Logically reversible measurements: Construction and application

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We show that for any von Neumann measurement, we can construct a logically reversible measurement such that Shannon entropies and quantum discords induced by the two measurements have compact connections. In particular, we prove that quantum discord for the logically reversible measurement is never less than that for the von Neumann measurement.

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I. INTRODUCTION

Measurement, as envisaged, plays an inevitable role in quantum mechanics, and lies at the heart of “interpretational problem” of quantum mechanics. Nonetheless, different views of measurement almost universally agree on the measurement outcomes. A quantum measurement is described in terms of a complete set of positive operators for the system to be measured. A few examples of quantum measurement are von Neumann measurement [1] which consists of orthogonal projectors, positive-operator-valued measure (POVM) [2], unitarily reversible measurement [3, 4], etc. The most general type of measurement that can be performed on a quantum system is known as a generalized measurement [5, 6]. Generalized measurements can be understood within the framework of quantum operations. Any measurement on a quantum state is inherently associated with wave function collapse and probability distribution. We recollect the necessary preliminaries briefly below.

Quantum measurements.— Let $\mathcal{H}$ be a finite dimensional complex Hilbert space, which represents some quantum system. The set of quantum states $\rho$ on $\mathcal{H}$ is denoted by $\mathcal{D}(\mathcal{H})$. A quantum measurement on $\mathcal{H}$ is a set $\Lambda \equiv \{\Lambda_x\}_{x \in X} \subseteq \mathcal{L}(\mathcal{H})$ of positive operators indexed by $X$ and satisfies $\sum_x \Lambda_x = 1_{\mathcal{H}}$. Given a quantum state $\rho \in \mathcal{D}(\mathcal{H})$ and a quantum measurement $\Lambda = \{\Lambda_x\}_{x \in X}$, then a probability distribution $p = \{p(x)\}_{x \in X}$ is induced where $p(x) = \text{Tr} (\Lambda_x \rho)$ is the probability of the outcome $x$ to occur. In this case, $\rho$ is transformed into the quantum state $\rho_x = \frac{\Lambda_x \rho \Lambda_x}{p(x)}$, where $\Lambda_x = A_x^2$. If $\Pi = \{\Pi_x\}_{x \in X}$ is a set of orthogonal projectors, then the measurement $\{\Pi_x\}_{x \in X}$ is said to be a von Neumann measurement [1]. The celebrated Neumark extension theorem [7, 8] states that each quantum measurement can be seen as a von Neumann measurement on a larger Hilbert space [9].

We know that in a generalized measurement process, the input state $\rho$ cannot always be retrieved with a nonzero success probability by a “reversing operation” on the state $\rho_x$. A measurement $\{\Lambda_x\}_{x \in X}$ is called logically reversible [10] if the premeasurement state $\rho$ of the measured system is uniquely determined from the postmeasurement state $\rho_x$ and the outcome of the measurement. Ueda et al. in Ref. [10] have shown that the measurement $\{\Lambda_x\}_{x \in X}$ is logically reversible if and only if each measurement

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operator $\Lambda_x$ is a reversible operator. Moreover, if for each measurement operator $\Lambda_x$, there exists a unitary operator $U_x$ such that

$$U_x \rho_s U_x^* = \rho,$$

(1.1)

for each state $\rho$ whose support lies on a subspace $M$ of $\mathcal{H}$, then $\{\Lambda_x\}_{x \in X}$ is called the unitarily reversible measurement [4]. It is clear that any von Neumann measurement $\{\Pi_x\}_{x \in X}$ is not logically reversible except $X$ has only a single element. Note that in a logically reversible measurement, the system’s information is preserved during the measurement process. Thus, the reversibility of a measurement is related to the information gained from that measurement. Quantum teleportation [11] can be seen as the problem of reversing a set of quantum operations [4].

