}}{r} \right| dx$$

$$\leq 2\delta^{-\delta} L \int_{B_{2r}(y)} |h(x)|^{-\delta} \varphi'\left(\varphi^{-1}(h(x))\right) \left| \frac{w-w_{B_{2r}(y)}}{r} \right| dx$$

$$\leq \varepsilon \int_{B_{2r}(y)} |h(x)|^{1-\delta} dx + c(\varepsilon) \int_{B_{2r}(y)} \left[ \varphi\left(\frac{w-w_{B_{2r}(y)}}{r}\right)\right]^{1-\delta} dx$$

By (4.13), (4.16), (4.17) and (4.18), we have

$$\int_{B_{r/2}(y)} |\varphi(|Dw|)|^{1-\delta} dx \leq c(\varepsilon) r^n \left( \int_{B_{2r}(y)} |\varphi(|Dw|)|^\theta dx \right)^{\frac{1-\delta}{\theta}} + c(\delta + 2\varepsilon + c(\varepsilon)\delta^{1-\delta}) \int_{B_{2r}(y)} |\varphi(|Dw|)|^{1-\delta} dx.$$

Therefore, taking $\varepsilon = \frac{1}{10}$ and $\delta < \delta_1 := \min\left\{ \frac{1}{8c}, \frac{1}{3c(\varepsilon)^2}, \frac{1-\theta}{2} \right\}$, we get

$$\int_{B_{r/2}(y)} |\varphi(|Dw|)|^{1-\delta} dx \leq c \left( \int_{B_{2r}(y)} |\varphi(|Dw|)|^\theta dx \right)^{\frac{1-\delta}{\theta}} + \frac{1}{2} \int_{B_{2r}(y)} |\varphi(|Dw|)|^{1-\delta} dx.$$

Since the constant $c$ in the above expression does not depend on $\delta$, using the standard Gehring’s argument, we get the desired conclusion. \qed

We next establish an a priori estimate to the following Dirichlet problem

$$\begin{cases} 
\text{div} A(x, Dw) = \text{div} \left( \frac{\varphi(|Dw|)}{|Dw|} \right) & \text{in } B_{\rho} \\
w \in w_0 + W^{1, p}_{0, \alpha, \lambda} (B_{\rho}) & \text{for the purpose of proving an existence result.}
\end{cases}$$

Lemma 4.2. Assume (1.2) and (1.3). Then there exists a small constant $\delta_1 = \delta_1(n, s_\varphi, \nu, L)$ such that the following holds: For any $\delta \in (0, \delta_1)$, if $w \in W^{1, \varphi^{1-\delta}}(B_{\rho})$ is a weak solution to (1.17) with $w_0 \in W^{1, \varphi^{1-\delta}}(B_{\rho})$ and $f \in L^{\varphi^{1-\delta}}(B_{\rho})$, then there holds

$$\int_{B_{\rho}} [\varphi(|Dw|)]^{1-\delta} dx \leq c \left[ \int_{B_{\rho}} [\varphi(|Dw_0|)]^{1-\delta} dx + \int_{B_{\rho}} [\varphi(|f|)]^{1-\delta} dx \right],$$

where $c > 0$ depends on $n, s_\varphi, \nu, L$. 

Proof. Suppose \( \delta \in (0, \frac{1}{n s_{\varphi}}) \) and define \( v := w - w_0 \in W^{1, \varphi^{1-\delta}}_0(B_\rho) \). Then we apply Lemma \ref{lemma:3.5} to get a Lipschitz truncation \( v_\lambda \in W^{1, \infty}_0(B_\rho) \) such that \( v_\lambda = v, Dv_\lambda = Dv \) a.e. on \( B_\rho \setminus E_\lambda \), where

\[
E_\lambda := \{ x \in B_\rho : \{ \mathcal{M}(\varphi(|Dv|)^\theta) \}^\frac{1}{\theta} > \lambda \}
\]

and we have that \( \varphi(|Dv_\lambda|) \leq c \lambda \) on \( B_\rho \) for some positive constant depending on \( n \) and \( s_{\varphi} \). Taking \( v_\lambda \) as a test function in the equation (4.17), we have

\[
\int_{B_\rho \setminus E_\lambda} \langle A(x, Dw), Dv_\lambda \rangle \, dx = -\int_{E_\lambda} \langle A(x, Dw), Dv_\lambda \rangle \, dx \]
\[
\quad + \int_{B_\rho \setminus E_\lambda} \frac{\varphi'(|f|)}{|f|} f \, Dv_\lambda \, dx + \int_{E_\lambda} \frac{\varphi'(|f|)}{|f|} f \, Dv_\lambda \, dx \]
\[
\leq c \left( \varphi^{-1}(\lambda) \int_{E_\lambda} \varphi'(|Dw|) \, dx \right.
\quad \left. + \int_{B_\rho \setminus E_\lambda} \varphi'(|f|)|Dv| \, dx + \varphi^{-1}(\lambda) \int_{E_\lambda} \varphi'(|f|) \, dx \right). \quad (4.19)
\]

Multiplying (4.19) by \( \lambda^{-(1+\delta)} \) and integrating from 0 to \( \infty \) with respect to \( \lambda \), we obtain

\[
I_1 := \int_0^\infty \lambda^{-(1+\delta)} \int_{B_\rho \setminus E_\lambda} \langle A(x, Dw), Dv_\lambda \rangle \, dx \, d\lambda
\]
\[
\leq c_* \left( \int_{B_\rho} \int_0^{g(x)} \lambda^{-(1+\delta)} \varphi^{-1}(\lambda) \varphi'(|Dw|) \, d\lambda \right) \, dx
\]
\[
\quad + \int_{B_\rho} \int_0^\infty \lambda^{-(1+\delta)} \varphi'(|f|)|Dv| \, d\lambda \, dx
\]
\[
\quad + \int_{B_\rho} \int_0^\pi \lambda^{-(1+\delta)} \varphi^{-1}(\lambda) \varphi'(|f|) \, d\lambda \, dx
\]
\[
\leq c_*(I_2 + I_3 + I_4), \quad (4.20)
\]

for some positive constant \( c_* \) depending on \( n, s_{\varphi} \) and \( L \), where we have defined

\[
g(x) := \{ \mathcal{M}(\varphi(|Dv|)^\theta) \}^\frac{1}{\theta} \quad (x \in B_\rho). \quad (4.21)
\]

