Geodesic congruences in exact plane wave spacetimes

and the memory effect

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Abstract

Displacement and velocity memory effects in the exact, vacuum, plane gravitational wave line element have been studied recently by looking at the behaviour of pairs of geodesics or via geodesic deviation. Instead, one may investigate the evolution of geodesic congruences. In our work here, we obtain the evolution of the kinematic variables which characterise timelike geodesic congruences, using chosen pulse profiles (square and sech-squared) in the exact, plane gravitational wave line element. We also analyse the behaviour of geodesic congruences in possible physical scenarios describable using derivatives (first, second and third) of one of the chosen pulses. Beginning with a discussion on the generic behaviour of such congruences and consequences thereof, we find exact analytical expressions for shear and expansion with the two chosen pulse profiles. Qualitatively similar numerical results are noted when various derivatives of the sech-squared pulse are used. We conclude that for geodesic congruences, a growth (or decay) of shear causes focusing of an initially parallel congruence, after the departure of the pulse. A correlation between the ‘focusing time (or $u$ value, $u$ being the affine parameter)’ and the amplitude of the pulse (or its derivatives) is found. Such features distinctly suggest a memory effect, named in recent literature as $B$ memory.

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I. INTRODUCTION

The memory effect in gravitational wave physics has been a topic of active research interest in recent times [1, 2]. Though yet to be observed in gravitational wave detectors there have been proposals [2, 3] about how it can be seen in advanced versions of the present-day detectors. The physics of memory is related to a net displacement (or a residual velocity) noted in freely falling detectors, caused by the passage of a pulse of gravitational radiation. This leads to a permanent change in the Minkowski spacetimes that exist before the arrival of the pulse and after its departure. The change is connected with spacetime diffeomorphisms taking one asymptotically flat spacetime to another, which do not tend to the identity at infinity. Asymptotically flat spacetimes before the arrival of the pulse and after its departure are therefore inequivalent. It is known that they may be related via BMS (Bondi-Metzner-Sachs) transformations (eg. super-translations) [4, 5].

The first report of such an effect appears in the context of gravitational collapse in globular clusters as noted by Zel’dovich and Polnarev [6]. Subsequently, Braginsky and Grishchuk, [7] while working within linearized gravity, defined the memory effect to be the difference between the quadrupole moments of the source at initial and final times. Later, Christodoulou [8] showed that there is a nonlinear contribution to the effect and argued that this is due to the effective stress energy of the gravitational waves transported to null infinity. Thorne [9] had argued that the non-linear contribution to Christodoulou memory could be attributed to gravitons sourced by a gravitational wave burst. Following this idea, Bieri et al. [10, 11] demonstrated a contribution to memory from other particles having zero rest mass. Thus, they were able to distinguish linear and non-linear contributions as ordinary and null memory. The stress energy travels to null infinity in the latter case only.

Apart from four-dimensional asymptotically flat spacetimes, there exists work on the memory effect in other spacetimes and in higher dimensions [10–12]. Memory effects have also been discussed in the context of modified gravity and massive gravity theories [13].

Very recently, Zhang et al. [14–16] have tried to arrive at the memory effect in the well-known exact plane gravitational wave spacetimes [17–19]. Apart from other analyses in their paper [14], they studied geodesics in this geometry by assuming certain specific forms of the functions which appear in the line element. In particular, they chose a Gaussian pulse (and its derivatives) and numerically solved the geodesic equations to obtain some qualitative
results on the displacement and velocity memory effects. *The appearance of a net relative displacement and/or a net relative velocity caused by the passage of a pulse are termed as the displacement and velocity memory effects respectively.* Building on these ideas we shall show in our work how the behaviour of geodesic congruences may also lead to a memory effect via a change in the shear and expansion of the congruence, caused by the pulse. This memory effect involving geodesic congruences is closer to velocity memory but not quite the same.

In general, a memory effect can therefore be arrived at in three different ways, eventually leading to qualitatively similar broad conclusions. Let us now briefly discuss each and note the differences between them too. We will confine ourselves to the sandwich pulse profiles in exact plane wave metrics \[17, 20\] while studying memory effects. From a motivational standpoint, one may argue that any pulse observed in a future detection of a binary merger event is likely to be finite for a certain range of $u$. The Fourier decomposition of such signals would exhibit a peak frequency (chirp) of the burst itself over a quasi-static low frequency background as observed in the detection events by LIGO \[21\]. However, it goes without saying, that the exact plane wave metric is largely theoretical and has no direct link with present-day GW observations.

The first among the three ways involves obtaining a net displacement between pairs of geodesics, caused by a gravitational wave pulse, after the pulse has left. This may be found by *directly integrating the geodesic equation* to obtain the evolution of the separation of each coordinate, for pairs of geodesics. A second way, largely related to the previous one, is to *directly integrate the geodesic deviation equation* and understand the evolution of the deviation vector. Both these approaches are associated with displacement memory. Additionally, the former may be used to arrive at a velocity memory as discussed in \[14–16\].

Here we choose the third way of arriving at a memory effect, namely by looking at the behaviour of geodesic congruences. This approach is covariant and has been proposed in a recent article by O’Loughlin and Demirchian \[22\] wherein the term $B$-memory ($B$ denotes the tensor $B^i_j$, the covariant gradient of the velocity field) is introduced in the context of impulsive gravitational waves. Our work supports and extends the proposal in \[22\] using the simplest class of pp wave spacetimes—the well-known exact plane gravitational waves.

