Simple model of propagating flame pulsations

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ABSTRACT

We present a simple model that exhibits dynamical flame properties in one dimension. This is investigated analytically and numerically. The results are applicable to problems of flame propagation in Type Ia supernovae.

Key words: supernovae: general.

1 INTRODUCTION

Observations of distant Type Ia supernovae (SNe Ia) explosions provide us with very important information for modern cosmology (Riess et al. 1998; Schmidt et al. 1998; Komatsu et al. 2009). One probable scenario of SNe Ia is the following: the explosions are caused by a sufficiently fast thermonuclear burning of white dwarfs close to the Chandrasekhar limit (Röpke, Hillebrandt & Bildstein 2006). After many years of astrophysical research, the accepted picture is that the SN Ia explosion usually starts as a flame front, propagating slowly because of the thermal conduction (subsonic deflagration). Then, the flame is transformed — for a reason as yet unknown — into a supersonic detonation wave (Hillebrandt & Niemeyer 2000). The main observational properties of the SN Ia explosions must depend significantly on the flame propagation process, on the transition to detonation and on the instabilities that are inherent to both slow and fast propagation regimes. These are presently far from being satisfactorily understood. They pose a challenge for theoretical physics, hydrodynamics and mechanics, because the SN Ia explosions cannot be investigated experimentally, and the interiors of white dwarfs are not directly observable.

In this paper, we present a very simple model of the pulsating instability of a subsonic deflagration under conditions typical for white dwarfs. We investigate this model analytically and by means of numerical simulations. The instability already occurs in a one-dimensional geometry, so our model is one-dimensional. Because of the strong quantum degeneracy of the electron gas in white dwarfs, there is only a very weak expansion of the two-component electron–nuclei gas at the deflagration front. Combined with the one-dimensional geometry, this means that the motion of the fuel plays only a minor role; thus, we consider the fuel as motionless. The reacting species in white dwarfs are nuclei, whereas heat is transported by relativistic electrons and photons. This means that the Lewis number $Le$, characterizing the relative role of the thermal conduction and fuel diffusion, is much higher than 1: $Le \gg 1$. It also means that the diffusion of reactants can be neglected. The only relevant dimensionless number, incorporated in our model, is the so-called Zel’dovich number $Ze$. This characterizes the steepness of the reaction-rate dependence on the temperature, or the thickness of the conductive region of the flame front relative to the effective thickness of a layer with reactions. Exact definitions of the Zel’dovich number in the context of our model are presented. Usually, the subsonic deflagration becomes pulsating for sufficiently high $Ze$. Our model might help us to obtain an insight into the mechanisms of the pulsations of the subsonic deflagration. It might also provide useful tests for more or less sophisticated numerical codes used for simulations of CO burning in white dwarfs. However, it cannot give a final answer about the existence of pulsations in real flames. Our main goal is to develop a fully analytically solvable model of burning in order to check whether the critical $Ze$ number for the instability, predicted by the theory, coincides with that obtained in a numerical experiment. For the sake of analytical solvability, we crudely simplify some properties of the matter in white dwarfs. Our model can be considered as a further simplification of the model considered by Matkowsky & Sivashinsky (1978).

The pulsating regimes of the subsonic deflagration have attracted considerable attention for a long time (Zel’dovich et al. 1980; Williams 1985; Bayliss & Matkowsky 1990). Our attention to this problem was inspired additionally by a long-standing controversy between the results of Timmes & Woosley (1992) and Bychkov & Liberman (1995). Bychkov & Liberman (1995) argued that the subsonic deflagration in white dwarfs undergoes a pulsating instability. This instability can have important consequences for the deflagration-to-detonation transition. However, the numerical simulations of Timmes & Woosley (1992) did not show any instability. The difference between these two results could be caused by different and complicated data bases for nuclear reactions and/or by the peculiarities of the numerical methods. Our simple model, introduced below, might help to separate these two quite different sources of this controversy.

