CONFORMAL ANOMALIES VIA CANONICAL TRACES

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ABSTRACT. Using Laurent expansions of canonical traces of holomorphic families of classical pseudodifferential operators, we define functionals on the space of Riemannian metrics and investigate their conformal properties, thereby giving a unified description of several conformal invariants and anomalies.

INTRODUCTION

In this paper, we use the Kontsevich-Vishik canonical trace to produce a series of conformal spectral invariants (or covariants or anomalies) associated to conformally covariant pseudodifferential operators. Although only one covariant is new, the use of canonical traces provides a systematic treatment of these covariants.

The search for conformal anomalies is motivated both by string theory and conformal geometry. Historically, the variation of functionals $F$ on the space of Riemannian metrics $\text{Met}(M)$ on a closed manifold $M$ under conformal transformations:

$$g \mapsto e^2f g, \quad f \in C^\infty(M, \mathbb{R})$$

has been a topic of interest to both mathematicians and physicists going back at least to Hermann Weyl (see Duff [Du] for a historical review of the physics literature, and Chang [C] for a survey of recent work in mathematics). In physics, the study of conformal invariants underwent a revival in the early 1980s with Polyakov’s work [Pol] on the conformal anomaly of bosonic strings, one of the motivating factors behind the development of determinant line bundles in mathematics.

The conformal anomaly of a Fréchet differentiable map $F : \text{Met}(M) \to \mathbb{C}$ at a given (background) metric $g$ is the differential at 0 of $F_g : C^\infty(M, \mathbb{R}) \to \mathbb{C}$, $F_g(f) := F(e^{2f} g)$. Thus the conformal anomaly in the direction $f$ is

$$\delta_f F := dF_{\gamma}(0).f = \frac{d}{dt} \bigg|_{t=0} F(e^{2tf} g).$$

A functional $F$ is conformally invariant if $\delta_f F = 0$ for any Riemannian metric $g$ and any smooth function $f$. If $\delta_f F_g = \int_M f(x)\delta_x F_g(x)dvol_g(x)$, then $\delta_x F_g(x)$ is called the the local (conformal) anomaly of $F_g$ (or equivalently of $F$ in the background metric $g$). A functional $F(g, x)$ on $\text{Met}(M) \times M$ is conformally covariant if, roughly speaking, $\delta_f F$ does not depend on derivatives of $f$ and $g$.

Conformal anomalies arise naturally in quantum field theory. A conformally invariant classical action $A(g)$ in a background metric $g$, for example the string theory
or nonlinear sigma model action, does not usually lead to a conformally invariant effective action $W(g)$, since the quantization procedure breaks the conformal invariance and hence gives rise to a conformal anomaly. In particular, in string theory the conformal invariance persists after quantization only in specific critical dimensions.

From a path integral point of view, the conformal anomaly of the quantized action is often said to arise from a lack of conformal invariance of the formal measure on the configuration space of the QFT. Whatever this means, we can detect the source of the conformal anomaly in the quantization procedure. In order to formally reduce the path integral to a Gaussian integral, one writes the classical action as a quadratic expression

$$A_g(\phi) = \langle A_g \phi, \phi \rangle_g$$

where $\phi$ is a field, typically a tensor on $M$, $A_g$ a differential operator on tensors and $\langle \cdot, \cdot \rangle_g$ the inner product induced by $g$. Because this inner product is not conformally invariant, the conformal invariance of $A_g(\phi)$ usually translates to a conformal covariance of the operator $A_g$. An operator $A_g$ is

**conformally covariant** of bidegree $(a,b) \in \mathbb{R}^2$ if

$$A_{\bar{g}} = e^{-bf} A_g e^{af}, \quad (0.1)$$

for $\bar{g} = e^{2f} g$. Thus this first step, which turns a conformally invariant quantity (the classical action) to a conformally covariant operator, already breaks the conformal invariance.

The second step in the computation of the path integral uses an Ansatz to give a meaning to the formal determinants that arise from the Gaussian integration. Mimicking finite dimensional computations, the effective action derived from a formal integration over the configuration space $C$ is

$$e^{-\frac{1}{2} W(g)} := \int_C e^{-\frac{1}{2} A(g)(\phi)} \mathcal{D}\phi = "\text{det}"(A_g)^{-\frac{1}{2}}.$$  

If there were a well defined determinant "$\text{det}$" on differential operators with the usual properties, (0.1) would yield

$$"\text{det}"(A_{e^{2f}g}) = "\text{det}"(e^{-bf} A_g e^{af})$$

$$= "\text{det}"(e^{-bf}) "\text{det}"(A_g) "\text{det"}(e^{af})$$

$$= "\text{det"}(e^{(a-b)f}) "\text{det"}(A_g),$$

where $e^{cf}$ is treated as a multiplication operator for $c \in \mathbb{R}$. Hence, even if a "good" determinant exists, the effective action $W(g)$ would still suffer a conformal anomaly, since $A_g$ is only conformally covariant:

$$\delta_f W(g) = \delta_f \log "\text{det"}(A_g) = \delta_f \log "\text{det"}(e^{(a-b)f}) = (a-b) \text{"tr"}(f),$$

where "$\text{tr}$" is a hypothetical trace associated to "$\text{det}$".

The $\zeta$-determinant $\text{det}_\zeta$ on operators is used by both physicists and mathematicians as an Ersatz for the usual determinant on matrices. Since the work of Wodzicki and Kontsevich–Vishik, we know the $\zeta$-determinant has a multiplicative anomaly, which
fortunately does not affect our rather specific situation. Indeed, the above heuristic derivation holds (Branson-Orsted [BO], Parker-Rosenberg [PR], Rosenberg [R]):

$$\delta_f \log \det_\zeta(A_g) = (a - b) \text{tr}^A_g(f),$$

if one replaces “tr”(f) with tr$_{A_g}$, the finite part in the heat-operator expansion tr$(f e^{-\epsilon A_g})$ when $\epsilon \to 0$. (Here and whenever the heat operator $e^{-\epsilon A_g}$ appears, we assume that $A_g$ is elliptic with non-negative leading symbol.) In summary, the regularization procedures involved in the $\zeta$-determinant and the finite part of the heat-operator expansion are not responsible for the conformal anomaly of the effective action $\mathcal{W}(g)$; the conformal anomaly appears as soon as one uses the conformally covariant operator $A_g$ associated to the originally conformally invariant action $\mathcal{A}(g)$.

These QFT arguments lead to the search for conformally covariant operators and associated spectral conformal covariants. There are four types of conformal covariants in the literature, in order of computational difficulty: (i) local covariants, those that depend only on the metric at a fixed point (ii) global invariants which are the integrals of (noncovariant) local quantities, (iii) global invariants which are not integrals of local expressions, but whose variation in any metric direction is local; (iv) global invariants which are not integrals of local expression, and whose variation in conformal directions is nonlocal. All four types have examples associated to spectral $\zeta$- and $\eta$-functions, as we now explain.

For (i), the residue at $z = 1$ of the local zeta function $\zeta_{A_g}(z, x)$, which turns out to be proportional to the local Wodzicki residue $\text{res}(A_g^{-1})$, is a pointwise conformal covariant for a conformally covariant operator $A_g$, under certain ellipticity and positivity conditions on the operator [PR]. (A classical example of a pointwise invariant is the length of the Weyl tensor [We].) For (ii), the value at $z = 0$ of the global $\zeta$-function $\zeta_{A_g}(z)$ of a conformally covariant operator $A_g$ is conformally invariant, again for certain operators, which may be pseudodifferential [PR, R]:

$$\delta_f \zeta_{A_g}(0) = 0.$$

In hindsight, this can be predicted by thinking of $\zeta_{A_g}(0)$ as an Ersatz for “tr”($\text{Id}$) in the heuristic notation above. It is well known that $\zeta_{A_g}(0)$ is the integral of the finite part of the pointwise heat kernel of $A_g$ (up to the nonlocal conformally invariant term $\dim \text{Ker}(A_g)$). When $A_g$ is a differential operator, $\zeta_{A_g}(0) = -\frac{1}{\text{ord}(A_g)} \text{res}(\log A_g)$, so the conformal invariance of $\zeta_{A_g}(0)$ is equivalent to the conformal invariance of the exotic determinant introduced by Wodzicki for zero order classical pseudodifferential operators and extended by Scott [Sc] to the residue determinant $\det_{\text{res}}(A_g) = e^{\text{res}(\log A_g)}$ on operators of any order. This gives another description of $\zeta_{A_g}(0)$ as the integral of a local quantity, namely the local Wodzicki residue of the logarithm of $A_g$.

Jumping to (iv), conformal anomalies arising from $\zeta$-determinants of conformally covariant operators vanish in certain cases, for one has [PR, R]

$$\delta_f \zeta'_{A_g}(0) = -\delta_f \log \det_\zeta(A_g) = (a - b) \int_M f(x) a_n(A_g, x) \text{dvol}_g(x),$$
where as \( \epsilon \rightarrow 0 \)
\[
\text{tr} \left( e^{-\epsilon A_g} \right) \sim \sum_{j=0}^{\infty} \left( \int_{M} a_j(A_g, x) d\text{vol}_g(x) \right) \epsilon^{\frac{j-n}{2n}},
\]
for \( \alpha = \text{ord}(A_g) \), \( n = \dim(M) \); here we assume \( A_g \) has all but finitely many eigenvalues nonnegative. The nonlocal nature of the functional determinant and its variation is well known; however, the above formula shows it gives rise to a local conformal anomaly \( (a - b)a_n(A_g, x) \). In particular, \( \zeta_A'(0) \) yields a conformal invariant in odd dimensions, as \( a_n(A_g) \) then vanishes. The conformal anomaly \( \delta_f \log \det \zeta(A_g) \), where \( \Delta_g \) is the Laplace-Beltrami operator on a closed Riemannian surface, is responsible for the conformal anomaly in bosonic string theory; since the coefficients \( a, b \) depend on the dimension of the manifold and the rank of auxiliary tensor bundles, combinations of such conformal anomalies cancel in certain critical dimensions, viz. the cancellation of conformal anomalies in 26 dimensions for bosonic string theory [Pol]. Further work on the conformal anomaly of functional determinants is in work of Branson and Orsted [B1, B2, BO].

For (iii), if \( A_g \) is a self-adjoint invertible elliptic operator, the phase of its \( \zeta \)-determinant can be expressed in terms of the \( \eta \)-invariant \( \eta_{A_g}(0) \) by
\[
\det_{\zeta}(A_g) := \det_{\zeta}(|A_g|) \cdot e^{i\pi \frac{\eta_{A_g}'(0) - \eta_{A_g}(0)}{2n}}.
\]
Again, only in certain dimensions is the phase conformally invariant; namely if \( \dim(M) \) and \( \text{ord}(A_g) \) have opposite parity [R].

