Phase Transitions in Softly Broken N=2 SQCD at Non-Zero $\theta$ Angle

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Abstract

We investigate the behavior of softly broken $N = 2$ SQCD at non-zero bare theta angle $\theta_0$, using superfield spurions to implement the SUSY breaking. We find a first-order phase transition as $\theta_0$ is varied from zero to $2\pi$, in agreement with a prediction of 't Hooft. The low-energy theta angle $\theta_{\text{eff}}$, which determines the effective charges of dyonic excitations, has a complicated dependence on $\theta_0$ and breaking parameters.

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1 Introduction

Surprisingly, very little is known about the behavior of QCD at non-zero bare theta angle, \( \theta_0 \). ‘t Hooft [1] argued that if QCD continues to exhibit confinement at all \( \theta_0 \), then a phase transition must occur at some finite \( \theta_0 \) – most likely \( \theta_0 = \pi \). Indeed, this possibility is supported by investigations using chiral Lagrangians [2]. Alternatively, he suggested that the theory could switch to a novel phase, such as one with ‘oblique confinement’, in some finite region of \( \theta_0 \) near \( \pi \). Recently, based on lattice investigations, Schierholz [3] has suggested that confinement breaks down completely even for infinitesimal values of \( \theta_0 \), leaving QCD in a Higgs phase.

Seiberg and Witten’s [4, 5] solution of low-energy \( \mathcal{N}=2 \) SQCD, combined with recent results extending the exact solutions to models with explicit, soft-SUSY breaking [6, 7], provides a useful laboratory for the study of strong gauge dynamics. The models in question exhibit electric confinement due to condensation of magnetic monopoles, a mechanism first proposed for QCD by ‘t Hooft and Mandelstam [8]. In their picture the relevant low-energy degrees of freedom are magnetic monopoles interacting via the long-range fields of the Abelian subgroup \( U(1)^2 \) of \( SU(3) \). ‘t Hooft’s argument [1] that the nature of confinement could depend on the \( \theta_0 \) angle is a direct consequence of the existence of magnetic excitations, and the dependence of their effective charge on \( \theta_0 \) (the ‘Witten effect’ [9]).

While it is still an open question whether QCD behaves in the manner envisaged in [8], the Seiberg-Witten model clearly has the correct ingredients to further investigate ‘t Hooft’s predictions: it remains in the Coulomb phase at low energies and exhibits light magnetic degrees of freedom. In this paper we study the behavior of softly broken SQCD models at non-zero \( \theta_0 \). We note that the highly non-trivial consistency conditions on the Seiberg-Witten ansatz are as equally valid at non-zero \( \theta_0 \) as at \( \theta_0 = 0 \). It is necessary to break supersymmetry completely to see any physical dependence on \( \theta_0 \). In both the \( \mathcal{N}=2 \) and \( \mathcal{N}=1 \) models studied in [1], the gaugino remains massless and hence any dependence on \( \theta_0 \) can be rotated away. (See [10] for a discussion of \( \theta_0 \) and CP invariance in this class of models.) In the models that we study the anomalous \( U(1)_R \) symmetry which may be used to generate shifts in \( \theta_0 \) is explicitly violated by SUSY breaking interactions such as masses for the gauginos and adjoint quarks. Hence the physics can and does depend on \( \theta_0 \) in an interesting way.

2 Review of Solved Models

In this section we review Seiberg and Witten’s solution of \( \mathcal{N}=2 \) SQCD and the spurion technique for studying the effects of soft breaking perturbations. This paper will concentrate
on the case of an $SU(2)$ gauge group and a single $N = 2$ vector multiplet $A^a$. In terms of $N = 1$ multiplets $A^a$ contains a vector multiplet $(A^a_\mu, \lambda^a)$ and a chiral multiplet $(\psi^a, \phi^a)$, all in the adjoint representation.

2.1 Pure Glue $N=2$ SQCD

$N = 2$ SUSY models are highly constrained and can be characterized completely by a prepotential $\mathcal{F}(A)$, which determines both the Kahler- and super-potentials of the model. At tree level the prepotential is given by $\mathcal{F}(A) = \frac{1}{2} \tau_{cd} A^2$ where $\tau_{cd}$ contains the bare gauge coupling and $\theta_0$ angle

$$\tau_{cd} = \frac{\theta_0}{2\pi} + i \frac{4\pi}{g_0^2}$$

(1)

The classical theory possesses a number of $U(1)_R$ symmetries under which the superspace coordinate $\theta$ has charge +1, $W_a$ charge +1 and the matter fields arbitrary charge. In the quantum theory the $U(1)_R$ symmetries are anomalous except for that with matter field charge 0. The anomalous Ward identities associated with the broken symmetries may be used to rotate $\theta_0$ onto the matter and gaugino mass terms and, since these are zero, to rotate $\theta_0$ away without physical consequences.

