HIGHER ORDER ANALOGUES OF EXTERIOR DERIVATIVE

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Abstract. We give new examples of linear differential operators of order $k = 2m + 1$ (any given odd integer) that are invariant under the isometries of $\mathbb{R}^n$ and satisfy so-called $L^1$-duality estimates and div/curl inequalities.

1. Introduction

The purpose of this note is to exhibit (elementary) examples of $k$th-order linear differential operators $\{S_k\}_k$ acting on $\mathbb{R}^n$ that can be regarded as higher order analogues of the exterior derivative complex

$$d : C^\infty_c(\mathbb{R}^n) \to C^\infty_c(\mathbb{R}^n), \quad 0 \leq q \leq n$$

(Here $C^\infty_c(\mathbb{R}^n)$ and $C^\infty_c(\mathbb{R}^n)$ stand for the $q$-forms and $(q + 1)$-forms on $\mathbb{R}^n$ whose coefficients are smooth and compactly supported.) More precisely we require that, for each $k$, $S_k$ map $q$-forms to $(q + 1)$-forms and $S_k \circ S_k = 0$; that the Hodge Laplacian for $S_k$, namely the operator $S_k S_k^* + S_k^* S_k$, be elliptic, and that the first-order operator in this family be the exterior derivative (that is, $S_1 = d$). We also require that $S_k$ and $S_k^*$ have non-trivial invariance properties and satisfy so-called $L^1$-duality estimates as well as div-curl inequalities (more on these below). While various operators satisfying one or more of these conditions were recently constructed for any order $k = 1, 2, 3, \ldots$, see [BB3], [LR] and [VS3]-[VS5], those operators fail to be invariant under
pullback by the rotations of $\mathbb{R}^n$ as soon as $k \geq 2$. By contrast, here we define linear differential operators $S_k$ of odd order

$$k = 2m + 1, \quad m = 0, 1, 2, \ldots,$$

that have the same invariance properties as the codifferential $d^*$ (the $L^2$-adjoint of exterior derivative) as soon as $k \geq 3$ (i.e. $m \geq 1$); that is

$$S_k \circ \psi^* = \psi^* \circ S_k \quad \text{and} \quad S_k^* \circ \psi^* = \psi^* \circ S_k^*$$

for any isometry $\psi : \mathbb{R}^n \to \mathbb{R}^n$ (as customary, $\psi^*$ denotes the pullback of $\psi$ acting on $q$-forms). While such invariance is non-trivial, it is far weaker than the invariance of $d$, which indeed is what should be expected of any linear differential operator of order greater than 1, see [P, Note 4] and [Te].

Specifically, given $m = 0, 1, 2, 3, \ldots$, we define

$$S_{(2m+1)} := d (d^*d)^m \quad \text{and, consequently,} \quad S_{(2m+1)}^* = (d^*d)^m d^*$$

It is clear that $S_{(1)} = d$ and, more generally, that $S_{(2m+1)}$ takes $q$-forms to $(q+1)$-forms and $S_{(2m+1)} \circ S_{(2m+1)} = 0$. It is also clear that the Hodge Laplacian for $S_{(2m+1)}$ is

$$\Box^{(2m+1)} = \Box^{2m+1} = \Box \circ \Box \circ \cdots \circ \Box$$

where the composition above is performed $(2m+1)$-many times and

$$\Box = dd^* + d^*d$$

is the Hodge Laplacian for the exterior derivative, so in particular $\Box^{(2m+1)}$ is elliptic because it is the composition of elliptic operators [Wo].

Note, however, that

$$d \circ S_{(2m+1)} = 0 \quad \text{and} \quad d^* \circ S_{(2m+1)}^* = 0$$

see (1), and so the natural compatibility conditions for the data of the Hodge system for $S_{(2m+1)}$ and $S_{(2m+1)}^*$ are the same as for the system for $d$ and $d^*$. As a consequence, the $L^1$-duality inequalities that are relevant to the Hodge system for $S_{(2m+1)}$ and $S_{(2m+1)}^*$ are the same as in [LS, page 61] and [VST], namely

**Proposition 1.1 (LS).** There is $C = C(n)$ such that for any $0 \leq q \leq n - 2$ and for any $f \in C_{q+1}^\infty (\mathbb{R}^n)$

\begin{equation}
(2) \quad df = 0 \quad \Rightarrow \quad |\langle f, h \rangle| \leq C \| f \|_{L_{q+1}^1(\mathbb{R}^n)} \| \nabla h \|_{L_{q+1}^n(\mathbb{R}^n)}
\end{equation}
for any $h \in L^{\infty}_{q+1}(\mathbb{R}^n)$ such that $\nabla h \in L^{n}_{q+1}(\mathbb{R}^n)$.

