On the Construction of Protograph-based Partially Doped GLDPC Codes

Jaewha Kim, Jae-Won Kim, and Jong-Seon No

Abstract

A generalized low-density parity-check (GLDPC) code is a class of codes, where single parity check nodes in a conventional low-density parity-check (LDPC) code are replaced by linear codes with higher parity check constraints. In this paper, we introduce a new method of constructing GLDPC codes by inserting the generalized check nodes for partial doping. While the conventional protograph GLDPC code dopes the protograph check nodes by replacing them with the generalized check nodes, a new GLDPC code is constructed by adding the generalized check nodes and partially doping the selected variable nodes to possess higher degrees of freedom, called a partially doped GLDPC (PD-GLDPC) code. The proposed PD-GLDPC codes can make it possible to do more accurate extrinsic information transfer (EXIT) analysis and the doping granularity can become finer in terms of the protograph than the conventional GLDPC code. We also propose the constraint for the typical minimum distance of PD-GLDPC codes and prove that the PD-GLDPC codes satisfying this condition have the linear minimum distance growth property. Furthermore, we obtain the threshold optimized protograph for both regular and irregular ensembles of the proposed PD-GLDPC codes over the binary erasure channel (BEC). Specifically, we propose the construction algorithms for both regular and irregular protograph-based PD-GLDPC codes that enable the construction of GLDPC codes with higher rates than the conventional ones. The block error rate performance of the proposed PD-GLDPC code shows that it has a reasonably good waterfall performance with low error floor and outperforms other LDPC codes for the same code rate, code length, and degree distribution.

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Index Terms

Binary erasure channel (BEC), doping, generalized low-density parity-check (GLDPC) code, low-density parity-check (LDPC) code, partially doped GLDPC (PD-GLDPC) code, protograph, protograph extrinsic information transfer (PEXIT), typical minimum distance.

I. INTRODUCTION

Low-density parity-check (LDPC) codes, first introduced in [1], have received many interests due to their low decoding complexity by the sparsity of the parity check matrix and Shannon capacity approaching error performance [2]. An LDPC code is defined over a bipartite graph consisting of variable nodes and single parity check (SPC) constraint nodes. A generalized LDPC (GLDPC) code was first introduced in [3], which is constructed by replacing (doping) some SPC nodes by generalized constraint (GC) nodes in a parity check matrix (PCM) of a linear code with a larger minimum distance [4]. This makes the LDPC code to have a larger minimum distance [5], faster decoding convergence [6], and a better asymptotic threshold at the cost of using additional decoding complexity and redundancy [7].

Many types of linear codes for GC nodes, also called as the component codes, are used in the GLDPC codes such as Hamming codes [8], Hadamard codes [9], Bose–Chaudhuri–Hocquenghem (BCH) codes, and Reed-Solomon (RS) codes [10]. The research of the GLDPC codes is extended to spatially coupled LDPC (SC-LDPC) codes [11]–[13] and doubly GLDPC (DGLDPC) codes [14]–[16]. While some capacity-approaching GLDPC codes were constructed using irregular random GLDPC codes [7], [17], there are limited works related to the capacity-approaching protograph-based GLDPC codes due to two difficulties of the code construction. First, since the rate-loss that is incurred during the doping process is not negligible, constructing a protograph GLDPC code for a medium to high code rate is difficult. Second, due to the high doping granularity, the optimization of protograph GLDPC codes is limited. In order to replace a specific SPC node with a GC node, the degree of the SPC node and length of the component code should be the same. Thus, the selection of SPC nodes to be replaced becomes limited, which makes the doping process difficult.

In this paper, we propose a new construction method of protograph-based GLDPC code, called partially doped GLDPC (PD-GLDPC) codes, which resolves the limitations for the construction of the conventional protograph GLDPC codes. In the proposed construction, constraint rows for a GC node are added, rather than replaced by partially doping the selected variable nodes,
to the parity check matrix after the lifting process of a protograph. Furthermore, we propose an optimized construction method of the protograph-based PD-GLDPC codes while the newly constructed code has a minimum distance that grows linearly with the block length of the code from the constraint for the existence of the typical minimum distance. Both the new optimized code construction method and its analysis of the proposed protograph-based PD-GLDPC codes are proposed in this paper, which makes the proposed protograph-based PD-GLDPC code approach the channel capacity.

Specifically, by focusing the partial doping on the protograph variable nodes of degree 2, we construct protograph-based PD-GLDPC codes with a good waterfall performance. Furthermore, the proposed protograph-based PD-GLDPC codes have more accurate protograph extrinsic information transfer (PEXIT) analysis compared to conventional protograph GLDPC codes, which is due to the exact computation of the extrinsic information transfer (EXIT) for a protograph GC node in the proposed protograph-based PD-GLDPC codes. While a GC node in conventional protograph GLDPC codes computes the average mutual informations of its neighboring protograph variable nodes, a GC node in the proposed protograph-based PD-GLDPC codes computes the mutual information of single protograph variable node as its neighborhood. Thus, the computation of the PEXIT of a GC node becomes more accurate in the proposed PD-GLDPC codes.

Since the proposed protograph-based PD-GLDPC codes are constructed through the doping process by adding new GC nodes of the error correcting codes connected to the subset of variable nodes lifted from single protograph variable node, the code construction can be made over any base matrix, regardless of the check node degree distribution of the original protograph. That is, the new rows for the doped GC nodes are added by the component code after lifting, where the lifting size is the multiple of the length of the component code and the variable nodes in each added row are lifted from single protograph variable node. Thus, the doping granularity can be much smaller than the conventional GLDPC codes, which enables much flexible construction of the proposed PD-GLDPC codes.

Finally, we propose the optimization method to construct large protographs for the proposed protograph-based PD-GLDPC codes. Due to the large rate loss from protograph GC nodes, it is difficult to construct GLDPC codes with medium to high code rates from small protographs. The large protographs of high code rates constructed from the degree distribution based on the random ensemble enable the larger number of doping of GC nodes. Also, the number of doping
of GC nodes can be varied according to the code rate of the initial protograph. The optimized balance between the number of doping nodes and the code rate of the initial protograph is given in the proposed protograph-based PD-GLDPC code. Furthermore, the construction of the proposed protograph-based PD-GLDPC codes with code rates $1/2$ and $1/4$ are presented in this paper.

The rest of the paper is organized as follows. In Section II, we introduce some preliminaries on the binary erasure channel (BEC) and protograph GLDPC codes. Section III illustrates the proposed protograph-based PD-GLDPC code structure and derives its PEXIT analysis with performance comparison with the conventional GLDPC codes. The construction and optimization of protograph-based PD-GLDPC codes from regular and irregular protographs are given in Section IV. Section V shows the optimized results and error performance of the proposed codes. Section VI concludes the paper with some discussion of the results.

II. Backgrounds

In this section, we introduce some notations and concepts of a binary erasure channel, protograph LDPC codes, and the construction method of conventional protograph GLDPC codes. The EXIT analysis and the decoding process of conventional GLDPC codes are also briefly introduced.

A. Protograph LDPC Code and BEC

Let $\mathbf{x} = \{x_1, \cdots, x_k\}, x_i \in \{0, 1\}$ be a $k$-bit binary message vector, which is encoded via an $(n, k)$ linear code, forming an $n$-bit codeword $\mathbf{c} = \{c_1, \cdots, c_n\}, c_i \in \{0, 1\}$. The codeword passes through a memoryless BEC, where each bit is either erased with probability $\epsilon$ or correctly received.

A protograph LDPC code is defined by a relatively small bipartite graph $G = (V, C, E)$ representing a protograph, where $V = \{v_1, \cdots, v_{n_v}\}$ is a set of variable nodes and $C = \{c_1, \cdots, c_{n_c}\}$ is a set of check nodes. Let $E$ be the set of edges $e$, where $e = (v, c)$ connects the variable node $v \in V$ and the check node $c \in C$. The bipartite graph can also be expressed in terms of an $n_c \times n_v$-sized base matrix $\mathbf{B}_{n_c \times n_v} = \{b_{i,j}\}, i \in [n_c], j \in [n_v]$, where $b_{i,j} \in \{0, 1, 2, \cdots\}$ and $[A]$ is a set of positive integers less than or equal to a positive integer $A$. The rows represent the check nodes and the columns represent the variable nodes in the protograph. Each entry $b_{i,j}$ of
**B. Construction of Protograph GLDPC Code** [18]

Conventionally, a protograph GLDPC code is constructed by replacing (doping) a check node of a protograph with a GC node that has a parity check constraint from an \((n_i, k_i, d_{min})\) linear code (component code), where \(n_i\) (\(k_i\)) is the code length (dimension) and \(d_{min}\) is the minimum distance of the component code for a check node \(c_i\). The condition for replacement is that the check node degree should be exactly equal to the length of the component code, i.e., \(deg(c_i) = n_i\). Note that the original check node has the parity check constraint of an \((n_i, k_i)\) SPC code. The code rate \(R\) of protograph GLDPC code is \(R = 1 - \frac{m_{proto}}{n_v}\), where \(m_{proto} = \sum_{i=1}^{n_c} (n_i - k_i)\). While the minimum distance of an SPC node is 2, the variable nodes connected to the GC node will be protected by parity check constraints of the component code with the minimum distance larger than two. Fig. 1 is the conventional protograph GLDPC code of code rate \(\frac{3}{7}\) by replacing an SPC node with the \((7, 4)\) Hamming code.

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Fig. 1: Conventional protograph GLDPC code construction by replacing an SPC node with a protograph GC node using the \((7, 4)\) Hamming code as the component code.
Algorithm 1 The PEXIT analysis of a protograph GLDPC code [20]

Step 1) Initialization

Initialize $I_{ch}(j) = \epsilon$ for $j \in [n_v]$.

