The Calderón–Zygmund Theorem with an $L^1$ Mean Hörmander Condition

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Abstract

In 2019, Grafakos and Stockdale introduced an $L^q$ mean Hörmander condition and proved a “limited-range” Calderón–Zygmund theorem. Comparing their theorem with the classical one, it requires weaker assumptions and implies the $L^p$ boundedness for the “limited-range” instead of $1 < p < \infty$. However, in this paper, we show that the $L^q$ mean Hörmander condition is actually enough to obtain the $L^p$ boundedness for all $1 < p < \infty$ even in the worst case $q = 1$. We use a similar method to that used by Fefferman (Acta Math 124:9–36, 1970): form the Calderón–Zygmund decomposition with the bounded overlap property and approximate the bad part. Also we give a criterion of the $L^2$ boundedness for convolution type singular integral operators under the $L^1$ mean Hörmander condition.

Keywords  Singular integrals · Calderón–Zygmund theory · Fourier analysis

Mathematics Subject Classification  42B20

1 Introduction

The Calderón–Zygmund theorem is a well-known tool to investigate the $L^p$ boundedness of singular integral operators. It was originally developed by Calderón and Zygmund [2] and later improved by Hörmander [6]. Today it is usually stated as follows:
Theorem A Let $T$ be a singular integral operator with a kernel $K$. Suppose that $T$ is bounded from $L^{p_0}(\mathbb{R}^d)$ to $L^{p_0,\infty}(\mathbb{R}^d)$ for some $1 < p_0 < \infty$ and its kernel $K$ satisfies the Hörmander condition;

$$[K]_H := \sup_{B \subset \mathbb{R}^d} \sup_{y \in B} \int_{x \in \mathbb{R}^d \setminus 2B} |K(x, y) - K(x, c(B))| \, dx < \infty, \quad (1.1)$$

where the supremum $\sup_{B \subset \mathbb{R}^d}$ is taken over all balls $B$ in $\mathbb{R}^d$, $c(B)$ is the center of $B$, $2B$ denotes the ball with the same center as $B$ and whose radius is twice as long. Then $T$ is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$. It follows that $T$ is bounded on $L^p(\mathbb{R}^d)$ for all $1 < p < p_0$. In 2019, Grafakos and Stockdale [5] introduced an $L^q$ mean Hörmander condition ($H_q$ condition for short);

$$[K]_{H_q} := \sup_{B \subset \mathbb{R}^d} \left( \frac{1}{|B|} \int_{y \in B} \left( \int_{x \in \mathbb{R}^d \setminus 2B} |K(x, y) - K(x, c(B))| \, dx \right)^q \, dy \right)^{1/q} < \infty \quad (1.2)$$

and proved the following:

Theorem B [5] Let $T$ be a singular integral with a kernel $K$. Suppose that $T$ is bounded from $L^{p_0}(\mathbb{R}^d)$ to $L^{p_0,\infty}(\mathbb{R}^d)$ for some $1 < p_0 < \infty$ and its kernel $K$ satisfies the $H_{q'}$ condition for some $1 \leq q < p_0$ where $q'$ denotes the Hölder conjugate of $q$. Then $T$ is bounded from $L^q(\mathbb{R}^d)$ to $L^{q,\infty}(\mathbb{R}^d)$. It follows that $T$ is bounded on $L^p(\mathbb{R}^d)$ for all $q < p < p_0$. Note that $[K]_{H_1} \leq [K]_{H_2}$ if $1 \leq q_1 \leq q_2 \leq \infty$ and the $H_\infty$ condition is the same as the classical Hörmander condition (1.1). They named Theorem B ‘limited-range Calderón–Zygmund theorem’ because it implies the $L^p$ boundedness not for all $1 < p < p_0$ but for the ‘limited-range’: $q < p < p_0$. However, as stated in [5], they did not find any operators that satisfy the assumption of Theorem B and not bounded on $L^q$. In this sense, there is no evidence that it is truly a limited-range theorem. In this paper, we show that it is not actually limited-ranged. In fact, the $H_q$ condition is enough for the $L^1 \rightarrow L^{1,\infty}$ boundedness even in the worst case $q = 1$.

