Localized and stationary light wave modes in dispersive media

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In recent experiments, localized and stationary pulses have been generated in second-order nonlinear processes with femtosecond pulses, whose asymptotic features relate with those of nondiffracting and nondispersing polychromatic Bessel beams in linear dispersive media. We investigate on the nature of these linear waves, and show that they can be identified with the X-shaped (O-shaped) modes of the hyperbolic (elliptic) wave equation in media with normal (anomalous) dispersion. Depending on the relative strengths of mode phase mismatch, group velocity dispersion with respect to a plane pulse, and of the defeated group velocity dispersion, these modes can adopt the form of pulsed Bessel beams, focus wave modes, and X-waves (O-waves), respectively.

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I. INTRODUCTION

Stationary, temporally and spatially localized, X-shaped optical wave packets, having a duration of a few tens of femtoseconds and spot size of a few microns, have been recently observed to be spontaneously generated in dispersive nonlinear materials from a standard laser wave packet [1,2,3]. Balancing between second-order or Kerr nonlinearity, group velocity dispersion (GVD), angular dispersion and diffraction, has been suggested to act as a kind of mode-locking mechanism that drives pulse reshaping and keeps the interacting waves trapped, phase and group-matched [4,5,6,7,8,9,10,11,12,13].

The purpose of the present paper is to investigate on the nature of these waves. The main hypothesis underlying our investigation is that these nonlinearly generated X-shaped waves behave asymptotically as linear waves. This assumption is based, first, on the observed stationarity, not only of the central hump of the wave packet, but also of its asymptotic, low-intensity, conical part [1,2,3], stationarity that cannot be attributed to nonlinear wave interactions, but to some linear mechanism of compensation between material and angular dispersion. Indeed, several kinds of linear polychromatic versions of Bessel beams [7,8,9,10,11], as Bessel-X pulses [12,13,14], pulsed Bessel beams [15,16,17,18], subcycle Bessel-X pulses or focus wave modes [12,19], and envelope X waves [13,14], with the capability of maintaining transversal and temporal (longitudinal) localization in dispersive linear media, have been described in recent years (for an unified description and understanding and predicting the spatiotemporal features of the nonlinear X waves generated in experiments. On the linear hand, this description allows us to predict the existence of new kinds of wave modes, and classify all them according to the values of a few physically meaningful parameters.

Each wave mode is specified by the values of the defeated material GVD, the mode group velocity mismatch (GVM) and phase mismatch (PM) with respect to a plane pulse of the same carrier frequency in the same medium. Wave modes are then shown (Section III) to belong to two broad categories: hyperbolic modes, with X-shaped spatiotemporal structure, if material dispersion is normal, or elliptic modes, with O-shaped structure, if material dispersion is anomalous [13,14]. In Section III we show that each wave mode can adopt the approximate form of 1) a pulsed Bessel beam (PBB), 2) an envelope focus wave mode (eFWM), or 3) an envelope X (eX) wave in normally dispersive medium [envelope O (eO) wave in anomalously dispersive medium], according that the mode bandwidth makes PM, GVM or defeated GVD, respectively, to be dominant mode characteristic on propagation. This classification allows us to understand the spatiotemporal features of wave modes in dispersive media in terms of a few parameters (the characteristic PM, GVM and GVD lengths), including modes with mixed pulsed Bessel, focus wave mode, and X-like (O-like) structure.

The above description is obtained from the paraxial approximation to wave propagation. We choose this approach because of its wider use in nonlinear optics, and
because it leads to simpler expressions in terms of parameters directly linked to the physically relevant properties of the mode and dispersive medium. In Section IV we compare the paraxial and the more exact nonparaxial approaches, to show that the paraxial approach is accurate enough for the description of wave modes currently generated by linear optical devices and in nonlinear wave mixing processes.

II. WAVE MODES OF THE PARAXIAL WAVE EQUATION

We start by considering the propagation of a three-dimensional wave packet $E(x, z, t) = A(x, z, t)$ \( \equiv \exp(-i\omega_0 t + ik_0 z) \) of a certain optical carrier frequency $\omega_0$, subject to the effects of diffraction and dispersion of the material medium. Within the paraxial approximation, and up to second order in dispersion, the propagation of narrow-band pulses is ruled by the equation

$$\partial_z A = \frac{i}{2k_0} \Delta_A - \frac{1}{2} k_0'' \partial^2_{\tau} A,$$

(1)

where $z$ is the propagation direction, $\tau = t - k_0 z$ is the local time, $\Delta_A \equiv \partial^2_{\tau} + \partial^2_{\tau'}$, and $k_0(\omega) \equiv k(\omega)|_{\omega_0}$, with $k(\omega)$ the propagation constant in the medium. Eq. (1) is valid for a narrow envelope spectrum $A(x, z, \Omega)$ around $\Omega \equiv \omega - \omega_0 = 0$, that is, for bandwidths

$$\Delta \Omega \ll \omega_0,$$

(2)

a condition that requires at least few carrier oscillations to fall within the envelope $A$.

We search for stationary and localized solutions of Eq. (1) in the wide sense that the intensity does not depend on $z$ in a reference frame moving at some velocity. These solutions must then be of the form

$$A(x, y, \tau, z) = \Phi(x, y, \tau + \alpha z) \exp(-i\beta z).$$

(3)

The free parameters $\alpha$ and $\beta$ are assumed to be small in the sense that

$$|\alpha| \ll k_0,$$

(4)

$$|\beta| \ll k_0,$$

(5)

so that the group velocity $1/(k_0' - \alpha)$ and phase velocity $\omega_0/(k_0 - \beta)$ of the wave differ slightly from those of a plane pulse of the same carrier frequency in the same material, $1/k_0'$ and $\omega_0/k_0$, respectively.

