A CONSECUTIVE LEHMER CODE FOR PARABOLIC QUOTIENTS OF
THE SYMMETRIC GROUP

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ABSTRACT. In this article we define an encoding for parabolic permutations that
distinguishes between parabolic 231-avoiding permutations. We prove that the
componentwise order on these codes realizes the parabolic Tamari lattice, and con-
clude a direct and simple proof that the parabolic Tamari lattice is isomorphic to a
certain $\nu$-Tamari lattice, with an explicit bijection. Furthermore, we prove that this
bijection is closely related to the map $\Theta$ used when the lattice isomorphism was
first proved in (Ceballos, Fang and Mühle, 2020), settling an open problem therein.

1. INTRODUCTION

A (right) inversion of a permutation $w$ is a pair of indices $(i, j)$ with $i < j$ such
that $w(i) > w(j)$. The number of inversions of $w$ can therefore be regarded as a
degree of disorder of $w$. The Lehmer code associated with $w$ is the integer tuple
whose $i$th entry counts the number of inversions of $w$ of the form $(i, \cdot)$. Björner
and Wachs defined a “consecutive” version of the Lehmer code in [2, Section 9],
which we shall call the BW-code of $w$. This encoding associates an integer tuple
whose $i$th entry counts the number of inversions of $w$ of the form $(i, \cdot)$. 

In contrast to the original Lehmer code, the BW-code no longer uniquely deter-
mines a permutation. However, the permutations with the same BW-code form
an interval in the (left) weak order on the group of all permutations, the
symmetric group [2, Proposition 9.10].

Another consequence of [2, Proposition 9.10] is that among all permutations
with the same BW-code, there is a unique permutation $w$ which avoids the pattern 231,
i.e., in which no three indices $i < j < k$ exist such that $w(k) < w(i) < w(j)$,
and this permutation minimizes the number of inversions among all permutations
with the same BW-code as $w$.

Let us denote the symmetric group of degree $n$ by $S_n$, and its subset of all
231-avoiding permutations by $S_n(231)$. The (left) weak order on $S_n$ is a lattice,
i.e., every two elements have a unique lower bound and a unique upper bound [6,
14]. The restriction of this lattice to $S_n(231)$ constitutes a sublattice [2, Theo-
rem 9.6(i)] and a quotient lattice [12, Theorem 5.1]. In fact, the resulting lattice
incarnates the famous Tamari lattice denoted by $T_n$ [13]. We can thus see the BW-
code as a concrete and simple way to quotient the weak order on $S_n$ into $T_n$.

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An analogue of 231-avoiding permutations for parabolic quotients of $S_n$ was introduced in [9], and it was shown that these permutations constitute a quotient lattice (but no longer a sublattice) of the corresponding (left) weak order, the \textit{parabolic Tamari lattice} [9, Theorem 1]. Since any parabolic quotient of $S_n$ is naturally indexed by a composition $\alpha$ of $n$, we call the resulting lattice the $\alpha$-\textit{Tamari lattice} $T_\alpha$.

The main purpose of this article is to define a parabolic analogue of the BW-code; see Definition 3.1. We prove that the componentwise order on these parabolic BW-codes is isomorphic to $T_\alpha$.

Let us denote the set of parabolic BW-codes by $C_\alpha$, and let us denote the componentwise order on integer tuples (of the same length) by $\leq_{\text{comp}}$. Our first main result now reads as follows.

\textbf{Theorem 1.1.} For every $n > 0$ and every integer composition $\alpha$ of $n$ it holds that $T_\alpha \cong (C_\alpha, \leq_{\text{comp}})$.

Originally, the Tamari lattice was defined in terms of a “rotation” operation on parenthesizations, binary trees or equivalently Dyck paths. A \textit{northeast path} is a lattice path in $\mathbb{N}^2$ comprised of north steps (marked by $N$) and east steps (marked by $E$) of unit length. A \textit{Dyck path} of semilength $n$ is equivalent to a northeast path that stays weakly above the staircase path $(NE)^n$ and uses $n$ north and $n$ east steps.

A \textit{rotation} of a northeast path exchanges two portions of the path under certain conditions, and $T_n$ arises as the rotation order on the set of Dyck paths of semilength $n$. An extension of this construction was introduced in [11]. In that paper, the set of all northeast paths weakly above a fixed northeast path $\nu$, which start and end at the same coordinates as $\nu$, was considered. Ordering this set by rotation produces another lattice, the $\nu$-\textit{Tamari lattice} [11, Theorem 1.1].

For any composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ of $n$, we can define the $\alpha$-bounce path $\nu_\alpha = N^{\alpha_1}E^{\alpha_1}N^{\alpha_2}E^{\alpha_2} \cdots N^{\alpha_r}E^{\alpha_r}$. Theorem II in [4] established that $T_\alpha$ is isomorphic to the $\nu_\alpha$-Tamari lattice. The proof of this result is rather technical, using some deep lattice-theoretic properties of $T_\alpha$. The second main contribution of this article is a much simpler and direct proof of this result.

In general, the $\nu$-Tamari lattice admits a simple encoding as the componentwise order on so-called $\nu$-\textit{bracket vectors} [4, Theorem 4.2]. If $\nu = \nu_\alpha$, then the corresponding bracket vectors can be converted in a simple way into parabolic BW-codes. Since both parabolic BW-codes and bracket vectors are ordered componentwise, the proof of the next result follows readily.

\textbf{Theorem 1.2 ([4, Theorem II])}. For every $n > 0$ and every integer composition $\alpha$ of $n$, the $\nu_\alpha$-Tamari lattice is isomorphic to $T_\alpha$.

The original proof of Theorem 1.2 in [4] did not provide an explicit map between the two lattices, but rather passed through their Galois graphs, whose elements are related by a map $\Theta$ between the two lattices. In [4], it was postulated as Open Problem 2.23 to prove that $\Theta$ extends to a full lattice isomorphism, not limited to elements of the Galois graphs. Using parabolic BW-codes, by introducing a stack processing procedure on $(\alpha, 231)$-avoiding permutations, we settle this open problem affirmatively, while giving another interpretation of parabolic BW-codes; see Corollary 6.11.

In Section 2, we recall the basic definitions regarding parabolic quotients of the symmetric group, parabolic pattern avoidance and the weak order. In Section 3, we define the parabolic BW-codes and prove Theorem 1.1.
In Section 4, we recall the definitions of Dyck paths and northeast paths, as well as ordinary Tamari lattices and ν-Tamari lattices. We then sketch a proof of the fact that ν-Tamari lattices are intervals of an ordinary Tamari lattice using a map that has appeared in [11]. While this map was originally described in terms of trees, we take the perspective of Dyck paths, which makes its description much simpler; see Section 4.3. In Section 4.4, we use the Dyck path-language to review that the anti-isomorphism on the Tamari lattice exchanges two sequence statistics on Dyck paths. This property is well-known to experts, but only appears in print in the more general framework of Tamari interval posets [10]. The specialization to the setting of Dyck paths can be deduced from [10, Theorem 23].

We then prove Theorem 1.2 in Section 5 by describing an explicit conversion from parabolic BW-codes to να-bracket vectors. Finally, in Section 6, we give a combinatorial interpretation of the map Θ mentioned after Theorem 1.2 in terms of a certain stack-processing procedure, and relate the bijection between parabolic BW-codes and να-bracket vectors to Θ, thus solving [4, Open Problem 2.23].

2. α-permutations and the α-Tamari lattice

Throughout this article, we fix an integer n > 0 and define \( [n] = \{1, 2, \ldots, n\} \).

2.1. α-permutations. Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \) be a composition of n. For \( a \in [r] \), we define

\[
s_a \overset{\text{def}}{=} a_1 + a_2 + \cdots + a_r,
\]

and we set \( s_0 = 0 \). The set \( \{s_{\alpha-1}+1, s_{\alpha-1}+2, \ldots, s_{\alpha}\} \) is the \( \alpha \)-region.

The indicator map \( \varrho_\alpha : [n] \to [r] \) is defined by \( \varrho_\alpha(i) = a \) if, and only if \( s_{\alpha-1} < i \leq s_{\alpha} \). In other words, \( \varrho_\alpha(i) \) is the index of the \( \alpha \)-region containing \( i \). When no confusion will arise, we will drop the subscript \( \alpha \). For three indices \( i < j < k \) with \( \varrho(i) < \varrho(j) < \varrho(k) \), we say that \( j \) is in an \( \alpha \)-region strictly between \( i \) and \( k \).

Let \( \mathcal{S}_n \) denote the symmetric group of degree \( n \). We consider the subset of \( \alpha \)-permutations, defined by

\[
\mathcal{S}_n^\alpha \overset{\text{def}}{=} \{ w \in \mathcal{S}_n \mid \text{if } \varrho(i) = \varrho(i+1), \text{ then } w(i) < w(i+1) \}.
\]

Clearly, if \( \alpha = (1, 1, \ldots, 1) \), then \( \mathcal{S}_n^\alpha = \mathcal{S}_n \).

Remark 2.1. If we consider the subgroup \( G = \mathcal{S}_{|\alpha_1|} \times \mathcal{S}_{|\alpha_2|} \times \cdots \times \mathcal{S}_{|\alpha_r|} \) of \( \mathcal{S}_n \), then we may identify \( \mathcal{S}_n^\alpha \) with the set of minimal-length representatives of the left cosets in \( \mathcal{S}_n/G \).

An \( \alpha \)-permutation \( w \in \mathcal{S}_n^\alpha \) has an \((\alpha, 231)\)-pattern if there are three indices \( i < j < k \)—each in different \( \alpha \)-regions—such that \( w_i < w_j \) and \( w_i = w_k + 1 \). If \( w \) does not have an \((\alpha, 231)\)-pattern, then \( w \) is \((\alpha, 231)\)-avoiding. Let \( \mathcal{S}_n(231) \) denote the set of \((\alpha, 231)\)-avoiding permutations.

Remark 2.2. In the case \( \alpha = (1, 1, \ldots, 1) \), the \((\alpha, 231)\)-avoiding permutations are exactly the classical 231-avoiding permutations: one can either ask \( w_i = w_k + 1 \) or not, since if \( w \) has any 231-pattern, then one can find one with the extra condition \( w_i = w_k + 1 \). In the general case, these notions differ since \( i \) could belong to the same \( \alpha \)-region as \( j \). For example, 3 24 1 belongs to \( \mathcal{S}_{(1,2,1)}(231) \) whereas it has a classical 231-pattern spread out over different \( \alpha \)-regions.
2.2. The weak order. For \( w \in \mathfrak{S}_a \), we define its (right) inversion set by

\[
\text{Inv}(w) \overset{\text{def}}{=} \{(i, j) \mid i < j \text{ and } w_i > w_j\}.
\]

This enables us to define a partial order—the (left) weak order—on \( \mathfrak{S}_a \) by setting

\[
u \leq_L \upsilon \quad \text{if, and only if} \quad \text{Inv}(u) \subseteq \text{Inv}(v).
\]

Two permutations \( u, \upsilon \in \mathfrak{S}_a \) form a cover relation—denoted by \( u <_L \upsilon \)—if \( u <_L \upsilon \) and there is no \( w \in \mathfrak{S}_a \) with \( u <_L w <_L \upsilon \). One easily checks that \( u <_L \upsilon \) if, and only if there are two indices \( i < j \) in different \( \alpha \)-regions, such that \( u_i = u_j - 1 \), and

\[
v_k = \begin{cases} 
u_j, & \text{if } k = i, \\ 

u_i, & \text{if } k = j, \\ 

\nu_k, & \text{otherwise.}
\end{cases}
\]

The partially ordered set \((\mathfrak{S}_n, \leq_L)\) is a lattice by [14, Theorem 2.1]; see also [6]. For an arbitrary composition \( \alpha \) of \( n \), it follows from [1, Theorem 4.1] that \((\mathfrak{S}_n, \leq_L)\) is an interval of \((\mathfrak{S}_n, \leq_L)\), and thus also a lattice.

