Generating the Mapping Class Group by Two Torsion Elements

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Abstract. We prove that the mapping class group of a closed connected orientable surface of genus \( g \geq 6 \) is generated by two elements of order \( g \). Moreover, for \( g \geq 7 \), we obtain a generating set of two elements, of order \( g \) and \( g' \), where \( g' \) is the least divisor of \( g \) greater than 2. We also prove that the mapping class group is generated by two elements of order \( g/\gcd(g, k) \) for \( g \geq 3k^2 + 4k + 1 \) and any positive integer \( k \).

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1. Introduction

The mapping class group \( \text{Mod}(\Sigma_g) \) of a closed, connected orientable surface \( \Sigma_g \) is the group of orientation-preserving diffeomorphisms of \( \Sigma_g \to \Sigma_g \) up to isotopy. Dehn [3] showed that \( \text{Mod}(\Sigma_g) \) is generated by \( 2g(g-1) \) many Dehn twists. Afterwards, Lickorish [12] decreased this number to \( 3g-1 \). Humphries [6] introduced a generating set consisting of \( 2g+1 \) many Dehn twists and proved that this is the least such number.

Note that, the above-generating sets contain only elements of infinite order. Maclachlan [15] proved that \( \text{Mod}(\Sigma_g) \) can also be generated by only using torsions. Wajnryb [20] proved that \( \text{Mod}(\Sigma_g) \) can be generated by two elements; one of order \( 4g+2 \) and the other a product of opposite Dehn twists. In this paper, we study the problem of generating \( \text{Mod}(\Sigma_g) \) by two torsion elements of small orders. Korkmaz [8] found a generating set for \( \text{Mod}(\Sigma_g) \) consisting of two torsion elements of order \( 4g+2 \). He also posed the following problem [10]: for which \( k < 4g+2 \), \( \text{Mod}(\Sigma_g) \) can be generated by two elements of order \( k \) (A similar question is also asked by Margalit [16])? In particular, what is the smallest such \( k \)?

We first prove that \( \text{Mod}(\Sigma_g) \) is generated by two elements of order \( g \) if \( g \geq 6 \).

Theorem 1. The mapping class group \( \text{Mod}(\Sigma_g) \) is generated by two elements of order \( g \) for \( g \geq 6 \).
We also obtain generating sets consisting of the elements of smaller orders.

**Theorem 2.** For \( g \geq 7 \) the mapping class group \( \text{Mod}(\Sigma_g) \) is generated by two elements of order \( g \) and order \( g' \) where \( g' \) is the least divisor of \( g \) such that \( g' > 2 \).

**Theorem 3.** For \( g \geq 3k^2 + 4k + 1 \) and any positive integer \( k \), the mapping class group \( \text{Mod}(\Sigma_g) \) is generated by two elements of order \( g/\gcd(g,k) \).

Since there is a surjective homomorphism from \( \text{Mod}(\Sigma_g) \) onto the symplectic group \( \text{Sp}(2g,\mathbb{Z}) \), we have the following immediate result:

**Corollary 4.** The symplectic group \( \text{Sp}(2g,\mathbb{Z}) \) is generated by two elements of order \( g \) for \( g \geq 6 \).

See [2, 7, 15, 17] or [14] for generating sets consisting of involutions, [11, 13, 18] or [4] for generating sets consisting of torsions and [19] or [1] for other generating sets for the mapping class groups.

2. Preliminaries

Throughout the paper, we always consider \( \Sigma_g \), where all genera are depicted as in Fig. 1. Note that the rotation by \( 2\pi/g \) degrees about \( z \)-axis, denoted by \( R \), is a well-defined self-diffeomorphism of \( \Sigma_g \). Following the notation in [21], we denote simple closed curves by lowercase letters \( a_i, b_i, c_i \) and corresponding positive Dehn twists by uppercase letters \( A_i, B_i, C_i \) or with the usual notation \( t_{ai}, t_{bi}, t_{ci} \), respectively. All indices should be considered modulo \( g \). For the composition of diffeomorphisms, \( f_1f_2 \) means that \( f_2 \) is first and then \( f_1 \) comes second as usual.

Commutativity, braid relation and the following basic facts on the mapping class group are used throughout the paper for many times: For any simple closed curves \( c_1 \) and \( c_2 \) on \( \Sigma_g \) and diffeomorphism \( f : \Sigma_g \to \Sigma_g \), \( ft_{c_1}f^{-1} = t_{f(c_1)} \); \( c_1 \) is isotopic to \( c_2 \) if and only if \( t_{c_1} = t_{c_2} \) in \( \text{Mod}(\Sigma_g) \); and if \( c_1 \) and \( c_2 \) are disjoint, then \( t_{c_1}(c_2) = c_2 \). We always refer to [5] for all the remaining properties of the mapping class groups.

Now, let us present Humphries minimal generating set for \( \text{Mod}(\Sigma_g) \):

**Theorem 5.** (Dehn–Lickorish–Humphries) The mapping class group \( \text{Mod}(\Sigma_g) \) is generated by the set \( \{A_1, A_2, B_1, B_2, \ldots, B_g, C_1, C_2, \ldots, C_{g-1}\} \).

It is easy to see that the rotation \( R \) satisfies that \( R(a_k) = a_{k+1} \), \( R(b_k) = b_{k+1} \) and \( R(c_k) = c_{k+1} \). Deducing from Theorem 5, Korkmaz [9] showed that the mapping class group is generated by four elements. Note that his first element is the rotation \( R \) and others are products of one positive and one negative Dehn twists.

**Theorem 6.** If \( g \geq 3 \), then the mapping class group \( \text{Mod}(\Sigma_g) \) is generated by the four elements \( R, A_1A_2^{-1}, B_1B_2^{-1}, C_1C_2^{-1} \).

The next result easily follows from Theorem 6.
Corollary 7. If $g \geq 3$, then the mapping class group $\text{Mod}(\Sigma_g)$ is generated by the four elements $R, A_1B_1^{-1}, B_1C_1^{-1}, C_1B_2^{-1}$.

Proof. Let $H$ be the subgroup of $\text{Mod}(\Sigma_g)$ generated by the set 
\{ $R, A_1B_1^{-1}, B_1C_1^{-1}, C_1B_2^{-1}$ \}.

It is enough to show that $H$ contains the elements $A_1A_2^{-1}, B_1B_2^{-1}$ and $C_1C_2^{-1}$ by Theorem 6.

It is easy to see that $B_2A_2^{-1} \in H$ since $B_2A_2^{-1} = R(B_1A_1^{-1})R^{-1} \in H$ and $B_2C_2^{-1} = R(B_1C_1^{-1})R^{-1} \in H$.

One can also show that $B_1B_2^{-1} = (B_1C_1^{-1})(C_1B_2^{-1}) \in H$. Similarly, we have that $C_1C_2^{-1} = (C_1B_2^{-1})(B_2C_2^{-1}) \in H$ and we also have that $A_1A_2^{-1} = (A_1B_1^{-1})(B_1B_2^{-1})(B_2A_2^{-1}) \in H$.

It follows from Theorem 6 that $H = \text{Mod}(\Sigma_g)$, completing the proof of the corollary. □

3. Twelve New Generating Sets for $\text{Mod}(\Sigma_g)$

In this section, we introduce twelve new generating sets consisting of two elements of small orders for the mapping class group. Following the ideas in [9], we construct generating sets consisting of $R$, an element of order $g$, and another element which can be expressed as a product of Dehn twists.

The corollaries in this section are mainly the corollaries of Theorem 6. We use the first four corollaries to create generating sets of elements of order $g$. We use Corollaries 12, 13, 14, 15, 16 and 20 to create generating sets of elements of order $g$ and $g'$, where $g'$ is the least divisor of $g$ greater than 2. In the following, we give four new generating sets to prove Theorem 1.

Corollary 8. If $g = 6$, then the mapping class group $\text{Mod}(\Sigma_6)$ is generated by the two elements $R$ and $C_1B_4A_4A_1^{-1}B_5^{-1}C_2^{-1}$.

Proof. Let $F_1 = C_1B_4A_4A_1^{-1}B_5^{-1}C_2^{-1}$. Let us denote by $H$ the subgroup of $\text{Mod}(\Sigma_6)$ generated by the set \{ $R, F_1$ \}.

