TWO-SIDED ESTIMATES OF TOTAL BANDWIDTH FOR SCHRÖDINGER OPERATORS ON PERIODIC GRAPHS

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Abstract. We consider Schrödinger operators with periodic potentials on periodic discrete graphs. Their spectrum consists of a finite number of bands. We obtain two-sided estimates of the total bandwidth for the Schrödinger operators in terms of geometric parameters of the graph and the potentials. In particular, we show that these estimates are sharp. It means that these estimates become identities for specific graphs and potentials. The proof is based on the Floquet theory and trace formulas for fiber operators. The traces are expressed as finite Fourier series of the quasimomentum with coefficients depending on the potentials and cycles of the quotient graph from some specific cycle sets. In order to obtain our results we estimate these Fourier coefficients in terms of geometric parameters of the graph and the potentials.

1. Introduction

Laplace and Schrödinger operators on graphs have a lot of applications in physics, chemistry and engineering. They are used to study properties of different periodic media, e.g. nanomedia, see [25] (two authors of this survey, Novoselov and Geim, won the Nobel Prize for discovering graphene). We consider Schrödinger operators \( H = -\Delta + V \) on periodic graphs, where \( \Delta \) is the Laplacian and \( V \) is a periodic potential. On a discrete graph, the Laplace operator acts in the space of functions defined on the set of vertices of the graph. Here vertices of the graph play a crucial role, while edges show the interaction between the vertices. Basically, two types of the discrete Laplace operator have been studied: the combinatorial Laplacian and the normalized Laplacian. The spectrum of \( H \) is a finite union of spectral bands \( \sigma_j, j = 1, \ldots, \nu \). Our main goal is to obtain two-sided estimates for the total bandwidth \( \sum_{j=1}^{\nu} |\sigma_j| \) in terms of geometric parameters of the graphs and the potentials.

There are a lot of results about spectral properties for the one-dimensional case. Most of the results were obtained for the lattice \( \mathbb{Z} \), since there are applications to Toda lattices, see, e.g., [28]. Here corresponding operators of the Lax pairs are Jacobi matrices with periodic coefficients. We mention that Last [21] determined two-sided sharp estimates (2.10) for the total bandwidth of Schrödinger operators with periodic potentials on \( \mathbb{Z} \), see also [5], [12] and references therein.

In the continuous case we know only estimates for Schrödinger operators with periodic potentials on the real line. Korotyaev determined two-sided estimates for gap lengths of Schrödinger operators with periodic potentials [8], [9]. In [10] Korotyaev obtained various estimates of spectral bands and, in particular, estimates of action variables for the KdV equation on the circle. The case of matrix-valued potentials was discussed by Chelkak and Korotyaev [4]. It is important that the proof for these cases was based on the conformal mapping theory (associated with quasimomentum) and trace formulas.

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We discuss the multidimensional case. We do not know estimates for the continuous case. We consider the discrete case, where there are estimates of the total bandwidth. It is known that the Lebesgue measure $|\sigma(\Delta)|$ of the spectrum of the combinatorial Laplacian $\Delta$ on periodic graphs satisfies $|\sigma(\Delta)| \leq 2\kappa_+$, where $\kappa_+$ is the maximum vertex degree of the graph. But, in contrast to the one-dimensional case, the bands may overlap. Thus, the total bandwidth may exceed this number. Upper estimates of the total bandwidth for the Schrödinger operator with a periodic potential in terms of geometric parameters of the graph were obtained in [14, 17]. We do not know results about lower estimates of the total bandwidth of the Laplace and Schrödinger operators on periodic graphs.

In this paper we consider Laplace and Schrödinger operators with periodic potentials on periodic discrete graphs. We obtain two-sided estimates of the total bandwidth for the Schrödinger operator in terms of geometric parameters of the graph and the potentials. We show that these estimates are sharp, i.e., they become identities for specific graphs and potentials. The proof is based on the Floquet theory and recent trace formulas for fiber operators from [20].

1.1. Schrödinger operators on periodic graphs. Let $G = (\mathcal{V}, \mathcal{E})$ be a connected infinite graph, possibly having loops and multiple edges and embedded into the space $\mathbb{R}^d$. Here $\mathcal{V}$ is the set of its vertices and $\mathcal{E}$ is the set of its unoriented edges. Considering each edge in $\mathcal{E}$ to have two orientations, we introduce the set $\mathcal{A}$ of all oriented edges. An edge starting at a vertex $x$ and ending at a vertex $y$ from $\mathcal{V}$ will be denoted as the ordered pair $(x, y) \in \mathcal{A}$ and is said to be incident to the vertices. Let $e = (y, x)$ be the inverse edge of $e = (x, y) \in \mathcal{A}$. Vertices $x, y \in \mathcal{V}$ will be called adjacent and denoted by $x \sim y$, if $(x, y) \in \mathcal{A}$. We define the degree $\kappa_x$ of the vertex $x \in \mathcal{V}$ as the number of all edges from $\mathcal{A}$, starting at $x$.

Let $\Gamma$ be a lattice of rank $d$ in $\mathbb{R}^d$ with a basis $\{a_1, \ldots, a_d\}$, i.e.,

$$\Gamma = \{a : a = \sum_{s=1}^{d} n_s a_s, (n_s)_{s=1}^{d} \in \mathbb{Z}^d\},$$

and define the fundamental cell $\Omega$ of the lattice $\Gamma$ by

$$\Omega = \{x \in \mathbb{R}^d : x = \sum_{s=1}^{d} x_s a_s, (x_s)_{s=1}^{d} \in [0, 1)^d\}. \quad (1.1)$$

We introduce the equivalence relation on $\mathbb{R}^d$: $x \equiv y \ (\text{mod } \Gamma) \iff x - y \in \Gamma \quad \forall x, y \in \mathbb{R}^d$.

We consider locally finite $\Gamma$-periodic graphs $G$, i.e., graphs satisfying the following conditions:

1) $G = G + a$ for any $a \in \Gamma$;
2) the quotient graph $G_\Gamma = G/\Gamma$ is finite.

The basis $a_1, \ldots, a_d$ of the lattice $\Gamma$ is called the periods of $G$. We also call the quotient graph $G_\Gamma = G/\Gamma$ the fundamental graph of the periodic graph $G$. The fundamental graph $G_\Gamma$ is a graph on the $d$-dimensional torus $\mathbb{R}^d/\Gamma$. The graph $G_\Gamma = (\mathcal{V}_\Gamma, \mathcal{E}_\Gamma)$ has the vertex set $\mathcal{V}_\Gamma = \mathcal{V}/\Gamma$, the set $\mathcal{E}_\Gamma = \mathcal{E}/\Gamma$ of unoriented edges and the set $\mathcal{A}_\Gamma = \mathcal{A}/\Gamma$ of oriented edges which are finite.

Let $L^2(\mathcal{V})$ be the Hilbert space of all square summable functions $f : \mathcal{V} \to \mathbb{C}$ equipped with the norm

$$\|f\|^2_{L^2(\mathcal{V})} = \sum_{x \in \mathcal{V}} |f(x)|^2 < \infty.$$
We consider the Schrödinger operator $H$ acting on the Hilbert space $l^2(V)$ and given by

$$H = H_o + V, \quad H_o = -\Delta, \quad (1.2)$$

where the potential $V$ is real valued and satisfies for all $(x,a) \in V \times \Gamma$:

$$(Vf)_x = V_x f_x, \quad V_{x+a} = V_x, \quad (1.3)$$

and $\Delta$ is the combinatorial Laplacian on $f \in l^2(V)$ defined by

$$(\Delta f)_x = \sum_{(x,y) \in A} (f_x - f_y), \quad x \in V. \quad (1.4)$$

The sum in (1.4) is taken over all oriented edges starting at the vertex $x$. It is well known that $\Delta$ is self-adjoint and the point $0$ belongs to its spectrum $\sigma(\Delta)$ contained in $[0, 2\kappa_+)$, i.e.,

$$0 \in \sigma(\Delta) \subseteq [0, 2\kappa_+), \quad \text{where} \quad \kappa_+ = \max_{x \in V} \kappa_x < \infty, \quad (1.5)$$

and $\kappa_x$ is the degree of the vertex $x$, see, e.g., [24].

1.2. Edge and cycle indices. Here we will define an edge index. It was introduced in [14] in order to express fiber Laplacians and Schrödinger operators on the fundamental graph in terms of edge indices, see (1.18). For each $x \in \mathbb{R}^d$ we introduce the vector $x_\Lambda \in \mathbb{R}^d$ by

$$x_\Lambda = (x_1, \ldots, x_d), \quad \text{where} \quad x = \sum_{s=1}^{d} x_s a_s, \quad (1.6)$$

i.e., $x_\Lambda$ is the coordinate vector of $x$ with respect to the basis $\Lambda = \{a_j\}_{j=1}^{d}$ of the lattice $\Gamma$. For any vertex $x \in V$ of a $\Gamma$-periodic graph $\mathcal{G}$ the following unique representation holds true:

$$x = x_0 + [x], \quad \text{where} \quad (x_0, [x]) \in \Omega \times \Gamma, \quad (1.7)$$

and $\Omega$ is a fundamental cell of the lattice $\Gamma$ defined by (1.1). In other words, each vertex $x$ can be obtained from a vertex $x_0 \in \Omega$ by a shift by a vector $[x] \in \Gamma$. For any oriented edge $e = (x,y) \in A$ we define the edge index $\tau(e)$ as the vector of the lattice $\mathbb{Z}^d$ given by

$$\tau(e) = [y]_\Lambda - [x]_\Lambda \in \mathbb{Z}^d, \quad (1.8)$$

where $[\cdot] \in \Gamma$ is defined by (1.7) and the vector $[x]_\Lambda \in \mathbb{Z}^d$ is given by (1.6).

On the set $A$ of all oriented edges of the $\Gamma$-periodic graph $\mathcal{G}$ we define the surjection

$$f : A \to A_s = A/\Gamma, \quad (1.9)$$

which maps each $e \in A$ to its equivalence class $e_s = f(e)$ which is an oriented edge of the fundamental graph $\mathcal{G}_s$. For any oriented edge $e_s \in A_s$ we define the edge index $\tau(e_s) \in \mathbb{Z}^d$ by

$$\tau(e_s) = \tau(e) \quad \text{for some} \quad e \in A \quad \text{such that} \quad e_s = f(e), \quad e_s \in A_s, \quad (1.10)$$

where $f$ is defined by (1.9). Thus, edge indices of the fundamental graph $\mathcal{G}_s$ are induced by edge indices of the periodic graph $\mathcal{G}$. The edge index $\tau(e_s)$ is uniquely determined by (1.10), since

$$\tau(e + a) = \tau(e), \quad \forall (e,a) \in A \times \Gamma.$$

A path $p$ in a graph $\mathcal{G} = (V,A)$ is a sequence of consecutive edges

$$p = (e_1, e_2, \ldots, e_n), \quad \text{where} \quad e_s = (x_{s-1}, x_s) \in A, \quad s = 1, \ldots, n, \quad (1.11)$$

for some vertices $x_0, x_1, \ldots, x_n \in V$. The vertices $x_0$ and $x_n$ are called the initial and terminal vertices of the path $p$, respectively. If $x_0 = x_n$, then the path $p$ is called a cycle. The number
$n$ of edges in a cycle $c$ is called the length of $c$ and is denoted by $|c|$, i.e., $|c| = n$. The reverse of the path $p$ given by (1.11) is the path $p^* = (e_n, \ldots, e_1)$.