Step 2) Message update from variable node to check node

Update $I_{EV}(i,j) = 1 - \epsilon \prod_{t \in N(v_j), t \neq i} (1 - I_{AV}(t,j))$ for all $j \in [n_v]$, where $I_{EV}(i,j) = 0$ if $b_{i,j} = 0$. If $c_i$ is an SPC node, $I_{AV}(i,j) = I_{EC}(i,j)$ and if $c_i$ is a GC node, $I_{AV}(i,j) = I_{EGC}(i,j)$.

Step 3) Message update from check node to variable node

For all $i$, if $c_i$ is an SPC node, go to Step 3-1) and if $c_i$ is a GC node, go to Step 3-2).

Step 3-1) $I_{EC}(i,j) = \prod_{t \in N(c_i), t \neq j} I_{AC}(i,t)$, where $I_{AC}(i,t) = I_{EV}(i,t)$.

Step 3-2) For all $j \in N(c_i)$, compute

$$I_{EGC}(i,j) = \frac{1}{n_i} \sum_{h=1}^{n_i} (1 - I_{AGC}(i))^{h-1} (I_{AGC}(i))^{n_i-h} [h\tilde{e}_h - (n_i - h + 1)\tilde{e}_{h-1}],$$

(1)

where $I_{AGC}(i) = \frac{1}{n_i} \sum_{j=1}^{n_i} I_{EV}(i,j)$.

Step 4) APP mutual information computation

For all $j \in [n_v]$, $I_{APP}(j) = 1 - \epsilon \prod_{t \in N(v_j)} (1 - I_{AV}(t,j))$. If $c_t$ is an SPC node, $I_{AV}(t,j) = I_{EC}(t,j)$ and if $c_t$ is a GC node, $I_{AV}(t,j) = I_{EGC}(t,j)$.

Step 5) Convergence check of variable nodes

Repeat Step 2–4) until $I_{APP}(j) = 1$, for all $j \in [n_v]$.

C. PEXIT Analysis and Decoding Process of Protograph GLDPC Code

The asymptotic performance of the conventional protograph GLDPC codes is evaluated by the PEXIT analysis. We use the EXIT and PEXIT analyses of the GLDPC code over the BEC introduced in [20]–[22]. The PEXIT analysis tracks down the mutual information of extrinsic messages and a priori error probabilities of the variable nodes, SPC check nodes, and GC nodes of a protograph GLDPC code. Let $I_{ch}(j)$ be the channel information from the erasure channel for the protograph variable node $v_j$. In addition, $I_{EV}(i,j)$ ($I_{EC}(i,j)$) is the extrinsic information sent from $v_j$ ($c_i$) to $c_i$ ($v_j$) and $I_{AV}(i,j)$ ($I_{AC}(i,j)$) is the a priori mutual information of $v_j$ ($c_i$) sent from $c_i$ ($v_j$), where $c_i$ is an SPC node. For GC nodes, we use the notations $I_{AGC}(i)$ and $I_{EGC}(i,j)$ for a priori and extrinsic information. Let $N(c_i)$ ($N(v_j)$) be the set of variable (check) nodes incident to $c_i$ ($v_j$), i.e., neighborhood of $c_i$ ($v_j$). Finally, $I_{APP}(j)$ is a posteriori probability of $v_j$. To explain [1], if $c_i$ is a GC node with the $(n_i, k_i)$ Hamming code, the PEXIT
of the GC node is computed from a closed form using the property of the simplex code, which is the dual code of a Hamming code. Also, $I_{AGC}(i) = \frac{1}{n_i} \sum_{j=1}^{n_i} I_{EV}(i,j)$ is the average a priori mutual information for a GC node to compute the PEXIT message. In [1], we have

$$\bar{c}_h = \sum_{t=1}^{h} \sum_{u=0}^{t-1} (-1)^u 2^u \left( \begin{array}{c} k_i \\ t \end{array} \right) \left( \begin{array}{c} 2t-u \\ h \end{array} \right).$$

For two positive integers $a$ and $b$, we also have $\binom{a}{b} = \prod_{i=0}^{b-1} \frac{a-i}{b-i}$ and $[a]_b = \prod_{i=0}^{b-1} \frac{2a-2^i}{2^i}$, where $\binom{a}{0} = 1$ and $[a]_0 = 1$. The PEXIT process is given in Alg. [1] which searches for the minimum $\epsilon$ to successfully decode, i.e., $I_{APP}(j) = 1$, for all $j \in [n_v]$, in asymptotic sense.

Now, we briefly explain the decoding process of the conventional GLDPC codes [13]. The variable nodes process the conventional message-passing decoding over the BEC by sending correct extrinsic messages to the check nodes if any of the incoming bits from its neighborhood is not erased. The SPC nodes send correct extrinsic messages to the variable nodes if all of its incoming messages are correctly received, and send an erasure message otherwise. In this paper, the decoding of GC nodes is processed by the maximum likelihood (ML) decoder. For each iteration, a GC node $c_i$ with the $(n_i, k_i)$ component code receives the set of erasure locations $\{e_i\}$ from $N(c_i)$. Let $H_{GC}$ be the parity check matrix of the component code and $H_e$ be the submatrix of $H_{GC}$ indexed with $\{e_i\}$. The decoder computes the Gaussian-elimination operation of $H_e$, making it into a reduced row echelon form $H_{e\text{ reduced}}$. If $\text{rank}(H_{e\text{ reduced}}) = |\{e_i\}|$, the GC node solves all the input erasures and otherwise, the decoder corrects the erasures corresponding to the rows with weight 1 from $H_{e\text{ reduced}}$. The decoding complexity can be further reduced if the GC node exploits bounded distance decoding; however, the degradation of asymptotic performance is not negligible, as shown in [7].

**III. THE PROPOSED PROTOGRAPH-BASED PD-GLDPC CODE**

In this section, a new construction method of protograph-based GLDPC codes is proposed. While the conventional protograph GLDPC codes are constructed by replacing some protograph SPC nodes in the original protograph by GC nodes using the component code, the proposed protograph-based PD-GLDPC code is constructed by adding the GC nodes for the subset of variable nodes using component codes after the lifting process of the original protograph, where each GC node is connected to the variable nodes copied from single protograph variable node. A block diagram of the construction process of both codes together with the conventional random GLDPC code is given in Fig. [2].
Fig. 2: Block diagram of the construction process of conventional GLDPC codes and the proposed protograph-based PD-GLDPC codes.

Fig. 3: An example of a proposed \((B_{2 \times 3}, 7, 4, 14, 1)\) protograph-based PD-GLDPC code construction, where \(B_{2 \times 3} = [1 \ 1 \ 1; 1 \ 0 \ 1]\) and \(\pi\) is the \(14 \times 14\) sized permutation matrix.

A. Construction Method of Protograph-based PD-GLDPC Code

First, we define the partial doping for variable nodes using the addition of GC nodes to a lifted protograph, that is, the addition of rows for the parity check matrix by the component code, where each GC node is incident to variable nodes copied from single protograph variable node. Also, we define a partially doped protograph variable node as a protograph variable node incident to the added GC nodes. While the term doping in conventional GLDPC codes is used in the perspective of check nodes, we use the term partial doping in the perspective of variable nodes. Let \(B_{n_c \times n_v}\) be an \(n_c \times n_v\) base matrix, where some protograph variable nodes are partially doped with a \((\mu, \kappa)\) component code after the lifting process. Let \(x = 0, 1, 2, \cdots\) be the number of the partially doped protograph variable nodes, where each protograph variable node is doped by \(N/\mu\)
component codes after the lifting process. Then, a proposed protograph-based PD-GLDPC code is defined with parameters \((B_{n_c \times n_v}, \mu, \kappa, N, x)\). We assume that the component code used in the paper is the \((\mu, \kappa)\) Hamming code and that \(\mu\) divides the lifting factor \(N\) such that \(N = \mu \beta\), where \(\beta\) is a non-negative integer. The variable nodes copied from \(x\) protograph variable nodes in the base matrix are partially doped by the GC nodes. That is, in the proposed PD-GLDPC code construction, the \(N/\mu\) GC nodes are connected to the \(N\) variable nodes lifted from each protograph variable node. Thus, the proposed construction method can choose the protograph variable nodes to protect by partial doping. An example of the proposed construction is given in Fig. 3 which illustrates the doping process by a \((7, 4)\) component Hamming code over a \(2 \times 3\) base matrix.

The basic concept of the proposed construction is to focus on the protection of the variable nodes lifted from a protograph variable node. Since \(N\) is the multiple of the component code length, all the variable nodes lifted from single protograph variable node can be protected by
Fig. 5: An example of the Tanner graph representation of the proposed PD-GLDPC code from the protograph $B_{2\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, where $x = 1$.

using $\beta$ GC nodes. The parity check matrix for $\beta$ GC nodes, connected to the variable nodes lifted from a protograph variable node, $H_H$ is shown in Fig. 4(a), where $H_{Hamm}$ is the parity check matrix of the Hamming code. The constructed protograph-based PD-GLDPC code has a PCM $H_{PD-GLDPC}$ as in Fig. 4(b), where the upper part is the PCM of the added $\beta x$ GC nodes and the lower part $H_{proto}$ refers to the PCM of the LDPC code lifted from the original protograph. Since the doping proceeds after the lifting process, a protograph-based PD-GLDPC code cannot be expressed in terms of a protograph. Although for the proposed construction method, the number of partial doping doesn’t necessarily have to be a multiple of $\mu$, in order to make the number of added rows the multiple of the lifting size, similar to the size of the conventional one, we partially dope the protograph variable nodes by a granularity $\mu$, i.e., $x = 0, \mu, 2\mu, 3\mu, \ldots$. Assuming that $x$ is a multiple of $\mu$, let $y$ be a non-negative integer representing the number of bulks of doped protograph variable nodes, where each bulk is a set of protograph variable nodes with size $\mu$, i.e., $y = \frac{x}{\mu}$.

