Geometric algebras for euclidean geometry

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Abstract

We discuss and compare existing GA models for doing euclidean geometry. We begin by clarifying a set of fundamental terms which carry conflicting meanings in the literature, including $\mathbb{R}^n$, euclidean, homogeneous model, and duality. Equipped with these clarified concepts, we establish that the dual projectivized Clifford algebra $\mathbf{P}(\mathbb{R}^*_n,0,1)$ deserves the title of standard homogeneous model of euclidean geometry. We then turn to a comparison with the other main candidate for doing euclidean geometry, the conformal model. We establish that these two algebras exhibit the same formal “feature set” for doing euclidean geometry. We then compare them with respect to a set of practical criteria.

1 Introduction

Although noneuclidean geometry of various sorts plays a fundamental role in theoretical physics and cosmology, the overwhelming volume of practical science and engineering finds its field of activity within classical euclidean space $\mathbb{E}^n$. For this reason it is of no small interest to establish the best computational model for this space. The growing interest in geometric algebra\(^1\) as a tool for such modelling is the point of departure for the following observations. In particular, we want to explore the question, which form of geometric algebra is best-suited for modeling euclidean geometry? For practical reasons, we focus on the cases $n = 2$ and $n = 3$; the situation in higher dimensions is not qualitatively different.

\(^1\)Although some authors attempt to distinguish the two terms, we use geometric and Clifford algebra interchangeably in this article.
2 Review of existing models

Most of the current applications of geometric algebra to euclidean geometry use the so-called conformal model, based on the Clifford algebra $\mathbf{P}(\mathbb{R}_{n+1,1,0})$ (see for example [DL11], [DFM07], [Per09], and [DL03]).

The other model featured in this article is a homogeneous model $\mathbf{P}(\mathbb{R}_{n,0,1})$. [Gun11a] handles $\mathbf{P}(\mathbb{R}_{n,0,1})$ within a metric-neutral framework that includes $\mathbf{P}(\mathbb{R}_{n+1,0,0})$ for elliptic geometry, $\mathbf{P}(\mathbb{R}_{n,1,0})$ for hyperbolic geometry, and $\mathbf{P}(\mathbb{R}_{n,0,1})$ for dual euclidean geometry, and includes full treatment of kinematics and rigid body motion in these spaces using these algebras. For the current discussion focused on the euclidean metric, [Gun11b] (and the extended version [Gun11c]) is a more accessible reference. We adopt here the notation used in [Gun11a] that these four Clifford algebras are denoted by $\mathcal{C}l^n_\kappa$, where $\kappa$ can take on the four values $(0, 1, -1, \infty)$ corresponding to the euclidean, elliptic, hyperbolic, and dual euclidean case, resp. We introduce here the term projective geometric algebra to refer to this family of algebras; the term homogeneous has a similar usage in the literature. We prefer the term projective due to the central role which projective geometry plays in the underlying concepts – as the following discussion hopefully makes clear.

We begin with a brief review of the two algebras under consideration. For the purposes of brevity the basic ideas are introduced without much discussion. In the subsequent section Sect. 4, which discusses how the homogeneous model is portrayed in the contemporary literature, several of these ideas will be treated in more detail, in order to clear up a set of misconceptions. Only then can we carry out a comparison of the two approaches, using a range of criteria arising from practical applications involving euclidean geometry, kinematics, and dynamics.

2.1 $\mathbf{P}(\mathbb{R}_{n,0,1})$

We first sketch the mathematical ingredients involved in the euclidean algebra $\mathcal{C}l^n_0$. Although its roots can be traced back to Clifford and Study ([Gun11a], §7.13), this algebra first appeared in the modern literature in [Sc00] and [Sc05], and was then extended and embedded in the metric-neutral toolkit described in [Gun11a]. We describe its construction in some detail since there are several unfamiliar steps involved.

Projectivized exterior algebra. Begin with the standard real Grassmann or exterior algebra $\wedge \mathbb{R}^{n+1}$ which encapsulates the subspace structure of the vector space $\mathbb{R}^{n+1}$. It is a graded associative algebra with non-zero grades from 0 (the scalars) to $n + 1$, the pseudoscalars. The 1-vectors represent the vectors of $\mathbb{R}^n$. The higher grades are constructed via the wedge product, an anti-symmetric, associative product which is additive on the grade of its operators, and represents the join operator on subspaces. Projectivize this algebra to obtain the projectivized exterior algebra $\mathbf{P}(\wedge \mathbb{R}^{n+1})$ which in a natural way represents the subspace structure of projective $n$-space $\mathbb{R}P^n$ as built up out of points by joining them to form higher-dimensional subspaces.
Dual projectivized exterior algebra. The above process can also be carried out with the dual vector space \((\mathbb{R}^{n+1})^*\) to produce the dual projectivized exterior algebra \(P(\bigwedge \mathbb{R}^{n+1})\). It also models the subspace structure of \(\mathbb{R}P^n\) but dually, so that the 1-vectors represent hyperplanes and the wedge product corresponds to the meet or intersection of subspaces. To avoid confusion we write the wedge operator in \(P(\bigwedge \mathbb{R}^{n+1})\) as \(\land\) (meet) and the wedge operator in \(P(\bigwedge \mathbb{R}^{n+1})\) as \(\lor\) (join). The choice of symbols, which differs from some of the contemporary literature, is motivated by the affinity of meet to set intersection \(\cap\) and of join to set union \(\cup\). We can in fact assume that both products are available in the same algebra, as the next paragraph makes clear.

