On Global Asymptotic Stability for the Diffusive Carr–Penrose Model

Joseph G. Conlon\textsuperscript{1} \textsuperscript{a} · Michael Dabkowski\textsuperscript{2}

Received: 28 February 2022 / Accepted: 5 August 2022 / Published online: 23 August 2022
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2022

Abstract
This paper is concerned with large time behavior of the solution to a diffusive perturbation of the linear LSW model introduced by Carr and Penrose. Like the LSW model, the Carr–Penrose model has a family of rapidly decreasing self-similar solutions, depending on a parameter $\beta$ with $0 < \beta \leq 1$. It is shown that if the initial data have compact support, then the solution to the diffusive model at large time approximates the $\beta = 1$ self-similar solution. This result supports the intuition that diffusion provides the mechanism whereby the $\beta = 1$ self-similar solution of the LSW model is the only physically relevant one.

Keywords Nonlinear PDE · Linear diffusions · Coarsening

Mathematics Subject Classification 35F05 · 82C70 · 82C26

1 Introduction

In this paper we continue our study of the diffusive Carr–Penrose (CP) model introduced in Conlon et al. (2016). The model is obtained by adding a second-order diffusion term with coefficient $\epsilon/2 > 0$ to the Carr–Penrose equation (Carr and Penrose 1998). The density function $c_\epsilon(x, t)$ evolves according to a linear diffusion equation, subject to the linear mass conservation constraint as follows:

Communicated by Rustum Choksi.

\textsuperscript{a} Joseph G. Conlon
conlon@umich.edu

Michael Dabkowski
mgdabkowski@umich.edu

1 Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1109, USA
2 Department of Mathematics and Statistics, University of Michigan-Dearborn, Dearborn, MI 48128, USA
\[
\frac{\partial c_\varepsilon(x,t)}{\partial t} = \frac{\partial}{\partial x} \left\{ \left[ 1 - \frac{x}{\Lambda_\varepsilon(t)} \right] c_\varepsilon(x,t) \right\} + \varepsilon \frac{\partial^2 c_\varepsilon(x,t)}{\partial x^2}, \quad x > 0, \quad (1.1)
\]

\[
\int_0^\infty xc_\varepsilon(x,t)dx = 1. \quad (1.2)
\]

We also need to impose a boundary condition at \( x = 0 \) to ensure that (1.1), (1.2) with given initial data \( c_\varepsilon(\cdot,0) \), satisfying the constraint (1.2) has a unique solution. We impose the Dirichlet boundary condition \( c_\varepsilon(0,t) = 0, \quad t > 0. \) This condition has the advantage that the parameter \( \Lambda_\varepsilon(t) > 0 \) in (1.1) is given by a simple formula

\[
\Lambda_\varepsilon(t) = \int_0^\infty xc_\varepsilon(x,t)dx \int_0^\infty c_\varepsilon(x,t)dx = 1 / \int_0^\infty c_\varepsilon(x,t)dx. \quad (1.3)
\]

In Sect. 2 where we discuss the more physically relevant Becker–Döring (BD) (Becker and Döring 1935) and Lifschitz–Slyozov–Wagner (LSW) (Lifschitz and Slyozov 1961; Wagner 1961) models, we give some justification for diffusive models with zero Dirichlet condition. Penrose (1997) argued that solutions of the super-critical BD model are at large time approximate solutions to the LSW model. This claim was given some rigorous justification by Niethammer (2003), but much remains to be understood. In particular, the large time behavior of solutions to the LSW model is conjectured to be given by one of a family of self-similar solutions to LSW (Conlon and Dabkowski 2020). These can be conveniently parametrized by a single number \( \beta \) with \( 0 < \beta \leq 1 \). Already in the original papers (Lifschitz and Slyozov 1961; Wagner 1961) it was claimed that the only physically relevant self-similar solution to the LSW model is the critical one with \( \beta = 1 \). Therefore one expects following Penrose (1997) that solutions to the super-critical BD model converge at large time to the critical equilibrium, which forms a boundary layer close to the origin, with the time evolution of the excess mass being well approximated by the \( \beta = 1 \) self-similar solution to the LSW model. The only rigorous result known in this direction was obtained in Ball et al. (1986), where for the super-critical BD model weak convergence to the critical equilibrium is established. This implies that the excess mass drifts to infinity at large time, but nothing more precise. The problem of determining on what time scale the LSW time evolution becomes a good approximation for the time evolution of the excess mass appears to be extremely subtle. This has been illustrated by some examples in the paper of Penrose (1989).