Vacua of the classical theory are described by a moduli space with the adjoint scalar vacuum expectation value (parameterized by the gauge invariant quantity $u = tr[a^2]$) as coordinates. At energy scales below $u$ the classical low energy theory contains only the $U(1)$ subgroup of the gauge group (a photon and its gaugino) and the neutral components of the matter fields. At $u = 0$ the full $SU(2)$ symmetry is restored. The low energy $U(1)$ theory may be written in $N = 1$ superspace notation in terms of $\tau_{cd}$ as

$$\mathcal{L} = \frac{1}{4\pi} Im \left[ \int d^4\theta \frac{\partial \mathcal{F}}{\partial A} \bar{A} + \frac{1}{2} \int d^2\theta \frac{\partial^2 \mathcal{F}}{\partial A \partial W} W^a W^a \right].$$

(2)

Seiberg and Witten’s ansatz for the solution of the quantum theory also possesses a moduli space in $u$ and is described below the scale $u$ by a $U(1)$ theory with neutral matter fields, $(\psi, a)$. In the quantum theory the $SU(2)$ symmetry is not restored at $u = 0$. The solution gives two descriptions of the low energy theory, one in the orginal electric variables and one in the electro-magnetic dual variables. The explicit form of $a(u)$, $a_D(u)$ are given in terms of the periods of a meromorphic differential of the second kind on a genus one surface described by the equation:

$$y^2 = (x^2 - \Lambda^4)(x - u),$$

(3)

describing the double covering of the plane branched at $\pm \Lambda^2, u, \infty$. If one chooses the cuts $\{-\Lambda^2, \Lambda^2\}, \{u, \infty\}$, then the solution is represented by elliptic functions [7]

$$a(u) = \frac{4\Lambda}{\pi k} E(k), \quad a_D(u) = \frac{4\Lambda}{i\pi} \frac{E'(k) - K'(k)}{k}, \quad k^2 = \frac{2}{1 + u/\Lambda^2}.$$  

(4)
where
\[ K(k) = \frac{\pi}{2} F(1/2, 1/2, 1; k^2); \quad K'(k) = K(k'); \]
\[ E(k) = \frac{\pi}{2} F(-1/2, 1/2, 1; k^2); \quad E'(k) = E(k'), \quad k'^2 + k^2 = 1, \]
(5)

Note that \( a_D = \frac{\partial F}{\partial a} \) and the effective coupling constants \( \tau \) is given by
\[ \tau = \frac{1}{2} \frac{\partial^2 F}{\partial a^2} = \frac{\partial a_D}{\partial a} = \frac{a_D/dk}{da/dk} = \frac{iK'}{K} \]
(6)

The scale \( \Lambda \) generated by the theory is related to the bare coupling constant \( g_0 \) and bare \( \theta_0 \)-parameter by
\[ \Lambda^2 = \Lambda_0^2 \exp\left(-\frac{4\pi^2}{g_0^2} + \frac{i\theta_0}{2}\right) \]
(7)

This result is derived using the perturbatively exact one-loop \( \beta \) function which depends holomorphically on the bare chiral superfield \( \tau \). An important consequence is the dependence of \( \Lambda \) on \( \theta_0 \).

In addition the above relations imply the following useful form for \( u \)
\[ u = i\pi \left( F - \frac{1}{2} a \frac{\partial F}{\partial a} \right) \]
(8)

In fig. 1 we show the real and imaginary parts of the effective coupling \( \tau \) as a function of \( u \), which correspond to the effective theta angle and the gauge coupling respectively. The singularities in the \( \tau \) function are consistent with the hypothesis that a magnetic monopole (with electric and magnetic charges \((0, 1)\)) becomes massless at the point \(+\Lambda^2\) and a \((-1, 1)\) (or \((1, 1)\), depending on whether one approaches from above or below the branch cut) dyon becomes massless at the point \(-\Lambda^2\). Note that at the latter point the effective angle, \( \theta_{eff} \), is \(2\pi\) and hence by the Witten effect the dyon behaves like a \((0, 1)\) monopole at \( \theta_{eff} = 0 \). The theory in fact possesses a \( Z_2 \) symmetry that leaves the physics invariant under \( u \to -u \). The extra dyonic matter multiplets are introduced into the effective theory close to the singular points. Their interactions are constrained by \( N = 2 \) SUSY to be of the form
\[ \mathcal{L}_M = \int d^4\theta (M^\dagger e^{2V_D} M + \tilde{M}^\dagger e^{-2V_D} \tilde{M}) + 2\sqrt{2} Re \int d^2\theta A_D M \tilde{M}. \]
(9)
The theory possesses an $SL(2, Z)$ symmetry acting on $(a_D, a)$ and hence also on $(\tau, 1)$. The two generators of the transformations are

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$  \hspace{1cm} (10)$$

which corresponds to shifting $\theta_{\text{eff}}$ by $2\pi$, and

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$  \hspace{1cm} (11)$$

which corresponds to transforming to the electromagnetic dual variables. There are thus a number of different descriptions of the theory at any particular value of $u$. We shall make use of the following descriptions: at large $u$ the orginal electric variables in (4) are the weakly coupled description (we shall use superscript $w$); near the singularity at $+\Lambda^2$ with a light monopole field the electric variables are strongly coupled but the magnetic variables are weakly coupled (superscript $m$); and near the singularity $-\Lambda^2$ the magnetic variables are again weakly coupled but in order to make the couplings of the ($-1, 1$) (or $(1, 1)$) dyon to the photon local we shift $\theta_{\text{eff}}$ by $2\pi$ using the generator $T$ (these variables have superscript $d$).