The partially ordered set \( \mathcal{T}_\alpha \overset{\text{def}}{=} (\mathfrak{S}_\alpha(231), \leq_L) \) is the \( \alpha \)-Tamari lattice. This name is justified by the following result.

**Theorem 2.3** ([9, Theorem 1]). \( \mathcal{T}_\alpha \) is a lattice for every \( n > 0 \) and every integer composition \( \alpha \) of \( n \).

3. A generalized Lehmer code for \( \mathfrak{S}_a \)

3.1. Encoding \( \alpha \)-permutations. We consider the following set of integer tuples.

**Definition 3.1.** Let \( \mathcal{C}_\alpha \) denote the set of all integer tuples \( (c_1, c_2, \ldots, c_n) \) with the following properties:

- (C1): \( 0 \leq c_i \leq n - s_\varphi(i) \) for all \( i \in [n] \);
- (C2): \( c_i \leq c_{i+1} \) for all \( i \in [n - 1] \) such that \( \varphi(i) = \varphi(i + 1) \);
- (C3): \( c_a \leq c_i - s_a + s_\varphi(i) \) for all \( i \in [s_r - 2] \) and all \( a \in \{ \varphi(i) + 1, \varphi(i) + 2, \ldots, r - 1 \} \) such that \( c_j \geq s_a - s_\varphi(i) \).

The set \( \mathcal{C}_{\{1,1,\ldots,1\}} \) is precisely the set of integer tuples defined in [2, Definition 9.1].

**Remark 3.2.** The statement of (C3) is directly true if \( a = r \) and trivial if \( i > s_{r-2} \), hence the restriction to \( i \in [s_{r-2}] \) and \( a < r \).

Indeed, by (C1), \( c_n = 0 \) so that the implication required by (C3) is trivially satisfied when \( a = r \). If \( i > s_{r-2} \), then \( \varphi(i) \geq r - 1 \), so that the only case one could consider is again \( a = r \).

For example, with \( n = 3 \) and \( \alpha = (2,1) \), all conditions boil down to \( 0 \leq c_1 \leq 1 \) and \( c_3 = 0 \), hence three solutions. With \( n = 3 \) and \( \alpha = (1,2) \), one gets \( 0 \leq c_1 \leq 2 \) and \( 0 \leq c_2 \leq c_3 \leq 0 \), again providing three solutions. One can check that they are indeed the codes obtained in Table 1 (right column).

To see all conditions of the definition play a role, one has to consider compositions of at least three parts and at least one greater than one. For example, if \( \alpha = (1,2,1) \), one gets the following set of relations: \( 0 \leq c_1 \leq 3, 0 \leq c_2 \leq c_3 \leq 1, c_4 = 0 \), and the extra condition coming from (C3): \( c_1 \geq 2 \Rightarrow c_3 \leq c_1 - 2 \). In practice, we have twelve tuples satisfying all conditions except the last one and this last condition gets rid of (2,0,1,0) and (2,1,1,0), hence providing a total of
ten solutions. One can check that these solutions are exactly the codes obtained in Figure 2 (bottom elements in each cell of the drawing).

Given two tuples $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ we write $a \leq_{\text{comp}} b$ if $a_i \leq b_i$ for all $i \in [n]$. We claim in Theorem 1.1 that the poset $(C_n, \leq_{\text{comp}})$ is isomorphic to $T_n$. For example, one can check, again on Figure 2 that the bottom elements are indeed (partially) ordered by the componentwise order on their tuples.

As a first step towards proving Theorem 1.1, we associate an integer tuple with each $w \in \mathfrak{S}_n$.

Definition 3.3. For $w \in \mathfrak{S}_n$, we define its $\alpha$-code by

$$\text{code}_\alpha(w) \overset{\text{def}}{=} (c_1, c_2, \ldots, c_n),$$

where

$$c_i \overset{\text{def}}{=} \max\{k \mid w_i > w_{s_{\ell}(i)+1}, w_i > w_{s_{\ell}(i)+2}, \ldots, w_i > w_{s_{\ell}(i)+k}\}.$$ 

In other words, $c_i$ counts the number of consecutive entries in the one-line notation of $w$ that are smaller than $w_i$, starting from the first entry in the $\alpha$-region immediately after that of $i$. For $\alpha = (1, 1, \ldots, 1)$, Definition 3.3 agrees with [2, Definition 9.9].

If $\text{code}_\alpha(w) = (c_1, c_2, \ldots, c_n)$, then we say that $w_i$ sees $w_k$ if $0 < k - s_{\ell}(i) \leq c_i$. Clearly, if $w_i$ sees $w_k$, then $(i, k) \in \text{Inv}(w)$, and $w_i$ sees exactly $c_i$ elements for each index $i$.

In terms of patterns, $c_i$ is the number of 21-patterns where the 2 is at position $i$ that are not 231-patterns. Examples of codes of $\alpha$-permutations are shown in Table 1 and Figure 2.

3.2. Properties of the encoding.

Lemma 3.4. For $w \in \mathfrak{S}_n$ it holds that $\text{code}_\alpha(w) \in C_n$.

Proof. Let $w \in \mathfrak{S}_n$ and $\text{code}_\alpha(w) = (c_1, c_2, \ldots, c_n)$. Let $i \in [n]$. The maximal number of inversions of the form $(i, k)$ is $n - s_{\ell}(i)$, because $w_i < w_k$ for all $k \in \{i+1, i+2, \ldots, s_{\ell}(i)\}$. Hence, $c_i \leq n - s_{\ell}(i)$, which establishes (C1). If $s_{\ell}(i) = q(i+1)$, then $w_i < w_{i+1}$ by construction, and thus $c_i \leq c_{i+1}$. This establishes (C2).

Now let $k \in \{q(i)+1, q(i)+2, \ldots, r\}$ be such that $c_i \geq s_k - s_{\ell}(i)$. In particular, $w_i$ sees $w_{s_k}$, meaning that $w_i > w_{s_k}$. Since $w_{s_k}$ is the rightmost hence largest element of its region, $w_i$ also sees any $w_j$ which is seen by $w_{s_k}$. This implies that $c_i \geq c_{s_k} + s_k - s_{\ell}(i)$, which is (C3).

Theorem 1.1 in [9] establishes that the $\alpha$-Tamari lattice arises as a quotient lattice of the weak order on $\mathfrak{S}_n$. This is established by proving that for every $w \in \mathfrak{S}_n$ there exists a unique maximal $(\alpha, 231)$-avoiding permutation below $w$ in the weak order. The next lemma records this fact.

Lemma 3.5 ([9, Lemma 3.8]). For $w \in \mathfrak{S}_n$, the set $\{w' \in \mathfrak{S}_n(231) \mid w' \leq_L w\}$ has a greatest element denoted by $\pi^+_1(w)$.

We may thus regard $\pi^+_1$ as a map from $\mathfrak{S}_n$ to $\mathfrak{S}_n(231)$. The next lemma characterizes the preimages of this map.
Lemma 3.6 ([9, Lemma 3.16]). Let \( u, v \in \mathcal{G}_\alpha \) with \( u \leq_L v \). The following are equivalent.

(i) There are indices \( i < j < k \), each in different \( \alpha \)-regions, such that \( v_k < v_i < v_j \), \( v_i = v_k + 1 \) and \( \text{Inv}(v) \setminus \text{Inv}(u) = \{ (i, k) \} \).
(ii) \( \pi_k(u) = \pi_k(v) \).

We now prove that code\(_\alpha\) is an order-preserving map from \((\mathcal{G}_\alpha, \leq_L)\) to \((\mathcal{C}_\alpha, \leq_{\text{comp}})\).

Lemma 3.7. Let \( u, v \in \mathcal{G}_\alpha \) with \( u \leq_L v \). Then \( \text{code}_{\alpha}(u) \leq_{\text{comp}} \text{code}_{\alpha}(v) \), and these tuples differ by at most one element. Moreover, \( \text{code}_{\alpha}(u) = \text{code}_{\alpha}(v) \) if and only if \( \pi_k(u) = \pi_k(v) \).

Proof. Let \( u \leq_L v \) and \( \text{code}_{\alpha}(u) = (a_1, a_2, \ldots, a_n) \) and \( \text{code}_{\alpha}(v) = (b_1, b_2, \ldots, b_n) \).

By assumption, \( \text{Inv}(v) \setminus \text{Inv}(u) = \{ (i, k) \} \) for some indices \( i < k \) in different \( \alpha \)-regions such that \( v_i = v_k + 1 \). It follows that any entry which sees \( v_k \) must be greater than \( v_i \), and any entry which does not see \( v_i \) must be smaller than \( v_k \). Thus, \( a_j = b_j \) for all \( j \neq i \).

By construction, \( u_i = v_k \) and \( u_k = v_i \). Since \( u_i < u_k \), we conclude that \( u_i \) does not see \( u_k \).

If \( v_i \) sees \( v_k \), then \( a_i < b_i \). This is the case precisely when every \( j \) in \( \alpha \)-regions strictly between \( i \) and \( k \) satisfies \( v_j < v_i \), which by Lemma 3.6 means that \( \pi_k(u) \neq \pi_k(v) \).

If \( v_i \) does not see \( v_k \), then there exists an index \( j \) in an \( \alpha \)-region strictly between \( i \) and \( k \) such that \( v_i < v_j \), which by Lemma 3.6 is equivalent to \( \pi_k(u) = \pi_k(v) \). If we choose \( j \) as small as possible with this property, then any \( j' < j \) in an \( \alpha \)-region strictly between \( i \) and \( k \) satisfies \( v_i > v_{j'} \), and thus \( u_i = v_k > v_{j'} = u_j \), which entails \( a_i = b_i \).

Corollary 3.8. If \( u \leq_L v \), then \( \text{code}_{\alpha}(u) \leq_{\text{comp}} \text{code}_{\alpha}(v) \).

Proof. This follows from repeated application of Lemma 3.7.

Lemma 3.9. Let \( u, v \in \mathcal{G}_\alpha(231) \). If \( \text{code}_{\alpha}(u) \leq_{\text{comp}} \text{code}_{\alpha}(v) \), then \( u \leq_L v \).

Proof. Let \( \text{code}_{\alpha}(u) = (a_1, a_2, \ldots, a_n) \) and \( \text{code}_{\alpha}(v) = (b_1, b_2, \ldots, b_n) \) such that \( \text{code}_{\alpha}(u) \leq_{\text{comp}} \text{code}_{\alpha}(v) \).

Assume that there exists \( (i, k) \in \text{Inv}(u) \setminus \text{Inv}(v) \), and among all these inversions choose \( (i, k) \) such that \( u_i - u_k \) is minimal. Since \( (i, k) \) is not an inversion of \( v \), we have \( v_i < v_k \), so that \( v_i \) does not see \( v_k \). Since \( a_i \leq b_i \) it follows that \( u_i \) does not see \( u_k \) either. Since \( u_i > u_k \), \( \varphi(i) < \varphi(k) \) and there exists a smallest index \( j \) with \( \varphi(i) < \varphi(j) < \varphi(k) \) and \( u_i < u_j \). Since \( u \in \mathcal{G}_\alpha(231) \), we have that \( u_j > u_k + 1 \).