Theorem 1 Let $T$ be a singular integral operator with a kernel $K$. Suppose that $T$ is bounded from $L^{p_0}(\mathbb{R}^d)$ to $L^{p_0,\infty}(\mathbb{R}^d)$ for some $1 < p_0 < \infty$ and its kernel $K$ satisfies the $H_1$ condition;

$$[K]_{H_1} = \sup_{B \subset \mathbb{R}^d} \frac{1}{|B|} \int_{y \in B} \int_{x \in \mathbb{R}^d \setminus 2B} |K(x, y) - K(x, c(B))| \, dx \, dy < \infty. \quad (1.3)$$

Then $T$ is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ with a constant proportional to $\|T\|_{L^{p_0} \rightarrow L^{p_0,\infty}} + [K]_{H_1}$. 

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Our proof is motivated by Fefferman’s proof of the $L^1 \to L^{1,\infty}$ boundedness of strongly singular integral operators (see [4, Theorem 2']). In the proof, we form the Calderón–Zygmund decomposition of $f$: $f = g + b$, and approximate the bad part $b$ by a certain function $\tilde{b}$.

Also we will give a criterion of the $L^2$ boundedness for convolution type singular integral operators under the $H_1$ condition.

**Theorem 2** Let $K \in \mathcal{S}'(\mathbb{R}^d) \cap L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ be such that

$$A := \sup_{0 < a < b < \infty} \left| \int_{a < |x| < b} K(x) \, dx \right| < \infty,$$

$$B := \sup_{a > 0} \frac{1}{a} \int_{|x| < a} |x||K(x)| \, dx < \infty,$$

$$[K]_{H_1} := \sup_{r > 0} \frac{1}{V_d r^d} \int_{|y| \leq r} \int_{|x| \geq 2r} |K(x - y) - K(x)| \, dx \, dy < \infty,$$

where $V_d$ denotes the volume of the $d$ dimensional unit ball, and define $K_{\varepsilon,R} := K \chi_{\{\varepsilon < |x| < R\}}$ for any $0 < \varepsilon < R < \infty$. Then $K_{\varepsilon,R}$ satisfies

$$\sup_{0 < \varepsilon < R < \infty} \sup_{\xi \in \mathbb{R}^d} |\hat{K}_{\varepsilon,R}(\xi)| < \infty$$

with a constant proportional to $A + B + [K]_{H_1}$.

This is a natural generalization of the classical result stated by using the $H_\infty$ condition (see [1, Theorem 3], [3, Proposition 5.5]).

Note that it remains an open question: is the $H_1$ condition actually weaker than the classical one? As of this writing, we have no examples of $K$ such that $[K]_{H_\infty} = \infty$ but $[K]_{H_1} < \infty$.

This paper is organized as follows. We prove Theorem 1 in Sect. 2 and Theorem 2 in Sect. 3. In Sect. 4, we will remark on the $H^1 \to L^1$ boundedness under the assumption of Theorem 1.

## 2 Proof of Theorem 1

We use the following lemma:

**Lemma A** [4, Decomposition Lemma] Let $f \in L^1(\mathbb{R}^d)$ and $\lambda > 0$. Then there exists a family of disjoint dyadic cubes $(Q_j)_j$ such that

$$|f(x)| \leq \lambda \text{ a.e. } x \in \mathbb{R}^d \setminus \Omega,$$

$$\frac{1}{|B_j|} \int_{B_j} |f(x)| \, dx \leq 9^d \lambda,$$

$$|\Omega^*| \leq d^{d/2} V_d |\Omega| \leq C_d d^{d/2} V_d \frac{\|f\|_1}{\lambda}.$$

(2.1)
\[ \sum_j \chi_{2B_j} \leq (33\sqrt{d}/2)^d V_d, \quad (2.2) \]

where
- \( C_d \) denotes a constant which depends only on the dimension \( d \),
- \( B_j \) denotes the smallest ball circumscribing \( Q_j \),
- \( \Omega := \bigcup_j Q_j \),
- \( \Omega^* := \bigcup_j 2B_j \).

