Quasinormal modes for arbitrary spins in the Schwarzschild background

I.B. Khriplovich and G.Yu. Ruban

Budker Institute of Nuclear Physics
630090 Novosibirsk, Russia,
and Novosibirsk University

Abstract

The leading term of the asymptotic of quasinormal modes in the Schwarzschild background, \( \omega_n = -i n/2 \), is obtained in two straightforward analytical ways for arbitrary spins. One of these approaches requires almost no calculations. As simply we demonstrate that for any odd integer spin, described by the Teukolsky equation, the first correction to the leading term vanishes. Then, this correction for half-integer spins is obtained in a slightly more intricate way. At last, we derive analytically the general expression for the first correction for all spins, described by the Teukolsky equation.

1 Introduction

The investigation of perturbations of various fields in the Schwarzschild background was started in [1, 2]. Quasinormal modes (QNM) are the eigenmodes of the homogeneous wave equations, describing these perturbations, with the boundary conditions corresponding to outgoing waves at the spatial infinity and incoming waves at the horizon. The interest to QNMs was initiated by [3, 4].

Two boundary conditions make the frequency spectrum \( \omega_n \) of QNMs discrete. The asymptotic form of this spectrum for gravitational and scalar perturbations of the Schwarzschild background was found at first numerically in [5, 6]:

\[
\omega_n = -\frac{i}{2} \left( n + \frac{1}{2} \right) + 0.087424, \quad n \to \infty, \quad s = 0, 2. 
\] (1)

Here and below the gravitational radius \( r_g \) is put to unity; \( s \) is the spin of the perturbation. This result up to now serves as a touch stone for investigations in the field.

A curious observation was made in [7]: the real constant in (1) can be presented as

\[
\text{Re} \omega_n = \frac{\ln 3}{4\pi} = T_H \ln 3,
\] (2)

where \( T_H \) is the Hawking temperature \( (T_H = 1/(8\pi kM) \) in the common units\)\(^3\). Then, expression (2) for the asymptotic of \( \text{Re} \omega_n \) was derived in [9] by solving approximately the
recursion relations used previously in the numerical calculations. In the next paper \cite{10} formula (2) was derived analytically. Besides, in \cite{9} the following result was obtained for the asymptotic of QNMs for spin 1:

$$\omega_n = -\frac{i}{2} n, \quad \Re \omega_n \to 0, \quad n \to \infty, \quad s = 1.$$  (3)

Asymptotic (3) was also obtained numerically in \cite{11}.

While the results (1) and (3) for integer $s$ are firmly established now, it is not the case for spin $1/2$. Two different approaches\footnote{We believe that one of them, despite being rather popular, can be dismissed at once. It is based on the analysis of the location of the poles of the scattering amplitude, which by itself causes no objections. However, following \cite{14}–\cite{16}, the authors of \cite{12} analyze the poles of the corresponding Born amplitude. Meanwhile, the Born approximation by itself implies that the amplitude of the scattered wave is small. Therefore, its poles have no real meaning. Any coincidence between their position and that of the poles of a true amplitude is an accident only.} used in \cite{12} result in the interval two times smaller than those for integer spins, namely:

$$\omega_n = -\frac{i}{4} n, \quad n \to \infty, \quad s = 1/2. \quad (4)$$

On the other hand, numerical calculations in \cite{13} result in spectrum

$$\omega_n = -\frac{i}{2} n, \quad \Re \omega_n \to 0, \quad n \to \infty, \quad s = 1/2. \quad (5)$$

One of the motivations of our work was the resolution of this discrepancy; we not only confirm below equation (5), but find also first nonvanishing correction to it.

We consider the QNM problem in various analytical approaches. Two of them, rather simple and straightforward, give in fact only the leading asymptotic, $\omega_n = -i n/2$ for any spin. In the third approach, based on the Teukolsky equation, we at first demonstrate as easily that equation (3) is accurate for arbitrary odd integer spins, i.e. that first subleading correction to it vanishes. Then, with somewhat more efforts, we obtain this correction for half-integer spins. At last, we derive, in a more involved way, the unified general expression for the next term in the asymptotic values of the QNMs for all spins, which was conjectured previously in \cite{10}.

