Cohomological supports of tensor products of modules over commutative rings

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**Abstract**

This work concerns cohomological support varieties of modules over commutative local rings. The main result is that the support of a derived tensor product of a pair of differential graded modules over a Koszul complex is the join of the supports of the modules. This generalizes, and gives another proof of, a result of Dao and the third author dealing with Tor-independent modules over complete intersection rings. The result for Koszul complexes has a broader applicability, including to exterior algebras over local rings.

**Keywords:** Koszul complex, Dg modules, Cohomological support, Tensor products, Join, BGG correspondence

**Mathematics Subject Classification:** 13D02, 16E45 (primary); 13D07, 13H10 (secondary)

**Introduction**

Throughout, we fix a Koszul complex \(E\) over a (commutative noetherian) local ring \((R, m, k)\) on a list of elements \(f = f_1, \ldots, f_c\) in \(m\). As explained in [25] studying the homological properties of differential graded (abbreviated to dg), \(E\)-modules allows one to unify and extend the results about quotients \(R \to R/(f)\) when \(f\) is an \(R\)-regular sequence as well as those about exterior algebras over \(R\). The dg \(E\)-modules perfect when regarded as \(R\)-complexes—in the sense that they are quasi-isomorphic to a bounded complex of finite rank free \(R\)-modules—are the ones that exhibit especially structured homological phenomena; see, for example, [1,4,6,9,11,16,20,26]. The homological properties of such a dg \(E\)-module \(M\) are often encoded in its cohomological support, denoted \(V_E(M)\), which is a naturally associated Zariski closed subset of \(\mathbb{P}^{c-1}_k\); cf. 5.2.

The main result of this article is the following.

**Theorem** For dg \(E\)-modules \(M, N\) that are perfect over \(R\), there is an equality

\[
V_E(M \otimes^L_E N) = \text{Join}(V_E(M), V_E(N)).
\]

The join of closed subsets \(U, V\) of \(\mathbb{P}^{c-1}_k\), denoted \(\text{Join}(U, V)\), is the closure of the union of lines connecting a point from \(U\) to a point from \(V\); see 1.1 for details. Specializing the
theorem above to the case where \( R \) is a regular ring, \( f \) is an \( R \)-regular sequence, and \( M, N \) are finitely generated \( R/(f) \)-modules satisfying \( \text{Tor}^R_i(M, N) = 0 \) for all \( i \geq 1 \) recovers [15, Theorem 3.1]. The proof in loc. cit. involves a series reductions and ad hoc geometric arguments. Besides generalizing this result, a main point of this article is to offer a simpler proof by a passage to an exterior algebra, as briefly described below.

The theorem above is proved in Sect. 6. As a corollary, we deduce that when \( R \) is Gorenstein and \( \text{RHom}_E(M, N) \) is perfect as an \( R \)-complex, there is an equality

\[
V_E(\text{RHom}_E(M, N)) = \text{Join}(V_E(M), V_E(N)).
\]

This is Corollary 6.4 and it generalizes [15, Theorem 4.7]. Theorem 6.6 relates the support of the dg module \( M \otimes^L_E N \) to that of its homology modules, namely, \( \text{Tor}^E_i(M, N) \), thereby providing a positive answer to [15, Question 2].

The key ingredient in our work is a functor, denoted \( t \), from the derived category of dg \( E \)-modules \( D(E) \) to the derived category of dg \( \Lambda_1 \)-modules \( D(\Lambda_1) \) where \( \Lambda_1 \) is an exterior algebra on \( \Sigma k^c \); see Sect. 5. The relevance of this functor arises from Lemma 5.3 which identifies \( V_E(M) \) with \( V_{\Lambda_1}(tM) \), and that as dg \( \Lambda_1 \)-modules

\[
t(M \otimes^L_E N) \simeq tM \otimes^L_{\Lambda_1} tN.
\]

The expression for \( V_E(M \otimes^L_E N) \) in the theorem above is a consequence of Proposition 4.4 that asserts if \( X, Y \) are dg \( \Lambda_1 \)-modules with finite-dimensional homology, then

\[
V_{\Lambda_1}(X \otimes^L_{\Lambda_1} Y) = \text{Join}(V_{\Lambda_1}(X), V_{\Lambda_1}(Y)).
\]

This equality is in turn deduced using a contravariant version, from [2], of the Bernstein-Gelfand-Gelfand correspondence functor:

\[
d: D(\Lambda) \to D(S),
\]

where \( S \) is the symmetric algebra on \( \Sigma^{-2}k^c \). The main calculation in the proof of (†) is the interaction between tensor products and the functor \( d \), namely: Given dg \( \Lambda \)-modules \( X, Y \) with homology finite dimensional over \( k \), there is an isomorphism of dg \( S \)-modules

\[
d(X \otimes^L_{\Lambda} Y) \simeq dX \otimes_k dY,
\]

where the right-hand side is regarded as a dg \( S \)-module through a natural map of \( k \)-algebras \( \Delta: S \to S \otimes_k S \), which makes \( S \) into a Hopf algebra; see (1.1.1). Given this result, (†) follows by a standard argument concerning supports of modules over polynomial rings, discussed in Sect. 1; see especially Lemma 1.4. The isomorphism above, which is folklore, is contained in Proposition 4.4.

1 Joins and supports

Let \( k \) be a field. In what follows, we encounter graded \( k \)-vector spaces \( W \) whose natural grading is lower and also those whose natural grading is upper. It is convenient to adopt the convention that \( W \) has both an upper and a lower grading, with \( W^i = W_{-i} \) for each integer \( i \). We indicate the primary grading when necessary.