**Remark.** A path $p$ is uniquely defined by the sequence of its oriented edges $(e_1, e_2, \ldots, e_n)$. The sequence of its vertices $(x_0, x_1, \ldots, x_n)$ does not uniquely define $p$, since multiple edges are allowed in the graph $G$.

Let $C$ be the set of all cycles of the fundamental graph $G_\ast$. For any cycle $c \in C$ we define the cycle index $\tau(c) \in \mathbb{Z}^d$ by

$$\tau(c) = \sum_{e \in c} \tau(e), \quad c \in C.$$  

(1.12)

From the definition of indices it follows that

$$\tau(e) = -\tau(e), \quad \forall e \in A, \text{ and } \tau(c) = -\tau(c), \quad \forall c \in C.$$  

(1.13)

![Figure 1](image)

**Figure 1.** The hexagonal lattice $G$ and its fundamental graph $G_\ast$ with edge indices; $a_1, a_2$ are the periods of $G$. The fundamental cell $\Omega$ is shaded. The vertices $v_1, v_2$ of $G$ from $\Omega$ are black points.

Edge indices depend on the choice of the embedding of the periodic graph into $\mathbb{R}^2$ (i.e., the choice of $\Omega$). Cycle indices do not depend on this choice.

**Remarks.** 1) Edge indices depend on the choice of the embedding of the periodic graph $G$ into $\mathbb{R}^d$ (see Fig. 1). Cycle indices do not depend on this choice. This property of cycle indices has the following simple explanation. Any cycle $c$ in the fundamental graph $G_\ast$ is obtained by factorization of a path in the $\Gamma$-periodic graph $G$ connecting some $\Gamma$-equivalent vertices $x \in V$ and $x + a \in V$, $a \in \Gamma$. The index of the cycle $c$ is equal to $m = (m_j)^{d-1}$, where $a = m_1a_1 + \ldots + m_4a_4$, and, therefore, does not depend on the choice of the embedding.

2) The index of a cycle $c \in C$ represents the coordinates of $c$ with respect to some basis of the space $C^+ \subset C$ of cycles with non-zero indices.

1.3. **Invariant $I$.** Edges with non-zero indices are called bridges. They provide connectivity of the periodic graph and removal of all bridges disconnects the periodic graph into infinitely many connected components (see Fig. 2). Let $B_\ast$ be the set of all bridges of the fundamental graph $G_\ast = (V_\ast, A_\ast)$, i.e., $B_\ast = \text{supp } \tau$, where $\tau : A_\ast \rightarrow \mathbb{Z}^d$ is the index form defined by (1.8), (1.10). The number of the fundamental graph bridges depends on the choice of the embedding.
of the periodic graph $G$ into the space $\mathbb{R}^d$, i.e., this number is not an invariant for $G$. We define the number

$$I = \frac{1}{2} \min_{G \subset \mathbb{R}^d} \#B_*, \quad B_* = \text{supp} \tau,$$

(1.14)

where the minimum is taken over all embeddings of the periodic graph $G$ into $\mathbb{R}^d$, and $\#M$ denotes the number of elements in a set $M$. This number $I$ exists, since the fundamental graph is finite and we described $I$ in Theorem 3.2 from [19]. For simple periodic graphs (the $d$-dimensional lattice, the hexagonal lattice, the Kagome lattice, etc.) it is not difficult to determine this number. But for an arbitrary periodic graph this may be a rather complicated problem.

We recall some properties of the number $I$ (see Theorem 2.1.v and Proposition 2.2 in [19]).

**Proposition 1.1.** i) The number $I$ is an invariant of the $\Gamma$-periodic graph $G$, i.e., it does not depend on the choice of

- the embedding of $G$ into $\mathbb{R}^d$;
- the basis $a_1, \ldots, a_d$ of the lattice $\Gamma$.

ii) The invariant $I$ satisfies

$$d \leq I \leq \beta, \quad \beta = \#E_* - \#V_* + 1,$$

(1.15)

where $d$ is the rank of the lattice $\Gamma$ and $\beta$ is the Betti number of the fundamental graph $G_* = (V_*, E_*)$.

Moreover, for any $n \in \mathbb{N}$ there exists a periodic graph such that $\beta - I = n$.

### 1.4. Spectrum of Schrödinger operators

We introduce the Hilbert space

$$\mathcal{H} = L^2\left(T^d, \frac{dk}{(2\pi)^d}, \ell^2(V_*)\right) = \int_{T^d} \ell^2(V_*) \frac{dk}{(2\pi)^d}, \quad T^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d,$$

(1.16)

i.e., a constant fiber direct integral, equipped with the norm $\|g\|_{\mathcal{H}}^2 = \int_{T^d} \|g(k)\|^2_{\ell^2(V_*)} \frac{dk}{(2\pi)^d}$, where the function $g(k) \in \ell^2(V_*)$ for almost all $k \in T^d$. Here $\ell^2(V_*) = \mathbb{C}^\nu$ is the fiber space, $\nu = \#V_*$. The parameter $k \in T^d$ is called the quasimomentum. We recall Theorem 1.1 from [14].
Theorem 1.2. The combinatorial Schrödinger operator \( H = -\Delta + V \) on \( \ell^2(\mathcal{V}) \) has the following decomposition into a constant fiber direct integral

\[
UHU^{-1} = \int_{\mathbb{T}^d} H(k) \frac{dk}{(2\pi)^d},
\]

for some unitary operator \( U : \ell^2(\mathcal{V}) \to \mathcal{H} \) (the Gelfand transform). The fiber Schrödinger operator \( H(k) \) on \( \ell^2(\mathcal{V}_s) \) is given by

\[
H(k) = H_o(k) + V, \quad H_o(k) = -\Delta(k), \quad \forall k \in \mathbb{T}^d.
\]

Here \( V \) is the potential on \( \ell^2(\mathcal{V}_s) \), and \( \Delta(k) \) is the fiber Laplacian having the form

\[
(\Delta(k)f)_x = \sum_{e=1}^{\#A_s} (f_x - e^{i\langle\tau(e),k\rangle} f_y), \quad f \in \ell^2(\mathcal{V}_s), \quad x \in \mathcal{V}_s,
\]

where \( \tau(e) \) is the index of the edge \( e \in A_s \) defined by (1.8), (1.10), and \( \langle \cdot, \cdot \rangle \) denotes the standard inner product in \( \mathbb{R}^d \).

Remarks. 1) The fiber Laplacian is expressed in terms of edge indices which are not invariant and depend on the choice of the embedding of the periodic graph \( \mathcal{G} \) into \( \mathbb{R}^d \). But the invariance of indices of the fundamental graph cycles yields that the fiber Laplacians (1.18) written with respect to distinct embeddings of \( \mathcal{G} \) into \( \mathbb{R}^d \) are unitarily equivalent under some gauge transform. Nevertheless, the proper embedding simplifies analysis of the operator \( H \).

2) We can consider the fiber Laplacian \( \Delta(k) \), \( k \in \mathbb{T}^d \), as a discrete magnetic Laplacian with the magnetic vector potential \( \alpha(e) = \langle \tau(e), k \rangle, e \in A_s \), on the fundamental graph \( \mathcal{G}_s \).

3) The minimal number of exponents \( e^{i\langle\tau(e),\cdot\rangle} \neq 1, e \in A_s \), in the identities (1.18) for the fiber Laplacians \( \Delta(\cdot) \) is equal to \( 2\mathcal{I} \), where the invariant \( \mathcal{I} \) is defined by (1.14).

Each fiber operator \( H(k), k \in \mathbb{T}^d \), has \( \nu = \#\mathcal{V} \) real eigenvalues \( \lambda_j(k), j \in \mathbb{N}_\nu = \{1, \ldots, \nu\} \), which are labeled in non-decreasing order (counting multiplicities) by

\[
\lambda_1(k) \leq \lambda_2(k) \leq \ldots \leq \lambda_\nu(k), \quad \forall k \in \mathbb{T}^d.
\]

Each \( \lambda_j(\cdot), j \in \mathbb{N}_\nu \), is a real and piecewise analytic function on the torus \( \mathbb{T}^d \) and creates the spectral band \( \sigma_j(H) \) given by

\[
\sigma_j(H) = [\lambda_j^-, \lambda_j^+] = \lambda_j(\mathbb{T}^d).
\]

Note that \( \lambda_j^+ = \lambda_j(0) \) (see [26]). If \( \lambda_j(\cdot) = \Lambda_j = \text{const} \) on some subset of \( \mathbb{T}^d \) of positive Lebesgue measure, then the operator \( \hat{H} \) on \( \mathcal{G} \) has the eigenvalue \( \Lambda_j \) of infinite multiplicity. We call \( \{\Lambda_j\} \) a flat band. Thus, the spectrum of the Schrödinger operator \( H \) on the periodic graph \( \mathcal{G} \) has the form

\[
\sigma(H) = \bigcup_{k \in \mathbb{T}^d} \sigma(H(k)) = \bigcup_{j=1}^{\nu} \sigma_j(H) = \sigma_{ac}(H) \cup \sigma_{fb}(H),
\]

where \( \sigma_{ac}(H) \) is the absolutely continuous spectrum, which is a union of non-degenerate bands from (1.19), and \( \sigma_{fb}(H) \) is the set of all flat bands (eigenvalues of infinite multiplicity).

We remark that the last spectral band \( \sigma_{ac}(H) \) of the Schrödinger operator \( \hat{H} = -\Delta + V \) on periodic graphs is non-degenerate, see [13]. Moreover, if \( \nu \geq 2 \) and there are no loops with non-zero index in the fundamental graph \( \mathcal{G}_s \), then at least two spectral bands of \( H \) are non-degenerate (see Proposition 2.5 in [14]).
The paper is organized as follows. In Section 2 we formulate our main results:
- estimates of the total bandwidth for the combinatorial Laplace and Schrödinger operators with periodic potentials on periodic graphs (Theorem 2.1 and Corollary 2.2);
- estimates of the total bandwidth for the normalized Laplace operators (Theorem 2.4).

Section 3 is devoted to trace formulas for Laplace and Schrödinger operators on periodic graphs which are used in the proof of the main results. In Section 4 we prove the estimates of the total bandwidth for the combinatorial Laplace and Schrödinger operators. In Section 5 we estimate the total bandwidth for the normalized Laplace operators. In Section 6 we apply the obtained results and estimate the total bandwidth for Laplacians on some specific periodic graphs.

2. Main results

2.1. Overview of the total bandwidth estimates. In order to present our results we describe known total bandwidth estimates for Schrödinger operators $H = H_o + V$ with periodic potentials $V$, where the unperturbed operator $H_o = -\Delta$. There are only upper estimates determined in [14, 17, 19]. We do not know lower estimates.

Define the total bandwidth $S(H)$ of an operator $H$ by

$$S(H) = \sum_{j=1}^{\nu} |\sigma_j(H)|.$$  

In [14] the authors estimated the total bandwidth $S(H)$ of the Schrödinger operator $H$ in terms of geometric parameters of the graph:

$$S(H) \leq 4b, \quad \text{where} \quad b = \frac{1}{2} \# \text{supp} \tau,$$

and $\tau : A_s \to \mathbb{Z}^d$ is the index form defined by (1.8), (1.10). This number $b$ depends essentially on the choice of the embedding of the periodic graph $\mathcal{G}$ into the space $\mathbb{R}^d$, i.e., $b$ is not an invariant for $\mathcal{G}$.

