BOUNDEDNESS OF DIFFERENTIAL TRANSFORMS FOR HEAT SEMIGROUPS GENERATED BY FRACTIONAL LAPLACIAN

XINYU REN AND CHAO ZHANG

Abstract. In this paper we analyze the convergence of the following type of series

\[ T_N f(x) = \sum_{j=N_1}^{N_2} v_j \left( e^{-a_j} (\Delta)^{\alpha} f(x) - e^{-a_j} (\Delta)^{\alpha} f(x) \right), \quad x \in \mathbb{R}^n, \]

where \{e^{-t(\Delta)^{\alpha}}\}_{t>0} is the heat semigroup of the fractional Laplacian \((-\Delta)^{\alpha}\), \(N = (N_1, N_2) \in \mathbb{Z}^2\) with \(N_1 < N_2\), \(\{v_j\}_{j \in \mathbb{Z}}\) is a bounded real sequence and \(\{a_j\}_{j \in \mathbb{Z}}\) is an increasing real sequence. Our analysis will consist in the boundedness, in \(L^p(\mathbb{R}^n)\) and in \(BMO(\mathbb{R}^n)\), of the operators \(T_N\) and its maximal operator \(T^*_N f(x) = \sup_N |T_N f(x)|\).

It is also shown that the local size of the maximal differential transform operators is the same with the order of a singular integral for functions \(f\) having local support.

1. Introduction

Let \(\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}\) be the Laplace operator in \(\mathbb{R}^n\). Its heat semigroup is defined by

\[ e^{t\Delta} \varphi(x) = \int_{\mathbb{R}^n} e^{t\Delta}(x-y)\varphi(y) dy, \quad x \in \mathbb{R}^n, \quad t > 0, \]

where \(e^{t\Delta}(x)\) denotes the Gauss-Weierstrass kernel

\[ e^{t\Delta}(x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}. \]

And the fractional Laplacian can be defined as a pseudo-differential operator via the Fourier transform

\[ \mathcal{F}((-\Delta)^{\alpha} f)(\xi) = |\xi|^{2\alpha} \mathcal{F}(f)(\xi), \]

where \(\mathcal{F}\) is the Fourier transform. The corresponding fractional heat semigroup is defined as

\[ \mathcal{F} \left( e^{-t(-\Delta)^{\alpha}} f \right)(\xi) := e^{-t|\xi|^{2\alpha}} \mathcal{F}(f)(\xi), \quad \alpha \in (0, 1). \]

When \(\alpha = 1/2\), it is the Poisson semigroup. In the literature, the fractional heat semigroup \(\{e^{-t(-\Delta)^{\alpha}}\}_{t>0}\) has widely been used in the study of partial differential equations,

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harmonic analysis, potential theory and modern probability theory. For example, the semigroup \( \{e^{-t(-\Delta)^\alpha}\}_{t>0} \) is usually applied to construct the linear part of solutions to fluid equations in the mathematic physics, e.g. the generalized Naiver-Stokes equation, the quasi-geostrophic equation, the MHD equations. In fact, \( e^{-t(-\Delta)^\alpha}f(x) \) is the solution of the heat equation related to the fractional Laplacian:

\[
\begin{cases}
\partial_t u(x,t) + (-\Delta)^\alpha u(x,t) = 0, & (x,t) \in \mathbb{R}^{n+1}_+,
 u(x,0) = f(x), & x \in \mathbb{R}^n.
\end{cases}
\]

And, in the field of probability theory, the researchers use \( \{e^{-t(-\Delta)^\alpha}\}_{t>0} \) to describe some kind of Markov processes with jumps. For further information and the related applications of fractional heat semigroups \( \{e^{-t(-\Delta)^\alpha}\}_{t>0} \), we refer the reader to [4, 9]. In [12], by an invariant derivative technique and the Fourier analysis method, the authors concluded that the kernel, \( e^{-t(-\Delta)^\alpha}(x) \) satisfy the following pointwise estimate

\[
0 < e^{-t(-\Delta)^\alpha}(x) \leq \frac{t}{(t^{1/2\alpha} + |x|^{n+2\alpha})}, \quad (x,t) \in \mathbb{R}^{n+1}.
\]

In this article, we will introduce the heat semigroup \( \{e^{-t(-\Delta)^\alpha}\}_{t>0} \) into the analysis of martingale transforms in probability. Martingale transforms was considered firstly by D. L. Burkholder in 1966; see [3]. In [4], the author proved the almost everywhere convergence of the martingale transforms. In martingale theory, we always treat the martingale transforms as a corresponding tool of the singular integral operators in harmonic analysis. In fact, we want to analyze the behavior of the following type sum

\[
(1.1) \quad \sum_{j \in \mathbb{Z}} v_j(e^{-a_{j+1}(-\Delta)^\alpha}f(x) - e^{-a_j(-\Delta)^\alpha}f(x))
\]

where \( \{v_j\}_{j \in \mathbb{Z}} \) is a bounded sequence of real numbers and \( \{a_j\}_{j \in \mathbb{Z}} \) is an increasing sequence of positive numbers. Observe that in the case \( v_j \equiv 1 \), the above series is telescopy, and their behavior coincide with \( e^{-t(-\Delta)^\alpha}f(x) \). This way of analyzing convergence of sequences was considered by Jones and Rosenblatt for ergodic averages(see [10]), and latter by Bernardis et al. for differential transforms(see [2]). The authors considered the differential transforms related to the one-sided fractional Poisson type operator sequence and the heat semigroup generated by Laplacian(see [5, 6]).

To better understand the behavior of the sum in (1.1), we shall analyze its “partial sums” defined as follows. For each \( N \in \mathbb{Z}^2 \), \( N = (N_1, N_2) \) with \( N_1 < N_2 \), we define the partial sum operators

\[
(1.2) \quad T_N f(x) = \sum_{j=N_1}^{N_2} v_j(e^{-a_{j+1}(-\Delta)^\alpha}f(x) - e^{-a_j(-\Delta)^\alpha}f(x)), \quad x \in \mathbb{R}^n.
\]

We shall also consider the maximal operator

\[
(1.3) \quad T^* f(x) = \sup_N |T_N f(x)|, \quad x \in \mathbb{R}^n,
\]

where the supremum are taken over all \( N = (N_1, N_2) \in \mathbb{Z}^2 \) with \( N_1 < N_2 \). In [5], the authors proved the boundedness of the above operators related with the one-sided fractional Poisson type operator sequence. And the same results was gotten for the above operators related with the heat semigroup generated by Laplacian in [6].
Some of our results will be valid only when the sequence \(\{a_j\}_{j \in \mathbb{Z}}\) is lacunary. That means that there exists a \(\lambda > 1\) such that \(\frac{a_{j+1}}{a_j} \geq \lambda\), \(j \in \mathbb{Z}\). In particular, we shall prove the boundedness of the operator \(T^*\) in the weighted spaces \(L^p(\mathbb{R}^n, \omega)\), where \(\omega\) is the usual Muckenhoupt weight on \(\mathbb{R}^n\). We refer the reader to the book by J. Duoandikoetxea \cite[Chapter 7]{duoandikoetxea} for definitions and properties of the \(A_p\) classes. We shall also analyze the boundedness behavior of the operators in \(L^\infty\) and \(BMO\) spaces. The space \(BMO(\mathbb{R}^n)\) is defined as the set of functions \(f\) such that \(f^* \in L^\infty(\mathbb{R}^n)\), where

\[
 f^*(x) = \sup_{x \in B} \left\{ \frac{1}{|B|} \int_B |f(z) - \frac{1}{|B|} \int_B f|dz\right\}.
\]

We define \(\|f\|_{BMO(\mathbb{R}^n)} = \|f^*\|_{L^\infty(\mathbb{R}^n)}\). In fact, we have the following results:

**Theorem 1.1.** Assume that the sequence \(\{a_j\}_{j \in \mathbb{Z}}\) is a \(\lambda\)-lacunary sequence with \(\lambda > 1\). Let \(T^*\) be the operator defined in (1.3). Then

(a) for any \(1 < p < \infty\) and \(\omega \in A_p\), there exists a constant \(C\) depending on \(n, p, \omega, \lambda, \|v\|_{L^\infty(\mathbb{Z})}\) and \(\alpha\) such that

\[
\|T^*f\|_{L^p(\mathbb{R}^n, \omega)} \leq C \|f\|_{L^p(\mathbb{R}^n, \omega)},
\]

for all functions \(f \in L^p(\mathbb{R}^n, \omega)\).

