Nash implementation by stochastic mechanisms: 
a simple full characterization*

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Abstract

We study Nash implementation by stochastic mechanisms, and provide a surprisingly simple full characterization, which is in sharp contrast to the classical, albeit complicated, full characterization (of Nash implementation by deterministic mechanisms) in Moore and Repullo (1990) and Sjöström (1991). Our current understanding on the following four pairs of notions in Nash implementation is limited: “mixed-Nash-implementation VS pure-Nash-implementation,” “ordinal-approach VS cardinal-approach,” “almost-full-characterization (in Maskin (1999)) VS full-characterization (in Moore and Repullo (1990)),” and “Nash-implementation VS rationalizable-implementation.” Our results build a bridge connecting the two notions in each of the four pairs.

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1 Introduction

Mechanism design studies how to achieve a social goal in the presence of decentralized decision making, and one important subfield is Nash implementation.\(^1\) Maskin (1977, 1999)\(^2\) propose the famous notion of Maskin monotonicity, and prove that it is necessary for Nash implementation. Given an additional assumption of no-veto power, Maskin (1977, 1999) further prove that Maskin monotonicity is also sufficient.

The gap between necessity and sufficiency of Nash implementation is finally eliminated by Moore and Repullo (1990), Danilov (1991) and Dutta and Sen (1991), which provide necessary and sufficient conditions. Throughout the paper, we focus on Moore and Repullo (1990).\(^3\) Sjöström (1991) provides algorithms to check the conditions in Moore and Repullo (1990).

Compared to the simple and intuitive Maskin monotonicity, the full characterization in Moore and Repullo (1990) is complicated and hard to interpret. It is one of the most celebrated result, and yet, our understanding on it is still limited.

Most papers in the literature use a canonical mechanism as in Maskin (1999) to achieve Nash implementation. There are three cases in the canonical mechanism: (1) consensus, (2) unilateral deviation, (3) multi-lateral deviation. The difference between Maskin (1999) and Moore and Repullo (1990) is how to eliminate bad equilibria when case (2) (or case (3)) is triggered in the canonical mechanism. Maskin (1999) solves this problem by an exogenous condition of no-veto power, which is essentially equivalent to requiring all equilibria in case (2) to be good. Instead, Moore and Repullo (1990) consider all possible equilibria in case (2), and identify endogenous necessary conditions (of Nash

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\(^1\)There are two paradigms in mechanism design: full implementation (e.g., Nash implementation) and partial implementation (e.g., auction design). The former requires all solutions deliver the social goal, while the latter requires one solution only. By adopting different solution concepts, we may have different full implementation notions, e.g., Nash implementation (i.e., adopting Nash equilibria), and rationalizable implementation (i.e., adopting rationalizability).

\(^2\)Maskin (1977) is published as Maskin (1999).

\(^3\)Dutta and Sen (1991) focuses on 2-agent environments, and we assume three or more agents. Our results can be easily extended to 2-agent environments. Danilov (1991)’s full characterization hinges on a domain assumption, while Moore and Repullo (1990)’s does not, and that is why we focus on the latter only.
implementation), and then embed them into the canonical mechanism, which achieves Nash implementation.—Therefore, such conditions are both necessary and sufficient.

From a normative view, the result in Moore and Repullo (1990) is better than that in Maskin (1999), because the former implies the latter. However, the advantage of the Maskin approach is that the characterization is much simpler and more intuitive. Furthermore, no-veto power is usually considered as a weak condition. Thus, from a practical view, Maskin’s characterization is usually considered as an almost full characterization. Because of this, almost all of the papers in the literature on full implementation follow the Maskin approach, i.e., identify a Maskin-monotonicity-style necessary condition, and prove that it is sufficient, given no-veto power, e.g., Mezzetti and Renou (2012), Kartik and Tercieux (2012).

In this paper, we study Nash implementation by stochastic mechanisms, and provide a surprisingly simple full characterization. By taking full advantage of the convexity structure of lotteries, we show that the complicated Moore-Repullo-style full characterization collapses into a Maskin-monotonicity-style condition. That is, not only does our simple full characterization have a form similar to Maskin (1999), but also it has an interpretation parallel to Moore and Repullo (1990). In this sense, we build a bridge connecting Maskin (1999) and Moore and Repullo (1990).⁴

More importantly, our results show a conceptual advantage of the Moore-Repullo approach, which is not shared by the Maskin approach: the full characterization rigorously pin down the logical relation between different notions. For example, the notion of mixed-Nash implementation in Mezzetti and Renou (2012) does not require existence of pure-strategy equilibria, but the notion in Maskin (1999) does. Both Mezzetti and Renou (2012) and Chen, Kunimoto, Sun, and Xiong (2022) argue that this is a significant difference in their setups. However, how much impact does this difference induce? With only almost full characterization in both Maskin (1999) and Mezzetti and Renou (2012), we cannot find an answer for this question. Given the full characterization in our paper, we prove that this difference alone actually does not induce any impact (Theorem 4).

⁴The outcome space in Moore and Repullo (1990) is not convex, while it is convex in our environment, and our full characterization hinges crucially on this convexity assumption. As a result, the full characterization in Moore and Repullo (1990) does not imply ours, even though we share the same conceptual idea. See more discussion in Section 9.
In a second example, Bergemann, Morris, and Tercieux (2011) try to compare rationalizable implementation to mixed-Nash implementation. Bergemann, Morris, and Tercieux (2011) observe that the necessary condition of the former is stronger than the necessary condition of the latter, which, clearly, sheds limited light on their rigorous relation, generally. Only with no-veto power, we can conclude that the former is stronger than the latter—This may be misleading. First, as illustrated in Bergemann, Morris, and Tercieux (2011) and Xiong (2022b), no-veto power has no role in rationalizable implementation, and hence, there is not much justification to impose it, when we compare the two implementation notions. Second, no-veto power is not the reason that rationalizable implementation is stronger than mixed-Nash implementation, because with our full characterization, we are able to prove that the former always implies the latter with or without no-veto power (Theorem 12). The intuition is that mixed-Nash implementation is fully characterized by a condition on agents’ modified lower-contour sets (Theorem 7), which is also shared by rationalizable implementation. The difference between mixed-Nash equilibria and rationalizability is whether agents have correct beliefs in the corresponding solutions, and this difference has no impact on the identified condition on agents’ modified lower-contour sets.

Bochet (2007) and Benoit and Ok (2008) are the first two papers which propose to use stochastic mechanisms to achieve Nash implementation. Their results are orthogonal to ours, because of two differences. First, they impose weaker assumptions on agents’ preference on lotteries, and in this sense, their results are stronger. Second, they impose exogenous assumptions on agents’ preference on deterministic outcomes, which immediately makes no-veto power vacuously true on non-degenerate lotteries.\footnote{That is, they do not impose no-veto power on deterministic outcomes, but (implicitly) impose no-veto power on non-degenerate lotteries.} Allowing for non-degenerate lotteries only in case (2) and case (3) in the canonical mechanism, Bochet (2007) and Benoit and Ok (2008) establish an almost full characterization as Maskin (1999) does i.e., Bochet (2007) and Benoit and Ok (2008) take the Maskin approach. In contrast, we establish our full characterization without any assumption on agents’ preference on deterministic outcomes, and in this sense, our results are stronger. More importantly, we take the Moore-Repullo approach, i.e., no-veto power may fail for both degenerate and no-degenerate lotteries, and in spite of this, we find necessary and sufficient conditions. To the best of our knowledge, we are the first paper after Moore and Repullo (1990) and
Sjöström (1991), which takes the Moore-Repullo approach and establishes a simple full characterization for Nash implementation.

The remainder of the paper proceeds as follows: we describe the model in Section 2 and fully characterize Nash implementation for social choice functions in Section 3; we compare pure-Nash and mixed-Nash implementation in Section 4; we compare the ordinal and the cardinal approaches in Section 5; we focus on social choice correspondences in Section 6 and study ordinal and rationalizable implementation in Sections 7 and 8, respectively; we establish connection to Moore and Repullo (1990) and Sjöström (1991) in Section 9 and Section 10 concludes.

2 Model

2.1 Environment

We take a cardinal approach, and a (cardinal) model consists of

\[
\left\{ I = \{1, \ldots, I\}, \Theta, Z, f : \Theta \rightarrow Z, Y \equiv \Delta(Z), \left( u_i^\theta : Z \rightarrow \mathbb{R}\right)_{(i, \theta) \in I \times \Theta} \right\},
\]

where \( I \) is a finite set of \( I \) agents with \( I \geq 3 \), \( \Theta \) a finite or countably-infinite set of possible states, \( Z \) a finite set of pure social outcomes, \( f \) a social choice function (hereafter, SCF) which maps each state in \( \Theta \) to an outcome in \( Z \), \( Y \) the set of all possible lotteries on \( Z \), \( u_i \) the Bernoulli utility function of agent \( i \) at state \( \theta \). Throughout the paper, we assume that agents have expected utility, i.e.,

\[
U_i^\theta(y) = \sum_{z \in Z} y_z u_i^\theta(z), \quad \forall y \in Y,
\]

where \( y_z \) denotes the probability of \( z \) under \( y \), and \( U_i^\theta(y) \) is agent \( i \)'s expected utility of \( y \) at state \( \theta \). Without loss of generality, we impose the following assumption throughout the paper.

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\(^6\)Here is a difference between the Maskin approach and the Moore-Repullo approach: there is an exogenously given subset of outcomes that satisfies no-veto power in the former, but such an exogenous subset does not exist in the latter.

\(^7\)For simplicity, we focus on social choice functions first. We will introduce social choice correspondences in Section 4, and we extend our results in Section 6.
Assumption 1 (non-triviality) \( |f(\Theta)| \geq 2.8 \)

For each \( z \in Z \), we regard \( z \) as a degenerate lottery in \( Y \). With abuse of notation, we write \( z \in Z \subset Y \). Throughout the paper, we use \( -i \) to denote \( \mathcal{I} \setminus \{i\} \). For any \((\alpha, i, \theta) \in Y \times \mathcal{I} \times \Theta\), define
\[
\mathcal{L}_i^Y (\alpha, \theta) \equiv \left\{ y \in Y : U_i^\theta (\alpha) \geq U_i^\theta (y) \right\},
\]
\[
\mathcal{L}_i^Z (\alpha, \theta) \equiv \left\{ z \in Z : U_i^\theta (\alpha) \geq U_i^\theta (z) \right\}.
\]

For any finite or countably-infinite set \( E \), we use \( \triangle (E) \) to denote the set of probabilities on \( E \). For any \( \mu \in \triangle (E) \), we let \( \text{SUPP} [\mu] \) denote the support of \( \mu \), i.e.,
\[
\text{SUPP} [\mu] \equiv \{ x \in E : \mu (x) > 0 \}.
\]
Furthermore, define
\[
\triangle^o (E) \equiv \{ \mu \in \triangle (E) : \text{SUPP} [\mu] = E \}.
\]
For any finite set \( E \), we use \( \text{UNIF}(E) \) to denote the uniform distribution on \( E \), and use \(|E|\) to denote the number of elements in \( E \).

2.2 Mechanisms and Nash implementation

A mechanism is a tuple \( \mathcal{M} = \langle M \equiv \times_{i \in \mathcal{I}} M_i, g : M \rightarrow Y \rangle \), where each \( M_i \) is a countable set, and it denotes the set of strategies for agent \( i \) in \( \mathcal{M} \). A profile \((m_i)_{i \in \mathcal{I}} \in \times_{i \in \mathcal{I}} M_i \) is a pure-strategy Nash equilibrium in \( \mathcal{M} \) at state \( \theta \) if and only if
\[
U_i^\theta [g (m_i, m_{-i})] \geq U_i^\theta [g (m'_i, m_{-i})], \forall i \in \mathcal{I}, \forall m'_i \in M_i.
\]
Let \( \text{PNE}(\mathcal{M}, \theta) \) denote the set of pure-strategy Nash equilibria in \( \mathcal{M} \) at \( \theta \). Furthermore, a profile \( \lambda \equiv (\lambda_i)_{i \in \mathcal{I}} \in \times_{i \in \mathcal{I}} \triangle (M_i) \) is a mixed-strategy Nash equilibrium in \( \mathcal{M} \) at \( \theta \) if and only if
\[
\Sigma_{m \in M} \left[ \lambda_i (m_i) \times \Pi_{j \in \mathcal{I} \setminus \{i\}} \lambda_j (m_j) \times U_i^\theta [g (m_i, m_{-i})] \right] \geq \Sigma_{m \in M} \left[ \lambda'_i (m_i) \times \Pi_{j \in \mathcal{I} \setminus \{i\}} \lambda_j (m_j) \times U_i^\theta [g (m_i, m_{-i})] \right], \forall i \in \mathcal{I}, \forall \lambda'_i \in \triangle (M_i),
\]
\footnotetext{8}{If \( |f(\Theta)| = 1 \), e.g., \( f(\Theta) = \{z\} \) for some \( z \in Z \), the implementation problem can be solved trivially, i.e., we implement \( z \) at every state.}
where \( \lambda_j(m_j) \) is the probability that \( \lambda_j \) assigns to \( m_j \). Let \( \text{MNE}(\mathcal{M}, \theta) \) denote the set of mixed-strategy Nash equilibria in \( \mathcal{M} \) at state \( \theta \).

For any mechanism \( \mathcal{M} = \langle M, g : M \rightarrow Y \rangle \), and any \( \lambda \equiv (\lambda_i)_{i \in I} \in \times_{i \in I} \Delta(M_i) \), we use \( g(\lambda) \) to denote the lottery induced by \( \lambda \).

**Definition 1 (mixed-Nash-implementation)** An SCF \( f : \Theta \rightarrow Z \) is mixed-Nash-implemented by a mechanism \( \mathcal{M} = \langle M, g : M \rightarrow Y \rangle \) if

\[
\bigcup_{\lambda \in \text{MNE}(\mathcal{M}, \theta)} \text{SUPP}(g[\lambda]) = \{ f(\theta) \}, \forall \theta \in \Theta.
\]

\( f \) is mixed-Nash-implementable if there exists a mechanism that mixed-Nash-implements \( f \).

**Definition 2 (pure-Nash-implementation)** An SCF \( f : \Theta \rightarrow Z \) is pure-Nash-implemented by a mechanism \( \mathcal{M} = \langle M, g : M \rightarrow Y \rangle \) if

\[
\bigcup_{\lambda \in \text{PNE}(\mathcal{M}, \theta)} \text{SUPP}(g[\lambda]) = \{ f(\theta) \}, \forall \theta \in \Theta.
\]

\( f \) is pure-Nash-implementable if there exists a mechanism that pure-Nash-implements \( f \).

## 3 Mixed-Nash-implementation: a full characterization

In this section, we focus on mixed-Nash-implementation,\(^9\) and provide a surprisingly simple full characterization. As a benchmark, we first list the theorem of Maskin (1999) in Section 3.1. We present the full characterization in Section 3.2, which also contains the necessity part of the proof. The sufficiency part of the proof is more complicated, which is provided in Section 3.3.

\(^9\)We will show that mixed-Nash-implementation is equivalent to pure-Nash-implementation in Theorem 3.
3.1 Maskin’s theorem

By applying the ideas in Maskin (1999) to our environment with stochastic mechanisms, we adapt Maskin monotonicity as follows.\footnote{\textsuperscript{10}With stochastic mechanisms, there are two ways to define Maskin monotonicity (or related monotonicity conditions): (i) it is defined on the outcome space of $Y$ (e.g., Bergemann, Morris, and Tercieux (2011) and Chen, Kunimoto, Sun, and Xiong (2022)) and (ii) it is defined on the outcome space of $Z$ (e.g., Mezzetti and Renou (2012)). We follow the tradition of the former.}

**Definition 3 (Maskin monotonicity)** Maskin monotonicity holds if
\[
\left[ \mathcal{L}_i^Y (f (\theta), \theta) \subset \mathcal{L}_i^Y (f (\theta'), \theta'), \forall i \in \mathcal{I} \right] \implies f (\theta) = f (\theta'), \forall (\theta, \theta') \in \Theta \times \Theta.
\]

**Definition 4 (no-veto power)** No-veto power holds if for any $(a, \theta) \in Z \times \Theta$, we have
\[
\left| \left\{ i \in \mathcal{I} : a \in \arg\max_{z \in Z} u_i^\theta (z) \right\} \right| \geq |\mathcal{I}| - 1 \implies f (\theta) = a.
\]

**Theorem 1 (Maskin (1999))** Maskin monotonicity holds if $f$ is pure-Nash implementable. Furthermore, $f$ is pure-Nash implementable if Maskin monotonicity and no-veto power hold.

3.2 A simple full characterization

The following easy-to-check notion plays a critical role in our full characterization.

**Definition 5 (i-max set)** For any $(i, \theta) \in \mathcal{I} \times \Theta$, a set $E \in 2^Z \setminus \{\emptyset\}$ is an i-$\theta$-max set if
\[
E \subset \arg\max_{z \in E} u_i^\theta (z) \quad \text{and} \quad E \subset \arg\max_{z \in Z} u_j^\theta (z), \forall j \in \mathcal{I} \setminus \{i\}.
\]

Furthermore, $E$ is an i-max set if $E$ is an i-$\theta$-max set for some $\theta \in \Theta$.

This immediately leads to the following lemma, which sheds light on mechanisms that mixed-Nash-implement $f$. The proof is relegated to Appendix A.1.
Lemma 1  Suppose that an SCF $f$ is mixed-Nash implemented by $\mathcal{M} = (\mathcal{M}, g : \mathcal{M} \rightarrow Y)$. For any $(i, \theta) \in \mathcal{I} \times \Theta$ and any $\lambda \in \text{MNE}^{\mathcal{M}}(\mathcal{M}, \theta)$, we have

$$f(\theta) \in \arg \min_{z \in Z} u^\theta_i(z), \quad \text{and } \mathcal{L}^Z_i(f(\theta), \theta) \text{ is an i-max set} \quad \implies \bigcup_{m_i \in M_i} \text{SUPP}[g(m_i, \lambda_{-i})] = \{f(\theta)\}. \quad (2)$$

$\bigcup_{m_i \in M_i} \text{SUPP}[g(m_i, \lambda_{-i})]$ is the set of outcomes that can be induced with positive probability by $i$’s unilateral deviation from $\lambda$. Lemma 1 says that any unilateral deviation of $i$ from $\lambda$ must induce $f(\theta)$ if the condition on the left-hand side of “$\implies$” in (2) holds. In light of Lemma 1, we refine lower-counter sets as follows. For each $(i, \theta, a) \in \mathcal{I} \times \Theta \times Z$,

$$\hat{\mathcal{L}}^Y_i(a, \theta) \equiv \begin{cases} \{a\}, & \text{if } a = f(\theta) \in \arg \min_{z \in Z} u^\theta_i(z) \text{ and } \mathcal{L}^Z_i(f(\theta), \theta) \text{ is an i-max set,} \\ \mathcal{L}^Y_i(a, \theta), & \text{otherwise} \end{cases}. \quad (3)$$

Definition 6 ($\hat{\mathcal{L}}^Y$-monotonicity) $\hat{\mathcal{L}}^Y$-monotonicity holds if

$$\left[ \hat{\mathcal{L}}^Y_i(f(\theta), \theta) \subset \mathcal{L}^Y_i(f(\theta), \theta'), \quad \forall i \in \mathcal{I} \right] \implies f(\theta) = f(\theta'), \forall (\theta, \theta') \in \Theta \times \Theta. \quad (4)$$

This leads to a simple full characterization of mixed-Nash-implementation.

Theorem 2  An SCF $f : \Theta \rightarrow Z$ is mixed-Nash-implementable if and only if $\hat{\mathcal{L}}^Y$-monotonicity holds.

Proof of the “only if” part of Theorem 2: Suppose that $f$ is mixed-Nash-implemented by $\mathcal{M} = (\mathcal{M}, g : \mathcal{M} \rightarrow Y)$. Fix any $(\theta, \theta') \in \Theta \times \Theta$ such that

$$\left[ \hat{\mathcal{L}}^Y_i(f(\theta), \theta) \subset \mathcal{L}^Y_i(f(\theta), \theta'), \quad \forall i \in \mathcal{I} \right], \quad (4)$$

and we aim to show $f(\theta) = f(\theta')$, i.e., $\hat{\mathcal{L}}^Y$-monotonicity holds.
We prove \( f(\theta) = f(\theta') \) by contradiction. Suppose \( f(\theta) \neq f(\theta') \). Pick any \( \lambda \in MNE(\mathcal{M}, \theta) \), and we have \( \text{SUPP} [g(\lambda)] = \{f(\theta)\} \) because \( f \) is implemented by \( \mathcal{M} \). Since \( f(\theta) \neq f(\theta') \), we have \( \lambda \notin MNE(\mathcal{M}, \theta') \). As a result, there exists an agent \( j \) who has a profitable deviation \( m_j \in M_j \) from \( \lambda \) at \( \theta' \), i.e.,

\[
\exists j \in \mathcal{I}, \exists m_j \in M_j, g(m_j, \lambda_{-j}) \in \mathcal{L}_j^Y (f(\theta), \theta) \setminus \mathcal{L}_j^Y (f(\theta), \theta'),
\]

where \( g(m_j, \lambda_{-j}) \in \mathcal{L}_j^Y (f(\theta), \theta) \) and \( g(m_j, \lambda_{-j}) \notin \mathcal{L}_j^Y (f(\theta), \theta') \) follow from \( \lambda \in MNE(\mathcal{M}, \theta) \) and \( \lambda \notin MNE(\mathcal{M}, \theta') \), respectively.

In particular, \( g(m_j, \lambda_{-j}) \notin \mathcal{L}_j^Y (f(\theta), \theta') \) implies \( g(m_j, \lambda_{-j}) \neq f(\theta) \), and hence, Lemma 1 implies failure of the following condition:

\[
\left( f(\theta) \in \arg \min_{z \in Z} u_j^\theta (z),
\text{ and } \mathcal{L}_j^Z (f(\theta), \theta) \text{ is an } j \text{-max set} \right),
\]

which, together with (3), further implies

\[
\hat{\mathcal{L}}_j^Y (f(\theta), \theta) = \mathcal{L}_j^Y (f(\theta), \theta).
\]

(5) and (6) imply

\[
g(m_j, \lambda_{-j}) \in \hat{\mathcal{L}}_j^Y (f(\theta), \theta) \setminus \mathcal{L}_j^Y (f(\theta), \theta'),
\]

contradicting (4).■

3.3 Sufficiency of \( \hat{\mathcal{L}}^Y \)-monotonicity

3.3.1 Preliminary construction

In order to build our canonical mechanism to implement \( f \), we need to take three preliminary constructions. First, for each \((i, \theta) \in \mathcal{I} \times \Theta\), we define

\[
\hat{\Gamma}_i(\theta) \equiv \bigcup_{y \in \hat{\mathcal{L}}_i^Y (f(\theta), \theta)} \text{SUPP}[y] = \left\{ z \in Z : \exists y \in \hat{\mathcal{L}}_i^Y (f(\theta), \theta), \exists z \in \text{SUPP}[y] \right\},
\]

i.e., \( \hat{\Gamma}_i(\theta) \) is the set of outcomes that can be induced with positive probability by lotteries in \( \hat{\mathcal{L}}_i^Y (f(\theta), \theta) \). This leads to the following lemma, and the proof is relegated to Appendix A.2.
Lemma 2 Suppose that $\widehat{LY}^\gamma$-monotonicity holds. Then, $Z$ is not an i-max set for any $i \in I$ and

$$\widehat{\Gamma}_j(\theta)$$ is a $j\theta'$-max set $$\Rightarrow \widehat{\Gamma}_j(\theta) = \{ f(\theta') \}, \forall (j, \theta, \theta') \in I \times \Theta \times \Theta.$$

Second, for each $(\theta, j) \in \Theta \times I$, fix any function $\phi^\theta_j : \Theta \rightarrow Y$ such that

$$\phi^\theta_j(\theta') \in \left( \arg \max_{y \in \widehat{L}^\gamma_j(f(\theta), \theta)} \mathcal{U}^\theta_{\theta'}[y] \right), \forall \theta' \in \Theta,$$ (8)

Suppose that the true state is $\theta' \in \Theta$. In a canonical mechanism that implements $f$, when agent $j$ unilaterally deviates from “all agents reporting $\theta$,” we let $j$ choose any lottery in $\widehat{L}^\gamma_j(f(\theta), \theta)$, in order to ensure that truthful reporting is always a Nash equilibrium. Thus, $\phi^\theta_j(\theta')$ in (8) is an optimal lottery in $\widehat{L}^\gamma_j(f(\theta), \theta)$ for $j$ at $\theta' \in \Theta$.

Furthermore, by (7), we have

$$\phi^\theta_j(\theta') \in \left( \arg \max_{y \in \widehat{L}^\gamma_j(f(\theta), \theta)} \mathcal{U}^\theta_{\theta'}[y] \right) \cap \Delta(\widehat{\Gamma}_j(\theta)), \forall \theta' \in \Theta.$$ (9)

Finally, the following lemma completes our third construction, and the proof is relegated to Appendix A.3.

Lemma 3 For each $(\theta, j) \in \Theta \times I$, there exist $\varepsilon^\theta_j > 0$ and $y^\theta_j \in \widehat{L}^\gamma_j(f(\theta), \theta)$, such that

$$\left[ \varepsilon^\theta_j \times y + \left( 1 - \varepsilon^\theta_j \right) \times y^\theta_j \right] \in \widehat{L}^\gamma_j(f(\theta), \theta), \forall y \in \Delta(\widehat{\Gamma}_j(\theta)).$$ (10)

3.3.2 A canonical mechanism

Let $\mathbb{N}$ denote the set of positive integers. We use the mechanism $M^* = \langle \times_{i \in I} M_i, g : M \rightarrow Y \rangle$ defined below to implement $f$. In particular, we have

$$M_i = \left\{ (\theta_i, k^2_i, k^3_i, \gamma_i, b_i) \in \Theta \times \mathbb{N} \times \mathbb{N} \times (\mathbb{Z}[2^Z \setminus \varnothing]) \times Z : \gamma_i(E) \in E, \forall E \in [2^Z \setminus \varnothing] \right\}, \forall i \in I,$$

and $g[m = (m_i)_{i \in I}]$ is defined in three cases.
Case (1): consensus if there exists $\theta \in \Theta$ such that
\[
\left( \theta_i, k_i^2 \right) = (\theta, 1), \forall i \in I,
\]
then $g[m] = f(\theta)$;

Case (2), unilateral deviation: if there exists $(\theta, j) \in \Theta \times I$ such that
\[
\left( \theta_i, k_i^2 \right) = (\theta, 1) \text{ if and only if } i \in I \setminus \{j\},
\]
then
\[
g[m] = \left( 1 - \frac{1}{k_j^2} \right) \times \phi_j^\theta(\theta_j) + \frac{1}{k_j^2} \times \left( \varepsilon_j^\theta \times \left( 1 - \frac{1}{k_j^2} \right) \times \gamma_j \left( \hat{\Gamma}_j(\theta) \right) + \frac{1}{k_j^2} \times \text{UNIF} \left( \hat{\Gamma}_j(\theta) \right) \right),
\]
where $(\varepsilon_j^\theta, y_j^\theta)$ are chosen for each $(\theta, j) \in \Theta \times I$ according to Lemma 3. Note that $\gamma_j \left( \hat{\Gamma}_j(\theta) \right) \in \hat{\Gamma}_j(\theta)$ by (11) and UNIF$\left( \hat{\Gamma}_j(\theta) \right) \in \triangle \left( \hat{\Gamma}_j(\theta) \right)$;

Case (3), multi-lateral deviation: otherwise,
\[
g[m] = \left( 1 - \frac{1}{k_j^2} \right) \times b_j^* + \frac{1}{k_j^2} \times \text{UNIF}(Z),
\]
where $j^* = \max \left( \text{arg max}_{i \in I} k_i^2 \right)$, i.e., $j^*$ is the largest-numbered agent who submits the highest number on the second dimension of the message.

Each agent $i$ uses $k_i^2$ to show intention to be a whistle-blower, i.e., $i$ voluntarily challenges agents $-i$’s reports if and only if $k_i^2 > 1$. In Case (1), agents reach a consensus, i.e., all agents report the same state $\theta$, and choose not to challenge voluntarily (or equivalently, $k_i^2 = 1$ for every $i \in I$). In this case, $f(\theta)$ is assigned by $g$.

Case (2) is triggered if any agent $j$ unilaterally deviates from Case (1) (in the first two dimensions of $j$’s message): either $j$ challenges voluntarily (i.e., $k_j^2 > 1$), or $j$ challenges involuntarily (i.e., $k_j^2 = 1$ and $j$ reports a different state $\theta_j (\neq \theta)$ in the first dimension). In this case, $g$ assigns the compound lottery in (12), which is determined by the state $\theta$ being agreed upon by $-j$, and by $\left( \theta_j, k_j^2, k_j^3, \gamma_j \right)$ of $j$’s message. By (8) and (10), the compound
lottery in (12) is an element in \( \hat{\mathcal{L}}^Y (f(\theta), \theta) \), which ensures that truth-reporting is always a Nash equilibrium at each state \( \theta \in \Theta \). Note that \( k_j^2 \) and \( k_j^3 \) determine the probabilities in the compound lottery in (12). Furthermore, \( (\theta, \theta_j) \) determines \( j \)'s challenge scheme \( \phi_j^\theta (\theta_j) \) in (12)—revealing the true state is always \( j \)'s best challenge scheme due to (8). Finally, \( \theta \) determines the set \( \hat{\Gamma}_j(\theta) \), and \( j \) is entitled to pick an optimal outcome in \( \hat{\Gamma}_j(\theta) \) via \( \gamma_j \) (i.e., the fourth dimensions in \( j \)'s message): the picked outcome \( \gamma_j \left( \hat{\Gamma}_j(\theta) \right) \) occurs with probability \( \frac{1}{k_j^2} \times \varepsilon_j^\theta \times \left( 1 - \frac{1}{k_j^3} \right) \), and the outcome UNIF(\( \hat{\Gamma}_j(\theta) \)) occurs with probability \( \frac{1}{k_j^2} \times \varepsilon_j^\theta \times \frac{1}{k_j^3} \).—The higher \( k_j^3 \), the more probability is shifted from “UNIF(\( \hat{\Gamma}_j(\theta) \))” to “\( \gamma_j \left( \hat{\Gamma}_j(\theta) \right) \).”

