ALGEBRAIC FIBRE SPACES WITH STRICTLY NEF RELATIVE ANTI-LOG CANONICAL DIVISOR

JIE LIU, WENHAO OU, JUANYONG WANG, XIAOKUI YANG, AND GUOLEI ZHONG

ABSTRACT. Let $(X, \Delta)$ be a projective klt pair, and $f : X \to Y$ a fibration to a smooth projective variety $Y$ with strictly nef relative anti-log canonical divisor $-(K_{X/Y} + \Delta)$. We prove that $f$ is a locally constant fibration with rationally connected fibres, and the base $Y$ is a canonically polarized hyperbolic projective manifold. In particular, when $Y$ is a single point, we establish that $X$ is rationally connected. Moreover, when $\dim X = 3$ and $-(K_X + \Delta)$ is strictly nef, we prove that $-(K_X + \Delta)$ is ample, which confirms the singular version of the Campana-Peternell conjecture for threefolds.

1. INTRODUCTION

In this paper, we study the geometric structures of a projective klt pair $(X, \Delta)$ admitting a fibration $X \to Y$ with strictly nef relative anti-log canonical divisor $-(K_{X/Y} + \Delta)$. First, we propose the following question, which is the singular version of Campana and Peternell’s conjecture:

**Conjecture 1.1.** Let $(X, \Delta)$ be a projective klt pair. If $-(K_X + \Delta)$ is strictly nef, then $-(K_X + \Delta)$ is ample.
Recently, Li, the second author and the fourth author proved in [LOY19] that smooth projective varieties with strictly nef anti-canonical divisors are rationally connected ([LOY19, Theorems 1.2 and 1.3]). By the well-known result of Campana and Kollár-Mori-Miyaoka ([Cam92, KMM92]), this provides important evidences for an affirmative answer to the Campana and Peternell conjecture, i.e., the smooth case of Conjecture 1.1 in all dimensions. As motivated by [LOY19, Theorem 1.3], we consider a particular case of Conjecture 1.1 in the singular pair setting (cf. [Zha06]).

**Conjecture 1.1.** Let \((X, \Delta)\) be a projective klt pair. If \(-(K_X + \Delta)\) is strictly nef, then \(X\) is rationally connected.

Before giving an affirmative answer of Conjecture 1.2, we recall that the augmented irregularity \(q^*(X)\) of a normal projective variety \(X\) is defined as the maximum of the irregularities \(h(X, O_X)\), where \(X \to X\) runs over the (finite) quasi-étale covers of \(X\) (cf. [NZ10, Definition 2.6]). The first main result of this paper is:

**Theorem A.** Let \(X\) be a normal projective variety. Suppose that there is an effective \(\mathbb{Q}\)-divisor \(\Delta\) on \(X\) such that the pair \((X, \Delta)\) is klt and that \(-(K_X + \Delta)\) is strictly nef. Then \(X\) is rationally connected. In particular, the augmented irregularity \(q^*(X) = 0\).

In the following, we study the relative case of the above setting by considering a fibration \(f : X \to Y\) between projective manifolds. Miyaoka proved in [Miy93, Theorem 2] that, the relative anti-canonical divisor \(-K_{X/Y}\) is never ample if \(f\) is smooth and \(Y\) is not a single point. In the past few decades, many generalizations have been established by dropping the smoothness of \(X\) and \(f\) (e.g. [Zha96], [ADK08], [AD13]). A remarkable one is the following result in [AD13, Theorem 5.1] established by Araujo and Druel:

**Theorem 1.3 (Araujo-Druel).** Let \(X\) be a normal projective variety and let \(f : X \to C\) be a fibration onto a smooth projective curve \(C\). Assume that there exists an effective \(\mathbb{Q}\)-Weil divisor \(\Delta\) on \(X\) such that \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier.

1. If \((X, \Delta)\) is log canonical over the generic point of \(C\), then \(- (K_{X/C} + \Delta)\) is not ample.
2. If \((X, \Delta)\) is klt over the generic point of \(C\), then \(- (K_{X/C} + \Delta)\) is not nef and big.

One may wonder whether \(-K_{X/Y}\) can be strictly nef in the relative setting. Indeed, there is an example constructed by Mumford. Let \(C\) be a smooth curve of genus at least two. There exists a vector bundle \(E\) of rank two over \(C\) such that the first Chern class \(c_1(E) = 0\) and the tautological bundle \(\mathcal{O}_{PE}(1)\) of the projectivization \(X := PE\) is strictly nef. It is easy to see that \(-K_{X/C}\) is strictly nef. The second main result in this paper is to investigate the geometric structures for such fibrations. More precisely, we obtain:

**Theorem B.** Let \(f : X \to Y\) be a fibration from a normal projective variety \(X\) onto a smooth projective variety \(Y\). Suppose that there exists an effective \(\mathbb{Q}\)-Weil divisor \(\Delta\) on \(X\) such that \((X, \Delta)\) is a klt pair and \(- (K_{X/Y} + \Delta)\) is strictly nef. Then \(f\) is a locally constant fibration with rationally connected fibres and \(Y\) is a canonically polarized hyperbolic projective manifold.

Recall that, a projective manifold \(Y\) is said to be **canonically polarized** if its canonical divisor \(K_Y\) is ample and to be **hyperbolic** if any holomorphic map \(C \to Y\) from the complex line \(C\) is constant. As an application of Theorem B and techniques developed in [Ou21] (see also [Dru17, Theorem 1.1]), we obtain characterizations of the geometric structures of regular foliations on projective manifolds with strictly nef anti-canonical divisor.
Corollary C. Let $\mathcal{F}$ be a foliation on a projective manifold $X$. Assume that either $\mathcal{F}$ is regular, or $\mathcal{F}$ has a compact leaf.

1. If $-K_{\mathcal{F}}$ is strictly nef, then there exists a locally constant fibration $f : X \to Y$ with rationally connected fibres over a canonically polarized hyperbolic projective manifold $Y$ such that $\mathcal{F}$ is induced by $f$. In particular, $\mathcal{F}$ is algebraically integrable.

2. If the fundamental group $\pi_1(X)$ is virtually solvable (i.e., a finite index subgroup of $\pi_1(X)$ is solvable), then $-K_{\mathcal{F}}$ cannot be strictly nef.

Finally, we show that Conjecture 1.1 holds when $\dim X = 3$, which extends results in [Ser95, Theorem 3.9] and [Ueh00, Main Theorem] (see also [HL20, Conjecture 1.2]).

Theorem D. Let $X$ be a normal projective threefold. If there is an effective $\mathbb{Q}$-divisor $\Delta$ on $X$ such that the pair $(X, \Delta)$ is klt and that $-(K_X + \Delta)$ is strictly nef, then $-(K_X + \Delta)$ is ample.

To end up this section, we propose a singular version of Serrano’s conjecture ([Ser95]), which has an affirmative answer when $\dim X \leq 2$ (cf. [HL20, Corollary 1.8]).

Question 1.4. Let $(X, \Delta)$ be a projective klt pair, and $L$ be a strictly nef $\mathbb{Q}$-divisor on $X$. Is $K_X + \Delta + tL$ ample for sufficiently large $t \gg 1$?

Acknowledgement. J. Liu is supported by the NSFC grants No. 11688101 and No. 12001521. J. Wang is supported by the National Natural Science Foundation of China project ‘Geometry, analysis, and computation on manifolds’. X. Yang is supported by the NSFC grant 12171262. G. Zhong is supported by a President’s Scholarship of NUS.

2. Preliminary results

Throughout this paper, we refer to [KM98, Chapter 2] for different kinds of singularities. By a projective klt (resp. canonical, dlt) pair $(X, \Delta)$, we mean that $X$ is a normal projective variety, $\Delta \geq 0$ is an effective $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and $(X, \Delta)$ has only klt (resp. canonical, dlt) singularities.

A $\mathbb{Q}$-Cartier divisor $L$ on a projective variety is said to be strictly nef if $L \cdot C > 0$ for every (complete) curve $C$ on $X$. We also recall the definition of almost strictly nef divisors for the convenience of the proofs in the later sections.

Definition 2.1 (cf. [CCP08, Definition 1.1]). A $\mathbb{Q}$-Cartier divisor $L$ on a normal projective variety $X$ is called almost strictly nef, if there exist a birational morphism $\pi : X \to Y$ to a normal projective variety $Y$ and a strictly nef $\mathbb{Q}$-divisor $L_Y$ on $Y$ such that $L = \pi^*L_Y$.

One of the motivation to introduce almost strictly nef divisors is to study the non-uniruled case of Question 1.4 via the descending of Iitaka fibrations, in which case, the pullback of a strictly nef divisor to a higher model which resolves the indeterminacy is almost strictly nef (cf. [CCP08, Theorem 2.6]).

Now, we show that, under the condition of Question 1.2, such $X$ is uniruled. This permits us to take a first glance on the geometric property of a projective klt pair with (almost) strictly nef anti-log canonical divisor.

Proposition 2.2. Let $(X, \Delta)$ be a projective pair such that $-(K_X + \Delta)$ is almost strictly nef. Then $X$ is uniruled, i.e., $X$ is covered by rational curves.

Proof. Suppose the contrary that $X$ is not uniruled. Then $K_X$ is pseudo-effective as a Weil divisor (cf. [BDPP13]) by considering a resolution of $X$. In other words, $K_X$
lies in the closure of the cone generated by effective Weil divisors on $X$ (cf. [MZ18, Definition 2.2]). So $-\Delta = -(K_X + \Delta) + K_X$ is also pseudo-effective (as a Weil divisor) and thus $\Delta = 0$. This in turn implies the almost strict nefness of $-K_X$, a contradiction to the pseudo-effectivity of $K_X$. □

In the remaining part of this section, we review the definition and basic properties of locally constant fibrations and flat vector bundles for the convenience of readers.

Let $\phi : X \to Y$ be a proper morphism between normal varieties. We say that $\phi$ is a fibration if $\phi^* \mathcal{O}_X = \mathcal{O}_Y$.

**Definition 2.3** ([MW21, Definition 2.6]; cf. [Wan20, Definition 1.6]). Let $\phi : X \to Y$ be a surjective morphism with connected fibres between analytic varieties, and let $\Delta$ be a Weil $\mathbb{Q}$-divisor on $X$.

1. The morphism $\phi : X \to Y$ is said to be a locally constant fibration with respect to the pair $(X, \Delta)$ if it satisfies the following conditions:
   - The morphism $\phi : X \to Y$ is a (locally trivial) analytic fibre bundle with the fibre $F$.
   - Every component $\Delta_i$ of $\Delta$ is horizontal (i.e., $\phi(\Delta_i) = Y$).
   - There exist a representation $\rho : \pi_1(Y) \to \text{Aut}(F)$ of the fundamental group $\pi_1(Y)$ of $Y$ to the automorphism group $\text{Aut}(F)$ of $F$ and a Weil $\mathbb{Q}$-divisor $\Delta_\rho$ on $F$ which is invariant under the action of $\pi_1(Y)$, so that $(X, \Delta)$ is isomorphic to the quotient of $(Y^\text{univ} \times F, \text{pr}_Y^* \Delta_\rho)$ over $Y$ by the action of $\pi_1(Y)$ on $Y^\text{univ} \times F$. Here $Y^\text{univ}$ is the universal cover of $Y$, and the action of $\gamma \in \pi_1(Y)$ is defined to be
     $$\gamma \cdot (y, z) := (\gamma \cdot y, \rho(\gamma)(z))$$
     for any $(y, z) \in Y^\text{univ} \times F$.
2. The morphism $\phi : X \to Y$ is simply said to be locally constant if it satisfies the above conditions for $\Delta = 0$.
3. For the convenience of the notation, we denote by $\text{Aut}(F, \Delta_\rho)$ the group of automorphisms of $F$ which leave $\Delta_\rho$ invariant, counted with the multiplicities.

One of the most important examples of locally constant families are flat vector bundles, which plays an important role in this paper.

**Definition 2.4.** Let $E \to Y$ be a holomorphic vector bundle of rank $r$ over a normal variety. Then we say that $E$ is flat if $E \to Y$ is a locally constant family corresponding to a representation $\rho : \pi_1(Y) \to \text{GL}(r, \mathbb{C})$.

Let $E \to Y$ be a holomorphic vector bundle over a normal projective variety. Recall that $E$ is called nef if the tautological line bundle $\mathcal{O}_E(1)$ over $\mathbb{P}E$ is nef, and $E$ is called numerically flat if both $E$ and its dual bundle $E^*$ are nef. According to [DPS94, Theorem 1.18 and Corollary 1.19] and [Sim92, Section 3], a numerically flat vector bundle $E \to Y$ over a normal projective variety is a flat vector bundle.

