Zero-Determinant strategies in finitely repeated $n$-player games

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Abstract—In two-player repeated games, Zero-Determinant (ZD) strategies are a class of strategies that can enable a player to unilaterally enforce a linear payoff relation between her own and her opponent’s payoff irrespective of the opponent’s strategy. This manipulative nature of the ZD strategies attracted significant attention from researchers due to its close connection to controlling distributively the outcome of evolutionary games in large populations. In this paper, we study the existence of ZD strategies in repeated $n$-player games with a finite but undetermined time horizon. Necessary and sufficient conditions are derived for a linear relation to be enforceable by a ZD strategist in $n$-player social dilemmas, in which the expected number of rounds can be captured by a fixed and common discount factor ($0 < \delta < 1$). Thresholds exist for such a discount factor above which generous, extortionate and equalizer payoff relations can be enforced. For the first time in the studies of repeated games, ZD strategies are examined in the setting of finitely repeated $n$-player, two-action games. Our results show that depending on the group size and the ZD-strategist’s initial probability to cooperate, for finitely repeated $n$-player social dilemmas, it is possible for extortionate, generous and equalizer ZD-strategies to exist. The threshold discount factors rely on the slope and baseline payoff of the desired linear relation and the variation in the “one-shot” payoffs of the $n$-player game. To show the utility of our general results, we apply them to a linear $n$-player public goods game.

Index Terms—Game theory, $n$-player games, repeated games, zero-determinant strategies.

I. INTRODUCTION

The functionalities of many complex social systems rely on their composing individuals’ willingness to set aside their personal interest for the benefit of the greater good [12]. One mechanism for the evolution of cooperation is known as direct reciprocity: even if in the short run it pays off to be selfish, mutual cooperation can be favoured when the individuals encounter each other repeatedly. Direct reciprocity is often studied in the standard model of repeated games and it is only recently, inspired by the discovery of a novel class of strategies, called zero-determinant (ZD) strategies [14], that repeated games began to be examined from a new angle by investigating the level of control that a single player can exert on the average payoffs of its opponent. In [14] Press and Dyson showed that in infinitely repeated $2 \times 2$ prisoners dilemma games, if a player can remember the actions in the previous round, this player can unilaterally impose some linear relation between its own expected payoff and that of its opponent. It is emphasized that this enforced linear relation cannot be avoided even if the opponent employs some intricate strategy with a large memories. Such strategies are called zero-determinant because they enforce a part of the transition matrix to have a determinant that is equal to zero. Later, ZD strategies were extended to games with more than two possible actions [15], continuous action spaces [11], and alternative moves [10]. The success of ZD strategies in an evolutionary setting was examined in [16], [9]. For a given population size, in the limit of weak selection it was shown in [17] that all ZD strategies that can survive an invasion of any memory-one strategy must be “generous”, namely enforcing a linear payoff relation that favors others. This surprising fact was tested experimentally in [4]. Most of the literature focuses on two-player games; however, in [13] the existence of ZD-strategies in infinitely repeated public goods games was shown by extending the arguments in [14] to a symmetric public goods game. Around the same time, characterization of the feasible ZD strategies in multiplayer social dilemmas and those strategies that maintain cooperation in such $n$-player games were reported in [6]. Both in [13] and [6] it was noted that group size $n$ imposes restrictive conditions on the set of feasible ZD strategies and that alliances between co-players can overcome this restrictive effect of the group size.

The evolutionary success of ZD strategies in such $n$-player games was studied in [7] and the results show that sustaining large scale cooperation requires the formation of alliances. ZD strategies for finitely repeated $2 \times 2$ games with discounted payoffs were defined and characterized in [5]. The threshold discount factors above which the ZD strategy can exist were derived in [8]. In this paper we use the framework of ZD strategies in infinitely repeated multiplayer social dilemmas from [6] and extend it to the finitely repeated case in which future payoffs are discounted. We build upon our results in [2], in which enforceable payoff relations were characterized, by developing new theory that allows us to express threshold discount factors that determine how fast a strategic player can enforce a desired linear payoff relation. These general results are applicable to multiplayer and two player games and can be applied to a variety of complex social dilemma settings including the famous prisoner’s dilemma, the public goods game, the volunteer’s dilemma, the $n$-player snowdrift game and much more. These additional results can be also be used to determine ones possibilities for exerting control given a constraint on the expected number of interactions, and thus provide novel insights for one’s level of influence in real-world repeated interactions. The results in this paper can be used to investigate, both analytically and experimentally, the effect of the group size and the initial condition on the level of control that a single player can exert in finitely repeated $n$-player social dilemma games. Thus, our results may open the door for novel control techniques that seek to achieve or sustain cooperation in large social systems that evolve under evolutionary forces. The paper is organized as follows. In
Assumption 1 \( n \)-player social dilemma games and it ensures that there is a conflict between the interest of each individual and that of the group as a whole. Thus, those games whose payoffs satisfy Assumption 1 can model a social dilemma that results from selfish behaviors in a group. Consider the following examples.

**Example 1.** As an example of a game that satisfies Assumption 1 consider a public goods game in which each cooperator contributes an amount \( c > 0 \) to a public good. The sum of the contributions get multiplied by an enhancement factor \( 1 < r < n \) and then divided evenly among all group members. This results in the following payoffs:

\[
\frac{rc(z + 1)}{n} - c, \quad \frac{rz}{n}, \quad \text{for } z = 0, \ldots, n - 1.
\]

**Example 2** (\( n \)-player stag hunt game). In the public goods of Example 7 a single player can create a benefit. In some other social dilemma games only a group of cooperators can create a benefit. For example, in the \( n \)-player stag hunt game, players obtain the benefit \( b \) if and only if all players cooperate [18]. This results in the following payoffs:

\[
b_z = 0, \quad \text{for all } 0 \leq z \leq n - 1,\]

\[
a_z = \begin{cases} b - c, & \text{if } z = n - 1; \\ -c, & \text{otherwise}. \end{cases}
\]

**C. Strategies**

In repeated games the players must choose how to update their actions as the game interactions are repeated over rounds. A **strategy** of a player determines the conditional probabilities with which actions are chosen by the player. To formalize this concept we introduce some additional notation. A history of plays up to round \( t \) is denoted by \( h^t = (\sigma^0, \sigma^1, \ldots, \sigma^{t-1}) \in \mathcal{A}^t \) such that each \( \sigma^k \in \mathcal{A} \) for all \( k = 0 \ldots t - 1 \). The union of possible histories is denoted by \( \mathcal{H} = \bigcup_{t=0}^\infty \mathcal{A}^t \), with \( \mathcal{A}^0 = \emptyset \) being the empty set. Finally, let \( \Delta(\mathcal{A}) \) denote the probability distribution over the action set \( \mathcal{A} \). As is standard in the theory of repeated games, a strategy of player \( i \) is then defined by a function \( \rho : \mathcal{H} \to \Delta(\mathcal{A}) \) that maps the history of play to the probability distribution over the action set. An interesting and important subclass of strategies are those that only take into account the action profile in round \( t - 1 \), (i.e. \( \sigma^{t-1} \in \mathcal{H}^t \)) to determine the conditional probabilities to choose some action in round \( t + 1 \). Correspondingly these strategies are called **memory-one strategies** and are formally defined as follows.

**Definition 1** (Memory-one strategy, [5]). A strategy \( \rho \) is a memory-one strategy if \( \rho(h^t) = \rho(h^{t-1}) \) for all histories \( h^t = (\sigma^0, \ldots, \sigma^{t-1}) \) and \( h^{t-1} = (\sigma^0, \ldots, \sigma^{t-2}) \) with \( t, t' \geq 1 \) and \( \sigma^{t-2} = \sigma^{t'-2} \).