Now, there cannot be any element between \( u_k + 1 \) and \( u_i - 1 \) in the same \( \alpha \)-region as \( u_j \). Indeed, if this was the case, since \( u_j > u_i, u_{i-1} \) would be such an element. But, since it is seen by \( u_i \) by minimality of \( j \), and since \( b_i \geq a_i \), the value \( v_{j-1} \) would be seen by \( v_i \), so that \( v_k > v_i > v_{j-1} \). In that case, \( (j - 1, k) \) would be an inversion of \( u \), not an inversion of \( v \) and would violate the minimality of \( (i, k) \) among such elements as defined earlier. So all elements between \( u_k \) and \( u_i \) belong to \( \alpha \)-regions different from the \( \alpha \)-region containing \( u_j \). Thus, among those, there is a smallest one \( u_i \) (which is not \( u_k \) but can be \( u_j \)) that is on the left of \( u_j \). This element belongs to an \((\alpha, 231)\)-pattern in \( u\): \((\ell, j, \ell')\), where \( \ell' \) is the position of \( u_\ell - 1 \) in \( u \), which is a contradiction.
Therefore, our assumption must have been wrong, and it follows $\text{Inv}(u) \subseteq \text{Inv}(v)$, thus $u \leq_L v$ by definition.

Note that we never used in the previous proof that $v$ is $(a, 231)$-avoiding. This makes sense thanks to Lemma 3.5: its property has no equivalent going upwards so $u$ and $v$ do not play symmetrical roles.

3.3. Decoding $a$-codes. We proceed to prove that code$_a$ is a bijection from $\mathcal{S}_a(231)$ to $C_a$.

Lemma 3.10. If $w \in \mathcal{S}_a(231)$, then the leftmost 0 in code$_a(w)$ corresponds to the position of the 1 in the one-line notation of $w$.

Proof. Let code$_a(w) = (c_1, c_2, \ldots, c_n)$, and let $j_0 \in [n]$ be such that $w_{j_0} = 1$. Moreover, if $j = \min\{i \mid c_i = 0\}$, then $j \leq j_0$, since $c_{j_0} = 0$. Let $w_j = a$. Since the entries in an $a$-region are ordered increasingly, $a$ is the smallest element in its $a$-region.

Now, define $m = \min(w_1, w_2, \ldots, w_j)$. Then, if $m \neq a$, since $m$ is strictly to the left of $a$ in $w$, it cannot be 1 either, so that we have a 231 pattern with the values $m, a,$ and $m - 1$ necessarily in that order in $w$ and in different regions. Otherwise $m = a$. If $m \neq 1$, the region of $a$ cannot be the rightmost region of $w$ since $a - 1$ did not appear in this prefix of $w$. Let $b$ be the smallest element in the $(\varrho(j) + 1)$-st $a$-region, and let $k = s_{\varrho(j)} + 1$, i.e., $w_k = b$. Since $c_j = 0$, we have $a < b$. We then have a 231 pattern with the values $a, b,$ and $a - 1$.

So $a = 1$ and $j = j_0$. \hfill $\square$

Proposition 3.11. For $c \in C_a$ there exists a unique $w \in \mathcal{S}_a(231)$ such that code$_a(w) = c$.

Proof. We proceed by induction on $n$. For $n \leq 3$, the claim can be checked directly (see Table 1), which establishes the induction base. Assume that the claim holds for all compositions of $n' < n$.

Let $c = (c_1, c_2, \ldots, c_n) \in C_a$. By definition, $c_n = 0$, which enables us to define $j_0 = \min\{j \in [n] \mid c_j = 0\}$. By (C2), $j_0 = s_{a - 1} + 1$ for some $a \in [r]$, meaning that $j_0$ is the first element in the $a$th $a$-region.

Let $a' = (a'_1, a'_2, \ldots, a'_r)$ be the unique composition of $n - 1$ which is obtained by subtracting 1 from $a_d$. (If $a_d = 1$, then we simply remove this part.) We define $s'_b = a'_1 + a'_2 + \cdots + a'_r$, and we obtain

$$s'_b = \begin{cases} s_b, & \text{if } b < a, \\ s_b - 1, & \text{if } b \geq a. \end{cases}$$

We define $c' = (c'_1, c'_2, \ldots, c'_{n-1})$ by setting

$$c'_i = \begin{cases} c_i, & i < j_0 \text{ and } c_i < s_{a - 1} - s_{a_d(i)}, \\ c_i - 1, & i < j_0 \text{ and } c_i \geq s_{a - 1} - s_{a_d(i)}, \\ c_{i+1}, & i \geq j_0. \end{cases}$$

It is straightforward to check that $c' \in C_{a'}$. By induction hypothesis, there exists a unique $w' \in \mathcal{S}_{a'}(231)$ with code$_{a'}(w') = c'$. \hfill $\square$
\begin{table}
\centering
\begin{tabular}{|c|c|c|}
\hline
\(\alpha\) & \(w\) & \(\text{code}_\alpha(w)\) \\
\hline
\(1\) & 1 & (0) \\
\hline
\(2\) & 12 & (0,0) \\
(1,1) & 12 & (0,0) \\
& 21 & (1,0) \\
\hline
\(3\) & 123 & (0,0,0) \\
(2,1) & 123 & (0,0,0) \\
& 132 & (0,1,0) \\
& 231 & (1,1,0) \\
(1,2) & 123 & (0,0,0) \\
& 213 & (1,0,0) \\
& 312 & (2,0,0) \\
(1,1,1) & 123 & (0,0,0) \\
& 132 & (0,1,0) \\
& 213 & (1,0,0) \\
& 231 & (0,1,0) \\
& 312 & (2,0,0) \\
& 321 & (2,1,0) \\
\hline
\end{tabular}
\caption{The \(\alpha\)-permutations for any composition \(\alpha\) of \(n \leq 3\) together with their corresponding \(\alpha\)-codes.}
\end{table}

We now “inject” 1 into \(w'\) to construct a permutation \(w \in \mathfrak{S}_n\) via
\[
w_i = \begin{cases} 
  w'_i + 1, & \text{if } i < j_0, \\
  1, & \text{if } i = j_0, \\
  w'_{i-1} + 1, & \text{if } i > j_0.
\end{cases}
\]

Since \(j_0\) is the first element in the \(d\)th \(\alpha\)-region, it follows that \(w \in \mathfrak{S}_\alpha\). Assume that \(w\) has an \((\alpha, 231)\)-pattern \((i, j, k)\). Since \(w' \in \mathfrak{S}_{\alpha'}(231)\), it must be that \(k = j_0\), and \(w_i = 2\). By construction, \(w'_i = 1\), implying that \(c'_i = 0\). Since \(i < j_0\), it follows that \(c_i = 0\), contradicting the choice of \(j_0\). Thus, \(w \in \mathfrak{S}_\alpha(231)\). By construction, it follows that \(w\) is the only \((\alpha, 231)\)-avoiding permutation with \(\text{code}_\alpha(w) = c\).

Figure 1 illustrates the procedure described in the proof of Proposition 3.11. We may now conclude the proof of our first main theorem.

**Proof of Theorem 1.1.** Proposition 3.11 establishes that \(\mathfrak{S}_{\alpha}(231)\) and \(\mathcal{C}_\alpha\) are in bijection, and Corollary 3.8 and Lemma 3.9 establish that for \(u, v \in \mathfrak{S}_{\alpha}(231)\) we have \(u \leq_L v\) if, and only if \(\text{code}_\alpha(u) \leq_{\text{comp}} \text{code}_\alpha(v)\). This finishes the proof.

In fact, the preimages of the maps \(\text{code}_\alpha: \mathfrak{S}_\alpha \to \mathcal{C}_\alpha\) and \(\pi_{\alpha}^L: \mathfrak{S}_\alpha \to \mathfrak{S}_{\alpha}(231)\) coincide.

**Lemma 3.12.** For \(u, v \in \mathfrak{S}_\alpha\) we have \(\text{code}_\alpha(u) = \text{code}_\alpha(v)\) if, and only if \(\pi_{\alpha}^L(u) = \pi_{\alpha}^L(v)\).
(2, 6, 0, 1, 3, 1, 1, 0) → 1 1 1
(1, 5, 1, 3, 1, 0) → 1 1 2
(1, 4, 1, 0, 0, 0) → 1 3 2
(1, 3, 0, 1, 0, 0) → 1 3 2
(0, 2, 1, 0, 0) → 5 1 3 2
(0, 2, 1, 0, 0) → 5 1 3 2
(0, 0, 0, 0) → 5 8 1 4 7 3 6 2

Figure 1. Decoding the (2, 3, 2, 1)-code (2, 6, 0, 1, 3, 1, 1, 0). The arrows indicate the left-most zero in each step; the red digits indicate the positions that see the left-most zero.

Proof. Let \( u, v \in \mathcal{S}_n \). Let \( \text{code}_a(u) = (a_1, a_2, \ldots, a_n) \) and \( \text{code}_a(v) = (a_1, a_2, \ldots, a_n) \).

If \( u \leq_L v \), then the desired equivalence follows from repeated application of Lemma 3.7.

Otherwise, \( u \) and \( v \) are incomparable. By [1, Theorem 4.1], \((\mathcal{S}_n, \leq_L)\) is a lattice and thus the meet \( w = u \wedge_L v \) exists and satisfies \( w \leq_L u \) and \( w \leq_L v \). If \( \pi^L_a(u) = \pi^L_a(v) \), then \( \pi^L_a(u) = \pi^L_a(w) \), by Lemma 3.5. It follows that \( \text{code}_a(u) = \text{code}_a(v) \) by Lemma 3.7.

Conversely, let \( \text{code}_a(u) = \text{code}_a(v) \). Lemma 3.5 implies \( \pi^L_a(u) \leq_L u \) and \( \pi^L_a(v) \leq_L v \). In view of the previous reasoning we find \( \text{code}_a(\pi^L_a(u)) = \text{code}_a(u) = \text{code}_a(v) = \text{code}_a(\pi^L_a(v)) \). Proposition 3.11 thus implies \( \pi^L_a(u) = \pi^L_a(v) \). \( \square \)

Figure 2 shows the weak order on \( \mathcal{S}_{(1,2,1)} \) with the preimages of the map \( \pi^L_{(1,2,1)} \) indicated; the bottom elements per highlighted region are exactly the \(((1,2,1), 231)\)-avoiding permutations. The elements are labeled by their corresponding \((1,2,1)\)-codes, too.

4. Lattice paths and various Tamari lattices

Among other things, [9] introduces a bijection \( \Theta \) from \( \mathcal{S}_n(231) \) to a certain family of northeast paths, denoted by \( \mathcal{L}_{v_a} \). This map was simplified in [4] and used to show that the lattice \( T_{v_a} \) is isomorphic to a certain lattice on \( \mathcal{L}_{v_a} \), denoted by \( \mathcal{T}_{v_a} \) [4, Theorem II]. The proof of this result is only partially bijective, and relies on structural properties of both lattices.

Our parabolic BW-codes will eventually enable us to show that \( \Theta \) is in fact an anti-isomorphism from \( T_{v_a} \) to \( T_{\text{Flip}(v_a)} \), where \( \text{Flip}(v_a) \) is essentially the northeast path \( v_a \) read backwards.
Figure 2. The weak order on $\mathfrak{S}_{(1,2,1)}$, where the permutations are labeled by their $(1,2,1)$-codes.

In order to prove this result, we will take a detour through certain families of lattice paths on $\mathbb{N} \times \mathbb{N}$. Even though the objects presented here and the bijections relating them mostly belong to the combinatorial folklore and are known to many combinatorialists, we shall present them in form of a unified framework in which their definitions and main properties fit together fairly.

