The Dirichlet problem for a prescribed mean curvature equation

Yuki Tsukamoto
(Received October 2, 2019)
(Revised June 30, 2020)

Abstract. We study a prescribed mean curvature problem where we seek a surface whose mean curvature vector coincides with the normal component of a given vector field. We prove that the problem has a solution near a graphical minimal surface if the prescribed vector field is sufficiently small in a dimensionally sharp Sobolev norm.

1. Introduction

In this paper, we consider the following prescribed mean curvature problem with the Dirichlet condition,

\[
\left\{ \begin{array}{ll}
\text{div} \left( \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = H(x, u(x), \nabla u(x)) & \text{in } \Omega, \\
u = \phi & \text{on } \partial \Omega,
\end{array} \right.
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \). The function \( H(x, t, z) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) is given and we seek a solution \( u \) satisfying (1). Since the left hand side of (1) is the mean curvature of the graph of \( u \), (1) is a prescribed mean curvature equation whose prescription depends on the location of the graph as well as the slope of the tangent space.

Prescribed mean curvature problems in a wide variety of formulation have been studied by numerous researchers. In the most classical case of \( H = H(x) \), (1) has a solution if \( H \) and \( \phi \) have suitable regularity and the mean curvature of \( \partial \Omega \) satisfies a certain geometric condition (see [3, 4, 6, 7, 8, 11], for example). Giusti [5] determined a necessary and sufficient condition that a prescribed mean curvature problem without boundary conditions has solutions. In the case of \( H = H(x, t) \), Getherdt [2] constructed \( H^{1,1} \) solutions, and Miranda [10] constructed BV solutions. In those papers, assumptions of the boundedness \( |H| < \infty \) and the monotonicity \( \frac{\partial H}{\partial t} \geq 0 \) play an important role. If \( |H| < \Gamma \) where \( \Gamma \) is determined by \( \Omega \), there exist solutions of (1), and the uniqueness of solutions is guaranteed by the monotonicity, that is, \( \frac{\partial H}{\partial t} \geq 0 \). Under the
assumptions of boundedness, monotonicity and the convexity of $W$, Bergner [1] solved the Dirichlet problem in the case of $H = H(x, u, v(Vu))$ using the Leray-Schauder fixed point theorem. Here, $v$ is the unit normal vector of $u$, that is, $v(z) = \frac{1}{\sqrt{1 + |z|^2}}(z, -1)$. For the same problem as [1], Marquardt [9] gave a condition on $qW$ depending on $H$ which guarantees the existence of solutions even for a non-convex domain $\Omega$.

The motivation of the present paper comes from a singular perturbation problem studied in [12], where one considers the following problem on a domain $\bar{\Omega} \subset \mathbb{R}^{n+1}$,

$$-\varepsilon \Delta \phi_\varepsilon + \frac{W'(\phi_\varepsilon)}{\varepsilon} = \varepsilon V \phi_\varepsilon \cdot f_\varepsilon, \quad (2)$$

Here, $W$ is a double-well potential, for example $W(\phi) = (1 - \phi^2)^2$ and $\{f_\varepsilon\}_{\varepsilon>0}$ are given vector fields uniformly bounded in the Sobolev norm of $W^{1,p}(\bar{\Omega})$, $p > \frac{n+1}{2}$. In [12], we proved under a natural assumption

$$\int_{\Omega} \left( \frac{\varepsilon |V \phi_\varepsilon|^2}{2} + \frac{W(\phi_\varepsilon)}{\varepsilon} \right) dx + ||f_\varepsilon||_{W^{1,p}(\bar{\Omega})} \leq C \quad (3)$$

that the interface $\{\phi_\varepsilon = 0\}$ converges locally in the Hausdorff distance to a surface whose mean curvature $H$ is given by $f \cdot v$ as $\varepsilon \to 0$. Here, $f$ is the weak $W^{1,p}$ limit of $f_\varepsilon$. If the surface is represented locally as a graph of a function $u$ over a domain $\Omega \subset \mathbb{R}^n$, the corresponding relation between the mean curvature and the vector field is expressed as

$$\text{div} \left( \frac{Vu}{\sqrt{1 + |Vu|^2}} \right) = v(Vu(x)) \cdot f(x, u(x)) \quad \text{in } \Omega, \quad (4)$$

where $f \in W^{1,p}(\Omega \times \mathbb{R}; \mathbb{R}^{n+1})$ with $p > \frac{n+1}{2}$. Note that $f$ is not bounded in $L^\infty$ in general, unlike the cases studied in [1, 9]. In this paper, we establish the well-posedness of the perturbative problem including (4) which has a $W^{1,p}$ norm control on the right-hand side of the equation. The following theorem is the main result of this paper.

