Dynamic characteristics of a periodically forced periodic system with a clearance in low frequency vibration

Yuqing Shi, Sanshan Du, Xifeng Zhu, Fengwei Yin and Guanwei Luo

School of Mechatronic Engineering, Lanzhou Jiaotong University, Lanzhou 730070, China
Email: shiyq@mail.lzjtu.cn, 2041384@qq.com, zhuxf@mail.lzjtu.cn, 65518280@qq.com, luogw@mail.lzjtu.cn

Abstract. A two-degree-of-freedom periodically forced system with a clearance is considered. Pattern types, diversity, regularity and bifurcation characteristics of the fundamental group of impact vibrations and subharmonic impact vibrations in the parameter plane, under low frequency vibration, are analyzed. The transition irreversibility of adjacent impact vibrations with fundamental period and two types of transition regions (narrow hysteresis and tongues-shaped regions), and a series of singular points are found by multi-target and multi-parameter co-simulation analysis. The occurrence mechanism and distribution characteristics of the hysteresis and tongue-shaped regions, as well as pattern types and regularity of subharmonic impact vibrations in the tongue-shaped regions are studied. The transition law from impact vibrations with fundamental period to incomplete and complete chattering-impact vibrations is studied.

1. Introduction
Dynamics research of mechanical systems with clearance and restraint is of great significance for improving the performance of mechanical equipment, realizing effective control of vibration and noise of complex equipment and effective utilizing of vibration. In recent years, qualitative analysis, numerical simulation and experiments have been carried out by scholars to in-depth study the dynamic stability, bifurcation, grazing singularity, chattering-impact vibration and chaos control of such systems. Complex dynamic behaviors exhibiting in various types of vibro-impact systems neighbouring the grazing bifurcation point were reported in documents [1-5]. The vibro-impact systems usually exhibit chattering-impact vibration with sticking in low exciting frequency range, the main characteristic of which is that an accumulation of a series of impact events of the constrained vibratory block occurs in finite time and the chattering-impact sequence attenuates successively till it finally sticks to the impact surface. Complex dynamic characteristics induced by chattering-impact vibration with sticking were found to exist in low exciting frequency range of the vibro-impact systems [6-11]. The main purpose of this paper aims at researching the vibration characteristics of the vibration system with a clearance in low frequency vibration, and analyzing the pattern types and bifurcation characteristics of the fundamental group of impact vibrations and subharmonic impact vibrations under low frequency vibration.

2. Mechanical Model
A two-degree-of-freedom harmonically forced system with a clearance is schematically presented in Fig.1, which consists of two vibratory blocks with masses $M_1$ and $M_2$. Although the model itself is
highly simplified, it captures features that are common to many mechanical systems with clearances in engineering. Both the two vibratory blocks are attached to the supporting base by linear spring-damper elements $K_1$, $K_i$ and $C_i$, $C_i$ respectively, and the harmonic exciting forces $P_i \sin(\Omega t + \tau)$ and $P_i \sin(\Omega t + \tau)$ act on them. Where $P_i$ (i=1, 2), $\Omega$ and $\tau$ are the amplitudes, exciting frequency and phase angle respectively. The moving displacements of the vibratory blocks $M_i$ are represented by $X_i$. As for small forcing amplitudes, the system will present simple linear vibration. With increasing $P_i$, when the difference of displacements of two vibratory blocks equals the clearance $B$, i.e. $X_i - X_j = B$, the system will occur rigid collision. The coefficient of restitution is $R$, and $\dot{X}_i$ and $\dot{X}_j$, (i=1, 2) represent the impacting velocities of before and after the instant of the impact, respectively. The clearance leads to the vibration system exhibiting complex non-smooth dynamic characteristics. The non-dimensional differential equations of vibration are given by

$$\ddot{x}_i + 2\zeta \dot{x}_i + x_i = (1 - f) \sin(\omega t + \tau), \quad \frac{m}{1-m} \ddot{x}_i + 2\zeta \frac{c}{1-c} \dot{x}_i + \frac{k}{1-k} x_i = f \sin(\omega t + \tau), \quad x_i - x_j < \delta$$

