SOBOLEV ORTHOGONAL POLYNOMIALS ON THE CONIC SURFACE

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ABSTRACT. Orthogonal polynomials with respect to the weight function \( w_{\beta, \gamma}(t) = t^\beta (1-t)^\gamma \), \( \gamma > -1 \), on the conic surface \( \{ (x, t) : \|x\| = t, x \in \mathbb{R}^d, t \leq 1 \} \) are studied recently, and they are shown to be eigenfunctions of a second order differential operator \( D_\gamma \) when \( \beta = -1 \). We extend the setting to the Sobolev inner product, defined as the integration of the \( s \)-th partial derivatives in \( t \) variable with respect to \( w_{\beta, \gamma} \) over the conic surface plus a sum of integrals over the rim of the cone. Our main results provide an explicit construction of an orthogonal basis and a formula for the orthogonal projection operators; the latter is used to exploit the interaction of differential operators and the projection operator, which allows us to study the convergence of the Fourier orthogonal series. The study can be regarded as an extension of the orthogonal structure to the weight function \( w_{\beta, -s} \) for a positive integer \( s \). It shows, in particular, that the Sobolev orthogonal polynomials are eigenfunctions of \( D_\gamma \) when \( \gamma = -1 \).

1. Introduction

Orthogonal polynomials on the conic surface of the revolution were studied recently, which are shown to possess properties parallel to those of spherical harmonics on the unit sphere. Let \( \mathbb{V}_0^{d+1} \) be the conic surface
\[
\mathbb{V}_0^{d+1} = \{ (x, t) \in \mathbb{R}^{d+1} : \|x\| = t, x \in \mathbb{R}^d, 0 \leq t \leq 1 \}
\]
in \( \mathbb{R}^{d+1} \). For the weight function \( w_{\beta, \gamma}(t) = t^\beta (1-t)^\gamma \), \( \beta > -d \) and \( \gamma > -1 \), the orthogonal polynomials with respect to the inner product
\[
\langle f, g \rangle_{\beta, \gamma} = b_{\beta, \gamma} \int_{\mathbb{V}_0^{d+1}} f(x, t)g(x, t)w_{\beta, \gamma}(t)dm(x, t)
\]
are called the Jacobi polynomials on the cone. These polynomials are studied in \[12, 21, 22, 23\]. It was shown in \[21\] that these polynomials share many properties of spherical harmonics, including explicit orthogonal basis and an addition formula, which provides essential tools for an extensive study in approximation theory and computational analysis over the cone in \[23\]. Another remarkable property of the Jacobi polynomials on the cone is that they are eigenfunctions of a second-order linear
differential operator $D_\gamma$ when $\beta = -1$, which is an analog of the Laplace-Beltrami operator on the unit sphere.

The purpose of the present paper is to study Sobolev orthogonal polynomials on the conic surface, which are orthogonal with respect to an inner product that contains derivatives. The first case is

$$\langle f, g \rangle_{\beta, -1} = \frac{1}{\omega_d} \int_{S^{d-1}} \frac{\partial}{\partial t} f(x, t) \frac{\partial}{\partial t} g(x, t) t^{\beta + 1} \, dm(x, t) + \frac{\lambda}{\omega_d} \int_{S^{d-1}} f(\xi, 1) g(\xi, 1) \, d\sigma(\xi),$$

and we also consider $\langle f, g \rangle_{\beta, -s}$ that involves derivatives up to $s$ order for a positive integer $s$. There is a reason that we consider only derivatives in the $t$ variable but not the $x$ variable; see the discussion in Section 4. Like in the case of $\gamma > -1$, our main result provides explicit construction of orthogonal bases and a closed-form formula for the orthogonal projection operator. The study requires an extension of the Jacobi polynomials with a parameter being a negative integer, which needs to satisfy the Sobolev orthogonality of one variable that is inherited from $\langle \cdot, \cdot \rangle_{\beta, -s}$ when we restrict the inner product to polynomials depending only on the $t$ variable. Such Sobolev orthogonal polynomials of one variable have been studied by several authors; see, for example, [1, 2, 7, 15, 16, 20] and [11]. We shall follow the approach in [20] since it is more convenient for studying orthogonal projection operators and provides a link, in particular, between the Sobolev orthogonal structure and the ordinary orthogonal structure, which is useful for studying the convergence of the Fourier orthogonal series in the Sobolev orthogonal polynomials. In the framework of polynomial approximation theory on the ball and standard or Sobolev orthogonal polynomials, we can refer to [3, 4, 5, 8, 9, 10, 14, 18], among others.

In more than one way, our study extends the Jacobi polynomials for $w_{\beta, \gamma}$ on the cone from $\gamma > -1$ to $\gamma = -s$ with $s \in \mathbb{N}$. We will show, in particular, that the spectral operator $D_{-s}$ has the Sobolev orthogonal polynomials as eigenfunctions if $s = 1$. While the latter fails for $s > 1$, we do have a clear understanding of what the eigenspaces of $D_{-s}$ are. For orthogonal polynomials in several variables, our study is also closely related to the Sobolev orthogonal polynomials on the unit ball, which have been extensively studied (see [6] and its references therein). In particular, the description of the eigenspaces of $D_\gamma$ is similar to the study on the unit ball in [13].

The paper is organized as follows. The next section is preliminary, in which we recall two essential ingredients needed for our study, the Jacobi polynomials with negative parameters and spherical harmonics. In Section 3 we review results on ordinary orthogonal polynomials on the conic surface and discuss further properties of the orthogonal projection operators. The Sobolev orthogonal polynomials are defined and studied in Section 4. Finally, the eigenspaces of the operator $D_\gamma$ are discussed in Section 5.

## 2. Preliminary

The study for orthogonal polynomials on the conic surface follows that of spherical harmonics on the unit sphere. The latter will also be essential for constructing an orthogonal basis on the cone. Another ingredient is the Jacobi polynomial, which we often need an extension to negative parameters in the study of the Sobolev orthogonal polynomials.
2.1. Jacobi polynomials with negative parameters. The Jacobi polynomials $P_n^{(\alpha, \beta)}$ are given explicitly by the hypergeometric function

\begin{equation}
P_n^{(\alpha, \beta)}(t) = \frac{(\alpha + 1)_n}{n!} \binom{n + \alpha + \beta + 1}{\alpha + 1} F_1 \left( -n, n + \alpha + \beta + 1, 1 - x \middle| \frac{1}{2} \right)
\end{equation}

for $\alpha, \beta > -1$ and $n = 0, 1, 2, \ldots$. They are orthogonal with respect to the weight function $w_{\alpha, \beta}(t) = (1 - t)^\alpha (1 + t)^\beta$ on $[-1, 1]$ with $\alpha, \beta > -1$, and they satisfy

\begin{equation}
\frac{c_{\alpha, \beta}}{2^{\alpha + \beta + 1}} \int_{-1}^{1} P_n^{(\alpha, \beta)}(t) P_n^{(\alpha, \beta)}(t) w_{\alpha, \beta}(t) dt = h_n^{(\alpha, \beta)} \delta_{n,m},
\end{equation}

where $c_{\alpha, \beta}$ is the constant so that $h_0^{(\alpha, \beta)} = 1$,

\begin{equation}
c_{\alpha, \beta} = \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1) \Gamma(\beta + 1)} \quad \text{and} \quad h_n^{(\alpha, \beta)} = \frac{(\alpha + 1)_n (\beta + 1)_n (\alpha + \beta + n + 1)}{n!(\alpha + \beta + 2)_n (\alpha + \beta + 2n + 1)}.
\end{equation}

For studying the Sobolev orthogonal polynomials, we often need the parameters $\alpha$ or $\beta$ to be negative integers. Such polynomials are discussed already in [17] but they are no longer orthogonal with respect to $w_{\alpha, \beta}$, since the weight function $(1 - t)^a (1 + t)^b$ is no longer integrable if $a$ or $b \leq -1$. Moreover, $P_n^{(\alpha, \beta)}$ has a degree reduction if $n + \alpha + \beta$ is a negative integer between 1 to $n$, which causes problems for studying the Sobolev orthogonal polynomials, especially in several variables, since such polynomials are needed for all $n \in \mathbb{N}_0$.

What we need in this paper are the polynomials $P_n^{(\alpha, -s)}$ with $\alpha > -1$ and $s \in \mathbb{N}$. These polynomials are well defined if $n \geq s$ and satisfy [17, Section 4.22]

\begin{equation}
P_n^{(\alpha, -s)}(t) = \frac{(-\alpha - n)_s}{2^s (-n)_s} (1 + t)^s P_n^{(\alpha, s)}(t), \quad n \geq s,
\end{equation}

which follows from [17 (4.22.2)] by using $P_n^{(\alpha, \beta)}(-t) = (-1)^n P_n^{(\beta, \alpha)}(t)$. The definition of $P_n^{(\alpha, -s)}$ for $n < s$ could be problematic because of the degree reduction. Such polynomials have been studied in the setting of the Sobolev orthogonal polynomials; for example, the Sobolev orthogonality defined via the inner product

\begin{equation}
[f, g]_{\alpha, \beta}^{-s} := \int_{-1}^{1} f^{(s)}(t) g^{(s)}(t) w_{\alpha, \beta}(t) dt + \sum_{k=0}^{s-1} \mu_k f^{(k)}(1) g^{(k)}(1),
\end{equation}

where $\mu_k$ are fixed positive constants and $\alpha, \beta > -1$. The study of such polynomials and their orthogonality has appeared in several papers; see, for example, [11, 17, 2, 15, 19, 20] and [11]. There are several ways to define a complete set of orthogonal polynomials for the inner product (2.4). We shall follow the approach given in [20], see also [15, 16], which is more suitable for studying the Fourier orthogonal series. We now recall the necessary result from [20].

