Bethe Ansatz for the $SU(4)$ Extension of the Hubbard Model

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December 4, 2017

Abstract

We apply the nested algebraic Bethe ansatz method to solve the eigenvalue problem for the $SU(4)$ extension of the Hubbard model. The Hamiltonian is equivalent to the $SU(4)$ graded permutation operator. The graded Yang-Baxter equation and the graded Quantum Inverse Scattering Method are used to obtain the eigenvalue of the $SU(4)$ extension of the Hubbard model.

PACS: 75.10.Jm, 71.10.Fd, 05.30.Fk.
Keywords: Strongly correlated electrons, $SU(4)$ extension of the Hubbard model, Algebraic Bethe ansatz, Yang-Baxter equation.

1 Introduction

Since the discovery of high-temperature superconductivity, much more attention has been paid to the theoretical mechanism for such phenomena. Most proposals concern about the Hubbard model and the $t-J$ model [1]. C.N.Yang [2] advocated the importance of $\eta$-pairing mechanism and the property of off-diagonal long-range order (ODLRO) for the eigenfunctions in superconductivity. And it is stated that the wave function of BCS theory do have the ODLRO. Essler, Korepin and Schoutens [3] proposed an $SU(2|2)$ extended Hubbard model with the $\eta$-pairing symmetry. They showed that the eigenstates of this extended Hubbard model exhibit ODLRO and is thus superconducting. A more general extension of the Hubbard model with $\eta$-pairing is also proposed in ref.[4], see ref.[5] for some related works. One of the authors showed that the $SU(2|2)$ Hamiltonian can be constructed by the coupled $SU(1|1)$ chains, and proposed a new $SU(4)$ extension of the Hubbard model[6]. It can be proved that the eigenstate which shows ODLRO of the $SU(2|2)$ case is also an eigenstate for the $SU(4)$ extension of the Hubbard model. Thus we present another extension of the Hubbard model, its eigenstate possesses the ODLRO property. The $SU(4)$ extension of the Hubbard model is constructed by two coupled $SU(2)$ chains. To some extent, it is connected with the spin-ladders which are introduced to describe the quasi-one-dimensional materials in the superconductivity[8]. Recently, a series of exact solvable spin-ladders including coupled $t-J$ model were proposed[8]. Different from the $SU(4)$ spin-ladder model, the $SU(4)$ extension of the Hubbard model has the Hubbard-like interactions instead of the rung interactions in the ladder models. And another
motivation for us to study the $SU(4)$ extension of the Hubbard model is that it is the fermion version of the $SU(4)$ spin chain with orbital degeneracy[9, 10].

In this paper, we shall give a detailed calculation for the eigenvalues of the $SU(4)$ extension of the Hubbard model. This extension of the Hubbard model is a fermion chain, so we naturally introduce the graded Quantum Inverse Scattering Method (QISM) to solve the eigenvalue problem. The Hamiltonian is presented to be an $SU(4)$ graded permutation operator. We thus start from the rational $SU(4)$ R-matrix. By using the graded method, we show that the Hamiltonian can be obtained from the transfer matrix with periodic boundary conditions. Then, we can apply the graded nested Bethe ansatz method to find exactly the eigenvalues of the Hamiltonian.

The paper is organized as follows: We shall introduce our model and fix notations in the next section. We then shall prove that the one-dimensional Hamiltonian is equal to the graded permutation operator and the system is integrable in section 3. In section 4, the graded algebraic Bethe ansatz method will be used to obtain the eigenvalue of the Hamiltonian. the corresponding Bethe ansatz equations will be presented. Finally, we give a brief analysis of the Bethe ansatz equations in the last section.

2 Description of the Model

The Hamiltonian of the $SU(4)$ extension of the Hubbard model is written as:

$$H = -H^0 + U \sum_{j=1}^{N} (n_{j,\uparrow} - \frac{1}{2})(n_{j,\downarrow} - \frac{1}{2}),$$

where $H^0 = \sum_{jk} H_{jk}^0$ is the two coupled $SU(2)$ fermion chains, and the summation takes for nearest-neighbour sites. The local Hamiltonian of the coupled $SU(2)$ fermion chain has the form

$$H_{jk}^0 = [c_{j\uparrow}^\dagger c_{j\uparrow} + c_{k\downarrow}^\dagger c_{k\downarrow} + 2n_{j\uparrow}n_{k\downarrow} - n_{j\uparrow} - n_{k\downarrow} + 1]$$

$$\times [c_{j\downarrow}^\dagger c_{j\downarrow} + c_{k\uparrow}^\dagger c_{k\uparrow} + 2n_{j\downarrow}n_{k\uparrow} - n_{j\downarrow} - n_{k\uparrow} + 1].$$

The operator $c_{j\sigma}$ and $c_{j\sigma}^\dagger$ represent the annihilation and creation operators of electrons with spin $\sigma = \uparrow, \downarrow$ on a lattice site $j$. These operators are canonical Fermion operators satisfying anticommutation relations expressed as

$$\{c_{i\sigma}^\dagger, c_{j\tau}\} = \delta_{ij}\delta_{\sigma\tau}, \quad \{c_{i\sigma}, c_{j\tau}\} = \{c_{i\sigma}^\dagger, c_{j\tau}\} = 0.$$  

We denote by $n_{j\sigma} = c_{j\sigma}^\dagger c_{j\sigma}$ the number operator for the electrons on a site $j$ with spin $\sigma$.

