ROGUE WAVES ON THE PERIODIC WAVE BACKGROUND
IN THE FOCUSING NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. We present exact solutions for rogue waves arising on the background of periodic waves in the focusing nonlinear Schrödinger equation. The exact solutions are obtained by characterizing the Lax spectrum related to the periodic waves and by using the one-fold Darboux transformation. The magnification factor of the rogue waves is computed in the closed analytical form. We relate explicitly the rogue wave solutions to the modulation instability of the periodic wave background.

1. INTRODUCTION

We address rogue waves described by the focusing nonlinear Schrödinger (NLS) equation:

\begin{equation}
\frac{i}{2} \psi_t + \frac{1}{2} \psi_{xx} + |\psi|^2 \psi = 0.
\end{equation}

The canonical rogue wave was derived by Peregrine [24] in the exact form:

\begin{equation}
\psi(x, t) = \left[1 - \frac{4(1 + 2it)}{1 + 4x^2 + 4t^2}\right] e^{it}.
\end{equation}

As $|t| + |x| \to \infty$, the rogue wave (1.2) approaches the constant wave background $\psi_0(t) = e^{it}$ and it is related to the modulation instability of the constant wave background [25]. The rogue wave reaches its maximum amplitude $|\psi(0, 0)| = 3$ at $(x, t) = (0, 0)$, hence it brings a triple magnification over the constant wave background. Other rational solutions for rogue waves in the NLS equation (1.1) were constructed by using Darboux transformations in [3, 17, 22]. Further connection between rogue waves and the modulationally unstable constant wave background was investigated by using the inverse scattering method [6, 7, 8], asymptotic analysis [9], and the finite-gap theory [19, 20].

Periodic waves in the focusing NLS equation are known to be modulationally unstable with respect to long-wave perturbations [10, 15, 16]. Rogue waves have been observed numerically on the background of the modulationally unstable periodic waves [1, 2]. Construction of solutions of the NLS equation (1.1) for such rogue waves on the periodic background was first performed numerically in [11, 21] by applying the one-fold Darboux transformation to the numerically constructed solutions of the Lax equations. It was only recently in [13, 14] when the exact solutions for such rogue waves were constructed in the closed form for the dn-periodic and cn-periodic elliptic waves. A more complete analysis of rogue waves on the background of periodic elliptic waves was developed in [18]. Rogue waves on the multi-phase solutions and their magnification factors were studied in [4, 5] by using Riemann Theta functions.

Let us now give a mathematical definition of a rogue wave on the periodic background. If $\psi_0(x, t)$ is a periodic solution of the focusing NLS equation (1.1), then $\psi(x, t)$ is said to be a rogue wave on the periodic background if it satisfies

\begin{equation}
\inf_{x_0, t_0, \alpha_0 \in \mathbb{R}} \sup_{x \in \mathbb{R}} |\psi(x, t) - \psi_0(x - x_0, t - t_0)e^{i\alpha_0}| \to 0 \quad \text{as} \quad t \to \pm \infty.
\end{equation}

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The magnification factor of the rogue wave can be defined as a ratio of the maximum of $\psi$ to the maximum of $\psi_0$.

The purpose of this work is to construct the exact solutions for rogue waves on the periodic wave background in a closed analytical form. We improve the previous work [14] in several ways. First, we develop a general scheme for integrability of the Hamiltonian system which arises in the algebraic method with one eigenvalue [26, 27] (nonlinearization of the linear equations was pioneered in [12]). This scheme allows us to integrate the traveling wave reduction of the focusing NLS equation (1.1) and to relate parameters of the periodic waves to eigenvalues in the Lax spectrum. Second, we introduce a new representation for the second growing solution to the linear equations that is applicable to every admissible eigenvalue of the Lax spectrum. Third, we compute the exact value of the magnification factors for a general elliptic solution with nontrivial phase that coincides with the constant wave background, the dn-periodic or cn-periodic elliptic waves in particular cases. Fourth, we relate the existence of such rogue wave solutions to the modulation instability of the periodic wave background studied in [15]. Fifth, we visualize numerically the Lax spectrum, the admissible eigenvalues, the modulation instability bands, and the rogue waves.

2. Algebraic method with one eigenvalue

The NLS equation (1.1) with $\psi \equiv u$ appears as a compatibility condition $\varphi_{xt} = \varphi_{tx}$ of the following pair of linear equations on $\varphi \in \mathbb{C}^2$:

\begin{equation}
\varphi_x = U(\lambda, u)\varphi,
\end{equation}

and

\begin{equation}
\varphi_t = V(\lambda, u)\varphi,
\end{equation}

where $\bar{u}$ is the conjugate of $u$ and $\lambda \in \mathbb{C}$ is a spectral parameter.

The procedure of computing a new solution $\hat{\psi} \equiv \hat{u}$ of the NLS equation (1.1) from another solution $\psi \equiv u$ is well-known [3, 14, 18]. Let $\varphi = (p_1, q_1)^T$ be any nonzero solution to the linear equations (2.1) and (2.2) for a fixed value $\lambda = \lambda_1$. The new solution is given by the one-fold Darboux transformation

\begin{equation}
\hat{u} = u + \frac{2(\lambda_1 + \bar{\lambda}_1)p_1\bar{q}_1}{|p_1|^2 + |q_1|^2},
\end{equation}

and in particular, it may represent the rogue wave $\hat{u}$ on the background $u$ satisfying the limits (1.3). The main question is which value $\lambda_1$ to fix and which nonzero solution $\varphi$ to the linear equations (2.1)–(2.2) to take. For a periodic wave background $u$, we show that the admissible values of $\lambda_1$ are defined by the algebraic method with one eigenvalue, which is a particular case of a general method of nonlinearization of the linear equations on $\varphi$ developed in [12] and [26, 27].

Let $\varphi = (p_1, q_1)^T$ be a nonzero solution of the linear equations (2.1)–(2.2) with fixed $\lambda = \lambda_1$ such that $\lambda_1 + \bar{\lambda}_1 \neq 0$. We introduce the following relation between the solution $u$ to the NLS equation (1.1) and the squared eigenfunction $\varphi = (p_1, q_1)^T$:

\begin{equation}
u = p_1^2 + q_1^2.
\end{equation}

We intend to determine the value of $\lambda_1$ for which the relation (2.4) holds in $(x, t)$. Assuming (2.4), the spectral problem (2.1) is nonlinearized into the Hamiltonian system given by

\begin{equation}
\frac{dp_1}{dx} = \frac{\partial H}{\partial q_1}, \quad \frac{dq_1}{dx} = -\frac{\partial H}{\partial p_1},
\end{equation}
where the Hamiltonian function is
\begin{equation}
H = \lambda_1 p_1 q_1 + \bar{\lambda}_1 \bar{p}_1 \bar{q}_1 + \frac{1}{2} |p_1^2 + \bar{q}_1^2|^2.
\end{equation}

The value of $H$ is a constant of motion for the system (2.5)–(2.6), which is also Liouville integrable since there exists another constant of motion:
\begin{equation}
F = i(p_1 q_1 - \bar{p}_1 \bar{q}_1).
\end{equation}

The $x$-derivatives of $u$ are related to the squared eigenfunctions because of the constraint (2.4) and the Hamiltonian system (2.5)–(2.6). By differentiating (2.4) in $x$ and using (2.5), (2.6), and (2.7), we obtain
\begin{equation}
\frac{du}{dx} + 2iF u = 2(\lambda_1 p_1^2 - \bar{\lambda}_1 \bar{q}_1^2)
\end{equation}
and
\begin{equation}
\frac{d^2 u}{dx^2} + 2|u|^2 u + 2iF \frac{du}{dx} - 4Hu = 4(\lambda_1 p_1^2 + \bar{\lambda}_1 \bar{q}_1^2).
\end{equation}

