A Nonconservative Earthquake Model of Self-Organized Criticality on a Random Graph

Stefano Lise and Maya Paczuski
Department of Mathematics, Huxley Building, Imperial College of Science, Technology, and Medicine, London UK SW7 2BZ
(March 22, 2022)

We numerically investigate the Olami-Feder-Christensen model on a quenched random graph. Contrary to the case of annealed random neighbors, we find that the quenched model exhibits self-organized criticality deep within the nonconservative regime. The probability distribution for avalanche size obeys finite size scaling, with universal critical exponents. In addition, a power law relation between the size and the duration of an avalanche exists. We propose that this may represent the correct mean-field limit of the model rather than the annealed random neighbor version.

PACS numbers: 05.65.+b, 45.70.Ht, 89.75.-k

The idea of self-organized criticality (SOC) was introduced as a possible explanation for the widespread occurrence in nature of long range correlations in space and time \([1]\). The term refers to the intrinsic tendency of a large class of spatially extended dynamical systems to spontaneously organize into a dynamical critical state. In general, SOC systems are driven externally at a very slow rate and relax with bursts of activity, avalanches, on a very fast time scale. One signature of SOC is a scale free, e.g. power law, distribution of avalanche sizes. This is normally related to some long range spatial and temporal correlations within the system. Typical natural realizations of this phenomena include, among others, earthquakes, forest fires, and biological evolution (for reviews, see \([2,3]\)).

A problem that has attracted a lot of attention, but is still poorly understood, is that of identifying fundamental mechanisms leading to SOC behavior. In particular, much effort has been directed at understanding how conservation of the transported quantity (e.g. sand) in the avalanche dynamics affects criticality \([4,5]\). For instance, it is well known that the Abelian sandpile model \([1]\), is subcritical if dissipation is introduced \([2]\). On the other hand, nonconservative sandpile models which display criticality have since been introduced \([3]\). Although both the analytical and numerical evidence in favor of criticality are quite convincing, the role played in these models by the non-conservative dynamics is not clear. In fact, dissipation is a dynamical variable and does not always occur.

A model, which in the context of SOC in nonconservative systems has played an important role, is the Olami-Feder-Christensen (OFC) model of earthquakes \([4]\). In the OFC model a finite fraction, controlled by a fixed parameter \(\alpha\), of the transported quantity is dissipated in each relaxation event. The presence of criticality in the non-conservative OFC model has been controversial since the very introduction of the model \([5,16,17]\) and it is still debated \([18,19]\). Recent numerical investigations, though, have shown that the OFC model on a square lattice displays scaling behavior, up to lattice sizes presently accessible by computer simulations \([13,14]\). The avalanche size distribution is described by a power law, characterized by a universal exponent \(\tau \approx 1.8\), independent of the dissipation parameter. This distribution does not display finite size scaling, however.

To overcome the limitation of relying almost exclusively on computer simulation results, it has sometimes been useful to consider an annealed random neighbor (RN) version of the model \([15,16]\), where each site interacts with randomly chosen sites instead of its nearest neighbors on the lattice. This considerably simplifies the problem. In the past, RN models have usually been considered as mean-field descriptions of their fixed lattice counterparts, since spatial correlations are absent. Analogous to other RN models, the RN OFC model can be solved analytically \([16,17]\). It displays criticality only in the conservative case, where it becomes equivalent to a critical branching process. As soon as some dissipation is introduced the avalanches become localized, although the mean avalanche size diverges exponentially as dissipation tends to zero. The absence of criticality, together with the exponential divergence has cast some doubt on whether the OFC model on a fixed lattice is critical.

However, it is important to point out that the RN model may not describe the behavior of the OFC model on a fixed lattice in any dimension, and thus may not correspond to the mean field limit of the model. Usually, mean field behavior describes the high dimensional behavior of the system (e.g. the behavior above an upper critical dimension); this is not exactly the same limit as a model without any spatial correlations in it.

In fact, criticality in the OFC model on a lattice has been ascribed to a mechanism of partial synchronization \([19]\). The system has a tendency to order into a periodic state \([19,20]\) which is frustrated by the presence of inhomogeneities such as the boundaries. In addition, inhomogeneities induce partial synchronization of the elements of the system building up long range spatial correlations and thereby creating a critical state. The mechanism of synchronization requires an underlying spatial structure and therefore cannot operate in an annealed...
RN model, where each site is assigned new random neighbors at each update.