Duality. It is convenient to be able to carry out the meet operator in \(P(\bigwedge \mathbb{R}^{n+1})\). (This is often in the context of Grassmann algebra called the regressive product.) To do this we take advantage of Poincare duality ([Gre67b], Sec. 6.8), which provides an algebra isomorphism \(J : P(\bigwedge \mathbb{R}^{n+1}) \leftrightarrow P(\bigwedge \mathbb{R}^{n+1})^*\) mapping a geometric entity of \(P(\bigwedge \mathbb{R}^{n+1})\) to the same geometric entity in \(P(\bigwedge \mathbb{R}^{n+1})^*\). In this sense it is an identity map. Equipped with this map we define a meet operation \(\land\) in \(P(\bigwedge \mathbb{R}^{n+1})\) by \(X \land Y := J(J(X) \land J(Y))\), and similarly we define a join operator \(\lor\) for \(P(\bigwedge \mathbb{R}^{n+1})^*\). Note that this solution to the duality problem does not require any metric. The common misconception to the contrary appears to have originated in [HZ91], as well as the unfortunate reversal of the \(\land\) and \(\lor\) operators.

Introducing a metric. In \(E^2\), consider two lines \(m_1 : a_1x + b_1y + c_1 = 0\) and \(m_2 : a_2x + b_2y + c_2 = 0\). Assuming WLOG \(a_1^2 + b_1^2 = 1\), then \(\cos \alpha = a_1a_2 + b_1b_2\), where \(\alpha\) is the angle between the two lines: changing the \(c\) coefficient translates the line but does not change the angle it makes to other lines. This generalizes to the angle between two hyperplanes in \(E^n\). Accordingly, we attach the degenerate metric \((n,0,1)\) to the dual projectivized exterior algebra \(P(\bigwedge \mathbb{R}^{n+1})\) to obtain the geometric algebra \(P(\mathbb{R}^n,0,1)\). This provides a faithful model for euclidean geometry in dimension \(n\). For details see [Gun11b] and [Gun11c].

3 The conformal model

The Clifford algebra \(P(\mathbb{R}_{n+1,1,0})\) provides a projective model for \((n + 1)\)-dimensional hyperbolic geometry \(H^{n+1}\) ([Gun11a], Chapter 7). The absolute quadric \(Q\) in this case is an \(n\)-dimensional sphere; the interior points form the model of \(H^{n+1}\). The exterior points, on the other hand, provide a model for the space of spheres in \(E^n\). Given any point \(P \in \mathbb{R}P^{n+1}\) outside \(Q\), the polar plane \(P^\perp\) cuts the absolute quadric \(Q\) in the points of the sphere associated to \(P\). A harmonic homology in \(P(\mathbb{R}_{n+1,1,0})\) which preserves \(Q\) (see Sect. 4.6 [Gun11a]) induces a self-mapping of \(E^n\) which is the inversion in the sphere represented by the center of the homology. Since inversion in spheres generates the group of conformal self-maps of \(E^n\), the underlying geometry is also called conformal. Details of
this theme outside the scope of this article, see for example [HJ03].

The null points – points on the quadric $Q$ – represent the points of $E^n$ (spheres of radius 0). $Q$ can also be identified with the points of $E^n$ via stereographic projection. In brief: at the price of providing a curved embedding of $E^n$, one obtains a model in which spheres in $E^n$ are represented as linear objects (hyperplanes). By restricting attention to $Q$, one can derive a model of euclidean geometry. This model, naturally called the conformal model of euclidean geometry, has gained a significant following since it was introduced in [HLR01]. There are a variety of introductory treatments of the conformal model of euclidean geometry. We rely mostly on [DFM07], Chapters 13+, since it is the most widely available and the most detailed.

The conformal model also contains a model of $n$-dimensional elliptic space and for $n$-dimensional hyperbolic space by restricting the ambient metric to appropriate hyperplanes. Note that in contrast to $Cl^n_{κ}$, these Clifford algebras are constructed in the standard way so 1-vectors represent points. We return to this later in Sect. 7.

4 Which homogeneous model?

We have mentioned above that the algebra $P(R^*_{n,0,1})$ is sometimes referred to as a homogeneous model of euclidean geometry since it is based on projective rather than vector space, hence homogeneous coordinates are used. It turns out that the phrase “homogeneous model of euclidean geometry” has a variety of interpretations, not restricted to $P(R^*_{n,0,1})$. Our first task is to consider these other interpretations.

For example, Chapter 11 of [DFM07], entitled The homogeneous model, describes one such model. We discuss this in some detail, as it is representative of several other textbooks ([Per09], [DL03]. In §11.1 one reads:

...[the homogeneous model of euclidean geometry] embeds $R^n$ in a space $R^{n+1}$ with one more dimension and then uses the algebra of $R^{n+1}$ to represent those elements of $R^n$ in a structured manner.

