Black hole Area-Angular momentum inequality in non-vacuum spacetimes

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We show that the area-angular momentum inequality $A \geq 8\pi |J|$ holds for axially symmetric closed outermost stably marginally trapped surfaces. These are horizon sections (in particular, apparent horizons) contained in otherwise generic non-necessarily axisymmetric black hole spacetimes, with non-negative cosmological constant and whose matter content satisfies the dominant energy condition.

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Introduction. Isolated and stationary black holes cannot rotate arbitrarily fast. The total angular momentum $J$ in Kerr solutions that are consistent with cosmic censorship (i.e. without naked singularities) is bounded from above by the square of the total mass $M$. The heuristic standard picture of gravitational collapse [1] emphasizes the physical relevance of this bound and suggests its generic validity beyond idealized situations. The total mass-angular momentum inequality $J \leq M^2$ has been indeed extended to the dynamical case of vacuum axisymmetric black hole spacetimes [2–7]. However, this inequality involves global quantities. In order to gain further insight into the gravitational collapse process in the presence of matter and/or multiple horizons, it is also desirable to have a quasi-local version of the inequality. This attempt encounters immediately, though, the ambiguities associated with the quasi-local definition of gravitational mass and angular momentum. In this context, an alternative (but related) bound on the angular momentum can be formulated in terms of a horizon area-angular momentum inequality $A \geq 8\pi |J|$ [8]. This inequality was conjectured for the non-vacuum axisymmetric stationary case (actually, including the charged case) with matter surrounding the horizon in [8] and then proved in [9–11], whereas its validity in the vacuum axisymmetric dynamical case was conjectured and discussed in [12] in a partial results were given in [13, 14] and a complete proof in [15]. Equality holds in the extremal case. Here we reconcile and extend both results by proving the validity of the inequality in fully general dynamical non-vacuum spacetimes (on the horizon allowed), only requiring axisymmetry on the horizon. The rest of this letter is devoted to prove this result.

The dynamical non-vacuum case. Proofs of $A \geq 8\pi |J|$ require some kind of geometric stability condition characterizing the surface $S$ for which the inequality is proved. On the one hand, in the non-vacuum stationary case discussed in [9–11] surfaces $S$ are taken to be sections of black hole horizons modeled as outer trapping horizons [16]. This entails, first, the vanishing of the expansion $\theta^{(\ell)}$ associated with light rays emitted from $S$ along the (outgoing) null normal $k^\mu$ [i.e. $S$ is a marginally trapped surface] and, second, that when moving towards the interior of the black hole one finds fully trapped surfaces, so that the variation of $\theta^{(\ell)}$ along some future ingoing null normal $k^\mu$ is negative (outer condition): $\delta_k \theta^{(\ell)} < 0$ (see [17] for a detailed discussion of this condition in the context of black hole extremality). The latter inequality acts as a stability condition on $S$ and, actually, is closely related to the stably outermost condition imposed on marginally trapped surfaces contained in spatial 3-slices $\Sigma$ when proving the existence of dynamical trapping horizons [18]. Such stably outermost condition means that the variation of $\theta^{(\ell)}$ along some outward deformation of $S$ in the slice $\Sigma$ is non-negative. That is, $\delta_S \theta^{(\ell)} \geq 0$ for some spacelike outgoing vector $v$ tangent to $\Sigma$ (see the generalization to spacetime normal vectors in [19]; we refer to [18] for a discussion on operator $\delta_S$). Regarding now the vacuum dynamical case in [13], the inequality $A \geq 8\pi |J|$ is first proved for stable minimal surfaces $S$ in a spatial maximal slice $\Sigma$, i.e. $S$ is a local minimum of the area when considering arbitrary deformations of $S$ in $\Sigma$ and then generalized for arbitrary surfaces, in particular horizon sections.

The present discussion of the inequality $A \geq 8\pi |J|$ closely follows the strategy and steps in [13], adapting them to the use of a stability condition in the spirit of those in [2–7, 19], i.e. based on marginally trapped surfaces rather than on minimal surfaces. In the line of [13, 15] we will refer to a marginally trapped surface $S$ as (spacetime) stably outermost (see Definition 1 below) if for some outgoing spacelike vector or outgoing past null vector $X^\mu$ it holds $\delta_X \theta^{(\ell)} \geq 0$. Then, it follows:

**Theorem 1.** Given an axisymmetric closed marginally trapped surface $S$ satisfying the (axisymmetry-compatible) spacetime stably outermost condition, in a spacetime with non-negative cosmological constant and fulfilling the dominant energy condition, it holds the inequality

$$A \geq 8\pi |J| \quad ,$$

where $A$ and $J$ are the area and gravitational (Komar) angular momentum of $S$. If equality holds, then $S$ has the geometry of an extreme Kerr throat sphere and, in addition, if vector $X^\mu$ in the stability condition can be found to be spacelike then $S$ is a section of a non-expanding horizon.