In the exact plane gravitational wave spacetime, there arises free profile functions ($A_+(u)$ or $A_\times(u)$). Apart from generic results obtained without choosing specific functional forms for
the profiles, we also find exact results with simple pulse profiles (eg. a square pulse and a sech-squared pulse). Further, for the sech-squared pulse we analyse memory using the first, second and third derivatives (which may arise in different physical contexts) of the pulse. ¹

As is well known, a timelike geodesic congruence is studied through the behaviour of the expansion, shear and rotation which comprise the trace, symmetric traceless and anti-symmetric parts of the $B$-tensor. The nature of evolution of these kinematical variables associated with the congruence are first obtained qualitatively using simple inequality arguments. Subsequently, using the pulse profiles mentioned earlier, we obtain the kinematic variables exactly or using numerical methods. By noting the point of convergence, we are able to relate the amplitude of the pulse with the time at which it focuses. We demonstrate shear-induced focusing which causes a permanent change in the expansion after the departure of the pulse. This is $B$-memory as introduced in [22]. In other words, it is the expansion, shear and rotation which may undergo a permanent change caused by the appearance of the pulse. Since the $B$-tensor is the gradient of the normalised velocity field, $B$-memory, as mentioned earlier, has a connection with velocity memory, though it is not quite the same. An important feature of our work is analytical solvability. Unlike the Gaussian pulse or its derivatives, for our choices of the profiles, quite a bit can be done by exactly solving the Raychaudhri equations. We explicitly illustrate the memory effect using the behaviour of shear and expansion, for the chosen pulse profiles, through our largely analytical results. However, the results for the derivatives of one of the pulses are numerically obtained.

In Section II, we write the line element of the vacuum, plane wave spacetimes in Brinkmann coordinates and obtain the geodesic equations. As an illustration, we show the displacement and velocity memory effects using a square pulse. Section III deals with the evolution of the expansion, shear and rotation of geodesic congruences given by the Raychaudhuri equations. A qualitative analysis for both the pulse and its derivatives is followed by exact solutions for the case of a pulse. In Section IV, we numerically analyse the physically interesting cases involving the derivatives of the continuous sech-squared pulse. Finally, Section V is a summary of our results with some comments on future work.

¹ In a recent paper, [23], Shore has looked at the square pulse briefly in an Aichelburg-Sexl impulsive gravitational wave line element, which is different from the spacetime we work with here.
II. GEODESICS IN EXACT PLANE WAVE SPACETIMES

A. Brinkmann coordinates

The exact plane wave spacetimes are a class among general pp-wave spacetimes which solve the vacuum Einstein field equations of General Relativity \[18, 19, 24\]. The metric components are the same at every point on each wave surface. The coordinate system employed in our calculation is the standard Brinkmann coordinates which is both harmonic and global. The line element in Brinkmann coordinates is given by the form:

\[
ds^2 = \delta_{ij}dx^i dx^j + 2dudV + K_{ij}(u)x^i x^j du^2
\]  

(1)

The gravitational field is encoded in the term \(K_{ij}(u)\). which satisfies the wave equation:

\[
\Box (K_{ij}(u)x^i x^j) = 0
\]  

(2)

\(K_{ij}(u)\) is a tracefree, \(2 \times 2\) matrix having two independent components which are known as the polarizations of the plane gravitational wave (+,×). We have,

\[
K_{ij}(u)x^i x^j = \frac{1}{2}A_+(u)[x^2 - y^2] + A_\times(u)xy
\]  

(3)

The polarizations \(A_+(u)\) (plus), \(A_\times(u)\) (cross) are functions of retarded time variable \(u\). Another coordinate system used for this metric is the BJR (Baldwin-Jeffrey-Rosen) coordinate system \[25\] which however suffers from the presence of coordinate singularities.

B. The geodesic equations

The geodesic equations in Brinkmann coordinates having both non-zero polarizations are given as:

\[
\frac{d^2 x}{du^2} = \frac{1}{2}A_+(u)x + \frac{1}{2}A_\times(u)y
\]  

(4)

\[
\frac{d^2 y}{du^2} = -\frac{1}{2}A_+(u)y + \frac{1}{2}A_\times(u)x
\]  

(5)

\[
\frac{d^2 V}{du^2} + \frac{1}{4} \frac{dA_+(u)}{du}(x^2 - y^2) + A_+(u)\left(\frac{dx}{du} - y\frac{dy}{du}\right) + A_\times(u)\left(\frac{dy}{du} + x\frac{dx}{du}\right) + \frac{1}{2}\frac{dA_\times(u)}{du}xy = 0
\]  

(6)
Notice that we have used $u$ as an affine parameter. This is easily checked by writing down the geodesic equation for the $V$ coordinate. The general form for $V(u)$ and $\dot{V}(u)^2$ is obtained by performing some algebra on Eq.(6) and from the geodesic Lagrangian (derived from the metric) in Eq.(1).

\[
\frac{dV}{du} = -\frac{1}{4} A_+(x^2 - y^2) - \frac{1}{2} \left[ \left( \frac{dx}{du} \right)^2 + \left( \frac{dy}{du} \right)^2 \right] - \frac{1}{2} A_x(u) xy - \frac{k}{2} \tag{7}
\]

\[
V(u) = -\frac{1}{2} \left( x \frac{dx}{du} + y \frac{dy}{du} \right) - \frac{k}{2} u + C_1 \tag{8}
\]

The solution for $V(u)$ contains the integration constant $C_1$ and $k$, which is 0 or 1 for null or timelike geodesics, respectively. Thus, for any pulse of a given polarization, if Eqs.(4) and (5) for $x(u)$ and $y(u)$ are analytically solvable, then $V(u)$ also can be analytically obtained.

