Some pseudo-Kähler Einstein 4-symmetric spaces with a “twin” special almost complex structure.

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Abstract

On 4-symmetric symplectic spaces, invariant almost complex structures - up to sign- arise in pairs. We exhibit some 4-symmetric symplectic spaces, with a pair of “natural” compatible (usually not positive) invariant almost complex structures, one of them being integrable and the other one being maximally non integrable (i.e. the image of its Nijenhuis tensor at any point is the whole tangent space at that point). The integrable one defines a pseudo-Kähler Einstein metric on the manifold, and the non integrable one is Ricci Hermitian (in the sense that the almost complex structure preserves the Ricci tensor of the associated Levi Civita connection) and special in the sense that the associated Chern Ricci form is proportional to the symplectic form.
Introduction

An almost complex structure $j$ on a manifold $M$ is a smooth field of endomorphisms of the tangent bundle whose square is equal to minus the identity. It is said to be integrable if it is induced by a complex structure on $M$; this means that one can locally define complex coordinates on $M$ and that the changes of coordinates are holomorphic; the associated almost complex structure is then given by $j \frac{\partial}{\partial x^k} = \frac{\partial}{\partial y^k}$ if $x^k$ and $y^k$ are the real and imaginary part of the local complex coordinates $\{z^k = x^k + iy^k | k = 1, \ldots, n\}$.

The Newlander-Nirenberg theorem asserts that an almost complex structure $j$ on a manifold $M$ is integrable if and only if its Nijenhuis torsion $N^j$ vanishes identically. Recall that the Nijenhuis torsion associated to a smooth field $A$ of endomorphisms of the tangent bundle is the tensor of type $(1,2)$ defined by

$$N^A(X,Y) := [AX, AY] - A[AX, Y] - A[X, AY] + A^2[X, Y] \quad \forall X, Y \in \mathfrak{X}(M), \quad (0.1)$$

where $\mathfrak{X}(M)$ is the Lie algebra of $C^\infty$ vector fields on $M$. The Image distribution associated to an almost complex structure $j$ is

$$\text{Im}\ N^j := \text{Span}\{N^j_x(X,Y) | X, Y \in T_xM\}. \quad (0.2)$$

A way to measure the non integrability of an almost complex structure $j$ is to look at the dimension of this image distribution; remark that this distribution is smooth but its dimension may vary from point to point.

An almost complex structure $j$ is called maximally non integrable if $\text{Im}\ N^j$ is the whole tangent bundle.

The condition to be maximally non integrable is open in the space of almost complex structures and generic in high dimension; in a recent preprint [10], R. Coelho, G. Placini, and J. Stelzig show that they exist on any $2n$-dimensional almost complex manifold when $2n \geq 10$. We want here to provide such maximally non integrable almost complex structures which are geometrically “natural”. The first example of a geometrical non integrable almost complex structure appears in Eells and Salamon [13]; it arises on a twistor space when flipping the sign of the vertical part of the standard integrable almost complex structure on this space. We showed in [9] that many twistor spaces are endowed in this way with one integrable and one maximally non integrable almost complex structures.

On a symplectic manifold $(M, \omega)$, an almost complex structure $j$ is said to be compatible with $\omega$ if the tensor $g_j$ defined by

$$g_j(X,Y) := \omega(X,jY) \quad (0.3)$$

is a metric.
is symmetric, hence yields an associated pseudo-Riemannian metric. A compatible almost complex structure is said to be positive when the associated metric $g_j$ is Riemannian.

A triple $(M, \omega, j)$ with $(M, \omega)$ symplectic and $j$ a compatible almost complex structure is equivalent to the data of an almost pseudo-Kähler manifold $(M, g, j)$, i.e. an almost pseudo-Hermitian manifold (which is a pseudo-Riemannian manifold $(M, g)$ with an almost complex structure $j$ which is compatible in the sense that the tensor $\omega$ defined by $\omega(X, Y) = g(jX, Y)$ is skew-symmetric), with the extra condition that $d\omega = 0$.

It is well known that there exist compatible positive almost complex structures on any symplectic manifold, and there have been various attempts to get procedures to select some of those.

D. Blair and S. Ianus [6] studied the restriction of the Hilbert functional, usually defined on the space of Riemannian metrics on $M$, to the space of metrics built from positive compatible almost complex structures, i.e.

$$ \mathcal{F}(j) = \int_M s^{g_j} \frac{\omega^N}{n!} $$

where $s^{g_j}$ is the scalar curvature of the Levi Civita connection associated to the metric $g_j$. Extrema of the Hilbert functional are the Einstein metrics. The extrema of $\mathcal{F}$ are those $j$’s which are Ricci Hermitian in the following sense:

**Definition 0.1.** An admissible almost complex structure $j$ on a symplectic manifold $(M, \omega)$ is said to be Ricci Hermitian iff

$$ Ric^{g_j}(jX, jY) = Ric^{g_j}(X, Y), $$

(0.4)

where $Ric^{g_j}$ is the Ricci tensor of the Levi Civita connection associated to the pseudo-Riemannian metric $g_j$.

This is automatically satisfied if $j$ is integrable since the integrability is equivalent to the fact that $j$ is parallel for the Levi Civita connection. Examples of non Kählerian triples $(M, \omega, j)$ satisfying this condition have been given in 1990 par Davidov et Muskarov [11] on some twistor spaces over Riemannian manifolds.

V. Apostolov and T. Draghici have introduced in [2] the notion of special almost complex structures on a symplectic manifold.
**Definition 0.2.** An admissible almost complex structure $j$ on a symplectic manifold $(M, \omega)$ is said to be *special* if its Chern Ricci form is proportional to $\omega$. The Chern Ricci form is the 2-form on $M$ defined by

$$ChernRicci^j(X, Y) := \text{Tr} \, j R^C(X, Y) \tag{0.5}$$

where $R^C$ is the curvature of the Chern connection $\nabla^C$ defined by

$$\nabla^C_X Y := \nabla^g_X Y - \frac{1}{2} j \left( \nabla^g_X j \right) Y \tag{0.6}$$

with $\nabla^g$ denoting the Levi Civita connection associated to the pseudo-Riemannian metric $g_j$. It is the only connection for which $\omega$ and $j$ are parallel and whose torsion is proportional to $N^j$.

Positive admissible special almost complex structures were studied by Alberto Della Vedova in [12] in a homogeneous context. Homogeneous almost Kähler manifolds are also studied by D. V. Alekseevsky and Fabio Podestà in [1].

There exist two natural almost complex structures $j^\pm$ on the twistor space $T$ over the hyperbolic space $H_{2n}$. This twistor space can be viewed as an adjoint orbit in the semisimple Lie algebra $\mathfrak{so}(1, 2n)$, and thus carries a Kirillov-Kostant-Souriau symplectic structure $\omega$. It was shown in [8] that both $j^\pm$ are compatible with $\omega$, $j^-$ is positive and maximally non integrable whereas $j^+$ is integrable but not positive. This twistor space is an example of a 4-symmetric space and we study in this paper a generalization of this construction on some 4-symmetric spaces; many examples appeared in the thesis of Manar Hayyani [16].

Symplectic and almost complex structures on $k$-symmetric spaces have been considered, and many results exist on 3-symmetric spaces. In [7], Maciej Bochenski and Aleksy Tralle study symplectic structures on $k$-symmetric spaces, which are invariant by the symmetries, and a list of all symplectic 3-symmetric manifolds with simple groups of transvections is given. This extends our previous work about symmetric symplectic spaces [4] and the results obtained by Pierre Bieliavsky [5]. Cecilia Ferreira studies in [14] some necessary and some sufficient conditions for the integrability of a canonical almost complex structure on a $k$-symmetric space. These always exist when $k$ is odd. J.A. Jiménez [17] has given a classification of compact simply connected Riemannian 4-symmetric spaces; these spaces are homogeneous for a connected compact semisimple Lie group with an automorphism of order four. Geometrically, they can be regarded as fiber bundles over Riemannian 2-symmetric spaces with totally geodesic fibers isometric to a Riemannian 2-symmetric space.
Vitaly Balashchenko studies in canonical distributions on Riemannian homogeneous \( k \)-symmetric spaces; those are distributions associated to canonical affinor structures (an affinor structure is a field of endomorphisms \( A \) which is a product structure, i.e. \( A^2 = \text{Id} \), an almost complex structure, i.e. \( A^2 = -\text{Id} \), a \( f \)-structure, i.e. \( A^3 - A = 0 \), or a \( h \)-structure, i.e. \( A^3 + A = 0 \)).

Here we study a class of 4-symmetric spaces which carry two associated invariant almost complex structures, which are natural in a sense that we define. One of them is integrable and we state conditions for the other one to be maximally non integrable.

Some of those spaces carry a symplectic structure for which the almost complex structures are compatible, in general not positive.

We get families of examples of Einstein pseudo-Kähler manifolds, admitting another natural almost complex structure which is maximally non integrable, Ricci Hermitian in the sense of definition 0.1, and special in the sense of definition 0.2.

In section 1, we recall basic facts about \( k \)-symmetric spaces and invariant structures. In section 2, we construct a class of 4-symmetric spaces with two “twin” invariant almost complex structures \( j^\pm \). We indicate conditions under which a symplectic 2-form \( \omega \) can be defined, for which \( j^\pm \) are compatible. The examples given in section 3 are summarized in:

**Proposition 0.3.** Let \( G \) be a connected Lie subgroup of \( \text{Gl}(m, \mathbb{R}) \), with \( m = k + 2n \), whose Lie algebra \( \mathfrak{g} \) is stable under \( \text{ad} \rho \) where \( \rho = \begin{pmatrix} 0 & J_{2n} \\ 0 & -\text{Id}_n \end{pmatrix} \) with \( J_{2n} = \begin{pmatrix} 0 \ & \text{Id}_n \\ 0 \ & 0 \end{pmatrix} \).

Denote by \( \sigma \) the automorphism of \( \mathfrak{g} \) of order 4 given by \( \sigma = \exp(\frac{1}{4}\pi \text{ad} \rho) \).

\( G \) is stable by the automorphism \( \tilde{\sigma} : G \to G : g \mapsto RgR^{-1} \) for \( R := \begin{pmatrix} \text{Id}_k & 0 \\ 0 & J_{2n} \end{pmatrix} \) and the homogeneous space \( G/G_0^{\tilde{\sigma}} \), where \( G_0^{\tilde{\sigma}} \) is the connected component of the subgroup of elements fixed by \( \tilde{\sigma} \), has a natural structure of 4-symmetric space.

The tangent space at the base point identifies with \( \mathfrak{g}_{-1}^\tau \oplus \mathfrak{g}_{-1}^{\sigma_2} \) where \( \mathfrak{g}_{-1}^{\sigma} \) denotes the \( \tau \) eigenspace of eigenvalue \( \lambda \) in \( \mathfrak{g} \). Two natural invariant almost complex structures \( j^\pm \) on \( G/G_0^{\tilde{\sigma}} \) are defined by their value at the base point, \( j^\pm \in \text{End}(\mathfrak{g}_{-1}^\tau \oplus \mathfrak{g}_{-1}^{\sigma_2}) \), given by

\[
J^\pm|_{\mathfrak{g}_{-1}^{\sigma_2}} := \sigma|_{\mathfrak{g}_{-1}^{\sigma_2}}, \quad J^\pm|_{\mathfrak{g}_{-1}^\tau} = \pm \exp(\frac{1}{4}\pi \text{ad} \rho);
\]

\( j^+ \) is always integrable; the image of the Nijenhuis torsion of \( j^- \) is the \( G \)-invariant distribution whose value at the base point is given by \( [\mathfrak{g}_{-1}^{\tau}, \mathfrak{g}_{-1}^{\sigma_2}] + [\mathfrak{g}_{-1}^{\sigma_2}, \mathfrak{g}_{-1}^{\sigma_2}] \cap \mathfrak{g}_{-1}^\tau \).

The \( G \)-invariant 2-form \( \tilde{\Omega} \) on \( G/G_0^{\tilde{\sigma}} \) whose value at the base point is given by

\[
\tilde{\Omega}(X, Y) := \text{Tr}(\rho[X, Y]) \quad \forall X, Y \in \mathfrak{g}_{-1}^\tau + \mathfrak{g}_{-1}^{\sigma_2}
\]
is an invariant symplectic structure, if and only if it is non degenerate, i.e. iff $g': g_{-1}^2 \times g_{-1}^2 \to \mathbb{R} : X, Y \mapsto \text{Tr}(XY)$ and $\beta'' : g_{-1}^2 \times g_{-1}^2 \to \mathbb{R} : X, Y \mapsto \text{Tr}(XY)$ are non degenerate; this is automatically true if $g$ is simple. The almost complex structures $j^\pm$ are compatible with $\omega$, in general not positive. When $\omega$ is non degenerate, one considers the associated metrics, $g^\pm(X, Y) = \omega(X, j^\pm Y)$. The Ricci tensors of the associated Levi Civita connections are $j^\pm$ hermitian and, under some conditions on $g$, the structure $j^-$ is special. In particular, we have the following examples.

The space $Sl(k + 2n, \mathbb{R})/S(Gl(k, \mathbb{R}) \times Gl(n, \mathbb{C}))$, where $Gl(n, \mathbb{C})$ is the subset of elements in $Gl(2n, \mathbb{R})$ commuting with $J_{2n}$, with the pair $(\omega, j^+)$, is a pseudo-Kähler 4-symmetric space which is Einstein $(\text{Ric}^g = (k + n) g^+)$; with the pair $(\omega, j^-)$, it is an almost pseudo-Kähler 4-symmetric space, $j^-$ is maximally non integrable and special $(\text{ChernRicci}^j = 2(n - k) \omega)$.