In Conlon and Schlichting (2019) we introduced a continuous nonlinear Fokker–Planck (NFP) model which bears many similarities to the discrete BD model. In fact it is shown in Conlon and Schlichting (2019) that there is a continuous interpolation from the BD to the NFP model. The interpolated models are like BD also discrete, but on the chain \( \varepsilon \mathbb{Z}^+ \) with \( 0 < \varepsilon \leq 1 \). In this interpolation the BD model is given by \( \varepsilon = 1 \), and the NFP model by the limit as \( \varepsilon \to 0 \). The main result of Conlon and Schlichting (2019) is the proof of convergence to equilibrium, so a continuous analogue of the main result of Ball et al. (1986). The methodology is also similar, using the fact that a free energy functional exists, which decreases along trajectories of the solution to NFP. Comparison of the BD system to a continuous diffusive system was first considered by Velázquez (1998). It was argued further in the physics literature.
(Meerson 1999; Rubinstein and Zaltzman 2000) that diffusion may be the mechanism of the selection principle, whereby the $\beta = 1$ self-similar solution of LSW describes the asymptotic behavior of the excess mass in the super-critical BD model and some other models of Ostwald ripening (Pego 2007).

In the NFP model the Fokker–Planck equation is solved with a nonzero Dirichlet boundary condition, which couples to a parameter in the PDE and conservation law. In Sect. 2 we observe that one can introduce a second parameter into the model, which does not affect the Fokker–Planck dynamical law or conservation law, but sends the Dirichlet boundary condition to zero as this parameter goes to infinity. Therefore it is reasonable to expect that the study of the zero Dirichlet boundary condition model yields some insight into the NFP model. Furthermore, since it is also a diffusive model one expects that the selection principle for the $\beta = 1$ self-similar solution of LSW continues to operate. In the present paper we verify this is the case for the much simpler diffusive CP model (1.1), (1.2).

The system (1.1), (1.2) with $\varepsilon \geq 0$ can be interpreted as an evolution equation for the probability density function (pdf) of random variables. Thus let us assume that the initial data $c_\varepsilon(x, 0) \geq 0$, $x > 0$, for (1.1), (1.2) satisfy $\int_0^\infty c_\varepsilon(x, 0) \, dx < \infty$, and let $X_{\varepsilon,0}$ be the non-negative random variable with pdf $c_\varepsilon(\cdot, 0)/\int_0^\infty c_\varepsilon(x, 0) \, dx$. The conservation law (1.2) implies that the mean $\langle X_{\varepsilon,0} \rangle$ of $X_{\varepsilon,0}$ is finite, and this is the only absolute requirement on the variable $X_{\varepsilon,0}$. If for $t > 0$ the variable $X_{\varepsilon,t}$ has pdf $c_\varepsilon(\cdot, t)/\int_0^\infty c_\varepsilon(x, t) \, dx$, then (1.3) implies that $\Lambda_\varepsilon(t) = \langle X_{\varepsilon,t} \rangle$, and hence (1.1) becomes an evolution equation for the pdf of $X_{\varepsilon,t}$.

There is an infinite one-parameter family of self-similar solutions to (1.1), (1.2) with $\varepsilon = 0$. Using the normalization $\langle X_0 \rangle = 1$, the initial data for these solutions are given by

$$P(X_0 > x) = \begin{cases} \left[1 - (1 - \beta)x\right]^{\beta/(1-\beta)}, & 0 < x < 1/(1 - \beta), \\ e^{-x}, & \beta = 1, \\ \left[1 + (\beta - 1)x\right]^{\beta/(1-\beta)}, & 0 < x < \infty, \end{cases} \quad \text{if } 0 < \beta < 1, \beta = 1, \beta > 1. \quad (1.4)$$

The random variable $X_t$ corresponding to the evolution (1.1), (1.2) with $\varepsilon = 0$ and initial data (1.4) is then given by

$$X_t = \langle X_t \rangle X_0, \quad \frac{d}{dt}\langle X_t \rangle = \beta. \quad (1.5)$$