B. The PEXIT Analysis of Protograph-based PD-GLDPC Codes

The PEXIT of the proposed protograph-based PD-GLDPC codes is similar to that of the conventional protograph GLDPC codes in Alg. 1 except for the EXIT of a GC node. Since the incoming mutual information of each GC node is obtained from only one protograph variable node in the proposed code, the average mutual information sent to each GC node is the same as the extrinsic message of the protograph variable node connected to the GC node. Let $b_j, j \in [x]$
be the virtual node representing the set of $\beta$ GC nodes connected to the protograph variable node $v_j$. An example of the representation of a virtual node over a protograph is given in Fig. 5. Note that although $b_j$ is not a protograph node itself, it is possible to compute the PEXIT of a protograph-based PD-GLDPC code. Also, let $I_{EV}^{(b_j)}(j)$ be the extrinsic information from $v_j$ to $b_j$ expressed as

$$I_{EV}^{(b_j)}(j) = 1 - \epsilon \prod_{t \in N(v_j)} (1 - I_{AV}(t, j)), j \in [x].$$

Since $b_j$ is solely connected to $v_j$, the index term for the extrinsic information from $v_j$ to $b_j$ is expressed by the notation of $j$ only. In order to compute the EXIT of $b_j$, let $I_{AGC}^{(b_j)}(j)$ and $I_{EGC}^{(b_j)}(j)$ be the a priori and extrinsic mutual informations of $b_j$, respectively. Note that the EXIT of each GC node is computed from single a priori mutual information to process single value of extrinsic mutual information for the neighboring variable nodes. For the conventional protograph GLDPC codes, the a priori mutual information of a protograph GC node is computed by averaging the extrinsic mutual information of its neighboring variable nodes. However, for the proposed protograph-based PD-GLDPC codes, since $b_j$ receives the extrinsic mutual information of $v_j$ only, it is clear that $I_{AGC}^{(b_j)}(j) = I_{EV}^{(b_j)}(j), j \in [x]$. We also compute the extrinsic mutual information from $b_j$ to $v_j$ denoted as $I_{EGC}^{(b_j)}(j)$ using (1), given the a priori mutual information $I_{AGC}^{(b_j)}(j)$, which is given as

$$I_{EGC}^{(b_j)}(j) = \frac{1}{h} \sum_{h=1}^{\mu} (1 - I_{AGC}^{(b_j)}(j))^{h-1} \left( I_{AGC}^{(b_j)}(j) \right)^{\mu-h} [h \tilde{e}_h - (\mu - h + 1) \tilde{e}_{h-1}]. \quad (2)$$

Note that for the proposed protograph-based PD-GLDPC codes, the a priori (extrinsic) EXIT of the GC node is computed from the extrinsic (a priori) EXIT of single protograph variable node. While the EXIT of variable nodes and SPC nodes for the proposed protograph-based PD-GLDPC codes is the same as that of the conventional protograph GLDPC codes described in Alg. 1, the EXIT of the GC nodes in the proposed codes is changed to (2) whereas the conventional protograph GLDPC codes use (1).

In general, as the portion of degree-2 variable nodes in the LDPC codes increases, the asymptotic performance is enhanced [23], but their minimum distance decreases and then the error floor becomes worse. Although the proposed protograph-based PD-GLDPC code construction method enables the partial doping for any variable nodes in the given protographs, we focus only on partial doping for degree-2 variable nodes as follows. First, we construct the original base matrix $H_{\text{proto}}$ with the large portion of degree-2 variable nodes. Then, we partially dope some of
the protograph variable nodes of degree-2 to increase the minimum distance and improve their performance. Thus, regular protographs of degree-2 variable nodes and irregular protographs with many degree-2 variable nodes are used for the construction of the proposed protograph-based PD-GLDPC codes. In terms of irregular ensemble LDPC codes, a large portion of degree-2 variable nodes enables the LDPC code to achieve the capacity approaching performance [24]. On the other hand, by reasonably selecting the number of partially doped variable nodes for degree-2, the property of the linear minimum distance growth with the length of the LDPC code can be guaranteed. Thus, when we design the proposed protograph-based PD-GLDPC codes, balancing the partial doping over degree-2 variable nodes enables both the existence of a typical minimum distance and a good asymptotic performance.

C. Condition for Typical Minimum Distance of Protograph-based PD-GLDPC Code

The existence of a typical minimum distance in the given LDPC codes guarantees that their minimum distance grows linearly with the block length of the code in an asymptotic sense [25]. It was proved in [26] that a protograph LDPC code has a typical minimum distance if there is no cycle consisting of only degree-2 variable nodes in the protograph. Furthermore, in [27], the condition for a typical minimum distance of the conventional protograph GLDPC codes was given as follows. Assuming that there are no degree-2 variable nodes with double edges, i.e., no type 1 degree-2 variable nodes defined in [27], the neighborhoods of a check node $c_i$ that satisfy $d^i_{\min} \geq N^{(dv=2)}(c_i)$ are removed along with its edges, where $N^{(dv=2)}(c_i)$ is the number of degree-2 variable nodes among the neighborhoods of $c_i$. This process is repeated until no further degree-2 variable nodes remain. Then, the GLDPC code has a typical minimum distance if all degree-2 variable nodes are removed.

The proposed protograph-based PD-GLDPC code also has a similar approach to that of the conventional protograph GLDPC codes in [27]. However, since a GC node of the proposed protograph-based PD-GLDPC code is not well defined by a protograph node, the derivation of the weight enumerator of the proposed codeword is quite different from that of the conventional GLDPC code. Thus, the condition for the existence of the typical minimum distance of the proposed protograph-based PD-GLDPC codes is slightly different from that of the conventional protograph LDPC code. That is, the degree-2 variable nodes to be partially doped are initially excluded before the degree-2 variable node removing process stated in [26], because those variable nodes should be changed to the variable nodes of degrees higher than two by partial
doping. Thus, we can regard the degree-2 variable nodes to be partially doped as the variable nodes with higher degrees. Then, we have the following theorem for the proposed protograph-based PD-GLPDC codes.

**Theorem 1.** For a $(B_{n_c \times n_v}, \mu, \kappa, N, x)$ protograph-based PD-GLDPC code, if the undoped degree-2 variable nodes in the protograph have no cycles among themselves, the proposed protograph-based PD-GLDPC code has a typical minimum distance.

**Proof.** The proof is given in Appendix A.

The existence of the typical minimum distance of the proposed protograph-based PD-GLDPC code guarantees that the minimum distance of the proposed code grows linearly with the code length of the LDPC code, and thus the proposed code has low error floor for the large code length. In the next section, we use Theorem 1 as the constraint to optimize the protograph in order to guarantee the existence of the typical minimum distance of the proposed protograph-based PD-GLDPC code.

**D. Comparison between the Proposed PD-GLDPC Code and the Conventional Protograph GLDPC Code**

The main difference between the proposed protograph-based PD-GLDPC codes and the conventional protograph GLDPC codes is the perspective of doping. While the conventional protograph GLDPC code replaces an entire row, i.e., a protograph check node by the parity check matrix of the component code, the proposed protograph-based PD-GLDPC code appends some rows incident to the variable nodes copied from single protograph variable nodes. The focus of the conventional protograph GLDPC code is to choose a certain protograph check node, whereas the protograph-based PD-GLDPC code is focusing on choosing which protograph variable nodes are further protected by partial doping. The constraint for the conventional protograph GLDPC code is that the check nodes to be replaced should have the degree equal to the component code length, while the constraint for the proposed protograph-based PD-GLDPC codes is that the lifting size of a protograph should be the multiple of the component code length.

The selection of the check nodes to be replaced in the conventional protograph GLDPC codes is generally difficult since it requires a combinatory search of either the best threshold, minimum distance, or convergence speed. Also, since many SPC nodes of a given base matrix connect the variable nodes with various degrees, it is very difficult to select the variable nodes to be partially
doped for further protection in the conventional protograph GLDPC codes.

Furthermore, the conventional protograph GLDPC code construction has two limitations in terms of PEXIT analysis and doping granularity. First, the PEXIT analysis of the conventional protograph GLDPC codes is not accurate because the PEXIT of the GC node is derived from averaged EXIT of \( \mu \) (component code length) protograph variable nodes. Using a MAP-oriented computation as in Alg. 1, the GC node outputs a single value that represents the extrinsic mutual information for the \( \mu \) protograph variable nodes. Thus, a GC node averages up the a priori mutual information of the \( \mu \) neighborhood protograph variable nodes and outputs uniform extrinsic mutual information, which is similar to the EXIT analysis of a random ensemble LDPC code [21]. Since each GC node receives a priori inputs from \( \mu \) different protograph variable nodes, the EXIT analysis is not accurate. Thus, the higher variance of the a priori mutual information from the average, the greater the deviation of the code between the threshold and the actual decoding performance. On the other hand, since the EXIT of a GC node in the proposed protograph-based PD-GLDPC code requires the a priori mutual information of the same protograph variable node, there is no deviation since the \( \mu \) mutual information has the same value, which makes the PEXIT of the proposed code more accurate.

The second limitation is that the conventional protograph GLDPC codes have large doping granularity of protograph variable nodes compared to that of the proposed one. By replacing a single protograph GC node by a component code with parameters \((\mu, \kappa)\), the \( \mu \) protograph variable nodes are doped. Whereas, for every partial doping of \( \beta \) GC nodes in the proposed protograph-based PD-GLDPC code, the variable nodes copied from single protograph variable node are partially doped. In other words, the doping granularity is one, which is smaller than the conventional protograph GLDPC codes.

In summary, compared to the conventional protograph GLDPC codes, the proposed protograph-based PD-GLDPC code is more accurate in the PEXIT analysis and has the smaller doping granularity.

