Effective potential and stability of the rigid membrane

E. Elizalde and S.D. Odintsov

Department E.C.M., Faculty of Physics, University of Barcelona, Diagonal 647, 08028 Barcelona, Spain
e-mail: eli @ ebubecm1.bitnet

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Abstract

The calculation of the effective potential for fixed-end and toroidal rigid p-branes is performed in the one-loop as well as in the 1/d approximations. The analysis of the involved zeta-functions (of inhomogeneous Epstein type) which appear in the process of regularization is done in full detail. Assymptotic formulas (allowing only for exponentially decreasing errors of order $\leq 10^{-3}$) are found which carry all the dependences on the basic parameters of the theory explicitly. The behaviour of the effective potential (specified to the membrane case $p = 2$) is investigated, and the extrema of this effective potential are obtained.

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1On leave from Department of Mathematics and Physics, Pedagogical Institute, 634041 Tomsk, Russia.
1 Introduction

The theory of the rigid string [1,2] is interesting because of the number of its applications to quantum chromodynamics (see [1-3] and references therein) and to statistical physics. Using the same idea, it is not difficult to construct the action for the rigid membrane (or $p$-brane). During the last few years, there has been some activity in the study of quantum extended objects (like membranes, see ref. [4] for a review). However, already the semiclassical quantization of a so highly nonlinear system as a membrane is a very difficult task [5-7]. Nevertheless, some interesting issues, like the study of the Casimir energy, the large-$d$ approximation and the tachyon problem can be addressed already at the semiclassical level [8,9].

In the present paper, we study the Casimir energy and the static potential for the rigid $p$-brane (at the classical level, this theory has been considered in ref. [10]), specifying afterwards our results to the membrane case ($p = 2$). We shall start from the following action, which is multiplicatively renormalized only in the string case ($p = 1$),

$$S = \int d^{p+1} \xi \sqrt{g} \left( k + \frac{1}{2\rho^2} \left[ \Delta(g)X^i \right]^2 \right),$$

where $g_{\alpha\beta} = \partial_\alpha X^i \partial_\beta X^i$, $\alpha = 0, 1, \ldots, p$, $i = 1, 2, \ldots, d$, $\Delta(g) = g^{-1/2} \partial_\alpha g^{1/2} g^{\alpha\beta} \partial_\beta$, the constant $k$ is the analog of the usual string tension, and $1/\rho^2$ is the coupling constant corresponding to the rigid term.

Let us note that the $p$-brane is an interacting system, without a free part in the action. Hence, one must start from some classical solution for the ground state, and then study the quantum fluctuations on such background. In this framework we can understand how the tachyon appears (if that is the case), if the background is stable, and also address some other issues. Owing to the fact that string theory can be obtained as some compactification of the membrane [11], we can also expect to find in this way some new features of string physics.

The paper is organized as follows. In section 2 we calculate the potential corresponding to two cases: fixed-end and periodic boundary conditions. In section 3 we obtain the static potential, that is, the effective potential in the limit of large spacetime dimensionality.
Owing to the difficulty of the exact expressions, a saddle point analysis is carried out in section 4. In section 5 we provide a short summary of very useful mathematical results on the inhomogeneous Epstein-type zeta functions and apply them to the expressions that appear in the process of regularization. Finally, in section 6 we study the general case and provide some discussion of the results obtained.

2 Calculation of the potential

We consider as the background the classical solutions of the field equations [8,9] (which are the same for the rigid as for the usual $p$-brane)

$$X^0_{cl} = \xi_0, \quad X^1_{cl} = 0, \quad X^{d-1}_{cl} = \xi_1, \ldots, X^{d-p}_{cl} = \xi_p,$$

with $X^0_{cl} = (X^1, \ldots, X^{d-p})$ and $(\xi_1, \ldots, \xi_p) \in \mathcal{R} \equiv [0, a_1] \times \cdots \times [0, a_p]$. We also use the axial gauge of ref. [8], i.e.,

$$X^0 = X^0_{cl}, \quad X^{d-1} = X^{d-1}_{cl}, \ldots, X^{d-p} = X^{d-p}_{cl},$$

where the Faddeev-Popov ghosts are absent.

