Extending wavelet filters. Infinite dimensions, the non-rational case, and indefinite-inner product spaces

Daniel Alpay, Palle Jorgensen and Izchak Lewkowicz

Abstract In this paper we are discussing various aspects of wavelet filters. While there are earlier studies of these filters as matrix valued functions in wavelets, in signal processing, and in systems, we here expand the framework. Motivated by applications, and by bringing to bear tools from reproducing kernel theory, we point out the role of non-positive definite Hermitian inner products (negative squares), for example Krein spaces, in the study of stability questions. We focus on the non-rational case, and establish new connections with the theory of generalized Schur functions and their associated reproducing kernel Pontryagin spaces, and the Cuntz relations.

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1 Introduction

Roughly speaking, systems whose inputs and outputs may be viewed as signals are called filters. Mathematically, filters are often presented as operator valued functions of a complex variable. In applications, filters are used in areas as (i) prediction, (ii) signal processing, (iii) systems theory and (iv) Lax-Phillips scattering theory [55].

Daniel Alpay
Department of Mathematics, Ben Gurion University of the Negev, P.O.B. 653, Be’er Sheva 84105, Israele-mail: dany@math.bgu.ac.il

Palle Jorgensen
Department of Mathematics, 14 MLH, The University of Iowa Iowa City, IA 52242-1419 USA e-mail: jorgen@math.uiowa.edu

Izchak Lewkowicz
Department of Electrical Engineering, Ben Gurion University of the Negev, P.O.B. 653, Be’er Sheva 84105, Israel e-mail: izchak@ee.bgu.ac.il
There, one is faced with spectral theoretic questions which can be formulated and answered with the use of a suitable choice of an operator valued function defined on a domain in complex plane; in the case of scattering theory, the scattering operator and the scattering matrix; in the other areas, the names used include polyphase matrix, see e.g., [43, 49]. We also mention that more recently, filters are used in (iv) multiresolution analysis in wavelets. We follow standard conventions regarding time-frequency duality, i.e., the correspondence between discrete time on one side and a complex frequency variable on the other. In the simplest cases, one passes from a time series to a generating function of a complex variable. These frequency response functions fall in various specific classes of functions of a complex variable; the particular function spaces in turn are dictated by applications. Again, motivated by applications, in our present study, we adopt a wider context for both sides of the duality divide. On the frequency side, we work with operator valued functions. This framework is relevant to a host of applications, and we believe of independent interest in operator theory. From the literature, we mention [57], [22] (see also [21]), and the papers referenced below.

We here consider the set of $\mathbb{C}^{N\times N}$-valued functions meromorphic in the open unit disk $\mathbb{D}$ and define two subsets of it: We shall denote by $\mathcal{G}_N$ the family satisfying the symmetry

$$W(\varepsilon_N z) = W(z)P_N,$$

where $\varepsilon_N = e^{i\frac{2\pi}{N}}$ and $P_N$ denotes the permutation matrix

$$P_N = \begin{pmatrix} 0_{1\times(N-1)} & 1 \\ I_{N-1} & 0_{(N-1)\times1} \end{pmatrix}.$$  \hspace{1cm} (2)

We shall also denote by $\mathcal{W}^{ln}$ the set of $\mathbb{C}^{N\times N}$-valued functions which take unitary values on the unit circle $\mathbb{T}$.

Classically wavelet filters, denoted by $\mathcal{W}_N$, are characterized by rational functions satisfying both symmetries, i.e.

$$\mathcal{W}_N = \mathcal{W}^{ln} \cap \mathcal{G}_N.$$  \hspace{1cm} (3)

In a previous paper, see [9], we have provided an easy-to-compute characterization of $\mathcal{W}_N$ as both a set of rational functions, and in terms of state space realization.

The aim of this work is to explore the possibility of extending the notion of wavelet filters, described in (3). The functions considered still satisfy the symmetry in (1), but:

- The functions are not necessarily rational or finite dimensional.
- The functions are not necessarily unitary on the unit circle $\mathbb{T}$.

1 Classically, in the engineering literature, the functions are analytic, or more generally meromorphic, outside the closed unit disk. The map $z \mapsto 1/z$ relates the two settings.

2 For rational functions, the term *para-unitary* is also used in the engineering literature.
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- The functions are meromorphic (rather than analytic) in \( \mathbb{D} \).

To explain our strategy, first recall the following: If \( W \) is a \( \mathbb{C}^{N \times N} \)-valued function which is rational and takes unitary values on the unit circle, the kernel

\[
K_W(z, w) = \frac{I_N - W(z)W(w)^*}{1 - zw^*}
\]

is positive definite in the open unit disk \( \mathbb{D} \) if \( W \) has no poles there, or more generally has a finite number of negative squares in \( \mathbb{D} \). See Definition 3.4 below for the latter. In our approach, unitarity on the unit circle is replaced by the requirement that \( W \) is a generalized Schur function, in the sense that \( W \) is meromorphic in \( \mathbb{D} \) and the associated kernel \( K_W(z, w) \) has a finite number of negative squares there. This family includes in particular the case of matrix-valued rational functions which take contractive values on the unit circle. We will also consider the case where the values on the unit circle are, when defined, contractive with respect to indefinite metrics. These kernels are of the form

\[
\frac{J_2 - W(z)J_1 W(w)^*}{1 - zw^*}
\]

when \( W \) is \( \mathbb{C}^{p_2 \times p_1} \)-valued and analytic in a neighborhood of the origin, and where \( J_1 \) and \( J_2 \) are signature matrices, respectively in \( \mathbb{C}^{p_1 \times p_1} \) and \( \mathbb{C}^{p_2 \times p_2} \), which have the same number of strictly negative eigenvalues:

\[
\nu_-(J_1) = \nu_-(J_2),
\]

and such that the kernel \( K_W \) has a finite number of negative squares. In [9] we studied the realization of wavelet filters in the \( \mathbb{C}^{N \times M} \)-valued (with \( M \geq N \)) rational case. The above approach allows us to extend these results to the case where the filter is not necessarily rational and \( M \) may be smaller than \( N \). Furthermore, the conditions in [9] of the function being analytic in the open unit disk, and taking coisometric values on the unit circle, are both relaxed (in particular, in the previous case, in (5), we had \( J_1 = I_M \) and \( J_2 = I_N \)).

The paper is organized as follows. Since we address different audiences, Sections 2, 3 and 4 are of a review nature. In Section 2 we give background on the use of filters in mathematics. We note that the more traditional framework in the literature has so far been unnecessarily restricted by two kinds of technical assumptions: (i) restricting to rational operator valued functions, and (ii) restricting the range of the operator valued functions considered. In Section 3 we address indefinite inner product spaces, and survey the theory of Pontryagin and Krein spaces. This overview allows us in Section 4 to describe a setting that expands both the above mentioned restrictions in (i) and (ii), namely the theory of generalized Schur functions. Our results in Sections 5 and 6 (Theorems 5.3, 5.4, and 6.4) deal with representations. We use these results in obtaining classifications, and decomposition theorems. In Section 7 we employ these theorems in the framework of wavelets.
2 Some background

2.1 Cuntz relations

The Cuntz relations were realized by J. Cuntz in \cite{24} as generators of a simple purely infinite $C^*$-algebra. Since then, they found many applications, and the related literature about Cuntz relations has flourished. Since Cuntz’s paper \cite{24}, the study of their representations has mushroomed, and now makes up a big literature, see for example \cite{19, 18, 20, 25, 13, 37, 39}, and some of their applications \cite{38, 40, 41, 42, 32}, for example to fractals \cite{31}.

In the initial framework, one is given a finite set $S_1, \ldots, S_N$ of isometries with orthogonal ranges adding up to the whole Hilbert space. Their representations play a role in a variety of applications, for example wavelets, and more generally multi-scale phenomena. The study of what are called non-type I $C^*$-algebras was initiated in the pioneering work of Glimm \cite{34, 35} and Dixmier \cite{28}. This in turn was motivated by use of direct integrals in representation theory, both in the context of groups and $C^*$-algebras. Direct integrals of representations are done practically with the use of Borel cross sections. Glimm proved that there are purely infinite $C^*$-algebras which do not admit Borel cross sections as a parameter space for the set of equivalence classes of irreducible representations; the Cuntz algebra(s) $O_N$ is the best known examples, \cite{24}. Nonetheless, it was proved in \cite{18} that there are families of equivalence classes of representations of $O_N$ indexed by wavelet filters, the latter in turn being indexed by infinite-dimensional groups.

One illustration of the need for expanding the framework of $O_N$ from Hilbert space to the case of Krein spaces is illustrated by applications to scattering theory for the automorphic wave equation \cite{54}. The initial study was restricted to the case when the operators $S_i$ act on Hilbert space, and when they act isometrically. However, since then, there has been a need for generalizing the Cuntz relations. It was noted in \cite{19} that the isometric case adapts well to the restricted framework of orthogonal wavelet families \cite{26}. Nonetheless, applications to engineering dictate much wider families, such as wavelet frames.

In this work we extend what is known in the literature in a number of different directions, including to the case of Pontryagin spaces. We obtain Cuntz relations for isometries between certain reproducing kernel Pontryagin spaces of analytic functions.

2.2 Wavelet filters

In electrical engineering terminology, systems whose inputs and outputs may be viewed as signals are called filters. By filter, we here mean functions $W(z)$ defined on the disk in the complex plane and taking operator values, i.e., linear operators
mapping between suitable spaces.

While filters (in the sense of systems and signal processing) have already been used with success in analysis of wavelets, so far some powerful tools from systems theory have not yet been brought to bear on wavelet filters. The traditional restriction placed on these functions \( W(z) \) is that they are rational, and take values in the unitary group when \( z \) is restricted to have modulus 1. In models from systems theory, the complex variable \( z \) plays the role of complex frequency. A reason for the recent success of wavelet algorithms is a coming together of tools from engineering and harmonic analysis. While wavelets now enter into a multitude of applications from analysis and probability, it was the incorporation of ideas from signal processing that offered new and easy-to-use algorithms, and hence wavelets are now used in both discrete problems, as well as in harmonic analysis decompositions. It is our purpose to use tools from systems theory in wavelet problems and also show how ideas from wavelet decompositions shed light on factorizations used by engineers.

Each of the various wavelet families demands a separate class of filters, for the case of compactly supported biorthogonal wavelets, see for example Resnikoff, Tian, Wells [60] and Sebert and Zou [63]. By now there is a substantial literature on the use of filters in wavelets (see e.g., [18, 26, 57, 39]). For filters in wavelets, there are two pioneering papers [50, 51], and the book [56].

In a previous work [9] we characterized all rational wavelet filters attaining unitary values on the unit circle. It turned out that this family is quite small (and in particular the subset of Finite Impulse Response filters, commonly used in engineering).

Thus, we here remove both restrictions on the filters, i.e., rational and unitary, and consider \( W(z) \) which are generalized Schur functions, and use reproducing kernel Pontryagin spaces associated with \( W \). See [6] for background.

We hope that this message will be useful to practitioners in their use of these rigorous mathematics tools.

