This paper investigates the conditions for the formation of local bumps in the activity of binary attractor neural networks with spatially dependent connectivity. We show that these formations are observed when asymmetry between the activity during the retrieval and learning is imposed. Analytical approximation for the order parameters is derived. The corresponding phase diagram shows a relatively large and stable region, where this effect is observed, although the critical storage and the information capacities drastically decrease inside that region. We demonstrate that the stability of the network, when starting from the bump formation, is larger than the stability when starting even from the whole pattern. Finally, we show a very good agreement between the analytical results and the simulations performed for different topologies of the network.

PACS numbers: 87.18.Sn, 64.60.Cn, 07.05.Mh

I. INTRODUCTION

The bump formations in recurrent neural networks have been recently reported in several investigations concerning linear-threshold units, binary units, Small-World networks of Integrate and Fire neurons and in a variety of spatially distributed neural networks with excitatory and inhibitory couplings between the cells.

As has been shown, the localized retrieval is due to the short-range connectivity of the networks and could explain the behavior in structures of biological relevance as the neocortex, where the probability of connections decreases with the distance.

In the case of linear-threshold neural network model, the signal-to-noise analysis has been recently adapted to spatially organized networks and has shown that the retrieval states of the connected network have non-uniform activity profiles when the connections are short-range enough, even without any spatial structure of the stored patterns. The increase of the gain or the saturation level of neurons enhance the level of localization of the retrieval states and do not lead to a drastic decrease of the storage capacity in these networks even for very localized solutions.

An interesting investigation of the spontaneous activity bumps in Small-World networks (SW) of Integrate-and Fire neurons, has recently shown that the network retrieves when its connectivity is close to the random and displays localized bumps of activity, when its connectivity is close to the ordered. The two regimes are mutually exclusive in the range of the parameter governing the proportion of the long-range connections on the SW topology of Integrate-and-Fire network, while the two behaviors coexist in the case of linear-threshold and smoothly saturated units. Moreover, it has been stated that the transition between localization and retrieval regimes, in the case of SW networks of Integrate-and-Fire neurons, can occur at a degree of randomness beyond the SW regime and it is not related, in general, to the SW properties of the network.

The bump formations have been also investigated in spatially distributed neural network models with excitatory and inhibitory synaptic couplings. Recently it has been shown that a neural network only with excitatory couplings can exhibit localized activity from an initial transient synchrony of a localized group of cells, followed by desynchronized activity within the group. This activity may grow or may be reduced when depression with a given size of the frequency of the inputs is introduced. It is also very sensitive to the initial conditions and the range of the parameters of the network.

The result of bump formations have been recently reported by us, in the case of binary Hebb model for associative network. We stated out that these spatially asymmetric retrieval states (SAS) can be observed if an asymmetry between the learning and the retrieval states is imposed. This means that the network is constrained to have a different activity compared to that induced by the patterns.

In the present investigation we regard a symmetric and distance dependent connectivity for all neurons within an attractor neural network (NN) of Hebbian type formed by
$N$ binary neurons $\{S_i\}, S_i \in \{-1, 1\}, i = 1, \ldots, N$, storing $p$ binary patterns $\eta^{(\mu)}_i, \mu \in \{1, \ldots, p\}$. The connectivity between the neurons is symmetric $c_{ij} = c_{ji} \in \{0, 1\}, c_{ii} = 0$, where $c_{ij} = 1$ means that neurons $i$ and $j$ are connected.

We are interested only on connectivities in which the fluctuations between the individual connectivity are small, e.g. $\forall i \sum_j c_{ij} \approx cN$, where $c$ is the mean connectivity.

The learned patterns are drawn from the following distribution:

$$P(\eta^{(\mu)}_i) = \frac{1 + a}{2} \delta(\eta^{(\mu)}_i - 1) + \frac{1 - a}{2} \delta(\eta^{(\mu)}_i + 1),$$

where the parameter $a$ is the sparsity of the code $[11, 12$.

We study the Hopfield model $[13$:

$$H = \frac{1}{N} \sum_{ij} J_{ij} S_i S_j,$$

with Hebbian learning rule $[14$:

$$J_{ij} = \frac{1}{c} \sum_{\mu=1}^{p} c_{ij}(\eta^{(\mu)}_i - a)(\eta^{(\mu)}_j - a).$$

Further in this article we will work in terms of variables $\xi^{(\mu)}_i \equiv \eta^{(\mu)}_i - a$.

In the case of binary network and symmetrically distributed patterns, the only asymmetry between the retrieval and the learning states that can be imposed, independent on the position of the neurons, is the total number of the neurons in a given state. Having in mind that there are only two possible states, this condition leads to a condition on the mean activity of the network, which is introduced by adding an extra term $H_a$ to the Hamiltonian

$$H_a = NR(\sum_i S_i / N - a).$$

This term favors states with lower total activity $\sum_i S_i$ that is equivalent to decrease the number of active neurons, creating asymmetry between the learning and the retrieval states.