The space $SO(k + 2n, \mathbb{R})/(SO(k, \mathbb{R}) \times U(n))$, where $U(n)$ is the subset of elements in $SO(2n, \mathbb{R})$ commuting with $J_{2n}$, with the pair $(-\omega, j^+)$, is a Kähler 4-symmetric space which is Einstein $(\text{Ric}^g = \frac{1}{2}(k + n - 1) g^+, g^+ \text{ negative definite})$; with the pair $(\omega, j^-)$, it is an almost pseudo-Kähler 4-symmetric space, $j^-$ is maximally non integrable and special $(\text{ChernRicci}^j = (n - 1 - k) \omega)$. The case where $k = 1$ corresponds to the twistor space $SO(1 + 2n, \mathbb{R})/U(n)$ on the sphere $SO(1 + 2n, \mathbb{R})/SO(2n, \mathbb{R})$.

The space $SO_0(k, 2n, \mathbb{R})/(SO(k, \mathbb{R}) \times U(n))$, with the pair $(\omega, j^+)$, is a pseudo-Kähler 4-symmetric space which is Einstein $(\text{Ric}^g = \frac{1}{2}(k + n - 1) g^+, g^+ \text{ negative definite})$. The case where $k = 1$ corresponds to the twistor space $SO_0(1, 2n, \mathbb{R})/U(n)$ on the hyperbolic space $SO_0(1, 2n, \mathbb{R})/SO(2n, \mathbb{R})$.

The space $U(k' + n)/ (U(k') \times U(n))$ with the pair $(-\omega, j^+)$ is a Kähler symmetric space which is Einstein $(\text{Ric}^g = \frac{1}{2}(k' + n) g^+, g^+ \text{ negative definite})$. The space $U(k', n)/ (U(k') \times U(n))$ with the pair $(\omega, j^+)$ is a Kähler symmetric space which is Einstein $(\text{Ric}^g = \frac{1}{2}(k' + n) g^+)$. Both spaces carry only one natural almost complex structure (up to sign), $j^+$, defined by $\sigma$ on $p = g_{-1}^2$.

The space $Sp(\mathbb{R}^{2(k' + n)}, \tilde{\Omega})/(Sp(\mathbb{R}^{2k'}, \Omega_k) \times U(n))$, where $\tilde{\Omega} = \left( \begin{array}{cc} \Omega_{2r} & 0 \\ 0 & \Omega_{2n} \end{array} \right)$ with $\Omega_{2r} := -J_{2r}$ and where $U(n)$ is the subset of elements in $Sp(\mathbb{R}^{2n}, \Omega_{2n})$ commuting with $J_{2n}$, with $(\omega, j^+)$, is a pseudo-Kähler 4-symmetric space which is Einstein $(\text{Ric}^g = \frac{1}{2}(2k' + n + 1) g^+)$; with the pair $(\omega, j^-)$, it is an almost pseudo-Kähler 4-symmetric space, $j^-$ is maximally non integrable and special $(\text{ChernRicci}^j = 2 n - 2 k = \frac{1}{2}(2k' + n + 1) g^+)$.
For all those spaces, the symplectic structure coincides with the Kirillov-Kostant-Souriau symplectic form on the coadjoint orbit of the element \( \rho^* : \mathfrak{g} \to \mathbb{R} : X \mapsto \text{Tr} \rho X \) in \( \mathfrak{g}^* \).

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### 1 \( k \)-symmetric spaces

The notion of \( k \)-symmetric space appeared in the late sixties in the works of Gray, Wolf, Ledger, and Obata \cite{gray, wolf, ledger, obata}, as a generalization of the notion of symmetric space which corresponds to \( k = 2 \). General results are summarized in a monograph written by Kowalski \cite{kowalski} in 1980.

**Definition 1.1.** A \( k \)-symmetric space (with \( k \) an integer \( \geq 2 \)) is a pair \((M, S)\), where \( M \) is a smooth manifold and \( S : M \times M \to M : (x, y) \mapsto S(x, y) =: s_x y \) is a smooth map such that

- for each \( x \in M \), \( s_x \) -which will be called the symmetry at \( x \)- is a diffeomorphism of order \( k \) of \( M \) (i.e. \( s_x^k = \text{Id} \) and \( k \) is the smallest positive integer with that property), which fixes \( x \), and the differential of \( s_x \) at the point \( x \) has no non-zero fixed vector (which implies that \( x \) is an isolated fixed point);
- \( s_x \circ s_y = s_{s_x y} \circ s_x \) for all \( x, y \in M \).

An automorphism of a \( k \)-symmetric space \((M, S)\) is a diffeomorphism \( \varphi : M \to M \) such that \( \varphi(s_x(y)) = s_{\varphi(x)} \varphi(y) \) \( \forall x, y \in M \). Each symmetry is an automorphism.

It is known that, if \( M \) is connected, the group of automorphisms \( \text{Aut}(M, S) \) is a Lie group which acts transitively on \( M \); we shall only consider \( k \)-symmetric spaces for which \( \text{Aut}(M, S) \) acts transitively. Choosing a point \( p_0 \in M \), one defines the automorphism \( \tilde{\sigma} \) of \( \text{Aut}(M, S) \) given by \( g \mapsto \tilde{\sigma}(g) := s_{p_0} \circ g \circ (s_{p_0})^{-1} \). The stabilizer \( H \) of \( p_0 \) in \( \text{Aut}(M, S) \) is contained in the subgroup \( \text{Aut}(M, S)_{\tilde{\sigma}} \) of elements fixed by \( \tilde{\sigma} \), and the connected component of \( \text{Aut}(M, S)_{\tilde{\sigma}} \) is contained in \( H \).

**Definition 1.2.** A \( k \)-symmetric triple (also called a homogeneous \( k \)-symmetric space for the Lie group \( G \)) is a triple \((G, \tilde{\sigma}, H)\) where \( G \) is a Lie group, \( \tilde{\sigma} \) is an automorphism
of $G$ of order $k$ and $H$ is a subgroup of $G$ such that

$$G^\sigma_\circ \subset H \subset G^\sigma \quad (1.1)$$

where $G^\sigma_\circ$ is the subgroup of elements of $G$ which are fixed by $\sigma$ and $G^\sigma_\circ$ its connected component of the identity.

A $k$-symmetric triple defines a $k$-symmetric space $(M, S)$ where $M$ is the homogeneous space $M = G/H$, and $S : M \times M \to M$ is defined by

$$s_{\pi(g)}\pi(g') := \pi(g\tilde{\sigma}(g^{-1}g')), \quad \forall g, g' \in G,$$

where $\pi : G \to G/H$ denotes the canonical projection. Remark that $G$ acts by automorphisms of this symmetric space.

Hence, a $k$-symmetric triple defines a $k$-symmetric space in a unique way, but a connected $k$-symmetric space can be associated to different $k$-symmetric triples. Since we study invariant structures, we only deal with homogeneous spaces and the precision of the group of invariance considered will be of crucial importance; we shall thus use the description of a (homogenous) $k$-symmetric space in terms of a chosen $k$-symmetric triple.

A $k$-symmetric triple defines a pair $(g, \sigma)$ where $g$ is the Lie algebra of $G$ and where $\sigma : g \to g$ is the automorphism of $g$ of order $k$ defined by the differential of $\tilde{\sigma}$ at the neutral element $e \in G$. The Lie algebra $\mathfrak{h}$ of the group $H$ is always equal to the subalgebra $g^\sigma$ of elements which are fixed by $\sigma$.

**Definition 1.3.** A $k$-symmetric algebra (where $k$ is an integer $\geq 2$) is a pair $(g, \sigma)$, where $g$ is a Lie algebra and $\sigma$ is an automorphism of $g$ of order $k$.

A $k$-symmetric algebra defines a $k$-symmetric triple $(G, \tilde{\sigma}, H)$ for any Lie group $G$ with Lie algebra $\mathfrak{g}$ which admits an automorphism $\tilde{\sigma}$ lifting $\sigma$ (for instance the simply connected group) and for any $H$ satisfying relation (1.1) (for instance $H = G^\sigma_\circ$).

Observe that, for any k-symmetric space corresponding to a $k$-symmetric algebra $(\mathfrak{g}, \sigma)$, the tangent space to $M = G/H$ at the base point $p_0 = \pi(e)$ identifies to $\mathfrak{g}/\mathfrak{g}^\sigma$ and the differential of the symmetry $s_{p_0}$ at the base point identifies with the linear map on $\mathfrak{g}/\mathfrak{g}^\sigma$ induced by $\sigma$.

**Definition 1.4.** An invariant almost complex structure (resp. invariant symplectic structure, invariant pseudo-riemannian structure,..) on a homogeneous $k$-symmetric space described by a $k$-symmetric triple $(G, \tilde{\sigma}, H)$ is an almost complex structure (resp. symplectic structure, pseudo-riemannian structure,..) on $M = G/H$ which is invariant by the symmetries and by the action of $G$. 

8
Such a structure is completely determined by its value at the base point, and that restriction must be invariant under the action of $H$ and under the symmetry at $p_0$. Observe that the differential at the base point of the action of an element $h \in H$ is the action, still denoted $\text{Ad} \ h$, induced by $\text{Ad} \ h$ on the quotient $\mathfrak{g}/\mathfrak{g}^\sigma$. The differential at the base point of the symmetry at the base point is given by the action, still denoted $\sigma$, induced by $\sigma$ on $\mathfrak{g}/\mathfrak{g}^\sigma$. The action of $\sigma$ and the action of $\text{Ad} \ h$ commute since $H \subset \tilde{G}^\tilde{\sigma}$. If $H$ is connected, a structure is invariant iff its value at the base point is invariant under the action induced on the quotient $\mathfrak{g}/\mathfrak{g}^\sigma$ by $\sigma$ and by $\text{ad} \ X$ for all $X \in \mathfrak{g}^\sigma$.

An invariant almost complex structure $j$ corresponds thus bijectively to a $J: \mathfrak{g}/\mathfrak{g}^\sigma \to \mathfrak{g}/\mathfrak{g}^\sigma$ such that $J \circ \sigma = \sigma \circ J$ and $J \circ \text{Ad} \ h = \text{Ad} \ h \circ J$ for all $h \in H$.

To study the integrability of an invariant almost complex structure on a homogeneous $k$-symmetric space, we shall use the following description:

**Proposition 1.5.** ([13], thm 6.4, page 217) Let $j$ be a $G$-invariant almost complex structure on a homogeneous space $M = G/H$, let $J: \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$ be its value at the base point $p_0 = \pi(e)$, with $\pi: G \to G/H$ the canonical projection. Let $\hat{J}: \mathfrak{g} \to \mathfrak{g}$ be any linear map such that $\pi_*\hat{J}(X) = J(\pi_*X)$ with $\pi_*$ the differential of $\pi$ at the neutral element $e$, i.e. the canonical projection $\pi_*: \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$. Then

$$N^j_{p_0}(\pi_*X, \pi_*Y) = \pi_*\tilde{N}^J(X, Y)$$

where $\tilde{N}^J(X, Y) := [\hat{J}X, \hat{J}Y] - \hat{J}[[\hat{J}X, Y] - \hat{J}[X, \hat{J}Y] + \hat{J}^2[X, Y]$ for all $X, Y \in \mathfrak{g}$.

We are particularly interested to build homogeneous 4-symmetric (almost)-pseudo-Kähler manifolds, hence study invariant complex structures $j$ on a 4-symmetric symplectic space which are compatible with the invariant symplectic 2-form $\omega$.

**Definition 1.6.** A homogeneous $k$-symmetric symplectic space is a $G$-homogeneous $k$-symmetric space endowed with a symplectic 2-form which is $G$-invariant and invariant by the symmetries; it is defined by a $k$-symmetric symplectic quadruple $(G, \tilde{\sigma}, H, \tilde{\Omega})$ with $(G, \tilde{\sigma}, H)$ a $k$-symmetric triple and

$$\tilde{\Omega}: \mathfrak{g}/\mathfrak{g}^\sigma \times \mathfrak{g}/\mathfrak{g}^\sigma \to \mathbb{R}$$

a non degenerate bilinear skewsymmetric map such that

- $\tilde{\Omega}(\pi_*(\text{Ad} \ hX), \pi_*(\text{Ad} \ hY)) = \tilde{\Omega}(\pi_*(X), \pi_*(Y))$ for all $X, Y \in \mathfrak{g}$, for all $h \in H$;
\begin{itemize}
\item $\tilde{\Omega}(\pi_*(\sigma X), \pi_*(\sigma Y)) = \tilde{\Omega}(\pi_*(X), \pi_*(Y))$ for all $X, Y \in \mathfrak{g}$;
\item $\bigoplus_{XYZ} \tilde{\Omega}(\pi_*(\sigma [X,Y]), \pi_*(\sigma Z)) = 0$ for all $X, Y, Z \in \mathfrak{g}$,
\end{itemize}

$\bigoplus$ denoting the sum over cyclic permutations and $\pi_* : \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}^\sigma$ being, as before, the canonical projection.