### 3 Pontryagin spaces and Krein spaces

For a number of problems in the study of signals and filters (for example stability considerations), it is necessary to work with Hermitian inner products that are not positive definite. This view changes the Hermitian quadratic forms, allowing for negative squares, as well as the associated linear spaces. But more importantly, this wider setting also necessitates changes in the analysis, for example in the meaning of the notion of the adjoint operator, as well as the reproducing kernels. There are a number of subtle analytic points involved, as well as a new operator theory. We turn to these details below.
3.1 Krein spaces

A Krein space is a pair $(V, [\cdot, \cdot])$, where $V$ is a linear vector space on $\mathbb{C}$ endowed with an Hermitian form $[\cdot, \cdot]$, and with the following properties: $V$ can be written as $V = V_+ + V_-$, where:

1. $V_+$ endowed with the Hermitian form $[\cdot, \cdot]$ is a Hilbert space.
2. $V_-$ endowed with the Hermitian form $-\left[\cdot, \cdot\right]$ is a Hilbert space.
3. It holds that $V_+ \cap V_- = \{0\}$.
4. For all $v_\pm \in V_\pm$, 
   \[ [v_+, v_-] = 0. \]

The representation $V = V_+ + V_-$ is called a fundamental decomposition, and is highly non unique as soon as $\dim V_- > 0$. Given such a decomposition, the map

\[ \sigma(v_+ + v_-) = v_+ - v_- \]

is called a fundamental symmetry. Note that the space $V$ endowed with the Hermitian form (where $w = w_+ + w_-$ is also an element of $V$, with $w_\pm \in V_\pm$)

\[ \langle v, w \rangle = [v, \sigma w] = [v_+, w_+] - [v_-, w_-] \]

is a Hilbert space. These norms are called natural norms, and they are all equivalent. The Hilbert space topologies associated to any two such decompositions are equivalent, and $V$ is endowed with any of them; see [15, p. 102]. When $V_-$ is finite dimensional, $V$ is called a Pontryagin space and the dimension of $V_-$ is called the negative index (or the index for short) of the Pontryagin space. We refer to the books [12], [15], [36], [6] for more information on Krein and Pontryagin spaces. Note that in [36] it is the space $V_+$ rather than $V_-$ which is assumed finite dimensional in the definition of a Pontryagin space. Surveys may be found in for instance in [30], [29], [7]. It is interesting to note that Laurent Schwartz introduced independently the notion of Krein and Pontryagin spaces (he used the terminology Hermitian spaces for Krein and Pontryagin spaces) in his paper [62]. For applications of Krein spaces to the study of boundary conditions for hyperbolic PDE, including wave equations, and exterior domains, see for example [23, 52, 53, 58]. We now give two examples, which will be important in the sequel.

Example 3.1 Let $J \in \mathbb{C}^{p \times p}$ be an Hermitian involution, i.e.

\[ J = J^{-1} = J^*. \]

Such a matrix is called a signature matrix. We denote by $\mathbb{C}_J$ the space $\mathbb{C}^p$ endowed with the associated indefinite inner product

\[ [x, y]_J = y^* J x, \quad x, y \in \mathbb{C}^p. \]

It is a finite dimensional Pontryagin space.
Example 3.2 Let $J$ be a signature matrix. We consider the space $H_2(\mathbb{D})^p$ of functions analytic in $\mathbb{D}$ and with values in $\mathbb{C}^p$:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}^p,$$

such that

$$\sum_{n=0}^{\infty} a_n^* a_n < \infty.$$  

Then, $H_2(\mathbb{D})^p$ endowed with the Hermitian form

$$[f, g]_J = \sum_{n=0}^{\infty} b_n^* J a_n$$  

(is a Krein space, which we denote by $H_2,J(\mathbb{D})$).

In the above example, if $p = 1$ and $J = 1$ (as opposed to $J = -1$) the space $H_2,J(\mathbb{D})$ is equal to the classical Hardy space $H_2(\mathbb{D})$ of the unit disk.

3.2 Operators in Krein and Pontryagin spaces

When one considers a bounded operator $A$ between two Krein spaces $(\mathcal{K}_1, [\cdot, \cdot])_1$ and $(\mathcal{K}_2, [\cdot, \cdot])_2$ (in this paper, it will be most of the time between two Pontryagin spaces) the adjoint can be computed in two different ways, with respect to the Hilbert spaces inner products, (and then we use the notation $A^*$) and with respect to the Krein spaces inner products (and then we use the notation $A^{[\cdot]}$). More precisely, if $\sigma_1$ and $\sigma_2$ are fundamental symmetries in $\mathcal{K}_1$ and $\mathcal{K}_2$ which define the Hilbert spaces inner products

$$\langle f_1, g_1 \rangle_1 = [\sigma_1 f_1, g_1]_1 \quad \text{and} \quad \langle f_2, g_2 \rangle_2 = [\sigma_2 f_2, g_2]_2,$$

(with $f_1, g_1 \in \mathcal{K}_1$ and $f_2, g_2 \in \mathcal{K}_2$), we have for $f_1 \in \mathcal{K}_1$ and $f_2 \in \mathcal{K}_2$

$$[Af_1, f_2]_2 = \langle \sigma_2 A f_1, f_2 \rangle_2 = \langle f_1, A^* \sigma_2 f_2 \rangle_1 = [f_1, A^{[\cdot]} f_2]_1,$$

with

$$A^{[\cdot]} = \sigma_1 A^* \sigma_2.$$  

(6)

In the case of $\mathbb{C}_J$ (see Example 3.1) we have

$$A^{[\cdot]} = JA^* J.$$  

(7)

The operator $A$ from $\mathcal{D}(A) \subset \mathcal{K}_1$, where $(\mathcal{K}_1, [\cdot, \cdot])_1$ is a Krein space, into the Krein space $(\mathcal{K}_2, [\cdot, \cdot])_2$ is a contraction if
\[ [Ak_1, Ak_1]_2 \leq [k_1, k_1], \quad \forall k_1 \in \mathcal{D}(A). \]

A densely defined contraction, or even isometry, operator \( A \) between Krein spaces need not be continuous, let alone have a continuous extension. See for instance [29, Theorem 1.1.7]. In the case of Pontryagin spaces with same negative index, \( A \) has a continuous extension to all of \( \mathcal{K} \), see [6, Theorem 1.4.1, p. 27], and Theorem 3.3 below. Even when it is continuous and has a well-defined adjoint, this adjoint need not be a contraction. The operator is called a bicontraction if both it and its adjoint are contractions. When the Krein spaces are Pontryagin spaces with same negative index, a contraction is automatically continuous and its adjoint is also a contraction.

An important notion in the theory of Pontryagin spaces is that of relation. Given two Pontryagin spaces \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \), a relation is a linear subspace of \( \mathcal{P}_1 \times \mathcal{P}_2 \). For instance the graph of an operator is a relation. The domain of the relation \( \mathcal{R} \) is the set of \( f \in \mathcal{P}_1 \) such that there is a \( g \in \mathcal{P}_2 \) for which \( (f, g) \in \mathcal{R} \). A relation \( \mathcal{R} \) is called contractive if,

\[ [g, g]_2 \leq [f, f]_1 \quad \forall (f, g) \in \mathcal{R}. \]

A key result is the following theorem of Shmulyan (see [6, Theorem 1.4.1, p. 27]).

**Theorem 3.3** A densely defined contractive relation between Pontryagin spaces with same negative index extends to the graph of a uniquely defined contraction operator from \( \mathcal{P}_1 \) into \( \mathcal{P}_2 \).

### 3.3 Kernels

Recall that a (say, matrix-valued) function \( K(z, w) \) of two variables, defined for \( z \) and \( w \) in a set \( \Omega \) is called a positive definite kernel if it is Hermitian: \( K(z, w)^* = K(w, z) \) for all \( z, w \in \Omega \), and if for every choice of \( M \in \mathbb{N} \) and \( w_1, \ldots, w_M \in \Omega \) the \( M \times M \) Hermitian block matrix with \( (\ell, j) \) block entry \( K(w_\ell, w_j) \) is non negative. For instance, if \( b \) is a finite Blaschke product,

\[
b(z) = \prod_{n=1}^{m} \frac{z - a_n}{1 - z a_n^*}
\]

for some \( a_1, \ldots, a_m \) in the open unit disk, the kernel

\[
k_b(z, w) = \frac{1 - b(z)b(w)^*}{1 - zw^*}
\]

is positive definite, as can be seen from the formula

\[
k_b(z, w) = \langle k_b(\cdot, w), k_b(\cdot, z) \rangle_{H_2(\mathbb{D})}.
\]

When \( b \) is replaced with a function \( s \) analytic and contractive in the open unit disk, the corresponding kernel \( k_s(z, w) = \frac{1 - s(z)(w)^*}{1 - zw^*} \) is still positive definite in \( \mathbb{D} \), see [10].
This follows, for instance, from the fact that the operator of multiplication by \(s\) is a contraction from \(H^2(\mathbb{D})\) into itself. In the special case of a finite Blaschke product (or more generally, of an inner function), this multiplication operator is an isometry. This makes the underlying computations much easier. More generally, the kernels which appear in the following section can be seen as far reaching generalizations of the kernels \(k_p(z, w)\).

The notion of positive definite kernel has been extended by Krein as follows:

**Definition 3.4** Let \(\kappa \in \mathbb{N}_0\). A (say, matrix-valued) function \(K(z, w)\) defined on a set \(\Omega\) has \(\kappa\) negative squares if it is Hermitian, and if for every choice of \(M \in \mathbb{N}\) and \(w_1, \ldots, w_M \in \Omega\) the \(M \times M\) Hermitian block matrix with \((i, j)\) block entry \(K(w_i, w_j)\) has at most \(\kappa\) strictly negative eigenvalues, and exactly \(\kappa\) strictly negative eigenvalues for some choice of \(M, w_1, \ldots, w_M\). When \(\kappa = 0\), the function is positive definite.

The one-to-one correspondence between positive definite kernels and reproducing kernel Hilbert spaces was first extended to the indefinite case by L. Schwartz; see [62]. There is a one-to-one correspondence between reproducing kernel Pontryagin spaces and kernels with a finite number of negative squares. For completeness, we mention that such a result fails if the number of negative squares is not finite. A necessary and sufficient condition for a function to be the reproducing kernel of a Krein space is that this function is the difference of two positive kernels, but the associated Krein space need not be unique. Here too we refer to Schwartz [62], and also to the paper [1]. Realization of operator-valued analytic functions (without assumptions on an associated kernel, but with some symmetry hypothesis) have also been considered. See for instance [27]. The \(\mathbb{C}^{p \times p}\)-valued function \(K(z, w)\) defined for \(z, w\) in an open set \(\Omega\) of the complex plane will be called an analytic kernel if it is Hermitian and if it is analytic in \(z\) and \(w^\ast\). If it has moreover a finite number of negative squares, the elements of the associated reproducing kernel Pontryagin space are analytic in \(\Omega\). See [6, Theorem 1.1.2, p. 7].

There are two important classes of operators between reproducing kernel spaces, namely multiplication and composition operators. We conclude this section with three results on these operators.

**Theorem 3.5** Let \((\mathcal{H}_1, \langle \cdot, \cdot \rangle_1)\) and \((\mathcal{H}_2, \langle \cdot, \cdot \rangle_2)\) be two reproducing kernel Krein spaces of vector-valued functions, defined in \(\Omega\), and with reproducing kernels \(K_1(z, w)\) and \(K_2(z, w)\), respectively \(\mathbb{C}^{p_1 \times p_1}\) and \(\mathbb{C}^{p_2 \times p_2}\)-valued. Let \(m\) be a \(\mathbb{C}^{p_2 \times p_1}\)-valued function and let \(\varphi\) be a map from \(\Omega\) into itself. Assume that the map

\[
(T_{m, \varphi}f)(z) = m(z) f(\varphi(z))
\]

defines a bounded operator from \((\mathcal{H}_1, \langle \cdot, \cdot \rangle_1)\) into \((\mathcal{H}_2, \langle \cdot, \cdot \rangle_2)\). Then, for every \(z, w \in \Omega\), and \(\xi_2 \in \mathbb{C}^{p_2}\),

\[
\left( T_{m, \varphi}^\dagger K_2(\cdot, w) \xi_2 \right)(z) = K_1(z, \varphi(w)) m(w)^\ast \xi_2.
\]
Proof: Let $z,w \in \Omega$, $\xi_2 \in \mathbb{C}^{p_2}$ and $\xi_1 \in \mathbb{C}^{p_1}$. We have

$$\xi_1^* \left( T_{m,\varphi}^* K_2(\cdot, w) \xi_2 \right)(z) = [T_{m,\varphi}^* K_2(\cdot, w) \xi_2, K_1(\cdot, z) \xi_1]_1$$

$$= [K_2(\cdot, w) \xi_2, T_{m,\varphi} (K_1(\cdot, z) \xi_1)]_2$$

$$= [K_2(\cdot, w) \xi_2, m(\cdot) K_1(\varphi(\cdot), z) \xi_1]_2$$

$$= [m(\cdot) K_1(\varphi(\cdot), z) \xi_1, K_2(\cdot, w) \xi_2]_2$$

$$= (\xi_2^* m(w) K_1(\varphi(w), z) \xi_1)^*$$

$$= \xi_1^* K_1(z, \varphi(w)) m(w)^* \xi_2.$$  

\[\Box\]

As a corollary we have the following result:

**Theorem 3.6** Assume in the preceding theorem that $\mathcal{K}_1$ and $\mathcal{K}_2$ are Pontryagin spaces with same negative index. Then, $T_{m,\varphi}$ is a contraction if and only if the kernel

$$K_2(z,w) - m(z) K_1(\varphi(z), \varphi(w)) m(w)^*$$  \hspace{1cm} (10)

is positive definite in $\Omega$.

