On a Class of Reflected Backward Stochastic Volterra Integral Equations and Related Time-Inconsistent Optimal Stopping Problems

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9th colloquium on Backward Stochastic Differential Equations and Mean Field Systems, June 27-July 1, 2022
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Forward stochastic Volterra integral equations (FSVIE):

\[ X(t) = \phi(t) + \int_0^t b(t, s, X(s)) \, ds + \int_0^t \sigma(t, s, X(s)) \, dW(s) \]

They represent:

- Systems with memory
- Fractional Brownian motion

\[ W^H(t) = \int_0^t K(t, s) \, dW(s) \]

Fredholm, Bellman, Wiener, Volterra...
Motivation

Consider the equation

\[ dY(t) = 0 \quad Y(T) = \xi \]

According to \( \xi \), we distinguish two cases:

- If \( \xi \in \mathbb{R} \) (BODE), \( Y(t) = \xi \)
- If \( \xi \in L^2(\Omega, \mathcal{F}_T) \) (BSDE), \( Y(t) = \xi \)

Problem!!

\[ \mathcal{F}_t \text{-adapted} \quad \mathcal{F}_T \text{-measurable} \]
The process $Y$ is required to be $\mathcal{F}_t$-adapted. Set

$$Y(t) = \mathbb{E}[\xi | \mathcal{F}_t]$$

**MRT, $\exists! Z \in L^2$**

$$Y(t) = \mathbb{E}[\xi | \mathcal{F}_t] = \mathbb{E}[\xi] + \int_0^t Z(s)dW(s)$$

Simple calculations apply

$$Y(t) = \xi - \int_t^T Z(s)dW(s).$$
BSDE (Y, Z)

\[-dY(t) = f(t, Y(t), Z(t)) dt - Z(t) dW(t), \quad Y(T) = \xi\]

equivalent to

\[Y(t) = \xi + \int_t^T f(s, Y(s), Z(s)) ds - \int_t^T Z(s) dW(s)\]
BSVIE \(^1\)

Consider the following equation (still \(\xi(t) \mathcal{F}_T\)-measurable)

\[
Y(t) = \xi(t) \quad \text{BSDE} \quad Y(t) = \xi
\]

Define \(M_t(u) = \mathbb{E}[\xi(t)|\mathcal{F}_u]\), MRT, \(\exists! Z(t, \cdot) \in L^2\)

\[
M_t(u) = \mathbb{E}[\xi(t)] + \int_0^u Z(t, s)dW(s)
\]

Putting \(u = t\) and \(Y(t) = M_t(t)\)

\[
Y(t) = \mathbb{E}[\xi(t)] + \int_0^t Z(t, s)dW(s)
\]

After some computation, we get

\[
Y(t) = \xi(t) + \int_t^T f(t, s, Y(s), Z(t, s))dt - \int_t^T Z(t, s)dW(s)
\]

\(^1\) Yong

N. Agram

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KTH 8/27
Reflected BSDE \((Y, Z, L)\)

- \(-dY(t) = f(t, Y(t), Z(t))dt - Z(t)dW(t), \quad Y(T) = \xi\)
- \(Y(t) \geq L(t)\)
- \(K(t)\) is an increasing, continuous process, \(K(0) = 0\) and
  \[
  \int_0^T (Y(s) - L(s)) \, dK(s) = 0, \quad \mathbb{P}\text{-a.s.}
  \]
Question

What is the corresponding reflected BSVIE?
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We define the following spaces for the solution.