**Lemma 2.5** ([LOY19, Lemma 4.4]). Let $V$ be a finite dimensional representation of $G := \pi_1(Y)$ with $E$ being the flat vector bundle on $Y$ induced by this representation. Then there is a one-to-one correspondence between the set of $G$-fixed points $z \in \mathbb{P}V$ and the set of codimension one flat subbundles of $E$.

This is nothing but a tautology fact (functorial definition of projective spaces); see [LOY19, Lemma 4.4] for the case when $Y$ is a smooth projective variety. Indeed, the
projectivity and smoothness of $Y$ are not needed in the proof. For readers’ convenience, we include a detailed proof.

**Proof of Lemma 2.5.** Clearly, there is a one-to-one correspondence between the set of $G$-fixed-points $z \in \mathbb{P}V$ (which corresponds to a hyperplane $V_z \subset V$) and the set of quotient representations $V \rightarrow V/V_z$. By the bijection between the isomorphism classes of local systems on $Y$ and isomorphism classes of representations of the fundamental group $\pi_1(Y)$, there is a one-to-one correspondence between the set of $G$-fixed-points $z \in \mathbb{P}V$ and line bundle quotients $E \rightarrow F$. Since the kernel of a surjective morphism between locally free sheaves is locally free, our lemma follows. \[□\]

**Lemma 2.6 ([LOY19, Lemma 4.5]).** Let $Y$ be a normal projective variety and $\pi : Y^{\text{univ}} \rightarrow Y$ the universal cover with Galois group $G = \pi_1(Y)$. Let $\rho : G \rightarrow GL(V, \mathbb{C})$ be a linear representation of $G$ and denote by $E$ the corresponding flat vector bundle over $Y$. If $z \in \mathbb{P}(V)$ is a $G$-fixed-point, then it induces a section $\sigma : Y \rightarrow \mathbb{P}E$ such that $\sigma^*\Theta_{\mathbb{P}E}(1)$ is numerically trivial.

**Proof.** Let $F$ be the co-rank one flat subbundle of $E$ corresponding to $z$ (cf. Lemma 2.5). Let $\sigma : Y \rightarrow \mathbb{P}E$ be the section defined by the following short exact sequence

$$0 \rightarrow F \rightarrow E \rightarrow E/F \rightarrow 0.$$  

Then we get $\sigma^*\Theta_{\mathbb{P}E}(1) \cong E/F$. On the other hand, as both $F$ and $E$ are numerically flat, it follows that $E/F$ and hence $\sigma^*\Theta_{\mathbb{P}E}(1)$ are also numerically flat. In particular, the line bundle $\sigma^*\Theta_{\mathbb{P}E}(1)$ is numerically trivial. \[□\]

**Remark 2.7.** In the above lemma, the image $\sigma(Y)$ is just the image of $Y^{\text{univ}} \times \{z\}$ under the natural morphism $Y^{\text{univ}} \times \mathbb{P}(V) \rightarrow \mathbb{P}E$. Here, we identify $\mathbb{P}E$ with the quotient of $Y^{\text{univ}} \times \mathbb{P}V$ by the induced diagonal action of $G$.

Let $f : X \rightarrow Y$ be a locally constant fibration with respect to the pair $(X, \Delta)$, and let $\mu : Y' \rightarrow Y$ be a morphism from a normal variety $Y'$. Then there exists a locally constant fibration $f' : X' \rightarrow Y'$ with respect to a pair $(X', \Delta')$ induced by $f$. Indeed, we consider the natural composition

$$\pi_1(Y') \xrightarrow{\mu_*} \pi_1(Y) \xrightarrow{\rho} \text{Aut}(F, \Delta_f),$$  

where $\rho$ is the representation associated to $f$. Then $X' = X \times_Y Y'$.

With this base change kept in mind, we prove the following lemma.

**Lemma 2.8.** Let $f : X \rightarrow Y$ be a locally constant fibration between normal projective varieties with respect to a pair $(X, \Delta)$. Assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $Y$ is $\mathbb{Q}$-Gorenstein. Let $\mu : Y' \rightarrow Y$ be a morphism from a normal projective variety $Y'$ with $\mathbb{Q}$-Gorenstein singularities. Denote by $f' : X' \rightarrow Y'$ the induced locally constant fibration by $f$ and let $g : X' \rightarrow X$ be the natural morphism. Then we have

$$g^*(K_{X/Y} + \Delta) \sim_{\mathbb{Q}} K_{X'/Y'} + \Delta'.$$

**Proof.** Let $F$ be a general fibre of $f$ and let $\Delta_F$ be the $\mathbb{Q}$-Weil divisor on $F$ as in Definition 2.3. Let $Y^{\text{univ}}$ and $(Y')^{\text{univ}}$ be the universal coverings of $Y$ and $Y'$, respectively. Then the morphism $\mu : Y' \rightarrow Y$ can be lifted to a morphism $\tilde{\mu} : (Y')^{\text{univ}} \rightarrow Y^{\text{univ}}$ satisfying the following commutative diagram

$$
\begin{array}{ccc}
(Y')^{\text{univ}} \times F & \xrightarrow{\tilde{g}} & Y^{\text{univ}} \times F \\
\downarrow \pi' & & \downarrow \pi \\
X' & \xrightarrow{g} & X
\end{array}
$$
where \( \bar{g} := (\bar{\mu}, \text{id}) \). Since \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier, the \( \mathbb{Q} \)-Weil divisor \( K_F + \Delta_F \) is also \( \mathbb{Q} \)-Cartier. Moreover, since \( \pi' \) and \( \pi \) are both étale, we get

\[
(\pi')^*(K_{X'}/Y' + \Delta') = (p'_2)^*(K_F + \Delta_F) \quad \text{and} \quad \pi^*(K_{X/Y} + \Delta) = p_2^*(K_F + \Delta_F),
\]

where \( p'_2 \) and \( p_2 \) are the projections \( (Y')^{univ} \times F \to F \) and \( Y^{univ} \times F \to F \), respectively. In particular, we obtain

\[
(g \circ \pi')^*(K_{X/Y} + \Delta) \sim (\pi \circ \bar{g})^*(K_{X/Y} + \Delta) \sim (\pi')^*(K_{X'/Y'} + \Delta').
\]

Note that, by our assumption, the line bundle \( (p'_1)^* \Theta_F (mK_F + m\Delta_F) \) is equivariant with respect to the induced action of \( \pi_1(Y') \) on \( (Y')^{univ} \times F \) for for \( m \) sufficiently divisible. Thus we have \( g^*(K_{X/Y} + \Delta) \sim K_{X'/Y'} + \Delta' \). \( \square \)

### 3. Periodic Points of a Special Kähler Group

In this section, we study the periodic points of an automorphism group, as a linear quotient of some special Kähler group, with the main results Proposition 3.4 and Proposition 3.11. As an application, we apply Proposition 3.11 to the locally constant MRC fibration \( f : X \to Y \) in [MW21, Theorem 1.2], which plays a significant role to find a section of such \( f \) (cf. Corollary 3.13 and Lemma 2.5). We refer readers to [LOY19, Section 4] which considered the Albanese map onto the complex torus; however, the fundamental group of our \( Y \) (as the object of the MRC fibration) here is neither abelian nor solvable.

To begin with this section, we briefly recall some basic facts about the Shafarevich map, which will be used in the proof of this and also later sections.

**Definition 3.1** ([Kol95, Definition 3.5]). Let \( Y \) be a normal variety and \( H \triangleleft \pi_1(Y) \) a normal subgroup of the fundamental group of \( Y \). A normal variety \( \text{Sh}^H(Y) \) and a rational map \( \text{sh}^H : Y \to \text{Sh}^H(Y) \) are called the \( H \)-Shafarevich variety and the \( H \)-Shafarevich map of \( Y \) if

1. \( \text{sh}^H \) has connected fibers, and
2. there are countably many closed subvarieties \( D_i \subseteq Y \) (\( D_i \neq Y \)) such that for every closed, irreducible subvariety \( Z \subseteq Y \) with \( Z \not\subseteq \cup D_i \), we have

\[
\text{sh}^H(Z) = \text{point if and only if } \text{im}[\pi_1(Z) \to \pi_1(X)] \text{ has finite index in } H,
\]

where \( Z \) is the normalization of \( Z \).

The following proposition guarantees the existence of the Shafarevich maps.

**Proposition 3.2** (cf. [Kol95, Theorem 3.6]). Let \( X \) be a normal variety, and \( H \triangleleft \pi_1(X) \) a normal subgroup. Then

1. The \( H \)-Shafarevich map \( \text{sh}^H_X : X \to \text{Sh}^H(X) \) exists.
2. For every choice of \( \text{Sh}^H(X) \) (within its birational equivalence class) there are open subsets \( X_o \subseteq X \) and \( W_o \subseteq \text{Sh}^H(X) \) with the following properties:
   - \( \text{sh}^H_X : X_o \to W_o \) is everywhere defined on \( X_o \).
   - Every fibre of \( \text{sh}^H_X|_{X_o} \) is closed in \( X \).
   - \( \text{sh}^H_X|_{X_o} \) is a topologically locally trivial fibration.
3. If \( X \) is proper, then \( \text{sh}^H_X : X_o \to W_o \) is proper and the very general points of \( X \) is contained in \( X_o \) for a suitable choice of \( \text{Sh}^H(X) \) and \( X_o \).
Now, we apply Definition 3.1 and Proposition 3.2 to our specific setting. Let $Y$ be a normal projective variety and let $\chi : \pi_1(Y) \to \text{GL}(r, \mathbb{C})$ be a representation. Let $\pi : Y' \to Y$ be a finite étale cover such that the algebraic Zariski closure $\overline{G_{Y'}}$, of the image $G_{Y'} := \chi(\pi_1(Y'))$ of the following composed map

$$\chi' : \pi_1(Y') \xrightarrow{\pi_*} \pi_1(Y) \xrightarrow{\chi} \text{GL}(r, \mathbb{C})$$

is connected. Moreover, denote by $\text{Rad}(\overline{G_{Y'}}) \triangleleft \overline{G_{Y'}}$ for the solvable radical of $\overline{G_{Y'}}$. Then the quotient $\overline{G_{Y'}}/\text{Rad}(\overline{G_{Y'}})$ is torsion-free. Denote by $K \triangleleft \pi_1(Y')$ the kernel of the map $\pi_1(Y') \to \overline{G_{Y'}}/\text{Rad}(\overline{G_{Y'}})$, which is normal in $\pi_1(Y')$. Then it follows from Proposition 3.2 that there exists a dominant almost holomorphic map (which is the $K$-Shafarevich map) $\text{sh}_{Y'}^K : Y' \to \text{Sh}(Y')$ to a smooth projective variety. In other words, there exists a dense Zariski open subset $Y'_0 \subset Y'$ such that the restriction $\text{sh}_{Y'}^K|_{Y'_0}$ is well-defined and proper. Take a very general point $x \in Y'_0$ and let $Z$ be a subvariety through $x$, with normalisation $n : \overline{Z} \to Z$. Then, the rational map $\text{sh}_{Y'}^K$ maps $Z$ to a point if and only if the composed map

$$\pi_1(\overline{Z}) \xrightarrow{n_*} \pi_1(Y') \to \overline{G_{Y'}} \to \overline{G_{Y'}}/\text{Rad}(\overline{G_{Y'}})$$

has finite image (cf. Definition 3.1). We refer readers to [CCE15] for more information on the Shafarevich maps and Shafarevich varieties.

The following theorem is crucial for our proof of Theorem B.

**Theorem 3.3 ([CCE15, Theorem 1]).** With the notation above, assume that $Y'$ has at worst klt singularities. Then the smooth projective variety $\text{Sh}^K(Y')$ is of general type.

**Proof.** Let $\eta : \overline{Y}' \to Y'$ be a resolution. By [Tak03, Theorem 1.1], the push-forward $\eta_* : \pi_1(\overline{Y}') \to \pi_1(Y')$ is an isomorphism. In particular, the composed map

$$\overline{Y}' \to Y' \to \text{Sh}^K(Y')$$

is a Shafarevich map for $K \triangleleft \pi_1(\overline{Y}')$. Then the statement follows immediately from [CCE15, Théorème 1] and the discussion above. \qed

In the sequel of this section, we study the periodic point of certain groups acting on a projective varieties. By the Borel fixed point theorem, we generalize [LOY19, Theorem 4.1] to the solvable group case to get the first main result of this section. Compared with [LOY19, Theorem 4.1], our argument is in the line of the theory of algebraic group.

