Superlogarithmic estimates on pseudoconvex domains and CR manifolds

By J. J. Kohn*

Abstract

This paper is concerned with proving superlogarithmic estimates for the operator $\square_b$ on pseudoconvex CR manifolds and using them to establish hypoellipticity of $\square_b$ and of the $\bar{\partial}$-Neumann problem. These estimates are established under the assumption that subellipticity degenerates in certain specified ways.

1. Introduction

Let $\Omega \subset X$ be a domain with a smooth boundary in a complex manifold $X$. If $h$ is a holomorphic function on $\Omega$ which is smooth up to the boundary we denote by $\hat{h}$ the restriction of $h$ to the boundary of $\Omega$. Then $\hat{h}$ satisfies the tangential Cauchy-Riemann equations: $\bar{\partial}_b \hat{h} = 0$. The operator $\bar{\partial}_b$ can be extended to forms on the boundary; this extension is useful in studying the boundary values of $\bar{\partial}$-closed forms (see [KR]). In [K1] I defined $\bar{\partial}_b$ and the associated Laplacian

\begin{equation}
\square_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b
\end{equation}

on forms on abstract CR manifolds. The operator $\square_b$ is then used to study $\bar{\partial}_b$. The analysis of $\square_b$ on $(p, q)$-forms on compact strongly pseudoconvex CR manifolds $M$ of dimension greater than three with $0 < q < 1/2(\dim M - 1)$ is carried out in [K1]. To analyze the operator $\square_b$ on $(p, q)$-forms with $0 \leq q \leq 1/2(\dim M - 1)$ and $\dim M \geq 3$ we need to use microlocal analysis; see [K3]. Microlocal analysis is also used to prove that the operators $\bar{\partial}_b$ and $\square_b$ have a closed range if $M$ is the boundary of a domain in a Stein manifold (see [K4]). In this paper we show how microlocal estimates for $\square_b$ imply estimates for the $\bar{\partial}$-Neumann problem. In particular we investigate superlogarithmic estimates but the technique presented here works for subelliptic and other types of estimates as well.

*Research was partially supported by NSF Grant DMS-9801626.
Let $M$ be a pseudoconvex CR manifold of dimension $2n - 1$. We say that $\Box_b$ is hypoelliptic at $x_0 \in M$ if whenever $\Box_b \phi$ is $C^\infty$ in some neighborhood of $x_0$ then $\phi$ is $C^\infty$ in a neighborhood of $x_0$. If $\Box_b$ is hypoelliptic on $(0, 1)$ forms then we conclude that: 1. The restriction of $\bar{\partial}_b$ to the orthogonal complement of square integrable CR functions (i.e. functions in the null space of $\bar{\partial}_b$) is hypoelliptic. 2. If the square integrable function $f$ is $C^\infty$ in a neighborhood of $x_0 \in M$ then $S_b f$, the orthogonal projection of $f$ to the space of square-integrable CR functions, is also $C^\infty$ in the same neighborhood.

Hypoellipticity is a consequence of subellipticity (see [KN]). The operator $\Box_b$ is subelliptic at $x_0$ if there exist a neighborhood $U$ and positive constants $\varepsilon$ and $C$ such that

$$\|\phi\|^2_2 \leq C(\Box_b \phi, \phi) = CQ_b(\phi, \phi),$$

for all $\phi \in C_0^\infty(U)$. Such estimates for the closely related $\bar{\partial}$-Neumann problem have been studied extensively. The necessary and sufficient condition for such an estimate to hold for the $\bar{\partial}$-Neumann problem near a point in the boundary of a pseudoconvex domain is that the D'Angelo type at the point is finite (see [D] and [C]). The approach to the problem taken in [K2] is via subelliptic multipliers; this leads to the condition of "finite ideal type", which is sufficient for subellipticity. This condition is probably equivalent to the finite D'Angelo type. In the appendix there is a discussion of these matters and of their extension to CR manifolds.

Throughout this paper we will deal with $(p, q)$-forms. However, in all proofs we will restrict ourselves to the case of $(0, q)$-forms, since we consider only local properties where the extension from type $(0, q)$ to type $(p, q)$ is trivial. To prove a priori estimates for $(0, q)$-forms we will write the decomposition $\phi = \phi^0 + \phi^+ + \phi^-$; the precise definition will be given in the next section. The method is to prove estimates analogous to (1.2) with $\phi$ replaced by $\phi^0$, $\phi^+$, and $\phi^-$, respectively. The case of $\phi^0$ being particularly simple, we have

$$\|\phi^0\|^2_1 \leq CQ_b(\phi^0, \phi^0);$$

this is the "elliptic" case with $\varepsilon = 1$ and holds on all CR manifolds. To study the other microlocalizations we consider the ideals $I_q^+(x_0)$ and $I_q^-(x_0)$ defined (approximately) as follows (the precise definition will be given in Section 5). The ideal $I_q^+(x_0)$ consists of all germs of $C^\infty$ functions at $x_0$ such that $\rho \in I_q^+(x_0)$ if

$$\|\rho \phi^+\|^2_2 \leq C\left(Q_b(\phi^+, \phi^+) + \|\phi^+\|^2\right),$$

and $\rho \in I_q^-(x_0)$ if

$$\|\rho \phi^-\|^2_2 \leq C\left(Q_b(\phi^-, \phi^-) + \|\phi^-\|^2\right).$$
Thus if $1 \in I^+_q(x_0) \cap I^-_q(x_0)$ we obtain (1.2). In the appendix we give explicit formulas for elements of these ideals and also the definition of finite ideal $q$-type. In general we have $I^+_q(x_0) \subset I^+_{q'}(x_0)$ if $q \leq q'$ and $I^-_q(x_0) = I^-_{n-q-1}(x_0)$. Hence,

$$I^+_q(x_0) \cap I^-_q(x_0) = \begin{cases} I^+_q(x_0) & \text{if } q \leq \frac{1}{2}(n-1), \\ I^-_q(x_0) & \text{if } q \geq \frac{1}{2}(n-1). \end{cases}$$

We also have $I^+_0(x_0) = I^+_{n-1}(x_0) = \{0\}$ and so the estimate (1.2) cannot hold for $(0,0)$-forms, i.e. functions, or for $(0,n-1)$-forms. In fact, for pseudoconvex compact CR manifolds, the nullspaces of $\Box_b$ on functions and on $(0,n-1)$-forms are in general infinite-dimensional (on boundaries of domains in complex manifolds $\Box_b h = 0$ is equivalent to the condition that $h$ is the restriction of a holomorphic function). It is then natural to impose the global condition of orthogonality to the nullspace of $\Box_b$ but even this assumption is not sufficient. Rossi (see [R]) has given an example of a strongly pseudoconvex three-dimensional CR manifold for which the Szegö projection does not preserve smoothness and hence the restriction of $\Box_b$ to the orthogonal complement of its nullspace is not hypoelliptic. In [K3] it is shown that for compact pseudoconvex CR manifolds on which the range of $\bar{\partial}_b$ is closed in $L^2$ and for which $1 \in I^+_q(x_0)$ the restriction of $\Box_b$ to the orthogonal complement of its nullspace is globally hypoelliptic. This closed-range property is proved for compact CR manifolds that are boundaries of domains in Stein manifolds (see [K4]).

The problem is to find conditions under which hypoellipticity holds when subellipticity fails. The most desirable condition is a purely geometric condition such as in Fedii’s theorem (see [F]), which states that the operator $\frac{\partial^2}{\partial x^2} + a(x)\frac{\partial^2}{\partial y^2}$ with $a \in C^\infty$ is hypoelliptic if $a(x) > 0$ when $x \neq 0$. In [K6] Fedii’s result is generalized to degenerate subelliptic operators in any dimension. This phenomenon appears on CR manifolds; thus on a pseudoconvex CR manifold $\Box_b$ is hypoelliptic in a neighborhood $U$ of $x_0$ if it is subelliptic at all $x$ with $x \neq x_0$. The natural geometric condition for hypoellipticity seems to be that the points where subellipticity fails are contained in a real curve transversal to the “good” directions (i.e. the tangent vectors to the curve do not lie in $T^{(1,0)} + T^{(0,1)}$); this is true for certain classes of CR manifolds (see [K5] and [K6]) but Christ has shown that it is false in general (see [Ch2]). The example of Kusuoka and Stroock (see [KS]) illustrates that, in general, the geometry alone of the set on which the operator degenerates does not give sufficient information to determine whether it is hypoelliptic. They study the operator $\frac{\partial^2}{\partial x^2} + a^2(x)\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in $\mathbb{R}^3$ with $a(x) \neq 0$ if $x \neq 0$ and prove that it is hypoelliptic if and only if $\lim_{x \to 0} x \log a(x) = 0$. Hypoellipticity of such operators is obtained by use of superlogarithmic estimates (see [Ch1] and [M]). Superlogarithmic estimates for $\Box_b$ are formulated as follows.
Definition 1.1. If $M$ is a CR manifold and $x_0 \in M$ then a superlogarithmic estimate holds for $\square_b$ on positively microlocalized $(p,q)$-forms at $x_0$ if there exists a neighborhood $U$ of $x_0$ such that for each $\delta > 0$ there exists $C_\delta$ such that
\begin{equation}
\|(\log \Lambda)\varphi^+\|^2 \leq \delta Q_b(\varphi^+,\varphi^+) + C_\delta\|\varphi^+\|^2,
\end{equation}
for all $(p,q)$ forms $\varphi$ with support in $U$. Here $\Lambda$ denotes the square root of the Laplacian. A superlogarithmic estimate holds for $\square_b$ on negatively microlocalized $(p,q)$-forms at $x_0$ if the above holds with $\varphi^+$ replaced by $\varphi^-$. In that case
\begin{equation}
\|(\log \Lambda)\varphi^-\|^2 \leq \delta Q_b(\varphi^-,\varphi^-) + C_\delta\|\varphi^-\|^2.
\end{equation}

Here we will prove superlogarithmic estimates of the form (1.6) and (1.7) when there exists a subelliptic multiplier $\rho$ which may go to zero on a submanifold $S$ at a controlled rate. The condition which we will use is the following:
\begin{equation}
\lim_{x \to S} d(x,S) \log |\rho(x)| = 0,
\end{equation}
where $d(x,S)$ denotes the distance from $x$ to $S$. Our results will impose conditions on the holomorphic dimension of $S$; this concept will be recalled in Section 4.

Theorem 1.2. Let $M$ be a pseudoconvex CR manifold. Let $S \subset M$ be a manifold such that the holomorphic dimension of $S$ at each point is less than or equal to $q - 1$. Let $x_0 \in S$ and $\rho \in \mathcal{I}_q^+(x_0)$. Suppose that $\rho$ satisfies (1.8); then there exists a neighborhood $U$ of $x_0$ on which the superlogarithmic estimate (1.6) is satisfied for all $(0,q)$-forms with support in $U$.

Theorem 1.3. Let $M$ be a pseudoconvex CR manifold. Let $S \subset M$ be a manifold such that the holomorphic dimension of $S$ at each point is less than or equal to $n - q - 2$. Let $x_0 \in S$ and $\rho \in \mathcal{I}_q^-(x_0)$. Suppose that $\rho$ satisfies (1.8); then there exists a neighborhood $U$ of $x_0$ on which the superlogarithmic estimate (1.7) is satisfied for all $(0,q)$-forms with support in $U$.

One of the crucial steps in proving these results is the following localization lemma, which may be of independent interest.

Lemma 1.4 (Localization Lemma). Let $M$ be pseudoconvex CR manifold and let $S \subset M$ be a submanifold of holomorphic dimension less than or equal to $m$. Let $x_0 \in S$. For small positive $a$ let $S_a$ denote the set of $x \in M$ such that $d(x,S) < a$. Then there exists a neighborhood $U$ of $x_0$ and a constant $C$ such that
\begin{equation}
\|\varphi^+\|_{S_a}^2 \leq C \left( a^2 Q_b(\varphi^+,\varphi^+) + \|\varphi^+\|_{M-S_a}^2 + \|\varphi^+\|_{L^2}^2 \right),
\end{equation}
for all $(p,q)$-forms $\varphi$ with $q \geq m+1$ supported in $U$. Furthermore,

$$
\|\varphi^-\|_{S_a}^2 \leq C \left( a^2 Q_b(\varphi^-, \varphi^-) + \|\varphi^-\|_{M-S_a}^2 + \|\varphi^+\|_{-1}^2 \right),
$$

for all $(p,q)$-forms $\varphi$ with $q \leq n-m-2$ supported in $U$. Here $\|\cdot\|_{S_a}$ and $\|\cdot\|_{M-S_a}$ denote $L_2$-norms over $S_a$ and $M-S_a$, respectively.

These theorems imply that the superlogarithmic estimate

$$
\|\log \Lambda \varphi\|^2 \leq \delta Q_b(\varphi, \varphi) + C_{\delta} \|\varphi\|^2
$$

holds, when $M$ is pseudoconvex, for $(0,q)$-forms supported in $U$ whenever $(1.8)$ with $\rho \in T^*_q(x_0) \cap T^{-}_q(x_0)$ holds for $S$ of holomorphic dimension $m \leq \min\{q-1, n-q-2\}$. Note that the estimate (1.11) cannot hold if $q = 0$ or $q = n-1$.

The superlogarithmic estimates (1.6), (1.7), and (1.11) imply the following results concerning hypoellipticity.

**Theorem 1.5.** Let $M$ be a pseudoconvex CR manifold and let $x_0 \in M$ such that (1.11) holds for $(p,q)$-forms supported in a neighborhood $U$ of $x_0$. Suppose that $\varphi$ and $\alpha$ are $(p,q)$-forms on $M$ in $L_2(M)$ such that $\Box_b(\varphi) = \alpha$. Suppose further that the restriction of $\alpha$ to $U$ is in $C^\infty(U)$. Then the restriction of $\varphi$ to $U$ is also in $C^\infty(U)$.

Denote by $\mathcal{H}_{p,q}^b(M)$ the nullspace of $\Box_b$ on $(p,q)$-forms. If $M$ is compact and if (1.11) holds for $(p,q)$-forms in some neighborhood of every point of $M$ then $\mathcal{H}_{p,q}^b(M)$ consists of $C^\infty$ forms and is finite-dimensional. Furthermore, under this assumption, we have the following hypoellipticity consequences:

(a) Whenever $\alpha \in L_2(M)$ and $\alpha \perp \mathcal{H}_{p,q}^b(M)$ the equation $\Box_b(\varphi) = \alpha$ has a unique solution $\varphi \perp \mathcal{H}_{p,q}^b(M)$.

(b) If $\bar{\partial}_b \alpha = 0$ then there exists a unique $(p, q-1)$-form $\psi$ such that $\bar{\partial}_b \psi = \alpha$ and $\psi$ is perpendicular to the nullspace of $\bar{\partial}_b$; the restriction of $\psi$ to any open set on which $\alpha$ is $C^\infty$ is also $C^\infty$.

(c) Dually, if $\bar{\partial}^*_b \alpha = 0$ then there exists a unique $(p, q+1)$-form $\theta$ such that $\bar{\partial}^*_b \theta = \alpha$ and $\theta$ is perpendicular to the nullspace of $\bar{\partial}^*_b$; the restriction of $\theta$ to any open set on which $\alpha$ is $C^\infty$ is also $C^\infty$.

(d) The orthogonal projection operators on $\mathcal{H}_{p,q}^b(M)$ and on the nullspaces of $\bar{\partial}_b$ and $\bar{\partial}^*_b$ are pseudolocal (that is, the operator applied to a form is $C^\infty$ on the open set on which the form is $C^\infty$).

The next theorem deals with the cases of $(p,n-1)$ and $(p,0)$-forms. To deal with these we will assume that the range of $\bar{\partial}_b$ on functions is closed. We will also assume that the estimate (1.6) holds for $(p,n-1)$-forms which is
are satisfied when \( M \) is equivalent to (1.7) for (\( p, 0 \))-forms. In view of the above results these conditions are satisfied when \( M \) is pseudoconvex and there exists \( \rho \in I^1(x_0) \) which satisfies (1.8) with \( S \) of holomorphic dimension zero (i.e. \( S \) totally real).

**Theorem 1.6.** Let \( M \) be a CR manifold such that the range of \( \bar{\partial}_b \) on functions is closed. Let \( x_0 \in M \) such that (1.6) holds for (\( p, n - 1 \))-forms and (1.7) holds for (\( p, 0 \))-forms supported in a neighborhood \( U \) of \( x_0 \). Suppose that \( \varphi \) and \( \alpha \) are (\( p, n - 1 \))-forms, on \( M \) in \( L_2(M) \), which are orthogonal to \( \mathcal{H}_{b}^{0, n-1}(M) \), such that \( \Box_b(\varphi) = \alpha \). Suppose further that the restriction of \( \alpha \) to \( U \) is in \( C^\infty(U) \). Then the restriction of \( \varphi \) to \( U \) is also in \( C^\infty(U) \). Dually, the same holds when \( \varphi \) and \( \alpha \) are (\( p, 0 \))-forms, on \( M \) in \( L_2(M) \), which are orthogonal to \( \mathcal{H}_{b}^{0, 0}(M) \), such that \( \Box_b(\varphi) = \alpha \).