The main result of Carr and Penrose (1998) is that a solution of (1.1), (1.2) with $\varepsilon = 0$ converges at large time to the self-similar solution with parameter $\beta$, provided the initial data and the self-similar solution of parameter $\beta$ behave in the same way at the end of their supports. The main result of this paper is that a similar result holds for the diffusive model (1.1), (1.2) with $\varepsilon > 0$ and initial data of compact support. In this case a selection principle operates, so that the asymptotic behavior is given by the $\beta = 1$ self-similar solution (1.4) corresponding to the exponential variable:

**Theorem 1.1** Assume the initial data $c_\varepsilon(\cdot, 0)$ for (1.1), (1.2) are non-negative, integrable and has compact support. Let $X_{\varepsilon,t}$, $t > 0$, be the random variable...
corresponding to the solution \( c_\varepsilon(\cdot, t) \) of (1.1), (1.2) with Dirichlet boundary condition \( c_\varepsilon(0, t) = 0 \). Then

\[
\frac{X_{\varepsilon,t}}{\langle X_{\varepsilon,t} \rangle} \xrightarrow{D} X \quad \text{as} \quad t \to \infty , \quad \lim_{t \to \infty} \frac{d}{dt} \langle X_{\varepsilon,t} \rangle = 1 , \tag{1.6}
\]

where \( X \) is the exponential random variable with mean 1 and \( \xrightarrow{D} \) denotes convergence in distribution.

A result analogous to Theorem 1.1 was proved in Conlon et al. (2016) for a reduced model which we denoted the inviscid CP model, since a corresponding viscous CP model is equivalent to the diffusive CP model (1.1), (1.2). The evolution of the inviscid model is determined by solving a first-order nonlinear PDE. This PDE is essentially the \( \varepsilon = 0 \) CP Eq. (1.1) with the addition of a quadratic nonlinearity. The method of characteristics may be used to solve this Burgers’ type equation in the case of no shocks. Hence no boundary condition is needed at \( x = 0 \) to guarantee uniqueness of the solution. In Sect. 3 we introduce a diffusive model, also without boundary condition at \( x = 0 \), by solving (1.1) on the whole line \(-\infty < x < \infty\) with initial data which is zero on the half line \( x < 0 \) and non-negative for \( x > 0 \). The parameter \( \Lambda_\varepsilon(t) \) in (1.1) is given by the first formula on the RHS of (1.3). The conservation law (1.2) now no longer holds. We prove a result analogous to Theorem 1.1 for this model.

Our approach in Sect. 3 of the current paper to proving (1.6) for the whole line model follows along similar lines to that in §3 of Conlon et al. (2016) for proving (1.6) in the inviscid model. A key tool in both proofs is the beta function of a random variable, which is defined by (3.23), (3.24). For any positive random variable \( X \) with finite mean one associates a non-negative function \( \beta_X(\cdot) \) with domain \([0, \|X\|_\infty)\). This function satisfies an invariance property under affine transformations of \( X \). Furthermore, the rate of coarsening in the \( \varepsilon = 0 \) CP model at time \( t \) is given by a simple formula in terms of the beta function of the distribution of the solution of the CP model at time \( t \) (see (2.36), (2.37) of Conlon et al. (2016)). This formula also extends to the LSW model (see (1.4) of Conlon and Dabkowski (2020)). The invariant distributions (1.4) for the \( \varepsilon = 0 \) CP model have \( \beta_{X_0}(\cdot) \equiv \beta \).

There is a close relationship between boundedness properties of the beta function of a random variable \( X \) and log concavity properties of its pdf. The log concavity property for \( X \) holds if and only if \( \sup \beta_X(\cdot) \leq 1 \). Equivalently, the function \( x \mapsto E[X - x \mid X > x], \ 0 < x < \|X\|_\infty \), is decreasing if and only if \( \sup \beta_X(\cdot) \leq 1 \). In Conlon et al. (2016) the limits (1.6) are established in the inviscid model for initial data random variable \( X_{\varepsilon,0} \) such that \( \sup \beta_{X_{\varepsilon,0}}(\cdot) \leq 1 \). This corresponds to the situation in which the solution of the Burgers’ type equation contains no shocks, and hence the equation may be solved by the method of characteristics. In the case of initial random variable \( X_{\varepsilon,0} \) with density of compact support as in the assumptions for Theorem 1.1, the solution of the Burgers’ type equation does in general contain shocks. However viewed from large time the initial data seem like a delta distribution, corresponding to \( \beta \to 0 \) in (1.4).

