CONFORMALLY FLAT TANGENT BUNDLES WITH GENERAL
NATURAL LIFTED METRICS

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Abstract. We study the conditions under which the tangent bundle \((TM, G)\)
of an \(n\)-dimensional Riemannian manifold \((M, g)\) is conformally flat, where \(G\) is
a general natural lifted metric of \(g\). We prove that the base manifold must have
constant sectional curvature and we find some expressions for the natural lifted
metric \(G\), such that the tangent bundle \((TM, G)\) become conformally flat.

Mathematics Subject Classification 2000: Primary 53C55, 53C15,
53C05

Key words and phrases: tangent bundle, Riemannian metric,
general natural lift, conformal curvature.

1. Introduction

The possibility to consider vertical, complete and horizontal lifts on the tangent
bundle \(TM\) of a smooth \(n\)-dimensional Riemannian manifold \((M, g)\), leads to some
interesting geometric structures, studied in the last years (see [1], [2], [3], [9], [10]),
and to interesting relations with some problems in Lagrangian and Hamiltonian
mechanics. One uses several Riemannian and pseudo-Riemannian metrics, induced
by the Riemannian metric \(g\) on \(M\). Among them, we may quote the Sasaki metric,
the Cheeger-Gromoll metric and the complete lift of the metric \(g\). On the other hand,
the natural lifts of \(g\) to \(TM\), introduced in the papers [6] and [7], induce some new
Riemannian and pseudo-Riemannian geometric structures with many nice geometric
properties ([5], [6]).

Professor Oproiu has studied some properties of a natural lift \(G\), of diagonal type,
of the Riemannian metric \(g\) and a natural almost complex structure \(J\) of diagonal
type on \(TM\) (see [12], [13], [14], and see also [16], [17]). In the paper [11], the same
author has presented a general expression of the natural almost complex structures
on \(TM\). In the definition of the natural almost complex structure \(J\) of general
type there are involved eight parameters (smooth functions of the density energy
on \(TM\)). However, from the condition for \(J\) to define an almost complex structure,
four of the above parameters can be expressed as (rational) functions of the other
four parameters. A Riemannian metric \(G\) which is a natural lift of general type of
the metric \(g\) depends on other six parameters.

In [4], the present author got the conditions and the unique form of the matrix
associated to the general natural lifted metric \(G\), such that the the tangent bundle
\(TM\), with respect to the metric \(G\) has constant sectional curvature. In [15] we have
found the conditions under which the Kählerian manifold \((TM, G, J)\) has constant
holomorphic sectional curvature.

* Partially supported by the Grant ET 5871; 2006,2007, CEEX, Ministerul Educației și
Cercetării, România.
In the present paper we study the conformal curvature of the tangent bundle of a Riemannian manifold \((M, g)\). Namely, we are interested in finding the conditions under which the Riemannian manifold \((TM, G)\), where \(G\) is the general natural lifted metric of \(g\), is conformally flat.

2. Preliminary results

Consider a smooth \(n\)-dimensional Riemannian manifold \((M, g)\) and denote its tangent bundle by \(\tau : TM \rightarrow M\). Recall that \(TM\) has a structure of a \(2n\)-dimensional smooth manifold, induced from the smooth manifold structure of \(M\). This structure is obtained by using local charts on \(TM\) induced from usual local charts on \(M\). If \((U, \varphi) = (U, x^1, \ldots, x^n)\) is a local chart on \(M\), then the corresponding induced local chart on \(TM\) is \(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \ldots, x^n, y^1, \ldots, y^n)\), where the local coordinates \(x^i, y^j, i, j = 1, \ldots, n\), are defined as follows. The first \(n\) local coordinates of a tangent vector \(y \in \tau^{-1}(U)\) are the local coordinates in the local chart \((U, \varphi)\) of its base point, i.e. \(x^i = x^i \circ \tau\), by an abuse of notation. The last \(n\) local coordinates \(y^i, j = 1, \ldots, n\), of \(y \in \tau^{-1}(U)\) are the vector space coordinates of \(y\) with respect to the natural basis in \(T_{\tau(y)}M\) defined by the local chart \((U, \varphi)\). Due to this special structure of differentiable manifold for \(TM\), it is possible to introduce the concept of \(M\)-tensor field on it (see [8]). The \(M\)-tensor fields are defined by their components with respect to the induced local charts on \(TM\) (hence they are defined locally), but they can be interpreted as some (partial) usual tensor fields on \(TM\). However, the essential quality of an \(M\)-tensor field on \(TM\) is that the local coordinate change rule of its components with respect to the change of induced local charts is the same as the local coordinate change rule of the components of an usual tensor field on \(M\) with respect to the change of local charts on \(M\). More precisely, an \(M\)-tensor field of type \((p, q)\) on \(TM\) is defined by sets of \(n^{p+q}\) components (functions depending on \(x^i\) and \(y^j\)), with \(p\) upper indices and \(q\) lower indices, assigned to induced local charts \((\tau^{-1}(U), \Phi)\) on \(TM\), such that the local coordinate change rule of these components (with respect to induced local charts on \(TM\)) is that of the local coordinate components of a tensor field of type \((p, q)\) on the base manifold \(M\) (with respect to usual local charts on \(M\)), when a change of local charts on \(M\) (and hence on \(TM\)) is performed (see [8] for further details); e.g., the components \(y^i, i = 1, \ldots, n\), corresponding to the last \(n\) local coordinates of a tangent vector \(y\), assigned to the induced local chart \((\tau^{-1}(U), \Phi)\) define an \(M\)-tensor field of type \((1, 0)\) on \(TM\). A usual tensor field of type \((p, q)\) on \(M\) may be thought of as an \(M\)-tensor field of type \((p, q)\) on \(TM\). If the considered tensor field on \(M\) is covariant only, the corresponding \(M\)-tensor field on \(TM\) may be identified with the induced (pullback by \(\tau\)) tensor field on \(TM\). Some useful \(M\)-tensor fields on \(TM\) may be obtained as follows. Let \(u : [0, \infty) \rightarrow \mathbb{R}\) be a smooth function and let \(\|y\|^2 = g_{\tau(y)}(y, y)\) be the square of the norm of the tangent vector \(y \in \tau^{-1}(U)\). If \(\delta^i_j\) are the Kronecker symbols (in fact, they are the local coordinate components of the identity tensor field \(I\) on \(M\)), then the components \(u(\|y\|^2)\delta^i_j\) define an \(M\)-tensor field of type \((1, 1)\) on \(TM\). Similarly, if \(g_{ij}(x)\) are the local coordinate components of the metric tensor field \(g\) on \(M\) in the local chart \((U, \varphi)\), then the components \(u(\|y\|^2)g_{ij}\) define a symmetric \(M\)-tensor field of type \((0, 2)\) on \(TM\). The components \(g_{0i} = y^k g_{ki}\) define an \(M\)-tensor field of type \((0, 1)\) on \(TM\).