For $R = 0$, this model has been intensively studied $[11, 12, 15$ and shows no evidence of spatial asymmetric states.

For $R > 0$, the energy of the system increases with the number of active neurons ($S_i = 1$) and the term $H_a$ tends to limit the number of active neurons below $aN$.

In the present work we show explicitly that the constraint on the network level of activity is a sufficient condition for observation of spatially asymmetric states in binary Hebb neural network. Similar observations have been reported in the case of linear-threshold network $[1]$, where in order to observe bump formations, one has to constrain the activity of the network. The same is true in the case of smoothly saturating and binary networks $[2]$, when the highest activity level, they can achieve, is above the maximum activity of the units in the stored pattern.

In the present paper we discuss in details the binary neural network model, giving a complete analytical derivation of its properties. We explain the minimal conditions for the formations of spatially asymmetric states in a recurrent network with metrically organized connectivity.

We compare the results for three different topologies of the network and show interesting multi-bump formations in the case of the SW-like topology. Further extension of the analysis to the information, carried by the network, reveals that the retrieved states can be well recovered by a limited amount of information. We show that the stability of the network’s retrieval, when starting from the bump, is larger than the stability, when starting even from the whole pattern. This could be useful in practical retrieval tasks with minimal information transmitted.

The paper is organized as follows: In section Analytical analysis and using replica theory, we present our detailed mean-field analysis in the case of distance-dependent connectivities. We derive the equations for the corresponding order parameters (OP) for finite and zero temperatures. In section Simulations we present the results of the simulations, done for the case of different topologies of the network, and we compare them with the analytical results. In section Discussion we focus on the stability of the phase diagram of the network and especially on the dependence of the SAS region on the parameters of the model. We discuss the behavior of the critical storage capacity when bump formations are present, as well as their effect on the information capacity of the network. The conclusion, drawn in the last part, shows that only the asymmetry between the learning and the retrieval states is sufficient to observe spatially asymmetric states.

II. ANALYTICAL ANALYSIS

For the analytical analysis of the SAS states, we consider the decomposition of the connectivity matrix $c_{ij}$ by its eigenvectors $a^{(k)}_i$:

$$c_{ij} = \sum_k \lambda_k a^{(k)}_i a^{(k)}_j, \quad \sum_i a^{(k)}_i a^{(k)}_i = \delta_{kl},$$

where $\lambda_k$ are the corresponding (positive) eigenvalues. For convenience we denote $b^{(k)}_i = a^{(k)}_i \sqrt{\lambda_k}$, having

$$c_{ij} = \sum_k b^{(k)}_i b^{(k)}_j.$$  

We will assume that $a^{(k)}_i$ are ordered by their eigenvalues in decreasing order, e.g. for $k > l \Rightarrow \lambda_k \geq \lambda_l$. To get some intuition of what $a^{(k)}_j$ look like, we plot in Fig. 11 the first two eigenvectors.

For a wide variety of connectivities, the first three eigenvectors approximate

$$a^{(0)}_k = \sqrt{1/N},$$

(3)
and its eigenvectors $a_i$. The decomposition of the sites, by using an expansion of the connectivity matrix

Introducing variables $m_{pl}$, at each replica $\rho$, each configuration and each eigenvalue, we get:

\[ \langle\langle Z^n \rangle\rangle = e^{-\betapn/2c+\beta RaN} \]

where we have split the sums over the first $s$-patterns and the remaining (infinite) $p-s$ ones.
The “condensed” order parameters \( m_{\mu l}^{\nu} \) have a clear physical meaning as an overlap between the pattern \( \nu \) and the state of the neuron in the replica \( \rho \) of the system, modulated by the \( l \)-th eigenvalue of the connectivity matrix:

\[
m_{\mu l}^{\nu} = \frac{1}{cN} \sum_{i} \xi_{\nu}^{\mu} S_{\rho i}^{\mu} b_{i}^{l}.
\] (11)

In the present investigation, the main physical hypothesis is that only a finite number, \( k \), of the order parameters \( m_{\mu l}^{\nu} \), \( l = 1, \ldots, k \) are macroscopically different from zero. As we can see later in this paper the simulations also confirm this hypothesis. This avoids the introduction of infinite number of parameters, dependent on the state of the system, that would make impossible the analytical solution of the problem.