For convenience, we first define a renormalization of the Jacobi polynomials,

\begin{equation}
\tilde{P}_n^{(\alpha, \beta)}(t) = A_n^{(\alpha, \beta)} P_n^{(\alpha, \beta)}(t) \quad \text{with} \quad A_n^{(\alpha, \beta)} = \frac{2^n}{(n + \alpha + \beta + 1)_n}
\end{equation}

for $\alpha, \beta > -1$. This normalization has the advantage that it satisfies, by [17 (4.5.5)],

\begin{equation}
\frac{d}{dt} \tilde{P}_n^{(\alpha, \beta)}(t) = \tilde{P}_n^{(\alpha + 1, \beta + 1)}(t).
\end{equation}
Now, for $\alpha, \beta > -1$, $s \in \mathbb{N}$ and $n = 0, 1, 2, \ldots$, we define a new sequence of polynomials

$$J_n^{(\alpha-s, \beta-s)}(t) := \begin{cases} \frac{(t+1)^n}{n!}, & 0 \leq n \leq s-1, \\ \int_{-1}^{t} \frac{(t-u)^{s-1}}{(s-1)!} \hat{P}_{n-s}^{(\alpha, \beta)}(u) du, & n \geq s. \end{cases}$$

(2.7)

It is easy to see that $J_n^{(\alpha-s, \beta-s)}$ is a polynomial of degree $n$ and it satisfies

$$\partial^s J_n^{(\alpha-s, \beta-s)}(t) = \hat{P}_{n-s}^{(\alpha, \beta)}(t), \quad n \geq s;$$

(2.8)

$$\partial^k J_n^{(\alpha-s, \beta-s)}(-1) = \begin{cases} \delta_{k,n}, & n \leq s-1, \\ 0, & n \geq s, \end{cases} \quad 0 \leq k \leq s-1,$$

(2.9)

where $\partial^k$ denotes the $k$-th derivative. These are our polynomials that extend the definition of the Jacobi polynomials to allow negative parameters, which are also orthogonal polynomials with respect to the Sobolev inner product (2.4). More precisely, we have the following [24]:

**Theorem 2.1.** For $\alpha, \beta > -1$ and $s \in \mathbb{N}$. The polynomial $J_n^{(\alpha-s, \beta-s)}$ is orthogonal with respect to the inner product $(\cdot, \cdot)_{\alpha, \beta}^s$ and its norm square is given by

$$\left[ J_n^{(\alpha-s, \beta-s)}, J_n^{(\alpha-s, \beta-s)} \right]_{\alpha, \beta}^{-s} = \begin{cases} \mu_n \hat{h}_{n-s}^{(\alpha, \beta)} & 0 \leq n \leq s-1, \\ \hat{h}_{n-s}^{(\alpha, \beta)} & n \geq s, \end{cases}$$

where $\mu_n$ comes from (2.4), and $\hat{h}_{n-s}^{(\alpha, \beta)}$ is the norm square of $\hat{P}_{n-s}^{(\alpha, \beta)}$, which is given in terms of $h_{n-s}^{(\alpha, \beta)}$ by

$$\hat{h}_{n-s}^{(\alpha, \beta)} = \frac{2^{\alpha+\beta+1}}{c_{\alpha, \beta}} \left[ A_n^{(\alpha, \beta)} \right]^2 h_{n-s}^{(\alpha, \beta)}.$$

Our next proposition shows that, if $\alpha > -1$ and $s \in \mathbb{N}$, then the definition $J_n^{(\alpha-s)}$ in (2.7) agrees with that of (2.3) when $n \geq s$.

**Proposition 2.2.** For $\alpha > -1$ and $s \in \mathbb{N}$,

$$J_n^{(\alpha-s)}(t) = \frac{(n-s)!}{n!} (1+t)^s \hat{P}_{n-s}^{(\alpha, s)}(t), \quad n \geq s.$$ 

(2.10)

In particular, for $n \geq s$,

$$J_n^{(\alpha-s)}(t) = \frac{(-1)^s 2^s}{(-\alpha - n)_s} A_n^{(\alpha, s)} P_n^{(\alpha-s)}(t).$$

(2.11)

**Proof.** We use the hypergeometric expression of the Jacobi polynomials,

$$P_n^{(\alpha, \beta)}(t) = \frac{(\alpha + 1)_n}{n!} 2F_1 \left( -n, n + \alpha + \beta + 1; \frac{1-t}{2} \right)$$

where $2F_1$ is the hypergeometric function.
and the fact that \( P_n^{(\alpha,\beta)}(t) = (-1)^n P_n^{(\beta,\alpha)}(-t) \). It follows that

\[
\int_{-1}^{1} \frac{(t-u)^{s-1}}{(s-1)!} P_n^{(\alpha+s,0)}(u)du = (-1)^n \int_{-1}^{1} \frac{(t-u)^{s-1}}{(s-1)!} P_n^{(0,\alpha+s)}(-u)du
\]

\[
= (-1)^n \sum_{k=0}^{n} \frac{(-n)_k(n+\alpha+s+1)_k}{k!k!} \int_{-1}^{1} \frac{(t-u)^{s-1}}{(s-1)!} \left( \frac{1+u}{2} \right)^k du
\]

\[
= \frac{(-1)^n}{s!} (1+t)^s \sum_{k=0}^{n} \frac{(-n)_k(n+\alpha+s+1)_k}{k!(s+1)_k} \left( \frac{1+t}{2} \right)^k
\]

\[
= \frac{(-1)^n}{s!} (1+t)^s \frac{n!}{(s+1)n} P_n^{(s,\alpha)}(-t) = \frac{n!}{(n+s)!} (1+t)^s P_n^{(s,\alpha)}(t).
\]

Replacing \( n \) by \( n-s \) and using \( A_n^{(\alpha,s)} = A_n^{(\alpha+s,0)} \), the identity (2.10) then follows from (2.7). The second identity follows from the identity (2.3).

It should be pointed out that the integral expression of \( J_n^{(\alpha,-s)} \) for \( n \geq s \) in (2.7) is more convenient for studying the Fourier orthogonal series, as shown in [20] and as we shall see in Section 4 below.

2.2. Spherical harmonics. A homogeneous polynomial \( Y \) of \( d \) variables is called a solid harmonic if \( \Delta Y = 0 \), where \( \Delta \) is the Laplace operator on \( \mathbb{R}^d \). We denote by \( \mathcal{H}^{d,0}_m \) the space of homogeneous solid harmonics of degree \( m \) in \( d \) variables. Thus, if \( Y \in \mathcal{H}^{d,0}_m \), then \( Y(r\xi) = r^m Y(\xi) \) for \( \xi \in S^{d-1} \). Spherical harmonics are restrictions of solid harmonics on the unit ball. We denote the space of spherical harmonics of degree \( m \) by \( \mathcal{H}^d_m \). Thus, \( \mathcal{H}^{d}_m = \mathcal{H}^{d,0}_m |_{S^{d-1}} \). It is a common practice to identify \( \mathcal{H}^{d,0}_m \) and \( \mathcal{H}^d_m \), we distinguish them to emphasize the dependence on variables for the orthogonal polynomials on the conic surface. It is well known that

\[
\dim \mathcal{H}^d_m = \binom{m+d-2}{n} + \binom{m+d-3}{n-1}, \quad m = 1, 2, 3, \ldots,
\]

and spherical harmonics of different degrees are orthogonal with respect to the surface measure on the unit sphere. Throughout the paper, we denote by \( \{Y^m_\ell : 1 \leq \ell \leq \dim \mathcal{H}^d_m \} \) an orthonormal basis of \( \mathcal{H}^d_m \), so that

\[
\frac{1}{\omega_d} \int_{S^{d-1}} Y^m_\ell(\xi) Y^m_\ell(\xi) d\sigma(\xi) = \delta_{\ell,\ell} \delta_{m,m},
\]

where \( d\sigma \) is the surface measure of \( S^{d-1} \) and \( \omega_d \) is the surface area of \( S^{d-1} \).

Let \( \text{proj}^{\mathcal{H}^{d-1}}_n : L^2(S^{d-1}) \rightarrow \mathcal{H}^d_n \) be the orthogonal projection operator from \( L^2(S^{d-1}) \) onto \( \mathcal{H}^d_n \). If \( \{Y^m_\ell : 1 \leq \ell \leq \dim \mathcal{H}^{d-1}_m \} \) is an orthonormal basis of \( \mathcal{H}^{d-1}_m \), then

\[
\text{proj}^{\mathcal{H}^{d-1}}_n f(x) = \sum_{\ell=1}^{\dim \mathcal{H}^d_n} \hat{f}_\ell Y^m_\ell, \quad \hat{f}_\ell = \frac{1}{\omega_d} \int_{S^{d-1}} f(\xi) Y^m_\ell(\xi) d\sigma(\xi).
\]

For \( f \in L^2(S^{d-1}) \), its Fourier expansion in spherical harmonics is defined by