Gray and Wolf have introduced in [22] the notion of a canonical field of endomorphisms:

**Definition 1.7.** A field of endomorphisms on a $k$-symmetric space described by a $k$-symmetric triple $(G, \tilde{\sigma}, H)$ is said to be canonical if it is invariant and if its value at the base point is given by the action on $\mathfrak{g}/\mathfrak{g}^\sigma$ of a polynomial in $\sigma$.

Remark that any polynomial in $\sigma$ is automatically invariant under the action induced on the quotient $\mathfrak{g}/\mathfrak{g}^\sigma$ by $\sigma$ and by $\text{Ad}h$ for all $h \in H$ so yields a canonical field of endomorphisms.

Any endomorphism of order $k$ of a finite dimensional vector space being semisimple, one can identify the complexification of the tangent space at the base point, $(\mathfrak{g}/\mathfrak{g}^\sigma)^C$, to the sum of all eigenspaces for $\sigma$, corresponding to eigenvalues (which are of the form $\lambda = e^{2\pi i r/k}$ since $\sigma^k = \text{Id}$) which are different from 1.

To have a canonical almost complex structure on a $k$-symmetric space, $-1$ cannot be an eigenvalue for $\sigma$. This is of course automatic if $k$ is odd. In this paper we want to study almost complex structures on 4-symmetric spaces and we have to go beyond this canonical condition.

**Definition 1.8.** Let $(G, \tilde{\sigma}, H)$ be a 4-symmetric triple and write the decomposition of the Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_{\sigma^2} + \mathfrak{g}_{-1}^{\sigma^2} = \mathfrak{g}^\sigma + \mathfrak{g}_{-1}^\sigma + \mathfrak{g}_{-1}^{\sigma^2},$$

where $\mathfrak{g}_{\lambda}^k$ is the eigenspace for $\sigma^k$ of eigenvalue $\lambda$. One identifies, via $\pi_*$, the tangent space to $G/H$ at the base point $eH$ with $\mathfrak{g}_{-1}^\sigma + \mathfrak{g}_{-1}^{\sigma^2}$. Any invariant almost complex structure $J$ on $G/H$ is defined by

$$J : \mathfrak{g}_{-1}^{\sigma^2} + \mathfrak{g}_{-1}^{\sigma^2} \to \mathfrak{g}_{-1}^\sigma + \mathfrak{g}_{-1}^{\sigma^2}$$

such that $J|_{\mathfrak{g}_{-1}^{\sigma^2}} = \hat{\sigma} : \mathfrak{g}_{-1}^{\sigma^2} \to \mathfrak{g}_{-1}^{\sigma^2}$ and $J|_{\mathfrak{g}_{-1}^\sigma} = \tau : \mathfrak{g}_{-1}^\sigma \to \mathfrak{g}_{-1}^{\sigma^2}$ satisfy

$$(\hat{\sigma})^2|_{\mathfrak{g}_{-1}^{\sigma^2}} = -\text{Id}|_{\mathfrak{g}_{-1}^{\sigma^2}}, \quad \hat{\sigma} \circ \sigma|_{\mathfrak{g}_{-1}^{\sigma^2}} = \sigma|_{\mathfrak{g}_{-1}^{\sigma^2}} \circ \hat{\sigma}, \quad \hat{\sigma} \circ \text{Ad}(h) = \text{Ad}(h) \circ \hat{\sigma} \quad \forall h \in H;$$
\[ \tau^2 = \sigma_{\tilde{g}^{-1}} = -\text{Id}_{\tilde{g}^{-1}} \quad \text{and} \quad \tau \circ \text{Ad}(h) = \text{Ad}(h) \circ \tau \quad \forall h \in H. \]

An invariant almost complex structure on a 4-symmetric space is said to be natural if \( \tilde{\sigma} = \sigma \), i.e. if \( J|_{\tilde{g}^{-2}} = \sigma|_{\tilde{g}^{-2}} \).

When \( \tilde{g}^{-1} \neq 0 \), invariant almost complex structures on 4-symmetric spaces arise in pairs (called twins) \( j^\pm \) corresponding to \( J^\pm \) defined by

\[ J^\pm|_{\tilde{g}^{-2}} = \hat{\sigma} \quad \text{and} \quad J^\pm|_{\tilde{g}^{-1}} = \pm \tau. \]

**Remark 1.9.** Any homogeneous 4-symmetric space \((G, \tilde{\sigma}, H)\) fibers over the symmetric space \( G/G^{\tilde{\sigma}^2} \) since \((\tilde{\sigma}^2)^2 = \text{Id}\) and \( H \subset G^{\tilde{\sigma}} \subset G^{\tilde{\sigma}^2} \).

Observe that \( \tilde{g}^{\tilde{\sigma}^2} = \tilde{g}^{\sigma} + \tilde{g}^{-2} \) and that \( G \) is contained in the group of automorphisms of the symmetric space \( G/G^{\tilde{\sigma}^2} \). The Lie algebra \( \tilde{g}' \) of the transvection group for this symmetric space, which is given by

\[ \tilde{g}' = p + [p, p] \quad \text{with} \quad p = \tilde{g}^{-2}, \]

is an ideal in \( \tilde{g} \).

The fiber of the projection \( G/H \to G/G^{\tilde{\sigma}^2} \) identifies with the symmetric space \( G^{\tilde{\sigma}^2}/H \) defined by the triple \((G^{\tilde{\sigma}^2}, \tilde{\sigma}, H)\).

A natural almost complex structure \( j \) on \( G/H \) induces an invariant almost complex structure on the symmetric space \( G/G^{\tilde{\sigma}^2} \) if and only if \( \sigma \) commutes with \( \text{Ad} k \) for any \( k \in G^{\tilde{\sigma}^2} \); in that case the fibration \( G/H \to G/G^{\tilde{\sigma}^2} \) is a pseudo-holomorphic map.

The fiber \( G^{\tilde{\sigma}^2}/H \) always carries an invariant complex structure \( j^\nu \) induced by \( \tau \) since the tangent space to the fiber at \( eH \) identifies to \( \tilde{g}^{\sigma^2}/h = \tilde{g}^{\sigma^2}/\tilde{g}^{\sigma} \simeq \tilde{g}^{-1} \); it is integrable since \([\tilde{g}^{-1}, \tilde{g}^{-1}] \subset \tilde{g}^{\sigma} \).

**Definition 1.10.** A homogeneous 4-symmetric natural (almost)-pseudo-Kähler manifold is a homogeneous 4-symmetric space \((G, \tilde{\sigma}, H)\), endowed with a natural almost complex structure \( j \), defined by \( \tau \) as in definition 1.8 and with a symplectic structure \( \omega \) on \( G/H \) invariant by \( G \) and by the symmetries, defined by a \( \tilde{\Omega} \) as in definition 1.6 such that \( j \) is compatible with it. Since the tangent space to \( G/H \) at the base point identifies via \( \pi_* \) with \( \tilde{g}^{-1} + \tilde{g}^{-2} \), the data of \( \tilde{\Omega} \) with the compatibility with \( J \) implies the compatibility with \( J^\pm \) and is equivalent to the data of two non degenerate skewsymmetric bilinear maps

\[ \tilde{\Omega}^\nu : \tilde{g}^{-1} \times \tilde{g}^{-1} \to \mathbb{R} \quad \text{and} \quad \tilde{\Omega}' : \tilde{g}^{-2} \times \tilde{g}^{-2} \to \mathbb{R} \]
such that

\[
\tilde{\Omega}^v(\tau X, \tau X') = \tilde{\Omega}^v(X, X') = \tilde{\Omega}^v(\text{Ad } hX, \text{Ad } hX') \quad (1.2)
\]

\[
\tilde{\Omega}'(\sigma Y, \sigma Y') = \tilde{\Omega}'(Y, Y') = \tilde{\Omega}'(\text{Ad } hY, \text{Ad } hY') \quad (1.3)
\]

\[
\tilde{\Omega}'([X, Y], Y') + \tilde{\Omega}'(Y, [X, Y']) + \tilde{\Omega}^v\left(\frac{1}{2}([Y, Y'] - \sigma([Y, Y']))\right), X) = 0 \quad (1.4)
\]

for all \(X, X' \in g_{\sigma^{-1}}, \ h \in H\) and \(Y, Y' \in g_{\sigma^{-2}}\).

The fiber \(G_{\sigma^2}/H\) is then a symmetric pseudo-Kähler manifold with the symplectic structure defined by \(\tilde{\Omega}^v\) and the complex structure defined by \(\tau\).

The basis of the fibration, i.e. the symmetric space \(G/G_{\sigma^2}\), is endowed with an invariant symplectic structure defined by \(\tilde{\Omega}'\) iff \(\tilde{\Omega}'(X, X') = \tilde{\Omega}^v(\text{Ad } kX, \text{Ad } kX')\) for all \(k \in G_{\sigma^2}\). This condition, in view of equation (1.4) and the fact that \(g_{\sigma^{-1}} \subset g_{\sigma^2}\), implies that \([g_{\sigma^{-1}}, g_{\sigma^2}] \subset g_{\sigma}\).

## 2 A class of 4-symmetric spaces

Let \(D\) be a derivation of a Lie algebra \(g\) such that \(\exp 2\pi D = \text{Id}\). Remark that this implies that \(D\) is semisimple and that all eigenvalues of \(D\) are contained in \(i\mathbb{Z}\). Assume also that \(\exp tD \neq \text{Id}\) for all \(0 < t < 2\pi\). Let

\[
\sigma = \exp \frac{1}{2\pi} D.
\]

Clearly \(\sigma\) is an automorphism of order 4 and we consider the 4-symmetric algebra \((g, \sigma)\). We have, as above, the splitting

\[
g = g^{\sigma} + g^{\sigma-1} + p \quad \text{with} \quad p = g^{\sigma^2} \quad \text{i.e.} \quad \sigma^2|_p = -\text{Id}|_p.
\]

We identify \(g/g^{\sigma}\) to \(g^{\sigma^{-1}} + p\) and we define, as in definition (1.8), two natural complex structures \(J^\pm\) on \(g^{\sigma^{-1}} + p\) by

\[
J^\pm|_p := \sigma|_p \quad \text{and} \quad J^\pm|_{g^{\sigma^{-1}}} = \pm \exp \frac{1}{4\pi} D.
\]

They are different iff \(g^{\sigma^{-1}} \neq \{0\}\). For these to define invariant almost complex structures on \(G/G_{\sigma^2}\), they have to commute with \(\sigma\)-which is obvious- and to commute with the action of \(\text{ad } X\) for each \(X \in g^{\sigma}\); so one has only to check if \(\exp \frac{1}{4\pi} D\) commutes with \(\text{ad } X\) on \(g^{\sigma^{-1}}\) for all \(X \in g^{\sigma}\). Clearly

\[
g^{\sigma} = g^{\sigma}_{\sigma} = \bigoplus_{m \in \mathbb{Z}} g^{D}_{4mi} \cap g \quad \text{and} \quad g^{\sigma^{-1}} = \left(\bigoplus_{m \in \mathbb{Z}} g^{D}_{4m+2i}\right) \cap g
\]
where \( g^{D}_{m_i} \) is the eigenspace of \( D \) of eigenvalue \( m_i \) in the complexified Lie algebra \( g^C \).

We have, for all \( X \in g^{D}_{4m_i} \) and \( Y \in g^{D}_{(4m_i+2)i} \):

\[
(\text{ad} X \circ \exp \frac{1}{4} \pi D) Y = \left( \exp(m' + \frac{1}{2}i \pi) \right) [X, Y] = (-1)^{m'} i [X, Y]
\]

\[
(\exp \frac{1}{4} \pi D \circ \text{ad} X) Y = \left( \exp(m' + m + \frac{1}{2}i \pi) \right) [X, Y] = (-1)^{m' + m} i [X, Y].
\]

The structures \( J^\pm \) define thus invariant almost complex structure on \( G/G_0^\sigma \) if \( 4m_i \) is not an eigenvalue of \( D \) for \( m \) odd.

We consider the particular case where \( D \) is a semisimple derivation of \( g \) whose eigenvalues are precisely \( 0, i, -i, 2i \) and \(-2i \), and, as above, \( \sigma = \exp \frac{1}{2} \pi D \). In that case

\[
g^\sigma = g \cap g^D_0, \quad g^\sigma_{-1} = g \cap (g^D_{2i} \oplus g^D_{-2i}) \quad \text{and} \quad p = g \cap (g^D_i \oplus g^D_{-i}).
\]

The invariant almost complex structures on \( G/G_0^\sigma \) defined by \( J^\pm|_p := \sigma|_p \) and \( J^\pm|_{g^\sigma_{-1}} = \pm \exp \frac{1}{4} \pi D \) have Nijenhuis torsions which are \( G \)-invariant tensors. The maps \( J^\pm \) extend \( \mathbb{C} \)-linearly to \( T_{p_0}(G/G_0^\sigma)^C = g^D_i \oplus g^D_{-i} \oplus g^D_{2i} \oplus g^D_{-2i} \) as

\[
J^\pm|_{g^D_i} = i \text{Id} |_{g^D_i}, \quad J^\pm|_{g^D_{-i}} = -i \text{Id} |_{g^D_{-i}} \quad \text{and} \quad J^\pm|_{g^D_{2i}} = \pm i \text{Id} |_{g^D_{2i}} \quad J^\pm|_{g^D_{-2i}} = -\pm i \text{Id} |_{g^D_{-2i}}.
\]

Using proposition 1.5 and extending \( N^j_{p_0} \) to the complexified tangent space at the base point, we have

\[
N^j_{p_0}(\pi_* X, \pi_* Y) = \begin{cases} 
0 & \text{for } X, Y \in g^D_{2i} \text{ or } X, Y \in g^D_{-2i} \\
0 & \text{for } X \in g^D_{2i}, Y \in g^D_{-2i} \\
0 & \text{for } X \in g^D_i, Y \in g^D_{-i} \\
(-2 \pm 2)\pi_* [X, Y] & \text{for } X, Y \in g^D_i \text{ or } X, Y \in g^D_{-i} \\
(\pm 2 - 2)\pi_* [X, Y] & \text{for } X \in g^D_i, Y \in g^D_{-i} \text{ or } X \in g^D_{2i}, Y \in g^D_{-2i}
\end{cases}
\]

so that \( N^j_{p_0}(\pi_* X, \pi_* Y) = 0 \), \( N^j_{p_0}(\pi_* X, \pi_* Y) = -4 \pi_* [X, Y] \) for all \( X, Y \in g^\sigma_{-1} \oplus p \).