2.2 $N = 1$ Solution

The $N = 2$ solution can be perturbed by the addition of a small $N = 1$ preserving mass for the adjoint matter field \[4\]

$$L_{\text{pert}} = 2Re \int d^2\theta \ mU(A)$$  \hspace{1cm} (12)$$

The $N = 1$ theory exhibits a classical $U(1)_R$ symmetry with charges $\theta + 1$, $W_\alpha + 1$ and $U + 2$. It is anomalous in the quantum theory and can be used to rotate $\theta_0$ onto the gaugino mass term and hence eliminate it from the theory.
The dependence of the effective theory’s superpotential on the bare mass term is that of
the bare theory up to gauge renormalization. The minima of the potential are given by the
extrema of the superpotential and are hence determined by

\[ \sqrt{2} a_M a_{\tilde{M}} + m \kappa = 0 \]  

\[ a^{(m)} a_M = a^{(m)} a_{\tilde{M}} = 0 \]  

where

\[ \kappa = \frac{\partial u}{\partial a^{(m)}} = i \pi \left( a_D^{(m)} - \frac{1}{2} a^{(m)} \tau^{(m)} \right) \]  

(15)

and we have used (8). We find

\[ a_D = 0 \]  

\[ a_M = a_{\tilde{M}} = \left( -m \kappa(0)/\sqrt{2} \right)^{1/2} \]  

The vacuum energy is now minimized at the singular points where there is a condensate of
the magnetic monopole field and \( \theta_{\text{eff}} = 0 \). In terms of the original electric variables there is
confinement by the ’t Hooft–Mandelstam mechanism. We note that since confinement sets
in on the scale of the \( N = 2 \) breaking we must have \( m \ll \Lambda \) to retain a \( U(1) \) phase and
hence the validity of the effective theory.

### 2.3 Softly Broken \( N = 2 \) SQCD

Soft SUSY breaking interactions may be introduced into the exact solution of \( N = 2 \) SQCD
through the \( F \)-components of the ‘spurion’ coupling fields. We treat the couplings \( \tau \) and \( m \)
as the vevs of the lowest components of chiral superfields and their occurrence in the effective
theory is hence constrained by SUSY and the spurions’ \( U(1) \) couplings. The requirement
that the theory returns to the SUSY limit as the \( F \)-components are turned off constrains
how the couplings may occur in the theory up to possible D-terms that vanish in the SUSY
limit which we shall discuss below where appropriate.

#### 2.3.1 \( N = 1 \) Spurions

A first example of a solvable (at order \( p^0 \)) softly broken model is to start from the \( N = 1 \)
solution described above and allow the spurion field \( m \) to acquire a non-zero \( F \)-component
vev \( \bar{m} \). At tree level this induces the additional \( N = 1 \) violating interaction

\[ 2 \text{Re}(f_m u) \]  

(18)

This term explicitly breaks the \( Z_2 \) and anomalous \( U(1)_R \) symmetries of the supersymmetric
\( N = 1, 2 \) models. There is no anomalous \( U(1) \) symmetry in the theory and hence \( \theta_0 \) can
have physical consequences.
By requiring the $N = 1$ limit as $f_m \rightarrow 0$ we observe that the superpotential is not renormalized in the quantum theory. Treating $m$ as a spurion field there is a non-anomalous $U(1)_R$ symmetry under which $m$ transforms with charge +2 and hence may only occur in the D-terms as $m^\dagger m/\Lambda^2$. These unknown terms are an order down in the momentum expansion (we take $p \sim m << \Lambda$) compared to the soft breaking terms from the superpotential. Hence (18) is the leading order correction to the low-energy effective potential when $f_m/\Lambda^2 << a_m/\Lambda << 1$. We obtain the following potential for the softly-broken model

$$V = 2|a^{(m)}|^2(|a_M|^2 + |a_{\tilde{M}}|^2) + \frac{1}{2b}(|a_M|^2 + |a_{\tilde{M}}|^2)^2 + \frac{1}{b}(|\kappa m|^2 + 2\sqrt{2}Re(a_Ma_{\tilde{M}}\kappa^* m^*) - 2Re(f_m u)$$  \hspace{1cm} (19)

with

$$b = \frac{1}{4\pi} Im\tau^{(m)}, \quad \kappa = \frac{\partial u}{\partial a^{(m)}}$$  \hspace{1cm} (20)

Minimizing the potential with respect to the monopole fields yields $|a_M|^2 = |a_{\tilde{M}}|^2$. We have

$$a_M = i\rho e^{i\alpha}, \quad a_{\tilde{M}} = i\rho e^{i\beta}$$  \hspace{1cm} (21)

where for a minimum we have $\alpha + \beta = \text{phase}(m\kappa)$. From which we find

$$V = -\frac{2}{b}\rho^4 + \frac{|\kappa m|^2}{b} - 2Re f_m u$$  \hspace{1cm} (22)

with the monopole condensate

$$\rho^2 = \begin{cases} -b|a^{(m)}|^2 & \frac{1}{\sqrt{2}}|\kappa m| > 0 \\ 0 & \end{cases}$$  \hspace{1cm} (23)