Note that axisymmetry is only required on the horizon surface (this includes the intrinsic geometry of $S$ and a certain component of its extrinsic geometry, see below), so that
Let us consider null vectors $\xi\ell$ accounts solely for the angular momentum of the black hole (horizon) in an otherwise generically non-axisymmetric-- and normalized as

$S\xi\ell = 1$. We can then write the induced metric on $S$ as $g_{ab} = \frac{1}{\eta}\eta g_{ab} + \xi\ell\delta_{ab}$ with $\eta = \eta\xi\ell$, so that

$$
\Omega_a^{(f)} = \Omega_a^{(g)} + \Omega_a^{(l)}
$$

with $\Omega_a^{(g)} = \eta\ell\xi\ell/\eta$ and $\Omega_a^{(l)} = \xi\ell\Omega_a^{(g)}\xi\ell$. In addition, we demand $\Omega_a^{(f)}$ to be also axisymmetric, $\ell\eta\xi\ell = 0$. Second, $S$ is taken to be a marginal trapped surface; $\theta^{(f)} = 0$. We will refer to $\ell\eta$ as the **outgoing null vector**, Third, a stability condition must be imposed on $S$, namely we demand the marginally trapped surface to be a **space-time stably outermost** in the following sense:

**Definition 1.** Given a closed marginally trapped surface $S$ we will refer to it as space-time stably outermost if there exists an outgoing $(-k^a$-oriented) vector $X^a = \gamma^a - \psi k^a$, with $\gamma > 0$ and $\psi > 0$, such that the variation of $\theta^{(f)}$ with respect to $X^a$ fulfills the condition

$$
\delta_X \theta^{(f)} \geq 0.
$$

If, in addition, $X^a$ (in particular $\gamma$, $\psi$) and $\Omega^{(f)}$ are axisymmetric, we will refer to $\delta_X \theta^{(f)} > 0$ as an (**axisymmetry-compatible**) space-time stably outermost condition.

Here $\delta$ denotes a variation operator associated with a deformation of the space-time $S$ (c.f. for example [18, 24]). Two remarks are in order. First, note that the characterization of a marginally trapped surface as space-time stably outermost is independent of the choice of future-oriented null normals $\ell^a$ and $k^a$. Indeed, given $f > 0$, for $\ell^a = f \ell^a$ and $k^a = f^{-1} k^a$ we can write $X^a = \gamma^a - \psi k^a = \gamma^a f \ell^a - \psi f k^a$ (with $\gamma^a = f^{-1} \gamma > 0$ and $\psi = f \psi > 0$), and it holds $\delta_X \theta^{(f)} = f \cdot \delta_X \theta^{(f)} > 0$. Second, the proof of inequality (1) would only require the vector $X^a$ in the stability condition to be outgoing past null [14, 21]. We have, however, kept a more generic characterization in Definition 1 that directly extends the stably outermost condition in [18] (in particular, $S$ is space-time stably outermost if there exists a $(-k^a$-oriented) vector for which $S$ is stably outermost in the sense of [19]).

We can now establish the lower bound on the horizon area by following analogous steps to those in [15]. First, we derive a generic inequality on $S$, provided by the following lemma.

**Lemma 1.** Given a closed marginally trapped surface $S$ satisfying the space-time stably outermost condition for an axisymmetric $X^a$, then for all axisymmetric $\alpha$ it holds

$$
\int_S \left[ D_a \alpha D^a \alpha + \frac{1}{2} \alpha^2 \cdot 2 R \right] S d\eta \geq 0
$$

where $\beta = \alpha \gamma / \psi$.

To prove it we basically follow the discussion in section 3.3. of [22], allowing one to essentially reduce the non-time-symmetric case to the time-symmetric one (c.f. Th. 2.1 in
From the axisymmetry of the vector, we can write
\[ \frac{1}{\psi} \delta \chi \theta^{(\ell)} = \frac{1}{\psi} \left[ \sigma^{(\ell)} + G_{ab} \theta^{(\ell)} \right] \]

\[ \int \left[ \sigma^{(\ell)} + G_{ab} \theta^{(\ell)} \right] dS \geq \int \left[ -\Delta \psi \theta^{(\ell)} + 2G_{ab} \theta^{(\ell)} \right] dS. \]

