Algebraic number fields and the LLL algorithm

M. J. Uray (Uray M. János)
uray.janos@inf.elte.hu
ELTE – Eötvös Loránd University Budapest
Faculty of Informatics
Department of Computer Algebra

Abstract

In this paper we analyze the computational cost of various operations performed symbolically in real algebraic number fields where the elements are represented as polynomials of a primitive element of the field. We give bounds on the costs in terms of several parameters, including the degree of the field and the representation size of the input. Beyond the basic field operations we also analyze the cost of the less-than comparison and the integer rounding functions. As an important application we give a polynomial bound on the running time of the LLL lattice reduction algorithm when the vector coordinates are from an algebraic number field and the computations are performed exactly.

1 Introduction

Exact symbolic computation with algebraic expressions or specifically, algebraic numbers is an important feature that most computer algebra systems provide. They use efficient algorithms for the calculations, described in several papers and books, for example: [1], [2], [3] or [4]. However, to our knowledge, no complete account of the computational costs is given in these works. Present paper provides explicit bounds on the costs of many operations in algebraic number fields using several parameters, including the size of the input and the parameters of the number field. The costs are computed in terms of word operations, taking into account the increasing cost when multi-precision representation is needed for large integers.
The obtained explicit formulas enable us to calculate the running time of several well-known algorithms if they use exact arithmetic with algebraic numbers. For example consider Gaussian elimination, where the number of operations on the entries is easily shown to be polynomial, but if we use exact arithmetic on rational numbers or algebraic numbers, then the growing size of the entries can cause concerns. The Bareiss algorithm \cite{7} is a modification for rational numbers which deals with this problem by certain simplifications to ensure polynomial running time (although with larger exponent). The idea has a straightforward generalization to algebraic number fields, which this paper presents briefly.

As a more important application of our algebraic number field results, we prove the polinomiality of the LLL lattice reduction algorithm when it is performed symbolically with algebraic numbers. The analysis of the running time of the LLL algorithm requires much more care than that of the Bareiss algorithm, and relies on several subproblems, which include finding a bound on the number of main steps, and examining how the sizes of the entries grow during these iterations. The original paper describing this algorithm \cite{5} solves these problems for integer-valued vectors, but these calculations fail when algebraic numbers are considered. The present paper solves this by giving more general answers to these questions.

For several practical purposes, the execution of the LLL algorithm with the usual (e.g. 64-bit) floating-point numbers seems sufficient, since the goal is finding a well-reduced basis or a short vector. Still, we think that the analysis of the LLL algorithm using symbolic algebraic numbers deserves interest. First, it is interesting from a theoretical point of view. Second, there are applications when the exact values in the reduction are needed. For example in \cite{6}, algebraic integers are represented by ultimately periodic series of integer vectors, obtained by a repeated application of the LLL algorithm. This representation is a generalization of continued fractions, and as with continued fractions, the exact representation is only guaranteed to be obtained if we use symbolic calculation.

The paper is built up as follows: Section 2 analyzes the computational costs of several operations in algebraic number fields; Section 3 gives a brief calculation about the Bareiss algorithm with algebraic numbers; Section 4 covers the running time of the LLL algorithm; and in Section 5 we summarize the results.

2 Algebraic number fields

In this section we discuss the complexity of various operations in algebraic number fields. Let $F$ be a real algebraic number field of degree $m$ and $\alpha \in F$ a primitive element, i.e. $F = \mathbb{Q}(\alpha)$. Without loss of generality we can assume that $\alpha$ is an algebraic integer. Denote its minimal polynomial by $f(x) = x^m + f_{m-1}x^{m-1} + \ldots + f_1x + f_0 \ (f_i \in \mathbb{Z})$. We will consider $f$, $\alpha$ and $m$ as fixed throughout this article.
Elements in this field can be represented by rational linear combinations of $1, \alpha, \alpha^2, \ldots, \alpha^{m-1}$. However in order to minimalize the problems with rational numbers like simplification, we use an integer linear combination and a common denominator. Furthermore, we consider only the numerator, i.e. the ring $\mathbb{Z}[\alpha]$, because dealing with the single denominator is trivial, and in many algorithms using algebraic numbers they can be cleared in the beginning.

For most operations, representing $\alpha$ by its minimal polynomial suffices, because the algebraic properties do not change when different conjugates of $\alpha$ are used. However for some operations like the less-than comparison, additional information is needed to distinguish conjugates. For this, we use isolating intervals, i.e. intervals with rational endpoints that contain exactly one root of $f(x)$, namely $\alpha$.

In subsection 2.1, we give bounds on the growth of the representation size of the numbers in $\mathbb{Z}[\alpha]$ during the operations, and in 2.2 we give bounds on the running time of these operations.

### 2.1 Coefficient size growth

For an algebraic integer $a \in \mathbb{Z}[\alpha]$, $a = a_0 + a_1\alpha + a_2\alpha^2 + \ldots + a_{m-1}\alpha^{m-1}$ ($a_i \in \mathbb{Z}$), we will use the following norm-like function to measure its coefficient size:

$$c(a) := \max_{i=0}^{m-1} |a_i|.$$  

This quantity (or rather its logarithm) together with the field degree $m$ (which is constant for a fixed field) indicates the storage size needed by the algebraic integer $a$. The following result shows how this size can grow during several operations.

**Lemma 2.1.** Let $a, b \in \mathbb{Z}[\alpha]$ and $s \in \mathbb{Z}$. Write $\frac{1}{b} \in \mathbb{Q}(\alpha)$ in the following form (if $b \neq 0$):

$$\frac{1}{b} = \frac{\tilde{b}}{N(b)}, \quad \tilde{b} \in \mathbb{Z}[\alpha], \quad N(b) \in \mathbb{Z},$$

where $N(b)$ is the norm of $b$. Let $A := \log c(a)$, $B := \log c(b)$, $S := \log |s|$ and $F := \log \|f\|_{\infty} := \max_{i=0}^{m-1} |f_i|$. Then there exist positive constants $M_\alpha, P_\alpha, Q_\alpha, S_\alpha$...
such that:

(2.2) \( c(0) = 0; \)
(2.3) \( c(s) = |s|; \)
(2.4) \( c(a \pm b) \leq c(a) + c(b), \quad \log c(a \pm b) = O(\max(A, B)); \)
(2.5) \( c(sa) = |s|c(a), \quad \log c(sa) = O(S + A); \)
(2.6) \( c(ab) \leq M_a c(a) c(b), \quad \log c(ab) = O(A + B + mF); \)
(2.7) \( c(\tilde{b}) \leq P_a c(b)^{m-1}, \quad \log c(\tilde{b}) = O(mB + m^2 F); \)
(2.8) \( |N(b)| \leq Q_a c(b)^m, \quad \log |N(b)| = O(mB + mF + m \log m); \)
(2.9) \( |a| \leq S_a c(a), \)
(2.10) \( |a| \geq \frac{1}{P_a S_a c(a)^{m-1}}, \quad S_a := 1 + |\alpha| + |\alpha|^2 + \ldots + |\alpha|^{m-1}; \)

and we have:

(2.11) \( M_a \leq m \left( 1 + \|f\|_\infty \right)^{m-1}, \quad \log M_a = O(mF); \)
(2.12) \( P_a \leq m\|f\|_2^{-1} (M_a + \sqrt{m})^{m-1}, \quad \log P_a = O(m^2 F); \)
(2.13) \( Q_a \leq m \|f\|_2^{-1}, \quad \log Q_a = O(mF + m \log m); \)
(2.14) \( S_a \leq m \max(1, |\alpha|)^{m-1}, \quad \log S_a = O(mF). \)

(2.2), (2.3), (2.4) and (2.5) are trivial, the others are proved below.

**Proof of (2.6) and (2.11)**

Let \( c := ab, \) and \( c = c_0 + c_1 \alpha + c_2 \alpha^2 + \ldots + c_{m-1} \alpha^{m-1}. \) Then:

\[
c = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} a_i b_j \alpha^{i+j} = \sum_{k=0}^{m-1} \left( \sum_{j=0}^{k} a_j b_{k-j} \right) \alpha^k + \sum_{k=0}^{m-2} \left( \sum_{j=k+1}^{m-1} a_j b_{k+m-j} \right) \alpha^{m+k}
\]

In order to get the \( c_i \)’s, we need to write the \( \alpha^{m+k} \)’s in terms of lower powers of \( \alpha: \)

\[
\alpha^{m+k} = r_{k,0} + r_{k,1} \alpha + r_{k,2} \alpha^2 + \ldots + r_{k,m-1} \alpha^{m-1}
\]

Substituting this to \( c \) above, we get:

\[
c_i = \sum_{j=0}^{l} a_j b_{l-j} + \sum_{k=0}^{m-2} \left( \sum_{j=k+1}^{m-1} a_j b_{k+m-j} \right) r_{k,l}
\]

For calculating \( c(c) \), we need upper bounds for \( c_i \)’s, and first for the \( r_{k,l} \)’s.