Furthermore, if we define

\[
g := f \chi_{\mathbb{R}^d \setminus \Omega}, \quad (2.3)
\]

\[
b_j := f \chi_{Q_j}, \quad b := f \chi_\Omega = \sum_j b_j, \quad (2.4)
\]

then immediately it follows that

\[
\|g\|_{L^p_0} \leq \lambda^{p_0 - 1} \|f\|_1, \quad (2.5)
\]

\[
\frac{1}{|B_j|} \|b_j\|_1 \leq 9^d \lambda, \quad (2.6)
\]

\[
\|b\|_1 = \sum_j \|b_j\|_1 \leq \|f\|_1. \quad (2.7)
\]

Lemma A is essentially the Whitney decomposition of \( \{ x \in \mathbb{R}^d : Mf(x) > \lambda \} \), where \( M \) is the Hardy–Littlewood maximal function with uncentered balls.\(^1\)

Note that our good part \( g \) (2.3) and bad part \( b \) (2.4) are different from usual ones. Ordinarily, they are defined by

\[
g := f \chi_{\mathbb{R}^d \setminus \Omega} + \sum_j \left( \frac{1}{|Q_j|} \int_{Q_j} f \right) \chi_{Q_j},
\]

\[
b_j := \left( f - \frac{1}{|Q_j|} \int_{Q_j} f \right) \chi_{Q_j}, \quad b := \sum_j b_j
\]

to guarantee the zero mean condition \( \int b_j = 0 \). However, our proof does not require it, hence we use our simpler definition.

Now we are going to give the proof of Theorem 1.

**Proof of Theorem 1** Fix \( f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \), \( t, \lambda > 0 \) and form the Calderón–Zygmund decomposition of \( f \) at height \( t\lambda \) (where \( t \) is given later to set appropriate estimates). In addition, fix \( \varphi \in C_c^\infty(\mathbb{R}^d) \) such that

\[
\text{supp} \varphi \subset B(0, 1), \quad \int \varphi = 1, \quad \varphi \geq 0 \quad (2.8)
\]

\(^1\) The constant \( C_d \) in Lemma A is the \( L^1(\mathbb{R}^d) \to L^{1,\infty}(\mathbb{R}^d) \) bound of \( M \), hence it can be taken to be \( 3^d \).
and write $\varphi_j(x) := s_j^{-d} \varphi(s_j^{-1} x)$ where $r_j$ is the radius of $B_j$ and $s_j := r_j/2$. We approximate $b_j$ by $\tilde{b}_j := b_j \ast \varphi_j$ and $b$ by $\tilde{b} := \sum_j \tilde{b}_j$.

Now we have $f = g - (\tilde{b} - b) + \tilde{b}$ and it suffices to show the following inequalities.

$$\lambda |\{ x \in \mathbb{R}^d : |Tg(x)| > \lambda \} | \lesssim \| f \|_1, \quad (2.9)$$

$$\lambda |\{ x \in \mathbb{R}^d : |T(\tilde{b} - b)(x)| > \lambda \} | \lesssim \| f \|_1, \quad (2.10)$$

$$\lambda |\{ x \in \mathbb{R}^d : |T\tilde{b}(x)| > \lambda \} | \lesssim \| f \|_1. \quad (2.11)$$