Similar estimates for the normalized Laplacian $\Delta_n$ (see (2.11)) were obtained in [18]:

$$S(\Delta_n) \leq 2b, \quad b = \sum_{x \in V_s} b_x \kappa_x,$$

where $b_x$ is the number of the fundamental graph bridges (i.e., edges with non-zero indices) starting at the vertex $x \in V_s$, and $\kappa_x$ is the degree of $x$.

For the Schrödinger operators $H$ we derived the following estimate in [17]:

$$S(H) \leq 4\beta,$$

where $\beta = \#E_s - \#V_s + 1$ is the Betti number of the fundamental graph $\mathcal{G}_s = (V_s, E_s)$. Similar estimates were obtained for magnetic Schrödinger operators with periodic magnetic potentials in [17].

Finally in [19] the authors introduced the new invariant $I$, see (1.14), and proved that

$$S(H) \leq 4I.$$

Let us compare the estimates (2.1), (2.3) and (2.4). Due to (1.14) and (1.15), the invariant $I$ satisfies

$$I = \min_{G \subset \mathbb{R}^d} b \quad \text{and} \quad I \leq \beta.$$
Moreover, the difference between the Betti number $\beta$ and the invariant $I$ may be any non-negative integer number (for specific graphs), see Proposition [1.1 ii]). The number $b$ can be less, equal or significantly greater (dependently on the embedding of the periodic graph into $\mathbb{R}^d$) than the Betti number $\beta$. Thus, (2.4) gives the best upper estimate. Remark that (2.4) is sharp since we have an identity for specific graphs and potentials, and all estimates do not depend on potentials $V$.

In [18] it was shown that the last band $\sigma_\nu(H) = [\lambda^-_\nu, \lambda^+_\nu]$ of the Schrödinger operator $H = H_o + V$ is non-degenerate, i.e., $\lambda^-_\nu < \lambda^+_\nu$, and the following estimate holds:

$$0 < C^{-1} |\sigma_\nu(H_o)| \leq |\sigma_\nu(H)| \leq C |\sigma_\nu(H_o)|,$$

for a constant $C > 0$ depending on geometric parameters of the graph and the potential $V$.

Estimates on effective masses for Laplace and Schrödinger operators on periodic graphs were determined in a series of papers [11], [16], [18]. A band localization for Laplace and Schrödinger operators on periodic graphs was discussed in [6], [15], [22].

2.2. Estimates for combinatorial Laplace and Schrödinger operators. In order to describe our main results we need the following definitions. Let $w = (w_x)_{x \in V_*}$ be a real potential on $V_*$. Define the minimum and maximum and the diameter of the potential $w$ by

$$w_- = \min_{x \in V_*} w_x, \quad w_+ = \max_{x \in V_*} w_x, \quad \text{diam } w = w_+ - w_- \quad (2.5)$$

We present two-sided estimates of the total bandwidth for the Schrödinger operators $H$.

Our main goal is to estimate the total bandwidth for $H$ from below.

**Theorem 2.1.** Let $H = -\Delta + V$ be the Schrödinger operator defined by (1.2) – (1.4) on a periodic graph $\mathcal{G}$. Then its total bandwidth $\mathcal{G}(H) = \sum_{j=1}^\nu |\sigma_j(H)|$ satisfies

$$\frac{2d_*}{v_*} \leq \mathcal{G}(H) \leq 4I, \quad (2.6)$$

where the invariant $I$ is defined by (1.12),

$$v_* = \kappa_+ + \text{diam}(V - \kappa), \quad d_* = \begin{cases} d, & \text{if } d \text{ is even} \\ d + 1, & \text{if } d \text{ is odd} \end{cases}, \quad (2.7)$$

$\kappa_+$ and $\text{diam}(V - \kappa)$ are given by (2.5). In particular, the Lebesgue measure $|\sigma(H)|$ of the spectrum of $H$ satisfies

$$\frac{2d_*}{v_*} \leq |\sigma(H)| \leq 4I. \quad (2.8)$$

**Remarks.** 1) The lower estimate in (2.6) gives another proof of the known fact that the absolutely continuous spectrum of $H$ is not empty.

2) The upper estimate in (2.6) was proved in [19]. The proof of the lower estimate in (2.6) consists of the following steps:

- The fiber Schrödinger operators are represented as fiber adjacency operators on a modified quotient graph with additional loops and with specific weights.
- In Corollary 4.4 using the spectral theorem, we estimate the total bandwidth for the Schrödinger operator $H$ in terms of the total bandwidth of $H^n$ for any $n \in \mathbb{N}$.
- In Theorem 4.3 using the trace formulas for the $n$-th power of the fiber Schrödinger operator obtained in [20], we show that the total bandwidth of $H^n$ is more than $2N_n^{\text{odd}}$, where $N_n^{\text{odd}}$ is
the number of all cycles of length \( n \) with an odd sum of index components in the fundamental graph \( G_\ast \).

- In Lemma 4.3, we show that the cycle indices form the lattice \( \mathbb{Z}^d \). Using properties of cycle indices, we show that the number \( \mathcal{N}_n^{\text{odd}} \geq n d_\ast \) for some optimal \( n \leq \nu \), where \( d_\ast \) is given in (2.7).

3) Trace formulas for the discrete Laplacians on (mostly regular) graphs and the numbers of graph cycles were analyzed by various authors, see [1, 2, 3, 23, 27] and references therein.

Taking \( V = 0 \) in (2.6), we obtain similar estimates for the unperturbed operator \( H_0 = -\Delta \). The eigenvalues of the unperturbed fiber operator \( H_0(k) \) will be denoted by \( \lambda_j^\nu(k) \), \( j \in \mathbb{N}_\nu \). The spectral bands \( \sigma_j(H_0) \), \( j \in \mathbb{N}_\nu \), for the operator \( H_0 \) have the form \( \sigma_j(H_0) = [\lambda_j^\nu - \kappa_j^\nu, \lambda_j^\nu + \kappa_j^\nu] = \lambda_j^\nu(\mathbb{T}^d) \).

**Corollary 2.2.** Let \( H_0 = -\Delta \), where \( \Delta \) is the combinatorial Laplacian defined by (1.4) on a periodic graph \( G \). Then its total bandwidth \( \mathcal{S}(H_0) \) satisfies

\[
\frac{2d_\ast}{\kappa_{\ast}^{-1}} \leq \mathcal{S}(H_0) \leq 4I, \quad \text{where} \quad \kappa_\ast = 2\kappa_+ - \kappa_-, \quad (2.9)
\]

and \( \kappa_\pm \) have the form (2.5).

**Remark.** If \( G \) is a regular graph of degree \( \kappa_+ \), i.e., all vertices of \( G \) have the same degree \( \kappa_+ \), then \( \kappa_\ast = \kappa_+ \). For non-regular graphs \( \kappa_\ast > \kappa_+ \). Thus, among all periodic graphs with the same \( d_\ast \), \( \kappa_+ \) and \( \nu \), the regular one has the largest lower bound on the total bandwidth \( \mathcal{S}(H_0) \) in (2.9). This agrees with the fact that deleting an edge from the fundamental graph \( G_\ast \) increases the lower endpoint \( \kappa_j^{-\nu} \) of each spectral band \( \sigma_j(H_0) \), \( j \in \mathbb{N}_\nu \), of the operator \( H_0 = -\Delta \) (in particular, the last spectral band \( \sigma_\nu(H_0) = [\lambda_\nu^\nu, 0] \) shrinks), see, e.g. Corollary 4.2 in [7].

In order to show that the estimates (2.6) are sharp we consider Schrödinger operators \( H = -\Delta + V \) with periodic potentials \( V \) on the simplest periodic graph, the one-dimensional lattice \( \mathbb{Z} \). We estimate the total bandwidth of \( H \) using Theorem 2.1. The following corollary shows that in this case the estimates (2.6), (2.7) coincide with the known Last’s estimates [21] for Schrödinger operators on \( \mathbb{Z} \).

**Corollary 2.3.** Consider the Schrödinger operator \( H = -\Delta + V \) on \( \mathbb{Z} \), where \( \Delta \) is the Laplacian on \( \mathbb{Z} \) given by

\[
(\Delta f)_x = f_{x+1} + f_{x-1} - 2f_x, \quad f \in \ell^2(\mathbb{Z}), \quad x \in \mathbb{Z},
\]

and the potential \( V \) is real \( \nu \)-periodic, \( V_{x+\nu} = V_x , x \in \mathbb{Z}, \nu \geq 3 \). Then

\[
\frac{4}{\nu_\ast - 1} \leq |\sigma(H)| = \mathcal{S}(H) \leq 4, \quad \text{where} \quad \nu_\ast = 2 + \text{diam } V. \quad (2.10)
\]

**Remarks.** 1) Last’s proof of the lower estimate in (2.10) (see [21]) is based on the theory of periodic scalar Jacobi operators. For such operators the relation between the energy \( \lambda \) and the one-dimensional quasimomentum \( k \) (the dispersion relation) has a very specific form:

\[
D(\lambda) = 2 \cos k,
\]

where \( D(\lambda) \) is the so called discriminant (or Lyapunov function). For an arbitrary periodic graph the dispersion relation may have a very complicated form and is difficult to analyze. Our proof of the lower estimates is based on the trace formulas for the Schrödinger operators.
on periodic graphs and bounds on the number of the fundamental graph cycles from some specific sets.

2) Consider the Schrödinger operator \( H_t = -\Delta + tV \) from Corollary 2.3, where the coupling constant \( t > 1 \) and in addition \( V_x \neq V_y \) for all distinct \( x, y \in \mathbb{N}_\nu \). Then each band \( \sigma_j(H_t) \), \( j \in \mathbb{N}_\nu \), of the operator \( H_t = -\Delta + tV \) has the following asymptotics

\[
|\sigma_j(H_t)| = \frac{4}{t^{\nu - 1}} \prod_{x \in \mathbb{N}_\nu, x \neq j} |V_x - V_j|^{-1} + \frac{O(1)}{t^\nu}, \quad \text{as} \quad t \to \infty,
\]

see [13]. Thus, the estimate (2.10) and the lower estimate in (2.6) are sharp.

3) Consider the Schrödinger operators \( H_t = -\Delta + tV \) with periodic potentials \( V \) on a periodic graph \( \mathcal{G} \), where the coupling constant \( t > 1 \) and in addition \( V_x \neq V_y \) for all distinct \( x, y \in \mathcal{G}_* \). Then the following asymptotics holds true (see Theorem 2.2 in [14]):

\[
|\sigma(H_t)| = C + O(1/t), \quad \text{as} \quad t \to \infty,
\]

and \( C = 0 \) iff there are no loops with non-zero index in the fundamental graph \( \mathcal{G}_* \).

