Equivalence Principle and the Principle of Local Lorentz Invariance

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Abstract

In this paper we scrutinize the so called Principle of Local Lorentz Invariance (PLLI) that many authors claim to follow from the Equivalence Principle (EP). Using rigorous mathematics we introduce in the General Theory of Relativity two classes of reference frames (PIRFs and LLRF$\gamma$s) which natural generalizations of the concept of the inertial reference frames of the Special Relativity Theory. We show that it is the class of the LLRF$\gamma$s that is associated with the PLLI. Next we give a definition of physically equivalent reference frames. Then, we prove that there are models of General Relativity Theory (in particular on a Friedmann universe) where the PLLI is false. However our find is not in contradiction with the many experimental claims vindicating the PLLI, because theses experiments do not have enough accuracy to detect the effect we found. We prove moreover that PIRFs are not physically equivalent.

1 Introduction

In this paper we scrutinize the so called Principle of Local Lorentz Invariance (PLLI) that some authors claim to follow from the Equivalence Principle (EP). We show that PLLI is false according to General Theory of Relativity (GRT), but nevertheless it is a very good approximation in the physical world we live in. In order to prove our claim, we recall the mathematical definition of reference frames in GRT which are modelled as certain unit timelike vector fields. We study the classification of reference frames and give a physically motivated and mathematical rigorous definition of physically equivalent reference frames. We investigate next which are the reference frames in GRT which share some of the properties of the inertial reference frames (IRFs) of the Special Theory of Relativity (SRT). We found that there are two kind of frames that appear as generalizations of the IRFs of SRT. These are the pseudo inertial reference frames (PIRFs) and the local Lorentz reference frames (LLRF$\gamma$s). Now, PLLI is a statement that LLRF$\gamma$s are physically equivalent. We show that PLLI is false by expliciting showing that there are models of GRT (explicitly a Friedmann Universe) containing LLRF$\gamma$s which are not physically equivalent.

We emphasize that our finding is not in contradiction with the many experimental proofs offered as vindicating the PLLI, since all this proofs do not have enough accuracy to detect the effect we have found which is proportional to $2av^2$, where $a << 1$ and $v$ is the initial metric velocity of the LLRF$\gamma L'$ in relation to a LLRF$\gamma L$ in a Friedmann Universe model of GRT. Indeed, for this model we showed that any point of the world manifold $p$ there is a LLRF$\gamma L$, for which $V|_\gamma = L|_\gamma$ (where $V$ is a fundamental PIRF such that the center of
mass of each galactic cluster follows one of its integral lines) and such that its expansion ratio at \( p \) is null and that if \( L' \) is a LLRF \( \gamma' (\gamma \cap \gamma' = p \in M) \) moving with initial metric velocity \( v \) at \( p \) relative \( L \) then the expansion ratio of \( L' \) is \( 2av^2 \).

We prove also that there are models of \( GRT \) where \( PIRFs \) are not physically equivalent also.

## 2 Some Basic Definitions

### 2.1 Relativistic Spacetime Theories

In this subsection we recall what we mean by a relativistic spacetime theory [1], a key concept necessary to prove our claim that the so called \( PLLI \) is not a fidedigence law of nature.

In our approach a physical theory \( \tau \) is characterized by:

(i) a theory of a certain \textit{species of structure} in the sense of Boubarki [2];

(ii) its physical interpretation;

(iii) its present meaning and present applications.

We recall that in the mathematical exposition of a given physical theory \( \tau \), the postulates or basic axioms are presented as definitions. Such definitions mean that the physical phenomena described by \( \tau \) behave in a certain way. Then, the definitions require more motivation than the pure mathematical definitions. We call coordinative definitions the physical definitions, a term introduced by Reichenbach [3]. Also, according to Sachs and Wu [4] it is necessary to make clear that completely convincing and genuine motivations for the coordinative definitions cannot be given, since they refer to nature as a whole and to the physical theory as a whole.

The theoretical approach to physics behind (i), (ii) and (iii) above is then to admit the mathematical concepts of the \textit{species of structure} defining \( \tau \) as primitives, and define coordinatively the observation entities from them. Reichenbach assumes that “physical knowledge is characterized by the fact that concepts are not only defined by other concepts, but are also coordinated to real objects”. However, in our approach, each physical theory, when characterized as a species of structure, contains some \textit{implicit} geometric objects, like some of the reference frame fields defined below, that cannot in general be coordinated to real objects.  

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**Definition 1.** A general relativistic \textit{spacetime} theory is as a theory of a species of structure such that, if \( \text{Mod} \ \tau \) is the class of models of \( \tau \), then each \( \Upsilon \in \text{Mod} \ \tau \) contains as substructure a Lorentzian spacetime \( ST = (M, D, g) \). We recall here that \( g \) is a Lorentz metric and \( D \) is the Levi-Civita connection of \( g \) on \( M \) [4]. More precisely, we have

\[
\Upsilon = (M, D, g, T_1, \ldots, T_m), \quad (1)
\]

\[1\]Indeed, it would be an absurd to suppose that all the infinity of IRFs (observation 3) that exist in a Minkowski spacetime are simultaneously realized as physical systems.
The $\mathbf{T}_i \in \sec \tau M$ (the tensor bundle), $i = 1, \ldots, m$ are (explicit) geometrical objects defined in $U \subseteq M$ characterizing the physical fields and particle trajectories that cannot be geometrized in the theory. Here, to be geometrizable means to be a metric field or a connection on $M$ or objects derived from these concepts as, e.g., the Riemann tensor (or the torsion tensor in more general spacetime theories). The $\mathbf{T}_i$, $i = 1, \ldots, m$ are supposed to satisfy a set of differential equations involving also $D$ and $g$ called the dynamical laws of the theory.

As already said above, each spacetime theory has some implicit geometrical that do not appear explicitly in eq. (1). These objects are the reference frame fields which we now study and analyze in detail.

2.2 Reference Frames

Definition 2. Let $ST$ be a relativistic spacetime. A moving frame at $x \in M$ is a basis for the tangent space $T_x M$. An orthonormal frame for $x \in M$ is a basis of orthonormal vectors for $T_x M$.

Proposition 3. Let $Q \in \sec T U \subseteq \sec T M$ be a time-like vector field such that $g(Q, Q) = 1$. Then, there exist, in a coordinate neighborhood $U$, three space-like vector fields which together with $Q$ form an orthogonal moving frame for $x \in U$. The proof is trivial [7].

Definition 2. A non-spinning particle on $ST$ is a pair $(m, \sigma)$ where $\sigma : \mathcal{R} \supset I \to M$ is a future pointing causal curve [4-6] and $m \in [0, +\infty)$ is the mass. When $m = 0$ the particle is called a photon. When $m \in (0, +\infty)$ the particle is said to be a material particle. $\sigma$ is said to be the world line of the particle.

Definition 3. An observer in $< M, D, g >$ is a future pointing time-like curve $\gamma : \mathcal{R} \supset I \to M$ such that $g(\gamma_u, \gamma_u) = 1$. The inclusion parameter $I \to \mathcal{R}$ in this case is called the proper time along $\gamma$, which is said to be the world line of the observer.