**Proof of (2.9)** Since $T$ is bounded from $L^{p_0}(\mathbb{R}^d)$ to $L^{p_0,\infty}(\mathbb{R}^d)$, it follows that

$$\lambda |\{ x \in \mathbb{R}^d : |Tg(x)| > \lambda \} | = \frac{1}{\lambda^{p_0-1}} \lambda^{p_0} |\{ x \in \mathbb{R}^d : |Tg(x)| > \lambda \} |$$

$$\leq \frac{1}{\lambda^{p_0-1}} \| Tg \|_{L^{p_0,\infty}}$$

$$\leq \frac{1}{\lambda^{p_0-1}} \| T \|_{L^{p_0} \to L^{p_0,\infty}} \| g \|_{L^{p_0}}$$

$$\leq (2.5) \frac{1}{\lambda^{p_0-1}} \| T \|_{L^{p_0} \to L^{p_0,\infty}} (t\lambda)^{p_0-1} \| f \|_1$$

$$= t^{p_0-1} \| T \|_{L^{p_0} \to L^{p_0,\infty}} \| f \|_1.$$

**Proof of (2.10)** Since

$$\{ x \in \mathbb{R}^d : |T(\tilde{b} - b)(x)| > \lambda \} \subset \Omega^* \cup \{ x \in \mathbb{R}^d \setminus \Omega^* : |T(\tilde{b} - b)(x)| > \lambda \},$$

it follows that

$$\lambda |\{ x \in \mathbb{R}^d : |T(\tilde{b} - b)(x)| > \lambda \} |$$

$$\leq \lambda |\Omega^*| + \lambda |\{ x \in \mathbb{R}^d \setminus \Omega^* : |T(\tilde{b} - b)(x)| > \lambda \} |$$

$$\leq t^{-1} C_d d^{d/2} V_d \| f \|_1 + \| T(\tilde{b} - b) \|_{L^1(\mathbb{R}^d \setminus \Omega^*)}.$$
\[
T \tilde{b}_j (x) = \int_{y \in \mathbb{R}^d} K (x, y) \tilde{b}_j (y) \, dy \\
= \int_{y \in \mathbb{R}^d} K (x, y) \int_{z \in B(0, s_j)} b_j (y - z) \varphi_j (z) \, dz \, dy \\
= \int_{z \in B(0, s_j)} \int_{y \in \mathbb{R}^d} K (x, y) b_j (y - z) \, dy \varphi_j (z) \, dz \\
= \int_{z \in B(0, s_j)} \left( \int_{y \in B_j} K (x, y + z) \, dy \right) \varphi_j (z) \, dz
\]

since \( T \) is a singular integral operator with a kernel \( K \). Therefore, for each \( j \), we have

\[
\| T (\tilde{b}_j - b_j) \|_{L^1 (\mathbb{R}^d \setminus \Omega^* )} \\
\leq \int_{x \in \mathbb{R}^d \setminus 2B_j} | T \tilde{b}_j (x) - T b_j (x) | \, dx \\
= \int_{x \in \mathbb{R}^d \setminus 2B_j} \left| \int_{z \in B(0, s_j)} \left( \int_{y \in B_j} (K (x, y + z) - K (x, y)) b_j (y) \, dy \right) \varphi_j (z) \, dz \right| \, dx \\
\leq \int_{y \in B_j} \left( \int_{z \in B(0, s_j)} \int_{x \in \mathbb{R}^d \setminus 2B_j} | K (x, y + z) - K (x, y) | \, dx \varphi_j (z) \, dz \right) | b_j (y) | \, dy \\
\leq \| \varphi_j \|_{\infty} \int_{y \in B_j} \left( \int_{z \in B(0, s_j)} \int_{x \in \mathbb{R}^d \setminus B(\gamma, r_j)} | K (x, y + z) - K (x, y) | \, dx \, dz \right) | b_j (y) | \, dy \\
= V_d \| \varphi \|_{\infty} \int_{y \in B_j} | [K]_{H_1} | b_j (y) | \, dy \\
= V_d \| \varphi \|_{\infty} [K]_{H_1} \| b_j \|_1.
\]

