**Delineating the conformal window**

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We identify and characterise the conformal window in gauge theories relevant for beyond the standard model building, e.g. Technicolour, using the criteria of metric confinement and causal analytic couplings, which are known to be consistent with the phase diagram of supersymmetric QCD from Seiberg duality. Using these criteria we find perturbation theory to be consistent throughout the predicted conformal window for several of these gauge theories and we discuss recent lattice results in the light of our findings.

**THE CONFORMAL WINDOW**

In a generic non-Abelian gauge theory with gauge group $G$ and $N_f$ fermions transforming according to a representation $R$ of $G$ we expect there to be a conformal window, i.e. a region $N_f^I < N_f < N_f^I$ for which the theory is asymptotically free at short distances while the long distance physics is scale-invariant and typically governed by a non-trivial fixed-point. In this paper we consider such theories with fermions in a single representation of the gauge groups $SU, SO, Sp$.

The upper boundary of the conformal window is determined in perturbation theory from the β-function:

$$\beta(x) \equiv \frac{dx}{d\ln(Q^2)} = -\left(\beta_0 x^2 + \beta_1 x^3 + \cdots\right),$$

(1)

at a small value of the coupling $x \equiv \alpha_s/\pi$. The first two coefficients of the expansion $[2,3]$ are universal and independent of the renormalisation group scheme:

$$4\beta_0 = \frac{11}{3} C_2(G) - \frac{4}{3} T(R) N_f$$

(2)

$$16\beta_1 = \frac{34}{3} C_2^2(G) - \frac{20}{3} C_2(G) T(R) N_f - 4 C_2(R) T(R) N_f.$$  

(3)

When $\beta_0$ changes sign, from positive to negative at

$$N_f^I = \frac{11 C_2^2(G)}{4 T(R)},$$

(4)

the theory changes from the asymptotically free conformal phase to the infrared free phase. This is the upper boundary of the conformal window, coinciding with the loss of asymptotic freedom (LOAF), and the transition point in $N_c = 3$ QCD is at $N_f^I = 16.5$. For $N_f$ just below this upper boundary, Eqs. (3,4) imply that $\beta_1 < 0$, and so $\beta(x)$ will have a non-trivial zero at $x_{FP} \simeq -\beta_0/\beta_1 > 0$. The fixed point coupling $x_{FP}$ approaches zero as $N_f$ approaches $N_f^I$ from below. The smallness of $x_{FP}$ just below $N_f^I$ justifies the use of the 2-loop β-function. Thus the transition to the infrared free phase is always via a conformal phase and this is independent of the fermion representation.

The lower boundary of the conformal window, $N_f^{II}$, below which confinement and chiral symmetry breaking typically set in, is much harder to determine. From the two-loop β-function, the fixed point is lost and the lower boundary of the conformal window would be reached from above when $\beta_1 = 0$. However, this not only ignores higher order corrections but also neglects non-perturbative effects which, generally, are expected to become important towards the lower end of the conformal window, where the 2-loop estimate of the fixed point coupling is becoming large, $x_{FP} \gtrsim 1$.

While the lower boundary of the conformal window is of theoretical interest in its own right, its current importance arises from its central role in technicolour models [4] with walking dynamics [5,6] and, in particular, of more recent models such as minimal walking technicolour [7] and conformal technicolour [8]. Therefore, a lot of effort has recently gone into exploring this region, using both lattice [9,21] and approximate analytical [22–25] methods. In principle the former should provide a definitive answer: however, it has become clear, from the pioneering lattice calculations, that identifying and characterising (near-)conformal theories on a lattice is a very challenging problem. So it remains important to try and gain as much analytical insight as possible.

Since it is the chiral symmetry breaking of technicolour that drives the interesting ‘walking’ scenarios, it is natural to look to analytic methods that estimate its onset. The standard technique involves the use of Schwinger-Dyson (SD) equations in a ladder-like approximation [26,27]. While this does make a prediction for the value of $N_f$ at which chiral symmetry is spontaneously broken, the credibility of the estimate is called into question by the fact that in the case of $N_c = 1$ supersymmetric QCD (SQCD), where Seiberg duality [38] allows us to calculate the value of $N_f^{II}$ exactly, the SD estimate is far above the known value [37]. Thus it is useful to look for other analytical estimates which can help determine where conformality may be lost.

