Wall divisors on irreducible symplectic orbifolds of Nikulin-type

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Abstract
We determine the wall divisors on irreducible symplectic orbifolds which are deformation equivalent to a special type of examples, called Nikulin orbifolds. The Nikulin orbifolds are obtained as partial resolutions in codimension 2 of a quotient by a symplectic involution of a Hilbert scheme of 2 points on a K3 surface. This builds on the previous article [MR20] in which the theory of wall divisors was generalized to orbifold singularities.

1 Introduction

1.1 Motivations and main results
During the last years, many efforts have been made to extend the theory of smooth compact varieties with trivial first Chern class to a framework of varieties admitting some singularities. Notably, let us cite, the generalization of the Bogomolov decomposition theorem [BGL22]. One of the motivations for such generalizations is given by the minimal model program in which certain singular varieties appear naturally.

More specifically, in the theory of irreducible symplectic varieties, many generalizations can be mentioned. One of the most important concerns the global Torelli theorem which allows to obtain geometrical information on the variety from its period ([BL18], [Men20] and [MR20]).

In this paper, we are considering a specific kind of singularities: quotient singularities. A complex analytic space with only quotient singularities is called an orbifold. An orbifold $X$ is called irreducible holomorphically symplectic if $X \setminus \text{Sing} X$ is simply connected, admits a unique (up to a scalar multiple), non-degenerate holomorphic 2-form and $\text{Codim} \text{Sing} X \geq 4$ (Definition 2.1). The framework of irreducible symplectic orbifolds appears to be very favorable. In particular, general results about the Kähler cone have been generalized for the first time in this context (see [MR20]). This is particularly important, since knowledge on the Kähler cone is needed to be able to apply the global Torelli theorem (see Theorem 2.8) effectively. The key tool used to study the Kähler cone of irreducible symplectic orbifolds are wall divisors (originally introduced for the smooth case in [Mon13]).

Definition 1.1 ([MR20] Definition 4.5]). Let $X$ be an irreducible symplectic orbifold and let $D \in \text{Pic}(X)$. Then $D$ is called a wall divisor if $q(D) < 0$ and $g(D^+) \cap \mathcal{B}_{K,X} = \emptyset$, for all $g \in \text{Mon}_{2}^2 \text{Hdg}(X)$, where, $q$ denotes the famous Beauville–Bogomolov form on $H^2(X, \mathbb{Z})$.

In particular, we recall that the Kähler classes can be characterized by their intersections with the wall divisors (see Corollary 2.3). The definitions of the birational Kähler cone $\mathcal{B}_{K,X}$ and the Hodge monodromy group $\text{Mon}_{2}^2 \text{Hdg}(X)$ are recalled in Section 2.3 and Definition 2.6 respectively.

A very practical feature of wall divisors is their deformation invariance. More precisely, let $\Lambda$ be a lattice of signature $(3, \text{rk } \Lambda - 3)$ and $(X, \varphi)$ a marked irreducible symplectic orbifold with $\varphi : H^2(X, \mathbb{Z}) \simeq \Lambda$. Then there exists a set $\mathcal{W}_X \subset \Lambda$ such that for all $(Y, \psi)$ deformation equivalent to $X$, the set $\psi^{-1}(\mathcal{W}_X) \cap H^{1,1}(Y, \mathbb{Z})$ is the set of wall divisors of $Y$ (see Theorem 2.14). We call the set $\mathcal{W}_X$ the set of wall divisors of the deformation class of $X$.

In this paper, we are going to provide the first description of the wall divisors of a deformation class of singular irreducible symplectic varieties. The most "popular" singular irreducible symplectic variety, in the literature (see [Fuj83] Section 13, table 1, [2], [MT07], [Men14], [Men15].

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We can prove Proposition 1.5 (i). Our main tool to determine wall divisors is Proposition M schemes of two points on K3 surfaces. In Section 4, we determined the wall divisors of an orbifold orbifold. In Section 3, we provide some reminders about the orbifold symplectic orbifolds, especially from [MR20] where the theory of the Kähler cone has been developed. We obtain M” as the blow-up of X/ι in the image of Σ (see Example [2.2]; we denote by Σ′ the exceptional divisor. The orbifolds deformation equivalent to this variety will be called orbifolds of Nikulin-type. We also recall that the Beauville–Bogomolov lattice of the orbifolds of Nikulin-type is U(2)^3 ⊕ E_8(-1) ⊕ (-2)^2 (see Theorem 3.6).

**Theorem 1.2.** Let \( \Lambda := U(2)^3 \oplus E_8(-1) \oplus (-2)^2 \).

The set \( \mathcal{W} = \{ D \mid D \in \Lambda, \ \text{div}(D) = 2, \ \text{div}(D') = 2, \ \text{and} \ \text{div}(D'') = 2 \} \), where \( D_{U(2)^3} \) is the projection of D to the U(2)^3-part of the lattice.

The divisibility \( \text{div} \) is defined in Section 1.3 below.

**Remark 1.3.** Note that if one chooses an automorphism \( \varphi \) of the lattice \( \Lambda \), the conditions that \( D_{U(2)^3} \) and \( \varphi(D)_{U(2)^3} \) are divisible by 2 are equivalent for elements with \( D^2 = -12 \) and \( \text{div}(D) = 2 \).

Combined with the global Torelli theorem (Theorem 2.8), the previous theorem can be used for studying automorphisms on orbifolds of Nikulin-type. As an example of application, we construct a symplectic involution on orbifolds of Nikulin-type which is not induced by a symplectic involution on a Hilbert scheme of 2 points on a K3 surface (non-standard involution) (see Section 8.2).

**Proposition 1.4.** Let X be an irreducible symplectic orbifold of Nikulin-type such that there exists \( D \in \text{Pic}(X) \) with \( q(D) = -2 \) and \( \text{div}(D) = 2 \). Then, there exists an irreducible symplectic orbifold Y bimeromorphic to X and a symplectic involution \( \iota \) on Y such that:

\[
H^2(Y, \mathbb{Z})^c \simeq U(2)^3 \oplus E_8(-1) \oplus (-2) \text{ and } H^2(Y, \mathbb{Z})^c \simeq (-2).
\]

The proof of this Proposition is given in Section 8.2.

For the proof of Theorem 1.2 we need to show that the following two operators are monodromy operators. The reflections \( R_D \) on the second cohomology group are defined in Section 1.3 below.

**Proposition 1.5** (Compare Corollaries 4.6, 6.18, and 6.19).

(i) The reflection \( R_{\Sigma'} \) is a monodromy operator of \( M' \).

(ii) More generally, let X be an orbifold of Nikulin-type and \( \alpha \in H^2(X, \mathbb{Z}) \) which verifies one of the two numerical conditions:

\[
\begin{align*}
q(\alpha) &= -2 \text{ and } \text{div}(\alpha) = 2, \text{ or } \\
qu(\alpha) &= -4 \text{ and } \text{div}(\alpha) = 2.
\end{align*}
\]

Then \( R_\alpha \) is a monodromy operator.

**Remark 1.6.** Note that Proposition 1.5 (i) can also be obtained from the recent result of Lehn–Mongardi–Pacienza [LMP22] Theorem 3.10.

### 1.2 Organization of the paper and sketch of the proof

The paper is organized as follows. In Section 2, we provide some reminders related to irreducible symplectic orbifolds, especially from [MR20] where the theory of the Kähler cone has been developed. In Section 3, we provide some reminders about the orbifold \( M' \) especially from [Men15]; moreover, we investigate the monodromy operators of \( M' \) inherited from the ones on the Hilbert schemes of two points on K3 surfaces. In Section 4, we determined the wall divisors of an orbifold \( M' \) obtained from a very general K3 surfaces endowed with a symplectic involution (\( S, i \)). As a corollary, we can prove Proposition 1.5(i). Our main tool to determine wall divisors is Proposition 2.
which says that the dual divisor of an extremal ray of the cone of classes of effective curves (the Mori cone) is a wall divisor. The proof of Theorem 1.2 is then divided into two parts. The first part (Section 5) consists in finding enough examples of extremal rays of Mori cones in several different \(M'\)-orbifolds; the second part (Section 6) consists in using our knowledge on the monodromy group of \(M'\) to show that we have find all possible wall divisors. Finally, Section 8.2 is devoted to the proof of Proposition 1.4.

1.3 Notation and convention

- Let \(\Lambda\) be a lattice of signature \((3, \text{rk } \Lambda - 3)\). Let \(x \in \Lambda\) such that \(x^2 < 0\). We define the reflection \(R_x\) associated to \(x\) by:
  \[
  R_x(\lambda) = \lambda - \frac{2\lambda \cdot x}{x^2} x,
  \]
  for all \(\lambda \in \Lambda\).
- In \(\Lambda\), we define the divisibility of an element \(x \in \Lambda\) as the integer \(a \in \mathbb{N}^*\) such that \(x \cdot \Lambda = a \mathbb{Z}\). We denote by \(\text{div}(x)\) the divisibility of \(x\).
- Let \(X\) be a manifold and \(C \subset X\) a curve. We denote by \([C]_X\) the class in \(X\) of the curve \(C\).

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2 Reminders on irreducible symplectic orbifolds

2.1 Definition

In this paper an orbifold is a complex space with only quotient singularities.

Definition 2.1. An irreducible symplectic orbifold (or hyperkähler orbifold) is a compact Kähler orbifold \(X\) such that:

(i) \(\text{Codim} \text{Sing } X \geq 4\),
(ii) \(H^{2,0}(X) = \mathbb{C}\sigma\) with \(\sigma\) non-degenerated on \(X_{\text{reg}} := X \setminus \text{Sing } X\),
(iii) \(\pi(X_{\text{reg}}) = 0\).

We refer to [Cam05 Section 6], [Men20 Section 3.1], [FM21 Section 3.1] and [MR20 Section 2.1] for discussions about this definition.

Example 2.2 ([Men20 Section 3.2]). Let \(X\) be a hyperkähler manifold deformation equivalent to a Hilbert scheme of 2 points on a K3 surfaces and \(\iota\) a symplectic involution on \(X\). By [Mon12 Theorem 4.1], \(\iota\) has 28 fixed points and a fixed K3 surface \(\Sigma\). We denote by \(M'\) the blow-up of \(X/\iota\) in the image of \(\Sigma\). The orbifold \(M'\) is irreducible symplectic (see [Men20 Proposition 3.8]).

Definition 2.3. An orbifold \(M'\) constructed as before is called a Nikulin orbifold. An irreducible symplectic orbifold deformation equivalent to a Nikulin orbifold is called an orbifold of Nikulin-type.
2.2 Moduli space of marked irreducible symplectic orbifolds

Let \( X \) be an irreducible symplectic orbifold. As explained in [Men20, Section 3.4], \( H^2(X, \mathbb{Z}) \) is endowed with a quadratic form of signature \((3, b_2(X) - 3)\) called the Beauville–Bogomolov form and denoted by \( q_X \) (the bilinear associated form is denoted by \( \langle \cdot, \cdot \rangle_{q_X} \) or \( \langle \cdot, \cdot \rangle \) when there is no ambiguity). Let \( \Lambda \) be a lattice of signature \((3, \text{rk} \Lambda - 3)\). We denote \( \Lambda_X := \Lambda \otimes \mathbb{K} \) for \( \mathbb{K} \) a field. A marking of \( X \) is an isometry \( \varphi : H^2(X, \mathbb{Z}) \to \Lambda \). Let \( \mathcal{M}_\Lambda \) be the set of isomorphism classes of marked irreducible symplectic orbifolds \((X, \varphi)\) with \( \varphi : H^2(X, \mathbb{Z}) \to \Lambda \). As explained in [Men20, Section 3.5], this set can be endowed with a non-separated complex structure such that the period map:

\[
\varphi : \quad \mathcal{M}_\Lambda \longrightarrow \mathcal{D}_\Lambda \\
(X, \varphi) \quad \mapsto \varphi(\sigma_X)
\]

is a local isomorphism with \( \mathcal{D}_\Lambda := \{ \alpha \in \mathbb{P}(\Lambda) \mid \alpha^2 = 0, \alpha \cdot \bar{\alpha} > 0 \} \). The complex manifold \( \mathcal{M}_\Lambda \) is called the moduli space of marked irreducible symplectic orbifolds of Beauville–Bogomolov lattice \( \Lambda \).

Moreover there exists a Hausdorff reduction of \( \mathcal{M}_\Lambda \).

**Proposition 2.4** ([Men20, Corollary 3.25]). There exists a Hausdorff reduction \( \overline{\mathcal{M}}_\Lambda \) of \( \mathcal{M}_\Lambda \) such that the period map \( \varphi \) factorizes through \( \overline{\mathcal{M}}_\Lambda \):

\[
\xymatrix{ \mathcal{M}_\Lambda \ar[r] & \overline{\mathcal{M}}_\Lambda \ar[r] & \mathcal{D}_\Lambda. }
\]

Moreover, two points in \( \mathcal{M}_\Lambda \) map to the same point in \( \overline{\mathcal{M}}_\Lambda \) if and only if they are non-separated in \( \mathcal{M}_\Lambda \).

2.3 Global Torelli theorems

**Theorem 2.5** ([Men20, Theorem 1.1]). Let \( \Lambda \) be a lattice of signature \((3, b - 3)\), with \( b \geq 3 \). Assume that \( \mathcal{M}_\Lambda \neq \emptyset \) and let \( \mathcal{M}^3_\Lambda \) be a connected component of \( \mathcal{M}_\Lambda \). Then the period map:

\[
\varphi : \quad \overline{\mathcal{M}}^3_\Lambda \longrightarrow \mathcal{D}_\Lambda \\
\psi \quad \mapsto \varphi(\sigma_X)
\]

is an isomorphism.

There also exists a Hodge version of this theorem, which we state in the following.

**Definition 2.6.** Let \( X_1 \) and \( X_2 \) be two irreducible symplectic orbifolds. An isometry \( f : H^2(X_1, \mathbb{Z}) \to H^2(X_2, \mathbb{Z}) \) is called a parallel transport operator if there exists a deformation \( s : X \to B \), two points \( b_1, b_2 \in B \), two isomorphisms \( \psi_i : X_i \to X_{b_i} \), \( i = 1, 2 \) and a continuous path \( \gamma : [0, 1] \to B \) with \( \gamma(0) = b_1 \), \( \gamma(1) = b_2 \) and such that the parallel transport in the local system \( R_s \mathbb{Z} \) along \( \gamma \) induces the morphism \( \psi_{2, *} \circ f \circ \psi_{1, *} : H^2(X_{b_1}, \mathbb{Z}) \to H^2(X_{b_2}, \mathbb{Z}) \).

Let \( X \) be an irreducible symplectic orbifold. If \( f : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}) \) is a parallel transport operator from \( X \) to \( X \) itself, \( f \) is called a monodromy operator. If moreover \( f \) sends a holomorphic 2-form to a holomorphic 2-form, \( f \) is called a Hodge monodromy operator. We denote by \( \text{Mon}^2(X) \) the group of monodromy operators and by \( \text{Mon}^2_{\text{Hdg}}(X) \) the group of Hodge monodromy operators.

**Remark 2.7.** If \( (X, \varphi) \) and \( (X', \varphi') \) are in the same connected component \( \mathcal{M}_\Lambda^3 \) of \( \mathcal{M}_\Lambda \), then \( \varphi^{-1} \circ \varphi' \) is a parallel transport operator.

**Theorem 2.8** ([Mir20, Theorem 1.1]). Let \( X \) and \( X' \) be two irreducible symplectic orbifolds.

(i) The orbifolds \( X \) and \( X' \) are bimeromorphic if and only if there exists a parallel transport operator \( f : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z}) \) which is an isometry of integral Hodge structures.

(ii) Let \( f : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z}) \) be a parallel transport operator, which is an isometry of integral Hodge structures. There exists an isomorphism \( \tilde{f} : X \to X' \) such that \( f = \tilde{f}_* \) if and only if \( f \) maps some Kähler class on \( X \) to a Kähler class on \( X' \).
2.4 Twistor space

Let \( \Lambda \) be a lattice of signature \((3, \text{rk} - 3)\). We denote by "\(\cdot\)" its bilinear form. A positive three-space is a subspace \( W \subset \Lambda \oplus \mathbb{R} \) such that \( \gamma_W \) is positive definite. For any positive three-space, we define the associated twistor line \( T_W \subset \mathcal{D}_\Lambda \) by:

\[
T_W := \mathcal{D}_\Lambda \cap \mathbb{P}(W \otimes \mathbb{C}).
\]

A twistor line is called generic if \( W^\perp \cap \Lambda = 0 \). A point of \( \alpha \in \mathcal{D}_\Lambda \) is called very general if \( \alpha^\perp \cap \Lambda = 0 \).

**Theorem 2.9** ([Men20], Theorem 5.4). Let \((X, \varphi)\) be a marked irreducible symplectic orbifold with \( \varphi : H^2(X, \mathbb{Z}) \to \Lambda \). Let \( \alpha \) be a Kähler class on \( X \), and \( W_\alpha := \text{Vect}_\mathbb{Z}(\varphi(\alpha), \varphi(\text{Re} \sigma_X), \varphi(\text{Im} \sigma_X)) \). Then:

(i) There exists a metric \( g \) and three complex structures (see [Men20, Section 5.1] for the definition) \( I, J \) and \( K \) in quaternionic relation on \( X \) such that:

\[
\alpha = [g(\cdot, I \cdot)] \quad \text{and} \quad g(\cdot, J \cdot) + ig(\cdot, K \cdot) \in H^{0,2}(X).
\]

(ii) There exists a deformation of \( X \):

\[
\mathcal{X} \to T(\alpha) \simeq \mathbb{P}^1,
\]

such that the period map \( \mathcal{P} : T(\alpha) \to T_{W_\alpha} \) provides an isomorphism. Moreover, for each \( s = (a, b, c) \in \mathbb{P}^1 \), the associated fiber \( \mathcal{X}_s \) is an orbifold diffeomorphic to \( X \) endowed with the complex structure \( aI + bJ + cK \).

**Remark 2.10.** Note that if the irreducible symplectic orbifold \( X \) of the previous theorem is endowed with a marking then all the fibers of \( \mathcal{X} \to T(\alpha) \) are naturally endowed with a marking. Therefore, the period map \( \mathcal{P} : T(\alpha) \to T_{W_\alpha} \) is well defined.

**Remark 2.11.** Let \( X \) be an irreducible symplectic orbifold endowed with a finite symplectic automorphisms group \( G \) (i.e. \( G \) fixes the holomorphic 2-form of \( X \)). Let \( \alpha \) be a Kähler class of \( X \) fixed by \( G \) and \( \mathcal{X} \to T(\alpha) \) the associated twistor space. Then \( G \) extends to an automorphism group on \( \mathcal{X} \) and restricts on each fiber to a symplectic automorphism group. Indeed, since \( G \) is symplectic \( G \) fixes all the the complex structures \( I, J, K \).

We provide the following lemma which will be used several times in this paper. It is a generalization of [MR20, Lemma 2.17].

**Lemma 2.12.** Let \( \Lambda' \subseteq \Lambda \) be a sublattice of rank \( b' \), which also has signature \((3, b' - 3)\). Consider the inclusion of period domains \( \mathcal{D}_{\Lambda'} \subseteq \mathcal{D}_\Lambda \). Suppose that a very general points of \( (\tilde{X}, \tilde{\varphi}) \in \mathcal{M}_\Lambda \cap \mathcal{P}^{-1} \mathcal{D}_{\Lambda'} \) with \( \tilde{\varphi}(\text{Pic}(\tilde{X})) = \Lambda' \) satisfies that \( K_{\tilde{X}} = C_{\tilde{X}} \). Let \((X, \varphi)\) and \((Y, \psi)\) in \( \mathcal{M}_\Lambda \) be any two marked irreducible symplectic orbifolds which satisfy \( \mathcal{P}(X, \varphi) \in \mathcal{D}_{\Lambda'} \) and \( \mathcal{P}(Y, \psi) \in \mathcal{D}_{\Lambda'} \). If \( \varphi(K_X) \cap \Lambda' \neq \emptyset \) and \( \psi(K_Y) \cap \Lambda' \neq \emptyset \). Then \((X, \varphi)\) and \((Y, \psi)\) can be connected by a sequence of generic twistor spaces whose image under the period domain is contained in \( \mathcal{D}_{\Lambda'} \). That is: there exists a sequence of generic twistor spaces \( f_i : \mathcal{X}_i \to \mathbb{P}^1 \simeq T(\alpha_i) \) with \((x_i, x_{i+1}) \in \mathbb{P}^1 \times \mathbb{P}^1 \), \( i \in \{0, \ldots, k\}, k \in \mathbb{N} \) such that:

- \( f_0^{-1}(x_0) = (X, \varphi), f_i^{-1}(x_{i+1}) = f_{i+1}^{-1}(x_{i+1}) \) and \( f_k^{-1}(x_{k+1}) = (Y, \psi) \), for all \( 0 \leq i \leq k - 1 \).

- \( \mathcal{P}(T(\alpha_i)) \subset \mathcal{D}_{\Lambda'} \) for all \( 0 \leq i \leq k + 1 \).

**Proof.** We split the proof in two steps.

**First case:** We assume that \((X, \varphi)\) and \((Y, \psi)\) are very general in \( \mathcal{M}_\Lambda \cap \mathcal{P}^{-1} \mathcal{D}_{\Lambda'} \) (hence \( C_{\tilde{X}} = K_{\tilde{X}} \) and \( C_{\tilde{Y}} = K_{\tilde{Y}} \)). By [Huy12, Proposition 3.7] the period domain \( \mathcal{D}_{\Lambda'} \) is connected by generic twistor lines. Note that the proof of [Huy12, Proposition 3.7] in fact shows that the twistor lines can be chosen in a such a way that they intersect in very general points of \( \mathcal{D}_{\Lambda'} \). In particular, we can connect \( \mathcal{P}(Y, \psi) \) and \( \mathcal{P}(X, \varphi) \) by such generic twistor lines in \( \mathcal{D}_{\Lambda'} \). Since for a very general element \((\tilde{X}, \tilde{\varphi})\) of \( \mathcal{M}_\Lambda \cap \mathcal{P}^{-1} \mathcal{D}_{\Lambda'} \) we know \( K_{\tilde{X}} = C_{\tilde{X}} \), Theorem 2.9 shows that all these twistor lines can be lifted to twistor spaces. Moreover, by Theorem 2.8 (ii) the period map \( \mathcal{P} \) is injective.
on the set of points \((\tilde{X}, \tilde{\varphi}) \in \mathcal{M}_\Lambda\) such that \(\mathcal{K}_X = \mathcal{C}_X\). Therefore, all these twistor spaces intersect and connect \((X, \varphi)\) to \((Y, \psi)\).