\[
f = \sum_{n=0}^{\infty} \sum_{\ell=1}^{\dim \mathcal{H}^d_n} \hat{f}_\ell Y^m_\ell = \sum_{n=0}^{\infty} \text{proj}^{\mathcal{H}^{d-1}}_n f.
\]
Let $\Delta_0$ be the Laplace-Beltrami operator, which is the restriction of the Laplacian $\Delta$ on the unit sphere. Under the spherical polar coordinates $x = r\xi$, $r > 0$, $\xi \in \mathbb{S}^{d-1}$,
\[
\Delta = \frac{d^2}{dr^2} + \frac{d-1}{r} \frac{d}{dr} + \frac{1}{r^2} \Delta_0.
\]
The spherical harmonics are the eigenfunctions of $\Delta_0$. More precisely,
\[(2.12)\quad \Delta_0 Y = -n(n+d-2)Y, \quad Y \in \mathcal{H}_n^d, \quad n = 0, 1, 2, \ldots.
\]
The $\Delta_0$ is a second-order differential operator on the unit sphere. One can also consider spherical gradient $\nabla_0$, the first-order differential operators, on the sphere, which is defined by
\[
\nabla = \frac{1}{r} \nabla_0 + \xi \frac{d}{dr}, \quad x = r\xi, \quad \xi \in \mathbb{S}^{d-1}.
\]
The integration by parts formula holds and gives [5, (1.8.14)],
\[(2.13)\quad \int_{\mathbb{S}^{d-1}} \nabla_0 f(\xi) \cdot \nabla_0 g(\xi) d\sigma(\xi) = -\int_{\mathbb{S}^{d-1}} \Delta_0 f(\xi) g(\xi) d\sigma(\xi).
\]
Together with (2.2), this identity implies that if $\{Y^n_\ell : 1 \leq \ell \leq \dim \mathcal{H}_n^d\}$ is an orthogonal basis of $\mathcal{H}_n^d$, then
\[
\int_{\mathbb{S}^{d-1}} \nabla_0 Y^n_\ell(\xi) \cdot \nabla_0 Y^n_{\ell'}(\xi) d\sigma(\xi) = \lambda_n \int_{\mathbb{S}^{d-1}} Y^n_\ell(\xi) \cdot Y^n_{\ell'}(\xi) d\sigma(\xi) = \lambda_n \delta_{\ell,\ell'} \delta_{n,n'},
\]
where $\lambda_n = n(n+d-2)$, which implies that $\{Y^n_\ell : 1 \leq \ell \leq \dim \mathcal{H}_n^d\}$ is also a family of orthogonal polynomials for the Sobolev inner product defined by, for example,
\[(2.14)\quad (f, g)_\mathcal{V} = \sum_{k=1}^s \mu_k \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} \nabla_0^k f(\xi) \cdot \nabla_0^k g(\xi) d\sigma(\xi) + \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(\xi) g(\xi) d\sigma(\xi),
\]
where $\mu_k \geq 0$ and $s$ is a fixed positive integer, and
\[
\nabla_0^{2m} := \Delta_0^m \quad \text{and} \quad \nabla_0^{2m+1} = \Delta_0^m \nabla_0.
\]
In other words, the Sobolev orthogonal polynomials for $\langle \cdot, \cdot \rangle_\mathcal{V}$ on the unit sphere are trivially spherical harmonics themselves.

3. Orthogonal polynomials on the conic surface

Orthogonal polynomials on the conic surface $\mathcal{Y}_0^{d+1}$ are studied in [21] for the inner product defined by
\[(3.1)\quad (f, g)_{\mathcal{Y}_0^{d+1}} = b_{\beta,\gamma} \int_{\mathcal{Y}_0^{d+1}} f(x,t) g(x,t) w_{\beta,\gamma}(t) dm(x,t),
\]
where $dm$ is the Lebesgue measure on the conic surface and the weight function $w_{\beta,\gamma}$ is the Jacobi weight function on $[0, 1]$,
\[
w_{\beta,\gamma}(t) = t^\beta (1-t)^\gamma, \quad \beta > -d, \quad \gamma > -1
\]
and $b_{\beta,\gamma}$ is the normalization constant so that $(1,1)_{\beta,\gamma} = 1$, which is determined by
\[
\int_{\mathcal{Y}_0^{d+1}} f(x,t) dm(x,t) = \int_0^1 t^{d-1} \int_{\mathbb{S}^{d-1}} f(t\xi,\xi) d\sigma(\xi) dt,
\]
where $d\sigma$ denotes the Lebesgue measure on the unit sphere $\mathbb{S}^{d-1}$. Then,
\[
b_{\beta,\gamma} = \frac{1}{\omega_d} c_{\beta+d-1,\gamma} \quad \text{with} \quad c_{\beta,\gamma} = \frac{\Gamma(\beta + \gamma + 2)}{\Gamma(\beta + 1)\Gamma(\gamma + 1)} \quad \text{and} \quad \omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)},
\]
where \(\omega_d\) is the surface area of \(S^{d-1}\). The inner product is well defined for the space \(\mathbb{R}[x, t]/(t^2 - \|x\|^2)\) of polynomials the ideal generated by \(t^2 - \|x\|^2\).

For \(n \in \mathbb{N}_0\), let \(\mathcal{V}_n^d(w_{\beta,\gamma})\) denote the space of orthogonal polynomials of degree \(n\). Since \(\mathcal{V}_0^d\) is a quadratic surface, so the dimension of the space is the same as that of \(\mathcal{H}_n^d\). Thus, \(\dim \mathcal{V}_0^d(w_{\beta,\gamma}) = 1\) and

\[
\dim \mathcal{V}_n^d(w_{\beta,\gamma}) = \binom{n + d - 1}{n} + \binom{n + d - 2}{n - 1}, \quad n = 1, 2, 3, \ldots.
\]

An orthogonal basis for \(\mathcal{V}_n^d(w_{\beta,\gamma})\) is given in [24] in terms of the Jacobi polynomials and spherical harmonics. Let \(\mathcal{Y}_m^d : 1 \leq \ell \leq \dim \mathcal{H}_m^d\) be an orthonormal basis of \(\mathcal{H}_m^d\).

Define the polynomials, called the Jacobi polynomials on the conic surface, by

\[
S_{m,\ell}^n(x, t) := P_{(2m+\beta+d-1,\gamma)}^{(2m+\beta+d-1,\gamma)}(1-2t)Y_m^d(x),
\]

where, for \((x, t) \in \mathcal{V}_0^d, Y_m^d(x)\) is a solid harmonic in \(\mathcal{H}_m^d\) and \(Y_m^d(t\xi) = t^mY_m^d(\xi)\).

Then \(\{S_{m,\ell}^n(x, t) : 0 \leq m \leq n, 1 \leq \ell \leq \dim \mathcal{H}_m(S^{d-1})\}\) is an orthogonal basis of \(\mathcal{V}_n^d(w_{\beta,\gamma})\), which satisfies

\[
\delta_{\beta,\gamma} \int_{\mathcal{V}_0^d} S_{m,\ell}^n(x, t)S_{m',\ell'}^{n'}(x, t)w_{\beta,\gamma}(t)dm(x, t) = h_{\beta,\gamma}^{n,n'}\delta_{n,n'}\delta_{n,n}\delta_{\ell,\ell'},
\]

where \(h_{\beta,\gamma}^{n,n'}\) is the square of the \(L^2(\mathcal{V}_0^d(w_{\beta,\gamma}))\) norm of \(S_{m,\ell}^n\) and

\[
h_{\beta,\gamma}^{n,n'} = \frac{(\beta + d)_{2m}}{(\beta + \gamma + d + 1)_{2m}}h_{n,m}^{(2m+\beta+d-1,\gamma)}
\]

with \(h_{n-m}^{(2m+\beta+d-1,\gamma)}\) defined in [22].

Of particular interest is the case \(\beta = -1\), for which the space \(\mathcal{V}_n^d(w_{-1,\gamma})\) is an eigenspace of a second order linear differential operator. Parametrizing the space \(\mathcal{V}_0^d\) by \((x, t) = (t\xi, t)\) and let \(\Delta_0^{(\xi)}\) be the Laplace-Beltrami operator on the unit sphere in the \(\xi\) variable, which is the restriction of the Laplace operator \(\Delta\) on \(S^{d-1}\). For \(\gamma > -1\), define

\[
\mathcal{D}_\gamma = t((1-t)\frac{d^2}{dt^2} + (d - 1 - (d + \gamma)t)\frac{d}{dt} + t^{-1}\Delta_0^{(\xi)}).
\]

**Theorem 3.1.** Let \(d \geq 2\) and \(\gamma > -1\). The orthogonal polynomials in \(\mathcal{V}_n(\mathcal{V}_0^d, w_{-1,\gamma})\) are eigenfunctions of \(\mathcal{D}_\gamma\); more precisely,

\[
\mathcal{D}_\gamma u = -n(n + \gamma + d - 1)u, \quad \forall u \in \mathcal{V}_n(\mathcal{V}_0^d, w_{-1,\gamma}).
\]

The differential operator \(\mathcal{D}\) plays an important role in the study of the Fourier orthogonal series and the best approximation by polynomials on the conic surface [25]. For \(f \in L^2(\mathcal{V}_0^d(w_{\beta,\gamma}))\), the Fourier orthogonal series of \(f\) is defined by

\[
f = \sum_{n=0}^\infty \text{proj}_n^{\beta,\gamma} f,
\]

where \(\text{proj}_n^{\beta,\gamma} : L^2(\mathcal{V}_0^d(w_{\beta,\gamma})) \to \mathcal{V}_n^d(w_{\beta,\gamma})\) is the orthogonal projection operator, which satisfies, using the orthogonal basis given above,

\[
\text{proj}_n^{\beta,\gamma} f = \sum_{m=0}^n \sum_{\ell=1} \dim \mathcal{H}_m(S^{d-1}) f_{m,\ell}^{n,1}(\beta,\gamma) S_{m,\ell}^n(\beta,\gamma), \quad \text{where } f_{m,\ell}^{n,1}(\beta,\gamma) := \frac{\langle f, S_{m,\ell}^n(\beta,\gamma) \rangle_{\beta,\gamma}}{h_{m,n}^{\beta,\gamma}}.
\]
In the rest of this section, we consider a property on the derivative of the reproducing kernel. Parametrizing the function \( f : \mathbb{V}_0^{d+1} \to \mathbb{R} \) by \((x, t) = (t\xi, t)\) with \( \xi \in S^{d-1} \), it follows that

\[
\frac{d}{dt} f(x, t) = \frac{d}{dt} f(t\xi, t) = \xi \cdot \nabla_x f(t\xi, t) + \frac{\partial}{\partial t} f(t\xi, t),
\]

where \( \frac{\partial}{\partial t} = \partial_{d+1} \) denotes the partial derivative with respect to the \( d + 1 \) variable of \( f \), which should not be confused with the \( \frac{d}{dt} \) on the left-hand side. Throughout the rest of this paper, we shall adopt the notation

\[
\partial_t = \partial_{d+1} = \frac{\partial}{\partial t}
\]

for functions \( f(x, t) \). We consider the action of \( \partial_t \) on the projection operator.