Explicitly, the local Hamiltonian (2) can be written as

$$H_{jk}^0 = \sum_{\sigma=\uparrow, \downarrow} [(c_{j\sigma}^\dagger c_{k\sigma} + c_{k\sigma}^\dagger c_{j\sigma})(1 - n_{j,\sigma})(1 - n_{k,\sigma}) + (c_{j\sigma}^\dagger c_{k\sigma} + c_{k\sigma}^\dagger c_{j\sigma})n_{j,\sigma}n_{k,\sigma}]$$

$$+\sum_{\sigma=\uparrow, \downarrow} [(c_{j\sigma}^\dagger c_{j\sigma})(1 - n_{j,\sigma})(1 - n_{k,\sigma}) + (c_{j\sigma}^\dagger c_{j\sigma})n_{j,\sigma}n_{k,\sigma}]$$

$$+\sum_{\sigma=\uparrow, \downarrow} [(c_{j\sigma}^\dagger c_{j\sigma})(1 - n_{j,\sigma})(1 - n_{k,\sigma}) + (c_{j\sigma}^\dagger c_{j\sigma})n_{j,\sigma}n_{k,\sigma}]$$

$$+\frac{1}{2}(n_{j\uparrow} - n_{j\downarrow})(n_{k\downarrow} - n_{k\uparrow}) + \frac{1}{2}(n_{j\uparrow} - 1)(n_{k\uparrow} - 1)$$

$$+\frac{1}{2}(n_{j\downarrow} - 1)(n_{k\downarrow} - \frac{1}{2})(n_{k\uparrow} - \frac{1}{2}) + \frac{1}{4}.$$  

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where we denote by $n_j = \sum_{\sigma = \uparrow, \downarrow} n_{j\sigma}$ the number operator for the electrons on a site $j$. The Hamiltonian $H^0$ is invariant under spin-reflection $c_{j\uparrow} \leftrightarrow c_{j\downarrow}$. But it does not have the invariant property under particle-hole replacement $c_{j\sigma}^\dagger \leftrightarrow c_{j\sigma}$. There are four kinds of state at a given site:

$$|0 >, |\uparrow >, |\downarrow >, |\uparrow \downarrow >, |\downarrow \uparrow >,$$

(5)

two of them are fermionic and the other two are bosonic. The state $| \uparrow \downarrow >$ represents that an electron-pair is localized on a single lattice site. The formation of such pairs, called localons, is considered to be a mechanism to create ‘Cooper pairs’.

We introduce here some generators. The spin operators $S = \sum_{j=1}^{N} S_j$, $S^\dagger$ and $S^z$,

$$S^\dagger_j = c_{j\uparrow}^\dagger c_{j\downarrow}, \quad S^\dagger_j = c_{j\downarrow}^\dagger c_{j\uparrow}, \quad S^z_j = \frac{1}{2}(n_{j\uparrow} - n_{j\downarrow}),$$

(6)

form an $SU(2)$ algebra $[S, S^\dagger] = 2S^z$, $[S^\dagger, S^\dagger] = S^z$, $[S, S^z] = -S$. Since the fermion operator is grassmann odd, we find that the above spin operators are grassmann even (bosonic). The $\eta$-pairing generators also form an $SU(2)$ algebra $[\eta, \eta^\dagger] = 2\eta^2$, $[\eta^\dagger, \eta^\dagger] = \eta^\dagger$, $[\eta, \eta^\dagger] = -\eta$ with

$$\eta_j = c_{j\uparrow}c_{j\downarrow}, \quad \eta_j^\dagger = c_{j\downarrow}^\dagger c_{j\uparrow}^\dagger, \quad \eta_j^2 = -\frac{1}{2}\eta_j + \frac{1}{2},$$

(7)

We see that those operators be grassmann even. The generator $X_j = (n_{j\uparrow} - \frac{1}{2})(n_{j\downarrow} - \frac{1}{2})$ is also a grassmann even operator. We further introduce eight grassmann odd generators. They take the following form:

$$Q_{j\sigma} = (1 - n_{j, -\sigma})c_{j\sigma}^\dagger, \quad Q_{j\sigma}^\dagger = (1 - n_{j, -\sigma})c_{j\sigma},$$

$$\tilde{Q}_{j\sigma} = n_{j, -\sigma}c_{j\sigma}, \quad \tilde{Q}_{j\sigma}^\dagger = n_{j, -\sigma}c_{j\sigma}^\dagger,$$