**Second case:** If \((X, \varphi)\) is not very general, we consider a very general Kähler class \(\alpha \in \mathcal{K}_X \cap \Lambda_0^+ \neq \emptyset\). Then the associated twistor space \(\mathcal{K} \to T(\alpha)\) have a fiber which is a very general marked irreducible symplectic orbifold in \(\mathcal{M}_\Lambda \cap \mathcal{P}^{-1} \mathcal{D}_\Lambda\). Hence we are back to the first case.

\[
\Box
\]

### 2.5 Kähler cone

Let \(X\) be an irreducible symplectic orbifold of dimension \(n\). We denote by \(\mathcal{K}_X\) the Kähler cone of \(X\).

We denote by \(\mathcal{C}_X\) the connected component of \(\left\{ \alpha \in H^{1,1}(X, \mathbb{R}) \mid q_X(\alpha) > 0 \right\}\) which contains \(\mathcal{K}_X\); it is called the **positive cone**. Let \(\mathcal{B} \mathcal{K}_X\) be the **birational Kähler cone** which is the union \(\cup f^* \mathcal{K}_X\), for \(f\) running through all birational maps between \(X\) and any irreducible symplectic orbifold \(X'\).

In [MR20, Definition 4.5], we define the wall divisors in the same way as Mongardi in [Mon15, Definition 1.2].

**Definition 2.13.** Let \(X\) be an irreducible symplectic orbifold and let \(D \in \text{Pic}(X)\). Then \(D\) is called a wall divisor if \(q(D) < 0\) and \(g(D^\perp) \cap \mathcal{B} \mathcal{K}_X = \emptyset\), for all \(g \in \text{Mon}_{\text{Hdg}}(X)\).

We denote by \(\mathcal{W}_X\) the set of primitive wall divisors of \(X\) (non divisible in \(\text{Pic}(X)\)). By [MR20, Corollary 4.8], we have the following theorem.

**Theorem 2.14.** Let \(\Lambda\) be a lattice of signature \(\langle 3, \text{rk}\Lambda - 3 \rangle\) and \(\mathcal{M}_\Lambda^3\) a connected component of the associated moduli space of marked irreducible symplectic orbifolds. Then there exists a set \(\mathcal{W}_\Lambda \subset \Lambda\) such that for all \((X, \varphi) \in \mathcal{M}_\Lambda^3:\

\[
\mathcal{W}_X = \varphi^{-1}(\mathcal{W}_\Lambda) \cap H^{1,1}(X, \mathbb{Z}).
\]

**Definition 2.15.** The set \(\mathcal{W}_\Lambda\) will be called the set of wall divisor of the deformation class of \(X\).

**Example 2.16** ([Mon15, Proposition 2.12] and [Huy16, Theorem 5.2 Chapter 8]). If \(\mathcal{M}_\Lambda^3\) is a connected component of the moduli space of marked K3 surface, then:

\[
\mathcal{W}_\Lambda = \left\{ D \in \Lambda \mid D^2 = -2 \right\}.
\]

If \(\mathcal{M}_\Lambda^3\) is a connected component of the moduli space of marked irreducible symplectic manifolds equivalent by deformation to a Hilbert scheme of 2 points on a K3 surface, then:

\[
\mathcal{W}_\Lambda = \left\{ D \in \Lambda \mid D^2 = -2 \right\} \cup \left\{ D \in \Lambda \mid D^2 = -10 \text{ and } D \cdot \Lambda \subset 2 \mathbb{Z} \right\}.
\]

**Remark 2.17.** Let \(\beta \in H^{2n-1, 2n-1}(X, \mathbb{Q})\). We can associate to \(\beta\) its dual class \(\beta^\vee \in H^{1,1}(X, \mathbb{Q})\) defined as follows. By [Men20, Corollary 2.7] and since the Beauville–Bogomolov form is integral and non-degenerated (see [Men20, Theorem 3.17]), we can find \(\beta^\vee \in H^2(X, \mathbb{Q})\) such that for all \(\alpha \in H^2(X, \mathbb{C})\):

\[
(\alpha, \beta^\vee)_q = \alpha \cdot \beta,
\]

where the dot on the right hand side is the cup product. Since \((\beta^\vee, \sigma_X)_q = \beta \cdot \sigma_X = 0\), we have \(\beta^\vee \in H^{1,1}(X, \mathbb{Q})\).

We also define the **Mori cone** as the cone of classes of effective curves in \(H^{2n-1, 2n-1}(X, \mathbb{Z})\).

**Proposition 2.18** ([MR20, Proposition 4.12]). Let \(X\) be an irreducible symplectic orbifold. Let \(R\) be an extremal ray of the Mori cone of \(X\) of negative self intersection. Then any class \(D \in \mathbb{Q} R^\vee\) is a wall divisor.

It induces a criterion for Kähler classes.

**Definition 2.19.** Given an irreducible symplectic orbifold \(X\) endowed with a Kähler class \(\omega\). Define \(\mathcal{W}_X^+ := \left\{ D \in \mathcal{W}_X \mid (D, \omega)_q > 0 \right\}\), i.e. for every wall divisor, we choose the primitive representative in its line, which pairs positively with the Kähler cone.
Corollary 2.20 ([MR20], Corollary 4.14). Let $X$ be an irreducible symplectic orbifold such that either $X$ is projective or $b_2(X) \geq 5$. Then

$$K_X = \{ \alpha \in C_X \mid (\alpha, D) > 0 \forall D \in \mathcal{W}_X^+ \}.$$ 

Finally, we recall the following proposition about the birational Kähler cone.

Proposition 2.21 ([MR20], Corollary 4.17). Let $X$ be an irreducible symplectic orbifold. Then $\alpha \in H^{1,1}(X, \mathbb{R})$ is in the closure $\overline{BK}_X$ of the birational Kähler cone $BK_X$ if and only if $\alpha \in \mathcal{T}_X$ and $(\alpha, [D])_F \geq 0$ for all uniruled divisors $D \subset X$.

3 The Nikulin orbifolds

3.1 Construction and description of Nikulin orbifolds

In order to enhance the readability, we recall the construction of the Nikulin orbifold from Example 2.22 and Definition 2.3. Let $X$ be a (smooth) irreducible symplectic 4-fold deformation equivalent to the Hilbert scheme of two points on a K3 surface (called manifold of $\text{K3}^{[2]}$-type). Suppose that $X$ admits a symplectic involution $\iota$. By [Mon20] Theorem 4.1, $\iota$ has 28 fixed points and a fixed K3 surface $\Sigma$. We define $M := X/\iota$ and $r : M' \to M$ the blow-up in the image of $\Sigma$. As mentioned in Example 2.22, the orbifolds $M'$ constructed in this way are irreducible symplectic orbifolds (see [Mon20] Proposition 3.8) and are named Nikulin orbifolds.

A concrete example of such $X$ can be obtained in the following way: Let $S$ be a K3 surface endowed with a symplectic involution $\iota$. It induces a symplectic involution $\iota^{[2]}$ on $S^{[2]}$ the Hilbert scheme of two points on $S$. Then the fixed surface $\Sigma$ of $\iota^{[2]}$ is the following:

$$\Sigma = \left\{ \xi \in S^{[2]} \mid \text{Supp } \xi = \{ s, \iota(s) \}, s \in S \right\}.$$ (1)

Remark 3.1. Let us describe this surfaces $\Sigma$: Consider as usual $S \times S \xrightarrow{\nu} S \times S \xrightarrow{\iota} S^{[2]}$, where $\nu$ is the blow-up of the diagonal $\Delta_S \subset S \times S$, and $\tilde{\iota}$ the double cover induced by permutation of the two factors. Consider the surface $S_i := \{(s, \iota(s)) \mid s \in S\} \subset S \times S$, which is preserved by the involution $\iota \times \iota$. Restricted to $S_i$, the permutation of the two factors in $S \times S$ corresponds to the action of $\iota$ on $S$ (via the isomorphism $S_i \cong S$ induced by the first projection), and thus $S_i \cap \Delta_S$ corresponds to the fixed points of $\iota$ in $S$. Therefore, the strict transform $\tilde{S}_i$ of $S_i$ is isomorphic to the blow-up $\text{Bl}_{\text{Fix}_S} S$ of $S$ in the fixed points of $\iota$. Denote

$$\Sigma := \tilde{\rho}(\tilde{S}_i) \cong \text{Bl}_{\text{Fix}_S} S / \tau,$$

where $\tau$ is the involution on $\text{Bl}_{\text{Fix}_S} S$ which is induced by $\iota$. Then $\Sigma$ is a K3 surface, which is fixed by $\iota^{[2]}$ and admits the description in (1) by construction.

Note that the existence of a symplectic involution on a K3 surfaces or on $\text{K3}^{[2]}$-type manifold can be checked purely on the level of lattices. We will need the following lemma.

Lemma 3.2. Let $X$ be a K3 surface or an irreducible symplectic manifold of $\text{K3}^{[2]}$-type. Assume that there is a primitive embedding $E_8(-2) \hookrightarrow \text{Pic } X$, then there exists no wall divisor in $E_8(-2)$. In particular under the additional assumption that $\text{Pic } X \cong E_8(-2)$, then $C_X = K_X$.

Proof. All elements of $E_8(-2)$ are of square divisible by 4. Hence by Example 2.10 $E_8(-2)$ cannot contain any wall divisor. Then the lemma follows from Corollary 4.20.

Proposition 3.3. Let $X$ be a K3 surface or a manifold of $\text{K3}^{[2]}$-type. Then there exists a symplectic involution $\iota$ on $X$ if and only if $X$ satisfies the following conditions:

(i) There exists a primitive embedding $E_8(-2) \hookrightarrow \text{Pic } X$.

(ii) The intersection $K_X \cap E_8(-2)^+ \neq \emptyset$.

In this case the pullback $\iota^*$ to $H^2(X, \mathbb{Z})$ acts on $E_8(-2)$ as $-\text{id}$ and trivially on $E_8(-2)^+$. 
Propositions 10] we know that the natural map \( \text{Aut}(\iota) \) is a symplectic automorphism, which extends to an involution on \( X \) such that \( \iota^*(\alpha) \) is also a Kähler class.

For the other implication assume (i) and (ii). We consider the involution \( i \) on \( E_8(-2) \oplus E_8(-2) \oplus i \), defined by \(-id\) on \( E_8(-2) \) and id on \( E_8(-2) \). By [Nik80 Corollary 1.5.1], \( i \) extends to an involution on \( H^2(X, \mathbb{Z}) \).

By [Mar11 Section 9.1.1], \( i \) is a monodromy operator. Moreover, by (ii), we can find a Kähler class of \( X \) in \( E_8(-2) \). It follows from the global Torelli theorem (see [Mar11 Theorem 1.3 (2)] or Theorem 2.8 (ii)) that there exists a symplectic automorphism \( \iota \) on \( X \) such that \( \iota^* = i \). However by [Bea83a Proposition 10], we know that the natural map \( \text{Aut}(X) \to O(H^2(X, \mathbb{Z})) \) is an injection. Hence \( i \) is necessarily an involution.

Remark 3.4. Fix a primitive embedding of \( E_8(-2) \) in the \( K3^{[2]} \)-lattice \( \Lambda : = U^3 \oplus E_8(-1)^2 \oplus (-2) \). Let \( \mathcal{M}_{K3^{[2]}} \) be the moduli space of marked \( K3^{[2]} \)-type manifolds endowed with a symplectic involution such that the anti-invariant lattice is identified with the chosen \( E_8(-2) \). Denote by \( \Lambda' \cong U^3 \oplus E_8(-2) \oplus (-2) \) the orthogonal complement of \( E_8(-2) \). From Proposition 3.3 we observe that the period map restricts to \( \mathcal{P}^*: \mathcal{M}_{K3^{[2]}} \to \mathcal{D}_{\Lambda^*} := \{ \sigma \in \mathbb{P}(\Lambda' \otimes \mathbb{C}) \mid \sigma^2 = 0, \sigma \cdot \sigma > 0 \} \).

Note that the fibers of \( \mathcal{P}^* \) are in one to one correspondence with the chambers cut out by wall divisors (no wall divisor can be contained in the orthogonal complement of \( \Lambda^* \) see Example 2.16). In particular, this is given by the chamber structure inside \( \Lambda^* \) given by the images of the wall divisors under the orthogonal projection \( \Lambda_{K3^{[2]}} \to \Lambda^* \).

3.2 The lattice of Nikulin orbifolds starting from \( S^{[2]} \)

From now on we restrict ourselves to the case \( X = S^{[2]} \) for a suitable K3 surface \( S \) with an involution \( \iota \). We consider the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{O}_1^{[2]} \cong \mathbb{C}^2 & \xrightarrow{\iota_1^{[2]}} & S^{[2]} \\
N_1 \xrightarrow{r_1} & \xrightarrow{\pi} & \pi \\
M' \xrightarrow{\pi} & M, & \\
\end{array}
\]

where \( \pi : S^{[2]} \longtwoheadrightarrow S^{[2]}/\iota_1^{[2]} =: M \) is the quotient map, \( r_1 \) is the blow-up in \( \Sigma \) of \( S^{[2]} \), \( \iota_1^{[2]} \) is the involution induced by \( \iota^{[2]} \) on \( N_1 \), \( \pi_1 : N_1 \longtwoheadrightarrow N_1/\iota_1^{[2]} \cong M' \) is the quotient map and \( r \) is the blow-up in \( \pi_1(\Sigma) \) of \( M \). We also denote by \( j : H^2(S, \mathbb{Z}) \hookrightarrow H^2(S^{[2]}, \mathbb{Z}) \) the natural Hodge isometric embedding (see [Bea83b Proposition 6 Section 6 and Remark 1 Section 9]).

We fix the following notation for important divisors:

- \( \Delta \) the class of the diagonal divisor in \( S^{[2]} \) and \( \delta := \frac{1}{2} \Delta \);
- \( \delta_1 := r_1^*(\delta) \) and \( \Sigma_1 \) the exceptional divisor of \( r_1 \);
- \( \delta' := \pi_1^*(\delta) \) and \( \Sigma' := \pi_1^*(\Sigma_1) \) the exceptional divisor of \( r \).

Here we use the definition of the push-forward given in [AP06]. In particular \( \pi_* \) verifies the following equations (see [AP06 Theorem 5.4 and Corollary 5.8]):

\[
\pi_* \circ \pi^* = 2 \text{id} \quad \text{and} \quad \pi^* \circ \pi_* = \text{id} + \iota_1^{[2]*}.
\]

As a consequence, we have (see [Men18 Lemma 3.6]):

\[
\pi_*(\alpha) \cdot \pi_*(\beta) = 2\alpha \cdot \beta,
\]

with \( \alpha \in H^k(S^{[2]}, \mathbb{Z})^{[2]} \) and \( \beta \in H^8-k(S^{[2]}, \mathbb{Z})^{[2]} \), \( k \in \{0, ..., 8\} \). Of course, the same equations are also true for \( \pi_{1*} \).
Remark 3.5. Note that the commutativity of diagram \(2\) and equations \(3\) imply \(\pi_1, r_1^* (x) = r^* \pi_*(x)\) for all \(x \in H^2(S^{[2]}, \mathbb{Z})\).

We denote by \(q_{M'}\) and \(q_{S^{[2]}}\) respectively the Beauville–Bogomolov form of \(M'\) and \(S^{[2]}\). We can also define a Beauville–Bogomolov form on \(M\) by:

\[ q_M(x) := q_{M'}(r^*(x)), \]

for all \(x \in H^2(M, \mathbb{Z})\). We recall the following theorem.

**Theorem 3.6.** (i) The Beauville–Bogomolov lattice of \(M'\) is given by \((H^2(M', \mathbb{Z}), q_{M'}) \simeq U(2)^3 \oplus E_8(-1) \oplus (-2)^2\) where the Fuji constant is equal to 6.

(ii) \(q_M(\pi_*(x)) = 2q_{S^{[2]}}(x)\) for all \(x \in H^2(S^{[2]}, \mathbb{Z})^{[2]}\).

(iii) \(q_{M'}(\delta') = q_{M'}(\Sigma') = -4\).

(iv) \((r^*(x), \Sigma')_{M'} = 0\) for all \(x \in H^2(M, \mathbb{Z})\).

(v) \(H^2(M', \mathbb{Z}) = r^* \pi_*(j(H^2(S, \mathbb{Z}))) \oplus \mathbb{Z} \frac{\delta' + \Sigma'}{2} \oplus \mathbb{Z} \frac{\delta' - \Sigma'}{2}\).

**Proof.** This theorem corresponds to several results in \cite{Men15}. We want to emphasize that our notation are slightly different from \cite{Men15}. In \cite{Men15}, we also define a Beauville–Bogomolov form on \(M'\) provided by \cite{Men15, Theorem 2.39}. As in Section 3.1, we can consider the associated period domain:

\[ D_{\Lambda}^{[2]} := \left\{ \sigma \in \mathbb{P}(\Lambda^{[2]} \otimes \mathbb{C}) \ \middle| \ \sigma^2 = 0, \ \sigma \cdot \pi > 0 \right\}. \]

Remark 3.7. In the previous theorem the Beauville–Bogomolov lattice of \(M'\) is obtained as follows:

- \(r^* \pi_*(j(H^2(S, \mathbb{Z}))) \simeq U(2)^3 \oplus E_8(-1)\),
- \(\mathbb{Z} \frac{\delta' + \Sigma'}{2} \oplus \mathbb{Z} \frac{\delta' - \Sigma'}{2} \simeq (-2)^2\).

We recall that the divisibility \(\text{div}\) of a lattice element is defined in Section 1.3.

**Remark 3.8.** Theorem 3.6 shows that \(\text{div}(\Sigma') = \text{div}(\delta') = 2\).

### 3.3 Monodromy operators inherited from \(\text{Mon}^2(S^{[2]})\)

We keep the notation from the previous subsection. The monodromy group is defined in Section 1.3.

**Proposition 3.9.** Let \(f \in \text{Mon}^2(S^{[2]}), \) (resp. \(f \in \text{Mon}^{2, \text{Hdg}}(S^{[2]})\)) be a monodromy operator such that \(f \circ i^{[2]} \equiv i^{[2]} \circ f\) on \(H^2(S^{[2]}, \mathbb{Z})\). We consider \(f' : H^2(M', \mathbb{Z}) \to H^2(M', \mathbb{Z})\) such that \(f'(\Sigma') = \Sigma'\) and:

\[ f'(r^*(x)) = \frac{1}{2} r^* \circ \pi_\ast \circ f \circ \pi^*(x), \]

for all \(x \in H^2(M, \mathbb{Z})\). Then \(f' \in \text{Mon}^2(M')\), (resp. \(f' \in \text{Mon}^{2, \text{Hdg}}(M')\)).

**Proof.** Let \(\varphi\) be a marking of \(S^{[2]}\). Since \(f\) is a monodromy operator, we know that \((S^{[2]}, \varphi)\) and \((S^{[2]}, \varphi \circ f)\) are in the same connected component of their moduli space (see Section 2.2 for the definition of the moduli space). We consider:

\[ \Lambda^{[2]} := \varphi \left( H^2(S^{[2]}, \mathbb{Z})^{[2]} \right). \]

We know that \(\Lambda^{[2]} \simeq U^3 \oplus E_8(-2) \oplus (-2)\) which is a lattice of signature \((3, 12)\) (see for instance \cite{Men15} Proposition 2.6). As in Section 3.1 we can consider the associated period domain:

\[ D_{\Lambda^{[2]}} := \left\{ \sigma \in \mathbb{P}(\Lambda^{[2]} \otimes \mathbb{C}) \ \middle| \ \sigma^2 = 0, \ \sigma \cdot \pi > 0 \right\}. \]
By Lemma 3.2, a very general K3$^{[2]}$-type manifold mapping to $D_{\Lambda^{[2]}}$ satisfies that the Kähler cone is the entire positive cone. Furthermore, by Proposition 3.3 (iii) the intersection $\varphi(K_{S^{[2]}}) \cap \Lambda^{[2]} \neq \emptyset$ and therefore also $\varphi \circ f(K_{S^{[2]}}) \cap \Lambda^{[2]} \neq \emptyset$ is non-empty. We can apply Lemma 2.12 to see that $(S^{[2]}, \varphi)$ and $(S^{[2]}, \varphi \circ f)$ can be connected by a sequence of twistor spaces $\mathcal{I}_i$ such that $\varphi \circ f \in \text{Aut}(\mathcal{I}_i)$. By construction and Remark 2.11 all these twistor spaces are endowed with an involution $\mathcal{I}_i$ which restricts on each fiber to a symplectic involution. Hence we can consider for each twistor space the blow-up $\tilde{\mathcal{I}}_i/\mathcal{I}_i \to \mathcal{I}_i/\mathcal{I}_i$ of the quotient $\mathcal{I}_i/\mathcal{I}_i$ in the codimension 2 component of its singular locus. We obtain $\mathcal{I}_i/\mathcal{I}_i \to Y_1$ a sequence of families of orbifolds deformation equivalent to $\mathcal{I}_i$. This sequence of families provides a monodromy operator of $\mathcal{I}_i$ that we denote by $f'$. We need to verify that $f'$ satisfies the claimed properties. First note that by construction $f'((\Sigma')) = \Sigma'$. All fibers of a twistor space are diffeomorphic to each other and hence the monodromy operator $f$ is provided by a diffeomorphism $u : S^{[2]} \to S^{[2]}$ such that $u^* = f$. Moreover, by construction this diffeomorphism commutes with $f$. It induces a homeomorphism $\overline{\pi}$ on $\mathcal{I}_i$ with the following commutative diagram:

$$
\begin{array}{cccc}
S^{[2]} & \xrightarrow{u} & S^{[2]} \\
\downarrow{\pi} & & \downarrow{\pi} \\
M & \xrightarrow{\overline{\pi}} & M \\
\downarrow{r} & & \downarrow{r} \\
M' & \xrightarrow{\overline{\pi}} & M'.
\end{array}
$$

Note that, by construction $f' = \overline{\pi}^*$. We can use the commutativity of the previous diagram to check that $f'$ verifies the properties from the proposition. Let $x \in H^2(M, \mathbb{Z})$. We have:

$$f'(r^*(x)) = \overline{\pi}^*(r^*(x)) = r^*(\overline{\pi}(x)). \quad (5)$$

Moreover:

$$\pi^*(\overline{\pi}(x)) = u^*(\pi^*(x)).$$

Taking the image by $\pi_*$ and using (3) we obtain that:

$$2\pi^*(x) = \pi_* u^* \pi^*(x).$$

Combining this last equation with (5), we obtain the statement of the proposition.

