The $q$-Onsager algebra and its alternating central extension

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Abstract

The $q$-Onsager algebra $O_q$ has a presentation involving two generators $W_0, W_1$ and two relations, called the $q$-Dolan/Grady relations. The alternating central extension $\tilde{O}_q$ has a presentation involving the alternating generators $\{W_k\}_{k=0}^\infty, \{\check{W}_k\}_{k=0}^\infty$, $\{G_{k+1}\}_{k=0}^\infty, \{\check{G}_{k+1}\}_{k=0}^\infty$ and a large number of relations. Let $\langle W_0, W_1 \rangle$ denote the subalgebra of $O_q$ generated by $W_0, W_1$. It is known that there exists an algebra isomorphism $O_q \to \langle W_0, W_1 \rangle$ that sends $W_0 \mapsto W_0$ and $W_1 \mapsto W_1$. It is known that the center $Z$ of $O_q$ is isomorphic to a polynomial algebra in countably many variables. It is known that the multiplication map $\langle W_0, W_1 \rangle \otimes Z \to O_q, w \otimes z \mapsto wz$ is an isomorphism of algebras. We call this isomorphism the standard tensor product factorization of $O_q$. In the study of $O_q$ there are two natural points of view: we can start with the alternating generators, or we can start with the standard tensor product factorization. It is not obvious how these two points of view are related. The goal of the paper is to describe this relationship. We give seven main results; the principal one is an attractive factorization of the generating function for some algebraically independent elements that generate $Z$.

Keywords. $q$-Onsager algebra; $q$-Dolan/Grady relations; alternating central extension.

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1 Introduction

This paper is part of a sequence [14], [15], [16] concerning the $q$-Onsager algebra and its alternating central extension. We refer to those papers for background information and historical remarks. Let us recall a few main points. The $q$-Onsager algebra $O_q$ is associative and infinite-dimensional. It has a presentation with two generators $W_0, W_1$ and two relations, called the $q$-Dolan/Grady relations:

$$[W_0, [W_0, [W_0, W_1]_q]_q^{-1}] = (q^2 - q^{-2})^2[W_1, W_0],$$
$$[W_1, [W_1, [W_1, W_0]_q]_q^{-1}] = (q^2 - q^{-2})^2[W_0, W_1].$$

In [1] Theorem 4.5], Baseilhac and Kolb obtain a Poincaré-Birkhoff-Witt (or PBW) basis for $O_q$. They obtain this PBW basis by using an approach of Damiani [6] along with two
automorphisms of $O_q$ that are reminiscent of the Lusztig automorphisms for $U_q(\hat{\mathfrak{sl}}_2)$. The PBW basis elements are denoted

$$\{B_{n\delta+\alpha_0}\}_{n=0}^\infty, \quad \{B_{n\delta+\alpha_1}\}_{n=0}^\infty, \quad \{B_{n\delta}\}_{n=1}^\infty.$$  \hfill (1)

We will be discussing the generating functions

$$B^-(t) = \sum_{n=0}^\infty B_{n\delta+\alpha_0} t^n, \quad B^+(t) = \sum_{n=0}^\infty B_{n\delta+\alpha_1} t^n,$$

$$B(t) = \sum_{n=0}^\infty B_{n\delta} t^n, \quad B_{0\delta} = q^{-2} - 1.$$

In [3] Baseilhac and Koizumi introduce a current algebra for $O_q$, in order to solve boundary integrable systems with hidden symmetries. Following [16] we denote this current algebra by $O_q$. In [5, Definition 3.1] Baseilhac and Shigechi give a presentation of $O_q$ by generators and relations. The generators, said to be alternating, are denoted

$$\{W_{-k}\}_{k=0}^\infty, \quad \{W_{k+1}\}_{k=0}^\infty, \quad \{G_{k+1}\}_{k=0}^\infty, \quad \{\tilde{G}_{k+1}\}_{k=0}^\infty.$$

The relations are given in (50)–(60) below. The alternating generators form a PBW basis for $O_q$, see [15, Theorem 6.1] or Proposition 6.3 below. We will be discussing the generating functions

$$W^-(t) = \sum_{n=0}^\infty W_{-n} t^n, \quad W^+(t) = \sum_{n=0}^\infty W_{n+1} t^n,$$

$$G(t) = \sum_{n=0}^\infty G_n t^n, \quad \tilde{G}(t) = \sum_{n=0}^\infty \tilde{G}_n t^n, \quad G_0 = \tilde{G}_0 = -(q - q^{-1})[2]_q^2.$$

Next we describe how $O_q$ and $Q_q$ are related. In this description we will refer to the polynomial algebra $\mathbb{F}[z_1, z_2, \ldots]$, where $\mathbb{F}$ is the ground field and $\{z_n\}_{n=1}^\infty$ are mutually commuting indeterminates. For notational convenience define $z_0 = 1$.

The algebras $O_q$ and $Q_q$ are related as follows.

- Let $\langle W_0, W_1 \rangle$ denote the subalgebra of $O_q$ generated by $W_0, W_1$. By [15, Theorem 10.3] there exists an algebra isomorphism $O_q \rightarrow \langle W_0, W_1 \rangle$ that sends $W_0 \mapsto W_0$ and $W_1 \mapsto W_1$.

- By [15, Theorem 10.2] the center $Z$ of $O_q$ is isomorphic to $\mathbb{F}[z_1, z_2, \ldots]$.

- By [15, Theorem 10.4] the multiplication map

$$\langle W_0, W_1 \rangle \otimes Z \rightarrow O_q$$

$$w \otimes z \mapsto wz$$  \hfill (2)

is an isomorphism of algebras.
By the above bullets points or \[15\] Theorem 9.14, the algebra \(O_q\) is isomorphic to \(O_q \otimes \mathbb{F}[z_1, z_2, \ldots]\). Motivated by this and as explained in \[15\], we call \(O_q\) the alternating central extension of \(O_q\). We call \(\otimes\) the standard tensor product factorization of \(O_q\).

For the rest of this section, we identify the algebra \(O_q\) with \(\langle \mathcal{W}_0, \mathcal{W}_1 \rangle\) via the isomorphism in the first bullet point above.

In the study of \(O_q\) there are two natural points of view: we can start with the alternating generators, or we can start with the standard tensor product factorization. It is not obvious how these two points of view are related. The goal of the paper is to describe this relationship.

In this paper we obtain seven main results, that are summarized as follows.

Our first and second main results relate the generating functions for \(O_q\) and \(O_q\). We show that

\[
\frac{q + q^{-1}}{t + t^{-1}} W^-(\frac{q + q^{-1}}{t + t^{-1}}) = \frac{q^{-1} t B^+(q^{-1} t) + B^-(q t)}{(q^2 - q^{-2})(t - t^{-1})} \tilde{G} \left(\frac{q + q^{-1}}{t + t^{-1}}\right),
\]

\[
\frac{q + q^{-1}}{t + t^{-1}} W^+(\frac{q + q^{-1}}{t + t^{-1}}) = \frac{B^+(q^{-1} t) + q t B^-(q t)}{(q^2 - q^{-2})(t - t^{-1})} \tilde{G} \left(\frac{q + q^{-1}}{t + t^{-1}}\right),
\]

and also

\[
\frac{q + q^{-1}}{t + t^{-1}} W^-(\frac{q + q^{-1}}{t + t^{-1}}) = \tilde{G} \left(\frac{q + q^{-1}}{t + t^{-1}}\right) \frac{q t B^+(q t) + B^-(q^{-1} t)}{(q^2 - q^{-2})(t - t^{-1})},
\]

\[
\frac{q + q^{-1}}{t + t^{-1}} W^+(\frac{q + q^{-1}}{t + t^{-1}}) = \tilde{G} \left(\frac{q + q^{-1}}{t + t^{-1}}\right) \frac{B^+(q t) + q^{-1} t B^-(q^{-1} t)}{(q^2 - q^{-2})(t - t^{-1})}.
\]

Our third main result concerns some elements \(\{Z_n^\vee\}_{n=1}^\infty\) from \[15\] Theorem 10.2 that are algebraically independent and generate \(Z\). By \[15\] Definitions 8.4, 8.6, the generating function

\[
Z^\vee(t) = \sum_{n=0}^\infty Z_n^\vee t^n, \quad Z_0^\vee = 1
\]

satisfies \(Z^\vee(t) = (q + q^{-1})^{-2} \Psi(t)\), where

\[
\Psi(t) = t^{-1} S T W^-(S) W^+(T) + t S T W^+(S) W^-(T) - q^2 S T W^-(S) W^-(T)
- q^{-2} S T W^+(S) W^+(T) + (q^2 - q^{-2})^{-2} \tilde{G}(S) \tilde{G}(T)
\]

and

\[
S = \frac{q + q^{-1}}{q^{-1} t + q t^{-1}}, \quad T = \frac{q + q^{-1}}{q t + q^{-1} t^{-1}}.
\]  

We obtain a factorization

\[
Z^\vee(t) = \xi \tilde{G}(S) B(t) \tilde{G}(T), \quad \xi = -q(q - q^{-1})(q^2 - q^{-2})^{-4}.
\]
In our fourth main result, we introduce the alternating generators for $O_q$. These generators are denoted

$$\{W_{-k}\}_{k=0}^{\infty}, \quad \{W_{k+1}\}_{k=0}^{\infty}, \quad \{G_{k+1}\}_{k=0}^{\infty}, \quad \{\tilde{G}_{k+1}\}_{k=0}^{\infty}$$

and defined as follows. We display a surjective algebra homomorphism $\gamma : O_q \to O_q$ that sends

$$W_0 \mapsto W_0, \quad W_1 \mapsto W_1, \quad \mathcal{Z}_{n}^{\vee} \mapsto 0, \quad n \geq 1.$$ 

The map $\gamma$ sends

$$W_{-k} \mapsto W_{-k}, \quad W_{k+1} \mapsto W_{k+1}, \quad G_{k+1} \mapsto G_{k+1}, \quad \tilde{G}_{k+1} \mapsto \tilde{G}_{k+1}$$

for $k \in \mathbb{N}$. The existence of $\gamma$ was previously conjectured by Baseilhac and Belliard in [2, Conjecture 2].

In our fifth main result, we prove [14, Conjecture 6.2]. The essential ingredients in our proof are the factorization [11] and the existence of the alternating generators of $O_q$.

In our sixth main result, we establish an algebra isomorphism $\varphi : O_q \to O_q \otimes \mathbb{F}[z_1, z_2, \ldots]$ that sends

$$W_{-n} \mapsto \sum_{k=0}^{n} W_{k-n} \otimes z_k, \quad W_{n+1} \mapsto \sum_{k=0}^{n} W_{n+1-k} \otimes z_k,$$

$$G_n \mapsto \sum_{k=0}^{n} G_{n-k} \otimes z_k, \quad \tilde{G}_n \mapsto \sum_{k=0}^{n} \tilde{G}_{n-k} \otimes z_k$$

for $n \in \mathbb{N}$. In particular $\varphi$ sends

$$W_0 \mapsto W_0 \otimes 1, \quad W_1 \mapsto W_1 \otimes 1.$$ 

For $n \in \mathbb{N}$ let $\mathcal{Z}_n$ denote the preimage of $1 \otimes z_n$ under $\varphi$. We have $\mathcal{Z}_0 = 1$. By construction the elements $\{\mathcal{Z}_n\}_{n=1}^{\infty}$ are algebraically independent and generate $\mathcal{Z}$.

In our seventh main result, we show that the elements $\{\mathcal{Z}_n\}_{n=1}^{\infty}$ are related to the elements $\{\mathcal{Z}_n^{\vee}\}_{n=1}^{\infty}$ in the following way. The generating function $Z(t) = \sum_{n=0}^{\infty} \mathcal{Z}_n t^n$ satisfies

$$Z^{\vee}(t) = Z(S)Z(T),$$

where $S, T$ are from (3).

Our seven main results are contained in Theorems 9.8, 9.9, 10.1, 11.6, 12.4, 13.5, 13.14.

We remark that [11], [12] contain analogs of the above results that apply to the alternating central extension for the positive part of $U_q(\hat{sl}_2)$; see also [13].

At the end of the paper we give some conjectures and open problems.

The paper is organized as follows. Section 2 contains some preliminaries. In Section 3 we give the definition and basic properties of $O_q$. In Sections 4, 5 we describe the PBW
basis for $O_q$ due to Baseilhac and Kolb. In Sections 6, 7 we give the definition and basic properties of $O_q$. In Section 8 we recall an algebra isomorphism $\phi : O_q \otimes F[z_1, z_2, \ldots] \to O_q$ and the generating function $Z(t)$. In Section 9 we compare our generating functions for $O_q$ and $O_q$. In Section 10 we obtain our factorization of $Z(t)$. In Section 11 we introduce the algebra homomorphism $\gamma : O_q \to O_q$ and use it to obtain the alternating generators of $O_q$. In Section 12 we use the factorization of $Z(t)$ and the alternating generators of $O_q$ to prove [14, Conjecture 6.2]. In Section 13 we obtain the algebra isomorphism $\phi : O_q \to O_q \otimes F[z_1, z_2, \ldots]$ and describe how it is related to the inverse of $\phi$. We also obtain the generating function $Z(t)$ and describe how it is related to $Z(t)$. In Sections 14, 15 we tie up some loose ends from earlier sections. In Section 16 we give some conjectures and open problems. In Appendix A we describe some features of the polynomial algebra $F[z_1, z_2, \ldots]$ that are used in the main body of the paper.

2 Preliminaries

We now begin our formal argument. Throughout the paper, the following notational conventions are in effect. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Let $F$ denote a field. Every vector space and tensor product discussed in this paper is over $F$. Every algebra discussed in this paper is associative, over $F$, and has a multiplicative identity. A subalgebra has the same multiplicative identity as the parent algebra. Let $\mathcal{A}$ denote an algebra. By an automorphism of $\mathcal{A}$ we mean an algebra isomorphism $\mathcal{A} \to \mathcal{A}$. The algebra $\mathcal{A}^{\text{opp}}$ consists of the vector space $\mathcal{A}$ and the multiplication map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $(a, b) \to ba$. By an antiautomorphism of $\mathcal{A}$ we mean an algebra isomorphism $\mathcal{A} \to \mathcal{A}^{\text{opp}}$.

Throughout the paper, let $s$ and $t$ denote commuting indeterminates.

**Definition 2.1.** (See [3, p. 299].) Let $\mathcal{A}$ denote an algebra. A Poincaré-Birkhoff-Witt (or PBW) basis for $\mathcal{A}$ consists of a subset $\Omega \subseteq \mathcal{A}$ and a linear order $<$ on $\Omega$ such that the following is a basis for the vector space $\mathcal{A}$:

$$a_1a_2\cdots a_n \quad n \in \mathbb{N}, \quad a_1, a_2, \ldots, a_n \in \Omega, \quad a_1 \leq a_2 \leq \cdots \leq a_n.$$  

We interpret the empty product as the multiplicative identity in $\mathcal{A}$.