**Theorem 1.** Let $\Omega$ be a $C^{1,1}$ bounded domain in $\mathbb{R}^n$ and fix constants $\varepsilon > 0$, $\frac{n+1}{2} < p < n+1$ and $q = \frac{np}{n+1-p}$. Suppose $h \in W^{2,\infty}(\Omega)$ satisfies the minimal surface equation, that is,

$$\text{div} \left( \frac{Vh}{\sqrt{1 + |Vh|^2}} \right) = 0. \quad (5)$$
Then there exists a constant $\delta_1 > 0$ which depends only on $n$, $p$, $\Omega$, $\|h\|_{W^{2,q}(\Omega)}$, and $\varepsilon$ with the following property. Suppose $G \in W^{1,p}(\Omega \times \mathbb{R})$ and $\phi \in W^{2,q}(\Omega)$ satisfy

$$
\|G\|_{W^{1,p}(\Omega \times \mathbb{R})} + \|\phi\|_{W^{2,q}(\Omega)} \leq \delta_1, \quad (6)
$$

and a measurable function $H(x,t,z): \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is such that $H(x,\cdot,\cdot)$ is a continuous function for a.e. $x \in \Omega$, and for all $(t,z) \in \mathbb{R} \times \mathbb{R}^n$,

$$
|H(x,t,z)| \leq |G(x,t)| \quad \text{for a.e. } x \in \Omega. \quad (7)
$$

Then, there exists a function $u \in W^{2,q}(\Omega)$ such that $u - h - \phi \in W^{1,q}_0(\Omega)$ and

$$
\text{div} \left( \frac{Vu}{\sqrt{1 + |Vu|^2}} \right) = H(x,u(x),Vu(x)) \quad \text{in } \Omega, \quad (8)
$$

$$
\|u - h\|_{W^{2,q}(\Omega)} < \varepsilon. \quad (9)
$$

The claim proves that there exists a solution of (1) in a neighbourhood of any minimal surface if $H$ and $\phi$ are sufficiently small in these norms. In particular, if we take $H(x,t,z) = v(z) \cdot f(x,t)$ and $G(x,t) = |f(x,t)|$, where $\|f\|_{W^{1,p}(\Omega \times \mathbb{R})}$ is sufficiently small, above conditions on $G$ and $H$ in Theorem 1 are satisfied and we can guarantee the existence of a solution for (1) nearby the given minimal surface (see Corollary 1). The method of proof is as follows. We prove that the linearized problem of (1) has a unique solution in $W^{2,q}(\Omega)$ and the norm of this solution is controlled by $G$ and $\phi$. When (6) is satisfied, there exist a suitable function space $\mathcal{A}$ and a mapping $T : \mathcal{A} \rightarrow \mathcal{A}$, and a fixed point of $T$ is a solution of (8) with $u - h - \phi \in W^{1,q}_0(\Omega)$. We show that $T$ satisfies assumptions of the Schauder fixed point theorem, and Theorem 1 follows.

2. Proof of Theorem 1

Throughout the paper, $\Omega$ is a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary $\partial \Omega$. We define functions $A_{ij} : \mathbb{R}^n \rightarrow \mathbb{R}$ $(i,j = 1, \ldots, n)$ as

$$
A_{ij}(z) := \frac{1}{\sqrt{1 + |z|^2}} \left( \delta_{ij} - \frac{z_i z_j}{1 + |z|^2} \right)
$$

and the operator

$$
L[z](u) := A_{ij}(z)D_{ij}u(x) \quad \text{for any } u \in W^{2,1}(\Omega),
$$
where we omit the summation over $i, j = 1, \ldots, n$. By the Cauchy–Schwarz inequality, for any $\xi \in \mathbb{R}^n$,

$$A_j(\xi)_i,_{\xi_j} = \frac{1}{\sqrt{1 + |\xi|^2}} \left( \delta_{ij} - \frac{z_i z_j}{1 + |\xi|^2} \right) \xi_i,_{\xi_j}$$

$$= \frac{1}{\sqrt{1 + |\xi|^2}} \left[ |\xi|^2 - \left( \frac{z_i}{\sqrt{1 + |\xi|^2}} \right)^2 \right]$$

$$\geq \frac{1}{\sqrt{1 + |\xi|^2}} \left[ |\xi|^2 - \left( \frac{|x|^2}{1 + |\xi|^2} \right) \xi_i^2 \right]$$

$$= \frac{1}{(1 + |\xi|^2)^{3/2}} |\xi|^2. \quad (10)$$

Hence, as is well-known, the operator $L[\xi]$ is elliptic.