It is only left to prove that if $f \in \text{Aut}_{\text{Hdg}}(S^{[2]})$ then also $f' \in \text{Aut}_{\text{Hdg}}(M')$. The maps $\pi$ and $r$ are holomorphic maps between Kähler orbifolds, hence induce morphisms $\overline{\pi}^*$ and $r^*$ which respect the Hodge structure. Then $\pi_*$ respects the Hodge structure because of (3). Since $f'$ is a composition of morphisms which respect the Hodge structure, we therefore obtain that $f' \in \text{Aut}_{\text{Hdg}}(M')$.

**Remark 3.10.** The previous proposition can be generalized to other irreducible symplectic orbifolds obtained as partial resolutions in codimension 2 of quotients of irreducible symplectic manifolds.

**Corollary 3.11.** The reflection $R_{\delta'}$ as defined in Section 1.3 is an element of the Monodromy group $\text{Mon}_{\text{Hdg}}^2(M')$.

**Proof.** By [Mar11, Section 9], we know that $R_{\delta} \in \text{Mon}_{\text{Hdg}}^2(S^{[2]})$. By Proposition 3.9 and Theorem 3.6 (iv), we only have to check that:

$$R_{\delta'}(r^*(x)) = \frac{1}{2} r^* \circ \pi_* \circ R_{\delta} \circ \pi^*(x),$$

for all $x \in H^2(M, \mathbb{Z})$. We have:

$$R_{\delta} \circ \pi^*(x) = \pi^*(x) - \frac{2(\delta, \pi^*(x))q_{S^{[2]}}(\delta)}{q_{S^{[2]}}(\delta)} \delta.$$

Taking the image by $\pi_*$, applying (3) and Theorem 3.3 (ii), we obtain:

$$\pi_* R_{\delta} \circ \pi^*(x) = 2x - \frac{4(\pi_*(\delta), 2x)_{\text{QM}}}{2q_{\text{M}}(\pi_* \delta)} \pi_*(\delta) = 2 \left( x - \frac{2(\pi_*(\delta), x)_{\text{QM}}}{q_{\text{M}}(\pi_* \delta)} \pi_*(\delta) \right).$$

Then dividing by 2, taking the image by $r^*$, and using $q_{\text{M}} = q_{\text{M'}} \circ r^*$ (compare Section 3.2) concludes the computation.
4 A first example: the very general Nikulin orbifolds

4.1 Wall divisors of a Nikulin orbifold constructed from a K3 surface without effective curves

Let \( S \) be a K3 surface admitting a symplectic involution, which does not contain any effective curves. Such a K3 surface exists by Proposition 3.3 and the surjectivity of the period map. Then, we consider \( M' \) the Nikulin orbifold associated to \( S^{[2]} \) and the induced involution \( \iota^{[2]} \) as in Section 3.1 (we keep the same notation as earlier in this section).

**Proposition 4.1.** The wall divisors of \( M' \) are \( \delta' \) and \( \Sigma' \) which are both of square \(-4\) and divisibility 2.

This section is devoted to the proof of this proposition. The idea of the proof is to study the curves in \( M' \) and use Proposition 2.18.

Consider the following diagram:

\[
\begin{array}{ccc}
S^{[2]} & \xrightarrow{\nu} & S^{(2)} \\
\downarrow \rho & & \downarrow \\
S \times S & \xrightarrow{\overline{\nu}} & S^2 \\
\downarrow p_1 & & \downarrow p_2 \\
\overline{\Sigma} & \leftarrow & S
\end{array}
\]

where \( p_1, p_2 \) are the projections, \( \rho \) the quotient map and \( \nu \) the blow-up in the diagonal in \( S^{(2)} \). By assumption \( S \) does not contain any effective curve. Hence considering the image by the projections \( p_1, p_2 \) and \( \rho \), we deduce that \( S^{(2)} \) does not contain any curve either. Hence all curves in \( S^{[2]} \) are contracted by \( \nu \), i.e. fibers of the exceptional divisor \( \Delta \to \Delta_{S^{[2]}} \), where \( \Delta_{S^{[2]}} \) is the diagonal in \( S^{(2)} \). We denote such a curve by \( \ell_3^s \), where \( s \in S \) keeps track of the point \((s, s) \in S^{(2)} \). To simplify the notation, we denote the cohomology class \( \ell_3 : = [\ell_3^s] \), since it does not depend on \( s \in S \).

Our next goal is to determine the irreducible curves in \( N_1 \). Recall that \( r_1 : N_1 \to S^{[2]} \) is the blow-up in the fixed surface \( S \). Let \( C \) be an irreducible curve in \( N_1 \). There are three cases:

(i) The image of \( C \) by \( r_1 \) does not intersect \( \Sigma \) and is of class \( \ell_3 \). Therefore, \( C \) is of class \( r_1^*(\ell_3) \).

(ii) The image of \( C \) by \( r_1 \) is contained in \( \Sigma \) and of class \( \ell_3 \).

(iii) The image of \( C \) by \( r_1 \) is a point. Then \( C \) is of class \( \ell_2\Sigma \) (the class of a fiber of the exceptional divisor \( \Sigma_1 \to \Sigma \)).

Note that \( \ell_2^s \) is contained in \( \Sigma \) if \( s \in S \) is a fixed point of the involution \( \iota \), and otherwise the intersection \( \ell_3^s \cap \Sigma = \emptyset \) is empty (this follows from the description of \( \Sigma \) in Remark 3.1). Therefore, there cannot be a case, where the image of \( \ell_3^s \) intersects \( \Sigma \) in a zero-dimensional locus.

It remains to understand the case (ii).

**Lemma 4.2.** Consider a curve \( \ell_3^s \) contained in \( \Sigma \) (i.e. when \( s \in S \) is a fixed point of \( \iota \)). The surface \( H_0 := r_1^{-1}(\ell_3^s) \) is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \).

**Proof.** The surface \( H_0 \) is a Hirzebruch surface. Since \( r_1 \) is the blow-up along \( \Sigma \), observe that \( H_0 \cong \mathbb{P}(N_{\Sigma/S}|_{\ell_3^s}) \). Therefore, we need to compute \( N_{\Sigma/S}|_{\ell_3^s} \).

Keeping the notation from Remark 3.1 recall that \( \tilde{S}_i := \tilde{\rho}^{-1}(\Sigma) \) and \( S_i := \tilde{\nu}(\tilde{S}_i) \). For simplicity, we also denote by \( \ell_3^s \) the preimage of \( \ell_3^s \) by \( \tilde{\rho} \). (Note for this, that \( \tilde{\rho} \) restricts to an isomorphism on the preimage of \( \Delta \), and therefore, it makes sense to identify \( \ell_3^s \) with its preimage).

Observe that:

\[ N_{\Sigma/S}|_{\ell_3^s} \cong \tilde{\rho}^*(N_{\Sigma/S}|_{\ell_3^s}) \cong N_{\tilde{S}_i/S\times S}|_{\ell_3^s} \cong \tilde{\nu}^*(N_{\tilde{S}_i/S\times S}|_{\ell_3^s}) \cong \mathcal{O}_{\ell_3^s} \oplus \mathcal{O}_{\ell_3^s}, \]

where we identify \( s \in S \cong \tilde{S}_i \). It follows that \( H_0 \cong \mathbb{P}^1 \times \mathbb{P}^1 \). \( \square \)
It follows that the extremal curves in case (ii) have classes $r_1^*(\ell_\delta)$.

**Remark 4.3.** In particular, considering cases (i) and (ii), we see that the extremal curves $C$ such that $r_1(C) = \ell_\delta$ for some $s \in S$ have classes $r_1^*(\ell_\delta)$.

Hence, we obtain that the extremal curves in $N_1$ have classes $r_1^*\ell_\delta$ and $\ell_\Sigma$. This implies that the extremal curves in $M'$ have classes $\pi_1 r_1^*\ell_\delta$ and $\pi_1 \ell_\Sigma$.

We can compute their dual divisors.

**Lemma 4.4.** The dual divisors in $H^2(M', \mathbb{Q})$ of $\pi_1 r_1^*\ell_\delta$ and $\pi_1 \ell_\Sigma$ are respectively $\frac{1}{2}\delta'$ and $\frac{1}{2}\Sigma'$ in $H^2(M', \mathbb{Q})$.

**Proof.** Write

$$H^2(N_1, \mathbb{Z}) = r_1^*\nu^*H^2(S^{[2]}, \mathbb{Z}) \oplus \mathbb{Z}[\delta_1] \oplus \mathbb{Z}[\Sigma_1].$$

We denote by $p_{\delta_1} : H^2(N_1, \mathbb{Z}) \to \mathbb{Z}[\delta_1] \cong \mathbb{Z}$ and $p_{\Sigma_1} : H^2(N_1, \mathbb{Z}) \to \mathbb{Z}[\Sigma_1] \cong \mathbb{Z}$ the projections. Let $x \in H^2(N_1, \mathbb{Z})^{[2]}$. Since $\ell_\delta \cdot \alpha = 0$ for all $\alpha \in r_1^*\nu^*H^2(S^{[2]}, \mathbb{Z}) \oplus \mathbb{Z}[\Sigma_1]$, we have

$$\ell_\delta \cdot x = (\ell_\delta \cdot \delta_1)p_{\delta_1}(x) = -p_{\delta_1}(x)$$

and similarly $\ell_\Sigma \cdot x = (\ell_\Sigma \cdot \Sigma_1)p_{\Sigma_1}(x) = -p_{\Sigma_1}(x)$. It follows from (4):

$$\pi_1(\ell_\delta) \cdot \pi_1(x) = -2p_{\delta_1}(x) \text{ and } \pi_1(\ell_\Sigma) \cdot \pi_1(x) = -2p_{\Sigma_1}(x).$$

Therefore, Theorem 3.6 (iii) shows that $\frac{1}{2}\delta'$ and $\frac{1}{2}\Sigma'$ in $H^2(M', \mathbb{Q})$ are the duals of $\pi_1(\ell_\delta)$ and $\pi_1(\ell_\Sigma)$ respectively.

By Proposition 2.13 this proves that $\delta'$ and $\Sigma'$ are wall divisors in $M'$. Their claimed numerical properties are given by Theorem 3.6 (iii) and Remark 3.8.

It remains to show that $\delta'$ and $\Sigma'$ are the only wall divisors in $M'$. Let us assume for contradiction that there is another wall divisor $D$. By Theorem 3.6 (v), we have $D = a\delta' + b\Sigma' + K$, with $(a, b) \in \mathbb{Z}[\frac{1}{2}] \times \mathbb{N}[\frac{1}{2}]$, (up to replacing $D$ by $-D$) and $K$ a divisor orthogonal to $\delta'$ and $\Sigma'$. Since $\delta'$ and $\Sigma'$ correspond to the duals of the extremal rays of the Mori cone, all classes $\alpha \in C_{M'}$ such that $(\alpha, \delta')_q > 0$ and $(\alpha, \Sigma')_q > 0$ are Kähler classes by [MR20, Theorem 4.1].

Hence, we cannot have $a = b = 0$. Indeed, $D$ would be orthogonal to the Kähler classes with orthogonal projection on Pic $M'$ equal to $-(\Sigma' + \delta')$ which is impossible by definition of wall divisors.

Therefore, $a$ or $b$ are non trivial. It follows that $K = 0$. Indeed, if $K \neq 0$, as before, the class $-(\Sigma' + \delta') - \frac{4(a+b)}{(K, K)}K$ is the projection on Pic $M'$ of a Kähler class; however it is orthogonal to $D$ which is impossible.

Now, we can assume that $a \neq 0$ and $b \neq 0$ (indeed, if $a = 0$ or $b = 0$, then $D \in \mathbb{Z}\delta'$ or $D \in \mathbb{Z}\Sigma'$). If $a < 0$, $D$ would be orthogonal to $a\Sigma' - b\delta'$ the projection on Pic $M'$ of a Kähler class; this is impossible by definition of wall divisors. Hence we assume that $a > 0$. By Corollary 3.11 $R_{\delta'}$ is a monodromy operator. Moreover, $R_{\delta'}(D) = -a\delta + b\Sigma'$; as previously $R_{\delta'}(D)$ is orthogonal to some Kähler class. This gives a contradiction and thus concludes the proof.

### 4.2 Application: an example of non-natural symplectic involution on a Nikulin orbifold

Using the results from the previous subsection, we will prove the existence of a non-natural symplectic involution on our example. We recall that the reflections $R_x$ with $x \in H^2(M', \mathbb{Z})$ are defined in Section 1.3

**Proposition 4.5.** Let $(S, \iota)$ be a very general $K3$ surface endowed with a symplectic involution (that is Pic $S \simeq E_8(-2)$). Let $M'$ be a Nikulin orbifold constructed from $(S, \iota)$ as in Section 7.5. There exists $\kappa'$ a symplectic involution on $M'$ such that $\kappa'^* = R_{\frac{1}{2}(\delta' - \Sigma')}$.

**Proof.** We consider the following involution on $S \times S$:

$$\kappa : S \times S \longrightarrow S \times S \quad (x, y) \longmapsto (x, \iota(y)).$$
We consider

$$V := S \times S \setminus (\Delta_{S^2} \cup \Delta, \cup (\Fix \tau \times \Fix \tau)).$$

We denote by $\sigma_2$ the involution on $S \times S$ which exchange the two K3 surfaces and $\tau \times \tau$ the involution which acts as $\tau$ diagonally on $S \times S$. Then we consider

$$U := V/ \langle \sigma_2, \tau \times \tau \rangle.$$  

This set can be seen as an open subset of $M'$ and $V$ can also be seen as an open subset of $\widetilde{S \times S}$. Moreover, the map $\pi_1 \circ \tilde{\rho} : V \to U$ is a four to one non-ramified cover. For simplicity, we denote $\gamma := \pi_1 \circ \tilde{\rho}$.  

First, we want to prove that $\kappa$ induces an involution $\kappa'$ on $U$ with a commutative diagram:

$$\begin{array}{ccc}
V & \xrightarrow{\kappa} & V \\
\gamma \downarrow & & \downarrow \gamma \\
U & \xrightarrow{\kappa'} & U.
\end{array}$$  

If such a map $\kappa'$ would exist then it will necessarily verify the following equation:

$$\kappa' \circ \gamma = \gamma \circ \kappa.$$

The map $\gamma$ being surjective, to be able to claim that the previous equation provides a well defined map from $U$ to $U$, we have to verify that:

$$\kappa' \circ \gamma(x_1, y_1) = \kappa' \circ \gamma(x_2, y_2),$$

when $\gamma(x_1, y_1) = \gamma(x_2, y_2)$. That is:

$$\kappa' \circ \gamma(x_1, y_1) = \gamma \circ \kappa(x_1, y_1) = \gamma \circ \kappa(x_2, y_2),$$

for all $((x_1, y_1), (x_2, y_2)) \in S^4$ such that $\gamma(x_1, y_2) = \gamma(x_2, y_2)$. Let $(x, y) \in S^2$. We have:

$$\gamma^{-1}(\gamma(x, y)) = \{((x, y), (y, x), (\iota(x), \iota(y)), (\iota(y), \iota(x)))\}.$$  

We also have:

$$\kappa(\gamma^{-1}(\gamma(x, y))) = \{(x, \iota(y)), (y, \iota(x)), (\iota(x), y), (\iota(y), x)\} = \gamma^{-1}((\gamma(x, y)) = \gamma^{-1}(\gamma(x, y))).$$

This shows $\kappa'$. Hence $\kappa'$ is set theoretically well defined. Since $\gamma$ is a four to one non-ramified cover, it is a local isomorphism; therefore $\kappa'$ inherits of the properties of $\kappa$. In particular $\kappa'$ is a holomorphic symplectic involution.

It follows that $\kappa'$ induces a bimeromorphic symplectic involution on $M'$. By [MR20, Lemma 3.2], $\kappa'$ extends to bimeromorphic symplectic involution which is an isomorphism in codimension 1 (we still denote by $\kappa'$ this involution). In particular, $\kappa'$ is now well defined on $H^2(M', \mathbb{Z})$ (see [MR20, Lemma 3.4]).  

Now, we are going to prove that $\kappa'$ extends to a regular involution. We recall from Theorem 3.6 (v) that:

$$H^2(M', \mathbb{Z}) = r^* \pi_*(j(H^2(S, \mathbb{Z}))) \oplus \mathbb{Z} \delta' + \mathbb{Z} \delta'.$$

Since $\kappa'$ is symplectic $\kappa'$ acts trivially on $r^* \pi_*(j(H^2(S, \mathbb{Z})))$. Indeed $\Pic M' = \mathbb{Z} \delta' + \mathbb{Z} \delta'$. Moreover, $\kappa'$ exchanges $\Delta_{S^2}$ and $S'$. Hence by continuity and commutativity of diagram (7), we have that $\kappa'$ exchanges the divisors $\delta'$ and $\Sigma'$. By Proposition 3.1 and Corollary 2.20 it follows that $\kappa'$ sends Kähler classes to Kähler classes. Hence by [MR20, Proposition 3.3], $\kappa'$ extends to an involution on $M'$.

**Corollary 4.6.** Let $M'$ be a Nikulin orbifold. Then:

$$R_{\Sigma'} \in \Mon^2(M').$$
Proof. Let \((X, \iota)\) be any manifold of \(K3^{[2]}\)-type endowed with a symplectic involution. Let \((S, \iota)\) be a very general \(K3\) surface endowed with a symplectic involution. With exactly the same argument as in proof of Proposition \(5.3\) we can connect \(X\) and \(S^{[2]}\) by a sequence of twistor spaces; each twistor space being endowed with an involution which restricts to a symplectic involution on its fibers. This sequence of twistor spaces provides a sequence of twistor spaces between \(M_X^{[2]}\) and \(M_{S^{[2]}}^{[2]}\) the irreducible symplectic orbifolds associated to \(X\) and \(S^{[2]}\) respectively. This sequence of twistor spaces provides a parallel transport operator \(f : H^2(M_X^{[2]}, \mathbb{Z}) \to H^2(M_{S^{[2]}}^{[2]}, \mathbb{Z})\) which sends \(\Sigma_X\) to \(\Sigma_{S^{[2]}}^{[2]}\) (respectively the exceptional divisors of the blow-ups \(M_X^{[2]} \to X/\tilde{\iota}\) and \(M_{S^{[2]}}^{[2]} \to S^{[2]}/\tilde{\iota}^{[2]}\)).

By Proposition \(5.5\) \(\kappa^* \in \text{Mon}^2(M_{S^{[2]}}^{[2]}).\) Moreover by Corollary \(3.11\) \(R_{\delta^*} \in \text{Mon}^2(M_{S^{[2]}}^{[2]}).\) Hence \(R_{\Sigma_{S^{[2]}}^{[2]}} = \kappa^* \circ R_{\delta^*} \circ \kappa^* \in \text{Mon}^2(M_{S^{[2]}}^{[2]}).\) Therefore \(R_{\Sigma_X} = f^{-1} \circ R_{\Sigma_{S^{[2]}}^{[2]}} \circ f \in \text{Mon}(M_X^{[2]}).\)

\(\Box\)

Remark 4.7. Let \((S, \iota)\) be a \(K3\) surface endowed with a symplectic involution. Let \(M'\) be a Nikulin orbifold constructed from \((S, \iota)\) as in Section \(5.2\) The previous proof also shows that \(R_{\Sigma_X}^{(\delta^* - \Sigma')}\) is a parallel transport operator.

5 In search of wall divisors in special examples

In this section we study some explicit examples of \(K3\) surfaces with symplectic involutions and their associated Nikulin orbifolds. This will be used in Section \(7\) in order to determine which divisors on \(K3\)-type orbifolds are wall divisors.

5.1 When the \(K3\) surface \(S\) used to construct the Nikulin orbifold contains a unique rational curve

Objective

In this section we assume that \(\text{Pic} S \simeq E_8(-2) \oplus (-2).\) By Riemann–Roch \(S\) contains only one curve which is rational. We denote this curve \(C.\) In particular in this case, \(K_S \cap E_8(-2)^+ \neq \emptyset.\) Hence, by Proposition \(5.3\) there exists a symplectic involution \(\iota\) on \(S\) whose anti-invariant lattice is the \(E_8(-2) \subset \text{Pic}(S).\) Moreover, the curve \(C\) is fixed by \(\iota.\) The objective of this section is to determine wall divisors of the Nikulin orbifold \(M'\) obtained from the couple \((S^{[2]}, \iota^{[2]}))\) (see Section \(5.4)).

Notation

We keep the notation from Section \(4.1\) and we consider the following notation in addition.