**Theorem 3.2.** For \( \beta > -d \) and \( \gamma > -1 \), let \( f \) be a differentiable function such that \( f \in L^2(\mathbb{V}_0^{d+1}, \mathbb{W}_{\beta, \gamma}) \) and \( \partial_t f \in L^2(\mathbb{V}_0^{d+1}, \mathbb{W}_{\beta+1, \gamma+1}) \). Then, for \( n = 0, 1, 2, \ldots \),

\[
\frac{\partial}{\partial t} \proj_n^{\beta, \gamma} f(x, t) = \proj_{n-1}^{\beta+1, \gamma+1} \partial_t f(x, t).
\]

**Proof.** For \( m = n \), the polynomial \( \mathbb{S}_{n,\ell}^{n,(\beta, \gamma)}(x, t) = Y_{\ell}^n(x) \), so that \( \frac{\partial}{\partial t} \mathbb{S}_{n,\ell}^{n,(\beta, \gamma)}(x, t) = 0 \). For \( 0 \leq m \leq n - 1 \), using the well-known formula for the derivative of the Jacobi polynomials, we obtain

\[
\frac{\partial}{\partial t} \mathbb{S}_{m,\ell}^{n,(\beta, \gamma)}(x, t) = \frac{\partial}{\partial t} P_{n-m}^{(2m+\beta+d-1, \gamma)}(1-2t)Y_{\ell}^m(x)
= \tau_{n,m} P_{n-m-1}^{(2m+\beta+d, \gamma+1)}(1-2t)Y_{\ell}^m(x) = \tau_{n,m} \mathbb{S}_{m,\ell}^{n-1,(\beta+1, \gamma+1)}(x, t),
\]

where \( \tau_{n,m} = -(n+m+\beta+\gamma+d) \). As a consequence of these identities, we see that \( \partial_t \proj_n^{\beta, \gamma} f \in \mathbb{V}_0^{d-1}(\mathbb{W}_{\beta+1, \gamma+1}) \). Consequently, it follows that

\[
\left\langle \partial_t f, \mathbb{S}_{m,\ell}^{n-1,(\beta+1, \gamma+1)} \right\rangle_{\beta+1, \gamma+1} = \left\langle \partial_t \proj_n^{\beta, \gamma} f, \mathbb{S}_{m,\ell}^{n-1,(\beta+1, \gamma+1)} \right\rangle_{\beta+1, \gamma+1} = \sum_{k=0}^{n-1} \sum_{\nu} \mathbb{S}_{k,\nu}^{n,(\beta, \gamma)} \tau_{n,k} \left\langle \mathbb{S}_{k,\nu}^{n-1,(\beta+1, \gamma+1)}, \mathbb{S}_{m,\ell}^{n-1,(\beta+1, \gamma+1)} \right\rangle_{\beta+1, \gamma+1},
\]

which implies immediately that

\[
\widehat{\partial_t f}_{\beta+1, \gamma+1} = \tau_{n,m} \mathbb{S}_{m,\ell}^{n,(\beta, \gamma)}, \quad 0 \leq m \leq n - 1.
\]

Consequently, we obtain

\[
\frac{\partial}{\partial t} \proj_n^{\beta, \gamma} f(x, t) = \sum_{m=0}^{n-1} \sum_{\ell} \frac{\partial}{\partial t} \mathbb{S}_{m,\ell}^{n-1,(\beta+1, \gamma+1)}(x, t)
= \sum_{m=0}^{n-1} \sum_{\ell} \tau_{n,m} \mathbb{S}_{m,\ell}^{n-1,(\beta+1, \gamma+1)}(x, t)
= \proj_{n-1}^{\beta+1, \gamma+1} \partial_t f(x, t),
\]

where in the first step we used again \( \partial_t \mathbb{S}_{m,\ell}^{n}(x, t) = 0 \). \( \square \)
For $1 \leq p \leq \infty$, let $\|f\|_{p,\omega,\gamma}$ denote the norm of the space $L^p(\mathbb{V}_d^{d+1}, w_{\beta, \gamma})$, and we adopt the convention that the space is $C(\mathbb{V}_d^{d+1})$ with the norm taken as the uniform norm when $p = \infty$. Let $\Pi_n(\mathbb{V}_d^{d+1})$ denote the space of polynomials of degree $n$ restricted on the $\mathbb{V}_d^{d+1}$. For $f \in L^p(\mathbb{V}_d^{d+1}, w_{\beta, \gamma})$, the quantity
\[ E_n(f)_{p,\omega,\gamma} := \inf_{P \in \Pi_n(\mathbb{V}_d^{d+1})} \|f - P\|_{p,\omega,\gamma} \]
is the error of the best approximation by polynomials of degree at most $n$ in the norm of $L^p(\mathbb{V}_d^{d+1}, w_{\beta, \gamma})$. We call $\eta \in C^\infty$ an admissible cut-off function if it is supported on $[0, 2]$ and satisfies $\eta(t) = 1$ if $0 \leq t \leq 1$. Let $\eta$ be such a function; we define
\[ Q^{(\beta, \gamma)}_{n, \eta} f = \sum_{k=0}^{2n} \eta \left( \frac{k}{n} \right) \text{proj}_k^{\beta, \gamma} f. \]Then it is known $[\text{23}]$ that $Q^{(\beta, \gamma)}_{n, \eta}$ is a bounded operator in $L^p(\mathbb{V}_d^{d+1}, w_{\beta, \gamma})$ and it is a polynomial of near best approximation in the sense that
\[ \|f - Q^{(\beta, \gamma)}_{n, \eta} f\|_{p,\omega,\gamma} \leq c E_n(f)_{p,\omega,\gamma}, \]where $c$ is a constant that depends only on $\eta$, $p$, $\beta$ and $\gamma$. In particular, we obtain the following as a corollary of Theorem 3.2.

**Corollary 3.3.** Let $\beta > -d$ and $\gamma > -1$. Let $r$ be a positive integer and let $f \in C^r(\mathbb{V}_d^{d+1})$ such that $\partial^k f \in L^p(\mathbb{V}_d^{d+1}, w_{\beta+k, \gamma+k})$ for $0 \leq k \leq r$. Then, for $1 \leq p \leq \infty$,
\[ \|\partial^k f - \partial^k Q^{(\beta, \gamma)}_{n, \eta} f\|_{p,\omega,\beta+k, \gamma+k} \leq c_k E_n(\partial^k f)_{p,\omega,\beta+k, \gamma+k}, \quad 0 \leq k \leq r. \]

**Proof.** This follows immediately from
\[ \partial^k Q^{(\beta, \gamma)}_{n, \eta} f = \sum_{j=0}^{2n} \eta \left( \frac{j}{n} \right) \partial^j \text{proj}_j^{\beta, \gamma} f = \sum_{j=k}^{2n} \eta \left( \frac{j}{n} \right) \text{proj}_j^{\beta+k, \gamma+k} f \]
\[ = \sum_{j=0}^{2n-k} \eta \left( \frac{j+k}{n} \right) \text{proj}_j^{\beta+k, \gamma+k} \partial^k f = Q^{(\beta+k, \gamma+k)}_{n, \eta} \partial^k f, \]
where we define
\[ Q^{(\beta, \gamma)}_{n, \eta} g = \sum_{j=0}^{2n-k} \eta \left( \frac{j+k}{n} \right) \text{proj}_j^{\beta, \gamma} g, \quad 0 \leq k < 2n. \]
For fixed $k \leq r$ independent of $n$, the function $\eta \left( \frac{j+k}{n} \right)$ plays essentially the same role as $\eta \left( \frac{j}{n} \right)$, so that $[\text{3.3}]$ holds with $Q^{(\beta, \gamma)}_{n, \eta} \partial^k f$ in place of $Q^{(\beta, \gamma)}_{n, \eta} f$, form which the proof follows readily. \hfill \Box

## 4. SOBOLEV ORTHOGONAL POLYNOMIALS ON THE CONIC SURFACE

As seen in the differential operator $[\text{3.3}]$, the derivatives on the conic surface $\mathbb{V}_d^{d+1}$ are partial derivatives with respect to $\xi$ and $t$ with $x = t \xi$ for $(x, t) \in \mathbb{V}_d^{d+1}$. For the Sobolev inner product, the derivatives in $\xi \in \mathbb{S}^{d-1}$ will act on the spherical harmonics and lead to the same orthogonal basis as discussed at the end of Subsection 2.2. Hence,
we consider only the derivatives with respect to $\partial_t$; see Remark 4.1 below. For $s \in \mathbb{N}$, let us define
\[ W^s_p(\mathcal{V}^{d+1}_0, \mathcal{W}_{\beta+s,0}) = \{ f \in C(\mathcal{V}^{d+1}_0) : \partial_t^s f \in L^p(\mathcal{V}^{d+1}_0, \mathcal{W}_{\beta+s,0}) \} , \]
where $1 \leq p \leq \infty$ and the space is taken as the $C^s(\mathcal{V}^{d+1}_0)$ if $p = \infty$.

Let $s$ be a positive integer and $\beta > -d - s$. Let $\lambda_1, \ldots, \lambda_{r-1}$ be fixed positive numbers. We consider the Sobolev inner product defined by
\[
\langle f, g \rangle_{\beta, -s} = \frac{1}{\omega_d} \int_{\mathcal{V}^{d+1}_0} \frac{\partial^s f}{\partial t^s} f(x, t) \frac{\partial^s g}{\partial t^s} g(x, t) t^{\beta+s} \text{dm}(x, t)
\]
\[
+ \sum_{k=0}^{s-1} \lambda_k \frac{1}{\omega_d} \int_{\mathcal{S}^{d-1}} \frac{\partial^k f}{\partial t^k} (\xi, 1) \frac{\partial^k g}{\partial t^k} (\xi, 1) d\sigma(\xi),
\]
which is evidently an inner product on the space $W^s_p(\mathcal{V}^{d+1}_0, \mathcal{W}_{\beta+s,0})$. We denote by $\mathcal{V}^{d}(\mathcal{W}_{\beta,-s})$ the space of orthogonal polynomials of degree $n$ with respect to this inner product. Our first task is to find an orthogonal basis for this space.