(1)

$$\frac{m}{1-m} \ddot{x}_i + \ddot{x}_j = \frac{m}{1-m} \dot{x}_i + \dot{x}_j, \quad R = (\dot{x}_+ - \dot{x}_+)/(\dot{x}_+ - \dot{x}_-), \quad (x_i - x_j = \delta),$$

(2)

in which dimensionless time $t$, variables $x_i$ and dimensionless parameters are given by

$$t = T \sqrt{\frac{K_1}{M_1}}, \quad x = \frac{X_i}{K_i P_i + P_i}, \quad m = \frac{M_i}{M_i + M_j}, \quad \frac{k}{K_i + K_j}, \quad \frac{c}{C_i + C_j}, \quad \omega = \Omega \sqrt{\frac{M_i}{K_i}}, \quad \zeta = \frac{C_i}{2\sqrt{K_i M_i}},$$

$$\delta = \frac{B K_i}{P_i + P_i}, \quad i = 1, 2$$

(3)

Mechanical model shown in Fig. 1 is a non-smooth dynamic system, and there exists complex and abundant periodic and subharmonic impact vibrations in its parameter space, which can be visually summarized by introducing the symbol $p/n$, where $n$ represents the number of the exciting force periods $T_0 = 2\pi/\omega$ in the vibration period $T_n = 2\pi n/\omega$ and $p$ means the number of impacts in the vibration period $T_n = 2\pi n/\omega$, $p = 0, 1, 2, 3, \ldots, n = 1, 2, 3, \ldots$. For $n = 1$, i.e., $p/1$ vibration group, we call it the fundamental group of periodic impact vibrations which has the period of the exciting force ($n = 1$), and for the convenience of analysis, the period of the $p/1$ vibration group is also named as the fundamental period in the following discussion. For $p = 0$, it corresponds to the impact less vibration $0/1$, which is also included in the fundamental group of impact vibrations $p/1$, because the transition characteristics between adjacent $0/1$ and $1/1$ vibrations are exactly the same as those of adjacent $p/1$ and $(p+1)/1$ vibrations, $(p > 0)$. Based on the symbol feature $p/n$ of periodic and subharmonic impact vibrations, the Poincaré maps of the system can be established by choosing the Poincaré sections

$$\sigma_p = \{(x_i, \dot{x}_i, x_j, \dot{x}_j, t) \in \mathbb{R}^4 \times T \mid x_i - x_j = \delta, \dot{x}_i - \dot{x}_j > 0\}; \quad \sigma_n = \{(x_i, \dot{x}_i, x_j, \dot{x}_j, t) \in \mathbb{R}^4 \times T \mid x_i = x_{i_{\text{in}}, \text{mod}(t = 2\pi/\omega)}\}$$

(4)

For periodic and subharmonic impact vibrations, the impact number $p$ can be determined by the number of branches of the impact map corresponding to $\sigma_p$ and the number $n$ can be ascertained by the number of branches of the map corresponding to $\sigma_n$. Both combining $\sigma_p$ and $\sigma_n$, as well as the symbol $p/n$, the multi-objective and multi-parameter simulation analysis of the system shown in Fig. 1 can be carried out from system level. The impact Poincaré map corresponding to $\sigma_p$ is expressed by

$$X^{(p)} = f(X^{(0)}, \mu)$$

(5)
where \( X^{(i)} = (r^{(i)}, x_2^{(i)}, \dot{x}_2^{(i)}, \ddot{x}_2^{(i)})^T \), \( X^{(i+1)} = (r^{(i+1)}, x_2^{(i+1)}, \dot{x}_2^{(i+1)}, \ddot{x}_2^{(i+1)})^T \), \( X \in \mathbb{R}^4 \), \( \mu \) are parameters, \( \mu \in \mathbb{R}^+ \), \( m=8 \).