To extend the proof of Theorem 1.1 beyond initial data with compact support, it seems that the first step should be to establish (1.6) in the case of log concave initial...
data. Some results along these lines have already been proved in Conlon et al. (2016). We show there that if the initial data \( c_\varepsilon(\cdot, 0) \) for (1.1) with zero Dirichlet condition satisfy the log concavity property, then \( c_\varepsilon(\cdot, t) \) also has the log concavity property for all \( t > 0 \). Equivalently, if the beta function for \( X_{\varepsilon, 0} \) is bounded by 1 then the beta function for \( X_{\varepsilon, t} \) is bounded by 1 for all \( t > 0 \). Since the coarsening rate \( \Lambda_1(0) \) can be estimated in terms of the beta function of \( X_{\varepsilon, t} \), this implies a uniform upper bound on the rate of coarsening as \( t \to \infty \). Finally, one expects that (1.6) continues to hold for initial data \( X_{\varepsilon, 0} \) which is a “small” perturbation of log concave data.

The remainder of the paper consists of generalizing the results of the whole line diffusive CP model to the half line model with zero Dirichlet condition at \( x = 0 \). One can most easily see how this introduces additional subtlety into the problem, by comparing the formulas for the rates of coarsening in the \( \varepsilon = 0 \) CP and diffusive \( \varepsilon > 0 \) models. On differentiating (1.3) we see that if \( \varepsilon = 0 \), then

\[
\frac{d\Lambda_0(t)}{dt} = c_0(0, t) \left[ \int_0^\infty c_0(x, t) \, dx \right]^2,
\]

(1.7)

whereas if \( \varepsilon > 0 \), the formula is given by

\[
\frac{d\Lambda_\varepsilon(t)}{dt} = \varepsilon \frac{\partial c_\varepsilon(0, t)}{\partial x} \left[ \int_0^\infty c_\varepsilon(x, t) \, dx \right]^2.
\]

(1.8)

It clearly follows from (1.7) that the function \( t \to \Lambda_0(t) \) is increasing. We also see from (1.8) and the maximum principle (Protter and Weinberger 1984) applied to (1.1) that the function \( t \to \Lambda_\varepsilon(t) \) is strictly increasing if \( \varepsilon > 0 \).

In comparing the CP to the diffusive CP model, an obvious question to ask is if the limit of the solution to the diffusive model on a fixed time interval \( 0 < t \leq T \), with parameter \( \varepsilon > 0 \) and given initial data independent of \( \varepsilon \), converges as \( \varepsilon \to 0 \) to the solution of the CP model with the same initial data. A strong convergence result was proven in Conlon et al. (2016), and a similar result for a diffusive LSW model in Conlon (2010). In Sect. 2 we explain how the main theorem of Niethammer (2003) may be interpreted as an analogous result for the BD model. In the proof of convergence for the diffusive CP and LSW models a boundary layer analysis is necessary. We can see this from (1.7), (1.8) since these equations suggest that

\[
\lim_{\varepsilon \to 0} \frac{\varepsilon}{2} \frac{\partial c_\varepsilon(0, t)}{\partial x} = c_0(0, t).
\]

(1.9)

From (1.9) we expect there is a boundary layer of size \( O(\varepsilon) \) at the origin, within which the function \( x \to c_\varepsilon(x, t) \) increases rapidly from 0 to \( c_0(0, t) \). The analysis of this boundary layer is the main source of difficulty in proving convergence.