By using that \( f \) is a minimal polynomial of \( \alpha \), one can get a recursive formula for the \( r_{k,l} \) coefficients [1, p. 159]:

(2.15) \( r_{0,l} = -f_l; \)
(2.16) \( r_{k+1,l} = r_{k,l-1} - f r_{k,m-1}, \)

by defining the coefficients with negative indices to zero. Then one can easily show
by induction that:

\[ |r_{k,l}| \leq \|f\|_\infty (1 + \|f\|_\infty)^k. \]

We can get then a bound for \(c_l\)’s:

\[
|c_l| \leq \sum_{j=0}^l |a_j| |b_{l-j}| + \sum_{k=0}^{m-2} \left( \sum_{j=k+1}^{m-1} |a_j| |b_{k+m-j}| \right) |r_{k,l}| \leq \]

\[
\leq m c(a) c(b) + \sum_{k=0}^{m-2} m c(a) c(b) \|f\|_\infty (1 + \|f\|_\infty)^k = \]

\[
= m c(a) c(b) + m c(a) c(b) \|f\|_\infty \frac{(1 + \|f\|_\infty)^{m-1} - 1}{(1 + \|f\|_\infty) - 1} = \]

\[
= m (1 + \|f\|_\infty)^{m-1} c(a) c(b), \]

and this is (2.6) with (2.11).

**Proof of (2.7), (2.8), (2.12) and (2.13)**

Let \(g(x)\) be the polynomial for which \(b = g(\alpha)\), and consider the following polynomial:

\[ h(x) := \text{res}_y (f(y), x - g(y)) = x^m + h_{m-1}x^{m-1} + \ldots + h_2x^2 + h_1x + h_0, \]

which is called the characteristic polynomial of \(b\) in \(\mathbb{Q}(\alpha)\), and it is either the minimal polynomial of \(b\) or its positive integer power \([1, \text{p. 162-164}]. \) Therefore \(h(b) = 0\), i.e.:

\[
b^m + h_{m-1}b^{m-1} + \ldots + h_2b^2 + h_1b + h_0 = 0, \]

which can be arranged as:

\[ \frac{1}{b} = \frac{-h_{m-1}b^{m-2} - \ldots - h_2b - h_1}{h_0}. \]

We know that the constant term of the characteristic polynomial is the norm, i.e. \(h_0 = N(b)\), therefore the numerator on the right hand side is \(\tilde{b}\).

First we give bounds on the coefficients of \(h\). Using the Sylvester matrix
representation of resultants, \( h(x) = \text{res}_y(f(y), x - g(y)) \) can be written as:

\[
\begin{vmatrix}
1 & f_{m-1} & f_{m-2} & \cdots & f_1 & f_0 \\
1 & f_{m-1} & \cdots & \cdots & f_1 & f_0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
1 & f_{m-1} & f_{m-2} & \cdots & f_1 & f_0 \\
-g_{m-1} & -g_{m-2} & \cdots & -g_1 & x-g_0 \\
-g_{m-1} & -g_{m-2} & \cdots & -g_1 & x-g_0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
-g_{m-1} & -g_{m-2} & \cdots & -g_1 & x-g_0 \\
-g_{m-1} & -g_{m-2} & \cdots & -g_1 & x-g_0 
\end{vmatrix}
\]

Split this determinant into the sum of two by a \( g \)-row, where the first contains only the \( x \) and the second contains the \(-g_k\)'s. This can be done on each \( g \)-row, which gives a sum of \( 2^m \) determinants. Then, for a specific \( k \), the term \( h_k x^k \) is the sum of those determinants that contain \( x \) exactly \( k \) times. There are \( \binom{m}{k} \) such, and each can be bounded by Hadamard’s inequality [1, p. 51] for rows after removing the rows and columns of the \( x \)'es. This gives the following bound for the \( h_k \)'s:

\[
|h_k| \leq \binom{m}{k} \|f\|_2^{m-1} \|g\|_2^{m-k} \leq \binom{m}{k} \|f\|_2^{m-k} \|g\|_2^{m-k} \leq \binom{m}{k} \|f\|_2^{m-k} \|g\|_\infty \leq \binom{m}{k} \|f\|_2^{m-k} \|g\|_\infty \leq \sqrt{m} \|f\|_2^{m-k} \|g\|_\infty \leq \sqrt{m} c(b)^{m-k},
\]

using that \( \|g\|_2 \leq \sqrt{m} \|g\|_\infty = \sqrt{m} c(b) \).

For \( k = 0 \), since \( h_0 = N(b) \), this is exactly (2.8) with (2.13).

To bound \( \tilde{b} \), we use the formula in the numerator of (2.17), and write it in a recursive way:

\[
d_0 := 1 \\
d_{k+1} := d_k b + h_{m-k-1} \\
\tilde{b} = -d_{m-1}
\]

Then we prove the following bounds by induction:

\[
c(d_k) \leq \|f\|_2^{m-1} \|c(b)\|_2^k \sum_{j=0}^k \binom{m}{j} M_{\alpha}^{k-j} m^j.
\]

6
We use the coefficient bounds (2.4)-(2.6) for addition and multiplication:
\[
c(\tilde{d}_0) = c(1) = 1 \leq \|f\|_{2}^{m-1}
\]
\[
c(d_{k+1}) = c(d_k b + h_{m-k-1}) \leq M_\alpha c(d_k) c(b) + |h_{m-k-1}| \leq \|f\|_{2}^{m-1} c(b)^{k+1} \left( M_\alpha \sum_{j=0}^{k} \binom{m}{j} M_\alpha^{k-j} m^{\frac{k-j}{2}} + \binom{m}{k+1} m^{\frac{k+1}{2}} \right) = \|f\|_{2}^{m-1} c(b)^{k+1} \sum_{j=0}^{k+1} \binom{m}{j} M_\alpha^{k+1-j} m^{\frac{j}{2}}.
\]

Then we can get the bound for \(c(\tilde{b})\):
\[
c(\tilde{b}) = c(d_{m-1}) \leq \|f\|_{2}^{m-1} c(b)^{m-1} \sum_{j=0}^{m-1} \binom{m}{j} M_\alpha^{m-1-j} m^{\frac{j}{2}} \leq m \|f\|_{2}^{m-1} c(b)^{m-1} \sum_{j=0}^{m-1} \binom{m-1}{j} M_\alpha^{m-1-j} m^{\frac{j}{2}} = m \|f\|_{2}^{m-1} c(b)^{m-1} (M_\alpha + \sqrt{m})^{m-1},
\]
which is (2.7) with (2.12).

**Proof of (2.9), (2.10) and (2.14)**

**Upper bound (2.9):**
\[
|a| = \sum_{k=0}^{m-1} a_k \alpha^k \leq \sum_{k=0}^{m-1} |a_k| |\alpha|^k \leq c(a) \sum_{k=0}^{m-1} |\alpha|^k = c(a) S_\alpha,
\]

**Lower bound (2.10) comes from the upper bound of the inverse:**
\[
\left| \frac{1}{a} \right| = \frac{|\tilde{a}|}{|N(a)|} \leq \frac{S_\alpha c(\tilde{a})}{1} \leq P_\alpha S_\alpha c(a)^{m-1}.
\]

A simple upper bound for \(S_\alpha\):
\[
S_\alpha = \sum_{k=0}^{m-1} 1^{m-k-1} |\alpha|^k \leq \sum_{k=0}^{m-1} \max(1, |\alpha|)^{m-1} = m \max(1, |\alpha|)^{m-1}.
\]

It is well-known that for any complex root \(\alpha\) of the polynomial \(f(x) = x^m + f_{m-1}x^{m-1} + \ldots + f_1 x + f_0\):
\[
|\alpha| \leq 1 + \max\{|f_0|, |f_1|, \ldots, |f_{m-1}|\},
\]
therefore \(\log |\alpha| = O(F)\), and substituting this into the bound for \(S_\alpha\), we get (2.14).
2.2 Running time of field operations

In this section we give bounds on the running time of several operations in algebraic number fields. As the field elements are represented by integers, these calculations rely on the running time of integer operations, especially multiplication and division, for which different algorithms exist with different time complexity. For the sake of generality, we give our results in terms of the complexity of integer multiplication, using the following notation.

Let $\text{Mul}(A, B)$ be the running time of the multiplication of two integers $a, b \in \mathbb{Z}$ whose sizes are bounded by $A$ and $B$ (i.e. $\log |a| \leq A$ and $\log |b| \leq B$), and let $\text{Mul}(A) := \text{Mul}(A, A)$. The value depends on the actual integer multiplication algorithm used, for example:

- basic multiplication: $\text{Mul}(A, B) = O(AB)$,
- Karatsuba multiplication: $\text{Mul}(A) = O(A^{\log_2 3})$,
- Schönhage–Strassen algorithm: $\text{Mul}(A) = O(A \log A \log \log A)$.

