On Order, Disorder and Coherence

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Abstract

We provide a brief survey of quantum statistical characterisations of order, disorder and coherence in systems of many degree of freedom. Here, order and coherence are described in terms of symmetry breakdown, while disorder is described in terms of entropy and algorithmic complexity, whose interconnection has been recently extended from the classical to the quantum domain. We see that, in the present physical context, the concepts of order and disorder are not mutually antithetical but bear an interrelationship similar to that between signals and noise.

1. Introduction.

This article is designed to provide a succinct account of the prevailing quantum statistical pictures of order and disorder in systems with many degrees of freedom. In this context, order essentially signifies organisation of the microscopic components of such systems to produce macroscopic fields, or signals, as exemplified by the polarisation of a ferromagnet; while disorder amounts to randomness.

Here we formulate mathematical pictures of order and disorder within the framework of operator algebraic statistical mechanics [Em, Th, Se1], which provides a natural setting for their descriptions. We start in Sec. 2 with a brief sketch of the structure of algebraic quantum theory. We then pass on, in Sec. 3, to both the probabilistic formulation of disorder in terms of Von Neumann’s entropy [VN] and the intrinsic description thereof by Kolmogorov’s algorithmic complexity [Ko]. In particular, we discuss Brudno’s theorem [Br] and its recent quantum generalisation [BKMSS], which shows that these two characterisations of disorder essentially yield the same picture. In Sec. 4, we formulate the concept of order due to symmetry breakdown. In Sec. 5, we refine this formulation of order to an extreme version thereof, namely coherence, in the sense proposed by Glauber [Gl]. We provide some concrete examples both of order, in Sec. 4, and of coherence in Sec. 5. We conclude in Sec. 6 with some further brief observations about order, disorder and coherence, and discuss the need to widen the concept of order to the description of organisational structures that are not covered by existing theories.

2. The Operator Algebraic Framework

We employ the standard operator algebraic description [Em, Th, Se1] of a quantum
mechanical system, \( \Sigma \), as a triple \((\mathcal{A}, \mathcal{S}, \alpha)\) representing its observables, states and dynamics, respectively. Specifically, \( \mathcal{A} \) is a \( C^\ast \)-algebra, whose self-adjoint elements represent the bounded observables of \( \Sigma \), and \( \alpha \) is a homomorphism of either the additive group \( \mathbb{R} \) into the automorphisms of \( \mathcal{A} \) or of the semigroup \( \mathbb{R}_+ \) into completely positive, identity preserving, linear contractions of this algebra, according to whether \( \Sigma \) is a conservative system or an open Markovian dissipative one. The state space \( \mathcal{S} \) is a norm closed, convex subset of the positive, normalised, linear functionals on \( \mathcal{A} \) that is stable under the action of the dual of \( \alpha \). We shall denote the expectation value of \( \Lambda \) (\( \in \mathcal{A} \)) for the state \( \rho \) by \( \rho(\Lambda) \). The pure states are the extremal elements of \( \mathcal{S} \). Thus, the model is specified by the structures of \( \mathcal{A} \), \( \mathcal{S} \) and \( \alpha \). We note that this generic model also covers the case of classical mechanical systems, which are distinguished by the condition that \( \mathcal{A} \) is abelian. In this case, by the Gelfand isomorphism, \( \mathcal{A} \) is the algebra of continuous functions on a compact space \( K \), \( \mathcal{S} \) is a set of probability measures on \( K \) and the transformations \( \alpha_t \) are implemented by transformations \( \tau_t \) of \( K \). Here \( K \) is the ‘phase space’ of the model.

The Finite System Model [VN]. For this, \( \mathcal{A} \) is the \( W^\ast \)-algebra of bounded operators in a Hilbert space \( \mathcal{H} \) and \( \mathcal{S} \) is the set of normal, i.e. ultraweakly continuous, states, \( \rho \), on \( \mathcal{A} \): these correspond to density matrices, denoted by the same symbol, according to the formula \( \rho(A) \equiv \text{Tr}(\rho A) \). Thus \( \mathcal{S} \) is a convex set and its extremal elements, representing the pure states, are those whose density matrices are one-dimensional projectors. In the case where \( \Sigma \) is conservative, its dynamical automorphisms, \( \alpha_t \), are unitary transformations \( A \rightarrow \exp(iHt)A \exp(-iHt) \) of \( \mathcal{A} \), where \( H \) is the Hamiltonian operator of \( \Sigma \) in units for which \( \hbar = 1 \). In the case where the system is dissipative and its dynamical semigroup \( \alpha \) is strongly continuous, its generator \( L \) takes the following form [Li].

\[
LA = i[H, A]_\pm + \sum_r (V^*_r AV_r - \frac{1}{2}[V^*_r V_r, A]_\pm),
\]

where \( H (= H^\ast) \), \( V_r \) and \( \sum_r V^*_r V_r \) belong to \( \mathcal{A} \) and \([, , ]_\mp \) denote commutator and anti-commutator, respectively.