Here is a plan of the remainder of the paper. Section 2 is devoted to the formulation of our simple model. It also contains a travelling wave solution for arbitrary $Ze$. The linear stability of the travelling wave solution is investigated in Section 3. There, we find the critical Zel’dovich number $Ze_{cr}$, below which the travelling wave is stable against small perturbations. Section 4 is devoted to the numerical simulations of the front propagation below and above the threshold. The results of this section make it clear that adequate codes are needed for simulations of the subsonic deflagration, and for a better...
understanding of non-linear pulsating regimes of flame propagation. Our conclusions are presented in the final section of the paper.

2 MODEL

Our model includes two dependent variables: the temperature $T$ and the deficient reagent fraction $c$. There are also two independent variables: time $t$ and distance $x$. Neglecting the diffusion of the deficient reagent (which is reasonable, because the kinetic coefficients in a white dwarf are such that $Le \gg 1$), we introduce the following system of equations for $T$ and $c$:

$$
\begin{align*}
C\partial_t T &= \kappa \partial_x^2 T + W_0 \omega \Theta(T - T_0), \\
\partial_t c &= -\omega \Theta(T - T_0).
\end{align*}
$$

(1)

Here, $\Theta(\ldots)$ is the step function. These equations describe deflagration burning in solid propellants, because the two main physical processes that drive a slow front are present: the thermal conductivity and the burning itself. The equations include the following constants: $C$ is the thermal capacity of the fuel per unit volume; $W$ is the energy per unit volume of the deficient reagent fraction, released as a result of the reaction; $\omega$ is the reaction rate, with dimensions $1/s$.

Assuming that the deficient reagent fraction before ignition is equal to $c_0$, we can see that the fuel temperature after complete burnout will increase by the value

$$
T_f = \frac{Wc_0}{C}.
$$

(2)

Assuming further that the temperature $T(0)$ before ignition is much lower than $T_f$, we neglect the former, setting $T(0) = 0$.

In real white dwarfs, the thermal capacity depends on the temperature: for high density $\rho \sim 10^8 - 10^9 \text{ g cm}^{-3}$, all parameters are determined by the relativistic degenerate electrons (i.e. $C \propto T$). However, here we omit the dependence in order to make the analytical analysis possible. This approximation also concerns the coefficient of thermal conduction $\kappa$, which is determined by both electrons and photons.

The step-wise approximation of the temperature dependence of nuclear reaction rates is used in order to have linear equations in both domains $T < T_0$ and $T > T_0$. The intrinsic non-linearity of the problem is moved only boundary conditions between the two regions. This simplification allows us to investigate the problem analytically. However, we believe that our approximation of the temperature dependence represents, to some extent, a very sharp temperature dependence of nuclear reaction rates, especially for high values of the Zel’dovich number: $Ze \gg 1$ (see Fig. 1).

Usually, the Zel’dovich number $Ze$ is defined assuming the Arrhenius law for the temperature dependence of reactions rates: $w \propto \exp(-T_s/T)$. In this case, the Zel’dovich number is usually defined as $Ze = T_s/T_f$. To approximate this temperature dependence, we can equate the temperatures for both dependences where the rates become e-fold lower than their maximum values at $T = T_f$. Then, we have the following definition of $T_0$ through $T_s$ and $T_f$:

$$
Ze = \frac{T_s}{T_f}, \quad \frac{T_0}{T_f - T_0} \text{ or } T_0 = T_f \frac{T_s}{T_f + T_s} = T_f \frac{Ze}{1 + Ze}.
$$

(3)

Applying the same procedure to the power law for the reaction rate ($w \propto T^n$), we would obtain the following relationship: $T_0 = T_1 e^{-1/n} \approx T_1 (1 - n^{-1})$ (for $n \gg 1$).

