MOMENTS AND ONE LEVEL DENSITY OF SEXTIC HECKE $L$-FUNCTIONS

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Abstract. Let $\omega = \exp(2\pi i/3)$. In this paper, we study moments of central values of sextic Hecke $L$-functions of $\mathbb{Q}(\omega)$ and one level density result for the low-lying zeros of sextic Hecke $L$-functions of $\mathbb{Q}(\omega)$. As a corollary, we deduce that, assuming GRH, at least 2/45 of the members of the sextic family do not vanish at $s = 1/2$.

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1. Introduction

Due to important arithmetic information encoded by the central values of $L$-functions, their possible zeros there have been investigated extensively. In general, one expects an $L$-function to be non-vanishing at the central point unless there is a good reason for it. This can be further expounded by instancing an non-vanishing result for this family. A more general result for the first moment of central $L$-functions, its possible zeros there have been investigated extensively. In general, one expects an $L$-function to be non-vanishing at the central point unless there is a good reason for it. This can be further expounded by instancing an non-vanishing result for this family. A more general result for the first moment of central $L$-functions, its possible zeros there have been investigated extensively. In general, one expects an $L$-function to be non-vanishing at the central point unless there is a good reason for it. This can be further expounded by instancing an non-vanishing result for this family. A more general result for the first moment of central $L$-functions, its possible zeros there have been investigated extensively. In general, one expects an $L$-function to be non-vanishing at the central point unless there is a good reason for it. This can be further expounded by instancing an non-vanishing result for this family. A more general result for the first moment of central $L$-functions, its possible zeros there have been investigated extensively.

There are two methods of studying these potential central zeros of $L$-functions. One typical approach to achieve the non-vanishing result is to study the moments of a family of $L$-functions. In this way, H. Iwaniec and P. Sarnak [20] showed that $L(1/2, \chi) \neq 0$ for at least 1/3 of the primitive Dirichlet characters $\chi \pmod{q}$. M. Jutilia [21] showed that there are $\gg X/\log X$ fundamental discriminants $d$ with $|d| < X$ such that $L(1/2, \chi_d) \neq 0$. This result is improved by K. Soundararajan [23], who showed that $L(1/2, \chi_{8d}) \neq 0$ for at least 87.5% of the real characters $\chi_{8d}$. For the family of $L$-functions of primitive cubic Dirichlet characters, S. Baier and M. P. Young [11] showed that for any $\varepsilon > 0$, there are $\gg X^{6/7-\varepsilon}$ primitive cubic Dirichlet characters $\chi$ with conductor $\leq X$ such that $L(1/2, \chi) \neq 0$.

The result of Baier and Young [11] is similar to another one of W. Luo [20], who studied moments of the family of $L(1/2, \chi)$ associated to cubic Hecke characters $\chi$ with square-free modulus in $\mathbb{Q}(\omega)$ with $\omega = \exp(2\pi i/3)$ and derived a non-vanishing result for this family. A more general result for the first moment of central $L$-values associated to the $n$-th order Hecke characters of any number field containing the $n$-th root of unity was obtained by S. Friedberg, J. Hoffstein and D. Lieman [6]. Corresponding non-vanishing results were obtained by V. Blomer, L. Goldmakher and B. Louvel in [2].

In [10,11], the authors studied moments of $L(1/2, \chi)$ for families of quadratic and quartic Hecke characters in $\mathbb{Q}(i)$ and $\mathbb{Q}(\omega)$. In this paper, we first extend our studies above to the family of $L$-functions associated to sextic Hecke characters in $\mathbb{Q}(\omega)$. We have

Theorem 1.1. Let $W : (0, \infty) \to \mathbb{R}$ be a smooth, compactly supported function. For $y \to \infty$ and any $\varepsilon > 0$,

$$\sum_{c \equiv 1 \pmod{36}}^* L \left( \frac{1}{2}, \chi_c \right) W \left( \frac{N(c)}{y} \right) = A \hat{W}(1)y + O(y^{\frac{6}{7} + \varepsilon}),$$

where $\chi_c = (\frac{\cdot}{c})_6$ is the sextic residue symbol in $\mathbb{Q}(\omega)$ (defined in Section 2.1)

$$\hat{W}(1) = \int_0^{\infty} W(x) \, dx.$$

Moreover, $A$ is an explicit constant given in (3.2) and $\sum^*$ denotes summation over squarefree elements of $\mathbb{Z}[\omega]$. 


We note here that for \( c = 1 \), \( \chi_c \) is the principal character instead of a sextic one. It follows from \([2]\) Corollary 1.4 that
\[
\sum_{c \equiv 1 \mod 36}^* \left| L\left(\frac{1}{2}, \chi_c\right) \right|^2 W\left(\frac{N(c)}{y}\right) \ll \varepsilon y^{1+\varepsilon}.
\]

From this and Theorem 1.1 we readily deduce, via a standard argument (see \([24]\)), the following

**Corollary 1.2.** For \( y \to \infty \) and any \( \varepsilon > 0 \), we have

\[\# \left\{ c \in \mathbb{Z}[\omega] : c \equiv 1 \pmod{36}, c \text{ square-free}, N(c) \leq y, \ L\left(\frac{1}{2}, \chi_c\right) \neq 0 \right\} \gg \varepsilon y^{1-\varepsilon}.
\]

The main different feature of Theorem 1.1 and the result in \([6]\) is that our result is over square-free integers. Our proof of Theorem 1.1 is similar to that of the main result in \([25]\), but our choice of the smooth weight follows the treatment in \([1]\).

The other approach, alluded to earlier, towards establishing the non-vanishing result is via the study of the \( n \)-level densities of low-lying zeros of families of \( L \)-functions. The density conjecture of N. Katz and P. Sarnak \([22, 23]\) suggests that the distribution of zeros near \( 1/2 \) of a family of \( L \)-functions is the same as that of eigenvalues near 1 of a corresponding classical compact group. For the family of quadratic Dirichlet \( L \)-functions, the density conjecture implies that \( L(1/2, \chi) \neq 0 \) for almost all such \( \chi \). Assuming GRH, A. E. Özlük and C. Snyder \([20]\) computed the one level density for this family to show that \( L(1/2, \chi_d) \neq 0 \) for at least \( 15/16 \) of the fundamental discriminants \( |d| \leq X \). Following the computations carried out in \([3, 11]\), one can improve this percentage to \((19 - \cot(1/4))/16\).

Our next result is on the one level density of low-lying zeros of a family of sextic Hecke \( L \)-functions in \( \mathbb{Q}(\omega) \). In \([14]\), A. M. Güloğlu studied the one level density of the low-lying zeros of a family of cubic Hecke \( L \)-functions in \( \mathbb{Q}(\omega) \). The result was extended by C. David and A. M. Güloğlu in \([5]\) to hold for test functions whose Fourier transforms are supported in \((-13/11, 13/11)\). This allows them to deduce that at least \( 2/3 \) of the corresponding \( L \)-functions do not vanish at the central point under GRH.

Our result here is motivated by the above-mentioned works. To state it, we first need to set some notations. Let
\[
C(X) = \{ c \in \mathbb{Z}[\omega] : (c, 6) = 1, \text{ c squarefree}, \ X \leq N(c) \leq 2X \}.
\]
We shall define in Section 2.1 the primitive quadratic Kronecker symbol \( \chi^{(72c)} \) for \( c \in C(X) \). We denote the non-trivial zeroes of the Hecke \( L \)-function \( L(s, \chi^{(72c)}) \) by \( 1/2 + i\gamma_{\chi^{(72c)}, j} \). Without assuming GRH, we order them as
\[
\ldots \leq \Re\gamma_{\chi^{(72c)}, -2} \leq \Re\gamma_{\chi^{(72c)}, -1} < 0 \leq \Re\gamma_{\chi^{(72c)}, 1} \leq \Re\gamma_{\chi^{(72c)}, 2} \leq \ldots .
\]
We set
\[
\tilde{\gamma}_{\chi^{(72c)}, j} = \frac{\gamma_{\chi^{(72c)}, j}}{2\pi} \log X
\]
and define for an even Schwartz class function \( \phi \),
\[
S(\chi^{(72c)}, \phi) = \sum_j \phi(\tilde{\gamma}_{\chi^{(72c)}, j}).
\]

We further let \( \Phi_X(t) \) be a non-negative smooth function supported on \((1, 2)\), satisfying \( \Phi_X(t) = 1 \) for \( t \in (1 + 1/U, 2 - 1/U) \) with \( U = \log \log X \) and such that \( \Phi_X^{(j)}(t) \ll j! \) for all integers \( j \geq 0 \). We refer the reader to \([29]\) Chapter 13 for the construction of such functions. Our result is as follows:

**Theorem 1.3.** Suppose that GRH is true for the Hecke \( L \)-functions \( L(x, \chi^{(72c)}) \) discussed above. Let \( \phi(x) \) be an even Schwartz function whose Fourier transform \( \hat{\phi}(u) \) has compact support in \((-45/43, 45/43)\), then
\[
\lim_{X \to +\infty} \frac{1}{\#C(X)} \sum_{(c, 6) = 1}^* S(\chi^{(72c)}, \phi) \Phi_X\left(\frac{N(c)}{X}\right) = \int_{\mathbb{R}} \phi(x) dx.
\]
Here, as earlier, the “\( * \)” on the sum over \( c \) means that the sum is restricted to squarefree elements \( c \) of \( \mathbb{Z}[\omega] \).

