A study and an application of the concentration compactness type principle

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Abstract

In this article we develop a concentration compactness type principle in a variable exponent setup. As an application of this principle we discuss a problem involving fractional ‘$(p(x), p^+)$-Laplacian’ and power nonlinearities with exponents $(p^+)^*$, $p_*(x)$ with the assumption that the critical set $\{ x \in \Omega : p_*(x) = (p^+)^* \}$ is nonempty.

**keywords**: Concentration compactness principle, fractional Sobolev space with variable exponent, fractional $p(x)$-Laplace operator, critical exponent.

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1 Introduction

One of the most important theoretical developments in the theory of elliptic PDEs is due to the work of P. L. Lions ([16] in 1984, [17] and [18] in 1985). In his work he introduced the notion of concentration compactness principle (CCP) which became a fundamental method to show the
existence of solutions of variational problems involving critical Sobolev exponents. A strong reason for the popularity of this principle is because it could address a way to compensate for the lack of compact embeddings amongst certain function spaces, which mostly resulted due to the presence of a critical exponent or due to the consideration of an unbounded domain. It aided to examine the nature of weakly convergent subsequence and determine the energy levels of variational problems below which the Palais-Smale condition is satisfied. Lions [17] gave a systematic theory to handle the issue of loss of compactness not only when it is lost due to translations but also because of the invariance of $\mathbb{R}^N$, for instance, by the non-compact group of dilations.

Later, in 1995, Chabrowski [9] extended the result of Lions for semilinear elliptic equations with critical and subcritical Sobolev exponent but at infinity. Palatucci [22] developed a CCP which can be applicable to study a PDE involving a fractional Laplacian and a critical exponent term. At this point, we also refer the reader to the noteworthy work of Dipierro et al [10] (Proposition 3.2.3). Mosconi et al, further generalized the result due to [22] which can be used to analyse equations involving fractional $p$-Laplacian with a critical growth [20] and with a nearly critical growth [21]. A CCP was recently proposed by Bonder et al. [7], which can be used to study problems involving the fractional $p$-Laplacian operator for $1 < p < \frac{N}{s}$ in unbounded domain. It is also noteworthy to refer to the problem addressed by the authors in [3] where they have discussed the existence of multiple nontrivial solutions of $(p, q)$ fractional Laplacian equations involving concave-critical type nonlinearities using CCP. An advanced version of CCP of P. L. Lions is obtained by Fu [12] for variable exponent case dealing with Dirichlet problems involving $p(x)$-Laplacian with critical exponent $p^*(x) = \frac{Np(x)}{N-p(x)}$. Moreover, Bonder and Silva [6] developed a more general result for the variable exponent case where the exponent does not require to be critical everywhere. The author worked with the exponent $q(x)$ considering the set $\{x \in \Omega : q(x) = p^*(x)\}$ to be nonempty.

In the recent years, an increased interest among the researchers has been observed to the study of the following type of elliptic equations.

$$(-\Delta)^s_{p(x)} u = g(x, u) \text{ in } \Omega,$$
$$u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,$$

where $(-\Delta)^s_{p(x)}$ is the fractional $p(x)$-Laplacian, the domain $\Omega$ is bounded in $\mathbb{R}^N$, $p(., .)$ is a bounded, continuous symmetric real valued function over $\mathbb{R}^N \times \mathbb{R}^N$ and the function $g$ has a subcritical growth. The solution space for the problem in (1.1) is the fractional Sobolev space with variable exponent which is defined in Section 2. Readers may refer [1], [2], [5], [14], [15] and the references therein for further readings on problems of the type as in (1.1). The present work is new in the sense that- to our knowledge- there is no existence result for the problem

$$(-\Delta)^s_{p(x)} u + (-\Delta)^s_{p(x)} u = |u|^{(p^+)^*-2}u + |u|^{p^*_s(x)-2}u + \lambda |u|^\beta(x)-2u \text{ in } \Omega,$$
$$u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,$$

where $s \in (0, 1)$, $\lambda > 0$, $1 < p(x, y) \leq p^+ = \sup_{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N} p(x, y) < \infty$, $p^*_s(x) = \frac{Np(x, x)}{N-sp(x, x)}$, $(p^+)^* = \frac{Np^+}{N-sp^+}$ and the function $\beta$ appears with a subcritical growth. Due to the lack of
continuous embedding in case of variable fractional critical exponent $p^*_s(x)$, i.e., between $W^{s,q(\cdot)}(\Omega)$ and $L^{p^*_s(\cdot)}(\Omega)$, it is difficult to prove the concentration compactness principle for fractional Sobolev space with variable exponent. The novelty of this work lies in our usage of two critical exponents $q^*, r(x)$ with the assumptions that $1 < q < \inf r(x) \leq r(x) \leq q^* < \infty$ and the critical set $\{ x \in \Omega : r(x) = q^* \}$ is nonempty in deriving a concentration compactness type principle (CCTP). This is an important finding owing to its importance in studying the existence results in elliptic PDEs of the type $(1.2)$. We have further discussed the problem $(1.2)$ in Section 4 as an application to this principle for a Dirichlet problem involving fractional-\'(1.2) in Section 4 as an application to this principle for a Dirichlet problem involving fractional-\' and the constant $\mu$ measures the functional '($1.2) in Section 4 as an application to this principle for a Dirichlet problem involving fractional-\' (CCTP). This is an important finding owing to its importance in studying the existence results in elliptic PDEs of the type $(1.2)$. We have further discussed the problem $(1.2)$ in Section 4 as an application to this principle for a Dirichlet problem involving fractional-\' with the critical exponents $(p^+)^*$ and $p^*_s(x)$ assuming the critical set $\{ x \in \Omega : p^*_s(x) = (p^+)^* \}$ to be nonempty.

1.1 Statements of the main results

Following the original method discovered by P. L. Lions [17], we derive a concentration compactness type principle which is given in Theorem 1.1 below. The proof of this is in Section 3.

Theorem 1.1. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $s \in (0, 1)$, $q \in (1, \infty)$, $sq < N$ and $r(\cdot)$ be a bounded continuous function in $\mathbb{R}^N$ such that

$$1 < q < r^- = \inf_{x \in \mathbb{R}^N} r(x) \leq r(x) \leq \sup_{x \in \mathbb{R}^N} r(x) = r^+ \leq q^* = \frac{Nq}{N-sq} < \infty.$$ 

Let $\{u_n\}$ be a bounded sequence in $W^{s,q}_0(\Omega)$, then there exist $u \in W^{s,q}_0(\Omega)$ and bounded regular measures $\mu, \nu_1, \nu_2$ such that, up to a subsequence,

$$u_n \rightharpoonup u \text{ weakly in } W^{s,q}_0(\Omega) \text{ and strongly in } L^{\beta(x)}(\mathbb{R}^N) \text{ for every } 1 < \beta(x) < q^*;$$

$$\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^q}{|x-y|^{N+sq}} dy \leq \mu, \quad |u_n|^q \rightharpoonup \nu_1, \quad |u_n|^{r(x)} \rightharpoonup \nu_2$$

(1.3)

where $\rightharpoonup$ denotes the tight convergence. Define a measure $\nu$ as $\nu = \nu_1 + \nu_2$ and assume that the critical set $A_r = \{ x \in \Omega : r(x) = q^* \} \neq \emptyset$. Then for some countable set $I$ we have

$$\mu = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^q}{|x-y|^{N+sq}} dy + \sum_{i \in I} \mu_i \delta_{x_i}, \quad \mu_i = \mu(\{x_i\});$$

(1.4)

$$\nu = |u|^{q^*} + |u|^{r(x)} + \sum_{i \in I} \nu_i \delta_{x_i}, \quad \nu_i = \nu(\{x_i\});$$

(1.5)

$$2^{-\frac{q^*}{q}} S \min \left( \nu_i^{\frac{1}{q^*}}, \nu_i^{\frac{1}{r}} \right) \leq \mu_i^{\frac{1}{q}}, \quad \forall i \in I$$

(1.6)

where $\{x_i : i \in I\}$ is a set of distinct points in $\mathbb{R}^N$, $\{\nu_i : i \in I\} \in (0, \infty)$, $\{\mu_i : i \in I\} \in (0, \infty)$ and the constant $S = S(N, s, p, r, \Omega) > 0$ is a Sobolev constant defined as

$$S = \inf_{u \in W^{s,q}_0(\Omega) \setminus \{0\}} \frac{\|u\|_{s,q}^2}{\|u\|_{L^{q^*}(\Omega)} + \|u\|_{L^{r(x)}}(\Omega)}.$$ 

(1.7)
Remark 1.2. The definition of tight convergence is given in the Section 3.

As an application of Theorem 1.1 we will prove the existence of nontrivial weak solution of the following nonlocal problem with critical growth,

\[
(-\Delta)^{s}_{p(x)} u + (-\Delta)^{s}_{p(x)} u = |u|^{(p^+)^*-2} u + |u|^{p^*_s(x)-2} u + \lambda |u|^\beta(x)-2 u \quad \text{in } \Omega,
\]

\[u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,\]

in Section 4 where \((-\Delta)^{s}_{p(x)}\) is the fractional \(p(x)\)-Laplacian defined as

\[
(-\Delta)^{s}_{p(x)} u = P.V. \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))}{|x-y|^{N+sp(x,y)}} dy.
\]

The result is stated in the form of a theorem as follows.

Theorem 1.3. Let \(s \in (0,1), \lambda > 0, p(\cdot, \cdot)\) be a continuous symmetric function in \(\mathbb{R}^N \times \mathbb{R}^N\) such that \(1 < p^- = \inf_{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N} p(x,y) \leq p(x,y) \leq p^+ = \sup_{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N} p(x,y) < \infty, sp^+ < N\) and \(p^+ < (p^*_s)^- = \inf_{x \in \mathbb{R}^N} p^*_s(x), \) where \(p^*_s(x) = \frac{Np(x,x)}{sp^*(x)}\). Further, we assume that the critical set \(A = \{x \in \Omega : p^*_s(x) = (p^*)^s\} \neq \emptyset\) and \(\beta \in C_+(\overline{\Omega})\) such that \(p^+ < \beta^- \leq \beta^+ < (p^*_s)^-\). Then there exists a \(\Lambda > 0\) depending on \(p, \beta, N, s, \Omega, S\) such that for \(\lambda > \Lambda\), the problem (1.8) admits a nontrivial weak solution in \(W_0 \cap W^{s,p^+}_0(\Omega)\).

The notations and important results will be discussed in the succeeding section.

2 Important results on Sobolev spaces with variable exponent

Let \(\Omega\) be a bounded domain in \(\mathbb{R}^N\) and denote

\[
C_+([-\Omega \times \Omega]) = \{f \in C([-\Omega \times \Omega]) : 1 < f^- \leq f(x,y) \leq f^+ < \infty, \forall (x,y) \in [-\Omega \times \Omega]\}
\]

where \(f^+ = \sup_{[-\Omega \times \Omega]} f(x,y) \) (or \(\sup_{[-\Omega \times \Omega]} f(x)\)), \(f^- = \inf_{[-\Omega \times \Omega]} f(x,y) \) (or \(\inf_{[-\Omega \times \Omega]} f(x)\)).

