Dynamic Stiction Caused by Friction Vector Rotation

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We numerically study a simple sliding system: a rigid mass pulled by a spring having strong in-plane stiffness anisotropy and a small misalignment angle. Simulations show that the apparent “stick” phase in this system is in reality a phase of very slow creep, followed by rapid sliding, slip. Surprisingly, the absolute value of the friction force remains almost constant from the very beginning of the “stick” phase, merely rotating in the sliding plane. We call this specific mechanism of apparent stick due to rotation of the force vector “dynamic stiction”. Another interesting consequence of our model is the presence of high-frequency oscillations perpendicular to the pulling direction in the “dynamic stick” phase.

Introduction.—Dry friction plays an essential role in many physical processes and numerous engineering, biological and medical applications [1]-[3]. Often, the simplest friction “law” (Amontons’ law [4],[5]) is used to describe dry friction. It states that the body remains at rest until the driving force exceeds some critical value called static friction. After that, sliding occurs at almost constant force \(F\), which is roughly proportional to the normal force, \(W\): \(F = \mu W\). The proportionality coefficient \(\mu\) is called coefficient of friction. At the latest since Coulomb [6], one distinguishes between static and kinetic friction. The present paper is devoted to the analysis of the transition between static and kinetic friction, or in other words transition from stick to slip. This transition plays an essential role in many technological [7] and geological [8] processes and has been studied intensively along two main lines, which can be characterized as static and kinetic. In the static approach, the tangentially loaded contact is considered as being divided into sliding and sticking parts, while at increasing tangential load, the fraction of stick decreases until complete sliding begins. The prominent representative of this approach is the theory of partial sliding by Cattaneo [9] and Mindlin [10]. In the kinetic view, both stick and slip are considered as sliding with different velocities. The prominent representative of this approach are the rate- and state-dependent laws of friction developed in the 1970s in context of geotectonics [11],[12]. A similar, but purely phenomenological approach has been developed in context of pre-sliding [13],[14] which is of basic importance for precision positioning and feedback control systems [15]. Further approaches combining both of the above views have been developed, e.g. rapid propagation of slip or detachment fronts in the contact plane [16].

All of the above approaches are entirely focused on describing the magnitude of friction force. Its direction is assumed to be opposite to the direction of sliding. This, however, is not necessarily the case due to finite stiffness of the junction between the macroscopic drive and the immediate contact spot. Consideration of the direction of friction force introduces an additional degree of freedom and opens a completely new view on the old problem of transition from stick to slip. Here we show that both stick and slip phases can be naturally understood in a purely mechanical way as emerging from rotation of the friction vector. The importance of the rotation of the direction of friction force was highlighted in [17] in context of active control of friction by transverse ultrasonic vibrations.

Model.—Figure 1a shows a side view of the model: a rigid mass \(m\) is coupled to a spring and is in contact with a horizontal flat floor (XY plane) under the normal load \(W\). The right end of the spring is driven along the X-axis with velocity \(V\). In the top view of the system (Figure 1b), the spring is characterized by the stiffness tensor. Let \(x\) and \(y\) be the principal axes of the stiffness tensor (as shown with green dashed lines in Figure 1b) and \(k_x\) and \(k_y\), the corresponding principal values. We assume that the principal axes are inclined relative to the direction of the drive velocity by a small angle \(\varphi\). Such a misalignment is not an unrealistic assumption in tribological apparatus, since angles less than one degree are practically unavoidable but are sufficient to...
produce friction vector rotation if the anisotropy of the spring is large enough. The projections of the spring forces onto the principal axes are \( k_{x} \) and \( k_{y} \) for the principal axes, where \( u_{x} \) and \( u_{y} \) are the spring elongations in the \( x \) and \( y \) directions, respectively. The velocity of the immediate contact point relative to the support will be denoted as “slip velocity”, \( v_{\text{slip}} \). The direction of the friction force, \( F \), was assumed to be opposite to the direction of \( v_{\text{slip}} \) and its magnitude a function of the magnitude of \( v_{\text{slip}} \). A velocity-weakening law \( F = \mu_{\infty} \exp(-v_{\text{slip}}/v_{f}) \) \( F \) was used, where \( \mu_{0} \) and \( \mu_{\infty} \) are the friction coefficients for \( v_{\text{slip}} = 0 \) and \( \infty \) respectively, and \( v_{f} \) is a velocity constant. Note that we do not assume the existence of a finite static friction (which vanishes).

