PyChEst: a Python package for the consistent retrospective estimation of distributional changes in piece-wise stationary time series

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Abstract

We introduce PyChEst, a Python package which provides tools for the simultaneous estimation of multiple changepoints in the distribution of piece-wise stationary time series. The nonparametric algorithms implemented are provably consistent in a general framework: when the samples are generated by unknown piece-wise stationary processes. In this setting, samples may have long-range dependencies of arbitrary form and the finite-dimensional marginals of any (unknown) fixed size before and after the changepoints may be the same. The strength of the algorithms included in the package is in their ability to consistently detect the changes without imposing any assumptions beyond stationarity on the underlying process distributions. We illustrate this distinguishing feature by comparing the performance of the package against state-of-the-art models designed for a setting where the samples are independently and identically distributed.

Keywords: Changepoint estimation, time series, stationary ergodic, long-range dependence, consistency, Python

1 Introduction

Retrospective multiple changepoint estimation is a classical problem in statistics, with a vast literature concerning both parametric and nonparametric settings, (Basseville and Nikiforov, 1993; Brodsky and Darkhovsky, 1993; Müller and Siegmund, 1994; Brodsky and Darkhovsky, 2000; Csörgö and Horváth, 1998) . The problem can be introduced as follows. A given sample

\[ X = X_1, \ldots, X_{\tau_1}, X_{\tau_1+1}, \ldots, X_{\tau_2}, \ldots, X_{\tau_\kappa+1}, \ldots, X_n \]

of length \( n \in \mathbb{N} \) is formed as the concatenation of \( \kappa + 1 \) non-overlapping segments, where \( \kappa \in 1, \ldots, n \) and \( 0 < \tau_1 < \cdots < \tau_\kappa < 1 \). Each segment is generated by some unknown stochastic process. The process distributions of every pair of consecutive segments are different. The index \( \tau_k, \ k \in 1, \ldots, \kappa \) where one segment ends and another starts is called a changepoint. The changepoints \( \tau_k \) as well as the number of changes \( \kappa \) are unknown and the objective is to estimate them given the sample \( X \).

The changepoint estimation problem is usually considered in a framework where the samples in each segment \( X_{\tau_k+1}, \ldots, X_{\tau_{k+1}} \) are assumed to be independently and identically distributed (i.i.d.) and the changes are in the mean or in some fixed known moment, (Yao, 1988; Carlstein, 1988; Pan and Chen, 2006; Lebarbier, 2005; Harchaoui and Lévy-Leduc, 2010; Ciuperca, 2011; Chen et al., 2011; Fryzlewicz, 2014). In the literature concerning methods for dependent time series the form of the change and the nature of dependence are usually restricted; for instance,
the process distributions are assumed to have finite-memory or satisfy strong mixing conditions (Kokoszka and Leipus, 1998; Lavielle, 1999; Kokoszka and Leipus, 2002; Davis et al., 2006; Hariz et al., 2007; Lavielle and Teyssiére, 2007; Brodsky and Darkhovsky, 2008, 2013). Even if no restrictions are imposed on the memory of the processes, the finite-dimensional marginals before and after the changes are almost exclusively assumed to be different (Carlstein and Lele, 1993; Giraitis et al., 1995). On the other hand, existing changepoint estimation packages are typically concerned with independent observations, where occasionally additional constraints such as parametric assumptions and moment conditions are placed on the distributions. In some cases the included methods are designed to detect changes in the mean or variance only, and the settings considered do not allow for consistency considerations of the estimators (Killick and Eckley, 2014; Ross, 2015; James and Matteson, 2015; Truong et al., 2020).

We have implemented two consistent multiple changepoint estimators, proposed by Khaleghi and Ryabko (2012, 2014), which are designed to locate changes in the process distributions of highly dependent time series. The only assumption made is that each segment $X_{\tau_k+1}, \ldots, X_{\tau_{k+1}}$ is generated by an (unknown) stationary ergodic process. In particular, no such assumptions as independence, finite memory or mixing are placed on the samples. Moreover, the marginal distributions of any fixed size before and after the change may be the same: the changes refer to those in the time-series process distributions. For example the mean, variance etc. may remain unchanged throughout the entire sequence. This means that the most general and perhaps the most natural notion of change is considered, that is, change in the distribution of the data. We also do not place any conditions on the densities of the marginal distributions (the densities may not exist). Since little assumptions are made on how the data are generated, this setting is especially suitable for highly dependent time-series, potentially accommodating a variety of new applications. To the best of our knowledge, PyChEst is the first software package to provide tools for the consistent estimation of distributional changes in time series with long-range dependencies.

The remainder of this paper is organized as follows. In Section 2 we give a mathematical formulation of the changepoint problem considered, and provide an overview of the methodology, as well as the possibilities and limitations of the framework that we consider. In Section 3 we introduce the PyChEst package, and describe how it can be installed and used, and provide some examples and use-cases of the package. In Section 4 we illustrate the robustness of our algorithms against long-range dependencies.

## 2 Methodology

In this section we briefly introduce the setting considered, and provide an overview of the algorithms included in PyChEst, as well as a discussion of our implementation. We refer to (Khaleghi and Ryabko, 2012, 2014) for detailed arguments and technical proofs.