2.3. Estimates for normalized Laplacians. We define a normalized Laplacian \( \Delta_n \) on \( \ell^2(\mathcal{V}) \) by

\[
(\Delta_n f)_x = \frac{1}{\sqrt{\kappa_x}} \sum_{(x,y) \in A} \left( \frac{f_x}{\sqrt{\kappa_x}} - \frac{f_y}{\sqrt{\kappa_y}} \right), \quad f \in \ell^2(\mathcal{V}), \quad x \in \mathcal{V}. \tag{2.11}
\]

It is known (see, e.g., [21]) that the normalized Laplacian \( \Delta_n \) is a bounded self-adjoint operator on \( \ell^2(\mathcal{V}) \) and its spectrum \( \sigma(\Delta_n) \) is a subset of the segment \( [0, 2] \), containing the point 0, i.e.,

\[
0 \in \sigma(\Delta_n) \subseteq [0, 2].
\]

Remark. If \( \mathcal{G} \) is a regular graph of degree \( \kappa_+ \), then the normalized Laplacian \( \Delta_n \) and the combinatorial Laplacian \( \Delta \) are related by the simple identity \( \Delta = \kappa_+ \Delta_n \). However, in the case of an arbitrary graph the spectra of these operators, in spite of many similar properties, may have significant differences.

The normalized Laplacian \( \Delta_n \) has the decomposition into a constant fiber direct integral (5.1). The precise expression of the fiber Laplacian \( \Delta_n(k) \) is given by (5.2), (5.3). The eigenvalues of the fiber Laplacian \( \Delta_n(k) \) will be denoted by \( \mu_j(k), j \in \mathbb{N}_\nu \). The spectral bands \( \sigma_j(\Delta_n), j \in \mathbb{N}_\nu \), for the normalized Laplace operator \( \Delta_n \) have the form \( \sigma_j(\Delta_n) = [\mu_{j-}, \mu_{j+}] = \mu_j(\mathbb{T}^d) \).

Now we estimate the total bandwidth for the normalized Laplacian \( \Delta_n \).

Theorem 2.4. Let \( \Delta_n \) be the normalized Laplacian defined by (2.11) on a periodic graph \( \mathcal{G} \). Then its total bandwidth \( \mathcal{S}(\Delta_n) \) satisfies

\[
\frac{2d_*}{\kappa_-} \leq \mathcal{S}(\Delta_n) \leq \frac{4I}{\kappa_+}, \tag{2.12}
\]

where the invariant \( I \) is defined by (1.14), \( d_* \) is given in (2.7), and \( \kappa_\pm \) have the form (2.5).

Remarks. 1) The upper estimate in (2.12) is a direct consequence of the estimate (2.2) proved in [18] and the definition (1.14) of the invariant \( I \). The proof of the lower estimate in (2.12) is similar to the proof of the lower estimate in (2.6).

2) The lower estimates in (2.6), (2.9) and (2.12) are expressed in terms of only the dimension \( d \) of the periodic graph, the number \( \nu \) of fundamental graph vertices, vertex degrees, and the potential \( V \).
2.4. Examples. In this subsection we apply the obtained estimates of the total bandwidth to the Laplacians on some simple periodic graphs. In addition we compute the total bandwidth.

Example 2.5. Let $\Delta$ be the combinatorial Laplacian defined by (1.4) on the Kagome lattice $K$ (see Fig. 3a). Then the total bandwidth $\mathcal{S}(\Delta) = 6$, but the estimate (2.9) is given by

$$\frac{1}{4} \leq \mathcal{S}(\Delta) \leq 12. \quad (2.13)$$

Remarks. 1) The spectrum of the combinatorial Laplacian $\Delta$ on the Kagome lattice is given by $\sigma(\Delta) = \sigma_{ac}(\Delta) \cup \sigma_{fb}(\Delta)$, where $\sigma_{fb}(\Delta) = \sigma_3 = \{6\}$ and $\sigma_{ac}(\Delta) = \sigma_1 \cup \sigma_2$, $\sigma_1 = [0, 3]$, $\sigma_2 = [3, 6]$. Then we obtain that the total bandwidth $\mathcal{S}(\Delta) = \sum_{j=1}^{3} |\sigma_j(\Delta)| = 6$.

2) More accurate lower estimates for the Kagome lattice will be given in Example 6.1.

Example 2.6. Let $\Delta_n$ be the normalized Laplacian defined by (2.11) on the periodic graph $G$ shown in Fig. 4a. Then the total bandwidth $\mathcal{S}(\Delta_n) = \frac{4}{3}$, but the estimate (2.12) is given by

$$\frac{4}{31} \leq \mathcal{S}(\Delta_n) \leq 2. \quad (2.14)$$

Remarks. 1) The spectrum of the normalized Laplacian $\Delta_n$ on the graph $G$ shown in Fig. 4a is given by $\sigma(\Delta_n) = \sigma_{ac}(\Delta_n) \cup \sigma_{fb}(\Delta_n)$, where $\sigma_{ac}(\Delta_n) = \sigma_1 \cup \sigma_2 \cup \sigma_3$, and $\sigma_{fb}(\Delta_n) = \sigma_4 = \{1\}$, $\sigma_1 = [0, \frac{1}{3}]$, $\sigma_2 = [\frac{2}{3}, \frac{1}{3}]$, $\sigma_3 = [\frac{2}{3}, 2]$. Then we obtain that the total bandwidth $\mathcal{S}(\Delta_n) = \frac{4}{3}$.

2) More accurate lower estimates for this graph $G$ will be given in Example 6.2.
3. Trace formulas for Schrödinger operators

3.1. Trace formulas for adjacency operators. We consider the adjacency operator $A$ acting on the Hilbert space $\ell^2(\mathcal{V})$ and given by

\[(Af)_x = \sum_{(x,y) \in A} f_y, \quad f \in \ell^2(\mathcal{V}), \quad x \in \mathcal{V}.\]  

(3.1)

The fiber adjacency operator $A(k)$, $k \in \mathbb{T}^d$, has the form

\[(A(k)f)_x = \sum_{e=(x,y) \in A} e^{i(\tau(e),k)} f_y, \quad f \in \ell^2(\mathcal{V}_e), \quad x \in \mathcal{V}_e,\]  

(3.2)

where $\tau(e)$ is the edge index, defined by (1.8), (1.10). We recall Theorem 2.2 from [20].

**Theorem 3.1.** Let $A(k)$, $k \in \mathbb{T}^d$, be the fiber adjacency operator defined by (3.2) on the fundamental graph $\mathcal{G}_*$. Let $\mathcal{C}_n$ be the set of all cycles of length $n$ in $\mathcal{G}_*$, and $\tau(c)$ be the index of $c$ defined by (1.12). Then for each $n \in \mathbb{N}$ the trace of $A^n(k)$ satisfies

\[\text{Tr} A^n(k) = \sum_{c \in \mathcal{C}_n} \cos(\tau(c),k),\]  

(3.3)

where $N_0^n$ is the number of all cycles of length $n$ with zero index in $\mathcal{G}_*$.

3.2. Trace formulas for Schrödinger operators. In order to formulate trace formulas for the Schrödinger operator $H = -\Delta + V$ defined by (1.2) – (1.4), we need some modification of the fundamental graph $\mathcal{G}_*$. We add a loop $e_x$ with index $\tau(e_x) = 0$ at each vertex $x$ of the fundamental graph $\mathcal{G}_* = (\mathcal{V}_*, \mathcal{A}_*)$ and consider the modified fundamental graph $\tilde{\mathcal{G}}_* = (\mathcal{V}_*, \tilde{\mathcal{A}}_*)$, where

\[\tilde{\mathcal{A}}_* = \mathcal{A}_* \cup \{e_x\}_{x \in \mathcal{V}_*}.\]

We denote by $\tilde{\mathcal{C}}$ the set of all cycles in $\tilde{\mathcal{G}}_*$. For each cycle $c \in \tilde{\mathcal{C}}$ we define the weight

\[\omega(c) = \omega(e_1) \ldots \omega(e_n), \quad \text{where} \quad c = (e_1, \ldots, e_n) \in \tilde{\mathcal{C}},\]  

(3.4)

and $\omega(e)$ is given by

\[\omega(e) = \begin{cases} 1, & \text{if } e \in \mathcal{A}_*, \\ v_x, & \text{if } e = e_x, \quad v_x = V_x - \kappa_x, \end{cases}\]

(3.5)

where $\kappa_x$ is the degree of the vertex $x$.

**Remark.** Note that

\[\omega(c) = 1 \quad \text{for each cycle} \quad c \in \mathcal{C}.\]

We recall Theorems 2.4 and 2.5 from [20].

**Theorem 3.2.** Let $H(k)$, $k \in \mathbb{T}^d$, be the fiber Schrödinger operator defined by (1.17) – (1.18) on the fundamental graph $\mathcal{G}_*$. Let $\tilde{\mathcal{C}}_n$ be the set of all cycles of length $n$ in the modified fundamental graph $\tilde{\mathcal{G}}_*$. Then for each $n \in \mathbb{N}$ the trace of $H^n(k)$ satisfies

\[\text{Tr} H^n(k) = \mathcal{T}_n(k), \quad \mathcal{T}_n(k) = \sum_{c \in \tilde{\mathcal{C}}_n} \omega(c) \cos(\tau(c),k),\]  

(3.5)
where \( \tau(c) \) is the index of \( c \) defined by (1.12); \( \omega(c) \) is given by (3.3), and \( \tilde{C}_n^0 \) is the set of all cycles of length \( n \) with zero index in \( \tilde{G}_s \).

**Remarks.**

1) The formulas (3.3), (3.6) are trace formulas, where the traces of the fiber operators are expressed in terms of some geometric parameters of the graph (vertex degrees, cycle indices and lengths) and the potential \( V \).

2) The index of the cycle \( c \) in the fundamental graph \( G_s \) is equal to zero if and only if \( c \) corresponds to a cycle in the periodic graph \( G \), see Remark 1 after Fig. 1.

3) We sometimes write \( T_n(k, V), \omega(c, V), \ldots \) instead of \( T_n(k), \omega(c), \ldots \), when several potentials \( V \) are dealt with.

4) The trace formulas for the unperturbed operator \( H_0 = -\Delta \), where \( \Delta \) is the combinatorial Laplacian defined by (1.11) are given by the identities (3.5) and (3.6), where \( \omega(c) = \omega(c, 0) \).

4. Estimates for the combinatorial Schrödinger operators

In this section we estimate the total bandwidth for the combinatorial Schrödinger operators with periodic potentials.

4.1. Estimates for adjacency operators. Let \( C_n \) be the set of all cycles of length \( n \) in the fundamental graph \( G_s \). We define the following subsets of \( C_n \):

- \( C_n^+ \) is the set of all cycles of length \( n \) with non-zero indices, and \( N_n^+ \) is their number:
  \[
  C_n^+ = \{ c \in C_n : \tau(c) \neq 0 \}, \quad N_n^+ = |C_n^+| < \infty. \tag{4.1}
  \]

- \( C_n^{odd} \) is the set of all cycles of length \( n \) with odd sum of index components, and \( N_n^{odd} \) is their number:
  \[
  C_n^{odd} = \{ c \in C_n : \langle \tau(c), 1 \rangle \text{ is odd} \}, \quad N_n^{odd} = |C_n^{odd}| < \infty, \tag{4.2}
  \]
where \( \tau(c) \) is the index of \( c \) defined by (1.12), and \( 1 = (1, \ldots, 1) \in \mathbb{R}^d \).

**Remark.** Note that \( N_n^+ \geq N_n^{odd} \) for each \( n \in \mathbb{N} \), since \( C_n^{odd} \subseteq C_n^+ \).

We consider the adjacency operator \( A \).