Let \(p \in C_+(\overline{\Omega})\) and \(\nu\) be a complete, \(\sigma\)-finite measure in \(\Omega\). The Lebesgue space with variable exponent is defined as

\[
L^{p(\cdot)}_\nu(\Omega) = \{u : \Omega \to \mathbb{R} \text{ is } \nu \text{ measurable : } \int_{\Omega} |u|^{p(\cdot)} d\nu < \infty\}
\]

which is a Banach space endowed with the norm (see [11])

\[
\|u\|_{L^{p(\cdot)}_\nu(\Omega)} = \inf \{\eta \in \mathbb{R}^+ : \int_{\Omega} \frac{|u(x)|^{p(\cdot)}}{\eta} d\nu < 1\}.
\]

For \(d\nu = dx\) we will denote the Lebesgue space with variable exponent as \(L^{p(\cdot)}(\Omega)\) whose norm will be denoted by \(\|u\|_{L^{p(\cdot)}(\Omega)}\).

We now give a few more notations and state Propositions which will be referred to henceforth very often.
Proposition 2.1 ([11], Proposition 2.1). Let \( f, g \in C_+(\overline{\Omega}) \) with \( f(x) \leq g(x) \) for every \( x \in \overline{\Omega} \). Then
\[
\|u\|_{L^p(\Omega)} \leq 2[1 + \nu(\Omega)]\|u\|_{L^p(\Omega)}, \quad \forall u \in L^p(\Omega) \cap L^q(\Omega).
\]

Proposition 2.2 ([11]). 1. (Hölder Inequality) Let \( \alpha, \theta, \gamma : \overline{\Omega} \to [1, \infty] \) with \( \frac{1}{\alpha(x)} = \frac{1}{\theta(x)} + \frac{1}{\gamma(x)} \). If \( h \in L^{\alpha}(\Omega) \) and \( f \in L^{\theta}(\Omega) \), then
\[
\|hf\|_{L^{\gamma}(\Omega)} \leq C\|h\|_{L^{\alpha}(\Omega)}\|f\|_{L^{\theta}(\Omega)}.
\]
2. If \( p, q \in C_+(\overline{\Omega}) \) and \( p(x) \leq q(x) \), for \( x \in \overline{\Omega} \), then \( L^p(\Omega) \hookrightarrow L^q(\Omega) \) and this embedding is continuous.

We fix the exponents \( 0 < s < 1 \), \( p \in C_+(\overline{\Omega} \times \overline{\Omega}) \), \( q \in C_+(\overline{\Omega}) \) and assume that \( p(\cdot, \cdot) \) is a symmetric function, \( p(x, y) = p(y, x) \). We define the fractional Sobolev space with variable exponent and the corresponding Gagliardo seminorm as (see [2])
\[
W^{s, q(\cdot), p(\cdot)}(\Omega)
= \{ u \in L^q(\Omega) : \int_\Omega \int_{\Omega} \frac{|u(x) - u(y)|^{p(x, y)}\|x - y\|^{N+sp(x, y)}}{\eta^{p(x, y)}\|x - y\|^{N+sp(x, y)}} \, dy \, dx < \infty, \text{ for some } \eta \in \mathbb{R}^+ \}
\]
and
\[
[u]^{s, p(\cdot)}_\Omega = \inf \{ \eta \in \mathbb{R}^+ : \int_\Omega \int_{\Omega} \frac{|u(x) - u(y)|^{p(x, y)}}{\eta^{p(x, y)}\|x - y\|^{N+sp(x, y)}} \, dy \, dx < 1 \}.
\]
The space \( W^{s, q(\cdot), p(\cdot)}(\Omega) \) is a Banach space equipped with the norm
\[
\|u\|_{W^{s, q(\cdot), p(\cdot)}(\Omega)} = \|u\|_{L^q(\Omega)} + [u]^{s, p(\cdot)}_\Omega.
\]
One can continuously extend \( p \) to \( \mathbb{R}^N \times \mathbb{R}^N \) and \( q \) to \( \mathbb{R}^N \), using the Tietze extension theorem, such that \( p \in C_+(\mathbb{R}^N \times \mathbb{R}^N) \) and \( q \in C_+(\mathbb{R}^N) \) respectively. We now address another type of variable exponent fractional Sobolev space, denoted as \( W \), by
\[
W = \{ u : \mathbb{R}^N \to \mathbb{R} : u|_\Omega \in L^q(\Omega) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x, y)}}{\eta^{p(x, y)}\|x - y\|^{N+sp(x, y)}} \, dy \, dx < \infty, \text{ for some } \eta > 0 \}.
\]
The corresponding norm is given by
\[
\|u\|_W = \|u\|_{L^q(\Omega)} + \inf \{ \eta : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x, y)}}{\eta^{p(x, y)}\|x - y\|^{N+sp(x, y)}} \, dy \, dx < 1 \}.
\]
We define a special subspace of \( W \), denoted as \( W_0 \), as follows
\[
W_0 = \{ u \in W : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}.
\]
The norm on \( W_0 \) is defined as
\[
\|u\|_{W_0} = \inf \{ \eta \in \mathbb{R}^+ : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x, y)}}{\eta^{p(x, y)}\|x - y\|^{N+sp(x, y)}} \, dy \, dx < 1 \}.
\]
Lemma 2.3 ([5]). Let $u, u_k \in W_0$, $k \in \mathbb{N}$ and define the modular function as

$$
\rho_{W_0}(u) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^q}{|x - y|^{N + sq}} dydx.
$$

Then we have the following relation between the modular function and the norm.

1. $\|u\|_{W_0} = \eta \iff \rho_{W_0}(\frac{u}{\eta}) = 1$.
2. $\|u\|_{W_0} > 1 (< 1, = 1) \iff \rho_{W_0}(u) > 1 (< 1, = 1)$.
3. $\|u\|_{W_0} > 1 \implies \|u\|_{W_0}^p \leq \rho_{W_0}(u) \leq \|u\|_{W_0}^q$.
4. $\|u\|_{W_0} < 1 \implies \|u\|_{W_0}^p \leq \rho_{W_0}(u) \leq \|u\|_{W_0}^q$.
5. $\lim_{k \to \infty} \|u_k - u\|_{W_0} = 0 \iff \lim_{k \to \infty} \rho_{W_0}(u_k - u) = 0$.

Remark 2.4. The notion of fractional Sobolev space with variable exponent is a generalization of fractional Sobolev space with constant exponent. Let $q \in (1, \infty)$, then we denote the fractional Sobolev space

$$W_0^{s,q}(\Omega) = \{ u \in L^q(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^q}{|x - y|^{N + sq}} dydx < \infty, \ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \}$$

equipped with the norm

$$\|u\|_{s,q}^q = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^q}{|x - y|^{N + sq}} dydx.$$

Given below are a few well known Propositions and Theorems in the literature.

Proposition 2.5 ([5]). The spaces $(W_0, \|\cdot\|_{W_0})$ and $(W_0^{s,q}(\Omega), \|\cdot\|_{s,q})$ are reflexive, uniformly convex Banach spaces.

Theorem 2.6 (Theorem 6.5, [25]). Let $0 < s < 1$ and $q \in [1, \infty)$ with $sq < N$. Then there exists a constant $C > 0$ depending on $N, s, q$ such that for any measurable and compactly supported function $u : \mathbb{R}^N \to \mathbb{R}$ we have

$$C\|u\|_{L^r(\mathbb{R}^N)} \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^q}{|x - y|^{N + sq}} dydx$$

for any $r \in [q, q_s^*]$, where $q_s^* = \frac{Nq}{N - sq}$ is the fractional Sobolev critical exponent. Moreover, the space $W^{s,q}(\mathbb{R}^N)$ is continuously embedded in $L^r(\mathbb{R}^N)$ for every $r \in [1, q_s^*]$.

Theorem 2.7 ([5]). Let us assume $0 < s < 1$, $p \in C_+(\mathbb{R}^N \times \mathbb{R}^N)$, $q \in C_+(\mathbb{R}^N)$ such that $sp(x, y) < N$ for every $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ and $p(x, x) \leq q(x) < p_s^*(x) = \frac{Np(x, x)}{N - sp(x, x)}$ for every $x \in \mathbb{R}^N$. Then for any $\beta \in C_+(\mathbb{R}^N)$, $1 < \beta(x) < p_s^*(x)$, there exists $C > 0$ depending on $p, s, q, N, \Omega$, and $\beta$ such that for every $u \in W_0$,

$$\|u\|_{L^{\beta(\cdot)}(\mathbb{R}^N)} = \|u\|_{L^{\beta(\cdot)}(\Omega)} \leq C\|u\|_{W_0}.$$ 

Moreover, the embedding from $W_0$ to $L^{\beta(\cdot)}(\Omega)$ is continuous and also compact.
Lemma 2.9. 1. The space $W^{s,p}_0$ endowed with the norm
$$\|u\|_X = \|u\|_{s,p} + \|u\|_{W_0}.$$

2. The embedding $X \hookrightarrow L^{r}(\mathbb{R}^N)$ is continuous for any continuous function $1 < r(x) \leq (p^+)^*$ and is compact whenever $1 < r(x) < (p^+)^*$.

The proof of the above lemma is a straightforward application of Proposition 2.2, Proposition 2.3 and Theorem 2.6.

3 Concentration compactness type principle

In this section, we establish the concentration compactness type principle. We now prove our main result with the help of a few properties proved below. The following two lemmas provide the decay estimate and the scaling property of compactly supported nonlocal gradient of smooth functions.

Corollary 3.1 (7). Let $\phi \in W^{1,\infty}(\mathbb{R}^N)$ such that support of $\phi$ lies in the unit ball of $\mathbb{R}^N$ and given $\epsilon > 0$, $x_0 \in \mathbb{R}^N$ define $\phi_{\epsilon,x_0}(x) = \phi(\frac{x-x_0}{\epsilon})$. Then
$$\int_{\mathbb{R}^N} \frac{|\phi_{\epsilon,x_0}(x) - \phi_{\epsilon,x_0}(y)|^p}{|x - y|^{N+sp}} dy \leq C \min (\epsilon^{-sp}, \epsilon^N|x-x_0|^{-(N+sp)})$$
where $C$ depends on $N, s, p, \|\phi\|_{1,\infty}$.

We now state and prove the following Lemma.

Lemma 3.2. Let $1 < p^- \leq p(x,y) \leq p^+ < \infty$ for every $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, $sp^+ < N$, $\phi \in C^\infty_c(\mathbb{R}^N)$ such that $0 \leq \phi \leq 1$, $\phi(0) = 1$ and support of $\phi$ lies in the unit ball of $\mathbb{R}^N$. For some $x_0 \in \mathbb{R}^N$ and $\epsilon > 0$, define $\phi_{\epsilon,x_0}(x) = \phi(\frac{x-x_0}{\epsilon})$. Then
$$\int_{\mathbb{R}^N} \frac{|\phi_{\epsilon,x_0}(x) - \phi_{\epsilon,x_0}(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dy \leq C \min \left(\frac{1}{\epsilon^{sp^+}} + \frac{1}{\epsilon^{sp^-}}, (\epsilon^N + \epsilon^{N+(p^+)+})|x-x_0|^{-(N+sp^-)}\right)$$
where $C$ depends on $N, s, p, \|\phi\|_{1,\infty}$.