Let $\sigma : G \times F \to F$ be a group action of $G$ on a projective variety $F$. We say that $G$ has a fixed point $y \in F$, if $\sigma(g, y) = y$ for any $g \in G$. We say that $G$ has a periodic point $y \in F$, if there exists a positive integer $m$ such that $\sigma(g^m, y) = y$ for any $g \in G$.

**Proposition 3.4.** Let $G$ be a group acting on a projective variety $F$ via a group homomorphism $\rho : G \to \text{Aut}(F)$ to the automorphism group of $F$ such that the image $\rho(G)$ is virtually solvable and there is a $G$-linearized ample line bundle $L$ on $F$. Then $G$ has a periodic point on $F$.

**Proof.** After replacing $G$ by a finite index subgroup, we can assume that $\Gamma := \rho(G)$ is solvable. By hypothesis, $\rho(G)$ is a subgroup of the linear algebraic group $\text{Aut}_L(F)$. Denote by $[\Gamma, \Gamma]$ the commutator subgroup of $\Gamma$. Recall that for an algebraic group $H$, its commutator group $[H, H]$ is a closed subgroup of $H$, hence also algebraic by [Bor69, Chapter I, §2.3 Proposition, pp. 58-59]. Hence, we have $[\Gamma, \Gamma] = \overline{[\Gamma, \Gamma]}$ by [Bor69, Chapter I, §2.1(e), p. 57]. Inductively, the closure of the $p$-th derived group $\Gamma^{(p)}$ coincides with the $p$-th derived group $\overline{\Gamma}^{(p)}$ of $\overline{\Gamma}$. 

\[\Gamma^{(p)} \subseteq \overline{\Gamma}^{(p)} \subseteq \overline{\Gamma}\\]
By the solvability of $\Gamma$, the $p$-th derived group $\Gamma^{(p)}$ is trivial for some $p < \infty$ (i.e., the derived length of $\Gamma$ is finite). Hence, $\Gamma^{(p)}$ is also trivial for some $p < \infty$ and thus $\Gamma$ is a solvable algebraic group. With $G$ replaced by some subgroup of finite index if necessary, we can assume that $\Gamma$ is connected. By Borel’s fixed-point theorem (cf. [Bor69, Theorem 10.4, p. 137]), the action of $\Gamma$ on $\mathfrak{A}$ has a fixed point and our proposition is proved by considering the exact sequence $1 \to \ker \to \mathfrak{A} \to \mathfrak{A} / \mathfrak{A} \bar{\mathfrak{A}} \to 1$. □

In what follows, we recall the notion of special varieties in the sense of Campana. Roughly speaking, a special variety is a compact complex variety in the Fujiki class $\mathcal{C}$ (i.e., bimeromorphic to a compact Kähler manifold) that admits no orbifold-theoretic (in the sense of F. Campana) meromorphic fibration onto a positive-dimensional variety of (orbifold) general type. For the precise definition, see [Cam04, Definition 2.1(2)].

Note that there are two fundamental examples of special varieties ([Cam04, Theorem 3.22, Theorem 5.1]) as indicated in the following lemma; in fact, every special variety is essentially built up from them ([Cam04, Example 2.3, §6.5]).

**Lemma 3.5.** Let $X$ be a compact complex variety in the Fujiki class $\mathcal{C}$. Then $X$ is special if one of the following holds:

1. $X$ is rationally connected;
2. The Kodaira dimension vanishes, i.e., $\kappa(X) = 0$.

**Remark 3.6.** One should notice that the notion ‘rational connectedness’ in [Cam04] (cf. [Cam04, §3.3]) is usually called ‘rational chain connectedness’ (cf. [Kol96, 3.2 Definition]), and this is why in [Cam04, Theorem 3.22] the smoothness of $X$ is required. In this article, we take the usual definition of ‘rational connectedness’ (i.e., any two general points can be connected by a rational curve), and since this is by definition a bimeromorphic property for complex varieties, we do not need the smoothness condition in the lemma. Let us remark that ‘rational chain connectedness’ is however not a bimeromorphic property, this is why the lemma does not hold in this generality; yet the two notions ‘rational connectedness’ and ‘rational chain connectedness’ coincide for varieties with dlt singularities by [HM07, Corollary 1.5(2)].

Besides, special varieties have the following property, known as the ‘weak speciality’ (cf. [Cam04, §9]):

**Lemma 3.7** (cf. [Cam04, Proposition 9.27]). Let $X$ be a special variety. Then any finite étale cover of $X$ can never dominate a positive dimensional variety of general type.

In the sequel we will prove the main results of this section concerning the special Kähler groups. First recall that a group $G$ is said to be a Kähler group if $G$ can be realized as the fundamental group of a compact Kähler manifold. Further, a Kähler group $G$ is said to be special, if $G$ can be realized as the fundamental group of a special compact Kähler manifold.

Applying [Tak03] to the resolution of some projective klt pair, we immediately have the following lemma.

**Lemma 3.8.** Let $(X, \Delta)$ be a projective klt pair. If $X$ is special, then the fundamental group $\pi_1(X)$ is a special Kähler group.

**Proposition 3.9.** Any linear quotient of a special Kähler group is virtually abelian.

**Proof.** Let $X$ be a compact Kähler manifold and $G := \pi_1(X)$ the fundamental group. Let $\chi : G \to \text{GL}(r, \mathbb{C})$ be a linear representation. Let $G_X := \chi(G)$ be the image and
\(G_X\) its Zariski closure. We will prove that \(G_X\) is virtually abelian. To this end, we can freely replace \(G\) (and thus \(G_X\) and \(\overline{G_X}\)) by a finite index subgroup, and correspondingly replace \(X\) by a finite étale cover, so that we may assume that \(\overline{G_X}\) is a connected algebraic group. Denote by \(\overline{R}\) the solvable radical of \(G_X\) and let \(R := \overline{R} \cap G_X\). Let 
\[s : G_X \to G_X/R\] be its natural quotient.

By [CCE15, Théorème 6.5 and its proof], after replacing \(X\) by a further étale cover, there is a proper modification \(X' \to X\) such that \(X'\) dominates a variety which is bimeromorphic to the total space of a smooth fibration \(\text{Sh}^{K_i}(X) \to W\) over the Shafarviech variety \(\text{Sh}^{K_i}(X)\) such that \(W\) is of general type, where \(K_i\) is the kernel of \(\chi\) and \(K_2\) is the kernel of \(\circ \chi\). Since \(X\) is special, our \(W\) has to be a single point (cf. Lemma 3.7). This in turn implies that the Shafarviech variety \(\text{Sh}^{K_i}(X)\) is a single point and thus the image \(s(G_X) = \circ \chi(\pi_1(X))\) is finite. Hence, \(G_X\) itself is virtually soluble (also cf. [CCE15, Théorème 6.3]). Now, it follows from [CCE15, Corollaire 4.2] that \(G_X\) is virtually abelian, which gives our proposition.  

The following corollary is an immediately consequence of Proposition 3.9.

**Corollary 3.10.** Let \(X\) be a normal projective variety with at worst klt singularities. Suppose that \(X\) is special in the sense of Campana (this is the case when \(\kappa(X) = 0\); cf. Lemma 3.5 (2)). Then any linear quotient of \(\pi_1(X)\) is virtually abelian.

**Proof.** By Lemma 3.8, \(\pi_1(X)\) is a special Kähler group (cf. [Tak03]), noting that the specialness is a birational invariant. Then our corollary follows from Proposition 3.9. 

Now we are in the position to prove the second main result in this section.

**Proposition 3.11.** Let \(G\) be a special Kähler group. Suppose that \(G\) acts on a projective variety \(F\) via a group homomorphism \(\rho : G \to \text{Aut}(F)\) to the automorphism group of \(F\), such that there exists a \(G\)-linearized ample line bundle \(L\) on \(F\) (cf. [Bri18, Definition 3.2.3, Lemma 3.2.4]). Then \(G\) has a periodic point on \(F\).

**Proof.** It follows directly from Proposition 3.4 and Proposition 3.9. 

Note that the solvability of special Kähler groups has its own interests (cf. [Cam04, Conjecture 7.1]). Thus we may ask the following question.

**Question 3.12.** Let \(G\) be a special Kähler group. Suppose that \(G\) acts on a projective variety \(F\) via a group homomorphism \(G \to \text{Aut}(F)\) such that there is a \(G\)-linearized ample line bundle \(L\) on \(F\). Can we give some descriptions on the kernel of \(\rho : G \to G|_F\)? Will \(\ker \rho\) (and hence \(G\)) be virtually soluble?

In our case, the solvability of \(G\) follows from the solvability of \(\ker \rho\) due to the exact sequence \(1 \to \ker \rho \to G \to G|_F \to 1\) (cf. Proposition 3.9). Moreover, Question 3.12 has a negative answer if we don’t assume the speciality. For example, it is known that the group given by the presentation

\[
\Gamma_g = \left\langle \alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g : \prod_{i=1}^g [\alpha_i, \beta_i] = 1 \right\rangle
\]

is Kähler. Indeed, it is the fundamental group of a compact Riemann surface of genus \(g\). However, when \(g > 1\) the commutator subgroup \([\Gamma_g, \Gamma_g]\) is a free non-abelian group and hence \(\Gamma_g\) is not virtually soluble.

Applying Proposition 3.11 to a locally constant fibration, we end up this section with the following corollary.
Corollary 3.13. Let $f : X \to Y$ be a locally constant fibration with a fibre $F$. Suppose that the irregularity $q(F) = 0$ and $Y$ is special. Then the fundamental group $G := \pi_1(Y)$ acts on $F$ and has a periodic point on it.

Proof. To apply Proposition 3.11, we only need to verify that there is a $G$-linearized ample line bundle $L$ on $F$ (cf. [MW21, Definition 2.6]). Since $f$ is locally constant, we have the following diagram

$$
\begin{array}{ccc}
F \times Y^{\text{univ}} & \to & Y^{\text{univ}} \\
\downarrow & & \downarrow \\
X & \to & Y
\end{array}
$$

where $Y^{\text{univ}}$ is the universal cover of $Y$ and $F \times Y^{\text{univ}} \cong X \times_Y Y^{\text{univ}}$ with the induced projection $pr_1 : F \times Y^{\text{univ}} \to F$ and $pr_2 : F \times Y^{\text{univ}} \to Y^{\text{univ}}$. Let us fix an $f$-ample divisor $A$ on $X$ and denote by $p_X : F \times Y^{\text{univ}} \to X$ the natural projection. Applying [MW21, Proof of Lemma 2.7], we see that there exists a line bundle $A_Y$ on $Y$ such that

$$
p_X^*(A - f^*A_Y) \cong pr_1^*A_F
$$

for some line bundle $A_F$ on $F$. Since $X \cong (F \times Y^{\text{univ}})/G$, our $p_X^*(A - f^*A_Y)$ is $G$-linearized, and hence,

$$
g^*p_X^*(A - f^*A_Y) \sim p_X^*(A - f^*A_Y)
$$

for any $g \in G$ (regarded as an automorphism of $F$); see [Bri18, Definition 3.2.3, Lemma 3.2.4]. Hence, $pr_1^*A_F$ is also $G$-linearized. Moreover, since $G$ acts on $F \times Y^{\text{univ}}$ via the diagonal, our $A_F$ is a $G$-linearized ample line bundle, noting that $A_F \cong A|_F$. \hfill \square

4. Proofs of Theorem A, Theorem B and Corollary C

In this section, we shall prove our main results Theorem A and Theorem B. As an application of Theorem B, we show Corollary C.

Assumption 4.1. Throughout this section, we follow the following assumption and its notations.

- Let $(X, \Delta)$ be a projective klt pair, and $f : X \to Y$ a locally constant fibration (cf. Definition 2.3) onto a $\mathbb{Q}$-Gorenstein normal projective variety $Y$ such that the relative canonical divisor $-(K_{X/Y} + \Delta)$ is nef. If $Y$ is smooth, then it follows from Theorem 4.2 below that $f$ is a locally constant fibration.
- Let $F$ be a fibre of $f$ and $G := \pi_1(Y)$.
- Assume that there exists a very ample divisor $A_F$ on $F$ which is $G$-linearized (cf. [Bri18, Definition 3.2.3, Lemma 3.2.4]). If $Y$ is smooth, then the existence of such $A_F$ follows from Theorem 4.2.
- By Definition 2.3, our $G$ acts on $F$ via the homomorphism $G \to \text{Aut}_{A_F}(F)$.
- Denote by $\rho : \pi_1(Y) \to \text{GL}(V, \mathbb{C})$ the representation induced by $\pi_1(Y) \to \text{Aut}(F, A_F)$, where $V = H^0(F, A_F)$. Denote by $Y^{\text{univ}}$ the universal cover of $Y$. Then we have the following (Cartesian) commutative diagram:
Let \( G \) act on \( Y^\text{univ} \times F \) via diagonal and \( X \cong (Y^\text{univ} \times F)/G \). Let \( \text{pr}_2 : X \times_Y Y^\text{univ} \cong Y^\text{univ} \times F \to F \) be the second projection.