We can now introduce new dimensionless variables labelled by $\tilde{\ldots}$ and defined as follows:

$$
t = \tau \tilde{t}, \quad x = \xi \tilde{x}, \quad c = c_0 \tilde{c}, \quad T = T_r \tilde{T},
$$

(4)

where

$$
I = \sqrt{\frac{\kappa}{C_\omega} \left( \frac{T_f}{T_0} - 1 \right)}, \quad \tau = \frac{Cl^2}{\kappa}.
$$

(5)

Scales for $x, t, T$ and $c$ are chosen for all characteristic parameters of the travelling wave in dimensionless units to be equal to 1. This concerns the velocity of the travelling wave, the concentration and the characteristic width of the wave. Here, we use only the dimensionless variables, skipping the tilde $\tilde{\ldots}$.

Thus, our dimensionless system is

$$
\tilde{\partial}_t \tilde{T} = \tilde{\partial}_x^2 \tilde{T} + \tilde{w}_0 \tilde{c} \Theta(T - T_0), \quad \tilde{\partial}_t \tilde{c} = -\tilde{\omega} \Theta(T - T_0),
$$

(6)

where

$$
\tilde{w}_0 = \omega_0 \tau = T_0/(1 - T_0), \quad (0 < T_0 < 1).
$$

(7)

The dimensionless value of the igniting temperature $T_0$ is used in equation (7). In initial physical units, this is equal to $T_0/T_f$. Our effective Zel’dovich number (equation 3) in dimensionless units can be expressed as

$$
Ze = \omega_0 = \frac{T_0}{1 - T_0}.
$$

(8)

The stationary travelling wave depends on $x$ and $t$ only via the combination $\xi = x - vt$. For this reason, it is convenient to introduce new spatial coordinates $\xi = x - vt$ instead of $x$, where $v$ is the constant velocity of moving frame. Thus, $T$ and $c$ now depend on $\xi$ and $t$.

It is important that system (6) is linear in both $T > T_0$ and $T < T_0$. Non-linearity appears only at the points where the transition $0 \rightarrow 1$ in the $\Theta$ function occurs. Our equations (6) mean that $T$, $\partial_t T$ and $c$ are continuous at $T = T_0$. These conditions can be treated as matching conditions for solutions of the linear equations at $T < T_0$ and $T > T_0$. As a result, we can obtain the following system of equations for monotonic in $\xi$ functions, instead of system (6). This is convenient for the investigation of the stationary travelling wave and its linear stability: for $T_+ > T_0$,

$$
\partial_t T_+ = \nu \partial_\xi T_+ + \partial_\xi^2 T_+ + \omega_0 c_+, \quad \partial_t c_+ = \nu \partial_\xi c_+ - \omega_0 c_+;
$$

(9)

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Figure 1. Comparison of the Arrhenius law for the reaction rate and our step-wise function for the same $Ze = 9$.

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for \( T_- < T_0 \),
\[
\partial_t T_- = \nu \partial_x^2 T_- - c_- = 1. \tag{10}
\]
Here,
\[
f(\xi, t) = \begin{cases} f_-(\xi, t) & \text{for } \xi > \xi_t(t), \\ f_+(\xi, t) & \text{for } \xi < \xi_t(t), \end{cases}
\]
where \( f \) represents \( T \) or \( c \), and \( \xi_t(t) \) is the position of the front, where \( T = T_0 \). The matching conditions are
\[
T_+ [\xi_t(t), t] = T_- [\xi_t(t), t] = T_0, \
\partial_x T_+ [\xi_t(t), t] = \partial_x T_- [\xi_t(t), t], \
c_+ [\xi_t(t), t] = 1. \tag{12}
\]
The stationary travelling wave obeys the following boundary conditions
\[
\xi \to \infty: \ T = 0, \ c = 1, \\
\xi \to -\infty: \ \partial_x T = 0, \ c = 0, \tag{13}
\]
and the conditions
\[
\partial_x c = \partial_x T = \xi_t(t) = 0. \tag{14}
\]
The latter equality originates from a freedom because of the translation invariance of the problem, and hence from the possibility of setting the front at an arbitrary point of the \( \xi \)-space.

