Hermite–Gaussian model for quantum states

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Abstract

In order to characterize quantum states within the context of information geometry, we propose a generalization of the Gaussian model, which we called the \textit{Hermite–Gaussian model}. We obtain the Fisher–Rao metric and the scalar curvature for this model, and we show its relation with the one-dimensional quantum harmonic oscillator. Moreover, using this model we characterize some families of states of the quantum harmonic oscillator. We find that for eigenstates of the Hamiltonian, mixtures of eigenstates and even or odd superpositions of eigenstates the associated Fisher–Rao metrics are diagonal.

Keywords: Fisher–Rao metric, statistical models, Gaussian model, Hermite–Gaussian model

1. Introduction

The information geometry approach \cite{1-10} studies the differential geometric structure of statistical models. A statistical model consists of a family of probability distribution functions (PDFs) parameterized by continuous variables. In order to endow these models with a geometric structure, it is necessary to define the Fisher–Rao metric \cite{4}, which in turn, is linked with the concepts of entropy and Fisher information. Once we have a statistical manifold, the main goal of the information geometry approach is to characterize the family of PDFs using geometric quantities, like the geodesic equations, the Riemann curvature tensor, the Ricci tensor or the scalar curvature.

The geometrization of thermodynamics and statistical mechanics are some of the most important achievements in this field, expressed mainly by the foundational works of Gibbs \cite{11}, Hermann \cite{12}, Weinhold \cite{6}, Mrugała \cite{13}, Ruppeiner \cite{14}, and Caratheódory \cite{15}. These investigations lead to the Weinhold and Ruppeiner geometries, where a Riemannian metric tensor in the space of thermodynamic parameters is provided and a notion of distance between macroscopic states is obtained. However, the utility of information geometry is not only limited to those areas. For instance, it has been applied in quantum mechanics leading to a quantum generalization of the Fisher–Rao metric \cite{16}, and recently, also in nuclear plasmas \cite{17,18}. Moreover, generalized extensions of the information geometry approach to the non-extensive formulation of statistical mechanics \cite{19} have been also considered \cite{20,21}. Applications of information geometry to chaos can be also performed by considering complexity on curved manifolds \cite{24-28}, leading to a criterion for characterizing global chaos on statistical manifolds: the more negative is the curvature, the more chaotic is the dynamics; from which some consequences concerning dynamical systems have been explored \cite{29}. More generally, the curvature has been proved to be a quantifier which measures interactions in thermodynamical systems, where the positive or negative sign corresponds to repulsive or attractive correlations, respectively \cite{7}.

Motivated by previous works of some of us \cite{29,30}, we propose a generalization of the Gaussian model which we call the \textit{Hermite–Gaussian model}, and we show its relation with the one-dimensional quantum harmonic oscillator. The application of information geometry techniques to the study of quantum harmonic oscillators can be useful in many applications. For example, the translational modes in a quantum ion trap are quantum harmonic oscillators that...
need to be characterized and controlled in order to avoid coherence losses. Our contribution may serve as a tool for the characterization of unknown parameters in those scenarios. Furthermore, the present work can be also considered as a continuation of the recent Cafaro’s program [24–28] of global characterization of dynamics on curved statistical manifolds generated by Gaussian models.

The paper is organized as follows. In Section II, we review the main features of the information geometry approach. In Section III, we present the Hermite–Gaussian model, we obtain the Fisher–Rao metric and the scalar curvature for this model, and we show its relation with the one-dimensional quantum harmonic oscillator. Moreover, we use this model to characterize some families of states of the quantum harmonic oscillator. We focus in three different families of states: Hamiltonian eigenstates, mixtures of eigenstates and superposition of eigenstates. Finally, in Section IV, we present the conclusions and some future research directions.

2. Information geometry

The information geometry approach studies the differential geometric structure possessed by families of probability distribution functions (PDFs). In this section we introduce the general features of this approach, which will be used in the next sections. The presentation is based on the book of S. Amari and H. Nagaoka [8].

2.1. Statistical models

Information geometry applies techniques of differential geometry to study properties of families of probability distribution functions parameterized by continuous variables. These families are called statistical models. More specifically, a statistical model is defined as follows. We consider the probability distribution functions defined on \( X \subseteq \mathbb{R}^n \), i.e., the functions \( p : X \rightarrow \mathbb{R} \) which satisfy

\[
p(x) \geq 0, \quad \text{and} \quad \int_X p(x)dx = 1. \tag{1}
\]

When \( X \) is a discrete set the integral must be replaced by a sum. A statistical model is a family \( S \) of probability distribution function on \( X \), whose elements can be parameterized by appealing to a set of \( m \) real variables, i.e.,

\[
S = \left\{ p_\theta(x) = p(x|\theta) \left| \theta = (\theta^1, \ldots, \theta^m) \in \Theta \subseteq \mathbb{R}^m \right. \right\}, \tag{2}
\]

with \( \theta \mapsto p_\theta \) an injective mapping. The dimension of the statistical model is given by the number of real variables used to parameterized the family \( S \).