2 Quasinormal modes in Regge – Wheeler formalism

The Regge – Wheeler equation for the radial function $\Psi$ corresponding to the angular momentum $j$ of a field with integer spin $s$ ($s = 0, 1, 2; \ j \geq s$) is written usually as

$$\frac{d^2 \Psi}{dz^2} + \left\{ \omega^2 - \left(1 - \frac{1}{r}\right) \left[ \frac{j(j+1)}{r^2} + \frac{1-s^2}{r^3} \right] \right\} \Psi = 0. \quad (6)$$

Its analogue for $s = 1/2$ (again the angular momentum $j \geq s$), written for the standard representation of the Dirac $\gamma$-matrices and states of definite parity, is

$$\frac{d^2 \Psi}{dz^2} + \left\{ \omega^2 - \left(1 - \frac{1}{r}\right) \left( \frac{j+1}{r^2} \right)^2 + \frac{\kappa}{2r^3} \left(1 - \frac{1}{r}\right)^{1/2} - \frac{\kappa}{r^2} \left(1 - \frac{1}{r}\right)^{3/2} \right\} \Psi = 0; \quad (7)$$
κ = ±(j + 1/2), with the sign depending on the parity of the state considered (this sign is irrelevant for our problem). The presence of the terms with fractional powers of r and r−1 in equation (7) is quite natural since wave equations for half-integer spins are written via tetrads which are roughly square roots of metric.\(^5\)

In both equations, (6) and (7), r is treated as a function of the so-called “tortoise” coordinate z. They are related as follows: z = r + ln(r−1), so that z → ∞ for r → ∞, and z → −∞ for r → 1. The boundary conditions for QNMs of (6) and (7) are

\[\Psi(z) \sim e^{±iωz}, \quad z \to ±∞.\] (8)

Here, for our purpose, it is convenient to go over in both equations, (6) and (7), to the usual coordinate r and to the new radial function u(r) related to Ψ as follows:

\[Ψ = \frac{r^{1/2}}{(r−1)^{1/2}} u(r).\] (9)

The obtained equations for u(r) can be rewritten as

\[\frac{d^2u}{dr^2} + \left\{ω^2 + \frac{1}{r−1} \left[2ω^2 - \left(j + \frac{1}{2}\right)^2 + s^2 - \frac{1}{4}\right] + \frac{1}{(r−1)^2} \left(ω^2 + \frac{1}{4}\right) \right.\]

\[+ \frac{1}{r} \left[\left(j + \frac{1}{2}\right)^2 - s^2 + \frac{1}{4}\right] + \frac{1}{r^2} \left(-s^2 + \frac{1}{4}\right)\} u = 0, \quad s = 0, 1, 2; \] (10)

\[\frac{d^2u}{dr^2} + \left\{ω^2 + \frac{1}{r−1} \left[2ω^2 - \left(j + \frac{1}{2}\right)^2 + \frac{1}{2}\right] + \frac{1}{(r−1)^2} \left(ω^2 + \frac{1}{4}\right) \right.\]

\[+ \frac{1}{r} \left[\left(j + \frac{1}{2}\right)^2 - \frac{1}{2}\right] - \frac{3}{4} \frac{1}{r^2}\]

\[- \frac{κ}{r^{3/2}(r−1)^{1/2}} + \frac{1}{2} \frac{κ}{r^{3/2}(r−1)^{3/2}}\} u = 0, \quad s = 1/2. \] (11)

We are interested in the solutions of equations (10) and (11) in the interval 1 < r < ∞ for |ω| → ∞. Obviously, all the terms singular at r → 0, in both these equations, are relatively small in this interval if |ω| → ∞.\(^6\) Therefore, these terms can be safely omitted, and we arrive at the following universal truncated wave equation for all spins:

\[\frac{d^2u}{dr^2} + \left[ω^2 + \frac{2ω^2}{r−1} + \frac{ω^2 + 1/4}{(r−1)^2}\right] u = 0. \] (12)

We have omitted here also the terms −(j + 1/2)^2 + s^2 and -(j + 1/2)^2 + 1/2 in the coefficients at 1/(r−1) in (10) and (11), respectively. Though these terms could be easily

\(^5\)We mention here another rather popular, but false belief, namely, that equation (6) applies to half-integer s as well. The explicit difference between (6) and (7) demonstrates that this idea is wrong.

\(^6\)In particular, in equation (11)

\[|κ| r^{-3/2}(r−1)^{-1/2} \ll |ω^2|(r−1)^{-1}, \quad \text{and} \quad |κ| r^{-3/2}(r−1)^{-3/2} \ll |ω^2|(r−1)^{-2},\]

for the interval 1 < r < ∞.
included into the solutions, they would result in corrections to \( \text{Im}\omega_n \) on the order of \( 1/n \) only, which are negligible as compared to the leading term \( \sim 1/n \).

We retain however the term \( 1/4 \) in the coefficient at \( 1/(r-1)^2 \) in (12). Otherwise the wave function asymptotic for \( z \to -\infty \) would be \( e^{-i\omega z + 1/2} \), instead of \( e^{-i\omega z} \). In other words, the effective potential in the initial Regge – Wheeler equations (3), (7) would not vanish for \( z \to -\infty \), but would tend instead to \( 1/4 \). Indeed, the wave function asymptotic for \( z \to -\infty \) is determined by the discussed coefficient at \( 1/(r-1)^2 \) in (12). Since the coefficient \( \omega^2 + 1/4 \) at \( 1/(r-1)^2 \) in (12) corresponds to \( \omega^2 \) in equations (6), (7), then obviously the coefficient \( \omega^2 \) in (12) would correspond to \( \omega^2 - 1/4 \) in (6), (7).

