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Modelling non-Gaussianity from foreground contaminants

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Abstract. We introduce a general class of models for characterizing the non-Gaussian properties of foreground contaminants in the cosmic microwave background with view towards the removal of the non-primordial non-Gaussian signal from the primordial one. This is important not only for treating temperature maps but also for characterizing the nature and origin of the primordial cosmological perturbations and thus establishing a theory of the early universe.

1. Introduction

Cosmology is presently a very active field because of the large number of observations that are becoming available and that will allow us to characterize with great precision the nature and physical origin of the primordial cosmological perturbations. A key question is whether these primordial perturbations were Gaussian.

The currently favoured best-fit cosmological models, as supported by the recent Wilkinson Microwave Anisotropy Probe (WMAP) data, are in agreement with the primordial cosmological perturbations being Gaussian to great accuracy \cite{1}. With the WMAP data there has been a claimed detection of primordial non-Gaussianity that is currently disputed \cite{2}. However, even the conservative official analysis hints at something, the value of $f_{NL}$ lying within $1.9\sigma$ from the null value.

Either a detection or a more stringent constraint of $f_{NL}$ would have profound implications for our understanding of the physics of the early Universe. In particular, it would be an observational discriminant among competing models for the generation of primordial fluctuations. An $f_{NL} \neq 0$ would be a very interesting challenge for cosmology since it would require an extensive revision of the standard cosmological model. Present observations have ruled out a great number of cosmological models already. All currently favoured models would be ruled out by such a detection. A non-detection of $f_{NL}$ with an improved constraint on the allowed values would strongly restrict the fluctuation generation mechanisms.

\textsuperscript{1} Present address
Due to the great improvement in sensitivity of the European Space Agency’s Planck satellite over WMAP, understanding the nature and in particular non-Gaussian properties of foregrounds will be indispensable for the Planck $f_{NL}$ analysis. For WMAP, foregrounds were not an issue because no signal was detected. Currently estimators exist [3, 4] that are proven optimal in the absence of foreground contamination and much work has been devoted to their efficient implementation [5]. However, to date there has been no comprehensive study of the impact of foregrounds. Foreground sources distributed in an anisotropic way are susceptible of being misidentified as primordial non-Gaussianity. In order to extract from the data an accurate and significant measure of $f_{NL}$, new estimators are needed that divide the data and are designed to mask possible contaminants. First, however, it is required to characterize the statistical properties of these spurious sources of non-Gaussianity, which requires understanding the non-linear properties of foregrounds.

This contribution reports on work in progress where we formulate a general class of models for characterizing the non-Gaussian properties of foreground contaminants in the cosmic microwave background (CMB). [6].

2. Measuring non-Gaussianity

In linear perturbation theory, the temperature anisotropy of the CMB is derived from metric perturbations by the relation $\Delta T/T \propto g_T \Delta \Phi$, where the radiation transfer function $g_T$ depends on the scale considered. Linear perturbation theory is a valid approximation because the physics of the perturbations generated during inflation is extremely weakly coupled.

In the simplest models of inflation, metric perturbations are produced by quantum fluctuations of the inflaton field. If the physics that governs the evolution of the metric perturbations is linear, since quantum fluctuations are Gaussian then so will temperature fluctuations be Gaussian. However, non-linear coupling between long and short wavelength fluctuations of the inflaton can generate weak non-Gaussianities. These will propagate into the metric perturbations and consequently into the temperature anisotropy. The primordial metric perturbations $\Phi$ can be characterized approximately by a non-linear coupling parameter, $f_{NL}$, such that

$$\Phi(x) = \Phi_L(x) + f_{NL} \left[ \Phi_L^2(x) - \langle \Phi_L^2(x) \rangle \right],$$

where $\Phi_L(x)$ are Gaussian linear perturbations with $\langle \Phi_L(x) \rangle = 0$. In many models (e.g. single-field slow roll inflation [8, 9]), the non-linear corrections are too small to be detected. Other models (e.g. multiple field inflation [10, 11]) can generate stronger and potentially detectable non-linearity. A Gaussian distribution is characterized by vanishing odd-order moments and even-order moments defined only by the variance. Non-Gaussianity can thus be measured by correlations among an odd number of points. Here we shall use the angular three-point correlation function or bispectrum as our statistical tool since it is sensitive to weakly non-Gaussian fluctuations.

Expanding the temperature anisotropy in momentum space

$$\frac{\Delta T(\Omega)}{T} = \sum_{l,m} a_{lm} Y_{lm}(\Omega),$$

where the harmonic coefficients are

$$a_{lm} = 4\pi (-i)^l \int \frac{d^3k}{(2\pi)^3} \Phi(k) g_{T_l}(k) Y^*_{lm}(\Omega),$$

2 For temperature fluctuations on super-horizon scales the Sachs–Wolfe effect dominates and $g_T$ is a function of the parameter $\omega$ of the equation of state. At the decoupling epoch it reduces to $g_T = -1/3$ in the case of adiabatic fluctuations. On sub-horizon scales, however, $g_T$ oscillates and we need to solve the Boltzmann photon transport equations coupled to the Einstein equations for $g_T$, which is found to have as parameter the decoupling time.
the bispectrum is defined as

$$B_{l_{1}l_{2}l_{3}}^{m_{1}m_{2}m_{3}} \equiv \langle a_{l_{1}m_{1}} a_{l_{2}m_{2}} a_{l_{3}m_{3}} \rangle = G_{l_{1}l_{2}l_{3}}^{m_{1}m_{2}m_{3}} b_{l_{1}l_{2}l_{3}},$$

(4)

Here

$$G_{l_{1}l_{2}l_{3}}^{m_{1}m_{2}m_{3}} = \int d\Omega Y_{l_{1}m_{1}}(\Omega) Y_{l_{2}m_{2}}(\Omega) Y_{l_{3}m_{3}}(\Omega)$$