(b) for any \(\omega \in A_1\), there exists a constant \(C\) depending on \(n, \omega, \lambda, \|v\|_{\ell^\infty(\mathbb{Z})}\) and \(\alpha\) such that

\[
\omega\{x \in \mathbb{R}^n : |T^*f(x)| > \sigma\} \leq C \frac{1}{\sigma} \|f\|_{L^1(\mathbb{R}^n, \omega)}, \quad \sigma > 0,
\]

for all functions \(f \in L^1(\mathbb{R}^n, \omega)\).

(c) given \(f \in L^\infty(\mathbb{R}^n)\), then either \(T^*f(x) = \infty\) for all \(x \in \mathbb{R}^n\), or \(T^*f(x) < \infty\) for a.e. \(x \in \mathbb{R}^n\). And in this latter case, there exists a constant \(C\) depending on \(n, \lambda, \|v\|_{\ell^\infty(\mathbb{Z})}\) and \(\alpha\) such that

\[
\|T^*f\|_{BMO(\mathbb{R}^n)} \leq C \|f\|_{L^\infty(\mathbb{R}^n)}.
\]

(d) given \(f \in BMO(\mathbb{R}^n)\), then either \(T^*f(x) = \infty\) for all \(x \in \mathbb{R}^n\), or \(T^*f(x) < \infty\) for a.e. \(x \in \mathbb{R}^n\). And in this latter case, there exists a constant \(C\) depending on \(n, \lambda, \|v\|_{\ell^\infty(\mathbb{Z})}\) and \(\alpha\) such that

\[
\|T^*f\|_{BMO(\mathbb{R}^n)} \leq C \|f\|_{BMO(\mathbb{R}^n)}.
\]

The constants \(C\) appeared above all are independent of \(N\).

**Remark 1.2.** From the conclusions in Theorem 1.1 for \(f \in L^p(\mathbb{R}^n, \omega)\) with \(\omega \in A_p\), we can define \(Tf\) by the limit of \(T_Nf\) in \(L^p\) norm

\[
Tf(x) = \lim_{(N_1, N_2) \to (-\infty, +\infty)} T_Nf(x), \quad x \in \mathbb{R}^n.
\]

For more results related with the convergence of \(T_Nf\), see Theorem 3.4.

In classical harmonic analysis, if \(f = \chi_{(0,1)}\) and \(\mathcal{H}\) is the Hilbert transform, it is known that \(\frac{1}{r} \int_{-r}^{r} \mathcal{H}(f)(x)dx \sim \log \frac{r}{r}^+\) as \(r \to 0^+\). In general, this is the growth of a
singular integral applied to a bounded function at the origin. The following theorem shows that if $f$ is a bounded function, the growth of $T^* f$ at the origin is of the same order of a singular integral operator. Some related results about the local behavior of variation operators can be found in [1]. One dimensional results about the variation of convolution operators can be found in [11]. And one dimensional results about differential transforms of one-sided fractional Poisson type operator sequence is proved in [5].

The following theorem analyzes the local growth behavior of $T^*$ in $L^\infty$:

**Theorem 1.3.** Let $\{v_j\}_{j \in \mathbb{Z}} \in \ell^p(\mathbb{Z})$ for some $1 \leq p \leq \infty$. Let $\{a_j\}_{j \in \mathbb{Z}}$ be any increasing sequence and $T^*$ defined in (1.3). Then for every $f \in L^\infty(\mathbb{R}^n)$ with support in the unit ball $B = B(0,1)$, for any ball $B_r \subset B$ with $2r < 1$, we have

$$
\frac{1}{|B_r|} \int_{B_r} |T^* f(x)| \, dx \leq C \left( \log \frac{2}{r} \right)^{1/p'} \|v\|_{\ell^p(\mathbb{Z})} \|f\|_{L^\infty(\mathbb{R}^n)}.
$$

In the statement above, $p' = \frac{p}{p-1}$, and if $p = 1$, $p' = \infty$.

This article is organized as follows. In Section 2, by using Calderón-Zygmund theory, we prove the uniform boundedness of the operators $T_N$. In Section 3, we give the proof of Theorem 1.1. And we prove Theorem 1.3 in the last section.

Throughout this article, the letters $C, c$ will denote positive constants which may change from one instance to another and depend on the parameters involved. We will make a frequent use, without mentioning it in relevant places, of the fact that for a positive $A$ and a non-negative $a$,

$$
\sup_{t > 0} t^a e^{-At} = C_{a,A} < \infty.
$$

2. **Uniform $L_p$ boundedness of the operators $T_N$**

In this section, we will make some preparations to prove Theorem 1.1. In fact, we will prove the uniform boundedness of the operators $T_N$. The standard Calderón-Zygmund theory will be a fundamental tool in proving the $L^p$ boundedness of the operators $T_N$. For this theory, the reader can see some classical textbooks about harmonic analysis, for example, see [7, 8]. Nowadays it is well known that the fundamental ingredients in the theory are the $L^{p_0}(\mathbb{R}^n)$ boundedness for some $1 < p_0 \leq \infty$ and the smoothness of the kernel of the operator. Even more, the constants that appear in the results only depend on the boundedness constant in $L^{p_0}(\mathbb{R}^n)$ and the constants related with the size and smoothness of the kernel.

In the following proposition, we present and prove the $L^2$ boundedness of the operators $T_N$.

**Proposition 2.1.** There is a constant $C > 0$ depending on $n$ and $\|v\|_{\ell^\infty(\mathbb{Z})}$ (not on $N$) such that

$$
\|T_N f\|_{L^2(\mathbb{R}^n)} \leq C \|f\|_{L^2(\mathbb{R}^n)}.
$$
Proof. Let \( f \in L^2(\mathbb{R}^n) \). Using the Plancherel theorem, we have
\[
\| T_N f \|_{L^2(\mathbb{R}^n)}^2 = \left\| \sum_{j=N_1}^{N_2} v_j \left( e^{-a_j t} f - e^{-a_j} f \right) \right\|_{L^2(\mathbb{R}^n)}^2 \\
= \int_{\mathbb{R}^n} \left\{ \sum_{j=N_1}^{N_2} v_j \left( e^{-a_j t} f(x) - e^{-a_j} f(x) \right) \right\}^2 \, dx \\
= \int_{\mathbb{R}^n} \left( F \left\{ \sum_{j=N_1}^{N_2} v_j \left( e^{-a_j t} f(\cdot) - e^{-a_j} f(\cdot) \right) \right\}(\xi) \right)^2 \, d\xi \\
= \int_{\mathbb{R}^n} \left\{ \sum_{j=N_1}^{N_2} v_j \int_{a_j}^{a_{j+1}} \partial_t F \left( e^{-t} f \right) (\xi) \, dt \right\}^2 \, d\xi \\
\leq C \| v \|_{L^\infty(\mathbb{Z})}^2 \int_{\mathbb{R}^n} \left\{ \sum_{j=N_1}^{N_2} \int_{a_j}^{a_{j+1}} \partial_t F \left( e^{-t} f \right) (\xi) \, dt \right\}^2 \, d\xi \\
= C_v \int_{\mathbb{R}^n} \left\{ \sum_{j=N_1}^{N_2} \int_{a_j}^{a_{j+1}} |\xi|^{2 \alpha} e^{-t}|\xi|^{2 \alpha} |F(f)(\xi)| \, dt \right\}^2 \, d\xi \\
\leq C_v \int_{\mathbb{R}^n} \left\{ \sum_{j=N_1}^{N_2} \int_{a_j}^{a_{j+1}} |\xi|^{2 \alpha} e^{-t}|\xi|^{2 \alpha} \, dt |F(f)(\xi)| \right\}^2 \, d\xi \\
\leq C_v \int_{\mathbb{R}^n} \left\{ \int_0^\infty |\xi|^{2 \alpha} e^{-t}|\xi|^{2 \alpha} \, dt |F(f)(\xi)| \right\}^2 \, d\xi \\
\leq C_{v,n} \| f \|_{L^2(\mathbb{R}^n)}^2.
\]
Then the proof of the theorem is complete. \( \square \)

