Research Article
Prime Decomposition of Three-Dimensional Manifolds into Boundary Connected Sum

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1. Introduction

Since 2003 Matveev [1–3] had suggested a new version of the Diamond Lemma [4] of great importance for various fields of mathematics, which is suitable for and efficient solving topological problems. In this paper we apply this result to get a new conceptual proof of theorem on decomposition of three-dimensional manifolds into boundary connected sum of prime components.

2. Definition, Lemma, and 𝜕-Irreducible Manifolds

Diamond Lemma (see [5]). If an oriented graph Γ has the properties (FP) and (MF), then each of its vertices has a unique root.

Definition 1. Let D be a proper disk in a compact connected 3-manifold M. A disk reduction of M along D consists in cutting M along the disk D. Its result is a new manifold M′.

We apply nontrivial disk reductions to a given manifold M as many times as possible. If this process stops, then we obtain a set of new manifolds, which is called a root of M.

(1) If the disk is splitting, the manifold M is obtained by gluing the manifolds M₁ and M₂ together along disks on their boundaries. Then M is called a boundary connected sum of the manifolds M₁ and M₂ and denoted by M = M₁ # M₂.

(2) If the disk is nonsplitting, then M′ is also connected.

Definition 2. M is said to be 𝜕-irreducible if every properly embedded disk in M is trivial.

Lemma 3 (see [6]). Let M be a 𝜕-irreducible manifold. Let N be the manifold obtained from M by attaching a 1-handle to make the boundary connected. Then N is a prime which is not 𝜕 irreducible.

3. Proof of Theorem 4

Theorem 4. Any connected irreducible compact 3-manifold M different from a ball and with nonempty boundary is homeomorphism to a boundary connected sum M = M₁ # M₂ # … # 𝑛Mₙ of prime manifolds. All the summands are defined uniquely up to reordering and, if M is non-Orientale, replacing solid tori by solid Klein bottles S¹ × D².
We apply the universal scheme [4] in two stages. First, by considering reductions along all disks we establish uniqueness of the $\partial$-irreducible manifolds $M_j$. Then we restrict ourselves to reductions only along splitting disks and by lemma [3] complete the proof of the theorem.

We construct the graph $\Gamma$ [5] whose vertices are compact connected irreducible manifolds, considered up to addition or deletion of three-dimensional balls. The edges of the graph correspond to reductions along both splitting and nonsplitting disks.

Lemma 5 (see [7]). Each essential disks reduction strictly decreases $\gamma(A)$, where $\gamma(A) = \sum_j g^2(F_j) \ (A \in V(\Gamma))$, $g(F_j)$ is the genus of a component $F_j \subset \partial M$, and the sum is taken over all components of $\partial M$.

Since every set of vertexes of nonnegative integers has a minimal vertex, by Lemma 5, the process stops and we get a root. Then $\Gamma$ has the properties (FP).

Lemma 6. The graph $\Gamma$ has a mediator function; that is, $\Gamma$ has the properties (MF).

Proof. Let $(AB_1, AB_2) \in E^2(\Gamma)$ be a pair of edges with common beginning. Following the universal scheme, we define the value $\mu(AB_1, AB_2)$ of the mediator function $\mu : E^2(\Gamma) \to N \cup \{0\}$ to be the minimal number $|D_1 \cap D_2|$ of connected components in the intersection of disks $D_1, D_2 \subset A$, defining the edges $AB_1$ and $AB_2$. This minimum is taken over all pairs of such disks. As usual, we assume that the disks are in general position, so that their intersection consists of a finite number of circles and arcs.

We first consider the property (MF1), that is, the case $\mu(AB_1, AB_2) = 0$. When the reduction is carried out along disjoint disks $D_1$ and $D_2$, then each of these disks survives under the reduction along the other disk. We can therefore assume that $D_i \subset B_i$ for $(i, j) = (1, 2)$ and $(i, j) = (2, 1)$. By reducing each of the manifolds $B_i$ along the disk $D_i$, we obtain the same vertex $C \in V(\Gamma)$. This proves the property (MF1).

