ABSTRACT. In this article, we study the small gaps of the log-gas $\beta$-ensemble on the unit circle, where $\beta$ is any positive integer. The main result is that the $k$-th smallest gap, normalized by $n^{-\beta+1}$, has the limit proportional to $x^{k(\beta+1)-1}e^{-x^{\beta+1}}$. In particular, the result applies to the classical COE, CUE and CSE in random matrix theory. The essential part of the proof is to derive several identities and inequalities regarding the Selberg integral, which should have their own interest.

1. Introduction

The extremal spacings of random point processes are important quantities for statistical physics. In random matrix theory, the question was considered for the smallest gap by Vinson [9]; by a different method, Soshnikov also investigated the smallest gap for the determinantal point processes on the real line with translation invariant kernels [8]; Soshnikov’s technique was adapted and improved by Ben Arous-Bourgade [2], where they proved that the $k$-th smallest gap, normalized by $n^{\frac{1}{2}}$, has the limit proportional to $x^{3k-1}e^{-x^3}$ for the determinantal point processes of CUE and GUE. Ben Arous-Bourgade also derived the convergence of the largest gap for these two cases; and in [4], we further prove that the limiting density for the largest gap of CUE and GUE is given by the Gumbel distribution. In this paper, we will study the small gaps of the log-gas $\beta$-ensemble on the unit circle, here $\beta$ is any positive integer. Our results confirm the (numerical) prediction in physics [7], and recover Ben Arous-Bourgade’s results in the case of CUE (where $\beta = 2$). But our proof is different and very technical. One can not make use of the nice structure of the determinantal point processes any more (for example, when $\beta = 1, 4$, they are Pfaffian processes other than the determinantal point processes [1]), and we have to start from the Selberg integral to get the estimates regarding the point correlation functions, where we need to derive several asymptotic limits and inequalities (such as Lemma [1] and Lemma [3]) which should have their own interest in the Selberg integral theory. Other than the log-gas $\beta$-ensemble on the unit circle, in [3], we continue to study the small gaps of the log-gas $\beta$-ensemble on the real line (such as GOE, GUE and GSE). There are also many other interesting models one may study regarding the extremal spacings although they seem much harder to solve, such as the tensor product of $2 \times 2$ unitary matrices of qubit system in quantum information theory (see [7] for the numerical results) and the product of random Wigner matrices in random matrix theory.
1.1. Main results. For circular $\beta$-ensemble with $\beta > 0$, the density of the eige-
nangles $\theta_j \in [-\pi, \pi), 1 \leq j \leq n$ with respect to the Lebesgue measure is

$$J(\theta_1, \cdots, \theta_n) = \frac{1}{C_{\beta,n}} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^\beta,$$

with $\beta = 2$ corresponding to CUE and $\beta = 1$ for COE and $\beta = 4$ for CSE. The
normalization constant

$$C_{\beta,n} := \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_n \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^\beta$$

is derived by the Selberg integral as

$$C_{\beta,n} = (2\pi)^n \frac{\Gamma(1 + \beta n/2)}{(\Gamma(1 + \beta/2))^n}.$$

One interpretation of the density $J(\theta_1, \cdots, \theta_n)$ is as the Boltzmann factor for a
classical gas at inverse temperature $\beta$ with potential energy

$$- \sum_{1 \leq j < k \leq n} \ln |e^{i\theta_j} - e^{i\theta_k}|.$$

Because of the pairwise logarithmic repulsion (two-dimensional Coulomb law), such
a classical gas is referred to as a log-gas. This interpretation allows for a number
of properties of correlations and distributions to be anticipated using arguments
based on macroscopic electrostatics [5].

We also need the following integration constants for the two-compo
nent log-gas

(1) $C_{\beta,n_1,n_2} := \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_{n_1+n_2} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^{q_j q_k \beta},$

(2) $C_{\beta,n_1,n_2}(I) := \int_{(-\pi,\pi)^{n_1} \times I^{n_2}} d\theta_1 \cdots d\theta_{n_1+n_2} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^{q_j q_k \beta},$

where $q_j = 1$ for $1 \leq j \leq n_1$ and $q_j = 2$ for $n_1 + 1 \leq j \leq n_1 + n_2$.

Let’s denote

(3) $C_{\beta,n_1,(k)} := \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_{n_1+1} \prod_{j<l} |e^{i\theta_j} - e^{i\theta_l}|^{q_j q_l \beta}$

with $q_j = 1$ for $1 \leq j \leq n_1$ and $q_{n_1+1} = k$, then we have

$$C_{\beta,n_1,(2)} = C_{\beta,n_1,1}$$

and the following results.

**Lemma 1.** For $0 < k \leq n$, $\beta \geq 1$, we have

$$C_{\beta,n-k,(k)} \leq C_{\beta,n}(n\beta)^{k(k-1)\beta/2},$$

$$\lim_{n \to +\infty} \frac{C_{\beta,n-2,1}}{C_{\beta,n} n^{\beta}} = A_\beta, \quad \lim_{n \to +\infty} \frac{C_{\beta,n-k,(k)}}{C_{\beta,n} n^{k(k-1)\beta/2}} = A_{\beta,k},$$

where

$$A_{\beta,k} = \frac{(2\pi)^{-k}(\Gamma(\beta/2 + 1))^{k-1}}{\Gamma(k\beta/2 + 1)} \prod_{j=1}^{k-1} \frac{\Gamma(j\beta/2 + 1)}{\Gamma((k+j)\beta/2 + 1)} (\beta/2)^{k(k-1)\beta/2}.$$
and
\[ A_\beta = A_{\beta,2} = (2\pi)^{-1} (\beta/2)^\beta (\Gamma(\beta/2 + 1))^3 / \Gamma(3\beta/2 + 1) \Gamma(\beta + 1). \]

Now let’s consider the following point process on \( \mathbb{R}^2 \),
\[ \chi^{(n,\gamma)} = \sum_{i=1}^{n} \delta_{(n\gamma(-\theta(i_1) + \theta(i_0)))}, \quad \chi^{(n)} = \chi^{(n,\gamma)} \bigg|_{\gamma = \frac{\beta + 2}{\pi}}; \]
where \( \gamma > 0 \), \( \theta(i_0) (1 \leq i \leq n) \) is the increasing rearrangement of \( \theta_i (1 \leq i \leq n) \) and \( \theta(i+n) = \theta(i) + 2\pi \), i.e. the indexes are modulo \( n \). Regarding the point process \( \chi^{(n)} \), the main result is

**Theorem 1.** For the circular \( \beta \)-ensemble where \( \beta \) is a positive integer, as \( n \to +\infty \), the process \( \chi^{(n)} \) defined in (4) converges to a Poisson point process \( \chi \) with intensity
\[ E\chi(A \times I) = \frac{A_\beta |I|}{2\pi} \int_A u^\beta du. \]
for any bounded Borel sets \( A \subset \mathbb{R}_+ \) and \( I \subseteq (-\pi, \pi) \), and \( |I| \) is the Lebesgue measure of \( I \). In particular, the result holds for COE, CUE and CSE with
\[ A_1 = \frac{1}{24}, \quad A_2 = \frac{1}{24\pi}, \quad A_4 = \frac{1}{270\pi} \]
correspondingly.

As a direct consequence of the main result, we easily have (we refer to [2, 9] for the case when \( \beta = 2 \))

**Corollary 1.** Let’s denote \( m_k \) as the \( k \)-th smallest gap, and
\[ \tau_k = n^{(\beta+2)/(\beta+1)} \times (A_\beta/(\beta + 1))^{1/(\beta+1)} m_k, \]
then for any bounded interval \( A \subset \mathbb{R}_+ \), we have
\[ \lim_{n \to +\infty} \mathbb{P}(\tau_k \in A) = \int_A \frac{\beta + 1}{(k-1)!} x^{k(\beta+1)-1} e^{-x^{\beta+1}} dx. \]
In particular, the limiting density function for \( \tau_1 \) is
\[ (\beta + 1)x^\beta e^{-x^{\beta+1}}. \]

1.2. **Strategy and key lemmas.** Let’s first explain the main steps to prove Theorem 1. In the article, let \( \beta \) be a positive integer. As in [2, 8], we still need to reduce the problem to the convergence of the factorial moment (10), but the proof follows a quite different way. This is because, for the determinantal point processes as considered in [2, 8], there are many good structures one can make use of. For example, all the point correlation functions of the determinantal point processes are given explicitly and one can express the factorial moment by these correlation functions, one can also use Hadamard-Fischer inequality to control the estimates. But in the case of general \( \beta \)-ensemble, we have no choice but start from the Selberg integral. We need to take a more complicated strategy: In Lemma 3 we will find that (10) is equivalent to the convergence of \( C_{\beta,n} \delta_{k,k}(I) / C_{\beta,n,k}\beta \). The convergence for \( k = 1 \) is guaranteed by Lemma 1. For every positive integer \( k \geq 2 \), we find that the uniform bound (11) and the upper bound (12) will imply the convergence of...
\( \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n}n^k} \) (Lemma 4). The main tools are some comparison inequalities between random variables (Lemma 7) and integrations (Lemma 5). To be more precise, we will introduce another auxiliary point process in \( \beta \) which is proved to be equivalent to the factorial moments of \( \hat{\chi}^{(n)} \) (see Lemma 4 and the limit (30)), and the expectation of which can be expressed in terms of the integration of the density function \( J(\theta_1, \cdots, \theta_n) \) (see (23)). For the rest of the proof, we only use the expression of the density function \( J(\theta_1, \cdots, \theta_n) \) without knowing the asymptotic behavior of its \( 2k \)-dimensional correlation functions. The most difficult part of the whole proof is the upper bound (12), it requires the expression of the \( 2 \)-dimensional correlation function for the two-component log-gas. Using some formulas on the generalized hypergeometric function, we further have that this correlation function can be expressed as a \( 2\beta \)-dimensional integral (42). Then we use contour deformation method to evaluate this integral and give the uniform bound and the asymptotic limit in \( \beta \).

It is natural to consider another point process. We introduce \( \theta_{i,j} = \theta_i - \theta_j \) for \( \theta_i > \theta_j \), \( \theta_{i,j} = \theta_i - \theta_j + 2\pi \) for \( \theta_i < \theta_j \). For any \( \gamma > 0 \), let’s define

\[
\theta_{i,j,\gamma} = (n^\gamma \theta_{i,j}, \theta_j)
\]

and

\[
\hat{\chi}^{(n,\gamma)} = \sum_{i \neq j} \delta_{\theta_{i,j,\gamma}}, \quad \hat{\chi}^{(n)} = \hat{\chi}^{(n,\gamma)} \bigg|_{\gamma = \frac{\beta + 2}{\beta + 2}}.
\]

Then we have

\[
\chi^{(n)}(A \times I) = \sum_{j=1}^{n-1} \hat{\chi}^{(n,\gamma,j)}
\]

such that

\[
\hat{\chi}^{(n,\gamma,j)} = \sum_{i=1}^{n} \delta_{(n^\gamma(\theta_{i+j} - \theta_{(i)}), \theta_{(i)})}.
\]

Then we have

\[
\hat{\chi}^{(n,\gamma,1)} = \chi^{(n,\gamma)} \quad \text{and} \quad 0 \leq \hat{\chi}^{(n,\gamma,j)}(B) \leq n
\]

for every Borel set \( B \subset \mathbb{R}^2 \).

We need to show the following lemma which indicates that there is no successive small gaps, which is also considered in \( \beta \) for the determinantal point processes.

**Lemma 2.** For any bounded interval \( A \subset \mathbb{R}_+ \) and \( I \subseteq (-\pi, \pi) \), we have \( \chi^{(n)}(A \times I) - \hat{\chi}^{(n)}(A \times I) \to 0 \) in probability as \( n \to +\infty \).

Thanks to Proposition 2.1 in [2] which is the consequence of Kallenberg’s result on the convergence of point processes [6], for every positive integer \( k \) and bounded interval \( A \subset \mathbb{R}_+ \) and \( I \subseteq (-\pi, \pi) \), if we can prove the following convergence of the factorial moment

\[
\lim_{n \to +\infty} \mathbb{E} \left( \frac{C_{\beta,n}n^k (\hat{\chi}^{(n)}(A \times I))!}{(\hat{\chi}^{(n)}(A \times I) - k)!} \right) = \left( \int \left| A_{\beta} \right|^k \frac{1}{2\pi} \right)^k,
\]

(10)
then, together with Lemma 2, Theorem 1 will be proved.

Actually, (10) is the direct consequence of the following two lemmas.

**Lemma 3.** For any bounded interval \( A \subset \mathbb{R}^+ \), \( I \subseteq (-\pi, \pi) \) and any positive integer \( k \geq 1 \), we have

\[
E \left( \left( \frac{(\bar{\chi}(n)(A \times I))}{\bar{\chi}(n)(A \times I) - k} \right)! \right) - \left( \int_A u^k \, du \right)^k \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n} n^{k\beta}} \to 0
\]
as \( n \to +\infty \).

We also need the following asymptotic limit regarding the Selberg integral,

**Lemma 4.** For any interval \( I \subseteq (-\pi, \pi) \) and any positive integer \( k \geq 1 \), we have

\[
\lim_{n \to +\infty} C_{\beta,n-2k,k}(I) \cdot C_{\beta,n} n^{k\beta} = \left( \frac{|I| A_\beta}{2\pi} \right)^k.
\]

We will see that the proof of Lemma 4 is based on Lemma 3 and the following two inequalities

\[
(11) \lim_{n \to +\infty} \sup \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n} n^{k\beta}} < +\infty.
\]

\[
(12) \lim_{n \to +\infty} \sup \frac{C_{\beta,n-4,k}(I)}{C_{\beta,n} n^{2\beta}} \leq \left( \frac{|I| A_{\beta}}{2\pi} \right)^2.
\]

The proof of (11) in §6 is based on the estimate of the factorial moment of \( \bar{\chi}(n,1)((0,c_0) \times (-\pi, \pi)) \) for a fixed constant \( c_0 \in (0, 1/\beta) \). The proof of (12) in §8 is based on the Selberg integral and generalized hypergeometric functions.

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## 2. Proof of Lemma 1

Now we give the proof of Lemma 1 which is based on the Selberg integral.