Lemma 2.2. There exists a constant \( C > 0 \) depending on \( n \) and \( \alpha \) such that

(i) \[ 0 < e^{-t\Delta}(x) \leq C \frac{t}{(t^{\frac{n}{2\alpha}} + |x|)^{n+2\alpha}}, \quad x \in \mathbb{R}^n, \ t > 0, \]

(ii) \[ |\partial_t e^{-t\Delta}(x)| \leq C \frac{1}{(t^{\frac{1}{2\alpha}} + |x|)^{n+2\alpha}}, \quad x \in \mathbb{R}^n, \ t > 0, \]

(iii) \[ |\nabla_x e^{-t\Delta}(x)| \leq C \frac{1}{(t^{\frac{n}{2\alpha}} + |x|)^{n+1}}, \quad x \in \mathbb{R}^n, \ t > 0, \]

and

(iv) \[ |\partial_t \nabla_x e^{-t\Delta}(x)| \leq C \frac{1}{(t^{\frac{n}{2\alpha}} + |x|)^{n+2\alpha+1}}, \quad x \in \mathbb{R}^n, \ t > 0. \]
Proof. For (i), it was proved in [9, Lemma 5.4]. For the other estimations, we can get the proof easily by using the results in [12, Lemmas 2.1–2.2, Remark 2.1]. □

The following proposition contains the size description of the kernel and the smoothness estimates that are required in the Calderón-Zygmund theory.

Proposition 2.3. Let \( f \in L^p(\mathbb{R}^n), 1 \leq p \leq \infty \). Then

\[
T_N f(x) = \int_{\mathbb{R}^n} K_N(y) f(x - y) dy
\]

with

\[
K_N(y) = \sum_{j=N_1}^{N_2} v_j \left( e^{-a_{j+1}(-\Delta)^\alpha}(y) - e^{-a_j(-\Delta)^\alpha}(y) \right).
\]

Moreover, there exists constant \( C > 0 \) depending on \( n, \alpha \) and \( \|v\|_{\ell^\infty(\mathbb{Z})} \) (not on \( N \)) such that, for any \( y \neq 0 \),

1) \( |K_N(y)| \leq \frac{C}{|y|^n} \),

2) \( |\nabla_y K_N(y)| \leq \frac{C}{|y|^{n+1}} \).

Proof. i) Regarding the size condition for the kernel, by Lemma 2.2 we have

\[
|K_N(y)| \leq \sum_{j=N_1}^{N_2} |v_j| \left| e^{-a_{j+1}(-\Delta)^\alpha}(y) - e^{-a_j(-\Delta)^\alpha}(y) \right|
\]

\[
\leq C_{n,v} \sum_{j=-\infty}^{\infty} \left| \int_{a_j}^{a_{j+1}} \partial_t e^{-t(-\Delta)^\alpha}(y) dt \right|
\]

\[
\leq C_{n,v,\alpha} \int_{0}^{\infty} \frac{1}{(t^{\frac{1}{2\alpha}} + |y|)^{n+2\alpha}} dt = C_{n,v,\alpha} \frac{1}{|y|^{n}}.
\]

ii) With a similar argument as above in i), by Lemma 2.2 we get

\[
|\nabla_y K_N(y)| \leq \sum_{j=N_1}^{N_2} |v_j| \left| \nabla_y e^{-a_{j+1}(-\Delta)^\alpha}(y) - \nabla_y e^{-a_j(-\Delta)^\alpha}(y) \right|
\]

\[
\leq C_{n,v} \sum_{j=-\infty}^{\infty} \int_{a_j}^{a_{j+1}} \left| \partial_t \nabla_y e^{-t(-\Delta)^\alpha}(y) \right| dt
\]

\[
\leq C_{n,v,\alpha} \int_{0}^{\infty} \frac{1}{(t^{\frac{1}{2\alpha}} + |y|)^{n+2\alpha+1}} dt = C_{n,v,\alpha} \frac{1}{|y|^{n+1}}.
\]

The proof of the proposition is complete. □

Then, we have the following theorem about the uniform boundedness of \( T_N \):

Theorem 2.4. Let \( \{a_j\}_{j \in \mathbb{Z}} \) be a positive increasing sequence and \( \{T_N\}_{N=(N_1, N_2)} \) be the operator \( T_N \) defined in (1.3). We have the following statements:
(a) for any \(1 < p < \infty\) and \(\omega \in A_p\), there exists a constant \(C\) depending on \(n, p, \omega, \|v\|_{L^p(\mathbb{Z})}\) and \(\alpha\) such that

\[
\|T_N f\|_{L^p(\mathbb{R}^n, \omega)} \leq C \|f\|_{L^p(\mathbb{R}^n, \omega)},
\]

for all functions \(f \in L^p(\mathbb{R}^n, \omega)\).

(b) for any \(\omega \in A_1\), there exists a constant \(C\) depending on \(n, \omega, \|v\|_{L^\infty(\mathbb{Z})}\) and \(\alpha\) such that

\[
\omega \left( \{x \in \mathbb{R}^n : |T_N f(x)| > \sigma \} \right) \leq C \frac{1}{\sigma} \|f\|_{L^1(\mathbb{R}^n, \omega)}^\alpha, \quad \sigma > 0,
\]

for all functions \(f \in L^1(\mathbb{R}^n, \omega)\).

(c) there exists a constant \(C\) depending on \(n, \|v\|_{L^\infty(\mathbb{Z})}\) and \(\alpha\) such that

\[
\|T_N f\|_{BMO(\mathbb{R}^n)} \leq C \|f\|_{L^\infty(\mathbb{R}^n)}^\alpha,
\]

for all functions \(f \in L^\infty(\mathbb{R}^n)\).

(d) there exists a constant \(C\) depending on \(n, \|v\|_{L^\infty(\mathbb{Z})}\) and \(\alpha\) such that

\[
\|T_N f\|_{BMO(\mathbb{R}^n)} \leq C \|f\|_{BMO(\mathbb{R}^n)}^\alpha.
\]

The constants \(C\) appeared above all are independent of \(N\).

