Operator Content and Modular Property of Chern-Simons Coupled to a Massless Scalar

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The partition function of Abelian Chern-Simons coupled to massless charged scalar is evaluated in the infinite $k$ limit, on the geometry $S^2 \times S^1$. The expression is obtained by counting scaling operators and checked by calculating regularized determinant. It is shown that the partition function does not obey modular invariance in the form proposed by Cardy. A general form of modular invariance, if any, must be more involved.
1. Introduction

Conformal invariance serves as a powerful principle in two dimensions. It is tempting to generalize what we have learned in two dimensions to higher ones. Some have been done along this line [1], [2]. Recently, Cardy proposed generalizations of modular invariance to higher dimensions [3]. He demonstrated modular invariance for a free scalar in various dimensions, with geometry $S^d \times S^1$. Unfortunately, the form of modular invariance working for an anti-periodic free scalar in three dimensions fails in the 3d Ising model at the critical point. In this letter, we shall find that this form fails too for a Chern-Simons matter field theory in three dimensions. This failure, however, has not yet ruled out the possibility that there exist other forms of modular invariance working for this particular conformal field theory. A universal expression is even more desirable.

The quantized Chern-Simons, Abelian and non-Abelian, gauge theories coupling to matter fields are believed to be conformally invariant, provided the corresponding classical actions involve no any dimensional parameters, including matter masses. It has been shown in [4] that the beta functions of gauge couplings vanish (see [5], [6] as well for the Abelian cases) and massless matters remain massless after quantization. Furthermore, the induced non-local term for the gauge field by integrating over matter fields,

$$\int d^3x F_{\mu\nu} \frac{1}{\sqrt{\partial^2}} F_{\mu\nu},$$

is invariant under the special conformal transformations, $x \to x' = \frac{x + bx^2}{1 + 2b \cdot x + b^2 x^2}$. Being conformally invariant makes it possible to explore the models’ other nice features such as the modular invariance. For this purpose, let us define the theory with Abelian Chern-Simons coupled to massless charged scalar on the geometry $S^2 \times S^1$. A natural metric on it is

$$ds^2 = R^2 d\Omega^2 + L^2 d\tau^2,$$

where $d\Omega^2$ is the unit round metric on $S^2$, and $L$ the radius of $S^1$. This metric, viewed as a metric on $S^2 \times R$, has a map to the flat metric in 3d Euclidean space under a Weyl rescaling. As the model is conformally invariant, its Hamiltonian on $S^2$ has a map to the dilatation operator on $R^3$. Then, the problem of calculating the partition function of such a system on $S^2 \times S^1$ is equivalent to that of counting the scaling dimensions in the model.
Our main result in this paper is the following partition function for the model of interest, in the infinite $k$ (the statistical parameter) limit

$$Z(q) = \frac{1}{2\pi i} \oint z^{-1}dz \prod_{n=0}^{\infty} \frac{1}{(1 - q^{n+1/2})^{2n+1}(1 - z^{-1}q^{n+1/2})^{2n+1}},$$

where $q = \exp(-2\pi \frac{L}{R})$ is the modular parameter. The contour of the above integral is the unit circle on the complex plane. It is an interesting mathematics problem to evaluate this contour integral. It should involve certain analog of Rogers-Ramanujan identities. The evaluation of this integral is necessary in finding a possible form of modular invariance of the model.

For the Chern-Simons matter theories with finite $k$, expression of the partition function seems to be much more complicated, because the scaling dimensions of operators may not be the canonical ones. It has been seen in [4] that the scalar field has indeed an anomalous dimension, proportional to $\frac{1}{k^2}$, while the Abelian Chern-Simons gauge field and so the statistics parameter $k$ need no infinite renormalization. Renormalization of some composite gauge invariant operators with a dimension not larger than 3 has been investigated in a recent work [7]. It is remarkable that the anomalous dimensions of these composite operators have simple relations with that of the primary field. This perhaps can be attributed to that Chern-Simons matter theories have vanishing beta functions. If these simple relations remain for all gauge invariant operators, a similar result, of $k$ dependent, could be obtained. Investigations along this line are in progress.