The left-hand side of (1.1) represents the one level density of low-lying zeros of the sextic family of Hecke \( L \)-functions in \( \mathbb{Q}(\omega) \). On the other hand, the right-hand side of (1.1) shows that, in connection with the random matrix theory (see the discussions in \([7]\)), the family of sextic Hecke \( L \)-functions of \( \mathbb{Q}(\omega) \) is a unitary family.
Using the argument in the proof of [22 Corollary 1.4], we deduce readily a non-vanishing result for the family of sextic Hecke $L$-functions under our consideration.

**Corollary 1.4.** Suppose that GRH is true for the Hecke $L$-functions $L(x, \chi^{(72c)})$ and that $1/2$ is a zero of $L(s, \chi^{(72c)})$ of order $n_c \geq 0$. As $X \to \infty$,

$$\sum_{(c,6)=1} n_c \Phi_X \left( \frac{N(c)}{X} \right) \leq \left( \frac{43}{45} + o(1) \right) \#C(X).$$

Moreover, as $X \to \infty$

$$\# \{ c \in C(X) : L \left( \frac{1}{2}, \chi^{(72c)} \right) \neq 0 \} \geq \left( \frac{2}{45} + o(1) \right) \#C(X).$$

We note that, analogue to the situation in $Q(\omega)$, the authors previously studied in [12, 13] the one level densities of the low-lying zeros of families of quadratic Hecke $L$-functions in all imaginary quadratic number fields of class number one as well as a family of quartic Hecke $L$-functions in $Q(i)$. Our proof of Theorem [13, Corollary 1] is similar to those in [11, 12] and [22]. A two dimensional Poisson summation over $Z(\omega)$ is used to treat the character sums over the primes, then we use essentially a result derived from the theorem of S. J. Patterson in [27] to estimate certain sextic Gauss sums at the primes.

1.5. **Notations.** The following notations and conventions are used throughout the paper.

We write $\Phi(t)$ for $\Phi_X(t)$,

$e(z) = \exp(2\pi i z) = e^{2\pi i z}$.

$\tilde{f} = O(g)$ or $f \ll g$ means $|f| \leq cg$ for some unspecified positive constant $c$.

$\sim f(g)$ means $\lim_{x \to \infty} f(x)/g(x) = 0$.

$\mu[\omega]$ denotes the M"obius function on $Z[\omega]$.

$\zeta_Q(\omega)(s)$ stands for the Dedekind zeta function of $Q(\omega)$.

## 2. Preliminaries

### 2.1. Residue symbols and Kronecker symbol.

It is well-known that $K = Q(\omega)$ has class number 1. For $j = 2, 3, 6$, the symbol $(\frac{a}{\omega})_j$ is the quadratic ($j = 2$), cubic ($j = 3$) and sextic ($j = 6$) residue symbol in the ring of integers $O_K = Z[\omega]$. For a prime $\omega \in Z[\omega], (\omega, j) = 1$, we define for $a \in Z[\omega], (a, \omega) = 1$ by $(\frac{a}{\omega})_j \equiv a^{(N(\omega) - 1)/j} \pmod{\omega}$, with $(\frac{a}{\omega})_j \in (-\omega)^{6/j}$, the cyclic group of order $j$ generated by $(-\omega)^{6/j}$. When $\omega|a$, we define $(\frac{a}{\omega})_j = 0$. Then these symbols can be extended to any composite $n$ with $N(n, j) = 1$ multiplicatively. We further define $(\frac{a}{\omega})_j = 1$ when $n$ is a unit in $Z[\omega]$. We note that we have $(\frac{a}{\omega})_6^2 = (\frac{a}{\omega})_3, (\frac{a}{\omega})_6^3 = (\frac{a}{\omega})_2$ whenever $(\frac{a}{\omega})_6$ is defined.

We say that $n = a + b\omega$ in $Z[\omega]$ is primary if $n \equiv \pm 1 \pmod{3}$, which is equivalent to $a \not\equiv 0 \pmod{3}$, and $b \equiv 0 \pmod{3}$ (see [11 p. 209]).

Recall that (see [11 p. 883]) the following cubic reciprocity law holds for two co-prime primary $n, m$:

$$\left( \frac{n}{m} \right)_3 = \left( \frac{m}{n} \right)_3.$$

We also have the following supplementary laws for a primary $n = a + b\omega, n \equiv 1 \pmod{3}$ (see [24 Theorem 7.8]),

$$\left( \frac{\omega}{n} \right)_3 = \omega^{(1-a-b)/3} \quad \text{and} \quad \left( \frac{1-\omega}{n} \right)_3 = \omega^{(a-1)/3}.$$

Following the notations in [24 Section 7.3], we say that any primary $n = a + b\omega \in Z[\omega], (n, 6) = 1$ is $E$-primary if

$$a + b \equiv 1 \pmod{4}, \quad \text{if} \quad 2|b,$$

$$b \equiv 1 \pmod{4}, \quad \text{if} \quad 2|a,$$

$$a \equiv 3 \pmod{4}, \quad \text{if} \quad 2 \nmid ab.$$

It follows from [24 Lemma 7.9] that $n$ is $E$-primary if and only if $n^3 = c + d\omega$ with $c, d \in Z$ such that $6|d$ and $c + d \equiv 1 \pmod{4}$. This implies that products of $E$-primary numbers are again $E$-primary. Note that in $Z[\omega]$, every ideal co-prime to 6 has a unique $E$-primary generator. Furthermore, the following sextic reciprocity law holds for two $E$-primary, co-prime numbers $n, m \in Z[\omega]$:

$$\left( \frac{n}{m} \right)_6 = \left( \frac{m}{n} \right)_6 \left( -1 \right)^{(N(n)-1)/2}(N(m)-1)/2. $$

We also have the following supplementary laws for an $E$-primary $n = a + b\omega$ with $(n, 6) = 1$ (see [24] Theorem 7.10):

\[(2.2) \quad \left( -\frac{\omega}{n} \right)_6 = (-\omega)^{N(n) - 1}/6, \quad \left( \frac{1 - \omega}{n} \right)^2 = \left( \frac{a}{3} \right)_z \quad \text{and} \quad \left( \frac{2}{n} \right) = \left( \frac{2}{N(n)} \right)_z, \]

where $\left( \cdot \right)_z$ denotes the Jacobi symbol in $\mathbb{Z}$.