Now the spaces \( \mathcal{H}_{b}^{0, 0}(M) \) and \( \mathcal{H}_{b}^{p, n-1}(M) \) are, in general, infinite dimensional. Nevertheless, if (1.5) holds for (\( p, n - 1 \))-forms at each point of \( M \) we have the properties:

(a) The closed range of \( \bar{\partial}_b \) on functions implies that \( \Box_b \) has closed range on all forms. Hence, for \( q = 0 \) and \( q = n - 1 \), whenever \( \alpha \in L_2(M) \) and \( \alpha \perp \mathcal{H}_{b}^{0, n-1}(M) \) the equation \( \Box_b(\varphi) = \alpha \) has a unique solution \( \varphi = \perp \mathcal{H}_{b}^{0, n-1}(M) \).

(b) If \( \alpha \) is a (\( p, n - 1 \))-form orthogonal to \( \mathcal{H}_{b}^{p, n-1}(M) \) with \( \bar{\partial}_b\alpha = 0 \) then there exists a unique (\( p, n - 2 \))-form \( \psi \) such that \( \bar{\partial}_b\psi = \alpha \) and \( \psi \) is perpendicular to the nullspace of \( \bar{\partial}_b \); the restriction of \( \psi \) to the open set on which \( \alpha \) is \( C^\infty \) is also \( C^\infty \).

(c) Dually, if \( \alpha \) is a (\( p, 0 \))-form orthogonal to \( \mathcal{H}_{b}^{p, 0}(M) \) with \( \bar{\partial}_b^*\alpha = 0 \) then there exists a unique (\( p, 1 \))-form \( \theta \) such that \( \bar{\partial}_b^*\theta = \alpha \) and \( \theta \) is perpendicular to the nullspace of \( \bar{\partial}_b \); the restriction of \( \theta \) to the open set on which \( \alpha \) is \( C^\infty \) is also \( C^\infty \).

(d) The orthogonal projection operators on \( \mathcal{H}_{b}^{p, 0}(M) \), \( \mathcal{H}_{b}^{p, n-1}(M) \) and on the nullspaces of \( \bar{\partial}_b \) in (\( p, n - 2 \))-forms and \( \bar{\partial}_b \) in (\( p, 1 \))-forms are pseudolocal.

Finally we take up the applications of superlogarithmic estimates to the \( \bar{\partial} \)-Neumann problem. Let \( \Omega \) be a domain in a complex \( n \)-dimensional manifold \( X \). Assume that the boundary of \( \Omega \), denoted by \( b\Omega \), is \( C^\infty \). For (\( p, q \))-forms on \( \Omega \) we define the operator \( \Box \) by

\[
(1.12) \quad \Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial},
\]

where \( \bar{\partial}^* \) is the \( L_2 \) adjoint of \( \bar{\partial} \). Then \( \Box \varphi \) is a form whose components are Laplacians of the corresponding components of \( \varphi \) which satisfies the \( \bar{\partial} \)-Neumann boundary conditions: \( \varphi \) is contained in the domain of \( \bar{\partial}^* \) on (\( p, q \))-forms and \( \bar{\partial}\varphi \) is contained in the domain of \( \bar{\partial}^* \) on (\( p, q + 1 \))-forms. Let \( \mathcal{H}^{p,q}(\Omega) \) denote the null space of \( \Box \). Then we have:
Theorem 1.7. Suppose that $x_0 \in b\Omega$ and suppose that (1.6) holds for $(p, q)$-forms on $b\Omega$ supported in $U \cap b\Omega$, where $U$ is a neighborhood of $x_0$ in $X$ and $1 \leq q \leq n - 2$. Suppose further that $\square \varphi = \alpha$ with $\alpha$ a $(p, q)$-form in $L_2$ whose restriction to $U \cap \Omega$ is in $C^\infty$. Then the restriction of $\varphi$ to $U \cap \Omega$ is in $C^\infty$.

Note that the hypothesis of the above theorem is satisfied if $b\Omega$ is pseudoconvex and if there exists $\rho \in T^+_q(x_0)$ satisfying (1.8) with $S$ of holomorphic dimension less than or equal to $q - 1$. The theorem has consequences which are analogous to those of Theorem 1.5 listed above.

Theorem 1.8. Suppose that $x_0 \in b\Omega$ and suppose that (1.6) holds for $(p, n - 1)$-forms on $b\Omega$ supported in $U \cap b\Omega$, where $U$ is a neighborhood of $x_0$ in $X$. Suppose that $X$ is a Stein manifold and that $\Omega$ is pseudoconvex with $\bar{\Omega}$ compact. Suppose further that $\square \varphi = \alpha$ with $\alpha$ a $(p, n - 1)$-form in $L_2$ whose restriction to $U \cap \Omega$ is in $C^\infty$. Then, if $\varphi \perp H^{0, n-1}(\Omega)$, the restriction of $\varphi$ to $U \cap \Omega$ is in $C^\infty$.

Note that the hypotheses of this theorem are satisfied if there exists $\rho \in T^+_{n-1}(x_0)$ satisfying (1.8) with $S$ of holomorphic dimension less than or equal to $n - 1$. Again, the consequences of this theorem are entirely analogous to those of Theorem 1.6 listed above.

Here I wish to express my thanks to M. Christ; communication with him led to the research presented here. I am also indebted to him for supplying the example discussed at the end of Section 4. I also thank Siqi Fu and the referee for going through the original manuscript and suggesting several improvements and clarifications.

2. Definitions and preliminaries

Definition 2.1. Let $M$ be a $2n - 1$ dimensional differentiable manifold. Denote by $\mathbb{C}T(M)$ the complexified tangent bundle of $M$. An integrable CR structure on $M$ is given by a subbundle $T^{1,0} \subset \mathbb{C}T(M)$ with the following properties. If $T^{0,1}$ denotes the conjugate of $T^{1,0}$ then $T^{1,0}_x \cap T^{0,1}_x = \{0\}$ for all $x$. The subbundle $T^{1,0} \oplus T^{0,1}$ is of codimension one. If $L$ and $L'$ are local vector fields with values in $T^{1,0}$ then the commutator $[L, L'] = LL' - L'L$ also has values in $T^{1,0}$. We say $M$ is a CR manifold if $M$ has a given integrable CR structure.

Definition 2.2. A form of type $(p, q)$ at $x \in M$ is a skew-symmetric multilinear map

$$\varphi : \underbrace{T^{1,0}_x \times \cdots \times T^{1,0}_x}_{p} \times \underbrace{T^{0,1}_x \times \cdots \times T^{0,1}_x}_{q} \rightarrow \mathbb{C}.$$ 

The bundle of $(p, q)$-forms is denoted by $\mathcal{A}_{b}^{p,q}$.
Definition 2.3. The operator \( \overline{\partial}_b : \mathcal{A}^{p,q}_b \to \mathcal{A}^{p,q+1}_b \) is defined as follows. If \( \varphi \in \mathcal{A}^{p,q} \) let \( \varphi' \) be a \((p,q)\)-form which restricted to \( \prod_p T^{1,0} \times \prod_q T^{0,1} \) equals \( \varphi \). Then \( \overline{\partial}_b \varphi \) is the restriction of \( d\varphi' \) to \( \prod_p T^{1,0} \times \prod_q T^{0,1} \). The operator \( \overline{\partial}^*_b : \mathcal{A}^{p,q}_b \to \mathcal{A}^{p,q-1}_b \) is the \( L_2 \)-adjoint of \( \overline{\partial}_b \).

Definition 2.4. Denote by \( \theta \) a nonvanishing real 1-form which annihilates \( T^{1,0} \oplus T^{0,1} \). Then the Levi form on \( M \) is the hermitian form on \( T^{1,0} \) given by \( \sqrt{-1}d\theta(L, L') \), where \( L \) and \( L' \) are in \( T^{1,0} \). Now, \( M \) is pseudoconvex if for some choice of \( \theta \) the Levi form is nonnegative.

Let \( L_1, \ldots, L_{n-1} \) be a local basis for \((1, 0)\) vectorfields in a neighborhood \( U \) of \( x_0 \in M \) and let \( \omega_1, \ldots, \omega_{n-1} \) be the dual basis of \((1, 0)\)-forms. If \( u \in C^\infty(U) \),
\[
\overline{\partial}_b(u) = \sum_j \bar{L}_j(u) \bar{\omega}_j.
\]
If \( \varphi = \sum \varphi_j \bar{\omega}_j \) is a \((0, 1)\)-form on \( U \),
\[
\overline{\partial}_b \varphi = \sum_{i<j} \left( \bar{L}_j \varphi_i - \bar{L}_i \varphi_j + \sum_k a_{ij}^k \varphi_k \right) \bar{\omega}_i \wedge \bar{\omega}_j
\]
and
\[
\overline{\partial}^*_b \varphi = -\sum_i (L_i(\varphi_i) + a_i \varphi_i),
\]
where \( a_{ij}^k, a_i \in C^\infty(U) \).

In general the operator \( \overline{\partial}_b : \mathcal{A}^{0,q}_b \to \mathcal{A}^{0,q+1}_b \) is expressed as follows. Let \( \varphi \in \mathcal{A}^{0,q}_b \) be the form given locally by:
\[
\varphi = \sum_I \varphi_I \bar{\omega}_I,
\]
where \( I \) is the \( q \)-tuple of integers \( I = (i_1, \ldots, i_q) \) with \( 0 < i_1 < \cdots < i_q \leq n-1 \) and where \( \bar{\omega}_I = \bar{\omega}_{i_1} \wedge \cdots \wedge \bar{\omega}_{i_q} \). Then
\[
\overline{\partial}_b \varphi = \sum_K \left( \sum_{j \notin I} \epsilon^{jI}_K \bar{L}_j \varphi_I + a_{IK} \varphi_I \right) \bar{\omega}_K,
\]
where \( K \) runs through all strictly increasing \((q+1)\)-tuples and each of the coefficients \( \epsilon^{jI}_K \) is either 0, 1, or \(-1\) and is defined as follows. First, if \( j \notin I \) we denote by \( \langle j I \rangle \) the ordered \( q \)-tuple containing \( j \) and the elements of \( K \). Then we define
\[
\epsilon^{jI}_K = \begin{cases} 0 & \text{if } j \notin K \\ \text{sgn} \left( \frac{\langle j I \rangle}{K} \right) & \text{if } j \in K,
\end{cases}
\]
where \( \text{sgn}(\frac{\langle j I \rangle}{K}) \) denotes the sign of the permutation \( \{j, I\} \to K \). Further,
\[
\overline{\partial}^*_b \varphi = -\sum_H \left( \sum_{i, I \supseteq H} \epsilon^{iH}_I \bar{L}_i \varphi_I + a_H \varphi_H \right) \bar{\omega}_H,
\]
where \( H \) runs through all strictly increasing \((q-1)\)-tuples and \( a_{IK}, a_H \in C^\infty(U) \).
We choose real coordinates \{x_1, \ldots, x_{2n-1}\} with the origin at \(x_o\) such that, 
\[\theta(\frac{\partial}{\partial x_{2n-1}}) > 0\] and such that, setting \(z_i = x_i + \sqrt{-1}x_{i+n-1}\) for \(i = 1, \ldots, n-1\), we have \(L_i|_{x_o} = \frac{\partial}{\partial z_i}|_{x_o}\). Set \(T = -\sqrt{-1} \frac{\partial}{\partial x_{2n-1}}\). Then the Levi form can be written as \(c_{ij} = d\theta(L_i, \bar{L}_j)\) and we have 
\[|L_i, \bar{L}_j| = c_{ij}T + \sum_k (d_{ij}^k L_k + e_{ij}^k \bar{L}_k),\]
where \(d_{ij}^k, e_{ij}^k \in C^\infty(U)\).

Let \(\{\xi_1, \ldots, \xi_{2n-1}\}\) be the dual coordinates to \(\{x_1, \ldots, x_{2n-1}\}\) and \(|\xi|^2 = \sum_i \xi_i^2\). Let \(\psi^+\) and \(\tilde{\psi}^+\) be nonnegative functions in \(C^\infty(\{\xi \in \mathbb{R}^{2n-1} \mid |\xi| = 1\})\), with range in \([0, 1]\), such that
\[\supp(\psi^+) \subset \{ |\xi| = 1 \mid \xi_{2n-1} \geq \frac{1}{2} |\xi'| \}\]
\[\supp(\tilde{\psi}^+) \subset \{ |\xi| = 1 \mid \xi_{2n-1} \geq \frac{1}{4} |\xi'| \}\]
\(\psi^+(\xi) = 1\) when \(\xi_{2n-1} \geq \frac{3}{4} |\xi'|\) and \(\tilde{\psi}^+(\xi) = 1\) when \(\xi_{2n-1} \geq \frac{1}{3} |\xi'|\). Here \(\xi' = (\xi_1, \ldots, \xi_{2n-2})\). We extend these functions to \(\mathbb{R}^{2n-1}\) so that \(\psi^+(\xi) = \psi^+(\frac{\xi}{|\xi|})\) and \(\tilde{\psi}^+(\xi) = \tilde{\psi}^+(\frac{\xi}{|\xi|})\), for \(|\xi| \geq 1\) and so that \(\psi^+(\xi) = \tilde{\psi}^+(\xi) = 0\) for \(|\xi| < \frac{1}{2}\) with \(\tilde{\psi}^+(\xi) = 1\) on \(\supp(\psi^+)\). Set \(\psi^-(\xi) = \psi^+(-\xi), \tilde{\psi}^-(\xi) = \tilde{\psi}^+(-\xi)\) and \(\psi^0 = 1 - \psi^+ - \psi^-\). Define \(\tilde{\psi}^0\) so that \(\tilde{\psi}^0 = 1\) on a neighborhood of \(\supp(\psi^0)\) and so that 
\[\supp(\tilde{\psi}^0) \subset \{ |\xi| < 2 \} \cup \{ |\xi_{2n-1}| < \frac{3}{4} |\xi'| \}.\]
The operator \(\Psi\) is defined by 
\[\overline{\Psi u}(\xi) = \psi(\xi)\hat{u}(\xi).\]
The operators \(\Psi^+, \Psi^-, \psi^0, \tilde{\psi}^+, \tilde{\psi}^-,\) and \(\tilde{\psi}^0\) are defined as above with substitution of \(\psi^+, \psi^-, \psi^0, \tilde{\psi}^+, \tilde{\psi}^-\), and \(\tilde{\psi}^0\) for \(\psi\), respectively.

The microlocal decomposition \(\varphi = \varphi^+ + \varphi^- + \varphi^0\), alluded to in the introduction, is now interpreted as follows:
\[\varphi = \zeta \Psi^+ \varphi + \zeta \Psi^- \varphi + \zeta \Psi^0 \varphi,\]
for all \(\varphi \in C^\infty_0(U)\), where \(\zeta \in C^\infty_0(U')\), \(\bar{U} \subset U'\) and \(\zeta = 1\) on \(U\). The microlocalization \(\varphi^+\) is also sometimes interpreted as \(\zeta \tilde{\Psi}^+ \varphi\) and similarly with \(\varphi^-\) and \(\varphi^0\).

The following lemma is a consequence of the general Gårding inequality; here we give a proof which does not invoke the general case.

**Lemma 2.5.** Let \((a_{ij})\) be a matrix of \(C^\infty\) functions on \(U'\) which is nonnegative. Let \(U \subset \bar{U} \subset U', \zeta \in C^\infty_0(U')\) with \(\zeta = 1\) on \(U\) and \(\sigma \in C^\infty(U')\) with \(\sigma \leq 2\). Then, if \(U'\) is sufficiently small,
\[
\sum_{IJ} (a_{IJ} \sigma T \zeta \Psi^{+} \varphi_I, \sigma \zeta \Psi^{+} \varphi_J) \geq -C \left( \max_{\supp \zeta} |D(\sigma)| + 1 \right) \| \zeta \Psi^{+} \varphi \|^2 \\
+ C_\sigma \| \Psi^{+} \varphi \|_{-1} \| \zeta \Psi^{+} \varphi \| + \| \Psi^{+} \varphi \|_{-1}^2, 
\]
and
\[
\sum_{IJ} (a_{IJ} \sigma T \zeta \Psi^{-} \varphi_I, \sigma \zeta \Psi^{-} \varphi_J) \geq -C \left( \max_{\supp \zeta} |D(\sigma)| + 1 \right) \| \zeta \Psi^{-} \varphi \|^2 \\
+ C_\sigma \| \Psi^{-} \varphi \|_{-1} \| \zeta \Psi^{-} \varphi \| + \| \Psi^{-} \varphi \|_{-1}^2, 
\]
where
\[
C_\sigma = \max_{\supp \zeta} |D^2(\sigma)| + \max_{\supp \zeta} |D(\sigma)|^2 + 1,
\]
for all \( \varphi \) with \( \supp(\varphi) \subset U \). Here \( D \) denotes first partial derivatives.