The Infinite System Model [Em, Th, Se1]. This represents a system, \( \Sigma \), of particles that occupies an infinitely extended space \( X \), which we take to be either a Euclidean continuum, \( \mathbb{R}^d \), or a lattice, \( \mathbb{Z}^d \). We denote by \( \mathcal{L} \) the set of all bounded open regions of \( X \), and, for each \( \Lambda \) in \( \mathcal{L} \), we construct a \( W^\ast \)-algebra, \( \mathcal{A}_\Lambda \), of observables that is just that of a system, \( \Sigma_\Lambda \), of particles of the given species confined to \( \Lambda \). These local algebras are constructed so as to satisfy the natural demands that \( \mathcal{A}_\Lambda \) is isotonic with respect to \( \Lambda \) and that \( \mathcal{A}_\Lambda \) and \( \mathcal{A}_{\Lambda'} \) intercommute if \( \Lambda \) and \( \Lambda' \) are disjoint. It follows from the isotony property that \( \mathcal{A}_\mathcal{L} := \bigcup_{\Lambda \in \mathcal{L}} \mathcal{A}_\Lambda \), is well-defined normed \( * \)-algebra. We designate its norm completion, \( \mathcal{A} \), to be the \( C^\ast \)-algebra of the bounded observables of \( \Sigma \). We assume that this algebra is equipped with a representation, \( \gamma \), of the space translation group \( X \) in its automorphisms, which satisfies the covariance condition that \( \gamma(x)(\mathcal{A}_\Lambda) \equiv \mathcal{A}_{\Lambda+X} \).

We assume that the state space, \( \mathcal{S} \), is a convex set of positive, normalised, linear functionals on \( \mathcal{A} \), whose restrictions to the local algebras \( \mathcal{A}_\Lambda \) are normal: the local normality condition serves to exclude the possibility of finding an infinity of particles in a bounded spatial region [DDR].
The dynamics of $\Sigma$ is formulated as a natural infinite volume limit of that of the finite system $\Sigma_\Lambda$. In the conservative case, this latter dynamics is governed by the form of the Hamiltonian operator, $H_\Lambda$, affiliated* to $A_\Lambda$. Thus, if $A$ is an element of $A_\mathcal{L}$ and therefore of $A_\Lambda$ for $\Lambda$ sufficiently large, $\exp(iH_\Lambda t)A\exp(-iH_\Lambda t)$ is its evolve at time $t$ with respect to the dynamics of $\Sigma_\Lambda$. In the simplest cases, such as that of lattice systems with short range interactions [St, Ro], this converges in norm to a definite limit as $\Lambda$ increases to $\mathbf{X}$ over a sequence of suitably regular regions and thus yields a definition of the dynamical automorphisms $\alpha$ by the formula
\[
\alpha_t A = \text{norm}: \lim_{\Lambda \uparrow \mathbf{X}} \exp(iH_\Lambda t)A\exp(-iH_\Lambda t) \quad \forall A \in A_\mathcal{L}, \; t \in \mathbf{R}. \tag{2.2}
\]
More generally, when the convergence condition for this formula is not fulfilled, $S$ has to be formulated so as to comprise just those states that support a limit dynamics represented by a weaker form of Eq. (2.2) [Se1, 2]. In the case where $\Sigma$ is an open dissipative system, its dynamical semigroup is similarly defined as a limit of that of the corresponding finite system $\Sigma_\Lambda$. Thus the model of $\Sigma$, is represented by the quadruple $(A, S, \gamma, \alpha)$. The subset $S_X$ of $S$ comprising the space translationally invariant states is manifestly convex and we denote by $\mathcal{E}(S_X)$ the set of its extremal elements. These are termed the spatially ergodic states.

Affiliated Quantum Fields. Identifying the algebra $A$ with any faithful representation thereof, a quantum field, $\xi(x)$, of the model is defined to be a distribution valued operator that is covariant with respect to space translations and whose integral against a test function $f(x)$ with compact support is affiliated to the local algebras $A_\Lambda$ for which $\Lambda \supset \text{supp}(f)$.

Explicit Constructions. The local algebras $A_\Lambda$, the space translation group $\gamma$ and the local Hamiltonians $H_\Lambda$, on which the model of $\Sigma$ is based, are constructed as follows (cf. [St, Ro, HHW] or the general treatments [Em, Th, Se1]).