Eliminating the squared eigenfunctions $p_1^2$ and $\bar{q}_1^2$ from (2.4) and (2.8) and substituting the result into (2.9) yields a closed second-order equation:
\begin{equation}
u'' + 2|u|^2 u + 2icu' - 4bu = 0,
\end{equation}
where $c$ and $b$ are real parameters given by
\begin{equation}
\begin{cases}
c = F + i(\lambda_1 - \bar{\lambda}_1), \\
b = H + iF(\lambda_1 - \bar{\lambda}_1) + |\lambda_1|^2.
\end{cases}
\end{equation}

The differential equation (2.10) is a traveling wave reduction of the NLS equation (1.1) for the solutions in the form:
\begin{equation}
\psi(x, t) = u(x + ct)e^{2ibt}
\end{equation}
These solutions include the periodic traveling wave of the NLS equation (1.1).

In order to complete the algebraic method with one eigenvalue and to integrate the second-order equation (2.10), we represent the Hamiltonian system (2.5)–(2.6) by using the following Lax equation:
\begin{equation}
\frac{d}{dx} W(\lambda) = [U(\lambda, u), W(\lambda)],
\end{equation}
where $U(\lambda, u)$ is given by (2.1), $u$ is given by (2.4), and $W(\lambda)$ is represented in the form:
\begin{equation}
W(\lambda) = \begin{pmatrix}
W_{11}(\lambda) & W_{12}(\lambda) \\
\overline{W}_{12}(-\lambda) & -\overline{W}_{11}(-\lambda)
\end{pmatrix},
\end{equation}
with the entries
\begin{equation}
W_{11}(\lambda) = 1 - \left( \frac{p_1 q_1}{\lambda - \lambda_1} - \bar{p}_1 \bar{q}_1 \right), \quad W_{12}(\lambda) = \frac{p_1^2}{\lambda - \lambda_1} + \frac{\bar{q}_1^2}{\lambda + \lambda_1}.
\end{equation}

By using (2.4), (2.6), (2.7), and (2.8), we can rewrite the pole decompositions (2.15) in the form:
\begin{equation}
W_{11}(\lambda) = \frac{\lambda^2 + ic\lambda + \frac{1}{2}|u|^2 - b}{(\lambda - \lambda_1)(\lambda + \lambda_1)}, \quad W_{12}(\lambda) = \frac{u \lambda + \frac{1}{2}u' + icu}{(\lambda - \lambda_1)(\lambda + \lambda_1)}.
\end{equation}

It follows from (2.14) and (2.15) that $\det W(\lambda)$ contains only simple poles at $\lambda_1$ and $-\bar{\lambda}_1$. Since $U(\lambda, u)$ has zero trace, it follows from (2.13) and (2.14) that $\det W(\lambda)$ is constant in $x$. 

The \((1, 2)\)-component of the Lax equation (2.13) with (2.14) and (2.16) is equivalent to the second-order equation (2.10). On the other hand, constants of motion for the second-order equation (2.10) follow from the two properties of \(\det W(\lambda)\) listed above. By using (2.14) and (2.16), we obtain
\[
(2.17) \quad \det W(\lambda) = - [W_{11}(\lambda)]^2 - W_{12}(\lambda)\overline{W}_{12}(-\lambda) = - \frac{P(\lambda)}{(\lambda - \lambda_1)^2(\lambda + \lambda_1)^2},
\]
where
\[
(2.18) \quad P(\lambda) := \left( \lambda^2 + ic\lambda + \frac{1}{2}|u|^2 - b \right)^2 - \left( u\lambda + \frac{1}{2}u' + ic\lambda \right) \left( \bar{u}\lambda - \frac{1}{2}\bar{u}' + ic\lambda \right).
\]
Since \(P(\lambda)\) is independent on \(x\), expanding (2.18) in \(\lambda\) gives two constants of motion for the second-order equation (2.10) in the form:
\[
(2.19) \quad |u'|^2 + |u|^4 - 4b|u|^2 = 8d \quad \text{and} \quad (2.20) \quad i(u'\bar{u} - u\bar{u}') - 2c|u|^2 = 4e
\]
where \(d\) and \(e\) are real parameters appearing in the coefficients of the polynomial \(P(\lambda)\) given by
\[
(2.21) \quad P(\lambda) = \lambda^4 + 2ic\lambda^3 - (c^2 + 2b)\lambda^2 + 2i(e - bc)\lambda + b^2 - 2ec + 2d.
\]
If \(\lambda_1\) is a root of \(P(\lambda)\), so is \(-\lambda_1\), thanks to the symmetry of the coefficients in \(P(\lambda)\). The admissible values for \(\lambda_1\) are given by four roots of \(P(\lambda)\) which are symmetric about the imaginary axis. If \(\lambda_1\) is a root of \(P(\lambda)\), then \(\det W(\lambda)\) has simple poles at \(\lambda_1\) and \(-\lambda_1\) in the quotient (2.17).

3. Classification of periodic waves

Here we solve the second-order equation (2.10) by using Jacobian elliptic functions. The algebraic method with one eigenvalue allows us to define the admissible values of \(\lambda_1\) for the periodic wave solutions of the NLS equation (1.1).

First, we note that the transformation
\[
u(x) = \tilde{u}(x)e^{-ix}, \quad b = \tilde{b} + \frac{1}{4}c^2, \quad d = \tilde{d} + \frac{1}{2}ec, \quad e = \tilde{e}
\]
leaves the system (2.10), (2.19) and (2.20) invariant for tilde variables and eliminates the parameter \(c\). Similarly, \(P(\lambda)\) in (2.21) can be written as
\[
P(\lambda) = \left( \lambda + \frac{i}{2}c \right)^4 - 2\tilde{b} \left( \lambda + \frac{i}{2}c \right)^2 + 2i\tilde{e} \left( \lambda + \frac{i}{2}c \right) + \tilde{b}^2 + 2\tilde{d},
\]
which shows that \(\lambda + \frac{i}{2}c\) are independent of \(c\). Hence, one can set \(c = 0\) without loss of generality. This corresponds to the Lorentz transformation for solutions of the NLS equation (1.1).

The periodic waves are divided into two groups depending whether the phase of the complex-value function \(u(x)\) is trivial or non-trivial.