First recall the following structure theorem and criterion for locally constant fibrations. It is essentially proved in [CCM19] and [MW21] and it is the starting point for the proofs of our Theorem A and Theorem B.

**Theorem 4.2 ([CCM19], [MW21, Theorem 4.7]).** Let \( (X, \Delta) \) be a klt pair, and \( f : X \to Y \) a fibration to a smooth projective variety. Let \( F \) be a general fibre of \( f \). If the relative anti-log canonical divisor \( -(K_{X/Y} + \Delta) \) is nef, then \( f \) is a locally constant fibration induced by a representation \( \pi_1(Y) \to \text{Aut}(F, \Delta_F) \). Moreover, there exists a sufficiently ample line bundle \( A \) on \( F \) which is \( \pi_1(Y) \)-linearized (cf. [Bri18, Definition 3.2.3, Lemma 3.2.4]).

The smoothness assumption of \( Y \), however, cannot be removed in the theorem above:

**Example 4.3.** Let \( Y \subset \mathbb{P}^{n+1} \) be a two-dimensional projective cone over a rational normal curve \( C_n \subset \mathbb{P}^n \). Then \( Y \) is a Fano variety with klt singularities. Let \( f : X \to Y \) be the blow-up of \( Y \) at the vertex with exceptional divisor \( E \). Then \( X \) is smooth and \( E \) is a smooth rational curve such that

\[
f^* K_Y = K_X + \frac{1}{n} E.
\]

In particular, the pair \( (X, \frac{1}{n} E) \) is klt and \( -(K_{X/Y} + \frac{1}{n} E) \) is trivial and hence nef. In general, let \( Y \) be a projective variety with klt singularities, but not terminal. Let \( f : X \to Y \) be a terminal modification of \( Y \). Let \( \Delta \) be the \( \mathbb{Q} \)-Weil divisor on \( X \) defined as \( K_X + \Delta = f^* K_Y \). Then \( \Delta \) is effective, \( (X, \Delta) \) is klt and the relative anti-log canonical divisor \( -(K_{X/Y} + \Delta) \) is trivial and hence nef.

In the following, we first show that if the fundamental group of a subvariety \( Z \) of \( Y \) is virtually solvable, then \( -(K_{X/Y} + \Delta) \) cannot be strictly nef.

**Proposition 4.4.** Under Assumption 4.1, let \( \eta : Z \to Y \) be a morphism from another \( \mathbb{Q} \)-Gorenstein normal projective variety \( Z \). If \( \pi_1(Z) \) has a periodic point on \( F \) via the composed map

\[
\bar{\rho} : \pi_1(Z) = \pi_1(Y) \xrightarrow{\rho} \text{GL}(V, \mathbb{C})
\]

then up to replacing \( Z \) by some finite étale cover if necessary, there is a lifting \( \sigma : Z \to X \) of \( \eta : Z \to Y \) such that the pull-back \( \sigma^*(K_{X/Y} + \Delta) \) is numerically trivial. Further, if the image of \( \pi_1(Z) \) under \( \bar{\rho} \) is virtually solvable, then we can get such a periodic point on \( F \).

Before we prove Proposition 4.4, we prepare the following lemma, which is an analogue of Lemma 2.6.

**Lemma 4.5.** Under Assumption 4.1, we have the following statements.
(a) Every G-fixed point \( z \) (if exists) induces a section \( \sigma : Y \to X \) of the locally constant fibration \( f : (X, \Delta) \to Y \).

(b) Let \( A_X \) be the quotient bundle of \( A_f \) on \( X \); that is \( p^*_X A_X = pr^*_Y A_f \) (cf. Assumption 4.1).

Let \( E := f_* \mathcal{O}_X(A_X) \). Then every G-fixed point \( z \) induces a short exact sequences of flat vector bundles

\[
0 \to E' \to E \to \mathcal{O}_Y(\sigma^* A) \to 0.
\]

In particular, \( \sigma^* A_X \equiv 0 \).

**Proof.** Let \( i : X \subset \mathbb{P}E \) be the embedding over \( Y \) induced by \( A_X \), with \( \mathcal{O}_X(A_X) \cong i^* \mathcal{O}_{\mathbb{P}E}(1) \). For each fibre \( F \), we get a \( G \)-equivariant embedding \( i_F : F \subset \mathbb{P}(H^0(F, \mathcal{O}_F(1))) \). By Lemma 2.6, the \( G \)-fixed point \( z \in F \subset \mathbb{P}(H^0(F, \mathcal{O}_F(1))) \) induces a subbundle \( E' \) of \( E \) so that \( \mathbb{P}(E/E') \) is a flat line bundle on \( Y \) and the surjection \( E \to Q \) corresponds to a section \( \sigma_1 : Y \to \mathbb{P}E \) of the projective bundle \( \mathbb{P}E \to Y \) with \( \sigma^*_1 \mathcal{O}_{\mathbb{P}E}(1) = \mathcal{O} \equiv 0 \).

But for every \( y \in Y \), \( \sigma_1(y) \) corresponds to \( z \in F \) under the (canonical) identification \( X_y \cong F \). Hence, the image \( \sigma_1(Y) \) is contained in the image of \( X \) under the canonical embedding. Denote by \( \sigma \) the induced morphism \( Y \to X \) and we then have

\[
0 \equiv Q \cong \sigma^*_1 \mathcal{O}_{\mathbb{P}E}(1) \cong \sigma^* \mathcal{O}_X(A) = \mathcal{O}_Y(\sigma^* A),
\]

noting that \( \sigma_1 = i \circ \sigma \).

**Proof of Proposition 4.4.** Let \( X' \to Z \) be the locally constant fibration induced by \( f \) (cf. the arguments before Lemma 2.8). By Lemma 2.8, after replacing \( Y \) by \( Z \) and \( X \) by \( X' \), we may assume that \( Z = Y \). Denote by \( \overline{G}_Y \) the Zariski closure of the image \( G_Y : = \rho(\pi_1(Y)) \) in \( \text{GL}(V, \mathbb{C}) \).

First, we prove the second part of this proposition. Suppose that \( G_Y \) is virtually solvable. Then its closure \( \overline{G}_Y \) is a virtually solvable linear algebraic group, which has finitely many components, and both \( F \) and \( \Delta_F \) are invariant under the induced action \( \overline{G}_Y \) on \( \mathbb{P}(V) \). In particular, after replacing \( Y \) by an appropriate finite étale cover, we may assume that \( \overline{G}_Y \) itself is connected and solvable. Thus, by Borel’s fixed point theorem (cf. [Bor69, Theorem 10.4, p. 137]), there exists a \( \overline{G}_Y \)-fixed-point \( y \in F \subset \mathbb{P}(V) \) (cf. Proposition 3.4), which completes the second part of our proof.

Now we prove the first part of this proposition. Let \( m \in \mathbb{N} \) be a sufficiently divisible positive integer. Then \( -m(K_F + \Delta_F) + A_F \) is ample as \( -(K_F + \Delta_F) = -(K_{X/Y} + \Delta)|_F \) is nef (cf. Assumption 4.1). Since both \( K_F + \Delta_F \) and \( A_F \) are \( \overline{G}_Y \)-linearized, for any positive integer \( p \), the induced action of \( \overline{G}_Y \) on the vector space

\[
V_{m,p} := H^0(F, \mathcal{O}_F(p(-m(K_{X/Y} + \Delta) + A_F)))
\]

induces a flat vector bundle structure on the direct image sheaf

\[
E_{m,p} := f_* \mathcal{O}_X(p(-m(K_{X/Y} + \Delta) + A_X)).
\]

Recall that \( A_X \) is the quotient line bundle of \( A_f \) on \( X \), that is, we have \( p^*_X A_X = pr^*_Y A_F \), where \( p_X \) is the natural morphism and \( p_Y : Y^\text{univ} \times F \to F \) is the second projection (cf. Assumption 4.1). Moreover, by the ampleness of \( -m(K_F + \Delta_F) + A_F \), for sufficiently large \( p \gg 1 \) there exists an embedding \( F \subset \mathbb{P}(V_{m,p}) \) such that the restriction of the action \( \overline{G}_Y \) on \( \mathbb{P}(V_{m,p}) \) to \( F \) coincides with the restriction of the action \( \overline{G}_Y \) on \( \mathbb{P}(V) \) to \( F \). In particular, the point \( y \in F \subset \mathbb{P}(V_{m,p}) \) is a \( \overline{G}_Y \)-fixed point for the action of \( \overline{G}_Y \) on the projective space \( \mathbb{P}(V_{m,p}) \). Denote by \( \sigma : Y \to X \) the section induced by \( y \). Applying Lemma 4.5 to \( E_{m,1} \) (\( m \neq 0 \) and \( E_{0,1} \)), we see that the pull-backs

\[
\sigma^*(-m(K_{X/Y} + \Delta) + A_X) \quad \text{and} \quad \sigma^* A_X
\]
Proposition 4.4 follows.

As a consequence of Proposition 4.4, we are able to prove our first main result Theorem A. After that, we slightly generalize Theorem A to the case when the anti-log canonical divisor is almost strictly nef (cf. Proposition 4.7).

Remark 4.6. In the spirit of [LOY19, Section 4], the key step to show Theorem A is to find a $(K_X + Δ)$-trivial section of the maximal rationally connected fibration (MRC fibration for short), up to replacing $X$ by a quasi-étale cover, which is Proposition 4.4. In general, the MRC fibration for a projective variety may not be holomorphic. However, by a recent joint work of Matsumura and the third author, it has been shown that, if $(X, Δ)$ is a projective klt pair with the anti-log canonical divisor $-(K_X + Δ)$ being nef, then up to replacing $X$ by its quasi-étale cover, there is a locally constant (holomorphic) fibration $f : X → Y$ with respect to the pair $(X, Δ)$ such that $K_Y ≡ 0$ ([MW21, Theorem 1.1]; cf. [Cao19, CH19, CCM19, Wan20]).

Proof of Theorem A. First, we note that $X$ is uniruled (cf. Proposition 2.2). After replacing $(X, Δ)$ by a quasi-étale cover $(X', Δ')$, our pair $(X', Δ')$ is still klt, with $-(K_{X'} + Δ')$ being strictly nef (cf. [KM98, Proposition 5.20]). Furthermore, if $X'$ is rationally connected, then so is $X$. Therefore, we are free to replace $X$ by its quasi-étale cover.

By [MW21, Theorem 1.1], with $(X, Δ)$ replaced by a quasi-étale cover, the maximal rationally connected (MRC for short) fibration $f : (X, Δ) → Y$ of $X$ is a (holomorphic) locally constant fibration with respect to the pair $(X, Δ)$ such that $Y$ has only klt singularities and the canonical divisor $K_Y ≡ 0$. Then by taking a desingularization of $Y$ and considering the base change of $f$, from Theorem 4.2 and [Tak03, Theorem 1.1] we see that there is a $G$-linearized ample line bundle over $F$, and hence all the assumptions in Assumption 4.1 are satisfied.

Suppose the contrary that $X$ is not rationally connected. Then $\dim Y > 0$. By Proposition 4.4, up to replacing $X$ by a further étale cover, our $f$ admits a section $σ : Y → X$ such that $σ^*(K_X + Δ) ≡ 0$. We pick a curve $C$ in $Y$. Then $σ_*C ≠ 0$. Since $σ^*(K_X + Δ) ≡ 0$, it follows from the projection formula that

$$-σ^*(K_X + Δ) ⋅ C = -(K_X + Δ) ⋅ σ_*C = 0,$$

which contradicts the strict nefness of $-(K_X + Δ)$. Hence, $X$ is rationally connected and the first part of Theorem A follows.

Note that a rationally connected klt projective variety has the vanishing irregularity (cf. [Deb01, Corollary 4.18] and [KM98, Theorem 5.22]). Thus, to show the augmented irregularity $q^a(X)$ vanishing, we only need to show every quasi-étale cover is rationally connected. However, this follows from the first part of Theorem A and arguments in the beginning of our proof.