To summarize, it is only natural that equation (12), essentially semiclassical one (due to the assumption \( |\omega| \gg 1 \)), is universal, i.e. independent of spin \( s \). Moreover, even if one assumes that \( j \gg 1 \) as well (i.e. gives up the condition \( j \ll |\omega| \) used in (12)), the resulting, again semiclassical equation

\[
\frac{d^2u}{dr^2} + \left\{ \omega^2 + \frac{2\omega^2}{r-1} + \frac{1}{(r-1)^2} \left( \omega^2 + \frac{1}{4} \right) - \frac{1}{r(r-1)} \left( j + \frac{1}{2} \right)^2 \right\} u = 0 \quad (13)
\]

is still universal, i.e. spin-independent.

We address now the eigenvalues of equation (12). Its two independent solutions can be conveniently expressed via the Whittaker functions \( W_{\lambda,\mu}(x) \) (see, e.g., [17]). They are

\[
W_{i\omega,i\omega}(-2i\omega(r-1)), \quad W_{-i\omega,i\omega}(2i\omega(r-1)).
\]

With their different asymptotic for \( r \to \infty \),

\[
W_{i\omega,i\omega}(-2i\omega(r-1)) \to e^{i\omega[r+\ln(r-1)]} = e^{i\omega z},
\]

\[
W_{-i\omega,i\omega}(2i\omega(r-1)) \to e^{-i\omega[r+\ln(r-1)]} = e^{-i\omega z},
\]

these solutions are obviously independent. On the other hand, the second one does not comply with boundary condition (8) and therefore should be excluded.

As to the first solution, its limit for \( r \to 1 \) is

\[
W_{i\omega,i\omega}(-2i\omega(r-1)) \quad \rightarrow \quad \frac{\Gamma(-2i\omega)}{\Gamma(1/2 - 2i\omega)} \left[ -2i\omega(r-1) \right]^{i\omega + 1/2} + \frac{\Gamma(2i\omega)}{\Gamma(1/2)} \left[ -2i\omega(r-1) \right]^{-i\omega + 1/2}. \quad (14)
\]

When going over to the function \( \Psi \) used in the “tortoise” coordinate \( z \) (see (9)), the overall factor \( (r-1)^{1/2} \) in this expression cancels, and \( (r-1)^{\pm i\omega} \) goes over into \( e^{\pm i\omega z} \) for \( r \to 1 \). To comply with the boundary condition at the horizon, one should get rid of the first term in equation (14). To this end, recalling that \( \Gamma(-n) \) has poles for integer positive \( n \), we put \( 1/2 - 2i\omega = -n \), or \( \omega_n = -(i/2)(n+1/2) \).

In fact, equation (12) by itself was obtained from (10) and (11) under the assumption \( |\omega_n| \to \infty \), or \( n \gg 1 \). Therefore, in this way we can guarantee, for the initial problem, only that

\[
\omega_n = -\frac{i}{2}n, \quad n \gg 1, \quad (15)
\]

for all spins.
Though less accurate than quantization rules (1) and (3), this one is still quite sufficient for insisting that the correct leading term in the quantization rule for spin $1/2$ is (5), but not (4).

In conclusion of this section, we demonstrate, with a relatively simple example of truncated equation (12), an analytical approach, that will allow later, for more accurate treatment of the wave equations, to find not only the leading term $\sim n$ in the asymptotic of QNMs, but as well the next, constant one. The method goes back to [10] where it was applied to the Regge–Wheeler equation for $s = 0, 2$. After finding in the present section by this method the eigenvalues of equation (12), we will apply below the technique to the Teukolsky equation for arbitrary spins. Our line of reasoning differs from that of [10].

Equation (12) has two singular points, $r = 1$ and $r = \infty$. We connect them by a cut in the complex plane $r$ going, for instance, from $r = 1$ along the real axis to the right (solid line in Fig. 1). Let us consider the closed contour marked by the dashed line in Fig. 1. Since there is no singularity inside it, the solution at some point on this contour, after going around the contour, comes back to its initial value, which means that the phase of this solution changes by $2\pi n$, $n = 0, \pm1, \pm2, \ldots$.

When we follow an arc of a large radius $r \gg 1$, where the asymptotic solution is $e^{i\omega r} r^{i\omega}$, i.e. go around the singular point at infinity, the wave function acquires the phase $\delta(\infty) = 2\pi i \omega$.