Proof. Let \( \theta \) be a conical cutoff function with \( \supp \theta \subset \supp \tilde{\psi}^+ \) and \( \theta = 1 \) on \( \supp \psi^+ \); denoting by \( \Theta \) the corresponding pseudodifferential operator, we have
\[
\zeta \Psi^{+} \varphi = \zeta (\tilde{\psi}^{+})^2 \Theta \Psi^{+} \varphi = (\tilde{\psi}^{+})^2 \zeta \Psi^{+} \varphi + [\zeta, (\tilde{\psi}^{+})^2] \Theta \Psi^{+} \varphi.
\]
Then, since the supports of the symbols of \( \Theta \) and of \([\zeta, (\tilde{\psi}^{+})^2]\) are disjoint, the operator \([\zeta, (\tilde{\psi}^{+})^2] \Theta \) is of order \( -\infty \) and we have
\[
\sum_{IJ} (a_{IJ} \sigma T \zeta \Psi^{+} \varphi_I, \sigma \zeta \Psi^{+} \varphi_J) = \sum_{IJ} (a_{IJ} \sigma \tilde{\zeta} T (\tilde{\psi}^{+})^2 \zeta \Psi^{+} \varphi_I, \sigma \zeta \Psi^{+} \varphi_J) \\
+ O(\| \Psi^{+} \varphi \|_{-1}^2),
\]
where \( \tilde{\zeta} \in C_0^\infty(U') \) with \( \tilde{\zeta} = 1 \) on \( \supp \zeta \). Let \( R \) denote the pseudodifferential operator of order \( 1/2 \) whose symbol is \( \xi_{2n-1}^{1/2} \psi^+(\xi) \); then \( T(\tilde{\psi}^{+})^2 = R^* R \) and
\[
\sum_{IJ} (a_{IJ} \sigma T \zeta \Psi^{+} \varphi_I, \sigma \zeta \Psi^{+} \varphi_J) = \sum_{IJ} \left( (a_{IJ} \sigma \tilde{\zeta} R \zeta \Psi^{+} \varphi_I, \sigma \tilde{\zeta} R \zeta \Psi^{+} \varphi_J) \\
+ ([a_{IJ} \sigma \tilde{\zeta}^2, R^*] R \zeta \psi^+ \varphi_I, \zeta \psi^+ \varphi_J) \\
+ O(\| \Psi^{+} \varphi \|_{-1}^2) \right).
\]
The first term on the right is nonnegative and from the pseudodifferential operator calculus we obtain
\[
\| [a_{IJ} \sigma \tilde{\zeta}^2, R^*] R \zeta \psi^+ \varphi_I \| \leq C \left( \max_{\supp \zeta} |D(\sigma)| + 1 \right) \| \zeta \Psi^{+} \varphi \| + C_\sigma \| \zeta \Psi^{+} \varphi \|_{-1}^2 \right).
\]
The first inequality in the lemma then follows and the second is proved analogously.
3. Weighted microlocal estimates

We begin by discussing the estimate for $\zeta^+\varphi$ for $(0,1)$-forms; here the calculation is more transparent because we do not have to deal with complicated indices. We will then show how the proof generalizes to the case of $(p,q)$-forms.

**Lemma 3.1.** Let $M$ be a pseudoconvex CR manifold of dimension $2n-1$, let $x_0 \in M$ and let $\lambda$ be a real nonnegative $C^\infty$ function with $s \in \mathbb{R}^+$ such that $s\lambda \leq 1$ and $\text{Re}(L_iL_j(\lambda))$ is positive-definite. Then there exists a neighborhood $U$ of $x_0$ and $C > 0$ such that

$$s\|\zeta^+\varphi\|^2 \leq C\left(Q_b(\zeta^+\varphi, \zeta^+\varphi) + (s^2 \max_{\text{supp}(\zeta)} D(\lambda)^2 + 1)\|\zeta^+\varphi\|^2 + C(s, \lambda)\|\zeta^+\varphi\|_{-1} \|\zeta^+\varphi\| + \|\zeta^+\varphi\|_{-1}^2\right),$$

where

$$C(s, \lambda) = s \max_{\text{supp}(\zeta)} |D^2(\lambda)| + s^2 \max_{\text{supp}(\zeta)} D(\lambda)^2 + 1,$$

$\zeta \in C_0^\infty(U)$, $\varphi \in \mathcal{A}_{0,1}^0$, $\text{supp}(\varphi) \subset U'$, and $U' \supset \bar{U}$.

**Proof.** If $\varphi \in C_0^\infty(U)$ then:

$$\|\sigma\bar{\partial}_b\varphi\|^2 + \|\sigma\bar{\partial}_b^*\varphi\|^2 = \sum_{i<j} \|\sigma\tilde{L}_j\varphi_i - \tilde{L}_i\varphi_j\|^2 + \|\sigma\sum_{i} \tilde{L}_i\varphi_i\|^2 + O\left(\sum \|\sigma\tilde{L}_j\varphi_i\|\|\sigma\varphi\| + \|\sigma\varphi\|^2\right).$$

We have $\tilde{L}_i^* = -L_i + a_i$. Substituting this in the above we use the following integrations by parts to “convert” the $L$ into $\tilde{L}$:

$$(\sigma^2 L_i u, v) = -(\sigma^2 u, \tilde{L}_i v) + O(|\tilde{L}_i(\sigma)u||\sigma v| + |\sigma u||\sigma v|)$$

and

$$(\sigma^2 \tilde{L}_i^* u, \tilde{L}_i^* v) = \left(\sigma^2 \tilde{L}_i u, L_j v\right) + O(\|\sigma u\||\sigma v|$$

$$+ \sum \|\sigma\tilde{L}_k\sigma\||\sigma v| + \|\sigma u\||\sigma\tilde{L}_k\varphi\|)$$

$$= (\sigma\tilde{L}_i u, \tilde{L}_i v) + (\sigma c_{ij}Tu, \sigma v) + 2(L_i\tilde{L}_j(\sigma)u, \sigma v)$$

$$+ O\left(\|\sigma u\||\sigma v| + \|\sigma L_j u\||L_i(\sigma) v|$$

$$+ ||L_i(\sigma)u||\sigma L_j v| + \sum \|\sigma\tilde{L}_k\sigma\||\sigma v| + \|\sigma u\||\sigma\tilde{L}_k\varphi\|\right).$$

Let $\sigma_s = e^{s\lambda}$; then, substituting $\sigma_s$ for $\sigma$, we have
\[ \|\sigma_s \partial_b \varphi\|^2 = \|\sum_i \sigma_s L^*_i \varphi_i\|^2 \]
\[ = \sum_i \|\sigma_s L^*_i \varphi_i\|^2 + 2 \sum_{i<j} \text{Re}(\sigma_s^2 L^*_i \varphi_i, L^*_j \varphi_j) \]
\[ = \sum_i \|\sigma_s L_i \varphi_i\|^2 + 2 \sum_{i<j} \text{Re}(\sigma_s^2 L_j \varphi_i, L_i \varphi_j) \]
\[ + (\sigma_s c_{ij} T \varphi_i, \sigma_s \varphi_j) + (L_i L_j(\sigma_s) \varphi_i, \sigma_s \varphi_j) \]
\[ + O\left(\|\sigma_s \varphi\|^2 + \sum_{i,j} \|\sigma_s L_j \varphi_i\|\|L_i(\sigma_s) \varphi_j\| + \sum_k (\|\sigma_s L_k \varphi\|\|\sigma_s \varphi\|)\right). \]

Combining this with (3.2) we obtain
\[
\|\sigma_s \partial_b \varphi\|^2 + \|\sigma_s \partial_b^* \varphi\|^2 = \sum_{i,j} \left(\|\sigma_s L_i \varphi_j\|^2 + (\sigma_s c_{ij} T \varphi_i, \sigma_s \varphi_j) \right) \]
\[ + \text{Re}(L_i L_j(\sigma_s) \varphi_i, \sigma_s \varphi_j) \]
\[ + O\left(\|\sigma_s \varphi\|^2 + \sum_{i,j} \|\sigma_s L_j \varphi_i\|\|L_i(\sigma_s) \varphi_j\| \right) \]
\[ + \sum_k (\|\sigma_s L_k \varphi\|\|\sigma_s \varphi\|)\). \]

Since \( \sigma_s \) is bounded independently of \( s \),
\[
(3.3) \quad \sum_{i,j} \left( (\sigma_s c_{ij} T \varphi_i, \sigma_s \varphi_j) + \text{Re}(L_i L_j(\sigma_s) \varphi_i, \sigma_s \varphi_j) \right) \]
\[ \leq C\left(Q_b(\varphi, \varphi) + \|\varphi\|^2 + \sum_{i,j} \|L_i(\sigma_s) \varphi_j\|^2\right). \]

Note that \( D(\sigma_s) = sD(\lambda)\sigma_s \) and \( D^2(\sigma_s) = sD^2(\lambda)\sigma_s + s^2 D(\lambda)^2 \sigma_s \). Then we have
\[
\max_{\text{supp}(\zeta)} |D(\sigma_s)| \sim s \max_{\text{supp}(\zeta)} |D(\lambda)|, \]
\[
\max_{\text{supp}(\zeta)} |D^2(\sigma_s)| \sim s \max_{\text{supp}(\zeta)} |D^2(\lambda)| + s^2 \max_{\text{supp}(\zeta)} D(\lambda)^2, \]
\[
\text{Re}L_i L_j(\sigma_s) \sim s \text{Re}L_i L_j(\lambda) + s^2 D(\lambda)^2, \]
and
\[
C_{\sigma_s} \sim C(s, \lambda) = s \max_{\text{supp}(\zeta)} |D^2(\lambda)| + s^2 \max_{\text{supp}(\zeta)} D(\lambda)^2 + 1, \]
where \( C_{\sigma_s} \) is defined by (2.1).

When we substitute \( \zeta \Psi^+ \varphi \) for \( \varphi \) in (3.3) the first term is estimated from below by...
\[
\sum_{i,j} (\sigma_s c_{ij} T \zeta \Psi^+ \varphi_i, \sigma_s \zeta \Psi^+ \varphi_j) \\
\geq -C \left( (s \max_{\supp \zeta} |D(\lambda)| ||\zeta \Psi^+ \varphi||^2 + C(s, \lambda) ||\Psi^+ \varphi||^-1 ||\zeta \Psi^+ \varphi|| + ||\Psi^+ \varphi||^2 \right).
\]

The second term is estimated from below by
\[
(L_i \tilde{L}_j (\sigma_s) \zeta \Psi^+ \varphi_i, \sigma_s \zeta \Psi^+ \varphi_j) \geq sC' ||\sigma_s \zeta \Psi^+ \varphi||^2 - s^2 \max_{\supp \zeta} D(\lambda)^2 ||\zeta \Psi^+ \varphi||^2.
\]

Since \(\sigma_s\) is bounded away from zero independently of \(s\) the estimate (3.1) follows from the above and (3.3) thus proving the lemma.

To generalize the above to \((0, q)\)-forms \(\varphi = \sum \varphi I \tilde{\omega}_I\) we define
\[
A_{IJ}(\lambda) = \text{Re} \sum_{i,j,K} \epsilon_i^j \epsilon_j^K L_i \tilde{L}_j (\lambda)
\]
and
\[
c_{IJ} = \sum_{i,j,K} \epsilon_i^j \epsilon_j^K c_{ij},
\]
where \(K\) runs over all ordered \((q - 1)\)-tuples. Each of the coefficients \(\epsilon_i^j \epsilon_j^K\) is either 0, 1, or \(-1\) defined as follows. First, if \(i \notin K\) we denote by \(\langle iK \rangle\) the ordered \(q\)-tuple containing \(i\) and the elements of \(K\). Then we define
\[
\epsilon_i^j \epsilon_j^K = \begin{cases} 
0 & \text{if } i \in K \\
0 & \text{if } \langle iK \rangle \neq I \\
\text{sgn} \left( \frac{iK}{I} \right) & \text{if } \langle iK \rangle = I,
\end{cases}
\]
where \(\text{sgn} \left( \frac{iK}{I} \right)\) denotes the sign of the permutation \(\{i, K\} \to I\).

**Lemma 3.2.** Let \(M\) be a pseudoconvex CR manifold of dimension \(2n - 1\), let \(x_0 \in M\) and let \(\lambda\) be a real nonnegative \(C^\infty\) function, \(s \in \mathbb{R}^+ \) with \(s\lambda \leq 1\), and suppose that \(\text{Re} A_{IJ}(\lambda)\) is positive-definite. Then there exists a neighborhood \(U\) of \(x_0\) and \(C > 0\) such that
\[
(3.4) \quad s ||\zeta \Psi^+ \varphi||^2 \leq C \left( Q_b(\zeta \Psi^+ \varphi, \zeta \Psi^+ \varphi) + (s^2 \max_{\supp(\zeta)} D(\lambda)^2 + 1) ||\zeta \Psi^+ \varphi||^2 \\
+ C(s, \lambda) ||\Psi^+ \varphi||^-1 ||\zeta \Psi^+ \varphi|| + ||\Psi^+ \varphi||^2 \right),
\]
where
\[
C(s, \lambda) = s \max_{\supp(\zeta)} |D^2(\lambda)| + s^2 \max_{\supp(\zeta)} D(\lambda)^2 + 1,
\]
\(\zeta \in C^\infty(\bar{U}), \varphi \in \mathcal{A}_{0,q}^\infty, \supp(\varphi) \subset U', \text{ and } U' \supset \bar{U}.
\]
The proof of this lemma is entirely analogous to the case of \((0, 1)\)-forms. Integrating by parts as above we obtain the following in place of (3.2)

\[
\text{Re} \sum_{I, J} \left( (\sigma_s c_{IJ} T \varphi_I, \sigma_s \varphi_J) + (A_{I, J}(\sigma_s) \varphi_I, \sigma_s \varphi_J) \right)
\leq C \left( Q_b(\varphi, \varphi) + \|\varphi\|^2 + \|D(\sigma_s)\varphi\|^2 \right).
\]

Observe that \(c_{ij} \geq 0\) implies that \(c_{IJ} \geq 0\) since at a point \(x_o\) we can choose the \(\{L_i\}\) so that \(c_{ij}(x_o) = \delta_{ij} c_{ii}(x_o)\) and then \(c_{IJ}(x_o) = \delta_{IJ} \sum_{i \in I} c_{ii}(x_o)\). Hence, again substituting \(\zeta \Psi^\pm \varphi\) for \(\varphi\), we can apply Lemma 2.5 and conclude the proof of Lemma 3.2.

To microlocalize with \(\Psi^-\) we define the local conjugate-linear duality map \(F^q : \mathcal{A}^{0, q}_{b, n} \to \mathcal{A}^{0, n-q-1}_{b, n}\) as follows. If \(\varphi = \sum \varphi_I \tilde{\omega}_I\) then

\[
F^q(\varphi) = \sum \epsilon^I \varphi_I \tilde{\omega}_I,
\]

where \(I'\) denotes the strictly increasing \((n-q-1)\)-tuple consisting of all integers in \([1, n-1]\) which do not belong to \(I\) and \(\epsilon^I\) is the sign of the permutation \(\{I, I'\} \to \{\{1, \ldots, n-1\}\}\). Then \(F^{n-q-1} F^q \varphi = \varphi\) and we have

\[
\partial_b F^q \varphi = F^{q-1} \partial_b^* \varphi + \sum a_{IJ} \varphi_I \tilde{\omega}_J
\]

and

\[
\tilde{\partial}_b F^q \varphi = F^{q+1} \tilde{\partial}_b^* \varphi + \sum b_{HK} \varphi_H \tilde{\omega}_K.
\]

Hence

\[
Q_b(F^q \varphi, F^q \varphi) = O(Q_b(\varphi, \varphi) + \|\varphi\|^2).
\]

Thus replacing \(\varphi\) by \(F^q \varphi\) in (3.5), we obtain

\[
\text{Re} \sum_{I, J} \left( (\sigma_s c_{IJ} T \varphi_I, \sigma_s \varphi_J) + (A_{I, J}(\sigma_s) \varphi_I, \sigma_s \varphi_J) \right)
\leq C \left( Q_b(\varphi, \varphi) + \|\varphi\|^2 + \|D(\sigma_s)\varphi\|^2 \right).
\]

Now note that, since \(\bar{T} = -T\),

\[
\text{Re} \sum_{I, J} (\sigma_s c_{IJ} T \varphi_I, \sigma_s \varphi_J) = -\text{Re} \sum_{I, J} (\sigma_s c_{IJ} T \varphi_I, \sigma_s \varphi_J).
\]

Hence, substituting \(\zeta \psi^+ \varphi\) for \(\varphi\), we proceed as above and obtain the following result.
Lemma 3.3. Let $M$ be a pseudoconvex CR manifold of dimension $2n-1$, let $x_o \in M$ and let $\lambda$ be a real non-negative $C^\infty$ function, $s \in \mathbb{R}^+$ with $s\lambda \leq 1$, and suppose that $\text{Re} A_{\lambda'}(\lambda)$ is positive-definite. Then there exists a neighborhood $U$ of $x_o$ and $C > 0$ such that

$$s\|\zeta \Psi^{-}\|_2^2 \leq C \left( Q_b(\zeta \Psi^{-} \varphi, \zeta \Psi^{-} \varphi) + (s^2 \max_{\text{supp}(\zeta)} D(\lambda)^2 + 1) \|\zeta \Psi^{-} \varphi\|_2^2 \right)$$

$$+ C(s,\lambda)\|\Psi^{-} \varphi\| -1 \zeta \Psi^{-} \varphi\| + \|\Psi^{-} \varphi\|_2^2 - 1,$$

where

$$C(s,\lambda) = s \max_{\text{supp}(\zeta)} |D^2(\lambda)| + s^2 \max_{\text{supp}(\zeta)} D(\lambda)^2 + 1,$$

$\zeta \in C_0^\infty(U), \varphi \in \mathcal{A}_b^{0,q}$, supp$(\varphi) \subseteq U'$, and $U' \supset \bar{U}$.

