Abstract

We study the point of transition between complete and incomplete financial models thanks to Dirichlet Forms methods. We apply recent techniques, developed by Bouleau, to hedging procedures in order to perturbate parameters and stochastic processes, in the case of a volatility parameter fixed but uncertain for traders; we call this model Perturbed Black Scholes (PBS) Model. We show that this model can reproduce at the same time a smile effect and a bid-ask spread; we exhibit the volatility function associated to the local-volatility model equivalent to PBS model when vanilla options are concerned. Lastly, we present a connection between Error Theory using Dirichlet Forms and Utility Function Theory.

Key Words: Smiles, dynamic hedging, local volatility, stochastic volatility, error, Dirichlet form, carré du champs operator, bias, utility functions.

1 Introduction

In this article, we study the impact of uncertainty of volatility on the price of vanilla options.

A classical approach consists in perturbating volatility by a small random variable and in considering the associated truncated expansion, but this approach encounters difficulties in dealing with infinite dimensions. An alternative way, based on Dirichlet Forms (for reference see Albeverio [1], Bouleau et al. [3] and Fukushima et al. [12]), has been suggested by Bouleau [4]; this idea yields the same representation of small perturbations as that of the classical perturbation approach.

In classical theory of financial mathematics, we assume that all market securities have a definite price. Indeed, the hypothesis of completeness of the market (see Lamberton et al. [15]) forces a single price for a contingent claim. If we take into account an uncertainty on a parameter, we find that the price of the contingent claim is not unique but we have many possible prices, therefore we can reproduce the bid-ask spread by means of a utility function related to the uncertainty on prices.

If the uncertainty on parameter is small, we may neglect orders higher than the second, so we chose to work with Gaussian distributions. In Error Theory using Dirichlet Forms, we associate the variance to the “carré du champ” operator Γ and the shift to the generator of semigroup \( \mathcal{A} \).

Historical Black Scholes model for asset pricing assumes that the diffusion process for asset price is log-normal with a constant volatility; however, many works on empirical market data...
present a skewed structure of market implied volatilities with respect to the strike; this effect is
called smile of volatility or volatility skew (see Renault et al. \[18\] and Rubinstein \[19\]).

Implied volatility is convex as a function of the strike and generally exhibits a slope with
respect to the strike at forward money (see Perignon et al. \[17\]); to take this into account, we
propose a new model based on BS model, characterized by an uncertain volatility parameter,
called Perturbed Black-Scholes model. This is a stochastic volatility model with closed forms for
option pricing; we study some constraints to force a smile on implied volatility and define the
local volatility model, by means of its volatility function, equivalent to the PBS model for vanilla
options.

Summarizing, we propose a new financial model for securities pricing based on the Black
Scholes model with a random variable as volatility, we use a perturbative approach to preserve
closed forms for options prices and greeks; this model permits to reproduce a smile on implied
volatility and generate automatically a bid-ask spread.

The paper is organized as follows:

In section 2 and 3, we present two ways to perturb the volatility, the classical approach and
the one using Dirichlet Forms. In section 4, we present the PBS model and study the effect of
uncertainty on volatility for an underlying following Black Scholes model without drift. In section
5, we investigate the relations with the literature, while in section 6, we present an interpretation
of the relative index defined in section 4 by means of utility functions theory. Finally section 7
resumes and concludes.

2 Classical Approach

We consider a function \( F \), which denotes the payoff or the hedging value of an option and depends
on a parameter \( \sigma \) (the volatility for instance), then we consider a perturbation of the parameter
by means of a "normal" distribution and analyze the impact of this perturbation on \( F \); we suppose
that \( F \in C^2 \) with respect to the parameter \( \sigma \).

This approach may be performed with elementary limit calculations in a finite dimensional
framework:

One-dimensional Case

Assume that \( \sigma \in \mathbb{R} \). We model the perturbation on the parameter \( \sigma \) by means of the following
transformation:

\[
\sigma_0 \to \sigma_0 + \sqrt{\epsilon \gamma} g + \epsilon a
\]

where \( g \) is a random variable, following the centered reduced normal law and independent of all
the other random variables present in this financial problem, like the Brownian Motion; \( a \) is a
constant and \( \epsilon \) is a very small parameter. In other words we replace the parameter with a random
variable with mean \( \sigma_0 + \epsilon a \) and variance \( \epsilon \gamma \). We can interpret the term \( \gamma \) as the normalized
variance of the variable \( \sigma \), subject to the perturbation; \( a \) is the bias of the parameter induced by
the perturbation.

We evaluate the bias and the variance of the perturbation of \( F \), i.e. \( F(\sigma_0 + \sqrt{\epsilon \gamma} g + \epsilon a) - F(\sigma_0) \).
Finally we can remark:

**Remark 2.1** The bias and variance have the two following chain rules:

1. the bias of the function $F$, induced by the parameter uncertainty, depends both on the bias and the variance of the parameter. Indeed, this bias presents two terms. The first term, given by the bias of the perturbed parameter, is proportional to the first derivative. The second term is related to the convexity of the function $F$ and proportional to the variance of the parameter; it has a purely probabilistic origin (see Bouleau [4] and [5]).

2. the variance of the function $F$ is proportional to the variance of the perturbed parameter and to the first derivative of the function.

The figure 1 resumes the two impacts.

![Figure 1: Impact of uncertainty on a parameter through a non-linear function.](image)
Multi-dimensional Case

We can extend the previous results to the case of a \(d\)-dimensional parameter; assume \(\sigma_0 \in \mathbb{R}^d\) and \(F : \mathbb{R}^d \to \mathbb{R}\). In this case the part of the parameter is expressed as

\[
\sigma_0 \to \sigma_0 + \sqrt{\epsilon} \gamma + \epsilon A
\]

where

\[
G \sim \mathcal{N}(0, \mathbb{I}_d)
\]

\(\gamma \in M(d)\) symmetric and positive definite

\(\gamma^{\frac{1}{2}}\) is a square root of \(\gamma\)

\(a \in \mathbb{R}^d\)

We suppose again that \(\epsilon \in \mathbb{R}\) is a small parameter, so that we can approximate the bias and the variance

\[
\mathbb{E} \left[ F(\sigma_0 + \sqrt{\epsilon} \gamma^{\frac{1}{2}} G + \epsilon A) - F(\sigma_0) \right] = \mathbb{E} \left[ \sum_{i=1}^{d} \frac{\partial F}{\partial x_i} (\sigma_0) \left( \sqrt{\epsilon} \sum_{j=1}^{d} \gamma_{i,j} \frac{\partial}{\partial x_j} G_j + \epsilon A_i \right) \right]
\]

\[
+ \frac{1}{2} \sum_{i,k=1}^{d} \frac{\partial^2 F}{\partial x_i \partial x_k} \bigg|_{\sigma_0} \left( \sqrt{\epsilon} \sum_{j=1}^{d} \gamma_{i,j} \frac{\partial}{\partial x_j} G_j + \epsilon A_i \right) \times \left( \sqrt{\epsilon} \sum_{l=1}^{d} \gamma_{k,l} \frac{\partial}{\partial x_l} G_l + \epsilon A_k \right) + o(\epsilon) \right] = \epsilon \left\{ \vec{A} \cdot \mathbb{E} \left[ \vec{\nabla} F(\sigma_0) \right] \right\}
\]

