Thermodynamics of a collapsing shell in an expanding Universe

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Abstract

We describe the quasi-static collapse of a radiating, spherical shell of matter in de Sitter space-time using a thermodynamical formalism. It is found that the specific heat at constant area and other thermodynamical quantities exhibit singularities related to phase transitions during the collapse.

Because of the paradigm of inflation, de Sitter space-time has recently become the subject of great research interest. One aspect of research has been the conjectured dS-CFT correspondence [1] which is the analogue of Maldacena’s AdS-CFT conjecture [2] for the case of a space-time with a positive cosmological constant. As in the latter case, the “gravitational” (dS) side of the correspondence is expected to yield the macroscopic description of the system, whereas the CFT (conformal field theory) should be able to describe the microphysics. A well known example for the AdS-CFT is given by the microscopical description of the thermodynamical laws of black holes [3]. Furthermore, applications of the de Sitter space Bousso bound [4], related to the Bekenstein bound for flat space-times, have also drawn much interest.

The dS-CFT correspondence has not yet reached the degree of comprehension of the AdS-CFT, since the corresponding CFT has not been completely identified. Therefore, one usually proceeds by studying simplified “gravitational” models in order to obtain general features that the relevant CFT is required to possess. One such model is given by a thin spherical shell of matter in a de Sitter space-time, since it may represent a collection of D-branes forming the so called D-Sitter space-time of Ref. [5], or can be used as a tool for the study of the entropy bound in a very simplified physical environment [6]. Finally, one may also consider the “operational” approach to black hole entropy, as illustrated in Ref. [7], which leads one to the conclusion that the entropy of a black hole is the same as that stored in a shell of matter sitting just outside its Schwartzschild radius after it has collapsed from infinity. According to the AdS-CFT correspondence, a black hole is represented by a thermal state and its formation by the approach to such a state. A thermodynamical description of the gravitational collapse should thus be useful in the de Sitter case as well.

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It is in this framework that the study of the collapse of a spherical shell of matter in a de Sitter space-time, in analogy with our previous treatment of the anti-de Sitter case [8], may be useful and interesting since the former introduces an additional horizon. The application of a thermodynamical formalism to describe the collapse may give important insights on the process and the inclusion of the radiation coming out of the shell should reveal its importance in the identification of the diverse thermodynamical quantities. We shall compare our results and extend the evaluation of the specific heats with the approach of Ref. [7] and comment on other references [6, 7], where different choices for the definition of the thermodynamical observables are employed.

In particular, we shall examine the collapse of a radiating spherical shell of matter in de Sitter space-time with the assumption that collapse is a quasi-static process, that is, the shell contraction is sufficiently slow so that the system can be described as evolving through a sequence of equilibrium states. In fact a radiative process related to the velocity of the collapse (as one would expect) naturally leads to a quasi-static collapse [9]. This assumption has allowed us to introduce a thermodynamical formalism (see Refs. [8, 9, 10]) to describe the process. The properties of the system depend on the equation of state, that is a relation between the thermodynamically independent quantities. In order to obtain some explicit results we shall consider the general case of a power-law dependence of the shell temperature (introduced as usual through the second law of thermodynamics) on the inner horizon radius and also examine the particular choice corresponding to the Hawking temperature of the incipient black hole or other choices [7]. We use units for which $\hbar = c = k_B = 1$, with $k_B$ the Boltzmann constant.

The spherically symmetric space-time we consider is divided into an inner region and an outer one by a thin massive spherical shell. The inner region can be expressed in static coordinates as

$$ds^2_1 = -f_i(r) dt^2 + \frac{dr^2}{f_i(r)} + r^2 d\Omega^2,$$  

and will be taken to be described by a Schwarzschild metric, so that $f_i(r) = 1 - 2m/r$ where $m$ is a constant ADM mass. The outer region, because of the radiation emitted by the shell, is described by a Vaidya-dS space-time

$$ds^2_o = \frac{1}{f_o(r, t)} \left[ \left( \frac{\partial_t M(r, t)}{\partial_r M(r, t)} \right)^2 dt^2 - dr^2 \right] + r^2 d\Omega^2,$$  

with $f_o(r, t) = 1 - 2M(r, t)/r - r^2/\ell^2$, where $M(r, t)$ is the Bondi mass and its dependence on the time $t$ is related to the amount of radiation (energy) flowing out of the shell, $\partial_t M$ and $\partial_r M$ are the partial derivatives of $M(r, t)$ with respect to $t$ and $r$ respectively, and $\ell$ is the dS (cosmological) radius.