The theory of Press and Dyson showed that, for determining the best performing strategies in terms of expected payoffs in
two-action repeated games, it is sufficient to consider only the space of memory-one strategies \([14], [15]\).

III. MEAN DISTRIBUTIONS OF MEMORY-ONE STRATEGIES IN FINITELY REPEATED \(n\)-PLAYER GAMES

In this section we zoom in on a particular player that employs a memory-one strategy in the \(n\)-player game and refer to this player as the key player. In particular, we focus on the relation between the mean distribution of action profiles and the memory-one strategy of the key player. Let \(p_{x,z} \in [0,1]\) denote the probability that the key player cooperates in round \(t+1\) given that, in round \(t\), the player plays \(x \in \{C, D\}\) and \(z\) of the co-players cooperate. By stacking these probabilities for all \(2n\) possible outcomes into a vector, we obtain the memory-one strategy that determines the probability for the key player to cooperate in round \(t+1\):

\[
P = (p_{C,n-1}, \ldots, p_{C,0}, p_{D,n-1}, \ldots, p_{D,0}) \in [0,1]^{2n},
\]

where we have used the convention to order the conditional probabilities based on the key player’s decision and a descending number of cooperating co-players. Accordingly, the memory-one strategy \(p^{Rep} = (p_0, p_0)\), gives the probability to cooperate when the current action is simply repeated. That is, when the key player cooperates in round \(t\), by employing \(p^{Rep}\) she will cooperate in round \(t+1\) with probability one. Let \(v_{x,z}(t)\) denote the probability that the outcome of round \(t\) is \((x, z) \in A\), with \(x \in \{C, D\}\). And let \(v(t) = (v_{x,z}(t)) \in R^{2n}\) be the vector of outcome probabilities in round \(t\). As in \([5], [8], [11], [10]\) we focus on finitely repeated games. The finite number of rounds is determined by a fixed and common discount factor \(0 < \delta < 1\). Using the limit of the geometric sum of \(\delta\), the expected number of rounds is \(\frac{1}{1-\delta}\). As in \([5]\), the mean distribution of \(v(t)\) is:

\[
v = (1 - \delta) \sum_{t=0}^{\infty} \delta^t v(t).
\]

In this paper we are interested in the average discounted payoffs of the finitely repeated \(n\)-player game. Let \(g_{x,z}^i\) denote the payoff in a given round that player \(i\) receives by choosing \(x \in A\) and \(z\) of its co-players cooperated. By stacking the possible payoffs we obtain the vector \(g^i = (g_{x,z}^i) \in R^{2n}\) that contains all possible payoffs in a given round of player \(i\). The expected “one-shot” payoff of player \(i\) in round \(t\) is \(\pi^i(t) = g^i v(t)\). And the average discounted payoff in the finitely repeated game for player \(i\) is then:

\[
\pi^i = (1 - \delta) \sum_{t=0}^{\infty} \delta^t \pi^i(t) = (1 - \delta) \sum_{t=0}^{\infty} \delta^t g^i \cdot v(t) = g^i \cdot v.
\]

The following lemma relates the limit distribution \(v\) of the finitely repeated game to the memory-one strategy \(p\) of the key player. The presented lemma is a straightforward \(n\)-player extension of the 2-player case that is given in \([5]\) and relies on the fundamental results from \([1]\).

**Lemma 1 (Limit distribution).** Suppose the key player applies memory-one strategy \(p\) and the strategies of the other players are arbitrary, but fixed. For the finitely repeated \(n\)-player game, it holds that

\[
(\delta p - p^{Rep}) \cdot v = -(1-\delta)p_0,
\]

where \(p_0\) is the key player’s initial probability to cooperate.

**Proof:** The probability that \(i\) cooperates in round \(t\) is \(q_C(t) = p^{Rep} \cdot v(t)\). And the probability that \(i\) cooperates in round \(t+1\) is \(q_C(t+1) = p \cdot v(t)\). Now define

\[
u(t) := \delta q_C(t+1) - q_C(t) = (\delta p - p^{Rep}) \cdot v(t).
\]

Multiplying equation \((2)\) by \((1-\delta)\) \(\delta^t\) and summing up over \(t = 0, \ldots, \tau\) we obtain

\[
(1 - \delta) \sum_{t=0}^{\tau} \delta^t u(t) = (1 - \delta) (\delta q_C(1) - q_C(0)) + \delta^2 q_C(2) - \delta q_C(1) + \cdots = (1 - \delta) \delta \sum_{t=0}^{\tau} q_C(\tau + 1) - (1 - \delta)q_C(0).
\]

Because \(0 < \delta < 1\), it follows that

\[
\lim_{\tau \to \infty} (1 - \delta) \sum_{t=0}^{\tau} \delta^t u(t) = -(1 - \delta)p_0.
\]

And by the definition of \(v\) in equation \((1)\):

\[
\lim_{\tau \to \infty} (1 - \delta) \sum_{t=0}^{\tau} \delta^t (\delta p - p^{Rep}) = (\delta p - p^{Rep}) \cdot v.
\]

By substituting \(u(t)\) back into the equation we obtain

\[
(\delta p - p^{Rep}) \cdot v = -(1 - \delta)p_0.
\]

This completes the proof. \(\blacksquare\)

**Remark 1.** Note that in the limit \(\delta \to 1\), the infinitely repeated game is recovered. In this setting, the expected number of rounds is infinite. And, if the limit exists, the average payoffs are given by

\[
\pi^i = \lim_{\tau \to \infty} \frac{1}{\tau+1} \sum_{t=0}^{\tau} \pi^i(t).
\]

By Akins Lemma (see \([11], [6]\)), for the infinitely repeated game irrespective of the initial probability to cooperate, it holds that

\[
(p - p^{Rep}) \cdot v = 0.
\]

Hence, a key difference between the infinitely repeated and finitely repeated game is that \(p_0\) is important for the relation between the memory-one strategy \(p\) and the mean distribution \(v\) when the game is repeated a finite number of expected rounds. When the game is infinitely repeated, i.e. \(\delta \to 1\), the importance of the initial conditions on the relation between \(p\) and \(v\) disappears \([6]\).
IV. ZD-STRATEGIES IN FINITELY REPEATED n-PLAYER GAMES

Based on Lemma 1, we now formally define a ZD strategy for a finitely repeated n-player game.

Definition 2. A memory-one strategy \( p \) is a ZD-strategy for an n-player game if there exist constants \( \alpha, \beta, \gamma \), 1 \( \leq j \leq n \) with \( \sum_{j \neq i} \beta_j \neq 0 \) such that

\[
\delta p = p^{Rep} + \alpha g^i + \sum_{j \neq i} \beta_j g^j + (\gamma - (1 - \delta)p_0)I. \tag{3}
\]

Moreover, the linear payoff relation in equation (4) becomes

\[
\delta p = p^{Rep} + \alpha g^i + \sum_{j \neq i} \beta_j g^j + (\gamma - (1 - \delta)p_0)I. \tag{3}
\]

The following proposition shows how the ZD strategy can enforce a linear relation between the key players expected payoff and that of her co-players.

