From Quantum Query Complexity to
State Complexity

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Abstract. State complexity of quantum finite automata is one of the interesting topics in studying the power of quantum finite automata. It is therefore of importance to develop general methods how to show state succinctness results for quantum finite automata. One such method is presented and demonstrated in this paper. In particular, we show that state succinctness results can be derived out of query complexity results.

1 Introduction

An important way to get deeper insights into the power of various quantum resources and operations is to explore the power of various quantum variations of the basic models of classical automata. Of a special interest is to do that for various quantum variations of the classical automata, especially for those models that use very limited amounts of quantum resources: states, correlations, operations and measurements. This paper aims to contribute to such a line of research.

Number of (basis) states used is a natural complexity measure for (quantum) finite automata. The size of a (quantum) finite automaton is defined as the number of (basis) states of the (Hilbert) space on which the automaton will operate. In case of a hybrid, that is quantum/classical finite automata, it is natural to consider both complexity measures – the number of classical states and also the number of quantum (basis) states.

Quantum finite automata were introduced by Kondacs and Watrous\(^{28}\) and also by Moore and Crutchfields\(^{33}\), and since that time they were intensively explored\(^{11,13,34,39}\). State complexity and succinctness results are an important research area of the classical finite automata theory, see\(^{38}\), with a variety of applications. Once quantum versions of classical finite automata were introduced and explored, it started to be of large interest to find out, also through

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succinctness results, a relation between the power of classical and quantum finite automata models. This has turned out to be an area of surprising outcomes that again indicated that the relations between the classical and the corresponding quantum finite automata models are intriguing. In the past twenty years, state complexity of several variants of quantum finite automata were deeply and broadly studied [23,43,4,10,11,19,22,24,26,29,30,31,36,40,42,43,44].

State succinctness results were proved for some special languages and promise problems and for several automata models. The methods used to prove those results are various and often ad hoc. It is therefore natural to try to find out whether there are quite general methods to get state succinctness results for quantum finite automata. The answer is yes. We will show, in this paper, that state succinctness results can be derived in a nice way out of query complexity results. Here is the basic idea: State complexity is deeply related to communication complexity [27]. Buhrman et al. proved that various communication complexity results can be derived out of query complexity results [14]. If a communication protocol is simple enough, then we can use quantum finite automata to implement it. By using this line of thought, state succinctness results can be derived.

Quantum query complexity is the quantum generalization of the model of decision tree complexity. In this model, an algorithm to compute a Boolean function $f : \{0,1\}^n \to \{0,1\}$ is charged for “queries” to the input bits, while any intermediate computation is considered as free (see [13]).

Communication complexity was introduced by Yao [37] in 1979. In the setting of two parties, Alice is given $x \in \{0,1\}^n$, Bob is given $y \in \{0,1\}^n$ and their task is to communicate in order to determine the value of some Boolean function $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$, while exchanging as small number of bits as possible. In this model, local computation is considered to be free, but communication is considered to be expensive and has to be minimized. Moreover, for computation, Alice and Bob can use all power available. There are usually three types of communication complexities considered according to the models of protocols used by Alice and Bob: deterministic, probabilistic and quantum.

Query complexity and communication complexity are related to each other. By using a simulation technique that transforms quantum query algorithms to quantum communication protocols, Buhrman et al. [14,41] obtained new quantum communication protocols and showed the first impressively (exponential) gap between quantum and classical communication complexity. In the reverse direction, Buhrman et al. showed that how to use lower bounds for quantum communication protocols to derive lower bounds for quantum query algorithms.

State complexity of finite automata and communication complexity are also related to each other. We can use communication complexity results to prove lower bounds on state complexity [25,26,27]. On the other hand, if the communication protocols are easy enough, then they can be simulated by finite automata and obtain new state complexity results (upper bounds) for finite automata.

Therefore, we can build connections from query complexity to state complexity. This could be a potential framework to get state succinctness results for
quantum finite automata comparing to classical finite automata. We will demonstrate for several cases in this paper, that how to use quantum query complexity results to derive state succinctness results of finite automata.

We first consider the promise problem (partial function) studied in [32]. Namely, the problem

\[ DJ'(x) = \begin{cases} 
1 & \text{if } W(x) \in \{0, 1, n-1, n\} \\
0 & \text{if } W(x) = \frac{n}{2},
\end{cases} \]  

(1)

where \( W(x) \) is the Hamming weight of \( x \). Montanaro et al. [32] gave a quantum query algorithm for \( DJ' \) with 2 queries. However, their proof is quite complicated. Motivated by the method from [7], we give a simpler quantum query algorithm with 2 queries for \( DJ' \).

Based on this simple query algorithm, we design a quantum communication protocol for the following promise problem

\[ EQ'(x, y) = \begin{cases} 
1 & \text{if } H(x, y) \in \{0, 1, n-1, n\} \\
0 & \text{if } H(x, y) = \frac{n}{2},
\end{cases} \]  

(2)

where \( H(x, y) \) is the Hamming distance between bit strings \( x \) and \( y \). We further prove that the exact quantum communication complexity of \( EQ' \) is \( O(\log n) \) while the deterministic communication complexity is \( \Omega(n) \).