There is $C = C(n)$ such that for any $2 \leq q \leq n$ and for any $g \in C_{q-1}^{\infty}(\mathbb{R}^n)$

\[
\|g\|_{L^1_{q-1}(\mathbb{R}^n)} \leq C \|
abla h\|_{L^n_{q+1}(\mathbb{R}^n)}
\]

for any $h \in L^{\infty}_{q-1}(\mathbb{R}^n)$ such that $\nabla h \in L^{n}_{q-1}(\mathbb{R}^n)$.

Here $L^p_{q\pm 1}(\mathbb{R}^n)$ denote the spaces of $(q \pm 1)$-forms whose coefficients are in the Lebesgue class $L^p(\mathbb{R}^n)$, and $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2_{q\pm 1}(\mathbb{R}^n)$:

\[
\langle f, h \rangle = \int_{\mathbb{R}^n} f \wedge * h
\]

where $*$ denotes the Hodge-star operator for $\mathbb{R}^n$.

We take this opportunity to point out that these inequalities can be restated in the seemingly more invariant, in fact equivalent, fashion (see also [BB3, Theorem 1'])

**Proposition 1.2.** There is $C = C(n)$ such that for any $0 \leq q \leq n - 2$ and for any $f \in C_{q+1}^{\infty}(\mathbb{R}^n)$

\[
df = 0 \quad \Rightarrow \quad |\langle f, h \rangle| \leq C \|f\|_{L^1_{q+1}(\mathbb{R}^n)} \|d^* h\|_{L^n_q(\mathbb{R}^n)}
\]

for any $h \in L^{\infty}_{q+1}(\mathbb{R}^n)$ such that $d^* h \in L^n_q(\mathbb{R}^n)$.

There is $C = C(n)$ such that for any $2 \leq q \leq n$ and for any $g \in C_{q-1}^{\infty}(\mathbb{R}^n)$

\[
d^* g = 0 \quad \Rightarrow \quad |\langle g, h \rangle| \leq C \|g\|_{L^1_{q-1}(\mathbb{R}^n)} \|dh\|_{L^n_q(\mathbb{R}^n)}
\]

for any $h \in L^{\infty}_{q-1}(\mathbb{R}^n)$ such that $dh \in L^n_q(\mathbb{R}^n)$.

We show below that this result is equivalent to each of the following div/curl-type inequalities (one for any choice of $m = 0, 1, 2, \ldots$) which are proved with the methods of [LS]:

**Theorem 1.3.** Fix $0 \leq q \leq n$ and let $f \in L^1_{q+1}(\mathbb{R}^n)$ with $df = 0$, and $g \in L^1_{q-1}(\mathbb{R}^n)$ with $d^* g = 0$ be given. Then, for any $m = 0, 1, 2, 3, \ldots$, the (unique) $q$-form $v_{(m)}$ that solves the system

\[
\begin{align*}
S_{(2m+1)} v_{(m)} &= f \\
S^*_{(2m+1)} v_{(m)} &= g
\end{align*}
\]
belongs to the Sobolev space $W^{2m,r}_q(\mathbb{R}^n)$ with $r = n/(n-1)$ whenever $q$ is neither 1 (unless $g = 0$) nor $n-1$ (unless $f = 0$), and we have

\begin{equation}
\|v(m)\|_{W^{2m,r}_q(\mathbb{R}^n)} \leq C \left( \|f\|_{L^{q+1}_r(\mathbb{R}^n)} + \|g\|_{L^{q-1}_r(\mathbb{R}^n)} \right).
\end{equation}

Here $W^{2m,r}_q(\mathbb{R}^n)$ denotes the space of $q$-forms whose coefficients belong to the Sobolev space $W^{2m,r}(\mathbb{R}^n)$ of functions that are $2m$-many times differentiable in the sense of distributions and whose derivatives of any order $0 \leq |\alpha| \leq 2m$ are in the Lebesgue class $L^r(\mathbb{R}^n)$.