Denote by \(\nabla\) the Levi Civita connection of the Riemannian metric \(g\) on \(M\). Then we have the direct sum decomposition

\[
(1)\quad TTM = VTM \oplus HTM
\]
of the tangent bundle to $TM$ into the vertical distribution $VTM = \text{Ker } \tau_*$ and the horizontal distribution $HTM$ defined by $\nabla$. The set of vector fields $(\frac{\delta}{\delta y^1}, \ldots, \frac{\delta}{\delta y^n})$ on $\tau^{-1}(U)$ defines a local frame field for $VTM$ and for $HTM$ we have the local frame field $(\frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n})$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{0i} \frac{\partial}{\partial y^h}, \quad \Gamma^h_{0i} = y^k \Gamma^h_{ki},$$

and $\Gamma^h_{ki}(x)$ are the Christoffel symbols of $g$.

The set $(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n})$ defines a local frame on $TM$, adapted to the direct sum decomposition (1). Remark that

$$\frac{\partial}{\partial y^i} = (\frac{\partial}{\partial x^i})^V, \quad \frac{\delta}{\delta x^i} = (\frac{\partial}{\partial x^i})^H,$$

where $X^V$ and $X^H$ denote the vertical and horizontal lift of the vector field $X$ on $M$ respectively. We can use the vertical and horizontal lifts in order to obtain invariant expressions for some results in this paper. However, we should prefer to work in local coordinates since the formulas are obtained easier and, in a certain sense, they are more natural.

We can easily obtain the following

**Lemma 2.1.** If $n > 1$ and $u, v$ are smooth functions on $TM$ such that

$$ug_{ij} + vg_{0i}g_{0j} = 0,$$

on the domain of any induced local chart on $TM$, then $u = 0, v = 0$.

**Remark.** In a similar way we obtain from the condition

$$u\delta^i_j + vg_{0j}y^i = 0$$

the relations $u = v = 0$.

Consider the energy density of the tangent vector $y$ with respect to the Riemannian metric $g$

$$t = \frac{1}{2} \|y\|^2 = \frac{1}{2} g_{\tau(y)}(y, y) = \frac{1}{2} g(x)y^i y^k, \quad y \in \tau^{-1}(U).$$

Obviously, we have $t \in [0, \infty)$ for all $y \in TM$.

### 3. The Conformal Curvature of the Tangent Bundle with General Natural Lifted Metric

Let $G$ be the general natural lifted metric on $TM$, defined by

$$G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = c_1 g_{ij} + d_1 g_{0i}g_{0j} = G^{(1)}_{ij},$$

$$G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = c_2 g_{ij} + d_2 g_{0i}g_{0j} = G^{(2)}_{ij},$$

$$G(\frac{\delta}{\delta y^i}, \frac{\delta}{\delta y^j}) = G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^j}) = c_3 g_{ij} + d_3 g_{0i}g_{0j} = G^{(3)}_{ij},$$

where $c_1, c_2, c_3, d_1, d_2, d_3$ are six smooth functions of the density energy on $TM$.

