Abstract The Gaylord’s oscillator is a vibrating of a uniform rigid rod without slipping on a rigid circular surface with a definite radius. The dominant equation of motion was the outcome of a strongly nonlinear pendulum equation of the second order. The run article is interested in obtaining the frequency–amplitude equation and the approximate solution of Gaylord’s oscillator by a simpler approach. The frequency–amplitude relationship is derived in terms of the Bessel function. Quasi-exact periodic solution derived depends on a non-perturbative approach. The validation of the analytical solution is made through the comparison with the numerical solution which shows excellent approval. Finally, the non-perturbative method is of high accuracy besides simplicity if it is compared with the other perturbative techniques in analyzing the behavior of oscillators with strong nonlinearities.

Keywords Gaylord’s oscillator · Periodic solution · Non-perturbative method

1 Introduction

It is known that many engineering problems can be formulated by nonlinear ordinary or partial differential equations. With an exception of a few problems, their exact solutions seem to be extremely complex and sometimes unreachable. Therefore, many scientists have been interested in asymptotic solutions to deal with several nonlinear equations, such as the small parameter method and the averaging method for some weak nonlinear problems [1–4]. In addition, the multiple scales (MS) method [5] and the Lindstedt–Poincaré (LP) method have been used in obtaining the solutions of vibratory systems [6, 7]. However, these methods depend on a small parameter, and improper selection of this parameter leads to wrong solutions. The pivotal procedure in any asymptotic or perturbation method is the finding of the small parameter and the scaling needed to set the equations of interest in a more suitable form for the convenient application of the method. This is not always easy.

The methods of HPM [8] and HAM [9] may be considered the main outlines of iterative methods which have functionally played a fundamental role in obtaining approximate solutions that have good closeness to the closed-form solutions of a wide group of nonlinear problems. These methods are dependent upon the initial guess for the solution, so if the first guess is not nearly enough to the final solution of the problem, it may diverge and, consequently, the desired solution will not be performed at the end of the process. One of the troublesome arising in problems dealing with the nonlinear oscillators is the inquiry of an infinite number of the frequencies of vibration which has been treated by a lot of researchers. Deriving an exact or semi-exact solution to these nonlinear equations has been a demanding task for applied physicists and scientists due to the entity of nonlinearity.

Y. O. El-Dib
Department of Mathematics, Faculty of Education, Ain Shams University, Roxy, Cairo, Egypt
E-mail: yusryeldib52@hotmail.com
In the previous decades, various techniques have been suggested to bring the approximate analytic solution to vibrational problems such as higher-order HPM [10, 11], Taylor series [12], energy balance technique [13], residual harmonic balance method [13], Li–He modified HPM [14, 15], variational approach [16]. Moreover, there are many analytical techniques for acquiring the approximate solution to the nonlinear equations, for example, the He–Laplace method [17], global residual harmonic balance method [18], and Hamiltonian-based frequency–amplitude formulation for nonlinear oscillators [19].

The frequency formula was formulated by JH. He [20–22] represents a genius idea in converting a nonlinear equation into an approximately linear equation. The frequency of a nonlinear vibration system is nonlinearly related to its amplitude, and this relationship is critical in the design of a packaging system and a microelectromechanical system (MEMS) [23]. The application of He’s frequency leads to discuss the periodic property and the instability properties of a rotating pendulum system [24].

Since a linear equation often has a perfect solution. The solution of the linearized equation represents a near-perfect solution to the nonlinear equation which is called a quasi-exact solution. However, dealing with a linear equation, whatever it is, is easier than dealing with a nonlinear equation [25]. At least, the application of the homotopy perturbation method to the linearized equations with constant coefficients, always, leads to an exact solution [26].

