THREE-DIMENSIONAL WAKE FIELDS, GENERATED IN PLASMA
BY CYLINDRICAL ELECTRON BUNCH
S.S. Elbakian, E.V. Sekhpossian, A.G. Khachatryan

Yerevan Physics Institute, Alikhanian Brothers St. 2, Yerevan 375036, Republic of Armenia
E-mail: khachatr@moon.yerphi.am

Abstract
The expressions for wake fields, generated in plasma (in the plasma waveguide or unlimited plasma) by
relativistic electron bunch, was received and analyzed for the cases of the presence and absence of strong
external longitudinal magnetic field. For the both cases the comparative analysis of the dependence of
field amplitudes on the parameters of the electron bunch was done.

PACS number(s): 52.35.Mw, 52.40.Mj
1 Introduction

Presently the studies on new methods of charged particle acceleration by means of wake fields, generated in plasma by laser radiation (BWA (Beat-Wave Acceleration), LWFA (Laser Wake Field Acceleration)) and by bunches of relativistic particles (PWFA (Plasma Wake-Field Acceleration)), moving in plasma are intensively developed (see, e.g. reviews [1, 2] and cited there literature). The intensity of acceleration fields (in the order of $10^7 - 10^8 V/cm$), attained by these methods can be used both for the charge acceleration, and for focusing of electron (positron) bunches in order to obtain the beams of high density and to ensure high luminosity in linear colliders of next generation [1, 3].

Linear theory of wake field generating by two- and three-dimensional rigid bunches of charged particles in boundless and limited plasma was developed in many works [3-11]. Nonlinear theory of wake field generating by a rigid one-dimensional bunch of final extent and sequence of charge particle bunch was developed in [12-17]. It was shown that optimum condition for wave generating is $n_b = n_0/2$ ($n_b, n_0$ are the density values of bunch and plasma electrons, correspondingly). Important result of this theory is the demonstration that, in the case of nonlinear wake fields a transformation ratio $R = E_{ac}/E_{st}$ ($E_{ac}$ and $E_{st}$ are the extension of correspondingly accelerating and decelerating electric fields) depends on gamma-factor of accelerating bunch and may be significant without special bunch shaping, as it occurs in the case of linear wake fields generated by a rigid bunch. The conclusion of the results mentioned above is reaffirmed in [18] for the assumption $\beta_0 = v_0/c = 1, (\gamma_0 = (1 - \beta_0^2)^{-1/2} = \infty)$ (where $v_0$ is the bunch velocity), that brings to the incorrect expression for the maximum value of accelerating field $E_{ac}$ (when $n_b/n_0 = 1/2, E_{ac} = \infty$).

The influence of transverse sizes of the bunch on nonlinear wake field generated by short bunches ($d \ll \lambda_p, d$ is the bunch length, $\lambda_p$ is the wave length) was considered in [19].

Non-linear theory of wake field generated by two- and three-dimensional bunches for the general case is not yet developed.

This work contains the results and analysis of linear equation solution, describing the interaction of the axial symmetrical homogenous bunch of charged particles with plasma in the assumption of the plasma vorticity absence (laminar flow), as well as at the strong external constant magnetic field applied along the bunch motion, when the transverse movements of plasma electrons are suppressed.

2 Basic equations

Vector equation, describing the excitation of nonlinear three-dimensional wake fields by the rigid bunch of charged particles with electron density $n_b$ moving with constant velocity $v_0$ along the $z$ axis through the cold plasma at equilibrium with density $n_0$ in hydrodynamic description and in the assumption of
absence of the plasma vorticity

$$\text{rot} \left( \rho - \frac{e}{mc^2} A \right) = 0,$$

is given by the following formula [3]

$$\left( \nabla \nabla - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \rho + \frac{\rho}{\sqrt{1 + \rho^2}} \left[ \frac{\beta_0^2 k_p^2}{c^2} \left( 1 - \frac{n_b}{n_0} \right) \right] + \frac{\partial}{c \partial t} \nabla \rho + \nabla^2 \sqrt{1 + \rho^2} +$$

$$+ \frac{1}{c \partial t} \nabla \sqrt{1 + \rho^2} = - \beta_0^3 k_p^2 n_b / n_0,$$

where $\rho = p/mc$ is dimensionless momentum of plasma electrons, $A$-vector potential of electromagnetic field, $k_p = \omega_p / v_0$, $\omega_p = \sqrt{4 \pi n_0 e^2 / m}$-is the plasma frequency of electrons, $\beta_0 = v_0 / c$, and $n_b$-arbitrary function of coordinates and time.

The similar equation, describing interaction between laser pulse and the plasma, was obtained in [20-22].

Let as consider axial-symmetrical bunch, when $\rho$ depends only on variable $r$ and $\tilde{z} = z - v_0 t$ (steady state). In this case vector $\rho$ has only longitudinal $\rho_z$ and radial $\rho_r$ components, which do not depend on azimuthal angle $\varphi$, and $\rho_\varphi = 0$.