Suppose we are given a logically reversible measurement $\Lambda_u = \{\Lambda_{u,x}\}_{x \in X}$. Since each measurement operator $\Lambda_{u,x}$ is a positive (reversible) operator, then, by the spectral decomposition theorem,

$$\Lambda_{u,x} = \sum_{i \in \Sigma_x} a_x(i) \Pi_x(i),$$

(1.2)

where $\sum_{i \in \Sigma_x} \Pi_x(i) = 1_H$ and $a_x(i) > 0$ for any $i \in \Sigma_x$. In particular, if for all $x \in X$ there exist subsets $\{i_s\}_{s=1}^{m_x} \subseteq \Sigma_x$ such that $\sum_{s=1}^{m_x} \Pi_x(i_s)$ are the same projector onto a subspace $M$ and $a_x(i_1) = \cdots = a_x(i_{m_x})$, then the measurement $\Lambda_u$ is also a unitarily reversible on the subspace $M$ [4].

The success probability $p_s$ of reversing, after the measurement with result $x$, has the upper bound [12] [13]

$$p_s \leq \frac{\min_{i \in \Sigma_x} \{a_x(i)\}}{p_u(x)},$$

(1.3)

where $p_u(x) = Tr(\Lambda_{u,x} \rho)$. If we define the total success probability $p_{s_{\text{total}}}$ of reversing as

$$p_{s_{\text{total}}} = \sum_{x \in X} p_u(x) p_s,$$

(1.4)

then

$$p_{s_{\text{total}}} \leq \sum_{x \in X} \min_{i \in \Sigma_x} \{a_x(i)\}.$$

(1.5)

Note that the above bound is independent of the quantum state $\rho$.

**Shannon and von Neumann entropies.**– A classical state is described by a probability distribution. Shannon entropy $H(p)$, for the probability distribution $p = \{p(x)\}_{x \in X}$, is defined by [14]

$$H(p) = -\sum_{x \in X} p(x) \log_2 p(x).$$

(1.6)

For a quantum state $\rho \in D(\mathcal{H})$, the quantum analog of Shannon entropy is von Neumann entropy, and is given by

$$S(\rho) = -Tr(\rho \log_2 \rho).$$

(1.7)

An equivalent expression of $S(\rho)$ is [7],

$$S(\rho) = \min_{\{\{\psi_a\}, \rho_a\}} H(\{\rho_a\}),$$

(1.8)

where the minimum is taken over all pure state convex decompositions of $\rho$. A decomposition minimizes $\{H(\{\rho_a\}) : \{\{\psi_a\}, \rho_a\}\}$ if and only if it is a spectral decomposition of $\rho$. For an arbitrary ensemble $\{\rho_i, \eta_i\}$, which forms a convex decomposition of $\rho$, we have

$$S(\rho) \leq H(\{\eta_i\}) + \sum_i \eta_i S(\rho_i).$$

(1.9)

The equality is achieved if and only if $\{\rho_i\}$ has mutual orthogonal supports.

**Quantum discord.**– Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be (the Hilbert spaces of) two quantum systems, $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a quantum state, $\rho_A$ and $\rho_B$ be the reduced states of $\rho_{AB}$. In quantum information theory, quantum mutual information

$$I_{A:B}(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}),$$

(1.10)

is regarded as a measure of the total correlation [15] between $\mathcal{H}_A$ and $\mathcal{H}_B$. With the quantum conditional entropy, $S(\rho_B|\rho_A) = S(\rho_{AB}) - S(\rho_A)$, quantum mutual information becomes

$$I_{A:B}(\rho_{AB}) = S(\rho_B) - S(\rho_B|\rho_A).$$
Given a von Neumann measurement \( \Pi^A = \{\Pi^A_x\}_{x \in X} \) on the quantum system \( \mathcal{H}_A \), let us define a conditional entropy on the quantum system \( \mathcal{H}_B \) by \( S_{B|A}(\rho_{AB})|\{\Pi^A_x\}| = \sum_x \eta_x S(\rho_{B|x}) \), where \( \rho_{B|x} = \eta^{-1}_x \text{Tr}_A(\Pi^A_x \otimes 1_{\mathcal{H}_B} \rho_{AB}) \) and \( \eta_x = \text{Tr}(\Pi^A_x \otimes 1_{\mathcal{H}_B} \rho_{AB}) \). Moreover, we denote by

