A note on skew spectrum of graphs

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Abstract

We give some properties of skew spectrum of a graph, especially, we answer negatively a problem concerning the skew characteristic polynomial and matching polynomial in [M. Cavers et al., Skew-adjacency matrices of graphs, Linear Algebra Appl. 436 (2012) 4512–4529].

1 Introduction

We consider simple graphs. Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). An orientation of \( G \) is a sign-valued function \( \sigma \) on the set of ordered pairs \( \{(i, j), (j, i) | ij \in E(G)\} \) that specifies an orientation to each edge \( ij \) of \( G \): If \( ij \in E(G) \), then we take \( \sigma(i, j) = 1 \) when \( i \to j \) and \( \sigma(i, j) = -1 \) when \( j \to i \). The resulting oriented graph is denoted by \( G^\sigma \). Both \( \sigma \) and \( G^\sigma \) are called orientations of \( G \).

The skew adjacency matrix \( S^\sigma = S(G^\sigma) \) of \( G^\sigma \) is the \( \{0, 1, -1\} \)-matrix with \( (i, j) \)-entry equal to \( \sigma(i, j) \) if \( ij \in E(G^\sigma) \) and 0 otherwise. If there is no confusion, we simply write \( S = [s_{i,j}] \) for \( S^\sigma \). Thus \( s_{i,j} = 1 \) if \( ij \in E(G^\sigma) \), \( -1 \) if \( ji \in E(G^\sigma) \), and 0 otherwise. The (skew) characteristic polynomial of \( S = S^\sigma \) is

\[
p_S(x) = \det(xI - S) = x^n + s_1x^{n-1} + \cdots + s_{n-1}x + s_n,
\]

where \( n = |V(G)| \). Let \( \rho(D) \) be the spectral radius of a square matrix \( D \), i.e., the largest modulus of the eigenvalues of \( D \). The spectral radius of \( G \) is the spectral radius of its adjacency matrix. The maximum skew spectral radius of \( G \) is defined as \( \rho_s(G) = \max_S \rho(S) \), where the maximum is taken over all of the skew adjacency matrices \( S \) of \( G \).

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An odd-cycle graph is a graph with no even cycles (cycles of even lengths). In particular, a tree is an odd-cycle graph.

Let $G$ be a graph with $n$ vertices. Let $m_k(G)$ be the number of matchings in $G$ that cover $k$ vertices. Obviously, $m_k(G) = 0$ if $k$ is odd. The matching polynomial of $G$ is defined as

$$m(G, x) = \sum_{k=0}^{n} (-1)^{\frac{k}{2}} m_k(G) x^{n-k},$$

where $m_0(G) = 1$.

Let $G$ be a graph on $n$ vertices. After showing that $G$ is an odd-cycle graph if and only if $p_S(x) = (-i)^n m(G, ix)$ for all skew adjacency matrices $S$ of $G$ (see [1, Lemma 5.4]), Cavers et al. [1] posed the following question:

**Problem 1.** If $p_S(x) = (-i)^n m(G, ix)$ for some skew adjacency matrix $S$ of $G$, must $G$ be an odd-cycle graph?

After showing that if $G$ is an odd-cycle graph, then $\rho(S) = \rho(G)$ for all skew adjacency matrices $S$ of $G$ ([1, Lemma 6.2]), Cavers et al. [1] posed the following question:

**Problem 2.** If $G$ is a connected graph and $\rho(S)$ is the same for all skew adjacency matrices $S$ of $G$, must $G$ be an odd-cycle graph?

In this note we answer Problem 1 negatively by constructing a class of graphs, and when $G$ is a connected bipartite graph we answer Problem 2 affirmatively.