4. The localization lemma

Definition 4.1. Let $S \subset M$ be a submanifold of $M$; the holomorphic dimension of $S$ at $x_o \in S$ is the dimension of $T_{x_o}^{1,0}(M) \cap CT_{x_o}(S)$ and $S$ is called totally real if the holomorphic dimension of $S$ at $x$ is zero for all $x \in S$.

We are now in a position to formulate Lemma 1.4 precisely.

Lemma 4.2 (Localization Lemma). Let $M$ be a pseudoconvex CR manifold and let $S \subset M$ be a submanifold of holomorphic dimension less than or equal to $m$. Let $x_0 \in S$. For small positive $a$ let $S_a$ denote the set of $x \in M$ such that $d(x,S) < a$, where $d(x,S)$ denotes the distance from $x$ to $S$. Then there exist positive constants $a_o$, $C$ and neighborhoods $U, U'$ of $x_0$, with $\bar{U} \subset U'$, and a constant $C$ such that

$$\|\zeta \Psi^+ \varphi\|_{2}^2 \leq C \left( a^2 Q_b(\zeta \Psi^+ \varphi, \zeta \Psi^+ \varphi) + \|\zeta \Psi^+ \varphi\|_{M-S_a}^2 + \|\Psi^+ \varphi\|_{-2}^2 \right)$$

for $0 < a < a_o$, $\zeta \in C_0^\infty(U)$ and all $(p,q)$-forms $\varphi$ with $n - 1 \geq q \geq m + 1$ supported in $U'$. Furthermore

$$\|\zeta \Psi^- \varphi\|_{2}^2 \leq C \left( a^2 Q_b(\zeta \Psi^- \varphi, \zeta \Psi^- \varphi) + \|\zeta \Psi^- \varphi\|_{M-S_a}^2 + \|\Psi^- \varphi\|_{-2}^2 \right)$$

for all $(p,q)$-forms $\varphi$ with $0 \leq q \leq n - m - 2$ supported in $U'$. Here $\|\cdot\|_{S_a}$ and $\|\cdot\|_{M-S_a}$ denote $L_2$-norms over $S_a$ and $M-S_a$, respectively.

Proof. Let $f_1, \ldots, f_k$ be real-valued functions in a neighborhood of $x_o$ such that $x \in S$ if and only if $f_i(x) = 0$ for $i = 1, \ldots, k$. Suppose further that the gradients of the $f_i$ are linearly independent. Since the rank of $(L_i f_j |_{x_o})$ is greater than or equal to $n - 1 - m$, we can choose (after renumbering the $L_i$) functions $\{g_1, \ldots, g_{n-1-m}\}$ which are real linear combinations of the $f_i$, such
that \( L_i(g_j)|_{x_0} = \delta_{ij} \) when \( i = 1, \ldots, n - q - m \). Then, setting \( \lambda = \sum g_k^2 \) we have \( A_{IJ}(\lambda)|_{x_0} = 2\delta_{IJ}p_I \), where \( I \) and \( J \) are ordered \( q \)-tuples so that \( p_I \) is greater than or equal to the number of elements in \( I \cap \{1, \ldots, n - 1 - m\} \). Since \( m \leq q - 1 \) we have \( p_I > 0 \) and hence \((A_{IJ}(\lambda)|_{x_0})\) is positive definite. Thus there exists a neighborhood \( U' \) of \( x_o \) on which \( \text{Re}(A_{IJ}(\lambda)) \) is positive definite.

Let \( \zeta_o \in C_0^\infty(S_{2a}) \) with \( \zeta_o = 1 \) on \( S_a \) such that \(|D\zeta_o| \leq \frac{C}{a^2} \). Then \( \lambda \leq Ca \) and \(|L_i(\lambda)| \leq Ca \) in \( S_{2a} \). Setting \( s = \frac{\delta}{a^2} \) with \( \delta \) sufficiently small so that \( s \lambda \leq 1 \) in \( S_{2a} \) we can apply Lemma 3.2 and obtain

\[
(4.3) \quad s\|\zeta_o \Psi^+ \varphi\|^2 \leq C \left( Q_b(\zeta_o \Psi^+ \varphi, \zeta_o \Psi^+ \varphi) \right. \\
+ \left( s^2 \max_{\text{supp}(\zeta_o \zeta)} |D^2(\lambda)| + s^2 \max_{\text{supp}(\zeta_o \zeta)} D(\lambda)^2 + 1 \right) \|\zeta_o \Psi^+ \varphi\|^2 \\
+ C(s, \lambda) \|\Psi^+ \varphi\|_{-1} \|\zeta_o \Psi^+ \varphi\| + \|\Psi^+ \varphi\|_{-1}^2,
\]

where

\[
C(s, \lambda) = s \max_{\text{supp}(\zeta_o \zeta)} |D^2(\lambda)| + s^2 \max_{\text{supp}(\zeta_o \zeta)} D(\lambda)^2 + 1,
\]

\( \zeta \in C_0^\infty(U), \varphi \in A_0^{\alpha}, \text{supp}(\varphi) \subset U', \) and \( U' \supset \bar{U} \). Dividing both sides of (4.3) by \( s \) and with \( s = \frac{\delta}{a^2} \) and \( a_o \leq \delta \) we have

\[
s \max_{\text{supp}(\zeta_o \zeta)} D(\lambda)^2 + \frac{1}{s} \leq C\delta
\]

and

\[
\frac{1}{s} C(s, \lambda) \leq C.
\]

Hence

\[
\|\zeta_o \Psi^+ \varphi\|^2 \leq C \left( \frac{\alpha^2}{\delta} Q_b(\zeta_o \Psi^+ \varphi, \zeta_o \Psi^+ \varphi) + \delta \|\zeta_o \Psi^+ \varphi\|^2 \\
+ \|\Psi^+ \varphi\|_{-1} \|\zeta_o \Psi^+ \varphi\| + \|\Psi^+ \varphi\|_{-1}^2 \right).
\]

Thus, since

\[
\|\zeta \Psi^+ \varphi\|_{S_a} \leq \|\zeta_o \Psi^+ \varphi\|
\]

and

\[
a^2 Q_b(\zeta_o \Psi^+ \varphi, \zeta_o \Psi^+ \varphi) \leq a^2 Q_b(\Psi^+ \varphi, \Psi^+ \varphi) + C \|\Psi^+ \varphi\|_{M-S_a}
\]

then (4.1) follows by choosing \( \delta \) sufficiently small.

To prove (4.2) note that \( A_{IJ}(\lambda)|_{x_0} = 2\delta_{IJ}p_I \), where \( I' \) and \( J' \) are ordered \((n - 1 - q)\)-tuples so that \( p_I \) is greater than or equal to the number of elements in \( I' \cap \{1, \ldots, n - 1 - m\} \). Since \( m \leq n - q - 2 \) we have \( p_I > 0 \) and hence \((A_{IJ}(\lambda)|_{x_0})\) is positive definite. Then, proceeding as above, using Lemma 3.4 instead of Lemma 3.3 we obtain the desired result.
Note that for \(\zeta \Psi^0 \varphi\) the above estimates are a consequence of ellipticity and thus hold for all \(q\). Then, when the above estimates hold for \(\zeta \Psi^+ \varphi\) and \(\zeta \Psi^- \varphi\), we obtain the following:

\[
\| \varphi \|_{S_a}^2 \leq C \left( a^2 Q_b(\varphi, \varphi) + \| \varphi \|_{M-a}^2 + \| \varphi \|_{-1}^2 \right).
\]

Substituting \(2a\) for \(a\) and \(\zeta_a \varphi\) for \(\varphi\) we get

\[
\| \zeta_a \varphi \|^2 \leq C \left( a^2 Q_b(\zeta_a \varphi, \zeta_a \varphi) + \| \zeta_a \varphi \|_{-1}^2 \right).
\]

Choosing \(U'\) with sufficiently small diameter we get

\[
C\| \zeta_a \varphi \|_{-1}^2 \leq \frac{1}{2} \| \zeta_a \varphi \|_{-1}^2.
\]

Thus we obtain:

**Corollary 4.3.** Let \(M\) be pseudoconvex CR manifold and let \(S \subset M\) be a submanifold of holomorphic dimension less than or equal to \(m\) with \(m \leq \frac{1}{2}(n - 3)\). Let \(x_0 \in S\). For small positive \(a\) let \(S_a\) denote the set of \(x \in M\) such that \(d(x, S) < a\), where \(d(x, S)\) denotes the distance from \(x\) to \(S\). Then there exist positive constants \(a_o, C,\) and a neighborhood \(U\) such that

\[
\| \varphi \|_{S_a}^2 \leq C \left( a^2 Q_b(\varphi, \varphi) + \| \varphi \|_{M-a}^2 \right)
\]

for \(0 < a < a_o\) and all \((p, q)\)-forms \(\varphi\) with \(m + 1 \leq q \leq n - m - 2\) supported in \(U\).

M. Christ has pointed out that pseudoconvexity is a crucial assumption for the estimate (4.4). He has supplied the following example. Let \(M\) be a five-dimensional CR manifold with coordinates \((z_1, z_2, t)\), whose CR structure is defined by the vector fields

\[
L_1 = \partial_{z_1} - i \bar{z}_1 \partial_{t}, \quad L_2 = \partial_{z_2} + i \bar{z}_2 \partial_{t}.
\]

Then we have \(\tilde{L}_j = -L_j); setting \(T = i \partial_{t}\) we have \([L_1, \tilde{L}_1] = 2T\) and \([L_2, \tilde{L}_2] = -2T\). Let \(S\) be defined by \(z_1 = z_2 = 0\); then \(m\), the holomorphic dimension of \(S\), equals zero. Let \(S_a = \{(z_1, z_2, t) \mid |z| < a\}\), with \(|z|^2 = |z_1|^2 + |z_2|^2\). Let \(g = |z|^2 - it\); then \(L_1 g = L_2 g = 0\). For each \(\tau > 0\) we define the \((0,1)\)-form \(\varphi^\tau\) by \(\varphi^\tau = f^\tau \zeta \bar{\omega}_1\), where \(f^\tau(z_1, z_2, t) = \exp \tau(-g + g^2)\) and \(\zeta \in C_0^\infty(|z|^2 + t^2 < 2r^2)\) with \(\zeta = 1\) on \(|z|^2 + t^2 \leq r^2\). Note that \(L_1 f^\tau = L_2 f^\tau = 0\). We have

\[
Q_b(\varphi^\tau, \varphi^\tau) = \| f^\tau L_1 \zeta \|^2 + \| f^\tau L_2 \zeta \|^2.
\]

If \(a < r \leq \frac{1}{2}\) we have

\[
\Re(-g + g^2) = -|z|^2 - t^2 + |z|^4 \geq -\frac{a^2}{2}
\]
on the region \(|z|^2 + t^2 \leq \frac{a^2}{4}\) which is contained in \(S_a\) so that
\[ca^5 \exp\left(-\frac{1}{2} \tau a^2\right) \leq \|\varphi^\tau\|_{S_a}^2.\]
Furthermore
\[\Re(-g + g^2) = -|z|^2 - t^2 + |z|^4 \leq -\frac{3}{4} \tau^2 \leq -\frac{3}{4} a^2\]
on the support of the derivatives of \(\zeta\); thus
\[Q_b(\varphi^\tau, \varphi^\tau) + \|\varphi^\tau\|_{M-S_a}^2 \leq C \exp\left(-\frac{3}{4} \tau a^2\right).\]
This contradicts (4.3) for large \(\tau\).

5. Subelliptic multipliers

To study the operators \(\square_b, \bar{\partial}_b,\) and the \(\bar{\partial}\)-Neumann problem on \((p,q)\)-forms we define microlocal subelliptic multipliers below. In this section we develop those properties of the multipliers which are needed in the proofs of superlogarithmic estimates and the hypoellipticity results. Explicit formulas for subelliptic multipliers are given in the appendix.

**Definition 5.1.** For \(x_o \in M\) let \(I^+_q(x_o)\) be the subset of germs of \(C^\infty\) functions at \(x_o\) defined as follows. The germ \(\rho\) is in \(I^+_q(x_o)\) if and only if there exist \(U, U', \varepsilon, C,\) and \(\zeta\) such that
\[
\|\rho \zeta \Psi^+ \varphi\|_2^2 \leq C \left( Q_b(\zeta \Psi^+ \varphi, \zeta \Psi^+ \varphi) + \|\Psi^+ \varphi\|^2 \right)
\]
for all \((0,q)\)-forms \(\varphi\) with \(\text{supp}(\varphi) \subset U'\). Here, \(U\) is a neighborhood of \(x_o\) such that \(\bar{U} \subset U'\), \(\varepsilon\) and \(C\) are positive constants and \(\zeta \in C^\infty_0(U)\) such that \(\zeta = 1\) on a neighborhood of \(x_o\). We define \(I^-_q(x_o)\) and \(I^-_q(x_o)\) by replacing \(\Psi^+\) in the above by \(\Psi^0\) and \(\Psi^-,\) respectively.

Note that \(I^0_q(x_o)\) is the set of all germs of \(C^\infty\) functions at \(x_o\); in fact we have
\[
\|\rho \zeta \Psi^0 \varphi\|_2^2 \leq C \left( Q_b(\zeta \Psi^0 \varphi, \zeta \Psi^0 \varphi) + \|\Psi^0 \varphi\|^2 \right).
\]
Thus, if we denote the set of subelliptic multipliers satisfying (5.1) by \(I_q(x_o)\), we have \(I_q(x_o) = I^+_q(x_o) \cap I^-_q(x_o)\).

**Proposition 5.2.** If \(1 \leq q' \leq q \leq n - 1\) then \(I^+_q(x_o) \supset I^+_q(x_o)\).

**Proof.** Now, \(\rho \in I^+_q(x_o)\) and we want to show that \(\rho \in I^+_q(x_o)\). Let \(\varphi\) be a \((0,q)\)-form supported in \(U\),
\[\varphi = \sum \varphi_I \bar{\omega}_I,\]
where the $I$ are ordered $q$-tuples of integers between 1 and $n - 1$. We define the $(0, q - 1)$-forms $\beta^j$ by

$$\beta^j = \sum_K \varphi_{KJ} \bar{\omega}_K,$$

where $I = \langle KJ \rangle$ and $q \leq j \leq n - 1$. Thus we have

$$\|\rho \zeta \psi + \beta^j \|_\varepsilon \leq C \left( Q_b(\zeta \psi^+ \beta^j, \zeta \psi^+ \beta^j) + \|\varphi\|^2 \right).$$

Now

$$Q_b(\zeta \psi^+ \beta^j, \zeta \psi^+ \beta^j) = \sum \|\tilde{L}_k \psi^+ \beta^j_k\|^2 + \sum (b_{HK} T \zeta \psi^+ \beta^j_H, \zeta \psi^+ \beta^j_K) + O(\|\varphi\|^2),$$

where $b_{HK} = \sum c_{hK}^H c_{hk}$ and

$$Q_b(\zeta \psi^+, \zeta \psi^+) = \sum \|\tilde{L}_k \psi \|^2 + \sum (c_{IJ} T \zeta \psi^+, \zeta \psi^+) + O(\|\varphi\|^2),$$

where $c_{IJ} = \sum c_{iJ}^j c_{ij}$. To compare $a_{IJ}$ and $b_{HK}$ at a point $x$ we choose the $L_1, \ldots, L_{n-1}$ so that $c_{ij}(x) = \delta_{ij} c_{ii}(x)$. Then we have

$$c_{IJ}(x) = \delta_{IJ} \sum_{i \in I} c_{ii}(x)$$

and

$$b_{HK}(x) = \delta_{HK} \sum_{h \in H} c_{hh}(x).$$

If $I = \langle Hk \rangle$,

$$c_{II}(x_o) = b_{HH}(x_o) + c_{kk}(x_o) \geq b_{HH}(x_o);$$

hence

$$\sum_{H,K} b_{HK}(x) \beta^j_H(x) \beta^j_K(x) \leq \sum_{IJ} c_{IJ}(x) \varphi_I(x) \bar{\varphi}_J(x).$$

Since

$$\|\rho \zeta \psi^+ \varphi\|_\varepsilon \leq \sum_j \|\rho \zeta \psi^+ \beta^j\|_\varepsilon$$

we conclude (using Gårding’s inequality) that

$$\|\rho \zeta \psi^+ \varphi\|^2_\varepsilon \leq C \left( Q_b(\zeta \psi^+ \varphi, \zeta \psi^+ \varphi) + \|\varphi\|^2 \right)$$

so that $\rho \in \mathcal{I}^+_q(x_o)$. \hfill \Box

**Proposition 5.3.** $\mathcal{I}^-_q(x_o) = \mathcal{I}^+_{n-q-1}(x_o)$ for $0 \leq q \leq n - 2$.

**Proof.** If $\varphi$ is a $(0, q)$-form with support in $U$ given by $\varphi = \sum e^{I'} \varphi_I \bar{\omega}_I$, as in Section 3, we set $F_q \varphi = \sum \varphi_{I'} \bar{\omega}_{I'}$, where $I'$ denotes the ordered $(n - q - 1)$-tuple consisting of the complement of $I$. Then noting that $\|\psi^+ u\| = \|\psi^- u\|$, we have

$$\|\rho \zeta \psi^+ \varphi\|_\varepsilon = \|\rho \zeta \psi^- F_q \varphi\|_\varepsilon + O(\|\varphi\|)$$
and

\[ Q_b(\zeta \Psi^+ \varphi, \zeta \Psi^+ \varphi) = Q_b(\zeta \Psi^- F_q \varphi, \zeta \Psi^- F_q \varphi) + O(\|\varphi\|^2), \]

which concludes the proof.