When statistical models are applied to physical systems, the interpretation of \( X \) and \( \Theta \) is the following. \( X \) represents the microscopic variables of the system, which are typically difficult to determine, for instance the positions of the particles of a gas. \( \Theta \) represents the macroscopic variables of the system, which can be easily measured. The set \( X \) is called the microspace and the variables \( x \in X \) are the microvariables. The set \( \Theta \) is called the macrospace and the variables \( \theta^1, \ldots, \theta^m \) are the macrovariables.

Given a physical system, we can define many statistical models. First, we have to choose the microvariables to be considered, and then we have to choose the macrovariables which parametrized the PDFs defined on the microspace. All statistical models are equally valid, but no all of them are equally useful. In general, the choice of the statistical model would be based in pragmatic considerations.

2.2. Geometric structure of statistical models

In order to apply differential geometry to statistical models, it is necessary to endow them with a metric structure. This is accomplished by means of the Fisher–Rao metric

\[
I = I_{ij} = \int_X dx \ p(x|\theta) \frac{\partial \log p(x|\theta)}{\partial \theta^i} \frac{\partial \log p(x|\theta)}{\partial \theta^j}, \quad i, j = 1, \ldots, m. \tag{3}
\]

The metric tensor \( I \) gives to the macrospace a geometrical structure. Therefore, the family \( S \) results to be a statistical manifold, i.e., a differential manifold whose elements are probability distribution functions.
From the Fisher–Rao metric, we can obtain the line element between two nearby PDFs with parameters \( \theta' + d\theta' \) and \( \theta' \)

\[
ds = \sqrt{I_{ij} d\theta^i d\theta^j}, \quad i, j = 1, \ldots, m.
\]

Using the metric tensor we can obtain the geodesic equations for the macrovariables \( \theta_i \) along with relevant geometrical quantities, like the Riemann curvature tensor, the Ricci tensor or the scalar curvature.

**Geodesic equations:**

\[
\frac{d^2}{d\tau^2} + \Gamma^d_{ij} \frac{d\theta^i}{d\tau} \frac{d\theta^j}{d\tau} = 0,
\]

**Christoffel symbols:**

\[
\Gamma^d_{ij} = \frac{1}{2} I^{im} (I_{jm,k} + I_{jm,k} - I_{km,j}) \tag{5}
\]

**Riemann curvature tensor:**

\[
R_{iklm} = \frac{1}{2} (I_{ik,m} + I_{kl,m} - I_{lm,k} - I_{lm,j}) + I_{ip} (\Gamma^d_{ik} \Gamma^p_{jm} - \Gamma^d_{im} \Gamma^p_{jl}) \tag{6}
\]

**Ricci tensor:**

\[
R_{ik} = \Gamma^d_{ik} R_{dm} \tag{7}
\]

**Scalar curvature:**

\[
R = \Gamma^d_{ik} \Gamma^d_{jk} \tag{8}
\]

The comma in the sub-indexes denotes the partial derivative operation (of first and second orders), \( x^{\hat{l}} \) is the inverse of \( I_{ij} \), and \( \tau \) is a parameter that characterizes the geodesic curves.

Moreover, the Fisher–Rao metric gives information about the estimators of the macrovariables. Given an unbiased estimator \( T = (T_1, \ldots, T_m) \) of the parameters \( (\theta_1, \ldots, \theta_m) \), i.e., \( E(T) = (\theta_1, \ldots, \theta_m) \), the Cramér–Rao bound gives a lower bound for the covariance matrix of \( T \),

\[
\text{cov}(T) \geq \Gamma^{-1}, \tag{9}
\]

where the matrix inequality \( A \geq B \) means that the matrix \( A - B \) is positive semi-definite. In particular, this relation gives bounds for the variance of the unbiased estimators \( T_i \),

\[
\text{var}(T_i) \geq (\Gamma^{-1})_{ii}, \tag{10}
\]

This bound is important when looking for optimal estimators. In what follows, we present an important statistical model used in the geometry information approach, the Gaussian model.

### 2.3. Gaussian model

One of the most relevant statistical models used in the geometry information approach is the **Gaussian model**. The reason for that is the wide versatility of this model for describing multiple phenomena. The Gaussian model is obtained by choosing the family \( S \) as the set of multivariate Gaussian distributions. For instance, if \((x_1, \ldots, x_n) \in \mathbb{R}^n\) are the microvariables and there is no correlations between them, then \((\mu_1, \ldots, \mu_n, \sigma_1^2, \ldots, \sigma_n^2) \in \mathbb{R}^n \times \mathbb{R}^n\) are the set of macrovariables, where \( \mu_i \) and \( \sigma_i^2 \) correspond to the mean value and the variance of the microvariable \( x_i \).

If we consider only one microvariable \( x \), the Gaussian model is given by the following probability distribution functions

\[
p(x|\mu, \sigma) = \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \tag{11}
\]

which are parameterized by the mean value \( \mu \) and the standard deviation \( \sigma \). From equations (11) to (8), one can obtain the Fisher–Rao metric and the scalar curvature of this model,

\[
I_{ij} = \left( \begin{array}{cc} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{\sigma^2} \end{array} \right) \quad \text{with} \quad \alpha, \beta = \mu, \sigma, \tag{12}
\]

\[
R = -1. \tag{13}
\]

The Gaussian model is a curved manifold with constant curvature. In some contexts, the negative value of the curvature is interpreted as modeling attractive interactions, like in an ideal gas [7].