If $H$ contains the elements $A_1A_2^{-1}, B_1B_2^{-1}$ and $C_1C_2^{-1}$, then we are done by Theorem 6 (Fig. 2).
Figure 2. Proof of Corollary 8

Let

\[ F_2 = RF_1R^{-1} \]
\[ = R(C_1B_4A_6A_1^{-1}B_5^{-1}C_2^{-1})R^{-1} \]
\[ = RC_1R^{-1}RB_4R^{-1}RA_6R^{-1}RA_1^{-1}R^{-1}RB_5^{-1}R^{-1}RC_2^{-1}R^{-1} \]
\[ = Rt_{c_1}R^{-1}Rt_{b_4}R^{-1}Rt_{a_6}R^{-1}Rt_{a_1}^{-1}R^{-1}Rt_{b_5}^{-1}R^{-1}Rt_{c_2}^{-1}R^{-1} \]
\[ = t_{R(c_1)}^{-1}t_{R(b_4)}t_{R(a_6)}^{-1}t_{R(a_1)}^{-1}t_{R(b_5)}^{-1}t_{R(c_2)}^{-1} \]
\[ = t_{c_2}t_{b_5}t_{a_1}^{-1}t_{b_6}^{-1}t_{c_3}^{-1} \]
\[ = C_2B_5A_1A_2^{-1}B_6^{-1}C_3^{-1} \]

and

\[ F_3 = F_2^{-1} = C_3B_6A_2A_1^{-1}B_5^{-1}C_2^{-1}. \]
We have \( F_3 F_1(c_3, b_6, a_2, a_1, b_5, c_2) = (b_4, a_6, a_2, a_1, b_5, c_2) \) so that \( F_4 = B_4 A_6 A_2 A_1^{-1} B_5^{-1} C_2^{-1} \in H \). Note that \( F_3 F_1(c_3) = b_4 \) since

\[
t_{F_3 F_1(c_3)} = (F_3 F_1)t_{c_3}(F_3 F_1)^{-1}
\]

\[
= F_3 F_1 C_3 F_1^{-1} F_3^{-1}
\]

\[
= C_3 B_4 C_3 B_4^{-1} C_3^{-1}
\]

\[
= (t_{c_3} t_{b_4}) t_{c_3} (t_{c_3} t_{b_4})^{-1}
\]

\[
= t_{c_3} t_{b_4}(c_3)
\]

\[
= t_{b_4}.
\]

We get \( F_1 F_4^{-1} = C_1 A_2^{-1} \in H \). Hence, by conjugating \( C_1 A_2^{-1} \) with \( R \) iteratively, we get \( C_i A_{i+1}^{-1} \in H \) for all \( i \).

Let

\[
F_5 = F_4(C_2 A_3^{-1}) = B_4 A_6 A_2 A_1^{-1} B_5^{-1} A_3^{-1},
\]

\[
F_6 = RF_5 R^{-1} = B_5 A_1 A_3 A_2^{-1} B_6^{-1} A_4^{-1}
\]

and

\[
F_7 = F_5 F_6 = B_4 A_6 B_6^{-1} A_4^{-1}.
\]

Hence, \( (C_4 A_5^{-1}) F_7(c_4, a_5) = (b_4, a_5) \) so that \( B_4 A_5^{-1} \in H \). We then get \( B_i A_{i+1}^{-1} \in H \) for all \( i \) and \( B_i C_i^{-1} = (B_i A_{i+1}^{-1})(A_i+1C_i^{-1}) \in H \) for all \( i \).

Similarly, we see that \( (A_4 B_3^{-1}) F_7(a_4, b_3) = (b_4, b_3) \) so that \( B_4 B_3^{-1} \in H \) implying that \( B_i+1B_{i+1}^{-1} \in H \) for all \( i \).

In particular, we get \( B_1 B_2^{-1} \in H \).

Finally, we have \( C_1 C_2^{-1} = (C_1 B_1^{-1})(B_1 B_2^{-1})(B_2 C_2^{-1}) \in H \) and \( A_1 A_2^{-1} = (A_1 B_6^{-1})(B_6 B_5^{-1})(B_1 A_2^{-1}) \in H \).

It follows from Theorem 6 that \( H = \text{Mod}(\Sigma_6) \), completing the proof of the corollary.

\[\square\]

**Corollary 9.** If \( g = 7 \), then the mapping class group \( \text{Mod}(\Sigma_g) \) is generated by the two elements \( R \) and \( C_1 B_4 A_6 A_7^{-1} B_5^{-1} C_2^{-1} \) Fig. 3.

**Proof.** Let \( F_1 = C_1 B_4 A_6 A_7^{-1} B_5^{-1} C_2^{-1} \). Let \( H \) denote the subgroup of \( \text{Mod}(\Sigma_7) \) generated by the set \( \{R, F_1\} \).

Let

\[
F_2 = RF_1 R^{-1} = C_2 B_5 A_7 A_1^{-1} B_6^{-1} C_3^{-1}
\]

and

\[
F_3 = F_2^{-1} = C_3 B_6 A_1 A_7^{-1} B_5^{-1} C_2^{-1}.
\]

We have \( F_3 F_1(c_3, b_6, a_1, a_7, b_5, c_2) = (b_4, a_6, a_1, a_7, b_5, c_2) \) so that \( F_4 = B_4 A_6 A_1 A_7^{-1} B_5^{-1} C_2^{-1} \in H \).

Let

\[
F_5 = RF_4 R^{-1} = B_5 A_7 A_2 A_1^{-1} B_6^{-1} C_3^{-1}
\]

and

\[
F_6 = F_5^{-1} = C_3 B_6 A_1 A_2^{-1} A_7^{-1} B_5^{-1}.
\]
Figure 3. Proof of Corollary 9

We get $F_6F_4(c, b_6, a_1, a_2, a_7, b_5) = (b_4, a_6, a_1, a_2, a_7, b_5)$ so that $F_7 = B_4A_6A_1A_2^{-1}A_7^{-1}B_5^{-1} \in H$.

Let $F_8 = RF_7R^{-1} = B_5A_7A_2A_3^{-1}A_1^{-1}B_6^{-1}$
and

\[ F_9 = F_8^{-1} = B_6 A_1 A_3 A_2^{-1} A_7^{-1} B_5^{-1}. \]

Hence, we have \( F_9 F_7(b_6, a_1, a_3, a_2, a_7, b_5) = (a_6, a_1, a_3, a_2, a_7, b_5) \) so that \( F_{10} = A_6 A_1 A_3 A_2^{-1} A_7^{-1} B_5^{-1} \in H \).

We then see that \( F_{10} F_8 = A_6 B_6^{-1} \in H \) and by conjugating \( A_6 B_6^{-1} \) with \( R \) iteratively, we get \( A_i B_i^{-1} \in H \) for all \( i \).

Let

\[ F_{11} = (B_6 A_6^{-1}) F_4 = B_4 B_6 A_1 A_7^{-1} B_5^{-1} C_2^{-1} \]

and

\[ F_{12} = R^{-1} F_{11} R = B_3 B_5 A_7 A_6^{-1} B_4^{-1} C_1^{-1}. \]

We also have \( F_{12} F_1 = B_3 C_2^{-1} \in H \) and then \( B_{i+1} C_i^{-1} \in H \) for all \( i \).

Let

\[ F_{13} = (B_6 A_6^{-1}) F_1 (A_7 B_7^{-1}) = C_1 B_4 B_6 B_7^{-1} B_5^{-1} C_2^{-1} \]

and

\[ F_{14} = RF_{13} R^{-1} = C_2 B_5 B_7 B_1^{-1} B_6^{-1} C_3^{-1}. \]

Finally, \( F_{13} F_{14} (C_3 B_4^{-1}) = C_1 B_1^{-1} \in H \) which gives \( C_i B_i^{-1} \in H \) for all \( i \).

It follows from Corollary 7 that \( H = \text{Mod}(\Sigma_7) \), which finishes the proof. \( \square \)

**Corollary 10.** If \( g = 8 \), then the mapping class group \( \text{Mod}(\Sigma_8) \) is generated by the two elements \( R \) and \( B_1 C_4 A_7 A_8^{-1} C_5^{-1} B_2^{-1} \) Fig. 4.

**Proof.** Let \( F_1 = B_1 C_4 A_7 A_8^{-1} C_5^{-1} B_2^{-1} \) and let \( H \) be the subgroup of \( \text{Mod}(\Sigma_8) \) generated by the set \( \{R, F_1\} \).

Let us consider the elements

\[ F_2 = RF_1 R^{-1} = B_2 C_5 A_8 A_1^{-1} C_6^{-1} B_3^{-1} \]

and

\[ F_3 = F_2^{-1} = B_3 C_6 A_1 A_8^{-1} C_5^{-1} B_2^{-1}. \]

We have \( F_3 F_1(b_3, c_6, a_1, a_8, c_5, b_2) = (b_3, c_6, b_1, a_8, c_5, b_2) \) so that

\[ F_4 = B_3 C_6 B_1 A_8^{-1} C_5^{-1} B_2^{-1} \in H. \]

We get that \( F_3 F_4^{-1} = B_1 A_1^{-1} \) and then by conjugating \( B_1 A_1^{-1} \) with \( R \) iteratively, we get \( B_i A_i^{-1} \in H \) for all \( i \).

Let

\[ F_5 = R^2 F_1 R^{-2} = B_3 C_6 A_1 A_2^{-1} C_7^{-1} B_4^{-1}, \]

\[ F_6 = F_5^{-1} = B_4 C_7 A_2 A_1^{-1} C_6^{-1} B_3^{-1} \]

and

\[ F_7 = (B_2 A_2^{-1}) F_6 (A_1 B_1^{-1}) = B_4 C_7 B_2 B_1^{-1} C_6^{-1} B_3^{-1}. \]
Figure 4. Proof of Corollary 10

We also have $F_7F_1(b_4, c_7, b_2, b_1, c_6, b_3) = (c_4, c_7, b_2, b_1, c_6, b_3)$ so that $F_8 = C_4C_7B_2B_1^{-1}C_6^{-1}B_3^{-1} \in H$. It is easy to check that $F_8F_7^{-1} = C_4B_4^{-1} \in H$ and then we get $C_iB_i^{-1} \in H$ for all $i$.

