LONG-DISTANCE EFFECTS IN RARE K DECAYS

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Rare K decays provide for very clean tests of the Standard Model, and are especially suited to search for New Physics signal. In this talk, recent progresses in the estimation of long-distance effects induced by light quarks in $K_L \rightarrow \pi^0 \mu^+ \mu^-$ and $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ are reported.

1 Introduction

The rare $K$ decays considered in this talk proceed through Flavor Changing Neutral Currents, arising at loop-level in the Standard Model. What makes them specially interesting is that short-distance (SD) effects contribute significantly to their decay rates, and that long-distance (LD) hadronic effects are under theoretical control. They are thus ideal to constrain the Standard Model by precise extraction of CKM parameters. In addition, being suppressed in the SM and being driven by SD physics give them good sensitivity to possible New Physics, complementary to direct searches. The SM theoretical predictions are

$$B(K_L \rightarrow \pi^0 \nu \bar{\nu}) = (3.0 \pm 0.6) \times 10^{-11}$$
$$B(K_L \rightarrow \pi^0 e^+ e^-) = 3.7^{+1.1}_{-0.9} \times 10^{-11}$$
$$B(K^+ \rightarrow \pi^+ \nu \bar{\nu}) = (7.8 \pm 1.2) \times 10^{-11}$$
$$B(K_L \rightarrow \pi^0 \mu^+ \mu^-) = (1.5 \pm 0.3) \times 10^{-11}$$

(1)

Experimentally, KTeV has set upper limits for the neutral modes, and AGS-E787/E979 found three events for the charged one.

The leading parts of the effective Hamiltonians relevant to the study of these modes are

$$H_{eff} (\bar{s}d \rightarrow \nu \bar{\nu}) \sim \frac{G_F}{\sqrt{2}} (y_{\nu}(\bar{s}d) V_{-\bar{\nu}} V_{-\nu} + h.c.)$$

(2)

$$H_{eff} (\bar{s}d \rightarrow \ell^+ \ell^-) \sim \frac{G_F}{\sqrt{2}} (y_{\ell \nu}(\bar{s}d) V_{-\bar{\nu}} V_{-\nu} + y_{\ell A}(\bar{s}d) V_{-\nu} V_{-\nu} + h.c.)$$

(3)

$$y_{\nu} = \frac{\alpha}{2\pi} \sum \lambda_q \frac{X_0(x_q)}{\sin^2 \theta_W}, \quad y_{\ell A} = \frac{\alpha}{2\pi} \sum \lambda_q \frac{Y_0(x_q)}{\sin^2 \theta_W}, \quad y_{\ell \nu} = \frac{\alpha}{2\pi} \sum \lambda_q \left[ \frac{Y_0(x_q)}{\sin^2 \theta_W} - 4Z_0(x_q) \right]$$

(4)
with \( \lambda_q = V_{q6}^* V_{qd} \), \( x_q = m_q^2/M_W^2 \), summation over \( q = u, c, t \) and the Inami-Lim functions
\( X_0 = C_0^Z - 4B_0^W \), \( Y_0 = C_0^Z - B_0^W \) and \( Z_0 = C_0^Z + D_0^Z/4 \) depicted in Fig. 1.

In \( H_{eff} \), the operators are new interactions, while the Wilson coefficients \( y_i \), encoding the effects of heavy particles, their coupling constants. Combining the known scaling of CKM matrix elements with the behavior of the Inami-Lim functions as functions of quark masses, one can assess of the relative strengths of the \( u, c, t \) contributions, and thereby of the SD or LD nature of the process. For example, for the simplest and cleanest rare decay mode, \( K_L \rightarrow \pi^0 \nu \bar{\nu} \), taking the matrix element of Eq. 2,

\[
\mathcal{M}(K_L \rightarrow \pi^0 \nu \bar{\nu}) = G_F \sqrt{2} \langle \pi^0 | H_{eff} | K_L \rangle \langle \nu \bar{\nu} | V_{-A} \rangle = G_F \text{Im} y_\nu \langle \pi^0 | (\bar{s}d)_V | K^0 \rangle \langle \nu \bar{\nu} | V_{-A} \rangle
\]