4.1. Dyck paths and the ordinary Tamari lattice. We first consider up steps (of the form $U \triangleq (1, 1)$) and down steps (of the form $D \triangleq (1, -1)$). A Dyck path of semilength $n$ is a lattice path using only up and down steps, starting and ending on the $x$-axis and never going below it. Consequently, any Dyck path uses as many up steps as it uses down steps. Let $D_n$ denote the set of Dyck paths of semilength $n$.

The ordinate of a lattice point on a Dyck path is simply the value of its $y$-coordinate. If $P \in D_n$, then any up step $U$ on $P$ has a matching down step: this is the first down step $D$ on $P$ whose starting point has the same ordinate as the ending point of $U$. In particular, the portion of $P$ strictly between $U$ and $D$ is a Dyck path (of strictly smaller semilength) in its own right.

A valley of $P$ is a lattice point $V$ on $P$ preceded by a down step and followed by an up step. The rotation of $P$ by a valley $V$ is the Dyck path $P' \in D_n$ obtained by swapping the down step before $V$ and the portion of $P$ (weakly) between the up step after $V$ and its matching down step. The reflexive and transitive closure of this operation yields a partial order on $D_n$.

The set of Dyck paths ordered by this rotation order forms the (ordinary) Tamari lattice $T_n$, first described in [13].
Figure 3. Example of horizontal distance and cover relation in $T_v$ for northeast paths.

Remark 4.1. It may not be immediately clear from this definition, but the ordinary Tamari lattice is a particular instance of an $\alpha$-Tamari lattice (see Section 2.2), namely when $\alpha = (1, 1, \ldots, 1)$; see [2, Section 9].

4.2. Northeast paths and the $v$-Tamari lattice. Now, we consider north steps (of the form $N \overset{\text{def}}{=}(0,1)$) and east steps (of the form $E \overset{\text{def}}{=}(1,0)$). A northeast path of length $n$ is a lattice path using $k$ north and $n-k$ east steps which starts on the x-axis. The height of a lattice point on a northeast path is the value of its $y$-coordinate. A valley of a northeast path $\nu$ is a lattice point $V$ on $\nu$ preceded by an east step and followed by a north step.

Given a northeast path $\nu$, a $v$-path is a northeast path which shares the starting and ending points with $\nu$ and never goes below $\nu$. Let $L_v$ denote the set of $v$-paths. The horizontal distance of a lattice point $Q$ on $\mu \in L_v$, denoted by $\text{horiz}_v(Q)$, is the largest number of east steps that can be added to $Q$ without crossing to the other side of $v$.

The rotation of $\nu$ by a valley $V$ is the path $\nu' \in L_v$ obtained by exchanging the east step before $V$ with the portion of $\nu$ between $V$ and the next lattice point $W$ on $\nu$ satisfying $\text{horiz}_v(V) = \text{horiz}_v(W)$. In this situation, we write $\nu \leq_v \nu'$. See Figure 3 for an illustration. The reflexive and transitive closure of this operation yields a partial order on $L_v$.

For any northeast path $v$, the set of $v$-paths ordered by this rotation order forms the $v$-Tamari lattice $T_v$ introduced in [11].

Remark 4.2. Note that the definition that $T_v \cong T_n$ when $v = (NE)^n$. The isomorphism is given by substituting north steps by up steps and east steps by down steps.

In [3], it was shown that the $v$-Tamari lattice can be realized using the componentwise order on so-called $v$-bracket vectors. If $\nu$ is a northeast path of length $n$, then the minimal $v$-bracket vector is the vector $b^{\text{min}}$ consisting of $n+1$ entries, whose $i$-th entry is the height of the $i$-th lattice point on $\nu$. If $\nu$ has $k$ north steps, then the fixed positions are the entries $f_0, f_1, \ldots, f_k$, where $f_i$ is the position of the last appearance of $i$ in $b^{\text{min}}$. An integer vector $b$ with $n+1$ entries is a $v$-bracket vector, if it has the following three properties:

(B1): for $0 \leq s \leq k$, we have $b(f_s) = s$;

(B2): for $1 \leq i \leq n+1$, we have $b^{\text{min}}(i) \leq b(i) \leq k$;

(B3): if $b(i) = s$, then for all $j$ with $i < j < f_s$, we have $b(j) \leq s$.

The set of $v$-bracket vectors is denoted by $B_v$. The following theorem was proven by means of an explicit bijection in [3, Section 4].
Theorem 4.3 ([3, Theorem 21]). For any northeast path \( v \), the \( v \)-Tamari lattice \( T_v \) is isomorphic to \( (B_v, \leq_{\text{comp}}) \).

4.3. \( v \)-Tamari lattices are intervals of ordinary Tamari lattices. It was shown in [11, Theorem 3] in terms of binary trees that every \( v \)-Tamari lattice is isomorphic to an interval in some ordinary Tamari lattice. More precisely, the bijection in [11, Section 2 and 3] from binary trees to pairs of non-crossing lattice paths is done by a double reading of a word obtained from the contour of the binary tree, and the reverse direction is more complicated, involving a so-called “push-gliding” algorithm. Later, this bijection is transplanted from binary trees to Dyck paths in [5, Section 2], but only the direction from Dyck paths to pairs of non-crossing lattice paths, not the reverse direction, and without proof. Furthermore, the bijection given in [5] needs to keep track of how pairs of peaks appear in sub-paths of the Dyck path, making it seem non-trivial and hard to reverse.

We now propose a reformulation of the same bijection, given in both ways, along with proofs of equivalence to the original bijection. Our reformulation is much simpler than the original ones, as the bijectivity is trivial, and does not require any complicated algorithm in both directions.

Clearly, a northeast path is uniquely determined by the lengths of the runs of east steps at each height. In other words, if \( v \) is a northeast path, then we can write it uniquely as

\[
v = E_{a_0} NE_{a_1}^1 \cdots E_{a_{k-1}+1} NE_{a_k},
\]

By abuse of notation we will also write \( v = [a_0, a_1, \ldots, a_k] \). Now, if \( \mu \in \mathcal{L}_v \) with \( \mu = [b_0, b_1, \ldots, b_k] \), then we have \( \sum_{i=0}^{k} a_i = \sum_{i=0}^{k} b_i = m \) and \( \sum_{i=0}^{j} a_i \geq \sum_{i=0}^{j} b_i \) for all \( 0 \leq j \leq k \). We may have \( a_i = 0 \) or \( b_i = 0 \) for some indices \( i \).

Construction 4.4. Given a northeast path \( v \) composed of \( k \) north steps and \( n-k \) east steps, let \( \mu \in \mathcal{L}_v \) such that \( v = [a_0, a_1, \ldots, a_k] \) and \( \mu = [b_0, b_1, \ldots, b_k] \). We define the Dyck path \( \text{Dyck}(v, \mu) \) of semilength \( n+1 \) by

\[
\text{Dyck}(v, \mu) \overset{\text{def}}{=} U_{a_0+1} D_{b_0+1} \cdots U_{a_k+1} D_{b_k+1}.
\]

Conversely, if \( P \in \mathcal{D}_{n+1} \), we can recover the pair \( (v, \mu) \) satisfying \( P = \text{Dyck}(v, \mu) \) by looking at the lengths of the runs of up and down steps in \( P \), which determine the \( a_i \)’s and \( b_i \)’s, respectively.

The map \( \text{Dyck} \) is a bijection from \( \mathcal{D}_{n+1} \) to the set

\[
\{(v, \mu) \mid v \text{ has length } n \text{ and } \mu \in \mathcal{L}_v\},
\]

and is illustrated in Figure 4. The map \( \text{Dyck} \) is simple and clearly bijective, a quality absent from previous formulations.

Proposition 4.5. Let \( \mu, \mu' \in \mathcal{L}_v \), with \( v = [a_0, a_1, \ldots, a_k] \), \( \mu = [b_0, b_1, \ldots, b_k] \) and \( \mu' = [b'_0, b'_1, \ldots, b'_k] \). Then \( \mu <_v \mu' \) if, and only if \( b_i = b'_i \) except for two indices \( \ell < m \) such that \( b_\ell > 0 \) and that \( m \) is the first index after \( \ell \) satisfying \( \sum_{i=\ell}^{m} (b_i - a_i) \geq 0 \). In this case, we have \( b'_\ell = b_\ell - 1 \) and \( b'_m = b_m + 1 \).

Proof. We observe that the minimum of \( \text{horiz}_v \) for points with the same height occurs at the rightmost point. For any index \( 0 \leq j \leq k \), let \( V_j \) be the rightmost point of height \( j \) on \( \mu \). We have \( \text{horiz}_v(V_j) = \sum_{i=0}^{j} a_i - \sum_{i=0}^{j} b_i \).

Assume that \( \mu' \) covers \( \mu \) in \( T_v \), with \( V \) the valley of \( \mu \) leading to this covering relation, and \( W \) the first lattice point after \( V \) with \( \text{horiz}_v(W) = \text{horiz}_v(V) \). Let \( \ell \)
(resp. \( m \)) be the height of \( V \) (resp. \( W \)). We have \( V = V_\ell \), also \( b_\ell > 0 \) as \( V \) is a valley. By the definition of \( W \), we know that \( \ell < m \), \( \text{horiz}_V (V) = \text{horiz}_W (W) \geq \text{horiz}_V (V_m) \), and \( \text{horiz}_V (V) < \text{horiz}_V (V_m') \) for any \( m' \) such that \( \ell < m' < m \). Therefore, \( m \) is the first index such that \( \ell < m \) and \( \text{horiz}_V (V_\ell) \geq \text{horiz}_V (V_m) \). In other words,

\[
0 \leq \text{horiz}_V (V_\ell) - \text{horiz}_V (V_m) = \sum_{i=0}^{\ell} (a_i - b_i) - \sum_{i=0}^{m} (a_i - b_i) = \sum_{i=\ell+1}^{m} (a_i - b_i).
\]

In this case, we also have

\[
\mu' = [b_0, \ldots, b_{\ell-1}, b_\ell + 1, b_{\ell+1}, \ldots, b_{m-1}, b_m - 1, b_{m+1}, \ldots, b_k]
\]
as desired.

Conversely, suppose that we have two indices \( \ell \) and \( m \) satisfying the given conditions. Let \( V = V_\ell \) on \( \mu \) and \( W \) be the first lattice point after \( V \) with \( \text{horiz}_V (W) = \text{horiz}_V (V) \). By the conditions that \( m \) satisfies, for any \( m' \) such that \( \ell < m' < m \), we must have \( \text{horiz}_V (V_\ell) < \text{horiz}_V (V_{m'}) \), meaning that the height of \( W \) is at least \( m \). We also observe that the horizontal distance of the leftmost point of height \( m \) is \( \text{horiz}_V (V_{m-1}) + a_m > \text{horiz}_V (V_\ell) \), while \( \text{horiz}_V (V_m) \leq \text{horiz}_V (V_\ell) \) for \( V_m \) the rightmost point of height \( m \). Therefore, the height of \( W \) must be \( m \). We conclude the proof by plugging in the definition of \( \text{horiz}_V (V_\ell) \) in the condition.

The following proposition show that \textbf{Dyck} is the same bijection as that in [11], but from non-crossing lattice paths instead of binary trees.

**Proposition 4.6.** Given a northeast path \( \nu \) of length \( n \), the map \textbf{Dyck} is an isomorphism from \( T_v \) to an interval \( I_v \) of \( T_{n+1} \).