\[
J^{\text{SN}}_{B|A}(\rho_{AB}) = S(\rho_B) - \inf_{\Pi^A} \sum_x \eta_x S(\rho_{B|x}), \tag{I.11}
\]

which is interpreted as a measure of classical correlation [16, 17] between \( \mathcal{H}_A \) and \( \mathcal{H}_B \). In general, \( I_{AB}(\rho_{AB}) \) and \( J^{\text{SN}}_{B|A}(\rho_{AB}) \) are different, and the difference between them

\[
D^{\text{SN}}_{A}(\rho_{AB}) = I_{AB}(\rho_{AB}) - J^{\text{SN}}_{B|A}(\rho_{AB}) \tag{I.12}
\]

is called quantum discord, which is interpreted as a measure of quantum correlation [16, 17, 18]. It is an important information-theoretic measure of quantum correlation [19], beyond entanglement measures [20].

Moreover, if we replace the von Neumann measurement in (I.12) with the generalized quantum measurement \( M^A = \{M^A_x\}_{x \in Z} \) on \( \mathcal{H}_A \), then the general quantum discord can be defined as follows:

\[
D_A(\rho_{AB}) = S(\rho_A) - S(\rho_{AB}) + \inf_{M^A} \sum_z \eta_z S(\rho_{B|z}),
\]

where \( \rho_{B|z} = \eta_{z}^{-1} \text{Tr}_A(\Lambda^A_x \otimes 1_{\mathcal{H}_B} \rho_{AB}) \) and \( \eta_z = \text{Tr}(M^A_x \otimes 1_{\mathcal{H}_B} \rho_{AB}) \). Clearly, \( D_A(\rho_{AB}) \leq D^{\text{SN}}_{A}(\rho_{AB}) \).

Recall that, a purification \( \rho \in \mathcal{D}(\mathcal{H}_A) \) is any pure state \( \langle \phi_p | \phi_p \rangle \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B) \) such that \( \text{Tr}_B(\langle \phi_p | \phi_p \rangle) = \rho \). It, then, follows from Neumark theorem and the additivity of von Neumann entropy with respect to tensor products, that

\[
D_A(\rho_{AB}) = D^{\text{SN}}_{AE}(\rho_{AB} \otimes |e_0\rangle\langle e_0|) \tag{I.13}
\]

This paper is organized as follows. Section [II] deals with the construction of a class of logically reversible measurements based on a von Neumann measurement, and provides a relationship between Shannon entropies of the two measurements. Section [III] presents an inequality between quantum discord induced by the two measurements. Conclusion is presented in Section [IV].

II. LOGICALLY REVERSIBLE MEASUREMENTS

In this section, we show that it is possible to construct a logically reversible measurement from any given von Neumann measurement, and establish a compact relation between Shannon entropies induced by the two measurements.

Let \( \rho \in \mathcal{D}(\mathcal{H}) \) and \( \Pi = \{\Pi_x\}_{x \in X} \) be a von Neumann measurement with \( |X| = n \). Now, based on \( \Pi \) and any \( a \in (0, 1/n) \), we can construct the following logically reversible measurement \( \Lambda^{(a)}_u \) of \( X \):

\[
\Lambda^{(a)}_{u,x} = \{1 - (n - 1)a\} \Pi_x + \sum_{y \neq x} a \Pi_y. \tag{II.1}
\]