Case (3) includes all other scenarios, and as usual, agents compete in an integer game. The agent \( j^* \) who reports the highest integer in the second dimension wins, and \( j^* \) is entitled to pick an optimal outcome \( b_{j^*} \) in \( Z \). In this case, \( g \) assigns the compound lottery in (13): \( b_{j^*} \) occurs with probability \( \left( 1 - \frac{1}{k_{j^*}^2} \right) \) and UNIF(\( Z \)) occurs with probability \( \frac{1}{k_{j^*}^2} \).—The higher \( k_{j^*}^2 \), the more probability is shift from “UNIF(\( Z \))” to “\( b_{j^*} \).”

The following lemma substantially simplifies the analysis of mixed-strategy Nash equilibria in \( M^* \), and the proof is relegated to Appendix A.4.

**Lemma 4** Consider the canonical mechanism \( M^* \) above. For any \( \theta \in \Theta \) and any \( \lambda \in \text{MNE}(M^*, \theta) \), we have \( \text{SUPP}[\lambda] \subset \text{PNE}(M^*, \theta) \).

Lemma 4 says that every pure-strategy profile on the support of a mixed-strategy Nash equilibrium in \( M^* \) at \( \theta \) must also be a pure-strategy Nash equilibrium at \( \theta \). As a result, it suffers no loss of generality to focus on pure-strategy Nash equilibria.

### 3.3.3 Proof of “if” part of Theorem 2

Suppose that \( \hat{\mathcal{L}}^Y \)-monotonicity holds. Fix any true state \( \theta^* \in \Theta \). We aim to prove

\[
\bigcup_{\lambda \in \text{MNE}(M^*, \theta^*)} \text{SUPP}(g[\lambda]) = \{ f(\theta^*) \}.
\]
First, truth revealing is a Nash equilibrium, i.e., any pure strategy profile
\[ m^* = (\theta_i = \theta^*, k_i^1 = 1, k_i^2, \gamma_i, b_i)_{i \in \mathcal{I}} \]
is a Nash equilibrium, which triggers Case (1) and \( g[m^*] = f(\theta^*) \). Any unilateral deviation \( m_j = (\bar{\theta}_j, k_j^2, \bar{k}_j^3, \bar{\gamma}_j, \bar{b}_j) \) of agent \( j \in \mathcal{I} \) would either still trigger Case (1) and induce \( f(\theta^*) \), or trigger Case (2) and induce
\[
\begin{align*}
 g[m_j, m_\neq j] &= \left( 1 - \frac{1}{k_j^2} \right) \times \phi_{\theta^*}^j (\bar{\theta}_j) \\
 &\quad + \frac{1}{k_j^2} \times \left( \epsilon_{\theta^*} \times \left[ \left( 1 - \frac{1}{k_j^2} \right) \gamma_j \left( \hat{f}_j(\theta^*) \right) + \frac{1}{k_j^2} \times \text{UNIF} \left( \hat{f}_j(\theta^*) \right) \right] + \left( 1 - \epsilon_{\theta^*} \right) \times \gamma_{\theta^*}^j \right),
\end{align*}
\]
and by (9) and (10), we have
\[
g[m_j, m_\neq j] \in \hat{\mathcal{L}}_j^Y (f(\theta^*), \theta^*) \subset \mathcal{L}_j^Y (f(\theta^*), \theta^*), \forall m_j \in M_j.
\]
Therefore, any \( m_j \in M_j \) is not a profitable deviation and \( m^* \) is a Nash equilibrium which induces \( g[m^*] = f(\theta^*) \).

Second, by Lemma 4, it suffers no loss of generality to focus on pure-strategy equilibria. Fix any
\[
\tilde{m} = (\tilde{\theta}_i, \tilde{k}_i^2, \tilde{k}_i^3, \tilde{\gamma}_i, \tilde{b}_i)_{i \in \mathcal{I}} \in \text{PNE}(M^*, \theta^*),
\]
and we aim to prove \( g[\tilde{m}] = f(\theta^*) \). We first prove that \( \tilde{m} \) does not trigger Case (3). Suppose otherwise, i.e.,
\[
g[\tilde{m}] = \left( 1 - \frac{1}{k_j^{*^2}} \right) \times \tilde{b}_j^* + \frac{1}{k_j^{*^2}} \times \text{UNIF}(Z),
\]
where \( j^* = \max \left( \arg \max_{i \in \mathcal{I}} k_i^2 \right) \). By Lemma 2, \( Z \) is not an i-max set for any \( i \in \mathcal{I} \), and thus,
\[
\exists j \in \mathcal{I}, \min_{z \in Z} u_j^{\theta^*}(z) < \max_{z \in Z} u_j^{\theta^*}(z),
\]
and as a result,
\[
U_j^{\theta^*}(\text{UNIF}(Z)) < \max_{z \in Z} u_j^{\theta^*}(z),
\]
which further implies that agent \( j \) finds it strictly profitable to deviate to
\[
m_j = (\tilde{\theta}_j, k_j^2, \tilde{k}_j^3, \tilde{\gamma}_j, b_j) \text{ with } b_j \in \arg \max_{z \in Z} u_j^{\theta^*}(z), \text{ for sufficiently large } k_j^2,
\]
contradicting $\tilde{m} \in \text{PNE}(\mathcal{M}^*, \theta^*)$.

Thus, $\tilde{m}$ must trigger either Case (1) or Case (2). Suppose $\tilde{m}$ triggers Case (1), i.e.,

\[ \tilde{m} = \left( \tilde{\theta}_i = \tilde{\theta}, \tilde{k}_i^2 = 1, \tilde{k}_i^3, \tilde{\gamma}_i, \tilde{b}_i \right)_{i \in I} \text{ for some } \tilde{\theta} \in \Theta, \]

and $g[\tilde{m}] = f(\tilde{\theta})$. We prove $g[\tilde{m}] = f(\theta^*)$ by contradiction. Suppose $f(\tilde{\theta}) \neq f(\theta^*)$. By $\hat{\mathcal{L}}^\gamma$-monotonicity, there exists $j \in \mathcal{I}$ such that

\[ \exists y^* \in L^\gamma_j \left[ f(\tilde{\theta}), \tilde{\theta} \right] \setminus L^\gamma_j \left[ f(\tilde{\theta}), \theta^* \right], \]

which, together with (9), implies

\[ U^\theta_{i,j} \left[ \phi^\tilde{\theta}_{j} (\theta^*) \right] \geq U^\theta_{i,j} \left[ y^* \right] > U^\theta_{i,j} \left[ f(\tilde{\theta}) \right]. \]

Therefore, it is strictly profitable for agent $j$ to deviate to

\[ m_j = \left( \theta^*, k_j^2, \tilde{k}_j^3, \tilde{\gamma}_j, \tilde{b}_j \right) \text{ for sufficiently large } k_j^2, \]

contradicting $\tilde{m} \in \text{PNE}(\mathcal{M}^*, \theta^*)$.

Finally, suppose $\tilde{m}$ triggers Case (2), i.e., there exists $j \in \mathcal{I}$ such that

\[ \exists \tilde{\theta} \in \Theta, \tilde{m}_i = \left( \tilde{\theta}_i = \tilde{\theta}, \tilde{k}_i^2 = 1, \tilde{k}_i^3, \tilde{\gamma}_i, \tilde{b}_i \right), \forall i \in \mathcal{I} \setminus \{j\}, \]

and

\[
\begin{align*}
g[\tilde{m}] &= \left( 1 - \frac{1}{k_j^2} \right) \times \phi^\tilde{\theta}_{j} (\tilde{\theta}_j) \\
&\quad + \frac{1}{k_j^2} \times \left( \epsilon_{i,j} \times \left[ \left( 1 - \frac{1}{k_j^3} \right) \times \gamma_j \left( \hat{\Gamma}_j \left( \tilde{\theta}_j \right) \right) \right] + \frac{1}{k_j^3} \times \text{UNIF} \left( \hat{\Gamma}_j \left( \tilde{\theta}_j \right) \right) \right). \tag{14}
\end{align*}
\]

We now prove $g[\tilde{m}] = f(\theta^*)$. By (9) and (10), we have

\[ g[\tilde{m}] \in \Delta \left[ \hat{\Gamma}_j \left( \tilde{\theta}_j \right) \right]. \tag{15} \]

Since every $i \in \mathcal{I} \setminus \{j\}$ can deviate to trigger Case (3), and dictate her top outcome in $Z$ with arbitrarily high probability, $\tilde{m} \in \text{PNE}(\mathcal{M}^*, \theta^*)$ implies

\[ \hat{\Gamma}_j \left( \tilde{\theta}_j \right) \subset \arg \max_{z \in Z} u^\theta_j (z), \forall i \in \mathcal{I} \setminus \{j\}. \tag{16} \]
Inside the compound lottery $g[\tilde{m}]$ in (14), conditional on an event with probability $\frac{1}{k^3_j} \times \epsilon^j_\theta$, we have the compound lottery

$$
\left[ \left( 1 - \frac{1}{k^3_j} \right) \times \tilde{\gamma}_j \left( \tilde{\Gamma}_j \left( \tilde{\theta} \right) \right) + \frac{1}{k^3_j} \times \text{UNIF} \left( \tilde{\Gamma}_j \left( \tilde{\theta} \right) \right) \right],
$$

and hence, agent $j$ can always deviate to

$$m_j = \left( \tilde{\theta}_j, k^2_j, k^3_j, \gamma_j, \tilde{b}_j \right)_{i \in \mathcal{I} \setminus \{j\}} \text{ with } \gamma_j \left( \tilde{\Gamma}_j \left( \tilde{\theta} \right) \right) \in \arg \max_{z \in \tilde{\Gamma}_j(\tilde{\theta})} u^{\theta^*_j} (z)
$$

for sufficiently large $k^3_j$. Thus, $\tilde{m} \in \text{PNE}(\mathcal{M}^*, \theta^*)$ implies

$$\tilde{\Gamma}_j \left( \tilde{\theta} \right) \subset \arg \max_{z \in \tilde{\Gamma}_j(\tilde{\theta})} u^{\theta^*_j} (z). \quad (17)$$

(16) and (17) imply that $\tilde{\Gamma}_j \left( \tilde{\theta} \right)$ is a $j$-$\theta^*$-max set, which together Lemma 2, further implies

$$\tilde{\Gamma}_j \left( \tilde{\theta} \right) = \{ f (\theta^*) \}. \quad (18)$$

(15) and (18) imply $g[\tilde{m}] = f (\theta^*)$. \blacksquare

4 Discussion: implementation-in-PNE Vs implementation-in-MNE

The current literature has limited understanding on the difference between implementation in pure Nash equilibria and in mixed Nash equilibria. Compared to Maskin (1999), Mezzetti and Renou (2012) argue that mixed-Nash implementation substantially expand the scope of implementation.

What drives such difference? \hfill (19)

Both Mezzetti and Renou (2012) and Chen, Kunimoto, Sun, and Xiong (2022) argue that a significant difference between Maskin (1999) and Mezzetti and Renou (2012) is whether we require existence of pure Nash equilibria in mixed-Nash implementation. That is, Maskin (1999) actually adopts the notion of double implementation defined as follows.
Definition 7 (double-Nash-implementation, Maskin (1999)) An SCF $f : \Theta \rightarrow Z$ is double-Nash-implementable if there exists a mechanism $\mathcal{M} = \langle M, g : M \rightarrow Y \rangle$ such that

$$\bigcup_{\lambda \in \text{MNE}(\mathcal{M}, \theta)} \text{SUPP}(g[\lambda]) = \bigcup_{\lambda \in \text{PNE}(\mathcal{M}, \theta)} \text{SUPP}(g[\lambda]) = \{ f(\theta) \}, \forall \theta \in \Theta.$$

The following theorem says that this difference does not answer the question in (19).

Theorem 3 Consider any SCF $f : \Theta \rightarrow Z$. The following statements are equivalent.

(i) $f$ is pure-Nash-implementable;

(ii) $f$ is mixed-Nash-implementable;

(iii) $f$ is double-Nash-implementable;

(iv) $\hat{L}^Y$-monotonicity holds.

The proof in Section 3.3.3 shows sufficiency of $\hat{L}^Y$-monotonicity for all of (i), (ii) and (iii) in Theorem 3, while the necessity of $\hat{L}^Y$-monotonicity is described by (a slightly modified version of) the proof for the “only if” part of Theorem 2 in Section 3.2.\(^{11}\)

If we focus on SCFs, Theorem 3 implies that the analysis in Mezzetti and Renou (2012) remains the same if we replace mixed-Nash implementation in their setup with pure-Nash-implementation or double-Nash-implementation. However, Mezzetti and Renou (2012) considers social choice correspondences (hereafter, SCC) besides SCFs, and hence, Theorem 3 does not provide a full answer for the question in (19). An SCC is a set-valued function $F : \Theta \rightarrow 2^Z \setminus \{ \emptyset \}$, and we thus extend our definitions to SCCs.\(^{12}\)

Definition 8 (mixed-Nash-A-implementation, Mezzetti and Renou (2012)) An SCC $F$ is mixed-Nash-A-implementable if there exists a mechanism $\mathcal{M} = \langle M, g : M \rightarrow Y \rangle$ such that

$$\bigcup_{\lambda \in \text{MNE}(\mathcal{M}, \theta)} \text{SUPP}(g[\lambda]) = F(\theta), \forall \theta \in \Theta.$$

\(^{11}\)We omit the proof Theorem 3, because it is implied by Theorem 4, which is proved in Appendix A.11.

\(^{12}\)We will consider six different definitions of mixed-Nash-implementation, and we call them versions A, B, C, D, E and F.
Definition 9 A SCC $F$ is pure-Nash-implementable if there exists a mechanism $\mathcal{M} = \langle M, g : M \rightarrow Y \rangle$ such that

$$\bigcup_{\lambda \in \text{PNE}^{(M, \theta)}} \text{SUPP}(g[\lambda]) = F(\theta), \forall \theta \in \Theta.$$ 

Definition 10 (mixed-Nash-B-implementation) An SCC $F$ is mixed-Nash-B-implementable if there exists a mechanism $\mathcal{M} = \langle M, g : M \rightarrow Y \rangle$ such that

$$\bigcup_{\lambda \in \text{MNE}^{(M, \theta)}} \text{SUPP}(g[\lambda]) = \bigcup_{\lambda \in \text{PNE}^{(M, \theta)}} \text{SUPP}(g[\lambda]) = F(\theta), \forall \theta \in \Theta.$$ 

Theorem 4 Consider any SCC $F : \Theta \rightarrow 2^Z \setminus \{\emptyset\}$. The following statements are equivalent.

(i) $F$ is pure-Nash-implementable;

(ii) $F$ is mixed-Nash-A-implementable;

(iii) $F$ is mixed-Nash-B-implementable.

Theorem 7 will provide a full characterization of mixed-Nash-A-implementation, which will be used to prove Theorem 4. We will prove Theorem 4 in Appendix A.11, after we prove Theorem 7 in Appendix A.10.

Compared to Maskin (1999), Mezzetti and Renou (2012) introduce three new ingredients to the model: (I) a new class of mechanisms (i.e., stochastic mechanisms), (II) new solutions (i.e., mixed-strategy Nash equilibria, or pure-strategy Nash equilibria, or both) and (III) how to interpret “implementing $F(\theta)$” (see more discussion in Section 6.1). Given SCFs, (III) is the same in both Maskin (1999) and Mezzetti and Renou (2012), and Theorem 3 shows that (II) is also the same in the two papers, which immediately leads to an answer for the question in (19): the difference is solely driven by the new class of mechanisms, i.e., (I). Given SCCs, Theorem 4 shows that (II) is still the same in the two papers, and hence, the difference must be driven by (I) and (III).14

13Mezzetti and Renou (2012) observe that pure-Nash-implementation and mixed-Nash-A-implementation share the same necessary condition (i.e., set-monotonicity), but do not provide their relationship. Only with our full characterization, we are able to prove their equivalence.

14Given SCCs, (III) is not the same in the two papers. Maskin’s notion coresponds to mixed-Nash-D-implementation in Definition 13.
5 Discussion: cardinal approach Vs ordinal approach

We take a cardinal approach in this paper, i.e., agents have cardinal utility functions. However, an ordinal approach is usually adopted in the literature of implementation (e.g., Mezzetti and Renou (2012)). We show that our cardinal approach is more general than the ordinal approach.

Throughout this section, we fix an ordinal model in Mezzetti and Renou (2012), which consists of

\[ \mathcal{I} = \{1, \ldots, I\}, \Theta^*, Z, f : \Theta^* \rightarrow Z, Y \equiv \Delta (Z), \left( \succeq^\theta \right)_{(i, \theta) \in \mathcal{I} \times \Theta^*}, \]  

where each ordinal state \( \theta \in \Theta^* \) determines a profile of preferences \( \left( \succeq^\theta \right)_i \) on \( Z \). This ordinal model differs from our cardinal model on two aspects only. First, agents have ordinal preference (i.e., \( \left( \succeq^\theta \right)_i \)) compared to the cardinal utility functions (i.e., \( u^\theta \) ) in Section 2.1. Second, the ordinal state set, denoted by \( \Theta^* \), is finite, while the cardinal state set, denoted by \( \Theta \) in Section 2.1, is either finite or countably-infinite.

For each ordinal state \( \theta \in \Theta^* \), we say \( u^\theta \equiv \left( u^\theta_i : Z \rightarrow \mathbb{R} \right)_{i \in \mathcal{I}} \) is a cardinal representation of \( \left( \succeq^\theta \right)_i \) if and only if

\[ z \succeq^\theta z' \iff u^\theta_i (z) \geq u^\theta_i (z'), \forall (z, z', i) \in Z \times Z \times \mathcal{I}. \]

Each \( \left( u^\theta_i : Z \rightarrow \mathbb{R} \right)_{i \in \mathcal{I}} \) is called a cardinal state. That is, each ordinal state \( \theta \in \Theta^* \) can be represented by a set of cardinal states defined as follows.

\[ \Omega^{\left[ \succeq^\theta, \mathbb{R} \right]} \equiv \left\{ \left( u^\theta_i : Z \rightarrow \mathbb{R} \right)_{i \in \mathcal{I}} : \begin{array}{c} z \succeq^\theta z' \iff u^\theta_i (z) \geq u^\theta_i (z'), \forall (z, z', i) \in Z \times Z \times \mathcal{I}. \end{array} \right\} \subset \left( \left( \mathbb{R} \right)^Z \right)^\mathcal{I}. \]

Mixed-Nash ordinal-implementation in Mezzetti and Renou (2012) requires that \( f \) be implemented by a mechanism under any cardinal representation.

**Definition 11 (mixed-Nash-ordinal-implementation, Mezzetti and Renou (2012))** \( f \) is mixed-Nash-ordinally-implementable if there exists a mechanism \( \mathcal{M} = (M, g : M \rightarrow Y) \) such that

\[ \bigcup_{\lambda \in \text{MNE}^{\left[ M, u^\theta \right]}} \text{SUPP} (g [\lambda]) = \{ f (\theta) \}, \forall \theta \in \Theta^*, \forall u^\theta \in \Omega^{\left[ \succeq^\theta, \mathbb{R} \right]} . \]
Since $\Omega^{[x, R]}$ is uncountably infinite, our results do not apply directly. However, we may consider cardinal utility functions with rational values only.

$$\Omega^{[x, Q]} \equiv \left\{ u_i^\theta : Z \rightarrow Q \right\}_{i \in I} : \begin{array}{l} z \geq_i^x z' \iff u_i^\theta(z) \geq u_i^\theta(z'), \\
\forall (z, z', i) \in Z \times Z \times I. \end{array}$$

Clearly, $\Omega^{[x, Q]} \subset \Omega^{[x, R]}$, and $\Omega^{[x, Q]}$ is countably infinite.

**Theorem 5** $f$ is mixed-Nash-ordinally-implementable (or equivalently, $f$ is mixed-Nash-A-implementable with $\Theta = \bigcup_{\theta \in \Theta} \Omega^{[x, R]}$) if and only if $f$ is mixed-Nash-A-implementable with $\Theta = \bigcup_{\theta \in \Theta} \Omega^{[x, Q]}$.

The "only if" part of Theorem 5 is implied by $\Omega^{[x, Q]} \subset \Omega^{[x, R]}$, and the "if" part is immediately implied by the following lemma.

**Lemma 5** For any $\theta \in \Theta^*$ and any $\overline{u}^\theta \in \Omega^{[x, R]}$, there exists $(\widehat{u}^\theta, \overline{u}^\theta) \in \Omega^{[x, Q]} \times \Omega^{[x, Q]}$, such that

$$\mathcal{L}_i^Y(z, \widehat{u}^\theta) \subset \mathcal{L}_i^Y(z, \overline{u}^\theta) \subset \mathcal{L}_i^Y(z, \overline{u}^\theta), \forall (i, z) \in I \times Y. \tag{21}$$

The proof of Lemma 5 is relegated to Appendix A.5.

**Proof of the "if" part of Theorem 5:** We use the canonical mechanism $\mathcal{M}^* = \langle \times_{i \in I} M_i, g : M \rightarrow Z \rangle$ (with $\Theta = \bigcup_{\theta \in \Theta} \Omega^{[x, R]}$) in Section 3.3.2 to implement $f$, i.e.,

$$\bigcup_{\lambda \in \text{MNE}(\mathcal{M}^*, \overline{u}^\theta)} \text{SUPP} (g[\lambda]) = \{ f(\theta) \}, \forall \theta \in \Theta^*, \forall \overline{u}^\theta \in \Omega^{[x, Q]}. \tag{22}$$

Fix any $\theta \in \Theta^*$ and pick any $\overline{u}^\theta \in \Omega^{[x, R]}$. By Lemma 5, there exists $(\widehat{u}^\theta, \overline{u}^\theta) \in \Omega^{[x, Q]} \times \Omega^{[x, Q]}$, such that (21) holds. In particular, $\times_{i \in I} \mathcal{L}_i^Y(f(\theta), \widehat{u}^\theta) \subset \times_{i \in I} \mathcal{L}_i^Y(f(\theta), \overline{u}^\theta)$ immediately implies

$$\text{MNE}(\mathcal{M}, \underline{u}^\theta) \subset \text{MNE}(\mathcal{M}, \overline{u}^\theta). \tag{23}$$

Lemma 2 implies that $Z$ is not an $i$-max set for any $i \in I$. As a result, no equilibrium exists when Case (3) occurs. In fact, as the proof in Section 3.3.3 shows that, at state $\overline{u}^\theta \in \Omega^{[x, R]}$, an equilibrium exists in $\mathcal{M}^*$ only when either Case (1) occurs, or Case (2)
in which agents \(-i\) report \(\theta'\) with 
\[ \hat{L}^Y_i \left( f \left( \theta' \right), \theta' \right) = \{ f \left( \theta' \right) \} \] 
occurs. That is, in both cases, we have 
\[ g \left[ \lambda \right] \in Z, \forall \lambda \in MNE^{(M^*, \overline{u^\theta})}, \]
which, together with 
\[ \times_{i \in I} L^Y_i \left( z, \overline{u^\theta} \right) \subset \times_{i \in I} L^Y_i \left( z, \tilde{u}^\theta \right) \] 
for every \(z \in Z\), implies 
\[ MNE^{(M, \overline{u^\theta})} \subset MNE^{(M, \tilde{u}^\theta)}. \quad (24) \]

(22), (23) and (24) imply 
\[ \bigcup_{\lambda \in MNE^{(M, \tilde{u}^\theta)}} \text{SUPP}(g[\lambda]) = \{ f(\theta) \}. \]

Theorem 5 extends to SCCs (see e.g., Theorem 11).

6 Extension to social choice correspondences

6.1 Four additional definitions

Given any solution concept, what does it mean that an SCC \(F: \Theta \rightarrow 2^Z \setminus \{\emptyset\}\) is implemented in the solution? There are two views in the literature. The first view is that, at each state \(\theta \in \Theta\), each solution must induce a deterministic outcome and \(F(\theta)\) is the set of all such deterministic outcomes.—This view is adopted in Kunimoto and Serrano (2019). The second view is that, at each state \(\theta \in \Theta\), \(F(\theta)\) is the set of outcomes that can be induced with positive probability by some solution—This view is adopted in Mezzetti and Renou (2012) and Jain (2021). Furthermore, we may or may not require existence of pure Nash equilibria, when we define mixed-Nash-implementation. Definitions 8 and 10 follow the second view, while the former does not require existence of pure Nash equilibria, and the latter does. Besides these two definitions, we can define four alternative versions of mixed-Nash-implementation, with different combination of requirements. For any mechanism \(M = \langle M, g : M \rightarrow Y \rangle\), define 
\[ \Phi^M \equiv \{ (\lambda_i)_{i \in I} \in \times_{i \in I} \Delta(M_i) : |\text{SUPP} \left( g \left[ (\lambda_i)_{i \in I} \right] \right) | = 1 \}, \]
i.e., \(\Phi^M\) is the set of mixed strategy profiles that induces a unique deterministic outcome.
Definition 12 (mixed-Nash-C-implementation) An SCC \( F : \Theta \rightarrow 2^Z \setminus \{\emptyset\} \) is mixed-Nash-C-implementable if there exists a mechanism \( M = \langle M, g : M \rightarrow Y \rangle \) such that

\[
\bigcup_{\lambda \in \text{MNE}^{(M, \theta)}} \text{SUPP}(g[\lambda]) = g \left( \text{MNE}^{(M, \theta)} \cap \Phi^M \right) = F(\theta), \forall \theta \in \Theta.
\]

Definition 13 (mixed-Nash-D-implementation, Maskin (1999)) An SCC \( F : \Theta \rightarrow 2^Z \setminus \{\emptyset\} \) is mixed-Nash-D-implementable if there exists a mechanism \( M = \langle M, g : M \rightarrow Y \rangle \) such that

\[
\bigcup_{\lambda \in \text{MNE}^{(M, \theta)}} \text{SUPP}(g[\lambda]) = g \left( \text{PNE}^{M, \theta} \right) = F(\theta), \forall \theta \in \Theta.
\]

Definition 14 (mixed-Nash-E-implementation) An SCC \( F : \Theta \rightarrow 2^Z \setminus \{\emptyset\} \) is mixed-Nash-E-implementable if there exists a mechanism \( M = \langle M, g : M \rightarrow Y \rangle \) such that

\[
\bigcup_{\lambda \in \text{MNE}^{(M, \theta)}} \text{SUPP}(g[\lambda]) = F(\theta), \forall \theta \in \Theta,
\]

and \( \text{MNE}^{M, \theta} \subset \Phi^M \).

Definition 15 (mixed-Nash-F-implementation) An SCC \( F : \Theta \rightarrow 2^Z \setminus \{\emptyset\} \) is mixed-Nash-F-implementable if there exists a mechanism \( M = \langle M, g : M \rightarrow Y \rangle \) such that

\[
\bigcup_{\lambda \in \text{MNE}^{(M, \theta)}} \text{SUPP}(g[\lambda]) = \bigcup_{\lambda \in \text{PNE}^{(M, \theta)}} \text{SUPP}(g[\lambda]) = F(\theta), \forall \theta \in \Theta,
\]

and \( \text{MNE}^{(M, \theta)} \subset \Phi^M \).

6.2 Version E and version F: full characterization

It is straightforward to extend Theorem 2 to mixed-Nash-E-implementation and Mixed-Nash-F-implementation. Define

\[
\hat{L}_i^Y(a, \theta) \equiv \begin{cases} 
\{a\}, & \text{if } a \in F(\theta) \cap \arg\min_{z \in Z} u_i^\theta(z) \text{ and } \mathcal{L}_i^Z(a, \theta) \text{ is an } i\text{-max set}, \\
\mathcal{L}_i^Y(a, \theta), & \text{otherwise.} 
\end{cases}
\]
Definition 16 (**L**-monotonicity) **L**-monotonicity holds for an SCC \( F \) if

\[
\begin{align*}
\{a \in F(\theta), \quad \bar{L}^Y_i(a, \theta) \subset L^Y_i(a, \theta'), \, \forall i \in I\} \implies a \in F(\theta'), \, \forall (\theta, \theta', a) \in \Theta \times \Theta \times Z.
\end{align*}
\]

In the degenerate case that \( F \) is a social choice function, \( \bar{L}^Y_i(a, \theta) \) in (25) becomes \( \bar{L}^Y_i(a, \theta) \) in (3), and **L**-monotonicity in Definition 16 becomes **L**-monotonicity in Definition 6. Hence, we use the same notation.

**Theorem 6** Given an SCC \( F : \Theta \to 2^Z \setminus \{\emptyset\} \), the following three statements are equivalent.

(i) \( F \) is mixed-Nash-E-implementable;

(ii) \( F \) is mixed-Nash-F-implementable;

(iii) \( Z \) is not an \( i \)-max set for any \( i \in I \) and **L**-monotonicity holds for \( F \).

It is worth noting that we need the requirement of “\( Z \) is not an \( i \)-max set for any \( i \in I \)” in (iii) of Theorem 6. We do not need this requirement in Theorem 2, because it is implied by **L**-monotonicity when \( F \) is a degenerate SCF (see Lemma 2).

The proof of Theorem 6 is almost the same as that of Theorem 2, and the detailed proof is relegated to Xiong (2022a).

### 6.3 Version A and version B: full characterization

Define

\[
Z^* \equiv \begin{cases} 
\cup_{\theta \in \Theta} F\left(\theta\right), & \text{if } Z \text{ is an } i \text{-max set for some } i \in I, \\
Z, & \text{if } Z \text{ is not an } i \text{-max set for any } i \in I.
\end{cases}
\]

**Lemma 6** Suppose that an SCC \( F \) is mixed-Nash-A-implemented by \( \mathcal{M} = \langle M, g : M \to Y \rangle \). Then, we have \( g(M) \subset \triangle(Z^*) \).
Lemma 6 says that only lotteries in $\triangle (Z^*)$ can be used by a mechanism which mixed-Nash-A-implements an SCC, and the proof is relegated to Appendix A.6. The implication of Lemma 6 is that, in order to achieve mixed-Nash-A-implementation, we should delete $Z \setminus Z^*$ from our model.