The vibrating rigid long rod over a circular surface without sliding is an ancient oscillator whose dominant equation was first calculated by Gaylord [5]. This equation newly has been treated by many researchers; for example, Ghasemi et al. [27] applied the HPM, Ganji et al. [28] used the min–max approach and amplitude–frequency formulation, and Ebrahimi et al. [29] used He’s energy balance technique to discuss the motion of this oscillation. Khan et al. [30] utilized the Hamiltonian technique to determine the unknown frequency of vibrating a rod over a circular surface. Hosen [31] introduced an alternate harmonic balance technique to determine approximate solutions for a strongly nonlinear equation arising from the motion of a vibrating rigid long rod over a surface without sliding. El-Dib and Moatimid [32] obtained an approximate periodic solution of Gaylord’s equation using the coupled Laplace transform and HPM. Ganji et al. [33] used the AG approach which was introduced by Akbari and Ganji et al. [34]. The AG method depends on an initial guess solution having unknown coefficients with initial conditions (IC). By its successive derivatives concerning time, a system of nonlinear algebraic equations is obtained. This system was solved numerically using a suitable iterative scheme until getting the final solution.

A schematic diagram of the problem is shown in Fig. 1 [5]. According to Gaylord’s equation [5], the nonlinear equation of motion of the rigid rod rocks can be derived by Newton’s second law of motion as follows:

$$\ddot{\theta} + a(\dot{\theta}^2 + \dot{\theta}^2 \theta) + b \theta \cos \theta = 0,$$

where

$$a = \frac{12r^2}{l^2}, \quad b = \frac{12rg}{l^2}.$$  

The derivation of this equation can be found in the previous paper, see Ref. [32]. It is significant to determine periodic motions to Gaylord’s oscillation even when there is no linear frequency. This is because the inherent complex dynamics of Gaylord’s Eq. (1) help one to better understand the complex world. For non-polynomial and non-perturbative methods and nonlinear dynamical systems, one still has difficulty getting an adequate solution of periodic motions through generalized harmonic equations. However, in the present work, a green
light to derive a periodic solution with a simpler approach is offered. Such an approach can be used to get an analytical periodic solution without using Taylor expansion nor using a perturbation technique.

2 The methodology

Although Eq. (1) does not have a linear periodic term, the periodic solution will be derived depending on the nonlinear frequency which is assumed to be $\omega$. This frequency can be estimated as follows:

In terms of the potential function $f(\theta, \dot{\theta}, \ddot{\theta})$. Gaylord’s equation can be found in the form

$$\ddot{\theta} + f(\theta, \dot{\theta}, \ddot{\theta}) = 0,$$

where the function $f$ is an odd function defined by

$$f(\theta, \dot{\theta}, \ddot{\theta}) = a(\dot{\theta}^2 + \ddot{\theta}^2) + b \theta \cos \theta.$$

Begin with the consideration that Eq. (2) admits the solution

$$\theta(t) = A \cos \omega t,$$

where $A$ is a constant representing the amplitude of the oscillation and $\omega$ represents the total frequency of Gaylord’s oscillation. According to the hypothesis of the solution, as given by (4), Eq. (2) is assumed to have the harmonic form

$$\ddot{\theta} + \omega^2 \theta = 0.$$

This form is an approximate corresponding to the original Eq. (1). The frequency $\omega^2$ was estimated by He [10], Ren et al. [35], Alex [36] using the weighted residuals approach. To show the effectiveness of this approach, the following two trial solutions are considered:

$$\theta_1(t) = A \cos \omega_1 t \text{ and } \dot{\theta}_2(t) = A \cos \omega_2 t,$$

where $\omega_1$ and $\omega_2$ are trial frequencies. The above solutions represent solutions for the harmonic oscillations:

$$\ddot{\theta}_1 + \omega_1^2 \theta_1 = 0, \text{ and } \ddot{\theta}_2 + \omega_2^2 \theta_2 = 0.$$