To obtain the potential in the dyon region we note that only the final term in (22) violates the $Z_2$ symmetry of the $N = 1$ limit. In the dyon region we therefore have

$$V^{(d)}(u) = V^{(m)}(-u) - 4Re f_m u$$  \hspace{1cm} (24)

$$\rho^2_{(d)}(u) = \rho^2_{(m)}(-u)$$  \hspace{1cm} (25)

The potential may now be minimized numerically. The potential is shown for $\theta_0 = 0$, $m/\Lambda = 0.01$ and $f_m/\Lambda = 0.001$ as a function of $u$ in fig. 2. The $Z_2$ symmetry is explicitly broken and the global minimum lies close to the point $+\Lambda^2$ as we expect since we are only perturbing the $N = 1$ model. By choosing $f_m$ to be complex it is possible to perturb the minima away from the singularity in any direction on the $u$ plane. Since the singularity lies on the cut it is possible to vary $f_m$ smoothly and see a discontinuous jump in the value of $\theta_{eff}$ as the minima crosses the cut.
Fig. 2: Potential in $N = 1$ spurion model as a function of $u$ for $m = 0.01$ and $f_m = 0.001$.

2.3.2 $N = 2$ Spurion

Another method for introducing soft breakings in a controllable way is to consider a full $N = 2$ vector multiplet of spurion fields [7]. This has the advantage of retaining control over the D-terms, although it produces a very restricted set of breaking terms. Following [7], let us introduce the spurion field $S$ as a second vector multiplet

$$
L = \frac{1}{4\pi} Im \left[ \int d^4 \theta \left( \frac{\partial F}{\partial A} \bar{A} + \frac{\partial F}{\partial S} \bar{S} \right) + \frac{1}{2} \int d^2 \theta \left( \frac{\partial^2 F}{\partial A^2} W_\alpha W^{\alpha} + \frac{\partial^2 F}{\partial A \partial S} W_\alpha W'^{\alpha} + \frac{\partial^2 F}{\partial S^2} W'^{\alpha} W'^{\alpha} \right) \right].
$$

(26)

with bare prepotential

$$
F_{cl} = \frac{1}{2\pi} SA^2
$$

(27)

For convenience we define the $2 \times 2$ matrix of couplings:

$$
\tau_{11} = \frac{\partial^2 F}{\partial A^2}, \quad \tau_{01} = \frac{\partial^2 F}{\partial s \partial A}, \quad \tau_{00} = \frac{\partial^2 F}{\partial s^2}.
$$

(28)

Freezing the scalar component of the second multiplet’s matter field generates the coupling of the pure glue model with $s_{cl} = \pi \tau_{cl}$ (this normalization is convenient since $\Lambda \sim \exp(is)$). We may also freeze the F-component of the spurion matter field, $F_0$, and generate soft breaking interactions. At the bare level, this induces the following SUSY violating interactions in the $SU(2)$ model:

$$
L_{SSB} = \frac{1}{8\pi^2} Im(F_0^* \psi^\alpha_A \psi^\alpha_A + F_0 \lambda^\alpha \lambda^\alpha) - \frac{|F_0|^2}{4\pi Im\tau_{cl}} (Im a^\alpha)^2
$$

(29)

The gauginos and matter fermions acquire masses and hence it is again not possible to rotate away $\theta_0$. The imaginary components of the scalar fields also acquire a mass at tree level. Since SUSY is explicitly broken in the theory and there are thus no symmetries protecting the scalar masses there will be quadratic divergences in the theory below the SUSY breaking
scale that will radiatively generate masses for the real components of the scalar fields as well. Thus we expect that if \( F_0 \) were taken to infinity we would recover pure glue (non-SUSY) QCD.

The soft breaking terms again break the \( Z_2 \) symmetry of the pure N=2 model. At the quantum level, the effects of the SUSY breaking on the effective theory can be computed using the holomorphic dependence of the prepotential \( \mathcal{F}(A) \) on the spurion field; that is the quantum prepotential is just that of the pure glue theory.