The CKM structure and GIM hard breaking (\( X_0(x) \sim x \)) suppress all light-quark effects, leaving only the top quark contribution:

\[
\text{Im} y_\nu \sim \text{Im} \lambda_u X_0(x_u) + \text{Im} \lambda_c X_0(x_c) + \text{Im} \lambda_t X_0(x_t) \approx \text{Im} \lambda_t X_0(x_t)
\]

A second important fact is that the matrix element \( \langle \pi^0 | (\bar{s}d)_V | K^0 \rangle \) can be extracted from the well-measured decay \( K^+ \rightarrow \pi^0 \ell^+ \nu_\ell \) using isospin symmetry, hence hadronic uncertainties are small. From these properties, \( B(K_L \rightarrow \pi^0 \nu \bar{\nu}) \sim 2.6 \cdot 10^{-11} qCD \rightarrow (3.0 \pm 0.6) \cdot 10^{-11} \), with the error coming mostly from the uncertainty on the CKM element \( 3 \).

2 CP-conserving contributions to \( K_L \rightarrow \pi^0 \ell^+ \ell^- \)

Modes involving a charged lepton pair are more complicated as they interact with photons. Three types of processes are relevant: the direct CP-violating (DCPV), indirect CP-violating (ICPV) and CP-conserving (CPC) one, see Fig. 2. While the theoretical complexity increases for \( \ell^+ \ell^- \) modes, recent experimental results for \( K_S \rightarrow \pi^0 \ell^+ \ell^- \) \( 4 \) and \( K_L \rightarrow \pi^0 \gamma \) \( 5,6 \) permit reliable theoretical estimates for ICPV and CPC. The relative sizes of the three contributions are

\[
\begin{array}{|c|c|c|c|c|}
\hline
 & DCPV & ICPV & CPC-2^{++} & CPC-0^{++} \\
\hline
K_L \rightarrow \pi^0 \nu \bar{\nu} & 100\% & (\sim 1\%) & - & - \\
K_L \rightarrow \pi^0 e^+ e^- & 40\% & 60\% & (\sim 3\%) & - \\
K_L \rightarrow \pi^0 \mu^+ \mu^- & 30\% & 35\% & - & 35\% \\
\hline
\end{array}
\]

where \( - \) means \(< 0.1\% \). The CPC contribution proceeds through two photons which can be in a scalar \( 0^{++} \) (helicity suppressed for \( e^+ e^- \)) or tensor \( 2^{++} \) state (phase-space suppressed, especially for \( \mu^+ \mu^- \)). Our work was to estimate the \( 0^{++} \) CPC contribution \( 7 \).

Figure 1: The \( W^\pm \) box, \( Z \) penguin and photon penguin diagrams.

Figure 2: Direct CPV, Indirect CPV and CPC contributions.
**CPV contributions:** The photon penguin now plays a role. Since it behaves as $D_0^+(x) \to \ln x$ when $x \to 0$, light quarks may contribute significantly. For direct CPV, one should again look at the scaling of the $u,c,t$ contributions to the imaginary parts of $y_{T,L}^\nu$:

$$\text{Im} y_{7L} \sim \text{Im} \lambda_y Y_0(x_t) \to -(0.68 \pm 0.03) \text{Im} \lambda_t \times 10^{-4}$$

$$\text{Im} y_{7V} \sim \text{Im} \lambda_y D_0^+(x_c) + \text{Im} \lambda_y C_0^2(x_t) \to (0.73 \pm 0.04) \text{Im} \lambda_t \times 10^{-4}$$

with roughly equal $c$ and $t$-quark contributions for $y_{7V}$, and where NLO QCD effects are included. In terms of these, the DCPV contributions are given by:

$$B(K_L \to \pi^0 e^+ e^-)_{DCPV} = (2.67(\text{Im} y_{7V}^2 + \text{Im} y_{7A}^2) + 0 \text{ Im} y_{7A}) \cdot 10^{-12}$$

$$B(K_L \to \pi^0 \mu^+ \mu^-)_{DCPV} = (0.63(\text{Im} y_{7V}^2 + \text{Im} y_{7A}^2) + 0.85 \text{ Im} y_{7A}) \cdot 10^{-12}$$

An extra helicity suppressed piece appears for the $\mu^+ \mu^-$, giving different sensitivities to the two modes on the SD physics, and therefore also to possible New Physics.