The Levi-Civita connection $\nabla$ of the Riemannian manifold $(TM, G)$ is obtained from the formula

$$2G(\nabla_X Y, Z) = X(G(X, Z)) + Y(G(X, Z)) - Z(G(X, Y)) + G([X, Y], Z) - G([X, Z], Y) - G([Y, Z], X); \quad \forall X, Y, Z \in \chi(M)$$
and is characterized by the conditions
\[ \nabla G = 0, \quad T = 0, \]
where \( T \) is the torsion tensor of \( \nabla \).
In the case of the tangent bundle \( TM \) we can obtain the explicit expression of \( \nabla \).
The symmetric \( 2n \times 2n \) matrix
\[
\begin{pmatrix}
G_{ij}^{(1)} & G_{ij}^{(3)} \\
G_{ij}^{(3)} & G_{ij}^{(2)}
\end{pmatrix}
\]
associated to the metric \( G \) in the base \( (\delta_{x}^j, \ldots, \delta_{x}^n, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}) \) has the inverse
\[
\begin{pmatrix}
H_{ij}^{(1)} & H_{ij}^{(3)} \\
H_{ij}^{(3)} & H_{ij}^{(2)}
\end{pmatrix}
\]
where the entries are the blocks
\[
H_{(1)}^{kl} = p_1 g^{kl} + q_1 y^k y^l
\]
(4)
\[
H_{(2)}^{kl} = p_2 g^{kl} + q_2 y^k y^l
\]
\[
H_{(3)}^{kl} = p_3 g^{kl} + q_3 y^k y^l.
\]
Here \( g^{kl} \) are the components of the inverse of the matrix \((g_{ij})\) and \( p_1, q_1, p_2, q_2, p_3, q_3 : [0, \infty) \to \mathbb{R} \), some real smooth functions. Their expressions are obtained by solving the system:
\[
\begin{align*}
G_{ih}^{(1)} H_{(1)}^{hk} + G_{ih}^{(3)} H_{(3)}^{hk} &= \delta_i^k \\
G_{ih}^{(1)} H_{(3)}^{hk} + G_{ih}^{(3)} H_{(2)}^{hk} &= 0 \\
G_{ih}^{(3)} H_{(1)}^{hk} + G_{ih}^{(2)} H_{(3)}^{hk} &= 0 \\
G_{ih}^{(3)} H_{(3)}^{hk} + G_{ih}^{(2)} H_{(2)}^{hk} &= \delta_i^k,
\end{align*}
\]
in which we substitute the relations (3) and (4). By using Lemma 2.1 we get
\( p_1, p_2, p_3 \) as functions of \( c_1, c_2, c_3 \)
\[
(5) \quad p_1 = \frac{c_2}{c_1 c_2 - c_3^2}, \quad p_2 = \frac{c_1}{c_1 c_2 - c_3^2}, \quad p_3 = -\frac{c_3}{c_1 c_2 - c_3^2}
\]
and \( q_1, q_2, q_3 \) as functions of \( c_1, c_2, c_3, d_1, d_2, d_3, p_1, p_2, p_3 \)
\[
q_1 = -\frac{c_2 d_1 p_1 - c_3 d_3 p_1 - c_3 d_2 p_3 c_2 d_3 p_3 + 2 d_1 d_2 p_1 t - 2 d_1 d_3 p_3}{c_1 c_2 - c_3^2 + 2 c_2 d_1 t + 2 c_1 d_2 t - 4 c_3 d_3 t + 4 d_1 d_2 t^2 - 4 d_3^2 t^2},
\]
(6)
\[
q_2 = -\frac{d_2 p_2 + d_3 p_3}{c_2 + 2 d_2 t} + \frac{(c_3 + 2 d_3 t) [(d_3 p_1 + d_2 p_3) (c_1 + 2 d_1 t) - (d_1 p_1 + d_3 p_3) (c_3 + 2 d_3 t)]}{(c_2 + 2 d_2 t) [(c_1 + 2 d_1 t) (c_2 + 2 d_2 t) - (c_3 + 2 d_3 t)^2]}
\]
\[
q_3 = -\frac{(d_3 p_1 + d_2 p_3) (c_1 + 2 d_1 t) - (d_1 p_1 + d_3 p_3) (c_3 + 2 d_3 t)}{(c_1 + 2 d_1 t) (c_2 + 2 d_2 t) - (c_3 + 2 d_3 t)^2}.
\]
In the paper [15] we obtained the expression of the Levi Civita connection of the Riemannian metric \( G \) on \( TM \).
Theorem 3.1. The Levi-Civita connection $\nabla$ of $G$ has the following expression in the local adapted frame \((\frac{\partial}{\partial y^j}, \ldots, \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n})\)

\[
\begin{align*}
\nabla \frac{\partial}{\partial y^j} \frac{\partial}{\partial y^j} &= Q_{ij}^h \frac{\partial}{\partial y^j} + \hat{Q}_{ij}^h \frac{\delta}{\delta x^1} + \nabla \frac{\delta}{\delta x^1} \frac{\partial}{\partial y^j} = \left(\Gamma_{ij}^h + \widetilde{Q}_{ij}^h \right) \frac{\partial}{\partial y^j} + P_{ij}^h \frac{\delta}{\delta x^1}, \\
\nabla \frac{\partial}{\partial y^j} \frac{\delta}{\delta x^1} &= P_{ij}^h \frac{\delta}{\delta x^1} + \hat{P}_{ij}^h \frac{\partial}{\partial y^j}, \quad \nabla \frac{\delta}{\delta x^1} \frac{\partial}{\partial y^j} = \left(\Gamma_{ij}^h + \widetilde{S}_{ij}^h \right) \frac{\delta}{\delta x^1} + S_{ij}^h \frac{\partial}{\partial y^j},
\end{align*}
\]