In order to accommodate the new implementation notion, we need to further adapt the notion of $i$-max set as follow.

**Definition 17 ($i$-$Z^*$-$\theta$-max set and $i$-$Z^*$-max set)** For any $(i, \theta) \in \mathcal{I} \times \Theta$, a set $E \in 2^{Z^*} \setminus \{\emptyset\}$ is an $i$-$Z^*$-$\theta$-max set if

$$E \subset \arg \max_{z \in E} u^i_\theta (z) \text{ and } E \subset \arg \max_{z \in Z^*} u^i_\theta (z), \forall j \in \mathcal{I} \setminus \{i\}.$$ 

Furthermore, $E \in 2^{Z^*} \setminus \{\emptyset\}$ is an $i$-$Z^*$-max set if

$$\Lambda^{i-Z^*-\Theta} (E) \equiv \{\theta \in \Theta : E \text{ is an } i$-$Z^*$-$\theta$-max set} \neq \emptyset. \quad (27)$$

For each $E \in 2^{Z} \setminus \{\emptyset\}$, define

$$\mathcal{L}_E^Z (E, \theta) \equiv \bigcap_{z \in E} \mathcal{L}_{i}^Z (z, \theta).$$

For each $(i, \theta) \in \mathcal{I} \times \Theta$, define

$$\Theta_i^\theta \equiv \{\theta' \in \Theta : F (\theta) \text{ is an } i$-$Z^*$-$\theta'$-max set and } F (\theta) \subset F (\theta')\}, \quad (28)$$

$$\Xi_i (\theta) \equiv \left\{ K \in 2^{\Theta_i^\theta} \setminus \{\emptyset\} : \Theta_i^\theta \cap \bigg[ \Lambda^{i-Z^*-\Theta} \left( Z^* \cap \mathcal{L}_E^Z (F (\theta), \theta) \cap \left[ \bigcap_{\theta' \in K} F (\theta') \right] \right) = K \bigg] \right\}, \quad (29)$$

where $\Lambda^{i-Z^*-\Theta} (\cdot)$ is defined in (27).

It is worthy of noting that we may replace the definition of $\Theta_i^\theta$ in (28) with $\Theta_i^\theta \equiv \Theta$, and use it define $\Xi_i (\theta)$ and $\hat{\mathcal{L}}_{i}^{Y-A-B} (\text{UNIF} [F (\theta)], \theta)$ in (29) and (30), respectively. With this modification, our full characterization (i.e., Theorem 7) still holds. However, since $\Theta_i^\theta$ is a much smaller set than $\Theta$, our definition of $\Theta_i^\theta$ in (28) is much more computationally efficient, i.e., we need to check much fewer sets in (29) and (30).
The full characterization is established by a monotonicity condition which is defined on modified lower-contour sets. For each \( \theta \in \Theta \), define

\[
\hat{L}_i^{Y-A-B} \left( \text{UNIF} \left[ F(\theta) \right], \theta \right) \equiv \begin{cases} \\
\bigtriangleup \left[ Z^* \cap \mathcal{L}_i^Z \left( F(\theta), \theta \right) \cap \left( \bigcup_{K \in \Xi(\theta)} \bigcap_{\theta' \in K} F(\theta') \right) \right], & \text{if } \begin{cases} F(\theta) \subset \arg \min_{z \in Z^*} u_i^\theta(z), \\
\Xi_i(\theta) \neq \emptyset, \\
\text{and } Z^* \cap \mathcal{L}_i^Z \left( F(\theta), \theta \right) \text{ is an } i-Z^*-\text{max set} \end{cases} \\
\left[ \bigtriangleup (Z^*) \right] \cap \mathcal{L}_i^Y \left( \text{UNIF} \left[ F(\theta) \right], \theta \right), & \text{otherwise} \end{cases}
\]

Definition 18 (\( \hat{L}_i^{Y-A-B} \)-uniform-monotonicity) \( \hat{L}_i^{Y-A-B} \)-uniform-monotonicity holds for an SCC \( F \) if

\[
\left[ \hat{L}_i^{Y-A-B} \left( \text{UNIF} \left[ F(\theta) \right], \theta \right) \subset \mathcal{L}_i^Y \left( \text{UNIF} \left[ F(\theta) \right], \theta' \right), \forall i \in I \right] \implies F(\theta) \subset F(\theta'), \forall (\theta, \theta') \in \Theta \times \Theta.
\]

Theorem 7 Given an SCC \( F : \Theta \rightarrow 2^Z \setminus \{\emptyset\} \), the following three statements are equivalent.

(i) \( F \) is mixed-Nash-A-implementable;

(ii) \( F \) is mixed-Nash-B-implementable;

(iii) \( \hat{L}_i^{Y-A-B} \)-uniform-monotonicity holds for \( F \).

The necessity part of \( \hat{L}_i^{Y-A-B} \)-uniform-monotonicity in Theorem 7 is implied by the following lemma.

Lemma 7 Suppose that an SCC \( F \) is mixed-Nash-A-implemented by \( \mathcal{M} = (M, g : M \rightarrow Y) \). For any \((i, \theta) \in I \times \Theta\) and any \( \lambda \in \text{MNE}^{(\mathcal{M}, \theta)} \), we have

\[
\begin{cases} \\
F(\theta) \subset \arg \min_{z \in Z^*} u_i^\theta(z), \\
\Xi_i(\theta) \neq \emptyset \text{ and } \\
\text{and } Z^* \cap \mathcal{L}_i^Z \left( F(\theta), \theta \right) \text{ is an } i-Z^*-\text{max set} \\
\end{cases}
\implies \bigcup_{m_i \in M_i} \text{SUPP} \left[ g \left( m_i, \lambda_{-i} \right) \right] \subset \left[ Z^* \cap \mathcal{L}_i^Z \left( F(\theta), \theta \right) \cap \left( \bigcup_{E \in \Xi(\theta)} \bigcap_{\theta' \in E} F(\theta') \right) \right].
\]
Like Lemma 1 for SCFs, Lemma 7 is the counterpart for SCCs, and the proof of Lemma 7 is relegated to Appendix A.7. The sufficiency part of $\hat{L}_A$-uniform-monotonicity in Theorem 7 is implied by the following lemma.

**Lemma 8** Suppose that $\hat{L}_A$-uniform-monotonicity holds. We have

$$[Z^* \text{ is a } j-Z^*-\theta'-\text{max set}] \implies Z^* \subset F(\theta'), \forall (j, \theta') \in I \times \Theta,$$

and

$$[\hat{\Gamma}_j A-B (\theta)] \text{ is a } j-Z^*-\theta'-\text{max set}] \implies \hat{\Gamma}_j A-B (\theta) \subset F(\theta'), \forall (j, \theta, \theta') \in I \times \Theta \times \Theta,$$

where $\hat{\Gamma}_j A-B (\theta) \equiv \bigcup_{y \in \hat{L}_j A-B (\text{UNIF}(F(\theta)), \theta)} \text{SUPP } [y]$. \hfill (31)

Like Lemma 2 for SCFs, Lemma 8 is the counterpart for SCCs, and the proof of Lemma is relegated to Appendix A.9.

The detailed proof of Theorem 7 is relegated to Appendix A.10.

### 6.4 Version C and version D: full characterization

For each $(i, \theta) \in I \times \Theta$, define

$$\Theta_i^{\theta, C-D} \equiv \left\{ \theta' \in \Theta : F(\theta) \cap \arg \min_{z \in Z^*} u^\theta_i (z) \text{ is an } i-Z^*-\theta'-\text{max set,} \right. \left. \text{and } F(\theta) \cap \arg \min_{z \in Z^*} u^\theta_i (z) \subset F(\theta') \right\},$$

$$\Xi_i^{C-D} (\theta) \equiv \left\{ K \in 2^{\Theta_i^{\theta, C-D}} \setminus \{ \emptyset \} : \Theta_i^{\theta, C-D} \cap \left[ \Lambda^{i-Z^*-\Theta} \left( Z^* \cap \bigcup_{K \in \Xi_i^{C-D} (\theta)} \left[ \bigcap_{\theta' \in K} F(\theta') \right] \right) = K \right\}. \hfill (31)$$

For each $(i, \theta, a) \in I \times \Theta \times Z$, define

$$\hat{L}_i^{Y, C-D} (a, \theta)$$

$$\equiv \begin{cases} \triangle \left[ Z^* \cap \bigcup_{K \in \Xi_i^{C-D} (\theta)} \left( \bigcap_{\theta' \in K} F(\theta') \right) \right], & \text{if } \left\{ a \in F(\theta) \cap \arg \min_{z \in Z^*} u^\theta_i (z), \Xi_i^{C-D} (\theta) \neq \emptyset, \right. \\
\triangle (Z^*) \cap \bigcup_{K \in \Xi_i^{C-D} (\theta)} \left( \bigcap_{\theta' \in K} F(\theta') \right), & \text{otherwise} \right. \end{cases}$$

$$\text{otherwise}$$

$$\triangle (Z^*) \cap \bigcup_{K \in \Xi_i^{C-D} (\theta)} \left( \bigcap_{\theta' \in K} F(\theta') \right),$$
Definition 19 ($\hat{L}^{Y-C-D}$-Maskin-monotonicity)

$\hat{L}^{Y-C-D}$-Maskin-monotonicity holds for an SCC $F$ if

$$a \in F(\theta), \left[\hat{L}_{i}^{Y-C-D}(a, \theta) \subset L_{i}^{Y}(a, \theta), \forall i \in I\right] \implies a \in F(\theta'), \forall (\theta, \theta', a) \in \Theta \times \Theta \times Z.$$ 

Theorem 8

Given an SCC $F : \Theta \rightarrow 2^{Z} \setminus \{\emptyset\}$, the following three statements are equivalent.

(i) $F$ is mixed-Nash-C-implementable;

(ii) $F$ is mixed-Nash-D-implementable;

(iii) $\hat{L}^{Y-C-D}$-Maskin-monotonicity holds for $F$.

The proof of Theorem 8 is similar to that of Theorem 7, and it is relegated to Xiong (2022a).

7 Ordinal implementation: full characterization

Throughout this section, we fix an ordinal model

$$\left\langle I = \{1, \ldots, I\}, \Theta^{*}, Z, F : \Theta^{*} \rightarrow 2^{Z} \setminus \{\emptyset\}, Y \equiv \triangle (Z), \left(\succeq_{i}^{\theta}\right)_{(i, \theta) \in I \times \Theta^{*}}\right\rangle,$$

and show that it is straightforward to derive full characterization of ordinal mixed-Nash implementation à la Mezzetti and Renou (2012). For any $(a, i, \theta) \in Z \times I \times \Theta^{*}$, consider

$$L_{i}^{Z}(a, \theta) \equiv \left\{ z \in Z : a \succeq_{i}^{\theta} z \right\},$$

$$SL_{i}^{Z}(a, \theta) \equiv \left\{ z \in Z : a \succ_{i}^{\theta} z \right\}.$$ 

Definition 20 (set-monotonicity, Mezzetti and Renou (2012))

An SCC $F$ is set-monotonic if for any $(\theta, \theta') \in \Theta^{*} \times \Theta^{*}$, we have $F(\theta) \subset F(\theta')$ whenever for any $i \in I$, one of the following two condition holds: either (1) $Z \subset L_{i}^{Z}(F(\theta), \theta')$ or (2) for any $a \in F(\theta)$, both $L_{i}^{Z}(a, \theta) \subset L_{i}^{Z}(a, \theta')$ and $SL_{i}^{Z}(a, \theta) \subset SL_{i}^{Z}(a, \theta')$ hold.
Mezzetti and Renou (2012) prove that set-monotonicity is necessary for ordinal mixed-Nash-A-implementation.

**Theorem 9 (Mezzetti and Renou (2012))** Set-monotonicity holds if \( F \) is mixed-Nash-A-implementable on \( \Theta = \bigcup_{\theta \in \Theta} \Omega^{[\succeq_{\theta}, R]} \) (i.e., \( F \) is ordinally-mixed-Nash-implementable à la Mezzetti and Renou (2012)).

It is easy to show that \( L^Y \)-uniform-monotonicity defined below is necessary condition for mixed-Nash-A-implementation.\(^{15}\)

**Definition 21 (\( L^Y \)-uniform-monotonicity)** \( L^Y \)-uniform-monotonicity holds for an SCC \( F \) if

\[
\left\{ L^Y_i \left( \text{UNIF} \left[ F (\theta) \right], \theta \right), \forall i \in I \right\} \subset \left\{ L^Y_i \left( \text{UNIF} \left[ F (\theta') \right], \theta' \right), \forall i \in I \right\}
\]

where \( \Theta = \bigcup_{\theta \in \Theta} \Omega^{[\succeq_{\theta}, R]} \).

**Lemma 9 (Mezzetti and Renou (2012), Proposition 1)** The following statements are equivalent.

(i) set-monotonicity holds;

(ii) \( L^Y \)-uniform-monotonicity holds for \( F \) on \( \Theta = \bigcup_{\theta \in \Theta} \Omega^{[\succeq_{\theta}, R]} \);

(iii) \( L^Y \)-uniform-monotonicity holds for \( F \) on \( \Theta = \bigcup_{\theta \in \Theta} \Omega^{[\succeq_{\theta}, Q]} \).

(i) being equivalent to (ii) in Lemma 9 is Proposition 1 in Mezzetti and Renou (2012), which provides its proof. The same argument shows that (i) is equivalent (iii).

Given Lemma 9, the following theorem shows that Theorem 9 is immediately implied by Theorem 7, because \( L^Y \)-uniform-monotonicity is immediately implied by \( \hat{L}^{Y-A-B} \)-uniform-monotonicity. Theorem 10 is implied by Theorem 11 below.

**Theorem 10** An SCC \( F \) is mixed-Nash-A-implementable on \( \Theta = \bigcup_{\theta \in \Theta} \Omega^{[\succeq_{\theta}, Q]} \) if and only if \( F \) is mixed-Nash-A-implementable on \( \Theta' = \bigcup_{\theta \in \Theta} \Omega^{[\succeq_{\theta}, R]} \).

---

\(^{15}\)See Lemma 11 and the discussion in Section 9.4.
In light of Theorem 7, define
\[ \forall (i, a, \theta) \in Z \times \Theta, \]
\[ \tilde{L}_i^{Z^*-A-B}(a, \theta) \equiv \begin{cases} 
  Z^* \cap \mathcal{L}_i^{Z^*}(F(\theta), \theta) \cap \left( \bigcup_{K \in \Xi(\theta)} \bigcap_{\theta' \in K} F(\theta') \right), & \text{if } F(\theta) \subset \arg \min_{z \in Z^*} u_{i}^{\theta}(z), \\
  Z^* \cap \mathcal{L}_i^{Z}(a, \theta), & \text{otherwise}
\end{cases}, \]

\[ \tilde{S}_i^{Z^*-A-B}(a, \theta) \equiv \tilde{L}_i^{Z^*-A-B}(a, \theta) \cap S \mathcal{L}_i^{Z}(a, \theta). \]

**Definition 22 (strong set-monotonicity)** An SCC \( F \) is strongly set-monotonic if for any \((\theta, \theta') \in \Theta^* \times \Theta^*\), we have \( F(\theta) \subset F(\theta') \) whenever for any \( i \in \mathcal{I} \), one of the following two condition holds: either (1) \( Z^* \subset \mathcal{L}_i^{Z^*}(F(\theta), \theta') \) or (2) for any \( a \in F(\theta) \), both \( \tilde{L}_i^{Z^*-A-B}(a, \theta) \subset \mathcal{L}_i^{Z}(a, \theta') \) and \( \tilde{S}_i^{Z^*-A-B}(a, \theta) \subset S \mathcal{L}_i^{Z}(a, \theta') \) hold.

Using a similar argument as in the proof of Lemma 9 (or equivalently, Proposition 1 in Mezzetti and Renou (2012)), it is straightforward to show the following lemma.

**Lemma 10** The following statements are equivalent.

(i) strong set-monotonicity holds;

(ii) \( \hat{L}^{Y-A-B} \)-uniform-monotonicity holds for \( F \) on \( \Theta = \bigcup_{\theta \in \Theta} \Omega[z^{}, R] \);

(iii) \( \hat{L}^{Y-A-B} \)-uniform-monotonicity holds for \( F \) on \( \Theta = \bigcup_{\theta \in \Theta} \Omega[z^{}, Q] \).

This immediately leads to the following full characterization, and the proof is relegated to Appendix A.13.

**Theorem 11** The following statements are equivalent.

(i) strong set-monotonicity holds;
(ii) $F$ is mixed-Nash-$A$-implementable on $\Theta = \bigcup_{\theta \in \Theta} \Omega^{[\succeq, Q]}$;

(iii) $F$ is mixed-Nash-$A$-implementable on $\Theta = \bigcup_{\theta \in \Theta} \Omega^{[\succeq, R]}$.

A prominent class of preferences discussed in Mezzetti and Renou (2012) is the single-top preferences. Given single-top preferences, it is straightforward to show $Z = Z^*$, $L_i^Y (\text{UNIF}[F(\theta)], \theta) \equiv \widehat{L}_i^{Y,A-B} (\text{UNIF}[F(\theta)], \theta)$, and set-monotonicity $\iff$ strong set-monotonicity.

As a result, Theorem 11 implies that set-monotonicity fully characterizes ordinally-mixed-Nash-implementable à la Mezzetti and Renou (2012).

Similarly, we can easily derive full characterization of ordinal implementation for the other 5 versions of mixed-Nash implementation of SCCs.

8 Compared to rationalizable implementation

Given a mechanism $M = \langle M \equiv \times_{i \in I} M_i, g : M \rightarrow Y \rangle$, define $S_i \equiv 2^{M_i}$ and $S = \times_{i \in I} S_i$ for each $i \in I$. For each state $\theta \in \Theta$, consider an operator $b^{M, \theta} : S \rightarrow S$ with $b^{M, \theta} \equiv \left[ b_i^{M, \theta} : S \rightarrow S_i \right]_{i \in I'}$, where each $b_i^{M, \theta}$ is defined as follows. For every $S \in S$,

$$
\begin{align*}
\left. b_i^{M, \theta} (S) = \right\{ & m_i \in M_i : \\
(1) & \exists \lambda_{-i} \in \Delta (M_{-i}) \text{ such that } \lambda_{-i} (m_{-i}) > 0 \implies m_{-i} \in S_{-i}, \text{ and } \\
(2) & m_i \in \arg \max_{\lambda_{-i} \in \Delta (M_{-i})} \lambda_{-i} (m_{-i}) \sum_{m_{-i} \in M_{-i}} \lambda_{-i} (m_{-i}) u_i (g (m'_i, m_{-i}), \theta) \right\}.
\end{align*}
$$

Clearly, $S$ is a lattice with the order of “set inclusion,” and $b^{M, \theta}$ is monotonically increasing.\(^\text{16}\) Thus, Tarski’s fixed point theorem implies existence of a largest fixed point of $b^{M, \theta}$, and we denote it by $S^{M, \theta} \equiv \left( S_i^{M, \theta} \right)_{i \in I'}$. We say $m_i \in M_i$ is rationalizable in $M$ at state $\theta$ if and only if $m_i \in S_i^{M, \theta}$.

\(^{\text{16}}\)That is, $S \subset S'$ implies $b^{M, \theta} (S) \subset b^{M, \theta} (S')$. We say that $S \in S$ satisfies the best reply property in $M$ at $\theta$ if and only if $S \subset b^{M, \theta} (S)$. It is straightforward to show that $S \subset S^{M, \theta}$ if $S$ satisfies the best reply property.
Definition 23 (Jain (2021)) An SCC $F : \Theta \rightarrow 2^Z \setminus \{\emptyset\}$ is rationalizable implementable if there exists a mechanism $M = \langle M, g : M \rightarrow Y \rangle$ such that

$$\bigcup_{m \in S^M, \theta} \text{SUPP}[g(m)] = F(\theta), \forall \theta \in \Theta.$$ 

Theorem 12 An SCC $F : \Theta \rightarrow 2^Z \setminus \{\emptyset\}$ is mixed-Nash-A-implementable if $F$ is rationalizable-implementable.

The detailed proof of Theorem 12 is relegated to Xiong (2022a).

9 Connected to Moore and Repullo (1990) and Sjöström (1991)

In this section, we illustrate Moore and Repullo (1990) and Sjöström (1991). We show that our full characterization share the same conceptual ideas as those in Moore and Repullo (1990) and Sjöström (1991), and furthermore, we show why their full characterization is complicated, and why ours is simple.

9.1 A common conceptual idea

Maskin (1999) proves that Maskin monotonicity almost fully characterizes Nash implementation. As being showed in Moore and Repullo (1990) and illustrated in Sjöström (1991), in order to fully characterize Nash implementation, we need to take two additional steps before defining Maskin monotonicity. All of these papers use the canonical mechanism in Maskin (1999) to achieve Nash implementation, and the two additional steps corresponds to eliminating bad equilibria in Case (3) and Case (2) of the canonical
mechanism. Roughly, we have the following two additional steps.

Step (I): select \( \hat{Z} \in 2^\mathcal{Z} \setminus \{\emptyset\} \) such that \( \bigcup_{\theta \in \Theta} F(\theta) \subset \hat{Z} \),

and \( \hat{Z} \) satisfies a unanimity condition:

\[
\forall (\theta^*,y) \in \Theta \times \hat{Z},
\begin{bmatrix}
  y \in \arg \max_{z \in \hat{Z}} U_{\theta}^0(z), \\
  \forall i \in I
\end{bmatrix}
\implies \text{“} y \text{ is a good outcome at } \theta^*. \text{”}
\]

(32)

where \( \hat{Z} \) is the set of outcomes that agents can choose when Case (3) is triggered in the canonical mechanism. In particular, “\( y \) is a good outcome at \( \theta^* \)” in (32) may have different formalization in different environments and under different implementation notions, which will be illustrated in Sections 9.2, 9.3 and 9.4.

To see the necessity of Step (I), suppose an SCC \( F \) is Nash implemented by \( \mathcal{M} = \langle \mathcal{M}, g : \mathcal{M} \rightarrow Y \rangle \), and define \( \hat{Z} = g(\mathcal{M}) \). If \( y = g(m) \) satisfies the left-hand side of (32) for some \( \theta^* \in \Theta \), then, \( m \) must be a Nash equilibrium at \( \theta^* \), and hence, \( y = g(m) \) must be a good outcome \( \theta^* \), i.e., the right-hand side of (32) holds.

To see the sufficiency of Step (I), suppose the true state is \( \theta^* \). Consider any Nash equilibrium in Case (3) of the canonical mechanism, which induces \( y \in \hat{Z} \). Then, \( y \) must be a top outcome in \( \hat{Z} \) for all agents, i.e., the left-hand side of (32) holds. Thus, by (32), \( y \) is a good outcome at \( \theta^* \).

Step (II): for each \( (\theta,i) \in \Theta \times I \) and each \( a \in F(\theta) \), select \( \hat{L}_i(a,\theta) \in 2^{[\hat{Z} \cap L_i(a,\theta)]} \setminus \{\emptyset\} \)

such that \( a \in \hat{L}_i(a,\theta) \),

and \( \hat{L}_i(a,\theta) \) satisfies a weak no-veto condition:

\[
\forall (\theta^*,y) \in \Theta \times \hat{L}_i(a,\theta),
\begin{bmatrix}
  y \in \arg \max_{z \in \hat{Z}} U_{\theta}^0(z), \\
  \forall j \in I \setminus \{i\}, \\
  y \in \arg \max_{z \in \hat{L}_i(a,\theta)} U_{\theta}^0(z),
\end{bmatrix}
\implies \text{“} y \text{ is a good outcome at } \theta^*. \text{”}
\]

(33)

where \( \hat{L}_i(a,\theta) \) is the set of outcomes that agent \( i \) can choose when agent \( i \) is the whistle-blower and agents \( -i \) report \( (a,\theta) \), i.e., Case (2) is triggered in the canonical mechanism.

To see the necessity of Step (II), suppose an SCC \( F \) is Nash implemented by \( \mathcal{M} = \langle \mathcal{M}, g : \mathcal{M} \rightarrow Y \rangle \), and define \( \hat{L}_i(a,\theta) \in 2^{[\hat{Z} \cap L_i(a,\theta)]} \setminus \{\emptyset\} \)

such that \( a \in \hat{L}_i(a,\theta) \),

and \( \hat{L}_i(a,\theta) \) satisfies a weak no-veto condition:

\[
\forall (\theta^*,y) \in \Theta \times \hat{L}_i(a,\theta),
\begin{bmatrix}
  y \in \arg \max_{z \in \hat{Z}} U_{\theta}^0(z), \\
  \forall j \in I \setminus \{i\}, \\
  y \in \arg \max_{z \in \hat{L}_i(a,\theta)} U_{\theta}^0(z),
\end{bmatrix}
\implies \text{“} y \text{ is a good outcome at } \theta^*. \text{”}
\]

(33)
\( \langle M, g : M \rightarrow Y \rangle \), and define

\[
\hat{L}_i(a, \theta) = \left\{ m_i \in M_i, \begin{array}{l}
g(m_i, \lambda_{-i}) : g(\lambda_i, \lambda_{-i}) = a, \\
\text{and } (\lambda_i, \lambda_{-i}) \text{ is a Nash equilibrium}
\end{array} \right\}.
\]

If \( y = g(m_i, \lambda_{-i}) \) satisfies the left-hand side of (33) for some \( \theta^* \in \Theta \), \( (m_i, \lambda_{-i}) \) must be a Nash equilibrium at \( \theta^* \), and \( y = g(m_i, \lambda_{-i}) \) must be a good outcome at \( \theta^* \), i.e., the right-hand side of (33) holds.

To see the sufficiency of Step (II), suppose the true state is \( \theta^* \). Consider any Nash equilibrium in Case (2) of the canonical mechanism such that \( i \) is the whistle-blower and it induces \( y \in \hat{L}_i(a, \theta) \). Then, \( y \) must be a top outcome in \( \hat{Z} \) for agents \(-i\) (because they can deviate to Case (3)), and \( y \) must be a top outcome in \( \hat{L}_i(a, \theta) \) for agents \( i \) (because \( i \) can deviate to any outcome in \( \hat{L}_i(a, \theta) \)), i.e., the left-hand side of (33) holds. Thus, by (33), \( y \) is a good outcome at \( \theta^* \).

Given Steps (I) and (II), we define a modified Maskin monotonicity as follows.

\( \hat{L} \)-Maskin-monotonicity holds iff

\[
\left[ a \in F(\theta), \begin{array}{l}
\hat{L}_i(a, \theta) \subset L_i(a, \theta), \forall i \in I
\end{array} \right] \implies a \in F(\theta'), \forall (\theta, \theta', a) \in \Theta \times \Theta \times Z.
\]

This leads to a full characterization of Nash implementation: an SCC \( F \) is Nash implementable if and only if \( \hat{L} \)-Maskin-monotonicity holds. In particular, in the canonical mechanism, Step (I) eliminates bad equilibria in Case (3), and Step (II) eliminates bad equilibria in Case (2), and \( \hat{L} \)-Maskin-monotonicity eliminates bad equilibria in Case (1).
9.2 The full characterization in Moore and Repullo (1990) and Sjöström (1991)

In the environment of Moore and Repullo (1990), the two steps becomes

Step (I): select \( \hat{Z} \in 2^Z \backslash \{ \emptyset \} \) such that \( \forall \theta \in \Theta \), \( F(\theta) \subset \hat{Z} \) and
\[
\forall (\theta^*, y) \in \Theta \times \hat{Z}, \quad \left[ y \in \arg\max_{z \in \hat{Z}} U^*_i(z), \forall i \in I \right] \Rightarrow y \in F(\theta^*); \quad (34)
\]

Step (II): for each \( (\theta, i) \in \Theta \times I \) and each \( a \in F(\theta) \), select \( \hat{L}_i(a, \theta) \in 2^{[\hat{Z} \cap \mathcal{L}_i(a, \theta)] \backslash \{ \emptyset \}} \) such that \( a \in \hat{L}_i(a, \theta) \) and
\[
\forall (\theta^*, y) \in \Theta \times \hat{L}_i(a, \theta), \quad \left[ y \in \arg\max_{z \in \hat{Z}} U^*_i(z), \forall i \in I \backslash \{ i \}, \right. \quad \left. y \in \arg\max_{z \in \hat{L}_i(a, \theta)} U^*_i(z) \right] \Rightarrow y \in F(\theta^*) \quad (35)
\]

This leads to the full characterization in Moore and Repullo (1990): an SCC \( F \) is Nash implementable if and only if there exists such \( [\hat{Z}, (\hat{L}_i(a, \theta))_{i \in I, \theta \in \Theta, a \in F(\theta)}] \) and \( \hat{L} \)-Maskin-monotonicity hold. However, Moore and Repullo (1990) is silent regarding how to find such \( [\hat{Z}, (\hat{L}_i(a, \theta))_{i \in I, \theta \in \Theta, a \in F(\theta)}] \), while Sjöström (1991) provides an algorithm to find the largest such sets.\(^{17}\) In order to find \( \hat{Z} \), we need an iterative process of elimination. At round 1, define \( Z^1 \equiv Z \). If \( \hat{Z} = Z^1 \) does not satisfy (34), we eliminate any \( z \) that is a top outcome in \( Z^1 \) for all agents at some state \( \theta \), but \( z \notin F(\theta) \), and let \( Z^2 \) denote the set of outcomes that survive round 1. We have found the appropriate \( \hat{Z} \) if \( \hat{Z} = Z^2 \) satisfies (34). However, \( \hat{Z} = Z^2 \) may not satisfy (34), i.e., it may happen that some \( z \in Z^1 \) is not a top outcome of some agent in \( Z^1 \) at some state \( \theta \), but becomes a top outcome in \( Z^2 \) for all agents at \( \theta \), while \( z \notin F(\theta) \). In this case, we need another round of elimination. At round 2, if \( \hat{Z} = Z^2 \) does not satisfy (34), we eliminate any \( z \) that is a top outcome in \( Z^2 \) for all

\(^{17}\) There may be multiple candidates of \( \hat{Z} \) which satisfies (34). The union of these candidates is the largest \( \hat{Z} \) satisfying (34), which is identified by Sjöström (1991). Similarly, Sjöström (1991) identifies the largest such \( \hat{L}_i(a, \theta) \) for each \( i \in I, \theta \in \Theta \) and \( a \in F(\theta) \).
agents at some state $\theta$, but $z \notin F(\theta)$, and let $Z^3$ denote the set of outcomes that survive round 2.... We continue this process until we find $Z^n$ such that $\hat{Z} = Z^n$ satisfies (34).