- $S^2$ is the set of $\mathbb{R}$-valued $\mathbb{F}$-adapted processes $(Y(t))_{t \in [0, T]}$:
  \[
  \|Y\|_{S^2}^2 := \mathbb{E}\left[ \sup_{t \in [0, T]} |Y(t)|^2 \right] < \infty.
  \]

- $\mathcal{H}^2$ is the space of progressively measurable processes $(v(t))_{t \in [0, T]}$:
  \[
  \|v\|_{\mathcal{H}^2}^2 := \mathbb{E} \left[ \int_0^T |v(t)|^2 dt \right] < \infty.
  \]

- $L^2$ is the set of $\mathbb{R}$-valued processes $(Z(t,s))_{(t,s) \in [0, T] \times [0, T]}$: for a.a. $t \in [0, T]$ $Z(t, \cdot) \in \mathcal{H}^2$ and satisfy
  \[
  \|Z\|_{L^2}^2 := \mathbb{E} \left[ \int_0^T \int_t^T |Z(t,s)|^2 ds dt \right] < \infty.
  \]

- $K^2$ is the space of processes $K$ which satisfy
  - for each $t \in [0, T]$, $u \mapsto K(t, u)$ is an $\mathbb{F}$-adapted and increasing process with $K(t, 0) = 0$;
  - $(t, u) \mapsto K(t, u)$ is continuous and $K(\cdot, T) \in \mathcal{H}^2$. 
Let \((Y, Z, K)\) be the solution of

\[
Y(t) = \xi(t) + \int_t^T f(t, s, Y(s), Z(t, s)) \, ds + \int_t^T K(t, ds) - \int_t^T Z(t, s) \, dW(s)
\]  

(1)

(a) \(Y \in \mathcal{H}^2, t \mapsto Y(t)\) is continuous and \(Z \in \mathbb{L}^2\);

(b) \(Y(t) \geq L(t)\) \(\mathbb{P}\)-a.s., \(0 \leq t \leq T\);

(c) \(K(t, ds)\) is the Lebesgue-Stieltjes measure induced by the function \(s \mapsto K(t, s)\), it enjoys the following properties:

(c1) \(K \in \mathcal{K}^2\);

(c2) The Skorohod flatness condition holds: for each \(0 \leq \alpha < \beta \leq T\),

\[K(t, \alpha) = K(t, \beta)\] whenever \(Y(u) > L(u)\) for each \(u \in [\alpha, \beta]\) \(\mathbb{P}\)-a.s.
Remark

The Skorohod flatness condition (c2) implies

\[ \int_{t}^{T} K(t, ds) = 0 \text{ whenever } Y(t) > L(t) \text{ for each } t \in [0, T] \quad \mathbb{P}\text{-a.s.} \]

Mimicking Lin (2002), Yong (2006) and Wang & Zhang (2007) (for BSVIEs), we construct \( Y \) so that, for every \( t \in [0, T] \), \( Y(t) = \tilde{Y}(t, t) \) parametrized by \( t \):

Accompanying reflected BSVIE \( \tilde{Y}(t, \cdot) \)

\[
\tilde{Y}(t, u) = \xi(t) + \int_{u}^{T} f(t, s, Y(s), Z(t, s))ds + \int_{u}^{T} K(t, ds)
- \int_{u}^{T} Z(t, s)dW(s), \quad u \in [t, T]
\]
We make the following assumptions on \((f, \xi, L)\):

(A1) \(\xi(t)\) is a \(B([0, T]) \otimes \mathcal{F}_T\)-measurable map \(\xi : \Omega \times [0, T] \rightarrow \mathbb{R}:\)

\[
\sup_{0 \leq t \leq T} \mathbb{E}[|\xi(t)|^2] < \infty;
\]

(A2) The obstacle \((L(u), 0 \leq u \leq T)\) is a real-valued and \(\mathcal{F}\)-adapted continuous process:

\[
L(T) \leq \xi(t), \quad t \in [0, T] \quad \text{and} \quad \mathbb{E}\left[\sup_{0 \leq u \leq T} (L(u))^2\right] < \infty. \quad (2)
\]
The driver $f$ is a map from $\Omega \times [0, T] \times [0, T] \times \mathbb{R} \times \mathbb{R}$ onto $\mathbb{R}$, for any fixed $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, $f(t, \cdot, y, z)$ is progressively measurable.