In the case where $\Sigma$ is a system of particles, e.g. Pauli spins, on a lattice $X = \mathbf{Z}^d$, we assume that the algebra of observables, $A_0$, of each particle is that of the operators in a finite dimensional Hilbert space, $\mathcal{H}_0$. We take the local algebra $A_\Lambda$, for $\Lambda \in \mathcal{L}$, to be the tensor product $\otimes_{x \in \Lambda} A_x$ of copies $A_x$ of $A_0$ attached to the respective sites $x$ in $\Lambda$; and, for $\Lambda \subseteq \Lambda'$, we identify $A (\in A_\Lambda)$ with $A \otimes I_{\Lambda \setminus \Lambda'} (\in A_{\Lambda'})$. Under this identification, the algebras $\{A_\Lambda | \Lambda \in \mathcal{L}\}$ satisfy the conditions of isotony and local commutativity and thus permit the above definitions of $A_\mathcal{L}$ and $A$. Further, denoting by $a_x$ the copy in $A_x$ of the element $a_0$ of $A_0$, we define the space translational automorphism group $\gamma$ by the formula $\gamma(x)a_{x'} = a_{x + x'}$. Thus, the local algebras $\{A_\Lambda\}$ transform covariantly under this group. The local Hamiltonian $H_\Lambda$ is the element of $A_\Lambda$ representing the interaction energy involving only the particles in $\Lambda$.

In the case where $\Sigma$ is a system of particles of one species in the Euclidean continuum $X = \mathbf{R}^d$, we formulate its observables in second quantisation, as expressed in terms of a

* A possibly unbounded operator $Q$ in the representation space of a $W^*$-algebra $\mathcal{B}$ is said to be affiliated to $\mathcal{B}$ if it commutes with $\mathcal{B}'$, the commutant of $\mathcal{B}$.
quantised scalar or spinor field $\psi$, according to whether the particles are bosons or fermions. In either case, $\psi$ is a distribution valued operator in Fock space $\mathcal{H}_F$. Its algebraic properties are governed by the canonical commutation or anticommutation relations according to whether $\Sigma$ is composed of bosons or fermions. We define the local Hilbert space $\mathcal{H}_\Lambda$ to be the subspace of $\mathcal{H}_F$ generated by application to the Fock vacuum of the polynomials in the smeared fields obtained by integrating $\psi^\star$ against $\mathcal{D}(\Lambda)$-class test functions and we define the local algebra $\mathcal{A}_\Lambda$ to be that of the bounded operators in $\mathcal{H}_\Lambda$. The space translational automorphism group $\gamma$ is defined by the canonical formula $\gamma(x)\psi(x')\equiv\psi(x + x')$. The local Hamiltonian $H_\Lambda$ is just that of $\Sigma_\Lambda$ and we assume that it, and consequently also the automorphisms $\alpha_t$, is invariant under the gauge automorphisms $\psi(x)\to\psi(x)\exp(ic)$, with $c$ real and constant.

**Equilibrium States of Conservative Systems.** The equilibrium states of a conservative system $\Sigma$ at inverse temperature $\beta$ are characterised by the Kubo-Martin-Schwinger (KMS) condition, namely (cf. [HHW, Em, Th, Se1])

$$
\langle \rho; [\alpha_t A]B \rangle = \langle \rho; B\alpha_{t+i\beta}A \rangle \quad A, B \in \mathcal{A}, \quad t \in \mathbb{R}.
$$

(2.3)

This represents various conditions of dynamical and thermodynamical stability [Se1] and, in the case of an infinite system, it automatically ensures that $\rho$ is locally normal [TW]. In general, it follows from Eq. (2.3) that its equilibrium (KMS) states at the inverse temperature $\beta$ comprise a convex set, which we denote by $\mathcal{S}_\beta$. In the case where $\Sigma$ is a finite system, $\mathcal{S}_\beta$ consists of just the canonical state with density matrix $\exp(-\beta H)/\text{Tr(Ident)}$. By contrast, for an infinite system, $\mathcal{S}_\beta$ is a Choquet simplex, which may contain more than one element and whose decomposition into extremals is just the central one [Ru, EKV]. Thus the set $\mathcal{E}(\mathcal{S}_\beta)$, of its extremals consists of primary states and may naturally be interpreted as comprising the pure equilibrium phases [EKV] of the system. Moreover, as they are primary, they enjoy the clustering property that [Ru]

$$
\lim_{|x| \to \infty} [\langle \rho; A\gamma(x)B \rangle - \langle \rho; A \rangle \langle \rho; \gamma(x)B \rangle] = 0 \quad A, B \in \mathcal{A}.
$$

(2.4)

**Open Dissipative Systems.** The situation is different for these systems since they carry no natural counterpart of the KMS states. In particular, the model $(\mathcal{A}, S, \gamma, \alpha)$ does not necessarily have any stationary, dynamically stable primary states, which could be the counterparts of the pure phase equilibrium states of conservative systems. For example, as we shall discuss in Sec. 5, the stable primary states of a laser model, for a certain range of values of its parameters, are period functions of time [HL1, AS].