Finally, we consider the promise problem \( A(n) = (A_{yes}(n), A_{no}(n)) \), where \( A_{yes}(n) = \{x\#y\#x\#y\} \ | \ H(x, y) \in \{0, 1, n-1, n\}, x, y \in \{0, 1\}^n \) and \( A_{no}(n) = \{x\#y\#x\#y\} \ | \ H(x, y) = \frac{n}{2}, x, y \in \{0, 1\}^n \}. \) We will prove that the promise problem \( A(n) \) can be solved exactly by a one-way finite automata with quantum and classical state (1QCFA) with \( O(n^2) \) quantum basis states and \( O(n^3) \) classical states, whereas the sizes of the corresponding one-way deterministic finite automata (1DFA) are \( 2^{\Omega(n)} \).

The paper is structured as follows. In Section 2 basic concepts and notations are introduced and models involved are described in some details. A new quantum query algorithm is given for \( DJ' \) in Section 3. Communication complexity of \( EQ' \) is explored in Section 4. State complexity results for the promise problem \( A(n) \) are showed in Section 5.

2 Preliminaries

In this section, we recall some basic definitions about query complexity, communication complexity and quantum finite automata. Concerning basic concepts and notations of quantum information processing and finite automata, we refer the reader to [20,21,22,35].

2.1 Exact query complexity

Exact quantum query complexity for partial functions was dealt with in [12,17,23] and for total functions in [6,7,8,9,32]. Concerning more basic concepts and notations concerning query complexity, we refer the reader to [15].
An exact classical (deterministic) query algorithm for computing a Boolean function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \) can be described by a decision tree. A decision tree \( T \) is a rooted binary tree where each internal vertex has exactly two children, each internal vertex is labeled with a variable \( x_i \) and each leaf is labeled with a value 0 or 1. \( T \) computes a Boolean function \( f \) as follows: The start is at the root. If this is a leaf then stop. Otherwise, query the variable \( x_i \) that labels the root. If \( x_i = 0 \), then recursively evaluate the left subtree, if \( x_i = 1 \) then recursively evaluate the right subtree. The output of the tree is the value of the leaf that is reached at the end of this process. The depth of \( T \) is the maximal length of a path from the root to a leaf (i.e. the worst-case number of queries used on any input).

The exact classical query complexity (deterministic query complexity, decision tree complexity) is the minimal depth over all decision trees computing \( f \).

Let \( f : \{0, 1\}^n \rightarrow \{0, 1\} \) be a Boolean function and \( x = x_1x_2\ldots x_n \) be an input bit string. An exact quantum query algorithm for \( f \) works in a Hilbert space with some fixed number of basis states. It starts in a fixed starting state, then performs on it a sequence of transformations \( U_1, Q, U_2, Q, \ldots, U_t, Q, U_{t+1} \). Unitary transformations \( U_i \) do not depend on the input bits, while \( Q \), called the query transformation, does, in the following way. Each of the basis states corresponds to either one or none of the input bits. If the basis state \( |\psi\rangle \) corresponds to the \( i \)-th input bit, then \( Q|\psi\rangle = (-1)^{x_i}|\psi\rangle \). If it does not correspond to any input bit, then \( Q \) leaves it unchanged: \( Q|\psi\rangle = |\psi\rangle \). Finally, the algorithm performs a measurement in the standard basis. Depending on the result of the measurement, the algorithm outputs either 0 or 1 which must be equal to \( f(x) \). The exact quantum query complexity is the minimum number of queries made by any quantum algorithm computing \( f \).

### 2.2 Communication complexity

We recall here only very basic concepts and notations of communication complexity, and we refer the reader to [27] for more details. We will deal with the situation that there are two communicating parties and with very simple tasks of computing two inputs Boolean functions for the case one input is known to one party and the other input to the other party. We will completely ignore computational resources needed by parties and focus solely on the amount of communication need to be exchanged between both parties in order to compute the value of a given Boolean function.

More technically, let \( X, Y \) be finite subsets of \( \{0, 1\}^n \). We will consider two-input functions \( f : X \times Y \rightarrow \{0, 1\} \) and two communicating parties. Alice is given \( x \in X \) and Bob is given \( y \in Y \). They want to compute \( f(x,y) \). If \( f \) is defined only on a proper subset of \( X \times Y \), \( f \) is said to be a partial function or a promise problem.

The computation of \( f(x,y) \) will be done using a communication protocol. During the execution of the protocol, the two parties alternate roles in sending messages. Each of these messages is a bit-string. The protocol, whose steps are based on the communication so far, specifies also for each step whether the communication terminates (in which case it also specifies what is the output).
If the communication is not to terminate, the protocol specifies what kind of message the sender (Alice or Bob) should send next, as a function of its input and communication so far.

A deterministic communication protocol $P$ computes a (partial) function $f$, if for every (promised) input pair $(x, y) \in X \times Y$ the protocol terminates with the value $f(x, y)$ as its output. In a probabilistic protocol, Alice and Bob may also flip coins during the protocol execution and proceed according to its output and the protocol can also have an erroneous output with a small probability. In a quantum protocol, Alice and Bob may use quantum resources to produce the output or (qu)bits for communication.