The plan of this paper is as follows. In the next section, we shall compute the partition function by counting scaling operators. Modular invariance proposed by Cardy is numerically disproven. In section 3, we calculate the partition function in the infinite $k$ limit, upon regularizing a determinant. The result is compared to that obtained by counting of scaling dimensions in section 2. The agreement of the results indicates the consistency of the approach used.

2. Partition Function on the Geometry $S^2 \times S^1$

Just as in a two dimensional conformal field theory, the partition function of a higher dimensional conformally invariant field theory on a certain geometry encodes information of scaling operators. In two dimensions, one considers usually a torus. The partition function on a torus, modular certain conformal anomaly related terms, can be considered
as the generating function of counting of scaling operators. This can be generalized to higher dimensions with the geometry $S^d \times S^1$. The natural metric on $S^d \times S^1$ is the one introduced in the introduction, provided $d \Omega^2$ is replaced by the unit round metric $d \Omega^2_\mathbb{R}$ on $S^d$. Such a metric can serve as a metric as well on the cylinder $S^d \times \mathbb{R}$. The latter geometry, under a Weyl rescaling, is related to the flat geometry in the Euclidean space with the origin removed, $R^{d+1} \setminus \{0\}$. To see this, make a change of coordinates $\tau = (R/L)\log r$, the metric becomes
\[
ds^2 = \frac{R^2}{r^2} (dr^2 + r^2 d \Omega^2_\mathbb{R})
\]
which is conformal to the flat metric on the Euclidean space. The Hamiltonian $H$ on space $S^d$ is then proportional to the dilatation operator $D$ in the $d+1$ dimensional flat space, up to a conformal anomaly. The partition function on $S^d \times S^1$ is the generating function of scaling dimensions $x$ of the model:
\[
Z(q) = \sum_x q^x
\]
\[
q = \exp(-2\pi \delta) = \exp(-2\pi \frac{L}{R}), \quad (2.1)
\]
where the sum is over all scaling operators. In a recent interesting paper [3], generalizations of modular invariance to higher dimensions were made. The proposal of Cardy works for a free scalar field coupled conformally to the scalar curvature of the geometry. The particular form of modular invariance, which works for an anti-periodic free scalar in three dimensions, does not work for the 3d Ising model at the critical point. Here we shall see, to our disappointment, that this form does not work either for the model of Abelian Chern-Simons coupled to a massless complex scalar. This, though, has not ruled out yet a universal form of modular invariance for 3d conformal field theories. As we do not understand a possible underlying principle of modular invariance in higher dimensions, it is conceivable that a universal form is far from being simple.

Note, that the formula (2.1) works only for a strict conformal field theory. It is then a nontrivial check of the conformal invariance to calculate the partition function by independent means. We shall calculate the partition function of our model in this section, starting from (2.1). If our conjecture of finiteness is true, then this partition function is the one for all coupling $k$ and independent of $k$. In the next section, we shall calculate the partition function in the infinite $k$ limit, starting with a path integral. We shall see the results are same.
The Euclidean action of our working model is
\[ S = \int d^3x \left( (D^i \phi)^+(D_i \phi) + i \frac{k}{4\pi} \epsilon^{ijk} A_i \partial_j A_k \right), \tag{2.2} \]
where \( D^i = \partial^i + i A^i \). The scaling operators in a gauge theory are all gauge invariant operators. However, not all these operators in the model (2.2) are independent. The inter-relationship is specified by the equations of motion
\[ \frac{k}{4\pi} \epsilon^{ijk} F_{jk} = \left( (D^i \phi)^+ \phi - \phi^+ D^i \phi \right), \tag{2.3} \]
\[ D^i D_i \phi = (D^i D_i \phi)^+ = 0. \]
These equations of motion will simplify counting of gauge invariant scaling operators. The first equation in (2.3) means that the strength of the gauge field is not independent. So in constructing a gauge invariant operator we need not consider \( F \) and its covariant derivatives. The second equation in (2.3) implies that not all three \((D_i)^2 \phi\) (and their conjugates) are independent. We can forget, for instance, \((D_3)^2 \phi\) in our construction of scaling operators.