**Proof.** Previously, we have remarked that the constants in the \(L^p\) boundedness only depend on the initial constant in \(L^{p_0}(\mathbb{R}^n)\) (in our case \(p_0 = 2\)), the size constant and smoothness constant of the kernel. Hence the uniform boundedness of the operators \(T_N\) in \(L^p(\mathbb{R}^n)\) spaces is a direct consequence of the Calderón-Zygmund theory. The finiteness of \(T_N\) for functions in \(L^\infty(\mathbb{R}^n)\) is obvious, since for each \(N, K_N\) is an integrable function. On the other hand, if \(f \in BMO(\mathbb{R}^n)\), we can proceed as follows. Let \(B = B(x_0, r_0)\) and \(B^* = B(x_0, 2r_0)\) with some \(x_0 \in \mathbb{R}^n\) and \(r_0 > 0\). We decompose \(f\) to be

\[
f = (f - f_B)\chi_{B^*} + (f - f_B)\chi_{(B^*)^c} + f_B =: f_1 + f_2 + f_3.
\]

The function \(f_1\) is integrable, hence \(T_N f_1(x) < \infty, \text{a.e. } x \in \mathbb{R}^n\). For \(T_N f_2\), we note that, for any \(x \in B\) and \(t > 0\),

\[
e^{-t(-\Delta)^\alpha} f_2(x) = \int_{\mathbb{R}^n} e^{-t(-\Delta)^\alpha} (x - y) f_2(y) dy 
\leq C \sum_{k=1}^\infty \int_{2^k r_0 < |x - y| \leq 2^{k+1} r_0} \frac{t}{(t^\alpha + |x - y|)^{n+2\alpha}} |f(y) - f_B| dy
\leq C t \sum_{k=1}^\infty \frac{(2^k r_0)^{-2\alpha}}{(2^k r_0)^n} \int_{|x - y| \leq 2^{k+1} r_0} |f(y) - f_B| dy
\leq C t \sum_{k=1}^\infty (2^k r_0)^{-2\alpha} (1 + 2k) \|f\|_{BMO(\mathbb{R}^n)} < \infty.
\]

So, \(e^{-t(-\Delta)^\alpha} f_2(x)\) is finite for any \(x \in B\) and \(t > 0\). Since \(T_N f_2(x)\) is a finite summation and \(x_0, r_0\) is arbitrary, \(T_N f_2(x) < \infty, \text{a.e. } x \in \mathbb{R}^n\). Finally we note that \(T_N f_3(x) \equiv 0\), since \(e^{-t(-\Delta)^\alpha} f_3 \equiv f_B\) for any \(j \in \mathbb{Z}\). Hence, \(T_N f(x) < \infty, \text{a.e. } x \in \mathbb{R}^n\). Then, by Propositions 2.1 and 2.3 we get the proof of part (c) of Theorem 2.4. To get (d), since \(T_N 1 = 0\), the known arguments give the conclusion, see [11]. \(\square\)
3. Boundedness of the maximal operator $T^*$

In this section, we will give the proof of Theorem 1.1 related to the boundedness of the maximal operator $T^*$. The next lemma, parallel to Proposition 3.2 in [2] (also Proposition 3.1 in [3]), shows that, without lost of generality, we may assume that

\[
1 < \lambda \leq \frac{a_{j+1}}{a_j} \leq \lambda^2, \quad j \in \mathbb{Z}.
\]

**Lemma 3.1.** Given a $\lambda$-lacunary sequence $\{a_j\}_{j \in \mathbb{Z}}$ and a multiplying sequence $\{v_j\}_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$, we can define a $\lambda$-lacunary sequence $\{\eta_j\}_{j \in \mathbb{Z}}$ and $\{\omega_j\}_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ verifying the following properties:

(i) $1 < \lambda \leq \eta_{j+1}/\eta_j \leq \lambda^2$, $\|\{\omega_j\}\|_{\ell^\infty(\mathbb{Z})} = \|\{v_j\}\|_{\ell^\infty(\mathbb{Z})}$.

(ii) For all $N = (N_1, N_2)$, there exists $N' = (N'_1, N'_2)$ with $T_N = \tilde{T}_{N'}$, where $\tilde{T}_{N'}$ is the operator defined in (1.2) for the new sequences $\{\eta_j\}_{j \in \mathbb{Z}}$ and $\{\omega_j\}_{j \in \mathbb{Z}}$.

**Proof.** We follow closely the ideas in the proof of Proposition 3.2 in [2]. We include them here for completeness.

Let $\eta_0 = a_0$, and let us construct $\eta_j$ for positive $j$ as follows (the argument for negative $j$ is analogous). If $\lambda^2 \geq a_1/a_0 \geq \lambda$, define $\eta_1 = a_1$. In the opposite case where $a_1/a_0 > \lambda^2$, let $\eta_1 = \lambda a_0$. It verifies $\lambda^2 \geq \eta_1/\eta_0 = \lambda \geq \lambda$. Further, $a_1/\eta_1 \geq \lambda^2 a_0/\lambda a_0 = \lambda$. Again, if $a_1/\eta_1 \leq \lambda^2$, then $\eta_2 = a_1$. If this is not the case, define $\eta_2 = \lambda^2 a_0 \leq a_1$. By the same calculations as before, $\eta_0, \eta_1, \eta_2$ are part of a lacunary sequence satisfying (3.1).

To continue the sequence, either $\eta_3 = a_1$ (if $a_1/\eta_2 \leq \lambda^2$) or $\eta_2 = \lambda^3 \eta_0$ (if $a_1/\eta_2 > \lambda^2$). Since $\lambda > 1$, this process ends at some $j_0$ such that $\eta_{j_0} = a_1$. The rest of the elements $\eta_j$ are built in the same way, as the original $a_k$ plus the necessary terms put in between two consecutive $a_j$ to get (3.1). Let $J(j) = \{k : a_{j-1} < \eta_j \leq a_j\}$, and $\omega_k = v_j$ if $k \in J(j)$. Then

\[
v_j(e^{-a_{j+1}(-\Delta)^\alpha} f(x) - e^{-a_j(-\Delta)^\alpha} f(x)) = \sum_{k \in J(j)} \omega_k(e^{-a_{k+1}(-\Delta)^\alpha} f(x) - e^{-a_k(-\Delta)^\alpha} f(x)).
\]

If $M = (M_1, M_2)$ is the number such that $\eta_{M_2} = a_{N_2}$ and $\eta_{M_1-1} = a_{N_1-1}$, then we get

\[
T_N f(x) = \sum_{j=N_1}^{N_2} v_j(e^{-a_{j+1}(-\Delta)^\alpha} f(x) - e^{-a_j(-\Delta)^\alpha} f(x))
\]

\[
= \sum_{k=M_1}^{M_2} \omega_k(e^{-a_{k+1}(-\Delta)^\alpha} f(x) - e^{-a_k(-\Delta)^\alpha} f(x)) = \tilde{T}_M f(x),
\]

where $\tilde{T}_M$ is the operator defined in (1.2) related with sequences $\{\eta_k\}_{k \in \mathbb{Z}}$, $\{\omega_k\}_{k \in \mathbb{Z}}$ and $M = (M_1, M_2)$.

This proposition allows us to assume in the rest of the article that the lacunary sequences $\{a_j\}_{j \in \mathbb{Z}}$ satisfy (3.1) without saying it explicitly.

In order to prove Theorem 1.1 for the case of the fractional laplacian we shall need a Cotlar’s type inequality to control the operator $T^*$. Namely, we shall prove the following theorem:
Theorem 3.2. For each \( q \in (1, +\infty) \), there exists a constant \( C \) depending on \( n, \|v\|_{\ell^\infty(Z)} \) and \( \lambda \) such that, for every \( x \in \mathbb{R}^n \) and every \( M \in \mathbb{Z}^+ \),

\[
T^*_M f(x) \leq C \left\{ \mathcal{M}(T_{(-M,M)} f)(x) + \mathcal{M}_q f(x) \right\},
\]

where

\[
T^*_M f(x) = \sup_{-M \leq N_1 < N_2 \leq M} |T_N f(x)|
\]

and

\[
\mathcal{M}_q f(x) = \sup_{r > 0} \left( \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)|^q \, dy \right)^{\frac{1}{q}}, \quad 1 < q < \infty.
\]

For the proof of this theorem we shall need the following lemma:

Lemma 3.3. Let \( \{a_j\}_{j \in \mathbb{Z}} \) a \( \lambda \)-lacunary sequence and \( \{v_j\}_{j \in \mathbb{Z}} \) \( \in \ell^\infty(Z) \). Then