Theorem 1.1 concerns large time behavior of solutions to the diffusive problem. That is \( \varepsilon > 0 \) is fixed, and we are interested in the behavior of solutions \( c_\varepsilon(\cdot, t) \) to (1.1), (1.2) with Dirichlet condition \( c_\varepsilon(0, \cdot) = 0 \) as \( t \to \infty \). There is a close relation between this problem and the problem of the convergence of \( c_\varepsilon(\cdot, t) \) as \( \varepsilon \to 0 \) on a fixed time interval \( 0 < t \leq T \). The reason is that \( \lim_{t \to \infty} \Lambda_\varepsilon(t) = \infty \). Since the \( \varepsilon = 0 \)
CP model is scale invariant, rescaling of the PDE (1.1) so that the mean mass at large time $T$ is $O(1)$ makes the diffusion constant in (1.1) small while leaving the other terms the same. Therefore large time behavior $T \to \infty$ of $c_{\varepsilon}(\cdot, T)$ may be estimated by means of $\varepsilon \to 0$ analysis, provided we can establish certain uniformity properties on the solution as $T \to \infty$. Here as in previous papers (Conlon and Dabkowski 2020; Conlon et al. 2016) uniformity is measured in terms of the boundedness of $\beta_{X_{\varepsilon,t}}(\cdot)$ as $t \to \infty$.

The main technical issues of this paper are concerned with the ratio (4.1) of the Dirichlet Green’s function for (1.1) on the half line to the whole line Green’s function. The whole line Green’s function is explicitly given by the Gaussian (3.4). The relevant results for the ratio (4.1) are established in Conlon and Dabkowski (2021). The most crucial of these is the log concavity property (1.17) in Theorem 1.1 of Conlon and Dabkowski (2021). This allows us to generalize bounds on the rate of coarsening, established for the whole line problem in Lemma 3.1, to the half line problem with Dirichlet boundary condition. It also enables us to obtain uniform bounds on $\beta_{X_{\varepsilon,t}}(\cdot)$ as $t \to \infty$. To carry this out we need various results for conditional Gaussian expectations. These are proved in Appendix of the current paper.

In the whole line problem the convergence in distribution (1.6) to the exponential variable follows in a straightforward way from the coarsening bounds proved in Lemma 3.1. In order to extend this result to the half line problem we need good control over the ratio of Green’s functions in a neighborhood of the origin, which is uniform for all large time. This is the subject of Sect. 5. Our approach follows the method of Conlon and Dabkowski (2021), first proving results for the semiclassical limit $\varepsilon \to 0$ of the ratio. These results are then extended to any finite interval $0 < \varepsilon \leq \varepsilon_0$ of diffusivity. Hence the results of Sect. 5 are uniform in $\varepsilon$ for $0 < \varepsilon \leq \varepsilon_0$ and time $T$ in any range $0 < T_0 \leq T < \infty$.

The most technically involved argument in the paper is the proof of convergence in the rate of coarsening $\lim_{t \to \infty} d\Lambda_{\varepsilon}(t)/dt = 1$. The reason is that it involves showing convergence as $\varepsilon \to 0$ of a derivative of the ratio of Green’s functions, as one can see from (1.8). The relevant result in Conlon and Dabkowski (2021) is Theorem 1.2. Because of the uniform bounds and convergence properties already obtained on $\beta_{X_{\varepsilon,t}}(\cdot)$ as $t \to \infty$, it is sufficient by rescaling to reduce the problem to a question of convergence as $\varepsilon \to 0$ on a finite time interval $0 < T \leq T_0$.

## 2 Relation to the BD, NFP and LSW Models

The NFP model introduced in Conlon and Schlichting (2019) is a continuous version of the BD model in which the dynamics is given by a Fokker–Planck (FP) equation on the positive real line with a time varying parameter. The parameter is coupled to a Dirichlet boundary condition and a conservation law. The FP equation is given by

$$
\partial_t c(x, t) + \partial_x (b(x, t) c(x, t)) = \partial_x^2 (a(x) c(x, t)), \quad x, t > 0,
$$