Proposition 1. Suppose the key player employs a fixed ZD strategy with parameters \( \alpha, \beta, \gamma \) as in definition 2. Then, irrespective of the fixed strategies of the remaining \( n - 1 \) co-players, the payoffs obey the equation

\[
\alpha \pi^i + \sum_{j \neq i} \beta_j \pi^j + \gamma = 0. \tag{4}
\]

Proof:

\[
(\delta p - p^{Rep}) = \alpha g^i + \sum_{j \neq i} \beta_j g^j + (\gamma - (1 - \delta)p_0)I
\]

\[
(\delta p - p^{Rep}) \cdot v = \alpha \pi^i + \sum_{j \neq i} \beta_j \pi^j + \gamma - (1 - \delta)p_0
\]

\[
(\delta p - p^{Rep}) \cdot v + (1 - \delta)p_0 = \alpha \pi^i + \sum_{j \neq i} \beta_j \pi^j + \gamma
\]

\[
0 = \alpha \pi^i + \sum_{j \neq i} \beta_j \pi^j + \gamma.
\]

To be consistent with the earlier work on ZD strategies in infinitely repeated n-player games in [6], we introduce the parameter transformations:

\[
l = \frac{-\gamma}{(\alpha + \sum_{k \neq i} \beta_k)}, s = \sum_{k \neq i} \beta_k,
\]

\[
w_{j \neq i} = \frac{\beta_j}{\sum_{k \neq i} \beta_k}, \phi = -\sum_{k \neq i} \beta_k, w_i = 0.
\]

Using these parameter transformations, equation (3) can be written as

\[
\delta p = p^{Rep} + \phi \left[ sg^i - \sum_{j \neq i} w_{j} g^j + (1-s)I \right] - (1 - \delta)p_0 I, \tag{6}
\]

under the conditions that \( \phi \neq 0, w_i = 0 \) and \( \sum_{j \neq i} \beta_j = 1 \). Moreover, the linear payoff relation in equation (4) becomes

\[
\pi^{-1} = s \pi^i + (1-s)l,
\]

where \( \pi^{-1} = \sum_{j \neq i} w_{j} \pi^j \). The four most widely studied ZD strategies are given in Table II.

TABLE II: The four mostly studied ZD strategies. Depending on the parameter values \( s \) and \( l \), players may be fair, generous, extortionate or equalizers.

| ZD-Strategy | Parameter values | Enforced payoff relation |
|-------------|------------------|--------------------------|
| Fair        | \( s = 1 \)      | \( \pi^{-1} = \pi^i \)   |
| Generous    | \( l = a_n - 1 \), 0 < s < 1 | \( \pi^{-1} \geq \pi^i \) |
| Extortionate| \( l = b_0, 0 < s < 1 \) | \( \pi^{-1} \leq \pi^i \) |
| Equalizer   | \( s = 0 \)      | \( \pi^{-1} = \pi \)     |

Because the entries of the ZD-strategy correspond to conditional probabilities, they are required to belong to the unit interval. Hence, not every linear payoff relation with parameters \( s, l \) is valid. Let \( w = (w_i) \in \mathbb{R}^{n-1} \) denote the vector of weights that the ZD strategist assigns to her co-players. Consider the following definition that was given in [5] for two-player games.

Definition 3 (Enforceable payoff relations). Given a discount factor \( 0 < \delta < 1 \), a payoff relation \( (s, l) \in \mathbb{R}^2 \) with weights \( w \) is enforceable if there are \( \phi \in \mathbb{R} \) and \( p_0 \in [0, 1] \), such that each entry in \( p \) according to equation (3) is in \([0, 1]\). We indicate the set of enforceable payoff relations by \( E_{\delta} \).

An intuitive implication of decreasing the expected number of rounds in the repeated game (e.g. by decreasing \( \delta \)) is that the set of enforceable payoff relations will decrease as well. This monotone effect is formalized in the following proposition that extends a result from [5] to the n-player case.

Proposition 2 (Monotonicity of \( E_{\delta} \)). If \( \delta' \leq \delta'' \), then \( E_{\delta'} \subseteq E_{\delta''} \).

Proof: Albeit with different formulations of the \( p \), the proof follows from the same argument used in the two-player case [6]. It is presented here to make the paper self-contained. From Definition 3, \( (s, l) \in E_{\delta} \) if and only if one can find \( \phi \in \mathbb{R} \) and \( p_0 \in [0, 1] \) such that \( p \in [0, 1]^n \). We have

\[
0 \leq p \leq 1 \rightarrow 0 \leq \delta p \leq \delta I. \tag{7}
\]

Then by substituting (3) into the above inequality we obtain,

\[
p_0(1 - \delta)I \leq p^\infty \leq \delta I + (1 - \delta)p_0 I, \tag{8}
\]

with

\[
p^\infty = p^{Rep} + \phi \left[ sg^i - \sum_{j \neq i} w_{j} g^j + (1-s)I \right].
\]

Now observe that \( p_0(1 - \delta)I \) on the left-hand side of the inequality (8) is decreasing for increasing \( \delta \). Moreover, \( \delta I + (1 - \delta)p_0 I \) on the right-hand side of the inequality is increasing for increasing \( \delta \). The middle part of the inequality, which is exactly the definition of a ZD strategy for the infinitely repeated game in [6], is independent of \( \delta \). It follows that by increasing \( \delta \) the range of possible ZD parameters \( (s, l, \phi) \) and \( p_0 \) increases and hence if \( 0 \leq p \leq 1 \) is satisfied for some \( \delta' \), then it is also satisfied for some \( \delta'' \geq \delta' \).
Moreover, let \( \hat{r} \)
repeated fair ZD strategies. In the finitely repeated game with payoffs that satisfy Assumption 1 there do not exist
for the finitely repeated
stated in the following corollary.

To write the statement compactly, we let
payoffs as in Table I that satisfy Assumption 1, the payoff
\( \hat{r} \)

The enforceable payoff relations
n
ZD strategy that are necessary for the payoff relation to be enforceable in the finitely repeated n-player game.

Proposition 3. The enforceable payoff relations \((l, s, w)\) for the finitely repeated n-player game with \(0 < \delta < 1\), with payoffs as in Table II that satisfy Assumption 1 require the following necessary conditions:

\[
- \frac{1}{n-1} \leq - \min_{j \neq i} w_j < s < 1, \\
\phi > 0, \\
b_0 \leq l \leq a_{n-1},
\]

with at least one strict inequality in \(9\).

Because fair strategies are defined with the slope \(s = 1\) (see, Table III, an immediate consequence of Proposition 3 is stated in the following corollary.

Corollary 1. For the finitely repeated n-player social dilemma game with payoffs that satisfy Assumption 1 there do not exist fair ZD strategies.

In the following theorem we extend the results for infinitely repeated n-player games from [6] to finitely repeated games. To write the statement compactly, we let \( a_{-1} = b_n = 0 \). Moreover, let \( \bar{w}_z = \min_{w_h \in w} (\sum_{h=1}^{z} w_h) \) denote the sum of the \( z \) smallest weights and let \( w_0 = 0 \).

Theorem 1. For the finitely repeated n-player game with payoffs as in Table II that satisfy Assumption 1 the payoff relation \((s, l) \in \mathbb{R}^2\) with weights \( w \in \mathbb{R}^{n-1} \) is enforceable if and only if \(- \frac{1}{n-1} < s < 1\) and

\[
\max_{0 \leq z \leq n-1} \left\{ \frac{b_z - \bar{w}_z(b_z - a_{z-1})}{(1-s)} \right\} \leq l, \\
\min_{0 \leq z \leq n-1} \left\{ \frac{a_z + \bar{w}_{n-z-1}(b_{z+1} - a_z)}{(1-s)} \right\} \geq l,
\]

moreover, at least one inequality in \(10\) is strict.

Remark 2. For \(n = 2\) the full weight is placed on the single opponent i.e., \( w_j = 1 \). When the payoff parameters are defined as \( b_1 = T, b_0 = P, a_1 = R, a_0 = S \), the result in Theorem 1 recovers the earlier result obtained for the finitely repeated 2-player game in [6].

Theorem 1 does not stipulate any conditions on the key player’s initial probability to cooperate other than \( p_0 \in [0, 1] \). However, the existence of extortionate and generous strategies does depend on the value of \( p_0 \). This is formalized in the following proposition.

Proposition 4. For the existence of extortionate strategies it is necessary that \( p_0 = 0 \). Moreover, for the existence of generous strategies it is necessary that \( p_0 = 1 \).