The main purpose of the present work is the investigation of the OFC model on a quenched random graph. This formulation, which can be handled numerically, is worth analyzing to see if it presents critical or non-critical behavior. Since the largest distance between two sites in the random graph grows only as a logarithm of the number of sites, it can be considered to be a high dimensional limit of a lattice model, and thus may describe the correct mean field limit. Contrary to the RN case, in a random graph the choice of neighbors is not annealed but quenched, so that spatial correlations can develop. Indeed, we show that the OFC model on a random graph displays criticality even in the nonconservative regime.

A random graph is defined as a set of $N$ sites connected at random by bonds. Two connected sites are denoted as “nearest-neighbor”. Formally, the random graph can be constructed by considering all $N(N - 1)/2$ possible bonds between sites and occupying a certain number of them with equal probability. A constraint of fixed connectivity can also be imposed by requiring that each site has the same number of neighbors, $q$. We have mainly concentrated on this latter situation but the first case will also be discussed. The model is then defined as follows.

To each site of the graph is associated a real variable $F_i$, which initially takes some random value in the interval $(0, F_{th})$. All the forces are increased uniformly and simultaneously at the same speed, until one of them reaches the threshold value $F_{th}$ and becomes unstable ($F_i \geq F_{th}$). The uniform driving is then stopped and an “earthquake” (or avalanche) starts:

$$F_i \geq F_{th} \Rightarrow \begin{cases} F_i \to 0 \\ F_{nn} \to F_{nn} + \alpha F_i \end{cases}$$

(1)

where “$nn$” denotes the set of nearest-neighbor sites of $i$. The parameter $\alpha$ controls the level of conservation of the dynamics and, in the case of a graph with fixed connectivity $q$, it takes values between 0 and $1/q$ ($\alpha = 1/q$ corresponding to the conservative case). The topping rule (1) can possibly create an unstable site, producing a chain reaction. All sites that are above threshold at a given time step in the avalanche relax simultaneously according to (1) and the earthquake is over when there are no more unstable sites in the system ($F_i < F_{th}$, $\forall i$). The uniform growth then starts again. The number of topplings during an earthquake defines its size $s$, and we will be interested in the probability distribution $P_N(s)$. Another quantity of interest is the duration $\tau$ of an earthquake which will be identified with the number of time steps needed for the earthquake to finish.

We consider first a random graph where all sites have exactly the same number of nearest neighbors $q$. In this case, we have verified (both for $q = 4$ and $q = 6$) that the system organizes into a subcritical state. This is analogous to what happens in the OFC model on a lattice with periodic boundary conditions, where no critical behavior is observed [19, 21]. In order to observe scaling in the avalanche distribution, one has to introduce some inhomogeneities. In the lattice model this is generally achieved by considering open boundary conditions which imply that boundary sites have fewer neighbors and therefore cycle at a different frequency from bulk sites. This is an inhomogeneity with a diverging length scale in the thermodynamic limit. For the OFC model on a random graph, we have found that it suffices to consider just two sites in the system with coordination $q - 1$ [22]. When either of these sites topple according to rule (1), an extra amount $\alpha F_i$ is simply lost by the system.

After a sufficiently long transient time, the system settles into a statistically stationary state. We have verified that the statistical properties of the system (e.g. the avalanche distribution) are independent of the actual realization of the random graph, as long as the coordination number $q$ is the same. As a point of comparison, in figure 1 we report the probability distribution of avalanche sizes for (a) the annealed RN model and (b) for the OFC model on a random graph for various system sizes $N$. The dynamical rule for the annealed RN model are formally similar to (1), where, instead of the nearest-neighbor sites on the graph, $q$ new random sites are chosen at each relaxing event. In both cases of fig. 1, the number of neighbors is $q = 4$ and the parameter $\alpha = 0.10$. It is clear that no scaling is present in the RN model as the cut-off in the avalanche size distribution does not grow with system size. On the contrary for the model on a random graph, the distribution scales with system size, which is indicative of a critical state. In fact, the largest avalanche roughly coincides with system size. It is important to underline that we are considering a situation far away from the conservative case ($60\%$ of the force in the toppling site is dissipated) and therefore one could expect that if a finite length scale related to conservation existed in the system it should appear for system sizes we have considered.