**Proof.** We can write

\[
C_{\beta,n_1,1} = \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_{n_1+1} \prod_{1 \leq j < k \leq n_1} |e^{i\theta_j} - e^{i\theta_k}|^\beta \prod_{1 \leq j \leq n_1} |e^{i\theta_j} - e^{i\theta_{n_1+1}}|^{2\beta}
\]

\[
= \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_{n_1+1} \prod_{1 \leq j < k \leq n_1} |e^{i\theta_j} - e^{i\theta_k}|^\beta \prod_{1 \leq j \leq n_1} |e^{i\theta_j} + 1|^{2\beta}
\]

\[
=(2\pi)^{n_1+1} M_{n_1}(\beta, \beta, \beta/2),
\]

here we used changing of variables \( \theta_j \mapsto \theta_j + \theta_{n_1+1} \pm \pi \) (\( 1 \leq j \leq n_1 \)) and the formula (4.4) in [5]:

\[
M_n(a, b, \lambda) := \int_{-\pi/2}^{\pi/2} d\theta_1 \cdots \int_{-\pi/2}^{\pi/2} d\theta_n \prod_{l=1}^{n} e^{\pi i \theta(l-a-b)} \prod_{1 \leq j < k \leq n} |e^{2\pi i \theta_j} - e^{2\pi i \theta_k}|^{2\lambda}
\]
(13) \[ \prod_{j=0}^{n-1} \frac{\Gamma(\lambda j + a + b + 1)\Gamma(\lambda(j + 1) + 1)}{\Gamma(\lambda j + a + 1)\Gamma(\lambda j + b + 1)\Gamma(1 + \lambda)}. \]

Similarly, we have
(14) \[ C_{\beta,n_1,1}(I) = (2\pi)^{n_1} |I|M_{n_1}(\beta, \beta, \beta/2) = (2\pi)^{-1} |I|C_{\beta,n_1,1}, \]
and
(15) \[ C_{\beta,n_1,(k)} = (2\pi)^{n_1+1} M_{n_1}(k\beta/2, k\beta/2, \beta/2). \]

For every positive integer \( k \), we have
\[ M_{n}(k\lambda, k\lambda, \lambda) = \prod_{j=0}^{n-1} \frac{\Gamma(\lambda j + 2k + 1)\Gamma(\lambda j + 1) + 1}{\Gamma(\lambda j + 2k + 1)\Gamma(\lambda j + 1)\Gamma(1 + \lambda)} \]
\[ = \frac{1}{\Gamma(\lambda + 1)^n} \prod_{j=k}^{2k-1} \frac{\Gamma(\lambda(n + j) + 1)}{\Gamma(j\lambda + 1)} \prod_{j=1}^{k-1} \frac{\Gamma(j\lambda + 1)}{\Gamma(\lambda(n + j) + 1)}. \]

thus
\[ C_{\beta,n_1,(k)} = (2\pi)^{n_1+1} M_{n_1}(k\beta/2, k\beta/2, \beta/2) \]
\[ = \frac{(2\pi)^{n_1+1}}{\Gamma(\beta/2 + 1)^n_1} \prod_{j=k}^{2k-1} \frac{\Gamma(\beta(n + j)/2 + 1)}{\Gamma(j\beta/2 + 1)} \prod_{j=1}^{k-1} \frac{\Gamma(j\beta/2 + 1)}{\Gamma(\beta(n + j)/2 + 1)}. \]

And for \( n_1 = n - k > 0 \), we have
\[ \frac{C_{\beta,n-k,(k)}}{C_{\beta,n}} = \frac{2\pi(\Gamma(\beta/2 + 1))^k \prod_{j=1}^{k-1} \left( \frac{\Gamma(\beta(n + j)/2 + 1)}{\Gamma(j\beta/2 + 1)} \right)^{\text{sgn}(j-k)}}{\Gamma(n\beta/2 + 1)} \]
\[ = \frac{(2\pi)^{1-k} \Gamma(\beta/2 + 1)^k \prod_{j=1}^{k-1} \frac{\Gamma(\beta(n + j)/2 + 1)}{\Gamma((k + j)\beta/2 + 1)\Gamma(\beta(n - j)/2 + 1)}}{\Gamma(k\beta/2 + 1)\Gamma(\beta(n + j)/2 + 1)}. \]

As \( \ln \Gamma(x) \) is convex for \( x > 0 \), we have \( (\Gamma(\beta/2 + 1))^k \leq \Gamma(k\beta/2 + 1) \). For \( n > k - 1 \geq j \geq 1 \), we have \( k\beta/2 \geq 1 \), \( \beta j \geq 1 \) and
\[ \frac{\Gamma(j\beta/2 + 1)}{\Gamma(k\beta/2 + 1)} \leq \left( \frac{\Gamma(j\beta/2 + 1)}{\Gamma(j\beta/2 + 2)} \right)^{\beta j} = \left( \frac{1}{j\beta/2 + 1} \right)^{\beta j} \leq 1, \]
and
\[ \frac{\Gamma(\beta(n + j)/2 + 1)}{\Gamma(\beta(n - j)/2 + 1)} \leq \left( \frac{\Gamma(\beta(n + j)/2 + 1)}{\Gamma(\beta(n + j)/2)} \right)^{\beta j} = \left( \frac{\beta(n + j)/2 + 1}{\Gamma(\beta(n + j)/2 + 1)} \right)^{\beta j} \]
\[ = (\beta(n + j)/2 + 1)^{\beta j} \leq (\beta(n + j)/2)^{\beta j} \leq (n\beta)^{\beta j}, \]
therefore
\[ \frac{C_{\beta,n-k,(k)}}{C_{\beta,n}} \leq (2\pi)^{1-k} \prod_{j=1}^{k-1} (n\beta)^{\beta j} = (2\pi)^{1-k} (n\beta)^{k(k-1)/2}, \]
which will imply the first inequality. Using convexity of \( \ln \Gamma(x) \), we also have
\[ (\beta(n - j)/2 + 1)^{\beta j} \leq \frac{\Gamma(\beta(n + j)/2 + 1)}{\Gamma(\beta(n - j)/2 + 1)} \leq (\beta(n + j)/2)^{\beta j}, \]
which implies
\[ \lim_{n \to +\infty} \frac{\Gamma(\beta(n + j)/2 + 1)}{\Gamma(\beta(n - j)/2 + 1)n^{\beta j}} = (\beta/2)^{\beta j}. \]
And thus
\[ \lim_{n \to +\infty} \frac{C_{\beta,n-k,k}}{C_{\beta,n} n^{k(k-1)/2}} = (2\pi)^{1-k}(\Gamma(\beta/2 + 1))^{k} \prod_{j=1}^{k-1} \frac{\Gamma(j\beta/2 + 1)}{\Gamma((j+k)\beta/2 + 1)} \cdot \lim_{n \to +\infty} \prod_{j=1}^{k-1} \frac{\Gamma((\beta(n+j)/2 + 1))}{\Gamma((\beta(n-j)/2 + 1)) n^{\beta j}}. \]

\[ = (2\pi)^{1-k}(\Gamma(\beta/2 + 1))^{k} \prod_{j=1}^{k-1} \frac{\Gamma(j\beta/2 + 1)}{\Gamma((k+j)\beta/2 + 1)} \prod_{j=1}^{k-1} (\beta/2)^{\beta j} =: A_{\beta,k}. \]

As \( C_{\beta,n1,2} = C_{\beta,n1,1} \), we have
\[ \lim_{n \to +\infty} \frac{C_{\beta,n-2,1}}{C_{\beta,n} n^{\beta}} = \lim_{n \to +\infty} \frac{C_{\beta,n-2,2}}{C_{\beta,n} n^{\beta}} = A_{\beta,2}, \]

and the expression of \( A_{\beta} = A_{\beta,2} \) follows directly from that of \( A_{\beta,k} \).

\[ \square \]

3. ONE INTEGRAL LEMMA

Let’s prove the following crucial lemma which will imply the bounds of the integrations of the joint density on the small neighborhood around one variable.

**Lemma 5.** Let \( m, n, \beta \) be positive integers with \( m \leq n \). Given any \( c \) such that \( n\beta c \in (0, 1) \). Given \( \theta_j \in \mathbb{R}, j = 1, \cdots, m \), let’s define
\[ F(x) = \prod_{j=1}^{m} (e^{ix} - e^{i\theta_j}), \]

then we have
\[ \left( \frac{\sin(c/2)}{c/2} \right)^{\beta} \cos(n\beta c) \frac{\beta^{\beta+1}}{\beta+1} \int_{-\pi}^{\pi} dx_1 |F(x_1)|^{2\beta} \]
\[ \leq \int_{-\pi}^{\pi} dx_1 \int_{x_1+c}^{x_1+c+2c} dx_2 |e^{ix_1} - e^{ix_2}|^{\beta} |F(x_1)|^{\beta} |F(x_2)|^{\beta} \]
\[ \leq \frac{\beta^{\beta+1}}{\beta+1} \int_{-\pi}^{\pi} dx_1 |F(x_1)|^{2\beta}, \]

and for \( k \geq 1 \),
\[ \int_{-\pi}^{\pi} dx_1 \int_{x_1+c}^{x_1+c+2c} dx_2 \cdots dx_k \prod_{1 \leq j \leq l \leq k} |e^{ix_j} - e^{ix_l}|^{\beta} \prod_{j=1}^{k} |F(x_j)|^{\beta} \]
\[ \leq \beta^{\beta(k-1)/2+k-1} \int_{-\pi}^{\pi} dx_1 |F(x_1)|^{k\beta}. \]

For intervals \( A \subset (0, c), I \subset (-\pi, \pi), \) let’s denote
\[ \varphi(\beta, A) := \int_{A} |1 - e^{iu}|^{\beta} du, \]

then we have
\[ \left| \int_{I} dx_1 \int_{x_1+A} dx_2 |e^{ix_1} - e^{ix_2}|^{\beta} |F(x_1)|^{\beta} |F(x_2)|^{\beta} - \varphi(\beta, A) \int_{I} dx_1 |F(x_1)|^{2\beta} \right| \]
\[ \leq \varphi(\beta, A)(n\beta c) \int_{-\pi}^{\pi} dx_1 |F(x_1)|^{2\beta}, \]

\[ \square \]
Thus for $t$ and $c$ we have
\begin{equation}
\left(\frac{\sin(c/2)}{c/2}\right)^\beta \int_A u^\beta du \leq \varphi(\beta, A) \leq \int_A u^\beta du.
\end{equation}

**Proof.** We can write $F(x)^\beta = \sum_{j=0}^{m\beta} a_j e^{ijx}$. A change of variables $x_2 = x_1 + t$ shows
\begin{equation}
\int_{-\pi}^\pi dx_1 \int_{x_1}^{x_1+c} dx_2 e^{ix_1} - e^{ix_2} |F(x_1)|^\beta |F(x_2)|^\beta
= \int_0^c dt \int_{-\pi}^\pi |1 - e^{it}|^\beta |F(x_1)|^\beta |F(x_1 + t)|^\beta dx_1.
\end{equation}

As
\begin{equation}
F(x_1)^\beta = \sum_{j=0}^{m\beta} a_j e^{ijx_1}, \quad F(x_1 + t)^\beta = \sum_{j=0}^{m\beta} a_j e^{ijt} e^{ijx_1},
\end{equation}

by Parseval’s theorem, we have
\begin{equation}
\int_{-\pi}^\pi F(x_1)^\beta F(x_1 + t)^\beta dx_1 = 2\pi \sum_{j=0}^{m\beta} a_j a_j e^{ijt} = 2\pi \sum_{j=0}^{m\beta} |a_j|^2 e^{ijt},
\end{equation}

and
\begin{equation}
\int_{-\pi}^\pi |F(x_1)|^{2\beta} dx_1 = \int_{-\pi}^\pi |F(x_1)|^\beta dx_1 = 2\pi \sum_{j=0}^{m\beta} |a_j|^2.
\end{equation}

Thus for $t \in (0, c)$, $0 \leq j \leq m\beta \leq n\beta$, we have $0 \leq jt \leq n\beta c < 1$ and
\begin{equation}
\int_{-\pi}^\pi |F(x_1)|^\beta |F(x_1 + t)|^\beta dx_1
\geq \text{Re} \int_{-\pi}^\pi F(x_1)^\beta F(x_1 + t)^\beta dx_1
= 2\pi \sum_{j=0}^{m\beta} |a_j|^2 (\cos jt) \geq 2\pi \sum_{j=0}^{m\beta} |a_j|^2 \cos(n\beta c)
= \cos(n\beta c) \int_{-\pi}^\pi |F(x_1)|^{2\beta} dx_1,
\end{equation}

integrating for $t \in (0, c)$ gives
\begin{equation}
\int_{-\pi}^\pi dx_1 \int_{x_1}^{x_1+c} dx_2 |e^{ix_1} - e^{ix_2}|^\beta |F(x_1)|^\beta |F(x_2)|^\beta
\geq \int_0^c dt |1 - e^{it}|^\beta \cos(n\beta c) \int_{-\pi}^\pi |F(x_1)|^{2\beta} dx_1.
\end{equation}

As $(\sin x)/x$ is decreasing for $x \in (0, 1)$ and $0 < c \leq n\beta c < 1$, we further have
\begin{equation}
\int_0^c dt |1 - e^{it}|^\beta = \int_0^c dt |2 \sin(t/2)|^\beta \geq \int_0^c dt \left|\frac{\sin(c/2)}{c/2}\right|^\beta = c^{\beta+1} \left(\frac{\sin(c/2)}{c/2}\right)^\beta.
\end{equation}
Therefore, we have

\begin{align*}
\int_{-\pi}^{\pi} dx_1 \int_{x_1}^{x_1+c} dx_2 |e^{ix_1} - e^{ix_2}|^\beta |F(x_1)|^\beta |F(x_2)|^\beta \\
\geq \frac{c^{\beta+1}}{\beta + 1} \left( \frac{\sin(c/2)}{c/2} \right)^\beta \cos(n_\beta c) \int_{-\pi}^{\pi} |F(x_1)|^{2\beta} dx_1,
\end{align*}

which is the lower bound in the first inequality.