The system of equations for the component of momentum $\rho_z$ and $\rho_r$ has the following form:

$$\frac{\partial^2}{\partial \tilde{z}^2} \left( \beta_0 \rho_z - \sqrt{1 + \rho^2} \right) + \frac{\beta_0^2 k_p^2 \rho_z}{\beta_0 \sqrt{1 + \rho^2} - \rho_z} + \frac{\beta_0^2 k_p^2 n_b}{n_0} = \beta_0 \rho_z - \sqrt{1 + \rho^2} \frac{1}{\beta_0 \sqrt{1 + \rho^2} - \rho_z} \frac{\partial}{\partial \tilde{z}} \left( \frac{r \partial}{\partial r} \rho_z \right) -$$

$$- \frac{\rho_z}{\beta_0 \sqrt{1 + \rho^2} - \rho_z} \frac{1}{r \partial r} \left( r \partial \frac{\partial \rho_z}{\partial r} \right).$$

$$\frac{\partial^2}{\partial \tilde{z} \partial r} \left( \beta_0 \sqrt{1 + \rho^2} - \rho_z \right) + \frac{1}{\gamma_0} \frac{\partial^2 \rho_r}{\partial \tilde{z}^2}$$

$$= \frac{\rho_r}{\sqrt{1 + \rho^2}} \left\{ \beta_0^2 k_p^2 \left( 1 - \frac{n_b}{n_0} \right) - \frac{\partial^2}{\partial \tilde{z}^2} \left( \beta_0 \rho_z - \sqrt{1 + \rho^2} \right) - \right.$$

$$\left. - \beta_0 \frac{\partial}{\partial \tilde{z}} \left( r \rho_r \right) - \frac{1}{r \partial r} \left( r \partial \frac{\partial \rho_z}{\partial r} \right) \right\}. \quad (4)$$

Inserting in (3) and (4) $\rho_r = 0$ and $\rho_z = \rho_z(\tilde{z})$ (dependence on $r$ is absent) we come to the one-dimensional nonlinear equation, considered in [12-17]

$$\frac{\partial^2}{\partial \tilde{z}^2} \left( \beta_0 \rho_z - \sqrt{1 + \rho_z^2} \right) + \frac{\beta_0^2 k_p^2 \rho_z}{\beta_0 \sqrt{1 + \rho_z^2} - \rho_z} + \frac{\beta_0^2 k_p^2 n_b}{n_0} = 0. \quad (5)$$
Inserting in (4) $\rho_r = 0$ and expressing the derivative $\partial \rho_z / \partial \tilde{z}$ through derivatives of $\rho_z$ we receive for scalar potential $\varphi$ the following equation, describing interaction of electron bunch with plasma in the presence of the external magnetic field $B_0 (0, 0, B_0)$ [19]:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \chi}{\partial r} \right) \rho_r + \frac{1}{\gamma_0^2} \frac{\partial^2 \chi}{\partial z^2} + \beta_0^2 k_p^2 \rho_r \left( 1 - \frac{\beta_0 \chi}{\sqrt{\chi^2 - 1/\gamma_0^2}} \right) = \frac{\beta_0^2 k_p^2 n_b}{\gamma_0^2 n_0},$$

where

$$\chi = 1 + \frac{e \varphi}{mc \gamma_0^2} = \sqrt{1 + \rho_z^2} - \beta_0 \rho_z. \quad (7)$$

### 3 Wakefield generating at vorticity absence

Let us consider the problem of wake field generating by rigid cylindrical bunch of radius $a$ and horizontal dimension $d$ with homogeneous distribution of electrons of the bunch

$$n_b = \begin{cases} n_b, & 0 \leq r \leq a, \ 0 \leq \tilde{z} \leq d, \\ 0, & a \leq r \leq b, \ \tilde{z} \geq d, \ \tilde{z} \leq 0, \end{cases} \quad (8)$$

moving in conducting plasma waveguide of radius $b \geq a$.

Assuming in (3) and (4) $n_b / n_0 \ll 1$ (linear approximation) and linearizing the system on $\rho$ we shall obtain the following system of equations, describing the process under consideration:

$$\frac{\partial^2 \rho_r}{\partial \tilde{z} \partial r} + \frac{1}{r} \frac{\partial \rho_r}{\partial r} \rho_z \frac{\partial^2 \rho_z}{\partial r^2} - \frac{\partial \rho_z}{\partial r} \beta_0^2 \frac{\partial^2 \rho_r}{\partial r^2} + \beta_0^2 k_p^2 \rho_z = -\beta_0^2 k_p^2 \rho_z, \quad (9)$$

$$\frac{\partial^2 \rho_z}{\partial \tilde{z} \partial r} - \frac{1}{\gamma_0^2} \frac{\partial^2 \rho_z}{\partial z^2} + \beta_0^2 k_p^2 \rho_r = 0.$$