It follows that

\[
\| T (\tilde{b} - b) \|_{L^1 (\mathbb{R}^d \setminus \Omega^* )} \leq \sum_j \| T (\tilde{b}_j - b_j) \|_{L^1 (\mathbb{R}^d \setminus \Omega^* )} \leq V_d \| \varphi \|_{\infty} [K]_{H_1} \| f \|_1.
\]

**Proof of (2.11)** By the same argument as in the proof of (2.9), we have

\[
\lambda | \{ x \in \mathbb{R}^d : | T \tilde{b} (x) | > \lambda \} | = \frac{1}{\lambda} \| T \|_{L^p_0 \rightarrow L^{p_0} \infty} \| \tilde{b} \|_{p_0} \\
\leq \frac{1}{\lambda^{p_0-1}} \| T \|_{L^p_0 \rightarrow L^{p_0} \infty} \| \tilde{b} \|_{p_0}^{-1} \| \tilde{b} \|_1.
\]
Since it is obvious that
\[ \|\tilde{b}\|_1 \leq \sum_j \|b_j \ast \varphi_j\|_1 \leq \sum_j \|b_j\|_1 \leq (2.7) \|f\|_1, \]
it is enough to show that \( \|\tilde{b}\|_\infty \lesssim \lambda \). For each \( j \), we have
\[ \|b_j \ast \varphi_j\|_\infty \leq \|b_j\|_1 \|\varphi_j\|_\infty = \|\varphi\|_\infty \frac{\|b_j\|_1}{s_j} \leq (2.6) 18^d V_d \|\varphi\|_\infty t \lambda. \]
Therefore, it follows from the bounded overlap property (2.2),
\[ \|\tilde{b}\|_\infty \leq 297 d \|\varphi\|_\infty t \lambda. \]
Hence we conclude that
\[ \lambda \{|x \in \mathbb{R}^d : |T\tilde{b}(x)| > \lambda\} \leq t^{p_0-1} (297^d d^{d/2} V_d^2 \|\varphi\|_\infty)^{p_0-1} \|T\|_{L^{p_0} \rightarrow L^{p_0, \infty}} \|f\|_1. \]
\[ \square \]
Combining estimates above, we obtain
\[ \|Tf\|_{1, \infty} \leq 3 (t^{p_0-1} \|T\|_{L^{p_0} \rightarrow L^{p_0, \infty}} + (t^{-1} C d d^{d/2} V_d + V_d \|\varphi\|_\infty [K]_{H_1}) + t^{p_0-1} (297^d d^{d/2} V_d^2 \|\varphi\|_\infty)^{p_0-1} \|T\|_{L^{p_0} \rightarrow L^{p_0, \infty}} \|f\|_1. \]
Finally, remember that \( t \) and \( \varphi \) are arbitrary. Since \( \inf_{\varphi} \) satisfies (2.8), \( \|\varphi\|_\infty = V_d^{-1} \), we conclude that
\[ \|T\|_{L^1 \rightarrow L^{1, \infty}} \leq 3 \inf_{t > 0} (t^{-1} C d d^{d/2} V_d + [K]_{H_1} + t^{p_0-1} (1 + (297^d d^{d/2} V_d)^{p_0-1}) \|T\|_{L^{p_0} \rightarrow L^{p_0, \infty}} \|T\|_{L^{p_0} \rightarrow L^{p_0, \infty}} + \inf_{\varphi} \|\varphi\|_\infty [K]_{H_1}) \]
\[ \square \]
\section{The Proof of Theorem 2}
We use the following lemma:

**Lemma 1** If \( K \in L^1_{loc}(\mathbb{R}^d \setminus \{0\}) \) satisfies (1.5) and (1.6), then
\[ \sup_{0 < \varepsilon < R < \infty} [K_{\varepsilon, R}]_{H_1} \leq [K]_{H_1} + 7B. \] (3.1)
Proof of Lemma 1} It is obvious that
\[
\begin{align*}
\frac{1}{V_{d}r^{d}} \int_{|y| \leq r} \int_{|x| \geq 2r} |K_{\varepsilon, R}(x - y) - K_{\varepsilon, R}(x)| \, dx \, dy \\
= \frac{1}{V_{d}r^{d}} \int_{|y| \leq r} \int_{|x| \geq 2r, \varepsilon < |x - y| < R, \varepsilon < |x| < R} |K(x - y) - K(x)| \, dx \, dy \\
+ \frac{1}{V_{d}r^{d}} \int_{|y| \leq r} \int_{|x| \geq 2r, \varepsilon < |x - y| < R, |x| \leq \varepsilon} |K(x - y)| \, dx \, dy \\
+ \frac{1}{V_{d}r^{d}} \int_{|y| \leq r} \int_{|x| \geq 2r, \varepsilon < |x - y| < R, |x| \geq R} |K(x - y)| \, dx \, dy \\
+ \frac{1}{V_{d}r^{d}} \int_{|y| \leq r} \int_{|x| \geq 2r, |x - y| \leq \varepsilon, \varepsilon < |x| < R} |K(x)| \, dx \, dy \\
+ \frac{1}{V_{d}r^{d}} \int_{|y| \leq r} \int_{|x| \geq 2r, |x - y| \geq R, \varepsilon < |x| < R} |K(x)| \, dx \, dy
\end{align*}
\]
and the first term is bounded by $[K]_{H_{1}}$. To estimate other terms, note that (1.5) implies
\[
\sup_{a > 0} \int_{a < |x| < ca} |K(x)| \, dx \leq \sup_{a > 0} \int_{a < |x| < ca} \frac{|x|}{a} |K(x)| \, dx \leq cB
\]
for any $c > 1$. Since we have
\[
\begin{align*}
\varepsilon < |x - y| < R, |x| \leq \varepsilon &\quad \Rightarrow \quad |x - y| \leq |x| + |y| \leq 3|x| / 2 \leq 3\varepsilon / 2, \\
\varepsilon < |x - y| < R, |x| \geq R &\quad \Rightarrow \quad |x - y| \geq |x| - |y| \geq |x| / 2 \geq R / 2, \\
|x - y| \leq \varepsilon, \varepsilon < |x| < R &\quad \Rightarrow \quad |x| \leq 2(|x - y| + |y|) - |x| \leq 2|x - y| < 2\varepsilon, \\
|x - y| \geq R, \varepsilon < |x| < R &\quad \Rightarrow \quad |x| \geq 2(|x - y| - |y|) / 3 + |x| / 3 \geq 2|x - y| / 3 \geq 2R / 3
\end{align*}
\]
under the condition $2|y| \leq 2r \leq |x|$, the second and fifth terms are bounded by $3B / 2$, the third and fourth terms are bounded by $2B$.

\textbf{Proof of Theorem 2} Fix $0 < \varepsilon < R < \infty$ and $\xi \in \mathbb{R}^{d}$. Since it is obvious that
\[
|\hat{K}_{\varepsilon, R}(0)| = \left| \int_{|x| < \varepsilon} K(x) \, dx \right| \leq A,
\]
we assume $\xi \neq 0$ and write $s := |\xi|^{-1}$. If we decompose $\hat{K}_{\varepsilon, R}(\xi)$ as
\[
\hat{K}_{\varepsilon, R}(\xi) = \int_{x \in \mathbb{R}^{d}} K_{\varepsilon, R}(x) e^{-2\pi i x \cdot \xi} \, dx
\]
\[
= \int_{|x| < 2s} K_{\varepsilon, R}(x)(e^{-2\pi i x \cdot \xi} - 1) \, dx + \int_{|x| < 2s} K_{\varepsilon, R}(x) \, dx + \int_{2s \leq |x|} K_{\varepsilon, R}(x) e^{-2\pi i x \cdot \xi} \, dx
\]
\[
=: I_{1} + I_{2} + I_{3},
\]

