DOES THE JONES POLYNOMIAL DETECT THE UNKNOT?

STEPHEN BIGELOW

Abstract. We address the question: Does there exist a non-trivial knot with a trivial Jones polynomial? To find such a knot, it is almost certainly sufficient to find a non-trivial braid on four strands in the kernel of the Burau representation. I will describe a computer algorithm to search for such a braid.

1. Introduction

The Jones polynomial $V_K(q)$ of a knot $K$ is one of the most famous and important knot invariants. It is not hard to construct distinct knots with the same Jones polynomial. However the answer to the following question remains unknown.

Question 1.1. Does there exist a non-trivial knot $K$ with $V_K(q) \equiv 1$?

This is given as Problem 1 in [8]. There have been many attempts to find such a knot. A brute force approach was used in [5] to check all knots with up to seventeen crossings. Another approach used in [1] and [9] is to start with a complicated diagram of the unknot and apply mutations which do not alter the Jones polynomial but may alter the knot type.

The approach described in this paper comes from the theory of braids. Any knot $K$ can be obtained as the closure of some braid $\beta$. The Jones polynomial of $K$ is a trace function of the representation of $\beta$ into the Temperley-Lieb algebra. We are therefore led to ask the following question.

Question 1.2. Is the representation of the braid group into the Temperley-Lieb algebra faithful?

This is Problem 3 in [8]. A non-trivial braid in the kernel of the Temperley-Lieb representation could be used to construct a knot with Jones polynomial equal to one. I am not aware of any proof that the knot so obtained must be non-trivial, but this seems unlikely to pose a problem if a specific braid were known. The following conjecture is therefore widely assumed to be true.

Conjecture 1.3. If the Temperley-Lieb representation of the braid group is unfaithful then there exists a non-trivial knot with Jones polynomial equal to one.

The Temperley-Lieb representation of $B_n$ appears as a summand in a larger representation into the Hecke algebra $H(q, n)$ of type $A_{n-1}$. We will call this latter representation the Jones representation, although some authors use this term for what we are calling the Temperley-Lieb representation. The Jones representation was used by Ocneanu in [6] to define a two-variable generalisation of the Jones polynomial called the HOMFLY polynomial. The following conjecture is also widely assumed to be true.

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Conjecture 1.4. If the Jones representation of the braid group is unfaithful then there exists a non-trivial knot with HOMFLY polynomial equal to one.

We will focus on the braid group $B_4$. In this case the Jones and Temperley-Lieb representations both decompose into the Burau representation together with some very simple representations. Thus we have the following.

Proposition 1.5. The following are equivalent:
- the Jones representation of $B_4$ is faithful,
- the Temperley-Lieb representation of $B_4$ is faithful, and
- the Burau representation of $B_4$ is faithful.

We are therefore led to ask the following question.

Question 1.6. Is the Burau representation of $B_4$ faithful?

A negative answer would almost certainly lead to a non-trivial knot whose HOMFLY polynomial is equal to one. As far as I know, a positive answer would have no such dramatic consequences other than finally determining for which values of $n$ the Burau representation of $B_n$ is faithful. Krammer [10] has already shown that $B_4$ is be linear.

The Burau representation of $B_n$ is known to be faithful for $n \leq 3$ [4] and unfaithful for $n \geq 5$ [2]. The case $n = 4$ seems to lie very close to the border between faithfulness and unfaithfulness.

The main aim of this paper is to propose a computer search for a non-trivial braid in the kernel of the Burau representation of $B_4$. This might seem overly ambitious. After all, it amounts to a search for a very special case of a non-trivial knot whose HOMFLY polynomial is equal to one (assuming Conjecture 1.4). Many people have tried and failed to find a non-trivial knot whose weaker Jones polynomial is equal to one. However there is some reason for optimism. A knot constructed by the methods of this paper would have thousands of crossings. Thus we are searching in relatively unexplored territory which might contain unexpected treasures. This is probably enough to justify the expenditure of some computer time, but perhaps not too much human time or brain power.

2. The Burau Representation

We now define the braid groups $B_n$ and the Burau representation.