(5)

is the Gaunt integral and

$$b_{l_{1}l_{2}l_{3}} = \left( \frac{2}{\pi} \right)^{3} \int dx \, dk_{1} \, dk_{2} \, dk_{3} \, (xk_{1}k_{2}k_{3})^{2} B_{k_{1}, k_{2}, k_{3}}(k_{1}, k_{2}, k_{3})$$

$$\times j_{l_{1}}(k_{1}x)j_{l_{2}}(k_{2}x)j_{l_{3}}(k_{3}x) \, g_{T_{1}}(k_{1})g_{T_{2}}(k_{2})g_{T_{3}}(k_{3})$$

(6)

is the reduced bispectrum defined in terms of the radiation transfer functions $g_{T_{i}}$, the Bessel functions of fractional order $j_{l}$ and the primordial bispectrum $B_{k_{1}, k_{2}, k_{3}}$. In the flat-sky approximation, which amounts to $(l, m) \to \ell$, where $\ell$ is a two-dimensional wave vector in the sky, the bispectrum reduces to

$$\langle a(\ell_{1})a(\ell_{2})a(\ell_{3}) \rangle = (2\pi)^{2} \delta^{2}(\ell_{1} + \ell_{2} + \ell_{3}) B(\ell_{1}, \ell_{2}, \ell_{3})$$

(7)

which is non-vanishing only for closed triangle configurations in momentum space. Since in this approximation $G_{l_{1}l_{2}l_{3}}^{m_{1}m_{2}m_{3}} \to (2\pi)^{2} \delta^{2}(\ell_{1} + \ell_{2} + \ell_{3})$, then $b_{l_{1}l_{2}l_{3}} \to B(\ell_{1}, \ell_{2}, \ell_{3})$ [12].

The signal-to-noise squared for the bispectrum gives us an indication of the dominant triangle configuration to the non-Gaussian signal. We assume a scale invariant power spectrum $P(\ell) \propto \ell^{-2}$ in the flat sky approximation and

$$B(\ell_{1}, \ell_{2}, \ell_{3}) = f_{NL} [P(\ell_{1})P(\ell_{2}) + P(\ell_{2})P(\ell_{3}) + P(\ell_{3})P(\ell_{1})]$$

as a result of the local non-Gaussianity ansatz. Then

$$\left( \frac{S}{N} \right)^{2} \propto \int d^{2}\ell_{1} \int d^{2}\ell_{2} \int d^{2}\ell_{3} \delta^{2}(\ell_{1} + \ell_{2} + \ell_{3}) \frac{B^{2}(\ell_{1}, \ell_{2}, \ell_{3})}{P(\ell_{1})P(\ell_{2})P(\ell_{3})}$$

$$\propto f_{NL}^{2} \int d^{2}\ell_{1} \int d^{2}\ell_{2} \int d^{2}\ell_{3} \delta^{2}(\ell_{1} + \ell_{2} + \ell_{3}) \left( \frac{1}{\ell_{1}^{2}\ell_{2}^{2}} + \frac{1}{\ell_{2}^{2}\ell_{3}^{2}} + \frac{1}{\ell_{1}^{2}\ell_{3}^{2}} \right)^{2}$$

$$\propto f_{NL}^{2} \int \frac{d\ell_{<}}{\ell_{<}} \int d^{2}\ell_{>}.$$  

(8)

For squeezed triangles, where the length of the two large sides is of order $\ell_{\text{max}}$ and that of the smaller side $\ell_{\text{min}}$, we find that

$$\left( \frac{S}{N} \right)^{2} \propto f_{NL}^{2} \int \frac{d\ell_{<}}{\ell_{<}} \int d^{2}\ell_{>} \propto f_{NL}^{2} \ln \left[ \frac{\ell_{\text{max}}}{\ell_{\text{min}}} \right] \ell_{\text{max}}^{2}.$$  

(9)

For equilateral triangles with the length of the sides of order $\ell_{\text{max}}$

$$\left( \frac{S}{N} \right)^{2} \propto f_{NL}^{2} \ell_{\text{max}}^{2}.$$  

(10)

Comparing the two results we observe that the dominant contribution to the signal comes from the squeezed triangles, which introduce the logarithmic boost factor and thus couple the small-scale with the large-scale anisotropies. Should only equilateral triangles be considered, then the
description of the signal would be equivalent to the separate description from different patches
in the sky, which would mean losing most of the signal. The squeezed triangle configurations
correspond to the local case where the non-Gaussianity was created primarily on super-horizon
scales, as opposed to the equilateral case where the non-Gaussianity is primarily created at
horizon-crossing. Qualitatively an $f_{NL} \neq 0$ can be understood as a modulation of the small-
scale power (on scales near the resolution of the map) by the large-scale signal. Exploring further
this idea, we can construct a cubic estimator of $f_{NL}$ by filtering the signal into two contributions,
融为一体 $T_{\text{low} \ell}$ and $T_{\text{high} \ell}$, and considering the integral
\[ E = \int d\Omega \ T_{\text{low} \ell} T_{\text{high} \ell}^2. \] (11)

This is but a caricature of the optimal estimator given in Ref. [13] with the ideal weighting, which
here serves illustration purposes. The large-scale variation in $T_{\text{high} \ell}^2$ reflects spatial dependence
in the power spectrum $P(\ell, \Omega)$ where $\ell$ is large and the variation in $\Omega$ is slow. In a Gaussian
model, such perceived spatial dependence is simply noise from the cosmic variance and thus
fortuitous. In a model with $f_{NL} \neq 0$, however, it is correlated with the large-scale temperature
anisotropy.