These requirements on the key player’s initial probability to cooperate make intuitive sense. In a finitely repeated game, if the key player aims to be an extortioner that profits from the cooperative actions of others, she cannot start to cooperate because she could be taken advantage off by defectors. On the other hand, if she aims to be generous, she cannot start as a defector because this will punish both cooperating and defecting co-players.

VI. Thresholds on Discount Factors

In the previous section we have characterized the enforceable payoff relations of ZD strategies in finitely repeated n-player social dilemma games. Our conditions generalize those obtained for two-player games and illustrate how a single player can exert control over the outcome of an n-player repeated game with a finite number of expected rounds. The conditions that result from the existence problem do not specify requirements on the discount factor other than \( \delta \in (0, 1) \). In practice, one could be interested in how long it would take to enforce some desired payoff relation. In this section we address this problem.

Problem 2 (The minimum threshold problem). Suppose the desired payoff relation \((s, l) \in \mathbb{R}^2\) satisfies the conditions in Theorem 1. What is the minimum \( \delta \in (0, 1) \) under which the linear relation \((s, l)\) with weights \( w \) can be enforced by the ZD strategist?

We consider the three classes of ZD strategies separately. Before giving the main results it is necessary to introduce some additional notation. Define \( \bar{w}_z = \max_{w_h \in w} (\sum_{h=1}^{z} w_h) \) to be the maximum sum of weights for some permutation of \( \sigma \in \mathcal{A} \) with \( z \) cooperating co-players. Additionally, for some given payoff relation \((s, l) \in \mathbb{R}^2\) and \( w \in \mathbb{R}^{n-1} \) define

\[
\rho^E := \max_{0 \leq z \leq n-1} (1-s)(a_z - l) + \bar{w}_{n-z-1}(b_{z+1} - a_z), \\
\rho^C := \min_{0 \leq z \leq n-1} (1-s)(a_z - l) + \bar{w}_{n-z-1}(b_{z+1} - a_z), \\
\rho^D := \max_{0 \leq z \leq n-1} (1-s)(l - b_z) + \bar{w}_z(b_z - a_{z-1}), \\
\rho^P := \min_{0 \leq z \leq n-1} (1-s)(l - b_z) + \bar{w}_z(b_z - a_{z-1}).
\]

In the following, we will use these extrema to derive threshold discount factors for extortionate, generous and equalizer strategies in symmetric n-player social dilemma games. The proofs of our results can be found in Section VII.

A. Extortionate ZD strategies

We first consider the case in which \( l = b_0 \) and \( 0 < s < 1 \), such that the ZD strategy is extortionate. We have the following result.

Theorem 2. Assume \( p_0 = 0 \) and \((s, b_0) \in \mathbb{R}^2\) satisfy the conditions in Theorem 1 then \( \rho^E > 0 \) and \( \rho^P + \rho^C > 0 \).
Moreover, the threshold discount factor above which extortionate ZD strategies exist is determined by
\[ \delta_t = \max \left\{ \frac{\rho^C - \rho^C}{\rho^C}, \frac{\rho^D}{\rho^D + \rho^D} \right\}. \]

B. Generous ZD strategies

If a player instead aims to be generous, in general, different thresholds will apply. Thus, we now consider the case in which \( l = a_{n-1} \) and \( 0 < s < 1 \) such that the ZD strategy is generous.

**Theorem 3.** Assume \( p_0 = 1 \) and \((s, a_{n-1}) \in \mathbb{R}^2\) satisfy the conditions in Theorem 2. Then \( \rho^D > 0 \) and \( \rho^C + \rho^D > 0 \). Moreover, the threshold discount factor above which generous ZD strategies exist is determined by
\[ \delta_t = \max \left\{ \frac{\rho^D - \rho^D}{\rho^D}, \frac{\rho^C}{\rho^C + \rho^D} \right\}. \]

C. Equalizer ZD strategies

The existence of equalizer strategies with \( s = 0 \) does not impose any requirement on the initial probability to cooperate. In general, one can identify different regions of the unit interval for \( p_0 \) in which different threshold discount factors exist. For instance, the boundary cases can be examined in a similar manner as was done for extortionate and generous strategies and, in general, will lead to different requirements on the discount factor. In this section, we derive conditions for the discount factor such that the equalizer payoff relation can be enforced for a variable initial probability to cooperate that is within the open unit interval.

**Theorem 4.** Suppose \( s = 0 \) and \( l \) satisfies the bounds in Theorem 3. Then, the equalizer payoff relation can be enforced for \( p_0 \in (0, 1) \) if and only if the following inequalities hold
\[
\begin{align*}
\delta &\geq 1 - \frac{\rho^D}{\rho^D + (\rho^D - \rho^D)p_0}, \\
\delta &\geq 1 - \frac{\rho^C}{(1 - p_0)(\rho^C + \rho^D)}, \\
\delta &\geq 1 - \frac{\rho^C}{(1 - p_0)(\rho^C - \rho^D) + \rho^C}, \\
\delta &\geq 1 - \frac{\rho^D}{(\rho^C + \rho^D) p_0}. 
\end{align*}
\]

Based on Theorem 4, the following corollary provides relatively easy to check sufficient conditions that allow an equalizer strategy to enforce a desired linear relation for every initial probability to cooperate in the open unit interval. These sufficient conditions link thresholds for generous and extortionate strategies to those of equalizer strategies.

**Corollary 2.** Suppose \( s = 0 \) and \( l \) satisfies the bounds in Theorem 3. Then, the equalizer payoff relation can be enforced for all \( p_0 \in (0, 1) \) if
\[ \delta \geq \delta_t = \max \left\{ \frac{\rho^C - \rho^C}{\rho^C}, \frac{\rho^D - \rho^D}{\rho^D}, \frac{\rho^D}{\rho^C + \rho^D}, \frac{\rho^C}{\rho^C + \rho^D} \right\}. \]

**Proof.** The sufficient conditions are obtained by solving the conditions in Theorem 3 that are linear in \( p_0 \), for the smallest upper-bounds on the discount factor \( \delta \).

With Theorems 2, 3, and 4, we have provided expressions for deriving the minimum discount factor for some desired linear relation. Because the expressions depend on the “one-shot” payoff of the \( n \)-player game, in general, they will differ between social dilemmas. In order to determine these expressions, one needs to find the global extrema of a function over \( z \) that can be efficiently done for a large class of social dilemma games. The derived thresholds can, for example, be used as an indicator for a minimum number of rounds in a experiments on extortion and generosity in repeated games or simply as an indicator for how many expected interactions a single ZD strategists requires to enforce some desired payoff relation in a group of decision makers.

**VII. Proofs of the main results**

A. **Proof of Proposition 2**

Suppose \( \sigma(C, C, \ldots, C) \) and the payoffs given in Table 1 it follows that
\[ \delta_{P(C, C, \ldots, C)} = 1 + \phi(1 - s)(l - a_{n-1}) - (1 - \delta)p_0. \]

Now suppose that all players are defecting. Similarly, we have
\[ \delta_{P(D, D, \ldots, D)} = \phi(1 - s)(l - b_0) - (1 - \delta)p_0. \]

In order for these payoff relations to be enforceable, it needs to hold that both entries in equations 16 and 17 are in the interval \([0, \delta]\). Equivalently,
\[ (1 - \delta)(1 - p_0) \leq \phi(1 - s)(a_{n-1} - l) \leq 1 - (1 - \delta)p_0, \]
and
\[ 0 \leq p_0(1 - \delta) \leq \phi(1 - s)(l - b_0) \leq \delta + (1 - \delta)p_0. \]

Combining 18 and 19 it follows that \( 0 < (1 - \delta) \leq \phi(1 - s)(a_{n-1} - b_0) \). From the assumption that \( a_{n-1} = b_0 \) listed in Assumption 1 it follows that
\[ 0 < \phi(1 - s). \]

Now suppose there is a single defecting player, i.e., \( \sigma = (C, C, \ldots, D) \) or any of its permutations. In this case, the entries of the memory-one strategy are as given in equation 21. Again, for both cases we require \( \delta_{P_{\sigma}} \) to be in the interval \([0, \delta]\). This results in the inequalities given in equations 22 and 23.