(2.1)
where the drift \( b(\cdot, \cdot) \) is of the form
\[
b(x, t) = a(x)[\theta(t)W'(x) - V'(x)]. \tag{2.2}
\]
The functions \( V(\cdot), W(\cdot) \) are assumed to be positive \( C^1 \) and increasing. The conservation law is given by
\[
A\theta(t) + \int_0^\infty W(x)c(x, t) \, dx = \rho, \quad \text{where } A \geq 0 \text{ is constant.} \tag{2.3}
\]
It is easy to see that \( c(x, t) = c_\theta^{\text{eq}}(x) \) where
\[
c_\theta^{\text{eq}}(x) = a(x)^{-1} \exp(-V(x) + \theta W(x)), \tag{2.4}
\]
is an equilibrium solution to (2.1), (2.2). The Dirichlet boundary condition is given in terms of this equilibrium solution by
\[
c(0, t) = c_\theta^{\text{eq}}(0), \quad t > 0. \tag{2.5}
\]
The NFP model consists of solving (2.1) subject to the constraints (2.3), (2.5) with given non-negative initial data. Differentiating (2.3) with respect to \( t \) and using (2.1), we see that (2.1)–(2.5) may be considered an infinite dimensional dynamical system \( t \mapsto [\theta(t), c(\cdot, t)] \). Note that the boundary condition (2.5) also forms part of the dynamical law.

Observe that if we replace the potential \( V(\cdot) \) with \( V(\cdot) + M \) where \( M \) is a constant, the PDE (2.1) and conservation law (2.3) do not change. However the boundary condition (2.5) becomes
\[
c(0, t) = e^{-M a(0)^{-1} \exp(-V(0) + \theta(t)W(0))}. \tag{2.6}
\]
If we let \( M \to \infty \) then (2.6) yields the zero Dirichlet condition in the limit. Thus we have a continuous interpolation between the NFP model and a diffusive model with zero Dirichlet condition.

One can see from the formulation of the BD model in Conlon (2010), that if \( a(\cdot), V(\cdot), W(\cdot) \) are given by the formulae
\[
a(x) = (1 + x)^\alpha, \quad 0 < \alpha < 1, \quad W(x) = 1 + x, \quad a(x)V'(x) = 1, \quad V(0) = 0 \tag{2.7}
\]
then the NFP model can be considered a continuous version of the BD model. In the case of (2.7) equilibrium solutions (2.4) of the NFP model, which satisfy the conservation law (2.3), exist for all \( \theta \leq 0 \). The \( \theta = 0 \) equilibrium solution is the critical equilibrium. Let \( \rho_{\text{crit}} \) be the value of \( \rho \) in (2.3) which corresponds to the critical equilibrium, so
\[
\int_0^\infty W(x)c_0^{\text{eq}}(x) \, dx = \rho_{\text{crit}}. \tag{2.8}
\]
In Conlon and Schlichting (2019) it is shown that for \( \rho > \rho_{\text{crit}} \) in (2.3) the solution \( c_\varepsilon(\cdot, t) \) of (2.1)–(2.5) converges weakly as \( t \to \infty \) to \( c_0^{\text{eq}}(\cdot) \). The main tool in the proof is the free energy functional \( G \) defined by

\[
G(c(\cdot), \theta) = \int_0^\infty \left[ \log \frac{c(x)}{c_0^{\text{eq}}(x)} - 1 \right] c(x) \, dx + \frac{A\theta^2}{2},
\]  

which decreases along orbits \( t \to c(\cdot, t) \) of the solution to (2.1)–(2.5). One can easily see this by writing (2.1) as

\[
\partial_t c(x, t) = \partial_x \left( a(x) c(x, t) \partial_x \log \frac{c(x, t)}{c_0^{\text{eq}}(\theta(t))(x)} \right).
\]  

(2.10)

Letting \( D \) be the dissipation functional

\[
D(c(\cdot), \theta) = \int_0^\infty a(x) \left( \partial_x \log \frac{c(x)}{c_0^{\text{eq}}(\theta(t))(x)} \right)^2 c(x) \, dx,
\]  

(2.11)

we see from (2.3), (2.5) and (2.9), (2.10) that

\[
\frac{d}{dt} G(c(\cdot), \theta(t)) = -D(c(\cdot), \theta(t)).
\]  

(2.12)

Recently an alternative continuous model has been proposed by Goudon and Monasse (2020). This model is quite similar to the NFP model, but the time varying parameter which occurs in the conservation law and boundary condition also enters the second derivative term in the Fokker–Planck PDE, in addition to the first derivative term as in (2.1), (2.2). The parameter in Goudon and Monasse (2020) seems to represent mass concentrated at the origin, which is similar to the situation with the BD model. However there does not appear to be a free energy functional which decreases along orbits of the solution. Therefore a proof of convergence at large time to equilibrium is not available by Lyapunov’s second method as was the case in Ball et al. (1986) and Conlon and Schlichting (2019).