In the weakly coupled region (far from the singularities) the theory is best described in terms of the electric variables [7]:

\[
a_{D}^{(w)} = \frac{4\Lambda E'}{k}, \quad a^{(w)} = \frac{4\Lambda E(k)},
\]

\[
\tau_{11}^{(w)} = \frac{iK'}{K}, \quad \tau_{01}^{(w)} = \frac{2\Lambda}{kk}, \quad \tau_{00}^{(w)} = -\frac{8i\Lambda^2}{\pi} \left( \frac{E - K}{k^2K} + \frac{1}{2} \right).
\] (30)

In the region close to the singularity associated with a massless monopole the dual variables are weakly coupled. Using (30) in combination with \( SL(2, Z) \) and the residual \( Z_2 \) symmetry, one can derive expressions for the couplings in the monopole region

\[
a^{(m)} = a_{D}^{(w)}, \quad a_{D}^{(m)} = -a^{(w)},
\]

\[
\tau_{11}^{(m)} = -\frac{1}{\tau_{11}^{(w)}}, \quad \tau_{01}^{(m)} = i\tau_{01}^{(w)}, \quad \tau_{00}^{(m)} = \frac{8i\Lambda^2}{\pi} \left( \frac{E'}{k^2K'} - \frac{1}{2} \right).
\] (31)

and in the dyon region

\[
a^{(d)}(u) = i\left(a_{D}^{(m)}(-u) - a^{(m)}(-u)\right), \quad a_{D}^{(d)}(u) = ia^{(m)}(-u),
\]

\[
\tau_{11}^{(d)}(u) = \tau_{11}^{(m)}(-u) - 1, \quad \tau_{01}^{(d)}(u) = i\tau_{01}^{(m)}(-u), \quad \tau_{00}^{(d)}(u) = -\tau_{00}^{(m)}(-u).
\] (32)

Note that in the dyon region we have made an \( SL(2, Z) \) transformation to shift \( \theta_{\text{eff}} \) by 2\( \pi \), so that the dyon has a local coupling to the photon, with charge \((0,1)\). \( \tau^{(d)}(u) \) is then written in terms of \( \tau^{(m)}(-u) \) in order to avoid the branch cut to infinity when \( u \) is in the dyon region \( u \sim -\Lambda^2 \). The functions \( a^{(m)} \) and \( (a^{(d)} + a_{D}^{(d)}) \) disappear at the corresponding singular points, leading to massless monopoles and dyons.

The potential is obtained by eliminating the auxiliary fields \( F_M, F_{\bar{M}} \) and \( F_a \):

\[
V = \frac{1}{2b_{11}}(|a_M|^2 + |a_{\bar{M}}|^2)^2 + 2|a|^2(|a_M|^2 + |a_{\bar{M}}|^2)
\]

\[
+ \frac{1}{b_{11}} \sqrt{2}b_{01}(F_0a_Ma_{\bar{M}} + F_0a_{\bar{M}}a_M) - \frac{\det b_{ij}}{b_{11}}|F_0|^2,
\]

where

\[
b_{ij} \equiv \frac{1}{4\pi} \text{Im} \, \tau_{ij} = \frac{1}{4\pi} \text{Im} \frac{\partial^2 \mathcal{F}}{\partial a^i \partial a^j}.
\] (33)
$a_M, a_M$ are, as before, the scalar components of the (monopole or dyon) superfield, $M$, and all fields are taken in the corresponding patches.

Minimized with respect to the monopole (dyon) fields it becomes (here $f_0 = |F_0|$ and is real)

$$V = -\frac{2}{b_{11}} \rho^4 - \frac{\det b}{b_{11}} f_0^2$$

(34)

with the monopole (dyon) condensate

$$\rho^2 = \begin{cases} -b_{11}|a|^2 + \frac{1}{\sqrt{2}}|b_{01}|f_0 > 0 \\ 0 \end{cases}$$

(35)

The resulting potential is plotted in fig. 3 for $\theta_0 = 0$. The soft SUSY terms lift the degeneracy of the moduli space, and a unique vacuum appears near $+\Lambda^2$. The $Z_2$ symmetry is explicitly broken as can be seen in fig. 3. It is also explicitly broken by the regions of the $u$ plane in which the monopoles and dyons acquire vevs which we show in fig. 4. As $F_0$ increases the minima moves outwards along the real $u$ axis where $\theta_{eff} = 0$.

![Fig. 3: Potential for $N = 2$ spurion with $\theta_0 = 0, F_0/\Lambda = 0.3$.](image1)

![Fig. 4: Regions of the $u$ plane with monopole, $\langle M \rangle$, and dyon condensates, $\langle D \rangle$, in the $N = 2$ spurion model at $\theta_0 = 0, F_0/\Lambda = 0.3$.](image2)

Fig. 6.
3 Dependence On The Bare $\theta_0$ Angle

We now turn to our discussion of the dependence of the solutions described above on the bare theta angle, $\theta_0$. 't Hooft argued in [1] that theories that confine by monopole condensation must exhibit a phase transition as $\theta_0$ crosses $\pi$ and conjectured that new phases such as 'oblique' confinement (confinement due to the condensation of a purely magnetically charged mode which is the bound state of dyons of opposite electric charge) might occur in a patch around $\pi$. His argument is essentially that due to the 'Witten effect' as $\theta_0$ changes from 0 to $2\pi$ the charge of the (0,1) monopole causing confinement at $\theta_0 = 0$ must become (1,1). Since the theory must be $2\pi$ periodic in $\theta_0$ a dyon in the theory must become the (0,1) state at $\theta_0 = 2\pi$. As $\theta_0$ changes the vacuum must switch between having condensates of these two different dyons so one expects a first order phase transition, most likely at $\theta_0 = \pi$.