We will study these four types of conformal anomalies and covariants in the common framework of variations of Kontsevich-Vishik functionals of conformally covariant operators. Whereas previous work on conformal anomalies uses heat kernel expansions, we use \( \zeta \)-function techniques instead. Our starting point is canonical traces, which are cut-off integrals of symbols of non-integer order pseudodifferential operators, which extend to Laurent expansions of cut-off integrals of holomorphic families of symbols. These coefficients are universal expressions in the symbol expansion of the family (Paycha-Scott [PS]), so their regularity properties and their variation in terms of external parameters (here the metric) are easily controlled. We thereby avoid some technical difficulties in the variation of heat kernel asymptotic expansions. The main result of the paper is that the coefficients of the Laurent expansions give explicit conformal anomalies.

In more detail, the three functionals \( \zeta_{A_g}(0), \zeta_A'(0) \) and \( \eta_{A_g}(0) \) are all \( A_g \)-weighted traces in the notation of [P2], namely \( \text{tr}^3(A(I)), \text{tr}^3(\log A_g) \) and \( \text{tr}^3(A_g |A_g|^{-1}) \) respectively. Here, for a weight \( Q \) (i.e. an admissible positive order elliptic operator), the \( Q \)-weighted trace \( \text{Tr}^Q(A) \) of a classical pseudodifferential operator \( A \) is the finite part at \( z = 0 \) of the meromorphic map \( z \mapsto \text{TR}(A Q^{-z}) \) (up to a factor depending on the kernel of \( Q \)), where \( TR \) is the Kontsevich-Vishik canonical trace on noninteger order operators extending the usual trace on smoothing operators [KV]. (This definition
of weighted trace is equivalent to previous ones \([\text{P2]}\) by the discussion after Def. 3.) Thus all our spectral invariants are examples of canonical traces.

If the conformally covariant operator \(A_g\) is a weight, we may define functionals given by meromorphic functions \(z \mapsto F_h(g)(z) := \text{TR}(h(A_g) A_g^{-z})\) where \( h \) is a real or complex valued function defined on a subset \(W \subset \mathbb{C}\). In particular, the functionals \(\zeta_{A_g}(z)\) and \(\eta_{A_g}(z)\) correspond to choosing \(h(\lambda) = 1\) (with \(W = \mathbb{C}\)) and \(h(\lambda) = \frac{1}{|\lambda|}\) (with \(W = \mathbb{R}/\{0\}\)). Using results on the coefficients in the Laurent expansion \([\text{PS]}\) for \(z \mapsto F_h(g)(z)\) at \(z = 0\), we derive the conformal anomaly of these meromorphic functionals (Theorem 2.5):

\[
\delta_f \text{TR}(h(A_g) A_g^{-z}) = (a - b) \text{TR} \left( f h'(A_g) A_g^{-z+1} \right) - z (a - b) \text{TR} \left( f h(A_g) A_g^{-z} \right).
\]

This formula strongly depends on the tracial nature of the canonical trace \(\text{TR}\) on noninteger order operators \([\text{LS}]\). Identifying the coefficients on either side, we get a hierarchy of functionals and their conformal anomalies, the first one involving the Wodzicki residue \(\text{res}\):

\[
\begin{align*}
\delta_f \text{res}(h(A_g)) &= (a - b) \text{res} ( f h'(A_g) A_g) ; \\
\delta_f \text{tr}^A h(A_g) &= (a - b) \text{tr}^A ( f h'(A_g) A_g) + \frac{a - b}{\alpha} \text{res} ( f h(A_g)) ; \\
\delta_f \text{tr}^A (h(A_g) \log A_g)) &= (a - b) \text{tr}^A ( f h'(A_g) A_g \log A_g) \\
&\quad + \frac{b - a}{\alpha} \text{tr}^A ( f h(A_g)) ; \\
\vdots \\
\delta_f \text{tr}^A (h(A_g) \log^j A_g) &= (a - b) \text{tr}^A ( f h'(A_g) A_g \log^j A_g) \\
&\quad + j \frac{a - b}{\alpha} \text{tr}^A ( f h(A_g) \log^{j-1} A_g) .
\end{align*}
\]

Different choices for \(h\) lead to conformal covariants/anomalies of the four types mentioned above (Theorem 2.8). Applying this to explicit geometric conformally covariant operators such as the Dirac, Paneitz and Peterson operators (see \S 2.2) yields conformal anomalies and covariants, including a new example associated to the heat kernel asymptotics of conformally covariant pseudodifferential operators. The Laurent approach provides a natural hierarchy among these invariants/covariants: the most divergent term in the Laurent expansion is a conformal invariant; if this global invariant vanishes in a particular case, then the new “most divergent” term, if it is of the form \(\int_M \mathcal{I}(g,x)\text{dvol}_g(x)\) tends to give rise to a local conformal anomaly proportional to \(\mathcal{I}(g,x)\).
1. Regularized traces

In this section, we recall known results on regularized traces and the Wodzicki residue, and give some extensions to families of operators.

1.1. Preliminaries. Let $E \rightarrow M$ be a hermitian vector bundle over a closed Riemannian $n$-manifold $M$, and let $Cl(M, E)$ denote the algebra of classical pseudodifferential operators (ΨDOs) acting on smooth sections of $E$. $S^*M \subset T^*M$ denotes the unit cosphere bundle, and $tr_x$ denotes the trace on the fiber $E_x$ of $E$ over $x \in M$.

Definition 1.1. A positive order elliptic operator $Q \in Cl(M, E)$ is admissible if there is an angle with vertex 0 which contains the spectrum of the leading symbol $\sigma_L(Q)$ of $Q$. A choice of a half line $L_\theta = \{re^{i\theta}, r > 0\}$ which does not intersect the spectrum of $Q$ (which is discrete since $M$ is compact) is a spectral cut for $Q$, and $\theta$ is an Agmon angle. An admissible operator is also called a weight.

Ell$^{\text{adm}, >0}$($M, E$) (resp. Ell$^{\text{adm}, \ast, >0}$($M, E$)) is the class of admissible (resp. invertible admissible) elliptic operators of positive order in $Cl(M, E)$.

Examples of admissible elliptic operators are classical ΨDOs with positive leading symbol such as generalized Laplacians and formally self-adjoint elliptic classical ΨDOs such as Dirac operators in odd dimensions.

An admissible invertible elliptic operator of positive order and with spectral cut $L_\theta$ has well-defined complex powers (Seeley [Se]) defined for $\text{Re}(z)$ sufficiently negative by the contour integral

$$Q^z_{\theta} := \frac{i}{2\pi} \int_{C_\theta} \lambda^z (Q - \lambda I)^{-1} d\lambda$$

where $C_\theta$ is a contour encircling $L_\theta$. One then extends the complex power $Q^z_{\theta}$ to any half plane $\text{Re} \ z < k, k \in \mathbb{N}$ via the formula $Q^k_{\theta}Q^{z-k}_{\theta} = Q^z_{\theta}$. These complex powers clearly depend on the choice of spectral cut. Setting $z = 0$, we get

$$Q^0_{\theta} = I - \Pi_Q = \frac{i}{2\pi} \int_{C_\theta} (Q - \lambda I)^{-1} d\lambda,$$

where $\Pi_Q$ is the projection onto the generalized kernel of $Q$. The logarithm of $Q$, which also depends on the spectral cut, is defined by

$$\log_{\theta} Q := \frac{d}{dz} \Big|_{z=0} Q^z_{\theta}.$$ 

This dependence will be omitted from the notation from now on.

1.2. The Wodzicki residue. Let $A \in Cl(M, E)$ have order $\alpha$ and symbol $\sigma_A(x, \xi) \sim \sum_{j=0}^\infty \psi(\xi) \sigma_{\alpha-j}(A)(x, \xi)$, where $\sigma_{\alpha-j}$ is the positively homogeneous component of order $\alpha - j$ and $\psi$ is a smooth cut-off function which is one outside a ball around 0 and vanishes on a smaller such ball. Let $dx = dx^1 \wedge \ldots \wedge dx^n$ be the locally defined coordinate form on $M$, and let $d\xi$ be the volume form on $T^*M$ (or the restriction of
$d\xi$ to the unit cosphere bundle $S^*M \subset T^*M$ or to the unit cosphere $S^*_xM$ at a fixed $x \in M$). Then

$$\text{res}_x(A)dx := \left( \int_{S^*_xM} \text{tr}_x\sigma_{-n}(A)(x, \xi) d\xi \right) dx,$$

is (nontrivially) a global top degree form on $M$ whose integral

$$\text{res}(A) := \frac{1}{(2\pi)^n} \int_M \text{res}_x(A) dx$$

is the Wodzicki residue $\text{Wo}$ of $A$ (see Kassel $\text{K}$, Lesch $\text{L}$ for a review and further development).

The Wodzicki residue has several striking properties. From its definition, the Wodzicki residue vanishes on differential operators and operators of n onintegral order, but it is nonzero in general. The Wodzicki residue is local, in that it is integral over $M$ of a density which is computed pointwise from a homogeneous component of the symbol. Most importantly, the Wodzicki residue is cyclic on $\text{CL}(M, E)$ in the following sense:

$$\text{res}([A, B]) = 0, \quad \forall A, B \in \text{Cl}(M, E).$$

The Wodzicki residue extends to logarithms of admissible elliptic operators $Q$ by

$$\text{res}(\log Q) := \frac{1}{(2\pi)^n} \int_M \text{res}_x(\log Q)d\xi dx = \frac{1}{(2\pi)^n} \int_{S^*_xM} \text{tr}_x\sigma_{-n}(\log Q)(x, \xi)d\xi d\xi dx$$

(Okikiolu $\text{O}$). More generally, given $A \in \text{Cl}(M, E)$, if

$$\text{res}_x(A \log Q) dx := \left( \int_{S^*_xM} \text{tr}_x\sigma_{-n}(A \log Q)(x, \xi) d\xi \right) dx$$

defines a global form on $M$, we can integrate it over $M$ to define

$$\text{res}(A \log Q) := \frac{1}{(2\pi)^n} \int_{S^*_xM} \text{tr}_x\sigma_{-n}(A \log Q)(x, \xi)d\xi dx.$$

This holds in particular if $A$ is a differential operator [PS Thm. 2.5].

The cyclicity of the Wodzicki residue partially extends to logarithmic operators. The Wodzicki residue vanishes on brackets of the type $[A, B \log_\theta Q]$ where $A, B \in \text{Cl}(M, E), Q \in \text{Ell}^{*,\text{adm}}_0(M, E)$, and $[A, B]$ is a differential operator [O] [PS] (Thm. 4.9).