The above discussions allow us to define a sextic Dirichlet character $\chi^{(72c)} \pmod{72c}$ for any element $c \in \mathcal{O}_K, (c, 6) = 1$, such that for any $n \in (\mathcal{O}_K/72c\mathcal{O}_K)^*$,

\[\chi^{(72c)}(n) = \left( \frac{72c}{n} \right)_6.\]

One deduces from (2.2) and the sextic reciprocity that $\chi^{(72c)}(n) = 1$ when $n \equiv 1 \pmod{72c}$. It follows from this that $\chi^{(72c)}(n)$ is well-defined. As $\chi^{(72c)}(n)$ is clearly multiplicative, of order 6 and trivial on units, it can be regarded as a sextic Hecke character modulo 72c of trivial infinite type. We write $\chi^{(72c)}$ for this Hecke character as well and we call it the Kronecker symbol. Furthermore, if $c$ is square-free, $\chi^{(72c)}$ is non-principal and primitive. To see this, we write $c = u_c \cdot \omega_1 \cdots \omega_k$ with a unit $u_c$ and primes $\omega_j$. Suppose $\chi^{(72c)}$ is induced by some $\chi$ modulo 6 with $\omega_j \mid c'$, then by the Chinese Remainder Theorem, there exists an $n$ such that $n \equiv 1 \pmod{72c/\omega_j}$ and $\left( \frac{\omega_j}{n} \right)_6 \neq 1$. It follows from this that $\chi(n) = 1$ but $\chi^{(72c)}(n) \neq 1$, a contradiction. Thus, $\chi^{(72c)}$ can only be possibly induced by some $\chi$ modulo 36c or modulo $8(1 - \omega)^3c$. Suppose it is induced by some $\chi$ modulo 36c, then by the Chinese Remainder Theorem, there exists an $n$ such that $n \equiv 1 \pmod{9c}$ and $n \equiv 1 + 4\omega \pmod{8}$, then we have $\chi(n) = 1$ but $\chi^{(72c)}(n) = \left( \frac{2}{n} \right)_2 \neq 1$ by (2.2), a contradiction. Suppose it is induced by some $\chi$ modulo $8(1 - \omega)^3c$, then by the Chinese Remainder Theorem, there exists an $n$ such that $n \equiv 1 \pmod{8c}$ and $n \equiv 1 + 3(1 - \omega) \pmod{9}$, then we have $\chi(n) = 1$ but

\[\chi^{(72c)}(n) = \left( \frac{\omega(1 - \omega)}{n} \right)_3 \neq 1\]

by (2.2) (note that we have $3 = -\omega^2(1 - \omega)^2$), a contradiction. This implies that $\chi^{(72c)}$ is primitive. This also shows that $\chi^{(-72c)}$ is non-principal.

2.2. Gauss sums. Suppose that $n \in \mathbb{Z}[\omega]$ with $(n, 6) = 1$. For $j = 2, 3, 6$, the quadratic ($j = 2$), cubic ($j = 3$) and sextic ($j = 6$) Gauss sum is defined by

\[g_j(n) = \sum_{x \pmod{n}} \left( \frac{x}{n} \right)_j \bar{e}(\frac{x}{n}),\]

where $\bar{e}(z) = e((z - \bar{z})/\sqrt{-3})$.

We can infer from its definition that $g_j(1) = 1$. Moreover, the following well-known relation (see [27] p. 195)) holds for all $n$:

\[(2.3) \quad |g_j(n)| = \begin{cases} \sqrt{N(n)} & \text{if } n \text{ is square-free}, \\ 0 & \text{otherwise}. \end{cases}\]

More generally, for any $n, r \in \mathbb{Z}[\omega]$, with $(n, 6) = 1$, we set

\[g_j(r, n) = \sum_{x \pmod{n}} \left( \frac{x}{n} \right)_j \bar{e}(\frac{rx}{n}).\]

We need the following properties of $g_j(r, n)$:
Lemma 2.3. For $j = 2, 3, 6$, any prime $\varpi$ satisfying $(\varpi, 6) = 1$, we have

\begin{align}
\tag{2.4} g_j(rs, n) &= \left( \frac{r}{n} \right)_j g(r, n), \quad (s, n) = 1, \\
\tag{2.5} g_j(r, n_1 n_2) &= \left( \frac{n_2}{n_1} \right)_j \left( \frac{n_1}{n_2} \right)_j g_j(r, n_1)g_j(r, n_2), \quad (n_1, n_2) = 1,
\end{align}

where

\begin{align}
\tag{2.6} g_6(\varpi^k, \varpi^l) &= \begin{cases} N(\varpi)^k g_6(\varpi) & \text{if } l = k + 1, k \equiv 0 \pmod{6}, \\
N(\varpi)^k g_3(\varpi) & \text{if } l = k + 1, k \equiv 1 \pmod{6}, \\
N(\varpi)^k g_2(\varpi) & \text{if } l = k + 1, k \equiv 2 \pmod{6}, \\
N(\varpi)^k \left( \frac{-1}{\varpi} \right)_6 g_3(\varpi) & \text{if } l = k + 1, k \equiv 3 \pmod{6}, \\
N(\varpi)^k \left( \frac{-1}{\varpi} \right)_6 g_2(\varpi) & \text{if } l = k + 1, k \equiv 4 \pmod{6}, \\
-N(\varpi)^k, & \text{if } l = k + 1, k \equiv 5 \pmod{6}, \\
\varphi(\varpi^l) = \#(\mathbb{Z}[\varpi]/(\varpi^l))^* & \text{if } k \geq l, l \equiv 0 \pmod{6}, \\
0 & \text{otherwise.}
\end{cases}
\end{align}

Proof. Both (2.4) and (2.5) follow easily from the definition. For (2.6), the case $l \leq k$ is easily verified. If $l > k$, then

\begin{align}
\tag{2.7} \sum_{a \equiv \varpi \pmod{\varpi^l}} \left( \frac{a}{\varpi^l} \right)_6 \tilde{e} \left( \frac{a^k}{\varpi^l} \right) &= \sum_{b \equiv \varpi \pmod{\varpi}} \left( \frac{b}{\varpi} \right)_6 \sum_{c \equiv \varpi^{l-k} \pmod{\varpi^l}} \tilde{e} \left( \frac{c}{\varpi^{l-k}} \right) \\
&= \sum_{b \equiv \varpi \pmod{\varpi}} \left( \frac{b}{\varpi} \right)_6 \tilde{e} \left( \frac{b}{\varpi^{l-k}} \right) \sum_{c \equiv \varpi^{l-k} \pmod{\varpi^l}} \tilde{e} \left( \frac{c_2}{\varpi^{l-k}} \right) = 0,
\end{align}

where the last equality follows from [17, Lemma, p. 197]. This proves the last case when $l \geq k + 2$.

It thus remains to deal with the case in which $l = k + 1$. In this case, the right-hand side expression of (2.7) is

\begin{align}
N(\varpi)^{l-1} \sum_{b \equiv \varpi \pmod{\varpi}} \left( \frac{b}{\varpi} \right)_6 \tilde{e} \left( \frac{b}{\varpi} \right).
\end{align}

The expressions in (2.0) for $g_6(\varpi^k, \varpi^l)$ follow from this, taking into account the definitions of $g_j(\varpi)$ for $j = 2, 3$ and $6$. This completes the proof of the lemma. \qed

2.4. The approximate functional equation. Let $c \in O_K, c \equiv 1 \pmod{36}$ be square-free. It follows from (2.2) that $\chi_c = \chi_6$ is trivial on units, it can be regarded as a primitive Hecke character $\chi_c$ of trivial infinite type. The Hecke $L$-function associated with $\chi_c$ is defined for $\Re(s) > 1$ by

\begin{align}
L(s, \chi_c) = \sum_{0 \neq A \subset \mathcal{O}_K} \chi_c(A)(N(A))^{-s},
\end{align}

where $\mathcal{A}$ runs over all non-zero integral ideals in $K$ and $N(A)$ is the norm of $A$. As shown by E. Hecke, $L(s, \chi_c)$ admits analytic continuation to an entire function and satisfies a functional equation. We refer the reader to [10][11][14][23] for a more detailed discussion of these Hecke characters and $L$-functions.