**Theorem 4.1.** Let \( A \) be the adjacency operator defined by (1.1) on a periodic graph \( G \), and let \( n \in \mathbb{N} \). Then the total bandwidth \( \mathcal{G}(A^n) = \sum_{j=1}^n |\sigma_j(A^n)| \) for the operator \( A^n \) satisfies
\[
\mathcal{G}(A^n) \geq \max\{ N_n^+, 2N_n^{odd} \},
\]
where \( N_n^+ \) and \( N_n^{odd} \) are defined in (4.1) and (4.2).

We omit the proof, since it is similar to the proof of Theorem 4.3.

**Remark.** The numbers \( N_n^+ \) and \( N_n^{odd} \) defined by (4.1) and (4.2) are the numbers of all cycles from the corresponding sets including cycles with back-tracking parts, i.e., including cycles \( (e_1, \ldots, e_n) \) for which \( e_{s+1} = e_s \) for some \( s \in \mathbb{N} \) (\( e_{n+1} \) is understood as \( e_1 \)).

Now we estimate the total bandwidth \( \mathcal{G}(A) = \sum_{j=1}^\nu |\sigma_j(A)| \) for the adjacency operator \( A \).
Corollary 4.2. Let $A$ be the adjacency operator defined by \((3.1)\) on a periodic graph $G$. Then the total bandwidth $\mathcal{S}(A)$ satisfies
\[
\frac{1}{n\kappa_{+}} \max \{ \mathcal{N}_n^+, 2\mathcal{N}_n^{\text{odd}} \} \leq \mathcal{S}(A) \leq 4\mathcal{I}, \quad \forall n \in \mathbb{N},
\]
where the invariant $\mathcal{I}$ has the form \((1.14)\); $\mathcal{N}_n^+$ and $\mathcal{N}_n^{\text{odd}}$ are defined in \((4.1)\) and \((4.2)\), and $\kappa_{+}$ is given in \((1.3)\). In particular, the adjacency operator $A$ has at least one non-degenerate band.

We omit the proof, since it is similar to the proof of Corollary 4.4.

4.2. Estimates of bandwidths for Schrödinger operators. We discuss estimates of the total bandwidth for the Schrödinger operators $H = -\Delta + V$ on periodic graphs. Without loss of generality (adding a constant to the periodic potential $V$) we may assume that
\[
v_- = \min_{x \in \mathcal{V}_n} (V_x - \kappa_x) = 0, \quad v_+ = \max_{x \in \mathcal{V}_n} (V_x - \kappa_x) \geq 0, \quad (4.3)
\]
where $\kappa_x$ is the degree of the vertex $x$. From $\sigma(A) \subseteq [-\kappa_+, \kappa_+]$ and under the condition \((4.3)\) we deduce that $\sigma(H) \subseteq [-\kappa_+, \kappa_+ + v_+]$.

Theorem 4.3. Let $H = -\Delta + V$ be the Schrödinger operator defined by \((1.2) - (1.4)\) with a periodic potential $V$ satisfying \((4.3)\) on a periodic graph $G$ with the fundamental graph $G_* = (\mathcal{V}_*, \mathcal{A}_*)$, and let $n \in \mathbb{N}$. Then the total bandwidth $\mathcal{S}(H^n)$ for the operator $H^n$ satisfies
\[
\mathcal{S}(H^n) \geq \max \{ B_{n,1}, B_{n,2} \}, \quad (4.4)
\]
where
\[
B_{n,1} = \sum_{\substack{c \in \tilde{C}_n \\tau(c) \neq 0 \\nu}} \omega(c) \geq \mathcal{N}_n^+, \quad B_{n,2} = 2 \sum_{\substack{c \in \tilde{C} \\nu \\tau(c), k \text{ is odd}} \\nu} \omega(c) \geq 2\mathcal{N}_n^{\text{odd}}. \quad (4.5)
\]
Here $\tilde{C}_n$ is the set of all cycles of length $n$ in the modified fundamental graph $\tilde{G}_*$; $\tau(c)$ is the index of $c$ defined by \((1.12)\); $\omega(c)$ is given by \((3.2)\); $\nu = (1, \ldots, 1) \in \mathbb{R}^d$, and $\mathcal{N}_n^+$ and $\mathcal{N}_n^{\text{odd}}$ are defined by \((4.1)\) and \((4.2)\).

Proof. The definition of the spectral bands of $H^n$ and the identity \((3.3)\) imply
\[
\mathcal{S}(H^n) = \sum_{j=1}^{\nu} |\sigma_j(H^n)| = \sum_{j=1}^{\nu} \left( \max_{k \in \mathbb{T}^d} \lambda_j^n(k) - \min_{k \in \mathbb{T}^d} \lambda_j^n(k) \right) = \sum_{j=1}^{\nu} \lambda_j^n(0) - \sum_{j=1}^{\nu} \lambda_j^n(k_{**}) = \text{Tr} H^n(0) - \text{Tr} H^n(k_{**}) = \mathcal{T}_n(0) - \mathcal{T}_n(k_{**}) = \sum_{c \in \tilde{C}_n} \omega(c) \left( 1 - \cos \langle \tau(c), k_{**} \rangle \right) = 2 \sum_{c \in \tilde{C}_n} \omega(c) \sin^2 \frac{\langle \tau(c), k_{**} \rangle}{2}
\]
for any $k_{**} \in \mathbb{T}^d$. Thus,
\[
\mathcal{S}(H^n) \geq \mathcal{T}_n(0) - \mathcal{T}_n(k_{**}) = 2 \sum_{c \in \tilde{C}_n} \omega(c) \sin^2 \frac{\langle \tau(c), k_{**} \rangle}{2}, \quad \forall k_{**} \in \mathbb{T}^d. \quad (4.6)
\]
Due to (3.5) and (3.6), we have
\[ \mathcal{T}_{n,0} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \text{Tr} H^n(k) dk = \text{Tr} H^n(k_0) = \mathcal{T}_n(k_0) \quad \text{for some} \quad k_0 \in \mathbb{T}^d. \]

If \( k_* = k_0 \), then, using (3.5) and (3.6), the estimate (4.6) has the form
\[ \mathcal{G}(H^n) \geq \mathcal{T}_n(0) - \mathcal{T}_n(k_0) = \mathcal{T}_n(0) - \mathcal{T}_{n,0} = \sum_{c \in \mathcal{C}_n, \tau(c) \neq 0} \omega(c). \] (4.7)

Due to (3.3), (3.4) and (4.3), \( \omega(c) \geq 0 \) for all \( c \in \mathcal{C} \) and \( \omega(c) = 1 \) for each cycle \( c \in \mathcal{C} \). Then we obtain the following estimate
\[ B_{n,1} = \sum_{c \in \mathcal{C}_n, \tau(c) \neq 0} \omega(c) \geq \sum_{c \in \mathcal{C}_n, \tau(c) \neq 0} \omega(c) = N_n^+. \] (4.8)

Similarly, if \( k_* = \pi \mathbb{I} \), we obtain
\[ \mathcal{G}(H^n) \geq 2 \sum_{c \in \mathcal{C}_n} \omega(c) \sin^2 \left( \frac{\langle \tau(c), \pi \mathbb{I} \rangle}{2} \right) \geq 2 \sum_{c \in \mathcal{C}_n, \langle \tau(c), \pi \mathbb{I} \rangle \text{ is odd}} \omega(c) \geq 2 \sum_{c \in \mathcal{C}_n, \langle \tau(c), \pi \mathbb{I} \rangle \text{ is odd}} \omega(c) = 2\lambda_n^{\text{odd}}. \] (4.9)

Here we have also used that \( \tau(c) \in \mathbb{Z}^d \) for all cycles \( c \in \mathcal{C} \). Combining (4.7) – (4.9), we get (4.4), (4.5).

Now we estimate the total bandwidth for the Schrödinger operator \( H \).

**Corollary 4.4.** Let \( H = -\Delta + V \) be the Schrödinger operator defined by (1.2) – (1.4) with a periodic potential \( V \) satisfying (4.8) on a periodic graph \( \mathcal{G} \). Then its total bandwidth \( \mathcal{G}(H) \) satisfies
\[ \frac{\max\{B_{n,1}, B_{n,2}\}}{n(v_+ + \chi_+)^{n-1}} \leq \mathcal{G}(H) \leq 4\mathcal{I}, \quad \forall n \in \mathbb{N}, \] (4.10)

where the invariant \( \mathcal{I} \) is defined by (1.14); \( B_{n,s}, s = 1, 2, \) have the form (4.5), and \( v_+ \) and \( \chi_+ \) are given by (2.5).

**Proof.** Recall that under the condition (1.3) we have \( \sigma(H) \subseteq [-\chi_+, v_+ + \chi_+] \), where \( \chi_+ = \max_{x \in \mathcal{V}_*} \chi_x \). We have a simple estimate for the spectral bands \( \sigma_j(H^n) \) and \( \sigma_j(H), j = 1, \ldots, \nu; \)
\[ |\sigma_j(H^n)| = \max_{k \in \mathbb{T}^d} \lambda_j^n(k) - \min_{k \in \mathbb{T}^d} \lambda_j^n(k) = \lambda_j^n(k^+) - \lambda_j^n(k^-) \leq n(v_+ + \chi_+)^{n-1}|\lambda_j(k^+) - \lambda_j(k^-)| \leq n(v_+ + \chi_+)^{n-1} \sigma_j(H)|, \]
for some \( k^\pm \in \mathbb{T}^d \). Then using (1.4), we obtain the lower estimate in (4.10). The upper estimate in (4.10) was proved in [19].

In order to prove Theorem 2.1 we need Lemma 4.3. Recall that \( \mathcal{C} \) is the set of all cycles of the fundamental graph \( \mathcal{G}_* = (\mathcal{V}_*, \mathcal{A}_*) \), and for any cycle \( c \in \mathcal{C} \) the cycle index \( \tau(c) \in \mathbb{Z}^d \) is defined by (1.12). Repeating a cycle \( c \) \( m \) times, we obtain \( m \)-multiple \( c^n \) of \( c \). If \( c \) is not a \( m \)-multiple of a cycle with \( m \geq 2 \), \( c \) is called prime. A cycle \( c = (e_1, \ldots, e_n) \) has backtracking if \( e_{i+1} = e_{i+1} \) for some \( s \in \mathbb{N}_n \) (\( e_{i+1} \) is understood as \( e_1 \)). A cycle with no backtracking is called proper. We show that in \( \mathcal{G}_* \) there exist \( d \) prime proper cycles of length \( \nu = \# \mathcal{V}_* \) with indices forming an orthonormal basis of \( \mathbb{Z}^d \).
A graph is called bipartite if its vertex set is divided into two disjoint sets (called parts of the graph) such that each edge connects vertices from distinct parts.