Israel’s junction equations [10] for a static thin shell located at radius $r = R$, allow us to relate the proper mass of the shell $E$ to the inner and outer metrics through the equation

$$E(R, M) = 4\pi R^2 \rho = R \left( \sqrt{f_i(R)} - \sqrt{f_o(R)} \right),$$  

where $\rho$ is the surface energy density, and $M$ and $R$ are the dynamical independent variables.
(\(m\) and \(\ell\) are taken to be fixed). One may evaluate the surface tension, denoted by \(P\), as

\[
P(R, M) \equiv \frac{\partial E}{\partial A} = \frac{1}{8\pi R} \left[ \sqrt{f_i(R)} - \sqrt{f_o(R)} + \frac{1}{\sqrt{f_i(R)}} \frac{m}{R} - \frac{1}{\sqrt{f_o(R)}} \left( \frac{M}{R} - \frac{R^2}{\ell^2} \right) \right],
\]

where \(A = 4\pi R^2\) is the shell area. In our thermodynamical description the surface tension and the area correspond respectively to the intensive and extensive variables. The proper mass, on the other hand, corresponds to the internal energy (see Refs. [8, 10]) and Eq. (3) is a statement of the first law of thermodynamics (\(dE\) is an exact differential).

Given our previous definitions of the thermodynamical quantities, one may define the infinitesimal heat flow \(\delta Q\) by

\[
\delta Q = dE - P\,dA.
\]

and using the explicit expressions for the pressure and the internal energy one finds

\[
\delta Q = \frac{dM}{\sqrt{f_o(R)}}.
\]

The above is in agreement with the constraint associated with the continuity equation for matter in the form

\[
\frac{dL}{d\tau} = \frac{1}{\sqrt{f_o(R)}} \frac{dM}{d\tau},
\]

where \(L\) is the shell luminosity and \(\tau\) is the proper time of an observer sitting on the shell thus confirming our definitions of internal energy and surface tension \((P)\).

It is now possible to introduce a temperature \(T\) through the second law of thermodynamics, that is the existence of the entropy as the exact differential

\[
dS = \frac{\delta Q}{T}.
\]

The temperature appears as an integrating factor (see also Ref. [12]) which must satisfy the integrability condition

\[
\frac{\partial}{\partial R} \left( T \sqrt{f_o(R)} \right)^{-1} = 0,
\]

whose general solution, which exhibits the usual Tolman radial dependence, is

\[
T = \frac{B_h(R_h)}{\sqrt{f_o(R)}},
\]

where \(B_h = B_h(R_h)\) is an arbitrary function of the inner horizon radius \(R_h = \sqrt{4/3} \ell \cos(\theta/3)\) (where \(\cos\theta = -3\sqrt{3} M \ell\) and the outer horizon is given by \(R_\ell = \sqrt{4/3} \ell \cos(\theta/3 + 4\pi/3)\), with \(\pi < \theta < 3\pi/2\) and \(0 \leq 27 M^2 \ell^2 < 1\), which can be used as an independent thermodynamical variable instead of the Bondi mass \(M = (R_h/2) (1 - R_h^2/\ell^2)\). We finally obtain

\[
dS = \left( 1 - 3 \frac{R_h^2}{\ell^2} \right) \frac{dR_h}{2 B_h}.
\]
Given the expression for the temperature one may evaluate the specific heat at constant radius
\[
C_R \equiv T \left( \frac{\partial S}{\partial T} \right)_R = T \left( \frac{\partial S}{\partial R_h} \right)_R \left( \frac{\partial T}{\partial R_h} \right)_R^{-1}. \tag{12}
\]
With our definition (10) for the temperature we have
\[
C_R = \left[ \frac{2 \ell^2 B'_h}{\ell^2 - 3 R_h^2} + \frac{B_h}{R f_o(R)} \right]^{-1}, \tag{13}
\]
where \(B'_h = dB_h/dR_h\). Thus \(C_R\) shows a singularity for \(R\) satisfying
\[
(R_h - R) \left( R^2 + R R_h + R_h^2 - \ell^2 \right) = \frac{3 R_h^2}{\left( \ln B_h^2 \right)'}. \tag{14}
\]
The specific heat at constant tension takes the form
\[
C_P \equiv T \left( \frac{\partial S}{\partial T} \right)_P = \frac{T}{2 B_h} \left( 1 - 3 \frac{R_h^2}{\ell^2} \right) \left[ \left( \frac{\partial T}{\partial R_h} \right)_R \left( \frac{\partial T}{\partial P} \right)_{R_h} \left( \frac{\partial P}{\partial R_h} \right)_R \right]^{-1}
\]
\[
= \frac{H R f^7/2}{B_h \left( \frac{M}{R} - R^2 \ell^2 \right)^2 - f_0^{3/2} \left( R f_o B_h' \frac{4 R_h}{dM} + B_h \right) H}, \tag{15}
\]
where the function
\[
H(R) = \frac{1}{f_0^{3/2} \left( 1 - 3 \frac{m}{R} + 3 \frac{m^2}{R^2} \right) - \frac{1}{f_1^{3/2}} \left( 1 - 3 \frac{m}{R} + 3 \frac{m^2}{R^2} \right)}. \tag{16}
\]
Other thermodynamical quantities of interest, related to the second derivative of the Gibbs potential \[13\], are the change of the area with respect to the temperature for fixed tension \((\partial A/\partial T)_P\) and with respect to the tension for a fixed temperature \((\partial A/\partial P)_T\). All the above three quantities show a singular behavior if the denominator of Eq. (15) vanishes, that is if there is an \(R\) satisfying
\[
\frac{3 R_h}{2 R} \left( 1 - \frac{R_h^2}{\ell^2} \right) \left[ 1 - \frac{R_h}{2 R} \left( 1 - \frac{R_h^2}{\ell^2} + \frac{R^2}{\ell^2} \right) \right] + \left[ \frac{f_o(R)}{f_1(R)} \right]^{3/2} \left( 1 - 3 \frac{m}{R} + 3 \frac{m^2}{R^2} \right) \
= 1 - \frac{R_h^2}{4 R^2} \left( 1 - \frac{R_h^2}{\ell^2} - \frac{2 R^3}{R_h \ell^2} \right)^2 \left[ 1 + f_o(R) \left( \frac{\ell^2}{\ell^2 - 3 R_h^2} \left( \ln B_h^2 \right)' \right)^{-1} \right]. \tag{17}
\]
In order to have an explicit expression for the specific heats and to proceed further in our investigation, we need an equation of state, that is an expression for the function \(B_h\). Let
us examine a rather general case assuming a power-law dependence of the function $B_h$ on the horizon radius, leading to a temperature