Let

$$F_9 = RF_7R^{-1} = B_5C_8B_3B_2^{-1}C_7^{-1}B_4^{-1}$$

and

$$F_{10} = (C_4B_4^{-1})F_9^{-1}(B_5C_5^{-1}) = C_4C_7B_2B_3^{-1}C_8^{-1}C_5^{-1}.$$
Similarly, we see that $F_{10}F_8(c_4,c_7,b_2,b_3,c_8,c_5) = (c_4,c_7,b_2,b_3,b_1,c_5)$ so that $F_{11} = C_4C_5B_2B_3^{-1}B_1^{-1}C_5^{-1} = H$. Thus, $F_{10}^{-1}F_{11} = C_8B_1^{-1} ∈ H$ and then we get $C_iB_{i+1}^{-1} ∈ H$ for all $i$.

It follows from Corollary 7 that $H = \text{Mod}(\Sigma_8)$, completing the proof of the corollary.

\[\square\]

**Corollary 11.** If $g ≥ 9$, then the mapping class group $\text{Mod}(\Sigma_g)$ is generated by the two elements $R$ and $C_1B_4A_7A_8^{-1}B_5^{-1}C_2^{-1}$ (Fig. 5).

**Proof.** Let $F_1 = C_1B_4A_7A_8^{-1}B_5^{-1}C_2^{-1}$. Let us denote by $H$ the subgroup of $\text{Mod}(\Sigma_g)$ generated by the set $\{R,F_1\}$.

Let

$F_2 = RF_1R^{-1} = C_2B_5A_8A_9^{-1}B_6^{-1}C_3^{-1}$

and

$F_3 = F_2^{-1} = C_3B_6A_9A_8^{-1}B_5^{-1}C_2^{-1}$.

We have $F_3F_1(c_3,b_9,a_9,a_8,b_5,c_2) = (b_4,b_6,a_9,a_8,b_5,c_2)$ so that $F_4 = B_4B_6A_9A_8^{-1}B_5^{-1}C_2^{-1} ∈ H$.

Hence, we see that $F_4F_3^{-1} = B_4C_3^{-1} ∈ H$ and then by conjugating $B_4C_3^{-1}$ with $R$ iteratively, we get $B_{i+1}C_i^{-1} ∈ H$ for all $i$.

Let

$F_5 = F_4(C_2B_3^{-1}) = B_4B_6A_9A_8^{-1}B_5^{-1}B_3^{-1}$,

$F_6 = R^{-2}F_5R^2 = B_2B_4A_7A_6^{-1}B_3^{-1}B_1^{-1}$

and

$F_7 = F_6^{-1} = B_1B_3A_6A_7^{-1}B_4^{-1}B_2^{-1}$.

We get $F_7F_5(b_1,b_3,a_6,a_7,b_4,b_2) = (b_1,b_3,b_6,a_7,b_4,b_2)$ so that $F_8 = B_1B_3B_6A_7^{-1}B_4^{-1}B_2^{-1} ∈ H$.

We also have $F_8F_7^{-1} = B_6A_6^{-1} ∈ H$ and then $B_iA_i^{-1} ∈ H$ for all $i$.

Let

$F_9 = F_5(A_8B_8^{-1})(B_8C_7^{-1}) = B_4B_6A_9C_7^{-1}B_5^{-1}B_3^{-1}$,

$F_{10} = R^{-1}F_9R = B_3B_5A_8C_6^{-1}B_4^{-1}B_2^{-1}$

and

$F_{11} = F_{10}^{-1} = B_2B_4C_6A_8^{-1}B_5^{-1}B_3^{-1}$.

Hence, we have $F_{11}F_9(b_2,b_4,c_6,a_8,b_5,b_3) = (b_2,b_4,b_6,a_8,b_5,b_3)$ so that $F_{12} = B_2B_4B_6A_8^{-1}B_5^{-1}B_3^{-1} ∈ H$.

Finally, we see that $F_{12}F_{11}^{-1} = B_6C_6^{-1} ∈ H$ and then $B_iC_i^{-1} ∈ H$ for all $i$.

It follows from Corollary 7 that $H = \text{Mod}(\Sigma_g)$, completing the proof of the corollary.

\[\square\]

We introduce six new generating sets in Corollaries 12, 13, 14, 15, 16, and 20 to prove Theorem 2.

**Corollary 12.** If $g = 8$, then the mapping class group $\text{Mod}(\Sigma_g)$ is generated by the two elements $R$ and $B_1A_5C_5C_7^{-1}A_7^{-1}B_3^{-1}$.
Proof. Let \( F_1 = B_1 A_5 C_5 C_7^{-1} A_7^{-1} B_3^{-1} \). Let us denote by \( H \) the subgroup of \( \text{Mod}(\Sigma_g) \) generated by the set \( \{ R, F_1 \} \).

Let

\[
F_2 = RF_1 R^{-1} = B_2 A_6 C_6 C_8^{-1} A_8^{-1} B_4^{-1}
\]

and

\[
F_3 = F_2^{-1} = B_4 A_8 C_8 C_6^{-1} A_6^{-1} B_2^{-1}.
\]

Figure 5. Proof of Corollary 11
We have $F_3F_1(b_4,a_8,c_8,c_6,a_6,b_2) = (b_4,a_8,b_1,c_6,a_6,b_2)$ so that $F_4 = B_4A_8B_1C_6^{-1}A_6^{-1}B_2^{-1} \in H$.

We get $F_4F_3^{-1} = B_1C_8^{-1} \in H$ and then by conjugating $B_1C_8^{-1}$ with $R$ iteratively, we get $B_{i+1}C_i^{-1} \in H$ for all $i$.

Let

$$F_5 = RF_4R^{-1} = B_5A_1B_2C_7^{-1}A_7^{-1}B_3^{-1}.$$ 

We also have $F_5F_2(b_5,a_1,b_2,c_7,a_7,b_3) = (b_5,b_1,b_2,c_7,a_7,b_3)$ so that $F_6 = B_5B_1B_2C_7^{-1}A_7^{-1}B_3^{-1} \in H$.

Hence, we get $F_6F_5^{-1} = B_1A_1^{-1} \in H$ and then $B_iA_i^{-1} \in H$ for all $i$.

Let

$$F_7 = (C_4B_5^{-1})F_6(C_7B_8^{-1})(A_7B_7^{-1}) = C_4B_1B_2B_3^{-1}B_8^{-1}B_7^{-1},$$

$$F_8 = RF_7R^{-1} = C_5B_2B_3B_4^{-1}B_1^{-1}B_8^{-1}$$

and

$$F_9 = F_8^{-1} = B_8B_1B_4B_3^{-1}B_2^{-1}C_5^{-1}.$$ 

Similarly, check that $F_9F_7(b_8,b_1,b_4,b_3,b_2,c_5) = (b_8,b_1,c_4,b_3,b_2,c_5)$ so that $F_{10} = B_8B_1C_4B_3^{-1}B_2^{-1}C_5^{-1} \in H$.

Finally, we see that $F_{10}F_9^{-1} = C_4B_4^{-1} \in H$ and then $C_iB_i^{-1} \in H$ for all $i$.

It follows from Corollary 7 that $H = \text{Mod}(\Sigma_8)$, completing the proof of the corollary. 

\[ \square \]

**Corollary 13.** If $g = 9$, then the mapping class group $\text{Mod}(\Sigma_9)$ is generated by the two elements $R$ and $B_1A_3C_5C_8^{-1}A_6^{-1}B_4^{-1}$.

**Proof.** Let $F_1 = B_1A_3C_5C_8^{-1}A_6^{-1}B_4^{-1}$. Let us denote by $H$ the subgroup of $\text{Mod}(\Sigma_9)$ generated by the set $\{R,F_1\}$.

Let

$$F_2 = RF_1R^{-1} = B_2A_4A_5C_9^{-1}A_7^{-1}B_5^{-1}$$

and

$$F_3 = F_2^{-1} = B_5A_7C_9C_6^{-1}A_4^{-1}B_2^{-1}.$$ 

We have that $F_3F_1(b_5,a_7,c_9,c_6,a_4,b_2) = (c_5,a_7,b_1,c_6,b_4,b_2)$ so that $F_4 = C_5A_7B_1C_6^{-1}B_4^{-1}B_2^{-1} \in H$.

Let

$$F_5 = RF_4R^{-1} = C_6A_8B_2C_7^{-1}B_5^{-1}B_3^{-1}$$

and

$$F_6 = F_5^{-1} = B_3B_5C_7B_2^{-1}A_8^{-1}C_6^{-1}.$$ 

We get $F_6F_4(b_3,b_5,c_7,b_2,a_8,c_6) = (b_3,c_5,c_7,b_2,a_8,c_6)$ so that $F_7 = B_3C_5C_7B_2^{-1}A_8^{-1}C_6^{-1} \in H$.