\textit{Notation 5.1.} \(\text{• We denote by } D_C \text{ the following divisor in } S^{[2]}:\)

\[ D_C = \left\{ \xi \in S^{[2]} \mid \text{Supp} \xi \cap C \neq \emptyset \right\}. \]

Moreover, we set \(D_C' := \pi_{1*} r_1^*(D_C).\)

\(\text{• We denote by } \overline{C}^{(2)} \text{ the strict transform of } C^{(2)} \subset S^{(2)} \text{ by } \nu \text{ and } \overline{C}_{\overline{S}}^{(2)} \text{ the strict transform of } C_{\overline{S}}^{(2)} \subset S^{(2)} \text{ by } r_1.\) We denote by \(j^C_\overline{C} : \overline{C}_{\overline{S}}^{(2)} \hookrightarrow S_{\overline{S}}^{(2)}\) and \(j^C : C^{(2)} \hookrightarrow N_1\) the embeddings. Note that \(\overline{C}_{\overline{S}}^{(2)} \simeq C^{(2)} \simeq C^{(2)} \simeq \mathbb{P}^2.\)

\(\text{• We recall that } \Delta_{S^{(2)}} \text{ is the diagonal in } S^{(2)} \text{ and } \Delta \text{ the diagonal in } S^{[2]}.\) We also denote by \(\Delta_{S^{(2)}}\) the diagonal in \(S \times S\) and \(\Delta_{S^{[2]}} := \nu^{-1}(\Delta_{S^{(2)}})\) the exceptional divisor. Furthermore, we denote \(\Delta_{S^{(2)}}^C := C^{(2)} \cap \Delta_{S^{(2)}}\) and \(\Delta_{S^{[2]}}^C := C^{[2]} \cap \Delta_{S^{[2]}}.\) Moreover we denote \(\Delta_{S^{[2]}}^C := \{ \{ x, \iota(x) \} \mid x \in C \} \subset S^{[2]}\) and \(\Delta_{S^{[2]}}^C := \{ \{ x, \iota(x) \} \mid x \in C \} \subset S^{[2]}.\)

\(\text{• We denote } H_2 := \nu^{-1}(\Delta_{S^{(2)}}^C) \text{ and } \overline{H}_2 \text{ its strict transform by } r_1.\) We denote by \(j^{H_2} : H_2 \hookrightarrow S^{(2)}\) and \(j^{\overline{H}_2} : \overline{H}_2 \hookrightarrow N_1\) the embeddings.
We summarize our notation on the following diagram:

![Diagram](attachment:image.png)

**Lemma 5.2.** The surface $H_2$ is a Hirzebruch surface isomorphic to $\mathbb{P}(\mathcal{E})$, where $\mathcal{E} := \mathcal{O}_C(2) \oplus \mathcal{O}_C(-2)$. Let $f$ be a fiber of $\mathbb{P}(\mathcal{E})$. There exists a section $C_0$ which satisfies: $\text{Pic } H_2 = \mathbb{Z} C_0 \oplus \mathbb{Z} f$, $C_0^2 = -4$, $f^2 = 0$ and $C_0 \cdot f = 1$.

**Proof.** Let $\sigma_2$ be the involution on $S \times S$ which exchanges the two K3 surfaces and $\tilde{\sigma}_2$ the induced involution on $\tilde{S} \times \tilde{S}$. The involution $\tilde{\sigma}_2$ acts trivially on $\Delta_{\tilde{S}^2}$. It follows that $\tilde{\rho}$ induces an isomorphism $\Delta_{\tilde{S}^2} \cong \Delta$. In particular, it shows that:

$$H_2 \cong \mathbb{P}(\mathcal{N}_{\Delta_{\tilde{S}^2}/S \times S} \otimes \mathcal{O}_C) .$$

We consider the following commutative diagram:

$$\Delta_{\tilde{S}^2} \quad \Delta_{S^2} \quad \Delta_C$$

Under the isomorphism induced by (9), observe that:

$$\mathcal{N}_{\Delta_{\tilde{S}^2}/S \times S} \otimes \mathcal{O}_C \cong T_{S/C} .$$

To compute $T_{S/C}$, we consider the following exact sequence:

$$0 \rightarrow T_C \rightarrow T_{S/C} \rightarrow \mathcal{N}_{C/S} \rightarrow 0.$$ 

We have $T_C = \mathcal{O}_C(2)$ and $\mathcal{N}_{C/S} = \mathcal{O}_C(-2)$. Moreover, $\text{Ext}^1(\mathcal{O}_C(-2), \mathcal{O}_C(2)) = H^1(\mathcal{O}_C, \mathcal{O}_C(4)) = 0$. Hence:

$$T_{S/C} = \mathcal{O}_C(-2) \oplus \mathcal{O}_C(2).$$

As a consequence $H_2 \cong \mathbb{P}(\mathcal{E})$ as claimed.

Therefore, by [Har77 Chapter V, Proposition 2.3 and Proposition 2.9], we know that $\text{Pic } H_2 = \mathbb{Z} C_0 \oplus \mathbb{Z} f$, with $C_0^2 = -4$, $f^2 = 0$ and $C_0 \cdot f = 1$; $C_0$ being the class of a specific section and $f$ the class of a fiber. □

**Lemma 5.3.** We have $\ell \cdot \delta = -1$ and $C \cdot D_C = -2$ in $S^{[2]}$.

**Proof.** We denote by $\tilde{\ell}_g$ a fiber associated to the exceptional divisor $\Delta_{\tilde{S}^2} \rightarrow \Delta_{S^2}$. We know that $\tilde{\ell}_g \cdot \Delta_{\tilde{S}^2} = -1$. We can deduce for instance from [Men18 Lemma 3.6] that $\ell_g \cdot \Delta = -2$. That is $\ell_g \cdot \delta = -1$.

Similarly, we have $(C \times S + S \times C) \cdot (s \times C + C \times s) = -8$. By [Men18 Lemma 3.6] (see [4]):

\[
\rho_+(C \times S + S \times C) \cdot \rho_+(s \times C + C \times s) = -8
\]

\[
\rho_+(C \times S) \cdot \rho_+(s \times C) = -2.
\]

Then taking the pull-back by $\nu$, we obtain $C \cdot D_C = -2$. □
Strategy

We will need several steps to find the wall divisors on $M'$:

I. Understand the curves contained in $S^{[2]}$.

II. Understand the curves contained in $N_1$.

III. Deduce the corresponding wall divisors in $M'$ using Proposition 2.18, Corollaries 3.11 and 4.6.

Curves in $S^{[2]}$

The first step is to determine the curves contained in $S^{[2]}$. Before that, we can say the following about curves in $S^{(2)}$.

Lemma 5.4. There are only two cases for irreducible curves in $S^{(2)}$:

(1) the curves $C_s := \rho(C \times s) = \rho(s \times C)$ with $s \notin C$;

(2) curves in $C^{(2)} \cong \mathbb{P}^2$.

Proof. In $S \times S$, considering the images of curves by the projections $p_1$ and $p_2$ of diagram (6), there are only two possibilities:

- $s \times C$ or $C \times s$, with $s$ a point in $S$,
- curves contained in $C \times C$.

Then, we obtain all the curves in $S^{[2]}$ as images by $\rho$ of curves in $S \times S$. □

It follows four cases in $S^{[2]}$.

Lemma 5.5. We have the following four cases for irreducible curves in $S^{[2]}$:

(0) Curves which are fibers of the exceptional divisor $\Delta \to \Delta_S^{[2]}$. As in Section 4.4, we denote these curves by $\ell_s^\delta$, where $s = \nu(\ell_s^\delta)$ and we denote their classes by $\ell^\delta$.

(1) Curves which are strict transforms of $C_s$ with $s \notin C$. We denote the class of these curves by $C_s$. Note that $C = \nu^*(\lfloor C_s \rfloor)$ for $s \notin C$.

(2a) Curves contained in $H_2$.

(2b) Curves contained in $C^{(2)}$.

Proof. Let $\gamma$ be an irreducible curve in $S^{[2]}$. By Lemma 5.4 there are three cases for $\nu(\gamma)$:

(0) $\nu(\gamma)$ is a point and $\gamma$ is a fiber of the exceptional divisor;

(1) $\nu(\gamma) = C_s$, with $s \notin C$.

(2) $\nu(\gamma) \subset C^{(2)}$.

Moreover case (2) can be divided in 2 sub-cases: $\nu(\gamma) = \Delta_C$ or $\nu(\gamma) \notin \Delta_C$. This provides cases (2a) and (2b). □

Now, we are going to determine the classes of the extremal curves in cases (2a) and (2b) in two lemmas.

Lemma 5.6. We have $j^H_2(C_0) = 2(C - \ell^\delta)$ and $j^H_2(f) = \ell^\delta$. 

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Proof. It is clear that \( j^*_{\mathcal{H}^2} (f) = \ell \delta \), we are going to compute \( j^*_\mathcal{H}^2 (C_0) \). Necessarily, \( j^*_\mathcal{H}^2 (C_0) = aC + b\delta \). We can consider the intersection with \( \Delta \) and \( D_C \) and use Lemma 5.3 to determine \( a \) and \( b \). We consider the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{H}^2 & \xrightarrow{j^*_{\mathcal{H}^2}} & S \times S \\
\downarrow & & \downarrow \bar{\rho} \\
\mathcal{H}^2 & \xrightarrow{j^*_{\mathcal{H}^2}} & S[2].
\end{array}
\]

By commutativity of the diagram, we have:

\[ j^*_\mathcal{H}^2 (C_0) = \bar{\rho}^* j^*_{\mathcal{H}^2} (C_0). \quad (10) \]

By [Har77, Propositions 2.6 and 2.8], we have:

\[ C_0 \cdot j^*_{\mathcal{H}^2} (\Delta_{\mathcal{S}^2}) = C_0 \cdot \mathcal{O}_C (-1) = -C_0 \cdot (C_0 + 2f) = 4 - 2 = 2. \]

By projection formula that is:

\[ \bar{\rho}^* j^*_{\mathcal{H}^2} (C_0) \cdot \Delta_{\mathcal{S}^2} = 2. \]

Taking the push-forwards by \( \bar{\rho} \), we obtain by [Men18, Lemma 3.6] (see 3):

\[ \bar{\rho}^* j^*_{\mathcal{H}^2} (C_0) \cdot \bar{\rho}^* (\Delta_{\mathcal{S}^2}) = 4. \]

Hence by (10):

\[ j^*_\mathcal{H}^2 (C_0) \cdot \Delta = 4. \]

Hence by lemma 5.3

\[ b = -2. \]

We have \( D_C = \nu^* (\rho_*(C \times S)) \). So by projection formula:

\[ D_C \cdot j^*_\mathcal{H}^2 (C_0) = \rho_*(C \times S) \cdot [\Delta_{\mathcal{S}^2}] = \rho_*(C \times S) \cdot \rho_*(s \times C + C \times s) = 2 \rho_*(C \times S) \cdot \rho_*(s \times C). \]

Taking the pull-back by \( \nu \) of the last equality, we obtain:

\[ D_C \cdot j^*_\mathcal{H}^2 (C_0) = 2D_C \cdot C. \]

So \( a = 2 \) which concludes the proof.

Lemma 5.7. We have \( j^*_C (d) = C - \ell \delta \), where \( d \) is the class of a line in \( \mathcal{C}^{(2)} \simeq \mathbb{P}^2 \).

Proof. We consider the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{C}^{(2)} & \xrightarrow{j^*_C} & S[2] \\
\downarrow & & \downarrow \nu \\
\mathcal{C}^{(2)} & \xrightarrow{j^*_C} & S^{(2)}. \end{array}
\]

Let \( \gamma \) be an irreducible curve in \( \mathcal{C}^{(2)} \). Since \( \mathcal{C}^{(2)} \) is the strict transform of \( C^{(2)} \) by \( \nu \), \( j^*_C (\gamma) \) is the strict transform of \( j^*_\mathcal{C}^* (\gamma) \) by \( \nu \). Hence to compute \( j^*_C (d) \) for \( d \) the class of a line, it is enough to find a curve in \( C^{(2)} \) with class \( d \) and determine its strict transform by \( \nu \). For instance \( C^s \) with \( s \in C \) verifies \( [C^s] = d \) in \( C^{(2)} \). Moreover, \( C^s \) intersects \( \Delta_{\mathcal{S}^2} \) transversely in one point. It follows that \( j^*_C (d) = C - \ell \delta \). \( \square \)
Curves in \( N_1 \)

**Lemma 5.8.** We have the following cases for irreducible curves in \( N_1 \):

(00) Curves contracted to a point by \( r_1 \). They are fibers of the exceptional divisor \( \Sigma_1 \to \Sigma \) and their class is \( \ell_\Sigma \).

(0) Curves sent to \( \ell_\Sigma^2 \) by \( r_1 \). An extremal such a curve has class \( r_1^* (\ell_\Sigma^2) \) by Lemma 4.3.

(1) Curves sent to \( C^s \) by \( r_1 \) with \( s \notin C \). They are curves of class \( r_1^* (C) \).

(2a.i) Curves contained in \( r_1^{-1} (H_2 \cap \Sigma) \).

(2a.ii) Curves contained in \( \overline{H_2} \) the strict transform of \( H_2 \) by \( r_1 \).

(2b.i) Curves contained in \( r_1^{-1} (C^{(2)} \cap \Sigma) \).

(2b.ii) Curves contained in \( r_1^{-1} (C^{(2)}) \) the strict transform of \( C^{(2)} \) by \( r_1 \).

**Proof.** Let \( \gamma \) be an irreducible curve in \( N_1 \). If \( r_1 (\gamma) \) is a point, we are in case (00). If \( r_1 (\gamma) \) is a curve, we are in one of the cases of Lemma 5.3.

If \( r_1 (\gamma) = \ell_\Sigma^s \) for some \( s \in S \). This is cases (i) and (ii) of Section 4.1. It follows from Remark 4.3 that the extremal curves in case (0) have classes \( r_1^* (\ell_\Sigma) \). If \( r_1 (\gamma) = C^s \) with \( s \notin C \), then \( C^s \) does not intersects \( \Sigma \) and we have \( [\gamma] = r_1^* (C) \). The last four cases appear when \( r_1 (\gamma) \subset H_2 \) or \( r_1 (\gamma) \subset C^{(2)} \).

Now we are going to determine the classes of the curves in cases (2a.i), (2a.ii), (2b.i) and (2b.ii).

**Lemma 5.9.** The extremal curves in \( r_1^{-1} (H_2 \cap \Sigma) \) are of classes \( \ell_\Sigma \) or \( r_1^* (\ell_\Sigma) \).

**Proof.** Since \( C \) is the unique curve contained in \( S \). The involution \( \iota \) on \( S \) restricts to \( C \). Since \( \iota \) is a symplectic involution, \( \iota \) does not act trivially on \( C \). Moreover, since \( C \cong \mathbb{P}^1 \), \( \iota \) has two fixed points \( x \) and \( y \). It follows that \( \iota (C) \) also has two fixed points \( (x, x) \) and \( (y, y) \). Hence \( H_2 \cap \Sigma = \ell_\Sigma^x \cup \ell_\Sigma^y \). The surfaces \( r_1^{-1} (\ell_\Sigma^x) \) and \( r_1^{-1} (\ell_\Sigma^y) \) are Hirzebruch surfaces and by Lemma 4.2 they are isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \). Then the extremal curves of these Hirzebruch surfaces will have classes \( \ell_\Sigma \) or \( r_1^* (\ell_\Sigma) \) in \( N_1 \).

**Lemma 5.10.** Let \( C_0 \) be the class of the section in \( \overline{H_2} \) obtained in Lemma 5.3. Then \( j_{\overline{H_2}} (C_0) = 2 (r_1^* (C) - r_1^* (\ell_\Sigma) - \ell_\Sigma) \).

**Proof.** As explained in the proof of the previous lemma, \( H_2 \cap \Sigma \) corresponds to two fibers of the Hirzebruch surface \( H_2 \). Hence \( j_{\overline{H_2}} (C_0) \) and \( \Sigma \) intersect in two points. The class \( j_{\overline{H_2}} (C_0) \) corresponds to the class of the strict transform by \( r_1 \) of \( j_{\overline{H_2}} (C_0) \). By Lemma 5.4 \( [j_{\overline{H_2}} (C_0)] = 2 (C - \ell_\Sigma) \). Hence \( j_{\overline{H_2}} (C_0) = 2 (r_1^* (C) - r_1^* (\ell_\Sigma) - \ell_\Sigma) \).

**Lemma 5.11.** The curve \( \Delta_{S^{(2)}} \) is a line in \( C^{(2)} \cong \mathbb{P}^2 \).

**Proof.** The map

\[
\begin{array}{ccc}
\mathbb{P}^2 & \xrightarrow{2:1} & \mathbb{P}^1 \\
\mathbb{P}^2 & \xrightarrow{\rho} & \mathbb{P}^1 \\
\end{array}
\]

is a two to one ramified cover. We recall that \( \Delta_{S^{(2)}} = \{ (x, \iota (x)) \mid x \in C \} \). We have:

\[
[\Delta_{S^{(2)}}]_{S^{(2)}} = [C \times s]_{S^{(2)}} + [s \times C]_{S^{(2)}}.
\]

It follows:

\[
\rho_* \left( [\Delta_{S^{(2)}}]_{S^{(2)}} \right) = 2 [C^s]_{S^{(2)}}.
\]

However \( \rho : \Delta_{S^{(2)}} \to \Delta_{S^{(2)}} \) is a two to one cover; so \( [\Delta_{S^{(2)}}]_{S^{(2)}} = [C^s]_{S^{(2)}} \). Hence \( [\Delta_{S^{(2)}}]_{C^{(2)}} = [C^s]_{C^{(2)}} \) and \( [\Delta_{S^{(2)}}]_{C^{(2)}} \) is the class of a line in \( C^{(2)} \).
Lemma 5.12. The surface $r^{-1}_s(C^{(2)} \cap \Sigma)$ is a Hirzebruch surface that we denote by $H_1$. Let $f$ be a fiber of $H_1$. There exists a section $D_0$ which satisfies: $\text{Pic} H_1 = \mathbb{Z} D_0 \oplus \mathbb{Z} f$, $D_0^2 = -2$, $f^2 = 0$ and $D_0 \cdot f = 1$. Moreover, $j^{H_1}(D_0) = r^*_1(C) - \ell_\delta = \ell_\Sigma$, where $j^{H_1}: H_1 \hookrightarrow N_1$ is the embedding.

Proof. We denote by $\zeta$ the curve $C^{(2)} \cap \Sigma$. This curve is the strict transform of $\Delta^{S_{\Sigma}(2)}$ by $\nu$. In particular, it is a rational curve by Lemma 5.11. Moreover by Lemmas 5.11 and 5.4, we have:
\[
[\zeta]_{\Sigma} = C - \ell_\delta.
\] (11)

To understand which Hirzebruch surface $H_1$ is, we are going to compute $N_{S/\Sigma}|\zeta = \mathcal{O}_C(-k) \oplus \mathcal{O}_C(k)$. We consider $\zeta := \tilde{\rho}^{-1}(\zeta)$. We have:
\[
\rho^*(N_{S/\Sigma}|\zeta) = \tilde{\nu}^*(N_{S/\Sigma}|\Delta^{S_{\Sigma}(2)}).
\]

As in the proof of Lemma 5.2, we have:
\[
N_{S/\Sigma}|\Delta^{S_{\Sigma}(2)} \cong T_S|C \cong \mathcal{O}(\mathcal{C}(-2)) \oplus \mathcal{O}(\mathcal{C}(2)).
\]

Hence
\[
\rho^*(\mathcal{O}_C(-k) \oplus \mathcal{O}_C(k)) = \rho^*(N_{S/\Sigma}|\zeta) = \mathcal{O}_C(-2) \oplus \mathcal{O}_C(2).
\]

Since $\rho : \tilde{\zeta} \to \zeta$ is a two to one cover, we obtain $k = 1$, that is:
\[
N_{S/\Sigma}|\zeta = \mathcal{O}_C(-1) \oplus \mathcal{O}_C(1).
\]

By [Har77] Chapter V, Proposition 2.3 and Proposition 2.9, there exists a section $D_0$ such that $\text{Pic} H_1 = \mathbb{Z} D_0 \oplus \mathbb{Z} f$, with $D_0^2 = -2$, $f^2 = 0$ and $D_0 \cdot f = 1$; $f$ being the class of a fiber. Moreover, by (11) and using the projection formula, we know that:
\[
\ell_\Sigma = \downarrow(r^*_1(C) - \ell_\delta = a\ell_\Sigma).
\]

To compute $a$, we only need to compute $D_0 = j^{H_1}(\Sigma)$. We apply the same method used in the proof of Lemma 5.6. By [Har77] Propositions 2.6 and 2.8, we have:
\[
D_0 = j^{H_1}(\Sigma) = D_0 \cdot \mathcal{O}(\mathcal{C}(-1) \oplus \mathcal{O}_C(1))(-1) = -D_0 \cdot (D_0 + f) = 2 - 1 = 1.
\]

This proves that $a = -1$.  

\[\square\]

Lemma 5.13. Let $d$ be the class of a line in $C^{(2)}$. Then $j^{C^{(s)}}|d = r^*_1(C) - r^*_1(\ell_\delta) = -\ell_\Sigma$.

Proof. For instance $j^{C^{(s)}}|d$ corresponds to the class of the strict transform of $C^s$ by $r_1 \circ \nu$ for $s \in C$. Let $C^s$ be the strict transform of $C^*$ by $\nu$. As we have already seen in Lemma 5.7 $[C^s]_{\Sigma} = j^C_\delta(d) = C - \ell_\delta$. The intersection $C^{(2)} \cap \Sigma$ corresponds to the strict transform of $\Delta^{S_{\Sigma}(2)}$ by $\nu$ and by Lemma 5.11 it has the class of a line in $C^{(2)}$. Hence $C^s$ intersects $\Sigma$ transversely in one point and we obtain: $j^{C^{(s)}}|d = r^*_1(C - \ell_\delta = \ell_\Sigma$.

\[\square\]

Conclusion on wall divisors

Lemma 5.14. The extremal curves of $M'$ have classes $\pi_1 r^*_1 \ell_\delta$, $\pi_1 \ell_\Sigma$ and $\pi_1 (r^*_1(C) - r^*_1(\ell_\delta) = \ell_\Sigma$.