Figure 1. Mechanical model of a impact vibration system with a clearance

3. Pattern Types and Bifurcation Characteristics under Low Frequency Vibration

Based on formula (3), the value range of some dimensionless parameters can be easily identified as \( m \in (0, 1) \), \( k \in (0, 1) \), \( c \in (0, 1) \) and \( f \in (0, 1) \). Taking the dimensionless parameters: \( m=0.5 \), \( k=0.5 \), \( c=0.5 \), \( f=0.0 \), \( \zeta=0.1 \) and \( R=0.8 \) as the analyzing parameters, we mainly study the influence of clearance \( \delta \) and exciting frequency \( \omega \) on the dynamic characteristics of the vibro-impact system. Fig. 2(a) displays the pattern types and existence regions of periodic and subharmonic impact vibrations associated with the analyzing parameters of the vibro-impact system with a clearance shown in Fig. 1 in the \((\omega, \delta)\)-parameter plane. As shown in the lower right corner of Fig. 2(a), the system will exhibit subharmonic impact vibrations \( 1/n \) \((n \geq 1)\) in high exciting frequency and small clearance region or slightly large clearance region, which generally occur period doubling or grazing bifurcation with changing exciting frequency \( \omega \) or clearance \( \delta \). In high exciting frequency and large clearance region, the system presents impactless vibration \( 0/1 \), which transfer to \( 1/1 \) vibration by undergoing real grazing bifurcation. However, the transition from \( 0/1 \) vibration to subharmonic impact vibration \( 1/n \) is usually achieved by undergoing bare grazing or saddle-node bifurcation. In low exciting frequency range, the vibro-impact characteristics of the system are more complex and diverse. As for low exciting frequency range as well as \( \delta <1 \) region, the system mainly presents fundamental group of impact vibrations \( p/1 \), incomplete chattering-impact vibration and complete chattering-impact vibration with sticking. The impact vibrations with fundamental period \( p/1 \) distributed in the \((\omega, \delta)\)-parameter plane exhibit strip-shape, and the larger the impact number \( p \) is, the narrower the strip-shape of \( p/1 \) vibration is. As seen in the bottom left of Fig. 2(a), all vibration regions with \( p \geq 16 \) are attributed to the narrow region marked by dark blue uniformly. The impact velocity of \( p/1 \) vibration is attenuated successively in the fundamental period. As \( p \) becomes big enough, the system exhibits incomplete chattering-impact vibration, and further decreasing frequency, incomplete chattering-impact vibration will occur sliding bifurcation and then transfer to complete chattering-impact vibration with sticking by passing though the sliding bifurcation boundary of incomplete chattering-impact vibration. The incomplete and complete chattering-impact vibration are represented by the symbol \( 1/p \) and \( 1/p \) respectively, and the common characteristics of them is that the number of impact \( p \) is very large, as well as, the main difference of them is that the impact velocity of the former is attenuated successively in the fundamental period, but it can not fade to zero, so there is no sticking phenomenon between the two vibratory blocks. However, the impact velocity of the latter is attenuated successively to zero in the fundamental period, which leads to the two vibratory blocks occurring sticking until the pressure between them decreases to zero and then the next cycle of complete chattering-impact vibration with sticking begins. Fig.2(b)-(f) are the local detail description of Fig.2(a), which reveal the mutual transition characteristics between adjacent periodic-impact vibrations \( p/1 \) and \((p+1)/1 \). The complexity of low exciting frequency range of the vibro-impact system with a clearance is derived from the mutual transition irreversibility of adjacent periodic-impact vibrations \( p/1 \) and \((p+1)/1 \). With decreasing \( \omega \) or \( \delta \), \( p/1 \) vibration can occur real grazing bifurcation and transfer to \((p+1)/1 \) vibration by passing though the real grazing bifurcation boundary of
$p/1$ vibration, or occur bare grazing bifurcation and go through tongue-shape region and then embed in the existence region of $(p+1)/1$ vibration. With increasing $\omega$ or $\delta$, $(p+1)/1$ vibration can occur saddle-node bifurcation and transfer to $p/1$ vibration by passing through the saddle-node bifurcation boundary of $(p+1)/1$ vibration, or occur period doubling bifurcation and go through tongue-shape region and then embed in the existence region of $p/1$ vibration. The mutual transition between adjacent periodic-impact vibrations $p/1$ and $(p+1)/1$ is irreversible except a series of singular points, at which saddle-node and period doubling bifurcation boundaries of $(p+1)/1$ vibration, as well as, real-grazing and bare-grazing bifurcation boundaries of $p/1$ vibration alternately intersect and create inevitably two types of transition regions (hysteresis and tongue-shaped regions). Each tongue-shaped region has two singular points. Each singular point is the intersection point of real-grazing and bare-grazing bifurcation boundaries of $p/1$ vibration, and it is also the intersection point of saddle-node and period doubling bifurcation boundaries of $(p+1)/1$ vibration, therefore, The singular point is a co-dimension two grazing bifurcation point of $p/1$ vibration and a co-dimension two bifurcation point associated with the intersection of saddle-node and period doubling bifurcation boundaries of $(p+1)/1$ vibration, or the connection point of the two types of transition regions. One side of each singular point is hysteresis region, and the other side is tongue-shaped region. The tongue-shaped region consists of two boundaries. The upper boundary is bare grazing bifurcation boundary of $p/1$ vibration, and the lower boundary is period doubling bifurcation boundary of $(p+1)/1$ vibration. Series of tongue-shaped regions in the $(\omega, \delta)$-parameter plane gradually become smaller with decreasing $\omega$ or increasing $\delta$, but they exhibit typing characteristics, as well as, on both sides of each tongue-shaped region connect hysteresis region by singular points. The hysteresis region is sandwiched between two boundaries. The upper boundary is the saddle node bifurcation boundary of $(p+1)/1$ vibration, and the lower boundary is the real grazing bifurcation boundary of $p/1$ vibration. Both neighbouring $p/1$ and $(p+1)/1$ vibrations are stable and can coexist in the hysteresis region relying on the initial conditions of the system or on the change way of the exciting frequency $\omega$ and clearance $\delta$. Subharmonic impact vibrations $(np+1)/n$ dominate in every tongue-shaped region between adjacent existence regions of $p/1$ and $(p+1)/1$ vibrations, which undergo period doubling or bare grazing bifurcation with changing $\omega$ or $\delta$. The area of existence regions of subharmonic $(np+1)/n$ vibrations becomes smaller with increasing $n$, and the position of their existence regions in the tongue-shaped region rises higher with increasing $n$. 