\section*{C. Memory effects}

The memory effect can be easily realized in the above class of spacetimes by choosing suitable pulse profiles [15, 16]. A simple example is that of a square pulse which is shown in Fig.(1).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{square_pulse.pdf}
\caption{Square pulse with $a=0.5$, $A_0=1$.}
\end{figure}

The analytical form of the pulse chosen is: $A_+(u) = 2A_0^2 \left[ \Theta(u+a) - \Theta(u-a) \right]$. The solutions are obtained by solving the geodesic equations and then matching them at the boundaries. Initially parallel geodesics before the wave region (green vertical lines showing the boundary

\[2\text{In this paper, } \dot{f} = \frac{df}{du}, \text{ for any general } f. \text{ The two symbols are used interchangeably throughout the paper.}\]
of the wave region) are seen to have a non-zero finite separation even after the passage of
the pulse visible in the Figs. (2a) and (2b).

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig2a}
\includegraphics[width=0.4\textwidth]{fig2b}
\caption{(a) $A_0 = 1$, $a = 0.5$ \hspace{1cm} (b) $A_0 = 1$, $a = 0.5$}
\end{figure}

FIG. 2: Displacement memory effect along $x$ and $y$ directions for the first (orange) and
second (blue) geodesics respectively.

Velocity memory effect is obtained by plotting the first derivative of the solutions for geodesic
equations. The results appear in Fig. 3, as shown below:

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig3a}
\includegraphics[width=0.4\textwidth]{fig3b}
\caption{(a) $A_0 = 1$, $a = 0.5$ \hspace{1cm} (b) $A_0 = 1$, $a = 0.5$}
\end{figure}

FIG. 3: Velocity memory effect along $x$ and $y$ directions for the first (orange) and second
(blue) geodesics respectively.

The velocity memory effect as shown, displays a sharp change in the wave region which
settles to a nonzero finite value.

In the following sections, we attempt to explore the possibility of arriving at a somewhat
different memory effect from the evolution of the $B$-tensor.
III. EXPANSION, SHEAR AND ROTATION IN BRINKMANN COORDINATES
FOR EXACT PLANE GRAVITATIONAL WAVES

The geodesic Lagrangian for the exact plane gravitational wave line element written in
Brinkmann coordinates, with \( u \) as the affine parameter, is given as

\[
\mathcal{L} = \dot{x}^2 + \dot{y}^2 + \frac{1}{2}A_+(u)(x^2 - y^2) + A_x(u)xy + 2\dot{V}
\]  

(9)

It is clear that the last term in the R. H. S. of \( \mathcal{L} \) is a total derivative and hence has no
effect on equations of motion for the ‘generalised’ coordinates \( x \) and \( y \). Eq.(7) basically
reduces to an identity. The system becomes two dimensional for the parameter \( u \). The
general formalism for obtaining the evolution of the kinematic variables in two dimensions
(i.e. the Raychaudhuri equations) is available in [26]. The gradient of velocity (found
by differentiation of \( x(u) \) and \( y(u) \) w.r.t \( u \)) can be written as a second rank tensor which
can be decomposed into expansion (trace), shear (symmetric, traceless) and rotation (anti-
symmetric).

\[
\mathcal{B}_{ij} = \partial_j v_i = \begin{pmatrix} \frac{1}{2} \theta & 0 \\ 0 & \frac{1}{2} \theta \end{pmatrix} + \begin{pmatrix} \sigma_+ & \sigma_\times \\ \sigma_\times & -\sigma_+ \end{pmatrix} + \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}
\]  

(10)

The evolution equation for the gradient of velocity may be written as:

\[
v^k \partial_k (\partial_j v^i) = \partial_j f^i - (\partial_j v^k)(\partial_k v^i)
\]

(11)

\[
v^k \partial_k (\mathcal{B}^i_j) = \partial_j f^i - \mathcal{B}^k_j \mathcal{B}^i_k
\]

(12)

In Eqs.(11) and (12) the term \( f^i \) denotes the acceleration per unit mass. Hence \( f^x = \frac{\dot{\theta}}{\dot{u}} \)
and \( f^y = \frac{\dot{\omega}}{\dot{u}} \). The four kinematic variables \( \{ \theta, \sigma_+, \sigma_\times, \omega \} \) obey the evolution Eq.(12) which
lead to separate equations given as:

\[
\frac{d\theta}{du} + \frac{\dot{\theta}^2}{2} + 2(\sigma_+^2 + \sigma_\times^2 - \omega^2) = (\partial_x f^x + \partial_y f^y)
\]

(13)

\[
\frac{d\sigma_+}{du} + \theta \sigma_+ = \frac{1}{2}(\partial_x f^x - \partial_y f^y)
\]

(14)

\[
\frac{d\sigma_\times}{du} + \theta \sigma_\times = \frac{1}{2}(\partial_y f^x + \partial_x f^y)
\]

(15)

\[
\frac{d\omega}{du} + \theta \omega = \frac{1}{2}(\partial_y f^x - \partial_x f^y)
\]

(16)

Eqs.(13) - (16) may be solved for specific pulse profiles (square pulse, sech-squared pulse
and its derivatives) by substituting the values of \( f^x \) and \( f^y \) from their respective geodesic
equations given in Eqs.(11) and (12).
(a) $A_+(u) = 2A_0^2[\Theta(u + a) - \Theta(u - a)]$

(b) $A_+(u) = \frac{1}{2} \text{sech}^2(u)$

FIG. 4: Pulse profiles

Figs. (4a) and (4b) shows the two pulse profiles that we choose to work with in this paper. The nature of the pulses vary in their differentiable nature. We further study consequences for the derivatives of the continuous sech-squared pulse.