Proposition 1.4. With same hypotheses as Theorem 1.3, if $q = 1$ and $g \neq 0$ a substitute of (7) holds with $\|g\|_{L^1_1(\mathbb{R}^n)}$ replaced by $\|g\|_{H^1(\mathbb{R}^n)}$, where $H^1(\mathbb{R}^n)$ is the real Hardy space. If $q = n-1$ and $f \neq 0$, then (7) holds with $\|f\|_{H^1_n(\mathbb{R}^n)}$ in place of $\|f\|_{L^1_n(\mathbb{R}^n)}$, where $H^1_n(\mathbb{R}^n)$ is the space of $n$-forms whose coefficients are in $H^1(\mathbb{R}^n)$.

In the case when $m = 0$, Theorem 1.3 and Proposition 1.4 were proved in [LS], as in such case we have $S_{(1)} = d$ and $W^{0,r}_q(\mathbb{R}^n) = L^r_q(\mathbb{R}^n)$, and so Theorem 1.3 and Proposition 1.4 can be viewed as a generalization (actually, as we will see, a consequence) of those earlier results.

We remark in closing that one could also consider the operators

$$S_{(2m)} := (dd^*)^m \quad \text{and} \quad \tilde{S}_{(2m)} := (d^*d)^m$$

but these fail to map $q$-forms to $(q+1)$-forms and do not form a complex and as such are of not pertinent to this note.

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2. Proofs

We begin by recalling the elliptic estimates for $\Box^s = \Box \circ \cdots \circ \Box$, see [CZ] and e.g., [Wo], [SR].

Theorem 2.1. Given any $s \in \mathbb{Z}^+$, we have that

$$\Box^s : C^{\infty,\infty}_q(\mathbb{R}^n) \to C^{\infty,\infty}_q(\mathbb{R}^n)$$

is invertible, and

\begin{equation}
\|(|\Box^s|)^{-1} u\|_{W^{2s,r}_q(\mathbb{R}^n)} \lesssim \|u\|_{L^r_q(\mathbb{R}^n)}
\end{equation}

for any $1 < r < \infty$. 

Proof of Theorem 1.3. The case $m = 0$ was proved in [LS] and here we will show that the estimates in the case when $m \in \mathbb{Z}^+$ follow from the inequalities for $m = 0$. Without loss of generality we may assume: $f \in C_{q+1}^\infty(\mathbb{R}^n)$ and $g \in C_{q-1}^\infty(\mathbb{R}^n)$, so that each of $d^* f$ and $dg$ has smooth and compactly supported coefficients.

Applying the codifferential $d^*$ to the first equation in (6) and the exterior derivative $d$ to the second equation, and then adding the two equations, see (1), we find that

$$\Box^{m+1} v_{(m)} = d^* f + dg$$

Comparing $v_{(m)}$ with the solution $u$ of the Hodge system for $d$ and $d^*$ with same data as (6), namely

$$\begin{cases}
  du = f \\
  d^* u = g
\end{cases}$$

we find

$$\Box^m v_{(m)} = u$$

and so the elliptic estimate (8) (with $s := m$) grants

$$\|v_{(m)}\|_{W^{2m,r}(\mathbb{R}^n)} \lesssim \|u\|_{L^q(\mathbb{R}^n)}$$

for any $1 < r < \infty$. On the other hand, by [LS] we have that $u \in L^r_q(\mathbb{R}^n)$ with $r := n/(n-1)$ and

$$\|u\|_{L^r_q(\mathbb{R}^n)} \leq C(n)\left(\|f\|_{L^1_q(\mathbb{R}^n)} + \|g\|_{L^1_q(\mathbb{R}^n)}\right).$$

The desired conclusion (7) now follows by combining (11) and (12). $\square$

Proof of Proposition 1.4. The case $m = 0$ was proved in [LS] and here we will again only consider $m \in \mathbb{Z}^+$. As before, we may assume: $f \in C_{q+1}^\infty(\mathbb{R}^n)$ and $g \in C_{q-1}^\infty(\mathbb{R}^n)$. Now (11) holds as before, and if $q = 1$ and $g \neq 0$ it was proved in [LS] that a substitute of (12) holds with $\|g\|_{L^1_q(\mathbb{R}^n)}$ replaced by $\|g\|_{H^1(\mathbb{R}^n)}$, so the proof of Proposition 1.3 in the case $q = 1$ follows by combining (11) and the $H^1$-substitute for (12). (The case $q = n-1$ and $f \neq 0$ is proved in a similar fashion.) $\square$

Next we show that Theorem 1.3 (for any choice of $m = 0, 1, 2, \ldots$) is equivalent to Proposition 1.2.