3.1. Periodic waves with trivial phase. Let us set \(e = c = 0\). It follows from (2.20) that
\[
d \log \left( \frac{u}{\bar{u}} \right) = 0 \quad \Rightarrow \quad \frac{u}{\bar{u}} = e^{2i\theta},
\]
where real \(\theta\) is constant in \(x\). Thanks to the rotational symmetry of solutions to the NLS equation (1.1), one can take \(u(x)\) as a real function. The first-order quadrature (2.19) is rewritten for real \(u(x)\) as follows:
\[
(3.1) \quad \left( \frac{du}{dx} \right)^2 + V(u) = 0, \quad V(u) := u^4 - 4bu^2 - 8d.
\]
The four roots of \( V(u) \) are symmetric and can be enumerated as \( \pm u_1 \) and \( \pm u_2 \). If all four roots are real, one can order them as
\[
- u_1 \leq - u_2 \leq u_2 \leq u_1.
\]
Solutions of the first-order quadrature (3.1) exist either in \([-u_1, -u_2]\) or in \([u_2, u_1]\). The two solutions are related by the transformation \( u \mapsto -u \), hence without loss of generality, one can consider the positive solution in \([u_2, u_1]\). This solution is given by the explicit formula
\[
(3.3) \quad u(x) = u_1 \text{dn}(u_1 x; \kappa), \quad \text{where} \quad \kappa^2 = \frac{u_1^2 - u_2^2}{u_1^2}.
\]
If two roots of \( V(u) \) are real (e.g., \( \pm u_1 \)) and two roots are complex (e.g., \( \pm i \nu_2 \)), the solution in \([-u_1, u_1]\) is given by the explicit formula
\[
(3.4) \quad u(x) = u_1 \text{cn}(\alpha x; \kappa), \quad \text{where} \quad \alpha^2 = u_1^2 + \nu_2^2, \quad k^2 = \frac{u_1^2}{u_2^2 + \nu_2^2}.
\]
The connection formulas of roots \( \pm u_1, \pm u_2 \) with the parameters \((b, d)\) in (3.1) are given by
\[
(3.5) \quad \left\{ \begin{array}{l}
4b = u_1^2 + u_2^2, \\
8d = -u_1^2 u_2^2.
\end{array} \right.
\]
Substituting (3.5) into \( P(\lambda) \) given by (2.21) with \( c = e = 0 \) yields
\[
(3.6) \quad P(\lambda) = \lambda^4 - \frac{1}{2}(u_1^2 + u_2^2) \lambda^2 + \frac{1}{16}(u_1^2 - u_2^2)^2.
\]
The two admissible pairs of eigenvalues are given by
\[
(3.7) \quad \lambda_1^\pm = \pm \frac{u_1 + u_2}{2}, \quad \lambda_2^\pm = \pm \frac{u_1 - u_2}{2},
\]
where \( u_1 \) is real and \( u_2 \) is either real for (3.3) or purely imaginary for (3.4).

Both waves with the trivial phase can be simplified by using the scaling transformation of solutions to the NLS equation (1.1):
\[
(3.8) \quad \text{if} \ u(x, t) \ \text{is a solution, so is} \ au(ax, a^2t), \ \text{for every} \ a \in \mathbb{R}.
\]
The \( \text{dn} \)-periodic wave (3.3) with \( u_1 = 1 \) and \( u_2 = \sqrt{1-k^2} \) becomes
\[
(3.9) \quad u(x) = \text{dn}(x; \kappa), \quad \kappa \in (0, 1)
\]
and the two eigenvalue pairs in (3.7) are real-valued:
\[
(3.10) \quad \lambda_1^\pm = \pm \frac{1}{2}(1 + \sqrt{1-k^2}), \quad \lambda_2^\pm = \pm \frac{1}{2}(1 - \sqrt{1-k^2}).
\]
The \( \text{cn} \)-periodic wave (3.4) with \( u_1 = k \) and \( u_2 = \sqrt{1-k^2} \) becomes
\[
(3.11) \quad u(x) = k \text{ cn}(x; \kappa), \quad \kappa \in (0, 1)
\]
and the two eigenvalue pairs in (3.7) form a complex quadruplet:
\[
(3.12) \quad \lambda_1^\pm = \frac{1}{2}(\pm k + i \sqrt{1-k^2}), \quad \lambda_2^\pm = \frac{1}{2}(\pm k - i \sqrt{1-k^2}).
\]
These two cases agree with the outcomes of the algebraic method in [14].

Figs. 1 and 2 represent the Lax spectrum computed numerically by using the Floquet–Bloch decomposition of solutions to the spectral problem (2.1) with the potentials \( u \) in the form (3.9) and (3.11) respectively. The black curves represent the purely continuous spectrum whereas the red dots represent eigenvalues (3.10) and (3.12) respectively. Appendix A gives details of the Floquet–Bloch decomposition and the numerical method used to compute the Lax spectrum at the period waves.
3.2. Periodic waves with nontrivial phase. Let us set $c = 0$ but consider $e \neq 0$. Substituting the decomposition $u(x) = R(x)e^{i\Theta(x)}$ with real $R$ and $\Theta$ into (2.19) and (2.20) with $c = 0$ yields

\[
\begin{align*}
(R^2 + R^2\Theta^2 + R^4 - 4bR^2 &= 8d, \\
R^2\Theta' &= -2e. 
\end{align*}
\]

Expressing $\Theta' = -2e/R^2$ results in the following first-order equation:

\[
\left(\frac{dR}{dx}\right)^2 + 4e^2R^{-2} + R^4 - 4bR^2 = 8d. 
\]

The singularity $R = 0$ is unfolded with the transformation $\rho = R^2$ which yields

\[
\frac{1}{4} \left(\frac{d\rho}{dx}\right)^2 + W(\rho) = 0, \quad W(\rho) := \rho^3 - 4b\rho^2 - 8d\rho + 4e^2. 
\]

Since $\rho = R^2 \geq 0$ and $W(0) = 4e^2 \geq 0$, one of the roots of the cubic polynomial $W$ is negative. Therefore, the positive periodic solutions of the first-order quadrature (3.15) exist only if there exist
three real roots of \( W \). We denote the roots by \( \{\rho_1, \rho_2, \rho_3\} \) and order them as follows:

\[
\rho_3 \leq 0 \leq \rho_2 \leq \rho_1.
\]

The connection formulas of roots \( \rho_1, \rho_2, \) and \( \rho_3 \) to the parameters \((b, d)\) in (3.1) are given by

\[
\begin{cases}
4b = \rho_1 + \rho_2 + \rho_3, \\
8d = -\rho_1 \rho_2 - \rho_1 \rho_3 - \rho_2 \rho_3, \\
4e^2 = -\rho_1 \rho_2 \rho_3.
\end{cases}
\]

Only one square root for \( e \) must be used for a particular wave. In what follows, we shall use the negative square root with \( 2e = -\sqrt{\rho_1 \rho_2} \sqrt{-\rho_3} \), which is real-valued thanks to (3.16).

The positive periodic solution is located in the interval \([\rho_2, \rho_1]\) and is given by

\[
\rho(x) = \rho_1 - (\rho_1 - \rho_2) \sin^2(\alpha x; k),
\]

where \( \alpha \) and \( k \) are related to \((\rho_1, \rho_2, \rho_3)\) by

\[
\alpha^2 = \rho_1 - \rho_3, \quad k^2 = \frac{\rho_1 - \rho_2}{\rho_1 - \rho_3}.
\]

Thanks to the scaling transformation (3.8), we can set \( \alpha = 1 \) and use the parametrization \( \rho_1 = \beta, \rho_2 = \beta - k^2, \) and \( \rho_3 = \beta - 1 \) which yields the exact expression considered in [15]:

\[
\rho(x) = \beta - k^2 \sin^2(x; k).
\]

The exact solution (3.20) has two parameters \( \beta \) and \( k \) which belong to the following triangular region: \( k \in [0, 1] \) (since \( \rho_3 \leq \rho_2 \leq \rho_1 \), \( \beta \leq 1 \) (since \( \rho_3 \leq 0 \)), and \( \beta \geq k^2 \) (since \( \rho_2 \geq 0 \)). On the three boundaries, we have reductions to the \( \text{dn} \)-periodic wave (3.9) if \( \beta = 1 \) (\( \rho_3 = 0 \)), the \( \text{cn} \)-periodic wave (3.11) if \( \beta = k^2 \) (\( \rho_2 = 0 \)), and the constant wave background if \( k = 0 \) (\( \rho_1 = \rho_2 \)):

\[
u(x) = \sqrt{\beta} \exp[i \sqrt{1 - \beta} x], \quad \beta \in (0, 1).
\]