**Proof.** It is clear that \textbf{Dyck}(\( \nu, \mu \)) is a Dyck path of semilength \( n+1 \) for every \( \mu \in L_v \). We already know from [11, Theorem 3] that \( T_v \) is isomorphic to some interval \( I_v \) in \( T_{n+1} \). We only need to show that this interval is exactly \{\textbf{Dyck}(\( \nu, \mu \)) \mid \( \mu \in L_v \)\} under ordinary rotation order. To that end, we show that if \( \mu' \) covers \( \mu \) in \( T_v \), then \( P' = \textbf{Dyck}(\( \nu, \mu' \)) \) covers \( P = \textbf{Dyck}(\( \nu, \mu \)) \) in \( T_{n+1} \).

By abuse of notation, if we have \( P = D^{c_0}UD^{c_1}U \cdots UD^{c_{n+1}} \) for some \( (c_i)_{0 \leq i \leq n+1} \), then we also write \( P = [c_0, c_1, \ldots, c_{n+1}] \). We note that \( c_0 \) is always 0.

Now pick an arbitrary northeast path \( \nu \) of length \( n \) and some \( \mu \in L_v \). Suppose that \( \nu = [a_0, a_1, \ldots, a_k] \) and \( \mu = [b_0, b_1, \ldots, b_k] \). For \( d \in \{0, 1, \ldots, k\} \) we define

\[
f_\nu (d) \overset{\text{def}}{=} d + \sum_{i=0}^{d} a_i.
\]
Similarly, we transport these paths to \( \mathcal{T}_D \) and therefore correspond to an east step in \( D \). Then, \( m' \) covers \( m \) in \( \mathcal{T}_D \), denote by \( \ell \) and \( m \) the two indices satisfying the conditions of Proposition 4.5. Now, let \( \ell^* = f_\nu(\ell) + 1 \) and \( m^* = f_\nu(m) + 1 \). We now show that \( \ell^* \) and \( m^* \) are indices satisfying the conditions of Proposition 4.5 for \( P \) and \( P' \) in \( \mathcal{T}_{(NE)^{n+1}} \). It is clear that \( \ell^* < m^* \), and from the definition of Dyck, we know that \( c_i = c'_i \) except for \( \ell^* \) and \( m^* \). We also have \( c'_{\ell^*} = b'_\ell + 1 = b_\ell = c_{\ell^*} - 1 \). Similarly, \( c_{m^*} = c_{m^*} + 1 \).

For \( j^* > \ell^* \), let \( s(j^*) = \sum_{i=\ell+1}^{j^*} (c_i - 1) \). To guarantee that \( P' \) covers \( P \) in \( \mathcal{T}_{n+1} \), we transport these paths to \( \mathcal{T}_{(NE)^{n+1}} \) and use Proposition 4.5. We thus only need to show that \( m^* \) is the first index after \( \ell^* \) such that \( s(m^*) \geq 0 \), as \( (NE)^{n+1} = [0,1,\ldots,1] \). Suppose that \( j^* \) is the smallest index with \( j^* > \ell^* \) and \( s(j^*) \geq 0 \). If there is no index \( j \) such that \( j^* = f_\nu(j) + 1 \), then \( c_{j^*} = 0 \) and thus \( s(j^* - 1) \geq s(j^*) \geq 0 \), contradicting the minimality of \( j^* \). We thus have \( j^* = f_\nu(j) + 1 \) for some \( j \), and we observe that

\[
 s(j^*) = \sum_{i=\ell+1}^{j^*} (c_i - 1) = \left( \sum_{i=\ell+1}^{j} (b_i + 1) \right) - (j^* - \ell^*) \\
= \left( \sum_{i=\ell+1}^{j} b_i \right) - \left( (j^* - \ell^*) \right) = \sum_{i=\ell+1}^{j} (b_i - a_i). 
\]

The last equality follows from the fact that \( c_i = 0 \) indicates two consecutive up steps in \( P \) and therefore correspond to an east step in \( \nu \). Similarly, we have \( s(m^*) = \sum_{i=\ell+1}^{m} (b_i - a_i) \), and by Proposition 4.5, we have \( s(m^*) \geq 0 \). By the minimality of \( j^* \), we have \( j^* \leq m^* \).

As \( j \) satisfies the sum condition of Proposition 4.5 with respect to \( \ell \), we must have \( m \leq j \). This implies \( m^* \leq j^* \), and we obtain \( j^* = m^* \). We conclude that \( P' \) covers \( P \) in \( \mathcal{T}_{n+1} \).

4.4. Two sequence statistics of Dyck paths. In preparation of things to come, we now consider an anti-isomorphism of \( \mathcal{T}_D \) that exchanges two particular sequence statistics on Dyck paths. While this property is known to experts, we could not find an explicit reference stating it in the framework of Dyck paths. It can be deduced, however, from the recent work of Pons on Tamari interval posets [10]; see the paragraph after Theorem 23 therein. We translate the appropriate specialization of her result to the framework of Dyck paths. We then relate this map and these statistics to \( \nu \)-Tamari lattices via the map Dyck.

A rising contact of a Dyck path \( P \) is an up step in \( P \) that starts on the \( x \)-axis. Every Dyck path \( P \) of length \( n > 0 \) can be uniquely decomposed into \( P = P_0 \uparrow P_1 \downarrow \downarrow P_D \) with \( P_0, P_1, P_D \) both Dyck paths, by taking \( P_0 \) to be the sub-path before the last rising contact of \( P \). We denote by \( \epsilon \) the empty Dyck path, and we define an involution
Given a Dyck path \( P \) indexed from 0 to \( n \), we have \( \text{Cont}(P) = (2,4,0,2,2,0,1,0,0,0,1,0) \) and \( \text{Drun}(P) = (2,0,2,1,2,3,0,1,0,1,0,0) \).

We now define another sequence statistic. The contact sequence \( \text{Cont} \) is obtained by taking \( \text{Cont}(P) = (2,4,0,2,2,0,1,0,0,0,1,0) \) for all 0 \( \leq i \leq n \).

We define \( \text{Flip} \), the northeast path obtained by reversing \( v \) and exchanging north and east steps. Geometrically, \( \text{Flip}(v) \) is \( v \) reflected across a diagonal of slope \(-1\). It is known that, for \( (v', \mu') = \text{Conj}'(v, \mu) \), we have \( v = \text{Flip}(v) \) (see [11, Theorem 2 and 3]). We have the following corollary.

**Corollary 4.8.** The bijection \( \text{Conj}' \) is an anti-isomorphism between \( T_n \) and \( T_{\text{Flip}(v)} \).

We now consider two statistics on Dyck paths that are interchanged by \( \text{Conj} \). Given a Dyck path \( P \), we define its descent run sequence, denoted by \( \text{Drun}(P) \) and indexed from 0 to \( n \), as follows. We write \( P = D^0UD^1U \cdots UD^n \) with some \( c_i \geq 0 \) (again, noting \( c_0 = 0 \)), then we take \( \text{Drun}(P)_i = c_{n-i} \) for all 0 \( \leq i \leq n \). The map \( \text{Drun} \) is injective because \( P \) can be reconstructed from the \( c_i \)'s.

We now define another sequence statistic. The contact sequence of \( P \), denoted by \( \text{Cont}(P) \) and indexed from 0 to \( n \), is obtained by taking \( \text{Cont}(P)_0 \) the number of rising contacts of \( P \), and \( \text{Cont}(P)_i \) the number of rising contacts of the sub-Dyck path strictly between the \( i \)-th up step and its matching down step.
Examples for Cont and Drun are given in Figure 5. The following result, also illustrated in Figure 5, is well-known. In terms of binary trees, Cont (resp. Drun) describes maximal left (resp. right) descending paths, and the counterpart of Conj on binary trees is taking the vertical mirror image. The following can be proven inductively.

**Proposition 4.9.** For any Dyck path $P$, we have $\text{Cont}(P) = \text{Drun}(\text{Conj}(P))$.

As Dyck is bijective, let $(v, \mu) = \text{Dyck}^{-1}(P)$, and define $\text{Drun}(v, \mu) = \text{Drun}(P)$ and $\text{Cont}(v, \mu) = \text{Cont}(P)$. Suppose that $v = [a_0, a_1, \ldots, a_k]$ and $\mu = [b_0, b_1, \ldots, b_k]$. Then we have

$$\text{Drun}(v, \mu) = \text{Drun}(P) = (b_k + 1, 0^{b_k}, b_{k-1} + 1, 0^{b_{k-1}}, \ldots, b_0 + 1, 0^{b_0}, 0).$$

Here “$0^a$” means the entry 0 repeated $a$ times. For an expression of Cont, we need some more definitions. We define the reverse horizontal distance to $\mu$ of a lattice point $Q$ on $v$, denoted by $\text{horiz}'_{\mu}(Q)$, to be the number of west steps $(-1, 0)$ we can take from $Q$ before crossing to the other side of $\mu$. See Figure 5 for an example. This figure also illustrates the following result.

**Proposition 4.10.** Let $P = \text{Dyck}(v, \mu)$ with $v, \mu$ northeast paths of length $n$. Take the sequence $(d_i)_{0 \leq i \leq n}$ with $d_i = \text{horiz}'_{\mu}(Q_i)$, where $Q_i$ is the $(i + 1)$-st lattice point of $\mu$. Then we have

$$\text{Cont}(v, \mu)_0 = |\{\ell \mid 0 \leq \ell \leq n, d_\ell = 0\}|;$$

$$\text{Cont}(v, \mu)_i = |\{\ell \mid i < \ell \leq n, d_\ell = d_i + 1, \forall i < m \leq \ell, d_m > d_i\}|.$$

**Proof.** Assume that $v = [a_0, a_1, \ldots, a_k]$ and $\mu = [b_0, b_1, \ldots, b_k]$. For each $0 \leq i \leq k$, there are $a_i + 1$ lattice points of height $i$ on $v$, and the $(i + 1)$-st consecutive run of up steps of $P$ consists of $a_i + 1$ steps. According to the construction of Dyck, the $(i + 1)$-st up step in $P$ corresponds to a lattice point $Q_i$ on $v$. Moreover, $Q_i$ is the leftmost point with height $j$ on $v$ if, and only if the corresponding up step is the first in the $j$-th run of up steps in $P$.

We now prove that $d_i$ is the ordinate of the starting point of the $(i + 1)$-st up step in $P$. We proceed by induction. For $i = 0$, we have $d_0 = 0 = \text{horiz}'_{\mu}(Q_0)$, as $v$ and $\mu$ start at the origin. Now suppose that $d_{i-1}$ is the ordinate of the starting point of the $i$-th up step in $P$. We have two cases.

(i) If $Q_i$ is not the leftmost lattice point with its height, then its corresponding up step in $P$ directly comes after the one associated with $Q_{i-1}$. Thus, the ordinate of the starting point of this up step is $d_{i-1} + 1 = d_i$.

(ii) If $Q_i$ is the leftmost lattice point with height $\ell$, then its corresponding up step is the first up step of the next run after the up step associated with $Q_{i-1}$. The ordinate of the up step associated with $Q_i$ is thus $d_{i-1} + 1 - (b_1 + 1)$, taking into account the length of the $\ell$-th run of down steps in $P$. This is also equal to $d_i$, as we reach $Q_i$ from $Q_{i-1}$ by a north step, and it takes $b_1$ less west steps to cross $\mu$ from $Q_i$ than from $Q_{i-1}$.