Hence we have

**Proposition 2.1.** Let \( D \) be a semisimple derivation of a Lie algebra \( g \) whose eigenvalues are precisely \( 0, i, -i, 2i \) and \(-2i \). Let \( \sigma = \exp \frac{1}{2} \pi D \). Let \( G \) be a Lie group with Lie algebra \( g \) such that \( \sigma \) lifts to an automorphism \( \tilde{\sigma} \) of \( G \). Let \( j^\pm \) be the two
natural almost complex structure on the 4-symmetric space $G/G_0^\sigma$ defined, identifying the tangent space at the base point to $g_-^\sigma + p$, with $p = g^\sigma_{-1}$, by

$$J^\pm|_p := \sigma|_p \quad J^\pm|_{g_-^\sigma} = \pm \exp \frac{1}{4} \pi D.$$  

Then $j^+$ is always integrable. The image of the Nijenhuis torsion of $j^-$ is the $G$-invariant distribution whose value at the base point is given by

$$\pi_* ([g_-^\sigma + p, g_-^\sigma + p]) = \pi_* ([p, g_-^\sigma] + [p, p] \cap g_-^\sigma).$$

Thus $j^-$ is maximally non integrable iff $[p, g_-^\sigma] = p$ and $[p, p] \cap g_-^\sigma = g_-^\sigma$.

**Remark 2.2.** The equality $[p, p] \cap g_-^\sigma = g_-^\sigma$ will be true iff $p + [p, p] + g^\sigma = g$, in particular (since $p + [p, p]$ is an ideal in $g$ by remark 1.9) if $g$ is simple, or if $g$ is reductive with $p$ intersecting each simple factor and $\sigma = \text{Id}$ on the center.

### 2.1 Construction

**2.1.1 The derivation, the automorphism, and the almost complex structures**

Let $\rho$ be the $(k + 2n) \times (k + 2n)$ matrix, with $k$ and $n$ positive integers, defined by

$$\rho = \begin{pmatrix} 0 & 0 \\ 0 & J_{2n} \end{pmatrix} \quad \text{where} \quad J_{2n} = \begin{pmatrix} 0 & -\text{Id}_n \\ \text{Id}_n & 0 \end{pmatrix} \quad (2.1)$$

and let $D$ be the derivation of $\text{gl}(m, \mathbb{R})$, with $m = k + 2n$, defined by

$$D = \text{ad} \rho \quad \text{so that} \quad D \begin{pmatrix} A & B \\ B' & C \end{pmatrix} = \begin{pmatrix} 0 & -BJ_{2n} \\ J_{2n}B' & [J_{2n}, C] \end{pmatrix}. $$

Clearly $\rho$ is semisimple with eigenvalues $0, i$ and $-i$, so that $D$ is a semisimple derivation with eigenvalues $0, i, 2i$ and $-2i$ and we can apply the results above for any subalgebra $g$ of $\text{gl}(m, \mathbb{R})$ which is stable under $D$, and so that $g^C$ intersects the $2i$ eigenspace of $D$. Observe that

$$\sigma = \exp \frac{1}{2} \pi D = \exp \frac{1}{2} \pi \text{ad} \rho = \text{Ad} \exp \left( \frac{1}{2} \pi \rho \right) = \text{Ad} \begin{pmatrix} \text{Id}_k & 0 \\ 0 & J_{2n} \end{pmatrix}, \quad (2.2)$$

and we shall denote by $R$ the matrix $R = \begin{pmatrix} \text{Id}_k & 0 \\ 0 & J_{2n} \end{pmatrix}$ so that $\sigma = \text{Ad} R$. We have

$$\sigma \begin{pmatrix} A & B \\ B' & C \end{pmatrix} = \begin{pmatrix} A & -BJ_{2n} \\ J_{2n}B' & -J_{2n}CJ_{2n} \end{pmatrix} \quad \text{so that} \quad$$
\[ g^\sigma = \left\{ \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \in g \mid [C, J_{2n}] = 0 \right\} \]

\[ p = \left\{ \begin{pmatrix} 0 & B \\ B' & 0 \end{pmatrix} \in g \right\} \]

\[ g_{\sigma_1} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \in g \mid CJ_{2n} + J_{2n}C = 0 \right\} \]

\[ J^\pm \begin{pmatrix} 0 & B \\ B' & 0 \end{pmatrix} = \begin{pmatrix} 0 & -BJ_{2n} \\ J_{2n}B' & 0 \end{pmatrix} \]

The last equality following from \( \tau = \exp(\frac{1}{4}\pi \text{ad}(\rho)) = \text{Ad}(\exp(\frac{1}{4}\pi \rho)) \) so that

\[ \tau \begin{pmatrix} A & B' \\ B' & C \end{pmatrix} = \text{Ad} \left( \begin{pmatrix} \text{Id} & 0 \\ \frac{1}{\sqrt{2}}(\text{Id} + J_{2n}) \end{pmatrix} \right) \begin{pmatrix} A & B' \\ B' & C \end{pmatrix} \]

\[ = \begin{pmatrix} A & \frac{1}{\sqrt{2}}(B' + J_{2n}B') \\ \frac{1}{\sqrt{2}}(B' + J_{2n}B') & \frac{1}{2}(C + J_{2n}C - CJ_{2n} - J_{2n}CJ_{2n}) \end{pmatrix} \] (2.3)

and thus \( J^\pm \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} = \pm \tau \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \pm J_{2n}C \end{pmatrix} \) when \( CJ_{2n} + J_{2n}C = 0 \).

Let \( G \) be the connected Lie subgroup of \( \text{Gl}(m, \mathbb{R}) \) with Lie algebra \( g \); then

\[ \tilde{\sigma} : G \to G : g \mapsto RgR^{-1} \]

is an automorphism of \( G \) lifting \( \sigma \) and \((G, \tilde{\sigma}, G^\sigma_0)\) is a 4-symmetric triple.

On the corresponding homogeneous 4-symmetric space \( G / G^\sigma_0 \), following Proposition 2.1, \( J^+ \) defines an invariant complex structure \( j^+ \) and \( J^- \) an invariant almost complex structure \( j^- \) for which the image of the Nijenhuis tensor is the invariant distribution whose value at the base point is \( \text{Im} N_{j^-} = [p, g^\sigma_{-1}] + [p, p] \cap g^\sigma_{-1} \) and \( j^- \) is maximally non integrable iff \( [p, g^\sigma_{-1}] = p \) and \([p, p] \cap g^\sigma_{-1} = g^\sigma_{-1} \).

**Remark 2.3.** Since \([ \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}, \begin{pmatrix} 0 & B' \\ B' & 0 \end{pmatrix} \] = \begin{pmatrix} 0 & -BC \\ CB & 0 \end{pmatrix} \), the equality \([p, g^\sigma_{-1}] = p \) will automatically hold if there exist \( k \geq 1 \) elements \( \begin{pmatrix} 0 & 0 \\ 0 & \tilde{C}_i \end{pmatrix} \) in \( g^\sigma_{-1} \) such that

\[ \sum_{i=1}^{k} \tilde{C}_i^2 = \text{Id}_{2n} \]

The stabilizer of \( \rho \) under the action by conjugation of \( G \) in \( \text{gl}(m, \mathbb{R}) \) is given by

\[ \left\{ g \in G \mid g\rho g^{-1} = \rho \right\} = \left\{ g \in G \mid g\rho = \rho g \right\} \]

\[ = \left\{ \begin{pmatrix} E & 0 \\ 0 & D \end{pmatrix} \in G \mid DJ_{2n} = J_{2n}D \right\} \]

\[ = \{ g \in G \mid gR = Rg \} = \{ g \in G \mid \tilde{\sigma}(g) = g \} , \]
so coincides with $G^\sigma$. Since $J^\pm$ coincides with half the bracket with $\rho$ on $g^\sigma_{\sigma-1}$, it commutes with the adjoint action of any element in $G^\sigma$. Hence, on the homogeneous 4-symmetric space $G/G^\sigma$, corresponding to the 4-symmetric triple $(G, \tilde{\sigma}, G^\sigma)$, $J^+$ defines also an invariant complex structures $j^+$ and $J^-$ also an invariant almost complex structure $j^-$ for which the image of the Nijenhuis tensor is the invariant distribution whose value at the base point is given as above.

2.1.2 The symplectic structure

Since the stabilizer of $\rho$ under the action by conjugation of $G$ in $\mathfrak{gl}(m, \mathbb{R})$ coincides with $G^\sigma$, the homogeneous space $G/G^\sigma$ is diffeomorphic to the orbit of $\rho$ under the action of $G$ in $\mathfrak{gl}(m, \mathbb{R})$.

Let us define the $G$-invariant 2-form $\omega$ whose value at the base point is given by

$$\tilde{\Omega}(X, Y) := \text{Tr}(\rho[X, Y]) \quad \forall X, Y \in g^\sigma_{\sigma-1} + p.$$  \tag{2.4}$$

Observe that $\text{Tr}(\rho[X, Y]) = 0$ for any $X \in g^\sigma$ and any $Y \in g$. The 2-form is invariant by the symmetries since

$$\tilde{\Omega}(\sigma X, \sigma Y) = \text{Tr}(\text{Ad} R([X, Y])) = \text{Tr}(\text{Ad} R(\rho[X, Y])) = \text{Tr}(\rho[X, Y]) = \tilde{\Omega}(X, Y).$$

The 2-form $\omega$ is closed since, for all $X, Y, Z \in g^\sigma_{\sigma-1} + p$, denoting by $pr: g \to g^\sigma_{\sigma-1} + p$ the projection parallel to $g^\sigma_{\sigma-1}$, we have:

$$\bigoplus_{XYZ} \tilde{\Omega}(\rho[X, Y]_{pr}, Z) = \bigoplus_{XYZ} \text{Tr}(\rho[[X, Y]_{pr}, Z]) = \bigoplus_{XYZ} \text{Tr}(\rho[[X, Y], Z]) = 0.$$

The 2-form $\omega$ is an invariant symplectic structure on the 4-symmetric space defined by $(G, \tilde{\sigma}, G^\sigma)$ and on its cover defined by $(G, \tilde{\sigma}, G^\sigma_0)$, if and only if it is non degenerate. This will be true if and only if

$$\tilde{\Omega}' : p \times p \to \mathbb{R} \quad \text{with} \quad \tilde{\Omega}'\left(\begin{pmatrix} 0 & B_1 \\ B_1' & 0 \end{pmatrix}, \begin{pmatrix} 0 & B_2 \\ B_2' & 0 \end{pmatrix}\right) = \text{Tr}(J_{2n}(B_1'B_2 - B_2'B_1))$$

and

$$\tilde{\Omega}'' : g^\sigma_{\sigma-1} \times g^\sigma_{\sigma-1} \to \mathbb{R} \quad \text{with} \quad \tilde{\Omega}''\left(\begin{pmatrix} 0 & 0 \\ C_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & C_2 \end{pmatrix}\right) = 2 \text{Tr}(J_{2n}C_1C_2)$$

are non degenerate, i.e. iff

$$g' : p \times p \to \mathbb{R} : X, Y \mapsto \text{Tr}(XY) \quad \text{and} \quad \beta'' : g^\sigma_{\sigma-1} \times g^\sigma_{\sigma-1} \to \mathbb{R} : X, Y \mapsto \text{Tr}(XY)$$
are non degenerate. This we shall now assume.

If $\rho$ belongs to the Lie algebra $\mathfrak{g}$ and if the map $\beta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} : X, Y \mapsto \text{Tr}(XY)$ is non degenerate (which is the case if $\mathfrak{g}$ is simple), then $G/G^\rho$, identifies with the coadjoint orbit of the element $\rho^* : \mathfrak{g} \to \mathbb{R} : X \mapsto \rho^X$ and the 2-form $\omega$ is automatically non degenerate because it is the Kirillov-Kostant-Souriau symplectic 2-form on this orbit.