Observation 1. The physical meaning of proper time is discussed in details, e.g., in [5,6] which deals with the theory of time in relativistic theories.

Definition 4. An instantaneous observer is an element of $TM$, i.e., a pair $(z, Z)$, where $z \in M$, and $Z \in T_z M$ is a future pointing unit time-like vector.

The Proposition 1 together with the above definitions suggests:

Definition 5. A reference frame in $ST = < M, D, g >$ is a time-like vector field which is a section of $T U$, $U \subseteq M$ such that each one of its integral lines is an observer.

Observation 2. In [4-6] an arbitrary reference frame $Q \in \sec T U \subseteq \sec T M$ is classified as (i), (ii) below.

(i) according to its synchronizability. Let $\alpha Q = g(Q, \cdot)$. We say that $Q$ is locally synchronizable iff $\alpha Q \wedge da Q = 0$. $Q$ is said to be locally proper time synchronizable iff $da Q = 0$. $Q$ is said to be synchronizable iff there are $C^{\infty}$ functions $h, t : M \to \mathcal{R}$ such that $\alpha Q = h dt$ and $h > 0$. $Q$ is proper time synchronizable iff $\alpha Q = dt$. These definitions are very intuitive.

(ii) according to the decomposition of

$$Da Q = a Q \otimes a Q + \omega Q + \sigma Q + \frac{1}{3} \Theta Q h. \quad (2)$$
where
\[ h = g - \alpha_Q \otimes \alpha_Q \] (3)
is called the projection tensor (and gives the metric of the rest space of an instantaneous observer [17-19]), \( \alpha_Q \) is the (form) acceleration of \( Q \), \( \omega_Q \) is the rotation of \( Q \), \( \sigma_Q \) is the shear of \( Q \) and \( \Theta_Q \) is the expansion ratio of \( Q \). In a coordinate chart \( (U, x^\mu) \), writing \( Q = Q^\mu \partial/\partial x^\mu \) and \( h = (g_{\mu\nu} - Q_\mu Q_\nu)dx^\mu \otimes dx^\nu \) we have
\[ \omega_Q_{\mu\nu} = Q_{[\mu,\nu]}, \]
\[ \sigma_Q_{\alpha\beta} = [Q_{(\mu,\nu)}] - \frac{1}{3} \Theta_Q h_{\mu\nu} h_{\alpha\beta}, \]
\[ \Theta_Q = Q_{\mu,\mu}. \] (4)

We shall need in what follows the following result that can be easily proved:
\[ \alpha_Q \wedge d\alpha_Q = 0 \Leftrightarrow \omega_Q = 0. \] (5)

Eq. (3) means that rotating reference frames (i.e., frames for which \( \omega_Q \neq 0 \)) are not locally synchronizable. This result is the key in order to solve the misconceptions usually associated with rotating reference frames even in the SRT (see [8] for examples).

**Observation 3.** In Special Relativity where the space time manifold is \( < M = \mathbb{R}^4, g = \eta, D^n > \) an inertial reference frame (IRF) \( I \in \text{sec} TM \) is defined by \( D^n I = 0 \). We can show very easily that in GRT where each gravitational field is modelled by a spacetime \( < M, g, D > \) there are no frame \( \tilde{\Omega} \in \text{sec} TM \) satisfying \( D^n \tilde{\Omega} = 0 \). So, no IRF exist in any model of GRT.

The following question arises naturally: which characteristics a reference frame on a GRT spacetime model must have in order to reflect as much as possible the properties of an IRF of SRT?

The answer to the question is that there are two kind of frames in GRT [PIRFs (definition 6) and LLRFs (definition 9)], such that each frame in one of these classes share some important aspects of the IRFs of SRT. Both concepts are important and as we will see, it is important to distinguish between them in order to avoid misunderstandings.

### 2.2.1 Pseudo Inertial Reference Frames

**Definition 6.** A reference frame \( I \in \text{sec} TU, U \subset M \) is said to be a pseudo inertial reference frame (PIRF) if \( D^n I = 0 \) and \( \alpha_I \wedge d\alpha_I = 0 \), and \( \alpha_I = g(I,.) \).

This definition means that a PIRF is in free fall and is non rotating. It means also that it is at least locally synchronizable.

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\( \eta \) is a constant metric, i.e., there exists a chart \( \langle x^\mu \rangle \) of \( M = \mathbb{R}^4 \) such that \( \eta(\partial/\partial x^\mu, \partial/\partial x^\nu) = \eta_{\mu\nu} \), the numbers \( \eta_{\mu\nu} \) forming a diagonal matrix with entries \((1, -1, -1, -1)\). Also, \( D^n \) is the Levi-Civita connection of \( \eta \).
2.2.2 Naturally Adapted Charts to a Given Reference Frame

**Definition 7.** Let \( Q \in \text{sec} TU, U \subseteq M \) be a reference frame. A chart in \( U \) of the maximal oriented atlas of \( M \) with coordinate functions \( \langle x^\mu \rangle \) such that \( \partial/\partial x^0 \in \text{sec} TU \) is a timelike vector field and the \( \partial/\partial x^i \in \text{sec} TU \) (\( i = 1, 2, 3 \)) are spacelike vector fields is said to be a possible naturally adapted coordinate chart to the frame \( Q \) (denoted \( \langle \text{nacs—} Q \rangle \) in what follows) if the space-like components of \( Q \) are null in the natural coordinate basis \( \langle \partial/\partial x^\mu \rangle \) of \( TU \) associated with the chart.\(^3\)

2.2.3 Local Lorentzian Coordinate Chart

**Definition 8.** A chart \( (U, \xi^\mu) \) of the maximal oriented atlas of \( M \) is said to be a local Lorentzian coordinate chart (LLCC) and \( \langle \xi^\mu \rangle \) are said to be local Lorentz coordinates (LLC) in \( p_0 \in U \) iff

\[
g(\partial/\partial \xi^\mu, \partial/\partial \xi^\nu) \big|_{p_0} = \eta_{\mu\nu},
\]

\[
\Gamma_{\beta\mu}^\alpha(\xi^\mu) \big|_{p_0} = 0, \quad \Gamma_{\beta\gamma,\mu}^\alpha(\xi^\mu) \big|_p = -\frac{1}{3}(R_\beta^{\alpha\mu}(\xi^\mu) + R_\beta^{\alpha\mu}(\xi^\mu)) \big|_p, \quad p \neq p_0 \quad (7)
\]

Let \( (V, x^\mu) \) (\( V \cap U \neq \emptyset \)) be an arbitrary chart. Then, supposing that \( p_0 \) is at the origin of both coordinate systems the following relations holds (approximately)

\[
\xi^\mu = x^\mu + \frac{1}{2} \Gamma_{\alpha\beta}^\mu(p_0)x^\alpha x^\beta,
\]

\[
x^\mu = \xi^\mu - \frac{1}{2} \Gamma_{\alpha\beta}^\mu(p_0)\xi^\alpha \xi^\beta, \quad (8)
\]

where in eqs.\(^8\) \( \Gamma_{\alpha\beta}^\mu(p_0) \) are the values of the connection coefficients at \( p_0 \) expressed in the chart \( (V, x^\mu) \).