Recall the modified Jacobi polynomial $J^{(\alpha, -s)}_n$ defined in (2.7). Let \{ $Y^m_n : 1 \leq \ell \leq \dim \mathcal{H}^d_m$ \} be an orthonormal basis of $\mathcal{H}^d_m$. For $1 \leq \ell \leq \dim \mathcal{H}^d_m$, $0 \leq m \leq n$, we define
\[
S_{m,\ell}^{(\beta, -s)}(x, t) := J^{(2m+\beta+d-1, -s)}_{n-m}(1-2t) Y^m_{\ell}(x).
\]

**Theorem 4.1.** Let $s \in \mathbb{N}$ and $\beta > -d - s$. The polynomials $S_{m,\ell}^{(\beta, -s)}$, $1 \leq \ell \leq \dim \mathcal{H}^d_m$ and $0 \leq m \leq n$ consist of a basis of $\mathcal{V}^{d}(\mathcal{W}_{\beta,-s})$. Moreover, for all $\ell$, $h_{m,n}^{(\beta, -s)} = \langle S_{m,\ell}^{(\beta, -s)}, S_{m,\ell}^{(\beta, -s)} \rangle_{\beta, -s}$ satisfies
\[
h_{m,n}^{(\beta, -s)} = \begin{cases} 2^{2n-2m}\lambda_{n-m} & 0 \leq n-m \leq s-1, \\ 2^{s-2m-d}\lambda_{n-m-s} & n-m \geq s. \end{cases}
\]

**Proof.** Let $q_{n-m}(t) = J^{(2m+\beta+d-1, -s)}_{n-m}(t)$. Changing variable $x = t\xi$ and using the homogeneity of $Y^m_{\ell}$, we obtain
\[
\langle S_{m,\ell}^{(\beta, -s)}, S_{m,\ell'}^{(\beta, -s)} \rangle_{\beta, -s} = 2^{2s-\alpha+1} \int_0^1 t^{d-1} q_{n-m}^{(s)}(1-2t) q_{n-m}^{(s)}(1-2t) t^{\beta+s+m+m'} dt 
\]
\[
\times \frac{1}{\omega_d} \int_{\mathcal{S}^{d-1}} Y^m_{\ell}(\xi) Y^m_{\ell'}(\xi) d\sigma(\xi) 
\]
\[
+ \sum_{k=0}^{s-1} \lambda_k 2^{2k} q_{n-m}^{(k)} (-1) q_{n-m}^{(k)} (-1) 
\]
\[
\times \frac{1}{\omega_d} \int_{\mathcal{S}^{d-1}} Y^m_{\ell}(\xi) Y^m_{\ell'}(\xi) d\sigma(\xi). 
\]

Using the orthogonality of $Y^m_{\ell}$ and changing variable $t \mapsto (1-t)/2$ in the first integral, we see that the expression containing $q_{n-m}$ can be written in terms of the Sobolev inner product $[\cdot, \cdot]_{\alpha, 0}$ defined in (2.4). More precisely, we obtain
\[
\langle S_{m,\ell}^{(\beta, -s)}, S_{m,\ell'}^{(\beta, -s)} \rangle_{\beta, -s} = 2^{s-\alpha+1} [q_{n-m}, q_{n-m}]_{\alpha, 0}^{\beta, -s} \delta_{m,m'} \delta_{\ell,\ell'},
\]
where $\alpha = s + \beta + 2m + d - 1$ and the inner product $[\cdot, \cdot]_{\alpha, 0}$ is defined as in (2.4) but with $\mu_k = 2^{2k} \lambda_k/2^{2s-\alpha+1}$. As shown in Theorem 2.4, the polynomials $J^{(\alpha, -s)}_n$ are orthogonal with respect to this inner product, regardless of the values of $\mu_k$. This
proves the orthogonality of \( S^{n,\beta,-s}_{m,\ell} \). Moreover, the norm \( h^{(\beta,-s)}_{m,n} \) follows from the norm given in Theorem 2.1.

**Remark 4.1.** We could also include an additional term in the right-hand side of \( \langle \cdot, \cdot \rangle_{\beta,-s} \) defined by

\[
\int_{Y_0} \nabla^s_0 f(x,t) \cdot \nabla^s_0 g(x,t) \, d\mu(x,t),
\]

where \( \nabla_0 \) acts on \( \xi \). By the discussion for the inner product 2.14 on the unit sphere, the inner product with this additional term has the same orthogonal basis.

Recall that the constant \( A_n^{(\alpha,\beta)} \) is defined in 2.5.

**Corollary 4.2.** For \( 0 \leq m \leq n - s \),

\[
(4.3) \quad \frac{\partial^s}{\partial t^s} S^{n,\beta,-s}_{m,\ell}(x,t) = (-2)^s A_n^{(s+2\ell+\beta-d-1,0)} S^{n-s,\beta+s,0}_{m,\ell}(x,t),
\]

and it is equal to zero if \( m > n - s \). Moreover, for \( 1 \leq k \leq s - 1 \) and \( \xi \in \mathbb{S}^{d-1} \),

\[
(4.4) \quad \frac{\partial^k}{\partial t^k} S^{n,\beta,-s}_{m,\ell}(\xi,1) = \begin{cases} (-2)^k Y^m_{\ell}(\xi) \delta_{n,n-m}, & m > n - s \\ 0, & m \leq n - s \end{cases}
\]

**Proof.** If \( m \leq n - s \), by our convention of the derivative \( \partial_\ell \) over the conic surface and the identity 2.8,

\[
\frac{\partial^s}{\partial t^s} S^{n,\beta,-s}_{m,\ell}(x,t) = \frac{\partial^s}{\partial t^s} \left[ f_{n-m}^{(2\ell+\beta-d-1,-s)}(1-2t) \right] Y^m_{\ell}(x),
\]

from which 1.3 follows from 2.5 and the definition of \( S^{n,\beta+s,0}_{m,\ell} \). Moreover, since \( n - m \geq s \), for \( 1 \leq k \leq s - 1 \) it follows immediately by 2.9 that (4.3) holds. If \( m > n - s \), then the derivative in the identity 1.3 is evidently zero since the Jacobi polynomial in \( S^{n,\beta,-s}_{m,\ell} \) is of degree \( n - m < s \). Moreover, for \( 1 \leq k \leq s - 1 \), (4.4) follows from 2.4.

Our notation for the Sobolev orthogonal polynomials \( S^{n,\beta,-s}_{m,\ell} \) is the same as the one for the ordinary orthogonal polynomials \( S^{n,\beta,\gamma}_{m,\ell} \) with \( \gamma = -s \). This is intentional as can be seen in 2.11, which however only works for \( n \geq s \) or \( S^{n,\beta,-s}_{m,\ell} \) for \( n - m \geq s \).

We can, moreover, give an explicit expression of the Sobolev orthogonal polynomials by using the identity 2.10.

**Corollary 4.3.** For \( 0 \leq \ell \leq \dim \mathcal{H}_m \) and \( 0 \leq m \leq n \), the polynomials \( S^{n,\beta,-s}_{m,\ell} \) satisfy

\[
(4.5) \quad S^{n,\beta,-s}_{m,\ell}(x,t) = \begin{cases} b_{m,n}^s (1-t)^s S^{n-s,\beta,s}_{m,\ell}(x,t), & n - m \geq s, \\ (1-t)^{n-m} Y^m_{\ell}(x), & 0 \leq n - m \leq s - 1, \end{cases}
\]

where \( b_{m,n}^s = (-1)^s A_n^{(s+2\beta+\beta-d-1,0)} / (n-m)_s \). In particular, the space \( \mathcal{V}^d_n(\omega_{\beta,-s}) \) satisfies a decomposition

\[
(4.6) \quad \mathcal{V}^d_n(\omega_{\beta,-s}) = \bigoplus_{j=0}^{s-1} (1-t)^j \mathcal{V}^{d}_{n-j,0} \bigoplus (1-t)^s \mathcal{V}^{d}_{n-s}(\omega_{\beta,s}).
\]
Proof. This follows from the identity (2.3), which shows, together with (2.11), that
\[
J_n^{(\alpha,-s)}(1-2t) = \frac{(-1)^s}{(-n)^s} A_{n-s}^{(\alpha,s)} (1-t)^s P_{n-s}^{(\alpha,s)} (1-2t)
\]
for \( n \geq s \), which implies, with \( n \) replaced by \( n-m \), the identity (4.5) when \( n-m \geq s \).

For \( n-m \leq s-1 \), (4.5) follows immediately from the definition of \( J_n^{(\alpha,-s)} \) in (2.7).
Since \( \{Y_{\ell}^m : 1 \leq \ell \leq m \} \) is a basis of \( \mathcal{H}_n^{d,0} \), the decomposition (4.6) is an immediate consequence of (4.5).

With the expression (4.5) for \( S_{m,\ell}^{n,(\beta,-s)} \), we could bypass the integral definition of \( J_n^{(\alpha,-s)} \) in (2.7). However, the integral definition is more convenient for studying the Fourier orthogonal series, as shown below.

For \( f \in W_2^d(V_{0}^{d+1},w_{\beta+s,0}) \), the Fourier orthogonal series of \( f \) is defined by
\[
f = \sum_{n=0}^{\infty} \text{proj}_{n}^{\beta,-s} f,
\]
where \( \text{proj}_{n}^{\beta,-s} : W_2^d(V_{0}^{d+1},w_{\beta+1,0}) \to V_{n}^{d}(w_{\beta,-s}) \) is the orthogonal projection operator, which satisfies, using the orthogonal basis given above,
\[
\text{proj}_{n}^{\beta,-s} f = \sum_{m=0}^{n} \sum_{\ell} \hat{f}_{m,\ell}^{(\beta,-s)} S_{m,\ell}^{n,(\beta,-s)}, \quad \text{where} \quad \hat{f}_{m,\ell}^{(\beta,-s)} := \left< f, S_{m,\ell}^{n,(\beta,-s)} \right>_b.
\]
Using the basis in Theorem 4.1 and its corollary, we can derive an integral representation for the projection operator \( \text{proj}_{n}^{\beta,-s} f \).