Using \( \int S \frac{1}{\psi} \delta \chi \theta^{(\ell)} dS \geq 0 \), integrating by parts to remove boundary terms, we can write
\[ 0 \leq \int \left[ \sigma_{ab} - G_{ab} \theta^{(\ell)} \right] dS + \int \alpha^2 \left[ -\Omega^{(\ell)} - \frac{1}{2} R - G_{ab} \theta^{(\ell)} \right] dS \]

\[ + \int [2\alpha D_a \Omega^{(\ell)} - 2\alpha \Omega^{(\ell)}] dS. \]

From the axisymmetry of \( \alpha \) and \( \psi \), \( \Omega^{(\ell)} \) is a non-spacelike vector. Therefore
\[ 0 \leq \int \left[ \sigma_{ab} - G_{ab} \theta^{(\ell)} \right] dS + \int \alpha^2 \left[ -\Omega^{(\ell)} - \frac{1}{2} R - G_{ab} \theta^{(\ell)} \right] dS \]

\[ + \int [2\alpha D_a \Omega^{(\ell)} - 2\alpha \Omega^{(\ell)}] dS. \]

Making use of the Young’s inequality in the last integral
\[ D^a \alpha D_a \alpha \geq 2D^a \alpha (D_a \psi - \alpha \Omega^{(\ell)}) - |\omega| D_a \psi - \alpha \Omega^{(\ell)} \]

inequality follows for all axisymmetric \( \alpha \).

Inequality constitutes the first key ingredient in the present discussion and the counterpart of inequality (15) in [11] [inserting their Eqs. (30) and (31)]. In this spacetime version, the geometric meaning of each term in inequality (8) is apparent. For our present purposes, we first disregard the positive-definite gravitational radiation shear squared term. Imposing Einstein equations, we also disregard the cosmological constant and matter terms (29), under the assumption of non-negative cosmological constant \( \Lambda \geq 0 \) and the dominant energy condition (note that \( \alpha k^b + \beta k^b \) is a non-space like vector). Therefore
\[ \int D_a \alpha D^a \alpha \geq \int D_a \alpha D^a \alpha \]

This geometric inequality completes the first stage towards the lower bound on \( A \).

In a second stage, under the assumption of axisymmetry we evaluate inequality (12) along the lines in [11]. First, we note that the sphericity of \( S \) follows from Lemma 1 under the outermost stably and dominant energy conditions together with \( \Lambda \geq 0 \) since, upon the choice of a constant \( \alpha \) in Eq. (8), it implies (for non-vanishing angular momentum) a positive value for the Euler characteristic of \( S \). Then, the following form for the axisymmetric line element on \( S \) is adopted
\[ ds^2 = q_{ab} dx^a dx^b = e^\theta (e^{2\theta} d\theta^2 + \sin^2 \theta d\varphi^2), \]

with \( \sigma \) and functions on \( \theta \) satisfying \( \sigma + q = c \), where \( c \) is a constant. This coordinate system can always be found in axisymmetry [30]. We can then write \( dS = e^\theta dS_0 \), with \( dS_0 = \sin \theta d\theta d\varphi \). In addition, the squared norm \( \eta \) of the axial Killing vector \( \eta = (\partial_\varphi)^\alpha \) is given by \( \eta = e^\theta \sin^2 \theta \).

Regarding the left hand side in (12), we proceed exactly as in [11]. In particular, choosing \( \alpha = e^{-\varphi/2} \), the evaluation of the left-hand-side in inequality (12) results in (see [11])
\[ \int S \left[ D_a \alpha D^a \alpha + \frac{1}{2} \alpha^2 \right] dS \]

\[ = e^\theta \left[ 4\pi (c + 1) - \int S \left( \alpha + \frac{1}{8} \frac{d\alpha}{d\theta} \right)^2 \right] dS_0. \]

The second key ingredient in the present discussion concerns the evaluation of the right hand side in (12), in particular the possibility of making contact with the variational functional \( M \) employed in [11] [13].

Due to the \( S^2 \) topology of \( S \), we can always express \( \Omega^{(\ell)} \) in terms of a divergence-free and an exact form. Writing
\[ \Omega^{(\ell)} = \frac{1}{2\eta} e_{ab} D^b \varphi + D_a \lambda, \]

with \( \varphi \) and \( \lambda \) fixed up to a constant, from the axisymmetry of \( q_{ab} \) and \( \Omega^{(\ell)} \) (functions \( \varphi \) and \( \lambda \) are then axially symmetric) it follows that \( \Omega^{(\ell)} \) is the divergence-free part whereas \( \Omega^{(\ell)} \) is the exact (gauge) part. In particular, \( \eta \alpha \Omega^{(\ell)} = \frac{1}{2\eta} e_{ab} \eta^a D_b \varphi \) and expressing \( \xi^a \) as \( \xi^a = \eta^{-1/2} e_{ab} \eta^a \), we have
\[ \Omega^{(\ell)} \eta^a = \frac{1}{2\eta^{1/2}} \epsilon^a D_a \varphi. \]