A. Qualitative analysis on the evolution of kinematic variables

1. Generic pulse

For any generic pulse having both polarizations $A_+(u)$ and $A_\times(u)$ to be non-zero, the four evolution equations for these kinematic variables become:

\[
\frac{d\theta}{du} + \frac{\theta^2}{2} + 2(\sigma_+^2 + \sigma_\times^2 - \omega^2) = 0 \quad (17)
\]

\[
\frac{d\sigma_+}{du} + \theta \sigma_+ = \frac{1}{2} A_+(u) \quad (18)
\]

\[
\frac{d\sigma_\times}{du} + \theta \sigma_\times = \frac{1}{2} A_\times(u) \quad (19)
\]

\[
\frac{d\omega}{du} + \theta \omega = 0 \quad (20)
\]

A geodesic congruence starting out with initial values for all four variables set to zero, simplifies the Eqs. (17-20). There is no twist in the entire range $u \in (u_i, u_f)$ if $\omega = 0$ initially. We consider only ‘+’ polarization and therefore $\sigma_\times = 0$ always. The resulting

3The green vertical lines show the FWHM (Full Width Half Maximum) region of the pulse.
equations become:

\[
\begin{align*}
\frac{d\theta}{du} + \frac{\theta^2}{2} + 2\sigma_+^2 &= 0 \quad (21) \\
\frac{d\sigma_+}{du} + \theta\sigma_+ &= \frac{1}{2} A_+(u) \quad (22)
\end{align*}
\]

Eq. (21) implies that \(\frac{d\theta}{du}\) is always negative. Therefore, irrespective of its initial value at some \(u_i\), \(\theta\) will eventually diverge to negative infinity. Eq. (22) is multiplied with two on both sides and added/subtracted to/from Eq. (21) yielding a pair of uncoupled, first order, ordinary, differential equations. Then, defining \((\theta + 2\sigma_+) = \xi\) and \((\theta - 2\sigma_+) = \eta\), the corresponding equations turn out to be:

\[
\begin{align*}
\frac{d\xi}{du} + \frac{\xi^2}{2} &= A_+(u) \quad (23) \\
\frac{d\eta}{du} + \frac{\eta^2}{2} &= -A_+(u) \quad (24)
\end{align*}
\]

Note that the variables \(\xi\) and \(\eta\) are the eigenvalues of the \(B\)-matrix when \(\sigma_0\) and \(\omega\) are zero. From Eq. (24) it is clear that \(\dot{\eta}(u) < 0\) as the function representing the pulse is manifestly positive and asymptotically zero in value. Integrating \(\dot{\eta}(u)\) from \(u_i\) (\(u_i\) will be negative and located far from the region around the origin where the pulse is non-zero in value) to \(u_f\) (positive \(u\) value and located far down the positive \(u\) axis), we conclude that since the right hand side of Eq. (24) is the negative of a positive definite number, the change \(\lim_{u \to u_f} \eta(u) - \lim_{u \to u_i} \eta(u) < 0\) is always less than zero. Mathematically, the change in the value of \(\eta\) from \(u \to u_i\) to \(u \to u_f\) can never be set to zero due to the presence of the integral of the pulse which equals the area enclosed (between \(u_i\) and \(u_f\)), and hence it is always nonzero, positive and finite. But no such conclusion can be drawn for \(\xi(u)\) from Eq. (23).

This equation can further be analyzed as yielding:

\[
\xi\big|_{u_i}^{u_f} = -\frac{1}{2} \int_{u_i}^{u_f} \xi^2(u) \, du + \text{Area enclosed by the pulse} (= a_1) \quad (25)
\]

Eq. (25) implies that \(\xi\big|_{u_i}^{u_f} = 0\) only if \(\int_{u_i}^{u_f} \xi^2(u) \, du = a_1\)

(In the following we will denote (\(P\) may be \(\theta, \sigma_+, \xi\) or \(\eta\)), \(\lim_{u \to u_f} P(u)\) as simply \(P(u_f)\), assuming its value to be zero when \(u \to u_i\)). Thus, there are essentially two possibilities:

\textbf{(a) } \xi(u_f) \geq 0, \eta(u_f) < 0:

The resulting inequalities are:
\[ \theta(u_f) - 2\sigma_+(u_f) < 0, \quad \theta(u_f) + 2\sigma_+(u_f) \geq 0 \]

As obvious from the above, \( \theta \to -\infty \) and \( \sigma_+ \to \infty \) as \( u \to u_f \) (focusing) is possible and will be shown as a consequence in various examples given later. On the other hand \( \theta \to -\infty \) and \( \sigma_+ \to -\infty \) is not permissible because, as stated above, at \( u = u_f \) (focusing), \( \theta(u_f) + 2\sigma_+(u_f) \geq 0 \).

(b) \( \xi(u_f) \leq 0, \eta(u_f) < 0 \):

The resulting inequalities are:

\[ \theta(u_f) - 2\sigma_+(u_f) < 0, \quad \theta(u_f) + 2\sigma_+(u_f) \leq 0 \]

Hence,

\[ \theta(u_f) < 0, \quad \left( \sigma_+(u_f) > \frac{\theta(u_f)}{2} \right) \text{ or } \left( \sigma_+(u_f) \leq -\frac{\theta(u_f)}{2} \right) \quad (26) \]

Thus, \( \theta \to -\infty, \sigma_+ \to \infty \) as well as \( \theta \to -\infty, \sigma_+ \to -\infty \) are both permissible, modulo constraints. We will see examples of this too, later.

2. Derivatives of the pulse

Let us now analyse the nature of evolution of the kinematical variables for the first three derivatives of the pulse. The physical relevance which motivates us to look at consequences for the derivatives of a pulse are given in the following section [IV]. Eqs. (21) and (22) are now modified in their R. H. S. with the pulse being replaced by its derivative (first, second or third). Since we consider an even pulse (sech-squared pulse), the first and third derivatives are odd functions and the second derivative is even.