Astronomical microwave sources as well as instrumental effects can produce spurious non-
Gaussian signals which would be measured together with the primordial signal by the bispectrum
and thus susceptible of being confused for primordial non-Gaussianity. Examples of such sources
are [14]: the synchrotron emission, which results from the acceleration of cosmic ray electrons in
magnetic fields; the free-free emission, which is produced by electron-ion scattering and is thus
correlated with Hydrogen emission lines; and thermal galactic dust emission.

The separation of the CMB and foreground signal components has relied on their differing
spectral and spatial distribution. However, correlations at different scales of the power of
non-primordial foreground signals could not be separated from the CMB by the canonical
methods. Here we aim to show that a detectable non-Gaussian signal could be mimicked by
such correlations.

3. Faking non-Gaussianity

Since the $a_{lm}$’s are random quantities for the same sky realization, we must devise a way of
generating maps of the sky with built-in non-linear foregrounds. We use motifs of a fixed shape
and distributed in a probabilistic manner in the sky to model foreground sources. This resembles
the galaxy counting method for the study of the large-scale structure formation [15].

Foreground realizations are generated in two steps. First a point $\xi$ for the center of a motif
is randomly selected according to a probability density function $Q(\xi)$ in the simplest case in
an uncorrelated Poissonian manner. Then to each point $\theta$ we assign a conditional probability
$P(\theta, \xi)$ of finding the point $\theta$ for a motif centered at $\xi$. This probability is independent of how
many points are already inside the motif. We assign the value one to the motif interiors and
zero to the regions between motifs. Assuming that the foregrounds are optically thin, i.e. that
they do not mask the effects of each other, we take overlapping motifs to add. We also need
to properly normalize these quantities so as to relate to the foreground maps. Measuring the
mean number density of motifs, the probability density $Q(\xi)$ is such that when integrated over
the entire sky it yields the average number of motifs $\langle N \rangle$
\[ \int d^2 \xi \ Q(\xi) = \langle \text{number of motifs} \rangle. \] (12)

The conditional probability $P(\theta, \xi)$ yields the temperature at a point $\theta$ given that a motif
is centered at $\xi$. Thus correlations among the $\xi$’s as encoded in $Q$ will model the large-scale
structure, while correlations among the $\theta$’s in the same motif as encoded in $P$ will model the small-scale structure.

We calculate the two and three-point correlation functions of the statistical map in a perturbative manner using Feynman diagrams. These correlation functions are interpreted respectively as the power spectrum and the bispectrum by assigning a temperature to the motifs. This temperature is constrained: from below by the minimum temperature that a foreground distribution must have to account for the remaining signal after removal of the astrophysical sources described previously; and from above by the maximum value of the power spectrum that a spurious source could have so as not to overshadow the background radiation.

We define the two-point correlation function of the final map $P(\theta_1, \theta_2)$ by the sum over the two contributions to the probability that $\theta_1$ and $\theta_2$ are within a motif

$$P(\theta_1, \theta_2) = p_{1,1}(\theta_1, \theta_2) + p_2(\theta_1, \theta_2),$$

(13)

where

$$p_{1,1} = \frac{1}{2!} \int d^2 \xi_1 \int d^2 \xi_2 \ P(\theta_1, \xi_1)Q(\xi_1)P(\theta_2, \xi_2)Q(\xi_2)$$

$$p_2 = \frac{1}{2!} \int d^2 \xi_1 P(\theta_1, \xi_1)P(\theta_2, \xi_1)Q(\xi_1).$$

(14)

Here $p_{1,1}(\theta_1, \theta_2)$ is the probability that the points $\theta_1$ and $\theta_2$ are within different motifs centered at $\xi_1$ and $\xi_2$, and $p_2(\theta_1, \theta_2)$ is the probability that the points are within the same motif. Analogously we define the three-point correlation function $B(\theta_1, \theta_2, \theta_3)$ by the sum of the three contributions

$$B(\theta_1, \theta_2, \theta_3) = p_{1,1,1}(\theta_1, \theta_2, \theta_3) + p_{2,1}(\theta_1, \theta_2, \theta_3) + p_3(\theta_1, \theta_2, \theta_3)$$

(15)

where

$$p_{1,1,1} = \frac{1}{3!} \int d^2 \xi_1 \int d^2 \xi_2 \int d^2 \xi_3 \ P(\theta_1, \xi_1)Q(\xi_1)P(\theta_2, \xi_2)Q(\xi_2)P(\theta_3, \xi_3)Q(\xi_3)$$

$$p_{2,1} = \frac{1}{2!} \int d^2 \xi_1 \int d^2 \xi_2 P(\theta_1, \xi_1)P(\theta_2, \xi_1)Q(\xi_1)P(\theta_3, \xi_2)Q(\xi_2) + \text{(two permutations)}$$

$$p_3 = \frac{1}{3!} \int d^2 \xi_1 P(\theta_1, \xi_1)P(\theta_2, \xi_1)P(\theta_3, \xi_1)Q(\xi_1)$$

(16)

are defined analogously. The combinatorial factors preceding the integrals are the number of equivalent motif configuration for indistinguishable patches in the $\xi$-space which must be introduced to avoid multiple counting. We introduce a factor $1/nQ!$ where $nQ$ is the number of $Q$’s, and a factor $1/nP!$ for each $\xi$ where $nP$ is the number of $P$’s shared by the same $\xi$. For convenience we also define the one-point function

$$p_1(\theta_1) \equiv \int d^2 \xi_1 P(\theta_1, \xi_1)Q(\xi_1).$$

(17)