IV. CONSTRUCTIONS OF PROTOGRAPH-BASED PD-GLDPC CODES

In this section, a protograph-based PD-GLDPC code is constructed from both a regular protograph and an irregular protograph. Threshold optimization of protograph-based PD-GLDPC codes is also proposed.
Algorithm 2 The construction method of a protograph-based PD-GLDPC code from a variable node regular protograph

**Input:** $w_r$, $\mu$, $\kappa$, $n_v$, $\bar{R}$

**Output:** $y$, $B_{n_c \times n_v}$

**Step 1) Initialization**
For each $y$ in $\left[\frac{n_v - n_c + 1}{\mu}\right] \leq y \leq \left\lfloor \frac{n_v}{\mu} \right\rfloor$, construct a variable node regular base matrix $B$ of size $n_c \times n_v$ using the PEG method, where each variable node has degree $w_r$ and $n_c = n_v(1 - \bar{R}) - (\mu - \kappa)y$.

**Step 2) Construction of the candidate protograph-based PD-GLDPC code**
For each $y$, partially dope $(\mu, \kappa)$ Hamming codes on $y\mu$ blocks of variable nodes in the base matrix.

**Step 3) Check the existence of the typical minimum distance**
For each $y$, $\left[\frac{n_v - n_c + 1}{\mu}\right] \leq y \leq \left\lfloor \frac{n_v}{\mu} \right\rfloor$, if there exists any cycle for the submatrix induced by only undoped variable nodes of degree 2, re-initialize the base matrix as in **Step 1)** for that $y$.

**Step 4) Selection of the optimal protograph-based PD-GLDPC code**
For each $y$, compute the PEXIT threshold of the constructed protograph-based PD-GLDPC code and find $y$ with the best threshold.

A. Construction from a Variable Node Regular Protograph

We define a variable node regular protograph as a protograph, where all of its variable nodes have the same degree. Similar to the conventional protograph GLDPC code construction, we introduce the construction process of a proposed protograph-based PD-GLDPC code from a regular protograph. In this paper, we have strictly satisfied the variable node regularity for the protograph while making the check nodes as regular as possible using the progressive edge-growth (PEG) algorithm [28]. Since the PEG algorithm is a random construction method, we have considered doping for variable nodes from the left part of the base matrix, i.e., doping for $v_j$, $j = 1, 2, \cdots, y\mu$. After lifting by $N$, the number of partially doped variable nodes is $\mu yN$ and each subset of size $\mu$ variable nodes lifted from a protograph variable node constitutes a Hamming code with PCM $H_{Hamm}$ as a component code. Note that the number of dopings for the protograph variable nodes should be sufficient enough to satisfy the constraint for the existence of the typical minimum distance. The necessary condition of $y$ for the typical minimum distance...
Algorithm 3 Construction of $G_c$ and the protograph-based PD-GLDPC code

**Input:** $\mu$, $\kappa$, $n_v$, $R$, $l$, $r$, $y_{\text{max}}$

**Output:** $y_{\text{opt}}$, $G_c$, $G_p$

**Step 1** Optimize degree distribution of $G_c$

Optimize $\lambda_{G_c}(x) = \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3 + \lambda_5 x^4 + \lambda_6 x^5 + \lambda_l x^{l-1}$ and $\rho_{G_c}(x) = \rho_{r-1} x^{r-2} + \rho_r x^{r-1}$ using differential evolution under constraints (a)∼(c): (a) Rate constraint $R = 1 - \int_0^1 \rho_{G_c}(x) dx / \int_0^1 \lambda_{G_c}(x) dx$, $0 \leq \lambda_i \leq 1$, $0 \leq \rho_i \leq 1$, (b) Typical minimum distance constraint $\lambda_2 / 2 \times n_v \leq n_c - y_{\text{max}} (\mu - \kappa)$ $\leftrightarrow$ $\lambda_2 \leq \frac{2 \Sigma \{ n_v y_{\text{max}} (\mu - \kappa) \} }{n_v}$, and (c) $G_p$ existence constraint $\lambda_3 \geq \frac{12 \Sigma y_{\text{max}}}{n_v}$, $\lambda_4 \geq \frac{24 \Sigma y_{\text{max}}}{n_v}$, $\lambda_5 \geq \frac{20 \Sigma y_{\text{max}}}{n_v}$, $\lambda_6 \geq \frac{6 \Sigma y_{\text{max}}}{n_v}$.

**Step 2** Construction of $G_c$

From the optimized degree distribution and by using the random PEG algorithm, construct $G_c$ defined as $D^{(2,3,4,5,6,l)} = (a, b, c, d, e, f)$.

**Step 3** Optimization of $G_p$

For each $y = 1, 2, \ldots, y_{\text{max}}$, construct $G_p$ defined as $D^{(2,3,4,5,6,l)} = (a + 15 y, b - 4 y, c - 6 y, d - 4 y, e - y, f)$ and choose $y_{\text{opt}} \in \{y\}$ with the best threshold.

**Step 4** Typical minimum distance check of the protograph-based PD-GLDPC code

For the chosen $y_{\text{opt}}$ and $G_p$, if there exists any degree-2 cycle for the submatrix induced by undoped variable nodes, go to **Step 2**. Otherwise, output $y_{\text{opt}}$ and $G_p$.

condition is that the number of undoped degree-2 variable nodes is less than the number of check nodes of the original protograph, that is,

$$n_v - \mu y \leq n_c - 1 \leftrightarrow y \geq \frac{n_v - n_c + 1}{\mu}.$$  

Furthermore, in order to satisfy the sufficient condition for the typical minimum distance, we make sure that the undoped degree-2 variable nodes do not form a protograph cycle among themselves. The code rate $R$ of the proposed protograph-based PD-GLDPC is derived as $R = 1 - \frac{n_c + (\mu - \kappa) y}{n_v}$, where for a given code rate and $n_v$, a protograph-based PD-GLDPC code is constructed. The construction of a protograph-based PD-GLDPC code from a regular protograph is summarized in Alg. 2.
B. Differential Evolution-based Construction from Irregular Protograph

An optimization of an irregular protograph ensemble is made by initially obtaining the degree distribution of the random ensemble using differential evolution [29] and constructing the protograph via the PEG algorithm for the construction of the proposed protograph-based PD-GLPDC codes from irregular protographs. In this paper, in order to make the check node degrees as even as possible, we try to construct the protograph from the degree distribution of a random ensemble. We define $G_c$ as the optimized protograph of the conventional LDPC code and $G_p$ as the initial irregular protograph that is used to construct the protograph-based PD-GLDPC code. $G_c$ is constructed to have the same variable node degree distribution as the protograph-based PD-GLDPC code constructed from $G_p$ after lifting by $N$.

Let $\lambda_{G_c}(x)$ and $\rho_{G_c}(x)$ be the variable and check node degree distributions of irregular ensemble to construct $G_c$, which is the optimized protograph for the conventional LDPC codes. In this paper, we assume the degree distribution of the ensemble of $\lambda_{G_c}(x) = \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3 + \lambda_5 x^4 + \lambda_6 x^5 + \lambda_l x^{l-1}$ and $\rho_{G_c}(x) = \rho_r x^{r-2} + \rho_r x^{r-1}$, where $\lambda_i$ and $\rho_i$ are the portions of edges of variable and check nodes of degree-$i$. Using the optimized degree distributions of $\lambda_{G_c}(x)$ and $\rho_{G_c}(x)$, a protograph $G_c$ is constructed by the PEG algorithm. For the description of the protographs that construct the conventional LDPC codes and the proposed protograph-based PD-GLDPC codes, let $D^{dv} = (a_1, \cdots, a_{max})$ be a $|dv|$-sized vector defining the numbers of protograph variable nodes, where $a_i$ is the number of protograph variable nodes of degree $l_i$ and $dv = \{l_1, l_2, \cdots, l_{max}\}$ is a set of variable node degrees that exist in the protograph. That is, the protograph-based PD-GLDPC code is constructed from an $n_c \times n_v$-sized protograph with $D^{dv}$ by using the PEG algorithm. Unlike the regular protograph case, in order to make the same variable node degree distributions of $G_c$ and the protograph-based PD-GLDPC constructed from $G_p$ after lifting by $N$, an optimization of $\lambda_{G_c}(x)$ and $\rho_{G_c}(x)$ should be constrained by $y_{max}$, which is the maximum number of bulks of protograph variable nodes allowed to be partially doped in $G_p$. Then, we make sure that the check node degrees should be as even as possible for both codes. A protograph-based PD-GLDPC code is constructed by doping $y\mu$ protograph variable nodes in $G_p$. Construction of a protograph-based PD-GLDPC code from $G_p$ is optimized by ranging the doping bulk $y$, $1 \leq y \leq y_{max}$. That is, we search for the optimal value $y$ which maximizes the coding gain between the protograph-based PD-GLDPC codes constructed from $G_p$ and the conventional protograph LDPC codes constructed from $G_c$. Conditions for the degree
distributions in order to construct $G_c$ are derived as follows. The conditions need to guarantee two criteria: i) The variable node degree distributions of the protograph LDPC code constructed from $G_c$ and the protograph-based PD-GLDPC code constructed from $G_p$ after lifting by $N$ are the same and ii) a typical minimum distance exists for both codes. In this paper, we assume that $\mu y_{\text{max}}$ variable nodes are partially doped only from the degree-2 protograph variable nodes. For the maximum number of $y_{\text{max}}$ bulks of partially doped protograph variable nodes using the PCM of the $(15, 11)$ Hamming code, the degree distribution of $G_p$ should be

$$D^{(2,3,4,5,6,20)} = (a + 15y_{\text{max}}, b - 4y_{\text{max}}, c - 6y_{\text{max}}, d - 4y_{\text{max}}, e - y_{\text{max}}, f).$$

For the existence constraint of the typical minimum distance, the degree distribution parameters should be non-negative. The parameters $a \sim f$ are approximated by the PEG construction as

$$a \approx \left\lfloor \frac{n_v \lambda_2}{\Sigma} \right\rfloor, b \approx \left\lfloor \frac{n_v \lambda_3}{\Sigma} \right\rfloor, c \approx \left\lfloor \frac{n_v \lambda_4}{\Sigma} \right\rfloor, d \approx \left\lfloor \frac{n_v \lambda_5}{\Sigma} \right\rfloor, e \approx \left\lfloor \frac{n_v \lambda_6}{\Sigma} \right\rfloor, \text{ and } f \approx \left\lfloor \frac{n_v \lambda_l}{\Sigma} \right\rfloor,$$

where $\Sigma = \int_0^1 \lambda_{G_c}(x) dx$. For the realization of the protograph from the degree distribution using the PEG algorithm, if the summation $a + b + c + d + e + f$ is lower than $n_v$, the values of $a \sim f$ are added by 1 in order starting from the lowest variable node degree until the summation is equal to $n_v$.