• 1st derivative and 3rd derivative

Let us consider the case for the first derivative. The new set of equations are:

\[
\begin{align*}
\frac{d\xi}{du} + \frac{\xi^2}{2} &= \frac{dA_+}{du} \quad (27) \\
\frac{d\eta}{du} + \frac{\eta^2}{2} &= -\frac{dA_+}{du} \quad (28)
\end{align*}
\]
We employ the same trick as was done in the case of the pulse. The integral of the 1st derivative of the pulse is positive as long as we assume $u_i$ far down the negative $u$ axis and $u_f$ relatively closer to $u = 0$ on the positive $u$ direction, but reasonably away from the region where the function is clearly non-zero. Thus, in this case we land up with the conclusions similar to the case of the pulse discussed just above.

The same line of argument holds for the 3rd derivative too (odd function).

- **2nd derivative**

In this case, the resulting equations are:

$$\frac{d\xi}{du} + \frac{\xi^2}{2} = \frac{d^2 A_+}{du^2} \tag{29}$$

$$\frac{d\eta}{du} + \frac{\eta^2}{2} = -\frac{d^2 A_+}{du^2} \tag{30}$$

The second derivative of the pulse is an even function. The integral of the second derivative from $u_i$ to $u_f$ can be seen to be negative for a sech-squared pulsed though other examples do exist. We discuss this somewhat opposite behaviour because it is different. Here, we will find that, at $u_f$, $\xi(u_f) < 0$ but $\eta(u_f)$ may be greater or less than zero. Thus, we have,

$$\theta(u_f) + 2\sigma_+(u_f) < 0, \quad \theta(u_f) - 2\sigma_+(u_f) > 0$$

Hence, it is clear that $\theta \to -\infty$, $\sigma_+ \to -\infty$ is allowed, though $\theta \to -\infty$, $\sigma_+ \to +\infty$ is not. On the contrary, if $\xi < 0$, $\eta < 0$, then,

$$\theta(u_f) + 2\sigma_+(u_f) < 0, \quad \theta(u_f) - 2\sigma_+(u_f) < 0$$

Here, $\theta \to -\infty$ and $\sigma_+ \to -\infty$ and $\theta \to -\infty$ and $\sigma_+ \to +\infty$ may both arise and are allowed.

We will illustrate some of the above-mentioned features related to the case of the second derivative of a pulse, when we discuss the example of a sech-squared pulse later.

We now move on to specific examples in the following section.
B. Kinematic variables for square pulse in plus polarization

The results given below for the square pulse are fully analytical. The values of $f^x$ and $f^u$ are obtained from the geodesic equations of the pulse profile which is of the form given in Fig. (4a). We have,

$$f^x = \frac{d^2x}{du^2} = \begin{cases} 0 & u \leq -a \\ A_0^2x & -a \leq u \leq a \\ 0 & u \geq a \end{cases}$$  \hspace{1cm} (31)$$

$$f^y = \frac{d^2y}{du^2} = \begin{cases} 0 & u \leq -a \\ -A_0^2y & -a \leq u \leq a \\ 0 & u \geq a \end{cases}$$  \hspace{1cm} (32)$$

Eqs. (13)-(16) in the first ($u \leq -a$) and third ($u \geq a$) regions have zero value on their R. H. S. We assume initial values of all the kinematic variables to be zero. So, $\sigma_x$ and $\omega$ become zero in all the regions. Thus, the evolution equations in the second region ($-a < u < a$) become:

$$\frac{d\theta}{du} + \frac{\theta^2}{2} + 2\sigma_x^2 = 0$$  \hspace{1cm} (33)$$

$$\frac{d\sigma_+}{du} + \theta\sigma_+ = A_0^2$$  \hspace{1cm} (34)$$

Hence, using the transformed variables $\xi$ and $\eta$ we solve the Eqs. (23) and (24) in all three regions. The solutions in the second (wave) region turn out as:

$$\xi = 2A_0 \tanh[A_0(u + C_1)]$$  \hspace{1cm} (35)$$

$$\eta = 2A_0 \tan[A_0(C_2 - u)]$$  \hspace{1cm} (36)$$

Thus, $\theta$ and $\sigma_+$ can found from $\xi$ and $\eta$. Thereafter, matching values at $u = -a$ (which is zero for both variables) the value for $C_1, C_2$ is obtained. In the same way, the solution for the region $u \geq a$ is obtained by matching at $u = a$. The final solutions for expansion and shear are:

$$\theta(u) = \begin{cases} 0 & u \leq -a \\ A_0[\tanh[A_0(u + a)] - \tan[A_0(u + a)]] + \frac{1}{u-a+A_0^{-1}\coth[2aA_0]} & -a \leq u \leq a \\ \frac{1}{u-a-A_0^{-1}\cot[2aA_0]} & u \geq a \end{cases}$$  \hspace{1cm} (37)$$
\[
\sigma_+(u) = \begin{cases} 
0 & u \leq -a \\
\frac{A_0}{2} \left( \tanh[A_0(u + a)] + \tan[A_0(u + a)] \right) & -a \leq u \leq a \\
\frac{1}{2} \left( \frac{1}{u-a+A_0} - \frac{1}{u-a-A_0} \right) \cot[2aA_0] & u \geq a 
\end{cases}
\]

(a) \(A_0 = 1, a = 0.5\)

(b) \(A_0 = 1, a = 0.5\)

FIG. 5: Expansion and shear variation in case of square pulse

The plots in Figs. (5a) and (5b) have kinks at \(u = -0.5, 0.5\) because of the nature of \(A_+(u)\). The expansion \(\theta\) develops a negativity as \(u\) enters the region where the pulse is non-zero. This acquired negativity drives it towards a focal point after the pulse has departed, i.e. beyond \(u = a\). The appearance of the focal point is what we noted earlier as the intersection of the geodesics beyond \(u = a\). Similarly, the initially zero shear acquires a positivity on entering the region where the pulse is active and non-zero. Subsequently, even after the departure of the pulse the shear keeps on increasing. Thus, there is a permanent change in the shear and expansion (after the pulse departs), which is in contrast to their zero value before the arrival of the pulse. Further, we note that \(\theta \to -\infty\) at \(u = a + A_0^{-1} \cot[2aA_0]\), which clearly depends on the width and height of the pulse.