### 2.1 Problem formulation

In order to formalize the problem, we require some preliminary notation and basic definitions. Let $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ be a measurable space corresponding to the alphabet. Here, $\mathcal{X}$ could be a subset of $\mathbb{R}$ or more generally a subset of $\mathbb{R}^d$, $d \in \mathbb{N}$ equipped with its Borel $\sigma$-algebra. Let $\mathcal{X}^\mathbb{N}$ be the
set of all $\mathcal{X}$-valued infinite sequences equipped with the Borel $\sigma$-algebra $\mathcal{B}$ on $\mathcal{X}^\mathbb{N}$ generated by the cylinder sets. Stochastic processes are probability measures on $(\mathcal{X}^\mathbb{N}, \mathcal{B})$. Associated with a process $\mu$ is a sequence of random variables $X := (X_t), \ t \in \mathbb{N}$ with joint (process) distribution $\mu$ where for every $t \in \mathbb{N}, X_t : \mathcal{X}^\mathbb{N} \to \mathcal{X}$ is the coordinate projection of an in element of $\mathcal{X}^\mathbb{N}$ onto its $t^{th}$ element. We call a process $\mu$ stationary if $\mu(X_1, \ldots, X_u) = \mu(X_{1+j}, \ldots, X_{u+j})$ for all $B \in \mathcal{B}^{(u)}, u, j \in \mathbb{N}$. For each $n \in \mathbb{N}$ and $B \in \mathcal{B}^{(u)}$, $u \in \mathbb{N}$ define the empirical measure $\mu_n(X, B) : \mathcal{X}^\mathbb{N} \to [0, 1], n \in \mathbb{N}$ of $B$ as

$$\mu_n(X, B) := \frac{1}{n-u+1} \sum_{i=1}^{n-u+1} \mathbb{1}\{X_i, \ldots, X_{i+u} \in B\}$$

for $n \geq u$ and 0 otherwise, where $\mathbb{1}$ is the indicator function. A stationary process $\mu$ with corresponding sequence of random variables $X$ is (stationary) ergodic if for every $u \in \mathbb{N}$ and $B \in \mathcal{B}^{(u)}$ it holds that $\lim_{n \to \infty} \mu_n(X, B) = \mu(B), \ \mu - \text{a.s.}$. This is equivalent to the standard definition involving the triviality of invariant measurable sets, see e.g. (Gray, 2009).

The changepoint estimation problem considered in this paper can be formulated as follows. We are given a piece-wise stationary an $\mathcal{X}$-valued sample $X := X_1, \ldots, X_n$ of length $n \in \mathbb{N}$, formed as the concatenation of a number $\kappa + 1$ of stationary segments,

$$X_1, \ldots, X_{\tau_1}, X_{\tau_1+1}, \ldots, X_{\tau_2}, \ldots, X_{\tau_\kappa+1}, \ldots, X_n$$

with changepoints at $\tau_1 < \cdots < \tau_\kappa$ where each (stationary) segment is generated by one of $m \leq \kappa + 1$ unknown stationary ergodic processes $\mu_1, \ldots, \mu_\kappa$.

In Section 2 we described the changepoint problem, and informally stated that a piece-wise stationary sample $X$ is obtained as a concatenation of $\kappa + 1$ segments each generate by a stationary ergodic process. More formally, the process by which $X$ is obtained can be specified as follows. First, let $\rho$ be a measure on the space $(\mathcal{X}^{(\kappa+1)} \times \mathbb{N}, \mathcal{B}^{(\kappa+1)} \otimes \mathbb{N})$ where, $\mathcal{B}^{(\kappa+1)} := \sigma(\{B_1 \times \cdots \times B_\kappa : B_k \in \mathcal{B}, k = 1, \ldots, \kappa\})$.

Define the infinite matrix of $\mathcal{X}$-valued random variables $X := (X_{i,j}), i \in 1, \ldots, \kappa + 1, j \in \mathbb{N}$ where $X_{i,j} : \mathcal{X}^{(\kappa+1)} \times \mathbb{N} \to \mathcal{X}, i \in 1, \ldots, \kappa + 1, j \in \mathbb{N}$ are jointly distributed according to $\rho$, so that for $B \in \mathcal{B}^{(\kappa+1)}$ we have $\Pr(X \in B) = \rho(B)$.

For each $i \in 1, \ldots, \kappa + 1$, let $X_i := (X_{i,j}), j \in \mathbb{N}$ and define the projection map $\pi_i \mapsto X_i$ which maps $X$ on its $i^{th}$ row. The marginal distribution $\mu_i$ of $X_i$ is then defined as the distribution induced by $\rho$ on the $i^{th}$ row of $X$, i.e. $\mu_i := \rho \circ \pi_i^{-1}$. Next, define the mapping $c : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ as $c(j) \mapsto (t^*(j) + 1, j - \tau_{t^*(j)})$ where $t^*(j) := \max_{i \in 0, \ldots, \kappa+1} \tau_i \leq j$ picks out the changepoint $\tau_i$ that is closest to $j \in \mathbb{N}$ from the left, with the convention that $\tau_0 := 0$ and $\tau_{\kappa+1} := n$. The piecewise stationary sample $X$ generated by $\rho$ with changepoints at $\tau_1, \ldots, \tau_\kappa$ can be specified as a sequence of coordinate projections $X_t : \mathcal{X}^n \to \mathcal{X}, t = 1, \ldots, n$ such that for any $\ell \in 1, \ldots, n, t_1, \ldots, t_\ell \in 1, \ldots, n$ and $B_\ell \in \mathcal{B}_\mathcal{X}, i \in 1, \ldots, \ell$ it holds that

$$\Pr(X_{t_1} \in B_1, \ldots, X_{t_\ell} \in B_\ell) = \rho(X_{c(t_1)} \in B_1, \ldots, X_{c(t_\ell)} \in B_\ell).$$