On the other hand, since \( F \) is 2\( \pi \)-periodic, for \( t \in (0, c) \), we have

\begin{align*}
0 &\leq \int_{-\pi}^{\pi} |F(x_1)|^\beta - |F(x_1 + t)|^\beta |^2 dx_1 \\
&= \int_{-\pi}^{\pi} (|F(x_1)|^{2\beta} + |F(x_1 + t)|^{2\beta}) dx_1 - 2 \int_{-\pi}^{\pi} |F(x_1)|^\beta |F(x_1 + t)|^\beta dx_1 \\
&= 2 \int_{-\pi}^{\pi} |F(x_1)|^{2\beta} dx_1 - 2 \int_{-\pi}^{\pi} |F(x_1)|^\beta |F(x_1 + t)|^\beta dx_1,
\end{align*}

which implies

\begin{equation}
\int_{-\pi}^{\pi} |F(x_1)|^\beta |F(x_1 + t)|^\beta dx_1 \leq \int_{-\pi}^{\pi} |F(x_1)|^{2\beta} dx_1,
\end{equation}

and using (17) and \( 2 - 2 \cos(n_\beta c) \leq (n_\beta c)^2 \), we also have

\begin{equation}
\int_{-\pi}^{\pi} \|F(x_1)\|^\beta - |F(x_1 + t)|^\beta |^2 dx_1 \leq (n_\beta c)^2 \int_{-\pi}^{\pi} |F(x_1)|^{2\beta} dx_1.
\end{equation}

By (17) and (18), we have

\begin{align*}
\int_{-\pi}^{\pi} dx_1 \int_{x_1}^{x_1+c} dx_2 |e^{ix_1} - e^{ix_2}|^\beta |F(x_1)|^\beta |F(x_2)|^\beta \\
\leq \int_{0}^{c} dt |1 - e^{it}|^\beta \int_{-\pi}^{\pi} |F(x_1)|^{2\beta} dx_1 \\
\leq \int_{0}^{c} dt |t|^\beta \int_{-\pi}^{\pi} |F(x_1)|^{2\beta} dx_1 \\
= \frac{c^{\beta+1}}{\beta + 1} \int_{-\pi}^{\pi} dx_1 |F(x_1)|^{2\beta},
\end{align*}

which gives the upper bound in the first inequality.

If \( x_j \in (x_1, x_1 + c) \) for \( 1 < j \leq k \), then we have \( |e^{ix_j} - e^{ix_l}| \leq |x_j - x_l| < c \) for \( 1 \leq j < l \leq k \), therefore,

\begin{align*}
\int_{-\pi}^{\pi} dx_1 \int_{(x_1, x_1+c)^{k-1}} dx_2 \cdots dx_k &\prod_{1 \leq j < l \leq k} |e^{ix_j} - e^{ix_l}|^\beta \prod_{j=1}^{k} |F(x_j)|^\beta \\
\leq \int_{-\pi}^{\pi} dx_1 \int_{(x_1, x_1+c)^{k-1}} dx_2 \cdots dx_k \prod_{1 \leq j < l \leq k} c^\beta \prod_{j=1}^{k} |F(x_j)|^\beta \\
= c^{\beta(k-1)/2} \int_{(0,c)^{k-1}} dt_2 \cdots dt_k \int_{-\pi}^{\pi} dx_1 \prod_{j=1}^{k} |F(x_1 + t_j)|^\beta,
\end{align*}
Lemma 6. We first need the following estimate which is the second inequality, here we denote $t_1 = 0$.

By changing of variables, the definition of $\varphi(\beta, A)$, Hölder inequality and (19), we have

$$
\leq \frac{c^{\beta(k-1)/2}}{k} \int_{(0,c)^{k-1}} dt_2 \cdots dt_k \int_{-\pi}^\pi dx_1 \sum_{j=1}^k |F(x_1 + t_j)|^{\beta k} \lesssim \sum_{j=1}^k \int_{(0,c)^{k-1}} dt_2 \cdots dt_k \int_{-\pi}^\pi dx_1 |F(x_1)|^{\beta k}
$$

which is the third inequality. This completes the proof. □

4. No successive small gaps

In this section, we will prove Lemma 2 which implies that there is no successive small gaps. We first need the following estimate

Lemma 6. For $B = (0, c_0) \times (-\pi, \pi)$, $n \geq k > 1$, $n^{1-\gamma}\beta c_0 \in (0, 1)$, we have

$$
E\chi^{(n, \gamma, k-1)}(B) \leq n^{1-\gamma}(n^{1-\gamma}\beta c_0)^{\beta(k-1)/2 + k-1}.
$$
Proof. We consider the point process

\[ \xi^{(n)} = \sum_{i=1}^{n} \delta_{\theta_i}, \quad \xi^{(n,k)} = \sum_{i_1, \ldots, i_k \text{ all distinct}} \delta_{(\theta_{i_1}, \ldots, \theta_{i_k})}, \]

For \( B = (0, c_0) \times (-\pi, \pi) \), \( n \geq k > 1 \), let \( c_n = c_0/n^\gamma \), then we have

\[ \bar{\chi}^{(n,\gamma,j)}(B) = \sum_{i=1}^{n} \chi^{(n)}(\theta_i, (0, c_n)) \geq \frac{1}{j!} \xi^{(n,j+1)}(A_{j+1,c_n}), \]

here, the angles are modulo \( 2\pi \) and

\[ \Lambda_{k,c} = \{ (\theta_1, \ldots, \theta_k) : \theta_1 \in (-\pi, \pi), \theta_j - \theta_1 \in (0, c), \forall 1 < j \leq k \}. \]

Let

\[ \Lambda_{k,c,n} = \{ (\theta_1, \ldots, \theta_n) : \theta_j \in (-\pi, \pi), \forall 1 \leq j \leq n - k + 1, \theta_j - \theta_{n-k+1} \in (0, c), \forall n - k + 1 < j \leq n \}, \]

then by Lemma 1 and Lemma 3, we have

\[
\mathbb{E}\bar{\chi}^{(n,\gamma,k-1)}(B) \leq \frac{1}{(k-1)!} \mathbb{E}\xi^{(n,k)}(\Lambda_{k,c,n})
\]

\[
= \frac{1}{(k-1)! (n-k)!} \int_{\Lambda_{k,c,n}} J(\theta_1, \ldots, \theta_n) d\theta_1 \cdots d\theta_n
\]

\[
= \frac{1}{(k-1)! (n-k)!} \left( \frac{1}{C_{\beta,n}} \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_{n-k} \prod_{1 \leq j < m \leq n-k} |e^{i\theta_j} - e^{i\theta_m}|^\beta \right)
\]

\[
\times \int_{\Lambda_{k,c,n}} dx_1 \cdots dx_k \prod_{1 \leq j < m \leq k} |e^{ix_j} - e^{ix_m}|^\beta
\]

\[
\leq \frac{n^k}{(k-1)!} \frac{1}{C_{\beta,n}} \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_{n-k} \prod_{1 \leq j < m \leq n-k} |e^{i\theta_j} - e^{i\theta_m}|^\beta
\]

\[
\times C^n_{\beta,(k-1)/2+k-1} \int_{-\pi}^{\pi} dx_1 \prod_{m=1}^{n-k} |e^{ix_1} - e^{i\theta_m}|^{k\beta}
\]

\[
= \frac{n^k}{(k-1)!} \left( \frac{C^n_{\beta,(k-1)/2+k-1}}{C_{\beta,n}} \right)^{\beta(k-1)/2+k-1}
\]

\[
\leq \frac{n^k}{(k-1)!} (n\beta)^{k(k-1)/2+k-1} \left( \frac{c_n^{\beta(k-1)/2+k-1}}{(k-1)! \beta^k} \right)
\]

\[
\leq n^{(n\beta c_n)^{\beta(k-1)/2+k-1}} = n^{(n^1 - 1)^\beta c_0 \beta(k-1)/2+k-1},
\]

this completes the proof. \( \square \)

Now we can give the proof of Lemma 2.

Proof. Let \( c \) be such that \( A \subset (0, c) \), and \( B = (0, c) \times (-\pi, \pi) \), \( \gamma = \beta + \frac{2}{\beta+1} \). Then by definitions 11 and 13, \( \chi^{(n)}(A \times I) - \bar{\chi}^{(n)}(A \times I) \neq 0 \) implies \( \bar{\chi}^{(n,\gamma,j)}(A \times I) > 0 \) for some \( j > 1 \), and thus we must have \( \bar{\chi}^{(n,\gamma,\gamma-1)}(B) > 0 \). Since \( \gamma > 1 \), for \( n \) large enough we have \( n^{n^1 - 1} \beta c \in (0, 1) \), and by Lemma 5 with \( k = 3 \), we have

\[ \mathbb{P}(\chi^{(n)}(A \times I) - \bar{\chi}^{(n)}(A \times I) \neq 0) \leq \mathbb{P}(\bar{\chi}^{(n,\gamma,2)}(B) > 0) \]
\[ \leq E(\tilde{\chi}^{(n,\gamma,2)}(B)) \leq n(n^{1-\gamma}\beta c)^{3\beta+2} = n(n^{-\frac{1}{2+\beta}}\beta c)^{3\beta+2} \to 0, \]

this completes the proof. \qed

5. One more auxiliary point process: comparison inequalities

Now we can introduce another auxiliary point process as

\[ (20) \quad \rho^{(k,n,\gamma)} = \sum_{i_1, \ldots, i_{2k} \text{ all distinct}} \delta(\theta_{i_1}, \theta_{i_2}, \ldots, \theta_{2k-1}, \theta_{2k}), \quad \rho^{(k,n)} = \rho^{(k,n,\gamma)} \big|_{\gamma = \frac{\beta+2}{\beta+4}}, \]

where \( \theta_{i_{2j-1}, i_{2j}, \gamma} \) with \( 1 \leq j \leq k \) is as defined in (\ref{eq}).

Then we have the following comparison lemma which shows that the random variable \( \rho^{(k,n)} \) is equivalent to the factorial moment of \( \tilde{\chi}^{(n)} \) (see (\ref{eq}) also).

Lemma 7. For any bounded intervals \( A \subset \mathbb{R}_+ \) and \( I \subseteq (-\pi, \pi) \), let \( B = A \times I \), then we have

\[ \rho^{(k,n,\gamma)}(B^k) \leq \frac{(\tilde{\chi}^{(n,\gamma)}(B))^!}{(\tilde{\chi}^{(n,\gamma)}(B) - k)!}, \quad \gamma > 0, \]

Let \( c_1 \) be such that \( A \subset (0, c_1) \), \( c_n = c_1 n^{-\frac{\beta+2}{\beta+4}} \) and

\[ a = \max \{ i - j : i, j \in \mathbb{Z}, \theta(i) - \theta(j) \leq 2c_n \}, \]

if \( c_n \in (0, 1) \), then we have

\[ 0 \leq \frac{(\tilde{\chi}^{(n)}(B))^!}{(\tilde{\chi}^{(n)}(B) - k)!} - \rho^{(k,n)}(B^k) \leq k(k - 1)(a - 1)(\tilde{\chi}^{(n)}(B))^{k-1}, \]

and

\[ \rho^{(k,n)}(B^k) \geq (\tilde{\chi}^{(n)}(B))^k - k(k - 1)a(\tilde{\chi}^{(n)}(B))^{k-1}. \]

Proof. Let’s denote

\[ X_1 = \{(i_1, \cdots, i_{2k}) : i_j \in \mathbb{Z}, 1 \leq i_j \leq n, \forall 1 \leq j \leq 2k, \]

\[ i_{2j-1} \neq i_{2j}, \quad \forall 1 \leq j \leq k, \{i_{2j-1}, i_{2j}\} \neq \{i_{2l-1}, i_{2l}\}, \quad \forall 1 \leq j < l \leq k, \]

\[ X_2 = \{(i_1, \cdots, i_{2k}) : i_j \in \mathbb{Z}, 1 \leq i_j \leq n, \forall 1 \leq j \leq 2k, \]

\[ i_j \neq i_l, \quad \forall 1 \leq j < l \leq 2k, \]

\[ Y_{i,l} = \{(i_1, \cdots, i_{2k}) : \{i_{2j-1}, i_{2j}\} \cap \{i_{2l-1}, i_{2l}\} \neq \emptyset}, \]

then we have \( X_2 \subseteq X_1 \) and \( X_1 \setminus X_2 = \cup_{1 \leq j < l \leq k} Y_{j,l} \). Let

\[ X_{j,B} = \{(i_1, \cdots, i_{2k}) \in X_j : \theta_{i_{2j-1}, i_{2j}, \gamma} \in B, \forall 1 \leq j \leq k, j = 1, 2, \]

\[ Y_{j,l,B} = \{(i_1, \cdots, i_{2k}) \in Y_{j,l} : \theta_{i_{2j-1}, i_{2j}, \gamma} \in B, \forall 1 \leq j \leq k, j = 1, 2, \]

then we have

\[ (21) \quad \rho^{(k,n,\gamma)}(B^k) = |X_{2,B}|, \quad X_{2,B} \subseteq X_{1,B}, |X_{1,B}| = \frac{(\tilde{\chi}^{(n,\gamma)}(B))^!}{(\tilde{\chi}^{(n,\gamma)}(B) - k)!}, \]

which gives the first inequality, here \( |X| \) is the number of elements in the set \( X \).

We also have \( X_{1,B} \setminus X_{2,B} = \cup_{1 \leq j < l \leq k} Y_{j,l,B} \) and by symmetry \( |Y_{j,l,B}| = |Y_{1,2,B}| \) for \( 1 \leq j < l \leq k \), therefore

\[ (22) \quad |X_{1,B}| - |X_{2,B}| \leq \sum_{1 \leq j < l \leq k} |Y_{j,l,B}| = k(k - 1)|Y_{1,2,B}|/2. \]
Now we assume $\gamma = \frac{\beta + 2}{\beta + 1}$. If $a = 0$, then we have $\theta_{j,l} \geq n^{-\gamma}(2c_n) = 2c_1$ for every $1 \leq j < l \leq n$, thus $\theta_{j,l,\gamma} \not\in B$, and $\chi^{(n)}(B) = \rho^{(k,n)}(B^k) = 0$; if $k = 1$, then by definition $\chi^{(n)}(B) = \rho^{(k,n)}(B^k)$. Thus the second and third inequalities are clearly true in these two trivial cases, for the rest, we only need to consider the case $a > 0, k > 1$.