In the next section, we are going to introduce a generalization of the Gaussian model, based on the eigenstates of the harmonic oscillator Hamiltonian.
3. Hermite–Gaussian model

We propose a generalization of the Gaussian model, called the Hermite-Gaussian model, which is motivated by the quantum harmonic oscillator. Given the microspace $X = \mathbb{R}$ and the macrospace $\Theta = \{(\mu, \sigma)\}$, we define for each $n$ the Hermite–Gaussian model as the family of probability distribution functions given by

$$p_n(x|\mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} a_n H_n\left(\frac{x-\mu}{\sqrt{2}\sigma}\right), \quad a_n = \frac{1}{\sqrt{2^n n!}}.\quad (14)$$

In particular, if $n = 0$, the Gaussian model is recovered. The Fisher–Rao metric of the Hermite–Gaussian model takes the form

$$I_{\alpha\beta}^{(n)} = \int_X \frac{1}{p(x|\mu, \sigma)} \partial_\alpha p(x|\mu, \sigma) \partial_\beta p(x|\mu, \sigma) dx, \quad \alpha, \beta = \mu, \sigma.\quad (15)$$

and its explicit formula is the following (see Appendix B)

$$I_{\alpha\beta}^{(n)} = \begin{pmatrix} \frac{2n+1}{\sigma^2} & 0 \\ 0 & \frac{2(n^2+n+1)}{\sigma^2} \end{pmatrix}. \quad (16)$$

Taking into account that the scalar curvature is given by

$$R^{(n)} = - \frac{1}{n^2 + n + 1},\quad (17)$$

we can express the Fisher–Rao metric in terms of $R^{(n)}$

$$I_{\alpha\beta}^{(n)} = \begin{pmatrix} \frac{2n+1}{\sigma^2} & 0 \\ 0 & -\frac{2}{\sigma^2 R^{(n)}} \end{pmatrix}. \quad (18)$$

From the Fisher–Rao metric, we can compute the Cramér–Rao bound for unbiased estimators of the parameters $\mu$ and $\sigma$. The lower covariance matrix of any pair of unbiased estimators $T_1, T_2$ of the parameters $\mu, \sigma$, is given by

$$\text{cov} (T_1, T_2) \geq \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\frac{\sigma^2 R^{(n)}}{2} \end{pmatrix}. \quad (19)$$

For the covariance of the estimators we obtain

$$\text{var} (T_1) \geq \frac{\sigma^2}{2n + 1}, \quad (20)$$

$$\text{var} (T_2) \geq \frac{\sigma^2}{2(n^2 + n + 1)} = -\frac{\sigma^2 R^{(n)}}{2}. \quad (21)$$

In what follows, we show the connection between the Hermite–Gaussian model and the quantum harmonic oscillator. We use these model to characterize the PDFs given by quantum states of the harmonic oscillator. We focus on Hamiltonian eigenstates, mixtures of eigenstates and superposition of eigenstates.
3.1. Hamiltonian Eigenstates

The relation between the Hermite–Gaussian model and the quantum harmonic oscillator is straightforward. We start considering the Hamiltonian of the harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2(\hat{x} - x_0)^2,$$  \hspace{1cm} (22)

where $m$ is the mass, $\omega_0$ is the frequency, $x_0$ is the equilibrium position of the oscillator, and $\hat{x}$ and $\hat{p}$ are the position and momentum operators. Its eigenstates $|n\rangle$ satisfy the time-independent Schrödinger equation, $H|n\rangle = E_n|n\rangle$, with $E_n = h\omega_0(n + \frac{1}{2})$. Moreover, the eigenstates satisfy orthogonality and completeness relations

$$\langle n|m \rangle = \delta_{nm} \hspace{1cm} \text{orthogonality}$$
$$\sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{I} \hspace{1cm} \text{completeness}$$

where $\hat{I}$ is the identity operator.

The wave function of the eigenstate $|n\rangle$, in the coordinate representation, is given by

$$\varphi_n(x) = \langle x|n \rangle = \frac{1}{\sqrt{\sqrt{2\pi}\sigma}} a_n H_n \left(\frac{x - \mu}{\sqrt{2}\sigma}\right),$$  \hspace{1cm} (23)

with $\mu = x_0$, $\sigma^2 = \frac{h}{2ma_n}$, and $a_n = \frac{1}{\sqrt{\sqrt{2\pi}\sigma}}$. Then, the PDF of the position operator for the eigenstate $|n\rangle$ is $P_n(x) = |\varphi_n(x)|^2$.