The frequency $\omega^2$ can be computed approximately because of He’s frequency formulation as

$$\omega^2 = \frac{\omega_1^2 R_2(0) - \omega_2^2 R_1(0)}{R_2(0) - R_1(0)},$$

where $R_1(t)$ and $R_2(t)$ are the weighted residuals functions given by

$$R_1(t) = \frac{2\omega_1}{\pi} \int_0^{\pi/2\omega_1} \left( f\left(\theta_1, \dot{\theta}_1, \ddot{\theta}_1\right) - \omega_1^2 A \cos(\omega_1 t) \right) \cos(\omega_1 t) \, dt,$$

$$R_2(t) = \frac{2\omega_2}{\pi} \int_0^{\pi/2\omega_2} \left( f\left(\theta_2, \dot{\theta}_2, \ddot{\theta}_2\right) - \omega_2^2 A \cos(\omega_2 t) \right) \cos(\omega_2 t) \, dt.$$

Equation (8) is obtained by an ancient Chinese method, which was further developed into two submathematics branches: One is He’s frequency formulation for nonlinear collators [37] and the other is Chun-Hui He’s iteration method for numerical simulation [38].

Inserting (6) into (9) and (10) yields

$$R_1(t) = \frac{2\omega_1}{\pi} \int_0^{\pi/2\omega_1} \left( -\omega_1^2 A + a(\dot{\theta}_1^2 + \ddot{\theta}_1^2 + b \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} \dot{\theta}_1^{2n+1} \right) \cos(\omega_1 t) \, dt.$$

Inserting (6) into (9) and (10) yields

$$R_1(t) = \frac{2\omega_1}{\pi} \int_0^{\pi/2\omega_1} \left( -\omega_1^2 A + a(\dot{\theta}_1^2 + \ddot{\theta}_1^2) + b \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} \dot{\theta}_1^{2n+1} \right) \cos(\omega_1 t) \, dt.$$

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\[ R_2(t) = \frac{2\omega_1}{\pi} \int_0^{\pi/2 \omega_2} \left( -\omega_2^2 A + a(\dot{\theta}_2^2 + \dot{\theta}_2^2) + b \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}} \dot{\theta}_2^{2n+1} \right) \cos(\omega_2 t) \, dt \]

\[ = -\frac{1}{2} \omega_2^2 A \left( 1 + \frac{1}{2} a A^2 \right) + \frac{b}{2} \left( 1 - \frac{3}{2!} \left( \frac{A}{2} \right)^2 \right) + \frac{5}{2!} \frac{A}{3!} \left( \frac{A}{2} \right)^4 - \frac{7}{3!} \frac{A}{4!} \left( \frac{A}{2} \right)^6 + \ldots + \frac{(-)^r (2r+1)}{r!} \frac{A}{(r+1)!} \left( \frac{A}{2} \right)^{2r}. \]

(12)

Employing (11) and (12) into (8) becomes

\[ \omega^2 = \frac{b}{(1 + \frac{1}{2} a A^2)^2} \left( 1 - \frac{3}{2!} \left( \frac{A}{2} \right)^2 \right) + \frac{5}{2!} \frac{A}{3!} \left( \frac{A}{2} \right)^4 - \frac{7}{3!} \frac{A}{4!} \left( \frac{A}{2} \right)^6 + \ldots + \frac{(-)^r (2r+1)}{r!} \frac{A}{(r+1)!} \left( \frac{A}{2} \right)^{2r}. \]

(13)

The above series can be put in a summation form [39] as

\[ \omega^2 = \frac{b}{(1 + \frac{1}{2} a A^2)^2} \sum_{n=0}^{\infty} \frac{(-)^n (2n+1)}{n! \times (n+1)!} \left( \frac{A}{2} \right)^{2n}. \]

(14)

This can be read as

\[ \omega^2 = \frac{2b}{(1 + \frac{1}{2} a A^2)^2} \left( \frac{1}{A} \sum_{n=0}^{\infty} \frac{(-)^n}{n! \times (n+1)!} \left( \frac{A}{2} \right)^{2n+1} - \sum_{n=0}^{\infty} \frac{(-)^n}{n! \times (n+2)!} \left( \frac{A}{2} \right)^{2n+2} \right). \]