where $\Gamma_{ij}^h$ are the Christoffel symbols of the connection $\nabla$ and the M-tensor fields appearing as coefficients in the above expressions are given as

\[
\begin{align*}
Q_{ij}^h &= \frac{1}{2} (\partial_i G_{jk}^{(2)} + \partial_j G_{ik}^{(2)} - \partial_k G_{ij}^{(2)}) H_{(2)}^{kh} + \frac{1}{2} (\partial_i G_{jk}^{(3)} + \partial_j G_{ik}^{(3)}) H_{(3)}^{kh} \\
\hat{Q}_{ij}^h &= \frac{1}{2} (\partial_i G_{jk}^{(3)} + \partial_j G_{ik}^{(3)}) H_{(3)}^{kh}, \\
P_{ij}^h &= \frac{1}{2} (\partial_i G_{jk}^{(3)} - \partial_k G_{ij}^{(3)}) H_{(3)}^{kh} + \frac{1}{2} (\partial_i G_{jk}^{(1)} - \partial_j G_{ik}^{(1)}) H_{(1)}^{kh} + R^l_{0ij} G_{lk}^{(2)} H_{(2)}^{kh} + c_3 R_{0ij} H_{(3)}^{kh}, \\
\hat{P}_{ij}^h &= \frac{1}{2} (\partial_i G_{jk}^{(1)} + \partial_j G_{ik}^{(1)}) H_{(1)}^{kh} + \frac{1}{2} (\partial_i G_{jk}^{(2)} + \partial_j G_{ik}^{(2)}) H_{(2)}^{kh} + c_3 R_{0ij} H_{(1)}^{kh}, \\
S_{ij}^h &= -\frac{1}{2} (\delta_P G_{ij}^{(2)} + R^l_{0ij} G_{lk}^{(2)} H_{(2)}^{kh} + c_3 R_{0ij} H_{(3)}^{kh}), \\
\hat{S}_{ij}^h &= -\frac{1}{2} (\delta_P G_{ij}^{(1)} + R^l_{0ij} G_{lk}^{(1)} H_{(1)}^{kh} + c_3 R_{0ij} H_{(3)}^{kh}).
\end{align*}
\]

where $R^l_{0ij}$ are the components of the curvature tensor field of the Levi-Civita connection $\nabla$ of the base manifold $(M, g)$.

Taking into account the expressions (4), (5) and by using the formulas (5), (6) we can obtain the detailed expressions of $P_{ij}^h, Q_{ij}^h, S_{ij}^h, \hat{P}_{ij}^h, \hat{Q}_{ij}^h, \hat{S}_{ij}^h$.

The curvature tensor field $K$ of the connection $\nabla$ is defined by the well known formula

$$K(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \quad X, Y, Z \in \Gamma(TM).$$

By using the local adapted frame \((\frac{\delta}{\delta x^1}, \frac{\partial}{\partial y^j})\), $i, j = 1, \ldots, n$ we obtained in [15], after a standard straightforward computation

\[
\begin{align*}
K\left(\frac{\delta}{\delta x^1}, \frac{\partial}{\partial y^j}\right) \frac{\delta}{\delta x^1} \frac{\delta}{\delta x^1} &= XXXX_{ij}^h \frac{\delta}{\delta x^1} + XXXY_{ij}^h \frac{\partial}{\partial y^j} \\
K\left(\frac{\delta}{\delta x^1}, \frac{\partial}{\partial y^j}\right) \frac{\partial}{\partial y^j} \frac{\partial}{\partial y^j} &= XXX \frac{\delta}{\delta x^1} + XX Y \frac{\delta}{\delta x^1} + XYX \frac{\delta}{\delta x^1} + Y Y X \frac{\delta}{\delta x^1} \\
K\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^j}\right) \frac{\partial}{\partial y^j} \frac{\delta}{\delta x^1} &= Y Y X \frac{\delta}{\delta x^1} + Y Y X \frac{\delta}{\delta x^1} + Y Y X \frac{\delta}{\delta x^1} \\
K\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^j}\right) \frac{\delta}{\delta x^1} \frac{\delta}{\delta x^1} &= Y Y X \frac{\delta}{\delta x^1} + Y Y X \frac{\delta}{\delta x^1} + Y Y X \frac{\delta}{\delta x^1} + Y Y Y \frac{\delta}{\delta x^1} \\
K\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^j}\right) \frac{\delta}{\delta x^1} \frac{\delta}{\delta x^1} &= Y Y X \frac{\delta}{\delta x^1} + Y Y X \frac{\delta}{\delta x^1} + Y Y X \frac{\delta}{\delta x^1} + Y Y Y \frac{\delta}{\delta x^1}
\end{align*}
\]

where the M-tensor fields appearing as coefficients denote the horizontal and vertical components of the curvature tensor of the tangent bundle, and they are given by