Theorem 1.3 $\Rightarrow$ Proposition 1.2. To prove (11), it again suffices to consider the case when $f$ and $h$ have smooth and compactly supported coefficients; given $f$ as in (II), we consider the solution $v_{(m)}$ (for $m$ fixed arbitrarily) of the system (6) with $g := 0$, namely
\[
\begin{align*}
&\begin{cases}
  d(d^*d)^m v_{(m)} = f \\
  (d^*d)^m d v_{(m)} = 0
\end{cases}
\end{align*}
\]
see (1), so that
\[
\langle f, h \rangle = \langle d(d^*d)^m v_{(m)}, h \rangle
\]
Integrating by parts the right-hand side of this identity we obtain
\[
|\langle f, h \rangle| = |\langle v_{(m)}, (d^*d)^m d^*h \rangle|
\]
Hölder inequality for \(W^{2m,n/(n-1)}_q(\mathbb{R}^n)\) and its conjugate space \(W^{-2m,n}_q(\mathbb{R}^n)\) now grants
\[
|\langle f, h \rangle| \leq \|v_{(m)}\|_{W^{2m,n/(n-1)}_q(\mathbb{R}^n)} \|(d^*d)^m d^*h\|_{W^{-2m,n}_q(\mathbb{R}^n)}
\]
and by Theorem 1.3 it thus follows that
\[
|\langle f, h \rangle| \leq \|f\|_{L^{n/(n-1)}_q(\mathbb{R}^n)} \|(d^*d)^m d^*h\|_{W^{-2m,n}_q(\mathbb{R}^n)}
\]
On the other hand, we have
\[
\|(d^*d)^m d^*h\|_{W^{-2m,n}_q(\mathbb{R}^n)} = \sup_{\|\zeta\|_{W^{2m,n/(n-1)}_q(\mathbb{R}^n)} \leq 1} |\langle (d^*d)^m d^*h, \zeta \rangle|
\]
Integrating the latter by parts \(2m\)-many times and applying Hölder inequality for \(L^n_q(\mathbb{R}^n)\) and its dual space \(L^{n/(n-1)}_q(\mathbb{R}^n)\) we find
\[
|\langle (d^*d)^m d^*h, \zeta \rangle| \leq \|d^*h\|_{L^n_q(\mathbb{R}^n)} \|(d^*d)^m \zeta\|_{L^{n/(n-1)}_q(\mathbb{R}^n)}
\]
but
\[
\|(d^*d)^m \zeta\|_{L^{n/(n-1)}_q(\mathbb{R}^n)} \leq \|\zeta\|_{W^{2m,n/(n-1)}_q(\mathbb{R}^n)}
\]
which concludes the proof of (4). To prove (5) it suffices to apply (4) to \(f := *h \in C^{\infty,c}_{q+1}(\mathbb{R}^n)\) with \(\tilde{q} := n - q\) (recall that \(d^* \approx *d^*\) and that \(* : L^n_q(\mathbb{R}^n) \to L^{n/(n-1)}_{q+1}(\mathbb{R}^n)\) is an isometry). \(\square\)