In order to characterize the critical behavior of the model, a finite size scaling (FSS) ansatz is used, i.e.

$$P_N(s) \approx N^{-\beta} f(s/N^D)$$

(2)

where $f$ is a suitable scaling function and $\beta$ and $D$ are critical exponents describing the scaling of the distribution function. In figure 2, a FSS collapse of $P_N(s)$ for different values of $\alpha$ and for different $q$ is shown. The distribution $P_N(s)$ satisfies the FSS hypothesis reasonably well, with universal critical coefficients. The critical exponent derived from the fit of fig. 2 are $\beta = 1.65$ and $D = 1$, independent of the dissipation parameter $\alpha$ and the coordination number of the graph $q$. The FSS hypothesis implies that, for asymptotically large $N$, $P_N(s) \sim s^{-\tau}$ and the value of the exponent is $\tau = \beta/D \approx 1.65$. Due to the numerical uncertainty on
the estimate it is difficult to assert with certainty that \( \tau \) is a novel exponent, different from the one for the conservative RN model (\( \tau = 1.5 \)) or the lattice model in two dimensions (\( \tau \approx 1.8 \)).

The OFC model on a lattice does not show ordinary FSS [21]. Although the avalanche size distribution converges to a well-defined, universal power law, the cut-off in the distribution due to finite system sizes does not behave according to FSS [13]. In particular, the apparent numerical value for the exponent \( D \) determined through FSS would violate some exact bounds [10].

In fig. 3 we report the average size of an avalanche stopping at time \( t, < s >_t \), as a function of the rescaled time \( t = t + 2 \) (as we are mainly interested at large values of \( t \), the constant should be irrelevant). The curves for different system sizes overlap (deviations can be attributed to finite-size effects) and we observe that \( < s >_t \approx t^\gamma \), where \( \gamma \approx 2.1 \), providing further evidence of criticality in the nonconservative system.

An interesting question (of difficult solution though) is whether the model becomes subcritical below a certain non-zero value \( \alpha_c \) (for \( \alpha = 0 \) the system is clearly not critical as sites do not interact). In our simulations we have found that for \( \alpha \geq 0.10 \) and \( q = 4 \) the model displays scaling behavior with universal critical exponents (see fig. 1 and 2). For lower values of \( \alpha \) the analysis is more complicated as the extremely long transient times to stationarity prevent the investigation of large lattices. Nonetheless for very low values of \( \alpha \) (\( \alpha \approx 0.03 \)) the cut-off in the avalanche distribution does not appear to vary systematically with system size (even for relatively small systems), suggesting that a non-zero \( \alpha_c \) may exist.

We have also considered the OFC model on a random graph with variable local connectivity \( g_i \). In this case, the toppling rule [7] must be modified to take into account that different sites have a different coordination number \( g_i \). Each site consequently has a different \( \alpha_i \), which we determined by requiring that the total fraction \( \bar{\alpha} \) of the force transferred from the unstable site to the nearest-neighbors sites is constant in the system, i.e. \( \alpha_i = \bar{\alpha}/g_i \). In particular, we have studied a graph with average connectivity \( < g_i > = 4 \). We have found that there is no criticality in the system as the cut-off in the probability distribution does not scale with system size. In agreement with previous investigations [23,24], this result indicates that if the disorder is too strong (as for a completely random graph) the critical state is destroyed. On the other hand inhomogeneities are necessary to break the periodic state the system would otherwise reach. It is a difficult question to establish what is the maximum level of disorder that the system can sustain without loosing its critical properties.

In conclusion, in this paper we have investigated the OFC model on a quenched random graph. We have shown that the model is critical even in the nonconservative regime. This is in contrast to what happens in the annealed RN OFC model which displays criticality only in the conservative case. Contrary to the annealed case, a quenched random graph has an underlying spatial structure so that partial synchronization of the elements of the system can still occur. As a random graph can be regarded as a high dimensional limit of a regular lattice, we propose that this formulation represents the correct mean-field limit of the model rather than the annealed random neighbor version.

This work was supported by the EPSRC (UK), Grant No. GR/M10823/01 and Grant No. GR/R37357/01.