We shall consider the toroidal rigid $p$-brane which has the boundary conditions

$$X^\perp(0, \xi_1, \ldots, \xi_p) = X^\perp(T, \xi_1, \ldots, \xi_p) = 0$$

and

$$X^\perp(\xi_0, 0, \xi_2, \ldots, \xi_p) = X^\perp(\xi_0, a_1, \xi_2, \ldots, \xi_p),$$

$$\vdots$$

$$X^\perp(\xi_0, \xi_1, \ldots, \xi_{p-1}, 0) = X^\perp(\xi_0, \xi_1, \ldots, \xi_{p-1}, a_p).$$

For the fixed-end boundary conditions, equation (4) is exactly the same, while eqs. (5) are replaced by the following (of Dirichlet type)

$$X^\perp(\xi_0, 0, \xi_2, \ldots, \xi_p) = \cdots = X^\perp(\xi_0, \xi_1, \ldots, \xi_{p-1}, 0),$$

$$X^\perp(\xi_0, a_1, \xi_2, \ldots, \xi_p) = \cdots = X^\perp(\xi_0, \xi_1, \ldots, \xi_{p-1}, a_p).$$
The effective potential is given by

\[ V = - \lim_{T \to \infty} \frac{1}{T} \ln \int \mathcal{D}X^\perp \exp(-S). \] (7)

Restricting ourselves to the one-loop approximation, we need only consider the terms which are quadratic in the quantum fields (this applies to the usual membrane and p-brane, see [8,9]).

Integrating out \( X^\perp \) and using boundary conditions to read off the resulting \( \text{Tr} \ln \Delta \) (see [8,9,12]), we get

\[ V_{\text{fixed end}} = k \prod_{i=1}^{p} a_i + \frac{d - p - 1}{2} \left[ \sum_{n_1,\ldots,n_p=1}^{\infty} \left( \frac{\pi^2 n_1^2}{a_1^2} + \cdots + \frac{\pi^2 n_p^2}{a_p^2} \right)^{1/2} \right] + \sum_{n_1,\ldots,n_p=1}^{\infty} \left( \frac{\pi^2 n_1^2}{a_1^2} + \cdots + \frac{\pi^2 n_p^2}{a_p^2} + k \rho^2 \right)^{1/2}, \] (8)

and

\[ V_{\text{toroidal}} = k \prod_{i=1}^{p} a_i + \frac{d - p - 1}{2} \left[ \sum_{n_1,\ldots,n_p=-\infty}^{\infty} \left( \frac{4\pi^2 n_1^2}{a_1^2} + \cdots + \frac{4\pi^2 n_p^2}{a_p^2} \right)^{1/2} \right] + \sum_{n_1,\ldots,n_p=-\infty}^{\infty} \left( \frac{4\pi^2 n_1^2}{a_1^2} + \cdots + \frac{4\pi^2 n_p^2}{a_p^2} + k \rho^2 \right)^{1/2}. \] (9)

Observe that the contribution from the higher-derivative mode appears in (8) and (9) with a positive sign, as it follows from the path integral. In the second ref. [3] this sign has been taken to be negative ‘by hand’. Actually, some arguments based on the linearized approximations were given there in favor of the stability of such quantum field theory. Unfortunately, the expression for the Casimir energy obtained in [3] (second ref.) disagrees with the large-\( d \) approximation [2].

### 3 The limit of large spacetime dimensionality

We calculate first the static potential —that is, the effective potential in the limit of large spacetime dimensionality. Such calculation for the usual Nambu-Goto or Eguchi string [13] has been carried out in ref. [14] and for the rigid string in ref. [2]. Let us introduce the
composite fields $\sigma_{\alpha\beta}$ for $\partial_\alpha X^\perp \cdot \partial_\beta X^\perp$, and constrain $\sigma_{\alpha\beta} = \partial_\alpha X^\perp \cdot \partial_\beta X^\perp$ by introducing Lagrange multipliers $\lambda^{\alpha\beta}$:

$$Z = \int \mathcal{D}X^\perp \mathcal{D}\sigma \mathcal{D}\lambda \exp \left\{ -k \int d^{p+1}\xi \left[ \det(\delta_{\alpha\beta} + \sigma_{\alpha\beta}) \right]^{1/2} + \frac{1}{2\rho^2} \int d^{p+1}\xi \Delta_0 X^\perp \cdot \Delta_0 X^\perp - \frac{k}{2} \int d^{p+1}\xi \lambda^{\alpha\beta} \left( \partial_\alpha X^\perp \cdot \partial_\beta X^\perp - \sigma_{\alpha\beta} \right) \right\}, \quad \text{(10)}$$

where $\Delta_0 = \eta^{\alpha\beta} \partial_\alpha \partial_\beta$. Integrating over $X^\perp$, we get

$$Z = \int \mathcal{D}\sigma \mathcal{D}\lambda \exp(-S_{\text{eff}}), \quad \text{(11)}$$

with

$$S_{\text{eff}} = \frac{1}{2} (d - p - 1) \text{Tr} \ln \left( \frac{1}{\rho^2} \Delta_0^2 + k \partial_\alpha \lambda^{\alpha\beta} \partial_\beta \right) + kTR^p \left[ (1 + \sigma_0)^{1/2} (1 + \sigma_1)^{p/2} - \frac{1}{2} (\sigma_0 \lambda_0 + p \sigma_1 \lambda_1) \right], \quad \text{(12)}$$

where we have chosen $a_1 = \cdots = a_p = R$, and $\sigma_{\alpha\beta} = \text{diag}(\sigma_0, \sigma_1, \ldots, \sigma_1)$, $\lambda^{\alpha\beta} = \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_1)$ (compare with [8,9], where the case $1/\rho^2 = 0$ was considered).

In this case we obtain

$$\text{Tr} \ln \left[ \Delta_0^2 + k\rho^2 (\lambda_0 \partial_0^2 + \lambda_1 \partial_x^2) \right] = \text{Tr} \ln \left[ (\partial_0^2 + \partial_x^2) + k\rho^2 (\lambda_0 \partial_0^2 + \lambda_1 \partial_x^2) \right] = \text{Tr} \ln \left\{ \partial_0^2 + \frac{k\rho^2 \lambda_0}{2} + \sqrt{k\rho^2 (\lambda_0 - \lambda_1) \partial_x^2 + \frac{k^2 \rho^4 \lambda_0^2}{4}} \right\} \times \left[ \partial_0^2 + \frac{k\rho^2 \lambda_0}{2} + \sqrt{k\rho^2 (\lambda_0 - \lambda_1) \partial_x^2 + \frac{k^2 \rho^4 \lambda_0^2}{4}} \right]. \quad \text{(13)}$$

The spectrum for the boundary conditions (2)-(6) is known. Using this spectrum and evaluating the $\text{Tr}ln$ terms by means of analytic regularization for large $T$, we obtain (see [8,9] for details of this method)

$$S_{\text{eff}} = kTR^p \left\{ (1 + \sigma_0)^{1/2} (1 + \sigma_1)^{p/2} - \frac{1}{2} (\sigma_0 \lambda_0 + p \sigma_1 \lambda_1) \right\} + \frac{d - p - 1}{2kR^{p+1}} \left\{ \sum \left( \pi^2 n^2 + \frac{k\rho^2 \lambda_0 R^2}{2} - \sqrt{k\rho^2 (\lambda_0 - \lambda_1) n^2 R^2 + \frac{k^2 \rho^4 \lambda_0^2 R^4}{4}} \right) \right\}^{1/2} \quad \text{(14)}$$

$$+ \sum \left( \pi^2 n^2 + \frac{k\rho^2 \lambda_0 R^2}{2} + \sqrt{k\rho^2 (\lambda_0 - \lambda_1) n^2 R^2 + \frac{k^2 \rho^4 \lambda_0^2 R^4}{4}} \right)^{1/2} \right\}.$$
where for the fixed-end $p$-brane $\vec{n}^2 = n_1^2 + \cdots + n_p^2$ and $\sum_{\vec{n}}$ means $\sum_{n_1,\ldots,n_p=1}^{\infty}$ as in (8), while for the toroidal $p$-brane $\vec{n}^2 = 4(n_1^2 + \cdots + n_p^2)$ and $\sum_{\vec{n}}$ means $\sum_{n_1,\ldots,n_p=-\infty}^{\infty}$ as in (9).