(a) 

(b)
Figure 2. Pattern types and occurrence regions of various impact vibrations of the vibratory system with a clearance in the $(\omega, \delta)$-parameter plane: (a) occurrence regions of fundamental, subharmonic, chattering and sticking impact vibrations; (b)-(f) local detail description of Fig. 2(a).

The simulation results of Fig.2(b)-(f) describe the characteristics and regularities in tongue-shaped regions in detail. Where Fig. 2(b) shows a series of subharmonic vibrations $1/2, 1/3, 1/4, 1/5, \ldots, 1/n, \ldots$, impact orbits ($p=1, n \geq 2$), etc., in a tongue-shaped region between adjacent existence regions of 0/1 and 1/1 vibrations. Existence region of subharmonic $1/n$ vibration is surrounded at the top by its period doubling bifurcation boundary and at the bottom by its real grazing or bare grazing bifurcation boundary. Fig. 2(c) clearly displays a series of tongue-shaped regions between adjacent existence regions of $p/1$ and $(p+1)/1$ vibrations ($p \geq 1$). Fig. 2(d)-(f) describe the transition irreversibility of neighbouring $p/1$ and $(p+1)/1$ vibrations and the formation of two types of transition regions (hysteresis and tongue-shaped regions), $p=4-15$. The starting point of simulating in Fig. 2(d) is the vertex of the low left corner (0.365, 0.3), and the initial values of calculating are $X^{(0)}=(x^{(0)}, \dot{x}_1^{(0)}, \dot{x}_2^{(0)}, \dot{\delta}^{(0)})^T=(0, 0, 0, 0)^T$ and $\dot{\delta}^{(0)}=0.3$, and then the dynamic characteristics of the system can be obtained by scanning $(\omega, \delta)$-parameter plane line by line. The top mark “→” of the abscissa $\omega$ in Fig. 2(d) indicates that exciting frequency increases during simulating. The starting point of simulating in Fig. 2(e) is the vertex of the top right corner (0.4378, 0.573), and the initial values of calculating are the same as that in Fig. 2(d). The top mark “←”
of the abscissa $\omega$ in Fig. 2(e) indicates exciting frequency decreases during simulating. Fig. 2(f) is the superposition of Fig. 2 (d) and 2(e). Based on simulation results of Fig. 2(d)-(f), here we only discuss the mutual transition irreversibility of adjacent periodic-impact vibrations 5/1 and 6/1 and the formation process of two types of of transition regions. The related characteristics of other impact vibrations with fundamental period are similar. The singular points of the tongue-shaped region sandwiched between existence regions of adjacent periodic-impact vibrations 5/1 and 6/1 are marked as $S_1$ and $S_2$. The right lower side of $S_1$ is the saddle node bifurcation boundary $SN_{6/1}$ of 6/1 vibration, and the left side of $S_2$ is the real grazing bifurcation boundary $G_{5/1}$ of 5/1 vibration in Fig. 2(d). In Fig. 2(e), the right lower side of $S_1$ is real grazing bifurcation boundary $G_{5/1}$ of 5/1 vibration, and the left side of $S_2$ is the saddle node bifurcation boundary $SN_{6/1}$ of 6/1 vibration. As shown in Fig. 2(f), the saddle node bifurcation boundary $SN_{6/1}$ of 6/1 vibration and real grazing bifurcation boundary $G_{5/1}$ of 5/1 vibration constitute the hysteresis region $HR_{6}$ between adjacent periodic-impact vibrations 5/1 and 6/1. The period doubling bifurcation boundary $PD_{6/1}$ of 6/1 vibration and bare grazing bifurcation boundary $G^{b}_{5/1}$ of 5/1 vibration constitute the tongue-shaped region, and the subharmonic impact vibrations 11/2, 16/3 and 21/4 can be observed in the related tongue-shaped region.