To define $\rho_z (r, \tilde{z})$ and $\rho_r (r, \tilde{z})$ let us perform the Hankel transformation [23] of the equations system (9) on $r$ in the finite limits $(0, b)$ and solving the obtained equations on $\tilde{z}$ in the assumption of the continuity conditions of the momentum components $\rho_z$, $\rho_r$ and components of the electrical field $E_z = \frac{mcn_b}{e} \frac{\partial \varphi}{\partial \tilde{z}}$, $E_r = \frac{mcn_b}{e} \frac{\partial \varphi}{\partial z}$ at the front ($\tilde{z} = d$) and rear ($\tilde{z} = 0$) bunch boundaries we shall receive the following expression for the field components $E_z$ and $E_r$:

Inside the bunch $(0 \leq r \leq a, \ 0 \leq \tilde{z} \leq d)$

$$E_z^I = \frac{mv_0 \omega_p}{e} n_b \frac{n_b}{n_0} \sin k_p (d - \tilde{z}) \times$$

$$\times \left\{ 1 - (k_p a) \frac{I_0(k_p r)}{I_0(k_p b)} \left[ K_0(k_p b) I_1(k_p a) + I_0(k_p b) K_1(k_p a) \right] \right\} + E_{zH}, \quad (10)$$
\[ E'_r = -\frac{mv_0 \omega_p}{e} \frac{n_b}{n_0} (k_p a) \cos k_p (d - \tilde{z}) \frac{I_1(k_p r)}{I_0(k_p b)} \times \]
\[ \times [I_1(k_p a) K_0(k_p b) + I_0(k_p b) K_1(k_p a)] + E'_{rH}; \]

Over the bunch \((a \leq r \leq b, 0 \leq \tilde{z} \leq d)\)

\[ E'_z = -\frac{mv_0 \omega_p}{e} \frac{n_b}{n_0} \sin k_p (d - \tilde{z})(k_p a) \frac{I_1(k_p a)}{I_0(k_p b)} \times \]
\[ \times [K_0(k_p r) I_0(k_p b) - I_0(k_p r) K_0(k_p b)] + E'_{zH}, \]

\[ E''_r = -\frac{mv_0 \omega_p}{e} \frac{n_b}{n_0} (k_p a) \cos k_p (d - \tilde{z}) \frac{I_1(k_p a)}{I_0(k_p b)} \times \]
\[ \times [J_1(k_p r) K_0(k_p b) + I_0(k_p b) K_1(k_p r)] + E''_{rH}, \]

where

\[ E'_{zH}, E'_{rH} = \frac{2}{e} \frac{mv_0 \omega_p}{n_0} (k_p a) \sum_{n=1}^{\infty} \frac{\mu_n J_0 \left( \frac{\mu_n \gamma_d}{b} \right) J_1 \left( \frac{\mu_n a}{b} \right) e^{-\gamma_0 \kappa d/2}}{\kappa \left( k_p^2 b^2 + \mu_n^2 \right) J_1^2(\mu_n)} \text{sh}_{\gamma_0 \kappa} \left( \frac{d}{2} - \tilde{z} \right), \] (14)

\[ E''_{zH}, E''_{rH} = \frac{2}{e} \frac{mv_0 \omega_p}{n_0} (k_p a) \sum_{n=1}^{\infty} \frac{J_1 \left( \frac{\mu_n \gamma_d}{b} \right) J_1 \left( \frac{\mu_n a}{b} \right) e^{-\gamma_0 \kappa d/2}}{\kappa \left( k_p^2 b^2 + \mu_n^2 \right) J_1^2(\mu_n)} \text{ch}_{\gamma_0 \kappa} \left( \frac{d}{2} - \tilde{z} \right) \] (15)

in the corresponding areas on \(r\).

Before the bunch \((0 \leq r \leq a \leq b, d \leq \tilde{z} \leq \infty)\) we have the following expression for the field components:

\[ E''_{zH} = -2 e \frac{mv_0 \omega_p}{n_0} (k_p a) \sum_{n=1}^{\infty} \frac{\mu_n J_0 \left( \frac{\mu_n \gamma_d}{b} \right) J_1 \left( \frac{\mu_n a}{b} \right)}{\kappa \left( k_p^2 b^2 + \mu_n^2 \right) J_1^2(\mu_n)} e^{-\gamma_0 \kappa \left( \frac{d}{2} - \tilde{z} \right) \text{sh}_{\gamma_0 \kappa} \frac{d}{2}}, \] (16)

\[ E''_{rH} = -2 e \frac{mv_0 \omega_p}{n_0} (k_p a) \sum_{n=1}^{\infty} \frac{J_1 \left( \frac{\mu_n \gamma_d}{b} \right) J_1 \left( \frac{\mu_n a}{b} \right)}{\kappa \left( k_p^2 b^2 + \mu_n^2 \right) J_1^2(\mu_n)} e^{-\gamma_0 \kappa \left( \frac{d}{2} - \tilde{z} \right) \text{sh}_{\gamma_0 \kappa} \frac{d}{2}}. \] (17)