Let $D$ be a disk. Let $p_1, \ldots, p_n$ be distinct points in the interior of $D$. We call these “puncture points”. Let $D_n = D \setminus \{p_1, \ldots, p_n\}$. Let $d_0$ be a basepoint on $\partial D_n$. For concreteness, take $D$ to be the unit disk in the complex plane centred at the origin, take $p_1, \ldots, p_n$ to be real numbers satisfying $-1 < p_1 < \cdots < p_n < 1$, and take $d_0$ to be $-i$.

The braid group $B_n$ is defined to be the group of homeomorphisms from $D_n$ to itself which act as the identity on $\partial D_n$, taken up to isotopy relative to $\partial D_n$. It is generated by $\sigma_1, \ldots, \sigma_{n-1}$, where $\sigma_i$ exchanges $p_i$ and $p_{i+1}$ by a counterclockwise half twist.

The fundamental group $\pi_1(D_n, d_0)$ is a free group with basis $x_1, \ldots, x_n$, where $x_i$ is a loop based at $d_0$ which passes counterclockwise around $p_i$ and no other puncture points. Let $\phi: \pi_1(D_n, d_0) \rightarrow \langle q \rangle$ be the homomorphism given by $\phi(x_i) = q$. Let $\tilde{D}_n$ be the covering space corresponding to the subgroup $\ker(\phi)$ of $\pi_1(D_n)$. Fix a point $\tilde{d}_0$ in the fibre over $d_0$. 
A more concrete description of $\tilde{D}_n$ can be given as follows. Make a bi-infinite stack of $\mathbb{Z}$ copies of $D_n$. On each copy, make a series of vertical cuts connecting each of the puncture points $p_i$ to the boundary. Glue the left-hand side of each cut to the right-hand side of the corresponding cut on the copy of $D_n$ one level lower.

The group of covering transformations of $\tilde{D}_n$ is $\langle q \rangle$. The $\mathbb{Z}$-module $H_1(\tilde{D}_n)$ can be considered as a $\mathbb{Z}[q^{\pm 1}]$-module, where multiplication by $q$ is the induced action of the covering transformation $q$. Thought of in this way, $H_1(\tilde{D}_n)$ turns out to be a free $\mathbb{Z}[q^{\pm 1}]$-module of rank $n - 1$.

The Burau representation is the induced action of $B_n$ by $\mathbb{Z}[q^{\pm 1}]$-module homomorphisms on $H_1(\tilde{D}_n)$. We make this more precise as follows. Let $\beta: D_n \to D_n$ be a homeomorphism representing a braid $[\beta]$ in $B_n$. The induced action of $\beta$ on $\pi_1(D_n)$ satisfies $\phi \beta = \phi$. It follows by some basic algebraic topology that there exists a unique lift $\tilde{\beta}$ which makes the following diagram commute.

$$
\begin{array}{ccc}
(D_n, d_0) & \xrightarrow{\beta} & (D_n, d_0) \\
\downarrow & & \downarrow \\
(D_n, d_0) & \xrightarrow{\tilde{\beta}} & (D_n, d_0)
\end{array}
$$

Furthermore, $\tilde{\beta}$ commutes with the action of $q$ on $\tilde{D}_n$ by a covering transformation. Thus $\tilde{\beta}$ induces a $\mathbb{Z}[q^{\pm 1}]$-module homomorphism

$$
\tilde{\beta}_*: H_1(\tilde{D}_n) \to H_1(\tilde{D}_n).
$$

The Burau representation is the map

$$
\text{Burau}([\beta]) = \tilde{\beta}_*.
$$

For example, using an appropriate choice of basis for $H_1(\tilde{D}_4)$, the Burau representation of $B_4$ is given by

$$
\begin{align*}
\sigma_1 & \mapsto \begin{pmatrix} -q & q & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix}, \\
\sigma_2 & \mapsto \begin{pmatrix} 1 & 0 & 0 \\
1 & -q & q \\
0 & 0 & 1 \end{pmatrix}, \\
\sigma_3 & \mapsto \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & -q \end{pmatrix}.
\end{align*}
$$

3. The Jones and HOMFLY polynomials

In this section we make the connection between the Burau representation of $B_4$ and the Jones and HOMFLY polynomials of a knot. We start by proving Proposition 1.3, which states that in the case of $B_4$ the Jones, Temperley-Lieb, and Burau representations are either all faithful or all unfaithful.