In what follows, we slightly extend our Theorem A to the following Proposition 4.7 on the anti-log canonical divisor being almost strictly nef (cf. Definition 2.1).

Proposition 4.7. Let $(X, Δ)$ be a projective klt pair with $-(K_X + Δ)$ being almost strictly nef. Then, any quasi-étale cover of $X$ is rationally connected. In particular, the augmented irregularity $q^a(X) = 0$.

Proof. Let $π : (X, Δ) → (Z, Δ_Z)$ be the projective birational morphism such that $K_X + Δ = π^*(K_Z + Δ_Z)$ and $-(K_Z + Δ_Z)$ is strictly nef. Clearly, $(Z, Δ_Z)$ is also a projective klt pair. Let $(X', Δ') := τ^*(Δ) → (X, Δ)$ be any quasi-étale cover. Then $(X', Δ')$ is a...
Proposition 4.7
Theorem B
Proposition 4.4
and
Yam10
and its proof that any quasi-
to the pair
Theorem 4.2
and \(\text{Theorem B} \)
Assumption 4.1
Theorem B
that every resolution
\(\mathbb{u_1D44D}\)
Theorem 4.10.
Under
we shall prove that every irreducible subvariety
\(\mathbb{u1D44C}\)
rationally connected and
\(\mathbb{u1D26}\)
which completes the proof of the proposition.
□

As a consequence of Proposition 4.7 and [GKP16, Proposition 7.8], we end up the first part of this section with the following result on the finiteness of the algebraic fundamental group of the smooth locus when \(-(K_X + \Delta)\) is almost strictly nef.

**Proposition 4.8.** Let \((X, \Delta)\) be a projective klt pair with \(-(K_X + \Delta)\) being almost strictly nef. Then the algebraic fundamental group \(\pi_1^{\text{alg}}(X_{\text{reg}})\) of the smooth locus \(X_{\text{reg}} \subseteq X\) is finite.

**Proof.** Denote by \(\mathcal{R}\) the set of normal projective varieties \(X\) such that there exists a \(\mathbb{Q}\)-Weil divisor \(\Delta\) such that \((X, \Delta)\) is klt and the anti-log canonical divisor \(- (K_X + \Delta)\) is almost strictly nef. Then it follows from Proposition 4.7 and its proof that any quasi-étale Galois cover \(Y\) of \(X\) still lies in \(\mathcal{R}\). Moreover, since \(X\) is rationally connected by Proposition 4.7, and the algebraic fundamental group is a profinite completion of the topological fundamental group, we see that \(\pi_1^{\text{alg}}(X)\) is finite (cf. [Tak03]). In particular, the conditions of [GKP16, Proposition 7.8] are verified and thus \(\pi_1^{\text{alg}}(X_{\text{reg}})\) is finite. □

In the second part of this section, we aim to show Theorem B and Corollary C. In the view of Theorem 4.2, we already know that such \(f\) (in Theorem B) is a locally constant fibration. Thus, to prove Theorem B, it remains to show that the fibres of \(f\) are rationally connected and \(Y\) is a canonically polarized hyperbolic manifold. Indeed, we shall prove that every irreducible subvariety \(Z\) of \(Y\) is of general type in the sense that every resolution \(Z' \rightarrow Z\) is of general type.

Now, we want to apply Proposition 4.4 to the situation of Theorem B. First, the following lemma is an application of [Yam10, Theorem 1.1], which reveals the relationships between fundamental groups and degeneracy of entire curves.

**Lemma 4.9 ([LOY20, Lemma 9.1]).** Let \(Y\) be an irreducible projective variety. If there exists a representation \(\rho : \pi_1(Y) \rightarrow \text{GL}(r, \mathbb{C})\) such that its image \(\text{Im}(\rho)\) is not virtually solvable, then every holomorphic map \(f : \mathbb{C} \rightarrow Y\) is degenerate, i.e., \(f(\mathbb{C})\) is not Zariski dense in \(Y\).

The following theorem will be used in the proof of Theorem B. Before giving the statement, let us introduce some necessary notion. Let \(L\) be a nef \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-Weil divisor on a normal projective variety. We define \(\text{NT}(L)\) to be the Zariski closure of the union of curves \(C\) in \(X\) with \(L\)-degree 0, i.e., \(L \cdot C = 0\).

**Theorem 4.10.** Under Assumption 4.1, if the subvariety \(T := \text{NT}(-(K_{X/Y} + \Delta))\) does not dominate \(Y\) and \(- (K_{X/Y} + \Delta)\) is strictly nef, then \(Y\) is of general type and there exists a linear representation \(\chi : \pi_1(Y) \rightarrow \text{GL}(r, \mathbb{C})\) whose image is not virtually solvable.
Proof. Let \( \chi : \pi_1(Y) \to \text{GL}(r, \mathbb{C}) \) be the linear representation given in Assumption 4.1.

Step 1. Let \( Z \subseteq Y \) be an irreducible subvariety of positive dimension passing through a very general point of \( Y \), and let \( Z' \to Z \) be an arbitrary resolution. Then the image of the induced composed map

\[
\chi' : \pi_1(Z') \to \pi_1(Y) \xrightarrow{\chi} \text{GL}(r, \mathbb{C})
\]

is not virtually solvable.

Assume to the contrary that the image of \( \chi' \) is virtually solvable. Since \( T \) does not dominate \( Y \) and \( Z \) is very general, we may assume that \( Z \) is not contained in the image of \( T \) in \( Y \). Let us denote by \( f' : X' \to Z' \) the locally constant fibration with respect to \((X', \Delta')\) induced by \( \rho' \) and \( f \). By Lemma 2.8, we have

\[
g^*(K_{X/Y} + \Delta) = K_{X'/Z'} + \Delta',
\]

where \( g \) is the induced morphism \( X' \to X \). On the other hand, by Proposition 4.4, after replacing \( Z' \) by a finite étale cover if necessary, there exists a section \( \sigma : Z' \to X' \) such that the pull-back \( \sigma^*(K_{X'/Z'} + \Delta') \) is numerically trivial. Since \( Z' \to Z \) is birational and \( \dim(Z) > 0 \), there exists an irreducible curve \( C' \subseteq Z' \) such that it is birational onto its image \( C \) in \( Z \) such that \( C \) is not contained in the image of \( T \). In particular, the induced morphism \( C' \to \sigma(C') \to (g \circ \sigma)(C') \) is birational and \((g \circ \sigma)(C')\) is not contained in \( T \). Then by projection formula we get

\[
0 < -(K_{X/Y} + \Delta) \cdot (g \circ \sigma)(C') = -(K_{X'/Z'} + \Delta') \cdot \sigma(C) = 0,
\]

which is a contradiction.

Step 2. The base manifold \( Y \) is of general type.

Let \( Y' \to Y \) be the finite étale cover of \( Y \) given in the discussion before Theorem 3.3 with the Shafarevich map \( \text{sh}_{Y'}^K : Y' \to \text{Sh}_{Y'}^K \) corresponding to the representation \( \chi \).

Let \( G_{Y'} \) be the image of \( \pi_1(Y') \) under the composite map \( \pi_1(Y') \to \pi_1(Y) \xrightarrow{\chi} \text{GL}(r, \mathbb{C}) \) with \( G_{Y'} \) being its Zariski closure. To show that \( Y \) is of general type, it is enough to show that \( Y' \) is general. In particular, by Theorem 3.3, it remains to show that the Shafarevich map \( \text{sh}_{Y'}^K \) is birational. Let \( X' \to Y' \) be the locally constant fibration induced by \( f \). By Lemma 2.8, the relative anti-log canonical divisor \(-(K_{X'/Y'} + \Delta')\) is nef and it is clear that the closed subvariety \( T' := \text{NT}(-(K_{X'/Y'} + \Delta')) \) does not dominate \( Y' \). Let \( Z \) be a very general fibre of \( \text{sh}_{Y'}^K \). Then \( Z \) is normal as \( X' \) is normal. Let \( Z' \to Z \) be a resolution. Then by Definition 3.1 the image of the following composed map

\[
\pi_1(Z') \to \pi_1(Z) \to \pi_1(Y') \to \pi_1(Y) \xrightarrow{\chi} G_{Y'} \to \overline{G_{Y'}/\text{Rad}(\overline{G_{Y'}})}
\]

is finite and hence the image of the following composed map

\[
\pi_1(Z') \to \pi_1(Z) \to \pi_1(Y') \to \pi_1(Y) \xrightarrow{\chi} \text{GL}(r, \mathbb{C})
\]

is virtually solvable. Here, \( \text{Rad}(\overline{G_{Y'}}) \) denotes the solvable radical of \( \overline{G_{Y'}} \). Hence, by Step 1 above, we obtain \( \dim(Z) = 0 \) and it follows that \( \text{sh}_{Y'}^K \) is birational. \( \square \)

Now we are in the position to prove Theorem B.
Proof of Theorem B. By Theorem 4.2, it remains to show that $Y$ is a hyperbolic manifold with ample canonical divisor and the fibres of $f$ are rationally connected. Firstly note that $(F, \Delta_F)$ is a projective klt pair with $-(K_F + \Delta_F)$ being strictly nef, where $F$ is a general fibre of $f$. Hence, the variety $F$ is rationally connected by Theorem A. Moreover, by Theorem 4.10, it is known that the canonical divisor $K_Y$ is big. Thus to show that $K_Y$ is ample, it is enough to show that $Y$ does not contain any rational curves, which shall be a direct consequence of the hyperbolicity of $Y$. Indeed, if $K_Y$ is not nef, then it follows from the cone theorem (cf. [KM98, Theorem 3.7]) that $Y$ contains a rational curve; if $K_Y$ is nef (and big) but not ample, then it follows from the base-point-free theorem (cf. [KM98, Theorem 3.3]) that $Y$ admits a birational holomorphic Kodaira fibration onto its canonical model $Y^\text{can}$ (with only canonical singularities), in which case, the exceptional locus of $Y \to Y^\text{can}$ is covered by rational curves (cf. [KM98, Proof of Proposition 1.3]). Thus, it is enough to show that $Y$ is hyperbolic.

Let $h : \mathbb{C} \to Y$ be a holomorphic map and denote by $Z$ the Zariski closure of its image. Assume to the contrary that $\dim(Z) > 0$. Then $f(C)$ is not degenerate in $Z$. Let $Z'$ be a resolution of $Z$ and let $X' \to Z'$ be the locally constant fibration induced by $f$. Then applying Theorem 4.10, we see that there exists a linear representation of $\pi_1(Z')$ whose image is not virtually solvable. On the other hand, note that the holomorphic map $h$ can be lifted to a holomorphic map $h' : \mathbb{C} \to Z'$. In particular, the image $h'(\mathbb{C})$ is dense in $Z'$, which contradicts Lemma 4.9. Hence, we have $\dim(Z) = 0$; that is, $h$ is constant and consequently $Y$ is hyperbolic.

Remark 4.11. From the proof, one can easily derive the following result concerning the geometry of $Y$: for an irreducible closed subvariety $Z \subseteq Y$ of positive dimension and with a resolution $Z' \to Z$, then $Z'$ is of general type and $\pi_1(Z')$ admits a linear representation whose image is not virtually solvable.

Let $X$ be a projective manifold. A foliation on $X$ is a coherent subsheaf $\mathcal{F}$ of $T_X$ such that $\mathcal{F}$ is closed under the Lie bracket and the quotient $T_X/\mathcal{F}$ is torsion free. The canonical divisor $K_\mathcal{F}$ of $\mathcal{F}$ is a Weil divisor on $X$ such that $\mathcal{O}_X(K_\mathcal{F}) \cong \text{det } \mathcal{F}$. We call that $\mathcal{F}$ is regular if the quotient $T_X/\mathcal{F}$ is locally free and $\mathcal{F}$ is algebraically integrable if there exists a rational map $f : X \to Y$ such that $\mathcal{F}$ is induced by $f$.

Proof of Corollary C. Assume that either $\mathcal{F}$ is regular, or $\mathcal{F}$ has a compact leaf. By [Ou21, Theorem 1.1], there exists a locally trivial fibration $f : X \to Y$ with rationally connected fibres such that there exists a foliation $\mathcal{G}$ on $Y$ with $K_\mathcal{G} \equiv 0$ and $\mathcal{F} = f^{-1}(\mathcal{G})$. In particular, we have $K_\mathcal{F} \equiv K_{X/Y}$ and therefore $-K_{X/Y}$ is strictly nef.