It is straightforward to check that a set-function satisfying the above, extends to a probability measure on $(\mathcal{X}^n, \mathcal{B}^{(n)})$, with the property that the (marginal) distributions of the segments $X_{n+1}, \ldots, X_{n+1}$ are each given by a stationary ergodic process distribution $\mu_k, k \in 1, \ldots, \kappa + 1$. Since it is assumed that $\mu_k \neq \mu_{k+1}, i \in 1, \ldots, \kappa$, the indices $\tau_k, k \in 1, \ldots, \kappa$ are called changepoints. The consecutive segments are generated by different processes so that $\mu_k \neq \mu_{k+1}, k \in 1, \ldots, \kappa$. However, their means, variances, or, more generally, their
finite-dimensional marginal distributions may be the same up to some (unknown) fixed size. On the other hand, letting $m \leq \kappa + 1$ means that non-consecutive segments can have the same distribution. Observe that by this formulation, the finite-dimensional marginals of any fixed size before and after the changepoints $\tau_k$ may be the same. We consider a general notion, where the changes are in the process distributions. Our aim is to provide asymptotically consistent estimates $\widehat{\tau}_k(n)$ of the unknown changepoints $\tau_k$, $k = 1, \ldots, \kappa$. An estimate $\widehat{\tau}_k(n)$ is said to be asymptotically consistent if
\[ \lim_{n \to \infty} \frac{1}{n} |\widehat{\tau}_k(n) - \tau_k| = 0, \text{ a.s.} \] (1)
We seek changepoint estimation algorithms that provide an asymptotically consistent estimate for every changepoint $\tau_k$, $k = 1, \ldots, \kappa$. Observe that the asymptotic regime simply means that the estimation error is arbitrarily small if the sequence is sufficiently long. That is, the problem is to retrospectively locate the changes and the given sample $X$ does not grow with time. This differs, for example, from the problem of online changepoint detection where the observations arrive sequentially over time, and the objective is to detect a change as soon as possible. Denote by
\[ \lambda(n) := \min_{k = 1, \ldots, \kappa + 1} \frac{1}{n} (\tau_k - \tau_{k-1}) \]
with $k \in 1, \ldots, \kappa + 1$, $\tau_0 := 0$, $\tau_{\kappa + 1} := n$, the minimum normalized length of the stationary segments, and let
\[ \lambda_{\min} := \liminf_{n \to \infty} \lambda(n). \]
In order for the consistency conditions of the algorithms implemented as part of this package to hold we require that $\lambda_{\min} > 0$. Note that this linearity condition is standard in the changepoint literature, although it may be unnecessary when simpler formulations of the problem are considered. Indeed, an equivalent assumption is made by Fryzlewicz (2014, Assumption 3.2, 3.3) for the simpler piece-wise i.i.d. setting. However, conditions of this kind are inevitable in the general setting that we consider, where there may be arbitrary long-range dependencies: if the length of one of the segments is constant or sub-linear in $n$ then asymptotic consistency is not possible in this setting.

2.2 A distance between process distributions

Most non-parametric approaches to estimating a changepoint are usually based on the following idea. Initially, every possible index $i \in \{1, \ldots, n\}$ is considered a potential changepoint. The difference between the empirical expectation of the two segments $X_1, \ldots, X_i$ and $X_{i+1}, \ldots, X_n$ on either side of every fixed $i \in \{1, \ldots, n\}$ is calculated, and the changepoint estimate is chosen as the index that maximizes this difference in absolute value. A more general approach is based on maximizing the difference between the empirical distributions of the two segments under a given norm. Different norms give rise to different test statistics. The commonly used distances include the Kolmogorov-Smirnov statistics, obtained when the difference is calculated under the $L_\infty$ norm, the Cramér-von Mises statistics corresponding to the use of $L_2$ norm, and the generalizations thereof. We rely on empirical estimates of a distributional distance between the underlying process measures to locate the changepoints. The distance $d(\mu_1, \mu_2)$ between a pair of process distributions $\mu_1, \mu_2$ can be defined as
\[ d(\mu_1, \mu_2) := \sum_{i \in \mathbb{N}} u_i |\mu_1(A_i) - \mu_2(A_i)| \]
where, \( w_i \) are positive summable real weights, and \( A_i \) range over a countable field that generates the sigma-algebra of the underlying probability space. For a discrete alphabet \( A_i \) range over the set of all possible tuples. For example, in the case of binary alphabets, the distributional distance is the weighted sum of the differences of the probability values (measured with respect to \( \mu_1 \) and \( \mu_2 \)) of all possible tuples \( 0, 1, 00, 01, 10, 11, 000, 001, \ldots \). For real-valued processes the sets \( A_i \) range over the products of all intervals with rational endpoints (i.e. the intervals, all pairs of such intervals, triples, etc.). Asymptotically consistent estimates of this distance can be obtained by replacing unknown probabilities with the corresponding frequencies, provided that the corresponding process distributions are stationary ergodic. Although the distance involves infinite summations, the computational complexity of obtaining its empirical approximation is \( O(n \text{polylog} n) \), see (Khaleghi et al., 2016) for a discussion. Our implementation of this distance relies on the Aho-Corasick Algorithm (Aho and Corasick, 1975) for calculating empirical frequencies.