If $G_c$ is determined for a given $y_{\text{max}}$ as a variable node degree distribution of $D^{(2,3,4,5,6,20)} = (a, b, c, d, e, f)$, where $a + b + c + d + e + f = n_v$, $G_p$ defined by $D^{(2,3,4,5,6,20)} = (a + 15y, b - 4y, c - 6y, d - 4y, e - y, f)$ can be constructed for $y = 1, \cdots, y_{\text{max}}$. By allowing the PEG algorithm of the variable node degree distribution over a base matrix with size $\{n_c - (\mu - \kappa)y\} \times n_v$, both the code rate and the variable node degree distributions for $G_c$ and the proposed protograph-based PD-GLDPC codes constructed from $G_p$ after lifting by $N$ are matched. We search for the value of $y$, which has the best PEXIT threshold while having a typical minimum distance. The optimized doping value is denoted as $y^{\text{opt}}$. The construction of $G_c$ and the PD-GLDPC code is described in Alg. 3.

V. Numerical Results and Analysis

In this section, we propose the optimized protograph design and show the BLER of the proposed protograph-based PD-GLDPC codes. The performance of the conventional protograph LDPC code is compared with that of the proposed protograph-based PD-GLDPC code. In order to make this comparison, we construct a protograph ensemble that has the same variable node degree distribution as that of the proposed protograph-based PD-GLDPC code.
Fig. 6: Comparison of threshold and BLER for the regular LDPC codes, irregular protograph LDPC codes, and the protograph-based PD-GLDPC codes from a regular protograph for code rates $\frac{1}{2}$ and $\frac{1}{4}$.

A. Simulation Results for Optimized Protograph-based PD-GLDPC Code from Regular Protograph

For numerical analysis, we use a $(15, 11)$ systematic Hamming code as the component code. For a half-rate code, we construct a $(B_{(200-4y)\times 400}, 15, 11, 75, 15y)$ protograph-based PD-GLDPC code, for a given $y$. Since the doping of single protograph variable node requires $\mu - \kappa = 4$ additional protograph rows, the original protograph should have a total of $200 - 4y$ rows to construct a half-rate protograph-based PD-GLDPC code. In order to satisfy the typical minimum distance property for the proposed PD-GLDPC code, we make sure that there are no cycles among the undoped degree-2 variable nodes if $y \geq \frac{400 - (200 - 4y)}{15} + 1 \iff y \geq \frac{400 - 200 + 1}{11} \approx 18.27$ during the PEG process of regular base matrix construction. For $y = 19, 20, \cdots$, the best threshold is when $y = 19$, having a BEC threshold of 0.444. For comparison, we construct a $(3, 6)$ regular protograph with size $200 \times 400$ and lift it by 75, which has a threshold of 0.429. Also, by comparing an irregular protograph LDPC code that has the same degree distribution as the constructed protograph-based PD-GLDPC code, we can find the performance gain from BP decoding to MAP decoding over GC nodes. Since the doping of protograph variable nodes in the proposed protograph-based PD-GLDPC codes occurs only over degree-2 variable nodes, the degrees of $y\mu$ protograph variable nodes will change. Since the parity check matrix of a $(15, 11)$
systematic Hamming code has 4 degree-1 columns, 6 degree-2 columns, 4 degree-3 columns, and 1 degree-4 column, the conventional irregular LDPC protograph without doping having the same variable node degree distribution with the proposed protograph-based PD-GLDPC code is defined as $D^{(2,3,4,5,6)} = (115, 76, 114, 76, 19)$. Irregular protograph LDPC code and the (3,6) regular protograph LDPC code are constructed from the $200 \times 400$ protograph and the protograph-based PD-GLDPC code is constructed from the $124 \times 200$ protograph with $x = y\mu = 285$. The irregular protograph LDPC code also has the same variable node degree distribution as the proposed protograph-based PD-GLDPC code, where it is defined by $D^{(2,3,4,5,6)} = (115, 76, 114, 76, 19)$. Using the PEG algorithm, we construct the irregular protograph LDPC code, having a threshold of 0.4234. The BLER performances of the constructed irregular protograph LDPC code, protograph-based PD-GLDPC code, and a $(3,6)$ regular protograph LDPC code with $(n,k) = (30000, 15000)$ are shown in Fig. 6(a), where all three codes are half rate LDPC codes constructed by $N = 75$. The coding gain of the constructed protograph-based PD-GLDPC code is 0.021 and 0.015 compared to irregular protograph LDPC code and $(3,6)$ regular protograph LDPC code, respectively.

Similarly, we construct a $(B_{(300−4y)\times400}, 15, 11, 105, 15y)$ protograph-based PD-GLDPC code for code rate $1/4$. For $y \geq \frac{400−(300−4y)+1}{15} \leftrightarrow y \geq \frac{400−300+1}{11} \approx 9.18$, the optimized threshold is the highest value 0.6935 at $y = 10$. Similar to the half-rate case, we construct a $(3,4)$ regular protograph LDPC code using the PEG algorithm to have a threshold of 0.647. The irregular protograph ensemble has a variable degree distribution of $D^{(2,3,4,5,6)} = (250, 40, 60, 40, 10)$ and the threshold is 0.6883. The irregular protograph LDPC code and the $(3,4)$ regular protograph LDPC code are constructed from the $300 \times 400$ protograph and the protograph-based PD-GLDPC code is constructed from the $260 \times 400$ protograph with $x = y\mu = 150$. The irregular protograph also has the same variable node degree distribution as the protograph-based PD-GLDPC code, where it is defined by $D^{(2,3,4,5,6)} = (250, 40, 60, 40, 10)$. All the codes are lifted by 105 to make $(42000, 10500)$ codes and the numerical results are in Fig. 6(b). The coding gain of the constructed protograph-based PD-GLDPC code is 0.0052 and 0.0465 compared to irregular protograph LDPC code and $(3,4)$ regular protograph LDPC code, respectively.

B. Simulation Result for Optimized Protograph-based PD-GLDPC Code from Irregular Protograph

The construction for the protograph of the conventional protograph LDPC code, $G_c$ is made for $y_{\text{max}} = 5, 10, 15$ for the half-rate protograph ensemble. The numerical results are summarized
Fig. 7: Comparison of threshold and BLER for the irregular protograph LDPC code from $G_c$, the conventional random ensemble GLDPC code from [17], and the protograph-based PD-GLDPC code from $G_p$ for code rate $1/2$.

TABLE I: Simulation results for optimized protograph-based PD-GLDPC code from irregular protographs

| $y_{max}$ | $\lambda_{G_c}(x), \rho_{G_c}(x)$ (threshold) | $G_c$ protograph $D(2,3,4,5,6,20)$ \(/ G_c$ threshold $\) | $G_p$ protograph $D(2,3,4,5,20), y^{opt}$ \(/ PD$-GLDPC$\) threshold $\) | Coding gain |
|-----------|---------------------------------------------|-------------------------------------------------|--------------------------------|----------------|
| 5         | $\lambda_{G_c}(x) = 0.2049x + 0.2489x^2 + 0.1150x^3 + 0.074x^4 + 0.0210x^5 + 0.3363x^6$ \(/ G_c$ threshold $\) | (165, 134, 47, 23, 5, 26) / 0.4620 | (240, 114, 17, 3, 26), $y^{opt} = 5$ / 0.4599 | 0.0079 |
| 10        | $\lambda_{G_c}(x) = 0.1894x + 0.2255x^2 + 0.1431x^3 + 0.1191x^4 + 0.0357x^5 + 0.2872x^6$ \(/ G_c$ threshold $\) | (152, 121, 57, 38, 9, 23) / 0.4523 | (287, 85, 3, 2, 23), $y^{opt} = 9$ / 0.4638 | 0.0115 |
| 15        | $\lambda_{G_c}(x) = 0.1632x + 0.1758x^2 + 0.2143x^3 + 0.1827x^4 + 0.0543x^5 + 0.2098x^6$ \(/ G_c$ threshold $\) | (131, 94, 86, 59, 14, 16) / 0.4352 | (341, 38, 2, 3, 16), $y^{opt} = 14$ / 0.4534 | 0.0182 |

in Table I, where the coding gain given for the proposed PD-GLDPC code is compared to the conventional protograph LDPC code with equal degree distribution. The random ensemble-based GLDPC code with threshold 0.466 in [17] is represented as a variable node ensemble $\lambda(x) = 0.8x^2 + 0.01x^3 + 0.01x^5 + 0.18x^7$ and a doping portion of 40% by Hamming code. Fig. 7 shows the performance comparison of three half-rate codes which are the irregular protograph LDPC code constructed from $G_c$, a random ensemble-based GLDPC code in [17], and the proposed protograph-based PD-GLDPC code constructed from $G_p$, where $y_{max} = 5$. All three codes in Fig. 7 are $(n, k) = (30000, 15000)$ codes of half-rate, where $G_c$ is defined as $D(2,3,4,5,6,20) =$
Fig. 8: Comparison of threshold and BLER for the irregular protograph LDPC code from $G_c$ and the proposed protograph-based PD-GLDPC code from $G_p$ for code rate $1/4$.