A more closely related model is that studied by Niethammer and Pego (2005). This is a generalized LSW model, which can be obtained from the NFP model by eliminating the second derivative term in (2.1), yielding a first-order PDE. The boundary condition (2.5) is no longer required, but the conservation law remains as in (2.3). Global existence and uniqueness of solutions are proved, as well as continuous dependence on the initial data. An important tool is the energy functional \( E \) defined by

\[
E(c(\cdot), \theta) = \int_0^\infty V(x) c(x) \, dx + \frac{A\theta^2}{2}.
\]  

(2.13)
Defining the dissipation functional by

\[ \mathcal{D}(c(\cdot), \theta) = \int_0^\infty a(x)[\theta W'(x) - V'(x)]^2 c(x) \, dx, \]  

we see that along orbits \( t \to c(\cdot, t) \) of the solution one has

\[ \frac{d}{dt} \mathcal{E}(c(\cdot, t), \theta(t)) = -\mathcal{D}(c(\cdot, t), \theta(t)), \]  

provided \( V(0) = W(0) = 0. \)

We next introduce a small parameter \( \varepsilon > 0 \) into the NFP model with the goal of proving convergence on a fixed time scale to the LSW model as \( \varepsilon \to 0. \) Since we want this to be related to the problem of large time behavior of the NFP model, we scale the solution \( c(\cdot, \cdot) \) of (2.1)–(2.5) by setting

\[ c_{\varepsilon}(x, t) = \gamma(\varepsilon) - \frac{1}{\varepsilon} c(x/\varepsilon, t/\varepsilon), \]  

where

\[ a_{\varepsilon}(x) = \varepsilon a(x/\varepsilon), \quad b_{\varepsilon}(x, t) = b(x/\varepsilon, t/\varepsilon) = a_{\varepsilon}(x)[\theta_{\varepsilon}(t) W_{\varepsilon}(x) - V_{\varepsilon}(x)], \]  

\[ V_{\varepsilon}(x) = V(x/\varepsilon), \quad \theta_{\varepsilon}(t) W_{\varepsilon}(x) = \theta(t/\varepsilon) W(x/\varepsilon). \]  

This defines \( \theta_{\varepsilon}(\cdot), \ W_{\varepsilon}(\cdot) \) up to a multiplicative constant. The constant is determined from the scaled conservation law

\[ A_{\varepsilon} \theta_{\varepsilon}(t) + \int_0^\infty W_{\varepsilon}(x)c_{\varepsilon}(x, t) \, dx = \rho. \]  

Thus we have that

\[ A_{\varepsilon} = \varepsilon^{-1} \gamma(\varepsilon) A, \quad \theta_{\varepsilon}(t) = \varepsilon \gamma(\varepsilon)^{-1} \theta(t/\varepsilon), \quad W_{\varepsilon}(x) = \varepsilon^{-1} \gamma(\varepsilon) W(x/\varepsilon). \]  

The boundary condition (2.5) becomes

\[ c_{\varepsilon}(0, t) = [\gamma(\varepsilon) a(0)]^{-1} \exp[-V(0) + \theta(t/\varepsilon) W(0)] = c_{\varepsilon,0}(t, \varepsilon), \]  

where the equilibrium density \( c_{\varepsilon,0}(\cdot) \) is given by

\[ c_{\varepsilon,0}(x) = [\gamma(\varepsilon) \varepsilon^{-1} a_{\varepsilon}(x)]^{-1} \exp(-V_{\varepsilon}(x) + \theta W_{\varepsilon}(x)). \]  

Note that

\[ \int_0^\infty W_{\varepsilon}(x)c_{0,\varepsilon}(x) \, dx = \rho_{\text{crit}}, \]
where $\rho_{\text{crit}}$ is given by (2.8). We see from (2.9) that the free energy $G_\varepsilon$ corresponding to (2.16)–(2.21) is given by

$$G_\varepsilon(c(\cdot), \theta) = \int_0^\infty \left[ \log \frac{c(x)}{c_{0,\varepsilon}(x)} - 1 \right] c(x) \, dx + \frac{A_\varepsilon \theta^2}{2}. \quad (2.23)$$

