Quantum Gauge Theory Amplitude Solutions

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Abstract

The n-point amplitudes of gauge and gravity theory are given as a series in the coupling. The recursive derivative expansion is used to find all of the coupling coefficients. Initial conditions to any bare Lagrangian, or of an improved action, are used to compute quantum amplitudes.
Introduction

Gauge theory amplitudes and correlators have been well studied for many years. Many techniques have been developed so as to compute tree and one-loop amplitudes, and holographic duals to supersymmetric gauge theories also gave computational tools. The amplitude and correlator calculations in these theories, either in weak coupling or strong coupling, are typically given to low orders.

There are a variety of methods to extend the scope of these calculations, and to find complete expressions for amplitudes and correlators. This is a means in quantum field theory to find order unity, or large coupling results.

The amplitudes of pure gauge theories are given here as a series in the coupling, $g$ or $G_N$. The derivative expansion has been formulated and applied to many theories \[1\]-\[10\]. The solution to this iteration is presented for gauge theory models; the scalar field theory coupling expansion is given in \[11\]. There is a further mathematical simplification possible within the tensor formulation in the contributions.

The Lagrangians considered are,

\[ \mathcal{L}_{YM} = \int d^d x \frac{1}{g^2} \text{Tr} F^2 + \sum \mathcal{O}(F, \nabla) \]  
with the summation on all the possible higher dimension operators (e.g. so-called irrelevant due to their perturbative scaling). The gravitational theories are,

\[ \mathcal{L}_G = \int d^d x \sqrt{g} \frac{1}{G_N} R + \sum \mathcal{O}(R, \nabla) \]  
with a possibly infinite tower of higher dimension operators. The classical scattering of the $F^2$ and $R$ actions are given in \[12\] (those in scalar theory in \[13\]; the number basis representation is very useful for labeling all of the contributions, including tree graphs containing ghosts. These tree graphs and the numbers parameterizing them enter into the quantum solution through the initial conditions.

The derivative expansion is used to find the perturbative quantum amplitudes. Instantons are not included; their momentum structure is expected to have a number basis as do the classical scattering.

Amplitudes
The coupling expansions of scalar field theory amplitudes have appeared in [12] and contain the required integral and tensor calculations. The primary complications in the gauge theory calculation is due to the masslessness and tensor algebra.

The recursive approach to the amplitude calculations has a solution presented in [11]. The latter is represented in Figure 1 with the rainbow graphs; a sum of these graphs with any number of nodes is necessary, and with internal ghost lines in covariant gauges, subject to the coupling addition that

\[ q = \sum_{b=1}^{b_{\text{nodes}}} q_b \]  

(\( q_b \) is the coupling power of classical scattering at node \( b \) which is \( n - 2 \) for Yang-Mills theory).

The classical scattering is required to specify the tensors at the nodes. The \( \text{Tr}F^2 \) Yang-Mills scattering can be obtained in a well ordered manner with the use of string theory.

The \( \kappa(a; 1) \) and \( \kappa(b; 2) \) set of primary numbers used on the string inspired set of Greens functions numbers produces the contributions,

\[
(-\frac{1}{2})^{a_1}\frac{1}{2}^{n-a_2}\prod_{i=1}^{a_1}\varepsilon(\kappa(i; 1)) \cdot \varepsilon(\kappa(i; 1)) \times \prod_{j=a_1+1}^{a_2}\varepsilon(\kappa(j; 1)) \cdot k_{\kappa(j;2)}
\]  

(3)

\[
\times \prod_{p=a_2+1}^{n} k_{\kappa(p;1)} \cdot k_{\kappa(p;2)}
\]  

(4)

together with the permutations of \( 1, \ldots, n \). The permutations extract all possible combinations from the set of terms in the labeled \( \phi^3 \) diagram, after distributing the numbers into the three categories.

