Discovery of a phase transition behavior for supply chains against disruptive events

Qihui Yang1,*, Caterina Scoglio1, Don Gruenbacher1

1Department of Electrical and Computer Engineering, Kansas State University, Manhattan, Kansas, 66506, USA

* qihui@ksu.edu

In today’s global economy, supply chain (SC) entities have become increasingly interconnected with demand and supply relationships due to the need for strategic outsourcing. Such interdependence among firms not only increases efficiency, but also creates more vulnerabilities in the system. Natural and human-made disasters such as floods and transport accidents may halt operations and lead to tremendous economic losses. Due to the interdependence among firms, the adverse effects of any disruption can be amplified and spread throughout the systems. This paper aims at understanding the systemic behaviors of SC systems against cascading failures. Considering the upper and lower bound load constraints, i.e., surplus inventory and cost, we examine the fraction of failed entities under load decrease and load fluctuation scenarios through numerical simulations. We also provide analytic results through mean-field theory. Results indicate the occurrence of a first-order phase transition, and such systemic risk failures need to be considered in the future by policymakers in conjunction with SC risk management strategies to improve system resilience. With respect to the lower bound parameter, i.e., cost per output, the system is more robust under power law distributions than uniform distributions, for the studied scenarios.

Introduction

In the competitive economic market, SC entities often build business relationships with outsourcing partners to reduce the overall cost and promote productivity. As a result, SCs have become more complicated and geographically dispersed, increasing the frequencies of SC disruptions. Due to the increased dependencies among entities, disruptions of a company’s operation can result in revenue losses in its business partners, and cascade to other components in the SC, leading to a ripple effect. On the other hand, in recent years, sustainability concerns have pushed for higher efficiency in the use of resources and reduction in protective redundancies in SCs, thus making SCs more susceptible to disruptions. For example, since Australia food SC entities had reduced inventories to expedite the production cycle, it was difficult for them to restock during the Victorian bushfire disaster in 2009. Natural disasters, intentional attacks, and economic recessions can cause significant financial losses, and even market share decrease to the SC entities due to delays in the flow of goods. During the 2011 Japan earthquake and tsunami, Toyota Motor Company suffered at least a 140,000-vehicle production loss. This adverse effect spread to other countries including the UK and the US, which led to a massive collapse in the global automotive and electronics industry. In the following weeks after the disaster, Toyota in North America experienced shortages of over 150 parts, resulting in curtailed operations at only 30% of capacity. In light of these low probability and high impact disruptions, SC resilience has been the subject of many recent studies.

Many works have focused on developing risk management strategies to plan for catastrophic events, increase chain agility, and mitigate risks against SC disruptions. Several studies have analyzed the SC cascading failure process utilizing complex network theory. Tang et al. examined the robustness of an interdependent SC network model, which consists of an undirected cyber layer and a directed physical layer, subject to different node removal strategies. However, failures are overload-driven in this cascading failure model, like most other models built for smart grids, in which power flows exceeding the line capacity can cause power outages. Thus, it may not appropriately reflect the nature of cascading failures in SC systems, which are mostly underload-driven. When entities cannot fulfill the expected production requirement to overcome the fixed production costs, they will fail to gain profit and possibly exit the market. Tang et al. assessed the SC system robustness in the form of production capability losses, but it assumed that the failed node loads only propagate downstream, and entities fail without mitigation strategies. Wang et al. attempted to improve the cluster SC network resilience against cascading failures, leveraging insights from the ant colony’s spatial fidelity zones. In the model developed by Wang et al., a node can propagate the failure impact both upstream and downstream, and can dynamically change the strength of the business relationship with its neighbors. To the best of the authors’ knowledge, there are few works considering underload-driven failures, and most current
Given the tremendous damages caused by the disruptions, understanding the nature of the systemic failure in SC that goes beyond a single component behavior is a significant problem to be addressed. In this paper, we investigate if an abrupt breakdown would emerge in SC systems against underload cascading failures. First, we will build a generalized underload cascade failure model and test the system behavior under scenarios of load decrease and load fluctuations. Entities are modeled as network nodes, and links refer to the supply-demand relationships between the entities. The material flow of goods through an entity node defines its load. When a node is disrupted, its downstream and upstream neighbors will be affected due to supply shortage and demand losses respectively. If the neighbor node’s remaining load drops below the lower bound constraint, i.e., cost, new failure occurs and cascades to other nodes in the entire system. Second, we will provide analytic results based on the assumption of equal load redistribution upon failures. The goal of our work is to shed light on the qualitative behavior of real-world SC systems that policymakers can consider to avoid risks of systemic failure.

The contributions of this paper are: (i) We found numerically and analytically the first-order phase transition of the SC model under a load decrease scenario. In percolation theory, based on statistical physics, this indicates that a small fraction of failures can result in a sudden breakdown in the SC system. (ii) The system is relatively robust against load fluctuations, and we verify that strategies of backup suppliers and surplus inventories often adopted by SC managers to mitigate losses can also improve the system robustness from a systemic failure perspective. (iii) Regarding different distributions of the lower bound load parameter, reflecting cost per output, the underload cascading failure model is more robust under the power-law distribution case than the uniform distribution case, for the studied scenarios. These findings indicate that disruptive events such as load decrease can have catastrophic impacts to the system, and the systemic failure risks need to be considered by SC decision-makers in conjunction with the current SC risk management strategies.