The extra homogeneous coordinate is given the index 0; the existing $n$ coordinates are assigned the indices $i = 1$ to $i = n$, with the “inherited” inner products $e_i \cdot e_i = 1$ for $1 \leq i \leq n$. To specify the metric remains only to specify a value of $e_0 \cdot e_0$. Regarding this choice, the authors write:

Of course this quantity $[e_0 \cdot e_0]$ is not part of the real geometry we want to describe – that resides completely in the base space $R^n$. Since we have no geometrical reasons to choose a particular value of $e_0 \cdot e_0$, we should choose it for reasons of computational convenience. The choice $e_0 \cdot e_0 = 0$ would make $e_0$ noninvertible, which is inconvenient to many computations (such as taking

\footnote{presumably from the vector space inner product of $R^n$}
Figure 1: Step-by-step construction of the unique perpendicular through the point \( P \) to the line \( \Pi \). First (upper right) \( \Pi \cdot P \) is the perpendicular plane to \( \Pi \) passing through \( P \). Then (lower left) wedging this plane with \( \Pi \) gives the intersection point with \( \Pi \), and finally (lower right) joining this to \( P \) gives the desired line.

A dual. The other natural choices are \( e_0 \cdot e_0 = +1 \) and \( e_0 \cdot e_0 = -1 \); we support both in this book ...

Translating this into the language of the Cayley-Klein construction of metric spaces atop projective space ([Kle26] or [Gun11a], Chapter 4), this is equivalent to imposing an elliptic metric \((n+1,0,0)\) or hyperbolic metric \((n,1,0)\) on projective space. In fact, “euclidean” has dramatically different meanings in the vector space and the projective space settings. We discuss this below in more detail in Sect. 4.1 where three different meanings of the term \( \mathbb{R}^n \) are distinguished. Note also that the use of a degenerate metric is rejected out-of-hand as being “inconvenient” for calculating duals, a decision that has far-reaching consequences, also discussed below.

Before discussing the underlying ideas, let’s first compare this non-degenerate homogeneous model with \( P(\mathbb{R}^n,0,1) \) on a simple geometric construction, for example a geometric construction taken from §11.9 of [DFM07]:

Given a point \( P \) and a line \( \Pi \) in \( \mathbb{E}^3 \), find the unique line passing through \( P \) and intersecting \( \Pi \) orthogonally.

[DFM07] devotes two pages (pp. 310-311) to various strategies to solve the problem, frequently interrupting the calculation to remove unintended side-effects of the elliptic metric in intermediary stages and in the end does not produce a closed-form solution. In contrast, \( P(\mathbb{R}^3,0,1) \) yields directly the compact solution \((\Pi \cdot P) \land \Pi) \lor P\); Fig. 1 decomposes the solution in three steps

The above example is not isolated. For example, using a non-degenerate metric, it is impossible to express euclidean translations as sandwich operators (§11.8 of [DFM07]), a phenomena which turns up again below in Sect. 5.1. Faced with the failure of this model, the authors of [DFM07] themselves acknowledge that the metric \((n+1,0,0)\) is not satisfactory for euclidean geometry (p. 312):

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Note that this solution is metric-neutral hence valid for non-euclidean metrics also.
The main problem with using the metric of $\mathbb{R}^{n+1}$ is that you cannot use it directly to do Euclidean geometry, for it has no clear Euclidean interpretation.

The foregoing quote is a good motivation for the next section, where we attempt to clarify the situation by differentiating various meanings of $\mathbb{R}^n$ and “euclidean”.

4.1 Three meanings of $\mathbb{R}^n$

In this section we attempt to shed light on the confusion of metrics by differentiating three distinct meanings to the symbol $\mathbb{R}^n$:

**Vector space** In this form, $\mathbb{R}^n$ represents the vector space used to define real projective space $\mathbb{R}P^{n-1}$ (Gun11a, Sect. 2.1). It is an $n$-dimensional linear space with an addition operation, real scalar multiplication, and distributive law, but **without** inner product. One can develop a theory of linear mappings between such spaces, and from this the dual vector space $V^\ast$. The evaluation map $\mathbb{R}^n \otimes (\mathbb{R}^n)^\ast \to \mathbb{R}$ of a vector and a dual vector (linear functional), often written $\langle \mathbf{v}, \mu \rangle := \mu(\mathbf{v})$ is sometimes confused with an inner product but is not; it is sometimes called instead the **scalar product** [Gre67a]. We recommend using the terminology ($n$-dimensional) $V$ for this meaning of $\mathbb{R}^n$. See [Gre67a], Chapter 1-2 for details.

