LIE’S REDUCTION METHOD AND DIFFERENTIAL GALOIS THEORY IN THE COMPLEX ANALYTIC CONTEXT

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Abstract. This paper is dedicated to the differential Galois theory in the complex analytic context for Lie-Vessiot systems. Those are the natural generalization of linear systems, and the more general class of differential equations admitting superposition laws, as recently stated in [5]. A Lie-Vessiot system is automatically translated into an equation in a Lie group that we call automorphic system. Reciprocally an automorphic system induces a hierarchy of Lie-Vessiot systems. In this work we study the global analytic aspects of a classical method of reduction of differential equations, due to S. Lie. We propose a differential Galois theory for automorphic systems, and explore the relationship between integrability in terms of Galois theory and the Lie’s reduction method. Finally we explore the algebra of Lie symmetries of a general automorphic system.

1. Introduction. Differential Galois theory deals with the solvability and reducibility of differential equations by means of differential algebraic operations. It started with the earlier works of E. Vessiot [57] on linear differential equations. This framework, known as Picard-Vessiot theory, found a natural algebraic generalization in the theory of strongly normal extensions [22, 23] due to E. R. Kolchin. Modern presentations of the linear theory can be found in several articles and monographies, e.g. [21, 44, 56].

In a parallel way to differential Galois theory, the inquiries of S. Lie [25, 26, 27, 28, 29] were also concerned with the reduction of differential equations in geometrical terms. In particular, he developed a systematic method of reduction based of Lie

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group actions. The reader can find a modern and didactic explanation in the lecture notes by R. L. Bryant [6].

In this paper we explore the relationship between the differential Galois theory and the geometric approach by Lie. We find that those approaches are equivalent, and lead to the same algorithms for the integration of differential equations. As some previous investigations that put together the differential algebraic approach with the geometric differential approach to differential equations we should credit the excellent papers of C. Athorne [1, 2, 3] on the subject.

Most of this theory extends to the case of flat $G$-invariant connections over principal bundles on arbitrary dimensional manifolds. We restrict ourselves to the case of trivial bundles defined over Riemann surfaces for the sake of simplicity. Some of the constructions, like the Galois bundle (Definition 5.4) make us of the duality between left and right translations and therefore can not be done in a principal bundle, where only right translations are defined. However, the Galois group can be defined in the general case.

1.1. **Summary of results.** Throughout the first section we review the concept of Lie-Vessiot system, and we state the global version of Lie’s superposition theorem; it is a recallment of recently published results that relate superposition laws with differential Galois theory, the reader can consult [5] for further details. Second section is devoted to the notion of automorphic systems. We introduce the concept of Lie-Vessiot hierarchy, that relates an automorphic system with a family of Lie-Vessiot systems induced in homogeneous spaces. In the third section we introduce Lie’s reduction method (Theorem 4.4) in the complex analytic context. We note that Lie's reduction method is local in the time parameter, and there we explore the obstructions to the existence of a reduction method global in time (Proposition 5). The result is analogous to the one's obtained in the algebraic case [4] which are expressed in terms of Galois cohomology. In the fourth section we propose a Galois theory for automorphic systems, based on Lie’s reduction method. In our theory, the Galois group is the smallest group to which the equation can be reduced. We explore the relationship with the classical Picard-Vessiot theory. We prove that our Galois group is Zariski dense in the classical one (Theorem 5.5). We study the reduction of the automorphic system to its Galois group (Theorem 5.6, cf. [4] Theorem 5.5). Finally, in the last section we study the infinitesimal symmetries of automorphic systems under the light of differential Galois theory. We give a characterization of the algebra of Lie symmetries generalizing a result due to C. Athorne (Theorem 6.2, cf. [2]). We also find that the Lie algebra of meromorphic right invariant symmetries is contained in the centralizer of the Galois group (Theorem 6.4), which is a nonsymplectic version of the Morales-Ramis lemma [36].

1.2. **On the notion of general solution.** Let us consider a system of first order non-autonomous differential equations,

$$\frac{dx_i}{dt} = F_i(t,x_1,\ldots,x_n) \quad i = 1,\ldots,n$$

by the **general solution** of (1) we mean the space of all solutions. If we are interested in a very small range of variability of the variable $t$ for what the functions $F_i$ are regular, then we can identify the general solutions with the space of initial conditions. However, if we consider a bigger range of variability of $t$ and we include singularities we find many problems to study the structure of the general solution. The cornerstone of our approach is to use some additional geometrical properties
of the equation (1) that give to the general solution the structure of homogeneous space. As it is shown in [5] it can be done in the case the differential equation (1) admit a non-linear superposition law.

The possible algebraic structures of the general solution a differential equation is a problem that has been handled by A. Buium [7] K. Nishioka [40, 41], and leaded to the first proof of irreducibility of first Painlevé transcendental by H. Umemura [52]. Results in [7] (Theorems III.3.1 and III.4.1) and [40] establish that differential equations that allow some structure or projective variety on its solution space are solved by strongly normal extensions. We shown in [5] that strongly normal extensions correspond to differential equations in homogeneous spaces. Henceforth, there is no more projective structure allowed to the general solution of a differential equation beyond the one of homogeneous space, and there is no differential equations whose solution depends rationally upon arbitrary constants beyond Lie-Vessiot systems.

We shall point out that the condition of non-movable singularities in [7] is much stronger than the so-called Painlevé property. In the Painlevé equations the algebraic structure of the space of initial conditions is time dependent, and therefore there is not a canonical algebraic structured inherited by the general solution. The Painlevé equations are not, except for some degenerated cases, Lie-Vessiot systems.

2. Lie-Vessiot systems. The class of ordinary differential equations admitting fundamental systems of solutions was introduced by S. Lie in 1885 [25], as certain class of auxiliary equations appearing in his integration methods for ordinary differential equations. An ordinary differential equation admitting a fundamental system of solutions is, by definition, a system of non-autonomous differential equations (1), for which there exists a set of formulae,

\[ \varphi_i(x^{(1)}, \ldots, x^{(r)}, \lambda_1, \ldots, \lambda_n) \quad i = 1, \ldots, n, \]  

expressing the general solution as function of \( r \) particular solutions of (1) and some arbitrary constants \( \lambda_i \). This means that for \( r \) particular solutions \( x^{(1)}(t), \ldots, x^{(r)}(t) \) of the equations satisfying certain non-degeneracy condition, the expression:

\[ x_i(t) = \varphi_i(x^{(1)}(t), \ldots, x^{(r)}(t), \lambda_1, \ldots, \lambda_n) \]  

is the general solution of the equation (1). In [27] Lie also stated that the arbitrary constants \( \lambda_i \) parameterize the solution space, in the sense that for different constants, we obtain different solutions: there are not functional relations between the arbitrary constants \( \lambda_i \). The set of formulae \( \varphi_i \) is usually referred to as a superposition law for solutions of (1).

Lie’s superposition theorem (1) states that a differential equation locally admits a superposition law if and only if it has a finite dimensional Lie-Vessiot-Guldberg algebra (Definition 2.3). It was believed that the integration of this Lie algebra to a Lie group action lead to a global superposition law. Nowadays it is known that there are some other geometrical obstructions to the existence of a superposition law going beyond the integration of the Lie-Vessiot-Guldberg algebra: in a recent work [5] a characterization of differential equations admitting superpositon laws is given.

2.1. Non-autonomous Analytic Vector Fields.

Definition 2.1. A non-autonomous complex analytic vector field \( \tilde{X} \) in \( M \), depending on the Riemann surface \( S \), is an autonomous vector field in \( S \times M \), compatible
with ∇ in the following sense:

\[ \vec{X}f(t) = \partial f(t) \]

In each cartesian power \( M^r \) of \( M \) we consider the lifted vector field \( \vec{X}^r \). This is just the direct sum copies of \( \vec{X} \) acting in each component of the cartesian power \( M^r \). We have the local expression for \( \vec{X} \),

\[ \vec{X} = \partial + \sum_{i=1}^{n} F_i(t, x) \frac{\partial}{\partial x_i} \]

and also the local expression for \( \vec{X}^r \), which is a non-autonomous vector field in \( M^r \),

\[ \vec{X}^r = \partial + \sum_{i=1}^{n} F_i(t, x^{(1)}) \frac{\partial}{\partial x_i^{(1)}} + \ldots + \sum_{i=1}^{n} F_i(t, x^{(r)}) \frac{\partial}{\partial x_i^{(r)}}. \] (4)

2.2. Superposition Law.

Definition 2.2. A superposition law for \( \vec{X} \) is an analytic map

\[ \varphi: U \times M \to M, \]

where \( U \) is analytic open subset of \( M^r \), verifying:

(a) \( U \) is union of integral curves of \( \vec{X}^r \).
(b) If \( \vec{x}(t) \) is a solution of \( \vec{X}^r \), defined for \( t \) in some open subset \( S' \subset S \), then

\[ x_{\lambda}(t) = \varphi(\vec{x}(t), \lambda), \]

where \( \lambda \) varies in \( M \), is the general solution of \( \vec{X} \) for \( t \) varying in \( S' \).

Example 1 (Linear systems). Let us consider a linear system of ordinary differential equations,

\[ \frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)x_j, \quad i = 1, \ldots, n \]

linear combinations of solutions of this system are also a solutions. Thus, the solution of the system is a \( n \) dimensional vector space, and we can express the global solution as linear combinations of \( n \) linearly independent solutions. The superposition law is written down as follows,

\[ \mathbb{C}^{n \times n} \times \mathbb{C}^{n} \to \mathbb{C}^{n}, \quad (x^{(j)}_{i}, \lambda_{j}) \mapsto (y_{i}) \quad y_{i} = \sum_{j=1}^{n} \lambda_{j} x^{(j)}_{i}. \]

Example 2 (Riccati equations). Let us consider the ordinary differential equation,

\[ \frac{dx}{dt} = a(t) + b(t)x + c(t)x^2, \]

let us consider four different solutions \( x_1(t), x_2(t), x_3(t), x_4(t) \). A direct computation gives that the anharmonic ratio is constant,

\[ \frac{d}{dt} \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_3 - x_2)} = 0. \]
If \(x_1, x_2, x_3\) represent three known solutions, we can extract the unknown solution \(x\) of the expression,

\[
\lambda = \frac{(x_1 - x_2)(x_3 - x)}{(x_1 - x)(x_3 - x_2)}
\]

obtaining,

\[
x = \frac{x_3(x_1 - x_2) - \lambda x_1(x_3 - x_2)}{(x_1 - x_2) - \lambda(x_3 - x_2)}
\]

which is the general solution in function of the constant \(\lambda\), and then a superposition law for the Riccati equation. Note tat the above formula is not valid when two of the solutions \(x_1, x_2, x_3\) coincide. Therefore the open subset \(U\) is the complement of the diagonals in \(\mathbb{P}(1, \mathbb{C})^3\).