The probability distribution \( p^{(a)}(x) = \{p^{(a)}(x)\}_{x \in X} \) is induced, and the probability \( p^{(a)}(x) \) of the classical outcome \( x \) to occur is given by

\[
p^{(a)}(x) = \text{Tr}(\Lambda^{(a)}_{u,x} \rho) = (1 - na)p(x) + a, \tag{II.2}
\]

where \( p(x) = \text{Tr}(\Pi_x \rho) \). It is easy to show that the measurement \( \Lambda^{(a)}_u \) is not unitarily reversible on any subspace \( \mathcal{M} \) with \( \dim \mathcal{M} \neq 1 \) of \( \mathcal{H} \). Note that the total success probability of reversing, after the original von Neumann measurement \( \Pi \), is zero. However, by inequality (I.5), the total success probability \( p^{\text{total}} \) of reversing, after the logically reversible measurement \( \Lambda^{(a)}_u \), has the nonzero upper bound

\[
p^{\text{total}} \leq na. \tag{II.3}
\]

Below, in Proposition II.1, we give an important relationship between Shannon entropies induced by the two measurements. We will adopt the notation, \( H(p) := H(\{p(x)\}) \).
Proposition II.1. For \( \rho \in D(\mathcal{H}) \), and the logically reversible measurement \( \Lambda_u^{(a)} = \{ \Lambda_u^{(a)} \}_{x \in X} \) which is induced by a von Neumann measurement \( \Pi = \{ \Pi_x \}_{x \in X} \) where \( |X| = n \) and \( a \in (0, \frac{1}{n}) \), we observe the relation

\[
H(p_u^{(a)}) - n[\max \{ f(a), f(1-na + a) \}] \leq H(p) \leq H(p_u^{(a)}),
\]

where \( p_u^{(a)}(x) = Tr(\Lambda_u^{(a)} \rho) = (1-na)p(x) + a \), \( p(x) = Tr(\Pi_x \rho) \) and \( f(x) = -x \log_2 x \).

Proof. Let us consider the following two sets:

\[
A = \{ x | p(x) \leq \frac{1}{n} \}, \quad B = \{ x | p(x) > \frac{1}{n} \}.
\]

Note that for positive numbers \( p \leq \frac{1}{n} \) and \( q > \frac{1}{n} \), we have

\[
p \leq (1-na)p + a, \quad q > (1-na)q + a.
\]

Therefore,

\[
0 \leq \sum_{x \in A} (p_u^{(a)}(x) - p(x)) = \sum_{x \in B} (p(x) - p_u^{(a)}(x)).
\]

To prove the other half of Proposition II.1, we consider two cases separately.

Case 1: If \( p(x) > \frac{1}{n} \), let \( a_2 = (1-na)p(x) + a = p_u^{(a)}(x) \), \( \beta_2 = p(x) \) and \( \gamma_2 = 1-na + a \). Then \( \frac{1}{n} < a_2, \beta_2, \gamma_2 \leq 1, 0 < \gamma_2 - a_2 \leq 1 - \beta_2 \), and \( f(a_2) \leq f(\beta_2) + f(\gamma_2) \) (see Fig. 1(a)). Hence,

\[
-p_u^{(a)}(x) \log_2 p_u^{(a)}(x) \leq -p(x) \log_2 p(x) - (1-na + a) \log_2 (1-na + a).
\]

Case 2: If \( p(x) \leq \frac{1}{n} \), let \( a_3 = (1-na)p(x) + a, \beta_3 = p(x) \), and \( \gamma_3 = a \). Then \( 0 < a_3, \beta_3, \gamma_3 \leq \frac{1}{n}, a_3 \leq \beta_3 + \gamma_3 \), and \( f(a_3) \leq f(\beta_3) + f(\gamma_3) \) (see Fig. 1(b)). Hence,

\[
-p_u^{(a)}(x) \log_2 p_u^{(a)}(x) \leq -p(x) \log_2 p(x) - a \log_2 a.
\]

Now, summing (II.5) and (II.6) over allowed probabilities and adding them, we obtain

\[
H(p_u^{(a)}) - n \left[ \max \{ f(a), f(1-na + a) \} \right] \leq H(p).
\]

Combining (II.4) and (II.7), the proposition is proved.