Model formulation

Underload cascading failure model. In this paper, we model SCs as weighted directed networks, in which nodes are entities and edges denote the business relationships between the entities. Featured with multi-hierarchy, enterprises in SC networks are both demand-side and supply-side, while nodes in the same tier have similar network connections, core businesses, and competitive environments. Therefore, nodes in the same category generally compete for upstream suppliers and downstream clients. Due to the absence of empirical data sets, we generate synthetic networks for a four-tier SC network, i.e., suppliers, production centers, distribution centers, and customers. N nodes are generated and equally divided into four tiers, in which one enterprise belongs only to one layer. Links are created randomly with a given connection probability p, which is the likelihood of existing business relationships between nodes in two tiers. Additional efforts are made to ensure the network created is without self-loops. Despite the random placement of links, most nodes will have approximately the same number of connections in the generated system. For example, the average node outdegree of a network with 100 nodes in each tier (N=400) and p=0.1 will be around 10.

The cascading failure processes with and without recovery measures are shown as flowcharts in Fig. 1 with the following steps. Step 0: After model initialization, we uniformly decrease or fluctuate the initial loads to simulate the initial failures. Step 1: These newly failed nodes affect their upstream and downstream neighbors throughout the system, and all loads and flows affected are updated. Step 2: With recovery measures, the nodes affected can mitigate losses by altering flows with existing partners or building new partners. Then, if there are still nodes with load below their lower bound load, new failure occurs and continue the process from Step 1–2. Such a procedure is repeated until no new failure happens.

More specifically, in the model initialization, we set up the initial loads and load constraints of the nodes in the network. The sum of material flows that goes through an entity node defines its load $L_t$ (t−1), i.e., the total number of products that an enterprise has sold per time unit. The total number of products an enterprise currently has, i.e., inventory, defines its upper bound load, $A_t$. In normal condition, an enterprise’s load should be above the lower limit load, $B_t$, to make a profit and survive the competitive market. Each node i has a specific $A_i$ and $B_i$, which are proportional to the initial load $L_i(0)$, and are given by $A_i = aL_i(0)$ ($a > 1$) and $B_i = bL_i(0)$ ($0 < b < 1$), where $a$ and $b$ are tolerance parameters.
In the load propagation process, the impact of failed nodes spreads to the entire system. For example, in Fig. 2, when node 9 and 13 fail at time \( t \), the loads of the failed nodes first propagate to their neighbor nodes 4, 14, 15 and 8, 18 through connectivity links. Then, these affected nodes 14, 15 further impact their downstream nodes 18, 19 and 20 by reducing supply. Meanwhile, node 8 influences its upstream neighbor nodes 2, 3 and 4 by cutting back demand.

In the recovery process, the nodes, which have not been marked as failed, can recover their loads by requesting rush orders or building new business partnerships. To mitigate losses, node \( i \) will first select from the alive nodes in its upstream neighbor set \( \Gamma_i^U \) and downstream neighbor set \( \Gamma_i^D \). If all the neighbor nodes still cannot help it recover above \( B_i \), node \( i \) will build links with alive nodes in its upstream non-neighbor set \( \Gamma_i^U(t) \) and downstream non-neighbor set \( \Gamma_i^D(t) \).

Note that, while in this paper we conduct numerical simulations on a four-tier SC network, the underload cascading process is independent of the network topologies, and can be applied to other types of SC networks. More details can be found in Methods.

**Figure 1.** Flowcharts for the cascading failure process. Given the initial loads for nodes in tier 4, we calculate the initial loads \( L_i(0) \) and load constraints for all the nodes, and the initial flows on the links. Keeping the load constraints for all nodes unchanged, we decrease or fluctuate all the initial loads. The new initial load of node \( i \) is \( L'_i(0) \). If node \( i \) fails at time \( t \), i.e., \( L'_i(t) < B_i \), its load and all the flows through node \( i \) become 0. The new failure propagates throughout the system, and all the loads and flows affected get updated. With recovery measures, nodes can mitigate the losses by altering flows with other nodes under the upper bound load constraint \( A_i \) (see Methods). Such a procedure is repeated until no new failure happens.
Figure 2. Failed node load propagation in a four-tier supply chain. Red circles represent the initial failures caused by load decrease or fluctuations. When nodes 9 and 13 fail, the failure can propagate to their neighbor set 4, 14, 15, and set 8, 18 respectively. Meanwhile, the affected node load will further spread to other nodes in tier 1 and 4 such as nodes 2 and 20.

Notations. The notations frequently used in this study are summarized as follows.

| Notation | Description |
|----------|-------------|
| $A_i$ | Upper bound load of node $i$, i.e., inventory |
| $B_i$ | Lower bound load of node $i$, i.e., costs |
| $a$ | Upper bound parameter associated with $A_i$ |
| $b$ | Lower bound parameter associated with $B_i$, reflecting cost per output |
| $[b_{min}, b_{max}]$ | Range of lower bound parameter $b$ |
| $L_i(t)$ | Load of node $i$ at time $t$ |
| $L'_i(0)$ | New initial load of node $i$ under load decrease/ fluctuations scenario |
| $ΔL_i(t)$ | Absolute value of load loss of node $i$ at time $t$, which equals $L_i(t) - L_i(t-1)$ |
| $ψ_i(t)$ | Load increase of node $i$ at time $t$ during the recovery process |
| $RL_i(t)$ | Residual load of node $i$ at time $t$, which equals $A_i - L_i(t)$, i.e., surplus inventory |
| $F_{ij}(t)$ | Flow on edge $e_{ij}$ between nodes $i, j$ |
| $Γ^U_i(t)$ | Set of upstream nodes connected to node $i$, i.e., upstream neighbor set of node $i$ |
| $Γ^D_i(t)$ | Set of downstream nodes connected to node $i$, i.e., downstream neighbor set of node $i$ |
| $Γ^U_i(t)$ | Set of upstream nodes not connected to node $i$, i.e., upstream non-neighbor set of node $i$ |
| $Γ^D_i(t)$ | Set of downstream nodes not connected to node $i$, i.e., downstream non-neighbor set of node $i$ |
| $δ$ | Strength of load decrease |
| $σ$ | Strength of load fluctuation, i.e., variation size |
| $N$ | Total number of nodes in the network |
| $p$ | Connection probability that a node connects to the nodes in the downstream tier |
| $f_i$ | The fraction of failed nodes in the end |
| $f_t$ | The fraction of failed nodes until time $t$ in the mean-field model |
| $N_t$ | The number of surviving nodes until time $t$ in the mean-field model |
| $L_i(t)$ | Load of each node at time $t$ in the mean-field model |
| $L'_q(0)$ | New initial load of each node under load decrease in the mean-field model |

Table 1. Notations.