\[
\begin{align*}
XXX X_{ij}^h &= \delta_S^h \delta_{ij}^h - l_k^h l_{ij}^h - l_k^h l_{ij}^h + R_{0ij}^h P_{lk}^h \\
XXX Y_{ij}^h &= \delta_{ij}^h l_k^h + l_k^h l_{ij}^h - l_k^h l_{ij}^h - \hat{P}_{ij}^h l_{sk}^h - \hat{P}_{ij}^h l_{sk}^h + \hat{P}_{ij}^h l_{sk}^h - \hat{P}_{ij}^h l_{sk}^h - \frac{1}{2} \hat{\nabla}_i R_{0jk} G_{rl}^{(2)} H_{hl}^{(3)} + c_3 \hat{\nabla}_i R_{0jk} l_{rh}^{(2)} \delta_{ij}^h
\end{align*}
\]
Next we get the first order partial derivatives with respect to the tangential coordinates $y^i$ of the entries of the matrices $G$ and $H$

\[ \partial_i G_{jk}^{(a)} = c'_{a} g_{ai} g_{jk} + d'_{a} g_{ai} y^i g_{jk} + d_{a} g_{ij} g_{0k} + d_{a} g_{0i} g_{jk} \]

\[ \partial_i H_{jk}^{(a)} = p'_{a} g_{ijk} + a'_{a} g_{0i} y^i y^k + q_{a} \delta^j_i y^k + q_{a} y^j \delta^i_k \]

\[ \partial_i \partial_j G_{kl}^{(a)} = c''_{a} g_{0i} g_{jk} + c'_{a} g_{ij} g_{kl} + d''_{a} g_{0i} g_{jk} g_{0l} + d'_{a} g_{ij} g_{0k} g_{0l} + \]

\[ + d''_{a} g_{0i} g_{jk} g_{0l} + d''_{a} g_{0i} g_{0k} g_{jl} + d''_{a} g_{0i} g_{0k} g_{jl} + d_{a} g_{0j} g_{0k} g_{0l} + d_{a} g_{0j} g_{0k} g_{0l} + \]

\[ \alpha = 1, 2, 3. \]

Next we get the first order partial derivatives with respect to the tangential coordinates $y^i$ of the $M$-tensor fields $P_{ij}^{k}, Q_{ij}^{k}, \tilde{P}_{ij}^{k}, \tilde{Q}_{ij}^{k}, S_{ij}^{k},$ and $\tilde{S}_{ij}^{k}$

\[ \partial_i Q_{jk}^{h} = \frac{1}{2} \partial_i H_{(2)}^{hl} (\partial_j G_{kl}^{(2)} + \partial_k G_{jl}^{(2)} - \partial_l G_{jk}^{(2)}) + \frac{1}{2} H_{(2)}^{hl} (\partial_i \partial_j G_{kl}^{(2)} + \partial_i \partial_k G_{jl}^{(2)} - \partial_i \partial_l G_{jk}^{(2)}) + \]

\[ + \frac{1}{2} \partial_i h_{(3)}^{hl} (\partial_j G_{kl}^{(3)} + \partial_k G_{jl}^{(3)} ) + \frac{1}{2} H_{(3)}^{hl} (\partial_i \partial_j G_{kl}^{(3)} + \partial_i \partial_k G_{jl}^{(3)} ) \]

\[ \partial_i \tilde{Q}_{jk}^{h} = \frac{1}{2} \partial_i H_{(3)}^{hl} (\partial_j G_{kl}^{(3)} + \partial_k G_{jl}^{(3)} - \partial_l G_{jk}^{(3)} ) + \frac{1}{2} H_{(3)}^{hl} (\partial_i \partial_j G_{kl}^{(3)} + \partial_i \partial_k G_{jl}^{(3)} - \partial_i \partial_l G_{jk}^{(3)} ) + \]

\[ + \frac{1}{2} \partial_i h_{(1)}^{hl} (\partial_j G_{kl}^{(1)} + \partial_k G_{jl}^{(1)} ) + \frac{1}{2} H_{(1)}^{hl} (\partial_i \partial_j G_{kl}^{(1)} + \partial_i \partial_k G_{jl}^{(1)} ) \]

\[ \partial_i \tilde{P}_{jk}^{h} = \frac{1}{2} \partial_i H_{(3)}^{hl} (\partial_j G_{kl}^{(3)} + \partial_k G_{jl}^{(3)} - \partial_l G_{jk}^{(3)} ) + \frac{1}{2} H_{(3)}^{hl} (\partial_i \partial_j G_{kl}^{(3)} + \partial_i \partial_k G_{jl}^{(3)} - \partial_i \partial_l G_{jk}^{(3)} ) + \]

\[ + \frac{1}{2} \partial_i H_{(1)}^{hl} (\partial_j G_{kl}^{(1)} + \partial_k G_{jl}^{(1)} ) + \frac{1}{2} H_{(1)}^{hl} (\partial_i \partial_j G_{kl}^{(1)} + \partial_i \partial_k G_{jl}^{(1)} ) \]

\[ \partial_i S_{jk}^{h} = - \frac{1}{2} \{ (\partial_i \partial_r G_{jkr}^{(1)} + R_{ijkr}^{(2)} + R_{0ikr}^{(2)} \partial_r G_{tr}^{(2)} ) H_{(2)}^{hl} \} \]
where becomes $G$ general natural lifted metric.

conformally flat of second and third order, as constants, the tangent vector $y$.