(165, 134, 47, 23, 5, 26) and has the same variable node degree distribution as the PD-GLDPC code after lifting by $N = 105$. $x = y\mu = 75$ protograph variable nodes are partially doped in the PD-GLDPC code. The constructed protograph-based PD-GLDPC code for $y_{\text{max}} = 5$ has a coding gain of 0.0079 and 0.0039 compared to the GLDPC code in [17] and the irregular protograph LDPC from $G_c$, respectively. Fig. 7 shows that the proposed protograph-based PD-GLDPC code has a good performance both in the waterfall and low error floor region due to the fact that the code is optimized by increasing the doping as much as possible, and at the same time, the typical minimum distance constraint is satisfied. In terms of the asymptotic analysis, increasing the portion of degree-2 variable nodes increases the possibility of the code to approach the channel capacity [24]. However, the existence of a typical minimum distance of the protograph is also important, which upper bounds the portion of degree-2 variable nodes in the LDPC code. Thus, balancing of degree-2 variable nodes is needed in order to satisfy both a typical minimum distance condition and a good threshold. The proposed protograph-based PD-GLDPC code guarantees the balancing of the degree-2 variable nodes by carefully choosing the rate of the protograph code and the number of doping on degree-2 variable nodes.

Similarly, the construction of $G_c$ is made for the code rate $1/4$ case. We simulate the proposed protograph-based PD-GLDPC with the small doping case, where the optimization of random ensemble for $G_c$ is done under the constraint $y_{\text{max}} = 3$. The degree distribution of the optimized
results is obtained as

$$\lambda_{G_c}(x) = 0.2858x + 0.1218x^2 + 0.0533x^3 + 0.0369x^4 + 0.0095x^5 + 0.4927x^9,$$

$$\rho_{G_c}(x) = 0.9995x^5 + 0.0005x^6,$$

which makes $G_c$ with $D^{(2,3,4,5,6,30)} = (258, 74, 24, 13, 2, 29)$ via the PEG algorithm. The threshold of the proposed protograph-based PD-GLDPC code is best when $y = 2$ and for this instance, $G_p$ is then defined as $D^{(2,3,4,5,30)} = (288, 66, 12, 5, 29)$. Both codes are $(n, k) = (42000, 10500)$ codes of code rate $1/4$, where $G_c$ is defined as $D^{(2,3,4,5,6,30)} = (258, 74, 24, 13, 2, 29)$ and has the same variable node degree distribution as $G_p$ after lifting by $N = 105$. $x = y\mu = 30$ protograph variable nodes are partially doped in the protograph-based PD-GLDPC code. The simulation result of both the irregular protograph LDPC code and the proposed protograph-based PD-GLDPC code are shown for the same variable node degree distribution in Fig. 8. The thresholds of $G_c$ and the proposed PD-GLDPC code are 0.6702 and 0.7031, respectively, where the coding gain is 0.0329 compared to the irregular protograph LDPC code.

VI. Conclusion

We proposed a new construction method of a protograph-based PD-GLDPC code that has advantages the accuracy of the PEXIT analysis and a finer granularity of the protograph node doping compared to the conventional protograph GLDPC codes. Also, the proposed optimized PD-GLDPC codes constructed from regular and irregular protographs have good BLER performances compared to the regular protograph LDPC codes and conventional GLDPC code constructed from irregular random ensemble. While the PEXIT of a conventional GLDPC code over a GC node averages out the a priori and extrinsic mutual information, the proposed protograph-based PD-GLDPC code does not need such an averaging process because the mutual information exchanges over the same protograph variable node. Furthermore, since it is possible to partially dope the protograph variable nodes with a granularity 1, the design of a protograph-based PD-GLDPC code becomes more flexible in terms of balancing the rate loss and doping.
APPENDIX A

PROOF OF THEOREM 1: THE CONSTRAINT FOR THE EXISTENCE OF THE TYPICAL MINIMUM DISTANCE OF THE PROPOSED PROTOGRAPH-BASED PD-GLDPC CODES

A proof for the constraint of the existence of a typical minimum distance for the proposed protograph-based PD-GLDPC codes is given in this appendix. Similar to that in [27], a typical minimum distance is driven by the weight enumerator analysis over the lifted protograph. In order to use the notations in [27], we’ve distinguished the indexing notations during the enumeration for the partially doped variable nodes using ‘. Also, the $c_j$ and $v_i$ notations are used for the check nodes and the variable nodes, respectively. Suppose that the proposed protograph-based PD-GLDPC code is constructed from the protograph defined by $G = (V, C, E)$ and the $x$ variable nodes are partially doped, where component codes are identical with the parameters $(\mu, \kappa)$. We are given a variable node set $V = \{v_1, \cdots, v_{n_v}\}$ and a check node set $C_{PD-GLDPC} = B_{Hamm} \cup C = \{b_1, \cdots, b_x\} \cup \{c_1, \cdots, c_n\}$ for the protograph. It is important to note that the GC node set $B_{Hamm}$ is not defined over a protograph. However, the codeword enumeration can be made when the protograph is lifted, where $b_{i'}, i' \in [x]$ is a virtual check node that represents the Hamming check nodes used for partial doping for $v_{i'}$ in the original protograph. Although $b_{i'}$ is not a protograph check node, we define it for the enumeration of the partially doped protograph variable nodes. The protograph-based PD-GLDPC code is constructed by lifting the graph $G$ by $N$ times and permuting the replicated edges. Each $v_i$ ($c_j$) has degree $q_{v_i}$ ($q_{c_j}$) and each $b_{i'}$ has degree $\mu$. For the enumeration of the GC node $b_{i'}$, we can think of it as a protograph node of degree $\mu$ that is lifted by a factor of $\frac{N}{\mu}$. The upper bound of the weight enumerator of the proposed protograph-based PD-GLDPC code with weight $d$, denoted as $A_{d}^{PD-GLDPC}$ is derived as follows.

Let $w_{m,u}, u \in [q_{v_m}]$ be the $u$th edge weight from a variable node $v_m$. For a partially doped variable node $v_m, m \in [x]$, there are $\mu$ weights sent towards the incident GC node, where the $u$th weight is defined as $w_{m,u}, u \in [\mu]$. For a given input weight vector $d = (d_1, \cdots, d_{n_v})$, we need to calculate $A_{d}^{PD-GLDPC}(d)$ and sum it over every instance of $d$ that satisfies $d = d_1 + \cdots + d_{n_v}$. For input $d_{i'}, i' \in [x]$, it is clear that $\sum_{i=1}^{\mu} w_{m,i} = d_{i'}$ because the extrinsic weight $w_{m,i}$ consists of weights solely from $v_m$. We introduce the following notations:

- $A_{d_{i'}}^{\mu}(w_i) = \binom{N}{d_{i'}} \delta_{d_{i'},w_{i,1},\cdots,\delta_{d_{i'},w_{i,q_{v_i}}}} = \begin{cases} \binom{N}{d_{i'}}, & \text{if } w_{i,j} = d_{i'}, \forall j \in [q_{v_i}] \\ 0, & \text{otherwise} \end{cases}$ is the vector weight
enumerator for a variable node \( v_i \) of the protograph [27].

- \( A^{c_j}(z_j) \) is the vector weight enumerator for a check node \( c_j \) of the original protograph, for the incoming weight vector \( z_j = [z_{j,1}, \cdots, z_{j,q_c}] \) [27].

- \( B^v(w^v) = \begin{cases} 1, & \text{for } w^v,1 + \cdots + w^v,\mu = d^v \\ 0, & \text{otherwise} \end{cases} \) is the vector weight enumerator for partially doped variable nodes \( v_i, i' \in [x] \).

- \( B^{b_{i'}}(w^{b_{i'}}) \) is the vector weight enumerator for check nodes that are created during the lifting process given the weight vector \( w^{b_{i'}} \). \( A^{b_{i'}}(d_{i'}) \) is the summation of enumerators over all possible \( w^{b_{i'}} \) values given that \( w^{b_{i'}},1 + \cdots + w^{b_{i'}},\mu = d^v \) satisfying

\[
A^{b_{i'}}(d_{i'}) = \sum_{w^{b_{i'}}} B^{b_{i'}}(w^{b_{i'}}) = \sum_{w^{b_{i'}}} \sum_{\{m\}} C\left(\frac{N}{\mu}; m_1, \cdots, m_K\right),
\]

where \( w' = (w'_1, \cdots, w'_\mu) \) such that \( \sum_{i=1}^{\mu} w'_i = d'_k, w'_i \leq \frac{N}{\mu} \).

Then, the weight enumerator is given as

\[
A_d^{PD-GLDPC} = \sum_{\{d\}} A^{PD-GLDPC}(d),
\]

where

\[
A^{PD-GLDPC}(d) = \frac{\prod_{i=1}^{n_v} A_{d^v}^{v_i}(w_i) \prod_{j=1}^{n_c} A^{c_j}(z_j) \times \prod_{i',1}^{x} B^{v_i'}(w^{v_i'}) B^{b_{i'}}(w^{b_{i'}}) \prod_{s=1}^{q_{w_s'}} \prod_{r=1}^{q_{w_r'}} \left(\frac{N}{\mu}\right) \prod_{s'=1}^{x} \prod_{r'=1}^{x} \left(\frac{N}{\mu}\right)}{\prod_{i=1}^{n_v} \prod_{r=1}^{q_{w_r'}} \left(\frac{N}{\mu}\right) \prod_{s'=1}^{x} \prod_{r'=1}^{x} \left(\frac{N}{\mu}\right)}.}
\]