Remark II.2. Note that \( \lim_{a \to 0} \max \{ f(a), f(1-na + a) \} = 0 \). So, it follows from Proposition II.1 that \( \lim_{a \to 0} H(p_u^{(a)}) = H(p) \). This is expected because when \( a \to 0 \), \( \Lambda_u^{(a)} \to \Pi \).
III. QUANTUM DISCORD FOR LOGICALLY REVERSIBLE MEASUREMENTS

In this section, we study quantum discord with respect to logically reversible measurement $\Lambda^{(a,A)}_u$ on $\mathcal{H}_A$, where $\dim \mathcal{H}_A = n$ and $a \in (0, \frac{1}{n})$. Quantum discord of state $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ for the logically reversible measurement is defined by

$$D^{(a)}_{u,A}(\rho_{AB}) = I_{A:B}(\rho_{AB}) - J^{(a)}_{u,A|B}(\rho_{AB}),$$

where

$$J^{(a)}_{u,A|B}(\rho_{AB}) = S(\rho_B) - \inf_{\Lambda^{(a,A)}_u} \sum_x \eta_{u,x} S(\rho^{(a)}_{u,B|X}),$$

$$\rho^{(a)}_{u,B|X} = \eta_{u,x}^{-1} Tr_A(\Lambda^{(a,A)}_u \otimes I_B \rho_{AB}),$$

$$\eta_{u,x} = Tr(\Lambda^{(a,A)}_{u,x} \otimes I_B \rho_{AB}).$$

In the following, we establish an important relation between quantum discord for von Neumann measurement, $D^{vN}_A(\rho_{AB})$, and quantum discord for logically reversible measurement, $D^{(a)}_{u,A}(\rho_{AB})$. For this we need Lemma III.1.

Lemma III.1. Let $\rho, \rho_1, \rho_2 \in D(\mathcal{H}_A)$, $p_0 + p_1 + p_2 = 1$, $\rho = (p_0 + p_1)\rho_1 + p_2\rho_2$, and $H_0(r) = -r \log_2 r - (1-r) \log_2 (1-r)$ for any $r \in [0,1]$. Then,

$$S(\rho) \leq p_0 S(\rho_1) + (p_1 + p_2) S \left( \frac{p_1}{p_1 + p_2} \rho_1 + \frac{p_2}{p_1 + p_2} \rho_2 \right) - (p_1 + p_2) H_0 \left( \frac{p_1}{p_1 + p_2} \right) + H_0(p_2).$$

Proof. Let us introduce two quantum systems $\mathcal{H}_B$ and $\mathcal{H}_C$, and construct a quantum state $\rho_{ABC} \in D(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ as $\rho_{ABC} = p_0 \rho_1 \otimes |0\rangle_B \langle 0| \otimes |0\rangle_C \langle 0| + p_1 \rho_1 \otimes |0\rangle_B \langle 0| \otimes |1\rangle_C \langle 1| + p_2 \rho_2 \otimes |1\rangle_B \langle 1| \otimes |1\rangle_C \langle 1|$. Then, we have $S(\rho_A) = S(\rho)$, $S(\rho_{AB}) = H_0(p_2) + (p_0 + p_1) S(\rho_1) + p_2 S(\rho_2)$, $S(\rho_{AC}) = H_0(p_0) + p_0 S(\rho_1) + (p_1 + p_2) S \left( \frac{p_1}{p_1 + p_2} \rho_1 + \frac{p_2}{p_1 + p_2} \rho_2 \right)$, and $S(\rho_{ABC}) = H(p) + (p_0 + p_1) S(\rho_1) + p_2 S(\rho_2)$, where the probability distribution $p = (p_0, p_1, p_2)$. Now, exploiting the strong subadditivity of von Neumann entropy $\mathcal{B}$, $S(\rho_{ABC}) + S(\rho_A) \leq S(\rho_{AB}) + S(\rho_{AC})$, and simplifying we obtain the desired result.