Combining these propositions we see that \( \mathcal{I}^- (x_o) \subset \mathcal{I}^+_q (x_o) \), if \( n - q - 1 \geq q \), that is if \( q \leq \frac{1}{2} (n - 1) \). Hence \( \mathcal{I}^- (x_o) = \mathcal{I}^+_q (x_o) \) whenever \( q \leq \frac{1}{2} (n - 1) \). In particular \( \mathcal{I}^- (x_o) = \mathcal{I}^+_1 (x_o) \) when \( n \geq 2 \).

6. Superlogarithmic estimates

In this section we will prove superlogarithmic estimates under the assumption that there exists a subelliptic multiplier \( \rho \) satisfying the condition

\[ (*) \quad \lim_{x \to S} d(x, S) \log \rho(x) = 0. \]

We define the operator \( \log \Lambda \) by

\[ \log \Lambda u(\xi) = \frac{1}{2} (\log(1 + |\xi|^2) \hat{u}(\xi) \]

and we have the following result.

**Theorem 6.1.** Let \( M \) be a pseudoconvex CR manifold. Let \( S \subset M \) be a manifold such that the holomorphic dimension of \( S \) at each point is less than or equal to \( q - 1 \). Let \( x_o \in S \), let \( U' \) be a neighborhood of \( x_o \) and suppose that \( \rho \in C^\infty (U') \) with \( \rho \in \mathcal{I}^- (x_o) \) satisfying \((*)\). Let \( U \) be a neighborhood of \( x_o \) with \( \tilde{U} \subset U' \) and \( \zeta \in C_0^\infty (U) \). Then the following superlogarithmic estimate \((\mathrm{SL}^+)_q\) holds. For each \( \delta > 0 \) there exist \( C_\delta \), such that

\[ \{(\log \Lambda) \zeta \Psi^\pm \varphi\}^2 \leq \delta^2 Q_b(\zeta \Psi^+ \varphi, \zeta \Psi^+ \varphi) + C_\delta \|\Psi^\pm \varphi\|^2, \]

for all \( \varphi \in A_b^0 \) with support in \( U' \). Furthermore, if the holomorphic dimension of \( S \) at each point is less than or equal to \( n - q - 2 \), and if \( \rho \in \mathcal{I}^- (x_o) \) satisfying \((*)\) then the estimate \((\mathrm{SL}^-)_q\) given by

\[ \{(\log \Lambda) \zeta \Psi^- \varphi\}^2 \leq \delta^2 Q_b(\zeta \Psi^- \varphi, \zeta \Psi^- \varphi) + C_\delta \|\Psi^- \varphi\|^2, \]

for all \( \varphi \in A_b^0 \) with support in \( U' \).

**Proof.** Let \( \gamma_0, \tilde{\gamma}_0, \gamma, \tilde{\gamma} \) be nonnegative functions on \( C^\infty ([0, \infty)) \) such that \( \text{supp}(\gamma_0) \subset \text{supp}(\tilde{\gamma}_0) \subset [0, 2) \), \( \text{supp}(\gamma) \subset \text{supp}(\tilde{\gamma}) \subset [1, 3) \), \( \tilde{\gamma}_0 = 1 \) on \( \text{supp}(\gamma_0) \), \( \tilde{\gamma} = 1 \) on \( \text{supp}(\gamma) \), and

\[ \gamma_0(x)^2 + \sum_k \gamma(2^{-k} x)^2 = 1, \]

when \( x \geq 0 \). When \( k \geq 1 \) we set \( \gamma_k(x) = \gamma(2^{-k} x) \), and define the operators \( \Gamma_k \) and \( \tilde{\Gamma}_k \) by

\[ \tilde{\Gamma}_k u(\xi) = \gamma_k(|\xi|) \hat{u}(\xi) \]
and
\[ \hat{\Gamma}_k u(\xi) = \hat{\gamma}_k(|\xi|) \hat{u}(\xi). \]

Let \( \rho \) be a subelliptic multiplier, satisfying \((*)\). Then for each \( \delta > 0 \) there exists \( A_\delta > 0 \) such that
\[ \frac{1}{\rho(x)} \leq A_\delta \exp\left( \frac{\delta}{d(x, S)} \right). \]

Then, using the fact that \( |\xi| \sim 2^k \) when \( \xi \in \text{supp}(\gamma_k) \) and applying \((5.1)\), we have
\[ \|\Gamma_\kappa \zeta \Psi^+ \varphi\|_{M-S_n} \leq \max_{M-S_n} \frac{1}{\rho} \|\rho \Gamma_\kappa \zeta (\Psi^+ \varphi)\| \]
\[ \leq A_\delta \exp\left( \frac{\delta}{a} \left( \|\Gamma_\kappa \rho \zeta \Psi^+ \varphi\| + \|\rho \zeta, \Gamma_\kappa \Psi^+ \varphi\| \right) \right) \]
\[ \leq A_\delta \exp\left( \frac{\delta}{a} \left( 2^{-k\epsilon} \|\Gamma_\kappa (\rho \zeta \Psi^+ \varphi)\|_\varepsilon + 2^{k} \|\Psi^+ \varphi\| \right) \right) \]
\[ \leq A_\delta' 2^{(\frac{\delta}{k} - k \epsilon)} \sqrt{Q_b(\zeta \Gamma_\kappa \Psi^+ \varphi, \zeta \Gamma_\kappa \Psi^+ \varphi) + A_\delta'' 2^{(\frac{\delta}{k} - k \epsilon)} \|\Psi^+ \varphi\|}. \]

Set \( a = \frac{2^k}{\delta} \) with \( k > K_\delta \), where \( K_\delta \) is chosen large enough so that
\[ A_\delta' 2^{(\frac{\delta}{k} - k \epsilon)} \leq \frac{\delta}{k} \]
and
\[ A_\delta'' 2^{(\frac{\delta}{k} - k \epsilon)} \leq \left( \frac{2}{3} \right)^k. \]

Note that \( \|\Psi^+ \Gamma_\kappa \varphi\|_{-1} \sim 2^{-k} \|\Psi^+ \Gamma_\kappa \varphi\| \). Thus from Lemma 4.2 we obtain
\[ \|\Gamma_\kappa \zeta \Psi^+ \varphi\|_2 = \|\Gamma_\kappa \zeta \Psi^+ \varphi\|_{M-S_n}^2 + \|\Gamma_\kappa \zeta \Psi^+ \varphi\|^2_{M-S_n} \]
\[ \leq C \left( \frac{\delta^2}{k^2} Q_b(\zeta \Gamma_\kappa \Psi^+ \varphi, \zeta \Gamma_\kappa \Psi^+ \varphi) + \left( \frac{2}{3} \right)^k \|\Psi^+ \varphi\| \right) \]
\[ \leq C \left( \frac{\delta^2}{k^2} \|\Gamma_\kappa \partial_b \zeta \Psi^+ \varphi\|^2 + \|\Gamma_\kappa \partial_b^* \zeta \Psi^+ \varphi\|^2 \right) + \left( \frac{2}{3} \right)^k \|\Psi^+ \varphi\| \]
for \( k > K_\delta \). Then, since \( k \sim \log |\xi| \) for \( \xi \in \text{supp}(\gamma_k) \) we obtain, after multiplying by \( k^2 \) and summing over \( k \),
\[ \| (\log \Lambda) \zeta \Psi^+ \varphi \|_2 \sim \sum_{k \leq K_\delta} \| (\log \Lambda) \Gamma_\kappa \zeta \Psi^+ \varphi \|_2 + \sum_{k > K_\delta} k^2 \|\Gamma_\kappa \zeta \Psi^+ \varphi\|_2 \]
\[ \leq C \left( K_\delta^2 \|\Psi^+ \varphi\|_2 + \delta^2 \left( \|\Gamma_\kappa \partial_b \zeta \Psi^+ \varphi\|_2 + \|\Gamma_\kappa \partial_b^* \zeta \Psi^+ \varphi\|_2 \right) \right) \]
\[ \leq \delta^2 Q_b(\zeta \Psi^+ \varphi, \zeta \Psi^+ \varphi) + C_\delta \|\Psi^+ \varphi\|_2. \]
This concludes the proof for the superlogarithmic estimate in the + microlocalization; the proof in the − case is entirely analogous. \( \square \)

Combining the +, −, and 0 microlocalizations we obtain the following.
Corollary 6.2. Let \( M \) be a pseudoconvex manifold. Let \( S \subset M \) be a manifold such that the holomorphic dimension of \( S \) at each point is less than or equal to \( m \), where \( m = \min\{q-1, n-q-2\} \). Let \( x_o \in S \), let \( U \) be a neighborhood of \( x_o \) and suppose that \( \rho \in C^\infty(\bar{U}) \) with \( \rho \in \mathcal{I}_q(x_o) \) satisfying (*)

Then the superlogarithmic estimate \((\text{SL})_q \) holds. For each \( \delta > 0 \) there exists \( C_\delta \) such that

\[
(\text{SL})_q \| (\log \Lambda) \varphi \|^2 \leq \delta^2 Q_b(\varphi, \varphi) + C_\delta \| \varphi \|^2,
\]

for all \( \varphi \in \mathcal{A}^{0,q} \) with support in \( U \).

7. Hypoellipticity

In this section we show that the superlogarithmic estimate \((\text{SL})_q \) implies hypoellipticity of \( \Box_b \). Further we show that if the range of \( \bar{\partial}_b \) is closed in \( L^2 \) then the estimates \((\text{SL}^+)_1 \) on \((p,1)\)-forms and \((\text{SL}^-)_0 \) on \((p,0)\)-forms imply that the restrictions of \( \Box_b \) to \((p,0)\)-forms orthogonal to \( \mathcal{H}^{p,0} \) and to \((p,n-1)\)-forms orthogonal to \( \mathcal{H}^{p,n-1} \) are hypoelliptic.

Theorem 7.1. Assume that \((\text{SL})_q \) holds in a neighborhood \( U \) of \( x_o \in M \). Then if \( \varphi \) is a square integrable \((p,q)\)-form such that \( \Box_b \varphi = \alpha \) with \( \alpha \) square integrable whose restriction to \( U \) is in \( C^\infty(\bar{U}) \) then the restriction of \( \varphi \) to \( U \) is also in \( C^\infty(\bar{U}) \).

Proof. More precisely, we will show that, for any \( \zeta_0, \zeta_1 \in C^\infty_0(\bar{U}) \) with \( \zeta_1 = 1 \) in a neighborhood of the support of \( \zeta_0 \), if \( \zeta_1 \alpha \in H^s \) then \( \zeta_0 \varphi \in H^s \).

To do this we first prove the following \textit{a priori} estimate: given \( s \) there exists \( C_s > 0 \) such that

\[
\| \zeta_0 \varphi \|_s \leq C_s (\| \zeta_1 \alpha \|_s + \| \varphi \|),
\]

for all \( \varphi \in \mathcal{A}^{p,q}_b \). Let \( \sigma \in C^\infty_0(\bar{U}) \) such that \( \sigma = 1 \) in a neighborhood of \( \text{supp}(\zeta_0) \) and \( \zeta_1 = 1 \) in a neighborhood of \( \text{supp}(\sigma) \). Let \( \zeta' \in C^\infty_0(U') \) with \( \zeta' = 1 \) in a neighborhood of \( \bar{U} \). We define the operator \( R^s \) by

\[
R^s u(x) = \int e^{ix \cdot \xi} (1 + |\xi|^2)^{s \text{sgn}(\xi)} \hat{u}(\xi) d\xi,
\]

for \( u \in C^\infty_0(U') \). Since the symbol of \((\Lambda^s - R^s)\zeta_0 \) is zero,

\[
\| \zeta_0 \varphi \|_s \leq \| R^s (\zeta_0 \varphi) \| + C \| \varphi \| \leq \| [R^s, \zeta_0] (\zeta_1 \varphi) \| + \| R^s (\zeta_1 \varphi) \| + C \| \varphi \|.
\]

From the calculus of pseudodifferential operators we conclude that

\[
\| [R^s, \zeta_0] (\zeta_1 \varphi) \| \leq C (\| R^{s-1} (\zeta_1 \varphi) \| + \| \varphi \|)
\]
and
\[ \|R^s(\zeta_1 \varphi)\| = \|R^s(\zeta' \zeta_1 \varphi)\| \leq \|\zeta' R^s(\zeta_1 \varphi)\| + \|[(R^s, \zeta')]\| + O(\|R^{s-1}(\zeta_1 \varphi)\| + \|\varphi\|) \leq \|\zeta' R^s(\zeta_1 \varphi)\| + C\|\varphi\| \]
so that
\[ \|\zeta_0 \varphi\| \leq C(\|\log(\Lambda)\zeta' R^s(\zeta_1 \varphi)\| + \|\varphi\|). \]

Next, from (SL) with \(\varphi\) replaced by \(\zeta' R^s(\zeta_1 \varphi)\), we obtain
\[ ** (\|\log(\Lambda)\zeta' R^s(\zeta_1 \varphi)\|^2 \leq \delta^2 Q_b(\zeta' R^s(\zeta_1 \varphi), \zeta' R^s(\zeta_1 \varphi)) + C\delta\|\varphi\|^2. \]

The equation \(\square_b \varphi = \alpha\) is equivalent to
\[ Q_b(\varphi, \psi) = (\alpha, \psi), \]
for all \(\psi \in A_0^{b, 0}\). So we have
\[ Q_b(\zeta' R^s(\zeta_1 \varphi), \zeta' R^s(\zeta_1 \varphi)) = Q_b(\varphi, \zeta_1(\zeta^s) - (\zeta')^2 R^s(\zeta_1 \varphi) + \text{error} = (\alpha, \zeta_1(R^s) - (\zeta')^2 R^s(\zeta_1 \varphi)) + \text{error} = (\zeta' R^s(\zeta_1 \alpha), \zeta' R^s(\zeta_1 \varphi)) + \text{error}. \]
The “error” is given by \(\text{error} = I + II\), where
\[ I = (\partial_b, \zeta' R^s(\zeta_1 \varphi), \zeta' R^s(\zeta_1 \varphi)) + ((\partial_b, \zeta' R^s(\zeta_1 \varphi), \partial_b \zeta' R^s(\zeta_1 \varphi)) \]
and
\[ II = (\partial_b, [\zeta_1 R^{s+} \zeta', \partial_b] \zeta' R^s(\zeta_1 \varphi) + (\partial_b, [\zeta_1 R^{s+} \zeta', \partial_b] \zeta' R^s(\zeta_1 \varphi) = (\zeta_1 R^{s+} \zeta', \partial_b) \varphi, \partial_b \zeta' R^s(\zeta_1 \varphi) + (\zeta_1 R^{s+} \zeta', \partial_b) \varphi, \partial_b \zeta' R^s(\zeta_1 \varphi) + (\zeta_1 R^{s+} \zeta', \partial_b) \varphi, \partial_b \zeta' R^s(\zeta_1 \varphi). \]

By the Jacobi identity,
\[ [\partial_b, \zeta' R^s(\zeta_1)] = [\partial_b, \zeta'] R^s(\zeta_1) + \zeta' [\partial_b, R^s] \zeta_1 + \zeta' [\partial_b, R^s] \zeta_1. \]
Since the supports of the derivatives of \(\zeta_1\) and of \(\zeta'\) are disjoint from the support of \(\sigma\) the operator \([\partial_b, \zeta'] R^s(\zeta_1) + \zeta' [\partial_b, R^s] \zeta_1\) is bounded in \(L_2\). The principal symbol of \([\partial_b, R^s]\) is bounded by \(C(\log(1 + |\xi|))(1 + |\xi|^2)^{2\sigma(s)}\); hence
\[ \|\zeta' [\partial_b, R^s] \zeta_1 \varphi\| \leq C(\|\log(\Lambda)\zeta_1 \varphi\| + \|\varphi\|). \]
Arguing similarly we can bound all the terms in \(I\) and \(II\) and obtain
\[ \text{error}^2 \leq C(Q_b(\zeta' R^s(\zeta_1 \varphi), \zeta' R^s(\zeta_1 \varphi)) + ||(\log(\Lambda)\zeta_1 \varphi||^2 + \|\varphi\|^2). \]
After combining this with ** we see that \(\text{error}^2\) gets multiplied by \(\delta^2\) and thus, for suitably small \(\delta\), we obtain the \text{a priori} estimate *. To conclude the proof
of the theorem we must pass from the *a priori* estimate to showing regularity of the solution. This can be done by applying a standard smoothing operator to $\zeta \varphi$ or by using the method of elliptic regularization as in [KN].