$$T = \frac{1}{\sqrt{f_o(R)}} \frac{1}{4\pi R_h^a},$$

(18)

with $a$ a constant. We can now determine the specific heat at constant area

$$C_R = -4\pi f_o R_h^{a+1} \left( 1 - 3 \frac{R_h^2}{\ell^2} \right) \left( 2a f_o + R_h \frac{\partial f_o}{\partial R_h} \right)^{-1}.$$  \hspace{1cm} (19)

This implies that $C_R$ diverges for

$$0 = 2a f_o + R_h \frac{\partial f_o}{\partial R_h} = 2a \left( 1 - \frac{R_h^2}{\ell^2} \right) - \frac{R_h}{R} \left[ (1 + 2a) - (3 + 2a) \frac{R_h^2}{\ell^2} \right].$$

(20)

We have examined numerically the behavior of $C_R$: for the power-law case it shows two singularities between the black hole and the cosmological horizons and vanishes on both horizons (as shown in Fig. 1).

If we now examine the case for which the temperature is that of a black hole with horizon radius $R_h$ [14], one has

$$B_h = \frac{1}{4\pi R_h} \left( 1 - 3 \frac{R_h^2}{\ell^2} \right),$$

(21)

this seems to be the most natural choice if one assumes that at the end of the collapse the system behaves as if a black hole were being formed (for an analysis supporting the naturalness of this choice see Refs. [10, 15, 16]). On substituting for $B_h$ in Eq. (14) one obtains

$$R - \frac{R^3}{\ell^2} = \frac{R_h}{2} \left( 1 + \frac{3R_h}{\ell^2} \right) \left[ 3 - \frac{2R_h^2}{\ell^2} + \frac{3R_h^4}{\ell^4} \right],$$

(22)

which determines the singularity of the specific heat at constant area. Its behavior is analogous to that exhibited in the “power-law” case of Eq. (18) and is again shown in Fig. 1. We finally note that for the choice of a Hawking temperature, the entropy, using Eqs. (11) and (21), is given by

$$S = \int \frac{\delta Q}{T} = \pi R_h^2 = \frac{1}{4} \text{(horizon area)}.$$ \hspace{1cm} (23)

This expression will exhibit a simple additive property, in the sense that the entropy of two non-interacting (well separated) shells will just be the sum of the two entropies, as expected for usual thermodynamical systems [15]. Such an additive property also rules out any integration constant in Eq. (23). Concerning the specific heat for constant tension $C_P$ for a power-law equation of state [18] the results are again plotted in Fig. 1 and one finds only one singularity on the inner horizon. The case for a Hawking temperature has a similar behaviour.