We see that $F_7F_6^{-1} = C_5B_5^{-1} \in H$ and then by conjugating $C_5B_5^{-1}$ with $R$ iteratively, we get $C_iB_i^{-1} \in H$ for all $i$. 

\[ \square \]
Let
\[ F_8 = (B_7C_7^{-1})F_6(C_6B_6^{-1}) = B_3B_5B_7B_7^{-1}A_8^{-1}B_6^{-1}, \]
\[ F_9 = RF_8R^{-1} = B_4B_6B_8B_3^{-1}A_9^{-1}B_7^{-1} \]
and
\[ F_{10} = F_9^{-1} = B_7A_9B_3B_8^{-1}B_6^{-1}B_4^{-1}. \]

We also have \( F_{10}F_8(b_7, a_9, b_3, b_8, b_6, b_4) = (b_7, a_9, b_3, a_8, b_6, b_4) \) so that \( F_{11} = B_7A_9B_3A_8^{-1}B_6^{-1}B_4^{-1} \in H. \)

Finally, we have \( F_{11}^{-1}F_{10} = A_8B_8^{-1} \in H \) and then \( A_iB_i^{-1} \in H \) for all \( i \).

Check \( F_4(B_4A_4^{-1})F_2(B_5C_5^{-1}) = B_1C_9^{-1} \in H \) and then \( B_{i+1}C_i^{-1} \in H \) for all \( i \).

It follows from Corollary 7 that \( H = \text{Mod}(\Sigma_9) \), completing the proof of the corollary. \( \square \)

**Corollary 14.** If \( g = 10 \), then the mapping class group \( \text{Mod}(\Sigma_9) \) is generated by the two elements \( R \) and \( A_1C_1B_3B_7^{-1}C_5^{-1}A_5^{-1} \).

**Proof.** Let \( F_1 = A_1C_1B_3B_7^{-1}C_5^{-1}A_5^{-1} \). Let us denote by \( H \) the subgroup of \( \text{Mod}(\Sigma_{10}) \) generated by the set \( \{R, F_1\} \).

Let
\[ F_2 = RF_1R^{-1} = A_2C_2B_4B_8^{-1}C_6^{-1}A_6^{-1}. \]

We have \( F_2F_1(a_2, c_2, b_4, b_8, c_6, a_6) = (a_2, b_3, a_2, b_8, b_7, a_6) \) so that \( F_3 = A_2B_3B_4B_8^{-1}B_7^{-1}A_6^{-1} \in H. \)

Let
\[ F_4 = R^4F_3R^{-4} = A_6B_7B_8B_2^{-1}B_1^{-1}A_{10}^{-1} \]
and
\[ F_5 = F_4^{-1} = A_{10}B_1B_2B_8^{-1}B_7^{-1}A_6^{-1}. \]

We get \( F_5F_3(a_{10}, b_1, b_2, b_8, b_7, a_6) = (a_{10}, b_1, a_2, b_8, b_7, a_6) \) so that \( F_6 = A_{10}B_1A_2B_8^{-1}B_7^{-1}A_6^{-1} \in H. \)

We see that \( F_6F_5^{-1} = A_2B_2^{-1} \in H \) and then by conjugating \( A_2B_2^{-1} \) with \( R \) iteratively, we get \( A_iB_i^{-1} \in H \) for all \( i \).

Let
\[ F_7 = (B_2A_2^{-1})(A_3B_3^{-1})F_3(B_7A_7^{-1})(A_6B_6^{-1}) = B_2A_3B_4B_8^{-1}A_7^{-1}B_6^{-1}, \]
\[ F_8 = RF_2F_3^{-1}R^{-1}F_7 = B_2A_3C_3C_7^{-1}A_7^{-1}B_6^{-1}, \]
\[ F_9 = F_8^{-1} = B_6A_7C_7C_3^{-1}A_3^{-1}B_2^{-1} \]
and
\[ F_{10} = R^4F_9R^{-4} = B_{10}A_1C_1C_7^{-1}A_7^{-1}B_6^{-1}. \]

We also have \( F_{10}F_8(b_{10}, a_1, c_1, c_7, a_7, b_6) = (b_{10}, a_1, b_2, c_7, a_7, b_6) \) so that \( F_{11} = B_{10}A_1B_2C_7^{-1}A_7^{-1}B_6^{-1} \in H. \)

We then get \( F_{11}F_{10}^{-1} = B_2C_1^{-1} \in H \) and then \( B_{i+1}C_i^{-1} \in H \) for all \( i \).
Let
\[ F_{12} = (B_2A_2^{-1})F_3(A_6B_6^{-1})(B_6C_5^{-1})(B_7A_7^{-1})(B_8A_8^{-1}) = B_2B_3B_4A_8^{-1}A_7^{-1}C_5^{-1}, \]
\[ F_{13} = F_{12}^{-1} = C_5A_7A_8B_4^{-1}B_3^{-1}B_2^{-1} \]
and
\[ F_{14} = RF_{13}R^{-1} = C_6A_8A_9B_5^{-1}B_4^{-1}B_3^{-1}. \]

Hence, we have \( F_{14}F_{12}(c_6, a_8, a_9, b_5, b_4, b_3) = (c_6, a_8, a_9, c_5, b_4, b_3) \) so that
\[ F_{15} = C_6A_8A_9C_5^{-1}B_4^{-1}B_3^{-1} \in H. \]

Finally, we see that \( F_{15}^{-1}F_{14} = C_5B_5^{-1} \in H \) and then \( C_iB_i^{-1} \in H \) for all \( i \).

It follows from Corollary 7 that \( H = \text{Mod}(\Sigma_{10}) \), completing the proof of the corollary.

**Corollary 15.** If \( g \geq 13 \), then the mapping class group \( \text{Mod}(\Sigma_g) \) is generated by the two elements \( R \) and \( A_1B_4C_8C_{10}^{-1}B_6^{-1}A_3^{-1} \).

**Proof.** Let \( F_1 = A_1B_4C_8C_{10}^{-1}B_6^{-1}A_3^{-1} \). Let us denote by \( H \) the subgroup of \( \text{Mod}(\Sigma_g) \) generated by the set \( \{ R, F_1 \} \).

Let
\[ F_2 = RF_1R^{-1} = A_2B_5C_9C_{11}^{-1}B_7^{-1}A_4^{-1} \]
and
\[ F_3 = F_2^{-1} = A_4B_7C_{11}^{-1}C_9^{-1}B_5^{-1}A_2^{-1}. \]

We have \( F_3F_1(a_4, b_7, c_{11}, c_9, b_5, a_2) = (b_4, b_7, c_{11}, c_9, b_5, a_2) \) so that
\[ F_4 = B_4B_7C_{11}^{-1}C_9^{-1}B_5^{-1}A_2^{-1} \in H. \]

We see that \( F_4F_3^{-1} = B_4A_4^{-1} \) and then by conjugating \( B_4A_4^{-1} \) with \( R \) iteratively, we get \( B_iA_i^{-1} \in H \) for all \( i \).

Let
\[ F_5 = R^2F_1R^{-2} = A_3B_6C_{10}C_{12}^{-1}B_8^{-1}A_5^{-1} \]
and
\[ F_6 = F_5^{-1} = A_5B_8C_{12}C_{10}^{-1}B_6^{-1}A_3^{-1}. \]

We also have \( F_6F_1(a_5, b_8, c_{12}, c_{10}, b_6, a_3) = (a_5, c_8, c_{12}, c_{10}, b_6, a_3) \) so that
\[ F_7 = A_5C_8C_{12}C_{10}^{-1}B_6^{-1}A_3^{-1} \in H. \]

We get \( F_7F_6^{-1} = C_8B_8^{-1} \) and then \( C_iB_i^{-1} \in H \) for all \( i \).

Let
\[ F_8 = (A_4B_4^{-1})F_1(A_3B_3^{-1}) = A_1A_4C_8C_{10}^{-1}B_6^{-1}B_3^{-1}, \]
\[ F_9 = R^3F_8R^{-3} = A_4A_7C_{11}C_{13}^{-1}B_9^{-1}B_6^{-1} \]
and
\[ F_{10} = F_9^{-1} = B_6B_9C_{13}C_{11}^{-1}A_7^{-1}A_4^{-1}. \]

Hence, check that \( F_{10}F_8(b_6, b_9, c_{13}, c_{11}, a_7, a_4) = (b_6, c_8, c_{13}, c_{11}, a_7, a_4) \) so that
\[ F_{11} = B_6C_8C_{13}C_{11}^{-1}A_7^{-1}A_4^{-1} \in H. \]

Finally, we have \( F_{11}F_{10}^{-1} = C_8B_5^{-1} \) and then \( C_iB_i^{-1} \in H \) for all \( i \).

It follows from Corollary 7 that \( H = \text{Mod}(\Sigma_g) \), completing the proof of the corollary.
\[ \square \]
Corollary 16. If \( g \geq 12 \), then the mapping class group \( \text{Mod}(\Sigma_g) \) is generated by the two elements \( R \) and \( B_1A_3C_6C_{10}^{-1}A_7^{-1}B_5^{-1} \).

Proof. Let \( F_1 = B_1A_3C_6C_{10}^{-1}A_7^{-1}B_5^{-1} \). Let us denote by \( H \) the subgroup of \( \text{Mod}(\Sigma_g) \) generated by the set \( \{R, F_1\} \).

Let \( F_2 = RF_1R^{-1} = B_2A_4C_7C_{11}^{-1}A_8^{-1}B_6^{-1} \)
and \( F_3 = F_2^{-1} = B_6A_8C_{11}C_7^{-1}A_4^{-1}B_2^{-1} \).