After taking the averages over the pattern, for the first term we get:

\[
I = \exp \beta \left( -\frac{1}{2} \sum_{\mu pl} (m_{\mu l}^{\nu})^2 + \frac{\beta(1-a^2)}{2cN} \sum_{\rho \sigma l k \mu} m_{\mu l}^{\nu} m_{\sigma k l}^{\mu} S_{\rho i}^{\mu} S_{\sigma i}^{\mu} b_{i}^{l} b_{i}^{k} \right).
\] (12)

The integration of the last expression over the OP \( m_{\mu pl}^{\nu} \) gives:

\[
\int \prod_{\mu pl} \frac{dm_{\mu l}^{\nu}}{\sqrt{2\pi I}} I = \int \prod_{\rho \sigma l k} dq_{\rho l}^{k} \exp \left( -\frac{p}{2} Tr \ln[A_{\rho \sigma l}^{k}] \right) \prod_{\rho \sigma l k} \delta(q_{\rho l}^{k} - \frac{1}{cN} \sum_{i} S_{\rho i}^{\mu} S_{\sigma i}^{\mu} b_{i}^{l} b_{i}^{k}).
\] (13)

The expression within the \( \delta \)-function can be regarded as a definition of the OP \( q_{\rho \sigma l}^{k} \).

Now let us suppose that the order parameter \( q_{\rho \sigma l}^{k} \) can split as a product of two terms: one, which only depends on the replica’s and the eigenvalue’s indexes and another one that introduces the spatial dependence of the distribution of the eigenvalues and eigenvectors:

\[
y_{\rho \sigma l}^{k} = q_{\rho \sigma l}^{k} \sum_{i} b_{i}^{l} b_{i}^{k} / cN = q_{\rho \sigma}^{l} \delta l k (\lambda_{l} / cN).\]

In this way we can keep only one index for the OP \( q_{\rho \sigma l}^{k} \) when taking into account the spatial distribution of the matrix of interactions.

Introducing the parameter \( r_{\rho \sigma}^{l} \), conjugate to \( q_{\rho \sigma l}^{k} \), for the last expression we obtain:

\[
\int \prod_{\mu pl} \frac{dm_{\mu l}^{\nu}}{\sqrt{2\pi I}} I = \int \prod_{\rho \sigma l} dq_{\rho l}^{k} \prod_{\rho \sigma l} dr_{\rho \sigma}^{l} \exp \left( -\frac{p}{2} Tr \ln[A_{\rho \sigma l}^{k}] \right) \exp cN \left( -\frac{1}{2} \alpha \beta^2 (1-a^2) \sum_{\rho \sigma l} r_{\rho \sigma}^{l} q_{\rho l}^{k} + \frac{1}{2cN} \alpha \beta^2 (1-a^2) \sum_{i \rho \sigma l} r_{\rho \sigma}^{l} S_{\rho i}^{\mu} S_{\sigma i}^{\mu} b_{i}^{l} b_{i}^{k} \right),
\] (14)

where the parameter \( \alpha \equiv p/N \) is the storage capacity of the network.

The matrix \( A_{\rho \sigma l}^{k} \) is

\[
A_{\rho \sigma l}^{k} = \delta_{\rho \sigma} \left( 1 - \beta (1-a^2) \mu_{l} (1-q_{l}) \right) - \beta (1-a^2) q_{\rho \sigma} \mu_{l}
\] (15)

and \( \mu_{l} = \lambda_{l} / cN \).
For the replicated partition function \( \langle \langle Z^n \rangle \rangle \), after taking the limit \( h^\nu \to 0 \), we obtain:

\[
\langle \langle Z^n \rangle \rangle = e^{-\beta n(1-a^2)/2c + \beta R n} \int \prod \nu \, dm^{\nu}_{\nu} \int \prod_{\rho} \, dq^{\rho}_{\rho} \, dr^{l}_{\rho}\]

\[
\exp \, cN \left( -\frac{1}{2} \sum_{\nu, \rho} (m^{\nu}_{\nu})^2 - \frac{1}{2} Tr \ln [A^k_{\rho}] - \frac{1}{2} \alpha \beta^2 (1-a^2) \sum_{\rho \neq \sigma, l} r^{l}_{\rho} q^{l}_{\rho} \right) \]

\[
\left\langle \left\langle Tr_{S^\nu} \exp \, cN \left[ \frac{1}{2cN} \alpha \beta^2 \sum_{i, \rho, l} r^{l}_{\rho} S^{\nu}_{i} S^{\nu}_{i} + \beta \sum_{i, \rho} m^{\nu}_{\rho} \frac{1}{cN} \sum_{l} \xi^{\nu}_{i} S^{\nu}_{i} + \beta R \sum_{i} S^{\nu}_{i} \right] \right\rangle \right\rangle. \quad (16)