Fix a finite-dimensional graded \( k \)-space \( W := \{ W^i \}_{i \in \mathbb{Z}} \) concentrated in positive even degrees. Let \( S := \text{Sym}_k W \) be the symmetric algebra (over \( k \)) on \( W \) and \( \text{Proj} S \) the set of homogeneous prime ideals of \( S \) not containing the irrelevant maximal ideal \( S^{>0} \) of \( S \), equipped with the Zariski topology. In this section, we recall some basics on joins of
closed subsets of \( \text{Proj} \, S \) and of supports of graded \( S \)-modules. Our standard references are \cite[Sec. 1.3]{18}, for joins, and \cite{19}, for supports.

1.1 The map \( W \to W \oplus W \) given by \( w \mapsto (w, 0) + (0, w) \) induces a map

\[
\Delta : S \to S \otimes_k S
\]

(1.1.1)
of graded \( k \)-algebras and makes \( S \) into a graded Hopf algebra over \( k \). It also defines a rational map

\[
\delta : \text{Proj}(S \otimes_k S) \to \text{Proj} \, S
\]

that is defined (and regular) off of the anti-diagonal \( D \) in \( \text{Proj}(S \otimes_k S) \); here, \( D \) is the image of the embedding \( \text{Proj} \, S \hookrightarrow \text{Proj}(S \otimes_k S) \) determined by the map \( W \oplus W \to W \) given by \( (w_1, w_2) \mapsto w_1 + w_2 \).

Given closed subsets \( U := \mathcal{V}(I) \) and \( V := \mathcal{V}(J) \) of \( \text{Proj} \, S \), consider

\[
\text{J}(U, V) := \text{Proj}(S/I \otimes_k S/J)
\]

viewed as a closed subset of \( \text{Proj}(S \otimes_k S) \). The join of \( U \) and \( V \), denoted \( \text{Join}(U, V) \), is the closure in \( \text{Proj} \, S \) of the set

\[
\delta(\text{J}(U, V) \setminus D).
\]

When \( k \) is algebraically closed, the Nullstellensatz identifies \( \text{Proj} \, S \) with projective space \( \mathbb{P}^{d-1}_k \) where \( d = \dim_k W \). Under this identification, the join of \( U \) and \( V \) is the closure of the union of lines in \( \text{Proj} \, S \) containing a point \( u \) in \( U \) and a point \( v \) in \( V \).

**Remark 1.1** The join can also be defined as follows: Consider the rational map

\[
\delta' : \text{Proj}(S \otimes_k S) \to \text{Proj} \, S,
\]

that is regular of the diagonal in \( \text{Proj}(S \otimes_k S) \), induced by the \( k \)-algebra map \( S \to S \otimes_k S \) determined by \( w \mapsto w \otimes 1 - 1 \otimes w \). The linear automorphism \( \alpha \) of \( \text{Proj}(S \otimes_k S) \) determined by

\[
w \otimes 1 \mapsto w \otimes 1 \quad \text{and} \quad 1 \otimes w \mapsto -1 \otimes w
\]

fixes \( J(U, V) \) for any pair of closed subsets \( U, V \) of \( \text{Proj} \, S \), maps \( D \) bijectively to \( D' \) and \( \delta = \delta' \alpha \). Hence,

\[
\delta(J(U, V) \setminus D) = \delta'(J(U, V) \setminus D'),
\]

where the right-hand side is the definition of the join used in \cite[Sec. 1.3]{18}. That is the definitions of joins from loc. cit. and 1.1 coincide. We opt for the latter as the isomorphism in Proposition 4.4 respects \( \Delta \).

1.2 Let \( X \) be a graded \( S \)-module. The support of \( X \) over \( S \) is the subset

\[
\text{Supp}_S X := \{ p \in \text{Proj} \, S \mid X_p \neq 0 \},
\]

where \( X_p \) denotes the homogeneous localization of \( X \) at \( p \). Following Foxby \cite{19}, the small support of \( X \) is

\[
\text{supp}_S X := \{ p \in \text{Proj} \, S \mid X_\frac{1}{\kappa(p)} \neq 0 \},
\]

where \( \kappa(p) \) is the graded field \( S_p/pS_p \). Consider the closed subset

\[
\mathcal{V}(\text{ann}_S X) := \{ p \in \text{Proj} \, S \mid p \supseteq \text{ann}_S X \}
of Proj S. In general, there are inclusions
\[ \text{supp}_S X \subseteq \text{Supp}_S X \subseteq V(\text{ann}_S X). \quad (1.3.1) \]
Moreover, \( \text{Supp}_S X \) is the specialization closure of \( \text{supp}_S X \); see [10, Lemma 2.2]. Equalities hold when the \( S \)-module \( X \) is finitely generated.

**Lemma 1.4** Let \( X, Y \) be finitely generated graded \( S \)-modules. There is an equality
\[ \text{Supp}_S (X \otimes_k Y) = \text{Join}(\text{Supp}_S X, \text{Supp}_S Y), \]
where \( X \otimes_k Y \) is regarded as a graded \( S \)-module via \( (1.1.1) \).