After finding $\hat{Z}$, we need a similar iterative process to find each $\hat{L}_i(a, \theta)$. At round 1, define $L_i^1(a, \theta) \equiv L_i(a, \theta)$. If $\hat{L}_i(a, \theta) = L_i^1(a, \theta)$ does not satisfy (35), we eliminate any $z$ that is a top outcome in $\hat{Z}$ for agents $-i$ at some state $\theta^*$ and a top outcome in $L_i^1(a, \theta)$ for agent $i$ at $\theta^*$, but $z \notin F(\theta^*)$. Let $L_i^2(a, \theta)$ denote the set of outcomes that survive round 1. At round 2, if $\hat{L}_i(a, \theta) = L_i^2(a, \theta)$ does not satisfy (35), we eliminate any $z$ that is a top outcome in $\hat{Z}$ for agents $-i$ at some state $\theta^*$ and a top outcome in $L_i^2(a, \theta)$ for agent $i$ at $\theta^*$, but $z \notin F(\theta^*)$. Let $L_i^3(a, \theta)$ denote the set of outcomes that survive round 2.... We continue this process until we find $L_i^n(a, \theta)$ such that $\hat{L}_i(a, \theta) = L_i^n(a, \theta)$ satisfies (35).

Clearly, both the existential statement in Moore and Repullo (1990) and the iterative process in Sjöström (1991) make their full characterization complicated.

9.3 Our full characterization for SCFs

For simplicity, we focus on SCFs throughout this subsection. Given stochastic mechanisms, the insight of this paper is that we can easily select $\left[\hat{Z}, \left(\hat{L}_i(f(\theta), \theta)\right)_{i \in I, \theta \in \Theta}\right]$ in Step (I) and Step (II). Taking full advantage of the convexity structure of lotteries, we assign the following lottery, when Case (3) is triggered in the canonical mechanism.

$$g[m] = \left(1 - \frac{1}{k_{j_i}}\right) \times b_{j_i} + \frac{1}{k_{j_i}^2} \times \text{UNIF}(\hat{Z}).$$

In particular, the winner of the integer game, i.e., $j^*$, can increase the probability of her top outcome $b_{j^*}$ by increasing $k_{j^*}^2$. As a result, a Nash equilibrium in Case (3) at true state $\theta^*$ must require all agents be indifferent between any two outcomes in $\hat{Z}$ at $\theta^*$. Given SCFs,
this means\textsuperscript{18}

Step (I): select $\hat{Z} \in 2^Z \setminus \{\emptyset\}$ such that $f(\Theta) \subset \hat{Z}$ and

$$\forall \theta^* \in \Theta,$$

$$\begin{bmatrix}
    \hat{Z} \subset \arg \max_{z \in \hat{Z}} U_{i}^{\theta^*} (z), \\
    \forall i \in \mathcal{I}
\end{bmatrix}
\implies
\begin{bmatrix}
    \hat{Z} \subset \{f(\theta^*)\}, \\
    \text{and hence, } |\hat{Z}| = 1,
\end{bmatrix}
.$$  

Assumption 1 and $f(\Theta) \subset \hat{Z}$ imply that $|\hat{Z}| = 1$ always fails. As a result, Step (I) becomes

Step (I): select $\hat{Z} \in 2^Z \setminus \{\emptyset\}$ such that $f(\Theta) \subset \hat{Z}$ and

$$\forall \theta^* \in \Theta,$$

$$\begin{bmatrix}
    \hat{Z} \subset \arg \max_{z \in \hat{Z}} U_{i}^{\theta^*} (z), \\
    \forall i \in \mathcal{I}
\end{bmatrix}
\text{fails.}
$$

Since $\hat{Z} \subset Z$, it is straightforward to show

$$\begin{bmatrix}
    Z \text{ is not a } i\text{-max set} \\
    \forall i \in \mathcal{I}
\end{bmatrix}
\iff
\begin{bmatrix}
    Z \subset \arg \max_{z \in \hat{Z}} U_{i}^{\theta^*} (z), \\
    \forall i \in \mathcal{I}
\end{bmatrix}
\text{fails}
$$

$$\iff
\begin{bmatrix}
    \hat{Z} \subset \arg \max_{z \in \hat{Z}} U_{i}^{\theta^*} (z), \\
    \forall i \in \mathcal{I}
\end{bmatrix}
\text{fails}
.$$}

Therefore, without an iterative process of elimination, we have already found the largest such $\hat{Z}$, i.e., $\hat{Z} = Z$. In particular, a necessary condition for Nash implementation is: $Z$ is not a $i$-max set for any $i \in \mathcal{I}$.\textsuperscript{19}

Similarly, taking full advantage of the convexity structure of lotteries, we assign the

18 Given the canonical mechanism $\mathcal{M} = (\mathcal{M}, g : \mathcal{M} \rightarrow Y)$, all agents being indifferent between any two outcomes in $\hat{Z}$ at $\theta^*$ implies any $m \in \mathcal{M}$ is a Nash equilibrium at $\theta^*$, i.e., $\mathcal{M} \subset \text{MNE}^{\mathcal{M}, \theta^*}$. Since $\cup_{m \in \text{SUPP}[g(m)]} = \hat{Z}$, we conclude that $\hat{Z} = \cup_{m \in \text{SUPP}[g(m)]} = \{f(\theta^*)\}$, i.e., $|\hat{Z}| = 1$.

19 This necessary condition is explicitly stated in (iii) of Theorem 6, but omitted in Theorem 2, because, with SCFs, it is implicitly encoded in $\hat{L}^Y$-monotonicity (see Lemma 2).
following lottery, when Case (2) is triggered in the canonical mechanism.

\[ g[m] = \left(1 - \frac{1}{k_j^3}\right) \times \phi_j^\theta (\theta_j) + \frac{1}{k_j^2} \times \left(\epsilon_j^\theta \times \left[\left(1 - \frac{1}{k_j^3}\right) \times \gamma_j (\tilde{\Gamma}_j (\theta)) + \frac{1}{k_j^3} \times \text{UNIF} \left(\tilde{\Gamma}_j (\theta)\right)\right]\right) \]

In particular, the whistle-blower \( j \) can increase the probability of her top outcome \( \gamma_j (\tilde{\Gamma}_j (\theta)) \) in \( \tilde{\Gamma}_j (\theta) \) by increasing \( k_j^3 \). As a result, a Nash equilibrium in Case (2) at true state \( \theta^* \) must require all outcomes in \( \tilde{\Gamma}_j (\theta) \) be top for agents \( -j \) at \( \theta^* \) and agent \( j \) be indifferent between any two outcomes in \( \tilde{\Gamma}_j (\theta) \) at \( \theta^* \). Given SCFs, we thus have

Step (II): for each \((\theta, i) \in \Theta \times I\), select \( \tilde{L}^\gamma_i (f (\theta), \theta) \) such that

\[ f (\theta) \in \tilde{L}^\gamma_i (f (\theta), \theta) \quad \text{and} \quad \forall (\theta^*, y) \in \Theta \times \tilde{L}^\gamma_i (f (\theta), \theta), \]

\[ \left[ \tilde{\Gamma}_i (\theta) \subseteq \arg \max_{z \in Z} u_i^{\theta^*} (z), \forall j \in I \setminus \{i\}, \quad \tilde{\Gamma}_i (\theta) \subseteq \arg \max_{z \in \tilde{\Gamma}_i (\theta)} u_i^{\theta^*} (z), \right] \implies \begin{cases} \tilde{\Gamma}_i (\theta) \subseteq \{ f (\theta^*) \}, \\ \text{and hence, } |\tilde{\Gamma}_i (\theta)| = 1, \end{cases} \]

or equivalently, \([\tilde{\Gamma}_i (\theta) \text{ is an } i-\theta^*-\text{max set}] \implies \begin{cases} \tilde{\Gamma}_i (\theta) \subseteq \{ f (\theta^*) \}, \\ \text{and hence, } |\tilde{\Gamma}_i (\theta)| = 1, \end{cases} \tag{36}\]

where \( \tilde{\Gamma}_i (\theta) \equiv \bigcup_{y \in \tilde{L}^\gamma_i (f (\theta), \theta)} \text{SUPP} [y] \).

Thus, it is straightforward to find the largest such \( \tilde{L}^\gamma_i (f (\theta), \theta) \) by considering three scenarios:

\[
\begin{cases}
\text{scenario (I): } f (\theta) \in \arg \min_{z \in Z} u_i^{\theta} (z) \quad \text{and} \quad \tilde{L}^\gamma_i (f (\theta), \theta) \text{ is an } i-\text{max set}, \\
\text{scenario (II): } f (\theta) \in \arg \min_{z \in Z} u_i^{\theta} (z) \quad \text{and} \quad \tilde{L}^\gamma_i (f (\theta), \theta) \text{ is not an } i-\text{max set}, \\
\text{scenario (III): } f (\theta) \notin \arg \min_{z \in Z} u_i^{\theta} (z)
\end{cases}
\]

In scenario (I), we must have \( \tilde{L}^\gamma_i (f (\theta), \theta) = \{ f (\theta) \} \) by Lemma 1. In scenarios (II) and (III), \( \tilde{L}^\gamma_i (f (\theta), \theta) = \tilde{L}^\gamma_i (f (\theta), \theta) \) makes (36) hold vacuously.\(^{20}\) This leads to the definition of \( \tilde{L}^\gamma_i (f (\theta), \theta) \) in (3).

\(^{20}\)In scenario (III), \( f (\theta) \notin \arg \min_{z \in Z} u_i^{\theta} (z) \) implies \( \tilde{\Gamma}_i (\theta) = Z \), and \( Z \) is not an \( i-\theta^*-\text{max set} \) for any \( i \in I \), which would be implied by \( \tilde{L}^\gamma_i \)-monotonicity imposed later (see Lemma 2).
9.4 Our full characterization for mixed-Nash-A-implementation

We consider SCCs and focus on mixed-Nash-A-implementation throughout this subsection. We first illustrate why $\hat{L}^Y$-Maskin-monotonicity is defined on $\text{UNIF}[F(\theta)]$. Suppose that $F$ is mixed-Nash-A-implemented by $M = \langle M, g : M \rightarrow Y \rangle$, and consider any $\lambda \in MNE(M, \theta)$. By mixed-Nash-A-implementation, $\lambda$ could be any lotteries in $\triangle [F(\theta)]$, and as a result, we need to define $\hat{L}^Y$-Maskin-monotonicity as:

$$
\begin{align*}
\left[ \hat{L}^Y_i(\eta, \theta) \subset L^Y_i(\eta, \theta') \quad \forall \eta \in \triangle [F(\theta)], \forall i \in I \right] \implies F(\theta) \subset F(\theta'), \forall (\theta, \theta') \in \Theta \times \Theta.
\end{align*}
$$

However, the following lemma shows that it suffers no loss of generality to define $\hat{L}^Y$-Maskin-monotonicity on $\text{UNIF}[F(\theta)]$ only (i.e., Definition 18). The proof of Lemma 11 is relegated to Appendix A.12.

**Lemma 11** For any $E \in 2^Z \setminus \{\emptyset\}$ and any $(\gamma, i, \theta, \theta') \in \triangle^\circ (E) \times I \times \Theta \times \Theta$, we have

$$
\begin{align*}
\left( L^Y_i(\eta, \theta) \subset L^Y_i(\eta, \theta') \quad \forall \eta \in \triangle [E] \right) \iff L^Y_i(\gamma, \theta) \subset L^Y_i(\gamma, \theta').
\end{align*}
$$

(37)

Second, by a similar argument as in Section 9.3, mixed-Nash-A-implementation implies

Step (I): select $\hat{Z} \in 2^Z \setminus \{\emptyset\}$ such that $\bigcup_{\theta \in \Theta} F(\theta) \subset \hat{Z}$ and

$$
\forall \theta^* \in \Theta,
\begin{align*}
\left[ \hat{Z} \subset \arg\max_{z \in \hat{Z}} U^\theta_i(z) \quad \forall i \in I \right] \implies \hat{Z} \subset F(\theta^*).
\end{align*}
$$

(38)

Without an iterative process of elimination, it is straightforward to find the largest such $\hat{Z}$ by consider two scenarios. If $Z$ is not an $i$-max set for any $i \in I$, $\hat{Z} = Z$ makes (38) hold vacuously. Otherwise, we must have $\hat{Z} = \bigcup_{\theta \in \Theta} F(\theta)$.\(^{21}\) Therefore, we must consider $Z^*$ defined in (38) for mixed-Nash-A-implementation.

\(^{21}\)Given the canonical mechanism $M = \langle M, g : M \rightarrow Y \rangle$, all agents being indifferent between any two outcomes in $\hat{Z}$ at $\theta^*$ implies any $m \in M$ is a Nash equilibrium. As a result, we have $\hat{Z} = \bigcup_{m \in \text{SUPP}[g(m)]} F(\theta^*)$, which further implies $\bigcup_{\theta \in \Theta} F(\theta) \subset \hat{Z} \subset F(\theta^*) \subset \bigcup_{\theta \in \Theta} F(\theta)$, i.e., $\hat{Z} = \bigcup_{\theta \in \Theta} F(\theta)$.
Third, by a similar argument as in Section 9.3, mixed-Nash-A-implementation implies

Step (II): for each \((\theta, i) \in \Theta \times I\), select \(\hat{L}_i^Y (\text{UNIF}[F(\theta)], \theta) \in 2^{[\Delta(Z^*) \cap L_i^Y (\text{UNIF}[F(\theta)], \theta)] \setminus \{\emptyset\}}\)
such that \(\text{UNIF}[F(\theta)] \in \hat{L}_i^Y (\text{UNIF}[F(\theta)], \theta)\) and
\[
\forall (\theta^*, y) \in \Theta \times \hat{L}_i^Y (\text{UNIF}[F(\theta)], \theta),
\]
\[
\left[\hat{\Gamma}_i(\theta) \text{ is an } i\text{-}Z^*-\theta^*-\text{max set}\right] \implies \hat{\Gamma}_i(\theta) \subset F(\theta^*). \tag{39}
\]

Thus, we can find the largest such \(\hat{L}_i^Y (a, \theta)\) by considering three scenarios:

\[
\left(\begin{array}{l}
\text{scenario (I): } F(\theta) \subset \arg \min_{z \in Z^*} u_i^\theta(z) \quad \text{and } Z^* \cap L_i^Z (F(\theta), \theta) \text{ is an } i\text{-}Z^*-\text{max set,} \\
\text{scenario (II): } F(\theta) \subset \arg \min_{z \in Z^*} u_i^\theta(z) \quad \text{and } Z^* \cap L_i^Z (F(\theta), \theta) \text{ is not an } i\text{-}Z^*-\text{max set,} \\
\text{scenario (III): } F(\theta) \setminus \arg \min_{z \in Z^*} u_i^\theta(z) \neq \emptyset
\end{array}\right).
\]

In scenario (I), Lemma 7 implies that we must have
\[
\hat{L}_i^Y (\text{UNIF}[F(\theta)], \theta) = \Delta \left[ Z^* \cap L_i^Z (F(\theta), \theta) \cap \left( \bigcup_{K \in \Xi_i(\theta)} \bigcap_{\theta' \in K} F(\theta') \right) \right].
\]

In scenario (II), \(\hat{L}_i^Y (\text{UNIF}[F(\theta)], \theta) = L_i^Y (\text{UNIF}[F(\theta)], \theta)\) makes (39) hold vacuously. In scenario (III), \(\hat{L}_i^Y (\text{UNIF}[F(\theta)], \theta) = L_i^Y (\text{UNIF}[F(\theta)], \theta)\) implies \(\hat{\Gamma}_i(\theta) = Z^*\), and (39) holds by Lemma 8.—This leads to the definition of \(\hat{L}_i^{Y-A-B} (\text{UNIF}[F(\theta)], \theta)\) in (30).

10 Conclusion

We study Nash implementation by stochastic mechanisms, and provide a surprisingly simple full characterization. Even though our full characterization is of a form similar to Maskin monotonicity à la Maskin (1999), it has an interpretation parallel to Moore and Repullo (1990) and Sjöström (1991). In this sense, we build a bridge between Maskin (1999) and Moore and Repullo (1990) (as well as Sjöström (1991)).
Furthermore, our full characterization shed light on

\[
\begin{align*}
\text{"mixed-Nash-implementation VS pure-Nash-implementation,"} \\
\text{"ordinal-approach VS cardinal-approach,"} \\
\text{"Nash-implementation VS rationalizable-implementation"}
\end{align*}
\]
A Proofs

A.1 Proof of Lemma 1

Suppose that $f$ is mixed-Nash implemented by $M = \langle M, g : M \rightarrow Y \rangle$, and fix any $(i, \theta) \in I \times \Theta$ and any $\lambda \in MNE(M, \theta)$ such that

$$
\begin{bmatrix}
 f(\theta) \in \arg\min_{z \in Z} u_i(\theta) \quad \text{and} \\
 L^Z_i (f(\theta), \theta) \text{ is an } i\text{-max set}
\end{bmatrix},
$$

and we aim to show

$$
\bigcup_{m_i \in M_i} \text{SUPP}[g(m_i, \lambda_{-i})] = \{f(\theta)\}.
$$

First, $f(\theta) \in \arg\min_{z \in Z} u_i(\theta)$ implies

$$
L^Y_i (f(\theta), \theta) = \triangle \left[ L^Z_i (f(\theta), \theta) \right].
$$

Given $\lambda \in MNE(M, \theta)$, we have

$$
\{f(\theta)\} \subset \bigcup_{m_i \in M_i} \text{SUPP}[g(m_i, \lambda_{-i})] \subset L^Z_i (f(\theta), \theta).
$$

Furthermore, $L^Z_i (f(\theta), \theta)$ being an $i$-max set implies existence of $\theta' \in \Theta$ such that

$$
L^Z_i (f(\theta), \theta) \subset \arg\max_{z \in L^Z_i (f(\theta), \theta)} u_i(\theta'),
$$

$$
L^Z_i (f(\theta), \theta) \subset \arg\max_{z \in Z} u_i(\theta'), \forall j \in I \setminus \{i\},
$$

which, together with (40), further implies

$$
(m_i, \lambda_{-i}) \in MNE(M, \theta'), \forall m_i \in M_i.
$$

Since $f$ is mixed-Nash-implemented by $M$, we have

$$
\{f(\theta)\} \subset \bigcup_{m_i \in M_i} \text{SUPP}[g(m_i, \lambda_{-i})] \subset \{f(\theta')\},
$$

and as a result, $f(\theta) = f(\theta')$ and

$$
\bigcup_{m_i \in M_i} \text{SUPP}[g(m_i, \lambda_{-i})] = \{f(\theta)\}.
$$

A.2 Proof of Lemma 2

Suppose that $\hat{L}^Y$-monotonicity holds. We first prove that $Z$ is not an $i$-max set for any $i \in I$. Suppose otherwise, i.e., $Z$ is an $i$-max set for some $i \in I$, or equivalently, there
exists $\theta' \in \Theta$ such that all agents are indifferent between any two outcomes in $Z$, and hence,

$$\mathcal{L}_j^Y (z, \theta') = Y, \forall (j, z) \in \mathcal{I} \times Z.$$ 

As a result, for any $\theta \in \Theta$, we have

$$\hat{\mathcal{L}}_j^Y (f(\theta), \theta) \subset Y = \mathcal{L}_j^Y (f(\theta), \theta'), \forall j \in \mathcal{I},$$

which, together with $\hat{\mathcal{L}}^Y$-monotonicity, implies $f(\theta) = f(\theta')$ for any $\theta \in \Theta$, or equivalently, $f(\Theta) = \{f(\theta')\}$, contradicting Assumption 1.

Second, fix any $(j, \theta, \theta') \in \mathcal{I} \times \Theta \times \Theta$ such that

$$\left[\hat{\Gamma}_j (\theta) \text{ is a } j \text{-} \theta' \text{-max set}\right],$$

and we aim to prove $\hat{\Gamma}_j (\theta) = \{f(\theta')\}$. By the definition of $\hat{\Gamma}_j (\theta)$ in (7) and the definition of $\hat{\mathcal{L}}_j^Y (f(\theta), \theta)$ in (3), we have

$$\hat{\Gamma}_j (\theta) = \begin{cases} 
\{f(\theta)\}, & \text{if } f(\theta) \in \text{arg min}_{z \in Z} u^0_j (z) \text{ and } \mathcal{L}_j^Z (f(\theta), \theta) \text{ is an } j \text{-max set,} \\
\mathcal{L}_j^Z (f(\theta), \theta), & \text{if } f(\theta) \in \text{arg min}_{z \in Z} u^0_j (z) \text{ and } \mathcal{L}_j^Z (f(\theta), \theta) \text{ is not an } j \text{-max set,} \\
Z, & \text{if } f(\theta) \notin \text{arg min}_{z \in Z} u^0_j (z) 
\end{cases}$$

(42)

To see $\hat{\Gamma}_j (\theta) = Z$ when $f(\theta) \notin \text{arg min}_{z \in Z} u^0_j (z)$, pick any $z' \in \text{arg min}_{z \in Z} u^0_j (z)$, and we have $u^0_j (f(\theta)) > u^0_j (z')$, which further implies

$$u^0_j (f(\theta)) > U^0_j [(1 - \varepsilon) \times z' + \varepsilon \times \text{UNIF}(Z)] \text{ for sufficiently small } \varepsilon > 0,$$

i.e., $[(1 - \varepsilon) \times z' + \varepsilon \times \text{UNIF}(Z)] \in \mathcal{L}_j^Y (f(\theta), \theta)$ and $\hat{\Gamma}_j (\theta) = Z$.

As proved above, $Z$ is not a $j$-max set. As a result, (41) and (42) imply that we must have $\hat{\Gamma}_j (\theta) = \{f(\theta)\}$, which, together with (41), further implies

$$\begin{align*}
\hat{\mathcal{L}}_i^Y (f(\theta), \theta) & \subset Y = \mathcal{L}_i^Y (f(\theta), \theta'), \forall i \in \mathcal{I} \setminus \{j\}, \\
\hat{\mathcal{L}}_j^Y (f(\theta), \theta) & = \{f(\theta)\} \subset \mathcal{L}_j^Y (f(\theta), \theta').
\end{align*}$$

(43)

Finally, (43) and $\hat{\mathcal{L}}^Y$-monotonicity imply $f(\theta) = f(\theta')$, i.e., $\hat{\Gamma}_j (\theta) = \{f(\theta)\} = \{f(\theta')\}$. ■
A.3 Proof of Lemma 3

By the definition of $\hat{\Gamma}_j (\theta)$ in (7) and the definition of $\hat{L}_j^\gamma (f (\theta), \theta)$ in (3), we have

$$\hat{\Gamma}_j (\theta) = \begin{cases} \{ f (\theta) \}, & \text{if } f (\theta) \in \arg \min_{z \in Z} u_j^\theta (z) \text{ and } L_j^\gamma (f (\theta), \theta) \text{ is an } j\text{-max set}, \\ L_j^\gamma (f (\theta), \theta), & \text{if } f (\theta) \in \arg \min_{z \in Z} u_j^\theta (z) \text{ and } L_j^\gamma (f (\theta), \theta) \text{ is not an } j\text{-max set}, \\ Z, & \text{if } f (\theta) \notin \arg \min_{z \in Z} u_j^\theta (z) \end{cases}.$$ 

Fix any $(\theta, j) \in \Theta \times \mathcal{I}$, and we consider three cases. First, suppose $f (\theta) \in \arg \min_{z \in Z} u_j^\theta (z)$ and $L_j^\gamma (f (\theta), \theta)$ is an $j$-max set, i.e., $\hat{\Gamma}_j (\theta) = \{ f (\theta) \}$. Thus, we can choose $\epsilon_j^\theta = \frac{1}{2}$ and $y_j^\theta = f (\theta)$, and (10) holds.

Second, suppose $f (\theta) \in \arg \min_{z \in Z} u_j^\theta (z)$ and $L_j^\gamma (f (\theta), \theta)$ is not an $j$-max set, i.e., $\hat{\Gamma}_j (\theta) = L_j^\gamma (f (\theta), \theta)$ and

$$\hat{L}_j^\gamma (f (\theta), \theta) = L_j^\gamma (f (\theta), \theta) = \Delta \left[ L_j^\gamma (f (\theta), \theta) \right] = \Delta \left[ \hat{\Gamma}_j (\theta) \right].$$

Thus, we can choose $\epsilon_j^\theta = \frac{1}{2}$ and $y_j^\theta = f (\theta)$, and (10) holds.

Third, suppose $f (\theta) \notin \arg \min_{z \in Z} u_j^\theta (z)$, i.e., $\hat{\Gamma}_j (\theta) = Z$. By $f (\theta) \notin \arg \min_{z \in Z} u_j^\theta (z)$, we can pick any $y_j^\theta \in \arg \min_{z \in Z} u_j^\theta (z) \subset L_j^\gamma (f (\theta), \theta) = \hat{L}_j^\gamma (f (\theta), \theta)$, and we have

$$U_j^\theta (y_j^\theta) < u_j^\theta (f (\theta)).$$

Thus, there exists sufficiently small $\epsilon_j^\theta > 0$ such that

$$U_j^\theta \left( \epsilon_j^\theta \times y + \left( 1 - \epsilon_j^\theta \right) \times y_j^\theta \right) < u_j^\theta (f (\theta)), \forall y \in Y = \Delta (Z) = \Delta \left( \hat{\Gamma}_j (\theta) \right),$$

and hence,

$$\left[ \epsilon_j^\theta \times y + \left( 1 - \epsilon_j^\theta \right) \times y_j^\theta \right] \in L_j^\gamma (f (\theta), \theta) = \hat{L}_j^\gamma (f (\theta), \theta), \forall y \in \Delta \left( \hat{\Gamma}_j (\theta) \right),$$

i.e., (10) holds. □

A.4 Proof of Lemma 4

For each $(\theta, i) \in \Theta \times \mathcal{I}$, fix any

$$\hat{b}_i^\theta \in \arg \max_{z \in Z} u_i^\theta (z) \text{ and } \hat{\gamma}_i \in (Z)^{[2^Z \setminus \{\emptyset\}]} \text{ such } \hat{\gamma}_i (E) \in E, \forall E \in [2^Z \setminus \{\emptyset\}],$$

and hence,

$$\left[ \epsilon_j^\theta \times y + \left( 1 - \epsilon_j^\theta \right) \times y_j^\theta \right] \in L_j^\gamma (f (\theta), \theta) = \hat{L}_j^\gamma (f (\theta), \theta), \forall y \in \Delta \left( \hat{\Gamma}_j (\theta) \right),$$

i.e., (10) holds. □
i.e., $\hat{b}_i^\theta$ is a top outcome for $i$ at $\theta$. We need the following lemma to prove Lemma 4.

**Lemma 12** Consider the canonical mechanism $\mathcal{M}^*$ in Section 3.3.2. For any $(\theta, i) \in \Theta \times I$, define

$$m_i^n \equiv \left( \theta, k_i^2 = n, \tilde{k}_i, \hat{b}_i^\theta \right) \in M_i, \forall n \in \mathbb{N}.$$  

Then,

$$\lim_{n \to \infty} U_i^\theta [g (m_i^n, m_{-i})] \geq U_i^\theta [g (m_i, m_{-i})], \forall (m_i, m_{-i}) \in M_i \times M_{-i}.$$  

**Proof of Lemma 12:** Fix any $(\theta, i) \in \Theta \times I$ and any $(m_i, m_{-i}) \in M_i \times M_{-i}$. We consider two scenarios: (A) there exists $\theta' \in \Theta$ such that

$$m_j = \left( \theta', k_j^2 = 1, *, *, * \right), \forall j \in I \setminus \{i\},$$

i.e., $(m_i, m_{-i})$ triggers either Case (1) or Case (2), while $(m_i^n, m_{-i})$ triggers Case (2) if $n \geq 2$; and (B) otherwise, i.e., $(m_i^n, m_{-i})$ triggers Case (3).

In Scenario (A), $g (m_i, m_{-i}) \in \tilde{\mathcal{L}}^Y_i \left( f (\theta'), \theta' \right)$, and $g (m_i^n, m_{-i})$ induces $\phi_i^{\theta'} (\theta)$ with probability $\left( 1 - \frac{1}{n} \right)$. As a result, 

$$\lim_{n \to \infty} U_i^\theta [g (m_i^n, m_{-i})] = U_i^\theta \left[ \phi_i^{\theta'} (\theta) \right] = \max_{y \in \tilde{\mathcal{L}}^Y_i (f (\theta'), \theta')} U_i^\theta [y] \geq U_i^\theta [g (m_i, m_{-i})],$$

where the second equality follows from the definition of $\phi_i^{\theta'} (\theta)$ in (8) and the inequality from $g (m_i, m_{-i}) \in \tilde{\mathcal{L}}^Y_i \left( f (\theta'), \theta' \right)$.