After the bunch \((-\infty \leq \tilde{z} \leq 0)\) we have

\[ E''_z = \frac{mv_0 \omega_p}{e} \frac{n_b}{n_0} [\sin(k_p \tilde{z}) + \sin k_p (d - \tilde{z})] \times \]
\[ \times \left\{ 1 - (k_p a) \frac{I_0(k_p r)}{I_0(k_p b)} [K_0(k_p b) I_1(k_p a) + I_0(k_p b) K_1(k_p a)] \right\} + E''_{zH}, \]

\[ 0 \leq r \leq a, \] (18)
The components different from zero $B_\varphi$ of $E$-wave ($E_z$, $E_r$, $B_\varphi$) magnetic field is determining by the formula

$$B_\varphi = \frac{mc^2}{e} \left( \frac{\partial \rho_r}{\partial \bar{z}} - \frac{\partial \rho_z}{\partial r} \right)$$  \hspace{1cm} (22)$$

and is described by the following expressions

$$B_\varphi^{I,II} = -2\frac{m\nu_0\omega_p}{e} \beta_0(k_p a) \frac{n_b}{n_0} \sum_{n=1}^{\infty} \frac{J_1(\mu_n \bar{z}) J_1(\mu_n \bar{r})}{\kappa^2 b^2 J_1^2(\mu_n)} \times$$

$$\times \left[ 1 - e^{-\gamma_0 \kappa (\frac{d}{2} - \bar{z})} \right], \hspace{1cm} \text{at} \ 0 \leq \bar{z} \leq d, \ 0 \leq r \leq b,$$  \hspace{1cm} (23)$$

$$B_\varphi^{III} = -2\frac{m\nu_0\omega_p}{e} \beta_0(k_p a) \frac{n_b}{n_0} \sum_{n=1}^{\infty} \frac{J_1(\mu_n \bar{z}) J_1(\mu_n \bar{r})}{\kappa^2 b^2 J_1^2(\mu_n)} e^{\gamma_0 \kappa (\frac{d}{2} - \bar{z})} \text{sh} \left( \gamma_0 \kappa d \frac{d}{2} \right), \hspace{1cm} (24)$$

$$d \leq \bar{z} \leq \infty, \ 0 \leq r \leq b,$$

$$B_\varphi^{IV} = -2\frac{m\nu_0\omega_p}{e} \beta_0(k_p a) \frac{n_b}{n_0} \sum_{n=1}^{\infty} \frac{J_1(\mu_n \bar{z}) J_1(\mu_n \bar{r})}{\kappa^2 b^2 J_1^2(\mu_n)} e^{-\gamma_0 \kappa (\frac{4}{2} - \bar{z})} \text{sh} \left( \gamma_0 \kappa d \frac{d}{2} \right), \hspace{1cm} (25)$$

$$-\infty \leq \bar{z} \leq 0, \ 0 \leq r \leq b.$$
In formulas (10)-(25) \( J_0, J_1 \) are the Bessel functions, and \( I_0, I_1, K_0, K_1 \)-are the modified Bessel functions, \( \kappa^2 = k_p^2 \beta_0^2 + \mu_n^2 / b^2 \).

As it may be seen from the given expressions for field components \( E_z, E_r \) they consist of periodic (wake field) and non-periodic ("Coulomb") parts. Before the bunch the field has only non-periodic part, which exponentially falls of with the remote from the front boundary, so that part may be neglected. After the bunch the "Coulomb" part also falls off exponentially with the remote from the rear boundary \( \bar{z} = 0 \) and only periodic wake field remains. Magnetic field component \( B_\varphi \) has only non-periodic "Coulomb" part, which exponentially also decreases at the remote from the bunch boundaries and it can be ignored.

In the range of \( 0 \leq \bar{z} \leq d \), where magnetic field is not small, the radial force \( f_r = -eE_r + e\beta_0 B_\varphi \) acting on the bunch electrons, and this force in some ranges on \( \bar{z} \) can compress the bunch, focusing it. Before the bunch \((\bar{z} \geq d)\) and in the range of wake field \((\bar{z} \leq 0)\) the component of the magnetic field \( B_\varphi \) is small and the radial force is \( f_r \approx -eE_r \).