Therefore, if we consider the eigenstate $|n\rangle$ of an harmonic oscillator with parameters $\mu$ and $\sigma$, the PDF of the position operator $P_n(x)$ is equal to the probability distribution function $p_n(x|\mu, \sigma)$ of the Hermite–Gaussian model, given in equation (24). Moreover, the Fisher–Rao metric and the scalar curvature associated with the probability distribution function $P_n(x)$ are given in equations (16) and (17), respectively.

It is important to remark that the Fisher–Rao metric is diagonal, and the scalar curvature is always negative and decreases with the quantum number $n$, tending to zero in the limit of high quantum numbers. Moreover, from the Cramér–Rao bound we obtain that the minimal variance of the estimators of the parameter $\mu$ grows with $\sigma^2$ and decreases with the eigenstate number, and the minimal variance of estimators of the parameter $\sigma$ also grows with $\sigma^2$ but decreases with the square of the eigenstate number. Equivalently, the minimal variance of the estimators of $\sigma$ is proportional to the scalar curvature.

3.2. General states

We are going to consider the PDF of the position operator obtained from general states of the harmonic oscillator. Let us consider the basis of the Hamiltonian eigenstates $|\langle n\rangle|_{\text{vec}}$, and a state $\hat{\rho}$ of the form

$$\hat{\rho} = \sum_{n,m} \lambda_{nm} |n\rangle\langle m|.$$

The probability distribution function of the position operator is given by

$$P(x) = \langle x|\hat{\rho}|x \rangle = \sum_{n,m} \lambda_{nm} \varphi_n(x)\varphi_m(x) = \sum_{n,m} \frac{\lambda_{nm} a_n a_m}{\sqrt{2\pi}\sigma} e^{-\frac{(x - \mu)^2}{4\sigma^2}} H_n \left(\frac{x - \mu}{\sqrt{2}\sigma}\right) H_m \left(\frac{x - \mu}{\sqrt{2}\sigma}\right),$$  \hspace{1cm} (25)

where $\varphi_n(x)$ is the wave function of the eigenstate $|n\rangle$, given in equation (23).

For practical reasons, we define the function $f(y)$,

$$f(y) = \sum_{n,m} \frac{\lambda_{nm} a_n a_m}{\sqrt{2\pi}} e^{-y^2} H_n(y) H_m(y).$$  \hspace{1cm} (26)
Then, we have \( P(x) = \frac{f(x)}{\sqrt{2\pi \sigma^2}} \), with \( y(x) = \frac{x - \mu}{\sqrt{2\sigma^2}} \).

In order to calculate the Fisher–Rao metric associated with \( P(x) \), we need the partial derivatives \( \partial_\mu P(x) \) and \( \partial_\sigma P(x) \), which are given by

\[
\partial_\mu P(x) = \partial_\mu \left( \frac{f(y(x))}{\sigma} \right) = -\frac{f'(y(x))}{\sqrt{2\sigma^2}},
\]

\[
\partial_\sigma P(x) = \partial_\sigma \left( \frac{f(y(x))}{\sigma} \right) = -\frac{f(y(x))}{\sigma^2} - \frac{y(x)f'(y(x))}{\sigma^2},
\]

with \( f'(y) = \frac{df}{dy} \).

Replacing the PDF (25) and the partial derivatives (27) and (28) in the integral of equation (3), and making the change of variable \( y = y(x) \), we obtain the Fisher–Rao metric

\[
I_{\mu\sigma} = I_{\sigma\mu} = \int_{-\infty}^{\infty} \frac{\partial_\mu P(x) \partial_\sigma P(x)}{P(x)} dx = \frac{1}{\sigma^2} \int_{-\infty}^{\infty} y f'(y) dy + \int_{-\infty}^{\infty} \frac{y^2 f''(y)}{f(y)} dy,
\]

\[
I_{\mu\mu} = I_{\sigma\sigma} = \int_{-\infty}^{\infty} \left( \frac{\partial_\mu P(x)}{P(x)} \right)^2 dx = \frac{1}{2\sigma^2} \int_{-\infty}^{\infty} f'(y)^2 dy,
\]

\[
I_{\sigma\sigma} = \int_{-\infty}^{\infty} \left( \frac{\partial_\sigma P(x)}{P(x)} \right)^2 dx = \frac{1}{2\sigma^2} \int_{-\infty}^{\infty} f'(y)^2 dy - \frac{1}{\sigma^2},
\]

where in the first equation we used that \( \int_{-\infty}^{\infty} f'(y) dy = 0 \), and in the last equation we used that \( \int_{-\infty}^{\infty} f(y) dy = \frac{1}{\sqrt{2\pi}} \) and \( \int_{-\infty}^{\infty} y f(y) dy = 0 \).