The equations of motion of the mass \( m \) in coordinates \( x \) and \( y \) read:

\[
m \ddot{u}_{x} + k_{x} u_{x} = F \cos \theta, \quad m \ddot{u}_{y} + k_{y} u_{y} = F \sin \theta,
\]

where \( \theta = \psi + \phi \) is the angle between the friction force and the \( x \)-axis. Simple geometrical considerations show that

\[
\cos \theta = (v \cos \phi - \dot{u}_{x}) / v_{\text{slip}}, \quad \sin \theta = (v \sin \phi - \dot{u}_{y}) / v_{\text{slip}}
\]

with

\[
v_{\text{slip}} = \sqrt{(v \cos \phi - \dot{u}_{x})^2 + (v \sin \phi - \dot{u}_{y})^2}.
\]

With account of (2), (3), Eqs. (1) can be rewritten as

\[
\begin{align*}
    m \ddot{u}_{x} + k_{x} u_{x} &= F(v_{\text{slip}}) \frac{v \cos \phi - \dot{u}_{x}}{(v \cos \phi - \dot{u}_{x})^2 + (v \sin \phi - \dot{u}_{y})^2}, \\
    m \ddot{u}_{y} + k_{y} u_{y} &= F(v_{\text{slip}}) \frac{v \sin \phi - \dot{u}_{y}}{(v \cos \phi - \dot{u}_{x})^2 + (v \sin \phi - \dot{u}_{y})^2}.
\end{align*}
\]

This system of two non-linear second-order differential equations completely determines the dynamics of the system. They were solved numerically using the Runge-Kutta method. The time evolution of object position in the laboratory coordinates can be obtained by

\[
X = V t - u_{x} \cos \phi - u_{y} \sin \phi.
\]

**Apparent stick-slip motion of the system**—Figure 2 presents solutions to the equations of motion. A small in-plane misalignment of \( \phi = 1^\circ \) and a strong stiffness anisotropy of \( k_{x}/k_{y} = 10^4 \) were assumed corresponding to typical cantilever springs used in laboratory friction tests. Other parameters are listed in the caption. The drive started to move at \( t = 0 \); the time dependencies of the \( x \)- and \( y \)-components of the spring force are shown in Figure 2a,b. The longitudinal component of the spring force shows the classical stick-slip behavior consisting of a linear increase in time followed by a sharp drop (Figure 1a). The transverse component of the force, on the contrary, reveals an unexpected behavior: It jumps to the maximum value (equal to the magnitude of the friction force in slow sliding) and subsequently decreases to vanish at the start of the slip phase (Figure 2b). The magnitude of the force remains practically constant during the whole stick phase (Figure 2c) dropping only in the phases of rapid slip. Maintaining equilibrium in the pulling direction is possible due to the in-plane rotation of the friction vector described by the angle \( \psi \) between the direction of force and the driving direction, Figure 2d. The time dependency of the longitudinal coordinate \( X \) (Figure 2e) shows a pronounced stick-slip character. Although the stair-like object position (\( X \)) and the sawtooth-shaped spring force (\( k_{x} u_{x} \)) indicate typical stick-slip, in reality, the body never comes to a full stop. During the stick phases, the object is slowly slipping and gradually accelerating in the \( X \) direction (see the inset of Figure 2e), which reminds us of the so-called “slow creep” known from studies of the rate- and state-dependent friction laws [18]. To underline the dynamic nature of the apparent stick phase we call it “dynamic stiction.”
Another interesting feature seen in numerical simulations are high-frequency oscillations (Figure 3). These oscillations are not numerical artifacts but rather an inherent property of the described system, which will be discussed below. A closer look shows that the oscillation frequency coincides at the beginning of the "stick" phase with the natural frequency of the transverse oscillations, \( \omega_y = (k_y/m)^{1/2} \) (for the system parameters used, 1.0 kHz) and decreases when approaching the phase of rapid slip (Figure 3b,d). The high frequency dynamics of sliding systems is of significant interest for many technical applications [19], [20]. Their physical origin and influencing factors have been studied for many decades, but their nature often remains unclear [19]. Thus, the concept of the friction vector rotation offers a new perspective also on the problem of frictionally induced high-frequency oscillations.

In the following, we discuss in more detail the main features of the observed stick-slip motion: (a) **dynamic stiction**, (b) **slow creep**, (c) **high-frequency oscillations**, and (d) **low-frequency oscillations**.

**Dynamic stiction.**—Based on the results presented so far, we would like to describe schematically the mechanism of dynamic stiction. The simulations reveal that the body, when loaded, never stops but is moving with a small velocity. This leads to the fact that the magnitude of frictional force remains constant all the time. At the beginning of sliding, the projection of friction force on the sliding direction gradually increases from zero. This means that the vector of friction force should be initially directed perpendicular to the sliding direction. Indeed, we see that at the beginning of macroscopic sliding the friction force vector jumps to the direction perpendicular to the sliding direction \( (\psi \approx 90^\circ) \), see Figure 2d). When the spring force in sliding direction increases, the friction force vector rotates, but its absolute value \( F_0 = (F_x + F_y)^{1/2} \) remains constant and practically equal to the value of friction force at very small sliding velocities, \( F_0 = \mu W \) (Figure 2c). Thus, the perpendicular component of the friction force, \( F_y = (F_0^2 - F_x^2)^{1/2} \), is decreasing, gradually approaching zero (Figure 2b). The change in the pulling force can be supported by the rotation of the friction vector only until \( F_y \leq F_0 \). As soon as the pulling force exceeds this critical value, no static equilibrium is possible any more, and the phase of rapid slip starts.