**Proof of Proposition 1.3.** We will not define the Temperley-Lieb or Jones representations, but we will use some of their basic properties, all of which can be found in [7].
The Jones representation of $B_n$ can be decomposed into irreducible summands, one corresponding to each Young diagram with $n$ boxes. Let $V_\lambda$ denote the representation corresponding to the Young diagram $\lambda$. The Temperley-Lieb representation is the sum of those $V_\lambda$ for which $\lambda$ has one or two rows.

The Young diagrams with 4 boxes are (4), (3,1), (2,2), (2,1,1) and (1,1,1,1). The $V_\lambda$ which lie in the Temperley-Lieb representation are as follows.

- $V_{(4)}$ is one-dimensional.
- $V_{(3,1)}$ is the Burau representation.
- $V_{(2,2)}$ can be defined by composing the Burau representation of $B_3$ with the map from $B_4$ to $B_3$ given by $\sigma_1 \mapsto \sigma_1$, $\sigma_2 \mapsto \sigma_2$, and $\sigma_3 \mapsto \sigma_1$.

The Young diagram $(2,1,1)$ is a reflection of $(3,1)$. Reflection of the Young diagram has the effect of substituting $\sigma_i \mapsto -q\sigma_i^{-1}$ in the corresponding representation. It is shown in [11] that the kernel of the Burau representation is invariant under this substitution. Thus the kernel of $V_{(2,1,1)}$ is the same as that of the Burau representation. Finally, $V_{(1,1,1,1)}$ is one-dimensional.

If the Burau representation of $B_4$ is faithful then so are the Temperley-Lieb and Jones representations. Conversely, suppose $\beta$ is a non-trivial braid in the kernel of the Burau representation of $B_4$. Consider the commutator $[(\sigma_1\sigma_2)^3, \beta]$. This lies in the kernel of $V_{(3,1)}$, and hence $V_{(2,1,1)}$. Since this is a commutator, it lies in the kernel of any one-dimensional representation. Since $(\sigma_1\sigma_2)^3$ is central in $B_3$, it also lies in the kernel of the representation corresponding to (2,2). Thus it lies in the kernel of the Jones and Temperley-Lieb representations.

It remains to show that $[(\sigma_1\sigma_2)^3, \beta]$ must be non-trivial. This is not difficult, but would take us too far afield. We therefore omit this part of the proof.

Suppose $\beta$ lies in the kernel of the Temperley-Lieb representation. Then the closures of the braids

$$\beta\sigma_1\sigma_2\ldots\sigma_{n-1}$$

and

$$\sigma_1\sigma_2\ldots\sigma_{n-1}$$

have the same Jones polynomials. The closure of the braid $\sigma_1\sigma_2\ldots\sigma_{n-1}$ is the unknot. Thus the closure of $\beta\sigma_1\sigma_2\ldots\sigma_{n-1}$ has Jones polynomial equal to one. If we could be sure that this was a non-trivial knot then Conjecture 1.3 would be proved. If it is the unknot then all is not lost, since we could use any power $\beta^k$ in place of $\beta$. Thus the following conjecture implies Conjecture 1.3.

**Conjecture 3.1.** Let $\beta$ be a non-trivial braid in $B_n$. There exists some integer $k$ such that the closure of $\beta^k\sigma_1\sigma_2\ldots\sigma_{n-1}$ is a non-trivial knot.

As well as powers of $\beta$, we also have products of conjugates of $\beta$ at our disposal. And in place of $\sigma_1\sigma_2\ldots\sigma_{n-1}$ we could use any braid whose closure is the unknot. Thus we can weaken the above conjecture to the following.

**Conjecture 3.2.** Let $H$ be a non-trivial normal subgroup of $B_n$. Then there exists $\beta_1 \in H$ and $\beta_2 \in B_n$ such that the closure of $\beta_2$ is the unknot but the closure of $\beta_1\beta_2$ is a non-trivial knot.

The above discussion applies equally well to the Jones representation and the HOMFLY polynomial. Thus Conjecture 3.2 also implies Conjecture 1.4.

A counterexample to Conjecture 3.2 would be truly astonishing, implying an unprecedented correlation between the algebraic structure of $H$ and the geometric
structure of the knots constructed. However it might be quite difficult to prove this “obvious” conjecture. This problem would probably be easily overcome in the case of a specific non-trivial braid in the kernel of the Burau representation of $B_4$.