Simplify. We decided to consider these functions as well as their derivatives of first, second and third order, as constants, the tangent vector $y$ as a first order tensor, the components $G^{(1)}_{ij}, G^{(2)}_{ij}, G^{(3)}_{ij}$, $H^{(1)}_{ij}, H^{(2)}_{ij}, H^{(3)}_{ij}$ as second order tensors and so on, on the Riemannian manifold $M$, the associated indices being $h, i, j, k, l, r, s$.

We consider an $n$-dimensional Riemannian manifold $M$ with the fundamental metric $g$. The change of the metric

$$g^* = \rho^2 g,$$

where $\rho$ is a certain positive function, does not change the angle between two vectors at a point and so is called a conformal transformation of the metric.

The Weyl conformal curvature tensor is a tensor field invariant under any conformal transformation of the metric and it is given by the expression

$$C(X,Y)Z = K(X,Y)Z + L(Y,Z)X - L(X,Z)Y + g(Y,Z)NX - g(X,Z)NY,$$

with $g(NX,Y) = L(X,Y)$, for any vector fields $X, Y, Z$, where $L$ is a $(2,0)$-tensor field, called by some mathematicians the tensor of Brinkmann, given by

$$L(X,Y) = -\frac{1}{n-2}R(X,Y) + \frac{1}{2(n-1)(n-2)}rg(X,Y).$$

In local coordinates we have the expressions

$$C^h_{kij} = K^h_{kij} + \delta^h_k L_{ji} - \delta^h_j L_{ki} + L^h_k g_{ij} - L^h_j g_{ki},$$

$$L_{ji} = -\frac{1}{n-2}R_{ij} + \frac{1}{2(n-1)(n-2)}rg_{ji},$$

$$L^h_k = L_{kt} g^h.$$

The tensor $C$ vanishes identically for $n = 3$.

If a Riemannian metric $g$ is conformally related to a Riemannian metric $g^*$ which is locally Euclidian, then the Riemannian manifold with the metric $g$ is said to be conformally flat.

For the tangent bundle $TM$ of an $n$-dimensional Riemannian manifold, with the general natural lifted metric $G$, the expression of the tensor of conformal curvature becomes

$$C(X,Y)Z = K(X,Y)Z + L(Y,Z)X - L(X,Z)Y + G(Y,Z)NX - G(X,Z)NY,$$

where

$$L(X,Y) = -\frac{1}{2(n-1)}Ric(X,Y) + \frac{1}{4(n-1)(2n-1)}scal G(X,Y);$$

$$scal = G^{ji}R_{ji}; \quad G(NX,Y) = L(X,Y).$$

By using the local addapted frame, we obtain

$$C\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) \frac{\delta}{\delta x^k} = CXXX X^h_{kij} \frac{\delta}{\delta x^h} + CXXX Y^h_{kij} \frac{\partial}{\partial y^h}.$$
the horizontal and vertical components of the tensors $L$ and $N$ being

$$LXX_{ij} = -\frac{1}{2(n-1)} \left( \text{Ric}XX_{ij} - \frac{1}{2(2n-1)} \text{scal} G^{(1)}_{ij} \right)$$

$$LXY_{ij} = -\frac{1}{2(n-1)} \left( \text{Ric}XY_{ij} - \frac{1}{2(2n-1)} \text{scal} G^{(3)}_{ij} \right)$$

$$LYX_{ij} = -\frac{1}{2(n-1)} \left( \text{Ric}YX_{ij} - \frac{1}{2(2n-1)} \text{scal} G^{(3)}_{ij} \right)$$

$$LYY_{ij} = -\frac{1}{2(n-1)} \left( \text{Ric}YY_{ij} - \frac{1}{2(2n-1)} \text{scal} G^{(2)}_{ij} \right)$$

$$NXX_{j} = LXX_{jk}H^{kh}_{(1)} + LXY_{jk}H^{kh}_{(3)}$$

$$NXY_{j} = LXX_{jk}H^{kh}_{(3)} + LXY_{jk}H^{kh}_{(2)}$$

$$NYX_{j} = LÝ_{jk}H^{kh}_{(1)} + LY_{jk}H^{kh}_{(3)}$$

$$NYY_{j} = LÝ_{jk}H^{kh}_{(3)} + LY_{jk}H^{kh}_{(2)}.$$
curvature tensor. In this study it is useful the following generic result similar to the lemma 2.1.

**Lemma 3.2.** If $\alpha_1, \ldots, \alpha_{10}$ are smooth functions on $TM$ such that
\[
\alpha_1 \delta^h_{ij} g_{jk} + \alpha_2 g^{h}_{ik} g_{jk} + \alpha_3 \delta^h_{ij} g_{ij} + \alpha_4 g^{h}_{ik} g_{0j} + \alpha_5 \delta^h_{ij} g_{0j} g_{0k} + \\
+ \alpha_7 g_{ik} g_{0j} y^h + \alpha_8 g_{ik} g_{0j} y^h + \alpha_9 g_{ij} g_{0k} y^h + \alpha_{10} g_{0j} g_{0j} g_{0k} y^h = 0,
\]
then $\alpha_1 = \cdots = \alpha_{10} = 0$.

After the analysis of the values in $y = 0$ of several components of the Weyl tensor of conformal curvature, computed by using the RICCI package from Mathematica, we can formulate the next theorem.