Applying Lemma \ref{lemma:3.3} \( I_2 \) can be estimated as

\[
I_2 = \int_{B_\rho} \varphi^{-1}(g(x))[g(x)]^{-\delta} \varphi'(|Dw|) \, dx
\]
\[
\leq c \left( \int_{B_\rho} [g(x)]^{1-\delta} \, dx + \int_{B_\rho} [\varphi(|Dw|)]^{1-\delta} \, dx \right)
\]
\[
\leq c \left( \int_{B_\rho} [\varphi(|Dv|)]^{1-\delta} \, dx + \int_{B_\rho} [\varphi(|Dw|)]^{1-\delta} \, dx \right)
\]
\[
\leq c \left( \int_{B_\rho} [\varphi(|Dw|)]^{1-\delta} \, dx + \int_{B_\rho} [\varphi(|Dw_0|)]^{1-\delta} \, dx \right). \quad (4.22)
\]
Similarly, we estimate $I_3$ as follows:

$$I_3 = \frac{1}{\delta} \int_{B_p} \frac{\lambda}{g(x)^{\gamma}} \phi(x) |Dv| \, dx \leq \frac{C}{\delta} \int_{B_p} \phi^{-1}(g(x)) |g(x)|^{-\delta} \phi(x) \, dx$$

$$\leq \frac{c(\varepsilon)}{\delta} \int_{B_p} \lambda^{-1+\delta} \phi^{-1}(\lambda) \phi(x) \, dx$$

In a similar manner as we did for the estimate (4.4), we have

$$I_4 = \int_{B_p} \left[ \int_0^\infty \lambda^{-1+\delta} \phi^{-1}(\lambda) \phi(x) \, d\lambda \right] \, dx$$

Now we estimate $I_1$ from below. Since $Dv_\lambda = Dv$ on $B_p \setminus E_\lambda$, we have

$$I_1 = \int_{B_p} \left[ \frac{\lambda}{g(x)^{\gamma}} \phi(x) \phi'(|Dv|) \, dx \right]$$

$$= \frac{1}{\delta} \int_{B_p} \frac{\lambda}{g(x)^{\gamma}} \phi(x) \phi'(|Dv|) \, dx$$

To estimate $I_5$, we use Lemma 3.1 again to conclude that $g(x)^{-\delta} \in A_{1/\theta}$, and so

$$I_5 = \nu \int_{B_p} g^{-\delta} |V(Dv) - V(Dw_0)|^2 \, dx$$

$$\geq c \int_{B_p} g^{-\delta} \{M(|V(Dv) - V(Dw_0)|^2)\}^{\frac{1}{2}} \, dx.$$
Also, we observe that for $x \in B_\rho$,
\[
g(x) = \left\{ \mathcal{M}(|\varphi(Dv)|^q)(x) \right\}^{\frac{1}{q}} \leq c \left\{ \left( \mathcal{M}(V(Dw) - V(Dw_0)^{2q})(x) \right) \right\}^{\frac{1}{q}} + \left\{ \mathcal{M}(|\varphi(Dw_0)|^q)(x) \right\}^{\frac{1}{q}}. \tag{4.26}
\]
Comparing the last two terms in (4.26), we have
\[
\left\{ \mathcal{M}(V(Dw) - V(Dw_0)^{2q}) \right\}^{\frac{1}{1 - \delta}} \leq c g^{-\delta} \left\{ \left( \mathcal{M}(V(Dw) - V(Dw_0)^{2q}) \right) \right\}^{\frac{1}{q}} + c \left\{ \mathcal{M}(|\varphi(Dw_0)|^q)(x) \right\}^{\frac{1}{q}} \tag{4.27}
\]
on $B_\rho$. Therefore, combining (4.25) and (4.27), we have
\[
I_5 \geq c \int_{B_\rho} \left\{ \mathcal{M}(V(Dw) - V(Dw_0)^{2q}) \right\}^{\frac{1}{1 - \delta}} dx - c \int_{B_\rho} \left\{ \mathcal{M}(|\varphi(Dw_0)|^q) \right\}^{\frac{1}{q}} dx
\geq c \int_{B_\rho} |V(Dw) - V(Dw_0)|^{2 - 2\delta} dx - c \int_{B_\rho} |\varphi(|Dw_0|)|^{1 - \delta} dx
\geq c \int_{B_\rho} |\varphi(|Dw_0|)|^{1 - \delta} dx - c \int_{B_\rho} |\varphi(|Dw_0|)|^{1 - \delta} dx. \tag{4.28}
\]
Applying Lemma 3.3 we can estimates $I_6$ as follows.
\[
I_6 = L \int_{B_\rho} [g(x)]^{-\delta} \varphi'(|Dw_0|) |Dv| dx
\leq L \int_{B_\rho} [g(x)]^{-\delta} \varphi'(|Dw_0|) \varphi^{-1}(g(x)) dx
\leq \varepsilon \int_{B_\rho} [g(x)]^{1 - \delta} dx + c(\varepsilon) \int_{B_\rho} |\varphi(|Dw_0|)|^{1 - \delta} dx
\leq c(\varepsilon) \int_{B_\rho} |\varphi(|Dv|)|^{1 - \delta} dx + c(\varepsilon) \int_{B_\rho} |\varphi(|Dw_0|)|^{1 - \delta} dx
\leq c(\varepsilon) \int_{B_\rho} |\varphi(|Dw_0|)|^{1 - \delta} dx + c(\varepsilon) \int_{B_\rho} |\varphi(|Dw_0|)|^{1 - \delta} dx. \tag{4.29}
\]
We finally combine (4.22), (4.28) and (4.29) to conclude
\[
\int_{B_\rho} |\varphi(|Dw|)|^{1 - \delta} dx \leq c_\ast \{ \varepsilon + \delta \} \int_{B_\rho} |\varphi(|Dw|)|^{1 - \delta} dx
+ c_\ast \{ \varepsilon + \delta \} \int_{B_\rho} |\varphi(|Dw_0|)|^{1 - \delta} dx \tag{4.30}
\]
for some constant $c_\ast > 0$ depends on $n, s, \nu, L$. Taking $\varepsilon = \delta_1 = \frac{1}{4c_\ast}$, we get the desired estimate (4.18). \hfill \Box