### 4 A saddle point analysis

The functions that appear on the right hand side of (14) are rather complicated to analyze. To our knowledge, they have never been considered in the literature and will be the object of a separate investigation. (Note that in the case of the usual Dirac $p$-brane [8,9] these functions are simply constants). So we shall have here little to say about the corresponding effective potential, only, for example, that as $R \to \infty$, $V \sim V_{cl} = kR^p$. Rewriting the expression for the static potential identically as

$$V = kR^p \left[ (1 + \sigma_0)^{1/2}(1 + \sigma_1)^{p/2} - \frac{1}{2} (\sigma_0 \lambda_0 + p \sigma_1 \lambda_1) + \frac{d - p - 1}{2kR^{p+1}} K(k\rho^2 R^2, \lambda_0, \lambda_1) \right],$$

we are led to the four saddle-point equations:

$$\begin{align*}
\lambda_0 & = (1 + \sigma_1)^{p/2}(1 + \sigma_0)^{-1/2}, \\
\lambda_1 & = (1 + \sigma_1)^{p/2-1}(1 + \sigma_0)^{1/2}, \\
\sigma_0 & = \frac{d - p - 1}{kR^{p+1}} \frac{\partial K(k\rho^2 R^2, \lambda_0, \lambda_1)}{\partial \lambda_0}, \\
\sigma_1 & = \frac{d - p - 1}{kpR^{p+1}} \frac{\partial K(k\rho^2 R^2, \lambda_0, \lambda_1)}{\partial \lambda_1},
\end{align*}$$

By eliminating from these equations $\sigma_0$ and $\sigma_1$, we get

$$\begin{align*}
(\lambda_1 \lambda_0)^{p/(p-1)} \lambda_0^{-2} - 1 & = \frac{d - p - 1}{kR^{p+1}} \frac{\partial K(k\rho^2 R^2, \lambda_0, \lambda_1)}{\partial \lambda_0}, \\
(\lambda_1 \lambda_0)^{1/(p-1)} - 1 & = \frac{d - p - 1}{kpR^{p+1}} \frac{\partial K(k\rho^2 R^2, \lambda_0, \lambda_1)}{\partial \lambda_1}.
\end{align*}$$

In principle, if not analytically, it is of course possible to eliminate (let us say) $\lambda_0$ from these two equations by means of a numerical calculation, and to rewrite $V$ in terms of $\lambda_1$ only. After doing this, one can study $V$ as a function of $R$ and in terms of this parameter $\lambda_1$, in order to see if the tachyon is in the spectrum. We do not do this work here, since
our purpose is just to show the possibility, in principle, of calculating the static potential
for a rigid string. However, we are ready to look to some interesting limiting case of these
equations. Let us assume that \( \lambda_0 = \lambda_1 \equiv \lambda \) (such a choice has been taken for the rigid
string in [2]). Then

\[
V = kR^p \left\{ (1 + \sigma_0)^{1/2}(1 + \sigma_1)^{p/2} - \frac{1}{2} \lambda (\sigma_0 + p\sigma_1) \right\}.
\]

It follows from the saddle point equations that \( \sigma_0 = \sigma_1, \lambda = (1 + \sigma)^{(p-1)/2}, \) and

\[
V = kR^p \left[ \lambda^{(p+1)/(p-1)} - \frac{p + 1}{2} \lambda^{2/(p-1)} - 1 + \frac{(d - p - 1)\pi}{2kR^{p+1}} K(k^2 \lambda R^2) \right].
\]

Here we have defined

\[
K(k^2 \lambda R^2) \equiv \sum_{\vec{n}} \sqrt{\vec{n}^2} + \sum_{\vec{n}} (\vec{n}^2 + k(\rho/\pi)^2 \lambda R^2)^{1/2},
\]

and the last saddle point equation gives

\[- \frac{p + 1}{2} \lambda^{2/(p-1)} + \frac{(d - p - 1)\pi}{2kR^{p+1}} \frac{\partial K(k^2 \lambda R^2)}{\partial \lambda} = 0.\]