2.3. Lie’s Superposition Theorem.

**Definition 2.3.** The Lie-Vessiot-Guldberg algebra of \(\vec{X}\) is the Lie algebra of vector fields in \(\mathcal{M}\) spanned by the set vector fields \(\{\vec{X}_t\}_{t \in \mathcal{S}}\). The Lie-Vessiot-Guldberg algebra of \(\vec{X}\) is denoted \(\mathfrak{g}(\vec{X})\).

**Remark 1.** The Lie-Vessiot-Guldberg algebra of \(\vec{X}\) is finite dimensional if and only if there exist \(\vec{X}_1, \ldots, \vec{X}_s\) autonomous vector fields in \(\mathcal{M}\), spanning a finite dimensional Lie algebra, and holomorphic functions \(f_1(t), \ldots, f_s(t)\) in \(\mathcal{S}\) such that,

\[
\vec{X} = \partial + \sum_{i=1}^{s} f_i(t) \vec{X}_i.
\]

**Notation.** From now on, let us consider a complex analytic Lie group \(G\), and a faithful analytic action of \(G\) on \(\mathcal{M}\),

\[
\mathcal{R}(G) = \text{the Lie algebra of right-invariant vector fields in } G.
\]

We denote by \(\mathcal{R}(G, M)\) the Lie algebra of fundamental vector fields of the action of \(G\) on \(M\) (see, for instance, [43]).

A point \(x\) of \(M\) is called a principal point if its istropy subgroup \(H_x\) consist in the identity element only. It is clear that \(x\) is a principal point if and only if its orbit \(O_x\) is a principal homogeneous \(G\)-space. For each \(r \in \mathbb{N}\), \(G\) acts in the cartesian power \(M^r\) component by component. There is a minimum \(r\) such that there exist principal orbits in \(M^r\). If \(M\) is an homogeneous space then this number \(r\) is called the rank of \(M\). Most homogeneous spaces are of finite rank. For instance, algebraic homogeneous spaces have finite rank (see [4, 5]),

**Definition 2.4.** We say that the action of \(G\) is on \(M\) pretransitive if there exists \(r \geq 1\) and an analytic open subset \(U \subset M^r\) such that:

(a) \(U\) is union of principal orbits.

(b) The space of orbits \(U/G\) is a complex analytic manifold.

**Remark 2.** For non pretransitive Lie group actions there are are not good invariants, even after prolongation to a carterian power. From the existence relative geometric quotients (when we replace the \(G\)-space by a suitable \(G\)-invariant Zariski open subset, [45]) it follows that algebraic actions are pretransitive. However there are analytic non pretransitive actions, for instance:

(i) Consider the following action of \(\mathbb{Z}\) in \(\mathbb{Z} \times \mathbb{C}\). \(n \cdot (m, z) = (m, e^{2\pi in}z)\). There is no principal orbit, even if we shift the action to a cartesian power. Therefore this action is of infinite rank and it is not pre-transitive.
(ii) Consider a torus $T^2$ of complex dimension 2, and an action of $\mathbb{C}$ in $T^2$ by translations. If we choose a non-resonant direction, this action is faithful and all orbits are principal. However, those orbits are not closed and the quotient is not a complex manifold.

In particular, finite rank $G$-homogeneous spaces are pretransitive (see [4, 5]). The notions of Lie-Vessiot system (below) and pretransitive action characterize the differential equations admitting superposition laws.

**Definition 2.5.** A non-autonomous analytic vector field $\vec{X}$ in $M$ is called a Lie-Vessiot system, relative to the action of $G$, if its Lie-Vessiot-Guldberg algebra is spanned by fundamental fields of the action of $G$ on $M$; if and only if $g(\vec{X}) \subset \mathcal{R}(G, M)$.

**Theorem 2.6** (global Lie’s superposition theorem [4, 5]). The non-autonomous vector field $\vec{X}$ in $M$ admits a superposition law if and only if it is a Lie-Vessiot system related to certain pretransitive Lie group action in $M$.

An interesting point to remark is that the group underlying a Lie-Vessiot system is recovered from the superposition law (see [5], Lemma 6.4)

3. **Automorphic systems.** The automorphic system is the translation of a Lie-Vessiot system to the Lie group $G$. This approach is due to Vessiot. From now on, consider the a complex analytic connected Lie group $G$, and a faithful pre-transitive action of $G$ on $M$.

Let $\vec{X}$ be a Lie-Vessiot system in $M$ with coefficients in the Riemann surface $S$. Then,

$$\vec{X} = \partial + \sum_{i=1}^{s} f_i(t) \vec{X}_i,$$

where the vector fields $\vec{X}_i$ are fundamental vector fields in $M$. Consider the natural map,

$$\mathcal{R}(G) \rightarrow \mathcal{X}(M), \quad \vec{A} \mapsto \vec{A}^M,$$

applying right invariant vector fields to fundamental fields. Let us call $\vec{A}_i$ to the element of $\mathcal{R}(G)$ such that $\vec{A}_i^M = \vec{X}_i$.

**Definition 3.1.** We call automorphic vector field associated to $\vec{X}$ to the non-autonomous vector field in $G$,

$$\vec{A} = \partial + \sum_{i=1}^{s} f_i(t) \vec{A}_i.$$ 

Reciprocally let $N$ be an homogeneous $G$-space; we call Lie-Vessiot system induced in $N$ by $\vec{A}$ to the non-autonomous vector field,

$$\vec{A}^N = \partial + \sum_{i=1}^{s} f_i(t) \vec{A}_i^N.$$ 

Let us note that the original system $\vec{X}$ we departed from, is the Lie-Vessiot system $\vec{A}^M$ induced in $M$ by the automorphic system $\vec{A}$ en $G$.

The action of $G$ on itself is transitive. Right invariant vector fields in $G$ are fundamental fields of the action of $G$ by the left side. We should recall that right invariant and left invariant vector fields commute, and if $G$ is connected then a
vector field in \( G \) is left invariant if and only if it commutes with right invariant vector fields and vice-versa. Therefore we can interpret an automorphic system in two different ways:

(i) A non-autonomous vector field which is infinitesimally generated by right-invariant vector fields.

(ii) A non-autonomous vector field which is invariant under left-invariant vector fields (right translations).

3.1. Superposition Law for the Automorphic System. The automorphic system \( \vec{A} \) is a particular case of a Lie-Vessiot system. Hence, there is a superposition law for \( \vec{A} \), let us compute it.

Consider \( \sigma(t) \) a local solution of \( \vec{A} \). At \( t_0 \in S \), the tangent vector \( \sigma'(t_0) \) is \( (\vec{A}_{t_0})_{\sigma(t_0)} \). Consider any \( \tau \in G \); let us define a new curve \( \gamma(t) \) in \( G \) as the composition \( \sigma(t) \cdot \tau \). The tangent vector to the curve \( \gamma(t) \) at \( t_0 \) is \( \gamma'(t_0) = R_\tau(\sigma'(t_0)) = R_\tau((\vec{A}_{t_0})_{\sigma(t_0)}) = (\vec{A}_{t_0})_{\gamma(t_0)} \). Hence, \( \gamma(t) \) is another solution of \( \vec{A} \).

**Proposition 1.** The composition law \( G \times G \rightarrow G \) in \( G \) is the superposition principle for \( \vec{A} \).

**Proof.** Consider a solution \( \sigma(t) \). As stated above, for all \( \tau \in G \), \( \sigma(t) \cdot \tau \) is another solution of \( \vec{A} \). Let us see that this is the general solution. Consider \( t_0 \in S \). For each \( \tau \in G \), \( \sigma(t) \cdot \tau \) is the solution curve of initial conditions \( t_0 \mapsto \sigma(t_0) \cdot \tau \). The action of \( G \) on itself -by the right side- is free and transitive; and all initial conditions are obtained in this way.

**Corollary 1.** Consider \( \sigma(t) \) and \( \tau(t) \) two solutions of \( \vec{A} \). Then \( \sigma(t) \cdot \tau(t)^{-1} \) is a constant point of \( G \).

3.2. Structure of Solution Space. The particularity of Lie-Vessiot systems is that certain finite sets of solutions give us the general solution. For the automorphic systems, this structure is even simpler. Any particular solution gives us the general solution. There is no difference between the notion of general and particular solution.

**Proposition 2.** Let \( \vec{A} \) be an automorphic system in \( G \) depending on \( S \). Consider \( S' \subset S \) such that there exist an analytic solution \( \sigma : S' \rightarrow G \). Then the space of solutions of \( \vec{A} \) defined in \( S' \) is a principal homogeneous space with an action of \( G \) by the right side.

**Proof.** Consider \( \text{Sol}(\vec{A}) \) the space of solutions of \( \vec{A} \) defined in \( S \). The superposition law gives us have an action of \( G \) on \( \text{Sol}(\vec{A}) \) by the right side,

\[ \text{Sol}(\vec{A}) \times G \rightarrow \text{Sol}(\vec{A}), \quad (\sigma(t), \tau) \rightarrow R_\tau(\sigma(t)). \]

This action is free and transitive, by uniqueness of solutions. The space of solutions \( \text{Sol}(\vec{A}) \) is a principal homogeneous space.