(i) \[ \sup_{0 \leq t \leq T} \mathbb{E} \left[ \left( \int_{t}^{T} |f(t, s, 0, 0)| ds \right)^2 \right] < \infty, \]

(ii) \[ \exists c_f \geq 0, \text{ for all } (t, s) \in [0, T]^2 \text{ and } y, y', z, z' \in \mathbb{R}, \]
\[ |f(t, s, y, z) - f(t, s, y', z')| \leq c_f (|y - y'| + |z - z'|). \]

(iii) For some $\alpha \in (0, 1/2]$ and $c_1 > 0$, for all $(y, z) \in \mathbb{R} \times \mathbb{R}$ and all $0 \leq t, t' \leq s \leq T$,
\[ |f(t', s, y, z) - f(t, s, y, z)| \leq c_1 |t' - t|^{\alpha}, \]
and for some $\beta > 1/\alpha$ and $c_2 > 0$,
\[ \mathbb{E}[|\xi(t) - \xi(t')|^{\beta}] \leq c_2 |t' - t|^{\alpha \beta} \]
and
\[ \mathbb{E} \left( \left( \int_{0}^{T} |f(0, s, 0, 0)|^2 ds \right)^{\beta/2} \right) < \infty. \]
Existence and Uniqueness

Remark
Assumption (iii) yields the continuity of $Y$ and the bi-continuity of $K(\cdot, \cdot)$ which in turn guarantee the Skorohod flatness condition (c2).

Theorem
Under the above assumptions, the reflected BSVIE (1) associated with $(f, \xi, L)$ admits a unique solution $(Y, Z, K)$.
Moreover, for every $t \in [0, T]$,

$$Y(t) = \operatorname{ess sup}_{\tau \geq t} \mathbb{E} \left[ \int_t^\tau f(t, s, Y(s), Z(t, s))ds + L(\tau) \mathbb{1}_{\{\tau < T\}} + \xi(t) \mathbb{1}_{\{\tau = T\}} \right] \bigg| \mathcal{F}_t$$  (3)
Comparison Theorem

Let \((Y, Z, K)\) be a solution of the reflected BSVIE associated with \((f, \xi, L)\) and \((Y', Z', K')\) be a solution of the reflected BSVIE associated with \((f', \xi', L')\).

**Theorem**

Assume that \((f, \xi, L)\) and \((f', \xi', L')\) satisfy the assumptions (A1), (A2) and (A3) and that either the map \(y \mapsto f(t, s, y, z)\) or \(y \mapsto f'(t, s, y, z)\) is nondecreasing. Assume further that

(H1) \(\xi(t) \leq \xi'(t)\), \(\mathbb{P}\)-a.s., \(0 \leq t \leq T\),

(H2) \(f(t, s, y, z) \leq f'(t, s, y, z)\), for all \((t, y, z) \in [0, s] \times \mathbb{R} \times \mathbb{R}\), a.s., a.e. \(s \in [0, T]\),

(H3) \(L(t) \leq L'(t)\), \(0 \leq t \leq T\), \(\mathbb{P}\)-a.s.

Then

\[ Y(t) \leq Y'(t), \quad 0 \leq t \leq T, \quad \mathbb{P}\text{-a.s.} \]
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Consider

\[ f(t, s, X(s)) := \varphi(s - t)\psi(X(s)), \quad \xi(t) := \varphi(T - t)h(X(T)), \]

where \( \varphi \) is a (deterministic) discounting function (deflator).

Maximize \( J(t, \tau) := \mathbb{E} \left[ \int_t^\tau f(t, s, X(s))ds + L(\tau)1_{\{\tau < T\}} + \xi(t)1_{\{\tau = T\}} \right] \)

**Optimal stopping problem**

Find an \( \mathbb{F} \)-stopping time \( \tau_t^* \), indexed by \( t \), such that

\[ \tau_t^* = \arg \max_{\tau \geq t} J(t, \tau). \]

- \( J = \text{investment in a commodity} \)
- \( X(t) = \text{price of a commodity} \)
- \( f = \text{utility rate per unit time} \)
- \( L = \text{utility function at the stopping time } \tau, \xi = \text{utility at the final time } T \)
The associated value-function is

\[
Y(t) := \text{ess sup}_{\tau \geq t} \mathbb{E} \left[ \int_t^\tau f(t, s, X(s))ds + L(\tau) \mathbb{1}_{\{\tau < T\}} + \xi(t) \mathbb{1}_{\{\tau = T\}} \right| \mathcal{F}_t \right].
\]