We thus conclude the induction. Now (3) is a translation of the definition of Cont in terms of the $d_i$’s. \qed
5. The $\alpha$-Tamari Lattices are Certain $\nu$-Tamari Lattices

It was shown in [4, Theorem II], that the $\alpha$-Tamari lattice is isomorphic to a certain $\nu$-Tamari lattice, namely, when $\nu$ is the $\alpha$-bounce path, defined by

\begin{equation}
\nu_\alpha \defeq N^{a_1} E^{s_1} N^{a_2} E^{s_2} \cdots N^{a_n} E^{s_n},
\end{equation}

where $\alpha = (a_1, a_2, \ldots, a_n)$. The proof given in [4] is rather indirect and exploits certain lattice-theoretic properties of the $\alpha$- and the $\nu$-Tamari lattices. In this section, we give a direct proof using the realization of $T_\alpha$ as the componentwise order on $\alpha$-codes established in Theorem 1.1.

In particular, we construct a bijection from the $\alpha$-codes to the $\nu_\alpha$-bracket vectors. Let us adapt the definitions to the particular case of $\nu$. The $\nu_\alpha$-bracket vectors, denoted by $b^*_{\nu_\alpha}$, is defined by

\begin{equation}
b^*_{\nu_\alpha}(k) \defeq \begin{cases} 
    i + s_{a_i} - 1, & \text{if } k = 2s_{a_i} + i \text{ for } 0 < i \leq a_{a_i} \\
    s_{a_i}, & \text{if } k = 2s_{a_i} + a_{a_i} + i \text{ for } 0 < i \leq a_{a_i}, \\
    n, & \text{if } k = 2n + 1.
\end{cases}
\end{equation}

We write $B_\alpha$ instead of $B_{\nu_\alpha}$ for the set of $\nu_\alpha$-bracket vectors.

For $b \in B_\alpha$, by (B1), there are $n + 1$ positions with fixed value in a vector of length $2n + 1$. For simplification, we define a reduced version of $\nu_\alpha$-bracket vectors. For $b \in B_\alpha$, we define its $\nu_\alpha$-reduced vector $r$ by

$$r(s_{a_i} + i) \defeq b(2s_{a_i} + a_{a_i} + i)$$

for $1 \leq i \leq a_{a_i}$. It is clear that $r$ is obtained from $b$ by removing components whose indices are fixed positions. To recover $b$ from $r$, we only need to fill in the positions of the fixed positions according to (B1). Let $\Lambda_{\text{red}}$ denote the “reduction” map from $b$ to $r$, and let $\Lambda_{\text{ext}}$ be its inverse.

Such $\nu_\alpha$-reduced vectors thus inherit the following properties from $\nu_\alpha$-bracket vectors.

**Proposition 5.1.** A vector $r \in \mathbb{N}^n$ is a $\nu_\alpha$-reduced vector if, and only if:

\begin{itemize}
  \item[(R1)] for $1 \leq i \leq n$, we have $s^{\nu}(i) \leq r(i) \leq n$;
  \item[(R2)] for all $i, j$ with $i < j \leq s^{\nu}(r(i) + 1)$, we have $r(j) \leq r(i)$.
\end{itemize}

**Proof.** Let $b$ be the $\nu_\alpha$-bracket vector corresponding to $r$. We only need to show that the conditions for $r$ are equivalent to those for $b$.

Condition (B1) for $b$ is satisfied by construction. The equivalence between (B2) for $b$ and (R1) for $r$ is trivial given the definition of $b^*_{\nu_\alpha}$.

Now for the equivalence between (B3) for $b$ and (R2) for $r$, we observe that for (B3) to hold for $b$, for each $i$ with $b(i) = k$, we only need to check for all $j$ with $i < j \leq 2s^{\nu}(k) - 1$, since all indices from $2s^{\nu}(k) - 1$ to $f_k$ are fixed positions. \qed

We can thus take Proposition 5.1 as the definition of $\nu_\alpha$-reduced vectors without passing through $\nu_\alpha$-bracket vectors, and we denote by $R_\alpha$ the set of all $\nu_\alpha$-reduced vectors. By Proposition 5.1, $(R_\alpha, \leq_{\text{comp}})$ is isomorphic to the $\nu_\alpha$-Tamari lattice. We also have the following property.

**Proposition 5.2.** Given a $\nu_\alpha$-reduced vector $r$, for any indices $i < j$ with $q(i) = q(j)$, we have $r(i) \geq r(j)$.
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\[ \Delta \] show that, for any \( \varrho \), 

\[ \Delta \] be clear from the context. The transformation 

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\[ \Delta \] More intuitively, to obtain 

\[ \Delta \] given that it satisfies (C2), we only need to 

\[ \Delta \] of \( n \), the transformation 

\[ \Delta \] is a bijection from 

\[ \Delta \] relates \( \alpha \)-codes with \( \mathcal{R}_\alpha \). 

Figure 6 illustrates the map \( \Gamma_{R} \) on the \((2,3,2,1)\)-code \((2,6,0,1,3,1,1,0)\), and Table 2 illustrates this bijection for \( \alpha = (1,2,1) \).

**Table 2. Illustration of the map \( \Gamma_{R} \) for \( \alpha = (1,2,1) \).**

| \( c \in \mathcal{C}_{(1,2,1)} \) | \( \Gamma_{R}(c) \in \mathcal{R}_{(1,2,1)} \) | \( \Lambda_{\text{ext}} \circ \Gamma_{R}(c) \in \mathcal{B}_{(1,2,1)} \) |
|---|---|---|
| (0,0,0,0) | (1,3,3,4) | (0,1,1,2,3,3,4,4) |
| (1,0,0,0) | (2,3,3,4) | (0,2,1,2,3,3,4,4) |
| (0,0,1,0) | (1,4,3,4) | (0,1,1,2,4,3,4,4) |
| (2,0,0,0) | (3,3,3,4) | (0,3,1,2,3,3,4,4) |
| (1,0,1,0) | (2,4,3,4) | (0,2,1,2,4,3,4,4) |
| (0,1,1,0) | (1,4,4,4) | (0,1,1,2,4,3,4,4) |
| (3,0,0,0) | (4,3,3,4) | (0,4,1,2,3,3,4,4) |
| (3,0,1,0) | (4,4,3,4) | (0,4,1,2,4,3,4,4) |
| (1,1,1,0) | (2,4,4,4) | (0,2,1,2,4,3,4,4) |
| (3,1,1,0) | (4,4,4,4) | (0,4,1,2,4,3,4,4) |

**Proof.** Let \( k = r(i) \). By (R1), we have \( k \geq s_{\varrho(i)} \), and thus \( s_{\varrho(k+1)} - 1 \geq s_{\varrho(i)} - 1 \geq s_{\varrho(i)} \). Since \( \varrho(i) = \varrho(j) \), we have \( i < j \leq s_{\varrho(i)} \leq s_{\varrho(k+1)} - 1 \). Then (R2) in Proposition 5.1 states that \( r(j) \leq k = r(i) \). \( \square \)

For any composition \( \alpha \), we define a transform \( \Delta_{R} \) on \( \mathcal{R}_\alpha \) such that 

\[
(\Delta_{R}(r))_i \overset{\text{def}}{=} r(2s_{\varrho(i)} - \alpha_{\varrho(i)} - i + 1) - s_{\varrho(i)}.
\]

More intuitively, to obtain \( \Delta_{R}(r) \), we first split \( r \) into regions according to \( \alpha \), then reverse each region while subtracting \( s_{k} \) on the \( k^{th} \) region. We denote by \( \Gamma_{R} \) its inverse. Although both \( \Delta_{R} \) and \( \Gamma_{R} \) depend on \( \alpha \), the composition \( \alpha \) should always be clear from the context. The transformation \( \Delta_{R} \) relates \( \alpha \)-codes with \( \mathcal{R}_\alpha \).

Figure 6 illustrates the map \( \Gamma_{R} \) on the \((2,3,2,1)\)-code \((2,6,0,1,3,1,1,0)\), and Table 2 illustrates this bijection for \( \alpha = (1,2,1) \).

**Proposition 5.3.** Given a composition \( \alpha \) of \( n \), the transformation \( \Delta_{R} \) is a bijection from \( \mathcal{R}_\alpha \) to \( \mathcal{C}_\alpha \).

**Proof.** First, for \( r \in \mathcal{R}_\alpha \), let \( c = \Delta_{R}(r) \) and let us check that \( c \) satisfies the conditions in Definition 3.1 using those in Proposition 5.1 for \( r \). By (R1) for \( r \) and the definition of \( \Delta_{R} \), clearly \( c \) satisfies (C1). Proposition 5.2 and the definition of \( \Delta_{R} \) imply that \( c \) satisfies (C2). To check (C3) for \( c \) given that it satisfies (C2), we only need to show that, for any \( i \) and \( j \) such that \( \varrho(i) < \varrho(j) \), if \( c_i \geq s_{\varrho(j)} - s_{\varrho(i)} \), then we have \( c_j + s_{\varrho(j)} \leq c_i + s_{\varrho(i)} \). Translating to \( r \), we need to check that, for any \( i' \) and \( j' \) with \( \varrho(i') < \varrho(j') \), if \( r(i') \geq s_{\varrho(j')} \), then we have \( r(j') \leq r(i') \). Now, suppose that \( r(i') \geq s_{\varrho(j')} \). We have \( \varrho(r(i') + 1) > \varrho(j') \) by the definition of \( \varrho \). As the values are integers, we have \( \varrho(j') \leq \varrho(r(i') + 1) - 1 \), which means \( j' \leq s_{\varrho(j')} \leq s_{\varrho(r(i') + 1) - 1} \), and by (R2), we have \( r(j') \leq r(i') \). Therefore, \( c \) also satisfies (C3).
Now for the reverse direction, given \( c \in \mathcal{C}_n \), let \( r = \Gamma_R(c) \). It is clear that (C1) translates directly to (R1). We only need to show that (R2) holds for \( r \). Suppose that \( 1 \leq i < j \leq s_{q(i+1)} - 1 \). If \( q(i) = q(j) \), by the definition of \( \Gamma_R \) and (C2) on \( c \), we have \( r(j) \leq r(i) \). Now we check the case \( q(i) < q(j) \). When translated to \( c \), (R2) in this case means that we need to check for any \( i' < j' \) such that \( q(i') < q(j') \leq q(c_{i'} + s_{q(i')} + 1) - 1 \), we have \( c_{i'} + s_{q(i')} \leq c_{j'} + s_{q(j')} \). By (C2), we may assume that \( j' = s_a \) for some \( a \). By the definition of \( q \), we see that \( q(s_a) \leq q(c_{i'} + s_{q(i')} + 1) - 1 \) implies \( s_a < c_{i'} + s_{q(i')} + 1 \), thus \( s_a \leq c_{i'} + s_{q(i')} \) since they are integers. By (C3), we have \( s_a \leq c_{i'} - s_a + s_{q(i')} \). Therefore, (R2) holds for \( r \), meaning that \( r \in \mathcal{R}_n \).  

This allows us to conclude to the announced simple proof of Theorem 1.2.

**Proof of Theorem 1.2.** Let \( \mathcal{T}_v \) denote the \( v_\alpha \)-Tamari lattice. We have the following isomorphisms of lattices:

\[
\mathcal{T}_n \cong \left( \mathcal{C}_n, \leq \text{comp} \right) \cong \left( \mathcal{R}_n, \leq \text{comp} \right) \cong \left( \mathcal{B}_n, \leq \text{comp} \right) \cong \mathcal{T}_v. \tag{\ref*{thm:main}}
\]

Note that the proof of Theorem 1.2 in [4] relies on lattice-theoretic properties of \( \mathcal{T}_n \) and \( \mathcal{T}_v \), and is only partially bijective. Our proof here is fully bijective, which gives a clearer vision of the isomorphism.