Let $P(x, y)$ denote the output of the protocol $P$. For an exact protocol, that always outputs the correct answer, $Pr(P(x, y) = f(x, y)) = 1$.

The communication complexity of a protocol $P$ is the worst case number of (qu)bits exchanged. The communication complexity of $f$ is, with which respect to the communication mode used, the complexity of an optimal protocol for $f$.

We will use $D(f)$ to denote the deterministic communication complexity and $Q_E(f)$ to denote the exact quantum communication complexity.

### 2.3 One-way finite automata with quantum and classical states

In this subsection we recall the definition of 1QCFA.

Two-way finite automata with quantum and classical states were introduced by Ambainis and Watrous [1] and then explored in [40][41][42][43][44][45]. 1QCFA are one-way versions of 2QCFA, which were introduced by Zheng et al. [42].

Informally, a 1QCFA can be seen as a 1DFA which has access to a quantum memory of a constant size (dimension), upon which it performs quantum transformations and measurements. Given a finite set of quantum basis states $Q$, we denote by $\mathcal{H}(Q)$ the Hilbert space spanned by $Q$. Let $\mathcal{U}(\mathcal{H}(Q))$ and $\mathcal{O}(\mathcal{H}(Q))$ denote the sets of unitary operators and projective measurements over $\mathcal{H}(Q)$, respectively.

**Definition 1.** A one-way finite automaton with quantum and classical states $A$ is specified by a 10-tuple

$$A = (Q, S, \Sigma, \Theta, \Delta, \delta, |q_0\rangle, s_0, S_{acc}, S_{rej})$$

where:

1. $Q$ is a finite set of orthonormal quantum basis states.
2. $S$ is a finite set of classical states.
3. $\Sigma$ is a finite alphabet of input symbols and let $\Sigma' = \Sigma \cup \{\text{'}, \text{'}\}$, where $\text{'}, \text{'}$ will be used as the left end-marker and $\text{'}, \text{'}$ as the right end-marker.
4. $|q_0\rangle \in Q$ is the initial quantum state.
5. $s_0$ is the initial classical state.
6. $S_{acc} \subset S$ and $S_{rej} \subset S$, where $S_{acc} \cap S_{rej} = \emptyset$, are sets of the classical accepting and rejecting states, respectively.
7. $\Theta$ is a quantum transition function

$$\Theta : S \setminus (S_{\text{acc}} \cup S_{\text{rej}}) \times \Sigma' \to U(H(Q)),$$

assigning to each pair $(s, \gamma)$ a unitary transformation.

8. $\Delta$ is a mapping

$$\Delta : S \times \Sigma' \to O(H(Q)),$$

where each $\Delta(s, \gamma)$ corresponds to a projective measurement (a projective measurement will be taken each time a unitary transformation is applied; if we do not need a measurement, we denote that $\Delta(s, \gamma) = I$, and we assume the result of the measurement to be a fixed $c$).

9. $\delta$ is a special transition function of classical states. Let the results set of the measurement be $C = \{c_1, c_2, \ldots, c_s\}$, then

$$\delta : S \times \Sigma' \times C \to S,$$

where $\delta(s, \gamma)(c_i) = s'$ means that if a tape symbol $\gamma \in \Sigma'$ is being scanned and the projective measurement result is $c_i$, then the state $s$ is changed to $s'$.

Given an input $w = \sigma_1 \cdots \sigma_l$, the word on the tape will be $w = \epsilon w \$$. For convenience, we denote $\sigma_0 = \epsilon$ and $\sigma_{l+1} = \$$. Now, we define the behavior of 1QCFA $A$ on the input word $w$. The computation starts in the classical state $s_0$ and the quantum state $|\phi_0\rangle$, then the transformations associated with symbols in the word $\sigma_0 \sigma_1 \cdots, \sigma_{l+1}$ are applied in succession. The transformation associated with a state $s \in S$ and a symbol $\sigma \in \Sigma'$ consists of three steps:

1. Firstly, $\Theta(s, \sigma)$ is applied to the current quantum state $|\phi\rangle$, yielding the new state $|\phi'\rangle = \Theta(s, \sigma)|\phi\rangle$.
2. Secondly, the observable $\Delta(s, \sigma) = O$ is measured on $|\phi'\rangle$. The set of possible results is $C = \{c_1, \cdots, c_s\}$. According to quantum mechanics principles, such a measurement yields the classical outcome $c_k$ with probability

$$p_k = |\langle P(c_k)|\phi'\rangle|^2,$$

and the quantum state of $A$ collapses to $P(c_k)|\phi'\rangle/\sqrt{p_k}$.
3. Thirdly, the current classical state $s$ will be changed to $\delta(s, \sigma)(c_k) = s'$.

An input word $w$ is assumed to be accepted (rejected) if and only if the classical state after scanning $\sigma_{l+1}$ is an accepting (rejecting) state. We assume that $\delta$ is well defined so that 1QCFA $A$ always accepts or rejects at the end of the computation.