**3. Entropy, Algorithmic Complexity and Disorder.**

**Entropy and Disorder.** The entropy, $S(\rho)$, of a state $\rho$ is given by Von Neumann’s formula [VN], which, in units for which Boltzmann’s constant is $\log_2(e)$, takes the following form.

$$
S(\rho) = -\text{Tr}(\rho\log_2(\rho)).
$$

(3.1)
In order to expose the probabilistic character of this formula, we note that $\rho$ is a convex combination of mutually orthogonal one-dimensional projectors, $P_k$, of its eigenvectors, i.e.

$$\rho = \sum_k w_j P_k.$$  \hspace{1cm} (3.2)

Thus $w = \{w_k\}$ is a probability measure on the pure states $\{P_k\}$ and Eq. (3.1) is equivalent to the formula

$$S(\rho) = -\sum_k w_k \log_2(w_k),$$  \hspace{1cm} (3.3)

which is just Shannon’s formula [SW] for the entropy of the probability measure $w$. Indeed, in the case where $\Sigma$ is a classical system and $\rho$ is a probability measure on a discrete space $K$, the entropy $S(\rho)$ is given by Eq. (3.3), with $w_k$ the probability attached to the pure state represented by the point $k$ of $K$.

The formula (3.3), and hence also (3.1), has the natural interpretation [SW, Kh, Sz] that $-S(\rho)$ represents the information carried by the state $\rho$; or, equivalently, that the value of $S(\rho)$ is a measure of the degree of disorder of that state. To be precise, it is a strictly probabilistic measure of that disorder since, by Eq. (3.3), $S(\rho)$ depends exclusively on the probability measure $w$ and not at all on the structures of the pure states $P_j$.

Turning now to the case where $\Sigma$ is an infinite system, as formulated in Section 2, we denote by $\rho_\Lambda$ the restriction of a state $\rho$ to the local algebra $\mathcal{A}_\Lambda$. The entropy density induced by $\rho$ in the region $\Lambda$ is therefore $S(\rho_\Lambda)/|\Lambda|$, where the numerator is the Von Neumann entropy of $\rho_\Lambda$ and the denominator is the volume of $\Lambda$. It then follows from the strong subadditivity of entropy [LR] that, for any translationally invariant state $\rho$, this local entropy density converges to a limit $s(\rho)$ as $\Lambda$ increases to $X$ over a set of suitably regular regions, i.e.

$$\lim_{\Lambda \uparrow X} \frac{S(\rho_\Lambda)}{|\Lambda|} = s(\rho) \quad \forall \rho \in \mathcal{S}_X.$$  \hspace{1cm} (3.4)

Evidently, it follows from the discussion following Eq. (3.3) that $s(\rho)$ is a strictly probabilistic measure of the disorder of the state $\rho$.

**Algorithmic Complexity and Disorder.** A complementary, intrinsic characterisation of disorder, as applied to pure states, has been provided by Kolmogorov [Ko] in the classical regime, and quantum versions of this have subsequently been proposed by Berthiaume et al [BVL], Gacs [Ga] and Vitanyi [Vi]. This is based on the concept of *algorithmic complexity*, which was introduced by Kolmogorov [Ko] in the context of classical communication theory, in the following form. For any string, $k_N$, of $N$ symbols, drawn from the binary set $\{0, 1\}$, the algorithmic complexity $C(k_N)$ is defined to be the length of the shortest programme required for the precise specification of that string by a universal Turing machine. Such a string of 0’s and 1’s corresponds to a pure state of a one-dimensional classical lattice gas, $\Sigma_N$, whose phase space is $K_N := \{0, 1\}^{[1, N]}$.

In order to pursue the properties of this algorithmic complexity, we treat $\Sigma_N$ as a subsystem of the infinitely extended classical lattice gas, $\Sigma$, whose phase space is $K := \{0, 1\}^\mathbb{Z}$. Thus the elements of $K$ are the maps $k : x \rightarrow k_x$ of $\mathbb{Z}$ into $\{0, 1\}$ and the spatially
ergodic states of Σ are defined as in Section 2. The following theorem, due to Brudno [Br] provides a remarkable relationship between algorithmic complexity and entropy in terms of these definitions.