**Theorem 4.4.** Let \( s \) be a positive integer. For \( \beta > -d-s \), let \( f \) be a differentiable function such that \( \partial_x f \in L^2(V_{0}^{d+1},w_{\beta+s,0}) \). Then, for \( n = 0, 1, 2, \ldots \) and \( (x,t) \in V_0^{d+1} \),
\[
\text{proj}_{n}^{\beta,-s} f(x,t) = \sum_{m=0}^{s-1} \frac{(t-1)^m}{m!} \text{proj}_{n-m}^{\beta-d} \left[ \partial_x^m f(\cdot,1) \right](\xi)
\]
\[
+ (-1)^s \int_{t}^{1} \frac{(v-t)^{s-1}}{(s-1)!} \text{proj}_{n-s}^{\beta-s,0} \left( \partial_x^s f \right)(x,v)dv.
\]

**Proof.** Let \( \alpha = s + \beta + d - 1 \). For \( 0 \leq m \leq n - s \), it follows from (4.3) that
\[
\left< f, S_{m,\ell}^{n,(\beta,-s)} \right>_b = (-2)^s A_{n-m}^{(2m+\alpha,0)} \frac{1}{\omega_d} \int_{V_0^{d+1}} \frac{\partial_x^s f(x,t)S_{m,\ell}^{n-s,(\beta+s,0)}(x,t)\beta+s,0}{\partial_x^s f(\cdot,s)} dm(x,t)
\]
and the norm of \( S_{m,\ell}^{n,(\beta,-s)} \) satisfies
\[
h_{m,n}^{(\beta,-s)} = 2^{s-\beta-2m} \eta_{n-m-s}^{(\alpha,2m+0)} = 2^{2s} \frac{1}{\omega_d^{2m+0}} \left[ A_{n-m-s}^{(2m+\alpha,0)} \right]^{2} h_{n-s-m}^{(2m+\alpha,0)}
\]
\[
= \frac{1}{\omega_d^{2m+0}} \left[ A_{n-m-s}^{(2m+\alpha,0)} \right]^{2} \left( \frac{\alpha+1}{\alpha+2} \right)^{2m} h_{n-s-m}^{(2m+\alpha,0)}
\]
\[
= \frac{1}{\omega_d^{2m+0}} \left[ A_{n-m-s}^{(2m+\alpha,0)} \right]^{2} h_{n-s,m}^{(\beta+s,0)}.
\]
Consequently, it follows that

$$
\hat{f}_{m, \ell}^{(s, \beta - s)} = \frac{\Delta_{m+\alpha, 0}}{\lambda_{\beta + s, 0}} \langle \partial_{\xi}^{k} f, S_{m, \ell}^{n-s, (\beta + s, 0)} \rangle_{\beta + s, 0} \left( \frac{\Delta_{m, \ell}^{n-s, (\beta + s, 0)}}{(2\pi)^{n-1}} \right).
$$

Now, for \( n-s < m \leq n \), it follows by Corollary 4.2 that

$$
\langle f, S_{m, \ell}^{n, (\beta - s)} \rangle_{\beta - s} = \frac{1}{(2\pi)^{n-1}} \int_{S^{d-1}} \partial_{\xi}^{n-m} f(\xi, 1) Y^{m}_{\ell}(\xi) d\sigma(\xi), \quad 0 \leq n-m \leq s-1.
$$

Consequently, by (2.7) and using \( \alpha = s + \beta + d - 1 \) again, it follows that

$$
\text{proj}_{n-s}^{\beta-s} f(x, t) = \sum_{m=0}^{n-s} \frac{(t-1)^{m}}{m!} \sum_{\ell=1}^{\dim H_{\ell}^{n-s}} \frac{1}{\omega_{d}} \int_{S^{d-1}} \partial_{u}^{m} f(\eta, 1) V^{m}_{\ell}(\eta) Y^{n-m}(\eta) d\sigma(\eta) + \sum_{m=0}^{n-s} \sum_{\ell=1}^{\dim H_{\ell}^{n-s}} \frac{(t-1)^{m}}{m!} \int_{S^{d-1}} \partial_{u}^{m} f(\eta, 1) V^{m}_{\ell}(\eta) Y^{n-m}(\eta) d\sigma(\eta) Y^{n-m}(\xi),
$$

The inner sum of the first term in the right-hand side is exactly \( \text{proj}_{n-m}^{\beta-s} \partial_{t}^{m} f(\cdot, 1) \) for the function \( \xi \mapsto \partial_{t}^{m} f(\xi, 1) \) on the unit sphere. Changing variable \( u = 1-2t \) and using (2.6) in the integral, the second term on the right-hand side becomes,

$$
\sum_{m=0}^{n-s} \sum_{\ell=1}^{\dim H_{\ell}^{n-s}} \frac{(t-1)^{m}}{m!} \int_{S^{d-1}} \partial_{u}^{m} f(\eta, 1) V^{m}_{\ell}(\eta) Y^{n-m}(\eta) d\sigma(\eta) Y^{n-m}(\xi) = (-1)^{s} \int_{S^{d-1}} \frac{(v-t)^{s-1}}{(s-1)!} \sum_{m=0}^{n-s} \sum_{\ell=1}^{\dim H_{\ell}^{n-s}} \partial_{u}^{m} f(\eta, 1) V^{m}_{\ell}(\eta) Y^{n-m}(\eta) d\sigma(\eta) Y^{n-m}(\xi)
$$

Putting the two terms together completes the proof. \( \square \)

**Corollary 4.5.** Let \( s \) be a positive integer. For \( \beta > -d - s \), let \( f \) be a differentiable function such that \( \partial_{t}^{s} f \in L^{2}([0, \infty), w_{\beta+s, 0}) \). Then, for \( n \geq s \),

$$
\frac{\partial^{s}}{\partial t^{s}} \text{proj}_{n-s}^{\beta-s} f(x, t) = \text{proj}_{n-s}^{\beta+s, 0} (\partial_{t}^{s} f)(x, t).
$$

**Proof.** Taking the derivative \( \partial_{t}^{s} \) of (4.7), we see that the first term in the right-hand side is zero, while the second term gives the stated identity. \( \square \)

Another consequence of Theorem 4.3 is an expression for the error of approximation. Let \( \eta \in C^{\infty}(\mathbb{R}_{+}) \) be an admissible cut-off function. We denote by \( Q^{(\beta-s)}_{n, \eta} \) the near
best approximation operator defined in (3.7) but with $\gamma = -s$. Furthermore, let
\[
Q_{n-m}^{d-1} f(\xi) = \sum_{k=0}^{2(n-m)} \eta \begin{pmatrix} k \cr m \end{pmatrix} \text{proj}_k^{d-1} f(\xi), \quad 0 \leq m \leq n,
\]
be a near best approximation operator of degree $n - m$ on the unit sphere. For $f \in L^p(S^{d-1})$, let
\[
E_n^{d-1}(f)_p = \inf_{Y \in \Pi_n(S^{d-1})} \|f - Y\|_{L^p(S^{d-1})}, \quad 1 \leq p \leq \infty
\]
be the error of best approximation by polynomials on the unit sphere, where the norm is the uniform norm for $p = \infty$ and $\Pi_n(S^{d-1})$ denote the space of polynomial of degree at most $n$ restricted on the unit sphere $S^{d-1}$. Then (cf. [5, Theorem 2.6.3.])
\[
\|f - Q_n^{d-1} f\|_p \leq cE_n^{d-1}(f)_p, \quad 1 \leq p \leq \infty.
\]

**Theorem 4.6.** For $f \in C^s(V_0^{d+1})$,
\[
f(x, t) - Q_n^{(\beta, -s)} f(x, t) = \sum_{m=0}^{s-1} \frac{(t - 1)^m}{m!} \left[ \partial_t^m [f(\cdot, 1)](\xi) - Q_{n-m, \eta}^{d-1} [\partial_t^m f(\cdot, 1)](\xi) \right]
\]
\[+ (-1)^s \int_t^1 \left( \frac{v-t}{s-1} \right)^{s-1} \left[ \partial_t^s f(x, v) - Q_{n-s, \eta}^{(\beta+s,0)} (\partial_t^s f)(x, v) \right] \, dv.
\]

**Proof.** Multiplying (4.7) by $\eta \left( \frac{k}{n} \right)$ and summing up over $k$ gives an expression of $Q_n^{(\beta, -s)} f$. Furthermore, using the Taylor expansion of $t \to f(x, t)$ with respect to the $t$ variable at $t = 1$, together with its remainder formula, we can write
\[
f(x, t) = \sum_{m=0}^{s-1} \frac{(t - 1)^m}{m!} \partial_t^m [f(\cdot, 1)](\xi, 1) + (-1)^s \int_t^1 \left( \frac{v-t}{s-1} \right)^{s-1} \partial_t^s f(x, v) \, dv.
\]
Together the difference of the two identities gives the error formula for $f - Q_n^{(\beta, -s)} f$ and completes the proof.

Taking the $k$-th order derivative with respect to the $t$ variable for $0 \leq t \leq s - 1$, we obtain
\[
\partial_t^k \left( f - Q_n^{(\beta, -s)} f \right)(x, t) = \sum_{m=k}^{s-1} \frac{(t - 1)^m}{(m-k)!} \partial_t^m [f(\cdot, 1)](\xi) - Q_{n-m, \eta}^{d-1} [\partial_t^m f(\cdot, 1)](\xi)
\]
\[+ (-1)^{s-k} \int_t^1 \left( \frac{v-t}{s-k-1} \right)^{s-k-1} \left[ \partial_t^s f(x, v) - Q_{n-s, \eta}^{(\beta+s,0)} (\partial_t^s f)(x, v) \right] \, dv,
\]
moreover, taking the $s$-th derivative in the $t$ variable, we obtain
\[
(4.9) \quad \partial_t^s \left( f - Q_n^{(\beta, -s)} f \right)(x, t) = Q_{n-s, \eta}^{(\beta+s,0)} (\partial_t^s f)(x, t).
\]
In the above error formulas for $1 \leq k \leq s - 1$, the integral is over $v \in [t, 1]$ while the variable is $(x, v) = (\xi, v)$. As it is, we cannot deduce the error estimate for $\partial_t^k \left( f - Q_n^{(\beta, -s)} f \right)$ immediately from that of $f - Q_n^{(\beta+s,0)} (\partial_t^s f)$ over $V_0^{d+1}$ if $1 \leq k \leq s - 1$, although we can if $k = s$ by (4.9). We state the latter case as a corollary.