1.3. The canonical trace. By a procedure well known to physicists and mathematicians (see Paycha $\text{P}$ for a review), a classical symbol $\sigma$ on $\mathbb{R}^n$, has a cut-off integral in momentum space $\{\xi\}$. To set the notation, let $\psi$ be the cutoff function of $\text{I}2$ and set

$$\sigma_{(N)}(x, \xi) := \sigma(x, \xi) - \sum_{j=0}^N \psi(\xi) \sigma_{n(z)-j}(x, \xi).$$
Proposition 1.1. Let $\sigma$ be a classical symbol on an open subset $U \subset \mathbb{R}^n$ of order $\alpha$. For $x \in U$, let $B_x^*(R) \subset T^*_x U$ be the ball of radius $R$ centered at $0$. As $R \to \infty$,

$$
\int_{B_x^*(R)} \text{Tr}_x \sigma(x,\xi) \, d\xi \sim \sum_{j=0, \alpha-j+n \neq 0}^{\infty} a_{\sigma,j}(x) R^{\alpha-j+n} + b_\sigma(x) \log R + c_\sigma(x),
$$

(1.1)

with

$$
a_{\sigma,j}(x) = \alpha + n - j; \quad b_\sigma(x) = \int_{S^*_x U} \text{Tr}_x \sigma_{-n}(x,\xi) \, d\xi
$$

and with finite part/cut-off integral

$$
c_\sigma(x) := \int_{T^*_x U} \text{Tr}_x \sigma(x,\xi) \, d\xi
$$

$$
:= \text{fp}_{R \to \infty} \int_{B_x^*(R)} \text{Tr}_x \sigma(x,\xi) \, d\xi
$$

$$
= \int_{T^*_x U} \text{Tr}_x \sigma_N(x,\xi) \, d\xi + \sum_{j=0}^{N} \int_{B_x^*(1)} \psi(\xi) \text{Tr}_x \sigma_{\alpha-j}(x,\xi) \, d\xi
$$

$$
- \sum_{j=0, \alpha-j+n \neq 0}^{\infty} a_{\sigma,j}(x). \quad (1.2)
$$

The finite part is independent of reparametrization of $R$ provided $b_\sigma(x)$ vanishes.

Whenever $\alpha$ is nonintegral, via a partition of unity on $M$ one can patch the local cut-off integrals $\int_{T^*_x U} \text{Tr}_x \sigma_A(x,\xi) \, d\xi$ into a cut-off integral

$$
\omega_{KV}(A)(x) = \int_{T^*_x M} \text{Tr}_x \sigma_A(x,\xi) \, d\xi
$$

(1.3)

on $T^*_x M$ and then integrate over $M$ to get the Kontsevich-Vishik canonical trace $[KV]$

$$
\text{TR}(A) := \frac{1}{(2\pi)^n} \int_{M} \omega_{KV}(A)(x) \, dx = \frac{1}{(2\pi)^n} \int_{M} dx \int_{T^*_x M} \text{tr}_x \sigma(A)(x,\xi) \, d\xi. \quad (1.4)
$$

We consider holomorphic families of classical symbols $[KV]$.

Definition 1.2. A family of complex valued classical symbols $z \mapsto \sigma(z)$ on an open subset $U$ of $\mathbb{R}^n$ is holomorphic on a subset $W \subset \mathbb{C}$ if:

1. The order $\alpha(z)$ of $\sigma(z)$ is holomorphic$^1$; in $z \in W$;
2. For any nonnegative integer $j$, the map $(z, x, \xi) \mapsto \sigma(z)_{\alpha(z)-j}(x,\xi)$ is holomorphic in $z$ and the map $z \mapsto (\sigma(z))_{\alpha(z)-j}$ is a continuous map from $W$ to $C^\infty(T^*U)$ in the standard topology on $C^\infty(T^*U)$.

$^1$i.e. differentiable in $z$
3. For \(N \gg 0\), the truncated kernel

\[
K(z)^{(N)}(x, y) := \int_{T^*_xU} e^{i\xi \cdot (x-y)} \sigma(z)(N)(x, \xi) d\xi
\]

defines a holomorphic map \(W \to C^k(N)(U \times U)\), \(z \mapsto K(z)^{(N)}\) for some \(k(N)\) with \(\lim_{N \to \infty} k(N) = \infty\).

A family \(A(z) \in Cl(M, E)\) of classical \(\Psi\)DOs is holomorphic for \(z \in W \subset \mathbb{C}\) if it is defined in any local trivialization by a holomorphic family of classical symbols \(\sigma_A(z)\).

The cut-off integral \(\int_{T^*_xU} \text{Tr}_x \sigma(z)(x, \xi) d\xi\) is defined whenever the order \(\alpha(z)\) is non-integral. The following extends results of Kontsevich-Vishik on the explicit Laurent expansions of holomorphic families \([PS\text{ Thm. 2.4}]:\)

**Proposition 1.2.** Let \(\sigma(z)\) be a holomorphic family of classical symbols on an open set \(U \subset \mathbb{R}^n\) of linear order \(\alpha(z) = \alpha'(0) z + \alpha(0)\) with \(\alpha'(0) \neq 0\). Then the map \(z \mapsto \int_{T^*_xU} \text{Tr}_x \sigma(z)(x, \xi) d\xi\) is meromorphic with Laurent expansion at \(z = 0\) given by

\[
\int_{T^*_xU} \text{Tr}_x \sigma(z)(x, \xi) d\xi = \left( -\frac{1}{\alpha'(0)} \int_{S^*_xU} \text{Tr}_x \sigma(0)_{-n}(x, \xi) d\xi \right) \cdot \frac{1}{z} + \sum_{k=0}^{K} \frac{z^k}{k!} \left( \int_{T^*_xU} \text{Tr}_x \sigma^{(k)}(0)(x, \xi) d\xi \right.

\[
- \frac{1}{\alpha'(0)(k+1)} \int_{S^*_xU} \text{Tr}_x \sigma^{(k+1)}(0)_{-n}(x, \xi) d\xi \right)

\[
+ O(z^K),
\]

for \(K \geq 0\). Applying this to the symbols \(\sigma_A(z)\) of a holomorphic family \(A(z)\) of classical \(\Psi\)DOs, taking the fibrewise trace and replacing \(U\) by \(M\) via a partition of the unity provides an analogous formula for the first \(k + 1\) terms of the Laurent expansion around 0 of \(\omega_{KV}(A(z))(x)\) defined by \([1.3]\) with \(A\) replaced by \(A(z)\) and hence, after integration over \(M\), of the canonical trace \(\text{TR}(A(z))\).

**Remark 1.1.** (i) Even though \(\sigma(z)\) is a classical symbol, \(\sigma^{(k)}(0)\) need not be \([PS]\).

(ii) Since \(\omega_{KV}(A(z))(x)dx = \left( \int_{T^*_xM} \text{Tr}_x \sigma_{A(z)}(x, \xi) d\xi \right) dx\) defines a global form, the coefficient of \(z^k\) in the Laurent expansion of \(\omega_{KV}(A(z))(x)dx\) also gives rise to a global form.

(iii) For a classical \(\Psi\)DO \(A\) of order \(\alpha\), the operator \(A(z) = AQ^{-z}\) defines a holomorphic family of classical \(\Psi\)DOs of order \(\alpha(z) = -qz + \alpha\). From Proposition \([1.2]\) we recover the well known result relating the Wodzicki residue to a complex residue:

\[
\text{Res}_{z=0} \omega_{KV}(AQ^{-z})(x) = -\frac{1}{q} \text{Res}_x(A),
\]

(1.5)
which after integration over $M$ yields

$$\text{Res}_{z=0} \text{TR}(AQ^{-z}) = -\frac{1}{q}\text{res}(A).$$

If $A$ is a differential operator, \(\int_{T^*_x M} \text{Tr}_x \sigma(0)(x,\xi) \, d\xi = \int_{T^*_x M} \text{Tr}_x \sigma_A(0)(x,\xi) \, d\xi\) vanishes. Therefore \(\left(\int_{S^*_x M} \text{Tr}_x (\sigma'(0))_{-n}(x,\xi)\right) \, d\xi \, dx\) defines a global form, whose integral is \(-\text{res} (A \log \theta Q)\) [PS].

1.4. Weighted traces. For a weight $Q$ with spectral cut and a nonnegative integer $k$, set

$$A^k(M,E) := \left\{ \sum_{j=0}^{k} A_j \log^j Q, \ A_j \in \text{Cl}(M,E), \ 0 \leq j \leq k \right\}.$$

Operators in $A^k(M,E)$ coincide with Lesch’s log-polyhomogeneous operators [L]. $A^k(M,E)$ is in fact independent of the choice of $Q$ (Ducourtioux [D, PS]) and coincides with the class $\text{Cl}^{*,-k}(M,E)$ of Lesch [L]. Note that $A^0(M,E) = \text{Cl}(M,E)$. The order of $A_j \log^j Q$ is defined to be the order of $A_j$.

Cut-off integrals extend [L] to symbols of operators in $A^k(M,E)$, once (1.1) is extended to include the terms $d_{\sigma,j} \log^j R, j = 1, \ldots, k + 1$. As for classical operators, for a noninteger order $A \in A^k$, $\omega_{KV}(A)(x) \, dx := \int_{T^*_x M} \text{Tr}_x (\sigma(A)(x,\xi)) \, d\xi$ defines a global form, and one can define the canonical trace $\text{TR}(A)$ by (1.4). The linear functional $\text{TR}$ is cyclic:

$$\text{TR}([A,B]) = 0, \ \forall A \in A^k(M,E), B \in A^j(M,E), \ \text{ord}(A) + \text{ord}(B) \notin \mathbb{Z}. \quad (1.6)$$

Weighted traces are defined by the finite part in the Laurent expansion of the canonical trace of a holomorphic family; this is in contrast to the Wodzicki residue, which occurs as the residue in the Laurent expansion.

**Definition 1.3.** For $A \in A^k(M,E)$, the $Q$-weighted trace of $A$ is

$$\text{tr}^{Q}(A) := \text{fp}_{z=0} \text{TR}(AQ^{-z}) + \text{tr}(A \Pi_Q)$$

$$:= \lim_{z \to 0} \left( \text{TR}(AQ^{-z}) - \sum_{j=0}^{k} \frac{a_{j+1}}{z^{j+1}} \right) + \text{tr}(A \Pi_Q),$$

where $a_{j+1}$ is the residue of $\text{TR}(AQ^{-z})$ of order $j + 1$.