In the running time calculations later in this paper we use only the following assumptions about the Mul function:

- $\text{Mul}(A, B) = \text{Mul}(B, A)$,
- $B \leq C \Rightarrow \text{Mul}(A, B) \leq \text{Mul}(A, C)$,
- $\text{Mul}(A, B + C) \leq \text{Mul}(A, B) + \text{Mul}(A, C)$,
- $\text{Mul}(A, nB) \leq n \text{Mul}(A, B)$ ($n \in \mathbb{Z}^+$),
- $n \text{Mul}(A) \leq \text{Mul}(nA)$,
- $A \leq \text{Mul}(A) \leq A^2$.

We assume furthermore that the exact division $C := A/B$ of two integers (i.e. without remainder) can be performed in $\text{Mul}(B, C)$ time.

The following results use the Mul function to give running time bounds on the operations in $\mathbb{Z}[\alpha]$.

**Lemma 2.2.** Let again $a, b \in \mathbb{Z}[\alpha]$ and $s \in \mathbb{Z}$. Let $A := \log c(a)$, $B := \log c(b)$, $S := \log |s|$ and $F := \log \|f\|_{\infty} := \max_{i=0}^{m-1} |f_i|$. Then the operations in $\mathbb{Z}[\alpha]$ can be
calculated in the following time:

\[(2.18) \quad a \pm b : \quad O(m \max(A, B));\]
\[(2.19) \quad sa : \quad O(m \text{Mul}(S, A));\]
\[(2.20) \quad ab : \quad O(m^2 \text{Mul}(A, B) + m^2 \text{Mul}(mF, A + B + \log m));\]
\[(2.21) \quad \frac{1}{b} : \quad O(m^3 \text{Mul}(B + F + \log m));\]
\[(2.22) \quad a < b : \quad O(m^2 \text{Mul}(mA + mB + m^2F));\]
\[(2.23) \quad \lfloor \frac{a}{s} \rfloor : \quad O(m^2 \text{Mul}(m \max(A, S) + m^2F));\]
\[(2.24) \quad \lfloor \frac{a}{b} \rfloor : \quad O(m^2 \text{Mul}(mA + m^2B + m^3F)).\]

\[(2.18)\] and \[(2.19)\] are trivial, the others are proved below. Note that the same bounds work for \(\lceil \cdot \rceil\) and \(\lfloor \cdot \rfloor\) as for \(\lfloor \cdot \rfloor\).

**Proof of (2.20)**

The product of \(a, b \in \mathbb{Z}[\alpha]\) can be computed by the following steps:
1. Calculate the product of the polynomial of \(a\) and \(b\), i.e. calculate:
   \[d_l := \sum_j a_j b_{l-j} \quad (0 \leq l \leq 2m - 2).\]
2. Calculate its remainder modulo \(f\) by:
   \[c_l := d_l + \sum_{k=0}^{m-2} d_{m+k} r_{k,l} \quad (0 \leq l \leq m - 1).\]

The \(r_{k,l}\) coefficients can be precalculated from \(f\) by \[(2.15)\].

When calculating the running time, we ignore the additions and count only the multiplications, which dominate. The first step involves \(m^2\) multiplications between \(a\) and \(b\) coefficients:

\[T_1 = m^2 \text{Mul}(A, B)\]

For the second step, we need a bound for the lengths of \(d_l\) and \(r_{k,l}\) (for the latter, we use \[(2.16)\]):

\[
\log |d_l| \leq \log \left(\sum_j |a_j||b_{l-j}|\right) = O(A + B + \log m)
\]

\[
\log |r_{k,l}| \leq \log\|f\|_\infty + k \log (1 + \|f\|_\infty) = O(mF).
\]

Therefore:

\[T_2 = \sum_{l=0}^{m-1} \sum_{k=0}^{m-2} \text{Mul}(\log |d_{m+k}|, \log |r_{k,l}|) = O(m^2 \text{Mul}(mF, A + B + \log m)).\]

Putting the two together, \(T_1 + T_2\) is \[(2.20)\].
The multiplicative inverse of $b \in \mathbb{Z}[\alpha]$ can be calculated by the extended Euclidean algorithm (EEA). Let $g(x)$ be the polynomial for which $b = g(\alpha)$, then the EEA for $f$ and $g$ computes $s$ and $t$ such that $s(x)f(x) + t(x)g(x) = 1$, thus $t(\alpha)g(\alpha) = 1$.

The problem is that it can run in exponential time because of the growing coefficients, but we can use one of its variants, the subresultant algorithm, which runs in polynomial time. Brown calculated its running time in [8], which is for univariate polynomials (using our notation) [8] p. 500:

$$T = O \left( m^4 \log^2 \max(\|f\|_\infty, \|g\|_\infty) \right).$$

This form is however not suitable for our calculation for several reasons detailed below, therefore we need to make some modifications by which a similar calculation as in [8] gives a more appropriate result.

First, that calculation uses the same bound for the coefficients of $f(x)$ and $g(x)$. But in our application, the two polynomials play different roles: $f(x)$ is the minimal polynomial (which is fixed in a particular algebraic number field), and $g(x)$ depends on the actual algebraic number. [8] (22) bounds the coefficients of the intermediate polynomials by $m(2L + \log m)$, where $L := \log \max(\|f\|_\infty, \|g\|_\infty)$. We replace it by $m(\log\|f\|_\infty + \log\|g\|_\infty + \log m)$, which is a more specific bound, and both come from the Hadamard’s lemma used to the Sylvester-like determinant form of those coefficients. Also, the original calculation (in [8] (74)) ignored the logarithmic term $\log m$, but we preserve it to get the worst-case complexity. With these changes so far, the result is:

$$T = O \left( m^4 (\log\|f\|_\infty + \log\|g\|_\infty + \log m)^2 \right) = \quad \quad \quad \quad$$

\[ \quad \quad \quad \quad = O \left( m^4 (B + F + \log m)^2 \right). \]

Our next problem is that Brown used the standard integer multiplication algorithm, and not the more general Mul() function. Multiplication first arises in the running time of one pseudo-division, which is by [8] (68):

$$T_{\text{pdiv}} = O (m d LL'),$$

where $L$ is the coefficient size bound of the inputs of the pseudo division, $L'$ is of the pseudo-quotient, and $d$ is the degree of the pseudo-quotient. It can be easily seen that $L' = O(dL)$. Then changing to the Mul() function, one pseudo-division is:

$$T_{\text{pdiv}} = O (m d \text{Mul}(L, L')) = O (m d \text{Mul}(L, dL)) = O (m d^2 \text{Mul}(L)).$$

Next problem is that Brown assumed that the polynomial remainder sequence of the algorithm is always normal, i.e. the degree of the polynomials decrease by exactly one in each step. Abnormal sequences are rare, but we cannot ignore them in a worst-case complexity calculation. Removing that assumption and combining
the result with the improved bounds on $T_{\text{div}}$ above we get:

$$T = O \left( m^2 \max_j d_j \cdot \text{Mul}(m(B + F + \log m)) \right),$$

where $d_j$'s are the degree differences in the polynomial sequence (i.e. the degrees of the quotients). In the worst case, $d = O(m)$, so:

$$T = O \left( m^3 \cdot \text{Mul}(m(B + F + \log m)) \right).$$

Our last problem is that Brown considers the basic subresultant algorithm, but we need the extended one. The latter maintains two auxiliary polynomial sequences. It follows from [3, p. 290-291] that the coefficients of these polynomials can be written in similar Sylvester-like determinants as the basic polynomials, so their size have the same asymptotic bound: $m(B + F + \log m)$. Since they repeat all operations performed on the basic sequence, they need asymptotically the same time, and thus does not change the asymptotic bounds on the final running time.

**Proof of (2.22)**

Since $a < b$ is equivalent to $a - b < 0$, we need to consider only the $a < 0$ comparison. Assume the nontrivial case $a \neq 0$.