In principal, we can use any distance function between the samples, provided that the distance used reflects that between the underlying process distributions. In the case of changepoint analysis, the distance is required to satisfy convexity as well. It is important to note that the distinction between the underlying process distributions is not reflected by string metrics, such as the Hamming distance, or the Levenshtein distance, etc. The Hamming distance between two sequences is defined as the minimum number of substitutions required to transform one sequence into another, and the Levenshtein distance corresponds to the smallest number of deletions, insertions and substitutions needed to achieve the same objective, see e.g. (Stephen, 1994). More generally, a string distance between a pair of sequences is 0 if and only if the two sequences are exactly the same. However, what we require is for the distance to converge to 0 for long enough samples, if and only if both sequences are generated by the same process distribution. Consider a simple example where the elements of the sequences \( X := X_1, \ldots, X_n \) and \( Y := Y_1, \ldots, Y_n \) are drawn independently from a Bernoulli distribution with probability \( p := 1/2 \). Regardless of the value of \( n \), on average, the Hamming distance between the two sequences is 1/2 while they are generated by the same process distribution. At the same time, the empirical estimate of the distributional distance between \( X \) and \( Y \) becomes arbitrarily small for large enough \( n \).

### 2.3 Detecting the number of changepoints

In general, rates of convergence are provably impossible to obtain for stationary ergodic processes (Shields, 1996). As a result, the two-sample test does not have a consistent solution in this framework (Ryabko, 2010). This means that it is not possible to determine, even in the weakest asymptotic sense, whether or not two samples have been generated by the same or by different stationary ergodic distributions. An implication of this negative result is that it is provably impossible to determine the number of changes \( \kappa \) without making further assumptions. In light of this limitation, without placing any additional constraints on the underlying process distributions, in this software package we provide two options. With this theoretical impossibility result in mind, PyChEst provides two (consistent) algorithms as follows.

- The first algorithm is an implementation of the the so-called List-Estimator of Khaleghi and Ryabko (2012), which aims to produce a (sorted) list of candidate estimates whose first \( \kappa \) elements are consistent estimates of the true changepoints. More precisely, the
algorithm generates an exhaustive list of possibly more than $\kappa$ candidate estimates (but makes no attempt to estimate $\kappa$). The produced list must have the property that its first $\kappa$ elements are consistent estimates of the true changepoints. In order to achieve this goal, the list-estimator requires an additional parameter $\alpha \in (0, 1)$, which is a lower-bound on the minimum separation $\lambda_{\min}$. It is worth noting that if the correct value of $\kappa$ is known a-priori, it is then possible to consistently estimate the changepoints without any additional information, and in particular access to some $\alpha \leq \lambda_{\min}$ would no longer be required (Khaleghi and Ryabko, 2016).

- The second algorithm included in the package, is an implementation of the procedure proposed by Khaleghi and Ryabko (2014), which is able to consistently estimate the changepoints and detect the number of changes, provided that the total number $m$ of different process distributions is given. Note that in the specific case where all of the process distributions are different, knowing $m$ amounts to knowing the number of changepoints ($m = \kappa + 1$). However, the required additional parameter $m$ can be in general very different from $\kappa + 1$, and has a natural interpretation in many real-world applications. For instance, the sequence $X$ may correspond to the behavior of a system over time, which may have alternated $\kappa > m - 1$ times between a known number $m$ of states. In a simple case, the system may only take on $m = 2$ states, for example, “normal” and “abnormal”. Another application of this setting would be in speech segmentation where the total number $m$ of speakers is known, while the number $\kappa$ of alternations between the speakers is not available. Thus, the number $m$ of process distributions is provided as an intrinsic part of the problem, which is provably sufficient to estimate $\kappa$. See Section 2.4 for an informal overview of the algorithms as well as details concerning their implementation; more technical discussions can be found in the corresponding papers referenced above.

2.4 An overview of our implementations

In this section we give an overview of the changepoint estimation algorithms provided in the package focusing on our algorithmic implementations. Let us denote by $\hat{d}(\cdot, \cdot)$ the empirical estimate of the distributional distance $d(\cdot, \cdot)$; we refer to (Khaleghi et al., 2016) for a definition and a discussion on computational considerations.