4. A Simple Model: Uniform Distribution

For concreteness we assume the motifs to be circles of radius $\theta_{circ}$, with sharp edges $P(\theta, \xi) = N_P \Theta(\theta_{circ} - |\xi - \theta|)$ and uniformly distributed in the sky $Q(\xi) = const = N_Q$. Thus $N_Q = \langle N \rangle / A_{sky}$ and $N_P = T_{motif}$, which in this case are constant inside the motif and independent of the position of the motifs in the sky. [To be dimensionally correct, we must have $T_{motif}$ normalized to the average temperature of the sky $T_{sky}$] We will also discuss a non-uniform
distribution of the motifs in the sky, motivated by the variation of the distribution of the large-scale structure along the latitude. This will be the working case for introducing dependence of the temperature on the position of the motifs as encoded in the distribution probability. [For a refinement of the functional form see Ref. [6].] We compute the two and three-point correlator functions in real space for the statistical ensemble just described. The one-point function becomes

\[ p_1(\theta_1) = N_p N_Q \int d^2 x_1 \Theta[\theta_{circ} - |x_1|] = N_p N_Q A_{motif} \]  

(18)

where \( A_{motif} = \pi \theta_{circ}^2 \). For the power spectrum we must calculate \( p_{1,1} \) and \( p_2 \), which can be written as

\[ p_{1,1}(\theta_1, \theta_2) = \frac{1}{2!} N_p^2 N_Q^2 \int d^2 x_1 \Theta[\theta_{circ} - |x_1|] \int d^2 x_2 \Theta[\theta_{circ} - |x_2|] = \frac{1}{2!} p_1^2 \]  

(19)

\[ p_2(\theta_1, \theta_2) = \frac{1}{2!} N_p^2 N_Q \int d^2 x_1 \Theta[\theta_{circ} - |x_1|] \Theta[\theta_{circ} - |x_1 + \theta_{12}|]. \]  

(20)

Here \( x_i = \xi_i - \theta_i \) and \( \theta_{ij} = \theta_i - \theta_j \). For the bispectrum we observe that both \( p_{1,1,1} \) and \( p_{2,1} \) can be expressed in terms of \( p_1 \) and \( p_2 \) as follows

\[ p_{1,1,1}(\theta_1, \theta_2, \theta_3) = \frac{1}{3!} N_p^3 N_Q^3 \int d^2 x_1 \Theta[\theta_{circ} - |x_1|] \int d^2 x_2 \Theta[\theta_{circ} - |x_2|] \times \int d^2 x_3 \Theta[\theta_{circ} - |x_3|] \]

\[ = \frac{1}{3!} p_1^3 \]  

(21)

\[ p_{2,1}(\theta_1, \theta_2, \theta_3) = \frac{1}{2!} \frac{1}{2!} N_p^3 N_Q^2 \int d^2 x_1 \Theta[\theta_{circ} - |x_1|] \Theta[\theta_{circ} - |x_1 + \theta_{12}|] \times \int d^2 x_3 \Theta[\theta_{circ} - |x_3|] + \text{(two permutations)} \]

\[ = \frac{1}{2!} p_1 \left[ p_2(\theta_1, \theta_2) + p_2(\theta_2, \theta_3) + p_2(\theta_3, \theta_1) \right], \]  

(22)

so we have in addition to compute \( p_3 \) only

\[ p_3(\theta_1, \theta_2, \theta_3) = \frac{1}{3!} N_p N_Q \int d^2 x_1 \Theta[\theta_{circ} - |x_1|] \Theta[\theta_{circ} - |x_1 + \theta_{12}|] \Theta[\theta_{circ} - |x_1 + \theta_{13}|]. \]  

(23)

These quantities describe the probability that one, two or three points, denoted by \( \theta_1, \theta_2 \) and \( \theta_3 \), are within the same motif centered at points \( \xi \)'s which are distributed in the sky according to \( Q \). These probabilities can also be interpreted as the measure of the overlap of motifs centered at each of these points \( \theta \)'s. The integrals of the step functions about the distance from the centers at \( \xi \)'s can thus be visualized as the area of the intersection of motifs centered at the \( \theta \)'s. [For more details see Ref. [6].] We find that

\[ p_{1,1}(\theta_1, \theta_2) = \frac{1}{2!} p_1^2 \]  

(24)

\[ p_2(\theta_1, \theta_2) = \frac{1}{2!} N_p^2 N_Q A_{1\cap 2} \]  

(25)

where \( A_{1\cap 2} \) is the area of intersection between the motifs centered at \( \theta_1 \) and \( \theta_2 \)

\[ A_{1\cap 2} = \left( 2 \theta_{circ}^2 \arctan \left( \frac{4 \theta_{circ}^2}{|\theta_{12}|^2} - 1 \right) - \frac{|\theta_{12}|^2}{2} \sqrt{\frac{4 \theta_{circ}^2}{|\theta_{12}|^2} - 1} \right) \Theta [2 \theta_{circ} - |\theta_{12}|]. \]  

(26)
These contribute to the power spectrum. Moreover, we find that

\[ p_{1,1,1}(\theta_1, \theta_2, \theta_3) = \frac{1}{3!} p_1^3 \]  
(27)

\[ p_{2,1}(\theta_1, \theta_2, \theta_3) = \frac{1}{2!} p_1 [p_2(\theta_1, \theta_2) + p_2(\theta_2, \theta_3) + p_2(\theta_3, \theta_1)] \]  
(28)

\[ p_3(\theta_1, \theta_2, \theta_3) = \frac{1}{3!} N^2 N_Q A_{12\cap3} \]  
(29)

where \( A_{12\cap3} \) is the area of intersection among the three motifs

\[
A_{12\cap3} = \left[ \frac{1}{4} \sqrt{(|\theta_{12}|^2 + |\theta_{23}|^2 + |\theta_{13}|^2)^2 - 2(|\theta_{12}|^4 + |\theta_{23}|^4 + |\theta_{13}|^4) - \frac{1}{2} A_{motif}} \right]
\]

\[
\times \Theta [2\theta_{circ} - |\theta_{12}|] \Theta [2\theta_{circ} - |\theta_{23}|] \Theta [2\theta_{circ} - |\theta_{13}|]
\]

\[
+ \frac{1}{2} A_{12\cap2} \Theta [2\theta_{circ} - |\theta_{12}|] \Theta [2\theta_{circ} - |\theta_{23}|] \Theta [2\theta_{circ} - |\theta_{13}|]
\]

\[
+ \frac{1}{2} A_{23\cap1} \Theta [2\theta_{circ} - |\theta_{12}|] \Theta [2\theta_{circ} - |\theta_{23}|] \Theta [2\theta_{circ} - |\theta_{13}|]
\]

\[
+ \frac{1}{2} A_{13\cap2} \Theta [2\theta_{circ} - |\theta_{12}|] \Theta [2\theta_{circ} - |\theta_{23}|] .
\]  
(30)