**Theorem 2.** Suppose $v \in C^{1,\alpha}(\Omega)$ with $0 < \alpha < 1$, $B = (B_1, \ldots, B_n) \in L^\infty(\Omega; \mathbb{R}^n)$ with $\|B_i\|_{L^\infty(\Omega)} \leq K$ for all $i \in \{1, \ldots, n\}$, $f \in L^q(\Omega)$ and $\phi \in W^{2,q}(\Omega)$ with $q > n$. Then there exists a unique function $u \in W^{2,q}(\Omega)$ such that

$$\begin{cases}
L[\nabla v](u) + B \cdot \nabla u = f & \text{in } \Omega, \\
u - \phi \in W_0^{1,q}(\Omega).
\end{cases} \quad (11)$$

Moreover, there exists a constant $c_0$ which depends only on $n, q, \Omega, K$, and $\|v\|_{C^{1,\alpha}(\Omega)}$ such that

$$\|u\|_{W^{2,q}(\Omega)} \leq c_0(\|f\|_{L^q(\Omega)} + \|\phi\|_{W^{2,q}(\Omega)}). \quad (12)$$

**Proof.** By (10), for any $\xi \in \mathbb{R}^n$,

$$A_j(\nabla v)_{\xi_i} \xi_j \geq \frac{1}{(1 + \|v\|_{C^{1,\alpha}(\Omega)}^2)^{3/2}} |\xi|^2 =: \lambda |\xi|^2, \quad (13)$$

where the constant $\lambda$ depends only on $\|v\|_{C^{1,\alpha}(\Omega)}$. Since each $A_j$ is a smooth function of $\nabla v$, there exists a constant $A$ which depends only on $\|v\|_{C^{1,\alpha}(\Omega)}$ such that

$$\|A_j(\nabla v)\|_{C^{0,\alpha}(\Omega)} \leq A \quad \text{for all } i, j \in \{1, \ldots, n\}. \quad (14)$$

By (13) and (14), there exists a unique solution $u \in W^{2,q}(\Omega)$ satisfying (11) (cf. [4, Theorem 9.15]). Using [4, Theorem 9.13], we can know that there
exists a constant $c_1$ which depends only on $n$, $q$, $\Omega$, $\lambda$, $K$, and $A$ such that

$$\|u\|_{W^{2,q}(\Omega)} \leq c_1(\|u\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega)} + \|\phi\|_{W^{2,q}(\Omega)}).$$ \hfill (15)

Using the Aleksandrov maximum principle [4, Theorem 9.1], we can know that there exists a constant $c_2$ which depends only on $n$, $q$, $\Omega$, $K$, and $\lambda$ such that

$$k_u k_{W^2} \left\| \frac{q}{W} \right\| a c_1 (k_f k_{L^Q} (W) + k_f k_{W^2} \left\| \frac{q}{W} \right\|).$$ \hfill (16)

By the Hölder and Sobolev inequalities, $\phi \in C(\overline{\Omega})$ and

$$\|u\|_{L^q(\Omega)} \leq c \|u\|_{L^q(\Omega)} \leq c \left( \sup_{x \in \Omega} |\phi| + \|f\|_{L^q(\Omega)} \right) \leq c (\|\phi\|_{C(\overline{\Omega})} + \|f\|_{L^q(\Omega)}) \leq c_3 (\|f\|_{L^q(\Omega)} + \|\phi\|_{W^{2,q}(\Omega)}),$$ \hfill (17)

where $c_3$ depends only on $n$, $q$, and $\Omega$. By (15) and (17), there exists a constant $c_0$ which depends only on $n$, $q$, $\Omega$, $\lambda$, $K$, and $A$ such that

$$\|u\|_{W^{2,q}(\Omega)} \leq c_0 (\|f\|_{L^q(\Omega)} + \|\phi\|_{W^{2,q}(\Omega)}).$$ \hfill (18)

Thus this theorem follows. \hfill \square

To proceed, we need the following theorem (cf. [13, Theorem 5.12.4]).

**Theorem 3.** Let $\mu$ be a positive Radon measure on $\mathbb{R}^{n+1}$ satisfying

$$K(\mu) := \sup_{B_r(x) \subset \mathbb{R}^{n+1}} \frac{1}{r^n} \mu(B_r(x)) < \infty.$$

Then there exists a constant $c_4$ which depends only on $n$ such that

$$\left| \int_{\mathbb{R}^{n+1}} \phi \, d\mu \right| \leq c_4 K(\mu) \int_{\mathbb{R}^{n+1}} |\nabla \phi| \, d\mathcal{L}^{n+1}$$

for all $\phi \in C_c^1(\mathbb{R}^{n+1})$.