**Inner product space** One begins with a vector space and adds an **inner product** between pairs of vectors, which is a symmetric bilinear form on the vector space. This produces an **inner product space**. When the form is positive definite, it’s called a **euclidean** inner product space. We recommend retaining the use of $\mathbb{R}^n$ for this meaning. Consult [Gre67a], Chapter 7 for details on inner product spaces.\(^4\)

**Euclidean space** This is a simply-connected, constant-curvature (with curvature 0) **metric space** homeomorphic to $\mathbb{R}^n$ but equipped with the Euclidean distance function (discussed for example in Gun11a, Chapter 4) between its points. We recommend using the notation $E^n$ for this space. The points of $E^n$, with the exception of the origin, are in a 1:1 correspondence to the **vectors** of $\mathbb{R}^n$ (the origin of $E^n$ maps to the zero vector of $\mathbb{R}^n$), but $E^n$ is **not** a vector space, and the inner product discussed in the previous item has, **a priori**, nothing to do with $E^n$. Armed with these three different meanings which sometimes are attached to the same symbol $\mathbb{R}^n$, let’s return to the discussion of the homogeneous model. provides a decomposition (as point sets) of $\mathbb{R}P^n$ as the disjoint union $E^n \cup E\ell^{n-1}$. Here $E\ell^{n-1}$ is the ideal hyperplane, equipped with the elliptic metric. Considering this ideal hyperplane as equivalent to the set of free vectors of $E^n$, we can also write this – in quotation marks – as “$E^n \cup \mathbb{R}^n$”. Furthermore, Thm. 65

\(^4\)Though note his definition of euclidean differs from ours.
4.1.1 Rephrasing using the differentiated notation

When we apply this differentiated terminology to the initial quote from [DFM07], we arrive at the following:

...[the homogeneous model] embeds $\mathbb{E}^n$ in a real vector space $V$ of dimension $n+1$ and then uses the algebra of $V$ to represent the elements of $E^n$ in a structured manner.

In this form, $\mathbb{R}^n$ no longer occurs; there is no longer a specific real vector space nor metric signature implied by this definition, and hence it is compatible with the Cayley-Klein construction of metric spaces atop projective space [Kle26]. Consequently, one can use this modified description as a starting point for the search for the correct choice of $V$ and of the metric signature, in order to arrive at the desired Cayley-Klein space; we have briefly sketched in Sect. 2.1 above how one arrives at the solution $V = (\mathbb{R}^{n+1})^*$ with signature $(n,0,1)$ (or consult Ch. 3 and Ch. 4 of [Gun11a]). This solution produces the Clifford algebra $\mathbb{P}(\mathbb{R}^{n+1}_{n,0,1})$ as the correct homogeneous model for Euclidean geometry.

To sum up: this confusion of the three meanings of $\mathbb{R}^n$ means that many of the objections to “the” homogeneous model are simply justified complaints against using the wrong metric to model euclidean geometry. We leave it to the reader to verify that there is nothing particular about the metric $(n+1,0,0)$; any non-degenerate metric $(p,q,0)$ with $p+q = n+1$ will produce the same problems.

5 Homogeneous models using a degenerate metric

Faced with the difficulties ensuing on the use of the elliptic homogeneous model, [DFM07], p. 314, states:

We emphasize that the problem is not geometric algebra itself, but the homogeneous model and our desire to use it for euclidean geometry. It will be replaced by a much better model for that purpose in Chapter 13 [the conformal model].

In fact, what the authors of [DFM07] have shown is that the homogeneous model with non-degenerate metric is “the problem” – recall that they rejected the use of degenerate metrics a priori. Hence it remains to be seen whether a degenerate metric could provide a faithful model for euclidean geometry. We now turn to an analysis of three common objections to the use of such a degenerate metric.

5.1 Objection 1: lack of covariance

[Li08] is often cited as evidence that a degenerate metric is not appropriate for modelling euclidean geometry. This reference, on p. 11, identifies the Clifford algebra $\mathbb{R}_{n,0,1}$ (in our notation) as the appropriate algebra for euclidean geometry. It focuses on the case of $n = 3$
and remarks on the presence of dual quaternions in the even sub-algebra. This leads to
the claim:

However, the dual quaternion representations of primitive geometric objects such as points, lines, and planes in space are not covariant. More accurately, the representations are not tensors, they depend upon the position of the origin of the coordinate system irregularly.

The observant reader has hopefully noticed that the algebra referred to here, $\mathbb{R}_{n,0,1}$, is the algebra for dual euclidean space, which is built on the standard Grassmann algebra where 1-vectors represent points. This metric space has a single ideal point, and is radically different from euclidean space ([Gun11a], Ch. 10). As explained above, to obtain euclidean geometry one must use the dual algebra $\mathbf{P}(\mathbb{R}_{n,0,1}^*)$. It appears that the lack of covariance is a result of mixing up these two algebras (since each, considered in its own realm, acts covariantly). In order to determine whether this was the case, the author of this article engaged in an email exchange with Professor Li, whose outcome was that the latter acknowledged that $\mathbf{P}(\mathbb{R}_{n,0,1}^*)$ in fact represents all euclidean isometries covariantly. Of course the best proof that this objection cannot be sustained is to verify for oneself that the versor-based sandwich operators indeed realise all euclidean isometries ([Gun11b] or [Gun11c], §4.2).

5.1.1 Relation to dual quaternions

The dual quaternions appear in the above citation. Since they play an important role in the historical development of geometric algebra and in particular of euclidean geometric algebra, we take the opportunity to describe them briefly and indicate their significance.