Results
Researchers often study the phase transition behavior exhibited by the system through stressing external forces to it until the rupture point. In this work, we examine the system behavior of a four-tier SC network under stress of load decrease and fluctuations. Trade between SC entities often happen periodically, and goods need time to be transported to the buyer. Accordingly, entities usually may not be able to adjust business relationships with others, or switch to another supplier to purchase products in time under disruptive events.

Numerical simulations are conducted using Java programming following the cascading failure process in Fig. 1. Note that each node’s load constraints, $A_i$ and $B_i$, remained fixed as $\delta$ or $\sigma$ changes under each scenario. More specifically, as the failures are underload-driven, we conduct extensive simulations with different distributions of lower bound parameter $b$ (Fig. 3). Two commonly used families of distributions are considered: (i) Uniform and (ii) Power-law distribution, in which uniform distribution provides an intuitive baseline.

Uniformly distributed over $[b_{\text{min}}, b_{\text{max}}]$, denoted by $U[b_{\text{min}}, b_{\text{max}}]$, the probability density function for a random variable $b$ is given by

$$p(b) = \frac{1}{b_{\text{max}} - b_{\text{min}}} \cdot 1_{b_{\text{min}} \leq b \leq b_{\text{max}}}$$  \hspace{1cm} (1)

Following a power-law distribution, the probability density function for a random variable $b$ is of the form

$$p(b) = k \cdot (b)^{-\gamma} \text{ with } b \in [b_{\text{min}}, 1]$$  \hspace{1cm} (2)

![Histogram of b uniformly distributed over [0.2, 0.7]](image1)

(a) Histogram of $b$ uniformly distributed over [0.2, 0.7]

![Histogram of b when $p(b) \propto (b)^{-2}$](image2)

(b) Histogram of $b$ when $p(b) \propto (b)^{-2}$

Figure 3. Examples of lower bound parameter distributions.

**Load decrease.** In this scenario, we consider a negative demand shock, in which demand for goods or services shrinkages suddenly. It is not far away from reality; for instance, the 2008 financial crisis led to a drop in consumer spending, thus production decreased and companies went bankrupt in SCs throughout. Due to uncertain trade policy environments, exports of US soya beans to China dropped by 50% in 2018, which is predicted to have a broader impact both within nations and globally.

To model the load decrease scenario, we simultaneously decrease the initial loads for all nodes in tier 4 by a factor $\delta$, i.e., $L_i'(0) = (1 - \delta)L_i(0)$. Then, we calculate flows and loads for all nodes in tiers 1–3, following the model initialization procedure. Keeping the network topology unchanged, it is equivalent to a uniform decrease of all nodes by a factor $\delta$. In each realization, $\delta$ changes from 0 to 1 with a step size of 0.02. We record the fraction of failed nodes $f$ at the end of the simulation, and the results are averaged over 100 realizations for each network configuration.

As shown in Fig. 4, there is a discontinuous phase transition in all cases. When there are no recovery measures and $b$ is uniformly distributed over $[b_{\text{min}}, b_{\text{max}}]$, we found a sharp collapse of the system in Fig. 4a–c. More specifically, the critical point above which failure occurs is only determined by $b_{\text{max}}$, the upper limit of the uniform distribution. For example, in Fig. 4b, in which $B_i$ ranges between $[0.2L_i(0), 0.7L_i(0)]$, when $\delta > 0.3$, i.e., $L_i'(0) < 0.7L_i(0)$, initial failures start to appear. These initial failures will propagate throughout the system and result in system
collapse. In other words, initial failures when no mitigation strategies are adopted, can result in a sudden breakdown of the system.

Then, we include recovery measures to each case and conduct experiments with different values of the upper bound parameter, \( a \). Results show that the recovery strategy can reduce the scale of systemic failure, and higher surplus inventory yields a smaller fraction of failed nodes. This is because the reconfiguration of the trade flows among alive nodes can absorb losses of nodes affected by the initial failures. In Fig. 4b, when \( \delta > 0.8 \), i.e., \( L_i'(0) < 0.2L_i(0) \), all the loads fail at the beginning and are marked as failed, so they will not mitigate losses under the recovery process.

With \( b \) in the form of power distribution, the system collapses when \( \delta \) increases to 0.88 without recovery process. Compared to the uniform distribution case, most \( b \) values are relatively small (see Fig. 3b), and the system becomes more robust. With recovery measures, we observe almost no difference for the system behavior under different values of \( a \). This is because the surplus inventory, \( 0.2L_i(0) \), can absorb the losses of the new affected nodes.