**Proof** As a matter of notation, we write \( S^e \) for \( S \otimes_k S \) and use \( (\overline{\quad}) \) for closure in the Zariski topology. For any finite generated \( S^e \)-module \( N \) and \( p \in \text{Proj} S \), one has
\[ N \otimes_{S^e} \kappa(p) \simeq N \otimes_{S^e} (S^e \otimes_{S^e} \kappa(p)). \]
This leads to the following equivalences:
\[ p \in \text{supp}_S N \iff \text{supp}_{S^e}(N) \cap \text{supp}_{S^e}(S^e \otimes_{S^e} \kappa(p)) \neq \emptyset \iff \text{supp}_{S^e}(N) \cap (\delta^{-1}(p) \setminus D) \neq \emptyset \iff p \in \delta(\text{supp}_{S^e}(N) \setminus D). \]
Applying this observation to \( N := X \otimes_k Y \) justifies the last equality below:
\[
\text{Join}(\text{Supp}_S X, \text{Supp}_S Y) = \overline{\delta(\text{supp}_{S^e}(X \otimes_k Y) \setminus D)}
= \overline{\delta(\text{supp}_{S^e}(X \otimes_k Y) \setminus D)}
= \overline{\text{supp}_S(X \otimes_k Y)}.
\]
The first equality holds as \( X, Y \) are finitely generated over \( S \), while the second equality holds because \( X \otimes_k Y \) is finitely generated over \( S^e \). Thus, for the desired statement, it suffices to verify that
\[ \overline{\text{supp}_S(X \otimes_k Y)} = \text{Supp}_S(X \otimes_k Y). \]
To that end, given 1.2, it suffices to verify that \( \text{Supp}_S(X \otimes_k Y) \) is closed in \( \text{Proj} S \). As an \( S \)-module \( X \otimes_k Y \) need not be finitely generated, but it is finitely generated over \( S^e \), and that suffices.
Indeed, let \( G \) be a finite generating set for \( X \otimes_k Y \) over \( S^e \) and \( T \) the \( S \)-submodule of \( X \otimes_k Y \) generated by \( G \); here, \( S \) is acts via the diagonal map \( (1.1.1) \). Since \( T \) is finitely generated over \( S \), one gets the first equality below:
\[
\mathcal{V}(\text{ann}_S T) = \text{Supp}_S T
\subseteq \text{Supp}_S(X \otimes_k Y)
\subseteq \mathcal{V}(\text{ann}_S(X \otimes_k Y))
= \mathcal{V}(\text{ann}_S T).
\]
The containments are from 1.2; the last equality holds as \( \text{ann}_S(X \otimes_k Y) = \text{ann}_S T \). Thus, the inclusions above are equalities, as desired. \( \square \)
2 Dg modules over graded algebras

Let $A = \{A_i\}_{i \in \mathbb{Z}}$ be a strictly graded-commutative dg algebra. Its homology algebra, $H(A)$, is thus also strictly graded-commutative.

### 2.1 A dg $A$-module $F$ is semifree provided it admits an exhaustive filtration

$0 = F(-1) \subseteq F(0) \subseteq F(1) \subseteq \ldots \subseteq F,$

where each subquotient $F(i)/F(i-1)$ is a coproduct of suspensions of $A$. A semifree resolution of a dg $A$-module $M$ is a surjective quasi-isomorphism of dg $A$-modules $\tilde{F} \rightarrow M$ where $F$ is a semifree dg $A$-module. Such resolutions of $M$ exist and any two are unique up to homotopy equivalence; see, for example, [17, 6.6].

### 2.2 Let $M$ be a dg $A$-module and fix $F \tilde{\rightarrow} M$ a semifree resolution over $A$. By [17, ], the functors $F \otimes_A -$ and $\text{Hom}_A(F, -)$ preserve (surjective) quasi-isomorphisms. Hence by replacing objects with their semifree resolutions, we obtain bi-functors $- \otimes_A^L -$ and $\text{RHom}_A(-, -)$ on $\text{D}(A)$; that is to say,

$M \otimes_A^L := F \otimes_A -$ and $\text{RHom}_A(M, -) := \text{Hom}_A(F, -)$.

As usual, we set

$\text{Tor}_i^A(M, N) := H_i(M \otimes_A^L N)$ and $\text{Ext}_i^A(M, N) := H^i(\text{RHom}_A(M, N))$.

As $A$ is graded-commutative, these are graded $H(A)$-modules.

### 2.3 The derived category of dg $A$-modules is denoted $\text{D}(A)$, and it is regarded as a triangulated category in the standard way; see, for example, [5, Sect. 2]. The suspension functors associates with each dg $A$-module $M$ the dg module $\Sigma M$ with

$(\Sigma M)_i = M_{i-1}, \quad \delta \Sigma M = -\delta M \quad \text{and} \quad a \cdot \Sigma m = (-1)^{|a|} am,$

where $|a|$ denotes the degree $a$. A thick subcategory of a triangulated category is a triangulated subcategory that is closed under retracts.

### 2.4 Let $A$ be a dg algebra over a field $k$. We define several thick subcategories of $\text{D}(A)$ that will be of interest in what follows.

Let $\text{D}^b_+(A)$ denote the full subcategory of $\text{D}(A)$ consisting of dg $A$-modules $M$ with each $H_i(M)$ finite dimensional and $H_i(M) = 0$ for all $i \ll 0$; define $\text{D}^b_-(A)$ analogously where the second condition is replaced with $H_i(M) = 0$ for all $i \gg 0$. We let $\text{D}^b_0(A)$ denote $\text{D}^b_+(A) \cap \text{D}^b_-(A)$. That is, $\text{D}^b_0(A)$ consists exactly of those dg $A$-modules whose homology is finite-dimensional over $k$. We write $\text{Perf}(A)$ for the thick subcategory of $\text{D}(A)$ generated by $A$; see [5, Theorem 4.2] for an alternative characterization.

3 Exterior algebras

In this section, $V := \{V_i\}_{i \in \mathbb{N}}$ is a finite graded $k$-space concentrated in odd degrees. Set $(-)^\vee := \text{Hom}_k(-, k)$, the graded dual, and $W := \Sigma^{-1}(V^\vee)$. Let

$\Lambda := \bigwedge_k V$ and $\mathcal{S} := \text{Sym}_k W$;

the former being the exterior algebra, over $k$, on $V$. Set $\Gamma := \mathcal{S}^\vee$ with the standard $\mathcal{S}$-module structure: For $\alpha \in \Gamma$ and $\chi \in \mathcal{S}$, one has

$\chi \cdot \alpha := a(\chi \cdot -).$
We view $\Lambda$ as a graded Hopf algebra, with coproduct $\Lambda \to \Lambda \otimes_\Lambda \Lambda$ induced by the map of $k$-spaces $v \mapsto (v, 1) + (1, v)$, for $v \in V$. Hence for any left dg $\Lambda$-module, the antipode defines a dg $\Lambda$-module structure on $M^\vee$. Also, for a pair of dg $\Lambda$-modules $M, N$, their tensor product $M \otimes_\Lambda N$ is regarded as a dg $\Lambda$-module through the coproduct. See, for example, [7, Remark 5.2]. We also view $S$ as a graded Hopf algebra over $k$, with coproduct defined in 1.1.