5 Explicit expressions for the inhomogeneous Epstein functions

The expressions to be regularized are (in general) of the inhomogeneous Epstein form

\[
E_p^c(s) \equiv \sum_{n_1, \ldots, n_p=1}^{\infty} (n_1^2 + \cdots + n_p^2 + c^2)^{-s},
\]

allowing for \( c = 0 \). These functions are not easy to deal with (for \( p > 1 \), the case \( p = 1 \) is
the only one that has been investigated in the literature) and refs. [15] can be considered
as pioneering in this respect. There, very general formulas have been derived for the functions

\[
M_N^c(s; a_1, \ldots, a_N; \alpha_1, \ldots, \alpha_N) \equiv \sum_{n_1, \ldots, n_N=1}^{\infty} (a_1 n_1^{\alpha_1} + \cdots + a_N n_N^{\alpha_N} + c^2)^{-s},
\]

7
for any value of $c$, such as

$$M_N^c(s; a_1, \ldots, a_N; \alpha_1, \ldots, \alpha_N) = \frac{1}{a_N^s \Gamma(s)} \sum_{p=0}^{N-1} \sum_{r=1}^{p} \frac{b_{ir}^{-1/\alpha_r}}{\alpha_r} \Gamma \left( \frac{1}{\alpha_r} \right)$$

\begin{align}
\times \sum_{k_{j_1, \ldots, k_{j_{N-1}}=0}}^{\infty} \Gamma \left( s + \sum_{l=1}^{N-p-1} k_{j_l} - \sum_{r=1}^{p} \frac{1}{\alpha_r} \right) \\
\times \prod_{i=1}^{N-p-1} \left[ \frac{(-b_{jr})^{k_{j_r}}}{k_{j_r}!} \zeta(-\alpha_{j_r} k_{j_r}) \right] \zeta \left( \alpha_n \left( s + \sum_{l=1}^{N-p-1} k_{j_l} - \sum_{r=1}^{p} \frac{1}{\alpha_r} \right) \right) + \Delta_{ER},
\end{align}

(24)

with $b_{ir} \equiv a_{ir}/a_N$ (notice the slight erratum in eq. (3.22) of the second ref. [15]), and $1 \leq i_1 < \cdots < i_p \leq N - 1, 1 \leq j_1 < \cdots < j_{N-p-1} \leq N - 1$, being $i_1, \ldots, i_p, j_1, \ldots, j_{N-p-1}$ a permutation of $1, 2, \ldots, N - 1$. The sum on $C_{N-1,p}$ means sum over the $\binom{N-1}{p}$ choices of the indices $i_1, \ldots, i_p$ among the $1, 2, \ldots, N - 1$.

In our case, $a_1 = \cdots = a_p = 1$ and $\alpha_1 = \cdots = \alpha_p = 2$, and the involved general formula above, (24), simplifies considerably. We have, for $c = 0$,

$$E_p(s) = \frac{(-1)^{p-1}}{2^{p-1}} \frac{1}{\Gamma(s)} \sum_{j=0}^{p-1} (-1)^j \binom{p-1}{j} \Gamma(2s - j) \zeta \left( s - \frac{j}{2} \right) + \Delta_{ER},$$

(25)

and for $c \neq 0$,

$$E^c_p(s) = \frac{(-1)^{p-1}}{2^{p-1}} \frac{1}{\Gamma(s)} \sum_{j=0}^{p-1} (-1)^j \binom{p-1}{j} \Gamma(2s - j) E^c_1 \left( s - \frac{j}{2} \right) + \Delta_{ER}.$$  

(26)

Notice that the poles of this function arise from those of $E^c_1(s - j/2)$, which are obtained for the values of $s$ such that $s - j/2 = 1/2, -1/2, -3/2, \ldots$. They are poles of order one at $s = p/2, (p - 1)/2, p/2 - 1, \ldots$, except for $s = 0, -1, -2, \ldots$, then the function is finite (owing to the $\Gamma(s)$ in the denominator). These poles are removed by zeta-function regularization [10].