Substituting (3.17) into \( P(\lambda) \) given by (2.21) with \( c = 0 \) yields

\[
P(\lambda) = \lambda^4 - \frac{1}{2}(\rho_1 + \rho_2 + \rho_3) \lambda^2 - i \sqrt{\rho_1 \rho_2} \sqrt{-\rho_3} \lambda + \frac{1}{16}(\rho_1^2 + \rho_2^2 + \rho_3^2 - 2 \rho_1 \rho_2 + 2 \rho_1 \rho_3 - 2 \rho_2 \rho_3).
\]

The four roots of \( P(\lambda) \) can now be found in the explicit form (see also equation (88) in [15]):

\[
\lambda_1^\pm = \pm \frac{1}{2}(\sqrt{\rho_1} + \sqrt{\rho_2}) + \frac{i}{2} \sqrt{-\rho_3}, \quad \lambda_2^\pm = \pm \frac{1}{2}(\sqrt{\rho_1} - \sqrt{\rho_2}) - \frac{i}{2} \sqrt{-\rho_3}.
\]

Figure 3. Lax spectrum for the periodic wave (3.20) with \((\beta, k) = (0.85, 0.85)\) (left) and \((\beta, k) = (0.95, 0.9)\) (right). Red dots represent eigenvalues (3.23).
Fig. 3 represents the Lax spectrum computed numerically by using the Floquet–Bloch decomposition of solutions to the spectral problem (2.1) with the potentials \( u(x) = R(x)e^{i\Theta(x)} \). The transformation

\[
(3.24) \quad p_1(x) = \tilde{p}_1(x)e^{i\Theta(x)/2}, \quad q_1(x) = \tilde{q}_1(x)e^{-i\Theta(x)/2}
\]

is used to reduce the spectral problem (2.1) to the one with a periodic potential considered in Appendix A. The black curves represent the purely continuous spectrum whereas the red dots represent eigenvalues (3.23).

Fig. 4 shows boundaries of the triangular region on the \((\beta, k)\) plane where the periodic waves with nontrivial phase (3.20) exist (black curves). Blue dots represent the particular values of parameters \((\beta, k)\) chosen for illustration of the Lax spectra on Figs. 1, 2, and 3. The green curve given by the following explicit expression (see equation (100) in [15])

\[
(3.25) \quad \beta = -1 + k^2 + \frac{2E(k)}{K(k)},
\]

separates two regions on the \((\beta, k)\) plane. The Lax spectrum for the periodic waves in lower (upper) region includes bands intersecting at the imaginary (real) axis. The two choices on Figs. 2 and 3 correspond to two points on both sides of the boundary (3.25) on Fig. 4.

Figure 4. The black curves are boundaries of the triangular region where the periodic waves with nontrivial phase (3.20) exist. The blue dots show parameter values of \((\beta, k)\) for the solutions chosen for illustration on Figs. 1, 2, and 3. The green curve displays the boundary (3.25), whereas the red curve shows the boundary (4.19).

4. Eigenfunctions of the linear equations

Here we characterize squared eigenfunctions of the linear equations (2.1)–(2.2) in terms of the periodic wave \( u \). For each admissible eigenvalue \( \lambda_1 \) among the roots of the polynomial \( P(\lambda) \) in (2.21), the squared eigenfunctions \( p_1^2, \tilde{q}_1^2 \), and \( p_1q_1 \) after the transformation (3.24) are periodic functions with the same period as the periodic wave \( u \). The second linearly independent solution of the linear equations (2.1)–(2.2) exists for the same eigenvalues and is characterized in terms of the periodic
eigenfunctions. The second solution is not periodic and grows linearly in $x$ and $t$ almost everywhere on the $(x,t)$-plane.

Let us recall the representations (2.15) and (2.16) for $W_{11}(\lambda)$ and $W_{12}(\lambda)$ in terms of the squared eigenfunctions and the periodic wave $u$. By computing contributions at simple poles $\lambda = \lambda_1$ and $\lambda = -\lambda_1$, we obtain the following explicit expressions for $p_1^2$, $q_1^2$, and $p_1 q_1$:

\[
\begin{align*}
\begin{cases}
p_1^2 = \frac{1}{\lambda_1 + \lambda_1} \left[ \frac{1}{2} u u' + icu + \lambda_1 u \right], \\
q_1^2 = \frac{1}{\lambda_1 + \lambda_1} \left[ -\frac{1}{2} u u' - icu + \lambda_1 u \right], \\
p_1 q_1 = -\frac{1}{\lambda_1 + \lambda_1} \left[ \frac{1}{2} |u|^2 - b + i\lambda_1 c + \lambda_1^2 \right].
\end{cases}
\end{align*}
\]

Let $\varphi = (p_1, q_1)^T$ be a solution of the linear equations (2.1)–(2.2) for $\lambda = \lambda_1$. The second, linearly independent solution $\varphi = (\hat{p}_1, \hat{q}_1)^T$ of the same equations is obtained in the form:

\[
\begin{align*}
\hat{p}_1 &= p_1 \varphi_1 - \frac{2q_1}{|p_1|^2 + |q_1|^2}, \\
\hat{q}_1 &= q_1 \varphi_1 + \frac{2p_1}{|p_1|^2 + |q_1|^2},
\end{align*}
\]

where $\varphi_1$ is to be determined. Wronskian between the two solutions is normalized by $p_1 \hat{q}_1 - \hat{p}_1 q_1 = 2$. Substituting (4.2) into (2.1) and using (2.1) for $\varphi = (p_1, q_1)^T$ yields the following equation for $\varphi_1$:

\[
\frac{\partial \varphi_1}{\partial x} = \frac{-4(\lambda_1 + \lambda_1) \hat{p}_1 \hat{q}_1}{(|p_1|^2 + |q_1|^2)^2}.
\]

Similarly, substituting (4.2) into (2.2) and using (2.2) for $\varphi = (p_1, q_1)^T$ yields another equation for $\varphi_1$:

\[
\frac{\partial \varphi_1}{\partial t} = \frac{-4i(\lambda_1^2 - \lambda_1^2) \hat{p}_1 \hat{q}_1}{(|p_1|^2 + |q_1|^2)^2} + \frac{2i(\lambda_1 + \lambda_1) (w p_1^2 + \bar{u} q_1^2)}{(|p_1|^2 + |q_1|^2)^2}.
\]

The system of first-order equations (4.3) and (4.4) is compatible in the sense $\varphi_{1xt} = \varphi_{1tx}$ since it is derived from the compatible Lax system (2.1)–(2.2).