**Definition 2.2.** Let $\{z_n\}_{n=1}^{\infty}$ denote mutually commuting indeterminates. Let $F[z_1, z_2, \ldots]$ denote the algebra consisting of the polynomials in $z_1, z_2, \ldots$ that have all coefficients in $F$. For notational convenience, define $z_0 = 1$.

Some features of $F[z_1, z_2, \ldots]$ are explained in Appendix A.

Throughout the paper, we fix a nonzero $q \in F$ that is not a root of unity. Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n \in \mathbb{N}.$$  

5
3 The \(q\)-Onsager algebra \(O_q\)

In this section we recall the \(q\)-Onsager algebra \(O_q\).

For elements \(X, Y\) in any algebra, define their commutator and \(q\)-commutator by
\[
[X, Y] = XY - YX, \quad [X, Y]_q = qXY - q^{-1}YX.
\]

Note that
\[
[X, [X, [X, Y]_q]_{q^{-1}}] = X^3Y - [3]_qX^2YX + [3]_qXYX^2 - YX^3.
\]

**Definition 3.1.** (See [1, Section 2], [8, Definition 3.9].) Define the algebra \(O_q\) by generators \(W_0, W_1\) and relations
\[
[W_0, [W_0, [W_0, W_1]_q]_{q^{-1}}] = (q^2 - q^{-2})^2[W_1, W_0], \quad (5)
\]
\[
[W_1, [W_1, [W_1, W_0]_q]_{q^{-1}}] = (q^2 - q^{-2})^2[W_0, W_1]. \quad (6)
\]

We call \(O_q\) the \(q\)-Onsager algebra. The relations (5), (6) are called the \(q\)-Dolan/Grady relations.

**Remark 3.2.** In [4] Baseilhac and Kolb define the \(q\)-Onsager algebra in a slightly more general way that involves two scalar parameters \(c, q\). Our \(O_q\) is their \(q\)-Onsager algebra with \(c = q^{-1}(q - q^{-1})^2\).

We mention some symmetries of \(O_q\).

**Lemma 3.3.** There exists an automorphism \(\sigma\) of \(O_q\) that sends \(W_0 \leftrightarrow W_1\). Moreover \(\sigma^2 = \text{id}\), where \(\text{id}\) denotes the identity map.

**Lemma 3.4.** (See [10, Lemma 2.5].) There exists an antiautomorphism \(\dagger\) of \(O_q\) that fixes each of \(W_0, W_1\). Moreover \(\dagger^2 = \text{id}\).

**Lemma 3.5.** (See [16, Lemma 3.5].) The maps \(\sigma, \dagger\) commute.

**Definition 3.6.** Let \(\tau\) denote the composition of \(\sigma\) and \(\dagger\). Note that \(\tau\) is an antiautomorphism of \(O_q\) that sends \(W_0 \leftrightarrow W_1\). We have \(\tau^2 = \text{id}\).

Later in the paper we will make use of the following map.

**Lemma 3.7.** There exists an algebra homomorphism \(\vartheta: O_q \rightarrow F\) that sends
\[
W_0 \mapsto 0, \quad W_1 \mapsto 0.
\]

**Proof.** The \(q\)-Dolan/Grady relations (5), (6) hold if we set \(W_0 = 0\) and \(W_1 = 0\).
4 A PBW basis for $O_q$

In [4], Baseilhac and Kolb obtain a PBW basis for $O_q$ that involves some elements

$$\{B_{n\delta+\alpha_0}\}_{n=0}^{\infty}, \quad \{B_{n\delta+\alpha_1}\}_{n=0}^{\infty}, \quad \{B_{n\delta}\}_{n=1}^{\infty}. \quad (7)$$

These elements are recursively defined as follows. Writing $B_{\delta} = q^{-2}W_1W_0 - W_0W_1$ we have

$$B_{\alpha_0} = W_0, \quad B_{\delta+\alpha_0} = W_1 + \frac{q[B_{\delta}, W_0]}{(q - q^{-1})(q^2 - q^{-2})}, \quad (8)$$

$$B_{n\delta+\alpha_0} = B_{(n-2)\delta+\alpha_0} + \frac{q[B_{\delta}, B_{(n-1)\delta+\alpha_0}]}{(q - q^{-1})(q^2 - q^{-2})} \quad n \geq 2 \quad (9)$$

and

$$B_{\alpha_1} = W_1, \quad B_{\delta+\alpha_1} = W_0 - \frac{q[B_{\delta}, W_1]}{(q - q^{-1})(q^2 - q^{-2})}, \quad (10)$$

$$B_{n\delta+\alpha_1} = B_{(n-2)\delta+\alpha_1} - \frac{q[B_{\delta}, B_{(n-1)\delta+\alpha_1}]}{(q - q^{-1})(q^2 - q^{-2})} \quad n \geq 2. \quad (11)$$

Moreover for $n \geq 1$,

$$B_{n\delta} = q^{-2}B_{(n-1)\delta+\alpha_1}W_0 - W_0B_{(n-1)\delta+\alpha_1} + (q^{-2} - 1) \sum_{\ell=0}^{n-2} B_{\delta+\alpha_1}B_{(n-\ell-2)\delta+\alpha_1}. \quad (12)$$

By [4, Proposition 5.12] the elements $\{B_{n\delta}\}_{n=1}^{\infty}$ mutually commute.

**Lemma 4.1.** (See [4, Theorem 4.5].) Assume that $q$ is transcendental over $\mathbb{F}$. Then a PBW basis for $O_q$ is obtained by the elements (7) in any linear order.

We mention a variation on the formula (12). By [4, Section 5.2] the following holds for $n \geq 1$:

$$B_{n\delta} = q^{-2}W_1B_{(n-1)\delta+\alpha_0} - B_{(n-1)\delta+\alpha_0}W_1 + (q^{-2} - 1) \sum_{\ell=0}^{n-2} B_{\delta+\alpha_0}B_{(n-\ell-2)\delta+\alpha_0}. \quad (13)$$

Recall the antiautomorphism $\tau$ of $O_q$, from Definition 3.6.

**Lemma 4.2.** The antiautomorphism $\tau$ sends $B_{n\delta+\alpha_0} \leftrightarrow B_{n\delta+\alpha_1}$ for $n \in \mathbb{N}$, and fixes $B_{n\delta}$ for $n \geq 1$.

**Proof.** The first assertion is verified by comparing (8), (9) with (10), (11). The second assertion is verified by comparing (12), (13). \hfill \Box

**Lemma 4.3.** The algebra homomorphism $\vartheta : O_q \rightarrow \mathbb{F}$ from Lemma 3.7 sends $B_{n\delta+\alpha_0} \mapsto 0$ and $B_{n\delta+\alpha_1} \mapsto 0$ for $n \in \mathbb{N}$, and $B_{n\delta} \mapsto 0$ for $n \geq 1$.

**Proof.** The map $\vartheta$ sends $B_{\delta} \mapsto 0$ by Lemma 3.7 and since $B_{\delta} = q^{-2}W_1W_0 - W_0W_1$. The remaining assertions follow from (8)–(12). \hfill \Box
5 Generating functions for $O_q$

In this section, we describe the elements (7) using generating functions. The following definition is for notational convenience.

**Definition 5.1.** For an integer $n \leq 0$ define

$$B_{n\delta} = \begin{cases} 0, & \text{if } n < 0; \\ q^{-2} - 1, & \text{if } n = 0. \end{cases}$$

**Definition 5.2.** We define some generating functions in the indeterminate $t$:

$$B^-(t) = \sum_{n \in \mathbb{N}} B_{n\delta + \alpha_0} t^n, \quad B^+(t) = \sum_{n \in \mathbb{N}} B_{n\delta + \alpha_1} t^n, \quad (14)$$

$$B(t) = \sum_{n \in \mathbb{N}} B_{n\delta} t^n. \quad (15)$$

Observe that

$$B^-(0) = W_0, \quad B^+(0) = W_1, \quad B(0) = q^{-2} - 1. \quad (16)$$

**Lemma 5.3.** For the algebra $O_q$,

$$\frac{q[B_\delta, B^-(t)]}{(q - q^{-1})(q^2 - q^{-2})} = (t^{-1} - t)B^-(t) - t^{-1}W_0 - W_1, \quad (17)$$

$$\frac{q[B^+(t), B_\delta]}{(q - q^{-1})(q^2 - q^{-2})} = (t^{-1} - t)B^+(t) - W_0 - t^{-1}W_1. \quad (18)$$

**Proof.** Equation (17) expresses (8), (9) in terms of generating functions. Equation (18) expresses (10), (11) in terms of generating functions. □

**Lemma 5.4.** For the algebra $O_q$,

$$[W_0, B^+(t)]_q = -(q - q^{-1})t(B^+(t))^2 - qt^{-1}B(t) - (q - q^{-1})t^{-1}, \quad (19)$$

$$[B^-(t), W_1]_q = -(q - q^{-1})t(B^-(t))^2 - qt^{-1}B(t) - (q - q^{-1})t^{-1}. \quad (20)$$

**Proof.** The relation (19) (resp. (20)) expresses the relation (12) (resp. (13)) in terms of generating functions. □

The following notation will be useful. For an integer $k < 0$ define

$$B_{k\delta + \alpha_0} = B_{(-k-1)\delta + \alpha_1}, \quad B_{k\delta + \alpha_1} = B_{(-k-1)\delta + \alpha_0}.$$  

Note that

$$B_{k\delta + \alpha_0} = B_{\ell\delta + \alpha_1} \quad (k, \ell \in \mathbb{Z}, \ k + \ell = -1). \quad (21)$$
Proposition 5.5. (See [7, Proposition 2.6].) For \( k, \ell \in \mathbb{Z} \) we have
\[
q^{-1}[B_{\ell \delta + \alpha_1}, B_{(k+1)\delta + \alpha_1}] + q^{-1}[B_{k\delta + \alpha_1}, B_{(\ell+1)\delta + \alpha_1}]
= B_{(k-\ell-1)\delta} - B_{(k-\ell+1)\delta} + B_{(\ell-k-1)\delta} - B_{(\ell-k+1)\delta}.
\]

Proof. This follows from [7, Proposition 2.6] using Remark 5.7 below. \( \square \)

Proposition 5.6. (See [7, Proposition 2.8].) For \( m \in \mathbb{N} \) and \( \ell \in \mathbb{Z} \),
\[
[B_{(m+1)\delta}, B_{\ell \delta + \alpha_1}] - [B_{\ell \delta + \alpha_1}, B_{(m-1)\delta}] = [B_{m\delta}, B_{(\ell+1)\delta + \alpha_1}]q - [B_{(\ell-1)\delta + \alpha_1}, B_{m\delta}]q^2.
\]

Proof. This follows from [7, Proposition 2.8] using Remark 5.7 below. \( \square \)

Remark 5.7. The notation of [7] corresponds to ours in the following way.

| our notation | notation of [7] |
|--------------|------------------|
| \( q \) | \( v \) |
| \( W_0 \) | \( v^{1/2}(v - v^{-1})B_0 \) |
| \( W_1 \) | \( v^{1/2}(v - v^{-1})B_1 \) |
| \( B_{n\delta + \alpha_0} \) | \( v^{1/2}(v - v^{-1})B_{-1,n} \) |
| \( B_{n\delta + \alpha_1} \) | \( v^{1/2}(v - v^{-1})B_{1,n} \) |
| \( B_{n\delta} \) | \( -v^{-1}(v - v^{-1})^2 \Theta' \) |
| \( 1, 1 \) | \( \mathbb{K}_{n\delta + \alpha_1}, \mathbb{K}_\delta \) |
| \( [X, Y]_q \) | \( v[X, Y]_{v-2} \) |

Recall the commuting indeterminates \( s, t \). By the comment above Lemma 4.1, \( [B(s), B(t)] = 0 \).

Proposition 5.8. For the algebra \( O_q \),
\[
0 = (qs - q^{-1}t)B^+(s)B^+(t) + (qt - q^{-1}s)B^+(t)B^+(s) - (q - q^{-1})t(B^+(t))^2
- (q - q^{-1})s(B^+(s))^2 + \frac{q(s - t)}{1 - st}B(t) + \frac{q(t - s)}{1 - st}B(s),
\]  

(22)

\[
0 = (1 - q^{-2}st)B^-(s)B^+(t) + (st - q^{-2})B^+(t)B^-(s) + (1 - q^{-2})t(B^+(t))^2
+ (1 - q^{-2})s(B^-(s))^2 + \frac{1}{1 - st}B(s) - \frac{1}{s - t}B(t),
\]

(23)

\[
0 = (qs - q^{-1}t)B^-(s)B^-(t) + (qt - q^{-1}s)B^-(t)B^-(s) - (q - q^{-1})t(B^-(t))^2
- (q - q^{-1})s(B^-(s))^2 + \frac{q(s - t)}{1 - st}B(t) + \frac{q(t - s)}{1 - st}B(s).
\]

(24)

Proof. We first obtain (22). Observe that
\[
\sum_{k=-1}^{\infty} \sum_{\ell=-1}^{\infty} [B_{\ell \delta + \alpha_1}, B_{(k+1)\delta + \alpha_1}]q^{k\ell} = [t^{-1}W_0 + B^+(t), s^{-1}B^+(s)]q,
\]

(25)

\[
\sum_{k=-1}^{\infty} \sum_{\ell=-1}^{\infty} [B_{k\delta + \alpha_1}, B_{(\ell+1)\delta + \alpha_1}]q^{k\ell} = [s^{-1}W_0 + B^+(s), t^{-1}B^+(t)]q.
\]