\[
+ \frac{1}{2} \sum_{i,k=1}^{d} \gamma_{i,k} \mathbb{E} \left[ \frac{\partial^2 F}{\partial x_i \partial x_k} \bigg|_{\sigma_0} \right] + o(\epsilon)
\]

\[
\mathbb{E} \left[ \left( F(\sigma_0 + \sqrt{\epsilon} \gamma^{\frac{1}{2}} G + \epsilon A) - F(\sigma_0) \right)^2 \right] = \mathbb{E} \left[ \left\{ \sum_{i=1}^{d} \frac{\partial F}{\partial x_i} \bigg|_{\sigma_0} \left( \sqrt{\epsilon} \sum_{j=1}^{d} \gamma_{i,j} \frac{\partial}{\partial x_j} G_j + \epsilon A_i \right) \right\}^2 \right] + o(\epsilon)
\]

\[
= \epsilon \mathbb{E} \left[ t \left( \vec{\nabla} F(\sigma_0) \right) \gamma \vec{\nabla} F(\sigma_0) \right] + o(\epsilon)
\]

where \(\gamma_{k,l}^{\frac{1}{2}}\) indicate the element \((k, l)\) of matrix \(\gamma^{\frac{1}{2}}\), \(A_i\) indicate the \(i^{th}\) component of vector \(A\) and likewise \(G_i\) is the \(i^{th}\) row of random matrix \(G\).

Finally we find a similar result in one-dimensional case:

**Remark 2.2** The bias and variance have the two following chain rules:

1. the bias of the function \(F\) induced by the uncertainty of the parameter depends both on the bias and the variance of the parameter, as a matter of fact this bias presents two terms: a first term given by the scalar product of the perturbed parameter bias and the gradient of function \(F\); the second term is related to the convexity of the function \(F\) and proportional to the covariance of the parameter \(\sigma\);
2. the variance of the function $F$ induced by the uncertainty of the parameter is proportional to the norm of gradient of $F$ weighted by the covariance matrix of the perturbed parameter $\sigma$.

Several ways are conceivable in order to extend this approach to infinite dimensional parameters. This is compulsory since we are generally concerned with perturbation of a stochastic process. One way is based on Dirichlet Forms which possess several convenient features for computing the propagation of perturbations.

3 Dirichlet Forms Approach

Then we present the Dirichlet Forms approach, we start by recalling the essential ingredients of this method (see Bouleau [4]).

We define an error structure:

**Definition 3.1 (Error structure)** An error structure is a term
\[
\left( \tilde{\Omega}, \tilde{F}, \tilde{P}, D, \Gamma \right)
\]
where

- $(\tilde{\Omega}, \tilde{F}, \tilde{P})$ is a probability space;
- $D$ is a dense sub-vector space of $L^2(\tilde{\Omega}, \tilde{F}, \tilde{P})$;
- $\Gamma$ is a positive symmetric bilinear application from $D \times D$ into $L^1(\tilde{\Omega}, \tilde{F}, \tilde{P})$ satisfying the functional calculus of class $C^1 \cap \text{Lip}$, i.e. if $F$ and $G$ are of class $C^1$ and Lipschitzian, $u$ and $v \in D$, we have $F(u)$ and $G(v) \in D$ and
  \[
  \Gamma [F(u), G(v)] = F'(u)G'(v)\Gamma[u, v] \ \tilde{P} \text{ a.s.;}
  \]
- the bilinear form $\mathcal{E}[u, v] = \frac{1}{2} \tilde{E} [\Gamma[u, v]]$ is closed;
- the constant function $1$ belongs to $D$, i.e. the error structure is Markovian.

In mathematical literature the form $\mathcal{E}$ is known as a "local Dirichlet form" that possesses a “carré du champ” operator $\Gamma$.

We recall the definition of sharp operator associated with $\Gamma$

**Definition 3.2 (Sharp operator)** Let $(\tilde{\Omega}, \tilde{F}, \tilde{P}, D, \Gamma)$ an error structure and $(\hat{\Omega}, \hat{F}, \hat{P})$ a copy of the probability space $(\tilde{\Omega}, \tilde{F}, \tilde{P})$. Under the Mokobodzki hypothesis that the space $D$ is separable, there exists an operator sharp $(\cdot)^\#$ with these three properties:

- $\forall u \in D$, $u^\# \in L^2(\hat{P} \times \tilde{P})$;
- $\forall u \in D$, $\Gamma[u] = \hat{E} [(u^\#)^2]$;
- $\forall u \in D^n$ and $F \in C^1 \cap \text{Lip}$, $(F(u_1, ..., u_n))^\# = \sum_{i=1}^n \left( \frac{\partial F}{\partial x_i} \circ u \right) u_i^\#$.
From a numerical point of view, the sharp operator is an useful tool to compute $\Gamma$ because the sharp is linear whereas the carré du champ is bilinear. We can also associate to the error structure $\left( \tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \mathbb{D}, \Gamma \right)$ a unique strongly continuous contraction semi-group $(P_t)_{t \geq 0}$ via Hille Yosida theorem (see Albeverio [1] pages: 9-11 and 20-26). This semigroup has a generator $(A, \mathcal{D}A)$; it is a self-adjoint operator that satisfies, for $F \in \mathcal{C}^2$, $u \in \mathcal{D}A$ and $\Gamma[u] \in L^2(\tilde{\mathbb{P}})$:

$$A[F(u)] = F'(u)A[u] + \frac{1}{2}F''(u)\Gamma[u] \tilde{\mathbb{P}} \text{ a.s.}$$

The functional calculus extends the ideas of classical Gauss error theory, the idea is to consider the perturbation as an error. In analogy with the classical approach of error theory we associate the carré du champ operator $\Gamma$ to the normalized\(^1\) variance of the error, the sharp operator becomes a linear version of the standard deviation of the error. Similarly, the generator describes the error biases after normalization (for more details we refers to the book of Bouleau [4] chapters III and V or to [5]).

### 3.1 Perturbation on a parameter

Now we go back to the study of the impact of a perturbation. To do so we implement an error structure on the parameter. Therefore we consider the same function $F \in \mathcal{C}^2$ depending on a parameter, the volatility for instance. We model the perturbation on the parameter with the following transformation:

$$\sigma_0 \rightarrow \sigma = \sigma_0 + X.$$ 

Here $X$ is a random variable that represents the error on the parameter $\sigma$. We consider an error structure associated to $X$ with some hypotheses to allow the computations.

Assumptions:

1. $X \in \mathcal{D}, \Gamma[X]$ and $A[X]$ are known;
2. functions $x \mapsto \Gamma[X](x)$ and $x \mapsto A[X](x)$ belong to $L^1 \cap L^2$ are continuous at zero and not vanishing;
3. the error structure possesses a gradient operator, allowing to define the sharp operator which is a special case of gradient (see Bouleau et al. [3] chapter V § 5.2 or Bouleau [4] chapter V § 2).

Denoting $X^\#$ the sharp of the parameter $\sigma$, we have

$$F(\sigma^\#) = F'(\sigma)X^\#$$

$$\Gamma[F(\sigma)] = [F'(\sigma)]^2 \Gamma[X]$$

$$A[F(\sigma)] = F'(\sigma)A[X] + \frac{1}{2}F''(\sigma)\Gamma[X].$$

Following the idea of truncated development, we are interested in evaluating the bias and the variance at the value $\sigma = \sigma_0$; i.e. at $X = 0$:

$$\Gamma[F(\sigma)](\sigma_0) = [F'(\sigma_0)]^2 \Gamma[X]|_{X=0}$$

$$A[F(\sigma)](\sigma_0) = F'(\sigma_0)A[X]|_{X=0} + \frac{1}{2}F''(\sigma_0)\Gamma[X]|_{X=0}.$$

\(^1\)the variance divided by the small parameter $\epsilon$. 