Lemma 4.5. i) Let $\mathcal{C}$ be the set of all cycles of the fundamental graph $\mathcal{G}_s = (V_s, A_s)$. Then the image of the function $\tau: \mathcal{C} \rightarrow \mathbb{R}^d$ defined by $t_{12}$ is the lattice $\mathbb{Z}^d$, i.e.,

$$\tau(\mathcal{C}) = \mathbb{Z}^d.$$  \hfill (4.11)

ii) For some basis of the lattice $\Gamma$ there exist prime cycles $c_1, \ldots, c_d$ of lengths $\leq n = \#V_s$, such that the set $\{\tau(c_s)\}_{s \in \mathbb{N}_d}$ forms an orthonormal basis of $\mathbb{Z}^d$.

iii) There exists $n \leq \nu$ such that the number $N_n^{\text{odd}}$ defined by (4.2) satisfies

$$N_n^{\text{odd}} \geq nd_s, \quad d_s = \begin{cases} d, & \text{if } d \text{ is even} \\ d + 1, & \text{if } d \text{ is odd} \end{cases}.$$ \hfill (4.12)

If $\mathcal{G}$ is bipartite, then

$$N_n^{\text{odd}} \geq 2nd.$$ \hfill (4.13)

Proof. i) Let $\Gamma \subset \mathbb{R}^d$ be a lattice with a basis $\mathbb{A} = \{a_1, \ldots, a_d\}$. We consider a $\Gamma$-periodic graph $\mathcal{G} = (V, A)$. We take some vertex $x \in V$ of $\mathcal{G}$. Let $m = (m_j)_{j=1}^d \in \mathbb{Z}^d$. Then $a = \sum_{s=1}^d m_s a_s \in \Gamma$. Since $\mathcal{G}$ is connected, there exists an oriented path $p$ from $x$ to $x + a$ in $\mathcal{G}$. Then $c = p/\Gamma$ is a cycle in the fundamental graph $\mathcal{G}_s = \mathcal{G}/\Gamma$ with index $\tau(c) = m$. Thus, $m \in \tau(\mathcal{C})$. Conversely, let $m = (m_j)_{j=1}^d \in \mathbb{R}^d$ and $m = \tau(c)$ for some cycle $c \in \mathcal{C}$. Then there exists an oriented path $p$ in the periodic graph $\mathcal{G}$ such that $c = p/\Gamma$ and $p$ connects $\Gamma$-equivalent vertices $x$ and $x + a$ in $\mathcal{G}$, where $a = \sum_{s=1}^d m_s a_s$. Thus, $a \in \Gamma$ and, consequently, $m \in \mathbb{Z}^d$.

ii) Let $T = (V_s, E_T)$ be a spanning tree of the fundamental graph $\mathcal{G}_s$. Then for each $e \in S_T = E_s \setminus E_T$ there exists a cycle $c_e$ in $\mathcal{G}_s$ whose edges are all in $T$ except $e$. Each $c_e$ is prime and proper and the set of all cycles $B = \{c_e\}_{e \in S_T}$ forms a basis of the cycle space $\mathcal{C}$ of the graph $\mathcal{G}_s$. Note that the length of each cycle from $B$ is not more than $\nu$. Since $\tau(\mathcal{C}) = \mathbb{Z}^d$ (see (1.1)), we conclude that $\{\tau(c)\}_{c \in B}$ generates the group $\mathbb{Z}^d$. Consequently, there exist $d$ cycles $c_1, \ldots, c_d \in B$ such that the set

$$\{\tau(c_s) = (\tau_1(c_s), \ldots, \tau_d(c_s)) \in \mathbb{Z}^d : s \in \mathbb{N}_d\}$$

forms a basis of $\mathbb{Z}^d$. Then the set

$$\mathbb{A}' = \{a'_s = \tau_1(c_s)a_1 + \cdots + \tau_d(c_s)a_d : s \in \mathbb{N}_d\}$$

is a basis of the lattice $\Gamma$. Thus, under the change of the basis of the lattice $\Gamma$ from $\mathbb{A}$ to $\mathbb{A}'$ the indices $\{\tau(c_s)\}_{s \in \mathbb{N}_d}$ of the cycles $c_1, \ldots, c_d$ transform to an orthonormal basis of $\mathbb{Z}^d$.

iii) We divide the set $\{c_1, \ldots, c_d\}$ into two disjoint sets: the set $\mathcal{S}_e$ of cycles with even length and the set $\mathcal{S}_o$ of cycles with odd length. Denote by $p$ the maximum number of entries in the sets $\mathcal{S}_e$ and $\mathcal{S}_o$:

$$p = \max\{\#\mathcal{S}_e, \#\mathcal{S}_o\}, \quad p \geq \frac{d_s}{2},$$ \hfill (4.14)

where $d_s$ is given in (1.12). Let $p = \#\mathcal{S}_e$. The proof for the case $p = \#\mathcal{S}_o$ is similar. Denote by $n$ the maximal length of cycles from the set $\mathcal{S}_e$. If some cycle $c \in \mathcal{S}_e$ has the length less than $n$, then we consider the cycle $\overline{c} = (c, e, e, \ldots, e, e) \in \mathcal{C}$ of length $n$ with a back-tracking
part \((e, e, \ldots, e, e)\) for some \(e \in \mathcal{A}_v\). Due to item \(ii\), \(n \leq \nu\), and the cycle \(c\) is prime and proper. Then the cycle \(\tilde{c}\) is also prime and \(\tau(\tilde{c}) = \tau(c)\), since \(\tau(e) = -\tau(e)\) for all \(e \in \mathcal{A}_v\).

Using that each cyclic permutation of edges of a prime cycle \(c\) gives a prime cycle of the same length \(|c|\) and with the same index \(\tau(c)\), and the reverse \(\nu\) of a prime cycle \(c\) is also a prime cycle with \(|\nu| = |c|\) and \(\tau(\nu) = -\tau(c)\), we obtain

\[
\mathcal{N}_n^{\text{odd}} \geq 2np. \tag{4.15}
\]

Combining this and (4.14), we obtain (4.12).

If \(\mathcal{G}\) is bipartite, then without loss of generality we may assume that the fundamental graph \(\mathcal{G}_w\) is also bipartite. Then there are no cycles of odd length in \(\mathcal{G}_w\), and \(p = \#\mathcal{S}_\nu = d\). Combining this and (4.13), we obtain (4.19). \(\blacksquare\)

Now we prove Theorem 2.1 and Corollaries 2.2, 2.3.

**Proof of Theorem 2.1.** By Lemma 4.5, there exists \(n \leq \nu\) such that the number \(\mathcal{N}_n^{\text{odd}}\) defined by (4.2) satisfies \(\mathcal{N}_n^{\text{odd}} \geq nd_\nu\), where \(d_\nu\) is given in (2.7). Then, using the second inequality in (4.5), (4.10) and \(v_\nu = v_+ + \kappa_\nu \geq 2\) we have

\[
\mathcal{S}(H) = \sum_{j=1}^\nu |\sigma_j(H)| \geq \frac{B_{n,2}}{n v_\nu^{u-1}} \geq \frac{2\mathcal{N}_n^{\text{odd}}}{n v_\nu^{u-1}} \geq \frac{2nd_\nu}{n v_\nu^{u-1}} \geq \frac{2d_\nu}{v_\nu^{u-1}}.
\]

The upper estimate in (2.0) was proved in [19]. The estimate (2.8) follows from (2.6) and the inequalities

\[
\frac{1}{\nu} \mathcal{S}(H) \leq \max_{j \in \mathbb{N}_\nu} |\sigma_j(H)| \leq |\sigma(H)| \leq \mathcal{S}(H) = \sum_{j=1}^\nu |\sigma_j(H)|. \tag{4.16}
\]

**Proof of Corollary 2.2.** The estimate (2.9) for the operator \(H_o\) is obtained from the estimate (2.6) for the Schrödinger operator \(H = H_o + V\) as \(V = 0\). \(\blacksquare\)

**Remark.** If \(\mathcal{G}\) is bipartite, then, by (4.13), \(\mathcal{N}_n^{\text{odd}} \geq 2nd\) for some \(n \leq \nu\) and the lower estimates in (2.0) and (2.9) can be improved:

\[
\mathcal{S}(H) \geq \frac{2\mathcal{N}_n^{\text{odd}}}{n v_\nu^{u-1}} \geq \frac{4nd}{n v_\nu^{u-1}} \geq \frac{4d}{v_\nu^{u-1}}, \quad \mathcal{S}(H_o) \geq \frac{4d}{v_\nu^{u-1}}. \tag{4.17}
\]

**Proof of Corollary 2.3.** It is well known (see, e.g., [29]) that the spectrum of the one-dimensional Schrödinger operator \(H = -\Delta + V\) on \(\mathbb{Z}\) consists of \(\nu\) bands

\[
\sigma_n = [\lambda_n^-, \lambda_n^+], \quad n \in \mathbb{N}_\nu, \quad \lambda_1^- < \lambda_1^+ \leq \lambda_2^- < \ldots < \lambda_{\nu-1}^- \leq \lambda_\nu^- < \lambda_\nu^+, \tag{4.18}
\]

where \(\lambda_\nu^+, \lambda_\nu^-, \lambda_{\nu-1}^+, \ldots\) are the eigenvalues of the fiber operator \(H(\nu)\); \(\lambda_v, \lambda_{v-1}^+, \lambda_{v-2}^-, \ldots\) are the eigenvalues of \(H(\tau)\) (see Fig. 5). These bands are separated by gaps \((\lambda_n^+ - \lambda_{n+1}^-), n \in \mathbb{N}_{\nu-1}\). Some of the gaps may be degenerate, i.e., \(\lambda_n^+ = \lambda_n^-\).

For the one-dimensional lattice \(\mathbb{Z}\), the numbers \(d_\nu\) and \(v_\nu\) defined in (2.7) have the form

\[
v_\nu = \kappa_\nu + \text{diam}(V - \kappa) = 2 + \text{diam}V, \quad d_\nu = 2.
\]

The fundamental graph \(\mathcal{G}_w\) of the lattice \(\mathbb{Z}\) is just the cycle graph with \(\nu\) vertices \(1, \ldots, \nu\), and \(\nu\) edges

\[
e_1 = (1, 2), \quad e_2 = (2, 3), \quad \ldots, \quad e_{\nu-1} = (\nu - 1, \nu), \quad e_\nu = (\nu, 1).
\]
with indices
\[ \tau(e_1) = \ldots = \tau(e_{\nu-1}) = 0, \quad \tau(e_\nu) = 1, \]
and their inverse edges. Thus, \( I = d = 1 \), and the estimate (2.6) has the form (2.10).

\[ \begin{array}{cccccc}
\lambda_1^- & \lambda_1^+ & \lambda_2^- & \lambda_2^+ & \ldots & \lambda_{\nu-2}^- \\
\sigma_1 & \sigma_2 & \sigma_{\nu-2} & \sigma_{\nu-1} & \lambda_{\nu-1}^- & \lambda_{\nu-1}^+ \\
\end{array} \]

**Figure 5.** The spectrum of the Schrödinger operator \( H = -\Delta + V \) with a \( \nu \)-periodic potential \( V \) on \( \mathbb{Z} \).