Supposing Replica Symmetry (RS) ansatz, i.e., \( m^{\nu}_{\rho} = m^{\nu} \) for any replica index \( \rho \) and \( q^{l}_{\rho} = q^{l} \) for \( \rho \neq \sigma \), for the free energy, Eq. (7) we obtain:

\[
f = \frac{\alpha (1-a^2) + Ra}{2c} + \frac{\alpha}{2 \beta n} Tr \ln [A^k] + \frac{1}{2} \sum_{\nu, l} (m^{\nu}_{l})^2 - \frac{\alpha \beta^2 (1-a^2)}{2} \sum_{l, k} r_{l} q^{l} - \frac{1}{n \beta} \left\langle \left\langle \ln Tr_{S^\nu} \exp \left[ \frac{1}{2cN} \alpha \beta^2 (1-a^2) \sum_{i, \rho, l} r^{l}_{\rho} S^{\nu}_{i} S^{\nu}_{i} + \beta \sum_{i, \rho} m^{\nu}_{\rho} \frac{1}{cN} \sum_{l} \xi^{\nu}_{i} S^{\nu}_{i} + \beta R \sum_{i} S^{\nu}_{i} \right] \right\rangle \right\rangle. \quad (17)

The over-line in the last expression has its clear physical meaning as an average over the spatial distribution \( (.)_{\bar{i}} = \frac{1}{cN} \sum_{i} (.) \).

As a next step, let us suppose that the average over a finite number of patterns \( \xi^{\nu} \) can be self-averaged [12]. In our case this is expressed by the following decomposition \( S^{\nu}_{i} S^{\nu}_{i} b^{\nu}_{i} b^{\nu}_{i} \approx S^{\nu} S^{\nu} (b^{\nu})^2 \), which is reasonable to assume as a first approximation by taking into account the effect of the spatial distribution. This approximation permits to complete the analytical analysis.

After taking the trace \( Tr_{S^\nu} \) and the limit \( n \to 0 \), in Eq. (17) and using the saddle point method, we end up with the following expression for the free energy:

\[
f = \frac{1}{2c} \alpha (1-a^2) + \frac{1}{2} \sum_{k} (m_{k})^2 - \frac{\alpha \beta (1-a^2)}{2} \sum_{k} r_{k} q_{k} + \frac{\alpha \beta (1-a^2)}{2} \sum_{k} \mu_{k} r_{k} + \frac{\alpha}{2 \beta} \sum_{k} \ln (1-\beta (1-a^2) \mu_{k} + \beta (1-a^2) q_{k}) - \beta (1-a^2) q_{k} (1-\beta (1-a^2) \mu_{k} + \beta (1-a^2) q_{k})^{-1} - \frac{1}{\beta} \int \frac{dz e^{-z^2/2}}{\sqrt{2\pi}} \ln 2 \cosh \beta \left( \sqrt{\alpha (1-a^2)} \sum_{l} r_{l} b^{l}_{i} + \sum_{l} m_{l} \xi_{i} b^{l}_{i} + R b^{0}_{i} \right). \quad (18)

The equations for the OP \( r_{k}, m_{k} \) and \( q_{k} \) are respectively:

\[
r_{k} = \frac{q_{k} (1-a^2)}{(1-\beta (1-a^2) (\mu_{k} - q_{k}))^2}, \quad (19)

\[
m_{k} = \int \frac{dz e^{-z^2/2}}{\sqrt{2\pi}} \xi_{i} b^{k}_{i} \tanh \beta \left( \sqrt{\alpha (1-a^2)} \sum_{l} r_{l} b^{l}_{i} + \sum_{l} m_{l} \xi_{i} b^{l}_{i} + R b^{0}_{i} \right). \quad (20)
\]
and

\[ q_k = \int \frac{dze^{-z^2/2}}{\sqrt{2\pi}} (b_k^c)^2 \tanh^2 \beta \left( z \sqrt{\alpha(1-a^2)} \sum_l r_l b_l^c \sum_i m_i \xi_i^c b_i^c + \sum_i m_i \xi_i^c + R b_i^c \right). \]  \tag{21}