**Corollary 4.7.** Let $f \in W_p^s(W_{\beta+s,0})$. Then, for $1 \leq p \leq \infty$,
\[
\left\| \partial_t^s f - \partial_t^s Q_n^{(\beta, -s)} f \right\|_{p, W_{\beta+s,0}} \leq cE_n(\partial_t^s f)_{p, W_{\beta+s,0}}.
\]
We can derive another interesting property of the projection operator for the SOPs, which relies on the property that the Jacobi polynomial \( P_n^{(\alpha,s)} \) is related to \( P_n^{(\alpha,s)} \), as seen in [17 (4.22.2)]. The following lemma shows that our extension \( J_n^{s,-s} \) satisfies the same property.

**Theorem 4.8.** If \( f(x,t) = (1-t)^s g(x,t) \), then

\[
\text{proj}_{n}^{\beta,-s} f = (1-t)^s \text{proj}_{n}^{\beta,s} g.
\]

**Proof.** By the definition of the \( S_{m,\ell}^{(\beta,-s)} \), it follows from (2.10) that

\[
S_{m,\ell}^{(\beta,-s)}(x,t) = \sigma_{m,n}(1-t)^s S_{m,\ell}^{(\beta,s)}(x,t), \quad \sigma_{m,n} := \frac{(n-s-m)!}{(n-m)!} A_{n-m}^{2m+\beta+d-1,0}.
\]

Expanding \( g \) in terms of the Fourier orthogonal series in \( L^2(\mathbb{R}^{d+1}, w_{\beta,\alpha}) \), we obtain

\[
f(x,t) = (1-t)^s g(x,t) = (1-t)^s \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{\ell} \hat{g}_{m,\ell}^{n,\beta,s} S_{m,\ell}^{n,\beta,s}(x,t)
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{\ell} \hat{g}_{m,\ell}^{n,\beta,s} \sigma_{n+s,m,\ell}^{-1} S_{m,\ell}^{n+s,\beta,-s}(x,t).
\]

Applying the orthogonality in the Sobolev inner product, it follows immediately that

\[
\left\langle f, S_{m,\ell}^{n+s,\beta,-s} \right\rangle_{\beta,-s} = \sigma_{n+1,m}^{-1} S_{m,\ell}^{n,\beta,s} \left\langle S_{m,\ell}^{n+s,\beta,-s}, S_{m,\ell}^{n+1,\beta,-s} \right\rangle_{\beta,-s}, \quad 0 \leq m \leq n;
\]

in particular, with \( n \) replaced by \( n-s \), we then obtain

\[
\hat{f}_{m,\ell}^{n,\beta,-s} = \sigma_{n,m}^{-1} S_{m,\ell}^{n-s,\beta,s}, \quad 0 \leq m \leq n-s.
\]

For \( n-s < m \leq n \), the polynomial \( J_{n-m}^{2m+\beta+d-1,0}(t) \) is a polynomial of degree at most \( s-1 \), so that \( \partial_x^n S_{n,\ell}^{(\beta,-1)}(x,t) = 0 \); moreover, \( \partial_x^k f(\xi,1) = 0 \) for \( 0 \leq k \leq s-1 \), since \( f \) contains a factor \((1-t)^s\). It follows that \( \hat{f}_{m,\ell}^{n,\beta,-s} = 0 \) for \( n-s < m \leq n \) by the definition of the inner product. Consequently, by (2.10) and (4.10) we obtain

\[
(1-t) \text{proj}_{n-s}^{\beta,s} g = (1-t) \sum_{m=0}^{n-s} \hat{g}_{m,\ell}^{n-1,\beta,s} S_{m,\ell}^{n-s,\beta,s}
\]

\[
= \sum_{m=0}^{n} \sum_{\ell} \hat{g}_{m,\ell}^{n,\beta,s} S_{m,\ell}^{n,\beta,-s} = \text{proj}_{n}^{\beta,-s} f.
\]

The proof is completed. \( \square \)

**Corollary 4.9.** Let \( f \in L^p(\mathbb{R}^{d+1}, w_{\beta,\alpha}) \) and \( f(t) = (1-t)g(t) \). Then, for \( 1 \leq p \leq \infty \),

\[
\left\| f - Q_{n,\eta}^{(\beta,-s)} f \right\|_{p,w_{\beta,0}} \leq cE_n\left( g \right)_{p,w_{\beta,0}} \leq cE_n\left( g \right)_{p,w_{\beta,0}}
\]

**Proof.** Since \( f = (1-t)^s g \), we see that

\[
Q_{n,\eta}^{(\beta,-1)} f = \sum_{k=0}^{2n} \eta \left( \frac{k}{n} \right) (1-t)^s \text{proj}_{k-1}^{\beta,s} g = (1-t)^s \hat{Q}_{n,\eta}^{\beta,s} g,
\]
where \( \widetilde{Q}^{(\beta,s)}_{n,\eta} \) is defined as in the proof of Corollary 6.3. For \( p \geq 1 \), we then obtain
\[
\left\| f - Q^{(\beta,-s)}_{n,\eta} f \right\|_{p,w_{\beta,0}} = \left\| (1-t)^p \left[ g - \widetilde{Q}^{(\beta,s)}_{n,\eta} g \right] \right\|_{p,w_{\beta,0}} 
\leq \left\| g - \widetilde{Q}^{(\beta,s)}_{n,\eta} g \right\|_{p,w_{\beta^*}} \leq cE_n(g)_{p,w_{\beta,s}},
\]
By the definition of \( E_n(g)_{p,w} \), it follows immediately \( E_n(g)_{p,w_{\beta,s}} \leq E_n(g)_{p,w_{\beta,0}} \). The proof is completed.

5. Partial differential equation for the Sobolev orthogonal polynomials

This section considers partial differential equations satisfied by the Sobolev orthogonal polynomials. As we mentioned in Theorem 3.1 proved in [21], the ordinary or-

nal polynomials. As we mentioned in Theorem 3.1 proved in [21], the ordi-

one for the Sobolev orthogonal polynomials, we need to be negative integers. We first consider the action of the differential operator when \( \gamma \) is a real number.

5.1. Eigenfunctions of \( D_\gamma \) for \( \gamma \in \mathbb{R} \). Recall that the operator \( D_\gamma \) is given by
\[
D_\gamma = t(1-t)\frac{d^2}{dt^2} + (d-1-(d+\gamma)t)\frac{d}{dt} + t^{-1} \Delta_0(\xi).
\]
We start with a lemma for the action of this operator.

**Lemma 5.1.** Let \( p(t) \) be a polynomial in the \( t \) variable and \( Y(\xi) \) be a function defined on the unit sphere \( S^{d-1} \). For \( k \geq 1 \),
\[
(5.1) \quad D_\gamma \left[(1-t)^k p(t)Y(\xi)\right] = (1-t)^k D_{\gamma+2k} [p(t)Y(\xi)]
- k (1-t)^{k-1} [d - 1 - (d+\gamma + k - 1)t] p(t)Y(\xi).
\]
**Proof.** For \( k = 1 \), a quick computation from the definition of \( D_\gamma \) shows
\[
D_\gamma[(1-t)p(t)Y(\xi)] = (1-t) \left[ t(1-t)p''(t) + [d-1-(d+\gamma+2)t]p'(t) \right]
- [d-1-(d+\gamma)t]p(t)Y(\xi) + t^{-1} (1-t)p(t) \Delta_0(\xi) Y(\xi) 
= (1-t)D_{\gamma+2} [p(t)Y(\xi)] - [d-1-(d+\gamma)t]p(t)Y(\xi),
\]
which verifies (5.1) for \( k = 1 \). Assume that (5.1) holds for \( k \geq 1 \). Then
\[
D_\gamma[(1-t)^{k+1} p(t)Y(\xi)] = D_{\gamma}[(1-t)^k (1-t)p(t)Y(\xi)]
= (1-t)^k D_{\gamma+2k} [(1-t)p(t)Y(\xi)]
- k (1-t)^{k-1} [d-1-(d+\gamma+k-1)t] [(1-t)p(t)Y(\xi)]
= (1-t)^k \{ (1-t)D_{\gamma+2k+2} [p(t)Y(\xi)] - [d-1-(d+\gamma+2k)t]p(t)Y(\xi) \}
- k (1-t)^k [d-1-(d+\gamma+k-1)t] p(t)Y(\xi)
= (1-t)^{k+1} D_{\gamma+2k+2} [p(t)Y(\xi)] - (k+1)(1-t)^k [d-1-(d+\gamma+k)t] p(t)Y(\xi),
\]
so that (5.1) holds for \( k + 1 \) and the proof is completed by induction. \( \square \)
Theorem 5.2. For $0 \leq j \leq n$, let $Z_{j,n}(x,t) = p(t)Y(x)$ with $p$ being a polynomial of degree $j$ in one variable and $Y$ be a solid harmonic polynomial in $\mathcal{H}_{n-1}^{d,0}$. For $\gamma \in \mathbb{R}$, the only polynomial $p$ for which $Z_{j,n}$ satisfies

$$D_\gamma Z(x,t) = \lambda_n^{(\gamma)}Z(x,t), \quad \lambda_n^{(\gamma)} = -n(n + \gamma + d - 1),$$

is the Jacobi polynomial $p(t) = c_j P_j^{(2n-2j+d-2\gamma)}(1-2t)$, where $c_j$ is an appropriate constant, if $2n + \gamma + d - r - 1 \neq 0$ for $0 \leq r \leq j$.