Since the obtained $W^{1, \varphi^{1 - \delta}}(B_\rho)$-estimate only works for a very weak solution in $W^{1, \varphi^{1 - \delta}}(B_\rho)$, we need to do more works in order to get the desired gradient estimate $W^{2, \varphi^{2 \delta}}(B_\rho)$ with some higher exponent $q \geq 1 - \delta$. We then prove an existence result within $W^{1, \varphi^{1 - \delta}}(B_\rho)$ under the condition $\delta \in (0, \delta_2]$, where $\delta_2$ is defined as $\delta_2 := \min\{\sigma, \delta_1\}$ so that we can use Lemma 4.1 and Lemma 4.2. See also [23, Section 7].
Corollary 4.3. Assume (1.2) and (1.3). Then for all $w_0 \in W^{1,\varphi^{1-\delta}}(B_\rho)$ with $\delta \in (0, \delta_2]$, there exists a very weak solution $w \in W^{1,\varphi^{1-\delta}}(B_\rho)$ to the problem
\[
\begin{cases}
\text{div } A(x, Dw) = 0 & \text{in } B_\rho \\
w \in w_0 + W^{1,\varphi^{1-\delta}}_0(B_\rho),
\end{cases}
\]
such that we have the estimates
\[
\int_{B_\rho} [\varphi(|Dw|)]^{1-\delta} dx \leq c \int_{B_\rho} [\varphi(|Dw_0|)]^{1-\delta} dx
\]
where the positive constant $c$ depends on $n, s, \nu$ and $L$.

Proof. Let $\delta \in (0, \delta_2]$ and take a sequence of functions $w_{0,k} \in C^\infty(\overline{B_\rho})$ such that
\[
w_{0,k} \to w_0 \text{ strongly in } W^{1,\varphi^{1-\delta}}(B_\rho).
\]
Then, there exists a unique weak solution $w_k \in W^{1,\varphi}(B_\rho)$ to the boundary value problem
\[
\begin{cases}
\text{div } A(x, Dw_k) = 0 & \text{in } B_\rho \\
w_k \in w_{0,k} + W^{1,\varphi}_0(B_\rho).
\end{cases}
\]
The solvability follows from [33, Section 2]. According to Lemma 4.2, we get a uniform bound
\[
\int_{B_\rho} [\varphi(|Dw_k|)]^{1-\delta} dx \leq c \int_{B_\rho} [\varphi(|Dw_{0,k}|)]^{1-\delta} dx \leq c \int_{B_\rho} [\varphi(|Dw_0|)]^{1-\delta} dx,
\]
for sufficiently large $k$. Combining (4.35) with Lemma 3.4, we conclude that the norm $\|w_k\|_{W^{1,\varphi^{1-\delta}}(B_\rho)}$ is uniformly bounded. Thus, passing to a subsequence, we have a function $w \in W^{1,\varphi^{1-\delta}}(B_\rho)$ such that
\[
w_k \to w \text{ weakly in } W^{1,\varphi^{1-\delta}}(B_\rho), \quad w_k \to w \text{ strongly in } L^{\varphi^{1-\delta}}(B_\rho).
\]
Similarly, we consider a sequence $\tilde{w}_k := w_k - w_{0,k}$ and take $k \to \infty$ to observe that $w \in w_0 + W^{1,\varphi^{1-\delta}}_0(B_\rho)$.

Now, fix any test function $\phi \in C^\infty_0(B_\rho)$ and choose an open set $U$ such that $\text{supp } \phi \subset U \subset B_\rho$. Then in light of Lemma 4.1 we have a uniform bound of $\|w_k\|_{W^{1,\varphi}(U)}$. Passing to a subsequence, we find that
\[
w_k \to w \text{ weakly in } W^{1,\varphi}(U), \quad w_k \to w \text{ strongly in } L^\varphi(U).
\]
We recall the Minty-Browder technique to conclude that $w$ is a weak solution to $\text{div } A(x, Dw) = 0$ in $U$, which means that
\[
0 = \int_U \langle A(x, Dw), D\phi \rangle dx = \int_{B_\rho} \langle A(x, Dw), D\phi \rangle dx.
\]
Since $\phi$ is arbitrary, $w$ becomes a very weak solution of (4.31). □
5. Proof of Theorem 2.1

This section is devoted to proving the main result, Theorem 2.1. To do this, we need to compare a solution \( u \) to (1.1) under consideration with a solution \( w \) to the following homogeneous problem

\[
\begin{aligned}
\begin{cases}
\text{div} A(x,Dw) = 0 & \text{in } B_{2r}, \\
w \in u + W^{1,\varphi^{1-\delta_0}}_0(B_{2r}),
\end{cases}
\end{aligned}
\]  

(5.1)

where \( \delta_0 > 0 \) is to be selected later. We further assume that

\[
\int_{B_{2r}} [\varphi(|Du|)]^{1-\delta_0} \, dx \leq \Lambda, \quad \int_{B_{2r}} |\varphi(|f|)|^{1-\delta_0} \, dx \leq \delta_0 \Lambda
\]  

(5.2)

for some \( \Lambda > 0 \). These assumptions will be made during an exit time argument below. We will frequently use the universal constants \( \sigma, \delta_1 \) and \( \delta_2 \) given in the previous section.

Lemma 5.1. Assume 1.2 and 1.3. Then for any \( \varepsilon \in (0,1] \), there exists a constant \( \delta_0 = \delta_0(n,s_{\varphi},\nu,L,\varepsilon) \in (0,\frac{\sigma}{2}) \) such that the following holds: For any very weak solution \( u \in W^{1,\varphi^{1-\delta_0}}_0(\Omega) \) to (1.1) with \( f \in W^{1,\varphi^{1-\delta_0}}_0(\Omega) \), if (5.2) holds, then there exists a very weak solution \( w \in W^{1,\varphi^{1-\delta_0}}_0(B_{2r}) \) to the equation (5.1) such that

\[
\int_{B_{2r}} |V(Du) - V(Dw)|^{2-2\delta_0} \, dx \leq \varepsilon \Lambda
\]  

(5.3)

with the estimate

\[
\int_{B_{2r}} [\varphi(|Dw|)]^{1-\delta_0} \, dx \leq c \Lambda, \quad \left( \int_{B_{r}} [\varphi(|Dw|)]^{1+\sigma} \, dx \right)^{\frac{1-\delta_0}{1+\sigma}} \leq c \Lambda,
\]  

(5.4)

where the constant \( c > 0 \) depends only on \( n, s_{\varphi}, \nu \) and \( L \).