**Figure 4.** Effects of load decrease. Plots of the fraction of failed nodes \( f \) versus the relative load decrease \( \delta \) mimicking the demand shock. Results are averaged over 100 realizations with network size \( N=400 \) and connection probability \( p=0.1 \). Our results show a sudden breakdown of the system, and mitigation strategies can enhance system robustness.
**Load fluctuations.** We mimic the load fluctuations by setting final customers’ initial load as \( L'_1(0) = (1 + \sigma \xi_i)L_i(0) \), i.e., \( L'_i(0) \in [(1 - \sigma)L_i(0), (1 + \sigma)L_i(0)] \), where \( \xi_i \) is a random variable uniformly distributed in [-1, 1]. Then, we calculate the new initial loads for the nodes in tier 1–3 and flows on the edges. It is equivalent to allowing all the initial loads to fluctuate by a fraction of \( \sigma \). To ensure \( L'_i(0) < A_i \), with \( A_i = aL_i(0) \), upper bound parameter \( a \) is set to be two, when \( \sigma \) varies between [0, 1]. Results are averaged over 100 runs of the simulation.

Fig. 5 shows the changes in the fraction of failed nodes as variation size \( \sigma \) increases. Compared with the load decrease scenario, the system collapse happens less abruptly. When \( b \) is uniformly distributed over [0, 0.9], the fraction of failed nodes reaches a plateau of around 75\% as fluctuation escalates. In comparison, when \( b \) is uniformly distributed over [0, 0.8], about 20\% of nodes failed in the end, suggesting that the system is relatively robust. Recall that \( b \) reflect the cost per output. Results indicate that increasing homogeneity regarding the low cost per output in SC entities may help improve the system robustness against fluctuations.

![Figure 5](image)

**Figure 5.** Effects of load fluctuations. A plot of the fraction of failed nodes versus the relative strength of load fluctuations \( \sigma \). Results are averaged over 100 realizations with network size \( N=400 \) and connection probability \( p=0.1 \). The results indicate that the SC network is more robust under load fluctuations compared to the load decrease scenario.

**Mean-field analysis.** Next, we forecast a first-order transition in the SC system in the mean-field sense. SC disruptions can propagate throughout the system due to the long-chain feature, which is similar to the redistribution of power flows in the whole system upon failures according to the long-range nature of Kirchhoff’s law. This feature inspires us to leverage the equal load redistribution model that has been used in power systems. The assumption is that when a node fails, the load it carries before the failure will be redistributed equally among all the remaining nodes.

To obtain the fraction of failed nodes, we provide analytic solutions by numerically solving Eq. 9 and verify them by iterating the algorithm in Fig. 8. We assume the initial node load \( \bar{L}_0 = 1 \) before the disruptions, and thus lower bound load \( B_1 = b\bar{L}_0 = b \cdot 1 \) with \( b \in [b_{\text{min}}, b_{\text{max}}] \). Under disruption of load decrease, the initial load becomes \( \bar{L}_0 = (1 - \delta)\bar{L}_0 = (1 - \delta) \cdot 1 \). From Fig. 6, we observe a first-order phase transition in all cases. In the uniform distribution case, the critical point below which a sudden collapse happens is only determined by \( b_{\text{max}} \), the upper constraint of the uniform distribution. When the new initial load \( \bar{L}_0 \) is above \( B_1 \), there are no failures in the system. Once \( \bar{L}_0 \) falls below \( B_1 \), the system collapses because the fraction of failed node obtained from the equation will increase to 1. When \( b \) follows a power-law distribution, the system is more robust, and there is an abrupt breakdown of the system around \( \delta \approx 0.9 \). Interestingly, this is contrary of the mean-field result for the overload cascade model, in which the first-order jump happens with no precursors in power-law distributions, and with precursors in line capacity following a uniform distribution. This indicates that the scale of cascade failures in SCs could be significantly affected by the
shape of the $b$ distribution. Since $b = B/L_0$ is related to a company’s cost, we can conclude that the system with entities of similar low cost per output are more robust to SC disruptions, compared to the one with various cost.

![Figure 6](image.png)

**Figure 6.** Mean-field results. Plots of the fraction of failed nodes $f$ as a function of the relative load decrease $\delta$. The results obtained from the algorithm are average over 100 runs of simulation. As $\delta$ continues to increase, all the nodes in the network will eventually fail. Regarding the lower bound parameter, $b$, the system is more robust under power-law distributions than uniform distributions.

**Discussion**

In this paper, we constructed a four-tier SC system and explored its emergent behavior against underload cascading failures. When disruption occurs, the node with load below the lower bound fails, and propagates its failed node load to its upstream and downstream neighbors. The model proposed exhibits a first-order phase transition behavior under load decrease scenario as predicted by mean-field analysis. More specifically, under different distributions of lower bound parameter $b$, i.e., cost per output, the system is more robust when $b$ follows a power distribution compared to the uniform distribution, for the studied scenario. The results suggest that increasing homogeneity regarding low cost per output of SC entities would help reduce the possibility of a systemic failure.

The system is more robust against load fluctuations than load decrease. In the load fluctuation scenario, the discontinuous transition occurs only when variation size $\sigma$ is very high, which may rarely happen in reality. The emergent behaviors observed from the underload-driven model proposed in this work is different from the analytic
results derived from the overload-driven system by Pahwa et al.\textsuperscript{22}, in which power-law distribution of capacity results in a more abrupt system breakdown.

Besides, we include recovery strategies to the cascading failure process, and the simulation results show that surplus inventory and backup suppliers can significantly reduce the scale of the systemic failure when disruptive events happen. Therefore, it is essential to consider a systemic risk perspective of the SC in conjunction with the managerial decision making aspects of risk mitigation\textsuperscript{25}. Since SC networks are more vulnerable against disruptive events such as load decrease, i.e., demand shock, than load fluctuations, it is crucial for policymakers to pay attention to these events and maintain a stable trade environment. Besides, the discontinuous phase transition behavior of the system indicates it is hard to predict a catastrophic failure in SCs, and thus SC entity decision makers need to take proactive protection measures to avoid the impact of a demand shock.