Proof: Assume that $T$ is a contraction. Then, its adjoint is also a contraction since the Pontryagin spaces have the same negative index. Let $g \in \mathcal{K}_2$ be of the form

$$g(z) = \sum_{k=1}^{N} K_2(z, w_k) \xi_k,$$

where $N \in \mathbb{N}$, $w_1, \ldots, w_N \in \omega$ and $\xi_1, \ldots, \xi_N \in \mathbb{C}^{p_2}$. By (9) we have

$$\sum_{\ell,k=1}^{N} \xi_{\ell}^* m(w_{\ell}) K_1(\varphi(w_{\ell}), \varphi(w_k)) m(w_k)^* \xi_{\ell} =$$

$$= \sum_{k=1}^{N} K_1(z, \varphi(w_k)) m(w_k)^* \xi_k, \sum_{\ell=1}^{N} K_1(z, \varphi(w_\ell)) m(w_\ell)^* \xi_\ell]_1$$

$$= [T_{m,\varphi}^* g, T_{m,\varphi}^* g]_1$$

$$\leq [g, g]_2$$

$$= \sum_{\ell,k=1}^{N} \xi_{\ell}^* K_2(w_\ell, w_k) \xi_{\ell},$$

and hence the kernel (10) is positive definite. Conversely, assume that the kernel (10) is positive definite. Then the linear span of the pairs of functions

$$(K_2(\cdot, w) \xi, K_1(\cdot, \varphi(w)) m(w)^* \xi), \ w \in \Omega, \ \xi \in \mathbb{C}^{p_2},$$
defines a linear densely defined contractive relation in $\mathcal{K}_1 \times \mathcal{K}_2$. By Shmulyan’s theorem (see Theorem 3.3), this relation has an everywhere defined extension which is the graph of a bounded contraction: There is a unique contraction $X$ from $\mathcal{K}_2$ into $\mathcal{K}_1$ such that

$$X(K_2(\cdot, w)\xi) = K_1(\cdot, \phi(w))m(w)^*\xi, \quad w \in \Omega, \ \xi \in \mathbb{C}^p.$$  

By (9), we have $X = T_{m, \phi}$, and this concludes the proof. □

We will consider in the sequel special cases of this result, in particular when

$$m(z) = (1 \ z \cdots z^{N-1}),$$  

see Theorem 5.3, or more generally when

$$m(z) = (m_0(z) m_1(z) \cdots m_{N-1}(z)),$$

see Theorem 5.4. The operator $T_{m, \phi}$ defined by (8) is then a block operator, and its components satisfy, under appropriate supplementary hypothesis, the Cuntz relations formally defined in (18)-(19) below.

We conclude this section with a result on composition operators in reproducing kernel Pontryagin spaces.

**Theorem 3.7** Let $K(z, w)$ be a $\mathbb{C}^p \times \mathbb{C}^p$-valued function which has $\kappa$ negative squares in the set $\Omega$. The associated reproducing kernel Pontryagin space will be denoted by $\mathcal{P}(K)$. Let $\phi$ be a map from $\Omega$ into itself, and assume that:

1. $f(\phi(z)) \equiv 0 \implies f \equiv 0$

for $f \in \mathcal{P}(K)$. Then:

(a) The function $K_\phi(z, w) = K(\phi(z), \phi(w))$ has at most $\kappa$ negative squares in $\Omega$ and its associated reproducing Pontryagin space is the set of functions of the form $F(z) = f(\phi(z))$, with $f \in \mathcal{P}(K)$ and Hermitian form

$$[F, G]_{\mathcal{P}(K_\phi)} = [f, g]_{\mathcal{P}(K)}.$$  

(11)

(b) The map $f \mapsto f(\phi)$ is unitary from $\mathcal{P}(K)$ into itself if and only if

$$K(z, w) = K(\phi(z), \phi(w)), \quad \forall z, w \in \Omega.$$  

(12)

**Proof:** Set

$$\mathcal{M}_\phi = \{f(\phi(z)), \ f \in \mathcal{P}(K)\}.$$  

By hypothesis, we have $f(\phi(z)) \equiv 0$ if and only if $f \equiv 0$, and so the Hermitian form (11) is well defined and induces a Pontryagin structure on $\mathcal{M}_\phi$. Furthermore, with $c \in \mathbb{C}^p$ and $F(z) = f(\phi(z)) \in \mathcal{M}_\phi$, we have:
\[ [F(\cdot), K_{\varphi}(\cdot, w)]_{\mathcal{R}(K)} = [f(\cdot), K(\cdot, \varphi(w))]_{\mathcal{R}(K)} \]
\[ = c^* f(\varphi(z)) \]
\[ = F(w), \]

and hence the reproducing kernel property is in force. To prove \((b)\) we use the uniqueness of the kernel for a given reproducing kernel Pontryagin space. \(\square\)

To fine-tune the previous result, note that for \(\varphi(z) = z^N\), the composition map is an isometry from \(\mathcal{H}_2(\mathbb{D})\) into itself, but is not unitary (unless \(N = 1\)). We also note that the preceding theorem holds also for reproducing kernel Krein spaces. Indeed, the correspondence between functions which are difference of positive functions on a given set and reproducing kernel Krein spaces is not one-to-one, but a given reproducing kernel Krein space has a unique reproducing kernel.

### 4 Generalized Schur functions and associated spaces

In this section we review the main aspects of the realization theory of generalized Schur functions and of their associated reproducing kernel Pontryagin spaces.

#### 4.1 Generalized Schur functions

In the positive definite case, this theory originates with the works of de Branges and Rovnyak, see [16, 17]. In earlier work on models involving operators in Hilbert space, and matrix factorization, de Branges spaces have served as a surprisingly powerful tool. The theory was developed in the indefinite case in a fundamental series of papers by Krein and Langer, see for instance [44, 45, 46, 47, 48], and, using reproducing kernel methods in [7] and in the book [6]. It was later used in [6, p. 119] and in the paper [3] to study generalized Schur functions with some given symmetry. In this paper we use this setting to present non rational and non unitary wavelet filters. In [6] the case of operator valued functions is studied, but we here consider the case of \(\mathbb{C}^{p \times p}\)-valued functions. We now recall the definition of a generalized Schur function. A (say \(\mathbb{C}^{p \times p}\)-valued) function \(W\) is called a Schur function if it is analytic and contractive in the open unit disk, or, equivalently, if the associated kernel

\[ K_W(z, w) = \frac{I_p - W(z)W(w)^*}{1 - zw^*} \]  

(13)

is positive definite in a neighborhood of the origin. Then, it has a unique analytic extension to the open unit disk, and this extension is such that the kernel \(K_W\) is still positive definite in \(\mathbb{D}\). There are two other kernels associated to \(W\), namely the kernel \(K_{\tilde{W}}(z, w)\) (with \(\tilde{W}(z) \overset{\text{def}}{=} W(z^*)^*\)), and the kernel
Extending wavelet filters

\[ D_W(z,w) = \left( \frac{K_W(z,w)}{W(z) - W(w^*) z - w^*} \frac{W(z) - W(w^*)}{z - w^*} K_W(z,w) \right). \]

These three kernels are simultaneously positive definite in the open unit disk. The first is the state space for a unique coisometric realization of \( W \), the second is the state space for a unique isometric realization of \( W \), and the reproducing kernel Hilbert space with reproducing kernel \( D_W \) is the state space for a unique unitary realization of \( W \). In these three cases, uniqueness is up to an invertible similarity operator.

Let \( J \in \mathbb{C}^{p \times p} \) be a signature matrix. We now consider functions with values in \( \mathbb{C}_J \) defined in Example 3.1, denoted by \( \Theta \) (rather than \( W \)). A \( \mathbb{C}^{p \times p} \)-valued functions \( \Theta \) analytic in a neighborhood of the origin is called \( J \)-contractive if the associated kernel

\[ K_\Theta(z,w) = \frac{J - \Theta(z)J\Theta(w)^*}{1 - zw^*} \]

is positive definite. It has a unique meromorphic extension to the open unit disk, and this extension is such that the kernel \( K_\Theta \) is still positive definite in the domain of analyticity of \( \Theta \) in \( \mathbb{D} \). Here too, besides the kernel \( K_\Theta \) we have the kernel \( D_\Theta(z,w) \) and the kernel

\[ D_\Theta(z,w) = \left( \frac{K_\Theta(z,w)}{\Theta(z)J - \Theta(w)^*} \frac{J\Theta(z) - J\Theta(w^*)}{z - w^*} K_\Theta(z,w) \right). \]

We note that the kernel \( K_\Theta \) can be written as

\[ K_\Theta(z,w) = \frac{I_p - \Theta(z)\Theta(w)^*}{1 - zw^*}, \]

where \( [\cdot] \) denotes the adjoint in \( \mathbb{C}_J \). This conforms with the way these kernels and the two other related kernels are written down in [6].

As we already mentioned, Krein and Langer developed in [44, 45, 46, 47, 48], the theory of operator-valued functions such that the corresponding kernels \( K_\Theta \) (with a signature operator rather than a signature matrix) has a finite number of negative squares in some open subset of the open unit disk. Then, \( \Theta \) has a unique meromorphic extension to the open unit disk, and this extension is such that \( K_\Theta \) has the same number of negative squares in \( \Omega(\Theta) \), the domain of analyticity \( \Theta \) in \( \mathbb{D} \). The three kernels have simultaneously the same number of negative squares, and as in the positive definite case, are respectively state spaces for coisometric, isometric and unitary realizations of \( \Theta \).

In the special case \( J = I \) (we return to the notation \( W \) rather than \( \Theta \) for the function), Krein and Langer proved, see [44], that \( W \) can be written as \( W_0B_0^{-1} \), where \( W_0 \) is analytic and contractive in the open unit disk, and where \( B_0 \) is a finite matrix-valued Blaschke product. It follows that \( W \) has a finite number of poles in the open unit
disk. In the rational case, and when $W$ takes unitary values on the unit circle, $W$ is a quotient of two matrix-valued rational Blaschke product. Note however that when $J$ has mixed inertia, $W$ may have an infinite number of poles, even when $\kappa = 0$. For example, take

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad W(z) = \begin{pmatrix} 1 & 0 \\ 0 & b(z)^{-1} \end{pmatrix},$$

where $b$ is a convergent Blaschke product with an infinite number of zeros. Such examples originate with the work of Potapov [59].

**Definition 4.1** We denote by $\mathcal{H}_\kappa^{p,p}(\mathbb{D})$ the family of $\mathbb{C}^{p \times p}$-valued functions $W$ meromorphic in the open unit disk, and such that the kernel $K_W$ (defined by (13)) has $\kappa$ negative squares in the domain of analyticity of $W$ in $\mathbb{D}$.

Given a signature matrix $J$, we denote by $\mathcal{H}_\kappa^J(\mathbb{D})$ the family of $\mathbb{C}^{p \times p}$-valued functions $\Theta$ meromorphic in the open unit disk, and such that the kernel $K_\Theta$ (defined by (14)) has $\kappa$ negative squares in the domain of analyticity of $\Theta$ in $\mathbb{D}$.

We denote by $\mathcal{P}(W)$ and $\mathcal{P}(\Theta)$ respectively the associated reproducing kernel Pontryagin spaces.