Therefore, we can write the Fisher–Rao metric as follows:

\[
I_{\alpha\beta} = \frac{1}{\sigma^2} \left( \begin{array}{cc}
I_{\mu\mu} & I_{\mu\sigma} \\
I_{\mu\sigma} & I_{\sigma\sigma}
\end{array} \right),
\]

where \( I_{\mu\mu}, I_{\mu\sigma} \) and \( I_{\sigma\sigma} \) are independent of \( \mu \) and \( \sigma \), and they are given by

\[
I_{\mu\mu} = \int_{-\infty}^{\infty} \frac{y f'(y)^2}{f(y)} dy,
\]

\[
I_{\mu\sigma} = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{f'(y)^2}{f(y)} dy,
\]

\[
I_{\sigma\sigma} = \sqrt{2} \int_{-\infty}^{\infty} \frac{y^2 f''(y)}{f(y)} dy - 1.
\]

From the Fisher–Rao metric and using equations (5) to (8), we can obtain the scalar curvature

\[
R = \frac{2I_{\mu\mu}}{I_{\mu\mu}^2 + I_{\mu\sigma} I_{\sigma\sigma} - I_{\mu\sigma} I_{\mu\sigma}}.
\]

The Cramér–Rao bound gives the lower covariance matrix of any pair of unbiased estimators \( T_1, T_2 \) of the parameters \( \mu, \sigma \),

\[
\text{cov} (T_1, T_2) \geq \frac{\sigma^2}{I_{\mu\mu} I_{\sigma\sigma} - I_{\mu\sigma} I_{\sigma\mu}} \left( \begin{array}{cc}
I_{\sigma\sigma} & -I_{\mu\sigma} \\
-I_{\mu\sigma} & I_{\mu\mu}
\end{array} \right).
\]
Finally, we can express the variance of $T_2$ in terms of the scalar curvature,

$$\text{var}(T_2) \geq -\frac{\sigma^2 R}{2}. \quad (32)$$

**Corollary 1:** The Fisher–Rao metric for a general state of the harmonic oscillator is independent of the parameter $\mu$ and it only depends on the parameter $\sigma$ by a general factor $1/\sigma^2$.

**Corollary 2:** The scalar curvature for a general state of the harmonic oscillator is independent of the parameters $\mu$ and $\sigma$, and it only involves integrals of the dimensionless function $f(y)$ and its derivative $f'(y)$.

**Corollary 3:** The lower variance of unbiased estimators of the parameter $\sigma$ is proportional to $\sigma^2 R$.

### 3.3. Mixtures of Hamiltonian eigenstates

We consider quantum states which are mixtures of the Hamiltonian eigenstates. Mixtures of eigenstates are particular cases of the states given in equation (24), with $\lambda_{nm} = \delta_{nm} \lambda_n$, i.e., $\hat{\rho} = \sum_n \lambda_n |n\rangle\langle n|$. Therefore, the probability distribution function of the position operator, the Fisher–Rao metric and the scalar curvature can be obtained from the general expressions (25), (29) and (30), considering $\lambda_{nm} = \delta_{nm} \lambda_n$.

In this case, the PDF of the position operator takes the form

$$P(x) = \sum_n \lambda_n \varphi_n(x) = \sum_n \lambda_n p_n(x|\mu,\sigma).$$

The diagonal elements of the Fisher–Rao metric are zero, and the elements $I_{\mu\sigma} = I_{\sigma\mu}$ are given in equation (29),

$$I_{\mu\sigma} = I_{\sigma\mu} = \frac{1}{\sigma^2} \int_{-\infty}^{+\infty} \frac{y (f'(y))^2}{f(y)} dy. \quad (33)$$

with $f(y) = \sum_n \frac{\lambda_n^2}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} H_n^2(y)$. Since Hermite polynomials $H_n(y)$ are even or odd functions of the variable $y$, $H_n^2(y)$ are even functions. Then, $f(y)$ is also an even function and its derivative $f'(y)$ is an odd function. Finally, the integrand of equation (33) is an odd function of $y$. Therefore, $I_{\mu\sigma} = I_{\sigma\mu} = 0$.

Finally, the scalar curvature is obtained from equation (30),

$$R = -\frac{2}{I_{\sigma\sigma}}.$$

As an example, we consider the mixture state $\hat{\rho}_{01} = \frac{1}{2} |0\rangle\langle 0| + \frac{1}{2} |1\rangle\langle 1|$. The Fisher–Rao metric is given by

$$I_{(01)}^{01} = \frac{1}{\sigma^2} \begin{pmatrix}
2 + \sqrt{2\pi} \left( \text{Erf} \left( \frac{1}{\sqrt{2}} \right) - 1 \right) & 0 \\
0 & 2 + \sqrt{2\pi} \left( 1 - \text{Erf} \left( \frac{1}{\sqrt{2}} \right) \right)
\end{pmatrix}, \quad (34)$$

where Erf(x) is the Gauss error function, with Erf($\frac{1}{\sqrt{2}}$) = 0.317. The scalar curvature is approximately $R^{01} \approx -0.604$.

### 3.4. Superposition of Hamiltonian eigenstates

We consider quantum states which are superpositions of Hamiltonian eigenstates. Superpositions of eigenstates of the form $|\psi\rangle = \sum_n \alpha_n |n\rangle$ are particular cases of states given in equation (24), with $\lambda_{nm} = \alpha_n \alpha_m^\ast$, i.e., $\hat{\rho} = |\psi\rangle\langle \psi| = \sum_n \alpha_n \alpha_m^\ast |n\rangle\langle m|$. Therefore, the PDF of the position operator, the Fisher–Rao metric and the scalar curvature can be obtained from the general expressions (25), (29) and (30), considering $\lambda_{nm} = \alpha_n \alpha_m^\ast$. 