For the statement (1), by Theorem B, the base manifold $Y$ is a canonically polarized projective manifold. In particular, the tangent bundle $T_Y$ is semi-stable with respect to $K_Y$ by the theorem of Aubin-Yau ([Aub78, Yau78]). Observing that the slope of $T_Y$ with respect to $K_Y$ is strictly negative, we see that $\mathcal{G}$ is a fortiori the foliation in points and hence $\mathcal{F}$ is exactly the foliation induced by the fibration $f : X \to Y$.

Next we assume that the fundamental group $\pi_1(X)$ is virtually solvable. Then the fundamental group $\pi_1(Y) = \pi_1(X)$ of $Y$ is virtually solvable since the fibres of $f$ are rationally connected. In particular, the image of every linear representation of $\pi_1(Y)$ is again virtually solvable, which contradicts Theorem 4.10.

Example 4.12. We collect some examples to show the sharpness of Corollary C.

1. Let $\mathcal{F} \subseteq T_{\mathbb{P}^n}$ be the foliation of rank $r$ defined by a linear projection $\mathbb{P}^n \to \mathbb{P}^{n-r}$. Then $\text{det } \mathcal{F} \cong \mathcal{O}_{\mathbb{P}^n}(r)$ is ample. This shows that the regularity of $\mathcal{F}$ in Corollary C cannot be dropped.
(2) Let \(X\) be an abelian variety and let \(\mathcal{F}\) be a general linear foliation on \(X\), that is, the translation of a general linear space. Then \(-K_{\mathcal{F}}\) is numerically trivial and \(\mathcal{F}\) is not algebraically integrable. This shows that we cannot replace strict nefness by nefness in Corollary C (1).

(3) Let \(\mathcal{F}\) be the foliation induced by the fibration \(X = \mathbb{P}E \to C\), which is Mumford’s example discussed after Theorem 1.3. Then \(-K_{\mathcal{F}}\) is strictly nef. This shows that the assumption on the fundamental group of \(X\) cannot be removed in Corollary C (2).

5. Amplicity of \(-(K_X + \Delta)\), Proof of Theorem D

In this section, we are devoted to the verification of \(\text{Corollary D}\). Let \((X, \Delta)\) be a projective klt threefold pair such that the anti-log canonical divisor \(-\left(K_X + \Delta\right)\) is strictly nef.

5.1. \(\mathbb{Q}\)-effectivity of \(-(K_X + \Delta)\). To show the ampleness of \(-\left(K_X + \Delta\right)\), we aim to show that \(-\left(K_X + \Delta\right)\) is num-effective, i.e., numerically equivalent to an effective divisor, which is our main result Proposition 5.5 in this subsection. Before that, we recall and generalize some previous results for the convenience of our proofs.

First we give the following generalization of [Ser95, Lemma 1.1].

**Lemma 5.1** (cf. [Ser95, Lemma 1.1]). Let \((X, \Delta)\) be a projective klt pair, and \(L_X\) a strictly nef \(\mathbb{Q}\)-divisor on \(X\). Then \((K_X + \Delta) + tL_X\) is strictly nef for every \(t > 2m \dim X\) where \(m\) is the Cartier index of \(L_X\).

**Proof.** By the cone theorem (cf. [KM98, Theorem 3.7, p. 76]), every curve \(C\) in \(X\) is numerically equivalent to a linear combination \(M + \sum a_iC_i\) (a finite sum), where \(M\) is pseudoeffective such that \(M \cdot (K_X + \Delta) \geq 0\) and the \(C_i\)'s are (integral) rational curves satisfying \(0 < C_i \cdot (-K_X - \Delta) \leq 2\dim X\) with \(a_i > 0\). Since \(mL_X\) is Cartier and strictly nef, we have \(L_X \cdot M \geq 0\), and \(L_X \cdot C_i \geq \frac{1}{m}\) for all \(i\). Therefore, for each \(C_i\),

\[((K_X + \Delta) + tL_X) \cdot C_i \geq tL_X \cdot C_i - 2\dim X > 0.\]

If \((K_X + \Delta) \cdot C \geq 0\), then \((K_X + \Delta) + tL_X) \cdot C \geq 0\) since \(L_X \cdot C > 0\). If \((K_X + \Delta) \cdot C < 0\), then the decomposition of \(C\) contains some \(C_i\); hence our lemma follows from (\dagger). \(\square\)

We remark here that, when \(\dim X = 2\), the conclusion of Lemma 5.1 holds for a more optimal lower bound \(t > 3\) (cf. [Fuj12, Proposition 3.8]).

**Lemma 5.2.** Let \((X, \Delta)\) be a projective klt pair, and \(L_X\) a strictly nef \(\mathbb{Q}\)-divisor on \(X\). Suppose \(a(K_X + \Delta) + bL_X\) is big for some \(a, b \geq 0\). Then \((K_X + \Delta) + tL_X\) is ample for sufficiently large \(t\).

**Proof.** If \(a = 0\), then \(L_X\) is big. If \(a \neq 0\), then \((K_X + \Delta) + \frac{b}{a}L_X\) is big. In both cases, it follows from Lemma 5.1 that \((K_X + \Delta) + tL_X\) is strictly nef and big for \(t \gg 1\), noting that nef divisors are always pseudoeffective. Then \(2((K_X + \Delta) + tL_X) - (K_X + \Delta)\) is also nef and big. By the base-point-free theorem (cf. [KM98, Theorem 3.3, p. 75]), some multiple of \((K_X + \Delta) + tL_X\) defines a morphism \(X \to \mathbb{P}^N\), which is finite by the projection formula. Therefore, \((K_X + \Delta) + tL_X\) is ample, and our lemma is proved. \(\square\)

We denote by \(\text{NE}(X)\) (resp. \(\text{ME}(X)\)) the Mori cone (resp. movable cone) of a projective variety \(X\) (cf. [BDPP13]). The following lemma characterizes the situation when \(K + tL\) is not big.
Lemma 5.3. Let \((X, \Delta)\) be a projective klt pair of dimension \(n\), and \(L_X\) a strictly nef \(\mathbb{Q}\)-divisor on \(X\). Then \((K_X + \Delta) + uL_X\) is not big for some rational number \(u > 2mn\) with \(m\) the Cartier index of \(L_X\), if and only if \((K_X + \Delta)^i \cdot L_X^{n-i} = 0\) for any \(0 \leq i \leq n\). Moreover, if one of the equivalent conditions holds, there exists a class \(0 \neq \alpha \in \overline{\text{NE}}(X)\) such that \((K_X + \Delta) \cdot \alpha = L_X \cdot \alpha = 0\).

Proof. Suppose first that \((K_X + \Delta) + uL_X\) is not big for some rational number \(u > 2mn\). Set \(u' := u - 2mn > 0\), \(D_1 := (K_X + \Delta) + (u'/2 + 2mn)L_X\) and \(D_2 := (u'/2)L_X\). Then \((K_X + \Delta + uL_X)^n = (D_1 + D_2)^n = 0\). Since both \(D_1\) and \(D_2\) are nef (cf. Lemma 5.1), the vanishing \(D_1^i \cdot D_2^{n-i} = 0\) for every \(0 \leq i \leq n\) yields \((K_X + \Delta)^i \cdot L_X^{n-i} = 0\) for any \(0 \leq i \leq n\). Conversely, if \((K_X + \Delta)^i \cdot L_X^{n-i} = 0\) for any \(0 \leq i \leq n\), then \((K_X + \Delta) + uL_X)^n = 0\) for any \(u\); in this case, \((K_X + \Delta) + uL_X\) being not big for any \(u > 2mn\) follows from [KM98, Proposition 2.61, p. 68] and Lemma 5.1. Therefore, the first part of our theorem is proved.

Assume that the nef divisor \(K_X + \Delta + uL_X\) is not big for some \(u > 2mn\). Then, there exists a class \(\alpha \in \overline{\text{NE}}(X)\) such that \((K_X + \Delta + uL_X) \cdot \alpha = 0\) (cf. [BDPP13, Theorem 2.2 and the remarks therein]). Suppose that \(L_X \cdot \alpha \neq 0\). Then \(L_X \cdot \alpha > 0\) and thus \((K_X + \Delta) \cdot \alpha < 0\). By the cone theorem (cf. [KM98, Theorem 3.7, p. 76]), we have \(\alpha = M + \sum a_iC_i\) where \(M\) is a class lying in \(\overline{\text{NE}}(X)_{(K_X + \Delta) > 0}\), \(a_i > 0\), and the \(C_i\) are extremal rational curves with \(0 < -(K_X + \Delta) \cdot C_i \leq 2n\). Now, for each \(i\), the intersection \((K_X + \Delta + uL_X) \cdot C_i > 0\), a contradiction to the choice of \(\alpha\). So we have \((K_X + \Delta) \cdot \alpha = L_X \cdot \alpha = 0\).

In the following, we slightly generalize [Ser95, Theorem 2.3] and [HL20, Corollary 1.8] to the following (not necessarily normal) \(\mathbb{Q}\)-Gorenstein surface case (cf. [CCP08, Conjecture 1.3]). This is also mentioned at the end of [Ser95, Section 2].

For a \(\mathbb{Q}\)-factorial normal projective variety \(X\) and a prime divisor \(S \subseteq X\), the canonical divisor \(K_S \in \text{Pic}(S) \otimes \mathbb{Q}\) is defined by \(K_S := \frac{1}{m}(mK_X + mS)|_S\), where \(m \in \mathbb{N}\) is the smallest positive integer such that both \(mK_X\) and \(mS\) are Cartier divisors on \(X\).

Proposition 5.4. Let \((X, \Delta)\) be a \(\mathbb{Q}\)-factorial klt threefold pair, \(S\) a prime divisor on \(X\), and \(L_S\) a strictly nef divisor (resp. almost strictly nef) on \(S\). Then \(K_S + tL_S\) is ample (resp. big) for sufficiently large \(t\).

Proof. Since \(X\) is \(\mathbb{Q}\)-factorial, there exists some positive integer \(m\) such that \(m(K_X + S)\) is Cartier. Let \(f : T \to S\) be the normalization, and \(f : R \to T\) a minimal resolution.

First, we show the case when \(L_S\) is strictly nef. In view of [Ser95, Proof of Theorem 2.3], we only need to reprove the following injection in [Ser95, Claim 4 of Proof of Theorem 2.3] for sufficiently large \(r \gg 1\):

\[
f^*_s \mathcal{O}_R(rmK_R) \hookrightarrow j^* \mathcal{O}_S(rmK_S).
\]

Since \((X, \Delta)\) is a dlt pair, \(X\) is Cohen-Macaulay (cf. [KM98, Theorems 5.10 and 5.22]). By [KMM87, Lemma 5-1-9], there is a natural injective homomorphism

\[
\omega_T^{[m]} := (\omega_T^m)^{\vee} = \mathcal{O}_T(mK_T) \hookrightarrow j^* \mathcal{O}_X(m(K_X + S)) = j^* \mathcal{O}_S(mK_S).
\]

On the other hand, it follows from [Sak84, Theorem (2.1)] that

\[
f^*_s \mathcal{O}_R(f^*(mK_T)) \cong \mathcal{O}_T(mK_T).
\]

Here, our \(mK_T\) is only a Weil divisor. Replacing \(m\) by a multiple, we write \(f^*(mK_T) = mK_R + \Gamma\) with \(\Gamma\) being an effective (integral) divisor (cf. [Sak84, (4.1)]). Then

\[
f^*_s \mathcal{O}_R(mK_R) \subseteq f^*_s (\mathcal{O}_R(mK_R) \otimes \mathcal{O}_R(\Gamma)) = f^*_s \mathcal{O}_R(f^*(mK_T)) \cong \mathcal{O}_T(mK_T) \hookrightarrow j^* \mathcal{O}_S(mK_S).
\]
Hence, we get the inclusion (1) as desired and our proposition for the case $L_S$ being strictly nef is proved.

Second, if $L_S$ is almost strictly nef, then so is $f^*j^*L_S$. By [Cha20, Theorem 26], our $K_T + tf^*j^*L_S$ is big for sufficiently large $t \gg 1$. Applying the inclusion (1), we see that, for $t \gg 1$, $K_S + tL_S$ is the sum of an effective divisor and the pushforward of some big divisor along a generically finite morphism; hence $K_{F_i} + tL_{Y|F_i}$ is also big, and our proposition for the case $L_S$ being almost strictly nef is proved. 

In what follows, we show the main result of this subsection, which is a key to the proof of Theorem D.