The following analysis refers to the case \( T = 0 \). Keeping \( C_k = \beta(\mu_k - q_k) \) finite and limiting the above system only to the first two \( m_k \)'s, the Eqs. \( [18\text{ ] [21} \] read:

\[ m_0 = \frac{1-a^2}{4\pi} \int_{-\pi}^{\pi} g(\phi) d\phi \] \tag{22}

\[ m_1 = \sqrt{2\mu_1} \frac{1-a^2}{4\pi} \int_{-\pi}^{\pi} g(\phi) \sin \phi \ d\phi \] \tag{23}

\[ C_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_c(\phi) d\phi \] \tag{24}

\[ C_1 = \frac{\mu_1}{\pi} \int_{-\pi}^{\pi} g_c(\phi) \sin^2 \phi \ d\phi \] \tag{25}

\[ r_k = \frac{\mu_k (1-a^2)}{[1-(1-a^2)C_k]^2}. \] \tag{26}

Here

\[ g(\phi) = \text{erf}(x_1) + \text{erf}(x_2) \] \tag{27}

\[ g_c(\phi) = \frac{[1+a]e^{-(x_1)^2} + (1-a)e^{-(x_2)^2}}{[\sqrt{\pi} y]} \] \tag{28}

\[ x_1 = \frac{[(1-a)(m_0 + m_1) \sqrt{2\mu_1} \sin \phi + R]/y}{[1]} \] \tag{29}

\[ x_2 = \frac{[(1+a)(m_0 + m_1) \sqrt{2\mu_1} \sin \phi - R]/y}{[1]} \] \tag{30}

\[ y = \sqrt{2\alpha(1-a^2)(r_0 + 2\mu_1 r_1 - 1 + a^2) \sin^2 \phi}. \] \tag{31}

For values of the order parameters \( m_1 = m_2 = \ldots = 0 \) and \( R = 0 \), we obtain a result similar to that of Canning and Gardner [18]. The classical result of Amit et al [15] is obtained when \( \mu_1 = 0 \).

The result of the numerical solution of Eqs. \( [22\text{ ] [26} \) is show in Fig. 2c,d), where we have rescaled the order parameters \( m_0 \) and \( m_1 \) to belong to the interval \([0,1]\), instead of the interval \([0,1-a]\), for any sparsity \( a \). The sharp bound of the phase transition is a result of taking into account just two terms of \( m_k \), \( k = 0,1 \) and the lack of finite-size effects in the thermodynamic limit. The corresponding behavior of \( m_0 \) and \( m_1 \), obtained by simulation, is presented in the top panels of Fig. 2a,b). The good correspondence between the numerical result and the simulations confirm our hypothesis of the finite number of macroscopic OP \( m_l \neq 0 \). The physical reason for this suggestion lies on the fact that the bump formations are compact structures with a large size and therefore the low-frequency components, e.g those corresponding to large eigenvalues of the matrix of connectivities, are responsible for the observed behavior.

### III. SIMULATIONS

We performed simulations for a network in form of a circular ring, with a distance between the neurons \( i \) and \( j \):

\[ |i-j| \equiv \min(i-j+N \ mod \ N, j-i+N \ mod \ N), \]

using three different topologies, explained in this section.

To compare with the results of Ref. [1], we work with a typical connectivity distance \( \sigma_x N \). This Gaussian-like distribution defines the first topology we are studying.

\[ P(c_{ij} = 1) = \frac{1}{\sqrt{2\pi \sigma_x N}} e^{-((i-j)/N)^2/2\sigma_x^2} + p_0, \]  \tag{32}

where \( p_0 \) is chosen to normalize the expression in the brackets. When \( \sigma_x \) is small enough, then spatial asymmetry is expected.

The dynamics of the network at time \( t+1 \) and temperature \( T = 0 \) for our case is

\[ S_i(t+1) = \text{sign} \left( \frac{1}{N} \sum_j \sum_{\mu} \xi_i^\mu \xi_j^\mu c_{ij} S_j(t) - T_h \right), \]
where $T_h$ is the threshold of the system, which is in general nonzero, due to the extra energy term $H_a$.

If the retrieval state, corresponding to the pattern $\xi^0_i$ is $S_i$, the mean overlap is $m_0 = \sum_i \xi^0_i S_i / N$ and the local overlap at site $i$ is

$$m(i) = \xi^0_i S_i.$$  \hspace{1cm} (33)

In the case when $S_i$ follow a single sine wave, the ideal measure of spatial asymmetry would be

$$m_1 = \frac{1}{N} \left| \sum_k \xi^0_k S_k e^{2\pi i k / N} \right|,$$

which corresponds to the theoretically derived OP $m_1$ from Eqs. (11,20).

Simulations with more sharp localized connectivity

$$P(c_{ij} = 1) \propto \frac{1 - b \cos \varphi}{1 - 2b \cos \varphi + b^2},$$  \hspace{1cm} (34)

with $\varphi = 2\pi |i - j| / N$ and $b$ being some parameter, show similar results. This connectivity defines the second type of topology we are studying. It has the advantage that the eigenvectors of the connectivity matrix are cosine waves and the eigenvalues are known. The results of the simulations for different $\sigma$ and $a$ are shown in Figs. 2b, 3b. These two topologies, Eqs. (32,34), give very similar results.