The coordinates \( \langle \xi^\mu \rangle \) are also known as Riemann normal coordinates and the explicit methods for obtaining them are described in many texts of Riemannian geometry as e.g., \([9,10]\) and of GRT, as e.g., \([11,12]\).

**Observation 4.** Let \( \gamma \in U \subseteq M \) be the world line of an observer in geodetic motion in spacetime, i.e., \( D_{\gamma*} \gamma_* = 0 \). Then as it is well known \([11]\) we can introduce in \( U \) a LLC \( \langle \xi^\mu \rangle \) such that for every \( p \in \gamma \) we have

\[
\left. \frac{\partial}{\partial \xi^\mu} \right|_{p \in \gamma} = \gamma_\star |_p; \quad g(\partial/\partial \xi^\mu, \partial/\partial \xi^\nu) \big|_{p \in \gamma} = \eta_{\mu\nu},
\]

\[
\Gamma_{\nu\rho}^\mu(\xi^\mu) \big|_{p \in \gamma} = g^{\nu\alpha}g(\partial/\partial \xi^\alpha, D_{\partial/\partial \xi^\nu} \partial/\partial \xi^\rho) \big|_{p \in \gamma} = 0. \quad (9)
\]

Take into account for future reference that if the \( \langle \xi^\mu \rangle \) are LLC then it is clear from definition 8 that in general \( \Gamma_{\nu\rho}^\mu(\xi^\mu) \big|_p \neq 0 \) for all \( p \notin \gamma \).

\(^3\)We can be prove very easily that there is an infinity of different \( \langle \text{nacs—} Q \rangle \).
2.2.4 Local Lorentz Reference Frame

**Definition 9.** Given a geodetic line \( \gamma \subset U \subset M \) and LLCC \((U, \xi^\mu)\) we say that reference frame \( L = \partial / \partial h^0 \in \sec T U \) is a Local Lorentz reference frame associated to \( \gamma \) \((LLRF\gamma)\)\(^4\) iff

\[
L|_{p \in \gamma} = \left. \frac{\partial}{\partial \xi^0} \right|_{p \in \gamma} = \gamma_s|_p, \\
\alpha_L \wedge d\alpha_L|_{p \in \gamma} = 0. \tag{10}
\]

Moreover, we say also that the Riemann normal coordinate functions or Lorentz coordinate functions \((LLC) < \xi^\mu >\) are associated with the \(LLRF\gamma\).

**Observation 5.** It is very important to have in mind that for a \(LLRF\gamma\) \( L \) in general \( L|_{p \in \gamma} \neq 0 \) (i.e., only the integral line \( \gamma \) of \( L \) in free fall in general), and also eventually \( \alpha_L \wedge d\alpha_L|_{p \in \gamma} \neq 0 \), which may be a surprising result for many readers. In contrast, a \(PIRF\) \( \Im \) such that \( \Im|_\gamma = L|_\gamma \) has all its integral lines in free fall and the rotation of the frame is always null in all points where the frame is defined. Finally its is worth to recall that both \( \Im \) and \( L \) may eventually have shear and expansion even at the points of the geodesic line \( \gamma \) that they have in common. This last point will be important in our analysis of the \(PLL\) in section 6.

**Definition 10.** Let \( \gamma \) be a geodetic line as in definition 9. A section \( s \) of the orthogonal frame bundle \( FU, U \subset M \) is called an inertial moving frame along \( \gamma \) \((IMF\gamma)\) when the set

\[
s_\gamma = \{(e_0(p), e_1(p), e_2(p), e_3(p)) | p \in \gamma \cap U \} \subset s, \tag{11}
\]

it such that \( \forall p \in \gamma \)

\[
e_0(p) = \gamma_s|_p, \left. g(e_\mu, e_\nu) \right|_{p \in \gamma} = \eta_{\mu\nu} \tag{12}
\]

with

\[
\Gamma^\mu_{\nu\rho}(p) = g^\alpha\beta \left. g(e_\alpha(p), D_{e_\nu(p)}e_\rho(p)) \right|_p = 0 \tag{13}
\]

**Observation 6.** The existence of \( s \in \sec FU \) satisfying the above conditions can be easily proved [9]. Introduce coordinate functions \(< \xi^\mu >\) for \( U \) such that at \( p_0 \in \gamma, e_0(p_0) = \left. \frac{\partial}{\partial \xi^0} \right|_{p_0} = \gamma_s|_{p_0} \) and \( e_i(p_0) = \left. \frac{\partial}{\partial \xi^i} \right|_{p_0}, i = 1, 2, 3 \) (three orthonormal vectors) satisfying Eq.(9) and parallel transport the set \( e_\mu(p_0) \) along \( \gamma \). The set \( e_\mu(p_0) \) will then also be Fermi transported [4] since \( \gamma \) is a geodesic and as such they define the standard of no rotation along \( \gamma \).

**Observation 7.** Let \( \Im \in \sec TV \) be a \( PIRF \) and \( \gamma \subset U \subset V \) one of its integral lines and let \(< \xi^\mu >, U \subset M \) be a LLC through all the points of the world line \( \gamma \) such that \( \gamma_s = \Im|_\gamma \). Then, in general \(< \xi^\mu >\) is not a \((nacs)\) in \( U \), i.e., \( \Im|_{p \in \gamma} \neq \partial / \partial \xi^0|_{p \in \gamma} \) even if \( \Im|_{p \in \gamma} = \partial / \partial \xi^0|_{p \in \gamma} \).

**Observation 8.** Before concluding this section it is very much important to recall again that a reference frame field as introduced above is a *mathematical*...

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\(^4\)When no confusion arises and \( \gamma \) is clear from the context we simply write \( LLRF \).
instrument. It did not necessarily need to have a material substratum (i.e., to be realized as a material physical system) in the points of the spacetime manifold where it is defined. More properly, we state that the integral lines of the vector field representing a given reference frame do not need to correspond to worldlines of real particles. If this crucial aspect is not taken into account we may incur in serious misunderstandings. We observe moreover that the concept of reference frame fields has been also used since a long time ago by Matolsci [13], although this author uses a somewhat different terminology.

3 Physically Equivalent Reference Frames

The objective of this section is two recall the definition of physically equivalent reference frames in a spacetime theory and in particular in GRT [1] which will be used in section 6 to prove that the PLLI is false. In order to do that we need to recall some definitions. Let \((M, D, g)\) be a Lorentzian spacetime and let \(G_M\) be the group of all diffeomorphisms of \(M\), called the manifold mapping group. Let \(A \subseteq M\).