### 5. Estimates for normalized Laplacians

#### 5.1. Direct integral decomposition.** We recall Proposition 1.1 from [18].

**Theorem 5.1.** Let the Hilbert space \( \mathcal{H} \) be defined by (1.16). Then the normalized Laplacian \( \Delta_n \) on \( \ell^2(V) \) given by (2.11) has the following decomposition into a constant fiber direct integral

\[ U \Delta_n U^{-1} = \int_{T^d} \Delta_n(k) \, \frac{dk}{(2\pi)^d}, \tag{5.1} \]

for some unitary operator \( U : \ell^2(V) \rightarrow \mathcal{H} \) (the Gelfand transform). The fiber Laplacian \( \Delta_n(k), k \in T^d \), on \( \ell^2(V_x) \) is given by

\[ \Delta_n(k) = I - T(k). \tag{5.2} \]

Here \( I \) is the identity operator on \( \ell^2(V_x) \), and the fiber transition operator \( T(k) \) has the form

\[ (T(k)f)_x = \sum_{e = (x,y) \in A_x} e^{i\langle \tau(c), k \rangle} \frac{\lambda(x,y)}{\sqrt{\kappa_x \kappa_y}} f_y, \quad f \in \ell^2(V_x), \quad x \in V_x, \tag{5.3} \]

where \( \kappa_x \) is the degree of the vertex \( x \), and \( \tau(e) \) is the index of the edge \( e \in A_x \) defined by (1.8), (1.10).

#### 5.2. Trace formulas for transition operators.** Recall that \( \mathcal{C} \) is the set of all cycles in the fundamental graph \( G_\ast = (V_x, A_x) \). For each cycle \( c = (e_1, \ldots, e_n) \in \mathcal{C} \) we define the *weight*

\[ \omega_n(c) = \frac{1}{\kappa_{x_1} \ldots \kappa_{x_n}}, \quad \text{where} \quad e_s = (x_s, x_{s+1}) \in A_x, \quad s \in \mathbb{N}_n, \quad x_{n+1} = x_1, \tag{5.4} \]

and \( \kappa_x \) is the degree of the vertex \( x \). We recall Theorem 5.3 from [20].

**Theorem 5.2.** Let \( T(k), k \in T^d \), be the fiber transition operator defined by (5.3). Then for each \( n \in \mathbb{N} \)

i) The trace of \( T^n(k) \) has the form

\[ \text{Tr} T^n(k) = T_n(k), \quad T_n(k) = \sum_{c \in C_n} \omega_n(c) \cos(\tau(c), k), \]

where \( C_n \) is the set of all cycles of length \( n \) in the fundamental graph \( G_\ast \); \( \tau(c) \) is the index of \( c \) defined by (1.12), and \( \omega_n(c) \) is given by (5.4).
where $C_n^0$ is the set of all cycles of length $n$ with zero index in $G$.  

5.3. Estimates of bandwidths. We discuss estimates of the total bandwidth for the normalized Laplacian $\Delta_n$.

**Theorem 5.3.** Let $T = I - \Delta_n$ be the transition operator, where $\Delta_n$ is the normalized Laplacian defined by (2.11) on a periodic graph $G$ with the fundamental graph $G_* = (V_*, A_*)$, and let $n \in \mathbb{N}$. Then the total bandwidth $\mathcal{S}(T^n) = \sum_{j=1}^{\nu} |\sigma_j(T^n)|$, $j \in \mathbb{N}_\nu$, $\nu = \#V_*$, for the operator $T^n$ satisfies

$$
\mathcal{S}(T^n) \geq \max \{ B_{n,1}, B_{n,2} \},
$$

where

$$
\frac{N^+}{\kappa^n} \leq B_{n,1} = \sum_{c \in C_n \tau(c) \neq 0} \omega_n(c) \leq \frac{N^+}{\kappa^n},
$$

and

$$
\frac{2}{\kappa^n} N^{odd} \leq B_{n,2} = 2 \sum_{c \in C_n \tau(c), \omega(c) \text{ is odd}} \omega_n(c) \leq 2 \frac{N^{odd}}{\kappa^n}.
$$

Here $C_n$ is the set of all cycles of length $n$ in $G_*; \tau(c)$ is the index of $c$ defined by (1.12), $\omega_n(c)$ is given by (5.4); $\kappa_+ \text{ and } \kappa_-$ are defined by (2.3) and (4.1), and the numbers $N^+$ and $N^{odd}$ are defined in (4.1). For regular graphs of degree $\kappa$, the inequalities in (5.6) become identities.

The proof of Theorem 5.3 is similar to the proof of Theorem 4.3.

Now we estimate the total bandwidth for the Laplacian $\Delta_n$.

**Corollary 5.4.** Let $\Delta_n$ be the normalized Laplacian defined by (2.11) on a periodic graph $G$. Then its total bandwidth $\mathcal{S}(\Delta_n)$ satisfies

$$
\frac{1}{n} \max \{ B_{n,1}, B_{n,2} \} \leq \mathcal{S}(\Delta_n) \leq 2b,
$$

where

$$
b = \sum_{x \in V_\ast} \frac{b_x}{\kappa_x} \leq \frac{2T}{\kappa_-},
$$

$b_x$ is the number of the fundamental graph edges having non-zero indices and starting at the vertex $x \in V_\ast; \kappa_x$ is the degree of $x$; the invariant $T$ has the form (1.12); $\kappa_-$ is defined in (2.3), and $B_{n,s}, s = 1, 2$, are given in (5.6).

**Proof.** Denote by $\xi_j(k), j \in \mathbb{N}_\nu$, the eigenvalues of the fiber transition operator $T(k)$ defined by (5.3). Then the spectral bands $\sigma_j(T), j \in \mathbb{N}_\nu$, for the transition operator $T = I - \Delta_n$ have the form $\sigma_j(T) = [\xi^-_j, \xi^+_j] = \xi_j(T^d)$. The spectrum of $T$ satisfies $\sigma(T) \subseteq [-1, 1]$. We have a simple estimate for the spectral bands $\sigma_j(T^n)$ and $\sigma_j(T)$, $j = 1, \ldots, \nu$:

$$
|\sigma_j(T^n)| = \max_{k \in T^d} \xi^n_j(k) - \min_{k \in T^d} \xi^n_j(k) = \xi^n_j(k^+) - \xi^n_j(k^-) \leq n|\xi_j(k^+) - \xi_j(k^-)| \leq n|\sigma_j(T)|,
$$

for some $k^\pm \in T^d$. Using estimates (5.9), (5.5) and the identity $|\sigma_j(\Delta_n)| = |\sigma_{\nu-j+1}(T)|$, $j \in \mathbb{N}_\nu$, we obtain the lower estimate in (5.7).
The upper estimate in \(5.7\) was proved in [18]. We prove the inequality in \(5.8\). We have
\[
b = \sum_{x \in \mathcal{V}_s} \frac{b_x}{\kappa_x} \leq \frac{1}{\kappa_-} \sum_{x \in \mathcal{V}_s} b_x = \frac{1}{\kappa_-} \# \mathcal{B}_s,
\]
where \(\kappa_-\) is defined in \((2.5)\), and \(\mathcal{B}_s\) is the set of all bridges of the fundamental graph \(\mathcal{G}_s = (\mathcal{V}_s, \mathcal{A}_s)\). This estimate holds for any embedding of the periodic graph \(\mathcal{G}\) into \(\mathbb{R}^d\). Choosing the embedding with the minimal number of the fundamental graph bridges, due to \((1.14)\), we obtain \(b \leq \frac{\nu}{\kappa_-}\).

Now we prove Theorem \(2.3\).

**Proof of Theorem 2.4.** The proof is similar to the proof of Theorem 2.1. By Lemma \(4.5\ iii\), there exists \(n \leq \nu\) such that the number \(\mathcal{N}_n^{odd}\) defined by \((4.2)\) satisfies \(\mathcal{N}_n^{odd} \geq nd_s\), where \(d_s\) is given in \((2.7)\). Then, using \((5.7)\), the second inequality in \((5.6)\) and \(\nu_+ \geq 2\), we obtain
\[
\mathcal{G}(\Delta_n) \geq \frac{B_{n,2}}{n} \geq \frac{2\mathcal{N}_n^{odd}}{n \nu_+^2} \geq \frac{2nd_+}{n \nu_+^2} \geq \frac{2d_+}{\nu_+}
\]
The upper estimate in \((2.12)\) follows from the upper estimate in \((5.7)\) and \((5.8)\).

**Remark.** If \(\mathcal{G}\) is bipartite, then, by \((1.13)\), \(\mathcal{N}_n^{odd} \geq 2n d\) for some \(n \leq \nu\) and the lower estimate in \((2.12)\) can be improved:
\[
\mathcal{G}(\Delta_n) \geq \frac{2\mathcal{N}_n^{odd}}{n \nu_+^2} \geq \frac{4n d}{n \nu_+^2} \geq \frac{4d_+}{\nu_+^2}.
\]

6. **Examples**

First we prove Examples 2.5 and 2.6 about estimates of bandwidths for the Laplacians on the Kagome lattice and on the graph shown in Fig. 4a.

**Proof of Example 2.5.** The Kagome lattice \(K\) is a periodic regular graph of degree \(\nu_+ = 4\). The fundamental graph \(K_s\) of \(K\) consists of three vertices \(x_1, x_2, x_3\), six edges
\[
e_1 = (x_1, x_2), \quad e_2 = (x_2, x_3), \quad e_3 = (x_3, x_1), \quad e_4 = (x_2, x_1), \quad e_5 = (x_3, x_2), \quad e_6 = (x_1, x_3)
\]
with indices
\[
\tau(e_1) = (0, 0), \quad \tau(e_2) = (0, 0), \quad \tau(e_3) = (0, 0),
\]
\[
\tau(e_4) = (0, 1), \quad \tau(e_5) = (1, -1), \quad \tau(e_6) = (-1, 0),
\]
and their inverse edges, see Fig. 3b. Thus, \(\mathcal{I} = 3\), \(d_s = d = 2\), \(\nu_+ = 4\), and the estimate \((2.9)\) has the form \((2.13)\).

**Proof of Example 2.6.** We consider the \(Z\)-periodic graph \(G\) shown in Fig. 4b. Its fundamental graph \(G_s\) consists of four vertices \(x_1, x_2, x_3, x_4\) with degrees \(\kappa_{x_1} = \kappa_{x_3} = 3\) and \(\kappa_{x_2} = \kappa_{x_4} = 2\) and five edges
\[
e_1 = (x_1, x_4), \quad e_2 = (x_1, x_2), \quad e_3 = (x_2, x_4), \quad e_4 = (x_4, x_3), \quad e_5 = (x_3, x_1)
\]
with indices
\[
\tau(e_1) = 1, \quad \tau(e_2) = \tau(e_4) = \tau(e_4) = 0,
\]
and their inverse edges (see Fig. 4b). Thus, \(\mathcal{I} = d = 1\), \(\nu_- = 2\), \(\nu_+ = 3\), \(\nu = 4\), and the estimate \((2.12)\) has the form \((2.14)\).
The estimates \( (2.13) \) and \( (2.14) \) are quite rude. Now we obtain more accurate estimates for these graphs using Corollaries 4.3 and 5.4.

**Example 6.1.** Let \( \Delta \) be the combinatorial Laplacian defined by (1.4) on the Kagome lattice \( K \) (see Fig. 3a). Then its total bandwidth \( \mathcal{G}(\Delta) \) satisfies
\[
2 \leq \mathcal{G}(\Delta) \leq 12. \tag{6.1}
\]

**Proof.** We estimate the total bandwidth for the combinatorial Laplacian \( \Delta \) applying the inequalities (4.10) as \( n = 1, 2, 3 \). By (4.1), we have
\[
B_{n,1} \geq \mathcal{N}_n^+, \quad B_{n,2} \geq 2\mathcal{N}_{n,odd}^+, \quad \forall n \in \mathbb{N}, \tag{6.2}
\]
where \( \mathcal{N}_n^+ \) and \( \mathcal{N}_{n,odd}^+ \) are defined in (4.1) and (4.2).