![Diagram](image_url)

Figure 3: Switching to mesons for the LD-dominated ICVP piece.

For indirect CPV, $\mathcal{M}(K_L \to \pi^0 \ell^+ \ell^-) = \epsilon \mathcal{M}(K_S \to \pi^0 \ell^+ \ell^-)$, one has to analyze the real part of the $y_{T,L}^\nu$ (see Eq. 3). These are completely dominated by the $u$-quark contribution, both from the CKM scaling and the GIM-breaking induced by $D_0^+(x)$ and $\text{Re} y_{7V} \sim \text{Re} \lambda_y D_0^+(x_u) \gg \text{Re} y_{7A}$. Being LD dominated, one has to switch to the meson world, and the process is dealt with using Chiral Perturbation theory. It was found in that loops have a small effect and both decays are dominated by a common counterterm

$$B(K_S \to \pi^0 e^+ e^-) = 5.2a_S^3 \cdot 10^{-9}, \quad B(K_S \to \pi^0 \mu^+ \mu^-) = 1.2a_S^3 \cdot 10^{-9}$$

Recent NA48 measurements of both modes give $|a_s| = 1.2 \pm 0.2$.

**CPC contributions:** The leading order is obtained from a $\pi^\pm$ or $K^\pm$ loop followed by a $\gamma \gamma$ loop. This process can be factorized into a $K_L \to \pi^0 P^+ P^-$ ($P = \pi$, $K$) form factor convoluted with the two-loop amplitude for $(P^+ P^-)_{0++} \to \gamma \gamma \to \ell^+ \ell^-$, as long as the form factor depends on $z \sim \left(p_{P^+} + p_{P^-}\right)^2$ only (see Fig. 4). The crucial point is that for a large range of parametrization of this dependence, the ratio $R_{\gamma \gamma} = B(K_L \to \pi^0 \ell^+ \ell^-) / B(K_L \to \pi^0 \gamma \gamma)$ is stable even if both the $\ell^+ \ell^-$ and $\gamma \gamma$ spectra and rates vary much. For dynamical reasons, the $\ell^+ \ell^-$ and $\gamma \gamma$ modes react similarly to modulations in the distribution of momenta entering the scalar subprocess (i.e., to $a_1(z)$ in Fig. 4). Given this observation, we infer the branching ratios of $\ell^+ \ell^-$ modes from the experimental measurement of the $\gamma \gamma$ one. Some higher order chiral corrections are thus included in our result, in particular the $O(p^6)$ chiral counterterms (with their VMD contents) needed to describe both the rate and spectrum for $K_L \to \pi^0 \gamma \gamma$. The stability of $R_{\gamma \gamma}$ is the key
dynamical feature permitting such an extrapolation, and thereby getting a reliable estimation for $\ell^+\ell^-$ modes.

Numerically, taking $\mathcal{B}_{\text{exp}}^B (K_L \to \pi^0\gamma\gamma) = (1.42 \pm 0.13) \cdot 10^{-6}$ as the average of KTeV and NA48 measurements, we find $\mathcal{B}^B (K_L \to \pi^0\mu^+\mu^-)_{\text{CPC}}^{0^{++}} = (5.2 \pm 1.6) \cdot 10^{-12}$, with a conservative error estimate of 30%. For the $e^+e^-$ mode, the BR is $\mathcal{O}(10^{-14})$, hence completely negligible.

CPC-2$^{++}$ contributions were discussed in 8. These authors used strong constraints from the experimental low-energy part of the photon spectrum in $K_L \to \pi^0\gamma\gamma$ to set the upper limits $\mathcal{B} (K_L \to \pi^0e^+e^-)_{\text{CPC}}^{2^{++}} < 3 \cdot 10^{-12}$ and $\mathcal{B} (K_L \to \pi^0\mu^+\mu^-)_{\text{CPC}}^{2^{++}} < 5 \cdot 10^{-14}$. These bounds are very conservative, $2^{++}$ contributions are probably smaller and can be neglected.