Definition 11. The diffeomorphism \(G_M \ni h : A \to M\) induces a deforming mapping
\[
h_* : T \mapsto h_* T = \bar{T}
\]
such that,
(i) If \(f : M \supseteq A \to \mathcal{R}\), then
\[
h_* f = f \circ h^{-1} : h(A) \to \mathcal{R}.
\]
(ii) If \(T \subseteq T^{(r,s)}(A) \subseteq \text{sec} T(M)\), where \(T^{(r,s)}(A)\) is the sub-bundle of tensors of the type \((r, s)\) of the tensor bundle \(T(M)\), then
\[
(h_* T)_{h e}(h_* \omega_1, ..., h_* \omega_r, h_* X_1, ..., h_* X_s)
\]
\[
T_e(\omega_1, ..., \omega_r, X_1, ..., X_s)
\]
\[
\forall X_i \in \text{sec} T_e(A), i = 1, 2, ..., r, \forall \omega_j \in \text{sec} T^e A, j = 1, 2, ..., s, \forall e \in M.
\]
(iii) If \(D\) is the Levi-Civita connection of \(g\) on \(M\) and \(X, Y \in \text{sec} TM\), then
\[
(h_* D_{h_* X} h_* Y)_{h e} h_* f = (D_X Y)_e f, \forall e \in M
\]
\[
h_* D_{h_* X} h_* Y \equiv h_* (D_X Y).
\]
If \(\{f_\mu = \partial / \partial x^\mu\}\) is a coordinate basis for \(TA\) and \(\{\theta^\mu = dx^\mu\}\) is the corresponding dual basis for \(T^* A\) and if
\[
T = T^{\mu_1 \cdots \mu_r} \theta^\mu_1 \otimes ... \otimes \theta^\mu_r \otimes f_{\mu_1} \otimes ... \otimes f_{\mu_r},
\]
then
\[
h_* T = (T^{\mu_1 \cdots \mu_r} \circ h^{-1}) h_* \theta^\mu_1 \otimes ... \otimes h_* \theta^\mu_r \otimes h_* f_{\mu_1} \otimes ... \otimes h_* f_{\mu_r}.
\]
Suppose now that $A$ and $h(A)$ can be covered by the local chart $(U, \varphi)$ of the maximal atlas of $M$, and that $A \subseteq U$, $h(A) \subseteq U$. Let $\langle x^\mu \rangle$ be coordinate functions associated with $(U, \varphi)$. The mapping

$$x'^\mu = x^\mu \circ h^{-1} : h(U) \rightarrow \mathcal{R}$$

defines a coordinate transformation $\langle x^\mu \rangle \rightarrow \langle x'^\mu \rangle$ if $h(U) \supseteq A \cup h(A)$. Indeed, $\langle x^\mu \rangle$ are the coordinate functions associated with a local chart $(V, \chi)$ where $h(U) \subseteq V$ and $U \cap V \neq \emptyset$. Now, since under these conditions $h_* \partial/\partial x^\mu = \partial/\partial x'^\mu$ and $h_* dx^\mu = dx'^\mu$, eqs. (20) and (21) imply that

$$(h_* T)_{\langle x^\mu \rangle} (he) = T_{\langle x^\mu \rangle} (e).$$

In eq. (21) $T_{\langle x^\mu \rangle} (e) \equiv T^\mu_{\nu_1 \ldots \nu_s} (x^\mu (e))$ are the components of $T$ in the local coordinate basis $\{ \partial/\partial x^\mu \}$, $\{ dx^\mu \}$ at event $e \in M$, and $(h_* T)_{\langle x'^\mu \rangle} (he) \equiv T^\mu_{\nu_1 \ldots \nu_s} (x'^\mu (he))$ are the components of $T' = h_* T$ in the local coordinate basis $\{ h_* \partial/\partial x^\mu = \partial/\partial x', h_* dx^\mu = dx'^\mu \}$ at the point $he$. Then eq. (21) reads

$$T'^\mu_{\nu_1 \ldots \nu_s} (x'^\mu (he)) = T^\mu_{\nu_1 \ldots \nu_s} (x^\mu (e)).$$

Using eq. (20) we can also write

$$T'^\mu_{\nu_1 \ldots \nu_s} (x'^\mu (e)) = (\Lambda^{-1})^\alpha_1 \ldots (\Lambda^r)_{\alpha_s, \beta_1 \ldots \beta_s} T^\mu_{\alpha_1 \ldots \alpha_s} (x'^\mu (h^{-1} e))$$

where $\Lambda^\mu_\alpha = \partial x'^\mu / \partial x^\alpha$, etc.

**Definition 12.** Let $h \in G_M$. If for a geometrical object $T$ we have

$$h_* T = T$$

then $h$ is said to be a symmetry of $T$ and the set of all $\{ h \in G_M \}$ such that eq. (24) holds is said to be the symmetry group of $T$.

**Definition 13.** Let $\Upsilon, \Upsilon' \in \text{Mod} \tau$, $\Upsilon = (M, D, g, T_1, \ldots, T_m)$ and $\Upsilon' = (M, D', g', T'_1, \ldots, T'_m)$ with the $T_i$, $i = 1, \ldots, m$ defined in $U \subseteq M$ and $T'_i$, $i = 1, \ldots, m$ defined in $V \subseteq M$. We say that $\Upsilon$ is equivalent to $\Upsilon'$ (and denotes $\Upsilon \sim \Upsilon'$) if there exists $h \in G_M$ such that $\Upsilon' = h_* \Upsilon$, i.e., $V \subseteq h(U)$ and

$$D' = h_* D, g' = h_* g, T'_1 = h_* T_1, \ldots, T'_m = h_* T_m$$

**Theories satisfying definition 14 are called generally covariant and $\Upsilon, \Upsilon' \in \text{Mod} \tau$ represent indeed the same physical model.**

**Definition 14.** Let $\Upsilon, \tilde{\Upsilon} \in \text{Mod} \tau$, $\Upsilon = (M, D, g, T_1, \ldots, T_m)$, $\tilde{\Upsilon} = (M, h_* D, h_* g, h_* T_1, \ldots, h_* T_m)$ with the $T_i$, $i = 1, \ldots, m$ defined in $U \subseteq M$ and $T'_i$, $i = 1, \ldots, m$ defined in $V \subseteq h(U) \subseteq M$ and such that

$$D = h_* D, g = h_* g.$$

Then $\tilde{\Upsilon}$ is said to be the $h$-deformed version of $\Upsilon$.

**Definition 15.** Let $Q \in \sec TM \subseteq \sec TM, \tilde{Q} \in \sec TV \subseteq \sec TM, U \cap V \neq \emptyset$ and let $\langle x^\mu \rangle, \langle \tilde{x}^\mu \rangle$ (the coordinate functions associated respectively to the
charts \((U, \varphi)\) and \((V, \varphi)\) be respectively a \((nacs|Q)\) and a \((nacs|\bar{Q})\) and suppose that \(\bar{x}^\mu = x^\mu \circ h^{-1} : h(U) \to \mathcal{R}\). Thus, \(\bar{Q} = h, Q\) and \(\bar{Q}\) is said to be a \(h\)-deformed version of \(Q\).