**Lemma 1.** Suppose $v \in W^{1,\infty}(\Omega)$ with $\|v\|_{W^{1,\infty}(\Omega)} \leq V$ and $G \in W^{1,p}(\Omega \times \mathbb{R})$ with $\frac{n+1}{2} < p < n+1$. Let $q = \frac{np}{n+1-p}$. Then there exists a constant $c_5$ which depends only on $n$, $p$, $\Omega$, and $V$ such that

$$\|G(\cdot, v(\cdot))\|_{L^q(\Omega)} \leq c_5 \|G\|_{W^{1,p}(\Omega \times \mathbb{R})}.$$ \hfill (19)
Proof. Define
\[ G := \{(x, v(x)) \in \Omega \times \mathbb{R}\}. \]
A set \( B^n_r(x) \) is the open ball with center \( x \) and radius \( r \) in \( \mathbb{R}^n \). In the following, \( \mathcal{H}^n \) denotes the \( n \)-dimensional Hausdorff measure in \( \mathbb{R}^{n+1} \) and \( \mathcal{H}^n \Gamma \) is a Radon measure defined by
\[ \mathcal{H}^n \Gamma (A) := \mathcal{H}^n (A \cap \Gamma) \quad \text{for all } A \subset \mathbb{R}^{n+1}. \]
Then the support of \( \mathcal{H}^n \Gamma \) satisfies in particular \( \text{spt} \mathcal{H}^n \Gamma \subset \Omega \times (-2V, 2V) \).

For any \( B^{n+1}_r((x_0, x_0')) \subset \mathbb{R}^{n+1} \) with \( (x_0, x_0') \in \mathbb{R}^n \times \mathbb{R} \),
\[ \frac{1}{r^n} \mathcal{H}^n \Gamma (B^{n+1}_r((x_0, x_0'))) \leq \frac{1}{r^n} \int_{B_r(x_0) \cap \Omega} \sqrt{1 + |\nabla v|^2} \, d\mathcal{L}^n \leq (1 + V)\omega_n, \quad (20) \]
where \( \omega_n \) is the volume of \( n \)-dimensional unit open ball. Using the standard Extension Theorem, we can know that there exists a function \( \tilde{G} \in W^{1, p}(\mathbb{R}^{n+1}) \) such that \( \tilde{G} = G \) in \( \Omega \times (-2V, 2V) \) and
\[ \| \tilde{G} \|_{W^{1, p}(\mathbb{R}^{n+1})} \leq c_6 \| G \|_{W^{1, p}(\Omega \times (-2V, 2V))}, \quad (21) \]
where \( c_6 \) depends only on \( n, p, \Omega, \) and \( V \). By Theorem 3 and smoothly approximating \( \tilde{G} \),
\[
\int_{\Omega} |G(x, v(x))|^q \, dx \leq \int_{\Omega} |\tilde{G}(x, v(x))|^q \sqrt{1 + |\nabla v|^2} \, dx
\]
\[
= \int_{\Gamma} |\tilde{G}(x, x_{n+1})|^q \, d\mathcal{H}^n
\]
\[
\leq c(n, V) \int_{\mathbb{R}^{n+1}} |\nabla \tilde{G}| \, |\tilde{G}|^{q-1} \, d\mathcal{L}^{n+1}
\]
\[
\leq c(n, p, V) \| \nabla \tilde{G} \|_{L^p(\mathbb{R}^{n+1})} \| \tilde{G} \|_{W^{1, p}(\mathbb{R}^{n+1})}^{q-1}
\]
\[
\leq c(n, p, V) c_6 \| G \|_{W^{1, p}(\Omega \times (-2V, 2V))}^{q} \]
\[
\leq c(n, p, V) c_6 \| G \|_{W^{1, p}(\mathbb{R} \times \mathbb{R})}^{q} \quad (22) \]
This lemma follows. \( \square \)

We write the Schauder fixed point theorem needed later ([4, Corollary 11.2]).

Theorem 4. Let \( \mathcal{G} \) be a closed convex set in Banach space \( \mathcal{B} \) and let \( T \) be a continuous mapping of \( \mathcal{G} \) into itself such that the image \( T(\mathcal{G}) \) is precompact. Then \( T \) has a fixed point.
We first prove Theorem 1 in the case that $h = 0$.

**Theorem 5.** Assume that $G \in W^{1,p}(\Omega \times \mathbb{R})$ with $\frac{n+1}{p} < p < n+1$ and $\phi \in W^{2,q}(\Omega)$ with $q = \frac{np}{n+1-p}$. Then there exists a constant $\delta_2 > 0$ which depends only on $n$, $p$, and $\Omega$ such that, if

$$
\|G\|_{W^{1,p}(\Omega \times \mathbb{R})} + \|\phi\|_{W^{2,q}(\Omega)} \leq \delta_2, \tag{23}
$$

then, for any measurable function $H(x,t,z) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that $H(x,\cdot,\cdot)$ is a continuous function for a.e. $x \in \Omega$ and

$$
|H(x,t,z)| \leq |G(x,t)| \quad \text{for a.e. } x \in \Omega, \text{ any } (t,z) \in \mathbb{R} \times \mathbb{R}^n, \tag{24}
$$

there exists a function $u \in W^{2,q}(\Omega)$ such that $u - \phi \in W^{1,q}_0(\Omega)$ and

$$
\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = H(x,u(x),\nabla u(x)) \quad \text{in } \Omega. \tag{25}
$$