The dual quaternions were first introduced by William Clifford (who called them bi-quaternions) and developed further by Eduard Study [Stu03] (who gave them the name dual and generalised them to other metrics). One introduces a non-zero number $\epsilon$ such that $\epsilon^2 = 0$, then defines dual numbers to be numbers of the form $a + b\epsilon$ for $a, b \in \mathbb{R}$. When one considers the quaternions as an algebra over the dual numbers instead of over the real numbers, one obtains the dual quaternions, which form an 8-dimensional associative real algebra. Dual quaternions were invented in order to model all euclidean isometries as sandwich operators; not just rotations around the origin (which Hamilton had done with the quaternions) but also translations. Dual quaternions are in the standard toolkit of modern kinematics ([McC90]).

The dual quaternions are isomorphic to the even subalgebra $\mathbf{P}(\mathbb{R}_{3,0,1}^*)$. Under this isomorphism, $\epsilon$ corresponds to the pseudo-scalar $\mathbf{I}$ ([Gun11a], §7.6). This even sub-algebra is also sometimes called the motor algebra. Note however that even though the two algebras are isomorphic, the dual quaternions historically were not handled as a graded algebra, but

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5In Li’s own words ([Li]): “The model you describe is exactly dual to the affine model, which is also 4D for 3D geometry. Your dual model is nice. It is frame independent, and is not the same as the dual vector model. Good work!”
as a flat 8-dimensional algebra. This fact may have contributed to the current confusion regarding the algebra, as the following paragraphs attempts to explain.

Not only [Li08], cited above, but also [DFM07], Ex. 11.17, refer to transformation formulae for euclidean isometries involving dual quaternions. These formulas apparently stem from [Stu03] or [Bla42]; the fact that the formulas take slightly different form for points, lines, and planes is due to the lack of a graded structure in the dual quaternions and not to any flaw in the larger graded algebra. Recall that points and planes do not have natural representations in this even sub-algebra, since they are 1-vectors or 3-vectors. Study (or Blaschke?), who did not have access to the graded algebra structure, apparently first discovered how to give them ad hoc representations in the smaller algebra at his disposal. Such a representation is provided as \{scalar + dual bivector\} for a plane, and \{pseudoscalar + bivector\} for a point. This overloading of the 8-dimensional algebra to handle entities not naturally included in the algebra leads, not surprisingly, to minor differences in the form of the sandwich (versor) operator to express the euclidean motion.

Another source of misunderstanding arises out of the existence of dual algebras. Not only are the dual quaternions isomorphic to the even subalgebra of \(P(\mathbb{R}^3_{3,0,1})\); they are also isomorphic to the even Clifford subalgebra of dual euclidean space, \(P(\mathbb{R}^3_{3,0,1})\). It appears that some researchers, familiar with dual quaternions and searching for an embedding of them into a “euclidean” Clifford algebra, have mistakenly concluded that \(P(\mathbb{R}^3_{3,0,1})\) is the proper algebra for euclidean space. It is a mistake which is easily made, as this algebra builds space out of points and hence is consistent with a cultural prejudice to think of space as built out of points by the join operator, instead of out of planes by the meet operator. Of course mathematically either approach is valid, and when the metric is non-degenerate, both approaches lead to the same metric space. However, as Ch. 7 of [Gun11a] establishes, euclidean space can only be realized as a Cayley-Klein space by using the plane-based, dual approach.

This mistake, combined with the the irregular transformation laws of the dual quaternions, leads to a host of contradictions and difficulties which appear to have led these researchers to conclude – as in the quotation above – that the algebra is flawed. However, the transformation irregularities, as mentioned above, disappear as soon as the proper graded algebra structure is provided; both \(Cl^3_{\infty}\) and \(Cl^3_0\) provide covariant representations of their respective isometry groups.

### 5.2 Objection 2: lack of duality

Another common objection to the use of degenerate metrics is often expressed in terms of a “lack of duality”. Consider the following quote from [HLR01], an article often cited as the birth-moment of the modern conformal model for euclidean geometry ([DFM07], §13.8). There on p. 13 one can read:

Any degenerate algebra can be embedded in a non-degenerate algebra of larger dimension, and it is almost always a good idea to do so. Otherwise, there will
be subspaces without a complete basis of dual vectors, which will complicate algebraic manipulations.

As with the case of euclidean above, there are multiple meanings for the term dual in the literature which must be carefully differentiated. These two meanings are:

1. The ability to calculate the regressive product. As shown in Sect. 2.1 above, for this operation one only requires the dual coordinates of a geometric entity, and this is provided in a non-metric way by Poincaré duality in the exterior algebra.

2. The effect of multiplication by the pseudoscalar: \( \Pi : \Pi(X) := XI \). This operation is known in the projective geometry literature as the polarity on the metric quadric, and we extend this usage to the geometric algebra. When the metric is non-degenerate, this polarity is an algebra involution; when degenerate, the pseudoscalar has no inverse and this is not an algebra isomorphism. When non-degenerate, one can define the regressive product (analogously to the use of Poincaré duality above in Sect. 2.1): \( X \land Y := \Pi(\Pi(X) \lor \Pi(Y)) \) and similarly for join \( \lor \) in the dual Grassman algebra.