Microlocally we obtain the following.

**Lemma 7.2.** With the same notation as above, estimate (SL$^+$)$_q$ implies:

$$\| \Psi^+ R^s \zeta_1 \varphi \| \leq C(\| \Psi^+ R^s \zeta_1 \Box_b \varphi \| + \| \varphi \|)$$

and

$$\| \Psi^+ R^s \zeta_1 \tilde{\partial}_b \varphi \| + \| \Psi^+ R^s \zeta_1 \tilde{\partial}_b^* \varphi \| \leq C(\| \Psi^+ R^s \zeta_1 \Box_b \varphi \| + \| \Psi^+ R^s \zeta_1 \Box_b \tilde{\partial}_b^* \varphi \| + \| \varphi \|),$$

for all $\varphi \in A^p_q \cap L_2$. Analogously, (SL$^-$)$_q$ implies:

$$\| \Psi^- R^s \zeta_1 \varphi \| \leq C(\| \Psi^- R^s \zeta_1 \Box_b \varphi \| + \| \varphi \|)$$

and

$$\| \Psi^- R^s \zeta_1 \tilde{\partial}_b \varphi \| + \| \Psi^- R^s \zeta_1 \tilde{\partial}_b^* \varphi \| \leq C(\| \Psi^- R^s \zeta_1 \Box_b \varphi \| + \| \Psi^- R^s \zeta_1 \Box_b \tilde{\partial}_b^* \varphi \| + \| \varphi \|),$$

for all $\varphi \in A^p_q \cap L_2$.

**Proof.** We have

$$Q_b(\zeta \Psi^+ R^s \zeta' \varphi, \zeta_1 \Psi^+ R^s \zeta' \varphi) = \| \Psi^+ R^s \zeta_1 \tilde{\partial}_b \varphi \|^2 + \| \Psi^+ R^s \zeta_1 \tilde{\partial}_b^* \varphi \|^2 + \text{error}$$

$$= (\Psi^+ \zeta_1 \Box_b \varphi, \zeta_1 \Psi^+ \zeta' \varphi) + \text{error}$$

$$\leq C(\| \Psi^+ \zeta_1 \Box_b \varphi \|^2 + \| \zeta_1 \Psi^+ \zeta' \varphi \|^2) + \text{error}$$

$$\leq C(\| \Psi^+ \zeta_1 \tilde{\partial}_b \varphi \|^2 + \| \Psi^+ \zeta_1 \tilde{\partial}_b^* \varphi \|^2) + \text{error},$$

and

$$\| \Psi^+ \zeta_1 \tilde{\partial}_b \tilde{\partial}_b^* \varphi \|^2 = (\Psi^+ \zeta_1 \tilde{\partial}_b \tilde{\partial}_b^* \varphi, \Psi^+ \zeta_1 \tilde{\partial}_b \varphi) + \text{error}$$

$$\leq (\| \Psi^+ \zeta_1 \Box_b \varphi \| + \| \Psi^+ \zeta_1 \tilde{\partial}_b \varphi \|) + \text{error},$$

$$\| \Psi^+ \zeta_1 \tilde{\partial}_b \tilde{\partial}_b^* \varphi \|^2 = (\Psi^+ \zeta_1 \tilde{\partial}_b \tilde{\partial}_b^* \varphi, \Psi^+ \zeta_1 \tilde{\partial}_b^* \varphi) + \text{error}$$

$$\leq (\| \Psi^+ \zeta_1 \Box_b \varphi \| + \| \Psi^+ \zeta_1 \tilde{\partial}_b^* \varphi \|) + \text{error}.$$

The error terms arise from commutators such as $[\zeta_1 \Psi^+ R^s, \tilde{\partial}_b]$. These are analyzed as before except for the terms that involve the commutators $[\Psi^+, \tilde{\partial}_b]$ and $[\Psi^+, \tilde{\partial}_b^*]$. To bound such terms let $\psi^0$ be a symbol which equals one in a neighborhood of the support of the derivatives of $\psi^+$ and whose support is contained in a region of the form

$$\{\xi = (\xi', \xi_{2n-1}) \in \mathbb{R}^{2n-1} \mid A < \xi_{2n-1} < a|\xi'|\}.$$
and let $\tilde{\psi}^+$ be a symbol which equals one on the support of $\tilde{\psi}^0$ with support contained in $\{\xi_{n-1} > 0\}$. Denoting by $\tilde{\psi}^0$ and by $\tilde{\psi}^+$ the corresponding pseudodifferential operators we have

$$
\|\tilde{\psi}^0 R^s \xi_1 \varphi\| \leq C(\|\square_b \tilde{\psi}^0 R^s \xi_1 \varphi\|_2 + \|\tilde{\psi}^0 R^s \xi_1 \varphi\|_2)
$$

since the support of $\tilde{\psi}^0$ lies in the elliptic region. These are the terms that arise in the error. Thus we have

$$
\|\tilde{\psi}^0 R^s \xi_1 \varphi\| \leq C(\|\tilde{\psi}^0 R^s \xi_1 \varphi\|_2 + \|\varphi\|) \leq C(\|\tilde{\psi}^+ R^s \xi_1 \varphi\|_2 + \|\tilde{\psi}^+ R^s \xi_1 \varphi\|_2) + \|\tilde{\psi}^+ R^s \xi_1 \varphi\|_2 + \|\varphi\|).
$$

We feed this into the above estimates with $\Psi^+$ replaced by $\tilde{\psi}^+$ and thus obtain the desired estimates in the $+$ microlocalization and a parallel argument yields them in the $-$ microlocalization.

The following result deals with the cases of $(p,0)$-forms and of $(p, n - 1)$-forms, where the spaces $\mathcal{H}^p_0$ and $\mathcal{H}^{p,n-1}_0$ are infinite-dimensional. We assume that $(\text{SL}^+)_1$ and $(\text{SL}^-)_0$ hold. This assumption is equivalent to assuming that $(\text{SL}^-)_{n-2}$ and $(\text{SL}^+)(n-1)$ hold, as is seen from the following observation.

**Remark.** When we use the operator $\mathcal{F}_q$, as in the proof of Proposition 5.3, it follows that the estimate $(\text{SL}^+)_q$ holds if and only if $(\text{SL}^-)_{n-q-1}$ holds.

**Theorem 7.3.** Let $M$ be a CR manifold and assume that the operator $\partial_b$ on functions has closed range in $L_2(M)$. Assume that $(\text{SL}^+)_1$ and $(\text{SL}^-)_0$ hold on a neighborhood $U$. Suppose that $q = 0$ or $n - 1$, $\varphi \perp \mathcal{H}^p_0$, and $\square_b \varphi = \alpha$ with $\alpha$ restricted to $U$ in $C^\infty(U)$. Then the restriction of $\varphi$ is also in $C^\infty(U)$.

**Proof.** We will deal only with the case of functions, that is $(0,0)$-forms; the remaining cases then follow. We will show that if $u \in L_2(M)$, $u \perp \mathcal{H}^0_0$ and if $\square_b u = f$ with $f \in L_2(M)$ and the restriction of $f$ to $U$ is in $C^\infty(U)$ then the restriction of $u$ to $U$ is in $C^\infty(U)$. Assuming that $u \in C^\infty(M)$ and using the above lemma we obtain the following a priori estimate:

$$
\|\Psi^- R^s \xi_1 u\| \leq C(\|\Psi^- R^s \xi_1 f\| + \|u\|).
$$

Note that we have not used the assumption $u \perp \mathcal{H}^0_0$. We can now deduce that $\|\Psi^- R^s \xi_1 u\| < \infty$ whenever $\square_b u = f$. Since the range of $\partial_b$ is closed we conclude that the range of $\partial_b^\ast$ is closed and since on functions we have $\square_b u = \partial_b^\ast \partial_b u$ we conclude that the range of $\partial_b^\ast$ equals the orthogonal complement of $\mathcal{H}^0_0$. Thus there exists a $(0,1)$-form $\varphi$ such that $\partial_b^\ast \varphi = u$ and we choose $\varphi$ so that it also satisfies $\partial_b \varphi = 0$. Hence

$$
\|\Psi^+ R^s \xi_1 \partial_b^\ast \varphi\| \leq C(\|\Psi^+ R^s \xi_1 \square_b \partial_b^\ast \varphi\| + \|\varphi\|).
$$
By elliptic regularization we conclude that whenever the right-hand side above is finite then so is the left-hand side. Since the range of $\tilde{\partial}_b^* \partial$ is closed, $\|\varphi\| \leq C\|\tilde{\partial}_b^* \varphi\| = C\|\varphi\|$. Hence

$$\|\Psi^+ R^s \zeta_1 u\| \leq C(\|\Psi^+ R^s \zeta_1 f\| + \|u\|).$$

Again we conclude that $\|\Psi^+ R^s \zeta_1 u\| < \infty$ and combining this with the above and with the 0 microlocalization we have $\zeta_0 u \in H^s$ for all $s$, which concludes the proof.

**Remark.** The range of $\tilde{\partial}_b$ on functions is closed whenever $M$ is compact and $(\text{SL})_1$ holds at all points of $M$. As noted this condition cannot hold if $\dim(M) = 3$. Another condition for the closed range of $\tilde{\partial}_b$ is given in [K4], namely: if $X$ is an $n$-dimensional Stein manifold and if $\Omega \subset X$ is a relatively compact pseudoconvex domain with a smooth boundary $M$ then $\tilde{\partial}_b$ on $M$ has closed range (see [K4]).

8. The $\bar{\partial}$-Neumann problem

In this section we shall establish the relation between subellipticity, subelliptic multipliers, and superlogarithmic estimates between boundaries of pseudoconvex domains and the corresponding estimates for domains. These results can be extended to various other estimates. The method here is the type of microlocalization worked out in [K4]. Another reduction to the boundary was derived by Greiner and Stein (see [GS]).

Let $\Omega \subset X$ be a relatively compact domain with a smooth boundary in a complex hermitian manifold $X$ and let $M$ denote the boundary of $\Omega$. Let $r$ be a defining function for $M$; that is, $r$ is a $C^\infty$ function in a neighborhood of $M$ such that $r = 0$ on $M$ and $dr \neq 0$. We will further assume that $|dr| = 1$ on $M$, that $r < 0$ in $\Omega$, and that $r > 0$ outside of $\Omega$. To simplify formulas we will also assume that $|r|$ is the geodesic distance to $M$. We denote by $A^{p,q}$ the space of $(p,q)$-forms in $C^\infty(X)$ restricted to $\Omega$. As usual, $\bar{\partial} : A^{p,q} \to A^{p,q+1}$. We denote by $\bar{\partial}^*$ the $L_2$-adjoint of $\bar{\partial}$. Also, $\bar{\partial}^*$ is an unbounded operator in Hilbert space whose domain, denoted by $\text{Dom}(\bar{\partial}^*)$, consists of all $\varphi \in L_2^{p,q}(M)$ such that $(\varphi, \bar{\partial}^* u) = (\bar{\partial}^* \varphi, u)$ for all $u \in L_2^{p,q-1}(M)$ with $\bar{\partial} u \in L_2^{p,q}(M)$. If $x_0 \in M$ we choose local holomorphic coordinates $\{z_1, z_2, \ldots, z_n\}$ on a neighborhood $U$ of $x_0$ with origin at $x_0$ such that $dz_n|_0 = \bar{\partial} r|_0$. Let $\{\omega_1, \ldots, \omega_n\}$ be an orthonormal basis of the $(1,0)$-forms on $U$ with $\omega_n = \bar{\partial} r$, so that $L_n(r) = 1$. Let $\{L_1, \ldots, L_n\}$ be the dual basis to the $\omega_i$ and note that on $M$ we have $L_i r = 0$ for $i = 1, \ldots, n - 1$, so that the CR structure on $M$, and on the manifolds $r = \text{constant}$, is given by the $\{L_1, \ldots, L_{n-1}\}$. Setting $T = \frac{1}{\sqrt{2}}(L_n - L_{n-1})$, we have that $\{L_1, \ldots, L_{n-1}, \bar{L}_1, \ldots, \bar{L}_{n-1}, T\}$ is an orthonormal basis of vectors tangent to $M$. Also, $T$ satisfies $T(r) = 0$ and $T = -T$. 
Suppose that $\phi \in \mathcal{A}_{0}^{0,q} \cap C_{0}^{\infty}(U \cap \bar{\Omega})$, so that $\phi = \sum \phi_I \tilde{\omega}_I$. Then $\phi$ is in the domain of $\bar{\partial}^*$ if and only if $\phi_I = 0$ on $M$ whenever $n \in I$. The $\bar{\partial}$-Neumann problem consists of solving the equation $\Box \phi = \alpha$, where $\Box = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}$. Setting $Q(\phi, \psi) = (\bar{\partial} \phi, \bar{\partial}^* \psi) + (\bar{\partial}^* \phi, \bar{\partial} \psi)$ we see that the above equation is satisfied if and only if $\phi$ is in the domain of $\bar{\partial}^*$ and $Q(\phi, \psi) = (\alpha, \psi)$ for all $\psi$ in the domain of $\bar{\partial}^*$.

On $U \cap \bar{\Omega}$ we will use the coordinates $\{x_1, \ldots, x_{2n-1}, r\}$, where $x_i = \text{Re}(z_i)$ for $i = 1, \ldots, n-1$ and $x_{n+i} = \text{Im}(z_{i-n+1})$ for $i = 1, \ldots, 2n-1$. We will denote by $x = (x_1, \ldots, x_{2n-1})$ the tangential coordinates and by $\xi = (\xi_1, \ldots, \xi_{2n-1})$ the dual coordinates. We define the tangential Fourier transform of $u \in C_0^{\infty}(U \cap \bar{\Omega})$ by

$$\tilde{u}(\xi, r) = \int e^{ix \cdot \xi} u(x, r) dx.$$ 

This notation should not be confused with the one used in previous sections. We define the operator $\Lambda_{\text{tan}}^s$ by

$$\Lambda_{\text{tan}}^s u(\xi, r) = (|\xi|^2 + 1)^{s/2} \tilde{u}(\xi, r)$$

and the tangential Sobolev norms by

$$\|u\|_s^2 = \int |\Lambda_{\text{tan}}^s u|^2 d\xi dr.$$ 

In terms of these coordinates the operators $L_i$ can be written as

$$L_i = \sum a_i^k(x, r) \frac{\partial}{\partial x_k},$$

for $i = 1, \ldots, n-1$ and

$$L_n = \frac{\partial}{\partial r} + \sum a^k(x, r) \frac{\partial}{\partial x_k}.$$ 

We define the tangential symbols $\mu_i$ by

$$\mu_i(x, \xi, r) = \frac{1}{\sqrt{-1}} \sum a_i^k(x, r) \xi_k,$$

for $i = 1, \ldots, n-1$ and

$$\mu_n(x, \xi, r) = \frac{1}{\sqrt{-1}} \sum (a^k(x, r) - \bar{a}^k(x, r)) \xi_k.$$ 

Note that $\mu_n$ is real. We set $\bar{\mu}_i(x, \xi) = \mu_i(x, \xi, 0)$ and $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_n)$ for $i = 1, \ldots, n$, and

$$|\bar{\mu}(x, \xi)| = \sqrt{\sum |\bar{\mu}_i(x, \xi)|^2}.$$ 

Subelliptic multipliers for the $\bar{\partial}$-Neumann problem are defined as follows.
Definition 8.1. Suppose that \( x_0 \in M \) and that \( f \in C^\infty(U) \). Then \( f \) is a subelliptic multiplier on \((p,q)\)-forms for the \( \bar{\partial} \)-Neumann problem if and only if there exist \( \varepsilon > 0 \) and \( C > 0 \) such that

\[
\| f \varphi \|_\varepsilon^2 \leq C(\| \varphi \| + \| \varphi \|_\varepsilon^2),
\]

for all \( \text{Dom}(\bar{\partial}^*) \cap \varphi \in \mathcal{A}^{p,q} \cap C^\infty_0(U \cap \bar{\Omega}) \). We denote by \( \mathcal{J}_q(x_0) \) the ideal of germs of subelliptic multipliers for the \( \bar{\partial} \)-Neumann problem at \( x_0 \).