Let \(\Upsilon, \bar{\Upsilon} \in \text{Mod} \tau\) be as in definition 14. Call \(o = (D, g, T_1, \ldots, T_m)\) and \(\bar{o} = (D, g, \bar{T}_1, \ldots, \bar{T}_m)\). Now, \(o\) is such that it solves a set of differential equations in \(\varphi(U) \subset \mathcal{R}^4\) with a given set of boundary conditions denoted \(b^o(x^\mu)\), which we write as

\[
D^\alpha_{\langle x^\nu \rangle} (o_{\langle x^\nu \rangle})_e = 0 ; \quad b^o(x^\nu) ; \quad e \in U, \tag{27}
\]

and \(\bar{o}\) defined in \(\bar{h}(U) \subseteq V\) solves

\[
D^\alpha_{\langle x^\nu \rangle} (\bar{h} \cdot o_{\langle x^\nu \rangle})_{\bar{h}e} = 0 ; \quad b^{\bar{h}, o}(\bar{x}^\mu) ; \quad \bar{h} \in \bar{h}(U) \subseteq V. \tag{28}
\]

In eqs. (27) and (28) \(D^\alpha_{\langle x^\nu \rangle}\) and \(D^\alpha_{\langle x^\nu \rangle}\) mean \(\alpha = 1, 2, \ldots, m\) sets of differential equations in \(\mathcal{R}^4\).

How can an observers living on \(M\) discover that \(\Upsilon, \bar{\Upsilon} \in \text{Mod} \tau\) are deformed versions of each other? In order to answer this question we need additional definitions.

**Definition 16.** Let \(Q, \bar{Q}\) be as in definition 15. We say that \(Q\) and \(\bar{Q}\) are physically equivalent according to theory \(\tau\) (and we denote \(Q \sim \bar{Q}\)) iff

(i) \(DQ = D\bar{Q}\) \hfill (29)

(ii) the system of differential equations must have the same functional form as the system of differential equations \(b^o(x^\nu)\) and \(b^{\bar{h}, o}(\bar{x}^\mu)\) must be relative to \(\langle x^\mu \rangle\) the same as \(b^o(x^\nu)\) is relative to \(\langle x^\mu \rangle\) and if \(b^o(x^\nu)\) is physically realizable then \(b^{\bar{h}, o}(\bar{x}^\mu)\) must also be physically realizable.

**Definition 17.** Given a reference frame \(Q \in \sec TU \subseteq \sec TM\) the set of all diffeomorphisms \(\{h \in G_M\}\) such that \(h_s Q \sim Q\) forms a subgroup of \(G_M\) called the equivalence group of the class of reference frames of kind \(Q\) according to the theory \(\tau\).

**Observation 9.** We can easily verify using definitions 16 and 17 any two \(IRF\) in Minkowski space time \((M, D^o, \eta)\) (observation 3) are equivalent and that the equivalence group of the class of inertial reference frames is the Poincaré group. Of course, we can verify that the symmetry group (definition 12) of \(D^o\) and \(\eta\) is also the Poincaré group. It is the existence of this symmetry group that permits a mathematical definition of the Special Principle of Relativity.\(^5\)

We can also show without difficulties that two distinct rotating references frames (with have the same angular velocity relative to a given \(IRF\) and that have the same radius) are physically equivalent. Of course, no \(IRF\) is equivalent to any rotating frame. A comprehensive example of phenomena related as \(\Upsilon, \bar{\Upsilon} \in \text{Mod} \tau\) in definitions 14 and 15 is (in Minkowski spacetime) the electromagnetic field

\(^5\)See [14] where we point out that the definition of physically equivalent reference frames given above leads to contradictions in \(SRT\) if superluminal phenomena exist and we insist in maintaining the validity of the Special Principle of Relativity.
of a charge at rest relative to an IRF \( I \) and the field of a second charge in uniform motion relative to the same IRF \( I \) and its field relative to an IRF \( I' \) where the second charge is at rest.

4 \( LLRF\gamma s \) and the Equivalence Principle

There are many presentations of the EP and even very strong criticisms against it, the most famous being probably the one offered by Synge [15]. We are not going to bet on this particular issue. Our intention here is to prove that there are models of GRT where the so called Principle of Local Lorentz Invariance (PLLI) which according to several authors (see below) follows from the Equivalence Principle is not valid in general. Our strategy to prove this strong statement is to give a precise mathematical wording to the PLLI (which formalizes the PLLI as introduced by several authors) in terms of a physical equivalence of \( LLRF\gamma s \) (see below) and then prove that PLLI is a false statement according to GRT. We start by recalling formulations and comments concerning the EP and the PLLI.

According to Friedmann [16] the “Standard formulation of the EP characteristically obscure [the] crucial distinction between first order laws and second order laws by blurring the distinction between infinitesimal laws, holding at a single point, and local laws, holding on a neighborhood of a point”....

According to our point of view, in order to give a mathematically precise formulation of Einstein’s EP besides the distinctions mentioned above between infinitesimal and local laws, it is also necessary to distinguish between some very different (but related) concepts, namely, 6

(i) The concept of an observer (definition 1);
(ii) The general concept of a reference frame in GRT (Definition 4);
(iii) The concept of a natural adapted coordinate system to a reference frame (Definition 7);
(iv) The concept of \( PIRFs \) (definition 6) and \( LLRF\gamma s \) (definition 9) on \( U \subset M \);
(v) The concept of an inertial moving observer carrying a tetrad along \( \gamma \) (a geodetic curve), a concept we abbreviate by calling it an IMF (definition 10).

Einstein’s EP is formulated by Misner, Thorne and Wheeler (MTW) [17] as follows: “in any and every Local Lorentz Frame (LLF), anywhere and anytime in the universe, all the (non-gravitational) laws of physics must take on their familiar special relativistic forms. Equivalently, there is no way, by experiments confined to small regions of spacetime to distinguish one LLF in one region of spacetime from any other LLF in the same or any other region”. We comment here that these authors7 did not give a formal definition of a LLF. They try to make intelligible the EP by formulating its wording in terms of a LLCC

\[ \footnote{6These concepts are in general used without distinction by different authors leading to misunderstandings and misconceptions.} \]

\[ \footnote{7For the best of our knowledge no author gave until now the formal definition of a LLRF as in definition 9.} \]
(see definition 8) and indeed these authors as many others do not distinguish the concept of a reference frame \( Z \in \sec TM \) from that of a \((nacs|Z)\). This may generate misunderstandings. The mathematical formalization of a \( LLF \) used by \( MTW \) (and many other authors) corresponds to the concept of \( LLRF \) introduced in definition 9.

In [18] Ciufolini and Wheeler call the above statement of \( MTW \) the medium strong form of the \( EP \). They introduced also what they called the strong \( EP \) as follows: “in a sufficiently small neighborhood of any spacetime event, in a locally falling frame, no gravitational effects are observable”. Again, no mathematical formalization of a locally falling frame is given, the formulation uses only \( LLCCs \).

Following [17,18] recently several authors as, e.g., Will [19], Bertotti and Grishchuk [20] and Gabriel and Haugan [21] (see also Weinberg [22] claim that Einstein \( EP \) requires a sort of local Lorentz invariance. This concept is stated in, e.g., [20] with the following arguments.