Let $g(x)$ be the polynomial for which $a = g(\alpha)$. We approximate $\alpha$ by a refined isolating interval $\frac{u}{d} \leq \alpha \leq \frac{v}{d}$ where $d \in \mathbb{Z}^+$, $u := \lfloor ad \rfloor$ and $v = \lceil ad \rceil$. We need a sufficiently accurate approximation so that when we substitute the endpoints to $g(x)$ instead of $\alpha$, its sign remains the same. For this, it suffices that $d$ is so large that for every positive $\varepsilon \leq 1$

$$|g(\alpha \pm \varepsilon) - g(\alpha)| < |a|. \tag{2.25}$$

We prove that this holds if:

$$d \geq (m - 1) \left( 1 + c(a)^m P_a S_\alpha^2 \right). \tag{2.26}$$

The proof proceeds by finding an upper bound for $|g(\alpha \pm \varepsilon) - g(\alpha)|$:

$$|g(\alpha \pm \varepsilon) - g(\alpha)| = \left| \sum_{k=0}^{m-1} a_k ((\alpha \pm \varepsilon)^k - \alpha^k) \right| =$$

$$= \left| \sum_{k=0}^{m-1} a_k \sum_{j=1}^{k} \binom{k}{j} \alpha^{k-j}(\pm \varepsilon)^j \right| \leq c(a) \sum_{k=0}^{m-1} \sum_{j=1}^{k} \binom{k}{j} |\alpha|^{k-j} |\varepsilon|^j =$$

$$= c(a) \sum_{j=1}^{m-1} \sum_{l=0}^{m-j-1} \binom{l+j}{j} |\alpha|^l |\varepsilon|^j \leq c(a) \sum_{j=1}^{m-1} (m-1)^j S_\alpha |\varepsilon|^j <$$

$$< c(a) S_\alpha \frac{(m-1)\varepsilon}{1 - (m-1)\varepsilon} = c(a) S_\alpha \left( \frac{1}{1 - (m-1)\varepsilon} - 1 \right).$$

Continuing this by substituting any $\varepsilon \leq \frac{1}{d}$ with $d$ as in (2.26), and using (2.10),
we get:
\[ |g(\alpha \pm \epsilon) - g(\alpha)| < c(a)S_\alpha \left( \frac{1}{c(a)^mP_\alpha S^2_\alpha} \right) = \frac{1}{c(a)^{m-1}P_\alpha S_\alpha} \leq |a|, \]
which proves (2.25).

We calculate \( g(w/d) \), where \( w \) is the smaller of \( u \) and \( v \) in absolute value, therefore:
\[ |w| = \min(|u|, |v|) = \min(||\alpha d||, ||\alpha d||) \leq |\alpha d| = |\alpha|d. \]

Then:
\[ g\left(\frac{w}{d}\right) = \frac{a_0d^{m-1} + a_1wd^{m-2} + \ldots + a_{m-1}w^{m-1}}{d^{m-1}}. \]

Because \( d^{m-1} > 0 \), we need only the numerator (call it \( r \)), which can be calculated by the following recursive formula:
\[
\begin{align*}
  r_0 &:= 0 \\
  r_{k+1} &:= r_k w + a_{m-k-1}d^k \\
  r &:= r_m
\end{align*}
\]

One can easily bound the size of \( r_k \) by induction:
\[ |r_k| \leq c(a)d^{k-1}(1 + |\alpha| + \ldots + |\alpha|^{k-1}) \leq c(a)d^{k-1}S_\alpha, \]
\[ \log |r_k| = O(A + mD + mF), \]
where \( D := \log d \).

The total cost of calculating \( r \) is, considering again only multiplications and including iteratively calculating \( d^k \):
\[ T = \sum_{k=0}^{m-1} (\text{Mul}(|r_k|, |w|) + \text{Mul}(|a_i|, d^k) + \text{Mul}(d, \log d^{k-1})) \leq \]
\[ \leq \sum_{k=0}^{m-1} (\text{Mul}(A + mD + mF, F + D) + \text{Mul}(A, kD) + \text{Mul}(D, (k - 1)D)). \]

Then, by using the smallest \( d \) for which (2.26) holds, one can get an upper bound for \( D \):
\[ D = O (\log m + mA + \log P_\alpha + \log S_\alpha) = O (mA + m^2F). \]

Then \( T \) can be simplified by noticing that \( D \) dominates over \( A \) and \( F \):
\[ T = \sum_{k=0}^{m-1} O (\text{Mul}(D, mD)) = O (m^2 \text{Mul}(D)), \]
which gives the running time of \( a < 0 \), and combining it with the subtraction \( a - b \), we obtain (2.22).
Proof of (2.23)

Let \( s > 0 \) (or otherwise multiply \( a \) and \( s \) by \(-1\)). If \( a/s \in \mathbb{Z} \), the operation is trivial, so let \( a/s \notin \mathbb{Z} \).

Note that since \( \lceil a/s \rceil = \lfloor a/s \rfloor + 1 \) if \( a/s \notin \mathbb{Z} \), and \( \lfloor a/s \rfloor = \lfloor a/s + 1/2 \rfloor \), the other rounding functions have the same asymptotic bounds as \( \lfloor \cdot \rfloor \).

The proof is similar to the previous one, with the following differences. We need \( d \) to be so large that for every positive \( \epsilon \leq \frac{1}{a} \), both of the following inequalities hold:

\[
|g(\alpha \pm \epsilon) - g(\alpha)| < |a - s \left\lceil \frac{a}{s} \right\rceil|,
\]

(2.27)

\[
|g(\alpha \pm \epsilon) - g(\alpha)| < |a - s \left\lfloor \frac{a}{s} \right\rfloor|.
\]

We prove that this can be achieved if:

\[
d \geq (m - 1) \left( 1 + c(a, s)^m P_{a, S_{a}^{m+1}} \right),
\]

(2.28)

where \( c(a, s) := \max(c(a), s) \).

Let \( b \) be either \( a - s \lfloor a/s \rfloor \) or \( a - s \lceil a/s \rceil \) as in (2.27), and let us determine \( c(b) \). Since \( b - a \) is an integer, i.e. the representations of \( a \) and \( b \) differ only in their constant term, we need to calculate only \( |b_0| \). \( b_0 \) is either \( a_0 - s \lfloor a/s \rfloor \) or \( a_0 - s \lceil a/s \rceil \), so:

\[
|b_0| \leq |a_0 - a| + s = \sum_{k=1}^{m-1} a_k \alpha^k + s \leq c(a) (S_\alpha - 1) + s \leq c(a, s) S_\alpha,
\]

therefore \( c(b) \leq c(a, s) S_\alpha \).

In the same way as in the previous proof but with (2.28), we get:

\[
|g(\alpha \pm \epsilon) - g(\alpha)| < c(a, s) S_\alpha \left( \frac{1}{c(a, s)^m P_{a, S_{a}^{m+1}}} \right) \leq \frac{1}{c(b)^{m-1} P_{a, S_{a}}} \leq |b|,
\]

which proves (2.27).

Then again, by using the smallest \( d \) for which (2.28) holds, we get an asymptotic bound for \( D \):

\[
D = O \left( m \max(A, S) + m^2 F \right),
\]

and otherwise (2.23) is the same as (2.22).

Proof of (2.24)

We write \( a/b \) as \( \hat{a} \hat{b}/N(b) \), and since \( N(b) \in \mathbb{Z} \), we can use the previous lemma (2.23). By using the (2.6)-(2.8) properties of \( c(\cdot) \):

\[
c(\hat{a} \hat{b}) \leq M_\alpha P_{a} c(a) c(b)^{m-1},
\]

\[
|N(b)| \leq Q_\alpha c(b)^{m},
\]

\[
c(\hat{a} \hat{b}, N(b)) \leq \max(M_\alpha P_{a}, Q_\alpha) c(a) c(b)^{m}.
\]
Taking logarithms:
\[
\log c(ab, |N(b)|) = O \left( \max( mF + m^2F, mF + m \log m ) + A + mB \right) = \\
= O \left( A + mB + m^2F \right),
\]
and substituting this to (2.23), we get (2.24).

### 3 Bareiss algorithm

The Bareiss algorithm \[7\] is an integer-preserving modification of Gaussian elimination that maintains as small integers as generally possible by using provably exact divisions to reduce their sizes. It can be used to perform Gaussian elimination symbolically on a matrix with rational coefficients (by first multiplying through with the common denominator) without exponential coefficient growth or extensive GCD calculations for simplifications.

In this section we apply the algorithm to $\mathbb{Z}[^\alpha]$ and calculate its running time using the results of the previous section, and compare it with the running time in $\mathbb{Z}$. We consider the simplest case, when a square matrix is converted into an upper triangular form (e.g. to calculate its determinant).

**Theorem 3.1.** Let $A \in R^{n \times n}$ where $R \in \{ \mathbb{Z}, \mathbb{Z}[^\alpha] \}$, and $L := \max_{i,j=1}^n \log c(a_{ij})$ (in $\mathbb{Z}$, $c(\cdot)$ is equivalent to $| \cdot |$). Then the Bareiss algorithm on $A$ runs in the following time, depending on the ring $R$ and the integer multiplication algorithm used (Mul):

- $R = \mathbb{Z}$ : 
  \[ O \left( n^3 \cdot \text{Mul}(n(\log n + L)) \right), \]
- $R = \mathbb{Z}[\alpha]$ : 
  \[ O \left( n^3 m^3 \cdot \text{Mul}(nm(\log n + L + mF)) \right). \]

**Proof:** Only the case $R = \mathbb{Z}[\alpha]$ is proved, since for $\mathbb{Z}$ it is well-known (at least when $\text{Mul}(X) = O(X^2)$), furthermore it easily follows from a similar and easier argument than the following.