**Notation 3.1** Fix a basis $e_1, \ldots, e_c$ for $V$, and let $\chi_1, \ldots, \chi_c$ be the dual basis for $W$; thus $\chi_i$ has lower degree $-|e_i| - 1$. These determine isomorphisms

$$\Lambda \cong \bigwedge (ke_1 \oplus \ldots \oplus ke_c) \quad \text{and} \quad S \cong k[\chi_1, \ldots, \chi_c].$$

**3.2** For a dg $\Lambda$-module $M$, its universal resolution $uM$ is the dg $(\Lambda \otimes_\Lambda S)$-module with underlying graded $(\Lambda \otimes_\Lambda S)$-module $\Lambda \otimes_\Lambda \Gamma \otimes_\Lambda M$, with $\Lambda \otimes_\Lambda S$ acting by left multiplication on the two left factors, and differential

$$1 \otimes 1 \otimes \partial^M + \sum_{i=1}^c (1 \otimes \chi_i \otimes e_i - e_i \otimes \chi_i \otimes 1).$$

The canonical projection $uM \to M$ is a semifree resolution of $M$ over $\Lambda$; see [3, Proposition 2.6] or [5, Sect. 7]. Moreover, since $uM$ is a dg module over $\Lambda \otimes_\Lambda S$, the graded $k$-space $\text{Hom}_\Lambda(uM, -)$ retains a dg $S$-module structure and so

$$\text{Ext}^*_\Lambda(M, -) = \text{H}^*(\text{Hom}_\Lambda(uM, -))$$

is a graded $S$-module.

**3.3** Let $M$ be dg $\Lambda$-module with $M_i$ degreewise finite dimensional over $k$ for each $i$ and 0 for $i \ll 0$; up to quasi-isomorphism any object in $D^b_+(\Lambda)$ has this form. There is an isomorphism of dg $S$-modules

$$\text{Hom}_\Lambda(uM, k) \cong S \otimes_\Lambda M^\vee,$$ (3.3.1)

where the right-hand term has differential $1 \otimes \partial^{M^\vee} + \sum_{i=1}^c \chi_i \otimes e_i$; we denote the dg $S$-module on the right by $S_M$. From this isomorphism and [25, Proposition 1.2.8], $\text{Hom}_\Lambda(uM, k)$ is a semifree dg $S$-module.

The contravariant functor $\text{Hom}_\Lambda(u(-), k)$ induces the exact functor

$$d: D(\Lambda)^{op} \to D(S).$$

By [2], this restricts to an exact equivalence

$$d: D^b_+(\Lambda)^{op} \cong D^b_+(S),$$

that further restricts to equivalences

$$D^b_+(\Lambda)^{op} \cong \text{Perf}(S) \quad \text{and} \quad \text{Perf}(\Lambda)^{op} \cong D^b_+(S).$$

One has also the functor $\text{Hom}_\Lambda(uk, -)$ that induces an exact functor

$$b: D(\Lambda) \to D(S)$$

which restricts to equivalences

$$D^b_+(\Lambda) \cong \text{Perf}(S) \quad \text{and} \quad \text{Perf}(\Lambda) \cong D^b_+(S).$$
cf. [5]. There is the following commutative diagram

\[
\begin{array}{ccc}
D^+_f(\Lambda)^{\text{op}} & \xrightarrow{d} & D^-_f(S) \\
\downarrow & & \downarrow \circ \text{b} \\
D^-_f(\Lambda) & & \\
\end{array}
\]

3.4 The functors \(b, d\) defined above determine two notions of cohomological support for \(\text{dg } \Lambda\)-modules. Namely, for a \(\text{dg } \Lambda\)-module \(M\), consider subsets of \(\text{Proj } S\)

\[
V^b_{\Lambda}(M) := \text{Supp}_S H(bM) = \text{Supp}_S \text{Ext}_\Lambda(k, M),
\]

\[
V^d_{\Lambda}(M) := \text{Supp}_S H(dM) = \text{Supp}_S \text{Ext}_\Lambda(M, k).
\]

In [14], the supports \(V^b_{\Lambda}(−)\) are used to classify the thick subcategories of \(D^+_f(\Lambda)\). Our focus will be on \(V^d_{\Lambda}(−)\) but it is worth recording their relationship.

**Proposition 3.1** Let \(M\) be in \(D^+_f(\Lambda)\). There is an equality \(V^d_{\Lambda}(M) = V^b_{\Lambda}(\circ V)\). Moreover, if \(M\) is in \(D^-_f(\Lambda)\), then \(V^d_{\Lambda}(M) = V^b_{\Lambda}(M)\).

**Proof** The first equality is immediate from \(b(\circ V) = d\); see 3.3. The second equality follows from the first. Indeed, it is easy to check that if \(N\) is in the thick subcategory generated by \(N'\), then

\[
V^b_{\Lambda}(N) \subseteq V^b_{\Lambda}(N') \quad \text{and} \quad V^d_{\Lambda}(N) \subseteq V^d_{\Lambda}(N').
\]

When \(M\) is in \(D^+_f(\Lambda)\), the \(\text{dg } \Lambda\)-modules \(M\) and \(M'\circ V\) generate the same thick subcategory—see [23, Sect. 4]—so the second equality follows from the first. \(\square\)

4 Support for tensor products, I

As in the previous section, \(V := \{V_i\}_{i > 0}\) is a finite graded \(k\)-space concentrated in positive even degrees, and

\[
\Lambda := \bigwedge V \quad \text{and} \quad S := \text{Sym}_k W,
\]

where \(W := \Sigma^{-1}(V^\circ V)\). In this section, we analyze the interaction between the functor \(d: D^+_f(\Lambda)^{\text{op}} \to D^-_f(S)\), from 3.3, and the tensor products \(\otimes_k\) and \(\otimes^1_\Lambda\). The main results, Proposition 4.2 and Proposition 4.4, are folklore but we could not find adequate references, so we give complete proofs; see also Remark 4.3.