Since the kernels $K_W$ and $K_\Theta$ are analytic in $z$ and $w^*$, the elements of the associated reproducing kernel Pontryagin spaces are analytic in the domain of definition of $W$ or $\Theta$ respectively. See [6, Theorem 1.1.3, p. 7].

More generally, it is useful to consider non square generalized Schur functions. We consider $J_1 \in \mathbb{C}^{p_1 \times p_1}$ and $J_2 \in \mathbb{C}^{p_2 \times p_2}$ two signature matrices, of possibly different sizes, such that (5) is in form denoted by $\nu^-$:

$$\nu^- (J_1) = \nu^- (J_2).$$

Reproducing kernel Pontryagin spaces with reproducing kernel of the form (4):

$$\frac{J_2 - \Theta(z)J_1\Theta(w)^*}{1 - zw^*}$$

when $\Theta$ is $\mathbb{C}^{p_2 \times p_1}$-valued and analytic in a neighborhood of the origin, have been characterized in [6, Theorem 3.1.2, p. 85] (in fact, the result there is more general and considers operator-valued functions). In the statement below $R_0$ denotes the backward-shift operator

$$R_0f(z) = \frac{f(z) - f(0)}{z}.$$

**Theorem 4.2** Let $(\mathcal{P}, [\cdot, \cdot]_\mathcal{P})$ be a reproducing kernel Pontryagin space of $\mathbb{C}^{p_2}$-valued functions. It has a reproducing kernel of the form (4) if and only if it is invariant under the backward-shift operator $R_0$ and

$$[R_0f, R_0f]_\mathcal{P} \leq [f, f]_\mathcal{P} - (f(0))^*J_2f(0), \quad \forall f \in \mathcal{P}.$$
An example of such non square $\Theta$ appears in Section 6.2 below. See (37).

4.2 State spaces and realizations

We begin with recalling the following definition. Let $W$ be an operator-valued function analytic in a neighborhood of the origin. A realization of $W$ is an expression of the form

$$W(z) = D + zC(I-zA)^{-1}B,$$

(15)

where $D = W(0)$ and $A, B, C$ are operators between appropriate spaces. It is an important problem to connect the properties of $W$ and of the operator matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

(16)

When the values of $W$ are linear bounded operators between two Krein spaces, Azizov proved that a realization exists, and that $M$ can be chosen unitary. See [11], and see [27] for further discussion and additional references. When $W$ is a matrix-valued rational function without a pole at the origin, the spaces may be chosen finite dimensional, when no special structure is forced on the operator matrix $M.$

In Section 4.1, we have studied the correspondence between kernels and operator valued Schur functions. Here we then pass to the realizations of Schur functions. The introduction of Schur functions offers many advantages, relevant to algorithms and to computation. Case in point: In the next subsection, we give explicit formulas for realizations, i.e., for the computation of the four block operator entries $A$ through $D$ making up admissible realizations of a given Schur function, and therefore of a kernel. As we show, there are several such choices, the coisometric realization (Theorem 4.3), and the unitary realization of de Branges and Rovnyak (Theorem 4.4), among others. There is in turn a rich literature on Schur algorithms in various special cases, see for example [2] for an overview. In preparation of Section 4.3 we need some definitions. Let $\mathcal{P}$ denote the space where $A$ acts in (15). We say that the realization is closely inner connected if the span of the functions

$$(I-zA)^{-1}B\xi,$$

where $\xi$ runs through $\mathbb{C}^p$ (recall that $J \in \mathbb{C}^{p \times p}$) and $z$ runs through a neighborhood of the origin, is dense in $\mathcal{P}.$ With the same choices of $\xi$ and $z,$ it will be called closely outer connected if the span of the functions

$$(I-zA^s)^{-1}C^{s|}\xi$$

is dense in $\mathcal{P},$ and connected if the span of the functions
\((I - zA)^{-1}B_\xi\), and \((I - wA^{[*]})^{-1}C^{[*]}\eta\)

is dense in \(\mathcal{P}\) (\(\eta\) running through \(\mathbb{C}^p\) and \(w\) through the same neighborhood of the origin as \(z\)). Here the adjoints are between Pontryagin spaces. We note that the terminology is different from that of classical system theory. In the finite dimensional case, what is called here closely inner connected corresponds to observability, and what is called outer connected corresponds to controllability. The notion of being closely connected is specific to this domain, and is, in general, different from minimality.

### 4.3 Coisometric and unitary realizations

Let \(\Theta \in \mathcal{H}_K^J\) be a generalized Schur function, assumed analytic in a neighborhood of the origin. In this section we review how the spaces \(\mathcal{P}(\Theta)\) and \(\mathcal{D}(\Theta)\) are the state spaces for coisometric and unitary realizations respectively. For the following theorems, see [6, Theorem 2.2.1, p. 49] and [6, Theorem 2.1.3] respectively. In Theorems 4.3 and 4.4 below the notions of coisometry and unitarity means that \(M\) in (16) is an operator coisometric (resp. unitary) from the Pontryagin \(\mathcal{P}(\Theta) \oplus \mathbb{C}^J\) into itself (resp. from \(\mathcal{D}(\Theta) \oplus \mathbb{C}^J\) into itself).

**Theorem 4.3** Let \(J \in \mathbb{C}^{p \times p}\) be a signature matrix, and \(\Theta \in \mathcal{H}_K^J\) be analytic in a neighborhood of the origin. Then the formulas

\[
Af(z) = \frac{f(z) - f(0)}{z},
\]

\[(B_\xi)(z) = \frac{\Theta(z) - \Theta(0)}{z} \xi,
\]

\[Cf = f(0),\]

\[D\xi = \Theta(0)\xi,
\]

with \(f \in \mathcal{P}(\Theta)\) and \(\xi \in \mathbb{C}^p\), define a closely outer connected realization of \(\Theta\) which is coisometric. This realization is unique up to a continuous and continuously invertible similarity operator.

This coisometric realization was introduced by L. de Branges and J. Rovnyak in [16] for scalar Schur functions, and extended to the operator-valued case in [17]. We note that the coisometric realization is also known as the backward shift realization; see e.g. [33].

L. de Branges and J. Rovnyak also formulated the unitary realization below.

**Theorem 4.4** Let \(J \in \mathbb{C}^{p \times p}\) be a signature matrix, and \(\Theta \in \mathcal{H}_K^J\) be analytic in a neighborhood of the origin. The formulas
A \left( \frac{f}{g} \right) = \begin{pmatrix} f(z) - f(0) \\ zg(z) - \tilde{\Theta}(z) J f(0) \end{pmatrix}, \\
(B\xi)(z) = \begin{pmatrix} \Theta(z) - \Theta(0) \\ (J - \tilde{\Theta}(z) J \tilde{\Theta}(0)^*) \xi \end{pmatrix}, \\
C \left( \frac{f}{g} \right) = f(0), \\
D\xi = \Theta(0) \xi,

with f \in \mathcal{D}(\Theta) and \xi \in \mathbb{C}^p, define a closely connected realization of \Theta which is unitary. This realization is unique up to a continuous and continuously invertible similarity operator.

It is important to note that, in some cases, all three realizations are unitary. This is in particular the case when \Theta is rational and J-unitary on the unit circle. See Section 4.4.

4.4 Finite dimensional de Branges spaces

The finite dimensional case is of special importance, and the case J = I was considered in details in our previous work [9]. Then the three realizations are unitary, and it is easier to focus on the \mathcal{P}(\Theta) spaces. As proved in [7] Corollary p. 111 for the case J = I and in [7] Theorem 5.5, p. 112 for the general case, given \Theta \in \mathcal{S}_J^\kappa, the associated space \mathcal{P}(\Theta) is finite dimensional if and only if \Theta is rational and J unitary on the unit circle:

\Theta(e^{it})^* J \Theta(e^{it}) = J,

at all points e^{it} (t \in [0, 2\pi]) where it is defined. If moreover \Theta is analytic in a neighborhood of the closed unit disk, we have

\mathcal{P}(\Theta) = H_2 \odot \Theta H_2.

Rationality is not enough to insure that \mathcal{P}(\Theta) is finite dimensional, as illustrated by the case J = 1 and \Theta = 0. Then, \mathcal{P}(\Theta) = H_2(\mathbb{D}).

Definition 4.5 We will denote by \mathcal{U}_J^\kappa the multiplicative group of rational \mathbb{C}^{p \times p}-valued functions \Theta which take J-unitary values on the unit circle, and for which the corresponding kernel K_\Theta has \kappa negative squares. We set

\mathcal{U}_J = \bigcup_{\kappa=0}^\infty \mathcal{U}_J^\kappa.
The results and realizations presented in the previous section take now an easier form. The various operators can be seen as matrices. Unitarity above is with respect to the indefinite metric of $\mathcal{S}(\Theta) \oplus \mathbb{C}_J$, and we can rephrase Theorem 4.4 as follows:

**Theorem 4.6** Let $W$ be a rational $\mathbb{C}^{p \times p}$-valued function analytic at the origin, and let

$$W(z) = D + zC(I-zA)^{-1}B$$

be a minimal realization of $W$. Then, $W$ is $J$-unitary on the unit circle if and only if there exists an invertible Hermitian matrix $H$ (which is uniquely determined from the given realization) such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{pmatrix} H & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & J \end{pmatrix}. \tag{17}$$

The change of variable $z \mapsto 1/z$ yields:

**Theorem 4.7** Let $W$ be analytic at infinity, and let

$$W(z) = D + C(zI-A)^{-1}B.$$ 

be a minimal realization of $W$. Then, $W$ is $J$-unitary on the unit circle if and only if there exists an invertible Hermitian matrix $H$ (which is uniquely determined from the given realization) and such that (17) holds.

The matrix $H$ is called the associated Hermitian matrix (to the given minimal realization). This result was proved in [8, Theorem 3.10] for the case where $A$ is non-singular. For the approach using reproducing kernel Hilbert spaces, see [6, 7, 4].

5 Cuntz relations

5.1 Cuntz relations and the de Branges-Rovnyak spaces

The results of this section are related to [22] and [10]. In that last paper, the functions $1, \ldots, z^{N-1}$ below are replaced by the span of a finite dimensional backward-shift invariant subspace, but the discussion is restricted to the Hilbert space case and scalar-valued functions.

Normally by Cuntz relations we refer to a finite system of isometries $S_1, \ldots, S_N$ in a Hilbert space $\mathcal{H}$ satisfying two conditions:

(a) Different isometries in the system must have orthogonal ranges,

$$S_j^*S_k = 0, \quad j \neq k, \tag{18}$$
and

(b) The sum of the ranges equals $H$:

$$\sum_{j=1}^{N} S_j S_j^* = I_H.$$

(19)

Note that (a) already forces $H$ to be infinite dimensional. Indeed, if $H$ is finite dimensional, an isometry is unitary and the orthogonality of the ranges is not possible, see the discussion below and Section 5.3. If we allow the isometries to operate between two finite dimensional spaces of different dimensions, then one can find isometries which satisfy the Cuntz relations. It is the set of three conditions: Each $S_j$ is isometric in a Hilbert space $H$, and (a) and (b), together imply that every realization is a representation of a simple, purely infinite $C^*$-algebra, called $O_N$. In applications to filters, the $N$ individual subspaces represent frequency bands. This allows for versatile computational algorithms tailored to multiscale problems such as wavelet decompositions, and analysis on fractals. In our present paper, we relax some of the original very restrictive axioms, while maintaining the computational favorable properties. Our more general framework still allows for algorithms based on iteration of the operator family $S_1, \ldots, S_N$ in a particular representation.

If one allows isometries between two Hilbert spaces, then the finite dimensional case may occur, as illustrated by the following example:

$$H_1 = \mathbb{C}, \quad H_2 = \mathbb{C}^2,$$

and

$$S_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We have

$$S_1^* S_1 = S_2^* S_2 = 1, \quad S_1^* S_2 = S_2^* S_1 = 0,$$

and

$$S_1 S_1^* + S_2 S_2^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We go beyond the setting of Hilbert space, and relax the conditions (a) and (b) imposed in the original framework from $C^*$-algebra theory, allowing here isometric operators between two Pontryagin spaces. We still preserve the features of the representations of use in iterative algorithms.