Let $G(s)$ be any even function which is holomorphic and bounded in the strip $-4 < \Re(s) < 4$ satisfying $G(0) = 1$. We have the following expression for $L(1/2 + it, \chi_c)$ for $t \in \mathbb{R}$ (see [11, Section 2.4]):

\begin{align}
L \left( \frac{1}{2} + it, \chi_c \right) &= \sum_{0 \neq A \subset \mathcal{O}_K} \frac{\chi_c(A)}{N(A)^{1/2 + it}} V_t \left( \frac{2\pi N(A)}{x} \right) \\
&\quad + \frac{g_6(c)}{N(c)^{1/2}} \left( \frac{2\pi}{|D_k|N(c)} \right)^it \frac{\Gamma(1/2 - it)}{\Gamma(1/2 + it)} \sum_{0 \neq A \subset \mathcal{O}_K} \frac{\chi_c(A)}{N(A)^{1/2 - it}} V_t \left( \frac{2\pi N(A)x}{|D_k|N(c)} \right),
\end{align}

where $V_t(x)$ is the Poisson summation formula.
where $g_6(c)$ is the Gauss sum defined in Section 2.1. $D_K = -3$ is the discriminant of $K$ and

$$V_t(\xi) = \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma(s + 1/2 + it)}{\Gamma(1/2 + it)} G(s) \frac{\xi^{-s}}{s} \, ds.$$ 

We write $V$ for $V_0$ and note that for a suitable $G(s)$ (for example $G(s) = e^{s^2}$), we have for any $c > 0$ (see [15, Proposition 5.4]):

$$V_t(\xi) \ll \left(1 + \frac{\xi}{1 + |t|}\right)^{-c}.$$

On the other hand, when $G(s) = 1$, we have (see [28, Lemma 2.1]) for the $j$-th derivative of $V(\xi)$,

$$(2.9) \quad V(\xi) = 1 + O(\xi^{1/2-\epsilon}) \quad \text{for } 0 < \xi < 1 \quad \text{and} \quad V^{(j)}(\xi) = O(e^{-\xi}) \quad \text{for } \xi > 0, \ j \geq 0.$$

### 2.5. Analytic behavior of Dirichlet series associated with sextic Gauss sums

For any Hecke character $\chi$ (mod 36) of trivial infinite type, we let

$$(2.10) \quad h(r, s; \chi) = \sum_{n, \ E \text{primary}} \frac{\chi(n)g_6(r, n)}{N(n)^s}.$$

The following lemma gives the analytic behavior of $h(r, s; \chi)$ on $\Re(s) > 1$.

**Lemma 2.6.** Let $r$ be E-primary. The function $h(r, s; \chi)$ has meromorphic continuation to the complex plane. It is holomorphic in the region $\sigma = \Re(s) > 1$ except possibly for a pole at $s = 7/6$. For any $\varepsilon > 0$, letting $\sigma_1 = 3/2 + \varepsilon$, then for $\sigma_1 \geq \sigma \geq \sigma_1 - 1/2, |s - 7/6| > 1/12$ and we have

$$h(r, s; \chi) \ll N(r)^{\sigma_1 - \sigma + \varepsilon}/(1 + t^2)^{5(\sigma_1 - \sigma + \varepsilon)/2},$$

where $t = \Im(s)$. Moreover, the residue satisfies the bound

$$\text{Res}_{s=7/6} h(r, s; \chi) \ll N(r)^{1/6 + \varepsilon}.$$ 

Lemma 2.6 is shown in the same manner as that of [10, Lemma 2.5]. We use Lemma 2.7 to remove the condition $(n, r) = 1$ in (2.10), and the result of Lemma 2.6 then follows from the proof of the Lemma on [27, p. 200], taking into account the observation from (2.6) that $|g_6(\varpi^k, \varpi^l)| \leq N(\varpi)^{k+1/2}$ when $k < l$ for any E-primary prime $\varpi$.

To state the next lemma, we define for $a, c \in \mathbb{Z}[\omega], (ac, 6) = 1$,

$$\psi_a(c)(-1)^{(N(a)-1)/2}(N(c)-1)/2 = \left(\frac{-1}{c}\right)^{(N(a)-1)/2} \left(\frac{-1}{c}\right)^{(N(c)-1)/2}.$$ 

It is easy to see that $\psi_a(c)$ is a Hecke character (mod 36) of trivial infinite type.

Furthermore, for any square-free $a \in \mathbb{Z}[\omega]$, we let $\{\varpi_1, \ldots, \varpi_k\}$ be the set of distinct E-primary prime divisors of $a$ and we define for $1 \leq j \leq 4$,

$$P_j(a) = \prod_{i=1}^{k} \frac{(a/\varpi_i)^j}{\varpi_i^{j+1}}, \quad \Psi_j(a) = \prod_{i=1}^{k-1} \psi_{\varpi_i}^{j+1}(\varpi_{i+1}),$$

where we set the empty product to be 1. In particular, we have $P_j(a) = 1$ when $a$ is a unit and $\Psi_j(a) = 1$ when $a$ is a unit or a prime. As $\left(\frac{\varpi_1}{\varpi_1^{j+1}}\right)_6 = \left(\frac{\varpi_2}{\varpi_1^{j+1}}\right)_6$ for two distinct E-primary primes $\varpi_1, \varpi_2$, one checks easily by induction on the number of prime divisors of $a$ and the sextic reciprocity that $P_j(a)$ is independent of the order of $\{\varpi_1, \ldots, \varpi_k\}$ when $j \neq 3$. Similarly, as $\left(\frac{\varpi_1}{\varpi_2}\right)_3 = \left(\frac{\varpi_1}{\varpi_3}\right)_3, \psi_{\varpi_1}(\varpi_2) = \psi_{\varpi_2}(\varpi_1), P_3(a)$ and $\Psi_3(a), 1 \leq j \leq 4$ are also independent of the order of $\{\varpi_1, \ldots, \varpi_k\}$.

Now we have

**Lemma 2.7.** Let $(rf\alpha, 6) = 1$. Suppose $f, \alpha$ are square-free and $(r, f) = 1$, and set

$$h(r, f, s; \chi) = \sum_{(n, rf) = 1} \frac{\chi(n)g_6(r, n)}{N(n)^s}, \quad h_\alpha(r, s; \chi) = \sum_{(n, \alpha) = 1} \frac{\chi(n)g_6(r, n)}{N(n)^s}.$$
Furthermore suppose \( r = r_1 r_2^2 r_3^3 r_4^4 r_5^6, r^* = r_1 r_2 r_3^3 r_4^4 r_5^5 \) where \( r_1 r_2 r_3 r_4 r_5 \) is square-free, and let \( r_6^* \) be the product of primes dividing \( r_6 \). Let \( 1 \leq j \leq 4 \) and

\[
h_{r_i}^{-1} (r^*, s; \chi) = \sum_{a \mid r_i} \mu[\omega](a) \chi(a) a^{-j+1} N(a)^{-(j+1)} \frac{(r^*/a^j)}{a^{j+1}} P_j(a) \Psi_j(a) \left( \prod_{E \mid a} g_6(\omega^j, \omega^{j+1}) \right) \times h_{r_i}^{-1} (r^* a^{4-j}, s; \psi_{\omega^{j+1}}),
\]

where the empty product is understood to be 1. Then

(2.11)

\[
h(r, f, s; \chi) = \sum_{a \mid f} \frac{\mu[\omega](a) \chi(a) g_6(r, a)}{N(a)^s} h(a^2 r, s; \psi_a \chi),
\]

\[
h(r_1 r_2^2 r_3^3 r_4^4 r_5^6, s; \chi) = h(r^*, r_6^*, s; \chi),
\]

\[
h(r^*, s; \chi) = \prod_{E \mid r} (1 - \chi(\omega)^6 N(\omega)^{5-6s})^{-1} h(r_1 r_2 r_3 r_4 (r^*, s; \chi),
\]

\[
h_{r_i}^{-1} (r^*, s; \chi) = \prod_{E \mid r} (1 - \psi_{\omega^{j+1}}(\omega^{5-j}) \chi(\omega)^6 N(\omega)^{-6s} g_6(\omega^j, \omega^{j+1}) g_6(\omega^{4-j}, \omega^{5-j})^{-1} h_{r_i}^{-1} (r^*, s; \chi).
\]

Proof. As the proof is similar to that of [III, Lemma 3.6], we only give the proof for the last equality given in (2.11) here. To prove this, we let \( a = \prod_{i=1}^5 a_i \in \mathbb{Z}[\omega] \) and let \( \omega \) be an \( E \)-primary prime in \( \mathbb{Z}[\omega] \) such that \( a^* \omega \) is square-free, where \( a^* = \prod_{i=1}^5 a_i \). Then

\[
h_{a^* \omega}(a \omega^j, s; \chi) = \sum_{(n, a^* \omega) = 1} \frac{\chi(n) g_6(a \omega^j, n)}{N(n)^s}
\]

\[
= \sum_{(n, a^*) = 1} \frac{\chi(n) g_6(a \omega^j, n)}{N(n)^s} - \sum_{(n, a^*) = 1} \frac{\chi(n) g_6(a \omega^j, n)}{N(n)^s}.
\]