The form of the amplitudes are expressed as,

\[
A^n = \sum_{\sigma} C_{\sigma} g^{n-2} T_{\sigma} \prod_{i=1}^{n-1} t_{\sigma(i,p)}^{-1}
\]  

(5)

with \( T_{\sigma} \) in (4) derived from the tensor set of \( \kappa \), e.g. found from \( \phi_n \) or the momentum routing of the propagators with \( \sigma(i,p) \). The normalization is \( i(-1)^n \). The numbers \( a_1 \) and \( a_2 \) are summed so that \( a_1 \) ranges from 1 to \( n/2 \), with the boundary condition \( a_2 \geq a_1 + 1 \). Tree amplitudes in gauge theory must possess at least one \( \varepsilon_i \cdot \varepsilon_j \).

All \( \phi^3 \) diagrams are summed at \( n \)-point, which is represented by the sum in \( \sigma \) in (5). The color structure is \( \text{Tr}(T_{a_1} \ldots T_{a_n}) \), and the complete amplitude involves summing the permutations of \( 1, \ldots, n \).
The first $n - 2$ numbers in $\kappa_2$ are summed beyond those of the primary numbers in accord with the set $i$ to $i + p - 1$ for a given vertex label $i + p - 1$, which labels the vertex in $\phi_n$.

The propagators are in correspondence with $\phi^3$ diagrams,

$$D_\sigma = g^{n-2} \prod \frac{1}{t_{\sigma(i,p)} - m^2}.$$  

(6)

The Lorentz invariants $t_{\sigma(i,p)}$ are defined by $i_i^{[p]}$,

$$i_i^{[p]} = (k_i + \ldots + k_{i+p-1})^2.$$  

(7)

Factors of $i$ in the propagator and vertices are placed into the prefactor of the amplitude. The sets of permutations $\sigma$ are what are required in order to specify the individual diagrams. The full sets of $\sigma(i, p)$ form all of the diagrams, at any $n$-point order.

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The numbers $\phi_n(i)$ can be arranged into the numbers $(p_i, [p_i])$, in which $p_i$ is the repetition of the value of $[p_i]$. Also, if the number $p_i$ equals zero, then $[p_i]$ is not present in $\phi_n$. These numbers can be used to obtain the $t_i^{[q]}$ invariants without intermediate steps with the momenta. The branch rules are recognizable as, for a single $t_i^{[q]}$,

$$0) \ l_{\text{initial}} = [p_m] - 1$$
1) \[ r = 1 \] to \[ r = p_m \]

If \[ r + \sum_{j=1}^{m-1} p_j = [p_m] - l_{\text{initial}} \] then \[ i = [p_m] \]
\[ q = [p_m] - l_{\text{initial}} + 1 \]

Beginning conditions has no sum in \( p_j \)

2) Else \( l_{\text{initial}} \rightarrow l_{\text{initial}} - 1 \) : decrement the line number

\( l_{\text{initial}} > [p_l] \) else \( l \rightarrow l - 1 \) : decrement the \( p \) sum

3) goto 1) \( (10) \)

The branch rule has to be iterated to obtain all of the poles. This rule checks the number of vertices and matches to compare if there is a tree on it in a clockwise manner. If not, then the external line number \( l_{\text{initial}} \) is changed to \( l_{\text{initial}} \) and the tree is checked again. The \( i \) and \( q \) are labels to \( t_i^{[q]} \).

**Coupling Coefficients**

The recursive solution to the coupling coefficients are calculated in this section. The solution to the recursive formulae is represented in Figure 1, and the graphs do not show the permutations on the external lines. These diagrams were evaluated in scalar theory in \([11]\).