Proof. We first observe that
$$\int_{\mathbb{R}^N} \frac{|\phi_{\epsilon,x_0}(x) - \phi_{\epsilon,x_0}(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dy = \int_{\mathbb{R}^N} \frac{|\phi(\frac{x-x_0}{\epsilon}) - \phi(\frac{y-x_0}{\epsilon})|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dy \leq \left(\frac{1}{\epsilon^{sp^+}} + \frac{1}{\epsilon^{sp^-}}\right) \int_{\mathbb{R}^N} \frac{|\phi'(x') - \phi'(y')|^{p(x'+\epsilon_0,y'+\epsilon_0)}}{|x' - y'|^{N+sp(x'+\epsilon_0,y'+\epsilon_0)}} dy'. \quad (3.9)$$
Denote $\tilde{p}(x', y') = p(x' + \epsilon x_0, y' + \epsilon x_0)$ and decompose the integral on the right hand side of (3.9) as follows.

$$
\int_{\mathbb{R}^N} \frac{|\phi(x') - \phi(y')| \tilde{p}(x', y')}{|x' - y'|^{N + sp(x', y')}} dy' = \left( \int_{|x' - y'| \geq 1} + \int_{|x' - y'| < 1} \right) \frac{|\phi(x') - \phi(y')| \tilde{p}(x', y')}{|x' - y'|^{N + sp(x', y')}} dy' = I + II.
$$

We try to find $L^\infty$ bounds of these two integrals.

$$
I = \int_{|x' - y'| \geq 1} \frac{|\phi(x') - \phi(y')| \tilde{p}(x', y')}{|x' - y'|^{N + sp(x', y')}} dy' \leq C \| \phi \|_\infty \int_{|x' - y'| \geq 1} \frac{1}{|x' - y'|^{N + sp}} dy' \leq C \frac{\| \phi \|_\infty}{sp} \tag{3.10}
$$

and

$$
II = \int_{|x' - y'| < 1} \frac{|\phi(x') - \phi(y')| \tilde{p}(x', y')}{|x' - y'|^{N + sp(x', y')}} dy' \leq \| \nabla \phi \|_\infty \int_{|x' - y'| < 1} \frac{1}{|x' - y'|^{N + sp(x', y') - \tilde{p}(x', y')}} dy' \leq \| \nabla \phi \|_\infty \int_{|x' - y'| < 1} \frac{1}{|x' - y'|^{N - p(1 - s)}} dy' \leq C \frac{\| \nabla \phi \|_\infty}{p^- (1 - s)} \tag{3.11}
$$

In order to obtain a decay estimate, we restrict ourselves to the case where $|x'| > 2$ such that $\phi(x') = 0$. Hence, we observe that $|x' - y'| \geq |x'| - 1 \geq |x'|/2$ and

$$
\int_{\mathbb{R}^N} \frac{|\phi(x') - \phi(y')| \tilde{p}(x', y')}{|x' - y'|^{N + sp(x', y')}} dy' = \int_{\mathbb{R}^N} \frac{|\phi(y')| \tilde{p}(x', y')}{|x' - y'|^{N + sp(x', y')}} dy' \leq \int_{|y'| \leq 1} \frac{|\phi(y')| \tilde{p}(x', y')}{|x'|^{N + sp(x', y')}} dy' \leq C \frac{\| \phi \|_\infty}{|x'|^{N + sp^-}} \tag{3.12}
$$

Combining (3.9), (3.10), (3.11) and (3.12) we get

$$
\int_{\mathbb{R}^N} \frac{\phi_{\epsilon, x_0}(x) - \phi_{\epsilon, x_0}(y)}{|x - y|^{N + sp(x, y)}} dy \leq C \left( 1/\epsilon^{sp^+} + 1/\epsilon^{sp^-} \right) \min \left( 1, \frac{|x - x_0|}{\epsilon} \right) \left( |x - x_0|^{- (N + sp^-)} \right) \leq C \min \left( 1/\epsilon^{sp^+} + 1/\epsilon^{sp^-}, (\epsilon^N + \epsilon^{N + sp^+}) |x - x_0|^{- (N + sp^-)} \right)
$$

where $C > 0$ depends on $N, s, p$ and $\| \phi \|_\infty$.  \qed
The next lemma plays a major role in the proof of Theorem 1.

**Lemma 3.3.** Let $\mu \geq 0$, $\nu \geq 0$ be two bounded measures and $1 < q < r \leq q^* < \infty$. Assume that there exists $C > 0$ such that for all $\phi \in C_c^\infty(\mathbb{R}^N)$,

$$C \min \left( \|\phi\|_{L_q^*(\mathbb{R}^N)}, \|\phi\|_{L_q^*(\mathbb{R}^N)}^{\frac{q^*}{r}} \right) \leq \|\phi\|_{L_q^*(\mathbb{R}^N)}. \quad (3.13)$$

Then, we can find an atmost countable set $I$, a collection $\{x_i : i \in I\}$ of disjoint points in $\mathbb{R}^N$ and $\{\nu_i : i \in I\} \subset (0, \infty)$ such that

$$\nu = \sum_{i \in I} \nu_i \delta_{x_i}.$$ 

To prove the above theorem we need the help of the following lemma proved below.

**Lemma 3.4.** Let $\nu$ be a non-negative and bounded measure such that for any bounded measurable function $\psi$, there exists some constant $C > 0$ such that

$$C \min \left( \|\psi\|_{L_q^*(\mathbb{R}^N)}, \|\psi\|_{L_q^*(\mathbb{R}^N)}^{\frac{q^*}{r}} \right) \leq \|\psi\|_{L_q^*(\mathbb{R}^N)}.$$ 

Then there exists $\lambda > 0$ such that $\nu(B) = 0$ or $\nu(B) \geq \lambda$ for all Borel sets $B$.

**Proof.** Let us consider a Borel set $B$ and the characteristic function $\chi_B$. Then for $\psi = \chi_B$ we have

$$\|\chi_B\|_{L_t^1(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} |\chi_B(x)|^t \nu \, dx \right)^{\frac{1}{t}} = \nu(B)^{\frac{1}{t}}, \text{ for any } 1 \leq t < \infty.$$ 

From the hypothesis of the lemma we have

$$C \min \left( \|\psi\|_{L_q^*(\mathbb{R}^N)}, \|\psi\|_{L_q^*(\mathbb{R}^N)}^{\frac{q^*}{r}} \right) \leq \|\psi\|_{L_q^*(\mathbb{R}^N)}. \quad (3.14)$$

This yields

$$C \min(\nu(B)^{\frac{1}{t^*}}, \nu(B)^{\frac{1}{t}}) \leq \nu(B)^{\frac{1}{t^*}}.$$ 

For the case $\nu(B) \geq 1$, we get

$$C \nu(B)^{\frac{1}{t^*}} \leq \nu(B)^{\frac{1}{t^*}}.$$ 

Hence, either $\nu(B) = 0$ or $\nu(B) \geq C^{\frac{q}{q^*}}$. Working on similar lines for the case $\nu(B) \leq 1$ we obtain our desired result. \square

**Proof of Lemma 3.3.** Following the proof of Lemma 3.4 and using (3.13) we can see that the measure $\nu$ is absolutely continuous with respect to the measure $\mu$. By the Lebesgue decomposition of $\mu$ with respect to $\nu$ we can express $\mu$ as

$$\mu = f \nu + \xi,$$
where \( f \in L^1_\nu(\mathbb{R}^N) \), \( f \geq 0 \) and \( \xi \geq 0 \) is a bounded measure singular with respect to \( \nu \). Now with the choice of \( \phi = f^{q^* - q} \chi_{\{f \leq n\}} \psi \), for some bounded measurable function \( \psi \), we can rewrite equation (3.13) as follows

\[
C \min \left( \| \phi \|_{L^{q^*\nu}(\mathbb{R}^N)}, \| \phi \|_{L^{q^*\nu}(\mathbb{R}^N)}^{\frac{q}{q^*}} \right) \leq \| \phi \|_{L^{q^*\nu}(\mathbb{R}^N)}^{\frac{q}{q^*}} \chi_{\{f \leq n\}} \psi \|_{L^\nu(\mathbb{R}^N)}. \quad (\text{since } \xi \text{ is singular w.r.t } \nu)
\]

Denote \( d\nu_n = f^{q^* - q} \chi_{\{f \leq n\}} d\nu \), then we get

\[
C \min \left( \| \psi \|_{L^{q^*\nu_n}(\mathbb{R}^N)}, \| \psi \|_{L^{q^*\nu_n}(\mathbb{R}^N)}^{\frac{q}{q^*}} \right) \leq \| \psi \|_{L^{q^*\nu_n}(\mathbb{R}^N)}. \quad (3.15)
\]

Hence, from Lemma 3.4 we conclude that for a \( \lambda > 0 \) and for any Borel set \( B \), either \( \nu_n(B) = 0 \) or \( \nu_n(B) \geq \lambda \). Now following the proof of Lemma 5.1 (stated in the Appendix) we can guarantee that there exists collection of disjoint points \( \{x_i : i \in I\} \) (\( I \) countable) in \( \mathbb{R}^N \), \( \{a^n_i : i \in I\} \) in \( (0, \infty) \) such that \( \nu_n \) can be represented as

\[
\nu_n = \sum_{i \in I} a^n_i \delta_{x_i}.
\]

On the other hand, \( \nu_n \) converges to \( f^{q^* - q} \nu = f^{\frac{N}{m}} \nu \) in measure. So we have

\[
f^{\frac{N}{m}} \nu = \sum_{i \in I} a_i \delta_{x_i}, \quad \text{where } a_i = f(x_i)^{\frac{N}{m}} \nu(\{x_i\}),
\]

thereby proving the claim of the lemma. \( \square \)

### 3.1 Proof of Theorem 1.1

Before proving our main result we discuss some important results and definitions (refer [21]).

**Definition 3.5.** A bounded sequence \( \{u_n\} \) is said to be tight if for every \( \epsilon > 0 \), there exists a compact subset \( K \) of \( \mathbb{R}^N \) such that

\[
\sup_{n \in \mathbb{N}} \int_{K^c} |u_n| dx < \epsilon.
\]

**Prokhorov’s theorem:** Every bounded sequence \( \{u_n\} \) are relatively sequentially compact if and only if the sequence is tight.

**Definition 3.6.** A sequence of integrable functions \( \{u_n\} \) in \( \mathbb{R}^N \) converges tightly to a Borel regular measure \( \nu \) if

\[
\int_{\mathbb{R}^N} \varphi u_n dx \to \int_{\mathbb{R}^N} \varphi d\nu
\]

for all \( \varphi \in C_b(\mathbb{R}^N) \), the space of bounded continuous functions in \( \mathbb{R}^N \). We will symbolize the tight convergence by \( \overset{\ast}{\to} \).
Proof of Theorem 1.1. Let \( \{u_n\} \) be a bounded sequence in \( W_0^{s,q}(\Omega) \). Since \( u_n \equiv 0 \) in \( \mathbb{R}^N \setminus \Omega \), the sequences \( \{u_n\}^q \) and \( \{|u_n|^{r(x)}\} \) are tight. By the Prokhorov’s theorem the existence of two positive Borel measures \( \nu_1 \) and \( \nu_2 \) are guaranteed such that \( |u_n|^q \rightharpoonup \nu_1 \) and \( |u_n|^{r(x)} \rightharpoonup \nu_2 \). Apparently, \( \text{supp}(\nu_1), \text{supp}(\nu_2) \subset \overline{\Omega} \).

Denote
\[
|D^s u_n(x)|^q = \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^q}{|x - y|^{N+sq}} dy.
\]

Consider an open, bounded subset \( D \) of \( \mathbb{R}^N \) such that \( \overline{\Omega} \subset D \). Let \( \text{dist}(D^c, \Omega) = d > 0 \). Then for any \( x \in D^c \) and \( y \in \Omega \), there exists a constant \( C_d \) depending on \( d \) such that \( |x - y| \geq C_d|x| \) and
\[
|D^s u_n(x)|^q \leq \int_{\Omega} \frac{|u_n(y)|^q}{|x - y|^{N+sq}} dy \leq \int_{\Omega} \frac{|u_n(y)|^q}{(C_d|x|)^{N+sq}} dy \leq \frac{C}{|x|^{N+sq}}.
\]

So the tightness of the sequence \( \{|D^s u_n|^q\} \) follows from the above inequality. Hence, there exists a Borel measure \( \mu \) such that \( |D^s u_n|^q \rightharpoonup \mu \) and (1.3) is proved.