Proof. Let \( 0 < \delta_0 \leq \frac{\sigma}{2} \). Then according to Lemma 4.1 and Corollary 4.3, there exists a very weak solution \( w \in W^{1,\varphi^{1-\delta_0}}_0(B_{2r}) \) to (6.1) such that the estimates (5.4) holds, since we have

\[
\left( \int_{B_{r}} [\varphi(|Dw|)]^{1+\sigma} \, dx \right)^{\frac{1-\delta_0}{1+\sigma}} \leq c^{1-\delta_0} \left( \int_{B_{2r}} [\varphi(|Dw|)]^{1-\sigma} \, dx \right)^{\frac{1-\delta_0}{1-\sigma}} \leq c \int_{B_{2r}} [\varphi(|Dw|)]^{1-\delta_0} \, dx \leq c \Lambda.
\]  

(5.5)

We next write \( v := u - w \in W^{1,\varphi^{1-\delta_0}}_0(B_{2r}) \) and take a Lipschitz truncation \( v_\lambda \in W^{1,\varphi^{1-\delta_0}}_0(B_{2r}) \) with \( \lambda > 0 \) in light of Lemma 3.5. We have \( v_\lambda = v, \, Dv_\lambda = Dv \) a.e. on \( B_{2r} \setminus E_\lambda \) for

\[
E_\lambda := \{ x \in B_{2r} : \{ 2^{\lambda}(|Dv_\lambda|)^{\theta}(x) \}^{\frac{1}{\theta}} > \lambda \},
\]

and \( \varphi(|Dv_\lambda|) \leq c \lambda \) for \( x \in B_{2r} \). Taking \( v_\lambda \) as a test function to both (1.1) and (5.1), we get
\[
\int_{B_{2r}\setminus E_\lambda} \langle A(x, Du) - A(x, Dw), Dv_\lambda \rangle \, dx \\
= - \int_{E_\lambda} \langle A(x, Du) - A(x, Dw), Dv_\lambda \rangle \, dx + \int_{B_{2r}} \langle \varphi'(|f|), Dv_\lambda \rangle \, dx \\
\leq c \left( \varphi^{-1}(\lambda) \int_{E_\lambda} \varphi'(|Du|) \, dx + \varphi^{-1}(\lambda) \int_{E_\lambda} \varphi'(|Dw|) \, dx \right. \\
\left. + \int_{B_{2r}\setminus E_\lambda} \varphi'(|f|)|Dv| \, dx + \varphi^{-1}(\lambda) \int_{E_\lambda} \varphi'(|f|) \, dx \right). \quad (5.6)
\]

Multiplying (5.6) by \( \lambda^{-(1+\delta_0)} \) and integrating from 0 to \( \infty \) with respect to \( \lambda \), along with the similar calculations as in (4.19), we discover that

\[
\delta_0 I_1 := \delta_0 \int_0^\infty \lambda^{-(1+\delta_0)} \int_{B_{2r}\setminus E_\lambda} \langle A(x, Du) - A(x, Dw), Dv_\lambda \rangle \, dx \, d\lambda \\
\leq c \{ \varepsilon_1 + \delta_0 \} \left[ \int_{B_{2r}} \varphi(|Du|)^{1-\delta_0} \, dx + \int_{B_{2r}} \varphi(|Dw|)^{1-\delta_0} \, dx \right] \\
+ c \{ \varepsilon_1 + \delta_0 \} \int_{B_{2r}} \varphi(|f|)^{1-\delta_0} \, dx \quad (5.7)
\]

We first find a lower bound of \( I_1 \). Defining \( g(x) \) as

\[
g(x) := \left\{ M([\varphi(|Du|)]^\theta) \right\}^\frac{1}{\theta} \quad (x \in B_{2r}), \quad (5.8)
\]

we again have \( [g(x)]^{-\delta_0} \in A_{1/\theta} \) by Lemma [3.1]. Then it follows that

\[
I_1 = \int_{B_{2r}} \int_{g(x)}^\infty \lambda^{-(1+\delta_0)} \langle A(x, Du) - A(x, Dw), Dv \rangle \, d\lambda \, dx \\
= \frac{1}{\delta_0} \int_{B_{2r}} [g(x)]^{-\delta_0} \langle A(x, Du) - A(x, Dw), Dv \rangle \, dx \\
\geq \frac{\nu}{\delta_0} \int_{B_{2r}} g^{-\delta_0} |V(Du) - V(Dw)|^2 \, dx \\
\geq \frac{c}{\delta_0} \int_{B_{2r}} g^{-\delta_0} \left\{ M(|V(D_u) - V(Dw)|^2 \right\}^\frac{1}{\theta} \, dx. \quad (5.9)
\]

We now observe from (2.5) that for any \( x \in B_{2r} \) and for any \( \tilde{\varepsilon} \in (0, 1) \),

\[
g(x) = \left\{ M([\varphi(|Du|)]^\theta) \right\}^\frac{1}{\theta} \\
\leq \left\{ M([c(\tilde{\varepsilon})|V(Du) - V(Dw)|^2 + \tilde{\varepsilon}^\theta |\varphi(|Du|)]^\theta) \right\}^\frac{1}{\theta} \\
\leq c(\tilde{\varepsilon}) \left\{ M(|V(D_u) - V(Dw)|^2 \right\}^\frac{1}{\theta} + c\tilde{\varepsilon} \left\{ M(|\varphi(|Du|)]^\theta(x) \right\}^\frac{1}{\theta}. \quad (5.10)
\]