\( i \) \( \left| \sum_{j=m}^{M} v_j (e^{-a_{j+1}} - e^{-a_{j}})(x, y) \right| \leq \frac{C_{v,\lambda,a,n}}{a_m^{n/2a}} \)

\( ii \) if \( k \geq m \) and \( z, y \in \mathbb{R}^n \) with \( |z - y| \geq a_k^{1/2a} \), then

\[
\left| \sum_{j=-M}^{m-1} v_j (e^{-a_{j+1}}(z, y) - e^{-a_{j}}(z, y)) \right| \leq C_{\lambda,v,a,n} \frac{1}{a_k^{n/2a}} \lambda^{-(k-m+1)}.
\]

Proof. By the mean value theorem and Lemma 2.2, there exists \( a_j \leq \xi_j \leq a_{j+1} \) such that

\[
\left| \sum_{j=m}^{M} v_j (e^{-a_{j+1}} - e^{-a_{j}})(x, y) \right| \leq C \|v\|_{l^\infty(Z)} \sum_{j=m}^{M} (a_{j+1} - a_j) |\partial t^{-\Delta} e^{-\lambda t}(x, y)|_{t=\xi_j} \leq C v \sum_{j=m}^{M} (a_{j+1} - a_j) \frac{1}{(\xi_j^{2a} + |x - y|^{n+2a})} \leq C v \sum_{j=m}^{M} (\lambda^2 - 1) a_j^{-\frac{2a}{m}} \leq C \lambda v \frac{1}{a_m^{n/2a}} \sum_{j=m}^{M} \frac{1}{\lambda^{n(j-m)/2a}} \leq C v \lambda \frac{1}{a_m^{n/2a}},
\]

where we have used \( \lambda \leq \frac{a_{j+1}}{a_j} \leq \lambda^2 \).

Now we shall prove (ii). By the mean value theorem, there exist \( a_j \leq \xi_j \leq a_{j+1} \) such that

\[
\left| \sum_{j=-M}^{m-1} v_j (e^{-a_{j+1}} - e^{-a_{j}})(z, y) \right| \leq C \|v\|_{l^\infty(Z)} \sum_{j=m}^{M} (a_{j+1} - a_j) |\partial t^{-\Delta} e^{-\lambda t}(x, y)|_{t=\xi_j} \leq C \lambda \frac{1}{a_m^{n/2a}} \sum_{j=m}^{M} \frac{1}{\lambda^{n(j-m)/2a}} \leq C \lambda \frac{1}{a_m^{n/2a}},
\]
\[
\leq C_{\nu,\alpha,n} \sum_{j=-M}^{m-1} (\lambda^2 - 1) a_j \frac{1}{(\xi_j^{1/2\alpha} + |x-y|)^{n+2\alpha}}
\]

\[
\leq C_{\lambda,\nu,\alpha,n} \sum_{j=-M}^{m-1} a_j \frac{1}{(\xi_j^{1/2\alpha} + |x-y|)^{n+2\alpha}}
\]

\[
\leq C_{\lambda,\nu,\alpha,n} \sum_{j=-M}^{m-1} \frac{a_j}{a_k} \left( \frac{1}{a_k^{n/2\alpha}} \right) \leq C_{\lambda,\nu,\alpha,n} \frac{1}{a_k^{n/2\alpha}} \lambda^{-(k-m+1)},
\]

where we have used that \(k \geq m\).

\[\square\]

**Proof of Theorem 3.2.** Observe that, for any \(x_0 \in \mathbb{R}^n\) and \(N = (N_1, N_2)\),

\[T_N f(x_0) = T_{(N_1,M)}f(x_0) - T_{(N_2+1,M)}f(x_0),\]

with \(-M \leq N_1 < N_2 \leq M\). Then, it suffices to estimate \(|T_{(m,M)}f(x_0)|\) for \(|m| \leq M\) with constants independent of \(m\) and \(M\). Denote \(B_k = B(x_0, a_1^{1/2\alpha})\) for each \(k \in \mathbb{N}\).

Let us split \(f\) as

\[f = f \chi_{B_m} + f \chi_{B_m^c} =: f_1 + f_2.\]

Then, we have

\[|T_{(m,M)}f(x_0)| \leq |T_{(m,M)}f_1(x_0)| + |T_{(m,M)}f_2(x_0)| =: I + II.\]

For \(I\), by Lemma 3.3 (i), we have

\[I = |T_{(m,M)}f_1(x_0)| = \left| \int_{\mathbb{R}^n} \sum_{j=m}^{M} v_j \left( e^{-a_j(-\Delta)^{\alpha}}(x_0, y) - e^{-a_j(-\Delta)^{\alpha}}(x_0, y) \right) f_1(y) dy \right|
\]

\[\leq C_{n,\nu,\lambda,\alpha} \frac{1}{a_m^{n/2\alpha}} \int_{\mathbb{R}^n} |f_1(y)| dy \leq C_{\nu,\lambda,\alpha,\nu} \mathcal{M} f(x_0).\]

For part \(II\),

\[II = |T_{(m,M)}f_2(x_0)| = \frac{2n/2}{a_{n-1}^{n/2\alpha}} \int_{B(x_0, a_{n-1}^{1/2\alpha})} |T_{(-M,M)}f_2(x_0)| dz
\]

\[\leq \frac{2n/2}{a_{n-1}^{n/2\alpha}} \int_{B(x_0, a_{n-1}^{1/2\alpha})} |T_{(-M,M)}f(z)| dz + \frac{2n/2}{a_{n-1}^{n/2\alpha}} \int_{B(x_0, a_{n-1}^{1/2\alpha})} |T_{(M,M)}f_1(z)| dz
\]

\[+ \frac{2n/2}{a_{n-1}^{n/2\alpha}} \int_{B(x_0, a_{n-1}^{1/2\alpha})} |T_{(m,M)}f_2(z) - T_{(m,M)}f_2(x_0)| dz
\]

\[+ \frac{2n/2}{a_{n-1}^{n/2\alpha}} \int_{B(x_0, a_{n-1}^{1/2\alpha})} |T_{(-M,m-1)}f_2(z)| dz =: A_1 + A_2 + A_3 + A_4.
\]

(If \(m + 1 = -M\), we understand that \(A_4 = 0\).) It is clear that

\[A_1 \leq \mathcal{M}(T_{(-M,M)}f)(x_0).\]
For $A_2$, by the uniform boundedness of $T_N$, we get

$$A_2 \leq \left( \frac{2^{n/2}}{a_{m-1}^{n/2\alpha}} \int_{B_{m-1}} |T_{(-M,M)}f_1(z)|^q \, dz \right)^{1/q} \leq C \left( \frac{1}{a_{m-1}^{n/2\alpha}} \int_{\mathbb{R}^n} |f_1(z)|^q \, dz \right)^{1/q}$$

$$= C \left( \frac{1}{a_{m-1}^{n/2\alpha}} \int_{B_m} |f(z)|^q \, dz \right)^{1/q} \leq C \left( \frac{\lambda^{n/2\alpha}}{a_{m}^{n/2\alpha}} \int_{B_m} |f(z)|^q \, dz \right)^{1/q} \leq CM_q f(x_0).$$

For the third term $A_3$, with $z \in B(x_0, \frac{1}{2}a_{m-1}^{1/2\alpha})$, we have

$$|T_{(m,M)}f_2(z) - T_{(m,M)}f_2(x_0)|$$

$$= \left| \int_{B_m^c} K_{(m,M)}(z - y)f(y) \, dy - \int_{B_m} K_{(m,M)}(x_0 - y)f(y) \, dy \right|$$

$$\leq \int_{B_m^c} |K_{(m,M)}(z - y) - K_{(m,M)}(x_0 - y)| |f(y)| \, dy$$

$$= \sum_{j=1}^{\infty} \int_{B_{2j-1} \setminus B_{2j-1}} |K_{(m,M)}(z - y) - K_{(m,M)}(x_0 - y)| |f(y)| \, dy,$$

where $B_{2j} = B(x_0, 2^ja_m^{1/\alpha})$ for any $j \geq 1$.