The Bareiss algorithm uses the following formula \[7\] p. 570:

\[
a^{(0)}_{00} = 1, \quad a^{(1)}_{ij} = a_{ij}, \\
a^{(k+1)}_{ij} = \frac{a^{(k)}_{kk} a^{(k)}_{ij} - a^{(k)}_{ik} a^{(k)}_{kj}}{a^{(k-1)}_{k-1,k-1}},
\]

with $1 \leq k \leq n - 1$ and $k + 1 \leq i, j \leq n$. It is known that the division is exact, i.e. the result remains in $R$. We calculate the running time of the recursive formula (3.1). Let $D$ denote the maximum of $\log c(\cdot)$ of the variables in the formula. The calculation consists of the following main operations:

1. two multiplications (see (2.20)):
  \[ O \left( m^2 \cdot \text{Mul}(D) + m^2 \cdot \text{Mul}(mF, D + \log m) \right); \]

2. exact division:

(a) calculating the inverse of $a^{(k-1)}_{k-1,k-1}$ (in $\tilde{a}^{(k-1)}_{k-1,k-1}/N(a^{(k-1)}_{k-1,k-1})$ form) (see (2.21)):
\[ O\left(m^3 \text{Mul}(m(D + F + \log m))\right) ; \]

(b) multiplying the numerator by $\tilde{a}^{(k-1)}_{k-1,k-1}$, whose $c(\cdot)$ is by (2.7) $O(mD + m^2 F)$ (see (2.20)):
\[ O\left(m^2 \text{Mul}(D, mD + m^2 F) + m^2 \text{Mul}(mF, mD + m^2 F)\right) ; \]

(c) and dividing the resulting algebraic number exactly by $N(a^{(k-1)}_{k-1,k-1})$, which is an integer, and whose size is by (2.8)
\[ O\left(mD + mF + m \log m\right) : \]
\[ O\left(m \text{Mul}(mD + m^2 F, mD + mF + m \log m)\right) . \]

Now we give an asymptotic bound on $D := \max_{k=1}^{n-1} \max_{i,j=1}^{k+1} \log c(a_{ij})$. The variables $a_{ij}^{(k)}$ are determinants of order $k \leq n$ with the elements of $A$. Such determinants can be written as a sum of $k!$ terms, each is a product of $k$ elements (and possibly a sign). Therefore using (2.4) and (2.6):
\[ c(a_{ij}^{(k)}) \leq n!M_a^{n-1}\left(\max_{i,j=1}^{n} c(a_{ij})\right)^n , \]
\[ \log c(a_{ij}^{(k)}) = O(n \log n + nmF + nL) . \]

The latter is $D$, and it shows that $D$ dominates over $mF$, which can be used to simplify the formulas above and to observe that step 2. (a) dominates over the others, therefore the running time of the evaluation of formula (3.1) is:
\[ T = O\left(m^3 \text{Mul}(mD)\right) \]

This formula needs to be evaluated $n^3$ times, which, after substituting (3.2) to $D$, gives the result to be proved.

4 LLL algorithm

4.1 Introduction

The LLL algorithm is a lattice basis reduction algorithm invented by A. K. Lenstra, H. W. Lenstra and L. Lovász [5]. For a lattice $\Lambda$, it transforms any basis $b_1, b_2, \ldots, b_n \in \mathbb{R}^n$ to a reduced basis of $\Lambda$. It is known that it runs in polynomial time if the vectors are in $\mathbb{Z}^n$. In this section we show that it is also polynomial for $\mathbb{Z}[\alpha]^n$ vectors.

The algorithm first performs the Gram–Schmidt orthogonalization on the input
vectors:

(4.1) \[ b_i^* := b_i - \sum_{j=1}^{i-1} \mu_{ij} b_j^* \quad (1 \leq i \leq n) \]

(4.2) \[ \mu_{ij} := \frac{\langle b_i, b_j^* \rangle}{\langle b_j^*, b_j^* \rangle} \quad (1 \leq j < i \leq n) \]

When the algorithm terminates, the \( b_i \) vectors are LLL-reduced, which means the following two properties:

(4.3) \[ |\mu_{ij}| \leq \frac{1}{2} \quad (1 \leq j < i \leq n), \]

(4.4) \[ \|b_i^* + \mu_{i,i-1} b_{i-1}^*\|_2^2 \geq \delta^2 \|b_{i-1}^*\|_2^2 \quad (2 \leq i \leq n), \]

where \( \delta \) is a parameter \((\frac{1}{4} < \delta < 1, \text{ usually } \delta = \frac{3}{4})\).

The skeleton of the LLL algorithm is the following. This contains only the changes of \( b_i \)'s. The full algorithm updates the other variables after each \( b_i \)-change to preserve (4.1) and (4.2) above.

\[
k := 2
\]

while \( k \leq n \) do

\[
b_k := b_k - \lceil \mu_{kk-1} \rceil b_{k-1}
\]

if \( k \geq 2 \land \|b_k^* + \mu_{kk-1} b_{k-1}^*\|_2^2 < \delta^2 \|b_{k-1}^*\|_2^2 \) then

\[
\text{swap step:}
\]

\[
b_k \leftrightarrow b_{k-1}
\]

\[
k := k - 1
\]

else

\[
\text{reduction step:}
\]

\[
\text{for } l := k - 2 \text{ to } 1 \text{ do}
\]

\[
b_k := b_k - \lfloor \mu_{kl} \rfloor b_l
\]

\[
k := k + 1
\]

Theorem 4.1. Starting with any \( b_1, b_2, \ldots, b_n \in \mathbb{R}^n \), the LLL algorithm performs

\[
O \left( n^4 K \delta \log \frac{nB}{L_0} \right)
\]

arithmetic operations in \( \mathbb{R} \) (the meanings of the variables are described below). If the implementation uses fixed-size numbers (e.g. floating-point numbers), then the bit complexity is the same. On the other hand, if it uses variable-length type like integers or exact algebraic numbers, the bit complexity is higher. It depends on the
exact type ($\mathbb{Z}$ or $\mathbb{Z}[\alpha]$) and the integer multiplication algorithm used ($\text{Mul}(X)$):

$\mathbb{Z}$:

- $\text{Mul}(X)$: $O\left(n^4 \log B \text{ Mul}(n \log B) K_\delta\right)$,
- $X^2$: $O\left(n^6 \log^3 BK_\delta\right)$,
- $X^{\log_3 3}$: $\sim O\left(n^5 \log 2 BK_\delta\right)$,
- $X \log X \log \log X$: $O\left(n^5 \log^2 B \log(n \log B) \log \log(n \log B) K_\delta\right)$,

$\mathbb{Z}[\alpha]$:

- $\text{Mul}(X)$: $O\left(n^4 m HK_\delta \left(m \text{ Mul}(m^2 n^6 H^2 K_\delta) + n^5 HK_\delta \text{ Mul}(n^2 H)\right)\right)$,
- $X^2$: $O\left(n^{16} m^6 H^5 K_\delta^3\right)$,
- $X^{\log_3 3}$: $\sim O\left(n^{13.5} m^{5.2} H^{4.2} K_\delta^{2.6}\right)$,
- $X \log X \log \log X$: $O\left(n^{10}(n + m^3) m H^3 K_\delta^2 \log(mn HK_\delta) \log \log(mn HK_\delta)\right)$,

where:

- $n$: the number and the dimension of the vectors;
- $B := \max_{i=1}^n \|b_i\|_2^2$;
- $C := \max_{i=1}^n c(b_i)$, where $c(x) := \max_{j=1}^n c(x_j)$;
- $L_0 := \min \left\{ \|x\|_2^2 \mid x \in \Lambda(b_1, b_2, \ldots, b_n) \setminus \{0\} \right\}$,
  where $\Lambda(b_1, b_2, \ldots, b_n) := \{c_1 b_1 + c_2 b_2 + \ldots + c_n b_n \mid c_1, c_2, \ldots, c_n \in \mathbb{Z}\}$;
- $\delta$: the parameter of the LLL algorithm ($1/4 < \delta < 1$);
- $K_\delta := \frac{1}{\log \delta}$;
- $m$: the degree of $\alpha$;
- $F := \|f\|_{\infty} := \max_{i=0}^m |f_i|$, where $f$ is the minimal polynomial of $\alpha$.

Before the theorem is proved, some other lemmas follow.