It is not surprising that in Section 5.3 we have finite dimensional spaces. Now for the generalized theory, we must allow for de Branges and Rovnyak spaces, and for negative squares and signature matrix. The resulting modifications in the form of the Cuntz relations, in the case of Hilbert space, entails some non-trivial modifications addressed in the next two sections. Our main results for this are proved in the present section, and in Section 5.3 for the finite dimensional case.
The main result of this section is that one can associate in a natural way to an element \( \Theta \in \mathcal{S}_J^k(\mathbb{D}) \) a family of operators which satisfy the Cuntz relations. We begin with a preliminary result, which is a corollary of Theorem 3.7 with \( \varphi(z) = z^N \).

**Proposition 5.1** Let \( \Theta \in \mathcal{S}_J^k(\mathbb{D}) \), and let \( \mathcal{P}(\Theta) \) be the associated Pontryagin space, with reproducing kernel

\[
K_{\Theta}(z,w) = \frac{J - \Theta(z)J \Theta(w)^*}{1 - zw^*}.
\]

The function

\[
K_{\Theta}(z^N,w^N) = \frac{J - \Theta(z^N)J \Theta(w^N)^*}{1 - z^N w^N}
\]

has also \( \kappa \) negative squares in its domain of definition in \( \mathbb{D} \). The associated reproducing kernel Pontryagin space \( \mathcal{M}_N \) is equal to the space of functions of the form \( F(z) = f(z^N) \), where \( f \in \mathcal{P}(\Theta) \), with the following indefinite inner product

\[
[F,G]_{\mathcal{M}_N} = [f,g]_{\mathcal{P}(\Theta)},
\]

where \( g \in \mathcal{P}(\Theta) \) and \( G(z) = g(z^N) \).

We have:

**Theorem 5.2** Let \( \Theta \in \mathcal{S}_J^k(\mathbb{D}) \), and let \( \mathcal{P}(\Theta) \) be the associated Pontryagin space with reproducing kernel

\[
K_{\Theta}(z,w) = \frac{J - \Theta(z)J \Theta(w)^*}{1 - zw^*}.
\]

Then, for \( N \in \mathbb{N} \), the function \( \Theta_N \) defined by \( \Theta_N(z) = \Theta(z^N) \) belongs to \( \mathcal{S}_{N\kappa}^J \). Furthermore, \( \mathcal{P}(\Theta_N) \) consists of all the functions of the form

\[
f(z) = \sum_{j=0}^{N-1} z^j f_j(z^N), \quad f_j \in \mathcal{P}(\Theta).
\]

Any such representation is unique, and the inner product in \( \mathcal{P}(\Theta_N) \) is given by

\[
[f,g]_{\mathcal{P}(\Theta_N)} = \sum_{j=0}^{N-1} [f_j,g_j]_{\mathcal{P}(\Theta)},
\]

where \( g(z) = \sum_{j=0}^{N-1} z^j g_j(z^N) \) for some \( g_0, \ldots, g_{N-1} \in \mathcal{P}(\Theta) \).

**Proof:** We proceed in a number of steps.

**STEP 1:** It holds that \( \nu_-(\Theta_N) \leq N \cdot \kappa \).

Indeed,
\[ \frac{J - \Theta(z^N)J\Theta(w^N)^*}{1 - zw^*} = \frac{J - \Theta(z^N)J\Theta(w^N)^*}{1 - z^Nw^*} \cdot \frac{1 - z^Nw^*}{1 - zw^*} = \frac{J - \Theta(z^N)J\Theta(w^N)^*}{1 - z^Nw^*} \cdot \sum_{k=0}^{N-1} z^k w^k. \]

This expresses the kernel \( K_{\Theta N} \) as the sum of \( N \) kernels, each with \( \kappa \) negative squares.

Thus, \( v_-(\Theta_N) \leq N\kappa \). To show that there is equality, we need to show that the associated spaces have pairwise intersections which all reduce to the zero function.

### STEP 2

Let \( k, \ell \in \{0, \ldots, N-1\} \), such that \( k \neq \ell \). Then, with \( \mathcal{M}_N \) as in the previous theorem:

\[ \mathcal{M}_N \cap \mathcal{M}_\ell = \{0\}. \]

Indeed, assume that \( k > \ell \) and let \( f, g \in \mathcal{M}_N \) be such that

\[ z^k f(z^N) = z^\ell g(z^N). \]

Then, \( f \) and \( g \) will simultaneously be identically equal to \( 0_{p \times 1} \). Assume \( f \neq 0_{p \times 1} \). One of its components, say the first, with \( f = (x_1(z) \cdots x_p(z))^t \) is not identically equal to zero (\( p \) is the size of the signature matrix \( J \)). Then we obtain

\[ z^{k-\ell} = \frac{y_1(z^N)}{x_1(z^N)}, \]

where \( y_1 \) denotes the first component of \( g \). Since \( f \) and \( g \) are meromorphic in \( D \), the function \( y_1/x_1 \) has a Laurent expansion at the origin. Moreover the Laurent expansion of \( \frac{y_1}{x_1}(z^N) \) contains only powers which are multiple of \( N \). By the uniqueness of the Laurent expansion, this contradicts the fact that it is equal to \( z^{k-\ell} \), with \( |k-\ell| < N \).

### STEP 3

It holds that

\[ \mathcal{P}(\Theta_N) = \bigoplus_{j=0}^{N-1} z^j \mathcal{M}_N, \]

and it holds that \( v_{\Theta N} = N\kappa \).

This is because the spaces \( z^j \mathcal{M}_N \) have pairwise intersections which reduce to the zero functions in view of STEP 2. \( \Box \)

### Theorem 5.3

In the notation above, set

\[ (S_j f)(z) = z^j f(z^N) \quad \mathcal{P}(\Theta) \longrightarrow \mathcal{P}(\Theta_N). \]

Then,

\[ S_j^{|\tau|} f = f_j \quad \mathcal{P}(\Theta_N) \longrightarrow \mathcal{P}(\Theta), \quad (21) \]

and
\[ S_j^*[\ast] S_k = \delta_{j,k} I_{\mathcal{P}(\Theta)} \]
\[ \sum_{j=0}^{N-1} S_j S_j^*[\ast] = I_{\mathcal{P}(\Theta_N)}, \quad (22) \]

where the \([\ast]\) denotes adjoint between Pontryagin spaces.

**Proof:** We proceed in a number of steps.

**STEP 1:** The operators \( S_j \) are continuous.

The operators \( S_j \) are between Pontryagin spaces of different indices, and some care is required to check continuity. To this end, fix \( j \in \{0, \ldots, N-1\} \) and note that \( S_j \) is everywhere defined. Furthermore we claim that it is a closed operator. Indeed, let \( f_1, f_2, \ldots \) be a sequence of elements in \( \mathcal{P}(\Theta_N) \) converging strongly to \( f \in \mathcal{P}(\Theta_N) \) and such that the sequence \( S_j f_1, S_j f_2, \ldots \) converges strongly to \( g \in \mathcal{P}(\Theta_N) \). Strong convergence in a Pontryagin space implies weak convergence, and in a reproducing kernel Pontryagin space, weak convergence implies pointwise convergence. Therefore, for every \( w \) where \( \Theta \) is defined,
\[ \lim_{k \to \infty} f_k(w) = f(w), \]
and
\[ \lim_{k \to \infty} (S_j f_k)(w) = g(w). \]

Since \( (S_j f_k)(w) = w^j f_k(w) \), and thus \( g(w) = w^j f(w) \). Therefore \( g = S_j f \), and the operator \( S_j \) is closed, and hence continuous.

**STEP 2:** \( (21) \) is in force.

Let \( g(z) = \sum_{k=0}^{N-1} z^k g_k(z^N) \in \mathcal{P}(\Theta_N) \) where the \( g_k \in \mathcal{P}(\Theta_N) \), and let \( u \in \mathcal{P}(\Theta) \). Then,
\[ [S_j u, g]_{\mathcal{P}(\Theta_N)} = [z^j u(z^N), \sum_{k=0}^{N-1} z^k g_k(z^N)]_{\mathcal{P}(\Theta_N)} \]
\[ = [u, g]_{\mathcal{P}(\Theta)} \]
\[ = [u, S_j^*[\ast] g]_{\mathcal{P}(\Theta)}, \]

where \([,]_{\mathcal{P}(\Theta)}\) and \([,]_{\mathcal{P}(\Theta_N)}\) denote the indefinite inner products in the corresponding spaces. Hence, we have \( S_j^*[\ast] g = g_j \).

**STEP 3:** The Cuntz relations hold.

From \( (21) \) we have for \( u \in \mathcal{P}(\Theta) \)
Extending wavelet filters

\[ S_j^{[s]}S_ku = S_j^{[s]}(z^k u(z^N)) = \begin{cases} 0 & \text{if } j \neq k, \\ u & \text{if } j = k. \end{cases} \]

Furthermore, for \( f(z) = \sum_{j=0}^{N-1} z^j f_j(z^N) \in \mathcal{P}(\Theta_N) \) (where the \( f_j \in \mathcal{P}(\Theta) \)), we have

\[ S_kS_k^{[s]}f = S_k(f_k) = z^k f_k(z^N), \]

and thus

\[ \sum_{k=0}^{N-1} S_kS_k^{[s]} = I_{\mathcal{P}(\Theta_N)}. \]

\[ \square \]

We note that, with

\[ S = (S_0 \, S_1 \cdots S_{N-1}) \quad \mathcal{P}(\Theta)^N \rightarrow \mathcal{P}(\Theta_N), \]

the Cuntz relations (21) can be rewritten as

\[ SS^{[s]} = I_{\mathcal{P}(\Theta_N)} \quad \text{and} \quad S^{[s]}S = I_{\mathcal{P}(\Theta)^N}. \]

At this stage, let us introduce some more notation. We set

\[ \Theta_N^k(z) = \Theta(z^N), \]

and \( S_j^{(i)} = S_j \) for \( i = 0, \ldots, N - 1 \). We can reiterate the preceding analysis with \( \Theta_N \) instead of \( \Theta \). We then obtain \( N \) isometries \( S_0^{(1)}, \ldots, S_{N-1}^{(1)} \) from \( \mathcal{P}(\Theta_N) \) into \( \mathcal{P}(\Theta_N^2) \) satisfying the Cuntz relations. Iterating \( k \) times, one obtains \( k \) sets of isometries,

\[ S_0^{(j-1)}, \ldots, S_{N-1}^{(j-1)}, \quad j = 1, \ldots, k, \]

from \( \mathcal{P}(\Theta_{N^j}) \) into \( \mathcal{P}(\Theta_{N}) \), which also satisfy the Cuntz relations. This gives us \( N^k \) isometries

\[ S_1^{(0)}S_2^{(1)} \cdots S_k^{(k-1)}, \]

with \( (i_1, i_2, \ldots, i_k) \in \{0, \ldots, N - 1\}^k \), from \( \mathcal{P}(\Theta) \) into \( \mathcal{P}(\Theta_{N^k}) \), all satisfying the Cuntz relations.