Theorem III.2. For $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ with $\dim \mathcal{H}_A = n$, $a \in (0, \frac{1}{n})$, and the probability distribution $p_{u,a} = \{(1 - (n-1)a), \cdots, a\}$, we have

$$D^{(a)}_{u,A}(\rho_{AB}) = \frac{na J^{(a)}_{u,A|B}(\rho_{AB})}{1 - na} - H(p_{u,a}),$$

$$\leq D^{vN}_A(\rho_{AB}) = \frac{na J^{(a)}_{u,A|B}(\rho_{AB})}{1 - na}.$$

Proof. Let $\Lambda^{(a,A)}_u = \{\Lambda^{(a,A)}_{u,x}\}_{x \in X}$ be the logically reversible measurement induced by von Neumann measurement $\Pi^A = \{\Pi^A_x\}_{x \in X}$. That is,

$$\Lambda^{(a,A)}_{u,x} = (1 - (k - 1)a)\Pi^A_x + \sum_{y \neq x} a\Pi^A_y,$$

where $k = |X|$ and $a \in (0, \frac{1}{n})$. Then the conditional state is

$$\rho^{(a)}_{u,B|x} = \eta_{u,x}^{-1} Tr_A(\Lambda^{(a,A)}_{u,x} \otimes I_B \rho_{AB}) = \frac{(1 - ka)\eta_x}{\eta_{u,x}} \rho_{B|x} + \frac{a}{\eta_{u,x}} \rho_B,$$

where $\eta_{u,x} = \eta_{u,x}^{-1} \eta_x + a$, $\eta_x = Tr(\Pi^A_x \otimes I_B \rho_{AB})$, and $\rho_B = Tr_A(\rho_{AB})$. Thus, by the concavity of von Neumann entropy, we have

$$\frac{(1 - ka)\eta_x}{\eta_{u,x}} S(\rho_{B|x}) + \frac{a}{\eta_{u,x}} S(\rho_B) \leq S(\rho^{(a)}_{u,B|x}),$$

implying

$$(1 - ka) \sum_{x \in X} \eta_{u,x} S(\rho_{B|x}) + ka S(\rho_B) \leq \sum_{x \in X} \sum_{x \in X} \eta_{u,x} S(\rho^{(a)}_{u,B|x}).$$

(III.1)

Because $\inf_{\Pi^A} \sum_x \eta_{u,x} S(\rho_{B|x})$ is achieved on rank-one projectors, $k = |X| = \dim \mathcal{H}_A = n$. Therefore, using (III.1), we have

$$\frac{J^{(a)}_{u,A|B}(\rho_{AB})}{1 - na} \leq J^{vN}_{B|A}(\rho_{AB}).$$

(III.2)
Besides, if we denote by $\rho_{B|X\{x\}} = \frac{\rho_B - \eta p_B|x}{1 - \eta}$, then $\rho_{u,B|x} = \frac{(1 - (n - 1)a)\rho_B}{\eta_{u,x}} + \frac{a(1 - \eta)}{\eta_{u,x}} \rho_{B|X\{x\}}$, and $\rho_B = \eta_s p_B|x + (1 - \eta)\rho_{B|X\{x\}}$.