In all cases, the natural almost complex structures $j^\pm$ are compatible with $\omega$ since

$$\tilde{\Omega}((\begin{array}{cc} 0 & B_1 \\ B'_1 & C_1 \end{array}), (\begin{array}{cc} 0 & B_2 \\ B'_2 & C_2 \end{array})) = \text{Tr} (J_{2n}(2C_1C_2 + B'_1B_2 - B'_2B_1))$$

$$= \tilde{\Omega}((\begin{array}{cc} 0 & -B_1J_{2n} \\ J_{2n}B'_1 & \pm J_{2n}C_1 \end{array}), (\begin{array}{cc} 0 & -B_2J_{2n} \\ J_{2n}B'_2 & \pm J_{2n}C_2 \end{array})).$$

In general, they are neither positive nor negative :

$$\tilde{\Omega}((\begin{array}{cc} 0 & B \\ B' & C \end{array}), J^\pm (\begin{array}{cc} 0 & B' \\ B' & C \end{array})) = \text{Tr} (\begin{array}{ccc} 0 & 0 & 0 \\ 0 & J_{2n} & 0 \\ 0 & 0 & J_{2n} \end{array}) \left[ (\begin{array}{cc} 0 & B \\ B' & C \end{array}), (\begin{array}{cc} 0 & -BJ_{2n} \\ J_{2n}B' & \pm J_{2n}C \end{array}) \right]$$

$$= \text{Tr} (J_{2n}(-B'BJ_{2n} \pm C J_{2n}C - J_{2n}B'B \mp J_{2n}C^2))$$

$$= 2 \text{Tr} B'B \pm 2 \text{Tr} C^2,$$

and $\pm 2 \text{Tr} C^2 = \pm 4(\text{Tr}(c^2) + \text{Tr}(d^2))$ for $C = \begin{pmatrix} c & d \\ d & -c \end{pmatrix}$.

### 2.1.3 The Levi Civita connection and the Ricci Hermitian property

We consider as above the homogeneous space $G/G^\rho$ or $G/G_\rho^\rho$ and we assume that the 2-form $\tilde{\Omega}$ given in equation [2.4] is non degenerate. The pseudo-Riemannian metrics associated to $j^\pm$ will be denoted by $g^\pm$; they are the $G$-invariant metrics whose value at the base point are given by

$$G^\pm(X,Y) := \tilde{\Omega}(X, J^\pm Y) := \text{Tr}(\rho[X, J^\pm Y]) \quad \forall X, Y \in \mathfrak{g}_{-1} + \mathfrak{p},$$

thus

$$G^\pm((\begin{array}{cc} 0 & B_1 \\ B'_1 & C_1 \end{array}), (\begin{array}{cc} 0 & B_2 \\ B'_2 & C_2 \end{array})) = \text{Tr} (\begin{array}{ccc} 0 & 0 & 0 \\ 0 & J_{2n} & 0 \\ 0 & 0 & J_{2n} \end{array}) \left[ (\begin{array}{cc} 0 & B_1 \\ B'_1 & C_1 \end{array}), (\begin{array}{cc} 0 & -B_2J_{2n} \\ J_{2n}B'_2 & \pm J_{2n}C_2 \end{array}) \right]$$

$$= \text{Tr}(B'_1B_2 + B'_2B_1) \pm 2 \text{Tr} C_1C_2.$$ 

The corresponding Levi Civita connection are denoted by $\nabla^{\pm}$. Invariance implies, $(\mathcal{L}_A \cdot g^\pm)(B^*, C^*) = A^\ast g^\pm(B^*, C^*) - g^\pm([A^\ast, B^*], C^*) - g^\pm(B^*, [A^\ast, C^*]) = 0$, and

$$2g^\pm(\nabla^{\pm}_A B^*, C^*) = g^\pm([A^\ast, B^*], C^*) + g^\pm([A^\ast, C^*], B^*) + g^\pm([B^*, C^*], A^*),$$

17
for any \( A, B, C \in \mathfrak{g} \), where \( A^* \) denotes the fundamental vector field associated to \( A \in \mathfrak{g} \), i.e. \( A_p^* := \frac{d}{dt} \exp(-tA)p \big|_{t=0} \) and, at the base point \( p_0 = \pi(e) \), \( A^*_{p_0} = -\pi_* (A) \). In our case, it gives, at the base point, for all \( X \)'s in \( \mathfrak{g}^*_{-1} \) and all \( Y \)'s in \( \mathfrak{p} = \mathfrak{g}^*_{-2} \):

\[
\nabla^+_X A^* Y^* (p_0) = \nabla^+_X A^* Y^* (p_0) = \pm [X, Y]_{p_0}^\ast \quad \nabla^+_Y A^* Y^* (p_0) = \frac{1}{2} [Y, Y]_{p_0}^\ast
\]

and \( \nabla^g \) being torsion free, \( \nabla^g_{Y^*} X^* (p_0) = 0 \) whereas \( \nabla^g_{Y^*} Y^* (p_0) = -2[X, Y]_{p_0}^\ast \).

Since \( j^+ \) is integrable, we know that \( \nabla g^* j^+ = 0 \) hence \( j^+ \) commutes with the curvature \( R^g(X, Y) = \nabla^g_X \circ \nabla^g_Y - \nabla^g_Y \circ \nabla^g_X - \nabla^g_{[X, Y]} \).

Since \( g^+(R^g(X, Y) Z, T) = g^+(R^g(Z, T) X, Y) \), this implies that

\[
R^g(j^+ X, j^+ Y) = R^g(X, Y).
\]

The Ricci tensor \( Ric^g(X, Z) = \text{Tr}[Y \to R^g(X, Y) Z] \) is thus hermitian for \( j^+ \):

\[
Ric^g_j(j^+ X, j^+ Z) = Ric^g(X, Z) \quad \forall X, Z.
\]

A direct computation at the base point shows that we have a similar result for \( j^- \).

One uses the fact that the curvature of a torsion free connection in a homogeneous situation can be computed from

\[
R^g(A^*, B^*) C^* = \nabla^g_{C^* B^*} A^* - \nabla^g_{C^* A^*} B^* + \nabla^g_{[A, B]^*} C^* - [[A, B], C]^*.
\]

At the base point, for all \( X \)'s in \( \mathfrak{g}^*_{-1} \) and all \( Y \)'s in \( \mathfrak{p} = \mathfrak{g}^*_{-2} \), it gives:

\[
\begin{align*}
R^{g^+}_{p_0}(X_1, X_2) X_3^* &= -[[X_1, X_2], X_3]_{p_0}^\ast \\
R^{g^+}_{p_0}(X_1, X_2) Y_3^* &= -[[X_1, X_2], Y_3]_{p_0}^\ast \\
R^{g^-}_{p_0}(X_1, X_2) Y_3^* &= 3[[X_1, X_2], Y]_{p_0}^\ast \\
R^{g^+}_{p_0}(X_1, Y_3^*) X_2^* &= -[[X_1, Y_3], X_2]_{p_0}^\ast \\
R^{g^-}_{p_0}(X_1, Y_3^*) X_2^* &= (1[[X_1, Y_3], X_2] + 2[[X_1, X_2] Y_3])_{p_0}^\ast \\
R^{g^+}_{p_0}(X_1, Y_3^*) Y_2^* &= -\frac{1}{2}[[X, Y_1], Y_2]_{p_0}^\ast \\
R^{g^-}_{p_0}(X_1, Y_3^*) Y_2^* &= (-7[[X_1, Y_2], Y_3] + [\sigma][Y_3, Y_1], Y_2]_{p_0}^\ast \\
R^{g^+}_{p_0}(Y_1^*, Y_2^*) X^* &= -\frac{1}{2}[[Y_1, Y_2], X]_{p_0}^\ast \\
R^{g^-}_{p_0}(Y_1^*, Y_2^*) X^* &= \frac{3}{2}[[Y_1, Y_2], X]_{p_0}^\ast \\
R^{g^+}_{p_0}(Y_1^*, Y_2^*) Y_3^* &= \frac{1}{4}([\sigma][Y_1, Y_2], Y_3] + [\sigma][Y_3, Y_1], Y_2]_{p_0}^\ast \\
R^{g^-}_{p_0}(Y_1^*, Y_2^*) Y_3^* &= \frac{1}{4}(-7[[Y_1, Y_2], Y_3] + [\sigma][Y_3, Y_1], Y_2]_{p_0}^\ast.
\end{align*}
\]
The Ricci tensor is invariant and its value at the base point is given by

\[ \text{Ric}^g_{p_0}(X^1, X^2) = \text{Tr}_{g_{V_1}} \text{ad} X_2 \circ \text{ad} X_1 + \text{Tr}_{|p} \text{ad} X_2 \circ \text{ad} X_1 \]

\[ \text{Ric}^g_{p_0}(X^*, Y^*) = 0 \]

\[ \text{Ric}^g_{p_0}(Y^1, Y^2) = \frac{1}{2} \text{Tr}_{g_{V_1}} \frac{1}{2} (\text{Id} - \sigma) \circ \text{ad} Y_2 \circ \text{ad} Y_1 + \frac{1}{4} \text{Tr}_{|p} (\text{ad} Y_2 \circ \text{ad} Y_1 + \text{ad}(\sigma Y_2, Y_1)) + \text{ad} Y_1 \circ \text{ad} Y_2 + 2 \text{ad} Y_2 \circ \text{ad} Y_1) \]

\[ \text{Ric}^g_{p_0}(X^1, X^2) = \text{Tr}_{g_{V_1}} \text{ad} X_2 \circ \text{ad} X_1 + 2 \text{Tr}_{|p} \text{ad}[X_1, X_2] \]

\[ \text{Ric}^g_{p_0}(X^*, Y^*) = 0 \]

\[ \text{Ric}^g_{p_0}(Y^1, Y^2) = \text{Tr}_{g_{V_1}} \frac{1}{2} (\text{Id} - \sigma) (\frac{1}{2} \text{ad} Y_2 \circ \text{ad} Y_1 - \text{ad} Y_1 \circ \text{ad} Y_2) + \frac{1}{4} \text{Tr}_{|p} (7 \text{ad} Y_2 \circ \text{ad} Y_1 - \text{ad}(\sigma Y_2, Y_1)) - \text{ad} Y_1 \circ \text{ad} Y_2 - 2 \text{ad} Y_2 \circ \text{ad} Y_1). \]

The Ricci tensor \( \text{Ric}^g_{p_0} \) is \( j^+ \) Hermitian and the tensor \( \text{Ric}^g_{p_0} \) is \( j^- \) Hermitian because each term is invariant under \( J^\pm \) which coincide with \( \sigma \) on \( p \) and with \( \pm \tau = \pm \exp \frac{i}{4} D \) on \( g_{e1}^* \). Indeed, since \( \text{ad} \sigma Y = \sigma \circ \text{ad} Y \circ \sigma^{-1} \), on any subspace \( V \) stable by \( \sigma \) (in particular on \( p \) and on \( g_{e1}^* \)), one has

\[ \text{Tr}_{|V} \text{ad} \sigma Y_2 \circ \text{ad} \sigma Y_1 = \text{Tr}_{|V} \sigma \circ \text{ad} Y_2 \circ \text{ad} Y_1 \circ \sigma^{-1} = \text{Tr}_{|V} \text{ad} Y_2 \circ \text{ad} Y_1 \]

\[ \text{Tr}_{|V} \text{ad} Y_2 \circ \sigma \circ \text{ad} \sigma Y_1 = \text{Tr}_{|V} \text{ad} Y_2 \circ \text{ad} Y_1 \circ \sigma^{-1} = \text{Tr}_{|V} \text{ad} Y_2 \circ \sigma \circ \text{ad} Y_1 \]

so that \( \text{Ric}^g_{p_0}(j^+ Y^1, j^+ Y^2) = \text{Ric}^g_{p_0}(Y^1, Y^2) \), and, similarly, since \( \text{ad} \tau X = \tau \circ \text{ad} X \circ \tau^{-1} \), on any subspace \( V \) stable by \( \tau \) (in particular on \( p \) and on \( g_{e1}^* \)), one has

\[ \text{Tr}_{|V} \text{ad} \tau X_2 \circ \text{ad} \tau X_1 = \text{Tr}_{|V} \tau \circ \text{ad} X_2 \circ \text{ad} X_1 \circ \tau^{-1} = \text{Tr}_{|V} \text{ad} X_2 \circ \text{ad} X_1 \]

so that \( \text{Ric}^g_{p_0}(j^+ X^1, j^+ X^2) = \text{Ric}^g_{p_0}(X^1, X^2) \).