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then we easily get

\[ |I_1| \leq \int_{|x|<2s} |K_{\varepsilon,R}(x)||e^{-2\pi ix\cdot \xi} - 1| \, dx \leq 4\pi \frac{1}{2s} \int_{|x|<2s} |x||K_{\varepsilon,R}(x)| \, dx \leq 4\pi B, \tag{1.5} \]

\[ |I_2| = \left| \int_{\varepsilon < |x| < 2s} K_{\varepsilon,R}(x) \, dx \right| \leq A. \tag{1.4} \]

To estimate \( I_3 \), fix a radial function \( \varphi \in C^\infty_c(\mathbb{R}^d) \) such that

\[ \text{supp} \varphi \subset B(0,1), \quad \int \varphi = 1, \quad \varphi \geq 0, \quad |\hat{\varphi}(1)| < 1 \]

and define \( \varphi_s(x) := s^{-d} \varphi(s^{-1} x) \). Moreover, rewrite

\[ I_3 = \int_{|x| \geq 2s} \int_{|y| \leq s} K_{\varepsilon,R}(x)\varphi_s(y) \, dy \, e^{-2\pi ix\cdot \xi} \, dx \tag{3.2} \]

and introduce

\[ I_4 := \int_{|x| \geq 2s} \int_{|y| \leq s} K_{\varepsilon,R}(x-y)\varphi_s(y) \, dy \, e^{-2\pi ix\cdot \xi} \, dx, \tag{3.3} \]

\[ I_5 := \int_{|x| < 2s} \int_{|y| \leq s} K_{\varepsilon,R}(x-y)\varphi_s(y) \, dy \, e^{-2\pi ix\cdot \xi} \, dx \tag{3.4} \]

\[ = \int_{|x| < 2s} \int_{|x-y| \leq s} K_{\varepsilon,R}(y)\varphi_s(x-y) \, dy \, e^{-2\pi ix\cdot \xi} \, dx \tag{3.5} \]

\[ I_6 := \int_{|x| < 2s} \int_{|y| \leq 3s} K_{\varepsilon,R}(y)\varphi_s(x) \, dy \, e^{-2\pi ix\cdot \xi} \, dx \tag{3.6} \]

\[ = \hat{\varphi}_s(\xi) \int_{|y| \leq 3s} K_{\varepsilon,R}(y) \, dy. \tag{3.7} \]

We decompose \( I_3 \) into \((I_3 - I_4) + (I_4 + I_5) - (I_5 - I_6) - I_6\). By (3.2), (3.3) and Lemma 1, we get
\[ |I_4 - I_3| \]
\[
\leq (3.2), (3.3) \left| \int_{|x| \geq 2s} \int_{|y| \leq s} (K_{\xi, R}(x - y) - K_{\xi, R}(x)) \varphi_s(y) e^{-2\pi i x \cdot \xi} \, dy \, dx \right|
\]
\[
\leq \int_{|y| \leq s} \int_{|x| \geq 2s} |K_{\xi, R}(x - y) - K_{\xi, R}(x)| \varphi_s(y) \, dy \, dx
\]
\[
\leq V_d \|\varphi\|_{\infty} \frac{1}{V_d s^d} \int_{|y| \leq s} \int_{|x| < 2s} |K_{\xi, R}(x - y) - K_{\xi, R}(x)| \, dx \, dy
\]
\[
\leq V_d \|\varphi\|_{\infty} [K_{\xi, R}]_{H_1}
\]
\[
\leq V_d \|\varphi\|_{\infty} ([K]_{H_1} + 7B).
\]