Below one can find an expression for the radial force \( f_r \) acting on the bunch \((0 \leq \bar{z} \leq d)\) at \( \beta_0 \approx 1, \gamma_0 \gg 1 \):

\[
    f_r = -m \omega_p \nu_0 \frac{n_b}{n_0} k_p a \left[ 1 - \cos k_p (d - \bar{z}) \right] \times
    \left\{ \begin{array}{ll}
    \frac{I_1(k_p a)}{I_0(k_p a)} [I_1(k_p a)K_0(k_p b) + I_0(k_p b)K_1(k_p a)], & r \leq a, \\
    \frac{I_1(k_p a)}{I_0(k_p a)} [K_0(k_p b)I_1(k_p r) + I_0(k_p b)K_1(k_p r)], & a \leq r \leq b, \\
    0, & b \leq r.
    \end{array} \right.
\]

where \( O(\gamma_0^{-2}) \) is the smaller defocusing part of the force.

Let us also give an expressions for the fields \( E_z, E_r \) inside and after the bunch in the case when \( a = b \) and in the case when \( b \to \infty \) (unlimited plasma):

\[
    E^I_z = \frac{m \omega_p \nu_0 n_b}{e n_0} k_p \sin k_p (d - \bar{z}) \left[ 1 - \frac{I_0(k_p r)}{I_0(k_p a)} \right] + E^I_{zH},
\]

\[
    E^I_r = -\frac{m \omega_p \nu_0 n_b}{e n_0} k_p \cos k_p (d - \bar{z}) \frac{I_1(k_p r)}{I_0(k_p a)} + E^I_{rH},
\]

\[
    E^{IV}_{z} = \frac{m \omega_p \nu_0 n_b}{e n_0} [ \sin k_p (d - \bar{z}) + \sin (k_p \bar{z}) ] \left[ 1 - \frac{I_0(k_p r)}{I_0(k_p a)} \right] + E^{IV}_{zH},
\]

\[
    E^{IV}_{r} = -\frac{m \omega_p \nu_0 n_b}{e n_0} [ \cos k_p (d - \bar{z}) - \cos (k_p \bar{z}) ] \frac{I_1(k_p r)}{I_0(k_p a)} + E^{IV}_{rH},
\]

\[
    a = b,
\]

where

\[
    E^I_{zH} = \frac{2 m \omega_p \nu_0 n_b k_p}{e n_0} \gamma_0 \sum_{\mu_n=1}^{\infty} \frac{\mu_n J_0 (\mu_n \bar{z})}{\kappa J_1 (\mu_n) (k_p^2 a^2 + \mu_n^2)} e^{-\gamma_0 \bar{z}} \text{sh} \gamma_0 \kappa \left( \frac{d}{2} - \bar{z} \right),
\]

\[
    a = b,
\]

\[
    \text{sh} \gamma_0 \kappa \left( \frac{d}{2} - \bar{z} \right),
\]

\[
    7
\]
that e.g. they increase with the remote from the bunch centre inside the bunch and decrease with the remote

The linear equation for the potential

$$E_{rH}^I = \frac{2mv_0\omega_b}{e} n_b(k_p a) \sum_{n=1}^{\infty} \frac{J_1(\mu_n \tilde{z})}{(k_p^2 a^2 + \mu_n^2) J_1(\mu_n)} e^{-\gamma_0 \frac{d}{2}} \text{ch}\gamma_0 \kappa \left( \frac{d}{2} - \tilde{z} \right),$$

and $E_{zH}^{IV}, E_{rH}^{IV} \to 0$ at $\tilde{z} < 0$.

At the $a \to \infty I_0(k_p a) \to \infty$ and the expressions for $E_{z}^{I}$ and $E_{z}^{IV}$ coincide with the expressions for one-dimensional bunch, while $E_{r}^{I}$ and $E_{r}^{IV} \to 0$. At $k_p a \ll 1$ ($k_p r < k_p a \ll 1$) periodic parts of the fields $E_{z}^{I,IV}$ and $E_{r}^{I,IV}$ are proportional to $(k_p a)^2$ and $k_p r$. Non-periodic parts are small at $\gamma_0 \gg 1$.

The expressions for $E_{z}$ and $E_{r}$ inside the bunch and in the wake field at $b \to \infty$, $\gamma_0 \gg 1$ go over into corresponding expressions for the case with unlimited plasma. In this case at $k_p r \ll 1$, $k_p a \ll 1$ ($r > a$ or $r < a$) the longitudinal fields inside and over the bunch as well as in the wake are proportional to $(k_p a)^2$, while the radial components of the fields $E_{r}^{I,IV} \sim r/2a$ at $r < a$ and $E_{r}^{I,IV} \sim a/2r$ at $r > a$, e.g. they increase with the remote from the bunch centre inside the bunch and decrease with the remote from the bunch boundaries inside it.