Here we wish to discuss two such methods, both of which have been extensively discussed in the 1990’s in related and overlapping contexts. First we shall discuss the criterion of ‘metric confinement’ [11], which provides a lower bound on the value of $N_f$ at which confinement occurs and thus also for the value of $N_f^{II}$ at which conformality is lost. Secondly we discuss the range of validity of
perturbation theory within the conformal window following [28, 41, 42] and we compare our findings with lattice simulations of these theories.

**Metric confinement**

Metric confinement determines when transverse gluons are not part of the physical Hilbert space from the properties of the transverse gluon propagator, \(D(Q^2, \mu^2, g)\), where \(\mu^2\) is the renormalisation scale. We refer the reader to [40] for a detailed exposition of metric confinement. The condition can be formulated (working always in Landau gauge) in terms of a superconvergence relation for the absorptive part \(\rho(k^2, \mu^2, g) = (1/\pi) \text{Im} \{D(-k^2, \mu^2, g)\}\) of the gluon propagator [40]:

\[
\int_0^\infty dk^2 \rho(k^2, \mu^2, g) = 0. \tag{5}
\]

Because of the known analyticity properties of the propagator \(D\), Eq. (5) is equivalent to the vanishing of the integral of \(D\) around the contour at \(|k^2| = \infty\) [40]. Thus, if \(D(Q^2, \mu^2, g)\) vanishes fast enough as \(|Q^2| \to \infty\), one will indeed have metric confinement. Asymptotic freedom then allows us to determine whether it does so or not from the value of the appropriate anomalous dimension. The condition for metric confinement, in terms of the 1-loop anomalous dimension of the gluon propagator \(\gamma_{00}\) can be seen to be [40]:

\[
\gamma_{00} = -\frac{1}{4} \left( \frac{13}{6} C_2(G) - \frac{4}{3} T(R) N_f \right) < 0. \tag{6}
\]

Note that because we are interested in the value of \(D\) as \(|Q^2| \to \infty\), the 1-loop perturbative value of \(\gamma_{00}\) is exact for our purposes: when Eq. (6) holds the theory confines and conformality has been lost. Metric confinement is claimed to provide a sufficient but not necessary condition for confinement and therefore Eq. (6) provides a lower bound on the lower boundary of the conformal window:

\[
N_f^H \geq N_f^{MC} \equiv 13C_2(G)/8T(R). \tag{7}
\]

We also note from Eq. (4) that this bound is strictly less than the upper edge of the conformal window: \(N_f^{MC} < N_f^I\). So metric confinement always leaves a finite window of opportunity for conformality.

This lower bound on \(N_f^H\) [40] is plotted for \(SU\) and \(SO\) gauge theories with fermions in single- and two-index representations, as the thick dotted line, in Figs. 1 and 2. We discuss the implications later in the paper.

Just as with the SD estimates, it is useful to test this bound in SQCD. Remarkably, one finds that the lower bound on \(N_f^H\) from metric confinement coincides with the value of \(N_f^H\) that is determined from Seiberg duality [38]. This has been shown for both \(SU\) and \(SO\) gauge groups [26, 43, 44] and is also the case for \(Sp\) gauge groups, as we have checked ourselves. Such agreement is particularly significant in the case of SQCD as it is known [38] that here the loss of conformality is through the onset of confinement and not of chiral symmetry breaking – the latter occurring at a much smaller value of \(N_f\). (This provides a striking counterexample to the earlier wisdom that confinement necessarily entails chiral symmetry breaking.)

It is also interesting to consider supersymmetric Yang Mills with fermionic matter in higher representations where there is no known Seiberg dual. In these cases if one determines the lower boundary of the conformal window using the Novikov-Shifman-Vainshtein-Zakharov (NSVZ) beta function for supersymmetric theories [45] by setting \(\gamma = 1\) (the unitary bound in these theories) [29], which in the case of SQCD is known to reproduce the result from Seiberg duality, we find that even in these theories metric confinement coincides with this result.