4. The case $n = 3$

The aim of this section is to prove the following.

**Theorem 4.1.** The Burau representation of $B_3$ is faithful.

This is a well-known result and has been proved in many different ways (see, for example, [4]). The proof given here is a warm-up for the ideas that will be used later.

A *fork* is an embedded tree $F$ in $D$ with four vertices $d_0, p_i, p_j$ and $z$ such that

- $F$ meets the puncture points only at $p_i$ and $p_j$,
- $F$ meets the $\partial D_n$ only at $d_0$, and
- all three edges of $F$ have $z$ as a vertex.

The edge of $F$ which contains $d_0$ is called the *handle* of $F$. The union of the other two edges forms a single edge which we call the *tine edge* of $F$ and denote by $T(F)$.

Orient $T(F)$ so that the handle of $F$ lies to the right of $T(F)$.

A *noodle* is an embedded oriented edge $N$ in $D_n$ such that

- $N$ goes from $d_0$ to another point on $\partial D_n$,
- $N$ meets $\partial D_n$ only at its endpoints, and
- a component of $D_n \setminus N$ contains precisely one puncture point.

This last requirement was not included in the definition given in [3]. Without it, Theorem 5.1 is not true, as far as I know.

Let $F$ be a fork and let $N$ be a noodle. We define a pairing $\langle N, F \rangle$ in $\mathbb{Z}[q^{\pm 1}]$ as follows. If necessary, apply a preliminary isotopy of $F$ so that $T(F)$ intersects $N$ transversely. Let $z_1, \ldots, z_k$ denote the points of intersection between $T(F)$ and $N$ (in no particular order). For each $i = 1, \ldots, k$, let $\gamma_i$ be the arc in $D_n$ which goes from $d_0$ to $z_i$ along $F$, then back to $d_0$ along $N$. Let $a_i$ be the integer such that $\phi(\gamma_i) = q^{a_i}$. In other words, $a_i$ is the sum of the winding numbers of $\gamma_i$ around each of the puncture points $p_j$.

Let $\epsilon_i$ be the sign of the intersection between $N$ and $F$ at $z_i$. Let

$$\langle N, F \rangle = \sum_{i=1}^{k} \epsilon_i q^{a_i}.$$  

We should really check that this is independent of our choice of preliminary isotopy of $F$. This is easy enough to prove directly. It is also a special case of the following lemma.

**Lemma 4.2** (The Basic Lemma). Let $\beta: D_n \to D_n$ represent an element of the kernel of the Burau representation. Then $\langle N, F \rangle = \langle N, \beta(F) \rangle$ for any noodle $N$ and fork $F$.

**Proof.** We can assume that the tine edges of $F$ and $\beta(F)$ both intersect $N$ transversely.

Let $\tilde{F}$ be the lift of $F$ to $D_n$ which contains $\tilde{d}_0$. Let $\tilde{T}(F)$ be the corresponding lift of $T(F)$. Then $\tilde{T}(F)$ intersects $q^a \tilde{N}$ transversely for any $a \in \mathbb{Z}$. Let $(q^a \tilde{N}, \tilde{T}(F))$...
denote the algebraic intersection number of these two arcs. Then the following
definition of \(\langle N, F \rangle\) is equivalent to Equation (1).

\[
\langle N, F \rangle = \sum_{a \in \mathbb{Z}} (q^n N, \hat{T}(F))q^a.
\]

(2)

Suppose \(T(F)\) goes from \(p_i\) to \(p_j\). Let \(\nu\) be disjoint small regular
neighbourhoods of \(p_i\) and \(p_j\) respectively. Let \(\gamma\) be a subarc of \(T(F)\) which starts
in \(\nu\) and ends in \(\nu\). Let \(\delta_i\) be a loop in \(\nu\) based at \(\gamma(0)\) which passes
counterclockwise around \(p_i\). Similarly, let \(\delta_j\) be a loop in \(\nu\) based at \(\gamma(1)\) which
passes counterclockwise around \(p_j\). Let \(\hat{T}_2(F)\) be the “figure eight”
\[
\hat{T}_2(F) = \gamma \delta_j \gamma^{-1} \delta_i^{-1}.
\]