In Scenario (B), $(m_i^n, m_{-i})$ triggers Case (3), and with sufficient large $n$, $(m_i^n, m_{-i})$ induces $\hat{b}_i^\theta$ with probability $\left( 1 - \frac{1}{n} \right)$. As a result, 

$$\lim_{n \to \infty} U_i^\theta [g (m_i^n, m_{-i})] = U_i^\theta \left[ \hat{b}_i^\theta \right] = \max_{z \in Z} U_i^\theta [z] \geq U_i^\theta [g (m_i, m_{-i})].$$

■

**Proof of Lemma 4:** Fix any $\theta \in \Theta$, any $\lambda \in MNE(\mathcal{M}^*, \theta)$ and any $\hat{m} \in \text{SUPP}[\lambda]$, i.e.,

$$\Pi_{i \in I} \lambda_i (\hat{m}_i) > 0.$$  

We aim to prove $\hat{m} \in PNE(\mathcal{M}^*, \theta)$. Suppose $\hat{m} \notin PNE(\mathcal{M}^*, \theta)$, i.e., there exists $j \in I$ and $m'_j \in M_i$ such that 

$$U_j^\theta \left[ g (m'_j, \hat{m}_{-j}) \right] > U_j^\theta \left[ g (\hat{m}_j, \hat{m}_{-j}) \right].$$
which, together with Lemma 12, implies
\[
\lim_{n \to \infty} U_j^\theta \left[ g \left( m_j^n, \hat{m}_- \right) \right] \geq U_j^\theta \left[ g \left( m_j^\theta, \hat{m}_- \right) \right] \geq U_j^\theta \left[ g \left( \hat{m}_j, \hat{m}_- \right) \right]. \tag{45}
\]
We thus have
\[
\lim_{n \to \infty} U_j^\theta \left[ g \left( m_j^n, \lambda_\theta \right) \right] - U_j^\theta \left[ g \left( \lambda_j, \lambda_\theta \right) \right] \geq 0 + \lim_{n \to \infty} \left( \sum_{m \in M^* \setminus \{\hat{m}_j\}} \left( U_j^\theta \left[ g \left( m_j^n, \hat{m}_- \right) \right] - U_j^\theta \left[ g \left( \hat{m}_j, \hat{m}_- \right) \right] \right) \right) \\
\geq 0,
\]
where the first inequality follows from Lemma 12, and the second inequality from (44) and (45). As a result, there exists \( n \in \mathbb{N} \) such that
\[
U_j^\theta \left[ g \left( m_j^n, \lambda_\theta \right) \right] > U_j^\theta \left[ g \left( \lambda_j, \lambda_\theta \right) \right],
\]
contradicting \( \lambda \in MNE^{(\mathcal{M}^*, \theta)}. \)

\[\bbox{\text{A.5 } \text{Proof of Lemma 5}}\]

For each \( (i, z, \theta) \in \mathcal{I} \times Z \times \Theta^* \), define
\[
SU_i^{(z, \theta)} \equiv \{ z' \in Z : z' \succ_i^\theta z \}, \\
SL_i^{(z, \theta)} \equiv \{ z' \in Z : z' \succeq_i^\theta z \}, \\
ID_i^{(z, \theta)} \equiv \{ z' \in Z : z \sim_i^\theta z' \}.
\]
We need the following lemma to proceed.

\[\bbox{\text{Lemma 13 }} \text{For any } \theta \in \Theta^*, \text{ any } u^\theta \in \Omega^{[z^\theta, \mathbb{R}]} \text{, any } (i, z) \in \mathcal{I} \times Z, \text{ consider}
\]
\[
\rho_{(i,z)} \equiv \frac{u_i^\theta (z) - \max_{z' \in SL_i^{(z, \theta)}} u_i^\theta (z')}{\max_{z' \in SU_i^{(z, \theta)}} u_i^\theta (z')} \quad \text{and} \quad \rho_{(i,z)}^{(\theta)} \equiv \frac{u_i^\theta (z) - \min_{z' \in SL_i^{(z, \theta)}} u_i^\theta (z')}{\min_{z' \in SU_i^{(z, \theta)}} u_i^\theta (z')}.
\]

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Then, for any \( z \in Z \setminus \left[ \arg \max_{z' \in Z} u_i^\theta (z') \cup \arg \min_{z' \in Z} u_i^\theta (z') \right] \), we have

\[
\left\{ y \in \Delta [Z] \setminus \Delta \left[ ID_{i}^{(z,\theta)} \right] : \frac{\sum_{z' \in SU_i^{(z,\theta)}} y_{z'} z_{z' \in SU_i^{(z,\theta) \cup SL_i^{(z,\theta)}}}}{\sum_{z' \in SU_i^{(z,\theta) \cup SL_i^{(z,\theta)}}} y_{z'}} \leq \rho_{(i,z)} \right\} \supset L_i^{z} (z, u_i^\theta) \setminus \Delta \left[ ID_{i}^{(z,\theta)} \right],
\]

(46)

\[
\left\{ y \in \Delta [Z] \setminus \Delta \left[ ID_{i}^{(z,\theta)} \right] : \frac{\sum_{z' \in SU_i^{(z,\theta)}} y_{z'} z_{z' \in SU_i^{(z,\theta) \cup SL_i^{(z,\theta)}}}}{\sum_{z' \in SU_i^{(z,\theta) \cup SL_i^{(z,\theta)}}} y_{z'}} \leq \rho_{(i,z)} \right\} \subset L_i^{z} (z, u_i^\theta) \setminus \Delta \left[ ID_{i}^{(z,\theta)} \right].
\]

(47)

**Proof of Lemma 13:** Fix any \( z \in Z \setminus \left[ \arg \max_{z' \in Z} u_i^\theta (z') \cup \arg \min_{z' \in Z} u_i^\theta (z') \right] \). For any \( y \in \Delta [Z] \), we have

\[
\left( \sum_{z' \in SU_i^{(z,\theta)}} y_{z'} \right) \times \min_{z' \in SU_i^{(z,\theta)}} u_i^\theta (z') + \left( \sum_{z' \in SL_i^{(z,\theta)}} y_{z'} \right) \times \min_{z' \in SL_i^{(z,\theta)}} u_i^\theta (z') \leq \sum_{z' \in SU_i^{(z,\theta) \cup SL_i^{(z,\theta)}}} \left[ y_{z'} \times u_i^\theta (z') \right] + \sum_{z' \in SL_i^{(z,\theta)}} \left[ y_{z'} \times u_i^\theta (z') \right].
\]

(48)

Pick any \( y \in L_i^{z} (z, u_i^\theta) \setminus \Delta \left[ ID_{i}^{(z,\theta)} \right] \), we have

\[
\sum_{z' \in SU_i^{(z,\theta)}} \left[ y_{z'} \times u_i^\theta (z') \right] + \sum_{z' \in SL_i^{(z,\theta)}} \left[ y_{z'} \times u_i^\theta (z') \right] \leq \left( \sum_{z' \in SU_i^{(z,\theta) \cup SL_i^{(z,\theta)}}} y_{z'} \right) \times u_i^\theta (z).
\]

(49)

(48) and (49) imply

\[
\left( \sum_{z' \in SU_i^{(z,\theta)}} y_{z'} \right) \times \min_{z' \in SU_i^{(z,\theta)}} u_i^\theta (z') + \left( \sum_{z' \in SL_i^{(z,\theta)}} y_{z'} \right) \times \min_{z' \in SL_i^{(z,\theta)}} u_i^\theta (z') \leq \left( \sum_{z' \in SU_i^{(z,\theta) \cup SL_i^{(z,\theta)}}} y_{z'} \right) \times u_i^\theta (z),
\]

or equivalently,

\[
\sum_{z \in SU_i^{(z,\theta) \cup SL_i^{(z,\theta)}}} y_{z} \leq \frac{u_i^\theta (z) - \min_{z' \in SL_i^{(z,\theta)}} u_i^\theta (z')}{\min_{z' \in SU_i^{(z,\theta)}} u_i^\theta (z') - \min_{z' \in SL_i^{(z,\theta)}} u_i^\theta (z')} = \rho_{(i,z)},
\]

i.e., (46) holds.
Furthermore, for any \( y \in \triangle [Z] \), we have

\[
\sum_{z' \in SU_i^{(z,\theta)}} \left[ y_{z'} \times u_i^\theta (z') \right] + \sum_{z' \in SL_i^{(z,\theta)}} \left[ y_{z'} \times u_i^\theta (z') \right] \leq \sum_{z' \in SU_i^{(z,\theta)}} y_{z'} \times \max_{z'' \in SU_i^{(z,\theta)}} u_i^\theta (z') + \sum_{z' \in SL_i^{(z,\theta)}} y_{z'} \times \max_{z'' \in SL_i^{(z,\theta)}} u_i^\theta (z').
\]  

(50)

Pick any \( y \in \{ \bar{y} \in \triangle [Z] \setminus \triangle \left[ ID_i^{(z,\theta)} \right] : \frac{\sum_{z' \in SU_i^{(z,\theta)}} y_{z'}}{\sum_{z' \in SU_i^{(z,\theta)} \cup SL_i^{(z,\theta)}} y_{z'}} \leq \rho_{(i,z)} \} \), we have

\[
\frac{\sum_{z' \in SU_i^{(z,\theta)} \cup SL_i^{(z,\theta)}} y_{z'}}{\sum_{z' \in SU_i^{(z,\theta)} \cup SL_i^{(z,\theta)}} y_{z'}} \leq \rho_{(i,z)} = \frac{u_i^\theta (z) - \max_{z' \in SL_i^{(z,\theta)}} u_i^\theta (z')}{\max_{z' \in SU_i^{(z,\theta)}} u_i^\theta (z') - \max_{z' \in SL_i^{(z,\theta)}} u_i^\theta (z')},
\]

or equivalently,

\[
\left( \sum_{z \in SU_i^{(z,\theta)}} y_z \right) \times \max_{z' \in SU_i^{(z,\theta)}} u_i^\theta (z') + \left( \sum_{z' \in SL_i^{(z,\theta)}} y_{z'} \right) \times \max_{z' \in SL_i^{(z,\theta)}} u_i^\theta (z') \leq \left( \sum_{z' \in SU_i^{(z,\theta)} \cup SL_i^{(z,\theta)}} y_{z'} \right) \times u_i^\theta (z),
\]  

(51)

and (50) and (51) imply

\[
\sum_{z' \in SU_i^{(z,\theta)}} \left[ y_{z'} \times u_i^\theta (z') \right] + \sum_{z' \in SL_i^{(z,\theta)}} \left[ y_{z'} \times u_i^\theta (z') \right] \leq \left( \sum_{z' \in SU_i^{(z,\theta)} \cup SL_i^{(z,\theta)}} y_{z'} \right) \times u_i^\theta (z),
\]

and as a result, \( y \in L_i^\gamma (z, u^\theta) \setminus \triangle \left[ ID_i^{(z,\theta)} \right] \) i.e., (47) holds.■

**Proof of Lemma 5**: Fix any \( \theta \in \Theta^* \) and pick any \( \overline{u}^\theta \in \Omega^{[\gamma, \Theta]} \). For each \( i \in I \), define \( r_i^\theta : Z \rightarrow \mathbb{R} \) as follows.

\[
r_i^\theta (z) \equiv \left| \left\{ \overline{u}^\theta_i (z') : z' \in SL_i^{2} \right\} \right| + 1
\]

That is,

\[
\begin{align*}
r_i^\theta (z) &= 1 \text{ if } \overline{u}^\theta_i (z) \text{ achieves the lowest value in } \overline{u}^\theta_i (Z) ; \\
r_i^\theta (z) &= 2 \text{ if } \overline{u}^\theta_i (z) \text{ achieves the second-lowest value in } \overline{u}^\theta_i (Z) ; \\
& \quad \ldots
\end{align*}
\]

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Clearly, \((\vartheta^i)_i \in \Omega[\vartheta^\theta, Q]\).

We now define \(\hat{\vartheta} \in \Omega[\vartheta^\theta, Q]\) such that (21) holds.

\[
\hat{\vartheta}^i(z) = \vartheta^i(z) \times 10^{\vartheta^i(z) \times n},
\]

where \(n\) is a positive integer which will be determined later.

We first consider \(z \in Z \setminus \left[ \arg\max_{z' \in Z} u^i_{\vartheta} (z') \cup \arg\min_{z' \in Z} u^i_{\vartheta} (z') \right]\). As \(n \to \infty\), we have

\[
\frac{\hat{\vartheta}^i(z) - \min_{z' \in U^\theta} \hat{\vartheta}^i(z') - \min_{z' \in L^\vartheta} \hat{\vartheta}^i(z')} {\min_{z' \in U^\theta} \hat{\vartheta}^i(z') - \min_{z' \in L^\vartheta} \hat{\vartheta}^i(z')} \to 0, \forall z \in Z \setminus \left( \arg\max_{z' \in Z} u^i_{\vartheta} (z') \cup \arg\min_{z' \in Z} u^i_{\vartheta} (z') \right).
\]

We thus fix any positive \(n\) such that

\[
\frac{\hat{\vartheta}^i(z) - \min_{z' \in L^\vartheta} \hat{\vartheta}^i(z')}{\min_{z' \in U^\theta} \hat{\vartheta}^i(z') - \min_{z' \in L^\vartheta} \hat{\vartheta}^i(z')} < \frac{\overline{u}^i(z) - \max_{z' \in SL^\vartheta(i)} \overline{u}^i(z')}{\max_{z' \in U^\theta} \overline{u}^i(z') - \max_{z' \in SL^\vartheta(i)} \overline{u}^i(z')},
\]

\(\forall z \in Z \setminus \left[ \arg\max_{z' \in Z} u^i_{\vartheta} (z') \cup \arg\min_{z' \in Z} u^i_{\vartheta} (z') \right]\).

which, together with Lemma 13, implies

\[
L^Y_i(z, \hat{\vartheta}) \setminus \triangle [ID^z_i] \subset L^Y_i(z, \overline{u}) \setminus \triangle [ID^z_i].
\]

and as a result,

\[
L^Y_i(z, \hat{\vartheta}) = \left[ L^Y_i(z, \hat{\vartheta}) \setminus \triangle [ID^z_i] \right] \cup \triangle (ID^z_i) = \left[ L^Y_i(z, \overline{u}) \setminus \triangle [ID^z_i] \right] \cup \triangle (ID^z_i) = L^Y_i(z, \overline{u}).
\]

(52)

Second, consider \(z \in \left[ \arg\max_{z' \in Z} u^i_{\vartheta} (z') \cup \arg\min_{z' \in Z} u^i_{\vartheta} (z') \right]\), and we have

\[
L^Y_i(z, \hat{\vartheta}) = Y = L^Y_i(z, \overline{u}), \forall z \in \arg\max_{z' \in Z} u^i_{\vartheta} (z'),
\]

(53)

\[
L^Y_i(z, \hat{\vartheta}) = \triangle [ID^z_i] = L^Y_i(z, \overline{u}), \forall z \in \arg\min_{z' \in Z} u^i_{\vartheta} (z').
\]

(54)

(52), (53) and (54) prove (21) for \(\hat{\vartheta}\).

We now define \(\hat{\vartheta} \in \Omega[\vartheta^\theta, Q]\) such that (21) holds:

\[
\hat{\vartheta}^i(z) = -\frac{1}{\hat{\vartheta}^i(z)},
\]

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where \( n \) is a positive integer which will be determined later. As \( n \to \infty \), we have

\[
\frac{\tilde{u}^\theta_i(z) - \max_{z' \in SL_i^{(z,\theta)}} \tilde{u}^\theta_i(z')}{\max_{z' \in SU_i^{(z,\theta)}} \tilde{u}^\theta_i(z') - \max_{z' \in SL_i^{(z,\theta)}} \tilde{u}^\theta_i(z')} \to 1, \forall z \in Z \setminus \left[ \arg \max_{z' \in Z} u^\theta_i(z') \cup \arg \min_{z' \in Z} u^\theta_i(z') \right].
\]

We thus fix any positive \( n \) such that

\[
\frac{\bar{u}^\theta_i(z) - \min_{z' \in SU_i^{(z,\theta)}} \bar{u}^\theta_i(z')}{\min_{z' \in SU_i^{(z,\theta)}} \bar{u}^\theta_i(z') - \min_{z' \in SL_i^{(z,\theta)}} \bar{u}^\theta_i(z')} < \frac{\tilde{u}^\theta_i(z) - \max_{z' \in SL_i^{(z,\theta)}} \tilde{u}^\theta_i(z')}{\max_{z' \in SU_i^{(z,\theta)}} \tilde{u}^\theta_i(z') - \max_{z' \in SL_i^{(z,\theta)}} \tilde{u}^\theta_i(z')},
\]

\[\forall z \in Z \setminus \left[ \arg \max_{z' \in Z} u^\theta_i(z') \cup \arg \min_{z' \in Z} u^\theta_i(z') \right].\]

As above, Lemma 13, implies

\[\mathcal{L}_i^Y\left(z, \bar{u}^\theta\right) \subset \mathcal{L}_i^Y\left(z, \tilde{u}^\theta\right), \forall z \in Z \setminus \left[ \arg \max_{z' \in Z} u^\theta_i(z') \cup \arg \min_{z' \in Z} u^\theta_i(z') \right],\]

and

\[\mathcal{L}_i^Y\left(z, \bar{u}^\theta\right) = Y = \mathcal{L}_i^Y\left(z, \tilde{u}^\theta\right), \forall z \in \arg \max_{z' \in Z} \bar{u}^\theta_i(z'),\]

\[\mathcal{L}_i^Y\left(z, \bar{u}^\theta\right) = \Delta \left[ ID^{(z,\theta)} \right] = \mathcal{L}_i^Y\left(z, \tilde{u}^\theta\right), \forall z \in \arg \min_{z' \in Z} \tilde{u}^\theta_i(z').\]

i.e., (21) holds for \( \tilde{u}^\theta \).

\section*{A.6 Proof of Lemma 6}

Suppose that an SCC \( F \) is mixed-Nash-A-implemented by \( \mathcal{M} = (M, g : M \to Y) \). We consider two scenarios: (I) \( Z \) is not an \( i \)-max set for any \( i \in I \) and (II) \( Z \) is an \( i \)-max set for some \( i \in I \). In scenario (I), we have \( Z^* = Z \) by (26), and hence \( g(M) \subset \Delta(Z) = \Delta(Z^*) \). In scenario (II), we have \( Z^* = \bigcup_{\theta \in \Theta} F(\theta) \) by (26). \( Z \) being an \( i \)-max set implies existence of some state \( \theta' \in \Theta \) such that all agents are indifferent between any two elements in \( Z \) at \( \theta' \). Therefore, every \( m \in M \) is a Nash equilibrium at \( \theta' \), and hence

\[
\bigcup_{m \in M} \text{SUPP}[g(m)] \subset F(\theta') \subset \bigcup_{\theta \in \Theta} F(\theta) = Z^*,
\]

i.e., \( g(M) \subset \Delta(Z^*) \).
A.7 Proof of Lemma 7

Suppose that an SCC $F$ is mixed-Nash-A-implemented by $\mathcal{M} = \langle M, g : \mathcal{M} \rightarrow Y \rangle$. Fix any $(i, \theta) \in \mathcal{I} \times \Theta$ and any $\lambda \in \text{MNE}^{(\mathcal{M}, \theta)}$ such that

$$F (\theta) \subset \operatorname{arg\ min}_{z \in Z_i^*} u_i^\theta (z),$$  \hspace{1cm} (55)

$$\Xi_i (\theta) \neq \emptyset,$$

and $Z^* \cap \mathcal{L}_{i}^{Z} (F (\theta), \theta)$ is an $i$-$Z^*$-max set.  \hspace{1cm} (56)

First, with $Z^* \cap \mathcal{L}_{i}^{Z} (F (\theta), \theta)$ being an $i$-$Z^*$-max set, there exists $\tilde{\theta} \in \Theta$ such that $Z^* \cap \mathcal{L}_{i}^{Z} (F (\theta), \theta)$ is an $i$-$Z^*$-$\tilde{\theta}$-max set, i.e.,

$$F (\theta) \subset Z^* \cap \mathcal{L}_{i}^{Z} (F (\theta), \theta) \subset \operatorname{arg\ max}_{z \in Z_i^*} u_i^{\tilde{\theta}} (z), \forall j \in \mathcal{I} \setminus \{i\},$$  \hspace{1cm} (57)

$$F (\theta) \subset Z^* \cap \mathcal{L}_{i}^{Z} (F (\theta), \theta) \subset \operatorname{arg\ max}_{z \in Z^* \cap \mathcal{L}_{i}^{Z} (F (\theta), \theta)} u_i^{\tilde{\theta}} (z).$$  \hspace{1cm} (58)

(55) and Lemma 6 imply

$$g (m_i, \lambda_{-i}) \in \Delta \left[ Z^* \cap \mathcal{L}_{i}^{Z} (F (\theta), \theta) \right], \forall m_i \in M_i, \forall (\lambda_i, \lambda_{-i}) \in \text{MNE}^{(\mathcal{M}, \theta)},$$

which, together with (57) and (58), immediately implies $\text{MNE}^{(\mathcal{M}, \theta)} \subset \text{MNE}^{(\mathcal{M}, \tilde{\theta})}$, and hence, $F (\theta) \subset F (\tilde{\theta})$. Therefore, we have

$$\begin{align*}
\left( \begin{array}{c}
\tilde{\theta} \in \Theta_i^\theta \\
\text{and } Z^* \cap \mathcal{L}_{i}^{Z} (F (\theta), \theta) \text{ is an } i\text{-}Z^*-\tilde{\theta}\text{-max set}
\end{array} \right). \hspace{1cm} (59)
\end{align*}$$

Second, we define an algorithm.

Step 1: let $K^1 \equiv \left\{ \theta' \in \Theta_i^\theta : Z^* \cap \mathcal{L}_{i}^{Z} (F (\theta), \theta) \text{ is an } i\text{-}Z^*-\theta'\text{-max set} \right\}$,

Step 2: let $K^2 \equiv \left\{ \theta' \in \Theta_i^\theta : Z^* \cap \mathcal{L}_{i}^{Z} (F (\theta), \theta) \cap \bigcap_{\theta' \in K^1} F (\theta') \text{ is an } i\text{-}Z^*-\theta'\text{-max set} \right\}$,

$$\ldots$$

Step $n$: let $K^n \equiv \left\{ \theta' \in \Theta_i^\theta : Z^* \cap \mathcal{L}_{i}^{Z} (F (\theta), \theta) \cap \bigcap_{\theta' \in K^{n-1}} F (\theta') \text{ is an } i\text{-}Z^*-\theta'\text{-max set} \right\}$,

$$\ldots$$

(60)
(59) implies $\bar{\theta} \in K^1 \neq \emptyset$. Furthermore, we have $K^1 \subset K^2$, and inductively, it is easy to show

$$\forall n \geq 1, \quad K^n \subset K^{n+1}$$

and

$$Z^* \cap \mathcal{L}^Z_i (F (\theta), \theta) \cap \left[ \bigcap_{\theta' \in K^n} F (\theta') \right] \supset Z^* \cap \mathcal{L}^Z_i (F (\theta), \theta) \cap \left[ \bigcap_{\theta' \in K^{n+1}} F (\theta') \right]. \quad (61)$$

Since $Z$ is finite, (61) implies that there exists $n \geq 1$ such that

$$Z^* \cap \mathcal{L}^Z_i (F (\theta), \theta) \cap \left[ \bigcap_{\theta' \in K^n} F (\theta') \right] = Z^* \cap \mathcal{L}^Z_i (F (\theta), \theta) \cap \left[ \bigcap_{\theta' \in K^{n+1}} F (\theta') \right]. \quad (62)$$

and as a result,

$$K^{n+1} \equiv \left\{ \theta' \in \Theta^\theta_i : Z^* \cap \mathcal{L}^Z_i (F (\theta), \theta) \cap \left[ \bigcap_{\theta' \in K^n} F (\theta') \right] \text{ is an } i\text{-}Z^*\text{-}\theta'\text{-max set} \right\}$$

where the second inequality follows from (62). Therefore, $\emptyset \neq K^1 \subset K^{n+1}$ and

$$K^{n+1} \in \Xi_i (\theta) \neq \emptyset. \quad (63)$$

Third, we aim to prove

$$\bigcup_{m_i \in M_i} \text{SUPP} [g (m_i, \lambda_{-i})] \subset Z^* \cap \mathcal{L}^Z_i (F (\theta), \theta) \cap \left[ \bigcup_{\mathcal{K} \in \Xi_i (\theta)} \bigcap_{\theta' \in \mathcal{K}} F (\theta') \right]. \quad (64)$$

(63) and (64) implies that it suffices to prove

$$\bigcup_{m_i \in M_i} \text{SUPP} [g (m_i, \lambda_{-i})] \subset Z^* \cap \mathcal{L}^Z_i (F (\theta), \theta) \cap \left[ \bigcap_{\theta' \in K^{n+1}} F (\theta') \right]. \quad (65)$$

We prove (65) inductively. First,

$$K^1 \equiv \left\{ \theta' \in \Theta^\theta_i : Z^* \cap \mathcal{L}^Z_i (F (\theta), \theta) \text{ is an } i\text{-}Z^*\text{-}\theta'\text{-max set} \right\},$$

and as a result,

$$g (m_i, \lambda_{-i}) \in \text{MNE} (M, \theta'), \forall m_i \in M_i, \forall \theta' \in K^1,$$
and hence,
\[ g(m_i, \lambda_{-i}) \in \Delta [F(\theta')], \forall m_i \in M_i, \forall \theta' \in K^1, \]
which, together with Lemma 6, implies
\[
\bigcup_{m_i \in M_i} \text{SUPP}[g(m_i, \lambda_{-i})] \subset \left[ Z^* \cap L^2_i (F(\theta), \theta) \cap \left( \bigcap_{\theta' \in K^1} F(\theta') \right) \right].
\]
This completes the first step of the induction.

Suppose that for some \( t < k + 1 \), we have proved
\[
\bigcup_{m_i \in M_i} \text{SUPP}[g(m_i, \lambda_{-i})] \subset \left[ Z^* \cap L^2_i (F(\theta), \theta) \cap \left( \bigcap_{\theta' \in K^1} F(\theta') \right) \right].
\]
Consider
\[
K^{t+1} \equiv \left\{ \theta' \in \Theta_i^\theta : Z^* \cap L^2_i (F(\theta), \theta) \cap \left[ \bigcap_{\theta' \in K^1} F(\theta') \right] \text{ is an } i-Z^*-\theta'\text{-max set} \right\},
\]
and as a result,
\[ g(m_i, \lambda_{-i}) \in MNE(M, \theta'), \forall m_i \in M_i, \forall \theta' \in K^{t+1}, \]
and hence,
\[ g(m_i, \lambda_{-i}) \in \bigcap [F(\theta')], \forall m_i \in M_i, \forall \theta' \in K^{t+1}, \]
which, together with Lemma 6, implies
\[
\bigcup_{m_i \in M_i} \text{SUPP}[g(m_i, \lambda_{-i})] \subset \left[ Z^* \cap L^2_i (F(\theta), \theta) \cap \left( \bigcap_{\theta' \in K^{t+1}} F(\theta') \right) \right].
\]
This completes the induction process, and proves (65). \( \blacksquare \)

A.8 Two lemmas

**Lemma 14** For any \( E \in 2^Z \setminus \{\emptyset\} \) and any \( (\gamma, \eta) \in \Delta^\circ [E] \times \Delta [E] \) and there exist \( \beta \in (0, 1) \) and \( \mu \in \Delta [E] \) such that
\[ \gamma = \beta \times \eta + (1 - \beta) \times \mu. \] (66)
Proof of Lemma 14: With $\alpha \in (0, 1)$, consider $(-\alpha) \times \eta + (1+\alpha) \times \gamma$. Since $\gamma$ is in the interior of $\triangle [E]$, we have
\[
\mu = (-\alpha^*) \times \eta + (1+\alpha^*) \times \gamma \in \triangle^0 [E],
\]
for some sufficiently small $\alpha^* \in (0, 1)$. As a result, we have
\[
\gamma = \frac{\alpha^*}{(1+\alpha^*)} \times \eta + \frac{1}{(1+\alpha^*)} \mu,
\]
i.e., $\beta = \frac{\alpha^*}{(1+\alpha^*)} \in (0, 1)$ and (66) holds.■

Lemma 15 If $\hat{\mathcal{L}}_{Y, A, B}$-monotonicity holds, we have
\[
\begin{cases}
F(\theta) \subset \arg\min_{z \in Z^*} u_i^\theta(z), \\
\text{and } Z^* \cap \hat{\mathcal{L}}_i^Z(F(\theta), \theta) \text{ is an } i\text{-}Z^\text{-max set}
\end{cases} \implies \Xi_i(\theta) \neq \emptyset.
\]

Proof of Lemma 15: Suppose $\hat{\mathcal{L}}_{Y, A, B}$-monotonicity and
\[
\begin{cases}
F(\theta) \subset \arg\min_{z \in Z^*} u_i^\theta(z), \\
\text{and } Z^* \cap \hat{\mathcal{L}}_i^Z(F(\theta), \theta) \text{ is an } i\text{-}Z^\text{-max set}
\end{cases}
\]
hold, and we aim to show $\Xi_i(\theta) \neq \emptyset$.

