The Navier-Stokes equations in periodic domains

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In the present technical note, we establish that the setting of the primitive variables of the unsteady incompressible fluid dynamics is ill-formulated in spatially periodic domains as the specification of the boundary velocity is too broad to sidestep time-dependency and approximation errors. As an illustration, we show that the Taylor-Green solution in planes suffers from the Hadamard-divergence, and the ABC flow in cubes is non-unique. In direct numerical simulations of homogeneous turbulence with no corrective precautions on the boundary values, our assertion helps us understand the well-experienced nuisances, such as slow rates of convergence in energy dissipation, fluctuations in the statistics moments, or spontaneous surges in the time-averaged flow quantities.

Keywords: Navier-Stokes; Viscosity; Turbulence; Finite Energy

1. Background

In incompressible real fluids, the Navier-Stokes equations of motion are derived from the principles of momentum conservation, and mass

\[ \partial_t \mathbf{u} - \nu \Delta \mathbf{u} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \rho^{-1} \nabla p, \quad \nabla \cdot \mathbf{u} = 0, \]  

where \( \Delta \) stands for the Laplacian, the vector quantity \( \mathbf{u}(x, t) = (u, v, w)(x, t) \) is the velocity \( (x = (x, y, z)) \), the scalar \( p(x, t) \) the pressure. We also use the tensor notation, \( \mathbf{u}(x, t) = (u_i(x, t)), x = (x_i), i = 1, 2, 3. \) The symbol \( \nu = \mu/\rho \) denotes the kinematic viscosity, where \( \rho \) and \( \mu \) are the density and the viscosity of the fluid respectively. For inviscid flows \( \nu = 0 \), the system is the Euler equations. Taking divergence of (1.1) and using the continuity, we obtain a Poisson’s equation for the pressure

\[ \Delta p(x) = -\rho \nabla \cdot ((\mathbf{u} \cdot \nabla)\mathbf{u})(x), \]  

which holds at every instant of time \( t \). The initial condition for the velocity is solenoidal and smooth

\[ \mathbf{u}(x, 0) = \mathbf{u}_0(x) \in C^\infty(\Omega). \]  

We are interested in finite-energy initial value problems subject to the initial data (1.3). To focus on the key issues of our note, let us consider viscous flows in a periodic cube with period \( L \) in all three directions, \( 0 \leq x, y, z \leq L \). The boundary conditions take the form of

\[ \mathbf{u}(x) = \mathbf{u}(x + L). \]  

Here the actual magnitudes in the velocity are not fixed; they are allowed to vary over the flow evolution in general. As the pressure only plays an auxiliary role in fluid motion, there must be no boundary conditions on the pressure. Otherwise, the dynamics is over-specified since the pressure can be eliminated by the incompressibility constraint, reducing (1.1) to a consistent system of three equations in three unknown velocities.

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2. Taylor-Green vortices

Consider an example of viscous flow in a square box \((0 \leq x, y \leq 2\pi)\) with boundary data

\[
\begin{align*}
    u(x,0) &= u(x,2\pi) = 0, & u(0,y) &= u(2\pi,y), \\
    v(0,y) &= v(2\pi,y) = 0, & v(x,0) &= v(x,2\pi).
\end{align*}
\]

(2.1)

It has been known for long that the Taylor-Green eddy,

\[
\begin{align*}
    u(x,t) &= \cos(x) \sin(y) f(t), \\
    v(x,t) &= -\sin(x) \cos(y) f(t), \\
    p/\rho(x,t) &= -(\cos(2x) + \cos(2y) f^2(t))/4,
\end{align*}
\]

(2.2)

where \(f(t) = \exp(-2\nu t)\), is an exact solution of the planar equations (1.1). Given the velocity \((u,v)\), the vorticity and the stream function are found to be

\[
\begin{align*}
    \zeta &= -2\cos(2x) \cos(2y) f(t), & \psi &= -\cos(x) \cos(y) f(t)
\end{align*}
\]

respectively.