Then we go around the branch point $r = 1$ by following an arc of a small radius. Here, due to the asymptotic solution $v(r) = (r - 1)^{-i\omega + 1/2}$, the wave function acquires the phase $\delta(1) = 2\pi (i\omega - 1/2)$. As to the paths along the cut, they generate no phase at all. Indeed, since $r = 1$ is a regular singular point, the wave function can be written as $u(r) = v(r) w(r)$, where $w(r)$ is analytic at $r = 1$. The phase of $v(r) = (r - 1)^{-i\omega + 1/2}$ remains constant along the paths adjacent to the cut, as well as the phase of $r - 1$. As to the analytic function $w(r)$, it obviously cannot acquire any phase after going around the

Figure 1: Singular point of equation (12), cut, and closed contour

\[ r = 1 \quad \text{and} \quad r = \infty. \]
cut. In other words, effectively for our purpose, the regular singular point \( r = 1 \) behaves as if it were an isolated singularity.

Thus, going counter-clockwise around the considered closed contour in the complex plane, one obtains

\[ \delta(\infty) + \delta(1) = 4\pi i \omega - \pi = 2\pi n, \]

or the quantization rule

\[ \omega_n = -\frac{i}{2} (n + 1/2). \]

Being interested in the solutions decreasing in time, we choose here positive \( n \) (and of course large ones). Again, one can guarantee here the leading term only, \( \omega_n = -i n/2. \)

### 3 Teukolsky equation. Quasinormal modes of odd integer spin

Now we address the problem of the next, subleading correction, of zeroth order in \( n \), to formula (15). It is only natural to expect that this correction is spin-dependent. So, to investigate it we will use the Teukolsky equation. As distinct from the Regge – Wheeler equation, this one describes in a unified way both integer and half-integer spins, ranging at least from \( s = 0 \) to \( s = 2 \) [18]–[20]. Previously, the Teukolsky equation was used in [5] for numerical calculations of QNMs.

In the Schwarzschild background the Teukolsky equation for a massless field is

\[ \Delta \frac{d^2 R}{dr^2} + (1 - s)(2r - 1) \frac{dR}{dr} + U(r)R = 0, \tag{16} \]

where

\[ \Delta(r) = r(r - 1), \quad U(r) = -\frac{r(2r - 3) i \omega s + r^3 \omega^2}{r - 1} - A_{js}, \quad A_{js} = (j + s)(j - s + 1). \]

Obviously, for a given spin \( s \) the QNMs are independent of helicity.

With the tortoise coordinate \( z(r) = r + \ln(r - 1) \) and new function \( \chi(r) = r\Delta^{-s/2}R(r) \), one obtains the following standard form for this equation:

\[ \frac{d^2 \chi}{dz^2} + [\omega^2 - V(r)]\chi = 0, \tag{17} \]

with the effective potential

\[ V(r) = \frac{s^2 - 4}{4 r^4} - \frac{A_{js}^2 - s + s^2 - 1}{r^3} + \frac{A_{js}^2 - s + s^2 - 3 i \omega s}{r^2} + \frac{2 i \omega s}{r}. \tag{18} \]

Clearly, for \( s = 0 \) \( V(r) \) is real, and equation (17) coincides with the scalar version of the Regge – Wheeler equation [6]. On the other hand, the Teukolsky equation for \( s = 1/2 \) coincides with the second order equation for a massless Dirac field in the chiral representation; of course, the latter differs from equation (7) written in the standard representation.
The asymptotic behavior of QNMs of the Teukolsky equation is
\[ \chi \sim \Delta^{s/2} e^{\pm i \omega z} \quad \text{for} \quad \omega z \to \pm \infty. \]  

(19)

In principle, here the idea of calculating the eigenmodes will be the same as the second one above, in the case of truncated equation (12). We choose a closed contour without any singularity inside it (dashed line in Fig. 2), calculate the phase of the wave function acquired after going around the contour, and equate this phase to \(2\pi n, \ n = 0, \pm 1, \pm 2, \ldots\). However, this problem for the Teukolsky equation is in general much more involved than that for the truncated equation (12) since equation (17) has three singular points: \( r = 0, \ r = 1, \ \text{and} \ r = \infty \). As previously, we choose a cut in the complex plane \( r \) going from \( r = 1 \) to \( r = \infty \) along the real axis to the right (solid line in Fig. 2). As to the cut starting at \( r = 0 \) (another solid line in Fig. 2), the choice of its location will be discussed later.  

The treatment of the first cut is practically the same as in the simple case of equation (12). The present boundary condition (19) means that the asymptotic solution for \( r \to 1 \) looks here as \((r - 1)^{-i \omega + s/2}\). Correspondingly, when going around the branch point \( r = 1 \) along an arc of a small radius, the wave function acquires the phase
\[ \delta(1) = 2\pi (i \omega - s/2). \]  

(20)

In this case as well, the regular singular point \( r = 1 \) is effectively equivalent, for our purpose, to an isolated singularity.

Situation with the cut starting at \( r = 0 \) is more complicated. The problem is that we have no a priori boundary condition at \( r = 0 \), but still have to find the phase acquired when going around this cut.