Language acceptance is a special case of so called promise problem solving. A promise problem is a pair $A = (A_{\text{yes}}, A_{\text{no}})$, where $A_{\text{yes}}$, $A_{\text{no}} \subset \Sigma^*$ are disjoint sets. Languages may be viewed as promise problems that obey the additional constraint $A_{\text{yes}} \cup A_{\text{no}} = \Sigma^*$.

A promise problem $A = (A_{\text{yes}}, A_{\text{no}})$ is solved exactly by a finite automaton $A$ if

1. $\forall w \in A_{\text{yes}}, Pr[A \text{ accepts } w] = 1$, and
2. $\forall w \in A_{\text{no}}, Pr[A \text{ rejects } w] = 1$. 
3 An exact quantum query algorithm for DJ'(x)

Montanaro et al. [32] gave a quantum algorithm for DJ' with 2 queries. However, their proof is complicated. Motivated by the method from [7], we give a simpler algorithm with 2 queries for DJ' as follows:

We use basis states $|0,0\rangle$, $|i,0\rangle$, $|i,j\rangle$ and $|k\rangle$ with $0 \leq i < j \leq n$ and $1 \leq k \leq n-2$. A basis state $|i,j\rangle$ corresponds to an input bit $x_i$ for $1 \leq i \leq n$; a basis state $|k\rangle$ corresponds to an input bit $y_k$ for $1 \leq k \leq n-2$ ($y_k$ is some certain bit $x_i$) and the other basis states do not correspond to any input bit.

1. The algorithm $A$ begins in the state $|0,0\rangle$ and then a unitary mapping $U_1$ is applied on it:
   $$U_1|0,0\rangle = \sum_{i=1}^{n} \frac{1}{\sqrt{n}} |i,0\rangle.$$  
   (7)

2. $A$ then performs the query:
   $$\sum_{i=1}^{n} \frac{1}{\sqrt{n}} |i,0\rangle \rightarrow \sum_{i=1}^{n} \frac{1}{\sqrt{n}} (-1)^{x_i} |i,0\rangle.$$  
   (8)

3. $A$ performs a unitary mapping $U_2$ to the current state such that
   $$U_2|0,0\rangle = \sum_{j \geq i \geq 1} \frac{1}{\sqrt{n}} |i,j\rangle - \sum_{1 \leq j < i} \frac{1}{\sqrt{n}} |j,i\rangle + \frac{1}{\sqrt{n}} |0,0\rangle.$$  
   (9)

   and the resulting quantum state will be
   $$U_2 \sum_{i=1}^{n} \frac{1}{\sqrt{n}} (-1)^{x_i} |i,0\rangle = \frac{1}{n} \sum_{i=1}^{n} (-1)^{x_i} |0,0\rangle + \frac{1}{n} \sum_{1 \leq i < j} ((-1)^{x_i} - (-1)^{x_j}) |i,j\rangle.$$  
   (10)

4. $A$ measures the resulting state in the standard basis. If the outcome is $|0,0\rangle$, then $\sum_{i=1}^{n} (-1)^{x_i} \neq 0$ and DJ'($x$) = 1. Otherwise, suppose that we get the state $|i,j\rangle$, then we have $x_i \neq x_j$. Let $y = x \setminus \{x_i, x_j\}$, we have $W(y) \in \{0, n-2, \frac{n-2}{2}\}$. If $W(y) = \frac{n-2}{2}$, then DJ'($x$) = 0. If $W(y) \in \{0, n-2\}$, then DJ'($x$) = 1. The remaining question is exactly the Deutsch-Jozsa promise problem [17] and we can get the answer with 1 query as follows: we use the subalgorithm $B$ to solve the remaining promise problem using $n-2$ quantum basis states $|1\rangle, \ldots, |n-2\rangle$ that will work as follows:

   (a) $B$ begins in the state $|1\rangle$ and performs on it a unitary transformation $U_3$ such that
   $$U_3|1\rangle = \sum_{k=1}^{n-2} \frac{1}{\sqrt{n-2}} |k\rangle.$$  
   (11)

   (b) $B$ performs a query $Q$:
   $$\sum_{k=1}^{n-2} \frac{1}{\sqrt{n-2}} |k\rangle \rightarrow \sum_{k=1}^{n-2} \frac{1}{\sqrt{n-2}} (-1)^{y_k} |k\rangle.$$  
   (12)
(c) $B$ performs a unitary transformation $U_4 = U_3^{-1}$ and
\[
U_3^{-1} \sum_{k=1}^{n-2} \frac{1}{\sqrt{n-2}} (-1)^{y_k} |k\rangle = \frac{1}{n-2} \sum_{k=1}^{n-2} (-1)^{y_k} |1\rangle + \sum_{k=2}^{n-2} \beta_k |k\rangle,
\] (13)

where $\beta_k$ are amplitudes that we do not need to be specified exactly.
(d) $B$ measures the resulting state in the standard basis and outputs 1 if the measurement outcome is $|1\rangle$ and 0 otherwise.