(8)

with $\sigma = \uparrow, \downarrow$ representing spin up and spin down respectively. We shall focus in this paper on the one-dimensional case with periodic boundary conditions, that means the nearest-neighbour sites of site $j$ are $j \pm 1$. In terms of these generators, the one-dimensional Hamiltonian $H^0_{jj+1}$ is written as

$$H^0_{jj+1} = \sum_{\sigma = \uparrow, \downarrow} [Q_{j\sigma}^\dagger Q_{j+1\sigma} + Q_{j+1\sigma}^\dagger Q_{j\sigma} + \tilde{Q}_{j\sigma}^\dagger \tilde{Q}_{j+1\sigma} + \tilde{Q}_{j+1\sigma}^\dagger \tilde{Q}_{j\sigma}]$$

$$+ \eta_j^\dagger \eta_{j+1} + \eta_j \eta_{j+1}^\dagger - 2\eta_j^\dagger \eta_{j+1}^\dagger - S^\dagger_j S_{j+1} - S^z_j S^\dagger_{j+1} + 2S^z_j S^z_{j+1} + 4X_j X_{j+1} + \frac{1}{4}.$$  

(9)

Next, we shall show that the Hamiltonian $H^0_{jj+1}$ is equal to the graded $SU(4)$ permutation operator. Because it is the graded version of the permutation operator, note that this Hamiltonian does not preserve the property of $SU(4)$ invariant, for example, we do not have relation $[H^0, S] = 0$.

We can use $4 \times 4$ matrices to express the above generators. The represenation can be realized as

$$S_j = E^{21}_j, \quad S^\dagger_j = E^{12}_j, \quad S^z_j = \frac{1}{2} (E^{22}_j - E^{11}_j),$$

$$\eta_j = E^{34}_j, \quad \eta_j^\dagger = E^{43}_j, \quad \eta_j^2 = \frac{1}{2} (E^{33}_j - E^{44}_j),$$

$$Q_{j\uparrow} = E^{32}_j, \quad Q_{j\uparrow}^\dagger = E^{23}_j, \quad Q_{j\downarrow} = E^{31}_j, \quad Q_{j\downarrow}^\dagger = E^{13}_j,$$

$$\tilde{Q}_{j\uparrow} = -E^{14}_j, \quad \tilde{Q}_{j\uparrow}^\dagger = -E^{41}_j, \quad \tilde{Q}_{j\downarrow} = E^{24}_j, \quad \tilde{Q}_{j\downarrow}^\dagger = E^{42}_j,$$

(10)
where the matrix $E_\alpha^\beta$ acts on the $j$-th space with its elements defined as $(E_\alpha^\beta)_{ij} = \delta_{i,j}$. We remark that the matrix $E_\alpha^\beta$ is not a conventional matrix but a supermatrix with Grassmann number $\epsilon_\alpha + \epsilon_\beta$, where $\epsilon_\alpha = 0$ represents Grassmann even (boson), and $\epsilon_\alpha = 1$ represents Grassmann odd (fermion). From the representations (14), we find that we should choose the grading $\epsilon_1 = \epsilon_2 = 1$, $\epsilon_3 = \epsilon_4 = 0$ so that the representations have correct Grassmann number.

Considering the above representation, we find that the Hamiltonian can be written as

$$H^0_{jj+1} = \sum_{\alpha \neq \beta} (-1)^{\epsilon_\beta} E^\alpha_\beta \otimes E^\beta_\alpha + \sum_\alpha E^{\alpha \alpha} \otimes E^{\alpha \alpha}.$$  \hfill (11)

The right hand side this relation is just the graded $SU(4)$ permutation operator. Now, we define the super (graded) tensor-product as

$$[A \otimes B]_{mjk} = (-1)^{(\epsilon_m + \epsilon_k)\epsilon_j} A_{mk} B_{\beta\beta}. \hfill (12)$$

According to this definition, the non-zero elements of the local Hamiltonian read as:

$$[H^0_{jj+1}]_{\alpha\beta} = \begin{cases} (-1)^{\epsilon_a \epsilon_\beta} \delta_{\alpha a} \delta_{\beta \gamma}, & \alpha \neq \beta, \\ 1, & \alpha = \beta = \gamma = \nu. \end{cases} \hfill (13)$$

Thus we have proved that the Hamiltonian $H^0$ is equal to the graded permutation operator, and because the Hubbard interaction term $U \sum_j X_j$ commute with $H^0$, they have common eigenvectors. We then can use graded algebraic Bethe ansatz method to find the eigenvalue of the Hamiltonian (1). We point out that the Hamiltonian $H^0$ is neither the usual $SU(4)$ chain nor the super $SU(2)/2$ chain.

### 3 The Integrability of the Model

We begin with the $SU(4)$ rational R-matrix. The non-zero elements of the matrix have the form

$$\tilde{R}(u)^{ab}_{ac} = \begin{cases} u + i, & a = b, \\ (-1)^{\epsilon_a \epsilon_b} u, & a \neq b, \\ i, & a \neq b, \end{cases} \hfill (14)$$

where indices $a, b$ take values $1, \cdots, 4$, $i = \sqrt{-1}$ is the cross parameter. We know this R-matrix satisfies the usual Yang-Baxter equation

$$\tilde{R}_{12}(u_1 - u_2)\tilde{R}_{13}(u_1 - u_3)\tilde{R}_{23}(u_2 - u_3) = \tilde{R}_{23}(u_2 - u_3)\tilde{R}_{13}(u_1 - u_3)\tilde{R}_{12}(u_1 - u_2).$$  \hfill (15)