Since \( r(x) \leq q^* \), using the Proposition 2.1 and Theorem 2.6 stated in Section 2 for any \( \phi \in C_c^\infty(\mathbb{R}^N) \) with \( 0 \leq |\phi| \leq 1 \) we have the following Sobolev inequality.
\[
S\{\|u_n\phi\|_{L^{q^*}(\mathbb{R}^N)} + \|u_n\phi\|_{L^{r(x)}(\mathbb{R}^N)}\} \leq \|u_n\phi\|_{s,q} \tag{3.16}
\]
where \( S \) is the Sobolev constant given in (1.7).

Observe that
\[
\left(\int_{\mathbb{R}^N} \left(|u_n|^q + |u_n|^{r(x)}\right) |\phi|^q \right)^\frac{1}{q} \leq \left(\int_{\mathbb{R}^N} |u_n\phi|^q \right)^\frac{1}{q} + \left(\int_{\mathbb{R}^N} |u_n\phi|^{r(x)} \right)^\frac{1}{q} = \|u_n\phi\|_{L^{q^*}(\mathbb{R}^N)} + \left(\int_{\mathbb{R}^N} |u_n\phi|^{r(x)} \right)^\frac{1}{q^*}. \tag{3.17}
\]

We first consider the case when \( \|u_n\phi\|_{L^{r(x)}(\mathbb{R}^N)} \geq 1 \). Then the above inequality (3.17) becomes
\[
S \left(\int_{\mathbb{R}^N} \left(|u_n|^q + |u_n|^{r(x)}\right) |\phi|^q \right)^\frac{1}{q} \leq S \left(\|u_n\phi\|_{L^{q^*}(\mathbb{R}^N)} + \|u_n\phi\|_{L^{r(x)}(\mathbb{R}^N)}\right) \leq \|u_n\phi\|_{s,q}. \tag{3.18}
\]

For the case \( \|u_n\phi\|_{L^{r(x)}(\mathbb{R}^N)} < 1 \), we have
\[
\left(\int_{\mathbb{R}^N} \left(|u_n|^q + |u_n|^{r(x)}\right) |\phi|^q \right)^\frac{1}{q} \leq \|u_n\phi\|_{L^{q^*}(\mathbb{R}^N)} + \left(\frac{S}{1}\|u_n\phi\|_{s,q} \right)^{\frac{1}{q^*}} \leq \frac{1}{S}\|u_n\phi\|_{s,q} + \left(\frac{1}{S}\|u_n\phi\|_{s,q} \right)^{\frac{1}{q^*}}. \tag{3.19}
\]
If $\frac{1}{s} ||u_n,\phi||_{s,q} \geq 1$, then the inequality (3.19) becomes

$$\frac{S}{2} \left( \int_{\mathbb{R}^N} \left| |u_n|^q + |u_n|^{r(x)} \right| |\phi|^q \right)^{\frac{1}{q}} \leq ||u_n,\phi||_{s,q},$$

(3.20)

otherwise, we get

$$\frac{S}{2^q} \left( \int_{\mathbb{R}^N} \left( |u_n|^q + |u_n|^{r(x)} \right) |\phi|^q \right)^{\frac{1}{q}} \leq ||u_n,\phi||_{s,q}.$$  

(3.21)

Let us denote $M_n = \int_{\mathbb{R}^N} \left( |u_n|^q + |u_n|^{r(x)} \right) |\phi|^q dx$. On combining (3.18), (3.20) and (3.21) we establish the following

$$2^{-\frac{s}{r}} S \min \left( M_n^{\frac{1}{q}}, M_n^{-\frac{1}{q}} \right) \leq ||u_n,\phi||_{s,q}.$$  

(3.22)

Now using the Minkowski's inequality we get

$$||u_n,\phi||_{s,q} = \left( \int_{\mathbb{R}^N} \frac{|u_n(x)\phi(x) - u_n(y)\phi(y)|^q}{|x-y|^{N+sq}} dydx \right)^{\frac{1}{q}}$$

$$= \left( \int_{\mathbb{R}^N} \frac{|u_n(x)(\phi(x) - \phi(y)) + \phi(y)(u_n(x) - u_n(y))|^q}{|x-y|^{N+sq}} dydx \right)^{\frac{1}{q}}$$

$$\leq \left( \int_{\mathbb{R}^N} \frac{|u_n(x)|^q|\phi(x) - \phi(y)|^q}{|x-y|^{N+sq}} dydx \right)^{\frac{1}{q}}$$

$$+ \left( \int_{\mathbb{R}^N} \frac{|\phi(y)|^q|u_n(x) - u_n(y)|^q}{|x-y|^{N+sq}} dydx \right)^{\frac{1}{q}}$$

$$\leq \left( \int_{\mathbb{R}^N} |u_n(x)|^q |D^s\phi(x)|^q dx \right)^{\frac{1}{q}} + \left( \int_{\mathbb{R}^N} |\phi(x)|^q |D^s u_n(x)|^q dx \right)^{\frac{1}{q}}.$$  

Since $\{u_n\}$ is bounded in $W^{s,q}_0(\Omega)$, there exists $u \in W^{s,q}_0(\Omega)$ and a subsequence, still denoted as $\{u_n\}$, such that $u_n$ converges weakly to $u$ in $W^{s,q}_0(\Omega)$ and strongly in $L^p(\mathbb{R}^N)$ for any $1 < p < q^*$. By the Corollary 3.1 we observe $\int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^q}{|x-y|^{N+sq}} dy \in L^\infty(\mathbb{R}^N)$. Then using (1.3) we get

$$\limsup \frac{1}{n} ||u_n,\phi||_{s,q} \leq \left( \int_{\mathbb{R}^N} |u(x)|^q |D^s\phi(x)|^q dx \right)^{\frac{1}{q}} + \left( \int_{\mathbb{R}^N} |\phi(x)|^q d\mu \right)^{\frac{1}{q}}.$$  

(3.23)

We denote the measure $\nu = \nu_1 + \nu_2$ and hence $|u_n|^q + |u_n|^{r(x)} \overset{\ast}{\to} \nu$. So

$$\limsup \frac{1}{n} M_n = \limsup \frac{1}{n} \int_{\mathbb{R}^N} \left( |u_n|^q + |u_n|^{r(x)} \right) |\phi|^q dx = \int_{\mathbb{R}^N} |\phi|^q d\nu.$$  

(3.24)

Considering the inequalities (3.23), (3.24) and applying limit $n \to \infty$ in (3.22) we have

$$2^{-\frac{s}{r}} S \min \left( \|\phi\|_{L^q_s(\mathbb{R}^N)}, \|\phi\|_{L^q_s(\mathbb{R}^N)} \right) \leq \left( \int_{\mathbb{R}^N} |u(x)|^q |D^s\phi(x)|^q dx \right)^{\frac{1}{q}} + \|\phi\|_{L^p(\mathbb{R}^N)},$$  

(3.25)
Now suppose that $u = 0$ and $u_n \to 0$ weakly in $W^{s,q}_0(\Omega)$, then
\[
2^{-\frac{2}{q^*}} S \min \left( \| \phi \|_{L^q_0(\Omega)}, \| \phi \|_{L^{q^*}_0(\Omega)} \right) \leq \| \phi \|_{L^q_0(\Omega)}.
\] (3.26)

By the Lemma 3.3 there exists a set of distinct points $\{x_i : i \in I\} \subset \mathbb{R}^N$ and $\{\nu_i : i \in I\} \subset (0, \infty)$ such that
\[
\nu = \sum_{i \in I} \nu_i \delta_{x_i}.
\]

Suppose that $u \neq 0$. Then the sequence $\{v_n\}$, when $v_n = u_n - u$, is bounded in $W^{s,q}_0(\Omega)$ and there exists a subsequence of $\{v_n\}$ (named as $\{v_n\}$) which converges weakly to 0 in $W^{s,q}_0(\Omega)$. By the Brézis-Lieb lemma [Lemma 5.2 in the Appendix] we have the following
\[
\lim_{n \to \infty} \left( \int_{\mathbb{R}^N} |\phi|^q \left( |u_n|^q + |u_n|^{|r(x)}| \right) dx - \int_{\mathbb{R}^N} |\phi|^q \left( |v_n|^q + |v_n|^{|r(x)}| \right) dx \right) = \int_{\mathbb{R}^N} |\phi|^q \left( |u|^q + |u|^{|r(x)}| \right) dx
\] (3.27)
for every $\phi \in C_c^\infty(\mathbb{R}^N)$. Clearly the sequences $\{|v_n|^q\}$ and $\{|v_n|^{|r(x)}|\}$ are tight. Hence, on similar lines the representation of $\nu$ is given as
\[
\nu = |u|^q + |u|^{|r(x)}| + \sum_{i \in I} \nu_i \delta_{x_i}.
\] (3.28)

This proves (1.5). Let us consider $\phi \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \phi \leq 1$, $\phi(0) = 1$ and $\text{supp}(\phi) \subset B_1(0)$. For a fixed $j$, choose $\epsilon > 0$ such that for $i \neq j$, $B_\epsilon(x_i) \cap B_\epsilon(x_j) = \emptyset$. We define $\phi_{\epsilon,j}(x) = \phi(\frac{x-x_j}{\epsilon})$.

**Claim:**
\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}^N} |u(x)|^q |D^s \phi(x)|^q dx = 0.
\] (3.29)

Without loss of generality, we can assume $x_j = 0$ and denote $\phi_\epsilon = \phi_{\epsilon,j}$. Using Corollary 3.4 we have
\[
|D^s \phi(x)|^q = \int_{\mathbb{R}^N} \frac{|\phi_\epsilon(x) - \phi_\epsilon(y)|^q}{|x-y|^{N+sq}} dy \leq C \min \left( \frac{1}{\epsilon^{sq}}, \epsilon^N |x|^{-(N+sq)} \right).
\]

Thus,
\[
\int_{\mathbb{R}^N} |u(x)|^q |D^s \phi(x)|^q dx \leq C \left( \frac{1}{\epsilon^{sq}} \int_{|x| < 2\epsilon} |u(x)|^q dx + \epsilon^N \int_{|x| \geq 2\epsilon} \frac{|u(x)|^q}{|x|^{N+sq}} dx \right) = C(I + II).
\] (3.30)

The first term I can be estimated as
\[
I \leq \frac{1}{\epsilon^{sq}} \| |u|^q\|_{L^q(B_{2\epsilon}(0))} \|1\|_{L^{\infty}(B_{2\epsilon}(0))} \leq C \| |u|^q\|_{L^q(B_{2\epsilon}(0))}.
\]
Since $u \in L^q((\mathbb{R}^N))$, $I \to 0$ as $\epsilon \to 0$. For the second term we proceed as follows

$$II = \sum_{k=1}^{\infty} \epsilon^N \int_{2^{k-1} < |x| \leq 2^k} |u(x)|^q \frac{1}{|x|^{N+sq}} dx \leq \sum_{k=1}^{\infty} \frac{1}{2^k(N+sq)} \int_{|x| \leq 2^k} |u(x)|^q dx \leq \sum_{k=1}^{\infty} \frac{C}{2^k N} \|u\|_{L^q((\mathbb{R}^N)^*}(B(2^{k+1} \epsilon, 0)).$$

Now given $\delta > 0$ take $k_0 \in \mathbb{N}$ such that $\sum_{k=k_0+1}^{\infty} 2^{-Nk} < \delta$. So,

$$II \leq \|u\|_{L^q((\mathbb{R}^N)^*)} \frac{\delta}{2^k} + \frac{1}{2} \sum_{k=1}^{k_0} \|u\|_{L^q((\mathbb{R}^N)^*)} \frac{\delta}{2^k} \leq \|u\|_{L^q((\mathbb{R}^N)^*)} \frac{\delta}{2^k} + \frac{1}{2} \sum_{k=1}^{k_0} \|u\|_{L^q((\mathbb{R}^N)^*)} \frac{\delta}{2^k}.$$

This inequality is true for any $\delta$. Thus $II \to 0$ as $\epsilon \to 0$. Hence, the claim.