**Perturbation theory and analyticity**

At large momentum transfer \(Q^2\), the coupling constant behaves as \(x(Q^2) \sim \frac{1}{\text{Im}(Q^2/\Lambda^2)}\). At 1-loop this simple expression is valid for all \(Q^2\), so that \(x(Q^2)\) diverges at \(Q^2 = \Lambda^2\). Thus if we attempt to calculate some physical quantity in a convergent power series in the 1-loop running coupling, this physical quantity will inherit this Landau singularity. This, however, will in general violate the known analyticity properties of such a physical quantity, which typically involves specific poles and cuts corresponding to asymptotic states. Thus we see that perturbation theory in the 1-loop running coupling cannot be adequate and that this is immediately visible from the unphysical analytic structure of the coupling. This suggests that, more generally, the analytic structure of a running coupling can indicate whether there is any possibility of perturbation theory providing a complete description of the physics.

Here we are interested in studying the conformal window and, in this case, we have an infra-red fixed point, so the coupling is bounded by \(0 \leq x(Q^2) \leq x_{FP}\) for \(0 \leq Q^2 < \infty\) and so cannot have such a divergence. In particular this is the case if we use the 2-loop coupling and if \(\beta_1 < 0\). As we approach the upper bound, \(N_f \to N_f^I\), the coupling becomes weak on all scales and we may expect perturbation theory to work well. In that case, the coupling \(x(Q^2)\) should manifest the analytic structure of a typical physical quantity i.e. a cut for \(k^2 = -Q^2 \geq 0\) corresponding to the production of mass-
less particles, and no other unphysical singularities in the entire complex $Q^2$ plane. If this is so then it is said to be causal analytic and indeed this turns out to be the case for $N_f \rightarrow N_f^{1}$ \cite{11}. If we now decrease $N_f$ away from $N_f^{1}$ then, as long as the coupling remains causal analytic, it is consistent for the physics to be perturbative. As we continue decreasing $N_f$, at some point $x(Q^2)$ will acquire unphysical singularities in the complex $Q^2$ plane. These might be poles or cuts. At this point the coupling ceases to be causal analytic and signals the fact that there must now be non-perturbative contributions that will serve to restore the correct analytic structure to the quantity being calculated. These may lead to confinement and/or chiral symmetry breaking and hence the loss of conformality.

The two loop beta-function can be integrated explicitly in terms of the Lambert W-function \cite{46} defined by
\[
W(z) \exp [W(z)] = z,
\]
giving \cite{26, 41, 42}
\[
x(Q^2) = -\frac{1}{c} \frac{1}{1 + W(z)}, \quad c = \frac{\beta_1}{\beta_0},
\]
\[
z = -\frac{1}{c} \left( \frac{Q^2}{\Lambda^2} \right)^{-\beta_0/c}.
\]

While $W(z)$ is a multi-valued function with an infinite number of branches, the unique branch for $c < 0$ with a real coupling along the positive real $Q^2$ axis is the principal branch denoted $W_0(z)$ \cite{26, 41, 42}. The requirement for this coupling to be causal translates into the criterion
\[
0 < -\frac{\beta_0^2}{\beta_1} < 1
\]
(8)