4.1. Periodic waves with trivial phase. We set $c = e = 0$ in solutions of the system (2.10), (2.19), and (2.20). We also use the scaling transformation (3.8). There exist two admissible pairs of values of $\lambda_1$ given by (3.10) for the dn-periodic solution (3.9) and (3.12) for the cn-periodic solution (3.11). By separating the variables in

\[
u(x, t) = \varpi(x) e^{2i\lambda_1 x}, \quad p_1(x, t) = P_1(\varpi) e^{i\lambda_1 t}, \quad q_1(x, t) = Q_1(\varpi) e^{-i\lambda_1 t},
\]

with real $\varpi$, $b = \frac{1}{2} (u_1 + u_2)$, and $\lambda_1 = \frac{1}{2} (u_1 \pm u_2)$, we obtain

\[
\begin{align*}
P_1^2 &= \frac{1}{\lambda_1 + \lambda_1} \left( \frac{1}{2} U' + \lambda_1 U \right), \\
Q_1^2 &= \frac{1}{\lambda_1 + \lambda_1} \left( -\frac{1}{2} U' + \lambda_1 U \right), \\
P_1 Q_1 &= \frac{1}{\lambda_1 + \lambda_1} \left( \pm u_1 u_2 + U^2 \right).
\end{align*}
\]

The quantities $\lambda_1$, $P_1$ and $Q_1$ are real-valued for the dn-periodic wave (3.9). By using (4.6), we solve the first-order equations (4.3) and (4.4) in the closed form:

\[
\varphi_1(x, t) = 2x + 2i(1 \pm \sqrt{1 - k^2}) t \pm 2\sqrt{1 - k^2} \int_0^x \frac{dy}{\sqrt{\varpi^2(y; k)}}
\]

up to addition of a complex-valued constant. We note that

\[
K(k) \pm \sqrt{1 - k^2} \int_0^{K(k)} \frac{dy}{\sqrt{\varpi^2(y; k)}} = K(k) \pm \frac{E(k)}{\sqrt{1 - k^2}} \geq 0, \quad k \in (0, 1),
\]
where the inequality for the upper sign is trivial, whereas the inequality for the lower sign follows from the inequality 19.9.8 in [23],
\[ E(k) \frac{K(k)}{k} > \sqrt{1 - k^2}. \]

Thanks to (4.8), the function \( \phi_1(x, t) \) grows linearly as \( |x| + |t| \to \infty \) for both signs. Hence, the second solution \( \varphi = (\hat{p}_1, \hat{q}_1)^T \) given by (4.2) with \( \phi_1 \) in (4.7) grows linearly as \( |x| + |t| \to \infty \) everywhere on the \((x, t)\)-plane. Compared to the representation for \( \varphi = (\hat{p}_1, \hat{q}_1)^T \) used in [14], the new representation (4.2) has the advantage of being equally applicable to both eigenvalues \( \lambda_1 = \frac{1}{2}(u_1 + u_2) \) because the denominators in the new representation (4.2) with (4.7) never vanish.

For the cn-periodic wave (3.11), \( \lambda_1 \) is complex and so are \( P_1 \) and \( Q_1 \). It was found in [14] that
\[
|P_1(x)|^2 + |Q_1(x)|^2 = \text{dn}(x; k)
\]
and
\[
2P_1(x)Q_1(x) = -\text{cn}(x; k)\text{dn}(x; k) \mp i\sqrt{1 - k^2}\text{sn}(x; k).
\]

By using (4.6) and (4.9), we solve the first-order equations (4.3) and (4.4) in the closed form:
\[
\phi_1(x, t) = 2k^2 \int_0^x \frac{\text{cn}^2(y; k)dy}{\text{dn}^2(y; k)} \mp 2ik\sqrt{1 - k^2} \int_0^x \frac{dy}{\text{dn}^2(y; k)} + 2ikt
\]
up to addition of a complex-valued constant. It is clear from (4.11) that \( \phi_1(x, t) \) grows linearly as \( |x| + |t| \to \infty \), hence, the second solution \( \varphi = (\hat{p}_1, \hat{q}_1)^T \) given by (4.2) with \( \phi_1 \) in (4.11) also grows linearly as \( |x| + |t| \to \infty \) everywhere on the \((x, t)\)-plane. Compared to the representation used in [14], it is now easier to see the growth of \( \phi_1(x, t) \) at infinity.

### 4.2. Periodic waves with nontrivial phase.

We set \( c = 0 \) for solutions of the system (2.10), (2.19), and (2.20). Let the roots \( \{\rho_1, \rho_2, \rho_3\} \) satisfy the order (3.16). There exist two admissible pairs of values of \( \lambda_1 \) and they are given in the explicit form (3.23). Let us fix
\[
\lambda_1 = \frac{1}{2}(\sqrt{\rho_1} \pm \sqrt{\rho_2}) \pm \frac{i}{2}\sqrt{\rho_3}.
\]

By separating the variables in
\[
u(x, t) = U(x)e^{ibt}, \quad p_1(x, t) = P_1(x)e^{ibt}, \quad q_1(x, t) = Q_1(x)e^{-ibt},
\]
with complex \( U \) and \( b = \frac{1}{2}(\rho_1 + \rho_2 + \rho_3) \), we obtain
\[
\begin{cases}
P_1^2 = \frac{1}{\lambda_1 + \lambda_2} \left( \frac{1}{2}U' + \lambda_1 U \right), \\
Q_1^2 = \frac{1}{\lambda_1 + \lambda_3} \left( -\frac{1}{2}U' + \lambda_1 U \right), \\
P_1Q_1 = \frac{1}{2(\lambda_1 + \lambda_3)} \left( \pm \sqrt{\rho_1\rho_2} \pm i\sqrt{-\rho_3(\sqrt{\rho_1} \pm \sqrt{\rho_2})} + |U|^2 \right).
\end{cases}
\]

Next, we represent the solutions in the polar form:
\[
U(x) = R(x)e^{i\Theta(x)}, \quad P_1(x) = \tilde{P}_1(x)e^{i\Theta(x)/2}, \quad Q_1(x) = \tilde{Q}_1(x)e^{-i\Theta(x)/2},
\]
where \( R \) and \( \Theta \) are real, whereas \( \tilde{P}_1 \) and \( \tilde{Q}_1 \) are complex-valued. In Appendix B, we prove for the periodic wave (3.20) that
\[
|\tilde{P}_1(x)|^2 + |\tilde{Q}_1(x)|^2 = \text{dn}(x; k)
\]
and
\[
\tilde{P}_1(x)\tilde{Q}_1(x) = -\frac{1}{2(\sqrt{\rho_1} \pm \sqrt{\rho_2})} \left[ \frac{\text{dn}(x; k)}{R(x)}(R(x)^2 \pm \sqrt{\rho_1\rho_2}) \mp i\sqrt{-\rho_3}R(x) \right].
\]
By using (4.14) and (4.16), we solve the first-order equations (4.3) and (4.4) in the closed form:

\[
\phi_1(x,t) = 2 \int_0^x \frac{\rho(y) \pm \sqrt{\rho_1 \rho_2} \pm i \sqrt{-\rho_3 (\sqrt{\rho_1} \pm \sqrt{\rho_2})}}{\text{dn}^2(y;k)} dy + 2i(\sqrt{\rho_1} \pm \sqrt{\rho_2}) t
\]

up to addition of a complex-valued constant. When \( \rho_1 = 1, \rho_2 = 1 - k^2 \), and \( \rho_3 = 0 \), expression (4.18) is equivalent to (4.7) for the \( \text{dn} \)-periodic wave (3.9). When \( \rho_1 = k^2, \rho_2 = 0 \), and \( \rho_3 = -(1 - k^2) \), expression (4.18) is equivalent to (4.11) for the \( \text{cn} \)-periodic wave (3.11).

\[
\text{It is clear from (4.18) with the upper sign that} \phi_1(x,t) \text{ grows linearly as } |x| + |t| \to \infty. \text{ On the other hand, } \phi_1(x,t) \text{ with the lower sign may not grow everywhere as } |x| + |t| \to \infty \text{ if}
\]

\[
\int_0^K \frac{\rho(y) - \sqrt{\rho_1 \rho_2}}{\text{dn}^2(y;k)} dy = K(k) + \frac{\beta - 1 - \sqrt{\beta(\beta - k^2)}}{1 - k^2} E(k) = 0,
\]