First, we prove $\Theta_i^\theta \neq \emptyset$. With $Z^* \cap \hat{\mathcal{L}}_i^Z(F(\theta), \theta)$ being an $i\text{-}Z^\text{-max}$ set, there exists $\tilde{\theta} \in \Theta$ such that $Z^* \cap \hat{\mathcal{L}}_i^Z(F(\theta), \tilde{\theta})$ is an $i\text{-}Z^\tilde{\theta}$-max set, i.e.,
\[
Z^* \cap \hat{\mathcal{L}}_i^Z(F(\theta), \theta) \subset \arg\max_{z \in Z^*} u_i^{\tilde{\theta}}(z), \forall j \in \mathcal{I} \setminus \{i\},
\]
\[
Z^* \cap \hat{\mathcal{L}}_i^Z(F(\theta), \theta) \subset \arg\max_{z \in Z^* \cap \hat{\mathcal{L}}_i^Z(F(\theta), \theta)} u_i^{\tilde{\theta}}(z).
\]
As a result, $F(\theta) \subset \arg\min_{z \in Z^*} u_i^\theta(z)$ implies
\[
\hat{\mathcal{L}}_{Y, A, B}^{\theta}(\text{UNIF}[F(\theta)], \theta) \subset \triangle(Z^*) \subset \hat{\mathcal{L}}_j^Y(F(\theta), \tilde{\theta}), \forall j \in \mathcal{I} \setminus \{i\},
\]
\[
\hat{\mathcal{L}}_{Y, A, B}^{\theta}(\text{UNIF}[F(\theta)], \theta) \subset \triangle[Z^* \cap \hat{\mathcal{L}}_i^Z(F(\theta), \theta)] \subset \hat{\mathcal{L}}_i^Y(F(\theta), \tilde{\theta}),
\]
which, together with $\hat{\mathcal{L}}_{Y, A, B}$-monotonicity, implies $F(\theta) \subset F(\tilde{\theta})$. Therefore, we have
\[
\begin{cases}
\tilde{\theta} \in \Theta_i^\theta \neq \emptyset, \\
\text{and } Z^* \cap \hat{\mathcal{L}}_i^Z(F(\theta), \theta) \text{ is an } i\text{-}Z^\tilde{\theta}\text{-max set}
\end{cases},
\]
(67)
Second, we prove $\Xi_i(\theta) \neq \emptyset$. Consider the following algorithm.

$$
\begin{align*}
\text{Step 1: let } K^1 & \equiv \{ \theta' \in \Theta^\theta_i : Z^* \cap L^Z_i (F (\theta), \theta) \text{ is an } i-Z^* - \theta' - \text{max set} \}, \\
\text{Step 2: let } K^2 & \equiv \{ \theta' \in \Theta^\theta_i : Z^* \cap L^Z_i (F (\theta), \theta) \cap \left[ \bigcap_{\theta' \in K^1} F (\theta') \right] \text{ is an } i-Z^* - \theta' - \text{max set} \}, \\
\text{Step } n & \equiv \{ \theta' \in \Theta^\theta_i : Z^* \cap L^Z_i (F (\theta), \theta) \cap \left[ \bigcap_{\theta' \in K^{n-1}} F (\theta') \right] \text{ is an } i-Z^* - \theta' - \text{max set} \}, \\
\end{align*}
$$

(67) implies $\bar{\theta} \in K^1 \neq \emptyset$. Furthermore, we have $K^1 \subseteq K^2$, and inductively, it is easy to show

$$
\forall n \geq 1,
K^n \subseteq K^{n+1} \text{ and } Z^* \cap L^Z_i (F (\theta), \theta) \cap \left[ \bigcap_{\theta' \in K^n} F (\theta') \right] \supset Z^* \cap L^Z_i (F (\theta), \theta) \cap \left[ \bigcap_{\theta' \in K^{n+1}} F (\theta') \right]. \quad (68)
$$

Since $Z$ is finite, (68) implies that there exists $n \geq 1$ such that

$$
Z^* \cap L^Z_i (F (\theta), \theta) \cap \left[ \bigcap_{\theta' \in K^n} F (\theta') \right] = Z^* \cap L^Z_i (F (\theta), \theta) \cap \left[ \bigcap_{\theta' \in K^{n+1}} F (\theta') \right]. \quad (69)
$$

and as a result,

$$
K^{n+1} \equiv \{ \theta' \in \Theta^\theta_i : Z^* \cap L^Z_i (F (\theta), \theta) \cap \left[ \bigcap_{\theta' \in K^n} F (\theta') \right] \text{ is an } i-Z^* - \theta' - \text{max set} \} \\
= \{ \theta' \in \Theta^\theta_i : Z^* \cap L^Z_i (F (\theta), \theta) \cap \left[ \bigcap_{\theta' \in K^{n+1}} F (\theta') \right] \text{ is an } i-Z^* - \theta' - \text{max set} \},
$$

where the second inequality follows from (69). Therefore, we have $K^{n+1} \in \Xi_i (\theta) \neq \emptyset. \blacksquare$

### A.9 Proof of Lemma 8

Suppose that $L^{Y-A-B}$-monotonicity holds. We first we prove

$$
[Z^* \text{ is a } j-Z^* - \theta' - \text{max set}] \implies Z^* \subseteq F (\theta') , \forall (j, \theta') \in I \times \Theta. \quad (70)
$$
Fix any \((j, \theta') \in \mathcal{I} \times \Theta\) such that \(Z^*\) is a \(j\)-\(Z^*\)-\(\theta'\)-max set, and we aim to show \(Z^* \subseteq F(\theta')\).

\(Z^*\) being an \(j\)-\(Z^*\)-\(\theta'\)-max set implies that all agents are indifferent between any two deterministic outcomes in \(Z^*\) at state \(\theta'\). This leads to two implications: (i) \(Z^* = \cup_{\theta \in \Theta} F(\theta)\), because if \(Z^* \neq \cup_{\theta \in \Theta} F(\theta)\), (26) implies \(Z = Z^*\) and \(Z\) is not an \(i\)-max set for any \(i \in \mathcal{I}\), contradicting all agents being indifferent between any two deterministic outcomes in \(Z^* = Z\) at state \(\theta'\); (ii) we have

\[
\tilde{\mathcal{L}}_i^{Y, A-B} \left( \text{UNIF} \left[ F \left( \tilde{\theta} \right) \right], \tilde{\theta} \right) \subseteq \bigtriangleup \left( Z^* \right) = \mathcal{L}_i^{\tilde{Y}} \left( \text{UNIF} \left[ F \left( \tilde{\theta} \right) \right], \theta' \right), \forall \tilde{\theta} \in \Theta, \forall i \in \mathcal{I}, \quad (71)
\]

where the equality follows from all agents being indifferent between any two deterministic outcomes in \(Z^*\) at \(\theta'\). Thus, \(\tilde{\mathcal{L}}_i^{Y, A-B}\)-monotonicity and (71) imply

\[
\cup_{\tilde{\theta} \in \Theta} F \left( \tilde{\theta} \right) \subseteq F(\theta'),
\]

i.e., \(Z^* = \cup_{\tilde{\theta} \in \Theta} F \left( \tilde{\theta} \right) \subseteq F(\theta')\), and (70) holds.

Second, we prove

\[
\left[ \hat{\Gamma}_j^{A-B} (\theta) \text{ is a } j\text{-}Z^*\text{-}\theta'\text{-max set} \right] \implies \hat{\Gamma}_j^{A-B} (\theta) \subseteq F(\theta'), \forall (j, \theta, \theta') \in \mathcal{I} \times \Theta \times \Theta.
\]

Fix any \((j, \theta, \theta') \in \mathcal{I} \times \Theta \times \Theta\) such that \(\hat{\Gamma}_j^{A-B} (\theta)\) is a \(j\)-\(Z^*\)-\(\theta'\)-max set, and we aim to show \(\hat{\Gamma}_j^{A-B} (\theta) \subseteq F(\theta')\). We consider three scenarios:

1. **scenario 1:** \(F(\theta) \subseteq \arg \min_{z \in Z^*} u^\theta_j(z), \quad Z^* \cap \mathcal{L}^Z_j (F(\theta), \theta) \text{ is a } j\text{-}Z^*\text{-max set} \)
2. **scenario 2:** \(F(\theta) \subseteq \arg \min_{z \in Z^*} u^\theta_j(z), \quad Z^* \cap \mathcal{L}^Z_j (F(\theta), \theta) \text{ is not an } j\text{-}Z^*\text{-max set} \)
3. **scenario 3:** \(F(\theta) \setminus \arg \min_{z \in Z^*} u^\theta_j(z) \neq \emptyset\).

By Lemma 15, we have \(\Xi_j(\theta) \neq \emptyset\) if scenario 1 occurs. Given this, (30) implies

\[
\hat{\Gamma}_j^{A-B} (\theta) = \begin{cases} 
Z^* \cap \mathcal{L}^Z_j (F(\theta), \theta) \cap \left( \bigcup_{K \in \Xi_j(\theta)} \bigcap_{\tilde{\theta} \in K} F \left( \tilde{\theta} \right) \right), & \text{if scenario 1 occurs}, \\
Z^* \cap \mathcal{L}^Z_j (F(\theta), \theta), & \text{if scenario 2 occurs}, \\
Z^*, & \text{if scenario 3 occurs}
\end{cases} \quad (72)
\]

Since \(\hat{\Gamma}_j^{A-B} (\theta)\) is a \(j\)-\(Z^*\)-\(\theta'\)-max set, scenario 2 cannot happen.
Suppose scenario 3 occurs, i.e., $\hat{\Gamma}_j^{A-B}(\theta) = Z^*$ is an $j$-$Z^*$-$\theta'$-max set. By (70), we have $\hat{\Gamma}_j^{A-B}(\theta) = Z^* \subset F(\theta')$.

Suppose scenario 1 occurs. We thus have $\Xi_j(\theta) \neq \emptyset$ and

$$\hat{\Gamma}_j^{A-B}(\theta) = Z^* \cap L_j^Z (F(\theta), \theta) \cap \left( \bigcup_{K \in \Xi_j(\theta)} \bigcap_{\tilde{\theta} \in K} F(\tilde{\theta}) \right),$$

and $Z^* \cap L_j^Z (F(\theta), \theta)$ is an $j$-$Z^*$-max set. Pick any $K^* \in \Xi_j(\theta)$. Recall the definition of $\Xi_j(\theta)$ in (29), we have

$$\Theta_j^\theta \cap \left[ \Lambda^{j-Z^*}\Theta \left( Z^* \cap L_j^Z (F(\theta), \theta) \cap \left[ \bigcap_{\tilde{\theta} \in K^*} F(\tilde{\theta}) \right] \right) \right] = K^*. \quad (73)$$

Since

$$Z^* \cap L_j^Z (F(\theta), \theta) \cap \left( \bigcap_{\tilde{\theta} \in K^*} F(\tilde{\theta}) \right) \subset Z^* \cap L_j^Z (F(\theta), \theta) \cap \left( \bigcup_{K \in \Xi_j(\theta)} \bigcap_{\tilde{\theta} \in K} F(\tilde{\theta}) \right) = \hat{\Gamma}_j^{A-B}(\theta),$$

$\hat{\Gamma}_j^{A-B}(\theta)$ being an $j$-$Z^*$-$\theta'$-max set implies $Z^* \cap L_j^Z (F(\theta), \theta) \cap \left( \bigcap_{\tilde{\theta} \in K^*} F(\tilde{\theta}) \right)$ is a $j$-$Z^*$-$\theta'$-max set. As a result, we have

$$\theta' \in \left[ \Lambda^{j-Z^*}\Theta \left( Z^* \cap L_j^Z (F(\theta), \theta) \cap \left[ \bigcap_{\tilde{\theta} \in K^*} F(\tilde{\theta}) \right] \right) \right]. \quad (74)$$

We now show

$$\theta' \in \Theta_j^\theta. \quad (75)$$

Given $F(\theta) \subset \hat{\Gamma}_j^{A-B}(\theta)$, $\hat{\Gamma}_j^{A-B}(\theta)$ being an $j$-$Z^*$-$\theta'$-max set implies two things: (1) $F(\theta)$ is a $j$-$Z^*$-$\theta'$-max set, and (2)

$$\hat{\mathcal{L}}_{i}^{Y-A-B} \left( \text{UNIF} \left[ F(\theta) \right], \theta \right) \subset \bigtriangleup (Z^*) \subset L_i^Y (F(\theta), \theta'), \forall i \in \mathcal{I} \setminus \{j\},$$

$$\hat{\mathcal{L}}_j^Y (F(\theta)), \theta) = \bigtriangleup \left[ \hat{\Gamma}_j^{A-B} (\theta) \right] \subset L_j^Y (F(\theta), \theta'),$$

which, together with $\hat{\mathcal{L}}_j^Y$-$A-B$-monotonicity implies $F(\theta) \subset F(\theta')$. Therefore, (75) holds.
(73), (74) and (75) show \( \theta' \in K^* \). We thus have

\[
\hat{\Gamma}_j^{A-B}(\theta) = Z^* \cap L_j^Z(F(\theta), \theta) \cap \left( \bigcup_{K \in \Xi_i(\theta)} \bigcap_{\hat{\theta} \in K} F(\hat{\theta}) \right)
\]

\[
\subset Z^* \cap L_j^Z(F(\theta), \theta) \cap \left( \bigcap_{\hat{\theta} \in K^*} F(\hat{\theta}) \right)
\]

\[
\subset F(\theta').
\]

where the first ”\( \subset \)” follows from \( K^* \in \Xi_i(\theta) \) and the second ”\( \subset \)” follows from \( \theta' \in K^* \).

\[\blacksquare\]

A.10 Proof of Theorem 7

By their definitions, mixed-Nash-B-implementation implies mixed-Nash-A-implementable, i.e., (ii) \( \implies \) (i). We show (i) \( \implies \) (iii) and (iii) \( \implies \) (ii) in Appendix A.10.1 and A.10.2, respectively.

A.10.1 The proof of (i) \( \implies \) (iii)

Suppose that an SCC \( F \) is mixed-Nash-A-implemented by \( \mathcal{M} = \langle M, g : M \rightarrow Y \rangle \). Fix any \( (\theta, \theta') \in \Theta \times \Theta \) such that

\[
\left[ \begin{array}{c}
\tilde{\mathcal{L}}_i^{Y-A-B}(\text{UNIF}[F(\theta)], \theta) \subset \mathcal{L}_i^{Y}(\text{UNIF}[F(\theta)], \theta'), \\
\forall i \in \mathcal{I}
\end{array} \right], (76)
\]

and we aim to show \( F(\theta) \subset F(\theta') \), i.e., \( \tilde{\mathcal{L}}_i^{Y-A-B} \)-uniform-monotonicity holds. In particular, fix any \( a \in F(\theta) \), and we aim to show \( a \in F(\theta') \). Since \( a \in F(\theta) \), there exists some \( \lambda \in \text{MNE}(\mathcal{M}, \theta) \), such that \( a \) is induced with positive probability by \( \lambda \). Thus, \( \text{SUPP}[g(\lambda)] \subset F(\theta) \). We now prove \( \lambda \in \text{MNE}(\mathcal{M}, \theta') \) by contradiction, which further implies \( a \in F(\theta') \).

Suppose \( \lambda \notin \text{MNE}(\mathcal{M}, \theta') \), i.e., there exist \( i \in \mathcal{I} \), and \( m_i' \in M_i \) such that

\[
U_i^\theta[g(m_i', \lambda_{-i})] \leq U_i^{\theta}[g(\lambda)], \quad (77)
\]

\[
U_i'^{\theta'}[g(m_i', \lambda_{-i})] > U_i'^{\theta}[g(\lambda)]. \quad (78)
\]

Lemma 14 and \( \text{SUPP}[g(\lambda)] \subset F(\theta) \) imply existence of \( \mu \in \triangle [F(\theta)] \) and \( \beta \in (0, 1) \) such that

\[
\text{UNIF}[F(\theta)] = \beta \times g(\lambda) + (1 - \beta) \times \mu. \quad (79)
\]
Thus, (78) and (79) imply

\[ U_\theta^i \left[ \beta \times g \left( m_i', \lambda_{-i} \right) + (1 - \beta) \times \mu \right] > U_\theta^i \left[ \beta \times g (\lambda) + (1 - \beta) \times \mu \right] = U_\theta^i \left( \text{UNIF} [F (\theta)] \right). \]  

(80)

(77) and (79) imply

\[ U_\theta^i \left[ \beta \times g \left( m_i', \lambda_{-i} \right) + (1 - \beta) \times \mu \right] \leq U_\theta^i \left[ \beta \times g (\lambda) + (1 - \beta) \times \mu \right] = U_\theta^i \left( \text{UNIF} [F (\theta)] \right). \]  

(81)

We now consider two cases. First, suppose

\[
\begin{pmatrix}
F (\theta) \subset \arg \min_{z \in Z^*} u_i^\theta (z), \\
\Xi_i (\theta) \neq \emptyset, \\
\text{and } Z^* \cap \mathcal{L}_i^Z (F (\theta), \theta) \text{ is an } i^*-\text{max set}
\end{pmatrix},
\]

(82)

holds. Thus, by (30), we have

\[ \hat{\mathcal{L}}_i^{Y-A-B} \left( \text{UNIF} [F (\theta)], \theta \right) = \triangle \left[ Z^* \cap \mathcal{L}_i^Z (F (\theta), \theta) \cap \left( \bigcup_{K \in \Xi_i (\theta)} \bigcap_{\tilde{\theta} \in K} F (\tilde{\theta}) \right) \right], \]

and by Lemma 7, we have

\[ g \left( m_i', \lambda_{-i} \right) \in \triangle \left[ Z^* \cap \mathcal{L}_i^Z (F (\theta), \theta) \cap \left( \bigcup_{K \in \Xi_i (\theta)} \bigcap_{\tilde{\theta} \in K} F (\tilde{\theta}) \right) \right], \]

which, together with \( \mu \in \triangle [F (\theta)] \), implies\(^{22}\)

\[ \beta \times g \left( m_i', \lambda_{-i} \right) + (1 - \beta) \times \mu \in \hat{\mathcal{L}}_i^{Y-A-B} \left( \text{UNIF} [F (\theta)], \theta \right). \]  

(83)

(76) and (83) imply

\[ \beta \times g \left( m_i', \lambda_{-i} \right) + (1 - \beta) \times \mu \in \mathcal{L}_i^Y \left( \text{UNIF} [F (\theta)], \theta' \right), \]

contradicting (80).

Second, suppose that (82) does not hold, and by (30), we have

\[ \hat{\mathcal{L}}_i^{Y-A-B} \left( \text{UNIF} [F (\theta)], \theta \right) = [\triangle (Z^*)] \cap \mathcal{L}_i^Y \left( \text{UNIF} [F (\theta)], \theta \right), \]

\[^{22}F (\theta) \subset \arg \min_{z \in Z^*} u_i^\theta (z) \text{ implies } \triangle [F (\theta)] \subset \hat{\mathcal{L}}_i^{Y-A-B} \left( \text{UNIF} [F (\theta)], \theta \right).\]
which, together with (81) and Lemma 6, implies
\[
\beta \times g \left( m_i', \lambda_{-i} \right) + (1 - \beta) \times \mu \subset \left[ \triangle \left( Z^* \right) \right] \cap L^Y_i \left( \text{UNIF} [F(\theta)], \theta \right)
\]
\[
= \hat{L}^{Y-A-B}_i \left( \text{UNIF} [F(\theta)], \theta \right). \tag{84}
\]

(76) and (84) imply
\[
\beta \times g \left( m_i', \lambda_{-i} \right) + (1 - \beta) \times \mu \in L^Y_i \left( \text{UNIF} [F(\theta)], \theta' \right).
\]
contradicting (80). ■

A.10.2 The proof of (iii) $\implies$ (ii)

Preliminary construction In order to build our canonical mechanism to implement $F$, we need to take two preliminary constructions. First, for each $(\theta, j) \in \Theta \times I$, fix any function $\psi^j_\theta : \Theta \rightarrow Y$ such that
\[
\psi^j_\theta (\theta') \in \left( \arg \max_{y \in \hat{L}^{Y-A-B}_j (\text{UNIF} [F(\theta)], \theta)} U^j_{\theta'} [y] \right), \forall \theta' \in \Theta, \tag{85}
\]
and by (31), we have
\[
\psi^j_\theta (\theta') \in \left( \arg \max_{y \in \hat{L}^{Y-A-B}_j (\text{UNIF} [F(\theta)], \theta)} U^j_{\theta'} [y] \right) \cap \triangle \left( \hat{\Gamma}^{A-B}_j (\theta) \right), \forall \theta' \in \Theta. \tag{86}
\]
The following lemma completes our second construction.

Lemma 16 For each $(\theta, j) \in \Theta \times I$, there exist
\[
\epsilon^j_\theta > 0 \text{ and } y^j_\theta \in \hat{L}^{Y-A-B}_j (\text{UNIF} [F(\theta)], \theta),
\]
such that
\[
\left[ \epsilon^j_\theta \times y + \left( 1 - \epsilon^j_\theta \right) \times y^j_\theta \right] \in \hat{L}^{Y-A-B}_j (\text{UNIF} [F(\theta)], \theta), \forall y \in \triangle \left( \hat{\Gamma}^{A-B}_j (\theta) \right). \tag{87}
\]
The proof of Lemma 16 is similar to that of Lemma 3, and we omit it.
A canonical mechanism

Let \( \mathbb{N} \) denote the set of positive integers. We use the mechanism \( M^{A-B} = \{ M^{A-B}_i : i \in \mathcal{I} \} \) defined below to implement \( F \). In particular, we have

\[
M^{A-B}_i = \left\{ \left( \theta_i, k^2_i, k^3_i, \gamma_i, b_i \right) \in \Theta \times \mathbb{N} \times \mathbb{N} \times (\mathbb{Z}^*)^2 \times \mathbb{Z}^* : \gamma_i(E) \in E, \forall E \in (\mathbb{Z}^*)^2 \right\}, \forall i \in \mathcal{I},
\]

and \( m = (m_i)_{i \in \mathcal{I}} = (\theta_i, k^2_i, k^3_i, \gamma_i, b_i)_{i \in \mathcal{I}} \) is defined in three cases.

Case (1): consensus

If there exists \( \theta \in \Theta \) such that

\[
\left( \theta_i, k^2_i \right) = (\theta, 1), \ \forall i \in \mathcal{I},
\]

then \( g[m] = \text{UNIF}[F(\theta)] \);

Case (2), unilateral deviation:

If there exists \( (\theta, j) \in \Theta \times \mathcal{I} \) such that

\[
\left( \theta_i, k^2_i \right) = (\theta, 1) \text{ if and only if } i \in \mathcal{I} \setminus \{j\},
\]

then

\[
g[m] = \left( 1 - \frac{1}{k^2_j} \right) \times \psi^\theta_j(\theta_j)
\]

\[
+ \frac{1}{k^2_j} \times \left( \varepsilon^\theta_j \times \left( 1 - \frac{1}{k^2_j} \right) \times \gamma_j(\hat{\Gamma}^{A-B}(\theta)) + \frac{1}{k^2_j} \times \text{UNIF}(\hat{\Gamma}^{A-B}(\theta)) \right),
\]

where \( \left( \varepsilon^\theta_j, y^\theta_j \right) \) are chosen for each \( (\theta, j) \in \Theta \times \mathcal{I} \) according to Lemma 16;

Case (3), multi-lateral deviation:

Otherwise,

\[
g[m] = \left( 1 - \frac{1}{k^2_{j^*}} \right) \times b_{j^*} + \frac{1}{k^2_{j^*}} \times \text{UNIF}(\mathbb{Z}^*),
\]

where \( j^* = \max \arg \max_{i \in \mathcal{I}} k^2_i \), i.e., \( j^* \) is the largest-numbered agent who submits the highest number on the second dimension of the message.

Lemma 17

Consider the canonical mechanism \( M^{A-B} \) above. For any \( \theta \in \Theta \) and any \( \lambda \in \text{MNE}(M^{A-B}, \theta) \), we have \( \text{SUPP}[\lambda] \subset \text{PNE}(M^{A-B}, \theta) \).

The proof of Lemma 17 is similar to that of Lemma 4, and we omit it.
(iii) ⇒ (ii) in Theorem 7: a proof  Suppose that $\hat{L}^{Y-A,B}$-monotonicity holds. Fix any true state $\theta^* \in \Theta$. We aim to prove

$$\bigcup_{\lambda \in MNE(\lambda^{A,B}, \theta^*)} \text{SUPP}(g[\lambda]) = F(\theta^*).$$

First, truth revealing is a Nash equilibrium, i.e., any pure strategy profile

$$m^* = \left(\theta_i = \theta^*, k^2_i = 1, * , * , * \right)_{i \in I}$$

is a Nash equilibrium, which triggers Case (1) and $g[m^*] = \text{UNIF}[F(\theta^*)]$. Any unilateral deviation $m_j \in M_j^{A,B}$ of agent $j \in I$ would either still trigger Case (1) and induce $\text{UNIF}[F(\theta^*)]$, or trigger Case (2) and induce $g[m_j, m^*_{-j}] \in \hat{L}^{Y-A,B}(\text{UNIF}[F(\theta^*)], \theta^*) \subset \hat{L}_j^Y(\text{UNIF}[F(\theta^*)], \theta^*), \forall m_j \in M_j$.

Therefore, any $m_j \in M_j$ is not a profitable deviation.

Second, by Lemma 17, it suffers no loss of generality to focus on pure-strategy equilibria. Fix any $\tilde{m} = \left(\tilde{\theta}_i, k^2_i, k^3_i, \tilde{\gamma}_i, \tilde{b}_i \right)_{i \in I} \in \text{PNE}(\lambda^{A,B}, \theta^*)$,

and we aim to prove $g[\tilde{m}] \in \triangle [F(\theta^*)]$.

$\tilde{m}$ may trigger either Case (1) or Case (2) or Case (3). We first consider the scenarios in which $\tilde{m}$ triggers Case (1), i.e.,

$$\tilde{m} = \left(\tilde{\theta}_i = \tilde{\theta}, k^2_i = 1, k^3_i, \tilde{\gamma}_i, \tilde{b}_i \right)_{i \in I} \text{ for some } \tilde{\theta} \in \Theta,$$

and $g[\tilde{m}] = \text{UNIF}[F(\tilde{\theta})]$. We now show $F(\tilde{\theta}) \subset F(\theta^*)$ by contradiction. Suppose otherwise. By $\hat{L}^{Y-A,B}$-monotonicity, there exists $j \in I$ such that

$$\exists y^* \in \hat{L}_j^{Y-A,B} \left(\text{UNIF}[F(\tilde{\theta})], \tilde{\theta} \right) \setminus \hat{L}_j^Y \left(\text{UNIF}[F(\tilde{\theta})], \theta^* \right),$$

which, together with (86), implies

$$U_j^{\theta^*} \left[\psi_j(\theta^*) \right] > U_j^{\theta^*} \left[ y^* \right] > U_j^{\theta^*} \left(\text{UNIF}[F(\tilde{\theta})] \right) = U_j^{\theta^*} (g[\tilde{m}]).$$

Therefore, it is strictly profitable for agent $j$ to deviate to

$$m_j = \left(\theta^*, k^2_j, k^3_j, \tilde{\gamma}_j, \tilde{b}_j \right) \text{ for sufficiently large } k^2_j,$$
contradicting \( \tilde{m} \in PNE(M^{A-B}, \theta^*) \).

Consider the scenarios in which \( \tilde{m} \) triggers Case (2), i.e., there exists \( j \in \mathcal{I} \) such that
\[
\exists \tilde{\theta} \in \Theta, \tilde{m}_i = \left( \tilde{\theta}_i = \tilde{\theta}, \tilde{k}_i^2 = 1, \tilde{k}_i^3, \tilde{\gamma}_i, \tilde{\nu}_i \right), \forall i \in \mathcal{I} \setminus \{j\},
\]
and
\[
g[\tilde{m}] = \left( 1 - \frac{1}{k_j^2} \right) \times \psi_j(\tilde{\theta}_j)
+ \frac{1}{k_j^2} \times \left[ \epsilon_j^\phi \times \left( 1 - \frac{1}{k_j^3} \right) \times \gamma_j(\tilde{\Gamma}_j^{A-B}(\tilde{\theta})) + \frac{1}{k_j^3} \times \text{UNIF}(\tilde{\Gamma}_j^{A-B}(\tilde{\theta})) \right].
\]

We now prove \( g[\tilde{m}] \in \Delta[F(\theta^*)] \). By our construction,
\[
g[\tilde{m}] \in \Delta[\tilde{\Gamma}_j^{A-B}(\tilde{\theta})].
\]

Since every \( i \in \mathcal{I} \setminus \{j\} \) can deviate to trigger Case (3), and dictate her top outcome in \( Z^* \) with arbitrarily high probability, \( \tilde{m} \in PNE(M^{A-B}, \theta^*) \) implies
\[
\tilde{\Gamma}_j^{A-B}(\tilde{\theta}) \subset \arg\max_{z \in Z^*} u_i^{\theta^*}(z), \forall i \in \mathcal{I} \setminus \{j\}.
\]

Inside the the compound lottery \( g[\tilde{m}] \) in (90), conditional on an event with probability \( \frac{1}{k_j^2} \times \epsilon_j^\phi \), we have the compound lottery
\[
\left[ \left( 1 - \frac{1}{k_j^3} \right) \times \tilde{\gamma}_j(\tilde{\Gamma}_j^{A-B}(\tilde{\theta})) + \frac{1}{k_j^3} \times \text{UNIF}(\tilde{\Gamma}_j^{A-B}(\tilde{\theta})) \right],
\]
and hence, agent \( j \) can always deviate to
\[
m_j = \left( \tilde{\theta}_j, \tilde{k}_j^2, \tilde{k}_j^3, \tilde{\gamma}_j, \tilde{\nu}_j \right)_{i \in \mathcal{I} \setminus \{j\}} \text{ with } \gamma_j(\tilde{\Gamma}_j^{A-B}(\tilde{\theta})) \in \arg\max_{z \in \tilde{\Gamma}_j^{A-B}(\tilde{\theta})} u_i^{\theta^*}(z)
\]
for sufficiently large \( k_j^3 \). Thus, \( \tilde{m} \in PNE(M^{A-B}, \theta^*) \) implies
\[
\tilde{\Gamma}_j^{A-B}(\tilde{\theta}) \subset \arg\max_{z \in \tilde{\Gamma}_j^{A-B}(\tilde{\theta})} u_i^{\theta^*}(z).
\]

(92) and (93) imply that \( \tilde{\Gamma}_j^{A-B}(\tilde{\theta}) \) is a \( j \)-\( Z^* - \theta^* \)-max set, which together Lemma 8, further implies
\[
\tilde{\Gamma}_j^{A-B}(\tilde{\theta}) \subset F(\theta^*).
\]
(91) and (94) imply \( g [\tilde{m}] \in \triangle [F (\theta^*)] \).