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We can remark that we have found the same chain rules for bias and variance as in the classical approach; however it is worth notice that we had not to specify the dimension of the space where $X$ is defined. This dimension can be finite or infinite, as in the case of the Wiener space, where we are able to define coherent error structures (See Bouleau et al. [3] and [5]). Hence this framework is an extension of the classical approach.

Now we present a classical case where we can perform explicit computation.

Example 3.1 (Diffusion)

**Proposition 3.1** Let $X_t$ be a diffusion process of the form

$$dX_t = a(X_t)dB_t + b(X_t)dt$$

where $a(.)$ and $b(.)$ satisfy classical conditions and $B_t$ is a brownian motion. We use this SDE to perturb a parameter $X_0$, the time $t$ play the role of the small parameter $\epsilon$ and we are interested in studying the impact of this perturbation on a smooth function $F$ depending on the parameter $X_0$, we have the following bias and variance:

$$A[F(X)]_{X=X_0} = F'(X_0)b(X_0) + \frac{1}{2}F''(X_0)a^2(X_0)$$

$$\Gamma[F(X)]_{X=X_0} = [F'(X_0)]^2 a^2(X_0).$$

The bias is given by the generator of the diffusion and the variance by the martingale term.

We use a short lemma:

**Lemma 3.1** Let $M_t$ a stochastic integral:

$$M_t = \int_0^t K_s L_s dB_s$$

where $K_t$ and $L_t$ are continuous squared integrable processes and $N_t$ the following stochastic scaled brownian motion:

$$N_t = K_0 L_0 B_t.$$  

Then $N_t$ is an approximation of $M_t$ in $L^2$ when $t$ goes to zero.

Proof of lemma: Indeed

$$\frac{1}{t} \mathbb{E} \left[ (M_t - N_t)^2 \right] = \frac{1}{t} \int_0^t (K_s L_s - K_0 L_0)^2 ds$$

$$\leq \frac{2}{t} \int_0^t \left\{ [K_s]^2 [L_s - L_0]^2 + [K_s - K_0]^2 [L_0]^2 \right\} ds \to 0 \quad \square$$

Proof of proposition 3.1: By Itô formula

$$F(X_t) = F(X_0) + \int_0^t F'(X_s) a(X_s) dB_s + \int_0^t \left[ F'(X_s) b(X_s) + \frac{1}{2} F''(X_s) a^2(X_s) \right] ds$$

$$= F(X_0) + \int_0^t A[F(X_s)] ds + M_t$$
where $M_t$ is a martingale.

Thanks to the previous lemma, we approximate $M_t$ with $N_t$, given by

$$N_t = F'(X_0)a(X_0)B_t$$

Finally we can write the approximate law of $F(X)$ generated by the perturbation on the parameter $X_0$

$$F(X_t) \approx F(X_0) + tA[F(X_0)] + \sqrt{t}F'(X_0)a(X_0)B_1$$

where $F(X_0)$, $A[F(X_0)]$ and $F'(X_0)a(X_0)$ are $\mathcal{F}_0$-measurable.

**Remark 3.1** We note that the bias and the variance are proportional to $\epsilon$ and we emphasize that the bias is exactly the generator of diffusion computed for the function $F$ at the starting point $X_0$, the variance is the quadratic variation of diffusion associate to function $F$ at the starting point $X_0$; this example show the relation between functional analysis and perturbative theory.

Finally we can state:

**Theorem 3.1 (impact of uncertainty)** The impact of uncertainty on the parameter transforms a constant into a gaussian distribution of the form

$$F(\sigma_0) \rightarrow F(\sigma_0) + \epsilon A[F(\sigma)]|_{\sigma=\sigma_0} + \sqrt{\epsilon} \sqrt{\Gamma[F(\sigma)]|_{\sigma=\sigma_0}} G$$

where $G$ is an exogenous independent standard Gaussian variable.

**Remark 3.2** The bias $\epsilon A[F(\sigma)]|_{\sigma=\sigma_0}$ exists even though the parameter is unbiased; it suffices that the function $F$ is non linear.

This expansion explains the role of the generator and the carré du champ operator (see Bouleau [7]); the theoretical image is perturbed due to the uncertainty on the parameter. This effect is small, however it produces not only a noise but also it alters the mean.

In this article we study in particular this shift of mean and we search to reproduce the smile effect by means of this shift.

### 4 Perturbed Black Scholes Model

We start with the classical Black Scholes model (see Black et al. [2] and Lamberton et al. [15]); let $(\Omega, \mathcal{F}, \mathbb{P})$ the historical probability space and $B_t$ the associated brownian motion, we suppose that the dynamic of the risky asset under historical probability $\mathbb{P}$ is given by the following BS diffusion without drift

$$dS_t = S_t \sigma_0 dB_t$$

$$S_t = S_0 e^{\sigma_0 B_t - \frac{1}{2} \sigma_0^2 t}$$

In this framework, the price of a European vanilla option is well known (see Lamberton et al. [15]).

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2We can remark further in this article that the presence of a drift term has an impact otherwise from the classical BS model.
The B&S model present many advantages, in particular the pricing only depends on volatility and we find closed forms for premium and greeks of vanilla options; unluckily the B&S model cannot reproduce the market price of call options for all strikes at the same volatility, this effect is called smile.

We propose to consider a perturbation of this model by means of an error structure on volatility.

We make three hypotheses:

1. the real market follows a B&S model with fixed and non perturbed volatility $\sigma_0$;
2. the trader has to estimate the volatility, so its volatility contains intrinsic inaccuracies, we model this ambiguity by means of an error structure; nonetheless we assume that the stock price $S_t$ is not erroneous. We evaluate the impact of the perturbation, generated by the trader mishandling, on the profit and loss process used by trader to hedge the vanilla option;
3. the trader knows this perturbation and he wants to modify the option prices to take into account the bias induced by the perturbation on volatility.

4.1 “Mismatch” on trading hedging

We consider a trader that use an ”official” BS asset model in order to hedge vanilla options; he uses the market price to determine the ”fair” values of parameters in his model (in this case only the level of flat volatility $\sigma_0$) by inversion of pricing formula.

The trader finds an observed volatility process $\varsigma_t$, usually known as implied volatility. He hedges his portfolio according to his volatility, so the price of an option, that pays a payoff $\Phi$ is $F(\varsigma_0, x, 0)$.

We study the profit and loss process associated to the hedging position.

The profit and loss process at the maturity of a trader that follows the strategy associated with his volatility $\varsigma_t$ is given by:

\[
P\&L = F(\varsigma_0, x, 0) + \int_0^T \frac{\partial F}{\partial x}(\varsigma_t, S_t, t) dS_t - \Phi(S_T)
\]

We make two remarks:

Remark 4.1 The profit and loss process is stochastic, due to two random sources:

- First of all, the stochastic ”real” model since the trader cannot use the correct hedging portfolio.
- Second, the stochastic process $\varsigma_t$, that can depend on a random component independent to the brownian motion $B_t$.