However, for \( s = 1 \) (and for any other odd integer spin that is described by the Teukolsky equation) the discussed correction can be found easily. For \( r \ll 1 \), two independent
solutions of equation (17) with potential (18) are

\[ \chi_+ \sim r^{1+s/2} \quad \text{and} \quad \chi_- \sim r^{1-s/2}. \]  \hspace{1cm} (21)

Since \( r = 0 \) is a regular singular point of the Teukolsky equation, the exact general solution of this equation can be presented as follows:

\[ \chi(r) = r^{1-s/2} \left[ \sum_{k=0}^{\infty} a_k r^k + r^s \sum_{k=0}^{\infty} b_k r^k \right] = r^{-s/2} \left[ \sum_{k=0}^{\infty} a_k r^{k+1} + \sum_{k=0}^{\infty} b_k r^{k+s+1} \right]. \]  \hspace{1cm} (22)

With an odd integer \( s \), the singularity of this solution at \( r = 0 \) is due to the overall factor \( r^{-s/2} \), or to \( r^{s/2} \) if by some reasons \( \chi_- \) vanishes (we will see in the next section that just this is the case). Correspondingly, the phase acquired by solution (21) as a result of going around the branch point \( r = 0 \) is

\[ \delta(0) = \pm \pi s. \]  \hspace{1cm} (23)

In fact, the sign in this expression does not matter for our problem since the two options differ by \( 2\pi s \) with an integer \( s \). And if necessary, one can always shift the initial \( n \gg 1 \) in the quantization rule by an integer \( s \).

At last, the asymptotic solution at infinity

\[ \chi \sim r^{s+i\omega} e^{i\omega r}, \quad r \to \infty, \]  \hspace{1cm} (24)

after going by \( 2\pi \) around the arc of infinitely large radius, acquires the phase

\[ \delta(\infty) = 2\pi(i\omega + s). \]  \hspace{1cm} (25)

With the phase (20) generated by the branch point \( r = 1 \), the total result of going around the closed contour is

\[ \delta(1) + \delta(\infty) + \delta(0) = 4\pi i\omega \]

(we have chosen here the sign minus in (23)). Equating this expression, as above, with \( 2\pi n \), we arrive at the quantization rule already mentioned in Introduction (see (3)), but now for any odd spin:

\[ \omega_n = -\frac{i}{2} n, \quad s = 1, 3, \ldots. \]  \hspace{1cm} (26)

4 Quasinormal modes of half-integer spin

The situation is somewhat more complicated for half-integer spins. Here we have to find out which of the two solutions that behave at \( r \to 0 \) as \( r^{1+s/2} \) and \( r^{1-s/2} \), respectively, is the true one. To this end we need to match the solution for \( |r| \ll 1 \) to that for \( |r| \gg 1 \). Fortunately, in the limit \( |\omega| \gg 1 \) it can be done analytically (here we follow the idea used in [10] for finding the QNMs of Regge – Wheeler equation (6)).

Here and below, to investigate the singularity at \( r = 0 \), it is convenient to shift \( z \to z + i\pi \), so that now \( z(r) = r + \ln(1-r) \), and in the limit \( r \ll 1 \) we have \( z(r) = -r^2/2 \).
Introducing new variable \( \rho = \omega z(r) = -\omega r^2/2 \), we transform equation (18) in the limit \( |\omega| \gg 1 \) to 8
\[
\frac{d\chi^2}{d\rho^2} + \left[ 1 - \frac{3is}{2\rho} + \frac{4-s^2}{16\rho^2} \right] \chi = 0.
\] (27)

Independent solutions of equation (27) are the Whittaker functions
\[
W_{\frac{1}{2} \pm \frac{1}{4}}(-2i\rho) \quad \text{and} \quad W_{-\frac{1}{2} \pm \frac{1}{4}}(2i\rho).
\] (28)

Though derived for \( |r| \ll 1 \), these solutions are valid also for \( |\rho| = |\omega r^2/2| \gg 1 \), if \( |\omega| \) is sufficiently large. Their asymptotic behavior for \( |\rho| = |\omega z| \to \infty \) is, respectively,
\[
\rho^{|s|} e^{i\rho} \quad \text{and} \quad \rho^{-|s|} e^{-i\rho}.
\] (29)

To choose the appropriate solution we compare the asymptotic behavior (29) with that of the solution of exact equation (17) for \( |\omega z| \to \infty \), as given in (19). With the leading asymptotic for QNMs already established in section 2 for arbitrary spins, \( \omega_n \approx -i/n/2 \), the solution (19) is exponentially small in the left half-plane \( r \). Therefore, we have to choose here the exponentially small solution of equation (27). In this way we arrive at
\[
\chi(\rho) = W_{\frac{1}{2} + \frac{1}{4}}(-2i\rho) \sim \rho^{\frac{3}{4}} e^{i\rho}.
\] (30)

