A CYCLIC COCYCLE AND RELATIVE INDEX THEOREMS ON PARTITIONED MANIFOLDS

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Abstract. In this paper, we extend Roe’s cyclic 1-cocycle to relative settings. We also prove two relative index theorems for partitioned manifolds by using its cyclic cocycle, which are generalizations of index theorems on partitioned manifolds. One of these theorems is a variant of [4, Theorem 3.3].

Introduction

Let $M$ be a complete Riemannian manifold and assume that $M$ is partitioned by a closed submanifold $N$ of codimension 1 into two submanifolds $M^+$ and $M^-$ with common boundary $N = M^+ \cap M^- = \partial M^+ = \partial M^-$. In this setting, J. Roe [5] defined a cyclic 1-cocycle $\zeta_N$ and proved the following index theorem on partitioned manifolds.

Let $D$ be the Dirac operator over $M$ and $D_N$ the graded Dirac operator over $N$ which is induced by $D$. In [5], Roe defined a coarse index class $\text{c-ind}(D) = [u_D] \in K_1(C^*(M))$, which is a $K_1$-class of the Roe algebra $C^*(M)$ and represented by the Cayley transform $u_D$ of $D$. Roe’s cyclic 1-cocycle $\zeta_N$ induces an additive map $(\zeta_N)_*: K_1(C^*(M)) \to \mathbb{C}$ by using Connes’ pairing. By using these ingredients, Roe proved an index theorem on partitioned manifolds:

\[(\zeta_N)_*(\text{c-ind}(D)) = -\frac{1}{8\pi i} \text{index}(D_N^+),\]

here $\text{index}(D_N^+)$ in the right hand side is the Fredholm index of $D_N^+$.

On the other hand, because of the vanishing of the Fredholm index of the Dirac operator on closed manifolds of odd dimension, the value $(\zeta_N)_*(\text{c-ind}(D))$ is trivial when $M$ is of even dimension. The author [8, 9] proved an index theorem with a nontrivial value $(\zeta_N)_*(x)$ for some $x \in K_1(C^*(M))$ when $M$ is of even dimension. The index theorem is as follows.

Let $C_w(M)$ be a $C^*$-algebra generated by bounded and smooth functions on $M$ with which gradient is bounded. A “good” $GL_l(C)$-valued continuous function $\phi \in GL_l(C_w(M))$ defines a $K_1$-class $[\phi] \in K_1(C_w(M))$. The author defined a $KK$-class $[D] \in KK(C_w(M), C^*(M))$ for the graded Dirac operator $D$ and a coarse Toeplitz index

\[\text{c-ind}(\phi, D) = [\phi] \hat{\otimes}_{C_w(M)} [D] \in K_1(C^*(M))\]

by using the Kasparov product

\[\hat{\otimes}_{C_w(M)} : K_1(C_w(M)) \times KK(C_w(M), C^*(M)) \to K_1(C^*(M)).\]

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On the other hand, let $\mathcal{H}$ be the closed subspace of the $L^2$-sections which is generated by the non-negative eigenvectors of the Dirac operator $D_N$ and $P$ the projection onto $\mathcal{H}$. Define the Toeplitz operator $T_{\phi|N} : \mathcal{H} \to \mathcal{H}$ by $T_{\phi|N}(s) = P\phi|N s$. The Toeplitz operator $T_{\phi|N}$ is Fredholm since the values of $\phi|N$ are in $GL_1(\mathbb{C})$. Then the author proved

$$(2) \quad (\zeta_N)_*(c\text{-ind}(\phi, D)) = -\frac{1}{8\pi i} \text{index}(T_{\phi|N}).$$

In this paper, we generalize (1) and (2) to relative index settings partitioned by (possibly non-compact) submanifolds of codimension 1. For this purpose, we generalize three ingredients, the index class in $K_1(C^*(M))$, the cyclic cocycle $\zeta_N$ and the Fredholm index on $N$ to the case of relative index settings, respectively.

Let $M_1$ and $M_2$ be two complete Riemannian manifolds and $W_1 \subset M_1$ and $W_2 \subset M_2$ are closed subsets. Assume that there exists an isometry $\psi : M_2 \setminus W_2 \to M_1 \setminus W_1$ such that $\psi$ conjugates all ingredients, for example, $D_1 = (\psi^*)^{-1}D_2\psi^*$ for the Dirac operators. Denote by $C^*(W_1 \subset M_1)$ the relative ideal in the Roe algebra $C^*(M_1)$, which is generated by controlled and locally traceable operators which is supported near $W_1$. In this setting, Roe [7] defined a coarse relative index $c\text{-ind}(D_1, D_2) \in K_1(C^*(W_1 \subset M_1))$ (see also [4], [6] and Definition 1.4), which is a generalization of Roe’s odd index. In this paper, we also define a coarse relative Toeplitz index $c\text{-ind}(\phi_1, D_1, \phi_2, D_2) \in K_1(C^*(W_1 \subset M_1))$ (see Definition 1.5), which is a generalization of the coarse Toeplitz index $c\text{-ind}(\phi, D)$. Roughly speaking, these coarse relative index classes are given by the difference of odd index classes for non-relative settings, respectively.

We define the Roe type cyclic 1-cocycle on a dense subalgebra in the relative ideal $C^*(W_1 \subset M_1)$ when $M_i$ is partitioned by a (possibly non-compact) submanifold $N_i$; see Section 2. The cyclic cocycle $\zeta$ induces an additive map $\zeta_* : K_1(C^*(W_1 \subset M_1)) \to \mathbb{C}$. In our main theorems, we send above coarse relative index classes by $\zeta_*$, then we get relative topological indices on $N_i$ which are introduced by M. Gromov and H. B. Lawson [2]. These are generalizations of index theorems on partitioned manifolds. Note that, Theorem 1 is a variant of [4] Theorem 3.3] [12].

**Theorem 1.** (see Theorem 3.2) Let $(M_i, W_i, D_i)$ be a tuple of a complete Riemannian manifold $M_i$ partitioned by $N_i$, a closed subset $W_i$ and the Dirac operator $D_i$ as previously. Then the following formula holds:

$$\zeta_*(c\text{-ind}(D_1, D_2)) = -\frac{1}{8\pi i} \text{ind}_i(D_{N_1}, D_{N_2}),$$

here the right hand side is Gromov-Lawson’s relative topological index.

**Theorem 2.** (see Theorem 3.1) Let $(M_i, W_i, D_i)$ be a tuple of a complete Riemannian manifold $M_i$ partitioned by $N_i$, a closed subset $W_i$ and the Dirac operator $D_i$ as previously. Take $\phi_i \in GL_1(C^*(M_i))$ such that $\phi_2 = \phi_1 \circ \psi$. Then the following formula holds:

$$\zeta_*(c\text{-ind}(\phi_1, D_1, \phi_2, D_2)) = -\frac{1}{8\pi i} \text{ind}_i(\phi_{N_1}, D_{N_1}, \phi_{N_2}, D_{N_2}).$$

The strategy of the proof of the theorems is the following. Firstly, we reduce to the product case, which is similar to the case for index theorems on partitioned manifolds. Secondly, we prove the product case. In the second step, we use index theorems (1) and (2).
Note that, in the definition of relative topological indices in the right hand sides, we use compactifications of neighborhoods of $N_1 \cap W_1$ and $N_2 \cap W_2$. However, in our proof, we do not use the fact that relative topological indices do not depend on the choice of such compactifications. Thus our main theorems give a new proof of well definedness of relative topological indices, respectively.