Remark 4.2 The profit and loss process must be studied on historical probability $\mathbb{P}$, therefore the presence of a drift on the BS diffusion modifies the second term of equation 4.1. Without drift this term is a martingale and this fact simplifies the computation. The case of BS model with drift will be dealt in a next article.
In order to analyze the law of P&L process, it is sufficient to study the expectation on a class of regular test functions $h(P&L)$ and the error on them.

We will come back on the role and choice of a particular function $h$ in the next subsection.

We suppose by simplicity that the trader volatility $\varsigma_t$ is a time independent random variable:

$$\varsigma_t = \sigma.$$  

We define an error structure for the volatility $\sigma$, therefore this volatility admits the following expansion:

$$\sigma_0 \rightarrow \sigma_0 + \epsilon A[\sigma](\sigma_0) + \sqrt{\epsilon \Gamma[\sigma](\sigma_0)} \tilde{N}$$

where $\tilde{N}$ is a standard gaussian variable defined in a space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$ independent to $\Omega$. Furthermore, if the volatility has been estimated by means of a statistic on market data, we can specify the functional $\Gamma$. In fact, thanks to a result of Bouleau and Chorro [6], $\Gamma$ is related with the inverse of the Fisher information matrix.

We want to estimate the variance and bias error of $E[h(P&L)]$. To perform the calculus, we assume that $\sigma = \sigma_0$ is the right value of the random variable in the sense that $\varsigma_t = \sigma_0$ and $P&L(\sigma_0) = 0$. We have the following relation for the sharp of volatility:

$$\varsigma_t^\# = \sigma^\#$$

Then we can state

**Theorem 4.1** We have the following bias and variance:

$$E[h(P&L)] \approx \epsilon h'(0) \Upsilon_1^{BS}(\sigma_0) + \frac{1}{2} \epsilon h''(0) \Upsilon_2^{BS}(\sigma_0) + \sqrt{\epsilon} \left[ h'(0) \right]^2 \Lambda^{BS}(\sigma_0)$$

where

$$\Upsilon_1^{BS}(\sigma_0) = \left\{ \frac{\partial F}{\partial \sigma}(\sigma_0, x, 0) A[\sigma](\sigma_0) + \frac{1}{2} \frac{\partial^2 F}{\partial \sigma^2}(\sigma_0, x, 0) \Gamma[\sigma](\sigma_0) \right\}$$

$$\Upsilon_2^{BS}(\sigma_0) = \left\{ \left[ \frac{\partial F}{\partial \sigma}(\sigma_0, x, 0) \right]^2 + \sigma_0^2 \int_0^T \mathbb{E} \left[ S_t^2 \left( \frac{\partial^2 F}{\partial \sigma \partial x}(\sigma_0, S_s, t) \right)^2 \right] dt \right\} \Gamma[\sigma](\sigma_0)$$

$$\Lambda^{BS}(\sigma_0) = \left\{ \mathbb{E} \left[ \frac{\partial F}{\partial \sigma}(\sigma_0, x, 0) \right]^2 \right\} \Gamma[\sigma](\sigma_0)$$

and we have the following truncated expansion:

$$E[h(P&L)] \approx \epsilon h'(0) \Upsilon_1^{BS}(\sigma_0) + \frac{1}{2} \epsilon h''(0) \Upsilon_2^{BS}(\sigma_0) + \sqrt{\epsilon} \left[ h'(0) \right]^2 \Lambda^{BS} \tilde{N}(0, 1)$$

**Proof:**

We start with the study of the variance. A computation yields

$$\left( E[h(P&L)] \right)^\# = E \left[ h'(P&L) \left( \frac{\partial F}{\partial \sigma}(\sigma_0, x, 0) + \int_0^T \frac{\partial^2 F}{\partial \sigma \partial x}(\sigma_0, S_s, s) dS_s \right) \sigma^\# \right]$$
thus the quadratic error is equal to:

\[
\Gamma \left[ \mathbb{E} \left[ h(P&L) \right] \right] = |h'(0)|^2 \left\{ \mathbb{E} \left[ \frac{\partial F}{\partial \sigma}(\sigma_0, x, 0) + \int_0^T \frac{\partial^2 F}{\partial \sigma \partial x}(\sigma_0, S_s, s) dS_s \right] \right\}^2 \Gamma[\sigma](\sigma_0)
\]

and the second term vanishes since the stock price is a martingale, here the hypothesis on drift is crucial.

The study of the bias is more complicated, we start with the remark that the bias is a linear operator:

\[
A \left[ \mathbb{E} \left[ h(P&L) \right] \right] = \mathbb{E} \left[ A[h(P&L)] \right] = \mathbb{E} \left[ h'(P&L) A[P&L] + \frac{1}{2} h''(P&L) \Gamma[P&L] \right]
\]

We study the two terms separately; for the first, we find the expectation of quadratic error in the case \( \varsigma = \sigma_0 \):

\[
\mathbb{E} \left[ \Gamma \left[ P&L \right] \right] = \left\{ \mathbb{E} \left[ \left( \frac{\partial F}{\partial \sigma}(\sigma_0, x, 0) \right)^2 \right] \right. \int_0^T \mathbb{E} \left[ \sigma_0^2 S_s^2 \left( \frac{\partial^2 F}{\partial \sigma \partial x}(\sigma_0, S_s, s) \right)^2 ds \right] \right\} \Gamma[\sigma](\sigma_0)
\]

We study the bias operator; we must evaluate the expectation of bias of profit and loss process, and we find the following result always in the case \( \varsigma = \sigma_0 \):

\[
\mathbb{E}[A[P&L]] = \frac{\partial F}{\partial \sigma}(\sigma_0, x, 0) A[\sigma](\sigma_0) + \frac{1}{2} \frac{\partial^2 F}{\partial \sigma^2}(\sigma_0, x, 0) \Gamma[\sigma](\sigma_0)
\]

Finally the bias of expectation of a function of profit and loss process:

\[
A[\mathbb{E}[h(P&L)]] = h'(0) \left\{ \frac{\partial F}{\partial \sigma}(\sigma_0, x, 0) A[\sigma](\sigma_0) + \frac{1}{2} \frac{\partial^2 F}{\partial \sigma^2}(\sigma_0, x, 0) \Gamma[\sigma](\sigma_0) \right\} + \frac{1}{2} h''(0) \left\{ \left[ \frac{\partial F}{\partial \sigma}(\sigma_0, x, 0) \right]^2 \right. + \sigma_0^2 \int_0^T \mathbb{E} \left[ S_t^2 \left( \frac{\partial^2 F}{\partial \sigma \partial x}(\sigma_0, S_t, t) \right)^2 \right] dt \right\} \Gamma[\sigma](\sigma_0)
\]

The proof ends with the truncated expansion that is a consequence of the error theory using Dirichlet Forms (see Bouleau \[8\] and \[9\]).

In order to interpret this result in finance, we consider that the trader knows the presence of errors in his procedure and wants to neutralize this effect.

We associate:

- the variance of \( h(P&L) \) process to the bid-ask spread of options;
- the bias of \( h(P&L) \) process to a shift of prices of options asked by the trader to the buyer.
Indeed in the classical theory of financial mathematics we assume that all market securities have a single price, with the probability theory language we can associate at any derivative securities a Dirac distribution for its price. If we take into account uncertainty on volatility, we have found that the price of the contingent claim is not unique but we have many possible prices; thus the Dirac distribution changes into a continuous distribution, characterized by a variance and a shift of the mean with respect to the previous Dirac distribution (see figure 2).