We show that there exists one root of (4.19) in \( \beta \in (k^2,1) \) for every \( k \in (0,1) \). Indeed, we obtain

\[
\beta = k^2 : K(k) + \frac{\beta - 1 - \sqrt{\beta(\beta - k^2)}}{1 - k^2} E(k) = K(k) - E(k) > 0,
\]

\[
\beta = 1 : K(k) + \frac{\beta - 1 - \sqrt{\beta(\beta - k^2)}}{1 - k^2} E(k) = K(k) - \frac{E(k)}{\sqrt{1 - k^2}} < 0,
\]

where the first inequality is due to 19.9.6 in [23] and the second inequality is due to (4.8). Since the left-hand side of (4.19) is a continuous function of \( \beta \) and \( k \), for every \( k \in (0,1) \), there exists some \( \bar{\beta} \in (k^2,1) \) such that equation (4.19) is satisfied. For this value of \( \beta \), the function \( \phi_1(x,t) \) grows almost everywhere on the \( (x,t) \) plane except for a straight line where \( \phi_1(x,t) \) remains bounded as \( |x| + |t| \to \infty \). The red curve on Fig. 4 shows the only root of equation (4.19) on the \( (\beta,k) \) plane.

5. Rogue waves on the periodic background

Here we use the one-fold Darboux transformation (2.3) with the second solution \( \varphi = (\hat{p}_1, \hat{q}_1)^t \) of the linear equations (2.1)–(2.2) with \( \lambda = \lambda_1 \):

\[
\begin{align*}
\hat{u} &= u + \frac{2(\lambda_1 + \bar{\lambda}_1)p_1 \hat{q}_1}{|p_1|^2 + |\hat{q}_1|^2} \\
\hat{u}|_{\phi_1 \to \infty} &= u + \frac{2(\lambda_1 + \bar{\lambda}_1)p_1 \hat{q}_1}{|p_1|^2 + |\hat{q}_1|^2} =: \hat{u}.
\end{align*}
\]

We show below that \( \hat{u} \) is a translated version of the original periodic wave \( u \) due to the symmetries of the NLS equation (1.1). Hence, the one-fold transformation (5.1) generates a rogue wave \( \hat{u} \) on the background of the periodic wave \( u \) and the rogue wave satisfies the limits (1.3) if \( |\phi_1(x,t)| \to \infty \) as \( |x| + |t| \to \infty \) everywhere on the \( (x,t) \)-plane.

The magnification factor of the rogue wave is found from the computation at the origin since \( \phi_1(0,0) = 0 \) by the choice of the integration constant in (4.7), (4.11), and (4.18). Since

\[
\hat{u}|_{\phi_1 = 0} = u - \frac{2(\lambda_1 + \bar{\lambda}_1)p_1 \hat{q}_1}{|p_1|^2 + |\hat{q}_1|^2} = 2u - \hat{u}
\]

and \( \hat{u} \) is a translated version of \( u \), the magnification factor does not exceed the triple magnification factor of the canonical rogue wave (1.2) on the constant wave background.
5.1. Periodic waves with trivial phase. For the dn-periodic wave (3.9), we obtain from (4.2), (4.5), and (5.1):

\[
\hat{u} = \left[ U + (U \pm \bar{U})(4 - |\phi_1|^2U^2) + 2(\phi_1 + \bar{\phi}_1)U' + 4\lambda_1(\phi_1 - \bar{\phi}_1)U \right] e^{2ibt},
\]

where \( U(x) = \text{dn}(x; k) \),

\[
\bar{U}(x) = \frac{\sqrt{1 - k^2}}{\text{dn}(x; k)} = U(x + K(k)),
\]
as follows from Table 22.4.3 in [23], and \( \phi_1(x, t) \) is given by (4.7). Since \(|\phi_1(x, t)| \to \infty\) as \(|x| + |t| \to \infty\), we have \( \hat{u}(x, t) \sim \mp \bar{U}(x)e^{2ibt} \) as \(|x| + |t| \to \infty\), which is a translation of the original periodic wave \( u(x, t) = U(x)e^{2ibt} \) by half-period. On the other hand, since \( \phi_1(0, 0) = 0 \), we have \( \hat{u}(0, 0) = 2 \pm \sqrt{1 - k^2} \), which is the magnification factor obtained in [14].

Fig. 5 shows rogue waves (5.4) for both signs which correspond to two choices of \( \lambda = \frac{1}{2}(u_1 \pm u_2) \) with \( u_1 = 1 \) and \( u_2 = \sqrt{1 - k^2} \).

For the cn-periodic wave (3.11), we obtain from (4.2), (4.5), and (5.1):

\[
\hat{u} = \left[ U + (U \mp i\bar{U})(4 - |\phi_1|^2\text{dn}(x; k)^2) + 2(\phi_1 - i\bar{\phi}_1)U' + 4\lambda_1\phi_1 - \bar{\lambda}_1\bar{\phi}_1)U \right] e^{2ibt},
\]

where \( U(x) = k\text{cn}(x; k) \),

\[
\bar{U}(x) = -\frac{k\sqrt{1 - k^2}\text{sn}(x; k)}{\text{dn}(x; k)} = U(x + K(k)),
\]
as follows from Table 22.4.3 in [23], and \( \phi_1(x, t) \) is given by (4.11). Since \(|\phi_1(x, t)| \to \infty\) as \(|x| + |t| \to \infty\), we have \( \hat{u}(x, t) \sim \pm i\bar{U}(x)e^{2ibt} \) as \(|x| + |t| \to \infty\), which is a translation of the original periodic wave \( u(x, t) = U(x)e^{2ibt} \) by a quarter period. On the other hand, since \( \phi_1(0, 0) = 0 \), we have \( \hat{u}(0, 0) = 2k \), which is the double magnification factor obtained in [14].

Fig. 6 shows rogue wave (5.5) with the upper sign for two choices of \( k \) with qualitatively different Lax spectrum on Fig. 2. The two signs in (5.5) correspond to two choices of \( \lambda = \frac{1}{2}(u_1 \pm u_2) \) with \( u_1 = k \) and \( u_2 = \sqrt{1 - k^2} \). The rogue wave (5.5) with the lower sign propagates to the opposite direction on the \((x, t)\) plane compared to the rogue wave with the upper sign on Fig. 6.
5.2. Periodic waves with nontrivial phase. For the periodic wave (3.20), we obtain from (4.2), (4.13), (4.15), and (5.1):

\[
\hat{u} = \left[ R + \frac{2(\sqrt{\rho_1} \pm \sqrt{\rho_2})(\bar{P}_1 \bar{Q}_1(\phi_1^2 \text{dn}^2(x; k) - 4) + 2(\bar{P}_1^2 \phi_1 - \bar{Q}_1^2 \bar{\theta}_1)\text{dn}(x; k))}{(\phi_1^2 \text{dn}^2(x; k) + 4)\text{dn}(x; k)} \right] e^{i\theta} e^{2ibt},
\]

where \((\bar{P}_1, \bar{Q}_1)\) are defined by (4.15) and \(\phi_1(x, t)\) is given by (4.18). Since \(|\phi_1(x, t)| \to \infty\) as \(|x| + |t| \to \infty\), it follows from (4.17) that

\[
|\hat{u}|_{|\phi_1| \to \infty} = \mp \left( \frac{\sqrt{\rho_1 \rho_2}}{R(x)} - i\frac{\sqrt{-\rho_3} R'(x)}{\text{dn}^2(x; k)} \right) e^{i\Theta(x)} e^{2ibt} =: \tilde{u}.
\]