In the above quote as well as many popular texts ([DFM07], [Per09], [DL03]) one gains the false impression that a non-degenerate metric must be used to implement the regressive product. As noted above, one can avoid the problems presented by the degenerate euclidean metric, by using Poincaré duality in the Grassmann algebra. This provides a non-metric mathematical foundation for calculating the dual basis – and the regressive product – with an identical computational complexity. Hence this objection cannot be sustained.

5.3 Objection 3: Absence of invertible pseudoscalar

[Li08], also p. 11, raises a further objection to the use of a degenerate metric:

Because the inner product in \( \mathbb{R}_{n,0,1} \) is degenerate, many important invertibilities in non-degenerate Clifford algebras are lost.\(^6\)

We have discussed the invertibility of \( I \) as a condition for duality above in Objection 2. One can ask, beyond duality, which “important invertibilities” are meant here? Perhaps what is meant here involves a class of formulae which appear in a nicer form when one uses \( I^{-1} \) in them, for example, one can often omit or simplify a power of \(-1\). Note that when \( I^{-1} \neq 0, I^{-1} \) can be normalised to so that \( I^{-1} = \pm I \). In a projective model such as we are considering here, any non-zero multiple of an element is equivalent to the same element, so the use of \( I^{-1} \) to prettify formulae is purely cosmetic and \( I \) would serve just as well. For example [Hes10] defines the dual of a multi vector \( A^* := AI^{-1} \) and shows then that the inner and outer products obey the relation: \( a \cdot A^* = (a \land A)^* \). If instead one defines \( A^* := AI \) the relation remains true.

\(^6\)Since it is irrelevant to this object, we overlook the fact, discussed above, that the correct Clifford algebra here should be \( \mathbb{R}_{n,0,1} \).
In fact, the projective model described in Sect. 2.1 and encountered no problems related to lack of pseudoscalar invertibility. The theory developed there for euclidean geometry is equivalent in all respects to that developed for the non-degenerate, noneuclidean geometries. In fact, the non-invertibility of \( I \) for the euclidean case, far from being a problem, is an advantage, since it faithfully represents the metric relationships within euclidean geometry. The degenerate metric produces the correct answers for euclidean calculations. For example, the elegant calculation illustrated in Fig. 1 depends on the degenerate pseudoscalar. To see this, consider the sub-expression \( \Pi \cdot P \). Letting the point \( P \) move freely, one obtains a set of planes \( \Pi \cdot P \) which all have the same polar point, the ideal point of the line \( \Pi \), hence are parallel – as one can also easily confirm from the figure. But in a non-degenerate metric, the polar point of a plane is unique. Hence the degenerate metric is required to provide the correct answer.

5.4 The remedy to these objections: education

Each of the objections discussed above is based on a misunderstanding of the nature of the projective (homogeneous) model, and indirectly, of projective geometry.

1. The confusion of the elliptic metric with the euclidean one (as described in Sect. 4) reflects an underlying insensitivity to the different role the metric plays in the vector space and in the projective space settings.

2. Regarding Objection 1, it appears that the dual construction – building space from planes rather than points – required for the algebra \( \mathbb{R}_{n,0,1}^* \) is a stumbling block for many practitioners. A cursory acquaintance with projective geometry leads one to see the complete equivalence of the two approaches.

3. Similarly for Objection 2: when one is aware that the dual algebra provides alternative coordinates for the subspace structure of projective space, one is led immediately to the Poincaré isomorphism \( J \) as a non-metric way to obtain dual coordinates.

4. It appears that Objection 3 is based on a lack of recognition of projective invariance under non-zero scaling, that is, \( I \) is projectively equivalent to \( I^{-1} \) (when it’s defined).

Conclusion. All the misunderstandings described above can be traced back to the neglect of projective geometry in the mathematics curriculum, accompanied by an exaggerated reliance on Euclidean inner product spaces as a universal basis for mathematical modelling.

5.5 Summary

The above discussion has hopefully persuaded the reader that the other meanings of homogeneous model for euclidean geometry in the current literature have serious flaws in comparison to \( \mathbb{P}(\mathbb{R}_{n,0,1}^*) \). This is not surprising, as \( \mathbb{P}(\mathbb{R}_{n,0,1}^*) \) arose directly from the 19th
century stream of mathematical research of Plücker, Klein, Clifford, and Study which culminated in the invention of Clifford algebras originally. In this perspective, it has every right to be called the name “standard” or “classical homogeneous model” of euclidean geometry.

That has not prevented other authors ([Dor11], [Per09]) from using the same terms to describe the other “homogeneous” variants discussed above. While the use of such terms remains a matter of taste and judgement, the same does not apply to the conclusions which follow. For example in [DFM07], p. 364 one reads: “The conformal model is the smallest known algebra that can model Euclidean transformations in [a] structure-preserving manner.” In the next section we want to take a look at this claim, one that is met quite often in the contemporary literature.