Finally, consider the scenarios in which \( \tilde{m} \) triggers Case (3), i.e.,

\[
g [\tilde{m}] = \left( 1 - \frac{1}{k^2_{j^*}} \right) \times \tilde{b}_{j^*} + \frac{1}{k^2_{j^*}} \times \text{UNIF} (Z^*),
\]

where \( j^* = \max \left( \arg \max_{i \in I} \tilde{k}^2_i \right) \). Since every \( i \in I \) can increase their integer in the second dimension and dictate her top outcome in \( Z^* \) with arbitrarily high probability, \( \tilde{m} \in \text{PNE} (M_{A-B}, \theta^*) \) implies

\[
Z^* \subset \arg \max_{z \in Z^*} u_i^\theta (z), \forall i \in I,
\]

i.e., \( Z^* \) is a \( j \)-\( Z^* \)-\( \theta^* \)-max set, which together Lemma 8, further implies

\[
Z^* \subset F (\theta^*).
\]

Therefore, \( g [\tilde{m}] \in \triangle (Z^*) \subset \triangle [F (\theta^*)] \). \( \blacksquare \)

### A.11 Proof of Theorem 4

By their definitions, we have (iii) \( \Rightarrow \) (i). Theorem 7 implies (ii) \( \Rightarrow \) (iii). We will show (i) \( \Rightarrow \) (ii) to complete the proof. We need two additional lemmas, before proving “(i) \( \Rightarrow \) (ii).”

**Lemma 18** Suppose that an SCC \( F \) is pure-Nash-implementable by \( M = (M, g : M \rightarrow Y) \). We have \( g (M) \subset \triangle (Z^*) \).

**Proof of Lemma 18:** Suppose that an SCC \( F \) is pure-Nash-implemented by \( M = (M, g : M \rightarrow Y) \). We consider two scenarios: (I) \( Z \) is not an \( i \)-max set for any \( i \in I \) and (II) \( Z \) is an \( i \)-max set for some \( i \in I \). In scenario (I), we have \( Z^* = Z \) by (26), and hence \( g (M) \subset \triangle (Z) = \triangle (Z^*) \). In scenario (II), we have \( Z^* = \bigcup_{\theta \in \Theta} F (\theta) \) by (26). \( Z \) being an \( i \)-max set implies existence of some state \( \theta' \in \Theta \) such that all agents are indifferent between any two elements in \( Z \) at \( \theta' \). Therefore, every \( m \in M \) is a pure-strategy Nash equilibrium at \( \theta' \), and hence

\[
\bigcup_{m \in M} \text{SUPP} [g (m)] \subset F (\theta') \subset \bigcup_{\theta \in \Theta} F (\theta) = Z^*,
\]

i.e., \( g (M) \subset \triangle (Z^*) \). \( \blacksquare \)
Lemma 19 Suppose that an SCC $F$ is pure-Nash-implemented by $\mathcal{M} = (M, g : M \rightarrow Y)$. For any $(i, \theta) \in \mathcal{I} \times \Theta$ and any $\lambda \in \text{PNE}(\mathcal{M}, \theta)$, we have

$$
F(\theta) \subset \arg\min_{z \in Z^*} u^\theta_i(z),
$$

$\Xi_i(\theta) \neq \emptyset$ and

and $Z^* \cap \mathcal{L}^Z_i(F(\theta), \theta)$ is an $i$-$Z^*$-max set

$$
\implies \bigcup_{m_i \in M_i} \text{SUPP}[g(m_i, \lambda_{-i})] \subset \left[ Z^* \cap \mathcal{L}^Z_i(F(\theta), \theta) \cap \left( \bigcup_{E \in \Xi_i(\theta)} \bigcap_{\theta' \in E} F(\theta') \right) \right].
$$

Proof of Lemma 19: Suppose that an SCC $F$ is pure-Nash-implemented by $\mathcal{M} = (M, g : M \rightarrow Y)$. Fix any $(i, \theta) \in \mathcal{I} \times \Theta$ and any $\lambda \in \text{PNE}(\mathcal{M}, \theta)$ such that

$$
F(\theta) \subset \arg\min_{z \in Z^*} u^\theta_i(z),
$$

$\Xi_i(\theta) \neq \emptyset$,

and $Z^* \cap \mathcal{L}^Z_i(F(\theta), \theta)$ is an $i$-$Z^*$-max set.

First, with $Z^* \cap \mathcal{L}^Z_i(F(\theta), \theta)$ being an $i$-$Z^*$-max set, there exists $\tilde{\theta} \in \Theta$ such that $Z^* \cap \mathcal{L}^Z_i(F(\theta), \theta)$ is an $i$-$Z^*$-$\tilde{\theta}$-max set, i.e.,

$$
F(\theta) \subset Z^* \cap \mathcal{L}^Z_i(F(\theta), \theta) \subset \arg\max_{z \in Z^*} u^\tilde{\theta}_j(z), \forall j \in \mathcal{I} \setminus \{i\},
$$

$$
F(\theta) \subset Z^* \cap \mathcal{L}^Z_i(F(\theta), \theta) \subset \arg\max_{z \in Z^* \cap \mathcal{L}^Z_i(F(\theta), \theta)} u^\tilde{\theta}_i(z),
$$

which immediately implies $\text{PNE}(\mathcal{M}, \theta) \subset \text{PNE}(\mathcal{M}, \tilde{\theta})$, and hence, $F(\theta) \subset F(\tilde{\theta})$. Therefore, we have

$$
\tilde{\theta} \in \Theta^\theta_i \neq \emptyset,
$$

and $Z^* \cap \mathcal{L}^Z_i(F(\theta), \theta)$ is an $i$-$Z^*$-$\tilde{\theta}$-max set

$$
\implies \bigcup_{m_i \in M_i} \text{SUPP}[g(m_i, \lambda_{-i})] \subset \left[ Z^* \cap \mathcal{L}^Z_i(F(\theta), \theta) \cap \left( \bigcup_{E \in \Xi_i(\theta)} \bigcap_{\theta' \in E} F(\theta') \right) \right].
$$

(95)
Second, we define an algorithm.

\[
\begin{align*}
\text{Step 1: let } K^1 &\equiv \big\{ \theta' \in \Theta_i^\theta : Z^* \cap Z_i^* (F(\theta), \theta) \text{ is an } i-Z^* - \theta'\text{-max set} \big\}, \\
\text{Step 2: let } K^2 &\equiv \big\{ \theta' \in \Theta_i^\theta : Z^* \cap Z_i^* (F(\theta), \theta) \cap \bigcap_{\theta' \in K^1} F(\theta') \text{ is an } i-Z^* - \theta'\text{-max set} \big\}, \\
\ldots \text{ Step } n &\equiv \big\{ \theta' \in \Theta_i^\theta : Z^* \cap Z_i^* (F(\theta), \theta) \cap \bigcap_{\theta' \in K^{n-1}} F(\theta') \text{ is an } i-Z^* - \theta'\text{-max set} \big\}, \\
\ldots \end{align*}
\]  
(96)

(95) implies \( \bar{\theta} \in K^1 \neq \emptyset \). Furthermore, inductively, it is easy to show

\[
\forall n \geq 1, \quad K^n \subset K^{n+1}, \quad Z^* \cap Z_i^* (F(\theta), \theta) \cap \bigcap_{\theta' \in K^n} F(\theta') \supset Z^* \cap Z_i^* (F(\theta), \theta) \cap \bigcap_{\theta' \in K^{n+1}} F(\theta'). 
\]  
(97)

Since \( Z \) is finite, (97) implies that there exists \( n \geq 1 \) such that

\[
Z^* \cap Z_i^* (F(\theta), \theta) \cap \bigcap_{\theta' \in K^n} F(\theta') = Z^* \cap Z_i^* (F(\theta), \theta) \cap \bigcap_{\theta' \in K^{n+1}} F(\theta'). 
\]  
(98)

and as a result,

\[
K^{n+1} \equiv \big\{ \theta' \in \Theta_i^\theta : Z^* \cap Z_i^* (F(\theta), \theta) \cap \bigcap_{\theta' \in K^n} F(\theta') \text{ is an } i-Z^* - \theta'\text{-max set} \big\} = \big\{ \theta' \in \Theta_i^\theta : Z^* \cap Z_i^* (F(\theta), \theta) \cap \bigcap_{\theta' \in K^{n+1}} F(\theta') \text{ is an } i-Z^* - \theta'\text{-max set} \big\},
\]

where the second inequality follows from (98). Therefore, \( \emptyset \neq K^1 \subset K^{n+1} \) and

\[
K^{n+1} \in \Xi_i (\theta) \neq \emptyset. 
\]  
(99)

Third, we aim to prove

\[
\bigcup_{m_i \in M_i} \text{SUPP} [g(m_i, \lambda_{-i})] \subset \left[ Z^* \cap Z_i^* (F(\theta), \theta) \cap \left( \bigcup_{K \in \Xi_i (\theta)} \bigcap_{\theta' \in K} F(\theta') \right) \right]. 
\]  
(100)
(99) and (100) implies that it suffices to prove
\[
\bigcup_{m_i \in M_i} \text{SUPP} [g (m_i, \lambda_{-i})] \subset \left[ Z^* \cap L^Z_i (F (\theta), \theta) \cap \left( \bigcap_{\theta' \in K^{i+1}} F (\theta') \right) \right].
\] (101)

We prove (101) inductively. First,
\[
K^1 \equiv \left\{ \theta' \in \Theta_i^\theta : Z^* \cap L^Z_i (F (\theta), \theta) \text{ is an } i-Z^*-\theta'-\text{max set} \right\},
\]
and as a result,
\[
g (m_i, \lambda_{-i}) \in \text{PNE} (\mathcal{M}, \theta'), \forall m_i \in M_i, \forall \theta' \in K^1,
\]
and hence,
\[
g (m_i, \lambda_{-i}) \in \Delta [F (\theta')], \forall m_i \in M_i, \forall \theta' \in K^1,
\]
which, together with Lemma 6, implies
\[
\bigcup_{m_i \in M_i} \text{SUPP} [g (m_i, \lambda_{-i})] \subset \left[ Z^* \cap L^Z_i (F (\theta), \theta) \cap \left( \bigcap_{\theta' \in K^1} F (\theta') \right) \right].
\]
This completes the first step of the induction.

Suppose that for some \( t < k + 1 \), we have proved
\[
\bigcup_{m_i \in M_i} \text{SUPP} [g (m_i, \lambda_{-i})] \subset \left[ Z^* \cap L^Z_i (F (\theta), \theta) \cap \left( \bigcap_{\theta' \in K^1} F (\theta') \right) \right].
\]
Consider
\[
K^{t+1} \equiv \left\{ \theta' \in \Theta_i^\theta : Z^* \cap L^Z_i (F (\theta), \theta) \cap \left[ \bigcap_{\theta' \in K^t} F (\theta') \right] \text{ is an } i-Z^*-\theta'-\text{max set} \right\},
\]
and as a result,
\[
g (m_i, \lambda_{-i}) \in \text{PNE} (\mathcal{M}, \theta'), \forall m_i \in M_i, \forall \theta' \in K^{t+1},
\]
and hence,
\[
g (m_i, \lambda_{-i}) \in \Delta [F (\theta')], \forall m_i \in M_i, \forall \theta' \in K^{t+1},
\]
which, together with Lemma 6, implies
\[
\bigcup_{m_i \in M_i} \text{SUPP} [g (m_i, \lambda_{-i})] \subset \left[ Z^* \cap L^Z_i (F (\theta), \theta) \cap \left( \bigcap_{\theta' \in K^{t+1}} F (\theta') \right) \right].
\]
This completes the induction process, and proves (101).•
Proof of Theorem ”(i)⇒(ii)” in Theorem 4: Suppose that an SCC $F$ is pure-Nash-implemented by $\mathcal{M} = \langle M, g : M \rightarrow Y \rangle$. By Theorem 7, it suffices to show $\hat{\mathcal{L}}_{i}^{Y-A-B}$-uniform-monotonicity. Fix any $(\theta, \theta') \in \Theta \times \Theta$ such that

$$\left[ \hat{\mathcal{L}}_{i}^{Y-A-B} (\text{UNIF} [ F (\theta) ], \theta) \subset \mathcal{L}_{i}^{Y} (\text{UNIF} [ F (\theta) ], \theta'), \right.$$ \hspace{1cm} \forall i \in I \tag{102}$$

and we aim to show $F (\theta) \subset F (\theta')$, i.e., $\hat{\mathcal{L}}_{i}^{Y-A-B}$-uniform-monotonicity holds. In particular, fix any $a \in F (\theta)$, and we aim to show $a \in F (\theta')$. Since $a \in F (\theta)$, there exists some $\lambda \in PNE^{(M, \theta)}$, such that $a$ is induced with positive probability by $\lambda$. Thus, $\text{SUPP} [ g (\lambda) ] \subset F (\theta)$. We now prove $\lambda \in PNE^{(M, \theta')}$ by contradiction, which further implies $a \in F (\theta')$.

Suppose $\lambda \notin PNE^{(M, \theta')}$, i.e., there exist $i \in I$, and $m_{i}' \in M_{i}$ such that

$$U_{i}^{\theta} [ g (m_{i}', \lambda_{-i}) ] \leq U_{i}^{\theta} [ g (\lambda) ], \tag{103}$$

$$U_{i}^{\theta'} [ g (m_{i}', \lambda_{-i}) ] > U_{i}^{\theta} [ g (\lambda) ]. \tag{104}$$

Lemma 14 and $\text{SUPP} [ g (\lambda) ] \subset F (\theta)$ imply existence of $\mu \in \Delta [ F (\theta) ]$ and $\beta \in [0, 1]$ such that

$$\text{UNIF} [ F (\theta) ] = \beta \times g (\lambda) + (1 - \beta) \times \mu. \tag{105}$$

Thus, (104) and (105) imply

$$U_{i}^{\theta'} [ \beta \times g (m_{i}', \lambda_{-i}) + (1 - \beta) \times \mu ] > U_{i}^{\theta} [ \beta \times g (\lambda) + (1 - \beta) \times \mu ] = U_{i}^{\theta'} (\text{UNIF} [ F (\theta) ]). \tag{106}$$

(103) and (105) imply

$$U_{i}^{\theta} [ \beta \times g (m_{i}', \lambda_{-i}) + (1 - \beta) \times \mu ] \leq U_{i}^{\theta} [ \beta \times g (\lambda) + (1 - \beta) \times \mu ] = U_{i}^{\theta} (\text{UNIF} [ F (\theta) ]). \tag{107}$$

We now consider two cases. First, suppose

$$\left( \begin{array}{l} F (\theta) \subset \arg \min_{z \in Z^{*}} u_{i}^{\theta} (z), \\
\Xi_{i} (\theta) \neq \emptyset,
\end{array} \right) \tag{108}$$

and $Z^{*} \cap L_{i}^{Z} (F (\theta), \theta)$ is an $i$-$Z^{*}$-max set

holds. Thus, by (30), we have

$$\hat{\mathcal{L}}_{i}^{Y-A-B} (\text{UNIF} [ F (\theta) ], \theta) = \Delta \left[ Z^{*} \cap L_{i}^{Z} (F (\theta), \theta) \cap \left( \bigcup_{E \in \Xi_{i} (\theta)} \bigcap_{\tilde{\theta} \in E} F (\tilde{\theta}) \right) \right] \tag{109},$$

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and by Lemma 19, we have
\[
g (m_i', \lambda_{-i}) \in \triangle \left[ Z^* \cap \mathcal{L}_i^Z (F(\theta), \theta) \cap \left( \bigcup_{E \in \Xi(\theta)} \bigcap_{\tilde{\theta} \in E} F(\tilde{\theta}) \right) \right],
\]
which, together with \( \mu \in \triangle [F(\theta)] \), implies
\[
\beta \times g (m_i', \lambda_{-i}) + (1 - \beta) \times \mu \in \hat{\mathcal{L}}_{i}^{Y-A-B} (\text{UNIF} [F(\theta)], \theta) .
\]
(102) and (109) imply
\[
\beta \times g (m_i', \lambda_{-i}) + (1 - \beta) \times \mu \in \mathcal{L}_i^Y (\text{UNIF} [F(\theta)], \theta') .
\]
contradicting (106).

Second, suppose that (108) does not hold, and by (30), we have
\[
\hat{\mathcal{L}}_{i}^{Y-A-B} (\text{UNIF} [F(\theta)], \theta) = [\triangle (Z^*)] \cap \mathcal{L}_i^Y (\text{UNIF} [F(\theta)], \theta) ,
\]
which, together with (107), implies
\[
\beta \times g (m_i', \lambda_{-i}) + (1 - \beta) \times \mu \subset [\triangle (Z^*)] \cap \mathcal{L}_i^Y (\text{UNIF} [F(\theta)], \theta) \subset \hat{\mathcal{L}}_{i}^{Y-A-B} (\text{UNIF} [F(\theta)], \theta) .
\]
(102) and (110) imply
\[
\beta \times g (m_i', \lambda_{-i}) + (1 - \beta) \times \mu \in \mathcal{L}_i^Y (\text{UNIF} [F(\theta)], \theta') .
\]
contradicting (106).

A.12 Proof of Lemma 11

The “⇒” direction in (37) is trivial, and we prove the “⇐” direction by contradiction. Suppose \( \mathcal{L}_i^Y (\gamma, \theta) \subset \mathcal{L}_i^Y (\gamma, \theta') \), and for some \( \eta \in \triangle [E] \),
\[
\exists y^* \in \mathcal{L}_i^Y (\eta, \theta) \setminus \mathcal{L}_i^Y (\eta, \theta') ,
\]
or equivalently,
\[
U_i^\theta [y^*] \leq U_i^\theta [\eta] ,
\]
(111)
\[
U_i^\theta' [y^*] > U_i^\theta [\eta] .
\]
(112)
By Lemma 14, there exist \( \beta \in (0, 1) \) and \( \mu \in \triangle [E] \) such that
\[
\gamma = \beta \times \eta + (1 - \beta) \times \mu.
\] (113)

Thus, (111), (112) and (113) imply
\[
U_i^\theta [\beta \times y^* + (1 - \beta) \times \mu] \leq U_i^\theta [\beta \times \eta + (1 - \beta) \times \mu] = U_i^\theta [\gamma],
\]
\[
U_i^{\theta'} [\beta \times y^* + (1 - \beta) \times \mu] > U_i^{\theta'} [\beta \times \eta + (1 - \beta) \times \mu] = U_i^{\theta'} [\gamma],
\]
or equivalently,
\[
[\beta \times y^* + (1 - \beta) \times \mu] \in L^Y_i (\gamma, \theta) \backslash L^Y_i (\gamma, \theta'),
\]
contradicting \( L^Y_i (\gamma, \theta) \subset L^Y_i (\gamma, \theta') \).

\[ \square \]

A.13 Proof of Theorem 11

Clearly, "(i)\(\iff\) (ii)" is implied by Theorem 7 and Lemma 10. "(iii)\(\implies\) (ii)" is implied by \( \cup_{\theta \in \Theta} \Omega[\geq \theta, Q] \subset \cup_{\theta \in \Theta} \Omega[\geq \theta, R] \). We prove "(ii)\(\implies\) (iii)" by combining the techniques developed in both Mezzetti and Renou (2012) and this paper. Suppose \( F \) is mixed-Nash-A-implementable on \( \Theta \equiv \cup_{\theta \in \Theta} \Omega[\geq \theta, Q] \). We aim to show \( F \) is mixed-Nash-A-implementable on \( \tilde{\Theta} \equiv \cup_{\theta \in \Theta} \Omega[\geq \theta, R] \).

We need four constructions before we can define a canonical mechanism to implement \( F \). First, by Theorem 7, \( \tilde{L}^{Y-A-B} \)-uniform-monotonicity holds on \( \Theta \equiv \cup_{\theta \in \Theta} \Omega[\geq \theta, Q] \). The following result is adapted from Lemma 8.

\textbf{Lemma 20} Suppose that \( \tilde{L}^{Y-A-B} \)-monotonicity holds on \( \Theta \equiv \cup_{\theta \in \Theta} \Omega[\geq \theta, Q] \). We have

\[
[Z^* \text{ is a } j-Z^*-\theta'-\text{max set}] \implies Z^* \subset F (\theta'), \forall (j, \theta') \in I \times \Theta^*,
\]

and

\[
\tilde{\Gamma}_{j}^{A-B} (\theta) \text{ is a } j-Z^*-\theta'-\text{max set}] \implies \tilde{\Gamma}_{j}^{A-B} (\theta) \subset F (\theta'), \forall (j, \theta, \theta') \in I \times \Theta^* \times \Theta^*,
\]

where

\[
\tilde{\Gamma}_{j}^{A-B} (\theta) \equiv \left( \bigcup_{y \in \tilde{L}^{Y-A-B}(\text{UNIF}[F(\theta), u^\theta])} \text{SUPP}[y] \right) \text{ for any } u^\theta \in \Omega[\geq \theta, Q].
\]

The proof of Lemma 20 is the same as that of Lemma 8, and we omit it. The intuition is that "\( j-Z^*-\theta'-\text{max set} \)" and "\( \tilde{\Gamma}_{j}^{A-B} (\theta) \)" are ordinal notions (i.e., they depend on ordinal states only).

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Second, we define a matric on $$\bar{\Theta} \equiv \bigcup_{\theta \in \Theta^*} \Omega^{[\geq, \mathbb{R}]}$$:

$$\rho (u, \bar{u}) = \max_{(i, z) \in I \times Z^*} |u_i (z) - \bar{u}_i (z)|, \forall (u, \bar{u}) \in \bar{\Theta} \times \bar{\Theta}.$$  

Furthermore, consider any $$u \in \bigcup_{\theta \in \Theta^*} \Omega^{[\geq, \mathbb{R}]}$$ such that

$$\max_{i \in I} \left[ \max_{z \in Z^*} u_i^\theta (z) - \min_{z \in Z^*} u_i^\theta (z) \right] > 0,$$

denote

$$\gamma_u \equiv \min_{\{(i, a, b) \in I \times Z^* \times Z^* : u_i (a) \neq u_i (b)\}} |u_i (a) - u_i (b)| > 0, \forall u \in \bigcup_{\theta \in \Theta^*} \Omega^{[\geq, \mathbb{R}]}.$$

We need the following two lemmas to complete the proof, which describes how to use cardinal states in $$\Omega^{[\geq, \mathbb{Q}]}$$ to approximate cardinal states in $$\Omega^{[\geq, \mathbb{R}]}$$.

**Lemma 21** For each $$\theta \in \Theta^*$$ and each $$\bar{u}^\theta \in \Omega^{[\geq, \mathbb{Q}]}$$, there exists $$\zeta \bar{u}^\theta \in (0, 1)$$ such that for any $$(i, x, y) \in I \times Z^* \times Z^*$$ and any $$u^\theta \in \Omega^{[\geq, \mathbb{R}]}$$,

$$\begin{pmatrix}
\bar{u}_i^\theta (x) - \bar{u}_i^\theta (y) > 0, \\
\rho (\bar{u}^\theta, u^\theta) < \frac{1}{3} \times \gamma_u^\theta
\end{pmatrix} \implies u_i^\theta (x) - \left[ (1 - \zeta \bar{u}^\theta) u_i^\theta (y) + \zeta \bar{u}^\theta \max_{z \in Z^*} u_i^\theta (z) + \zeta \bar{u}^\theta (|Z^*| - 1) \left( \max_{z \in Z^*} u_i^\theta (z) - \min_{z \in Z^*} u_i^\theta (z) \right) \right] > 0.$$

**Lemma 22** For each $$(\theta, j) \in \Theta^* \times I$$ and each $$\bar{u}^\theta \in \Omega^{[\geq, \mathbb{Q}]}$$, there exist

$$\epsilon_{ij}^{\bar{u}^\theta} > 0$$ and $$y_{ij}^{\bar{u}^\theta} \in \mathcal{L}_{ij}^{Y-A-B} \left( \text{UNIF} \left[ F (\theta) \right], \bar{u}^\theta \right),$$

such that for any $$u^\theta \in \Omega^{[\geq, \mathbb{R}]}$$,

$$\begin{pmatrix}
\left[ \max_{z \in Z^*} \bar{u}_j^\theta (z) - \min_{z \in Z^*} \bar{u}_j^\theta (z) \right] > 0, \\
\rho (\bar{u}^\theta, u^\theta) < \frac{1}{3 \times |Z^*|} \times \gamma_u^\theta
\end{pmatrix} \implies \begin{pmatrix}
\left[ \epsilon_{ij}^{\bar{u}^\theta} + y_{ij}^{\bar{u}^\theta} \right] \in \mathcal{L}_j^{Y} \left( \text{UNIF} \left[ F (\theta) \right], u^\theta \right), \\
\forall y \in \triangle \left( \bar{\Gamma}_{ij}^{A-B} (\theta) \right)
\end{pmatrix}.$$

Third, we need a modified version of the lottery proposed in Mezzetti and Renou (2012), which is described as follows.
Mezzetti and Renou (2012) define a scheme for choosing lottery outcomes based on specific conditions and parameters. The function $T(\theta, u^\theta, j, \tau, z)$ is given by:

$$T(\theta, u^\theta, j, \tau, z) = \frac{1}{|F(\theta)|} \sum_{x \in F(\theta)} \left( \delta(x, \theta, j, \tau) \times \left[ (1 - \zeta(\theta, u^\theta, j, \tau)) \times \tau(x) \right] + \zeta(\theta, u^\theta, j, \tau) \times z \right) + (1 - \delta(x, \theta, j, \tau)) \times x,$$

where $\delta(x, \theta, j, \tau) = \begin{cases} \frac{1}{2}, & \text{if } \tau(x) \in \tilde{L}^{Z^*_{A-B}}(x, \theta), \\ 0, & \text{if } \tau(x) \not\in \tilde{L}^{Z^*_{A-B}}(x, \theta), \end{cases}$ and $\zeta(\theta, u^\theta, j, \tau) = \begin{cases} \zeta^u, & \text{if } \tau(x) \in \tilde{S}L^{Z^*_{A-B}}(x, \theta) \text{ for some } x \in F(\theta), \\ 0, & \text{if } \tau(x) \not\in \tilde{S}L^{Z^*_{A-B}}(x, \theta) \text{ for any } x \in F(\theta), \end{cases}$

where $\zeta^u$ is chosen according to Lemma 21.

Given $(\theta, u^\theta, j)$, we define $\psi_j^u(\theta, u^\theta)$ for each $u^{\theta'} \in \cup_{\theta \in \Theta^*} \Omega[\tau^\theta, Q]$ as follows. Fix any

$$\left(\tau(u^{\theta'}|\theta, u^\theta, j), z(u^{\theta'}|\theta, u^\theta, j)\right) \in \arg \max_{(\tau', z') \in (Z^*)^{Z^* \times Z^*}} u^{\theta'} \left[ T(\theta, u^\theta, j, \tau', z') \right].$$

By finiteness of $Z^*$, $\left(\tau(u^{\theta'}|\theta, u^\theta, j), z(u^{\theta'}|\theta, u^\theta, j)\right)$ is well-defined. Then, define

$$\psi_j^u(\theta, u^\theta) \equiv T\left(\theta, u^\theta, j, \tau(u^{\theta'}|\theta, u^\theta, j), z(u^{\theta'}|\theta, u^\theta, j)\right).$$

The interpretation is that agent $j$ is the whistle-blower, and agents $-j$ report $(\theta, u^\theta)$. Our canonical mechanism would pick the lottery $T(\theta, u^\theta, j, \tau, z)$ with some positive probability, while $j$ is allowed to choose any $(\tau, z)$. Suppose the true cardinal state is $u^{\theta'} \in \cup_{\theta \in \Theta^*} \Omega[\tau^\theta, Q]$. Then, $\left(\tau(u^{\theta'}|\theta, u^\theta, j), z(u^{\theta'}|\theta, u^\theta, j)\right)$ is an optimal choice for $j$ by (116).

Therefore, by (117), $\psi_j^u(\theta, u^\theta)$ is an optimal blocking scheme for $j$ in this scenario.

Fourth, we define a blocking selector as follows.

A optimal blocking-plan selector For each $(\theta, \theta', j) \in \Theta^* \times \Theta^* \times I$, each $(\theta, \theta') \in \Omega[\tau^\theta, Q] \times \Omega[\tau^{\theta'}, Q]$, consider the following condition:
\[ u_j^{\theta'} \left[ \psi_j^{\theta} \left( u^{\theta'} \right) \right] \geq \max_{\gamma_i \in (\mathbb{Z}^+)^{\mathbb{Z}}} \left( u_j^{\theta'} \left[ \epsilon_j^{\theta} \times \gamma_j \left( \hat{\Gamma}_j^{A-B} (\theta) \right) + \left( 1 - \epsilon_j^{\theta} \right) \times y_j^{\theta} \right] \right), \quad (118) \]

where \( \psi_j^{\theta} \left( u_j^{\theta'} \right) \) is defined in (117), and \( (\epsilon_j^{\theta}, y_j^{\theta}) \) are chosen for each \((\theta, j) \in \Theta \times \mathcal{I}\) according to Lemma 22. Define

\[
\eta_j^{\theta} \left( u^{\theta'} \right) \equiv \begin{cases} 1, & \text{if (118) holds;} \\ 0, & \text{otherwise.} \end{cases} \quad (119)
\]

The interpretation of \( \eta_j^{\theta} \left( u^{\theta'} \right) \) is provided in the next section, when we describe Case (2) of the canonical mechanism.

**A canonical mechanism** Let \( \mathbb{N} \) denote the set of positive integers. We use the mechanism \( M^Q = \left\langle M^Q = \prod_{i \in \mathcal{I}} M^Q_i, \; g : M^Q \rightarrow \triangle (\mathbb{Z}^+) \right\rangle \) defined below to implement \( F \). In particular, for each \( i \in \mathcal{I} \), define \( M^Q_i \) as:

\[
\left\{ \left( \theta_i, u_i^{\theta_i}, k_i^3, k_i^4, \gamma_i, b_i \right) \in \Theta^i \times \Theta \times \mathbb{N} \times \mathbb{N} \times (\mathbb{Z}^+)^{\mathbb{Z}^i \setminus \{\emptyset\}} \times \mathbb{Z}^i : \gamma_i (E) \in E, \forall E \subseteq \mathbb{Z}^i \setminus \{\emptyset\} \right\},
\]

and \( g \left( m = (m_i)_{i \in \mathcal{I}} = (\theta_i, u_i^{\theta_i}, k_i^3, k_i^4, \gamma_i, b_i)_{i \in \mathcal{I}} \right) \) is defined in three cases.