Let \(\hat{T}_2(F)\) be the lift of \(T_2(F)\) which is equal to \(1 - q\hat{T}(F)\) outside a small neigh-
bourhood of the puncture points. Then the following definition of \(\langle N, F \rangle\) is equiva-
alent to Equation (3).

\[
\langle N, F \rangle = \frac{1}{1 - q} \sum_{a \in \mathbb{Z}} (q^n N, \hat{T}_2(F))q^a.
\]

(3)

Note that \(\hat{T}_2(F)\) is a closed loop in \(D_n\). Since \(\beta\) is in the kernel of the Burau rep-
resentation, the loops \(\hat{T}_2(F)\) and \(\hat{T}_2(\beta(F))\) represent the same element of \(H_1(D_n)\).
They therefore have the same algebraic intersection number with any lift \(q^n N\) of
\(N\). Thus Equation (3) will give the same result for \(\langle N, \beta(F) \rangle\) as for \(\langle N, F \rangle\).

We now use the assumption that \(n = 3\).

**Lemma 4.3** (The Key Lemma). In the case \(n = 3\), \(\langle N, F \rangle = 0\) if and only if \(T(F)\) is
isotopic to an arc which is disjoint from \(N\).

**Proof.** Apply an isotopy to \(F\) so that \(T(F)\) intersects \(N\) at a minimum number of
points, which we denote \(z_1, \ldots, z_k\) (in no particular order). Recall the definition
given in Equation (1).

\[
\langle N, F \rangle = \sum_{i=1}^{k} \epsilon_i q^{a_i}.
\]

If \(k = 0\) then clearly \(\langle N, F \rangle = 0\). We now assume that \(k > 0\) and prove that
\(\langle N, F \rangle \neq 0\).

By applying a homeomorphism to our picture, we can take \(N\) to be a horizontal
straight line through \(D_3\) with two puncture points above it and one puncture point
below it. (The noodle has been pulled straight and the fork is twisted!) Let \(D_3^+\) and \(D_3^-\) be the upper and components of \(D_3 \setminus N\) respectively. Relabel the puncture
points so that \(D_3^+\) contains \(p_1\) and \(p_2\) and \(D_3^-\) contains \(p_3\).

Consider the intersection of \(T(F)\) with \(D_3^-\). This consists of a disjoint collection of
arcs which have both endpoints on \(N\), and possibly one arc with an endpoint
on \(p_3\). An arc in \(T(F) \cap D_3^-\) which has both endpoints on \(N\) must enclose \(p_3\),
since otherwise it could be slid off \(N\) to reduce the number of points of intersection
between \(T(F)\) and \(N\). Thus \(T(F) \cap D_3^-\) must consist of a collection of parallel arcs
enclosing \(p_3\), and possibly one arc with an endpoint on \(p_3\).

Similarly, each of the arcs in \(T(F) \cap D_3^+\) either enclose one of the puncture points
\(p_1\) or \(p_2\), or have an endpoint on one of \(p_1\) or \(p_2\). There can be no arc in \(T(F) \cap D_3^+\)
which encloses both \(p_1\) and \(p_2\), since the outermost such arc together with the
outermost arc in \(T(F) \cap D_3^-\) would form a closed loop.
An example of a noodle and a tine edge in $D_3$ is shown in Figure 1. We have omitted the handle of the fork, which plays no role in our argument.

Let $z_i$ and $z_j$ be two points of intersection between $T(F)$ and $N$ which are joined by an arc in $T(F) \cap D^n_+$ or $T(F) \cap D^n_-$. This arc, together with a subarc of $N$, encloses one puncture point. Thus

$$a_j = a_i \pm 1.$$ 

Also, $T(F)$ intersects $N$ with opposite signs at $z_i$ and $z_j$, so

$$\epsilon_j = -\epsilon_i.$$ 

Thus

$$\epsilon_j(-1)^{a_j} = \epsilon_i(-1)^{a_i}.$$ 

Proceeding along $T(F)$, we conclude that the values of $\epsilon_j(-1)^{a_j}$ are the same for all $i = 1, \ldots, k$. Thus $\langle N, F \rangle$ evaluated at $q = -1$ is equal to $\pm k$. Thus $\langle N, F \rangle$ is not equal to zero.