**Proof.** Define

$$
\mathcal{A} := \{v \in C^{1,1/2-n/2q}(\overline{\Omega}); \|v\|_{C^{1,1/2-n/2q}(\overline{\Omega})} \leq 1 \}. \tag{26}
$$

The set $\mathcal{A}$ is a closed convex set in Banach space $C^{1,1/2-n/2q}(\overline{\Omega})$. By (24) and Lemma 1, $H(\cdot, v(\cdot), \nabla v(\cdot)) \in L^q(\Omega)$ for any $v \in \mathcal{A}$. Using Theorem 2, we can know that there exist a unique function $w \in W^{2,q}(\Omega)$ and a constant $c_7 > 0$ which depends only on $n$, $p$, $\Omega$, and not on $v$ such that

$$
\begin{cases}
L[\nabla v](w) = H(x,v,\nabla v) \quad \text{in } \Omega, \\
w - \phi \in W^{1,q}_0(\Omega), \\
\|w\|_{W^{2,q}(\Omega)} \leq c_7(\|G\|_{W^{1,p}(\Omega \times \mathbb{R})} + \|\phi\|_{W^{2,q}(\Omega)}). \tag{27}
\end{cases}
$$

By the Sobolev inequality and (27), we obtain

$$
\|w\|_{C^{1,1/2-n/2q}(\overline{\Omega})} \leq c_8 \|w\|_{C^{1,1-n/q}(\overline{\Omega})} \leq c_9 \|w\|_{W^{2,q}(\Omega)} \leq c_{10}(\|G\|_{W^{1,p}(\Omega \times \mathbb{R})} + \|\phi\|_{W^{2,q}(\Omega)}), \tag{28}
$$

where $c_8, c_9, c_{10} > 0$ depend only on $n$, $p$, and $\Omega$. Suppose that

$$
\|G\|_{W^{1,p}(\Omega \times \mathbb{R})} + \|\phi\|_{W^{2,q}(\Omega)} \leq c_{10}^{-1} =: \delta_2(n,p,\Omega). \tag{29}
$$

Let us define an operator $T : \mathcal{A} \to \mathcal{A}$ by $T(v) = w$ which satisfies (27). We show that $T(\mathcal{A})$ is precompact and $T$ is a continuous mapping. For any
sequence \( \{v_m\}_{m \in \mathbb{N}} \subset \mathcal{A} \), we have \( \sup_{m \in \mathbb{N}} \|T(v_m)\|_{\mathcal{C}_1^{1-\frac{n}{q}(\Omega)}} \leq c_8^{-1} \) by (28) and (29). There exists a subsequence \( \{T(v_{m_k})\}_{k \in \mathbb{N}} \subset \{T(v_m)\}_{m \in \mathbb{N}} \) which converges to a function \( w_\infty \in \mathcal{C}_1^{1}(\Omega) \) in the sense of \( \mathcal{C}_1^{1}(\Omega) \) by the Ascoli-Arzelà theorem. We see that

\[
\lim_{k \to \infty} \frac{|V_{W_{\infty}}(x) - V_{W_{\infty}}(y)|}{|x-y|^{1-\frac{n}{q}}} = \lim_{k \to \infty} \frac{|V_{T(v_{m_k})}(x) - V_{T(v_{m_k})}(y)|}{|x-y|^{1-\frac{n}{q}}} \leq c_8^{-1}.
\]

Let \( \tilde{w}_k := T(v_{m_k}) - w_\infty \), and \( \tilde{w}_k \) converges to 0 in the sense of \( \mathcal{C}_1^{1}(\Omega) \). Then we have

\[
\frac{|V_{\tilde{w}_k}(x) - V_{\tilde{w}_k}(y)|}{|x-y|^{1-\frac{n}{q}}} \leq \left( \frac{|V_{\tilde{w}_k}(x) - V_{\tilde{w}_k}(y)|}{|x-y|^{1-\frac{n}{q}}} \right)^{1/2} \frac{|V_{\tilde{w}_k}(x) - V_{\tilde{w}_k}(y)|^{1/2}}{x-y} \leq 2c_8^{-1/2}(2\|V_{\tilde{w}_k}\|_{L^\infty(\Omega)})^{1/2}.
\]

Hence, \( \{T(v_{m_k})\}_{k \in \mathbb{N}} \) converges to a function \( w_\infty \) in the sense of \( \mathcal{C}_1^{1,1/2-n/2q}(\Omega) \), and the operator \( T \) is a compact mapping. In particular, the set \( T(\mathcal{A}) \) is precompact.