Let us define the plasma density for the mentioned four regions. From the Poisson equation it follows that

$$n_e = -\frac{1}{4\pi e} \left[ \frac{\partial E_r}{\partial r} + \frac{E_r}{r} + \frac{\partial E_z}{\partial z} - 4\pi e (n_0 - n_b) \right]. \quad (29)$$

In the regions outside the bunch in (29) we should suppose $n_b = 0$. Using the expressions (10)-(21) for the fields we shall find out the following expressions for the plasma densities:

$$n_e^{I,II} = \begin{cases} n_0 \left[ 1 - \frac{v_0}{v_0} (1 - \cos k_p (d - \tilde{z})) \right], & r \leq a, \\ n_0, & a \leq r \leq b, \\ 0 \leq \tilde{z} \leq d, \end{cases} \quad (30)$$

$$n_e^{III} = n_0, \quad a10 \leq r \leq b, \quad d \leq \tilde{z} < \infty, \quad (31)$$

$$n_e^{IV} = \begin{cases} n_0 \left[ 1 - \frac{v_0}{v_0} (\cos (k_p \tilde{z}) - \cos k_p (d - \tilde{z})) \right], & r \leq a, \\ n_0, & a \leq r \leq b, \quad -\infty \leq \tilde{z} \leq 0. \end{cases} \quad (32)$$

Thus, the density $n_e$ depends periodically on $\tilde{z}$ inside and after the electron bunch at $r \leq a$ and does not depend on $r$ and sizes $a$ and $b$.

For the regions outside the bunch $a \leq r \leq b$ and $0 \leq z \leq \infty$, $-\infty \leq \tilde{z} \leq 0$ $n_e^{II-IV} = n_0$ and the density $n_e$ undergoes change to the lateral surface of the bunch and in the wake at $r = a$. On the waveguide surface $r = b$, $n_e = n_0$.

4 Wake field excitation under strong external magnetic field

The linear equation for the potential $\varphi$, describing the interaction of cylindrical electron bunch of radius $a$ and length $d$ with unlimited cold plasma, follows from expressions (6), (7) in assumption $e\varphi/mc^2 \ll 1,$
In this case the expressions (35), (36) significantly simplify:

\[
\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{1}{\gamma_0^2} \frac{\partial^2 \varphi}{\partial z^2} + \frac{k_p^2}{\gamma_0^2} \varphi = k_p^2 \frac{m}{e} \frac{n_b}{n_0}, \tag{33}
\]

where \( n_b \) is given by the expression (8).

After making Hankel (Fourier-Bessel) transformation of the equation (33) on \( r \) in boundless limits \((0, \infty)\) under the condition that \( \varphi(\bar{z}, r \rightarrow \infty) = 0 \) we shall receive the following equation

\[
\frac{\partial^2 \varphi}{\partial z^2} - \lambda^2 \varphi = h, \tag{34}
\]

where \( \varphi(\alpha, \bar{z}) = \int_0^\infty \varphi(r, \bar{z}) J_0(\alpha r) r dr \), \( \lambda^2 = \gamma_0^2 (\alpha^2 - k^2) \), \( k^2 = \frac{k_p^2}{\gamma_0^2} \), \( h = \frac{m}{e} \frac{k_p^2}{\gamma_0^2} \frac{n_b}{n_0} \frac{J_1(\alpha a)}{J_0(\alpha a)} \), \( J_0 \) and \( J_1 \) are the Bessel functions.

Before the bunch \((\bar{z} \geq d)\), where \( n_b = 0, h = 0 \) the potential \( \varphi(r, \bar{z}) \) is assumed as equal to zero (we ignore the "Coulomb" field, see chapter 3).

Solving the equation under assumption of continuity of the potential \( \varphi(\alpha, \bar{z}) \) on the front \((\bar{z} = d)\) and rear \((\bar{z} = 0)\) boundaries of the bunch, we shall come to the following expressions for \( \varphi(r, \bar{z}) \) in the ranges inside and over the bunch \((0 \leq \bar{z} \leq d, 0 \leq r \leq a, a \leq r < \infty)\)

\[
\varphi_1 = -\frac{m}{e} \frac{k_p^2}{\gamma_0^2} \frac{n_b}{n_0} a \text{Re} \int_0^\infty \frac{J_0(\alpha r) J_1(\alpha a)}{\alpha^2 - k^2} \left[ 1 - e^{-\lambda (d - \bar{z})} \right] d\alpha, \tag{35}
\]

where \( k \) has the positive imaginary part \( \text{Im} k = k' > 0, \lambda = \gamma_0 \sqrt{\alpha^2 - k^2} \) at \( \alpha > k, \lambda = -i\gamma_0 \sqrt{k^2 - \alpha^2} \) at \( \alpha < k \).