By the mean value theorem, we know that there exists $\xi$ on the segment $\overline{x_0z}$ such that

$$|K_{m,M}(z - y) - K_{m,M}(x_0 - y)| \leq |\nabla \xi K_{m,M}(\xi - y)| |z - x_0|$$

$$\leq C \frac{|z - x_0|}{|\xi - y|^{n+1}} \leq C \frac{1}{2^j} \cdot a_{m-1}^{1/2\alpha} \cdot \frac{1}{|2^j a_{m}^{n/2\alpha}|},$$

where we have used that in each summand, $y \in B_{2j} \setminus B_{2j-1}$. Hence

$$|T_{m,M}f_2(z) - T_{m,M}f_2(x_0)| \leq C \sum_{j=1}^{\infty} \frac{1}{2^j} a_{m-1}^{1/2\alpha} a_{m}^{1/2\alpha} \frac{1}{|2^j a_{m}^{n/2\alpha}|} \int_{B_{2j}} |f(y)| \, dy$$

$$\leq CMf(x_0) \frac{a_{m-1}^{1/2\alpha} + \infty}{a_{m}^{n/2\alpha}} \sum_{j=1}^{\infty} \frac{1}{2^j} \leq CMf(x_0).$$

Then,

$$A_3 = \frac{2^{n/2}}{a_{m-1}^{n/2\alpha}} \int_{B_{m-1}} |T_{m,M}f_2(z) - T_{m,M}f_2(x_0)| \, dz \leq CMf(x_0).$$

For the latest one, $A_4$, we have

$$A_4 = \frac{2^{n/2}}{a_{m-1}^{n/2\alpha}} \int_{B(x_0, \frac{1}{2}a_{m-1}^{1/2\alpha})} |T_{(-M,M)}f_2(z)| \, dz$$

$$\leq \frac{2^{n/2}}{a_{m-1}^{n/2\alpha}} \int_{B(x_0, \frac{1}{2}a_{m-1}^{1/2\alpha})} \int_{B_m} |K_{(-M,M)}(z - y)f(y)| \, dy \, dz.$$
Then, we consider the inner integral appeared in the above inequalities first. Since $z \in B(x_0, \frac{1}{2}a_m^{1/2\alpha})$, $y \in B_m^c$ and the sequence \{\(a_j\)\}_{j \in \mathbb{Z}} is $\lambda$-lacunary sequence, we have \(|z - y| \sim |y - x_0|\). From this and by Lemma 3.3(ii), we get
\[
\int_{B_m^c} |K_{(-M,m-1)}(z - y)f(y)| \, dy = \sum_{k=m}^{+\infty} \int_{B_{k+1} \setminus B_k} \left| \sum_{j=-M}^{m-1} v_j \left( e^{-a_{j+1}(-\Delta)^n}(z - y) - e^{-a_j(-\Delta)^n}(z - y) \right) f(y) \right| \, dy \\
\leq C_{\lambda, v, \alpha, n} \sum_{k=m}^{+\infty} \lambda^{-(k-m+1)} \left( \frac{1}{a_k^{n/2\alpha}} \int_{B_{k+1} \setminus B_k} |f(y)| \, dy \right) \\
\leq C_{\lambda, v, \alpha, n} \mathcal{M}f(x_0) \sum_{k=m}^{+\infty} \lambda^{-(k-m+1)} \\
\leq C_{\lambda, v, \alpha, n} \mathcal{M}f(x_0).
\]
Hence,
\[
A_4 \leq C \mathcal{M}f(x_0).
\]
Combining the estimates above for $A_1, A_2, A_3$ and $A_4$, we get
\[
II \leq \mathcal{M}(T_{(-M,M)}f)(x_0) + C\mathcal{M}_qf(x_0).
\]
And then we have
\[
|T_{(m, M)}f(x_0)| \leq C \left( \mathcal{M}(T_{(-M,M)}f)(x_0) + \mathcal{M}_qf(x_0) \right).
\]
As the constants $C$ appeared above all only depend on $\|v\|_{L^\infty(\mathbb{Z}^n)}$, $\lambda$, $\alpha$ and $n$, we have proved that
\[
T_M^*f(x_0) \leq C \{ \mathcal{M}(T_{(-M,M)}f)(x_0) + \mathcal{M}_qf(x_0) \}.
\]
This complete the proof of the theorem. \(\square\)

Now, we can give the proof of Theorem 1.1.

Proof of Theorem 1.1. Given $\omega \in A_p$, we choose $1 < q < p$ such that $\omega \in A_{p/q}$. Then it is well known that, the maximal operators $\mathcal{M}$ and $\mathcal{M}_q$ are bounded on $L^p(\mathbb{R}^n, \omega)$, see [7]. On the other hand, since the operators $T_N$ are uniformly bounded in $L^p(\mathbb{R}^n, \omega)$ with $\omega \in A_p$, we have
\[
\|T_M^*f\|_{L^p(\mathbb{R}^n, \omega)} \leq C \left( \|\mathcal{M}(T_{(-M,M)}f)\|_{L^p(\mathbb{R}^n, \omega)} + \|\mathcal{M}_qf\|_{L^p(\mathbb{R}^n, \omega)} \right) \\
\leq C \left( \|T_{(-M,M)}f\|_{L^p(\mathbb{R}^n, \omega)} + \|f\|_{L^p(\mathbb{R}^n, \omega)} \right) \leq C \|f\|_{L^p(\mathbb{R}^n, \omega)}.
\]
Note that the constants $C$ appeared above do not depend on $M$. Consequently, letting $M$ increase to infinity, we get the proof of the $L^p$ boundedness of the maximal operator $T^*$. This completes the proof of part (a) of the theorem.

In order to prove (b), we consider the $L^\infty(\mathbb{Z}^n)$-valued operator $\mathcal{T}f(x) = \{T_Nf(x)\}_{N \in \mathbb{Z}^2}$. Since $\|\mathcal{T}f(x)\|_{L^\infty(\mathbb{Z}^2)} = T^*f(x)$, by using (a) we know that the operator $\mathcal{T}$ is bounded from $L^p(\mathbb{R}^n, \omega)$ into $L^p_{L^\infty(\mathbb{Z}^2)}(\mathbb{R}^n, \omega)$, for every $1 < p < \infty$ and $\omega \in A_p$. The kernel
of the operator $T$ is given by $K(x) = \{K_N(x)\}_{N \in \mathbb{Z}^2}$. Therefore, by the vector valued Calderón-Zygmund theory, the operator $T$ is bounded from $L^1(\mathbb{R}^n, \omega)$ into weak-$L^1_{\infty}(\mathbb{R}^n, \omega)$ for $\omega \in A_1$. Hence, as $\|Tf(x)\|_{l^\infty(\mathbb{Z}^2)} = T^*f(x)$, we get the proof of (b).