Now, although \( D_i \) and \( D_j \) with \( i \neq j \) do not commute, \( D_i D_j \) is not independent of \( D_j D_i \). This follows from \( D_j D_i = D_i D_j + [D_j, D_i] \) and the second term on the r.h.s. is proportional to \( F_{ij} \) which have been given away from the counting of operators, by use of the first equation in (2.3). Once all these have been taken into account, a general gauge invariant scaling operator takes the form
\[ \left( (D_1^{(1)} D_2^{(1)} D_3^{(1)} \phi)^+ D_1^{(1)} D_2^{(1)} D_3^{(1)} \phi \right) \ldots \left( (D_1^{(m)} D_2^{(m)} D_3^{(m)} \phi)^+ D_1^{(m)} D_2^{(m)} D_3^{(m)} \phi \right) \tag{2.4} \]
While \( l_1^{(i)} \), \( l_2^{(i)} \) \((n_1^{(i)}, n_2^{(i)})\) are arbitrary non-negative integers, \( l_3^{(i)} (n_3^{(i)}) = 0, 1 \) is restricted, because of the second equation in (2.3). Let us assume that the scaling dimension of the above operator coincides with its canonical one, as it is when \( k \) goes to infinity. The scaling dimension of the operator (2.4) is
\[ x = m + \sum_{i,a} (l_i^{(a)} + n_i^{(a)}). \]

An observation of (2.4) suggests that the partition function of interest can be written as a sum
\[ Z(q) = \sum \delta(\sum r_i - \sum s_i) q^{\sum_i \frac{r_i}{2} + n_1^{(i)} + n_2^{(i)} + n_3^{(i)} + \sum_i n_1^{(i)} + n_2^{(i)} + n_3^{(i)}} + \sum_i \frac{s_i}{2} + l_1^{(i)} + l_2^{(i)} + l_3^{(i)}), \tag{2.5} \]
where \( r_i \) and \( s_i \) are multiplicities. The introduction of a delta function is due to the fact that a gauge invariant operator must contain the same number of \( \phi \) and \( \phi^+ \). This delta function renders a direct calculation of the partition function difficult. Let us introduce another parameter \( z \), and define

\[
Z(q, z) = \sum z \sum_{r_i} q \sum_{s_i} q^{(1/2 + n_1^{(i)} + n_2^{(i)} + n_3^{(i)}) + \sum s_i (1/2 + t_1^{(i)} + t_2^{(i)} + t_3^{(i)})} = Z_1(q, z) Z_1(q, z^{-1}).
\]

(2.6)

where

\[
Z_1(q, z) = \sum z \sum_{r_i} q \sum_{s_i} q^{(1/2 + n_1^{(i)} + n_2^{(i)} + n_3^{(i)})}.
\]

It is easy to see that \( Z(q) \) is just the coefficient of the term \( z^0 \) in \( Z(q, z) \). Now \( Z_1(q, z) \) is easy to calculate, the result is

\[
Z_1(q, z) = \prod_{n_a} \frac{1}{1 - zq^{1/2 + n_1 + n_2 + n_3}} = \prod_{n=0}^{\infty} \frac{1}{(1 - zq^{n+1/2})^{2n+1}}.
\]

It is interesting to compare this with the partition function of a free scalar, obtained by Cardy in [3],

\[
Z_0(q) = \prod_{n=0}^{\infty} \frac{1}{(1 - q^{n+1/2})^{2n+1}},
\]

we see that \( Z_0(q) = Z_1(q, z = 1) \).

Finally, the partition function for the model (2.2) (at least in the infinite \( k \) limit) is

\[
Z(q) = \frac{1}{2\pi i} \oint dz z^{-1} \prod_{n=0}^{\infty} \frac{1}{(1 - zq^{n+1/2})^{2n+1}(1 - z^{-1}q^{n+1/2})^{2n+1}}.
\]

(2.7)

The contour integral in (2.7) is formally defined. We shall duplicate the formula in the next section by working out the path integral in the infinite \( k \) limit. We shall see that parameter \( z \) plays a role of the holonomy of a flat gauge field along the Euclidean time direction. (2.7) is the starting point to understand modular and other properties of the theory. An exact evaluation of it is on the way.