**Slow creep.**—Let us consider in more detail the "stick phase" (which in reality is a phase of slow creep). The movement during this stage is quasistatic, which means that the inertial terms can be neglected. However, this is valid only for the movement in the \( x \)-direction. The high transverse stiffness \( k_y \) guarantees very small deflections \( u_y \). The velocity \( \dot{u}_y \), on the contrary, is not necessarily small due to high natural frequency in the \( y \)-direction but it has zero average and can be set to zero while considering the creep process. Thus, in the creep phase, we can neglect in Eqs. (4),(5) the terms with \( \dot{u}_x \) and \( \dot{u}_y \). After some transformations this leads to

\[
\dot{u}_x = V \cos \phi - \frac{k_y u_y / F_0}{\sqrt{1 - (k_y u_y / F_0)^2}} V \sin \phi . \tag{7}
\]

This is an ordinary differential equation of the first order, which completely determines the dynamics of the degree of freedom \( u_x(t) \). The coordinate \( X \) in the driving direction can finally be found using Eq. (6). The resulting solution shows that at small misalignment the system shows an almost perfect stick, while it rapidly becomes blurred when increasing the misalignment angle. In the limiting case of very small misalignment angles, Eq. (7) takes the form \( \dot{u}_x = \dot{V} \) with the solution \( u_x = Vt \) (for \( Vt < l_c = F_0/k_x \)). For the creep velocity, Eq. (6) yields

\[
\dot{X} = V - \dot{u}_x = V \phi + \frac{(Vt/l_y)^2}{\sqrt{1 - (Vt/l_y)^2}} . \tag{8}
\]

It contains two contributions: One of the first order in pulling velocity and of second order in misalignment angle and a second one that is linear in pulling velocity and of second order in misalignment angle. This term in Eq. (5) transforms this equation to

\[
u u_y + k_y u_y + F_0 \text{sign}(u_y - V \sin \phi) = 0 . \tag{9}
\]

The average value of displacement is easily found by setting \( \dot{u}_y = 0 \) and \( \ddot{u}_y = 0 \) : \( k_y \langle u_y \rangle = F_0 \) while the amplitude of oscillations is determined by the non-linear term.
\( F_s \cdot \text{sign}(\dot{u}_s - V \sin \phi) \) and is bounded by the value \( \ddot{u}_s = V \sin \phi \). After a setting time, the system oscillates undamped with the frequency \( \omega_s = (k_s/m)^{1/2} \).

Low-frequency oscillations.—Note that the system also shows low-frequency oscillations in the x-direction, which can be used for dynamic probing of approaching the moment of slip. Dynamics in the x-direction can be described with one single equation in both “stick” and slip phases by setting \( \dot{u}_s = 0 \) in (4). With \( v_x = \dot{u}_s \), this equation leads to

\[
\ddot{v}_x + 2(\Lambda \sin^3 \theta) \dot{v}_x + v_x = 0,
\]

where \( \Lambda = \mu_0 W/2V \sin(m_k \phi) \) and \( (\cdot) \) means the derivative with respect to \( t = \omega_s t \). Thus, the damping properties of longitudinal oscillations are controlled solely by the factor \( \zeta = \Lambda \sin^3 \theta \). During the “stick” phase, the angle \( \theta \) slowly changes from 90° to \( \phi \) (as shown in Figure 4 with a red line for \( \phi = 1^\circ \) and with a blue line for \( \phi = 10^\circ \)). Depending on the value of parameter \( \Lambda \), the system can either achieve the underdamped state marked in Figure 4 with a green line \( \zeta = 1 \) or not. For \( \Lambda > 1 \) and very small misalignment angles, the damping factor is initially very high and drops to low values in the immediate vicinity of the stick to slip transition. Thus, the damping response could be used for probing how far the system is from the critical point. On the other hand, at larger misalignment angles the underdamped state is never achieved. In other words, large misalignment angles suppress fractionally induced instabilities [22].

Our results establish a new perspective on slow creep, the stick-to-slip transition and the nature of high-frequency oscillations in sliding systems. We are completely aware that the demonstrated mechanism of stick-to-slip transition does not exhaust all possible mechanisms of the stick-slip phenomenon. However, we would like to draw attention of researchers and engineers to the fact that the well-known and much debated properties of the transition from stick to slip, including slow creep, may have a completely different – and much simpler – purely mechanical origin.

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