3.3. Hierarchy of Lie-Vessiot Systems. Let us consider the following objects: two \( G \)-spaces \( M \) and \( N \), an automorphic vector field \( \vec{A} \) in \( G \) depending on the Riemann surface \( S \), and the induced Lie-Vessiot systems \( \vec{A}^M \) and \( \vec{A}^N \) in \( M \) and \( N \) respectively.
Let $f$ be a surjective morphism of $G$-spaces,

$$f: M \to N, \quad f(\sigma \cdot x) = \sigma \cdot f(x).$$

The map $f$ applies fundamental vector fields of the action of $G$ on $M$ to fundamental vector fields of the action of $G$ on $N$. Thus, $f$ transforms Lie-Vessiot systems in $\tilde{A}^M$ into Lie-Vessiot systems in $N$. It is clear that it transforms $\tilde{A}^M$ into $\tilde{A}^N$:

$$f_*(\tilde{A}^M) = \tilde{A}^N.$$

A solution curve $x(t)$ of $\tilde{A}^N$ gives us, by composition, a solution curve $f(x(t))$ of $\tilde{A}^M$. We have a surjective map:

$$\text{Sol}(\tilde{A}^M) \to \text{Sol}(\tilde{A}^N).$$

Let us assume that $\tilde{A}^M$ admits a superposition law $\varphi$, which is true if the action of $G$ in $M$ is pretransitive,

$$\varphi: U \times M \to M, \quad U \subset M^r.$$

We can also use this superposition law of $\tilde{A}^M$ in $M$ for expressing the general solution of $\tilde{A}^N$. The composition $\phi_1 = f \circ \varphi$ express the general solution of $\tilde{A}^N$ as function of $r$ particular solutions of $\tilde{A}^M$,

$$\phi_1: U \times M \to N, \quad U \subset M^r.$$

Consider $\bar{x}(t)$, and $x, y \in M$ such that $f(x) = f(y)$. Then $\phi_1(\bar{x}(t), x) = \phi_1(\bar{x}(t), y)$. Then $\phi_1$ factorizes and gives us a map,

$$\phi: U \times N \to N, \quad U \subset M^r \quad (5)$$

that gives the general solution of $\tilde{A}^N$ in function of $r$ particular solutions of $\tilde{A}^M$. It is not a superposition law, but a different object known as a representation formula [51]. A general theory of representation formulae of solutions of differential equations can be done under this point of view.

In particular, the associative property of the action $a: G \times M \to M$ means that it is a morphism of $G$ spaces. It is clear that $a$ sends the Lie-Vessiot system $\tilde{A}$ in $G \times M$ to the Lie-Vessiot system $\tilde{A}^M$.

If $M$ is a $G$-homogeneous space, then there is a surjective morphisms of $G$-spaces,

$$f: G \to M, \quad \sigma \mapsto \sigma \cdot x_0.$$

In this particular case, the representation formula (5) gives us that a particular solution of $\tilde{A}$ gives us the general solution of $\tilde{A}^M$ thorough the action of $G$ on $M$. If $\sigma(t)$ is a solution of $\tilde{A}$, then $\sigma(t) \cdot x_0$ is the general solution of $\tilde{X}$ when $x_0$ moves in $M$. In particular, if we take $\sigma(t)$ such that $\sigma(t_0) = \text{Id}$, then $\sigma(t) \cdot x_0$ is the solution of the problem of initial conditions $x(t_0) = x_0$.

**Example 3** (Linear systems). Let us consider the system,

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)x_j.$$

As a vector field it is written

$$\tilde{X} = \sum_{i,j} a_{ij}(t)\tilde{X}_{ij}, \quad \tilde{X}_{ij} = x_j \frac{\partial}{\partial x_i}.$$
these vector fields span the lie algebra gl(n, C) of the action of general linear group GL(n, C) on C^n. Let us take coordinates u_{ij} in GL(n, C). Direct computation gives us the right invariant vector fields,
\[ \tilde{A}_{ij} = \sum_{k=1}^{n} u_{jk} \frac{\partial}{\partial u_{ik}} \]
then,
\[ \tilde{A} = \partial + \sum_{i,j} a_{ij}(t) \tilde{A}_{ij}. \]
The automorphic system is written,
\[ \frac{du_{ij}}{dt} = \sum_{k} a_{ik}(t)u_{kj}, \]
or in matrix form,
\[ \frac{dU}{dt} = A(t)U, \quad U = (u_{ij}), \quad A(t) = (a_{ij}(t)). \]
The solutions of the automorphic system are the fundamental matrices of solutions of the linear system. If \[ U(t) \] is one such of these matrices, then for each \[ x_0 \in \mathbb{C}^n \], \[ x(t) = U(t)x_0 \] is a solution of the linear system. Moreover if we take \[ U(t) \] such that \[ U(t_0) = Id \] the previous formula gives us the global solution with initial conditions \[ x(t_0) = x_0. \]

Example 4 (Riccati equations). Let us consider the general Riccati equation,
\[ \frac{dx}{dt} = a(t) + b(t)x + c(t)x^2. \]
As a vector field it is written:
\[ \tilde{X} = a(t)\tilde{X}_1 + b(t)\tilde{X}_2 + c(t)\tilde{X}_3, \]
being,
\[ \tilde{X}_1 = \frac{\partial}{\partial x}, \quad \tilde{X}_2 = x \frac{\partial}{\partial x}, \quad \tilde{X}_3 = x^2 \frac{\partial}{\partial x}. \]
The Lie algebra spanned by these vector fields is the infinitesimal generator of the group PGL(2, C) of projective transformations of the projective line \[ \mathbb{P}(1, \mathbb{C}) \],
\[ x \mapsto \frac{u_{11}x + u_{12}}{u_{21}x + u_{22}}, \]
which is a Lie group of dimension 3. In order to make the computation easier, let us consider the following: modulo a finite group of order 2, the group PGL(2, C) is identified with SL(2, C) the group of \[ 2 \times 2 \] matrices with determinant 1. By this isogeny the automorphic system is transformed into the linear system,
\[ \frac{d}{dt} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} \frac{b}{2} & a \\ -c & -\frac{b}{2} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \] (6)
each solution \( (u_{ij}(t)) \) induces the global solution of the Riccati equation,
\[ x(t) = \frac{u_{11}(t)x_0 + u_{12}(t)}{u_{21}(t)x_0 + u_{22}(t)}. \]
3.4. Logarithmic Derivative. For each open subset $S'$ of the Riemann surface $s$ we denote by $O(S', G)$ the space of analytic maps from $S'$ to $G$; the elements of this space are complex analytic curves in $G$. For a curve $\sigma(t) \in O(S, G)$ and a point $t_0$ in $S'$ we denote by $\sigma'(t_0)$ to its tangent vector at $t_0$, which is the image of $\partial_t$ by the tangent morphism

$$\sigma'_t : T_{t_0} S \to T_{\sigma(t_0)} G.$$ 

As usually, we identify the Lie algebra $\mathcal{R}(G)$ with the tangent space at the identity element $T_{1d}G$. There is only an element of $\mathcal{R}(G)$ whose value at $\sigma(t_0)$ is $\sigma'(t_0)$. The value of this right invariant vector field at $Id$ is $R'_{\sigma - 1(t_0)}(\sigma'(t_0))$. In such way we can assign to $\sigma$ a map from $S'$ to $\mathcal{R}(G)$ that assigns to each $t_0 \in S'$ the right invariant vector field whose value at $\sigma(t_0)$ is $\sigma'(t_0)$. By the identification of $\mathcal{R}(G)$ with $T_{1d}G$ this map sends $t$ to $R'_{\sigma(t)-1}(\sigma'(t))$. This is precisely Kolchin’s logarithmic derivative (see [23]).

From the differential geometry point of view, the logarithmic derivative is nothing but the pullback of the Maurer-Cartan form through the section $\sigma$.

**Definition 3.2.** Let $\sigma \in O(S', G)$ be a curve in $G$. We call logarithmic derivative $l\partial(\sigma(t))$ of $\sigma(t)$ to the map from $S'$ to $\mathcal{R}(G)$ that assigns to each $t_0 \in S'$ the right invariant vector field whose value at $\sigma(t_0)$ is $\sigma'(t_0)$. The logarithmic derivative is a map,

$$l\partial : O(S', G) \to \mathcal{R}(G) \otimes_C O(S'),$$

$$\sigma(t) \to l\partial(\sigma(t)) = R'_{\sigma(t)-1}(\sigma'(t)).$$

Because of the construction of the logarithmic derivative the following result becomes self-evident.

**Proposition 3.** Let $\tilde{A}$ be an automorphic system in $G$ depending on $S$. Then $\sigma \in O(S', G)$ is a solution of $\tilde{A}$ if and only if,

$$l\partial(\sigma) = \tilde{A} - \partial$$

(7)

Expression (7) is know as the automorphic equation of the automorphic system $\tilde{A}$. Solving the automorphic vector field $\tilde{A}$ is equivalent to finding a particular solution for the automorphic equation.

Let us recall that the adjoint automorphism $\text{Adj}_{\sigma}$ is the tangent map at the identity element of the internal automorphism of $G$, $\xi \mapsto \sigma \cdot \xi \cdot \sigma^{-1}$. Logarithmic derivative satisfies the following property with respect to composition.

**Proposition 4** (Gauge change formula). Consider $\sigma(t)$ and $\tau(t)$ in $O(S, G)$. The composition $\sigma(t)\tau(t)$ is also an element of $O(S, G)$. We have:

$$l\partial(\sigma(t)\tau(t)) = l\partial(\sigma(t)) + \text{Adj}_{\sigma(t)}(l\partial(\tau(t))).$$

**Proof.** By direct computation,

$$l\partial(\sigma(t)\tau(t)) = R'_{\tau(t)-1\sigma(t)-1}(\sigma(t)\tau(t)) = R'_{\sigma(t)-1}(R'_{\tau(t)-1}(\sigma(t))) + L'_\sigma(\tau'(t)) = R'_{\sigma(t)-1}(\sigma'(t)) + L'_\sigma(\tau'(t)) = l\partial(\sigma(t)) + \text{Adj}_{\sigma(t)}(l\partial(\tau(t))).$$

**Corollary 2.** For $\sigma(t) \in O(S, G)$,

$$l\partial(\sigma(t)^{-1}) = -\text{Adj}_{\sigma(t)-1}(l\partial(\sigma(t))).$$

**Proof.** Apply the gauge change formula to the composition $\sigma(t) \cdot \sigma(t)^{-1} = Id$. 

4. Lie's reduction method. Sophus Lie developed a method of reduction of Lie-Vessiot systems when a particular solution is known. In despite of its lack of popularity, this method is the hearth underlying of most known methods of reduction of differential ordinary equations, as the classical reduction of Riccati equation (see [16] vol I. ch. I-IV), symplectic reduction (see [6] lec. 7), representation formulas of solution of matrix differential equation (see [50]), generalized Wei-Norman method (see [12]). Another interesting application is the D'Alambert reduction of variational equations to normal variational equations in dynamical systems (see [38]). It is also equivalent to the method of reduction shown by Carinena and Ramos (cf. [10]). Here this method is presented by means of gauge transformations and the automorphic equation.