The tensor integrals between two adjacent nodes are,

\[
\int d^d x e^{ik \cdot x} \prod_{\mu} \partial_{\mu} \Delta(m, x)^N = T^m_{\mu_j} \sum_{a=1}^{\infty} \Delta(N, a)(m^2)^{a-N\beta_2/2} \left( k^2 \right)^{-d/2-N(\beta_1-\beta_2/2)-a} \]

\[
= \gamma^n \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{p_a=1}^{m} \sum_{a}^{p_a} \frac{p!}{\prod_{p=1}^{m}(ap_a)!} \frac{\Gamma(N+1)}{\Gamma(N-m)} c_{p_a} \]

\[
\times \rho(\beta_1, \beta_2) \frac{\Gamma(d/2 + N(\beta_1 - \beta_2/2) + 1)}{\Gamma(d/2 + N(\beta_1 - \beta_2/2) + 1 - a)} \]

\[
\rho \sum_{l=0}^{n} \frac{n!}{l!(n-l)!} (2-d)^{n-l} \frac{\Gamma(a - N\beta_2/2 + 1)}{\Gamma(a - N\beta_2/2 + 1 - l)} (m^2)^{a-N\beta_2/2} \left( k^2 \right)^{-d/2-N(\beta_1-\beta_2/2)-a} \]

(10)
Figure 1: The product form solution to the recursive formulae defining the loop expansion. The nodes are classical scattering vertices.
\[
\rho(\beta_1, \beta_2) \frac{\Gamma(-N(\beta_1 - \beta_2/2) + 1 - a)}{\Gamma(-N(\beta_1 - \beta_2/2) + 1)} \frac{\Gamma(-d/2 - N(\beta_1 - \beta_2/2) + 1)}{\Gamma(-d/2 - N(\beta_1 - \beta_2/2) + 1 - a)} (15)
\]
\[
\times \frac{\Gamma(d/2 + N(\beta_1 - \beta_2/2) + a + 1 - n)}{\Gamma(d/2 + N(\beta_1 - \beta_2/2) + a + 1)} (16)
\]
\[
\sum_{\sigma_w, \tilde{\sigma}_w} \prod_{i=1}^{n-w} \eta^{\mu(\alpha)} \prod_{i=1}^{n-w} k^{\mu(\alpha)} 2^{-w} \frac{\Gamma(-d/2 - N(\beta_1 - \beta_2/2) - a - n + 1)}{\Gamma(-d/2 - N(\beta_1 - \beta_2/2) - a - 2n + 1 + w/2)} (17)
\]

The sums on \(l\) and \(a\) should be performed, in order to have a simplified expression at fixed tensor structure. The momentum \(k\) is the sum of the momenta on the exterior of the integral, i.e. \(k = \sum_{j=1}^{a} k_j\).

General massless integrals in the gauge theory are more complicated than those in (17) because the tree amplitudes have massless poles. The integral contains the two node factors,

\[
\prod_{a=b,b+1}^{a} \prod_{\sigma_a(i,p)}^{a} t_{\sigma_a(i,p)}^{-1}, (18)
\]

with a series of propagators and derivatives,

\[
(\prod_{j=1}^{a} \partial_{\mu_j}) \Delta^N. (19)
\]

The integrals include the singular terms in (18). The node momentum is defined by \(P_b = \sum_i k_i\), with \(i = \sigma(m_i - m_b)\) to \(i = \sigma(m_f + m_b)\). The nodal momentum spans the lines that are integrated; the indices also specify which invariants in (18) are internal-external and external-external to the loop.

The vertex tensors are,

\[
\left(-\frac{1}{2}\right)^{a_1} \left(\frac{1}{2}\right)^{n-a_2} \prod_{i=1}^{a_1} \varepsilon(\kappa_{\alpha}(i; 1)) \cdot \varepsilon(\kappa_{\alpha}(i; 1)) \times \prod_{j=a_1+1}^{a_2} \varepsilon(\kappa_{\alpha}(j; 1)) \cdot k_{\kappa_{\alpha}(j; 2)} \times \prod_{p=a_2+1}^{n} k_{\kappa_{\alpha}(p; 1)} \cdot k_{\kappa_{\alpha}(p; 2)}. (20)
\]