As in section 2, we still choose the grading $\epsilon_1 = \epsilon_2 = 1$, $\epsilon_3 = \epsilon_4 = 0$, that means the grading is FFBB. Introducing a diagonal matrix $P^{bd}_{ac} = (-1)^{\epsilon_a \epsilon_d} \delta_{ab} \delta_{cd}$, we modify the original R-matrix to the following form

$$R(u) = I \tilde{R}(u). \hfill (16)$$

For the non-zero elements of the R-matrix $R^{ab}_{cd}$, we have a relation $\epsilon_a + \epsilon_b + \epsilon_c + \epsilon_d = 0$. Thus we find the new defined R-matrix satisfies the graded Yang-Baxter equation which reads explicitely as

$$R(u - v)^{b_1b_2}_{a_1a_2} R(u)^{c_1c_2}_{b_1b_2} (v)^{c_3c_4}_{b_2b_3} (-)^{(\epsilon_{c_1} + \epsilon_{c_2}) \epsilon_{b_2}} = R(v)^{b_1b_2}_{a_1a_2} R(u)^{c_1c_2}_{a_1a_2} R(u - v)^{c_3c_4}_{b_1b_2} (-)^{(\epsilon_{c_1} + \epsilon_{c_2}) \epsilon_{b_2}}. \hfill (17)$$
We remark that we have already graded the R-matrix, that means the matrix is now a supermatrix with Grassmann numbers. For \( u = 0 \), \( R_{12}(0) = iP_{12} \), and the elements of \( P_{12} \) read

\[
P_{12}^{ab} = (-1)^{\epsilon_a \epsilon_b} \delta_{ad} \delta_{bc}.
\]

We know that this \( P_{12} \) is the super permutation operator corresponding to group \( SU(2|2) \) with FFBB grading.

In the framework of the QISM, we can construct the \( L \) operator from the R-matrix as:

\[
L_{aq}(u) \equiv R_{aq}(u),
\]

where \( a \) represents the auxiliary space and \( q \) represents the quantum space. Thus we have the graded Yang-Baxter relation

\[
R_{12}(u - v)L_{1}(u)L_{2}(v) = L_{2}(v)L_{1}(u)R_{12}(u - v).
\]

Here the tensor product is understood to be in the sense of super tensor product defined in relation (12). In the rest of this paper, all tensor products are in the super sense.

Following the standard method of QISM, the row-to-row monodromy matrix \( T_{N}(u) \) is defined as the matrix product over the \( N \) operators on all sites of the lattice,

\[
T_{a}(u) = L_{aN}(u)L_{aN-1}(u) \cdots L_{a1}(u),
\]

where \( a \) still represents the auxiliary space. Explicitly we write

\[
\begin{align*}
\{[T(u)]^{ab} & \}_{\alpha_{1} \cdots \alpha_{N}}^{\beta_{1} \cdots \beta_{N}} \\
& = L_{N}(u)_{\alpha_{a} \beta_{N}}^{cN \beta_{N-1}} L_{N-1}(u)_{\alpha_{N-1} \beta_{N-1}}^{cN-1 \beta_{N-1}} \cdots L_{1}(u)_{\alpha_{1} \beta_{1}}^{c\beta_{1}} (-1)^{\sum_{j=2}^{N}(\epsilon_{\alpha_{j}} + \epsilon_{\beta_{j}}) \sum_{i=1}^{j-1} \epsilon_{\alpha_{i}}}
\end{align*}
\]

By repeatedly using the Yang-Baxter relation (13), one can prove that the monodromy matrix also satisfies the Yang-Baxter relation

\[
R_{12}(u - v)T_{1}(u)T_{2}(v) = T_{2}(v)T_{1}(u)R_{12}(u - v).
\]

We consider in this paper the periodic boundary condition. The transfer matrix \( t(u) \) of this model is defined as the supertrace of the monodromy matrix in the auxiliary space. In general case, the supertrace is defined as

\[
t(u) = strT(u) = \sum (-1)^{\epsilon_{a}} T(u)_{aa}.
\]

As a consequence of the Yang-Baxter relation (13) and the unitarity property of the R-matrix

\[
R_{12}(u)R_{21}(-u) = (i + u)(i - u) \cdot id.,
\]

we can prove that the transfer matrix commutes each other for different spectral parameters.

\[
[ t(u), t(v) ] = 0
\]

Generally in this sense the model is integrable. Expanding the transfer matrix in the powers of \( u \), we can find conserved quantities, the first non-trivial conserved quantity is the Hamiltonian.
For the R-matrix under consideration, the Hamiltonian can be obtained by taking the logarithmic derivative of the transfer matrix at the zero spectral parameter,

$$H' = \sum_{j} H'_{jj+1} = \frac{d \ln[t(u)]}{du} \bigg|_{u=0} = \sum_{j} p_{jj+1} L'_{jj+1}(0).$$

(26)