From now on let us consider the following objects: a G-homogeneous space $M$, an automorphic system $\bar{A}$ in $G$ depending on the Riemann surface $S$, an origin point $x_0 \in M$. Denote by $H$ the isotropy subgroup of $x_0$, and by $\tilde{X}$ the induced Lie-Vessiot system $\bar{A}^M$ in $M$ by $\bar{A}$.

4.1. Gauge Transformations. We fiber the extended phase space $S \times G$ over the Riemann surface $S$. It is a trivial principal fiber bundle $\pi: S \times G \to S$. We perform the same operation for $M$; we consider then $\pi: S \times M \to M$ as an associated bundle of fiber $M$ (see [43]). A map $\sigma(t) \in \mathcal{O}(S, G)$ is considered as a section of $\pi$. This section induces an automorphism $L_{\sigma(t)}$ of the principal bundle,

$$L_{\sigma(t)}: S \times G \to S \times G, \quad (t, \tau) \mapsto (t, \sigma(t) \cdot \tau),$$

and an automorphism of the associated bundle that we denote by the same symbol,

$$L_{\sigma(t)}: S \times M \to S \times M, \quad (t, x) \mapsto (t, \sigma(t) \cdot x).$$

Definition 4.1. The above automorphisms are called gauge transformations induced by $\sigma(t)$. 

Example 5 (Logarithmic derivative of matrices). We take the canonical basis $\frac{\partial}{\partial u_{ij}}$ of the tangent space of $GL(n, \mathbb{C})$. Let us consider $U \in GL(n, \mathbb{C})$; the tangent space $T_U GL(n, \mathbb{C})$ is identified with the space of $n \times n$ complex square matrices. We identify the Lie algebra $gl(n, \mathbb{C})$ with the tangent space at $I$, the identity matrix. Let $U(t)$ be a time-dependent non-degenerate matrix. Then $U'(t) \in T_{U(t)} GL(n, \mathbb{C})$ is the matrix whose coefficients are the derivatives of coefficients of $U$. In order to identify it with an element of $gl(n, \mathbb{C})$ we have to apply a right transformation,

$$R_{U^{-1}(t)}: T_{U(t)} GL(n, \mathbb{C}) \to T_{I} GL(n, \mathbb{C}) = gl(n, \mathbb{C}).$$

As $R_{U^{-1}}$ is a linear map on the functions $u_{ij}$, then it is its own differential; therefore

$$\frac{d log(U(t))}{dt} = R_{U^{-1}(t)} U'(t) = U'(t) U^{-1}(t),$$

and then

$$\frac{d log(U)}{dt} = U' U^{-1}.$$

For a time-dependent matrix $A$, that we consider as a curve in the Lie algebra $gl(n, \mathbb{C})$, we set the automorphic equation,

$$\frac{d log U}{dt} = A.$$

It is equivalent to

$$U' U^{-1} = A, \quad U' = U A,$$

the linear system defined by $A$. 

The above automorphisms are called gauge transformations induced by $\sigma(t)$. 

...
This is nothing but Cartan’s notion of *repère mobile* on the bundle. These are the natural transformations for Lie-Vessiot systems. Gauge transformations are easily understand by terms of the logarithmic derivative.

**Theorem 4.2.** Let transforms automorphic systems onto automorphic vector systems, and Lie-Vessiot systems onto Lie-Vessiot system. A map \( \tau(t) \) is a solution of the automorphic equation (7) if and only if \( L_{\sigma(t)}(\tau(t)) = \tau(t) \cdot \sigma(t) \) verifies,

\[
\text{l}\partial(\tau(t) \cdot \sigma(t)) = \text{Adj}_{\sigma(t)}(\bar{A} - \partial) + \text{l}\partial(\sigma(t)).
\]

**Proof.** Assume that \( \tau(t) \) is a solution of the equation (7). Then by the fundamental property of logarithmic derivative, \( \text{l}\partial(\sigma(t)\tau(t)) = \text{Adj}_{\sigma(t)}(\bar{A} - \partial) + \text{l}\partial(\sigma(t)) \). The “if and only if” condition is attained by considering the inverse gauge transform \( L_{\sigma(t)^{-1}} \). It proves that \( L_{\sigma(t)} \) maps the automorphic system \( \bar{A} \) to the automorphic system \( \bar{B} \) defined by,

\[
\bar{B}_t = \text{Adj}_{\sigma(t)}(\bar{A}_t) + \text{l}\partial(\sigma(t)).
\]

and then it maps also the Lie-Vessiot system \( \bar{A}^M \) to \( \bar{B}^M \).

**Example 6.** As it is well known, if we consider a linear system

\[
x' = Ax
\]

and a change of variable \( z = Bx \), being \( B \) a time dependent invertible matrix, then \( z' = B'x + Bx = B'B^{-1}z + BAx = (B' + BA)B^{-1}z \), and \( z \) satisfies the transformed linear system,

\[
z' = (B' + BA)B^{-1}z
\]

where \( (B' + BA)B^{-1} = B'B^{-1} + \text{Adj}_B(\bar{A}) \), as above.

### 4.2. Lie’s Reduction Method.

Let us recall that we consider \( x_0 \) a point of the \( G \)-homogeneous space \( M \) as origin, and we denote by \( H \) the isotropy subgroup of \( x_0 \). We also denote by \( H^0 \) to the connected component of the identity of \( H \).

From the canonical inclusion of Lie algebras \( \mathcal{R}(H^0) \subset \mathcal{R}(G) \) we know that an automorphic system \( \bar{B} \) in \( H^0 \) is, in particular, an automorphic system in \( G \). The non-autonomous vector field \( \bar{B} \) in \( H^0 \) naturally extends to a non-autonomous vector field in \( G \) by right translations. In order to solve the extended non-autonomous vector field it is enough to find a particular solution of \( \bar{B} \) in \( H^0 \). Reciprocally, an automorphic system in \( G \) restricts to an automorphic system in \( H^0 \) if and only if its Lie-Guldberg-Vessiot algebra is contained in \( \mathcal{R}(H^0) \). The Lie’s method of reduction stands on the following key lemma that characterizes which automorphic systems in \( G \) are, in fact, automorphic systems in \( H^0 \).

**Lemma 4.3.** Assume that \( x_0 \) is a constant solution of \( \bar{A}^M \). Then, \( \bar{A} \) is an automorphic system in \( H^0 \): there exist right-invariant vector fields \( B_i \in \mathcal{R}(H^0) \) such that:

\[
\bar{A} = \partial + \sum_{i=1}^{s} f_i(t) \bar{B}_i.
\]

**Proof.** For each \( t_0 \) in \( S \) we take a local solution \( \sigma(t) \) of \( \bar{A} \), defined in some neighborhood \( S' \) of \( t_0 \), with initial condition \( t_0 \mapsto Id \). In \( S' \) we have \( \bar{A} - \partial = \text{l}\partial(\sigma(t)) \). As \( \sigma(t) \cdot x_0 = x_0, \sigma(t) \) is a curve in \( H \) and its logarithmic derivative takes values at \( \mathcal{R}(H) \). The Lie algebra \( \mathcal{R}(H) \) coincides with the Lie algebra \( \mathcal{R}(H^0) \) of the connected component of the identity. Finally we conclude that for all \( t_0 \in S \), \( \bar{A}_{t_0} \in \mathcal{R}(H) \). Then the Lie-Guldberg-Vessiot algebra of \( \bar{A} \) is included in \( \mathcal{R}(H) \).
Let us examine the general case of reduction. Assume that we know an analytic solution \( x(t) \) for the Lie-Vessiot system \( \vec{A}^M \). For each \( x \in M \) we denote,
\[
H_{x_0, x} = \{ \sigma \in G | \sigma \cdot x_0 = x \}.
\]
The isotropy group \( H \) acts in \( H_{x_0, x} \) free and transitively by the right side, therefore \( H_{x_0, x} \) is a principal homogeneous \( H \)-space.

We construct the following sub-bundle of \( \pi: S \times G \to S \):
\[
\pi_1: H \subset S \times G,
\]
and \( \pi_1: H \to S \) the restriction of \( \pi \) in such way that for \( t_0 \in S \) the stalk \( \pi_1^{-1}(t_0) \) is \( H_{x_0, x(t_0)} \). Then \( \pi_1 \) is a principal bundle modeled over \( H \). Let us take a section \( \sigma(t) \) of \( \pi_1 \) defined in some \( S' \subset S \). Thus, in \( S' \) we have that \( x(t) = \sigma(t) \cdot x_0 \). Let us consider the gauge transformation \( L_{\sigma(t)^{-1}} \). It maps the automorphic system \( \vec{A} \) to an automorphic system \( \vec{B} \),
\[
\vec{B} - \partial = \text{Adj}_{\sigma(t)^{-1}}(\vec{A} - \partial - l\partial(\sigma)),
\]
\( L_{\sigma(t)^{-1}}(x(t)) \) is a solution of \( \vec{B} \). But, \( L_{\sigma(t)^{-1}}(x(t)) = \sigma^{-1}(t)\sigma(t) \cdot x_0 = x_0 \). Thus, we are in the hypothesis of the previous lemma. We have proven the following result.

**Theorem 4.4** (Lie’s reduction method). Assume that there is a solution \( x(t) \) of \( \vec{A}^M \) defined in a neighborhood of \( t_0 \). Then there exists a neighborhood \( S' \) of \( t_0 \) and a gauge transformation defined in \( S' \times G \) that maps the automorphic system \( \vec{A} \) to an automorphic system \( \vec{B} \) in \( H^0 \).

For performing Lie’s reduction we need a section of a principal bundle. In general this bundle is not trivial, and then there are no global sections. We have to consider two different cases: compact and non-compact Riemann surfaces. For non-compact Riemann surfaces we use the following result due to Grauert (see [49]).

**Theorem 4.5** (Grauert theorem). Let \( S \) be a complex connected non-compact Riemann surface. Let \( F \to S \) be a locally trivial complex analytic principal bundle with a connected complex Lie group as structure group. Then there is a meromorphic section of \( F \) defined in \( S \).

For compact Riemann surfaces we use the correspondence between algebraic and analytic geometry. The next definition is given in [48].

**Definition 4.6.** An algebraic Lie group \( H \) is called special if all principal bundle modeled over \( H \) has a global meromorphic section.

Special groups are *linear and connected* (see [48] Theorem 1). It \( H \) is an algebraic special group, complex analytic principal bundles modeled over \( H \) on a compact Riemann surface algebraic (this is a consequence of [47] Proposition 20 taking into account that condition (R) therein is equivalent to Definition 4.6 as stated in [48] Lemma 1).