Suppose that \( \{v_m\}_{m \in \mathbb{N}} \) converges to \( v \) in the sense of \( \mathcal{C}_1^{1,1/2-n/2q}(\Omega) \). By (28) and (29), \( \sup_{m \in \mathbb{N}} \|T(v_m)\|_{W^{2,q}(\Omega)} \) is bounded. Hence, there exists a subsequence \( \{T(v_{m_k})\}_{k \in \mathbb{N}} \subset \{T(v_m)\}_{m \in \mathbb{N}} \) which weakly converges to a function \( w \in W^{2,q}(\Omega) \). We show \( T(v) = w \), that is,

\[
A_{\beta}(V_{v}(x))D_{\beta}w(x) = H(x,v,V_{v}).
\]

For any \( \psi \in \mathcal{C}_0^\infty(\Omega) \), by the weak convergence and the Hölder inequality,

\[
\int_{\Omega} \psi \{A_{\beta}(V_{v})D_{\beta}w - A_{\beta}(V_{v_{m_k}})D_{\beta}(T(v_{m_k}))\} \leq \int_{\Omega} \psi A_{\beta}(V_{v})(D_{\beta}w - D_{\beta}(T(v_{m_k}))) + \int_{\Omega} \psi D_{\beta}(T(v_{m_k}))(A_{\beta}(V_{v}) - A_{\beta}(V_{v_{m_k}})) \leq \int_{\Omega} \psi A_{\beta}(V_{v})(D_{\beta}w - D_{\beta}(T(v_{m_k}))) + \|T(v_{m_k})\|_{W^{2,q}(\Omega)} \|\psi(A_{\beta}(V_{v}) - A_{\beta}(V_{v_{m_k}}))\|_{L^\infty(\Omega)} \rightarrow 0 \quad (k \to \infty).\]
By (24) and \(\|v_m\|_{L^\infty(\Omega)} \leq 1\), we compute
\[
|H(x, v_m(x), \nabla v_m(x))| \\
\leq |G(x, v_m(x)) - G(x, v(x))| + |G(x, v(x))| \\
\leq \int_{-1}^{1} |D_{1}G(x, t)|dt + |G(x, v(x))|.
\]
(32)

\(\int_{-1}^{1} |D_{1}G(\cdot, t)|dt + |G(\cdot, v(\cdot))|\) is an integrable function by Lemma 1, \(\|v\|_{C^1(\Omega)} \leq 1\), and Fubini’s theorem. Since \(H\) is a continuous function with respect to \(t\) and \(z\), using the dominated convergence theorem, we have
\[
\int_{\Omega} \psi\{H(x, v(x), \nabla v(x)) - H(x, v_m(x), \nabla v_m(x))\} \rightarrow 0 \quad (k \rightarrow \infty).
\]
(33)

By (31) and (33),
\[
\int_{\Omega} \psi\{A_{ij}(\nabla v) D_{ij}w - H(x, v(x), \nabla v(x))\} \\
= \lim_{k \rightarrow \infty} \int_{\Omega} \psi\{A_{ij}(\nabla v_m) D_{ij}(T(v_m)) - H(x, v_m(x), \nabla v_m(x))\} \\
= 0.
\]
(34)

Using the fundamental lemma of the calculus of variations, we have
\[
A_{ij}(x, \nabla v) D_{ij}w - H(x, v(x), \nabla v(x)) = 0 \quad \text{for a.e. } x \in \Omega,
\]
and \(T(v) = w\). Hence, \(\{T(v_m)\}_{m \in \mathbb{N}}\) weakly converges to \(T(v)\) in \(W^{2, q}(\Omega)\).

By the compactness of \(T\) and the uniqueness of limit, we can show \(\{T(v_m)\}_{m \in \mathbb{N}}\) converges to \(T(v)\) in \(C^{1,1/2-n/2q}(\Omega)\), and \(T\) is a continuous mapping. Using Theorem 4, we obtain a function \(u \in W^{2, q}(\Omega)\) satisfying \(u - \phi \in W^{1, q}_0(\Omega)\) and (25).

**Proof (Proof of Theorem 1).** We should show that there exists a function \(\tilde{u} \in W^{2, q}(\Omega)\) such that
\[
A_{ij}(\nabla \tilde{u} + \nabla h) D_{ij}(\tilde{u} + h) = H(x, \tilde{u} + h, \nabla \tilde{u} + \nabla h),
\]
(35)
\[
\tilde{u} - \phi \in W^{1, q}_0(\Omega),
\]
(36)
\[
\|\tilde{u}\|_{W^{2, q}(\Omega)} < \varepsilon.
\]
(37)

Using the minimal surface equation (5) for \(h\), we convert (35) as
Define
\[ A_{ij}(\nabla \tilde{u} + \nabla h)D_{ij}\tilde{u} + \frac{D_{ij}h}{(1 + |\nabla \tilde{u} + \nabla h|^2)^{3/2}}((|\nabla \tilde{u}|^2 + 2\nabla \tilde{u} \cdot \nabla h)\delta_{ij} - D_{ij}\tilde{u}D_{ij}h - D_{ij}\tilde{u}D_{ij}h) = H(x, \tilde{u} + h, \nabla \tilde{u} + \nabla h). \] (38)