It follows from (3.20) that

\[
|\tilde{u}|^2 = \frac{\rho_1 \rho_2}{\rho(x)} \frac{\rho_3 |\rho'(x)|^2}{4 \rho(x) \text{dn}^4(x; k)}
= \frac{\beta(\beta - k^2) \text{dn}^2(x; k) + (1 - \beta) k^4 \text{sn}^2(x; k) \text{cn}^2(x; k)}{(\beta - k^2 \text{sn}^2(x; k)) \text{dn}^2(x; k)}
= \beta - k^2 \text{cn}^2(x; k)
\]

as follows from Table 22.4.3 in [23]. Hence \(|\tilde{u}(x, t)|\) is a translation of \(|u(x, t)| = R(x)|\) by a half period. On the other hand, since \(\phi_1(0, 0) = 0\), \(R(0) = \sqrt{\rho_1}\), and \(R'(0) = 0\), it follows from (4.17) that \(|\tilde{u}(0, 0)| = 2\sqrt{\rho_1} \pm \sqrt{\rho_2}\), which gives the magnification factor of the rogue wave:

\[
M(\beta, k) := \frac{|\tilde{u}(0, 0)|}{\text{max}_{x \in \mathbb{R}} R(x)} = 2 \pm \sqrt{1 - \frac{k^2}{\beta}}.
\]

When \(\beta = 1\), \(M(1, k) = 2 \pm \sqrt{1 - k^2}\) coincides with the magnification factor of the dn-periodic wave (3.9). When \(\beta = k^2\), \(M(k^2, k) = 2\) coincides with the double magnification factor of the cn-periodic wave (3.11).

Fig. 7 shows rogue waves (5.6) for the upper sign (upper panels) and the lower sign (lower panels). The left and right panels correspond to two choices of \(k\) with qualitatively different Lax spectrum on Fig. 3. The upper and lower signs correspond to two choices of \(\lambda_1\) in (4.12). The rogue wave for the lower sign at the point \((\beta, k) = (0.85, 0.85)\) (left lower panel) is computed near the red curve of Fig.
Figure 7. Rogue waves on the periodic wave with non-trivial phase for \((\beta, k) = (0.85, 0.85)\) (left) and \((\beta, k) = (0.95, 0.9)\) (right). The upper and lower panels show rogue waves (5.6) with the upper and lower signs respectively.

4 given by the implicit equation (4.19). In this exceptional case, \(|\phi_1(x, t)|\) remains bounded along a straight line in the \((x, t)\) plane. As a result, instead of a rogue wave localized on the \((x, t)\)-plane, we see an algebraic soliton propagating on the periodic wave background, similar to the case of the modified KdV equation [14]. Note that the algebraic soliton on the periodic wave background does not satisfy the limits (1.3).

6. Relation to the modulation instability of the periodic waves

Here we solve the linear equations (2.1)–(2.2) for other values of \(\lambda\) in the case of the periodic wave (2.12) with \(c = 0\). Separating variables by

\[
(6.1) \quad u(x, t) = U(x)e^{2ibt}, \quad \varphi_1(x, t) = \phi_1(x)e^{ibt + t\Omega}, \quad \varphi_2(x, t) = \phi_2(x)e^{-ibt + t\Omega},
\]

where \(\Omega \in \mathbb{C}\) is another spectral parameter, yields two spectral problems

\[
(6.2) \quad \phi_x = \begin{pmatrix} \lambda & U \\ -\bar{U} & -\lambda \end{pmatrix} \phi, \quad \Omega \phi = i \begin{pmatrix} \lambda^2 + \frac{1}{2}|U|^2 - b \\ \frac{1}{2}U' + \lambda U \end{pmatrix} \phi.
\]

Since the second spectral problem in (6.2) is a linear algebraic system, it admits a nonzero solution if and only if the determinant of the coefficient matrix is zero. The latter condition yields the \(x\)-independent relation between \(\Omega\) and \(\lambda\) in the form \(\Omega^2 + P(\lambda) = 0\), where \(P(\lambda)\) is given by (2.21) with \(c = 0\) and the constants of motion (2.19) and (2.20) have been used.
By Theorem 5.1 in \[15\], squared eigenfunctions $\phi_1^2$ and $\phi_2^2$ determine eigenfunctions of the linearized NLS equation at the periodic wave $U(x)$ with the eigenvalues given by
\[(6.3) \quad \Gamma = 2\Omega = \pm 2i\sqrt{P(\lambda)}.
\]
If $\text{Re}(\Gamma) > 0$ for $\lambda$ in the Lax spectrum, the periodic wave $U$ is spectrally unstable. Admissible values of $\lambda$ are defined by the first spectral problem in (6.2) with the help of the Floquet–Bloch decomposition in Appendix A. By Theorem 7.1 in \[15\], $i\mathbb{R}$ belongs to the Lax spectrum and it follows from (3.6) and (3.22) that $P(\lambda) > 0$ for every $\lambda \in i\mathbb{R}$, hence $\Gamma \in i\mathbb{R}$ in (6.3).

Therefore, the modulation instability of the periodic wave $U$ arises only if $\lambda$ is in the interior of the finite band(s) with $\text{Re}(\lambda) \neq 0$, see Figs. 1, 2, and 3.

For the dn-periodic wave (3.9) it follows from (3.6) that $P(\lambda) < 0$ for every $|\lambda| \in (\lambda_2^+, \lambda_1^+)$ and $P(\lambda_{1,2}^+) = 0$, where $\lambda_{1,2}^+$ are given by (3.10). Therefore, $\Gamma \in \mathbb{R}$ for $|\lambda| \in (\lambda_2^+, \lambda_1^+)$ with the finite band of the modulation instability. If $\lambda = \lambda_{1,2}^+$ for which the rogue waves on Fig. 5 are constructed, we have $\Gamma = 0$, hence the rogue waves on the dn-periodic background are constructed at the end points of the modulation instability band.

Similarly, for the cn-periodic wave (3.11), the trace of $\Gamma$ in (6.3) on the complex plane is shown on Fig. 8. The curves are obtained when $\lambda$ changes along the two bands of the Lax spectra in Fig. 2 with $\text{Re}(\lambda) \neq 0$. Note that each curve on Fig. 8 is covered twice.

\[\text{Figure 8. Modulation instability bands for the cn-periodic wave (3.11) with } k = 0.85 \quad \text{(left) and } k = 0.95 \quad \text{(right).}\]

For the periodic wave (3.20), the trace of $\Gamma$ in (6.3) is shown on Fig. 9 from $\lambda$ in the Lax spectrum on Fig. 3 with $\text{Re}(\lambda) \neq 0$. The symmetry of the Lax spectrum on Fig. 3 is broken and each curve on Fig. 9 is covered once. It follows from Fig. 4 that the point $(\beta, k) = (0.85, 0.85)$ is selected near the boundary (4.19). This boundary coincides with the condition (found in equation (106) in \[15\]) under which the second band of the modulation instability is tangential to the imaginary axis of $\Gamma$ at $\Gamma = 0$. It follows from different types of the rogue waves in Fig. 7 that the rogue wave on the periodic background satisfying the limits (1.3) exists if the band of modulation instability is transverse to the imaginary axis at $\Gamma = 0$, whereas the algebraic soliton on the periodic background exists if the band of modulation instability is tangential to the imaginary axis at $\Gamma = 0$. The algebraic soliton on the periodic background does not satisfy the limits (1.3).