**Theorem 3.1** [Br]. Let ρ be a spatially ergodic measure on K and, for k∈K, let k_N be the restriction of k to [1,N]. Then, for ρ-almost all k∈K,

$$\lim_{N \to \infty} N^{-1} C_N(k) = s(\rho). \quad (3.5)$$

**Comment.** This theorem signifies that, for ρ-almost all k in K, the algorithmic complexity density $C(k_N)/N$ induced by k on the segment $[1,N]$ of Z converges to the entropy density $s(\rho)$ as $N \to \infty$. Hence, as algorithmic complexity is an intrinsic measure of disorder, the theorem vindicates the standard representation of the disorder of a macroscopic system by its entropy, at least in the case of one-dimensional lattice gases. From the physical standpoint, this does not conflict with the fact that the entropy of a pure state is zero, for the following reason. The specification of a pure state of a system with N degrees of freedom would require the evaluation of N variables. In the case of a macroscopic system, for which N is extremely large (e.g. even in the case of a one-dimensional one, it is typically of the order of $10^8$), such a specification is out of the question. Indeed, for such a system, the only accessible information about its state is limited to the determination of a ‘few’ macroscopic variables. The state inferred therefrom is then a highly mixed one.

**Quantum Systems.** The above picture of entropy, complexity and disorder has been extended to quantum systems in the following way. Firstly, Berthiaume et al [BVL] have formulated the algorithmic complexity of a pure state, $p$, of a string of N qubits as a natural quantum analogue of Kolmogorov’s classical picture (namely the length of the shortest programme required for the determination of $p$ by a universal quantum Turing machine) and Benatti et al [BKMSS] have established a quantum version of Brudno’s theorem for this complexity. These works were formulated within the framework of quantum informatics, but they may easily be translated into statistical mechanical terms by noting that a string of qubits corresponds to a system of Pauli spins on a one-dimensional lattice. The infinitely extended version, Σ, of such a system is thus a particular case of the quantum lattice model of Sec. 2 for which $d = 1$ and the single particle Hilbert space $H_0$ is two-dimensional. For any natural number $N$, we denote by $\Sigma_N$ the subsystem of $\Sigma$ comprising the spins at the sites 1,2, ,N and by $\mathcal{A}_N$ its algebra of observables. For any state $\rho$ of $\Sigma$, we denote by $\rho_N$ the state of $\Sigma_N$ given by its restriction to $\mathcal{A}_N$ and, following [BKMSS], we term a sequence of projectors $\{P_N \in \mathcal{A}_N\}$ $\rho$-typical if $\lim_{N \to \infty} \rho(P_N) = 1$. Further, if $P_N$ ($\in \mathcal{A}_N$) is a one-dimensional projector we denote by $C(p_N)$ the algorithmic complexity of this state of $\Sigma_N$ that it represents, as formulated by [BVL]. The quantum version of Brudno’s theorem then takes the following form [BKMSS].

**Theorem 3.2.** Let $\rho$ be a spatially ergodic state of the infinite chain, $\Sigma$, of Pauli spins. Then there exists a sequence of $\rho$-typical projectors $\{P_N \in \mathcal{A}_N\}$ such that, for any $\epsilon > 0$, every one-dimensional projector $p_N < P_N$ satisfies the following inequality for $N$
sufficiently large.

\[ N^{-1}C(p_N) \in (s(\rho) - \epsilon, s(\rho) + \epsilon). \quad (3.6) \]

**Comment.** This theorem signifies that the comment following Theorem 3.1 carries through to its natural quantum analogue. Further, as pointed out in [BKMSS], the above Theorem 3.2 is extendible to any chain of atoms, the observables of each of which are irreducibly represented in a finite dimensional Hilbert space \( \mathcal{H}_0 \). Moreover, we can extend that theorem to any \( d \)-dimensional lattice system, \( \Sigma \), as formulated in Section 2, in the following way. We replace the string \([1, N]\) of the one-dimensional lattice by the block \( \Lambda_N \coloneqq [1, N]^d \) of \( \mathbb{Z}^d \) in the definitions of \( \Sigma_N \), \( \mathcal{H}_N \) and \( \mathcal{A}_N \), so that \( \Sigma_N \) is now the system of atoms occupying the block \( \Lambda_N \). The algorithmic complexity, \( C(p_N) \) of a pure state \( p_N \) of this system is then the length of the shortest programme required to specify this state by a universal quantum Turing machine and its complexity density is \( C(p_N)/N^d \). Further, if \( \rho \) is a state of the infinite system \( \Sigma \), a sequence of projectors \( \{P_N \in \mathcal{A}_N\} \) is again termed \( \rho \)-typical if \( \lim_{N \to \infty} \rho(P_N) = 1 \). With these definitions, the treatment of [BKMSS] can be carried through to yield the following \( d \)-dimensional generalisation of Theorem 3.2.