The existence of the Laurent expansion is known [L]. As usual, this definition depends on a choice of spectral cut for $Q$. For $A \in C\ell(M,E)$, the weighted trace can also be defined by the finite part of $\text{tr}(AQ^{-z})$, where $\text{tr}$ is the ordinary operator trace. Indeed, for $\text{Re}(z) \gg 0$, $AQ^{-z}$ is trace-class, in which case $\text{TR}(AQ^{-z}) = \text{tr}(AQ^{-z})$.

The known meromorphic continuation of the right hand side (Grubb and Seeley [GS]) gives the equivalence of the two definitions. We prefer our current definition of the
weighted trace, since \( \text{TR}(AQ^{-z}) \) is well defined outside a countable set of poles, and hence does not require a meromorphic continuation.

Weighted traces do not have the local properties of the Wodzicki residue in general. For example, a formally self-adjoint, positive order, invertible elliptic operator \( A \in \mathcal{C}^\ell(M, E) \) is admissible with Agmon angle \( \theta = \frac{\pi}{2} \), as is its modulus \( |A| := \sqrt{A^*A} \), which has positive leading symbol. Then \( A|A|^{-1} \in \mathcal{C}^\ell(M, E) \), and we can set

\[
\eta_A(z) := \text{TR} \left( A|A|^{-z-1} \right).
\]

The \( \eta \)-invariant of \( A \) is given by its finite part:

\[
\eta_A(0) := \text{tr} |A| \left( A|A|^{-1} \right),
\] (1.7)

which is not local in general.

**Remark 1.2.** The map \( z \mapsto \eta_A(z) \) is holomorphic at \( z = 0 \) since the Wodzicki residue of a \( \Psi \)DO projection such as \( \text{res}(A|A|^{-1}) \) vanishes. It follows that \( \eta_A(0) = \text{tr} A|A|^{-1} \), i.e. the \( \eta \)-invariant can be defined using the easier \( A \) as a weight (Cardona, Ducourtioux and Paycha [CDP, Prop. 1].

For differential operators \( A \), \( \text{res}(A \log Q) \) is well defined [PS Thm. 3.7], and

\[
\text{tr}^Q(A) = -\frac{1}{q} \text{res}(A \log Q) = -\frac{1}{q} \int_M \text{res}_x(A \log Q) dx.
\] (1.8)

In this case, \( \text{tr}^Q(A) \) has a partial locality as an integral of \( \sigma_{-n}(A \log Q) \). In particular, for \( A = I \) we have

\[
\text{tr}^Q(I) = -\frac{1}{q} \text{res}(\log Q),
\] (1.9)

an expression related to the exotic determinant \( \det_{\text{res}} Q = e^{\text{res}(\log Q)} \) [Sc] (and references therein). In turn,

\[
\text{tr}^Q_I(I) - \text{tr}(\Pi_Q) = \zeta_Q(0),
\] (1.10)

where the zeta function is given by the usual meromorphic continuation of

\[
\zeta_Q(z) = \text{TR}(Q^{-z}) = \text{tr}(Q^{-z}),
\]

which is well defined for \( \text{Re}(z) > \frac{n}{q} \) (\( n = \dim(M) \), \( q = \text{ord}(Q) > 0 \)).

Since the Wodzicki residue vanishes for differential operators \( Q \), \( \zeta_Q(z) \) is holomorphic at \( z = 0 \), and an easy computation yields

\[
\zeta'_Q(0) = -\text{tr}^Q(\log Q)
\] (1.11)

for an invertible weight \( Q \).

In summary, the key spectral invariants \( \eta_A(0), \zeta_Q(0), \zeta'_Q(0) \) all occur as weighted traces.

The following proposition will be used in §2.
Proposition 1.3. Let $A \in C^\ell(M,E)$ and let $Q$ be an invertible weight. We have the Laurent expansion

$$\text{tr}(A Q^{-z}) = \frac{\text{res}(A)}{q z} + \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \text{tr}^Q(A \log^j Q) z^j + o(z^j).$$

Proof: By Remark 1, the map $z \mapsto \text{tr}(A Q^{-z})$ is meromorphic with a simple pole at $z = 0$ with residue $\frac{\text{res}(A)}{q}$, so

$$\text{tr}(A Q^{-z}) = \frac{\text{res}(A)}{q z} + \sum_{j=0}^{\infty} a_j(A, Q) z^j + o(z^j).$$

Since Laurent expansions can be differentiated term by term away from their poles, we obtain

$$\text{tr}^Q(A \log^j Q) = \text{fp}_{z=0} \text{tr}(A \log^j Q Q^{-z}) = (-1)^j \text{fp}_{z=0} \left( \partial_z^j \text{tr}(A Q^{-z}) \right) = (-1)^j j! a_j(A, Q).$$

Remark 1.4. $z \mapsto \text{tr}(A \log^j Q Q^{-z})$ has a Laurent expansion with poles of order at most $j + 1$:

$$\text{tr}(A \log^j Q Q^{-z}) = \sum_{l=1}^{j+1} b_{j+l}(A, Q) z^l + \sum_{i=0}^{k} a_{j,i}(A, Q) z^i + o(z^k).$$

The $a$ and $b$ coefficients are related. For example, the identity $\partial_z \text{tr}(A \log^j QQ^{-z}) = -\text{tr}(A \log^{j+1} Q^{-z})$ implies

$$a_{j+1,i} = -(i+1) a_{j,i+1}(A, Q), b_{j+1,i+1}(A, Q) = l b_{j+l}(A, Q).$$

1.5. Differentiable families of canonical traces. The definition of a $C^k$ differentiable family of classical symbols is completely analogous to the the holomorphic definition. Namely, the one-parameter family of symbols $t \mapsto \sigma_t$, $t \in \mathbb{R}$, with $\sigma_t$ defined on an open set $U \subset \mathbb{R}^n$, is $C^k$ for a fixed $k \in \mathbb{Z}^+$ if (i) the order $\sigma_t$ of $\sigma_t$ is $C^k$ in $t$, (ii) each homogeneous component $\sigma_{t,\alpha-j}(x, \xi)$ is $C^k$ in $t$, (iii) for $N \gg 0$ and $K_t^{(N)}(x, y)$ the truncated kernel, the map $U \to C^{K(N)}$, $t \mapsto K_t^{(N)}(x, y)$ is $C^k$ for some $K(N)$ with $\lim_{N \to \infty} K(N) = \infty$. A family $t \mapsto A_t$ of classical $\Psi$DOs is $C^k$ if it is defined in any local trivialization by $C^k$ family of symbols.

Remark 1.4. By (iii), for a $C^k$ family $t \mapsto \sigma_t$, the map $t \mapsto (\sigma_t)_{(N)}$ is also $C^k$ in $C^\infty(T^*U)$, and then by (ii), this differentiability holds for $t \mapsto \sigma_t$. As a consequence, for a $C^k$ family $t \mapsto \sigma_t$ and for any compact set $K \subset T^*U$, $t \mapsto \|\partial_t^k \sigma_t\|_K := \sup_{(x, \xi) \in K} |\partial_t^k \sigma_t(x, \xi)|$ is continuous and hence uniformly bounded on any interval $[t_0 - \eta, t_0 + \eta]$, $\eta > 0$, as are the homogeneous components $(\sigma_t)_{a-j}$. Moreover, the
minus one order symbol \((|\xi|+1)^{N-\alpha}(\partial^k_t \sigma_t)_{(N)}(x,\xi)\) is bounded on \(T_x^*U\) and gives rise to a continuous map

\[
t \mapsto (|\xi|+1)^{N-\alpha}\|\partial^k_t \sigma_t\|_{T_x^*U} \quad := \sup_{\xi \in T_x^*U} \left( (|\xi|+1)^{N-\alpha} \left| \partial^k_t \sigma_t(x,\xi) \right| \right).
\]

Hence \((|\xi|+1)^{N-\alpha}\|\partial^k_t \sigma_t\|_{(N)}\) is uniformly bounded above on \([t_0-\eta,t_0+\eta]\) by a constant \(C_{t_0,\eta,(N)}\). Therefore, for fixed \(x\) the map \(\xi \mapsto |(\partial^k_t \sigma_t)_{(N)}(x,\xi)|\) is bounded above by a map \(\xi \mapsto C_{t_0,\eta,(N)}(|\xi|+1)^{\alpha-N}\), which lies in \(L^1(T_x^*U)\) for \(N \gg 0\).

This remark implies that the cut-off integral and the canonical trace commute with differentiation as long as the symbols and operators have constant noninteger order.

**Theorem 1.4.**

1. Let \(t \mapsto \sigma_t\) be a \(C^1\) family of symbols on \(U\) with constant noninteger order \(\alpha\). Then

\[
\frac{d}{dt} \int_{T_x^*U} \text{Tr}_x \sigma_t(x,\xi) \, d\xi = \int_{T_x^*U} \text{Tr}_x \dot{\sigma}_t(x,\xi),
\]

where \(\dot{\sigma}_t = \frac{d}{dt} \sigma_t\).

2. Let \(t \mapsto A_t \in \text{Cl}(M, E)\) be a \(C^1\) family of constant noninteger order operators. Then

\[
\frac{d}{dt} \text{TR}(A_t) = \text{TR}(\dot{A}_t). \tag{1.12}
\]

3. Assume that for fixed \(t\), \(z \mapsto \sigma_t(z)\) is a holomorphic family of classical symbols on \(U\) parametrized by \(z \in W \subset \mathbb{C}\) with holomorphic order \(\alpha(z)\) independent of \(t\) and that \(t \mapsto \sigma_t(z)\) is a \(C^1\) family for fixed \(z \in W\). Then \(z \mapsto \int_{T_x^*U} \text{Tr}_x \sigma_t(z)(x,\xi) \, d\xi\) and \(z \mapsto \int_{T_x^*U} \text{Tr}_x \dot{\sigma}_t(z)(x,\xi) \, d\xi\) are meromorphic in \(z\), and the Laurent expansion of \(\int_{T_x^*U} \text{Tr}_x \dot{\sigma}_t(z)(x,\xi) \, d\xi\) around \(z = 0\) is obtained by term by term \(t\)-differentiation of the Laurent expansion of \(\int_{T_x^*U} \text{Tr}_x \sigma_t(z)(x,\xi) \, d\xi\).