As SCs grow globally, systemic risk analysis of the SCs needs further exploration. In this paper, we qualitatively show the dynamic behavior of SCs against disruptions and do not take the whole complexity of SCs into account. For example, the four-tier SC model we built disregards the possibility of a business relationship which crosses the tiers, e.g., tier 1 and 3, and the possible internal connections between entities in the same tier. It also did not reflect dynamic entry and exit mechanisms of node enterprises in SCs, in which weaker enterprises may be eliminated and replaced by new members. Future work can incorporate these features into the current model and analyze the system behavior under disruptive events. Although many SC networks are found to follow power-law distributions\textsuperscript{26–28}, exceptions exist\textsuperscript{29} and there is no consensus on the SC network topologies. Besides, since the links in the SC network are randomly generated based on a connection probability and we disrupt all nodes with a factor $\delta$ or $\sigma$, the initial failures triggered in this work can be seen as caused by random attacks. More simulations can be conducted to examine the phase transition behavior of SC systems under targeted attack with various network topologies. In this work, we mainly focus on the discovery of a phase transition, thus designing an optimal recovery strategy is not our primary focus. The recovery process developed assumes that SC entities have full knowledge of the surplus inventory information in the whole system, and future work can include more realistic assumptions regarding mitigation strategies.

**Methods**

**Underload cascading failure process.** In a SC network, the load on a node is defined as the material flowing through the node per time unit. The initial load on a node $i$, denoted as $L_i(0)$, varies between the lower bound load $B_i$ and upper bound load $A_i$. More specifically, $A_i$, representing a firm’s inventory, is given by $A_i = a L_i(0)$, where $a$ is the upper bound parameter, $a > 1$. Reflecting costs such as labor cost and maintenance fee, $B_i$ is calculated by $B_i = bL_i(0)$, with $b$ denoting the lower bound parameter, $0 < b < 1$. Residual load of a node $i$ at time $t$, given by $RL_i(t) = A_i - L_i(t)$, indicates the additional available products an entity can provide, i.e., surplus inventory. We assume a node fails if its load $L_i(t)$ falls below $B_i$, meaning that the entity fails to gain profit and survive in the competitive market.

According to the definition of node load, the sum of incoming flows always equals the sum of outgoing flows for each node $i$, i.e., $\sum_{j \in \Gamma_i^+} F_{ij}(t) = \sum_{j \in \Gamma_i^-} F_{ji}(t)$, with demand and supply balanced. We utilize a four-tier network as an example and illustrate the cascading failure process as follows.

**Step 0: Model initialization.** In this step, we first calculate the node degree for all nodes and weight on all edges. Node degree of node $i$ is the sum of its indegree $d_i^+$ and outdegree $d_i^-$, i.e., $d_i = d_i^+ + d_i^-$. Consistent with reference\textsuperscript{14,15}, weight on $e_{ij}$ reflects the business relationship strength between node $i$ and $j$, and is calculated by $w_{ij} = (d_i \cdot d_j)^\theta$, with $\theta = 0.5$.

Then, we compute the initial loads and node loads in the network. Suppose the initial loads for nodes in tier 4, equivalent to the demand for final customers, are known and equal. We rescale the weight on incoming edges of node $j$ such that new weights on incoming edges of node $j$ sums to 1, i.e., $\sum_{i \in \Gamma_j^+} w'_{ij} = 1$:

$$w'_{ij} = \frac{w_{ij}}{\sum_{k \in \Gamma_j^+} w_{kj}} \quad \text{(3)}$$

where $\sum_{k \in \Gamma_j^+} w_{kj}$ is the sum of weights on incoming edges of node $j$. 


Given node loads in tier 4, \( L_j(0) \), we calculate the initial flow over \( e_{ij} \) between tier 3 and 4 by

\[
F_{ij}(0) = w'_{ij} \cdot L_j(0)
\]  

(4)

Accordingly, the load of node \( i \) in tier 3 is the sum of the outgoing flows of node \( i \)

\[
L_i(0) = \sum_{\text{out of } i} F_{ik}(0)
\]  

(5)

Following the same method, we sequentially calculate the initial loads of nodes in tier 2 and 1. After the determination of all nodes’ initial loads, we calculate the load constraints \( A_t \) and \( B_t \) of each node.

**Step 1: Failed node load propagation.** We assume that when a node fails, it can neither receive supplies from upstream neighbors, nor ship products to its downstream partners. Once a node \( i \) fails at time \( t \), both its load and the flows on its incoming and outgoing edges are set to 0. The impact of failed node \( i \) propagates upstream using Eq. 6. In the first propagation, we calculate the loss of each node \( j \) in its upstream neighbor set, \( \Gamma_i^U \), suffers at time \( t \), i.e., \( \Delta L_j(t) \), and update the corresponding loads and flows. Meanwhile, in the second propagation, affected nodes \( j \) further impact nodes \( k \) in its upstream neighbor set, \( \Gamma_j^U \) using Eq. 7.

\[
\begin{align}
\Delta L_j(t) & = L_j(t - 1) - \Delta L_j(t) \\
L_j(t) & = L_j(t - 1) - \Delta L_j(t) \\
F_{ji}(t) & = F_{ji}(t - 1) - \Delta L_j(t)
\end{align}
\]  

(6)

\[
\begin{align}
\Delta L_k(t) & = \Delta L_j(t) - \Delta L_k(t) \\
L_k(t) & = L_k(t - 1) - \Delta L_k(t) \\
F_{kj}(t) & = F_{kj}(t - 1) - \Delta L_k(t)
\end{align}
\]  

(7)

Similarly, we update the loads and flows for each node \( j \) in \( \Gamma_i^D \) and then the downstream neighbor nodes of node \( j \). This process continues to tier 1 and 4, mimicking the ripple effect spreading throughout the system.