Complete SM prediction: The final parametrizations are, in the Standard Model

$$\begin{align*}
\mathcal{B} (K_L \to \pi^0e^+e^-) &\approx (2.4\kappa^2 \pm 6.2|a_S|\kappa + 15.7|a_S|^2) \times 10^{-12} \\
\mathcal{B} (K_L \to \pi^0\mu^+\mu^-) &\approx (1.0\kappa^2 \pm 1.6|a_S|\kappa + 3.7|a_S|^2 + 5.2) \times 10^{-12}
\end{align*}$$

(11)

with $\kappa = 10^4 \text{Im} \lambda_t = 1.36 \pm 0.12$. The interference sign between DCPV and ICPV is not fixed by experiment, but two independent theoretical analyses point toward a positive sign 8, 10.

The different sensitivities to SD physics can be illustrated by taking a specific model 7. Enhanced electroweak penguins could lead to an enhancement of SD effects $y_{7V}^{EWP} = 1.2 \times y_{7V}^{SM}$, $y_{7A}^{EWP} = 4.7 \times y_{7A}^{SM}$, leading to the situation depicted in Fig. 5. A combined observation of the two modes increases the sensitivity to New Physics, and in addition, informations on its nature can be extracted: the difference in the vector and axial vector currents manifests itself in a central value away from the $\text{Im} \lambda_t$ curve.

Figure 5: BR of the $\mu^+\mu^-$ against the $e^+e^-$ mode ($\times 10^{-11}$).
3 Light-quarks in \( K^+ \to \pi^+ \nu\bar{\nu} \)

The general structure is \( \kappa^+ \sim 3 \alpha^2 BR \left( K^+ \to \pi^0 e^+ \nu \right) / 2\pi^2 \lambda^2 \sin^4 \theta_W \):

\[
B \left( K^+ \to \pi^+ \nu\bar{\nu} \right) = \kappa^+ \left( |\text{Im} \lambda_t X (x_t)|^2 + |\text{Re} \lambda_t X (x_t) + \text{Re} \lambda_c X (x_c)|^2 \right)
\]

(12)

For the \( \text{Im} \lambda_t \) part, see Eq. 6. For the real part, the CKM structure compensates the GIM suppression, and the \( t \) and \( c \) contributions are similar in size (68\% vs. 32\%). Let us analyze the various uncertainties, with the purpose of precision physics in mind. The top quark effects are known to within 3\%, \( X (x_t) \simeq 1.529 \pm 0.042 \). The \( c \)-quark effects are also known at NLO\(^8\) but given the low scale set by \( m_c \), this corresponds to a 18\% error, \( X (x_c) \simeq 1.529 \pm 0.07 \).

Our goal was to analyze two subleading effects, on which control is needed to get down to a few \% precision at the BR level\(^12\). First there are the \( c \)-quark dimension eight operators\(^13\), for which a naive estimate of the possible impact on \( X (x_c) \) would be \( O(m_K^2/m_c^2) \sim 15\% \). Then, residual \( u \)-quark effects\(^14\) which are purely LD, could lead to \( O(\Lambda_{QCD}^2/m_c^2) \sim 10\% \) corrections to \( X (x_c) \). In the general OPE expansion, these effects are schematically depicted as

![Schematic OPE structure including c-quark Q(8) operators and LD meson loops.](image)

Figure 6: Schematic OPE structure including c-quark \( Q_c^{(8)} \) operators and LD meson loops.