Note that as one approaches the upper bound to the conformal window, $\beta_0 \rightarrow 0^+$ while $\beta_1 < 0$, this bound is always satisfied, i.e. the coupling is causal analytic in this Bank-Zaks limit, as one might expect. Note also that this is a stronger criterion than just requiring that the two-loop beta-function have a fixed-point since, as $\beta_1 \rightarrow 0^-$ one violates the bound in Eq. (3). Reflecting this, the analytically continued coupling will acquire singularities in the complex plane at a larger value of $N_f$ than where the Landau singularity appears \cite{26, 41, 42}. We observe from Eq. (3) that the coupling is causal analytic all the way down to $N_f^{1\text{MC}}$ provided $C_2(R) > \frac{\gamma}{2} C_2(G)$, which is true in all cases, except for $SU(2)$ (and $Sp(4)$) with fundamental fermions. Hence it is also the case all the way down to $N_f^{1\text{MC}}$. For multi-flavor QCD this was already noted in \cite{26}. This demonstrates that while causal analyticity may be a necessary condition for non-perturbative physics to be important, it is not sufficient. In \cite{26} it was also shown that in SQCD (whose beta-function differs from Eqs. (2) because of the presence of scalars and gluinos) analyticity breaks down before $N_f^{1\text{MC}}$ is reached. This fits in with the requirements of the weak-strong coupling Seiberg duality \cite{32} where the lower and upper boundaries of the conformal windows of the dual theories are mapped into each other, which implies that near the lower boundary the theory must be strongly coupled. This demonstrates that when analyticity breaks down, so that non-perturbative physics must be present, this does not necessarily entail confinement, chiral symmetry breaking, or indeed the loss of conformality.

The analyticity bound in Eq. (3) is obtained from the 2-loop beta-function and so can only be regarded as approximate. (Although in \cite{12} it was shown that going to 3-loops, utilising a particular Padé approximant functional form, does not alter the conclusions, as long as the 3-loop coefficient of the beta-function is not very large.) Moreover, we expect that the perturbative expansion for $\beta(x)$ cannot be better than asymptotic, with corrections $\sim \exp\{-c/x\}$ that mimic non-perturbative contributions. Roughly speaking, we would expect the causal analyticity calculated at 2-loops to be reliable as long as the coupling $x(Q^2)$ is not too large anywhere in the complex $Q^2$ plane.

When judging whether a coupling is ‘small’ or ‘large’ it is in some sense more natural to use the scaled (‘t Hooft) coupling $N_c x$ instead of $x$ as, at large $N_c$, $x \sim N_c^{-1}$ while the $n$-th coefficient of the beta-function scales as $\beta_n \sim N_c^{n+1}$, and similarly for the anomalous dimension. As an example, the mass anomalous dimension of an adjoint fermions is given by $\gamma_{\lambda_{ij}} = \frac{3}{2} (N_c x) + O(N_c^2 x^2)$. We shall therefore calculate $\max_{Q^2 \in \mathbb{C}} |N_c x(Q^2)|$ using the correct analytic continuation of $x$ from the 2-loop beta-function and use the magnitude of the result as a supplementary criterion for judging the reliability of any argument from analyticity.

For the moment we simply plot the value of $N_f$ where analyticity is lost, and hence where perturbation theory signals its own breakdown according to the criterion in Eq. (3), as the black solid lines in Figs. 1 and 2. We interpret these results below.

**Analyticity with the all-orders beta-function conjecture**

Inspired by the NSVZ beta function \cite{45}, an all-orders (AO) beta function for $SU(N)$ gauge theories with any matter representation was conjectured in \cite{54} and further studied in \cite{34}. It reads:
\[
\beta(x) = -\beta_0 x \frac{1 - T(R) N_f \gamma(x)/(6 \beta_0)}{1 - \frac{x}{2} C_2(G) \left( 1 + \frac{2 \gamma}{3 \beta_0} \right) x},
\]
(9)

where,
\[
\gamma(x) = \frac{3}{2} C_2(R) x + O(x^2), \quad 4 \beta_0 = C_2(G) - T(R) N_f.
\]
(10)

Here, $\gamma \equiv \frac{4 \ln m}{\Lambda_{QCD}}$ is the fermion mass anomalous dimension, and solving for $\gamma$ at a fixed point, i.e $\beta = 0$, yields
\[
\gamma = \frac{11 C_2(G) - 4 T(R) N_f}{2 T(R) N_f},
\]
which increases as $N_f$ is decreased.
Since \( \gamma \leq 2 \) is a rigorous bound from unitarity \cite{47}, this provides a different lower bound on \( N_f^H \),

\[
N_f^\text{AO} = \frac{11 C_2(G)}{8 T(R)}
\]

which we see is slightly below the bound provided by metric confinement in Eq. \( \text{(7)} \).