### 5.2 Cuntz relation: The general case

We now wish to extend the results of Section [5.1] and in particular Theorem [5.2] to the case where the \( N \) functions \( 1, z, \ldots, z^{N-1} \) are replaced by prescribed functions \( m_0(z), m_1(z), \ldots, m_{N-1}(z) \), whose finite dimensional linear span we denote by \( \mathcal{L} \), and the kernel \( K_\Theta(z, w) \) is replaced by a given analytic \( \mathbb{C}^{N \times N} \)-valued kernel.
$K(z,w)$ and the kernel $K_{\theta_0}(z,w)$ is replaced by a kernel $\tilde{K}(z,w)$. Let as in Section 5.1 $K_N(z,w) = K(z^N,w^N)$. We address the following problem: Given $K$ and $\tilde{K}$ two Hermitian kernels defined on a set $\Omega$, and with a finite number of negative squares there, when can one find decompositions of the form

$$f(z) = \sum_{n=0}^{N-1} m_n(z) g_n(z^N). \quad (23)$$

where the functions $g_0, \ldots, g_{N-1}$ belong to $\mathcal{P}(K)$ for some, or all, elements in $\mathcal{P}(\tilde{K})$. We have:

**Theorem 5.4** Let $K(z,w)$ and $\tilde{K}(z,w)$ be two kernels defined on a set $\Omega$, and assume that $\nu_- (\tilde{K}) = N \nu_- (K)$. \quad (24)

Let $m_0, \ldots, m_{N-1}$ be $N$ functions on $\Omega$. Assume that the kernel

$$\tilde{K}(z,w) - \left( \sum_{n=0}^{N-1} m_n(z)m_n(w)^* \right) K(z,w)$$

is positive definite in $\Omega$. Then, with $\varphi(z) = z^N$, the choice $g_n = T_{m_n, \varphi} f_n$, $n = 0, \ldots, N-1$ solves (23).

**Proof:** We use Theorem 3.6 with $K_1(z,w) = \tilde{K}(z,w)$ and

$$K_1(z,w) = \begin{pmatrix}
K(z,w) & 0 & 0 & \cdots & 0 \\
0 & K(z,w) & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & K(z,w)
\end{pmatrix}.$$  

Then

$$\left( \sum_{n=0}^{N-1} m_n(z)m_n(w)^* \right) K(z,w) = m(z) K_1(z) m(w)^*,$$

and Theorem 3.6 with $K_1$ and $K_2$ as above, and

$$m(z) = (m_0(z) \ m_1(z) \ \cdots \ m_{N-1}(z)), \quad \text{and} \quad \varphi(z) = z^N,$$

leads to the fact that the map

$$f \mapsto m(z) f(z^N)$$

is a contraction from $(\mathcal{P}(K))^N$ into $\mathcal{P}(\tilde{K})$. $\square$

In applications, one uses the kernel $\tilde{K}(z,w) = K_N(z,w)$ in the above result.

**Proposition 5.5** A sufficient condition for (24) to hold is that

$$m_j \mathcal{P}(K) \cap m_k \mathcal{P}(K) = \{0\}, \quad (25)$$

where $m_j$ and $m_k$ are linear functionals on $\mathcal{P}(K)$.
for all $j, k \in \{0, \ldots, N - 1\}$ such that $j \neq k$.

**Proof:** Indeed, when this condition is in force, we have that the Pontryagin space with reproducing kernel $m(z)K_1(z,w)m(w)^*$ is the direct sum of the Pontryagin spaces with reproducing kernels $m_j(z)K(z,w)m_j(w)^*$, $j = 0, \ldots, N - 1$. \(\square\)

We note that there are similarity between (23) and the solution of Gleason’s problem: Gleason’s problem is the following: Given a linear space of functions $\mathcal{M}$ of functions analytic in a set $\Omega \subset \mathbb{C}^N$, and given $a \in \Omega$, Gleason’s problem is the following: when can we find functions $g_1, \ldots, g_N \in \mathcal{M}$ (which depend on $a$) such that

$$f(z) - f(a) = \sum_{n=1}^{N} (z_n - a_n)g_n(z, a)$$

### 5.3 Cuntz relations: Realizations in the rational case

Recall that for a given generalized Schur function $\Theta$, we presented in Theorems 4.3 and 4.4 coisometric and unitary realizations respectively. The unitary realization turns to be more involved than the coisometric backwards shift realization. In some cases, these two realizations are unitarily equivalent, in particular when $\Theta$ is rational and $J$-unitary on the unit circle. As we already discussed in Section 4.4, this is equivalent to having the space $\mathcal{P}(\Theta)$ finite dimensional. In this section we adopt this simplifying assumption and study the realization of $\Theta_N(z) = \Theta(z^N)$ in terms of the realization of $\Theta$.

We take the signature matrix $J$ to belong to $\mathbb{C}^{L \times L}$. We know (see [7, 6] and Theorem 4.3 above) that

$$\Theta(z^N) = \mathcal{D} + z \mathcal{C}(I - z \mathcal{A})^{-1} \mathcal{B}$$

where $\mathcal{D} = \Theta_N(0) = \Theta(0)$ and where $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are defined as follows: $\mathcal{C}$ is the evaluation at the origin, $\mathcal{C}f = f(0)$.

$\mathcal{B}$ is defined by

$$\mathcal{B} \xi = \frac{\Theta_N(z) - \Theta_N(0)}{z} \xi, \quad \xi \in \mathbb{C}^L,$$

and $\mathcal{A}$ is the backward shift in $\mathcal{P}(\Theta_N)$. The matrix, see [7],

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$$

is unitary in the $\mathcal{P}(\Theta_N)$ metric. We know from Theorem 5.2 that $\mathcal{P}(\Theta_N)$ is equal to the space of functions of the form
where the $f_k \in \mathcal{P}(\Theta)$ are uniquely defined. We will denote by $U$ the map

$$f \mapsto \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{pmatrix}$$

from $\mathcal{P}(\Theta_N)$ onto $\mathcal{P}(\Theta)^N$. In view of (22), $U$ is a unitary map (between Pontryagin spaces).

Let $T$ denote the following map from $\mathcal{P}(\Theta)^N$ into itself defined by

$$TU \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{pmatrix} = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ R_0 f_0 \end{pmatrix}.$$

**Proposition 5.6** Let $f \in \mathcal{P}(\Theta_N)$, with representation (26). It holds that

$$U \mathcal{A} f = (TU f)(z^N),$$

and it holds that

$$\langle \mathcal{A} f, \mathcal{A} g \rangle_{\mathcal{P}(\Theta_N)} = \langle TU f, TU g \rangle_{\mathcal{P}(\Theta)^N}.$$  \hfill (28)

**Proof:** Indeed, with $f$ is of the form (26), we have

$$\mathcal{A} f(z) = R_0 f(z) = \frac{f(z) - f(0)}{z} = \sum_{k=1}^{N-1} z^{k-1} f_k(z^N) + z^{N-1} \frac{f_0(z^N) - f_0(0)}{z},$$

so that $U \mathcal{A} U^* f$ is equal to

$$\begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{pmatrix} \mapsto \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ R_0 f_0 \end{pmatrix},$$

that is (27) in force. Finally (28) follows from the formula for the inner product in $\mathcal{P}(\Theta_N)$. \hfill \Box
Proposition 5.7  Let $f \in \mathcal{P}(\Theta_N)$ with representation (26). Then,

$$\mathcal{E} f = C \begin{pmatrix} I & 0 & \cdots & 0 \end{pmatrix} U f,$$

(29)

where $C$ is the evaluation at the origin in $\mathcal{P}(\Theta)$.

Proof: This is clear from

$$\mathcal{E} f = f_0(0) = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ \vdots \\ f_{N-1} \end{pmatrix}.$$ 

□

Proposition 5.8  We have

$$\mathcal{B} \xi = z^{N-1}(B \xi)(z^N)$$

where $B$ is the operator from $\mathbb{C}^L$ into $\mathcal{P}(\Theta)$:

$$B \xi = R_0 \Theta \xi$$

and we have

$$\langle \mathcal{B} \xi, \mathcal{B} \eta \rangle_{\mathcal{P}(\Theta_N)} = \langle B \xi, B \eta \rangle_{\mathcal{P}(\Theta)}, \quad \eta, \xi \in \mathbb{C}^L.$$ 

(30)

Proof: We have

$$\mathcal{B} \xi(z) = R_0 \Theta_N \xi(z) = \frac{\Theta(z^N) - \Theta(0)}{z} = z^{N-1}(B \xi)(z^N)$$

Equality (30) follows from the definition of the inner product in $\mathcal{P}(\Theta_N)$. □

These various formulas allow to show directly that the realization is indeed unitary, and to compute the associated Hermitian matrix in the finite dimensional case.

6 Decompositions

6.1 Generalized down-sampling and an Hermitian form

In the preceding section we considered decompositions of a function in the form (23). Here we consider different kind of decompositions. We consider matrices $P \in \mathbb{C}^{N \times N}$ satisfying
\[
\det(I_N - \epsilon_N P^\ell) \neq 0, \quad \ell = 1, \ldots, N-1, \quad \text{and} \quad P^N = I_N.
\] (31)

We do not assume that \(P^{N-1} \neq I_N\), and in particular the choice \(P = I_N\) is allowed. The special case \(P = \epsilon_N P_N\) plays also an important role.

**Theorem 6.1** Let \(W\) be a \(C^{N\times M}\)-valued function defined in the open unit disk (typically, \(M = 1\) or \(M = N\)). Let \(P \in C^{N\times N}\) satisfying (31), and let, for \(k = 0, \ldots, N-1,\)

\[
W_k(z) = \frac{1}{N} \sum_{\ell=0}^{N-1} (\epsilon_N P)^{k\ell} W(\epsilon_N^\ell z).
\] (32)

Then,

\[
W_k(\epsilon_N z) = (\epsilon_N P)^{-k}(W_k(z)), \quad k = 0, \ldots, N-1,
\] (33)

\[
W(z) = \sum_{k=0}^{N-1} W_k(z).
\] (34)

**Proof:** We have

\[
W_k(\epsilon_N z) = \frac{1}{N} \sum_{\ell=0}^{N-1} (\epsilon_N P)^{k\ell} W(\epsilon_N^\ell \epsilon_N^\ell z)
\]

\[
= (\epsilon_N P)^{-k} \left( \frac{1}{N} \sum_{\ell=0}^{N-1} (\epsilon_N P)^{k(\ell+1)} W(\epsilon_N^{\ell+1} z) \right)
\]

\[
= (\epsilon_N P)^{-k} W_k(z),
\]

since \((\epsilon_N P)^{kN} = I_N\), and this proves (33). To prove (34) we write

\[
\sum_{k=0}^{N-1} W_k(z) = \sum_{k=0}^{N-1} \left( \frac{1}{N} \sum_{\ell=0}^{N-1} (\epsilon_N P)^{k\ell} W(\epsilon_N^\ell z) \right)
\]

\[
= \frac{1}{N} \left( \sum_{\ell=0}^{N-1} \left( \sum_{k=0}^{N-1} (\epsilon_N P)^{k\ell} \right) W(\epsilon_N^\ell z) \right)
\]

\[
= W(z),
\]

since, in view of (31),

\[
\sum_{k=0}^{N-1} (\epsilon_N P)^{k\ell} = \begin{cases} 
N, & \text{if } \ell = 0, \\
(I_N - (\epsilon_N P)^{\ell})^{-1} & \text{if } \ell = 1, 2, \ldots, N-1.
\end{cases}
\]

\[\Box\]

When \(P = I_N\), the index \(k = 1\) corresponds to the down-sampling operator.
6.2 Orthogonal decompositions in Krein spaces

In some cases the decomposition (34) is orthogonal for the underlying Krein space (or Pontryagin space) structure. We will assume that the Krein space \((\mathcal{H}, [\cdot, \cdot]_{\mathcal{H}})\) consists of \(\mathbb{C}^N\)-valued functions and satisfies the following property:

**Hypothesis 6.2** Let \(P\) be a matrix satisfying (31), and let \(\varphi(z) = \varepsilon_N z\). We assume that:

1. The composition operator \(f \mapsto f(\varphi)\) is continuous and unitary from \(\mathcal{H}\) into itself.
2. The operator of multiplication by \(P\) on the left is continuous and unitary from \(\mathcal{H}\) into itself.

We note that, in particular, the operator \(T_{P\varphi}\) defined by (8),
\[
T_{P\varphi}f(z) = P f(\varepsilon_N z),
\]
is continuous and unitary from \(\mathcal{H}\) into itself. Note also that \(T_{P\varphi}^N = I_{\mathcal{H}}\).