Define
\[ \mathcal{A} := \{ v \in C^{1,1/2-n/2q}(\Omega); \|v\|_{C^{1,1/2-n/2q}(\Omega)} \leq \varepsilon \}. \] (39)
The set \( \mathcal{A} \) is a closed convex set in Banach space \( C^{1,1/2-n/2q}(\Omega) \). We consider the following differential equation,
\[ A_{ij}(\nabla v + \nabla h)D_{ij}w + \frac{D_{ij}h}{(1 + |\nabla v + \nabla h|^2)^{3/2}}((\nabla v \cdot \nabla w + 2\nabla w \cdot \nabla h)\delta_{ij} - D_{ij}vD_{ij}w - D_{ij}wD_{ij}h) = H(x, v + h, \nabla v + \nabla h). \] (40)

Define
\[ B(\nabla v) \cdot \nabla w := \frac{D_{ij}h}{(1 + |\nabla v + \nabla h|^2)^{3/2}}((\nabla v \cdot \nabla w + 2\nabla w \cdot \nabla h)\delta_{ij} - D_{ij}vD_{ij}w - D_{ij}wD_{ij}h). \]

Here, there exists a constant \( c_{11} > 0 \) which depends only on \( n, p, \Omega, \varepsilon, \) and \( \|h\|_{W^{2,\infty}(\Omega)} \) such that
\[ \|B_i(\nabla v)\|_{L^\infty(\Omega)} \leq c_{11} \quad \text{for all } i \in \{1, \ldots, n\}, \] (41)
where \( B_i(\nabla v) = (B_1(\nabla v), \ldots, B_n(\nabla v)) \in L^\infty(\Omega; \mathbb{R}^n) \).

Using Theorem 2, we obtain a unique function \( w \in W^{2,q}(\Omega) \) satisfying \( w - \phi \in W^{1,q}_0(\Omega) \) and (40). By (41), Theorem 2, Lemma 1, and the Sobolev inequality, there exists a constant \( c_{12} > 0 \) which depends only on \( n, p, \Omega, \varepsilon, \) and \( \|h\|_{W^{2,\infty}(\Omega)} \) such that
\[ \|w\|_{C^{1,1/2-n/2q}(\Omega)} \leq c_{12}(\|G\|_{W^{1,p}(\Omega \times \mathbb{R}^n)} + \|\phi\|_{W^{2,q}(\Omega)}). \] (42)

Suppose that we have
\[ \|G\|_{W^{1,p}(\Omega \times \mathbb{R}^n)} + \|\phi\|_{W^{2,q}(\Omega)} \leq c_{12}^{-1} \varepsilon := \delta_1. \] (43)

Let a operator \( T: \mathcal{A} \rightarrow \mathcal{A} \) be defined by \( T(v) = w \) which satisfies \( w - \phi \in W^{1,q}_0(\Omega) \) and (40). The compactness of \( T \) can be proved by the argument of Theorem 5. In particular, the set \( T(\mathcal{A}) \) is precompact.
Suppose that \( \{v_m\}_{m \in \mathbb{N}} \subset \mathcal{C} \) converges to \( v \) in the sense of \( C^{1,1/2-n/2q}(\Omega) \). Then there exists a subsequence \( \{T(v_{m_k})\}_{k \in \mathbb{N}} \subset \{T(v_m)\}_{m \in \mathbb{N}} \) which weakly converges to a function \( w \in W^{2,q}(\Omega) \). For any \( \psi \in C_0^\infty(\Omega) \),

\[
\int_{\Omega} \psi \left( B(\nabla v) \cdot \nabla w - B(\nabla v_m) \cdot \nabla T(v_{m_k}) \right) = \int_{\Omega} \psi B(\nabla v) \cdot (\nabla w - \nabla (T(v_{m_k}))) + \int_{\Omega} \psi \nabla (T(v_{m_k})) \cdot (B(\nabla v) - B(\nabla v_{m_k})) \rightarrow 0 \quad (k \to \infty),
\]

since \( B \) is a continuous function and \( T(v_{m_k}) \) converges weakly to \( w \). By (44) and the argument of Theorem 5, we can show that \( T \) is a continuous mapping. Using Theorem 4, we obtain a function \( \tilde{u} \in W^{2,q}(\Omega) \) satisfying (35) and (36). Moreover, \( \tilde{u} \) satisfies (37) by (42) and (43). Define \( u := \tilde{u} + h \). Then \( u \) satisfies \( u - h - \phi \in W^{1,q}_{0}(\Omega) \), (8), and (9), and the proof is complete.