To start we are told that to state the Einstein \( EP \) we need to consider a laboratory that falls freely through an external gravitational field, such that the laboratory is shielded, from external non-gravitational fields and is small enough such that effects due to the inhomogeneity of the field are negligible through its volume. Then, they say, that the local non-gravitational test experiments are experiments performed within such a laboratory and in which self-gravitational interactions play no significant part. They define: “The Einstein \( EP \) states that the outcomes of such experiments are independent of the velocity of the apparatus with which they are performed and when in the universe they are performed”. This statement is then called the \( Principle of Local Lorentz Invariance (PLLII) \) and ‘convincing’ proofs of its validity are offered, and not need to be repeated here. Prugovecki [23] (pg 62) endorses the \( PLLII \) and also said that it can be experimentally verified. In his formulation he translates the statements of [16-22] in terms of Lorentz and Poincaré covariance of measurements done in two different \( IMF_\gamma \) (see Definition 10). Based on these past tentative of formalization\(^8\) we give the following one.

**Einstein \( EP \):** Let \( \gamma \) be a timelike geodetic line on the world manifold \( M \). For any \( LLRF_\gamma \) (see definition 9) all non-gravitational laws of physics, expressed through the coordinate functions \( (\xi^\mu) \) which are \( LLC \) associated with the \( LLRF_\gamma \) (definition 9) should at each point along \( \gamma \) be equal (up to terms in first order in those coordinates) to their special relativistic counterparts when

---

\(^8\)Again, no mathematical formalization of a locally falling frame is given, the formulation uses only the concept of \( LLCCs \). Worse, if local means in a neighborhood of a given spacetime event this principle must be false. For, e.g., it is well known that the Riemann tensor couples locally with spinning particles. Moreover, the neighborhood must be at least large enough to contain an experimental physicist and the devices of his laboratory and must allow for enough time for the experiments. With a gradiometer builded by Hughes corporation which has an area of approximately 400 cm\(^2\) any one can easily discover if he is leaving in a region of spacetime with a gravitational field or if he is living in an accelerated frame in a region of spacetime free with a zero gravitational field.

\(^9\)See [24] for a history of the subject.
the mathematical objects appearing in these special relativistic laws are expressed through a set of Lorentz coordinate functions naturally adapted to an arbitrary inertial frame $I \in \text{sec} TM'$, $(M' = \mathbb{R}^4, \eta, D^0)$ being a Minkowski spacetime (observation 3).

Also, if the PLLI would be a true law of nature it could be formulated as follows:

**Principle of Local Lorentz invariance (PLLI):** Any two LLRF $\gamma$ and $\text{LLRF} \gamma'$ associated with the timelike geodetic lines $\gamma$ and $\gamma'$ of two observers such that $\gamma \cap \gamma' = p$ are physically equivalent at $p$.

Of course, if PLLI is correct, it must follow that from experiments done by observers inside some LLRF $\gamma'$—say $L'$ that is moving relative to another LLRF $L$—there is no means for that observers to determine that $L'$ is in motion relative to $L$.

Unfortunately the PLLI is not true. To show that it is only necessary to find a model of GRT where the statement of the PLLI is false. Before proving this result we shall need to prove that there are models for GRT were PIRFs are not physically equivalent also.

## 5 Physical non equivalence of PIRFs $V$ and $Z$ on a Friedmann Universe

Recall that $GRT \tau_E$ is a theory of the gravitational field [4,5] where a typical model $\tau \in \text{Mod}\tau_E$ is of the form

$$\tau = < M, g, D, (m, \sigma) >, \quad (30)$$

where $ST = < M, g, D >$ is a relativistic spacetime and $T \in \text{sec} T^*M \otimes T^*M$ is called the energy-momentum tensor. $T$ represents the material and energetic content of spacetime, including contributions from all physical fields (with exception of the gravitational field and particles). For what follows we do not need to know the explicit form of $T$. The proper axioms of $\tau_E$ are:

$$D(g) = 0; \quad G = \text{Ric} - \frac{1}{2} Sg = T, \quad (31)$$

where $G$ is the Einstein tensor, $\text{Ric}$ is the Ricci tensor and $S$ is the Ricci scalar. The equation of motion of a particle $(m, \sigma)$ that moves only under the influence of gravitation is:

$$D_{\sigma\sigma} = 0. \quad (32)$$

$ST$ is in general not flat, which implies that there do not exist any IRF $I$, i.e., a reference frame such that $DI = 0$.

Now, the physical universe we live in is reasonably represented by metrics of the Robertson-Walker-Friedmann type [17]. In particular, a very simple spacetime structure $ST = < M, g, D >$ that represents the main properties observed
(after the big-bang) is formulated as follows: Let \( M = R^3 \times I, I \subset R \) and \( R : I \to (0, \infty), t \to R(t) \) and define \( g \) in \( M \) (considering \( M \) as subset of \( R^4 \)) by:

\[
g = dt \otimes dt - R(t)^2 \sum dx^i \otimes dx^i, i = 1, 2, 3. \tag{33}
\]

Then \( g \) is a Lorentzian metric in \( M \) and \( V = \partial/\partial t \) is a time-like vector field in \((M, G)\). Let \( < M, g, D > \) be oriented in time by \( \partial/\partial t \) and spacetime oriented by \( dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \). Then \( < M, g, D > \) is a relativistic spacetime for \( I = (0, \infty) \).

Now, \( V = \partial/\partial t \) is a reference frame. Taking into account that the connection coefficients in a \((nacs|V)\) given by the coordinate system in eq.(33) are

\[
\Gamma^0_{kl} = 0, \quad \Gamma^0_{k1} = R\dot{R}\delta_{k1}, \quad \Gamma^k_{0l} = \frac{\dot{R}}{R}\delta^k_l, \\
\Gamma^0_{00} = \Gamma^0_{0l} = \Gamma^0_{l0} = 0, \tag{34}
\]

we can easily verify that \( V \) is a PIRF (according to definition 6) since \( D_V V = 0 \) and \( d\alpha_V \wedge \alpha_V = 0, \quad \alpha_V = g(V, \cdot) \). Also, since \( \alpha_V = dt \), \( V \) is proper time synchronizable.

**Proposition 2**: In a spacetime defined by Eq.(28) which is a model of \( \tau_E \) there exists a PIRF \( Z \in \text{sec} \\text{TU} \) which is not physically equivalent to \( V = \partial/\partial t \).

**Proof**: Let \( Z \in \text{sec} \text{TU} \) be given by

\[
Z = \left(\frac{R^2 + u^2}{R}\right)^{1/2}\partial/\partial t + \frac{u}{R^2}\partial/\partial x^1 \tag{35}
\]

where in eq.(34) \( u \neq 0 \) is a real constant.