Plugging this expression into Eq. (15) and using (14) we find
\[ J = \frac{1}{8} \int_0^\pi \partial_\theta \omega d\theta = \frac{1}{8} (\omega(\pi) - \omega(0)) \]

which is identical to the relation between \( J \) and the twist potential \( \omega \) in Eq. (12) of [11]. As a remark, we note that if the axial vector \( \eta^a \) on \( S \) extends to a spacetime neighbourhood of \( S \) (something not needed in the present discussion), we can define the twist vector of \( \eta^a \) as \( \omega_a = \epsilon_{abcde} \nabla^c \eta^d \) and the relation \( \xi^a \omega_a = \xi^a D_a \omega \) holds. In the vacuum case, a twist potential \( \omega \) satisfying \( \omega_a = \nabla_a \omega \) can be defined, so that \( \omega \) and \( \omega \) coincide on \( S \) up to a constant. Note however that \( \omega \) on \( S \) can be defined always.
From Eqs. (15) and (13) and the choice of \( \alpha \), we have

\[
\alpha^2 \Omega^{(0)}(\theta^{(0)})^a = \frac{\alpha^2}{4\kappa^2} D_a \omega D^a \omega = \frac{1}{4\kappa^2} \left( \frac{d\omega}{d\theta} \right)^2.
\]

Using this and (14) in (12) we recover exactly the bound

\[
A \geq 4\pi e^{M-8},
\]

with the action functional

\[
\mathcal{M} = \frac{1}{2\pi} \int_S \left( \left( \frac{d\sigma}{d\theta} \right)^2 + 4\sigma + \frac{1}{\eta^2} \left( \frac{d\omega}{d\theta} \right)^2 \right) d\sigma d\theta,
\]

in Ref. [15], so that the rest of the proof reduces to that in this reference. Namely, the upper bound in [13] for \( J \)

\[
e^{(M-8)/8} \geq 2|J|,
\]

together with inequality (19) lead to the area-angular momentum inequality (1) and, in addition, a rigidity result follows: if equality in (1) holds, first, the intrinsic geometry of \( S \) is that of an extreme Kerr throat sphere (13) and, second, the vanishing of the positive-definite terms in (8) implies in particular, for spacelike \( X^a \) in (7), the vanishing of the shear \( \sigma_{ab} \) so that \( S \) is an instantaneous (non-expanding) isolated horizon (24).

Discussion. We have shown that axisymmetric stable marginally trapped surfaces (in particular, apparent horizons) satisfy the inequality \( A \geq 8\pi |J| \) in generically dynamical, non-necessarily axisymmetric, spacetimes with ordinary matter that can extend to the horizon. There are two key ingredients enabling the remarkable shift from the initial data discussion of inequality (1) in [15] to a (purely quasi-local) spacetime result. First, the derivation of the geometric inequality (8) where the spacetime interpretation of each term in the right hand side is transparent and, more importantly, the global sign is controlled by standard physical assumptions on the matter energy content. This relaxes the counterpart maximal slicing hypothesis in [15]. Second, using the spherical topology of \( S \) we express the quadratic term controlling the angular momentum in the inequality in terms of a potential \( \omega \) living solely on the sphere and leading to an exact match with the key variational functional in [15]. This permits to avoid any further assumption on the spacetime geometry. A critical ingredient in the present derivation is the stability assumption (7), basically the stably outermost condition in [15] that naturally extends the stability condition in [18] (in the context of spatial 3-slices) to general spacetime embeddings of marginally trapped surfaces. This stability condition, essentially equivalent to the outer horizon condition in [16], implies that (axially symmetric)[31] outer trapping horizons [16] satisfy inequality (1), independently of a future/past condition on \( \theta^{(k)} \) (respectively, \( \theta^{(k)} < 0 \) or \( \theta^{(k)} > 0 \)).

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Note that this is essentially the coordinate system employed in the definition of isolated horizon mass multipoles in [28].

The non-axially symmetric case is of astrophysical interest. Ambiguities in the general definition of $J$, in particular involving the choice and normalization of the axial vector $\eta^a$, complicate the writing of a universal area-angular momentum inequality. The present analysis however suggests that such upper bounds on appropriate $J$’s are to be expected, in particular when close to axisymmetry, this being the subject of current research.

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