The relation between the ideals of subelliptic multipliers for CR manifolds and for the \( \bar{\partial} \)-Neumann problem is given in the following.

Theorem 8.2. Suppose that \( \Omega \subset X \) is a pseudoconvex domain in a hermitian manifold \( X \). Suppose that \( \Omega \) is compact and has a smooth boundary \( M \). Then, if \( x_0 \in M \), the ideals \( \mathcal{J}_q(x_0) \) and \( \mathcal{I}_q^+(x_0) \) are related as follows. If \( f \in \mathcal{J}_q(x_0) \) then the restriction of \( f \) to \( M \) is in \( \mathcal{I}_q^+(x_0) \). Furthermore, if \( \rho \in \mathcal{I}_q^+(x_0) \), if \( f \in C^\infty(\bar{\Omega}) \), and if the restriction of \( f \) to \( M \) equals \( \rho \) then \( f \in \mathcal{J}_q(x_0) \).

The fact that hypoellipticity of the \( \bar{\partial} \)-Neumann problem follows from the assumption that a superlogarithmic estimate holds is formulated as follows.

Theorem 8.3. As above with \( x_0 \in M \) and \( M \) the boundary of \( \Omega \), assume that condition \((\text{SL}^+)\) holds in a neighborhood \( U \cap M \) of \( x_0 \). Then if \( \alpha \) is a \((p,q)\)-form on \( \Omega \) which is in \( L_2 \), whose the restriction to \( U \cap \bar{\Omega} \) is in \( C^\infty(U \cap \bar{\Omega}) \) and if \( \varphi \) satisfies the equation \( \Delta \varphi = \alpha \) (this means, in particular, that \( \varphi \) and \( \bar{\partial} \varphi \) are in the domains of \( \bar{\partial}^* \) on \((p,q)\) and \((p,q+1)\)-forms, respectively), then the restriction of \( \varphi \) to \( U \cap \bar{\Omega} \) is in \( C^\infty(U \cap \bar{\Omega}) \).

The key to proving these theorems is a passage between forms in \( \mathcal{A}^{p,q}_0 \) and forms in \( \mathcal{A}^{p,q} \). This is done by introducing an “approximate” harmonic extension of \( u \) on \( U \cap M \) to \( U \cap \bar{\Omega} \), denoted by \( u^{(h)} \). Supposing that \( u \in C^\infty_0(U \cap M) \) we define \( u^{(h)} \in C^\infty((\{ (x,r) \in \mathbb{R}^{2n} | r \leq 0 \}) \) by

\[
\begin{align*}
\quad u^{(h)}(x,r) &= (2\pi)^{-2n+1} \int e^{ix_\xi \cdot \xi} e^{\rho(x,\xi)} \tilde{u}(\xi) d\xi,
\end{align*}
\]

so that \( u^{(h)}(x,0) = u(x) \).

In this section \( \| \| \) will denote the \( L_2 \) norm on \( \Omega \), \( \| \|_b^b \) the \( L_2 \) norm on \( M \), and \( \| \|_s^b \) the Sobolev \( s \)-norm on \( M \).

Lemma 8.4. For each \( k \in \mathbb{Z} \), \( k \geq 0 \), and \( s \in \mathbb{R} \) there exists \( C_{s,k} > 0 \) such that

\[
\| r^k u^{(h)} \|_s^b \leq C_{s,k} \| u^{(h)} \|_{s-k-\frac{1}{2}}^b,
\]

for all \( u \in C^\infty_0(U \cap \bar{\Omega}) \).
Proof. We have
\[ \|r^k u^{(h)}\|_s^2 \leq C \int r^{2k} e^{2r|\hat{\mu}(x, \xi)|} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi dr. \]
Substituting \( r' = r|\hat{\mu}(x, \xi)| \) we integrate first with respect to \( r' \), and note that \( |\hat{\mu}(x, \xi)|^2 \sim (1 + |\xi|^2) \) to obtain the result. \( \square \)

Setting \( \triangle = -\sum \frac{\partial^2}{\partial x_i \partial x_i} \) we have
\[ \triangle = -\sum_{i=1}^n L_i \bar{L}_i + \sum_{i=1}^{2n-1} a^i(x, r) \frac{\partial}{\partial x_i} + a(x, r) \frac{\partial}{\partial r} \]
\[ \triangle = -\frac{\partial^2}{\partial r^2} + T^2 - \sum_{i=1}^{n-1} L_i \bar{L}_i + \sum_{i=1}^{2n-1} b_i(x, r) \frac{\partial}{\partial x_i} + b(x, r) \frac{\partial}{\partial r}, \]

since \( \frac{\partial}{\partial r} = \frac{1}{\xi}(L_n + \bar{L}_n) \) and \( L_n \bar{L}_n = \frac{\partial^2}{\partial r^2} - T^2 + D \), where \( D \) is a first order operator. Hence if \( (x, r) \in U \cap \Omega \),
\[ \triangle(u^{(h)})(x, r) = \int e^{ix\xi} e^{r|\hat{\mu}(x, \xi)|} (p^1(x, r, \xi) + rp^2(x, r, \xi)) \hat{u}(\xi) d\xi + Eu(x, r), \]
where the \( p^k(x, r, \xi) \) are symbols of order \( k \), uniformly in \( r \). Note that \( E(u) \) denotes the error term which is an operator of order \( -\infty \). Abusing notation we will denote all such terms by \( E(u) \). Further,
\[ L_i u^{(h)}(x, r) = (L_i u)^{(h)} + K_i u(x, r) + Eu(x, r), \]
with
\[ K_i u(x, r) = \int e^{ix\xi} e^{r|\hat{\mu}(x, \xi)|} (\hat{\mu}_i(x, \xi) + p^0_i(x, r, \xi) + rp^1_i(x, r, \xi)) \hat{u}(\xi) d\xi \]
and
\[ \bar{L}_i u^{(h)}(x, r) = (\bar{L}_i u)^{(h)} + \bar{K}_i u(x, r) + Eu(x, r) \]
for \( i = 1, \ldots, n-1 \). Since \( L_n = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial r} + T \right) \)
\[ L_n u^{(h)}(x, r) = \frac{1}{\sqrt{2}} \int e^{ix\xi} e^{r|\hat{\mu}(x, \xi)|} \left( |\hat{\mu}(x, \xi)| \right. \]
\[ + \left. |\hat{\mu}(x, \xi)| + \hat{\mu}_n(x, \xi) + p^0_n(x, r, \xi) \right) \hat{u}(\xi) d\xi + Eu(x, r) \]
and
\[ L_n u^{(h)}(x, r) = \frac{1}{\sqrt{2}} \int e^{ix\xi} e^{r|\hat{\mu}(x, \xi)|} \left( |\hat{\mu}(x, \xi)| - \hat{\mu}_n(x, \xi) + p^0_n(x, r, -\xi) \right) \hat{u}(\xi) d\xi + Eu(x, r) \]
where the \( p^k \) are symbols of order \( k \). If \( v \in C_0^\infty(U \cap \Omega) \) we define \( \hat{v} \) to be the restriction of \( v \) to \( M \) and we have
\[ \|\hat{v}\|_{s}^6 \leq C \left( \|v\|_{s+\frac{1}{2}} + \|\frac{\partial v}{\partial r}\|_{s-\frac{1}{2}} \right). \]
We microlocalize \( v \) for each fixed \( r \), setting
\[
\tilde{\Psi} v(\xi, r) = \psi(\xi) \check{v}(\xi, r)
\]
and then we have:

**Lemma 8.5.** If \( U \) is sufficiently small then there exists \( C > 0 \) such that
\[
\| \Psi^+ \check{L}_n \check{v}^{(h)} \| \leq C \left( \sum_{i=1}^{n-1} \| \Psi^+ L_i v \|_{\mathcal{L}^b_{1,2}}^b + \| v \| \right)
\]
and
\[
\| \Psi^0 \check{v}^{(h)} \|_1 + \| \Psi^{-} \check{v}^{(h)} \|_1 \leq C \left( \sum_{i=1}^{n} \| L_i v \| + \| v \| \right),
\]
for all \( v \in C^\infty_0(U \cap \bar{\Omega}) \).

**Proof.** Choosing \( U \) sufficiently small we have \( \check{\mu}_n(x, \xi) \geq 0 \) when \( \xi \in \text{supp}(\psi^+) \) and \( \check{\mu}_n(x, \xi) \leq 0 \) when \( \xi \in \text{supp}(\psi^-) \). Then for \( x \in U \) and \( \xi \in \text{supp}(\psi^+) \),
\[
|\check{\mu}(x, \xi)| - \check{\mu}_n(x, \xi) = \sum_{i=1}^{n-1} \frac{\check{\mu}_i(x, \xi)}{|\check{\mu}(x, \xi)| + \check{\mu}_n(x, \xi)} \hat{P}_i(x, \xi);
\]
hence the right-hand side is the principal symbol of a pseudodifferential operator on \( M \) of the form \( \sum_{i=1}^{n-1} P_i \check{L}_i \), where the \( P_i \) are of order zero. This establishes the first inequality. The second inequality follows from the fact that for \( x \in U \) and \( \xi \in \text{supp}(\psi^-) \),
\[
|\check{\mu}(x, \xi)| - \check{\mu}_n(x, \xi) \geq C|\xi|,
\]
when \( U \) is sufficiently small. \( \square \)

If \( \phi \in \mathcal{A}^{p,q}_b \) we denote by \( \check{\phi} \) the restriction of \( \phi \) to \( M \). Note also that if \( \phi \) is in the domain of \( \check{\partial}^* \) then \( \phi \in \mathcal{A}^{p,q}_{b,0} \). Recall that on a pseudonconvex domain the following are equivalent:
\[
Q(\phi, \phi) + \| \phi \|^2 \sim \sum_M c_{IJ} \phi_I \check{\phi}_J dS + \sum_{i=1}^{n} \| \check{L}_i \phi \|^2 + \| \phi \|^2,
\]
for all \( \phi \in \mathcal{A}^{p,q} \cap C^\infty_0(U' \cap \Omega) \) intersected with the domain of \( \check{\partial}^* \). Also
\[
Q_b(\zeta \Psi^+ \phi, \zeta \Psi^+ \phi) + (\| \zeta \Psi^+ \phi \|^2)^b
\]
\[
\sim \sum_{i=1}^{n} (c_{IJ} \Lambda^\frac{1}{\lambda} \check{\phi}_I, \Lambda^\frac{1}{\lambda} \check{\phi}_J)^b + \sum_{i=1}^{n-1} (\| \check{L}_i \zeta \Psi^+ \phi \|^2 + (\| \phi' \|^2)^b)^b,
\]
for all \( \phi \in \mathcal{A}^{p,q}_{b,0} \) with support in \( U' \cap M \). Combining the above with Lemma 8.5, we obtain:
Lemma 8.6. Suppose that $\Omega \subset X$ is a pseudoconvex domain in a hermitian manifold $X$. Suppose that $\Omega$ is compact and has a smooth boundary $M$. Then, if $x_0 \in M$ and if $U$ is a sufficiently small neighborhood of $x_0$ then there exists a constant $C > 0$ such that

$$\|\Psi^*\varphi\|_1^2 + \|\Psi^0\varphi\|_1^2 \leq C(Q(\varphi, \varphi) + \|\varphi\|^2),$$

for all $\varphi \in \mathcal{A}^{p,q} \cap \{\text{domain of } \partial^*\}$ with support in $U \cap \Omega$.

Proof. We have

$$\|\Psi^*\varphi^{(h)}\|_1^2 + \|\Psi^0\varphi^{(h)}\|_1^2 \leq C \left( \sum_{k=1}^n \|\tilde{L}_k\varphi_k\|_2^2 + \|\varphi\|^2 \right).$$

Let $\varphi^{(0)} = \varphi - \varphi^{(h)}$, so that $\varphi^{(0)} = 0$ on $M$. Then, with $\zeta \in C_0^\infty(U')$ and $\zeta = 1$ on a neighborhood of $\bar{U}$,

$$\|\Psi^*\varphi^{(0)}\|_1^2 + \|\Psi^0\varphi^{(0)}\|_1^2 \leq C(Q(\zeta \Psi^*\varphi^{(0)}), \zeta \Psi^*\varphi^{(0)}) + Q(\zeta \Psi^0\varphi^{(0)}, \zeta \Psi^0\varphi^{(0)}) + \|\varphi\|^2 \leq C(Q(\varphi, \varphi) + \|\varphi\|^2),$$

which concludes the proof.

Now we are in a position to prove the theorems of this section.

Proof of Theorem 8.2. Suppose that $f \in \mathcal{J}_q(x_0)$ and $\dot{f} = \rho$. Then, if $\varphi \in \mathcal{A}^{p,q} \cap C_0^\infty(U' \cap \Omega)$,

$$\|\rho \zeta \Psi^+\varphi\|_b^2 \leq C\left( \|f \zeta \Psi^+\varphi^{(h)}\|_{\varepsilon + \frac{1}{2}}^2 + \|f \zeta \Psi^+\varphi^{(h)}\|_{\varepsilon - \frac{1}{2}}^2 \right) \leq C(Q_b(\zeta \Lambda^\frac{1}{2}\Psi^+\varphi^{(h)}, \zeta \Lambda^\frac{1}{2}\Psi^+\varphi^{(h)}) + \|\Lambda^\frac{1}{2}\Psi^+\varphi^{(h)}\|^2) \leq C(Q_b(\zeta \Psi^+\varphi, \zeta \Psi^+\varphi) + (\|\Psi^+\varphi\|^2)^2),$$

so that $\rho \in \mathcal{I}_q^+(x_0)$.

Next, assume that $\rho \in \mathcal{I}_q^+(x_0)$ and that $\dot{f} = \rho$. Then if $\varphi \in \mathcal{A}^{p,q} \cap C_0^\infty(U \cap \Omega) \cap \{\text{domain of } \partial^*\}$, we have

$$\|f \zeta \Psi^+\varphi\|_b^2 \leq C(\|f \zeta \Psi^+\varphi\|_{\varepsilon + \frac{1}{2}}^2 + \|f \zeta \Psi^+\varphi\|_{\varepsilon - \frac{1}{2}}^2) \leq C(Q_b(\zeta \Lambda^{-\frac{1}{2}}\Psi^+\varphi, \zeta \Lambda^{-\frac{1}{2}}\Psi^+\varphi) + \|\varphi\|^2) \leq C\left( \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} \|\tilde{L}_i\zeta \Lambda^{-\frac{1}{2}}\Psi^+\varphi\|_b^2 \right) + \sum_{i,j} \sum_{k=1} \sum_{m=1} \left( c_{ij} \zeta \Lambda^{-\frac{1}{2}}\Psi^+\varphi \right) \zeta \Lambda^{-\frac{1}{2}}\Psi^+\varphi \|_b^2 + \|\varphi\|^2).$$
To estimate the last two terms above we proceed as follows. For $i < n$,

$$
(\|L_i \zeta \Lambda^{-\frac{1}{2}} \Psi^+ \varphi\|)^2 = 2\left(\frac{\partial}{\partial r} L_i \zeta \Lambda^{-\frac{1}{2}} \varphi, \bar{L}_i \zeta \Lambda^{-\frac{1}{2}} \varphi\right)
= 2\left(\frac{\partial}{\partial r} \Lambda^{-1}_{\tan} \bar{L}_i \varphi, \bar{L}_i \varphi\right) + \text{error}
\leq C(\|\Lambda^{-1}_{\tan} \bar{L}_i \bar{L}_n \varphi, \bar{L}_i \varphi\|) + \|\Lambda^{-1} T \bar{L}_i \varphi, \bar{L}_i \varphi\| + \text{error}
\leq C(Q(\varphi, \varphi) + \|\varphi\|^2),
$$

since $\Lambda^{-1}_{\tan} \bar{L}_i$ and $\Lambda^{-1} T$ are tangential pseudodifferential operators of order zero. The error terms can be estimated since $\left[\frac{\partial}{\partial r}, \bar{L}_i\right]$ is a tangential first order operator. Finally, setting $P = (\zeta)^2 T \Lambda^{-1}_{\tan}(\Psi^+)^2$ we have

$$
\sum_{I,J} (c_{I,J} T \zeta \Lambda^{-\frac{1}{2}} \Psi^+ \varphi_I, \zeta \Lambda^{-\frac{1}{2}} \Psi^+ \varphi_J)^h = \sum_{I,J} (c_{I,J} P \varphi_I, \varphi_J)^b + \text{error}
\leq |\sum_{I,J} (c_{I,J} P \varphi_I, P \varphi_J)^b|
+ |\sum_{I,J} (c_{I,J} \varphi_I, \varphi_J)^b| + \text{error}
\leq C(Q(P \varphi, P \varphi) + Q(\varphi, \varphi) + \|\varphi\|^2)
\leq C(Q(\varphi, \varphi) + \|\varphi\|^2),
$$

which concludes the proof of the theorem.\[\square\]