In order to prove (1.6), we first observe that

$$\int_{\mathbb{R}^N} |\phi_{\epsilon,j}|^q d\nu = \int_{\mathbb{R}^N} |\phi_{\epsilon,j}|^q (|u|^q + |u|^r(x)) dx + \sum_{i \in I} \nu_i |\phi_{\epsilon,j}(x_i)|^q \geq \nu_j.$$

Then

$$\|\phi_{\epsilon,j}\|_{L^q_{\mathcal{F}}((\mathbb{R}^N)^*)} \geq \nu_j^{\frac{1}{q}} \quad \text{and} \quad \|\phi_{\epsilon,j}\|_{L^q_{\mathcal{F}}((\mathbb{R}^N)^*)} \geq \nu_j^{\frac{1}{q}}.$$

From (3.26) we find

$$2^{-\frac{q^*}{r}} S \min \left( \nu_j^{\frac{1}{q}}, \nu_j^{\frac{1}{q}} \right) \leq \|\phi_{\epsilon,j}\|_{L^q_{\mathcal{F}}((\mathbb{R}^N)^*)}.$$

On the other hand

$$\int_{\mathbb{R}^N} |\phi_{\epsilon,j}|^q d\mu \leq \mu(B_{\epsilon}(x_j)) \to \mu_j \quad \text{as} \quad \epsilon \to 0.$$

Thus, on rewriting the inequality (3.26) we establish the inequality in (1.6).

$$2^{-\frac{q^*}{r}} S \min \left( \nu_j^{\frac{1}{q}}, \nu_j^{\frac{1}{q}} \right) \leq \mu_j^{\frac{1}{q}}.$$

We are now left to prove (1.4). We already have $\mu \geq \sum_{i \in I} \mu_i \delta_{x_i}$. On using the weak convergence and the Fatou’s lemma we have

$$\int_{\mathbb{R}^N} \phi d\mu = \lim_{n \to \infty} \inf \int_{\mathbb{R}^N} |D^s u_n(x)|^q \phi dx \geq \int_{\mathbb{R}^N} |D^s u(x)|^q \phi dx.$$

Thus, $\mu \geq |D^s u(x)|^q$. Since, $\sum_{i \in I} \mu_i \delta_{x_i}$ and $|D^s u(x)|^q$ are orthogonal measures, we conclude

$$\mu \geq |D^s u(x)|^q + \sum_{i \in I} \mu_i \delta_{x_i}.$$

This completes the proof.
4 Application to Dirichlet problem with fractional \((p(x), p^+)-Laplacian\)- proof of Theorem 1.3

In this section we study the following problem

\[
(-\Delta)^s_{p(x)} u + (-\Delta)^{s+}_p u = |u|^{(p^+)-2} u + |u|^{p^*_s(x)-2} u + \lambda |u|^\beta(x)-2 u \text{ in } \Omega,
\]

\[u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,
\]

as an example to demonstrate our theoretical finding of CCTP. Here \(\Omega\) is a bounded domain in \(\mathbb{R}^N\), \(\lambda > 0\), the assumptions on \(s\), \(p(.,.)\) are same as in Theorem 1.3 \(\beta \in C_+(\mathbb{R})\) such that \(\beta(x) < p_s^*(x)\) for every \(x \in \Omega\) and \(p^+ < \beta^- \leq \beta^+ < (p_s^*)^-\). The fractional \(p(x)\)-Laplacian-\((-\Delta)^s_{p(x)}\) and the fractional \(p^+\)-Laplacian-\((-\Delta)^{s+}_{p} \) are defined as

\[
(-\Delta)^s_{p(x)} u = P.V. \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))}{|x-y|^{N+sp(x,y)}} dy
\]

and

\[
(-\Delta)^{s+}_p u = P.V. \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p^+-2}(u(x) - u(y))}{|x-y|^{N+sp}} dy
\]

respectively. We name \(\mathcal{L} u = (-\Delta)^s_{p(x)} u + (-\Delta)^{s+}_{p} u\) as a fractional \((p(x), p^+)-Laplacian\). We consider here the critical case, i.e. when \(A = \{x \in \Omega : p_s^*(x) = (p^+)^*\}\) is nonempty. The natural solution space of the problem (4.31) is \(X = W^{s,p^+}_0(\Omega) \cap W_0\), defined in Section 2 which is endowed with the norm

\[
||u||_X = ||u||_{s,p^+} + ||u||_{W_0}.
\]

Note that for any \(p \in C_+(\mathbb{R}^N \times \mathbb{R}^N), 1 < p^- \leq p^+ < \infty, \)

\[(p^-)^* \leq (p_s^*)^- = \inf_{x \in \mathbb{R}^N} p_s^*(x) \leq p_s^*(x) \leq \sup_{x \in \mathbb{R}^N} p_s^*(x) = (p_s^*)^+ \leq (p^+)^* .\]

Definition 4.1. A function \(u \in X\) is said to be a weak solution of problem (4.31), if \(u\) is a critical point of the corresponding energy functional

\[
\Psi(u) = \int_{\mathbb{R}^{2N}} \frac{1}{p(x,y)} |u(x) - u(y)|^{p(x,y)} |x-y|^{-N+sp(x,y)} \, dy \, dx + \int_{\mathbb{R}^{2N}} \frac{1}{p^+} |u(x) - u(y)|^{p^+} |x-y|^{-N+sp} \, dy \, dx
\]

\[-\frac{1}{(p^+)^*} \int_{\Omega} |u|^{(p^+)^*} \, dx - \int_{\Omega} \frac{1}{p_s^*(x)} |u|^{p_s^*(x)} \, dx - \lambda \int_{\Omega} \frac{1}{\beta(x)} |u|^{\beta(x)} \, dx. \tag{4.32}
\]

We will use the Mountain Pass geometry to prove the existence of weak solution to the problem (4.31). Hence, we show that the functional \(\Psi\) satisfies the Palais-Smale (P-S) condition below a certain energy level.

Lemma 4.2. Let \(\{u_n\}_{n \in \mathbb{N}} \subset X\) be a Palais-Smale sequence, then \(\{u_n\}\) is bounded in \(X\).
Proof. Let \( \{u_n\} \) be a Palais-Smale sequence of the functional \( \Psi \), i.e. \( \Psi(u_n) \to c \) and \( \Psi'(u_n) \to 0 \) as \( n \to \infty \). Hence, for large \( n \) we have

\[
c + 1 \geq \Psi(u_n) - \frac{1}{\beta^-} \langle \Psi'(u_n), u_n \rangle = \int_{\mathbb{R}^N} \left( \frac{1}{p(x, y)} - \frac{1}{\beta^-} \right) |u_n(x) - u_n(y)|^{p(x, y)} \frac{1}{|x - y|^{N + sp(x, y)}} dydx + \frac{1}{p^+} - \frac{1}{\beta^-} \int_{\mathbb{R}^N} |u_n(x) - u_n(y)|^{p^+} \frac{1}{|x - y|^{N + sp^+}} dydx
\]

\[
+ \left( \frac{1}{p^+} - \frac{1}{\beta^-} \right) \int_{\Omega} |u_n|^{p^+} dx + \int_{\Omega} \left( \frac{1}{\beta^-} - \frac{1}{p_{s}(x)} \right) |u_n|p_{s}'(x) dx
\]

\[
+ \lambda \int_{\Omega} \left( \frac{1}{\beta^-} - \frac{1}{\beta(x)} \right) |u_n|^{\beta(x)} dx
\]

\[
\geq \left( \frac{1}{p^+} - \frac{1}{\beta^-} \right) \left( \int_{\mathbb{R}^N} |u_n(x) - u_n(y)|^{p(x, y)} \frac{1}{|x - y|^{N + sp(x, y)}} dydx + \int_{\mathbb{R}^N} |u_n(x) - u_n(y)|^{p^+} \frac{1}{|x - y|^{N + sp^+}} dydx \right)
\]

\[
\geq \left( \frac{1}{p^+} - \frac{1}{\beta^-} \right) \left( \min \{ \|u_n\|_{W_0}^{p^+}, \|u_n\|_{W_0}^{p_{s}} \} + \|u_0\|_{s,p^+} \right).
\]

Thus, \( \{u_n\} \) is bounded in \( X \). \( \square \)

Lemma 4.3. The functional \( \Psi \) in (1.32) satisfies the Palais-Smale (P-S) condition for energy level

\[
c < \left( \frac{1}{p^+} - \frac{1}{p_{s}} \right) 2^{\frac{(p^+)^{p^+}}{(p_{s})^{p_{s}}} - p^+} \min \left( S_{\beta}^{-}, S_{\frac{(p_{s})^{p_{s}}}{p^+} - p^+} \right).
\]

where \( S \) is the Sobolev constant given in (1.7).

Proof. Let \( \{u_n\} \) be a (P-S) sequence in \( X \). Then by Lemma 4.2 \( \{u_n\} \) is bounded in \( X \) and thus there exists \( u \in X \) such that, up to a subsequence (still denoted as \( \{u_n\} \)), \( \{u_n\} \) converges weakly to \( u \) in \( X \). We need to show that \( u_n \rightarrow u \) strongly in \( X \). Since \( \{u_n\} \) is bounded in \( X \), \( \{u_n\} \) is also bounded in \( W_0 \) and in \( W_{0}^{s,p^+}(\Omega) \). Thus, using the concentration compactness type principle, stated in Theorem 1.1, we have

\[
\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p^+}}{|x - y|^{N + sp^+}} dy \mu \geq \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p^+}}{|x - y|^{N + sp^+}} dy + \sum_{i \in I} \mu_j \delta_{x_i},
\]

\[
|u_n|^{(p^+)^*} + |u_n|^{p_{s}'(x)} \to \nu = |u|^{(p^+)^*} + |u|^{p_{s}'(x)} + \sum_{i \in I} \nu_i \delta_{x_i}
\]

and

\[
2^{-\frac{(p^+)^*}{(p_{s})^{p_{s}}}} S \min \left( \nu_1^{\frac{1}{(p^+)^*}}, \nu_i^{\frac{1}{(p_{s})^{p_{s}}}} \right) \leq \mu_i^{\frac{1}{p^+}}, \forall i \in I.
\]

Let us denote

\[
|U_n(x)| = \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p(x, y)}}{|x - y|^{N + sp(x, y)}} dy
\]
and consider an open, bounded subset $D$ of $\mathbb{R}^N$, defined as in the proof of Theorem 1. Then for any $x \in D$ and $y \in \Omega$, there exists a constant $C_d$ such that $|x - y| \geq C_d|x|$ and

$$|U_n(x)| = \int_{\Omega} \frac{|u_n(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} dy$$

$$\leq \int_{\Omega} \frac{|u_n(y)|^{p(x,y)}}{(C_d|x|)^{N + sp(x,y)}} dy$$

$$\leq C \max \left( \frac{1}{|x|^{N + sp^+}}, \frac{1}{|x|^{N + sp^-}} \right).$$