The polarizations satisfy the on-shell identity

\[
\sum_{\lambda = \pm} \varepsilon_{\lambda, \mu} \varepsilon_{-\lambda, \nu} = -\eta_{\mu\nu} + \frac{k_{\mu} p_{\nu} + p_{\mu} k_{\nu}}{p \cdot k}. (22)
\]
The propagators can be used in conjunction with a massless form of the integral in (17) with the identity,

\[ t_{i}^{[p]} = (k_{i} + \ldots + k_{i+p-1})^{2} \]

\[ = (k_{i} + \ldots + k_{j} + \frac{1}{N_{1}}(i\partial_{k})^{-1})^{2} \]

with the \( N_{1} \) due to the fact that the derivative is taken on the \( N_{1} \) propagators \( \Delta^{N_{1}} \); these propagators are with the \( N_{1} \) momenta in the invariant. There is a \( (2 - d) \) scaled into the number \( N_{1} \) so that the result \( (2 - d) \) times an integer. The propagator satisfies the identity

\[ \partial_{\mu}\Delta = (2 - d)\frac{x_{\mu}}{x^{2}}\Delta = (2 - d)(\frac{i}{\partial_{\mu}})\Delta. \]

The derivatives in the set of propagators in (19) compound the complexity. The \( i \) and \( p \) are used in conjunction with the set of node momenta \( k_{a} \) for \( a = \sigma(m_{i}) \) to \( a = \sigma(m_{f}) \), to determine the derivatives in (33).

The massless integrals are

\[ \int d^{d}x e^{ik \cdot x} \Delta^{N} = \int d^{d}x e^{ik \cdot x} (a^{2})^{N\beta_{1}} = \rho(\beta_{1}, N)k^{2}^{-N\beta_{1}-d/2}. \]

The form in (26) is used with the derivatives in (19) and (25) to deduce the tensor integrals; the expansion of the propagators is also to be included.

The derivatives in (25) result in

\[ \int d^{d}x e^{ik \cdot x} \prod_{a} \partial_{\mu_{a}}\Delta^{N} \sim \prod_{i}^{a_{2}} \eta_{\mu(\alpha)\mu(\beta)} \prod_{j}^{a_{2}} k_{\mu(\beta)} \]

The next set of derivatives act on the propagators. The permutation set will be obtained as in the tree amplitude case, with direct sets of numbers.

Expansions of the invariants are,

\[ (k_{i} + \ldots + k_{j} + \frac{1}{N_{1}}(i\partial_{k})^{-1})^{-2} \]
\[ = \sum_a (-1)^a((k_1 + \ldots + k_j)^2 + \frac{1}{N_1}(k_1 + \ldots + k_j) \cdot (i\partial_k)^{-1})^a(-\partial_k^2)^{-a-1} \] (29)

and in the two-particle case,

\[ (k_1 + (i\partial_k)^{-1})^2 = (k_1 \cdot i\partial_k^{-1} - (\partial_k^2)^{-1}) \] (30)

\[ = \sum_a (-1)^a(k_1 \cdot (i\partial_k)^{-1})^a(-\partial_k^2)^{-a-1}. \] (31)

These expansions can be used to evaluate the tensor integrals that contain the propagators. The third type of invariant is \((k_1 + \ldots + k_j)^2 = 1/N_1(-\partial_k^2). \) The sums in the terms (29) and (31) then have to be performed. The action on of the \( \partial_k \) derivatives is taken on the scalar integral of the propagators, in (27). (The differential operators commute in x-space.)