6 Comparison: A feature-set for “doing geometry”

Having clarified what is meant by standard homogeneous model and disposed of a series of objections to it, we are now prepared to compare it to the conformal model. As a basis for this comparison, we rely on a recent tutorial on the conformal model [Dor11]. This tutorial describes the challenge of “doing geometry” on a computer. The tutorial lists seven “tricks” and three “bonuses” which the conformal model offers in this regard. We list them here:

- Trick 1: Representing euclidean points in Minkowski space.
- Trick 2: Orthogonal transformations as multiple reflections in a sandwiching representation.
- Trick 3: Constructing elements by anti-symmetry.
- Trick 4: Dual specifications of elements permits intersection.
- Bonus: The elements of euclidean geometry as blades.
- Bonus: Rigid body motions through sandwiching.
- Bonus: Structure preservation and the transfer principle.
- Trick 5: Exponential representation of versors.
- Trick 6: Geometric calculus.
- Trick 7: Sparse implementation at compiler level.

7 presumably meaning doing euclidean geometry.
How does the homogeneous model $P(\mathbb{R}^n, 0, 1)$ stand with respect to these features? In fact, $P(\mathbb{R}^n, 0, 1)$ offers every one of the ten features listed. Some slight editing is required to “translate” to the homogeneous model; for example, Trick 1 has to be rephrased as “Representing euclidean points in projective space”. Duality (trick 4) is implemented in a non-metric way in our homogeneous model as we have repeatedly pointed out above. There are naturally some elements of euclidean geometry which cannot be represented as blades in the homogeneous model (bonus 1), such as point pairs and spheres. But the basic elements required for euclidean geometry are present: points, lines, and planes; and these are also the elements which [Dor11] explicitly treats.

Hence, the two models for euclidean geometry cannot be differentiated on the basis of these standard features. One immediate corollary is that the projective mode $P(\mathbb{R}^*_n, 0, 1)$, not the conformal model, is “smallest known algebra that can model Euclidean transformations in [a] structure-preserving manner.” The importance of this result will become more apparent in the next section, which turns to a practical comparison of the two models.

7 Comparison: practical issues

We have differentiated the homogeneous model based on $P(\mathbb{R}^*_n, 0, 1)$ from the other, inferior homogeneous models, and defended it against a variety of objections raised against it. We have also demonstrated that it offers all 10 basic selling points of the conformal model, and is the smallest geometric algebra that does so. We now turn to a wider comparison of the two models for euclidean geometry drawn from a variety of practical contexts. We omit consideration of features which are not part of standard euclidean geometry.

- **Efficiency.** The non-linear embedding of $\mathbb{E}^n$ in a higher-dimensional space in the conformal model introduces extra complexity which results in less efficient algorithms for the same euclidean results.

  - Converting euclidean points from 4D (homogeneous) to 5D (conformal) is stereographic projection, a non-linear operation. This operation is required to interface the conformal model with standard visualization systems and languages such as OpenGL using homogenous coordinates.

  - Before optimizing, a multiplication in the homogeneous model is 4 times cheaper than in the conformal model (since the former is 16-dimensional and the latter is 32-dimensional).

  - The representation of euclidean geometric configurations is complicated by the presence of elements which do not belong to that geometry, notably the point $e_\infty$, a distraction absent in projective models.

- **Differential equations.** In the conformal model, care must be taken whenever applying any kind of transformation to euclidean elements, as these elements lie on
a quadric hypersurface of the ambient $\mathbb{R}P^{n+1}$, and numerical errors quickly result in “wandering off” this surface, so that the element is no longer euclidean. In particular, solving differential equations presents serious challenges in this regard. Consider the example of the euclidean equations of motion for a rigid body. The possibility in the homogeneous model for “wandering off” when solving the equations (§6.2.4 of [Gun11b]) can be avoided by normalizing the rotor solution $\mathbf{g}$ so that $\mathbf{g} \tilde{\mathbf{g}} = 1$; after all, the solution space is a 12-dimensional subspace of a 14-dimensional space. Compare the treatment of the euclidean rigid body equations of motion in the conformal model, where Lagrange multipliers are introduced to keep the solver “on track”. A recent treatment, [LLD11], §1.3.1, explains:

... The idea here is to work in an overall space that is two dimensions higher than the base space, using the usual conformal Euclidean setup. The penalty for doing this, i.e., using a Euclidean setup, is that the number of degrees of freedom is not properly matched to the problem in hand, and we have to introduce additional Lagrange multipliers to cope with this.

- **Kinematics, rigid body mechanics, and classical screw theory.** The metric-neutral mathematical theory of rigid body mechanics maps precisely onto the projective model presented here. Indeed, a series of renowned mathematicians in the 19th century (Plücker, Klein, Clifford, Study, and others), in their research on kinematics and rigid body mechanics, laid the foundations of the mathematical theory which later developed into these projective Clifford algebras ([Kle72], [Stu03]). All the features of classical screw theory, as presented in [Bal00], are exactly represented in the projective algebra presented here ([Gun11b], Sec. 5 or [Gun11a], Ch. 8), providing a natural basis for a rejuvenation of this domain. Compare [Hes10], which envisions a similar role for the conformal model. It would be worthwhile to compare the two approaches to screw theory with regard to such criteria as compactness of expression, practicality, and comprehensibility.