These contribute to the bispectrum. It follows that

\[
P(\theta_1, \theta_2) = \frac{1}{2!} p_1^2 + p_2(\theta_1, \theta_2)
\]  
(31)

\[
B(\theta_1, \theta_2, \theta_3) = \frac{1}{3!} p_1^3 + \frac{1}{2!} p_1 [p_2(\theta_1, \theta_2) + p_2(\theta_2, \theta_3) + p_2(\theta_3, \theta_1)] + p_3(\theta_1, \theta_2, \theta_3).
\]  
(32)

We now proceed to compute the value of \( f_{NL} \) produced. The parameter \( f_{NL} \) characterizes the amplitude of the temperature non-Gaussianity since it couples to the quadratic term of the expansion of the temperature fluctuation \( T \equiv \Delta T/T \) about a Gaussian distribution \( T_L \). Assuming the Sachs–Wolfe approximation \( (\Delta T)/T = -(1/3)\Delta \Phi/c^2 \) on all scales and for an infinitely thin surface of last scattering, we have in real space that

\[
T(\theta) = T_L(\theta) + 3 f_{NL} [T_L^2(\theta) - \langle T_L(\theta) \rangle^2].
\]  
(33)

Here the Gaussian distribution has zero mean, \( \langle T_L(\theta) \rangle = 0 \), from which it follows to leading order that

\[
P(\theta_1, \theta_2) \equiv \langle T(\theta_1)T(\theta_2) \rangle = \langle T_L(\theta_1)T_L(\theta_2) \rangle .
\]  
(34)

Since a Gaussian distribution has vanishing odd-order momenta, we find for the three-point correlation function that

\[
B(\theta_1, \theta_2, \theta_3) \equiv \langle T(\theta_1)T(\theta_2)T(\theta_3) \rangle = 6 f_{NL} [P(\theta_1, \theta_3)P(\theta_3, \theta_2) + P(\theta_2, \theta_1)P(\theta_1, \theta_3) + P(\theta_3, \theta_2)P(\theta_2, \theta_1)].
\]  
(35)

We distinguish three cases for the three possible relations among the distances between the points \( \theta \)'s, namely 1) \( \theta_{12}, \theta_{13}, \theta_{23} > \theta_{circ} \); 2) \( \theta_{12} < \theta_{circ} \) and \( \theta_{13}, \theta_{23} > \theta_{circ} \); 3) \( \theta_{12}, \theta_{13}, \theta_{23} < \theta_{circ} \). Let us fix the size of the motif and for concreteness take the radius to be of the size of the resolution of the map, i.e. \( \theta_{circ} \sim 10^{-6} \).
4.1. Case $\theta_{12}, \theta_{13}, \theta_{23} > 2\theta_{\text{circ}}$

This is the case where each point $\theta_i$ is within a different motif and consequently only the one-point function $p_1$, and the contributions to the two and three-point functions which can be expressed in terms of $p_1$, i.e. $p_{1,1}$ and $p_{1,1,1}$, do not vanish. We find that

$$\begin{align*}
\mathcal{P}(\theta_1, \theta_2) &= \mathcal{P}(\theta_1, \theta_3) = \mathcal{P}(\theta_2, \theta_3) = p_{1,1} = \frac{1}{2}p_1^2, \\
\mathcal{B}(\theta_1, \theta_2, \theta_3) &= p_{1,1,1} = \frac{1}{6}p_1^3,
\end{align*}$$

from which it follows that

$$f_{NL} = \frac{1}{6} \frac{\mathcal{B}(\theta_1, \theta_2, \theta_3)}{\mathcal{P}(\theta_1, \theta_2)\mathcal{P}(\theta_2, \theta_3)} = \frac{1}{27} \frac{1}{p_1}, \quad p_1 > 1. \quad (36)$$

4.2. Case $\theta_{12} < 2\theta_{\text{circ}}$ and $\theta_{13}, \theta_{23} > 2\theta_{\text{circ}}$

This is the case where two points, here for concreteness $\theta_1$ and $\theta_2$, are within the same motif. In addition to the one-point function and the contributions derived from it, we have $p_{2}(\theta_1, \theta_2) \neq 0$ and $p_{2,1}(\theta_1, \theta_2, \theta_3) \neq 0$ from the contribution to the three-point function of the two-point function between $\theta_1$ and $\theta_2$ only. Hence

$$\begin{align*}
\mathcal{P}(\theta_1, \theta_2) &= p_{1,1} + p_{2}(\theta_1, \theta_2) = \frac{1}{2}p_1^2 + p_2, \\
\mathcal{P}(\theta_1, \theta_3) &= \mathcal{P}(\theta_2, \theta_3) = p_{1,1} = \frac{1}{2}p_1^2, \\
\mathcal{B}(\theta_1, \theta_2, \theta_3) &= p_{1,1,1} + p_{2,1}(\theta_1, \theta_2, \theta_3) = \frac{1}{6}p_1^3 + \frac{1}{2}p_1 p_{2}(\theta_1, \theta_2),
\end{align*}$$

and

$$f_{NL} = \frac{1}{6} \frac{\mathcal{B}(\theta_1, \theta_2, \theta_3)}{\mathcal{P}(\theta_1, \theta_2)\mathcal{P}(\theta_2, \theta_3)} = \frac{1}{27} \frac{1}{p_1^2} + \frac{1}{p_1} \frac{p_1 p_2(\theta_1, \theta_2)}{p_{1,1,1} p_{2}(\theta_1, \theta_2)}. \quad (38)$$