The following two operators namely, the intra-subsequence distance $\Delta_X$ and the single-changepoint estimator $\Phi_X$ are used in our methods. Let $X = X_1, \ldots, X_n$ be a sample generated by a stochastic process and consider a subsequence $X_a, \ldots, X_b$ of $X$ with $a < b \in 1, \ldots, n$.

i. Define the intra-subsequence distance of $X_a, \ldots, X_b$ as

$$\Delta_X(a, b) := \hat{d}(X_a, \ldots, X_{\lceil \frac{a+b}{2} \rceil}, X_{\lfloor \frac{a+b}{2} \rfloor}, \ldots, X_b)$$ (2)

ii. For some $\alpha \in (0, 1)$, define the single-changepoint-estimator of $X_a, \ldots, X_b$ as

$$\Phi_X(a, b, \alpha) := \arg\max_{t \in a, \ldots, b} \hat{d}(X_{a-\lceil n\alpha \rceil}, \ldots, X_t, X_{t+1}, \ldots, X_{b+\lceil n\alpha \rceil})$$ (3)

For simplicity of notation, from this point on we assume the floor and ceiling functions implicit.
2.4.1 List-Estimator

The changepoint estimation procedure proposed by Khaleghi and Ryabko (2012) works as follows. Given a piece-wise stationary sample \( X = X_1, \ldots, X_n \) and a lower-bound \( \alpha \in (0, 1) \) on the minimum (normalized) stationary segment length, a sequence of evenly-spaced indices is formed. The index-sequence partitions \( X \) into consecutive segments of length \( \alpha/3 \). The intra-subsequence-distance \( \Delta_X \) is calculated in all but the first and last segments and is used as their performance scores. Next, an appropriate subset of the segments is selected through the following iterative procedure: initially, all but the first and last segment are marked as available, and at each iteration, an available segment of highest score is selected, and added to the final list. Noting that \( \alpha \) is a lower-bound on the minimum normalized distance between the changepoints, the available candidates within \( n\alpha/2 \) of the selected segment are made unavailable. The selection procedure continues until the set of available segments becomes empty. This is depicted below, where the downward arrows show the location of the indices.

Noting that the first and last segments are made unavailable at the outset, it is easy to see that by this procedure, the segments to the left and right of each available segment are made unavailable. The single-changepoint estimator \( \Phi_X \) is used to generate a candidate estimate within every selected segment. There may be more than \( \kappa \) candidate estimates produced; however, as shown by Khaleghi and Ryabko (2012) for long enough \( X \) the first \( \kappa \) candidate estimates in the sorted list are consistent. We would like to point out that since the candidate estimates can be computed in parallel, our implementation of this algorithm is multi-threaded.

An informal argument as to why this procedure is consistent is as follows. When \( \alpha \leq \lambda_{\min} \), the generated index-sequence partitions \( X \) in such a way that every three consecutive segments of the partition contain at most one changepoint. Also, the segments are of lengths \( \alpha n/3 \).

Therefore, if \( n \) is large enough, the single-changepoint-estimator \( \Phi_X \) produces consistent changepoint estimates within each of the segments that actually contains a true changepoint. Moreover, the performance scores of the segments without changepoints converge to 0, while each of those corresponding to the segments that contain a changepoint converge to a non-zero constant. Thus, the \( \kappa \) candidate estimates of highest performance score that are at least at a distance \( \alpha n \) from one another, each consistently estimate a unique changepoint. We refer to the above reference for a rigorous proof of consistency.

2.4.2 Changepoint Estimator

The function called Changepoint Estimator in our package is an implementation of the algorithm proposed by Khaleghi and Ryabko (2014). Given a piece-wise stationary sample \( X = X_1, \ldots, X_n \), a lower-bound \( \alpha \in (0, 1) \) on the minimum (normalized) stationary segment length, and the total number of different process distributions \( m \), the algorithm works as follows. First, the List-Estimator described is used to obtain an initial set of \( K \leq \alpha^{-1} \) changepoint candidates. The estimates are sorted in increasing order appearance to give \( \hat{\tau}_1, \ldots, \hat{\tau}_K \). The sorted changepoint estimates are in turn used to produce a set

\[
S := \{X_i := X_{\hat{\tau}_{i-1}}, \ldots, X_{\hat{\tau}_i} : i \in 1, \ldots, K + 1, \hat{\tau}_0 := 1, \hat{\tau}_K := n\}
\]
of at most $\lfloor \alpha^{-1} \rfloor + 1$ consecutive non-overlapping segments of $X$, which are of possibly different lengths. The identified segments of $X$ are depicted below.

$$X_1, X_2, X_3, \ldots X_1 \overset{\hat{\tau}_1}{\rightarrow} X_2 \overset{\hat{\tau}_2}{\rightarrow} \cdots X_i \overset{\hat{\tau}_i}{\rightarrow} \cdots X_K \overset{\hat{\tau}_K}{\rightarrow} \cdots, X_n$$

Next, a procedure based on time-series clustering is employed to remove redundant estimates. This is where the total number $m$ of process distributions is used. To this end, first a total of $m$ cluster centers are obtained as follows. The first segment $X_1$ is chosen as the first cluster center. Iterating over $j = 2, \ldots, m$ a segment is chosen as a cluster center if it has the highest minimum distance from the previously chosen cluster centers. Once the $m$ cluster centers are specified, the remaining segments are assigned to the closest cluster. As shown by Khaleghi and Ryabko (2014), for a long enough $X$, this clustering algorithm will put the segments together if and only if they have the same (unknown) process distribution. In each cluster, the changepoint candidate that joins a pair of consecutive segments of $X$ is naturally identified as redundant and is removed from the list. This step of the algorithm is pictured below where $C_i$ denotes the $i$th cluster, $i \in 1, \ldots, m$.

$$\ldots eC_1 \overset{\hat{\tau}_{j-1}}{\rightarrow} eC_1 \overset{\hat{\tau}_j}{\rightarrow} eC_2 \overset{\hat{\tau}_{j+1}}{\rightarrow} eC_2 \overset{\hat{\tau}_{j+2}}{\rightarrow} eC_1 \ldots$$

In the above example, $\hat{\tau}_{j-1}$ and $\hat{\tau}_{j+1}$ each separate segments that, as identified by the clustering sub-routine, are generate by the same unknown process, and are thus considered as redundant. Once all of the redundant candidates are removed, the algorithm outputs the remaining candidate estimates.