**Proposition 5.5.** Let $(X, \Delta)$ be a projective klt threefold pair. Let $L_X$ be a strictly nef $\mathbb{Q}$-divisor on $X$. Suppose that $L_X$ is numerically equivalent to a non-zero effective divisor. Then $K_X + \Delta + tL_X$ is ample for sufficiently large $t$.

Before proving Proposition 5.5, we give the lemma below, which can be easily deduced from Zariski decomposition on the surface (or Kodaira’s lemma) and the Hodge index theorem.

**Lemma 5.6.** Let $L$ be a nef $\mathbb{Q}$-Cartier divisor on a smooth projective surface $S$. Suppose that $L \cdot B = 0$ for some big $\mathbb{Q}$-Cartier divisor $B$ on $S$. Suppose further that $L^2 = 0$. Then $L \equiv 0$.

**Proof of Proposition 5.5.** Suppose the contrary that $K_X + \Delta + tL_X$ is not ample for one (and hence any) $t \gg 1$. By Lemma 5.2 and Lemma 5.3 we have

$$(K_X + \Delta)^3 = (K_X + \Delta) \cdot L_X^2 = (K_X + \Delta) \cdot L_X = L_X^3 = 0.$$ 

Let us take a $\mathbb{Q}$-factorialization $\pi : (Y, \Gamma) \to (X, \Delta)$, which is a small birational morphism (cf. [Kol13, Corollary 1.37, pp. 29-30]) and let $L_Y := \pi^*L_X$ which is nef but not big. Then $L_Y^3 = 0$.

Since $L_X$ is numerically equivalent to a non-zero effective divisor, our $L_Y$ is also numerically equivalent to an effective divisor $\sum n_i F_i$ such that $n_i > 0$ for each $i$ and none of $F_i$ is $\pi$-exceptional by the choice of $\pi$. So it is clear that $L_Y|F_i$ (as the pullback of $L_X|\pi(F_i)$ via $\pi|_{F_i}$) is almost strictly nef (cf. Definition 2.1). By Proposition 5.4, we see that $K_{F_i} + tL_Y|F_i$ is also big for $t \gg 1$. Since $L_Y^3 = 0$, we have $(L_Y|F_i)^2 = L_Y \cdot F_i = 0$ for every $i$ due to the nefness of $L_Y$. Then by Lemma 5.6, we have

$$0 < L_Y|F_i \cdot (K_{F_i} + tL_Y|F_i) = L_Y \cdot (K_Y + F_i) \cdot F_i.$$ 

Notice that $L_Y \cdot F_i^2 = 0$ for every $i$. In fact, otherwise if $L_Y \cdot F_i^2 > 0$ for some $i$, then

$$0 = F_{b_i} \cdot L_Y^2 = F_{b_i} \cdot \sum n_i F_i \cdot L_Y \geq n_{b_i} L_Y \cdot F_{b_i}^2 > 0,$$

which is absurd. Consequently, we have $L_Y \cdot K_Y \cdot F_i > 0$ for every $i$, and as a result,

$$(K_Y + tL_Y) \cdot L_Y^2 = K_Y \cdot L_Y^2 = K_Y \cdot L_Y \cdot \sum n_i F_i > 0.$$ 

Since $L_Y$ is nef, the above inequality further implies that

$$(K_X + \Delta + tL_X) \cdot L_X^2 = (K_Y + \Gamma + tL_Y) \cdot L_Y^2 > 0,$$

by the projection formula. This nevertheless contradicts the calculation $L_X^3 = (K_X + \Delta) \cdot L_X^2 = 0$, and the proposition is thus proved. 

5.2. **Proof of Theorem D.** In the beginning, let us recall the following conjecture on the numerical nonvanishing for generalized polarized pairs, which is closely related to our **Question 1.1** (cf. Proposition 5.5). Recall that the notion of **generalized polarized pairs** was first introduced in [BZ16] to deal with the effectiveness of Iitaka fibrations.

**Conjecture 5.7** ([HL20, Conjecture 1.2]). Let \((X, B + M)\) be a projective generalized log canonical pair. Suppose that

1. \(K_X + B + M_X\) is pseudo-effective; and
2. \(M = \sum \mu_j M_j\) where \(\mu_j \in \mathbb{R}_{>0}\) and \(M_j\) are nef Cartier \(b\)-divisors.

Then there exists an effective \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor \(D\) such that \(K_X + B + M_X \equiv D\).

We shall apply the following theorem to prove our **Theorem D.** Indeed, Han and Liu showed this theorem in a more general setting ([HL20, Theorem 4.5]) by proving the existence of a good minimal model after passing \(X\) to a higher model if necessary. We refer readers to [HL20, Section 4] for more technical details involved.

**Theorem 5.8** (cf. [HL20, Theorems 1.5 and 1.11]). Let \((X, \Delta + M_X)\) be a 3-dimensional projective generalized klt pair such that \(M_X\) is an \(\mathbb{R}_{>0}\)-linear combination of nef Cartier \(b\)-divisors. Suppose that \(K_X + \Delta + M_X\) is pseudo-effective and \(K_X + M_X\) is not pseudo-effective. Then \(K_X + \Delta + M_X\) is numerically equivalent to an effective \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor.

With all the preparations settled down, we begin to prove **Theorem D.** First, it follows from [BCHM10, Corollary 1.4.3] that we can take a \(\mathcal{Q}\)-factorial terminalization \(\tau : (Y, \Gamma) \to (X, \Delta)\) of \((X, \Delta)\) such that \((Y, \Gamma)\) is a \(\mathcal{Q}\)-factorial terminal pair.

**Theorem 5.9.** If \(\Gamma \neq 0\), then \(- (K_X + \Delta)\) is ample.

**Proof.** Let us define \(M_t := -t(K_Y + \Gamma)\) with \(t > 1\). Then the sum \(K_Y + \Gamma + M_t = (1 - t)(K_Y + \Gamma)\) is nef and hence pseudo-effective, and \(K_Y + M_t = (1 - t)(K_Y + \Gamma) - \Gamma\). Clearly, one can choose suitable \(t\) such that \(K_Y + M_t\) is not pseudo-effective. Therefore, applying **Theorem 5.8,** we see that \(- (K_Y + \Gamma)\) is num-effective. By the projection formula and the strict nefness of \(- (K_X + \Delta)\), we see that \(- (K_X + \Delta)\) is also num-effective. So our theorem follows from **Proposition 5.5** (with \(L_X\) replaced by \(- (K_X + \Delta)\) therein).

**Remark 5.10.** If \(\Gamma = 0\), then our boundary divisor \(\Delta = \tau_* \Gamma = 0\). In this case, \(\tau\) is crepant and thus our \(X\) has at worst canonical singularities (with strictly nef anticanonical divisor \(-K_X\)). Then our **Theorem D** in this case follows from [Ueh00, Main Theorem]. Since in this case, our **Theorem A** covers [Ueh00, Remark 3.8]. For the convenience of readers, we shall give a simplified proof of [Ueh00] here.

In the following, we may assume that \(\Gamma = 0\). Suppose the contrary that \(- K_X\) is not ample. Then we see have

\[ h^0(Y, -rK_Y) = h^0(X, -rK_X) = 0 \]

where \(r\) is the Cartier index of \(K_X\) (cf. **Proposition 5.5**). Besides, since \(- K_X\) is not ample, it is not big (cf. **Lemma 5.2**). Hence, \(- K_Y\) is nef but not big, which implies that \(K_Y^2 = 0\) (cf. [KM98, Proposition 2.61]). Moreover, since \(Y\) is a \(\mathcal{Q}\)-factorial terminal threefold such that \(- K_Y\) is nef but not big, it follows from [KMM94, Corollary 6.2] that \(- K_Y \cdot c_2(Y) \geq 0\) where \(c_2(Y)\) is the second Chern class of the sheaf \(\Omega_Y^{\otimes 2}\). We note that, when \(X\) is smooth in codimension two (which is the case when \(X\) has only terminal singularities), \(c_2(X)\) coincides with the birational section Chern class \(c_2(X)\) (cf. e.g. [GKP16, Proof of Claim 4.11]).
Let us recall the following symbols. For a nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $L$ on a normal projective variety $Y$, the \textit{numerical dimension} $\nu(Y, L)$ of $L$ is defined to be the maximum $t$ such that $L^t \neq 0$. We denote by $\kappa(Y, L)$ the \textit{Iitaka dimension} of $L$. Clearly, $\kappa(Y, L) \leq \nu(Y, L) \leq \dim Y$.

**Lemma 5.11.** If $\Gamma = 0$ and $-K_Y$ is not big, then $h^2(Y, -rK_Y) > 0$.

**Proof.** Since $Y$ is rationally connected by Theorem A, we see that the Euler characteristic $\chi(\mathcal{O}_Y) > 0$ (cf. [Deb01, Corollary 4.18] and [KM98, Theorem 5.22]). Applying the Riemann-Roch formula for $-K_Y$, we get

$$\chi(-rK_Y) = \frac{1}{12}(-rK_Y)\mathcal{O}_Y + \chi(\mathcal{O}_Y) > 0,$$

by noting that $K_Y^3 = 0$ and $\frac{1}{12}(-rK_Y)\mathcal{O}_Y \geq 0$ (cf. [KMM94, Corollary 6.2]). This together with $h^0(Y, -rK_Y) = 0$ (by our assumption that $-(K_Y + \Delta)$ is not ample) implies our lemma.

**Lemma 5.12.** If $\Gamma = 0$ and $-K_Y$ is not big, then for a sufficiently ample general hyperplane section $S$ on $Y$, we have $H^1(S, (-rK_Y + S)|_S) \simeq H^2(Y, -rK_Y)$.

**Proof.** We consider the short exact sequence

$$0 \to \mathcal{O}_Y(-rK_Y) \to \mathcal{O}_Y(-K_Y + S) \to \mathcal{O}_S((-rK_Y + S)|_S) \to 0$$

and the induced long exact sequence

$$\cdots \to H^1(Y, -rK_Y + S) \to H^1(S, (-rK_Y + S)|_S) \to H^2(Y, -rK_Y) \to H^2(Y, -rK_Y + S) \to \cdots$$

Since $S$ is ample and $-rK_Y$ is nef, by the Fujita vanishing theorem (or Kawamata-Viehweg vanishing theorem), we have $H^i(Y, -rK_Y + S) = 0$ for $i > 0$. Therefore, $H^1(S, (-rK_Y + S)|_S) \simeq H^2(Y, -rK_Y)$ and our lemma is thus proved.

**Lemma 5.13.** If $\Gamma = 0$ and $-K_Y$ is not big, then the numerical dimension $\nu(Y, -K_Y)$ of $-K_Y$ is 1.

**Proof.** If the numerical dimension $\nu(Y, -K_Y) = 3$, then $-K_Y$ and hence $-(K_Y + \Delta)$ are big, a contradiction to our assumption (cf. Lemma 5.2). Clearly, $-K_Y \neq 0$, and thus we only need to exclude the case $\nu(Y, -K_Y) = 2$. Take a sufficiently ample general hypersurface $S$ on $Y$. Then we have $K_Y^2 \cdot S \neq 0$ since $K_Y^2 \neq 0$. Therefore $-K_Y|_S$ is a nef and big divisor on $S$. Applying Serre duality and Kawamata-Viehweg vanishing theorem, we have $H^1(S, (-rK_Y + S)|_S) = 0$ (where $r$ is the Cartier index of $K_Y$), a contradiction to Lemma 5.12 and Lemma 5.11.