For fixed $\theta_{i_1,i_2,\gamma} \in B$, let

$$T_j = \{l : l \neq i_j, \theta_{i_j,l,\gamma} \in B\} \cup \{l : l \neq i_j, \theta_{i_j,i_j,\gamma} \in B\},$$
$$T_j' = \{l : l \neq i_j, \theta_{i_j,l} \in (0,c_n)\} \cup \{l : l \neq i_j, \theta_{i_j,i_j} \in (0,c_n)\}, \quad j = 1, 2.$$  

Then we have $T_j \subseteq T_j'$ since $\theta_{j,l,\gamma} \in B$ implies $n^\gamma \theta_{j,l} \in A \subset (0,c_1)$ and $\theta_{j,l} \in (0,n^{-\gamma}c_1) = (0,c_n)$. Assume $\theta_{i_1} = \theta_{(p)}$ then we have

$$\{\theta_l : l \in T_1' \cup \{i_1\}\} = \{\theta_{(q)}(\text{mod}2\pi) : |\theta_{(q)} - \theta_{(p)}| < c_n\}$$
$$= \{\theta_{(q)}(\text{mod}2\pi) : r \leq q \leq s\},$$

for some $r,s \in \mathbb{Z}$ such that $|\theta_{(r)} - \theta_{(p)}| < c_n$, $|\theta_{(s)} - \theta_{(p)}| < c_n$, therefore $|\theta_{(r)} - \theta_{(s)}| < 2c_n$, and by the definition of $a$ we have $s - r \leq a$. Since $i_1 \not\in T_j$, we have

$$|T_1'| + 1 = |\{\theta_l : l \in T_1' \cup \{i_1\}\}| = |\{\theta_{(q)}(\text{mod}2\pi) : r \leq q \leq s\}|$$
$$\leq s - r + 1 \leq a + 1,$$

and thus $|T_1| \leq |T_1'| \leq a$. Similarly, we have $|T_2| \leq |T_2'| \leq a$.

Now for $\theta_{i_1,i_2,\gamma} \in B$, by definition we have $i_2 \in T_1$ and $i_1 \in T_2$. If $\theta_{i_3,i_4,\gamma} \in B$, $\{i_1,i_2\} \cap \{i_3,i_4\} \neq \emptyset$, $\{i_3,i_2\} \neq \emptyset$, then we must have $\{i_3,i_4\} = \{i_1,l\}$, $l \in T_2 \setminus \{i_1\}$ or $\{i_3,i_4\} = \{i_2,l\}$, $l \in T_1 \setminus \{i_2\}$, and the order of $i_3,i_4$ is uniquely determined. In fact, by the definition of $\theta_{i,j}$, we have $\theta_{i_3,i_4} + \theta_{i_4,i_3} = 2\pi$, if $\theta_{i_3,i_4} \in B$, $\theta_{i_4,i_3} \in B$ then we have $n^\gamma \theta_{i_3,i_4} + n^\gamma \theta_{i_4,i_3} \in A \subset (0,c_1)$, and $\theta_{i_3,i_4} + \theta_{i_4,i_3} < 2n^{-\gamma}c_1 = 2c_n < 2\pi$, a contradiction. Thus for $\theta_{i_1,i_2,\gamma} \in B$, the number of $(i_3,i_4)$ satisfying $\theta_{i_3,i_4,\gamma} \in B$, $\{i_1,i_2\} \cap \{i_3,i_4\} \neq \emptyset$, $\{i_3,i_2\} \neq \emptyset$ is at most $|T_2| - 1 + |T_1| - 1 \leq 2(a - 1)$. Now there are $\chi^{(n)}(B)$ choices of $(i_1,i_2)$, for fixed $(i_1,i_2)$ there are at most $2(a - 1)$ choices of $(i_3,i_4)$ and $\chi^{(n)}(B)$ choices of $(i_2\ldots i_k)$, $3 \leq l \leq k$, to satisfy $(i_1,\ldots,i_{2k}) \in Y_{1,2,B}$, thus we have

$$|Y_{1,2,B}| \leq \chi^{(n)}(B) \times 2(a - 1) \times \chi^{(n)}(B)^{k-2} = 2(a - 1)\chi^{(n)}(B)^{k-1}.$$  

By (21) and (22), we have

$$0 \leq \frac{(\chi^{(n)}(B))!}{(\chi^{(n)}(B) - k)!} - \rho^{(k,n)}(B^k) = |X_{1,B}| - |X_{2,B}| \leq k(k - 1)|Y_{1,2,B}|/2$$
$$\leq k(k - 1)(a - 1)(\chi^{(n)}(B))^{k-1},$$

which is the second inequality.

The third inequality follows from the second inequality and the fact that

$$\frac{(\chi^{(n)}(B))!}{(\chi^{(n)}(B) - k)!} = \prod_{j=0}^{k-1} (\chi^{(n)}(B) - j) = (\chi^{(n)}(B))^{k} \prod_{j=0}^{k-1} (1 - j/\chi^{(n)}(B))$$
$$\geq (\chi^{(n)}(B))^k \left(1 - \sum_{j=0}^{k-1} j/\chi^{(n)}(B)\right)$$
$$= (\chi^{(n)}(B))^k - k(k - 1)(\chi^{(n)}(B))^{k-1}/2,$$
this completes the proof. \hfill \Box

6. Proof of the uniform bound (11)

We first use Lemma 5 to show that the uniform bound (11) is equivalent to the uniform $L^k$-bound (29), then we use Lemma 6 to prove (29).

Let $B = (0, c_0) \times (-\pi, \pi)$, $n > 2k$, by the definition of $\rho^{(k,n,\gamma)}$ (recall (20)), we have

$$E_{\beta,n,k}(c) = \frac{n!}{(n - 2k)!} \int_{\Sigma_{n,k,c}} J(\theta_1, \ldots, \theta_n) \, d\theta_1 \cdots d\theta_n \bigg|_{c=c_0/n},$$

where

$$\Sigma_{n,k,c} = \{ (\theta_1, \ldots, \theta_n) : \theta_j \in (-\pi, \pi), \forall 1 \leq j \leq n - k, \quad \theta_j - \theta_{j-k} \in (0, c), \forall n - k < j \leq n \}.$$

For $0 \leq l \leq k$, with assumptions in Lemma 5 let’s denote

$$E_{\beta,n,k,l}(c) := \int_{\Sigma_{n-k-l}} \, d\theta_1 \cdots d\theta_{n-l} \prod_{j \leq m} |e^{i\theta_j} - e^{i\theta_m}| \cdot \cdot \cdot |e^{i\theta_k} - e^{i\theta_m}|,$$

Then we have

$$\int_{\Sigma_{n,k,c}} J(\theta_1, \ldots, \theta_n) \, d\theta_1 \cdots d\theta_n = \frac{E_{\beta,n,k,0}(c)}{C_{\beta,n}},$$

and by definition we can check that

$$E_{\beta,n,k,k}(c) = C_{\beta,n-2k,k}.$$

We need to show that (for $0 < n\beta c < 1$)

$$E_{\beta,n,k,l-1}(c) \leq \frac{E_{\beta,n,k,l-1}(c)}{E_{\beta,n,k,l}(c)} \leq \frac{E_{\beta,n,k,l}(c)}{E_{\beta,n,k,l-1}(c)},$$

In fact, after changing the order of variables, we can write

$$E_{\beta,n,k,l-1}(c) = \int_{\Sigma_{n-l-1,k-l,c}} \, d\theta_1 \cdots d\theta_{n-l-1} \prod_{1 \leq j < m \leq n-l-1} |e^{i\theta_j} - e^{i\theta_m}| \cdot \cdot \cdot |e^{i\theta_k} - e^{i\theta_m}| \cdot \cdot \cdot |e^{i\theta_1} - e^{i\theta_m}|,$$

and

$$E_{\beta,n,k,l}(c) = \int_{\Sigma_{n-l-1,k-l,c}} \, d\theta_1 \cdots d\theta_{n-l-1} \prod_{1 \leq j < m \leq n-l-1} |e^{i\theta_j} - e^{i\theta_m}| \cdot \cdot \cdot |e^{i\theta_k} - e^{i\theta_m}| \cdot \cdot \cdot |e^{i\theta_1} - e^{i\theta_m}|,$$

then (26) is the direct consequence of Lemma 5 by taking $F(x) = \prod_{m=1}^{n-l-1} (e^{ix} - e^{i\theta_m})^{q_m}$. By (26) we also have

$$E_{\beta,n,k,l}(c) \leq \left( \frac{c^{\beta+1}}{\beta+1} \right)^{k-l} E_{\beta,n,k,k}(c) = \left( \frac{c^{\beta+1}}{\beta+1} \right)^{k-l} C_{\beta,n-2k,k},$$
and
\[(28) \quad \left(\frac{\sin(c/2)}{c/2}\right)^{k\beta} (\cos(n\beta c))^k \left(\frac{c^{\beta+1}}{\beta+1}\right)^k C_{\beta,n-2k,k} \leq E_{\beta,n,k,0}(c).\]

Let \(c_0\) be fixed such that \(\beta c_0 \in (0, 1)\) and \(B = (0, c_0) \times (-\pi, \pi)\). Thanks to the integral expression of \(E_{\beta,n,k,l}(B^k)\) in [23], the definition of \(E_{\beta,n,k,l}\) above and the upper bound \((28)\), with \(\gamma = 1\), we have
\[
E_{\beta,n,k,0}(c) = \frac{n!}{(n-2k)!} \left| C_{\beta,n} \right|_{c=c_0/n} \\
\geq \frac{n!}{(n-2k)!} \left(\frac{\sin(c/2)}{c/2}\right)^{k\beta} (\cos(n\beta c))^k \left(\frac{c^{\beta+1}}{\beta+1}\right)^k C_{\beta,n-2k,k} \leq E_{\beta,n,k,0}(c). \\
\]

By the first inequality in Lemma 7, we have
\[
\rho^{(k,n,1)}(B^k) \leq \frac{(\chi^{(n,1)}(B))^k}{(\chi^{(n,1)}(B) - k)!} \leq (\chi^{(n,1)}(B))^k.
\]

which implies
\[
\limsup_{n \to +\infty} \mathbb{E}(n^{-1}\chi^{(n,1)}(B))^k \\
\geq \limsup_{n \to +\infty} n^{-k} \mathbb{E}^{(k,n,1)}(B^k) \\
\geq \lim_{n \to +\infty} \frac{n!n^{-2k}}{(n-2k)!} \limsup_{n \to +\infty} \frac{C_{\beta,n-2k,k}}{C_{\beta,n+2k}} (\cos(\beta c_0))^k \left(\frac{c_0^{\beta+1}}{\beta+1}\right)^k \\
= \limsup_{n \to +\infty} \frac{C_{\beta,n-2k,k}}{C_{\beta,n+2k}} \left(\frac{c_0^{\beta+1} \cos(\beta c_0)}{\beta+1}\right)^k.
\]

Thus, to prove (11), we only need to prove
\[(29) \quad \limsup_{n \to +\infty} \mathbb{E}(n^{-1}\chi^{(n,1)}(B))^k < +\infty.
\]

As \(\chi^{(n,\gamma)} = \sum_{j=1}^{n-1} \chi^{(n,\gamma,j)}\), by Lemma 6 (since \(\beta c_0 \in (0, 1)\)), we have
\[
\mathbb{E}(n^{-1}\chi^{(n,1,j)}(B)) \leq (\beta c_0)^{\beta j(j+1)/2+j} \leq (\beta c_0)^{j}.
\]

Using \(0 \leq \chi^{(n,1,j)}(B) \leq n\), we have
\[
\mathbb{E}(n^{-1}\chi^{(n,1,j)}(B))^k \leq \mathbb{E}(n^{-1}\chi^{(n,1,j)}(B)) \leq (\beta c_0)^{j}.
\]

By Minkowski inequality, we finally have
\[
(\mathbb{E}(n^{-1}\chi^{(n,1)}(B))^k)^{1/k} \leq \sum_{j=1}^{n-1} (\mathbb{E}(n^{-1}\chi^{(n,1,j)}(B))^k)^{1/k} \leq \sum_{j=1}^{n-1} (\beta c_0)^{j/k} \\
\leq (1 - (\beta c_0)^{1/k})^{-1},
\]

thus \((29)\) is true, so is (11).
7. The Convergence of Factorial Moments

In this section, we will prove Lemma 3.

For \( B = A \times I \), we will use Lemma 7 to deduce that

\[
\lim_{n \to +\infty} \left( E \left( \frac{\hat{X}^{(n)}(B)!}{(\hat{X}^{(n)}(B) - k)!} - E\rho^{(k,n)}(B^k) \right) \right) = 0,
\]

and use Lemma 5 to deduce that

\[
\lim_{n \to +\infty} \left( E(\rho^{(k,n)}(A \times I)) - \left( \int_A u^{\beta} du \right)^k \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n}n^{\beta k}} \right) = 0,
\]

then Lemma 6 follows from (30) and (31).

Let \( A \subset \mathbb{R}_+ \) be any bounded interval, \( I \subseteq (-\pi, \pi) \) and \( B = A \times I \). Let \( c_1 \) be such that \( A \subset (0, c_1) \), and \( B_1 = (0, c_1) \times (-\pi, \pi) \) such that \( B \subset B_1 \). Let’s denote \( \gamma = \frac{\beta + 2}{\beta + 1} \) and \( c_n = c_1/n^\gamma \).

Since \( \gamma > 1 \), for \( n \) large enough we have \( n\beta c_n = n^{1-\gamma} \beta c_1 \in (0, 1) \). By the expression of \( E\rho^{(k,n,\gamma)}(B^k) \), \( E_{\beta,n,k,l} \) and (27) again, with \( \gamma(\beta + 1) = \beta + 2 \), as in [6] we have

\[
E\rho^{(k,n)}(B^k) \leq E\rho^{(k,n)}(B_1^k) = \frac{n!}{(n-2k)!} \frac{E_{\beta,n,k,0}(c_n)}{C_{\beta,n}} \leq \frac{n!}{(n-2k)!} \frac{C_{\beta,n-2k,k}}{C_{\beta,n}} \left( \frac{c_1^{\beta+1}}{\beta+1} \right)^k N^{-\gamma(\beta+1)k}.
\]

Using (11), we have

\[
\limsup_{n \to +\infty} E\rho^{(k,n)}(B^k) < +\infty.
\]

Let \( a \) be defined in Lemma 7 and assume \( n \) large enough such that \( 0 < c_n \leq n\beta c_n = n^{1-\gamma} \beta c_1 < 1/4 \). By definition, we have \( 0 \leq a < n \) and \( a \geq k \) is equivalent to \( X^{(n,\gamma,k)}(B_2) > 0 \), here, \( a, k \in \mathbb{Z} \), \( k > 0 \) and \( B_2 = (0, 2c_1) \times (-\pi, \pi) \).