1. Index classes

1.1. Relative index data. Let $M$ be a complete Riemannian manifold and $W \subset M$ a closed subset. In this subsection, we recall the notion of a relative index data over a pair $(M, W)$ and a relative ideal in the Roe algebra. Coarse relative indices are elements in $K$-theory of its ideal. See [7] for details of these notions.

**Definition 1.1.** Let $M_i (i = 1, 2)$ be a complete Riemannian manifold and $D_i$ the Dirac operator on a Clifford bundle $S_i \to M_i$. We call $(M_i, W_i, D_i)$ an odd relative index data over $(M, W)$ if the following holds:

- $W_i \subset M_i$ is a closed subset,
- there exists isometry $\psi : M_2 \setminus W_2 \to M_1 \setminus W_1$ which induces isometry of Clifford bundles $\psi^* : S_i|_{M_i\setminus W_i} \to S_2|_{M_2\setminus W_2}$,
- there exists a continuous coarse map $f_i : M_i \to M$ such that $f(W_i) = W$,

$$W_i = f^{-1}(W) \text{ and } f_1 \circ \psi = f_2.$$

An even relative index data is given by an odd relative index data and respects $\mathbb{Z}_2$-gradings. We omit odd or even when it is not important.

Coarse relative indices are constructed by using a relative index data and are elements in $K$-theory of the relative ideal of the Roe algebra. Let $S \to M$ be a Hermitian vector bundle and recall that a bounded operator $T : L^2(M, S) \to L^2(M, S)$ is controlled if there exists a constant $R > 0$ such that $\varphi T \psi = 0$ when $\varphi, \psi \in C_c(M)$ satisfy $d(\text{Supp}(\varphi), \text{Supp}(\psi)) > R$. The infimum of such $R > 0$ is called propagation of a controlled operator $T$. A bounded operator $T$ on $L^2(M, S)$ is locally traceable (resp. locally compact) if $\varphi T \psi$ is of trace class (resp. compact) for any $\varphi, \psi \in C_c(M)$. The Roe algebra $C^*(M)$ is defined to be the norm closure of the set of controlled and locally traceable operators on $L^2(M, S)$.

An operator $T$ is supported near $W$ if there exists constant $r > 0$ such that $\varphi T = 0$ and $T \varphi = 0$ when $\varphi \in C_c(M)$ satisfies $d(\text{Supp}(\varphi), W) > r$. We call $T$ is supported in $N_r(W)$ by using such a constant $r$. Here, we set $N_r(A) = \{ x \in M : d(x, A) \leq r \}$ for a subset $A \subset M$. Denote by $\mathcal{B}_W$ the set of controlled and locally traceable operators which is supported near $W$. The relative ideal $C^*(W \subset M)$ is an ideal in $C^*(M)$ generated by $\mathcal{B}_W$.

Let $(M_i, W_i, D_i)$ is a relative index data over $(M, W)$ and denote by $\pi_i : C^*(M_i) \to C^*(M_1)/C^*(W_1 \subset M_1)$ the projection onto the quotient. The isometry $\psi : M_2 \setminus W_2 \to M_1 \setminus W_1$ appeared in a relative index data induces an isomorphism of $C^*$-algebras:

$$\Psi : C^*(M_1)/C^*(W_1 \subset M_1) \to C^*(M_2)/C^*(W_2 \subset M_2).$$

As is well known, we have $f(D_i) \in C^*(M_i)$ for any $f \in C_0(\mathbb{R})$. The isomorphism $\Psi$ gives a correspondence of $\pi_1(f(D_1))$ and $\pi_2(f(D_2))$. 


Lemma 1.2. [7] Lemma 4.3] Let \((M_i, W_i, D_i)\) be a relative index data over \((M, W)\). For any \(f \in C_0(\mathbb{R})\), we have
\[
\Psi(\pi_1(f(D_1))) = \pi_2(f(D_2)).
\]

By Lemma 1.2, a pair \((f(D_1), f(D_2))\) for \(f \in C_0(\mathbb{R})\) defines an element in a \(C^*\)-algebra
\[
\mathcal{E} := \{(T_1, T_2) \in C^*(M_1) \oplus C^*(M_2) : \Psi(\pi_1(T_1)) = \pi_2(T_2)\}.
\]

There is a \(D^*\)-version of this discussion. Let \(D^*(M)\) be a \(C^*\)-algebra generated by controlled and pseudolocal operators, here a bounded operator \(T\) is pseudolocal if \([T, \varphi]\) is compact for any \(\varphi \in C_0(M)\). The relative ideal \(D^*(W \subset M)\) is an ideal in \(D^*(M)\) which is generated by controlled and pseudolocal operators which are supported near \(W\) and are locally compact on \(M \setminus W\). Denote by the same letter \(\pi_i : D^*(M_i) \to D^*(M_i)/D^*(W_i \subset M_i)\) the projection onto the quotient and \(\psi\) induces an isometry of \(C^*\)-algebras
\[
\Psi : D^*(M_1)/D^*(W_1 \subset M_1) \to D^*(M_2)/D^*(W_2 \subset M_2).
\]

A continuous odd function \(\chi : \mathbb{R} \to [-1, 1]\) is a chopping function (or normalizing function) if we have \(\chi(t) \to \pm 1\) as \(t \to \pm \infty\). Functional calculus gives an element \(\chi(D_i) \in D^*(M_i)\) and then a variant of Lemma 1.2 as follows.

Lemma 1.3. [7] Lemma 4.4] Let \((M_i, W_i, D_i)\) be a relative index data over \((M, W)\). For any chopping function \(\chi\), we have
\[
\Psi(\pi_1(\chi(D_1))) = \pi_2(\chi(D_2)).
\]

By Lemma 1.3, a pair \((\chi(D_1), \chi(D_2))\) for \(f \in C_0(\mathbb{R})\) defines an element in a \(C^*\)-algebra
\[
\mathcal{D} := \{(T_1, T_2) \in D^*(M_1) \oplus D^*(M_2) : \Psi(\pi_1(T_1)) = \pi_2(T_2)\}.
\]

1.2. Coarse relative index. Following [4] and [7], we define the coarse relative index. By Lemma 1.3 we have an element
\[
\hat{\partial} \left[ \frac{\chi(D_1) + 1}{2}, \frac{\chi(D_2) + 1}{2} \right] \in K_1(\mathcal{E})
\]
for a chopping function \(\chi\), where \(\hat{\partial}\) is the exponential map in the 6-term exact sequence of a short exact sequence \(0 \to \mathcal{E} \to \mathcal{D} \to \mathcal{D}/\mathcal{E} \to 0\). \(K\)-theory of \(\mathcal{E}\) can be decomposed as follows. Let \(V_i : L^2(W_i, S_1) \to L^2(W, S)\) be a unitary which covers surjective continuous coarse map \(f_i|_{W_i} : W_i \to W\). We can choose \(V_i\) with arbitrary small propagation, here \(V_i\) has propagation less than \(\delta > 0\) if we have \(\psi V_i \varphi = 0\) for any \(\varphi \in C_b(W)\) and \(\psi \in C_0(W_i)\) with \(d(\text{Supp}(\varphi), f_i(\text{Supp}(\psi))) > \delta\). We assume that \(V_1\) and \(V_2\) have propagation less than \(\delta/2 > 0\). Define a unitary operator \(U : L^2(M_1, S_1) \to L^2(M_2, S_2)\) by
\[
U = V_2^* V_1 \oplus \psi^* : L^2(W_1, S_1) \oplus L^2(M_1 \setminus W_1, S_1) \to L^2(W_2, S_2) \oplus L^2(M_2 \setminus W_2, S_2).
\]