(26)
and also
\[
\sum_{k=-1}^{\infty} \sum_{\ell=-1}^{\infty} B_{(k-\ell-1)\delta} s^k t^\ell = B(s) \frac{1}{t} \frac{1}{1-st}, \quad (27)
\]
\[
\sum_{k=-1}^{\infty} \sum_{\ell=-1}^{\infty} B_{(k+\ell+1)\delta} s^k t^\ell = B(s) \frac{1}{s^2 t} \frac{1}{1-st} + \frac{1-q^{-2}}{s^2 t}, \quad (28)
\]
\[
\sum_{k=-1}^{\infty} \sum_{\ell=-1}^{\infty} B_{(\ell-k-1)\delta} s^k t^\ell = B(t) \frac{1}{s} \frac{1}{1-st}, \quad (29)
\]
\[
\sum_{k=-1}^{\infty} \sum_{\ell=-1}^{\infty} B_{(\ell+\ell+1)\delta} s^k t^\ell = B(t) \frac{1}{s t^2} \frac{1}{1-st} + \frac{1-q^{-2}}{s t^2}. \quad (30)
\]
Let \(E\) denote the equation that is \(q^{-1}\) times (25) plus \(q^{-1}\) times (26) minus (27) plus (28) minus (29) plus (30). The left-hand side of \(E\) is equal to zero, by Proposition 5.5. In the right-hand side of \(E\), eliminate \([W_0, B^+(s)]_q\) and \([W_0, B^+(t)]_q\) using (19), and simplify the result. This yields (22). Next we obtain (23). We have a preliminary comment about Proposition 5.5. In that proposition, replace \(k\) by \(-k-2\) and use (21) to obtain
\[
q^{-1} [B_{k\delta+\alpha_1}, B_{k\delta+\alpha_0}]_q + q^{-1} [B_{(k+1)\delta+\alpha_0}, B_{(\ell+1)\delta+\alpha_1}]_q
= B_{-(k+\ell+3)\delta} - B_{-(k+\ell+1)\delta} + B_{(k+\ell+1)\delta} - B_{(k+\ell+3)\delta}
\]
for \(k, \ell \in \mathbb{Z}\). We are done with our preliminary comment. Observe that
\[
\sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} [B_{k\delta+\alpha_1}, B_{k\delta+\alpha_0}]_q s^k t^\ell = [B^+(t), B^-(s)]_q, \quad (31)
\]
\[
\sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} [B_{(k+1)\delta+\alpha_0}, B_{(\ell+1)\delta+\alpha_1}]_q s^k t^\ell = s^{-1} t^{-1} [B^-(s) - W_0, B^+(t) - W_1]_q \quad (32)
\]
and also
\[
\sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} B_{-(k+\ell+3)\delta} s^k t^\ell = 0, \quad (33)
\]
\[
\sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} B_{-(k+\ell+1)\delta} s^k t^\ell = 0, \quad (34)
\]
\[
\sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} B_{(\ell+\ell+1)\delta} s^k t^\ell = \frac{B(s) - B(t)}{s-t}, \quad (35)
\]
\[
\sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} B_{(\ell+\ell+3)\delta} s^k t^\ell = \frac{s^{-2} (B(s) - sB_\delta + 1 - q^{-2}) - t^{-2} (B(t) - tB_\delta + 1 - q^{-2})}{s-t}. \quad (36)
\]
Let \(F\) denote the equation that is \(q^{-1}\) times (31) plus \(q^{-1}\) times (32) minus (33) plus (34) minus (35) plus (36). The left-hand side of \(F\) is equal to zero, by our preliminary comment. In the right-hand side of \(F\), eliminate \([W_0, B^+(t)]_q\) using (19), and \([B^-(s), W_1]_q\) using (20), and \([W_0, W_1]_q\) using \([W_0, W_1]_q = -qB_\delta\). Simplify the result to obtain (23). The equation (24) is obtained (22) by applying the antiautomorphism \(\tau\). \(\square\)
Proposition 5.9. For the algebra $O_q$, 
\[ 0 = B(s)B^+(t)(1 - q^{-2}st)(t - q^2s) - B^+(t)B(s)(1 - q^2st)(t - q^{-2}s) \]
\[ + B(s)B^+(q^{-2}s)(q^2 - q^{-2})(1 - q^{-2}st)s + B^-(q^{-2}s)B(s)(q^2 - q^{-2})(t - q^{-2}s)s, \]
\[ 0 = B^-(t)B(s)(1 - q^{-2}st)(t - q^2s) - B(s)B^-(t)(1 - q^2st)(t - q^{-2}s) \]
\[ + B^-(q^{-2}s)B(s)(q^2 - q^{-2})(1 - q^{-2}st)s + B(s)B^+(q^{-2}s)(q^2 - q^{-2})(t - q^{-2}s)s. \]

Proof. We will use Proposition 5.6. Observe that
\[ \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} [B_{(m+1)\delta}, B_{\ell\delta + \alpha_1}]s^m t^\ell = s^{-1} [B(s), B^+(t)], \quad (37) \]
\[ \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} [B_{\ell\delta + \alpha}, B_{(m-1)\delta}]s^m t^\ell = s [B^+(t), B(s)], \quad (38) \]
\[ \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} [B_{m\delta}, B_{(\ell+1)\delta + \alpha_1}]q^2 s^m t^\ell = t^{-1} [B(s), B^+(t) - W_1]_{q^2}, \quad (39) \]
\[ \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} [B_{(\ell-1)\delta + \alpha_1}, B_{m\delta}]q^2 s^m t^\ell = [W_0 + tB^+(t), B(s)]_{q^2}. \quad (40) \]

Let $E$ denote the equation that is (37) minus (38) minus (39) plus (40). The left-hand side of $E$ is equal to zero, by Proposition 5.6. After some algebraic manipulation, $E$ becomes
\[ 0 = (1 - q^{-2}st)(t - q^2s)B(s)B^+(t) - (1 - q^2st)(t - q^{-2}s)B^+(t)B(s) \]
\[ + s[B(s), W_1]_{q^2} + st[W_0, B(s)]_{q^2}. \quad (41) \]

Setting $t = q^{-2}s$ in (41), we obtain
\[ 0 = (1 - q^{-4}s^2)(q^{-2} - q^2)sB(s)B^+(q^{-2}s) + s[B(s), W_1]_{q^2} + q^{-2}s^2[W_0, B(s)]_{q^2}. \quad (42) \]

Applying the antiautomorphism $\tau$ to the right-hand side of (41), (42) yields
\[ 0 = (1 - q^{-2}st)(t - q^2s)B^-(t)B(s) - (1 - q^2st)(t - q^{-2}s)B(s)B^-(t) \]
\[ + s[W_0, B(s)]_{q^2} + st[B(s), W_1]_{q^2} \quad (43) \]
and
\[ 0 = (1 - q^{-4}s^2)(q^{-2} - q^2)sB^-(q^{-2}s)B(s) + s[W_0, B(s)]_{q^2} + q^{-2}s^2[B(s), W_1]_{q^2}. \quad (44) \]

The equation (41) minus $(1 - q^{-2}st)(1 - q^{-4}s^2)^{-1}$ times (42) minus $(t - q^{-2}s)(1 - q^{-4}s^2)^{-1}$ times (44) gives the first equation in the proposition statement. The equation (43) minus $(t - q^{-2}s)(1 - q^{-4}s^2)^{-1}$ times (42) minus $(1 - q^{-2}st)(1 - q^{-4}s^2)^{-1}$ times (44) gives the second equation in the proposition statement.

Corollary 5.10. For the algebra $O_q$,
\[ [W_0, B^-(t)]_{q} = (q - q^{-1})(B^-(t))^2 + qB(t) + q - q^{-1}, \quad (45) \]
\[ [W_0, B(t)]_{q^2} = (q^2 - q^{-2})B^-(q^{-2}t)B(t) - q^{-2}(q^2 - q^{-2})tB(t)B^+(q^{-2}t), \quad (46) \]
\[ [B^+(t), W_1]_{q} = (q - q^{-1})(B^+(t))^2 + qB(t) + q - q^{-1}, \quad (47) \]
\[ [B(t), W_1]_{q^2} = (q^2 - q^{-2})B(t)B^+(q^{-2}t) - q^{-2}(q^2 - q^{-2})tB^-(q^{-2}t)B(t). \quad (48) \]
Proof. Set $s = 0$ in the first and third equation of Proposition 5.8. Set $t = 0$ in Proposition 5.9. Evaluate the results using (23).

We mention two special cases of (23) for later use.

Corollary 5.11. For the algebra $O_q$,

$$0 = q^{-1}t(qt^{-1} - q^{-1}t)B^-(qt)B^+(q^{-1}t) + q^{-1}t(qt - q^{-1}t)B^+(q^{-1}t)B^-(qt)$$
$$+ t(q - q^{-1})(B^-(qt))^2 + q^{-2}t(q - q^{-1})(B^+(q^{-1}t))^2$$
$$+ t - t^{-1}q^{-1}B(q^{-1}t) - t - t^{-1}B(qt),$$

$$0 = q^{-1}t(qt^{-1} - q^{-1}t)B^-(q^{-1}t)B^+(qt) + q^{-1}t(qt - q^{-1}t)B^+(qt)B^-(q^{-1}t)$$
$$+ t(q - q^{-1})(B^+(qt))^2 + q^{-2}t(q - q^{-1})(B^-(q^{-1}t))^2$$
$$+ t - t^{-1}q^{-1}B(q^{-1}t) - t - t^{-1}B(qt).$$

Proof. To obtain the first (resp. second) displayed equation, set $s = qt'$ and $t = q^{-1}t'$ (resp. $s = q^{-1}t'$ and $t = qt'$) in (23).

6 The algebra $O_q$

In the previous section we discussed the $q$-Onsager algebra $O_q$. In this section we discuss its alternating central extension $O_q$.

Definition 6.1. (See [3], [5], Definition 3.1.) Define the algebra $O_q$ by generators

$$\{W_{-k}\}_{k \in \mathbb{N}}, \quad \{W_{k+1}\}_{k \in \mathbb{N}}, \quad \{G_{k+1}\}_{k \in \mathbb{N}}, \quad \{\tilde{G}_{k+1}\}_{k \in \mathbb{N}}$$

and the following relations. For $k, \ell \in \mathbb{N}$,

$$[W_0, W_{k+1}] = [W_{-k}, W_1] = (\tilde{G}_{k+1} - G_{k+1})/(q + q^{-1}),$$

$$[W_0, G_{k+1}]_q = [\tilde{G}_{k+1}, W_0]_q = \rho W_{-k-1} - \rho W_{k+1},$$

$$[G_{k+1}, W_1]_q = [W_1, \tilde{G}_{k+1}]_q = \rho W_{k+2} - \rho W_{-k},$$

$$[W_{-k}, W_{-\ell}] = 0, \quad [W_{k+1}, W_{\ell+1}] = 0,$$

$$[W_{-k}, W_{k+1}] + [W_{k+1}, W_{-\ell}] = 0,$$

$$[W_{-k}, G_{\ell+1}] + [G_{-\ell}, W_{k+1}] = 0,$$

$$[W_{-k}, \tilde{G}_{\ell+1}] + [\tilde{G}_{-\ell}, W_{k+1}] = 0,$$

$$[W_{k+1}, G_{\ell+1}] + [G_{k+1}, W_{\ell+1}] = 0,$$

$$[G_{k+1}, G_{\ell+1}] = 0, \quad [\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0,$$

$$[\tilde{G}_{k+1}, G_{\ell+1}] + [G_{k+1}, \tilde{G}_{\ell+1}] = 0.$$
In the above equations \( \rho = -(q^2 - q^{-2})^2 \). The generators (19) are called alternating. We call \( \mathcal{O}_q \) the alternating central extension of \( O_q \). For notational convenience define

\[
\mathcal{G}_0 = -(q - q^{-1})[2]_q^2, \quad \tilde{\mathcal{G}}_0 = -(q - q^{-1})[2]_q^2.
\] (61)

**Note 6.2.** In earlier papers [2], [10], [14], [15] the algebra \( \mathcal{O}_q \) is denoted by \( A_q \).

**Proposition 6.3.** (See [15, Theorem 6.1].) A PBW basis for \( \mathcal{O}_q \) is obtained by its alternating generators in any linear order \(<\) such that

\[
\mathcal{G}_{i+1} < \mathcal{W}_{-j} < \mathcal{W}_{k+1} < \tilde{\mathcal{G}}_{\ell+1}, \quad i, j, k, \ell \in \mathbb{N}.
\] (62)

Next we describe some symmetries of \( \mathcal{O}_q \).

**Lemma 6.4.** (See [2, Remark 1].) There exists an automorphism \( \sigma \) of \( \mathcal{O}_q \) that sends

\[
\mathcal{W}_{-k} \mapsto \mathcal{W}_{k+1}, \quad \mathcal{W}_{k+1} \mapsto \mathcal{W}_{-k}, \quad \mathcal{G}_{k+1} \mapsto \tilde{\mathcal{G}}_{k+1}, \quad \tilde{\mathcal{G}}_{k+1} \mapsto \mathcal{G}_{k+1}
\]
for \( k \in \mathbb{N} \). Moreover \( \sigma^2 = \text{id} \).

**Lemma 6.5.** (See [10, Lemma 3.7].) There exists an antiautomorphism \( \dagger \) of \( \mathcal{O}_q \) that sends

\[
\mathcal{W}_{-k} \mapsto \mathcal{W}_{-k}, \quad \mathcal{W}_{k+1} \mapsto \mathcal{W}_{k+1}, \quad \mathcal{G}_{k+1} \mapsto \tilde{\mathcal{G}}_{k+1}, \quad \tilde{\mathcal{G}}_{k+1} \mapsto \mathcal{G}_{k+1}
\]
for \( k \in \mathbb{N} \). Moreover \( \dagger^2 = \text{id} \).

**Lemma 6.6.** (See [16, Lemma 4.6].) The maps \( \sigma, \dagger \) commute.

**Definition 6.7.** Let \( \tau \) denote the composition of the automorphism \( \sigma \) from Lemma 6.4 and the antiautomorphism \( \dagger \) from Lemma 6.5. Note that \( \tau \) is an antiautomorphism of \( \mathcal{O}_q \) that sends

\[
\mathcal{W}_{-k} \mapsto \mathcal{W}_{k+1}, \quad \mathcal{W}_{k+1} \mapsto \mathcal{W}_{-k}, \quad \mathcal{G}_{k+1} \mapsto \mathcal{G}_{k+1}, \quad \tilde{\mathcal{G}}_{k+1} \mapsto \tilde{\mathcal{G}}_{k+1}
\]
for \( k \in \mathbb{N} \). We have \( \tau^2 = \text{id} \).

Next we discuss how \( \mathcal{O}_q \) is related to \( O_q \).

**Lemma 6.8.** (See [15, Theorem 10.3].) There exists an algebra homomorphism \( \imath : O_q \to \mathcal{O}_q \) that sends \( W_0 \mapsto \mathcal{W}_0 \) and \( W_1 \mapsto \mathcal{W}_1 \). Moreover, \( \imath \) is injective.

**Lemma 6.9.** (See [16, Lemma 4.11].) The following diagrams commute:

\[
\begin{array}{ccc}
O_q & \xrightarrow{\imath} & \mathcal{O}_q \\
\downarrow \sigma & & \downarrow \sigma \\
O_q & \xrightarrow{\imath} & \mathcal{O}_q \\
\downarrow \dagger & & \downarrow \dagger \\
O_q & \xrightarrow{\imath} & \mathcal{O}_q \\
\downarrow \tau & & \downarrow \tau \\
O_q & \xrightarrow{\imath} & \mathcal{O}_q
\end{array}
\]
7 Generating functions for $O_q$

In Definition 6.1 the algebra $O_q$ is defined by generators and relations. In this section we describe the defining relations in terms of generating functions.

**Definition 7.1.** We define some generating functions in the indeterminate $t$:  
$$W^-(t) = \sum_{n \in \mathbb{N}} W^-_n t^n, \quad W^+(t) = \sum_{n \in \mathbb{N}} W^+_n t^n,$$
$$G(t) = \sum_{n \in \mathbb{N}} G_n t^n, \quad \bar{G}(t) = \sum_{n \in \mathbb{N}} \bar{G}_n t^n.$$

Observe that
$$W^-(0) = W_0, \quad W^+(0) = W_1, \quad G(0) = -(q - q^{-1})[2]_q^2, \quad \bar{G}(0) = -(q - q^{-1})[2]_q^2.$$

We now give the relations (50)–(60) in terms of generating functions.