- **Ideal elements** are an integral part of the projective model of euclidean space. This consists of the ideal plane and all its points and lines (sometimes called elements “at infinity”). In the simplest (purely projective) form, the availability of ideal elements yields correct results regarding incidence of parallel lines and planes. As noted in Sect. 4.4.4 of [Gun11a], $P(\mathbb{R}^n_{n,0,1})$ also contains within it a model for the elliptic metric on the ideal plane. This implies that one has the complete algebra of free vectors at one’s disposal, for example, the difference of two normalised points is a free vector, etc. Ideal elements play an important role in kinematics and mechanics: an ideal line represents, in statics, a force couple; in kinematics, the axis of a euclidean translation, and plays an important role throughout the theory of rotors presented in [Gun11a]. One the other hand, the conformal model of euclidean space features a
The one-point compactification of \( \mathbb{R}^n \) (the ideal plane maps to the point \( e_\infty \)) ([DFM07], p. 356), and hence, without compensatory measures, lacks direct access to these important conceptual and computational elements.

- **Metric-neutrality.** The discussion of Ch. 4 makes clear that the projective model of euclidean geometry is a natural limit of the projective models of non-euclidean geometry. The conformal model can also be used to model hyperbolic and elliptic geometry ([DFM07], Sec. 16.7). In contrast to its model of euclidean geometry, however, these models are identical to the projective models presented here (that is, they are embedded as flat 4-spaces). Hence, the conformal model of euclidean geometry is **not** embedded as a natural limit of its models for non-euclidean geometry. The three algebras do not form a natural family, preventing a metric-neutral treatment, and hindering the understanding of euclidean phenomena as limits of the non-euclidean case as featured throughout [Gun11a].

- **Straightforward representation of geometry.** The geometric representation in the homogeneous model agrees with the naive sense perceptions of the human being. In the homogeneous model, straight is straight. In the conformal model, a straight euclidean line is represented by a great circle on a 4-dimensional sphere. Or, to take another example, a point in the conformal model is actually a point pair, one of whose points is the point at infinity \( e_\infty \) ([DFM07], p. 309). In light of this, expressions for euclidean configurations are more compact and understandable in the homogeneous model.

- **Learning curve.** The simplicity of the homogeneous representation noted in the previous point leads to considerable savings in explaining the model, a significant advantage when considering the unfamiliarity of the underlying concepts. Since the conformal model is also a projective model, learning the homogeneous model is a natural intermediate step towards understanding the conformal model, but not vice-versa.

- **Insider’s view** of non-euclidean spaces is naturally computed. This important visualization feature of the projective model lies outside the scope of this article; a detailed discussion can be found in [Gun10]. It essentially means: the projective model (due to its intimate relation to perspective rendering dating back to the Renaissance) naturally renders pictures that agree with those produced by cameras embedded in these spaces. Any other more complicated model aiming to produce such images will end up producing the same images. Furthermore, since the projective group \( PGL(4) \) is implemented in hardware on modern GPU’s, such pictures are rendered very fast in the projective model.
8 Integrating the two algebras

As more practitioners realise that $P(\mathbb{R}^{n,0,1})$ is a practical alternative for euclidean geometry, one can expect a shift in usage to this algebra. As this happens, there will be good reasons to want to be able to shift back and forth between this and the conformal algebra, depending on the requirements of the particular application; for example a robotics application that mixes rigid body mechanics (where the simpler projective GA is appropriate) with robot arm collision testing (where the conformal model is called for). It appears to be straightforward to devise a framework whereby this shifting can be designed into software structures so that the end-user can activate this shift in a transparent manner. In any case the integration of these two algebras should be the object of further research and development.

What is good for the projective model must not necessarily be a loss for the conformal model. In fact, when this integration is achieved, one can look forward, in the long term, to increased interest in the conformal model. This is because the projective model dramatically lowers the threshold for getting started using geometric algebra to do euclidean geometry. It sits halfway between the vector space context, which is of limited interest for doing euclidean geometry, and the conformal model, which can do euclidean geometry and much more, but requires a steep learning curve. A much higher fraction of undergraduates are candidates to learn projective geometric algebra than the conformal model. As the pool of active practitioners using the projective model grows, so also will the pool of those naturally grow, who are ready to take the next step, up to the conformal model – depending of course on whether it is an appropriate tool for their application.

One can also hope that themes of current research, especially geometric calculus, will be introduced into the projective model.

9 Conclusion

In the first half of the article we have clarified a set of fundamental terms, included $\mathbb{R}^n$, “euclidean”, “homogeneous model”, and “duality”, which are key to a correct understanding of how geometric algebra can be applied to doing euclidean geometry. As a result of this clarification work, we have established that the dual projective geometric algebra $P(\mathbb{R}^{n,0,1})$ is the clear choice to bear the name standard homogeneous model of euclidean geometry. It exhibits all the attractive features with respect to doing euclidean geometry which modern geometric algebra users expect (structure-preserving, isometries as sandwiches, exponential and logarithmic forms, etc.). Furthermore, in regard to many practical criteria, $P(\mathbb{R}^{n,0,1})$ exhibits clear advantages over the higher-dimensional, more complicated conformal model. Of course the conformal model will remain the model of choice for those applications that make essential uses of euclidean spheres or conformal maps. We hope that, based on the foregoing comparison, more practitioners, researchers, and educators will be encouraged to experiment with $P(\mathbb{R}^{n,0,1})$ for doing euclidean geometry, kinematics, and mechanics.
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