4. Assume that for fixed \(t\), \(z \mapsto A_t(z) \in \text{Cl}(M, E)\) defines a holomorphic family on \(W \subset \mathbb{C}\) with holomorphic order \(\alpha(z)\) independent of \(t\), and assume that \(t \mapsto \partial^k_{z=0} A_t(z)\) is a \(C^1\) family for \(k \in \mathbb{Z}_{\geq 0}\). Then \(z \mapsto \text{TR}(A_t(z))\) and \(z \mapsto \text{TR}(\dot{A}_t(z))\) are meromorphic in \(z\), and the Laurent expansion of \(\text{TR}(A_t(z))\) around \(z = 0\) is obtained by term by term \(t\)-differentiation of the Laurent expansion of \(\text{TR}(A_t(z))\).
Proof: 1. Once we justify pushing the derivative past the integral, by \textbf{1.12} (and noting that we may choose $N$ independent of $t$ by our assumption on $\alpha(z)$), we have

$$
\frac{d}{dt} \int_{T^*_tU} \text{Tr}_x \sigma_t(x, \xi) \, d\xi
= \frac{d}{dt} \int_{T^*_tU} \text{Tr}_x (\sigma_t)_{(N)} (x, \xi) \, d\xi + \frac{d}{dt} \sum_{j=0}^N \int_{B^*_tU} \psi(\xi) \text{Tr}_x (\sigma_t)_{\alpha-j} (x, \xi) \, d\xi
+ \frac{d}{dt} \sum_{j=0, \alpha-j+n \neq 0}^{\infty} \int_{S^*_tU} \text{Tr}_x (\sigma_t)_{\alpha-j} (x, \xi) \, dS\xi
+ \frac{d}{dt} \sum_{j=0, \alpha-j+n \neq 0}^{\infty} \int_{S^*_tU} \frac{d}{dt} \text{Tr}_x (\sigma_t)_{\alpha-j} (x, \xi) \, dS\xi
= \int_{T^*_tU} \text{Tr}_x \dot{\sigma}_t(x, \xi) \, d\xi,
$$

Recall that for $\xi \in A \subset \mathbb{R}^n$, $t_0 \in \mathbb{R}$ and $\epsilon > 0$, if $|t - t_0| \leq \epsilon$ implies $|\frac{d}{dt} f(t, \xi)| \leq g(\xi)$ with $g \in L^1(A)$, then $\frac{d}{dt}|_{t=t_0} \int_A f(t, \xi) \, d\xi = \int_A \frac{d}{dt}|_{t=t_0} f(t, \xi) \, d\xi$. This applies to the compact subsets $A = B^*_tU$ and $A = S^*_tU$ and $f(t, \xi) = \text{Tr}_x (\sigma_t)_{\alpha-j} (x, \xi)$ and to $A = T^*_tU$ and $f(t, \xi) = \text{Tr}_x (\sigma_t)_{(N)} (x, \xi)$ (where the the required uniform estimates follow from Remark \textbf{1.4} with $k = 1$).

2. By 1,

$$
\frac{d}{dt} \int_{T^*_tM} \text{tr}_x \sigma(A_t) (x, \xi) \, d\xi = \int_{T^*_tM} \text{tr}_x \dot{\sigma}(A_t) (x, \xi) \, d\xi
= \int_{T^*_tM} \text{tr}_x \dot{\sigma}(A_t) (x, \xi) \, d\xi.
$$

Since $A_t$ has constant order $\alpha$, $\dot{A}_t$ has order $\alpha$ modulo integers. Therefore $\dot{A}_t$ has noninteger order, so $\left( \int_{T^*_tM} \text{tr}_x \sigma(\dot{A}_t) (x, \xi) \, d\xi \right) \, dx$ is a global form on $M$ and $\text{TR}(\dot{A}) = \frac{1}{(2\pi)^n} \int_M dx \left( \int_{T^*_tM} \text{tr}_x \sigma(\dot{A}_t) (x, \xi) \, d\xi \right) \, dx$ is well defined. Since $\text{TR}(A) = \frac{1}{(2\pi)^n} \int_M dx \left( \int_{T^*_tM} \text{tr}_x \sigma(A) (x, \xi) \, d\xi \right) \, dx$, integrating (1.13) over $M$ yields (1.12).

4. We now prove 4, leaving the similar proof of 3 to the reader. If $A_t(z)$ has noninteger order, by 2

$$
\frac{d}{dt} \text{TR}(A_t(z)) = \text{TR}(\dot{A}_t(z))
$$

except at the poles, so this is an equality of meromorphic functions. By Proposition \textbf{1.2} the coefficients of the Laurent expansion on either side can be expressed in terms
of the cut-off integral of $\text{Tr}_x \sigma (\partial^k_x A_t(z)|_{z=0})$ (resp. $\text{Tr}_x \sigma (\partial_x^k \dot{A}_t(z))|_{z=0}$) on $T^*_x U$ and ordinary integrals over compact sets of the $-n$ component of $\text{Tr}_x \partial^j_x \sigma (\dot{A}_t(z))|_{z=0}$ (resp. $\text{Tr}_x \partial_x^j \sigma (\dot{A}_t(z))|_{z=0}$), $j \in \mathbb{Z}^\geq 0$. As above $t \mapsto (\partial^k_x A_t(z))|_{z=0}$ is $C^1$, so we can push the derivative past the integral as desired. 

Let $h : W \subset \mathbb{C} \to \mathbb{R}$ be a $C^1$ map such that

$$h(A) := \frac{i}{2\pi} \int_{C_\theta} h(\lambda) (A - \lambda)^{-1} d\lambda; \quad h'(A) := \frac{i}{2\pi} \int_{C_\theta} h'(\lambda) (A - \lambda)^{-1} d\lambda$$

takes any weight $A$ to $h(A), h'(A) \in C^\ell(M, E)$. Here $\theta$ is an Agmon angle for $A$ and $C_\theta$ is the associated contour (where we assume $C_\theta \subset W$). Examples of such maps are

$$h(z) = \frac{z}{|z|}, \quad W = \mathbb{R}^*; \quad h(z) = z^c, \quad W = \mathbb{C}; \quad h(z) = \frac{z}{|z|}, \quad W = \mathbb{R}^*.$$ 

for fixed $c \in \mathbb{R}$.

**Proposition 1.5.** Let $t \mapsto A_t$ be a differentiable family of weights of constant non-integer order and with common Agmon angle. Then

$$\frac{d}{dt} \text{TR} (h(A_t) A_t^{-z}) = \text{TR} \left( h'(A_t) \dot{A}_t A_t^{-z} \right) - z \text{TR} \left( h(A_t) \dot{A}_t A_t^{-z-1} \right), \quad (1.14)$$

This is equivalent to the following set of equations:

$$\frac{d}{dt} \text{res} (h(A_t)) = \text{res} \left( h'(A_t) \dot{A}_t \right), \quad (1.15)$$

$$\frac{d}{dt} \text{tr}^A_t (h(A_t)) = \text{tr}^A_t \left( h'(A_t) \dot{A}_t \right) - \frac{1}{q} \text{res} \left( h(A_t) \dot{A}_t A_t^{-1} \right), \quad (1.16)$$

$$\frac{d}{dt} \text{tr}^A_t (h(A_t) \log^j A_t) = \text{tr}^A_t \left( h'(A_t) \dot{A}_t \log^j A_t \right) + j \text{tr}^A_t \left( h(A_t) \dot{A}_t A_t^{-1} \log^{j-1} A_t \right), \quad (1.17)$$

for $j \in \mathbb{Z}^+$. 

Proof: Applying Theorem 1.4 gives the following equalities of meromorphic functions:

\[
\frac{d}{dt} \text{TR} \left( h(A_t) A_t^{-z} \right) = \text{TR} \left( \frac{d}{dt} \left( h(A_t) A_t^{-z} \right) \right)
\]

\[
= \text{TR} \left( \frac{d}{dt} (h(A_t)) A_t^{-z} \right) + \text{TR} \left( h(A_t) \frac{d}{dt} (A_t^{-z}) \right)
\]

\[
= -i \frac{2\pi}{2\pi} \text{TR} \left( \int_{C_0} h(\lambda) \frac{1}{(A_t - \lambda)} \dot{A}_t \frac{1}{(A_t - \lambda)} d\lambda A_t^{-z} \right)
\]

\[
- i \frac{2\pi}{2\pi} \text{TR} \left( h(A_t) \int_{C_0} \frac{1}{(A_t - \lambda)} \dot{A}_t A_t^{-z} \right)
\]

\[
= -i \frac{2\pi}{2\pi} \text{TR} \left( \left( \int_{C_0} h(\lambda) \frac{1}{(A_t - \lambda)}^2 d\lambda \right) \dot{A}_t A_t^{-z} \right)
\]

\[
- i \frac{2\pi}{2\pi} \text{TR} \left( h(A_t) \left( \int_{C_0} \frac{1}{(A_t - \lambda)}^2 d\lambda \right) \dot{A}_t \right)
\]

\[
= \frac{i}{2\pi} \text{TR} \left( \left( \int_{C_0} h'(\lambda) \frac{1}{(A_t - \lambda)} d\lambda \right) \dot{A}_t A_t^{-z} \right)
\]

\[
- i \frac{z}{2\pi} \text{TR} \left( h(A_t) \left( \int_{C_0} \frac{1}{(A_t - \lambda)} d\lambda \right) \dot{A}_t \right)
\]

\[
= \text{TR} \left( h'(A_t) \dot{A}_t A_t^{-z} \right) - z \text{TR} \left( h(A_t) A_t^{-z-1} \dot{A}_t \right)
\]

\[
= \text{TR} \left( h'(A_t) \dot{A}_t A_t^{-z} \right) - z \text{TR} \left( h(A_t) \dot{A}_t A_t^{-z-1} \right).
\]

In (1.19), (1.21), we use the cyclicity of TR on noninteger order operators, and in (1.20) we use integration by parts. This proves (1.14).

By Theorem 1.4.3, the Laurent expansion of \( \frac{d}{dt} \text{TR} \left( h(A_t) A_t^{-z} \right) \) is the term by term derivative of the Laurent expansion of \( \text{TR} \left( h(A_t) A_t^{-z} \right) \). The rest of the Proposition then follows from identifying the coefficients in the Laurent expansions in (1.14) and using Proposition 1.3.

\[\square\]

2. Conformal invariants and anomalies

In this part of the paper, we use canonical traces to build functionals of conformally covariant operators and study their conformal properties.