**Lemma 4.1** For \(S\)-modules \(X, Y\) with finitely generated homology,

\[
\text{Supp}_S H(X \otimes^1_\Lambda Y) = \text{Supp}_S H(X) \cap \text{Supp}_S H(Y).
\]

**Proof** Since \(X, Y\) have finitely generated homology, there are equalities

\[
\text{Supp}_S H(X) = \{p \in \text{Proj } S \mid X \otimes^1_\Lambda \kappa(p) \neq 0\},
\]

\[
\text{Supp}_S H(Y) = \{p \in \text{Proj } S \mid Y \otimes^1_\Lambda \kappa(p) \neq 0\}.
\]
See, for instance, [14, Theorem 2.4]. Since $S$ has finite global dimension, the $S$-module $H(X \otimes_S^L Y)$ is also finitely generated and so

$$\text{Supp}_S H(X \otimes_S^L Y) = \{ p \in \text{Proj } S \mid X \otimes_S^L Y \otimes_S^L \kappa(p) \not\simeq 0 \}.$$ 

The desired equality follows from the ones above and the isomorphism

$$X \otimes_S^L Y \otimes_S^L \kappa(p) \simeq (X \otimes_S^L \kappa(p)) \otimes_{\kappa(p)} (Y \otimes_S^L \kappa(p)).$$

The result below records the relationship between $d$ and tensor products.

**Proposition 4.2** For $M, N$ in $D^+_\Lambda(S)$, there is an isomorphism of dg $S$-modules

$$d(M \otimes_k N) \simeq dM \otimes_S^L dN.$$

Furthermore if $M, N$ are in $D^b_{\Lambda_1}(S)$, then

$$V^d_A(M \otimes_k N) = V^d_A(M) \cap V^d_A(N).$$

**Proof** Replacing $M$ and $N$ with semifree resolutions over $\Lambda$, we may assume both $M$ and $N$ are bounded below and degree-wise finite dimensional over $k$, as in 3.3. Let $\Phi$ denote the composition of the isomorphisms of dg $S$-modules

$$(S \otimes_k M^\vee) \otimes_S (S \otimes_k N^\vee) \longrightarrow (S \otimes_S S) \otimes_k M^\vee \otimes_k N^\vee \longrightarrow S \otimes_k M^\vee \otimes_k N^\vee,$$

where the first one is the twist isomorphism given by

$$(s \otimes \alpha) \otimes (s' \otimes \beta) \mapsto (s \otimes s') \otimes (\alpha \otimes \beta)$$

and the second map is the multiplication isomorphism. It is straightforward to see

$$\Phi \circ \sum_{i=1}^c (X_i \otimes e_i) \otimes 1 + 1 \otimes (X_i \otimes e_i) = \sum_{i=1}^c X_i \otimes (e_i \otimes 1 + 1 \otimes e_i) \circ \Phi.$$ 

As $M, N$ are degree-wise finite dimensional and bounded below, there is a natural isomorphism of dg $\Lambda$-modules

$$(M \otimes_k N)^\vee \simeq M^\vee \otimes_k N^\vee.$$

Hence, $\Phi$ yields an isomorphism

$$S_M \otimes_S S_N \xrightarrow{\Phi} S_{M \otimes_k N}.$$

As a consequence, (3.3.1) establishes the isomorphisms in $D(S)$:

$$d(M \otimes_k N) \simeq dM \otimes_S dN \simeq dM \otimes_S^L dN.$$  \hspace{1cm} (4.2.1)

As for the statement regarding supports, consider the following equalities:

$$V^d_A(M \otimes_k N) = \text{Supp}_S H(d(M \otimes_k N))$$

$$= \text{Supp}_S H(dM \otimes_S^L dN)$$

$$= \text{Supp}_S H(dM) \cap \text{Supp}_S H^*(dN)$$

$$= V^d_A(M) \cap V^d_A(N).$$

The second equality is from (4.2.1), while the third is Lemma 4.1. \hspace{1cm} $\square$
Remark 4.3  Buchweitz proved that if $M, N$ are graded $\Lambda$-modules that are bounded below and are degreewise finite rank over $k$, then

$$\text{b}(M \otimes_k N) \simeq \text{b}M \otimes^L_k \text{b}N; \quad (3.3.2)$$

see [13, (9.4.10)]. It is easy to see that this isomorphism holds for all pairs of objects in $D^b_+(\Lambda)$. From the equality $\text{b}((-)^\vee) = d$, the isomorphisms in (3.3.2) can also be deduced from (and imply) the ones in Proposition 4.2.

Proposition 4.4  For $M, N$ in $D^b_+(\Lambda)$, there is an isomorphism of dg $S$-modules

$$d(M \otimes^L_\Lambda N) \simeq dM \otimes_k dN,$$

where the right-hand side is a dg $S$-module through the diagonal $\Delta: S \to S \otimes_k S$ described in (1.1.1). Furthermore, if $M, N$ are in $D^b_+(\Lambda)$, then

$$V^d(M \otimes^L_\Lambda N) = \text{join}(V^d(M), V^d(N)).$$

Proof  Replacing $M$ and $N$ with suitable resolutions, we can assume $M$ and $N$ are bounded above, degreewise finite dimensional over $k$, and semifree. As in 3.2, we consider $\Gamma = S^\vee$, regarded as a graded $S$-module. Forgetting differentials, one has a commutative diagram

$$\begin{array}{c}
\Lambda \otimes_k \Gamma \otimes^2 \otimes_k M \otimes_k N \\
\downarrow \phi \quad \downarrow \cong \\
\lambda \otimes_k \Gamma \otimes \lambda \otimes_k (M \otimes \lambda \Lambda) N
\end{array}$$

of graded $S$-modules, where the map on the bottom is defined using the multiplication $\mu: \Gamma \otimes_k \Gamma \to \Gamma$ map which is dual to the diagonal $\Delta: S \to S \otimes_k S$ in (1.1.1), and $\pi: M \otimes_k N \to M \otimes \lambda \Lambda N$ is the canonical projection. It is straightforward to check $\Phi$ is a $\Lambda$-linear morphism of complexes that is compatible with the canonical augmentations to $M \otimes \lambda \Lambda N$. Thus, $\Phi$ is a comparison map between semifree resolutions of $M \otimes^L_\Lambda N$ over $\Lambda$, and so it is a homotopy equivalence.