**Corollary 1.** Suppose \( f = (f_1, \ldots, f_{n+1}) \in W^{1,p}(\Omega \times \mathbb{R}; \mathbb{R}^{n+1}) \) with \( \frac{n+1}{2} < p < n+1 \) and \( \phi \in W^{2,q}(\Omega) \) with \( q = \frac{n+1}{n+1-p} \). Let \( \varepsilon > 0 \) be arbitrary. Suppose \( h \in W^{2,\infty}(\Omega) \) satisfies the minimal surface equation, that is,

\[
\text{div} \left( \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) = 0.
\]

Let \( \delta_1 > 0 \) be the constant as in Theorem 1. If

\[
\sum_{i=1}^{n+1} \|f_i\|_{W^{1,p}(\Omega \times \mathbb{R})} + \|\phi\|_{W^{2,q}(\Omega)} \leq \delta_1,
\]

then there exists a function \( u \in W^{2,q}(\Omega) \) such that \( u - h - \phi \in W^{1,q}_{0}(\Omega) \) and

\[
\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = v(\nabla u(x)) \cdot f(x, u(x)) \quad \text{in} \quad \Omega,
\]

\[
\|u - h\|_{W^{2,q}(\Omega)} < \varepsilon.
\]

**Proof.** Define

\[
H(x, t, z) := v(z) \cdot f(x, t).
\]
By \( f \in W^{1,p}(\Omega \times \mathbb{R}; \mathbb{R}^{n+1}) \), for a.e. \( x \in \Omega \), \( f(x, \cdot) \) is an absolutely continuous function. Hence \( H(x, \cdot, \cdot) \) is a continuous function for almost every \( x \in \Omega \). We have

\[
|H(x, t, z)| \leq \sum_{i=1}^{n+1} |f_i(x, t)| \quad \text{for a.e. } x \in \Omega, \text{ any } (t, z) \in \mathbb{R} \times \mathbb{R}^n,
\]

and \( \sum_{i=1}^{n+1} |f_i(x, t)| \in W^{1,p}(\Omega \times \mathbb{R}) \). By the Minkowski inequality,

\[
\left\| \sum_{i=1}^{n+1} |f_i(x, t)| \right\|_{W^{1,p}(\Omega \times \mathbb{R})} \leq \sum_{i=1}^{n+1} \|f_i\|_{W^{1,p}(\Omega \times \mathbb{R})}.
\]

Define

\[
G(x, t) := \sum_{i=1}^{n+1} |f_i(x, t)|.
\]

Then \( H \) and \( G \) satisfy the assumption of Theorem 1, and this corollary follows.

\[ \square \]

**Remark 1.** The uniqueness of solutions follows immediately using [4, Theorem 10.2]. Under the assumptions of Theorem 1, if we additionally assume that \( H \) is non-decreasing in \( t \) for each \( (x, z) \in \Omega \times \mathbb{R}^n \) and continuously differentiable with respect to the \( z \) variables in \( \Omega \times \mathbb{R} \times \mathbb{R}^n \), then the solution is unique in \( W^{2,q}(\Omega) \).

**References**

[1] M. Bergner, The Dirichlet problem for graphs of prescribed anisotropic mean curvature in \( \mathbb{R}^{n+1} \), Analysis (Munich) 28 (2008), 149–166.

[2] C. Gerhardt, Existence, regularity, and boundary behaviour of generalized surfaces of prescribed mean curvature, Math. Z. 139 (1974), 173–198.

[3] M. Giaquinta, On the Dirichlet problem for surfaces of prescribed mean curvature, Manuscripta Math. 12 (1974), 73–86.

[4] D. Gilbarg, N. Trudinger, Elliptic partial differential equations of second order, Second edition, Springer-Verlag, Berlin, 1983.

[5] E. Giusti, On the equation of surfaces of prescribed mean curvature. Existence and uniqueness without boundary conditions, Invent. Math. 46 (1978), no. 2, 111–137.

[6] K. Hayasida, M. Nakatani, On the Dirichlet problem of prescribed mean curvature equations without H-convexity condition, Nagoya Math. J. 157 (2000), 177–209.

[7] H. Jenkins, J. Serrin, The Dirichlet problem for the minimal surface equation in higher dimensions, J. Reine Angew. Math. 229 (1968), 170–187.

[8] G. P. Leonardi, G. Saracco, The prescribed mean curvature equation in weakly regular domains, NoDEA Nonlinear Differ. Equ. Appl. 25 (2018), no. 2, 25:9.
T. Marquardt, Remark on the anisotropic prescribed mean curvature equation on arbitrary domains, Math. Z. 264 (2010), 507–511.

M. Miranda, Dirichlet problem with $L^1$ data for the non-homogeneous minimal surface equation, Indiana Univ. Math. J. 24 (1974), 227–241.

J. Serrin, The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables, Phil. Trans. R. Soc. Lond. A 264 (1969), 413–496.

Y. Tonegawa, Y. Tsukamoto, A diffused interface with the advection term in a Sobolev space, arXiv:1904.00525.

W. P. Ziemer, Weakly differentiable functions, Springer-Verlag, 1989.

Yuki Tsukamoto

Department of Mathematics
Tokyo Institute of Technology
Tokyo 152-8551 Japan
E-mail: tsukamoto.y.ag@m.titech.ac.jp