After the bunch \((\bar{z} \leq 0, 0 \leq r < \infty)\) the potential \( \varphi \) has the following form:

\[
\varphi_2 = \frac{m}{e} \frac{k_p^2}{\gamma_0^2} \frac{n_b}{n_0} a \text{Re} \int_0^\infty \frac{J_0(\alpha r) J_1(\alpha a)}{\alpha^2 - k^2} \left[ e^{\lambda \bar{z}} - e^{\lambda (d - \bar{z})} \right] d\alpha. \tag{36}
\]

Because the "Coulomb" components \((\alpha^2 > k^2)\) are small one can ignore them and bring out the expression of square brackets from the integral in the point \( \alpha = 0 \), where it makes the main input into the integral. In this case the expressions (35), (36) significantly simplify:

\[
\varphi_1 = -\frac{m}{e} \frac{k_p^2}{\gamma_0^2} \frac{n_b}{n_0} a \text{Re} \left[ 1 - e^{ik_p (d - \bar{z})} \right] \int_0^\infty \frac{J_0(\alpha r) J_1(\alpha a)}{\alpha^2 - k^2} d\alpha, \tag{37}
\]

\[
\varphi_2 = -\frac{m}{e} \frac{k_p^2}{\gamma_0^2} \frac{n_b}{n_0} a \text{Re} \left[ e^{-ik_p \bar{z}} - e^{-ik_p (d - \bar{z})} \right] \int_0^\infty \frac{J_0(\alpha r) J_1(\alpha a)}{\alpha^2 - k^2} d\alpha. \tag{38}
\]

The components of the fields are determined from the expressions:

\[
E_z = -\frac{1}{\gamma_0^2} \frac{\partial \varphi}{\partial \bar{z}}, \quad E_r = \frac{\partial \varphi}{\partial r}, \quad B_\varphi = \beta_0 E_r. \tag{39}
\]

Inside the bunch \((0 \leq r \leq a, 0 \leq \bar{z} \leq d)\) we have:
The wake field is determined by the following expressions:

\[
E_z^I(r, \bar{z}) = \frac{mv_0 \omega_p}{e} \frac{n_b}{n_0} \left\{ \sin k_p (d - \bar{z}) \left[ 1 + \frac{\pi}{2} (ka) J_0(ka) J_1(ka) \right] - \frac{\pi}{2} (ka) \cos k_p (d - \bar{z}) J_0(ka) J_1(ka) \right\},
\]

\[
E_r^I(r, \bar{z}) = \frac{mv_0 \omega_p}{e} \frac{1}{\gamma_0} \frac{n_b}{n_0} \left\{ \frac{\pi}{2} [1 - \cos k_p (d - \bar{z})] (ka) J_1(ka) Y_1(ka) - \frac{\pi}{2} (ka) \sin k_p (d - \bar{z}) J_1(ka) J_1(ka) \right\},
\]

where \( k = k_p / \gamma_0 \).

Over the bunch (\( a \leq r < \infty, 0 \leq \bar{z} \leq d \)) the field components are given the following expressions:

\[
E_z^{II}(r, \bar{z}) = \frac{mv_0 \omega_p}{e} \frac{n_b}{n_0} \left\{ \frac{\pi}{2} (ka) \sin k_p (d - \bar{z}) J_1(ka) Y_0(ka) - \frac{\pi}{2} (ka) \cos k_p (d - \bar{z}) J_0(ka) J_1(ka) \right\},
\]

\[
E_r^{II}(r, \bar{z}) = \frac{mv_0 \omega_p}{e} \frac{1}{\gamma_0} \frac{n_b}{n_0} \left\{ \frac{\pi}{2} [1 - \cos k_p (d - \bar{z})] J_1(ka) Y_1(ka) - \frac{\pi}{2} (ka) \sin k_p (d - \bar{z}) J_1(ka) J_1(ka) \right\}.
\]

The wake field is determined by the following expressions:

\[
E_z^{IV}(r, \bar{z}) = \frac{mv_0 \omega_p}{e} \frac{n_b}{n_0} \left\{ [\sin (k_p \bar{z}) - \sin k_p (\bar{z} - d)] \left[ 1 + \frac{\pi}{2} (ka) J_0(ka) J_1(ka) \right] + \frac{\pi}{2} (ka) J_0(ka) J_1(ka) \right\},
\]

\[
0 \leq r \leq a,
\]

\[
E_z^{IV}(r, \bar{z}) = \frac{mv_0 \omega_p}{e} \frac{n_b}{n_0} \left\{ [\sin (k_p \bar{z}) - \sin k_p (\bar{z} - d)] \left[ 1 + \frac{\pi}{2} (ka) J_0(ka) J_1(ka) \right] + \frac{\pi}{2} (ka) J_1(ka) J_1(ka) \right\},
\]

\[
a \leq r < \infty,
\]

\[
E_r^{IV}(r, \bar{z}) = \frac{mv_0 \omega_p}{e} \frac{1}{\gamma_0} \frac{n_b}{n_0} \left\{ [\cos (k_p \bar{z}) - \cos k_p (\bar{z} - d)] \frac{\pi}{2} (ka) J_1(ka) Y_0(ka) + \frac{\pi}{2} (ka) \cos k_p (\bar{z} - d) J_1(ka) J_0(ka) \right\},
\]

\[
0 \leq r \leq a,
\]
\[ E_r^{IV}(r, \bar{z}) = \frac{mv_0 \omega_p}{e} \frac{1}{\gamma_0 n_0} \left\{ \left[ \cos (k_p \bar{z}) - \cos k_p (\bar{z} - d) \right] \frac{\pi}{2} (ka) J_1(ka) Y_1(kr) - \right\} \\
\left\{ \left[ \sin (k_p \bar{z}) - \sin k_p (\bar{z} - d) \right] \frac{\pi}{2} (ka) J_1(ka) J_1(ka) \right\}, \]

\( a \leq r < \infty. \)

The components of the magnetic field \( B_\varphi \) is determined from the expression \( B_\varphi = \beta_0 E_r \).