Fortunately, groups that appear in our integrability theory are special groups. In the general case there may not exist a meromorphic global section: algebraic bundles are not locally trivial in Zariski topology. They are locally *isotrivial*; isomorphic to trivial bundles *up to a ramified covering*.

There is close a relation between Galois cohomology and special groups. The first set of Galois cohomology \( H^1(H, K) \) classifies principal homogeneous \( H \)-spaces with coefficients in the field \( K \) (see [23]). There is a dictionary between meromorphic principal bundles over \( S \) modeled over \( H \) and principal homogeneous spaces.
with coefficients in the field \( \mathcal{M}(S) \) of meromorphic functions in \( S \). Hence, an algebraic special group is an algebraic group whose first Galois cohomology set with coefficients in any field of meromorphic functions vanish.

Let us cite the following result ([39] Theorem 8).

**Theorem 4.7.** Let \( Sp(2n, \mathbb{C}) \) be the symplectic group of \( n \) degrees of freedom.

(i) \( Sp(2n, \mathbb{C}) \) is special.

(ii) Every connected solvable linear algebraic group is special.

(iii) Let \( H < G \) be a normal subgroup. If \( H \) and \( G/H \) are special then \( G \) is special.

If the isotropy group \( H \) of \( x_0 \) in \( M \) is special or \( S \) is open then the bundle \( \pi_1: \mathcal{H} \to S \) has a global meromorphic section \( \sigma \). The gauge transformation \( L_{\sigma^{-1}} \) maps the automorphic system \( \tilde{A} \) to an automorphic system in \( H^0 \). We have proven the following result.

**Proposition 5.** Assume that there is a meromorphic solution \( x(t) \) of \( \tilde{A}^M \) defined in \( S \). Let us take a point \( t_0 \in S \) and denote by \( x_0 \) the point \( x(t_0) \). Let \( H \) be isotropy subgroup of \( x_0 \) in \( M \). Assume one of the following additional hypothesis,

(a) \( H \) is a special group.

(b) \( S \) is non compact and \( H \) is connected.

In such case there is a meromorphic gauge transformation in \( S \times G \) that reduces \( \tilde{A} \) to an automorphic system in \( H^0 \), the connected component of the identity element of \( H \).

**Example 7.** Consider the Riccati equation

\[
\frac{dx}{dt} = a(t) + b(t)x + c(t)x^2
\]

and suppose that we know a particular solution \( f \). Then let

\[
\sigma(t) = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}
\]

so that

\[
f = \sigma(t) \cdot 0
\]

using the linear fractional action of \( SL(2, \mathbb{C}) \) on the projective line \( \mathbb{P}(1, \mathbb{C}) \) of the above example. The isotropy of \( 0 \) is the subgroup \( H_0 \) of matrices of the form:

\[
H_0 = \left\{ \begin{pmatrix} \lambda & 0 \\ \mu & \lambda^{-1} \end{pmatrix} \right\}.
\]

By means of the gauge transformation induced by \( \sigma^{-1} \),

\[
\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} = \begin{pmatrix} u_{11} - fu_{21} & u_{12} - u_{22} \\ u_{21} & u_{22} \end{pmatrix},
\]

we transform the linear system (6) into the reduced system,

\[
\frac{d}{dt} \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + cf & 0 \\ -c & -\frac{1}{2} - cf \end{pmatrix} \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}
\]

which is in triangular form, and then integrable by quadratures. The induced gauge transformation in the projective line maps \( x \) to \( z = x - u \), and then \( z \) verifies the Riccati equation,

\[
\frac{dz}{dt} = (b + 2cf)z + cz^2
\]
if we set $w = \frac{1}{z} = \frac{1}{x-u}$ we find the classical transformation of the Riccati equation to inhomogeneous linear equation (cf. [16] Vol I, chapter II).

\[
\frac{dw}{dt} = -c - (2b + cf)w.
\]

5. **Analytic Galois theory.** Here we present a differential Galois theory for automorphic systems in $G$ that depends meromorphically on a Riemann surface $S$. Our approach is similar to the tannakian presentation of differential Galois theory. The difference is that here we use the category of $G$-homogeneous spaces instead of constructions by tensor products. In the tannakian approach, the Galois group stabilizes all meromorphic invariant tensors of the differential equation; equivalently in our approach the Galois group stabilizes all meromorphic solutions of Lie-Vessiot systems induced by $\vec{A}$. This presentation is even more direct than the classical tannakian approach: we work directly in the category of $G$-homogeneous spaces. The Galois group and its applications appear naturally. In this frame there is no need of some technical points that usually appear in other presentations of tannakian differential Galois theory. We construct our Galois group as the fiber of a geometrically defined object, the Galois bundle.

5.1. **Galois Bundle.** Let us consider the automorphic system in $G$,

\[
\vec{A} = \partial + \sum f_i(t) \vec{A}_i,
\]

and assume that the $f_i(t)$ are meromorphic functions in $S$. Let us consider

\[
S^\times = S \setminus \{\text{poles of } f_i\},
\]

the Riemann surface that we obtain from $S$ by removing the poles of the meromorphic functions $f_i$. Then $\vec{A}$ is a complex analytic automorphic system in $G$ depending on $S^\times$. For each $G$-homogeneous space $M$ we consider $\vec{A}^M$, the induced Lie-Vessiot system.

For each $M$, let us define $\mathcal{M}_0(\vec{A}^M)$ as the set of solutions of $\vec{A}^M$ defined in $S^\times$ that are meromorphic at $S$. Let $\mathcal{C}(G)$ be the set of conjugacy classes of closed analytic subgroups of $G$. To each $c \in \mathcal{C}(G)$ it corresponds an homogeneous space $M(c)$ isomorphic to $G/H$ being $H$ any closed analytic subgroup of $G$ whose class of conjugation is $c$. When $c$ varies along $\mathcal{C}(G)$, $M(c)$ varies along the set of different class of isomorphic $G$-homogeneous spaces. Finally, we define the set of meromorphic solutions associated to $\vec{A}$,

\[
\mathcal{M}(\vec{A}) = \bigcup_{c \in \mathcal{C}(G)} \mathcal{M}_0(\vec{A}^M(c))
\]

The set $\mathcal{M}(\vec{A})$ consists of all the different meromorphic solutions of all the different Lie-Vessiot systems induced in $G$-homogeneous spaces.

**Definition 5.1.** For $t_0 \in S^\times$ we define the analytic Galois group of $\vec{A}$ at $t_0$, $\text{Gal}_{t_0}(\vec{A})$ as the subgroup of $G$ that stabilizes the values at $t_0$ of all the meromorphic solutions of all the Lie-Vessiot systems induced by $\vec{A}$.

\[
\text{Gal}_{t_0}(\vec{A}) = \bigcap_{x(t) \in \mathcal{M}(\vec{A})} H_{x(t_0)}.
\]

The Galois group $\text{Gal}_{t_0}(\vec{A})$ is an intersection of closed complex analytic subgroups of $G$, and then it is a closed analytic subgroup of $G$. 

Lemma 5.2. The class of conjugation of $\text{Gal}_{t_0}(\bar{A})$ does not depend on $t_0 \in S^\times$. Moreover $\text{Gal}_{t_0}(\bar{A})$ depends analytically on $t_0$ in $S^\times$.

Proof. Consider $t_0$ and $t_1$ in $S^\times$. If $t_0$ and $t_1$ are close enough we can assume that there is a solution $\sigma(t)$ of $\bar{A}$ defined on a connected neighborhood including $t_0$ and $t_1$. By a right translation we can assume that $\sigma(t_0)$ is the identity element.

Consider $\tau$ in $\text{Gal}_{t_0}(\bar{A})$. It means that for all meromorphic solution $x(t) \in M(\bar{A})$, $\tau \cdot x(t_0) = x(t_0)$. We have $x(t) = \sigma(t) \cdot x(t_0)$. Hence, $x(t_1) = \sigma(t_1) \cdot x(t_0)$ and $\sigma(t_1) \cdot \tau \cdot \sigma(t_1)^{-1} \cdot x(t_1) = \sigma(t_1) \cdot \tau \cdot \sigma(t_1)^{-1} \sigma(t_1) \cdot x(t_0) = \sigma(t_1) \cdot \tau \cdot x(t_0) = \sigma(t_1) \cdot x(t_0)$. Then, $\sigma(t_1) \cdot \tau \cdot \sigma^{-1}(t_1) \in \text{Gal}_{t_1}(\bar{A})$ an we conclude that,

$$\text{Gal}_{t_1}(\bar{A}) = \sigma(t_1) \cdot \text{Gal}_{t_0}(\bar{A}) \cdot \sigma(t_1)^{-1},$$

the Galois groups at $t_0$ and $t_1$ are conjugated. It is proven that the conjugacy class of the Galois group it is locally constant; $S^\times$ is a connected Riemann surface therefore it is constant.

Now, let us consider $H$ a subgroup of $G$ of the same class of conjugation that $\text{Gal}_{t_0}(\bar{A})$. The normalizer subgroup

$$N(H) = \{ \sigma \in G \mid \sigma H = H \sigma \},$$

is the bigger intermediate group $H \subset Z \subset G$ such that $H \triangleleft Z$. Let $M$ be the quotient $G$-homogeneous space $G/N(H)$, and let us consider the natural projection $\pi_1 : G \rightarrow M$. Points of $M$ parameterize the class of conjugation of $H$, to $x = [\sigma]$ it corresponds the group $\sigma \cdot H \cdot \sigma^{-1}$, and that the natural action of $G$ on the quotient is nothing but the action of $G$ by conjugation on the conjugacy class of $H$.

This parametrization allow us to define a map $h : S^\times \rightarrow M$ that sends $t \mapsto h(t)$. The image $h(t_0)$ is the point of $N$ corresponding to the subgroup $\text{Gal}_{t_0}(\bar{A})$ in its conjugacy class. Near $t_0$ take any solution $\sigma(t)$ of $\bar{A}$ such that $\sigma(t_0)$ is the identity. Then, $\text{Gal}_{t_0}(\bar{A}) = \sigma(t) \cdot \text{Gal}_{t_0}(\bar{A}) \cdot \sigma(t)^{-1}$ or equivalently $h(t) = \sigma \cdot h(t_0)$. Thus, $h(t)$ is a solution for the Lie-Vessiot system $\bar{A}^M$ induced in $M$, so that it is an analytic function in $S^\times$.