According to [7], the unitary mapping $U_2$ exists. The rest of the proof is easy to verify. Obviously, the algorithm $A$ uses 2 queries.

4 Communication complexity of EQ'($x, y$)

In this section, we will prove that $Q_E(EQ')$ is $O(\log n)$ while $D(EQ')$ is $\Omega(n)$.

Theorem 1. $Q_E(EQ') \in O(\log n)$.

Proof. Assume that Alice is given an input $x = x_1, \ldots, x_n$ and Bob an input $y = y_1, \ldots, y_n$. The following quantum communication protocol $P$ computes EQ' using $O(n^2)$ quantum basis states $|0, 0\rangle$, $|i, 0\rangle$, $|i, j\rangle$ and $|k\rangle$ with $0 \leq i < j \leq n$ and $1 \leq k \leq n - 2$ as follows:

1. Alice begins with the quantum state $|\psi_0\rangle = |0, 0\rangle$ and performs on it the unitary map $U_1$. The quantum state is changed to
\[
|\psi_1\rangle = U_1|0, 0\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |i, 0\rangle.
\] (14)

2. Alice applies the unitary map $U_x$ to the current state such that $U_x|i, 0\rangle = (-1)^{x_i}|i, 0\rangle$ for $i > 0$ and the quantum state is changed to
\[
|\psi_2\rangle = U_x|\psi_1\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (-1)^{x_i} |i, 0\rangle.
\] (15)

3. Alice then sends her current quantum state $|\psi_2\rangle$ to Bob.
4. Bob applies the unitary map $U_y$ to the state that he has received such that $U_y|i, 0\rangle = (-1)^{y_i}|i, 0\rangle$ for $i > 0$ and the quantum state is changed to
\[
|\psi_3\rangle = U_y|\psi_2\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (-1)^{x_i+y_i} |i, 0\rangle.
\] (16)

5. Bob applies the unitary map $U_2$ to his quantum state and the quantum state is changed to
\[
|\psi_4\rangle = U_2|\psi_3\rangle = \frac{1}{n} \sum_{i=1}^{n} (-1)^{x_i+y_i} |0, 0\rangle + \frac{1}{n} \sum_{1 \leq i < j} ((-1)^{x_i+y_i} - (-1)^{x_j+y_j}) |i, j\rangle.
\] (17)
6. Bob measures the resulting state in the standard basis and outputs 1 if the measurement outcome is \(|0, 0\rangle\). Otherwise, suppose that the outcome is \(|i, j\rangle\). Bob sends \(i\) and \(j\) to Alice using classical bits.

7. After Alice receives \(i\) and \(j\), let \(x' = x_1 \ldots x_{i-1} x_{i+1} \ldots x_{j-1} x_{j+1} \ldots x_n\). (In convenience, we write \(x' = x'_1 \ldots x'_{n-2}\)). Alice applies \(U_3\) to the basis state \(|1\rangle\) such that the quantum state is changed to

\[
|\psi_5\rangle = U_3|1\rangle = \frac{1}{\sqrt{n-2}} \sum_{k=1}^{n-2} |k\rangle. \tag{18}
\]

8. Alice then applies \(U_{x'}\) to the current state such that \(U_{x'}|k\rangle = (-1)^{x'_k}|k\rangle\) for \(k > 0\) and the quantum state is changed to

\[
|\psi_6\rangle = \frac{1}{\sqrt{n-2}} \sum_{k=1}^{n-2} (-1)^{x'_k}|k\rangle. \tag{19}
\]

9. Alice sends her current quantum state \(|\psi_6\rangle\) to Bob.

10. Bob applies the unitary map \(U_{y'}\) to the state that he has received such that \(U_{y'}|k\rangle = (-1)^{y'_k}|k\rangle\) for \(k > 0\), where \(y' = y_1 \ldots y_{i-1} y_{i+1} \ldots y_{j-1} y_{j+1} \ldots y_n\). (In convenience, we write \(y' = y'_1 \ldots y'_{n-2}\)). The quantum state is changed to

\[
|\psi_7\rangle = \frac{1}{\sqrt{n-2}} \sum_{k=1}^{n-2} (-1)^{x'_k+y'_k}|k\rangle. \tag{20}
\]

11. Bob performs a unitary transformation \(U_4 = U_3^{-1}\) to the current state and the quantum state is changed to

\[
|\psi_8\rangle = U_4|\psi_7\rangle = \frac{1}{n-2} \sum_{k=1}^{n-2} (-1)^{x'_k+y'_k}|1\rangle + \sum_{k=2}^{n-2} \beta_k|k\rangle, \tag{21}
\]

where \(\beta_k\) are amplitudes that we do not need to be specified exactly.

12. Bob measures the resulting state in standard basis and outputs 1 if the measurement outcome is \(|1\rangle\) and outputs 0 otherwise.

The unitary transformations \(U_1, U_2, U_3\) and \(U_4\) are the same ones as defined in Section 3.