The solution to the equation \( w' = mM^C \) is given as \( m = \{m_1, \cdots, m_K\} \). The term \( \left(\frac{N}{\mu}\right) \) is lower bounded by \( \left(\frac{N}{\mu}\right)^{w'_1,1+\cdots+w'_\mu,\mu} e^{-w'_1,1-\ln w'_1,1}. Then, A^{PD-GLDPC}(d) can be upper bounded as
\[ A_{PD-GLDPC}^{\ast}(d) \leq \sum_{\{w'_i, w'_{i,1}, \ldots, w'_{i,\mu}\} = d_i} \frac{\prod_{j=1}^{n_c} A^{\ast}(d_j) \times \prod_{i'=1}^{x} B^{\ast}(w'_{i,1})}{\prod_{i=1}^{n_c} (\frac{N}{d_i})^{q_{i-1}} \times \prod_{i'=1}^{x} \left( \frac{N}{\mu} \right)^{d_{i'}+\mu} e^{-d_{i'} \ln w'_{i,1}}} \]

\[ \leq \sum_{\{w'_i, w'_{i,1}, \ldots, w'_{i,\mu}\} = d_i} \frac{\prod_{j=1}^{n_c} A^{\ast}(d_j) \times \prod_{i'=1}^{x} B^{\ast}(w'_{i,1})}{\prod_{i=1}^{n_c} (\frac{N}{d_i})^{q_{i-1}} \times \left( \frac{N}{\mu} \right) e^{-P \ln P}} \]

\[ = \frac{\prod_{j=1}^{n_c} A^{\ast}(d_j) \times \prod_{i'=1}^{x} B^{\ast}(d_{i'})}{\prod_{i=1}^{n_c} (\frac{N}{d_i})^{q_{i-1}} \times \left( \frac{N}{\mu} \right) e^{-P \ln P}} \quad \text{(3)} \]

where \( P = \sum_{i'=1}^{x} d_{i'} \) is the total weight of the \( x \) partially doped variable nodes. Then \( \sum_{t \cdot \ln t} \leq (\sum_{t} t) \cdot \ln (\sum_{t} t) \) is used for the second and the third inequalities in (3). It was shown in (18) of [27] that the inequality

\[ \frac{\prod_{j=1}^{n_c} A^{\ast}(d_j)}{\prod_{i=1}^{n_c} (\frac{N}{d_i})^{q_{i-1}}} \leq \prod_{i=1}^{n_c} e^{-q_{i-1} \ln \left( \frac{d_i}{d_{i_{\text{min}}}} \right) - \frac{q_{i-1}(2+k^{(c)}_{i_{\text{max}}}, \ln 2)}{k^{(c)}_{i_{\text{min}}}} d_i} \]

holds, where \( d^{(c)}_{i_{\text{min}}} \) is the minimum distance of an SPC component code for the original protograph and \( k^{(c)}_{i_{\text{max}}} \) is the maximum number of codewords of an SPC component code. Using the similar notations in [27], let \( d^{(b)}_{i_{\text{min}}} \) and \( k^{(b)}_{i_{\text{max}}} \) be the minimum distance and the number of codewords of the \( (\mu, \kappa) \) component code for the GC nodes. Then, \( \prod_{i'=1}^{x} A^{\ast}(d_{i'}) \) is upper bounded as

\[ \prod_{i'=1}^{x} A^{\ast}(d_{i'}) \leq \prod_{i'=1}^{x} \frac{\mu}{d^{(b)}_{i_{\text{min}}}} \left( \frac{N}{\mu} \right)^{\frac{1}{d^{(b)}_{i_{\text{min}}}^{(d-1)}}} e^{-\frac{2k^{(b)}_{i_{\text{min}}}}{d^{(b)}_{i_{\text{min}}}} - \frac{2k^{(b)}_{i_{\text{min}}}}{d^{(b)}_{i_{\text{min}}}}} \right) \]

\[ = \prod_{i'=1}^{x} \frac{\mu}{d^{(b)}_{i_{\text{min}}}} \left( \frac{N}{\mu} \right)^{\frac{1}{d^{(b)}_{i_{\text{min}}}^{(d-1)}}} e^{-\frac{2k^{(b)}_{i_{\text{min}}}}{d^{(b)}_{i_{\text{min}}}}} \right) \]

\[ \leq \prod_{i'=1}^{x} \left( \frac{d_{i'} + \mu - 1}{d_{i'}} \right) \left( \frac{N}{\mu} \right)^{\frac{1}{d^{(b)}_{i_{\text{min}}}^{(d-1)}}} e^{-\frac{2k^{(b)}_{i_{\text{min}}}}{d^{(b)}_{i_{\text{min}}}}} \left( \frac{1}{d^{(b)}_{i_{\text{min}}}^{(d-1)}} \right) \]

\[ \leq \prod_{i'=1}^{x} \left( d_{i'} + \mu - 1 \right) \left( \frac{N}{\mu} \right)^{\frac{1}{d^{(b)}_{i_{\text{min}}}^{(d-1)}}} e^{-\frac{2k^{(b)}_{i_{\text{min}}}}{d^{(b)}_{i_{\text{min}}}}} \left( \frac{1}{d^{(b)}_{i_{\text{min}}}^{(d-1)}} \right) \].

For the inequality in the third line of (4), we use the fact that \( \sum_{i=1}^{P} t_i \ln t_i \leq s \cdot \ln \frac{s}{P} \) with \( s = t_1 + \cdots + t_p \), which is clear by using the derivative on the multivariable function that
consists of independent $t_i$'s. The equality is satisfied when all $t_i$ values are the same. Going back to (3), let $f(P) = \left(\frac{N}{P} \right)^{1/P} e^{-P \ln P}$ for convenience. Then we have

$$A^{PD-GLDPC}(d) \leq \prod_{i=1}^{N_v} e^{(q_i-1)\frac{d_i}{d_{\text{min}}}} d_i \ln \frac{d_i}{d_{\text{min}}} \frac{q_i(2+k_{\text{max}} \ln 2)}{d_{\text{min}}} d_i$$

$$\times \prod_{i' = 1}^{x} e^{d_{i'} + \mu - 1} (\mu) \frac{1}{\frac{d_{\text{min}}}{d_{i'}}} d_{i'} \frac{(2+k_{i'} \ln 2)}{d_{\text{min}}} d_i - \frac{1}{\frac{d_{\text{min}}}{d_{i'}}} d_{i'} \ln \frac{d_{i'}}{\mu} \times f(P)$$

$$\leq \prod_{i=1}^{N_v} e^{(q_i-1)\frac{d_i}{d_{\text{min}}}} d_i \ln \frac{d_i}{d_{\text{min}}} \frac{q_i(2+k_{\text{max}} \ln 2)}{d_{\text{min}}} d_i$$

$$\times e^{x(\mu - 1)} e^{P} \prod_{i' = 1}^{x} (\frac{N}{\mu}) \frac{1}{\frac{d_{\text{min}}}{d_{i'}}} d_{i'} \frac{(2+k_{i'} \ln 2)}{d_{\text{min}}} d_i - \frac{1}{\frac{d_{\text{min}}}{d_{i'}}} d_{i'} \ln \frac{d_{i'}}{\mu} \times f(P).$$  (5)

We classify the variable nodes in the protograph into three groups before doping:

- Protograph variable nodes of degrees higher than 2
- Protograph variable nodes of degree 2 to be partially doped
- Protograph other variable nodes of degree 2.

We also separate the weights of codewords after lifting into three parts according to the three groups of variable nodes: $u_i$, $p_z$, and $l_j$, where $u_i$ is the weight of the sub-codeword corresponding to a protograph variable node $v_i$ of degree higher than 2 and $p_z$ and $l_j$ are the weights of codewords of each partially doped and undoped protograph variable node $v_z$ and $v_j$ of degree 2 from the protograph, respectively. The sum of sub-codeword weights for each group of variable nodes is given as $U = \sum_i u_i$, $P = \sum_z p_z$, and $L = \sum_j l_j$. It is clear that for the total codeword weight $d$, $d = U + P + L$. Then, the upper bound of the first term in (5) is written as

$$\prod_{i=1}^{N_v} e^{(q_i-1)\frac{d_i}{d_{\text{min}}}} d_i \ln \frac{d_i}{d_{\text{min}}} \frac{q_i(2+k_{\text{max}} \ln 2)}{d_{\text{min}}} d_i$$

$$\leq e^{\frac{2(2+k_{\text{max}} \ln 2)}{d_{\text{min}}} \ln \frac{d_{\text{min}}}{d_{\text{min}}}} \frac{2(2+k_{\text{max}} \ln 2)}{d_{\text{min}}} \frac{P}{d_{\text{min}}},$$  (6)

which is derived by using three weight groups of codewords similar to (20) of [27]. We share the same inequality $u_i < Ne^{-d_i d_{\text{min}}^{-1}}$ over the given codeword weight $d$ as in [27]. Using the
derivation in [27], the upper bound of the second term of (5) can be derived as
\[
\prod_{i'=1}^x \left( \frac{N}{d_{\min}} \right) \frac{d_{i'}}{d_{\min}} \left( 1 - \frac{1}{d_{\min}} \right) d_{i'} \ln d_{\min} \mu = \prod_{i'=1}^x \left( \frac{d_{i'}}{d_{\min}} \ln \frac{N}{d_{\min}} \right) \left( 1 - \frac{1}{d_{\min}} \right) d_{i'} \ln d_{\min} \mu \\
= \prod_{i'=1}^x \left( \frac{(2 + k(b) \ln 2)}{d_{\min}} d_{i'} \ln \frac{N}{d_{\min}} \right) \left( 1 - \frac{1}{d_{\min}} \right) d_{i'} \ln d_{\min} \mu \\
\leq \prod_{i'=1}^x \left( \frac{N}{d_{\min}^2} \right) \ln \frac{N}{d_{\min}} \left( 2 + (k_{\max}^{(c)}) \right) \left( P + L \right) \cdot e^{(\mu - 1)} e^P \cdot f(P).
\]

Using (6) and (7), the upper bound of $A^{PD/GLDPC}(d)$ is derived in terms of $E(d, P, L)$ as follows:
\[
A^{PD/GLDPC}(d) \leq e^{\frac{1}{d_{\min}} P \ln \frac{N}{P} \left( 2 + \frac{k(b) \ln 2}{d_{\min}} \right)} \cdot e^{(d - P - L) \ln \frac{d - P - L}{N}} e^{\left( 2 - \frac{3}{d_{\min}} \right) \ln \frac{d - P - L}{N}} e^{\left( 2 + \frac{k_{\max}^{(c)} \ln 2}{d_{\min}} \right) \ln \frac{d - P - L}{N}} \cdot e^{(\mu - 1)} e^P \cdot f(P).
\]