Under the assumption of asymptotic linear behavior of nonlinear X-waves, we can get some insight on the possible values of $\alpha$ and $\beta$ of nonlinear X-waves on the only basis of the linear dispersive properties of the medium. If, for instance, a pulse of frequency $\omega_F$ generates a stationary and localized second harmonic pulse ($\omega_0 = 2\omega_F$) travelling at the same group and phase velocities as the fundamental pulse, we must have $k_0' - \alpha = k_F'$ and $k_0 - \beta = 2k_F$, that is, $\alpha = k_F' - k_0'$ and $\beta = \Delta k \equiv k_0 - 2k_F$. For illustration, Fig. 1 shows the values of $\alpha$ and $\beta$ of the second harmonic pulse in lithium triborate (LBO) as a function of its carrier frequency $\omega_0$. Note also that $|\alpha|$ and $|\beta|$ satisfy the conditions (4) and (5) for any carrier frequency in the entire visible range and beyond.

![FIG. 1: Values of $\alpha$ and $\beta$ of the localized and stationary, second-harmonic waves of different carrier frequencies $\omega_0$, for phase and group matching with the fundamental wave in the process of oo-e second harmonic generation in LBO at room temperature. Dispersion formulas for the refraction index are taken from Ref. [10].](image)

In Section IV a nonparaxial approach to the problem stated above will be performed. It will be shown that the paraxial and nonparaxial descriptions yield substantially the same results if conditions (2), (4) and (5) are satisfied, as is the case of the experiments and numerical simulations demonstrating the spontaneous generation of X-type waves.

Equation (1) with ansatz (3) yields

$$\Delta_A \Phi - k_0 k_0'' \partial^2_\tau \Phi + 2i k_0 k_0' \partial_\tau \Phi + 2 k_0 k_0' \Phi = 0,$$

(6)

for the reduced envelope $\Phi$, or the Helmholtz-type equation $\Delta_A \Phi + K^2(\Omega) \Phi = 0$, for its temporal spectrum $\Phi(x, y, \Omega)$, where

$$K(\Omega) = \sqrt{2k_0 \left( \beta + \alpha \Omega + \frac{1}{2} k_0'' \Omega^2 \right)}$$

(7)

will be referred to as the (transversal) dispersion relation since it relates the modulus $K$ of the transversal component of the wave vector with the detuning $\Omega$ of each monochromatic wave component from the carrier frequency $\omega_0$. For $\Omega$ such that $K(\Omega)$ is real, the Helmholtz equation admits the bounded, cylindrically symmetric, Bessel-type solution $\Phi(r, \Omega) = f(\Omega) J_0[K(\Omega) r]$, where $f(\Omega)$ is an arbitrary spectral amplitude and $J_0(\cdot)$ the
Bessel function of zero order and first class \( J_0 \). By inverse Fourier transform we can write the expression
\[
\Phi_{\alpha,\beta}(r, \tau + \alpha z) = \frac{1}{2\pi} \int_{\Omega} K(\Omega) \text{ real} \ d\Omega \hat{f}(\Omega) \\
\times J_0[K(\Omega)r] \exp[-i\Omega(\tau + \alpha z)] \tag{8}
\]
for the reduced envelope of the cylindrically symmetric wave modes, or localized, propagation invariant solutions of the paraxial wave equation, in the sense explained above. As indicated, the integration domain extends over frequencies \( \Omega \) such that the dispersion curve \( K(\Omega) \) is real. According to Eq. (8), a wave mode \( \Phi_{\alpha,\beta} \) is composed of locked monochromatic Bessel beams whose frequencies and radial wave vectors are linked by a specific dispersion relation \( K(\Omega) \), and whose relative weights are determined by a certain spectral amplitude \( \hat{f}(\Omega) \).

As shown in Figs. 2(a) and (b), the form of the dispersion curve \( K(\Omega) \) reflects the underlying hyperbolic or elliptic geometries of the paraxial wave equation in the respective cases of propagation in media with normal or anomalous dispersion. For normal dispersion \((k_\Omega'' > 0)\), \( K(\Omega) \) is in fact a single-branch vertical hyperbola if \( \beta > \alpha^2/2k_\Omega'' \), and a two-branch horizontal hyperbola if \( \beta < \alpha^2/2k_\Omega'' \) [see Fig. 2(a)]. For anomalous dispersion \((k_\Omega'' < 0)\), \( K(\Omega) \) takes real values only if \( \beta > \alpha^2/2k_\Omega'' \), in which case the dispersion curve is an ellipse [see Fig. 2(b)]. It is also convenient to introduce the (real or imaginary) frequency gap
\[
\Omega_g = \sqrt{\frac{\alpha^2}{k_0''}} - \frac{2\beta}{k_0''}, \tag{9}
\]
and radial wavevector gap
\[
K_g = \sqrt{-k_0k_\Omega''\Omega_g^2}, \tag{10}
\]
When \( \Omega_g \) and \( K_g \) are real, they represent actual frequency and radial wavevector gaps in the dispersion curve \( K(\Omega) \), as illustrated in Fig. 2. In any case, their moduli characterize the scales of variation of the frequency and radial wavevector in the dispersion curves.