If $R = 0$, no asymmetry can be observed for any $\sigma_x$, up to the level of the network fragmentation (Fig. 3). No difference between asymmetric and symmetric connectivity is observable for any connectivity $c < 0.05$ and any of the topologies tested.

The sparse code increases the SAS effects: Fig. 4b, vs. Fig. 2a), but SAS can not be observed for any sparsity $a$ if the proportion of the firing neurons is kept to be equal to $a$, i.e. when the retrieval and the memorized states have the same level of activity. However, even without sparsity ($a = 0$) the bump states can be observed Fig. 4b).

One could expect that the bump formations actually will deteriorate the performance of the neural networks, because the bump is localized in a limited part of the network and this effectively shrinks the active part of the network. However, it results not to be so. Namely, if we start the retrieval process from the bump $m_0 < 1$, fixing $R$ in such a way that a bump will be produced, this retrieval is better than the retrieval achieved, starting from totally overlapping pattern $m_0 = 1$.

In the case of one-dimensional network given with Gaussian-like distribution of the connectivity on a circular ring, we observe the behavior, given in Fig. 5. In Fig. 5 we observe a pronounced stability of the position of the center and the boundaries of the bump, when using the bump initial condition, while in the case of using the whole pattern as an initial condition, this is no
longer true. In the last case, one observes a shrinking of the region where the SAS state is observed, given by the rapid decrease of the order parameter (red dashed line), which occurs for smaller values of the load compared to the case of initial bump attractor.

This behavior is repeated for three different values of the bump initial conditions $m_0 = 0.15, m_0 = 0.2$ and $m_0 = 0.3$, given in Figs. 3(a,b,c). The corresponding behaviors when starting from the whole pattern are given in Figs. 3(d,e,f).

Figs. 3 also show that the difference between the two behaviors is smaller when the size of the initial bump is smaller. For example, when an initial bump condition is $m_0 = 0.3$, then the region of stability of the solutions extends up to value of the load $\alpha = 0.35$, while it decreases up to $\alpha = 0.29$ for uniform initial condition. For $m_0 = 0.15$, these values are $\alpha = 0.25$ and $\alpha = 0.2$ respectively.

The last result clearly points out the role of the bump formations on the stability during the retrieval.

The comparison of the overlap in the case of initial conditions corresponding to bump of size $m_0 = 0.15$, uniform distribution $m_0 = 1$ and uniform initial condition with $m_0 = 0.15$ is given in Fig. 3.

From Fig. 3 it can be observed that the uniform initial condition with $m_0 = 0.15$ gives the worst result for the size of the retrieval solution. The overlap shrinks to zero for load less than $\alpha = 0.2$, while this value is $\alpha = 0.29$ when starting form the bump. We face again with the conclusion that the bumps are stable formations and enhance the performance of the network.

This result could serve as a basis of the biological relevance of the bump formations for the whole stability of the network to learn and to retrieve information efficiently.

Finally, we also performed simulations in the case of the small-world (SW) topology with re-wiring rate $\omega \in [0, 1]$:

$$P(c_{ij} = 1) = (1 - \omega)\theta(c - |i - j|/N) + \omega c, \quad (35)$$

where $\theta$ is the theta function. For $R = 0$, no SAS behavior is observed.

However, when $R \neq 0$, the simulations show more complex behavior with the appearance of several bump formations, as well as phases with no bumps but significant $m_1$. Fig. 4 It seems probable that the multiple bumps are due to the roughness of the SW eigenvalues distribution. This different behavior of the SW topology deserves future attention.
IV. DISCUSSION

The numerical analysis of Eqs. (22-26) gives a stable region for the solutions corresponding to bump formations for different values of the load $\alpha$, the sparsity $a$ and the retrieval asymmetry parameter $R$, shown in Fig. 5. As can be seen, the sparsity of the code $a$ enhances the SAS effect, although it is also observed for $a = 0$. 

FIG. 5: (Color online) The dynamics of the bump formations. On the vertical axis the center and the boundaries of the bump, as well as the asymptotic value of $m$ are given as a function of the load $\alpha$ for three different initial conditions $m_0 = 0.15, m_0 = 1$ (Fig. a and d), $m_0 = 0.2, m_0 = 1$ (Fig. b and e) and finally $m_0 = 0.3, m_0 = 1$ (Fig. c and f). More stable behavior is observed when starting from the bump (Fig. a, b, c), than starting from uniform initial condition (the pattern itself) (Fig. d, e, f). The parameters of the simulations are $N = 6400, c = 0.05$ and $\sigma_x = 500$.