For elements \( X_i \)'s in \( g_{e1}^* \) of the form \( X_i = (0 \ 0 \ C_i) \) and \( Y_i \)'s in \( p \) of the form
Y_i = \left( \begin{array}{cc} 0 & B_i \\ B'_i & 0 \end{array} \right),
one has

\begin{align*}
Ric^2_{\eta_p} (X_1^2, X_2^2) &= \text{Tr} \left|_{\eta_p} \right. \text{ad} X_2 \circ \text{ad} X_1 + \text{Tr} \left|_{\eta_p} \right. \text{ad} X_2 \circ \text{ad} X_1 \\
&= \text{Tr} \left( C \rightarrow (C_2 C_1 C + C C_2 C_2 - C_1 C C_2 - C_2 C C_2) \right) + \text{Tr} \left( (B, B') \rightarrow (B C_1 C_2, C_2 C_1 B') \right) \\
Ric^2_{\eta_p} (X_1^2, Y_1^2) &= 0 \quad \text{and} \quad \eta^2_p (X_1^2, Y_1^2) = 0; \\
Ric^2_{\eta_p} (Y_1^2, Y_2^2) &= \frac{1}{2} \text{Tr} \left|_{\eta_p} \frac{1}{2} (\text{Id} - \sigma) \right. \text{ad} Y_2 \circ \text{ad} Y_1 + \frac{1}{4} \text{Tr} \left|_{\eta_p} \right. \text{ad} Y_2 \circ \text{ad} (\sigma Y_2, Y_1)) \\
&= \frac{1}{2} \text{Tr} \left|_{\eta_p} \frac{1}{4} ((B'_2 B_1 - J_2n B'_2 B_1 J_2n) C + C (B'_1 B_2 - J_2n B'_1 B_2 J_2n)) \\
&\quad + \frac{1}{4} \text{Tr} \left|_{\eta_p} \left( (B, B') \rightarrow \left( (2B_2 B'_1 - B_1 B'_2) B + B (3B'_2 B_2 + B'_2 B_1 - J B'_1 B_2 J + 3B_2 B'_1 B_2 J) \\
- (B_1 B'_2 + 2B_2 J B'_1 B_2 - 2B_2 B'_1 B_2 + J B'_2 B_1 - 2B_2 B'_1 B_2 J) \right) \\
B' (2B_1 B'_2 - 2B'_1 B_2) + (3B'_2 B_2 + B'_2 B_1 - J B'_1 B_2 J - 3B_2 B'_1 B_2 J) B' \\
- J B' (B_2 B'_1 + 2B_1 B'_2 - 2B'_1 B_2 - 3B'_1 B B'_2 + J B'_2 B B'_1 B'_2 + J B_2 B B'_1 B'_2) \right) \right)
\end{align*}

where the traces are computed on elements $C \in Mat(2n \times 2n)$ of the form $C = \left( \begin{array}{cc} c & c' \\ c & c' \end{array} \right)$ such that $(0 \ 0 \ 0) \in \mathfrak{g}_{-1}^\eta$, and on elements $B \in Mat(k \times 2n), B' \in Mat(2n \times k)$ such that $(0 \ B' \ 0) \in \mathfrak{p}$. Observe that, for any $D_i = \left( \begin{array}{cc} d_i & d'_i \\ -d'_i & d_i \end{array} \right),$

\begin{align*}
\text{Tr} \left( C \rightarrow (D_1 C + CD_2) \right) &= \text{Tr} \left( (c, c') \rightarrow (d_1 c + d'_1 c' + c d_2 - c' d'_2, d_1 c' - d'_1 c + c d'_2 + c' d_2) \right) \\
\text{Tr} \left( C \rightarrow C_1 C_2 \right) &= \text{Tr} \left( (c, c') \rightarrow (c_1 c c_2 + c_1 c' c_2 + c_1 c' c_2 - c_1 c' c_2 + c_1 c' c_2 + c_1 c' c_2) \right)
\end{align*}

We shall use those identities to show that $g^+$ is Einstein in the examples given in section 3.

### 2.1.4 The Chern connection, the Chern Ricci form and the property of being special

The almost complex structure $j^+$ is integrable, so $\nabla^{g^+} j^+ = 0$, and the Chern connection $\nabla^{C^+}$ in that case coincides with the Levi Civita connection $\nabla^{g^+}$. The Chern Ricci form $(2.5)$ is thus given by

$$
\text{ChernRicci}^{2^+} (X, Y) := \text{Tr} \left( j^+ R^{g^+} (X, Y) \right) = \text{Ric}^{g^+} (X, j^+ Y) - \text{Ric}^{g^+} (Y, j^+ X).
$$

Since we have the Ricci Hermitian property, this yields

$$
\text{ChernRicci}^{2^+} (X, Y) := 2 \text{Ric}^{g^+} (X, j^+ Y).
$$
And $j^+$ is special (i.e. ChernRicci$j^+$ is proportional to $\omega$) if and only if $g^+$ is Einstein, i.e. $Ricg^+$ is proportional to $g^+$.

For the almost complex structure $j^-$, the Chern connection $\nabla^{C-}$ is given by
\[
\nabla_X^{C-}Y = \nabla_X Y - \frac{1}{2}j(\nabla^{g-}j)_XY = \frac{1}{2}\nabla_X Y - \frac{1}{2}j\nabla_X (jY) \quad \forall X, Y \in \mathfrak{X}(M);
\]
it is invariant and given, for all $A, B \in \mathfrak{g}$ by
\[
\nabla_{A^*} B^* = \frac{1}{2}\nabla_A B^* - \frac{1}{2}j^-(\nabla^{g-}_{j-B^*}A^*+[A^*, j^-B^*]) = \frac{1}{2}\nabla_A B^* + \frac{1}{2}[A^*, B^*] - \frac{1}{2}j^-\nabla^{g-}_{j-B^*}A^*.
\]
At the base point $p_0$, for all $X_i$'s in $\mathfrak{g}_{-1}$ and all $Y_i$'s in $\mathfrak{p} = \mathfrak{g}_{-1}^\perp$, the Chern connection is given by
\[
\begin{align*}
\nabla^{C-}_{X_i^*} X_k^*(p_0) &= 0, \\
\nabla^{C-}_{X_i^*} Y^*_j(p_0) &= \left(-\frac{1}{2}[X_i, Y_k] + \frac{1}{2}[X_i, Y_k] + \sigma[X_i, \sigma Y_k]\right)^*_{p_0} = [\sigma X_i, -Y_k]^*_{p_0} = [X_i, Y_k]^*_{p_0} \\
\nabla^{C-}_{Y_k^*} X_i^*(p_0) &= \left(\frac{3}{2}[Y_k, X_i] - \frac{1}{2}\sigma[\tau X_i, Y_k]\right)^*_{p_0} = [Y_k, X_i]^*_{p_0}, \\
\nabla^{C-}_{Y_i^*} Y^*_k(p_0) &= \left(\frac{3}{4}[Y_i, Y_k]^*_{p_0} + \frac{1}{4}\frac{3}{2}\sigma[\sigma Y_k, Y_i]\right)^*_{p_0} = [Y_i, Y_k]^*_{p_0}.
\end{align*}
\]
because, $\sigma[\tau X_i, Y_k] = -[X_i, Y_k]$ and $\tau(\text{Id} - \sigma)[\sigma Y_k, Y_i] = (\text{Id} - \sigma)[Y_k, Y_i]$.

Hence we have, for all $A, B \in \mathfrak{g}_{-1}^\perp + \mathfrak{p}$
\[
\nabla^{C-}_{A^*} B^*(p_0) = [A, B]^*_{p_0} \quad \text{and} \quad T_{p_0}^{\nabla^{C-}}(A^*, B^*) = [A, B]^*_{p_0} = \frac{1}{4}N_{p_0}^{j^-}(A^*, B^*). \quad (2.10)
\]
The curvature of a connection with torsion in a homogeneous situation can be computed from
\[
\begin{align*}
R^{C-}(A^*, B^*)C^* &= \nabla_{\nabla^{C-}_{B^*}C^*} A^* - \nabla^{C-}_{A^*} B^* + \nabla^{C-}_{[A, B]C^*} C^* + \nabla^{C-}_{[A, C]B^*} B^* - \nabla^{C-}_{[B, C]} A^* \\
&- [[A, B], C]^* + T_{p_0}^{\nabla^{C-}}(A^*, [B, C]C^*) - T_{p_0}^{\nabla^{C-}}(B^*, \nabla^{C-}_{A^*} C^*) \\
&+ T_{p_0}^{\nabla^{C-}}(B^*, [A, C]^*) - T_{p_0}^{\nabla^{C-}}(A^*, [B, C]^*).
\end{align*}
\]
At the base point $p_0$, for all $A, B, C \in \mathfrak{g}_{-1}^\perp + \mathfrak{p}$, we have
\[
\begin{align*}
R_{p_0}^{C-}(A^*, B^*)C^* &= \nabla^{C-}_{[B, C]^*} A^*(p_0) - \nabla^{C-}_{[A, C]^*} B^*(p_0) + \nabla^{C-}_{[A, B]^*} C^*(p_0) + \nabla^{C-}_{[A, C]^*} B^*(p_0) \\
&- \nabla^{C-}_{[B, C]^*} A^*(p_0) - [[A, B], C]^*_{p_0} + T_{p_0}^{\nabla^{C-}}(A^*, [B, C]^*) \\
&- T_{p_0}^{\nabla^{C-}}(B^*, [A, C]^*) + T_{p_0}^{\nabla^{C-}}(B^*, [A, C]^*) - T_{p_0}^{\nabla^{C-}}(A^*, [B, C]^*) \\
&= \nabla^{C-}_{[A, B]^*} C^*(p_0) - [[A, B], C]^*_{p_0} \\
&= ([pr[A, B], C] - [[A, B], C])^*_{p_0}
\end{align*}
\]
21
where \( pr : \mathfrak{g} \to \mathfrak{g}^{\tau}_{+1} + \mathfrak{p} \) is the projection parallel to \( \mathfrak{g}^{\tau} \). Hence, for all \( X_i \)'s in \( \mathfrak{g}^{\tau}_{+1} \) and all \( Y_i \)'s in \( \mathfrak{p} = \mathfrak{g}^{\tau}_{-1} \), since \( pr[X_i, X_j] = 0, pr[X_i, Y_j] = [X_i, Y_j] \) and \( pr[Y_i, Y_j] = \frac{1}{2}(Id - \sigma)[Y_i, Y_j] \), the Chern curvature for \( j^- \) is given by

\[
R^{C-}_{p_0}(X_i^*, X_j^*)X_k^* = -[[[X_i, X_j], X_k]]^*_{p_0} \\
R^{C-}_{p_0}(X_i^*, X_j^*)Y_k^* = -[[[X_i, X_j], Y_k]]^*_{p_0} \\
R^{C-}_{p_0}(X_i^*, Y_j^*)X_k^* = 0 \\
R^{C-}_{p_0}(X_i^*, Y_j^*)Y_k^* = 0 \\
R^{C-}_{p_0}(Y_i^*, X_j^*)X_k^* = -\frac{1}{2}([[Id + \sigma][Y_i, Y_j], X_k]]^*_{p_0} \\
R^{C-}_{p_0}(Y_i^*, Y_j^*)Y_k^* = -\frac{1}{2}([[Id + \sigma][Y_i, Y_j], Y_k]]^*_{p_0}.
\]

Hence the Chern Ricci form, \( ChernRicci^\tau(X, Y) := Tr j^- R^{C-}(X, Y) \), for elements \( X_i \)'s in \( \mathfrak{g}^{\tau}_{-1} \) of the form \( X_i = \begin{pmatrix} 0 & 0 \\ 0 & C_i \end{pmatrix} \) and \( Y_i \)'s in \( \mathfrak{p} \) of the form \( Y_i = \begin{pmatrix} 0 & B_i \\ B_i^t & 0 \end{pmatrix} \), reads

\[
ChernRicci^\tau_{p_0}(X_i^*, X_j^*) = Tr [\mathfrak{g}^{\tau}_{-1}] \sigma \circ ad[X_i, X_j] - Tr [\mathfrak{g}^{\tau}_{-1}] \sigma \circ ad[X_i, X_j] \\
= Tr (C \to D(X_i, X_j)C + CD(X_i, X_j)) - Tr ((B, B') \to (BD(X_i, X_j), D(X_i, X_j)B'))
\]

with \( D(X_i, X_j) = J_{2n}[C_i, C_j] \) \( (2.11) \)

\[
ChernRicci^\tau_{p_0}(X_i^*, Y_j^*) = 0
\]

\[
ChernRicci^\tau_{p_0}(Y_i^*, Y_j^*) = Tr [\mathfrak{g}^{\tau}_{-1}] \sigma \circ ad\left(\frac{Id + \sigma}{2}[Y_i, Y_j]\right) - Tr [\mathfrak{g}^{\tau}_{-1}] \sigma \circ ad\left(\frac{Id + \sigma}{2}[Y_i, Y_j]\right) \\
= Tr (C \to D(Y_i, Y_j)C + CD(Y_i, Y_j)) - Tr ((B, B') \to (BD(Y_i, Y_j), D(Y_i, Y_j)B'))
\]

with \( D(Y_i, Y_j) = \frac{1}{2}(J_{2n}(B_i^t B_j - B_j^t B_i) + (B_i^t B_j - B_j^t B_i)J_{2n}) \) \( (2.12) \)

\[
ChernRicci^\tau_{p_0}(Y_i^*, X_j^*) = Tr [\mathfrak{g}^{\tau}_{-1}] \sigma \circ ad\left(\frac{Id + \sigma}{2}[Y_i, X_j]\right) - Tr [\mathfrak{g}^{\tau}_{-1}] \sigma \circ ad\left(\frac{Id + \sigma}{2}[Y_i, X_j]\right) \\
= Tr (C \to D(Y_i, X_j)C + CD(Y_i, X_j)) - Tr ((B, B') \to (BD(Y_i, X_j), D(Y_i, X_j)B'))
\]

for \( C \) such that \( \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \in \mathfrak{g}^{\tau}_{-1} \) and \( (B, B') \) such that \( \begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix} \in \mathfrak{p} \). Remark that \( C \) is of the form \( \begin{pmatrix} \sigma C & 0 \\ 0 & \sigma \end{pmatrix} \) and \( D \) is of the form \( \begin{pmatrix} d & d' \\ -d & d \end{pmatrix} \).