In Figs. \( \text{1} \) and \( \text{2} \) we plot this lower bound, \( N_f^\text{AO} \), as a thick dashed line. For the adjoint representations this line is invisible because it exactly coincides with the thick solid line that represents the loss of causality in the two-loop \( \beta \)-function.

We observe that if we restrict the matter anomalous dimension \( \gamma \) to first order in \( x \) then this all orders \( \beta \)-function may be integrated exactly, yielding:

\[
x(Q^2) = \frac{1}{E_1} + \frac{1}{1 + G_1 W(z)}, \quad G_1 \equiv 1 - \frac{D}{E_1},
\]

where

\[
E_1 = C_2(r) T(R) N_f/(4\beta_0), \quad D = \frac{1}{2} C_2(G) \left( 1 + \frac{2\gamma_0}{\beta_0} \right).
\]

We can integrate the AO \( \beta \)-function in this approximation of \( \gamma \) as it has the same structure as a Padé approximant to the 3-loop \( \beta \)-function which is integrable in terms of the W-function \cite{12}. The condition for having a causal coupling thus becomes \( \beta_0 < E_1 - D \) which is identical to the criterion for the two loop coupling being causal.

Similarly the coupling is causal analytic all the way down to \( N_f^\text{AO} \) provided \( C_2(R) > \frac{16}{15} C_2(G) \), which for the theories considered here, is generally only the case for the two-index symmetric representation.

Comparing with lattice data and other methods

Both the criterion of metric confinement and that of causal analyticity are consistent with the properties of the conformal window in SQCD as predicted from Seiberg duality. It is therefore interesting to ask what these criteria predict for the non-supersymmetric theories that are being investigated using lattice techniques. These theories include SU(2) and SU(3) with a ‘large’ number of fundamental (F) fermions \cite{14,15}, SU(2) with 2 adjoint (Adj) fermions \cite{9,13}, and SU(3) with 2 sextet (2S) fermions \cite{21,22}. These theories are part of the larger family of theories whose properties are shown in Figs. \( \text{1} \) and \( \text{2} \). On each of these plots we show \( N_f^I \), as well as three curves related to the lower boundary of the conformal window: the curve \( N_f^\text{MC} \) where metric confinement sets in, the curve \( N_f^\text{AO} \) mapped out by the vanishing of the AO \( \beta \)-function with \( \gamma = 2 \), and the curve where causal analyticity breaks down. The first two provide lower bounds for the conformal window, while the third gives us an estimate of where non-perturbative effects must be important. We have also displayed in these figures the SD predictions for chiral symmetry breaking (in the usual ladder approximation). Where chiral symmetry breaking occurs will typically be the lower boundary of the conformal window and, in any case, will provide a lower bound for it. Unfortunately, although time-honoured, such SD estimates are known to fail in SQCD \cite{39}.

\( SU(2) \) and \( SU(3) \) theories with fundamental flavours

In the left panels of Figs. \( \text{1} \) and \( \text{2} \) we display estimates for the conformal window of \( SU \) and \( SO \) theories (\( Sp \) being qualitatively the same as \( SU \)) with fundamental fermions. It shows that the metric confinement and causal analyticity criteria almost coincide in all cases. With the exception of \( SU(2) \) (and \( Sp(4) \)), causal analyticity extends to a slightly lower \( N_f \) than metric confinement. So, in contrast to SQCD, the whole of the conformal window is causal analytic, suggesting that it represents a perturbative infra-red conformal phase.

For \( SU(3) \) this suggests that the conformal window begins with \( N_f = 10 \) and for \( SU(2) \) with \( N_f = 7 \). However, since the limits are close together it is important to check whether the coupling remains small at these limits. In Fig. \( \text{3} \) we plot the maximal value of the complex 2-loop coupling \( \max_{Q^2 \in \mathbb{C}} |N_c x(Q^2)| \) for \( SU(3) \), as a function of the scaled flavour variable \( \Delta N_f = (N_f - N_f^\text{MC})/(N_f^I - N_f^\text{MC}) \) taking values from 0 to 1 within the conformal window, and indicate with dots the \( N_f = 10, 12, 16 \) theories. We see that, as expected the coupling remains small for \( N_f = 16 \) and increases as \( N_f \) is lowered. In particular, the coupling is rather large at the lower end of the window, leaving room for a significant shift, either way, in our estimate of what is the true region of causal analyticity.