\(U\) has propagation less than \(\delta\) and it induces a map \(\text{Ad}(U) : C^*(M_1) \to C^*(M_2)\) and a split of a short exact sequence of \(C^*\)-algebras
\[
0 \to C^*(W_1 \subset M_1) \to \mathcal{E} \to C^*(M_2) \to 0.
\]

Here, the first map is an inclusion \(T_1 \mapsto (T_1, 0)\), the second one is the projection \((T_1, T_2) \mapsto T_2\) and the split map is \(C^*(M_1) \ni T_2 \mapsto (U^* T_2 U, T_2) \in \mathcal{E}\). Thus we have
a direct sum decomposition $K_\ast(\mathcal{E}) = K_\ast(C^\ast(W_1 \subset M_1)) \oplus K_\ast(C^\ast(M_2))$. Denote by

$$q : K_\ast(\mathcal{E}) \to K_\ast(C^\ast(W_1 \subset M_1))$$

the projection onto the first summand, which is independent of the choice of a direct sum decomposition data over $(M, W_1)$. Let $\chi$ be a chopping function. Then the projection onto the first summand, which is independent of the choice of a chopping function, induces a map $f$ such that $\tilde{f}$ is compactly supported $\text{Supp}(\hat{f}) \subset (r, r)$: $c\text{-ind}(D_1, D_2) = [f(D_1)] - [U^*f(D_2)U]$. Note that an operator $f(D_1) - U^*f(D_2)U$ is supported in $N_r(W_1)$ by the proof of [7, Lemma 4.3]. We also have

$$c\text{-ind}(D_1, D_2) = \left[ \frac{D_1 - i}{D_1 + i} \right] - \left[ \frac{U^*D_2 - i}{D_2 + i} \right] U \in K_1(C^\ast(W_1 \subset M_1)).$$

Similarly, an even relative index data defines an even index class in $K_0(C^\ast(W_1 \subset M_1))$; see [7]. We do not use the even class in this paper.

### 1.3. Coarse relative Toeplitz index

Let $(M_i, W_i, D_i)$ be an even relative index data over $(M, W)$. We define a coarse relative Toeplitz index $c\text{-ind}(\phi_1, D_1, \phi_2, D_2) \in K_1(C^\ast(W_1 \subset M_1))$ of $(M_i, W_i, D_i)$ and a function $\phi_i$ on $M_i$.

Let $\mathcal{W}(M)$ be a $C^\ast$-algebra generated by $\mathcal{W} = \mathcal{W}(M)$, which is the set of smooth and bounded functions with which gradient is bounded; see [8, Definition 2.1]. We define a relative version of this $C^\ast$-algebra. Denote by $\mathcal{W}$ a $C^\ast$-algebra generated by $(f_1, f_2) \in \mathcal{W}(M_1) \oplus \mathcal{W}(M_2)$ such that $f_1 \circ \psi = f_2$ on the complement of $W_2$, then we have

$$\mathcal{W} = \{(f_1, f_2) \in C_w(M_1) \oplus C_w(M_2) ; f_1 \circ \psi = f_2 \}.$$
Due to Lemma \[1.2\] \[1.3\] and \[8\] Remark 4.2, we have \((u_{\phi_1}, u_{\phi_2}) \in GL_i(\mathfrak{c})\). By the same calculation of the proof of \[8\] Proposition 4.3, we have

\[
[u_{\phi_1}, u_{\phi_2}] = \chi(D_1, \chi(D_2)) \in K_i(\mathfrak{c}).
\]

Note that an operator \(u_{\phi_1} - U^* u_{\phi_2} U\) is supported in \(N_{2r}(W_1)\) when the Fourier transform of \(\chi\) is compactly supported \(\text{Supp}(\chi) \subset (-r, r)\) by the proof of \[8\] Lemma 4.3. By using the map \(q : K_\ast(\mathfrak{c}) \to K_\ast(C^*(W_1 \subset M_1))\), we define a coarse relative Toeplitz index as follows.

**Definition 1.5.** Let \((M_i, W_i, D_i)\) be an even relative index data over \((M, W)\) and \((\phi_1, \phi_2) \in GL_i(\mathfrak{m})\). The coarse relative Toeplitz index is defined to be

\[
c\text{-ind}(\phi_1, D_1, \phi_2, D_2) = q([\phi_1, \phi_2] \hat{\otimes}_{\mathfrak{m}} [\mathfrak{c}, (\chi(D_1), \chi(D_2))]) \in K_1(C^*(W_1 \subset M_1)).
\]

Similarly, an odd relative index data defines an even Toeplitz index class in \(K_0(C^*(W_1 \subset M_1))\). We do not use the even class in this paper.

2. The Roe type cyclic 1-cocycle in the relative setting

Roe \[5\] defined a cyclic 1-cocycle on a complete Riemannian manifold partitioned by a closed submanifold of codimension 1. In this section, we generalize the cocycle to a pair \((M, W)\) partitioned by a (possibly non-compact) submanifold of codimension 1.

2.1. Definition of cyclic cocycle. Let \((M, W)\) be a pair of a complete Riemannian manifold \(M\) and a closed subset \(W \subset M\) and \(S \to M\) a Hermitian vector bundle. In this subsection, we define a cyclic 1-cocycle on a dense subalgebra of a relative ideal \(C^*(W \subset M)\), which is a generalization of the Roe cocycle.

**Definition 2.1.** Let \(M\) be a complete Riemannian manifold and \(W \subset M\) a closed subset. Assume that the triple \((M^+, M^-, N)\) satisfies the following conditions:

- \(M^+\) and \(M^-\) are submanifolds of \(M\) of the same dimension as \(M\), \(\partial M^\pm \neq \emptyset\)
- \(M = M^+ \cup M^-\)
- \(N\) is a submanifold of \(M\) of codimension 1,
- \(N = M^+ \cap M^- = -\partial M^+ = \partial M^-\)
- \(Z = N \cap W\) is compact,
- \(N\) and \(W\) are coarsely transversal, that is, for any \(r > 0\) there exists \(s > 0\) such that \(N_r(N) \cap N_r(W) \subset N_s(Z)\).

Then we call \((M^+, M^-, N)\) a partition of \((M, W)\). We also say \((M, W)\) is partitioned by \((M^+, M^-, N)\), or is partitioned by \(N\), for short.

Assume that \((M, W)\) is partitioned by \(N\) and set \(W^\pm = M^\pm \cap W\). Then for any \(r > 0\), there exists \(s > 0\) such that \(N_r(W^+) \cap N_r(W^-) \subset N_r(M^+) \cap N_r(W^-) \subset N_r(M^-) \subset N_r(N) \cap N_r(W) \subset N_s(Z)\).

In order to generalize Roe's cyclic 1-cocycle, we firstly prove the following. Denote by \(\Pi\) the characteristic function of \(M^+\) and set \(\Lambda = 2\Pi - 1\).