Explicitly, we write the local Hamiltonian $H'_{jj+1}$ obtained above in a matrix form

$$H'_{jj+1} = \begin{pmatrix}
E^{11}_{j+1} & -E^{21}_{j+1} & E^{31}_{j+1} & E^{41}_{j+1} \\
-E^{12}_{j+1} & E^{22}_{j+1} & E^{32}_{j+1} & E^{42}_{j+1} \\
E^{13}_{j+1} & E^{23}_{j+1} & E^{33}_{j+1} & E^{43}_{j+1} \\
E^{14}_{j+1} & E^{24}_{j+1} & E^{34}_{j+1} & E^{44}_{j+1}
\end{pmatrix},$$

(27)

where $E^{ab}_{j+1}$ acts on $j+1$-th space. We can find this Hamiltonian is exactly the Hamiltonian $H^0_{jj+1}$. We show explicitly here what we mentioned above, this Hamiltonian is not the usual $SU(4)$ Hamiltonian because we have negative signs in the above relation, it is also different from the Hamiltonian of the supergroup $SU(2|2)$ for all diagonal elements are positive here. We have proved the Hamiltonian under consideration can be obtained from the transfer matrix of the exactly solvable model and therefore is integrable.

### 4 Graded Algebraic Bethe Ansatz Method

We use the nested algebraic Bethe ansatz method to find the eigenvalues of the transfer matrix. We start from the graded rational $SU(4)$ R-matrix with FFBB grading. Explicit form of the R-matrix is

$$R(u) = \begin{pmatrix}
-u - i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & u & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\
0 & -i & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -u - i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & i & 0 & 0 & 0 \\
0 & 0 & i & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & u & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & u & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & u & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & u & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & u + i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u + i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u + i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u + i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$  

(28)

This R-matrix satisfies the graded Yang-Baxter equation. In the framework of the QISM, the L-operator is defined by

$$L_j(u) = \begin{pmatrix}
-u - b(u) E^{11}_j & -i E^{21}_j & i E^{31}_j & i E^{41}_j \\
-i E^{12}_j & u - b(u) E^{22}_j & i E^{32}_j & i E^{42}_j \\
i E^{13}_j & i E^{23}_j & u + i E^{33}_j & i E^{43}_j \\
i E^{14}_j & i E^{24}_j & i E^{34}_j & u + i E^{44}_j
\end{pmatrix},$$

(29)
where $E_j^{\alpha \beta}$ acts on the $j$-th quantum space, and we use notation $b(u) \equiv 2u + i$. Considering the representations of the operator $E_j^{\alpha \beta}$ ([1]), we can also write L-operator in the following form

$$L_j(u) = \begin{pmatrix} u - b(u)\left(\frac{1}{4} - X_j + S_j \right) & -iS_j & iQ_{j\downarrow} & -i\tilde{Q}_{j\uparrow} \\ -iS_j & u - b(u)\left(\frac{1}{4} - X_j + S_j \right) & iQ_{j\uparrow} & i\tilde{Q}_{j\downarrow} \\ iQ_{j\downarrow} & iQ_{j\uparrow} & u + i\left(\frac{1}{4} + X_j + \eta_j \right) & -i\eta_j \\ -i\tilde{Q}_{j\uparrow} & i\tilde{Q}_{j\downarrow} & i\eta_j & u + i\left(\frac{1}{4} + X_j - \eta_j \right) \end{pmatrix}.$$  

(30)

We can also write it as

$$L_j(u) = \begin{pmatrix} u - b(u)(n_{j\downarrow} - n_{j\uparrow}n_{j\downarrow}) & -ic_{j\uparrow}c_{j\downarrow} & i(1 - n_{j\uparrow})c_{j\downarrow} & -i(1 - n_{j\downarrow})c_{j\uparrow} \\ -ic_{j\downarrow}c_{j\uparrow} & u - b(u)(n_{j\uparrow} - n_{j\uparrow}n_{j\downarrow}) & i(1 - n_{j\downarrow})c_{j\downarrow} & i\eta_{j\downarrow}c_{j\uparrow} \\ i(1 - n_{j\uparrow})c_{j\uparrow} & i(1 - n_{j\downarrow})c_{j\downarrow} & u + i(1 - n_{j\uparrow} + n_{j\uparrow}n_{j\downarrow}) & ic_{j\downarrow}c_{j\uparrow} \\ -in_{j\downarrow}c_{j\uparrow} & in_{j\uparrow}c_{j\downarrow} & ic_{j\downarrow}c_{j\uparrow} & u + in_{j\uparrow}n_{j\downarrow} \end{pmatrix}.$$  

(31)

We choose the local vacuum state as $|vac\rangle_j = (0, 0, 0, 1)^t$. Actually this local vacuum state is equal to the state $|\uparrow\downarrow\rangle_j$. Acting the L-operator on this local vacuum state, we have

$$L_j(u)|vac\rangle_j = \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & u & 0 \\ -iE_{j\downarrow} & iE_{j\downarrow} & u + i \end{pmatrix} \begin{pmatrix} |\uparrow\downarrow\rangle_j \\ |\uparrow\downarrow\rangle_j \\ |\uparrow\downarrow\rangle_j \\ i|\uparrow\rangle_j \end{pmatrix}.$$  

(32)