The system (9)–(14) presents a non-linear eigenvalue problem, with \( v \) being the eigenvalue. There is a unique solution to this problem:
\[
v = 1 \\
\xi > 0: \ c = 1, \ T = T_0 e^{-\xi}, \\
\xi < 0: \ c = c_+ = e^{\omega t}, \ T = T_+ = 1 - \frac{1}{\omega_0 + 1} e^{\omega t}. \tag{15}
\]
As a result, we set \( v = 1 \) as a constant velocity of the moving frame for the equations (9) and (10).

We have for the travelling burning wave velocity in initial physical units:
\[
v = \sqrt{\frac{\kappa \omega}{C}} \left( \frac{T_0}{T_0} - 1 \right). 
\]

### 3 FLAME FRONT STABILITY

Here, we investigate the linear stability of the travelling wave solution obtained in the previous section. The solution (15) is now called unperturbed, and is designated by the superscript \(^{(0)}\). The full solution is presented in the form
\[
T = T^{(0)} + T^{(1)}, \quad c = c^{(0)} + c^{(1)}, \quad \xi_t = \xi^{(1)}_t, \tag{16}
\]
where \( T^{(1)} \ll T^{(0)}, c^{(1)} \ll c^{(0)} \) and \( \xi^{(1)}_t \ll 1 \). In this case, our system can be linearized. Because equations (9) and (10) are linear from the very beginning, their linearization leads only to the setting of the superscript \( ' \)\(^{(1)} \)\) and to the setting of \( v = 1 \) However, the linearization of the matching conditions is not so trivial. These are transformed to
\[
\xi^{(1)}_t \partial_t T^{(0)}_+(0, t) + T^{(1)}_+(0, t) = 0, \tag{17}
\]
\[
\xi^{(1)}_t \partial_t T^{(0)}_-(0, t) + T^{(1)}_-(0, t) = 0, \tag{18}
\]
\[
\xi^{(1)}_t \partial_t T^{(1)}_+(0, t) + \partial_t T^{(1)}_+(0, t) = \xi^{(1)}_t \partial_t T^{(0)}_+(0, t) + \partial_t T^{(1)}_+(0, t), \tag{19}
\]
\[
\xi^{(1)}_t \partial_t c^{(0)}_+(0, t) + c^{(1)}_+(0, t) = 0. \tag{20}
\]

The boundary conditions (13) give vanishing boundary conditions for \( T^{(1)}_+ \) and \( c^{(1)}_+ \) at \( \xi \to -\infty \) and for \( T^{(1)}_+ \) at \( \xi \to +\infty \). There is no source for the perturbation of \( c \) before the front, so we set \( c_- = 0 \).

Because equations (9) and (10) and the matching conditions (17)–(20), together with the boundary conditions at \( \xi \to \pm \infty \), do not contain explicit dependence on time, their solution should have the following dependence on \( t \): \( \propto e^{\omega t} \). Here, \( p \) is an eigenvalue of the corresponding eigenvalue linear boundary problem. The perturbation of the front position has a similar form. Thus
\[
\xi_t(t) = \chi e^{\omega t}. \tag{21}
\]

Equations (9) and (10) do not contain even an explicit dependence on \( \xi \). This property leads to a very simple form of the solutions at \( \xi > 0 \) and \( \xi < 0 \):
\[
T^{(1)}_- = \alpha e^{\omega t + \lambda \xi}, \tag{22}
\]
\[
c^{(1)}_+ = \beta e^{\omega t (\rho + \omega_0) \xi}, \tag{23}
\]
\[
T^{(1)}_+ = \gamma e^{\omega t (\rho - \lambda) \xi} - \frac{\omega_0}{(p + \omega_0)^2 + \omega_0} \beta e^{\omega t (\rho + \omega_0) \xi}. \tag{24}
\]

where
\[
p = \lambda^2 + \lambda, \tag{25}
\]
and the necessary condition \( \text{Re} \lambda < 0 \) to obey the boundary conditions at \( \pm \infty \).