By combining the equations 22 and 23 we obtain
\[ 0 < (1 - \delta) \leq \phi(s + w_i)(b_{n-1} - a_{n-2}). \]

Again, because of the assumption \( b_{z+1} > a_z \) it follows that
\[ 0 < \phi(s + w_j), \forall j \neq i. \]

The inequalities 25 and 20 together imply that
\[ 0 < \phi(1 + w_j), \forall j \neq i. \]
\[ \delta p_\sigma = \begin{cases} 1 + \phi(sa_{n-2} - (1 - w_j)a_{n-2} - w_jb_{n-1} + (1 - s)l) - (1 - \delta)p_0, & \text{if defector is } j \neq i; \\ \phi(sb_{n-1} - a_{n-2} + (1 - s)l) - (1 - \delta)p_0, & \text{if defector is } i. \end{cases} \] (21)

Because at least one \( w_j > 0 \), it follows that \( \phi > 0 \).

Combining with equation (20) we obtain
\[ s < 1. \] (28)

In combination with equation (20), it follows that
\[ \forall j \neq i : s + w_j > 0 \Leftrightarrow \forall j \neq i : w_j > -s \Leftrightarrow \min_{j \neq i} w_j > -s. \] (29)

The inequalities in the equations (28) and (29) finally produce the bounds on \( s \):
\[ -\min_{j \neq i} w_j < s < 1 \] (30)

Moreover, because it is required that \( \sum_{j=1}^{n} w_j = 1 \), it follows that \( \min_{j \neq i} w_j \leq \frac{1}{n-1} \). Hence the necessary condition turns into:
\[ -\frac{1}{n-1} \leq -\min_{j \neq i} w_j < s < 1. \] (31)

We continue to show the necessary upper and lower bound on \( l \). From equation (18) we obtain:
\[ \phi(1 - s)(l - a_{n-1}) \leq (1 - p_0)(\delta - 1) \leq 0. \] (32)

From equation (20) we know \( \phi(1 - s) > 0 \). Together with equation (32) this implies the necessary condition
\[ l - a_{n-1} \leq 0 \Leftrightarrow l \leq a_{n-1}. \] (33)

We continue with investigating the lower-bound on \( l \), from equation (19)
\[ 0 \leq p_0(1 - \delta) \leq \phi(1 - s)(l - b_0) \leq \delta + (1 - \delta)p_0. \] (34)

Because \( \phi(1 - s) > 0 \) (see equation (20)) it follows that
\[ l \geq b_0. \]

Naturally, when \( l = a_{n-1} \) by assumption it holds that \( l > b_0 \) and when \( l = b_0 \) then \( l < a_{n-1} \).

B. Proof of Proposition 3

For brevity, in the following proof we refer to equations that are found in the proof of Proposition 3. Assume the ZD strategy is extortionate, hence \( l = b_0 \). From the lower bound in (19) in order for \( l \) to be enforceable, it is necessary that \( p_0 = 0 \). This proves the first statement. Now assume the ZD strategy is generous, hence \( l = a_{n-1} \). From the lower bound in (18) in order for \( l \) to be enforceable, it is necessary that \( p_0 = 1 \). This proves the second statement and completes the proof.

C. Proof of Theorem 7

In the following we refer to the key player, who is employing the ZD strategy, as player \( i \). Let \( \sigma = (x_1, \ldots, x_n) \) such that \( x_k \in A \) and let \( \sigma^C \) be the number of \( i \)'s co-players that cooperate and let \( \sigma^D = n - 1 - \sigma^C \), be the number of \( i \)'s co-players that defect. Also, let \( |\sigma| \) be the total number of cooperators including player \( i \). Using this notation, for some action profile \( \sigma \) we may write the ZD strategy as
\[ \delta p_\sigma = p^{rep} + \phi(1 - s)(l - g_\sigma^i) + \sum_{j \neq i} w_j(g_\sigma^i - g_\sigma^j) - (1 - \delta)p_0. \] (35)

Also, note that
\[ \sum_{j \neq i} w_j g_\sigma^j = \sum_{k \in \sigma^D} w_k g_\sigma^k + \sum_{h \in \sigma^C} w_h g_\sigma^h, \] (36)

and because \( \sum_{j \neq i} w_j = 1 \) it holds that
\[ \sum_{l \in \sigma^C} w_l = 1 - \sum_{k \in \sigma^D} w_k. \]

Substituting this into equation (36) and using the payoffs as in Table II we obtain
\[ \sum_{j \neq i} w_j g_\sigma^j = a_{|\sigma|-1} + \sum_{j \in \sigma^D} w_j(b_j - a_{|\sigma|-1}). \]

Accordingly, the entries of the ZD strategy \( \delta p_\sigma \) are given by equation (38). For all \( \sigma \in A \) we require that
\[ 0 \leq \delta p_\sigma \leq \delta. \] (37)

This leads to the inequalities in equations (39) and (40). Because \( \phi > 0 \) can be chosen arbitrarily small, the inequalities in equation (39) can be satisfied for some \( \delta \in (0, 1) \) and \( p_0 \in [0, 1] \) if and only if for all \( \sigma \) such that \( x_i = C \) the inequalities in equation (41) are satisfied.

\[ 0 \leq (1 - s)(a_{|\sigma|-1} - l) + \sum_{j \in \sigma^D} w_j(b_j - a_{|\sigma|-1}). \] (41)

The inequality (41) together with the necessary condition \( s < 1 \) (see Proposition 3) implies that
\[ a_{|\sigma|-1} + \frac{\sum_{j \in \sigma^D} w_j(b_j - a_{|\sigma|-1})}{(1 - s)} \geq l, \] (42)

and thus provides an upper-bound on the enforceable baseline payoff \( l \). We now turn our attention to the inequalities in
\( \delta p_\sigma = \begin{cases} 1 + \varphi \left[ (1-s)(l - a_{|\sigma|-1}) - \sum_{j \in D} w_j(b_{|\sigma|} - a_{|\sigma|-1}) \right] - (1-\delta) p_0, & \text{if } x_i = C, \\ \phi \left[ (1-s)(l - b_{|\sigma|}) + \sum_{j \in D} w_j(b_{|\sigma|} - a_{|\sigma|-1}) \right] - (1-\delta) p_0, & \text{if } x_i = D. \end{cases} \) (38)

Equation (40) can be satisfied if and only if for all \( \sigma \) such that \( x_i = D \) the following holds

\[ 0 \leq (1-s)(l - b_{|\sigma|}) + \sum_{j \in D} w_j(b_{|\sigma|} - a_{|\sigma|-1}) \]

Then from equation (39) it follows that \( p_0 = 0 \) is required. Then, the inequalities in equation (39) require that

\[ 0 < (1-s)(l - b_{|\sigma|}) + \sum_{j \in D} w_j(b_{|\sigma|} - a_{|\sigma|-1}) \]

Hence, Equation (49) with \( p_0 = 0 \) implies that for all \( \sigma \) such that \( x_i = C \) it holds that

\[ \frac{1-\delta}{\rho^C(\sigma)} \leq \phi \leq \frac{1-\delta}{\rho^D(z, w_z)} \]

(48)

Naturally, \( \rho^C \geq \rho^D \). In the special case in which equality holds, it follows from equation (48) that \( \delta > 0 \), which is true by definition of \( \delta \). We continue to investigate the case in which \( \rho^C > \rho^D \). In this case, a solution to equation (48) for some \( \phi > 0 \) exists if and only if

\[ \frac{1-\delta}{\rho^C(z, w_z)} \leq \phi \leq \frac{1-\delta}{\rho^D(z, w_z)} \]

(49)

which leads to the first expression in the theorem. Now, from equation (40) with \( p_0 = 0 \), it follows that in order for the payoff relation to be enforceable it is necessary that

\[ \forall \sigma \text{ s.t. } x_i = D : 0 \leq \phi \rho^D(\sigma) \leq \delta \Rightarrow 0 \leq \phi \rho^D(z, w_z) \leq \delta. \]
Because $\phi > 0$ is necessary for the payoff relation to be enforceable, it follows that $\rho^{D}(\sigma) \geq 0$ for all $\sigma$ such that $x_{i} = D$. Let us first investigate the special case in which $p_{D}(z, \hat{w}_{z}) = 0$. Then (50) is satisfied for any $\phi > 0$ and $\delta \in (0, 1)$. Now, assume $\rho^{D}(z, \hat{w}_{z}) > 0$. Then, equations (50) and (53) imply

$$\frac{1 - \delta}{\rho^{C}(z, \hat{w}_{z})} \leq \phi \leq \frac{\delta}{\rho^{D}(z, \hat{w}_{z})}. \tag{51}$$

This completes the proof.