Hypothesis 6.2 hold in particular for the spaces \(H_{2,J}\) when \(P\) is \(J\)-unitary, that is, satisfies
\[
P^*JP = J.
\]

**Theorem 6.3** Let \((\mathcal{H}, [\cdot, \cdot])\) be a Krein space of \(\mathbb{C}^N\)-valued functions, satisfying Hypothesis 6.2. Let \(W \in \mathcal{H}\) and let
\[
W_k(z) = \frac{1}{N} \sum_{\ell=0}^{N-1} (\varepsilon_N P)^k\ell W(\varepsilon_N^\ell z).
\]
(35)

Then,
\[
[W_\ell, W_k] = 0, \quad \ell \neq k,
\]
\[
W(z) = W_0(z) + \cdots + W_{N-1}(z),
\]
and
\[
W_k(\varepsilon_N z) = (\varepsilon_N P)^{-k}W(z).
\]

**Proof:** The last two claims are proved in Theorem 6.1. The first claim takes into account the hypothesis on \(\mathcal{H}\), and is proved as follows: We take \(k_1\) and \(k_2\) in \(\{0, \ldots, N-1\}\), and assume that \(k_2 < k_1\). Taking into account the definition of \(W_k\), we see that the inner product \([W_{k_1}, W_{k_2}]_{\mathcal{H}}\) is a sum of \(N^2\) inner products, namely
\[
([\varepsilon_N P]^{k_1\ell_1}W(\varepsilon_N^\ell_1 z), [\varepsilon_N P]^{k_2\ell_2}W(\varepsilon_N^\ell_2 z))_{\mathcal{H}}, \quad \ell_1, \ell_2 \in \{0, \ldots, N-1\}.
\]
These \(N^2\) inner products can be rearranged as \(N\) sums of inner product, each sum being equal to 0. Indeed, consider first the inner products corresponding to \(\ell_1 = \ell_2\).
In view of the unitary of the operator $T_{P,\varphi}$ we have
\[
\sum_{\ell_1=0}^{N-1} \left[ (\varepsilon_N P)^{k_1 \ell_1} W(\varepsilon_N, z), (\varepsilon_N P)^{k_2 \ell_1} W(\varepsilon_N, z) \right]_{\mathcal{H}} = \left[ \left( \sum_{\ell_1=0}^{N-1} (\varepsilon_N P)^{k_1 - k_2} \right)^{\ell_1} \right]_{\mathcal{H}} W, W]_{\mathcal{H}} = 0.
\]
Indeed, using $0 < k_1 - k_2 \leq N - 1$, and so, by hypothesis on $P$, we have
\[
det(I_N - (\varepsilon_N P)^{k_1 - k_2}) \neq 1,
\]
and the sum
\[
\sum_{\ell_1=0}^{N-1} ((\varepsilon_N P)^{k_1 - k_2})^{\ell_1} = 0.
\]
Let us now regroup the factors of $[W(z), W(\varepsilon_N z)]_{\mathcal{H}}$. Taking into account that
\[
[p^k_1(N-1)W(\varepsilon_N^{N-1}z), W(z)]_{\mathcal{H}} = [p^k_1(N-1)W(z), W(\varepsilon_N z)]_{\mathcal{H}},
\]
we have
\[
\sum_{\ell=0}^{N-2} \left[ (\varepsilon_N P)^{k_1 \ell} W(\varepsilon_N, z), (\varepsilon_N P)^{k_2 (\ell+1)} W(\varepsilon_N^{\ell+1}, z) \right]_{\mathcal{H}} + \left[ \sum_{\ell=0}^{N-2} (\varepsilon_N P)^{(k_1 - (\ell+1))k_2} + (\varepsilon_N P)^{k_1(N-1)} \right] W(z), W(\varepsilon_N z)]_{\mathcal{H}}
\]
\[
= \left[ \sum_{\ell=0}^{N-2} ((\varepsilon_N P)^{k_1 - k_2})^{\ell} \right] W(z), W(\varepsilon_N z)]_{\mathcal{H}} = 0.
\]
The remaining terms are summed up to 0 in the same way. \(\Box\)

### 6.3 Decompositions in reproducing kernel spaces

We begin with a result in the setting of Schur functions, as opposed to generalized Schur functions.

**Theorem 6.4** Let $W$ be a $\mathbb{C}^{p \times q}$-valued Schur function and let $\varphi(z) = \varepsilon_N z$. Then the operator of composition by $\varphi$ is a contraction from $\mathcal{H}(W)$ into itself if and only if there exists a $\mathbb{C}^{q \times q}$-valued Schur function $X(z)$ such that
\[
W(z) = W(\varepsilon_N z)X(z).
\]
Proof: By Theorem 3.6, the map $T_\phi$ is a contraction if and only if the kernel
\[
K_W(z, w) - K_W(\varepsilon_N z, \varepsilon_N w) = \frac{W(\varepsilon_N z)W(\varepsilon_N w)^* - W(z)W(w)^*}{1 - \varepsilon w^*}
\]
is positive definite in the open unit disk. By Leech’s factorization theorem, see [61, p. 107], the above kernel is positive definite if and only if there is a Schur function $X(z)$ such that (36) is in force. □

As an example, take any Schur function $s$ and build
\[
W(z) = \frac{1}{\sqrt{N}} \left( s(z) s(\varepsilon_N z) \cdots s(\varepsilon_N^{N-1} z) \right). \tag{37}
\]
Then
\[
W(z) = W(\varepsilon_N z)P_N,
\]
where $P_N$ is defined by (3).

7 The family $C_N$

An effective approach to generating wavelet bases is the use of Multiresolution Analysis (MRA), see for example [13, 18, 26]. Traditionally one looks for a finite family of functions in $L^2(\mathbb{R}, dx)$, or $L^2(\mathbb{R}^d, dx)$ for some dimension $d$. If $d = 1$, one chooses a scale number, say $N$. If $d > 1$, instead one scales with a $d \times d$ matrix $A$ over the integers. We assume that $A$ is expansive, i.e., with eigenvalues bigger than 1 in modulus. If $A$ is given, let $N$ be the absolute value of its determinant. To create MRA wavelets we need an initial finite family $F$ of $N$ functions in $L^2(\mathbb{R})$, or $L^2(\mathbb{R}^d)$. One of the functions is called the scaling function ($\phi$ in the discussion below). For the moment, we will set $d = 1$, but the outline below easily generalizes to $d > 1$. An MRA wavelet basis is a basis for $L^2(\mathbb{R})$, or $L^2(\mathbb{R}^d)$ which is generated from the initial family $F$ and two operations: one operation is scaling by the number $N$ (or the matrix $A$ if $d > 1$), and the other is action by integer translates of functions. The special property for the finite family of functions $F$ is that if the $N$-scaling is applied each function $\psi$ in $F$ the result is in the closed span of the integer translates of the scaling function $\phi$. The corresponding coefficients are called masking coefficients. The reason for this is that the scaled functions represent refinements, and they are computed from masking points in a refinement. The role of the functions $m_0, m_1, \ldots, m_{N-1}$ are the frequency response functions corresponding to the system of masking coefficients. From these functions we then build a matrix valued function $W(z)$ as in (38). The question we address here is the characterization of the matrix valued function which arise this way. Now the wavelet filters we consider here go beyond those studied earlier in that we allow for wider families
of Multiresolution Analyses (MRAs). This includes more general wavelet families, allowing for example for wavelet frame bases, see e.g., \cite{13, 41, 42}, multi-scale systems in dynamics, and in analysis of fractals; see \cite{31}.

7.1 The family $\mathcal{C}_N$: characterization

The filters we consider are matrix-valued (or operator valued) functions of a complex variable. In general if a positive integer $N$ is given, and if a matrix function $W(z)$ is designed to take values in $\mathbb{C}^{N \times N}$, then of course, there are $N^2$ scalar-valued function occurring as matrix entries. However, in the case of filters arising in applications involving $N$ distinct frequency-bands, for example in wavelet constructions with scale number $N$, then we can take advantage of an additional symmetry for the given matrix function $W(z)$, see for example (1) in the Introduction. Here we point out that this $N$-symmetry condition (or $N$-periodicity) means that $W(z)$ is then in fact determined by only $N$ scalar valued functions, see (38) below. These functions play three distinct roles as follows: They are (i) the scalar valued filter functions, $\hat{s}_i$, for $i = 0, 1, \ldots, N - 1$, in generalized quadrature-mirror filter systems (the quadrature case corresponds to $N = 2$); they are (ii) scaling filters for scale-number $N$ with each of the $N$ scalar functions $\hat{s}_i$ generating an element in a wavelet system of functions on the real line and corresponding to scale-number $N$; and (iii) the system of scalar functions $(\hat{s}_i)_{i=0,\ldots,N-1}$ generates an operator family $(S_i)_{i=0,\ldots,N-1}$ constituting a representation of the Cuntz relations; thus generalizing Theorem 5.3 above. The results presented in this section are related to \cite{9}.

Recall that $\epsilon_N = e^{2\pi i/N}$. We shall say that a $\mathbb{C}^{N \times N}$-valued ($N \geq 2$) function $W$ meromorphic in the open unit disk $D$ belongs to $\mathcal{C}_N$ if it is of the form

$$W(z) = \frac{1}{\sqrt{N}} \begin{pmatrix} \delta_0(z) & \delta_0(\epsilon_N z) & \cdots & \delta_0(\epsilon_N^{N-1} z) \\ \delta_1(z) & \delta_1(\epsilon_N z) & \cdots & \delta_1(\epsilon_N^{N-1} z) \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{N-1}(z) & \delta_{N-1}(\epsilon_N z) & \cdots & \delta_{N-1}(\epsilon_N^{N-1} z) \end{pmatrix},$$

(38)

where $\delta_0, \ldots, \delta_{N-1}$ are complex-valued functions meromorphic in $D$. Note that such a function, when analytic at the origin, will never be invertible there. A special case of this analyticity restriction of course is when $W(z)$ has polynomial entries. Under the filter-to-wavelet correspondence \cite{18}, polynomial filters are the compactly supported wavelets. In the sequel, it will turn out that we shall concentrate on the opposite cases. Namely, not only $W(z)$ will not be analytic at the origin, in fact we shall have $W(z)|_{z=0} = 0_{N \times N}$.

Recall that we have denoted by $P_N$ the permutation matrix,

$^3$ This correspondence: polynomial filter to compactly supported wavelet even works if $d > 1$. 

\[ P_N = \begin{pmatrix} 0_{1 \times (N-1)} & 1 \\ I_{N-1} & 0_{(N-1) \times 1} \end{pmatrix} \]

(see (2)).

**Lemma 7.1** A \( \mathbb{C}^{N \times N} \)-valued function meromorphic in the open unit disk is of the form (38) if and only if it satisfies (1):

\[ W(\epsilon_N z) = W(z)P_N \]

**Proof:** Let \( W \) be a \( \mathbb{C}^{N \times N} \)-valued function meromorphic in \( \mathbb{D} \), and satisfying (1), and let \( s_1, \ldots, s_N \) denote its columns, i.e.

\[ W(z) = (s_1(z) \ s_2(z) \ \cdots \ s_N(z)). \quad (39) \]

Namely, from (38)

\[
s_j(z) := \frac{1}{\sqrt{N}} \begin{pmatrix} \hat{s}_0(\epsilon_N^{-1} z) \\ \hat{s}_1(\epsilon_N^{-1} z) \\ \vdots \\ \hat{s}_{N-1}(\epsilon_N^{-1} z) \end{pmatrix}, \quad j = 1, \ldots, N.
\]

Multiplying \( W \) by \( P_N \) from the right makes a cyclic shift of the columns to the left, namely

\[ W(z)P_N = (s_2(z) \ s_3(z) \ \cdots \ s_N(z) \ s_1(z)). \]

Equation (1) then leads to

\[
(s_1(\epsilon_N z) \ s_2(\epsilon_N z) \ \cdots \ s_{N-1}(\epsilon_N z) \ s_N(\epsilon_N z)) = (s_2(z) \ s_3(z) \ \cdots \ s_N(z) \ s_1(z)).
\]

Thus

\[ s_2(z) = s_1(\epsilon_N z), \quad s_3(z) = s_1(\epsilon_N^2 z), \ldots, s_N(z) = s_1(\epsilon_N^{N-1} z), \]

and so \( W \) is of the asserted form. The converse is clear. \( \square \)

Note that in contrast to Lemma 7.1 in equation (37) we did not assume that \( W \) is square.