(15)

In terms of the Bessel functions of the first kind, it becomes

\[ \omega^2 = \frac{2b}{(1 + \frac{1}{2} a A^2)^2} \left( \frac{1}{A} J_1(A) - J_2(A) \right). \]

(16)

where \( J_1(A) \) and \( J_2(A) \) are the first kind of the Bessel functions in the amplitude \( A \) of order one and two, respectively. The above frequency–amplitude relationship which depends on the coefficient \( b \) is derived for the first time. It is noted that the amplitude \( A^2 \) and the nonlinear coefficients \( a \) are working opposite to each other in the denominator of the frequency \( \omega^2 \).

Accordingly, the quasi-exact solution of the motion of a simple pendulum can now be written as

\[ \theta(t) = A \cos \sqrt{\frac{b(J_1(A) - A J_2(A))}{A(2 + a A^2)}} 2t. \]

(17)

It is noted that the frequency \( \omega^2 \) estimated in (16) is very accurate than all approaches mentioned before and listed in the references. The simplest form for this frequency can be derived using He’s simplified formula [40] and following El-Dib [25], the angular frequency \( \omega \) can be estimated approximately as

\[ \omega^2 = \frac{df(\theta, \dot{\theta}, \ddot{\theta})}{d\theta} \bigg|_{\theta=kA, \dot{\theta}=-k A \omega, \ddot{\theta}=-k A \omega^2} = \left( a(2\ddot{\theta} + \dot{\theta}^2) + b(\cos \theta - \theta \sin \theta) \right) \bigg|_{\theta=kA, \dot{\theta}=-k A \omega, \ddot{\theta}=-k A \omega^2}, \]

(18)

where the parameter \( k \) is estimated at [41]

\[ k = \frac{1}{2\sqrt{n-m}}; \]

(19)

\( n \) is the order of the oscillator, and \( m \) is the freedom degree. Consequently, employing \( k = \frac{1}{2} \) into (19) the frequency \( \omega \) is approximately evaluated in the form

\[ \omega^2 = \frac{b(\cos(\frac{1}{2} A) - \frac{1}{2} A \sin(\frac{1}{2} A))}{1 + \frac{1}{2} a A^2}. \]

(20)

At this stage, the periodic solution of Gaylord’s oscillator (1) is the quasi-exact solution (4). Although the above frequency relationship is very simple, it is worthwhile to observe that it has not been obtained before using the perturbation approaches mentioned above and presented in the references.
The periodic property of Gaylord’s oscillator with a non-perturbative method

3 Numerical validation

It is usual to compare the analytical solutions to the numerical solutions, to test the accuracy of the approximate solutions obtained by a certain non-perturbation method. It will utilize the outlined solution (4) to obtain numerical simulations for solving Eq. (2). To show the high accuracy of the present approach, the comparison of the numerical solution with the analytical solution is displayed together in Fig. 2-a. The calculations are made for a system of \( A = 1, b = 1 \) and \( a = 0.01 \). The graph in Fig. 2-a shows the excellent agreement between the numerical solution (red curve) and the analytical solution (blue dash curve). The polar representation for the system of Fig. 2-a is illustrated in Fig. 2-b. It is shown that the periodic solution is represented by a full circle. In Fig. 3-a, b, the coefficient \( b \) has increased to the value \( b = 3 \), which leads to a decreasing the wavelength. The decrease in the amplitude \( A \) to the value of \( A = 0.1 \) with fixed coefficient \( a \) and \( b \) leads to decreased wavelength as shown in Fig. 4-a, b. The increase in the parameter \( b \) with fixed both \( a \) and \( A \) means that the frequency \( \omega \) is increased. To examine the influence of increasing the coefficient \( b \), a parametric phase plane is established as shown in Fig. 5. This graph shows that the velocity \( \dot{\theta} \) has increased with increasing the parameter \( b \), while the displacement \( \theta \) is restricted to the value \( \theta = 1 \) with fixed both the parameters \( a \) and \( b \) as reading from Fig. 5. In addition, if the amplitude \( A \) has changed to \( A = 1.5 \), an increase in the displacement parameter \( \theta \) is noted, which is expected; on the other side, the increase in the amplitude \( A \) has suppressed the influence of the parameter \( b \) on the velocity parameter \( \dot{\theta} \) as shown in Fig. 6. This conclusion is observed for the comparison between the graphs of Figs. 5 and 6.