Lemma 5.3. Consider $H$ the Galois group of $\bar{A}$ at $t_0$ and $M$ the quotient space $G/H$. Then $\bar{A}^M$ has a meromorphic solution in $M$.

Proof. Consider $x_0 = [H]$ the origin point of $M$. The group $\text{Gal}_{t_0}(\bar{A})$ is the isotropy group of the values of all meromorphic solutions of induced Lie-Vessiot systems, as stated in formula (8). The complex analytic group $G$ is of finite dimension. The equations of $\text{Gal}_{t_0}(\bar{A})$ as subgroup of $G$ are:

$$\text{Gal}_{t_0}(\bar{A}) = \{ \sigma \in G \mid \sigma(x(t_0)) = x(t_0), \forall x(t) \in M(\bar{A}) \}.$$

Each $x(t) \in M(\bar{A})$ gives us some of the equations of $\text{Gal}_{t_0}(\bar{A})$. As a complex analytic manifold $G$ is of finite dimension, thus $\text{Gal}_{t_0}(\bar{A})$ is defined by a finite number of equations, at least locally. Then it suffices to consider a finite number of such meromorphic solutions, $y_1(t), \ldots, y_m(t)$ each one defined in a homogeneous space $y_k : S^\times \rightarrow M_k$. For each $M_k$, we have $\text{Gal}_{t_0}(\bar{A}) \subset H_{y_k(t_0)}$. By fixing $y_k(t_0)$ as the origin point, the homogeneous space $M_k$ is identified with the quotient $G/H_{y_k(t_0)}$. We have a natural projection of $G$-spaces $p_k : M \rightarrow M_k$ that maps the origin $x_0$ of $M$ onto $y_k(t_0)$. By considering the cartesian power of those projections, we construct

$$\pi_1 : M \rightarrow M_1 \times \cdots \times M_m,$$
which is an injective morphism of $G$-spaces, that identifies $M$ with an orbit in the cartesian product. The image of the origin point is $(y_1(t_0), \ldots, y_m(t_0))$. Finally, the meromorphic solution of the Lie-Vessiot system in the cartesian power, $(y_1(t), \ldots, y_m(t))$, is contained in the image of $\pi_1$, so that it is a solution of $\tilde{A}^M$ which is meromorphic in $S$.

**Corollary 3.** The Galois group $\text{Gal}_t(\tilde{A})$ depends meromorphically on $t \in S$.

**Proof.** Let us recall the proof of the Lemma 5.2. Let us denote by $G$ conjugated with $G$ by the adjoint action of $G$ equation in addition to the automorphic system: the Lie-Vessiot equation induced by the fiber of $t$. We assume that $\tilde{A}$ is a $\text{GL}(E)$, the group of linear automorphisms of a complex finite dimensional vector space. In this case, the considered automorphic system $\tilde{A}$ is a system of linear homogeneous differential equations with meromorphic coefficients.

For this section let us assume that $G$ is $\text{GL}(E)$, the group of linear automorphisms of a complex finite dimensional vector space. In this case, the considered automorphic system $\tilde{A}$ is a system of linear homogeneous differential equations with meromorphic coefficients.

The Galois group $\text{Gal}_t(\tilde{A})$ depends meromorphically on $S$. Thus, we can define an analytic sub-bundle $\text{Gal}(\tilde{A}) \subset S \times G$, which is meromorphic in $S$ and such that the fiber of $t_0 \in S^\times$ is precisely $\text{Gal}_{t_0}(\tilde{A})$.

**Definition 5.4.** We call Galois bundle of $\tilde{A}$ to the complex analytic in $S^\times$ and meromorphic in $S$ principal bundle,

$$\text{Gal}(\tilde{A}) = \bigcup_{t \in S^\times} \text{Gal}_t(\tilde{A}) \xrightarrow{\pi} S^\times.$$ 

**Remark 3.** In the algebraic case, one can consider just algebraic subgroups of $G$ and then algebraic homogeneous spaces. In such a case, we will obtain an algebraic Galois group. We fall in the Kolchin’s theory of $G$-primitive extensions, and the Galois group here coincides with the Galois group of the attained strongly normal extension. This connection has been partially examined in [4], chapter 4.

**Remark 4.** Another advantage of this geometric presentation of the theory, even in the algebraic case, is that the Galois group is well defined as a unique differential algebraic subgroup of $G \otimes \mathcal{M}(S)$. This group is endowed with a natural differential equation in addition to the automorphic system: the Lie-Vessiot equation induced by the adjoint action of $G$ on itself. $G \otimes \mathcal{M}(S)$ endowed with this Lie-Vessiot system is a differential algebraic group. It is clear that the Galois bundle can be seen as a differential algebraic subgroup of such group. Then we have a canonical representation of the Galois group as a differential algebraic subgroup of $G \otimes \mathcal{M}(S)$.

**5.2. Analytic Galois Bundle and Picard-Vessiot Bundle.** For this section let us assume that $G$ is $\text{GL}(E)$, the group of linear automorphisms of a complex finite dimensional vector space. In this case, the considered automorphic system $\tilde{A}$ is a system of linear homogeneous differential equations with meromorphic coefficients.
depends on the base point \( \omega \) on the automorphisms of the fiber functor

Let us consider \((\text{bundle, Picard-Vessiot})\) objects of the category \( \mathcal{T} \) spanned by \( E \) trough tensor products, arbitrary direct sums, and their linear subspaces. Denote by \( \mathcal{T}^\nabla \) the category of linear connections spanned by \((E \times S, \nabla_{E \times S})\) through tensor products, arbitrary direct sums, and linear subconnections. The objects of \( \mathcal{T} \) are linear subspaces of the tensor spaces,

\[
F \subset T_{n_1, \ldots, n_r}^m(E) = \bigoplus_{i=1}^r E^{\otimes n_i} \oplus (E^*)^{\otimes m_i}.
\]

The objects of \( \mathcal{T}^\nabla \) are linear subconnections of the induced connections in the tensor bundles

\[
(V, \nabla_V) \subset (T_{n_1, \ldots, n_r}^m(E) \times S, \nabla_{T_{n_1, \ldots, n_r}^m(E)}).
\]

As before, we define the Riemann surface \( S^\times \) by removing from \( S \) the poles of the coefficients of the differential equations. Each point \( t \in S^\times \) defines a fiber functor:

\[
\omega_t : \mathcal{T}^\nabla \to \mathcal{T}
\]

that sends vector bundles to their fibers in \( t \). For a subconnection \((V, \nabla_V)\) as in (9), the fiber on \( t \in S^\times \) is a linear subspace \( V_t \) of \( T_{n_1, \ldots, n_r}^m(E) \).

It is known that the algebraic differential Galois group is the group of automorphisms of the fiber functor \( \omega_t \). The representation of this differential Galois group depends on the base point \( t \). In this way we obtain a bundle, that we call the Picard-Vessiot bundle,

\[
PV(\tilde{A}) \to S^\times,
\]

whose fiber in \( t \) is the group \( PV_t(\tilde{A}) \) of automorphisms of \( \omega_t \).

We represent the algebraic Galois group \( PV_t(\tilde{A}) \) into \( GL(E) \) in the following way. Let us consider \((V, \nabla_V)\) an object of the category \( \mathcal{T}^\nabla \). Its fiber in \( t \in S \) is a vector subspace of certain tensor product

\[
T_{n_1, \ldots, n_r}^m(E) = \bigoplus_{i=1}^r E^{\otimes n_i} \otimes (E^*)^{\otimes m_i}.
\]

An element \( \sigma \in GL(E) \) induces linear transformations of the tensor product. It is known that \( \sigma \) is in the differential Galois group \( PV_t(\tilde{A}) \) if and only if is stabilizes the vector space \( V_t \) for all object \((V, \nabla_V)\) of the category \( \mathcal{T}^\nabla \).

\[
PV_t(\tilde{A}) = \{ \sigma \in GL(E) \mid \sigma(V_t) = V_t \ \forall (V, \nabla_V) \in \text{Obj}(\mathcal{T}^\nabla) \}.
\]

There is a dictionary between linear connections \((V, \nabla_V)\) and meromorphic solutions of Lie-Vessiot systems in associated to \( \tilde{A} \) in algebraic homogeneous spaces.

Let us consider \((V, \nabla_V)\) as above. Let \( k \) be the dimension of the fibers of \( V \); for each \( t \in S^\times \) the fiber \( V_t \) is a \( k \)-plane of \( T_{n_1, \ldots, n_r}^m(E) \). Let us consider the grassmanian variety \( Gr(k, T_{n_1, \ldots, n_r}^m(E)) \) of \( k \)-planes in \( T_{n_1, \ldots, n_r}^m(E) \); it is a \( GL(E) \)-space. The map,

\[
S^\times \to Gr(k, T_{n_1, \ldots, n_r}^m(E)) \quad t \mapsto V_t,
\]

is a meromorphic solution of the Lie-Vessiot system induced by \( \tilde{A} \) into the grassmanian variety. This solution is contained in a \( GL(E) \)-orbit, that we denote by \( M \). Hence, the map

\[
S^\times \to M, \quad t \mapsto V_t,
\]
is a meromorphic solution of the Lie-Vessiot system $\tilde{A}^M$.

This homogeneous space $M$ is isomorphic to the quotient $GL(E)/H_Vt$ where $H_Vt$ is the stabilizer subgroup of the linear subspace $V_t$; it is an algebraic subgroup of $GL(E)$. Reciprocally, by Chevalley’s theorem, any algebraic group is the stabilizer of certain vector subspace. This means that the algebraic Galois group $PVt(\tilde{A})$ is the group of linear transformations $\sigma \in GL(E)$ that fix the values in $t$ of all meromorphic solutions of associated Lie-Vessiot systems in algebraic homogenous spaces. We have proven the following.

**Theorem 5.5.** There is a canonical inclusion $\text{Gal}_t(\tilde{A}) \subseteq PVt(\tilde{A})$. The analytic Galois group is Zariski dense in the algebraic Galois group.