Phase transitions in the effective chiral lagrangian of QCD with non-zero $\theta_0$ have also been observed [2]. Here $\theta_0$ can be rotated onto the quark mass matrix as a phase $\theta_0/N_f$ where $N_f$ is the number of quark flavors. The theory is again $2\pi$ periodic in $\theta_0$ but the effective theory depends on $\theta_0/N_f$ which leads to $N_f$ distinct, degenerate vacua. Phase transitions between these vacua occur at $\theta = (\text{odd integer}) \cdot \pi$.

The exact solutions of N=2 SQCD models allow further investigations of the $\theta_0$ dependence of confinement since they confine by the mechanism envisioned by 't Hooft. However the behavior of the SUSY theories is more subtle than the basic argument suggested above since $\theta_0$ is renormalized and leads to different effective theta angles, $\theta_{\text{eff}}(u)$, at different points on the $N = 2$ theory’s moduli space.

The effective theories depend on $\theta_0$ through the dynamically generated scale $\Lambda$

$$\Lambda^2 \sim \exp(i\theta_0/2).$$  \hspace{1cm} (36)

The effect of changing $\theta_0$ in the pure glue $N = 2$ theory is therefore to rotate the positions of the singularities on the moduli space by an angle of $\theta_0/2$. For every point on the moduli space at one value of $\theta_0$ there is an equivalent point on the moduli space at any other value of $\theta_0$. The theory is therefore unchanged by changing $\theta_0$. This is as expected since the $N = 2$ theory has an anomalous $U(1)_R$ symmetry acting on the gauginos and fermionic matter field which are exactly massless. The anomalous Ward identity may be used to rotate $\theta_0$ away.

When the theory is perturbed to an $N = 1$ theory by adding a mass term for the scalar and fermionic matter fields as described in section 2.2 the theory is pinned at the singular points on the moduli space at $\pm \Lambda^2$. The analysis in section 2.2 is independent of the phase of $\Lambda$ as pointed out in [10] and hence the theory is pinned at the singularities for all $\theta_0$. The effective $\theta_{\text{eff}}$ angle at the singularities is 0 and $2\pi$ respectively at the points with (0,1) and (-1,1) massless dyons. Thus $\theta_{\text{eff}} = 0$ for any $\theta_0$. Again this is the expected result since the
gauginos of the theory are still massless and the anomalous $U(1)_R$ symmetry may be used to rotate away $\theta_0$.

The problem becomes more interesting when the theory is softly broken to $N=0$ since the ability to rotate away $\theta_0$ is lost. In the case of the $N=1$ spurion there is no anomalous $U(1)$ symmetry and in the case of the $N=2$ spurion all the fermions in the model obtain masses. In addition the $Z_2$ symmetry of the effective theory is broken. Using the potentials given in section 3.2, we can track the groundstate of the system as $\theta_0$ is varied. In addition to looking for the phase transition predicted by ’t Hooft (or the disappearance of confinement above some critical $\theta_0$ – which we do not observe), we are particularly interested in the value of $\theta_{\text{eff}}$ at the minimum of the potential, as this determines the effective charge of the dyon/monopole condensate. Since the $N=1$ minima at $\pm \Lambda^2$ are directly on the singular points in $\tau$, there is the possibility of discontinuous behavior in $\theta_{\text{eff}}$ as the soft breakings force the $N=0$ minima away from the singularities.

3.1 $\theta_0$ Dependence for the N=1 Spurion

The potential of section 2.3.1 may be used to plot the potential of the $N=1$ spurion model with changing $\theta_0$. For real $f_m$ and $m$ we show the results ($m/\Lambda = 0.01$ and $f_m/\Lambda = 0.001$ in fig. 5.

At $\theta_0 = 0$ the minima close to $+\Lambda^2$ lies below that at $-\Lambda^2$. As $\theta_0$ is raised towards $\pi$ the line connecting the singularities rotates in the $u$ plane by $\theta_0/2$ and the energy difference between the global and local minima decreases. At $\theta = \pi$ the two vacua at $u \sim \pm i\Lambda^2$ are degenerate in energy. For $\theta_0 > \pi$ the vacua at the singularity with the (-1,1) dyon becomes the new minima of the theory. There is thus a first order phase transition at $\theta_0 = \pi$ where distinct vacua interchange.

The degeneracy of the two vacua at $\theta_0 = \pi$ may be easily seen from (22) and (23). The potential is that of the $N=1$ model which possesses a reflection symmetry about the axis defined by $\exp(i\pi/2 + i\theta_0/2)$ plus the final term proportional to $f_m$ which possesses the symmetry $u \rightarrow u^*$. At $\theta_0 = \pi$ these two symmetries are identical and the whole theory acquires a symmetry $u \rightarrow u^*$. Thus, provided the minima are away from the origin, there will be two degenerate minima. The phase transition is marked not only by a discontinuous jump in $u$ but also in $\theta_{\text{eff}}$. We show this discontinuity, which grows with $f_m$, in fig. 6.
Fig. 5: Movement of minima with changing $\theta_0$ in the $N = 1$ spurion model.