**Case (1): consensus** if there exists \((\theta, u^\theta) \in \Theta^i \times \Theta \) such that

\[
\begin{align*}
\left( \theta_i, u_i^{\theta_i}, k_i^3 \right) &= \left( \theta, u^\theta, 1 \right), \forall i \in \mathcal{I}, \\
\text{then } g \left[ m \right] &= \text{UNIF}[F(\theta)];
\end{align*}
\]

**Case (2), unilateral deviation:** if there exists \((\theta, u^\theta, j) \in \Theta^i \times \Theta \times \mathcal{I} \) such that

\[
\begin{align*}
\left( \theta_i, u_i^{\theta_i}, k_i^3 \right) &= \left( \theta, u^\theta, 1 \right) \text{ if and only if } i \in \mathcal{I} \setminus \{j\},
\end{align*}
\]
then
\[
 g \[m\] = \nu_j^\theta \left( u_j^\theta \right) + [1 - \nu] \times \left( \left( 1 - \frac{1}{k^3_j} \right) \times \nu_j \left( \widehat{\Gamma}_{j}^{A-B} (\theta) \right) + \frac{1}{k^3_j} \times \text{UNIF} \left( \widehat{\Gamma}_{j}^{A-B} (\theta) \right) \right),
\]

where
\[
 \nu = \left[ \left( 1 - \frac{1}{k^3_j} \right) \times \gamma_j \left( \widehat{\Gamma}_{j}^{A-B} (\theta) \right) + \left( 1 - \gamma_j \left( \widehat{\Gamma}_{j}^{A-B} (\theta) \right) \right) \right] \in (0, 1), \quad (120)
\]

\( \psi_j^\theta \left( u_j^\theta \right) \) is defined in (117), and \( \left( \epsilon_j^\theta, y_j^\theta \right) \) are chosen for each \( (\theta, j) \in \Theta \times I \) according to Lemma 22, and \( \eta_j^\theta \left( u^\theta \right) \) is defined in (119). Two points are worthy of mentioning. First, by our construction,

\( g \[m\] \in \left( \Delta \left( \widehat{\Gamma}_{j}^{A-B} (\theta) \right) \right) \cap \mathcal{L}_j^Y \left( \text{UNIF} \left[ F (\theta) \right], u^\theta \right). \)

Second, the Mezzetti-Renou technique is described by the compound lottery \( \psi_j^\theta \left( u_j^\theta \right) \) (with probability \( \nu \)), while our technique is described by the compound lottery with probability \( [1 - \nu] \), which ensures that all equilibria triggering Case (2) deliver good outcomes. However, it is not clear which of the two compound lotteries is better for the whistle-blower,\(^{23}\) and hence, we need \( \nu \) defined in (120), so that the whistle-blower may use \( \eta_j^\theta \left( u^\theta \right) \) to choose a better one between the two.\(^{24}\) —This is crucial to prove Lemma 23 below.

**Case (3), multi-lateral deviation:** otherwise,

\[
 g \[m\] = \left( 1 - \frac{1}{k^3_j} \right) \times b_j^* + \frac{1}{k^3_j} \times \text{UNIF} \left( Z^* \right),
\]

where \( j^* = \max \left( \arg \max_{i \in I} k^2_i \right) \), i.e., \( j^* \) is the largest-numbered agent who submits the highest number on the second dimension of the message.

\(^{23}\)In the canonical mechanism which we use to prove Theorem 7, we define \( \psi_j^\theta \left( u_j^\theta \right) \) as a best blocking plan in \( \widehat{\mathcal{L}}_j^{Y-A-B} \left( \text{UNIF} \left[ F (\theta) \right], u^\theta \right) \) at \( u^\theta \). Here, we must adopt the Mezzetti-Renou lottery, which may not be a best blocking plan in \( \widehat{\mathcal{L}}_j^{Y-A-B} \left( \text{UNIF} \left[ F (\theta) \right], u^\theta \right) \) at \( u^\theta \).

\(^{24}\)More precisely, when the true cardinal state is \( u^\theta \) and (120) holds, the whistle-blower \( j \) finds the Mezzetti-Renou lottery better and use \( \eta_j^\theta \left( u^\theta \right) = 1 \) to choose it. Otherwise, \( j \) uses \( \eta_j^\theta \left( u^\theta \right) = 0 \) to choose our compound lottery.
Lemma 23 Consider the canonical mechanism \( \mathcal{M}^{\mathcal{Q}} \) above. For any \( u \in \cup_{\theta \in \Theta} \Omega^{[\geq, \mathbb{R}]} \) and any \( \lambda \in \text{MNE}(\mathcal{M}^{\mathcal{Q}}, u) \), we have \( \text{SUPP}[\lambda] \subset \text{PNE}(\mathcal{M}^{\mathcal{Q}}, u) \).

The proof of Lemma 23 is relegated to Appendix A.13.3.

Theorem 11: a proof We are now ready to prove Theorem 11. Fix any true ordinal state \( \theta^* \in \Theta^* \) and any true cardinal state \( u^{\theta^*} \in \Omega^{[\geq, \mathbb{R}]} \). By Lemma 23, it suffers no loss of generality to focus on pure-strategy Nash equilibrium, and thus, we aim to prove

\[
\bigcup_{\lambda \in \text{PNE}(\mathcal{M}^{\mathcal{Q}}, u^{\theta^*})} \text{SUPP}(g[\lambda]) = F(\theta^*). 
\]

First, suppose

\[
\max_{i \in \mathcal{I}} \left[ \max_{z \in Z^*} u_i^{\theta^*}(z) - \min_{z \in Z^*} u_i^{\theta^*}(z) \right] = 0,
\]

i.e., all agents are indifferent between any two outcomes in \( Z^* \). Pick any \( \hat{u}^{\theta^*} \in \Omega^{[\geq, \mathcal{Q}]} \), and any pure strategy profile

\[
m^* = \left( \theta_i = \theta^*, u_i^{\theta^*}, k_i^3 = 1, *, *, *, * \right)_{i \in \mathcal{I}}
\]

is a Nash equilibrium, which triggers Case (1) and \( g[m^*] = \text{UNIF}[F(\theta^*)] \).

Second, suppose

\[
\max_{i \in \mathcal{I}} \left[ \max_{z \in Z^*} u_i^{\theta^*}(z) - \min_{z \in Z^*} u_i^{\theta^*}(z) \right] > 0,
\]

and denote

\[
\gamma^{u^{\theta^*}} \equiv \min_{\{(i,a,b) \in \mathcal{I} \times Z^* \times Z^*: u_i(a) \neq u_i(b)\}} \left| u_i^{\theta^*}(a) - u_i^{\theta^*}(b) \right| > 0.
\]

Pick any \( \hat{u}^{\theta^*} \in \Omega^{[\geq, \mathcal{Q}]} \) such that

\[
\rho\left(u^{\theta^*}, \hat{u}^{\theta^*}\right) < \frac{1}{2} \times \frac{1}{(3 \times |Z^*| + 1)} \gamma^{u^{\theta^*}}, \tag{121}
\]

and as a result,

\[
\gamma^{\hat{u}^{\theta^*}} \equiv \min_{\{(i,a,b) \in \mathcal{I} \times Z^* \times Z^*: u_i(a) \neq u_i(b)\}} \left| \hat{u}_i^{\theta^*}(a) - \hat{u}_i^{\theta^*}(b) \right| > \left[ 1 - \frac{1}{(3 \times |Z^*| + 1)} \right] \gamma^{u^{\theta^*}} > 0, \tag{122}
\]
and hence,
\[
\frac{1}{3 \times |Z^*|} \gamma_{u^{\theta^*}} > \frac{1}{3 \times |Z^*|} \left[ 1 - \frac{1}{(3 \times |Z^*| + 1)} \right] \gamma_{u^{\theta^*}} = \frac{1}{(3 \times |Z^*| + 1)} \gamma_{u^{\theta^*}} > \rho(u^{\theta^*}, u^{\theta^*}),
\]

where the first inequality follows from (122), and the second inequality from (121). Then, any pure strategy profile
\[
m^* = \left( \theta_i = \theta^*, u_i^{\theta_i} = u^{\theta^*}, k_i = 1, \ast, \ast \right)_{i \in I}
\]
is a Nash equilibrium, which triggers Case (1) and \( g[m^*] = \text{UNIF}[F(\theta^*)] \). Any unilateral deviation \( \bar{m}_j \in M^Q_j \) of agent \( j \in I \) would either still trigger Case (1) and induce \( \text{UNIF}[F(\theta^*)] \), or trigger Case (2) and induce a mixture of
\[
\psi_{j}^{u^{\theta^*}}(u^{\theta^*}) \text{ and } \left( \varepsilon_{j}^{u^{\theta^*}} \times \left( \left( 1 - \frac{1}{k_j^*} \right) \times \gamma_j \left( \tilde{\Gamma}_{j}^{A-B}(\theta) \right) + \frac{1}{k_j^*} \times \text{UNIF} \left( \tilde{\Gamma}_{j}^{A-B}(\theta) \right) \right) \right).
\]

(123) and Lemma 22 imply
\[
\left( \varepsilon_{j}^{u^{\theta^*}} \times \left( \left( 1 - \frac{1}{k_j^*} \right) \times \gamma_j \left( \tilde{\Gamma}_{j}^{A-B}(\theta) \right) + \frac{1}{k_j^*} \times \text{UNIF} \left( \tilde{\Gamma}_{j}^{A-B}(\theta) \right) \right) \right) \in L_j^Y \left( \text{UNIF}[F(\theta^*)], u^{\theta^*} \right).
\]

We now show
\[
\psi_{j}^{u^{\theta^*}}(u^{\theta^*}) \in L_j^Y \left( \text{UNIF}[F(\theta^*)], u^{\theta^*} \right).
\]

We consider two scenarios:
\[
\begin{align*}
\text{(i) } & \tau \left( u^{\theta_j} | (\theta^*, u^{\theta_j}) \right) (x) \notin \tilde{S}_{j}^{Z^*-A-B} \left( x, \theta^* \right) \text{ for any } x \in F(\theta^*), \\
\text{(ii) } & \tau \left( u^{\theta_j} | (\theta^*, u^{\theta_j}) \right) (x) \in \tilde{S}_{j}^{Z^*-A-B} \left( x, \theta^* \right) \text{ for some } x \in F(\theta^*),
\end{align*}
\]

where \( \tau \left( u^{\theta_j} | (\theta^*, u^{\theta_j}) \right) (x) \) is defined in (116). In scenario (i) we have
\[
\zeta \left( \theta^*, u^{\theta^*}, j, \tau \left( u^{\theta_j} | (\theta^*, u^{\theta_j}) \right) \right) = 0 \text{ for any } x \in F(\theta^*) \text{ and }
\]
\[
\tilde{u}^{\theta^*} j (x) = u^{\theta^*} j \left( \tau \left( u^{\theta_j} | (\theta^*, u^{\theta_j}) \right) (x) \right) \text{ for any } x \in F(\theta^*),
\]

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which, together with \( \{u^{\theta^*}, \hat{u}^{\theta^*}\} \subseteq \Omega^{[\kappa^{\theta^*}, \mathbb{R}]} \), immediately implies

\[
u^{\theta^*}_j(x) = u^{\theta^*}_j \left( \tau \left( u^{\theta^*}_j | (\theta^*, \hat{u}^{\theta^*}) \right) \right) \text{ for any } x \in F(\theta^*).\]

Therefore, (124) holds.

In scenario (ii), (123) and Lemma 21 imply (124). Therefore, any \( \tilde{m}_j \in M^Q \) is not a profitable deviation.

Third, fix any

\[
\tilde{m} = \left( \tilde{\theta}_i, \tilde{u}^{\theta}_i, \tilde{k}^i, \tilde{\gamma}_i, \tilde{\beta}_i \right)_{i \in \mathcal{I}} \in \text{PNE}(M^Q, u^{\theta^*}),
\]

and we aim to prove \( g[\tilde{m}] \in \triangle [F(\theta^*)] \). If \( \tilde{m} \) triggers Case (1), by a similar argument as in Mezzetti and Renou (2012), strong set-monotonicity implies \( g[\tilde{m}] \in \triangle [F(\theta^*)] \). If \( \tilde{m} \) triggers Case (2) and \( j \) is the whistle-blower, i.e.,

\[
\exists \theta \in \Theta^*, \tilde{\theta}_i = \theta, \forall i \in \mathcal{I} \setminus \{j\}.
\]

As argued above, \( \Gamma_j^{A-B}(\theta) \) must be a \( j-Z^*-u^{\theta^*} \)-max set, which, together Lemma 20, implies \( g[\tilde{m}] \in \triangle \left[ \Gamma_j^{A-B}(\theta) \right] \subseteq \triangle [F(\theta^*)] \). Similarly, if \( \tilde{m} \) triggers Case (3), \( Z^* \) must be a \( j-Z^*-u^{\theta^*} \)-max set, which, together Lemma 20, implies \( g[\tilde{m}] \in \triangle \left[ Z^* \right] \subseteq \triangle [F(\theta^*)] \). □

A.13.1 Proof of Lemma 21

Fix any \( \theta \in \Theta^* \) and any \( \hat{u}^\theta \in \Omega^{[\kappa^\theta, \mathbb{Q}]} \). Consider any \((i, x, y) \in \mathcal{I} \times Z^* \times Z^* \) and any \( u^\theta \in \Omega^{[\kappa^\theta, \mathbb{R}]} \) such that

\[
\begin{align*}
\hat{u}_i^\theta(x) - \hat{u}_i^\theta(y) & > 0, \\
\rho(\hat{u}_i^\theta, u^\theta) & < \frac{1}{3} \times \gamma \hat{u}_i^\theta.
\end{align*}
\]

We have

\[
\begin{align*}
u_i^\theta(x) - & \left( 1 - \xi \hat{u}_i^\theta \right) u_i^\theta(y) + \xi \hat{u}_i^\theta \max_{z \in Z^*} u_i^\theta(z) + \xi \hat{u}_i^\theta (|Z^*| - 1) \left( \max_{z \in Z^*} u_i^\theta(z) - \min_{z \in Z^*} u_i^\theta(z) \right) \nonumber \\
\geq & \hat{u}_i^\theta(x) - \left( 1 - \xi \hat{u}_i^\theta \right) \hat{u}_i^\theta(y) + \xi \hat{u}_i^\theta \max_{z \in Z^*} \hat{u}_i^\theta(z) + \xi \hat{u}_i^\theta (|Z^*| - 1) \left( \max_{z \in Z^*} \hat{u}_i^\theta(z) - \min_{z \in Z^*} \hat{u}_i^\theta(z) \right) \\
& - \rho(\hat{u}_i^\theta, u^\theta) \times \left( 1 + \left[ \left( 1 - \xi \hat{u}_i^\theta \right) + \xi \hat{u}_i^\theta + 2\xi \hat{u}_i^\theta (|Z^*| - 1) \right] \right). \tag{125}
\end{align*}
\]
Furthermore, we have

\[
\lim_{\xi^{\theta} \to 0} \left( - \begin{bmatrix} \max_{z \in Z^*} \tilde{u}_i^\theta (z) - \min_{z \in Z^*} \tilde{u}_i^\theta (z) \\ (1 - \xi^{\theta}) \tilde{u}_i^\theta (y) - \rho \left( \tilde{u}^\theta, u^\theta \right) \times \left( 1 + \left( 1 - \xi^{\theta} \right) + \xi^{\theta} + 2\xi^{\theta} (|Z^*| - 1) \right) \end{bmatrix} \right)
\]

\[
= \tilde{u}_i^\theta (x) - \tilde{u}_i^\theta (y) - 2 \times \rho \left( \tilde{u}^\theta, u^\theta \right) 
> \tilde{u}_i^\theta (x) - \tilde{u}_i^\theta (y) - 2 \times \frac{1}{3} \times \gamma^{\theta} 
= \tilde{u}_i^\theta (x) - \tilde{u}_i^\theta (y) - 2 \times \frac{1}{3} \times \min_{\{(j,a,b) \in I \times Z^* \times Z^*: \tilde{u}_j^\theta (a) \neq \tilde{u}_j^\theta (b)\}} \left| \tilde{u}_j^\theta (a) - \tilde{u}_j^\theta (b) \right| > 0, \tag{126}
\]

(125) and (126) imply existence of \(\xi^{\theta} \in (0,1)\) such that

\[
u_i^\theta (x) - \left( \left( 1 - \xi^{\theta} \right) u_i^\theta (y) + \xi^{\theta} \max_{z \in Z^*} u_i^\theta (z) + \xi^{\theta} (|Z^*| - 1) \left( \max_{z \in Z^*} u_i^\theta (z) - \min_{z \in Z^*} u_i^\theta (z) \right) \right) > 0, \tag{127}
\]
i.e., (114) holds. 

A.13.2 Proof of Lemma 22

Fix any \((\theta, j) \in \Theta^* \times I\) and any \(\tilde{u}^\theta \in \Omega^{[\xi^\theta, Q]}\). If

\[
\left[ \max_{z \in Z^*} \tilde{u}_j^\theta (z) - \min_{z \in Z^*} \tilde{u}_j^\theta (z) \right] = 0,
\]

choose any \(\epsilon_j^{\theta} \in (0,1)\) and any \(\tilde{y}_j^{\theta} \in \mathcal{L}^{\tilde{Y} \cdot A \cdot B} (\text{UNIF} [F (\theta), \tilde{u}^\theta])\), and (115) holds vacuously. Suppose

\[
\left[ \max_{z \in Z^*} \tilde{u}_j^\theta (z) - \min_{z \in Z^*} \tilde{u}_j^\theta (z) \right] > 0,
\]

and consider any \(u^\theta \in \Omega^{[\xi^\theta, R]}\) such that

\[
\rho \left( \tilde{u}^\theta, u^\theta \right) < \frac{1}{2} \times \frac{1}{3 \times |Z^*|} \times \gamma^{\theta} \tag{127}
\]

Choose any \(y_j^{\theta} \in \arg \min_{z \in Z^*} \tilde{u}_j^\theta (z)\). We now consider two scenarios: (i) \(F (\theta) \subset \arg \min_{z \in Z^*} \tilde{u}_j^\theta (z)\), and (ii) \(F (\theta) \setminus \arg \min_{z \in Z^*} \tilde{u}_j^\theta (z) \neq \emptyset\). First, in scenario (i), we have

\[
\mathcal{L}^{A \cdot B} (\theta) \subset Z^* \cap \mathcal{L}^Z (F (\theta), \tilde{u}^\theta)
\]
Thus, \( \{\hat{u}^\theta, u^\theta\} \in \Omega^{z^\theta, R} \) implies

\[
\hat{\Gamma}_{j}^{A-B} (\theta) \subset Z^* \cap L_{j}^{Z} \left(F (\theta), u^\theta\right),
\]
\[
y_{j}^{\hat{u}^\theta} \in \arg \min_{z \in Z^*} \hat{u}^\theta (z) = \arg \min_{z \in Z^*} u^\theta (z).
\]

Choose \( \varepsilon_{j}^{\hat{u}^\theta} = \frac{1}{2} \), and we have

\[
\left[ \varepsilon_{j}^{\hat{u}^\theta} \times y + (1 - \varepsilon_{j}^{\hat{u}^\theta}) \times y_{j}^{\hat{u}^\theta} \right] \in \Delta \left[ Z^* \cap L_{j}^{Z} \left(F (\theta), u^\theta\right) \right] \subset L_{j}^{Y} \left(\text{UNIF}[F (\theta)], u^\theta\right),
\]

\[\forall y \in \Delta \left(\hat{\Gamma}_{j}^{A-B} (\theta)\right),\]

i.e., \((115)\) holds.

Second, in scenario (ii), we have

\[
y_{j}^{\hat{u}^\theta} \in \arg \min_{z \in Z^*} \hat{u}^\theta (z),
\]

\[F (\theta) \setminus \arg \min_{z \in Z^*} \hat{u}^\theta (z) \neq \emptyset.\]

Pick any \( z' \in Z^* \), and hence,

\[
\lim_{\varepsilon_{j}^{\hat{u}^\theta} \to 0} \left[ \hat{u}^\theta \left(\text{UNIF}[F (\theta)]\right) - \hat{u}^\theta \left(\left[ \varepsilon_{j}^{\hat{u}^\theta} \times z' + (1 - \varepsilon_{j}^{\hat{u}^\theta}) \times y_{j}^{\hat{u}^\theta} \right]\right) \right]
\]

\[= \sum_{x \in F(\theta)} \frac{\hat{u}^\theta (x)}{|F (\theta)|} \times \hat{u}^\theta \left( y_{j}^{\hat{u}^\theta} \right)\]
\[
\geq \frac{1}{|F (\theta)|} \times \min_{(i,a,b) \in \mathcal{I} \times Z^* \times Z^* : u_i(a) \neq u_i(b)} \left| \hat{u}^\theta_i \left( a \right) - \hat{u}^\theta_i \left( b \right) \right|
\]
\[
\geq \frac{1}{|Z^*|} \times \gamma^{\hat{u}^\theta} > 0.
\]

By finiteness of \( Z^* \), there exists \( \varepsilon_{j}^{\hat{u}^\theta} > 0 \) such that

\[
\forall y \in \Delta \left(\hat{\Gamma}_{j}^{A-B} (\theta)\right),
\]

\[
\left[ \hat{u}^\theta \left(\text{UNIF}[F (\theta)]\right) - \hat{u}^\theta \left(\left[ \varepsilon_{j}^{\hat{u}^\theta} \times y + (1 - \varepsilon_{j}^{\hat{u}^\theta}) \times y_{j}^{\hat{u}^\theta} \right]\right) \right] > \frac{1}{2 \times |Z^*|} \times \gamma^{\hat{u}^\theta} > 0,
\]

which, together with \((127)\), implies

\[
\forall y \in \Delta \left(\hat{\Gamma}_{j}^{A-B} (\theta)\right),
\]

\[
\left[ u^\theta \left(\text{UNIF}[F (\theta)]\right) - u^\theta \left(\left[ \varepsilon_{j}^{u^\theta} \times y + (1 - \varepsilon_{j}^{u^\theta}) \times y_{j}^{u^\theta} \right]\right) \right] > \left( \frac{1}{2 \times |Z^*|} - \frac{1}{3 \times |Z^*|} \right) \times \gamma^{\hat{u}^\theta} > 0,
\]

i.e., \((115)\) holds. ■
A.13.3 Proof of Lemma 23

We need the following result to prove Lemma 23.

**Lemma 24** Consider the canonical mechanism $\mathcal{M}^Q$ in Appendix A.13. For any $(\theta, i) \in \Theta^* \times \mathcal{I}$, any $u^\theta \in \Omega^{[\theta, \mathbb{R}]}$ and any $\varepsilon > 0$, there exists a sequence $\{m^u_{i,n} : n \in \mathbb{N}\}$ such that

\[
\lim_{n \to \infty} u^\theta_i \left[ g \left( m^u_{i,n}, m_{-i} \right) \right] \geq u^\theta_i \left[ g \left( m_i, m_{-i} \right) \right] - \varepsilon, \forall (m_i, m_{-i}) \in M_i \times M_{-i}.
\]  

(128)

**Proof of Lemma 24:** Fix any $(\theta, i) \in \Theta^* \times \mathcal{I}$. First, consider any $u^\theta \in \Omega^{[\theta, Q]}$, and we show (128) for $u^\theta$. Fix any $b^\theta_i \in \text{arg max}_{z \in Z} u^\theta_i (z)$ and any $\gamma_i (E) \in (Z^*)^{[2^{Z^*} \setminus \{\emptyset\}]}$ such that $\gamma_i (E) \in \text{arg max}_{z \in E} u^\theta_i (z)$, \forall $E \in [2^{Z^*} \setminus \{\emptyset\}]$, and define

\[ m^u_{i,n} \equiv \left( \theta, u^\theta, k^3_i = n, k^4_i = n, \gamma_i, b^\theta_i \right) \in M_i, \forall n \in \mathbb{N}. \]

It is straightforward to show

\[ \lim_{n \to \infty} u^\theta_i \left[ g \left( m^u_{i,n}, m_{-i} \right) \right] \geq u^\theta_i \left[ g \left( m_i, m_{-i} \right) \right], \forall (m_i, m_{-i}) \in M_i \times M_{-i}. \]

i.e., (128) holds.

Second, consider any $\tilde{u}^\theta \in \Omega^{[\tilde{\theta}, \mathbb{R}]}$, and we show (128) for $\tilde{u}^\theta$. For any $\varepsilon > 0$, pick any $u^\theta \in \Omega^{[\theta, Q]}$ such that

\[ \rho \left( \tilde{u}^\theta, u^\theta \right) \leq \frac{1}{2} \times \varepsilon, \]  

(129)

and pick any sequence $\{m^u_{i,n} : n \in \mathbb{N}\}$ such that

\[ \lim_{n \to \infty} u^\theta_i \left[ g \left( m^u_{i,n}, m_{-i} \right) \right] \geq u^\theta_i \left[ g \left( m_i, m_{-i} \right) \right] - \frac{1}{2} \times \varepsilon, \forall (m_i, m_{-i}) \in M_i \times M_{-i}. \]  

(130)

Define

\[ m^\tilde{u}_{i,n} \equiv m^u_{i,n}. \]  

(131)

Thus, for any $(m_i, m_{-i}) \in M_i \times M_{-i}$, we have

\[
\lim_{n \to \infty} \tilde{u}^\theta_i \left[ g \left( m^\tilde{u}_{i,n}, m_{-i} \right) \right] = \lim_{n \to \infty} \tilde{u}^\theta_i \left[ g \left( m_{i,n}, m_{-i} \right) \right] \\
\geq \lim_{n \to \infty} u^\theta_i \left[ g \left( m^u_{i,n}, m_{-i} \right) \right] - \frac{1}{2} \times \varepsilon \\
\geq u^\theta_i \left[ g \left( m_i, m_{-i} \right) \right] - \frac{1}{2} \times \varepsilon - \frac{1}{2} \times \varepsilon,
\]  

(132)
where the equality follows from (131), the first inequality follows from (129), the second inequality follows from (130). I.e., (128) holds.

**Proof of Lemma 23:** Fix any \( \theta \in \Theta^* \), any \( u^\theta \in \Omega^{[\Sigma_\theta, \mathbb{R}]} \), any \( \lambda \in \text{MNE}(\mathcal{M}_Q, u^\theta) \) and any \( \hat{m} \in \text{SUPP}[\lambda] \), i.e.,

\[
\Pi_{i \in I} \lambda_i (\hat{m}_i) > 0.
\]

We aim to prove \( \hat{m} \in \text{PNE}(\mathcal{M}_Q, \theta) \). Suppose \( \hat{m} \notin \text{PNE}(\mathcal{M}_Q, \theta) \), i.e., there exists \( j \in I \) and \( m'_j \in M_i \) such that

\[
u_j^\theta \left[ g\left( m'_j, \hat{m}_{-j} \right) \right] > u_j^\theta \left[ g\left( \hat{m}_j, \hat{m}_{-j} \right) \right],
\]

and define

\[
\epsilon \equiv \left( u_j^\theta \left[ g\left( m'_j, \hat{m}_{-j} \right) \right] - u_j^\theta \left[ g\left( \hat{m}_j, \hat{m}_{-j} \right) \right] \right) \times \Pi_{i \in I} \lambda_i (\hat{m}_i) > 0. \tag{132}
\]

By Lemma 24, there exists a sequence \( \{ m^\theta_n \in M_j : n \in \mathbb{N} \} \) such that

\[
\lim_{n \to \infty} u_j^\theta \left[ g\left( m^\theta_n, m_{-j} \right) \right] \geq u_j^\theta \left[ g\left( m_j, m_{-j} \right) \right] - \frac{1}{2} \times \epsilon, \forall (m_i, m_{-i}) \in M_i \times M_{-i}. \tag{133}
\]

We thus have

\[
\lim_{n \to \infty} u_j^\theta \left[ g\left( m^\theta_n, \lambda_{-j} \right) \right] - u_j^\theta \left[ g\left( \lambda_j, \lambda_{-j} \right) \right] = \lim_{n \to \infty} \left( \sum_{m \in M^\ast \setminus \{ \hat{m} \}} \left[ \Pi_{i \in I} \lambda_i (m_i) \times \left( u_j^\theta \left[ g\left( m^\theta_n, m_{-j} \right) \right] - u_j^\theta \left[ g\left( m_{-j} \right) \right] \right) \right] + \Pi_{i \in I} \lambda_i (\hat{m}_i) \times \left( u_j^\theta \left[ g\left( m^\theta_n, \hat{m}_{-j} \right) \right] - u_j^\theta \left[ g\left( \hat{m}_{-j} \right) \right] \right) \right) \\
\geq - \left( \sum_{m \in M^\ast \setminus \{ \hat{m} \}} \Pi_{i \in I} \lambda_i (m_i) \right) \times \frac{1}{2} \times \epsilon \\
- \Pi_{i \in I} \lambda_i (\hat{m}_i) \times \frac{1}{2} \times \epsilon + \left[ \Pi_{i \in I} \lambda_i (\hat{m}_i) \times \left( u_j^\theta \left[ g\left( m'_j, \hat{m}_{-j} \right) \right] - u_j^\theta \left[ g\left( \hat{m}_{-j} \right) \right] \right) \right] \\
= - \frac{1}{2} \times \epsilon + \epsilon > 0,
\]

where the first inequality follows from (133), and the second inequality follows from (132).

As a result, there exists \( n \in \mathbb{N} \) such that

\[
u_j^\theta \left[ g\left( m^\theta_n, \lambda_{-j} \right) \right] > u_j^\theta \left[ g\left( \lambda_j, \lambda_{-j} \right) \right],
\]

contradicting \( \lambda \in \text{MNE}(\mathcal{M}_Q, u^\theta) \).
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