**Theorem 5.14.** If $\Gamma = 0$, then $-(K_X + \Delta)$ is ample.

**Proof.** Note that $\Delta = 0$ in this case. In view of Lemma 5.13, we only need to show the case $\nu(Y, -rK_Y) = 1$ is impossible. Recall that $\tau : Y \to X$ is the $\mathbb{Q}$-factorial terminalization. Assume that $\nu(Y, -rK_Y) = 1$. Then, we have $(-K_Y|_S)^2 = K_Y^2 \cdot S = 0$. Denote by $U$ the regular locus of $Y$. By Proposition 4.8, $\pi_1^{alg}(U)$ is finite. Take a sufficiently general very ample divisor $S$ on $Y$, which is not contracted by $\tau$. Then $\pi_1^{alg}(U) \cong \pi_1^{alg}(S \cap U)$ is finite (cf. [HL85, Theorem 1.1.3]), noting that $U$ has the same homotopy type of the space obtained from $S \cap U$ by attaching cells of dimension $\geq 3$, but the fundamental group of a CW complex only depends on its 2-skeleton (and hence $\pi_1(U) \cong \pi_1(S \cap U)$). Since our $S$ is general and $Y$ has at worst terminal (and hence isolated) singularities (cf. [KM98, Corollary 5.18]), $S \cap U$ coincides with $S$. Therefore, $\pi_1^{alg}(S)$ is finite. On the one hand, since $\pi_1^{alg}(S)$ is finite, applying [Miy88]
for the nef divisor \(-(r+1)K_Y\mid_S\) with \(-(r+1)K_Y\mid_S\) \(= 0\) (cf. also \cite[Theorem 3.6]{Ser95}), we see that, either \(H^i(S, -(r+1)K_Y\mid_S) = 0\), or there exists a positive integer \(n\) such that \(H^0(S, -(r+1)nK_Y\mid_S) \neq 0\). On the other hand, it follows from \text{Lemma 5.12, Lemma 5.11} and Serre duality that \(H^i(S, -(rK_Y + S)\mid_S) \neq 0\). In conclusion, \(-K_Y\mid_S\) is \(\mathbb{Q}\)-linearly equivalent to a non-zero effective divisor. Let \(T := \tau(S)\), which is still a surface due to the choice of \(S\). Then it follows from the commutative diagram that \(-K_X\mid_T\) is \(\mathbb{Q}\)-linearly equivalent to a non-zero effective Weil \(\mathbb{Q}\)-divisor (as a curve) on \(T\). Hence, it follows from the projection formula and from the strict nefness of \(-K_X\) that \(0 < -K_X \cdot (-K_Y\mid_T) = -K_Y \cdot (-K_Y\mid_S) = K_Y^2 \cdot S = 0\), which gives rise to a contradiction. 

\textbf{Proof of Theorem D.} It follows directly from Theorem 5.9 and Theorem 5.14 (cf. Remark 5.10).

\begin{thebibliography}{99}
\bibitem[AD13]{AD13} Carolina Araujo and Stéphane Druel. On Fano foliations. \textit{Advances in Mathematics}, 238:70–118, 2013.
\bibitem[ADK08]{ADK08} Carolina Araujo, Stéphane Druel, and Sándor József Kovács. Cohomological characterizations of projective spaces and hyperquadrics. \textit{Inventiones Mathematicae}, 174(2):233–253, 2008.
\bibitem[Aub78]{Aub78} Thierry Aubin. Équation du type Monge-Ampère usr les variétés kählériennes compactes. \textit{Bulletin de la Société mathématique de France}, 102(1):63–95, 1978.
\bibitem[BCHM10]{BCHM10} Caucher Birkar, Paolo Cascini, Christopher Derek Hacon, and James McKernan. Existence of minimal models for varieties of log general type. \textit{Journal of the American Mathematical Society}, 23(2):405–468, 2010.
\bibitem[BDPP13]{BDPP13} Sébastien Boucksom, Jean-Pierre Demailly, Mihai Păun, and Thomas Peternell. The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension. \textit{Journal of Algebraic Geometry}, 22(2):201–248, 2013.
\bibitem[Bor69]{Bor69} Armand Borel. \textit{Linear algebraic groups}. W. A. Benjamin, Inc., New York-Amsterdam, 1969.
\bibitem[Bri18]{Bri18} Michel Brion. Linearization of Algebraic Group Actions. In Lizhen Ji, Athanase Papadopoulos, and Shing-Tung Yau, editors, \textit{Handbook of Group Actions, Volume IV}, volume 41 of \textit{Advanced Lectures in Mathematics}, pages 291–340, Somerville, MA & Beijing, 2018. International Press of Boston, Inc. & Higher Education Press.
\bibitem[BZ16]{BZ16} Caucher Birkar and De-Qi Zhang. Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs. \textit{Publications Mathématiques. Institut de Hautes Études Scientifiques}, 123:283–331, 2016.
\bibitem[Cam92]{Cam92} Frédéric Campana. Connexité rationnelle des variétés de Fano. \textit{Annales scientifiques de l’École normale supérieure}, 25(5):539–545, 1992.
\bibitem[Cam04]{Cam04} Frédéric Campana. Orbifolds, special varieties and classification theory. \textit{Université de Grenoble. Annales de l’Institut Fourier}, 54(3):499–630, 2004.
\bibitem[Cao19]{Cao19} Junyan Cao. Albanese maps of projective manifolds with nef anticanonical bundles. \textit{Annales scientifiques de l’École normale supérieure}, 52(5):1137–1154, 2019.
\bibitem[CCE15]{CCE15} Frédéric Campana, Benoît Claudon, and Philippe Eyssidieux. Représentations linéaires des groupes kählériens: factorisations et conjecture de shafarevich linéaire. \textit{Compositio Mathematica}, 151(2):351–376, 2015.
\bibitem[CCM19]{CCM19} Frédéric Campana, Junyan Cao, and Shin-ichi Matsumura. Projective klt pairs with nef anti-canonical divisor. Preprint \url{https://arxiv.org/abs/1910.06471}, 2019. to appear in \textit{Algebraic Geometry}.
\bibitem[CCP08]{CCP08} Frédéric Campana, Jungkai A. Chen, and Thomas Peternell. Strictly nef divisors. \textit{Mathematische Annalen}, 342(3):363–383, 2008.
\bibitem[CH19]{CH19} Junyan Cao and Andreas Höring. A decomposition theorem for projective manifolds with nef anticanonical bundle. \textit{Journal of Algebraic Geometry}, 28(3):567–597, 2019.
\bibitem[Cha20]{Cha20} Priyankur Chaudhuri. Strictly nef divisors and some remarks on a conjecture of Serrano. Preprint \url{https://arxiv.org/abs/2008.05009}, 2020.
\end{thebibliography}
Frédéric Campana and Thomas Peternell. Projective manifolds whose tangent bundles are numerically effective. *Mathematische Annalen*, 289(1):169–187, 1991.

Olivier Debarre. *Higher-Dimensional Algebraic Geometry*. Universitext. Springer-Verlag, New York, NY, 2001.

Jean-Pierre Demailly, Thomas Peternell, and Michael Schneider. Compact complex manifolds with numerically effective tangent bundles. *Journal of Algebraic Geometry*, 3(2):295–345, 1994.

Stéphane Druel. On foliations with nef anti-canonical bundle. *Transactions of the American Mathematical Society*, 369(11):7765–7787, 2017.

Osamu Fujino. Minimal model theory for log surfaces. *Publications of the Research Institute for Mathematical Sciences*, 48(2):339–371, 2012.

Daniel Greb, Stefan Kebekus, and Thomas Peternell. Étale fundamental groups of Kawamata log terminal spaces, flat sheaves and quotients of abelian varieties. *Duke Mathematical Journal*, 165(10):1965–2004, 2016.

Robin Hartshorne. *Ample subvarieties of algebraic varieties*. Lecture Notes in Mathematics, Vol. 156. Springer-Verlag, Berlin-New York, 1970.

Hamlut Arend Hamm and Dũng Tráng Lê. Lefschetz theorems on quasi-projective varieties. *Bulletin de la Société mathématique de France*, 113:123–142, 1985.

Jingjun Han and Wenfei Liu. On numerical nonvanishing for generalized log canonical pairs. *Documenta Mathematica*, 25:93–123, 2020.

Christopher Derek Hacon and James McKernan. On Shokurov’s rational connectedness conjecture. *Duke Mathematical Journal*, 138(1):119–136, 2007.

Yujiro Kawamata, Katsumi Matsuda, and Kenji Matsuki. Introduction to the minimal model problem. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 283–360. North-Holland, Amsterdam, 1987.

János Kollár, Yoichi Miyaoka, and Shigefumi Mori. Rational connectedness and boundedness of Fano manifolds. *Journal of Differential Geometry*, 36(3):765–779, 1992.

Sean Keel, Kenji Matsuki, and James McKernan. Log abundance theorem for threefolds. *Duke Mathematical Journal*, 75(1):99–119, 1994.

János Kollár. *Shafarevich maps and automorphic forms*. M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1995.

János Kollár. *Rational curves on algebraic varieties*, volume 3.Folge, Band 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin Heidelberg, 1996. Corrected Second Printing 1999.

János Kollár. *Singularities of the minimal model program*, volume 200 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013.

Duo Li, Wenhao Ou, and Xiaokui Yang. On projective varieties with strictly nef tangent bundles. *Journal de mathématiques pures et appliquées*, 128:140–151, 2019.

Jie Liu, Wenhao Ou, and Xiaokui Yang. Projective manifolds whose tangent bundle contains a strictly nef subsheaf. Preprint https://arxiv.org/abs/2004.08507, 2020.

Jie Liu, Wenhao Ou, and Xiaokui Yang. Strictly nef vector bundles and characterizations of $\mathbb{P}^n$. *Complex Manifolds*, 8(1):148–159, 2021.

Hidetoshi Maeda. A criterion for a smooth surface to be Del Pezzo. *Mathematical Proceedings of the Cambridge Philosophical Society*, 113(1):1–3, 1993.

Yoichi Miyaoka. On the Kodaira dimension of minimal threefolds. *Mathematische Annalen*, 281(2):325–332, 1988.

Yoichi Miyaoka. Relative deformations of morphisms and applications to fibre spaces. *Commentarii Mathematici Universitatis Sancti Pauli*, 42(1):1–7, 1993.

Shin-ichi Matsumura and Juanyong Wang. Structure theorem for projective klt pairs with nef anti-canonical divisor. Preprint https://arxiv.org/abs/2105.14308, 2021.

Sheng Meng and De-Qi Zhang. Building blocks of polarized endomorphisms of normal projective varieties. *Advances in Mathematics*, 325:243–273, 2018.

Noboru Nakayama and De-Qi Zhang. Polarized endomorphisms of complex normal varieties. *Mathematische Annalen*, 346(4):991–1018, 2010.

Wenhao Ou. Foliations whose first class is nef. Preprint https://arxiv.org/abs/2105.10309, 2021.
[Sak84] Fumio Sakai. Weil divisors on normal surfaces. *Duke Mathematical Journal*, 51(4):877–887, 1984.

[Ser95] Fernando Serrano. Strictly nef divisors and Fano threefolds. *Journal für die reine und angewandte Mathematik*, 464:187–206, 1995.

[Sim92] Carlos Tschudi Simpson. Higgs Bundles and Local Systems. *Publications mathématiciques de l'I.H.E.S.*, 75(1):5–95, 1992.

[Tak03] Shigeharu Takayama. Local Simple Connectedness of Resolution of Log-Terminal Singularities. *International Journal of Mathematics*, 14(8):825–836, 2003.

[Ueh00] Hokuto Uehara. On the canonical threefolds with strictly nef anticanonical divisors. *Journal für die reine und angewandte Mathematik*, 522:81–91, 2000.

[Wan20] Juanyong Wang. Structure of projective varieties with nef anticanonical divisor: the case of log terminal singularities. Mathematische Annalen (to appear) DOI: 10.1007/s00208-021-02275-7, 2020.

[Yam10] Katsutoshi Yamanoi. On fundamental groups of algebraic varieties and value distribution theory. *Université de Grenoble. Annales de l'Institut Fourier*, 60(2):551–563, 2010.

[Yau78] Shing-Tung Yau. On the Ricci curvature of a compact Kahler manifold and the complex Monge-Ampère equation. I. *Communications on Pure and Applied Mathematics*, 31:339–411, 1978.

[Zha96] Qi Zhang. On projective manifolds with nef anticanonical divisors. *Journal für die reine und angewandte Mathematik*, 478(3):57–60, 1996.

[Zha06] Qi Zhang. Rational connectedness of log Q-Fano varieties. *Journal für die reine und angewandte Mathematik*, 2006(590):131–142, 2006.

**JIE LIU, INSTITUTE OF MATHEMATICS, ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING, 100190, CHINA**

*Email address*: jliu@amss.ac.cn

**WENHAO OU, INSTITUTE OF MATHEMATICS, ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING, 100190, CHINA**

*Email address*: wenhaoou@amss.ac.cn

**JUANYONG WANG, HCMS, ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING, 100190, CHINA.**

*Email address*: juanyong.wang@amss.ac.cn

**XIAOKUI YANG, DEPARTMENT OF MATHEMATICS AND YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, BEIJING, 100084, CHINA**

*Email address*: xkyang@mail.tsinghua.edu.cn

**GUOLEI ZHONG, DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, 10 LOWER KENT RIDGE ROAD, SINGAPORE 119076, REPUBLIC OF SINGAPORE**

*Email address*: zhongguolei@u.nus.edu