Proof. Our previous investigation on curves in $N_1$ show that the extremal curves in $N_1$ have classes $r^*_1 \ell_\delta$, $\ell_\Sigma$ and $r^*_1(C) - r^*_1(\ell_\delta) = \ell_\Sigma$. This implies that the extremal curves in $M'$ have classes $\pi_1 r^*_1 \ell_\delta$, $\pi_1 \ell_\Sigma$ and $\pi_1 (r^*_1(C) - r^*_1(\ell_\delta) = \ell_\Sigma$.

We can compute their dual divisors to obtain wall divisors with Proposition 2.18.

Proposition 5.15. The divisors $\delta'$, $\Sigma'$, $D'_C$ and $D'_C - \frac{1}{2}(\delta' + \Sigma')$ are wall divisors in $M'$. Moreover, they verify the following numerical properties:

- $q_{M'}(\delta') = q_{M'}(\Sigma') = q_{M'}(D'_C) = -4$ and $\text{div}(\delta') = \text{div}(\Sigma') = \text{div}(D'_C) = 2$;
We obtain that
\[ Z \quad \text{and} \quad D \]
\[ \text{Proof.} \quad \text{By Lemma 3.3,} \quad \frac{1}{2} \delta' + \Sigma' \in H^2(M', \mathbb{Q}) \] are the duals of \( \pi_1(\ell_k) \) and \( \pi_1(\ell_{\Sigma}) \) respectively. Moreover by Lemma 5.3, we know that \( D_C \cdot C = -2 \), hence by (1):
\[ D_C \cdot \pi_1(\ell_k(C)) = -4. \]

Moreover, since \( D_C = \delta(C) \) where \( \delta \) is the isometric embedding \( H^2(S, \mathbb{Z}) \hookrightarrow H^2(S^{[2]}, \mathbb{Z}) \), we have: \( q_{S^{[2]}}(D_C) = -2 \). So by Theorem 3.6 (ii):
\[ q_M(D_C') = -4. \]

We obtain that \( D_C' \) is the dual of \( \pi_1(\ell_k(C)) \). Then \( D_C - \frac{1}{2} (\delta' + \Sigma') \) is the dual of \( \pi_1(\ell_k(C) - \ell_k(C)) \).

By Proposition 2.13 this proves that \( \delta', \Sigma' \) and \( D_C - \frac{1}{2} (\delta' + \Sigma') \) are wall divisors in \( M' \). Their claimed numerical properties are given by Theorem 3.6 (ii) and Remark 3.8.

It remains to show that \( D_C' \) is a wall divisor. By Proposition 2.21 since \( D_C' \) is a uniruled divisor, we have \( (D_C', \alpha)_{\mathbb{Q}_{M'}} \geq 0 \) for all \( \alpha \in \mathbb{Q}_{M'} \). Since \( \mathbb{Q}_{M'} \) is open, it follows that \( (D_C', \alpha)_{\mathbb{Q}_{M'}} > 0 \) for all \( \alpha \in \mathbb{Q}_{M'} \). Now, we assume that there exists \( g \in \text{Mon}^{-1}_{\text{div}}(M') \) and \( \alpha \in \mathbb{Q}_{M'} \) such that \( (g(D_C'), \alpha)_{\mathbb{Q}_{M'}} = 0 \) and we will find a contradiction. Since \( g \in \text{Mon}^{-1}_{\text{div}}(M') \) and \( \text{Pic}(M') = \mathbb{Z} D_C' \oplus \mathbb{Z} \Sigma' - \Sigma \), there are only 6 possibilities:
\[ g(D_C') = \begin{cases} 
\{ \pm D_C', \pm \delta', \pm \Sigma' \} & \text{or} \\
\{ \pm D_C', \pm \delta' \} & \text{or} \\
\{ \pm D_C', \pm \Sigma' \} & \text{or} \\
\{ \pm \delta', \pm \Sigma' \} & \text{or} \\
\{ \pm D_C', \pm \delta' \} & \text{or} \\
\{ \pm D_C', \pm \Sigma' \} & \text{or} \\
\{ \pm \delta', \pm \Sigma' \} & \text{or} \\
\{ \pm D_C', \pm \delta' \} & \text{or} \\
\{ \pm D_C', \pm \Sigma' \} & \text{or} \end{cases} \]

Since \( (D_C', \alpha)_{\mathbb{Q}_{M'}} \neq 0 \), \( (\delta', \alpha)_{\mathbb{Q}_{M'}} \neq 0 \) and \( (\Sigma', \alpha)_{\mathbb{Q}_{M'}} \neq 0 \). This leads to a contradiction. \( \square \)

5.2 When the K3 surface \( S \) used to construct the Nikulin orbifold contains two rational curves swapped by the involution

Framework

Let \( \Lambda_{K3} := U^3 \oplus \mathbb{Z} \) be the K3 lattice. We fix for all this section three embeddings in \( \Lambda_{K3} \) of three lattices \( U \cong U^3, E_8(-1) \) and \( E_{16} \) such that \( \Lambda_{K3} \cong U \oplus E_8(1) \). We consider \( i \) the involution on \( \Lambda_{K3} \) which exchanges \( E_8 \) and \( E_{16} \) and fixes the lattice \( U \). We consider \( C \in E_8 \) such that \( C^2 = -2 \). We define \( E_8^o := \{ e - i(e) \mid e \in E_8 \} \cong E_8(-2) \). By the surjectivity of the period map (see for instance Theorem 2.22), we can choose a K3 surface \( S \) such that

\[ \text{Pic} S = C \oplus E_8^o. \]

Then \( \text{Pic} S \) contains only two rational curves: one of class \( C \) and the other of class \( i(C) = C - (C - i(C)) \). It follows from Example 2.10 and Corollary 2.22 that there exists \( \alpha \in \mathbb{K}_S \) invariant under the action of \( i \). Hence by Theorem 2.22 (ii), the involution \( i \) extends to an involution \( \iota \) on \( S \) such that \( \iota^* = i \). Of course, we can refer to older results on K3 surfaces to construct \( \iota \), however we though simplest for the reader to refer to results stated in this paper.

As in Section 5.1, the objective is to determine wall divisors of the Nikulin orbifold \( M' \) obtained from the couple \( (S^{[2]}, \iota^{[2]}) \) (see Section 3.1).

Notation and strategy

We keep the same notation and the same strategy used in Section 5.1. We also still use the notation from Section 4.1. In particular, we denote by \( C \) and \( \iota(C) \) the two curves in \( S \).

Curves in \( S^{[2]} \)

First, we determine the curves in \( S^{[2]} \):

**Lemma 5.16.** There are 5 cases for irreducible curves in \( S^{[2]} \):
We have the following four cases for irreducible curves in Lemma 5.17.

Proof. Same proof as Lemma 5.4.

It follows four cases in $S^{[2]}$.

Lemma 5.17. We have the following four cases for irreducible curves in $S^{[2]}$:

(0) Curves which are fibers of the exceptional divisor $\Delta \to \Delta_{S^{[2]}}$. As in Section 4.4, we denote these curves by $H^s_\Delta$, where $s = \nu(H^s_\Delta)$ and we denote their classes by $\ell^s_\Delta$.

(1) Curves which are strict transforms of $C^\ast$ with $s \notin C \cup \iota(C)$. We denote the class of these curves by $C$. Note that $C = \nu^*([C^\ast])$ for $s \notin C$.

(2) Curves which are strict transforms of $\iota(C^\ast)$ with $s \notin C \cup \iota(C)$. The class of these curves is $\iota^*(C)$.

(3a) Curves contained in $H_2$.

(3b) Curves contained in $C^{[2]}$.

(4a) Curves contained in $\iota(H_2)$.

(4b) Curves contained in $\iota(C^{[2]})$.

(5) Curves in $\iota(C) \times C$.

Proof. The proof is similar as the one of Lemma 5.4. We only remark in addition that $\nu^{-1}(\iota(C) \times C) \simeq \iota(C) \times C$ because $C$ and $\iota(C)$ do not intersect; hence $\iota(C) \times C$ does not intersect $\Delta_{S^{[2]}}$. So by an abuse of notation, we still denote $\nu^{-1}(\iota(C) \times C)$ by $\iota(C) \times C$.

Curves in $N_1$

Lemma 5.18. We have the following cases for irreducible curves in $N_1$:

(00) Curves contracted to a point by $r_1$. They are fibers of the exceptional divisor $\Sigma_1 \to \Sigma$ and their class is $\ell^s_\Sigma$.

(0) Curves send to $\ell^s_\Delta$ by $r_1$. An extremal such a curve has class $r_1^*(\ell^s_\Delta)$ by Lemma 4.2.

(1) Curves send to $C^\ast$ by $r_1$ with $s \notin C \cup \iota(C)$. They are curves of class $r_1^*(C)$.

(2) Curves send to $\iota(C^\ast)$ by $r_1$ with $s \notin C \cup \iota(C)$. They are curves of class $r_1^*(\iota^*(C))$.

(3a) Curves contained in $H^s_{\Sigma}$ the strict transform of $H_2$ by $r_1$.

(3b) Curves contained in $C^{[2]}$ the strict transform of $C^{[2]}$ by $r_1$.

(4a) Curves contained in $\iota(H^s_{\Sigma})$ the strict transform of $\iota(H_2)$ by $r_1$.

(4b) Curves contained in $\iota(C^{[2]})$ the strict transform of $\iota(C^{[2]})$ by $r_1$.

(5a) Curves contained in $\iota(C) \times C$ the strict transform of $\iota(C) \times C$ by $r_1$.

(5b) Curves contained in $r_1^{-1}(\iota(C) \times C \cap \Sigma)$. 

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Proof. The proof is similar to the proof of Lemma 5.8; the difference is that $H_2$, $\iota(H_2)$, $C^{(2)}$ and $\iota(C^{(2)})$ do not intersect $\Sigma$. Only $\iota(C) \times C$ intersects $\Sigma$. 

Now, we are going to determine the classes of all these curves.

Lemma 5.19. (3a) The extremal curves of $H_2$ have classes $2(r_1^*(C) - r_1^*(\ell_3))$ and $r_1^*(\ell_3)$ in $N_1$.

(3b) The extremal curves of $C^{(2)}$ have class $r_1^*(C) - r_1^*(\ell_3)$ in $N_1$.

(4a) The extremal curves of $C^{(3)}_1$ have classes $2(\iota^*(C)) - r_1^*(\ell_3)$ and $r_1^*(\ell_3)$ in $N_1$.

(4b) The extremal curves of $C^{(3)}_2$ have class $\iota^*(r_1^*(C)) - r_1^*(\ell_3)$ in $N_1$.

(5a) The extremal curves of $\iota(C) \times C$ have class $r_1^*(C) - \ell_\Sigma$ and $r_1^*(\iota^*(C)) - \ell_\Sigma$ in $N_1$.

Proof. Since $H_2$ and $\iota(H_2)$ do not intersect $\Sigma$, (3a) and (4a) are consequences of Lemma 3.6.

• Similarly, since $C^{(2)}$ and $\iota(C^{(2)})$ do not intersect $\Sigma$, (3b) and (4b) are consequences of Lemma 3.6.

• We have $\iota(C) \times C \simeq C \times C \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Let $j^\prime : \iota(C) \times C \hookrightarrow N_1$ be the embedding in $N_1$. We want to compute the classes $j^\prime_!(\{x\} \times \mathbb{P}^1)$ and $j^\prime_!(\mathbb{P}^1 \times \{x\})$, where $\{x\}$ is just the class of a point in $\mathbb{P}^1$. This corresponds to compute the strict transform by $r_1$ of $C^s$ with $s \in \iota(C)$ and the strict transform of $C^s$ with $s \in C$. Since $C^s$ and $\iota(C)$ intersect $\Sigma$ in one point, we obtain our result.

Lemma 5.20. The surface $r_1^{-1}(\iota(C) \times C \cap \Sigma)$ is a Hirzebruch surface that we denote by $H_2'$. Let $f$ be a fiber of $H_2'$. There exists a section $C'_0$ which satisfies: $\text{Pic} H_1 = \mathbb{Z}C'_0 \oplus \mathbb{Z}f$, $C'_0 \cdot f = -4$, $f^2 = 0$ and $C'_0 \cdot f = 1$. Moreover, $j_*(H_2'(C'_0)) = r_1^!(C) + r_1^!(\iota^*(C)) - 2\ell_\Sigma$, where $j_! : H_2' \hookrightarrow N_1$ is the embedding.

Proof. We denote by $\zeta$ the curve $\iota(C) \times C \cap \Sigma$. This curve is the strict transform of $\Delta_{S^{(2)}}^C$ by $\nu$. Since $\Delta_{S^{(2)}}^C$ does not intersect $\Delta$, its class in $S^{(2)}$ is:

$$[\zeta] = C + \iota^*(C).$$

(12)

To understand which Hirzebruch surface $H_2'$ is, we are going to compute $N_{\Sigma/S^{(2)}}[\zeta]$. Let $\Delta_{S^{(2)}}^C$ be the strict transform of $\Delta_{S^{(2)}}^C$ by $\nu$. We have $\zeta \simeq \Delta_{S^{(2)}}^C \simeq \Delta_{S^{(2)}}^C$.

Hence:

$$N_{\Sigma/S^{(2)}}[\zeta] = \rho^*(N_{\Sigma/S^{(2)}})[\Delta_{S^{(2)}}^C] = \overline{\nu}^*(N_{S_2/S^2}[\Delta_{S^{(2)}}^C]).$$

As in the proof of Lemma 5.2, we have:

$$N_{S_2/S^2}[\Delta_{S^{(2)}}^C] \simeq T_{S_2}C \simeq O_C(-2) \oplus O_C(2).$$

Hence

$$N_{\Sigma/S^2}[\zeta] \simeq O_C(-2) \oplus O_C(2).$$

By [Har77] Chapter V, Proposition 2.3 and Proposition 2.9, we know that there exists a section $C'_0$ of $H_2'$ such that $\text{Pic} H_1 = \mathbb{Z}C'_0 \oplus \mathbb{Z}f$, with $C'_0 \cdot f = -4$, $f^2 = 0$ and $C'_0 \cdot f = 1$; $f$ being the class of a fiber.

By [L2] and using the projection formula, we know that:

$$j_*(H_2'(C'_0)) = r_1^!(C) + r_1^!(\iota^*(C)) + a\ell_\Sigma.$$

To compute $a$, we only need to compute $C'_0 \cdot j_*(\iota(C))$. We apply the same method used in the proof of Lemma 5.8. By [Har77] Propositions 2.6 and 2.8, we have:

$$C'_0 \cdot j_*(\iota(C)) = C'_0 \cdot O_{\mathbb{P}(\mathbb{C}(-2) \oplus O_{\mathbb{C}(2))}}(-1) = C'_0 \cdot (C'_0 + 2f) = 4 - 2 = 2.$$

This proves that $a = -2$. 

\[\square\]
Conclusion on wall divisors

**Lemma 5.21.** The extremal curves of $M'$ have classes $\pi_1.r_1(\ell_3, \pi_1.\ell_2, \pi_1.(r_1(C) - r_1(\ell_3))$ and $\pi_1.(r_1(C) - \ell_2)$.

**Proof.** It is obtain by taking the image by $\pi_1$ of the classes of the extremal curves in $N_1$. □

We can compute their dual divisors to obtain wall divisors with Proposition 2.18.

**Proposition 5.22.** The divisors $\delta', \Sigma', D'_C, 2D'_C - \delta'$ and $2D'_C - \Sigma'$ are wall divisors in $M'$.

Moreover, they verify the following numerical properties:

- $q_{M'}(\delta') = q_{M'}(\Sigma') = -4$ and $\text{div}(\delta') = \text{div}(\Sigma') = 2$;
- $q_{M'}(D'_C) = -2$ and $\text{div}(D'_C) = 1$;
- $q_{M'}(2D'_C - \delta') = q_{M'}(2D'_C - \Sigma') = -12$ and $\text{div}[2D'_C - \delta'] = \text{div}[2D'_C - \Sigma'] = 2$.

**Proof.** By Lemma 4.3, $\frac{1}{2}\delta'$ and $\frac{1}{2}\Sigma' \in H^2(M', \mathbb{Q})$ are the duals of $\pi_1.(r_1(\ell_3))$ and $\pi_1.(\ell_3)$ respectively. Moreover by Lemma 5.3, we know that $D_C \cdot C = -2$, hence by (4):

$$\pi_1.(r_1(D_C + \iota^*(D_C))) \cdot \pi_1.(r_1(C + \iota^*(C))) = -8.$$

So

$$D'_C \cdot \pi_1.(r_1(C)) = -2.$$

Moreover, since $D_C = j(C)$ where $j$ is the isometric embedding $H^2(S, \mathbb{Z}) \hookrightarrow H^2(S[2], \mathbb{Z})$, we have: $q_{S[2]}(D_C) = -2$. So by Theorem 3.6 (ii): $q_{M'}(\pi_1.(r_1(D_C + \iota^*(D_C)))) = -8$.

Hence:

$$q_{M'}(D'_C) = -2.$$ (13)

We obtain that $D'_C$ is the dual of $\pi_1.(r_1(C))$. Then $D'_C - \frac{1}{2}\delta'$ is the dual of $\pi_1.(r_1(C) - r_1(\ell_3))$ and $D'_C - \frac{1}{2}\Sigma'$ is the dual of $\pi_1.(r_1(C) - \ell_3)$.

By Proposition 2.18 this proves that $\delta', \Sigma', 2D'_C - \delta'$ and $2D'_C - \Sigma'$ are wall divisors in $M'$. Their claimed numerical properties are given by Theorem 3.6 (iii), Remark 5.8 and (13).

It remains to prove that $D'_C$ is a wall divisor. The proof is very similar to the one of Proposition 5.13. For the same reason, we have $(D_C, \alpha)_{\text{Hdg}} > 0$ for all $\alpha \in \text{Pic}(M')$. Now, we assume that there exists $g \in \text{Mon}^2_{\text{Hdg}}(M')$ and $\alpha \in \text{Pic}(M')$ such that $(g(D_C), \alpha)_{\text{Hdg}} = 0$ and we will find a contradiction. Since $C \in E_1$, we have $\text{div}(D'_C) = 1$. Since $g \in \text{Mon}^2_{\text{Hdg}}(M')$ and $\text{Pic}(M') = Z D'_C \oplus Z \frac{\delta + \Sigma}{2} \oplus Z \frac{\delta - \Sigma}{2}$, it follows that there are only 2 possibilities:

$$g(D'_C) = \pm D'_C,$$

because $\text{div}(D'_C) = 1$ and $\text{div}(\frac{\delta + \Sigma}{2}) = 2$. Since $(D'_C, \alpha)_{\text{Hdg}} \neq 0$, this leads to a contradiction. □

**Remark 5.23.** Note that in this case, $D'_C - \frac{\delta + \Sigma}{2}$ is not a wall divisor. Indeed, by Lemma 5.21 the class $-2D'_C - \delta' - \Sigma'$ is the projection on Pic $M'$ of a Kähler class. However, observe that

$$\left(D'_C - \frac{\delta + \Sigma}{2}, -2D'_C - \delta' - \Sigma' \right)_q = 0.$$

### 5.3 Wall divisors on a Nikulin orbifold constructed from a specific elliptic K3 surface

As before, we consider the K3 lattice $\Lambda_{K3} := U^3 \oplus E_8(-1) \oplus E_8(-1)$ with the three embedded lattices $U \simeq U^3$, $E_1 \simeq E_8(-1)$ and $E_2 \simeq E_8(-1)$ such that $\Lambda_{K3} \simeq U \oplus E_1 \oplus E_2$. The involution $i$ on $\Lambda_{K3}$ is still the involution which exchanges $E_1$ and $E_2$ and fixes the lattice $U$. As before, we keep $E_0' := \{ e - i(e) | e \in E_1 \} \simeq E_8(-2)$. For simplicity, we denote $E_8(-2) := \{ e + i(e) | e \in E_1 \}$ which is the invariant lattice. Let $L_1 \in U$ such that $L_1^2 = 2$ and $e_2^{(0)} \in E_1$ an element with...
$$(e_2^0)^2 = -4.$$ Using the surjectivity of the period map (see for instance Theorem 2.6), we choose a K3 surface $S$ such that:

$$\text{Pic } S = \mathbb{Z}(L_1 + e_2^0) \oplus E^a.$$ Note that the direct sum is not orthogonal. We denote:

$$v_{K3} := 2L_1 + e_2,$$

with $e_2 := e_2^0 + i(e_2^0)$. We have $v_{K3}^2 = 0$ and $\text{Pic } S \supset \mathbb{Z}v_{K3} \oplus E^a$.

As before, it follows from Example 2.10 and Corollary 2.20 that there exists $\alpha \in K_S$ invariant under the action of $i$. Hence by Theorem 2.8 (ii), the involution $i$ extends to an involution $\iota$ on $S$ such that $\iota^* = i$. We consider $M'$ constructed from the couple $(S, \iota)$.

In contrary to the two previous sections, we will not need to find all the extremal curves in this case. The wall divisors will be deduced from the investigation of this section and the numerical properties obtained in Section 7.

The K3 surface $S$ contains a $(2)$-curve of class $L_1 + e_2^0$. We denote this curve $\gamma$. The class of $\iota(\gamma)$ is $L_1 + i(e_2^0)$. Hence $\gamma \cup \iota(\gamma)$ has class $v_{K3}$ and provides a fiber of the elliptic fibration $f : S \to \mathbb{P}^1$. Moreover, we have:

$$[\gamma] : [\iota(\gamma)] = (L_1 + e_2^0) \cdot (L_1 + i(e_2^0)) = 2.$$ We denote by $\overline{\gamma}$ the class $\nu^*(\gamma^s)$, with $\gamma^s := \gamma \times \{s\}$. We also denote $D_{\gamma} := j(\gamma)$ and $D_{\gamma}' := \pi_1^*(r_{\gamma}^*(D_{\gamma}))$.

We consider the following divisor in $S^{(2)}$:

$$A := \{ (x, y) \in S^{(2)} \mid f(x) = f(y) \},$$

with $f$ the elliptic fibration $S \to \mathbb{P}$. We denote by $A'$ the image by $\pi_1$ of the strict transform of $A$ by $r_1 \circ \nu$.