2.1. The conformal anomaly and associated two-tensor. Let \( M \) be a Riemannian manifold and \( \text{Met}(M) \) denote the space of Riemannian metrics on \( M \). \( \text{Met}(M) \) is trivially a Fréchet manifold as the open cone of positive definite symmetric (covariant)
two-tensors inside the Fréchet space
\[ C^\infty(T^*M \otimes_s T^*M) := \{ h \in C^\infty(T^*M \otimes T^*M) : h_{ab} = h_{ba} \} \]
of all smooth symmetric two-tensors. The Weyl group \( W(M) := \{ e^f : f \in C^\infty(M) \} \) acts smoothly on \( \text{Met}(M) \) by Weyl transformations
\[ W(g, f) = \bar{g} := e^{2f} g, \]
and given a reference metric \( g \in \text{Met}(M) \), a functional \( F : \text{Met}(M) \rightarrow \mathbb{C} \) induces a map
\[ F_g = F \circ W(g, \cdot) : C^\infty(M) \rightarrow \mathbb{C}, \ f \mapsto F(e^{2f} g). \]

**Definition 2.1.** A functional \( F \) on \( \text{Met}(M) \) is conformally invariant for a reference metric \( g \) if \( F_g \) is constant on a conformal class, i.e.
\[ F(e^{2f} g) = F(g) \quad \forall f \in C^\infty(M). \]
A functional \( F \) on \( \text{Met}(M) \) is conformally invariant if it is conformally invariant for all reference metrics. A functional \( F : \text{Met}(M) \times M \rightarrow \mathbb{C} \) is called a pointwise conformal covariant of weight \( w \) if
\[ F(e^{2f} g, x) = w \cdot f(x) F(g, x) \quad \forall f \in C^\infty(M), \ \forall x \in M. \]

For conformal covariants, we always assume that \( F(g, x) \) is given by a universal formula in the components of \( g \) and their derivatives at \( x \).

For a fixed Riemannian metric \( g = (g_{ab}), C^\infty(M) \) has the \( L^2 \) metric
\[ (f, \tilde{f})_g = \int_M f(x) \tilde{f}(x) d\text{vol}_g(x). \]
This extends to the \( L^2 \) metric on \( \text{Met}(M) \) given by
\[ \langle h, k \rangle_g := \int_M g^{ac}(x) g^{bd}(x) h_{ab}(x) k_{cd}(x) d\text{vol}_g(x), \quad (2.1) \]
with \( (g^{ab}) = (g_{ab})^{-1} \). The \( L^2 \) metric induces a weak \( L^2 \)-topology on \( \text{Met}(M) \), and \( L^2(T^*M \otimes_s T^*M) \), the \( L^2 \)-closure of \( C^\infty(T^*M \otimes_s T^*M) \) with respect to \( \langle , \rangle_g \), is independent of the choice of \( g \) up to Hilbert space isomorphism. The choice of a reference metric yields the inner product \( \langle , \rangle_g \) on the tangent space \( T_g \text{Met}(M) = C^\infty(T^*M \otimes_s T^*M) \), giving the weak \( L^2 \) Riemannian metric on \( \text{Met}(M) \), and forming the completion of each tangent space.

The metric \( g \) allows us to contract a two-tensor via
\[ \text{tr}_g(h) := h^b_b = g^{ab} h_{ab}. \]
The various inner products are related as follows:

**Lemma 2.1.** For \( g \in \text{Met}(M), h \in C^\infty(T^*M \otimes_s T^*M) \) and \( f \in C^\infty(M) \), we have
\[ \langle h, f g \rangle_g = \langle \text{tr}_g(h), f \rangle_g. \]
Proof: We have
\[
\langle h, fg \rangle_g = \int_M f(x) g^{ac}(x) g^{bd}(x) h_{ab}(x) g_{cd}(x) \, d\text{vol}_g(x)
\]
\[
= \int_M f(x) g^{ab}(x) h_{ab}(x) \, d\text{vol}_g(x)
\]
\[
= (\text{tr}_g(h), f)_g. \quad \square
\]

A functional \( F : \text{Met}(M) \to \mathbb{C} \) which is Fréchet differentiable has a differential
\[
dF(g) : T_g\text{Met}(M) = C^\infty(T^*M \otimes_s T^*M) \to \mathbb{C},
\]
\[
dF(g).h := \frac{d}{dt} \bigg|_{t=0} \frac{F(g + th) - F(g)}{t}.
\]
For such an \( F \), the differentiability of the Weyl map implies that the composition \( F_g : C^\infty(M) \to \mathbb{C} \) is differentiable at 0 with differential \( dF_g(0) : T_0C^\infty(M) = C^\infty(M) \to \mathbb{C} \).

Definition 2.2. The conformal anomaly for the reference metric \( g \) of a differentiable functional \( F \) on \( \text{Met}(M) \) is \( dF_g(0) \). In physics notation, the conformal anomaly in the direction \( f \in C^\infty(M) \) is
\[
\delta_f F_g := dF_g(0).f = dF(g).2f g
\]
\[
= \lim_{t \to 0} \frac{F(g + 2tfg) - F(g)}{t} = \frac{d}{dt} \bigg|_{t=0} F(e^{2fg}).
\]

Remark 2.1. \( F \) is conformally invariant if and only if \( dF_g(0).f = 0 \) for all \( g \in \text{Met}(M), f \in C^\infty(M) \).

If the differential \( dF(g) : C^\infty(T^*M \otimes_s T^*M) \to \mathbb{C} \) extends to a continuous functional \( \overline{dF(g)} : L^2(T^*M \otimes_s T^*M) \to \mathbb{C} \), then by Riesz’s lemma there is a unique two-tensor \( T_g(F) \) with
\[
\overline{dF(g)}.h = \langle h, T_g(F) \rangle_g, \quad \forall h \in L^2(T^*M \otimes_s T^*M).
\]
\( T_g(F) \) is precisely the \( L^2 \) gradient of \( F \) at \( g \).

Proposition 2.2. Let \( F \) be a functional on \( \text{Met}(M) \) which is differentiable at the metric \( g \) and whose differential \( dF(g) \) extends to a continuous functional \( \overline{dF(g)} : L^2(T^*M \otimes_s T^*M) \to \mathbb{C} \). Then the differential \( dF_g(0) \) also extends to a continuous functional \( \overline{dF_g(0)} : L^2(M) \to \mathbb{C} \). Identifying the conformal anomaly at \( g \) with a function in \( L^2(M) \), we have
\[
\overline{dF_g(0)} = 2 \text{tr}_g(T_g(F)).
\]
In particular, the functional \( F \) is conformally invariant iff \( \text{tr}_g(T_g(F)) = 0 \) for all metrics \( g \).
Proof: The differential $d(F_g)_0$ extends to a continuous functional because
\[ dF_g(0).f = dF(g)(2f g) \Rightarrow dF_g(0).f = dF(g).2f g \]
By Lemma 2.1,
\[ dF_g(0).f = dF(g).2f g = \langle T_g(F), 2f g \rangle_g = 2 \langle \text{tr}_g(T_g(F)), f \rangle_g, \]
as desired. □

Definition 2.3. Under the assumptions of the Proposition, the function
\[ x \mapsto \delta_x F_g := 2 \text{tr}_g(T_g(F))(x) \]
is called the local anomaly of the functional $F$ at the reference metric $g$.

Example 2.3. In field theory, for a classical action $A$ on a configuration space $\text{Conf}(M)$ with respect to a background metric $g$, $A : \phi \mapsto A(g)(\phi)$, where $\text{Conf}(M)$ is usually a space of tensors on $M$, the associated two-tensor $T_g(A)$ is called the stress-energy momentum tensor. In the path integral approach to quantum field theory, the effective action $W(g)$ is the average over the configuration space of the exponentiated classical action
\[ W(g) := \langle A(g) \rangle := - \log \langle e^{-A(g)} \rangle, \]
where
\[ \langle F \rangle = \int_{\text{Conf}(M)} F(\phi) D\phi \]
is the average of $F$ over the fields $\phi$ with respect to some heuristic volume measure $D\phi$ on $\text{Conf}(M)$. The associated two-tensor $T_g(W)$ is interpreted as the quantized stress-energy momentum tensor and denoted by
\[ T_g(W) = T_g(\langle A \rangle). \]
The local conformal anomaly associated to $dA_g(f) = 2 \langle \text{tr}_g(T_g(A)), f \rangle_g$ is defined to be
\[ \text{tr}_g(T_g(\langle A \rangle)) - \langle \text{tr}_g(T_g(A)) \rangle. \]
If the classical action is conformally invariant, as in string theory, $\text{tr}_g T_g(A) = 0$ and the local conformal anomaly reduces to $\text{tr}_g T_g(\langle A \rangle)$.

In general, the classical action is quadratic: $A(g)(\phi) = \langle A_g \phi, \phi \rangle_g$, where $\langle \cdot, \cdot \rangle_g$ is the inner product on the tensor fields $\phi$ induced by the metric $g$. $A_g$ is a geometric differential operator, i.e. an operator depending smoothly on the metric $g$ (via the curvature, for example). For bosonic strings, the fields are $\mathbb{R}^d$-valued smooth functions on a Riemann surface $M$, and $A_g$ is the Laplace-Beltrami operator. As pointed out in the introduction, even if $A(g)$ is conformally invariant, $A_g$ is usually only conformally covariant.
2.2. Conformally covariant operators. Given a vector bundle $E$ over a closed manifold $M$, we consider maps

$$\text{Met}(M) \rightarrow Cl(M, E), \ g \mapsto A_g.$$ 

**Definition 2.4.** The operator $A_g \in Cl(M, E)$ associated to a Riemannian metric $g$ is conformally covariant of bidegree $(a, b)$ if the pointwise scaling of the metric $\bar{g} = e^{2f} g$, for $f \in C^\infty(M, \mathbb{R})$ yields

$$A_g = e^{-bf} A_g e^{af} = e^{(a-b)f} A_g', \quad \text{for } A_g' := e^{-af} A_g e^{af},$$

for constants $a, b \in \mathbb{R}$.

We survey known conformally covariant differential and pseudodifferential operators; more details are in Change [C].

**Operators of order 1.** (Hitchin [H]) For $M^n$ spin, the Dirac operator $D_g := \gamma^i \cdot \nabla g_i$ is a conformally covariant operator of bidegree $(\frac{n-1}{2}, \frac{n+1}{2})$.

**Operators of order 2.** If $\dim(M) = 2$, the Laplace-Beltrami operator $\Delta_g$ is conformally covariant of bidegree $(0, 2)$. It is well known that in dimension two

$$R_g = e^{-2f} \left( R_g + 2\Delta_g f \right),$$

and by the Gauss-Bonnet theorem

$$\int_M R_g \ d\text{vol}_g = 2\pi \chi(M),$$

with the Euler characteristic $\chi(M)$ (much more than) a conformal invariant.

On a Riemannian manifold of dimension $n$, the Yamabe operator, also called the conformal Laplacian,

$$L_g := \Delta_g + c_n R_g,$$

is a conformally covariant operator of bidegree $(\frac{n-2}{2}, \frac{n+2}{2})$, where $R_g$ is the scalar curvature and $c_n := \frac{n-2}{4(n-1)}$.