By Lemma 4 and (1-\gamma)(\beta + 1) = -1, for \( 1 \leq k < n \), we have

\[
P(a \geq k) = P(X^{(n,\gamma,k)}(B_2) > 0) \leq E(X^{(n,\gamma,k)}(B_2)) \leq n(2n^{1-\gamma} \beta c_1)^k \beta^{\delta + 1} + n(2n^{1-\gamma} \beta c_1)\beta^{\delta + (k+2)\beta/k + 1} \leq (2\beta c_1)^{\delta + 1}(2n^{1-\gamma} \beta c_1)^{k+2} \leq (2\beta c_1)^{\delta + 1}(1/2)^k.
\]

Since \( P(a \geq k) = 0 \) for \( k \geq n \), thus

\[
P(a \geq k) \leq (2\beta c_1)^{\delta + 1}(1/2)^k.
\]

is always true for \( k \geq 1 \).

The above argument also implies that for \( k > 1, k \in \mathbb{Z} \), we must have

\[
\lim_{n \to +\infty} P(a \geq k) = 0.
\]

And by dominated convergence theorem, we can further deduce that

\[
\lim_{n \to +\infty} E(a - 1)^p = 0, \ \forall \ p \in (0, +\infty),
\]
here, \( f_+ = \max(f, 0) \).

By Lemma 7 for any \( k \rightarrow 1 \), we have \( (\tilde{\chi}^{(n)}(B))^k \leq 2\rho^{(k,n)}(B^k) \) or \( (\tilde{\chi}^{(n)}(B))^k \leq 2k(k-1)a(\tilde{\chi}^{(n)}(B))^k \), therefore, we have
\[
(\tilde{\chi}^{(n)}(B))^k \leq \max(2\rho^{(k,n)}(B^k), (2k(k-1)a)^k),
\]
and
\[
\mathbb{E}(\tilde{\chi}^{(n)}(B))^k \leq 2\mathbb{E}(\rho^{(k,n)}(B^k)) + (2k(k-1))k\mathbb{E}(a^k).
\]
By (32) and (33), we have
\[
\text{lim sup}_{n \to +\infty} \mathbb{E}(\tilde{\chi}^{(n)}(B))^k < +\infty.
\]

By Lemma 7 Hölder inequality, (33) and (34), we have
\[
0 \leq \mathbb{E}\left( \frac{(\tilde{\chi}^{(n)}(B))!}{(\tilde{\chi}^{(n)}(B) - k)!} - \rho^{(k,n)}(B^k) \right)
\leq k(k-1)\mathbb{E}((a-1)_+(\tilde{\chi}^{(n)}(B))^{k-1})
\leq k(k-1)(\mathbb{E}((a-1)^{k-1/k}(\mathbb{E}(\tilde{\chi}^{(n)}(B))^k))^{1-1/k} \to 0
\]
as \( n \to +\infty \), which implies (30).

For \( B = A \times I \), \( n > 2k, \gamma > 0 \) we have
\[
\mathbb{E}\rho^{(k,n,\gamma)}(B^k) = \frac{n!}{(n-2k)!} \int_{\Sigma_{n,k,A,I}} J(\theta_1, \ldots, \theta_n) d\theta_1 \cdots d\theta_n |_{\gamma = n-\gamma},
\]
here,
\[
\Sigma_{n,k,A,I} = \{ (\theta_1, \ldots, \theta_n) : \theta_j \in (-\pi, \pi), \forall 1 \leq j \leq n-2k, \theta_{j-k} \in I, \theta_j - \theta_{j-k} \in A, \forall n-k < j \leq n \}.
\]

Let’s denote
\[
\Sigma_{n,k,A,I,I} = \{ (\theta_1, \ldots, \theta_{n-l}) : \theta_j \in (-\pi, \pi), \forall 1 \leq j \leq n-2k, \theta_j \in I, \forall n-2k < j \leq n-k, \theta_j - \theta_{j-k+l} \in A, \forall n-k < j \leq n-l \},
\]
and
\[
E_{\beta,n,k,l}(A,I) := \int_{\Sigma_{n,k,A,I,I}} d\theta_1 \cdots d\theta_{n-l} \prod_{j<p} |e^{i\theta_j} - e^{i\theta_{p}}| q_j q_{p,\beta},
\]
with \( q_s = 1 + \chi_{(n-2k<s < n-2k+l)} \), then we have
\[
\int_{\Sigma_{n,k,A,I}} J(\theta_1, \ldots, \theta_n) d\theta_1 \cdots d\theta_n = \frac{E_{\beta,n,k,0}(A,I)}{C_{\beta,n}},
\]
and
\[
E_{\beta,n,k,k}(A,I) = C_{\beta,n-2k,k}(I).
\]

We need inequalities similar to (26).

**Lemma 8.** \( A \subset (0,c) \) and \( I \subset (-\pi, \pi) \), \( n\beta c \in (0,1) \), \( n > 2k, n, \beta, k \) are positive integers, then we have
\[
\left| E_{\beta,n,k,0}(A,I) - \left( \int_A w^\beta \, du \right)^k \right|
\]
\[
\leq (kn\beta c + \beta kc^2/24) \left( \frac{c^{\beta+1}}{\beta+1} \right)^k C_{\beta,n-2k,k}.
\]

**Proof.** As before, after changing the order of variables, we can write

\[
E_{\beta,n,k,l-1}(A, I) = \int_{\Sigma_{n-2,k-1,A,I,l-1}} d\theta_1 \cdots d\theta_{n-l-1} \Delta^\beta
\]

\[
\times \int_I dx_1 \int_{x_1+A} dx_2 [e^{ix_1} - e^{ix_2}]^2 \prod_{j=1}^{n-l-1} \prod_{m=1}^{n-1} |e^{ix_j} - e^{i\theta_m}| q_m^\beta,
\]

and

\[
E_{\beta,n,k,l}(A, I) = \int_{\Sigma_{n-2,k-1,A,I,l-1}} d\theta_1 \cdots d\theta_{n-l-1} \Delta^\beta
\]

\[
\times \int_I dx_1 \prod_{m=1}^{n-1} |e^{ix_1} - e^{i\theta_m}| 2q_m^\beta,
\]

here,

\[
\Delta = \prod_{1 \leq j < m \leq n-l-1} |e^{i\theta_j} - e^{i\theta_m}| q_j q_m, \quad q_s = 1 + \chi_{\{n-2k<s<n-2k+l\}}.
\]

By Lemma 17, \(\Sigma_{n-2,k-1,A,I,l-1} \subset \Sigma_{n-l,k-l,c}\) and (27), we have

\[
|E_{\beta,n,k,l-1}(A, I) - \varphi(\beta, A)E_{\beta,n,k,l}(A, I)|
\]

\[
\leq \varphi(\beta, A)(n\beta c) \int_{\Sigma_{n-2,k-1,A,I,l-1}} d\theta_1 \cdots d\theta_{n-l-1} \Delta^\beta \int_{-\pi}^\pi dx_1 \prod_{m=1}^{n-1} |e^{ix_1} - e^{i\theta_m}| 2q_m^\beta
\]

\[
\leq \varphi(\beta, A)(n\beta c) \int_{\Sigma_{n-l,k-l,c}} d\theta_1 \cdots d\theta_{n-l-1} \int_{-\pi}^\pi dx_1
\]

\[
\prod_{1 \leq j < m \leq n-l-1} |e^{i\theta_j} - e^{i\theta_m}| q_j q_m \prod_{m=1}^{n-1} |e^{ix_1} - e^{i\theta_m}| 2q_m^\beta
\]

\[
= \varphi(\beta, A)(n\beta c) \int_{\Sigma_{n-l,k-l,c}} d\theta_1 \cdots d\theta_{n-l-1} \prod_{1 \leq j < m \leq n-l-1} |e^{i\theta_j} - e^{i\theta_m}| q_j q_m^\beta
\]

\[
= \varphi(\beta, A)(n\beta c)E_{\beta,n,k,l}(c)
\]

\[
\leq \varphi(\beta, A)(n\beta c) \left( \frac{c^{\beta+1}}{\beta+1} \right)^{k-l} C_{\beta,n-2k,k},
\]

where \(\varphi(\beta, A)\) is as in Lemma 17 and \(E_{\beta,n,k,l}(c)\) is as in (25).

Therefore (using Lemma 17 again), we have

\[
|E_{\beta,n,k,0}(A, I) - \varphi(\beta, A)^k E_{\beta,n,k,k}(A, I)|
\]

\[
\leq \sum_{l=1}^k \varphi(\beta, A)^{l-1} |E_{\beta,n,k,l-1}(A, I) - \varphi(\beta, A)E_{\beta,n,k,l}(A, I)|
\]

\[
\leq \sum_{l=1}^k \varphi(\beta, A)^l (n\beta c) \left( \frac{c^{\beta+1}}{\beta+1} \right)^{k-l} C_{\beta,n-2k,k}
\]
\[ \leq \sum_{i=1}^{k} (n\beta c) \left( \frac{e^{\beta+1}}{\beta+1} \right)^k C_{\beta,n-2k,k} = (kn\beta c) \left( \frac{e^{\beta+1}}{\beta+1} \right)^k C_{\beta,n-2k,k}. \]

As \( 1 \geq \frac{\sin x}{x} \geq 1 - x^2/6 > 0 \) for \( x \in (0,1) \), and by Lemma 5 we have
\[
0 \leq \left( \int_A u^\beta du \right)^k - \varphi(\beta, A)^k \leq \left( \int_A u^\beta du \right)^k \left( 1 - \left( \frac{\sin(c/2)}{c/2} \right)^{\beta k} \right) \leq \left( \frac{e^{\beta+1}}{\beta+1} \right)^k \left( 1 - (1 - e^{2/24})^{\beta k} \right) \leq \left( \frac{e^{\beta+1}}{\beta+1} \right)^k \beta k e^2/24.
\]

By definition,
\[ 0 \leq E_{\beta,n,k,k}(A,I) = C_{\beta,n-2k,k}(I) \leq C_{\beta,n-2k,k}, \]

therefore, we have
\[
|E_{\beta,n,k,0}(A,I) - \left( \int_A u^\beta du \right)^k C_{\beta,n-2k,k}(I)| \leq |E_{\beta,n,k,0}(A,I) - \varphi(\beta, A)^k E_{\beta,n,k,k}(A,I)| + |\left( \int_A u^\beta du \right)^k - \varphi(\beta, A)^k| C_{\beta,n-2k,k}(I) \leq (kn\beta c) \left( \frac{e^{\beta+1}}{\beta+1} \right)^k C_{\beta,n-2k,k} + \left( \frac{e^{\beta+1}}{\beta+1} \right)^k \beta k e^2/24 C_{\beta,n-2k,k},
\]

which completes the proof. \( \square \)

Now we prove (31). By the integral expression of \( E_{\rho^{(k,n)}}(B^k) \) with \( \gamma = \frac{\beta+2}{\beta+1} \), the definition of \( E_{\beta,n,k,l}(A,I) \) and changing of variables, we have
\[
\mathcal{E}(\rho^{(k,n)}(A \times I)) - \left( \int_A u^\beta du \right)^k \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n} k^\beta} = \frac{n!}{(n-2k)!} E_{\beta,n,k,0}(n^{-\gamma} A,I) - \left( \int_{n^{-\gamma} A} u^\beta du \right)^k \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n} k^{\beta-k} k(k+1) \gamma} \]
\[
= \frac{n^{2k}}{C_{\beta,n}} \left( E_{\beta,n,k,0}(n^{-\gamma} A,I) - \left( \int_{n^{-\gamma} A} u^\beta du \right)^k C_{\beta,n-2k,k}(I) \right) - \left( \frac{n!}{(n-2k)!} \right) \frac{E_{\beta,n,k,0}(n^{-\gamma} A,I)}{C_{\beta,n}} \]
\[
\leq n^{2k} - \frac{n!}{(n-2k)!} = n^{2k} - \prod_{j=0}^{2k-1} (n-j) = n^{2k} - n^{2k} \prod_{j=0}^{2k-1} (1 - j/n) \leq n^{2k} - n^{2k} \left( 1 - \sum_{j=0}^{2k-1} j/n \right) = n^{2k} \sum_{j=0}^{2k-1} j/n = n^{2k-1} k(2k - 1).\]
As \( n^{-\gamma} A \subset (0, n^{-\gamma} c_1) \), \( \Sigma_{n^{-2k-1}, n^{-\gamma} A, I, l-1} \subset \Sigma_{n^{-l-1}, k^{-l}, n^{-\gamma} c_1} \), for \( n \) large enough we have \( n^{1-\gamma} \beta c_1 \in (0, 1) \), then we infer from (27) that

\[
0 \leq E_{\beta, n, k, 0}(n^{-\gamma} A, I) \leq E_{\beta, n, k, 0}(n^{-\gamma} c_1) \leq C_{\beta, n^{-2k}} \left( \frac{(n^{-\gamma} c_1)^{\beta+1}}{\beta+1} \right)^k.
\]

Therefore, we have

\[
0 \leq \left( n^{2k} - \frac{n!}{(n-2k)!} \right) E_{\beta, n, k, 0}(n^{-\gamma} A, I) \leq \frac{n^{2k-1} k(2k-1)}{n^{2k}} \frac{C_{\beta, n^{-2k}} (n^{-\gamma} c_1)^{\beta+1}}{\beta+1} \left( \frac{n^{-\gamma} c_1}{\beta+1} \right)^k.
\]

\[
= \frac{n^{2k-1} k(2k-1)}{n^{2k}} \frac{C_{\beta, n^{-2k}} (n^{-\gamma} c_1)^{\beta+1}}{\beta+1} \left( \frac{n^{-\gamma} c_1}{\beta+1} \right)^k.
\]