**Lemma 2.2.** An operator \([\Lambda, A]\) is of trace class on \(L^2(M, S)\) for any \(A \in \mathcal{B}_W\).

**Proof.** Assume that propagation of \(A \in \mathcal{B}_W\) is less than \(R\) and \(A\) is supported in \(N_r(W)\). Take \(s > 0\) such that \(N_{r+R}(W^+) \cap N_{r+R}(W^-) \subset N_s(Z)\). Note that operators \(\Pi A(1 - \Pi)\) and \((1 - \Pi)A\Pi\) are supported in \(N_{r+R}(W^+) \cap N_{r+R}(W^-) \subset\)
Since $A$ is locally traceable and $N_s(Z)$ is compact, these operators $\Pi A(1 - \Pi)$ and $(1 - \Pi)\Pi$ are of trace class. Thus an operator

$$[\Lambda, A] = [2\Pi, A] = 2(\Pi A(1 - \Pi) - (1 - \Pi)\Pi)$$

is of trace class.

By Lemma 2.2, the following bilinear map is well-defined and is cyclic 1-cocycle.

**Definition 2.3.** Define a map $\zeta : \mathcal{B}_W \times \mathcal{B}_W \to \mathbb{C}$ by

$$\zeta(A, B) = \frac{1}{4} \text{Tr}(A[\Lambda, A][\Lambda, B]).$$

**Proposition 2.4.** The bilinear map $\zeta : \mathcal{B}_W \times \mathcal{B}_W \to \mathbb{C}$ in Definition 2.3 is cyclic 1-cocycle on $\mathcal{B}_W$.

**Proof.** By equalities $\Lambda[\Lambda, A] = -[\Lambda, A]\Lambda$ and $[\Lambda, AB] = A[\Lambda, B] + [\Lambda, A]B$ and trace property imply this proposition. This is essentially the same as Roe’s proof of [5, Proposition 1.6].

By Lemma 2.2, a pair $(L^2(M, S), \Lambda)$ is a Fredholm module over $C^*(W \subset M) \subset \mathcal{B}_W$. Thus a Banach algebra

$$\mathcal{A}_W = \{T \in C^*(W \subset M) : [\Lambda, T] \text{ is of trace class}\}$$

with norm $\|T\|_{\mathcal{A}_W} = \|T\| + \|[\Lambda, T]\|_1$ is holomorphically closed in $C^*(W \subset M)$ by [1, p.92 Proposition 3], here $\|\cdot\|_1$ is the trace norm. Moreover, $\mathcal{A}_W$ is dense in $C^*(W \subset M)$. Thus the inclusion $i : \mathcal{A}_W \to C^*(W \subset M)$ induces an isomorphism of $K$-theory $i_* : K_*(\mathcal{A}_W) \cong K_*(C^*(W \subset M))$. Since $\zeta$ can be extended to $\mathcal{A}_W$, we have the following additive map by Connes’ pairing of $K$-theory with cyclic cohomology.

**Definition 2.5.** [1, p.109] Define the map

$$\zeta_* : K_1(C^*(W \subset M)) \to \mathbb{C}$$

by $\zeta_*([u]) = \frac{1}{\pi i} \sum_{i,j} \zeta((u^{-1})_{ji}, u_{ij})$, where we assume $[u]$ is represented by an element $u \in GL_l(\mathcal{A}_W)$ and $u_{ij}$ is the $(i,j)$-component of $u$. We note that this is Connes’ pairing of cyclic cohomology with $K$-theory, and $1/8\pi i$ is a constant multiple in Connes’ pairing.

By standard calculation implies the following; see, for instance, [5, Proposition 1.13].

**Proposition 2.6.** For any $u \in GL_l(C^*(W \subset M))$, one has

$$\zeta_*([u]) = -\frac{1}{8\pi i} \text{index}(\Pi u \Pi : \Pi(L^2(M, S))^l \to \Pi(L^2(M, S))^l).$$

**Remark 2.7.** By an isomorphism [10, Proposition 4.3.12]

$$K_*(C^*(W)) \cong K_*(C^*(W \subset M))$$

induced by the inclusion $W \to M$, any $x \in K_1(C^*(W \subset M))$ can be represented by $u \in GL_l(C^*(W \subset M))$ such that an operator $u - 1$ is supported in $N_r(W)$ for arbitrary small $r > 0$. Denote by $p_r$ the characteristic function of $N_r(W)$. Under the notations, we have

$$\zeta_*([x]) = -\frac{1}{8\pi i} \text{index}(\Pi p_r \Pi : \Pi(L^2(M, S))^l \to \Pi(L^2(M, S))^l).$$
2.2. **Some properties.** Let \((M, W)\) be a pair of a complete Riemannian manifold \(M\) and a closed subset \(W \subset M\) and \(S \to M\) a Hermitian vector bundle. In this subsection, we prove properties of the map \(\zeta_*\) which we use. Firstly, we prove “cobordism” invariance of \(\zeta_*\).

**Lemma 2.8.** Let \(N\) and \(N'\) be two partitions of \((M, W)\). Denote by \(\zeta\) and \(\zeta'\) the cyclic cocycle introduced in Definition 2.3 by using the partitions \(N\) and \(N'\), respectively. Assume that \(N' \subset N_r(N)\) for some \(r > 0\). Then we have

\[
\zeta_* = \zeta'_* : K_1(C^*(W \subset M)) \to \mathbb{C}.
\]

**Proof.** Denote by \(\Pi'\) the characteristic function of \(M^{++}\) and set \(\phi = \Pi - \Pi'\). Take any \(x \in K_1(C^*(W \subset M))\), \(x\) can be represented by \(u = v + 1 \in GL_l(C^*(W \subset M))\) such that \(v \in M_l(C^*(W \subset M))\) is supported near \(W\) as in Remark 2.7. Then we have

\[
\zeta_*(x) = \frac{1}{8\pi i} \text{index}(\Pi u \Pi) \text{ on } \Pi (L^2(M, S))^l = \frac{1}{8\pi i} \text{index}(\Pi v + 1)
\]

and

\[
\zeta'_*(x) = \frac{1}{8\pi i} \text{index}(\Pi' u \Pi' \text{ on } \Pi' (L^2(M, S))^l) = \frac{1}{8\pi i} \text{index}(\Pi' v + 1)
\]

since operators \([\Pi, v]\) and \([\Pi', v]\) are compact. By the way, properties \(\text{Supp}(\phi) \subset N_r(N), v\) is locally compact and \(v\) is supported near \(W\) imply an operator \((\Pi v + 1) - (\Pi' v + 1) = \phi v\) is compact. Thus we have \(\zeta_*(x) = \zeta'_*(x)\) for any \(x \in K_1(C^*(W \subset M))\). \(\square\)

Secondly, we shall prove an analogue of Higson’s Lemma [3, Lemma 3.1].

**Lemma 2.9.** Let \((M, W)\) and \((M', W')\) be two pairs which is partitioned by \(N\) and \(N'\), respectively, and \(S \to M\) and \(S' \to M'\) two Hermitian vector bundles. Let \(\Pi\) and \(\Pi'\) be the characteristic function of \(M^+\) and \(M'^+\), respectively. We assume that there exists an isometry \(\gamma : M^+ \to M'^+\) which lifts an isomorphism \(\gamma^* : S|M^+ \to S'|M'^+\). We denote the Hilbert space isometry defined by \(\gamma^*\) by the same letter \(\gamma^* : \Pi (L^2(M, S)) \to \Pi' (L^2(M', S'))\). Take \(u \in GL_l(C^*(W \subset M))\) and \(u' \in GL_l(C^*(W' \subset M'))\) such that \(\gamma^* u \Pi \sim \Pi' u' \gamma^*\). Then one has \(\zeta_*([u]) = \zeta'_*([u'])\).