Since \( D_Z Z = 0 \) and \( d\alpha_Z \wedge \alpha_Z = 0, \quad \alpha_Z = g(Z, \cdot) \), it follows that \( Z \) is a PIRF11. All that is necessary in order to prove our proposition is to show that \( DZ \neq DV \). It is enough to prove that the expansion ratios \( \Theta_Z \neq \Theta_V \). Indeed, eq.(34) gives

\[
\Theta_V = 3\dot{R}/R, \\
\Theta_Z = \frac{\left[ R\ddot{R} + 2\dot{R}(R^2 + u^2)^{1/2} \right]}{R^2 (R^2 + u^2)^{1/2}}, \tag{36}
\]

where

\[
v = R\left(\frac{d}{dt}x^1 \circ \gamma \right) \bigg|_{t=0} = u(1 + u^2)^{-1/2} \tag{37}
\]

10The suggestion of the validity of a proposition like the one formalized by proposition 3 has been first proposed by Rosen [25]. However, he has not been able to identify the true nature of the \( V \) and \( Z \) which he thought as representing ‘inertial’ frames. He tried to show the validity of the proposition by analyzing the output of mechanical and optical experiments done inside the frames \( V \) and \( Z \). We present in section 7.3 a simplified version of his suggested mechanical experiment. It is important to emphasize here that from the validity of the proposition 3 he suggested that it implies in a breakdown of the PLLI. Of course, the PLLI refers to the physical equivalence of LLRF\(\gamma_s \). Also the proof of proposition 3 given above is original.

11Introducing the \((nacs|Z)\) given by eq.(34) we can show that \( \alpha_Z = dt' \) and it follows that is also proper time synchronizable.
is the initial metric velocity of $Z$ relative to $V$, since we choose in what follows the coordinate function $t$ such that $R(0) = 1$, $t = 0$ being taken as the present epoch where the experiments are done. Then, $\Theta_V(p_0) = 3a$, and for $v << 1$, $\Theta_Z(p_0) = 3a - av^2$.\[\square\]

### 5.1 Mechanical experiments distinguish PIRFs

If accepted, the PLLI says that LLRF $\gamma$s at $p \in M$ are physically equivalent and that there are no mechanical experiments that can distinguish between them.

We shall prove below that PLLI is false, at least, if one of these experiments refers to the measurement of the expansion ratio of the LLRFs $\gamma$s at $p \in M$.

The question arises: can mechanical experiments (distinct from the one designed to measure the expansion ratio) distinguish between the PIRFs $V$ and $Z$? The answer is yes. To prove our statement we proceed as follows.

(i) We start by finding a $(nacs|Z)$. To do that we note if $\gamma$ is an integral curve of $Z$, we can write

\[
Z\Big|_{\gamma} = \left[ \frac{d}{ds}(x^\mu \circ \gamma) \frac{\partial}{\partial x^\mu} \right]_{\gamma}
\]

where $s$ is the proper time parameter along $\gamma$. Then, we can write [taking into account eqs. (34)] its parametric equations as

\[
\begin{align*}
\frac{dx^t}{dt} \circ \gamma &= \left( \frac{dx^t}{ds} \circ \gamma \right) = \frac{u}{R(R^2 + u^2)^{1/2}}; \quad x^2 \circ \gamma = 0; \quad x^3 \circ \gamma = 0 \\
\end{align*}
\]

(The direction $x^1 \circ \gamma = 0$ is obviously arbitrary). We then choose for $(nacs|Z)$ the coordinate functions $(t', x'^1, x'^2, x'^3)$ given by:

\[
\begin{align*}
x'^1 &= x^1 - u \int_0^t dr \frac{1}{R(r)[R(r)^2 + u^2]^{1/2}}; \quad x'^2 = x^2; \\
x'^3 &= x^3; \quad t' = \int_0^t dr \frac{[R(r)^2 + u^2]^{1/2}}{R(r)} - ux^1
\end{align*}
\]

We then get:

\[
g = dt' \otimes dt' - \overline{R(t')}^2 \left\{ \left[ \frac{1 - v^2(1 - \overline{R(t')}^{-2})}{1 - v^2} \right] dx'^1 \otimes dx'^1 + dx'^2 \otimes dx'^2 + dx'^3 \otimes dx'^3 \right\},
\]

and the connection coefficients in the $(nacs|Z)$ are,

\[
\Gamma_{k1}^0 = \frac{\dot{R} \bar{R}}{(R^2 + u^2)^{1/2}} \delta_{k1}, \quad \Gamma_{01}^1 = \frac{\dot{R} \bar{R}}{(R^2 + u^2)^{1/2}}, \quad \bar{\Gamma}_{02}^2 = \bar{\Gamma}_{03}^3 = \frac{\dot{R}}{(R^2 + u^2)^{1/2}},
\]

\[
\bar{\Gamma}_{k1}^0 = 0, \quad \bar{\Gamma}_{00}^1 = \bar{\Gamma}_{01}^0 = \bar{\Gamma}_{02}^0 = \bar{\Gamma}_{03}^0 = 0.
\]

where $\overline{R(t')} = R(t(t'))$ and $v$ given by eq. (37) is the initial metric velocity of $Z$ relative to $V$, since we choose in what follows the coordinate function $t$ such
that $R(0) = 1$, $t = 0$ being taken as the present epoch where the experiments are done. $\mathbf{Z} = \partial/\partial t'$ is a proper time synchronizable reference frame and we can verify that $t'$ is the time shown by standard clocks at rest in the $\mathbf{Z}$ frame synchronized à l’Einstein. Notice that an observer at rest in $\mathbf{Z}$ does not know a priori the value of $v$. He can discover this value as follows:

(ii) The solution of the equation of motion for a free particle $(m, \sigma)$ in $\mathbf{V}$ with the initial conditions at $p_0 = (0, x^i \circ \sigma(0) = 0)$, $i = 1, 2, 3$ and $\frac{d}{dt} x^i \circ \sigma(0) = \vec{u}^i$ for a fixed $i$ and $\frac{d}{dt} x^j \circ \sigma(0) = 0, j \neq i$, is given by an equation analogous to Eq.(38). The accelerations are such that

$$\left. \frac{d^2}{ds^2} x^j \circ \sigma(t) \right|_{p_0} = 0, \ j \neq i.$$  (43)

(iii) The equation of motion for a free particle $(m, \sigma')$ in $\mathbf{Z}$, can be write as (we write for simplicity in what follows $\frac{d^2}{ds^2} x^1 \circ \sigma'(t') \equiv \frac{d^2}{ds^2} x^1 \equiv \frac{d^2}{ds^2} x^1$, etc...)

$$\frac{d^2 x^1}{ds^2} = -2 \frac{\hat{R} \bar{R}^2}{(R^2 + u^2)^2} \frac{dx^1}{ds} \left( \frac{dt'}{ds} \right)^2,$$

$$\frac{d^2 x^i}{ds^2} = -2 \frac{\hat{R} \bar{R}^2}{(R^2 + u^2)^2} \frac{dx^i}{ds} \left( \frac{dt'}{ds} \right)^2, \ i = 2, 3,$$

$$\frac{dt'}{ds} = \left[ 1 + \hat{R}^2 \left( \frac{dx^1}{dt'} \right)^2 + \bar{R}^2 \left( \frac{dx^2}{dt'} \right)^2 + \bar{R}^2 \left( \frac{dx^3}{dt'} \right)^2 \right]^{-\frac{1}{2}}.$$  (44)

where the dot over $R$ in eq.(44) means derivative with respect to $t'$ and $\hat{R}$ denotes the square root of the coefficient of $dx^1 \otimes dx^1$ term in eq.(41).