We are now ready to prove that the Burau representation of $B_3$ is faithful.

**Proof of Theorem 4.1.** Let $\beta : D_3 \to D_3$ be a homeomorphism which represents an element of the kernel of the Burau representation. We will show that $\beta$ is isotopic relative to $\partial D_n$ to the identity map, and so represents the trivial braid.

Let $N$ be a noodle. As before, take $N$ to be a horizontal line through $D_n$ such that the puncture points $p_1$ and $p_2$ lie above $N$ and $p_3$ lies below $N$. Let $F$ be a fork such that $T(F)$ is a straight line from $p_1$ to $p_2$ which does not intersect $N$. Then $\langle N, F \rangle = 0$. By the Basic Lemma, $\langle N, \beta(F) \rangle = 0$. By the Key Lemma, $\beta(T(F))$ is isotopic to an arc which is disjoint from $N$. By applying an isotopy to $\beta$ relative to $\partial D_n$, we can assume that $\beta(T(F)) = T(F)$.

By a similar argument using different noodles, we can assume that $\beta$ fixes the triangle with vertices $p_1$, $p_2$ and $p_3$. Thus $\beta$ must be some power of $\Delta$, the Dehn twist about a curve parallel to $\partial D_n$. It is easy to show that the Burau representation of $\Delta$ is the scalar matrix $q^3I$. Thus the only power of $\Delta$ which lies in the kernel of the Burau representation is the trivial braid. 

**5. The case $n = 4$**

We now address the question of whether the Burau representation of $B_4$ is faithful. If the Key Lemma holds for the case $n = 4$ then the same argument used for $B_3$
can be used to show that the Burau representation of $B_4$ is faithful. The converse is also true: if the Key Lemma is false for a given $n$ then the Burau representation of $B_n$ is unfaithful. In other words, the following theorem holds.

**Theorem 5.1.** The following are equivalent:

- the Burau representation of $B_n$ is faithful,
- if $N$ and $F$ are any noodle and fork in $D_n$ such that $\langle N, F \rangle = 0$ then $T(F)$ is isotopic to an arc which is disjoint from $N$.

A proof can be found in [2], although the terminology of noodles and forks is not used. The proof of one direction is much the same as our proof that the Burau representation of $B_3$ is faithful. The proof of the other direction is constructive. Suppose $\langle N, F \rangle = 0$ but $T(F)$ is not isotopic to an arc which is disjoint from $N$. Let $\gamma_1$ be a simple closed curve which is parallel to the boundary of the component of $D_n \setminus N$ containing all but one puncture point. Let $\gamma_2$ be the boundary of a regular neighbourhood of $T(F)$. It is shown that the commutator of the Dehn twists about $\gamma_1$ and $\gamma_2$ is a non-trivial braid in the kernel of the Burau representation of $B_n$.

We now define a standard form for a noodle $N$ and tine edge $T(F)$, similar to the one used in the proof of the Key Lemma. Let $N$ be a horizontal straight line through $D_4$ with $p_1, p_2$ and $p_3$ above it, and $p_4$ below it. Let $D^+_4$ and $D^-_4$ be the upper and lower halves of $D_4 \setminus N$, respectively. Then $D^-_4 \cap T(F)$ is a collection of disjoint arcs which enclose $p_4$, and possibly one arc with an endpoint on $p_4$. Each arc in $D^+_4 \cap T(F)$ either

- encloses one of $p_1, p_2$ or $p_3$,
- encloses $p_1$ and $p_2$, the two leftmost puncture points in $D^+_4$, or
- has an endpoint on a puncture point.

Figure 2 shows an example of a noodle and a tine edge in standard form in $D_4$.

Any noodle $N$ and tine edge $T(F)$ can be put into standard form by first isotoping $T(F)$ so as to intersect $N$ at a minimum possible number of points, and then applying some homeomorphism to the entire picture. The homeomorphism might need to be orientation-reversing. This would have the effect of substituting $q^{-1}$ for $q$ in $\langle N, F \rangle$, so would not affect whether $\langle N, F \rangle$ is zero.