Now we demonstrate the existence of a generalised vortex structure. First, we select the velocity and the pressure,

\[
\begin{align*}
    \tilde{u}(x,t) &= \cos(mx) \sin(my) f(t), \\
    \tilde{v}(x,t) &= -\sin(mx) \cos(my) f(t), \\
    \tilde{p}/\rho(x,t) &= -(\cos(2mx) + \cos(2my) f^2(t))/4,
\end{align*}
\]

(2.3)

where the time-dependent factor \(f(t) = \exp(-2m^2\nu t)\), and \(m\) is an arbitrary integer. Note that the periodicity of \((u,v)\) is preserved, and the homogenous boundary data are respected for \(t > 0\). When the viscosity is low, these solutions are mild oscillations at moderate values of \(m\). Second, we observe that the exponential decay has other choices as long as \(f(t)\) satisfies

\[
df/dt + 2m^2\nu f = 0.
\]

(2.4)

An interesting case is \(m \to m/\sqrt{\nu}\) (assumed an integer) so that the decay is independent of the viscosity. In particular, the solutions,

\[
\begin{align*}
    \tilde{u}(x,t) &= \cos(mx/\sqrt{\nu}) \sin(my/\sqrt{\nu}) \exp(-2m^2 t), \\
    \tilde{v}(x,t) &= -\sin(mx/\sqrt{\nu}) \cos(my/\sqrt{\nu}) \exp(-2m^2 t),
\end{align*}
\]

(2.5)

define a velocity field which may be rapidly oscillating with slow decay even for small values of \(m\), soon after the start at locations \(x \approx 0\) or \(y \approx 2\pi\) (say). At least for some viscosity, these high-frequency fluctuations must constitute poor representations for any genuine flow structure because they are the ill-defined solutions in the sense of Hadamard (1964).

In three space dimensions, the initial Taylor-Green velocity is given by

\[
u_0(x) = \begin{pmatrix} A \cos(ax) \sin(by) \sin(cz), & B \sin(ax) \cos(by) \sin(cz), & C \sin(ax) \sin(by) \cos(cz) \end{pmatrix},
\]

(2.6)

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where the incompressibility demands $Aa + Bb + Cc = 0$. Closed-form algebraic expressions of solution $(u, p)(x, t)$ are not known to exist (cf. Taylor & Green 1937; Goldstein 1940). On the other hand, nearly all the computations by numerical approximations are undertaken subject to the periodic boundary condition so that the subsequent solutions enjoy a high degree of symmetry and accuracy, as advocated by practitioners of spectral methods. In contrast to the planar case, a key consequence is that the initial homogeneous data at the edges of the planes normal to each direction are not preserved over time. At any $t > 0,$ the boundary velocities can be de facto functions of the spectral modes, viscosity, as well as numerical error amass. It is fair to assert that the dynamic evolution must be more vulnerable to the Hadamard-divergence, as characterised by the sine-cosine oscillations with relaxed constraints on the boundaries (cf. the scenarios implied in (2.5)). Does the initial-boundary value problem remain well-posed?

3. General spurts

Our discussion in the preceding section inspires a general time-dependent transform on the three-dimensional equations. In periodic domains with periodic boundary conditions, a ‘spurt’ transform, $(\tilde{u}, \tilde{p})$, is a set of special Navier-Stokes functions such that, if $(u, p)$ satisfies the Navier-Stokes dynamics, so does $(u + \tilde{u}, p + \tilde{p})$.