The increase in the nonlinear coefficient \( a \) with fixed \( A \) and \( b \) unity is displayed in Fig. 7. The investigation of this graph shows that increases of \( a \) produce a slight increase in the velocity parameter \( \dot{\theta} \). In Fig. 8, the examination of the amplitude \( A \) with fixed \( b = 1 \) and \( a = 0.1 \) is made. The graph shows that both the displacement parameter \( \theta \) and the velocity parameter \( \dot{\theta} \) are dependent on the amplitude \( A \) with the same magnitude.

4 Conclusion

In this work, although the linear frequency of Gaylord’s oscillator is absent, but the periodic solution has been derived. Based on El-Dib–He’s frequency formula, an approximate nonlinear frequency has been estimated and so a quasi-exact periodic solution for the nonlinear oscillation of Gaylord’s oscillator has been derived. This periodic solution is compared with the numerical solution of the Mathematica software. The comparison showed excellent agreements. Some numerical illustrations are made especially to the phase plane to explain the behavior of each coefficient of Gaylord’s oscillator. It is worthwhile to ensure that the obtained periodic
(a) The time–history representation for the comparison between the numerical solution of Eq. (1) and the analytical solution (4). For a system having $A = 1$, $b = 3$ and $a = 0.01$

(b) The polar representation for the system of Fig.(3-a)

Fig. 3 a: The time–history representation for the comparison between the numerical solution of Eq. (1) and the analytical solution (4). For a system having $A = 1$, $b = 3$ and $a = 0.01$. b: The polar representation for the system of (a)

(a) The time-history representation for the comparison between the numerical solution of Eq. (1) and the analytical solution (4). For a system having $A = 1$, $b = 3$ and $a = 0.01$

(b) The polar representation for the system of Fig.(4-a)

Fig. 4 a: The time–history representation for the comparison between the numerical solution of Eq. (1) and the analytical solution (4). For a system having $A = 0.1$, $b = 3$ and $a = 0.01$. b: The polar representation for the system of (a)

Fig. 5 The variation of the coefficient $b$ in the phase plane $(\dot{\theta} - \dot{\theta})$ of the solution (4) for the system of $A = 1$ and $a = 0.01$
The periodic property of Gaylord’s oscillator with a non-perturbative method

Fig. 6 The variation of the coefficient $b$ in the phase plane $(\theta - \dot{\theta})$ of the solution (4) for the system of Fig. 4 except that $A = 3$.

Fig. 7 The variation of the coefficient $a$ in the phase plane $(\theta - \dot{\theta})$ of the solution (4) for the system of $A = b = 1$.

Fig. 8 The variation of the amplitude $A$ in the phase plane $(\theta - \dot{\theta})$ of the solution (4) for the system of $a = 0.1$ and $b = 1$. 
solution in this work does not depend on any perturbation techniques. Therefore, the present approach is a genius idea in deriving a quasi-exact solution for the nonlinear oscillators and enables us to treat a lot of problems in the fields of physical engineering application of oscillators using a non-perturbative technique.

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**Data availability** In this study, there are no data used.

**Declarations**

**Conflict of interest** The author declares that there are no competing interests regarding the publication of the present paper.

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