\textbf{Proposition 1.2} \(\Rightarrow\) \textbf{Theorem 1.3} for any \(m = 0, 1, 2, \ldots\) Without loss of generality we may assume, as before, that \(f \in C^{\infty,c}_{q+1}(\mathbb{R}^n)\) and \(g \in C^{\infty,c}_{q-1}(\mathbb{R}^n)\). Fix \(m \in \{0, 1, 2, 3, \ldots\}\) arbitrarily and write
\[
v_{(m)} = X_{(m)} + Y_{(m)}
\]
where
\[
\begin{align*}
&\text{(13)} \quad \begin{cases}
  d(d^*d)^m X_{(m)} = f \\
  (d^*d)^m d^* X_{(m)} = 0
\end{cases}
\end{align*}
\]
and
\[
\begin{align*}
&\text{(14)} \quad \begin{cases}
  d(d^*d)^m Y_{(m)} = 0 \\
  (d^*d)^m d^* Y_{(m)} = g
\end{cases}
\end{align*}
\]
We claim that
\begin{equation}
\|X_{(m)}\|_{W^{2m,n/(n-1)}} \leq C\|f\|_{L^1_{q+1}}; \tag{15}
\end{equation}
and
\begin{equation}
\|Y_{(m)}\|_{W^{2m,n/(n-1)}} \leq C\|g\|_{L^1_{q-1}}. \tag{16}
\end{equation}
Note that if $Y_{(m)}$ solves (14) then $X_{(m)} := \ast Y_{(m)}$ solves (13) with $f := \ast g \in C_\infty^{\gamma+1}(\mathbb{R}^n)$ and $\bar{q} := n - q$, and so it suffices to prove (15) for $f$ and $X_{(m)}$ as in (13). (Note that the proof of (15) is non-trivial only for $q \neq n$, and the hypotheses of Theorem 1.3 require $q \neq n - 1$, so all together we may assume $0 \leq q \leq n - 2$.) By duality, proving (15) is equivalent to showing
\begin{equation}
\left| \langle D^\beta X_{(m)}, \varphi \rangle \right| \leq C\|f\|_{L^1_{q+1}}\|\varphi\|_{L_q^n} \tag{17}
\end{equation}
for any for any $\varphi \in C_\infty^{\gamma,\gamma}(\mathbb{R}^n)$ and for any multi-index $\beta$ of length $s$ (that is, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, $\beta_1 + \cdots + \beta_n = s$) and for any $0 \leq s \leq 2m$, where we have set
\[D^\beta X_{(m)} := \sum_{|I| = q} \left( \frac{\partial^s X_{(m)}}{\partial x^\beta} \right) dx^I.\]
To this end, write $\varphi = \Box^{m+1} \Phi$ for some $\Phi \in C_\infty^{\gamma,\gamma}(\mathbb{R}^n)$, see Theorem 2.1 then
\[\left| \langle D^\beta X_{(m)}, \varphi \rangle \right| = \left| \langle D^\beta X_{(m)}, \Box^{m+1} \Phi \rangle \right|\]
Integrating the right-hand side of this identity parts we find
\[\left| \langle D^\beta X_{(m)}, \varphi \rangle \right| = \left| \langle \Box^{m+1} X_{(m)}, D^\beta \Phi \rangle \right|\]
But $\Box^{m+1} X_{(m)} = d^* f$, see (13) and so
\[\left| \langle D^\beta X_{(m)}, \varphi \rangle \right| = \left| \langle d^* f, D^\beta \Phi \rangle \right| = \left| \langle f, dD^\beta \Phi \rangle \right|.
\]
Applying Proposition 1.2 to $h := dD^\beta \Phi \in C_\infty^{\gamma,\gamma}(\mathbb{R}^n)$ we conclude
\[\left| \langle D^\beta X_{(m)}, \varphi \rangle \right| \leq C(n)\|f\|_{L^1_{q+1}}\|d^* dD^\beta \Phi\|_{L_q^n} \leq C(n)\|f\|_{L^1_{q+1}}\|\Phi\|_{W^{2(m+1),n}}\]
On the other hand, since we had chosen $\Phi = (\Box^{m+1})^{-1} \varphi$, Theorem 2.1 grants
\[\|\Phi\|_{W^{2(m+1),n}_q} \lesssim \|\varphi\|_{L^n_q},\]
which combines with the previous estimates to give the desired inequality. \qed

It should by now be clear that Propositions 1.1 and 1.2 are equivalent to one another: on the one hand, it is obvious that Proposition 1.2 $\Rightarrow$ Proposition 1.1 (because $\nabla h \in L^1_{q+1} \Rightarrow dh \in L^n_{(q+1)\pm 1}$ and $d^* h \in W^{2(m+1),n}_q$).
and, moreover, $\|dh\|, \|d^* h\| \leq \|\nabla h\|$). On the other hand, it was proved in [LS, page 61] that Proposition [1.1] $\Rightarrow$ Theorem [1.3] in the case $m = 0$ which in turn, as we have just seen, gives Theorem [1.3] for arbitrary $m$ as well as Proposition [1.2].

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