**Theorem 4.2** If the perturbation in the volatility is small, we can neglect orders higher than the second, so we always work with Gaussian distributions; the trader must modify his prices in order to take into account the two previous effects, namely the variance and the bias, then he fixes a supportable risk probability $\alpha < 0.5$ and accepts to buy the option at the price

$$(\text{Bid Premium}) = (\text{BS Premium}) + \epsilon A[\mathbb{E}[h(P\&L)]] + \sqrt{\epsilon \Gamma[\mathbb{E}[h(P\&L)]]} \mathcal{N}_\alpha$$

where $\mathcal{N}_\alpha$ is the $\alpha$-quantile of the reduced normal law. Likewise, the trader accepts to sell the option at the price

$$(\text{Ask Premium}) = (\text{BS Premium}) + \epsilon A[\mathbb{E}[h(P\&L)]] + \sqrt{\epsilon \Gamma[\mathbb{E}[h(P\&L)]]} \mathcal{N}_{1-\alpha}$$

**Remark 4.3** We remark that the two previous prices are symmetric, since $\mathcal{N}_\alpha + \mathcal{N}_{1-\alpha} = 0$; therefore the mid-premium is

$$(\text{Mid Premium}) = (\text{BS Premium}) + \epsilon A[\mathbb{E}[h(P\&L)]]$$

We emphasize that with our model we can reproduce a bid-ask spread and we can associate its width to the trader’s risk aversion (the probability $\alpha$) and the volatility uncertainty (the term $\sqrt{\epsilon \Gamma[\mathbb{E}[h(P\&L)]]}$).

In the rest of this article we work directly with the mid-premium, but all results represent the center of a normal distribution, in order to reproduce the bid and the ask premium we need to specify the probability $\alpha$. 
We conclude this subsection with a remark.

**Remark 4.4** The presence of the perturbation induces a problem on the completeness of the market, the market with perturbed volatility is not complete, since the volatility depends on a second random source orthogonal to $\Omega$, and the presence of a bid-ask spread is a direct consequence of this fact; on the other hand the enforcement that $\sigma$ to be equal $\sigma_0$ cancels the impact of the second random source. This apparent contradiction is due to the fact that an argument acts precisely at the boundary between complete and incomplete markets.

### 4.2 Role and choice of $h$ functions and relative index

In this subsection we discuss on the choice of function $h$ in equation (4.5), because function $h$ defines the magnitude of correction on prices; this choice becomes simpler since we have to specify only the first and second derivative in zero; therefore we can consider that the function $h$ is a parabola that passes through the origin. Owing to the two degrees of freedom associated with $\epsilon$ and $\alpha$, we can take $h'(0) = 1$: this is easy to understand from the economical point of view because the trader wants to balance his portfolio, i.e. the $P&L$ process. If we look at equation (4.5) we find that the choice of $h'(0) = 1$ defines completely the term of variance. Since the second derivative of $h$ has an impact only on the bias and the coefficient $\Upsilon^{BS}_2(\sigma_0)$ is positive, this impact is a shift of the mean, as in the following figure.

![Figure 3: Impact of ambiguity: the convexity (resp. concavity) of function $h$ raises (resp. reduces) the mean of prices but leaves the variance unchanged.](image)

We suggest to interpret this impact as an asymmetry of the balance between supply and demand. Indeed, if $h''(0) = 0$ we find that the function $h$ is the identity: this means that the trader uses directly the process of profit and loss, and the bias is “neutral”, i.e. that we find the same result if we consider the buyer’s point of view (it is enough to take minus identity function as $h$). A surplus of the demand of an option with respect to the supply induces a raising of the prices: this is the classical case of market where banks sell options and private investors buy. We model this perturbation with a positive second derivative for $h$ and we consider that if $h''(0) > 0$ (resp. $h''(0) < 0$) the demand (resp. supply) exceeds the supply (resp. demand).

We define the following index of asymmetry of balance between supply and demand:
\[ r_{S/D} = \frac{h''(0)}{h'(0)} \]

We can identify this index by means of the classical utility theory: if we interpret \( h \) as a utility function, \( r_{S/D} \) is known as the absolute index of \( h \). In the next part we study the bias of profit and loss process in the case of a call option; to simplify matters, we suppose that \( h \) is the identity function i.e. \( r_{S/D} = 0 \), but before we must introduce an other index, very important in the continuation of this article.

We concentrate our attention on a problem; after the perturbation of a parameter \( \sigma_0 \), we have:

\[ \sigma_0 \rightarrow \sigma = \sigma_0 + \epsilon A[\sigma] + \sqrt{\epsilon \Gamma[\sigma]} N \text{ with } N \sim N(0, 1) \]

However \( \epsilon \) is generally unknown: the Error Theory via Dirichlet Forms cannot define this parameter. In order to deal with this question we propose to renormalize this problem; we consider the ratio between the bias and the variance, since the variance is almost surely strictly positive, therefore that the dependence on \( \epsilon \) is cancelled:

\[ \frac{\text{Bias } X}{\text{Variance } X} = \frac{\epsilon A[X]}{\epsilon \Gamma[X]} = \frac{A[X]}{\Gamma[X]} \]

This ratio is not homogeneous, because the generator is linear and the operator “carre du champs” is bilinear, so we define a relative index by:

\[ r_r(X) = 2 \frac{X A[X]}{\Gamma[X]} \]

The factor 2 will be justified in section \( \[6\] \) where we show a relation between Dirichlet forms and utility theory, since we can interpret \( r_r(X) \) as a relative index of an exogenous utility function.

### 4.3 Call options case

We concentrate on call option and we study the bias and its derivatives in order to determine some sufficient condition to force the presence of a smile on implied volatility.

We know the premium of a call option (see Lamberton et al. \[15\]) with strike \( K \) and spot value \( x \), and its hedging strategy:

\[
C(\sigma_0, x, 0) = F(\sigma_0, x, 0) = x N(d_1) - K N(d_2)
\]

\[
\text{Delta} = \frac{\partial F}{\partial x}(\sigma_0, x, 0) = N(d_1)
\]

where \( d_1 = \frac{\ln x - \ln K + \frac{\sigma^2 T}{2}}{\sigma_0 \sqrt{T}} \) and \( d_2 = d_1 - \sigma_0 \sqrt{T} \).

The following results are classical (see \[15\]):

\[
\frac{\partial F}{\partial \sigma_0}(\sigma_0, x, 0) = x \sqrt{T} e^{-\frac{1}{2} x^2} \frac{e^{-\frac{1}{2} x^2}}{\sqrt{2\pi}}
\]

\[
\frac{\partial^2 F}{\partial \sigma_0^2}(\sigma_0, x, 0) = \frac{x \sqrt{T} e^{-\frac{1}{2} x^2}}{\sigma_0 \sqrt{2\pi}} d_1 d_2
\]

\[
\frac{\partial^2 F}{\partial K^2}(\sigma_0, x, 0) = \frac{x e^{-\frac{1}{2} x^2}}{K^2 \sigma_0 \sqrt{T} \sqrt{2\pi}}
\]
Then the bias of the call premium is given by

\[
A[C]|_{\sigma=\sigma_0} = \frac{x}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \left\{ A\left[\sigma\sqrt{T}\right]|_{\sigma=\sigma_0} + \frac{d_1 d_2}{2\sigma_0\sqrt{T}} \Gamma\left[\sigma\sqrt{T}\right]|_{\sigma=\sigma_0} \right\}.
\]