\section*{E. Proof of Theorem 3}

The proof is similar to the extortionate case in the proof of Theorem 2. From Proposition 4, we know that in order for the generous payoff relation to be enforceable it is necessary that $p_{0} = 1$. By substituting this into equation (40), it follows that in order for the payoff relation to be enforceable it is required that for all $\sigma$ such that $x_{i} = D$ the following holds:

$$\rho^{D}(\sigma) = (1 - s)(l - b_{\sigma}) + \sum_{j \in \sigma^{C}} w_{j}(b_{\sigma} - b_{\sigma_{j} - 1}) > 0. \tag{53}$$

Hence, equation (40) with $p_{0} = 1$ implies that for all $\sigma$ such that $x_{i} = D$ it holds that

$$\frac{1 - \delta}{\rho^{D}(\sigma)} \leq \phi \leq \frac{1 - \delta}{\rho^{D}(D)} \leq \phi \leq \frac{1}{\rho^{D}(z, \hat{w}_{z})}. \tag{54}$$

If $\rho^{D} > 0$ this implies $\delta \geq 0$. Otherwise equation (54) implies that

$$\frac{1 - \delta}{\rho^{D}(z, \hat{w}_{z})} \leq \frac{1}{\rho^{D}(z, \hat{w}_{z})} \Rightarrow \delta \geq \frac{\rho^{D} - \rho^{D}(z, \hat{w}_{z})}{\rho^{D}}. \tag{55}$$

which leads to the first expression in the theorem. Moreover, from equation (39) we know that the following must hold:

$$\forall \sigma \text{ s.t. } x_{i} = C: \ 0 \leq \phi \rho^{C}(\sigma) \leq \delta \Rightarrow \delta \geq \phi \rho^{C}(z, \hat{w}_{z}) \leq \delta. \tag{56}$$

Because $\phi > 0$ it follows that $\rho^{C}(\sigma) \geq 0$ for all $\sigma$ such that $x_{i} = C$. Let us now consider the special case in which $\rho^{C}(z, \hat{w}_{z}) = 0$ Then, equation (55) is satisfied for any $\phi > 0$ and $\delta \in (0, 1)$. Now suppose $\rho^{C}(z, \hat{w}_{z}) > 0$. Then, (54) and (53) imply that in order for the generous strategy to be enforceable it is necessary that

$$\frac{1 - \delta}{\rho^{D}(z, \hat{w}_{z})} \leq \phi \leq \frac{\delta}{\rho^{C}(z, \hat{w}_{z})}. \tag{57}$$

Such a $\phi$ exists if and only if

$$\frac{1 - \delta}{\rho^{D}(z, \hat{w}_{z})} \leq \delta \leq \frac{\rho^{D} + \rho^{C}}{\rho^{D} + \rho^{C}}. \tag{58}$$

This completes the proof.

\section{F. Proof of Theorem 2}

For brevity, we refer to equations found in the proof of Theorem 1. From (39) and (40) it follows that in order for the payoff relation to be enforceable for any $p_{D}(z, \hat{w}_{z}) > 0$. For the existence of equalizer strategies this must also hold for the special case in which $s = 0$. Hence, we can rewrite (39) and (40) to obtain the following set of inequalities

$$\frac{(1 - \delta)(1 - p_{0})}{\rho^{C}(z, \hat{w}_{z})} \leq \phi \leq \frac{1 - (1 - \delta)p_{0}}{\rho^{C}(z, \hat{w}_{z})}, \tag{59}$$

$$\frac{(1 - \delta)p_{0}}{\rho^{D}(z, \hat{w}_{z})} \leq \phi \leq \frac{\delta + (1 - \delta)p_{0}}{\rho^{D}(z, \hat{w}_{z})}. \tag{60}$$

There exists such a $\phi > 0$ if and only if the following inequalities are satisfied

$$\frac{(1 - \delta)p_{0}}{\rho^{D}(z, \hat{w}_{z})} \leq \frac{\delta + (1 - \delta)p_{0}}{\rho^{D}(z, \hat{w}_{z})}, \tag{61}$$

$$\frac{(1 - \delta)p_{0}}{\rho^{D}(z, \hat{w}_{z})} \leq \frac{1 - (1 - \delta)p_{0}}{\rho^{D}(z, \hat{w}_{z})}, \tag{62}$$

$$\frac{(1 - \delta)(1 - p_{0})}{\rho^{C}(z, \hat{w}_{z})} \leq \frac{1 - (1 - \delta)p_{0}}{\rho^{C}(z, \hat{w}_{z})}, \tag{63}$$

$$\frac{(1 - \delta)(1 - p_{0})}{\rho^{C}(z, \hat{w}_{z})} \leq \frac{\delta + (1 - \delta)p_{0}}{\rho^{C}(z, \hat{w}_{z})}. \tag{64}$$

By collecting the terms in $p_{0}$ and $\delta$ for (61)–(64) the conditions can be derived as follows. The condition in (61) can be satisfied if and only if

$$p_{0}(1 - \delta)\left(\rho^{D}(z, \hat{w}_{z}) - \rho^{D}(z, \hat{w}_{z})\right) \leq \rho^{D}(z, \hat{w}_{z})\delta. \tag{65}$$

In the special case that $\rho^{D}(z, \hat{w}_{z}) - \rho^{D}(z, \hat{w}_{z}) = 0$, this is satisfied for every $p_{D}(z, \hat{w}_{z}) > 0$. On the other hand, if $\rho^{D}(z, \hat{w}_{z}) - \rho^{D}(z, \hat{w}_{z}) > 0$, then the inequality can be satisfied for every $p_{D}(z, \hat{w}_{z}) > 0$. In this case, the conditions for existence and the thresholds become relatively easy to solve. The proofs of this section are found in the Appendices.

\section{VIII. APPLICATION TO THE LINEAR PUBLIC GOODS GAME}

In this section we apply the theory developed in this paper to the linear public goods game in Example 2. The weights are assumed to be equal, that is $w_{j} = \frac{1}{n_{j}}$ for all $j \neq i$. In this case, the conditions for existence and the thresholds become relatively easy to solve. The proofs of this section are found in the Appendices.
Out\[10366\] = threshold discount factor

1.5 − 1.0

0.5 − 0.5

1.0

\(\rho_C(0)\)

\(\rho_C(n-1)\)

\(\rho_D(0)\)

\(\rho_D(n-1)\)

(b) Values of functions that determine thresholds for generous strategies

A. Threshold discount factors in the linear public goods game

The existence problem for the linear public goods game

We first apply Theorem 1 to the linear public goods game to characterize the enforceable slopes and baseline payoffs. Consider the following conditions on the slope and baseline payoff of the linear relation.