The result for the product of propagators is

\[ \prod_{\sigma} t_{\sigma(i)}^{-1,\sigma(p)} = \prod_{j=1}^a \sum_{a_j} (-1)(q_j^2 + \frac{1}{N_j}q_j \cdot (i\partial_k)^{-1})^{a_j}(\partial_k^2)^{-a_j-1}, \] (32)

\[ = \prod_{j=1}^a \sum_{a_j=0}^{\infty} \frac{a_j!}{d_j!(a_j - d_j)!} (-1)^{d_j}(q_j^2)^{d_j}(\frac{i}{N_j}q_j \cdot \partial_k)^{a_j-d_j}(\partial_k^2)^{-2a_j+d_j-1} \] (33)

with \( t_{\sigma(p)}^{[\sigma]} \) and \( q_j \) the external portion of the external-internal momenta, with the former defined by \( \sigma(i, p) \). The product is over all invariants; when \( q_j = 0 \) then only the \( a_j = 0 \) term contributes. There are a \( b \) count of the derivatives, i.e. \( b \) of the propagators with the internal momenta so that \( b = \sum(a_j - d_j) \).

The combination of (27) and (33) results in a series of derivatives as in (27). The derivatives from the tensor are

\[ \prod_{a}^a (i\partial_{k,\mu}) \frac{1}{\partial_k^2}, \] (34)

and combine with the (33). Then there are a total of \( a+b \) derivatives in the numerator, which result in a tensor calculation; there are \( b_2 \) boxes in the denominator.

The result for the derivatives with \( 2\bar{a}_1 + \bar{a}_2 = a + b \) is,
The tensor is to be contracted with the function in (33) from the propagator products and with vertex tensors in (21) at nodes 1 and 2. The propagators are $\eta_{\mu\nu}/p^2$. The vertex tensors are discussed in the prior section and in [11]. The tensor at node 2 used in the calculation contains the propagator expansions as at node 1, and the $\varepsilon$’s stripped and contracted with those at node 1. The remaining momenta, in (21), if they are internal are integrated over.

The loop integrations follow the form as illustrated in Figure 1. At each node the momenta $k_\sigma$, from $\sigma_b(m_i)$ to $\sigma_b(m_f)$, flow into the loop; these momenta define the nodal momentum $P_b$. These tensors with the integral between the first two nodes result in,

$$
\rho(\beta_1, N)2^{a_1/2+a_2-b_2/2} \frac{\Gamma(-N\beta - d/2 + b_2 + 1)}{\Gamma(-N\beta - d/2 - a_1/2 - a_2 + b_2 + 1)} \frac{\Gamma(N\beta + d/2 - b_2)}{\Gamma(N\beta + d/2)} ,
$$

(35)

$$
\times \rho(\beta_1, N)2^{a_1/2+a_2-b_2/2} \frac{\Gamma(-N\beta - d/2 + b_2 + 1)}{\Gamma(-N\beta - d/2 - a_1/2 - a_2 + b_2 + 1)} \frac{\Gamma(N\beta + d/2 - b_2)}{\Gamma(N\beta + d/2)} .
$$

(36)

in terms of the external line momenta $k_i$ and polarizations,

$$
(k^2)^{-N\beta - d/2 - a_1/2 - a_2 + b_2} \prod_{i=1}^{c_1} k_{\kappa_1(i)} \cdot k_{\kappa_2(i)} \prod_{j=1}^{c_2} \varepsilon(\beta_1(j)) \cdot \varepsilon(\beta_2(j)) \prod_{l=1}^{c_3} \varepsilon(\tilde{\beta}_1(l)) \cdot k_{\tilde{\beta}_2(l)} .
$$

(37)

The basis could be rewritten in terms of the propagator momentum $q_j$, but the tree amplitude forms and their divergences suggest to keep the same basis. Due to the number of polarizations, $2c_2 + c_3$ is equal to the number of external lines left of node 1; the number $c_1$ can be arbitrarily high due to the series in (39).