**Proof.** It suffices to show the case when \(l = 1\). Let \(v : (1 - \Pi) (L^2(M, S)) \to (1 - \Pi') (L^2(M', S'))\) be any invertible operator. Then \(V = \gamma^* u \Pi + v (1 - \Pi) : L^2(M, S) \to L^2(M', S')\) is also invertible. Hence we obtain

\[
V((1 - \Pi) + u \Pi) - ((1 - \Pi') + u' \Pi') V = \gamma^* u \Pi - \Pi' u' \gamma^* - \gamma^* u \Pi - \Pi' u' \gamma^* \sim 0.
\]

Therefore, we obtain \(\zeta_*([u]) = \zeta'_*([u'])\) since \(V\) is an invertible operator and one has \(-8\pi i \zeta_*([u]) = \text{index}(\Pi u \Pi) = \text{index}(1 - \Pi) + \Pi u \Pi)\) and \(-8\pi i \zeta'_*([u']) = \text{index}((1 - \Pi') + \Pi' u' \Pi')\). \(\square\)

Let \((M, W)\) and \((M', W')\) be two pairs which is partitioned by \(N\) and \(N'\), respectively, and \(S \to M\) and \(S' \to M'\) two Hermitian vector bundles. Let \(\Pi\) and \(\Pi'\) be the characteristic function of \(M^+\) and \(M'^+\), respectively. We assume that there exists an isometry \(\gamma : N_r(W) \to N_r(W')\) which lifts an isomorphism \(\gamma^* : S_1|N_r(W) \to S_2|N_r(W')\). We define an additive map \(\Gamma : K_1(C^*(W \subset M)) \to \cdots\)
K_1(C^*(W' \subset M')) as follows. Take any x ∈ K_1(C^*(W \subset M)). x can be represented by u = v + 1 ∈ GL_1(C^*(W \subset M)) such that v ∈ M_1(C^*(W \subset M)) is supported in N_r(W); see Remark 2.7. Then we have γ^*u(γ)^{-1} = γ^*v(γ)^{-1} + 1 ∈ GL_1(C^*(W' \subset M')), here γ^*v(γ)^{-1} ∈ M_1(C^*(W' \subset M')) is supported in N_r(W'). The K_1-class of γ^*u(γ)^{-1} does not depend on the choice of such an above u. Set Γ(x) = [γ^*u(γ)^{-1}] ∈ K_1(C^*(W' \subset M'))).

Lemma 2.10. Moreover, we assume that an isometry γ: N_r(W') → N_r(W) preserves partitions, that is, γ satisfies IV = Π ◦ γ on N_r(W'). Then we have ζ'_i ◦ Γ = ζ_i on K_1(C^*(W \subset M)).

Proof. Let p_r and p'_r be the characteristic function of N_r(W) and N_r(W'), respectively. Take any x ∈ K_1(C^*(W \subset M)). We represent x by u = v + 1 ∈ GL_1(C^*(W \subset M)) such that v ∈ M_1(C^*(W \subset M)) is supported in N_r(W). We have

ζ'_i ◦ Γ(x) = index ((1 − p'_r Π') + Π'p'_r γ^*u(γ)^{-1}p'_r Π').

Take any invertible operator ν : L^2((N_r(W) \cap M^+)^c, S) → L^2((N_r(W') \cap M'^+)^c, S') and set V = γ^* + ν. Then we have

((1 − p'_r Π') + Π'p'_r γ^*u(γ)^{-1}p'_r Π') V − V ((1 − p_r Π) + Π_p, up_r Π) = Π'p'_r γ^*u(γ)^{-1}p'_r Πγ^* − γ^*Π_p, up_r Π = 0,

here we used Π' = Π ◦ γ on N_r(W'). Thus we obtain

ζ'_i ◦ Γ(x) = − \frac{1}{8πi} \text{index} ((1 − p'_r Π') + Π'p'_r γ^*u(γ)^{-1}p'_r Π')

= − \frac{1}{8πi} \text{index} ((1 − p_r Π) + Π_p, up_r Π) = ζ_i(x).

□

3. Relative index formula on odd dimension

In this section, we state and prove an index theorem for an odd relative index data partitioned by submanifolds of codimension 1. This index formula is a variant of [4] Theorem 3.3.

3.1. Index theorem. Firstly, we introduce a partition of a relative index data over a pair (M, W). See also [4].

Definition 3.1. Let (M, W) be a pair of a complete Riemannian manifold M and W ⊂ M a closed subset. We say a relative index data (M_1, W_1, D_1) over (M, W) is partitioned by (N_1, N_2) if the following hold.

• (M_1, W_1) is partitioned by N_1,
• Π_2 = Π_1 ◦ ψ, here Π_1 is the characteristic function of M_1^c,
• there exists a closed submanifold N ⊂ M which partitions (M, W) such that N_1 = f_i^{-1}(N) and f_i(Z_i) = W \cap N.

Let (M_i, W_i, D_i) be an odd relative index data over (M, W) partitioned by (N_1, N_2). The Dirac operator D_i induces a graded Dirac operator D_{N_i} on S|_{N_i} → N_i and they satisfy (ψ_{N_2} \setminus Z_i)^* ◦ D_{N_1} = D_{N_2} ◦ (ψ_{N_2} \setminus Z_2)^*. Then the relative topological index ind_i(D_{N_1}, D_{N_2}) ∈ Z is obtained.

Following [2] Section 4, we recall the definition of the relative topological index ind_i(D_{N_1}, D_{N_2}) ∈ Z. Chop off the manifold N_i outside of Z_i by a closed submanifold
$H_i$ of codimension 1 to obtain a compact manifold $\Omega_i$ with boundary $\partial \Omega_i = H_i$ such that $N_i(\mathbb{Z}) \subset \Omega_i$ for some $r > 0$. Let $N^i$ be a closed manifold such that $\Omega_i \subset N^i$ and $S_{N^i} \to \tilde{N}_i$ a graded Clifford bundle. Assume that all structures on $\Omega_i \subset N^i$ are isomorphic to those on $\Omega_i \subset N_i$, respectively, and there exists an isometry $\psi : \tilde{N}_2 \setminus Z_2 \to \tilde{N}_1 \setminus Z_1$ such that $\psi$ induces an isometry of graded Clifford bundles $\tilde{\psi}^* : \tilde{S}_{N_1} |_{\tilde{N}_1 \setminus Z_1} \to \tilde{S}_{N_2} |_{\tilde{N}_1 \setminus Z_2}$. There is the graded Dirac operator $\tilde{D}_{N^i}$ on $\tilde{S}_{N^i}$. Set

$$\text{ind}_i(D_{N^i} \cup D_{N^2}) = \text{index}(\tilde{D}_{N_1}^-) \pm \text{index}(\tilde{D}_{N_2}^+) \in \mathbb{Z}.$$  

The value is independent of the choice of a compactification $\tilde{N}_i$ and a graded Clifford bundle $\tilde{S}_i$.

Our first main theorem is the following. This is a variant of [1 Theorem 3.3].

**Theorem 3.2.** Let $M$ be a complete Riemannian manifold and $W \subset M$ a closed subset. Let $(M_i, W_i, D_i)$ be an odd relative index data over $(M, W)$ which is partitioned by $(N_1, N_2)$. Then the following formula holds:

$$\zeta_\ast(c\text{-ind}(D_1, D_2)) = -\frac{1}{8\pi i} \text{ind}_i(D_{N^1}, D_{N^2}).$$

We prove Theorem 3.2 in Subsection 3.2. In the proof of Theorem 3.2, we do not use the fact that the relative topological index $\text{ind}_i(D_{N^i}, D_{N^2})$ does not depend on the choice of a compactification $\tilde{N}_i$ and a graded Clifford bundle $\tilde{S}_{N^i}$. Thus Theorem 3.2 gives a new proof of well definedness of $\text{ind}_i(D_{N^1}, D_{N^2})$.