Therefore, the sequence $\{|U_n|\}$ is tight and there exists a positive bounded Borel measure $\sigma$ such that $|U_n| \overset{\lambda}{\to} \sigma$. Consider $\phi \in C_0^\infty(\mathbb{R}^N)$ with $0 \leq \phi \leq 1$, $\phi(0) = 1$ and support in the unit ball of $\mathbb{R}^N$. Define $\phi_{\epsilon,j} = \phi(\frac{x - x_j}{\epsilon})$. Since $\{u_n\}$ is a (P-S) sequence, we have $\Psi(u_n) \to c$ and $\Psi'(u_n) \to 0$ as $n \to \infty$. On the other hand

$$\langle \Psi'(u_n), \phi_{\epsilon,j}u_n \rangle = \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y))(\phi_{\epsilon,j}u_n)(x) - (\phi_{\epsilon,j}u_n)(y)}{|x - y|^{N + sp(x,y)}} dx dy$$

$$+ \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y))(\phi_{\epsilon,j}u_n)(x) - (\phi_{\epsilon,j}u_n)(y)}{|x - y|^{N + sp^+}} dx dy$$

$$- \int_{\Omega} |u_n|^{p^+} - 2u_n(\phi_{\epsilon,j}u_n) dx - \int_{\Omega} |u_n|^{p^+} - 2u_n(\phi_{\epsilon,j}u_n) dx$$

$$- \lambda \int_{\Omega} |u_n|^{\beta(x) - 2}u_n(\phi_{\epsilon,j}u_n) dx$$

$$= \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y))u_n(y)(\phi_{\epsilon,j}u_n)(x) - (\phi_{\epsilon,j}u_n)(y)}{|x - y|^{N + sp(x,y)}} dx dy$$

$$+ \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y))u_n(y)(\phi_{\epsilon,j}u_n)(x) - (\phi_{\epsilon,j}u_n)(y)}{|x - y|^{N + sp^+}} dx dy$$

$$+ \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p(x,y)} \phi_{\epsilon,j}(x)}{|x - y|^{N + sp(x,y)}} dx dy + \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p^+} \phi_{\epsilon,j}(x)}{|x - y|^{N + sp^+}} dx dy$$

$$- \int_{\Omega} |u_n|^{p^+} \phi_{\epsilon,j} dx - \int_{\Omega} |u_n|^{p^+} \phi_{\epsilon,j} dx - \lambda \int_{\Omega} |u_n|^{\beta(x)} \phi_{\epsilon,j} dx. \quad (4.33)$$

We denote $H_n(x, y) = \frac{|u_n(x) - u_n(y)|}{|x - y|^{N + sp(x,y)}}$ and $\Phi_n(x, y) = \frac{|u_n(y)||(\phi_{\epsilon,j}(x) - (\phi_{\epsilon,j})(y))}{|x - y|^{N + sp(x,y)}}$. Since, $\{u_n\}$ is a bounded sequence in $W_0$, on using the Hölder’s inequality on the first term in the right hand
side of (4.33) we observe

\[
\left| \int_{\mathbb{R}^{2N}} |u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y))u_n(y)(\phi_{\epsilon,j}(x) - \phi_{\epsilon,j}(y)) \, dxdy \right|
\]

\[
\leq \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p(x,y)-1}|u_n(y)||\phi_{\epsilon,j}(x) - \phi_{\epsilon,j}(y)|}{|x - y|^{N+s p(x,y)}} \, dxdy
\]

\[
= \int_{\mathbb{R}^{2N}} \left( \frac{|u_n(x) - u_n(y)|}{|x - y|^{N+s p(x,y)}} \right)^{p(x,y)-1} \left( \frac{|u_n(y)||\phi_{\epsilon,j}(x) - \phi_{\epsilon,j}(y)|}{|x - y|^{N+s p(x,y)}} \right) \, dxdy
\]

\[
\leq C \|H_n\|_{p^{(\cdot)}-1} \|\Phi_n\|_{L^{p^{(\cdot)}}(\mathbb{R}^N \times \mathbb{R}^N)} \frac{1}{k}
\]

where \( k \) is either \( p^+ \) or \( p^- \). According to Lemma 3.2, \( \int_{\mathbb{R}^{2N}} |\phi_{\epsilon}(y) - \phi_{\epsilon,j}(y)|^{p(x,y)} \, dy \in L^\infty(\mathbb{R}^N) \). Thus, applying limit \( n \to \infty \) in the above inequality (4.34) we get

\[
\lim_{n \to \infty} \left| \int_{\mathbb{R}^{2N}} |u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y))u_n(y)(\phi_{\epsilon,j}(x) - \phi_{\epsilon,j}(y)) \, dxdy \right|
\]

\[
\leq C \left( \int_{\mathbb{R}^{2N}} \frac{|u(y)|^{p^+} + |u(y)|^{p^-}}{|x - y|^{N+s p(x,y)}} |\phi_{\epsilon,j}(x) - \phi_{\epsilon,j}(y)|^{p(x,y)} \, dxdy \right)^{\frac{1}{k}}.
\]

(4.35)

We now make the following claim.

Claim:

\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}^{2N}} \frac{|u(x)|^{p^+} + |u(x)|^{p^-}}{|x - y|^{N+s p(x,y)}} |\phi_{\epsilon,j}(x) - \phi_{\epsilon,j}(y)|^{p(x,y)} \, dxdy = 0.
\]

Without loss of generality, we assume that \( x_j = 0 \) and denote \( \phi_{\epsilon} = \phi_{\epsilon,j} \). Using Lemma 3.2 we have

\[
\int_{\mathbb{R}^{2N}} \frac{|\phi_{\epsilon}(x) - \phi_{\epsilon}(y)|^{p(x,y)}}{|x - y|^{N+s p(x,y)}} \, dy \leq C \min \left( \frac{1}{\epsilon^{sp^+}} + \frac{1}{\epsilon^{sp^-}}, (\epsilon^N + \epsilon^{N+s(p^-+p^+)})|x|^{-(N+s p^-)} \right).
\]
Thus,

\[
\int_{\mathbb{R}^{2N}} \left( |u(x)|^{p^+} + |u(x)|^{p^-} \right) \frac{|\phi_\epsilon(x) - \phi_\epsilon(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} \, dy \, dx
\leq C \left( \frac{1}{\epsilon^{sp^+}} + \frac{1}{\epsilon^{sp^-}} \right) \int_{|x| < 2\epsilon} |u(x)|^{p^+} + |u(x)|^{p^-} \, dx
\]

\[
+ C(\epsilon^N + \epsilon^{N+s(p^- - p^+)}) \int_{|x| > 2\epsilon} |u(x)|^{p^+} + |u(x)|^{p^-} \, dx
\]

\[= C(I + II). \tag{4.36}\]

We observe

\[
I = \left( \frac{1}{\epsilon^{sp^+}} + \frac{1}{\epsilon^{sp^-}} \right) \int_{|x| < 2\epsilon} |u(x)|^{p^+} + |u(x)|^{p^-} \, dx
\]

\[
\leq \left( \frac{1}{\epsilon^{sp^+}} + \frac{1}{\epsilon^{sp^-}} \right) \left( \|u\|^{p^+}_{L^{p^+} \epsilon} + \|u\|^{p^-}_{L^{p^-} \epsilon} \right) \leq C \epsilon^{sp^+} \left( \frac{1}{\epsilon^{sp^+}} + \frac{1}{\epsilon^{sp^-}} \right) \left( \|u\|^{p^+}_{L^{p^+} \epsilon} + \|u\|^{p^-}_{L^{p^-} \epsilon} \right).
\]

Hence, \( I \to 0 \) as \( \epsilon \to 0 \). Similarly the second term in the right hand side of (4.36) can be rewritten as follows.

\[
II = \sum_{k=1}^{\infty} (\epsilon^N + \epsilon^{N+s(p^- - p^+)}) \int_{2^{k-1} \epsilon \leq |x| \leq 2^{k} \epsilon} \frac{|u(x)|^{p^+} + |u(x)|^{p^-}}{|x|^{N+sp^-}} \, dx
\]

\[
\leq \sum_{k=1}^{\infty} \frac{1}{2^{k(N+sp^- - sp^+)}} \left( \frac{1}{\epsilon^{sp^-}} + \frac{1}{\epsilon^{sp^+}} \right) \int_{|x| < 2^{k+1} \epsilon} |u(x)|^{p^+} + |u(x)|^{p^-} \, dx
\]

\[
\leq \sum_{k=1}^{\infty} C \epsilon^{sp^+} \left( \frac{1}{\epsilon^{sp^-}} + \frac{1}{\epsilon^{sp^+}} \right) \left( \|u\|^{p^+}_{L^{p^+} \epsilon} + \|u\|^{p^-}_{L^{p^-} \epsilon} \right)
\]

\[
\leq \sum_{k=1}^{\infty} C \epsilon^{sp^+} \left( \frac{1}{\epsilon^{sp^-}} + \frac{1}{\epsilon^{sp^+}} \right) \left( \|u\|^{p^+}_{L^{p^+} \epsilon} + \|u\|^{p^-}_{L^{p^-} \epsilon} \right). \tag{4.37}\]

Now for any \( \gamma > 0 \), there exists a \( k' \in \mathbb{N} \) such that \( \sum_{k=k'+1}^{\infty} 2^{-ksp^-} < \gamma \). So,

\[
II \leq \gamma C \epsilon^{sp^+} \left( \frac{1}{\epsilon^{sp^-}} + \frac{1}{\epsilon^{sp^+}} \right) \left( \|u\|^{p^+}_{L^{p^+} \epsilon} + \|u\|^{p^-}_{L^{p^-} \epsilon} \right)
\]

\[
+ C \epsilon^{sp^+} \left( \frac{1}{\epsilon^{sp^-}} + \frac{1}{\epsilon^{sp^+}} \right) \sum_{k=1}^{k'} \frac{1}{2^{ksp^-}} \left( \|u\|^{p^+}_{L^{p^+} \epsilon} + \|u\|^{p^-}_{L^{p^-} \epsilon} \right). \tag{4.38}\]
Following the argument used in the proof of (4.34) and using (3.29) we find

\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y))u_n(y)(\phi_{\epsilon,j}(x) - \phi_{\epsilon,j}(y))}{|x - y|^{N+sp(x,y)}} \, dx \, dy \to 0.
\]

Following the argument used in the proof of (4.31) and using (3.29) we find

\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p^+ - 2}(u_n(x) - u_n(y))u_n(y)(\phi_{\epsilon,j}(x) - \phi_{\epsilon,j}(y))}{|x - y|^{N+sp^+}} \, dx \, dy \to 0.
\]

Passing the limit \( n \to \infty \) in the inequality (4.33) we get

\[
0 = \int_{\mathbb{R}^{N}} \phi_{\epsilon,j} \, d\mu + \int_{\mathbb{R}^{N}} \phi_{\epsilon,j} \, d\sigma - \int_{\Omega} \phi_{\epsilon,j} \, d\nu - \lambda \int_{\Omega} |u|^{\beta(x)} \phi_{\epsilon,j} \, dx.
\]

Since, \( \phi_{\epsilon,j}(x) \to 0 \) as \( \epsilon \to 0 \) for any \( x \neq x_j \) and \( \phi(0) = 1 \) we have

\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}^{N}} |u|^{\beta(x)} \phi_{\epsilon,j} \, dx \to 0, \quad \lim_{\epsilon \to 0} \int_{\Omega} \phi_{\epsilon,j} \, d\nu = \nu_j;
\]

\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}^{N}} \phi_{\epsilon,j} \, d\mu = \mu_j, \quad \lim_{\epsilon \to 0} \int_{\mathbb{R}^{N}} \phi_{\epsilon,j} \, d\sigma = \sigma_j
\]

where \( \nu_j = \nu(\{x_j\}) \), \( \mu_j = \mu(\{x_j\}) \) and \( \sigma_j = \sigma(\{x_j\}) \). Hence, \( \mu_i + \sigma_i = \nu_i \) for every \( i \in I \). This implies \( \mu_i \leq \nu_i \) and from (1.6) we have

\[
2^{-\frac{(p^+)^*}{(p^+)^* - p^+}} S \min \left( \nu_i^{\frac{1}{(p^+)^*}}, \nu_i^{\frac{1}{(p^+)^* - p^+}} \right) \leq \nu_i^{\frac{1}{p^+}}.
\]

This arises two cases.