Now we turn to the numerical investigation of modular invariance in the form proposed in [3]. Our conclusion is that this form of modular invariance does not work for the theory. As the first step, we check modular invariance associated with a partition function

\[
\tilde{Z}(q) = \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^n}.
\]
\( \hat{Z}(q) \) is not invariant under \( \delta \to 1/\delta \). Write

\[
\log \hat{Z}(q) = \oint_{C(2+\epsilon)} \delta^{-s} F(s) ds,
\]

where a factor \( 1/(2\pi i) \) is absorbed into the definition of the contour integral. The above formula can be obtained by using the inverse Mellin transform of the gamma function, as we shall do for the integrand in (2.7) in the next section. The contour \( C(2+\epsilon) \) is the one with \( \text{Res} = 2 + \epsilon \), and \( \epsilon \) is a small positive number. The above contour is chosen such that all poles of the integrand are located to its left. Explicitly, \( F(s) = (2\pi)^{-s} \Gamma(s) \zeta(s-1) \zeta(s+1) \). \( \zeta(s) \) is the Riemann zeta function. Now, the quantity \( I_3 \)

\[
I_3(\delta) = \oint_{C(3/2+\epsilon)} \delta^{-s} \frac{\Gamma(s/2-1/4)}{\Gamma(s/2+1/4)} F(s + \frac{1}{2}) ds
\]

is modular invariant, except a few additional terms in the form \( \delta^a \), \( a = \pm 3/2, \pm 1/2 \), will appear after the modular transformation \( \delta \to 1/\delta \). These terms appear due to the poles of the integrand in the above contour integral at \( s = \pm 3/2, \pm 1/2 \).

Note that, because of appearance of terms \( \delta^{\pm 1/2} \), the function \( \Phi_3 \) defined in an appendix in [3] is not modular invariant. \( \Phi_3 = (\partial_x^2 - 9/4) I_3 \) is defined such that additional terms \( \delta^{\pm 3/2} \) are removed, here \( x = \log \delta \). Instead, we define a new function

\[
\Phi(x) = (\partial_x^4 - \frac{5}{2} \partial_x^2) I_3 + \frac{9}{16} I_3,
\]

This function is invariant under \( x \to -x \), thus an even function of \( x \). If we have the following expansion

\[
\log \hat{Z}(q) = \sum_{\lambda} a_{\lambda} q^\lambda,
\]

then the function \( I_3 \) can be expanded as

\[
I_3(\delta) = \sqrt{\delta} \sum_{\lambda} a_{\lambda} K_1(2\pi \lambda \delta).
\]

To numerically check modular invariance, we expand \( Z(q) \) up to \( q^4 \):

\[
\hat{Z}(q) = 1 + q + 3q^2 + 6q^3 + 13q^4 + \ldots,
\]

and to the same order

\[
\log \hat{Z}(q) = q + \frac{5}{2} q^2 + \frac{10}{3} q^3 + \frac{21}{4} q^4 + \ldots.
\]
Because the modified Bessel function $K_1(z)$ decays exponentially with the growth of its argument, it is sufficient for us to take the first few terms. We used Mathematica to calculate function $\Phi(x)$. The result is

$$\Phi(x) = 0.3358 + 5.8 \times 10^{-7}x - 1.7267x^2 + 6.2 \times 10^{-5}x^3 + 4.2673x^4 + 1.6 \times 10^{-3}x^5 \ldots.$$  

It is seen that to a good approximation, $\Phi(x)$ is an even function. To see how inclusion of higher order terms improves the situation, we expanded $Z(q)$ to $q^6$ and calculated $\Phi(x)$. We found

$$\Phi(x) = 0.3358 + 1.4 \times 10^{-11}x - 1.7267x^2 + 3.4 \times 10^{-9}x^3 + 4.2676x^4 + 2.2 \times 10^{-7}x^5 + \ldots.$$  

This improves the vanishing of the coefficients of odd terms by a factor $10^{-4}$.