We note that matching of the two solutions, (19) and (30), is not precluded by the fact that the pre-exponential factors in their asymptotic, \( \Delta_{\omega} \sim (\omega z)^{\lambda} \) and \( \rho^{\frac{3}{4}} \sim (\omega z)^{\frac{3}{4}} \), respectively, are different. This difference is only natural since the factor \((\omega z)^{\lambda}\) is due to the term \( -3is/2\rho = 3is/\omega r^2 \) in equation (27), and the factor \((\omega z)^{\frac{3}{4}}\) is due to the term \(-3is/2\rho = 3is/\omega r^2 \) in equation (27). The coincidence of the exponentials themselves is quite sufficient reason to believe that it is just (30) that reproduces the behavior of the exact solution for \( |r| \to 0 \).

The functions \( W_{\lambda, \mu}(y) \) with a given asymptotic behavior for \( |y| \to \infty \) can be expressed via other linearly independent solutions \( M_{\lambda, \mu} \) of the Whittaker equation with a definite behavior for \( |y| \to 0 \). These solutions are [17]
\[
M_{\lambda, \mu}(y) = y^{\mu+\frac{1}{2}} e^{-\frac{1}{2} y} \Phi(1/2 + \mu - \lambda, 1 + 2\mu, y),
\] (31)

where \( \Phi(a, b, y) \) is the confluent hypergeometric function. Functions \( W_{\lambda, \mu} \) and \( M_{\lambda, \mu} \) are related as follows [17]:
\[
W_{\lambda, \mu}(y) = \frac{\Gamma(-2\mu)}{\Gamma(1/2 - \mu - \lambda)} M_{\lambda, \mu}(y) + \frac{\Gamma(2\mu)}{\Gamma(1/2 + \mu - \lambda)} M_{\lambda, -\mu}(y).
\] (32)

In the present case we have
\[
W_{\frac{1}{2} \pm \frac{1}{4}}(-2i\rho) = \frac{\Gamma(-s/2)}{\Gamma(1/2 - s)} M_{\frac{1}{2} \pm \frac{1}{4}}(-2i\rho) + \frac{\Gamma(s/2)}{\Gamma(1/2 - s/2)} M_{\frac{1}{2} \pm \frac{1}{4}}(-2i\rho).
\] (33)

If one retains one more term in the expansion of \( z(r) \), i.e., with \( z(r) = -r^2/2 - r^2/3 \), a correction of first order in \( |\omega|^{-1/2} \), arises in square brackets of equation (27). It coincides with the corresponding perturbation obtained in [21] that generates corrections \( \sim n^{-1/2} \to \omega_n \) for \( s = 0, 2 \) in the Regge – Wheeler formalism.
For half-integer $s$, the first term in this expression vanishes since $\Gamma(z)$ turns to infinity for negative integer $z$. So, here our solution (30), with the account for (31) and with $\rho \sim r^2$, behaves for $r \to 0$ as

$$\chi \sim r^{1-s/2}. \tag{34}$$

Here again (see (23)) the phase due to the branch point $r = 0$ is

$$\delta(0) = \pi s, \tag{35}$$

but now $s$ is half-integer. We have again the same $\delta(1)$ and $\delta(\infty)$ as those for odd spin (see (20) and (25), respectively), and the same quantization condition

$$\delta(1) + \delta(\infty) + \delta(0) = 2\pi n.$$

At last, shifting the initial $n$ by the integer part of (now half-integer) $s$, we arrive at the quantization rule for any half-integer spin:

$$\omega_n = -\frac{i}{2} \left(n + \frac{1}{2}\right), \quad s = 1/2, 3/2, \ldots. \tag{36}$$

Of course, equation (36) can be directly employed also for odd integer $s$. In this case the second term in this equation vanishes, and now obvious line of reasoning results in formula (26).

## 5 Quasinormal modes of arbitrary spin

The problem for arbitrary $s$, including the case of direct physical interest, that of $s = 0, 2$, requires more sophisticated approach. In this general case both terms in the rhs of equation (33) survive, so that the solution near the origin contains both powers of $r$:

$$\chi = ar^{1+s/2} + br^{1-s/2}. \tag{37}$$

Obviously, the rotation of this expression by $2\pi$ around the branch point $r = 0$ in no way can result in its multiplication by some factor, i.e. the limit (37) of the solution for small $r$ cannot transform into itself under this procedure.\footnote{An additional problem arises for even $s$. In this case $\Gamma(-s/2)$ in equation (36) turns to infinity (and for $s = 0$, $\Gamma(s/2)$ turns to infinity as well), so that this solution stays finite due to a delicate cancellation between two terms in the rhs of equation (36). Therefore, to obtain the explicit form of the solution for even $s$ one should perform a careful limiting transition. Anyway, this solution for small $r$ also does not transform into itself under the rotation around the origin.} However, such a transformation does exist for the solution far away from the origin. The rotation gets possible due to the Stokes phenomenon, rather well-known in mathematical physics.