**Step 2: Load recovery process.** Facing changes in the external environment, entities can request rush orders with existing partners or develop new partners to mitigate losses\(^{14}\). Since acquiring new partners will bring additional costs, we assume a node would first recover its load by selecting from the surviving nodes in its neighbor sets, \( \Gamma_i^U \) and \( \Gamma_i^D \). Only if all the neighbor nodes cannot help it recover above \( B_t \), the node will build links with alive nodes in its non-neighbor sets, \( \Gamma_i^U(t) \) and \( \Gamma_i^D(t) \).

In this work, we design a recovery process to maximize the resource allocation, while satisfying that (i) each node at most provides its residual, i.e., \( L_i(t) < A_t(t) \), and (ii) flows are balanced for each node. To imitate that node \( i \) requests rush orders from downstream neighbor node \( j \) in \( \Gamma_i^D \), we update the outgoing flow of node \( i \) and loads of both nodes with the same increase, \( \psi_i(t) = \psi_j(t) \). Meanwhile, node \( i \) needs to reconfigure its incoming flows with its neighbor nodes in \( \Gamma_i^U \) to balance its supply and demand. As a result, the sum of load increase, \( \sum_{\text{tier } s} \psi_i(t) \) in each tier \( s \) are equal. On the other hand, since the number of surviving nodes, which are eligible to recover, is not necessarily the same for each tier, load loss \( \sum_{\text{tier } s} (L_i(0) - L_i(t)) \) of all the surviving nodes in each tier can be different. Similarly, the sum of residual load for nodes in each tier \( s \), \( \sum_{\text{tier } s} RL_i(t) \), may be different.

Therefore, we need to identify tier TierA with the most loss \( \sum_{\text{tier } s} (L_i(0) - L_i(t)) \), i.e., \( \Delta L_{MAX} \), and tier TierB with the minimum \( \sum_{\text{tier } s} RL_i(t) \), called \( RL_{MIN} \) (Eq. 8). In each tier, the load increase \( \psi_i(t) \) allocated to each node should sum up to the minimum value of \( RL_{MIN} \) and \( \Delta L_{MAX} \), the tier of which is defined as \textit{minTier}. Once each tier reaches the load increase objective \( Q = \sum_{\text{tier } s} \psi_i(t) \), the recovery process stops.
\begin{align}
RL_{\text{MIN}} &= \sum_{i \in \text{TierA}} RL_i(t) \\
\Delta L_{\text{MAX}} &= \sum_{i \in \text{TierB}} (L'_i(0) - L_i(t)) \\
\sum_{i \in \text{Tier}} \psi_i(t) &= \min(RL_{\text{MIN}}, \Delta L_{\text{MAX}})
\end{align}

For implementation, we construct two recursive functions, i.e., \textit{askSupplier} and \textit{askNonSupplier}, to increase the load upstream in Fig 7. First, we allocate load increase \( Q \) to the nodes in \textit{minTier} tier. Then, the node \( i \) in \textit{minTier}, who get their load increased, \( \psi_i(t) \), distributes the load to its suppliers in \( \Gamma_i^U \). Note that the total load increase of nodes in \( \Gamma_i^U \) should equal \( \psi_i(t) \). If the first selected supplier \( j \)'s \( RL_j(t) \) cannot satisfy node \( i \)'s request, node \( i \) will continue to ask from the next supplier in \( \Gamma_i^U \). If \( \sum_{j \in \Gamma_i^U} \psi_j(t) < \psi_i(t) \), node \( i \) will call \textit{askNonSupplier} function to build links with nodes in non-neighbor set \( \Gamma_i^D(t) \), until objective \( \psi_i(t) \) is met. This process continues until all related suppliers in tier 1 have increased their loads. Similarly, we use \textit{askCustomer} and \textit{askNonCustomer} functions to update the loads of nodes downstream.

\begin{verbatim}
Input: nodes in all tiers
1 Find the tier with \( RL_{\text{MIN}} \) for surviving nodes, and save the index to \textit{TierA}
2 Find the tier with \( \Delta L_{\text{MAX}} \) for surviving nodes, and save the index to \textit{TierB}
3 if (\( RL_{\text{MIN}} \geq \Delta L_{\text{MAX}} \))
4 \( Q \leftarrow \Delta L_{\text{MAX}} \)
5 \textit{minTier} \leftarrow \textit{TierA}
6 \textit{flag} \leftarrow 0
7 else
8 \( Q \leftarrow RL_{\text{MIN}} \)
9 \textit{minTier} \leftarrow \textit{TierB}
10 \textit{flag} \leftarrow 1
11 end if
12 for each node in the unfailed node set of \textit{minTier}
13 if (\( \textit{flag} = 0 \))
14 \( \text{Req} \leftarrow \text{node.deltL}, Q/L'_i(0) - L_i(t) \)
15 else
16 \( \text{Req} \leftarrow \text{node.residual}; \)
17 end if
18 if (\( \text{Req} != 0 \))
19 \( Q \leftarrow \text{Q-Req}; \)
20 \text{node.load} \leftarrow \text{node.load}+ \text{Req};
21 \text{node.residual} \leftarrow \text{node.residual}-\text{Req};
22 \text{LoadLeft} \leftarrow \text{node.askSupplier(this, Req)};
23 if (\( \text{LoadLeft} != 0 \))
24 \text{node. askNonSupplier (this, LoadLeft)}
25 end if
26 \text{LoadLeft} \leftarrow \text{node. askCustomer (this, Req)};
27 if (\( \text{LoadLeft} != 0 \)) \{ //build new links with other surviving nodes
28 \text{node. askNonCustomer (this, LoadLeft)};
29 end if
30 end if
31 if (\( Q = 0 \))
32 break;
33 end if
34 end for
\end{verbatim}

\textbf{Figure 7.} Algorithm for the recovery process

\textbf{Mean-field analysis.} The effects of SC disruptions can spread to the entire system, which has some common characteristics with the long-range nature of cascading failures in power systems. The reconfiguration of the material flows in the SC system is dependent on the business relationships among entities, like the impedance of the power lines. Initially, after a disruption, some nodes could get underloaded and fail, i.e., the first stage of the cascade failure process. The first stage could result in more failures, constituting the second stage and so on. The system goes through multiple times of cascades until no new failures occur.