**Theorem 3.3.** Let \( \rho \) be a spatially ergodic state of the infinite system, \( \Sigma \), of atoms on the lattice \( \mathbb{Z}^d \). Then there exists a \( \rho \)-typical sequence of projectors \( \{P_N \in \mathcal{A}_N\} \) such that, for any \( \epsilon > 0 \), the algorithmic complexity, \( C(p_N) \), of any one-dimensional projector \( p_N < P_N \) satisfies the following inequality for \( N \) sufficiently large.

\[ N^{-d}C(p_N) \in (s(\rho) - \epsilon, s(\rho) + \epsilon). \quad (3.7) \]

**Comment.** This theorem signifies that the comments following Theorems 3.1 and 3.2 extend to quantum systems on lattices of arbitrary finite dimensionality and thus vindicates the standard picture wherein the disorder of a state of a quantum lattice system is given by its Von Neumann entropy.

4. Symmetry and Order.

**Symmetry Groups and G-fields.** A *symmetry group* of the model \( \Sigma = (\mathcal{A}, \mathcal{S}, \gamma, \alpha) \) is a group, \( G \), that has a faithful representation, \( \theta \), in \( \text{Aut}(\mathcal{A}) \). It is a *dynamical symmetry group* if \( \theta(G) \) commutes with the time-translations \( \alpha_t \). For any symmetry group \( G \), an \( n \)-component quantum field \( \xi = (\xi_1, \ldots, \xi_n) \), affiliated to \( \mathcal{A} \), is termed a *G-field* if the action of \( \theta(G) \) on \( \xi \) takes one of the following forms.

(a) In the case where this symmetry group is spatial,

\[ [\theta(g)\xi](x) = \xi(T_g x) \quad \forall \ g \in G, \quad (4.1a) \]

where \( T_g \) is a transformation of \( X \).
(b) In the case where $G$ is an internal symmetry group,

$$[\theta(g)\xi]_j(x) = \sum_{k=1}^n V_{g,jk} \xi_k(x) \quad \forall g \in G,$$

(4.1b)

where $V_g = [V_{g,jk}]$ is a unitary transformation of $\mathbb{R}^n$ or $\mathbb{C}^n$, according to whether the field $\xi$ is real or complex.

Thus, in either case, the action of $\theta(g)$ on $\xi$ is that of a linear transformation $\phi_g$ of this field alone, without involvement of other observables of the system, i.e.

$$[\theta(g)\xi](x) = [\phi_g\xi](x) := \xi_g(x).$$

(4.1)

**Symmetry Breakdown.** An stationary state $\rho$ of $\Sigma$ that is not invariant with respect to a dynamical symmetry group $G$ is said to spontaneously break that symmetry. In this case, the states $\{\rho_g := \rho \circ \theta(g) | g \in G\}$ are also $G$-symmetry breaking stationary states of the system. We term this set of states the $G$-orbit of $\rho$ and denote it by $O_G(\rho)$.

In the case where $\Sigma$ is conservative and $\rho \in \mathcal{E}(S_\beta)$, the set of extremal KMS states at inverse temperature $\beta$, it follows from the KMS condition (2.1) and the definition of $O_G(\rho)$ that $\mathcal{E}(S_\beta) \subset O_G(\rho)$. Further, defining

$$\bar{\xi}(x) := \langle \rho; \xi(x) \rangle \text{ and } \bar{\xi}_g(x) := \langle \rho_g; \xi(x) \rangle,$$

(4.2)

it follows from Eq. (4.1) that

$$\bar{\xi}_g(x) = [\phi_g \bar{\xi}](x),$$

(4.3)

where the action of $\phi_g$ on the classical field $\bar{\xi}$ is the same as that on the quantum field $\xi$. We assume that this action is non-trivial* unless $g$ is the identity element of $G$ and consequently that $\bar{\xi}_g \neq \bar{\xi}_{g'}$ if $g \neq g'$. Hence, by Eqs. (4.1)-(4.3), the $G$-field $\xi$ serves to separate the states of the orbit $O_G(\rho)$. We note here that, since these states are primary, it follows from Eqs. (2.4) and (4.1)-(4.3) that

$$\lim_{a \to \infty} [\langle \rho; \xi_g^\ast(x) . \xi_g(x+a) \rangle - \bar{\xi}_g^\ast(x) . \bar{\xi}_g(x+a)] = 0,$$

(4.4)

where the dot denotes the scalar product in $\mathbb{R}^n$ or $\mathbb{C}^n$.