Consequently, it follows that

\[
\left\{ M(|V(D_u) - V(Dw)|^2 \right\}^{1-\delta_0} \leq \frac{c(\varepsilon_2)g^{-\delta_0} \left\{ M(|V(D_u) - V(Dw)|^2 \right\}^\frac{1}{\theta} + \varepsilon_2 \left\{ M(|\varphi(|Du|)]^\theta(x) \right\}^{1-\delta_0}}{\theta} \quad (5.11)
\]
Applying Lebesgue differentiation theorem, we get

\[ y \text{ for a.e.} \]

for any \( \varepsilon_2 \in (0, 1] \). Recalling (5.9), we have

\[
I_1 \geq \frac{1}{c(\varepsilon_2)} \int_{B_{2r}} \left\{ \mathcal{M}(|V(Du) - V(Dw)|^{2\delta_0}) \right\}^{\frac{1-\delta}{2}} dx
-
\frac{\varepsilon_2}{c(\varepsilon_2)} \int_{B_{2r}} \left\{ \mathcal{M}(|\varphi(|Du|)|^{\delta_0}) \right\}^{\frac{1-\delta_0}{2}} dx
\geq \frac{1}{c(\varepsilon_2)} \int_{B_{2r}} |V(Du) - V(Dw)|^{2-2\delta_0} dx - \frac{\varepsilon_2}{c(\varepsilon_2)} \int_{B_{2r}} |\varphi(|Du|)|^{1-\delta_0} dx
\geq \frac{1}{c(\varepsilon_2)} \int_{B_{2r}} |V(Du) - V(Dw)|^{2-2\delta_0} dx - \frac{\varepsilon_2}{c(\varepsilon_2)} \Lambda.
\] (5.12)

Combining (5.2), (5.9) and (5.12), we conclude that

\[
\int_{B_{2r}} |V(Du) - V(Dw)|^{2-2\delta_0} dx \leq (\varepsilon_2 + c(\varepsilon_2)\varepsilon_1 + c(\varepsilon_2)\varepsilon_2\delta_0)\Lambda \leq 3\varepsilon\Lambda,
\]

by taking \( \varepsilon_2 = \varepsilon \), \( \varepsilon_1 = \frac{\varepsilon}{c(\varepsilon_2)} \) and then \( \delta_0 := \min \left\{ \frac{\varepsilon}{c(\varepsilon_2)\varepsilon_2}, \delta_2 \right\} \) for the last inequality. This completes the proof. \( \square \)

We are in a position to prove the main result of the paper.

**Proof of Theorem 2.1.** Our proof is based on the harmonic analysis-free technique which was first introduced in [2]. We are under the same assumption as in Theorem 2.1. Notice that if we choose \( \varepsilon \) in Lemma 5.1 depending only on \( n, s, \rho \) and \( L \), accordingly \( \delta_0 \) is to be selected depending only on \( n, s, \nu \) and \( L \). First we fix any ball \( B_R(z) \subset \mathbb{R}^n \) and write upper level sets as

\[
E^\Lambda_A := \{ x \in B_s(z) : |\varphi(|Du|)|^{1-\delta_0} > \Lambda \} \quad (\Lambda > 0),
\]

for \( \frac{R}{2} \leq s \leq R \). Consider the concentric balls \( B_r(z) \) and \( B_{\rho r}(z) \) with \( \frac{2R}{3} \leq r_1 < r_2 \leq R \). Then for each \( y \in E^\Lambda_A \), we define a continuous function \( \Phi_y : (0, r_2 - r_1] \rightarrow [0, \infty) \) by

\[
\Phi_y(\rho) := \int_{B_{\rho r}(y)} \left\{ |\varphi(|Du|)|^{1-\delta_0} + \frac{|\varphi(Du)|^{1-\delta_0}}{\delta_0} \right\} dx.
\] (5.13)

Applying Lebesgue differentiation theorem, we get

\[
\lim_{\rho \to 0} \Phi_y(\rho) = |\varphi(|Du|)|^{1-\delta_0} + \frac{|\varphi(Du)|^{1-\delta_0}}{\delta_0} \geq |\varphi(|Du|)|^{1-\delta_0} > \Lambda
\] (5.14)

for a.e. \( y \in E^\Lambda_A \). Note that if \( \frac{2R}{10} \leq \rho \leq r_2 - r_1 \), then

\[
\Phi_y(\rho) \leq \frac{1}{r_2 - r_1} \int_{B_{\rho r}(z)} \left\{ |\varphi(|Du|)|^{1-\delta_0} + \frac{|\varphi(Du)|^{1-\delta_0}}{\delta_0} \right\} dx =: \Lambda_0.
\] (5.15)

For \( \Lambda > \Lambda_0 \), since \( \Phi_y \) is continuous and \( \lim_{\rho \to 0} \Phi_y(\rho) = \Lambda > \Lambda_0 \), there exists an exit time radius \( \rho_0 \in (0, \frac{2R}{10}) \) such that

\[
\Phi_y(\rho_0) = \Lambda \quad \text{and} \quad \Phi_y(\rho) < \Lambda \quad \text{if} \quad \rho \in (\rho_0, r_2 - r_1).
\] (5.16)

Now we consider the family \( \{ B_{\rho_0}(y) : y \in E^\Lambda_A \} \) which covers the set \( E^\Lambda_A \). By Vitali’s covering lemma, we find a countable family of disjoint sets \( \{ B_{\rho_i}(y_i) : y_i \in E^\Lambda_A \} \) such that

\[
E^\Lambda_A \subset \bigcup_{i \geq 1} B_{\rho_i}(y_i) \cup \text{negligible set},
\] (5.17)
where we have denoted \( \rho_i = \rho_{y_i} \). We write \( B^\epsilon_i := B_{5\rho_i}(y_i) \) and so \( B^\epsilon_{2r} = B_{10\rho_i}(y_i) \).