Closely connected with the dispersion curve are the so-called impulse response wave modes \( \Phi^{(i)}_{\alpha,\beta}(r, \tau + \alpha z) \), or modes with \( \hat{f}(\Omega) = 1 \). As seen in Fig. 2 the structure of \( \Phi^{(i)}_{\alpha,\beta} \) in space and time closely resembles that of the dispersion curve in the \( K-\Omega \) plane, but at radial and temporal scales of variation determined by the reciprocal quantities \(|K_g|^{-1}\) and \(|\Omega_g|^{-1}\), respectively. Eq. (8) with \( \hat{f}(\Omega) = 1 \) and the change \( \Omega \rightarrow \Omega + \alpha/k_0'' \) yields
\[
\Phi^{(i)}_{\alpha,\beta}(r, z, \tau) = \frac{1}{2\pi} \int K(\Omega') \text{ real} \ d\Omega' \times J_0[K(\Omega')r] \exp[-i\Omega'(\tau + \alpha z)] \\
\times \exp \left[ \frac{i\alpha}{k_0''}(\tau + \alpha z) \right], \tag{11}
\]
where
\[
K(\Omega') = \sqrt{kk_0'' \left( \Omega'^2 + \frac{2\beta}{k_0''} - \frac{\alpha^2}{k_0''} \right)}. \tag{12}
\]
The integral in Eq. (11) can be performed in all possible cases from formulae 6.677.3 (for \( k_\Omega'' > 0 \), \( \beta > \alpha^2/2k_\Omega'' \)), 6.677.6 (for \( k_\Omega'' > 0 \), \( \beta < \alpha^2/2k_\Omega'' \)) and 6.677.6 (for \( k_\Omega'' < 0 \), \( \beta > \alpha^2/2k_\Omega'' \)) of Ref. [20], to yield the closed form expression for impulse response modes
\[
\Phi^{(i)}_{\alpha,\beta}(r, z, \tau) = \frac{1}{2\pi} \Omega_g^{-1} \left[ \frac{2\beta}{k_0''} - \frac{\alpha^2}{k_0''} \right]^{1/2} \exp \left[ \frac{i\alpha}{k_0''}(\tau + \alpha z) \right] \\
+ \text{ C. C.} \tag{13}
\]
or, in terms of the frequency and radial wavevector gaps
\[
\Phi^{(i)}_{\alpha,\beta} = \frac{1}{2\pi} \Omega_g^{-1} \left[ \frac{2\beta}{k_0''} - \frac{\alpha^2}{k_0''} \right]^{1/2} \exp \left[ \frac{i\alpha}{k_0''}(\tau + \alpha z) \right], \tag{14}
\]
where \( R = [K_g^2r^2 + \Omega_g^2(\tau + \alpha z)^2]^{1/2} \).
As shown in Fig. 2(a), for \( k_0'' > 0 \) and \( \beta > \alpha^2/2k_0'' \) (\( \Omega_g \) imaginary and \( K_g \) real), the impulse response wave mode is singular in the cone \( r = |(\tau + \alpha z)|/\sqrt{k_0k_0''} \), is zero for \( r < |(\tau + \alpha z)|/\sqrt{k_0k_0''} \) (within the cone), and decays as \( 1/r \) for \( r > |(\tau + \alpha z)|/\sqrt{k_0k_0''} \) (out of the cone). The radial beatings in this region, of period \( 2\pi/K_g \), are a consequence of the radial wave vector gap \( K_g \).

Figure 2(b) shows the impulse response mode for \( k_0'' < 0 \) and \( \beta < \alpha^2/2k_0'' \) (\( \Omega_g \) real and \( K_g \) imaginary). As in the previous case, the mode is singular at the cone \( r = |(\tau + \alpha z)|/\sqrt{k_0k_0''} \), but damped oscillations are now temporal, of period \( 2\pi/\Omega_g \), as corresponds to the frequency gap \( \Omega_g \) in the dispersion curve. Out of the cone \( |r| > |(\tau + \alpha z)|/\sqrt{k_0k_0''} \), the mode is exponentially localized.

Modes in media with anomalous dispersion, i.e., with \( k_0'' < 0 \) and \( \beta > \alpha^2/2k_0'' \) (real \( \Omega_g \) and \( K_g \)), exhibit rather different characteristics [Fig. 2(c)]. These modes are no longer singular and of X-type, but regular and, say, of O-type. The damped oscillations decay temporally and radially as \( 1/t \) and \( 1/r \), respectively, with periods \( 2\pi/\Omega_g \) and \( 2\pi/K_g \). The absence of singularities is a consequence of the actual limitation that the elliptic dispersion curve imposes on the uniform spectrum \( \tilde{f}(\Omega) = 1 \).

**III. CLASSIFICATION OF WAVE MODES**

Numerical integration of Eq. (8) with a given dispersion curve (specified by the values of \( \alpha \), \( \beta \) and \( k_0'' \)) but different (bell-shaped) spectral amplitude functions \( \tilde{f}(\Omega) \) having also different (but finite) bandwidths \( \Delta \Omega \) [alternatively, numerical integration of

\[
\Phi_{\alpha,\beta}(r,\tau + \alpha z) = \int_{-\infty}^{\infty} d\sigma \Phi_{\alpha,\beta}^{(i)}(r,\tau + \alpha z - \sigma)
\]  

where \( f(\tau) \) is the inverse Fourier transform of \( \tilde{f}(\Omega) \), shows much richer and complex spatiotemporal features in comparison with the case of infinite bandwidth. These features strongly depend on the choice of the spectral bandwidth \( \Delta \Omega \), while no essentially new properties arise from the specific choice of \( \tilde{f}(\Omega) \) (Gaussian, Lorentzian, two-side exponential…). Modes with finite bandwidth may exhibit mixed, more or less pronounced radial and temporal oscillations, along with incipient or strong X-wave (O-wave), focus wave mode or Bessel structure, as explained throughout this section (see also the following figures). The purpose of this section is to perform a simple, comprehensive classification of wave modes in dispersive media. In the remainder of this paper, \( \Delta \Omega \) will refer to any suitable definition of half-width of the bell-shaped spectral amplitude function \( \tilde{f}(\Omega) \).