Consider the following spurt transform:

$$\begin{align*}
\tilde{u}(x, t) &= C_1 g(t, \nu, L), \\
\tilde{v}(x, t) &= C_2 h(t, \nu, L), \\
\tilde{w}(x, t) &= C_3 k(t, \nu, L), \\
\tilde{p}(x, t) &= -\rho \left( C_1 (x - \alpha) \ g' + C_2 (y - \beta) \ h' + C_3 (z - \gamma) \ k' \right),
\end{align*}$$

where the prime denotes differentiation with respect to time $t$, and $C_1, C_2, C_3$ are arbitrary (finite) constants. To be definitive, the constants, $\alpha, \beta, \gamma$, are chosen so that none of $x - \alpha, y - \beta, z - \gamma$ vanishes at the boundary. They may be convenient points outside the domain, say, $\alpha = \beta = \gamma = 2L.$ The spurt functions, $g, h, k,$ are assumed to be smooth and bounded. They may be arbitrarily chosen as long as the energy is finite. In addition, we would like to fix the spurt functions so that the initial data $(u_0, p_0)$ remains unchanged. For example, we can set $g(t = 0) = g'(t = 0), h(t = 0) = h'(t = 0), k(t = 0) = k'(t = 0)$.

Substituting $(u + \tilde{u}, p + \tilde{p})$ into the Navier-Stokes equations, we assert that the solution $(u, p)$ is invariant to $(\tilde{u}, \tilde{p})$ because the time-dependent velocity spurts have zero effect on the divergence as well as the non-linearity. Evidently, any superposition of the above spurts, with distinct $g, h, k, C'$s, is also a spurt quadruplet. Lastly, the spurts are still effective for inviscid flows described by the Euler equations if viscosity is formally taken to be zero.

The 2d weak solutions constructed by Shnirelman (1997) and the generalised weak solutions with decreasing energy (Shnirelman 2000) are believed to intersect with a well-defined time-dependent smooth solution plus arbitrary numbers of the (finite) spurts. Also we have made a thorough check on the ‘blow-up’ yes-list of Gibbon (2008): all the finite-time singularity computations have been performed on periodic boxes with periodic boundary conditions. Moreover, mesh convergence...
validation on the events prior to the inception of a singularity was almost non-existent in these numerical works, and no efforts have been made to monitor the boundary values over the time-marching. (In those simulations where the periodic pressure is imposed, the dynamics is over-determined and hence mathematically inconsistent.) Similarly, in the method of analytic strip, the Euler equations are analytically continued into complex periodic domain, see, for instance, equations (7)-(9) of Frisch et al. (2003), if we generalise the primitive variables \((u, p)\) to \(u(x,t) = u_r(x,t) + i u_i(x,t)\), and \(p(x,t) = p_r(x,t) + i p_i(x,t)\). With periodic boundaries, we can devise, by analogy to equation (3.1), two distinct spurt quadruplets, \((\tilde{u}_r, \tilde{p}_r)\) and \((\tilde{u}_i, \tilde{p}_i)\), for the real and imaginary parts. Unavoidably, the analytic-strip approach to Euler flow inherits the spurt-inflicted non-uniqueness in its solutions.

In parallel, it is commonly observed in laboratory or Nature that, for given initial conditions, incompressible eddies mutually interact, agglomerate, break up, mingle, merge, diffuse, and attenuate, with finite enstrophy and energy. While recognising a dedicated endeavour of a mass-scale direct numerical simulation by Kaneda et al. (2003), the conclusion on the existence of a non-zero energy dissipation in the limit \(\nu \to 0\) must be considered as indicative, if not incorrect, as a static spurt may have been enmeshed in the discretisation. The viability of (3.1) extends to the computational theory of projection methods (see, for instance, Chorin 1969) where the convergence analyses must be reappraised, as an indeterminate velocity spurt may be consigned to each component at every iteration, thus rendering the finite-difference scheme unquantifiable. By the same token, the trefoil vortex rings (Kerr 2018) were simulated in a periodic box with free-slip boundaries. The circulation field cannot be immune to the spurts which corrupt any emerging vortices to all length scales (in light of the well-known density theorem), regardless of the box size; therefore, his results of vortex reconnection with local self-similarity, and the related discussion of scaling bounds, are utterly misleading.