**Case 1** If \( 2^{-\frac{(p^+)^*}{(p^+)^* - p^+}} S \nu_i^{\frac{1}{(p^+)^*}} \leq \nu_i^{\frac{1}{p^+}} \). Then either \( \nu_i = 0 \) or \( \left( 2^{-\frac{(p^+)^*}{(p^+)^* - p^+}} S \right)^{\frac{N}{p^+}} \leq \nu_i \).

**Case 2** If \( 2^{-\frac{(p^+)^*}{(p^+)^* - p^+}} S \nu_i^{\frac{1}{(p^+)^*}} \leq \nu_i^{\frac{1}{p^+}} \). Then either \( \nu_i = 0 \) or \( \left( 2^{-\frac{(p^+)^*}{(p^+)^* - p^+}} S \right) \left( \frac{p^+}{p^*} \right) \leq \nu_i \).

Then, from the above two cases we conclude

\[
2^{-\frac{(p^+)^* p^+}{(p^+)^* - p^+}} \min \left( S^{\frac{N}{p^+}}, S^{\frac{(p^+)^* - p^+}{(p^+)^* - p^+}} \right) \leq \nu_i.
\]
Then,
\[ c = \lim_{n \to \infty} \Psi(u_n) \]
\[ = \lim_{n \to \infty} \left( \Psi(u_n) - \frac{1}{p^+}(\Psi'(u_n), u_n) \right) \]
\[ = \lim_{n \to \infty} \left( \int_{\mathbb{R}^2N} \left( \frac{1}{p(x, y)} - \frac{1}{p^+} \right) \frac{|u_n(x) - u_n(y)|^{p(x, y)}}{|x - y|^{N + sp(x, y)}} dx dy + \int_{\Omega} \left( \frac{1}{p^+} - \frac{1}{p^*_s(x)} \right) |u_n|^{p^*_s(x)} dx \right) \\
+ \lim_{n \to \infty} \left( \frac{1}{p^+} - \frac{1}{(p^*_s)^+} \right) \int_{\Omega} |u_n|^{(p^+)^*} dx + \lambda \lim_{n \to \infty} \int_{\Omega} \left( \frac{1}{p^+} - \frac{1}{\beta(x)} \right) |u_n|^\beta(x) dx \]
\[ \geq \lim_{n \to \infty} \left( \frac{1}{p^+} - \frac{1}{(p^*_s)^+} \right) \int_{B_k(x_j)} \phi_n \left( |u_n|^{(p^+)^*} + |u_n|^{p^*_s(x)} \right) dx \]
\[ = \left( \frac{1}{p^+} - \frac{1}{(p^*_s)^+} \right) \left( \int_{B_k(x_j)} \phi_n \left( |u|^{(p^+)^*} + |u|^{p^*_s(x)} \right) dx + \sum_{i \in I} \nu_i \phi_n(x_i) \right) \]
\[ \geq \left( \frac{1}{p^+} - \frac{1}{(p^*_s)^+} \right) 2^{-\frac{(p^+)^*}{(p^*_s)^+} - p^+} \min \left( S^\frac{N}{2}, S^\frac{(p^*_s)^+ - p^+}{(p^*_s)^+ - p^+} \right). \]

This implies the indexing set \( I = \emptyset \) if we are to have
\[ c < \left( \frac{1}{p^+} - \frac{1}{(p^*_s)^+} \right) 2^{-\frac{(p^+)^*}{(p^*_s)^+} - p^+} \min \left( S^\frac{N}{2}, S^\frac{(p^*_s)^+ - p^+}{(p^*_s)^+ - p^+} \right). \]

Hence, \(|u_n|^{(p^+)^*} \overset{p^+}{\to} |u|^{(p^+)^*} \) and \(|u_n|^{p^*_s(x)} \overset{p^+}{\to} |u|^{p^*_s(x)} \). Therefore, using Prokhorov’s theorem we have \( u_n \to u \) in \( L^{(p^+)^*}(\Omega) \) and in \( L^{p^*_s(x)}(\Omega) \).

Define
\[ \langle I_1(u), v \rangle = \int_{\mathbb{R}^2N} \frac{|u(x) - u(y)|^{p(x, y) - 2} (u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + sp(x, y)}} dx dy \]
and
\[ \langle I_2(u), v \rangle = \int_{\mathbb{R}^2N} \frac{|u(x) - u(y)|^{p^+ - 2} (u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + sp^+}} dx dy. \]

Observe
\[ \langle \Psi'(u_n), (u_n - u) \rangle = \langle I_1(u_n), (u_n - u) \rangle + \langle I_2(u_n), (u_n - u) \rangle - \int_{\Omega} |u_n|^{(p^+)^* - 2} u_n (u_n - u) dx \]
\[ - \int_{\Omega} |u_n|^{p^*_s(x) - 2} u_n (u_n - u) dx - \lambda \int_{\Omega} |u_n|^{\beta(x) - 2} u_n (u_n - u) dx. \]

Since \( \{u_n\} \) is bounded in \( W_0 \), \( \{u_n\} \) is also bounded in \( L^{(p^+)^*}(\Omega) \) and \( L^{p^*_s(x)}(\Omega) \). Thus, on
applying the Hölder’s inequality we get
\[
\left| \int_{\Omega} |u_n|^{(p^+)^*-2} u_n (u_n - u) dx \right| \leq \int_{\Omega} |u_n|^{(p^+)^*-1}|u_n - u| dx \\
\leq C \|u_n\|^{(p^+)^*-1} L^{(p^+)^*-1} \|u_n - u\|_{L^{(p^+)^*}(\Omega)} \\
\leq C \|u_n - u\|_{L^{(p^+)^*}(\Omega)} = o_n(1). \tag{4.42}
\]

Similarly we can show that
\[
\lim_{n \to \infty} \left( \int_{\Omega} \lambda |u_n|^{\beta(x)^{-2}} u_n (u_n - u) dx + \int_{\Omega} |u_n|^{p^+(x)^{-2}} u_n (u_n - u) dx \right) \to 0.
\]

Passing the limit \( n \to \infty \) in (4.41) we get \( \langle I_1(u_n), (u_n - u) \rangle + \langle I_2(u_n), (u_n - u) \rangle \to 0 \). Therefore,
\[
\lim_{n \to \infty} (\langle I_1(u_n) - I_1(u), (u_n - u) \rangle + \langle I_2(u_n) - I_2(u), (u_n - u) \rangle) = 0. \tag{4.43}
\]

Recall the Simon’s inequality [23] given as
\[
|x - y|^{p} \leq \frac{1}{p - 1} \left( \left( |x|^{p-2} |y|^{p-2} \right) |x - y|^{2} \right)^{\frac{p}{2}} \left( |x|^{p} + |y|^{p} \right)^{\frac{2-p}{2}}, \text{ if } 1 < p < 2.
\]
\[
|x - y|^{p} \leq 2^{p} \left( |x|^{p-2} |y|^{p-2} \right) |x - y|^{2}, \text{ if } p \geq 2. \tag{4.44}
\]

Let us first consider the case \( p^+ > 2 \). Then using the Simon’s inequality we have
\[
\|u_n - u\|_{s,p^+}^{p^+} = \int_{\mathbb{R}^N} \frac{|(u_n - u)(x) - (u_n - u)(y)|^{p^+}}{|x - y|^{N + sp^+}} dxdy \\
\leq \frac{1}{(p^+ - 1)} \int_{\mathbb{R}^N} \left\{ \frac{|u_n(x) - u_n(y)|^{p^+ - 2}(u_n(x) - u_n(y))((u_n - u)(x) - (u_n - u)(y))}{|x - y|^{N + sp^+}} \\
- \frac{|u(x) - u(y)|^{p^+ - 2}(u(x) - u(y))((u_n - u)(x) - (u_n - u)(y))}{|x - y|^{N + sp^+}} \right\} \\
\leq C_1 \langle I_2(u_n) - I_2(u), (u_n - u) \rangle. \tag{4.45}
\]

Similarly for \( 1 < p^+ < 2 \), using Hölder’s inequality and the boundedness of \( \{u_n\} \) in \( W^s_{0,p^+}(\Omega) \), we establish the following.
\[
\|u_n - u\|_{s,p^+}^{p^+} \leq 2^{p^+} \langle I_2(u_n) - I_2(u), (u_n - u) \rangle^{p^+} \left( \|u_n\|_{s,p^+}^{p^+} + \|u\|_{s,p^+}^{p^+} \right)^{\frac{2-p^+}{2}} \\
\leq C_2 \langle I_2(u_n) - I_2(u), (u_n - u) \rangle^{p^+} \left( \|u_n\|_{s,p^+}^{p^+ - \frac{2}{2}} + \|u\|_{s,p^+}^{p^+ - \frac{2}{2}} \right) \\
\leq C_3 \langle I_2(u_n) - I_2(u), (u_n - u) \rangle^{p^+}. \tag{4.46}
\]

Considering both the inequalities (4.43) and (4.46), from (4.43) we obtain
\[
\lim_{n \to \infty} \langle I_1(u_n) - I_1(u), (u_n - u) \rangle \leq 0.
\]
Since $I_1$ is of $(S_+)$-type by Lemma 4.3 (refer Appendix), we conclude that $u_n \to u$ strongly in $W_0$ and hence by simple calculation we get $\lim_{n \to \infty} \langle I_1(u_n) - I_1(u), (u_n - u) \rangle = 0$. Therefore, from (4.43), $\lim_{n \to \infty} \langle I_2(u_n) - I_2(u), (u_n - u) \rangle = 0$. By (4.45) and (4.46) we obtain that $u_n \to u$ strongly in $W_0^{s,p^+}(\Omega)$ and hence $u_n \to u$ strongly in $X$.

Clearly, $\Psi$ is a $C^1$ functional and also well-defined. Now to prove Theorem 1.3 we first show that $\Psi$ satisfies the hypotheses of Mountain Pass theorem.

Proof of Theorem 1.3. It can be concluded from the Lemma 4.3 that $\Psi(0) = 0$ and $\Psi$ satisfies the P-S condition below a certain energy level.

Claim 1. For every $u \in X$, there exists $r > 0$ and $0 < \delta \leq 1$ such that $\Psi(u) \geq r > 0$ for $\|u\|_X = \delta$.

Let $\|u\|_X \leq 1$, then using Theorem 2.9 we observe

$$
\Psi(u) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{p^+} |u(x) - u(y)|^{p(x,y)} dy dx + \frac{1}{p^+} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)|^{p^+} dx dy - \frac{1}{(p^+)^*} \int_{\Omega} |u|^{(p^+)^*} dx - \frac{1}{\lambda} \int_{\Omega} \beta(x) |u|^{\beta(x)} dx
$$

$$
\geq \frac{1}{p^+} \left( \|u\|_{W_0^{s,p^+}}^p + \|u\|_{s,p^+}^p \right) - C_1 \|u\|_{(p^+)^*}^p - C_2 \|u\|_{\beta^*}^p - \frac{\lambda C_3}{\beta^*} \|u\|_{\beta^*}^p,
$$

where $C_i > 0$, $i = 1, 2, 3$. Since, $p^+ < \beta < (p^*_s)^* \leq (p^+)^*$, there exists $r > 0$ and $0 < \delta \leq 1$ such that $\Psi(u) \geq r > 0$ for $\|u\|_X = \delta$.