Given a mode of parameters \( \alpha \) and \( \beta \) satisfying conditions 4 and 5, propagating in a dispersive material with GVD \( k_0'' \), and some spectral bandwidth \( \Delta \Omega \) satisfying 2, we have found it convenient to define the three following characteristic lengths: 1) the mode PM length

\[
L_p \equiv \frac{1}{\beta^2}.
\]

2) the mode walk-off, or GVM length

\[
L_w \equiv \frac{1}{\alpha \Delta \Omega},
\]

measuring, respectively, the axial distances at which the mode becomes phase mismatched and walks off with respect to a plane pulse of the same spectrum in the same medium, and 3) the GVD length

\[
L_d \equiv \frac{1}{k_0''(\Delta \Omega)^2},
\]

or distance at which the mode (invariable) duration differs significantly from that of the (broadening) plane
where \( \Omega_n \) they are the values of the mode lengths \( L \) ranges in \([-1, +1]\) for \( \Omega \) within the bandwidth \( \Delta \Omega \). Then, they are the values of the mode lengths \( L_p, L_w \) and \( L_d \) that determine the form of the dispersion curve within the spectral bandwidth, and hence the parameters that determine the spatiotemporal structure of the mode, as shown throughout this section. We analyze here three extreme cases, namely,\

- \( |L_p| \ll |L_w|, |L_d| \) PM-dominated case\
- \( |L_w| \ll |L_p|, |L_d| \) GVM-dominated case\
- \( |L_d| \ll |L_p|, |L_w| \) GVD-dominated case\

that represent three well-defined, opposite experimental situations, and that allow us also to understand, at least qualitatively, the features of general, intermediate cases.

For illustration, we have evaluated the characteristic lengths of wave modes of different frequencies \( \omega_0 \) that propagate in LBO at the phase and group velocities of the corresponding fundamental waves of half-frequency. In Figs. 4, the bandwidths \( \Delta \Omega = \omega_0 / 2\pi N \) correspond to “\( N \)-cycle” pulses [duration \( (\Delta \Omega)^{-1} \sim NT_0, T_0 = 2\pi / \omega_0 \) period] at each frequency \( \omega_0 \). The value \( N = 10 \) in Fig. 4(a) leads to a pulse duration \( (\Delta \Omega)^{-1} \sim 20 \) fs at \( \omega_0 = 3.55 \) fs\(^{-1} \) (\( \lambda = 0.53 \) \( \mu \)m), of the same order as in previous experiments and numerical simulations. Fig. 4(b) shows, in contrast, the extreme case of “single-cycle” wave modes. Generally speaking, modes of long enough duration belong to, or participate mostly of, the PM-dominated case [as in Fig. 4(a) for most frequencies], modes of some (still unspecified) intermediate duration belong to the GVM-dominated case, and extremely short modes to the GVD-dominated case, since \( L_p \) is independent on bandwidth, but \( L_p \) and \( L_d \) are inversely proportional to \( \Delta \Omega \) and \( \Delta \Omega^2 \), respectively. Depending, however, on the relative values of \( \alpha, \beta \) and \( k_0'' \) (particularly when one or two of them are very small), the GVM-dominated case, even the PM-dominated case, can extend down to the single-cycle regime [as in Fig. 4(b) for most frequencies], or, on the contrary, the GVM-dominated case, even the GVD-dominated case, apply to considerably long modes [as in the vicinity of the two singularities of the \( L_p \)-curve of Fig. 4(a)].

A. Phase-mismatch-dominated case: Pulsed Bessel beam type modes

Consider first modes with \( |L_p| \ll |L_w|, |L_d| \). When \( L_p > 0 \), the dispersion curve within the spectral bandwidth can be approached by the real constant value \( K(\Omega) \sim (2k_0L_p^{-1})^{1/2} \), or,

\[
K(\Omega) \simeq \sqrt{2k_0^3} \quad \text{(if \( \beta > 0 \)), (20)}
\]

[see Fig. 5(a)] regardless the exact dispersion curve is an actual hyperbola or ellipse [as in Fig. 5(b)], that is, independently of the sign of material group velocity dispersion. Wave modes under these conditions can only have superluminal phase velocity (\( \beta > 0 \)), but super- or subluminal group velocity (\( \alpha > 0 \) or \( \alpha < 0 \), respectively), and will adopt, from Eqs. 5 and 20, the approximate factorized form

\[
\Phi_{\alpha,\beta}(r, \tau + az) \simeq f(\tau + az)J_0 \left( \sqrt{2k_0^3}r \right) \quad (21)
\]
of a PBB of transversal size of the order of \((2k_0\beta)^{-1/2}\).

Figure 5(c) shows the prototype PBB of this kind of wave modes [Eq. 21] with a Gaussian spectrum \( \tilde{f}(\Omega) \), that is, the limiting case \( |L_p/L_w| = 0, |L_p/L_d| = 0 \), or horizontal thick lines of Figs. 5(a) and (b). In Fig. 5(d) we show, for comparison, the wave mode with \( |L_w/L_d| = 0.25, |L_p/L_d| = 0.25 \) and with the same Gaussian spectrum, obtained numerically from Eq. 3.

We see that the wave mode preserves a spatiotemporal structure similar to that of the prototype PBB of Fig. 5(c), even if \( |L_p| \) is not much smaller, but simply smaller than \( |L_w| \) and \( |L_d| \). Small differences can be understood as incipient focus wave mode and O-wave type behavior, as described in the following sections.

B. Group-velocity-mismatch-dominated case: Envelope focus wave modes
FIG. 5: (a) Dispersion curve within the bandwidth for $|L_p|/|L_w| \to 0$, $|L_p|/|L_d| \to 0$ (thick curve), and for $L_p/L_w = -0.25$, $L_p/L_d = -0.25$ (thin curve). (b) The same as in (a) but also outside the bandwidth of the Gaussian spectrum (in arbitrary units) $f(\Omega) = \exp[-(\Omega/\Delta\Omega)^2]$. (c) and (d) Gray-scale plots of the amplitude $|\Phi_{\alpha,\beta}|$ of (c) the PBB of Eq. (21) with spectrum $f(\Omega) = \exp[-(\Omega/\Delta\Omega)^2]$ (i.e., $f(\tau) \propto \exp[-(2\Delta\Omega\tau)^2]$) and (d) of the mode with $L_p/L_w = -0.25$, $L_p/L_d = -0.25$ and same spectrum as in (c), numerically calculated from Eq. (8). Normalized coordinates are $\sigma = (\tau + \alpha z)\Delta\Omega$, $\rho = r/r_0$, with $r_0 = (2k_0\alpha\beta)^{-1/2}$.