**Theorem 3.3.** Let $(M, g)$ be a Riemannian manifold. If the tangent bundle $TM$ with the general natural lifted metric $G$ is conformally flat, then the base manifold is of constant sectional curvature.

**Proof:** From the vanishing condition for the component $C_{kij}^h$ computed in $y = 0$, we find the expression of the Ricci tensor of the base manifold and if we replace this expression in the component $C_{kij}^h$ computed in $y = 0$, we obtain that the Riemannian curvature of the base manifold is given by
\[\text{Ric}_{kij}^h = c(g_{jk} \delta^h_i - g_{ik} \delta^h_j),\]
so, the base manifold is of constant sectional curvature.

4. **Conformally flat tangent bundles**

A detailed analysis of the annulation of all the components of the conformal curvature computed with RICCI, leads to the study of several cases, which give rise to the next theorems.

**Theorem 4.1.** Let $(M, g)$ be a Riemannian manifold and let $G$ be the general natural lifted metric to the tangent bundle, given by the relations (7). Assume that $c_2 + 2dt_2 \neq 0$, $c_3 + 2dt_3 \neq 0$, and $c_1c_2 - c_3^2 + 2c_2^2t + 2c_1d_2 - 4c_3d_3 + 4c_2d_2^2 - 4c_1d_3^2 \neq 0$. Then the Riemannian manifold $(TM, G)$ is conformally flat if and only if the base manifold is flat and the associated matrix of the natural lifted metric $G$ has one of the forms:
\[
\begin{pmatrix}
0 & \beta g_{ij} + \gamma g_{0i}g_{0j} & \alpha g_{ij} + (\alpha' + \frac{\beta \gamma}{\beta - \gamma}) g_{0i}g_{0j} \\
\beta g_{ij} + \gamma g_{0i}g_{0j} & 0 & \alpha g_{ij} + (\alpha' + \frac{\beta \gamma}{\beta - \gamma}) g_{0i}g_{0j} \\
\alpha g_{ij} + (\alpha' + \frac{\beta \gamma}{\beta - \gamma}) g_{0i}g_{0j} & \alpha g_{ij} + (\alpha' + \frac{\beta \gamma}{\beta - \gamma}) g_{0i}g_{0j} & 0
\end{pmatrix},
\]
where $k$ is a nonzero arbitrary real constant, $\alpha, \beta, \gamma$ are some arbitrary real smooth functions depending on the energy density, $\alpha \neq \frac{\beta^2}{k}$ and $\beta$ is nonnull.

**Proof:** In the proposition 4.3 we proved that the base manifold of the conformally flat tangent bundle must have constant sectional curvature, c. By using the RICCI package of the program Mathematica, we replace the corresponding expressions of the components of $K$ in all the components of the Weyl conformal curvature tensor of $TM$. After a quite long computation we find some components in which the third terms are of one of the forms:
\[
\frac{c_1c_3(c_2 - d_1)}{4(c_1c_2 - c_3^2)} \delta^h_{k} y^j y^j \quad \text{in} \quad CYXXX_{kij}^h.
\]
\[
\frac{c_3(c_2 - d_1)}{2(-c_1c_2 + c_3^2)} g_{ij} \delta_h^i \delta_k^h g_{ij} \quad \text{in} \quad CXXXY^h_{kij}
\]
\[
\frac{c_1(c_2 - d_1)}{2(c_1c_2 - c_3^2)} g_{ij} \delta_h^i \quad \text{in} \quad CYXXY^h_{kij}
\]
\[
\frac{c_2(c_2 - d_1)}{2(c_1c_2 - c_3^2)} g_{ij} \delta_h^i \quad \text{in} \quad CYXY^h_{kij}
\]
\[
\frac{c_3(c_2 - d_1)}{2(c_3^2 - c_1c_2)} g_{ij} \delta_h^i \quad \text{in} \quad CYXY^h_{kij}.
\]

Since the metric \( G \) must be non-degenerate, \( c_1, c_2 \) and \( c_3 \) cannot vanish at the same time, so we must have \( d_1 = cc_2 \). Replacing this expression of \( d_1 \) in all the components of the Weyl conformal curvature tensor we get some simpler expressions.

The annulation of the first two coefficients which appear in the new expression of \( CXXXXY^h_{kij} \) implies the annulation of the product
\[
4c(c_1c_2 - c_3^2)t(c_1c_3 + 2cc_2c_3 + 4cc_3d_2t + 2c_1d_3t),
\]
which leads to two cases: \( c = 0 \) or \( c_1 = \frac{2cc_2 + 2d_2t}{c_1 + 2d_3t} \).

The above theorem refers to the first case, namely at the case when the base manifold is flat. The second case will be discussed in the theorem \[4.4\].

In the first case, from the vanishing condition for the third term of the component \( CXXXXY^h_{kij} \), we obtain that
\[
c_1' = \frac{c_1}{c_3}(c_3' - d_3)
\]
and the only coefficient that remains in \( CXXXXY^h_{kij} \) and which must vanish is
\[
-c_1'(c_3' - d_3)(c_3 + 2td_3) = 0.
\]

The annulation of \( c_1 \) leads, after quite long computations, to the first form of the matrix and the condition \( d_3 = c_3' \) leads to the second form presented in the theorem.