Define the vacuum state as $|vac\rangle = \otimes_{j=1}^N |vac\rangle_j$. This vacuum state is the full-filled state just like the ‘Dirac sea’. Using the standard QISM, we denote the monodromy matrix defined in (20) as follows,

$$T(u) = \begin{pmatrix} A_{11}(u) & A_{12}(u) & A_{13}(u) & B_1(u) \\ A_{21}(u) & A_{22}(u) & A_{23}(u) & B_2(u) \\ A_{31}(u) & A_{32}(u) & A_{33}(u) & B_3(u) \\ C_1(u) & C_2(u) & C_3(u) & D(u) \end{pmatrix}.$$  

(33)

The transfer matrix with periodic boundary conditions is thus written explicitly as

$$t(u) = D(u) + A_{33}(u) - A_{22}(u) - A_{11}(u) = D(u) + \sum_a (-1)^a A_{aa}(u).$$  

(34)

We write the last equation for convenience in using the nested Bethe ansatz method later. With the help of the definition of the monodromy matrix (24) and the result in relation (32), we find that the action of
the monodromy matrix on the vacuum state is
\[
T(u)|\text{vac} > = \begin{pmatrix} u^N & 0 & 0 & 0 \\ 0 & u^N & 0 & 0 \\ 0 & 0 & u^N & 0 \\ C_1(u) & C_2(u) & C_3(u) & (u + i)^N \end{pmatrix} |\text{vac} > .
\] (35)

From the Yang-Baxter relation (22), we can find the commutation relations which are necessary for the algebraic Bethe ansatz method,
\[
D(u)C_c(v) = \frac{v - u + i}{v - u} C_c(v)D(u) - i \frac{1}{v - u} C_c(u)D(v),
\] (36)
\[
A_{ab}(u)C_c(v) = (-1)^{(\epsilon_a + \epsilon_b)v} \frac{R^{(1)}(u - v)_{bc}}{u - v} C_{d_1}(v)A_{ab}(u) - (-1)^{\epsilon_a \epsilon_b} i \frac{1}{u - v} C_b(u)A_{ac}(v),
\] (37)
\[
C_{c_1}(v)C_{c_2}(u) = (-)^{\epsilon_{c_1} \epsilon_{c_2}} \frac{R^{(1)}(v - u)_{c_1c_2}}{v - u} C_{d_2}(u)C_{b_1}(v).
\] (38)

Here all indices take values 1,2,3, and matrix $R^{(1)}$ is equal to the original R-matrix (28) with indices being just 1,2,3. We assume the eigenvectors of the transfer matrix as
\[
C_{c_1}(v_1)C_{c_2}(v_2)\cdots C_{c_m}(v_m)|\text{vac} > F^{c_1\cdots c_m},
\] (39)
where $F^{c_1\cdots c_m}$ is a function of the spectral parameters $v_j$. Operating the transfer matrix (34) on this assumed eigenvector, applying repeatedly the commutation relations and using the result (35), we have
\[
t(u)C_{c_1}(v_1)C_{c_2}(v_2)\cdots C_{c_m}(v_m)|\text{vac} > F^{c_1\cdots c_m},
\]
\[
= [D(u) + (-1)^{\epsilon_a A_{aa}(u)}C_{c_1}(v_1)C_{c_2}(v_2)\cdots C_{c_m}(v_m)|\text{vac} > F^{c_1\cdots c_m}
\]
\[
= (u + i)^N \prod_{j=1}^{m} \left( \frac{v_j - u + i}{v_j - u} \right) C_{c_1}(v_1)C_{c_2}(v_2)\cdots C_{c_m}(v_m)|\text{vac} > F^{c_1\cdots c_m}
\]
\[
+ u^N \prod_{j=1}^{m} \left( \frac{1}{u - v_j} \right) t^{(1)}(u, \{v_k\})|d_1\cdots d_m\rangle C_{d_1}(v_1)C_{d_2}(v_2)\cdots C_{d_m}(v_m)|\text{vac} > F^{c_1\cdots c_m}
\]
\[
+ u.t.,
\] (40)
where u.t. means unwanted terms, and $t^{(1)}$ is the nested transfer matrix. The eigenvalue problem of $t(u)$ becomes now the eigenvalue problem of $t^{(1)}$. Denote the eigenvalues of $t$ and $t^{(1)}$ by $\Lambda$ and $\Lambda^{(1)}$. In order to cancel the unwanted terms, we need the following Bethe ansatz equations
\[
\prod_{j=1}^{m} (v_k - v_j - i)(v_k + i)^N + v_k^N \Lambda^{(1)}(v_k) = 0, \quad k = 1,2,\cdots, m.
\] (41)

The nested transfer matrix is written explicitly as
\[
t^{(1)}(u, \{v_k\})_{d_1\cdots d_m} = \begin{pmatrix}
-(-1)^{\epsilon_a R^{(1)}(u - v_1)_{aa} R^{(1)}(u - v_2)_{b_1b_2} \cdots R^{(1)}(u - v_m)_{ad_m-1}c_m} \\
-(-1)^{\epsilon_a \epsilon_{b_1} \epsilon_{d_1} + (\epsilon_a + \epsilon_{b_1}) \epsilon_{d_2} + \cdots + (\epsilon_a + \epsilon_{b_{m-1}}) \epsilon_{d_{m-1}}}
\end{pmatrix}
\] (42)

