TOPOLOGICAL $([\prod^\omega \ell_2, \sum^\omega \ell_2])$-FACTORS OF DIFFEOMORPHISM GROUPS OF NON-COMPACT MANIFOLDS

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ABSTRACT. Suppose $M$ is a non-compact connected smooth $n$-manifold. Let $\mathcal{D}(M)$ denote the group of diffeomorphisms of $M$ endowed with the compact-open $C^\infty$-topology and $\mathcal{D}^c(M)$ denote the subgroup consisting of diffeomorphisms of $M$ with compact support. Let $\mathcal{D}(M)_0$ and $\mathcal{D}^c(M)_0$ be the connected components of $\text{id}_M$ in $\mathcal{D}(M)$ and $\mathcal{D}^c(M)$ respectively. In this paper we show that the pair $(\mathcal{D}(M), \mathcal{D}^c(M))$ admits a topological $(\prod^\omega \ell_2, \sum^\omega \ell_2)$-factor. In the case $n = 2$, this enables us to apply the characterization of $(\prod^\omega \ell_2, \sum^\omega \ell_2)$-manifolds and show that the pair $(\mathcal{D}(M)_0, \mathcal{D}^c(M)_0)$ is a $(\prod^\omega \ell_2, \sum^\omega \ell_2)$-manifold and determine its topological type. We also obtain a similar result for groups of homeomorphisms of non-compact topological $2$-manifolds.

1. INTRODUCTION

This article is a continuation of study of topological properties of groups of homeomorphisms and diffeomorphisms of non-compact manifolds with the compact-open ($C^\infty$-) topology $[21, 22, 23, 24]$. Suppose $G$ is a transformation group acting on a space $M$ continuously and effectively. Each $g \in G$ induces a homeomorphism $\hat{g}$ of $M$. When $M$ is non-compact, the group $G$ contains the normal subgroup $G_c$ consisting of $g \in G$ such that $\hat{g}$ has a compact support. Let $G_0$ and $(G_c)_0$ denote the connected components of the unit element $e$ in $G$ and $G_c$ respectively. In this paper we are concerned with the topological type of the pair $(G_0, (G_c)_0)$ in the case where $G$ has a weak topology.

Typical examples of the transformation groups $G$ are the group $\mathcal{H}(M)$ of homeomorphisms of a topological manifold (or a locally compact polyhedron) $M$ endowed with the compact-open topology and the group $\mathcal{D}(M)$ of diffeomorphisms of a smooth manifold $M$ endowed with the compact-open $C^\infty$-topology. In $[2]$ and $[3]$ it is shown that both the pairs $(\mathcal{H}(M)_0, \mathcal{H}^c(M)_0)$ for any countable infinite locally finite connected graph $M$ and $(\mathcal{D}(\mathbb{R})_0, \mathcal{D}^c(\mathbb{R})_0)$ for the real line $\mathbb{R}$ are homeomorphic to the pair $(\prod^\omega \ell_2, \sum^\omega \ell_2)$. Here $\prod^\omega \ell_2$ is the countable product of the separable Hilbert space $\ell_2$ and $\sum^\omega \ell_2$ is the countable weak product of $\ell_2$ defined by

$$\sum^\omega \ell_2 = \{ (x_i) \in \prod^\omega \ell_2 \mid x_i = 0 \text{ except finitely many } i \}.$$ 

In this paper, we show that the pairs $(\mathcal{H}(M)_0, \mathcal{H}^c(M)_0)$ and $(\mathcal{D}(M)_0, \mathcal{D}^c(M)_0)$ for a non-compact $2$-manifold $M$ are $(\prod^\omega \ell_2, \sum^\omega \ell_2)$-manifolds (Theorem 1.2) and determine their topological types from their homotopy types (Corollary 1.4).

To establish these results, first we deduce a characterization of $(\prod^\omega \ell_2, \sum^\omega \ell_2)$-manifolds under the stability property (Theorem 2.2) from a general criterion $[20, \text{Theorem 2.9}]$. A pair $(X, A)$ is said to be $(\prod^\omega \ell_2, \sum^\omega \ell_2)$-stable if $(X \times \prod^\omega \ell_2, A \times \sum^\omega \ell_2) \cong (X, A)$. Stability properties of homeomorphism...
groups of topological manifolds and their subgroups have already been studied by many authors (cf. [9, 10, 12, 13, 18] etc). In particular, in [22] we have treated the non-compact case in detail. On the other hand, the Moser’s theorem for volume forms [10] (cf. [23]) exhibits the $\ell_2$-stability property of diffeomorphism groups. We modify these arguments and show the $(\prod^{\omega} \ell_2, \sum^{\omega} \ell_2)$-stability property of the pairs $(\mathcal{H}(M), \mathcal{H}^c(M))$ and $(\mathcal{D}(M), \mathcal{D}^c(M))$ for non-compact (separable metrizable) $n$-manifolds $M$.

**Theorem 1.1.** The pair $(G,G_c)$ is $(\prod^{\omega} \ell_2, \sum^{\omega} \ell_2)$-stable in the following cases:

1. $G = \mathcal{D}(M)$ for a non-compact smooth $n$-manifold $M$ possibly with boundary ($n \geq 1$).
2. $G = \mathcal{H}(M)$ for a non-compact topological $n$-manifold $M$ possibly with boundary ($n \geq 1$).
3. $G = \mathcal{H}(M,\mu)$ for a non-compact topological $n$-manifold $M$ possibly with boundary ($n \geq 2$) and a good Radon measure $\mu$ on $M$.

See Section 5.2 for the group $\mathcal{H}(M,\mu)$ of $\mu$-preserving homeomorphisms of $M$.

In dimension $n = 2$, combined with the ANR-property and homotopy density ([21] Corollary 1.1], [22] Theorem 3.2, Corollary 3.1, [24] Theorems 1.1, 1.2), this stability property enables us to apply the characterization of $(\prod^{\omega} \ell_2, \sum^{\omega} \ell_2)$-manifolds (Theorem 2.2) to the pairs $(\mathcal{D}(M)_0, \mathcal{D}^c(M)_0)$ and $(\mathcal{H}(M)_0, \mathcal{H}^c(M)_0)$. For a subgroup $H$ of a transformation group $G$ on $M$ we use the following notations: $H_X = \{g \in H \mid \hat{g}|_X = \text{id}_X\}$ for $X \subset M$ and $H_c = H \cap G_c$. Let $H_1$ denote the path component of the unit element $e$ in $H$, and set

$$\langle H_e \rangle_1^* = \cup \{ (H_{M-K})_1 \mid K \text{ is a compact subset of } M \}.$$  

**Theorem 1.2.** Suppose $G$ is one of the following groups:

1. $\mathcal{D}_X(M)$ for a non-compact connected smooth $2$-manifold $M$ and a compact submanifold $X$ of $M$.
2. $\mathcal{H}_X(M)$ for a non-compact connected $2$-manifold $M$ possibly with boundary and a compact subpolyhedron $X$ of $M$ with respect to some triangulation of $M$.

Then the pair $(G_0, H)$ is a $(\prod^{\omega} \ell_2, \sum^{\omega} \ell_2)$-manifold for any subgroup $H$ of $G_0$ such that $H$ is $F_{\sigma}$ in $G$ and $(G_c)_1^* \subset H \subset (G_0)_c$.

Note that the subgroups $H = (G_c)_1^*, (G_c)_0$ and $(G_0)_c$ satisfy the conditions in Theorem 1.2.

The topological type of any $(\prod^{\omega} \ell_2, \sum^{\omega} \ell_2)$-manifold $(X, A)$ is classified by the homotopy type of $X$ (Theorem 2.2). Hence, by [21] Theorem 1.1] and [24] Theorem 1.1] we have the conclusion on the global topological type. Consider the next two cases:

(I) $(M, X) \cong (\mathbb{R}^2, \emptyset), (\mathbb{R}^2, 1pt), (\mathbb{S}^1 \times \mathbb{R}^1, \emptyset), (\mathbb{S}^1 \times [0,1], \emptyset)$ or $(\mathbb{P}^2 \setminus 1pt, \emptyset)$.

(II) $(M, X)$ is not the case (I) (in the cases (1) and (2) in Theorem 1.2).

Here $\mathbb{R}^n$ is the Euclidean $n$-space, $\mathbb{S}^n$ is the $n$-sphere and $\mathbb{P}^2$ is the projective plane.

**Corollary 1.1.** In Theorem 1.2 we have $(G_0, H) \cong \begin{cases} (\prod^{\omega} \ell_2, \sum^{\omega} \ell_2) \times \mathbb{S}^1 & \text{in the case (I)}, \\ (\prod^{\omega} \ell_2, \sum^{\omega} \ell_2) & \text{in the case (II)}. \end{cases}$
This paper is organized as follows. In Section 2 we deduce the characterization of \((\prod^\omega \ell_2, \sum^\omega \ell_2)\)-manifolds based upon the stability property (Theorem 2.2). In Section 3 we obtain the results on the diffeomorphism groups in Theorems 3.1–3.2 and Corollary 3.1 while Section 4 includes the results on the homeomorphism groups.

2. Characterization of topological \((\prod^\omega \ell_2, \sum^\omega \ell_2)\)-manifolds

In [20] we obtained a general characterization of infinite-dimensional manifold tuples based upon the stability property (cf. [7, 15, 19], [1, 4, 5, 6], [11], etc.). In this section we deduce a characterization of \((\prod^\omega \ell_2, \sum^\omega \ell_2)\)-manifolds from this general characterization theorem.

2.1. General characterization of infinite-dimensional manifold pairs under the stability property.

We begin with the definition of basic terminology. In this paper spaces are assumed to be separable metrizable and maps are continuous. The symbol \(\cong\) means a homeomorphism, while \(\sim\) means a homotopy equivalence. A pair of spaces means a pair \((X, A)\) of a topological space \(X\) and a subset \(A\) of \(X\). We say that two pairs \((X, A)\) and \((Y, B)\) are homeomorphic and write \((X, A) \cong (Y, B)\) if there exists a homeomorphism \(h : X \to Y\) with \(h(A) = B\). For a model space \(E\), an \(E\)-manifold means a space \(X\) locally homeomorphic to \(E\). More generally, for a model pair \((E, E_1)\), by an \((E, E_1)\)-manifold we mean a pair \((X, X_1)\) of spaces such that each point \(x\) of \(X\) admits an open neighborhood \(U\) of \(x\) in \(X\) and an open subset \(V\) of \(E\) such that \((U, U \cap X_1) \cong (V, V \cap E_1)\).

A closed subset \(A\) of a space \(X\) is called a \(Z\)-set of \(X\) if for any open cover \(U\) of \(X\) there exists a map \(f : X \to X - A\) which is \(U\)-close to \(\text{id}_X\). A \(\sigma\) \(Z\)-set of \(X\) means a countable union of \(Z\)-sets of \(X\). A subset \(A\) of \(X\) is said to be homotopy dense (HD) if there exists a homotopy \(h_t : X \to X\) \((t \in [0, 1])\) such that \(h_0 = \text{id}_X\) and \(h_t(X) \subset A\) for \(t \in (0, 1]\).

Consider the countable product \(s = \prod_{k \in \mathbb{N}} \mathbb{R}\), which is a topological linear space under the coordinatewise sum and scalar product. Since \(s\) is a separable Fréchet space, it follows that \(s \cong \ell_2\). Suppose \(s_1\) is a linear subspace of \(s\). For \(I \subset \mathbb{N}\) we set \(c(I) = \mathbb{N} \setminus I\) and \(s(I) = \prod_{k \in I} \mathbb{R}\), and let \(\pi_I : s \to s(I)\) denote the projection. We set \(s_1(I) = \pi_I(s_1) \subset s(I)\). Let \(\mathcal{M} = \mathcal{M}(s, s_1)\) denote the class of pairs \((X, A)\) which admit a closed embedding \(h : X \to s\) such that \(h^{-1}(s_1) = A\).