Claim 2. For any $v \in W_0$, we have $\Psi(tv) \to -\infty$ as $t \to +\infty$.

Let $v \in W_0$ with $t > 1$. Then it is easy to check that $\Psi(tv) \to -\infty$ as $t \to +\infty$. So the functional $\Psi$ satisfies the Mountain Pass geometry.

Claim 3. There exists a $\Lambda > 0$ such that

$$
0 < \inf_{u \in X \setminus \{0\}} \sup_{t \geq 0} \Psi(tu) < 2 \left( \frac{p^+}{(p^*_s)^* - p^+} \right) \min \left( S^N, S_\Lambda \right), \text{ for } \lambda > \Lambda.
$$

Let us choose $v \in X$. Clearly, $\lim_{t \to \infty} \Psi(tv) = -\infty$ and $\lim_{t \to 0^+} \Psi(tv) = 0$. Thus, by Claim-1 and Claim-2, there exists a $t_{\lambda} > 0$ such that $\sup_{t \geq 0} \Psi(tu) = \Psi(t_{\lambda}v)$. Now

$$
\Psi(tv) \leq \frac{p^\pm}{p^-} \|v\|_{W_0^p}^p + \frac{p^\pm}{p^+} \|v\|_{s,p^+}^p - \frac{t(p^\pm)^*}{(p^+)^*} \|v\|_{L^{(p^+)^*}(\Omega)}^p - \frac{t(p^\pm)^*}{(p^+)^*} \|v\|_{L^{(p^+)^*}(\Omega)}^p - \frac{\lambda \beta^\pm}{\beta^+} \|v\|_{\beta^\pm}^\pm(\Omega).
$$

Here $U^\alpha$, for $\alpha > 0$, denotes the maximum of $U^\alpha$ and $U^\alpha$. Therefore,

$$
0 \leq \frac{p^\pm}{p^-} t_{\lambda}^\pm - \|v\|_{W_0^p}^p + t_{\lambda}^\pm - \|v\|_{s,p^+}^p - \frac{t_{\lambda}^*(p^\pm)^* - 1}{(p^+)^*} \|v\|_{L^{(p^+)^*}(\Omega)}^p \|v\|_{L^{(p^+)^*}(\Omega)}^p - \frac{\lambda \beta^\pm}{\beta^+} t_{\lambda}^\pm - \|v\|_{\beta^\pm}^\pm(\Omega), \tag{4.47}
$$

[5.53]
So we have
\[
\frac{1}{\beta^+} \|v\|_{L^\beta(\Omega)}^{\beta^+} \leq \frac{p^+}{p^-} t_{\lambda}^{\beta^+} \|v\|^p_{L^p(\Omega)} + t_{\lambda}^{\beta^-} \|v\|_{L^p(\Omega)}^{\beta^-} - \frac{\lambda}{\beta^+} \left( \frac{p^+}{p^-} - \beta^+ \right) \|v\|_{L^p(\Omega)}^{\beta^+} - \frac{\lambda}{\beta^-} \left( \frac{p^+}{p^-} - \beta^- \right) \|v\|_{L^p(\Omega)}^{\beta^-}.
\]
(4.48)

Since \( p^- \leq p^+ < \beta^- < (p^*_s)^- \leq (p^*_s)^+ \), we observe that \( t_{\lambda} \to 0 \) as \( \lambda \to \infty \). Therefore, there exists a \( \Lambda > 0 \) such that for any \( \lambda > \Lambda \),
\[
\sup_{t \geq 0} \Psi(tu) < 2 \frac{(p^*_s)^+}{(p^*_s)^-} \left( \frac{1}{p^+} - \frac{1}{(p^*_s)^-} \right) \min \left( \frac{S_+}{S_+^N}, \frac{S_+^N}{S_+^N} \right).
\]
and hence the claim. Thus, we conclude that there exists a critical point \( u \) of \( \Psi \) in \( X \) which is also a weak solution of the problem \( (\ref{L8}) \).

\[\square\]

5 Appendix

Following are a few Lemmas and results that have been used at several places in the manuscript.

Lemma 5.1 (\[17\]). Let \( \mu \) and \( \nu \) are two positive bounded measures on \( \mathbb{R}^N \) satisfying
\[
\left( \int_{\mathbb{R}^N} |\phi|^t d\nu \right)^{\frac{1}{t}} \leq C \left( \int_{\mathbb{R}^N} |\phi|^t d\mu \right)^{\frac{1}{t}}, \quad \forall \phi \in C_c^\infty(\mathbb{R}^N),
\]
for \( 1 \leq t < h \leq \infty \) and for some \( C > 0 \). Then there exist a countable set \( I \), a collection of distinct points \( \{x_i : i \in I\} \subset \mathbb{R}^N \) and \( \{\nu_i : i \in I\} \subset (0, \infty) \) such that
\[
\nu = \sum_{i \in I} \nu_i \delta_{x_i}, \quad \mu \geq C^{-t} \sum_{i \in I} \nu_i^\frac{1}{t} \delta_{x_i}.
\]

Lemma 5.2 (Brézis-Lieb lemma, \[8\]). Let \( u_n \to u \) a.e. and \( u_n \to u \) weakly in \( L^p(\Omega) \) for all \( n \) where \( \Omega \subset \mathbb{R}^N \) and \( 0 < p < \infty \). Then
\[
\lim_{n \to \infty} \left( \int_{\Omega} |u_n|^p - \int_{\Omega} |u_n - u|^p \right) = \int_{\Omega} |u|^p.
\]

Lemma 5.3. Consider the mapping \( I_1 : W_0 \to W_0^* \) defined as
\[
\langle I_1(u), v \rangle = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p(x,y) - 2(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+sp(x,y)}} dx dy,
\]
for every \( u, v \in W_0 \). Then the following properties hold for \( I_1 \).

1. \( I_1 \) is a bounded and strictly monotone operator.
2. \( I_1 \) is a mapping of \((S_+)\) type, i.e. if \( u_n \to u \) weakly in \( W_0 \) and \( \lim_{n \to \infty} \sup \langle I_1(u_n) - I_1(u), u_n - u \rangle \leq 0 \), then \( u_n \to u \) strongly in \( W_0 \).
3. \( I_1 : W_0 \to W_0^* \) is a homeomorphism.

Remark 5.4. The Lemma 5.3 is a generalization of the Lemma 4.2 in \[2\] where the authors worked with the space \( \{u \in W^{s,q}(\Omega) : u = 0 \text{ in } \partial \Omega\} \). The proof here follows exactly the same arguments even in this case.
Conclusion

We have developed a concentration compactness type principle in a variable exponent setup. This has been applied to a problem involving fractional \((p(x), p^+)-\)Laplacian to guarantee the existence of a nontrivial solution.

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References

[1] A. Bahrouni, Comparison and sub-supersolution principles for the fractional \(p(\cdot)-\)Laplacian, J. Math. Anal. Appl. 458 (2), 2018, 1363-1372.

[2] A. Bahrouni, V. Rădulescu, On a new fractional Sobolev space and applications to non-local variational problems with variable exponent, Discrete Contin. Dyn. Syst. Ser. S 11 (3), 2018, 379389.

[3] M. Bhakta, D. Mukherjee, Multiplicity results for \((p, q)\) fractional elliptic equations involving critical nonlinearities, Adv. Differential Equations, 3/4(4), 185-228, 2019.

[4] G. M. Bisci, V. D. Rădulescu and R. Servadei, Variational Methods for Nonlocal Fractional Problems (Encyclopedia of Mathematics and its Applications), Cambridge University Press, Cambridge, 162, 2016.

[5] R. Biswas, S. Tiwari, Multiplicity and uniform estimate for a class of variable order fractional \(p(x)-\)Laplacian problems with concave-convex nonlinearities, arXiv:1810.12960v3 [math.AP], 2019.

[6] J. F. Bonder and A. Silva, Concentration-compactness principle for variable exponent spaces and applications, Elec. Jour. of Differ. Equ., 141, 2010, 1-18.

[7] J. F. Bonder, N. Saintier, A. Silva, The concentration-compactness principle for fractional order Sobolev spaces in unbounded domains and applications to the generalized fractional BrezisNirenberg problem, Nonlinear Differ. Equ. Appl., 2018, 25-52.

[8] H. Brézis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Am. Math. Soc., 88(3), 1983, 486490.

[9] J. Chabrowski, Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents, Calc. Var. Partial Differ. Equ, 3(4), 1995, 493512.
[10] S. Dipierro, M. Medina, E. Valdinoci, Fractional elliptic problems with critical growth in the whole of $\mathbb{R}^N$, arXiv:1506.01748v2 [math.AP], 2016.

[11] X. L. Fan, D. Zhao, On the generalized Olicz-Sobolev space $W^{1,p(x)}(\Omega)$, J. Gansu Educ. college, 12(1), 1998, 1-6.

[12] Y. Fu, The principle of concentration compactness in $L^p(x)(\Omega)$ spaces and its application, Nonlinear Anal., 71(5-6), 2009, 18761892.

[13] J. Giacomoni, S. Tiwari and G. Warnault, Quasilinear parabolic problem with $p(x)$-laplacian: existence, uniqueness of weak solutions and stabilization, NoDEA Nonlinear Differential Equations Appl., 23, 2016, 24.

[14] K. Ho and Y. Kim, A-priori bounds and multiplicity of solutions for nonlinear elliptic problems involving the fractional $p()$-Laplacian, arXiv:1810.04818v1 [math.AP] 11 Oct 2018.

[15] U. Kaufmann, J. D. Rossi, R. Vidal, Fractional Sobolev spaces with variable exponents and fractional $p(x)$-Laplacians, Electronic Journal of Qualitative Theory of Differential Equations, 76, 2017, 1-10.

[16] P. L. Lions, The concentration-compactness principle in the calculus of variations, locally compact case, Part 1,2. Ann. Inst. H. Poincaré, 1 (1 and 4), 1984, 109-145, 223-283.

[17] P. L. Lions, The concentration-compactness principle in the calculus of variations, The limit case., part I, Rev. Mat. Iberoamericana, 1(1), 1985, 145201.

[18] P. L. Lions, The concentration-compactness principle in the calculus of variations, The limit case., part II, Rev. Mat. Iberoamericana, 1(2), 1985, 45121.

[19] P. Mironescu, W. Sickel, A Sobolev non embedding, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei, (9) Mat. Appl., 26 (3), 2015, 291298.

[20] S. Mosconi, K. Perera, M. Squassina, Y. Yang, The Brezis-Nirenberg problem for the fractional $p$-Laplacian, Calc. Var. Partial Differ. Equ., 55(4), 2016, 125.

[21] S. Mosconi, M. Squassina, Nonlocal problems at nearly critical growth, Nonlinear Analysis, 136, 2016, 84101.

[22] G. Palatucci, A. Pisante, Improved Sobolev embeddings, profile decomposition and concentration-compactness for fractional Sobolev spaces. Calc. Var. Partial Differ. Equ. 50(34), 2014, 799829.

[23] L. D. Pezzo and J. D. Rossi, Traces for fractional Sobolev spaces with variable exponents, Advances in Operator Theory, 2, 2017, 435-446.

[24] J. Simon, Régularité de la solution d’une équation non-linéaire dans $\mathbb{R}^n$, Journée d’Analyse Non Linéaire. Proceedings, Besancon, France, Lectures notes in Mathematics. Springer-Verlag, 1997.
[25] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhikers guide to the fractional Sobolev spaces, Bull. Sci. Math, 136, 2012, 521-573.