We will not discuss this phenomenon in general, but instead will demonstrate directly how it works, by solving our problem. We consider the approximate solution $W_{\pm \frac{s}{2}}(-2i\rho)$ for generic $s$ and $\rho$ (though confine of course to small $r$). The analytic continuation to the specific values of spin will be performed only in the final result which is a smooth function of $s$.

At first we discuss the position of those lines in the complex plane $r$ where $\text{Im} \rho = \text{Im}(\omega z) = 0$. These four level lines in the complex $r$ plane are presented in Fig. 2.
(dotted lines therein). Their behavior corresponds to the leading asymptotic for QNMs, already established in section 2 for arbitrary spins: \( \omega_n \approx -i n/2 \). For small \( |r| \) (still, \( |\rho| = |\upsilon r^2/2| \) can be large!) the definition of level lines in the complex \( r \) plane reduces to \( \text{Im}(\omega r^2) = 0 \), or \( \text{arg} r = -1/2 \text{arg} \omega \). Two of them, \( c \) and \( d \), going to the right from the origin, with \( \text{arg} r = \pm \pi/4 \), are of no special interest to us. We will be interested mainly in the two level lines, \( a \) and \( b \), that become vertical at infinity. In the sector between these lines the exact solution is exponentially small for \( r \to \infty \) (see (24)).

Our solution (30) has a cut going from the branch point \( r = 0 \) to infinity. This cut should be chosen in a judicious, self-consistent way.

For instance, it cannot be drawn in the sector where the solution is exponentially small. Indeed, as it was demonstrated above, if we went around such a cut starting from the small solution (30), we would arrive in the result at a linear combination of both small and large solutions. But a large solution should not exist in this sector.

By the same reason, if starting from the real positive \( r \) axis we go in the positive direction, i.e. counter-clockwise, along the contour of large \( r \), the cut cannot be drawn along the level line \( a \) or in the sector to the right of it.

Neither, with this direction, should we draw the cut in the sector between the level lines \( b \) and \( c \) where the solution is exponentially large at infinity. In this sector we cannot guarantee the absence of an exponentially small admixture to the right of the level line \( b \); then, after going around the cut, the correct solution would be completely distorted.

Thus, for the counter-clockwise direction, the only consistent choice for the cut is that along the level line \( b \). Just in this way we will proceed. 10

So, let us go from the real \( r \) axis in the counter-clockwise direction along a contour of large \( r \). We reach the level line \( b \), and then proceed along its upper side. At a small distance from the origin, we follow an arc of a radius \( r \ll 1 \). Then we come back along the lower side of the cut to the arc of a large radius \( r \gg 1 \). At last, we close the contour by going along this arc, and then around the cut starting at \( r = 1 \).

As to the cut starting from the origin, here again the contributions to the acquired phase from the upper and lower sides of the cut cancel. So, we have to find only the phase generated by the rotation around the branch point \( r = 0 \). The rotation angle here is \(-2\pi \) in the \( r \) plane, which corresponds to \(-4\pi \) in the \( \rho \) plane. To calculate the mentioned phase, we use solution (30) in the limit \( \rho \gg 1 \), i.e. we work in the interval \( |\omega|^{-1/2} \ll r \ll 1 \) (recall that \( |\omega| \gg 1 \)).

To perform the rotation by \( 2\pi \) in the \( r \)-plane, or by \( 4\pi \) in the \( \rho \)-plane, we note first of all that functions \( M_{\lambda, \mu} \) transform under the rotations in a simple way. Indeed, according to equation (31),

\[
M_{\lambda, \mu}(y e^{-4i\pi}) = e^{-4i\pi\mu} M_{\lambda, \mu}(y).
\]

Then we need the relation inverse to (32), to express back \( M_{\lambda, \mu} \) via \( W_{\lambda, \mu} \). Its form depends on \( \text{arg} y \) (see [17], 9.233.1, 9.233.2). In our case, the required initial value of \( y = -2i\rho \) corresponds to the upper side of the cut along the level line \( b \). Since \( \text{arg} y \) remains constant along a level line, it can be found most easily for \( |r| \to \infty \) where on this line \( \text{arg} r = 3\pi/2 \). In such a way, we have here

\[
\text{arg} y = \text{arg} (-2i\rho) = -\frac{\pi}{2} + \text{arg} \omega + \text{arg} r = \frac{\pi}{2}.
\]