**Proof of Theorem 8.3.** Assume that the condition $(\text{SL}^+)^q$ holds in $U \cap M$. Let $\varphi$ be a $(p, q)$-form that satisfies $\Box \varphi = \alpha$ with $\alpha \in L_2$ and the restriction of $\alpha$ to $U \cap \Omega$ in $C^\infty(U \cap \Omega)$. We will show that the restriction of $\varphi$ to $U \cap \Omega$ is in $C^\infty(U \cap \Omega)$. First we establish the following a priori estimate for the tangential derivatives of $\varphi$. Given $s \in \mathbb{R}$ and $\zeta_0, \zeta_1 \in C^\infty_0(U \cap \Omega)$ with $\zeta_1 = 1$ on the support of $\zeta_0$, there exists a constant $C_s$ such that

$$(\ast)_{\tan} \quad \|\zeta_0 \varphi\|_s \leq C_s(\|\zeta_1 \alpha\|_s + \|\varphi\|),$$

for all $\varphi \in \mathcal{A}^{p, q} \cap \{\text{domain of } \bar{\partial}^*\}$. Now let $\zeta' \in C^\infty_0(U' \cap \bar{\Omega})$ with $\zeta' = 1$ on $U \cap \Omega$. We will abuse notation and use $\varphi$ to denote both $\zeta' \varphi$ and $\varphi$; it will be clear from the context which is which and the errors committed will be controlled as in Section 7. Again we set $\varphi = \varphi^{(h)} + \varphi^{(0)}$. Since $\varphi^0 = 0$ on $M$,

$$\|\zeta_0 \varphi^{(h)}\|_{s+2} \leq C_s(\|\zeta_1 \Box \varphi^{(0)}\|_s + \|\varphi^{(0)}\|)$$

and

$$\Box \varphi^{(0)} = \Box \varphi - \Box \varphi^{(h)} = \alpha + P_1 \varphi,$$

where $P_1$ is a tangential pseudodifferential operator of order one. Since the above inequality holds for all suitable pairs $\zeta_0, \zeta_1$ we deduce that

$$\|\zeta_0 \varphi^{(0)}\|_{s+2} \leq C_s(\|\zeta_1 \alpha\|_s + \|\varphi\|).$$
Then, Lemma 8.6 implies that
\[ \| \Psi^- \zeta \varphi \|_{s+2} + \| \Psi^0 \zeta \varphi \|_{s+2} \leq C_s(\| \alpha \|_s + \| \varphi \|). \]

Hence, in order to prove \((*)_{\tan}\), it will suffice to prove \((**)_{\tan}\)
\[ \| \Psi^+ \zeta \varphi^{(b)} \|_s \leq C_s(\| \alpha \|_s + \| \varphi \|). \]

Now, following the argumentation of Section 7, in order to simplify the formulas, we denote by \(C\) and \(C_\delta\) constants which may differ in different lines:
\[ \| \Psi^+ \zeta \varphi^{(b)} \|_s \leq C \left( \| \Psi^+ \zeta \varphi \bigr|_{s-\frac{1}{2}} \right)^2 + \| \varphi \|^2 \]
\[ \leq C \left( \| \log(\Lambda_{\tan}) R^{s-\frac{1}{2}} \Psi^+ \varphi \bigr|_{b} ^2 + \| \varphi \|^2 \right) \]
\[ \leq \delta^2 Q_b(\zeta_1 R^{s-\frac{1}{2}} \Psi^+ \varphi, \zeta_1 R^{s-\frac{1}{2}} \Psi^+ \varphi) + C_\delta \| \varphi \|^2 \]
\[ \leq C \delta^2 \left( \sum_{1}^{n-1} (\| L_1 \zeta_1 R^{s-\frac{1}{2}} \Psi^+ \varphi \|_b )^2 \right. \]
\[ + \sum_{ij} \left. c_{ij} T \zeta_1 R^{s-\frac{1}{2}} \Psi^+ \varphi_{i} , \zeta_1 R^{s-\frac{1}{2}} \Psi^+ \varphi_{j} \right) + C_\delta \| \varphi \|^2 \]
\[ \leq C \delta^2 \left( \sum_{1}^{n-1} \| L_1 \zeta_1 R^{s} \Psi^+ \varphi \|^2 \right. \]
\[ + \sum_{ij} \left. \int_{M} c_{ij} T \zeta_1 R^{s} \Psi^+ \varphi_{i} \zeta_1 R^{s} \Psi^+ \varphi_{j} dS \right) + C_\delta \| \varphi \|^2 \]
\[ \leq C \delta^2 Q(\zeta_1 R^{s} \Psi^+ \varphi, \zeta_1 R^{s} \Psi^+ \varphi) + C_\delta \| \varphi \|^2 \]
\[ \leq C \delta^2 (R^{s} \Psi^+ \zeta \alpha, \zeta_1 R^{s} \Psi^+ \varphi) + C_\delta \| \varphi \|^2. \]

The estimate \((*)_{\tan}\) then follows and implies the next inequalities:
\[ \| \frac{\partial}{\partial r} (\zeta_0 \varphi) \|_{s-1} \leq C(\| \zeta_1 \alpha \|_s + \| \varphi \|) \]
and
\[ \| \frac{\partial}{\partial r}^{k+2} (\zeta_0 \varphi) \|_{s-k-2} \leq C \left( \sum_{0}^{k} \| \frac{\partial}{\partial r}^j (\zeta_1 \alpha) \|_{s-j} + \| \varphi \| \right), \]
for integers \(k \geq 0\). The first inequality follows from
\[ \| \frac{\partial}{\partial r} (\zeta_0 \varphi) \|^2 \leq \| \tilde{L}_n (\zeta_0 \varphi) \|^2 + \| T (\zeta_0 \varphi) \|^2 \]
\[ \leq C(Q(\zeta_0 \varphi, \zeta_0 \varphi) + \| \zeta_0 \varphi \|^2 + \| \varphi \|^2), \]
and is then obtained by substituting \( \Lambda_{\tan}^{-1} \varphi \) for \( \varphi \). The second inequality is obtained as follows. Since \( \Box \) is elliptic we can solve the equation \( \Box \varphi = \alpha \) for the second derivatives with respect to \( r \):
\[ \frac{\partial^2 \varphi_I}{\partial r^2} = \sum_{K} a^K_I \alpha_K + \sum_{K,i,j} b^K_{ij} \frac{\partial^2 \varphi_K}{\partial x_i \partial x_j} + \sum_{K,i} c^K_i \frac{\partial^2 \varphi_K}{\partial x_i \partial r} + \text{first order}. \]
The second inequality is then obtained by applying $\zeta_0 \Lambda^{s-k-2} \frac{\partial^k}{\partial r^k}$ to the above equation and taking $L_2$ norms.

Using elliptic regularization one sees that all partial derivatives of $\zeta_0 \varphi$ are square-integrable and the theorem follows. \qed

Appendix

Here we will recall briefly the notion of ideal finite type and the explicit expressions for subelliptic multipliers as introduced in [K2] and [K3]; this material is also explained in [DK]. Given an $(n-1)$-tuple $\rho_1, \ldots, \rho_{n-1}$ of germs of $C^\infty$ functions at $x_o \in M$ we denote by $\mathcal{M}(\rho_1, \ldots, \rho_{n-1})$ the $(n-1) \times 2(n-1)$ matrix defined by

$$
\mathcal{M}(\rho_1, \ldots, \rho_{n-1}) = \begin{pmatrix}
\begin{array}{cccc}
c_{11} & c_{12} & \cdots & c_{1,n-1} \\
c_{21} & c_{22} & \cdots & c_{2,n-1} \\
& \vdots & \ddots & \vdots \\
c_{n-1,1} & c_{n-1,2} & \cdots & c_{n-1,n-1} \\
L_1 \rho_1 & L_2 \rho_1 & \cdots & L_{n-1} \rho_1 \\
L_1 \rho_2 & L_2 \rho_2 & \cdots & L_{n-1} \rho_2 \\
& \vdots & \ddots & \vdots \\
L_1 \rho_{n-1} & L_2 \rho_{n-1} & \cdots & L_{n-1} \rho_{n-1}
\end{array}
\end{pmatrix}
$$

**Theorem 8.7.** If $M$ is a pseudoconvex CR manifold of dimension $2n-1$ and if $x_o \in M$ then the $I^+_q(x_o)$ have the following properties:

A. $I^+_q(x_o)$ is an ideal.

B. The real radical of $I^+_q(x_o)$, denoted by $\sqrt{I^+_q(x_o)}$, is contained in $I^+_q(x_o)$. The ideal $\sqrt{I^+_q(x_o)}$ consists of all germs $g$ such that there exist $f \in I^+_q(x_o)$ and $m \in \mathbb{Z}$ with $|g|^m \leq |f|$.

C. $\text{Det}^{n-q-1} \mathcal{M}(\rho_1, \ldots, \rho_{n-1})$ is the ideal generated by the $(n-q-1) \times (n-q-1)$ subdeterminants of $\mathcal{M}(\rho_1, \ldots, \rho_{n-1})$. If $\rho_i \in I^+_q(x_o)$ then $\text{Det}^{n-q-1} \mathcal{M}(\rho_1, \ldots, \rho_{n-1}) \subset I^+_q(x_o)$.

**Definition 8.8.** The ideals $I^+_{q,k}(x_o)$ are defined by induction on $k$ as follows:

$$
I^+_{q,1}(x_o) = \sqrt[\sqrt{\text{Det}^{n-q-1} \mathcal{M}(0, \ldots, 0)}}
$$

$$
I^+_{q,k+1}(x_o) = \sqrt[\sqrt{\text{Det}^{n-q-1} \mathcal{M}(\rho_1, \ldots, \rho_{n-1})}]{I^+_{q,k}(x_o), \text{Det}^{n-q-1} \mathcal{M}(\rho_1, \ldots, \rho_{n-1})}
$$
Here \( (A, B, C, \cdots) \) denotes the ideal generated by \( \{A \cup B \cup C \cup \cdots\} \).

The ideals \( \mathcal{I}_{q,k}(x_0) \) are defined by setting \( \mathcal{I}_{q,k}(x_0) = \mathcal{I}_{n-q-1,k}^+(x_0) \). We set \( \mathcal{I}_{q,k}(x_0) = \mathcal{I}_{q,k}^+(x_0) \cap \mathcal{I}_{q,k}^-(x_0) \).

**Remark.** It then follows that \( \mathcal{I}_{q,k}^+(x_0) \subset \mathcal{I}_{q,k}^+(x_0) \cap \mathcal{I}_{q,k}^-(x_0) \), and \( \mathcal{I}_{q,k}^-(x_0) \subset \mathcal{I}_{q,k}^-(x_0) \). Furthermore, \( \mathcal{I}_{q,k}^+(x_0) \subset \mathcal{I}_{q+1,k}^+(x_0) \) and \( \mathcal{I}_{q,k}^-(x_0) \supset \mathcal{I}_{q+1,k}^-(x_0) \), so that \( \mathcal{I}_{q,k}(x_0) = \mathcal{I}_{m,k}^+(x_0) \), where \( m = \min\{q, n - q - 1\} \).

**Definition 8.9.** If \( \Omega \) is a pseudoconvex domain in a hermitian manifold \( X \) with a smooth boundary \( M \) then \( x_0 \in M \) is of finite ideal \( q \)-type (for \( \bar{\partial} \)) if \( 1 \in \mathcal{I}_{q,k}^+(x_0) \) for some \( k \). If \( M \) is a pseudoconvex CR manifold and \( x_0 \in M \) then \( x_0 \in M \) is of finite ideal \( q \)-type (for \( \bar{\partial}b \)) if \( 1 \in \mathcal{I}_{q,k}(x_0) \) for some \( k \).

Thus, both for domains and CR manifolds finite ideal \( q \)-type implies that subellipticity holds for \((p,q)\)-forms. The question is whether this condition is also necessary. For domains in two-dimensional manifolds, necessity was proved by Greiner (see [G]). In this case the ideal finite type condition can be expressed in terms of commutators of vector fields. The proof is easily generalized to the case when the matrix \( (c_{IJ}) \) is diagonalizable in a neighborhood of \( x_0 \). It also can be easily generalized to CR manifolds for which both matrices \( (c_{IJ}) \) and \( (c'_{IJ}) \) can be diagonalized in a neighborhood of \( x_0 \). In case the defining function for the boundary \( M \) is real analytic the results of Diederich and Forneš (see [DF]) were used in [K2] to prove that finite ideal type is equivalent to subellipticity. In the general case Catlin proved that subellipticity is equivalent to finite D’Angelo type (see [C] and [D]). Thus the problem is to prove that finite D’Angelo type is equivalent to finite ideal type.

**References**

[C] D. Catlin, Subelliptic estimates for the \( \bar{\partial} \)-Neumann problem on pseudoconvex domains, *Ann. of Math.* **126** (1987), 131–191; Necessary conditions for the subellipticity of the \( \bar{\partial} \)-Neumann problem, *Ann. of Math.* **117** (1983), 147–171.

[Ch 1] M. Christ, Hypoellipticity in the infinitely degenerate regime, in *Complex Analysis and Geometry, Proc. Conf. Ohio State Univ.*, Walter de Gruyter, New York, 2001, 59–84.

[Ch 2] __________, Spiraling and nonhypoellipticity, in *Complex Analysis and Geometry, Proc. Conf. Ohio State Univ.*, Walter de Gruyter, New York, 2001, 85–101.

[D] J. P. D’Angelo, Real hypersurfaces, orders of contact and applications, *Ann. of Math.* **115** (1982), 615–637.

[DF] K. Diederich and J. E. Forneš, Pseudoconvex domains with real-analytic boundary, *Ann. of Math.* **107** (1978), 371–384.
[DK] J. P. D’Angelo and J. J. Kohn, Subelliptic estimates and finite type, in *Several Complex Variables* (Berkeley, 1995–1996), *M. S. R. I. Publ.* 37, Cambridge Univ. Press, Cambridge (1999), 199–232.

[F] V. S. Fedii, A certain criterion for hypoellipticity, *Mat. Sb.* 85 (1971), 18–48.

[G] P. C. Greiner, Subelliptic estimates of the $\bar{\partial}$-Neumann problem in $\mathbb{C}^2$, *J. Differential Geom.* 9 (1974), 239–250.

[GS] P. C. Greiner and E. M. Stein, Estimates for the $\bar{\partial}$-Neumann problem, *Math. Notes* 19, Princeton Univ. Press, Princeton, NJ, 1977.

[K1] J. J. Kohn, Boundaries of complex manifolds, *Proc. Conf. Complex Analysis* (Minneapolis, 1964), Springer-Verlag, New York, 1965, 81–94.

[K2] ———, Subellipticity of the $\bar{\partial}$-Neumann problem on pseudo-convex domains: sufficient conditions, *Acta Math.* 142 (1979), 79–122.

[K3] ———, Microlocalization of CR structures, *Proc. Several Complex Variables* (Hangzhou, 1981), Birkhauser, Boston, 1984, 29–36.

[K4] ———, The range of the tangential Cauchy-Riemann operator, *Duke Math. J.* 53 (1986), 525–545.

[K5] ———, Hypoellipticity of some degenerate subelliptic operators, *J. Funct. Anal.* 159 (1998), 203–216.

[K6] ———, Hypoellipticity at points of infinite type, in *Analysis, Geometry, Number Theory: The Mathematics of Leon Ehrenpreis* (Philadelphia, PA 1998), *Contemp. Math.* 251 (2000), 393–398.

[KN] J. J. Kohn and L. Nirenberg, Non-coercive boundary value problems, *Comm. Pure Appl. Math.* 18 (1965), 443–492.

[KR] J. J. Kohn and H. Rossi, On the extension of holomorphic functions from the boundary of a complex manifold, *Ann. of Math.* 81 (1965), 451–472.

[KS] S. Kusuoka and D. Stroock, Applications of the Mallavain calculus II, *J. Fac. Sci. Univ. Tokyo* 32 (1985), 1–76.

[M] Y. Morimoto, A criterion for hypoellipticity of second order differential operators, *Osaka J. Math.* 24 (1987), 651–675.

[R] H. Rossi, Attaching analytic spaces to an analytic space along a pseudoconcave boundary, *Proc. Conf. Complex Analysis* (Minneapolis, 1964), Springer-Verlag, New York, 1965, 242–256.

(Received June 30, 2000)