The comparison of the expressions for the components of the fields (10)-(25), generated by the electron bunch in plasma without magnetic field, with the corresponding expressions (40)-(47) with the constant longitudinal strong magnetic field \( B_0 \) shows, that in the last case the character of the fields qualitatively varies from the case of plasma without the field. On the one hand, the dependence on the transversal coordinates \( r \) and \( a \) is determined by the Bessel functions \( Y \) and \( J \), having the oscillating character, and on the other hand, there is more distinctly expressed dependence on \( \gamma \)-factor of the bunch. Besides, magnetic wake field \( B_\varphi \) is equal to zero in plasma without the field and defers from zero in the case of plasma with \( B_0 \neq 0 \).

Acknowledgment

The work was supported by the ISTC Grant A – 013.

References

[1] E. Esarey, P. Sprangle, J. Krall, IEEE Trans. Plasma Sci. 24, 252 (1996).
[2] Ya. Feinberg, Fizika Plazmi 23, 275 (1997) (in Russian).
[3] A.Ts. Amatuni, S.S. Elbakian, A.G. Khachatryan, E.V. Sekhpossian, Part. Acc. 51, 1 (1995).
[4] P. Chen, Part. Acc. 20, 171 (1987).
[5] P. Chen, J.M. Dawson, R.W. Huff, T. Katsouleas, Phys. Rev. Lett. 54, 693 (1985).
[6] R.D. Ruth, A.W. Chao, P.L. Morton, P.B. Wilson, Part. Acc. 17, 171 (1985).
[7] R. Keinigs, M.E. Jones, Phys. Fluids 30, 252 (1987).
[8] V. Balakirev, Preprint 87-40, Kharkov: KhFTI, 1987 (in Russian).
[9] A. Amatuni, E. Sekhpossian, A. Khachatryan, S. Elbakian, Plasma Phys. Rep 21, 945 (1995).
[10] A. Khachatryan, A. Amatuni, E. Sekhpossian, S. Elbakian, Plasma Phys. Rep 22, 576 (1996).
[11] Ya. Feinberg, N. Ayzatskij, V. Balakirev et al., Proc. of XV Int. Workshop on Charged Part. Linear Acc., 1997, Alushta, p. 16 (in Russian).
[12] A. Amatuni, M. Magomedov, E. Sekhpossian, S. Elbakian, Fizika Plasmi 5, 85 (1979) (in Russian).

[13] A. Amatuni, E. Sekhpossian, S. Elbakian, Fizika Plasmi 12, 1145 (1986) (in Russian).

[14] A.Ts. Amatuni, S.S. Elbakian, E.V. Sekhpossian, Prep. YerPhI-935(86)-86, Yerevan, 1986.

[15] A. Amatuni, E. Sekhpossian, S. Elbakian, Proc. of XIII Int. Conf. on High Energy Part. Acc., 1987, Novosibirsk, Nauka Publ., v. 1, p. 175 (in Russian).

[16] A.Ts. Amatuni, S.S. Elbakian, E.V. Sekhpossian, R.O. Abramian, Part. Acc. 41, 153 (1993); Preprint EFI-1365(60), Yerevan, 1991.

[17] A.Ts. Amatuni, E.V. Sekhpossian, A.G. Khachatriyan, S.S. Elbakian, Izvestiya Akademii Nauk Armenii, Fizika 28, 8 (1993): Soviet J. of Cont. Phys., Allerton Press Inc. 28, 7 (1994).

[18] J.B. Rosenzweig, Phys. Rev. Lett. 88, 555 (1987).

[19] A.Ts. Amatuni, E.V. Sekhpossian, S.S. Elbakian, Izvestiya Akademii Nauk Armenii, Fizika 25, 308 (1990): Soviet J. of Cont. Phys., Allerton Press Inc. 25, 1 (1990).

[20] W.B. Mori, T. Katsouleas, Proc. of EPAC-90, Nice-Paris, 1990, v. 1, p. 603.

[21] J. Krall, E. Esarey, P. Sprangle, G. Joice, Phys. Plasmas 1, 1738 (1994).

[22] N. Andrejev, L. Gorbunov, R. Ramazashvili, Fizika Plasmi 23, 303 (1997) (in Russian).

[23] I. Snedon, Preobrazovanija Fourier, Moscow, Inostr. Lit. Publ., 1955 (in Russian).