It turns out that \(\xi \geq 0\) and \(\eta < 0\) in region 3 (i.e. \(u \geq a\)), thereby obeying the inequality obtained for case (a) in (III A 1). The solutions and conclusions for cross-polarization are exactly the same as for plus polarization with \(\sigma_+\) replaced by \(\sigma_\times\).

C. Kinematic variables for sech-squared pulse in plus and cross polarizations

Here, the pulse profile is a continuous function and hence the set of two coupled equations can be solved for the entire range of the affine parameter. We show here the results for
plus polarization (for $\times$ polarization the analytical results and plots are similar). From the geodesic equations, we obtain $f^x = \frac{1}{4} x \text{sech}^2(u)$ and $f^y = -\frac{1}{4} y \text{sech}^2(u)$ for the pulse profile as given in Fig. (4b). Subsequently, we solve Eqs. (21), (22) by using Eqs. (23), (24).

\[ \frac{d\xi}{du} + \frac{1}{2} \xi^2 = \frac{1}{2} \text{sech}^2(u) \]  
\[ \frac{d\eta}{du} + \frac{1}{2} \eta^2 = -\frac{1}{2} \text{sech}^2(u) \]  

Eqs. (39) and (40) can be solved analytically by substituting $\xi = 2\dot{\alpha}/\alpha$, $\eta = 2\dot{\beta}/\beta$, which lead to:

\[ \frac{d^2\alpha}{du^2} = \frac{1}{4} \text{sech}^2(u)\alpha \]  
\[ \frac{d^2\beta}{du^2} = -\frac{1}{4} \text{sech}^2(u)\beta \]  

The solutions of Eqs. (41) and (42) are:

\[ \alpha(u) = C_1 K\left[\frac{1}{2}(1 - \tanh(u))\right] + C_2 Q_{-\frac{1}{2}}\left(\tanh(u)\right) \]  
\[ \beta(u) = C_3 P_{\frac{1}{2}(\sqrt{2}-1)}\left(\tanh(u)\right) + C_4 Q_{\frac{1}{2}(\sqrt{2}-1)}\left(\tanh(u)\right) \]  

where, in Eq. (43), the first function is a complete elliptic integral of the first kind and the second is a Legendre function of the second kind. In Eq. (44) we have Legendre functions of first and second kinds respectively. The relationship between the kinematic variables \( \{\theta, \sigma_+\} \) and \( \{\alpha, \beta\} \) are:

\[ \theta = \left(\frac{\dot{\alpha}}{\alpha} + \frac{\dot{\beta}}{\beta}\right) \]  
\[ \sigma_+ = \frac{1}{2} \left(\frac{\dot{\alpha}}{\alpha} - \frac{\dot{\beta}}{\beta}\right) \]  

Substituting back the functional forms of $\alpha$, $\beta$ as obtained from Eqs. (43) and (44) in Eq. (45) we get analytic expressions of $\theta$ and $\sigma_+$. Since Eqs. (41) and (42) are second order ordinary differential equations, we end up with a total of four constants. However, our initial Eqs. (39) and (40) were first order and hence we only have two arbitrary constants which are fixed by setting the value of $\theta, \sigma_+$ to be zero at an initial value of $u$ (i.e. we have an initially parallel geodesic congruence). Thus, among the four constants in Eqs. (43) and (44) we choose two freely and the other two are fixed by constraining $\dot{\alpha} = \dot{\beta} = 0$ at an initial $u$. The plots generated for the kinematic variables are shown in Fig. 6.

4The same relationships between the variables \( \{\theta, \sigma_+\} \) and \( \{\xi, \eta\} \) are used.
Conclusions from Figs. (6a), (6b) are largely the same as found earlier for a square pulse. A permanent change in the expansion and shear of the congruence is noted here too. The plot for shear does not exhibit any kink (as seen in the square pulse)—the smoothness due to the continuous nature of the pulse.

In both the cases (square and sech-squared), expansion is always negative and \( \sigma_+ \) is monotonically increasing. It may be checked that the constraints imposed from the analysis of a generic pulse as discussed in Section III A, hold.

**IV. DERIVATIVES OF SECH-SQUARED PULSE**

We now move on towards applying the above formalism for calculating kinematic variables, when we have various derivatives of a sech-squared pulse. The nature of derivatives and its integrals over the duration of the pulse have been discussed previously in [27] and much later by Zhang et al. in [14]. In linear theory, the source quadrupole moment is related to the curvature tensor via the formula:

\[
R_{i0j0} = \frac{G}{3r} \frac{d^4D_{ij}}{dt^4} \tag{46}
\]

The above-mentioned authors defined integrals over the Riemann tensor in the limit where the wave is localised and subsequently looked at their values for the first three derivatives of chosen pulse profiles.
\[ I^{(3)} = \int_{t_i}^{t_f} dt \int_{t_i}^{t'} dt' \int_{t_i}^{t''} dt'' R_{0i0j}(t'') \]  
\[ I^{(2)} = \int_{t_i}^{t_f} dt \int_{t_i}^{t'} dt' R_{0i0j}(t') \]  
\[ I^{(1)} = \int_{t_i}^{t_f} dt R_{0i0j}(t) \]  

Depending upon the physical scenario (such as, collapse or flybys) the initial and final quadrupole moment would differ and hence one can obtain the nature of an incoming pulse by analysing the number of times the Riemann tensor has changed sign (i.e. by evaluating \( I^{(1)} \), \( I^{(2)} \) and \( I^{(3)} \)). Even in full nonlinear theory, one can guess the approximate nature of a pulse by knowing the values of these integrals, although Eq.(46) does not hold. Here we calculate these integrals for the derivatives of a sech-squared pulse and explain the corresponding physical scenario. Subsequently, we analyse the nature of expansion and shear for geodesic congruences, in the presence of the various derivatives of the sech-squared pulse.