**Assumption 2.1.** We assume that the model pair \((s, s_1)\) satisfies the following conditions:

1. \((s_1)\) \(s_1\) is a linear subspace of \(s\) and \(s_1\) is a \(\sigma\) \(Z\)-set of \(s_1\) itself.
2. \((s_2)\) \(s_1\) is homotopy dense in \(s\).
3. \((s_3)\) There exists a sequence \(I_n\) \((n \geq 1)\) of disjoint infinite subsets of \(\mathbb{N}\) such that for each \(n \geq 1\)
   a. \(\min I_n > n\),
   b. \(s_1 = s_1(I_n) \times s_1(c(I_n))\) and
   c. \((s(I_n), s_1(I_n)) \cong (s, s_1)\).

Under Assumption 2.1 we have the following characterization and homotopy invariance of \((s, s_1)\)-manifolds. This is exactly the case that \(\ell = 1\) in [20, Theorem 2.9, Corollary 2.10].

**Theorem 2.1.** A pair \((X, A)\) is an \((s, s_1)\)-manifold iff
Corollary 2.1. Suppose \((X, A)\) and \((Y, B)\) are \((s, s_1)\)-manifolds. Then \((X, A) \cong (Y, B)\) iff \(X \simeq Y\).

2.2. Characterization of \((\prod^\omega \ell_2, \sum^\omega \ell_2)\)-manifolds.

Next we deduce a characterization and classification of \((\prod^\omega \ell_2, \sum^\omega \ell_2)\)-manifolds from Theorem 2.1 and Corollary 2.1.

Theorem 2.2.

(1) A pair \((X, A)\) is a \((\prod^\omega \ell_2, \sum^\omega \ell_2)\)-manifold iff it satisfies the following conditions:

(i) \(X\) is a separable completely metrizable ANR.

(ii) \(A\) is \(F_\sigma\) in \(X\), \(A\) is homotopy dense in \(X\).

(iii) \((X, A)\) is \((\prod^\omega \ell_2, \sum^\omega \ell_2)\)-stable.

(2) Suppose \((X, A)\) and \((Y, B)\) are \((\prod^\omega \ell_2, \sum^\omega \ell_2)\)-manifolds. Then \((X, A) \cong (Y, B)\) iff \(X \simeq Y\).

Since \(s \cong \ell_2\), it follows that \((\prod^\omega \ell_2, \sum^\omega \ell_2) \cong (\prod^\omega s, \sum^\omega s)\). The latter is also denoted by the symbol \((s^\infty, s_f^\infty)\) for notational simplicity. Since \(s^\infty = \prod_{k \in \mathbb{N}} s = \prod_{n \in \mathbb{N}} (\prod_{k \in \mathbb{N}} \mathbb{R}) = \prod_{(n, k) \in \mathbb{N}^2} \mathbb{R}\), any bijection \(\alpha : \mathbb{N}^2 \cong \mathbb{N}\) induces a linear homeomorphism \(\varphi_\alpha : s^\infty \cong s\) and a linear subspace \(s_1 = \varphi_\alpha(s_f^\infty)\) of \(s\). Hence Theorem 2.2 follows from Theorem 2.1, Corollary 2.1 and the next two lemmas.

Lemma 2.1. The pair \((s, s_1)\) satisfies Assumption 2.1.

Proof. \((*_1)\), \((*_2)\) For each \(n \geq 1\) the closed subset \(s^n = s^n \times \{(0, 0, \ldots)\}\) of \(s^\infty\) satisfies the condition that \(s_f^\infty - s^n\) is homotopy dense in \(s^\infty\). In fact, with replacing the interval \([0, 1]\) by \([n, \infty]\) in the opposite orientation, an absorbing homotopy \(\psi : s^\infty \times [n, \infty] \to s^\infty\) is defined by

\[
\psi((x_i)_{i \in \mathbb{N}}, t) = \begin{cases} 
(x_1, \ldots, x_k, (t-k)x_{k+1}, k + 1 - t, t-k, 0, \ldots) & (t \in [k, k+1], k \geq n) \\
(x_1)_{i \in \mathbb{N}} & (t = \infty).
\end{cases}
\]

This implies that \(s_f^\infty\) is homotopy dense in \(s^\infty\) and that \(s^n\) is a \(Z\)-set of \(s_f^\infty\) for each \(n \geq 1\). Since \(s_f^\infty = \cup_{n=1}^\infty s^n\), it follows that \(s_f^\infty\) is a \(Z\)-set of \(s_f^\infty\) itself. Since \((s, s_1) \cong (s^\infty, s_f^\infty)\), this implies the conditions \((*_1)\) and \((*_2)\) for the pair \((s, s_1)\).

\((*_3)\) For any infinite subset \(J\) of \(\mathbb{N}\) it is easily seen that the subset \(J' = \mathbb{N} \times J\) of \(\mathbb{N}^2\) satisfies the conditions: \((b') \ s_f^\infty = s_f^\infty(J') \times s_f^\infty(c(J'))\) and \((c') \ (s^\infty(J'), s_f^\infty(J')) \cong (s^\infty, s_f^\infty)\). Thus the subset \(I = \alpha(J')\) of \(\mathbb{N}\) satisfies the corresponding conditions: \((b) \ s_1 = s_1(I) \times s_1(c(I))\) and \((c) \ (s(I), s_1(I)) \cong (s, s_1)\). Inductively we can find a sequence \(J_n\) (\(n \geq 1\)) of disjoint infinite subsets of \(\mathbb{N}\) with \(\min \alpha(J_n') > n\). Then the subsets \(I_n = \alpha(J_n')\) (\(n \geq 1\)) of \(\mathbb{N}\) satisfy the required condition.

Lemma 2.2. \((X, A) \in \mathcal{M}(s_f^\infty)\) iff \(X\) is separable completely metrizable and \(A\) is \(F_\sigma\) in \(X\).

Proof. Recall that \((X, A) \in \mathcal{M}(s_f^\infty)\) iff there exists a closed embedding \(f : X \to s^\infty\) such that \(f^{-1}(s_f^\infty) = A\). Since \(s^\infty\) is separable completely metrizable and \(s_f^\infty\) is \(F_\sigma\) in \(s^\infty\), any \((X, A) \in \mathcal{M}(s^\infty, s_f^\infty)\) satisfies the same conditions.
Conversely, suppose $X$ is separable completely metrizable and $A$ is $F_e$ in $X$. Then we can find a closed embedding $e : X \to s = s^1 \subset s^\infty$ and a map $g : X \to s$ such that $g^{-1}(\sigma) = A$, where $\sigma = \sum^\infty \mathbb{R} = \mathbb{R}_F^\infty$. A suitable change of indices induces a homeomorphism of pairs $\chi : (s \times s^\infty, \sigma \times s^\infty_f) \cong (s^\infty, s^\infty_f)$. The required embedding $f : X \to s^\infty$ is defined by $f = \chi \circ (g, e)$. Indeed, (i) since $e$ is a closed embedding, so is $f$, and (ii) since $e(X) \subset s^1 \subset s^\infty_f$, it follows that $f^{-1}(s^\infty_f) = (g, e)^{-1} \chi^{-1}(s^\infty_f) = (g, e)^{-1}(\sigma \times s^\infty_f) = A$. □

3. Stability property of $(G, G_e)$-spaces

To treat the groups of homeomorphisms and their subgroups systematically, we formulate our argument to transformation groups. If $E \cong F \times B$ and $B$ is $\ell_2$-stable, then $E$ itself is $\ell_2$-stable. Thus the study of stability property is reduced to seeking for infinite-dimensional factors.

3.1. Factorization of $G$-spaces.

In this subsection we give a simple criterion that a $G$-space admits a product decomposition. Suppose $E$ is a space and $G$ is a topological group which acts continuously on $E$ from the right. We seek a condition that $E$ factors to a product of a subspace $F$ of $E$ and a space $B$. Consider three maps $p : E \to B$, $f : E \to F$ and $g : B \to G$, which induce two maps

$$
\varphi : E \to B \times F; \varphi(x) = (p(x), f(x)) \quad \text{and} \quad \psi : B \times F \to E; \psi(b, y) = y \cdot g(b).
$$

Lemma 3.1. The maps $\varphi$ and $\psi$ are reciprocal homeomorphisms iff

$$
(*) \quad f(x) \cdot g(p(x)) = x \quad (\forall x \in E) \quad \text{and} \quad p(y \cdot g(b)) = b \quad (\forall (b, y) \in B \times F).
$$

Proof. From the definition of the maps $\varphi$ and $\psi$, we have the next identities:

$$
\psi \varphi(x) = \psi(p(x), f(x)) = f(x) \cdot g(p(x)).
$$

$$
\varphi \psi(b, y) = \varphi(y \cdot g(b)) = (p(y \cdot g(b)), f(y \cdot g(b))).
$$

The condition $(*)$ implies that $\psi \varphi(x) = x$ and $\varphi \psi(b, y) = (b, f(y \cdot g(b))) = (b, y)$, since

$$
f(y \cdot g(b)) = (y \cdot g(b)) \cdot g(p(y \cdot g(b)))^{-1} = (y \cdot g(b)) \cdot g(b)^{-1} = y.
$$

This means that $\psi = \varphi^{-1}$. The converse is obvious. □

Complement 3.1. In addition, if (a) the maps $p$, $f$ and $g$ are maps of pairs

$$
p : (E, E_1) \to (B, B_1), \quad f : (E, E_1) \to (F, F_1) \quad \text{and} \quad g : (B, B_1) \to (G, G_1),
$$

(b) $F_1 \subset E_1$, and (c) $G_1$ is a subgroup of $G$ such that $E_1$ is $G_1$-invariant (i.e., $E_1 \cdot G_1 \subset E_1$), then the maps $\varphi$ and $\psi$ induce the maps of pairs

$$
\varphi : (E, E_1) \to (B, B_1) \times (F, F_1) \quad \text{and} \quad \psi : (B, B_1) \times (F, F_1) \to (E, E_1).
$$

By Lemma 3.1 if the maps $p$, $f$ and $g$ satisfy the condition $(*)$, then the maps $\varphi$ and $\psi$ are reciprocal homeomorphisms of pairs (and $F_1 = F \cap E_1$).

The next lemma is the simplest case of Complement 3.1.
Lemma 3.2. Suppose the maps \( p : E \to B, f : E \to F \) and \( g : B \to G \) satisfy the condition \((*)\), so that the map \( \varphi : E \to B \times F \) is a homeomorphism. If \( E' \) is a \( G \)-invariant subspace of \( E \), then \( \varphi(E') = B \times (E' \cap F) \).

Proof. In Complement 3.1 we can take \((E_1, B_1, F_1, G_1) = (E', B, E' \cap F, G)\). For the condition \( f(E_1) \subseteq F_1 \) note that \( f(x) = x \cdot g(p(x))^{-1} \in E_1 \cdot G = E_1 \) for \( x \in E_1 \). \qed

3.2. Transformation groups on non-compact spaces.

A transformation group on a space \( M \) is a topological group \( G \) which acts on \( M \) continuously and faithfully. Each \( g \in G \) induces a homeomorphism \( \hat{g} \) of \( M \). For a subset \( H \subseteq G \) and a subset \( K \subseteq M \), let \( H_K = \{ h \in H \mid \hat{h} = \text{id} \text{ on } K \} \) and \( H(K) = H_{K_1} \).

A support function for a space \( E \) on \( M \) is a function which assigns to each \( f \in E \) a closed subset \( \text{supp } f \) of \( M \). When a space \( E \) is equipped with a support function on \( M \), for any subspace \( F \) of \( E \) we obtain the subspace \( E^c = \{ f \in F \mid \text{supp } f \text{ is compact} \} \). For the transformation group \( G \) on \( M \) the support of \( g \in G \) is canonically defined by \( \text{supp } g = \text{supp } \hat{g} \equiv \text{cl}_M \{ x \in M \mid \hat{g}(x) \neq x \} \). In this case \( G^c \) is a normal subgroup of \( G \).