**Lemma 7.2.** (See [15, Lemma 3.6].) For the algebra $O_q$ we have
$$[W_0, W^+(t)] = [W^-(t), W_1] = t^{-1}(\bar{G}(t) - G(t))/(q + q^{-1}), \quad (63)$$
$$[W_0, G(t)]_q = [\bar{G}(t), W_0]_q = \rho W^-(t) - \rho t W^+(t), \quad (64)$$
$$[G(t), W_1]_q = [W_1, \bar{G}(t)]_q = \rho W^+(t) - \rho t W^-(t), \quad (65)$$
$$[W^-(s), W^+(t)] = 0, \quad [W^+(s), W^+(t)] = 0, \quad (66)$$
$$s [W^-(s), G(t)] + t [G(s), W^-(t)] = 0, \quad (67)$$
$$s [W^-(s), \bar{G}(t)] + t [\bar{G}(s), W^-(t)] = 0, \quad (68)$$
$$s [W^+(s), G(t)] + t [G(s), W^+(t)] = 0, \quad (69)$$
$$s [W^+(s), \bar{G}(t)] + t [\bar{G}(s), W^+(t)] = 0, \quad (70)$$
$$[G(s), G(t)] = 0, \quad [\bar{G}(s), \bar{G}(t)] = 0, \quad (71)$$
$$[G(s), \bar{G}(t)] = 0, \quad [\bar{G}(s), G(t)] = 0. \quad (72)$$

**Proof.** Use (50)–(60) and Definition 7.1

8 The algebra isomorphism $\phi : O_q \otimes \mathbb{F}[z_1, z_2, \ldots] \rightarrow O_q$

Let $Z$ denote the center of $O_q$. In this section we describe $Z$ from various points of view. Using this description we will obtain an algebra isomorphism $\phi : O_q \otimes \mathbb{F}[z_1, z_2, \ldots] \rightarrow O_q$.

For notational convenience define
$$S = \frac{q + q^{-1}}{q^{-1}t + qt^{-1}}, \quad T = \frac{q + q^{-1}}{qt + q^{-1}t^{-1}}. \quad (74)$$

We view $S$ and $T$ as power series
$$S = (q + q^{-1}) \sum_{\ell \in \mathbb{N}} (-1)^\ell q^{-2\ell - 1} t^{2\ell + 1} \quad T = (q + q^{-1}) \sum_{\ell \in \mathbb{N}} (-1)^\ell q^{2\ell + 1} t^{2\ell + 1}. \quad (74)$$
**Definition 8.1.** (See [15] Definition 8.4.) For the algebra $O_q$ define

$$
\Psi(t) = t^{-1}STW^- (S)W^+ (T) + tSTW^+ (S)W^- (T) - q^2STW^- (S)W^- (T)
- q^{-2}STW^+ (S)W^+ (T) + (q^2 - q^{-2})^{-2}G(S)\tilde{G}(T).
$$

(75)

**Note 8.2.** In [15] Definition 8.4] the generating function $\Psi(t)$ is called $Z(t)$.

The following normalization is sometimes convenient.

**Definition 8.3.** Define

$$
Z^\vee (t) = [2]_q^{-2}\Psi(t).
$$

(76)

**Definition 8.4.** For $n \in \mathbb{N}$ define $Z^\vee_n \in O_q$ such that

$$
Z^\vee (t) = \sum_{n \in \mathbb{N}} Z^\vee_n t^n.
$$

By [15] Lemma 8.18] we have $Z^\vee_0 = 1$.

**Lemma 8.5.** (See [15] Lemma 8.10 and Proposition 8.12].) For $n \geq 1$ we have $Z^\vee_n \in Z$. Moreover $Z^\vee_n$ fixed by $\sigma$ and $\dagger$ and $\tau$.

**Definition 8.6.** Let $\langle W_0, W_1 \rangle$ denote the subalgebra of $O_q$ generated by $W_0, W_1$.

**Proposition 8.7.** (See [15] Theorems 10.2–10.4.) For the algebra $O_q$ the following (i)–(iii) hold:

(i) there exists an algebra isomorphism $O_q \to \langle W_0, W_1 \rangle$ that sends $W_0 \mapsto W_0$ and $W_1 \mapsto W_1$;

(ii) there exists an algebra isomorphism $\mathbb{F}[z_1, z_2, \ldots] \to Z$ that sends $z_n \mapsto Z^\vee_n$ for $n \geq 1$;

(iii) the multiplication map

$$
\langle W_0, W_1 \rangle \otimes Z \to O_q
$$

$$
w \otimes z \mapsto wz
$$

is an isomorphism of algebras.

Note that the isomorphism in Proposition 8.7(i) is induced by the map $\iota$ from Lemma 6.8

**Proposition 8.8.** (See [15] Theorem 9.14].) There exists an algebra isomorphism $\phi : O_q \otimes \mathbb{F}[z_1, z_2, \ldots] \to O_q$ that sends

$$
W_0 \otimes 1 \mapsto W_0,
W_1 \otimes 1 \mapsto W_1,
1 \otimes z_n \mapsto Z^\vee_n, \quad n \geq 1.
$$
Proposition 8.9. The following diagrams commute:

\[
\begin{align*}
O_q \otimes \mathbb{F}[z_1, z_2, \ldots] &\xrightarrow{\phi} O_q & O_q \otimes \mathbb{F}[z_1, z_2, \ldots] &\xrightarrow{\phi} O_q \\
\sigma \otimes \text{id} &\downarrow & \sigma \otimes \text{id} &\downarrow \\
O_q \otimes \mathbb{F}[z_1, z_2, \ldots] &\xrightarrow{\phi} O_q & O_q \otimes \mathbb{F}[z_1, z_2, \ldots] &\xrightarrow{\phi} O_q \\
\tau \otimes \text{id} &\downarrow & \tau &\downarrow \\
O_q \otimes \mathbb{F}[z_1, z_2, \ldots] &\xrightarrow{\phi} O_q
\end{align*}
\]

Proof. Chase the generators \( W_0 \otimes 1, W_1 \otimes 1, \{1 \otimes z_n\}_{n=1}^{\infty} \) around each diagram using Lemmas 3.3, 3.4, 6.4, 6.5 and Definitions 3.6, 6.7 along with Lemma 8.5 and Proposition 8.8.

We emphasize a few points.

Corollary 8.10. The following (i)–(iii) hold:

(i) the algebra \( O_q \) is generated by \( W_0, W_1, \mathcal{Z} \);

(ii) the elements \( \{\mathcal{Z}_n^\vee\}_{n=1}^{\infty} \) are algebraically independent and generate \( \mathcal{Z} \);

(iii) everything in \( \mathcal{Z} \) is fixed by \( \sigma \) and \( \dagger \) and \( \tau \).

Proof. (i) By Proposition 8.7(iii).
(ii) By Proposition 8.7(ii).
(iii) By (ii) above and Lemma 8.5.

9 Comparing the generating functions for \( O_q \) and \( \mathcal{O}_q \)

In this section we investigate how the generating functions \( B^\pm(t), B(t) \) for \( O_q \) are related to the generating functions \( W^\pm(t), \mathcal{G}(t), \tilde{\mathcal{G}}(t) \) for \( \mathcal{O}_q \).

Throughout this section, we identify \( O_q \) with \( \langle W_0, W_1 \rangle \) via the map \( \imath \) from Lemma 6.8.

Lemma 9.1. (See [16, Lemma 11.3].) The element \( \tilde{\mathcal{G}}_1 + qB_\delta \) is central in \( \mathcal{O}_q \).

Lemma 9.2. For the algebra \( O_q \),

\[
[B_\delta, \tilde{\mathcal{G}}_n] = 0, \quad n \in \mathbb{N}.
\]

Proof. By Lemma 9.1 and since \( [\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_n] = 0 \) for \( n \in \mathbb{N} \).

Lemma 9.3. For the algebra \( O_q \),

\[
[B_\delta, \mathcal{G}(t)] = 0.
\]
Proof. By Lemma 9.2.

Lemma 9.4. (See [16] Lemma 11.5.) For \( n \geq 1 \) the following hold in \( \mathcal{O}_q \):

\[
\mathcal{W}_{-n} = \mathcal{W}_n - \frac{(q - q^{-1})W_0 \tilde{G}_n}{(q^2 - q^{-2})^2} + \frac{q^2 [B_\delta, W_{1-n}]}{(q^2 - q^{-2})^2}, \quad (79)
\]

\[
\mathcal{W}_{n+1} = \mathcal{W}_{1-n} - \frac{(q - q^{-1})W_1 \tilde{G}_n}{(q^2 - q^{-2})^2} - \frac{[B_\delta, W_n]}{(q^2 - q^{-2})^2}. \quad (80)
\]

Lemma 9.5. For the algebra \( \mathcal{O}_q \),

\[
\mathcal{W}^-(t) = t\mathcal{W}^+(t) - \frac{(q - q^{-1})W_0 \tilde{G}(t)}{(q^2 - q^{-2})^2} + \frac{q^2 t[B_\delta, \mathcal{W}^-(t)]}{(q^2 - q^{-2})^2}, \quad (81)
\]

\[
\mathcal{W}^+(t) = t\mathcal{W}^-(t) - \frac{(q - q^{-1})W_1 \tilde{G}(t)}{(q^2 - q^{-2})^2} - \frac{t[B_\delta, \mathcal{W}^+(t)]}{(q^2 - q^{-2})^2}. \quad (82)
\]

Proof. The equation \( (81) \) (resp. \( (82) \)) expresses the recurrence \( (79) \) (resp. \( (80) \)) in terms of generating functions.

Recall \( S, T \) from (74). We will be discussing \( (\tilde{G}(S))^{-1} \) and \( (\tilde{G}(T))^{-1} \). These inverses exist by [14] Lemmas 4.1, 4.6.

Proposition 9.6. For the algebra \( \mathcal{O}_q \),

\[
B^-(t) = (q^2 - q^{-2})S \left( q^{-1} \mathcal{W}^+(S) - qt^{-1} \mathcal{W}^-(S) \right) (\tilde{G}(S))^{-1}, \quad (83)
\]

\[
B^+(t) = (q^2 - q^{-2})T \left( q\mathcal{W}^-(T) - q^{-1}t^{-1} \mathcal{W}^+(T) \right) (\tilde{G}(T))^{-1}. \quad (84)
\]

Proof. Let \( b^-(t) \) denote the expression on the right in (83). We show that \( B^-(t) = b^-(t) \). Using [14] Sections 4, 5 we examine the terms on the right in (83), and find that \( b^-(t) \) has the form \( b^-(t) = \sum_{n \in \mathbb{N}} b_n t^n \) with \( b_n \in \mathcal{O}_q \) for \( n \in \mathbb{N} \). We show that \( b_n = B_{n\delta + \alpha_0} \) for \( n \in \mathbb{N} \). Using Lemmas 9.3, 9.4 we obtain

\[
\frac{q[B_\delta, b^-(t)]}{(q - q^{-1})(q^2 - q^{-2})} = (t^{-1} - t)b^-(t) - t^{-1}W_0 - W_1. \quad (85)
\]

From (85) we obtain the recursion

\[
b_0 = W_0, \quad b_1 = W_1 + \frac{q[B_\delta, W_0]}{(q - q^{-1})(q^2 - q^{-2})}, \quad b_n = b_{n-2} + \frac{q[B_\delta, b_{n-1}]}{(q - q^{-1})(q^2 - q^{-2})} \quad n \geq 2.
\]

Comparing this recursion with (84), (14) we obtain \( b_n = B_{n\delta + \alpha_0} \) for \( n \in \mathbb{N} \). Therefore \( B^-(t) = b^-(t) \), so (83) holds. A similar argument that yields (84) is summarized as follows. Let \( b^+(t) \) denote the generating function on the right in (84). Using Lemmas 9.3, 9.5 we obtain

\[
\frac{q[b^+(t), B_\delta]}{(q - q^{-1})(q^2 - q^{-2})} = (t^{-1} - t)b^+(t) - W_0 - t^{-1}W_1. \quad (86)
\]
Comparing (18) and (86), we find that the coefficients of \( B^+(t) \) and \( b^+(t) \) satisfy the same recurrence and initial conditions. Therefore these coefficients coincide, so \( B^+(t) = b^+(t) \) and (84) holds.

Next we give a variation on Proposition 9.6.

**Proposition 9.7.** For the algebra \( \mathcal{O}_q \),

\[
B^-(t) = (q^2 - q^{-2})(\tilde{G}(T))^{-1}\left(qW^+(T) - q^{-1}t^{-1}W^-(T)\right)T,
\]
\[ (87) \]

\[
B^+(t) = (q^2 - q^{-2})(\tilde{G}(S))^{-1}\left(q^{-1}W^-(S) - qt^{-1}W^+(S)\right)S.
\]
\[ (88) \]

**Proof.** Apply the antiautomorphism \( \tau \) to everything in Proposition 9.6.

**Theorem 9.8.** For the algebra \( \mathcal{O}_q \),

\[
\frac{q + q^{-1}}{t + t^{-1}} W^-(\frac{q + q^{-1}}{t + t^{-1}}) = \frac{q^{-1}tB^+(q^{-1}t) + B^-(qt)}{(q^2 - q^{-2})(t - t^{-1})} \tilde{G}\left(\frac{q + q^{-1}}{t + t^{-1}}\right),
\]
\[ (89) \]

\[
\frac{q + q^{-1}}{t + t^{-1}} W^+(\frac{q + q^{-1}}{t + t^{-1}}) = \frac{B^+(q^{-1}t) + qtB^-(qt)}{(q^2 - q^{-2})(t - t^{-1})} \tilde{G}\left(\frac{q + q^{-1}}{t + t^{-1}}\right).
\]
\[ (90) \]

**Proof.** To verify these equations, evaluate \( B^+(q^{-1}t) \) and \( B^-(qt) \) using Proposition 9.6 and simplify the result using (74).

**Theorem 9.9.** For the algebra \( \mathcal{O}_q \),

\[
\frac{q + q^{-1}}{t + t^{-1}} W^-(\frac{q + q^{-1}}{t + t^{-1}}) = \tilde{G}\left(\frac{q + q^{-1}}{t + t^{-1}}\right)\frac{qtB^+(qt) + B^-(q^{-1}t)}{(q^2 - q^{-2})(t - t^{-1})},
\]
\[ (91) \]

\[
\frac{q + q^{-1}}{t + t^{-1}} W^+(\frac{q + q^{-1}}{t + t^{-1}}) = \tilde{G}\left(\frac{q + q^{-1}}{t + t^{-1}}\right)\frac{B^+(qt) + q^{-1}tB^-(q^{-1}t)}{(q^2 - q^{-2})(t - t^{-1})}.
\]
\[ (92) \]

**Proof.** To verify these equations, evaluate \( B^+(qt) \) and \( B^-(q^{-1}t) \) using Proposition 9.7 and simplify the result using (74). Alternatively, apply the antiautomorphism \( \tau \) everywhere in Theorem 9.8.

In the next two results we give some consequences of Theorems 9.8, 9.9.