Writing in the latter sum \( n = \omega^h n' \) with \( (n, \omega) = 1 \), then (2.5) gives that

\[
g_6(a \omega^j, \omega^h n') = \left( \frac{\omega^h}{n'} \right)_6 \left( \frac{n'}{\omega^h} \right)_6 g_6(a \omega^j, \omega^h) g_6(a \omega^j, n').
\]

Using (2.4) and (2.6) we see that \( g_6(a \omega^j, \omega^h) = 0 \) unless \( h = j + 1 \), in which case we deduce from sextic reciprocity and (2.4) that

\[
g_6(a \omega^j, \omega^{j+1}) = g_6(\omega^j, \omega^{j+1}) \left( \frac{a}{\omega^{j+1}} \right)_6 \psi_{\omega^{j+1}}(n') g_6(a \omega^{4-j}, n').
\]

This implies that

(2.12)

\[
h_{a^* \omega}(a \omega^j, s; \chi) \equiv h_{a^*}(a \omega^j, s; \chi) - \chi(\omega^{j+1}) N(\omega)^{-(j+1)} s g_6(\omega^j, \omega^{j+1}) \left( \frac{a}{\omega^{j+1}} \right)_6 h_{a^* \omega}(a \omega^{4-j}, s; \psi_{\omega^{j+1}} \chi).
\]

On the other hand,

\[
h_{a^* \omega}(r^* \omega^{4-j}, s; \psi_{\omega^{j+1}} \chi) = \sum_{(n, a^* \omega) = 1} \frac{\psi_{\omega^{j+1}}(n) \chi(n) g_6(a \omega^{4-j}, n)}{N(n)^s}
\]

\[
= \sum_{(n, a^*) = 1} \frac{\psi_{\omega^{j+1}}(n) \chi(n) g_6(a \omega^{4-j}, n)}{N(n)^s} - \sum_{(n, a^*) = 1} \frac{\psi_{\omega^{j+1}}(n) \chi(n) g_6(a \omega^{4-j}, n)}{N(n)^s}.
\]
Again, writing in the latter sum \( n = \varpi^h n' \) with \( (n', \varpi) = 1 \), then (2.13) yields that
\[
g_6(a\varpi^{4-j}, \varpi^{h}n') = \left( \frac{\varpi^{h}}{n'} \right)_6 g_6(a\varpi^{4-j}, \varpi^{h})g_6(a\varpi^{4-j}, n').
\]

Using the same treatment as earlier, (2.4) and (2.6) imply that \( g(a\varpi^{4-j}, \varpi^{h}) = 0 \) unless \( h = 5 - j \). Then in that case the sextic reciprocity, together with (2.4), yields that
\[
g_6(a\varpi^{4-j}, \varpi^{5-j}n') = g_6(a\varpi^{4-j}, \varpi^{5-j})\left( \frac{a}{\varpi^{5-j}} \right)_6 \psi_{\varpi^{5-j}}(n')g_6(a\varpi^{j}, n').
\]

This implies that
\[
h_{a_{\varpi}}(a\varpi^{4-j}, s; \psi_{\varpi^{j+1}}) = h_{a_{\varpi}}(a\varpi^{4-j}, s; \psi_{\varpi^{j+1}}) - \psi_{\varpi^{j+1}}(\varpi^{5-j})\chi(\varpi^{5-j})N(\varpi)^{-s}g_6(\varpi^{4-j}, \varpi^{5-j})\left( \frac{a}{\varpi^{5-j}} \right)_6 h_{a_{\varpi}}(a\varpi^{j}, s; \chi),
\]
as one checks easily that \( \psi_{\varpi^{j+1}}(\varpi^{5-j}) \) is principal.

Combining (2.12) and (2.13), we get
\[
h_{a_{\varpi}}(a\varpi^{j}, s; \chi) = (1 - \psi_{\varpi^{j+1}}(\varpi^{5-j})\chi(\varpi^{5-j})N(\varpi)^{-s}g_6(\varpi^{4-j}, \varpi^{5-j})^{-1}(1_{a_{\varpi}}(a\varpi^{j}, s; \chi)
\]
\[
 = \chi(\varpi^{j+1})N(\varpi)^{-s}g_6(\varpi^{4-j}, \varpi^{j+1})\left( \frac{a}{\varpi^{j+1}} \right)_6 h_{a_{\varpi}}(a\varpi^{4-j}, s; \chi)).
\]

Note that when \( j \neq 3 \), we have \( \left( \frac{\varpi^{4-j}}{\varpi^{j+1}} \right)_6 = 1 \) for two distinct primes \( \varpi, \varpi' \) as \( (4-j)(j+1) \equiv 0 (\text{mod } 6) \). When \( j = 3 \), we have \( \left( \frac{\varpi}{\varpi} \right)_6 = 1 \) when \( a \) is a cube. These observations together with an induction argument on the number of prime divisors of \( a_j \) leads to the last equality given in (2.11).

2.8. The large sieve with sextic symbols. One important input of this paper is the following large sieve inequality for sextic Hecke characters. The study of the large sieve inequality for characters of a fixed order has a long history. We refer the reader to \([12, 15, 16]\).

Lemma 2.9. \([2]\) Theorem 1.3] Let \( M, N \) be positive integers, and \( (a_n) \) be an arbitrary sequence of complex numbers, where \( n \) runs over \( \mathbb{Z}[\varpi] \). Then we have
\[
\sum_{m \in \mathbb{Z}[\varpi]}^s \left| \sum_{n \in \mathbb{Z}[\varpi]} \frac{a_n (m/n)}{M} \right|^{2} \ll \varepsilon (M + N + (MN)^{2/3})(MN)^{\varepsilon} \sum_{N(n) \leq N} |a_n|^2,
\]

for any \( \varepsilon > 0 \), where the asterisks indicate that \( m \) and \( n \) run over square-free \( E \)-primary elements of \( \mathbb{Z}[\varpi] \) and \( (\varpi/m)_6 \) is the sextic residue symbol.

2.10. Poisson Summation. The proof of Theorem 1.3 requires the following Poisson summation formula.

Proposition 2.11. Let \( \varpi \in \mathbb{Z}[\varpi] \) be \( E \)-primary and \( (\varpi)_6 \) be the sextic residue symbol \( (\text{mod } n) \). For any Schwartz class function \( \Phi \), we have
\[
\sum_{c \in \mathbb{Z}[\varpi]} \left( \frac{c}{n} \right)_6 \Phi \left( \frac{N(c)}{X} \right) = \mu_{\varpi}(m) \left( \frac{m}{n} \right)_6 \frac{X}{N(m)N(n)} \sum_{k \in \mathbb{Z}[\varpi]} g_6(k, n) \overline{\Phi} \left( \sqrt{\frac{N(k)X}{(n)N(n)}} \right),
\]

where
\[
\overline{\Phi}(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(N(x + y\varpi))e(-t(x + y\varpi)) \, dx \, dy, \quad t \geq 0.
\]

The above result is an easy consequence of a variation of \([9\) Lemma 2.6], we omit its proof here. When \( \Phi(t) \) is the one given in Theorem 1.3, we have the following estimations for \( \overline{\Phi} \) and its derivatives \([13\) (2-15)]: \( \overline{\Phi}(t) \in \mathbb{R} \) for any \( t \geq 0 \) and
\[
\overline{\Phi}(\mu(t) \ll_j \min\{1, U^{-1}t^{-j}\}
\]

for all integers \( \mu \geq 0, j \geq 1 \) and all \( t > 0 \).
3. Proof of Theorem 1.1

We derive readily from \([28]\), the approximate functional equation, with \(G(s) = 1, t = 0\) and \(x = 3N(c)/z\) that

\[
\sum_{c \equiv 1 \mod 36}^{*} L \left( \frac{1}{2}, \chi_{c} \right) W \left( \frac{N(c)}{y} \right) = M_1 + M_2,
\]

where

\[
M_1 = \sum_{c \equiv 1 \mod 36}^{*} \sum_{0 \neq A \subset O_K} \frac{\chi_{c}(A)}{N(A)^{1/2}} V \left( \frac{2\pi N(A)z}{3N(c)} \right) W \left( \frac{N(c)}{y} \right), \quad \text{and}
\]

\[
M_2 = \sum_{c \equiv 1 \mod 36}^{*} g_6(c) \frac{N(c)^{1/2}}{N(c)^{1/2}} \sum_{0 \neq A \subset O_K} \frac{\chi_{c}(A)}{N(A)^{1/2}} V \left( \frac{2\pi N(A)}{z} \right) W \left( \frac{N(c)}{y} \right).
\]

3.1. Evaluating \(M_1\), the main term. Since any integral non-zero ideal \(A\) in \(\mathbb{Z}[\omega]\) has a unique generator \(2^{r_1}(1-\omega)^{r_2}a\), with \(r_1, r_2 \in \mathbb{Z}, r_1, r_2 \geq 0\) and an \(E\)-primary \(a \in \mathbb{Z}[\omega]\), it follows from \([24]\) and \([22]\) that \(\chi_c(2) = \chi_c(1-\omega) = 1\), which implies that \(\chi_c(A) = \chi_c(a)\).