Definition 3.1. We say that

\[ G \text{ has a weak topology if for each neighborhood } U \text{ of } e \text{ in } G \text{ there exists a compact subset } K \text{ of } M \text{ such that } G_K \subseteq U, \]

\[ G \text{ has the multiplication supported by a discrete family } \{ E_i \}_{i \in \Lambda} \text{ of compact subsets in } M \text{ if for any } (g_i)_{i \in \Lambda} \in \prod_{i \in \Lambda} G(E_i) \text{ there exists a } g \in G(\bigcup_i E_i) \text{ such that } \hat{g} = \hat{g}_i \text{ on } E_i \text{ for each } i \in \Lambda. \]

The element \( g \) is denoted by \( \prod_{i \in \Lambda} g_i \).

Remark 3.1. Suppose \( G \) has the multiplication supported by a discrete family \( \{ E_i \}_{i \in \Lambda} \) of compact subsets in \( M \).

1. Since the action of \( G \) on \( M \) is faithful, each \( g \in G \) is uniquely determined by \( \hat{g} \). Thus the element \( \prod_{i \in \Lambda} g_i \) is uniquely determined by the defining property.

2. The multiplication map

\[ \eta : \prod_{i \in \Lambda} G(E_i) \to G(\bigcup_i E_i); \quad \eta((g_i)_i) = \prod_{i \in \Lambda} g_i \]

is a group homomorphism and \( \eta^{-1}(G^c(\bigcup_i E_i)) = \sum_{i \in \Lambda} G(E_i) \).

3. Any disjoint partition \( \Lambda = \Lambda_0 \cup \Lambda_1 \) yields the product decomposition \( \prod_{i \in \Lambda} G(E_i) = \prod_{i \in \Lambda_0} G(E_i) \times \prod_{i \in \Lambda_1} G(E_i) \), by which the group \( \prod_{i \in \Lambda_0} G(E_i) \) is regarded as a subgroup of \( \prod_{i \in \Lambda} G(E_i) \). Thus, for any \( (g_i)_{i \in \Lambda_0} \in \prod_{i \in \Lambda_0} G(E_i) \) we obtain the product \( \prod_{i \in \Lambda_0} g_i \in G(\bigcup_{i \in \Lambda_0} E_i) \). When \( \Lambda_0 \) is a finite subset, the element \( \prod_{i \in \Lambda_0} g_i \) coincides with the usual product of \( g_i \)'s in \( G \), which is independent of the order of \( g_i \)'s.

Lemma 3.3. If \( G \) has a weak topology and has the multiplication supported by a discrete family \( \{ E_i \}_{i \in \Lambda} \) of compact subsets in \( M \), then the multiplication map \( \eta \) is continuous.

Proof. Since \( \eta \) is a group homomorphism between topological groups, it suffices to show that the map \( \eta \) is continuous at the unit element \( e_\Lambda = (e)_i \) of \( \prod_i G(E_i) \). Given any neighborhood \( U \) of \( e \) in
G, there exists a neighborhood V of e in G and a compact subset K of M such that \( V^2 \subset U \) and \( G_K \subset V \). Since \( \{ E_i \}_{i} \) is discrete, there exists a finite subset \( \Lambda_0 \) of \( \Lambda \) such that \( E_i \cap K = \emptyset \) for any \( i \in \Lambda_1 \equiv \Lambda - \Lambda_0 \). This partition induces the decomposition \( \prod_{i \in \Lambda} G(E_i) = \prod_{i \in \Lambda_0} G(E_i) \times \prod_{i \in \Lambda_1} G(E_i) \).

Since the finite multiplication map
\[
\eta_0 : \prod_{i \in \Lambda_0} G(E_i) \to G; \quad \eta_0((g_i)_i) = \prod_{i \in \Lambda_0} g_i
\]
is continuous, there exists a neighborhood \( W_0 \) of the unit element \( e_{\Lambda_0} \) in \( \prod_{i \in \Lambda_0} G(E_i) \) such that \( \eta_0(W_0) \subset V \). Then, \( W = W_0 \times \prod_{i \in \Lambda_1} G(E_i) \) is a neighborhood of \( e_{\Lambda} \) in \( \prod_{i \in \Lambda} G(E_i) \) and for any \( g_{\Lambda} = (g_{\Lambda_0}, g_{\Lambda_1}) \in W \) it follows that (a) \( g_{\Lambda_0} \in W_0 \) and \( \eta(g_{\Lambda_0}) = \eta_0(g_{\Lambda_0}) \in V \); (b) \( \eta(g_{\Lambda_1}) \in G(\cup_{i \in \Lambda_1} E_i) \subset G_K \subset V \), so that \( \eta(g_{\Lambda}) = \eta(g_{\Lambda_0})\eta(g_{\Lambda_1}) \in V^2 \subset U \). This completes the proof.

\[\square\]

3.3. Factorization of \((G, G^c)\)-pairs.

In this subsection we incorporate the arguments in the previous subsections and deduce a practical criterion, Proposition 3.1, which is used in Sections 4 and 5. Now consider the following data:

**Assumption 3.1.**

1. \( M \) is a space, \( \{ E_i \}_{i \in \Lambda} \) is a discrete family of compact subsets in \( M \) and \( D_i \) is a compact subset of \( E_i \) for each \( i \in \Lambda \).
2. \( G \) is a transformation group on \( M \) which has a weak topology and has the multiplication supported by the family \( \{ E_i \}_{i \in \Lambda} \).
3. \( (\mathcal{E}, f_0) \) is a pointed space equipped with a support function on \( M \) and a continuous right action of \( G \). Suppose that
   (i) \( \text{supp} f_0 = \emptyset \) and (ii) \( \text{supp} f g \subseteq \text{supp} f \cup \text{supp} g \) for any \( (f, g) \in \mathcal{E} \times G \).
4. For each \( i \in \Lambda \) (a) \( (B_i, \alpha_i) \) is a pointed space and
   (b) \( P_i : (\mathcal{E}, f_0) \to (B_i, \alpha_i) \) and \( G_i : (B_i, \alpha_i) \to (G(E_i), e) \) are pointed maps.

Assume that these maps satisfy the following conditions.
   (i) \( P_i(f) = \alpha_i \) if \( \text{supp} f \cap D_i = \emptyset \),
   (ii) \( P_i(f g^{-1}) = \alpha_i \) for any \( (f, g) \in \mathcal{E} \times G \) with \( \tilde{g} = G_i(P_i(f)) \) on \( D_i \),
   (iii) \( P_i(f g) = \nu \) for any \( (f, g, \nu) \in \mathcal{E} \times G \times B_i \) with \( P_i(f) = \alpha_i \) and \( \tilde{g} = G_i(\nu) \) on \( D_i \).

Assumption 3.1 yields the following conclusions; By (2) and Lemma 3.3 the multiplication map
\[
\eta : \prod_{i \in \Lambda} G(E_i) \to G(\cup_i E_i)
\]
is continuous. By (3)(ii) the subspace \( \mathcal{E}^c \) is \( G^c \)-invariant (i.e., \( \mathcal{E}^c \cdot G^c = \mathcal{E}^c \)). For simplicity, we use the symbol
\[
(\mathcal{B}, \mathcal{B}^c, \alpha) = \left( \prod_{i \in \Lambda} B_i, \sum_{i \in \Lambda} B_i, (\alpha_i)_i \right).
\]
Recall that \( \sum_{i \in \Lambda} B_i = \{ (\mu_i)_i \in \prod_{i \in \Lambda} B_i \mid \mu_i = \alpha_i \text{ except finitely many } i \in \Lambda \} \). The maps \( P_i, G_i \)
\( i \in \Lambda \) in (4) are combined to yield the following maps between pointed pairs:

5. \( P : (\mathcal{E}, \mathcal{E}^c, f_0) \to (\mathcal{B}, \mathcal{B}^c, \alpha), \quad G : (\mathcal{B}, \mathcal{B}^c, \alpha) \to (G, G^c, e) \) and \( F : (\mathcal{E}, \mathcal{E}^c, f_0) \to (\mathcal{F}, \mathcal{F}^c, f_0) \).

These maps are defined by the formula:
\[
P(f) = (P_i(f))_i \in \mathcal{E} \quad (f \in \mathcal{E}), \quad G(\mu) = \eta((G_i(\mu_i))_i) \quad (\mu = (\mu_i)_i \in \mathcal{B}),
\]
\[
\mathcal{F} = P^{-1}(\alpha) \subset \mathcal{E}, \quad F(f) = f \cdot G(P(f))^{-1} \quad (f \in \mathcal{E}).
\]
If \( f \in \mathcal{E}^c \), then the compact set \( \text{supp} \, f \) meets only finitely many \( D_i \)'s. Thus by (4)(i) \( P_i(f) = \alpha_i \) except finitely many \( i \)'s and so \( P(f) \in \mathcal{B}^c \). Since \( G_i(\alpha_i) = e \) and \( \eta(\sum_{i \in \Lambda} G_i(\alpha_i)) \subset \mathcal{G}(\cup_i E_i) \), we have \( G(\mathcal{B}^c) \subset \mathcal{G}^c \). For each \( i \in \Lambda \), since \( G_i(\mathcal{P}(f)) = G_i(P_i(f)) \) on \( D_i \), the condition (4)(ii) implies that \( P_i(F(f)) = P_i(f \cdot G(P(f))^{-1}) = \alpha_i \).

This means that \( P \cdot F(f) = \alpha \) and \( F(f) \in \mathcal{F} \). If \( f \in \mathcal{E}^c \), then \( G(P(f)) \in \mathcal{G}^c \) and \( F(f) \in \mathcal{E}^c \cdot \mathcal{G}^c = \mathcal{E}^c \).

This implies that \( F(f) \in \mathcal{F}^c \).

The maps \( P, F \) and \( G \) determine two maps

\[ (6) \quad \Phi : (\mathcal{E}, \mathcal{E}^c) \rightarrow (\mathcal{B}, \mathcal{B}^c) \times (\mathcal{F}, \mathcal{F}^c) \quad \text{and} \quad \Psi : (\mathcal{B}, \mathcal{B}^c) \times (\mathcal{F}, \mathcal{F}^c) \rightarrow (\mathcal{E}, \mathcal{E}^c). \]

These maps are defined by \( \Phi(f) = (P(f), F(f)) \) and \( \Psi(\mu, h) = h \cdot G(\mu) \).

**Proposition 3.1.** (i) The maps \( \Phi \) and \( \Psi \) are reciprocal homeomorphisms of pairs.

(ii) \( a \) Suppose \((\mathcal{E}_1, \mathcal{E}_2)\) is a subpair of \((\mathcal{E}, \mathcal{E}^c)\) and \( \mathcal{E}_1 \) is \( \mathcal{G} \)-invariant and \( \mathcal{E}_2 \) is \( \mathcal{G}^c \)-invariant respectively. Then the homeomorphism \( \Phi \) restricts to the homeomorphism of the subpairs

\[ \Phi : (\mathcal{E}_1, \mathcal{E}_2) \rightarrow (\mathcal{B}, \mathcal{B}^c) \times (\mathcal{E}_1 \cap \mathcal{F}, \mathcal{E}_2 \cap \mathcal{F}). \]

(b) In particular, if \( \mathcal{E}_1 \) is a \( \mathcal{G} \)-invariant subspace of \( \mathcal{E} \), then the homeomorphism \( \Phi \) induces the homeomorphism of the subpairs

\[ \Phi : (\mathcal{E}_1, \mathcal{E}_1^c) \rightarrow (\mathcal{B}, \mathcal{B}^c) \times (\mathcal{E}_1 \cap \mathcal{F}, \mathcal{E}_1^c \cap \mathcal{F}). \]

**Proof.** (i) By Complement 3.1 it suffices to verify the following conditions:

\begin{align*}
(\ast_1) \quad & F(f) \cdot G(P(f)) = f \quad \text{for any } f \in \mathcal{E}, \quad (\ast_2) \quad P(h \circ G(\mu)) = \mu \quad \text{for any } (\mu, h) \in \mathcal{B} \times \mathcal{F}.
\end{align*}

The condition \((\ast_1)\) follows from the definition of the map \( F \). Since \( P(h) = \alpha \) and \( \widehat{G(\mu)} = \widehat{G_i(\mu_i)} \) on \( D_i \), by (4)(iii) it follows that \( P_i(h \circ G(\mu)) = \mu_i \). This implies the condition \((\ast_2)\).