Proof. Without losing generality, we assume $p$ is a monic polynomial,

$$p(t) = \sum_{i=0}^{j} (1-t)^i a_{j,i}, \quad a_{j,j} = 1.$$

Our objective is to determine the coefficients $a_{j,i}$ such that $Z_j$ satisfies (5.2). First, we observe that (5.2) holds if $p(t) = 1$. In this case $Y \in \mathcal{H}_{n-1}^{d,0}$ and $Y(x) = t^nY(\xi)$. Hence, taking the derivative over $t$ and using the relation (2.12), a quick computation shows that

$$D_\gamma Y(x) = D_\gamma [t^nY(\xi)] = n(n + d - 2)Y(\xi) - m(m + \gamma + d - 1)t^mY(\xi) + t^{m-1}\Delta_0 Y(\xi) = -m(m + d + \gamma - 1)t^mY(\xi) = -m(m + d + \gamma - 1)Y(x).$$

For $j \geq 0$, we use the identity (5.1) to obtain

$$D_\gamma Z_j(x, t) = \sum_{i=0}^{j} a_{j,i} D_\gamma [(1-t)^i t^{n-j} Y(\xi)]$$

$$= \sum_{i=0}^{j} a_{j,i} [(1-t)^i D_\gamma [t^{i+2}] Y(x)] = i(1-t)^{i-1}[d - 1 - (d + \gamma + i - 1)t]Y(x)$$

and then apply (5.3) for $Y \in \mathcal{H}_{n-1}^{d,0}$ and simplify to obtain

$$D_\gamma Z_j(x, t) = \sum_{i=0}^{j} (1-t)^i a_{j,i} \{ \lambda_n^{(\gamma+2i)} - i(d + \gamma + i - 1) \} Y(x)$$

$$+ \sum_{i=0}^{j-1} (i+1)(1-t)^i (\gamma + i + 1) a_{j,i+1} Y(x)$$

$$= (1-t)^j \lambda_n^{(\gamma)} Y(x) + \sum_{i=0}^{j-1} (1-t)^i a_{j,i} \{ \lambda_n^{(\gamma+2i)} - i(d + \gamma + i - 1) \} + a_{j,i+1}(i+1)(\gamma + i + 1) Y(x),$$

since $a_{j,j} = 1$. Thus, in order to have $Z_{j,n}(x,t)$ satisfying (5.2), we must have

$$a_{j,i} \{ \lambda_n^{(\gamma+2i)} - i(d + \gamma + i - 1) \} + a_{j,i+1}(i+1)(\gamma + i + 1) = a_{j,i} \lambda_n^{(\gamma)}$$

for $0 \leq i \leq j - 1$, which leads to the recurrence relation

$$a_{j,i} = -\frac{(i+1)(\gamma + i + 1)}{(j-i)(2n + \gamma + d - 1 - j + i)} a_{j,i+1}, \quad i = 0, 1, \ldots, j - 1.$$
Solving this recurrence relation and using $a_{j,j} = 1$, we obtain

$$a_{j,i} = (-1)^{j-i} \frac{(i+1)_{j-i}(\gamma+i+1)_{j-i}}{(j-i)!((2n+\gamma+d-1-j+i)_{j-i})}, \quad 0 \leq i \leq j.$$  

(5.4)

Using the identity $(a+i)_{j-i} = (a)_j/(a)_i$ and $(j-i)! = (-1)^j j! / (-j)_i$, we can rewrite it as

$$a_{j,i} = (-1)^i c_j (2n+\gamma+d-j-1)_i, \quad c_j = \frac{(\gamma+1)_j}{(2n+\gamma+d-j-1)_j},$$

which is well-defined if $(2n+\gamma+d-j-1)_i \neq 0$. Consequently, the polynomial $p$ must be given by

$$p(x) = (-1)^i c_j \, \mathcal{F}_1 \left( \frac{-j, 2n+\gamma+d-j-1}{\gamma+1}; 1-t \right)$$

$$= (-1)^i c_j P^{(\gamma, 2n-2j+d-2)}_j (2t-1) = c_j P^{(2n-2j+d-2, \gamma)}_j (1-2t),$$

which completes the proof. \hfill \square

For $\gamma > -1$, the theorem recovers Theorem 3.1 established in [21]. We discuss the case when $\gamma = -s$ and $s \in \mathbb{N}$ in the next subsection.

5.2. Eigenfunctions for $D_{-s}$. As a corollary of Theorem 5.2, we obtain the following result for $D_{-s}$.

**Proposition 5.3.** Let $s \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then the polynomials $S^{n,(1-s)}_{m,\ell}$ are eigenfunctions of $D_{-s}$ if and only if $m = n - s$ and $m = n$.

**Proof.** By (2.11), the polynomials $S^{n,(1-s)}_{m,\ell}$ can be written as

$$S^{n,(1-s)}_{m,\ell}(x,y) = c_{m-m} P^{(2n+d-2,-s)}_m (1-2t) Y^m_n (x), \quad n - m \geq s,$$

so that Theorem 5.2 applies for $n-m \geq s$, thus $S^{n,(1-s)}_{m,\ell}$ satisfies (5.2). If $n = m$, then $S^{n,(1-s)}_{m,\ell}(x,t) = Y^m_n (x)$. As a result, (5.2) holds with $p(z) = 1$. For $1 \leq n-m \leq s-1$, however, the polynomial $P^{m+d-2,-s}_m (t) = (1-t)^{n-m}/(n-m)!$ is not the Jacobi polynomial $P^{(2m+d-2,-s)}_m (t)$, hence $S^{n,(1-s)}_{m,\ell}$ is not a solution of (5.2).

In the case $s = 1$, the set $\{m: n - m > s\} \cup \{n\} = \{m: 1 \leq m \leq n\}$. Hence, we follow the following corollary.

**Theorem 5.4.** For $\gamma = -1$, all elements in $V^d_n (w_{-1,-1})$ are eigenfunctions of $D_{-1}$; that is,

$$D_{-1} Z = -n(n+d-2) Z, \quad \forall Z \in V^d_n (w_{-1,-1}).$$

Moreover, the space $V^d_n (w_{-1,-1})$ satisfies a decomposition

$$V^d_n (w_{-1,-1}) = H^d_{n,0} \cup (1-t) V^d_{n-1} (w_{-1,1}), \quad n = 0, 1, \ldots.$$

**Proof.** We only need to prove the decomposition. If $m \leq n - 1$, it follows from (2.11) that $S^{n,(1-s)}_{m,\ell} (c(1-t) S^{n-1,0}_{m,\ell}) = (1-t) V^d_{n-1} (w_{1,1})$, whereas if $m = n$, then $S^{n,(1-s)}_{n,\ell} = Y^m_n \in H^d_{n,0}$.

The theorem shows, in particular, that $D_{\gamma}$ has the complete eigenspaces if $\gamma \geq -1$. For $s = 2, 3, \ldots$, however, not every element of the space $V^d_n (w_{-1,-s})$ is an eigenfunction of $D_{-s}$ by Theorem 5.2. We can, however, identify the eigenspaces of the operator as follows.
Theorem 5.5. For \( s = 2, 3, \ldots \), define
\[
\mathcal{U}_n^d(w_{-1,-s}) := \mathcal{H}_n^d \bigcup_{j=1}^{s-1} P_j^{(2n-2j+d-3,-s)}(1-2t)\mathcal{H}_{n-j}^d \cup (1-t)^s \mathcal{V}_{n-s}^d(w_{-1,s}).
\]

Then \( \mathcal{U}_n^d(w_{-1,-s}) \) is the eigenspace of \( \mathcal{D}_{-s} \); more precisely, \( \mathcal{D}_{-s} Z = \lambda_n^{(-s)} Z \) for all \( Z \in \mathcal{U}_n^d(w_{-1,-s}) \). Moreover, the space satisfies
\[
\dim \mathcal{U}_n^d(w_{-1,-s}) = \binom{n + d - 1}{d}
\]
if \(-2n - d + j + s + 2\) is not a positive integer between 1 and \( j \) for \( 1 \leq j \leq s - 1 \) and \( n \geq j \).

Proof. As in the case of \( s = 1 \), using \( \text{(2.11)} \), the statement on the eigenspace follows readily from Theorem \( \text{(5.2)} \). As it is shown for the dimension of \( \mathcal{V}_n^d(w_{\beta,-s}) \), the space \( \mathcal{U}_n^d(w_{-1,-s}) \) has the full dimension \( \binom{n + d - 1}{d} \) if the Jacobi polynomials \( P_j^{(2n-2j+d-3,-s)} \) do not have the degree reduction, which holds if \( 2n - j + d - 2 - s + k = 0 \) for a certain integer \( k \), \( 1 \leq k \leq j \), by \([17\text{ p. 64}]\). \( \square \)

The theorem shows that the operator \( \mathcal{D}_{-s} \) has a complete basis of polynomials as eigenfunctions only if the restriction on \(-2n - d + j + s + 2\) as stated holds.

Example 5.6. If \( s = 2 \), then the space \( \mathcal{U}_n^d(w_{-1,-2}) \) is given by
\[
\mathcal{U}_n^d(w_{-1,-2}) = \mathcal{H}_n^d \cup (1 + (2n + d - 4)(1-t)) \mathcal{H}_{n-1}^d \cup (1-t)^s \mathcal{V}_{n-2}^d(w_{-1,2})
\]
and it has the full dimension if \(-2n - d + 5 \neq 0 \) or \( 2n - d + 4 \neq 0 \) for \( n \geq 1 \). Thus, \( \mathcal{D}_{-2} \) has a complete basis of polynomials as eigenfunctions if \( d \) is odd. In the case that \( d \) is even, \( \mathcal{U}_n^d(w_{-1,-2}) \) is not well-defined for \( n = d/2 - 2 \).

We note that the space \( \mathcal{U}_n^d(w_{-1,-s}) \) does not coincide with \( \mathcal{V}_n^d(w_{-1,-s}) \) for \( s = 2, 3, \ldots \). In the case of \( s = -2 \) and \( d \) is odd, one may ask if there is a Sobolev inner product for which the polynomials in the space \( \mathcal{U}_n^d(w_{-1,-s}) \) are orthogonal.

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