For the non-zero elements of the $R^{(1)}(u)_{ab}$, we still have $\epsilon_a + \epsilon_b + \epsilon_c + \epsilon_d = 0$, so we can prove the following relation in (12),
\[
(\epsilon_a + \epsilon_{b_1})\epsilon_{d_1} + (\epsilon_a + \epsilon_{b_2})\epsilon_{d_2} + \cdots + (\epsilon_a + \epsilon_{b_{m-1}})\epsilon_{d_{m-1}} = \sum_{j=2}^{m} (\epsilon_{c_j} + \epsilon_{d_j}) \sum_{k=1}^{j-1} \epsilon_{d_k}
\] (43)
That means the $R^{(1)}$ matrices in the above relation is still in the graded tensor product form which agrees with the definition in [21]. And the above defined nested transfer matrix can be defined as the supertrace on the auxiliary space for the reduced monodromy matrix which satisfies the Yang-Baxter relation.

\[ t^{(1)}(u, \{v_k\}) = \text{str} T^{(1)}(u, \{v_k\}) \]

\[ = (-1)^s R^{(1)}(u-v_1)_{b_1d_1} R^{(1)}(u-v_2)_{b_2d_2} \cdots R^{(1)}(u-v_m)_{b-md_m} (-1)^{ \sum_{j=2}^{m} (e_{j} + e_{d_j}) \sum_{k=1}^{j-1} e_{d_k} }, \]

\[ R_{12}^{(1)}(u-v) T_{1}^{(1)}(u, \{v_k\}) T_{2}^{(1)}(v, \{v_k\}) = T_{2}^{(1)}(v, \{v_k\}) T_{1}^{(1)}(u, \{v_k\}) R_{12}^{(1)}(u-v). \]  

(45)

Thus we deal with almost the same problem as the original one. Repeating the graded algebraic Bethe ansatz equations are listed as follows, where $v_j$ should satisfy their corresponding Bethe ansatz equations. To summarize, the Bethe ansatz equations are listed as follows,

\[ \Lambda^{(1)}(u) = \prod_{j=1}^{m} \left( \frac{\mu_j - u + i}{\mu_j - u} \right)^{m^{(1)}} (u - v_k + i) \prod_{k=1}^{m} \left( \frac{1}{u - \mu_j} \right) \prod_{k=1}^{m} (u - v_k) \Lambda^{(2)}(u). \]  

(46)

The eigenvalue $\Lambda^{(2)}(u)$ for grading FF can be obtained almost in the same way for the normal six-vertex model. We have

\[ \Lambda^{(2)}(u) = - \prod_{j=1}^{m^{(2)}} \left( \frac{\nu_j - u + i}{\nu_j - u} \right)^{m^{(1)}} (u - \nu_j - i) \prod_{k=1}^{m^{(2)}} \left( \frac{\nu_j - u - i}{\nu_j - u} \right) \prod_{k=1}^{m^{(2)}} (u - \mu_k). \]  

(47)

In addition to these relations, we denote by $\Lambda(u)$ the eigenvalue of the original transfer matrix $t(u)$. We have

\[ \Lambda(u) = \prod_{j=1}^{m} \left( \frac{\nu_j - u + i}{\nu_j - u} \right)^{m^{(1)}} (u + i)^{N} + \prod_{j=1}^{m^{(2)}} \left( \frac{1}{u - \nu_j} \right) u^{N} \Lambda^{(1)}(u), \]

(48)

where $v_j, \mu_j, \nu_j$ should satisfy their corresponding Bethe ansatz equations. To summarize, the Bethe ansatz equations are listed as follows,

\[ \left( \frac{v_k - \frac{i}{2}}{v_k + \frac{i}{2}} \right)^{N} = \prod_{j=1}^{m^{(1)}} \frac{\mu_j - v_k - i}{\mu_j - v_k + i} \prod_{l=1, l \neq k}^{m^{(1)}} \frac{v_k - v_l + i}{v_k - v_l - i}, \]  

(49)

\[ \prod_{j=1}^{m^{(1)}} \frac{\mu_k - v_j - i}{\mu_k - v_j + i} = \prod_{j=1}^{m^{(1)}} \frac{\mu_j - \mu_k - i}{\mu_j - \mu_k + i} \prod_{l=1}^{m^{(2)}} \frac{\mu_k - v_l - i}{\mu_k - v_l + i}, \]  

(50)

\[ \prod_{j=1}^{m^{(1)}} \frac{\nu_k - \nu_j - i}{\nu_k - \nu_j + i} = \prod_{j=1, l \neq k}^{m^{(2)}} \frac{\nu_j - \nu_k + i}{\nu_j - \nu_k - i}. \]  

(51)

where $k$ takes values from 1 to $m, m^{(1)}$ and $m^{(2)}$ in [19], [20], [21] respectively. Note that we have already redefined the parameters $v_j, \mu_j, \nu_j$ by shifting $\frac{i}{2}, i$ and $\frac{3i}{4}$ respectively in the above relations. According to the definition of Hamiltonian in section 3, we can find the energy of the Hamiltonian $H^0$ as

\[ E = N - \sum_{j=1}^{m} \frac{1}{\nu_j^2 + \frac{3}{4}}. \]  