**Proof.** Let $\mathcal{A}(\tilde{A})$ be the set of all the different meromorphic solution of all the different Lie-Vessiot systems induced by $\tilde{A}$ in algebraic homogeneous $GL(E)$-spaces. Then, $\mathcal{A}(\tilde{A}) \subseteq M(\tilde{A})$. We have that,

$$\text{Gal}_t(\tilde{A}) = \bigcap_{x \in M(\tilde{A})} H_x(t) \subseteq PVt(\tilde{A}) = \bigcap_{x \in \mathcal{A}(\tilde{A})} H_x(t).$$

Let us see that $\text{Gal}_t(\tilde{A})$ is Zariski dense. Let $H$ be the Zariski closure of $\text{Gal}_t(\tilde{A})$. It is an intermediate algebraic subgroup,

$$\text{Gal}_t(\tilde{A}) \subseteq H \subseteq PVt(\tilde{A}).$$

Let $M$ be $G/H$. By Lemma 5.3 there is a meromorphic solution of the Lie-Vessiot system in $GL(E)/\text{Gal}_t(\tilde{A})$; the algebraic homogeneous space $M$ is a quotient of such space. Therefore, there is a meromorphic solution of $\tilde{A}^M$ and $PVt(\tilde{A}) \subseteq H$. \qed

**Example 8.** Consider the differential equation,

$$\dot{x} = \frac{1}{t}.$$

It is an automorphic equation in the additive group $\mathbb{C}$. There is an analytic action of $\mathbb{C}$ on $\mathbb{C}^*$, $\mathbb{C} \times \mathbb{C}^* \to \mathbb{C}^*$, $(x, y) \mapsto x \cdot y = e^x y$. The associated Lie-Vessiot system is $\dot{y} = y/t$. It has meromorphic solutions $y = \lambda t$. The analytic Galois is contained in the isotropy group $2\pi i \mathbb{Z} \subseteq \mathbb{C}$; in fact they coincide. However, the algebraic Galois group is the whole additive group.

**Remark 5.** Example above shows that the analytic differential Galois group may contain more information than the algebraic differential Galois group. There arise an interesting question: Are there cases in which the analytic differential Galois group is defined but the algebraic one is not? This will happen exactly when the structural group $G$ is analytic but not algebraic, that is, a non-algebraic complex torus.

### 5.3. Integration by Quadratures.

The Lie’s reduction method, applied to an specific case of homogeneous space, gives us an analytic version of Kolchin theorem on reduction to the Galois group.

**Theorem 5.6.** Assume that the fiber of the Galois bundle $\pi: \text{Gal}(\tilde{A}) \to S^\times$, is contained a connected group $H \subset G$. Assume one of the additional hypothesis:

1. $H$ is an special group.
2. $S$ is a non-compact Riemann surface.
Then there is a meromorphic gauge transform of $G \times S$ that reduces $\vec{A}$ to an automorphic system in $H$.

Proof. Let $M$ be the homogeneous space $\text{Gal}_{t_0}(\vec{A})$. We can apply an internal automorphism of $G$ in order to ensure that $\text{Gal}_{t_0}(\vec{A}) \subset H$. There is a natural projection $M \to G/H$. Because of Lemma 5.3, there is a meromorphic solution of $\vec{A}^M$. This solution projects onto a meromorphic solution in $G/H$. By Lie’s reduction method, Theorem 4.4, there exist a gauge transformation reducing $\vec{A}$ to $\mathcal{R}(H)$. \hfill \square

5.4. Quadratures in Abelian Groups. If $G$ is a connected abelian group, it is known that the exponential map, $\mathcal{R}(G) \to G$, $\vec{A} \mapsto \exp(\vec{A})$, is the universal covering of $G$; in fact it is a group morphism if we consider the Lie algebra $\mathcal{R}(G)$ as a vector group. The integration of an automorphic equation in the vector space $\mathcal{R}(G)$ is done by a simple quadrature in $S$; thus the integration of an automorphic equation in $G$ is done by composition of the exponential map with this quadrature:

$$\sigma(t) = \exp \left( \int_{t_0}^t \vec{A}(\tau) d\tau \right),$$

where $d\tau$ is the meromorphic 1-form in $S$ such that $\langle d\tau, \partial \rangle = 1$.

5.5. Solvable Groups. Assume that there is a subgroup $G' \lhd G$ such that the quotient $\bar{G} = G/G'$ is an abelian group. We have an exact sequence of groups,

$$G' \to G \to \bar{G}.$$

The automorphic vector field $\vec{A}$ on $G$ is projected onto an automorphic vector field $\vec{B}$ on $\bar{G}$. $\bar{G}$ is a abelian, and then we can find the general solution of $\vec{B}$ by means of the exponential of a quadrature. The quadrature is of the form

$$\int_{t_0}^t \vec{B}(\tau) d\tau,$$

where $\vec{B}(\tau)d\tau$ is a closed 1-form with vectorial values in $\mathcal{R}(G)$. This 1-form is holomorphic in $S^\times$, and meromorphic in $S$. In the general case this closed 1-form is not exact. We need to consider the universal covering $\bar{S}^\times \to S^\times$. $\bar{S}^\times$ is simply connected, and by Poincare’s lemma every closed 1-form is exact. Then we can define,

$$\sigma: \bar{S}^\times \to \bar{G}, \quad t \mapsto \exp \left( \int_{t_0}^t \vec{B}(\tau) d\tau \right).$$

Let $t_0$ be a point of $S^\times$ and $\bar{t}_0$ a point of $\bar{S}^\times$ in the fiber of $t_0$. There is natural action of the Poincare’s fundamental group $\pi_1(\bar{S}^\times, \bar{t}_0)$ on the space of sections $\mathcal{O}(\bar{S}^\times, G)$; this is the monodromy representation. Let $H_\sigma$ be the isotropy of $\sigma$ for this action. There is a minimal intermediate covering $S^\times(\sigma)$ such that the section $\sigma$ factorizes. The fiber of the such covering $S^\times(\sigma) \to S^\times$ is isomorphic to the quotient
Some of the ramification points $S^\times(\sigma) \to S^\times$ have finite index; we add them to $S(\sigma)$ obtaining a bigger surface $S_1(\sigma)$. The projection of $S_1(\sigma)$ onto $S$ is a ramified covering of certain intermediate surface $S^\times \subset S_1 \subset S$. The section $\sigma(t)$ is meromorphic in $S_1(\sigma)$. Then, we substitute the Riemann surface $S_1(\sigma)$ for $S$; $\sigma(t)$ is a meromorphic solution of $\vec{B}$ in $S_1(\sigma)$. We apply Lie's reduction method 5, and reduce our equation to an automorphic equation in $G'$ with meromorphic coefficients in $S_1(\sigma)$. We can iterate this process and we obtain then the following theorem:

**Theorem 5.7.** Assume that $G$ is a connected solvable group, and one of the following hypothesis:

(a) $G$ is a special group.
(b) $S$ is a non-compact Riemann surface.

Then the automorphic system $\vec{A}$ on $G$ is integrable by quadratures of closed meromorphic 1-forms in $S$ and the exponential map in $G$.

**Proof.** Consider a resolution chain $G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_{n-1} \triangleleft G$. We can consider the process above with respect $G_{n-1} \triangleleft G$. If $S$ is non-compact, we are under the hypothesis of Grauert theorem. In the compact case, if $G$ is special, then it is a connected linear solvable group, so $G_{n-1}$ is also special. In both cases we can apply Proposition 5. We reduce the automorphic system to an automorphic system in $G_{n-1}$ and take coefficients functions in the corresponding covering of $S$. We iterate this process until we reduce $\vec{A}$ to canonical form $\partial$.

6. **Infinitesimal symmetries.** Let us consider the extended phase space $S^\times \times G$ for the automorphic system $\vec{A}$. We are looking for vector field symmetries of the system. It means, vector fields $\vec{L}$ in $S^\times \times G$ such that the Lie bracket verifies $[\vec{L}, \vec{A}] = \lambda(t, \sigma)\vec{A}$. This equation defines a sheaf of Lie algebras of infinite dimension of vector fields in $S^\times \times G$. As stated in [2], we can differentiate between characteristic and non-characteristic symmetries. Characteristic symmetries are them that are tangent to the solutions, *id est* proportional to $\vec{A}$. The sheaf of Lie algebra of characteristic symmetries is generated by $\vec{A}$: for any complex analytic function $F$ in $S^\times \times G$, $F\vec{A}$ is a characteristic symmetry of $\vec{A}$. Characteristic symmetries form a sheaf of ideals of the sheaf symmetries, and then there is a quotient, the sheaf of non-characteristic symmetries (see [3], also [2] for the linear case). On the other hand, we can also consider the sheaf of Lie algebras of transversal symmetries. We say that a vector field in $S^\times \times G$ is transversal if it is tangent to the fibers of the projection onto $S$. A vector field is transversal if and only if it can be written in the form,

$$\vec{L} = \sum_{i=1}^{s} F_i(t, \sigma)\vec{A}_i, \quad F_i \in \mathcal{O}_{S^\times \times G},$$

(10)
where the $\vec{A}_i$ form a basis of the tangent bundle to $G$. For example, they can form a basis of the Lie algebra $\mathcal{R}(G)$ of right invariant vector fields; or alternatively, they can form a basis of the Lie algebra $\mathcal{L}(G)$ of left invariant vector fields in $G$. Both cases lead us to interesting conclusions.

Let $\vec{L}$ be a transversal symmetry of $\vec{A}$; direct computation gives that the Lie bracket $[\vec{L}, \vec{A}]$ is a transversal vector field. Then, transversal symmetries are defined by the more restrictive equations,

$$\langle \vec{L}, dt \rangle = 0, \quad [\vec{L}, \vec{A}] = 0.$$ 

Any symmetry can be reduced to a transversal symmetry by adding a multiple of $\vec{A}$. For any non-transversal symmetry $\vec{L}$, we have that $\vec{L} - \langle dt, \vec{L} \rangle \vec{A}$ is a transversal symmetry. The kernel of such projection is precisely the space of characteristic symmetries. Thus, the sheaf of Lie Algebras of non-characteristic symmetries is isomorphic with the sheaf of transversal symmetries. For this reason, we restrict our studies to the sheaf of transversal symmetries.

**Definition 6.1.** We denote by $\text{Trans}(\vec{A})$ the sheaf of Lie algebras of transversal symmetries of $\vec{A}$.

6.1. **Equation of Transversal Symmetries in Function of Left Invariant Vector Fields.** Let $\vec{L}$ be an analytic vector field in $G$; $\vec{L}$ is a left invariant vector field if and only if $[\vec{L}, \vec{R}] = 0$ for all right invariant vector field $\vec{R} \in \mathcal{R}(G)$.

The right invariant vector fields are symmetries of the left invariant vector fields and vice versa. This property leads us to some interesting conclusions.