It follows from (5.16) that
\[
\int_{B^\epsilon_{2r}} [\varphi(|Du|)]^{1-\delta_0} \, dx \leq \Lambda, \quad \int_{B^\epsilon_{2r}} [\varphi(|f|)]^{1-\delta_0} \, dx \leq \delta_0 \Lambda, \quad (5.18)
\]
which verifies the assumptions (5.2). According to Lemma 5.1 we find that for \( \Lambda > \Lambda_0 \) and for any \( T \geq 1 \),
\[
\int_{\{x \in B^\epsilon_i : [\varphi(|Du|)]^{1-\delta_0} > TA \}} [\varphi(|Du|)]^{1-\delta_0} \, dx \\
\leq 2 \int_{\{x \in B^\epsilon_i : [\varphi(|Du|)]^{1-\delta_0} > TA \}} |V(Du) - V(Dw)|^{2-2\delta_0} \, dx \\
+ 2 \int_{\{x \in B^\epsilon_i : [\varphi(|Du|)]^{1-\delta_0} > TA \}} [\varphi(|Dw|)]^{1-\delta_0} \, dx, \quad (5.19)
\]
where the last term can be estimated as
\[
\int_{\{x \in B^\epsilon_i : [\varphi(|Du|)]^{1-\delta_0} > TA \}} [\varphi(|Dw|)]^{1-\delta_0} \, dx \\
\leq \left\{ \left\{ x \in B^\epsilon_i : [\varphi(|Du|)]^{1-\delta_0} > TA \right\} \right\} \frac{4n+\frac{\nu}{\nu+\sigma}}{\frac{1}{T^\delta}}, \left( \int_{B^\epsilon_i} [\varphi(|Dw|)]^{1+\sigma} \, dx \right)^{\frac{\delta+\frac{\nu}{\nu+\sigma}}{\frac{1}{T^\delta}}} \Lambda|B^\epsilon_{2r}|^{\frac{1-\delta_0}{1+\sigma}}.
\]

where we have used Chebyshev’s inequality for the second inequality. Then we find
\[
\int_{\{x \in B^\epsilon_i : [\varphi(|Du|)]^{1-\delta_0} > TA \}} [\varphi(|Du|)]^{1-\delta_0} \, dx \leq (4\varepsilon + c^* T^{-\frac{\delta+\frac{\nu}{\nu+\sigma}}{\frac{1}{T^\delta}}}) \Lambda|B^\epsilon_{2r}| \\
\leq 40^\nu (\varepsilon + c^* T^{-\frac{\delta+\frac{\nu}{\nu+\sigma}}{\frac{1}{T^\delta}}}) \Lambda|B^\epsilon_{\rho_i}(y_i)|, \quad (5.21)
\]
for some \( c^* \) depending only on \( n, s, \varphi, \nu, L \). Now we estimate
\[
|B^\epsilon_{\rho_i}(y_i)| = \frac{1}{\Lambda} \int_{B^\epsilon_{\rho_i}(y_i)} \left\{ [\varphi(|Du|)]^{1-\delta_0} + \frac{[\varphi(|f|)]^{1-\delta_0}}{\delta_0} \right\} \, dx \\
\leq \frac{1}{\Lambda} \left\{ \int_{\{x \in B^\epsilon_{\rho_i}(y_i) : [\varphi(|Du|)]^{1-\delta_0} > \frac{T}{2} \}} [\varphi(|Du|)]^{1-\delta_0} \, dx \\
+ \frac{1}{\Lambda} \int_{\{x \in B^\epsilon_{\rho_i}(y_i) : [\varphi(|f|)]^{1-\delta_0} > \frac{T}{2} \}} [\varphi(|f|)]^{1-\delta_0} \, dx + \frac{|B^\epsilon_{\rho_i}(y_i)|}{2}. \quad (5.22)
\]
Combining (5.21) with (5.22), we find
Recall that the set \( \{ x \in B_r : |\varphi(Du)|^1 - \delta_0 > \Lambda \} \) is covered by the family \( \{ B_i \}_{i=1}^\infty \), where \( \{ B_r(y_i) \}_{i=1}^\infty \) is a disjoint family of balls. Summing up over the covering \( \{ B_i \}_{i=1}^\infty \), we get

\[
\int_{E^*_\Lambda} |\varphi(Du)|^{1-\delta_0} \, dx 
\leq 80^n (\varepsilon + c^* T^{-\frac{4\gamma + \sigma}{1-\delta_0}}) \left[ \int_{\{ x \in B_r(y) : |\varphi(Du)|^1 - \delta_0 > \frac{\Lambda}{4} \}} |\varphi(Du)|^{1-\delta_0} \, dx \right. 
+ \left. \int_{\{ x \in B_r(y) : |\varphi(Du)|^1 - \delta_0 > \frac{\Lambda}{4} \}} \frac{|\varphi(f)|^{1-\delta_0}}{\delta_0} \, dx \right] 
\leq 80^n (\varepsilon + c^* T^{-\frac{4\gamma + \sigma}{1-\delta_0}}) \left[ \int_{\cup_{i=1}^{\infty} \{ x \in B_r(y) : |\varphi(Du)|^1 - \delta_0 > \frac{\Lambda}{4} \}} |\varphi(Du)|^{1-\delta_0} \, dx \right. 
+ \left. \int_{\cup_{i=1}^{\infty} \{ x \in B_r(y) : |\varphi(Du)|^1 - \delta_0 > \frac{\Lambda}{4} \}} \frac{|\varphi(f)|^{1-\delta_0}}{\delta_0} \, dx \right] 
\leq 80^n (\varepsilon + c^* T^{-\frac{4\gamma + \sigma}{1-\delta_0}}) \left[ \int_{E^*_\Lambda/4} |\varphi(Du)|^{1-\delta_0} \, dx + \int_{E^*_\Lambda/4^*} \frac{|\varphi(f)|^{1-\delta}}{\delta_0} \, dx \right].
\]

Thus, change of variable with respect to \( \Lambda \) leads to

\[
\int_{E^*_\Lambda} |\varphi(Du)|^{1-\delta_0} \, dx 
\leq 80^n (\varepsilon + c^* T^{-\frac{4\gamma + \sigma}{1-\delta_0}}) \left[ \int_{E^*_\Lambda/4T} |\varphi(Du)|^{1-\delta_0} \, dx + \int_{E^*_\Lambda/4T} \frac{|\varphi(f)|^{1-\delta_0}}{\delta_0} \, dx \right] \tag{5.24}
\]

for \( \Lambda > T\Lambda_0 \).
We next introduce truncation functions
\[[\varphi(|Du|)]^{1-\delta_0} \mathcal{I}_t := \min\{|\varphi(|Du|)]^{1-\delta_0}, t\} \quad (t > 0)\]