### 4.2 Properties in $\mathbb{R}^n$

**Lemma 4.2.** Consider the LLL algorithm over $\mathbb{R}^n$. Then the variables in the algorithm after any number of iterations (at the beginning or end of the body of the main while-loop) can be bounded by expressions depending only of the initial
values (using the notations above):
(4.5) \[ \|b_i\|_2^2 \leq B \]
(4.6) \[ \|b_i\|_2^2 \leq nB \quad (i \neq k) \]
(4.7) \[ |\mu_{ij}| \leq \frac{1}{2} \quad (i < k) \]
(4.8) \[ |\mu_{ij}| \leq 2^{n-i} \sqrt{n} \left( \frac{nB}{L_0} \right)^{\frac{n+1}{4}} \quad (i = k) \]
(4.9) \[ |\mu_{ij}| \leq \sqrt{n} \left( \frac{jB}{L_0} \right)^{\frac{1}{j}} \quad (i > k) \]
(4.10) \[ d_j \leq B^j \]
(4.11) \[ d_j \geq \left( \frac{L_0}{j} \right)^j \]

where
(4.12) \[ d_j := \|b_i\|_2^2 \|b_{i+1}\|_2^2 \ldots \|b_{j-1}\|_2^2 \]

**Proof:** Most of these inequalities are similar to those in [5], especially to [5, (1.30)-(1.34)], but those are for vectors in \( \mathbb{Z}^n \), and many of them uses the fact that \( d_j \geq 1 \) since \( d_j \) is both integer and positive. But in our case \( d_j \) is not necessarily an integer, so the first task is to prove a different lower bound for \( d_j \), namely (4.11).

It follows from Minkowski’s theorem that if \( S \) is a \( j \)-dimensional convex body that is symmetrical to the origin, and has no other common point with the \( \Lambda_j := \Lambda(b_1, b_2, \ldots, b_j) \) lattice than the origin, then:

\[ \text{Vol}(S) \leq 2^j d(\Lambda_j) \]

where \( \text{Vol}(S) \) is the \( j \)-dimensional hypervolume of \( S \) and \( d(\Lambda_j) \) is the determinant of \( \Lambda_j \), i.e. \( d(\Lambda_j) = \sqrt{d_j} \) [9, III. 5.3. (p. 81.)]. Let \( S \) be a hypercube with side \( 2r/\sqrt{j} \) where \( r^2 < L_0 \) and \( r^2 \to L_0 \), then:

\[ d(\Lambda_j) \geq \left( \frac{r}{\sqrt{j}} \right)^j, \]

\[ d_j \geq \left( \frac{L_0}{j} \right)^j, \]

which is (4.11).

(4.5), (4.6), (4.7) and (4.10) are either trivial or proved in [5] without the use of the integer property.

Using the Cauchy–Schwarz inequality and the other inequalities of this lemma, we can give a bound for \( |\mu_{ij}| \) by \( \|b_i\|_2 \):

\[ |\mu_{ij}|^2 \leq \frac{\|b_i\|_2^2 \|b_j\|_2^2}{\|b_j\|_2^2} = \frac{d_j-1}{d_j} \|b_i\|_2^2 \leq \frac{B^{j-1}}{(L_0/j)^{j}} \|b_i\|_2^2 \leq \left( \frac{jB}{L_0} \right)^j \|b_i\|_2^2 B, \]
from which \((1.9)\) follows directly from \((1.6)\). \((1.8)\) can be proved from \((1.9)\) in the same way as in \([5]\), but with the new values. ■

**Lemma 4.3.** For any \(b_1, \ldots, b_n \in \mathbb{R}^n\) basis, their Gram–Schmidt coefficients \((\mu_{ij})\) can be expressed explicitly with the \(b_i\)'s, by the quotient of the following two \(j \times j\) determinants:

\[
\mu_{ij} = \frac{\lambda_{ij}}{d_j}
\]

with

\[
d_j = \begin{vmatrix}
\langle b_1, b_1 \rangle & \cdots & \langle b_1, b_{j-1} \rangle & \langle b_1, b_j \rangle \\
\vdots & \ddots & \vdots & \vdots \\
\langle b_{j-1}, b_1 \rangle & \cdots & \langle b_{j-1}, b_{j-1} \rangle & \langle b_{j-1}, b_j \rangle \\
\langle b_j, b_1 \rangle & \cdots & \langle b_j, b_{j-1} \rangle & \langle b_j, b_j \rangle \\
\end{vmatrix}, \quad \lambda_{ij} = \begin{vmatrix}
\langle b_1, b_1 \rangle & \cdots & \langle b_1, b_{j-1} \rangle & \langle b_1, b_j \rangle \\
\vdots & \ddots & \vdots & \vdots \\
\langle b_{j-1}, b_1 \rangle & \cdots & \langle b_{j-1}, b_{j-1} \rangle & \langle b_{j-1}, b_j \rangle \\
\langle b_j, b_1 \rangle & \cdots & \langle b_j, b_{j-1} \rangle & \langle b_j, b_j \rangle \\
\end{vmatrix}
\]

Note that the two determinants differ only in their last column.

**Proof:** \([1\text{ p. 93}]\) shows that:

\[
\begin{pmatrix}
\langle b_1, b_1 \rangle & \cdots & \langle b_1, b_{j-1} \rangle & \langle b_1, b_j \rangle \\
\vdots & \ddots & \vdots & \vdots \\
\langle b_{j-1}, b_1 \rangle & \cdots & \langle b_{j-1}, b_{j-1} \rangle & \langle b_{j-1}, b_j \rangle \\
\langle b_j, b_1 \rangle & \cdots & \langle b_j, b_{j-1} \rangle & \langle b_j, b_j \rangle \\
\end{pmatrix} \begin{pmatrix}
\xi_1 \\
\vdots \\
\xi_{j-1} \\
\xi_j \\
\end{pmatrix} = \begin{pmatrix}
\langle b_1, b_1 \rangle \\
\vdots \\
\langle b_{j-1}, b_1 \rangle \\
\langle b_j, b_1 \rangle \\
\end{pmatrix}
\]

with \(\xi_j = \mu_{ij}\) (although the other \(\xi_i\)'s might differ from the \(\mu_{il}\)'s). Solving the system for \(\xi_j\) by the Cramer’s rule gives the statement of the lemma. ■

Note: this means that if \(b_1, b_2, \ldots, b_n \in \mathbb{R}^n\), where \(R \subseteq \mathbb{R}\) is an integral domain (e.g. \(R = \mathbb{Z}\) or \(R = \mathbb{Z}[[a]]\)), then \(\lambda_{ij}, d_j \in R\) as well.

**Lemma 4.4.** Consider the LLL algorithm on \(\mathbb{R}^n\), and let \(t\) be the number of main iterations (which are either reduction steps or swap steps). Then:

\[
(4.13) \quad t = O \left( n^2 \log \frac{nB}{L_0} K \right)
\]

**Proof:** Let \(r\) be the number of reduction steps, and \(s\) be the number of swap steps. Since the former adds, and the latter subtracts 1 from \(k\), and since the algorithm starts with \(k = 2\) and finishes when \(k = n + 1\), therefore \(r - s = n - 1\), so \(t = r + s = 2s + n - 1\).

Let \(D := d_1d_2 \ldots d_n\), and let \(D^{(s)}\) be the value of \(D\) after \(s\) swap steps.\([5]\) proves that for integer values (i.e. for \(b_1, \ldots, b_n \in \mathbb{Z}^n\)) there are at most \(O(n^2 \log B)\) iterations, but this uses the fact that \(D\) is an integer, hence \(D \geq 1\). We replace this with an other lower bound for \(D\). We also need an upper bound
for $D$, and for both cases we use the bounds of the $d_j$’s in (4.11) and (4.10):

$$D = \prod_{j=1}^{n} d_j \geq \prod_{j=1}^{n} \left( \frac{L_0}{j} \right) ^{j} \geq \prod_{j=1}^{n} \left( \frac{L_0}{n} \right) ^{j} = \left( \frac{L_0}{n} \right) ^{\frac{1}{2} \frac{n(n+1)}{2}}$$

$$D = \prod_{j=1}^{n} d_j = \prod_{j=1}^{n} \prod_{i=1}^{j} \|b_i^*\|_2^2 \leq B^{\frac{n(n+1)}{2}}$$

These bounds are true after any number of iterations, i.e. for any $D^{(s)}$. Further, we use the fact that a reduction step does not change $D$, and that a swap step reduces $D$ by at least $\delta$: $D^{(s+1)} < \delta D^{(s)}$ – both are proved in [5] without the use of the integer property. By induction, it follows that $D^{(s)} < \delta^s D^{(0)}$. Putting these inequalities together:

$$\left( \frac{L_0}{n} \right) ^{\frac{1}{2} \frac{n(n+1)}{2}} \leq D^{(s)} < \delta^s D^{(0)} \leq \delta^s B^{\frac{n(n+1)}{2}}.$$ 

After taking logarithms from both ends and rearranging, we get:

$$s < \frac{1}{\ln \frac{B}{\delta}} \frac{n(n+1)}{2} (\ln n + \ln B - \ln L_0),$$

and the statement follows from this, because $t = 2s + n - 1$. ■

### 4.3 Coefficient size in $\mathbb{Z}[\alpha]^n$

**Lemma 4.5.** Consider the LLL algorithm for $b_1, b_2, \ldots, b_n \in \mathbb{Z}[\alpha]^n$. The coefficient size of the variables after $t$ iterations can be bounded as follows:

(4.14) $c \left( b_i^{(t)} \right) \leq C^{(t)} \leq \left( 2^n \sqrt{n} \left( \frac{nB}{L_0} \right)^{n+1} \right) ^{t} C,$

(4.15) $c \left( d_j^{(t)} \right) \leq n^j j! M_{\alpha}^{j-1} \left( C^{(t)} \right)^{2j},$

(4.16) $c \left( \lambda_{ij}^{(t)} \right) \leq n^j j! M_{\alpha}^{j-1} \left( C^{(t)} \right)^{2j},$

where $C^{(t)} := \max_{i=1}^{n} c \left( b_i^{(t)} \right)$ and $C := C^{(0)}$.