We have \( F_3F_1(b_6, a_8, c_{11}, c_7, a_4, b_2) = (c_6, a_8, c_{11}, c_7, a_4, b_2) \) so that \( F_4 = C_6A_8C_{11}C_7^{-1}A_4^{-1}B_2^{-1} \in H \).

We get \( F_4F_3^{-1} = C_6B_6^{-1} \) in \( H \) and then by conjugating \( C_6B_6^{-1} \) with \( R \) iteratively, we get \( C_6B_6^{-1} \) for all \( i \).

Let \( F_5 = F_1(C_{10}B_{10}^{-1})(B_5C_5^{-1}) = B_1A_3C_6B_{10}^{-1}A_7^{-1}C_5^{-1} \)
and \( F_6 = R^2F_5R^{-2} = B_3A_5C_8B_{12}^{-1}A_9^{-1}C_7^{-1} \).

We also have \( F_6F_5(b_3, a_5, c_8, b_{12}, a_9, c_7) = (a_3, a_5, c_8, b_{12}, a_9, c_7) \) so that \( F_7 = A_3A_5C_8B_{12}^{-1}A_9^{-1}C_7^{-1} \in H \).

We get \( F_7F_6^{-1} = A_3B_3^{-1} \) in \( H \) and then \( A_1B_1^{-1} \) in \( H \) for all \( i \).

Let \( F_8 = (C_1B_1^{-1})(B_3A_3^{-1})F_1(B_5C_5^{-1}) = C_1B_3C_6C_{10}^{-1}A_7^{-1}C_5^{-1} \)
and \( F_9 = RF_8R^{-1} = C_2B_4C_7C_{11}^{-1}A_8^{-1}C_6^{-1} \).

Then check that \( F_9F_8(c_2, b_4, c_7, c_{11}, a_8, c_6) = (b_3, b_4, c_7, c_{11}, a_8, c_6) \) so that \( F_{10} = B_3B_4C_7C_{11}^{-1}A_8^{-1}C_6^{-1} \in H \).

Finally, we have \( F_{10}F_9^{-1} = B_3C_2^{-1} \) in \( H \) and then \( B_{i+1}C_i^{-1} \) in \( H \) for all \( i \).

It follows from Corollary 7 that \( H = \text{Mod}(\Sigma_g) \), completing the proof of the corollary.

\[ \square \]

Lemma 17. If \( g \geq 11 \), then the mapping class group \( \text{Mod}(\Sigma_g) \) is generated by the two elements \( R \) and \( A_1B_2C_4C_{g-1}^{-1}B_{g-3}^{-1}A_{g-4}^{-1} \).

Proof. Let \( F_1 = A_1B_2C_4C_{g-1}^{-1}B_{g-3}^{-1}A_{g-4}^{-1} \). Let us denote by \( H \) the subgroup of \( \text{Mod}(\Sigma_g) \) generated by the set \( \{R, F_1\} \).

Let \( F_2 = RF_1R^{-1} = A_2B_3C_5C_{g-1}^{-1}B_{g-2}^{-1}A_{g-3}^{-1} \).

We have \( F_2F_1(a_2, b_3, c_5, c_g, b_{g-2}, a_{g-3}) = (b_2, b_3, c_5, c_g, b_{g-2}, b_{g-3}) \) so that \( F_3 = B_2B_3C_5C_{g-1}^{-1}B_{g-2}^{-1}B_{g-3}^{-1} \in H \).

Let \( F_4 = R^{-1}F_3R = B_1B_2C_4C_{g-1}^{-1}B_{g-3}^{-1}B_{g-4}^{-1} \).
and

\[ F_5 = F_3^{-1} = B_{g-3}B_{g-2}C_gC_5^{-1}B_3^{-1}B_2^{-1}. \]

We also have \( F_5F_4(b_{g-3}, b_{g-2}, c_g, c_5, b_3, b_2) = (b_{g-3}, b_{g-2}, b_1, c_5, b_3, b_2) \) so that \( F_6 = B_{g-3}B_{g-2}B_1C_5^{-1}B_3^{-1}B_2^{-1} \in H. \)

We see that \( F_6F_5^{-1} = B_1C_g^{-1} \in H \) and then by conjugating \( B_1C_g^{-1} \) with \( R \) iteratively, we get \( B_{i+1}C_g^{-1} \in H \) for all \( i \).

Let

\[ F_7 = (C_{g-3}B_{g-2})(C_gB_{g-3})F_6 = C_{g-3}C_g-4B_1C_5^{-1}B_3^{-1}B_2^{-1} \]

and

\[ F_8 = R^2F_7R^{-2} = C_{g-1}C_{g-2}B_3C_7^{-1}B_5^{-1}B_4^{-1}. \]

We have \( F_8F_7(c_{g-1}, c_{g-2}, b_3, c_7, b_5, b_4) = (c_{g-1}, c_{g-2}, b_3, c_7, c_5, b_4) \) so that \( F_9 = C_{g-1}C_{g-2}B_3C_7^{-1}C_5^{-1}B_1^{-1} \in H. \)

We then get \( F_9F_8^{-1} = C_5B_5^{-1} \in H \) and then \( C_iB_1^{-1} \in H \) for all \( i \).

Let

\[ F_{10} = F_1(B_{g-3}C_{g-3}) = A_1B_2C_4C_{g-1}C_{g-3}A_{g-4}^{-1} \]

and

\[ F_{11} = RF_{10}R^{-1} = A_2B_3C_5C_{g-1}C_{g-2}A_{g-3}. \]

Hence, we see \( F_1F_{10}(a_2, b_3, c_5, c_g, c_{g-2}, a_{g-3}) = (b_2, b_3, c_5, c_g, c_{g-2}, a_{g-3}) \) so that \( F_1 = B_2B_3C_5C_{g-1}C_{g-2}A_{g-3}^{-1} \in H. \)

Finally, we have \( F_1F_{11}^{-1} = B_2A_2^{-1} \in H \) and then \( B_iA_i^{-1} \in H \) for all \( i \).

It follows from Corollary 7 that \( H = \text{Mod}(\Sigma_g) \), completing the proof of the lemma.  

\[ \square \]

**Lemma 18.** If \( g \geq 13 \), then the mapping class group \( \text{Mod}(\Sigma_g) \) is generated by the two elements \( R \) and \( A_1B_2C_4C_{g-2}B_{g-4}A_{g-5}^{-1} \).

**Proof.** Let \( F_1 = A_1B_2C_4C_{g-2}B_{g-4}A_{g-5}^{-1} \). Let us denote by \( H \) the subgroup of \( \text{Mod}(\Sigma_g) \) generated by the set \( \{R, F_1\} \).

Let

\[ F_2 = RF_1R^{-1} = A_2B_3C_5C_{g-1}B_{g-3}A_{g-4}^{-1}. \]

We have \( F_2F_1(a_2, b_3, c_5, c_{g-1}, b_{g-3}, a_{g-4}) = (b_2, b_3, c_5, c_{g-1}, b_{g-3}, b_{g-4}) \) so that \( F_3 = B_2B_3C_5C_{g-1}B_{g-3}B_{g-4} \in H. \)

Let

\[ F_4 = F_2F_3^{-1} = A_2B_2^{-1}A_{g-4}B_{g-4}, \]

\[ F_5 = RF_4R^{-1} = A_3B_3^{-1}A_{g-3}B_{g-3}, \]

\[ F_6 = F_5F_3 = B_2A_3C_5C_{g-1}A_{g-3}B_{g-4}, \]

\[ F_7 = R^{-2}F_6R^2 = B_9A_1C_3C_{g-3}A_{g-5}B_{g-6}^{-1}, \]

and

\[ F_8 = F_7^{-1} = B_{g-6}A_{g-5}C_{g-3}A_{g-1}B_{g-1}^{-1}. \]
We get $F_8 F_6(b_{g-6}, a_{g-5}, c_{g-3}, c_3, a_1, b_g) = (b_{g-6}, a_{g-5}, c_{g-3}, c_3, a_1, c_{g-1})$ so that $F_9 = B_{g-6} A_{g-5} C_{g-3} C_{g-3} A_1 C_{g-1}^{-1} C_{g-1}^{-1} \in H$.

We see that $F_9 F_8^{-1} = C_{g-1} B_{g-1}^{-1} \in H$ and then by conjugating $C_{g-1} B_{g-1}^{-1}$ with $R$ iteratively, we get $C_i B_i^{-1} \in H$ for all $i$.

Let
\[
 F_{10} = F_3 (C_{g-1} B_g^{-1}) = B_2 B_3 C_5 B_g^{-1} B_{g-3} B_{g-4}^{-1}
\]
and
\[
 F_{11} = R^2 F_1 R^{-2} = B_4 B_5 C_7 B_2^{-1} B_{g-1} B_{g-2}^{-1}.
\]

We also have $F_{11} F_1 = (b_4, c_5, b_7, b_2, b_{g-1}, b_{g-2}) = (b_4, c_5, b_7, b_2, b_{g-1}, b_{g-2})$ so that $F_{12} = B_4 C_5 C_7 B_2^{-1} B_{g-1} B_{g-2}^{-1} \in H$.

We then get $F_{12} F_{11}^{-1} = C_5 B_5^{-1} \in H$ and then $C_i B_i^{-1} \in H$ for all $i$.