For (c) and (d), we shall first prove that, if $f \in BMO(\mathbb{R}^n)$ and there exists $x_0 \in \mathbb{R}^n$ such that $T^*f(x_0) < \infty$, then $T^*f(x) < \infty$ for a.e. $x \in \mathbb{R}^n$. Set $B = B(x_0, 4|x - x_0|)$ with $x \neq x_0$. And we decompose $f$ to be

$$f = (f - f_B)\chi_B + (f - f_B)\chi_{B^c} + f_B =: f_1 + f_2 + f_3.$$ 

Note that $T^*$ is $L^p$ bounded for any $1 < p < \infty$. Then $T^*f_1(x) < \infty$, because $f_1 \in L^p(\mathbb{R}^n)$, for any $1 < p < \infty$. And $T^*f_3 = 0$, since $e^{-a_j(-\Delta)^{\alpha}}f_3 = f_3$ for any $j \in \mathbb{Z}$. On the other hand by the smoothness properties of the kernel, we have

$$|T_Nf_2(x) - T_Nf_2(x_0)| = \left| \int_{B^c} (K_N(x - y) - K_N(x_0 - y)) f_2(y) dy \right|$$

$$\leq C \int_{B^c} \frac{|x - x_0|}{|y - x_0|^{n+1}} |f(y) - f_B| dy$$

$$\leq C \sum_{k=1}^{+\infty} |x - x_0| \int_{2^kB \setminus 2^{k-1}B} \frac{|f(y) - f_B|}{|y - x_0|^{n+1}} dy$$

$$\leq C \sum_{k=1}^{+\infty} |x - x_0| 2^{k+1} |x - x_0| |f(y) - f_B| dy$$

$$\leq C \sum_{k=1}^{+\infty} 2^{-(k+1)} \frac{1}{|2^kB|} \int_{2^kB} \left( |f(y) - f_{2^kB}| + \sum_{l=1}^{k} |f_{2^lB} - f_{2^{l-1}B}| \right) |f|_{BMO(\mathbb{R}^n)} dy$$

$$\leq C \sum_{k=1}^{+\infty} 2^{-(k+1)} (1 + 2k) |f|_{BMO(\mathbb{R}^n)}$$

where $2^kB = B(x_0, 2^k \cdot 4|x - x_0|)$ for any $k \in \mathbb{N}$. Hence

$$\|T_Nf_2(x) - T_Nf_2(x_0)\|_{l^\infty(\mathbb{Z}^2)} \leq C \|f\|_{BMO(\mathbb{R}^n)},$$

and therefore $T^*f(x) = \|T_Nf(x)\|_{l^\infty(\mathbb{Z}^2)} \leq C < \infty$.

Finally, the estimate (14) can be proved in a parallel way to the proof of part (b) of Theorem 2.4, since $T1(x) = \{T_N1(x)\} = 0$(also see [11]).

From Theorem 3.4 we can get the following consequence:

**Theorem 3.4.** (a) If $1 < p < \infty$ and $\omega \in A_p$, then $T_Nf$ converges a.e. and in $L^p(\mathbb{R}^n, \omega)$ norms for all $f \in L^p(\mathbb{R}^n, \omega)$ as $N = (N_1, N_2)$ tends to $(-\infty, +\infty)$.

(b) If $p = 1$ and $\omega \in A_1$, then $T_Nf$ converges a.e. and in measure for all $f \in L^1(\mathbb{R}^n, \omega)$ as $N = (N_1, N_2)$ tends to $(-\infty, +\infty)$. 

Proof. First, we shall see that if \( \varphi \) is a test function, then \( T_N \varphi(x) \) converges for all \( x \in \mathbb{R}^n \). In order to prove this, it is enough to see that for any \((L, M)\) with \(0 < L < M\), the series

\[
A = \sum_{j=L}^{M} v_j (e^{-a_{j+1}(-\Delta)^\alpha} \varphi(x) - e^{-a_j(-\Delta)^\alpha} \varphi(x))
\]

and

\[
B = \sum_{j=-M}^{-L} v_j (e^{-a_{j+1}(-\Delta)^\alpha} \varphi(x) - e^{-a_j(-\Delta)^\alpha} \varphi(x))
\]

converge to zero, when \( L, M \to +\infty \). By Lemma 2.2 we have

\[
|A| = \left| \sum_{j=L}^{M} v_j \int_{\mathbb{R}^n} (e^{-a_{j+1}(-\Delta)^\alpha}(x - y) - e^{-a_j(-\Delta)^\alpha}(x - y)) \varphi(y) dy \right|
\]

\[
\leq C \int_{\mathbb{R}^n} \left( \sum_{j=L}^{M} \int_{a_j}^{a_{j+1}} \frac{1}{(t^{1/2\alpha} + |x-y|)^{n+2\alpha}} dt \right) |\varphi(x-y)| dy
\]

\[
\leq C (a_{M+1}^{-\frac{n}{2\alpha}} - a_L^{-\frac{n}{2\alpha}}) \|\varphi\|_{L^1(\mathbb{R}^n)} \to 0, \quad \text{as } L, M \to +\infty.
\]

On the other hand, since \( \int_{\mathbb{R}^n} (e^{-a_{j+1}(-\Delta)^\alpha}(x - y) - e^{-a_j(-\Delta)^\alpha}(x - y)) dy = 0 \) for any \( j \in \mathbb{Z} \), we can write

\[
B = \int_{\mathbb{R}^n} \sum_{j=-M}^{-L} v_j (e^{-a_{j+1}(-\Delta)^\alpha}(y) - e^{-a_j(-\Delta)^\alpha}(y)) (\varphi(x-y) - \varphi(x)) dy.
\]

And, then proceeding as in the case \( A \), and by using the fact that \( \varphi \) is a test function, we have

\[
|B| \leq C \|v\|_{L^\infty(\mathbb{Z})} \int_{\mathbb{R}^n} \sum_{j=-M}^{-L} \left| e^{-a_{j+1}(-\Delta)^\alpha}(y) - e^{-a_j(-\Delta)^\alpha}(y) \right| |\varphi(x-y) - \varphi(x)| dy
\]

\[
\leq C \|\nabla \varphi\|_{L^\infty(\mathbb{R}^n)} \sum_{j=-M}^{-L} \int_{\mathbb{R}^n} \frac{a_{j+1}|y|}{(a_j^{1/2\alpha} + |y|)^{n+2\alpha}} dy
\]

\[
\leq C \|\nabla \varphi\|_{L^\infty(\mathbb{R}^n)} \sum_{j=-M}^{-L} a_j^{1/2\alpha} \int_{\mathbb{R}^n} \frac{1}{a_j^{2\alpha}} \frac{1}{(1 + |y|^{n+2\alpha})^{n+2\alpha}} dy
\]

\[
\leq C_{n,\alpha} \|\nabla \varphi\|_{L^\infty(\mathbb{R}^n)} \frac{\lambda^{1/2\alpha}}{\lambda^{1/2\alpha} - 1} a_{-L}^{1/2\alpha} \sum_{j=-M}^{-L} a_j^{1/2\alpha} \to 0, \quad \text{as } L, M \to +\infty,
\]

where we have used the assumption \( 1/2 < \alpha < 1 \) to make the integral convergent.
For the case $0 < \alpha < \frac{1}{2}$, we can write

\[
B = \int_{B(0,1)} \sum_{j=-M}^{-L} v_j (e^{-a_j + \alpha} (y) - e^{-a_j} (y)) (\varphi(x-y) - \varphi(x)) \, dy \\
+ \int_{B(0,1)^c} \sum_{j=-M}^{-L} v_j (e^{-a_j + \alpha} (y) - e^{-a_j} (y)) (\varphi(x-y) - \varphi(x)) \, dy \\
:= B_1 + B_2.
\]

For $B_1$, we have

\[
|B_1| \leq C \|\nabla \varphi\|_{L^\infty(\mathbb{R}^n)} \sum_{j=-M}^{-L} \int_{B(0,1)} \frac{a_j + 1}{a_j^{1/2\alpha} + |y|^{n+2\alpha}} \, dy \\
\leq C \|\nabla \varphi\|_{L^\infty(\mathbb{R}^n)} \sum_{j=-M}^{-L} a_j^{1/2\alpha} \int_{B(0,1)} \frac{|y|}{a_j^{1/2\alpha} (1 + |y|^{n+2\alpha})} \, dy \\
\leq C \|\nabla \varphi\|_{L^\infty(\mathbb{R}^n)} \sum_{j=-M}^{-L} \frac{a_j}{a_j^{1/2\alpha}} r^{n} \lambda^{-1} a_{-L} \rightarrow 0, \quad \text{as} \ L, M \rightarrow +\infty.
\]