We note that $p_1^2 \propto N_Q A_{\text{motif}}^2$ and $p_2 \propto N_Q A_{1/2} < N_Q A_{\text{motif}}$, where $N_Q$ is the mean number density of motifs in the sky. The relative magnitude of $p_1^2$ and $p_2$ depends on the relation between the mean number of motifs $\langle N \rangle$ and the area of the motif $A_{\text{motif}}$. For convenience we define $\alpha \equiv N_Q A_{\text{motif}} = \langle N \rangle A_{\text{motif}} / A_{\text{sky}}$. We must further distinguish between the following two cases. If $\alpha < \alpha^2$, i.e. $\langle N \rangle > A_{\text{sky}} / A_{\text{motif}} \sim 10^{13}$, then $p_2 < \alpha < p_1^2$. This suggests that a minimum density of motifs is required in order for the one-point function to dominate, i.e. in order for two points which are close enough to be within the same motif to be also within a second, and thus necessarily overlapping, motif. On the other hand if $\alpha > \alpha^2$, i.e. $\langle N \rangle < A_{\text{sky}} / A_{\text{motif}} \sim 10^{13}$, then both $p_1^2, p_2 < \alpha$. Here in order to discriminate the relative magnitude, we need in addition to precise the relation between $A_{1/2}$ and $A_{\text{motif}}$, which depends on $\theta_{12}$. If $\theta_{12}$ is sufficiently small so that $A_{1/2} \sim A_{\text{motif}}$, then $p_2 \sim \alpha$ and consequently $p_2 > p_1^2$. However, if $\theta_{12}$ is very close to $2\theta_{\text{circ}}$, then $A_{1/2} \ll A_{\text{motif}}$ and consequently $p_2 \ll \alpha$, so that if $p_2 < \alpha^2$ then $p_2 < p_1^2$. This is equivalent to the case $\alpha < \alpha^2$.

Hence, if $\langle N \rangle > 10^{13}$, or $\langle N \rangle < 10^{13}$ and $\theta_{12} \sim 2\theta_{\text{circ}}$, then

$$f_{NL} \sim \frac{1}{6} \frac{p_1^3}{p_2^2} = \frac{1}{27} \frac{1}{p_1}, \quad p_1 > 1, \quad (40)$$

whereas if $\langle N \rangle < 10^{13}$ and $\theta_{12} \ll 2\theta_{\text{circ}}$, then

$$f_{NL} \sim \frac{1}{6} \frac{p_1 p_2(\theta_1, \theta_2)}{p_2(\theta_1, \theta_2)^2} = \frac{1}{12} \frac{1}{p_1}, \quad p_1 < 1. \quad (41)$$
4.3. Case \( \theta_{12}, \theta_{13}, \theta_{23} < 2\theta_{\text{circ}} \)

This is the case where the three points are within the same motif. All the terms contribute to the two and three-point functions. In order to determine which contribution dominates for a mean number of motifs \( \langle N \rangle \) distributed in the sky according to \( Q \), we do an analysis similar to that above. Thus if \( \langle N \rangle > 10^{13} \)

\[
f_{NL} \sim \frac{1}{6} \frac{p_1^2}{3p_1^1} = \frac{1}{27} p_1, \quad p_1 > 1. \tag{42}
\]

However, if \( \langle N \rangle < 10^{13} \) we find for \( \theta_{12} < \theta_{13}, \theta_{23} \) that

\[
f_{NL} \sim \frac{1}{6} \frac{p_3(\theta_1, \theta_2, \theta_3)}{2p_2(\theta_1, \theta_2)p_2(\theta_2, \theta_3)} = \frac{1}{18} \frac{-\frac{1}{2}A_{\text{motif}} + \frac{3}{2}A_{1\cap2}}{N_P N_Q A_{1\cap2} A_{2\cap3}}, \tag{43}
\]

while for \( \theta_{12} \sim \theta_{23} \sim \theta_{13} \) we find that

\[
f_{NL} \sim \frac{1}{6} \frac{p_3(\theta_1, \theta_2, \theta_3)}{3p_2(\theta_1, \theta_2)p_2(\theta_2, \theta_3)} = \frac{1}{27} \frac{\sqrt{2} \theta_{12}^2 - \frac{1}{2}A_{\text{motif}} + \frac{3}{2}A_{1\cap2}}{N_P N_Q A_{1\cap2} A_{2\cap3}}. \tag{44}
\]

For small \( \theta_{12} \) the term \( (\sqrt{3}/4)\theta_{12}^2 \) is subdominant, so for both cases we need only compare \( A_{1\cap2} \) with \( A_{\text{motif}} \). We present the results for the former case, with those for the latter following straightforwardly. For \( \theta < \theta_{eq} \), where \( \theta_{eq} \) is the distance such that \( (3/2)A_{1\cap2} = (1/2)A_{\text{motif}} \), we find that \( (3/2)A_{1\cap2} > (1/2)A_{\text{motif}} \) and hence that

\[
f_{NL} \sim \frac{1}{18} \frac{A_{1\cap2}}{N_P N_Q A_{1\cap2} A_{2\cap3}} = \frac{1}{18} \frac{1}{N_P N_Q A_{2\cap3}}. \tag{45}
\]

For \( \theta_{eq} < \theta_{12} < 2\theta_{\text{circ}} \) the contribution of \( (\sqrt{3}/4)\theta_{12}^2 \) becomes more important but the dominant contribution is that of \( (1/2)A_{\text{motif}} \). Moreover since \( A_{1\cap2} \) is a decreasing function of \( \theta_{12} \) except for \( \theta_{12} \) very close to \( 2\theta_{\text{circ}} \), then \( A_{1\cap2} A_{2\cap3} > A_{2\cap3}^2 \) and consequently \( -1/A_{1\cap2} A_{2\cap3} > -1/A_{2\cap3}^2 \). Thus

\[
f_{NL} \gtrsim \frac{1}{18} \frac{-\frac{1}{2}A_{\text{motif}}}{N_P N_Q A_{1\cap2} A_{2\cap3}} > \frac{1}{18} \frac{-\frac{1}{2}A_{\text{motif}}}{N_P N_Q A_{2\cap3}^2}. \tag{46}
\]

A similar calculation would follow for the case \( \theta_{12} \sim \theta_{23} \sim \theta_{13} \).