**Symmetry Breakdown and Order.** At a qualitative level, we conceive a state of a complex system to be ordered if its components cooperate in such a way as to produce a macroscopic field or signal. Thus, a prototype example of ordering in this sense is that of the alignment of the spins (mini-magnets!) of a ferromagnetic material so as to produce a resultant polarisation. Hence, assuming that the dynamics of the material is rotationally

* In fact, this condition is effectively fulfilled if the subgroup, $H$, of $G$ under which the field $\bar{\xi}$ is invariant is a normal one. For in that case we may replace the group $G$ of the present treatment by the factor group $G/H$, which then satisfies our demands.
invariant, this magnetic ordering amounts to a breakdown of its rotational symmetry, as manifested by the direction of the polarisation field.

The generalisation of this concept of order through symmetry breakdown is straightforward. Thus, an equilibrium state $\rho$ of $\Sigma$ is said to be ordered if it is not invariant with respect to a dynamical symmetry group $G$, and its ordering is then represented by the $g$-dependence of the classical field $\xi_g$. Among the numerous proven examples of such order are the following.

**Example 1.** The two-dimensional Ising model, whose pure equilibrium phases at any temperature below its critical point are polarised [MM]. For this model, $G$ is the binary group $(e, r)$, where $e$ is the identity element and $\theta(r)$ is the spin reversal automorphism. The $G$-field $\xi(x)$ is the Ising spin $s_x$ at the site $x$.

**Example 2.** The antiferromagnetic phase of the Heisenberg model [DLS]. For this, $G$ is the three dimensional rotation group and $\theta(G)$ represents its action on the Pauli spins constituting the model. The $G$ field $\xi(x)$ is the Pauli spin vector $\sigma(x)$ at the site $x$ and so $\xi_g(x)$ is a spatially periodic vector filed whose direction is determined by $g$.

**Example 3.** A crystalline phase of the generic model of a continuous system $\Sigma$ that occupies the space $X = \mathbb{R}^d$. For this system, $G$ is the factor group $X/Y$, where $X$ is the additive space translation group and $Y$ is the normal subgroup of $X$ corresponding to the crystal structure [EKV]. The $G$-field $\xi(x)$ may be chosen to be the particle density $\psi^*(x)\psi(x)$.

**Example 4.** This is the order corresponding to a generalised version of Bose-Einstein (BE) condensation, which was first proposed by O. Penrose and Onsager [PO] as a characterisation of the superfluidity of HeII and subsequently extended by Yang [Ya] to superconductors. In fact, this order corresponds to the breakdown of gauge symmetry*, the relevant dynamical symmetry group, $G$, being that of the gauge transformations $\psi(x) \to \psi(x)(\exp(ic))$, where $c$ runs through the reals. In the case of bosons, the $G$-field $\xi(x)$ is just $\psi(x)$; in the case of fermions it is the pair field $\psi^\uparrow(x)\psi^\downarrow(x)$, the two factors being the components of $\psi(x)$ with spin parallel and antiparallel to some fixed axis.

**Example 5.** This is the pumped phonon model, which was proposed by H. Froehlich [Fr] in a biological context and put onto a rigorous footing by Duffield [Du]. It is an open dissipative system, consisting of $N$ phonon modes that are coupled to energy pumps and sinks and that exchange quanta with one another in conformity with the principle of detailed balance at the prevailing environmental temperature. Remarkably, the model is driven by these forces into a state wherein a macroscopic number, of order $N$, of its quanta condense into the mode of lowest frequency when the pumping strength exceeds a critical value. Thus, in the limit $N \to \infty$, it exhibits a BE condensation into a nonequilibrium steady state. This amounts to a gauge symmetry breakdown, far from thermal equilibrium. The source of this phenomenon stems is the competition between the pumping and the discharge of quanta into the sinks, which fixes the total number of quanta and thereby renders the state of the

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* See [Se1, Ch.9] for a detailed discussion of this symmetry breakdown and its connections with both superfluidity and the so-called ‘off-diagonal long range order’ of O. Penrose and Onsager [OP] and Yang [Ya].
model similar to an equilibrium state of an ideal Bose gas with a fixed number of particles. By contrast, a phonon system at equilibrium with a thermostat at fixed temperature has a Planck distribution of its quanta and thus does not experience a BE condensation.