**Remark 3.3.** By the definition of the relative topological index and the vanishing of the Fredholm index of the Dirac operator on closed manifolds of odd dimension, the relative topological index vanishes when $M_i$ is of even dimension. Thus the value $\zeta_\ast(c\text{-ind}(D_1, D_2))$ also vanishes. We prove another relative index theorem for partitioned manifolds with non-vanishing the value $\zeta_\ast(x)$ when $M_i$ is of even dimension, in Section 7.

3.2. **Proof.** In this subsection, we prove Theorem 3.2. There are 2 steps to prove it, the first one is the reduction to the product case and the second one is the proof of the product case.

Firstly, we reduce the product case. Take a tubular neighbourhood of $N_i$ diffeomorphic to $(-1, 1) \times N_i$ such that $[0, 1) \times N_i \subset M_+^i$. By Lemma 2.9, we can replace $W_i$ by $W_i \cup (Z_i \times [0, 1))$ without changing the value $\zeta_\ast(c\text{-ind}(D_1, D_2))$. Fix small $r > 0$. Take a submanifold $N'_i \subset M_i$ which partitions $(M_i, W'_i)$ such that $N'_i \subset N_r(M^-_i)$ and $N'_i \cap M^-_i = \emptyset$. Denote by $\zeta_\ast$ the cyclic cocycle defined by using this new partition. By Lemma 2.8 we have

$$\zeta_\ast(c\text{-ind}(D_1, D_2)) = (\zeta_\ast)_\ast(c\text{-ind}(D_1, D_2)).$$

Next we take a relative index data $(M'_i, W'_i, D'_i)$ over $(M', W')$ as follows. We set $M'_i = (\mathbb{R} \times N_i) \cup M^+_i$ and $W'_i = (\mathbb{R} \times Z_i) \cup W^+_i$, here $i = 1, 2$ or empty and a metrizable product on $(-\infty, -r] \times N_i$. A Clifford bundle $S'_i \to M'_i$ satisfies $S'_i |_{M^+_i} = S_i |_{M^+_i}$ and $S'_i = (-\infty, -r] \times S_i |_{N_i}$. An isometry $\psi' : M'_2 \setminus W'_2 \to M'_1 \setminus W'_1$ satisfies $\psi'|_{M^+_2} = \psi|_{M^+_1}$ and $\psi' = \text{id} \times \psi|_{N}$ on $(-\infty, -r] \times (Z_2)^c$ and a continuous coarse map $f'_1 : M'_1 \to M'$ satisfies $f'_1 |_{M^+_1} = f |_{M^+_1}$ and $f'_1 = \text{id} \times f|_{N}$ on $(-\infty, -r] \times N_2$. A pair $(M'_1, W'_1)$ is partitioned by $N'_1$. Denote by $\zeta'$ the cyclic cocycle on $M'_1$. Note
Lemma 3.5. By using the map
Then (\definition of the relative topological index. Set
W defined by the identity map
N, \zeta)
and a compact subset
defined by using
Take unitaries
U, \zeta, M, \zeta
that we do not have to care the relative index data (\zeta, \zeta) is partitioned or not, that is, we do not have to care \(N' = (f')^{-1}(N')\) holds or not for some \(N'\).

Lemma 3.4. We have
\((\zeta_\ast)( \c-ind(D_1, D_2)) = \zeta_\ast( \c-ind(D_1', D_2'))\).

Proof. Take unitaries \(U\) and \(U'\) appeared in the definition of the projection \(q\) such that \(U = U'\) on \(L^2(M_i^+, S_i')\) and propagation of \(U\) and \(U'\) is less than \(r/4\). Take a function \(f \in U_1(C_0(\mathbb{R}))\) such that \(\text{Supp}(\hat{f}) \subset (-r/4, r/4)\) and \([f] = \left[\begin{smallmatrix} e^{i \pi} \\ e^{-i \pi} \end{smallmatrix}\right] \in K_1(C_0(\mathbb{R}))\). We have
\[
c-ind(D_1, D_2) = [f(D_1)U^* f(D_2)^* U]
\]
and
\[
c-ind(D_1', D_2') = [f(D_1')U'^* f(D_2')^* U']\).
\]
Since propagation of \(f(D_1)U^* f(D_2)^* U\) and \(f(D_1')U'^* f(D_2')^* U'\) is less than \(r\), we have
\[
\Pi_{1, r} f(D_1)U^* f(D_2)^* U \Pi_{1, r} = \Pi_{1}' f(D_1')U'^* f(D_2')^* U' \Pi_{1}',
\]
here \(\Pi_{1, r}\) (resp. \(\Pi_{1}'\)) is the characteristic function of \(M_{1, r}\) (resp. \(M_{1}' = (M_{1, r})\)). By using Lemma 2.10, we complete the proof.

We apply the same argument to \((M_i', W_i', D_i')\), so that the proof is reduced to the following product case. Let \((N, Z)\) be a pair of a complete Riemannian manifold \(N\) and a compact subset \(Z \subset N\) and \((N_i, Z_i, D_{N_i})\) an even relative index data over \((N, Z)\). Then \((M_i = \mathbb{R} \times N_i, W_i = \mathbb{R} \times Z_i, D_i)\) is an odd relative index data over \((M = \mathbb{R} \times N, W = \mathbb{R} \times Z)\), here the Dirac operator \(D_i\) on \(\mathbb{R} \times N_i\) is canonically defined by using \(D_{N_i}\). \((M_i, W_i, D_i)\) is partitioned by \((\{0\} \times N_1, \{0\} \times N_2)\). An isometry \(\psi\) and a continuous coarse map \(f_i\) are given by the product \(\text{id}_{\mathbb{R}} \times \psi_i N\) and \(\text{id}_{\mathbb{R}} \times f_i\), respectively.

Let us prove the product case. Take closed manifolds \(\tilde{N}_1\) and \(\tilde{N}_2\) as in the definition of the relative topological index. Set \(\tilde{M}_i = \mathbb{R} \times \tilde{N}_i\) and \(\tilde{W}_i = \mathbb{R} \times Z_i = W_i\). Then \((\tilde{M}_i, \tilde{W}_i, \tilde{D}_i)\) is a relative index data over \((\mathbb{R}^2, \mathbb{R})\), here a continuous coarse map \(\tilde{f}_i : \tilde{M}_i \to \mathbb{R}^2\) is defined to be \(\tilde{f}_i(x, y) = (x, \text{dist}(y, Z_i))\). In order to use Lemma 2.10, we prove the following. This is based on a concept that coarse relative index depends only on a neighborhood of \(W_i\); see also [7, Proposition 4.7].