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3.4.1. Even or odd superpositions

In this section we focus on a family of superpositions that yield analytic expressions. If we consider a superposition of eigenstates with only even or odd eigenstates, i.e.,

$$\hat{\rho} = \sum_{\text{even indexes}} \alpha_n \alpha_m^* |n\rangle \langle m|,$$

or

$$\hat{\rho} = \sum_{\text{odd indexes}} \alpha_n \alpha_m^* |n\rangle \langle m|,$$

we obtain that the diagonal elements of the Fisher–Rao metric are zero. The proof is similar to the case of mixtures of eigenstates. The diagonal elements are given in equation (29),

$$I_{\mu\sigma} = I_{\sigma\mu} = \frac{1}{\sigma^2} \int_{-\infty}^{+\infty} \frac{y (f'(y))^2}{f(y)} \, dy,$$

with

$$f(y) = \sum_{\text{even or odd indexes}} \alpha_n \alpha_m^* a_n a_m \sqrt{\frac{2}{\pi}} e^{-y^2} H_n(y) H_m(y).$$

If the indexes $n, m$ can only take even or odd values, then the product $H_n(y) H_m(y)$ is always an even function of the variable $y$. Then, $f(y)$ is also an even function and its derivative $f'(y)$ is an odd function. Finally, the integrand of equation (35) is an odd function of $y$, and the result of the integral is zero.

Again, we obtain that the scalar curvature, given in equation (30), is

$$R = -\frac{2}{I_{\mu\sigma}}.$$ 

3.4.2. Real or imaginary superpositions

Analytic expressions can also be obtained for superpositions of eigenstates that involve only real coefficients, i.e.,

$$\hat{\rho} = \sum_{n,m} \alpha_n \alpha_m^* |n\rangle \langle m|. $$

In order to compute the Fisher–Rao metric, we need the function $f(y)$, given in (26), and its derivative $f'(y)$,

$$f(y) = \sqrt{\frac{e^{-y^2}}{2\pi}} \left( \sum_n \alpha_n a_n H_n(y) \right)^2,$$

$$f'(y) = \sqrt{\frac{2e^{-y^2}}{2\pi}} \left( \sum_n \alpha_n a_n H_n(y) \right) \left[ \sum_n \alpha_n a_n \left( n H_{n-1}(y) - \frac{H_{n-1}(y)}{2} \right) \right],$$

where in the last equation we have used the recurrence relations of the Hermite polynomials (A.2). Replacing expres-
sions (36) in the Fisher–Rao metric (29), and taking into account relations (A.1) and (A.2), we obtain

\[ I_{\alpha\beta} = I_{\beta\alpha} = \frac{1}{\sigma^2} \int_{-\infty}^{+\infty} \frac{4y^2 e^{-\frac{y^2}{2}}}{\sqrt{\pi}} \left[ \sum_n \alpha_n a_n \left( nH_{n-1}(y) + \frac{H_{n+1}(y)}{2} \right) \right]^2 dy = \]

\[ \frac{1}{\sigma^2} \sum_n \alpha_n \left( \alpha_{n-2} \sqrt{n(n-1)(n-2)} + \alpha_n n \sqrt{n} + \alpha_{n+1} (n+1) \sqrt{n+1} + \alpha_{n+3} \sqrt{(n+3)(n+2)(n+1)} \right), \]

\[ I_{\mu\mu} = \frac{1}{\sigma^2} \int_{-\infty}^{+\infty} \frac{2e^{-\frac{y^2}{2}}}{\sqrt{\pi}} \left[ \sum_n \alpha_n a_n \left( nH_{n-1}(y) + \frac{H_{n+1}(y)}{2} \right) \right]^2 dy = \]

\[ \frac{1}{\sigma^2} \sum_n \alpha_n \left( -\alpha_{n-2} \sqrt{n(n-1)} + \alpha_n (2n+1) - \alpha_{n+2} \sqrt{(n+2)(n+1)} \right), \]

\[ I_{\sigma\sigma} = \frac{1}{\sigma^2} \int_{-\infty}^{+\infty} \frac{4y^2 e^{-\frac{y^2}{2}}}{\sqrt{\pi}} \left[ \sum_n \alpha_n a_n \left( nH_{n-1}(y) + \frac{H_{n+1}(y)}{2} \right) \right]^2 dy = \frac{1}{\sigma^2} - \]

\[ \frac{1}{\sigma^2} \sum_n \alpha_n \left( -\alpha_{n-4} \sqrt{n(n-1)(n-2)(n-3)} + \alpha_n (2n^2 + 2n + 3) - \alpha_{n+4} \sqrt{(n+4)(n+3)(n+2)(n+1)} \right) - \frac{1}{\sigma^2}. \]

If we consider a superposition of eigenstates with only imaginary coefficients, we obtain a similar result, but replacing the coefficients \( \alpha_n \) by its imaginary part, i.e., by \( \text{Im}(\alpha_n) \).