Next, we verify the property (MF2), that is, the case $\mu(AB_1, AB_2) > 0$, when any two disks $D_1$ and $D_2$ defining the edges $AB_1$ and $AB_2$ intersect. Then these disks lie in the same connected manifold $Q \subset A$. According to the universal scheme, it is enough to show that there exists a mediator disk, that is, a nontrivial disk $D \subset A$ satisfying $|D \cap D| < |D_1 \cap D_2|$ for $i = 1, 2$.

Case 1. Among the circles in $D_1 \cap D_2$ we choose one, denoted by $c$, which is innermost with respect to the disk $D_1$. This means that the circle $c$ bounds a disk $D$ in $D_1$ such that $D_1 \cap D_2 = c$. We cut $D$ along $c$ and glue up the boundaries of the cut by two parallel copies of the disk $D$. By applying a small perturbation we obtain a new disk $D'$ and a sphere $S''$ whose intersection with $D_2$ is empty and whose intersection with $D_1$ consists of a smaller number of circles (since the circle $c$ has disappeared). The disks $D'$ must be nontrivial in $Q$, since otherwise the disk $D_2$ would be trivial. Therefore, $D'$ can be taken as a mediator disk.

Case 2. Among the arcs in $D_1 \cap D_2$ we choose one, denoted by $c$, which is outermost with respect to the disk $D_1$. We cut $D_2$ along $c$ and glue up the boundaries of the cut by two parallel copies of the disk $D$. By applying a small perturbation we obtain a new disk $D'$ and a sphere $D''$ whose intersection with $D_2$ is empty and whose intersection with $D_1$ consists of a smaller number of arcs (since the circle $c$ has disappeared). At least one of these two disks (say $D'$) must be nontrivial in $Q$, since otherwise the disk $D_2$ would be trivial. Therefore, $D'$ can be taken as a mediator disk.

By applying the Diamond Lemma we prove that a root of any vertex of the graph exists and is unique.

We therefore obtain the following theorem, which is a particular case of Theorem 4 for manifolds without nonsplitting proper disks.

Theorem 7. Any connected irreducible compact three-dimensional manifold $M$ different from a ball and with nonempty boundary is decomposed by disk reductions into a union of $\partial$-irreducible parts. These parts are defined uniquely up to reordering.

In order to finish the proof of Theorem 4 we change the graph $\Gamma$ constructed by keeping the vertex set intact while disallowing reductions along nonsplitting disks. The edges of the new graph $\Gamma'$ therefore correspond to reductions only along splitting disks, so that the root of any vertex consists exactly of the prime summands of the manifolds corresponding to the vertex. The properties (FP) and (MF1) of $\Gamma$ are automatically inherited by the graph $\Gamma'$. The only difficulty with the proof of the property (MF2) is that after applying to the disk $D_2$ a surgery along the innermost circle $c \subset D_2$, both new spheres $S'$ and $S''$ and disk $D'' \subset Q$ may be nonsplitting and therefore can not be taken as mediator disks.

Case 1. Assume that $D_1 \cap D_2$ consists of $n > 3$ circles. Then we connect the spheres $S'$ and $S''$ by a tube which does not intersect $D_1$.

Since each of these spheres is obtained by connecting $S'$ and $S''$ by tubes contained in one of the parts into which $D'' \cup S' \cup S''$ divides $Q$, the disk $D_1$ is splitting and nontrivial (the latter follows from the simple observation that each simple arc connecting the sphere and disk on the boundary is trivial). It is important to note that, after a suitable small perturbation, $D_2$ intersects $D_1$ in $n - 1$ circles and intersects $D_2$ in two circles. Thus, $D_2$ is a mediator sphere.

The cases $n = 1$ and $n = 2$ need to be considered separately. We show that in each of these cases there exists a mediator disk. If $n = 1$, then the disks $D_1$ and $D_2$ split $Q$ into 4 parts $X_i, 1 \leq i \leq 4$. These parts are different, since they are separated by $D_1$ and $D_2$: any two parts lie on different sides with respect to at least one of the disks. The boundary of each part consists of a single splitting disk. At least one of
these disks is nontrivial, since both $D_1$ and $D_2$ are nontrivial. Hence, it can be taken as a mediator disk.