$\rho_{u,B|x} = \frac{(1 - (n - 1)a)\rho_B}{\eta_{u,x}} + \frac{a(1 - \eta)}{\eta_{u,x}} \rho_{B|X\{x\}}$

Let $p_0 = \frac{(1-na)\rho_B}{\eta_{u,x}}$, $p_1 = \frac{a\rho_B}{\eta_{u,x}}$, $p_2 = \frac{a(1-\eta)}{\eta_{u,x}}$, $p_1 = \rho_{B|x}$, $\rho_2 = \rho_{B|X\{x\}}$. Then, by Lemma III.1 we have

$S(\rho_{u,B|x}) \leq \frac{(1-na)\eta_s}{\eta_{u,x}}S(\rho_{B|x}) + \frac{a}{\eta_{u,x}} S(\rho_B) - \frac{a}{\eta_{u,x}} H_0(\eta_s) + H_0(\frac{a(1-\eta)}{\eta_{u,x}})$.

After simple algebra, and using (II.4), we obtain

$$\sum_{x \in X} \eta_{u,x} S(\rho_{u,B|x}) \leq (1 - na) \sum_{x \in X} \eta_s S(\rho_{B|x}) + naS(\rho_B) + H(p_{n,a}) - naH(\eta_s),$$

where $H(p_{n,a}) = -(1 - (n - 1)a) \log_2(1 - (n - 1)a) - (n - 1)a \log_2 a$ and $H(\eta_s) = -\sum_{x \in X} \eta_{u,x} \log_2 \eta_{u,x}$. Also, since $a \leq \eta_{u,x} \leq 1 - (n - 1)a$ for all $x$, we have $H(p_{n,a}) \leq H(\eta_s)$.

Therefore,

$$\sum_{x \in X} \eta_{u,x} S(\rho_{u,B|x}) \leq (1 - na) \sum_{x \in X} \eta_s S(\rho_{B|x}) + naS(\rho_B) + (1 - na)H(p_{n,a}),$$

and

$$ \mathcal{J}_{AB}^{vn}(\rho_{AB}) = S(\rho_{AB}) - \inf_{\Pi_A} \sum_{x \in X} \eta_s S(\rho_{B|x}) \leq \frac{\mathcal{J}_{u,B|A}^{na}(\rho_{AB})}{1 - na} + H(p_{n,a}).$$

Now,

$$D_{u,A}(\rho_{AB}) - \frac{na\mathcal{J}_{u,B|A}^{na}(\rho_{AB})}{1 - na} - H(p_{n,a})$$

$$= I_{A:B}(\rho_{AB}) - \frac{\mathcal{J}_{u,B|A}^{na}(\rho_{AB})}{1 - na} - H(p_{n,a})$$

$$\leq I_{A:B}(\rho_{AB}) - \mathcal{J}_{B|A}^{vn}(\rho_{AB}) = D_{A}^{vn}(\rho_{AB})$$

$$\leq I_{A:B}(\rho_{AB}) - \frac{\mathcal{J}_{u,B|A}^{na}(\rho_{AB})}{1 - na}$$

$$= D_{u,A}(\rho_{AB}) - \frac{na\mathcal{J}_{u,B|A}^{na}(\rho_{AB})}{1 - na},$$

where the first inequality is due to (II.4), and the second inequality follows from (III.2). Hence, the proof.

Remark III.3. Note that $\mathcal{J}_{u,B|A}^{na}(\rho_{AB}) > 0$, and from

Theorem III.2 we have

$$D_{u,A}(\rho_{AB}) - \frac{na\mathcal{J}_{u,B|A}^{na}(\rho_{AB})}{1 - na} \geq D_{A}^{vn}(\rho_{AB}),$$

where $a \in (0, 1/n)$. Hence,

$$D_{u,A}(\rho_{AB}) - D_{A}^{vn}(\rho_{AB}) \geq \frac{na\mathcal{J}_{u,B|A}^{na}(\rho_{AB})}{1 - na} > 0.$$

Thus, with the logically reversible measurement, one can extract more quantum discord than with the von Neumann measurement. See also [21].

IV. CONCLUSION

In summary, we constructed the logically reversible measurement based on the von Neumann measurement. We then established relationships for
Shannon entropies, and quantum discord with respect to these two measurements. In particular, we showed that quantum discord for the logically reversible measurement exceeds that for the von Neumann measurement.

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