For \(t > T \Lambda_0 =: t_0\) and \(q \in [1 - \delta_0, 1 + \delta_0]\), Fubini’s theorem gives
\[
\int_{B_{r_1}(z)} |\varphi(|Du|)]^{1-\delta_0} \int_{E_{A_1}^t} |\varphi(|Du|)]^{1-\delta_0} dx d\Lambda
\]
\[
= \left(\frac{q-1+\delta_0}{1-\delta_0}\right) \int_{t_0}^{t} \Lambda_{A_1}^{1-\delta_0-2} \int_{E_{A_1}^t} |\varphi(|Du|)]^{1-\delta_0} dx d\Lambda
\]
\[
\leq \left(\frac{q-1+\delta_0}{1-\delta_0}\right) \int_{t_0}^{t} \Lambda_{A_1}^{1-\delta_0-2} \int_{E_{A_1}^t} |\varphi(|Du|)]^{1-\delta_0} dx d\Lambda
\]
\[
+ \left(\frac{q-1+\delta_0}{1-\delta_0}\right) \int_{t_0}^{t} \Lambda_{A_1}^{1-\delta_0-2} \int_{E_{A_1}^t} |\varphi(|Du|)]^{1-\delta_0} dx d\Lambda
\]
\[
\leq (T \Lambda_0)^{1-\delta_0-1} \int_{B_{r_2}(z)} |\varphi(|Du|)]^{1-\delta_0} dx + 80^n (e + c^2 T^{1-\delta_0}) \left[ \left(\frac{q-1+\delta_0}{1-\delta_0}\right) \int_{t_0}^{t} \Lambda_{A_1}^{1-\delta_0-2} \int_{E_{A_1}^t} |\varphi(|Du|)]^{1-\delta_0} dx d\Lambda
\]
\[
+ \left(\frac{q-1+\delta_0}{1-\delta_0}\right) \int_{t_0}^{t} \Lambda_{A_1}^{1-\delta_0-2} \int_{E_{A_1}^t} |\varphi(|Du|)]^{1-\delta_0} dx d\Lambda \right]. \tag{5.25}
\]

To estimate the last two integrals, we again use Fubini’s theorem along with change of variable, to discover that
\[
\int_{t_0}^{t} \Lambda_{A_1}^{1-\delta_0-2} \int_{E_{A_1}^t} |\varphi(|Du|)]^{1-\delta_0} dx d\Lambda
\]
\[
\leq (4T)^{1-\delta_0-1} \left(\frac{1-\delta_0}{q-1+\delta_0}\right) \int_{B_{r_2}(z)} |\varphi(|Du|)]^{1-\delta_0} \int_{E_{A_1}^t} |\varphi(|Du|)]^{q-1+\delta_0} dx d\Lambda
\]
\[
\leq (4T)^{1-\delta_0-1} \left(\frac{1-\delta_0}{q-1+\delta_0}\right) \int_{B_{r_2}(z)} |\varphi(|Du|)]^{1-\delta_0} \int_{E_{A_1}^t} |\varphi(|Du|)]^{q-1+\delta_0} dx. \tag{5.26}
\]

Likewise, we have
\[
\int_{t_0}^{t} \Lambda_{A_1}^{1-\delta_0-2} \int_{E_{A_1}^t} |\varphi(|Du|)]^{1-\delta_0} dx d\Lambda
\]
\[
\leq (4T)^{1-\delta_0-1} \left(\frac{1-\delta_0}{q-1+\delta_0}\right) \int_{B_{r_2}(z)} |\varphi(|Du|)]^{q-1+\delta_0} dx. \tag{5.27}
\]

Combining (5.25), (5.26) and (5.27), it follows that
\[
\int_{B_{r_1}(z)} [\varphi(|Du|)]^{1-\delta_0} [\varphi(|Du|)]_{t}^{q-1+\delta_0} \, dx \\
\leq T^{-\frac{q}{q-\sigma_0}} \Lambda_0^{-\frac{q}{q-\sigma_0}} |B_{r_2}(z)| \\
+ 320^n (\varepsilon T^{-\frac{q}{q-\sigma_0}} + c^* T^{-\frac{q-1-\sigma_0}{2}} \int_{B_{r_2}(z)} [\varphi(|Du|)]^{1-\delta_0} [\varphi(|Du|)]_{t}^{q-1+\delta_0} \, dx \\
+ 320^n (\varepsilon T^{-\frac{q}{q-\sigma_0}} + c^* T^{-\frac{q-1-\sigma_0}{2}} \int_{B_{r_2}(z)} [\varphi(|f|)]^q \, dx \\
\leq T^{-\frac{q}{q-\sigma_0}} \Lambda_0^{-\frac{q}{q-\sigma_0}} |B_{r_2}(z)| \\
+ 320^n (\varepsilon T^2 + c^* T^{-\frac{q}{2}}) \int_{B_{r_2}(z)} [\varphi(|Du|)]^{1-\delta_0} [\varphi(|Du|)]_{t}^{q-1+\delta_0} \, dx \\
+ 320^n (\varepsilon T^2 + c^* T^{-\frac{q}{2}}) \int_{B_{r_2}(z)} [\varphi(|f|)]^q \, dx, \quad (5.28)
\]

since \( q \leq 1 + \delta_0 \leq 1 + \sigma_0 \). Now we choose \( T = (2560^n c^*)^{\frac{q}{2}} \) and \( \varepsilon = \frac{1}{2560^n T^2} \) in order to have the following estimate.

\[
\int_{B_{r_1}(z)} [\varphi(|Du|)]^{1-\delta_0} [\varphi(|Du|)]_{t}^{q-1+\delta_0} \, dx \\
\leq \left( \frac{c R^n}{(r_2 - r_1)^n} \int_{B_{r_1}(z)} \left\{ [\varphi(|Du|)]^{1-\delta_0} + \frac{[\varphi(|f|)]^{1-\delta_0}}{\delta_0} \right\} \, dx \right)^{\frac{1}{q-\sigma_0}} \\
+ \frac{1}{2} \int_{B_{r_2}(z)} [\varphi(|Du|)]^{1-\delta_0} [\varphi(|Du|)]_{t}^{q-1+\delta_0} \, dx + c \int_{B_{r_2}(z)} [\varphi(|f|)]^q \, dx.
\]