For $B_2$, we have

\[
|B_2| \leq C \|\varphi\|_{L^\infty(\mathbb{R}^n)} \sum_{j=-M}^{-L} \int_{B(0,1)^c} \frac{a_j + 1}{a_j^{1/2\alpha} + |y|^{n+2\alpha}} \, dy \\
\leq C \|\varphi\|_{L^\infty(\mathbb{R}^n)} \sum_{j=-M}^{-L} \int_{B(0,1)^c} \frac{1}{a_j^{1/2\alpha} (1 + |y|^{n+2\alpha})} \, dy \\
\leq C \|\varphi\|_{L^\infty(\mathbb{R}^n)} \sum_{j=-M}^{-L} \frac{a_j}{a_j^{1/2\alpha}} r^{-2\alpha - 1} \lambda^{-1} a_{-L} \rightarrow 0, \quad \text{as} \ L, M \rightarrow +\infty.
\]

So, when $0 < \alpha < 1/2$, we proved that $|B| \rightarrow 0$, as $L, M \rightarrow +\infty$. And for the case $\alpha = \frac{1}{2}$, it has been proved in [13]. Hence, we proved that $T_N \varphi(x)$ converges for all $x \in \mathbb{R}^n$ with $\varphi$ being a test function.
As the set of test functions is dense in $L^p(\mathbb{R}^n)$, by Theorem 1.1 we get the a.e. convergence for any function in $L^p(\mathbb{R}^n)$. Analogously, since $L^p(\mathbb{R}^n) \cap L^p(\mathbb{R}^n, \omega)$ is dense in $L^p(\mathbb{R}^n, \omega)$, we get the a.e. convergence for functions in $L^p(\mathbb{R}^n, \omega)$ with $1 \leq p < \infty$. By using the dominated convergence theorem, we can prove the convergence in $L^p(\mathbb{R}^n, \omega)$ norm for $1 < p < \infty$, and also in measure. □

4. Proof of the Local Growth of the Maximal Operator $T^*$

In this section, we will give the proof of the local growth of the maximal operator $T^*$.

Proof of Theorem 2.2. We will prove it only in the case $1 < p < \infty$. For the case $p = 1$ and $p = \infty$, the proof is similar and easier. Since $2r < 1$, we know that $B \backslash B_{2r} \neq \emptyset$. Let $f(x) = f_1(x) + f_2(x)$, where $f_1(x) = f(x)\chi_{B_{2r}(x)}$ and $f_2(x) = f(x)\chi_{B \backslash B_{2r}}(x)$. Then

$$|T^*f(x)| \leq |T^*f_1(x)| + |T^*f_2(x)|.$$

By Theorem 1.1 we have

$$\frac{1}{|B_r|} \int_{B_r} |T^*f_1(x)| \, dx \leq \left( \frac{1}{|B_r|} \int_{B_r} |T^*f_1(x)|^2 \, dx \right)^{1/2} \leq C \left( \frac{1}{|B_r|} \int_{\mathbb{R}^n} |f_1(x)|^2 \, dx \right)^{1/2} \leq C \|f\|_{L^\infty(\mathbb{R}^n)}.$$

On the other hand, applying Hölder’s inequality on the integers and on $\mathbb{R}^n$, Fubini’s Theorem and Lemma 2.2 for $1 < p < \infty$ and any $N = (N_1, N_2)$, we have

$$\left| \sum_{j=N_1}^{N_2} v_j \left( e^{-a_j+1(-\Delta)^{\alpha}} f_2(x) - e^{-a_j(-\Delta)^{\alpha}} f_2(x) \right) \right|$$

$$\leq C \sum_{j=N_1}^{N_2} \left| v_j \int_{\mathbb{R}^n} \left( e^{-a_j+1(-\Delta)^{\alpha}}(y) - e^{-a_j(-\Delta)^{\alpha}}(y) \right) f_2(x-y) \, dy \right|$$

$$\leq C \|v\|_{L^p(\mathbb{Z})} \left( \sum_{j=N_1}^{N_2} \left( \int_{\mathbb{R}^n} \left| e^{-a_j+1(-\Delta)^{\alpha}}(y) - e^{-a_j(-\Delta)^{\alpha}}(y) \right| |f_2(x-y)| \, dy \right)^{p'} \right)^{1/p'}$$

$$\leq C \|v\|_{L^p(\mathbb{Z})} \left( \sum_{j=N_1}^{N_2} \left( \int_{\mathbb{R}^n} \left| e^{-a_j+1(-\Delta)^{\alpha}}(y) - e^{-a_j(-\Delta)^{\alpha}}(y) \right| |f_2(x-y)| \, dy \right)^{p'/p} \right)^{1/p'}$$

(4.1)$$\leq C \|v\|_{L^p(\mathbb{Z})} \left( \sum_{j=N_1}^{N_2} \left( \int_{\mathbb{R}^n} \left| e^{-a_j+1(-\Delta)^{\alpha}}(y) - e^{-a_j(-\Delta)^{\alpha}}(y) \right| |f_2(x-y)| \, dy \right)^{p'/p} \right)^{1/p'}$$

$$\leq C \|v\|_{L^p(\mathbb{Z})} \left( \int_{\mathbb{R}^n} \sum_{j=-\infty}^{+\infty} \left| e^{-a_j+1(-\Delta)^{\alpha}}(y) - e^{-a_j(-\Delta)^{\alpha}}(y) \right| |f_2(x-y)| \, dy \right)^{1/p'}$$

$$\leq C \|v\|_{L^p(\mathbb{Z})} \left( \int_{\mathbb{R}^n} \sum_{j=-\infty}^{+\infty} \left| e^{-a_j+1(-\Delta)^{\alpha}}(y) - e^{-a_j(-\Delta)^{\alpha}}(y) \right| |f_2(x-y)| \, dy \right)^{1/p'}$$
\[
\leq C \|v\|_{L^p(Z)} \left( \int_{\mathbb{R}^n} \left( \int_0^{+\infty} \frac{t}{(1/2\alpha + |y|^{n+2\alpha})} \, dt \right) |f_2(x-y)|^{p'} \, dy \right)^{1/p'} \\
\leq C \|v\|_{L^p(Z)} \left( \int_{\mathbb{R}^n} \frac{1}{|y|^n} |f_2(x-y)|^{p'} \, dy \right)^{1/p'}.
\]

Hence
\[
\frac{1}{|B_r|} \int_{B_r} |T^* f_2(x)| \, dx \leq C \frac{1}{|B_r|} \int_{B_r} \left( \int_{\mathbb{R}^n} \frac{1}{|y|^n} |f_2(x-y)|^{p'} \, dy \right)^{1/p'} \, dx \\
= C \frac{1}{|B_r|} \int_{B_r} \left( \int_{\mathbb{R}^n} \frac{1}{|x-y|^n} |f_2(y)|^{p'} \, dy \right)^{1/p'} \, dx \\
\leq C \frac{\|f\|_{L^\infty(\mathbb{R}^n)}}{|B_r|} \int_{B_r} \left( \int_{r \leq |x-y| \leq 2r} \frac{1}{|x-y|^n} \, dy \right)^{1/p'} \, dx \\
\sim \left( \log \frac{2}{r} \right)^{1/p'} \|f\|_{L^\infty(\mathbb{R}^n)},
\]
where we have used the fact \( y \in B \setminus B_{2r} \) and \( x \in B_r \). Therefore we arrive to
\[
\frac{1}{|B_r|} \int_{B_r} |T^* f(x)| \, dx \leq C \left( 1 + \left( \log \frac{2}{r} \right)^{1/p'} \right) \|f\|_{L^\infty(\mathbb{R}^n)} \leq C \left( \log \frac{2}{r} \right)^{1/p'} \|f\|_{L^\infty(\mathbb{R}^n)}.
\]

Then we get the proof of Theorem 1.3.

\[ \square \]

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