FIG. 6: (a) Dispersion curve within the bandwidth for $L_w = 10/k_0$, $L_w/L_p \to 0$, $L_w/L_d \to 0$ (thick curve), for $L_w = 10/k_0$, $L_w/L_p = 1/8$, $L_w/L_d = 1/8$ (thin curve, label 1), and for $L_w = 10k_0$, $L_w/L_p = 1/3$, $L_w/L_d = 1/3$ (thin curve, label 2). (b) The same as in (a) but also outside the bandwidth of the Gaussian spectrum (in arbitrary units) $f(\Omega) = \exp[-(\Omega/\Delta\Omega)^2]$, and of (d) of the mode with $L_w/L_p = 1/3$, $L_w/L_d = 1/3$ [thin dispersion curve 2 in (a)], numerically calculated from Eq. (8). Normalized coordinates are $\sigma = (\tau + \alpha z)\Delta\Omega$, $\rho = r/r_0$, with $r_0 = (2/k_0\Delta\Omega\alpha)^{1/2}$.

The case $|L_w| \ll |L_p|, |L_d|$ leads to a new kind of wave modes that has not been reported. The dispersion curve within the bandwidth is now of the form of the horizontal parabola $K(\Omega) \simeq (2k_0L_w^{-1}\Omega_\alpha)^{1/2}$ with vertex at $\Omega = 0$, or,

$$K(\Omega) \simeq \sqrt{2k_0\alpha\Omega}, \quad (22)$$

[see Fig. 6(a)], regardless material dispersion is normal.
[as in Fig. 6(b)] or anomalous. For modes with superluminal group velocity ($\alpha > 0$), the horizontal parabola is right-handed [as in Figs. 6(a) and (b)], and left-handed for subluminal modes ($\alpha < 0$). Independently of the group velocity, phase velocity can be superluminal ($\beta > 0$) or subluminal ($\beta < 0$). In any case, their spatiotemporal form can be approached by Eq. 5 with $K(\Omega)$ given by Eq. 22. Moreover, with the two-sided exponential spectrum $f(\Omega) = (2\pi/\Delta \Omega) \exp(-|\Omega|/\Delta \Omega)$, Eq. 8 yields

$$\Phi_{\alpha,\beta}(r, \tau + \alpha z) \simeq \frac{-i\tau_0}{\tau + \alpha z - i\tau_0} \exp\left[ \frac{i k_0 |\alpha|^2}{2(\tau + \alpha z - i\tau_0)} \right]$$

(23)

for superluminal modes ($\alpha > 0$), and the complex conjugate of the r.h.s. of Eq. 23 for subluminal modes ($\alpha < 0$). In Eq. 24, $\tau_0 \equiv (\Delta \Omega)^{-1}$ characterizes the mode duration. The mode spot size at pulse center ($\tau + \alpha z = 0$) can be characterized by $\sigma_0 = (2/k_0 \Delta \Omega |\alpha|)^{1/2}$.

The functional form of the reduced envelope in Eq. 24 is similar to the fundamental Brittigham-Ziolkowski focus wave mode (FWM) 21, 22, and as such will be called envelope focus wave mode (eFWM). There are, however, important physical differences between them, which can be understood for the respective expressions of the complete fields $E$ of both kind of waves, namely,

$$E_{\alpha,\beta}(r, z, t) \simeq \frac{-i\tau_0}{\tau - i\tau_0} \exp\left[ \frac{i k_0 \tau^2}{2(\tau - i\tau_0)} \right] \exp(-i\omega_0 t + i k_0 z),$$

(24)

for the envelope focus wave mode,

with $k_0 = \omega_0/c$, for the fundamental FWM 22. The fundamental FWM is a localized, stationary free-space wave whose envelope propagates at luminal group velocity $c$, whereas the carrier oscillations back-propagate at the same velocity $c$. The eFWM is also a stationary, localized wave with the same intensity distribution as the fundamental FWM, but propagates in a dispersive medium with super- or subluminal group velocity $1/(|k_0' - \alpha|)$. The carrier oscillations propagate in the same direction at super- or subluminal phase velocity $\omega_0/(k_0 - \beta)$.

Figure 6(c) shows the prototype eFWM of this kind of wave modes, obtained from numerical integration of Eq. 8 with the approximate dispersion curve $K(\Omega) = \sqrt{2k_0 \Omega}$ [thick curves in Figs. 6(a) and (b)] i.e., in the limiting case $|L_w/L_p| = 0$, $|L_w/L_d| = 0$, and a Gaussian spectrum. To pursue the validity of the model eFWM to describe this kind of wave modes, we have also evaluated the wave mode field in some non-limiting cases with the same Gaussian spectrum. For $|L_w/L_p| = 1/8$, $|L_w/L_d| = 1/8$ [thin curves in Figs. 6(a) and (b), label 1], the mode is nearly undistinguishable from the prototype eFWM, despite the dispersion curve differs significantly from the limiting one. Even for the relatively large ratios $|L_w/L_p| = 1/3$, $|L_w/L_d| = 1/3$ [thin curves in Figs. 6(a) and (b), label 2], the calculated wave mode [see Fig. 6(d)] exhibits the same eFWM structure, with some incipient eX-wave behavior because of the actual hyperbolic form (not parabolic) of the dispersion curve, as explained in the next section.