The presentation of solution (21)–(24) contains four arbitrary constants: \( \chi, \alpha, \beta \) and \( \gamma \). However, the solution should obey the matching conditions (17)–(20). As a result, we have a homogeneous system of four linear equations for four variables. For the system of equations that has a non-trivial solution, its determinant should equal to 0. Thus, we obtain the following equation for the eigenvalue \( p \) expressed through \( \lambda \) in accordance with equation (25) under the condition that \( p + \omega_0)^2 + \omega_0 \neq 0 \):
\[
2 \omega_0 \lambda^3 + (\omega_0^2 + 2 \omega_0) \lambda^2 + 2 \omega_0^2 \lambda + \omega_0^3 = 0. \tag{26}
\]

The cubic polynomial entering into the nominator of equation (26) has a root corresponding to \( p = 0 \), which can be assumed beforehand (see below). As a result, the solution of equation (26) is reduced to a quadratic equation. Thus, we have the following set of eigenvalues in terms of \( \lambda \):
\[
\lambda_1 = -\frac{\sqrt{\omega_0^2 - 8 \omega_0} + \omega_0}{4}, \\
\lambda_2 = -\frac{\sqrt{\omega_0^2 - 8 \omega_0} - \omega_0}{4}, \\
\lambda_3 = -1. \tag{27}
\]

The third root \( \lambda_3 = -1 \) gives \( p_3 = 0, \alpha_3 = -\omega_0(1 + \omega_0), \beta_3 = 1 \) and \( \gamma_3 = 0 \). In accordance with equation (15), this eigenvalue corresponds to the small shift of the stationary travelling front as a whole. The existence of such a solution can be supposed in advance because of the translational symmetry of the problem. Thus, the physical roots are \( \lambda_1, \lambda_2 \). The real and imaginary parts of \( p_{1,2} \) versus \( \omega_0 \) are plotted in Figs 2 and 3. If \( \omega_0 \geq 8 \), so that both \( \lambda_1 \) and \( \lambda_2 \) are real, then both eigenvalues \( p_{1,2} \) are positive. If, however, \( \omega_0 < 8 \), then
\[
p_{1,2} = \frac{\omega_0^2 - 6 \omega_0 \pm \sqrt{8 \omega_0^2 - \omega_0^2}}{8}. 
\]

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4 NUMERICAL RESULTS

4.1 Original model

Such a simple system can be calculated numerically, and this is a good test for analytical predictions. We use the Crank–Nicolson finite-difference method to solve system (6). The initial and boundary conditions are set in two ways. First, a task of self-formation of the flame by a hot wall is set. In the whole region of calculation \([0; L]\), cold unburned matter is put, but the left wall is hot:

\[
T(t, x = 0) = 1, \quad T(t, x = L) = 0, \quad T[t = 0, x \in (0, L)] = 0, \quad c[t = 0, x \in (0, L)] = 1. \quad (28)
\]

If in this case a stationary front appears, then it is natural for the system. The second way is to set a distribution of the temperature of the flame by a hot wall is set. In the whole region of calculation \([0; L]\), cold unburned matter is put, but the left wall is hot:

\[
T(t, x = 0) = 1, \quad T(t, x = L) = 0, \quad T[t = 0, x \in (0, L)] = 0, \quad c[t = 0, x \in (0, L)] = 1. \quad (28)
\]

Figure 2. The complex growth rate \(p_1\) versus \(\omega_0 = Ze - 1\).

Figure 3. The complex growth rate \(p_2\) versus \(\omega_0 = Ze - 1\).

Hence, the travelling wave solution is stable against small perturbations at \(\omega_0 < 6\) and unstable in the opposite case \(\omega_0 > 6\). The eigenvalues at the threshold \(\omega_0 = 6\) are equal to \(p_{1,2} = \pm 4i/\sqrt{3}\). Thus, the perturbations at the threshold become purely oscillating.

Expressing this result in terms of the effective Zel’dovich number \(Ze\), introduced above, we can say that subsonic deflagration, having the form of the travelling wave, is stable in the frame of our model against small perturbations at \(Ze < Ze_{cr} = 6\). There is no stable stationary moving flame front at \(Ze > Ze_{cr} = 6\) in the frame of our model. As the time dependence of perturbations at the threshold \(Ze = Ze_{cr}\) is purely oscillating, we can assume that the subsonic deflagration at \(Ze > Ze_{cr} = 6\) becomes oscillating.