Let
\[
 F_{13} = F_1 (B_{g-4} C_{g-4}^{-1}) = A_1 B_2 C_4 C_{g-2} C_{g-4}^{-1} A_{g-5}^{-1}
\]
and
\[
 F_{14} = R F_{13} R^{-1} = A_2 B_3 C_5 C_{g-1}^{-1} C_{g-3} A_{g-4}^{-1}.
\]

Hence, $F_{14} F_{13}(a_2, b_3, c_5, c_{g-1}, c_{g-3}, a_{g-4}) = (b_2, b_3, c_5, c_{g-1}, c_{g-3}, a_{g-4})$ so that $F_{15} = B_2 B_3 C_5 C_{g-1}^{-1} C_{g-3} A_{g-4}^{-1} \in H$.

Finally, we have $F_{15} F_{14}^{-1} = B_2 A_2^{-1} \in H$ and then $B_i A_i^{-1} \in H$ for all $i$.

It follows from Corollary 7 that $H = \text{Mod}(\Sigma_g)$, completing the proof of the corollary. \(\square\)

**Lemma 19.** If $k \geq 7$ and $g \geq 2k + 1$, then the mapping class group $\text{Mod}(\Sigma_g)$ is generated by elements $R$ and $A_1 B_2 C_4 C_{g-1}^{-1} B_{g-2} A_{g-1}^{-1}$.

**Proof.** Let $F_1 = A_1 B_2 C_4 C_{g-k+4}^{-1} B_{g-k+2}^{-1} A_{g-k+1}^{-1}$. Let us denote by $H$ the subgroup of $\text{Mod}(\Sigma_g)$ generated by the set $\{R, F_1\}$.

Let
\[
 F_2 = R^{k-3} F_1 R^{3-k} = A_{k-2} B_{k-1} C_{k+1} C_{g-1}^{-1} B_{g-1} A_{g-2}^{-1}
\]
and
\[
 F_3 = F_2^{-1} = A_{g-2} B_{g-1} C_{k+1} C_{k+1}^{-1} B_{k-1} A_{k-2}^{-1}.
\]

$F_3 F_1(a_{g-2}, b_{g-1}, c_{k+1}, b_{k-1}, a_{k-2}) = (a_{g-2}, b_{g-1}, b_2, c_{k+1}, b_{k-1}, a_{k-2})$ so that $F_4 = A_{g-2} B_{g-1} B_2 C_{k+1} C_{k+1}^{-1} A_{k-2}^{-1} \in H$.

We get $F_4 F_3^{-1} = B_2 C_1^{-1} \in H$ and then by conjugating $B_2 C_1^{-1}$ with $R$ iteratively, we get $B_i C_i^{-1} \in H$ for all $i$.

Let
\[
 F_5 = F_1 (B_{g-k+2} C_{g-k+1}^{-1}) = A_1 B_2 C_4 C_{g-k+1}^{-1} C_{g-k+1} A_{g-k+1}^{-1}
\]
and
\[
 F_6 = R F_5 R^{-1} = A_2 B_3 C_5 C_{g-k+5} C_{g-k+2} A_{g-k+2}^{-1}.
\]

$F_6 F_5(a_2, b_3, c_{g-k+5}, c_{g-k+2}, a_{g-k+2}) = (b_2, b_3, c_5, c_{g-k+5}, c_{g-k+2}, a_{g-k+2})$.\]
so that $F_7 = B_2B_3C_5C_{g-k+5}A_{g-k+2}^{-1}C_{g-k+2}^{-1}$.

We then get $F_7F_6^{-1} = B_2A_2^{-1} \in H$ and then $B_iA_i^{-1} \in H$ for all $i$.

Let

$$F_8 = R^{k-2}F_6R^{2-k} = A_kB_{k+1}C_{k+3}C_{g-1}^{-1}A_g^{-1}$$

and

$$F_9 = F_8^{-1} = A_gC_gC_{k-3}B_{k+1}^{-1}A_k^{-1}.$$

We have $F_9 = (a_g, c_g, c_g, c_{k+3}, b_{k+1}, a_k) = (a_g, c_g, b_3, c_{k+3}, b_{k+1}, a_k)$ so that $F_{10} = A_gC_2B_{k-3}B_{k+1}^{-1}A_k^{-1} \in H$.

Finally, we see that $F_{10}F_9^{-1} = B_3C_{g-1}^{-1} \in H$ and then $B_iC_i^{-1} \in H$ for all $i$.

It follows from Corollary 7 that $H = \text{Mod}(\Sigma_g)$, completing the proof of the lemma.

\textbf{Corollary 20.} If $k \geq 5$ and $g \geq 2k+1$, then the mapping class group $\text{Mod}(\Sigma_g)$ is generated by elements $R$ and $A_1B_2C_4C_{g-k+4}B_{g-k+2}^{-1}A_{g-k+1}^{-1}$.

\textbf{Proof.} It directly follows from Lemmas 17, 18 and 19.

\section{4. Main Results}

In this section, we prove the main results of this paper. The following Lemma is useful to decide the order of an element.

\textbf{Lemma 21.} If $R$ is an element of order $k$ in a group $G$ and if $x$ and $y$ are elements in $G$ satisfying $RxR^{-1} = y$, then the order of $Rxy^{-1}$ is also $k$.

\textbf{Proof.} $(Rxy^{-1})^k = (yRy^{-1})^k = yR^ky^{-1} = 1$.

On the other hand, if $(Rxy^{-1})^l = 1$ then $(Rxy^{-1})^l = (yRy^{-1})^l = yR^ly^{-1} = 1$ i.e. $R^l = 1$ and hence $k \mid l$.

\hfill $\Box$

Now, we are ready to prove Theorem 2.

\textbf{Proof.} For $g = 10$, we let $H_{10}$ be the subgroup of $\text{Mod}(\Sigma_{10})$ generated by the set $\{R, R^4A_1C_1B_3B_7^{-1}C_5^{-1}A_g^{-1}\}$. We get $H_{10} = \text{Mod}(\Sigma_{10})$ by Corollary 14. Then we are done by Lemma 21 since $R^4(A_1C_1B_3)R^{-4} = A_5C_7B_7$.

Note that, order of $R^4$ is clearly 5 and hence order of the element $R^4(A_1C_1B_3)(A_5C_5B_7)^{-1}$ is also 5 by Lemma 21 since $R^4(a_1) = a_5$, $R^4(c_1) = c_5$ and $R^4(b_3) = b_7$ implies $R^4(A_1C_1B_3)R^{-4} = A_5C_5B_7$.

For $g = 9$, we let $H_9$ be the subgroup of $\text{Mod}(\Sigma_9)$ generated by the set $\{R, R^3B_1A_3C_5C_7^{-1}A_6^{-1}B_4^{-1}\}$. We have $H_9 = \text{Mod}(\Sigma_9)$ by Corollary 13. Then we are done by Lemma 21 since $R^3(B_1A_3C_5)R^{-3} = B_4A_6C_8$.

For $g = 8$, we let $H_8$ be the subgroup of $\text{Mod}(\Sigma_8)$ generated by the set $\{R, R^2B_1A_5C_7^{-1}A_7^{-1}B_3^{-1}\}$. Hence, $H_8 = \text{Mod}(\Sigma_8)$ by Corollary 12. Then we are done by Lemma 21 since $R^2(B_1A_5C_5)R^{-2} = B_3A_7C_7$.

For $g = 7$, we let $H_7$ be the subgroup of $\text{Mod}(\Sigma_7)$ generated by the set $\{R, RC_1B_4A_6A_7^{-1}B_5^{-1}C_2^{-1}\}$. We have $H_7 = \text{Mod}(\Sigma_7)$ by Corollary 9. Then we are done by Lemma 21 since $R(C_1B_4A_6)R^{-1} = C_2B_5A_7$. 

The remaining part of the proof is the case of $g \geq 11$. Let $k = g/g'$ so that $k$ is the greatest divisor of $g$ such that $k$ is strictly less than $g/2$. Clearly, the number $k$ can be any positive integer but three.

If $k = 2$, let $K_2$ be the subgroup of $\text{Mod}(\Sigma_g)$ generated by the set \{ $R, R^2A_1B_4C_8C_{10}^{-1}B_6^{-1}A_3^{-1}$ \}. We get $K_2 = \text{Mod}(\Sigma_g)$ by Corollary 15. Then we are done by Lemma 21 since $R^2(A_1B_4C_8)R^{-2} = A_3B_6C_{10}$.

If $k = 4$, let $K_4$ be the subgroup of $\text{Mod}(\Sigma_g)$ generated by the set \{ $R, R^4B_1A_3C_6C_{10}^{-1}A_7^{-1}B_5^{-1}$ \}. We get $K_4 = \text{Mod}(\Sigma_g)$ by Corollary 16. Then we are done by Lemma 21 since $R^4(B_1A_3C_6)R^{-4} = B_5A_7C_{10}$.

If $k = 5$, let $K_5$ be the subgroup of $\text{Mod}(\Sigma_g)$ generated by the set \{ $R, R^5A_1B_2C_4C_g^{-1}B_g^{-1}A_g^{-1}$ \}. We get $K_5 = \text{Mod}(\Sigma_g)$ by Corollary 20. Then we are done by Lemma 21 since $R^5(A_1B_2C_4)R^{-5} = A_{g-4}B_{g-3}C_{g-1}$.