(ii) By the conditions on \((\mathcal{E}_1, \mathcal{E}_2)\) the map \( F \) induces the map between subpairs, \( F : (\mathcal{E}_1, \mathcal{E}_2) \rightarrow (\mathcal{E}_1 \cap \mathcal{F}, \mathcal{E}_2 \cap \mathcal{F}). \) Thus the assertion follows from (i) and Complement 3.1.

(b) Since \( \mathcal{E}_1^c \) is \( \mathcal{G}^c \)-invariant, the statement follows from (a). \( \square \)

4. **Stability property of diffeomorphism groups**

In this section we study the stability proeprty of diffeomorphism groups (Theorem 1.1(1)) and prove Theorem 1.2(1).

4.1. **Preliminaries on volume forms and volume densities.**

Suppose \( M \) is a smooth (separable metrizable) \( n \)-manifold possibly with boundary. When \( M \) is orientable, the volume forms on \( M \) serves our purpose. However, to include the non-orientable case, it is necessary to recall the notion of volume density.

Suppose \( V \) is a 1-dimensional real vector space. The dual space \( V^* \) consists of all linear functions \( f : V \rightarrow \mathbb{R} \), while its variant \( V^# \) is defined by

\[ V^# = \{ f \in V \rightarrow \mathbb{R} \mid f(av) = |a|f(v) \, (\forall v \in V, \forall a \in \mathbb{R}) \} = \{ \pm |f| \mid f \in V^* \}. \]
These spaces form 1-dimensional real vector spaces under the usual sum and scalar product of real-valued functions. When $V$ is oriented, by $V_+$ we denote the connected component of $V - \{0\}$ consisting of positive vectors. Even if $V$ itself is not oriented, the space $V^\#$ always admits a canonical orientation with the positive vectors $V^+_\# = \{|f| \mid f \in V^*\}$. In addition, if $V$ is oriented, then $V^*$ admits the corresponding orientation and a canonical orientation-preserving isomorphism $V^* \cong V^\#$.

This construction extends to real line bundles. For any smooth real line bundle $L \to M$, we have the associated real line bundles $L^* = \cup_{p \in M}(E_p)^* \to M$ and $L^\# = \cup_{p \in M}(L_p)^\# \to M$. The line bundle $L^\#$ has a canonical orientation, so that it is trivial since $M$ is paracompact. If $L$ itself is oriented (i.e., each fiber $L_p$ ($p \in M$) is equipped with an orientation $o_p$ which varies continuously in $p \in M$), then $L^*$ also admits the corresponding orientation and there exists a canonical isomorphism $L^* \cong L^\#$ of oriented vector bundles over $M$.

For a smooth manifold $N$ possibly with boundary, let $C^\infty(M,N)$ denote the space of $C^\infty$ maps $f : M \to N$. More generally, for a smooth fiber bundle $E \to M$, let $\Gamma(E)$ denote the space of global sections of $E$. These spaces are endowed with the compact-open $C^\infty$-topology. Note that $\Gamma(M \times N) \cong C^\infty(M,N)$ for the product bundle $M \times N \to M$. For an oriented real line bundle $L \to M$, the subspace of positive sections of $L$ is defined by

$$\Gamma_+(L) = \{s \in \Gamma(L) \mid s(p) \in (L_p)_+ \ (\forall p \in M)\}.$$

**Lemma 4.1.** $\Gamma_+(L) \cong \ell_2$.

**Proof.** The trivial fiber bundle $L$ includes the sub-bundle $L_+ = \cup_{p \in M}(L_p)_+$, which is also a trivial fiber bundle with fiber $(0,\infty) \cong \mathbb{R}$. Since $\Gamma_+(L) = \Gamma(L_+) \cong C^\infty(M,\mathbb{R})$ and the latter is an infinite-dimensional separable Frechet space, we have the conclusion. \hfill \Box

Now we apply the above arguments to the line bundle $\wedge^n TM$. Any section $\omega \in \Gamma((\wedge^n TM)^\#)$ is called a density on $M$, since its components over coordinate charts transform by the absolute value of Jacobian under coordinate transitions and hence the integral $\int_M \omega \in \mathbb{R}$ is well-defined whenever $\omega$ has a compact support and the $\omega$-volume $\omega(M) = \int_M \omega \in (0,\infty]$ is defined as an improper integral for any positive density $\omega \in \Gamma_+((\wedge^n TM)^\#)$. To simplify the notations, let

$$V_{(+)\#}(M) = \Gamma_+((\wedge^n TM)^\#) \quad \text{and} \quad \forall_{(+)\#}(M;m) = \{\omega \in V_{(+)\#}(M) \mid \omega(M) = m\} \ (m \in (0,\infty]).$$

Suppose $N$ is another smooth $n$-manifold possibly with boundary and $f : N \to M$ is a $C^\infty$ map. Then the differential of $f$, $df : TN \to TM$, induces the pull-back $f^* \omega \in \forall_{\#}(N)$ for each $\omega \in \forall_{\#}(M)$. This defines a continuous map

$$f^* : \forall_{\#}(M) \to \forall_{\#}(N).$$

Moreover, if $f$ is an immersion, then $f^*(\forall_{\#}(M)) \subset \forall_{\#}(N)$. It is seen that the group $\mathcal{D}(M)$ acts continuously on the space $\forall_{\#}(M)$ by the pull-back and the subspace $\forall_{\#}(M)$ is invariant under this action. For the inclusion $i : N \subset M$, the pull-back $i^* \omega$ is also denoted by $\omega|_N$.

For $\mu, \nu \in \forall_{\#}(M)$ we write $\mu \sim_1 \nu$ if $\nu = c\mu$ for some $c > 0$. This is an equivalence relation and preserved by the pull-back.
When $M$ is oriented (i.e., $\wedge^n TM$ is oriented), the line bundle $(\wedge^n TM)^\ast$ also admits a canonical orientation and there exist canonical isomorphisms of oriented vector bundles

$$(\wedge^n TM)^\# \cong (\wedge^n TM)^\ast \cong \wedge^n T^* M.$$ 

This enables us to identify the space $V_+^\#(M)$ with the space $V_+(M) = \Gamma_+(\wedge^n T^* M)$ of positive volume forms on $M$. The induced homeomorphism $\eta_M : V_+^\#(M) \cong V_+(M)$ is compatible with the pull-back, that is, for any orientation-preserving diffeomorphism $h \in D_+(M)$ we have the commutative diagram:

$$\begin{align*}
V_+^\#(M) & \xrightarrow{h^*} V_+^\#(M) \\
\eta_M & \cong \eta_M \\
V_+(M) & \xrightarrow{h^*} V_+(M)
\end{align*}$$

Suppose $F$ is a compact smooth $n$-manifold possibly with boundary. For $\mu \in V_+^\#(F)$ we define $\overline{\mu} = \frac{1}{\mu(F)} \mu \in V_+^\#(F; 1)$. Then, $\mu \sim_1 \nu$ iff $\overline{\mu} = \overline{\nu}$ for $\mu, \nu \in V_+^\#(F)$.

**Lemma 4.2.** $V_+^\#(F; 1) \cong \ell_2$.

**Proof.** The space $V_+^\#(F; 1)$ is a convex subset of the separable Fréchet space $V^\#(F) = \Gamma((\wedge^n TF)^\#)$, and it admits a canonical homeomorphism

$$\chi : V_+^\#(F) \cong V_+^\#(F; 1) \times (0, \infty), \ \chi(\mu) = (\overline{\mu}, \mu(F)).$$

The inverse is given by $\chi^{-1}(\mu, c) = c\mu$. Since $V_+^\#(F) = \Gamma_+((\wedge^n TF)^\#) \cong \ell_2$ by Lemma 4.1, it follows that $V_+^\#(F; 1)$ is nowhere locally compact. Hence, $V_+^\#(F; 1) \cong \ell_2$ by [7].

In the next subsection we need a version of Moser’s theorem for volume forms ([16, Theorem 2] cf. [23, Theorem 2.1]). Suppose $M$ is a smooth $n$-manifold possibly with boundary, $L \subset N$ are compact connected oriented smooth $n$-submanifolds of $M$ possibly with boundary such that $L \subset \text{Int} N$ and $L$ inherits the orientation from $N$, and $\alpha \in V_+^\#(L; 1)$. Consider the following subgroup of $D(M)$:

$$G(N, L) = \{ h \in D_{M-\text{Int} N}(M) \mid h(L) = L \}.$$ 

For any subgroup $H$ of $D(M)$, we define a subgroup $H_1^\#$ of $H$ by

$$H_1^\# = \{ h \in H \mid h : (\natural)_h : \text{There exists a smooth isotopy } H : M \times [0, 1] \to M \\
such that } H_0 = \text{id}_M, H_1 = h \quad \text{and } H_t \in H \ (\forall t \in [0, 1]).$$

**Theorem 4.1.** There exists a map $\varphi : V_+^\#(L) \to G(N, L)_1^\#$ such that

$$(\varphi(\mu)|_L)^\ast \alpha \sim_1 \mu \ (\forall \mu \in V_+^\#(L)) \quad \text{and } \varphi(\alpha) = \text{id}_M.$$ 

**Proof.** Under the canonical homeomorphism $\eta_L : V_+^\#(L) \cong V_+(L)$, the map $\varphi$ corresponds with the map $\varphi' : V_+(L) \to G(N, L)_1^\#$ such that

$$(\varphi'(\mu)|_L)^\ast \alpha' \sim_1 \mu \ (\forall \mu \in V_+(L)) \quad \text{and } \varphi'(\alpha') = \text{id}_M,$$

where $\alpha' = \eta_L(\alpha) \in V_+(L; 1)$. Below we construct the map $\varphi'$.

There exists a map $s : V_+(L) \to V_+(N)$ such that

$$s(\alpha) = \begin{cases} 
\alpha & \text{if } \alpha \in V_+(L), \\
\eta_L^{-1} \circ \varphi'(s(\alpha)) & \text{if } \alpha \in V_+(N). 
\end{cases}$$
(i) \( s(\mu)|_L = \mu \ (\forall \mu \in \mathcal{V}_+(L)) \).

Let \( \beta = s(\alpha') \in \mathcal{V}_+(N) \). Choose a small bicollar \( E = \partial L \times [-1, 1] \) of \( \partial L \) in \( \text{Int} \ N \) with \( \partial L \times [-1, 0] \subset L \).