(52)
We shall use the notations introduced in ref. [1]. We denote respectively $N_{\uparrow}$ and $N_{\downarrow}$ as the number of single electrons with spin up and spin down, and $N_{a}$, number of local electron pairs. As we have pointed out, the reference state is chosen to be $|\uparrow\downarrow\rangle$, and the vacuum state is the full-filled state. Therefore, an electron means a hole. The numbers $m, m^{(1)}, m^{(2)}$ appeared in above relations can be written as $m = N_{\uparrow} + N_{\downarrow} + N_{a}, \ m^{(1)} = N_{\uparrow} + N_{\downarrow}, \ m^{(2)} = N_{a}$. Since the Hamiltonian commutes with the Hubbard term $U \sum_{j} X_{j}$, this term will not change the eigenvectors and the Bethe ansatz equations.

5 Analysis of the Bethe Ansatz Equations and Discussions

Using the standard method [12], we next analyze briefly the Bethe ansatz equations. By taking the logarithm of the Bethe ansatz equations, we obtain the following set of equations,

$$N\Phi(v_{k}, \frac{1}{2}) - \sum_{l=1}^{m} \Phi(v_{k} - v_{l}, 1) - \sum_{j=1}^{m^{(1)}} \Phi(\mu_{j} - v_{k}, \frac{1}{2}) = 2\pi I_{k},$$

$$\sum_{l=1}^{m} \Phi(\mu_{\alpha} - v_{l}, \frac{1}{2}) - \sum_{\beta=1}^{m^{(1)}} \Phi(\mu_{\alpha} - \mu_{\beta}, 1) - \sum_{a=1}^{m^{(2)}} \Phi(\nu_{a} - \mu_{\alpha}, \frac{1}{2}) = 2\pi J_{\alpha},$$

$$\sum_{\beta=1}^{m^{(1)}} \Phi(\mu_{\beta} - \nu_{c}, \frac{1}{2}) - \sum_{c=1}^{m^{(2)}} \Phi(\nu_{c} - \nu_{c}, 1) = 2\pi K_{b},$$

where $\Phi(\lambda, \alpha) = 2\arctan(\lambda/\alpha)$. $I_{k}, J_{\alpha}$ and $K_{b}$ are integer or half-odd integer depending on $m, m^{(1)}$ and $m^{(2)}$. In the thermodynamic limit $N \to \infty$, $v_{j}, \mu_{j}, \nu_{j}$ become continuous variables, $I_{k}, J_{\alpha}$ and $K_{b}$ are connected with the distribution functions of roots and holes, the Bethe ansatz equations are rewritten in the following form

$$\sigma(v) + \sigma_{h}(v) = K_{1/2}(v) - \int_{-B}^{B} dv' K_{1}(v - v') \sigma(v') + \int_{-B}^{B} d\mu K_{1/2}(v - \mu) \omega(\mu),$$

$$\omega(\mu) + \omega_{h}(\mu) = \int_{-B}^{B} dv K_{1/2}(\mu - v) \sigma(v) - \int_{-B}^{B} d\mu' K_{1}(\mu - \mu') \omega(\mu') + \int_{-B}^{B} dv K_{1/2}(\mu - v) \tau(v),$$

$$\tau(v) + \tau_{h}(v) = \int_{-B}^{B} d\mu K_{1/2}(\nu - \mu) \omega(\mu) - \int_{-B}^{B} dv' K_{1}(\nu - v') \tau(v'),$$

where $K_{\alpha}(\lambda) = \frac{\alpha}{\pi(\lambda + \alpha - \alpha)}$. These results are completely the same as those for SU(4) spin chain, see ref. [10]. The energy is written as

$$E = N - 2\pi \int_{-B}^{B} K_{1/2}(v) \sigma(v) dv.$$  

The ground state energy per site is obtained as

$$E_{0} = 1 - \left( \frac{3}{2} \ln 2 + \frac{\pi}{4} \right).$$

Because the coupled integral equations (56) are the same as for the case of normal SU(4) spin chain, the properties of the low-lying excitations of the system under consideration remain to be the same as those in ref. [10].
In conclusion, we have presented in this paper the Bethe ansatz equations for the SU(4) extension of the Hubbard model. We have proved that the Hamiltonian is equivalent to the graded SU(4) permutation operator. Using the graded algebraic Bethe ansatz method, we have found the eigenvalues of the transfer matrix.

In this paper, we have only dealt with FFBB grading, and there are only one matrix representations for the Fermion and Boson generators. We can consider other representations as presented in ref.[1] and different gradings. The SU(4) extension of the Hubbard model is different from the SU(2|2) case, but we use in this paper the same 16 generators as in ref.[3] to represent the Hamiltonian. The structure of the Bethe ansatz eigenvectors for SU(4) case seems to be the same as the SU(2|2) case. We can also give a detailed calculation of the thermodynamic Bethe ansatz for the SU(4) case as that of SU(2|2)[1].

Acknowledgements: One of the authors, H.F. is supported by JSPS. He thanks M.Batchelor, J.Links, S.Saito, Y.Umeno, Y.Z.Zhang and H.Q.Zhou for useful discussions and communications.
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