Let $E(d, P, L)$ be the parameter satisfying $A^{PD/GLDPC}(d) \leq e^{\frac{1}{d_{\min}} P \ln \frac{N}{P} \left( 2 + \frac{k(b) \ln 2}{d_{\min}} \right)} \cdot e^{(d - P - L) \ln \frac{d - P - L}{N}} e^{\left( 2 - \frac{3}{d_{\min}} \right) \ln \frac{d - P - L}{N}} e^{\left( 2 + \frac{k_{\max}^{(c)} \ln 2}{d_{\min}} \right) \ln \frac{d - P - L}{N}} \cdot e^{(\mu - 1)} e^P \cdot f(P)$. Then, from the upper bound in (8), $E(d, P, L)$ is given as
\[
E(d, P, L) = \frac{1}{d_{\min}} P \ln \frac{N}{P} + \frac{(2 + k(b) \ln 2)}{d_{\min}} P + \frac{(2 - \frac{3}{d_{\min}})}{d_{\min}} (d - P - L) \ln \frac{d - P - L}{N} \\
+ \frac{3(2 + k_{\max}^{(c)})}{d_{\min}} (d - P - L) + \frac{2(2 + k_{\max}^{(c)})}{d_{\min}} (P + L) + P + \ln P - P \ln \frac{N}{\mu}.
\]

Assuming that there are no type 1 degree-2 variable nodes defined in [27] and there are no cycles consisting only of undoped variable nodes of degree-2, we use the result of (22) in [27] such that the inequality $l_{2,k}^{(c)} \leq \frac{1}{d_{\min}^{(c)}} (L_{2}^{(c)} + \sum_i u_{i}^{(c)})$ is satisfied for all $j \in [n_c]$, where $l_{2,k}^{(c)}$ is the weight of the degree-2 undoped variable node of $G_p$ and the total weight of them is denoted as $L_{2}^{(c)}$ for check node $c_j$. Similar to the result in [27], we can derive the upper bound $L \leq \gamma(U + P)$, which is the same as $L \leq \frac{\gamma}{1 + \gamma} d$. Now, the upper bound of $E(d, P, L)$ needs to be derived for independent values $L$ and $P$. The first and second partial derivatives of $E(d, P, L)$ by $P$ are given as
\[
\frac{dE}{dP} = \frac{1}{d_{\min}^{(b)}} \ln \frac{N}{e*P} + \frac{(2 + k(b) \ln 2)}{d_{\min}^{(b)}} - \frac{3}{d_{\min}^{(c)}} \ln \frac{e(d - P - L)}{N} - \frac{(2 + k_{\max}^{(c)} \ln 2)}{d_{\min}^{(c)}} \\
+ 1 + \ln P - 1 - \ln \frac{N}{\mu} < 0,
\]
\[
\frac{d^2E}{dP^2} = - \frac{1}{d_{\min}^{(b)}} + \frac{(2 - \frac{3}{d_{\min}^{(c)}})}{d_{\min}^{(c)}} \cdot \frac{1}{P} + \frac{1}{P} > 0.
\]
Since the first derivative over $P$ is negative and the second derivative is positive, $E(d, P, L)$ is upper bounded by
\[
\lim_{P\to 0^+} E(d, P, L) = (2 - \frac{3}{d_{\min}^{(c)}})(d - L)\ln \frac{d - L}{N} + \frac{3(2 + k_{\max}^{(c)})}{d_{\min}^{(c)}} \cdot (d - L) + \frac{2(2 + k_{\max}^{(c)}\ln 2)}{d_{\min}^{(c)}} L.
\]
Since the resulting upper bound of $E(d, L)$ is the same as (37) in [27], the rest of the proof is the same as that in [27] and thus the proposed constraint guarantees the existence of typical minimum distance of the proposed protograph-based PD-GLDPC code.

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**REFERENCES**

[1] R. G. Gallager, “Low-density parity-check codes,” *IRE Trans. Inf. Theory*, vol. 8, no. 1, pp. 21–28, Jan. 1962.
[2] D. J. C. MacKay and R. M. Neal, “Near shannon limit performance of low density parity check codes,” *IEEE Electron. Lett.*, vol. 32, no. 18, pp. 1645–1646, 1996.
[3] R. M. Tanner, “A recursive approach to low complexity codes,” *IEEE Trans. Inf. Theory*, vol. 27, no. 5, pp. 533–547, Sep. 1981.
[4] G. Liva and W. E. Ryan, “Short low-error-floor Tanner codes with Hamming nodes,” in *Proc. IEEE MILCOM*, 2005.
[5] S. Abu-Surra, D. Divsalar, and W. E. Ryan, “Enumerators for protograph-based ensembles of LDPC and generalized LDPC codes,” *IEEE Trans. Inf. Theory*, vol. 57, no. 2, pp. 858–886, Feb. 2011.
[6] I. P. Mulholland, E. Paolini, and M. F. Flanagan, “Design of LDPC code ensembles with fast convergence properties,” in *Proc. IEEE BlackSeaCom*, 2015.
[7] Y. Liu, P. M. Olmos, and T. Koch, “A probabilistic peeling decoder to efficiently analyze generalized LDPC codes over the BEC,” *IEEE Trans. Inf. Theory*, vol. 65, no. 8, pp. 4831–4853, Aug. 2019.
[8] M. Lentmaier and K. S. Zigangirov, “On generalized low-density parity-check codes based on Hamming component codes,” *IEEE Commun. Lett.*, vol. 3, no. 8, pp. 248–250, Aug. 1999.
[9] G. Yue, L. Ping, and X. Wang, “Generalized low-density parity-check codes based on Hadamard constraints,” *IEEE Trans. Inf. Theory*, vol. 53, no. 3, pp. 1058–1079, Mar. 2007.
[10] N. Miladinovic and M. Fossorier, “Generalized LDPC codes with Reed-Solomon and BCH codes as component codes for binary channels,” in *Proc. IEEE GLOBECOM*, 2005.
[11] D. G. Mitchell, M. Lentmaier, and D. J. Costello, “On the minimum distance of generalized spatially coupled LDPC codes,” in *Proc. IEEE ISIT*, Jul. 2013.
[12] P. M. Olmos, D. G. M. Mitchell, and D. J. Costello, “Analyzing the finite-length performance of generalized LDPC codes,” in *Proc. IEEE ISIT*, Jun. 2015.
[13] D. G. M. Mitchell, P. M. Olmos, M. Lentmaier, and D. J. Costello, “Spatially coupled generalized LDPC codes: Asymptotic analysis and finite length scaling,” *IEEE Trans. Inf. Theory*, vol. 67, no. 6, pp. 3708–3723, Jun. 2021.
[14] E. Paolini, M. P. C. Fossorier, and M. Chiani, “Generalized and doubly generalized LDPC codes with random component codes for the binary erasure channel,” *IEEE Trans. Inf. Theory*, vol. 56, no. 4, pp. 1651–1672, Apr. 2010.
[15] Y. Wang and M. Fossorier, “Doubly generalized LDPC codes,” in Proc. IEEE ISIT. 2006.
[16] Y. Wang and M. Fossorier, “Doubly generalized LDPC codes over the AWGN channel,” IEEE Trans. Commun., vol. 57, no. 5, pp. 1312–1319, May 2009.
[17] R. Guan and L. Zhang, “Hybrid Hamming GLDPC codes over the binary erasure channel,” in Proc. IEEE ASID, 2017.
[18] G. Liva, W. E. Ryan, and M. Chiani, “Quasi-cyclic generalized LDPC codes with low error floors,” IEEE Trans. Commun., vol. 56, no. 1, pp. 49–57, Jan. 2008.
[19] J. Thorpe, “Low-density parity-check (LDPC) codes constructed from protographs,” Jet Propulsion Lab., Pasadena, CA, INP Progress Rep. 42–154, 2003.
[20] Y. Yu, Y. Han, and L. Zhang, “Hamming-GLDPC codes in BEC and AWGN channel,” in Proc. IEEE ICWMMN, 2015.
[21] A. Ashikhmin, G. Kramer, and S. ten Brink, “Extrinsic information transfer functions: Model and erasure channel properties,” in IEEE Tran. Inf. Theory, vol. 50, no. 11, pp. 2657–2673, Nov. 2004.
[22] G. Liva and M. Chiani, “Protograph LDPC codes design based on EXIT analysis,” in Proc. IEEE GLOBECOM, 2007.
[23] A. K. Pradhan, A. Thangaraj, and A. Subramanian, “Construction of near-capacity protograph LDPC code sequences with block-error thresholds,” IEEE Trans. Commun., vol. 64, no. 1, pp. 27–37, Jan. 2016.
[24] C. Di, T. J. Richardson, and R. L. Urbanke, “Weight distribution of low-density parity-check codes,” IEEE Trans. Inf. Theory, vol. 52, no. 11, pp. 4839–4855, Nov. 2006.
[25] R. G. Gallager, Low-density parity-check codes. Cambridge, MA: MIT Press, 1963.
[26] S. Abu-Surra, D. Divsalar, and W. E. Ryan, “On the existence of typical minimum distance for protograph-based LDPC codes,” in Proc. IEEE ITA, 2010.
[27] S. Abu-Surra, D. Divsalar, and W. E. Ryan, “On the typical minimum distance of protograph-based generalized LDPC codes,” in Proc. IEEE ISIT, 2010.
[28] Xiao-Yu Hu, E. Eleftheriou, and D. M. Arnold, “Regular and irregular progressive edge-growth Tanner graphs,” IEEE Trans. Inf. Theory, vol. 51, no. 1, pp. 386–398, Jan. 2005.
[29] A. Shokrollahi and R. Storn, “Design of efficient erasure codes with differential evolution,” in Proc. IEEE ISIT, 2000.