Alternatively, a very useful and exact recurrent formula is the following [13]

$$E^c_N(s; a_1, \ldots, a_N) \equiv \sum_{n_1, \ldots, n_N=1}^{\infty} \left( a_1 n_1^2 + \cdots + a_N n_N^2 + c^2 \right)^{-s}$$

\begin{align}
= -\frac{1}{2} E^c_{N-1}(s; a_2, \ldots, a_N) + \frac{1}{2} \sqrt{\frac{\pi}{a_1}} \frac{\Gamma(s - 1/2)}{\Gamma(s)} E^c_N(s - 1/2; a_2, \ldots, a_N) \\
+ \frac{\pi^s}{\Gamma(s)} a_1^{-s/2} \sum_{k=0}^{\infty} \frac{a_1^{k/2}}{k!(16\pi)^k} \prod_{j=1}^{\infty} \left[ (2s - 1)^2 - (2j - 1)^2 \right] \sum_{n_1, n_2, a_N=1}^{\infty} n_1^{s-k-1} \\
\times \left( a_2 n_2^2 + \cdots + a_N n_N^2 + c^2 \right)^{-(s+k)/2} \exp \left[ -\frac{2\pi}{\sqrt{a_1}} n_1 (a_2 n_2^2 + \cdots + a_N n_N^2 + c^2)^{1/2} \right].
\end{align}

(27)
The recurrence starts from expression

\[ E_1^c(s; 1) = -\frac{c^{-2s}}{2} + \frac{\sqrt{\pi} \Gamma(s - 1/2)}{2 \Gamma(s)} c^{-2s+1} \]

\[ + \frac{2\pi^s c^{-s+1/2}}{\Gamma(s)} \sum_{n=1}^{\infty} n^{s-1/2} K_{s-1/2}(2\pi n c). \] \tag{28} \]

In order to deal with the derivative of the function \( K \) above, one can follow two equivalent procedures: either do first the usual analytic continuation, and then take \( s = -1/2 \) and the derivative afterwards, or else take first the derivative of (22), perform the analytic continuation and put \( s = +1/2 \) at the end. The result is exactly the same. In either way other non-trivial series commutations have to be performed (see [17] and references therein). We get, in particular, for \( c \neq 0 \)

\[ E_2^c(s) = -\frac{1}{2} E_1^c(s) + \frac{\sqrt{\pi}}{2} \frac{\Gamma(s - 1/2)}{\Gamma(s)} E_1^c(s - 1/2) + \Delta_{ER}, \] \tag{29} \]

\[ E_3^c(s) = \frac{1}{4} E_1^c(s) - \frac{\sqrt{\pi}}{2} \frac{\Gamma(s - 1/2)}{\Gamma(s)} E_1^c(s - 1/2) + \frac{\pi}{4(s-1)} E_1^c(s-1) + \Delta_{ER}, \]

and similar expressions for \( c = 0 \), as it is clear from (25) and (26) (it has been proven in [16] that this case can be obtained from the former by analytically continuing in the parameter \( c \)). Remember that \( \Delta_{ER} \) is the well-known term (here exponentially small), which was found in [17]. As our final interest is numerical approximation (see above), we will not take into account exponentially small terms (let us point out that these expansions are asymptotic and very quickly convergent) [15]. Notice, moreover, that for \( c = 0 \) there is no dependence on \( \lambda \), so that the corresponding term does not contribute to eq. (22).