C. Group-velocity-dispersion-dominated case: Envelope X and envelope O type modes

1. Normal group velocity dispersion: Envelope X waves

We consider finally modes with $|L_d| \ll |L_p|, |L_w|$, or modes of short enough duration, or propagating in a medium with large enough GVD. When material dispersion is normal ($k_0'' > 0$), the dispersion curve within the bandwidth approaches the X-shaped curve [see Fig. 7(a)]

$$K(\Omega) \simeq \sqrt{k_0'' |\Omega|}$$

(26)

of the limiting case $|L_d/L_p|, |L_d/L_w| = 0$. The actual dispersion curve of a mode may be slightly shifted towards negative frequencies [as in Figs. 7(a) and (b), labels 1 and 2] or positive frequencies for modes with superluminal ($\alpha > 0$) or subluminal ($\alpha < 0$) group velocity, respectively. For modes with superluminal phase velocity ($\beta > 0$), $K(\Omega)$ is real everywhere [Fig. 7(b), label 1], but for modes with subluminal phase velocity there is a narrow frequency gap about $\Omega = 0$ [Fig. 7(b), label 2]. A prototype wave mode for this case can be obtained by introducing the approximate dispersion curve of Eq. 20 into Eq. 8. With the two-sided exponential spectrum $f(\Omega) = (2\pi/\Delta \Omega) \exp(-\Omega/\Delta \Omega)$ we obtain

$$\Phi_{\alpha,\beta}(r, \tau + \alpha z) \simeq \Re\left\{ \frac{-\tau_0}{\sqrt{k_0'' |\Omega|} + |\tau_0 + i(\tau + \alpha z)^2|} \right\}$$

(27)

where $\tau_0 \equiv (\Delta \Omega)^{-1}$ measures the pulse duration. Equation 27 is the X-wave recently described in Ref. 13 as an exact, stationary and localized solution of the paraxial wave equation with luminal phase and group velocities ($\alpha = \beta = 0$) in media with normal GVD. The X wave 27 is understood here as an approximate expression for modes with $\alpha, \beta$ such that $|L_d/L_p| \ll 1, |L_d/L_w| \ll 1$. The spatiotemporal form of the X wave is shown in Fig. 7(c). For $L_d/L_p = 1/6$ ($\beta > 0$), $L_d/L_w = 1/6$ [thin curves in Figs. 7(a) and (b), label 1], the mode retains an X-shaped structure [Fig. 7(d)] despite the dispersion curve differ significantly from the limiting one. Incipient PBB behavior, or radial oscillations, originates from the nearly horizontal dispersion curve in the central part of the spectrum. For $L_d/L_p = -1/6$ ($\beta < 0$), $L_d/L_w = 1/6$ [thin curves in Figs. 7(a) and (b), label 2], the X-shaped mode [Fig. 7(d)] shows instead incipient eFWM behavior (light is within the cone), together with temporal oscillations arising from the frequency gap in the dispersion curve.
FIG. 7: (a) Dispersion curve within the bandwidth for $L_d = 10/k_0, L_d / L_p \to 0, L_d / L_w \to 0$, with $L_d > 0$ (thick curve), for $L_d = 10/k_0, L_d / L_p = 1/6, L_d / L_w = 1/6$ (thin curve, label 1), and for $L_d = 10k_0, L_d / L_p = -1/6, L_d / L_w = 1/6$ (thin line, label 2). (b) The same as in (a) but also outside the bandwidth of the spectrum (in arbitrary units) $\hat{f}(\Omega) \propto \exp(-|\Omega|/\Delta \Omega)$. (c-e) Gray-scale plots of the amplitude $|\Phi_{\alpha,\beta}|$ of (c) the prototype eX [thick dispersion curve in (a)] with exponential spectrum $\hat{f}(\Omega) \propto \exp(-|\Omega|/\Delta \Omega)$, (d) the mode with $L_d / L_p = 1/6, L_d / L_w = 1/6$ [thin dispersion curve 1 in (a)], and (e) of the mode with $L_d / L_p = -1/6, L_d / L_w = 1/6$ [thin dispersion curve 2 in (a)]. Normalized coordinates are $\sigma = (\tau + \alpha z)\Delta \Omega, \rho = r/r_0$, with $r_0 = (k_0 k''_0 \Delta \Omega^2)^{-1/2}$.

2. Anomalous group velocity dispersion: Envelope O waves

When $|L_d| \ll |L_p|, |L_w|$ but GVD is anomalous, the dispersion curve within the bandwidth can be approached by the ellipse centered on $\Omega = 0$ [Figs. 8(a) and (b), thick curves] given by the expression

$$K(\Omega) \simeq \sqrt{2k_0(\beta - |k''_0|\Omega^2/2)}.$$  \hspace{1cm} (28)

Note that the term with $\beta$, no matter how small it is, must be retained to reproduce the real-valued part of the dispersion curve. The group velocity of the mode can be slightly subluminal ($\alpha < 0$) or superluminal ($\alpha > 0$), as in Fig. 8(a) and (b) (thin curves), but the phase velocity of these modes is always superluminal ($\beta > 0$). An approximate analytical expression for this type of modes can be obtained by introducing the approximate dispersion curve of Eq. (28) into Eq. (8). Under condition $|L_d| \ll |L_p|$, the frequency gap $\Omega_g \simeq \sqrt{2\beta/|k''_0|}$ is much smaller than $\Delta \Omega$, so that the amplitude spectrum $\hat{f}(\Omega)$ can be assumed to take a constant value in the integration domain of integral in Eq. (8), which then yields the expression

$$\Phi_{\alpha,\beta} \simeq \frac{1}{\sqrt{k_0|k''_0||r|^2 + (\tau + \alpha z)} \times \sin \left[ \sqrt{2\beta/|k''_0|} \sqrt{k_0|k''_0||r|^2 + (\tau + \alpha z)} \right],$$  \hspace{1cm} (29)

of the same form as the O-type impulse response mode in media with anomalous dispersion. Figure 8(c) shows its spatiotemporal form. For comparison, the wave mode with $L_d / L_p = -1/6, L_d / L_w = -1/8$ [Fig. 8(a), thin curve] and the two-sided exponential spectrum [Fig. 8(b)] was calculated from Eq. (8), and its O-shaped spatiotemporal form is depicted in Fig. 8(d).