An intuitive argument concerning the consistency of this procedure is as follows. First, note that for a long enough $X$ the List-Estimator produces an initial set of at least $\kappa$ estimates whose first $\kappa$ elements each uniquely and consistently estimate one of the $\kappa$ changepoints. (Since $\kappa$ is unknown, the fact that the correct estimates are located first in the list is irrelevant: what matters is that the correct changepoint estimates are somewhere in the list.) Therefore, if $X$ is long enough, the largest portion of each segment in $S$ is generated by a single process distribution. Since the List-Estimator ensures that the initial changepoint candidates are at least $n\alpha$ apart, the lengths of the segments in $S$ are linear in $n$. Thus, it can be shown that for large enough $n$, the empirical estimate of the distributional distance between a pair of segments in $S$ converges to 0 if and only if the same process distribution generates most of the two segments. Given the total number $m$ of process distributions, for long enough $X$ the clustering algorithm groups together those and only those segments in $S$ that are generated by the same process. This lets the algorithm identify and remove the redundant candidates. As a result of the consistency of the List-Estimator, the remaining candidates are consistent estimates of the true changepoints. The proof can be found in (Khaleghi and Ryabko, 2014).

### 3 Installation and Use

In this section we outline how to install and use PyChEst.
3.1 Installing PyChEst

The easiest way to install the Python package is to use Python’s package-management system pip, which can be achieved by typing the following segment into a command line of their choice:

```
pip install PyChEst
```

3.2 Using PyChEst

The PyChEst package provides two main functions, namely

1. list_estimator
2. find_changepoints

both described in the previous section. The list_estimator takes a sample and a parameter min_distance corresponding to a lower bound on the minimum normalized distance $\lambda_{\text{min}}$ between the changepoints. Note that this parameter is denoted by $\alpha$ in the mathematical descriptions given in Sections 2.3 and 2.4. The function can be called as follows.

```
>>> estimates = PyChEst.list_estimator(seq, min_distance)
```

The function find_changepoints requires the same parameters as well as an additional integer corresponding to the total number of processes that generate the data; it can be called as described below.

```
>>> estimates = PyChEst.find_changepoints(seq, min_distance, process_count)
```

Next, we demonstrate how the package can be used. To this end, first we load the required packages as follows.

```
>>> import PyChEst
>>> import numpy as np
```

We also fix a random seed for reproducibility.

```
>>> np.random.seed(1)
```

We generate a piece-wise i.i.d. Bernoulli sample with changepoints at 2000 and 6500, where the parameter corresponding to the first and last segments is 0.2 and that of the middle segment 0.7, i.e.

```
>>> seq = []
>>> seq.append(np.random.binomial(1, p=0.2, size=2000))
>>> seq.append(np.random.binomial(1, p=0.7, size=4500))
>>> seq.append(np.random.binomial(1, p=0.2, size=1500))
>>> seq = np.concatenate(sequence)
```
Notice that the data so-generated is a piece-wise i.i.d. sample with changepoints at 2000 and 6500, with the minimum length of the stationary segment equal to 1500. There are a total two processes present, since the first and last segments have been generated by the same process. We can now execute PyChest to find the changepoints. Both the \texttt{list\_estimator} and \texttt{find\_changepoints} a lower bound on the minimum normalized distance $\lambda_{\text{min}}$ between changepoint. The shortest stationary segment in this example is of length 1500, and given that the sequence is of length 8000, the minimum normalized stationary segment length equals $\lambda_{\text{min}} = 0.1875$. We assume this value unknown, and arbitrarily provide a parameter $\alpha = 0.125 \leq \lambda_{\text{min}}$ as input.

```python
>>> min_distance = 0.125
>>> estimates = PyChest.list_estimator(seq, min_distance)
```

The \texttt{list\_estimator} returns a Python of changepoint estimates at least $0.125 \times 8000$ apart, which are ordered in decreasing sequence of their performance scores $\Delta X$ as given by (2). As previously discussed, the theoretical consistency results guarantee that for a long enough sequence, with probability one, the first 2 estimates in the list are arbitrarily close to the true changepoints.

```python
>>> print(estimates)
[1997, 6502, 4572]
```

As can be seen above, the first two elements of the array returned by \texttt{list\_estimator} are good estimates of the true changepoints $\tau_1 = 2000$ and $\tau_2 = 6500$ in our example.