From the above expressions, for any value of \( p \) there is no difficulty in obtaining the value of \( \lambda \) which solves (22). In particular, for \( p = 2 \) we have:

\[ \lambda \approx \left( \frac{3 - d}{12\pi} \right)^{2/3} k^{1/3} \rho^2 \] \tag{30} \]

this is a sensible root for \( k \cdot \rho \) big (specifically, for \( 1/R << k^{2/3} \rho^2 \)) and \( d \neq 3 \). For \( p = 3 \) the result is:

\[ \lambda \approx \frac{1}{2^{10}} \left( \frac{3(4 - d)}{\pi} \right)^2 \frac{k \rho^6}{R^2}, \] \tag{31} \]
which is a sensible root for \( k \cdot \rho \) big (specifically, for \( k\rho^4 \gg 1 \)) and \( d \neq 4 \). Let us now substitute these values into the expression of \( V \), and look for the derivative \( \partial V / \partial R \). We get, for \( p = 2 \), an expression of the form

\[
\frac{\partial V_2}{\partial R} = c_1 R + \frac{c_{-1}}{R} + \frac{c_{-2}}{R^2},
\]

which has always one real root (at least), \( R_2 \). It corresponds to a minimum of \( V \) for \( d > 3 \) and reasonable values of the constants involved. Substituting back into \( V(R) \) we see that the minimum is attained at

\[
V_2(R_2) = kR_2^2 \left[ 3(1 + \sigma_0)^{1/2}(1 + \sigma_1) - \frac{3}{2} \left( \frac{3 - d}{12\pi} \right)^{2/3} (\sigma_0 + 2\sigma_1)k^{1/3}\rho^2 + \frac{(d - 3)^2}{48\pi^2}\rho^6 \right],
\]

i.e., for \( k \) big it is obtained at a negative value of \( V \), while for \( \rho \) big it is reached at a positive value of \( V \).

The case \( p = 3 \) is quite different. We get then

\[
V_3(R) = \alpha_1 R^3 - \alpha_2 R + \frac{\beta}{R},
\]

so that its derivative has two real roots

\[
R_{\pm} = \pm \left[ \frac{\alpha_2 + (\alpha^2 + 12\alpha\beta)^{1/2}}{6\alpha_1} \right]^{1/2},
\]

one of which is seen to correspond to a maximum and the other to a minimum of \( V \). Moreover, two additional, complex roots appear. The minimum for \( V \) is now attained at a negative value of \( V \) when either \( k \) or \( \rho \) are big enough and, conversely, at a positive value of the potential for \( k \) or \( \rho \) small. Note that in order to find the critical radius at which the potential becomes complex (so that the static approximation breaks down and tachyons appear) it is necessary to do the analysis with the general expression (15) directly. One can conjecture from our results here that the rigid membrane is tachyon-free (no critical radius exists), as it is also the case with rigid strings (see the second ref. [2] and the first ref. [3]). At least for the limiting situation discussed above, this is in fact the case.
6 Discussion of the general case

Having done the calculation for this special case, corresponding to the limit of large spacetime dimensionality, and armed with the full equation (24), we can now be more ambitious and consider the one-loop effective potential, (8)-(9), without further limit or approximation. For the sake of conciseness, we shall restrict ourselves to $p = 2$—but it is obvious that we could consider as well any other value of $p$. We rely on equations (27) and (28), which specialized to $p = 2$ yield, after some work,

$$
\sum_{n_1,n_2=1}^{\infty} \sqrt{\left(\frac{n_1}{a_1}\right)^2 + \left(\frac{n_2}{a_2}\right)^2} = \frac{1}{24} \left( \frac{1}{a_1} + \frac{1}{a_2} \right) - \frac{\zeta(3)}{8\pi^2} \left( \frac{a_1}{a_2^2} + \frac{a_2}{a_1^2} \right) - \frac{\pi^{3/2}}{2\sqrt{a_1a_2}} \left[ \exp \left( -2\pi \frac{a_1}{a_2} \right) \left( 1 + O(10^{-3}) \right) \right],
$$

and (this one after additional regularization, see above)

$$
\sum_{n_1,n_2=1}^{\infty} \sqrt{\left(\frac{n_1}{a_1}\right)^2 + \left(\frac{n_2}{a_2}\right)^2 + c^2} = \frac{c}{4} - \frac{\pi}{6} a_1 a_2 c^3 + \left( \frac{1}{4\pi} \sqrt{\frac{c}{a_2}} - \frac{ca_1}{4\pi a_2} \right) \left[ \exp \left( -2\pi ca_2 \right) \left( 1 + O(10^{-3}) \right) \right].
$$

In both cases we have assumed (this is, of course, no restriction) that $a_2 \leq a_1$.