FIG. 6: (Color online) The behavior of the overlap for three values of the initial conditions: starting from the bump formation with $m_0 = 0.15$, (Fig. a), from the pattern itself (uniform distribution $m_0 = 1$), (Fig. b) and from uniform distribution with $m_0 = 0.15$, (Fig. c). A clear advantage of the first initial condition is observed with respect to the the size of the region of retrieval. The parameters of the simulations are $N = 6400, c = 0.05$ and $\sigma_x = 500$.

FIG. 7: (Color online) Smoothed local activity for small-world connectivity. In the panel (a) no bumpiness, but significant fluctuations are present ($m_1 = 0.1m_0$). In the panel (b) single bump is presented. And the last panel (c) shows multiple bump activity. $N = 6400, c = 0.05$ and $a = 0.6$ (a), $a = 0.7$ (b) and $a = 0.8$ (c), re-wire rate $\omega = 0.06, R = 0.5$. The same effect can be achieved by changing $(c, \omega)$. 
FIG. 8: (Color online) Phase diagram $R$ versus $a$ for $\alpha = 0.001$ ($\alpha/c = 0.02$). The SAS region, with $m_0 \neq 0, m_1 \neq 0$ is relatively large and decreases by increasing the load (bottom). Note that not all of the values of $\alpha$ are feasible with arbitrary mean connectivity of the network $c$.

FIG. 9: (Color online) The critical storage capacity $\alpha_c$ for $a = 0.4$. A drop of the capacity is observed on the transition between normal retrieval state and state with localized activity.

As we expected, the asymmetry factor $R$ between the stored and retrieved patterns is very important in order to have spatial asymmetry. The diagram in Fig. 8 shows a clear phase transition with $R$. For small values of $R$, the effect (SAS) is not present. If $R > 1$, then the only stable state is the trivial $Z$ state, as all the nonzero solutions are suppressed. Only for intermediate values of $R$, the bump solution exists.

When increasing the load, the area corresponding to SAS states shrinks as is shown in Fig. 8(b). This is due primary to the fact that the critical capacity $\alpha_c$ of the network drops when the system enters into the SAS state. More precisely, the behavior of the critical storage capacity as a function of the asymmetry parameter $R$, shown in Fig. 9 presents a drastic drop of its value at the transition from homogeneous retrieval (symmetric) state to spatially localized (asymmetric state). Effectively, only the fraction of the network in the bump can be excited and the storage capacity drops proportionally to the size of the bump. For the values of the sparsity of the code $a = 0.4$, where the effect of the asymmetry parameter is maximal, the decrease of the critical capacity is approximately twofold.

We have also investigated the effect of the parameters $R$ and $a$ on the bump formation. For this, we introduce a parameter that can give a criterion for the bumpiness as the relative ratio between the two order parameters $m_0$ and $m_1$, given by Eqs. (22-26):

$$B = \sqrt{\frac{m_1^2}{m_0^2 + m_1^2}}.$$  

The behavior of $B$ is presented in Fig. 10. It is seen that the simultaneous increase of both $R$ and $a$ inside the area where the SAS behavior is observed, favors the bumpiness in the binary network. This behavior is due to the stronger localization of the activity as the extra term $H_a$ favors states with lower total activity. From the other side, increasing $a$ makes the code sparser, which favors the metrical organization of both patterns and neurons.

Finally we have investigated the information capacity of the network. For this aim we calculated the corresponding expression for the mutual information in the case of order parameters $m_0$ and $m_1$:
Here \( F[\cdot] \) is the hyper geometric function \([21]\). The first term is the normal expression for the mutual information (the sparsity of the code eliminates the symmetry between \( m \) and \(-m\)). The next two lines of the equation can be regarded as a correction to the usual expression due to OP \( m_1 \neq 0 \).

With the aim of Eqs. \([22, 23]\) and the above expression, for the information capacity \( I_M(cN) \) we obtained the behavior, presented in Fig.[11] which has been also reproduced by simulations. This behavior is non trivial and shows a well pronounced maximum for intermediate values of the sparsity of the code \( a \). The corresponding behavior of the order parameters \( m_0 \) and \( m_1 \), presented in Fig.[11]b, shows that inside the SAS region, their values are significantly different form zero, but the information capacity is very small although non zero. If from one side, the information capacity \( I_M(cN) \), measures the number of bits, needed to represent the state and if from the other side, the overlap parameters \( (m_0, m_1) \) measure the degree to which one learned state can be recovered, then having small information capacity and large OP \( m_0, m_1 \) means that the original state can be recovered well with very small amount of information. This result could be useful for effective retrieval of patterns by using very small amount of information, especially in combination with the observed improvement of the stability of the network, provided that the retrieval starts from the bump.

Similar results have been reported in \([2]\) with respect to the increase of the computational power of the structured network. The last is expressed by the ability of the network to retrieve several patterns, each in a different location, whose combination could form the global combinatorially large pattern.