If one can find a basis \( \{e_k\} \) of the 2n × 2n matrices \( C \) such that \( \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \in \mathfrak{g}^{\tau}_{-1} \) and of the pairs of \( k \times 2n \) and \( 2n \times k \) matrices \( (B, B') \) such that \( \begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix} \in \mathfrak{p} \), for which each element is a linear combination of elementary matrices \( E_{ij} \) (the matrix whose only non vanishing entry is a 1 at the intersection of the \( i \)th row and the \( j \)th column) with different indices \( i \) of lines and different indices \( j \) of columns, then, since the only coefficient of \( E_{ij} \) in \( D\epsilon_k \) with \( \epsilon_k = E_{ij} + \ldots \) is \( D_{ij} \) and in \( \epsilon_k D \) is \( D_{ij} \), we obtain that \( Tr (C \to DC + CD) \) and \( Tr ((B, B') \to (BD, DB')) \) are two multiples of \( Tr D \). Since

\[
Tr D(X_i, X_j) = Tr J_{2n}[C_i, C_j] = Tr \rho[X_i, X_j] = \tilde{\Omega}(X_i, X_j) \\
Tr D(Y_i, Y_j) = Tr J_{2n}(B_i^t B_j - B_j^t B_i) = Tr \rho[Y_i, Y_j] = \tilde{\Omega}(Y_i, Y_j),
\]

we conclude that \( ChernRicci^\tau \) is proportional to \( \omega \), hence \( j^- \) is special. This will be done explicitly in the examples given in the next section.
3 Examples

3.1 $G = GL(k + 2n, \mathbb{R})$ or $SL(k + 2n, \mathbb{R})$

For the group $G = GL(k + 2n, \mathbb{R})$ (resp. $G = SL(k + 2n, \mathbb{R})$), $\rho$ lies in its Lie algebra and the map $\beta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} : X, Y \mapsto Tr(XY)$ is non degenerate. In both cases, the 4-symmetric space $M$ associated to the triple $(G, \tilde{\sigma}, G^\mathfrak{g})$,

$$G/G^\mathfrak{g} = GL(k + 2n, \mathbb{R})/(Gl(k, \mathbb{R}) \times GL(n, \mathbb{C})) = SL(k + 2n, \mathbb{R})/S(GL(k, \mathbb{R}) \times GL(n, \mathbb{C})),$$

identifies with the coadjoint orbit of the element $\rho^\mathfrak{g} : \mathfrak{g} \to \mathbb{R} : X \mapsto Tr \rho X$. The 2-form $\omega$ is the Kirillov-Kostant-Souriau symplectic 2-form on this orbit.

With the pair $(\omega, j^+)$, this orbit is an invariant pseudo-Kähler 4-symmetric space $M$. With the pair $(\omega, j^-)$ it is an invariant almost-pseudo-Kähler 4-symmetric space, and $j^-$ is maximally non integrable, since $[p, p] \cap \mathfrak{g}^{\mathfrak{G}^-} = \mathfrak{g}^{\mathfrak{G}^-}$ and $[p, \mathfrak{g}^{\mathfrak{G}^-}] = \mathfrak{p}$.

These equalities follow from remarks 2.2 and 2.3, the fact that $\mathfrak{sl}(k + 2n, \mathbb{R})$ is simple, and the fact that $\tilde{C} = \begin{pmatrix} 0 & Id_n \\ 0 & 0 \end{pmatrix}$ satisfies $\tilde{C}^2 = Id_{2n}$ and $\begin{pmatrix} 0 & 0 \\ 0 & \tilde{C} \end{pmatrix}$ is in $\mathfrak{g}^{\mathfrak{G}^-}$.

Since $\mathfrak{g}^{\mathfrak{G}^-} = \{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \mid CJ_{2n} + J_{2n}C = 0 \}$, a basis of the corresponding $C'$ is given by $\{ E_{i+j} - E_{n+i+j}, E_{n+i+j} + E_{i+n+j}, 1 \leq i, j \leq n \}$ so that, for any $D = \begin{pmatrix} d & d' \\ -d & d' \end{pmatrix}$,

$$Tr(C \to CD) = \sum_{1 \leq i, j \leq n} (d_{ij} + d_{ji}) = 2n Tr d = n Tr D, \quad Tr(C \to DC) = \sum_{1 \leq i, j \leq n} (d_{ii} + d_{ii}) = n Tr D.$$

Since $p = \{ \begin{pmatrix} B \\ B' \end{pmatrix} \}$, a basis of the corresponding set of pairs $(B, B')$ is given by $\{(E_{r, s}, E_{s, r}) \leq \ell, \ell' \leq 2n, 1 \leq r, s \leq k \}$ and, for any $F_1, F_2 \in \text{Mat}(2n \times 2n), G_1, G_2 \in \text{Mat}(k \times k)$

$$Tr ((B, B') \to (G_1BF_1, F_2B'G_2)) = \sum_{1 \leq \ell \leq 2n, 1 \leq r \leq k} ((G_1)_{\ell r} (F_1)_{\ell r} + (F_2)_{\ell r} (G_2)_{r r}) = Tr G_1Tr F_1 + Tr F_2 Tr G_2.$$

The metric $g^+$ is Einstein : $\text{Ric}g^+ = (k + n) g^+$. Indeed, using the identities (2.5), (2.6) and (2.7), one has :

$$\text{Ric}_{p_0}^+(X_i^+, X_j^+) = 2(k + n) Tr C_2 C_1 = (k + n) g^+_{p_0}(X_i^+, X_j^+);$$

$$\text{Ric}_{p_0}^+(X_i^+, Y_i^+) = 0 \quad \text{and} \quad g^+_{p_0}(X_i^+, Y_i^+) = 0;$$

$$\text{Ric}_{p_0}^+(Y_i^+, Y_j^+) = \begin{pmatrix} n \\ 2 \end{pmatrix} Tr (B_2'B_1 + B_1'B_2) + \begin{pmatrix} 1 \\ 4 \end{pmatrix} (4k Tr (B_1'B_2 + B_2'B_1) + 2n Tr (B_2'B_1 + B_1'B_2)) = (k + n)(Tr B_1'B_2 + Tr B_2'B_1) = (k + n) g^+_{p_0}(Y_i^+, Y_j^+).$$

The almost complex structure $j^-$ is special : $\text{Chern Ricci}j^- = 2(n - k) \omega$. Indeed, using the identities (2.11), (2.12) and (2.13), one has :

$$\text{Chern Ricci}_{p_0}^j(X_i^+, X_j^+) = 2n Tr J_{2n}[C_i, C_j] - 2k Tr J_{2n}[C_i, C_j] = 2(n - k) \omega_{p_0}(X_i^+, X_j^+);$$

$$\text{Chern Ricci}_{p_0}^j(X_i^+, Y_i^+) = 0 \quad \text{and} \quad \omega_{p_0}(X_i^+, Y_i^+) = 0;$$

$$\text{Chern Ricci}_{p_0}^j(Y_i^+, Y_j^+) = (2n - 2k) Tr J_{2n}(B_1'B_j - B_j'B_1) = 2(n - k) \omega_{p_0}(Y_i^+, Y_j^+).$$
3.2 \( G = O(k+2n, \mathbb{R}), O(k, 2n, \mathbb{R}), SO(k+2n, \mathbb{R}) \) or \( SO_0(k, 2n, \mathbb{R}) \) with \( k, n \geq 1 \)

The orthogonal groups \( O(k+2n, \mathbb{R}) = \{ g \in Gl(k+2n, \mathbb{R}) \mid tr g g = 1 \} \), the pseudo-orthogonal groups \( O(k, 2n, \mathbb{R}) = \{ g \in Gl(k+2n, \mathbb{R}) \mid tr g (\begin{smallmatrix}1 & 0 \\ 0 & -1 \end{smallmatrix}) g = (\begin{smallmatrix}1 & 0 \\ 0 & -1 \end{smallmatrix}) \} \), and their connected components are \( \tilde{\sigma} \)-stable, since \( R \in O(k; 2n, \mathbb{R}) \cap O(k+2n, \mathbb{R}) \). Clearly, \( \rho \) belongs to the Lie algebras \( o(k+2n, \mathbb{R}) = \{ X \in gl(k+2n, \mathbb{R}) \mid tr X + X = 0 \} \) and \( o(k, 2n, \mathbb{R}) = \{ X \in gl(k+2n, \mathbb{R}) \mid tr X (\begin{smallmatrix}1 & 0 \\ 0 & -1 \end{smallmatrix}) + (\begin{smallmatrix}1 & 0 \\ 0 & -1 \end{smallmatrix}) X = 0 \} \). These algebras are simple for \( k + 2n \neq 4 \) and in each case the map \( \beta : g \times g \rightarrow \mathbb{R} \) is non degenerate. The subspace \( p = g^\sigma_{-1} = \{ (\begin{smallmatrix}0 & B \\ B & 0 \end{smallmatrix}) \in g \} \) does not vanish. Hence the 4-symmetric space \( G/G^{\sigma} \) identifies with the coadjoint orbit of \( \rho^\flat \) in the corresponding \( g^* \) and the 2-form \( \omega \) is symplectic. Thus

\[
\begin{align*}
O(k+2n, \mathbb{R})/O(k, \mathbb{R}) & \times U(n) & \text{SO}(k+2n, \mathbb{R})/\text{SO}(k, \mathbb{R}) \times U(n) \\
O(k, 2n, \mathbb{R})/O(k, \mathbb{R}) \times U(n) & \text{SO}(k, 2n, \mathbb{R})/\text{SO}(k, \mathbb{R}) \times U(n),
\end{align*}
\]

where \( U(n) = \{ A \in O(2n, \mathbb{R}) \mid AJ_{2n} = J_{2n}A \} \), endowed with \( \omega \) and \( j^+ \) are pseudo-Kähler 4-symmetric spaces. Endowed with \( \omega \) and \( j^- \), these are almost pseudo-Kähler 4-symmetric spaces, with the invariant natural almost complex structure \( j^- \) being maximally non integrable. Indeed, \( [p, p] \cap g^\sigma_{-1} = g^\sigma_{-1} \) and \( [p, g^\sigma_{-1}^\sigma] = p \), using remarks 2.2 and 2.3 and the fact that the \( 2n \times 2n \) matrices \( C_{ij} = E_{ij} - E_{ni+n-j} + E_{n+j+n-i} \), for \( 1 \leq i < j \leq n \), (where \( E_{ij} \) is the \( 2n \times 2n \) matrix whose only non vanishing entry is a 1 at the intersection of the \( i^{th} \) row and the \( j^{th} \) column), satisfy \( \sum_{ij} C_{ij}^2 = -(n-1) \text{Id}_{2n} \), and \( (\begin{smallmatrix}0 & 0 \\ 0 & c_{ij} \end{smallmatrix}) \in g^\sigma_{-1} \) since

\[
\begin{align*}
g^\sigma_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \mid C = \left( \begin{smallmatrix} \sigma & \sigma' \\ \sigma' & -\sigma \end{smallmatrix} \right), \ tr \sigma = -c & \quad tr \sigma' = -c' \right\}.
\end{align*}
\]

For \( g = o(k+2n, \mathbb{R}) \), then \( g^\sigma_{-1} + p = (\begin{smallmatrix}0 & B \\ B & 0 \end{smallmatrix}) \) with \( B \) a \( k \times 2n \)-matrix and \( C \) as in formula 3.1. For such an \( X \in g^\sigma_{-1} + p \),

\[
-\omega_{po}(\pi_+ X, j^\pm_{po} \pi_+ X) = -\tilde{\Omega}(X, J^\pm X) = 2 \text{Tr}(tr BB) \mp 2 \text{Tr}(C^2) = 2 \sum_{i=1}^{k} \sum_{j=1}^{2n} (B_{ij})^2 \mp 4 \text{Tr}(c^2 + c'^2)
\]

\[
= 2 \sum_{i=1}^{k} \sum_{j=1}^{2n} (B_{ij})^2 \mp 4 \sum_{i=1}^{n} \sum_{j=1}^{n} ((c'_{ij})^2 + (c_{ij})^2).
\]

24
Thus, the adjoint orbit of $\rho$ in $\mathfrak{o}(k+2n, \mathbb{R})$, $SO(k+2n, \mathbb{R})/SO(k, \mathbb{R}) \times U(n)$, endowed with $-\omega$ and the invariant complex structure $j^+$, is a Kähler manifold.

The case where $k = 1$ corresponds to the twistor space $SO(1+2n, \mathbb{R})/U(n)$ on the sphere $S^{2n} = SO(1+2n, \mathbb{R})/SO(2n, \mathbb{R})$.

For $\mathfrak{g} = \mathfrak{o}(k, 2n, \mathbb{R})$, then $\mathfrak{g}^o_{-1} + p = \{X = (0 \ 0 \ B)\}$ with $B$ a $k \times 2n$-matrix and $C$ as in formula 3.1. For such an $X \in \mathfrak{g}^o_{-1} + p$,

$$\omega_{p_0}(\pi_* X, j_{p_0}^\pm \pi_* X) = \tilde{\Omega}(X, J^\pm X) = 2 \text{Tr}(t^r B B) \pm 2 \text{Tr}(C^2)$$

$$= 2 \sum_{i=1}^k \sum_{j=1}^{2n} (B^i_j)^2 \pm 4 \sum_{i=1}^n \sum_{j=1}^n (c^i_j)^2 + (c'^i_j)^2).$$

Thus, on the adjoint orbit of $\rho$ in $\mathfrak{o}(k, 2n, \mathbb{R})$, $SO_0(k, 2n, \mathbb{R})/SO(k, \mathbb{R}) \times U(n)$, endowed with $\omega$, the invariant structure $j^-$, is positive admissible and maximally non integrable; we have a maximally non integrable almost Kähler manifold.