We can compute the first derivative with respect to the strike:

\[
\frac{\partial A[C]}{\partial K}|_{\sigma=\sigma_0} = \frac{x}{K\sigma_0 \sqrt{T} \sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \left\{ d_1 A\left[\sigma\sqrt{T}\right]|_{\sigma=\sigma_0} - \frac{d_1 + d_2 - d_1^2 d_2}{2\sigma_0 \sqrt{T}} \Gamma\left[\sigma\sqrt{T}\right]|_{\sigma=\sigma_0} \right\}
\]

\[
= \frac{d_1 A[C]|_{\sigma=\sigma_0}}{K\sigma_0 \sqrt{T}} - \frac{x}{2K\sigma_0^2 \sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} (d_1 + d_2) \Gamma\left[\sigma\sqrt{T}\right]|_{\sigma=\sigma_0}
\]

We find that the first derivative vanishes at the forward money if and only if the bias of call vanishes at the same strike.

\[
A[C]|_{K=x, \sigma=\sigma_0} = \frac{x}{\sqrt{2\pi}} e^{\frac{\sigma^2 T}{2}} \left\{ A\left[\sigma\sqrt{T}\right]|_{\sigma=\sigma_0} - \frac{\sigma_0 \sqrt{T}}{8} \Gamma\left[\sigma\sqrt{T}\right]|_{\sigma=\sigma_0} \right\}
\]

\[
\frac{\partial A[C]}{\partial K}|_{K=x, \sigma=\sigma_0} = \frac{1}{2x} A[C]|_{K=x, \sigma=\sigma_0}
\]

We can remark that the bias and its first derivative are positive (resp negative) at the money if and only if

\[
r^*_{\text{BS}}\left(\sigma\sqrt{T}\right)|_{\sigma=\sigma_0} = 2\sigma_0 \sqrt{T} A\left[\sigma\sqrt{T}\right]|_{\sigma=\sigma_0} > (\text{resp. } <) \frac{\sigma_0^2 T}{4}
\]

We find three cases:

1. if \(r_r\left(\sqrt{\sigma_0^2 T}\right) < \frac{1}{4}\sigma_0^2 T\), then the bias of call and his first derivative are negative at the money.
2. if \(r_r\left(\sqrt{\sigma_0^2 T}\right) = \frac{1}{4}\sigma_0^2 T\), then the bias of call and his first derivative vanish at the money.
3. if \(r_r\left(\sqrt{\sigma_0^2 T}\right) > \frac{1}{4}\sigma_0^2 T\), then the bias of call price and his first derivative are positive at the money.

**Remark 4.5** This bound increases with maturity; if we suppose a constant relative index (CRI) we can define a bound on maturity \(T_{\text{bias}}\).

Therefore if we study an option with maturity smaller (resp. greater) than \(T_{\text{bias}}\) we have that the bias associated to the hedging "profit and loss" process and his first derivative are positive (resp. negative).

\(^3\)In interest-free case the forward money is for \(K = x\), in this case we have \(d_1 = -d_2\).
We calculate the second derivative:

\[
\frac{\partial^2 A[C]}{\partial K^2} \bigg|_{\sigma=\sigma_0} = \frac{d_2}{K \sigma_0 \sqrt{T}} \frac{\partial A[C]}{\partial K} \bigg|_{\sigma=\sigma_0} - \frac{x}{2 \sigma_0 \sqrt{T} \sqrt{2\pi}} \left\{ A \left[ \sigma \sqrt{T} \right] \bigg|_{\sigma=\sigma_0} + \frac{d_1^2 + 2d_1d_2 - 2}{2\sigma \sqrt{T}} \Gamma \left[ \sigma \sqrt{T} \right] \bigg|_{\sigma=\sigma_0} \right\}
\]

and we evaluate the second derivative at the forward money

\[
\frac{\partial^2 A[C]}{\partial K^2} \bigg|_{K=x, \sigma=\sigma_0} = - \frac{1}{K \sqrt{2\pi}} \left\{ \left( \frac{1}{4} + \frac{1}{\sigma_0^2 T} \right) A \left[ \sigma \sqrt{T} \right] \bigg|_{\sigma=\sigma_0} - \left[ \frac{\sigma_0^2 T}{32} + \frac{1}{8} + \frac{1}{\sigma_0^2 T} \right] \Gamma \left[ \sigma \sqrt{T} \right] \bigg|_{\sigma=\sigma_0} \right\}.
\]

If we force the bias of the call to be convex, we find:

\[
(4.10) \quad r_r^{BS} \left( \sigma \sqrt{T} \right) \bigg|_{\sigma=\sigma_0} = 2\sigma \sqrt{T} A \left[ \sigma \sqrt{T} \right] \bigg|_{\sigma=\sigma_0} < \frac{\sigma_0^4 T^2 + 4\sigma_0^2 T + 32}{4\sigma_0^2 T + 16} = \Theta \left( \sigma_0 \sqrt{T} \right).
\]

**Remark 4.6** Previous bound \( \Theta(x) \) is strictly positive, decreasing in \([0, 4\sqrt{2}]\) and increasing if \( x > 4\sqrt{2} \), and we find that \( \Theta(4\sqrt{2}) = \frac{\sqrt{2}}{2} - 3 \).

If the relative index is constant (CRI) we have an always convex bias if \( r_r < \Theta(4\sqrt{2}) \)

In the previous relations we note that the constraint depends on the volatility by means of cumulated volatility \( \sigma_0 \sqrt{T} \). For more generality in this study we can assume that the erroneous parameter is not the volatility but the cumulated variance \( \int_0^T \sigma_0^2(s)ds \) that appears in general Black & Scholes model when the volatility is deterministic but depends on time.

Now we study the evolution of slope as a function of maturity at the money

\[
\frac{\partial A}{\partial T} \bigg|_{\sigma=\sigma_0} = \frac{x}{2T \sqrt{2\pi}} \left\{ (1 + d_1d_2) A \left[ \sigma \sqrt{T} \right] \bigg|_{\sigma=\sigma_0} + \frac{4d_1^2d_2^2 - 3\sigma_0^2 T - (d_1 + d_2)^2}{8\sigma_0 \sqrt{T}} A \left[ \sigma \sqrt{T} \right] \bigg|_{\sigma=\sigma_0} \right\}
\]

\[
\frac{\partial A}{\partial T} \bigg|_{K=x, \sigma=\sigma_0} = \frac{x e^{\frac{\sigma_0^2 T}{8}}}{8T \sqrt{2\pi}} \left\{ (4 - \sigma_0^2) A \left[ \sigma \sqrt{T} \right] \bigg|_{\sigma=\sigma_0} + \frac{\sigma_0 \sqrt{T}}{8} \left( \sigma_0^2 T - 12 \right) \Gamma \left[ \sigma \sqrt{T} \right] \bigg|_{\sigma=\sigma_0} \right\}
\]

\[
\frac{\partial^2 A}{\partial K \partial T} \bigg|_{\sigma=\sigma_0} = - \frac{x}{2\sigma_0 \sqrt{T} \sqrt{K} \sqrt{2\pi}} \left\{ d_2(1 - d_1^2) A \left[ \sigma \sqrt{T} \right] \bigg|_{\sigma=\sigma_0} + \frac{d_1(4d_1d_2 - 3d_1\sigma_0 T + (d_1 + d_2)(4 - 9d_1d_2 - d_1^2))}{8\sigma_0 \sqrt{T}} \Gamma \left[ \sigma \sqrt{T} \right] \bigg|_{\sigma=\sigma_0} \right\}
\]