Applying $\text{Hom}_\lambda(-, k)$ to $\Phi$ yields the top map in the commutative diagram

$$\begin{array}{c}
\text{Hom}_\lambda(u(M \otimes \lambda \Lambda N), k) \\
\downarrow \phi^u \quad \downarrow \cong \\
\text{Hom}_\lambda(u(M \otimes \lambda \Lambda uN, k)
\end{array}$$

of dg $S$-modules, where $S_M \otimes_k S_N$ is viewed as a dg $S$-module through $\Delta$. The vertical parallel maps are isomorphisms by (3.3.1); the one on the right also uses the standard isomorphisms

$$\text{Hom}_\lambda(uM \otimes \lambda \Lambda uN, k) \cong \text{Hom}_\lambda(uM, \text{Hom}_\lambda(uN, k)) \cong \text{Hom}_\lambda(uM, k) \otimes_k \text{Hom}_\lambda(uN, k).$$

This is where the assumption that both $M$ and $N$ are degreewise finite rank and bounded below is needed. As $\Phi$ is a homotopy equivalence, from the commutativity of the diagram above it follows that $\Delta \otimes \pi^*\lambda$ is a homotopy equivalence of dg $S$-modules justifying the first assertion; cf. 3.3.

With this in hand, we have

$$\text{H}(d(M \otimes^L_\Lambda N)) \cong \text{H}(dM \otimes_k dN) \cong \text{H}(dM) \otimes_k \text{H}(dN),$$
where the second map is the Künneth isomorphism. This gives the second of the following equalities:

\[
V^d_A(M \otimes \Lambda_1 N) = \text{Supp}_S H(d(M \otimes \Lambda_1 N))
= \text{Supp}_S (H(dM) \otimes_k H(dN))
= \text{Join}(\text{Supp}_S H(dM), \text{Supp}_S H(dN))
= \text{Join}(V^d_A(M), V^d_A(N)).
\]

The third equality is Lemma 1.4. □

5 Passage to the exterior algebra

Throughout this section and the next \((R, m, k)\) is a commutative noetherian local ring. Fix a list of elements \(f = f_1, \ldots, f_c\) in \(m\) and set

\[
E := R(e_1, \ldots, e_c | \partial e_i = f_i),
\]

the Koszul complex on \(f\) over \(R\), regarded as a local dg \(R\)-algebra in the standard way. One could take \(R\) to be a local dg algebra where \(f\) is a list of cycles in even degrees, contained in the maximal ideal of \(R\); we stick to the situation above for ease of exposition. Two special cases are worth mention.

Remark 5.1 When \(f\) forms an \(R\)-regular sequence, the augmentation \(E \xrightarrow{\sim} R/(f)\) is a quasi-isomorphism of dg algebras and the map \(R \to R/(f)\) is complete intersection. When \(f\) is the zero sequence, \(E\) is the exterior algebra over \(R\) on \(c\) generators of degree one.

Set \(\Lambda := k \otimes_R E\) and \(V := \Lambda_1\). We identify \(e_1, \ldots, e_c\) with their images in \(\Lambda\); they are a basis for the \(k\)-space \(V\). Set \(W := \Sigma^{-1}(V^\vee)\), and

\[
S := \text{Sym}_k W.
\]

Let \(\chi_1, \ldots, \chi_c\) be the basis of \(W\) dual to \(e_1, \ldots, e_c\).

5.2 Let \(M\) be a dg \(E\)-module whose homology is finitely generated over \(R\). Let \(F\) be a dg \(E\)-module that is semifree as a dg \(R\)-module, and \(F \xrightarrow{\sim} M\) an \(E\)-linear quasi-isomorphism. By [25, Proposition 4.2.8], \(\text{RHom}_E(M, k)\) can be equipped with a dg \(S\)-module structure through the isomorphism

\[
\text{RHom}_E(M, k) \simeq S \otimes_k \text{Hom}_R(F, k),
\]

where the differential of the complex on the right is

\[
1 \otimes \partial^{\text{Hom}_E(F, k)} + \sum_{i=1}^c \chi_i \otimes \text{Hom}(e_i, k);
\]

we let \(C_F\) denote this dg \(S\)-module. Following [24, Definition 3.3.1], the cohomological support of \(M\) over \(E\) is

\[
V_E(M) = \text{Supp}_S \text{Ext}^*_E(M, k) = \text{Supp}_S H^*(C_F).
\]

A bridge to exterior algebras has been used effectively to acquire cohomological information on these support varieties when \(R\) is regular and \(f\) is an \(R\)-regular sequence; see, for
instance, [7,14,23]. This path is still sensible at this generality and can be used to establish results over \( E \), as we do now.

Consider the functor \( t: \mathcal{D}(E) \to \mathcal{D}(\Lambda) \) given by \( k \otimes \frac{1}{R} \). In the statement below, the construction of the dg \( S \)-module \( S_{tF} \) is given in 3.3.

**Lemma 5.3** Let \( M \) be a dg \( E \)-module with finitely generated homology over \( R \) and fix \( F \cong M \) a quasi-isomorphism of dg \( E \)-modules where \( F \) is semifree when regarded as dg \( R \)-module. There is the following isomorphism of dg \( S \)-modules

\[
C_{tF} \cong S_{tF}.
\]

In particular, \( V_E(M) = V_{\Lambda}^{d}(tM) \).