(4)

Examples of hyperbolic discounting functions used in utility maximization include

- Loewenstein and Drazen (1992), Laibson (1997), Loewenstein and O’Donoghue (2002):

  \[
  \phi(s - t) := (1 + \alpha(s - t))^{-\frac{\delta}{\alpha}}, \quad \alpha, \delta > 0, \quad s > t.
  \]

  Note that \( \phi(s - t) \to e^{-\delta(s-t)} \) as \( \alpha \to 0 \).

- Strulik (2017):

  \[
  \phi(t, s) := \left( \frac{1 + \alpha t}{1 + \alpha s} \right)^\beta, \quad \alpha \geq 0, \quad \beta \geq 1, \quad s \geq t.
  \]
• It is easily seen that, apart from $e^{-\delta t}$, for any other choice of the discounting function $\varphi$, $\left( Y(t) + \int_0^t f(t, s, X(s))\,ds \right)_{t \leq T}$ will not be a supermartingale. i.e. the optimal stopping problem is time-inconsistent. We approach the problem as follows.

• Note that

$$\sup_{\tau \geq t} J(t, \tau) \leq \mathbb{E}[Y(t)]. \quad (5)$$

Now, if we can find an $\mathcal{F}$-stopping time $\tau_t^*$ such that

$$Y(t) = \mathbb{E} \left[ \int_t^{\tau_t^*} f(t, s, X(s))\,ds + L(\tau_t^*)1_{\{\tau_t^* < T\}} + \xi(t)1_{\{\tau_t^* = T\}} \mid \mathcal{F}_t \right]$$

then $\tau_t^*$ is optimal for $J(t, \cdot)$ since

$$J(t, \tau_t^*) = \mathbb{E}[Y(t)] \leq \sup_{\tau \geq t} J(t, \tau) \leq \mathbb{E}[Y(t)]. \quad (6)$$
It is tempting to suggest that \( Y \) solves a Reflected BSVIE of the form

\[
Y(t) = \xi(t) + \int_t^T f(t, s, X(s))ds + \int_t^T K(t, ds) - \int_t^T Z(t, s)dW(s)
\]

for some processes \((Z(t, s), K(t, s))\), where \(K(t, ds)\) is the Lebesgue-Stieltjes measure induced by the 'increasing function' \(s \mapsto K(t, s)\).
Proposition

Suppose the assumptions (A1), (A2) and (A3) are satisfied. For each $t \in [0, T]$, denote by $\tau^*_t$ the stopping time

$$\tau^*_t = \inf\{ t \leq u \leq T; \tilde{Y}(t, u) = L(u) \}$$

with the convention that $\tau^*_t = T$ if $\tilde{Y}(t, u) > L(u), \ t \leq u \leq T$. Then $\tau^*_t$ is optimal in the sense that

$$Y(t) = \mathbb{E} \left[ \int_t^{\tau^*_t} f(t, s, X(s)) ds + L(\tau^*_t)1_{\{\tau^*_t < T\}} + \xi(t)1_{\{\tau^*_t = T\}} \middle| \mathcal{F}_t \right].$$

(7)

Moreover, $\tau^*_t$ is an optimal strategy for $\mathcal{J}(t, \cdot)$ i.e.

$$\tau^*_t = \arg \max_{\tau \geq t} \mathcal{J}(t, \tau).$$
Remark

The choice of the optimal stopping time $\tau^*_t$ as the first hitting time of the accompanying Snell envelope $\tilde{Y}(t, \cdot)$ of the obstacle $L$ instead of the value function $Y$ (as it is the case for standard reflected BSVIE) is simply due to the fact that

$$Y(t) \neq Y(u) + \int_u^T f(t, s, X(s))ds + \int_u^T K(t, ds) - \int_u^T Z(t, s)dW(s), \quad u \geq t.$$
Thank you for your attention