Then, by the parametrized version of [14, Lemma A2] (cf. [23, Lemma 2.3]) we can find maps \( \lambda, \kappa \)

Consider two maps \( \lambda, \kappa: \mathcal{V}_+(N) \to \mathcal{V}_+(L; 1) \) defined by

\[
\lambda(\nu) = ((\psi(\nu)|_N)^*\beta)|_L = (\psi(\nu)|_L)^*\alpha' \quad \text{and} \quad \kappa(\nu) = \overline{\nu}|_L.
\]

Thus, the parametrized version of Moser’s Theorem [16, Theorem 2] (cf. [23, Theorem 2.1]) yields a map

\[
\chi: \mathcal{V}_+(N) \to \mathcal{D}_{M-\text{Int} E}(M)_1^2 \subset \mathcal{G}(N, L)_1^2
\]

such that

\[
(\text{iv}) \ (\chi(\nu)|_L)^*\lambda(\nu) = \kappa(\nu) \ (\forall \nu \in \mathcal{V}_+(N)) \quad \text{and} \quad (\text{iv}') \ \chi(\beta) = \text{id}_M.
\]

The required map \( \varphi': \mathcal{V}_+(L) \to \mathcal{G}(N, L)_1^2 \) is defined by

\[
\varphi'(\mu) = \psi(s(\mu))\chi(s(\mu)).
\]

Indeed, one sees that,

\[
(\text{v}) \ (\varphi'(\mu)|_L)^*\alpha' = (\psi(s(\mu)|_L \chi(s(\mu))|_L)^*\alpha' = (\chi(s(\mu))|_L)^*\psi(s(\mu)|_L)^*\alpha'
\]

\[
= (\chi(s(\mu)|_L)^*\chi(s(\mu)) = \kappa(\mu) = \overline{\mu} \quad (\forall \mu \in \mathcal{V}_+(L)) \quad \text{and} \quad (\text{v}') \ \varphi'(\alpha') = \psi(\beta)\chi(\beta) = \text{id}_M.
\]

This completes the proof. \( \square \)

4.2. \( (\prod_{i \in \Lambda} \ell_2, \sum_{i \in \Lambda} \ell_2) \)-stability of diffeomorphism groups.

Suppose \( M \) and \( N \) are smooth \( n \)-manifolds possibly with boundary. We fix a section \( \omega \in \mathcal{V}_+\ell_1^\mathbb{R}(N) \). Let \( \{E_i\}_{i \in \Lambda} \) be a discrete family of oriented smooth closed \( n \)-disks in \( M \). Since \( M \) is assumed to be separable, the index set \( \Lambda \) is at most countable and when \( M \) is non-compact, we can take \( \Lambda \) to be an infinite countable set. For each \( i \in \Lambda \) take an \( n \)-subdisk \( D_i \) in \( \text{Int} E_i \), which inherits the orientation from \( E_i \).

Consider the subspace \( \mathcal{E} \) of \( C^\infty(M, N) \) defined by

\[
\mathcal{E} = \{ f \in C^\infty(M, N) \mid f|_{D_i}: D_i \to N \text{ is a } C^\infty\text{-immersion for each } i \in \Lambda \}.
\]

Below we study the stability property of the space \( \mathcal{E} \) and its subspaces, based upon the arguments in Section 3.3.

Define the subgroup \( \mathcal{G} \) of \( \mathcal{D}(M) \) by \( \mathcal{G} = (\mathcal{G'})_1^2 \), where

\[
\mathcal{G'} = \{ h \in \mathcal{D}(M) \mid h = \text{id}_M \text{ on } M - \bigcup_{i \in \Lambda} \text{Int} E_i, \ h(D_i) = D_i \text{ for each } i \in \Lambda \}.
\]
The group $\mathcal{G}$ acts continuously on the space $\mathcal{E}$ by the right composition, and as a transformation group on $M$ it has a weak topology and also has the multiplication supported by the family $\{E_i\}_{i \in \Lambda}$. Hence, the composition map $\eta : \prod_{i \in \Lambda} \mathcal{G}(E_i) \to \mathcal{G}$ is continuous by Lemma 3.3.

Since $\mathcal{E} \neq \emptyset$, we can choose a distinguished element $f_0 \in \mathcal{E}$. A support function for $\mathcal{E}$ on $M$ is defined by

$$\text{supp}_{f_0}f = cl_M\{x \in M \mid f(x) \neq f_0(x)\}.$$ 

Note that it satisfies the condition in Assumption 3.1 (3) and the subspace $\mathcal{E}^c$ is $\mathcal{G}^c$-invariant (i.e., $\mathcal{E}^c \cdot \mathcal{G}^c = \mathcal{E}^c$).

For each $i \in \Lambda$ define a pointed space $(B_i, \alpha_i)$ and two maps $P_i$ and $G_i$ as follows: Let

$$(B_i, \alpha_i) = \left(\mathcal{V}^\#_i(D_i; 1), (f_0|_{D_i})^*\omega\right).$$

Theorem 4.1 yields a map $\varphi_i : \mathcal{V}^\#_i(D_i) \to \mathcal{G}$ such that

(i) $(\varphi_i(\lambda)|_{D_i})^*\alpha_i \sim_1 \lambda$ ($\forall \lambda \in \mathcal{V}^\#_i(D_i)$) and (ii) $\varphi_i(\alpha_i) = \text{id}_M$.

The maps $P_i$ and $G_i$ are defined by

$P_i : (\mathcal{E}, f_0) \to (B_i, \alpha_i)$; \hspace{1em} $P_i(f) = (f|_{D_i})^*\omega$,

$G_i : (B_i, \alpha_i) \to (\mathcal{G}, \text{id}_M)$; \hspace{1em} $G_i(\lambda) = \varphi_i(\lambda)$.

Claim. The maps $P_i$ and $G_i$ satisfy the conditions (i) - (iii) in Assumption 3.1 (4).

Proof. (i) If $(\text{supp}_{f_0}f) \cap D_i = \emptyset$, then $P_i(f) = (f|_{D_i})^*\omega = (f_0|_{D_i})^*\omega = \alpha_i$.

(ii) Suppose $(f, g) \in \mathcal{E} \times \mathcal{G}$ and $g = G_i(P_i(f))$ on $D_i$. For $\lambda = P_i(f) = (f|_{D_i})^*\omega$, it is seen that

$$(f|_{D_i})^*\omega \sim_1 \lambda \text{ and } (\varphi_i(\lambda)|_{D_i})^*\alpha_i \sim_1 \lambda, \text{ so that}$$

$$(\varphi_i(\lambda)|_{D_i})^{-1}(f|_{D_i})^*\omega \sim_1 ((\varphi_i(\lambda)|_{D_i})^{-1})^*\sim_1 \alpha_i.$$ 

Since $(f g^{-1})|_{D_i} = f|_{D_i}(g|_{D_i})^{-1} = f|_{D_i}(G_i(\lambda)|_{D_i})^{-1} = f|_{D_i}(\varphi_i(\lambda)|_{D_i})^{-1}$, it follows that

$$P_i(f g^{-1}) = ((f g^{-1})|_{D_i})^*\omega = (f|_{D_i}(\varphi_i(\lambda)|_{D_i})^{-1})^*\omega = ((f|_{D_i})^*\omega)^{-1}(f|_{D_i})^*\omega = \alpha_i.$$ 

(iii) Suppose $(f, g, \lambda) \in \mathcal{E} \times \mathcal{G} \times B_i$, $P_i(f) = \alpha_i$ and $g = G_i(\lambda)$ on $D_i$. Then one has

$$P_i(f) = \alpha_i, \hspace{1em} (\varphi_i(\lambda)|_{D_i})^*\alpha_i \sim_1 \lambda \text{ and } (f g)|_{D_i} = f|_{D_i}g|_{D_i} = f|_{D_i}G_i(\lambda)|_{D_i} = f|_{D_i}(\varphi_i(\lambda)|_{D_i}).$$

This implies that $(f|_{D_i})^*\omega \sim_1 \alpha_i$ and

$$(f g)|_{D_i})^*\omega = (f|_{D_i}(\varphi_i(\lambda)|_{D_i})^*\omega = (\varphi_i(\lambda)|_{D_i})^*(f|_{D_i})^*\omega \sim_1 (\varphi_i(\lambda)|_{D_i})^*\alpha_i \sim_1 \lambda,$$

so that $P_i(f g) = ((f g)|_{D_i})^*\omega = \lambda$. \hfill $\square$

Hence, we can apply the arguments in Section 3.3 to this setting. Two pointed pairs $(\mathcal{B}, \mathcal{B}^c, \alpha)$ and $(\mathcal{F}, \mathcal{F}^c, f_0)$ and three maps $P$, $G$ and $F$ are defined by

$$(\mathcal{B}, \mathcal{B}^c, \alpha) = \left(\prod_{i \in \Lambda} B_i, \sum_{i \in \Lambda} B_i, (\alpha_i)_i\right),$$

$P : (\mathcal{E}, \mathcal{E}^c, f_0) \to (\mathcal{B}, \mathcal{B}^c, \alpha)$; \hspace{1em} $P(f) = (P_i(f))_{i \in \Lambda}$ ($f \in \mathcal{E}$),

$G : (\mathcal{B}, \mathcal{B}^c, \alpha) \to (\mathcal{G}, \mathcal{G}^c, \text{id}_M)$; \hspace{1em} $G(\mu) = \eta((G_i(\mu_i))_i)$ ($\mu = (\mu_i)_i \in \mathcal{B}$),

$\mathcal{F} = P^{-1}(\alpha)$ \hspace{1em} $(\mathcal{F}^c = \mathcal{F} \cap \mathcal{E}^c),$: 

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Thus the pair $(\mathcal{E}, \mathcal{E}^c, f_0) \rightarrow (\mathcal{F}, \mathcal{F}^c, f_0)$; $F(f) = f \cdot G(P(f))^{-1}$ ($f \in \mathcal{E}$).

These maps determine two maps $\Phi$ and $\Psi$ by

$$\Phi : (\mathcal{E}, \mathcal{E}^c, f_0) \rightarrow (\mathcal{B}, \mathcal{B}^c, \alpha) \times (\mathcal{F}, \mathcal{F}^c, f_0); \quad \Phi(f) = (P(f), F(f)),$$

$$\Psi : (\mathcal{B}, \mathcal{B}^c, \alpha) \times (\mathcal{F}, \mathcal{F}^c, f_0) \rightarrow (\mathcal{E}, \mathcal{E}^c, f_0); \quad \Psi(\mu, g) = g \cdot G(\mu).$$

Lemma 4.3. $(\mathcal{B}, \mathcal{B}^c) \cong (\prod_{i \in \Lambda} \ell_2, \sum_{i \in \Lambda} \ell_2)$. Thus, the pair $(\mathcal{B}, \mathcal{B}^c)$ is $(\prod_{i \in \Lambda} \ell_2, \sum_{i \in \Lambda} \ell_2)$-stable.

Proof. From Lemma 4.2 it follows that $B_i = \oplus_+ (D_i; 1) \cong \ell_2$ for each $i \in \Lambda$ and $(\mathcal{B}, \mathcal{B}^c) \cong (\prod_{i \in \Lambda} \ell_2, \sum_{i \in \Lambda} \ell_2)$. Note that the pair $(\prod_{i \in \Lambda} \ell_2, \sum_{i \in \Lambda} \ell_2)$ itself is $(\prod_{i \in \Lambda} \ell_2, \sum_{i \in \Lambda} \ell_2)$-stable, since

$$(\ell_2)^2 \cong \ell_2 \text{ and } (\prod_{i \in \Lambda} \ell_2, \sum_{i \in \Lambda} \ell_2)^2 \cong (\prod_{i \in \Lambda} (\ell_2)^2, \sum_{i \in \Lambda} (\ell_2)^2) \cong (\prod_{i \in \Lambda} \ell_2, \sum_{i \in \Lambda} \ell_2).$$

Thus the pair $(\mathcal{B}, \mathcal{B}^c)$ is also $(\prod_{i \in \Lambda} \ell_2, \sum_{i \in \Lambda} \ell_2)$-stable. \qed

The next proposition follows from Proposition 3.1.