Let us consider $\{\vec{L}_1, \ldots, \vec{L}_s\}$ a basis of $\mathcal{L}(G)$, the Lie algebra of left invariant vector fields in $G$. An arbitrary transversal vector field in $S \times G$ is written in the form

$$\vec{L} = \sum_{i=1}^{s} g_i(\sigma, t) \vec{L}_i.$$ 

Let us set out the equations of transversal symmetries,

$$[\vec{L}, \vec{A}] = 0,$$

we expand the Lie bracket,

$$[\vec{L}, \vec{A}] = - \sum_{j=1}^{s} (\vec{A}g_j(t, \sigma)) \vec{L}_j,$$

and then we obtain the equation for the coefficients $g_j(t, \sigma)$,

$$\vec{A}g_j(\sigma, t) = 0, \quad j = 1, \ldots, s.$$ 

We have recovered the following result, which seems to be well known by differential geometers. From a differential algebraic point of view it was stated in the linear case by Athorne [2].

**Theorem 6.2.** Consider $\mathcal{O}_{S \times G}^\vec{A}$ the sheaf of first integrals of $\vec{A}$. Then, transversal symmetries of $\vec{A}$ are left invariant vector fields with coefficients in $\mathcal{O}_{S \times G}^\vec{A}$,

$$\text{Trans}(\vec{A}) = \mathcal{L}(G) \otimes_{\mathbb{C}} \mathcal{O}_{S \times G}^\vec{A}.$$ 

Note that the algebra of left invariant vector fields is a finite dimensional Lie algebra of dimension $s$ contained in $\text{Trans}(\vec{A})$. 
6.2. **Equation of Transversal Symmetries in Function of Right Invariant Vector Fields.** Consider,

\[ \tilde{A} = \partial + \sum_{i=1}^{s} f_i(t) \tilde{A}_i, \quad \tilde{L} = \sum_{i=1}^{s} g_i(t, \sigma) \tilde{A}_i, \]

where \( \{ \tilde{A}_1, \ldots, \tilde{A}_s \} \) is a basis of \( \mathcal{R}(G) \). Let us denote by \( c^k_{ij} \) the constants of structure of the Lie algebra \( \mathcal{R}(G) \), \( [\tilde{A}_i, \tilde{A}_j] = \sum_{k=1}^{c} c^k_{ij} \tilde{A}_k \).

Let us write the equations for transversal symmetries,

\[ [\tilde{L}, \tilde{A}] = \sum_{k=1}^{s} \left( -\partial g_k(t, \sigma) - \sum_{i=1}^{k} f_i(t) \left( \tilde{A}_i g_k(t, \sigma) - \sum_{j=1}^{s} g_j(t, \sigma) c^k_{ij} \right) \right) \tilde{A}_k = 0, \]

giving us the \( s \) partial differential equations satisfied by the coefficients \( g_k(\sigma, t) \) of \( \tilde{L} \),

\[ \partial g_k = -\sum_{i=1}^{s} f_i(t) \left( \tilde{A}_i g_k - \sum_{j=1}^{s} g_j c^k_{ij} \right). \tag{11} \]

6.3. **Right Invariant Symmetries.** Consider \( \mathcal{R}(G) \) as a \( \mathbb{C} \)-vector space. The group \( G \) acts in \( \mathcal{R}(G) \) through the adjoint representation,

\[ G \times \mathcal{R}(G) \to \mathcal{R}(G), \quad (\sigma, A) \mapsto \text{Adj}_\sigma(A) = L_\sigma A, \]

where \( \text{Adj}_\sigma(A) \) is the vector field \( A \) altered by a left translation of ratio \( \sigma \). Note that the value of \( \text{Adj}_\sigma(A) \) at the identity is \( R'_{\sigma^{-1}} L'_\sigma(A_{Id}) \). By the adjoint representation, \( \mathcal{R}(G) \) is a \( G \)-space. Then the automorphic system \( \tilde{A} \) induces a Lie-Vessiot system \( \tilde{R} \) in \( \mathcal{R}(G) \). Let us analyze what is the nature of the solutions of \( \tilde{R} \). A local solution of \( \tilde{R} \) defined in \( S' \subset S \) is an analytic map \( S' \to \mathcal{R}(G) \). We interpret this map as a vector field in \( S' \times G \) tangent to the fibers of the projection onto \( S' \). Such a solution is written in form,

\[ \tilde{V}(t) = \sum_{i=1}^{n} g_i(t) A_i, \]

and the differential equations for the coefficients \( g_i(t) \) are,

\[ \partial g_i(t) = \sum_{i,j=1}^{s} f_i(t) g_j c^k_{ij} \tag{12} \]

which is precisely a particular case of equation (11). Then we can state:

**Lemma 6.3.** The solutions of \( \tilde{R} \) in \( \mathcal{R}(G) \times S \) are the transversal symmetries of \( \tilde{A} \) whose restriction to fibers of \( G \times S \to S \) are right-invariant vector fields.

Let us consider, as before, that \( \tilde{A} \) is meromorphic in \( S \) complex analytic in \( S^s \). Denote \( \text{Right}(\tilde{A}) \) the set of transversal symmetries of \( \tilde{A} \) meromorphic in \( G \times S \) whose restriction to fibers of \( G \times S \to S \) are right invariant vector fields. In other words, the space of meromorphic solutions of \( \tilde{R} \). The space \( \text{Right}(\tilde{A}) \) is a finite dimensional Lie subalgebra of the algebra of sections of \( \text{Trans}(\tilde{A}) \) of dimension less or equal than \( s \). On the other hand, for each \( t \in S \) we can consider \( \text{Right}_t(\tilde{A}) \), the space of the values at \( t \) of elements of \( \text{Right}(\tilde{A}) \). We know that \( \text{Right}_t(\tilde{A}) \) is a Lie subalgebra of \( \mathcal{R}(G) \), and its class of conjugacy depends meromorphically on \( t \) in \( S \).
Theorem 6.4. For all \( t \) in \( S^\times \) the group \( \text{Gal}_t(\vec{A}) \) is contained in the centralizer of \( \text{Right}_t(\vec{A}) \).

Proof. Consider \( \vec{X}_1(t), \ldots, \vec{X}_r(t) \) a basis of \( \text{Right}(\vec{A}) \). Then \( \vec{X}_i(t) \) is a set of meromorphic solutions of the adjoint equation induced by \( \vec{A} \) in \( R(G) \). For each \( t \in S^\times \) we have that \( \sigma \in \text{Gal}_t(\vec{A}) \) verifies,

\[
\text{Adj}_\sigma(\vec{X}(t)) = \vec{X}(t),
\]

and then \( \sigma \) is in the centralizer of \( \text{Right}_t(\vec{A}) \).

Corollary 4. Let us \( \vec{A} \) automorphic system in the symplectic group \( \text{Sp}(2n, \mathbb{C}) \).

If \( \text{Right}(\vec{A}) \) contains an abelian algebra of dimension \( n \), then for all \( t \) in \( S^\times \) the component of the identity element of the analytic Galois group \( \text{Gal}_t^0(\vec{A}) \) is abelian.

Proof. Let us consider the Lie algebra \( \text{sp}(2n, \mathbb{C}) \). It is the Lie algebra of linear Hamiltonian autonomous vector fields in \( \mathbb{C}^{2n} \). Consider \( P \) the space of homogeneous polynomials of degree 2 in the canonical coordinates in \( \mathbb{C}^{2n} \). The space \( P \) is a Poisson algebra and Hamilton equations gives us an isomorphism of \( P \) with \( \text{sp}(2n, \mathbb{C}) \). For each \( t \) in \( S^\times \) we can consider both the Lie algebra \( \text{gal}_t(\vec{A}) \) of the Galois group and \( \text{Right}_t(\vec{A}) \) as Poisson subalgebras of \( P \). Theorem 6.4 implies that \( \text{Gal}_t(\vec{A}) \) is contained in the centralizer of \( \text{Right}_t(\vec{A}) \), and it implies that the Lie algebra of the Galois group \( \text{gal}_t(\vec{A}) \) commutes with \( \text{Right}_t(\vec{A}) \). In terms of Poisson brackets:

\[
\{\text{gal}_t(\vec{A}), \text{Right}_t(\vec{A})\} = 0.
\]

Thus, \( \text{gal}_t(\vec{A}) \) and \( \text{Right}_t(\vec{A}) \) are orthogonal Poisson subalgebras of \( P \). If we assume that \( \text{Right}_t(\vec{A}) \) is an abelian subalgebra of \( P \) of maximal dimension \( n \) then, by [36], the orthogonal of an abelian subalgebra of maximal dimension is also abelian. Hence \( \text{gal}_t(\vec{A}) \) is abelian, and the connected component \( \text{Gal}_t^0(\vec{A}) \) is an abelian group.

Remark 6. The adjoint action of \( G \) in \( R(G) \) is algebraic. In the case of the Picard-Vessiot theory, Theorem 6.4 also holds. The group \( \text{PV}_t(\vec{A}) \) is contained in the centralizer of \( \text{Right}_t(\vec{A}) \). It implies an stronger version of Corollary 4, because the connected component of the algebraic differential Galois group \( \text{PV}_t^0(\vec{A}) \) is bigger than \( \text{Gal}_t^0(\vec{A}) \).

7. Final comments.

Remark 7. The theory presented in this paper is finite dimensional. However, the differential Galois theory naturally extend to the infinite dimensional case. This has been already done by H. Umemura [53, 54] and B. Malgrange [31, 32, 33]. Those theories have been recently proven to be equivalent [55], and a number on applications have been developed (see e.g. [13], [14], [39]). In particular, G. Casale (see [15]) computed the Malgrange-Galois groupoid of the first Painlevé equation and found it to be the groupoid of all area-preserving transformations.

Remark 8. The monodromy of an automorphic system \( \vec{A} \) can be thought as a family of group morphisms,

\[
\text{mon}_t : \pi_1(S^\times, t) \to G.
\]

for \( t \in S^\times \). It is not difficult to check that its image \( \text{Mon}_t(\vec{A}) \), called monodromy group at \( t \), is a subgroup of the \( \text{Gal}_t(\vec{A}) \). In [24], the monodromy of some class of
multivalued functions is defined, and applied to the representability of such functions by quadratures. If we check Example 8, we see that the analytic differential Galois group coincide with the monodromy of the solutions. If there are no singularities or they are \textit{singular regular} then it is possible to prove that the analytic Galois group is the smallest one containing the monodromy, however this does not hold in the case in which the solutions have essential singularities.

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