The case where $k = 1$ corresponds to the twistor space $SO_0(1, 2n, \mathbb{R})/U(n)$ on the hyperbolic space $SO_0(1, 2n, \mathbb{R})/SO(2n, \mathbb{R})$.

In all cases, a basis of the $C$'s corresponding to elements in $\mathfrak{g}^o_{-1}$ is given by

$$\{E_{i j} - E_{j i} - E_{n+i n+j} + E_{n+j n+i}, E_{n+i j} - E_{n+j i} + E_{i n+j} - E_{j n+i}, 1 \leq i < j \leq n\}$$

so that, for any $D = (d_{i j})$, and for any $C_1, C_2$ of the form $\left[\begin{array}{ll} c_i & c'_i \\ c'_i & c_i \end{array}\right]$

$$\text{Tr} \left( C \rightarrow ((D)C + C(t^r D)) \right) = \sum_{1 \leq i < j \leq n} (d_{i j} + d_{j i} + d_{i j} + d_{j i}) = 2(n-1) \text{Tr} d = (n-1) \text{Tr} D$$

$$\text{Tr} \left( C \rightarrow (C_1 C_2 + (t^r C_2)C(t^r C_1)) \right) = 0.$$
The metric $g^+$ is Einstein $Ric^g = \frac{1}{2}(k + n - 1)g^+$. Indeed, the identities (2.5), (2.6) and (2.7) become:

\[
Ric^g_{\rho_0}(X_1^*, X_2^*) = \frac{1}{2} \text{Tr}(C \rightarrow C_{\rho_0}) + \frac{1}{2}(k + n - 1)g^\rho_0(X_1^*, X_2^*)
\]

The almost complex structure $j^-$ is special: $ChernRic^j = (n - 1)\omega$.

This results directly from the identities (2.11), (2.12) and (2.13), the computations given above, and the fact that $D(X_i, X_j) = J_{2n}[C_i, C_j]$ and $D(Y_i, Y_j) = \frac{1}{2}(J_{2n}B_j - B_j) + (B_iB_j - B_jB_i)J_{2n}$ are both symmetric.

### 3.3 $G = U(k', n)$ or $U(k' + n)$

The groups $U(k', n)$ (resp $U(k' + n)$) are seen as the subgroups of the elements of $O(2k' + 2n, \mathbb{R})$ (resp. $O(2k' + 2n, \mathbb{R})$) which commute with \( \left( \frac{J_{2k'} 0}{0 J_{2n}} \right) \).

The element $\rho$ belongs to their Lie algebra and $\beta: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R} : X, Y \mapsto \text{Tr}(XY)$ is non degenerate. In both cases, $\mathfrak{g}^{-1} = \{0\}$ and $G^\sigma = G^{\sigma^2}$. The 4-symmetric spaces are thus the symmetric spaces,

\[
U(k' + n)/ (U(k') \times U(n)) \quad U(k', n)/ (U(k') \times U(n))
\]

which identify with the coadjoint orbit of the element $\rho^\beta: \mathfrak{g} \rightarrow \mathbb{R} : X \mapsto \text{Tr} \rho X$.

There is only one natural almost complex structure (up to sign), $j^+$, defined by $\sigma$ on $\mathfrak{p}$; it is integrable.
For $g = u(k' + n)$, $p = \{ (\begin{smallmatrix} 0 & B \\ -B^t & 0 \end{smallmatrix}) | B = (\begin{smallmatrix} b & b' \\ -b' & b \end{smallmatrix}) \}$, and for such a $X \in p$:

$$-\omega_{p_0}(\pi_* X, j^+_{p_0} \pi_* X) = -\hat{\Omega}(X, J^+ X) = -2 \text{Tr}(-\text{tr} B B) = 2 \sum_{i=1}^{k} \sum_{j=1}^{2n} (B^i_j)^2$$

so that $(-\omega, J^+)$ define a Kähler structure on $U(k' + n) / (U(k') \times U(n))$.

For $g = u(k', n)$, $p = \{ (\begin{smallmatrix} 0 & B \\ -B^t & 0 \end{smallmatrix}) | B = (\begin{smallmatrix} b & b' \\ -b' & b \end{smallmatrix}) \}$, and for such a $X \in p$:

$$\omega_{p_0}(\pi_* X, j^+_{p_0} \pi_* X) = \hat{\Omega}(X, J^+ X) = 2 \text{Tr}(\text{tr} B B) = 2 \sum_{i=1}^{k} \sum_{j=1}^{2n} (B^i_j)^2$$

so that $(\omega, J^+)$ define a Kähler structure on $U(k', n) / (U(k') \times U(n))$.

In both case, the action of the stabilizer $U(k') \times U(n)$ on $p \simeq \text{Mat}(k' \times n, \mathbb{C}) \simeq \mathbb{C}^{k'} \otimes \mathbb{C}^n$ is irreducible so the Kähler metric is Einstein.

The curvature at the base point is given by $R^g_{\rho_0}(Y_1^*, Y_2^*)Y_3^* = -([[Y_1, Y_2], Y_3])_{p_0}$, so

$$\text{Ric}^g_{\rho_0}(Y_1^*, Y_2^*) = \text{Tr}(\text{ad} Y_2 \circ \text{ad} Y_1)|_p$$

$$= \text{Tr}_p \left( B \rightarrow (B_2 B_1 B + B B_2 B_2 - 2 B_2 B_1 B_1 - B_1 B_2 B_2) \right) \text{ with } B' = \overline{\text{Ad}} B$$

$$= \text{Tr}(b, b') \rightarrow \left( (b_2 b_1 - b_1 b_2) + b_1 b_2 - \tilde{b}_1 b', -b_2 b_1 + b_1 b_2 + \tilde{b}_1 b' + b_1 b' \right)$$

$$= (n + k') \text{Tr}(B_2 B_1') = \frac{1}{2}(n + k') \text{Tr}(B_2 B_2 + B_1 B_1').$$

avec $b, b' \in \text{Mat}(k' \times n)$ and $B' = (\begin{smallmatrix} b & b' \\ -b' & b \end{smallmatrix}).$

### 3.4 $G = Sp(k + 2n, \mathbb{R}) \simeq Sp(\mathbb{R}^{2(k'+n)}, \tilde{\Omega})$

We set $k = 2k'$ and we choose a basis of $\mathbb{R}^{k+2n} = \mathbb{R}^{2(k'+n)}$ in which the non-degenerate skew-symmetric 2-form writes

$$\tilde{\Omega} = \begin{pmatrix} \Omega_{2k'} & 0 \\ 0 & \Omega_{2n} \end{pmatrix} \quad \text{with} \quad \Omega_{2r} := -J_{2r} = \begin{pmatrix} O & \text{Id}_r \\ -\text{Id}_r & 0 \end{pmatrix}.$$

The subgroup $Sp(\mathbb{R}^{2(k'+n)}, \tilde{\Omega}) = \{ g \in GL(k + 2n, \mathbb{R}) | \text{tr} \ g \tilde{\Omega} \ g = \tilde{\Omega} \}$ is $\tilde{\sigma}$-stable, because $R \in Sp(\mathbb{R}^{2(k'+n)}, \tilde{\Omega})$. The element $\rho$ is in the Lie algebra $\mathfrak{sp}(\mathbb{R}^{2(k'+n)}, \tilde{\Omega})$:

$$\mathfrak{sp}(\mathbb{R}^{2(k'+n)}, \tilde{\Omega}) = \{ (A \begin{smallmatrix} \Omega_{2k'} \text{tr} B \Omega_{2n} \\ C \end{smallmatrix}) | \text{tr} A \Omega_{2k'} + C \Omega_{2n} = 0, \quad \text{tr} C(n + \Omega_{2n} + \Omega_{2n} C = 0) \}.$$
This Lie algebra \( \mathfrak{sp}(\mathbb{R}^{2(k'+n)}, \tilde{\Omega}) \simeq \mathfrak{sp}(k + 2n, \mathbb{R}) \) is simple, thus the 2-form \( \beta \) is non degenerate. The subspace \( \mathfrak{p} \) is given by

\[
\mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ \Omega_{2n} & 0 \end{pmatrix} \mid B \in \text{Mat}(2n \times 2k') \right\} \neq \{0\}.
\]

The 4-symmetric space

\[
G/G_0^\sim = \text{Sp}(\mathbb{R}^{2(k'+n)}, \tilde{\Omega}) / (\text{Sp}(\mathbb{R}^{2k'}, \Omega_k) \times U(n)),
\]

is the adjoint orbit of \( \rho \) in \( \mathfrak{sp}(\mathbb{R}^{2(k'+n)}, \tilde{\Omega}) \), or the coadjoint orbit of \( \rho^\flat \) in the dual Lie algebra, and the form \( \omega \) coincides with the canonical symplectic form of Kirillov-Kostant-Souriau.

The almost complex structure \( j^+ \) is an invariant complex structure and the almost complex structure \( j^- \) is maximally non integrable since \( g \) is simple and \( g_{-1} = \{ \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \mid \text{tr} C \Omega_{2n} + \Omega_{2n} C = 0 \text{ and } CJ_{2n} + J_{2n} C = 0 \} \),

contains the element \( \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \) where \( C = \begin{pmatrix} \text{Id}_n & 0 \\ 0 & -\text{Id}_n \end{pmatrix} \) satisfies \( C^2 = \text{Id}_{2n} \).

A basis of the \( C \)'s corresponding to elements in \( g_{-1} \) is given by

\[
\{ E_{ij} + E_{ji} - E_{n+i+n+j} - E_{n+j+n+i}, E_{n+i+j} + E_{n+j+i} + E_{i+n+j} + E_{j+n+i}, 1 \leq i \leq j \leq n \}
\]

so that, for any \( D = \begin{pmatrix} d & d' \\ -d' & d \end{pmatrix} \),

\[
\text{Tr} (C \rightarrow DC + CD) = \sum_{1 \leq i \leq j \leq n} (d_{ii} + d_{jj} + d_{ij} + d_{ji}) = 2(n+1) \text{Tr } d = (n+1) \text{Tr } D.
\]

Since \( \mathfrak{p} = \{ \begin{pmatrix} 0 & B \\ B' \end{pmatrix} \mid B' = \Omega_{2n} \text{tr } B \Omega_{2k'} \} \), a basis of the \( (B, B') \) is given by

\[
\{ (E_{\ell r}, \Omega_{2n} E_{\ell r} \Omega_{2k'}), 1 \leq \ell \leq 2n, 1 \leq r \leq 2k' \}
\]

and for any \( D \) as above such that \( \text{tr } D \Omega_{2n} = \Omega_{2n} D \),

\[
\text{Tr} ((B, B') \rightarrow (BD, DB')) = \sum_{1 \leq \ell \leq 2n, 1 \leq r \leq 2k'} D_{\ell \ell} = 2k' \text{Tr } D.
\]

The almost complex structure \( j^- \) is special : \( \text{ChernRicci}^{j-} = (n + 1 - k)\omega \).

28
The metric $g^+$ is Einstein $\text{Ric}g^+ = \frac{1}{2}(2k' + n + 1)g^+$. Indeed, the identities (2.5), (2.6) and (2.7) become:

\[
\text{Ric}_{g^+}^p(X_1^*, X_2^*) = \text{Tr}(C \to (C_2C_1 + CC_2 - C_1CC_2 - C_2CC_1)) + \text{Tr}((B, B') \to (BC_2C_1, C_2C_1B'))
\]
\[
= (n + 1)\text{Tr}C_2C_1 + 2k'\text{Tr}C_2C_1 = \frac{1}{2}(2k' + n + 1)g^+_p(X_1^*, X_2^*)
\]
\[
\text{Ric}_{g^+}^p(X_1^*, Y_1^*) = 0 \quad \text{and} \quad g^+_p(X_1^*, Y_1^*) = 0;
\]
\[
\text{Ric}_{g^+}^p(Y_1^*, Y_2^*) = \frac{1}{2}\text{Tr}(C \to \frac{1}{2}((B_2'B_1 - J_{2n}B_2'Y_1^*)C + C(B_1'B_2 - J_{2n}B_1'B_2Y_1^*))
\]
\[
+ \frac{1}{4}\text{Tr}(B \to ((2B_2B_1 - B_1B_2')B + B(3B_1'B_2 + B_2'B_1 - JB_1'B_2 + JB_2'B_1))
\]
\[
- (B_1JB_2' + 2B_2JB_1')B - 2B_2B'B_2 + B_1JB'B_2 + B_1JB'B_2 + 2B_2JB'B_1,B))
\]
\[
= \frac{1}{2}(n + 1)\text{Tr}((B_2'B_1 - J_{2n}B_2'B_1))
\]
\[
+ \frac{1}{4}(2n\text{Tr}(2B_2B_1' - B_1B_2') + 2k'\text{Tr}(3B_1'B_2 + B_2'B_1 - JB_1'B_2 + JB_2'B_1))
\]
\[
+ \frac{1}{4}\text{Tr}(2B_2B_1'J_{2n}B_2' + 3B_1'B_2JB_1 - JB_1'B_2 - 2J'B_2'B_1J)
\]
\[
= (\frac{1}{2}(n + 1) + \frac{1}{4}(2n + 8k' + 2))\text{Tr}(B_2'B_1) = (n + 1 + 2k')\frac{1}{2}g^+_p(Y_1^*, Y_2^*).
\]

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