**Proposition 4.1.** (1) The maps $\Phi$ and $\Psi$ are reciprocal homeomorphisms of pairs.

(2) (i) If $(\mathcal{E}_1, \mathcal{E}_2)$ is a subpair of $(\mathcal{E}, \mathcal{E}^c)$, $\mathcal{E}_1$ is $\mathcal{G}$-invariant and $\mathcal{E}_2$ is $\mathcal{G}^c$-invariant, then the map $\Phi$ restricts to the homeomorphism of the subpairs

$$\Phi : (\mathcal{E}_1, \mathcal{E}_2) \rightarrow (\mathcal{B}, \mathcal{B}^c) \times (\mathcal{E}_1 \cap \mathcal{F}, \mathcal{E}_2 \cap \mathcal{F}).$$

In particular, the pair $(\mathcal{E}_1, \mathcal{E}_2)$ is $(\prod_{i \in \Lambda} \ell_2, \sum_{i \in \Lambda} \ell_2)$-stable.

(ii) If $\mathcal{E}_1$ is a $\mathcal{G}$-invariant subspace of $\mathcal{E}$, then the map $\Phi$ induces the homeomorphism of the subpairs

$$\Phi : (\mathcal{E}_1, \mathcal{E}_1^c) \rightarrow (\mathcal{B}, \mathcal{B}^c) \times (\mathcal{F}_1, \mathcal{F}_1^c) \quad \text{where } \mathcal{F}_1 = \mathcal{E}_1 \cap \mathcal{F}.$$

**Example 4.1.** The space $\mathcal{E}$ includes the following $\mathcal{G}$-invariant subspaces;

$$\mathcal{E}_1 := \text{Imm}^\infty (M, N) = \{f \in C^\infty (M, N) \mid f \text{ is a } C^\infty \text{-immersion}\},$$

$$\mathcal{E}_2 := \text{Emb}^\infty (M, N) = \{f \in C^\infty (M, N) \mid f \text{ is a } C^\infty \text{-embedding}\},$$

$$\mathcal{E}_3 := \text{Cov}^\infty (M, N) = \{f \in C^\infty (M, N) \mid \text{f is a } C^\infty \text{-covering projection}\}.$$

For each $i = 1, 2, 3$, the map $\Phi$ induces the homeomorphism between the subpairs

$$\Phi : (\mathcal{E}_i, \mathcal{E}_i^c) \cong (\mathcal{B}, \mathcal{B}^c) \times (\mathcal{F}_i, \mathcal{F}_i^c), \quad \text{where } \mathcal{F}_i = \mathcal{E}_i \cap \mathcal{F}.$$

Thus, the pair $(\mathcal{E}_i, \mathcal{E}_i^c)$ is $(\prod_{i \in \Lambda} \ell_2, \sum_{i \in \Lambda} \ell_2)$-stable.

Next we consider the case where $M = N$. As a base point of the space $\mathcal{E}$ we take $f_0 = \text{id}_M$. Then the support function $\text{supp}_{f_0}$ reduces to the ordinary support function. The space $\mathcal{E}$ includes the group $\mathcal{D}(M)$ as a $\mathcal{G}$-invariant subspace. Below we discuss the $(\prod_{i \in \Lambda} \ell_2, \sum_{i \in \Lambda} \ell_2)$-stability of the pair $(\mathcal{D}(M), \mathcal{D}^c(M))$ and its subpairs. For any subset $\mathcal{H}$ of $\mathcal{D}(M)$ with $\text{id}_M \in \mathcal{H}$, the symbols $\mathcal{H}_0$ and $\mathcal{H}_1$ denote the connected component and the path component of $\text{id}_M$ in $\mathcal{H}$ respectively.

From Proposition 4.1(2) we have the following criterion.

**Proposition 4.2.** Suppose $(\mathcal{H}, \mathcal{K})$ is a pair of subgroups of $\mathcal{D}(M)$ with $(\mathcal{G}, \mathcal{G}^c) \subset (\mathcal{H}, \mathcal{K}) \subset (\mathcal{D}(M), \mathcal{D}^c(M))$. Then the map $\Phi$ induces the homeomorphism

$$\Phi : (\mathcal{H}, \mathcal{K}) \cong (\mathcal{B}, \mathcal{B}^c) \times (\mathcal{H} \cap \mathcal{F}, \mathcal{K} \cap \mathcal{F}).$$
Thus, the pair \((\mathcal{H},\mathcal{K})\) is \((\prod_{i\in\Lambda}\ell_2,\sum_{i\in\Lambda}\ell_2)\)-stable.

The next example includes Theorem 1.1(1).

**Example 4.2.** The pairs \((\mathcal{D}_X(M),\mathcal{D}^c_X(M))\) and \((\mathcal{D}_X(M),\mathcal{D}^c_X(M))\) (\(i = 0, 1\)) are \((\prod_{i\in\Lambda}\ell_2,\sum_{i\in\Lambda}\ell_2)\)-stable for any subset \(X\) of \(M - \cup_{i\in\Lambda}\text{Int} E_i\). Indeed, by the definition of \(\mathcal{G}\) we have \(\mathcal{G} \subset \mathcal{D}_X(M)\) and \(\mathcal{G}^c \subset \mathcal{D}^c_X(M)\). Hence, the conclusion follows from Proposition 4.2.

Now we have come to the position to apply the characterization of \((\prod^\omega\ell_2,\sum^\omega\ell_2)\)-manifolds (Theorem 2.2) based upon the stability (Proposition 4.2).

**Proof of Theorem 1.2(1).** It suffices to prove that the pair \((G_0,H)\) satisfies the conditions (i) \(\sim\) (iii) in Theorem 2.2(1).

(i) By [24, Theorem 1.1] the group \(G_0\) is a separable completely metrizable ANR.

(ii) (a) By the assumption \(H\) is \(F_\sigma\) in \(G_0\). (b) Since \((G_c)^1\subset H\) and \((G_c)^1\) is homotopy dense in \(G_0\) by [24, Theorem 1.2], the subgroup \(H\) is also homotopy dense in \(G_0\).

(iii) Since \(\mathcal{G}^c \subset (G_c)^1\) and so \((\mathcal{G},\mathcal{G}^c) \subset (G_0,H) \subset (\mathcal{D}(M),\mathcal{D}^c(M))\), the \((\prod^\omega\ell_2,\sum^\omega\ell_2)\)-stability of \((G_0,H)\) follows from Proposition 4.2.

This completes the proof. \(\square\)

5. Stability property of homeomorphism groups

Stability properties of homeomorphism groups and their subgroups have already been studied by many authors [9, 10, 12, 13, 18, 22]. K. Sakai and R.Y. Wong [18] showed that for Euclidean polyhedra \(X\) and \(Y\) the triples of homeomorphism groups and spaces of embeddings:

\[
(\mathcal{H}(X),\mathcal{H}^{\text{LIP}}(X),\mathcal{H}^{\text{PL}}(X)) \quad \text{and} \quad (\text{Emb}(X,Y),\text{Emb}^{\text{LIP}}(X,Y),\text{Emb}^{\text{PL}}(X,Y))
\]

are \((s,\Sigma,\sigma)\)-stable. In [22, Section 3] we discussed the case that \(X\) is a non-compact polyhedron and showed that, for instance, the tuple

\[
(\mathcal{H}(X),\mathcal{H}^{\text{loc-LIP}}(X),\mathcal{H}^{\text{LIP}}(X),\mathcal{H}^{\text{LIP, c}}(X),\mathcal{H}^{\text{PL}}(X),\mathcal{H}^{\text{PL, c}}(X))
\]

is \((s^\infty,\Sigma^\infty,s_0^\infty,\Sigma_f^\infty,\sigma^\infty,\sigma_f^\infty)\)-stable. These arguments are based upon the Morse’s \(\mu\)-length of arcs.

In this section we retrace these arguments due to the formulation in Section 3.3 and show that the pairs \((\mathcal{H}(X),\mathcal{H}^c(X))\) and \((\mathcal{H}(X;\mu),\mathcal{H}^c(X;\mu))\) are stable with respect to the pair \((s^\infty,s_f^\infty) \cong (\prod^\omega\ell_2,\sum^\omega\ell_2)\) (Theorem 1.1(2)(3)). Comparing with [18, 22], here our emphasis is put on measure-preserving homeomorphisms.

The following notations are used below; \(C(X,Y)\) is the space of continuous maps \(f : X \to Y\) endowed with the compact-open topology, \(\mathcal{E}(X,Y)\) denotes the subspace of \(C(X,Y)\) consisting of topological embeddings. For a subset \(A\) of \(X\) let

\[
C\mathcal{E}(X,A;Y) = \{ f \in C(X,Y) \mid f|_A : A \to Y \text{ is a topological embedding} \}
\]

and \(\mathcal{H}(X,A) = \{ h \in \mathcal{H}(X) \mid h(A) = A \}\). For any subset \(\mathcal{F}\) of \(\mathcal{H}(X)\) with \(\text{id}_X \in \mathcal{F}\), the symbols \(\mathcal{F}_0\) and \(\mathcal{F}_1\) denote the connected component and the path component of \(\text{id}_X\) in \(\mathcal{F}\) respectively. We
regard as $s = (-1, 1)^\infty$ instead of $\mathbb{R}^\infty$ if necessary, and use the symbol $(s^\Lambda, s^\Sigma)$ to denote the pair $(\prod_{i \in \Lambda} s, \sum_{i \in \Lambda} s) \cong (\prod_{i \in \Lambda} \ell_2, \sum_{i \in \Lambda} \ell_2)$ for notational simplicity.

5.1. Morse’s $\mu$-length of arcs.

Suppose $(X, d)$ is a metric space and $A$ is an arc in $X$. The arc $A$ admits a canonical linear order $\leq$ unique up to the reversion. For each $k \geq 1$ set

$$S_k = \{a = (a_0, a_1, \cdots, a_k) \in A^{k+1} \mid a_0 \leq a_1 \leq \cdots \leq a_k\},$$

$$\delta_k(a) = \min\{d(a_{i-1}, a_i) \mid i = 1, \cdots, k\} \quad (a \in S_k) \quad \text{and} \quad \mu_k(A) = \sup \{\delta_k(a) \mid a \in S_k\}.$$

The $\mu$-length of $A$ is defined by

$$\mu(A) = \sum_{k=1}^{\infty} 2^{-k} \mu_k(A).$$

We use the following property of the quantity $\mu(A)$.

Lemma 5.1. ([18] §1. pp. 197–202, cf. [22] Lemma 4.3)

(i) For each $f \in \mathcal{E}([-1, 1], X)$ there is a unique $t_f \in (-1, 1)$ such that $\mu(f([-1, t_f])) = \mu(f([t_f, 1]))$.

(ii) The function $\gamma : \mathcal{E}([-1, 1], X) \to (-1, 1)$, $f \mapsto t_f$, is continuous.

5.2. Selection theorem for good Radon measures.

Next we recall some basic facts on good Radon measures. A Radon measure on a space $X$ is a Borel measure $\mu$ on $X$ such that $\mu(K) < \infty$ for any compact subset $K$ of $X$. The measure $\mu$ is called good if $\mu(p) = 0$ for any point $p \in X$ and $\mu(U) > 0$ for any nonempty open subset $U$ of $X$. Let $\mathcal{M}(X)$ denote the space of all Radon measures $\nu$ on $X$ endowed with the weak topology. For $\mu, \nu \in \mathcal{M}(X)$ we say that $\nu$ is $\mu$-biregular if $\mu(A) = 0$ iff $\nu(A) = 0$ for any Borel subset $A$ of $X$. For $\mu \in \mathcal{M}(X)$ and a Borel subset $A$ of $X$ let

$$\mathcal{M}\_g^\Lambda(X) = \{\nu \in \mathcal{M}(X) \mid \nu(A) = 0, \nu \text{ is good}\} \quad \text{and} \quad \mathcal{M}\_g^\Lambda(X; \mu\text{-reg}) = \{\nu \in \mathcal{M}\_g^\Lambda(X) \mid \nu(A) = \mu(A), \nu \text{ is } \mu\text{-biregular}\}.$$

We say that $h \in \mathcal{H}(X)$ is $\mu$-biregular provided $\mu(h(B)) = 0$ iff $\mu(B) = 0$ for any Borel subset $B$ of $X$. The group $\mathcal{H}(X)$ includes two subgroups

$$\mathcal{H}(X; \mu\text{-reg}) = \{h \in \mathcal{H}(X) \mid h \text{ is } \mu\text{-biregular}\} \quad \text{and} \quad \mathcal{H}(X; \mu) = \{h \in \mathcal{H}(X) \mid h_\ast \mu = \mu \text{ (i.e., } h \text{ is } \mu\text{-preserving)}\}.$$

We need Oxtoby-Ulam theorem ([17]) and Fathi’s selection theorem ([8]).