**Proposition 9.10.** For the algebra \( \mathcal{O}_q \),

\[
q\tilde{G}\left(\frac{q + q^{-1}}{t + t^{-1}}\right) W_0 - q^{-1}W_0\tilde{G}\left(\frac{q + q^{-1}}{t + t^{-1}}\right) = \frac{q(q - q^{-1})(qt - q^{-1}t^{-1})B^-(qt) - t(q - q^{-1})(q^{-1}t - qt^{-1})B^+(q^{-1}t)}{t - t^{-1}} \tilde{G}\left(\frac{q + q^{-1}}{t + t^{-1}}\right)
\]
\[
= \tilde{G}\left(\frac{q + q^{-1}}{t + t^{-1}}\right)\frac{q^{-1}(q - q^{-1})(q^{-1}t - qt^{-1})B^-(-q^{-1}t) - t(q - q^{-1})(qt - q^{-1}t^{-1})B^+(qt)}{t - t^{-1}}.
\]
\[ 18 \]
and also
\[ q\mathcal{W}_1\tilde{G}\left( \frac{q + q^{-1}}{t + t^{-1}} \right) - q^{-1}\tilde{G}\left( \frac{q + q^{-1}}{t + t^{-1}} \right)\mathcal{W}_1 = \frac{q^{-1}(q - q^{-1})(q^{-1}t - qt^{-1})B^+(qt) - t(q^{-1}t - qt^{-1})B^-(qt)}{t - t^{-1}}\tilde{G}\left( \frac{q + q^{-1}}{t + t^{-1}} \right)\]
\[ = \tilde{G}\left( \frac{q + q^{-1}}{t + t^{-1}} \right)\frac{q(q - q^{-1})(q^{-1}t - qt^{-1})B^+(qt) - t(q^{-1}t - qt^{-1})B^-(qt)}{t - t^{-1}}.\]

**Proof.** In the equation on the right in (64), replace \( t \) by \( (q + q^{-1})(t + t^{-1})^{-1} \) to obtain
\[ q\tilde{G}\left( \frac{q + q^{-1}}{t + t^{-1}} \right)\mathcal{W}_0 - q^{-1}\mathcal{W}_0\tilde{G}\left( \frac{q + q^{-1}}{t + t^{-1}} \right) = \rho\mathcal{W}^\leftarrow\left( \frac{q + q^{-1}}{t + t^{-1}} \right) - \rho\frac{q + q^{-1}}{t + t^{-1}}\mathcal{W}^\rightarrow\left( \frac{q + q^{-1}}{t + t^{-1}} \right).\]

In the above equation, evaluate the right-hand side using Theorem 9.8 (resp. Theorem 9.9) to obtain the the first (resp. second) equation in the proposition statement. Using \( \tau \) we obtain the third and fourth equation in the proposition statement. \( \square \)

**Proposition 9.11.** For the algebra \( \mathcal{O}_q \), the generating function
\[ \tilde{G}\left( \frac{q + q^{-1}}{t + t^{-1}} \right) \]
is equal to each of the following:
\[ \left( \frac{B(qt)}{q^{-2} - 1} + \frac{qt(q - q^{-1})(B^-(qt))^2 - t(q^{-1}t - qt^{-1})B^-(qt)B^+(qt)}{t - t^{-1}} \right)\tilde{G}\left( \frac{q + q^{-1}}{t + t^{-1}} \right),\]
\[ \left( \frac{B(q^{-1}t)}{q^{-2} - 1} - \frac{q^{-1}t(q - q^{-1})(B^+(qt))^2 + t(qt - q^{-1}t^{-1})B^+(qt)B^-(qt)}{t - t^{-1}} \right)\tilde{G}\left( \frac{q + q^{-1}}{t + t^{-1}} \right),\]
\[ \tilde{G}\left( \frac{q + q^{-1}}{t + t^{-1}} \right)\left( \frac{B(qt)}{q^{-2} - 1} + \frac{qt(q - q^{-1})(B^+(qt))^2 - t(q^{-1}t - qt^{-1})B^-(qt)B^+(qt)}{t - t^{-1}} \right),\]
\[ \tilde{G}\left( \frac{q + q^{-1}}{t + t^{-1}} \right)\left( \frac{B(q^{-1}t)}{q^{-2} - 1} - \frac{q^{-1}t(q - q^{-1})(B^-(qt))^2 + t(qt - q^{-1}t^{-1})B^+(qt)B^-(qt)}{t - t^{-1}} \right).\]

**Proof.** We first show that the generating function \( \tilde{G}\left( \frac{q + q^{-1}}{t + t^{-1}} \right) \) is equal to the first of the four given expressions. By (63),
\[ [\mathcal{W}_0, \mathcal{W}^+(t)] = \frac{\tilde{G}(t) - \tilde{G}(t)}{t(q + q^{-1})}. \]

In (93), replace \( t \) by \( (q + q^{-1})(t + t^{-1})^{-1} \) to obtain
\[ \mathcal{W}_0\mathcal{W}^\leftarrow\left( \frac{q + q^{-1}}{t + t^{-1}} \right) - \mathcal{W}^\rightarrow\left( \frac{q + q^{-1}}{t + t^{-1}} \right)\mathcal{W}_0 = \frac{t + t^{-1}}{(q + q^{-1})^2}\left( \tilde{G}\left( \frac{q + q^{-1}}{t + t^{-1}} \right) - \tilde{G}\left( \frac{q + q^{-1}}{t + t^{-1}} \right) \right).\]

In the above equation, eliminate the \( \mathcal{W}^+ \) terms using (90), and pull the resulting \( \tilde{G} \) terms to the right using the first equation in Proposition 9.10. In the resulting equation, eliminate
$[W_0, B^+(q^{-1}t)]_q$ and $[W_0, B^-(qt)]_q$ using (19) and (45), respectively. In the resulting equation, eliminate $B(q^{-1}t)$ using the first equation in Corollary 5.11. By the resulting equation, the generating function (93) is equal to the first of the four given expressions. The first and second given expressions are equal by the first equation in Corollary 5.11. Using the anti-automorphism $\tau$ we find that the generating function (93) is equal to the third and fourth given expressions.

10 A factorization of $Z^\vee(t)$

Throughout this section we identify $O_q$ with $\langle W_0, W_1 \rangle$ via the map $\iota$ from Lemma 6.8. Recall the generating function $Z^\vee(t)$ from Definition 8.3. We will prove the following result.

**Theorem 10.1.** For the algebra $O_q$ we have

$$Z^\vee(t) = \xi \tilde{G}(S)B(t)\tilde{G}(T),$$

where

$$\xi = -q(q - q^{-1})(q^2 - q^{-2})^{-4}.$$  

Let us consider what is needed to prove Theorem 10.1. Evaluating the left-hand side of (95) using Definitions 8.1, 8.3 and then rearranging terms, we find that Theorem 10.1 asserts the following: zero is equal to

$$t^{-1}\left(\left(\tilde{G}(S)\right)^{-1}W^-(S)S\right)\left(TW^+(T)\tilde{G}(T)\right)^{-1} + t\left(\left(\tilde{G}(S)\right)^{-1}W^+(S)S\right)\left(TW^-(T)\tilde{G}(T)\right)^{-1} - q^2\left(\left(\tilde{G}(S)\right)^{-1}W^-(S)S\right)\left(TW^-(T)\tilde{G}(T)\right)^{-1} - q^{-2}\left(\left(\tilde{G}(S)\right)^{-1}W^+(S)S\right)\left(TW^+(T)\tilde{G}(T)\right)^{-1} + \frac{\left(\tilde{G}(S)\right)^{-1}G(S)}{(q^2 - q^{-2})^2}B(t) - \frac{B(t)}{(q^2 - q^{-2})^2(q^2 - 1)}.$$  

We will verify the above assertion. To prepare for this, we evaluate the terms in (97)–(101).

**Lemma 10.2.** For the algebra $O_q$,

$$\left(\tilde{G}(S)\right)^{-1}W^-(S)S = \frac{tb^+(t) + B^-(q^{-2}t)}{(q^2 - q^{-2})(q^{-1}t - qt^{-1})},$$

$$\left(\tilde{G}(S)\right)^{-1}W^+(S)S = \frac{B^+(t) + q^{-2}tB^-(q^{-2}t)}{(q^2 - q^{-2})(q^{-1}t - qt^{-1})}.$$  

**Proof.** In Theorem 9.9 replace $t$ by $q^{-1}t$ and evaluate the result using (74).
Lemma 10.3. For the algebra $O_q$,\[ TW^-(T)(\tilde{G}(T))^{-1} = \frac{tB^+(t) + B^-(q^2t)}{(q^2 - q^{-2})(qt - q^{-1}t^{-1})}, \]

\[ TW^+(T)(\tilde{G}(T))^{-1} = \frac{B^+(t) + q^2tB^-(q^2t)}{(q^2 - q^{-2})(qt - q^{-1}t^{-1})}. \]

Proof. In Theorem 9.8 replace $t$ by $qt$ and evaluate the result using (74). \[ \square \]

Lemma 10.4. For the algebra $O_q$,\[ (\tilde{G}(S))^{-1}G(S) = B(t) - \frac{(q - q^{-1})t(B^+(t))^2 - q^{-1}t(q^{-2}t - q^2t^{-1})B^-(q^{-2}t)B^+(t)}{q^{-1}t - qt}. \]

Proof. In the third equation of Proposition 9.11 replace $t$ by $q^{-1}t$ and evaluate the result using (74). \[ \square \]

We can now easily prove Theorem 10.1.

Proof of Theorem 10.1. It suffices to show that the sum of (97)–(102) is zero. This is routinely shown by evaluating the terms using Lemmas 10.2, 10.3, 10.4. \[ \square \]

Next, we give some consequences of Theorem 10.1.

Corollary 10.5. For the algebra $O_q$ we have\[ B(t) = \xi^{-1}(\tilde{G}(S))^{-1}Z^\vee(t)(\tilde{G}(T))^{-1}, \]

where $\xi$ is from (96).\[ \]

Proof. Rearrange the terms in (95). \[ \square \]

Corollary 10.6. For the algebra $O_q$,\[ [\tilde{G}(s), B(t)] = 0. \]

Proof. The generating function $\tilde{G}(s)$ commutes with each factor on the right in (107). \[ \square \]

Corollary 10.7. For the algebra $O_q$,\[ [\tilde{G}_{k+1}, B_{n\delta}] = 0 \quad k, n \in \mathbb{N}. \]

Proof. By Corollary 10.6. \[ \square \]

Corollary 10.8. The generating function $Z^\vee(t)$ is equal to each of\[ \xi\tilde{G}(S)B(t)\tilde{G}(T), \quad \xi B(t)\tilde{G}(S)\tilde{G}(T), \quad \xi\tilde{G}(S)\tilde{G}(T)B(t), \]

\[ \xi\tilde{G}(T)B(t)\tilde{G}(S), \quad \xi B(t)\tilde{G}(T)\tilde{G}(S), \quad \xi\tilde{G}(T)\tilde{G}(S)B(t). \]

Proof. Evaluate (95) using Corollary 10.7 and the equation on the right in (59). \[ \square \]
11 The algebra homomorphism $\gamma : \mathcal{O}_q \to \mathcal{O}_q$

In this section we construct a surjective algebra homomorphism $\gamma : \mathcal{O}_q \to \mathcal{O}_q$. We describe $\gamma$ in various ways. We apply $\gamma$ to the alternating generators of $\mathcal{O}_q$, and obtain some elements called the alternating generators of $\mathcal{O}_q$.

Recall from Proposition 8.8 the algebra isomorphism $\phi : \mathcal{O}_q \otimes \mathbb{F}[z_1, z_2, \ldots] \to \mathcal{O}_q$.

We will be discussing the algebra homomorphism $\theta : \mathbb{F}[z_1, z_2, \ldots] \to \mathbb{F}$ from Definition 17.17.

**Lemma 11.1.** There exists a unique algebra homomorphism $\gamma : \mathcal{O}_q \to \mathcal{O}_q$ that makes the following diagram commute:

\[
\begin{array}{ccc}
\mathcal{O}_q & \xleftarrow{\phi} & \mathcal{O}_q \otimes \mathbb{F}[z_1, z_2, \ldots] \\
\gamma & \downarrow & \downarrow \text{id} \otimes \theta \\
\mathcal{O}_q & \xrightarrow{x \mapsto x \otimes 1} & \mathcal{O}_q \otimes \mathbb{F}
\end{array}
\]

**Proof.** Concerning existence, define the map $\gamma$ to be the composition

\[
\gamma : \mathcal{O}_q \xrightarrow{\phi^{-1}} \mathcal{O}_q \otimes \mathbb{F}[z_1, z_2, \ldots] \xrightarrow{\text{id} \otimes \theta} \mathcal{O}_q \otimes \mathbb{F} \xrightarrow{x \mapsto x \otimes 1} \mathcal{O}_q.
\]

In this composition, each factor is an algebra homomorphism. Therefore $\gamma$ is an algebra homomorphism. By construction, $\gamma$ makes the diagram commute. We have shown that $\gamma$ exists. By construction $\gamma$ is unique. \hfill $\Box$

**Lemma 11.2.** The algebra homomorphism $\gamma : \mathcal{O}_q \to \mathcal{O}_q$ sends

\[W_0 \mapsto W_0, \quad W_1 \mapsto W_1, \quad Z_n^\vee \mapsto 0, \quad n \geq 1.\]

Moreover, $\gamma$ is surjective.

**Proof.** Chase the $\mathcal{O}_q$-generators $W_0, W_1, \{Z_n^\vee\}_{n=1}^\infty$ around the diagram in Lemma 11.1 using Proposition 8.8 and Definition 17.17. The map $\gamma$ is surjective since $W_0, W_1$ generate $\mathcal{O}_q$. \hfill $\Box$

Next we describe how $\gamma$ is related to the algebra homomorphism $\iota : \mathcal{O}_q \to \mathcal{O}_q$ from Lemma 6.8.

**Lemma 11.3.** The composition

\[
\mathcal{O}_q \xrightarrow{\iota} \mathcal{O}_q \xrightarrow{\gamma} \mathcal{O}_q
\]

is equal to the identity map on $\mathcal{O}_q$.

**Proof.** By Lemmas 6.8, 11.2 the given composition is an algebra homomorphism that fixes the $\mathcal{O}_q$-generators $W_0$ and $W_1$. \hfill $\Box$

**Lemma 11.4.** The following diagrams commute:

\[
\begin{array}{ccc}
\mathcal{O}_q & \xrightarrow{\gamma} & \mathcal{O}_q \\
\sigma & \downarrow & \sigma \\
\mathcal{O}_q & \xrightarrow{\gamma} & \mathcal{O}_q
\end{array}
\quad
\begin{array}{ccc}
\mathcal{O}_q & \xrightarrow{\gamma} & \mathcal{O}_q \\
\iota & \downarrow & \iota \\
\mathcal{O}_q & \xrightarrow{\gamma} & \mathcal{O}_q
\end{array}
\quad
\begin{array}{ccc}
\mathcal{O}_q & \xrightarrow{\gamma} & \mathcal{O}_q \\
\tau & \downarrow & \tau \\
\mathcal{O}_q & \xrightarrow{\gamma} & \mathcal{O}_q
\end{array}
\]
Proof. Chase the $O_q$-generators $W_0$, $W_1$, $\{Z^\gamma_n\}_{n=1}^\infty$ around each diagram, using Lemmas 3.3, 3.4, 6.4, 6.5 and Definitions 3.6, 6.7 along with Lemmas 8.5, 11.2.