If $k = 6$, let $K_6$ be the subgroup of $\text{Mod}(\Sigma_g)$ generated by the set \{ $R, R^6A_1B_2C_4C_g^{-1}B_g^{-1}A_g^{-1}$ \}. We get $K_6 = \text{Mod}(\Sigma_g)$ by Corollary 20. Then we are done by Lemma 21 since $R^6(A_1B_2C_4)R^{-6} = A_{g-5}B_{g-4}C_{g-2}$.

If $k \geq 7$, let $K$ be the subgroup of $\text{Mod}(\Sigma_g)$ generated by the set \{ $R, R^{-k}A_1B_2C_4C_g^{-1}B_g^{-1}A_g^{-1}$ \}. We get $K = \text{Mod}(\Sigma_g)$ by Corollary 20. Then we are done by Lemma 21 since $R^{-k}(A_1B_2C_4)R^k = A_{g-k+1}B_{g-k+2}C_{g-k+4}$.

Finally, we prove Theorem 1.

Proof. If $g = 6$, let $H_g$ be the subgroup of $\text{Mod}(\Sigma_6)$ generated by the set \{ $R, RC_1B_3A_6A_1^{-1}B_5^{-1}C_2^{-1}$ \}. We get $H_6 = \text{Mod}(\Sigma_6)$ by Corollary 8. Then we are done by Lemma 21 since $R(C_1B_3A_6)R^{-1} = C_2B_5A_1$. Note that, since $R(c_1) = c_2$, $R(b_4) = b_5$ and $R(a_6) = a_1$, we have $R(C_1B_3A_6)R^{-1} = C_2B_5A_1$ which implies order of the element $R(C_1B_3A_6)(C_2B_5A_1)^{-1}$ is $g$.

If $g = 7$, let $H_7$ be the subgroup of $\text{Mod}(\Sigma_7)$ generated by the set \{ $R, RC_1B_4A_6A_7^{-1}B_5^{-1}C_2^{-1}$ \}. We get $H_7 = \text{Mod}(\Sigma_7)$ by Corollary 9. Then we are done by Lemma 21 since $R(C_1B_4A_6)R^{-1} = C_2B_5A_7$.

If $g = 8$, let $H_8$ be the subgroup of $\text{Mod}(\Sigma_8)$ generated by the set \{ $R, RB_1C_4A_7A_8^{-1}C_5^{-1}B_2^{-1}$ \}. We get $H_8 = \text{Mod}(\Sigma_8)$ by Corollary 10. Then we are done by Lemma 21 since $R(B_1C_4A_7)R^{-1} = B_2C_5A_8$.

If $g \geq 9$, let $H_9$ be the subgroup of $\text{Mod}(\Sigma_g)$ generated by the set \{ $R, RC_1B_4A_7A_8^{-1}B_5^{-1}C_2^{-1}$ \}. We get $H_9 = \text{Mod}(\Sigma_g)$ by Corollary 11. Then we are done by Lemma 21 since $R(C_1B_4A_7)R^{-1} = C_2B_5A_8$.

5. Further Results

In this section, we prove Theorem 3 which states as: for $g \geq 3k^2 + 4k + 1$ and any positive integer $k$, the mapping class group $\text{Mod}(\Sigma_g)$ is generated by two elements of order $g/\gcd(g,k)$.

Korkmaz showed the following result in the proof of Theorem 6.

Theorem 22. If $g \geq 3$, then the mapping class group $\text{Mod}(\Sigma_g)$ is generated by the elements $A_iA_j^{-1}, B_iB_j^{-1}, C_iC_j^{-1}$ for all $i, j$. 
Sketch of the proof is as follows: $A_1A_2^{-1}B_1B_2^{-1}(a_1,a_3) = (b_1,a_3)$. $B_1A_3^{-1}C_1C_2^{-1}(b_1,a_3) = (c_1,a_3)$. Korkmaz then showed that $A_3$ can be generated by these elements using lantern relation. Hence, $A_i = (A_iA_3^{-1})A_3$, $B_i = (B_iB_3^{-1})(B_iA_3^{-1})A_3$ and $C_i = (C_iC_1^{-1})(C_1A_3^{-1})A_3$ are generated by given elements. This finishes the proof.

Now, we prove the next statement as a corollary to Theorem 22.

**Corollary 23.** If $g \geq 3$, then the mapping class group $\text{Mod}(\Sigma_g)$ is generated by the elements $A_iB_i^{-1}, C_iB_i^{-1}, C_iB_{i+1}^{-1}$ for all $i$.

**Proof.** Let us denote by $H$ the subgroup generated by the elements $A_iB_i^{-1}, C_iB_i^{-1}, C_iB_{i+1}^{-1}$ for all $i$.

We have $B_iB_j^{-1} = (B_iC_i^{-1})(C_iB_{i+1}^{-1}) \cdots (B_jC_j^{-1})(C_jB_{j+1}^{-1}) \in H$ for all $i,j$, we also have $C_iC_j^{-1} = (C_iB_i^{-1})(B_iC_i^{-1}) \in H$ for all $i,j$ and $A_iA_j^{-1} = (A_iB_i^{-1})(B_iA_j^{-1}) \in H$ for all $i,j$.

It follows from Theorem 22 that $H = \text{Mod}(\Sigma_g)$, completing the proof of the lemma. □

**Theorem 24.** If $g \geq 21$, then the mapping class group $\text{Mod}(\Sigma_g)$ is generated by the elements $R^2, B_1B_2A_5A_8C_{11}C_{14}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1}$.

**Proof.** Let $F_1 = B_1B_2A_5A_8C_{11}C_{14}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1}$. Let us denote by $H$ the subgroup of $\text{Mod}(\Sigma_g)$ generated by the set $\{R^2, F_1\}$.

Let

$$F_2 = R^2F_1R^{-2} = B_3B_4A_7A_{10}C_{13}C_{16}^{-1}C_{13}^{-1}A_{12}^{-1}A_9^{-1}B_6^{-1}B_5^{-1}$$

and

$$F_3 = F_2^{-1} = B_5B_6A_9A_{12}C_{15}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1}.$$  

We have $F_3F_1(b_5, b_6, \ldots, b_3) = (a_5, b_6, \ldots, b_3)$ so that

$F_4 = A_5B_6A_9A_{12}C_{15}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1} \in H$.

We also have $F_4F_3^{-1} = A_5B_5^{-1} \in H$ and then by conjugating $A_5B_5^{-1}$ with $R^2$ iteratively, we get $A_{2i+1}B_{2i+1}^{-1} \in H$ for all $i$.

Let

$$F_5 = R^4F_1R^{-4} = B_5B_6A_9A_{12}C_{15}C_{16}^{-1}C_{17}^{-1}A_{14}^{-1}A_{11}^{-1}B_8^{-1}B_7^{-1}$$

and

$$F_6 = (A_7B_7^{-1})F_5^{-1}(B_5A_5^{-1})$$

$$= A_7B_8A_{11}A_{14}C_{17}C_{20}^{-1}C_{18}^{-1}C_{15}^{-1}A_{12}^{-1}A_9^{-1}B_6^{-1}A_5^{-1}.$$  

We then have $F_6F_1(a_7, b_8, a_{11}, \ldots, b_6, a_5) = (a_7, a_8, a_{11}, \ldots, b_6, a_5)$ so that

$F_7 = A_7B_8A_{11}A_{14}C_{17}C_{20}^{-1}C_{18}^{-1}C_{15}^{-1}A_{12}^{-1}A_9^{-1}B_6^{-1}A_5^{-1} \in H$.

We have $F_7F_6^{-1} = A_8B_8^{-1} \in H$ and then by conjugating $A_8B_8^{-1}$ with $R^2$ iteratively, we get $A_{2i}B_{2i}^{-1} \in H$ for all $i$.

Hence, we get $A_iB_i^{-1} \in H$ for all $i$. 


Let 
\[ F_8 = (B_{12}A_{12}^{-1})F_4 = A_5B_6A_9B_{12}C_{15}C_{18}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1}. \]

We then get \[ F_8F_1(\ldots, b_{12}, \ldots) = (\ldots, c_{11}, \ldots) \] so that
\[ F_9 = A_5B_6A_9C_{11}C_{15}C_{18}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1} \in H. \]

We have \[ F_9F_8^{-1} = C_{11}B_{12}^{-1} \in H \] and then by conjugating \( C_{11}B_{12}^{-1} \) with 
\( R^2 \) iteratively, we get \( C_{2i+1}B_{2i+2}^{-1} \in H \) for all \( i \).

Let 
\[ F_{10} = (B_{11}A_{11}^{-1})F_7 = A_7A_8B_{11}A_{14}C_{17}C_{20}C_{18}^{-1}C_{15}^{-1}A_{12}^{-1}A_9^{-1}B_6^{-1}A_5^{-1}. \]

Similarly, we have \[ F_{10}F_1(\ldots, b_{11}, \ldots) = (\ldots, c_{11}, \ldots) \] so that
\[ F_{11} = A_7A_8C_{11}A_{14}C_{17}C_{20}C_{18}^{-1}C_{15}^{-1}A_{12}^{-1}A_9^{-1}B_6^{-1}A_5^{-1} \in H. \]

Hence, we get \( F_{11}F_{10}^{-1} = C_{11}B_{11}^{-1} \in H \) and we get \( C_{2i+1}B_{2i+1}^{-1} \in H \) for all \( i \).