Inside the conformal window the coupling does not decrease linearly with \( N_f \) but rather increases rapidly as \( N_f^\text{MC} \) is approached. This behaviour is plotted in Fig. \( \text{3} \). Although in \( SU(3) \) the coupling rapidly increases below \( N_f = 10 \) it should be noted that the coupling is already somewhat large by this point.

The so-called 1-family models of technicolour are based on an \( SU(2) \) gauge theory with \( N_f = 8 \) in the fundamental representation, see e.g. \cite{48}. This theory is well above the bound on \( N_f^\text{MC} \) that follows from metric confinement and within the window of causal analyticity with a relatively small coupling shown in Fig. \( \text{3} \), suggesting that the theory is conformal and weakly coupled.
so for various values of \( h \). Happens for gauge theories with adjoint fermions. We do 1-loop estimate \[12\].

Relative small anomalous mass dimension, close to the of this theory suggests that it is conformal \[9–12\] with a

Current lattice simulations of \( SU^7, 49\) is based on

\( SU \) groups that lattice calculations have so far focused upon. Results for \( Sp \) are identical to those of \( SU \). We note that the results look similar for the \( SU \) and \( SO \) groups and that there is no dependence on \( N_c \) for a fixed number of adjoint fermions. This is no surprise, since all our predictions involve some aspect of the perturbative running. Finally, and most interestingly, we see from Fig. \( \| \) that \( N_f = 2 \) is well above the bound on \( N_f^{MC} \) that follows from metric confinement and also well within the window of causal analyticity. (Which here coincides with \( N_f^{AO} \), the \( \gamma = 2 \) bound from the AO \( \beta \)-function.) This strongly suggests that the \( N_f = 2 \) theory is conformal.

One might be perturbed by the fact that, as we see in Fig. \( \| \) causal analyticity extends into the region where metric confinement already holds. However, the gap between the two curves is small and is presumably consistent with the uncertainty that higher order corrections would bring to the location of the breakdown of causal analyticity. Following on from the fundamental case, we calculate the value of \( x \) over the whole complex \( Q^2 \) plane, so as to see if it is everywhere ‘small’ and that our 2-loop analysis can be trusted or if it is somewhere ‘large’, increasing the uncertainty in our analysis.

The result \( \max_{\arg(Q^2)} |x(Q^2)| \) for the maximum value of \( |x| \) at fixed \( |Q^2| \) for the interesting case of \( N_f = 2 \) is shown in Fig. \( \| \) and \( \max_{Q^2 \in C} |N_c x(Q^2)| \) for general \( N_f \) in Fig. \( \| \). We observe that, while the maximum value of \( |x(Q)| \) for \( N_f = 2 \) is not as small as it is near the \( N_f^{CA} = 2.75 \) LOAF limit, it is certainly small compared to its value at the point near which causality is lost, \( N_f^{CA} = 1.38 \). This gives us confidence that at \( N_f = 2 \) the theory really is causally analytic and that it is in a perturbative (infra-red) conformal phase. It is thus consistent with the observation \[12\] that \( \gamma \) is close to the one-loop prediction.

On the other hand, at \( N_f = 1.5 \) the value of \( |x(Q)| \) is large enough that it is entirely plausible that a higher order calculation could shift the loss of analyticity from just below that value of \( N_f \) to above it, so ensuring that metric confinement does not take place within the region of causal analyticity.
The Next to Minimal Walking technicolour (NMWT) model [7, 50] is based on an SU(3) gauge theory with $N_f = 2$ in the two-index symmetric (sextet) representation. Current lattice simulations of this theory suggest that it is conformal or near-conformal [21, 23, 24] and that once again metric confinement sets in within the analyticity window. However, in contrast to the SU(2) case with adjoint fermions, metric confinement sets in very close to $N_f = 2$. (See table I). Thus we expect that the $N_f = 2$ theory is very close to the lower boundary of the conformal window.