The wide-range interactions of SC disruption propagation allow us to use a mean-field analysis. Here, we adopted the equal load redistribution model that has been widely used in power systems\textsuperscript{21,22,30}. The equal load redistribution
assumption is originated from the widely used democratic fiber bundle model\textsuperscript{11}, in which $N$ parallel fibers with different failure capacity share an applied force equally. In the following, we analyze the load decrease scenario using a simple equal-load redistribution model to predict a first-order phase transition.

We consider $N$ nodes with a lower bound load $B_i$ characterized by a probability distribution $p(B)$. Suppose failures happen in discrete time steps $t=0, 1, \ldots$. The fraction of failed nodes and the number of surviving nodes until cascade stage $t$ is denoted as $f_t$ and $N_t$ respectively. When the load of a node goes below $B_i$, the node fails and its load gets redistributed equally among the remaining surviving nodes. For simplicity, we did not consider the upper bound load $A$, and only focus on the threshold of the failure $B_i$ in the mean-field analysis.

There is no failure before the disruptions, thus $f_0 = 0$ and $N_0 = N$. Under disruptions, we assume all the nodes initially carried the same load $L_0^*$. In the mean-field sense, a fraction of nodes $f_1 = \int_{B_i}^{\infty} p(B) dB$ immediately fails, since their load $L_0^*$ is below the lower bound load. After the first stage, the number of surviving nodes equals to $N_1 = (1 - f_1)N$, and the new load per node becomes $L_1 = L_0^* - \int_{f_1 B_0}^{L_0^*} (1 - f_1) N_t = \left(1 - \frac{f_1}{1 - f_1} \right) L_0^*$. The cascade failure process continues recursively, and the mean-field equations for the $(t + 1)^{th}$ stage are as follows:

\[
\begin{align*}
    f_{t+1} &= \int_{B_i}^{\infty} p(B) dB \\
    N_{t+1} &= (1 - f_{t+1})N \\
    L_{t+1} &= L_t - \frac{(f_{t+1} N - f_t N)L_t}{N_{t+1}} = \left(1 - \frac{L_{t+1} - L_t}{1 - f_{t+1}} \right) L_t
\end{align*}
\]

where $\frac{f_{t+1} N - f_t N}{N_{t+1}}$ is the fraction of failed nodes.

Eq. 9 can be simplified as

\[
    f_{t+1} = F(L_0^*) \prod_{t=1}^{t} \left(1 - \frac{f_t - f_{t-1}}{1 - f_t} \right)
\]

where $F(x) = \int_{x}^{\infty} p(B) dB$.

```
Input: number of nodes $N$, initial load per node $L_0^*$
Output: fraction of failed nodes $f$
1 Create a new ArrayList of $M$ nodes nodeSet
2 for each node $m$ in nodeSet
3     $m.B \leftarrow$ a random variable $b$ following probability distribution $p(b)$
4     $m.load \leftarrow L_0^*$
5 end for
6 Create a null ArrayList of nodes called failnodeSet
7 while (nodeSet.size() != 0){
8     loadReduce $\leftarrow$ 0
9     for each node $m$ in nodeSet
10        if (m.load < m.B)
11           loadReduce $\leftarrow$ loadReduce + m.load;
12           failnodes.add(m);
13        end if
14     end for
15     if (loadReduce==0)
16         break;
17     end if
18     nodeSet.removeAll(failnodeSet)
19     for each node $m$ in nodeSet
20         $m.load \leftarrow m. load - Math.min(m.load, loadReduce/nodeSet.size());$
21     end for
22 end while
23 Return $f \leftarrow (N - nodeSet.size())/N$
```

**Figure 8.** Algorithm for the mean-field model under load decrease scenario
From Eq. 10, we can see that the critical point \( f^* \) is mainly dependent on the distribution of \( B \). Before the disruption, we assume the initial node load \( L_0 = 1 \), and thus \( B = bL_0 \). \( b = 1 \). To examine the effects of distributions, we compare the results under uniform distribution and power distribution of \( b \) by numerically solving Eq. 9. The fraction of failed nodes is also calculated through iterating the algorithm in Fig. 8, and both methods yield the same results.