## 2 Preliminaries

Let $\mathcal{U}_k$ be the set of all collections $U$ of (vertex) disjoint edges and even cycles in $G$ that cover $k$ vertices ($\mathcal{U}_k^c$ was used for this set in [1]). A routing $\overrightarrow{U}$ of $U \in \mathcal{U}_k$ is obtained by replacing each edge in $U$ by a digon and each (even) cycle in $U$ by a dicycle. For an orientation $\sigma$ of a graph $G$ and a routing $\overrightarrow{U}$ of $U \in \mathcal{U}_k$, let $\sigma(\overrightarrow{U}) = \prod_{(i,j) \in E(\overrightarrow{U})} \sigma(i,j)$. We say that $\overrightarrow{U}$ is positively (resp. negatively) oriented relative to $\sigma$ if $\sigma(\overrightarrow{U}) = 1$ (resp. $\sigma(\overrightarrow{U}) = -1$). For $U \in \mathcal{U}_k$, let $c^+(U)$ (resp. $c^-(U)$) be the number of cycles in $U$ that are positively (resp. negatively) oriented relative to $\sigma$ when $U$ is given a routing $\overrightarrow{U}$. Then $c(U) = c^+(U) + c^-(U)$ is the (total) number of cycles of $U$.

**Lemma 1.** [1, eq. (8)] Let $S$ be a skew adjacency matrix of $G$. Then $s_k = 0$ if $k$ is odd and

$$s_k = m_k(G) + \sum_{U \in \mathcal{U}_k, c(U) > 0} (-1)^{c^+(U)} 2^{c(U)}$$

if $k$ is even.
The following lemma is obtained from parts 2 and 3 of Lemma 6.2 in \cite{1}.

**Lemma 2.** Let $G$ be a connected bipartite graph,

$$A = \begin{bmatrix} O & B \\ B^\top & O \end{bmatrix}$$

the adjacency matrix of $G$, and

$$S = \begin{bmatrix} O & B \\ -B^\top & O \end{bmatrix}$$

and

$$\tilde{S} = \begin{bmatrix} O & \tilde{B} \\ -\tilde{B}^\top & O \end{bmatrix}$$

two skew adjacency matrices of $G$. Then $\rho(A) = \rho_s(G)$, and $\rho(S) = \rho(\tilde{S})$ if and only if $\tilde{S} = DSD^{-1}$ for some $\{-1,1\}$-diagonal matrix $D$.

**Lemma 3.** \cite{1} Theorem 4.2. The skew adjacency matrices of a graph $G$ are all cospectral if and only if $G$ is an odd-cycle graph.

## 3 Results

First we give a negative answer to Problem 1.

**Theorem 1.** For integer $m \geq 2$, let $G$ be the graph consisting of two $2m$-vertex cycles $C_1$ and $C_2$ with exactly one common vertex. Let $\sigma$ be an orientation of $G$ such that the cycle $C_1$ (resp. $C_2$) is positively (resp. negatively) oriented relative to $\sigma$. Let $S = S(G^\sigma)$ and let $n = 4m - 1$. Then $p_S(x) = (-i)^n m(G, ix)$.

**Proof.** It is sufficient to show that $s_k = m_k(G)$ for even $k$. By Lemma \cite{1} we only need to show that

$$\sum_{U \in \mathcal{U}_k \atop c(U) > 0} (-1)^{c(U)} 2^{c(U)} = 0$$

for even $k$.

This is obvious when $k < 2m$. Suppose that $k$ is even with $2m \leq k \leq 4m - 2$. Let $C_1 = v_1v_2 \ldots v_2m v_1$ and $C_2 = v'_1 v'_2 \ldots v'_2m v'_1$ with $v_1 = v'_1$.