(a) 1st derivative:

Recall the pulse profile given before. Its derivative would lead to (Fig. 7),

\[ A_1(u) = \frac{dA_+}{du}(u) = \frac{1}{2} \frac{d}{du} \text{sech}^2(u) = -\text{sech}^2(u) \tanh(u) \]  

![Fig. 7: 1st derivative of sech-squared pulse, flyby scenario](image)

The integrals given by Eqs.(47)-(49) are evaluated for this pulse. The values are:

\[ I^{(1)} = 0 \quad I^{(2)} = 1 \quad I^{(3)} \rightarrow \infty \]  

This could correspond to the case of a flyby leading to gravitational bremsstrahlung. Kovacs and Thorne [28] gave an analytic expression for the metric perturbation as a
function of time and other parameters related to the binary (viz. mass, inclination angle, impact parameter). At initial times \((t \to -\infty)\), the dominant contribution is constant. Hence, the quadrupole moment at initial instant is proportional to a quadratic function of time. Thus, both \(I^{(2)}\) and \(I^{(3)}\) are nonzero, following from Eq.(46). The non-zero kinematic variables are solved numerically from Eqs.(21) and (22) in Mathematica 10. The plots turn out to be:

(a) Expansion, \(-10 < u < 7\)

(b) Shear, \(-10 < u < 7\)

FIG. 8: Expansion and shear variation in case of first derivative of sech-squared pulse

Figs. (8a) and (8b) are in accordance with the constraint imposed by the condition \(\theta \to -\infty\) and \(\sigma_+ \to +\infty\) as discussed previously in Section III A.

(b) 2nd derivative:

In this case, the pulse profile is:

\[
A_2(u) = \frac{d^2 A_+(u)}{du^2} = \frac{1}{2} \frac{d^2}{du^2} \text{sech}^2(u) = -\text{sech}^4(u) + 2 \tanh^2(u) \text{sech}^2(u) \quad (52)
\]

FIG. 9: 2nd derivative of sech-squared pulse

\textsuperscript{5} A_+(u) is now replaced with \(A_1(u)\). Also, in the case of second and third derivatives \(A_+(u)\) is replaced by \(A_2(u)\) and \(A_3(u)\) respectively in the right hand side of Eq.(22).
As done for the 1st derivative, we find the integrals given by Eqs.(47)-(49) for the pulse given in Eq.(52). The values are:

\[ I^{(1)} = 0 \quad I^{(2)} = 0 \quad I^{(3)} = 1 \] (53)

This scenario was considered by Braginsky and Thorne [29] where they distinguished between bursts with or without memory within linearized gravity. In the latter case, the metric perturbation vanishes beyond the wave region and hence \( I^{(2)} \) vanishes and thus, no memory effect is possible\(^6\). In order to have a finite value of \( h_{ij} \) beyond the wave region, \( I^{(2)} \) has to be nonzero and finite. The plots are obtained after solving numerically\(^5\) in Mathematica 10.

(a) Expansion, \(-10 < u < 7\)  
(b) Shear, \(-10 < u < 5\)

FIG. 10: Expansion and shear variation in case of second derivative of sech-squared pulse

In this case, both expansion and shear diverge to minus infinity, which is permissible from the generic analysis (Section III A).

(c) 3rd derivative:

In this case, the pulse profile is given as:

\[
A_3(u) = \frac{d^3 A_1(u)}{du^3} = \frac{1}{2} \frac{d^3}{du^3} \text{sech}^2(u) = 8 \text{sech}^4(u) \tanh(u) - 4 \tanh^3(u) \text{sech}^2(u) \] (54)

\(^6\)Exact plane wave spacetimes are exact solutions in full nonlinear GR. Hence, even in this case we observe a memory effect.
The integrals for this pulse (as given in Eq. (54)) become:

\[ I^{(1)} = 0 \quad I^{(2)} = 0 \quad I^{(3)} = 0 \]  \hspace{5cm} (55)

This scenario is of gravitational collapse \[ [27] \]. The quadrupole moment tensor is initially and finally time independent. Hence, the first derivative of the quadrupole moment tensor vanishes. From Eq. (46) one finds that \( I^{(3)} \) vanishes. This implies that the minimum number of turning points for a pulse from gravitational collapse has to have at least three sign changes. The numerical solution obtained after solving Eqs. (21) and (22) leads to the following plots:

(a) Expansion, \(-10 < u < 4\)  \hspace{5cm} (b) Shear, \(-10 < u < 4\)

The nature of plots in Fig. (12) also obey the constraint \( \theta \rightarrow -\infty \) and \( \sigma_+ \rightarrow +\infty \). Here too, like in the case of second derivative, both expansion and shear diverges to minus infinity (allowed via the qualitative analysis in III A above).