Lemma 3.5. By using the map \(\Gamma : K_1(C^*(W_1 \subset M_1)) \to K_1(C^*(\tilde{W}_1 \subset \tilde{M}_1))\) defined by the identity map \(N_r(W_1) \to N_r(\tilde{W}_1)\), we have
\[
\Gamma(\c-ind(D_1, D_2)) = \c-ind(\tilde{D}_1, \tilde{D}_2).
\]

Proof. Take unitaries \(U\) and \(\tilde{U}\) appeared in the definition of the projection \(q\) such that \(U = \tilde{U}\) on \(L^2(N_r(W_1), S_1)\) and propagation of \(U\) and \(\tilde{U}\) is less than \(r/8\). Take a function \(f \in U_1(C_0(\mathbb{R}))\) such that \(\text{Supp}(\hat{f}) \subset (-r/4, r/4)\) and \([f] = \left[\begin{smallmatrix} e^{i \pi} \\ e^{-i \pi} \end{smallmatrix}\right] \in K_1(C_0(\mathbb{R}))\). We have
\[
c-ind(D_1, D_2) = [f(D_1)U^* f(D_2)^* U]
\]
and
\[
c-ind(\tilde{D}_1, \tilde{D}_2) = [f(\tilde{D}_1)\tilde{U}^* f(\tilde{D}_2)^* \tilde{U}].
\]
Since operators $f(D_1) - U^*f(D_2)U$ and $f(D_1)^* - U^*f(D_2)^*U$ are supported in $N_{r/4}(W_1)$, an operator $f(D_1)U^*f(D_2)^*U - 1$ is supported in $N_{r/4}(W_1)$. Since propagation of $f(D_1)U^*f(D_2)^*U$ is less than $3r/4$, we have

$$\Gamma(c\text{-}ind(D_1, D_2)) = [f(D_1)U^*f(D_2)^*U] = [f(D_1)\tilde{U}^*f(\tilde{D}_2)^*\tilde{U}] = c\text{-}ind(\tilde{D}_1, \tilde{D}_2).$$

Finally, we complete the proof of Theorem 3.2 for the product case. By Lemma 2.10, we have

$$\zeta_r(c\text{-}ind(D_1, D_2)) = \tilde{\zeta}_r(\Gamma(c\text{-}ind(D_1, D_2))).$$

By Lemma 3.5, the value equals $\tilde{\zeta}_r(c\text{-}ind(\tilde{D}_1, \tilde{D}_2))$. Since $\tilde{N}_1$ and $\tilde{N}_2$ is closed and we can take a unitary $\tilde{U}$ satisfies $\tilde{U}\Pi_1 = \Pi_2\tilde{U}$, we have

$$\tilde{\zeta}_r(c\text{-}ind(\tilde{D}_1, \tilde{D}_2)) = \frac{1}{8\pi i}\text{ind}\left(\Pi_1 \tilde{D}_1 - i\tilde{U}^*\tilde{D}_2 + i\tilde{U}\Pi_1\right)\left(\Pi_2 \tilde{D}_2 - i\Pi_2\right) = \frac{1}{8\pi i}\text{ind}\left(\Pi_1 \tilde{D}_1 - i\Pi_1\right) + \frac{1}{8\pi i}\text{ind}\left(\Pi_2 \tilde{D}_2 - i\Pi_2\right).$$

By an index theorem on partitioned manifold [3], Theorem 3.3, the value equals

$$-\frac{1}{8\pi i}\left(\text{ind}(\tilde{D}_{N_1}^+) - \text{ind}(\tilde{D}_{N_2}^+)\right).$$

This is nothing but the relative topological Toeplitz index, so that the proof is completed.

4. RELATIVE INDEX FORMULA ON EVEN DIMENSION

In this section, we state and prove an index theorem for an even relative index data partitioned by submanifolds of codimension 1. This index formula is a counterpart of Theorem 3.2.

4.1. INDEX THEOREM. Let $(M_i, W_i, D_i)$ be an even relative index data over $(N_1, N_2)$ partitioned by $(N_1, N_2)$. The Dirac operator $D_i$ induces a Dirac operator $D_{N_i}$ on a Clifford bundle $S_{N_i} = S^+|_{N_i} \to N_i$ and they satisfies $(\psi_{N_i}|_{Z_2})^* \circ D_{N_1} = D_{N_2} \circ (\psi_{N_i}|_{Z_2})^*$. We denote by $(\phi_{N_1}, \phi_{N_2})$ the pair of restriction of functions $(\phi_1, \phi_2) \in GL_2(\mathbb{H})$ to $N_1$ and $N_2$, respectively. Then we have $\phi_{N_i} \circ \psi|_{Z_2} = \phi_{N_2}|_{Z_2}$.

We define the relative topological Toeplitz index $\text{ind}(\phi_{N_1}, D_{N_1}, \phi_{N_2}, D_{N_2})$. Let $\tilde{N}_i$ be a closed manifold such that $\Omega_i \subset \tilde{N}_i$ and $S_{N_i} \to \tilde{N}_i$ a Clifford bundle as in subsection 4.1. Namely, we assume that all structures on $\Omega_i \subset N_i$ are isomorphic to those on $\Omega_i \subset \tilde{N}_i$, respectively, and there exists an isometry $\tilde{\psi} : \tilde{N}_2 \setminus Z_2 \to \tilde{N}_1 \setminus Z_1$ such that $\tilde{\psi}$ induces isometry of graded Clifford bundles $\psi^* : S_{\tilde{N}_1}|_{\tilde{N}_1 \setminus Z_1} \to S_{\tilde{N}_2}|_{\tilde{N}_2 \setminus Z_2}$. There is the Dirac operator $\tilde{D}_{N_i}$ on $S_{\tilde{N}_i}$. Take $\phi_{N_i} \in GL_2(C(\tilde{N}_i))$ such that $\phi_{N_i}|_{\Omega_i} = \tilde{\phi}_{N_i}|_{\Omega_i}$ and $\phi_{N_1} \circ \tilde{\psi} = \phi_{N_2}|_{\tilde{N}_2 \setminus Z_2}$. Denote by $\mathcal{H}_i$ the subspace of $L^2(\tilde{\tilde{N}}_i, S_{\tilde{N}_i})$ generated by the non-negative eigenvectors of $\tilde{D}_i$ and let $P_i : L^2(\tilde{\tilde{N}}_i, S_{\tilde{N}_i})^l \to \mathcal{H}_i^l$ be the projection. Then for any $s \in \mathcal{H}_i^l$, we define
the Toeplitz operator $T_{\tilde{\phi}_N} : \mathcal{H}_d^1 \rightarrow \mathcal{H}_d^1$ by $T_{\tilde{\phi}_N}(s) = P_t \tilde{\phi}_N s$. The Toeplitz operator $T_{\tilde{\phi}_N}$ is Fredholm since the values of $\tilde{\phi}_N$ are in $GL_1(\mathbb{C})$. Set
\[
\text{ind}_t(\phi_{N_1}, D_{N_1}, \phi_{N_2}, D_{N_2}) = \text{index} \left( T_{\tilde{\phi}_{N_1}} \right) - \text{index} \left( T_{\tilde{\phi}_{N_2}} \right) \in \mathbb{Z}.
\]
The value is independent of the choice of a compactification $\tilde{N}_i$, a Clifford bundle $\tilde{S}_{N_i}$ and a function $\tilde{\phi}_{N_i}$. This is essentially due to [2, Proposition 4.6].

Our second main theorem is the following. This is a counterpart of Theorem 3.2 and also a generalization of [8, Theorem 2.6].