The correlated sets $\kappa_i, \beta_i, \tilde{\beta}_i$ (of dimension $c_1, c_2$, and $c_3$) are determined from: 1) the $n$-point scattering $\phi_n$ set of numbers, 2) expansion of the propagators that contain internal loop momentum, and 3) the inner product of the momenta and $\varepsilon_j$ with the integrated tensors (36) including external momenta and metrics. These sets
are derived from numbers parameterizing the classical tensors as in (4); these can be imported from the numbers labeling the propagators on the affiliated scalar diagram.

These tensors of dimension $c_i$ in principle are determined from the well-defined set of numbers such as the $\phi_n$ are, which label the $\phi^3$ diagrams, and from the node momenta $P_a = \sum k_i$ together with the polarization numbers entering into the two nodes.

In a given diagram, illustrated in Figure 1, there are potentially external lines at all of the nodes $b_{\text{nodes}}$. The node numbers $b_{\text{nodes}}$ range from 2 to a maximum set by the coupling order; $n$-point tree amplitudes are of $g^{n-2}$ coupling order (without possible higher dimension terms in the classical action). These couplings constrain the number of internal lines within the integrals and the nodes; $\sum_{b=1}^{b_{\text{nodes}}}(n_b - 2)$ is a fixed order, $n-2+2L$ for an $L$ loop $n$-point. The tree amplitudes with ghost lines also have to be included, in the covariant gauge with covariant $\varepsilon$’s (there are light-cone $\varepsilon$’s but the tree amplitudes are different).

The quantum numbers that specify the diagram in Figure 1 are the propagator labels $\sigma(i,p)$, or the equivalent $\phi_n$ numbers. The node momentum indices $\sigma_b(m_i)$ to $\sigma_b(m_f)$ labels the external lines at each vertex. The ordering of the lines is required also in the $\sigma$.

The integration from nodes 1–2, 2–3, etc..., at a fixed order in their couplings, can be performed with a systematic implementation of the previous quantum numbers and node momentum. The integration along the chain of vertices is influenced in a sequential manner because the tensor is altered sequentially. The integration removes the propagators containing components of the node’s momenta in the $j+1$ side of $j-j+1$, and the tensor (analogous to (4)) then contains inner products which are non-tree like. However, the individual tensors in (4) do factor into the left node and right node with the use of metrics. This property makes all of the integrals independent, at the computational cost of specifying the metrics.

The result form of the $b_{\text{nodes}}$ integrations contains the expansion of the external-internal momenta at each node,

$$
\rho(\beta_1, N)2^{\tilde{a}_1/2+\tilde{a}_2-b_2/2} \frac{\Gamma(-N\beta - d/2 + b_2 + 1)}{\Gamma(-N\beta - d/2 - \tilde{a}_1/2 - \tilde{a}_2 + b_2 + 1)} \frac{\Gamma(N\beta + d/2 - b_2)}{\Gamma(N\beta + d/2)}
$$

$$
\times \prod_{b=1} \prod_{j=1} \Theta(N_j^{\sigma(i,p)}) \sum_{a_j=0}^{\infty} \sum_{d_j=1}^{a_j} \frac{a_j!}{d_j!(a_j - d_j)!} (-1)^{i} \left( \frac{i}{N_j} \right)^{a_j-d_j}
$$

(40)
and the kinematic factors,
\begin{equation}
\prod_{b=1}^{b_{\text{nodes}}} (P_b^2)^{-n_b\beta - d/2 - \tilde{\alpha}_1/2 - \tilde{\alpha}_2 + b_2} \tag{42}
\end{equation}

\begin{equation}
\sum_{\kappa, \beta, \beta'} \prod_{i=1}^{d_1} k_{\kappa_1(i)} \cdot k_{\kappa_2(i)} \prod_{j=1}^{d_2} \varepsilon(\beta_1(j)) \cdot \varepsilon(\beta_2(j)) \prod_{l=1}^{d_3} \varepsilon(\tilde{\beta}_1(l)) \cdot k_{\tilde{\beta}_2(l)} \tag{43}
\end{equation}