The evolution of the front position \(x(t)\) when \(\omega_0 = 7\) is shown in Fig. 5. According to the theory, such a regime should be unstable, which occurs in the simulation: a regime of moving with and the concentration according to the analytical solution (15), and to observe its evolution.

Numerical methods suppose discretization of space \(dx\) and time \(dt\). To obtain the physically correct solution, we should choose these quantities properly. The first condition is \(dt = 10^{-2}/\omega_0\). This implies that only 1 per cent of matter will burn at every numerical step, and therefore it prohibits abrupt changes in the solution, and in such a way controls fluctuations. The second condition comes from the analytical solution. It follows from equation (15) that there are two typical lengths in the system: 1 and \(1/\omega_0\). To resolve every change, we need \(dx \ll dx_0 = \min(1, 1/\omega_0)\). During the numerical simulation, several calculations are made with \(dx > dx_0\) to determine a consequence of wrong discretization.

The front coordinate \(x\) is determined by point \(c = 0.5\) (the definition diverges with the theoretical definition \(T = T_0\), but in the case of the stationary wave the points are located a constant distance from each other). The dependence of \(x(t)\) for the simulation with \(\omega_0 = 1\) (\(dx \ll dx_0\)) is shown in Fig. 4. A good linear curve means that the front moves with a constant velocity and is stable. By fitting \(dx/dt\), the front velocity can be found (i.e. \(v = 1.00\)), which is a very good agreement with the theory.

However, it must be noted that despite good linear dependence, observed on large scales in Fig. 4, small scales exhibit pulsations. These pulsations are a fully numerical effect, because their period is exactly \(T = dx/v\) (this can easily be tested by simulations with different \(dx\)).

The evolution of the front position \(x(t)\) when \(\omega_0 = 7\) is shown in Fig. 5. According to the theory, such a regime should be unstable, which occurs in the simulation: a regime of moving with
non-constant velocity interchanges with the regime of the front standing. For the model (6), the front will not move after \( t = 20 \). This fact is interpreted below.

Table 1 presents the results for the first way of setting the initial and boundary conditions. The comments column in the table shows the propagation regime: ‘flame’ means the flame propagation; ‘therm’ is the evolution of medium parameters as in Fig. 6 (which corresponds to the unstable regime). In this case, the temperature undergoes evolution like thermoconductivity without burning. A combination of the terms ‘flame’ and ‘therm’ in Table 1 means that there are transition stages in these cases before an ultimate regime is established. The ultimate regime is the last word in the combination. The results for the analytical solution as an initial condition are shown in Table 2. From those two tables, we see that \( \omega_0 = 6 \) is a critical point for the system parameters. When \( \omega_0 < 6 \), a stationary front can exist in a system. When \( \omega_0 > 6 \), the front appears, but it has non-constant velocity and lives for a limited period of time. The dependence \( x(t) \) for such a front (in the case of the first boundary condition) is shown in Fig. 5.

Here, we should emphasize simulations when \( dx > dx_0 \) is set. Two runs (\( \omega_0 = 1 \), \( dx = 1.5 \) and \( \omega_0 = 4 \), \( dx = 0.5 \)) show that a wrong \( dx \) results in wrong behaviour, such as an incorrect front velocity or an instability-like regime.