Theorem 5.1. Suppose $N$ is a compact connected $n$-manifold possibly with boundary and $\mu \in \mathcal{M}\_g^\Lambda(N)$.

1. For any $\nu \in \mathcal{M}\_g^\Lambda(N)$ with $\nu(N) = \mu(N)$ there is $h \in \mathcal{H}(N)$ such that $h_\ast \mu = \nu$.

2. There exists a continuous map

$$\sigma : (\mathcal{M}\_g^\Lambda(N; \mu\text{-reg}), \mu) \to (\mathcal{H}(N; \mu\text{-reg})_1, \text{id}_N)$$

such that $\sigma(\nu)_\ast \mu = \nu$ for any $\nu \in \mathcal{M}\_g^\Lambda(N; \mu\text{-reg})$. 

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A typical example of a good Radon measure is the Lusignov measure $m$ on $\mathbb{R}^n$ ($n \geq 1$). For any subpolyhedron $K$ of $\mathbb{R}^n$ it is seen that $H_{PL}(K) \subset H(K; m\text{-reg})$. Consider the $n$-cube $B^n = [-1, 1]^n$ in $\mathbb{R}^n$. We identify the interval $[-1, 1]$ with the segment in $B^n$ connecting the two points $(\pm 1, 0, \cdots, 0)$. (This means that $t \in [-1, 1]$ represents the point $(t, 0, \cdots, 0) \in B^n$.)

Consider a triple $(E, A, \mu)$, where (a) $E$ is a closed $n$-disk and $A$ is an arc in $E$ such that $(E, A) \equiv (B^n, [-1, 1])$ (i.e., $A$ is an unknot proper arc in $E$) and (b) $\mu \in M_{g,h}^\partial(E)$ for $n \geq 2$ and it is an empty structure for $n = 1$ (so that the symbol $\mu$ can be eliminated from the notation).

**Lemma 5.2.** For any homeomorphism $\theta : [-1, 1] \cong A$ (i.e., a parametrization of $A$ by $[-1, 1]$) there exists a map

$$\zeta : ((-1, 1), 0) \rightarrow (H_\partial(E, A; \mu)_1, \text{id}_{E})$$

such that $\zeta(t)(\theta(t)) = \theta(0)$ for each $t \in (-1, 1)$.

**Proof.** When $n = 1$, the assertion is obvious. Below we assume that $n \geq 2$.

(1) First we treat the case where $(E, A) = (B^n, [-1, 1])$, $\theta = \text{id}_{[-1, 1]}$ and $\mu$ is $m$-biregular. Consider the decomposition of the $n$-cube, $B^n = B_+ \cup B_0 \cup B_-$, where

$$B_+ = \{x \in B^n \mid x_n \geq 0\}, \quad B_- = \{x \in B^n \mid x_n \leq 0\}, \quad B_0 = \{x \in B^n \mid x_n = 0\} = B_+ \cap B_1.$$

We can easily find a map

$$\xi : ((-1, 1), 0) \rightarrow (H_{PL}^\partial(B^n, [-1, 1]), \text{id}_{B^n})$$

such that $\xi(t)(t) = 0$ and $\xi(t)(B_\pm) = B_\pm$. For $\mu_\pm := \mu|B_\pm \in M_{g_\pm}^\partial(B_\pm)$, Theorem 5.1(2) induces maps

$$\sigma_\pm : (M_{g_\pm}^\partial(B_\pm; \mu_{\pm\text{-reg}}), \mu_\pm) \rightarrow (H_\partial(B_\pm; \mu_{\pm\text{-reg}})_1, \text{id}_{B_\pm})$$

such that $\sigma_\pm(\nu)_* \mu_\pm = \nu$ for any $\nu \in M_{g_\pm}^\partial(B_\pm; \mu_{\pm\text{-reg}})$. Since

$$\xi(t)|_{B_\pm} \in H_{PL}(B_\pm) \subset H(B_\pm; m\text{-reg}) = H(B_\pm; \mu_{\pm\text{-reg}})$$

we can define the map $\zeta : (-1, 1) \rightarrow H_\partial(B^n, [-1, 1]; \mu)$ by

$$\zeta(t)|_{B_\pm} = (\sigma_\pm((\xi(t)|_{B_\pm})_* \mu_\pm))^{-1} \xi(t)|_{B_\pm}.$$

(2) To reduce the general case to the case (1), we construct a homeomorphism $\overline{\theta} : (B^n, [-1, 1]) \cong (E, A)$ such that $\overline{\theta}|_{[-1, 1]} = \theta$ and the pull-back $\overline{\theta}^* \mu$ is $m$-biregular. Since $A$ is unknoted in $E$, the homeomorphism $\theta : [-1, 1] \cong A$ admits an extension $\overline{\theta}_1 : B^n \cong E$. For $\mu_1 := \overline{\theta}_1^* \mu \in M_{g_1}^\partial(B^n)$, since $\mu_1([(-1, 1)] = 0$, we can find $\overline{\theta}_2 \in H_{[-1, 1], \partial}^0(B^n)$ such that $\mu_1(\overline{\theta}_2(0)) = 0$. Then $\mu_2 := \overline{\theta}_2^* \mu_1 \in M_{g_2}^\partial(B^n)$ restricts to $\mu_2|B_\pm \in M_{g_\pm}^\partial(B_\pm)$ and Theorem 5.1(1) yields homeomorphisms $\overline{\theta}_3 \in H_\partial(B_\pm)$ such that $(\overline{\theta}_3)_* \mu_2|B_\pm = c_\pm m|B_\pm$, where $c_\pm = \mu_2(B_\pm)/m(B_\pm)$. Define $\overline{\theta}_3 \in H_{B_0, \partial}^0(B^n)$ by $\overline{\theta}_3|B_\pm = \overline{\theta}_3^\pm$. Then $\overline{\theta}_3 \mu_2$ is $m$-biregular and hence $\overline{\theta} = \overline{\theta}_1 \overline{\theta}_2 \overline{\theta}_3$ satisfies the required conditions.

Since $\mu' := \overline{\theta}^* \mu$ is $m$-biregular, the case (1) yields a map

$$\zeta' : ((-1, 1), 0) \rightarrow (H_\partial(B^n, [-1, 1]; \mu'), \text{id}_{B^n})$$

such that $\zeta'(t)(t) = 0$ for any $t \in (-1, 1)$. The required map $\zeta$ is defined by

$$\zeta(t) = \overline{\theta} \zeta'(t) \overline{\theta}^{-1} \quad (t \in (-1, 1)).$$
Suppose $X$ is a metric space.

**Lemma 5.3.** For any $f_0 \in \mathcal{CE}(E, A; X)$ there exist maps

$$\varphi : (\mathcal{CE}(E, A; X), f_0) \rightarrow (s, 0) \quad \text{and} \quad \psi : (s, 0) \rightarrow (\mathcal{H}_\partial(E, A; \mu)_1, \text{id}_E),$$

which satisfy the following conditions:

(i) $\varphi(f) = \varphi(f')$ for any $f, f' \in \mathcal{CE}(E, A; X)$ with $f = f'$ on $A$.

(ii) $\varphi(f \circ \psi(\varphi(f))^{-1}) = 0$ for any $f \in \mathcal{CE}(E, A; X)$.

(iii) $\varphi(\psi(t)) = t$ for any $(f, t) \in \varphi^{-1}(0) \times s$.

**Proof.** There exists a disjoint family of closed $n$-disks $D_k$ ($k \in \mathbb{N}$) in $E$ such that $\text{diam } D_k \rightarrow 0$ ($k \rightarrow \infty$), $\mu(\partial D_k) = 0$ and $(D_k, D_k \cap A) \cong (\mathbb{B}^n, [-1, 1])$. Recall the map $\gamma$ in Lemma 5.1. For each $k \in \mathbb{N}$ choose a homeomorphism $\theta_k : [-1, 1] \cong D_k \cap A$ such that $\gamma(f \theta_k) = 0$. Then we can apply Lemma 5.2 to the triple $(D_k, D_k \cap A, \mu|_{D_k})$ and $\theta_k$ to obtain the map

$$\zeta_k : ((-1, 1), 0) \rightarrow (\mathcal{H}_\partial(D_k, D_k \cap A; \mu|_{D_k})_1, \text{id}_{D_k})$$

such that $\zeta_k(t)(\theta_k(t)) = \theta_k(0)$ for each $t \in (-1, 1)$.

Define the maps $\varphi$ and $\psi$ by

(a) $\varphi(f) = (\gamma(f \theta_k))_{k \in \mathbb{N}}$ ($f \in \mathcal{CE}(E, A; X)$),

(b) $\psi|_{D_k} = \zeta_k(t_k)$ ($k \in \mathbb{N}$) and $\psi(t)|_{E \setminus \cup_k D_k} = \text{id}$ ($t = (t_k) \in s$).

It remains to verify the properties (i) $\sim$ (iii). First note that for each $k \in \mathbb{N}$

$$f \psi(t) \theta_k = f \psi(t)|_{D_k} \theta_k = f \zeta_k(t_k) \theta_k \quad \text{and} \quad f \psi(t)^{-1} \theta_k = f(\psi(t)|_{D_k})^{-1} \theta_k = f \zeta_k(t_k)^{-1} \theta_k.$$

(i) Since $f \theta_k = f|_A \theta_k$, one sees that $\varphi(f)$ depends only on $f|_A$.

(ii) Let $t = \varphi(f)$. Then we have $t_k = \gamma(f \theta_k)$ and $\zeta_k(t_k)^{-1} \theta_k(0) = \theta_k(t_k)$. Thus, from the definition of the map $\gamma$ it follows that

$$\gamma(f \psi(\varphi(f))^{-1} \theta_k) = \gamma(f \psi(t)^{-1} \theta_k) = \gamma(f \zeta_k(t_k)^{-1} \theta_k) = 0 \quad \text{and} \quad \varphi(f \psi(\varphi(f))^{-1}) = 0.$$

(iii) Note that $\zeta_k(t_k)(\theta_k(t_k)) = \theta_k(0)$ and $\gamma(f \theta_k) = 0$ since $\varphi(f) = 0$. Thus, it follows that

$$\gamma(f \psi(t) \theta_k) = \gamma(f \zeta_k(t_k) \theta_k) = t_k \quad \text{and} \quad \varphi(f \psi(t)) = t.$$

\[ \square \]

5.3. $(s^\infty, s_f^\infty)$-stability of homeomorphism groups.