\[
\frac{\partial^2 A}{\partial K \partial T} \bigg|_{K=x, \sigma=\sigma_0} = - \frac{e^{\frac{\sigma_0^2 T}{8}}}{16T \sqrt{2\pi}} \left\{ (\sigma_0^2 T - 4) A \left[ \sigma \sqrt{T} \right] \bigg|_{\sigma=\sigma_0} - \sigma_0 \sqrt{T} \sigma_0^2 T - 12 \frac{8}{8} \Gamma \left[ \sigma \sqrt{T} \right] \bigg|_{\sigma=\sigma_0} \right\}
\]

to find a slope that increases with increasing maturity we have to impose:

\[
(4.11) \quad r_r \left( \sigma \sqrt{T} \right) \bigg|_{\sigma=\sigma_0} \geq \frac{\sigma_0^2 T - 12}{4} \frac{\sigma_0^2 T}{4 - \sigma_0^2 T} \quad \text{with} \quad \sigma_0^2 T < 4
\]
We study the evolution of smile as a function of maturity.

\[
\frac{\partial^3 A}{\partial K^2 \partial T} \bigg|_{K=\sigma=\sigma_0} = \frac{e^{-\frac{\sigma^2 T}{2}}}{x\sigma_0^2 T^2 \sqrt{2\pi}} \cdot \left\{ \frac{16 + \sigma_0^2 T^2}{32} A \left[ \sigma \sqrt{T} \right] \bigg|_{\sigma=\sigma_0} - \frac{\sigma_0^2 T (\sigma_0^2 T - 4)^2 + 128}{256} \Gamma \left[ \sigma \sqrt{T} \right] \bigg|_{\sigma=\sigma_0} \right\}
\]

This term is positive if and only if:

\[
(4.12) \quad r_r \left( \sigma \sqrt{T} \right) \bigg|_{\sigma=\sigma_0} > \frac{1}{4} \frac{\sigma_0^2 T (\sigma_0^2 T - 4)^2 + 128}{16 + \sigma_0^4 T^2}
\]

4.4 Example

Now we study a particular case, in order to reproduce the smile of volatility present in market data.

**Theorem 4.3** We fix the relative index at

\[
(4.13) \quad r_r^{BS} (\sigma \sqrt{T}) \bigg|_{\sigma=\sigma_0} = \frac{\sigma_0^2 T}{4}
\]

This choice fix the values of the bias and its derivatives at the money, in particular this choice force the bias and its first derivative to be zero at the money (see equation 4.9); the second derivative becomes positive at the money thanks to equation 4.10 for any maturity \( T \). Therefore the bias is strictly convex around the money and vanishes at the money, therefore it is positive in a neighbourhood of the money. Now if the bias vanishes at the money the implied ATM volatility is \( \sigma_0 \), but, since the bias is positive around, the implied volatility becomes greater than \( \sigma_0 \) around the money. Finally we have reproduced a smile effect around the money.

4.5 Dupire formula and Implicit Local Volatility Model

In this section we want to specify the Local Volatility Model equivalent to Perturbed Black Scholes Model. We know that the knowledge of prices of options for all strikes and maturities defines a single local volatility model that reproduces these prices; the Dupire formula (see Dupire [10]) defines the local volatility function:

\[
(4.14) \quad \sigma^2_{\text{imp}} (T, K) = \frac{\partial C}{\partial \sigma^2} \bigg|_{\sigma=\sigma_0}
\]

Our model is a perturbation of BS model, so we can consider the following expansion:

\[
\begin{align*}
C(\sigma, x, K, t, T) &= C(\sigma_0, x, K, t, T) + \epsilon A[C](\sigma, x, K, t, T) \bigg|_{\sigma=\sigma_0} \\
\frac{\partial C}{\partial T}(\sigma, x, K, t, T) &= \frac{\partial C}{\partial T}(\sigma_0, x, K, t, T) + \epsilon \frac{\partial A[C]}{\partial T}(\sigma, x, K, t, T) \bigg|_{\sigma=\sigma_0} \\
\frac{\partial^2 C}{\partial K^2}(\sigma, x, K, t, T) &= \frac{\partial^2 C}{\partial K^2}(\sigma_0, x, K, t, T) + \epsilon \frac{\partial^2 A[C]}{\partial K^2}(\sigma, x, K, t, T) \bigg|_{\sigma=\sigma_0}
\end{align*}
\]

\footnote{We recall that the vega is positive for call options.}
But in fact, if $\epsilon$ vanishes, the model is a Black Scholes model with volatility $\sigma_0$. Thanks to equation (4.7), we can rewrite:

$$\sigma^2\text{imp}(T, K) \approx \sigma_0^2 + \frac{\epsilon}{2} K^2 \frac{\partial^2 C}{\partial K^2} \bigg|_{\sigma=\sigma_0} - \frac{1}{2} K^2 \sigma_0^2 \frac{\partial^2 A[C]}{\partial K^2} \bigg|_{\sigma=\sigma_0}$$

**Theorem 4.4** The local volatility model equivalent to the PBS model for vanilla options, has the following local volatility function:

$$\sigma^2\text{imp}(T, K) \approx \sigma_0^2 \left\{ 1 + \frac{\epsilon}{2} \left[ 4 \frac{A\left[\frac{\sigma \sqrt{T}}{\sigma_0 \sqrt{T}}\right]}{\sigma_0 \sqrt{T}} \bigg|_{\sigma=\sigma_0} - \left[ 4 \ln \left( \frac{x}{K} \right) \right]^2 \frac{\Gamma \left[ \frac{\sigma \sqrt{T}}{\sigma_0 \sqrt{T}} \right]}{\sigma_0^2 T} \right] \right\}$$

**Remark 4.7** Local volatility $\sigma(T, K)$ has a minimum at forward money $K = x$, and present a logarithmic behavior as $K$ approaches zero and infinity.

We must preserve the positivity of the square of volatility, so we fix the following constraint:

$$\sigma_0^2 T + 2 - 2r_r \left( \sigma \sqrt{T} \right) \big|_{\sigma=\sigma_0} < \left\{ \frac{\Gamma \left[ \frac{\sigma \sqrt{T}}{\sigma_0 \sqrt{T}} \right]}{\sigma_0^2 T} \right\}^{-1}$$

5 **Extension and relation with literature**

We have proved that the PBS can reproduce at the same time the bid-ask spread and the volatility smile; but in literature many authors (see Perignon et al. [17], Renault et al. [18] and Rubinstein [19]) have remarked that, generally, the volatility presents a skewed structure (the graph of implied volatility is downward sloping); besides in this article we have limited the study of the PBS model
at the martingale case, when the extra returns of the stock are zero; however, a simple argument of risk aversion induce leads us to suppose that the parameter $\mu$ (the extra-returns term in the BS model) must be positive. In a next paper we will show that the extra returns term has an impact in PBS model, contrary to BS model, and we can use this effect to generate a slope in implied volatility. We want emphasize that the PBS model uses a perturbative approach, i.e. we start with a simple model (the Black Sholes model) and we adjust the model at the market data through a perturbation of the principal parameter (the volatility).

Clearly in literature some authors introduce a perturbative approach in finance, we recall the papers of Hagan et al. [14] and the book of Fouque et al. [11]; our approach is however different, it is a probabilistic approach and it can relate the bid-ask spread and the volatility smile through the simple economic argument of the existence of uncertainty in the market.