The subcase \( c_3 + 2td_3 = 0 \) will be treated separately (it has been excluded above), because it is a singular case, and the form obtained for the associated matrix in this case will be presented in the theorem \[4.2\].

**Remark (see [4]):** If the tangent bundle \( (TM,G) \) of a Riemannian manifold \((M,g)\) is of constant sectional curvature, that means the associated matrix of \( G \) has the form
\[
\begin{pmatrix}
kg_{ij} & \beta g_{ij} + \beta' g_{i0}g_{0j} + \alpha g_{ij} + \frac{\alpha'^2 + 2\alpha' \beta' - 2\alpha \beta'}{\beta^2} g_{i0}g_{0j} \\
\beta g_{ij} + \beta' g_{i0}g_{0j} & kg_{ij}
\end{pmatrix},
\]
where \( \alpha, \beta \) are some arbitrary real smooth functions depending on the energy density, \( \beta \) nonzero, and \( k \) is an arbitrary real constant, then the tangent bundle is conformally flat if the constant \( k \) becomes zero.

**Theorem 4.2.** Let \((M,g)\) be an \( n \)-dimensional Riemannian manifold and let \( G \) be the general natural lifted metric to \( TM \), having the associated matrix of the form obtained in the singular case \( c_3 + 2td_3 = 0 \), namely
\[
\begin{pmatrix}
c_1g_{ij} + d_1g_{i0}g_{0j} & -2td_3g_{ij} + d_3g_{i0}g_{0j} \\
-2td_3g_{ij} + d_3g_{i0}g_{0j} & c_2g_{ij} + d_2g_{i0}g_{0j}
\end{pmatrix},
\]
where \( c_1, d_1, c_2, d_2, d_3 \) are smooth real functions depending on the density of energy.
Assume that \( c_1^4c_2^2 + 2c_1^4c_2^2t + c_1^4c_2^2t^2 + c_1^4c_2^2t^2 - 32c_1^2d_1^4d_3^6 \neq 0 \). Then the bundle of
nonzero tangent vectors to $M$, $T_{M0}$, is conformally flat with respect to the natural lifted metric $G$, if and only if the base manifold is flat and the matrix becomes

$$
\begin{pmatrix}
k g_{ij} & \frac{\varepsilon^2}{t^2}(2g_{ij} + 2g_0 g_{0j}) \\
\frac{\varepsilon^2}{t^2}(2g_{ij} + 2g_0 g_{0j}) & \alpha g_{ij} + \frac{k\alpha'(2\alpha' + 4\alpha' + 4\alpha^2)}{2(k + 4\alpha')^2} g_{0i} g_{0j}
\end{pmatrix},
$$

where $k$ and $\varepsilon$ are two arbitrary real constants, $k \neq 0$ and $\alpha$ is a smooth real function depending on the density of energy, $\alpha \neq \frac{4e^{2/kt}}{kt}$.

**Corollary 4.3.** Let $(M, g)$ be an $n$-dimensional Riemannian manifold. The bundle of nonzero tangent vectors to $M$, $T_{M0}$, is conformally flat with respect to the natural lifted metric $G$ of diagonal type, if and only if the base manifold is flat, and the associated matrix of $G$ has the form

$$
\begin{pmatrix}
k g_{ij} & 0 \\
0 & \alpha g_{ij} + \frac{k\alpha'(2\alpha' + 4\alpha' + 4\alpha^2)}{2(k + 4\alpha')^2} g_{0i} g_{0j}
\end{pmatrix},
$$

where $k$ and $\varepsilon$ are two arbitrary real constants, $k \neq 0$ and $\alpha$ is a smooth real function depending on the density of energy, $\alpha \neq \frac{4e^{2/kt}}{kt}$.

The next theorem presents the form of the matrix associated to the general natural lifted metric $G$, obtained in the second case mentioned in the proof of the theorem namely $c_1' = \frac{2c_0(c_2 + 2d_2t)}{c_1 + 2d_2t}$. By using RICCI, we obtain that all its subcases reduce to some cases that we have already treated, except the singular case $c_2 + 2td_2 = 0$. So, the next theorem can be formulated as follows:

**Theorem 4.4.** Let $(M, g)$ be a Riemannian manifold, and let $G$ be the natural lifted metric to $TM$, having the associated matrix of the form obtained in the singular case $c_2 + 2td_2 = 0$, namely

$$
\begin{pmatrix}
c_1 g_{ij} + d_1 g_0 g_{0j} & c_3 g_{ij} + d_3 g_0 g_{0j} \\
c_3 g_{ij} + d_3 g_0 g_{0j} & -2td_2 g_{ij} + d_2 g_0 g_{0j}
\end{pmatrix},
$$

where $c_1, d_1, c_2, d_2, c_3, d_3$ are some arbitrary smooth real functions of the energy density. The tangent bundle $TM$ is conformally flat with respect to the natural metric $G$ if and only if the base manifold is flat and the matrix associated to $G$ becomes of the antidiagonal form

$$
\begin{pmatrix}
0 & c_3 g_{ij} + d_3 g_0 g_{0j} \\
c_3 g_{ij} + d_3 g_0 g_{0j} & 0
\end{pmatrix}.
$$

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