If \(H(x, y) \in \{0, n\}\), then the quantum state in Step 5 is

\[
|\psi_4\rangle = \frac{1}{n} \sum_{i=1}^{n} (-1)^{x_i+y_i}|0, 0\rangle = \pm|0, 0\rangle. \tag{22}
\]

Bob will get the quantum state \(|0, 0\rangle\) after the measurement in Step 6 and output 1 as the result of \(E_{Q'}(x, y)\).

If \(H(x, y) \in \{1, n - 1\}\), then there are two cases:

a) If the measurement outcome in Step 6 is \(|0, 0\rangle\) and Bob outputs 1 as the result of \(E_{Q'}(x, y)\).
If the measurement outcome in Step 6 is \(|i, j\rangle\), then \(H(x', y') \in \{0, n - 2\}\) and the quantum state in Step 11 is

\[
|\psi_b\rangle = \frac{1}{\sqrt{n - 2}} \sum_{k=1}^{n-2} (-1)^{x_k' + y_k'} |1\rangle = \pm |1\rangle
\]  

(23)

Bob will get the quantum state \(|1\rangle\) after the measurement in Step 12 and output 1 as the result of \(EQ'(x, y)\).

If \(H(x, y) = \frac{n}{2}\), then Bob will output 0 as the result of \(EQ'(x, y)\) in Step 12.

In Step 3 Alice sends \([\log(n^2)]\) qubits, in Step 6 Bob sends \(2[\log(n)]\) bits and in Step 9 Alice sends \([\log(n - 2)]\) qubits. Since we can use qubits to send bits, it is clear that this protocol uses only \(O(\log n)\) qubits for communication.

The proof for deterministic communication lower bound is similar to the ones in 15\(16\). In order to obtain an exponential quantum speed-up in 16\(16\), \(\frac{n}{2}\) must be an even integer in the distributed Deutsch-Jozsa promise problem (see 23 for argument). However, \(\frac{n}{2}\) can be arbitrary integer in the promise problem \(EQ'\) in this paper.

We use so called “rectangles” lower bound method 27 to prove the result.

A rectangle in \(X \times Y\) is a subset \(R \subseteq X \times Y\) such that \(R = A \times B\) for some \(A \subseteq X\) and \(B \subseteq Y\). A rectangle \(R = A \times B\) is called 1(0)-rectangle of a function \(f : X \times Y \rightarrow \{0, 1\}\) if for every \(x \in A\) and \(y \in B\) the value of \(f(x, y)\) is 1 (0).

Moreover, \(C^i(f)\) is defined as the minimum number of \(i\)-rectangles that partition the space of \(i\)-inputs (such inputs \(x\) and \(y\) that \(f(x, y) = i\)) of \(f\).

**Lemma 1.** 27 \(D(f) \geq \max\{\log C^1(f), \log C^0(f)\}\).

**Remark 1.** For a partial function \(f : X \times Y \rightarrow \{0, 1\}\) with domain \(D\), a rectangle \(R = A \times B\) is called 1(0)-rectangle if the value of \(f(x, y)\) is 1(0) for every \((x, y) \in D \cap (A \times B)\) – we do not care about values for \((x, y) \notin D\). The above lemma still holds for promise problems (that is for partial functions).

**Theorem 2.** \(D(EQ') \in \Omega(n)\).

**Proof.** Let \(P\) be a deterministic protocol for \(EQ'\). There are two cases:

**Case 1:** \(\frac{n}{2}\) is even. We consider the set \(E = \{(x, x), (x, \pi) | W(x) = \frac{n}{2}\}\). For every \((x, y) \in E\), we have \(P(x, y) = 1\). Suppose there is a 1-monochromatic rectangle \(R = A \times B \subseteq \{0, 1\}^n \times \{0, 1\}^n\) such that \(P(x, y) = 1\) for every promise pair \((x, y) \in R\). Let \(S = R \cap E\). For \(x, y \in \{0, 1\}^n\), let us denote \(x \land y = \sum_{i=1}^{n} x_i \land y_i\). We now prove that for any distinct \((x, x'), (y, y') \in S\), \(x \land y \neq \frac{n}{2}\).

According to the assumption that \((y, y') \in S \subseteq E\), we have \(y' = y\) or \(y' = \pi\).

If \(x \land y = \frac{n}{2}\), then \(H(x, y) = 2(\frac{n}{2} - \frac{n}{2}) = \frac{n}{2} = H(x, \pi)\) and \(P(x, y') = 0\). Since \((x, x') \in R\) and \((y, y') \in R\), we have \((x, y') \in R\) and \(P(x, y') = 0\), which is a contradiction.