Fig. 6: a) $\theta^{(w)}_{eff}$ as a function of $\theta_0$ at the minima of the potential in the $N = 1$ spurion model, b) the discontinuity in $\Delta \theta^{(w)}_{eff}$ at the phase transition as a function of the soft breaking $f_m$. 
Note that in the above discussion we have taken \( f_m \) real. In the case of \( f_m \) complex the phase transition can occur at \( \theta_0 \neq \pi \). For example, when \( \theta_0 \) is purely imaginary the minima are exactly degenerate at \( \theta_0 = 0 \), and the phase transition occurs as \( \theta_0 \) passes through zero.

### 3.2 \( \theta_0 \) Dependence for the \( N = 2 \) Spurion

For the \( N=2 \) spurion we may plot the potential of section 2.3.2 for varying \( \theta_0 \). The effective theory again has a first order phase transition at \( \theta_{\text{eff}} = \pi \). We demonstrate this in fig. 7 by showing the potential for varying \( \theta_0 \). Again, the transition is a result of the breaking of the \( Z_2 \) symmetry of the model and the fact that the effect of changing \( \theta_0 \) is effectively to rotate the \( u \) plane by \( \theta_0/2 \). The bare theory is \( 2\pi \) periodic in \( \theta_0 \) so the singularity with a massless (1,1) dyon (at \( \theta_0 = 0 \)) must become the new minima of the theory at \( \theta = 2\pi \). It is interesting to plot the region in which there is a monopole condensate for varying \( \theta_0 \) since the single region at \( \theta_0 = 0 \) must transmute to the two regions of condensation, observed for the dyon at \( \theta_0 = 0 \), as \( \theta_0 \) changes to \( 2\pi \). We show this transformation in fig. 8.

![Potential for N=2 spurion with \( \theta_0 = 0, \pi/2, 3\pi/2, \) and \( 2\pi \), showing the first order phase transition.](image.png)
Fig. 8: Evolution of the area of moduli space with monopole condensate as $\theta_0$ changes from 0 to $2\pi$

It can be shown again analytically that there are two degenerate minima at $\theta = \pi$ in the effective theory. As in the $N = 1$ spurion case the extra terms induced by the soft breaking parameter $F_0$ in (29) have a $u \rightarrow u^*$ symmetry. At $\theta_0 = \pi$ the $Z_2$ symmetry of the $N = 2$ potential also takes the form $u \rightarrow u^*$ and the whole potential is invariant under this transformation. Any minima will therefore be replicated at $u^*$.

In fig. 9a, b we show the behavior of $\theta_{\text{eff}}$ as $\theta_0$ is varied. The first order phase transition at $\theta = \pi$ is again apparent. Note the discontinuity in $\theta_{\text{eff}}$ as $\theta_0$ passes through $\pi$. This discontinuity grows with the size of the soft breaking, as shown in fig. 9b.

![Fig. 9](image)

**a)** $\theta^{(w)}_{\text{eff}}$ at the minimum as a function of $\theta_0$ for $F_0/\Lambda = 0.5$; **b)** the discontinuity in $\theta^{(w)}_{\text{eff}}$ at the phase transition as a function of the soft breaking parameter $F_0/\Lambda$.

4 Conclusions

In this paper we studied the effects of a non-zero $\theta_0$ angle on softly broken $N = 2$ SQCD. The soft breakings were necessary because in their absence the $\theta_0$ parameter is unphysical and can be rotated away through an anomalous $U(1)_R$ symmetry. We studied two types of controllable breakings, resulting from $N = 1$ and $N = 2$ supersymmetric spurion analysis.
The general phenomena we observed is illustrated in fig. 10 below. Once the soft breakings are added, the degeneracy of the $N = 2$ moduli space is lifted and the theory is pinned at a global minimum near one of the singular points at $u_{\pm} = \pm \Lambda^2$. Since the $Z_2$ symmetry is broken by the perturbation, the energy at one of the singular points is slightly higher than at the other. To see the origin of this dependence on $\theta_0$, one notes that $\Lambda^2$, regarded as a spurion field itself, has a non-trivial dependence on $\theta_0$ through the beta function (see equation (37)). As $\theta_0$ is varied from zero to $2\pi$, there is a first order phase transition at a critical value of $\theta_0$ in which the minimum jumps from near one singular point to the other. In the $N = 2$ spurion case, as well as in the $N = 1$ case with $f_m$ real, the critical value is $\theta_0 = \pi$.