(similar expansions of the \( W^\pm \) box are also considered). For dim. 8 operators \( Q_c^{(8)} \) like \((\bar{s}d)_{V-A} \partial^2 (\nu\bar{\nu})_{V-A}\), we have confirmed by an OPE at the charm scale the basis found in\(^13\). However, to estimate their matrix elements \( \langle \pi^+ | Q_c^{(8)} | K^+ \rangle \), we performed an (approximate) matching with ChPT. Symbolically, if QCD is turned off, \( Q_c^{(8)} \) and \( Q_u^{(8)} \) scale as

\[
\lambda_c \frac{q^2}{M_W^2} (\log x_c - \log x_u) \rightarrow \lambda_c \frac{q^2}{M_W^2} \left( \log \frac{m_c}{\mu_{IR}} - \log \frac{m_u}{\mu_{UV}} \right)
\]

(13)

with \( \mu_{IR} \equiv \mu_{UV} \) and \( \lambda_c = -\lambda_u \). If this naive picture survives to hadronization, there will be an exact cancellation of the scale dependences from the Chiral loop UV divergences and the c-quark IR one. Even if an exact matching cannot be expected (the ChPT amplitude exhibits the \( \Delta I = 1/2 \) enhancement), current estimates can be much improved.

For \( u \)-quark effects, the amplitude is computed at one-loop in ChPT. An important improvement over previous analyses\(^14\) was to include in the \( O(G_F^2 p^2) \) operator basis all the effective
interactions arising from the integration of heavy modes, in particular the local $Z^\mu (\bar{s}d)_{V-A}$ non-gauge invariant one. Alternatively, one can enforce GIM mechanism on the direct transition $K_L \rightarrow Z$, leading to the $\Delta S = 1$ ChPT Lagrangian\cite{12}

$$L^{(2)}_{\Delta S=1} = G_\Phi F_\pi^4 \left\{ \langle \lambda_6 L_\mu L^\mu \rangle - 2i \frac{g}{\cos \theta_W} \langle \lambda_6 L_\mu T_3 \rangle Z^\mu \right\}$$

Then, contrary to earlier estimation, $K^+ \rightarrow \pi^+ Z^*$ is not vanishing at tree-level, and does not depend on the singlet part of the $Z$ current. At one-loop, a divergence structure that matches the short-distance $c$-quark one is found, and estimating $\langle \pi^+ | Q^{(8)}_c | K^+ \rangle$ is possible.

Numerically, the combined effect of $c$-quark dim. 8 operators and of non-local long-distance $u$-quark loops can be written as $X(x_c) \rightarrow X(x_c) + \delta X(x_c)$ with $X(x_c)$ given earlier, and $\delta X(x_c) = \lambda^4 (0.04 \pm 0.02)$. This amounts to an enhancement of about 6% for the total rate, and is completely dominated by the long-distance $Z$-penguin.

**Conclusion**

Long-distance effects in rare $K$ decays are now under control. For $K_L \rightarrow \pi^0 \ell^+ \ell^-$, the theoretical analysis of\cite{12} combined with recent measurements by NA48\cite{4} permits the estimation of the indirect CPV contributions. Interference with direct-CPV violation has been argued to be positive by two independent groups. For pure long-distance CP-conserving contributions, the tensor $2^{++}$ one was estimated from the photon spectrum in $K_L \rightarrow \pi^0 \gamma \gamma$ and is negligible. For the scalar one, CPC-0$^{++}$, we found\cite{7} that a reliable estimate is possible thanks to the dynamical stability of the ratio $B(K_L \rightarrow \pi^0 \ell^+ \ell^-)/B(K_L \rightarrow \pi^0 \gamma \gamma)$. Obviously, extensive use is made of experimental inputs in these analyses. In particular, the main source of uncertainties on $B(K_L \rightarrow \pi^0 \ell^+ \ell^-)$ at present is in the parameter $a_S$ (see Fig.5), so any improvement in the measurement of $B(K_S \rightarrow \pi^0 \ell^+ \ell^-)$ would be much welcomed.

Concerning $K^+ \rightarrow \pi^+ \nu \bar{\nu}$, two subleading sources of potentially large theoretical uncertainties were analyzed and brought under control, namely dimension eight $c$-quark operators, and long-distance $u$-quark loops. Taken together, they amount to an increase of the BR by 6%. Theoretical uncertainties are then dominated by the $c$-quark dimension six OPE at NLO, and a NNLO analysis would presumably lead to a theoretical error of less than 5%. The ability of $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ in constraining the Standard Model is therefore lying on strong theoretical grounds.

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