4. Conclusions

In this work we have proposed a generalization of the Gaussian model -namely, the \textit{Hermite-Gaussian model}- and we have studied many of its properties from the point of view of the information geometry approach. We have shown its relation with the probabilities associated to the one-dimensional quantum harmonic oscillator model and analytic expressions for some particular classes of states were provided. Specifically, we found that for finite mixtures of eigenstates and finite superpositions of (even or odd) eigenstates the Fisher metric is always diagonal. Real and imaginary superpositions of eigenstates do not imply a diagonal Fisher metric and the matrix elements are given in terms of a series sum. An analytic expression for the scalar curvature was only obtained when the Fisher metric is diagonal, being negative and inversely proportional to the \( \sigma^2 \) element.

Due to the relevance of this model in many applications, our contribution may serve to extend the scope of information geometry techniques into a wider class of physical problems. For example, since in irreversible processes the final (reduced) state of a system (after interacting with the environment) is typically a mixture of their eigenstates, which for the case of Hermite-Gaussian models has a diagonal Fisher metric, the results obtained could be used for determining if the process involved is irreversible or not by simple inspection of the diagonal elements of the Fisher metric. In this context and considering that the states of the system can be expressed by means of Hermite-Gaussian models, if the Fisher metric of the final reduced state results non-diagonal then by Sections 3.3 and 3.4 it is not mixture of harmonic oscillator eigenstates, and thus the process cannot be irreversible.

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References

References

[1] R. Fisher, \textit{Phil. Trans. R. Soc. Lond. A} 222, 309-368 (1922).
In order to obtain the elements of the metric tensor, we need to calculate the partial derivatives of the probability distribution. For parameters $\mu$ and $\sigma$, the probability distribution of the $n$–Hermite–Gaussian model is

$$p_n(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma} a_n^2 H_n^2(y)},$$

with $a_n = \frac{1}{\sqrt{2^n n!}}, y = \frac{x - \mu}{\sqrt{2\sigma}}$.

In order to obtain the elements of the metric tensor, we need to calculate the partial derivatives of the probability distribution. It easy to show that

$$\partial_\mu p_n(x) = \frac{p_n'(x)}{\sqrt{2\sigma}},$$

$$\partial_\sigma p_n(x) = -\frac{p_n'(x) + y \sigma p_n'(y)}{\sigma},$$

where $H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} e^{-y^2}$, and their orthogonality relation is

$$\int_{-\infty}^{\infty} e^{-y^2} H_n(y) H_m(y) dy = \sqrt{\pi} 2^n n! \delta_{n,m}.$$  

An important feature of these polynomials is that if $n$ is even, $H_n(y)$ is an even function; and if $n$ is odd, $H_n(y)$ is an odd function.

Some relevant recurrence relations are the following:

$$H_n'(y) = 2n H_{n-1}(y), \quad H_{n+1}(y) = 2y H_n(y) - 2n H_{n-1}.$$  

### Appendix B. Hermite–Gaussian model

For parameters $\mu$ and $\sigma$, the probability distribution of the $n$–Hermite–Gaussian model is

$$p_n(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma} a_n^2 H_n^2(y)},$$

with $a_n = \frac{1}{\sqrt{2^n n!}}, y = \frac{x - \mu}{\sqrt{2\sigma}}$.
Replacing expressions (B.2) and (B.3) in (B.5) and doing some easy manipulations, we obtain

\[ p_n'(y) = \frac{2a_n^2}{\sqrt{2\pi} \sigma} e^{-y^2} H_n(y) \left( nH_{n-1}(y) - \frac{1}{2} H_{n+1}(y) \right), \]  \hspace{1cm} (B.4)

where we have used the recurrence relations (A.2). It should be noted that \( p_n(y) \) is even an function of \( y \), thus \( p_n'(y) \) is an odd function of \( y \).

Also, we will need to express \( yp_n'(y) \) in terms of Hermite polynomials,

\[ yp_n'(y) = \frac{2a_n^2}{\sqrt{2\pi} \sigma} e^{-y^2} H_n(y) \left( nyH_{n-1}(y) - \frac{1}{2} yH_{n+1}(y) \right) = \frac{2a_n^2}{\sqrt{2\pi} \sigma} e^{-y^2} H_n(y) \left( n(n-1)H_{n-2}(y) - \frac{1}{4} H_{n+2}(y) \right), \]

where we have used expression (B.4) and the recurrence relations (A.2).