The kinematic factors are due to various contractions at the nodes. There are also the remaining propagators from the tree amplitudes, from the external momenta,

\begin{equation}
\prod_{b=1}^{b_{\text{nodes}}} \prod_{\sigma_b} t_{\sigma}^{-1} \tag{44}
\end{equation}

The node momenta $P_a$ is used with the tree numbers (e.g. $\phi_n$) to restrict the invariants to those of the $\tilde{t}$ type and the $N_j$ numbers. The $N_j$ count the number of internal momenta within an invariant $t_i^{[p]}$, scaled by $1/(2 - d)$; the latter is part of a tree used to define a node. The number of internal lines is $\prod_{j=1}^{b_{\text{nodes}}} N_j$, with $n_j$ counted to the right-hand side of the node. All of this information is encoded in the numbers $\phi_n$, or an equivalent set such as $\sigma(i, p)$, that define the tree combinations.

The sets of $\beta$, $\tilde{\beta}$, and $\kappa$ depend on the tree contributions at the nodes and the external momenta configuration; there is likely a set theoretic definition of these groups of numbers.

Last, the summation on the internal lines has to be performed in accord with the coupling conservation, i.e. $\sum (n_j - 2) = n - 2 + 2L$ and the sum over the node number and internal line number in (43). The external lines have to be permuted in line with a fixed color structure.

**Gravity Amplitudes**

The gravity amplitudes have the almost the same form as in (43) except the kinematic factor has twice as many polarizations,

\begin{equation}
\sum \prod_{\kappa, \beta, \beta', \tilde{\beta}'} k_{\kappa_1(i)} \cdot k_{\kappa_2(i)} \prod_{j=1}^{d_2} \varepsilon(\beta_1(j)) \cdot \varepsilon(\beta_2(j)) \prod_{l=1}^{d_3} \varepsilon(\tilde{\beta}_1(l)) \cdot k_{\tilde{\beta}_2(l)} \tag{45}
\end{equation}

\begin{equation}
\prod_{j=1}^{d_1'} \varepsilon(\beta_1'(j)) \cdot \varepsilon(\beta_2'(j)) \prod_{l=1}^{d_3'} \varepsilon(\tilde{\beta}_1'(l)) \cdot k_{\tilde{\beta}_2'(l)} \tag{46}
\end{equation}
and the sums in $\kappa$, $(\beta, \bar{\beta})$, $(\beta^\prime, \bar{\beta}^\prime)$ are altered. The sums have a different dependence on $a_j$ and $d_j$.

In the derivation, the permutations on the external lines are completely symmetrized in the kinematics as there are no color quantum attached to the gravitons. Also, mixed particle type scattering can be obtained, between gauge and gravity modes, and in general including varying spin types if the classical scattering is known.

\textit{Concluding remarks}

The derivative expansion and its recursion is solved for both gauge and gravity theories. Expressions for the amplitudes are obtained, and a closed form for them is given. At any given order in derivatives, the operator’s prefactor $f(\lambda)$ can be determined. All the perturbative integrals are computed.

Three sets of similar tensor indices are required to make the expressions explicit, apart from a recursive integral. Six sets are required for gravity. The sets of indices are dependent on a number of parameters, and can be solved for recursively. However, it is very suggestive that there is a group theory or topological determination in their determination, which is applicable to mixed particle scattering.

There is a power series in the momenta of the external lines. The power series and its summation are likely to be relevant in certain kinematical regimes. The tensor indices, and their possible topological determination, are relevant in this calculation.

The amplitudes are written in terms of the polarizations, and thus they are not in a spinor helicity format. There are further simplifications with the former. Covariant tree amplitudes in the gauge and gravity theories are found in \cite{12,13}.

Knowledge of the coefficients in the perturbative sector can be used to find non-perturbative formalisms, i.e. strong coupling. The global tensor determination, and its symmetry, is necessary for this.
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