According to Corollary 1.2 from 13, we have \(|S| \leq 1.99^n\). Therefore, the minimum number of 1-monochromatic rectangles that partition the space of
inputs is
\[
C^1(\text{EQ}') \geq \frac{|E|}{|S|} \geq \frac{2\binom{n}{2}}{(1.99)^n} > \frac{2^{n+1}/n}{(1.99)^n}.
\]  
(24)

The deterministic communication complexity is then
\[
D(\text{EQ}') > \log C^1(\text{EQ}') > \log \frac{2^{n+1}/n}{(1.99)^n} > 0.0073n.
\]  
(25)

**Case 2:** \( \frac{n}{2} \) is odd. We assume that \( n = 4k + 2 \). We consider the set \( E = \{(x,x') \mid W(x) = \frac{n}{2}\} \), where \( x'_n = x_n = 1 \) and \( x'_i = 1 - x_i \) for \( i < n \). For every \( (x,x') \in E \), we have \( H(x,x') = n - 1 \) and \( \mathcal{P}(x,x') = 1 \). Suppose there is a 1-monochromatic rectangle \( R = A \times B \subseteq \{0,1\}^n \times \{0,1\}^n \) such that \( \mathcal{P}(x,y) = 1 \) for every promise pair \( (x,y) \in R \). Let \( S = R \cap E \). We now prove that for any distinct \( (x,x'),(y,y') \in S \), \( \sum_{i=1}^{n-1} |x_i \land y_i| \neq k \), that is \( |x \land y| \neq k + 1 \).

If \( |x \land y| = k+1 \), without a lost of generality, let \( x = k \underbrace{1 \cdots 1}_{k} \cdot \underbrace{0 \cdots 1}_{k+1} \) and \( y = 1 \underbrace{0 \cdots 0}_{k} \cdot \underbrace{1 \cdots 1}_{k+1} \). We have \( H(x,y') = k + k + 1 = \frac{n}{2} \) and \( \mathcal{P}(x,y') = 0 \). Since \( (x,x') \in R \) and \( (y,y') \in R \), we have \( (x,y') \in R \) and \( \mathcal{P}(x,y') = 0 \), which is a contradiction.

According to Corollary 1.2 from [15], we have \( |S| \leq 1.99^n \). Therefore, the minimum number of 1-monochromatic rectangles that partition the space of inputs is
\[
C^1(\text{EQ}') \geq \frac{|E|}{|S|} \geq \frac{\binom{n-1}{n/2-1}}{(1.99)^{n-1}} > \frac{2^{n-1}/(n-1)}{(1.99)^{n-1}}.
\]  
(26)

The deterministic communication complexity is then
\[
D(\text{EQ}') > \log C^1(\text{EQ}') > \log \frac{2^{n-1}/(n-1)}{(1.99)^{n-1}} > 0.0073n.
\]  
(27)

Therefore the theorem has been proved.

5 State succinctness results

Now we are ready to derive the state succinctness result.

**Theorem 3.** The promise problem \( A(n) \) can be solved exactly by a 1QCFA \( A(n) \) with \( O(n^2) \) quantum basis states and \( O(n^3) \) classical states, whereas the sizes of the corresponding 1DFA are \( 2^{O(n)} \).

**Proof.** Let \( x = x_1 \cdots x_n \) and \( y = y_1 \cdots y_n \) be in \( \{0,1\}^n \). The input word on the tape will be \( w = \varepsilon x \# y \# \# x \# y \$\). Let us consider a 1QCFA \( A(n) \) with \( O(n^2) \) quantum basis states \( \{0,0], [i,0], [i,j], [k] : 0 \leq i < j \leq n, 1 \leq k \leq n - 2 \} \) and \( O(n^3) \) classical states \( \{S_{i,j,p} : 0 \leq i,j,p \leq n + 1 \} \) (some of the states may be not used in the automaton actions).
\( \mathcal{A}(n) \) starts in the initial quantum state \( |0, 0\rangle \) and the initial classical state \( S_{000} \). We use classical states \( S_{ijp} \) (\( 1 \leq p \leq n + 1 \)) to point out the positions of the tape head that will provide some information for quantum transformations. If the classical state of \( \mathcal{A}(n) \) will be \( S_{ijp} \) (\( 1 \leq p \leq n \)) that will mean that the next scanned symbol of the tape head is the \( p \)-th symbol of \( x(y) \) and \( S_{ijn+1} \) means that the next scanned symbol of the tape head is \( # \). \( \)  

The behavior of \( \mathcal{A}(n) \) is composed of two parts. The first part is the behavior of \( \mathcal{A}(n) \) when reading the prefix of the input, namely \( \epsilon x y # \). In this part, \( \mathcal{A}(n) \) uses quantum basis state \( \{ |0, 0\rangle, |i, 0\rangle, |i, j\rangle : 0 \leq i < j \leq n \} \) and classical states \( S_{00p} \) (\( 0 \leq p \leq n + 1 \)) to simulate Steps 1 to 6 in the proof of Theorem 1. After the measurement at the end of the first part, if the outcome is \( |0, 0\rangle \), then the input is accepted. Otherwise, suppose that the outcome is \( |i, j\rangle \), the classical state will be changed to \( S_{ij0} \) (\( 1 \leq i < j \leq n \)), which means that \( H(x_i x_j y_i y_j) = 1 \) and the input bits \( x_i, x_j, y_i, y_j \) will be skipped during the second part of the behavior of \( \mathcal{A}(n) \). The second part is the behavior of the automation when reading the second part of the input \( #x y # \). In this part, \( \mathcal{A}(n) \) uses quantum basis states \( \{ |k\rangle : 1 \leq k \leq n - 2 \} \) and classical states \( S_{ijp} \) (\( 0 \leq p \leq n + 1 \)) to simulate Steps 7 to 12 in the proof of Theorem 1. The automaton proceeds as follows:

1. \( \mathcal{A}(n) \) reads the left end-marker \( \epsilon \), performs \( U_1 \) on the initial quantum state \( |0, 0\rangle \), changes its classical state to \( \delta(S_{000}, \epsilon) = S_{001} \), and moves the tape head one cell to the right.
2. Until the currently scanned symbol \( \sigma \) is not \( \# \), \( \mathcal{A}(n) \) does the following:
   (a) Applies \( \Theta(S_{00p}, \sigma) = U_{p, \sigma} \) to the current quantum state.
   (b) Changes the classical state \( S_{00p} \) to \( S_{00p+1} \) and moves the tape head one cell to the right.
3. \( \mathcal{A}(n) \) changes the classical state \( S_{00p+1} \) to \( S_{001} \) and moves the tape head one cell to the right.
4. Until the currently scanned symbol \( \sigma \) is not \( \# \), \( \mathcal{A}(n) \) does the following:
   (a) Applies \( \Theta(S_{00p}, \sigma) = U_{p, \sigma} \) to the current quantum state.
   (b) Changes the classical state \( S_{00p} \) to \( S_{00p+1} \) and moves the tape head one cell to the right.
5. When \( \# \) is reached, \( \mathcal{A}(n) \) performs \( U_2 \) on the current quantum state.
6. \( \mathcal{A}(n) \) measures the current quantum state in the standard basis. If the outcome is \( |0, 0\rangle \), \( \mathcal{A}(n) \) accepts the input; otherwise, suppose that the outcome is \( |i, j\rangle \), \( \mathcal{A}(n) \) changes the classical state to \( S_{ij0} \), moves the tape head one cell to the right.
7. \( \mathcal{A}(n) \) reads \( \# \), applies \( \Theta(S_{ij0}, \#) = U_{ij} \) to the current quantum state, changes its classical state to \( S_{ij1} \), and moves the tape head one cell to the right.
8. Until the currently scanned symbol \( \sigma \) is not \( \# \), \( \mathcal{A}(n) \) does the following:
   (a) Applies \( \Theta(S_{ijp}, \sigma) = U_{ijp, \sigma} \) to the current quantum state.
   (b) Changes the classical state \( S_{ijp} \) to \( S_{ijp+1} \) and moves the tape head one cell to the right.
9. \( \mathcal{A}(n) \) changes the classical state \( S_{ijp+1} \) to \( S_{ij1} \) and moves the tape head one cell to the right.
10. While the currently scanned symbol $\sigma$ is not the right end-marker $\$, $A(n)$ does the following:
   (a) Applies $\Theta(S_{ijp}, \sigma) = U_{ijp, \sigma}$ to the current quantum state.
   (b) Changes the classical state $S_{ijp}$ to $S_{ijp+1}$ and moves the tape head one cell to the right.

11. When the right end-marker is reached, $A(n)$ performs $U_4$ on the current quantum state.

12. $A(n)$ measures the current quantum state in the standard basis. If the outcome is $|1\rangle$, $A(n)$ accepts the input; otherwise, rejects the input.

where unitary transformations $U_1, U_2, U_3, U_4$ are the ones defined in the proof of Theorem 1

\[
U_{p, \sigma}|i, j\rangle = (-1)^\sigma|i, j\rangle \text{ if } i = p;
\]
\[
U_{p, \sigma}|i, j\rangle = |i, j\rangle \text{ if } i \neq p;
\]
\[
U_{ij}|i, j\rangle = |1\rangle;
\]
\[
U_{ijp, \sigma}|k\rangle = (-1)^\sigma|k\rangle \text{ if } p < i \text{ and } k = p;
\]
\[
U_{ijp, \sigma}|k\rangle = |k\rangle \text{ if } p < i \text{ and } k \neq p;
\]
\[
U_{ijp, \sigma}|k\rangle = |k\rangle \text{ if } p = i;
\]
\[
U_{ijp, \sigma}|k\rangle = (-1)^\sigma|k\rangle \text{ if } i < p < j \text{ and } k = p - 1;
\]
\[
U_{ijp, \sigma}|k\rangle = |k\rangle \text{ if } i < p < j \text{ and } k \neq p - 1;
\]
\[
U_{ijp, \sigma}|k\rangle = |k\rangle \text{ if } p = j;
\]
\[
U_{ijp, \sigma}|k\rangle = (-1)^\sigma|k\rangle \text{ if } p > j \text{ and } k = p - 2;
\]
\[
U_{ijp, \sigma}|k\rangle = |k\rangle \text{ if } p > j \text{ and } k \neq p - 2.
\]

It is easy to verify that unitary transformation $U_{p, \sigma}$, $U_{ij}$ and $U_{ijp, \sigma}$ exist. The rest of the proof is analogues to the proof in Theorem 1.

According to Theorem 2, $D(EQ') \in \Omega(n)$. Therefore, it is easy to see that the sizes of the corresponding 1DFA for $A(n)$ are $2^{\Omega(n)}$ [27].

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