**Lemma 5.24.** We have:

(i) the class of the strict transform of $\gamma^s$ by $r_1 \circ \nu$ is $r_1^*(\overline{\gamma} - \ell_k) - \ell_{\Sigma}$, for $s \in \gamma \cap \iota(\gamma)$.

(ii) The dual of $\pi_1^*(r_{\gamma}^*(\overline{\gamma}))$ is $D_{\gamma}'$.

(iii) The divisor $D_{\gamma}'$ has square $0$ and divisibility $1$.

(iv) The divisor $A'$ has class $D_{\gamma}' - E^a + \Sigma'$.

**Proof.** (i) The statement follows directly from the fact that $\gamma^s$ intersects $\Delta_{S^{(2)}}$ and $\Sigma$ in one point.

(ii) Let $w \in E_S(-2)$ be an invariant element under the action of $i$ such that $\gamma \cdot w = k$. Then

$$\overline{\gamma} \cdot j(w) = (D_{\gamma}, j(w))_{q_{S^{(2)}}} = k. \quad (14)$$

We set $w' := \pi_1^*(r_{\gamma}^*(j(w)))$. It follows by (11) that $\pi_1^*(r_{\gamma}^*(\overline{\gamma} + i^{[2]}(\overline{\gamma}))) \cdot w' = 4k$. Hence

$$\pi_1^*(r_{\gamma}^*(\overline{\gamma})) \cdot w' = 2k. \quad (15)$$

Moreover by Theorem 3.3 (ii), Remark 3.5 and (11), we have:

$$(\pi_1^*(D_{\gamma} + i^{[2]}(D_{\gamma})), \pi_1^*(r_{\gamma}^*(j(w))))_{q_{S^{(2)}}} = 4k.$$ Hence:

$$(D_{\gamma}', w')_{q_{S^{(2)}}} = 2k. \quad (16)$$

We obtain that $D_{\gamma}'$ is the dual of $\pi_1^*(r_{\gamma}^*(\overline{\gamma}))$. 

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(iii) We have by Theorem 3.6 (ii) and Remark 3.5

\[ q_{M'}(\pi_1^*, r_1^*(D_\gamma + i|2|^* (D_\gamma))) = 2q_{S|\Sigma}(D_\gamma + i|2|^*(D_\gamma)) = 0. \]

Hence \( q_{M'}(\pi_1^*, r_1^*(D_\gamma)) = 0 \). To prove that \( \text{div}(D'_\gamma) = 1 \), we choose a specific \( w : w = w^{(0)} + i(w^{(0)}) \) such that \( w^{(0)} \in E_1 \) and \( w^{(0)} \cdot c_2^{(0)} = 1 \); then \( w \cdot \gamma = 1 \). Then by (16) we have \( (D'_{\gamma}, w')_{M'} = 2 \). However \( w' \) is divisible by 2. We obtain our result.

(iv) Since \( A \) is invariant under the action of \( e \) and considering the intersection with \( \rho_+(w \times \{pt\}) \), we see that the class of \( A \) is given by \( \rho_+([S \times v_{K3}]) \). Then the strict transform \( \tilde{A} \) by \( r_1 \circ \nu \) has class \( r_1^*(j(v_{K3}) - \delta) - \Sigma_1 \) because \( A \) contains the surfaces \( \Delta_{S(Z)} \) and \( \Sigma \). We recall that \( \pi_1^*(j(v_{K3})) = 2D'_\gamma \). Therefore and since \( A \to A' \) is a double cover, \( A' \) has class \( h(2D'_\gamma - \delta' - \Sigma'_1) \).

\[ \square \]

**Lemma 5.25.** The reflexion \( R_{A'} \) is a monodromy operator and \( R_{A'}(\Sigma') = 2D'_\gamma - \delta' \).

**Proof.** The reflexion \( R_{A'} \) is a monodromy operator by [LMP22] Theorem 3.10,

\[ R_{A'}(\Sigma') = \Sigma' - \frac{2(\Sigma', A')}{q(A')}A' = \Sigma' + 2A' = 2D'_\gamma - \delta'. \]

\[ \square \]

**Lemma 5.26.** There exists an extremal curve of \( M' \) that is written \( a\pi_1^*, r_1^*(\gamma) + b\pi_1^*, r_1^*(\delta_3) + c\pi_1^*(\ell_2) \) with \( (a, b, c) \in \mathbb{Z}^3 \) and \( a > 0 \).

**Proof.** The curves \( \pi_1^*, r_1^*(\gamma) \), \( \pi_1^*, r_1^*(\delta_3) \) and \( \pi_1^*, (\ell_2) \) are primitive in \( H^{3,3}(M', \mathbb{Z}) \). Indeed, we have seen in the proof of Lemma 5.24 that we can choose \( k = 1 \) in equation (15). Moreover \( w' \) is divisible by 2. We obtain: \( \pi_1^*, r_1^*(\gamma) - \frac{1}{2}w' = 1 \). Similarly, by Lemma 1.4 we know that \( \pi_1^*, r_1^*(\delta_3) \) and \( \pi_1^*, (\ell_2) \) are primitive by considering the intersection with \( \Delta_{S(Z)} \). Therefore, the class of a curve in \( M' \) will be written \( a\pi_1^*, r_1^*(\gamma) + b\pi_1^*, r_1^*(\delta_3) + c\pi_1^*, (\ell_2) \), with \( (a, b, c) \in \mathbb{Z}^3 \). If we consider the pull-back by \( \pi_1 \) and the push-forward by \( r_1 \circ \nu \) of the class \( a\pi_1^*, r_1^*(\gamma) + b\pi_1^*, r_1^*(\delta_3) + c\pi_1^*, (\ell_2) \), we obtain \( 2a\gamma \). For a curve, there are two possibilities \( a = 0 \) or \( a > 0 \); it is not possible to have \( a = 0 \) for every curves, hence there exists an extremal curve as mentioned in the statement of the lemma.

\[ \square \]

**Remark 5.27.** Let \( a\pi_1^*, r_1^*(\gamma) + b\pi_1^*, r_1^*(\delta_3) + c\pi_1^*, (\ell_2) \) be the class of the extremal curve obtained from Lemma 5.24. By Proposition 2.15 the dual of this curve class is a wall divisor. According to Lemmas 5.24 and 1.4 that is \( aD'_\gamma + b\delta' + c\Sigma' \).

**Lemma 5.28.** Let \( E = aD'_\gamma + \frac{b}{2}\delta' + \frac{c}{2}\Sigma' \) be the previous wall divisor obtained from Remark 5.27 eventually renormalized such that \( E \) is primitive in \( H^2(M', \mathbb{Z}) \). Moreover, we assume that \( E \) verifies one of the numerical conditions listed in the statement of Theorem 1.2. Then:

\[ E = D'_\gamma - \frac{\delta' + \Sigma'}{2}. \]

**Proof.** We have:

\[ q_{M'}(E) = -(b^2 + c^2). \]

Considering the numerical conditions of Theorem 1.2 there are three possibilities:

(i) \( b = \pm 1 \) and \( c = \pm 1 \) or

(ii) \( b = \pm 2 \) and \( c = 0 \) or

(iii) \( b = 0 \) and \( c = \pm 2 \).
In the possibilities (ii) and (iii) \( q_M(E) = -4 \). In this case, following the conditions in Theorem 1.2 we know that \( E \) has divisibility 2. Hence by Lemma 5.24 (iii), \( a \) is divisible by 2. This corresponds to the extremal rays of curves \( \alpha \pi_1 r_1^\ast(\gamma) \pm 2\pi_1 r_1^\ast(\ell) \) or \( a\pi_1 r_1^\ast(\gamma) \pm 2\pi_1(\ell) \). However, by Lemma 5.24 (i) these rays cannot be extremal.

Therefore, \( E = a\Delta + \frac{1}{2}\theta + \frac{1}{2}a\Sigma' \). Moreover, the extremal ray associated to \( E \) has class:
\[
\alpha\pi_1 r_1^\ast(\gamma) \pm \pi_1 r_1^\ast(\ell) \pm \pi_1(\ell).
\]
But, we know by Lemma 5.24 (i) that \( \pi_1 r_1^\ast(\gamma) - \pi_1 r_1^\ast(\ell) - \pi_1(\ell) \) is the class of a curve. Hence the only possibility for the previous extremal ray is \( \pi_1 r_1^\ast(\gamma) - \pi_1 r_1^\ast(\ell) - \pi_1(\ell) \).

\[ \square \]

6 Monodromy orbits

In this section, we study the orbit of classes in the lattice \( H^2(X, \mathbb{Z}) \) under the action of \( \text{Mon}^2(X) \) for an irreducible symplectic orbifold \( X \) of Nikulin-type. Note that since the property of being a wall divisor is deformation invariant (see Theorem 2.14) we may assume without loss of generality, that \( X \) is the Nikulin orbifold \( M' \) for a given K3 surface with symplectic involution.

The main result of this section is Theorem 6.15 which describes a set of representatives in each monodromy orbit. This will enable us to determine wall divisors for Nikulin-type in the next section by checking only the representatives.

For completeness, note that we did not determine the precise monodromy orbit of each element. Since, we only used a subgroup of the actual monodromy group, it could happen that more than one of the elements in Theorem 6.15 belong to the same orbit.

6.1 Equivalence of lattices

\textbf{Lemma 6.1.} For the questions on hand the consideration of the following two lattices are equivalent:
\[
\Lambda := \Lambda_{M'} = U(2)^3 \oplus E_8(-1) \oplus (-2) \oplus (-2)
\]
and
\[
\hat{\Lambda} := U^3 \oplus E_8(-2) \oplus (-1) \oplus (-1).
\]

More precisely, there is a natural correspondence between lattice automorphisms for both lattices, and a natural identification between the rays in both lattices.

This is a special case of the following:

\textbf{Lemma 6.2.} Let \( M \) and \( N \) be two unimodular lattices. Then \( L := M \oplus N(2) \) and \( \hat{L} := M(2) \oplus N \) satisfy the following properties: There exists a natural identification between lattice automorphisms for both lattices, and the rays in both lattices can be naturally identified.

\textbf{Proof.} Observe that by multiplying the quadratic form of the lattice \( L \) by 2, we obtain a lattice \( L(2) \cong M(2) \oplus N(4) \), which obviously satisfies that the rays and automorphisms are naturally identified for \( L \) and \( L(2) \).

Notice that \( N(4) \) can be identified with the sublattice of \( N \) consisting of elements of the form \( \{2n \mid n \in N\} \). Therefore, we can naturally include \( L(2) \subset \hat{L} \). This immediately implies the natural identification of rays in \( L \) and \( L(2) \).

For the identification of automorphisms, observe that any automorphism \( \hat{\varphi} \in \text{Aut}(\hat{L}) \) preserves the sublattice \( L(2) \): In fact \( L(2) \subset \hat{L} \) consists precisely of those (not necessarily primitive) elements whose divisibility is even, and this subset needs to be preserved by any automorphism. This yields a natural inclusion \( \text{Aut}(\hat{L}) \subset \text{Aut}(L(2)) \cong \text{Aut}(L) \).

The inverse inclusion is given by the same argument from considering \( \hat{L}(2) \subset L \).

\[ \square \]

Fix a K3 surface \( S \) with a symplectic involution \( \iota \) and consider the induced Nikulin orbifold \( M' \) associated to \( S \). Further, fix a marking \( \varphi_S : H^2(S, \mathbb{Z}) \rightarrow \Lambda_{K3} = U^3 \oplus E_8(-1)^2 \) of \( S \) such that \( \iota^\ast \) corresponds to swapping the two copies of \( E_8(-1) \). Then the fixed part of \( \iota^\ast \) is isomorphic to \( U^3 \oplus E_8(-2) \).
In this subsection, we consider the sublattice operator \( \hat{\Lambda} \).

6.2 Monodromy orbits in \( \hat{\Lambda} \)

Therefore, the sublattice \( U^3 \oplus \mathbb{E}_8(-2) \) in \( \hat{\Lambda} \) can naturally be identified with the fixed part of \( H^2(S, \mathbb{Z}) \), whereas the generators of square \(-1\) correspond to \( \delta, \Sigma \) for the corresponding elements \( \delta, \Sigma \in \hat{\Lambda} \).

With this notation, we can define the group of \( \text{Mon}^2(\hat{\Lambda}) \) of monodromy operators for the lattice \( \hat{\Lambda} \): An automorphism \( \varphi \in \text{Aut}(\hat{\Lambda}) \) is in \( \text{Mon}^2(\hat{\Lambda}) \) if the corresponding automorphism \( \varphi \in \text{Aut}(\Lambda) \) is identified with an element of \( \text{Mon}^2(X) \) via the marking \( \varphi_X \).

In the following we will frequently consider the sublattice

\[
\hat{\Lambda}_1 := U^3 \oplus \mathbb{E}_8(-2) \oplus (-2) \oplus (-2) \subset \hat{\Lambda},
\]

which replaces the \((-1) \oplus (-1)\)-part by the sublattice generated by \( \delta \) and \( \Sigma \).

Define

\[
\text{Mon}^2(\hat{\Lambda}_1) := \{ f \in \text{Aut}(\hat{\Lambda}_1) \mid \exists \hat{f} \in \text{Mon}^2(\hat{\Lambda}) : f = \hat{f}|_{\hat{\Lambda}_1} \}.
\]

Remark 6.3. Note that while there exists an identification \( \text{Mon}^2(X) = \text{Mon}^2(\hat{\Lambda}) \), there exists a natural inclusion \( \text{Mon}^2(\hat{\Lambda}_1) \subseteq \text{Mon}^2(\hat{\Lambda}) \) but a priori this is not an equality.

Note that Proposition 5.9 can be reformulated in terms of the lattice \( \hat{\Lambda}_1 \):

Corollary 6.4. Let \( f \in \text{Mon}^2(S^{[2]}) \) be a monodromy operator such that \( f \circ \iota^{[2]} \ast = \iota^{[2]} \ast \circ f \) on \( H^2(S^{[2]}, \mathbb{Z}) \). Let \( \hat{f} \in \text{Aut}(\hat{\Lambda}_1) \) be the automorphism defined via the following properties: Via the marking described above, \( \hat{f} \) restricted to \( U^3 \oplus \mathbb{E}_8(-2) \oplus (-2) \) coincides with the restriction of \( f \) to the invariant part of the lattice \( \langle i \rangle \). Then \( \hat{f} \in \text{Mon}(\hat{\Lambda}_1) \).

Proof. This is a straightforward verification: Proposition 5.9 gives the inherited monodromy operator \( f' \in \text{Mon}^2(\Lambda) \) and \( \hat{f} \in \text{Aut}(\hat{\Lambda}_1) \) is precisely the restriction to \( \Lambda_1 \) of the corresponding automorphism of \( \hat{\Lambda} \) obtained via Lemma 6.1 \( \Box \)

For the proof of Theorem 6.15 we will study monodromy orbits with respect to successively increasing lattices:

\[
\hat{\Lambda}_3 \subset \hat{\Lambda}_2 \subset \hat{\Lambda}_1,
\]

where \( \hat{\Lambda}_3 := U^3 \oplus (-2) \) with the generator \( \delta \) for the \((-2)\)-part, \( \hat{\Lambda}_2 := U^3 \oplus \mathbb{E}_8(-2) \oplus (-2) \), and \( \hat{\Lambda}_1 \) is as defined above.

Define the following monodromy groups for these lattices.

\[
\text{Mon}^2(\hat{\Lambda}_2) = \{ f \in \text{Aut}(\hat{\Lambda}_2) \mid \exists f_1 \in \text{Mon}^2(\hat{\Lambda}_1) : f = f_1|_{\hat{\Lambda}_2}, f_1|_{\hat{\Lambda}_2} = \text{id} \}
\]

\[
\text{Mon}^2(\hat{\Lambda}_3) = \{ f \in \text{Aut}(\hat{\Lambda}_3) \mid \exists f_1 \in \text{Mon}^2(\hat{\Lambda}_1) : f = f_1|_{\hat{\Lambda}_3}, f_1|_{\hat{\Lambda}_3} = \text{id} \}.
\]

Note that with this definition, there exist natural inclusions \( \text{Mon}^2(\hat{\Lambda}_3) \subseteq \text{Mon}^2(\hat{\Lambda}_2) \subseteq \text{Mon}^2(\hat{\Lambda}_1) \).

6.2 Monodromy orbits in \( \hat{\Lambda}_3 \)

In this subsection, we consider the sublattice \( \hat{\Lambda}_3 = U^3 \oplus (-2) \subset \hat{\Lambda}_1 \), where the generator of \((-2)\) is the class \( \delta \).

Notation 6.5. For the rest of this article, fix elements \( L_i \in U \subset \hat{\Lambda}_1 \) of square \( 2i \) for each \( i \in \mathbb{Z} \). E.g. one can choose the elements \( ie + f \), where \( e, f \) is a standard basis for which \( U \) has intersection matrix

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]
Lemma 6.6. The $\text{Mon}^2(\hat{\Lambda}_3)$-orbit of a primitive element in $U^3 \oplus (-2)$ is uniquely determined by its square and its divisibility.

More precisely, we prove the following: Let $v \in \hat{\Lambda}_3$ be a primitive element. Then there exists a monodromy operator $f \in \text{Mon}^2(\hat{\Lambda}_3)$ such that $v$ is moved to an element of the following form:

$$f(v) = \begin{cases} L_i & \text{with } i = \frac{1}{2} q(v) \\ 2L_i - \delta & \text{with } i = \frac{1}{2} q(v) + 1 \end{cases} \text{ if } \text{div}(v) = 1$$

$$f(v) = \begin{cases} L_i & \text{if } \text{div}(v) = 2 \end{cases} \text{ if } \text{div}(v) = 2$$

The proof will make use of the following two well-known statements:

The Eichler criterion, which we will frequently use in this section (see [GHS10, Lemma 3.5], originally due to [Eic74, Chapter 10]).

Lemma 6.7. Let $\Gamma$ be a lattice with $U^2 \subseteq \Gamma$. Fix two elements $v, w \in \Gamma$ which satisfy

1. $q(v) = q(w), \quad$ 
2. $\text{div}(v) = \text{div}(w) =: r, \quad$ 
3. $\frac{v}{w} \in \text{Aut}(\Gamma) =: \Gamma'/\Gamma.$

Then there exists $\varphi \in \text{Aut}(\Gamma)$ such that $\varphi(v) = w$, and such that the induced action $\varphi_A$ on the discriminant group $\Gamma$ is the identity.

Furthermore, recall the following description of the monodromy group of varieties of $K3^{[2]}$-type:

Theorem 6.8 (see [Mar11, Lemma 9.2]). For a $K3$ surface $S$ the monodromy group $\text{Mon}^2(S^{[2]})$ coincides with $O^+(\Lambda_{K3^{[2]}})$, which is the order 2 subgroup of $\text{Aut}(\Lambda_{K3^{[2]}})$ which consists of automorphisms preserving the positive cone.

Proof of Lemma 6.6. The proof of Lemma 6.6 is an immediate consequence of the previous statements.

First apply the Eichler criterion (Lemma 6.7) for $\Gamma = \hat{\Lambda}_3$: Observe that for a given element $v \in U^3 \oplus (-2)$ the claimed image element has the same square and divisibility.

In the case of $\text{div}(v) = 1$, note that $\frac{v}{w} = 0 \in A_{\hat{\Lambda}_3}$ (since $\frac{v}{w} \in \hat{\Lambda}_3$). Therefore, the Eichler criterion applies automatically in this case.

For the case of $\text{div}(v) = 2$, note that $v$ can be written as $v = aL + b\delta$ with a primitive element $L \in U^3$ and $\delta$ as before. The fact that $\text{div}(v) = 2$ implies that $a$ is divisible by two. Since we assumed that $v$ is primitive, $b$ is odd. Note that $A_{\hat{\Lambda}_3} \cong \mathbb{Z}/2\mathbb{Z}$ is spanned by the image of $\frac{\delta}{2}$, since $U$ is unimodular. Therefore, we can also apply the Eichler criterion in this case.

In both cases, the Eichler criterion shows that there is $f \in \text{Aut}(\hat{\Lambda}_3)$ where $f(v)$ coincides with the claimed image.

Extending this by the identity on the respective orthogonal complements, we will by abuse of notation consider $f \in \text{Aut}(\Lambda_{K3^{[2]}})$ resp. $f \in \text{Aut}(\hat{\Lambda}_4)$. Applying Theorem 6.8 we observe that up to potentially swapping a sign in one of the copies of $U$, $f \in \text{Aut}(\Lambda_{K3^{[2]}})$ is in fact an element of $\text{Mon}^2(\Lambda_{K3^{[2]}})$. And therefore, Corollary 6.4 implies that $f \in \text{Aut}(\hat{\Lambda}_4)$ is in $\text{Mon}^2(\hat{\Lambda}_4)$ as claimed.

6.3 Monodromy orbits in the lattice $\hat{\Lambda}_2$

In this section, we study the monodromy group for the lattice $\hat{\Lambda}_2 = U^3 \oplus E_8(-2) \oplus (-2)$. Notice, that via the identifications described in Section 5.1 the lattice $\hat{\Lambda}_2$ corresponds to the $i^{[2]}$-invariant lattice of $\Lambda_{K3^{[2]}}$. Let us refine the methods from the previous section to describe properties of the monodromy orbits in this lattice $\hat{\Lambda}_2$. For this we need to deal with the $E_8(-2)$-part of the lattice.

For the basic notions considering discriminant groups of lattices, we refer to [Nik80]. Recall that $E_8$ is a unimodular lattice and therefore, the discriminant group $A_{E_8} = 0$ is trivial. Pick a
There exist precisely three \( \text{Aut}(E) \) orbits in \( E \): They are given by 0, and by non-zero elements \( \overline{v}, \overline{w} \in \text{Aut}(E) \) with \( \overline{v} = 0 \) respectively, \( \overline{w} = 0 \).

**Proof.** This lemma is a direct consequence of results in [31] Chapter 4. Let

\[
L_0 = \{0\}, \quad L_{-4} = \{ \alpha \in E \mid \alpha^2 = -4 \}, \quad L_{-8} = \{ \alpha \in E \mid \alpha^2 = -8 \}.
\]

As explained in [31] just after Notation 4.2.32, the natural map \( b : L_0 \cup L_{-4} \cup L_{-8} \rightarrow \text{Aut}(E) \) is surjective. The images \( b(L_{-4}) \) and \( b(L_{-8}) \) correspond respectively to the elements of square 1 and the non-trivial elements of square 0 in \( \text{Aut}(E) \).