**Figure 3.** Bifurcation diagram of fundamental impact vibrations 4/1 and 5/1 and bifurcation diagram of subharmonic $(4n+1)/n$ vibrations in a relative tongue-shaped region, $\omega=0.4287$: (a) bifurcation diagram $x_{imp}(\delta)$; (b) bifurcation diagram $\dot{x}_i(\delta)$; (a1) and (b1) local details of Fig. 3(a) and (b).
For more subharmonic impact vibrations existing in the tongue-shaped region, we can obtain them by enlarging the local region of the tongue-shaped region, or by calculating single parameter bifurcation diagrams. Subharmonic impact vibrations 9/2, 13/3 and 19/4 can be easily observed in the tongue-shaped region between existence regions of adjacent periodic-impact vibrations 4/1 and 5/1 from Fig.2(d)-(f). Fig.3(a) and (b) are respectively the bifurcation diagram of adjacent periodic-impact vibrations 4/1 and 5/1 and bifurcation diagram of subharmonic \((4n+1)/n\) vibration in a related tongue-shaped region in the case of \(\omega = 0.4287\). Fig. 3(a) is the diagram about the minimum displacements \(\x_{\text{imp}}\) of the vibratory block M1 appearing in the fundamental period corresponding to the value of clearance \(\delta\), and Fig. 3(b) is the diagram about the relative impact velocity \(\dot{x}_1 - \dot{x}_2\) of the two vibratory blocks corresponding to the value of clearance \(\delta\). For periodic and subharmonic impact vibrations, the number of branches in Fig. 3(a) determines the number \(n\) of the exciting force periods in the vibration period, and the number of branches in Fig. 3(b) determines the impact number \(p\) in the vibration period. Fig. 3(a1) and 3(b1) are the local details of Fig. 3(a) and 3(b) respectively, and they clearly show the subharmonic impact vibrations of 9/2, 13/3, 17/4, 21/5, 25/6, 29/7 and 33/8 in the related tongue-shaped region, which undergo period doubling bifurcation with increasing clearance \(\delta\) and bare grazing bifurcation with decreasing clearance \(\delta\).