**Proof:** Consider one reduction step of the algorithm. It performs the following $k - 1$ reductions, including the changes of the $\mu_{ij}$’s (in the full algorithm presented earlier, the $\mu_{ij}$’s are omitted, and the case for $l = k - 1$ is written separately):

for $l := k - 1$ to 1 do

\[ b_k := b_k - [\mu_{kk}] b_i \]

for $j := 1$ to $l - 1$ do

\[ \mu_{kj} := \mu_{kj} - [\mu_{kl}] \mu_{ij} \]

\[ 20 \]
Denote the value of $b_k$ and $\mu_{kj}$ in the start of each $l$-iteration by $b_k^{[l]}$ and $\mu_{kj}^{[l]}$. The values in the beginning and the end of the reduction step are for $l = k - 1$ and $l = 0$, respectively. Then:

$$c(b_k^{[l-1]}) = c(b_k^{[l]} - \mu_{ki}^{[l]}b_i) \leq c(b_k^{[l]}) + \left|\mu_{ki}^{[l]}\right| c(b_i)$$

$$\left|\mu_{kj}^{[l-1]}\right| = \left|\mu_{kj}^{[l]} - \mu_{kj}^{[l]}\right| \leq \left|\mu_{kj}^{[l]}\right| + \left|\mu_{kj}^{[l]}\right| \mu_{ij} \leq \left|\mu_{kj}^{[l]}\right| + (2\left|\mu_{ki}^{[l]}\right|) \frac{1}{2} \leq 2 \max_{j=1}^{k-1} \left|\mu_{kj}^{[l]}\right|.$$ 

By induction, and using (4.8) (which was proved only for the beginning of the reduction step, i.e. for $l = k - 1$):

$$\left|\mu_{kj}^{[l]}\right| \leq 2^{k-l-1} \max_{j=1}^{k-1} \left|\mu_{kj}^{[l-1]}\right| \leq 2^{n-l-1} \sqrt{n} \left(\frac{nB}{L_0}\right)^{\frac{n-1}{2}}.$$ 

Also by induction:

$$c(b_k^{[l-1]}) \leq c(b_k^{[l-1]}) + \sum_{i=1}^{k-1} \left|\mu_{ki}^{[l]}\right| c(b_i) \leq c(b_k^{[l-1]}) + \sum_{i=1}^{k-1} 2 \left|\mu_{ki}^{[l]}\right| c(b_i) \leq c(b_k^{[l-1]}) + \sum_{i=1}^{k-1} 2^{n-l} \sqrt{n} \left(\frac{nB}{L_0}\right)^{\frac{n-1}{2}} c(b_i) \leq \left(1 + (2^n - 2^{n-k+1}) \sqrt{n} \left(\frac{nB}{L_0}\right)^{\frac{n-1}{2}}\right) \max_{i=1}^{k} c(b_i^{[l-1]}) \leq 2^n \sqrt{n} \left(\frac{nB}{L_0}\right)^{\frac{n-1}{2}} \max_{i=1}^{k} c(b_i^{[k-1]}).$$

In the last inequality we increased 1 to cancel out the negative part with $2^{n-k+1}$. That increasing would fail if $\frac{nB}{L_0}$ were too small, so we prove that $\frac{nB}{L_0}$ is too large. Indeed, from (4.11) we know that $L_0 \leq j d_j^{\frac{1}{j}}$ for any $1 \leq j \leq n$, and using this for $j = 1$:

$L_0 \leq d_1 \leq B$, i.e. $\frac{B}{L_0} > 1$.

This was one reduction step of the algorithm, which changed $b_k$ as above and left the others unchanged. Therefore, the maximum of $c(b_i)$’s is increased at most as $c(b_k)$. A swap step however performs only the first of the $k - 1$ reductions on $b_k$ and exchanges two $b_i$’s. The single reduction does not increase the maximum by more than the $k - 1$ reductions, and the exchange does not change it at all. Therefore we can use the following upper bound in both cases:

$$C(t) \leq 2^n \sqrt{n} \left(\frac{nB}{L_0}\right)^{\frac{n-1}{2}} C(t-1),$$

21
and a simple induction finishes the proof of (4.14).

From Lemma 4.3 we know that $\lambda_{ij}$ and $d_j$ (omitting the $^{(t)}$ indices) are $j \times j$ determinants whose elements are of the form $\langle b_{ij'}, b_{j'j} \rangle$. The coefficient size of each element is, using the properties of the $c(\cdot)$ operator:

$$c(\langle b_{ij'}, b_{j'j} \rangle) \leq nM_\alpha c(b_{ij'}) c(b_{j'j}) \leq nM_\alpha C^2.$$  

A $j \times j$ determinant can be written as a sum of $j!$ terms, each is a product of $j$ elements of the matrix (and a sign), therefore:

$$c(d_j) \leq j! M_\alpha^{j-1} \max_{i' < j'} c(\langle b_{i'j'}, b_{j'j} \rangle)^j \leq j! M_\alpha^{j-1} (nM_\alpha C^2)^j,$$

which is equivalent to (4.15). We get (4.16) in the same way. ■

**Corollary 4.6.** The coefficients of these variables have the following asymptotic bounds after any number of iterations:

$$(4.17) \quad \log c(b_i) = O(n^5 H^2 K_\delta),$$

$$(4.18) \quad \log c(d_j) = O(n^6 H^2 K_\delta),$$

$$(4.19) \quad \log c(\lambda_{ij}) = O(n^6 H^2 K_\delta),$$

where:

$$H := \frac{1}{n} \log \left( \frac{nB}{L_0} \right) = O(\log B + m \log C + m \log n + m^2 F) .$$

**Proof:** In $\mathbb{Z}[\alpha]$, we can give a lower bound for $L_0$ using the coefficient bound $C$ of the initial $b_i$ vectors.

It can easily be proved that $L_0 \geq \min_{i=1}^n \|b_i^*\|_2^2$, and we know by (4.12) that $\|b_i^*\|_2^2 = d_i/d_{i-1}$. By using the lower bound by coefficient size (2.10) and the upper bound for $c(d_i)$ (4.15) at the beginning ($t = 0$):

$$d_i \geq \frac{1}{P_\alpha S_\alpha c(d_i)^{m-1}} \geq \frac{1}{P_\alpha S_\alpha (n^n n! M_\alpha^{2n-1} C^{2n})^{m-1}},$$

so by the upper bound of $d_{i-1}$ (4.10):

$$L_0 \geq \min_{i=1}^n \|b_i^*\|_2^2 \geq \frac{1}{P_\alpha S_\alpha M_\alpha^{(2n-1)(m-1)}} \frac{1}{(n^n n!)^{m-1} B^{n-1} C^{2n(m-1)}} .$$

Taking the logarithm of this, and using the logarithmic bounds for the constants $M_\alpha$, $P_\alpha$ and $S_\alpha$ (see (2.11), (2.12) and (2.14)), we can conclude that:

$$\log \frac{nB}{L_0} = O(n(\log B + m \log C + m \log n + m^2 F)) .$$

By taking the logarithm of (4.14), (4.15) and (4.16), substituting the iteration bound (4.13) and substituting the expression above for $\log \frac{nB}{L_0}$, we get (4.17), (4.18) and (4.19). ■
4.4 Running time of the LLL algorithm

Now we have enough information to calculate the running time of the LLL algorithm with algebraic numbers.

The original version of the algorithm uses floating-point numbers, but there is a modification for integer (or rational) input which uses only exact integer arithmetic \([1, \text{ p. } 94]\). Instead of maintaining the \(\mu_{ij}\) and the \(B_i := ||b_i'||^2\) variables, which can be rational fractions, this modification maintains the \(\lambda_{ij}'\)’s and the \(d_j\)’s, which are always integers, and their quotients give the original variables. We use the same formulas but with algebraic integers in \(\mathbb{Z}[\alpha]\).

First consider one swap step of the algorithm. It first swaps \(b_k\) and \(b_{k-1}\), and then swaps \(\lambda_{k,j}\) and \(\lambda_{k-1,j}\) for each \(j \in \{1, 2, \ldots, k-2\}\). These are \(O(nD)\) operations, where \(D = \log c(d_j)\) after any number of iterations, i.e. \(D = O(n^6H^2K_\delta)\) as in (4.18) (which is greater than the bound of \(\log c(b_k)\)).