The simple parity argument used to prove the Key Lemma in $D_3$ will not work for $D_4$ because of the existence of arcs enclosing two puncture points. In fact, in $D_4$ there can be some cancellation in the calculation of $\langle N, F \rangle$, whereas our argument showed that this cannot happen in $D_3$. We might attempt a more sophisticated
argument which shows that there cannot be complete cancellation. Unfortunately, none of the obvious approaches seem to work. For example, it is possible to have complete cancellation of all of the highest and lowest powers of $q$ that occur in the calculation of $\langle N, F \rangle$.

Conversely, we could attempt a computer search to find a counterexample to the Key Lemma for $n = 4$, and hence a non-trivial braid in the kernel of the Burau representation of $B_4$. This approach has worked for $B_5$ [2].

A tine edge $T(F)$ in standard form is determined up to isotopy by the following:

- four non-negative integers specifying the number of arcs in $T(F) \cap D_n^+$ of each of the four possible types, and
- which of the puncture points are endpoints of $T(F)$.

The handle of $F$ can be ignored because it has no effect on $\langle N, F \rangle$ up to sign and multiplication by a power of $q$.

By some of the basic theory of curves on surfaces, if $T(F)$ is in standard form and intersects $N$ then it is not isotopic to an arc which is disjoint from $N$. Given data defining $T(F)$, it is easy to compute $\langle N, F \rangle$ up to sign and multiplication by a power of $q$. We can thus embark upon an exhaustive open-ended search for a tine edge $T(F)$ in standard form which intersects $N$ but gives $\langle N, F \rangle = 0$. We now discuss issues of speed.

The polynomial $\langle N, F \rangle$ can be stored as an array of integers. Working with this array takes a significant amount of computer time. There is a simple trick which can be used to eliminate this problem. Let $M$ be a large integer. Consider a map $\mathbb{Z} [q^{\pm 1}] \to \mathbb{Z} / M \mathbb{Z}$ sending $q$ to some unit in $\mathbb{Z} / M \mathbb{Z}$. Instead of computing $\langle N, F \rangle$ we can compute its image in $\mathbb{Z} / M \mathbb{Z}$. This allows us to work with a single integer instead of an array. There will be some “false alarms” for which $\langle N, F \rangle$ is non-zero but its image in $\mathbb{Z} / M \mathbb{Z}$ is zero. However these are infrequent and easily checked separately.

This trick speeds up the search considerably. I have used it to check all forks for which $T(F)$ intersects $N$ at up to 2000 points. By comparison, the example in $D_5$ consists of a noodle and a tine edge which intersect at 100 points.

There are some possibilities for further improvements in the algorithm. Perhaps the simplest way to speed up the search is to increase the number of searchers. I would like to take this opportunity to advertise my webpage

[http://www.ms.unimelb.edu.au/~bigelow](http://www.ms.unimelb.edu.au/~bigelow)

where, at the time of writing, it is possible to donate computer time to this noble and possibly futile search.

6. Specialising $q$

We conclude this paper with an aside concerning the “false alarms” mentioned in the previous section. Recall that a false alarm occurs when $\langle N, F \rangle$ is non-zero but maps to zero in $\mathbb{Z} / M \mathbb{Z}$ when $q$ is assigned some unit $q_0$. Usually this is not very interesting, since $M$ was fairly arbitrary. But some false alarms occur when the integer $q_0$ is a root of $\langle N, F \rangle$. At first I thought that these more interesting false alarms should give rise to a non-trivial element of the kernel of the specialisation of the Burau representation to $q = q_0$. However it turns out that the correct theorem is as follows.
Theorem 6.1. Let $q_0$ be a complex number which is not zero or a root of unity. The following are equivalent:

- the Burau representation of $B_n$ is faithful when $q$ is specialised to $q_0$,
- if $N$ and $F$ are any noodle and fork in $D_n$ such that both $q_0$ and $1/q_0$ are roots of $\langle N, F \rangle$ then $T(F)$ is isotopic to an arc which is disjoint from $N$.

A computer search took about half a minute to find the following.

Corollary 6.2. The Burau representation of $B_4$ is not faithful at $q = 2$.

Proof. Let $T(F)$ be the tine edge in standard form as shown schematically in Figure 3. The endpoints of $T(F)$ at $p_1$ and $p_3$ are shown. Segments of $T(F)$ are labelled with numbers to indicate the number of parallel copies required.