**Proof** For the isomorphism, since \( m \text{Hom}_R(F, k) = 0 \) the \( E \)-action on \( \text{Hom}_R(F, k) \) factors through \( \Lambda \). It is immediate to check the adjunction isomorphism

\[
\alpha: \text{Hom}_R(F, k) \cong \to \text{Hom}_k(tF, k)
\]

is one of \( \Lambda \)-modules. Therefore from the definitions of \( C_{tF} \) and \( S_{tF} \) in 5.2 and 3.3, respectively, the map

\[
1 \otimes \alpha: C_{tF} \to S_{tF}
\]

is an isomorphism of dg \( S \)-modules. The equality of supports follows:

\[
V_E(M) = \text{Supp}_S H(C_{tF}) = \text{Supp}_S H(S_{tF}) = \text{Supp}_S H(S_M) = \text{Supp}_S H(dtM) = V_{\Lambda}^{d}(tM);
\]

the second equality holds by the established isomorphism above and the others are clear from the various definitions.

**Remark 5.4** Suppose \( f \) is an \( R \)-regular sequence and \( M \) a finitely generated \( R \)-module such that \( fM = 0 \). The cohomological support of \( M \) over \( E \) agrees with support variety of \( M \) introduced by Avramov in [1], and further developed in the work of Avramov and Buchweitz [4].

More generally, without the assumption \( f \) is regular, \( V_E(M) \) specializes to the support sets of Jorgensen [22] and Avramov and Iyengar [8]; cf. [25, Sect. 6.2]. When \( M \) has finite projective dimension over \( R \), the cohomological support \( V_E(M) \) agrees with those above; hence Lemma 5.3 reveals how, in this setting, all of these supports are cohomological supports over an exterior algebra.

**5.5** Let \( D_b(E/R) \) denote the full subcategory of \( D(E) \) consisting of objects \( M \) such that \( M \) is perfect when regarded as an object of \( D(R) \) via restriction of scalars. That is, if \( \eta: R \to E \) is the structure map and \( \eta_*: D(E) \to D(R) \) denotes the restriction of scalars functor along \( \eta \), then \( M \) is in \( D_b(E/R) \) if and only if \( \eta_*(M) \) is isomorphic in \( D(R) \) to a bounded complex of finite rank free \( R \)-modules. In particular, \( M \) has bounded and finitely generated homology over \( R \). When \( R \) is regular \( D_b(E/R) \) is just the bounded derived category of dg \( E \)-modules.
The result below is a particular case of a theorem of Gulliksen [21] and Avramov, Gashasrov, and Peeva [6].

**Proposition 5.6** For a dg $E$-module $M$, the following conditions are equivalent:

1. $\text{Tor}^R(k,M)$ is finitely generated over $k$;
2. $\text{Ext}_A(tM,k)$ is finitely generated over $S$;
3. $\text{Ext}_A(k,tM)$ is finitely generated over $S$.

Moreover, when $H(M)$ is finite over $R$, the conditions above are equivalent to:

4. $M$ is in $\text{Db}(E/R)$.

**Proof** The equivalence of (1), (2), and (3) is from a special case of [25, Theorem 4.3.2], and the fact (1) and (4) are equivalent when $H(M)$ is finite over $R$, is classical; see, for example, [12, Corollary 1.3.2].

---

**6 Support for tensor products, II**

The notation in this section is as in the previous one. The result below is the theorem announced in the introduction.

**Theorem 6.1** Suppose $E$ is a Koszul complex over a local ring $(R, m, k)$ on a finite list of elements in $m$. For $M, N$ in $\text{Db}(E/R)$,

$$V_E(M \otimes^L_E N) = \text{Join}(V_E(M), V_E(N)).$$

**Proof** We need only pass to the exterior algebra:

$$V_E(M \otimes^L_E N) = V_A^0(t(M \otimes^L_E N)) = V_A^0(tM \otimes^L_A tN) = \text{Join}(V_A^0(tM), V_A^0(tN))$$

$$= \text{Join}(V_E(M), V_E(N)).$$

The first and fourth equalities hold by Lemma 5.3; the second one follows from the isomorphism

$$t(M \otimes^L_E N) \cong tM \otimes^L_A tN.$$ 

By Proposition 5.6, the dg $A$-modules $tM, tN$ are in $\text{Db}(A)$ and so the third equality holds by Proposition 4.4. 

**Remark 6.2** There is an alternative proof of Theorem 6.1, using the Hopf algebra structure on $\text{Ext}_E^*(k, k)$. The key point is that for any dg $E$-modules the maps

$$\text{Ext}_E^*(M, k) \otimes_k \text{Ext}_E^*(N, k) \to \text{Ext}_E^*(M \otimes^L_E N, k) \to \text{Ext}_E^*(M \otimes^L_E N, k)$$

are $\text{Ext}_E^*(k, k)$-linear. The second map is induced by multiplication, $k \otimes^L_E k \to k$. In (6.2.1), the graded Ext-module on the left is given an $\text{Ext}_E^*(k, k)$-module structure through the diagonal

$$\text{Ext}_E^*(k, k) \to \text{Ext}_E^*(k, k) \otimes_k \text{Ext}_E^*(k, k).$$
This is a straightforward calculation. However, this approach requires a bit of background on dg algebras with divided powers and suitably adapting classical material to this more general setting; cf. [21]. The main point is that $\text{Ext}_E^*(k, k)$ is generated, as a $k$-algebra, by primitives induced by derivations that respect divided powers on the minimal semifree resolution of $k$ over $E$.

One can identify $S$ as a Hopf subalgebra of $\text{Ext}_E^*(k, k)$ so the maps in (6.2.1) are also $S$-linear. When $M, N$ are in $\text{Db}(E/R)$, the $S$-modules $\text{Ext}_E^*(M, k), \text{Ext}_E^*(N, k)$ are finite over $S$, see Proposition 5.6, so the assertion of Theorem 6.1 follows directly from Lemma 1.4 once noting the composition in (6.2.1) is an isomorphism.