References

1. Bode, C. & Wagner, S. M. Structural drivers of upstream supply chain complexity and the frequency of supply chain disruptions. J. Oper. Manag. 36, 215–228 (2015).
2. Ivanov, D., Sokolov, B. & Dolgui, A. The Ripple effect in supply chains: trade-off ‘efficiency-flexibility-resilience’ in disruption management. Int. J. Prod. Res. 52, 2154–2172 (2014).
3. Reyes Levalle, R. & Nof, S. Y. Resilience by teaming in supply network formation and re-configuration. Int. J. Prod. Econ. 160, 80–93 (2015).
4. Ivanov, D., Sokolov, B., Solovyeva, I., Dolgui, A. & Jie, F. Ripple Effect in the Time-Critical Food Supply Chains and Recovery Policies. IFAC-Pap. 48, 1682–1687 (2015).
5. Ivanov, D., Sokolov, B., Solovyeva, I., Dolgui, A. & Jie, F. Dynamic recovery policies for time-critical supply chains under conditions of ripple effect. Int. J. Prod. Res. 54, 7245–7258 (2016).
6. Hosseini, S., Ivanov, D. & Dolgui, A. Review of quantitative methods for supply chain resilience analysis. Transp. Res. Part E Logist. Transp. Rev. 125, 285–307 (2019).
7. Hosseini, S. et al. Resilient supplier selection and optimal order allocation under disruption risks. Int. J. Prod. Econ. 213, 124–137 (2019).
8. Canis, B. Motor Vehicle Supply Chain: Effects of the Japanese Earthquake and Tsunami. (DIANE Publishing, 2011).
9. Ivanov, D. & Dolgui, A. Low-Certainty-Need (LCN) supply chains: a new perspective in managing disruption risks and resilience. Int. J. Prod. Res. 0, 1–18 (2018).
10. Salehi Sadghiani, N., Torabi, S. A. & Sahebjamnia, N. Retail supply chain network design under operational and disruption risks. Transp. Res. Part E Logist. Transp. Rev. 75, 95–114 (2015).
11. Tang, L., Jing, K., He, J. & Stanley, H. E. Complex interdependent supply chain networks: Cascading failure and robustness. Phys. Stat. Mech. Its Appl. 443, 58–69 (2016).
12. Zeng, Y. & Xiao, R. Modelling of cluster supply network with cascading failure spread and its vulnerability analysis. Int. J. Prod. Res. 52, 6938–6953 (2014).
13. Tang, L., Jing, K., He, J. & Stanley, H. E. Robustness of assembly supply chain networks by considering risk propagation and cascading failure. Phys. Stat. Mech. Its Appl. 459, 129–139 (2016).
14. Wang, Y. & Xiao, R. An ant colony based resilience approach to cascading failures in cluster supply network. Phys. Stat. Mech. Its Appl. 462, 150–166 (2016).
15. Wang, Y. & Zhang, F. Modeling and analysis of under-load-based cascading failures in supply chain networks. Nonlinear Dyn. 92, 1403–1417 (2018).
16. Drzymalski, J., Odrey, N. G. & Wilson, G. R. Aggregating performance measures of a multi-echelon supply chain using the analytical network and analytical hierarchy process. Int. J. Serv. Econ. Manag. 2, 286–306 (2010).
17. Ivanov, D., Sokolov, B. & Kaeschel, J. A multi-structural framework for adaptive supply chain planning and operations control with structure dynamics considerations. Eur. J. Oper. Res. 200, 409–420 (2010).
18. Fang, H. et al. Network evolution model for supply chain with manufactures as the core. PLOS ONE 13, e0191180 (2018).
19. Zapperi, S., Ray, P., Stanley, H. E. & Vespignani, A. First-Order Transition in the Breakdown of Disordered Media. Phys. Rev. Lett. 78, 1408–1411 (1997).
20. Yang, Q. et al. Developing an agent-based model to simulate the beef cattle production and transportation in southwest Kansas. Phys. Stat. Mech. Its Appl. 526, 120856 (2019).
21. Zhang, Y. & Yağan, O. Optimizing the robustness of electrical power systems against cascading failures. Sci. Rep. 6, 27625 (2016).
22. Pahwa, S., Scoglio, C. & Scala, A. Abruptness of Cascade Failures in Power Grids. Sci. Rep. 4, 3694 (2014).
23. Dolgui, A., Ivanov, D. & Sokolov, B. Ripple effect in the supply chain: an analysis and recent literature. Int. J. Prod. Res. 56, 414–430 (2018).
24. Fuchs, R. et al. Why the US–China trade war spells disaster for the Amazon. Nature 567, 451 (2019).
25. Scheibe, K. P. & Blackhurst, J. Supply chain disruption propagation: a systemic risk and normal accident theory perspective. *Int. J. Prod. Res.* **56**, 43–59 (2018).

26. Sun, J., Tang, J., Fu, W. & Wu, B. Hybrid modeling and empirical analysis of automobile supply chain network. *Phys. Stat. Mech. Its Appl.* **473**, 377–389 (2017).

27. Büttner, K., Krieter, J., Traulsen, A. & Traulsen, I. Static network analysis of a pork supply chain in Northern Germany—Characterisation of the potential spread of infectious diseases via animal movements. *Prev. Vet. Med.* **110**, 418–428 (2013).

28. Perera, S., Bell, M. G. H. & Bliemer, M. C. J. Network science approach to modelling the topology and robustness of supply chain networks: a review and perspective. *Appl. Netw. Sci.* **2**, 33 (2017).

29. Kito, T., Brintrup, A., New, S. & Reed-Tsochas, F. *The Structure of the Toyota Supply Network: An Empirical Analysis*. (Social Science Research Network, 2014).

30. Yağan, O. Robustness of power systems under a democratic-fiber-bundle-like model. *Phys. Rev. E* **91**, 062811 (2015).

31. Daniels Henry Ellis & Jeffreys Harold. The statistical theory of the strength of bundles of threads. I. *Proc. R. Soc. Lond. Ser. Math. Phys. Sci.* **183**, 405–435 (1945).

Acknowledgements
The authors acknowledge the financial support by the U.S. National Science Foundation under Grant Award CMMI-1744812. The contents of the paper do not necessarily reflect the position or the policy of funding parties.

Author contributions
Q.Y., C.S. and D.G. conceived and designed the study. Q.Y. conducted the simulation experiments. Q.Y., C.S. and D.G. analyzed the results and contributed to the writing. All authors reviewed the manuscript.

Additional Information
Competing Interests: The authors declare no competing interests.