In addition to $\texttt{seq}$ and $\texttt{min\_distance}$, the \texttt{find\_changepoints} function also requires the total number of different processes that generate the data. Since there are three segments in our example with the first and last segments generated by the same process, the total number of processes here is 2. We refer to this parameter as $\texttt{process\_count}$ and note that it has been mathematically denoted by $m$ in Section 2.4.

```python
>>> process_count = 2
>>> estimates = PyChest.find_changepoints(seq, min_distance, process_count)
```

```python
>>> print(estimates)
[1997, 6502]
```

Next, we generate a real-valued sequence from the above binary-valued sequence by calculating its running mean (of window size $N = 25$) as follows.

```python
>>> N = 25
>>> seq = np.convolve(seq, np.ones(N)/N, mode='valid')
```

Applying \texttt{list\_estimator} and \texttt{find\_changepoints} to the generated data we obtain the following results.
Figure 1: An illustration of the changepoint estimates produced by `find_changepoints`.

```python
>>> min_distance = 0.125
>>> estimates = PyChest.list_estimator(seq, min_distance)

>>> print(estimates)
[1989, 6489, 4582]

>>> process_count = 2
>>> estimates = PyChest.find_changepoints(seq, min_distance, process_count)

>>> print(estimates)
[1989, 6489]
```

The result is visualized in Figure 1. Observe that while the illustration in this example refers to the estimation of changes in the mean, the algorithms are more generally designed to detect changes in the distribution. This is shown in the next section.

## 4 Robustness against long-range dependencies

The strength of `PyChEst` is in its ability to consistently estimate changepoints in the presence of long-range dependencies. In this section we showcase this feature by comparing `PyChEst` to the state of the art algorithm Wild Binary Segmentation (WBS) proposed by Fryzlewicz (2014) which is provably consistent in the piece-wise i.i.d. setting; we rely on the implementation provided by the CRAN package `wbs` of Baranowski and Fryzlewicz (2019).
4.1 Stationary samples which possess long-range dependencies

As part of the experiments outlined in Section 4.2 we obtain the stationary segments of some of our piece-wise stationary samples as sample-paths of a subclass of stationary ergodic processes that do not belong to any “simpler” class. More specifically, we consider Irrational Rotations outlined below, which are classical examples of a stationary ergodic process that is not a $B$-process, and that cannot be modelled by a hidden Markov model with a finite or countably infinite set of states, see, e.g. (Shields, 1996).

4.1.1 Irrational Rotation

For a fixed $t \in \mathbb{N}$, a binary-valued sample $Y := Y_1, Y_2, \ldots, Y_t$ is generated from an Irrational Rotation as follows. First, we fix a parameter $\beta \in (0, 1)$. Then, we sample $R_0$ uniformly at random from $[0, 1]$, and for each $i = 1, \ldots, t$ we obtain $R_i$ by shifting $R_{i-1}$ by $\beta$ to the right and removing the integer part, i.e.

$$R_i := R_{i-1} + \beta - \lfloor R_{i-1} + \beta \rfloor, \quad i = 1, \ldots, t$$

The sequence $Y$ is obtained from $R_i$, $i \in 1, \ldots, t$ by thresholding at 0.5 i.e., for each $i \in 1, \ldots, t$ we set

$$Y_i := I\{R_i \geq 0.5\} \quad (4)$$

It is well-known that if $\beta$ is irrational then $Y$ forms a binary-valued stationary ergodic sample, that possesses long-range dependencies: it is not even mixing in the ergodic-theoretic sense (Shields, 1996). We simulate $\beta$ by a floating point number with a long mantissa.

4.1.2 Hidden Irrational Rotation

In order to obtain a sample $Z := Z_1, Z_2, \ldots, Z_t$, $t \in \mathbb{N}$ from a real-valued stationary ergodic process that is not mixing, we generate what we call a “Hidden Irrational Rotation”. To this end, we first simulate an irrational number $\beta$ by a floating point number with a long mantissa, and generate a binary-valued Irrational Rotation sample $Y := Y_1, \ldots, Y_t$ given by (4). Next, we draw two independent samples $U_1, \ldots, U_t$ and $V_1, \ldots, V_t$, each generated uniformly i.i.d. from $[0, 1]$ and $[0.9, 1.9]$ respectively. We obtain $Z$ as

$$Z_i := U_i(1 - Y_i) + V_i Y_i, \quad i = 1, \ldots, t \quad (5)$$

This forms a real-valued (non-mixing) stationary ergodic sample $Z$. Observe that if we deterministically set $U_i = 0$ and $V_i = 1$ for all $i = 1, \ldots, t$ we recover the binary-valued sample $Y$. Figure 2 illustrates an example of a piece-wise stationary sample, with a changepoint at $\tau = 1500$ such that the segments before and after it correspond to sample-paths of two different Hidden Irrational Rotations with parameters $0.45..^1$ and $0.63..^2$ respectively. Observe that while there is a change in the process distributions, it cannot be located visually and without appropriate statistical tools.
Figure 2: An illustration of a piece-wise stationary sample generated by the concatenation of two different Hidden Irrational Rotations. There is a changepoint at 1500 and the segments before and after the change correspond to sample-paths of different Hidden Irrational Rotations.

4.2 Illustrations

In this section we compare the performance of PyChESt against that of wbs. As mentioned previously, the latter provides a consistent changepoint estimation method for the case where the samples are piece-wise i.i.d.