We now define for \(a\) being \(E\)-primary, \((c, 6) = 1\),

\[
\chi^{(a)}(c) = \left( \frac{a}{c} \right)_6.
\]

Similar to our discussions in Section 2 one can check easily that \(\chi^{(a)}\) is a Hecke character modulo 36\(a\) of trivial infinite type.

The above discussions allow us to recast \(M_1\) as

\[
M_1 = \sum_{r_1, r_2 \geq 0}^{*} \frac{1}{2^{r_1+3r_2}2N(a)^{1/2}} M(r, a),
\]

where

\[
M(r, a) = \sum_{c \equiv 1 \mod 36}^{*} \chi^{(a)}(c)V \left( \frac{\pi 2^{r_1+1}3^{r_2}N(a)z}{y} \frac{N(c)}{N(c)} \right) W \left( \frac{N(c)}{y} \right).
\]

Now we use Möbius inversion to detect the square-free condition of \(c\), getting

\[
M(r, a) = \sum_{l \equiv 1 \mod 36}^{*} \mu(l) \chi^{(a)}(l^2) M(l, r, a),
\]

with

\[
M(l, r, a) = \sum_{c \equiv 1 \mod 36}^{*} \chi^{(a)}(c)V \left( \frac{\pi 2^{r_1+1}3^{r_2}N(a)z}{y} \frac{N(c^2)}{N(c^2)} \right) W \left( \frac{N(c^2)}{y} \right).
\]

By Mellin inversion, we have

\[
V \left( \frac{\pi 2^{r_1+1}3^{r_2}N(a)z}{y} \frac{N(c^2)}{N(c^2)} \right) W \left( \frac{N(c^2)}{y} \right) = \frac{1}{2\pi i} \int_{(2)} \left( \frac{y}{N(c^2)} \right)^s \hat{f}(s) \, ds,
\]

where

\[
\hat{f}(s) = \int_{0}^{\infty} V \left( \frac{\pi 2^{r_1+1}3^{r_2}N(a)z}{xy} \right) W(x)x^{s-1}dx.
\]

Integrating by parts together with \([2,1]\) shows \(\hat{f}(s)\) is a function satisfying the bound

\[
\hat{f}(s) \ll (1 + |s|)^{-E} \left( 1 + \frac{2^{r_1+1}3^{r_2}N(a)z}{y} \right)^{-E},
\]

for all \(\Re(s) > 0\) and any integer \(E > 0\).
With this notation, we have

\[ M(l, r, a) = \frac{1}{2\pi i} \int_{c^2 \equiv 1 \mod 36} \hat{f}(s) \left( \frac{y}{N(l^2)} \right)^s \sum_{\psi \equiv c \mod 36} \chi(a)(c) N(c)^s ds. \]

Recall that for any \( c \), the ray class group \( h(c) \) is defined to be \( I_c / P_c \), where \( I_c = \{ A \in I : (A, c) = 1 \} \) and \( P_c = \{(a) \in P : a \equiv 1 \mod c\} \) with \( I \) and \( P \) denoting the group of fractional ideals in \( K \) and the subgroup of principal ideals, respectively. We now use the ray class characters to detect the condition that \( c^2 \equiv 1 \mod 36 \), getting

\[ M(l, r, a) = \frac{1}{\#h(36)} \sum_{\psi \mod 36} \psi(2) \int_{c^2 \equiv 1 \mod 36} \hat{f}(s) \left( \frac{y}{N(l^2)} \right)^s L(s, \psi \chi(a)) ds, \]

where \( \psi \) runs over all ray class characters \( \mod 36 \), \( \#h(36) = 108 \) and

\[ L(s, \psi \chi(a)) = \sum_{A \neq 0} \psi(A) \chi(a)(A) N(A)^s. \]

We compute \( M_1 \) by shifting the contour to the half line. If \( \psi \chi(a) \) is principal, the Hecke \( L \)-function has a pole at \( s = 1 \). We set \( M_0 \) to be the contribution to \( M_1 \) of these residues, and \( M_1' \) to be the remainder.

We first evaluate \( M_0 \). Note that \( \psi \chi(a) \) is principal if and only if both \( \psi \) and \( \chi(a) \) are principal. Hence \( a \) must be a sixth power. We denote \( \psi_0 \) for the principal ray class character \( \mod 36 \). Then we have

\[ L(s, \psi_0 \chi(a^6)) = \zeta_Q(\omega)(s) \prod_{\omega | (6a)} (1 - N(\omega)^{-s}). \]

Let \( c_0 = \sqrt{3}/9 \), the residue of \( \zeta_Q(\omega)(s) \) at \( s = 1 \). Then we have

\[ M_0 = \frac{\#h(36)}{\zeta_Q(\omega)(2)} \sum_{r_1, r_2 \geq 0} \frac{\hat{f}(1)}{(2 \pi r_1)^2 N(a)^3} \text{Res}_{s=1} L(s, \psi_0 \chi(a^6)) \sum_{l \text{ E-primary}} \frac{\mu_{36}(l) \chi(a^6)(l^2)}{N(l^2)} \]

\[ = \frac{c_0 y}{\#h(36) \zeta_Q(\omega)(2)} \sum_{r_1, r_2 \geq 0} \frac{\hat{f}(1)}{(2 r_1)^2 N(a)^3} \prod_{\omega | (6a)} (1 - N(\omega)^{-1}) \prod_{\omega | (6a)} (1 - N(\omega)^{-2})^{-1} \]

\[ = \frac{c_0 y}{\#h(36) \zeta_Q(\omega)(2)} \sum_{r_1, r_2 \geq 0} \frac{\hat{f}(1)}{(2 r_1)^2 N(a)^3} \prod_{\omega | (6a)} (1 + N(\omega)^{-1})^{-1}. \]

Set

\[ Z(u) = \sum_{a \text{ E-primary}} \frac{1}{\pi u 2^{r_1+1} 3^{r_2+1} N(a)^3} \prod_{\omega | (6a)} \frac{(1 - N(\omega)^{-1})^{-1}}{(1 + N(\omega)^{-1})^{-1}}, \]

which is holomorphic and bounded for \( \Re(u) \geq -1/3 + \delta > -1/3 \).

Note that using the Mellin convolution formula shows

\[ \hat{f}(1) = \int_0^\infty \left( \frac{\pi 2^{r_1+1} 3^{r_2-1} N(a)^6 x^s}{x y} \right) W(x) dx = \frac{1}{2\pi i} \int_{c^2 \equiv 1 \mod 36} \left( \frac{y}{\pi 2^{r_1+1} 3^{r_2-1} N(a)^6} \right)^s \hat{W}(1+s) \frac{G(s) \Gamma(s+1/2)}{s \Gamma(1/2)} ds, \]

where

\[ \hat{W}(s) = \int_0^\infty W(x) x^{s-1} dx. \]

Then

\[ M_0 = \frac{c_0 y}{\#h(36) \zeta_Q(\omega)(2)} \frac{1}{2\pi i} \int_{c^2 \equiv 1 \mod 36} \left( \frac{y}{z} \right)^s Z(s) \hat{W}(1+s) \frac{G(s) \Gamma(s+1/2)}{s \Gamma(1/2)} ds. \]
We move the contour of integration to $-1/3 + \varepsilon$, crossing a pole at $s = 0$ only. The new contour contributes $O((y/z)^{-1/3+\varepsilon} y)$, while the pole at $s = 0$ gives

\begin{equation}
Ay\tilde{W}(1), \quad \text{where} \quad A = \frac{c_0}{#h(36)\mathbb{Q}(\omega)(2)}Z(0).
\end{equation}