Applying Lemma 3.6 with

\[ \phi(s) := \int_{B_{r_1}(z)} [\varphi(|Du|)]^{1-\delta_0} [\varphi(|Du|)]_{t}^{q-1+\delta_0} \, dx \]

and \( \beta = \frac{q_0}{1-\delta_0} \in [1, 3\eta] \), we finally obtain

\[
\int_{B_{r_2}(z)} [\varphi(|Du|)]^{1-\delta_0} [\varphi(|Du|)]_{t}^{q-1+\delta_0} \, dx \\
\leq c \left( \int_{B_{r_1}(z)} [\varphi(|Du|)]^{1-\delta_0} \, dx \right)^{\frac{1}{q-\sigma_0}} + c \int_{B_{r_1}(z)} [\varphi(|f|)]^q \, dx. \quad (5.29)
\]

Letting \( t \to \infty \), we obtain the desired estimate.

\[
\int_{B_{r_2}(z)} [\varphi(|Du|)]^q \, dx \\
\leq c \left[ \left( \int_{B_{r_1}(z)} [\varphi(|Du|)]^{1-\delta_0} \, dx \right)^{\frac{1}{q-\sigma_0}} + \int_{B_{r_1}(z)} [\varphi(|f|)]^q \, dx \right]. \quad (5.30)
\]

This completes the proof of Theorem 2.1. □
References

1. E. Acerbi and N. Fusco, *Semicontinuity problems in the calculus of variations*, Arch. Rational Mech. Anal., 86 (2) (1984), 125-145.
2. E. Acerbi and G. Mingione, *Gradient estimates for a class of parabolic systems*, Duke math. J., 136 (2) (2007), 285-320.
3. K. Adimurthi and N. Phuc, *Global Lorentz and Lorentz-Morrey estimates below the natural exponent for quasilinear equations*, Calc. Var. Partial Differential Equations, 54 (3) (2015), 3107-3139.
4. L. Boccardo and T. Gallouët, *Nonlinear elliptic and parabolic equations involving measure data*, J. Funct. Anal., 87 (1) (1989), 149-169.
5. L. Boccardo and T. Gallouët, *Nonlinear elliptic equations with right-hand side measures*, Comm. Partial Differential Equations, 17 (3-4) (1992), 403-419.
6. S. Byun, Y. Cho, *Nonlinear gradient estimates for generalized elliptic equations with non-standard growth in nonsmooth domains*, Nonlinear Anal., 140 (3-4) (2016), 145-165.
7. S. Byun and L. Wang, *Nonlinear gradient estimates for elliptic equations of general type*, Nonlinear Anal., 194 (2020), 111364.
8. Y. Cho, *Global gradient estimates for divergence-type elliptic problems involving general non-linear operators*, J. Differential Equations, 264 (10) (2018), 6152-6190.
9. A. Cianchi, *Some results in the theory of Orlicz spaces and applications to variational problems*, Nonlinear analysis, function spaces and applications. vol.6(Prague, 1998), Acad. Sci. Czech Repub. Inst. Math., Prague, (1999), 50-92.
10. A. Cianchi and V. Maz’ya, *Quasilinear elliptic problems with general growth and merely integrable, or measure, data*, Nonlinear Anal., 164 (2017), 189-215.
11. L. Diening, K. Christian and E. Suli, *Finite element approximation of steady flows of incompressible fluids with implicit power-law-like rheology*, SIAM J. Numer. Anal., 51 (2) (2013), 984-1015.
12. L. Diening and F. Ettwein, *Fractional estimates for non-differentiable elliptic systems with general growth*, Forum Math., 120 (4) (2008), 523-556.
13. L. Diening, S. Schwarzacher, B. Stroffolini and A. Verde, *Parabolic Lipschitz truncation and caloric approximation*, 56 (4) (2017), Paper No. 120, 27pp.
14. L. Diening, B. Stroffolini and A. Verde, *The φ-harmonic approximation and the regularity of φ-harmonic maps*, J. Differential Equations, 253 (7) (2012), 1943-1958.
15. T. K. Donaldson and N.S. Trudinger, *Orlicz-Sobolev spaces and imbedding theorems*, J. Functional Analysis, 8 (1971), 52-75.
16. E. Giusti, *Direct methods in the Calculus of variations*, World Scientific Publishing Co., Inc, River Edge, NJ, 2003.
17. T. Iwaniec, *Projections onto gradient fields and Lp-estimates for degenerated elliptic operators*, Studia Math., 75 (3) (1983), 293-312.
18. T. Iwaniec and C. Sbordone, *Weak minima of variational integrals*, J. Reine Angew. Math., 454 (1994), 143-161.
19. J. Kinnunen and J.L. Lewis, *Very weak solutions of parabolic systems of p-Laplacian type*, Ark. Mat., 40 (1) (2002), 105-132.
25. J. Kinnunen and S. Zhou, *A local estimate for nonlinear equations with discontinuous coefficients*, Comm. Partial Differential Equations, **24** (11-12) (1999), 2043-2068.

26. J.L. Lewis, *On very weak solutions of certain elliptic systems*, Comm. Partial Differential Equations, **18** (9-10) (1993), 1515-1537.

27. G.M. Lieberman, *The natural generalization of the natural conditions of Ladyzhenskaya and Ural’tseva for elliptic equations*, Comm. Partial Differential Equations, **16** (2-3) (1991), 311-361.

28. G. Mingione, *Gradient estimates below the duality exponent*, Math. Ann., **346** (3) (2010), 571-627.

29. G. Mingione, *Nonlinear aspects of Calderón-Zygmund theory*, Jahresber. Dtsch. Math.-Ver., **112** (3) (2010), 159-191.

30. N.C. Phuc, *On Calderón-Zygmund theory for p- and A-superharmonic functions*, Calc. Var. Partial Differential Equations, **46** (1-2) (2013), 165-181.

31. M. M. Rao and J. D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker Inc., New York, 1991.

32. E. M. Stein, *Harmonic analysis real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol 43, Princeton University Press, Princeton, NJ, 1993.

33. R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, Mathematical Surveys and Monographs, vol 49, AMS, Providence, RI, 1997.

34. A. Verde, *Calderón-Zygmund estimates for systems of φ-growth*, J. Convex Anal., **18** (1) (2011), 67-84.

**Sun-Sig Byun**: Seoul National University, Department of Mathematical Sciences and Research Institute of Mathematics, Seoul 151-747, Korea

*Email address*: byun@snu.ac.kr

**Minkyu Lim**: Seoul National University, Department of Mathematical Sciences, Seoul 151-747, Korea

*Email address*: mk0314@snu.ac.kr