A laborious computation or a short computer program can be used to check that

$$ \langle N, F \rangle = -(q - 1)(q - 2)(2q - 1)(q^2 - q + 1)(q^2 + 1), $$

up to multiplication by a power of $q$. Both 2 and 1/2 are roots of this polynomial.

We can construct a specific non-trivial braid $\beta$ in the kernel of the Burau representation of $B_4$ at $q = 2$. To make things more readable, let $a = \sigma_1$, $b = \sigma_2$, and $c = \sigma_3$. Then

$$ [(ba)^3, \psi^{-1}b\psi], $$

where

$$ \psi = a^{-3}b^{-2}c^{-1}bcabcb^{-1}cbabc^{-2}b^{-1}c^{-2}. $$

Note, this uses the convention that braids compose from right to left.

The noodle and fork shown in Figure 3 are the simplest possible example in the sense that they have the fewest points of intersection. They also have the curious property that none of the subarcs of $T(F)$ above $N$ enclose two puncture points, so there is no cancellation in the calculation of $\langle N, F \rangle$. I can think of no explanation for this.

The Burau representation of $B_4$ is also unfaithful at 1/2 and at any root of unity. Despite hundreds of hours of computer time I know of no other values at which it is unfaithful, and certainly none at which it is faithful.

This is to be contrasted with the situation for $B_3$, where we have the following.
Lemma 6.3. Let $N$ and $F$ be a noodle and a fork such that $T(F)$ is not isotopic to an arc which is disjoint from $N$. Then the highest and lowest powers of $q$ in the polynomial $\langle N, F \rangle$ both occur with coefficient $\pm 1$.

Corollary 6.4. If the Burau representation of $B_3$ is unfaithful at $q = q_0$, then both $q_0$ and $1/q_0$ are roots of a monic polynomial. In particular, the Burau representation of $B_3$ is faithful at any rational number other than 0 or $\pm 1$.

Proof of Lemma 6.3. Put $N$ and $F$ in the standard form as in Figure 1. Thus $N$ is a horizontal straight line with $p_1$ and $p_2$ above it and $p_3$ below it. Assume that $d_0$ is the left endpoint of $N$. We show that the lowest power of $q$ in $\langle N, F \rangle$ occurs with coefficient $\pm 1$. The highest power of $q$ and the case where $d_0$ is the right endpoint of $N$ are handled similarly.

Let $z_1, \ldots, z_k$ be the points of intersection between $N$ and $T(F)$. Recall Equation (1), which states that

$$\langle N, F \rangle = \sum_{i=1}^{k} \epsilon_i q^{a_i}.$$  

Let $z_1$ be such that $a_1$ is minimal. We will show that there is only one such $z_1$. We proceed by induction $k$. The case $k = 1$ is trivial, so assume $k > 1$.

If $z_i$ were to the right of $p_3$ then there would be a subarc of $T(F)$ going from $z_i$ around $p_3$ in the clockwise sense to intersect $N$ at a point $z_j$. Then $a_j = a_i - 1$, which contradicts the minimality of $a_i$. Thus $z_i$ must lie to the left of $p_3$.

Let $P$ be a vertical line from the top of the disk to a point on $N$ between the puncture points $p_1$ and $p_2$ such that $P$ does not intersect $T(F)$. If $z_i$ were to the left of $p_3$ but to the right of $P$ then there would be a subarc of $T(F)$ going from $z_i$ around $p_2$ in the clockwise sense, once again contradicting the minimality of $a_i$. Thus $z_i$ lies to the left of $P$.

Let $N'$ be the union of $P$ with the portion of $N$ which lies to the left of $P$. This is a noodle which intersects $T(F)$ at fewer than $k$ points. The pairing $\langle N', F \rangle$ is the sum of those monomials $\epsilon_j q^{a_j}$ for which $z_j$ lies to the left of $P$. Thus $z_i$ is such that $a_i$ is minimal in the calculation of $\langle N', F \rangle$. By the induction hypothesis, there is only one such $z_i$, so we are done. \qed

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MELBOURNE, PARKVILLE VICTORIA 3052, AUSTRALIA
E-mail address: bigelow@unimelb.edu.au