Consider the equivalence

$(-)^+: \text{Db}(E/R) \to \text{Db}(E/R)$

where $M^+: = \text{RHom}_E(M, E)$ for each $M$.

**Lemma 6.3** If $M$ is in $\text{Db}(E/R)$, then $V_E(M) = V_E(M^+)$. 

**Proof.** As $M$ is perfect over $R$ there is an isomorphism of dg $\Lambda$-modules

$$t(M^+) \simeq \text{RHom}_\Lambda(tM, \Lambda).$$

By [23, Theorem 4.1], $tM$ and $\text{RHom}_\Lambda(tM, \Lambda)$ generate the same thick subcategory in $\text{D}(\Lambda)$. Thus the second equality below holds:

$$V_E(M) = V^d_\Lambda(tM) = V^d_\Lambda(\text{RHom}_\Lambda(tM, \Lambda)) = V^d_\Lambda(t(M^+)) = V_E(M^+). \quad \square$$

**Corollary 6.4** If $R$ is Gorenstein and $\text{RHom}_E(M, N)$ belongs to $\text{Db}(E/R)$, then

$$V_E(\text{RHom}_E(M, N)) = \text{Join}(V_E(M), V_E(N)).$$

**Proof** As $R$ is Gorenstein and the $R$-modules $H(M), H(N)$ are finitely generated, there is an isomorphism

$$\text{RHom}_E(M, N)^+ \simeq M \otimes_E^L N^+. $$

Thus, the second equality below holds

$$V_E(\text{RHom}_E(M, N)) = V_E(\text{RHom}_E(M, N)^+)$$

$$= V_E(M \otimes_E^L N^+)$$

$$= \text{Join}(V_E(M), V_E(N^+))$$

$$= \text{Join}(V_E(M), V_E(N));$$

the first and fourth equalities are by Lemma 6.3 and the third is by Theorem 6.1. \quad \square

**Remark 6.5** In light of Theorem 6.1, it would be interesting to determine whether Corollary 6.4 holds without the assumption that $\text{RHom}_E(M, N)$ is in $\text{Db}(E/R)$.

The result below relates the cohomological support of $M \otimes_E^L N$ to those of its homology modules. Specializing to the case $R$ is regular and $f$ is an $R$-regular sequence, yields a positive answer to [15, Question 2]. The containment in the statement of the theorem can be strict; see [15, Example 5.3].
Theorem 6.6 Let $M, N$ be in $D_b(E/R)$ and suppose the $S$-modules $\text{Ext}_E^*(M, k)$ and $\text{Ext}_E^*(N, k)$ are generated in cohomological degrees at most $s$ and $t$, respectively. There is a containment of closed subsets

$$V_E(M \otimes E N) \subseteq \bigcup_{i \leq s+t} V_E(\text{Tor}_i^E(M, N)).$$

Proof By Proposition 4.4 and Lemma 5.3, one may identify $\text{Ext}_E^*(M \otimes E N, k)$ with $\text{Ext}_E^*(M, k) \otimes k \text{Ext}_E^*(N, k)$ viewed as a graded $S$-module via restriction along the diagonal map (1.1.1). Let $T$ denote its graded $S$-submodule of $\text{Ext}_E^*(M \otimes E N, k)$ generated by

$$\bigoplus_{i+j \leq u} \text{Ext}_E^i(M, k) \otimes k \text{Ext}_E^j(N, k),$$

where $u = s + t$. The $(S \otimes k S)$-module generated by $T$ is $\text{Ext}_E^*(M \otimes E N, k)$, so arguing as in the proof of Lemma 1.4, one gets an equality

$$V_E(M \otimes E N) = \text{Supp}_S T.$$ \hfill (6.6.1)

Fix a semifree resolution $F \xrightarrow{\sim} M \otimes E N$ over $E$, and let $F'$ be the soft truncation of $F$ in lower degrees at most $u$. Thus there is morphism of dg $E$-modules $\tau: F \rightarrow F'$ with the property that

$$\tau_{\leq u}: F_{\leq u} \rightarrow F'_{\leq u}$$

is the identity map. Hence,

$$\text{Ext}(\tau, k): \text{Ext}_E^*(F', k) \rightarrow \text{Ext}_E^*(M \otimes E N, k)$$

is an isomorphism in upper degrees at most $u$. In particular, under the identification discussed above

$$T \subseteq \text{Im} \left( \text{Ext}_E^*(F', k) \xrightarrow{\text{Ext}(\tau, k)} \text{Ext}_E^*(M \otimes E N, k) \right),$$

and hence, one has an inclusion

$$\text{Supp}_S T \subseteq \text{Supp}_S \text{Ext}_E^*(F', k).$$

Since $F'$ has bounded homology, it is in the thick subcategory of $D(E)$ generated by $H(F')$ regarded as dg $E$-module via the augmentation $E \rightarrow H_0(E)$. Thus,

$$\text{Supp}_S T \subseteq \text{Supp}_S \text{Ext}_E^i \left( \bigoplus_{i \leq u} \text{Tor}_i^E(M, N), k \right) = \bigcup_{i \leq u} V_E(\text{Tor}_i^E(M, N)),$$

where for the first containment, we are also using the equality

$$H(F') = \bigoplus_{i \leq u} \text{Tor}_i^E(M, N).$$

Combining this with (6.6.1) finishes the proof. \hfill \Box

Remark 6.7 Theorem 6.6 implies that when $R$ is regular and $M \otimes E N$ has finitely generated homology over $R$, the complexity of $M \otimes E N$, in the sense of [1, Sect. 3], is bounded above by the maximum of the complexities of $\text{Tor}_i^E(M, N)$ for $i \leq s + t$, where $s$ and $t$ are from Theorem 6.6.
Data availability
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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Received: 4 January 2022 Accepted: 14 March 2022 Published online: 15 April 2022

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