**Theorem 4.1.** Let $M$ be a complete Riemannian manifold and $W \subset M$ a closed subset. Let $(M_i, W_i, D_i)$ be an even relative index data over $(M, W)$ which is partitioned by $(N_1, N_2)$. Take $(\phi_1, \phi_2) \in GL_1(\mathbb{C})$. Then the following formula holds:
\[
\zeta_t(\text{c-ind}(\phi_1, D_1, \phi_2, D_2)) = -\frac{1}{8\pi i} \text{ind}_t(\phi_{N_1}, D_{N_1}, \phi_{N_2}, D_{N_2}).
\]

We prove Theorem 4.1 in Subsection 4.2. In the proof of Theorem 4.1 we do not use the fact that the relative topological Toeplitz index $\text{ind}_t(\phi_{N_1}, D_{N_1}, \phi_{N_2}, D_{N_2})$ does not depend on the choice of a compactification $\tilde{N}_i$, a Clifford bundle $\tilde{S}_{N_i}$ and a function $\tilde{\phi}_{N_i}$. Thus Theorem 4.1 gives a new proof of well definedness of $\text{ind}_t(\phi_{N_1}, D_{N_1}, \phi_{N_2}, D_{N_2})$. That is the same as the case of $\text{ind}_t(D_{N_1}, D_{N_2})$.

### 4.2. Proof.
There are 2 steps to prove it, which is similar to Subsection 3.2. Namely, the first step is the reduction to the product case and the second one is the proof of the product case.

Similar argument in Subsection 3.2 implies we can reduce the product case. A function $\phi_i'$ on $M_i'$ is taken as $\phi_i' \circ \psi' = \phi_i'$ on $M_i' \setminus W_i'$, $\phi_i' = \phi_i$ on $M^+$ and $\phi_i' = 1 \otimes \phi_{N_i}$ on $(-\infty, -r] \times N_i$. The counterpart of Lemma 3.4 is as follows.

**Lemma 4.2.** We have
\[
(\zeta_t)_*(\text{c-ind}(\phi_1, D_1, \phi_2, D_2)) = \zeta_t'_*(\text{c-ind}(\phi_i', D_i', \phi_i', D_i')).
\]

**Proof.** Take unitaries $U$ and $U'$ with propagation less than $r/4$ as in the proof of Lemma 3.4. Take a chopping function $\chi$ such that $\text{Supp}(\chi) \subset (-r/8, r/8)$. Then operators $u_{\phi_i} U^* u_{\phi_i}^{-1} U$ and $u_{\phi_i'} U'^* u_{\phi_i'}^{-1} U'$ have propagation less than $r$. Thus we have
\[
\Pi_{1, r} u_{\phi_i} U^* u_{\phi_i}^{-1} U \Pi_{1, r} = \Pi_{1}' u_{\phi_i'} U'^* u_{\phi_i'}^{-1} U' \Pi_{1}'.
\]
By the way, we recall that
\[
\text{c-ind}(\phi_1, D_1, \phi_2, D_2) = \left[ u_{\phi_1} U^* u_{\phi_2}^{-1} U \right].
\]
Therefore, we complete the proof by using Lemma 2.9.

Therefore, we reduced to the product case $(M_i = \mathbb{R} \times N_i, W_i = \mathbb{R} \times Z_i, D_i)$ similar to Subsection 3.2. Here, a function $\phi_i$ is given by $\phi_i = 1 \otimes \phi_{N_i}$.

Let us prove the product case. Take a closed manifold $\tilde{N}_i$ and a function $\tilde{\phi}_N$, as in the definition of the relative topological Toeplitz index. Set $\tilde{M}_i = \mathbb{R} \times \tilde{N}_i$, $\tilde{W}_i = \mathbb{R} \times Z_i = W_i$ and $\tilde{\phi}_i = 1 \otimes \phi_{N_i}$. Then $(\tilde{M}_i, \tilde{W}_i, \tilde{D}_i)$ is an even relative index data over $(\mathbb{R}^2, \mathbb{R})$, here a continuous coarse map $f_i : M_i \rightarrow \mathbb{R}^2$ is defined to be $f_i(x, y) = (x, \text{dist}(y, Z_i))$. The counterpart of Lemma 3.5 is as follows.
Lemma 4.3. By using the map $\Gamma : K_1(C^*(W_1 \subset M_1)) \to K_1(C^*(\tilde{W}_1 \subset \tilde{M}_1))$ defined by the identity map $N_r(W_1) \to N_r(\tilde{W}_1)$, we have
\[
\Gamma(c\text{-}\text{ind}(\phi_1, D_1, \phi_2, D_2)) = c\text{-}\text{ind}(\tilde{\phi}_1, \tilde{D}_1, \tilde{\phi}_2, \tilde{D}_2).
\]

Proof. Take unitaries $U$ and $\tilde{U}$ such that $U = \tilde{U}$ on $L^2(N_r(W_1), S_1)$ and propagation of $U$ and $\tilde{U}$ is less than $r/8$ as in the proof of Lemma 3.3. Take a chopping function $\chi$ such that $\text{Supp}(\chi) \subset (-r/8, r/8)$. Then operators $u_{\phi_1} - U^*u_{\phi_2}U$ and $u_{\phi_1}^{-1} - U^*u_{\phi_2}^{-1}U$ are supported in $N_{r/4}(W_1)$, so that an operator $u_{\phi_1}U^*u_{\phi_2}^{-1}U - 1$ is supported in $N_{r/4}(W_1)$. Similarly, an operator $u_{\phi_1}^{-1}\tilde{U}^*u_{\phi_2}^{-1}\tilde{U} - 1$ is supported in $N_{r/4}(\tilde{W}_1) = N_{r/4}(W_1)$. Since propagation of $u_{\phi_1}U^*u_{\phi_2}^{-1}U$ and $u_{\phi_1}^{-1}\tilde{U}^*u_{\phi_2}^{-1}\tilde{U}$ is less than $3r/4$, we have
\[
\Gamma(c\text{-}\text{ind}(\phi_1, D_1, \phi_2, D_2)) = c\text{-}\text{ind}(\tilde{\phi}_1, \tilde{D}_1, \tilde{\phi}_2, \tilde{D}_2).
\]

Finally, we complete the proof of Theorem 4.1 for the product case. By Lemma 2.10 we have
\[
\zeta_*\left(c\text{-}\text{ind}(\phi_1, D_1, \phi_2, D_2) = \zeta_*\left(\Gamma(c\text{-}\text{ind}(\phi_1, D_1, \phi_2, D_2))\right)\right).
\]

By Lemma 4.3 the value equals $\zeta_*\left(c\text{-}\text{ind}(\tilde{\phi}_1, \tilde{D}_1, \tilde{\phi}_2, \tilde{D}_2)\right)$. Since $\tilde{N}_1$ and $\tilde{N}_2$ are closed and we can take a unitary $\tilde{U}$ satisfies $\tilde{U}\tilde{\Pi}_1 = \tilde{\Pi}_2\tilde{U}$, we have
\[
\tilde{\zeta}_*(c\text{-}\text{ind}(\tilde{\phi}_1, \tilde{D}_1, \tilde{\phi}_2, \tilde{D}_2)) = \frac{1}{8\pi i}\text{index}\left(\tilde{\Pi}_1u_{\phi_1}^{-1}\tilde{U}^*u_{\phi_2}^{-1}\tilde{U}\tilde{\Pi}_1\right) = \frac{1}{8\pi i}\text{index}\left(\tilde{\Pi}_1u_{\phi_1}^{-1}\tilde{\Pi}_1\right) + \frac{1}{8\pi i}\text{index}\left(\tilde{\Pi}_2u_{\phi_2}^{-1}\tilde{\Pi}_2\right).
\]

By an index theorem for Toeplitz operators on partitioned manifolds [5, Theorem 2.6], the value equals
\[
-\frac{1}{8\pi i}\left(\text{index}\left(T_{\phi_{\tilde{N}_1}}\right) - \text{index}\left(T_{\phi_{\tilde{N}_2}}\right)\right).
\]

This is nothing but the relative topological Toeplitz index, so that the proof is completed.

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