Since the effective theta angle is a function of $\theta_0$ the mode of confinement changes with $\theta_0$; it occurs by the condensation of dyons with charges

$$\left(q_e, q_m\right) = \left(\frac{\theta^{(w)}_{\text{eff}}}{2\pi}, 1\right), \left(-\frac{\theta^{(w)}_{\text{eff}}}{2\pi} + 1, 1\right).$$

(37)

For small values of $\Delta\theta^{(w)}_{\text{eff}}$, the discontinuity in $\theta^{(w)}_{\text{eff}}$ between the two phases, this means that the transition at $\theta_0 = \pi$ is between phases with condensates of charges approximately $(0, 1)$ and $(1, 1)$. For $\Delta\theta_{\text{eff}} = 0$, the two phases are related by T-duality (10). However, for non-zero $\Delta\theta_{\text{eff}}$ the phases are physically distinct. Using the properties of hypergeometric functions, one can also show for both the $N = 1$ and $N = 2$ spurion models that when $\theta_0 = \pi$,

$$\theta^{(w)}_{\text{eff}}(u) + \theta^{(w)}_{\text{eff}}(u^*) = -2\pi.$$

(38)

In the context of QCD, 't Hooft [1] suggested that two distinct types electrically and magnetically charged dyons could co-exist at $\theta = \pi$ leading him to propose that a condensate of a bound state of the two dyons would be energetically preferred, leading to ‘oblique confinement’. In the models investigated here though, for small soft breaking, the two regions with different dyon condensates are well separated on the $u$ plane and hence oblique confinement is not possible. However, in [4], it has been argued that in the QCD limit, $|F_0| \rightarrow \infty$, the vacuum exhibits simultaneous condensates of both types of excitations (37).

Fig.10: Movement of minima with changing $\theta_0$ in the $N = 1, 2$ spurion models.
The relation (38) then implies that when $\theta_0 = \pi$ the two types of dyons can form bound states of charge $(0, 2)$ (i.e. purely magnetically charged). This phenomena is reminiscent of oblique confinement.

Although we have concentrated on $N = 2$ SQCD without matter fields in the fundamental representation (i.e. $N_f = 0$), it is also possible to address the case of two massless flavors of quarks in the fundamental representation using the same analysis [5, 7]. When $N_f = 2$ and the quarks are massless, the elliptic curve specifying the low-energy solution is identical to that of the $N_f = 0$ case, as long as one rescales the charges so that the $W^\pm$ bosons have charge $\pm 2$ rather than $\pm 1$. Thus, one can describe the physics of the Coulomb phase (i.e. with vanishing fundamental squark vevs) of this model using the results already described in this paper. The main difference is the relation between the scale $\Lambda^2$ and $\theta_0$. By noting the charge rescaling, or by integrating the beta function $b_0 = 2N_c - N_f$, one obtains

$$\Lambda^2 \sim e^{i\theta_0}.$$  

This implies that as $\theta_0$ is varied from zero to $2\pi$, the $N_f = 2$ model undergoes two first order phase transitions, at $\theta_0 = \pi/2, 3\pi/2$.

Finally, we make some speculative remarks concerning the QCD limit, which requires taking the soft breakings to be large compared to $\Lambda$. In this limit the minimum of the potential must approach $u \simeq 0$, apparently forcing $\theta_{\text{eff}} \to \pi$ regardless of the value of $\theta_0$. This is interesting because the effective theta angle felt by the dyonic excitations is then independent of the initial value of the bare theta angle. Unfortunately, things are somewhat more complicated than this because there may be additional corrections to the effective theta angle which come from higher dimension D-terms such as $\int d^4\theta F_0^+ F_0/\Lambda^2 (DW)^2$. These are of course unsuppressed in the QCD limit, so we do not know precisely how the low-energy theta angle behaves as we continue to increase the soft breakings. However, the complicated relation between $\theta_{\text{eff}}$ and $\theta_0$ exhibited in the models studied here suggests that something similar, and more subtle than usually imagined, may also occur in QCD.
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References

[1] G. ‘t Hooft, Phys. Scr. 24, 841 (1981); Phys. Scr. 25, 133 (1981); Nucl. Phys. B190, 455 (1981).

[2] M. Creutz, Phys. Rev. D52, 2951 (1995);
N. Evans, S. D.H. Hsu, A. Nyffler and M. Schwetz (in preparation).

[3] G. Schierholz, Theta vacua, confinement and the continuum limit, Talk given at LATTICE 94: 12th International Symposium on Lattice Field Theory, Bielefeld, Germany, 27 Sep - 1 Oct 1994. Nucl. Phys. B, Proc. Suppl. 42, 270 (1995).

[4] N. Seiberg and E. Witten, Nucl. Phys. B426, 19 (1994).

[5] N. Seiberg and E. Witten, Nucl. Phys. B431, 484 (1994).

[6] N. Evans, S. D.H. Hsu and M. Schwetz, Phys. Lett. B355, 475 (1995); N. Evans, S. D.H. Hsu, M. Schwetz and S. B. Selipsky, Nucl. Phys. B456, 205 (1995).

[7] L. Álvarez-Gaumé, J. Distler, C. Kounnas and M. Mariño, hep-th/9604004;
L. Álvarez-Gaumé and M. Mariño, hep-th/9606191.

[8] G. ‘t Hooft, in: Proc. Europ. Phys. Soc. Conf. on High Energy Physics (1975) p.1225;
S. Mandelstam, Phys. Rev. D19, 2391 (1979).

[9] E. Witten, Phys.Lett. B86, 283 (1979).

[10] M. Di Pierro and K. Konishi, hep-th/9605178.

[11] M. Matone, Phys.Lett. B357, 342, (1995).