The study of SABR model, a stochastic volatility model introduced by Hagan et al., is based on a small volatility of volatility expansion. Now the principal difference between our perturbation method and the Hagan’s one is that our model is based on a probabilistic point of view, therefore our perturbative approach takes into account the second order derivative and, in particular, it can estimate the bias induced by non-linearity. In the paper of Hagan et al. the approach is based on analysis, so that they implicitly suppose that an exact value for the volatility and, by consequence, for the option price exists, their perturbative approach is therefore a first order formulation; they cannot then justify the presence of a bid-ask spread directly through their model.

Moreover Hagan et al. start with a more complex model (a stochastic volatility model with a volatility driven by a brownian diffusion with an independent component) and they simplify their model with the perturbative approach in order to find a closed form, therefore in SABR model the perturbative approach is a computation tool. Our approach allows to justify the perturbative approach with an economics reason, the problem for the traders to estimate the value of volatility. Therefore in our method the perturbation is naturally a consequence of uncertainty on volatility, and it has a "physics" nature; finally our model start by the classic, well-known and accepted Black Scholes model, in one phrase the SABR model is a downward perturbative approach the PBS is an upward one. The principal advantage of the SABR model with respect to PBS model
is that the SABR model can predict the dynamics of the smile when the forward price of stock changes; this capacity is related with the parameter $\beta$, and has no relation with the perturbative approach.

In the book of Fouque, Papanicolau and Sircar the authors study a model based on the Black-Scholes model with a stochastic volatility with a fast mean reversion, in this case the perturbative approach is based on the presence of two time scales the mean reversion time and the maturity of options, the first is very small compared the other time scaled, therefore, in their book, Fouque et al. introduce an expansion of the solution of their prices PDE (based on a hierarchical PDE system). Therefore, in their book, Fouque et al. use a perturbative approach with a “physical” nature, i.e. the presence of a two time scales; but their approach is based on analysis and it is not (directly) related with an uncertainty on volatility; finally the Fouque’s approach cannot reproduce a bid-ask spread.

6 Risk Aversion

In this section we make some recalls on the theory of utility functions and we show a connection with Error Theory using Dirichlet Forms. We consider a utility function $U(x): \mathbb{R} \to \mathbb{R}$ and $U(x) \in C^2$: let $\rho$ be defined by the following relation

$$\mathbb{E}[U(X)] = U(\mathbb{E}[X] - \rho). \quad (6.1)$$

We find, in the case of a small variance, that:

$$\rho = -\frac{\sigma^2}{2} \frac{U''(\mathbb{E}[X])}{U'(\mathbb{E}[X])} = \frac{\sigma^2}{2} r_a(X). \quad (6.2)$$

In a similar way we define $\hat{\rho}$ so that:

$$\mathbb{E}[U(X)] = U(\{\mathbb{E}[X](1 - \hat{\rho})\} : \text{and we find}$$

$$\hat{\rho} = -\frac{\sigma^2}{2} \mathbb{E}[X] \frac{U''(\mathbb{E}[X])}{U'(\mathbb{E}[X])} = \frac{\sigma^2}{2} r_r(X). \quad (6.3)$$

We can name the three objects:

1. $\rho$ is the risk price;
2. $r_a(X)$ is the absolute index of aversion at the wealth $X$;
3. $r_r(X)$ is the relative index of aversion at the wealth $X$.

We write the relations of error and bias, given an error structure on $X$:

$$A[U(X)] = U'(X)A[X] + \frac{1}{2} U''(X)\Gamma[X]$$

$$\Gamma[U(X)] = (U'(X))^2 \Gamma[X]$$

We observe that the bias of $U(X)$, $A[U(X)]$ is zero if and only if $A[X] = \rho$; in this case we find
\[ A[X] = \frac{r_0}{2} \Gamma[X] \]
\[ \frac{A[X]}{X} = \frac{r_r \Gamma[X]}{2 X^2}. \]

(6.4)

We have found a relation between the utility function theory and the error calculus using Dirichlet Forms. We suppose that all traders buy and sell according to their risk aversion, and this aversion is represented via a utility function \( U(x) \); then the utility function is defined by its relative index of aversion.

We make an hypothesis:

Hypothesis (\( \ast \)): for a trader the bias of the utility of a traded wealth vanishes.

We can interpret this hypothesis from an economics point of view in two ways:

1. the utility function of trader, supposed to be known, is like a lens traders look the market through. Traders don’t add any effect to balance their aversion;

2. the vector \((X, A, \Gamma)\) is supposed to be known, we can define the utility function of a trader as function \( U(x) \) that cancels the bias of \( U(X) \) where \( X \) is the considered wealth.

Under hypothesis (\( \ast \)) we have two relations between the utility function and Dirichlet Forms:

1. \( A[X] = \rho(X) \), where \( \rho \) is the risk price;

2. \( r_r(X) = 2 \frac{X A[X]}{\Gamma[X]} \)

Remark 6.1 Thanks to relation 4.13 we can define the class of utility function that preserve vanishing bias and slope at the money.

\[ r_r \left( \sigma \sqrt{T} \right) = \frac{\sigma^2 T}{4} \]
\[ X \frac{U''(X)}{U'(X)} = -\frac{X^2}{4} \]
\[ U'(X) = e^{-\frac{X^2}{4}} \]
\[ U(X) = \mathcal{N} \left( \frac{X}{2} \right) \]

Where \( \mathcal{N}(X) \) is the distribution function of the normal law. This utility function is concave if the wealth is positive and convex otherwise.

7 Conclusion and Economics Interpretation

In this paper we have studied the impact of a perturbation on volatility in Black-Scholes model in absence of drift and term structure; in particular we have dealt with the problem of call hedging.

We have proposed a new model for option pricing, called Perturbed Black-Scholes model; the basic idea is take into account the effect of uncertainty of volatility value in order to reproduce, at the same time, the spread bid-ask and the smile on implied volatility; the mainly advantage of this model is that it is based on the classic Black Scholes model and the price of an option in
PBS model is the BS price plus a small perturbation that depends only on the greeks founded with BS formula, therefore the computation of PBS price is given by a closed form.

The PBS model depends on four parameters, naturally on the volatility of stock; but also on the variance of the estimated volatility $\epsilon \Gamma[\sigma](\sigma_0)$, on a relative index $r_\varepsilon(\sigma)$, that represents the ratio between the bias and the variance of the estimated volatility, and, finally, contrary to Black Scholes model, on the drift $\mu$ that represents the extra returns of stock; the impact of the drift rate $\mu$ will be studies in a next paper.

In particular, if we set the relative index to be equal to $\frac{1}{4} \sigma_0^2 T$, we have proved that the implied volatility present a smile around the money.

Finally we have defined a Local Volatility Model equivalent to Perturbed Black Scholes Model as far as vanilla options are concerned; the related local volatility function is defined by Dupire formula:

$$
\sigma^2(T, K) \approx \sigma_0^2 \left\{ 1 + \frac{\epsilon}{2} \left[ A \left[ \frac{\sigma \sqrt{T}}{\sigma_0 \sqrt{T}} \right] - \left[ \sigma^2 T + 2 - \frac{4 \ln \left( \frac{\sigma}{K} \right)^2}{\sigma_0^2 T} \right] \Gamma \left[ \frac{\sigma \sqrt{T}}{\sigma_0 \sqrt{T}} \right] \right] \right\}
$$

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