By Lemma 8, we have

\[
\frac{n^{2k}}{C_{\beta, n}} \left( \frac{k n^{\beta c + \beta k c^2/24}}{\beta+1} \right)^k \left( \frac{C_{\beta, n^{-2k}} (n^{-\gamma} c_1)^{\beta+1}}{\beta+1} \right)^k C_{\beta, n^{-2k}} \left| \left( \int_{n^{-\gamma} A} u^{\beta} du \right) C_{\beta, n^{-2k, k}}(I) \right|
\]

\[
\leq \frac{n^{2k}}{C_{\beta, n}} \left( \frac{k n^{\beta c + \beta k c^2/24}}{\beta+1} \right)^k \left( \frac{C_{\beta, n^{-2k}} (n^{-\gamma} c_1)^{\beta+1}}{\beta+1} \right)^k C_{\beta, n^{-2k}} \left| \left( \int_{n^{-\gamma} A} u^{\beta} du \right) C_{\beta, n^{-2k, k}}(I) \right|
\]

\[
= \frac{n^{2k}}{C_{\beta, n}} \left( \frac{k n^{\beta c + \beta k c^2/24}}{\beta+1} \right)^k \left( \frac{C_{\beta, n^{-2k}} (n^{-\gamma} c_1)^{\beta+1}}{\beta+1} \right)^k C_{\beta, n^{-2k}} \left| \left( \int_{A} u^{\beta} du \right) C_{\beta, n^{-2k, k}}(I) \right|
\]

\[
\leq \left( k n^{1-\gamma} \beta c_1 + \beta k n^{-2\gamma} c_1^2/24 + n^{-1} k(2k-1) \right) \left( \frac{C_{\beta, n^{-2k}} (n^{-\gamma} c_1)^{\beta+1}}{\beta+1} \right)^k C_{\beta, n^{-2k}} \left| \left( \int_{A} u^{\beta} du \right) C_{\beta, n^{-2k, k}}(I) \right|
\]

\[
\leq (k n^{1-\gamma} \beta c_1 + \beta k n^{-2\gamma} c_1^2/24 + n^{-1} k(2k-1)) \left( \frac{C_{\beta, n^{-2k}} (n^{-\gamma} c_1)^{\beta+1}}{\beta+1} \right)^k C_{\beta, n^{-2k}} \left| \left( \int_{A} u^{\beta} du \right) C_{\beta, n^{-2k, k}}(I) \right|
\]

\[
\leq (k n^{1-\gamma} \beta c_1 + \beta k n^{-2\gamma} c_1^2/24 + n^{-1} k(2k-1)) \left( \frac{C_{\beta, n^{-2k}} (n^{-\gamma} c_1)^{\beta+1}}{\beta+1} \right)^k C_{\beta, n^{-2k}} \left| \left( \int_{A} u^{\beta} du \right) C_{\beta, n^{-2k, k}}(I) \right|
\]

Now (31) follows from (11) of the uniform boundedness of \( C_{\beta, n^{-2k}} \) and

\[
\lim_{n \to +\infty} (k n^{1-\gamma} \beta c_1 + \beta k n^{-2\gamma} c_1^2/24 + n^{-1} k(2k-1)) = 0.
\]

8. PROOF OF THE UPPER BOUND

Now we consider (12). We will make use of several formulas, especially these on the generalized hypergeometric functions \( 2F_1^{(a)} \), where we refer Chapter 13 of [15] for more details.
By definition, we can rewrite the two-component log-gas as
\[
C_{\beta,n,2}(I) = \int_{I^2} dr_1 dr_2 |e^{ir_1} - e^{ir_2}|^{4\beta} I_{n,2}(\beta; r_1, r_2),
\]
here
\[
I_{n,2}(\beta; r_1, r_2) := \int_{(-\pi, \pi)^n} d\theta_1 \cdots d\theta_n \prod_{j=1}^{n} \prod_{k=1}^{2} (1 - e^{i(\theta_j - r_k)})^{2\beta} \prod_{1 \leq j < k \leq n_1} |e^{i\theta_j} - e^{i\theta_k}|^\beta.
\]

Now the uniform upper bound \[(37)\] is a direct consequence of the following lemma, together with the integral expression \[(35)\] (with \(n_1 = n - 4\) and Fatou’s Lemma).

**Lemma 9.** There exists a constant \(C\) depending only on \(\beta\) such that
\[
I_{n-4,2}(\beta; r_1, r_2)|e^{ir_1} - e^{ir_2}|^{4\beta} \leq C C_{\beta,n}^{2\beta}, \quad \forall \; n > 4, \; r_1, r_2 \in [-\pi, \pi],
\]
and
\[
\limsup_{n \to +\infty} C_{\beta,n}^{-1} n^{-2\beta} I_{n-4,2}(\beta; r_1, r_2)|e^{ir_1} - e^{ir_2}|^{4\beta} \leq (2\pi)^{-2} A_{\beta}^{2}.
\]

We need to prepare a lot in order to prove Lemma 9. By Proposition 13.1.2 in [5], we have the following relation between the generalized hypergeometric function \(2F_1^{(\alpha)}\) and the Selberg type integrals,
\[
\frac{1}{M_n(a, b, 1/\alpha)} \int_{-1/2}^{1/2} d\theta_1 \cdots \int_{-1/2}^{1/2} d\theta_n \prod_{l=1}^{n} \left( e^{\pi i\theta_l (a-b)} |1 + e^{2\pi i\theta_l}|^{a+b} \right)
\]
\[
\prod_{l' = 1}^{m} (1 + t_l e^{2\pi i\theta_l}) \prod_{1 \leq j < k \leq n} |e^{2\pi i\theta_j} - e^{2\pi i\theta_k}|^{2/\alpha} =_{2} F_1^{(1/\alpha)}(-n, ab; -(n - 1) - \alpha(1 + a); t_1, \cdots , t_m)
\]
\[
=_{2} F_1^{(1/\alpha)}(-n, ab; \alpha(a + b + m); 1 - t_1, \cdots , 1 - t_m),
\]
\[
_{2} F_1^{(1/\alpha)}(-n, ab; \alpha(a + b + m); (1)^m)
\]
\[
(36)
\]
here, \(M_n(a, b, 1/\alpha)\) is defined as in [13] and we have used the following formula (Proposition 13.1.7 in [5]):
\[
_{2} F_1^{(\alpha)}(a, b; c; t_1, \cdots , t_m) = _{2} F_1^{(\alpha)}(a, b; a + b + 1 + (m - 1)/\alpha - c; 1 - t_1, \cdots , 1 - t_m),
\]
\[
_{2} F_1^{(\alpha)}(a, b; a + b + 1 + (m - 1)/\alpha - c; (1)^m)
\]
By Proposition 13.1.4 in [5], we have
\[
\frac{1}{S_n(\lambda_1, \lambda_2, 1/\alpha)} \int_{0}^{1} dx_1 \cdots \int_{0}^{1} dx_n \prod_{l=1}^{n} x_l^{\lambda_1} (1 - x_l)^{\lambda_2} (1 - sx_l)^{-r}
\]
\[
\times \prod_{1 \leq j < k \leq n} |x_j - x_k|^{2/\alpha}
\]
\[
=_{2} F_1^{(\alpha)} \left( r, \frac{1}{\alpha}(n - 1) + \lambda_1 + 1; \frac{2}{\alpha}(n - 1) + \lambda_1 + \lambda_2 + 2; (s)^n \right),
\]
\[
(37)
\]
here, by (4.1) and (4.3) in [5], the Selberg integral is
\[
S_n(\lambda_1, \lambda_2, \lambda) := \int_0^1 dt_1 \cdots \int_0^1 dt_n \prod_{i=1}^n t_i^{\lambda_1} (1 - t_i)^{\lambda_2} \prod_{1 \leq j < k \leq n} |t_j - t_k|^{2\lambda}
\]
(38)

Now we change variables \(\theta_j \mapsto \theta_j + r_1 \pm \pi\) to obtain
\[
I_{n_1,2}(\beta; r_1, r_2) = \int_{(-\pi,\pi)^{n_1}} d\theta_1 \cdots d\theta_{n_1} \prod_{j=1}^{n_1} (1 + e^{i\theta_j}|^{2\beta} |1 + e^{i(\theta_j+r_1-r_2)}|^{2\beta})
\]
\[
\times \prod_{1 \leq j < k \leq n_1} |e^{\theta_j} - e^{\theta_k}|^{\beta}.
\]

For \(\beta\) positive integer, we have
\[
|1 + e^{i(\theta_j+r_1-r_2)}|^{2\beta} = e^{-i\beta(\theta_j+r_1-r_2)}(1 + e^{i(\theta_j+r_1-r_2)})^{2\beta},
\]
which shows
\[
I_{n_1,2}(\beta; r_1, r_2) = e^{-i\beta n_1(r_1-r_2)} \int_{(-\pi,\pi)^{n_1}} d\theta_1 \cdots d\theta_{n_1} \prod_{j=1}^{n_1} \left(1 + e^{i\theta_j}|^{2\beta} (1 + e^{i(\theta_j+r_1-r_2)})^{2\beta}\right) \prod_{1 \leq j < k \leq n_1} |e^{\theta_j} - e^{\theta_k}|^{\beta}.
\]

Comparing with (39) and changing variables \(\theta_j \mapsto 2\pi \theta_j\), this integral is of the type therein with
\[
n = n_1, \quad m = 2\beta, \quad a - b = -2\beta, \quad a + b = 2\beta, \quad 2/\alpha = \beta,
\]
and
\[
t_k = t := e^{i(r_1-r_2)} \quad \text{for} \quad 1 \leq k \leq m.
\]

Thus (39) shows that \(I_{n_1,2}\) is proportional to
\[
t^{-\beta n_1} 2F_1^{(\beta/2)}(-n_1, 4; 8; ((1 - t)^{2\beta}),
\]
and by (13), \(2F_1^{(\beta/2)}\) equals to 1 at the origin, thus by considering the case of \(t_k = t = 1 \quad (1 \leq k \leq 2\beta)\) for \(r_1 = r_2\), we will have
\[
(39) \quad I_{n_1,2}(\beta; r_1, r_2) = I_{n_1,2}(\beta; r_1, r_1) t^{-\beta n_1} 2F_1^{(\beta/2)}(-n_1, 4; 8; ((1 - t)^{2\beta}),
\]
where
\[
(40) \quad I_{n_1,2}(\beta; r_1, r_1) = \int_{(-\pi,\pi)^{n_1}} d\theta_1 \cdots d\theta_{n_1} \prod_{j=1}^{n_1} |1 + e^{i\theta_j}|^{4\beta}
\]
\[
\times \prod_{1 \leq j < k \leq n_1} |e^{\theta_j} - e^{\theta_k}|^{\beta} = (2\pi)^{n_1} M_{n_1}(2\beta, 2\beta, \beta/2).
\]

Comparison with (37) shows that \(2F_1^{(\beta/2)}\) is of the type therein with
\[
r = -n_1, \quad a = \beta/2, \quad n = 2\beta, \quad \lambda_1 = \lambda_2 = 4 - \frac{1}{\alpha}(n - 1) - 1 = \frac{2}{\beta} - 1, \quad s = 1 - t,
thus by (37), we have

\[ (41) \quad 2F_{1}^{(\beta/2)}(-n_1, 4; 8; \{(1 - t)^{2\beta}) = \frac{1}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} \times \int_{[0,1]^{2\beta}} du_1 \cdots du_{2\beta} \prod_{j=1}^{2\beta} u_j^{2/\beta - 1} (1 - u_j)^{2/\beta - 1}(1 - (1 - t)u_j)^{n_1} \times \prod_{1 \leq j < k \leq 2\beta} |u_j - u_k|^{4/\beta}. \]

Using (39), (40), (41), we have

\[ (42) \quad \prod_{j=1}^{2\beta} u_j^{2/\beta - 1} (1 - u_j)^{2/\beta - 1}(1 - (1 - t)u_j)^{n_1} \prod_{1 \leq j < k \leq 2\beta} |u_j - u_k|^{4/\beta}. \]

Now we rewrite (42) as

\[ I_{n_1, 2}(\beta; r_1, r_2) = \frac{(2\pi)^{n_1} M_{n_1}(2\beta, 2\beta, \beta/2)t^{-\beta n_1}}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} \int_{[0,1]^{2\beta}} du_1 \cdots du_{2\beta} \prod_{j=1}^{2\beta} u_j^{2/\beta - 1} (1 - u_j)^{2/\beta - 1}(1 - (1 - t)u_j)^{n_1} \prod_{1 \leq j < k \leq 2\beta} |u_j - u_k|^{4/\beta}, \]

here \( t = e^{(r_1 - r_2)} \) and we denote

\[ F_{n_1, \beta}(t) := \int_{[0,1]^{2\beta}} du_1 \cdots du_{2\beta} \prod_{j=1}^{2\beta} u_j^{2/\beta - 1} (1 - u_j)^{2/\beta - 1}(1 - (1 - t)u_j)^{n_1} \prod_{1 \leq j < k \leq 2\beta} |u_j - u_k|^{4/\beta}, \]

then \( F_{n_1, \beta} \) is an analytic function (in fact a polynomial) of \( t \). As \( |1 - (1 - t)u_j| = |1 - u_j + tu_j| \leq |1 - u_j| + |tu_j| = 1 \) for \( u_j \in [0,1], |t| = 1 \), we have

\[ |F_{n_1, \beta}(t)| \leq \int_{[0,1]^{2\beta}} du_1 \cdots du_{2\beta} \prod_{j=1}^{2\beta} u_j^{2/\beta - 1} (1 - u_j)^{2/\beta - 1}|1 - (1 - t)u_j|^{n_1} \prod_{1 \leq j < k \leq 2\beta} |u_j - u_k|^{4/\beta} \leq \int_{[0,1]^{2\beta}} du_1 \cdots du_{2\beta} \prod_{j=1}^{2\beta} u_j^{2/\beta - 1}(1 - u_j)^{2/\beta - 1} \prod_{1 \leq j < k \leq 2\beta} |u_j - u_k|^{4/\beta} = S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta), \]

which together with (15) implies

\[ (43) \quad I_{n_1, 2}(\beta; r_1, r_2) = \frac{(2\pi)^{n_1} M_{n_1}(2\beta, 2\beta, \beta/2)}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} |F_{n_1, \beta}(t)| \leq (2\pi)^{n_1} M_{n_1}(2\beta, 2\beta, \beta/2) = (2\pi)^{-1} C_{\beta, n_1}(4). \]