Then we calculate the following expressions (here \(d_j'\) etc. denote the new value of the \(d_j\) etc. variables):

\[
\begin{align*}
\lambda_{i,k-1}' &:= \frac{d_{k-2}\lambda_{i,k} + \lambda_{k,k-1}\lambda_{i,k-1}}{d_{k-1}} \quad i \in \{k+1, k+2, \ldots, n\}; \\
\lambda_{i,k}' &:= \frac{d_k\lambda_{i,k-1} - \lambda_{k,k-1}\lambda_{i,k}}{d_{k-1}} \quad i \in \{k+1, k+2, \ldots, n\}; \\
d_{k-1}' &:= \frac{d_k d_{k-2} + \lambda_{k,k-1}^2}{d_{k-1}}.
\end{align*}
\]

Note that these formulas are very similar to the recursive formula of the Bareiss algorithm (3.1), so a similar calculation can be used to show that they require \(O(m^3 \text{ Mul}(mD))\) time, but now \(D\) is different (but it still dominates over \(mF\)).

The total time of one swap step is therefore:

\[
(4.20) \quad T_{\text{swap}} = O(nm^3 \text{ Mul}(mD)).
\]

Now consider one reduction step of the algorithm. Its main step is to calculate \([\mu_{kl}]\), i.e. \([\lambda_{kl}/d_l]\). After rounding, no matter how big \(c(\lambda_{kl})\) and \(c(d_l)\) were, \([\lambda_{kl}/d_l]\) is an integer, and by (4.8), its size can be much smaller:

\[
\log \left| \frac{\lambda_{kl}}{d_l} \right| = O\left(n \log \frac{nB}{L_0}\right) = O\left(n^2H\right).
\]

The reduction step is performed as follows for each \(l\) from \(k - 2\) to 1:

1. calculating \([\mu_{kl}]\) (see (2.24)):

\[
O\left(m^2 \text{ Mul}(m^2D + m^3F)\right);
\]

2. multiplying \([\mu_{kl}]\) by \(\lambda_{kj}\) for each \(j \in \{1, 2, \ldots, l-1\}\) (see (2.19)):

\[
O\left(nm \text{ Mul}(n^2H, D)\right);
\]

3. multiplying \([\mu_{kl}]\) by \(b_l\):

\[
O\left(nm \text{ Mul}(n^2H, D)\right).
\]
So the total reduction step is:

\[ T_{\text{red}} = O \left( n m^2 \text{Mul}(m^2 D) + n^2 m \text{Mul}(n^2 H, D) \right). \]

Since the number of iterations is \( t = O (n^3 H K_\delta) \) by (4.13), the total running time of the LLL algorithm is:

\[
T^Z_{\text{LLL}} [\alpha] = O (t (T_{\text{swap}} + T_{\text{red}})) = \\
= O \left( n^4 m H K_\delta \left( m \text{Mul}(m^2 D) + n \text{Mul}(n^2 H, D) \right) \right) = \\
= O \left( n^4 m H K_\delta \left( m \text{Mul}(m^2 n^6 H^2 K_\delta) + n^6 H K_\delta \text{Mul}(n^2 H) \right) \right), \\
H = O \left( \log B + m \log C + m \log n + m^2 F \right).
\]

For comparison, we can also calculate the running time for integers using \( \text{Mul}(X) \). We know from [5, (1.26)] that all integer variables are of \( O(n \log B) \) size, from which we can easily calculate that:

\[
t = O \left( n^2 \log B K_\delta \right), \\
T_{\text{swap}} = O \left( n \text{Mul}(n \log B) \right), \\
T_{\text{red}} = O \left( n^2 \text{Mul}(n \log B) \right),
\]

and we can conclude that:

\[ T^Z_{\text{LLL}} = O \left( n^4 \log B \text{Mul}(n \log B) K_\delta \right). \]

If we substitute the different possibilities for \( \text{Mul}(X) \) into \( T^Z_{\text{LLL}} \) and \( T^Z_{\text{LLL}} [\alpha] \), it completes the proof of Theorem 4.1 about the running time of the LLL algorithm.

4.5 Notes

In this section we proved that the LLL algorithm is polynomial also for algebraic numbers, but the result for \( T^Z_{\text{LLL}} [\alpha] \) is a very pessimistic upper bound for the worst-case complexity. In practice the algorithm can be much faster. We describe some reasons of this difference.

First, we calculated the maximum number of iterations in the algorithm (4.13), but this is only a theoretical limit, and in practice it can be often just a few (i.e. \( O(n) \)) steps.

Then, the calculation used \( L_0 \), the size of the shortest vector in the lattice. We used a worst-case theoretical upper bound for \( \log \frac{1}{L_0} \) using the coefficient size of the input, and it was greater than \( O(n) \) (see Corollary 4.6), but in practice it is often not so small, and if we make an assumption that it is constant (i.e. \( O(1) \)), then the running time can be reduced by several powers.

For rounding an algebraic number to integer, which is calculated by first approximating it with a rational number, we calculated how long integer coefficients are needed for this approximation to provably get the correct result (in the proof of (2.23)). But that matters only when the number is very close to an integer, which is rare in practice, and usually much smaller coefficients suffice.
When we calculated the running time of the extended Euclidean algorithm (in the proof of (2.21)), we did not exclude the rare case of abnormal polynomial sequences, i.e. when the degree differences are not always one, but this possibility increased the power of $m$ by one. In most of the cases however the polynomial sequences are normal, i.e. the degree difference is usually one in each step.

If we assume these simplifications, the $n^{16}$ factor in the running time for basic multiplication ($\text{Mul}(X) = X^2$) can be reduced to $n^8$.

5 Summary

In this paper we discussed symbolic computation in algebraic number fields. We represented field elements as polynomials of a primitive element, and calculated computational costs of operations and algorithms using this representation. We presented our results in terms of several parameters, including the size of the inputs, some constants depending on the field like the degree, and the integer multiplication algorithm used. We used the bit length of the coefficients of the representing polynomial to measure the size of the numbers.

First, we examined the field operations – addition, subtraction, multiplication and division – and some other functions like the less-than comparison and integer rounding functions. We gave bounds on the size of the outputs as well as on the running time of these operations.

Next, we used the Bareiss algorithm as a simple example to demonstrate an application of these results. We calculated the running time of the algorithm when it is extended to algebraic integers, and compared this result to the original one using integers. We found the expected result that it has a similar asymptotic bound but with additional constants regarding the algebraic number field.

The next main part of this paper was the proof of polynomiality of the LLL algorithm when likewise extended to algebraic numbers and the calculations are exact. This generalization was, unlike the Bareiss algorithm, rather nontrivial and required new ideas. We generalized several known properties of the algorithm for not necessarily integer inputs. A crucial problem with this was finding substitutes for inequalities like $d \geq 1$ which are true for positive integers, but not generally true for positive algebraic numbers. Then we calculated bounds on the coefficient size of the intermediate vector components during the algorithm. This enabled us to finally calculate the running time of the algorithm using our formulas for algebraic number fields. We found that this bound is far greater than for integers, but it is only a theoretical limit and it is believed that the algorithm is much faster in practice. Note that for the original integer version the well-known bound is also said to be worse than the practical running time.

It is out of scope of the present theoretical article but is a subject of future
research to perform measurements on the running time of the LLL algorithm with algebraic numbers to confirm the claim above regarding the practical running time. Another interesting question regarding this algorithm is to search examples for which calculating symbolically is superior to using numerical approximations, i.e. for which the numerical calculation fails to give the correct answer e.g. to the generalized continued fraction problem in [6], which gave an important motivation for this article.

References

[1] H. Cohen: A Course in Computational Algebraic Number Theory. Springer-Verlag Berlin Heidelberg, 1996

[2] M. Pohst, H. Zassenhaus: Algorithmic Algebraic Number Theory. Cambridge University Press, 1997

[3] K. O. Geddes, S. R. Czapor, G. Labahn: Algorithms for Computer Algebra. Kluwer Academic Publishers, 1992

[4] N. P. Smart: The Algorithmic Resolution of Diophantine Equations. Cambridge University Press, 1998

[5] A. K. Lenstra, H. W. Lenstra, L. Lovász: Factoring Polynomials with Rational Coefficients Mathematische Annalen 261. (1982), p. 515-534

[6] A. Pethő, M. E. Pohst, Cs. Bertók: On multidimensional Diophantine approximation of algebraic numbers. Journal of Number Theory 171. (2017), p. 422-448

[7] E. H. Bareiss: Sylvester’s identity and multistep integer-preserving Gaussian elimination. Mathematics of Computation 22 (1968), p. 565-578

[8] W. S. Brown: On Euclid’s Algorithm and the Computation of Polynomial Greatest Common Divisors Journal of the Association for Computing Machinery 18. (1971), p. 478-504

[9] J. W. S. Cassels: An Introduction to the Geometry of Numbers. Springer-Verlag Berlin Heidelberg, 1997