However by [31] Corollary 4.2.41 and [31] Lemma 4.2.46 (2) respectively, \( \text{Aut}(E) \) acts transitively on \( L_{-8} \) and on \( L_{-4} \). The result follows.

Consider \( E < E \oplus E \) consisting of elements of the form \( (e, c) \) for \( e \in E \) and denote the sublattice consisting of elements of the form \( (e, -e) \) by \( E^0 \). Note that this gets naturally identified with the anti-invariant lattice of \( S \) via the marking described in Section 6.1.

**Lemma 6.10.** With this notation, any lattice isometry \( \varphi \in \text{Aut}(\hat{\Lambda}) \) can be extended to a lattice isometry \( \Phi \in \text{Aut}(U^3 \oplus E) \), \( \overline{v} \in \text{Aut}(E^0 \oplus \hat{\Lambda}) \) with the additional property, that \( \Phi \) preserves the sublattice \( E^0 \).

**Proof.** This is an immediate consequence of [30] Corollary 1.5.2 (applied twice to the lattice \( \hat{\Lambda} \) and the surjection \( \text{Aut}(E) \rightarrow \text{Aut}(E^0) \)) which enables us to choose an appropriate extension on the orthogonal complement \( E^0 \) of \( \hat{\Lambda} \).

Fix two elements in \( E \): one element \( e_1 \) of square \( -4 \) and one element \( e_2 \) of square \( -8 \). Note that according to Lemma 6.9 the residue classes of \( \overline{e_1} \) and \( \overline{e_2} \) in \( \text{Aut}(E) \) represent the two non-zero orbits under the action of the isometry group. For coherence of the notation adopt the choices such that the residue of \( \overline{e_1} \) in the discriminant is \( \overline{e_1} \) and the residue of \( \overline{e_2} \) is \( \overline{e_2} \).

**Proposition 6.11.** Let \( \nu \in \hat{\Lambda} \) be a primitive non-zero element. Denote by \( \nu \nu_\nu \) the projection of \( \nu \) to the \( E \)-part of the lattice, and let \( \nu \nu_\nu \) be the image of \( \nu \nu \nu \) in the discriminant group \( A_E \). Then there exists a monodromy operator \( \nu \nu_\nu \) such that

\[
f(v) = \begin{cases} 
1) & L_i, \quad \nu_\nu = 1, \quad \nu_\nu_\nu = \nu, \\
2) & 2L_1 + \delta, \quad \nu_\nu = 2, \quad \nu = 8\nu_\delta - 2, \quad \nu_\nu = 0, \\
3) & 2L_1 + e_1, \quad \nu_\nu = 2, \quad \nu = 8\nu_\nu_\nu, \\
4) & 2L_1 + e_2, \quad \nu_\nu = 2, \quad \nu = 8\nu_\nu, \\
5) & 2L_1 + e_1 - \delta, \quad \nu_\nu = 2, \quad \nu = 8\nu - 6, \\
6) & 2L_1 + e_2 - \delta, \quad \nu_\nu = 2, \quad \nu = 8\nu_\nu_\nu, \quad \nu_\nu_\nu \neq 0.
\end{cases}
\]
Remark 6.12. Note that the values of $q$ and div uniquely distinguish the orbit, except from the cases 2 and 6, where the additional condition on $\bar{v}_{E_8}$ is needed to determine the known representative of the orbit.

Proof. Let us first observe that if $\text{div}(v) = 1$ the exact same proof as for Lemma 6.6 for the case of divisibility 1 applies.

Therefore, we only need to deal with the case, where $\text{div}(v) = 2$. Start by observing that the discriminant group of $\Lambda_2$ is $A_{E_8(-2)} \times \mathbb{Z}/2\mathbb{Z}$.

For our given element $v \in \Lambda_2$, denote by $\bar{v}$ the image of $\frac{v}{2}$ in the discriminant group, and let $\bar{v}_e$ be the $A_{E_8(-2)}$-part of this. By Lemma 6.9 there exists $\varphi \in \text{Aut}(E_8(-2))$ such that $\varphi(\bar{v}_e) \in A_{E_8(-2)}$ coincides with one of $\{0, \bar{v}_7, \bar{v}_7\}$. For the corresponding $\varphi_1 \in \text{Aut}(E_8(-1))$ consider $(\varphi_1, \varphi_1) \in \text{Aut}(E_8(-1) \oplus E_8(-1))$, which obviously commutes with the swapping of the two factors and induces $\varphi$ on $E_8(-2) \subset E_8(-1) \oplus E_8(-1)$. Extend this to $\Phi \in \text{Aut}(\Lambda_{E_8^{(2)}})$ via the identity on the other direct summands. By Theorem 6.8 the operator $\Phi \in \text{Mon}^2(S^{(2)})$ is in the monodromy group and therefore the induced action on $\Lambda_2$ is an element of $\text{Mon}^2(\Lambda_2)$ by Proposition 5.4.

By construction, this restricts to $\varphi \in \text{Aut}(E_8(-2))$. Therefore, up to first applying the above monodromy operator, we may assume that $\bar{v} \in \{0, \bar{v}_7, \bar{v}_7\} \times \mathbb{Z}/2\mathbb{Z}$.

For the second step, observe that cases 2) to 6) listed in the proposition correspond precisely to the non-zero elements of $\{0, \bar{v}_7, \bar{v}_7\} \times \mathbb{Z}/2\mathbb{Z}$. By varying the parameter $i$, the elements in the list can furthermore achieve all possible values for $q(v)$ with the prescribed residue in $\{0, \bar{v}_7, \bar{v}_7\} \times \mathbb{Z}/2\mathbb{Z}$.

Therefore, for our given element $v \in \Lambda_2$ with $\bar{v} \in \{0, \bar{v}_7, \bar{v}_7\} \times \mathbb{Z}/2\mathbb{Z}$, we can choose $v_0$ from the above list (for appropriate choice of $i$) such that $q(v) = q(v_0)$ and $\bar{v} = \bar{v}_0 \in \Lambda_2$ and $\text{div}(v) = \text{div}(v_0) = 2$ follows automatically. Therefore, by the Eichler criterion (Lemma 6.7), there exists an automorphism $\varphi \in \text{Aut}(\Lambda_2)$ such that $\varphi(v) = v_0$. This can be extended to an automorphism $\Phi \in \text{Aut}(U^3 \oplus E_8(-1) \oplus E_8(-1) \oplus (-2))$ of the lattice $\Lambda_{E_8^{(2)}}$ by Lemma 6.10. Observe that up to changing a sign in one of the copies of $U$, we can assume that $\Phi \in \text{Mon}^2(\Lambda_{E_8^{(2)}})$ by Theorem 6.8. Since this monodromy operator commutes with $i^{[2]}$ (it preserves the invariant lattice and the anti-invariant lattice by construction) Proposition 6.9 shows that it induces a monodromy operator on $\Lambda_{M}$ which in turn corresponds to $\varphi$ extended by the identity via Lemma 6.11. Therefore, again up to potentially changing a sign in one of the copies of $U$, the automorphism $\varphi \in \text{Mon}^2(\Lambda_2)$. This completes the proof.

6.4 Induced monodromy orbits on the lattice $\hat{\Lambda}_1$

Recall that $\hat{\Lambda}_1 = U^3 \oplus E_8(-2) \oplus (-2) \oplus (-2)$.

Theorem 6.13. Let $v \in \hat{\Lambda}_1$ be a primitive non-zero element. Denote by $v_{E_8}$ the projection of $v$ to the $E_8(-2)$-part of the lattice, and let $\bar{v}_{E_8}$ be its image in the discriminant group $A_{E_8(-2)}$. Then there exists a monodromy operator $f \in \text{Mon}^2(\hat{\Lambda}_1)$ such that

$$f(v) = \begin{cases} 1) L_i & \text{if } \text{div}(v) = 1 \text{ with } i = \frac{1}{2} q(v) \\ 2) 2L_i - \delta & \text{if } \text{div}(v) = 2, q(v) = 8i - 2, \text{ and } \bar{v}_{E_8} = 0 \\ 3) 2L_i + e_2 - \delta & \text{if } \text{div}(v) = 2, q(v) = 8i - 2, \text{ and } \bar{v}_{E_8} \neq 0 \\ 4) 2L_i - \delta - \hat{\Sigma} & \text{if } \text{div}(v) = 2, q(v) = 8i - 4, \text{ and } \bar{v}_{E_8} = 0 \\ 5) 2L_i + e_2 - \delta - \hat{\Sigma} & \text{if } \text{div}(v) = 2, q(v) = 8i - 4, \text{ and } \bar{q}(v_{E_8}) = 0, \bar{v}_{E_8} \neq 0 \\ 6) 2L_i + e_1 & \text{if } \text{div}(v) = 2, q(v) = 8i - 4, \text{ and } \bar{q}(v_{E_8}) = 0, \bar{v}_{E_8} = 0 \\ 7) 2L_i + e_1 - \delta & \text{if } \text{div}(v) = 2, q(v) = 8i - 6, \text{ and } \bar{q}(v_{E_8}) = 0, \bar{v}_{E_8} = 0 \\ 8) 2L_i + e_1 - \delta - \hat{\Sigma} & \text{if } \text{div}(v) = 2, q(v) = 8i - 8, \text{ and } \bar{q}(v_{E_8}) = 0, \bar{v}_{E_8} = 0 \\ 9) 2L_i + e_2 & \text{if } \text{div}(v) = 2, q(v) = 8i, \text{ and } \bar{q}(v_{E_8}) = 0, \bar{v}_{E_8} = 0. \end{cases}$$

Remark 6.14. Observe that whenever $L_0$ is involved in the statement of Theorem 6.13, it can be replaced by 0 (apart from case 1) since both elements are in the same monodromy orbit.

Proof. The proof of this theorem consists of a series of applications of Proposition 6.11 and the existence of the monodromy operator $R_{i - \frac{1}{2}}$ (compare Remark 1.7 and notation of Section 1.3). First note that since $v$ is primitive, it can be expressed as $v = k\gamma + a\delta + b\hat{\Sigma}$, where $\gamma \in U^3 \oplus E_8(-2)$ is a primitive element, and $\gcd(a, b, k) = 1$. 

30
First let us assume that $\text{div}(v) = 1$. The element $k\gamma + a \delta \in \hat{A}_2$ corresponds to $\gcd(k, a)$ times a primitive element of divisibility 1 inside $\hat{A}_2$. Therefore, by Proposition 6.11 there exists an element $f_1 \in \text{Mon}^2(\hat{A}_2)$ such that $f_1(k\gamma + a \delta) = \gcd(k, a) \cdot L_{q_1}$ for a suitable choice of $q_1$. By extending $f_1$ to $\text{Mon}^2(\hat{A}_2) \supseteq \text{Mon}^2(\hat{A}_2)$, observe that $f_1(v) = \gcd(k, a) \cdot L_{q_1} + b \delta$. Apply the monodromy operator $R_{i - \infty}$ to obtain $\gcd(k, a) \cdot L_{q_1} + b \delta$, which is a primitive element of divisibility 1 in $\hat{A}_2$. Using once again Proposition 6.11 find $f_2 \in \text{Mon}^2(\hat{A}_2) \subset \text{Mon}^2(\hat{A}_1)$ such that $f_2(\gcd(k, a) \cdot L_{q_1} + b \delta) = L_{q_2}$.

The composition of these monodromy operators is therefore the claimed $f \in \text{Mon}^2(\hat{A}_2)$ and concludes the proof under the assumption that $\text{div}(v) = 1$.

Therefore, we only need to deal with the case that $\text{div}(v) = 2$. Let $v$ be the residue class of $\frac{\alpha}{2}$ in $A_{\hat{A}_2}$. Let us first work under the additional assumption that $\gcd(k, a)$ is odd (while still assuming $\text{div}(v) = 2$). Under this assumption, the element $k\gamma + a \delta \in \hat{A}_2$ corresponds to $\gcd(k, a)$ times a primitive element $v_1 \in \hat{A}_2$, satisfies that $\bar{v}_1 = \bar{v}_{\hat{A}_2} \in A_{\hat{A}_2}$, where $\bar{v}_1$ is the residue of $\frac{\alpha}{2}$, and $\bar{v}_{\hat{A}_2}$ is the $A_{\hat{A}_2}$-part of $\bar{v}$. Then there exists a monodromy operator $f_1 \in \text{Mon}^2(\hat{A}_2) \subset \text{Mon}^2(\hat{A}_1)$ such that $f_1(v_1)$ is one of the cases 2) to 6) from Proposition 6.11. After applying the operator $R_{i - \infty}$, we obtain an element $v_2$ of one of the following forms:

$$v_2 = \begin{cases} a) & 2 \gcd(k, a)L_{q_1} + b \delta - \gcd(k, a) \hat{\Sigma} \\ b) & 2 \gcd(k, a)L_{q_1} + \gcd(k, a)e_1 + b \delta \\ c) & 2 \gcd(k, a)L_{q_1} + \gcd(k, a)e_2 + b \delta \\ d) & 2 \gcd(k, a)L_{q_1} + \gcd(k, a)e_1 + b \delta - \gcd(k, a) \hat{\Sigma} \\ e) & 2 \gcd(k, a)L_{q_1} + \gcd(k, a)e_2 + b \delta - \gcd(k, a) \hat{\Sigma} \end{cases}$$

for suitable choice of $q_1$. Since $\gcd(k, a, b) = 1$, note that the $\hat{A}_2$-component of $v_2$ is primitive unless we are dealing with case a) from above and at the same time $b$ is even. We will separately consider these two situations:

Case 1: If the $\hat{A}_2$-component of $v_2, \hat{\Lambda}_2$, of $v_2$ is primitive, then one can find a monodromy operator moving $v_2, \hat{\Lambda}_2$, to one of the cases from Proposition 6.11. In cases b) and c) from above the resulting element attains the form of one of the cases claimed in our theorem. In the other cases, apply the operator $R_{i - \infty}$ once again, followed by Proposition 6.11 to conclude the proof of case 1.

Case 2: Assume that $v_2, \hat{\Lambda}_2$, is non-primitive, and therefore we are in case a) from above with the additional assumption that $b = 2b'$ is even. This means that $v_2 = 2(\gcd(k, a)L_{q_1} + b' \delta) - \gcd(k, a) \hat{\Sigma}$. Since $\gcd(k, a)$ is odd, $v_2, \hat{\Lambda}_2$, is twice a primitive element of divisibility 1, and therefore $v_2, \hat{\Lambda}_2$, can be moved to an element of the form $2L_{q_2} - \gcd(k, a) \hat{\Sigma}$. Applying the operator $R_{i - \infty}$ and using Proposition 6.11 completes the proof of case 2 (since $\gcd(k, a)$ is odd by assumption).

The only remaining case, which we have not yet been analyzed is if $\text{div}(v) = 2$ and $\gcd(k, a)$ is even. Notice that under these assumptions $b$ is odd and in particular $\gcd(k, b)$ is odd. Therefore, after application of the operator $R_{i - \infty}$, we find ourselves in the above setting, which concludes the final case of the proof.

From this we can easily deduce a corresponding statement for the original lattice $\Lambda_{M'}$. We need to fix some notation in order to formulate the statement. Consider an irreducible symplectic orbifold $X$ of Nikulin-type, with a given marking $\varphi: H^2(X, \mathbb{Z}) \xrightarrow{\cong} \Lambda_{M'}$. Let $L_i^{(2)} \in U(2)$ be an element of square 4i (corresponding to the element $L_i \in U$). Furthermore, fix elements $e_1^{(i)}$ and $e_2^{(i)} \in E_8(-1)$ with squares $q(e_1^{(i)}) = -2$ and $q(e_2^{(i)}) = -4$ (these elements correspond to the elements $e_1$ and $e_2 \in E_8(-2)$).

**Theorem 6.15.** Let $v \in \Lambda_{M'}$ be a primitive non-zero element. Denote by $\bar{v}_{E_8}$ the projection of $v$ to the $E_8(-1)$-part of the lattice, and let $\bar{v}_{E_8}$ be its image in the $\mathbb{Z}/4\mathbb{Z}$-module $E_8(-1)/4E_8(-1)$. 

$$\text{div}(v) = 1$$
Then there exists a monodromy operator \( f \in \text{Mon}^2(X) \) such that

\[
f(v) = \begin{cases}
  & \text{If } v \text{ corresponds to a ray of divisibility } 1 \text{ in } \hat{\Lambda}_1 \text{ (see below for checkable condition):} \\
  1) & L^{(2)}_1 \quad \text{with } \text{div}(v) = 2 \text{ and } q(v) = 4i. \\
 2) & 2L^{(2)}_1 - \delta' \quad \text{if } \text{div}(v) = 2, \ q(v) = 16i - 4, \text{ and } \bar{v}_{E_8} = 0 \\
 3) & 2L^{(2)}_1 + 2e^{(1)}_2 - \delta' \quad \text{if } \text{div}(v) = 2, \ q(v) = 16i - 4, \text{ and } \bar{v}_{E_8} \neq 0 \\
 4) & L^{(2)}_1 - \delta e^{\pm} + \Sigma' \quad \text{if } \text{div}(v) = 2, \ q(v) = 4i - 2, \text{ and } \bar{v}_{E_8} = 0 \\
 5) & L^{(2)}_1 + e^{(1)}_2 - \delta' \quad \text{if } \text{div}(v) = 1, \ q(v) = 4i - 2, \text{ and } q(v_{E_8}) \equiv 0 \pmod{4} \\
 6) & L^{(2)}_1 + e^{(2)}_1 \quad \text{if } \text{div}(v) = 1, \ q(v) = 4i - 2, \text{ and } q(v_{E_8}) \equiv 2 \pmod{4} \\
 7) & 2L^{(2)}_1 + 2e^{(1)}_1 - \delta' \quad \text{if } \text{div}(v) = 2, \ q(v) = 16i - 12, \text{ and } \bar{v}_{E_8} \neq 0 \\
 8) & L^{(2)}_1 + e^{(1)}_2 - \delta e^{\pm} + \Sigma' \quad \text{if } \text{div}(v) = 1, \ q(v) = 4i, \text{ and } q(v_{E_8}) \equiv 0 \pmod{4} \\
 9) & L^{(2)}_1 + e^{(2)}_1 \quad \text{if } \text{div}(v) = 1, \ q(v) = 4i, \text{ and } q(v_{E_8}) \equiv 0 \pmod{4}.
\end{cases}
\]

The condition that \( v \) corresponds to a ray of divisibility 1 in \( \hat{\Lambda}_1 \) is equivalent to satisfying the following three conditions inside \( \Lambda \):

1. The restriction \( v_{U^3(2)} \) of \( v \) to \( U^3(2) \) is not divisible by 2,
2. The restriction \( v_{E_8} \) of \( v \) to \( E_8(-1) \) is divisible by 2, and
3. The restriction \( v_{(2)-\otimes(-2)} \) to \( \langle \delta' + \Sigma', \delta' + \Sigma' \rangle \) is contained in the sublattice \( \langle \delta', \Sigma' \rangle \).

**Proof.** Similar to Lemma \[4.2\] use the inclusion \( \hat{\Lambda}_1(2) \subset \Lambda_{M'} \) and notice that under this correspondence \( L_i \) is sent to \( L^{(2)}_i e_i \), is sent to \( 2e^{(1)}_i - \delta' \) to \( \delta' \), and \( \Sigma \) to \( \Sigma' \). Then passing to the primitive element in the ray and determining the new square and divisibility gives the new cases.

For the part of the condition involving \( v_{E_8} \), simply check that under the assumptions on \( q \) and \( \text{div} \) these are equivalent to the corresponding ones in \( \hat{\Lambda}_1 \) from Theorem \[6.13\]. The same formalism admits a straightforward verification of the characterization when \( v \) is corresponding to a ray of divisibility 1 in \( \hat{\Lambda}_1 \).

**Corollary 6.16.** There are at most 3 monodromy orbits of primitive non-zero elements with prescribed square and divisibility (both in \( \Lambda_{M'} \) and in \( \hat{\Lambda}_1 \)).

**Proof.** Since the values of \( q \) and \( \text{div} \) are given, this can be read of immediately from the statements of Theorems \[6.13\] and \[6.15\].

**Remark 6.17.** Again, one can replace \( L^{(2)}_1 \) by 0 in all cases except from Case 1), since both elements in question lie in the same monodromy orbit.

Let us conclude this section by the following observation:

**Corollary 6.18.** For every element \( v \in \Lambda_{M'} \) of square \(-4\) and divisibility 2 the reflection (defined by \( R_v(\alpha) := \alpha - e(\omega_v)/q(v^2)v \)) gives an element in the monodromy group.

**Proof.** Begin by observing that this property is equivalent for different elements in the same monodromy orbit. Therefore, it suffices to check it for one representative of each orbit. By the list from Theorem 6.15, the orbits of square \(-4\) and divisibility 2 have one of the following representatives: \( L^{(2)}_{-1} \) (Case 1), \( \delta' \) (Case 2), or \( 2L^{(2)}_1 + 2e^{(1)}_2 - \delta' \) (Case 3).

The associated elements \( L_{-1} \), \( \delta \), and \( 2L_1 + e_2 - \delta \) in the invariant part of \( \Lambda_{K(2)} \) all have square \(-2\) and thus their reflections correspond to monodromy operators on \( \Lambda_{K(2)} \) (e.g. by Theorem \[6.3\]) which commute with \( i^{(2)} \). Therefore, Proposition \[3.9\] applies to show that the claimed reflections in \( \Lambda_{M'} \) are indeed monodromy operators.

**Corollary 6.19.** For every element \( v \in \Lambda_{M'} \) of square \(-2\) and divisibility 2 the reflection (defined by \( R_v(\alpha) := \alpha - e(\omega_v)/q(v^2)v \)) gives an element in the monodromy group.