The plots of expansion and shear in Figs. (8, 10, 12) have certain differences with the plots in Fig. (6). The expansion is relatively of the same nature in all of these three cases as it is
always negative (given that the general inequality constraint is similar for the pulse and its derivatives). No such definite constraint can be imposed on the sign of shear except that it diverges either to $+\infty$ or $-\infty$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig13}
\caption{Plot of amplitude versus $u_f$ (focusing value). The plot is done in a log scale along both the axis.}
\end{figure}

In order to distinguish between visible effects for the different derivatives of the pulse (including the pulse itself) we have shown in Fig.[13], the variation of $u_f$ (value of $u$ where focusing occurs) with the amplitude of the pulse\footnote{Amplitude here denotes the coefficient with which the pulse is multiplied. For example, $A_+ (u) = a \text{ sech}^2(u)$ in the case of the pulse. In the Fig.\ref{fig13} $a$ ranges from $10^{-4}$ to 1 (arbitrary units).} for all cases. We note that as the amplitude increases, focusing occurs earlier (lower values of $u$, the affine parameter) and vice-versa. Physically, it means that the focusing value $u_f$ is determined by the peak amplitude of the gravitational wave pulse acting as a converging lens. It is also shown from the plot that for a particular amplitude the focusing for the pulse happens at lower values. Moreover, for the derivatives of the sech-squared pulse, with the third derivative of the pulse, focusing happens earlier as compared to the cases involving the first two derivatives (in fact, the $u_f$ values for the 1st and 2nd derivative cases are quite close). Therefore, in some sense, the $u_f$ versus amplitude plots may be a way to quantify and differentiate between the effects due a pulse and its various derivatives.

Finally, if we have an \textit{initially expanding congruence} (unlike the ones discussed above where we looked at an initially parallel congruence), we have checked (not shown) that focusing occurs and is dependent on the gravitational wave amplitude. As the amplitude increases,
one needs to prescribe the initial data at lower values of $u$ and vice versa. Similarly if we have an initially converging congruence, there is always focusing. Thus, for all types of initial configurations, we observe focusing as well as a permanent distortion (shear) for timelike geodesic congruences encountering a localised gravitational wave pulse (or its derivatives).

V. CONCLUSIONS

We have, in this article, tried to arrive at an understanding of a memory effect using the kinematic variables that define a geodesic congruence, namely the expansion, shear and rotation. In the exact, vacuum plane parallel gravitational wave line element, the Raychaudhuri equations for timelike geodesic congruences are written down and solved. We exploit the known fact that geodesic motion in such spacetimes reduces to a motion in an effective two dimensional $(x, y)$ mechanical system where the coordinate $u$ acts like ‘time’. The ‘geodesic Lagrangian’ becomes that of a non-relativistic oscillator along $x$ and an inverted oscillator along $y$, with ‘time’ ($u$)-dependent frequency $\omega$. The equation for the coordinate $V(u)$ is redundant since its geodesic equation reduces to an identity. Following standard methods, we find the behaviour of $\theta$ and $\sigma_+ \sigma_\times$ which are non-zero for plus (cross) polarization for general as well as specific choices of the pulse profiles. There is no rotation ($\omega$) involved in the congruences we have worked with here.

In contrast to noting a memory effect through geodesics or geodesic deviation, we have shown how kinematic variables like expansion and shear can carry information about memory. Qualitative treatment of the case of a generic pulse or its derivatives lead to constraints on the values of expansion and shear as $u \to u_f$ (focusing). These have been discussed in detail. Quantitative solutions (for specific pulses) for expansion and shear obtained above are in full agreement with the inequality constraints found in the qualitative analysis. The plots for expansion and shear for both the pulse and its derivatives show divergences at specific values of $u = u_f$. The exact location of $u_f$, expectedly, depends on the functional forms representing the pulse or its derivatives.

Furthermore, we note that the value of $u$ where focusing happens (along with the growth/decay of shear) after the pulse has departed, depends on the amplitude and the width of the pulse. We have plotted $u_f$ as a function of the amplitude of the pulse. We observe that as the amplitude increases the value of $u_f$ shrinks, showing that the pulse fo-
cuses more acutely. Even for an initial expanding congruence, there is focusing dependent on the pulse amplitude thereby resembling what is seen in a converging lens. Considering this analogy with geometric optics, we may also associate the amplitude of a pulse with the inverse of the focal length. Hence, pulse induced focusing coupled with a change of shear can act as yardsticks for understanding $B$-memory.

The fact that null geodesics do form caustics in exact, plane gravitational wave spacetimes of fixed polarization had been shown many years ago in the work of Bondi and Pirani [20]. However, their work does not involve studying the behaviour of the kinematic variables of geodesic congruences in order to figure out focusing effects or a change in shear. Further, in our studies, we demonstrate how such benign focusing for timelike geodesic congruences occur as induced by the appearance of a pulse. There is no real singularity in the spacetime (the invariant scalars are finite everywhere). The point of intersection of the geodesics implies deviation going to zero (which coincides with the location in $u$ where the expansion $\theta$ diverges to negative infinity) and is thus a critical point where the coordinate singularity of the metric (when written in BJR coordinates) appears [15]. Our explicit and detailed analysis of shear-induced focusing and its association with $B$-memory are both new and so are the numerous exact analytical solutions showcasing the $B$-memory effect directly.

A more complete and detailed treatment of the Raychaudhuri equations with both the $+$ and $\times$ profiles simultaneously present can be an extension of our work. It will also be interesting to note if rotation ($\omega$), when initially present, has any role to play in controlling the eventual focusing of geodesics. It is possible that rotation may prohibit focusing leading to finite changes in the expansion and shear. Studying the influence of a gravitational wave pulse on the evolution of the full $B$-matrix can also be an elegant and unified approach towards arriving at an associated memory effect. We conclude with the hope that in future, such studies on the kinematic evolution of geodesic congruences in relevant spacetimes of interest will be able to throw more light on newer aspects of $B$-memory.
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