Definition 11.5. By an alternating generator of $O_q$ we mean the $\gamma$-image of an alternating generator for $O_q$. Our notation for an alternating generator of $O_q$ is given in the table below. For $k \in \mathbb{N}$,

\[
\begin{array}{cccc}
  u & W_{-k} & W_{k+1} & G_{k+1} \\
\gamma(u) & W_{-k} & W_{k+1} & G_{k+1}
\end{array}
\]

For notational convenience, define

\[
G_0 = -(q - q^{-1})[2]_q^2, \quad \tilde{G}_0 = -(q - q^{-1})[2]_q^2, \quad (108)
\]

Theorem 11.6. The alternating generators of $O_q$ satisfy the following relations. For $k, \ell \in \mathbb{N}$,

\[
\begin{align*}
[W_0, W_{k+1}] &= [W_{-k}, W_1] = (\tilde{G}_{k+1} - G_{k+1})/(q + q^{-1}), \quad (109) \\
[W_0, G_{k+1}] &= [\tilde{G}_{k+1}, W_0] = \rho W_{-k} - \rho W_{k+1}, \quad (110) \\
[G_{k+1}, W_1] &= [W_1, \tilde{G}_{k+1}] = \rho W_{k+2} - \rho W_{-k}, \quad (111) \\
[W_{-k}, W_{-\ell}] &= 0, \quad [W_{k+1}, W_{\ell+1}] = 0, \quad (112) \\
[W_{-k}, W_{\ell+1}] + [W_{k+1}, W_{-\ell}] &= 0, \quad (113) \\
[W_{-k}, G_{\ell+1}] + [G_{k+1}, W_{-\ell}] &= 0, \quad (114) \\
[W_{-k}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{-\ell}] &= 0, \quad (115) \\
[W_{k+1}, G_{\ell+1}] + [G_{k+1}, W_{\ell+1}] &= 0, \quad (116) \\
[W_{k+1}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{\ell+1}] &= 0, \quad (117) \\
[G_{k+1}, G_{\ell+1}] &= 0, \quad [\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0, \quad (118) \\
[\tilde{G}_{k+1}, G_{\ell+1}] + [G_{k+1}, \tilde{G}_{\ell+1}] &= 0. \quad (119)
\end{align*}
\]

We are using the notation $\rho = -(q^2 - q^{-2})^2$.

Proof. Apply $\gamma$ to everything in (50)–(60), and evaluate the results using Definition 11.5.

Next, we describe how $\sigma, \tilde{\tau}, \tau$ act on the alternating generators of $O_q$.

Lemma 11.7. The automorphism $\sigma$ of $O_q$ sends

\[
W_{-k} \mapsto W_{k+1}, \quad W_{k+1} \mapsto W_{-k}, \quad G_{k+1} \mapsto \tilde{G}_{k+1}, \quad \tilde{G}_{k+1} \mapsto G_{k+1}
\]

for $k \in \mathbb{N}$.

Proof. By Lemmas 6.4, 11.4 and Definition 11.5.

Lemma 11.8. The antiautomorphism $\tilde{\tau}$ of $O_q$ sends

\[
W_{-k} \mapsto W_{-k}, \quad W_{k+1} \mapsto W_{k+1}, \quad G_{k+1} \mapsto \tilde{G}_{k+1}, \quad \tilde{G}_{k+1} \mapsto G_{k+1}
\]

for $k \in \mathbb{N}$.
Proof. By Lemmas 6.5, 11.4 and Definition 11.5.

**Lemma 11.9.** The antiautomorphism \( \tau \) of \( O_q \) sends

\[
W_{-k} \mapsto W_{k+1}, \quad W_{k+1} \mapsto W_{-k}, \quad G_{k+1} \mapsto G_k, \quad \tilde{G}_{k+1} \mapsto \tilde{G}_k
\]

for \( k \in \mathbb{N} \).

Proof. By Definition 6.7 along with Lemma 11.4 and Definition 11.5.

Next, we describe the kernel of \( \gamma \) in several ways.

**Proposition 11.10.** The following are the same:

(i) the kernel of \( \gamma \);

(ii) the 2-sided ideal of \( O_q \) generated by \( \{ Z_n \} \infty_{n=1} \).

Proof. We invoke the commuting diagram in Lemma 11.1. Let \( J \) denote the kernel of \( \theta \). By Lemma 17.18, \( J \) is the ideal of \( \mathbb{F}[z_1, z_2, \ldots] \) generated by \( \{ z_n \} \infty_{n=1} \). For the map id \( \otimes \theta \) from the commuting diagram, the kernel is \( O_q \otimes J \) and this is the 2-sided ideal of \( O_q \otimes \mathbb{F}[z_1, z_2, \ldots] \) generated by \( \{ 1 \otimes z_n \} \infty_{n=1} \). The algebra isomorphism \( \phi \) sends \( 1 \otimes z_n \mapsto Z_n^\gamma \) for \( n \geq 1 \). The result follows from these comments and the commuting diagram in Lemma 11.1.

**Proposition 11.11.** The vector space \( O_q \) is the direct sum of the following:

(i) the kernel of \( \gamma \);

(ii) the subalgebra \( \langle W_0, W_1 \rangle \) of \( O_q \).

Proof. By Lemmas 6.8, 11.3 and linear algebra.

## 12 More generating functions for \( O_q \)

In this section we use generating functions to describe the alternating generators of \( O_q \).

**Definition 12.1.** We define some generating functions in the indeterminate \( t \):

\[
W^-(t) = \sum_{n \in \mathbb{N}} W_{-n} t^n, \quad W^+(t) = \sum_{n \in \mathbb{N}} W_{n+1} t^n,
\]

\[
G(t) = \sum_{n \in \mathbb{N}} G_n t^n, \quad \tilde{G}(t) = \sum_{n \in \mathbb{N}} \tilde{G}_n t^n.
\]

Observe that

\[
W^-(0) = W_0, \quad W^+(0) = W_1, \quad G(0) = -(q - q^{-1})[2]^2_{i_q}, \quad \tilde{G}(0) = -(q - q^{-1})[2]^2_{i_q}.
\]

We now give the relations (109)–(119) in terms of generating functions.
Lemma 12.2. For the algebra $O_q$ we have

\begin{align}
[W_0, W^+(t)] &= [W^-(t), W_1] = t^{-1}(\tilde{G}(t) - G(t))/(q + q^{-1}), \\
[W_0, G(t)]_q &= [\tilde{G}(t), W_0]_q = \rho W^-(t) - \rho t W^+(t), \\
[G(t), W_1]_q &= [W_1, \tilde{G}(t)]_q = \rho W^+(t) - \rho t W^-(t), \\
[W^-(s), W^-(t)] &= 0, \quad [W^+(s), W^+(t)] = 0, \\
[W^-(s), W^+(t)] + [W^+(s), W^-(t)] &= 0, \\
s[W^-(s), G(t)] + t[G(s), W^-(t)] &= 0, \\
s[W^+(s), G(t)] + t[G(s), W^+(t)] &= 0, \\
s[W^+(s), \tilde{G}(t)] + t[\tilde{G}(s), W^+(t)] &= 0, \\
[G(s), G(t)] &= 0, \quad [G(s), \tilde{G}(t)] = 0, \\
[\tilde{G}(s), G(t)] + [G(s), \tilde{G}(t)] &= 0.
\end{align}

Proof. Apply $\gamma$ to everything in Lemma 7.2. \hfill $\Box$

So far, it appears that $O_q$ resembles $O_q$. However, this resemblance extends only so far. The next result holds for $O_q$ but not $O_q$.

Proposition 12.3. For the algebra $O_q$,

\begin{equation}
B(t)\tilde{G}(T)\tilde{G}(S) = -q^{-1}(q - q^{-1})^3[2]_q^4.
\end{equation}

Proof. For notational convenience, we identify $O_q$ with $\langle W_0, W_1 \rangle$ via the map $\iota$ from Lemma 6.8. By Corollary 10.8 the following holds for $O_q$:

\begin{equation}
B(t)\tilde{G}(T)\tilde{G}(S) = \xi^{-1}Z^{\nu}(t).
\end{equation}

We apply $\gamma$ to each side of (132). By Lemma 11.3 the map $\gamma$ fixes everything in $O_q$, so $\gamma$ fixes $B(t)$. By Definition 11.5 the map $\gamma$ sends $\tilde{G}(S) \mapsto \tilde{G}(S)$ and $\tilde{G}(T) \mapsto \tilde{G}(T)$. By construction $Z^{\nu}(t) = \sum_{n \in \mathbb{N}} Z_n^{\nu} t^n$ and $Z_0^{\nu} = 1$. By Lemma 11.2 the map $\gamma$ sends $Z_n^{\nu} \mapsto 0$ for $n \geq 1$. Therefore $\gamma$ sends $Z^{\nu}(t) \mapsto 1$. The result follows in view of (96). \hfill $\Box$

Theorem 12.4. The conjecture [14] Conjecture 6.2 is true.

Proof. The equation (131) is the same as [14] Eqn. (41)], in view of [14]. Consequently [14] Conjecture 6.2 is true by the discussion above [14] Eqn. (41)]. \hfill $\Box$

The article [14] discusses the meaning of (131). Below we summarize a few points.

Lemma 12.5. (See [14] Eqn. (43).) For the algebra $O_q$ the elements $\{\tilde{G}_n\}_{n=1}^\infty$ and $\{B_n\}_{n=1}^\infty$ are recursively obtained from each other as follows. For $n \geq 1$,

\begin{equation}
0 = [n]_q B_n \tilde{G}_0 + \sum_{j+k+2\ell+1=n, j,k,\ell \geq 0} (-1)\ell \binom{k+\ell}{\ell} [2n-j]q^{k+1} B_{j+2\ell} \tilde{G}_{k+1}.
\end{equation}

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See [14 Appendix A] for a detailed discussion of (133). Next we give a consequence of (133).

Lemma 12.6. (See [14 Lemma 4.10].) For the algebra \( O_q \) and \( n \geq 1 \),

(i) the element \( \tilde{G}_n \) is a polynomial of total degree \( n \) in \( B_\delta, B_{2\delta}, \ldots, B_{n\delta} \), where we view \( B_{k\delta} \) as having degree \( k \) for \( 1 \leq k \leq n \). For this polynomial the constant term is 0, and the coefficient of \( B_{n\delta} \) is \(-q[n]_q[2n]_q^{-1}[2]_q^{n-2} \);

(ii) the element \( B_{n\delta} \) is a polynomial of total degree \( n \) in \( \tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_n \), where we view \( \tilde{G}_k \) as having degree \( k \) for \( 1 \leq k \leq n \). For this polynomial the constant term is 0, and the coefficient of \( \tilde{G}_n \) is \(-q^{-1}[n]_q^{-1}[2n]_q[2]_q^{n-2} \).

Proof. By Lemma 12.5 and induction on \( n \). \( \square \)

13. The algebra isomorphism \( \varphi : O_q \to O_q \otimes \mathbb{F}[z_1, z_2, \ldots] \)

In this section we introduce an algebra isomorphism \( \varphi : O_q \to O_q \otimes \mathbb{F}[z_1, z_2, \ldots] \). As we will see, \( \varphi \) is closely related to the inverse of the map \( \phi \) from Proposition 8.8.

Lemma 13.1. There exists an algebra homomorphism \( \varphi : O_q \to O_q \otimes \mathbb{F}[z_1, z_2, \ldots] \) that sends

\[
\begin{align*}
W_{n-1} &\mapsto \sum_{k=0}^{n} W_{k-n} \otimes z_k, & W_{n+1} &\mapsto \sum_{k=0}^{n} W_{n+1-k} \otimes z_k, \\
G_n &\mapsto \sum_{k=0}^{n} G_{n-k} \otimes z_k, & \tilde{G}_n &\mapsto \sum_{k=0}^{n} \tilde{G}_{n-k} \otimes z_k
\end{align*}
\]

for \( n \in \mathbb{N} \). In particular \( \varphi \) sends

\[
W_0 \mapsto W_0 \otimes 1, \quad W_1 \mapsto W_1 \otimes 1.
\] (134)

Proof. One checks using (109)–(119) that for the alternating generators of \( O_q \) their \( \varphi \)-image candidates satisfy the defining relations for \( O_q \) given in (50)–(60). \( \square \)

Recall the algebra homomorphism \( \iota : O_q \to O_q \) from Lemma 6.8.

Lemma 13.2. The following diagram commutes:

\[
\begin{array}{ccc}
O_q & \xrightarrow{x \mapsto x \otimes 1} & O_q \otimes \mathbb{F}[z_1, z_2, \ldots] \\
\downarrow \iota & & \downarrow \text{id} \\
O_q & \xrightarrow{\varphi} & O_q \otimes \mathbb{F}[z_1, z_2, \ldots]
\end{array}
\]

Proof. Chase the \( O_q \)-generators \( W_0, W_1 \) around the diagram, using Lemma 6.8 and (134). \( \square \)

Our next goal is to show that \( \varphi \) is an algebra isomorphism. Recall the central elements \( \{Z_n\}_{n \in \mathbb{N}} \) for \( O_q \), from Definition 8.4. In Appendix A we describe an algebra isomorphism \( \vee : \mathbb{F}[z_1, z_2, \ldots] \to \mathbb{F}[z_1, z_2, \ldots] \). We will be discussing the images \( \{z_v\}_{n \in \mathbb{N}} \) of \( \{z_n\}_{n \in \mathbb{N}} \).
Lemma 13.3. The map $\varphi$ sends $Z_n^\vee \mapsto 1 \otimes z_n^\vee$ for $n \in \mathbb{N}$.

Proof. For notational convenience, we identify $O_q$ with $\langle W_0, W_1 \rangle$ via the map $i$ from Lemma 6.8. We will work with generating functions. It suffices to show that $\varphi$ sends $Z^\vee(t) \mapsto 1 \otimes Z^\vee(t)$, where $Z^\vee(t)$ is from Definition 17.3. By Corollary 10.8,

$$Z^\vee(t) = \xi B(t) \tilde{G}(T) \tilde{G}(S).$$

By Lemma 13.2, $\varphi$ sends $B_n \delta \mapsto B_n \delta \otimes 1$ for $n \in \mathbb{N}$. Therefore $\varphi$ sends $B(t) \mapsto B(t) \otimes 1$. By Lemma 13.1, $\varphi$ sends $\tilde{G}(t) \mapsto \tilde{G}(t) \otimes Z(t)$, where $Z(t)$ is from Definition 17.1. Therefore $\varphi$ sends $\tilde{G}(T) \mapsto \tilde{G}(T) \otimes Z(T)$ and $\tilde{G}(S) \mapsto \tilde{G}(S) \otimes Z(S)$. By these comments, $\varphi$ sends

$$Z^\vee(t) \mapsto \xi B(t) \tilde{G}(T) \tilde{G}(S) \otimes Z(S).$$

By Proposition 12.3, $\xi B(t) \tilde{G}(T) \tilde{G}(S) = 1$. By Proposition 17.6, $Z(S) Z(T) = Z(t)$. Consequently $\varphi$ sends $Z^\vee(t) \mapsto 1 \otimes Z^\vee(t)$. The result follows. \qed

Proposition 13.4. The following diagram commutes:

$$
\begin{array}{ccc}
O_q \otimes F[z_1, z_2, \ldots] & \xrightarrow{\phi} & O_q \\
\downarrow{\text{id} \otimes \vee} & & \downarrow{\text{id}} \\
O_q \otimes F[z_1, z_2, \ldots] & \xleftarrow{\varphi} & O_q
\end{array}
$$

Proof. Chase the generators $W_0 \otimes 1, W_1 \otimes 1, \{1 \otimes z_n\}_{n=1}^\infty$ around the diagram, using Proposition 8.8 along with (134) and Lemma 13.3. \qed

Theorem 13.5. The map $\varphi$ from Lemma 13.1 is an algebra isomorphism.