When one assumes that the function \( W \) in the previous lemma is a generalized Schur function, the symmetry condition (1) can be translated into the realization. We present the result for the closely outer connected coisometric realization, but similar results hold for the closely inner connected isometric realization and connected unitary realizations as well (see Section 4.2 for these notions). In the statement, recall that the state space \( \mathcal{S} \) will in general be infinite dimensional and endowed with a Pontryagin space structure.
**Theorem 7.2** Let $W$ be a generalized Schur function, and let
\[ W(z) = D + zC(I - zA)^{-1}B \]
be a closely-inner coisometric realization of $W$, with state space $\mathcal{S}$. Then, $W$ satisfies (1) if and only if there is a bounded invertible operator $T$ from $\mathcal{H}$ into itself such that
\[ \begin{pmatrix} \epsilon_N A & B \\ \epsilon_N C & D \end{pmatrix} \begin{pmatrix} T & 0 \\ 0 & I_N \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & I_N \end{pmatrix} \begin{pmatrix} T & 0 \\ 0 & I_N \end{pmatrix} \] (40)
Furthermore, the operator $T$ satisfies:
\[ T^N = I. \] (41)

**Proof:** The first equation follows from the uniqueness of the closely connected coisometric realization. Iterating (40) and taking into account that $\epsilon_N = 1$ we get
\[ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} T^N & 0 \\ 0 & I_N \end{pmatrix} = \begin{pmatrix} T^N & 0 \\ 0 & I_N \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \]
By uniqueness of the similarity operator we have $T^N = I$. \qed

**Proposition 7.3** Let $W_1$ and $W_2$ be in $\mathcal{C}_N$. Then the functions
\[ W_1(z)W_2(\overline{z})^* \quad \text{and} \quad W_1(z)W_2(1/\overline{z})^* \]
are meromorphic functions of $z^N$.

**Proof:** Let $W(z) = W_1(z)W_2(\overline{z})^*$. Since $P_NP_N^* = I_N$, we have
\[ W(\epsilon_Nz) = W_1(\epsilon_Nz)W_2(\overline{\epsilon_Nz})^* = W_1(z)P_NP_N^*W_2(\overline{z})^* = W_1(z)W_2(\overline{z})^* = W(z), \]
that is
\[ W(\epsilon_Nz) = W(z). \] (42)
The function $W_1$ and $W_2$ are meromorphic in the open unit disk, and so is the function $W$. We denote by $\Lambda$ the set of poles of $W$ and by $\Lambda_N$ the set of points $w$ in the open unit disk such that $w^N \in \Lambda$. Let for $z = re^{i\theta}$ with $r > 0$ and $\theta \in (-\pi, \pi]$,
\[ R(z) = W\left(\sqrt[2N]{r}e^{i\theta/2N}\right). \]
The function $R$ is analytic in $D \setminus \{\Lambda_N \cup (-1, 0)\}$. Thanks to (42) it is continuous across the negative axis at those points in $(-1, 0)$ which are not in $D \setminus \Lambda_N$. It follows
that $R$ is analytic in $\mathbb{D} \setminus \Lambda_N \cup \{0\}$. Furthermore, $W(z) = R(z^N)$. Any singular point of $R$ is a pole (otherwise its roots of order $N$ would be essential singularities of $W$), and so $R$ is meromorphic in $\mathbb{D}$. \[\square\]

In the rational case, the previous result has an easier and more precise proof. Indeed consider the Laurent expansion at the origin of $W$:

$$W(z) = \sum_{-m_0}^{\infty} W_k z^k.$$  

It converges in a punctured disk $0 < |z| < r$ for some $r > 0$. Equation (42) implies that

$$\sum_{-m_0}^{\infty} W_k z^k = \sum_{-m_0}^{\infty} W_k e_N z^k.$$  

By uniqueness of the Laurent expansion we get that

$$W_k = 0, \quad \text{for} \quad k \notin N\mathbb{Z}.$$  

Thus, if $m > 0$, we may assume without loss of generality that $m_0 = Nn_0$ for some $n_0 \in \mathbb{N}$. The function

$$W_-(z) = \sum_{k=-m_0}^{-1} W_k z^k$$  

is rational, and so is the function

$$W_+(z) = \sum_{k=0}^{\infty} W_k z^k.$$  

We see that

$$W_-(z) = \sum_{-m_0 \leq nN \leq -N} W_{nN} z^{nN}$$  

and so $W_-(z) = R_-(z^N)$, where the function

$$R_-(z) = \sum_{-m_0 \leq nN \leq -N} W_{nN} z^n$$  

is rational and analytic at infinity. The function $W_+$ is analytic at the origin, and thus can be written in realized form as:

$$W_+(z) = D + zC(I_p - zA)^{-1}B.$$  

Comparing with

$$W_+(z) = \sum_{n=0}^{\infty} W_{nN} z^{nN}$$  

we have that
$$CA^pB = \begin{cases} 0 & \text{if } p+1 \not\in \mathbb{N}, \\ W_{nN} & \text{if } p+1 = nN, \ n \in \mathbb{N}. \end{cases}$$

It follows that $W_+(z) = R_+(z^N)$, where $R_+$ is the rational function defined by

$$R_+(z) = D + \sum_{n=1}^{\infty} z^n CA^{n-1}B$$

$$= D + \sum_{n=1}^{\infty} z^n CA^{(n-1)N} A^{N-1}B$$

$$= D + zC(I_p - zA^{N-1})^{-1} A^{N-1}B.$$

The function

$$R(z) = R_-(z) + R_+(z)$$

is rational.

The proof of the preceding proposition can be mimicked to obtain the following result:

**Proposition 7.4** Let $W_1$ and $W_2$ be in $\mathcal{C}_N$, with non identically vanishing determinant. Then there exists a meromorphic function $R$ such that

$$W_1(z)W_2(z)^{-1} = R(z^N). \quad (43)$$

To this end, recall that the unitary matrix $F_N$,

$$F_N := \frac{1}{\sqrt{N}} \begin{pmatrix} \epsilon_N^{-(0,0)} & \epsilon_N^{-(0,1)} & \epsilon_N^{-(0,2)} & \cdots & \epsilon_N^{-(0,(N-1))} \\ \epsilon_N^{-(1,0)} & \epsilon_N^{-(1,1)} & \epsilon_N^{-(1,2)} & \cdots & \epsilon_N^{-(1,(N-1))} \\ \epsilon_N^{-(2,0)} & \epsilon_N^{-(2,1)} & \epsilon_N^{-(2,2)} & \cdots & \epsilon_N^{-(2,(N-1))} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \epsilon_N^{-(N-1,0)} & \epsilon_N^{-(N-1,1)} & \epsilon_N^{-(N-1,2)} & \cdots & \epsilon_N^{-(N-1,(N-1))} \end{pmatrix},$$

generates the discrete Fourier transform. Namely, the discrete Fourier transform of $x \in \mathbb{C}^N$ is given by $X = F_Nx$ and the inverse discrete Fourier transform is given by $x = F_N^*X$. Let furthermore

$$\hat{W}_N(z) := \text{diag}\{1, z^{-1}, \ldots, z^{1-N}\}F_N. \quad (44)$$

With this special choice of $W_2$ the previous proposition becomes:

**Proposition 7.5** $W \in \mathcal{C}_N$ and $\det W \neq 0$ if and only if it can be written as

$$W(z) = R(z^N)\hat{W}_N(z),$$

where $R$ and $\hat{W}_N$ are as in (43) and (44) respectively.
7.2 A connection with periodic systems

Let
\[ D_N(z) = \text{diag}(z^N, z^{N-1}, \varepsilon_1 z^{N-2}, \ldots, \varepsilon_N), \]
so that
\[ D_N(1) = \text{diag}(1, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N). \]

Functions which satisfy the related symmetry
\[ W(\varepsilon_N z) = D_N(1)^{-1} W(z) P_N \] (45)
appear in the theory of periodic systems. A function \( W \) satisfies (45) if and only if it is of the form
\[
W(z) = \frac{1}{\sqrt{N}} \begin{pmatrix}
\delta_0(z) & \delta_0(\varepsilon_N z) & \cdots & \delta_0(\varepsilon_{N-1} z) \\
\delta_1(z) & \frac{1}{\varepsilon_1} \delta_1(\varepsilon_N z) & \cdots & \frac{1}{\varepsilon_N} \delta_1(\varepsilon_{N-1} z) \\
\vdots & & & \\
\delta_{N-1}(z) & \frac{1}{\varepsilon_{N-1}} \delta_{N-1}(\varepsilon_N z) & \cdots & \frac{1}{\varepsilon_N} \delta_{N-1}(\varepsilon_{N-1} z)
\end{pmatrix}
\] (46)

See [5, Theorem 4.1, p. 381]. We note that the corresponding general bitangential interpolation problem (see [14] for references) was solved in [5] for functions analytic and contractive in the open unit disk (that is, for Schur functions). Let us denote by \( \mathcal{P}_{en} \) the family of functions meromorphic in the open unit disk and which satisfy (45).

**Proposition 7.6** The map \( W \mapsto D_N W \) is one-to-one from \( \mathcal{P}_{en} \) onto \( \mathcal{C}_N \). If \( W \) is analytic and contractive in the open unit disk so is \( D_N W \).

**Proof:** We first note that
\[ D_N(\varepsilon_N z) = D_N(z) D_N(1). \] (47)
Let now \( W \in \mathcal{P}_{en} \). In view of (47) and (45) we have
\[
D_N(\varepsilon_N z) W(\varepsilon_N z) = D_N(z) D_N(1)^{-1} W(z) P_N = D_N(z) W(z) P_N,
\]
and so \( D_N W \in \mathcal{C}_N \).  

**Epilogue:** A reason for the recent success of wavelet algorithms is a coming together of tools from engineering and harmonic analysis. While wavelets now enter into a multitude of applications from analysis and probability, it was the incorporation of ideas from signal processing that offered new and easy-to-use algorithms, and hence wavelets are now used in both discrete problems, as well as in harmonic analysis decompositions. Following this philosophy we here employed tools from
system theory to wavelet problems and tried to show how ideas from wavelet decompositions throw light on factorizations used by engineers.

Since workers in wavelet theory often are not familiar with filterers in general, and FIR filters (short for Finite Impulse Response) in particular, widely used in the engineering literature, we have taken the opportunity to include a section for mathematicians about filters. Conversely (in the other direction), engineers are often not familiar with wavelet analysis, and we have included a brief exposition of wavelet facts addressed to engineers. We showed that there are explicit actions of infinite-dimensional Lie groups which accounts for all the wavelet filters; as well as for other classes of filters used in systems theory. Moreover, we described these groups, and explained how they arise in systems. The corresponding algorithms, including the discrete wavelet algorithms are used in a variety of multi-scale problems, as used for example in data mining. These are the discrete algorithms, and we described their counterparts in harmonic analysis in standard $L^2$ Lebesgue spaces, as well as in reproducing kernels Hilbert spaces. We also outlined the role of Pontryagin spaces in the study of stability questions.

In the engineering literature the study of filters is mostly confined to FIR filters. Recall that FIR filters correspond to having the spectrum at the origin. In our previous work [9] we have explained that the set of FIR wavelet filters is small in a sense we made precise. This suggests two possible conclusions,

(i) It is unrealistic to offer optimization schemes, over all FIR wavelet filters as part of the design procedure.

(ii) It calls upon using, at least in some circumstances, also stable IIR (short for infinite impulse response) wavelet filters, i.e. the spectrum is confined to the open unit disk.

The above extension to $\mathbb{U}^{\text{in}}$ allows us to consider filters whose spectrum is in $\mathbb{C} \setminus \mathbb{T}$. The generalization to $\mathbb{U}^{\text{I}}$ permits the spectrum to be everywhere in the complex plane.

Roughly, we hope that this message will be useful to practitioners in their use of these rigorous mathematics tools. We offer algorithms hopefully improving on those used before.

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