Once again we compute the value of $|x(Q)|$ from the 2-loop $\beta$-function in the whole of the $Q^2$ complex-plane, but this time for SU(3) with 2 sextet fermions. The result for $\max_{arg(Q^2)} |x(Q^2)|$ is shown in Fig. 4 for $N_f = 2$ and $\max_{Q^2 \in \mathbb{C}} [N_f x(Q^2)]$ for general $N_f$ in Fig. 5 where we also indicate $N_f = 3$ which is near the upper boundary of the conformal window. We observe that the maximum value of $|N_f x|$ for $N_f = 2$ is relatively small, compared to the SU(3) theory with 10 fundamental flavors, although significantly larger than it is in the case of adjoint fermions. The corresponding value of $\alpha_s = \pi x$ is also larger than the value $\alpha_s \sim 0.5$ at which, in QCD, one typically begins to worry about the convergence of perturbation theory, while for MWT the coupling is indeed slightly smaller. (Though, it is not obvious how to compare the size of the couplings across theories with fermions in different representations.)

This leaves it unclear whether, at the point at which metric confinement sets in and conformality is lost, the theory is still consistently perturbative.

Conclusions

In this letter we have discussed the implications of ‘metric confinement’ and ‘causal analyticity’ for theories that are being actively studied using lattice techniques in the search for walking near-conformal field theories.

We noted that in the case of SQCD, where Seiberg duality gives us a precise description of the conformal window, both these criteria work very well: metric confinement predicts the precise location of the lower boundary of that window while causal analyticity predicts that the theory becomes strongly coupled in the lower part of the window, as required by the weak-strong duality.

On the other hand, the widely used SD calculations for the lower boundary of the conformal window from metric confinement $N_f^{CA}$, and loss of asymptotic freedom $N_f^I$ for theories considered in the text.

TABLE I: The $N_f$ values for loss of causal analyticity $N_f^{CA}$, the lower boundary of the conformal window from metric confinement $N_f^{MC}$, and loss of asymptotic freedom $N_f^I$ for theories.

| $G$ | $R$ | $N_f^{CA}$ | $N_f^{MC}$ | $N_f^I$ |
|-----|-----|-------------|-------------|---------|
| SU(2) | $F$ | 6.60 | 6.5 | 11 |
| Adj | 1.38 | 1.63 | 2.75 |
| SU(3) | $F$ | 9.68 | 9.75 | 16.5 |
| 2S | 1.61 | 1.95 | 3.3 |

Two flavor SU(3) sextet theory

FIG. 3: The maximal value of the 2-loop coupling $|N_f x(Q)|$ in the complex plane $Q \in \mathbb{C}$, excluding the negative real axis, with $\Delta N_f \equiv (N_f - N_f^{MC})/(N_f^{MC} - N_f^I)$ taking values from 0 to 1 within the conformal window for the gauge groups and representations indicated. The location of the theories of Fig. 3 are indicated in dots.

FIG. 4: The maximal value of the 2-loop coupling $|x(Q)|$ in the complex plane $Q \in \mathbb{C}$, excluding the negative real axis, for the theories indicated. A maximum away from $|Q|^2 = 0$ indicates that the theory is close to the limit of causal analyticity.

The Next to Minimal Walking technicolour (NMWT) model [7, 50] is based on an SU(3) gauge theory with $N_f = 2$ in the two-index symmetric (sextet) representation. Current lattice simulations of this theory suggests that it is conformal or near-conformal [21, 23, 24] and that it has relatively small anomalous mass dimension, close to the 1-loop estimate [24].

We show in the right panels of Figs. 1 and 2 what happens for SU and SO gauge theories with fermions in the two-index symmetric representation at various values of $N_c$ (The symmetric representation of Sp is identical to the adjoint of Sp). We note that there is a significant dependence on $N_c$ and that once again metric confinement sets in within the analyticity window. However, in contrast to the SU(2) case with adjoint fermions, metric confinement sets in very close to $N_f = 2$. (See table I). Thus we expect that the $N_f = 2$ theory is very close to the lower boundary of the conformal window.