### 4.2 Modification of the simple model

The model suffers from some non-physical effects: when \( T < T_0 \), the burning rate drops to zero. This is why the flame stops at a certain moment of time in the unstable regime and never runs again (only heating by the left wall could ignite further burning). Let us consider a small modification of the burning rate:

\[
\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + R(c, T), \quad \frac{\partial c}{\partial t} = -R(c, T),
\]

(29)

\[
R(c, T) = \omega_0 c \Theta(T - T_0) + \omega_1 c T^2 \Theta(T_0 - T) \Theta(T),
\]

(30)

with \( \omega_1 \ll \omega_0 \). This modification allows burning at all temperatures, which is more physically correct. The condition \( \omega_1 \ll \omega_0 \) implies that the model correction does not influence the stationary flame and the previous theory. So, when \( \omega < \omega_0 \), the flame spreads with a constant velocity. When \( \omega > \omega_0 \), the evolution changes. First, according to previous calculations, the front decays and smooths in the ‘thermoconductivity’ regime. However, after this, a slow burning in the region \( T < T_0 \) raises the temperature to the critical value and the flame blazes up again. An example of such propagation for \( \omega_0 = 8 \) is shown in Fig. 7; the front position versus time is shown in Fig. 8.

This evolution is called front pulsation. Such a regime is slightly different from ‘classical’ pulsations: \( t_{fr} = t_0 + v_1 \sin at \). Here, the flame stops, blazes up and stops again. It moves by jerks. It is interesting that the intervals between the jerks are considerably longer than the value \( 2\pi/\text{Imp} \) determined in the linear theory presented above. This relationship, as well as the jerky motion of the front, takes place even for \( \omega_0 \), which is only slightly higher than the threshold \( \omega_0 = 6 \).

### Table 1. Results for wall ignition.

| \( \omega_0 \) | \( dx \) | \( v \) | Comments |
|---|---|---|---|
| 1.0 | 0.05 | 1.000 | Flame |
| 4.0 | 0.05 | 0.996 | Flame |
| 5.0 | 0.05 | 0.992 | Flame |
| 5.5 | 0.05 | 0.993 | Therm–flame |
| 5.8 | 0.05 | 0.993 | Therm–flame |
| 6.0 | 0.05 | 6.15–1.24 | Therm–flame |
| 6.5 | 0.05 | 5.56–0.99 | Therm–flame–therm |
| 7.0 | 0.05 | 6.40–1.04 | Therm–flame–therm |
| 8.0 | 0.05 | 4.32–1.08 | Therm–flame–therm |
| 1.0 | 1.5 | 0.711 | Flame |
| 4.0 | 0.5 | 4.74–0.94 | Therm–flame–therm |

### Table 2. Analytical initial conditions.

| \( \omega_0 \) | \( dx \) | \( v \) | Comments |
|---|---|---|---|
| 1.0 | 0.05 | 1.000 | Flame |
| 4.0 | 0.05 | 0.996 | Flame |
| 5.5 | 0.02 | 1.006 | Flame |
| 5.8 | 0.02 | 1.010 | Flame |
| 6.0 | 0.01 | 1.019 | Flame |
| 7.0 | 0.01 | – | Therm |
| 8.0 | 0.01 | – | Therm |
| 9.0 | 0.01 | – | Therm |
| 1.0 | 1.5 | 0.711 | Flame |
| 4.0 | 0.5 | – | Therm |
Figure 8. Front propagation in terms of $x(t)$ for $\omega_0 = 8$ in the modified model (solid line). For comparison, the dotted line shows moving with constant velocity $v = 1$.

5 DISCUSSION AND CONCLUSIONS

In this paper, we have presented a simple analytically solvable model for flame propagation, which exhibits a different behaviour depending on the dimensionless parameter $Ze$ – the effective Zel’dovich number expressed through our dimensionless parameter $\omega_0$. The theory is compared with numerical simulations, and a good agreement is observed. When $Ze < 6$, both analytical and numerical solutions exhibit a constant-velocity front. Our analytical theory shows that the travelling flame front becomes unstable at $Ze > 6$, whereas our numerical simulations show that the front is destroyed and passes to jerk-like pulsations.

We believe that, in the case of pulsating instability in a real flame, the characteristic behaviour will be the same as in this model, but the critical $Ze$ can be different. Therefore, the proposed model can mimic the behaviour of numerical methods, and serve as a test for them. Our present work cannot say anything about the existence of such pulsations in real white dwarfs. However, if such a pulsating jerk-like regime of slow flame propagation indeed takes place in white dwarfs, then it could be able to trigger the transition to detonation.

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