**Operators of order 4.** (Paneitz [Pa, BO]) In dimension $n$, the Paneitz operators

$$P^n_g := \tilde{P}^n_g + (n-4)Q^n_g$$

are conformally covariant scalar operators of bidegree $(\frac{n-4}{2}, \frac{n+4}{2})$. Here $\tilde{P}^n_g := \Delta^2_g + d^* \left( (n-2) J_g g - 4 A_g \cdot \right) d$ with

$$J_g := \frac{R_g}{2(n-1)}, \ A_g = \frac{Ric_g - \frac{R_g}{n} g}{n-2} + \frac{J_g}{n} g,$$

and $A_g \cdot$ the homomorphism on $T^*M$ given by $\phi = (\phi_i) \mapsto (A_g)_{ij}^i \phi_j$, and $Q^n_g := n \left( J_g^2 - 4 |A_g|^2 + 2\Delta_g J_g \right)$ is Branson’s $Q$-curvature [B1], a local scalar invariant that is a polynomial in the coefficients of the metric tensor and its inverse, the scalar curvature.
and the Christoffel symbols. Note that \(A_g = \frac{1}{n} J_g g\) precisely when \(g\) is Einstein.

The \(Q\)-curvature generalizes the scalar curvature \(R_g\) in the following sense. On a 4-manifold, we have

\[
Q^4_g = e^{-4f} \left( Q^4_g + \frac{1}{2} P^4_g f \right)
\]

(cf. (2.3)), and \(\int_M Q^4_g d\text{vol}_g\) is a conformal invariant (cf. (2.4)), as is \(\int_M Q^n_g d\text{vol}_g\) in even dimensions [B2].

**Operators of order \(2k\).** (Graham, Jenne, Mason and Sparling [GJMS]) Fix \(k \in \mathbb{Z}^+\) and assume either \(n\) is odd or \(k \leq n\). There are conformally covariant (self-adjoint) scalar differential operators \(P^n_{g,k}\) of bidegree \((\frac{n-2k}{2}, \frac{n+2k}{2})\) such that the leading part of \(P^n_{g,k}\) is \(\Delta^k_g\) and such that \(P^n_{g,k} = \Delta^k_g\) on \(\mathbb{R}^n\) with the Euclidean metric.

\(P^n_{g,k}\) generalizes \(P^n_{g}\), since \(P^n_{g,k} = P^n_{g,2}\), and satisfies

\[
P^n_{g,k} = P^n_{g} + \frac{n - 2k}{2} Q^n_g
\]

where \(P^n_{g} = d^* S^n_g d\) for a natural differential operator \(S^n_g\) on 1-forms.

Note that \(P^n_{g,k}\) has bidegree \((a, b)\) with \(b - a = 2k\) independent of the dimension and in particular has bidegree \((0, 2k)\) in dimension \(2k\).

**Pseudodifferential Operators.** (Branson and Gover [BG], Petersen [Pe]) Peterson has constructed \(\Psi\)DOs, \(P^n_{g,k}, k \in \mathbb{C}\), of order \(2\text{Re}(k)\) and bidgree \((\frac{n-2k}{2}, \frac{n+2k}{2})\) on manifolds of dimension \(n \geq 3\) with the property that \(P^n_{g,k} - e^{-bf} P^n_{g,k} e^{af}\) is a smoothing operator. Thus any conformal covariant built from the total symbol of \(P^n_{g,k}\) is a conformal covariant of \(P^n_{g,k}\) itself. The family \(P^n_{g,k}\) contains the previously discovered conformally covariant \(\Psi\)DOs associated to conformal boundary value problems [BG].

2.3. A hierarchy of functionals and their conformal anomaly. Since the known conformal invariants for conformally covariant operators \(A_g\)

\[
\zeta_{A_g}(0) = \text{tr} A_g (I), \quad \log \det_c (A_g) = \text{tr} A_g (\log A_g), \quad \eta_{A_g} = \text{tr} A_g (A_g | A_g|^{-1})
\]

arise as weighted/canonical traces by (1.7), (1.10), (1.11), it is natural to look for a general prescription to derive conformal invariants from canonical traces.

Let \(A_g \in Cl(M, E)\) be an operator associated to a Riemannian metric \(g\) on \(M\). For \(f \in C^\infty(M, \mathbb{R})\), set \(g_t := e^{2ft} g, t \in \mathbb{R}\), and set \(A_t = A_g\). We always assume that the map \(g \mapsto A_g\) is smooth in the appropriate topologies, so that \(A_t\) is automatically a smooth curve in \(Cl(M, E)\).

**Lemma 2.4.** \(A_g \in Cl(M, E)\) is conformally covariant of bidegree \((a, b)\) if and only if for all \(f \in C^\infty(M, \mathbb{R})\),

\[
\dot{A}_t = (a - b) f A_t - a \{ f, A_t \}.
\]

(2.5)
Proof: This follows from differentiating applied to the family $g_t$. \qed

Theorem 2.5. Let $A_g$ be a conformally covariant weight of bidegree $(a, b)$ and whose order $\alpha$ and spectral cut $\theta$ are independent of the metric $g$. The meromorphic map
\[ F_h(g) : z \mapsto \text{TR} \left( h(A_g) A_g^{-z} \right) \]
has conformal anomaly
\[ \delta_f \text{TR} \left( h(A_g) A_g^{-z} \right) = (a - b) \text{TR} \left( f h'(A_g) A_g^{-z+1} \right) - z(a - b) \text{TR} \left( f h(A_g) A_g^{-z} \right) \]
as an identity of meromorphic functions.
This is equivalent to the following system of equations.
1. The conformal anomaly of $\text{res}(h(A_g))$:
\[ \delta_f \text{res} (h(A_g)) = (a - b) \text{res} (f h'(A_g) A_g) . \] (2.7)
2. The conformal anomaly of $\text{tr}^{A_g}(h(A_g))$:
\[ \delta_f \text{tr}^{A_g} (h(A_g)) = (a - b) \text{tr}^{A_g} \left( f h'(A_g) A_g \right) + \frac{b - a}{\alpha} \text{res}(f h(A_g)), \] (2.8)
3. The conformal anomaly of $\text{tr}^{A_g}(h(A_g) \log^j A_g)$ for $j \in \mathbb{Z}^+$:
\[ \delta_f \text{tr}^{A_g} \left( h(A_g) \log^j A_g \right) = (a - b) \text{tr}^{A_g} \left( f h'(A_g) A_g \log^j A_g \right) \]
\[ + j (a - b) \text{tr}^{A_g} \left( f h(A_g) \log^{j-1} A_g \right). \] (2.9)

Proof: Equations (2.6), (2.7), (2.8), (2.9) follow from (1.14), (1.15), (1.16), (1.17), respectively. In the computation, we use the cyclicity of TR on noninteger order operators, which eliminates the second term on the right hand side of (2.5). \qed

We collect these formulas for special choices of $h$. We assume $A_g$ and hence $A_{g\bar{}}$ is invertible. This allows us to ignore terms depending on the kernel of $A_g$, which can be easily treated as in the proof of part 1 below. All invariants and covariants are understood to be conformal.

Corollary 2.6. We have the following conformal anomalies for conformally covariant weights $A_g$ of order $\alpha$:
1. Anomalies associated to $h \equiv 1$:
\[ \delta_f \zeta_{A_g}(0) = \delta_f \text{Tr}^{A_g}(I) = 0, \]
\[ \delta_f \zeta'_ {A_g}(0) = -\delta_f \text{Tr}^{A_g} \left( \log A_g \right) = (a - b) \text{Tr}^{A_g}(f). \] (2.10)
Hence \( \zeta_{A_g}(0) = -\frac{1}{\alpha} \text{res}(\log A_g) \) is an invariant. \( \zeta'_{A_g}(0) \) has local anomaly \(-\frac{a-b}{\alpha} \text{res}_x(\log A_g)\) and is an invariant whenever \( \text{res}_x(\log A_g) \) vanishes for all \( x \in M \).

2. Anomalies associated to \( h(\lambda) = \lambda \):

\[
\delta_f \text{res}(A_g) = (a - b) \text{res}(f A_g),
\]

\[
\delta_f \text{Tr}^{A_g}(A_g) = (a - b) \text{Tr}^{A_g}(f A_g) + \frac{b - a}{\alpha} \text{res}(f A_g), \tag{2.11}
\]

\[
\delta_f \text{Tr}^{A_g}(A_g \log A_g) = (a - b) \text{Tr}^{A_g}(f A_g \log A_g) + (a - b) \text{Tr}^{A_g}(f A_g). \tag{2.12}
\]

\( \text{res}_x(A_g) \) is a pointwise covariant of weight \( a - b \). If \( A_g \) is a differential operator, then \( \text{res}_x(A_g) \) vanishes and \( \text{Tr}^{A_g}(A_g) = -\frac{1}{\alpha} \text{res}(A_g \log A_g) \) has local anomaly given by \( \frac{b - a}{\alpha} \text{res}_x(A_g \log A_g) \).

3. Anomalies associated to \( h(\lambda) = \lambda c, \ c \in \mathbb{R} \): assuming \( A_g \) is admissible for fixed \( c \), we have

\[
\delta_f \text{res}(A_g^c) = c(a - b) \text{res}(f A_g^c),
\]

\[
\delta_f \text{Tr}^{A_g}(A_g^c) = c(a - b) \text{Tr}^{A_g}(f A_g^c) + \frac{b - a}{\alpha} \text{res}(f A_g^c). \tag{2.13}
\]

If \( A_g^c \) is a differential operator (e.g. if \( A_g \) is differential and \( c \in \mathbb{Z}^+ \)), then \( \text{res}_x(A_g^c) \) vanishes and \( \text{Tr}^{A_g}(A_g^c) = -\frac{1}{\alpha} \text{res}(A_g^c \log A_g) \) has local anomaly given by \( \frac{c(b-a)}{\alpha} \text{res}_x(A_g^c \log A_g) \).

4. Anomalies associated to \( h(\lambda) = \lambda/|\lambda| \) and invertible \( A_g \):

\[
\delta_f \text{Tr}^{A_g}(A_g/|A_g|) = \frac{b - a}{\alpha} \text{res} \left( f A_g \right) \left| A_g \right|. \tag{2.14}
\]

**Proof:** Much of the Corollary follows immediately from the Theorem. In 1, the invariance of \( \text{res}_x(\log A_g) \) follows from (1.9). The last statement follows from \( \text{Tr}^{A_g}(f) = -\frac{1}{\alpha} \text{res}(f \log A_g) \) (1.13), since multiplication by \( f \) is a differential operator, and (1.11). If \( \text{Ker}(A) \) is nontrivial, \( \zeta_{A_g}(0) \) is still a conformal invariant: by (1.10), (2.10), the new terms cause no trouble, as \( \text{Tr}(\Pi A) = \dim(\text{Ker} A) \) is a conformal invariant and \( \text{res}(I) = 0 \).