5. Coherence.

Coherence is an extreme version of order. Specifically, a state \( \rho \) of the system \( \Sigma \) is said to be coherent with respect to a \( G \)-field, \( \xi(x) \), if it satisfies Glauber’s [Gl] condition that

\[
\langle \rho; \xi^\#(x_1) \cdots \xi^\#(x_n) \rangle = \Pi_{j=1}^{n} \langle \rho : \xi^\#(x_j) \rangle,
\]

where each \( \xi^\# \) is either \( \xi \) or \( \xi^* \) and \( \langle \rho; \xi(x) \rangle \) is a non-trivial function of \( x \), which is simple in that it involves just a ‘few’ parameters: the simplicity condition serves to exclude cases where the classical field represented by this function varies chaotically with \( x \).

The natural dynamical version of this coherence condition is that obtained by the replacement, in Eq. (5.1), of \( \xi(x) \) by its evolute \( \xi_t(x) \) (\( = \alpha_t \xi(x) \)). Thus, the resultant dynamical coherence condition is that

\[
\langle \rho; \xi_{t_1}^\#(x_1) \cdots \xi_{t_n}^\#(x_n) \rangle = \Pi_{j=1}^{n} \langle \rho : \xi_{t_j}^\#(x_j) \rangle,
\]

where \( \langle \rho; \xi_t(x) \rangle \) is a non-trivial, simple function of both \( x \) and \( t \).

It follows from these specifications that coherence corresponds to the behaviour of the quantum field \( \xi(x) \) or \( \xi_t(x) \), in the state \( \rho \), as a classical, dispersion-free field \( \overline{\xi}(x) := \langle \rho; \xi(x) \rangle \) or \( \overline{\xi}_t(x) := \langle \rho; \xi_t(x) \rangle \).

The following examples of coherence have been established in different variants of the Dicke model [Di] of matter interacting with a single radiative mode.

**Example 1.** Hepp and Lieb [HL2] have proved that the Dicke model has a low temperature equilibrium phase characterised by super-radiance, i.e. by coherence of the pure equilibrium phase with respect to the radiation field, with breakdown of the gauge symmetry of that field. They also treated [HL1] the open version of the Dicke model in which each atom of the matter was coupled to an energy pump and a sink and the radiation mode was coupled to a sink. This model undergoes a phase transition, far from thermal equilibrium, at a critical value, \( p_c \), of the pumping strength, \( p \). Specifically, for \( p < p_c \), its stable state is stationary and its radiation incoherent; while for \( p > p_c \), the stable state is simply periodic in time and its radiative mode coherent, again with breakdown of gauge symmetry. Thus the transition is from normal light to laser light.

**Example 2.** Alli and Sewell [AS] extended the open version of the Dicke model to one with many modes, each with its own sink, and obtained the following picture of its phase structure in terms two critical values, \( p_1 \) and \( p_2 \) (\( > p_1 \)) of the pumping strength. For \( p < p_1 \), there is a unique stable stationary state of the model, and the radiation is normal, i.e. incoherent. For \( p_1 < p < p_2 \), the stable state varies periodically with time and the radiation is coherent and monochromatic, again with breakdown of gauge symmetry.
For $p > p_2$, the radiation is chaotic according to either the mechanism of Ruelle-Takens [RT] (strange attractor) or that of Landau [LL] (multimode turbulence), depending on the parameters of the model.

**Example 3.** Pule, Verbeure and Zagrebnov [PVZ] constructed a model of a system of interacting two level bosonic atoms that is coupled to a radiative mode. This coupling was shown to lead to a rich equilibrium phase structure. In particular, for sufficiently large values of the chemical potential, the radiation is coherent and the matter exhibits a two-fold BE condensation, one for each of its atomic levels. This supports the experimental observation that the action of a laser field on a bosonic condensate of atoms with internal structure leads to enhancements of both the laser field and the BE condensation [KI].

6. Concluding Remarks

We may summarise the contents of this article as follows.

Disorder corresponds to randomness, of which entropy and algorithmic complexity provide probabilistic and intrinsic measures, respectively. Remarkably, Brudno’s theorem and its quantum analogues have established the essential equivalence of these measures in the thermodynamic limit (cf. Theorems 3.1-3.3).

Order, on the other hand, constitutes organisation manifested by a macroscopic field or signal, and this can prevail amidst high disorder. Thus, in the present physical context, the concepts of order and disorder are certainly not antitheses of one another. The particular types of order and coherence described here stem from symmetry breakdown and BE condensation, and certainly do not cover all the kinds of organisation that arise in complex systems. For example, it is still true that, as pointed out by Schroedinger many years ago [Sc], the presently available pictures of order do not cover biological organisation. Thus, a major challenge of statistical physics is to characterise other kinds of ordering that exist in nature.

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