Changing variables \( u_j \mapsto t_j/(1 + t_j) \), we obtain

\[ F_{n_1, \beta}(t) = \int_{(0, +\infty)^{2\beta}} \frac{dt_1 \cdots dt_{2\beta}}{(1 + t_1)^{2} \cdots (1 + t_{2\beta})^{2}}. \]
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\[
\prod_{j=1}^{2\beta} \frac{t_j^{2/\beta-1}}{(1 + t_j)^{2(2/\beta-1)}} \left( \frac{1 + t_j}{1 + t_j} \right)^{n_j} \prod_{1 \leq j < k \leq 2\beta} \left| \frac{t_j - t_k}{(1 + t_j)(1 + t_k)} \right|^{4/\beta} = \int_{(0, +\infty)^{2\beta}} dt_1 \cdots dt_{2\beta} \prod_{j=1}^{2\beta} \frac{t_j^{2/\beta-1}}{(1 + t_j)^{2(2/\beta-1)+2+n_j+4/\beta(2\beta-1)}} \times \prod_{1 \leq j < k \leq 2\beta} |t_j - t_k|^{4/\beta}.
\]

Since \(2(2/\beta - 1) + 2 + 4/\beta \cdot (2\beta - 1) = 4/\beta + 8 - 4/\beta = 8\), we have

\[
F_{n_1, \beta}(-z^2) = \int_{(0, +\infty)^{2\beta}} dt_1 \cdots dt_{2\beta} \prod_{j=1}^{2\beta} \frac{t_j^{2/\beta-1}}{(1 + t_j)^{2(2/\beta-1)+2+n_j+4/\beta(2\beta-1)}} \times \prod_{1 \leq j < k \leq 2\beta} |t_j - t_k|^{4/\beta}.
\]

For \(z \in (0, +\infty)\), a simple changing of variables \(zt_j \mapsto s_j\) shows that

\[
F_{n_1, \beta}(-z^2) = z^{-8\beta} \int_{(0, +\infty)^{2\beta}} ds_1 \cdots ds_{2\beta} \prod_{j=1}^{2\beta} \frac{s_j^{2/\beta-1}}{(1 + z^{-1}s_j)^{8+n_j}} \times \prod_{1 \leq j < k \leq 2\beta} |s_j - s_k|^{4/\beta}.
\]

Since both sides are analytic functions of \(z\) for \(\text{Re} z > 0\), this identity is always true for \(\text{Re} z > 0\), moreover, we can decompose \((0, +\infty)\) into \((0, 1] \cup [1, +\infty)\) and use the symmetry of \(s_j\) to obtain

\[
(44)\quad F_{n_1, \beta}(-z^2) = z^{-8\beta} \sum_{l=0}^{2\beta} \binom{2\beta}{l} F_{n_1, \beta, l}(z), \quad \text{Re} z > 0,
\]

where

\[
F_{n_1, \beta, l}(z) := \int_{(0, 1]^l \times (1, +\infty)^{2\beta-l}} ds_1 \cdots ds_{2\beta} \prod_{j=1}^{2\beta} \frac{s_j^{2/\beta-1}}{(1 + z^{-1}s_j)^{8+n_j}} \times \prod_{1 \leq j < k \leq 2\beta} |s_j - s_k|^{4/\beta}.
\]

The changing of variables \(s_j \mapsto s_j^{-1}\) for \(l < j \leq 2\beta\) shows that

\[
F_{n_1, \beta, l}(z) = \int_{(0, 1]^l} ds_1 \cdots ds_{2\beta} \prod_{j=1}^{2\beta} \frac{s_j^{2/\beta-1}}{(1 + z^{-1}s_j)^{8+n_j}} \times \prod_{j=l+1}^{2\beta} \frac{s_j^{-2/\beta + 1}}{(1 + z^{-1}s_j^{-1})^{8+n_j}} \times \prod_{1 \leq j < k \leq l} |s_j - s_k|^{4/\beta} \prod_{1 \leq j < k \leq 2\beta} |s_j^{-1} - s_k^{-1}|^{4/\beta} \\
\times \prod_{j=1}^{l} \prod_{k=l+1}^{2\beta} |s_j - s_k^{-1}|^{4/\beta}.
\]

\[
= \int_{(0, 1]^l} ds_1 \cdots ds_{2\beta} \prod_{j=1}^{2\beta} \frac{s_j^{2/\beta-1}}{(1 + z^{-1}s_j)^{8+n_j}} \times \prod_{j=l+1}^{2\beta} \frac{s_j^2}{(1 + s_j - z)^{8+n_1}} \prod_{j=1}^{2\beta} \frac{s_j^2}{(s_j + z - 1)^{8+n_1}}.
\]
\[ \times \prod_{1 \leq j < k \leq l} |s_j - s_k|^{4/\beta} \prod_{l < j < k \leq 2\beta} |s_j - s_k|^{4/\beta} \prod_{j = 1}^{l} \prod_{k = l + 1}^{2\beta} |1 - s_j s_k|^{4/\beta}, \]

here, \( a = -2/\beta + 1 + 8 - 4/\beta \cdot (2\beta - 1) = 2/\beta - 1 \). For \( z = e^{i\theta}, \theta \in (-\pi/2, \pi/2) \) i.e., \( \text{Re} \ z > 0 \), and for \( s > 0 \), we have \( |1 + z^{-1}s|^2 = |s + z^{-1}|^2 = 1 + s^2 + 2s \cos \theta > 1 \) and \( |1 - zs| = |s - z| \), therefore, we have

\[ |F_{n_1, \beta, l}(e^{i\theta})| \leq \int_{(0,1]^{2\beta}} ds_1 \cdots ds_{2\beta} \prod_{j = 1}^{l} \sum_{1 \leq j < k \leq l} |s_j|^{2/\beta - 1} |1 - e^{i\theta} s_j|^{n_1} \prod_{l < j \leq 2\beta} |s_j - s_k|^{4/\beta} \prod_{1 \leq j < k \leq l} |s_j - s_k|^{4/\beta}, \]

(45)

here, we used \( |1 - s_j s_k| \leq 1 \) and we denote

\[ F_{n_1, \beta, l}(\theta) := \int_{(0,1]^{l}} ds_1 \cdots ds_l \prod_{j = 1}^{l} \sum_{1 \leq j < k \leq l} |s_j|^{2/\beta - 1} |1 - e^{i\theta} s_j|^{n_1} \prod_{1 \leq j < k \leq l} |s_j - s_k|^{4/\beta}. \]

As \( \frac{1 - s}{1 + s} \leq e^{-2s} \) for \( s \in (0, 1) \), we have

\[ \frac{|1 - e^{i\theta} s|^{n_1}}{|1 + e^{-i\theta} s|^{n_1}} = \frac{1 + s^2 - 2s \cos \theta}{1 + s^2 + 2s \cos \theta} \leq e^{\frac{2s \cos \theta}{1 + s^2}}, \]

which implies

\[ F_{n_1, \beta, l}(\theta) \leq \int_{(0,1]^{l}} ds_1 \cdots ds_l \prod_{j = 1}^{l} \sum_{1 \leq j < k \leq l} |s_j|^{2/\beta - 1} e^{\frac{2s_j n_1 \cos \theta}{1 + s_j^2}} \prod_{1 \leq j < k \leq l} |s_j - s_k|^{4/\beta} \]

\[ \leq \int_{(0,1]^{l}} ds_1 \cdots ds_l \prod_{j = 1}^{l} \sum_{1 \leq j < k \leq l} |s_j|^{2/\beta - 1} e^{-s_j n_1 \cos \theta} \prod_{1 \leq j < k \leq l} |s_j - s_k|^{4/\beta}. \]

Let's denote

\[ J_{n, \beta}(z) := \int_{(0,+\infty)}^{n} \prod_{j=1}^{n} t_j^{2/\beta - 1} e^{-z t_j} \prod_{1 \leq j < k \leq n} |t_j - t_k|^{4/\beta} dt_1 \cdots dt_n, \]

then we have

\[ J_{n, \beta}(z) = z^{-2n^2/\beta} J_{n, \beta}(1). \]

According to Proposition 4.7.3 in [3], we have the explicit evaluation

(46)

\[ J_{n, \beta}(1) = \prod_{j=1}^{n} \frac{\Gamma(1 + 2 j/\beta) \Gamma(2 j/\beta)}{\Gamma(1 + 2/\beta)}. \]

By the definition of \( J_{n, \beta} \), we first easily have the upper bound

(47)

\[ F_{n_1, \beta, l}(\theta) \leq J_{l, \beta}(n_1 \cos \theta) = (n_1 \cos \theta)^{-2l^2/\beta} J_{l, \beta}(1). \]

We change of variables \( n_1 s_j \mapsto t_j \) to get

\[ F_{n_1, \beta, l}(\theta) \leq n_1^{-2l^2/\beta} \int_{(0,n_1]}^{l} dt_1 \cdots dt_l \]
By the dominated convergence theorem, we further have
\[
\limsup_{n_1 \to +\infty} n_1^{2\beta / \beta} F_{n_1, \beta, (l)}(\theta) \leq \int_{(0, +\infty)^l} dt_1 \cdots dt_l \prod_{j=1}^l t_j^{2/\beta - 1 - 2 t_j \cos \theta} \prod_{1 \leq j < k \leq l} |t_j - t_k|^{4/\beta}.
\]

Therefore, we have
\[
(48) \quad \limsup_{n_1 \to +\infty} (2n_1 \cos \theta)^{2\beta / \beta} F_{n_1, \beta, (l)}(\theta) \leq J_{l, \beta}(1).
\]

8.1. Proof of Lemma 9

Now we are ready to give the proof of Lemma 9.

Proof. If |e^{ir_1} - e^{ir_2}| \leq n^{-1}, then the first inequality holds by (13) with n_1 = n - 4 and Lemma 1 i.e.,
\[
I_{n-4, 2}(\beta; r_1, r_2)|e^{ir_1} - e^{ir_2}|^{4\beta} \leq (2\pi)^{-1} C_{\beta, n-4, (4)} n^{-4\beta} \leq CC_{\beta, n^{2\beta}}.
\]

If |e^{ir_1} - e^{ir_2}| \geq n^{-1}, as t = e^{i(r_1 - r_2)}, we have |t - 1| = |e^{ir_1} - e^{ir_2}| \geq n^{-1} and we can write t = -e^{2i\theta} for some \( \theta \in (-\pi / 2, \pi / 2) \), then by (13) and (13), we have
\[
I_{n-4, 2}(\beta; r_1, r_2)|e^{ir_1} - e^{ir_2}|^{4\beta} = \frac{(2\pi)^{n_1} M_{n-4}(2\beta, 2\beta, \beta / 2)}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} |F_{n-4, \beta}(t)||1 - t|^{4\beta}
\]
\[
= \frac{(2\pi)^{-1} C_{\beta, n-4, (4)} [1 - t]^{4\beta}}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} |F_{n-4, \beta}(t)|.
\]

By (14) and (15), we have
\[
|F_{n-4, \beta}(t)| \leq \sum_{l=0}^{2\beta} \binom{2\beta}{l} |F_{n-4, \beta, l}(e^{i\theta})| \leq \sum_{l=0}^{2\beta} \binom{2\beta}{l} |F_{n-4, \beta, (l)}(\theta)|^{2\beta} |F_{n-4, \beta, (2\beta-l)}(\theta)|,
\]
thus
\[
I_{n-4, 2}(\beta; r_1, r_2)|e^{ir_1} - e^{ir_2}|^{4\beta} \leq \sum_{l=0}^{2\beta} I_{n-4, 2, l}(\beta; r_1, r_2),
\]
where
\[
I_{n-4, 2, l}(\beta; r_1, r_2) = \frac{(2\pi)^{-1} C_{\beta, n-4, (4)} [1 - t]^{4\beta}}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} \binom{2\beta}{l} |F_{n-4, \beta, (l)}(\theta)|^{2\beta} |F_{n-4, \beta, (2\beta-l)}(\theta)|.
\]

As t = -e^{2i\theta}, we know that |1 - t| = 2 \cos \theta \geq n^{-1}, by (14) and Lemma 1, we have
\[
I_{n-4, 2, l}(\beta; r_1, r_2) \leq \frac{CC_{\beta, n^{2\beta}}(2\cos \theta)^{4\beta}}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} \binom{2\beta}{l} \times
\]
\[
(n_1 \cos \theta)^{-2^{2/\beta} J_{1, \beta}(1)} (n_1 \cos \theta)^{-2(2\beta-l)^2/\beta} J_{2\beta-1, \beta}(1)
\]
\[
\leq CC_{\beta, n^{2\beta}}(2n \cos \theta)^{4\beta} (n_1 \cos \theta)^{-2^{2/\beta} - 2(2\beta-l)^2/\beta}
\]

Therefore, by (48), Lemma 1 and Lemma 10 below, we have

\[ \lim_{n \to +\infty} C_{\beta, n}^{-1} n^{-2\beta} I_{n-4,2}^\beta (\beta; r_1, r_2) = 0, \]

thus

\[ (49) \quad \lim_{n \to +\infty} C_{\beta, n}^{-1} n^{-2\beta} I_{n-4,2}^\beta (\beta; r_1, r_2) \leq \lim_{n \to +\infty} C_{\beta, n}^{-1} n^{-2\beta} I_{n-4,2}^\beta (\beta; r_1, r_2). \]

Notice that

\[ I_{n-4,2}^\beta (\beta; r_1, r_2) = \frac{(2\pi)^{-1} C_{\beta, n-4, (4)} 1 - t |e^{i\theta} - e^{i\pi/2}|^4}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} |F_n|^2, \]

we have

\[ C_{\beta, n}^{-1} n^{-2\beta} I_{n-4,2}^\beta (\beta; r_1, r_2) = \frac{(2\pi)^{-1} C_{\beta, n-4, (4)} 1 - t |e^{i\theta} - e^{i\pi/2}|^4}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} \left( \frac{2\beta}{\beta} \right) |F_n|^2, \]

Thus, we can write

\[ \frac{(2\pi)^{-1} C_{\beta, n-4, (4)} 1 - t |e^{i\theta} - e^{i\pi/2}|^4}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} \left( \frac{2\beta}{\beta} \right) |F_n|^2 \]

This, together with (49), will complete the proof of Lemma 9 other than Lemma 10.

Now we prove the following identity to complete Lemma 9.