We do not know a model with a partition function investigated above. Cardy considered the partition function of an antiperiodic scalar. The antiperiodic condition is crucial for modular invariance in that case. Here in our model of Chern-Simons coupled to a massless complex scalar, we need not consider such boundary condition, since all gauge invariant operators given by (2.4) is even under this boundary condition. For this model, the candidate of a modular invariant function is $\Phi$ defined above, while not $\Phi_3$ introduced by Cardy. The reason is the following. First notice that to fix the additional finite terms under modular transformation, we need pick out poles of the integrand in (2.8) between $C(3/2+\epsilon)$ and $C(-3/2-\epsilon)$ [3]. It is readily seen that one pole is $s = 1/2$, provided it is not cancelled by a zero of $F(s+1/2)$. It happens that for a free scalar, this pole is cancelled. Then one looks for a possible pole of $F(s+1/2)$ at $s = -1/2$. Precisely because of the anti-periodic condition, this pole does not occur. Now for our toy model discussed above, it occurs. If one lift the anti-periodic condition, just as in the model under consideration, one normally expects that this pole must be there. Modular invariance requires that the integrand (excluding $\delta^{-s}$) in (2.8) is invariant under $s \rightarrow -s$. So one expects another pole at $s = 1/2$. Other poles at $s = \pm 3/2$ will normally occur. We do not expect poles at integers.

Even when there are no poles $s = \pm 1/2$, it is safe to use $\Phi$. This is because if $\Phi_3$ defined in [3] is invariant, $\Phi$ is invariant too. The partition function (2.7), up to order $q^6$, is

$$Z(q) = 1 + q + 7q^2 + 26q^3 + 82q^4 + 233q^5 + 657q^6 \ldots \tag{2.10}$$
We first use the expansion up to \( q^4 \) in calculating \( \Phi(x) \). The result is

\[
\Phi(x) = 0.4094 - 0.5971x + 0.2459x^2 - 3.0990x^3 + 5.7426x^4 + 1.9280x^5 + \ldots.
\]  

(2.11)

The coefficients of odd terms are of the same magnitude as the ones of even terms. This clearly rules out modular invariance. Next we use (2.10) up to order \( q^6 \) to calculate \( \Phi(x) \). We find

\[
\Phi(x) = 0.4094 - 0.5971x + 0.2460x^2 - 3.0994x^3 + 5.7453x^4 + 1.9163x^5 + \ldots.
\]

So inclusion of higher order terms merely improves higher terms in \( \Phi(x) \), within our approximation. The term \( x \), say, is stable when more and more higher terms are included.

3. Regularizing Determinant

In this section, we shall compute the partition function of our model by regularizing the determinant arising from the path integral in the infinite \( k \) limit. To compare the result in this section with (2.7), we shall first recast the logarithm of the integrand in (2.7) into a contour integral. We notice first that

\[
\log Z(q, z) = \sum_{n=0, r=1} (2n+1) \frac{1}{r} (z^r + z^{-r}) q^{r(n+1/2)}.
\]

Using the inverse Mellin transform of the gamma function

\[
e^{-\tau} = \int_C \tau^{-s} \Gamma(s) ds,
\]

where the contour \( C \) is parallel to the imaginary axis and right to it, and again a factor \( 1/(2\pi i) \) is absorbed into the definition of the contour integral, we find

\[
\log Z(q, z) = \int_{C(2+\epsilon)} \delta^{-s}(2\pi)^{-s}(2^s - 2) \Gamma(s) \zeta(s-1)(F(z, s+1) + c.c.) ds,
\]  

(3.1)

where \( \zeta(s-1) \) is the Riemann zeta function. The generalized zeta function \( F(z, s+1) \) is

\[
F(z, s+1) = \sum_{n=1}^{\infty} \frac{z^n}{n^{s+1}}.
\]  

(3.2)

The contour in (3.1) has been chosen so that all poles of the integrand are located to its left.
We now calculate the partition function with the path integral. In the large $k$ limit, only flat gauge field configurations contribute to the path integral. In the spacetime with topology $S^2 \times S^1$, these configurations are parametrized by an angle variable $\theta \in [0, 2\pi]$:

$$A_i = 0 \quad A_\tau = \frac{\theta}{2\pi L} = \delta^{-1} \frac{\theta}{2\pi}.$$  

Note, that we have assumed that the radius of $S^2$ is 1 so the parameter $\delta = L/R = L$. The circumference of $S^1$ is $2\pi L$. We will see that the complex parameter $z$ introduced previously is related to $\theta$ through $z = \exp(i\theta)$.