IV. NONPARAXIAL DESCRIPTIONS OF WAVE MODES

The purpose of this section is to show that the preceding classification of wave modes in dispersive media in terms of the characteristic lengths remains essentially un-
altered when performed from the more exact nonparaxial approach, if condition 2 of quasi-monochromatic, and 1 and 3 of quasi-luminality group and phase velocities are satisfied.

We consider now the polychromatic Bessel beam,

$$E(r, z, t) = \frac{1}{2\pi} \int K \text{ real} \, d\omega \hat{f}(\omega - \omega_0)$$

$$\times \, J_0(Kr) \exp(ik_z z) \exp(-i\omega_0 t),$$

(30)

where $K$ and $k_z$ must be related by $K = \sqrt{k^2(\omega) - k_z^2}$ for each monochromatic Bessel beam component to satisfy the Helmholtz equation $\Delta \hat{E} + k^2(\omega) \hat{E} = 0$. Stationarity of the intensity in some moving reference frame requires the axial propagation constant $k_z$ to be a linear function of frequency $\Omega$, a condition that is suitably expressed as

$$k_z(\Omega) = (k_0 - \beta) + (k_0' - \alpha)\Omega.$$  

(31)

Equation (30) can be then rewritten in the form

$$E(r, z, t) = \Phi_{\alpha, \beta}(r, \tau + \alpha z) \exp(-i\beta z) \exp(-i\omega_0 t + ik_0 z),$$

where the reduced envelope is given by the same expression as in the paraxial case, namely,

$$\Phi_{\alpha, \beta}(r, \tau + \alpha z) = \frac{1}{2\pi} \int K(\Omega) \text{ real} \, d\Omega \hat{f}(\Omega)$$

$$\times \, J_0[K(\Omega) r] \exp(-i\Omega(\tau + \alpha z)],$$

(32)

but with a transversal dispersion relation $K(\Omega) = \sqrt{\kappa^2(\Omega) - k_z^2}$ given now by

$$K(\Omega) = \left[ (2k_0\beta - \beta^2) + 2(k_0\alpha + k_0'\beta - \alpha\beta)\Omega \right.$$

$$+ \left. (k_0k_0'' + 2k_0'\alpha - \alpha^2)\Omega^2 \right]^{1/2}$$

(33)

up to second order in dispersion $[k(\Omega) = k_0 + k_0'\Omega + k_0''\Omega^2/2]$.  

In the case of propagation in free-space $(k_0 = \omega_0/c, k_0' = 1/c, k_0'' = 0)$, with $c$ the speed of light in vacuum), Eqs. (32) and (33) yield Eq. (7) of Ref. 23 for general free-space FWMs, if the identifications $\alpha = (1 - \gamma)/c$ and $\beta = \omega_0\alpha + 2\gamma\beta_s$ are made ($\gamma$ and $\beta_s$ being the the free parameters defined in Ref. 23). In particular, the case with $\alpha = 0$ yields the original Brittigham’s FWM 21, 22, and the case with $\beta = \omega_0\alpha$ yields the Bessel-X pulse of cone angle $\theta = (2\alpha)^{1/2}$, or X wave with narrow spectral amplitude centered at an optical frequency, introduced by Saari in Ref. 17, and demonstrated in Ref. 17.

In a dispersive media, and under conditions 1 and 5 of quasi-luminality, we can neglect in Eq. the terms $\beta^2$, $\alpha\beta$ and $\alpha^2$ in comparison with $2k_0\beta$, $k_0\alpha$, and $2k_0'\alpha$, respectively, to obtain the approximate expression

$$K(\Omega) \simeq \sqrt{2(k_0 + k_0') (\beta + \alpha\Omega) + k_0k_0''\Omega^2}$$

(34)

for the nonparaxial dispersion relation of quasi-luminal modes. The first conclusion is then that the paraxial dispersion curve [Eq. 7] may significantly differ from the nonparaxial one [Eq. 34], even if conditions 1 and 5 are satisfied. In fact, it is not difficult to find set of parameters for which the nonparaxial dispersion curve is, for instance, a vertical hyperbola, whereas the paraxial dispersion curve is an horizontal hyperbola [see Fig. 8(a)].

From a physical point of view, however, it is only the portion of the dispersion curve within the mode bandwidth that is of relevance for the spatiotemporal mode
that the nonparaxial dispersion curve is, from Eq. (34),
\[ \text{Fig. 10(a) and (b) for propagation in fused silica}, \]
so \( K \) or, in terms of the mode characteristic lengths,
\[ K(\Omega) = K \equiv \sqrt{\frac{k_0^3 k_{0''}'}{k_0^2 + k_0 k_{0''}'}} \approx \sqrt{\frac{k_0^3 k_{0''}'}{k_0^2}}. \] (39)

FIG. 9: (a) Paraxial and nonparaxial transversal dispersion
curves of the modes of carrier frequency \( \omega_0 = 4 \text{ fs}^{-1} \) with \( \alpha = 300 \text{ mm}^{-1}\text{fs} \) and \( \beta = 400 \text{ mm}^{-1} \) in fused silica \( (k_0 = 19530 \text{ mm}^{-1}, k_0' = 4988 \text{ mm}^{-1} \text{fs} \) and \( k_0'' = 77 \text{ mm}^{-1} \text{fs}^2 \)). (b) The same as in (a) but only within the bandwidth of the shortest (widest spectrum), single-cycle wave mode \( (\Delta \Omega = \omega_0/2\pi) \).