**Proof.** The proof is similar to the one of Corollary \[6.18\]. From Theorem \[6.15\] we note that all such elements \( v \) are in the same monodromy orbit (Case 4) which contains the element \( \delta' - \Sigma' \). Hence the result follows from Remark \[4.7\].
7 Determining the wall divisors

In this section, we combine the results from the last sections to prove the main theorem of this paper: Theorem 1.2, which gives a complete description of the wall divisors for Nikulin-type orbifolds.

For the proof of the theorem let us start from some $X_0$ which is the Nikulin orbifold associated to some K3 surface $S_0$ obtained by the construction in Section 5.1. Fix a marking $\varphi_0: H^2(S_0, \mathbb{Z}) \to \Lambda_{M'} = U(2)^3 \oplus E_6(-1) \oplus (-2) \oplus (-2)$, where as usual $U(2)^3 \oplus E_6(-1)$ corresponds to the part coming from the invariant lattice of $S_0$ and the two generators of the $(-2)$-part are $\frac{\hat{\alpha}_{M'}}{2}$ and $\frac{\hat{\beta}}{2}$. Let us recall the details of this identification: For the K3 surface $S_0$ with a symplectic involution $\iota$ the $-\iota$-anti-invariant part of the lattice is isomorphic to $E_6(-1)$ and one can choose a marking $\varphi_{\mathcal{S}_0}: H^2(S_0, \mathbb{Z}) \to \Lambda_{\mathcal{S}_0} \cong U^3 \oplus E_6(-1)^2$, such that the $\iota^*$ acts by exchanging the two copies of $E_6(-1)$. Therefore, the invariant lattice of $\iota$ corresponds to $\Lambda_{\mathcal{S}_0} \cong U^3 \oplus E_6(-2)$, where the elements of $E_6(-2)$ are of the form $e + \iota^*(e)$ for elements $e$ in the first copy of $E_6(-1)$. Similarly, the anti-invariant lattice of $\iota$ is $E_6(-2)$ consisting of elements of the form $e - \iota^*(e)$. We will denote the anti-invariant part of the lattice by $E^\circ$. With this convention, the lattice $U(2)^3 \oplus E_6(-1)$ corresponds to the invariant via a twist as described in Lemma 5.2 of $\Lambda_{K3}$.

In order to prove the main theorem, we need to determine for each ray in $\Lambda_{M'}$ whose generator is of negative Beauville-Bogomolov square, whether it corresponds to a wall divisor for Nikulin-type orbifolds. Obviously, this notion is invariant under the monodromy action by the deformation invariance (see Theorem 2.14). It therefore suffices to pick one representative for each monodromy orbit and to determine it for this choice.

By Lemma 6.2 the rays of $\Lambda_{M'}$ are in (1:1)-correspondence with rays in the lattice $\hat{\Lambda}_1$, and obviously the property that the generator has negative square coincides in both cases. Therefore, we only need to deal with the cases from Theorem 6.13 (respectively Theorem 6.15), for which $i$ is chosen such that the square is negative.

Case 1: As a warm-up, let us start with Case 1 of Theorem 6.13 separately (i.e. the ray in question is generated by the element $L_i$ with $i < 0$). Note that $L_i$ naturally corresponds to an element $\varphi_{S_0}^{-1}L_i \in H^2(S_0, \mathbb{Z})$. Let $(S, \varphi_S)$ be a marked K3 surface such that the Picard lattice of $S$ is $\text{Pic}(S) = \varphi_S^{-1}(L_i \oplus E^\circ)$ (which exists by the surjectivity of the period map). If $i < -1$, then $S$ does not contain any effective curve (since $\text{Pic}(S)$ only has non-zero elements of square smaller than $-2$). Therefore, we are in the situation of Section 5.1 and one observes that $L_i$ does not correspond to a wall divisor for Nikulin-type orbifolds if $i < -1$: In fact, $L_i \in \hat{\Lambda}_1$ corresponds to $L_i^{(2)} \in \Lambda$, which is not a wall divisor by Proposition 1.1. Note that the divisors $L_i^{(2)} \in \Lambda$ satisfy $q(L_i^{(2)}) = -4i$, $\text{div}(L_i^{(2)}) = 2$, and $(L_i^{(2)})(U(2)^3)$ is not divisible by 2, which confirms Theorem 1.2 for Case 1 if $i < -1$.

If $i = -1$ and therefore $q(L_i) = -2$, we are in the situation of Section 5.1 and one can deduce from Proposition 6.15 that $L_i^{(2)} \in \Lambda$ (which is precisely $D_C'$) is a wall-divisor for Nikulin-type, which confirms Theorem 1.2 in this case.

Cases 2, 4: As in the proof for Case 1, choose a K3 surface $S$ such that $\text{Pic}(S) = \varphi_S^{-1}(L_i \oplus E^\circ)$. As before, Section 5.1 applies and Proposition 1.1 implies that for $i < -1$, the only rays corresponding to wall-divisors are $\delta'$ and $\Sigma'$, and therefore there are no additional wall divisors of the forms given in Cases 2 and 4 in this example. Similarly, the results from Section 5.1 imply that for $i = -1$ the wall divisors are $\delta'$, $\Sigma'$, $L_i^{(2)}$, and $L_i^{(2)} - \frac{1}{2}(\delta' + \Sigma')$ (compare Proposition 6.15). Therefore, Case 4 provides precisely a wall divisor of square $-6$ and divisibility 2, thus confirming Theorem 1.2 in this case.

However, for Cases 2 and 4, we also need to consider $i = 0$ since the total square will still be negative. By Remark 6.14 the monodromy orbits of $2L_0 + \hat{\delta}$ (resp. $2L_0 + \hat{\delta} + \hat{\Sigma}$) coincide with those of $\hat{\delta}$ and $\hat{\delta} + \hat{\Sigma}$, and therefore we can instead deform towards a very general K3 surface $S$ with a symplectic involution (i.e. $\text{Pic}(S) = E^\circ$) and apply the results from Section 4.1 to observe that $\delta'$ is a wall divisor of square $-4$ and divisibility 2, whereas $\frac{1}{2}(\delta' + \Sigma')$ is not.

Cases 6, 7, and 8: Similar to the previous situation, the element $2L_i + e_1$ naturally corresponds to an element $\varphi_S^{-1}(2L_i + e_1) \in H^2(S_0, \mathbb{Z})$. Notice, that we are only interested in the cases, where $q(2L_i + e_1) < 0$, which corresponds to $i \leq 0$. Under this condition, the direct sum $(2L_i + e_1) \oplus E^\circ$
is a negative definite sublattice of $\Lambda_{K3}$. However, notice that this in itself cannot be realized as the Picard lattice of a K3 surface, since it is not a saturated sublattice: By definition $e_1 \in E_8(-2)$ is an element of square $-4$, where $E_8(-2)$ is part of the invariant lattice. Therefore, by the above observation, there exists an element $e_1^{(0)}$ in the first copy of $E_8(-1)$ of square $-2$ such that $e_1 = e_1^{(0)} + \iota^*(e_1^{(0)})$. With this notation the element $2L_i + 2e_i^{(0)} = 2L_i + (e_1^{(0)} + \iota^*(e_1^{(0)})) + (e_1^{(0)} - \iota^*(e_1^{(0)})) \in (2L_i + e_1) \oplus E^a$, but the element $L_i + e_1^{(0)}$ is not part of this direct sum. In fact, $(L_i + e_1^{(0)}) \oplus E^a$ is the saturation.

With this knowledge, let us choose a marked K3 surface $(S, \varphi_S)$ such that $\text{Pic}(S) = \varphi_S^{-1}((L_i + e_1^{(0)}) \oplus E^a)$. Note that if $i < 0$, then $S$ does not contain any effective curve (since every non-zero element has square smaller than $-2$). Therefore, the results from Section 4.1 apply, and one observes that non of these cases provides wall divisors.

If $i = 0$, then $S$ contains exactly two elements of square $-2$ which are exchanged by $\iota^*$: The elements $L_0 + e_1^{(0)}$ and $L_0 + \iota^*(e_1^{(0)})$. In this case according to Remark 6.14 we can choose $L_0 = 0$. Thus for $i = 0$ we find ourselves in the setting of Section 5.2 with $e_1^{(0)} = C$. Note that the element $D_C$ from Section 5.2 corresponds precisely to the element $e_1^{(1)}$ with our notation. We can therefore deduce from Proposition 5.22 that for $i = 0$ the Cases 6 and 7 provide wall divisors ($e_1^{(1)}$ with square $-2$ and divisibility 1, and $2e_1^{(1)} - \delta'$ with square $-12$ and divisibility 2), whereas by Remark 5.23 Case 8 does not provide a wall divisor, thus confirming Theorem 1.2.

Cases 3, 5, and 9: Again, the element $2L_{i+1} + e_2$ corresponds to an element in $\varphi_{S_0}(H^2(S_0, \mathbb{Z}))$. We need to consider $i \leq 0$ to cover all possibilities for wall divisors with negative squares.

If $i < 0$, then the lattice $(2L_{i+1} + e_2) \oplus E^a \subseteq \Lambda_{K3}$ is negative definite, and again its saturation is $(L_{i+1} + e_2^{(0)}) \oplus E^a$ for the corresponding element $e_2^{(0)}$ in the first copy of $E_8(-1)$ (we remind that $e_2^{(0)}$ has square $-4$). Similar to the above, deform to a marked K3 surface $(S, \varphi_S)$ such that $\varphi_S(\text{Pic}(S)) = (L_{i+1} + e_2^{(0)}) \oplus E^a$. Observe that all non-zero elements of this lattice have squares smaller than $-2$. Therefore, we can apply the results from Section 4.1 to observe that we do not find any further wall divisors in these cases.

For the remaining case $i = 0$, we need to prove that both $2L_1^{(2)} + 2e_2^{(1)} - \delta'$ (with square $-4$ and divisibility 2) and $L_1^{(2)} + e_2^{(1)} - \delta' + \gamma$ (with square $-2$ and divisibility 1) correspond to wall divisors. If $i = 0$, then $S$ contains exactly one element of square 0: $2L_1 + e_2$. Thus for $i = 0$ we find ourselves in the setting of Section 5.3. Note that the element $D_2$ from Section 5.3 corresponds precisely to the element $L_1^{(2)} + e_2^{(1)}$ with our notation. Let $M'$ constructed as in Section 5.3 From the investigations of the current section, we know that a wall divisor on $M'$ has the numerical properties of a wall divisor that we already found or possibly of $2L_1^{(2)} + 2e_2^{(1)} - \delta'$ or $L_1^{(2)} + e_2^{(1)} - \delta' + \gamma$. That is: we have proved that a wall divisor necessarily has one of the numerical properties listed in Theorem 1.2. Therefore, Lemma 5.23 shows that $L_1^{(2)} + e_2^{(1)} - \delta' + \gamma$ is a wall divisor. Finally, $2L_1^{(2)} + 2e_2^{(1)} - \delta'$ is also a wall divisor by Lemma 5.23.

This concludes the analysis of all possible cases and thus the proof of Theorem 1.2.

8 Application

8.1 A general result about the automorphisms of Nikulin-type orbifolds

Proposition 8.1. Let $X$ be an orbifold of Nikulin-type and $f$ an automorphism on $X$. If $f^* = \text{id}$ on $H^2(X, \mathbb{Z})$, then $f = \text{id}$.

This section is devoted to the proof of this proposition. We will adapt Beauville’s proof [Bea83, Proposition 10].

Lemma 8.2. Let $S$ be a K3 surface such that $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}E_8(-2)$ with $H^2 = 4$. According to Proposition 8.3 or from [Sm07, Proposition 2.3], the K3 surface $S$ is endowed with a symplectic involution $\iota$. Let $f \in \text{Aut}(S)$ such that $f$ commutes with $\iota$. Then $f = \iota$ or $f = \text{id}$.

Proof. We adapt the proof of [Huy16, Corollary 15.2.12]. Let $f \in \text{Aut}(S)$ which commutes with $\iota$. It follows that $f^*(H) = H$. By [Huy16, Corollary 3.3.5], $f$ acts on $T(S)$ (the transcendental lattice
of $S$) as $-\id$ or $\id$. However, the actions of $f^*$ on $A_T(S)$ and on $A_{\Pic}(S)$ have to coincide. This forces $f^*_T(S) = \id$. Moreover, we can consider $f^*_E(S)^{-2}$ as an isometry of $E_8(-2)$. By [Gri11, Theorem 4.2.39], the isometries group of $E_8(-2)$ is finite, hence $f^*_E(S)^{-2}$ is of finite order. Therefore by [Huy10, Chapter 15 Section 1.2], there are only two possibilities for $f$: $\id$ or a symplectic involution. Moreover, by [Huy10, Proposition 15.2.1], there is at most one symplectic involution on $S$.

**Lemma 8.3.** Let $(S, \iota)$ be a K3 surface endowed with a symplectic involution such that $\Pic S = \mathbb{Z}H \oplus \oplus E_8(-2)$ with $H^2 = 4$. Let $M'$ be the Nikulin orbifold constructed from $(S, \iota)$ as in Section 8.4. Let $(g, h) \in \Aut(S)^2$ such that $g \times h$ induces a bimeromorphism on $M'$ via the non-ramified cover

$$\gamma : S \times S \setminus (\Delta_{S^2} \cup S_i \cup (\Fix \iota \times \Fix \iota)) \to M' \setminus (\delta' \cup \Sigma' \cup \Sing M')$$

introduced in Section 4.2 (i.e: there exists a bimeromorphism $\rho$ on $M'$ such that $\rho \circ \gamma = \gamma \circ f \times g$).

Then $g$ and $h$ commute with $\iota$.

**Proof.** It is enough to prove that $g$ commutes with $\iota$; the proof for $h$ being identical. Let

$$A := \left\{ \eta = \eta_1 \circ \eta_2 \circ \eta_3 \mid (\eta_1, \eta_2) \in \{\id, \iota\}^2 \text{ and } \eta_2 \in \{\id, g, h\} \right\}.$$ 

Let $V := S \times S \setminus (\Delta_{S^2} \cup S_i \cup (\Fix \iota \times \Fix \iota))$; we consider the following open subset of $V$:

$$V^\circ := \left\{ (a, b) \in V \mid g(a) \neq \eta(b), \ g \circ \iota(a) \neq \eta(b), \ \forall \eta \in A \text{ and } a \notin g^{-1}(\Fix \iota) \right\}.$$ 

Since $g \times h$ induces a bimeromorphism on $M'$, there exist an open subset $W$ of $S \times S$ such that for all $(a, b) \in W$:

$$g \times h \left( \{(a, b), (b, a), (\iota(a), \iota(b)), (\iota(b), \iota(a))\} \right) = \{(g(a), h(b)), (h(b), g(a)), (\iota \circ g(a), \iota \circ h(b)), (\iota \circ h(b), \iota \circ g(a))\}.$$ 

That is:

$$\{(g(a), h(b)), (g(b), h(a)), (g \circ \iota(a), h \circ \iota(b)), (g \circ \iota(b), h \circ \iota(a))\} = \{(g(a), h(b)), (h(b), g(a)), (\iota \circ g(a), \iota \circ h(b)), (\iota \circ h(b), \iota \circ g(a))\}.$$ 

If we choose in addition $(a, b) \in V^\circ$, then there are only one possibility:

$$g \circ \iota(a) = \iota \circ g(a).$$

It follows that $g$ commutes with $\iota$ on an open set of $S$, so on all $S$. 

We are now ready to prove Proposition 5.7.

**Proof of Proposition 5.7.** We consider $X$ an orbifold of Nikulin-type and $f$ an automorphism on $X$ such that $f^* = \id$. In particular, $f$ is a symplectic automorphism. Let $(S, \iota)$ be a K3 surface, endowed with a symplectic involution, verifying the hypothesis of Lemma 8.2. We consider the Nikulin orbifold $M'$ constructed from $(S, \iota)$ as in Section 8.4. By [MR20, Lemma 2.17.1], there exists two markings $\varphi$ and $\psi$ such that $(X, \varphi)$ and $(M', \psi)$ are connected by a sequence of twistor spaces. Moreover by Remark 2.11, $f$ extends to an automorphism on all twistor spaces. In particular $f$ induces an automorphism $f'$ on $M'$. We consider $\gamma$, the non ramified cover of Lemma 8.3

$$\gamma : S \times S \setminus (\Delta_{S^2} \cup S_i \cup (\Fix \iota \times \Fix \iota)) \to M' \setminus (\delta' \cup \Sigma' \cup \Sing M').$$

Since $V = S \times S \setminus (\Delta_{S^2} \cup S_i \cup (\Fix \iota \times \Fix \iota))$ is simply connected, it is the universal cover of $U := M' \setminus (\delta' \cup \Sigma' \cup \Sing M')$.

Since $f^*$ acts as id on $H^2(M', \mathbb{Z})$, we have that $f'$ preserves $\delta'$ and $\Sigma'$ (it also preserves the set $\Sing M'$). Hence $f'$ induces an automorphism on $U$ and then on $V$. Therefore, it induces a bimeromorphism $\mathcal{T}$ on $S \times S$. Let $s_2 : S \times S \to S \times S : (a, b) \mapsto (b, a)$. By [Ogu10, Theorem 4.1 (d)], $\mathcal{T}$ can be written as a sequence of compositions between $s_2$ and automorphisms of the form $g_i \times h_i$, where $g_i, h_i$ are in $\Aut(S)$. Since, we are interested in the automorphism $f'$ on $M'$, without loss of generality, we can assume that $\mathcal{T} = g \times h$, with $g, h$ in $\Aut(S)$.

Therefore, by Lemma 8.3, $g$ and $h$ commute with $\iota$. It follows from Lemma 8.2 that $(g, h) \in \{\id, \iota\}^2$. So, the unique possibility for $\mathcal{T}$ to induces a non-trivial morphism on $U$ is $\mathcal{T} = \id \times \iota$ (or $\iota \times \id$). However, in this case, as seen in Section 4.2, $f'$ would interchange $\delta'$ and $\Sigma'$. This is a contradiction with the fact that $f^* = \id$ on $H^2(M', \mathbb{Z})$. Therefore, we obtain $f' = \id$.
8.2 Construction of a non-standard symplectic involution on orbifolds of Nikulin-type

Adapting the vocabulary introduced in [Mon13], we state the following definition.

Definition 8.4. Let $Y$ be an irreducible symplectic manifold of $K3^{[2]}$-type endowed with a symplectic involution $\iota$. Let $M'$ be the Nikulin orbifold constructed from $(Y, \iota)$ as in Example 8.3. Let $G \subset \text{Aut}(Y)$ such that all $g \in G$ commute with $\iota$. Then $G$ induces an automorphism group $G'$ on $M'$. The group $G'$ is called a natural automorphism group on $M'$ and $(M', G')$ is called a natural pair.

Let $X$ be an irreducible symplectic orbifold of Nikulin-type and $H \subset \text{Aut}(X)$. The group $H$ will be said standard if the couple $(X, H)$ is deformation equivalent to a natural pair $(M', G')$; in this case, we say that the couple $(X, H)$ is a standard pair.

Theorem 8.5. Let $X$ be an irreducible symplectic orbifold of Nikulin-type such that there exists $D \in \text{Pic}(X)$ with $D^2 = -2$ and $\text{div}(D) = 2$. Then there exists an irreducible symplectic orbifold $Z$ bimeromorphic to $X$ and a non-standard symplectic involution $\iota$ on $Z$ such that:

$$H^2(Z, \mathbb{Z})^\iota \simeq U(2)^3 \oplus E_8(-1) \oplus (-2) \text{ and } H^2(Z, \mathbb{Z})^{\iota \perp} \simeq (-2).$$

Proof. By Theorem 1.2, $D$ is not a wall divisor, hence there exists $\beta \in B_K(X)$ and $g \in \text{Mon}^2_{\text{td}}(X)$ such that $(g(D), \beta)_g = 0$. Let $f : X \to Z$ be a bimeromorphic map such that $f_*\beta$ is a Kähler class on $Z$. We set $D' := f_* \circ g(D)$. By Corollary 6.19, the involution $R_{D'}$ is a Hodge monodromy operator on $H^2(Z, \mathbb{Z})$. Moreover, by Theorem 2.8, there exists $\iota$ an automorphism of $Z$ such that $\iota^* = R_{D'}$. Moreover, by Proposition 5.1, $\iota$ is an involution. Since $\iota^* = R_{D'}$, we have $H^2(Z, \mathbb{Z})^{\iota} = D'^{\perp}$. It follows from Theorem 5.6 that: $H^2(Z, \mathbb{Z})^{\iota} \simeq U(2)^3 \oplus E_8(-1) \oplus (-2)$ and $H^2(Z, \mathbb{Z})^{\iota \perp} \simeq (-2)$.

Now, we show that $\iota$ is non-standard. We assume that $\iota$ is standard and we will find a contradiction. If $\iota$ is standard, there exists a natural pair $(M', \iota')$ deformation equivalent to $(Z, \iota)$. Since $\iota'$ is natural, $\iota'^*(\Sigma') = \Sigma'$. Moreover, since $(M', \iota')$ is deformation equivalent to $(Z, \iota)$, there exists $D' \in \text{Pic}(M')$ such that $q_{M'}(D') = -2$, $\text{div}(D') = 2$ and $H^2(M', \mathbb{Z})^{\iota'} = \mathbb{Z} D'$. However, since $\Sigma' \in H^2(M', \mathbb{Z})^{\iota'}$, we obtain by Theorem 5.6 that:

$$D' \in \Sigma'^{\perp} \simeq U(2)^3 \oplus E_8(-1) \oplus (-4).$$

For the rest of the proof, we identify $\Sigma'^{\perp}$ with $U(2)^3 \oplus E_8(-1) \oplus (-4)$. If follows that $D'$ can be written:

$$D' = \alpha + \beta,$$

with $\alpha \in U(2)^3 \oplus (-4)$ and $\beta \in E_8(-1)$. Since $\text{div}(D') = 2$, we have

$$D' = \alpha + 2\beta',$$

with $\beta' \in E_8(-1)$. If follows that $q_{M'}(D') \equiv 0 \mod 4$. This is a contradiction with $q_{M'}(D') = -2$. $\square$

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