Proof. By Proposition 13.4 and since the maps $\phi$ and $\vee$ are algebra isomorphisms. \qed

Lemma 13.6. The algebra isomorphism $\varphi$ sends $\langle W_0, W_1 \rangle$ onto $O_q \otimes 1$.

Proof. By Lemma 13.2. \qed

Lemma 13.7. The algebra isomorphism $\varphi$ sends $Z$ onto $1 \otimes F[z_1, z_2, \ldots]$.

Proof. By Propositions 8.8, 13.4. \qed

Definition 13.8. For $n \in \mathbb{N}$ let $Z_n$ denote the unique element in $Z$ that $\varphi$ sends to $1 \otimes z_n$. Note that $Z_0 = 1$.

Lemma 13.9. The elements $\{Z_n\}_{n=1}^\infty$ are algebraically independent and generate $Z$.

Proof. The elements $\{z_n\}_{n=1}^\infty$ are algebraically independent and generate $F[z_1, z_2, \ldots]$. So the elements $\{1 \otimes z_n\}_{n=1}^\infty$ are algebraically independent and generate $1 \otimes F[z_1, z_2, \ldots]$. The result follows in view of Lemma 13.7 and Definition 13.8. \qed
We have seen that $\varphi : O_q \rightarrow O_q \otimes \mathbb{F}[z_1, z_2, \ldots]$ is an algebra isomorphism that sends

$$Z_n \mapsto 1 \otimes z_n, \quad Z_n^\vee \mapsto 1 \otimes z_n^\vee, \quad n \in \mathbb{N}.$$ 

In the next two results, we clarify how $\{Z_n\}_{n=1}^\infty$ and $\{Z_n^\vee\}_{n=1}^\infty$ are related.

**Lemma 13.10.** For the algebra $O_q$ and $n \geq 1$,

(i) the element $Z_n^\vee$ is a polynomial of total degree $n$ in $Z_1, Z_2, \ldots, Z_n$, where we view $Z_k$ as having degree $k$ for $1 \leq k \leq n$. For this polynomial the constant term is 0, and the coefficient of $Z_n$ is $(q + q^{-1})^n(q^n + q^{-n})$;

(ii) the element $Z_n$ is a polynomial of total degree $n$ in $Z_1^\vee, Z_2^\vee, \ldots, Z_n^\vee$, where we view $Z_k^\vee$ as having degree $k$ for $1 \leq k \leq n$. For this polynomial the constant term is 0, and the coefficient of $Z_n^\vee$ is $(q + q^{-1})^{-n}(q^n + q^{-n})^{-1}$.

**Proof.** By the comment below Lemma 13.9, along with Lemmas 17.8, 17.16.

**Lemma 13.11.** For the algebra $O_q$ the following are the same:

(i) the 2-sided ideal generated by $\{Z_n\}_{n=1}^\infty$;

(ii) the 2-sided ideal generated by $\{Z_n^\vee\}_{n=1}^\infty$.

**Proof.** By Lemma 13.10.

**Lemma 13.12.** For the algebra $O_q$ and $n \in \mathbb{N}$,

$$W_n = \sum_{k=0}^n W_{k-n} Z_k, \quad W_{n+1} = \sum_{k=0}^n W_{n+1-k} Z_k,$$

$$G_n = \sum_{k=0}^n G_{n-k} Z_k, \quad \tilde{G}_n = \sum_{k=0}^n \tilde{G}_{n-k} Z_k.$$

In the above lines we identify $O_q$ with $\langle W_0, W_1 \rangle$ via $\iota$.

**Proof.** By Lemmas 13.1, 13.2 and Definition 13.8.

**Definition 13.13.** Define the generating function

$$Z(t) = \sum_{n \in \mathbb{N}} Z_n t^n.$$ 

(135)

**Theorem 13.14.** We have

$$Z^\vee(t) = Z(S)Z(T),$$

(136)

where $S, T$ are from (74).

**Proof.** By the comment below Lemma 13.9 along with Proposition 17.6.
Proposition 13.15. For the algebra $O_q$ we have
\[
W(t) = W(t)Z(t), \quad W(t) = W(t)Z(t),
\]
\[
G(t) = G(t)Z(t), \quad G(t) = G(t)Z(t).
\]

In the above lines we identify $O_q$ with $\langle W_0, W_1 \rangle$ via $\iota$.

Proof. By Lemma 13.12 and Definitions 12.1, 13.13.

The following result is a variation on Proposition 13.8.

Proposition 13.16. The following diagrams commute:
\[
\begin{array}{c}
O_q \xrightarrow{\varphi} O_q \otimes F[z_1, z_2, \ldots] \\
\downarrow \sigma \quad \downarrow \sigma \otimes \text{id} \\
O_q \xrightarrow{\varphi} O_q \otimes F[z_1, z_2, \ldots] \\
\downarrow \tau \quad \downarrow \tau \otimes \text{id} \\
O_q \xrightarrow{\varphi} O_q \otimes F[z_1, z_2, \ldots]
\end{array}
\]
\[
\begin{array}{c}
O_q \xrightarrow{\varphi} O_q \otimes F[z_1, z_2, \ldots] \\
\downarrow \uparrow \downarrow \uparrow \otimes \text{id} \\
O_q \xrightarrow{\varphi} O_q \otimes F[z_1, z_2, \ldots] \\
\downarrow \uparrow \downarrow \uparrow \otimes \text{id} \\
O_q \xrightarrow{\varphi} O_q \otimes F[z_1, z_2, \ldots]
\end{array}
\]

Proof. Chase the $O_q$-generators $W_0, W_1, \{Z_n\}_{n=1}^{\infty}$ around each diagram using Lemmas 3.3, 3.4, 6.4, 6.5 and Definitions 3.6, 6.7 along with Lemmas 8.5, 13.1, 13.3.

The following result is a variation on Lemma 11.1.

Lemma 13.17. The following diagram commutes:
\[
\begin{array}{c}
O_q \xrightarrow{\varphi} O_q \otimes F[z_1, z_2, \ldots] \\
\downarrow \gamma \quad \downarrow \text{id} \otimes \theta \\
O_q \xrightarrow{\varphi} O_q \otimes F
\end{array}
\]

Proof. Chase the $O_q$-generators $W_0, W_1, \{Z_n\}_{n=1}^{\infty}$ around the diagram, using Lemmas 11.2, 17.18 along with (134) and the comment below Lemma 13.9.

14 The algebra homomorphism $\eta : O_q \rightarrow F[z_1, z_2, \ldots]$

In this section, we introduce a surjective algebra homomorphism $\eta : O_q \rightarrow F[z_1, z_2, \ldots]$. We use $\eta$ to illuminate how $O_q$ is related to $O_q$. We describe the kernel of $\eta$ in several ways.

Lemma 14.1. There exists an algebra homomorphism $\eta : O_q \rightarrow F[z_1, z_2, \ldots]$ that sends
\[
W_n \mapsto 0, \quad W_{n+1} \mapsto 0, \quad G_n \mapsto G_0 z_n, \quad \tilde{G}_n \mapsto \tilde{G}_0 z_n \quad (137)
\]
for $n \in \mathbb{N}$. Moreover $\eta$ is surjective.
Proof. The algebra homomorphism $\eta$ exists because the defining relations (50)-(60) for $O_q$ hold if we make the assignments (137). The map $\eta$ is surjective since $\{z_n\}_{n=1}^\infty$ generate $F[z_1, z_2, \ldots]$.

Shortly we will describe how $\eta$ is related to the isomorphism $\varphi$ from Theorem 13.5. We will use the following result.

Lemma 14.2. The map $\eta$ sends $Z_n^\vee \mapsto z_n^\vee$ for $n \in \mathbb{N}$.

Proof. We will work with generating functions. It suffices to show that $\eta$ sends $Z_n^\vee(t) \mapsto Z(t)$. By Definition 8.3 we have $Z_n^\vee(t) = [2]^2 q^{-2} \Psi(t)$, where $\Psi(t)$ is from Definition 8.1. By Lemma 14.1, the map $\eta$ sends $W_0(t) \mapsto 0$, $W_1(t) \mapsto 0$, $G(t) \mapsto G_0 Z(t)$, $\tilde{G}(t) \mapsto \tilde{G}_0 Z(t)$.

By this and Definition 8.1, $\eta$ sends $\Psi(t) \mapsto (q^2 - q^{-2})^{-2} G_0 \tilde{G}_0 Z(S) Z(T)$.

By (61) we have $(q^2 - q^{-2})^{-2} G_0 \tilde{G}_0 = [2]^2_q^2$, and by Proposition 17.6 we have $Z(t) = Z(S) Z(T)$.

By these comments, $\eta$ sends $Z_n^\vee(t) \mapsto Z(t)$. $\square$

Recall the algebra homomorphism $\vartheta : O_q \rightarrow F$ from Lemma 3.7.

Proposition 14.3. The following diagram commutes:

$$
\begin{array}{ccc}
O_q & \xrightarrow{\varphi} & O_q \otimes F[z_1, z_2, \ldots] \\
\eta \downarrow & & \downarrow \vartheta \otimes \text{id} \\
F[z_1, z_2, \ldots] & \xrightarrow{x \mapsto 1 \otimes x} & F \otimes F[z_1, z_2, \ldots]
\end{array}
$$

Proof. Chase the $O_q$-generators $W_0$, $W_1$, $\{Z_n^\vee\}_{n=1}^\infty$ around the diagram, using (134) and Lemmas 3.7, 13.3, 14.1, 14.2. $\square$

Corollary 14.4. The map $\eta$ sends $Z_n \mapsto z_n$ for $n \in \mathbb{N}$.

Proof. By the comment below Lemma 13.9 along with Proposition 14.3. $\square$

Next, we describe the kernel of $\eta$ in several ways.

Proposition 14.5. The following are the same:

(i) the kernel of $\eta$;

(ii) the 2-sided ideal of $O_q$ generated by $W_0$, $W_1$.

Proof. We invoke the commuting diagram in Proposition 14.3. Let $K$ denote the kernel of the algebra homomorphism $\vartheta : O_q \rightarrow F$. By Lemma 3.7, $K$ is the 2-sided ideal of $O_q$ generated by $W_0$, $W_1$. For the map $\vartheta \otimes \text{id}$ from the commuting diagram, the kernel is $K \otimes F[z_1, z_2, \ldots]$ and this is the 2-sided ideal of $O_q \otimes F[z_1, z_2, \ldots]$ generated by $W_0 \otimes 1$, $W_1 \otimes 1$. The algebra isomorphism $\varphi$ sends $W_0 \mapsto W_0 \otimes 1$ and $W_1 \mapsto W_1 \otimes 1$. The result follows from these comments and the commuting diagram in Proposition 14.3. $\square$
Proposition 14.6. The vector space $O_q$ is the direct sum of the following:

(i) the center $Z$ of $O_q$;

(ii) the kernel of $\eta$.

Proof. By Lemma 13.7 we have $\varphi(Z) = 1 \otimes \mathbb{F}[z_1, z_2, \ldots]$. By this and the commuting diagram in Proposition 14.3, the restriction of $\eta$ to $Z$ gives a bijection $Z \to \mathbb{F}[z_1, z_2, \ldots]$. The result follows from this and linear algebra.

15 The algebra homomorphism $\vartheta : O_q \to \mathbb{F}$, revisited

For the sake of completeness, we show how the algebra homomorphism $\vartheta : O_q \to \mathbb{F}$ from Lemma 3.7 acts on the alternating generators of $O_q$.

Lemma 15.1. The map $\vartheta$ sends

$$W_{-n} \mapsto 0, \quad W_{n+1} \mapsto 0, \quad G_{n+1} \mapsto 0, \quad \tilde{G}_{n+1} \mapsto 0$$

for $n \in \mathbb{N}$.

Proof. We show that $\vartheta(W_{-n}) = 0$. Without loss of generality, we may assume that $\vartheta(W_{k-n}) = 0$ for $1 \leq k \leq n$. We chase $W_{-n}$ around the diagram in Proposition 14.3 using Lemmas 13.1 and 14.1. For one path in the diagram the outcome is 0, and for the other path in the diagram the outcome is $\vartheta(W_{-n}) \otimes 1$. Therefore $\vartheta(W_{-n}) = 0$. The remaining assertions are obtained in a similar manner.

16 Directions for future research

In this section we give some suggestions for future research.

Conjecture 16.1. Lemma 4.1 remains valid if we remove the assumption that $q$ is transcendental, and require only that $q$ is not a root of unity.

The following conjecture is a variation on [2, Conjecture 1].

Conjecture 16.2. A PBW basis for $O_q$ is obtained by the elements

$$\{W_{-i}\}_{i \in \mathbb{N}}, \quad \{\tilde{G}_{j+1}\}_{j \in \mathbb{N}}, \quad \{W_{k+1}\}_{k \in \mathbb{N}}$$

in any linear order $<$ that satisfies one of (i)–(vi) below:

(i) $W_{-i} < \tilde{G}_{j+1} < W_{k+1}$ for $i, j, k \in \mathbb{N}$;

(ii) $W_{k+1} < \tilde{G}_{j+1} < W_{-i}$ for $i, j, k \in \mathbb{N}$;

(iii) $W_{k+1} < W_{-i} < \tilde{G}_{j+1}$ for $i, j, k \in \mathbb{N}$;

(iv) $W_{-i} < W_{k+1} < \tilde{G}_{j+1}$ for $i, j, k \in \mathbb{N}$;
(v) \( \tilde{G}_{j+1} < W_{k+1} < W_{-i} \) for \( i, j, k \in \mathbb{N} \);
(vi) \( \tilde{G}_{j+1} < W_{-i} < W_{k+1} \) for \( i, j, k \in \mathbb{N} \).

We motivate the next problem with some comments.

Let \( \mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z} \) denote the cyclic group of order 4.