### 6. A COMBINATORIAL ANTI-ISOMORPHISM ON THE \( v_\alpha \)-TAMARI LATTICE

#### 6.1. Two ways from (\( \alpha, 231 \))-avoiding permutations to \( \alpha \)-paths

Recall that we have fixed a composition \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \) of \( n \), and that \( s_\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_r \) for \( \alpha \in [r] \). Moreover, recall the definition of the \( \alpha \)-bounce path from (4). We usually say \( \alpha \)-path rather than \( v_\alpha \)-path.

We now define two bijections from \( (\alpha, 231) \)-avoiding permutations to \( \alpha \)-paths. The first one uses the \( \alpha \)-code from Section 3.1, and sends \( w \in \mathbb{S}_n(231) \) to \( \varphi(w) \in \mathcal{L}_v \), satisfying \( \varphi(w) = [f_0, f_1, \ldots, f_n] \), where

\[
f_i \overset{\text{def}}{=} \left\{ j \mid 1 \leq j \leq n, \text{code}_\alpha(w)_j + s_{q(j)} = i \right\}.
\]

For example, for \( \alpha = (1, 3, 1, 2) \) and \( w = [5 \, 3 \, 4 \, 7 \, 2 \, 6] \in \mathbb{S}_n(231) \), we have \( \text{code}_\alpha(w) = (2, 2, 2, 3, 0, 0, 0) \) and \( \varphi(w) = [0, 0, 0, 1, 0, 1, 2, 3] \). Note that the first entry of \( \varphi(w) \) is always 0.

**Proposition 6.1.** The map \( \varphi \) is an isomorphism from \( \mathcal{T}_n \) to \( \mathcal{T}_v \).

**Proof.** We write \( \varphi(w) = [f_0, f_1, \ldots, f_n] \). By the definition of \( \Delta_R \), we have

\[
f_i = \left\{ j \mid 1 \leq j \leq n, \Delta_R^{-1}(\text{code}_\alpha(w))_j = i \right\}.
\]

We may rephrase this using \( v_\alpha \)-reduced vectors. Let \( r = \Gamma_R(\text{code}_\alpha(w)) \) denote the \( v_\alpha \)-reduced bracket vector associated with \( w \). Then,

\[
f_i = \left\{ j \mid 1 \leq j \leq n, r_j = i \right\} - 1.
\]

Now, if \( b = \Lambda_{\text{ext}}(r) \) is the associated \( v_\alpha \)-bracket vector, then the number of entries equal to \( i \) in \( b \) is \( f_i + 1 \). According to [3, Definition 26], there exists a unique \( v_\alpha \)-path with as many lattice points of height \( i \) as there are entries equal to \( i \) in \( b \). (See also item (ii) in the proof of [3, Proposition 27].) We conclude that the \( v_\alpha \)-path associated with \( b \) is precisely \( \varphi(w) \).
\[ w = 5 \quad 3 \quad 4 \quad 10 \quad 1 \quad 2 \quad 7 \quad 6 \quad 9 \quad 13 \quad 14 \quad 8 \quad 11 \quad 12 \in S_\alpha(231) \]

![Diagram](image)

**Figure 7.** Example of \( \Theta(w) \) for \( w \in S_\alpha(231) \) with \( \alpha = (1,3,1,2,4,3) \).

Therefore, \( \varphi \) is precisely the isomorphism used in the proof of Theorem 1.2. \qed 

The second bijection, denoted by \( \Theta \), from \((\alpha, 231)\)-avoiding permutations to \( \alpha \)-paths was first defined in [9]. We will use an equivalent definition derived from [4], using a family of trees called \( \alpha \)-trees, which we will not explicitly define.

**Construction 6.2.** Given \( w \in S_\alpha(231) \), we construct a labeled plane tree \( T(w) \) by an insertion procedure. We start with a node labeled \( n + 1 \) as the root, and we read the elements of \( w \) from left to right. Upon reading of an element \( w(i) \), we start a walk from the root. When we reach a node \( v \) with label \( \ell \), if \( w(i) < \ell \), then we move to the left-most child of \( v \); otherwise, we move to the first sibling of \( v \) on its right. When the destination node does not exist, we add it with label \( w(i) \) and terminate the walk. The labeled plane tree thus obtained is denoted by \( T(w) \).

Now we construct a northeast path \( P \) from \( T(w) \). If the root of \( T(w) \) has \( k \) children, then we start \( P \) with \( k \) north steps. Then, for each \( a \in [r] \), we inspect the elements \( w(i) \) in the \( a \)-th \( \alpha \)-region from right to left, i.e., \( i \) runs from \( s_a \) down to \( s_{a - 1} + 1 \). For each such \( w(i) \), let \( v_i \) be the node with label \( w(i) \) in \( T(w) \), and we append \( EN^{k_i} \) to \( P \), where \( k_i \) is the number of children of \( v_i \). We define \( \Theta(w) \) to be the path \( P \) thus obtained. See Figure 7 for an example.

For \( w \in S_\alpha(231) \), the tree \( T(w) \) from Construction 6.2 has the following immediate property.

**Proposition 6.3 ([4]).** For \( w \in S_\alpha(231) \), let \( T(w) \) be the labeled plane tree constructed in Construction 6.2. Reading the labels of \( T(w) \) in postorder (i.e., for each node \( u \), the children of \( u \) are increasing in order from left to right, all greater than \( u \)) gives \( 1, 2, \ldots, n \).

**Remark 6.4.** The map \( \Theta \) in Construction 6.2 is in fact \( \Theta^{-1} \circ \text{Flip} \) in [4]. We have altered the definition here for simplicity. In Construction 6.2, the tree \( T(w) \) is a labeled version of an \( \alpha \)-tree, and the map from \( w \) to \( T(w) \) is the map \( \Lambda_{\text{perm}} \) in [4]. Moreover, the map from \( T(w) \) to \( \Theta(w) \) is \( E_n \) in that article, but our definition here is adapted from Lemma 1.31 of the same article. The validity of our definition of \( \Theta \) is ensured by [4, Propositions 1.33 and 1.34]. The property in Proposition 6.3 follows from [4, Construction 1.14].
6.2. A stack-processing procedure. We now give another combinatorial definition of $\varphi$. For $w \in \mathfrak{S}_\alpha(231)$, we define the companion of an element $w(i)$ in $w$ to be the last element it sees, or $w(s_{\varphi(i)})$ when $w(i)$ sees no element. Then we can define $\varphi(w) = [f_0, f_1, \ldots, f_n]$, where $f_i$ is the number of elements in $w$ with $w(i)$ as its companion. We check that this definition of $f_i$ is the same as (5). We define the following stack processing that can be used to compute the companions of elements of $w$.

Construction 6.5. Given $w \in \mathfrak{S}_\alpha(231)$, we start with an empty stack $S$ and then perform the following steps on the $\alpha$-regions in reverse order, i.e., $k$ runs from $r$ down to 1.

- (Popping) For $i$ from 1 to $\alpha_k$, consider the $i$-th element $w(s_k + i)$ in region $k$. Pop elements from the stack until the top one is larger than $w(s_k + i)$.
- (Pushing) For $i$ from 1 to $\alpha_k$, push the element $w(s_k - i + 1)$ into the stack.

There are $n$ elements in $w$, and each element passes through two steps, totaling to $2n$ steps. See Figure 8 for an example.

Remark 6.6. Note that in terms of popping elements we only need the popping step for the last element in each region, as it is also the largest. However, taking the popping step for each element into account is important to understand the link between $\varphi$ and $\Theta$. Namely, given $w \in \mathfrak{S}_\alpha(231)$, the number of elements popped out in the popping step of $w(i)$ is the number of children of the node with label $w(i)$ in $T(w)$ (Proposition 6.9), which is in turn the length of the corresponding run of north steps in $\Theta(w)$.

Lemma 6.7. For $w \in \mathfrak{S}_\alpha(231)$, at each step of the stack processing of $w$ with the stack $S$, we have:

(i) the elements of $S$ are increasing from top to bottom;
(ii) for elements in $S$, their indices in $w$ are increasing from top to bottom.

Proof. For the first point, we proceed by induction on the number of steps. The claim is clearly satisfied at the beginning, when $S$ is empty. The popping step maintains the claim. Suppose that we are now pushing an element $w(i)$ into $S$. If
$w(i)$ is the last element in its region, then by induction hypothesis, all elements smaller than $w(i)$ should have been popped out; otherwise, the top of the stack is $w(i + 1) > w(i)$. In both cases, pushing $w(i)$ maintains our claim. We thus conclude the induction.

For the second point, we observe that the claim is valid for the empty stack, and the pushing steps maintain the claim, since all the elements in the same region are pushed consecutively starting with the last element in the region. The popping steps clearly also maintains the claim. □

**Proposition 6.8.** For $w \in \mathcal{S}_n(231)$, we consider the popping step of an element $w(i)$ in the stack processing of $w$ with stack $S$. If after that popping step $S$ is not empty with top element $w(j)$, then the companion of $w(i)$ is $w(j - 1)$; if $S$ is empty, then the companion of $w(i)$ is $w(n)$.

**Proof.** Assume that there is an index $\ell$ such that $s_{w(i)} < \ell < j$ and $w(\ell) > w(i)$. We take the smallest such index $\ell$. Then $w(\ell)$ cannot be popped out of $S$ before the treatment of the region $q(i)$, since an element $w(\ell')$ that pops $w(\ell)$ out must have $s_{w(\ell')} < \ell' < \ell$ and $w(\ell') > w(\ell) > w(i)$, violating the minimality of $\ell$. At this moment, $w(j)$ is also in $S$. By Lemma 6.7(ii), $w(j)$ is below $w(\ell)$. Since $w(j)$ is on top of $S$ after the popping step of $w(i)$, there is some $w(\ell')$ in the same region of $i$ with $\ell' < i$ that popped $w(\ell)$ out, meaning that $w(\ell') < w(\ell) < w(i)$, contradicting our hypothesis. Therefore, such $w(\ell)$ does not exist. From the definition of the popping step, we have $w(j) > w(i)$. Thus, the companion of $w(i)$ is $w(j - 1)$. In the case of empty stack, it means that no element in previously inserted regions is larger than $w(i)$, thus the companion of $w(i)$ is $w(n)$. □

### 6.3. Stack processing and the bijection $\Theta$

We now describe a link between stack processing of $w \in \mathcal{S}_n(231)$ and the tree $T(w)$ in Construction 6.2. Given $1 \leq k \leq r$, the nodes of region $k$ of $T(w)$ are those with labels corresponding to the values of $w$ in the $k$-th $\alpha$-region. We say that the root is of region 0, and that the region $r + 1$ is empty. From the insertion procedure, for a node of region $k$ with its parent of region $k'$, we have $k > k'$. The active nodes for region $k$ are nodes of region $k' \geq k$ whose parent is of region $k'' < k$.

**Proposition 6.9.** For $w \in \mathcal{S}_n(231)$, consider the stack processing of $w$ with stack $S$. For $1 \leq k \leq r + 1$, the elements in $S$ after processing the $k$-th $\alpha$-region are exactly the labels of the active nodes for region $k$ in $T(w)$.

Furthermore, for a node $u$ in $T(w)$, the labels of its children are exactly the elements popped out by the label of $u$ in the stack processing of $w$.

**Proof.** Let $E_k$ be the set of elements in $S$ after processing the $k$-th $\alpha$-region, and let $L_k$ be the set of labels of the active nodes of region $k$ in $T(w)$. We show that $E_k = L_k$. We proceed by induction on $k$ from $r + 1$ to 1. For $k = r + 1$, the set $E_{r+1}$ is empty, and there are no active nodes for region $r + 1$, so we have $E_{r+1} = L_{r+1}$. Suppose that $E_{k+1} = L_{k+1}$. We observe that, by Construction 6.5 and Lemma 6.7(i), we have

$$E_k = (E_{k+1} \setminus R_k) \cup \{ w(i) \mid q(i) = k \},$$

where $R_k = \{ w(i) \mid q(i) > k, w(i) < w(s_k) \}$.