Let $\mathcal{U}_k^1$ be the subset of $\mathcal{U}_k$ consisting of $C_1$ and $\frac{1}{2}(k - 2m)$ disjoint edges in $C_2$ and $\mathcal{U}_k^2$ the subset of $\mathcal{U}_k$ consisting of $C_2$ and $\frac{1}{2}(k - 2m)$ disjoint edges in $C_1$. Obviously, $\mathcal{U}_k^1 \cap \mathcal{U}_k^2 = \emptyset$. For any $U \in \mathcal{U}_k$ with $c(U) > 0$, $U \in \mathcal{U}_k^1$ or $U \in \mathcal{U}_k^2$. There is a bijection from $\mathcal{U}_k^1$ to $\mathcal{U}_k^2$ which maps $U \in \mathcal{U}_k^1$ consisting of
$C_1$ and $\frac{1}{2}(k-2m)$ disjoint edges, say $v'_{i_1}, v'_{i_1+1}, \ldots, v'_{i_s}, v'_{i_s+1}$ in $C_2$ to $U' \in U_k^2$ consisting of $C_2$ and $\frac{1}{2}(k-2m)$ disjoint edges $v_i, v_{i+1}, \ldots, v_s, v_{s+1}$ in $C_1$, where $s = \frac{1}{2}(k-2m)$ and $2 \leq i_1 < \cdots < i_s \leq 2m-1$. Thus $|U_k^1| = |U_k^2|$. Note that

$$\sum_{U \in U_k^1} (−1)^{c^+(U)}2^{c(U)} = (−1)^1 \cdot 2^1 \cdot |U_k^1|$$

and

$$\sum_{U \in U_k^2} (−1)^{c^+(U)}2^{c(U)} = (−1)^0 \cdot 2^1 \cdot |U_k^2|. $$

Thus

$$\sum_{U \in U_k^0} (−1)^{c^+(U)}2^{c(U)} = \sum_{U \in U_k^1} (−1)^{c^+(U)}2^{c(U)} + \sum_{U \in U_k^2} (−1)^{c^+(U)}2^{c(U)} = 0,$$

as desired. $\square$

See Fig. 1 for an example with 7 vertices for Theorem 1 and its proof.

![Fig. 1. A graph on 7 vertices with an orientation.](image)

Now we give an observation on Problem 2, i.e., affirmative answer when $G$ is a connected bipartite graph.

**Theorem 2.** Let $G$ be a connected bipartite graph. If $\rho(S)$ is the same for all skew adjacency matrices $S$ of $G$, then $G$ is a tree.

**Proof.** Let

$$A = \begin{bmatrix} O & B \\ B^\top & O \end{bmatrix}$$

and

$$\overline{S} = \begin{bmatrix} O & B \\ -B^\top & O \end{bmatrix}$$


be an (ordinary) adjacency and a skew adjacency matrix of $G$. Let $S$ be a skew adjacency matrix of $G$. Then $\rho(S) = \rho(S)$.

By Lemma 2 there is a $\{-1, 1\}$-diagonal matrix $D$ such that $S = DSD^{-1}$, i.e., $S$ is similar to $S$, which implies that all skew adjacency matrices of $G$ are cospectral. Thus by Lemma 3 $G$ is an odd-cycle graph. Since $G$ is connected and bipartite, $G$ is a tree.

Let $G$ be a connected bipartite graph on $n$ vertices. Let $K_{r,s}$ be the complete bipartite graph with $r$ and $s$ vertices in its two partite sets, respectively. Note that $\rho(G) < \rho(G + e)$ for an edge of the complement of $G$ (following from the Perron-Frobenius theorem) and that $\rho(K_{r,s}) = \sqrt{rs}$.

By Lemma 2, $\rho_s(G) = \rho(G) \leq \sqrt{\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil}$ with equality if and only if $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, cf. [1, Example 6.1].

Let $G$ be a connected graph on $n$ vertices. Let $P_n$ be the path on $n$ vertices. By [1, Lemma 6.4], $\rho_s(G) > \rho_s(G - e)$ for an edge $e$ of $G$. Let $T$ be a spanning tree of $G$. Then by Lemma 2 and a result of Collatz and Sinogowitz [2], $\rho_s(G) \geq \rho_s(T) = \rho(T) \geq \rho(P_n)$ with equality if and only if $G = P_n$, cf. [1, Example 6.3].

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