**Proposition 5** (Existence of extortionate strategies). Suppose \(p_0 = 0, l = 0\) and \(0 < s < 1\). For a public goods game with \(r > 1\), every slope \(s \geq \frac{r-1}{r}\) can be enforced independent of \(n\). If \(s < \frac{r-1}{r}\), the slope can be enforced if and only if

\[ n \leq \frac{r(1-s)}{r(1-s) - 1}. \]

**Proof:** The proof can be found in Appendix A.

**Proposition 6** (Existence of generous strategies). Suppose \(p_0 = 1, l = rc - c\) and \(0 < s < 1\). For a public goods game with \(1 < r < n\), the region of enforceable slopes of generous strategies is equivalent to the region of the enforceable slopes for extortionate strategies.

**Proof:** The proof can be found in Appendix B.

For the linear public goods game, the set of enforceable slopes for extortionate and generous strategies are equivalent. In general, however, the sets of enforceable slopes differ between the classes of ZD strategies.

B. Threshold discount factors in the linear public goods game

Let us now examine the threshold discount factors of extortionate strategies such that \(l = 0\) and \(0 < s < 1\). In this case the parameters in equation (1) result from the extreme points of the functions

\[\rho_C^e(z) := (1-s)\left(\frac{rc(z+1)}{n} - c\right) + \frac{n-z-1}{n-1} - c, \quad (65)\]

\[\rho_D^e(z) := -(1-s)\left(\frac{rcz}{n}\right) + \frac{z}{n-1} - c. \quad (66)\]

From Proposition 4 we know that if \(-\frac{1}{n-1} < s \leq 1 - \frac{n}{r(n-1)}\) no extortionate strategies can exist. Therefore, suppose that the slope is sufficiently large, i.e. \(s \geq 1 - \frac{n}{r(n-1)}\). Then, the extreme points of \(\rho_C^e(z)\) and \(\rho_D^e(z)\) are determined as

\[\rho_C^e = \rho_C^e(0), \quad \rho_C^e = \rho_C^e(n-1), \quad \rho_D^e = \rho_D^e(n-1), \quad \rho_D^e = \rho_D^e(0). \quad (67)\]

In the public goods game, next to the region of enforceable slopes, also the threshold discount factors for generous and extortionate strategies are equivalent, as highlighted in the following proposition.

**Proposition 7** (Thresholds for extortion and generosity). For the enforceable slopes \(s \geq 1 - \frac{n}{r(n-1)}\), in the public goods game the threshold discount factor for extortionate and generous strategies is determined as

\[\delta_r = \frac{1 - (1-s)(r - \frac{1}{n})}{1 - (1-s)(1 - \frac{1}{n})}. \quad (68)\]

**Proof:** The proof can be found in Appendix C.

A numerical example of threshold discount factors for extortionate and generous strategies in the linear public goods game is shown in Figure 1(a). The figures represent the values of the fractions in the expression for \(\delta_r\) in Theorem 2 and Theorem 3 using the extreme points of the functions in equations (65) and (66). The threshold discount factor for a particular slope \(s\) can be determined from Figure 1(a) by looking at the value of the solid red line at \(s\). Different colors correspond to expressions that are dominant before and after the critical existence point \(s = 1 - \frac{n}{r(n-1)} = 3/8\) that is indicated by the first vertical line. This critical value of \(s\) indicates the point at which the maxima and minima over \(z\) of \(\rho_C^e(z)\) and \(\rho_D^e(z)\) are as in equation (67).

For the existence of generous and extortionate strategies, this is a crucial point, namely, beyond this point and up to \(s < 1\) all the functions \(\rho_C^e(z)\) and \(\rho_D^e(z)\) in equations (65) and (66) (and those of generous strategies) are non-negative as can be seen from the vertical line in Figure 1(b). An equivalent requirement is formulated in Proposition 5 that states that for any slope \(s < \frac{r-1}{r}\) for existence of extortionate and generous strategies...
it is necessary that \( n \leq \frac{r(1-s)}{r(1-s)-1} \). Before the critical point, this requirement cannot be satisfied and hence no generous or extortionate strategies with such slopes can exist in the linear public goods game. The second vertical line indicated in Figure [1] indicates the point \( s = \frac{r-1}{r} = 1/2 \), after which any slope can be enforced independent of \( n \), as formulated in Proposition [5].

IX. FINAL REMARKS

We have extended the existing results for ZD strategies in finitely repeated two-player two-action games to \( n \)-player two-action games. We focused on \( n \)-player social dilemma games because of their importance to the current literature. However, the fundamental relation between the memory-one strategy and the limit distribution is independent of the structure of the game and thus the results in this paper can be extended by considering \( n \)-player games that are not social dilemmas. Our theory supports the finding that due to the finite number of expected rounds or discounting of the payoffs, the initial probability to cooperate of the key player remains important, and we have shown that for the existence of generous strategies the key player must start to cooperate with probability one. Likewise, for extortionate strategies this initial probability must be zero. Based on these necessary conditions on the initial probability to cooperate we derived expressions for the minimum discount factors above which a ZD strategists can enforce some desired generous or extortionate payoff relation. Because equalizer strategies do not impose such conditions on the initial probability to cooperate, one can identify a multitude of \( p_0 \) regions in the unit interval for which there exist different threshold discount factors. Consequently, we have derived an expression that ensures the desired equalizer strategy is enforceable for any initial probability to cooperate in the open unit interval. The derived necessary and sufficient conditions for existence and the thresholds discount factors presented in this paper may be helpful in designing novel control techniques for repeated decision making processes. Furthermore, our results can aid the design of human experiments that aim to test the level of control a decision maker can have in a real life setting.

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APPENDIX A

PROOF OF PROPOSITION 5

Let us first formulate the following Lemma that characterizes the enforceable baseline payoffs in the linear public goods game.

Lemma 2 (Enforceable baseline payoffs). For the public goods game the enforceable baseline payoffs are determined by

\[
\max \left\{ 0, \frac{rc(n-1)}{n} - \frac{c}{1-s} \right\} \leq l, \quad (69)
\]

\[
\min \left\{ \frac{rc}{n} - c + \frac{c}{1-s}, rc - c \right\} \geq l, \quad (70)
\]

with at least one strict inequality.

Proof: The bounds are obtained by substituting the single-round payoffs \( a_z \) and \( b_z \) of Example 1 into the inequalities of Theorem 1 and use the fact that the bounds are linear in \( z \). The obvious details are omitted.

We now continue to prove Proposition 5. For extortionate strategies \( l = 0 \) and \( 0 < s < 1 \). The inequalities in equations (69) and (70) in Lemma 2 become

\[
\max \left\{ 0, \frac{rc(n-1)}{n} - \frac{c}{1-s} \right\} \leq 0 \quad (71)
\]

\[
\min \left\{ \frac{rc}{n} - c + \frac{c}{1-s}, rc - c \right\} \geq 0 \quad (72)
\]

Solving for \( s \) will yield the enforceable slopes in the extortionate ZD strategy. Observe that a necessary condition for equation (71) to hold is that the left hand side is equal to 0 and in order for this to hold it is required that

\[
\frac{rc(n-1)}{n} - \frac{c}{1-s} \leq 0 \iff rc(n-1) - n \frac{c}{1-s} \leq 0. \quad (73)
\]

Equivalently,

\[
n \left( r - \frac{1}{1-s} \right) \leq r \iff n(r(1-s) - 1) \leq r(1-s). \quad (74)
\]
The conditions \(-\frac{1}{r-1} < s < 1\) in Theorem 1 and the assumption that \(r\) is positive implies that \(r(1-s)\) in the right-hand side of equation (72) is required to be strictly positive. It follows that if \(r(1-s) - 1 \leq 0\) the inequalities in equation (73) are always satisfied. To obtain the criteria on the slope \(s\) we may write,

\[
r(1-s) - 1 \leq 0 \iff -r s \leq 1 - r \iff s \geq \frac{r-1}{r}.
\] (75)

Note that if \(s \geq \frac{r-1}{r}\) is satisfied, the left-hand side of the inequality in equation (72) reads as \(rc - c\). The requirement \(0 \leq rc - c\) leads to \(r \geq 1\), which is very natural and satisfied for the payoff of the public goods game. It follows that for every \(r > 1\), every \(s \geq \frac{r-1}{r}\) can be enforced independent of \(n\).