4. Mistimed ABC flow

A specific type of incompressible flow is discussed in Dombre et al. (1986) with initial conditions \((L = 2\pi)\)

\[
\begin{align*}
\mathbf{u}_0(x) &= (\sin(mx) + \cos(my), \sin(mx) + \cos(my), \sin(mx) + \cos(mz)).
\end{align*}
\]

The initial-boundary value problem can be exactly solved subject to non-zero periodic boundary data. A straightforward computation shows that the solutions are given by

\[
\begin{align*}
\mathbf{u}(x,t) &= (\sin(mx) + \cos(my), \sin(mx) + \cos(my), \sin(mx) + \cos(mx)) \cdot \mathbf{f}(t), \\
p(x,t)/\rho &= -\left(\cos(mx) \sin(my) + \sin(mx) \cos(mz) + \cos(my) \sin(mz)\right) f^2(t),
\end{align*}
\]  

where \(f(t)\) denotes one of the solutions of (2.1), and \(f(t=0)=1\). Evidently, the ABC flows are susceptible to the instability à la Hadamard which is seen only as a possibility. What is less understood is the fact that every flow field of (4.2) is invariant to the spurt transform (3.1) because the addition of the spurt functions preserves the \(\mathbf{u}\)-boundary periodicity. Also the suitably-chosen \((\tilde{u}, \tilde{p})\) solutions do not modify the

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initial data. The essence is that we cannot rely on extensive computations or symmetry arguments to justify ABC’s simplicity and diversity without fair appreciation of the spurts. No matter how sophisticated our numerical algorithms appear to be, the numerical outputs largely contain unquantified temporal ingredients of obscurity. Within the parameters of practical interests, the claimed singular, chaotic or unpredictable phenomena exhibited in the ABC flows must have limited meaning, if not irrelevant to the continuum physics.

5. Remarks

The significance of the spurts is that the incompressible fluid dynamics in periodic domains with periodic boundary conditions is not well-determined in the formulation of the primitive pair \((u, p)\). Any theoretical treatment of the dynamics necessitates further justifications on the nature of the boundary values. The present author has been unaware of any definitive work on this apparently trivial but critical matter in the technical literature. It has been a (mis)belief that, on the basis of large-scale numerical computations, the Navier-Stokes dynamics blows up in finite time for suitable initial data of finite energy. Some functional analyses do result in diverged solutions which are often classified into the categories of ‘weak’ non-uniqueness or ‘wild’ solutions.\(^\dagger\) Evidently, any analysis without mathematical devices to filter out the spurts cannot be regarded as complete. Our exposition has implications in the numerical simulations of homogeneous turbulence in the box with period \(L\). In practice, the spurt functions may be understood as the representation for the numerical errors arising from truncation, spatial discretisation, mesh resolution, finite-precision arithmetic or aliasing procedures. Because the boundary velocities are loosely fixed in \(\text{(1.4)}\), the numerics may have appeared successfully in calculations from time \(t_n\) to the next step \(t_{n+1}\). The problem is that the errors may initiate and propagate in non-transparent manners. Then the converged ‘solutions’ would contain unquantified jumps in the boundary values due to accumulations. If the numerical surges are frequent and repeat over time, one is tempted to view fluid motions as topologies of multi-fractals which contradict the underlying principles of the Navier-Stokes dynamics.

To put potentially awkward spurts into perspective, let us consider the examples

\[
g(t) = t^2 \exp \left( \frac{t^3}{\nu} \right) \quad \text{or} \quad h(t) = L \exp \left( t^{1+\epsilon} \cos(m/\sqrt{\nu}) \right) - L, \quad (\epsilon > 0). \quad (5.1)
\]

In actual computations, we may not have no full knowledge of the way the spurt functions creep into our numerical approximations, small-viscosity motions starting from \(\text{(1.3)}\) subject to \(\text{(1.4)}\) may well run amok over a tiny instant \(t > 0^+\). In other words, the existence of the spurts may well render a solution \((u, p)\) into the class of Hadamard-divergence. Indeed, a solution containing a fraction of \(h(t)\) of \(\text{(5.1)}\) must appear as incongruous and confusing, possibly giving rise to the familiar characters of a temporal intermittency.