**Appendix B.1. Off–diagonal elements**

Since the metric tensor is symmetric, it is enough to calculate the element \( I_{\mu \nu}^{(n)} \), given by

\[ I_{\mu \nu}^{(n)} = \int_{-\infty}^{+\infty} \frac{1}{p_\mu(x)} \partial_\mu p_\nu(x) dx. \]  \hspace{1cm} (B.5)

Replacing expressions (B.4) and (B.3) in (B.5) and doing some easy manipulations, we obtain

\[ I_{\mu \nu}^{(n)} = \int_{-\infty}^{+\infty} \frac{1}{\sigma} \left( p_\mu'(y(x)) + y(x) \left[ \frac{p_\nu'(y(x))}{p_\nu(y(x))} \right]^2 \right) dx = \int_{-\infty}^{+\infty} \frac{\sqrt{2\pi}}{\sigma} \left( p_\nu'(y) + y \left[ \frac{p_\nu'(y)}{p_\nu(y)} \right]^2 \right) dy, \]  \hspace{1cm} (B.6)

where in the last equation we changed from variable \( x \) to the variable \( y = \frac{x - \mu}{\sqrt{2} \sigma} \). Since \( p_\nu(y) \) and \( p_\nu'(y) \) are even and odd functions of \( y \), respectively, then the integrand of (B.6) is an odd function. Therefore, \( I_{\mu \nu}^{(n)} = 0 \).

**Appendix B.2. Element \( I_{\mu \mu}^{(n)} \)**

The element \( I_{\mu \mu}^{(n)} \) is given by

\[ I_{\mu \mu}^{(n)} = \int_{-\infty}^{+\infty} \frac{1}{p_\mu(x)} \left[ \partial_\mu p_\mu(x) \right]^2 dx. \]  \hspace{1cm} (B.7)

Replacing expression (B.2) in (B.7), we obtain

\[ I_{\mu \mu}^{(n)} = \int_{-\infty}^{+\infty} \frac{1}{2 \sigma^2} \left( \frac{p_\mu'(y(x))}{p_\mu(y(x))} \right)^2 dx = \int_{-\infty}^{+\infty} \frac{1}{2 \sigma^2} \left( \frac{p_\mu'(y)}{p_\mu(y)} \right)^2 dy, \]  \hspace{1cm} (B.8)

In the last step, we have changed from variable \( x \) to the variable \( y \). Then, if we replace expressions (B.1) and (B.4) in (B.8) and we rearrange the expression, we obtain

\[ I_{\mu \mu}^{(n)} = \frac{2a_n^2}{\sqrt{2\pi} \sigma^2} \left[ n^2 \int_{-\infty}^{+\infty} e^{-y^2} H_n^2(y) dy - n \int_{-\infty}^{+\infty} e^{-y^2} H_{n-1}(y) H_n(y) dy + 1 \int_{-\infty}^{+\infty} e^{-y^2} H_n^2(y) dy \right] = \]
\[ = \frac{2}{\sqrt{2\pi} \sigma^2} \frac{1}{2n!} \left( n^2 \sqrt{\pi} 2^{n-1} (n-1)! + \frac{1}{4} \sqrt{\pi} 2^{n+1} (n+1)! \right). \]

In the last step we have used the orthogonality relation (A.1). Finally, we obtain \( I_{\mu \mu}^{(n)} = \frac{2n+1}{2\sigma^2} \).
Appendix B.3. Element $I^{(n)}_{r\sigma r}$

The element $I^{(n)}_{r\sigma r}$ is given by

$$I^{(n)}_{r\sigma r} = \int_{-\infty}^{+\infty} \frac{1}{p_n(x)} \left[ \partial_\sigma p_n(x) \right]^2 dx.$$  \hspace{1cm} (B.9)

Replacing expression (B.3) in (B.9), we obtain

$$I^{(n)}_{r\sigma r} = \int_{-\infty}^{+\infty} \frac{1}{p_n(y(x))} \left( p_n(y(x)) + y(x)p'_n(y(x)) \right)^2 dx = \int_{-\infty}^{+\infty} \frac{\sqrt{2}}{\sigma} \left[ \frac{p_n(y) + y'p'_n(y)}{p_n(y)} \right]^2 dy.$$  \hspace{1cm} (B.10)

In the last equation we have changed from variable $x$ to the variable $y$. Then, if we replace expressions (B.1) and (B.5) in (B.10) and we rearrange the expression, we obtain

$$I^{(n)}_{r\sigma r} = \int_{-\infty}^{+\infty} \frac{a_n^2}{\sigma^{n+2}} e^{-\frac{y^2}{\sigma^2}} \left( 2n(n-1)H_{n-2}(y) - \frac{1}{2}H_{n+2}(y) \right)^2 dy =$$

$$= \int_{-\infty}^{+\infty} \frac{a_n^2}{\sqrt{\pi}\sigma^2} e^{-\frac{y^2}{\sigma^2}} \left( 4n^2(n-1)^2H_{n-2}^2(y) + \frac{1}{4}H_{n+2}^2(y) - 2n(n-1)H_{n-2}(y)H_{n+2}(y) \right) dy =$$

$$= \frac{1}{\sqrt{\pi}\sigma^2} \frac{1}{2n!} \left( 4n^2(n-1)^2 \sqrt{\pi} 2^{n-2} (n-2)! + \frac{1}{4} \sqrt{\pi} 2^{n+2} (n+2)! \right).$$

In the last step, we have used the orthogonality relation (A.1). Finally, we obtain $I^{(n)}_{r\sigma r} = \frac{2(n^2+n+1)}{\sigma^2}$. 

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