4. Summary

The periodically-forced vibration system with a clearance mainly presents the fundamental group of impact vibrations \(p/1\), incomplete chattering-impact vibration \(\tilde{p}/1\), complete chattering-impact vibration with sticking \(\overline{p}/1\) and subharmonic impact vibrations in low exciting frequency range. The \(p/1\) vibration can happen real grazing bifurcation and transfer to \((p+1)/1\) vibration, or happen bare grazing bifurcation and pass though tongue-shape region and then transfer to \((p+1)/1\) vibration with decrease in the exciting frequency \(\omega\) and clearance \(\delta\). The \((p+1)/1\) vibration can happen saddle-node bifurcation and transfer to \(p/1\) vibration, or happen period doubling bifurcation and pass though tongue-shape region and then transfer to \(p/1\) vibration with increase in the exciting frequency \(\omega\) and clearance \(\delta\). The mutual transition between adjacent periodic-impact vibrations \(p/1\) and \((p+1)/1\) is irreversible except a series of singular points, at which saddle-node and period doubling bifurcation boundaries of \((p+1)/1\) vibration, real-grazing and bare-grazing bifurcation boundaries of \(p/1\) vibration alternately intersect and create inevitably two types of transition region - hysteresis region and tongue-shaped region. Both \(p/1\) and \((p+1)/1\) vibrations are stable and can coexist in the hysteresis region, whether it exists or not depends on the moving initial conditions of the system or on the change direction of the exciting frequency \(\omega\) and clearance \(\delta\). The tongue-shaped region exhibits obvious typing characteristics with decreasing exciting frequency \(\omega\) or increasing clearance \(\delta\) in the \((\omega, \delta)\)-parameter plane, and holds a series of subharmonic impact vibration \((np+1)/n\) dominates, which undergoes period doubling or bare grazing bifurcation with changing exciting frequency \(\omega\) or clearance \(\delta\). The existence region of the the impact vibration with fundamental period \(p/1\) distributing in the \((\omega, \delta)\)-parameter plane presents strip-shape, and the larger \(p\) is, the narrower the strip-shape of \(p/1\) vibration is. As \(p\) becomes big enough, the system exhibits incomplete chattering-impact vibration \(\tilde{p}/1\), and further decreasing frequency, incomplete chattering-impact vibration \(\tilde{p}/1\) will occur sliding bifurcation and then transfer to complete chattering-impact vibration with sticking \(\overline{p}/1\).

Acknowledgement

The authors gratefully acknowledge the support by National Natural Science Foundation (11672121, 11462012), Innovation and Entrepreneurship Talents Training Project of Lanzhou city (2014-RC-33) and Youth Science Foundation of Lanzhou Jiaotong University (2015019).

References

[1] A.B. Nordmark. Non-periodic motion caused by grazing incidence in an impact oscillator [J]. Journal of Sound and Vibration, 1991, 145 (2): 279–297.
[2] G.S. Whiston. Singularities in vibro-impact dynamics [J], Journal of Sound and Vibration, 1992, 152
(3) 427–460.

[3] A.P. Ivanov. Stabilization of an impact oscillator near grazing incidence owing to resonance [J]. Journal of Sound and Vibration 1993, 162 (3): 562–565.

[4] H.Y. Hu. Detection of grazing orbits and incident bifurcations of a forced continuous, piecewise-linear oscillator [J]. Journal of Sound and Vibration, 1994, 187(3): 485–493.

[5] M. di Bernardo, C.J. Budd, A.R. Champneys. Grazing and border-collision in piecewise-smooth systems: a unified analytical framework [J]. Physical Review Letters, 2001, 86 (12): 2553–2556.

[6] C.J. Budd, F. Dux. Chattering and related behaviour in impact oscillators [J]. Philos Trans Roy Soc Lond A, 1994, 347: 365–389.

[7] C. Toulemonde, C. Gontier. Sticking motions of impact oscillators [J]. European Journal of Mechanics A/Solids, 1998, 17 (2): 339–366.

[8] D. J. Wagg. Multiple non-smooth events in multi-degree-of-freedom vibro-impact systems [J]. Nonlinear Dynamics, 2006, 43: 137–148.

[9] A.B. Nordmark, P.T. Piiroinen. Simulation and stability analysis of impacting systems with complete chattering [J]. Nonlinear Dynamics, 2009, 58: 85–106.

[10] Albert C.J. Luo, Dennis O'Connor. Mechanism of impacting chatter with stick in a gear transmission system [J]. International Journal of Bifurcation and Chaos, 2009, 19(6): 2093–2105.

[11] Csaba Hős, Alan R. Champneys. Grazing bifurcations and chatter in a pressure relief valve model [J]. Physica D, 2012, 241: 2068–2076.