Since these distances must be such that \( 0 \leq \theta_{12}, \theta_{13}, \theta_{23} \leq 2\theta_{\text{circ}} \), then \( -\infty < f_{NL} < \infty \), i.e. we can generate arbitrary non-Gaussianity. We need, however, to be able to constrain \( f_{NL} \) to a finite interval. This requires the use of the foreground maps to constrain the parameters for the case of the uniform model, as well as the functional form of the probability functions for more realistic models.

5. Constraining the Model

Constraints on the parameters of the model can be extracted from the properties of foreground maps. Spatial templates from the WMAP data produced for synchrotron, free-free and dust emission show a temperature distribution strongly dependent on the latitude. In order to account for this observation we must consider a non-uniform probability density that can capture the qualitative aspects of these dependence. The simplest case of a non-uniform distribution of
motifs in the sky is that where $Q(\theta)$ is a slowly varying function across the scale of the motif, so that the probabilities are changed to

$$p_{1,1}(\theta_1, \theta_2) = \frac{1}{2!} N_P^2 A_{\text{motif}}^2 Q(\theta_1)Q(\theta_2)$$

and to

$$p_{2,1}(\theta_1, \theta_2, \theta_3) = \frac{1}{3!} N_P^3 A_{\text{motif}}^3 Q(\theta_1)Q(\theta_2)Q(\theta_3)$$

and

$$p_{3}(\theta_1, \theta_2, \theta_3) = \frac{1}{3!} N_P^3 A_{\text{motif}}^3 Q(\theta_1)Q(\theta_2)Q(\theta_3)$$

The changes in the two and three-point correlation functions, as well as in $\int_{N\ell}$, are straightforward. Assuming that the contribution of the overlapping motifs add in the intersection region (what we called the “optically thin” approximation), then the value of $Q$ in the mid distance between the centers of two overlapping motifs is the sum of the value of $Q$ at the center of each motif, i.e.

$$Q \left( \frac{\theta_1 + \theta_2}{2} \right) = Q(\theta_1) + Q(\theta_2).$$

Since the scale on which $Q(\xi)$ varies is large compared to $A_{\text{motif}}$, then

$$Q(\xi) = \frac{\langle T(\xi) \rangle}{T_{\text{motif}}} \frac{1}{A_{\text{sky}}}$$

so that the same normalization condition holds. Possible functional forms for $Q(\xi)$ will be explored in Ref. [6]. For the purpose of this section, however, we will not need to be more specific. The conditional probability becomes $P(\theta, \xi) = T(\theta)Q[\theta_{\text{circ}} - |\xi - \theta|]$ which will be simply the average temperature at the point $\xi$ where the motif on which $\theta$ lies is centered.

By quantifying the clean regions of these maps, we can also set bounds on the foreground temperature. From a careful examination of the five-year data maps [5] we extracted the minimum temperature $T_0$ that a foreground source should have in order to saturate the map with as close to a single temperature value as possible. Taking the minimum value of the temperature of the three foregrounds for each of the five frequency bands analyzed, we found by eye inspection that saturation of the maps was achieved for $T_0$ a few $\mu K$. Hence the temperature at each point of the motif distribution must be such that $T(\xi) > T_0$. On the other hand, the temperature of a single spot must be highly constrained so as not to stand out. This observation is captured by the condition that the power spectrum generated by the motif distribution does not outshine the background radiation. If the motif distribution produced a large power spectrum, as that produced by small and bright sources, then non-Gaussianity would be obvious. We thus set $P(\theta_i, \theta_j) < (T_0/T_{\text{sky}})^2$.

The three cases discriminated in the previous section will now be combined with the constraints on the temperature obtained from the foreground maps. Apart from factors of order $O(1)$, the results will be the same for both a uniform and a non-uniform probability distribution of motifs in the sky. For simplicity we will analyze the results from the uniform distribution.
5.1. Case $\theta_{12}, \theta_{13}, \theta_{23} > 2\theta_{\text{circ}}$

Here for any two pair of motif centers $i, j$, the constraints on the temperature yield that

$$P(\theta_i, \theta_j) = \frac{1}{2} N_P A_{\text{motif}}^2 N_Q N_{\text{motif}}^2 > \frac{1}{2} \left( \frac{T_0}{T_{\text{sky}}} \right)^2 A_{\text{motif}}^2 N_Q^2$$  \hspace{1cm} (54)

and simultaneously that

$$P(\theta_i, \theta_j) < \left( \frac{T_0}{T_{\text{sky}}} \right)^2$$  \hspace{1cm} (55)

from which it follows that

$$\frac{1}{2} \langle N \rangle^2 \left( \frac{A_{\text{motif}}}{A_{\text{sky}}} \right)^2 < 1.$$  \hspace{1cm} (56)