For \( I_5 - I_6 \), use (3.5), (3.6) and the mean value theorem to obtain

\[
|I_5 - I_6| \]
\[
= (3.5), (3.6) \left| \int_{|x| < 2s} \int_{|y| \leq 3s} K_{\xi, R}(y)(\varphi_s(x - y) - \varphi_s(x)) e^{-2\pi i x \cdot \xi} \, dy \, dx \right|
\]
\[
\leq \int_{|x| < 2s} \int_{|y| \leq 3s} |K_{\xi, R}(y)| |\varphi_s(x - y) - \varphi_s(x)| \, dy \, dx
\]
\[
\leq \int_{|x| < 2s} \int_{|y| \leq 3s} |K_{\xi, R}(y)| s^{-d-1} |y| \|\nabla \varphi\|_{\infty} \, dy \, dx
\]
\[
= 3s^{-d} \|\nabla \varphi\|_{\infty} \int_{|x| < 2s} \left( \frac{1}{3s} \int_{|y| \leq 3s} |y| |K_{\xi, R}(y)| \, dy \right) \, dx
\]
\[
\leq (1.5) \left( 3s^{-d} \|\nabla \varphi\|_{\infty} \int_{|x| \leq 2s} B \, dx \right)
\]
\[
= 3 \cdot 2^d V_d \|\nabla \varphi\|_{\infty} B.
\]

For \( I_4 + I_5 \) and \( I_6 \), remark that \( \hat{\varphi}_s(\xi) = \hat{\varphi}(s \xi) = \hat{\varphi}(1) \) because \( \varphi \) is radial and \( s = |\xi|^{-1} \). Then it follows immediately that

\[ I_4 + I_5 \]
\[
(3.3), (3.4) K_{\xi, R} * \varphi_s(\xi) = \hat{\varphi}(1) \widehat{K_{\xi, R}(\xi)},
\]
\[
|I_6| \]
\[
(3.7) |\hat{\varphi}_s(\xi) \int_{|x| \leq 3s} K_{\xi, R}(y) \, dy| \leq (1.4) |\hat{\varphi}(1)| A.
\]

Now we have

\[
|\widehat{K_{\xi, R}(\xi)}|
\]
\[
\leq |I_1| + |I_2| + |I_3 - I_4| + |I_4 + I_5| + |I_5 - I_6| + |I_6|
\]
\[
\leq 4\pi B + A + V_d \|\varphi\|_{\infty} ([K]_{H_1} + 7B) + |\hat{\varphi}(1)| |\widehat{K_{\xi, R}(\xi)}|
\]
\[
+ 3 \cdot 2^d V_d \|\nabla \varphi\|_{\infty} B + |\hat{\varphi}(1)| A
\]
for any $\xi \in \mathbb{R}^d$ (it is still valid in the case $\xi = 0$). Finally, remember $|\hat{\varphi}(1)| < 1$ to conclude that

$$|\hat{K}_{\varepsilon, R}(\xi)| \leq \frac{(1 + |\hat{\varphi}(1)|)A + (4\pi + V_d(7\|\varphi\|_{\infty} + 3\cdot2^d\|\nabla\varphi\|_{\infty}))B + V_d\|\varphi\|_{\infty}[K]_{H_1}}{1 - |\hat{\varphi}(1)|}. $$

\[\square\]

4 Remark

We can also obtain the $H^1 \to L^1$ boundedness under the assumption of Theorem 1.

**Theorem 3** Let $T$ be a singular integral operator with a kernel $K$. Suppose that $T$ is bounded from $L^{p_0}(\mathbb{R}^d)$ to $L^{p_0, \infty}(\mathbb{R}^d)$ for some $1 < p_0 < \infty$ and its kernel $K$ satisfies the $H_1$ condition. Then $T$ is bounded from $H^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$.

To see this, note that Theorem 1 implies that $T$ is bounded on $L^p(\mathbb{R}^d)$ for any $1 < p < p_0$. Hence we assume that $T$ is bounded on $L^{p_0}(\mathbb{R}^d)$ for some $1 < p_0 < \infty$ without loss of generality. Now we can show the $H^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ boundedness. We do not prove it here because its proof is the almost same as that of the classical theorem (see [3, Proposition 6.2, Corollary 6.3]).

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