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1This expression is analogous to Eq. (A.2) of Ref. [5] where a factor $(1 - 3R_h^2/\ell^2)^{-1}$ was inadvertently omitted.
Let us briefly compare our results with those of other approaches. In Ref. [7] a basic statement is that the temperature attributed to the shell is that of a static observer standing just outside the shell, that is

\[ T(R) = \frac{a(R)}{2\pi} = \frac{1}{4\pi} \frac{\partial f_o(R)}{\partial R}, \tag{24} \]

where \( a(R) \) is the acceleration of a static observer. A chemical potential must be introduced in order to obtain the entropy as an exact differential. Indeed, by defining the temperature as related to the acceleration of a static observer on the shell, one is led to the definition of the entropy differential

\[ dS = \beta dM + \beta P dA - \alpha dN, \tag{25} \]

where \( \beta = T^{-1}, \alpha = \mu/T \) with \( \mu \) the chemical potential, and \( N \) can be interpreted as the number of particles in the shell. Using the Gibbs-Duhem relation [7], one has

\[ S = \beta (M + PA) - \alpha N, \tag{26} \]

or

\[ n d\alpha = \beta dP + (\rho + P) d\beta, \tag{27} \]

with \( n = N/A \). This definition, after some calculations, gives an explicit expression for the entropy density as

\[ s = S/A = \beta P = \frac{1}{4} \left( 1 - \frac{1}{\gamma^2} \right), \tag{28} \]

with

\[ \gamma^2 = \frac{1}{8\pi\rho\sqrt{f_o(R)}} \left( \frac{\partial f_o}{\partial R} \right), \tag{29} \]

and we note that \( S \) becomes 1/4 of the horizon area as the shell reaches the inner horizon radius. Further we observe that, on choosing the temperature as suggested in Ref. [7] and using the definition [12], one can see that there is only one singularity in \( C_R \) (see Fig. 2).

Finally we mention the approach of Ref. [6] where an expression for the entropy is derived as a integral over a “Bousso lightsheet” [4] within the framework of the approach to the entropy bounds for asymptotically de Sitter space-times. In this formalism one can obtain the expression for the entropy as a function of the shell radius. The result obtained for the entropy coincides with our definition of the internal energy \( E(R, M) \). This leads to an expression for the temperature as given by Eq. [8],

\[ T = \frac{\sqrt{f_o(R)}}{\pi R}, \tag{30} \]

which does not seem to be related to any property of the shell or of the space-time and does not lead to any significant thermodynamical description of the collapse process. Indeed we note that this temperature vanishes as the shell approaches the black hole horizon. From the above
considerations the only approach compatible with ours is that of Ref. [7] where, however, one does not have a radiative metric and the temperature is determined by the Unruh effect.

Let us then illustrate our approach before comparing our results with those of Ref. [7]. We have analyzed the thermodynamical behavior for the collapse of a radiating shell in de Sitter space-time, under the assumption that the evolution consists of a succession of equilibrium states, that is the process is quasi-static. We note that a radiative process related to the velocity of the collapse (as one would expect) naturally leads to a quasi-static collapse [9]. On identifying the internal energy (3) and surface tension (4) of the shell, we were able to evaluate the specific heats at constant area ($C_R$) and tension ($C_P$) and other related thermodynamical quantities when the temperature is given by a power-law of the horizon radius as in Eq. (18) or by the Hawking value. Since we are just considering the “gravitational” part of the dS-CFT correspondence, we can only study the macroscopic structure of the system and obtain general results. Indeed from Fig. 1 we see that $C_R$ vanishes on the black hole and cosmological horizons and exhibits two singularities located between them. This structure is clearly related to the existence of two horizons, as can be inferred by comparing with the anti-de Sitter case of Ref. [8]. In the case of Ref. [7] there is only one singularity between the horizons and $C_R$ again vanishes on the horizons. The sign of $C_R$ is generally associated with thermodynamical stability, in particular a negative sign implies instability (see Ref. [17]). One can also argue that the singularities of
$C_R$ correspond to second order phase transitions insofar as the collapse (and state of the shell) is expected to be continuous \[13\]. Naturally, should the internal dynamics of the shell exhibit a discontinuity, the above considerations are not valid and one could also have a first order phase transition. However, as we mentioned previously, such considerations would require the knowledge of the shell microscopic structure which should be described by a CFT.

Concerning our results for $C_P$, we note that often phase transitions are associated with the behaviour of the Gibbs potential since they occur for constant temperature and pressure (and chemical potential) \[13\]. Indeed we found that all the second derivatives of the Gibbs potential, and in particular $C_P$, exhibit a singularity at the same point (see Eq. (17)) implying a second order phase transition at the inner horizon (this means that there is no change of entropy as the shell approaches the inner horizon). Let us at this point note the difference between our case and that of Ref. \[7\]. Even with a definition of the temperature as being related to the acceleration of a static observer, the specific heat at constant area shows a singularity (see Fig. 2). However, the nature of the transition is not clear in this case, because of the introduction of a chemical potential.

Let us end with some comments about the Bekenstein-Bousso bound, which, for our system, states that

$$S \leq 2\pi ER ,$$

where $E$ is the total energy of the system and $R$ is its maximum dimension. Eq. (31) is easily satisfied both for Eq. (23) and (28), whereas for the power-law case of Eq. (18) one can use the bound (31) in order to constrain the integration constant for the entropy.

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