**Definition 16.3.** (See [9, Definition 5.1].) Let \( \square_q \) denote the algebra with generators \( \{x_i\}_{i \in \mathbb{Z}_4} \) and the following relations. For \( i \in \mathbb{Z}_4 \),

\[
\frac{qx_ix_{i+1} - q^{-1}x_{i+1}x_i}{q - q^{-1}} = 1,
\]
\[
x_i^3x_{i+2} - [3]_q x_i^2x_{i+2}x_i + [3]_q x_{i+2}x_i^2x_i - x_{i+2}x_i^3 = 0.
\]

**Lemma 16.4.** (See [9, Proposition 5.6, Theorem 10.33].) Pick nonzero \( a, b \in \mathbb{F} \). Then there exists an injective algebra homomorphism \( \sharp : O_q \to \square_q \) that sends

\[
W_0 \mapsto ax_0 + a^{-1}x_1, \quad W_1 \mapsto bx_2 + b^{-1}x_3.
\]

**Problem 16.5.** Find the image of each alternating generator of \( O_q \), under the homomorphism \( \sharp \) from Lemma 16.4.

**Problem 16.6.** Let \( V \) denote a finite-dimensional irreducible \( O_q \)-module on which each of \( W_0, W_1 \) is diagonalizable. Consider the four types of alternating generators for \( O_q \):

\[
\{W_{-k}\}_{k \in \mathbb{N}}, \quad \{W_{k+1}\}_{k \in \mathbb{N}}, \quad \{G_{k+1}\}_{k \in \mathbb{N}}, \quad \{\tilde{G}_{k+1}\}_{k \in \mathbb{N}}.
\]

For each type the alternating generators mutually commute; find their eigenvalues and common eigenvectors in \( V \). Perhaps start by assuming that \( V \) has shape \((1, 2, 1)\) in the sense of Vidar [17].

**Conjecture 16.7.** For \( x \in O_q \) the following are equivalent:

(i) \( x \) commutes with \( W_{-k} \) for \( k \in \mathbb{N} \);
(ii) \( x \) is contained in the subalgebra of \( O_q \) generated by \( \{W_{-k}\}_{k \in \mathbb{N}} \).

**Conjecture 16.8.** For \( x \in O_q \) the following are equivalent:

(i) \( x \) commutes with \( \tilde{G}_{k+1} \) for \( k \in \mathbb{N} \);
(ii) \( x \) is contained in the subalgebra of \( O_q \) generated by \( \{\tilde{G}_{k+1}\}_{k \in \mathbb{N}} \).
17 Appendix A: the algebra $\mathbb{F}[z_1, z_2, \ldots ]$

In this appendix, we explain some features of the polynomial algebra $\mathbb{F}[z_1, z_2, \ldots ]$ that are used in the main body of the paper. Recall the notation $z_0 = 1$.

**Definition 17.1.** Define the generating function

$$Z(t) = \sum_{n \in \mathbb{N}} z_n t^n. \quad (139)$$

**Lemma 17.2.** (See [14, Lemma 4.5].) We have

$$Z \left( \frac{q + q^{-1}}{t + t^{-1}} \right) = \sum_{n \in \mathbb{N}} z_n^+ [2]_q^n t^n, \quad (140)$$

where $z_0^+ = 1$ and

$$z_n^+ = \sum_{\ell = 0}^{\lfloor (n-1)/2 \rfloor} (-1)^\ell \binom{n-1-\ell}{\ell} [2]_q^{-2\ell} z_{n-2\ell}, \quad n \geq 1. \quad (141)$$

**Proof.** This is routinely checked. □

**Example 17.3.** In the table below, we display $z_n^+$ for $1 \leq n \leq 9$.

| $n$ | $z_n^+$                      |
|-----|------------------------------|
| 1   | $z_1$                        |
| 2   | $z_2$                        |
| 3   | $z_3 - [2]_q^{-2} z_1$        |
| 4   | $z_4 - 2 [2]_q^{-2} z_2$      |
| 5   | $z_5 - 3 [2]_q^{-2} z_3 + [2]_q^{-4} z_1$ |
| 6   | $z_6 - 4 [2]_q^{-2} z_4 + 3 [2]_q^{-4} z_2$ |
| 7   | $z_7 - 5 [2]_q^{-2} z_5 + 6 [2]_q^{-4} z_3 - [2]_q^{-6} z_1$ |
| 8   | $z_8 - 6 [2]_q^{-2} z_6 + 10 [2]_q^{-4} z_4 - 4 [2]_q^{-6} z_2$ |
| 9   | $z_9 - 7 [2]_q^{-2} z_7 + 15 [2]_q^{-4} z_5 - 10 [2]_q^{-6} z_3 + [2]_q^{-8} z_1$ |

Recall the functions $S$, $T$ from (74).

**Lemma 17.4.** We have

$$Z(S) = \sum_{n \in \mathbb{N}} z_n^+ [2]_q^{-n} [2]_q^n t^n, \quad Z(T) = \sum_{n \in \mathbb{N}} z_n^+ q^n [2]_q^n t^n. \quad (142)$$

**Proof.** By Lemma 17.2 with $t$ replaced by $q^{-1} t$ or $qt$. □
Definition 17.5. For \( n \in \mathbb{N} \) define
\[
z_n^\vee = 2^n \sum_{k=0}^{n} q^{n-2k} z_k z_{n-k}.
\] (142)

Note that \( z_0^\vee = 1 \). Further define
\[
Z^\vee(t) = \sum_{n \in \mathbb{N}} z_n^\vee t^n.
\] (143)

Proposition 17.6. We have
\[
Z^\vee(t) = Z(S)Z(T).
\] (144)

Proof. By Lemma \[17.3\] and Definition \[17.5\].

Example 17.7. We have
\[
\begin{align*}
z_1^\vee &= (q + q^{-1})^2 z_1, \\
z_2^\vee &= (q + q^{-1})^2 (q^2 + q^{-2}) z_2 + (q + q^{-1})^2 z_1^2, \\
z_3^\vee &= (q + q^{-1})^3 (q^3 + q^{-3}) z_3 + (q + q^{-1})^4 z_1 z_2 - (q + q^{-1})(q^3 + q^{-3}) z_1.
\end{align*}
\]

Lemma 17.8. For \( n \geq 1 \) the element \( z_n^\vee \) is a polynomial of total degree \( n \) in \( z_1, z_2, \ldots, z_n \), where we view \( z_k \) as having degree \( k \) for \( 1 \leq k \leq n \). For this polynomial the constant term is 0, and the coefficient of \( z_n \) is \((q + q^{-1})^n(q^n + q^{-n})\).

Proof. By \[(141)\] and \[(142)\].

The following comments will help us interpret Lemma \[17.8\]. We describe a basis for the vector space \( \mathbb{F}[z_1, z_2, \ldots] \). For \( n \in \mathbb{N} \), a partition of \( n \) is a sequence \( \lambda = \{ \lambda_i \}_{i=1}^{\infty} \) of natural numbers such that \( \lambda_i \geq \lambda_{i+1} \) for \( i \geq 1 \) and \( n = \sum_{i=1}^{\infty} \lambda_i \). The set \( \Lambda_n \) consists of the partitions of \( n \). Define \( \Lambda = \cup_{n \in \mathbb{N}} \Lambda_n \). For \( \lambda \in \Lambda \) define \( z_\lambda = \prod_{i=1}^{\infty} z_{\lambda_i} \). The elements \( \{z_\lambda\}_{\lambda \in \Lambda} \) form a basis for the vector space \( \mathbb{F}[z_1, z_2, \ldots] \).

Next we construct a grading for the algebra \( \mathbb{F}[z_1, z_2, \ldots] \). For notational convenience abbreviate \( P = \mathbb{F}[z_1, z_2, \ldots] \). For \( n \in \mathbb{N} \) let \( P_n \) denote the subspace of \( P \) with basis \( \{z_\lambda\}_{\lambda \in \Lambda_n} \). For example \( P_0 = \mathbb{F}1 \). The sum \( P = \sum_{n \in \mathbb{N}} P_n \) is direct. Moreover \( P_r P_s \subseteq P_{r+s} \) for \( r, s \in \mathbb{N} \). By these comments the subspaces \( \{P_n\}_{n \in \mathbb{N}} \) form a grading of the algebra \( P \). For \( n \in \mathbb{N} \), the dimension of \( P_n \) is equal to the number of partitions of \( n \). Observe that
\[
P_n = \mathbb{F}z_n + \sum_{k=1}^{n-1} \text{Span}(P_k P_{n-k}), \quad n \geq 1.
\] (145)

We now interpret Lemma \[17.8\] in light of the above comments.

Lemma 17.9. For \( n \geq 1 \) we have \( z_n^\vee \in \sum_{k=1}^{n} P_k \) and
\[
z_n^\vee - (q + q^{-1})^n(q^n + q^{-n})z_n \in \sum_{k=1}^{n-1} P_k + \sum_{k=1}^{n-1} \text{Span}(P_k P_{n-k}).
\] (146)
Proof. By Lemma [17.8] \[ \square \]

**Lemma 17.10.** There exists an algebra homomorphism \( \vee : \mathbb{F}[z_1, z_2, \ldots] \to \mathbb{F}[z_1, z_2, \ldots] \) that sends \( z_n \mapsto z_n^{\vee} \) for \( n \geq 1 \).

**Proof.** Since \( \{z_n\}_{n=1}^{\infty} \) are algebraically independent and generate \( \mathbb{F}[z_1, z_2, \ldots] \). \[ \square \]

Our next goal is to show that the map \( \vee \) from Lemma [17.10] is an algebra isomorphism.

**Definition 17.11.** For \( z \in \mathbb{F}[z_1, z_2, \ldots] \) let \( z^{\vee} \) denote the image of \( z \) under the homomorphism \( \vee \) from Lemma [17.10].

**Lemma 17.12.** We have \( P_0^{\vee} = P_0 \) and for \( n \in \mathbb{N} \),

\[
P_1^{\vee} + P_2^{\vee} + \cdots + P_n^{\vee} = P_1 + P_2 + \cdots + P_n. \tag{147}
\]

**Proof.** We have \( P_0^{\vee} = P_0 \) since \( P_0 = \mathbb{F}1 \) and \( 1^{\vee} = 1 \). Next we prove (147) by induction on \( n \). For notational convenience, define \( F_n = P_1 + \cdots + P_n \). We show that \( F_n^{\vee} = F_n \). This holds vacuously for \( n = 0 \), so assume that \( n \geq 1 \). By induction, \( F_\ell^{\vee} = F_\ell \) for \( 0 \leq \ell \leq n - 1 \). Adjusting (145), we obtain

\[
F_n = F_{n-1} + \mathbb{F}z_n + \sum_{k=1}^{n-1} \text{Span}(F_k F_{n-k}). \tag{148}
\]

Upon applying \( \vee \) to each side of (148),

\[
F_n^{\vee} = F_{n-1}^{\vee} + \mathbb{F}z_n^{\vee} + \sum_{k=1}^{n-1} \text{Span}(F_k^{\vee} F_{n-k}^{\vee}) \tag{149}
\]

By (146) and the construction,

\[
z_n^{\vee} - (q + q^{-1})^n(q^n + q^{-n})z_n \in F_{n-1} + \sum_{k=1}^{n-1} \text{Span}(F_k F_{n-k}).
\]

Consequently, the right-hand side of (148) is equal to the right-hand side of (149). By these comments \( F_n^{\vee} = F_n \). \[ \square \]

**Proposition 17.13.** The map \( \vee \) from Lemma [17.10] is an algebra isomorphism.

**Proof.** By construction \( \vee \) is an algebra homomorphism. We show that \( \vee \) is a bijection. For \( n \in \mathbb{N} \), the restriction of \( \vee \) to \( \sum_{k=0}^{n} P_k \) gives a bijection \( \sum_{k=0}^{n} P_k \to \sum_{k=0}^{n} P_k \) by Lemma [17.12] and since \( \sum_{k=0}^{n} P_k \) has finite dimension. We have \( \mathbb{F}[z_1, z_2, \ldots] = \sum_{k=0}^{\infty} P_k \). Therefore \( \vee \) is a bijection. \[ \square \]

**Corollary 17.14.** The elements \( \{z_n^{\vee}\}_{n=1}^{\infty} \) are algebraically independent and generate \( \mathbb{F}[z_1, z_2, \ldots] \).
Proof. The elements \( \{ z_n \}_{n=1}^{\infty} \) are algebraically independent and generate \( \mathbb{F}[z_1, z_2, \ldots] \). The result follows from this and Proposition 17.13.

For \( n \geq 1 \), we now seek to express \( z_n \) as a polynomial in \( z_1^\vee, z_2^\vee, \ldots, z_n^\vee \). For \( n = 1, 2, 3 \) we obtain the following from Example 17.7.

Example 17.15. We have

\[
\begin{align*}
z_1 &= \frac{z_1^\vee}{(q + q^{-1})^2}, \\
z_2 &= \frac{(q + q^{-1})^2 z_2^\vee - (z_1^\vee)^2}{(q + q^{-1})^4 (q^2 + q^{-2})}, \\
z_3 &= \frac{(q + q^{-1})^2 (q^2 + q^{-2}) z_3^\vee - (q + q^{-1})^2 z_1^\vee z_2^\vee + (z_1^\vee)^3 + (q + q^{-1}) (q^2 + q^{-2}) (q^3 + q^{-3}) z_1^\vee}{(q + q^{-1})^5 (q^2 + q^{-2}) (q^3 + q^{-3})}.
\end{align*}
\]

Lemma 17.16. For \( n \geq 1 \) the element \( z_n \) is a polynomial of total degree \( n \) in \( z_1^\vee, z_2^\vee, \ldots, z_n^\vee \), where we view \( z_k^\vee \) as having degree \( k \) for \( 1 \leq k \leq n \). For this polynomial the constant coefficient is 0, and the coefficient of \( z_n^\vee \) is \( (q + q^{-1})^{-n} (q^n + q^{-n})^{-1} \).

Proof. This follows from Lemmas 17.9, 17.12. Alternatively, in Lemma 17.8 solve for \( z_n \) and evaluate the result using induction on \( n \).

Definition 17.17. Let \( \theta : \mathbb{F}[z_1, z_2, \ldots] \to \mathbb{F} \) denote the algebra homomorphism that sends \( z_n \mapsto 0 \) for \( n \geq 1 \).

Lemma 17.18. For the algebra \( \mathbb{F}[z_1, z_2, \ldots] \) the following (i)–(iv) are the same:

(i) the kernel of \( \theta \);

(ii) the sum \( \sum_{n=1}^{\infty} P_n \), where \( \{ P_n \}_{n \in \mathbb{N}} \) is the grading of \( \mathbb{F}[z_1, z_2, \ldots] \) from below Lemma 17.8;

(iii) the ideal generated by \( \{ z_n \}_{n=1}^{\infty} \);

(iv) the ideal generated by \( \{ z_n^\vee \}_{n=1}^{\infty} \).

Proof. It is routine to check that (i)–(iii) are the same; denote this common value by \( J \). Comparing (iii), (iv) we see that (iv) is equal to \( J^\vee \). Applying \( \vee \) to (ii) and using Lemma 17.12 we obtain \( J^\vee = J \).

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