For the statement about \( \text{res}_x(A_g) \) in 2, if \( \phi \) is a smooth function on \( M \), then \( \phi \cdot A_g \) is conformally covariant if \( A_g \) is. By (2.11),

\[
\delta_f \int_M \phi(x) \left( \int_{S^*_M} \text{Tr}_x \sigma_{-n}(A_g)(x, \xi) d\xi \right) dx = \delta_f \text{res}(\phi \cdot A_g) = (a - b) \text{res}(f \cdot \phi \cdot A_g) = (a - b) \int_M f(x) \phi(x) \left( \int_{S^*_M} \text{Tr}_x \sigma_{-n}(A_g)(x, \xi) d\xi \right) dx.
\]
Letting $\phi$ approach a delta function at $x$ and using the compactness of $M$ to push this limit past $\delta_f$ gives
\[
\delta_f \text{res}_x(A_g) = (a - b)f(x)\text{res}_x(A_g).
\]

The last statement in 2 follows as in the proof of 1 from $\text{Tr} A_g(A_g) = -\frac{1}{\alpha} \text{res}(A_g \log A_g)$. In 3, the last statement follows from $\text{Tr} A_g(fA_g^c) = -\frac{1}{\alpha} \text{res}(fA_g^c \log A_g)$.

**Remark 2.2.** The conformal anomaly in string theory boils down to a finite linear combination of local conformal anomalies $\frac{b_n}{\alpha} \text{res}_x(\log A_g)$ where the $A_g$ are Laplacians on forms. Their bidegree involving the dimension of spacetime, so this local conformal anomaly vanishes for a certain well chosen dimension.

As stated in the introduction, the Corollary and the Laurent expansion of Theorem 2.5 provide a natural hierarchy among these invariants/covariants. The most divergent term in the Laurent expansion is a conformal invariant; if this global invariant vanishes in a particular case, then the new “most divergent” term, if it is of the form $\int_M I(g,x) \text{dvol}_g(x)$, tends to give rise to a local conformal anomaly proportional to $I(g,x)$. This is confirmed by the more refined analysis for weights with nonnegative leading order symbol, i.e. weights with smoothing heat kernels.

**Lemma 2.7.** Let $A_g$ be a weight of order $\alpha$ with nonnegative leading symbol and let the heat kernel for $A = A_g$ have the asymptotic expansion 
\[
\text{Tr}_x e^{A}(\epsilon, x, x) \sim \sum_{j=0}^{\infty} a_j(A, x) \epsilon^{\frac{j-n}{\alpha}} + \sum_{k=0}^{\infty} b_k(A, x) \epsilon^k \log \epsilon + \sum_{\ell=1}^{\infty} c_\ell(A, x) \epsilon^\ell. \tag{2.15}
\]

Then
\[
\text{res}_x(A^k) = \begin{cases} 
(\frac{\alpha}{\alpha - k-1})^k a_{n+\alpha k}(A, x), & k \in \mathbb{Z}^+, \\
(-1)^{k+1} k! \alpha b_k(A, x), & k \in \mathbb{Z}^0,
\end{cases} \tag{2.16}
\]

with the understanding that $a_{n+\alpha k} = 0$ if $\alpha k \notin \mathbb{Z}$. In particular, $\text{res}_x(A) = \alpha b_1(A, x)$.

The last sum in (2.15) appears only if $(j - n)/\alpha$ is never integral. In particular, this sum does not appear if $A_g$ is a differential operator.

**Proof:** Setting $\zeta_A(z, x) := \omega_{KV}(A^{-z})(x)$ we have
\[
\text{Res}_{z=-k} \zeta_A(z, x) = \text{Res}_{z=0} \omega_{KV}(A^{-z+k})(x) = -\frac{1}{\alpha'}(0)^{\alpha} \text{res}_x(A^k) = \frac{\text{res}_x(A^k)}{\alpha},
\]
(see (1.3), (1.5)), where we use the same symbol for an operator and its kernel and assume $A$ is invertible for simplicity. Let us compute this complex residue. We have
\[
\zeta_A(z, x) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e_A(t, x) dt \sim \frac{1}{\Gamma(z)} \int_1^\infty t^{z-1} \left( \sum_{j=0}^\infty a_j(A, x) t^{\frac{z-n}{n}} + \sum_{k=0}^\infty b_k(A, x) t^k \log t + \sum_{\ell=0}^\infty c_{\ell}(A, x) t^\ell \right) dt + \frac{1}{\Gamma(z)} \int_1^\infty t^{z-1} e_A(t, x) dt.
\]
Since the last term is analytic in $z$, an easy integration on the first term yields the result. In particular, $\text{res}_x(A) = \alpha b_1$.

To state the final theorem, let the kernel $\tilde{e}(\epsilon, x, y)$ of $A e^{-|A|}$ have the asymptotic expansion
\[
\text{Tr}_x \tilde{e}(\epsilon, x, x) \sim \sum_{j=0}^\infty \tilde{a}_j(A, x) \epsilon^{\frac{z+n}{n}} + \sum_{k=0}^\infty \tilde{b}_k(A, x) e^k \log \epsilon + \sum_{\ell=1}^\infty \tilde{c}_{\ell}(A, x) e^\ell.
\]
Set $a_j(A) = \int_M a_j(A, x) d\text{vol}_g$, etc.

**Theorem 2.8.** Let $A = A_g$ be a conformally covariant weight of bidegree $(a, b)$, with nonnegative leading order symbol, and whose order $\alpha$ is independent of the metric $g$.

1. $a_n(A) + c_0(A)$ is a conformal invariant.
2. We have
   \[
   \delta_f \log \det \zeta A = -\delta_f(x) \zeta_A'(0) = (b - a) \int_M f(x) (a_n(A, x) + c_0(A, x)) d\text{vol}_g(x),
   \]
   so $\log \det \zeta(A)$ has local conformal anomaly given by $(b - a)(a_n(A, x) + c_0(A, x))$.
   In particular, $\det \zeta A$ is a conformal invariant if $A$ is a differential operator and $\dim(M)$ is odd.
3. $b_1(A, x)$ is a conformal covariant of weight $a - b$.
4. $a_n - \alpha(A, x)$ is a pointwise conformal covariant of weight $b - a$.
5. We have
   \[
   \delta_f \text{Tr}^A(A) = -\delta_f(c_1(A) + a_{n+\alpha}(A)) = (b - a) \int_M f(x) [c_1(A, x) + a_{n+\alpha}(A, x) - b_1(A, x)] d\text{vol}_g(x),
   \]
   so that $\text{Tr}^A(A)$ has a local conformal anomaly given by
   \[(b - a)(c_1(A, x) + a_{n+\alpha}(A, x) - b_1(A, x))\]
(with the understanding that \(a_{n+\alpha}(A, x) = 0\) if \(\alpha \notin \mathbb{Z}\), and \(c_1(A, x) = 0\) if \(A\) is differential). In particular, if \(A\) is a differential operator, then \(\text{Tr}^A(A)\) has a local conformal anomaly given by \((b - a)a_{n+\alpha}(A, x)\), and if \(A\) is a noninteger order \(\Psi DO\), it has a local conformal anomaly given by \((b - a)c_1(A, x)\).

6. The results of 5 generalize to \(\text{Tr}^A(A^k)\) for \(k \in \mathbb{Z}^+\), replacing \(c_1\) by \(c_k\), \(a_{n+\alpha}\) by \(a_{n+\alpha k}\), \(b_1\) by \(b_k\), and \((a - b)\) by \(k(a - b)\).

7. \(\delta_f\eta_A(0) = \frac{b-k}{a}\res\left(f\frac{A}{|A|}\right) = -(b - a)\int_M f(x)\tilde{a}_n(A, x)\). In particular, \(\eta_A(0)\) is a conformal invariant if \(n\) and \(\alpha\) have opposite parity.

**Proof:**

1. This follows from the first point in Corollary 2.6 and the fact that \(\zeta_{A_0}(0) = a_n(A) + c_0(A)\).

2. This follows from (1.11), (2.10), and the fact that \(\text{Tr}^A(f) = \int_M f(x)a_n(A, x)\).

It is well known that the only the \(k^{\dim(M) - \frac{1}{2}}\) terms in the heat kernel asymptotics are nonzero for differential operators, so \(a_n(A, x) = 0\) in odd dimensions.

3. This was shown in the second point of Corollary 2.6

4. If \(A\) is conformally covariant, so is \(A^{-1}\). The result now follows from 3 and the Lemma.

5. \(\text{Tr}^A(A) = f.p._{\epsilon=0} \text{Tr}(Ae^{-\epsilon A})\) is the finite part of \(\text{Tr}(Ae^{-\epsilon A}) = -\partial_\epsilon \text{Tr}(e^{-\epsilon A})\) as \(\epsilon \to 0\), so \(\text{Tr}^A(A) = -\int_M a_{n+\alpha}(A, x) + c_1(A, x)\). The first statement now follows from the second equation in (2.11) and the fact that \(\frac{b-k}{a}\res(fA) = b_1(A, x)\) (2.10). If \(A\) is a differential operator, then \(c_1\) is replaced by \(a_{n+\alpha}\) and \(\res_x(fA) = 0\). If \(A\) is a non-integral order \(\Psi DO\), then again \(\res_x(fA) = 0\).

6. By the first equation in (2.13), \(\delta_f\res(A^k) = k(a-b)\res(fA^k), k \in \mathbb{Z}^+\). The results for \(b_k\) follow as in 5, using (2.10). We can use the second equation in (2.13) and \(\text{Tr}^A(fA^k) = (-1)^k f.p._{\epsilon=0} \partial_\epsilon \text{Tr}(fe^{-\epsilon A})\) to prove the result for the \(a\) and \(c\) coefficients.

7. The first equality follows from (2.11) and Remark 2. For the second equality, we have

\[
\res(A/|A|) = \res_{s=0} \text{Tr}(A|A|^{-s}A^{-s}) = \res_{s=1} \text{Tr}(A|A|^{-s}).
\]

Using the pointwise version of the Mellin transform \(A|A|^{-s} = \Gamma(s)^{-1} \int_0^\infty t^{s-1} A e^{-t|A|} dt\) as in the Lemma, we get \(\res\left(f\frac{A}{|A|}\right) = -(b - a)\int_M f(x)\tilde{a}_n(A, x)\). The last statement follows from a careful computation [R, Prop. 3] of the the residue of \(A/|A|\). Note that since \(|A|\) has nonnegative symbol, this restriction on the symbol of \(A\) can be dropped here. 

**Remark 2.3.** (i) 1, 2, and 4 are known for the conformal Laplacian \([BO, PR]\). 3 is new to our knowledge. Related results for contact geometry are in Ponge \([Pon]\).

(ii) The results above involving Wodzicki residues can be proved directly, where the cyclicity is valid for all order operators.
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