Note that $Z(u)$ converges absolutely at $u = 0$ so it is easy to express $Z(0)$ explicitly as an Euler product, if desired. We then conclude that

\begin{equation}
M_0 = Ay\tilde{W}(1) + O\left(\left(\frac{y}{z}\right)^{-1/3+\varepsilon}\right).
\end{equation}

3.2. Estimating $M'_1$, the remainder term. To deal with $M'_1$, we bound everything by absolute values and use [23] to get that for any $E > 0$,

\begin{equation}
M'_1 \ll y^{1/2} \sum_{N(l) \leq \sqrt{y}} \frac{1}{N(l)} \sum_{\psi \bmod 36} \sum_{r_1, r_2 \geq 0} \frac{1}{2^{r_1}3^{r_2/2}N(a)^1/2} \left(1 + \frac{2^{r_1+1}3^{r_2-1}N(a)z}{y}\right)^{-E}
\end{equation}

\begin{equation}
\times \int_{-\infty}^{\infty} \left|L\left(\frac{1}{2} + it, \chi^{(a)}\right)\right| (1 + |t|)^{-E} dt.
\end{equation}

We now need the following estimation to bound the sum over $a$:

\begin{equation}
\sum_{N(a) \leq N} N(a)^{-1/2} \left|L\left(\frac{1}{2} + it, \chi^{(a)}\right)\right| \ll (N(1 + |t|))^{1/2+\varepsilon}.
\end{equation}

The proof of [33] is similar to that of [1] (39) and we will give a sketch of the arguments. We factor $a$ as $a_1a_2^2a_3^3a_4^4a_5^5a_6^6$ where $a_1a_2a_3a_4a_5$ is square-free. Then $\psi\chi^{(a)}$ equals $\psi\chi^{(a_1)}\chi^{(a_2)}\chi^{(a_3)}\chi^{(a_4)}\chi^{(a_5)}\chi^{(a_6)}$ times a principal character. Here $(\chi^{(a_2)})^2$ (respectively $(\chi^{(a_3)})^3$) can be regarded as a cubic Hecke character (mod 36a_2) (respectively (mod 36a_3)) of trivial infinite type and $(\chi^{(a_3)})^3$ can be regarded as a quadratic Hecke character (mod 36a_3) of trivial infinite type. For each fixed $a_1, 2 \leq i \leq 5$, it suffices to show that

\begin{equation}
\sum_{(a_1, a_2a_3a_4a_5) = 1}^{*} \left|L(1/2 + it, \chi^{(a_1)}\chi^{(a_2)})\chi^{(a_3)}\chi^{(a_4)}\chi^{(a_5)}\chi^{(a_6)}\right|^2 \ll N_1^{1+\varepsilon}N(a_2a_3a_4a_5)^{1/2+\varepsilon}(1 + |t|)^{1+\varepsilon},
\end{equation}

where the asterisk indicates that $a_1$ runs over $E$-primary square-free elements of $\mathbb{Z}[[\omega]]$. With this bound and the convergence of the sums over $a_2, a_3, a_4, a_5, a_6$, a use of Cauchy’s inequality gives [33].

Note that as $a_1a_2a_3a_4a_5$ is square-free, the character $\chi^{(a_1)}\chi^{(a_2)}\chi^{(a_3)}\chi^{(a_4)}\chi^{(a_5)}\chi^{(a_6)}$ is primitive with conductor $f$ satisfying

\begin{equation}
\frac{a_1a_2a_3a_4a_5}{(6, a_1a_2a_3a_4a_5)} f \quad \text{and} \quad f|36a_1a_2a_3a_4a_5.
\end{equation}

We may now further assume that $\psi\chi^{(a_1)}\chi^{(a_2)}\chi^{(a_3)}\chi^{(a_4)}\chi^{(a_5)}\chi^{(a_6)}$ is primitive. Thus the Hecke $L$-function

\begin{equation}
L(s, \psi\chi^{(a_1)}\chi^{(a_2)}\chi^{(a_3)}\chi^{(a_4)}\chi^{(a_5)}\chi^{(a_6)}),
\end{equation}

viewed as a degree two $L$-function over $\mathbb{Q}$, has analytic conductor $\ll N_1N(a_2a_3a_4a_5)(1 + t^2)$.

We then apply the approximate functional equation [28] with $G(s) = e^{s^2}$ for Hecke $L$-functions, removing the weight using the Mellin transform to reduce the problem of estimating [3.6] to bounding

\begin{equation}
\sum_{N(a_1) \leq N_1} \left|\sum_{N(n) \leq Q} \frac{\psi\chi^{(a_1)}\chi^{(a_2)}\chi^{(a_3)}\chi^{(a_4)}\chi^{(a_5)}\chi^{(a_6)}(n)}{N(n)^{1/2+it}}\right|^2.
\end{equation}

Moreover, by [18 Proposition 5.4], we may truncate the sum over $n$ so that $Q \ll (N_1N(a_2a_3a_4a_5)(1 + t^2))^{1/2+\varepsilon}$ with a negligibly small error.
In the inner sum above, writing \( n = n_1 n_2^2 \) with \( n_1, n_2 \) \( E \)-primary, \( n_1 \) square-free and using the Cauchy-Schwarz inequality, it is enough to estimate
\[
\sum_{N(a_1) \leq N_1} \sum_{N(n_1) \leq Q \atop n \text{ \( E \)-primary}} \psi^{\lambda(a_1)}(\chi(a_2))^{2}(\chi(a_3))^{3} \frac{(\chi(a_4))^{2} \chi(a_5)(n_1)}{N(n_1)^{1/2+it}}
\]
where the asterisk in the inner sum above indicates that \( n_1 \) runs over square-free elements of \( \mathbb{Z}[\omega] \). The bound from Lemma 2.9 then gives the desired estimate for (3.6).

We note that we may truncate the sums over \( r_1, r_2, a \) so that \( 2^{2r_1+1}3^{r_2-1}N(a)z \ll y^{1+\varepsilon} \) in (3.4) with a negligibly small error. We now apply (3.5) with \( N = y^{1+\varepsilon}(2^{2r_1+1}3^{r_2-1}N(a)z)^{-1} \) (we may assume that \( y \) is large enough) and treat all the sums trivially to see that
\[
M_1 \ll y^{1/2} \left( \frac{y}{z} \right)^{1/2+\varepsilon}.
\]

3.3. Estimating \( M_2 \). From the discussions at the beginning of Section 3.1, we have
\[
M_2 = \sum_{r_1, r_2 \geq 0 \atop a \text{ \( E \)-primary}} \frac{1}{2^{2r_1+1}3^{r_2/2}N(a)^{1/2}} V \left( \frac{\pi 2^{2r_1+1}3^{r_2}N(a)}{z} \right) \sum_{\psi \equiv 1 \bmod 36} \frac{g_0(c)\chi(a)}{N(c)^{1/2}} W \left( \frac{N(c)}{y} \right).
\]
Note that we can drop the restriction * in sum over \( c \) above, as it follows from (2.23) that \( g_0(c) = 0 \) unless \( c \) is square-free.