Let 
\[ F_{12} = (B_{15}C_{15}^{-1})F_4 = A_5B_6A_9A_{12}B_{15}C_{18}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1}. \]

We also have \[ F_{12}F_1(\ldots, b_{15}, \ldots) = (\ldots, c_{14}, \ldots) \] so that
\[ F_{13} = A_5B_6A_9A_{12}C_{14}C_{18}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1} \in H. \]

Check that \[ F_{13}F_{12}^{-1} = C_{14}B_{15}^{-1} \in H \] and then we get \( C_{2i}B_{2i+1}^{-1} \in H \) for all \( i \). Hence, we have \( C_iB_{i+1}^{-1} \in H \) for all \( i \).

Let 
\[ F_{14} = F_7(C_{15}B_{16}^{-1}) = A_7A_8A_{11}A_{14}C_{17}C_{20}C_{18}^{-1}B_{16}^{-1}A_{12}^{-1}A_9^{-1}B_6^{-1}A_5^{-1}. \]

We then get \( F_{14}F_1(\ldots, b_{16}, \ldots) = (\ldots, c_{16}, \ldots) \) so that
\[ F_{15} = A_7A_8A_{11}A_{14}C_{17}C_{20}C_{18}^{-1}C_{16}^{-1}A_{12}^{-1}A_9^{-1}B_6^{-1}A_5^{-1} \in H. \]

Hence, we see that \( F_{15}^{-1}F_{14} = C_{16}B_{16}^{-1} \in H \) and then we get \( C_{2i}B_{2i}^{-1} \in H \) for all \( i \). Finally, we have \( C_iB_{i}^{-1} \in H \) for all \( i \).

It follows from Corollary 23 that \( H = \text{Mod}(\Sigma_g) \), completing the proof of the theorem.

\[ \square \]

**Corollary 25.** If \( g \) is even and \( g \geq 22 \), then the mapping class group \( \text{Mod}(\Sigma_g) \) is generated by two elements of order \( g/2 \).

**Proof.** Let \( H \) be the subgroup of \( \text{Mod}(\Sigma_g) \) generated by the set \( \{R^2, R^2B_1B_2A_5A_8C_{11}C_{14}C_{16}^{-1}C_{13}^{-1}A_{10}^{-1}A_7^{-1}B_4^{-1}B_3^{-1}\} \). We get \( H = \text{Mod}(\Sigma_g) \) by Theorem 24. Then we are done by Lemma 21 since \( R^2(B_1B_2A_5A_8C_{11}C_{14})R^{-2} = B_3B_4A_7A_{10}C_{13}C_{16} \).

**Generalization of Theorem 24 and Corollary 25** is as follows:

**Theorem 26.** For \( k \geq 2 \) and \( g \geq 3k^2 + 4k + 1 \), the mapping class group \( \text{Mod}(\Sigma_g) \) is generated by the elements \( R^k, R^kF(R^kF^{-1}R^{-k}) \) where \( F = B_1B_2 \ldots B_kA_{2k+1}A_{3k+2} \ldots A_{k^2+2k}C_{k^2+3k+1}C_{k^2+4k+2} \ldots C_{2k^2+3k} \) Fig. 6.

**Proof.** We define an algorithm to prove the desired result.
Let \( F = B_1B_2 \ldots B_kA_{2k+1}A_{3k+2} \ldots A_{k^2+2k}C_{k^2+3k+1}C_{k^2+4k+2} \ldots C_{2k^2+3k} \) and \( F_1 = F(R^kF^{-1}R^{-k}) \). Let us denote by \( H \) the subgroup of \( \text{Mod}(\Sigma_g) \) generated by the set \( \{R^k, F_1\} \).
A) Use conjugation of $F_1$ with $R^k, R^{2k}, \ldots, R^{k^2}$ with proper multiplications to get $A_{k+1}B_{k+1}^{-1} \in H$, $A_{k+2}B_{k+2}^{-1} \in H$, $\ldots$, $A_{2k-1}B_{2k-1}^{-1} \in H$, $A_{2k}B_{2k}^{-1} \in H$, respectively. Hence, we have $A_i B_i^{-1} \in H$ for all $i$.

B) Follow the next $k$ steps.

1) Use conjugation of $F_1$ with $R^{kl}$ for some positive integers $l$’s with proper multiplications to get $C_{i_{k+1}}B_{i_{k+1}}^{-1} \in H$ and $C_{i_{k+1}}B_{i_{k+2}}^{-1} \in H$ for all $i$.

2) Use conjugation of $F_1$ with $R^{kl}$ for some positive integers $l$’s with proper multiplications to get $C_{i_{k+2}}B_{i_{k+2}}^{-1} \in H$ and $C_{i_{k+2}}B_{i_{k+3}}^{-1} \in H$ for all $i$.

\ldots

k) Use conjugation of $F_1$ with $R^{kl}$ for some positive integers $l$’s with proper multiplications to get $C_{i_{k}}B_{i_{k}}^{-1} \in H$ and $C_{i_{k}}B_{i_{k+1}}^{-1} \in H$ for all $i$.

Hence, $C_{i}B_{i}^{-1} \in H$ and $C_{i}B_{i+1}^{-1} \in H$ for all $i$.

It follows from Corollary 23 that $H = \text{Mod}(\Sigma_g)$, completing the proof of the theorem.

See Theorem 24 for an example application of the algorithm. □

Now, we prove Theorem 3.
Proof. For $k \geq 2$ and $g \geq 3k^2 + 4k + 1$, let $H$ be the subgroup of $\text{Mod}(\Sigma_g)$ generated by the set $\{R^k, R^kF(R^kF^{-1}R^{-k})\}$. Then $H = \text{Mod}(\Sigma_g)$ by Theorem 26. Hence, we are done by Lemma 21 since the orders of $R^k$ and $R^kF(R^kF^{-1}R^{-k})$ are $g/d$ where $d$ is the greatest common divisor of $g$ and $k$. If $k = 1$, we are done by Theorem 1. □

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References

[1] Baykur, I., Korkmaz, M.: Mapping class group is generated by two commutators. J. Algebra 574, 278–291 (2021)
[2] Brendle, T.E., Farb, B.: Every mapping class group is generated by 6 involutions. J. Algebra 278, 187–198 (2004)
[3] Dehn, M.: The group of mapping classes. In: Papers on Group Theory and Topology. Springer, Berlin (1987). (Translated from the German by J. Stillwell (Die Gruppe der Abbildungsklassen, Acta Math. 69 (1938), 135–206))
[4] Du, X.: Generating the extended mapping class group by torsions. J. Knot Theory Ramifications 26, 17500378 (2017)
[5] Farb, B., Margalit, D.: A Primer on Mapping Class Groups. Princeton University Press, Princeton (2011)
[6] Humphries, S.: Generators for the mapping class group. In: Topology of Low-Dimensional Manifolds, Proceedings of Second Sussex Conference, Chelwood Gate, 1977, Lecture Notes in Math., vol 722, Springer, pp. 44–47 (1979)
[7] Kassabov, M.: Generating mapping class groups by involutions. arXiv:math.GT/0311455, v1 (2003)
[8] Korkmaz, M.: Generating the surface mapping class group by two elements. Trans. Am. Math. Soc. 357, 3299–3310 (2005)
[9] Korkmaz, M.: Mapping class group is generated by three involutions. Math. Res. Lett. 27, 1095–1108 (2020)
[10] Korkmaz, M.: Minimal generating sets for the mapping class group of a surface. Handb. Teichmüller Sp. Vol. II I, 441–463 (2012)
[11] Lanier, J.: Generating mapping class groups with elements of fixed finite order. J. Algebra 511, 455–470 (2018)
[12] Lickorish, W.B.R.: A finite set of generators for the homeotopy group of a 2-manifold. Proc. Camb. Philos. Soc. 60, 769–778 (1964)
[13] Lu, N.: On the mapping class groups of the closed orientable surfaces. Topol. Proc. 13, 293–324 (1988)
[14] Luo, F.: Torsion elements in the mapping class group of a surface. arXiv:math.GT/0004048, v1 (2000)
[15] Maclachlan, C.: Modulus space is simply-connected. Proc. Am. Math. Soc. 29, 85–86 (1971)
[16] Margalit, D.: Problems, questions, and conjectures about mapping class groups. In: Proceedings of Symposia in Pure Mathematics, Vol 102, p. 20 (2019)
[17] McCarthy, J.D., Papadopoulos, A.: Involutions in surface mapping class groups. Enseign. Math. (2) 33, 275–290 (1987)

[18] Monden, N.: Generating the mapping class group by torsion elements of small order. Math. Proc. Camb. Philos. Soc. 154, 41–62 (2013)

[19] Stukow, M.: Small torsion generating sets for hyperelliptic mapping class groups. Topol. Appl. 145, 83–90 (2004)

[20] Wajnryb, B.: Mapping class group of a surface is generated by two elements. Topology 35, 377–383 (1996)

[21] Yildiz, O.: Generating the mapping class group by three involutions. arXiv:200209151v1 [math.GT] (2020)

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