Once again we compute the value of $|x(Q)|$ from the 2-loop $\beta$-function in the whole of the $Q^2$ complex-plane, but this time for SU(3) with 2 sextet fermions. The result for $\max_{arg(Q^2)} |x(Q^2)|$ is shown in Fig. 4 for $N_f = 2$ and $\max_{Q^2 \in \mathbb{C}} [N_f x(Q^2)]$ for general $N_f$ in Fig. 5 where we also indicate $N_f = 3$ which is near the upper boundary of the conformal window. We observe that the maximum value of $|N_f x|$ for $N_f = 2$ is relatively small, compared to the SU(3) theory with 10 fundamental flavors, although significantly larger than it is in the case of adjoint fermions. The corresponding value of $\alpha_s = \pi x$ is also larger than the value $\alpha_s \sim 0.5$ at which, in QCD, one typically begins to worry about the convergence of perturbation theory, while for MWT the coupling is indeed slightly smaller. (Though, it is not obvious how to compare the size of the couplings across theories with fermions in different representations.)

This leaves it unclear whether, at the point at which metric confinement sets in and conformality is lost, the theory is still consistently perturbative.
where chiral symmetry breaking sets in, are very badly off in SQCD. This is part of our motivation for bringing these other criteria into play.

It is interesting that for the theories considered here, generically perturbation theory is consistent all the way down to the lower end of their conformal window as determined by metric confinement, and so the mass anomalous dimension at the fixed point can be plausibly estimated in 1-loop perturbation theory. Doing so we find \( \gamma(x_{FP}) = 0.6, 1.34 \) for the MWT and NMWT theories respectively. Going to the next order in \( \overline{MS} \) the values of \( \gamma \) change by about 10% while the corresponding predictions from the AO \( \beta \)-function, setting \( \beta(x_{FP}) = 0 \) in Eq. 9 are \( \gamma(x_{FP}) = 0.75, 1.3 \). This can be compared to the results of lattice simulations [12, 13, 22, 24] which suggest anomalous dimensions consistent with the 1-loop result, albeit with the caveat that for the MWT model the simulations fix a finite point which is a factor two smaller than the two-loop result we have used.

In the case of MWT both criteria suggest that this theory lies well within a perturbative infra-red conformal phase. By contrast, NMWT appears to be almost on the boundary of the lower conformal window. This is certainly consistent with the mixed messages one has been getting from different lattice calculations on this theory [22, 24]. The possibility that this theory lies just outside the conformal window, which is possible because, strictly speaking, metric confinement provides a lower bound on where confinement sets in, makes it an interesting candidate walking technicolour model in itself. For example, the presence of four fermion operators, arising from extended technicolour interactions, can modify the conformal window and anomalous dimensions (indeed it can do so in all the theories we consider here [30]).

As already observed in [26], metric confinement suggests that the conformal window for \( SU(3) \) with \( N_f \) fundamental fermions begins at \( N_f = 10 \), as we can infer from Fig. 1. As pointed out in [26] causal analyticity extends just below \( N_f = 10 \), suggesting that the whole conformal window is weakly coupled. However if one actually looks at the coupling \( x \) in the \( N_f = 10 \) theory, one finds that its value is quite large, as shown in Fig. 4. So if it turns out that the \( N_f = 10 \) theory does not, in fact, lie in the conformal window then this again opens the possibility of the kind of large anomalous dimension that walking phenomenology needs. On the other hand, there appears to be little doubt that the \( N_f = 12 \) theory does lie well inside the conformal window, and \( N_f = 9 \) well outside.

Very similar remarks apply to \( SU(2) \) with \( N_f \) fundamental fermions. The conformal window should begin at \( N_f = 7 \), which is similar to \( N_f = 10 \) in \( SU(3) \). \( N_f = 8 \) is very similar to \( N_f = 12 \) in \( SU(3) \), while \( N_f = 6 \) lies just inside the region of metric confinement, albeit still in the region of causal analyticity.

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