[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. 59, 381 (1987); Phys. Rev. A. 38, 364 (1988).
[2] P. Bak, How Nature Works: The Science of Self-Organized Criticality (Copernicus, New York, 1996).
[3] H. Jensen, Self-Organized Criticality (Cambridge University Press, New York, 1998).
[4] T. Hwa and M. Kardar, Phys. Rev. Lett. 62, 1813 (1989).
[5] G. Grinstein, D.-H. Lee, and S. Sachdev, Phys. Rev. Lett. 64, 1927 (1990).
[6] S.S. Manna, L.B. Kiss, and J. Kertesz, J. Stat. Phys. 61, 923 (1990).
[7] A.A. Ali, Phys. Rev. E 52, R4595 (1995).
[8] S.S. Manna, A.D. Chakrabarti, and R. Cafiero, Phys. Rev. E 60, R5005 (1999).
[9] Z. Olami, H.J.S. Feder, and K. Christensen, Phys. Rev. Lett. 68, 1244 (1992); K. Christensen and Z. Olami, Phys. Rev. A 46, 1829 (1992).
[10] W. Klein and J. Rundle, Phys. Rev. Lett. 71, 1288 (1993); K. Christensen, Phys. Rev. Lett. 71, 1289 (1993).
[11] J.X. Carvalho and C.P.C. Prado, Phys. Rev. Lett. 84, 4006 (2000)
[12] K. Christensen, D. Hamon, H.J. Jensen, and S. Lise, Phys. Rev. Lett. 87, 039801 (2001); J.X. Carvalho and C.P.C. Prado, Phys. Rev. Lett. 87, 039802 (2001).
[13] S. Lise and M.Paczuski, Phys. Rev. E, 63, 036111 (2001).
[14] S. Lise and M.Paczuski, Phys. Rev. E, 64, 046111 (2001).
[15] S. Lise and H.J. Jensen, Phys. Rev. Lett. 76, 2326 (1996).
[16] M.L. Chabanol and V. Hakim, Phys. Rev. E, 56, 2343 (1997).
[17] H.M. Broker and P. Grassberger, Phys. Rev. E, 56, 3944 (1997).
[18] O. Kinouchi, S.T.R. Pinho, and C.P.C. Prado, Phys. Rev. E, 58, 3997 (1998).
[19] A.A. Middleton and C. Tang, Phys. Rev. Lett. 74, 742 (1995).
[20] J.E.S. Socolar, G. Grinstein, and C. Jayaprakash, Phys. Rev. E, 47, 2366 (1993).
[21] P. Grassberger, Phys. Rev. E, 49, 2436 (1994).
[22] We have also considered defects whose number scale as $\sqrt{N}$ (analogously to boundary sites in the lattice model). The model still displays scaling but the results appear easier to interpret with just two defects.
[23] H. Ceva, Phys. Rev. E, 52, 154 (1995).
[24] M. Mousseau, Phys. Rev. Lett. 77, 968 (1996).

FIG. 1. Probability distribution (a) for the RN OFC model and (b) for the OFC model on a random graph. In both cases, $q = 4$ and $\alpha = 0.10$. System sizes are (a) $N = 10^3$, $4 \cdot 10^3$ and (b) $N = 10^3$, $16 \cdot 10^3$, $256 \cdot 10^3$.

FIG. 2. Finite-size scaling plots for $P_N(s)$ for (a) $q = 4$, $\alpha = 0.15$, (b) $q = 4$, $\alpha = 0.20$ and (c) $q = 6$, $\alpha = 0.10$. System sizes are $N = 4 \cdot 10^3$, $16 \cdot 10^3$, $64 \cdot 10^3$ and $256 \cdot 10^3$. The critical exponents are $\beta = 1.65$ and $D = 1$. For visual clarity, curves (a) and (c) have been shifted along the x axis, $x \to x - 1$ and $x \to x + 1$, respectively.

FIG. 3. Average size of an avalanche lasting $t$ time steps as a function of $t$ for $q = 4$ and (a) $\alpha = 0.15$ and (b) $\alpha = 0.20$. Different curves correspond, from bottom to top, to system sizes $N = 1 \cdot 10^3$, $4 \cdot 10^3$, $16 \cdot 10^3$, $64 \cdot 10^3$ and $256 \cdot 10^3$. The slope of the straight line is $\gamma = 2.1$. Curve (b) has been shifted along the x axis, $x = x + 1$, for visual clarity.