Suppose $M$ and $N$ are metric spaces and $\{(E_i, A_i, \mu_i)\}_{i \in A}$ is a family of triples such that

(i) $\{E_i\}_{i \in A}$ is discrete family of topological closed disks in $M$,

(ii) for each $i \in A$,

(a) $n_i := \dim E_i \geq 1$ and $\text{Int } E_i$ is open in $M$,

(b) $A_i$ is an arc in $E_i$ such that $(E_i, A_i) \cong (\mathbb{B}^{n_i}, [-1, 1])$, and

(c) if $n_i \geq 2$, then $\mu_i \in \mathcal{M}_g^{A_i}(E_i)$ and if $n_i = 1$, then $\mu_i$ is an empty structure (so it can be eliminated from the notations).
Consider the subspace $\mathcal{E}$ of $C(M, N)$ and the subgroup $\mathcal{G}$ of $\mathcal{H}(M)$ defined by
\[
\mathcal{E} = \{ f \in C(M, N) \mid f|_{A_i} : A_i \to N \text{ is an embedding for each } i \in \Lambda \},
\]
\[
\mathcal{G} = \{ h \in \mathcal{H}(M) \mid h = \text{id}_M \text{ on } M - \bigcup_{i \in \Lambda} \text{Int } E_i, \ h|_{E_i} \in \mathcal{H}_\partial(E_i, A_i; \mu_i) = \text{1} \text{ for each } i \in \Lambda \}.
\]
The group $\mathcal{G}$ acts continuously on the space $\mathcal{E}$ by the right composition. Moreover, $\mathcal{G}$ is path connected. In fact, the pair $(\mathcal{G}, \mathcal{G}^c)$ is isomorphic to $\prod_{i \in \Lambda} \mathcal{H}_\partial(E_i, A_i; \mu_i)$ as a pair of a topological group and its subgroup. As a transformation group on $M$, the group $\mathcal{G}$ has a weak topology and has the multiplication supported by the family $\{E_i\}_{i \in \Lambda}$. Hence, the multiplication map $\eta : \prod_{i \in \Lambda} \mathcal{G}(E_i) \to \mathcal{G}$ is continuous.

We assume that $N$ includes an arc, so that $\mathcal{E} \neq \emptyset$ and we can fix a distinguished element $f_0 \in \mathcal{E}$. The associated support function for $\mathcal{E}$ on $M$ is defined by
\[
\text{supp}_{f_0} f = \text{cl}_M \{ x \in M \mid f(x) \neq f_0(x) \} \quad (f \in \mathcal{E}).
\]
It satisfies the conditions (i), (ii) in Assumption 3.1 (3) and the subspace $\mathcal{E}^c$ is $\mathcal{G}^c$-invariant.

For each triple $(E_i, A_i, \mu_i)$ and $f_0|_{E_i} \in \mathcal{E}^c(E_i, A_i; N)$, Lemma 5.3 yields two maps
\[
\varphi_i : (\mathcal{C}\mathcal{E}(E_i, A_i; N), f_0|_{E_i}) \to (s, 0) \quad \text{and} \quad \psi_i : (s, 0) \to (\mathcal{H}_\partial(E_i, A_i; \mu_i) = \text{1}, \text{id}_{E_i}),
\]
such that
(i) $\varphi_i(f) = \varphi_i(f')$ for any $f, f' \in \mathcal{C}\mathcal{E}(E_i, A_i; N)$ with $f = f'$ on $A_i$.
(ii) $\varphi_i(f \circ \psi_i(\varphi_i(f))^{-1}) = 0$ for any $f \in \mathcal{C}\mathcal{E}(E_i, A_i; N)$.
(iii) $\varphi_i(f \psi_i(t)) = t$ for any $(f, t) \in \varphi_i^{-1}(0) \times s$.

We define two maps $P_i$ and $G_i$ by
\[
P_i : (\mathcal{E}, f_0) \to (s, 0); \quad P_i(f) = \varphi_i(f|_{E_i}) \quad \text{and} \quad G_i : (s, 0) \to (\mathcal{G}(E_i), \text{id}_M); \quad G_i(t)|_{E_i} = \psi_i(t).
\]
The next claim follows directly from the properties of the maps $\varphi_i$ and $\psi_i$.

**Claim.** The maps $P_i$ and $G_i$ satisfy the following conditions.

(i) $P_i(f) = 0$ if $f \in \mathcal{E}$ and $(\text{supp}_{f_0} f) \cap A_i = \emptyset$.
(ii) $P_i(fg^{-1}) = 0$ if $(f, g) \in \mathcal{E} \times \mathcal{G}$ and $g = G_i(P_i(f))$ on $A_i$.
(iii) $P_i(fg) = t$ if $(f, g, t) \in \mathcal{E} \times \mathcal{G} \times s$, $P_i(f) = 0$ and $g = G_i(t)$ on $A_i$.

This claim means that the maps $P_i$ and $G_i$ satisfy the conditions (i) - (iii) in Assumption 3.1 (4). Hence, we can apply the arguments in Section 3.3 to this situation. The pointed pairs $(\mathcal{B}, \mathcal{B}^c, \alpha)$ and $(\mathcal{F}, \mathcal{F}^c, f_0)$ and three maps $P$, $G$ and $F$ are defined by
\[
(\mathcal{B}, \mathcal{B}^c, \alpha) = (s^\Lambda, s^\Lambda, 0) = (\prod_{i \in \Lambda} s_i, \sum_{i \in \Lambda} s_i, 0),
\]
\[
P : (\mathcal{E}, \mathcal{E}^c, f_0) \to (s^\Lambda, s^\Lambda, 0); \quad P(f) = (P_i(f))_{i \in \Lambda} \quad (f \in \mathcal{E}),
\]
\[
G : (s^\Lambda, s^\Lambda, 0) \to (\mathcal{G}, \mathcal{G}^c, \text{id}_M); \quad G(t) = \eta((G_i(t))_i) \quad (t = (t_i)_i \in s^\infty),
\]
\[
\mathcal{F} = P^{-1}(0) \quad \text{and} \quad F : (\mathcal{E}, \mathcal{E}^c, f_0) \to (\mathcal{F}, \mathcal{F}^c, f_0); \quad F(f) = f \cdot G(P(f))^{-1} \quad (f \in \mathcal{E}).
\]
These maps determine two maps $\Phi$ and $\Psi$ by
\[ \Phi : (\mathcal{E}, \mathcal{E}^e, f_0) \longrightarrow (s^A, s_f^A, 0) \times (\mathcal{F}, \mathcal{F}^e, f_0); \quad \Phi(f) = (P(f), F(f)), \]

\[ \Psi : (s^A, s_f^A, 0) \times (\mathcal{F}, \mathcal{F}^e, f_0) \longrightarrow (\mathcal{E}, \mathcal{E}^e, f_0); \quad \Psi(t, g) = g \cdot G(t). \]

The next proposition follows from Proposition 3.1.

**Proposition 5.1.**

1. The maps \( \Phi \) and \( \Psi \) are reciplocal homeomorphisms of pairs.

2. (i) If \((\mathcal{E}_1, \mathcal{E}_2)\) is a subpair of \((\mathcal{E}, \mathcal{E}^e)\), \(\mathcal{E}_1\) is \(G\)-invariant and \(\mathcal{E}_2\) is \(G^c\)-invariant, then the map \( \Phi \) restricts to the homeomorphism of the subpairs

\[ \Phi : (\mathcal{E}_1, \mathcal{E}_2) \longrightarrow (s^A, s_f^A) \times (\mathcal{E}_1 \cap \mathcal{F}, \mathcal{E}_2 \cap \mathcal{F}). \]

In particular, the pair \((\mathcal{E}_1, \mathcal{E}_2)\) is \((s^A, s_f^A)\)-stable.

(ii) If \(\mathcal{E}_1\) is a \(G\)-invariant subspace of \(\mathcal{E}\), then the map \( \Phi \) induces the homeomorphism of the subpairs

\[ \Phi : (\mathcal{E}_1, \mathcal{E}_1^e) \longrightarrow (s^A, s_f^A) \times (\mathcal{E}_1 \cap \mathcal{F}, \mathcal{E}_1^e \cap \mathcal{F}). \]

**Example 5.1.** The pair \((\mathcal{E}(M, N), \mathcal{E}^e(M, N))\) is \((s^A, s_f^A)\)-stable. Indeed, the space \(\mathcal{E}(M, N)\) is a \(G\)-invariant subspace of \(\mathcal{E}\) and the map \(\Phi\) induces the homeomorphism between the subpairs

\[ \Phi : (\mathcal{E}(M, N), \mathcal{E}^e(M, N)) \cong (s^A, s_f^A) \times (\mathcal{F}(M, N), \mathcal{F}^e(M, N)), \quad \text{where} \quad \mathcal{F}(M, N) = \mathcal{E}(M, N) \cap \mathcal{F}. \]

In the case where \(M = N\), we can take \(\text{id}_M\) as a base point \(f_0\) of the space \(\mathcal{E}\). Then the support function \(\text{supp}_{f_0}\) reduces to the ordinary support function. The space \(\mathcal{E}\) includes the group \(\mathcal{H}(M)\) as a \(G\)-invariant subspace. From Proposition 5.1(2) we have the following criterion.

**Proposition 5.2.** Suppose \((\mathcal{L}, \mathcal{K})\) is a pair of subgroups of \(\mathcal{H}(M)\) with \((G, G^c) \subset (\mathcal{L}, \mathcal{K}) \subset (\mathcal{H}(M), \mathcal{H}^c(M))\).

Then the map \( \Phi \) induces the homeomorphism

\[ \Phi : (\mathcal{L}, \mathcal{K}) \cong (s^A, s_f^A) \times (\mathcal{L} \cap \mathcal{F}, \mathcal{K} \cap \mathcal{F}). \]

Thus, the pair \((\mathcal{L}, \mathcal{K})\) is \((s^A, s_f^A)\)-stable.

The next example includes Theorem 1.1(2).

**Example 5.2.** The pairs \((\mathcal{H}_X(M), \mathcal{H}^e_X(M))\) and \((\mathcal{H}_X(M)_i, \mathcal{H}^e_X(M)_i)\) \((i = 0, 1)\) are \((s^A, s_f^A)\)-stable for any subset \(X\) of \(M - \bigcup_{i \in A} \text{Int} E_i\). Indeed, since \((G, G^c) \subset (\mathcal{H}_X(M)_1, \mathcal{H}^e_X(M)_1)\), this follows from Proposition 5.2.

Now we can apply the characterization of \((s^\infty, s_f^\infty)\)-manifolds (Theorem 2.2) based upon the stability (Proposition 5.2).

**Proof of Theorem 1.2(2).** It suffices to prove that the pair \((G_0, H)\) satisfies the conditions (i) \sim (iii) in Theorem 2.2(1).

1. By [21] Corollary 1.1 the group \(G_0\) is a separable completely metrizable ANR.

2. (i) By the assumption \(H\) is \(F_\sigma\) in \(G_0\). (ii) Since \((G_c)_1^1 \subset H\) and \((G_c)_1^1\) is homotopy dense in \(G_0\) by [22] Theorem 3.2 [and its proof], the subgroup \(H\) is also homotopy dense in \(G_0\).

3. From the definition of \(G\), it is seen that \(G^c \subset (G_c)_1^1\). Since \((G, G^c) \subset (G_0, H) \subset (\mathcal{H}(M), \mathcal{H}^c(M))\), the \((s^\infty, s_f^\infty)\)-stability of \((G_0, H)\) follows from Proposition 5.2.
This completes the proof. □

Finally we deduce the \((s^\infty, s^\infty_f)\)-stability of groups of measure-preserving homeomorphisms. The next example includes Theorem 5.1(3).

**Example 5.3.** Suppose \(X\) is a subset of \(M - \bigcup_{i \in \Lambda} \text{Int } E_i\) and \(\mu \in \mathcal{M}_{\mu}(\mathcal{A}(\bigcup \partial E_i)(M))\). We assume that \(n_i \geq 2\) and \(\mu_i = \mu|E_i\) for each \(i \in \Lambda\). Then the following pairs are \((s^A, s^A_f)\)-stable:

\[
(\mathcal{H}_X(M, \mu), \mathcal{H}^c_X(M, \mu)), (\mathcal{H}_X(M, \mu)_i, \mathcal{H}^c_X(M, \mu)_i) \quad (i = 0, 1) \text{ and } (\mathcal{H}_X(M, \mu)_1, \mathcal{H}^c_X(M, \mu)_1).
\]

This follows from Proposition 5.2 since \((\mathcal{G}, \mathcal{G}^c) \subset (\mathcal{H}_X(M, \mu)_1, \mathcal{H}^c_X(M, \mu)_1)\).

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