structure, and, as our second conclusion, this portion is
approximately the same in the paraxial and nonparaxial
approaches if the additional condition (4) of quasi-
monochromaticity is also satisfied: Writing, for trans-
dispersive materials, \( k_0/k_0' \approx \omega_0 \), we obtain
\[ K(\Omega) \approx \sqrt{2k_0 (1 + \Omega/\omega_0)} (\beta - \alpha \Omega) + k_0 k_{0''}' \Omega^2, \] (35)

or, in terms of the mode characteristic lengths,
\[ K(\Omega) \approx \sqrt{2k_0 \left( \frac{\Delta \Omega}{\omega_0} r, \tau + \alpha z \right) \left( L_p^{-1} + L_w^{-1} \Omega_n \right) + \frac{1}{2} L_d^{-1} \Omega_n^2}. \] (36)

Since \( (\Delta \Omega/\omega_0)|\Omega_n| < 1 \), the nonparaxial dispersion
curve within the bandwidth can be approached by the
paraxial one, that is, by Eq. (19), as illustrated in
Fig. 10(b) for the extreme case (widest possible bandwidth)
of a single-cycle mode \( (\Delta \Omega/\omega_0 = 1/2\pi) \). In particular, we can
affirm that the description performed in Section 1 is
of quasi-monochromatic, quasi-luminal modes in terms of
their characteristic lengths is independent of the approach
used.

To illustrate the relationship between the paraxial and
nonparaxial approaches, and the type of results we can
expect from the paraxial one, we consider wave modes of
any bandwidth \( \Delta \Omega \) propagating in normally dispersive
media \( (k_0'' > 0) \) with
\[ \alpha = k_0' - \sqrt{k_0^2 + k_0 k_{0''}'} \approx -k_0 k_{0''}'/2k_0, \] (37)
\[ \beta = -k_0 \left( k_0' - \sqrt{k_0^2 + k_0 k_{0''}'} \right) \approx k_0^3 k_{0''}'/2k_0^2, \] (38)

[see Figs. 10(a) and (b) for propagation in fused silica], so
that the nonparaxial dispersion curve is, from Eq. (34),
the (exactly) horizontal straight line
\[ K(\Omega) = K \equiv \sqrt{\frac{k_0^3 k_{0''}'}{k_0^2 + k_0 k_{0''}'}} \approx \sqrt{\frac{k_0^3 k_{0''}'}{k_0^2}}. \] (39)

and the corresponding nonparaxial wave modes are the
dispersion-free, diffraction-free PBBs \( \Phi_{\alpha,\beta}(r, \tau + \alpha z) = f(\tau + \alpha z) J_0(Kr) \) studied in Ref. [11]. The approximate
equalities in Eqs. (37), (38) and (39) hold for weakly
dispersive materials such that \( k_0'' \ll k_0^2/k_0 \), in which case \( \alpha \) and \( \beta \) satisfy conditions (1) and (3) of quasi-luminality for the group and phase velocities. As seen in Figs. 10(a) and (b), this is the case of fused silica at any visible
carrier frequency.

For these PBBs, it is easy to see that the paraxial and
nonparaxial descriptions become indistinguishable,
in spite of the apparent drawback that PBBs are no
longer exact solutions of the paraxial wave equation in
dispersive media \( [\text{when } k_0'' \neq 0, \text{ the paraxial dispersion curve \( (7) \) is never an horizontal straight line}] \). In fact, when \( k_0'' \ll k_0^2/k_0 \), the relationship \( |L_p| \ll |L_w| \ll |L_d| \) is satisfied for any mode bandwidth down to the single-
cycle limit [see Fig. 10(c) for the case of fused silica].
Accordingly, these modes are of PBB type, that is, the
paraxial dispersion curve within the bandwidth can be
approached by an horizontal straight line [see Fig. 10(d)
for \( \omega_0 = 2 \text{ fs}^{-1} \) in fused silica]. Finally, the paraxial
prototype PBB for these modes is given, from Eq. (21), by
\( \Phi_{\alpha,\beta}(r, \tau + \alpha z) = f(\tau + \alpha z) J_0(Kr) \), with \( K = \sqrt{k_0^3 k_{0''}'/k_0^2} \),
that is, by the same expression as in the nonparaxial
approach.

V. CONCLUSIONS

Summarizing, we have described and classified the
pulsed versions of Bessel beams with the property of be-
ing localized and remaining stationary (diffraction-free
and dispersion-free) during propagation in a dispersive
material with slightly super- or subluminal phase and
group velocities. As for the wave mode description,
we have found the analysis of the transversal dispersion
curve \( K(\Omega) \) to be an useful tool to understand the spa-
tiotemporal mode structure. Wave modes have been classi-
cified into three broad categories: PBB-like, eFWM-like,
and eX-like (eO-like) modes, depending on the relative
strength of their phase and group velocity mismatch with
respect to a plane pulse, and defeated GVD, as measured
by the mode phase-mismatch length \( L_p \), group-mismatch
length \( L_w \) and the dispersion length \( L_d \).

We have verified that the paraxial description leads
to the same description and classification as would be
obtained from the more accurate nonparaxial approach
when the conditions of narrow bandwidth (2) and of
quasi-luminality (4,5) are satisfied. All previously re-
ported optical Bessel beams, X waves, Bessel-X waves,
or focus wave modes generated by linear or nonlinear
means satisfy indeed these requirements.
FIG. 10: (a) and (b) Values of $\alpha$ and $\beta$ from Eqs. (37) and (38) at different carrier frequencies in fused silica, with the refraction index obtained from Ref. 19. (c) Characteristic lengths for the limiting case of single-cycle modes ($\Delta \Omega = \omega_0/2\pi$), with $\alpha$ and $\beta$ given by Eqs. (37) and (38) at different frequencies in fused silica. (d) For $\omega_0 = 2$ fs$^{-1}$ and $\Delta \Omega = \omega_0/2\pi$, comparison between the paraxial and non-paraxial dispersion curves within the bandwidth, given, respectively, by Eqs. (7) and (33).

VI. ACKNOWLEDGEMENTS

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