Now let $n = 2$. Then the disks $D_1$ and $D_2$ split $Q$ either into 5 parts $X_i$, $1 \leq i \leq 5$ (if the parts $X_1$ and $X_3$ are different), or into 4 parts (if the parts $X_1$ and $X_3$ coincide). None of the other parts can coincide, since only the parts $X_1$ and $X_3$ are not separated by $D_1$ and $D_2$. Since $X_1 \neq X_3$, then each of the parts $X_2$ and $X_4$ is bounded by a sphere $X_1$, $X_4$ is bounded by a disk, and at least one of these disks is nontrivial, since $D_2$ is nontrivial. This disk can be taken as a mediator. We also note that the part $X_3$ is bounded by a torus.

Assume that $X_1$ and $X_3$ coincide and constitute a single connected part $X$. Then we have three candidates for a mediator sphere: the spheres $\partial X_2$ and $\partial X_4$ and the new disk $D = \partial X_1 \cap \partial X_3 \subset X$. The latter is obtained by connecting $\partial X_1$ and $\partial X_3$ by a tube inside $X$. We assert that at least one of the three disks is nontrivial and therefore is a mediator.

Arguing by contradiction, assume that all three disks are trivial. This means that $X_2$ is ball and $X$ is homeomorphism to $D_2 \times I$. Then the reduction along the disk $D_1$ produces two manifolds $Y_1 = (X \cup X_4) \cup D_1$ and $Y'_1 = (X_3 \cup X_2) \cup D'_1$, and the reduction along the disk $D_2$ also produces two manifolds $Y_2 = (X \cup X_2) \cup D_2$ and $Y'_2 = (X_2 \cup X_3) \cup D'_2$. Here the balls $X_2$ are viewed as handle of index 1 (if they are attached to $X$) or as handle of index 2 (if they are attached to $X_3$). The balls $D_1, D'_1, D_2, D'_2, D_3, D'_3$ glue up the disks on the boundaries of the corresponding manifolds. Handles $X_2$ of index 1 connect different disks on the boundary of the manifold $X \approx D_2 \times I$, which implies that $Y_1 \approx Y'_1 \approx S^1 \times D^2$. On the other hand, the bases of handles $X_2$ of index 2 in the torus $\partial X_3$ are isotopic; hence $Y'_1 \approx Y'_2$. Therefore, the reductions along the disks $D_1$ and $D_2$ give the same result. This contradicts the fact that in our situation when $\mu(\overline{AB_1} - \overline{AB_2}) > 0$, the vertices $B_1$ and $B_2$ must be different.

**Case 2.** Assume that $D_1 \cap D_2$ consists of $n > 3$ arcs. The proof can be finished by an argument similar to that used in the proof of Case 1.

The arguments above are also applicable in the case of prime decompositions of non-Orientale manifolds, with the only difference that handles $X_2$ of index 1 can now be attached to the boundary of the manifold $X \approx D_2 \times I$ in two different ways. In one case the result is the direct product $S^1 \times D^2$, and in the other case we get the twisted product $S^1 \times D^2$. Nevertheless, since the manifold $M$ is non-Orientale, all summands $S^1 \times D^2$ can be replaced by $S^1 \times D^2$.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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**References**

[1] S. Matveev, *Algorithmic Topology and Classification of 3-Manifolds*, vol. 9 of Algorithms and Computation in Mathematics, Springer, Berlin, Germany, 2nd edition, 2007.

[2] C. Hog-Angeloni and S. Matveev, "Roots in 3-manifold topology," in *The Zieschang Gedenkschrift*, vol. 14 of Geometry & Topology Monographs, pp. 295–319, Geometry & Topology, 2008.

[3] S. Matveev and V. Turaev, "A semigroup of theta-curves in 3-manifolds," *Moscow Mathematical Journal*, vol. 11, no. 4, pp. 805–814, 2011.

[4] M. H. A. Newman, "On theories with a combinatorial definition of 'equivalence,'" *Annals of Mathematics*, vol. 43, pp. 223–243, 1942.

[5] G. A. Swarup, "Some properties of 3-manifolds with boundary," *The Quarterly Journal of Mathematics*, vol. 21, pp. 1–23, 1970.

[6] S. V. Matveev, "Roots and decompositions of three-dimensional topological objects," *Russian Mathematical Surveys*, vol. 67, no. 3, pp. 459–507, 2012.

[7] C. Hog-Angeloni and S. Matveev, "Roots of 3-manifolds and cobordisms," in *press*, http://arxiv.org/abs/math/0504223.