Now the partition function in the large $k$ limit reads:

$$Z(q) = \int \frac{d\theta}{2\pi} \det^{-1/2}[-(\partial + iA(\theta))(\partial + iA(\theta)) + \xi R],$$  \hspace{1cm} (3.3)

where we assumed that the scalar is conformally coupled to the scalar curvature, therefore, $\xi = 1/8$. The eigenvalues of $\Delta$ on the unit two sphere are $l(l + 1)$, $l$ being non-negative integer. Formally, the partition function takes the form

$$Z(q) = \int \frac{d\theta}{2\pi} \exp \left( - \sum_{l=0, n=-\infty}^{\infty} \frac{(l + 1/2)\log((l + 1/2)^2 + \delta^{-2}(n + \theta/(2\pi))^2))}{(l + 1/2)^2 + \delta^{-2}(n + \theta/(2\pi))^2)} \right).$$  \hspace{1cm} (3.4)

In it, the boundary condition along the time direction is not important, since it can always be put into the $\theta$ parameter. In other words, one can say that the holonomy of the flat gauge field shifts the boundary condition.

To be well-defined, (3.4) needs to be regularized. We use the $\zeta$-function regularization to do so for the sum. Define

$$\zeta(s, \theta) = \sum_{l=0, n=-\infty}^{\infty} \frac{l + 1/2}{((l + 1/2)^2 + \delta^{-2}(n + \theta/(2\pi))^2)}. $$

The regularized partition function is

$$Z(q) = \int \frac{d\theta}{2\pi} e^{\zeta'(0, \theta)}. $$

Using formula

$$x^{-s}\Gamma(s) = \int_0^\infty e^{-xt}t^{s-1}dt,$$

we rewrite the zeta function as

$$\zeta(s, \theta)\Gamma(s) = \int_0^\infty t^{s-1}F_1(\delta^{-2}t)F_2(t)dt,$$  \hspace{1cm} (3.5)
where
\[
F_1(t) = \sum_{n=-\infty}^{\infty} e^{-(n+\theta/(2\pi))^2t}
\]
\[
F_2(t) = \sum_{l=0}^{\infty} (l + \frac{1}{2}) e^{-(l+1/2)^2t}.
\]

\(\zeta(s, \theta)\), defined as above, has obviously a pole at \(s = \frac{3}{2}\). On the other hand, \(s = 0\) is a regular point of \(\zeta(s, \theta)\). Actually, \(\zeta(0, \theta) = 0\), since the r.h.s. of (3.5) is regular while \(\Gamma(s)\) is divergent at \(s = 0\). Denote the r.h.s. of (3.5) by \(g(s)\). Then
\[
\zeta'(s, \theta) = g'(s) - \zeta(s, \theta) \frac{\Gamma'(s)}{\Gamma(s)}.
\]

We shall see that \(g'(0)\) is regular, so that the first term on the r.h.s. of the above equation is zero at \(s = 0\). At \(s = 0\), the second term is
\[
-g(s) \frac{\Gamma'(s)}{(\Gamma(s))^2} \rightarrow g(0).
\]

We thus find
\[
\zeta'(0, \theta) = g(0).
\]

This is what we need to calculate. Thus, we need to extend the function \(g(s)\) in (3.5) to \(s = 0\). When \(t \rightarrow 0\), both \(F_i(t)\) are singular. To extract the singular behaviors of these functions, one can use integrals to approximate sums:
\[
F_1(t) \rightarrow \int_0^{\infty} dx e^{-x^2t} \rightarrow \frac{1}{\sqrt{t}}
\]
\[
F_2(t) \rightarrow \int_0^{\infty} dx x e^{-x^2t} \rightarrow \frac{1}{t}.
\]

To go beyond this approximation and actually calculate \(F_i(t)\) for small \(t\), we apply the inverse Mellin transform and obtain,
\[
F_1(t) = \int_{C(1/2+\epsilon)} t^{-s} \Gamma(s)(\zeta(2s, u) + \zeta(2s, 1-u)) ds = \frac{1}{2} \int_{C(1+\epsilon)} t^{-s/2} \Gamma(s/2)(\zeta(s, u) + \zeta(s, 1-u)) ds,
\]
\[
F_2(t) = \int_{C(1+\epsilon)} t^{-s} (2^{2s-1} - 1) \Gamma(s)\zeta(2s - 1) ds = \frac{1}{2} \int_{C(2+\epsilon)} t^{-s/2} (2^{s-1} - 1) \Gamma(s/2)\zeta(s - 1) ds,
\]

(3.6)
where \( u = \theta / (2\pi) \), \( \zeta(s, u) \) is the generalized Riemann zeta function [3]. The above contours are chosen such that all poles of integrands are located to the left of the contours.