**Lemma 10.** It holds that

\[ (2\pi) A_{\beta, 4} \left( \frac{2\beta}{\beta} \right) \frac{|J_{\beta, \beta}(1)|^2}{S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta)} = A^2_\beta. \]
Proof. Notice that the Selberg integral
\[
S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta) = \prod_{j=0}^{2\beta-1} \frac{\Gamma(2(j+1)/\beta)}{\Gamma(2(2\beta + j + 1)/\beta)} \frac{\Gamma(1 + 2(j+1)/\beta)}{\Gamma(1 + 2/\beta)}
\]
= \prod_{j=1}^{2\beta} \frac{(\Gamma(2j/\beta))^2 \Gamma(1 + 2j/\beta)}{\Gamma(2j/\beta + 4)\Gamma(1 + 2/\beta)} = \prod_{j=1}^{2\beta} \frac{(\Gamma(2j/\beta))^2}{\prod_{k=1}^{2\beta} (2j/\beta + k)\Gamma(1 + 2/\beta)},
\]
that
\[
\prod_{j=1}^{2\beta} \prod_{k=1}^{3} (2j/\beta + k) = (2/\beta)^{6\beta} \prod_{k=1}^{3} (j + k\beta/2) = (2/\beta)^{6\beta} \prod_{k=1}^{3} \frac{(1 + (k + 4)\beta/2)}{(1 + k\beta/2)},
\]
and that (using (46))
\[
\prod_{j=1}^{2\beta} \frac{(\Gamma(2j/\beta))^2 \Gamma(1 + 2j/\beta)}{\Gamma(2j/\beta + 4)\Gamma(1 + 2/\beta)} = \prod_{j=1}^{2\beta} \frac{(\Gamma(2j/\beta))^2}{\prod_{k=1}^{2\beta} (2j/\beta + k)\Gamma(1 + 2/\beta)} \prod_{j=1}^{\beta} (2j/\beta + 1)^2
\]
= \prod_{j=1}^{\beta} \left( \frac{(\Gamma(2j/\beta))^2}{\Gamma((1 + 2/\beta)^2)} \right) = |J_{\beta,\beta}(1)|^2 \prod_{j=1}^{\beta} (2j/\beta + 1)^2
\]
we have
\[
(2\pi)A_{\beta,4} \left( \frac{2\beta}{\beta} \right) S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta) \frac{|J_{\beta,\beta}(1)|^2}{(2/\beta)^{6\beta}}\prod_{j=1}^{3} \frac{(1 + (j + 4)\beta/2)}{(1 + j\beta/2 + 1)},
\]
as in Lemma [1]
\[
A_{\beta,4} = \frac{(2\pi)^{-3}(\Gamma(\beta/2 + 1))^4}{\Gamma(2\beta + 1)}(\beta/2)^{6\beta} \prod_{j=1}^{3} \frac{(j\beta/2 + 1)}{(4 + j)\beta(2\beta + 1)}
\]
we have
\[
(2\pi)A_{\beta,4} \left( \frac{2\beta}{\beta} \right) S_{2\beta}(2/\beta - 1, 2/\beta - 1, 2/\beta) \frac{|J_{\beta,\beta}(1)|^2}{(2/\beta)^{6\beta}}\prod_{j=1}^{3} \frac{(1 + (j + 4)\beta/2)}{(1 + j\beta/2 + 1)}
\]
= \frac{(2\pi)^{-2}(\Gamma(\beta/2 + 1))^4}{\Gamma(2\beta + 1)}(\beta/2)^{6\beta} \frac{(\Gamma(1 + 2\beta)) (\Gamma(1 + \beta/2))^2}{(\Gamma(1 + \beta))^2 (\Gamma(1 + 3\beta/2))^2}
\]
= \frac{(2\pi)^{-2}(\Gamma(\beta/2 + 1))^6}{(\Gamma(1 + \beta))^2 (\Gamma(1 + 3\beta/2))^2}(\beta/2)^{2\beta} = A_{\beta}^3,
\]
this completes the proof. □

9. Proof of Lemma [4]

Now we give the proof of Lemma [4]
Proof. As \( C_{\beta,n-2,1}(I) = |I| C_{\beta,n-2,1}/(2\pi) \) (recall [14]), by Lemma [1] we have
\[
\lim_{n \to +\infty} \frac{C_{\beta,n-2,1}(I)}{C_{\beta,n}n^2} = \frac{|I|}{2\pi} \lim_{n \to +\infty} \frac{C_{\beta,n-2,1}}{C_{\beta,n}n^2} = \frac{|I|A_{\beta}}{2\pi},
\]

i.e., Lemma 3 is true for \( k = 1 \). Now we assume \(|I| > 0\), then for every \( \lambda > 0 \), we can find \( A = (0, a(\lambda)) \) such that

\[
\lambda = \int_A u^\beta \, du \times \frac{|I|A_\beta}{2\pi}.
\]

Let’s denote

\[
X_n := \chi^{(n)}(A \times I),
\]

then by Lemma 3 with \( k = 1 \) and (12), we have

\[
\lim_{n \to +\infty} E X_n = \lim_{n \to +\infty} \left( \int_A u^\beta \, du \right) \frac{C_{\beta,n-1}(I)}{C_{\beta,n} n^\beta} = \lambda;
\]

and with \( k = 2 \) in Lemma 3 we have

\[
\lim_{n \to +\infty} E [X_n(X_n - 1)] = \lim_{n \to +\infty} \left( \int_A u^\beta \, du \right)^2 \frac{C_{\beta,n-2}(I)}{C_{\beta,n} n^{2\beta}}.
\]

On the other hand, by Hölder inequality, we have \( E(X_n)^2 \geq (E X_n)^2 \) and \( E(X_n(X_n - 1)) \geq (E X_n)^2 - E X_n \), and thus

\[
\lim_{n \to +\infty} \mathbb{E}(X_n(X_n - 1)) \geq \lim_{n \to +\infty}(\mathbb{E}(X_n)^2 - \mathbb{E}X_n) = \lambda^2 - \lambda.
\]

Therefore, we have

\[
\lim_{n \to +\infty} \frac{C_{\beta,n-2}(I)}{C_{\beta,n} n^{2\beta}} \geq \left( \int_A u^\beta \, du \right)^2 (\lambda^2 - \lambda) = (1 - \lambda^{-1}) \left( \frac{|I|A_\beta}{2\pi} \right)^2.
\]

Letting \( \lambda \to +\infty \), we have

\[
\lim_{n \to +\infty} \frac{C_{\beta,n-2}(I)}{C_{\beta,n} n^{2\beta}} \geq \left( \frac{|I|A_\beta}{2\pi} \right)^2,
\]

which along with (12) gives Lemma 3 for \( k = 2 \).

Moreover, since

\[
E(X_n - \lambda)^2 = E(X_n(X_n - 1)) - (2\lambda - 1)(E X_n) + \lambda^2,
\]

by Lemma 3 and (12), we have

\[
\limsup_{n \to +\infty} E(X_n(X_n - 1)) = \limsup_{n \to +\infty} \left( \int_A u^\beta \, du \right)^2 \frac{C_{\beta,n-2}(I)}{C_{\beta,n} n^{2\beta}} \leq \left( \int_A u^\beta \, du \right)^2 \left( \frac{|I|A_\beta}{2\pi} \right)^2 = \lambda^2,
\]

and thus

\[
\limsup_{n \to +\infty} E(X_n - \lambda)^2 \leq \lambda^2 - (2\lambda - 1)\lambda + \lambda^2 = \lambda.
\]

Now we denote by \( C \) a constant independent of \( n, \lambda \), which may be different from line to line. As \( X_n^k \leq \frac{2X_n!}{(X_n - k)!} + C \) \((-C \text{ can be chosen as the lower bound of the polynomial } 2x(x-1) \cdots (x-k+1) - x^k \text{ for } x \geq 0\), by Lemma 3 and (11), we have

\[
\limsup_{n \to +\infty} E(X_n^k) \leq 2 \limsup_{n \to +\infty} E \left( \frac{X_n!}{(X_n - k)!} \right) + C,
\]

\[
\leq 2 \limsup_{n \to +\infty} \left( \int_A u^\beta \, du \right)^k \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n} n^{k\beta}} + C.
\]
then for any integer \( k \)

\[
\limsup_{n \to +\infty} \frac{(X_n - \lambda)^2 X_n!}{(X_n - k + 1)!} \leq 2 \left( \int_A u^\beta du \right)^k \limsup_{n \to +\infty} \frac{C_{\beta,n-2k,k}}{C_{\beta,n}^k} + C
\]

\[
\leq C \left( \int_A u^\beta du \right)^k + C \leq C \lambda^k + C.
\]

By Hölder inequality, we have

\[
\mathbb{E} \left( \frac{(X_n - \lambda)^2 X_n!}{(X_n - k + 1)!} \right) \leq \mathbb{E} \left( (X_n - \lambda)^2 X_n^{k-1} \right)
\]

\[
\leq \left( \mathbb{E}(X_n - \lambda)^2 \right)^{\frac{1}{2}} \left( \mathbb{E} \left( (X_n - \lambda)^2 X_n^{2k-2} \right) \right)^{\frac{1}{2}}
\]

\[
\leq \left( \mathbb{E}(X_n - \lambda)^2 \right)^{\frac{1}{2}} \left( \mathbb{E} \left( X_n^{2k} + \lambda^2 X_n^{2k-2} \right) \right)^{\frac{1}{2}}
\]

and thus for any positive integer \( k \), we have

\[
\limsup_{n \to +\infty} \mathbb{E} \left( \frac{(X_n - \lambda)^2 X_n!}{(X_n - k + 1)!} \right) \leq \left( \limsup_{n \to +\infty} \mathbb{E}(X_n - \lambda)^2 \right)^{\frac{1}{2}} \left( \limsup_{n \to +\infty} \mathbb{E} \left( X_n^{2k} + \lambda^2 X_n^{2k-2} \right) \right)^{\frac{1}{2}}
\]

\[
\leq \lambda^{\frac{k}{2}} \left( (\lambda^2 + C) + \lambda^2 (\lambda^{2k-2} + C) \right)^{\frac{1}{2}} \leq C \lambda^{k} \lambda^{k+1}.
\]

(51)

Now we can prove the result by induction. Assume \( j \geq 2 \) and Lemma 4 is true for \( k = j, j - 1 \), then by Lemma 3 we further have

\[
\limsup_{n \to +\infty} \mathbb{E} \left( \frac{X_n!}{(X_n - j)!} \right) = \limsup_{n \to +\infty} \left( \int_A u^\beta du \right)^k \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n}^k}
\]

\[
= \left( \int_A u^\beta du \right)^k \left( \frac{[I]_a^2}{2\pi} \right)^k = \lambda^k, \quad k = j - 1, j.
\]

We note that \( (X_n - \lambda)^2 = (X_n - k)(X_n - k - 1) - (2\lambda - 2k - 1)(X_n - k) + (\lambda - k)^2 \)

then for any integer \( k \geq 2 \), we have the identity

\[
\frac{(X_n - \lambda)^2 X_n!}{(X_n - k)!} = \frac{X_n!}{(X_n - k - 2)!} - \frac{(2\lambda - 2k - 1)X_n!}{(X_n - k - 1)!} + \frac{(\lambda - k)^2 X_n!}{(X_n - k)!}.
\]

(52)

Now by induction, (51) (52) and Lemma 3 we have

\[
C \lambda^{\frac{k}{2}} \lambda^{j+1} \geq \limsup_{n \to +\infty} \mathbb{E} \left( \frac{(X_n - \lambda)^2 X_n!}{(X_n - j + 1)!} \right)
\]

\[
= \limsup_{n \to +\infty} \mathbb{E} \left( \frac{X_n!}{(X_n - j)!} \right) - \frac{(2\lambda - 2j + 1)X_n!}{(X_n - j)!} + \frac{(\lambda - j + 1)^2 X_n!}{(X_n - j + 1)!}
\]

\[
= \limsup_{n \to +\infty} \mathbb{E} \left( \frac{X_n!}{(X_n - j + 1)!} \right) - (2\lambda - 2j + 1)\lambda^j + (\lambda - j + 1)^2 \lambda^{j-1}
\]

\[
= \limsup_{n \to +\infty} \left( \int_A u^\beta du \right)^k \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n}^k} \lambda^j \lambda^{j-1},
\]

where we denote \( k = j + 1 \) in the last line. Therefore, as \( \lambda \) large enough, we have

\[
\limsup_{n \to +\infty} \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n}^k} \leq \left( \int_A u^\beta du \right)^{-k} \left( \lambda^{j+1} + C \lambda^{k} \lambda^{j+1} \right)
\]
\[
= \left( \frac{|I| A_\beta}{2\pi} \right)^k (1 + C\lambda^{-\frac{j}{2}} + C\lambda^{-j-\frac{1}{2}}).
\]

Letting \( \lambda \to +\infty \), we have
\[
\limsup_{n \to +\infty} \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n} n^{k\beta}} \leq \left( \frac{|I| A_\beta}{2\pi} \right)^k.
\]

Similarly, as \( \frac{(X_n - \lambda)^2 X_n!}{(X_n - j + 1)!} \geq 0 \), by induction and Lemma 3 again, we have
\[
0 \leq \liminf_{n \to +\infty} E \left( \frac{X_n!}{(X_n - j)!} \frac{(2\lambda - 2j + 1)X_n!}{(X_n - j)!} + \frac{(\lambda - j + 1)^2 X_n!}{(X_n - j)!} \right)
\]
\[
= \liminf_{n \to +\infty} \left( \int_A u^\beta du \right)^k \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n} n^{k\beta}} - (\lambda^2 - (j - 1)^2 - \lambda)\lambda^{j-1},
\]
where \( k = j + 1 \) again. Therefore, we have
\[
\liminf_{n \to +\infty} \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n} n^{k\beta}} \geq \left( \int_A u^\beta du \right)^{-k} (\lambda^2 - (j - 1)^2 - \lambda)\lambda^{j-1}
\]
\[
= \left( \frac{|I| A_\beta}{2\pi} \right)^k (1 - \lambda^{-1} - (j - 1)^2\lambda^{-2}).
\]

Letting \( \lambda \to +\infty \) again, we have
\[
\liminf_{n \to +\infty} \frac{C_{\beta,n-2k,k}(I)}{C_{\beta,n} n^{k\beta}} \geq \left( \frac{|I| A_\beta}{2\pi} \right)^k,
\]
thus Lemma 4 is also true for \( k = j + 1 \). This completes the proof. \( \square \)

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