From these equations it is easy to verify that the two situations:

(a) motion in the $(x^1', x^2')$ plane with initial conditions at $p_0$ with coordinates $(t' = 0, x^1 = x^2 = 0 = x^3)$ given by

$$\left. \frac{dx^1'(t')}{dt'} \right|_{p_0} = v_1', \left. \frac{dx^2'(t')}{dt} \right|_{p_0} = 0,$$  (45)

and

(b) motion in the $(x^1', x^2')$ plane with initial conditions at $p_0$ with coordinates $(t' = 0, x^1 = x^2 = 0 = x^3)$ given by

$$\left. \frac{dx^1'(t')}{dt'} \right|_{p_0} = 0, \left. \frac{dx^2'(t')}{dt'} \right|_{p_0} = v_2',$$  (46)

produce asymmetrical outputs for the measured accelerations along $x^1'$ and $x^2'$. The explicit values depends of course of the function $R(t)$. If we take
$R(t) = 1 + at$, the asymmetrical accelerations will be given in terms of $a < < 1$ and $v$. This would permit in principle for the eventual observers living in the $PIRF Z$ to infer the value of $u$ (or $v$).

6 \textbf{LLRF}\gamma \text{ and LLRF}\gamma'$ are not Physically Equivalent on a Friedmann Universe.

\textbf{Proposition 3}. There are models of GRT for which two \textit{Local Lorentz Reference Frames} are not physically equivalent.

\textbf{Proof}: Take as model of GRT the one just described above where $g$ is given by eq. (33) and take as before, $R(t) = 1 + at$. Consider two integral lines $\gamma$ and $\gamma'$ of $V$ and $Z$ such that $\gamma \cap \gamma' = p$.

We can associate with these two integral lines the $LLRF_\gamma L$ and the $LLRF_{\gamma'} L'$ as in definition 9. Observe that $V|_\gamma = L|_\gamma$ and $Z|_{\gamma'} = L'|_{\gamma'}$. Definition 17 says that if $L$ and $L'$ are physically equivalent then we must have $DL = DL'$. However, a simple calculation shows that in general $DL \neq DL'$ even at $p$! Indeed, we have

$$\Theta_L = -3t \left( \frac{\dot{R}}{R} \right)^2,$$

$$\Theta_{L'} = 2\dot{R}(\dot{R}^2 + u^2)^{1/2} + \frac{\ddot{R} R^2}{(R^2 + u^2)^{3/2}} - \frac{2\dot{R}}{(R^2 + u^2)^{1/2}} - \frac{2\dot{R} R^4}{(R^2 + u^2)^3} tx^1 - \frac{2\ddot{R}^2}{(R^2 + u^2)^{1/2}} - \frac{2\ddot{R} R^4}{(R^2 + u^2)^3} tx^3.$$

\[47\] \[48\]

\begin{equation}
\end{equation}

From equations (47) and (48) we see that the expansions ratios $\Theta_L$ and $\Theta_{L'}$ are different in our model and then it follows our result. At $p$, we have $\Theta_L(p) = 0$ and $\Theta_{L'}(p) = 2av^2$. ■

\textbf{Observation 10}. Proposition 3 establishes that in a Friedmann universe there is a $LLRF_\gamma$ (say $L$) whose expansion ratio at $p$ is zero. Any other $LLRF_{\gamma'}$ (say $L'$) at $p$ will have an expansion ratio at $p$ given by $2av^2$, where $a < < 1$ and $v$ is the metric velocity of $L'$ relative to $L$ at $p$. This expansion ratio can in principle be measured and this is the reason for the nonvalidity of the PLLI as formulated by many contemporary physicists and formalized above. Note that all experimental verifications of the PLLI mentioned by the authors that endorse the PLLI have been obtained for $LLRF_\gamma$s moving with $v < < 1$, and have no accuracy in order to contradict the result we found. We do not know of any experiment that has been done on a $LLRF_\gamma$ which enough precision to verify the effect. Anyway the non physical equivalence between $L$ and $L'$ is a prediction of GRT and must be accepted if this theory is right. PLLI is only
approximately valid.

We conclude this section by recalling that Friedman [16] formulates the PLLI by saying that if \( \langle U, \xi^\mu \rangle, \langle U', \xi'^\mu \rangle \) are LLCC adapted to the \( L \) and \( L' \) respectively, then the PLLI implies that two experiments whose initial conditions read alike in terms of \( \langle \xi^\mu \rangle \) and \( \langle \xi'^\mu \rangle \) will also have the same outcome in terms of these coordinate charts.

Friedman’s statement is not correct, of course, in view of proposition 3 above, for measurement of the expansion ratio of a reference frame is something objective and, of course, it is a physical experiment. However, for experiments different from this one of measuring the expansion ratio we can accept Friedman’s formulation of the PLLI as an approximately true statement.

**Observation 11.** Recall the expansion ratios calculated for \( V, Z, L, L' \). Now, \( a << 1 \). Then, if \( v << 1 \) the LLRF\( \gamma \ L \) and the LLRF\( \gamma \ L' \) will be almost ‘rigid’ whereas the \( V \) and \( Z \) are expanding. In other words, the \( L \) and \( L' \) frames can be thought as being physically materialized in their domain by real solid bodies and thus correspond to small real laboratories, the one used by physicists. On the other hand it is well known that the \( V \) frame is an idealization, since only the center of mass of the galactic clusters are supposed to be comoving with the \( V \) frame, i.e., each center of mass of a galactic cluster follows some particular integral line of \( V \). Concerning the \( Z \) frame, in order for it to be realized as a physical system it must be build with a special matter that suffers in all points of its domain an expansion a little bit greater than the cosmic expansion. Of course, such a frame would be a very artificial one, and we suspect that such a special matter cannot be prepared in our universe.

7 Conclusions

In this paper we presented a careful analysis of the concept of a reference frames in GRT which are modelled as certain unit timelike vector fields and gave a physically motivated and mathematical rigorous definition of physically equivalent reference frames. We investigate which are the reference frames in GRT which share some of the properties of the inertial reference frames of SRT. We found that in GRT there are two classes of frames that appears as generalizations of the inertial frames of SRT. These are the class of the pseudo inertial reference frames (PIRFs) and the class of the Local Lorentz reference frames (LLRF\( \gamma \)s) . We showed that LLRF\( \gamma \)s are not physically equivalent in general and this implies that the so called Principle of Local Lorentz invariance (PLLI) which several authors state as meaning that LLRF\( \gamma \)s are equivalent is false. It can only be used as an approximation in experiments that do not have enough accuracy to measure the effect we found. We prove moreover that there are models of GRT where PIRFs are not physically equivalent also.

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