We then consider \( t \to 1/t \):

\[
F_1(1/t) = \frac{1}{2} \int_{C(1+\epsilon)} t^{s/2} \Gamma(s/2)(\zeta(s, u) + \zeta(s, 1-u))ds
\]

\[
= \Gamma(1/2) \sqrt{t} + \frac{1}{2} \int_{C(-1-\epsilon)} t^{s/2} \Gamma(s/2)(\zeta(s, u) + \zeta(s, 1-u))ds \tag{3.7}
\]

\[
= \sqrt{\pi} t + \frac{1}{2} \int_{C(1+\epsilon)} t^{-s/2} \Gamma(-s/2)(\zeta(-s, u) + \zeta(-s, 1-u))ds.
\]

Denote the second term in (3.7) by \( f_1(t) \). It is easy to see that when \( t \) gets large, \( f_1(t) \) decreases exponentially.

Similarly, we have

\[
F_2(1/t) = t + \frac{1}{2} \int_{C(3+\epsilon)} t^{-s/2} \pi^{-s/2} \Gamma(s/2+1) \zeta(s+2)ds.	ag{3.8}
\]

It is easy to show that the last term behaves like \( 1/t^2 \) when \( t \) gets large. Denote this function by \( f_2(t) \).

We are ready to extend \( g(s) \). It is

\[
g(s) = \int_0^\infty t^{s-1} F_1(\delta-2t) F_2(t)dt
\]

\[
\int_0^1 t^{s-1} F_1(\delta-2t) F_2(t)dt + \int_1^\infty t^{s-1} F_1(\delta-2t) F_2(t)dt
\]

\[
\int_1^\infty \{ t^{-s-1} [\delta \sqrt{\pi} t + f_1(\delta^2 t)] \} dt + \frac{1}{2} + \frac{1}{24} - \frac{7}{960} t^{-1} + f_2(t) \] + \( t^{-s-1} F_1(\delta-2t) F_2(t)dt \}
\]

\[
= \sqrt{\pi} \delta \left( \frac{1}{2s-3} + \frac{1}{12(2s-1)} - \frac{7}{480(2s+1)} \right)
\]

\[
+ \int_1^\infty \{ t^{-s-1} [\delta \sqrt{\pi} t f_2(t) + (t/2 + 1/24 - 7/960 t^{-1} + f_2(t)) f_1(\delta^2 t)] + t^{-s-1} F_1(\delta-2t) F_1(t) \} dt.	ag{3.9}
\]

And

\[
g(0) = -\frac{207}{480} \sqrt{\pi} \delta
\]

\[
+ \int_1^\infty t^{-1} [F_1(\delta-2t) F_2(t) + \delta \sqrt{\pi} t f_2(t) + (t/2 + 1/24 - 7/960 t^{-1} + f_2(t)) f_1(\delta^2 t)] dt.	ag{3.10}
\]
According to the properties of functions $f_i(t)$ at large $t$ mentioned above, the integral on the r.h.s. of (3.10) is well defined. We shall evaluate this integral term by term. First, we calculate

$$
\int_1^{\infty} t^{-1} F_1(\delta^{-2}t)F_2(t)dt = a\sqrt{\pi}\delta
$$

\[3.11\]

where

$$a = \int_{C(2+\epsilon)} \frac{1}{s+1} (2^{s-1} - 1)\Gamma(s+1)\zeta(s-1)ds.\]

We have used Hurwitz formula [9] in evaluating (3.11)

$$\zeta(-s, u) = 2(2\pi)^{-s-1}\Gamma(s+1) \sum_{n=1}^{\infty} n^{-s-1} \sin(2\pi nu - \pi/2).$$

For the first time we have seen $F(z, s+1)$ introduced before in (3.2), $z = \exp(i\theta)$. Other terms can be calculated similarly. The final result is

$$
\zeta'(0, \theta) = \int_{C(2+\epsilon)} \delta^{-s} (2^{s-2} - 2)(2\pi)^{-s}\Gamma(s)\zeta(s-1)(F(z, s+1) + c.c.). \quad (3.12)
$$

This formula is exactly the same as in (3.1).

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