We conclude the paper by reiterating the question on how to define the parameter $\lambda_1$ in the one-fold Darboux transformation (2.3) in order to generate the rogue waves on a periodic background satisfying the limits (1.3). If $\lambda_1 \in i\mathbb{R}$, then $\hat{u} = u$ and no new solution is obtained. If $\lambda_1 \notin i\mathbb{R}$ is
outside the Lax spectrum, then the one-fold transformation (2.3) generates the recurrent pattern of rogue waves. Such recurrent rogue waves on the constant wave background are usually referred to as the Kuznetsov–Ma breathers. These rogue waves do not satisfy the limits (1.3).

On the other hand, if $\lambda_1 \notin i\mathbb{R}$ is inside the Lax spectrum, then the one-fold transformation (2.3) generates a periodic perturbation on the periodic wave background which develops and decays exponentially in time due to modulation instability with the growth rate $\Gamma$ in (6.3). In the context of the constant wave background, the space-periodic and time-localized solutions are usually referred to as the Akhmediev breathers. Since the perturbation period is different from the period of the periodic wave background, such solutions are generally quasi-periodic in space and exponentially localized in time. These rogue waves also satisfy the limits (1.3) but do not represent an isolated rogue wave.

Isolated rogue waves on the periodic wave background are generated by picking the value of $\lambda_1$ exactly at the end points of the Lax spectrum for $\lambda_1 \notin i\mathbb{R}$. Isolated rogue waves satisfy the limits (1.3) with the only exception when $|\phi_1(x, t)|$ remains bounded along a straight line in the $(x, t)$-plane. This exception which generates an algebraic soliton on the periodic wave background corresponds to the case when the modulation instability band is tangential to the imaginary axis at $\Gamma = 0$. The precise values of $\lambda_1$ at the end points of the Lax spectrum are captured by the algebraic method with one eigenvalue, which has been developed in this work.

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Appendix A. Floquet–Bloch decomposition of the Lax spectrum

If the entries of the matrix $U$ in the linear equation (2.1) are periodic in $x$ with the same period $L$, then Floquet’s Theorem guarantees that bounded solutions of the linear equation (2.1) can be represented in the form:

$$\varphi(x) = \begin{pmatrix} \tilde{p}_1(x) \\ \tilde{q}_1(x) \end{pmatrix} e^{i\mu x},$$

where $\tilde{p}_1(x) = \tilde{p}_1(x + L)$, $\tilde{q}_1(x) = \tilde{q}_1(x + L)$, and $\mu \in \left[ -\frac{\pi}{L}, \frac{\pi}{L} \right]$. The Lax spectrum in the linear equation (2.1) is formed by all admissible values of $\lambda$, for which the solutions are bounded in the
form (A.1), where $i\mu$ is referred to as the Floquet exponent. When $\mu = 0$ and $\mu = \pm \frac{\pi}{2}$, the solutions (A.1) are periodic and anti-periodic, respectively.

Substituting (A.1) into the linear equation (2.1) and re-arranging the terms yields the eigenvalue problem:

\[
(A.2) \quad \left( \frac{d}{dx} + i\mu \right) \begin{pmatrix} \rho_1 \\ -u \end{pmatrix} - \frac{d}{dx} \begin{pmatrix} -u \\ i\mu \end{pmatrix} = \lambda \begin{pmatrix} \rho_1 \\ \hat{q}_1 \end{pmatrix},
\]

for which we are looking for periodic solutions $(\rho_1, \hat{q}_1)$ at a discrete set of admissible values of $\lambda_1$.

The numerical scheme of computing the eigenvalues $\lambda$ is based on the discretization of the interval $[0, L]$ with $N + 1$ equally spaced grid points and using the highly accurate central difference approximation of derivatives (up to the 12th order of accuracy). MATLAB’s eigenvalue solver is used to compute all eigenvalues $\lambda$ in the discretization of the eigenvalue problem (A.2). Tracing the set of eigenvalues $\lambda$ for $\mu \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ gives the band of the Lax spectrum in the $\lambda$ plane shown on Figs. 1, 2, and 3.

Appendix B. Proof of identities (4.16) and (4.17)

We substitute (4.15) into (4.14), recall that $R^2\Theta' = -2e = \sqrt{\rho_1\rho_2} \sqrt{-\rho_3}$, and obtain

\[
(B.1) \quad \begin{cases}
\hat{P}_1^2 = \frac{1}{2(\sqrt{\rho_1} \pm \sqrt{\rho_2})} \left( R' + (\sqrt{\rho_1} \pm \sqrt{\rho_2})R + i\sqrt{-\rho_3}((\pm R + \sqrt{\rho_1\rho_2} R^{-1})) \right), \\
\hat{Q}_1^2 = \frac{1}{2(\sqrt{\rho_1} \pm \sqrt{\rho_2})} \left( -R' + (\sqrt{\rho_1} \pm \sqrt{\rho_2})R + i\sqrt{-\rho_3}((\pm R + \sqrt{\rho_1\rho_2} R^{-1})) \right), \\
\hat{P}_1\hat{Q}_1 = \frac{1}{2(\sqrt{\rho_1} \pm \sqrt{\rho_2})} \left( R^2 \pm \sqrt{\rho_1\rho_2} \pm i\sqrt{-\rho_3}(\sqrt{\rho_1} \pm \sqrt{\rho_2}) \right).
\end{cases}
\]

By using (3.14) with (3.17) and $\rho = R^2$, we obtain $(|\hat{P}_1|^2 + |\hat{Q}_1|^2)^2 = \rho(x) - \rho_3 = \text{dn}^2(x; k)$, where the last identity follows from (3.20). Extracting the square root yields (4.16).

In order to prove (4.17), we compute from (B.1):

\[
\hat{P}_1\hat{Q}_1 = \frac{-(R')^2 + (\sqrt{\rho_1} \pm \sqrt{\rho_2})^2 R^2 - \rho_3(\pm R^2 + \sqrt{\rho_1\rho_2} R^{-1})^2 - 2i\sqrt{-\rho_3}(\pm R^2 + \sqrt{\rho_1\rho_2} R^{-1})R'}{4(\sqrt{\rho_1} \pm \sqrt{\rho_2})^2}.
\]

By using (3.14) with (3.17) and (3.20), we check directly that

\[-(R')^2 + (\sqrt{\rho_1} \pm \sqrt{\rho_2})^2 R^2 - \rho_3(\pm R^2 + \sqrt{\rho_1\rho_2} R^{-1})^2 = \frac{\text{dn}^2(x; k)}{R^2}(\pm R^2 + \sqrt{\rho_1\rho_2})^2 + \frac{\rho_3(R')^2}{\text{dn}^2(x; k)},\]

which yields

\[
\hat{P}_1\hat{Q}_1 = \frac{1}{4(\sqrt{\rho_1} \pm \sqrt{\rho_2})^2} \left[ \frac{\text{dn}(x; k)}{R}(\pm R^2 + \sqrt{\rho_1\rho_2}) - \frac{i\sqrt{-\rho_3}R'}{\text{dn}(x; k)} \right]^2.
\]

Extracting the square root and picking the negative sign by using the limiting expression (4.10) for $\rho_1 = k^2$, $\rho_2 = 0$, and $\rho_3 = -(1 - k^2)$ yields the expression (4.17).

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