This relation constrains the size of the motifs given their average number $\langle N \rangle$ distributed on the sky. Using the relations above in the expression for $f_{NL}$ we find that

$$f_{NL} = \frac{1}{27} N_P A_{\text{motif}}^2 N_Q < \frac{1}{27} \langle N \rangle \left( A_{\text{motif}} / A_{\text{sky}} \right) \left( T_0 / T_{\text{sky}} \right).$$  \hspace{1cm} (57)

5.2. Case $\theta_{12} < 2\theta_{\text{circ}}$ and $\theta_{13}, \theta_{23} > 2\theta_{\text{circ}}$

Here for $\langle N \rangle > 10^{13}$, or $\langle N \rangle < 10^{13}$ and $\theta_{12} \sim 2\theta_{\text{circ}}$

$$P(\theta_1, \theta_2) \sim P(\theta_1, \theta_3) = P(\theta_2, \theta_3) = \frac{1}{2} N_P A_{\text{motif}}^2 N_Q N_{\text{motif}}^2 > \frac{1}{2} \left( \frac{T_0}{T_{\text{sky}}} \right)^2 A_{\text{motif}}^2 N_Q^2,$$  \hspace{1cm} (58)

which reduces to the previous case. For $\langle N \rangle < 10^{13}$ and $\theta_{12} \ll \theta_{\text{circ}}$, however,

$$P(\theta_1, \theta_2) \sim \frac{1}{2} N_P A_{1<2}^2 N_Q > \left( \frac{T_0}{T_{\text{sky}}} \right)^2 A_{1<2} N_Q$$  \hspace{1cm} (59)

which together with Eqn. (55) yields

$$\frac{1}{2} \langle N \rangle A_{1<2} / A_{\text{sky}} < 1.$$  \hspace{1cm} (60)

This relation constrains the overlapping between any two motifs given the average number of motifs. Then for $f_{NL}$ we find that

$$f_{NL} \sim \frac{1}{12} N_P A_{\text{motif}}^2 N_Q < \frac{1}{12} \langle N \rangle \left( A_{\text{motif}} / A_{\text{sky}} \right) \left( T_0 / T_{\text{sky}} \right).$$  \hspace{1cm} (61)

This relation sets a constraint on $f_{NL}$ of the same order of magnitude as the previous case.
5.3. Case $\theta_{12}, \theta_{13}, \theta_{23} < 2\theta_{\text{circ}}$

Here for $\langle N \rangle > 10^{13}$ we find the same result as in the first case where the constraint on the power spectrum is given by Eqn. (56). For $\langle N \rangle < 10^{13}$ the constraint on the power spectrum is the same as in the second example of the second case and given by Eqn. (60). We further discriminate between two cases, namely $\theta_{12} < \theta_{13}, \theta_{23}$ and $\theta_{12} \sim \theta_{13} \sim \theta_{23}$, and each case for two regimes of the parameter $\theta_{12}$. Thus for $\theta_{12} < \theta_{13}, \theta_{23}$ we find that

$$f_{NL} \sim \frac{1}{18} \frac{A_{1\gamma 2\gamma 3}}{N \rho A_{1\gamma 2} A_{2\gamma 3} N Q} < \frac{1}{18} \frac{1}{\langle N \rangle A_{1\gamma 2}(A_{2\gamma 3}/A_{\text{sky}})(T_0/T_{\text{sky}})}.$$  (62)

which for $\theta_{12} < \theta_{eq}$ becomes

$$f_{NL} < \frac{1}{18} \frac{1}{\langle N \rangle (A_{2\gamma 3}/A_{\text{sky}})(T_0/T_{\text{sky}})}.$$  (63)

whereas for $\theta_{eq} < \theta_{12} < 2\theta_{\text{circ}}$

$$f_{NL} \gtrsim \frac{1}{18} \frac{1}{\langle N \rangle A_{1\gamma 2}(A_{2\gamma 3}/A_{\text{sky}})(T_0/T_{\text{sky}})} > \frac{1}{18} \frac{1}{\langle N \rangle A_{2\gamma 3}(A_{2\gamma 3}/A_{\text{sky}})(T_0/T_{\text{sky}})}.$$  (64)

A similar calculation would follow straightforwardly for the case $\theta_{12} \sim \theta_{13} \sim \theta_{23}$, Eqn. (63) sets a weaker constraint on $f_{NL}$ than Eqn. (61) by the order of magnitude of $A_{2\gamma 3}/A_{\text{motif}}$. Since $A_{1\gamma 2}$ is predominantly a decreasing function of $\theta_{12}$, Eqn. (64) sets a weaker still constrain while allowing for a negative correlation among the three motifs.

6. Discussion

We propose a simple family of models for mimicking foreground sources that could contaminate the non-Gaussian signal of the CMB. This contamination could lead to the misidentification of a detection of a non-Gaussian signal for primordial when in fact we would be looking at the spurious signal from late-time, non-linear sources. Qualitatively non-Gaussian aspects of foregrounds that are likely to give a significant signal of $f_{NL} \neq 0$ result from the modulation of the small-scale power by the large-scale power.

Our model allows to generate foreground maps by distributing motifs in the sky according to a probability density $Q(\xi)$ and correlating them according to a conditional probability $P(\theta, \xi)$ of finding one, two or three points inside the same motif. The statistical properties of the resulting maps are determined by the two and three-point correlation functions, which are calculated by simply evaluated tree-level Feynman diagrams. We find the expression for $f_{NL}$ in terms of the parameters of $Q(\xi)$ and $P(\theta, \xi)$, namely the mean number of motifs and the intersection areas among two and three motifs. We suggest a prescription for introducing temperature in order to interpret the statistical properties of the motif ensemble as statistical properties of the temperature anisotropies. We also indicate how to use the foreground maps to constrain the normalization factors and accordingly we constrained the values for $f_{NL}$.

In the forthcoming paper we will use the model to generate concrete mock foreground maps consistent with the level of foreground contamination observed and make detailed analysis of the impact on $f_{NL}$ for Planck and other experiments.

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