These expressions are really valuable. They are asymptotic, the last term (already of exponential kind) being of order $10^{-3}$ with respect to the two first ones, and the not explicitly written contributions being of order $10^{-6}$. To our knowledge, the second expression—which can be termed as of inhomogeneous Epstein type—has never been discussed in the literature.

For fixed-end boundary conditions and not taking into account exponentially-small terms, we obtain

$$
V_{f.e.} \simeq k a_1 a_2 + \frac{(d-3)\pi}{24} \left[ \frac{1}{2} \left( \frac{1}{a_1} + \frac{1}{a_2} \right) - \frac{3\zeta(3)}{2\pi^2} \left( \frac{a_1}{a_2^2} + \frac{a_2}{a_1^2} \right) \right] + \frac{3}{\pi} \sqrt{k\rho} - \frac{2}{\pi^2} k^{3/2} \rho^3 a_1 a_2.
$$

It is now straightforward to perform the analysis of extrema of $V$. For brevity, we shall only discuss here some particular cases. First, the one which is obtained from the two
Lagrange equations for the extrema of $V$ as a function of $a_1$ and $a_2$ only, for $a_1 = a_2 \equiv a$, which is reached for

$$a = \left( \frac{3-d}{5} \frac{[3\zeta(3) - \pi^2]}{12\pi k + (3-d) k^{3/2} \rho^3} \right)^{1/3}. \quad (39)$$

It can be seen that for $12\pi k + (3-d) k^{3/2} \rho^3 < 0$ this point is a minimum. On the contrary, it is a maximum for $12\pi k + (3-d) k^{3/2} \rho^3 > 0$. Consistency with the range of validity of the series expansion above is obtained for

$$2\pi \sqrt{k\rho} > \frac{6}{\rho^2} >> 1. \quad (40)$$

typical values for which this is valid are: $\rho \simeq 2/3$, $k \simeq 4$, $2\pi c \simeq 8$.

Keeping now $a_1$ and $a_2$ fixed (but arbitrary), we see that (for $d > 3$) in terms of $\rho$, $V$ is unbounded from below, being always negative for $\rho$ big enough. Considering $V$ as a function of $k$, the situation is similar. Finally, in a sense the analysis of ref. [8] is still valid here, when we fix the values of $k$, $\rho$ and of the area $A = a_1 a_2$: the minima of the potential are obtained for elongated (stringy) membranes ($a_1/a_2$ small). Notice, however, that even for the particular case considered in [8], our asymptotic expansion provides a more universal expression, because it is valid for any value of $a_2 \leq a_1$ (this is again not restrictive, in the end). It also goes without saying that, from our general formula (38) for the potential $V = V(a_1, a_2, k, \rho, d)$, one can perform a simultaneous analysis on all the different parameters at the same time —e.g. in order to look for local minima of the potential hypersurface— the explicit dependences on $k$ and $\rho$ being also basic contributions of the present work.

In the case of toroidal boundary conditions, again neglecting exponentially-small contributions, we get (for a very detailed discussion of the relations between the different boundary conditions see the last of references [15])

$$V_{tor} \simeq k a_1 a_2 + \frac{(d-3)\pi}{2} \left[ -\frac{\zeta(3)}{\pi^2} \left( \frac{a_1}{a_2} + \frac{a_2}{a_1} \right) \frac{1}{\pi} \sqrt{k\rho} - \frac{1}{6\pi^2} k^{3/2} \rho^3 a_1 a_2 \right]. \quad (41)$$

The particular extremum for $a_1 = a_2 \equiv a$ is a minimum of $V$ provided that $\sqrt{k\rho} > 12\pi$ (it is a maximum for $\sqrt{k\rho} < 12\pi$). Consistency with the series expansion implies now

$$\sqrt{k\rho} > \frac{12\pi}{\rho^2} >> 1. \quad (42)$$
which can be met typically for values of $\rho \simeq 3$, $k \simeq 9$, $2\pi c \simeq 9$ —but, of course, as in the former case, the range of allowed values is much wider.

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