We further use the ray class characters to detect the condition that \( c \equiv 1 \pmod{36} \) to obtain
\[
M_2 = \frac{1}{\#h(36)} \sum_{r_1, r_2 \geq 0 \atop a \text{ \( E \)-primary}} \frac{1}{2^{2r_1+1}3^{r_2/2}N(a)^{1/2}} V \left( \frac{\pi 2^{2r_1+1}3^{r_2}N(a)}{z} \right) \sum_{\psi \equiv 1 \bmod 36} H(a, \psi, y),
\]
where
\[
H(a, \psi, y) = \sum_{c \text{ \( E \)-primary}} \psi(c)g_0(c)\chi(a)W \left( \frac{N(c)}{y} \right).
\]
We estimate \( H \) with the following:

**Lemma 3.4.** For any \( E \)-primary \( a \) and any ray class character \( \psi \pmod{36} \), we have
\[
H(a, \psi, y) \ll y^{1/2+\varepsilon} N(a)^{1/4} + y^{2/3} N(a)^{1/6+\varepsilon}.
\]

**Proof.** Note that the identity (2.3) implies \( g_0(c)\chi(a) = g_0(a, c) \) for \( (a, c) = 1 \). Introducing the Mellin transform of \( w \), we get
\[
H(a, \psi, y) = \frac{1}{2\pi i} \int_{(2)} \tilde{W}(s)y^{s}h \left( a, \frac{1}{2} + s; \psi \right) ds.
\]

We move the line of integration in (3.8) to \( \Re(s) = \frac{1}{3} + \varepsilon \), crossing a pole at \( s = 2/3 \), which contributes by Lemma 2.6
\[
\ll y^{2/3} N(a)^{1/6+\varepsilon}.
\]
The main contribution comes from the new line of integration, which by Lemma 2.6 again gives
\[
\ll y^{1/2+\varepsilon} N(a)^{1/4}.
\]
This completes the proof of Lemma 3.4. \( \square \)

Now, to estimate \( M_2 \), we note that we may truncate the sums over \( r_1, r_2, a \) so that \( 2^{2r_1+1}3^{r_2-1}N(a) \ll z^{1+\varepsilon} \) with a negligibly small error. By summing trivially over \( r_1, r_2, a \), one easily deduces that
\[
M_2 \ll y^{1/2+\varepsilon} z^{3/4+\varepsilon} + y^{2/3} z^{2/3+\varepsilon}.
\]

3.5. Conclusion. Combining (3.9) with (3.6) and (3.7), we obtain
\[
M = cy\tilde{W}(1) + O \left( y^{1/2} \left( \frac{y}{z} \right)^{1/2+\varepsilon} + y^{1/2+\varepsilon} z^{3/4+\varepsilon} + y^{2/3} z^{2/3+\varepsilon} \right).
\]
Setting \( z = y^{2/7} \), the proof of Theorem 1.1 is completed.
4. Proof of Theorem 1.3

The proof of Theorem 1.3 is similar to that of [12 Theorem 1.3] and [13 Theorem 1.1]. First, as in [11 Section 3.1], we can show that \( \#C(X) \sim cX \) for some constant \( c \) as \( X \to \infty \). Next, we take \( Z = \log^5 X \) and write \( \mu_2^2(c) = M_Z(c) + R_Z(c) \) where

\[
M_Z(c) = \sum_{(l), t^2 | c, N(t) \leq Z} \mu_1(l) \quad \text{and} \quad R_Z(c) = \sum_{(l), t^2 | c, N(t) > Z} \mu_1(l).
\]

We shall write \( \Phi(t) \) for \( \Phi_X(t) \) throughout. We define \( S(X, Y; \hat{\phi}, \Phi) = S_M(X, Y; \hat{\phi}, \Phi) + S_R(X, Y; \hat{\phi}, \Phi) \) with

\[
S_M(X, Y; \hat{\phi}, \Phi) = \sum_{(c, 6) = 1} M_Z(c) \sum_{\frac{\log N(z)}{\sqrt{N(z)}} \left( \frac{72c}{\sigma} \right) \hat{\phi} \left( \frac{\log N(z)}{\log X} \right) \Phi \left( \frac{N(c)}{X} \right)},
\]

and

\[
S_R(X, Y; \hat{\phi}, \Phi) = \sum_{(c, 6) = 1} R_Z(c) \sum_{\frac{\log N(z)}{\sqrt{N(z)}} \left( \frac{72c}{\sigma} \right) \hat{\phi} \left( \frac{\log N(z)}{\log X} \right) \Phi \left( \frac{N(c)}{X} \right)}.
\]

Here \( \hat{\phi}(u) \) is smooth and has its support in the interval \((-45/43 + \varepsilon, 45/43 - \varepsilon)\) for some \( 0 < \varepsilon < 1 \). To emphasize this condition, we shall set \( Y = X^{45/43 - \varepsilon} \) and write the condition \( N(z) \leq Y \) explicitly throughout this section.

Analogue to what is shown in the proof of [12 Theorem 1.2], we see that in order to establish Theorem 1.3, it suffices to show that

\[
\lim_{X \to \infty} \frac{S(X, Y; \hat{\phi}, \Phi)}{X \log X} = 0.
\]

Using standard techniques (see [12 Section 3.3]), we have that

\[
S_R(X, Y; \hat{\phi}, \Phi) = o(X \log X), \quad \text{as} \quad X \to \infty.
\]

Indeed, with the truth of GRH, the inner-most sum of \( S_R(X, Y; \hat{\phi}, \Phi) \), a character sum over primes, can be bounded very sharply (see [12 Lemma 2.5]) and we immediately get the estimate in (4.1).

To bound \( S_M(X, Y; \hat{\phi}, \Phi) \), we rewrite it as

\[
S_M(X, Y; \hat{\phi}, \Phi) = \sum_{\frac{\log N(z)}{\sqrt{N(z)}} \left( \frac{72c}{\sigma} \right) \hat{\phi} \left( \frac{\log N(z)}{\log X} \right) \Phi \left( \frac{N(c)}{X} \right)}.
\]

Applying Lemma 2.11 and noting that Lemma 2.3 gives

\[
g_6(k, \sigma) = \left( \frac{k}{\sigma} \right) g_6(\sigma),
\]

we can recast \( S_M(X, Y; \hat{\phi}, \Phi) \) further as

\[
S_M(X, Y; \hat{\phi}, \Phi) = X \sum_{N(l) \leq Z} \frac{\mu_1(l)}{N(l)} \sum_{m | l} \mu_1(m) \sum_{m \leq \frac{X}{e}} \mu_1(m) \frac{\log N(z)}{\sqrt{N(z)} \left( \frac{72k_m^2l^4}{\sigma} \right) g_6(\sigma) \hat{\phi} \left( \frac{\log N(z)}{\log X} \right) \Phi \left( \frac{N(k)}{N(m \sigma^2) \sigma} \right)}.
\]

In essentially the same manner (with some minor modifications) as in the proof of [12 Lemma 4.2], we derive from Lemma 2.16 the following:
Lemma 4.1. Let \((h, 6) = 1\). For any \(d \in \mathbb{Z}[\omega]\), we have

\[
\sum_{\substack{N(c) \leq x \\ c \equiv \ell \mod b}} \left(\frac{d}{c}\right)_6 g_6(c) N(c)^{-1/2} = O \left( (N(d))^{1/6+\varepsilon} N(b)^{5/12+\varepsilon} + (N(d))^{1/14} N(b)^{-4/7} x^{6/7+\varepsilon} \right).
\]

Using Lemma 4.1 instead of Proposition 1 of [27, p. 198] in the sieve identity in Section 4 of [27] and noting that in our case Proposition 2 on [27, p. 206] is still valid, we obtain that (taking \(u_3 = X/u_1, u_1 = X^{10/(5n+2R)} N(b)^{-5/(5n+2R)}\) as in [27] and noting that we have \(n = 6, R = 7\) in our case)

\[
E(x; m, k, l) := \sum_{\substack{N(\omega) \leq x \\ \omega \equiv E\text{-primary}}} \left(\frac{72^5 m l k}{\omega}\right) g_6(\omega) \Lambda(\omega) \ll x^\varepsilon \left( N(km^{54})^{7/132} x^{59/66} + N(km^{54})^{1/44+\varepsilon} x^{5/16-1/22} \right).
\]

It follows from this, \([13]\) and partial summation that

\[
\sum_{k \in \mathbb{Z}[\omega]} \sum_{\substack{N(\omega) \leq Y \\ \omega \equiv E\text{-primary}}} \log N(\omega) \left(\frac{72^5 m l k}{\omega}\right) g_6(\omega) \frac{\hat{\phi}}{\log X} \left( \frac{\sqrt{N(k)X}}{N(m^2 \omega)} \right) \ll \frac{N(m^{54})^{7/132+\varepsilon} N(m^{2})^{139/132+\varepsilon} Y^{125/132+\varepsilon} U^3}{X^{139/132}} + \frac{N(m^{54})^{1/44+\varepsilon} N(m^{2})^{45/44+\varepsilon} Y^{43/44+\varepsilon} U^3}{X^{45/44}}.
\]

We then conclude from \([11]\) and \([12]\) that

\[
S(X, Y; \hat{\phi}, \Phi) = o (X \log X), \quad \text{as} \quad X \to \infty.
\]

mindful of \(U = \log \log X, \; Z = \log^5 X\). This completes the proof of Theorem \([13]\).

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