To calculate the performance of the algorithms, we rely on their estimation errors calculated as follows. Suppose that there are a total of \( \kappa \) changepoints \( \tau_1(n), \ldots, \tau(n) \) in a sample of length \( n \in \mathbb{N} \). We calculate the estimation error of an algorithm that produces a total of \( K \) candidate estimates \( \hat{\tau}_1(n), \ldots, \hat{\tau}_K(n) \) as

\[
e_n := \begin{cases} \frac{1}{\kappa} \sum_{i=1}^{\kappa} |\tau_i(n) - \hat{\tau}_i(n)|, & \text{if } K = \kappa \\ 1, & \text{otherwise.} \end{cases} \tag{6}
\]

In our first experiment we examine the convergence of the estimation error \( e_n \) as sample-length \( n \) increases. First, we consider the problem of locating a single change in distribution so that \( \kappa = 1 \). We start with a simple setting where we generate piece-wise i.i.d. samples of length \( n = 500, 1000, 1500, 4500 \), each with a changepoint at \( \tau(n) := \lfloor 0.4n \rfloor \); the samples before the changepoint are generated i.i.d. according to a Bernoulli distribution with parameter 0.8 and independently, those after the changepoint are generated i.i.d. according to a Bernoulli distribution with parameter 0.5. We set \( \alpha := 0.21 \) and provide find_changepoints with \texttt{min_distance=0.21, and process_count=2}. We provide wbs with an upper-bound of 2 on \( 0.45234164353462432 \) and \( 0.634535465623456234234 \).
the number of changepoints together with the Strengthened Schwarz Information Criterion (SIC) penalty term as recommended by Baranowski and Fryzlewicz (2019). As can be seen in Figure 2(Left), the mean estimation error calculated by (6) (averaged over 20 iterations) converges to 0 for both algorithms. Next we repeated the same experiment, but this time we let \( \tau(n) := \lfloor 0.3n \rfloor \), \( n = 5000, 10000, 15000, 20000, 25000, 30000 \) and generated Hidden Irrational Rotations with shift parameters \( 0.452341643253462432 \) and \( 0.6345354645623456234234 \) before and after the changepoint respectively. To run the algorithms, we pass min_distance=0.21, and process_count=2 to find_change_points, and provided wbs with the same parameters as in the previous case, namely an upper-bound of 4 on the number of changepoints and the Strengthened SIC penalty term. As shown in Figure 2 (Right), the mean error rate (calculated over 20 iterations) of PyChEst approaches 0 with increased sample length \( n \), while that of wbs remains consistently high. This experiment shows that the rate of convergence of \( \epsilon_n \) for an algorithm such as wbs with consistency guarantees for the piece-wise i.i.d. setting can be better than that of PyChEst’s when the i.i.d. assumption holds. However, PyChEst is robust against long-range dependencies, and performs well on both piece-wise i.i.d. samples as well as piece-wise stationary samples.

In our second experiment, we consider the problem of locating \( \kappa = 2 \) changepoints in piece-wise stationary samples of length \( N = 60000 \). First, we generate piece-wise i.i.d. Bernoulli samples with changepoints at \( 0.4N \) and \( 0.7N \), so that \( \lambda_{\text{min}} = 0.3 \). The three stationary segments are drawn independently where the first and last segments are generated i.i.d. according to a Bernoulli distribution with parameter 0.8 and the middle segment is generated i.i.d. according to a Bernoulli distribution with parameter 0.5, leading to \( m = 2 \) as the total number of process distributions. We pass process_count=2 and min_distance=0.21 to PyChEst’s find_change_points. We run wbs with an upper-bound of 4 on \( \kappa \) together with the Strengthened SIC penalty term. Next, we consider the same problem, with the
piece-wise i.i.d. vs. piece-wise stationary process with long-range dependencies

|                          | PyChEst   | wbs        |
|--------------------------|-----------|------------|
| (piece-wise i.i.d.) Bernoulli | 0.000151  | 0.000124   |
| (piece-wise) Hidden Irrational Rotation | 0.000143  | 1          |

Table 1: Average (over 200 iterations) error (6) on piece-wise stationary samples of length 60000. In each case, the samples have $\kappa = 2$ changepoints where $\lambda_{\min} = 0.28$; each stationary segment is generated by one of $m = 2$ different processes i.e. the first and last processes are identical. A lower-bound $\alpha = 0.21$ is provided as input to find_changepoints together with the correct number of processes $m = 2$. An upper-bound of 4 on $\kappa$ together with the Strengthened SIC penalty term are passed to wbs. Both algorithms perform well in the piece-wise i.i.d. (Bernoulli) setting, while wbs is consistently confused by the long-range dependencies in piece-wise Hidden Irrational Rotations.

difference that the changepoints are located at $0.3N$ and $0.6N$ and the samples are piece-wise stationary ergodic processes with long-range dependencies. More specifically, the first and last segments are identically generated by a Hidden Irrational Rotation with shift parameter $0.452341643253462432$ and the middle segment is generated by a different Hidden Irrational Rotation with $0.6345354645623456234234$ as its shift parameter. We run both algorithms with the same input parameters as those provided in the piece-wise i.i.d. case above. The average (over 200 iterations) errors (6) is reported in Table 1. Both algorithms perform well in the piece-wise i.i.d. setting. However, in the case of piece-wise Hidden Irrational Rotations wbs is unable to provide a single changepoint estimate, while PyChEst continues to perform well. As a result, this experiment serves as another demonstration of the robustness of PyChEst against long-range dependencies.

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