On the other hand, when \(s < \frac{r-1}{r}\) in order for equation (73) to be satisfied it must hold that

\[
n \leq \frac{r(1-s)}{r(1-s)-1}.
\] (76)

Note that \(s < \frac{r-1}{r}\) implies \(r(1-s) - 1 \neq 0\) so the above inequality in well-defined. If (76) does not hold and \(s < \frac{r-1}{r}\) than

\[
\frac{rc(n-1)}{n} - c + \frac{c}{1-s} > 0,
\] (77)

thus the lower-bound in equation (71) is not satisfied and consequently there cannot exist extortionate strategies. We now investigate the inequality in equation (72). We already know that when \(s \geq \frac{r-1}{r}\) the upper-bound reads as \(0 < rc - c\) and is satisfied for any \(r > 1\). On the other hand, the left-hand side of equation (72) is equal to \(\frac{rc}{n} - c + \frac{c}{1-s}\) if

\[
\frac{rc}{n} - c + \frac{c}{1-s} \leq rc - c \iff n[(1-s)r - 1] \geq r(1-s).
\]

Because \(r(1-s) > 0\), these inequalities can only be satisfied if \(s < \frac{r-1}{r}\) and

\[
n \geq \frac{r(1-s)}{r(1-s)-1}.
\] (78)

Note that the only possibility for an enforceable payoff relation is the equality case in which \(n = \frac{r(1-s)}{r(1-s)-1}\), otherwise the lower-bound is not satisfied and there cannot exist extortionate strategies.

Finally, we check the necessary condition for the existence of solutions of equations (71) and (72) that the lower-bound cannot exceed the upper-bound. We already know that when \(s \geq \frac{r-1}{r}\) the lower and upper-bound read as \(0 \leq rc - c\) and is satisfied for any \(r > 1\). When \(s < \frac{r-1}{r}\) for existence, \(n\) cannot exceed \(\frac{r(1-s)}{r(1-s)-1}\). When equality holds note that we have

\[
if n = \frac{r(1-s)}{r(1-s)-1} and s < \frac{r-1}{r}:
\]

\[
0 = \frac{rc(n-1)}{n} - \frac{c}{1-s} \leq 0 \leq \frac{rc}{n} - c + \frac{c}{1-s} = rc - c,
\]

which is satisfied with a strict upper-bound if \(r > 1\). We conclude that the lower-bound never exceeds the upper-bound and this condition does not limit the existence of extortionate ZD strategies in the public goods game.

\[\Box\]

\section*{Appendix B}

\textbf{Proof of Proposition 6}

For generous strategies \(l = rc - c\) and \(0 < s < 1\), the inequalities in equations (69) and (70) in Lemma 2 become

\[
\max \left\{ \frac{0}{n}, \frac{rc(n-1)}{n} - c \frac{1}{r} \right\} \leq rc - c, \quad (79)
\]

\[
\min \left\{ \frac{rc}{n} - c + \frac{c}{1-s}, rc - c \right\} \geq rc - c. \quad (80)
\]

Clearly in order for generous strategies to exist it is necessary that the left hand side of equation (80) reads as \(rc - c\). Therefore it is required that

\[
\frac{rc}{n} - c + \frac{c}{1-s} \geq rc - c \iff n(r(1-s) - 1) \leq (1-s)r.
\]

Hence, this condition is equivalent to the condition in equation (73) and thus this condition gives the same feasible region for the existence of extortionate strategies. Now suppose that, \(s < \frac{r-1}{r}\) and \(n \geq \frac{r(1-s)}{r(1-s)-1}\). Also in this case, only equality is possible i.e. \(n = \frac{r(1-s)}{r(1-s)-1}\) because otherwise the upper-bound is not satisfied. Next to this, if \(s < \frac{r-1}{r}\) and \(n = \frac{r(1-s)}{r(1-s)-1}\) in order for the lower-bound to be satisfied it is required that

\[
rc - c \geq \frac{rc}{n} - c + \frac{c}{1-s} \geq rc - c \geq 0,
\]

which is satisfied with a strict lower-bound for any \(r > 1\). We conclude that, in the linear public goods game, the region of feasible slopes for generous strategies is equivalent to the region of feasible sloped for extortionate strategies.

\[\Box\]

\section*{Appendix C}

\textbf{Proof of Proposition 7}

For the linear public goods game the parameters in equation (71) can be obtained from the extrema of the following functions

\[
\rho^C(z) = (1-s)(\frac{rc(z+1)}{n} - c - l) + \frac{n-z-1}{n-1} c, \quad (81)
\]

\[
\rho^D(z) = (1-s)(l - \frac{rc}{n} z) + \frac{z}{n-1} c
\]

We focus first on the case in which \(l = 0\) and \(0 < s < 1\), and thus the strategy is extortionate. In this case the equations in (81) become

\[
\rho^C_z(z) := (1-s)(\frac{rc(z+1)}{n} - c + \frac{n-z-1}{n-1} c) \quad (82)
\]

\[
\rho^D_z(z) := -(1-s)(\frac{rc}{n} z + \frac{z}{n-1} c) \quad (83)
\]

We continue to obtain the maximizers and minimizers of equation (81), that because of linearity in \(z\) can only occur at the extreme points \(z = 0\) and \(z = n - 1\). When \(n > r\) and \(r > 1\), as is the case when the linear public goods game is a social dilemma, we have the following simple conditions on the slope of the extortionate strategy. If \(-\frac{1}{n-1} < s < 1 - \frac{1}{r(n-1)}\) no extortionate or generous strategies can exist. Hence assume \(s \geq 1 - \frac{n}{r(n-1)}\). Then,
\[\rho_C^e = \rho_C^e(0) = (1 - s)\left(\frac{rc}{n} - c\right) + c,\]
\[\rho_C^e = \rho_C^e(n - 1) = (1 - s)(rc - c) > 0,\]
\[\rho_D^e = \rho_D^e(n - 1) = -(1 - s)\left(\frac{rc(n - 1)}{n}\right) + c,\]
\[\rho_D^e = \rho_D^e(0) = 0.\]

(84)

The fractions in Proposition 2 become
\[\frac{\rho_D^e}{\rho_D^e + \rho_C^e} = \frac{\rho_C^e - \rho_C^e}{\rho_C^e} = (1 - s)(\frac{c}{n} - r) + 1\]

(85)

We focus now on the case in which \(l = rc - c\) and \(0 < s < 1\), and hence the strategy is generous. If \(l = rc - c\) the equations in (81) become
\[\rho_g^C(z) := (1 - s)\left(\frac{rc(z + 1)}{n} - rc\right) + \frac{n - z - 1}{n - 1}c\]
\[\rho_g^D(z) := (1 - s)(rc - c - \frac{rcz}{n}) + \frac{z}{n - 1}c\]

(86)

The extreme points of these functions read as
\[\rho_g^C = \rho_g^C(0) = \rho_D^e,\]
\[\rho_g^C = \rho_g^C(n - 1) = \rho_D^e,\]
\[\rho_g^D = \rho_g^D(n - 1) = \rho_D^e,\]
\[\rho_g^D = \rho_g^D(0) = \rho_D^e.\]

(87)

It follows that the fractions in Theorem 3 are equivalent to those in Theorem 2.

\[\blacksquare\]