It has been reported that, in direct numerical simulations of homogeneous turbulence in periodic boxes, certain higher-order moments of enstrophy and energy

\(^\dagger\) If we relax the smoothness requirement on the spurts, we can actually construct a singular Navier-Stokes solution consisting of the Cantor function or the Devil’s staircase over time \([0, 1]\) (say), because every stair height (normalised) preserves the boundary periodicity, as long as we choose the spurt pressure to be zero almost everywhere.
dissipation converge slowly, if at all. The poor rates of convergence are likely the numerical pathologies related to the presence of mild spurt functions trapped in various spectral modes. It is also known, that these global time-averaged quantities can fluctuate by several orders of magnitude over a particular period of time. One plausible explanation is that the spurt arbitrariness has been out-of-control in the numerics (cf. $g(t)$ in example (5.1)).

In the vorticity formulation, \( \omega(x) = \nabla \times u(x) \), the scalar pressure is eliminated from the system (1.1) so that a compensator for potential velocity arbitrariness has been removed, as expressed in the conservation law of angular momentum

\[
\partial_t \omega - \nu \Delta \omega = (\omega, \nabla) u - (u, \nabla) \omega.
\]

However, the velocity must be recovered from the di-vorticity \( \Delta u = -\nabla \times \omega \) or from the stream function vector \( u = \nabla \times \Psi \), where \( \Delta \Psi - \nabla (\nabla \cdot \Psi) = -\omega \). In either scheme, the boundary \( u \) must be prescribed and fixed over time. In periodic domains, this specification is tantamount to a loss of periodicity as, now, the problem is to find solutions in bounded domains with Dirichlet data.† By comparison, theoretical evaluation of the vorticity on \( \mathbb{R}^3 \) is less troublesome. For turbulence, the assumption of homogeneity no longer makes sense.

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† In general, there exist no boundary conditions for vorticity in the Navier-Stokes dynamics, see a review by Gresho (1991). The numerical computations of Ayala, Doering & Simon (2018) employed (artificial) periodic boundary conditions for the vorticity. Following this (erratic) approach, we may specify vorticity spurts in the periodic setting, for example, \( \tilde{\omega} = f(Re_0) \), where function \( f \) is non-zero arbitrary but bounded. Alternatively, we may assign \( \tilde{\omega} = f(\varepsilon_d) \), where \( \varepsilon_d \) denotes the maximum size of the discretisation errors in each mesh resolution. The existence of these spurts implies that \( \Delta \tilde{u} = 0 \), which generates one or more spurt velocity fields at every instant \( t \) in the numerics. There are many harmonic functions available for constructing solenoidal velocity spurts in the square domain \( L \times L \). We notice that no consistent convergence was demonstrated in their simulations. In numerous practical works at low \( \nu \sim O(10^{-5}\sim 10^{-4}) \) with proper velocity boundary data (for example, the no-slip on \( C^2 \) boundaries), it is well-documented that the palinstrophy in a flow may surge to an order of \( O(10^8\sim 10^9) \) in the early phase of evolution. The numerical size of \( |\nabla \omega|^2 \) is not a matter of concern, as long as the flow remains incompressible; the key criterion is whether the energy, enstrophy as well as palinstrophy, have converged or not in the light of systematically refined mesh-resolutions. Briefly, their results are rather crude and tentative. Above all, their ideas for finite-time singularities directly contradict the well-established global regularity for two-dimensional incompressible viscous flows, see Leray (1934); and Ladyzhenskaya (1959).

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