10Quite analogous arguments demonstrate that if one goes from the real \( r \) axis in the negative, clockwise direction, the cut should be chosen along the level line \( a \).
With this arg $y$, the inverse relation reads

$$M_{\lambda, \mu}(y) = \frac{\Gamma(1 + 2\mu)}{\Gamma(1/2 + \mu - \lambda)} e^{-i\pi\lambda} W_{-\lambda, \mu}(e^{-i\pi}y)$$

$$+ \frac{\Gamma(1 + 2\mu)}{\Gamma(1/2 + \mu + \lambda)} e^{i\pi(1/2 + \mu - \lambda)} W_{-\lambda, \mu}(y); \quad -\frac{\pi}{2} < \arg y < \frac{\pi}{2}. \quad (38)$$

Now we go back in the rotated solution

$$\chi(\rho e^{-4i\pi}) = \frac{\Gamma(-2\mu)}{\Gamma(1/2 - \mu - \lambda)} e^{-4i\pi\mu} M_{\lambda, \mu}(-2i\rho)$$

$$+ \frac{\Gamma(2\mu)}{\Gamma(1/2 + \mu - \lambda)} e^{4i\pi\mu} M_{-\lambda, -\mu}(-2i\rho) \quad (39)$$

to functions $W_{\pm\lambda, \pm\mu}(\pm2i\rho)$ by means of (38). To simplify the result of this transformation, we note first of all that the contributions to the result originating from the first term in equation (38) (containing $W_{-\lambda, \mu}(e^{-i\pi}y)$), are exponentially small along all the path leading from the line $b$ to the real axis; besides they vanish of course on this axis. So, these terms can be neglected at all.\(^{11}\) Then, we need the result only in the limit $|\rho| \gg 1$, where $W_{\lambda, \mu}(y) = W_{-\lambda, -\mu}(y)$. In this way, we arrive at relation

$$\chi(\rho e^{-4i\pi}) = -e^{-i\pi s}(1 + 2\cos \pi s) W_{\frac{3s}{4}, \frac{s}{4}}(-2i\rho). \quad (40)$$

The coefficient $-e^{-i\pi s}(1 + 2\cos \pi s)$ here results from trivial, but rather tedious transformations with $\Gamma$-functions, sines, and cosines. This coefficient can be rewritten as $e^{i\delta(0)}$, where

$$\delta(0) = \pi - \pi s - i \ln(1 + 2\cos \pi s) \quad (41)$$

is the phase acquired by the solution when following the arc of a small radius $r \ll 1$ around the origin $r = 0$.

As usual, the quantization condition for $\omega_n$ is

$$\delta(1) + \delta(\infty) + \delta(0) = 2\pi n. \quad (42)$$

Finally, with (20), (25), and (41), we obtain analytically the universal formula

$$\omega_n = -\frac{i}{2} \left( n + \frac{1}{2} \right) + \frac{1}{4\pi} \ln(1 + 2\cos \pi s), \quad n \to \infty \quad (43)$$

for eigenmodes of any spin $s$ described by the Teukolsky equation.\(^{12}\)

\(^{11}\)In the region where $|y| \gg 1$ and the asymptotic form of the function is used, we in fact have neglected already small power-like corrections to this form. So much the more, we can and even should neglect exponentially small corrections to it. This is the Stokes phenomenon at work.

\(^{12}\)Of course, the same result arises when going in the opposite direction along the contour in the complex $r$ plane, with a cut made along the level line $a$. However, in this case equation (38) modifies to:

$$M_{\lambda, \mu}(y) = \frac{\Gamma(1 + 2\mu)}{\Gamma(1/2 + \mu - \lambda)} e^{i\pi\lambda} W_{-\lambda, \mu}(e^{i\pi}y) + \frac{\Gamma(1 + 2\mu)}{\Gamma(1/2 + \mu + \lambda)} e^{-i\pi(1/2 + \mu - \lambda)} W_{-\lambda, \mu}(y);$$

$$-\frac{3\pi}{2} < \arg y < \frac{\pi}{2}.$$
6 Conclusions

Now few comments on our results.

It is not clear whether the Teukolsky equation is valid for $s > 2$.

For $s = 0, 2$ formula (43) was derived previously in [9, 10] in the Regge – Wheeler formalism. For these spins it gives

$$\omega_n = -\frac{i}{2} \left( n + \frac{1}{2} \right) + \frac{1}{4\pi} \ln 3, \quad n \to \infty.$$  

The result

$$\omega_n = -\frac{i}{2} n, \quad n \to \infty,$$

for $s = 1$ was previously obtained in [9, 11]. It is derived in an elementary way in section 3, confirmed in section 4, and follows immediately from (43).

For half-integer spins, simple calculation in section 4, as well as formula (43), give

$$\omega_n = -\frac{i}{2} \left( n + \frac{1}{2} \right), \quad n \to \infty, \quad s = 1/2, 3/2.$$  

For $s = 1/2$ it not only confirms the conclusion of [13], thus resolving the controversy on the matter, but contains also first nonvanishing correction to the leading term; the result for this correction is new. Our total result for $s = 3/2$ is also new. Quite recently, both results were confirmed in [22].

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