Now, by the definition of active nodes, we split $L_k$ into two parts, $L_k^{(1)}$ for nodes of region $k$, and $L_k^{(2)}$ for other nodes. It is clear that $L_k^{(1)} = \{ w(i) \mid q(i) = k \}$. A
Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \). For \( w \in S_n(231) \), we have

\[
(v_a, \varphi(w)) = \text{Conj}'(\text{Flip}(v_a), \text{Flip}(\Theta(w)))
\]

**Proof.** By definition of \( \text{Conj}' \) and Proposition 4.9, we only need to show that

\[
\text{Drun}(v_a, \varphi(w)) = \text{Cont}(\text{Flip}(v_a), \text{Flip}(\Theta(w)))
\]

node with label in \( L_k^{(2)} \) must be in some region \( k' > k \), and its parent in some other region \( k'' < k < k + 1 \). Therefore, \( L_k^{(2)} \subseteq L_{k+1} \). Conversely, let \( u \) be a node with labels in \( L_{k+1} \), and let \( v \) be the parent of \( u \) in region \( k_v \). If \( k_v < k \), then \( u \) is also in \( L_k^{(2)} \); otherwise, if \( k_v = k \), then \( u \) is not in \( L_k^{(2)} \). We thus have

\[
L_k = (L_{k+1} \setminus R_k') \cup \{w(i) \mid \varphi(i) = k\}
\]

Here, \( R_k' \) is the set of labels of nodes whose parents are of region \( k \).

Now, by the construction of \( T(w) \) in Construction 6.2, labels in \( R_k' \) must be in some region \( k' > k \), and they are smaller than some element in region \( k \) of \( w \), therefore smaller than \( w(s_k) \). We thus have \( R_k' \subseteq R_k \). Conversely, suppose that \( R_k \setminus R_k' \) is not empty, and let \( w(i) \in R_k \setminus R_k' \) and \( u \) the node in \( T(w) \) with \( w(i) \) as label. We have \( \varphi(i) > k \) and \( w(i) < w(s_k) \). Let \( v \) be the parent of \( u \) in \( T(w) \), and \( w(j) \) the label of \( v \). As \( w(i) \in R_k' \), we know that \( v \) is in some region \( k_v < k \). Suppose that \( v' \) is the node with label \( w(s_k) \). As \( w(i) < w(s_k) \), by Proposition 6.3, \( u \) precedes \( v' \) in postorder. By the construction of \( T(w) \), we know that the region of a node is always strictly smaller than that of its children, and weakly smaller than that of its siblings to the right. Therefore, the node \( v \) of region \( k_v < k \) must not be a descendant of \( v' \) of region \( k \). If \( v' \) is a descendant of \( v \), as \( u \) precedes \( v' \) in postorder, meaning that \( v' \) must be a sibling of \( u \) to the right, or a descendant of such a sibling, which is impossible because \( u \) is in region \( \varphi(i) > k \). Therefore, \( v' \) and \( v \) are not comparable in \( T(w) \), and along with the fact that \( u \) precedes \( v' \) in postorder, \( v \) also precedes \( v' \) in postorder. Now take the rightmost child of \( v \), say \( u' \) with label \( w(i') \), which also precedes \( v' \) in postorder. We have \( w(j) = w(i') + 1 \) and \( w(s_k) > w(j) \) by Proposition 6.3. Furthermore, \( v \) is in region \( k_v < k \), while \( v' \) is in region \( k \) and \( u' \) is in a region \( k_v > \varphi(i) > k \). We thus have an \((a, 231)\)-pattern \( w(j), w(s_k), w(i') \) in \( w \), which is not possible. Therefore, \( w(i) \) cannot exist, and we have \( R_k' = R_k \).

Comparing (6) and (7), along with \( R_k' = R_k \) and the induction hypothesis \( E_{k+1} = L_{k+1} \), we have \( E_k = L_k \), concluding the induction. Therefore, the first part of our claim holds for all \( 1 \leq k \leq r + 1 \).

For the second part, let \( w(i) \) be the label of \( u \) and \( w(j) \) an element popped out by \( w(i) \) in the stack processing, and \( v \) the node with \( w(j) \) as label. Suppose that \( u \) is of region \( k_u \). By the first part of our claim, \( v \) is an active node for region \( k_u - 1 \) but not for region \( k_u \). Therefore, the parent of \( v \) is of region \( k_u \). By Proposition 6.3, the label of the parent of \( v \) must be the first element in region \( k_u \) larger than \( w(j) \), which is \( w(i) \) according to the popping step of region \( k_u \). Thus, \( u \) is the parent of \( v \), and we have the second part of our claim. \( \square \)

We now prove that the isomorphism \( \varphi \) from \( \mathcal{T}_a \) to \( \mathcal{T}_{v_a} \) is closely related to \( \Theta \) defined in [4]; see Section 6.1.

**Theorem 6.10.** Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \). For \( w \in S_n(231) \), we have

\[
(v_a, \varphi(w)) = \text{Conj}'(\text{Flip}(v_a), \text{Flip}(\Theta(w)))
\]

**Proof.** By definition of \( \text{Conj}' \) and Proposition 4.9, we only need to show that

\[
\text{Drun}(v_a, \varphi(w)) = \text{Cont}(\text{Flip}(v_a), \text{Flip}(\Theta(w)))
\]
as \( \text{Drun} \) is injective. For the pair \((v_a, \varphi(w))\), we observe that
\[
  v_a = [0, 0^{a_1-1}, a_1, 0^{a_2-1}, a_2, \ldots, 0^{a_r-1}, a_r],
\]
with \(0^k\) standing for \(k\) entries of 0. Suppose that \(\varphi(w) = [f_0, f_1, \ldots, f_n]\). We have
\[
  \text{Drun}(v_a, \varphi(w)) = (f_{s_1} + 1, 0^{s_1}, f_{s_1-1} + 1, f_{s_1-2} + 1, \ldots, f_{s_r-1} + 1, 1, 0^{s_r-1}, \ldots, f_{s_1-1} + 1, \ldots, f_0 + 1, 0^{a_1}, 0).
\]
(8)

Now for the pair \((v_a, \Theta(w))\), for \(0 \leq i \leq 2n\), let \(Q'_i\) the \((i+1)\)-st lattice point on \(v_a\) in reverse order. We define \(d'_i\) to be the number of north steps \((0,1)\) we can take from \(Q'_i\) without crossing to the other side of \(\Theta(w)\). It is clear that \(d'_i\) is also the reverse horizontal distance of the \((i+1)\)-st lattice point of \(\text{Flip}(v_a)\) with respect to \(\text{Flip}(\Theta(w))\). By Proposition 4.10,
\[
\text{Cont}(\text{Flip}(v_a), \text{Flip}(\Theta(w)))_0 = |\{\ell \mid 1 \leq \ell \leq 2n, d'_\ell = 0\}|;
\]
\[
\text{Cont}(\text{Flip}(v_a), \text{Flip}(\Theta(w)))_i = |\{\ell \mid i < \ell \leq 2n, d'_\ell = d'_i + 1, \forall i < m \leq \ell, d'_m > d'_i\}|.
\]

Consider the stack processing of \(w\) with stack \(S\). We now show that the number of elements in the stack after \(i\) steps of stack processing is \(d'_i\). We proceed by induction on the number of steps we have taken in the stack processing. In the initial stage, \(d'_0 = 0\) agrees with the empty stack. When dealing with region \(k\), we first perform the popping step. By the construction of \(\Theta(w)\), for the \(i\)-th element in region \(k\), the number of children of its correspondent node, which is also the number elements popped out by \(w(s_k-1 + i)\) by Proposition 6.9, is the number of north steps of \(\Theta(w)\) on abscissa \(s_k - i + 1\), which is exactly \(d'_{2n-s_k+i+1} - d'_{2n-2s_k+i}\). Then for the pushing step, the stack size increases by 1 at each step, just as when we pass from \(d'_{2n-s_k+i+1}\) to \(d'_{2n-s_k+i}\). We thus conclude the induction.

We now show that \(\text{Drun}(v_a, \varphi(w)) = \text{Cont}(\text{Flip}(v_a), \text{Flip}(\Theta(w)))\). First, we know that \(d'_i\) is weakly decreasing for \(i\) from \(2(n - s_k) + 1\) to \(2(n - s_k) + s_k\) for all \(1 \leq i \leq r\), and by definition, \(\text{Cont}(\text{Flip}(v_a), \text{Flip}(\Theta(w)))\) takes the form
\[
  (g_{s_1}, 0^{s_1}, g_{s_1-1}, g_{s_1-2}, \ldots, g_{s_r-1}, 0^{s_r-1}, \ldots, g_{s_1-1}, \ldots, g_0, 0^{a_1}, 0).
\]
Here, \((g_i)_{0 \leq i \leq n}\) is a sequence of positive integers. The last 0 is from the last point, because it does not have any lattice point after it. In comparison to \(\text{Drun}(v_a, \varphi(w))\), it is clear that we only need to prove \(g_\ell = f_\ell + 1\) for all \(\ell\).

For \(\ell = n\), according to Proposition 6.8, an element \(w(j)\) has \(w(n)\) as its companion if, and only if the stack is empty after its popping step, which is equivalent to \(d'_n = 0\). Therefore, \(g_n = f_n + 1\). For \(\ell = 0\), it is clear that \(g_0 = 1 = f_0 + 1\), since \(v_a\) starts with a north step.

For \(0 < \ell < n\), we know that \(f_\ell\) is the number of nodes with \(w(\ell)\) as companion, which is also the number of times we see \(w(\ell+1)\) at the top of the stack during a popping step according to Proposition 6.8. Suppose that \(w(\ell+1)\) is the \(i\)-th element in region \(k\), thus \(\ell + 1 = s_{k-1} + i\). We know that \(w(\ell+1)\) is pushed into the stack at step \(2(n - s_{k-1}) - i + 1\). Suppose that there are \(p = d'_{2(n-s_{k-1})-i}\) elements before \(w(\ell+1)\) is pushed down, we have \(d'_{2(n-s_{k-1})-i+1} = p + 1\). Then \(w(\ell+1)\) is popped out once \(d'_i \leq p\). When we see \(w(\ell+1)\) on top of the stack,
we must have $d'_j = p + 1$ before it is popped. This is exactly the definition of \( \text{Cont}(\text{Flip}(v_α), \Theta(w)) \) which is also \( g_{k-1} + i - 1 \). We thus know that \( g_{k-1} + i - 1 \) is the number of times we see \( w(\ell + 1) \) on the top of the stack, the first time it is pushed, the other times we have an element whose companion is \( w(\ell) \). We thus have \( g_\ell = g_{k-1} + i - 1 = f_\ell + 1 \). It follows that \( \text{Drun}(v_α, \varphi(w)) = \text{Cont}(\text{Flip}(v_α), \text{Flip}(\Theta(w))) \), which concludes the proof.

An example of the proof of Theorem 6.10 can be seen in Figure 9. We thus solve [4, Open Problem 2.23].

**Corollary 6.11.** The map \( \text{Flip} \circ \Theta \) is an anti-isomorphism between \( T_α \) and \( T_{\text{Flip}(v_α)} \).

**Proof.** This is a consequence of Theorem 6.10, and Propositions 4.8 and 6.1.

**Remark 6.12.** There is a typo in [4, Open Problem 2.23]. It should be “lattice anti-isomorphism” instead of “lattice isomorphism”, as we can also see in Figure 11 therein.

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