There are no cycles of length one (i.e., loops) in the fundamental graph \( K_+ \) (see Fig. 3b), i.e.,
\[
\mathcal{N}_1^+ = \mathcal{N}_{1,odd}^+ = 0. \tag{6.3}
\]
Each oriented edge \( e \in A_+ \) generates the cycle \( c_e = (e, e) \) of length 2 consisting of 1 backtrack \( (e, e) \), but index of \( c_e \) is zero, since, by (1.1) and (1.3),
\[
\tau(c_e) = \tau(e) + \tau(e) = \tau(e) - \tau(e) = 0.
\]
The graph \( K_+ \) also has the following proper cycles (i.e., cycles without backtracking) of length 2:
\[
c_{2,1} = (e_5, e_3), \quad c_{2,2} = (e_1, e_4), \quad c_{2,3} = (e_2, e_5)
\]
with indices
\[
\tau(c_{2,1}) = (-1, 0), \quad \tau(c_{2,2}) = (0, 1), \quad \tau(c_{2,3}) = (1, -1),
\]
their cyclic permutations (two permutations for each cycle \( c_{2,s}, s = 1, 2, 3 \)) and their reverse cycles. Thus,
\[
\mathcal{N}_2^+ = 12, \quad \mathcal{N}_{2,odd}^+ = 8. \tag{6.4}
\]
There are no cycles of length 3 with backtracking in \( K_+ \), since there are no loops and each backtracking contributes 2 in the cycle length. Thus, all cycles of length 3 are proper cycles
\[
c_{3,1} = (e_1, e_2, e_3), \quad c_{3,2} = (e_1, e_2, e_3), \quad c_{3,3} = (e_2, e_3, e_4), \quad c_{3,4} = (e_3, e_1, e_5), \quad c_{3,5} = (e_6, e_5, e_4), \quad c_{3,6} = (e_6, e_5, e_4), \quad c_{3,7} = (e_5, e_4, e_6), \quad c_{3,8} = (e_4, e_6, e_2)
\]
with indices
\[
\tau(c_{3,1}) = (0, 0), \quad \tau(c_{3,2}) = (1, 0), \quad \tau(c_{3,3}) = (0, -1), \quad \tau(c_{3,4}) = (-1, 1), \quad \tau(c_{3,5}) = (0, 0), \quad \tau(c_{3,6}) = (0, -1), \quad \tau(c_{3,7}) = (1, 0), \quad \tau(c_{3,8}) = (-1, 1),
\]
their cyclic permutations (three permutations for each cycle \( c_{3,s}, s \in \mathbb{N}_3 \)) and their reverse cycles. Then,
\[
\mathcal{N}_3^+ = 36, \quad \mathcal{N}_{3,odd}^+ = 24. \tag{6.5}
\]
Using \( (6.2) - (6.5) \) and \( v_+ = 0 \), we obtain that the lower estimates in (4.10) for \( n = 1, 2, 3 \) have the form
\[
\mathcal{G}(\Delta) \geq 0, \quad \mathcal{G}(\Delta) \geq 2, \quad \mathcal{G}(\Delta) \geq 1,
\]
respectively. For \( n = 1 \) the lower estimate is trivial. For \( n = 2 \) the lower estimate is better than for \( n = 3 \). The upper estimate in (6.1) was proved in Example 2.5. ■

\[\text{TWO-SIDED ESTIMATES OF TOTAL BANDWIDTH FOR SCHröDINGER OPERATORS} \tag{21} \]
Example 6.2. Let $\Delta_n$ be the normalized Laplacian defined by (2.11) on the periodic graph $G$ shown in Fig. 4a. Then its total bandwidth $\mathcal{G}(\Delta_n)$ satisfies

$$
\frac{4}{9} \leq \mathcal{G}(\Delta_n) \leq \frac{4}{3}.
$$

(6.6)

Proof. We estimate the total bandwidth of $\Delta_n$ applying the inequalities (5.7) as $n = 1, 2, 3, 4$. There are no cycles of length 1 (i.e., loops) in the fundamental graph $G^*$ (see Fig. 4b). Thus, (5.6) gives

$$
B_{1,1} = B_{2,1} = 0 \quad \text{as} \quad n = 1.
$$

There are no proper cycles (i.e., cycles without backtracking) of length 2 in $G^*$. Thus, all cycles of length 2 have the form $c_e = (e, e)$ for some edge $e \in A^*$ and $\tau(c_e) = 0$. Then (5.6) gives

$$
B_{2,1} = B_{2,2} = 0 \quad \text{as} \quad n = 2.
$$

There are no cycles of length 3 with backtracking in $G^*$, since there are no loops and each backtracking contributes 2 in the cycle length. Thus, all cycles of length 3 are proper cycles

$$
c_1 = (e_1, e_3, e_2), \quad c_2 = (e_1, e_4, e_5)
$$

with indices $\tau(c_1) = \tau(c_2) = 1$ and weights $\omega_n(c_1) = \omega_n(c_2) = \frac{1}{18}$, their cyclic permutations (three permutations for each cycle $c_s$, $s = 1, 2$) and their reverse cycles. Then, (5.6) gives

$$
B_{3,1} = \frac{2}{3}, \quad B_{3,2} = \frac{4}{3} \quad \text{as} \quad n = 3.
$$

At last, the graph $G^*$ has the following prime cycle of length 4:

$$
c = (e_2, e_3, e_4, e_5),
$$

and 8 prime cycles of the form $(e, e', e', e)$ for all pairs of edges $e, e' \in A^*$ such that the terminal vertex of $e$ coincides with the initial vertex of $e'$ and $e' \neq e$. All remaining cycles of length four are 2-multiple of prime cycles of length 2. Each cycle of length 4 has zero index. Then, by (5.6) as $n = 4$, we obtain

$$
B_{4,1} = B_{4,2} = 0.
$$

For $n = 3$ the lower estimate in (5.7) has the form

$$
\mathcal{G}(\Delta_n) \geq \frac{4}{9}.
$$

For $n = 1, 2, 4$ the lower estimate in (5.7) is trivial.

Since the number $b$ defined in (5.8) is equal to $\frac{2}{3}$, the upper estimate in (5.7) yields

$$
\mathcal{G}(\Delta_n) \leq 2 \cdot \frac{2}{3} = \frac{4}{3}.
$$

Remark. Comparing the estimates (2.13) and (2.14) with the estimates (6.1) and (6.6), we see that the last ones are better. Moreover, the upper estimate in (6.6) is sharp, see remark after Example 2.6.

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References

[1] Ahumada, G. Fonctions periodiques et formule des traces de Selberg sur les arbres, C. R. Acad. Sci. Paris, 305 (1987), 709–712.
[2] Brooks, R. The Spectral Geometry of \(k\)-Regular Graphs, J. d’Analyse, 57 (1991), 120–151.
[3] Chinta, G.; Jorgenson, J.; Karlsson, A. Heat kernels on regular graphs and generalized Ihara zeta function formulas. Monatsh. Math., 178 (2015), 171–190.
[4] Chelkak, D.; Korotyaev, E. Spectral estimates for Schrödinger operators with periodic matrix potentials on the real line. Int. Math. Res. Not., 2006 (2006), 1–41.
[5] Deift, P.; Simon, B. Almost periodic Schrödinger operators III. The absolutely continuous spectrum in one dimension. Commun. Math. Phys., 90, (1983), 389–411.
[6] Fabila-Carrasco, J.S.; Lledó, F.; Post, O. Spectral gaps and discrete magnetic Laplacians, Linear Algebra Appl., 547 (2018), 183–216.
[7] Fabila-Carrasco, J.S.; Lledó, F.; Post, O. Spectral preorder and perturbations of discrete weighted graphs. Math. Ann. (2020), 49 pp.
[8] Korotyaev, E. Estimates of periodic potentials in terms of gap lengths. Commun. Math. Phys., 197 (1998), 521–526.
[9] Korotyaev, E. Characterization of the spectrum of Schrödinger operators with periodic distributions, Int. Math. Res. Not. 2003 (2003), 2019–2031.
[10] Korotyaev, E. Estimates for the Hill operator. I. J. Differ. Equ., 162 (2000), 1–26.
[11] Korotyaev, E. Effective masses for zigzag nanotubes in magnetic fields, Lett. Math. Phys., 83.1 (2008), 83–95.
[12] Korotyaev, E; Krasovsky, I. Spectral estimates for periodic Jacobi matrices. Commun. Math. Phys., 234 (2003), 517–532.
[13] Korotyaev, E.; Kutsenko, A. Inverse problem for the discrete periodic Schrödinger operator, Zapiski Nauchnych Seminarov POMI, 315 (2004), 96–101.
[14] Korotyaev, E.; Saburova, N. Schrödinger operators on periodic discrete graphs, J. Math. Anal. Appl., 420 (2014), no. 1, 576–611.
[15] Korotyaev, E.; Saburova, N. Spectral band localization for Schrödinger operators on periodic graphs, Proc. Amer. Math. Soc., 143 (2015), 3951–3967.
[16] Korotyaev, E.; Saburova, N. Effective masses for Laplacians on periodic graphs, J. Math. Anal. Appl., 436 (2016), 104–130.
[17] Korotyaev, E.; Saburova, N. Magnetic Schrödinger operators on periodic discrete graphs, J. Funct. Anal., 272 (2017), 1625–1660.
[18] Korotyaev, E.; Saburova, N. Spectral estimates for Schrödinger operators on periodic discrete graphs, St. Petersburg Math. J., 30 (2019), 667–698.
[19] Korotyaev, E.; Saburova, N. Invariants for Laplacians on periodic graphs, Math. Ann., 377 (2020), 723–758.
[20] Korotyaev, E.; Saburova, N. Trace formulas for Schrödinger operators on periodic graphs, J. Math. Anal. Appl., 508 (2022), 33pp.
[21] Last, Y. On the measure of gaps and spectra for discrete 1D Schrödinger operators, Commun. Math. Phys., 149 (1992), 347–360.
[22] Lledó, F.; Post, O. Eigenvalue bracketing for discrete and metric graphs, J. Math. Anal. Appl., 348 (2008), 806–833.
[23] Mnëv, P. Discrete path integral approach to the Selberg trace formula for regular graphs, Commun. Math. Phys., 274 (2007), 233–241.
[24] Mohar, B.; Woess, W. A survey on spectra of infinite graphs, Bull. London Math. Soc., 21 (1989), 209–234.
[25] Novoselov, K.S.; Geim, A.K. et al, Electric field effect in atomically thin carbon films, Science, 306 (2004), 666–669.
[26] Sy, P.W.; Sumada, T. Discrete Schrödinger operator on a graph, Nagoya Math. J., 125 (1992), 141–150.
[27] Terras, A.; Wallace, D. Selberg’s trace formula on the \(k\)-regular tree and applications, Int. J.Math. Math. Sci., 2003, 501–526 (2003).
[28] M. Toda, Theory of Nonlinear Lattices, 2nd. ed., Springer, Berlin, 1989.
[29] van Moerbeke, P. The spectrum of Jacobi matrices. Invent. Math., 37 (1976), 45–81.

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