Apart of the advantage of the localized states in terms of the minimal cost in transferring information, such structures could have important biological relevance for the cortical modules of the mammalian brain, which are characterized by geometrical distribution of the synaptic connections. Recent experiments of the activity of a macaque temporal cortex \([22]\) have demonstrated that the receptive fields of the visual evoked activity patterns are restricted up to three fold, when several patterns are present together. The last result seems to be coherent with the picture of localized activity in a neural network with metric organization of the synaptic activities.

V. CONCLUSION

In this paper we have studied the spatially dependent activity in a binary neural network model when retrieval and learning states have certain degree of asymmetry. We have shown that neither asymmetry of the connections, nor asymmetry of the code are necessary to achieve bump solutions for the activity of the network, but rather different symmetry between the retrieval and the learning states.

The extension of the classical method of Amit et al. \([13]\) in the case of spatially dependent connectivity has permitted, within some approximations, the derivation of the above results analytically. The good correspondence with simulations, tested for different topologies of the network, is also discussed through the paper.

We have shown that the region of the spatially asymmetric states is relatively large in the parameter space \((R, a)\) and we have measured quantitatively how the bumpiness of the network depends on these two parameters.

A drop of the critical storage capacity is observed in the transition form symmetric to SAS states, which is due to the fact that effectively only the fraction of the network in the bump can be excited and the storage capacity drops.
proportionally to the size of the bump. When the recovering of the pattern starts from a state corresponding to the bump formation, the network behaves in a better way in terms of stability and capacity, compared to the case of uniform initial conditions. This happens even if the overlap with the uniform initial conditions is significantly larger. The latter result could argue the biological relevance of the bump formations for effective retrieval of information.

Finally we have discussed the behavior of the information capacity versus the asymmetry retrieval factor $R$ and we have discussed its sharp decrease in the parametric region, where bump formations are present.

Acknowledgments

The authors thank A.Treves and Y.Roudi for stimulating discussions at the early stage of work as well as the Abdus Salam Center for Theoretical Physics for the financial support at the beginning of the present investigation. E.K. and K.K. also acknowledge the financial support from the Spanish Grants DGI.M.CyT.FIS2005-1729 and TIN 2004-07676-G01-01 respectively.

[1] Y.Roudi and A.Treves, JSTAT, P07010, 1 (2004).
[2] Y.Roudi and A.Treves, cond-mat/0505349.
[3] K.Koroutchev and E.Korutcheva, Preprint ICTP, Trieste, Italy, IC/2004/91, 1 (2004).
[4] K.Koroutchev and E.Korutcheva, Central Europ. J.Phys., 3, 409 (2005).
[5] A.Anishchenko, E.Bienenstock and A.Treves, q-bio.NC/0502003.
[6] J.Rubin and A.Bose, Network: Comput.Neural Syst., 15, 133 (2004).
[7] N.Brunel, Cereb.Cortex, 13, 1151 (2003).
[8] V.Breitenberg and A.Schuz, Anatomy of the Cortex, (Springer, Berlin,1991).
[9] D.J.Watts and S.H.Strogatz, Nature, 393, 440 (1998).
[10] D.J.Watts, Small Worlds: The Dynamics of Networks Between Order and Randomness(Princeton Review in Complexity) (Princeton University Press, Princeton, New Jersey, 1999).
[11] M.Tsodyks and M.Feigel’man, Europhys.Lett., 6, 101 (1988).
[12] D.Bollé, G.Shim, B.Vinck and V.A.Zagrebnov, Journal of Stat.Phys., 14, 565 (1994).
[13] J.Hopfield, Proc.Natl.Acad.Sci.USA, 79, 2554 (1982).
[14] D.Hebb, The Organization of Behavior: A Neuropsychological Theory, (Wiley, New York, 1949).
[15] D.Amit, H.Gutfreund and H.Sompolinsky, Ann.Phys., 173, 30 (1987).
[16] N.N.Bogolyubov, Physica(Suppl.), 26, S1 (1960).
[17] M.Mézard, G.Parisi and M.-A.Virasoro, Spin-glass theory and beyond, (World Scientific, Singapore, 1987).
[18] A.Canning and E.Gardner, Phys.A:Math.Gen., 21, 3275 (1988).
[19] I.Farkas et al., Phys. Rev.E 64, 026704 (2001).
[20] R.Blahut, Principle and Practice of Information Theory, (Addison-Wesley, Cambridge MA, 1988).
[21] M.Abramowitz and I.A.Stegun, Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables, (National Bureau of Standards, Applied Mathematics Series-55, 1972).
[22] E.Rolls, N.Aggelopoulos and F.Zheng, J.Neurosci. 23, 339 (2003).