Note from (2.9), (2.23) that

$$G_\varepsilon(c_\varepsilon(\cdot, t), \theta_\varepsilon(t)) = \varepsilon \gamma(\varepsilon)^{-1} G(c(\cdot, t/\varepsilon), \theta(t/\varepsilon)). \quad (2.24)$$

We consider now the situation where the coefficients are given by (2.7). We choose $\gamma(\varepsilon)$ such that $\lim_{\varepsilon \to 0} W_\varepsilon(\cdot) = W_0(\cdot)$ exists. Evidently we should take $\gamma(\varepsilon) = \varepsilon^2$, in which case $W_0(x) = x$. Next we need that $\lim_{\varepsilon \to 0} a_\varepsilon(x) \theta_\varepsilon(t) W'_\varepsilon(x)$ exists. Defining $\tilde{\theta}_\varepsilon(t) = \varepsilon^{1-\alpha} \theta_\varepsilon(t)$, we see the limit exists if $\lim_{\varepsilon \to 0} \tilde{\theta}_\varepsilon(t) = \tilde{\theta}_0(t)$ exists. The conservation law (2.18) then becomes

$$A_\varepsilon \alpha \tilde{\theta}_\varepsilon(t) + \int_0^\infty (\varepsilon + x) c_\varepsilon(x, t) \, dx = \rho. \quad (2.25)$$

Letting $\varepsilon \to 0$ and assuming $\lim_{\varepsilon \to 0} c_\varepsilon(x, t) = c_0(x, t)$, $\lim_{\varepsilon \to 0} \tilde{\theta}_\varepsilon(t) = \tilde{\theta}_0(t)$, we see that $c_0(\cdot, \cdot)$ is formally a solution to the LSW equation

$$\frac{\partial c_0(x, t)}{\partial t} + \frac{\partial}{\partial x} \left[ \tilde{\theta}_0(t) x^\alpha - 1 \right] = 0. \quad (2.26)$$

The formal limit of the conservation law (2.25) yields

$$\int_0^\infty x \, c_0(x, t) \, dx = \rho. \quad (2.27)$$

The issue of the limiting behavior as $\varepsilon \to 0$ of solutions $c_\varepsilon$ to (2.16)–(2.21) with $a(\cdot), V(\cdot), W(\cdot)$ as in (2.7) is actually more subtle than indicated due to the fact that the boundary condition $c_\varepsilon(0, t) = c_{eq, \varepsilon}(\theta_\varepsilon(t), \varepsilon(0))$ tends to force $c_\varepsilon$ to blow up as $\varepsilon \to 0$ since it is given by the formula

$$c_\varepsilon(0, t) = \varepsilon^{-2} \exp[\varepsilon^\alpha \tilde{\theta}_\varepsilon(t)]. \quad (2.28)$$

This happens unless the initial data within the boundary layer $x = O(\varepsilon)$ is close to the equilibrium $c_{0,\varepsilon}(\cdot)$, hence depending on $\varepsilon$. Outside the boundary layer it may be taken independent of $\varepsilon$. Assuming the initial data have this property, we expect that for $t > 0$ one has

$$\int_0^{M_\varepsilon} (\varepsilon + x) c_\varepsilon(x, t) \, dx = \rho_{\text{crit}} + O(1/M) \quad \text{if } M \text{ is large}. \quad (2.29)$$
Letting $\varepsilon \to 0$ in (2.25) and assuming $\lim_{\varepsilon \to 0} c_\varepsilon(x, t) = c_0(x, t)$ for $x > 0$ and $\lim_{\varepsilon \to 0} \tilde{\theta}_\varepsilon(t) = \tilde{\theta}_0(t)$, we see from (2.29) that the limit of the conservation law (2.25) is given by

$$
\int_0^\infty x \, c_0(x, t) \, dx = \rho - \rho_{\text{crit}},
$$

(2.30)

instead of (2.27).

Niethammer (2003) (and Schlichting (2019) by a different method) has rigorously shown in the context of the Becker–Döring model that a weak limit $\lim_{\varepsilon \to 0} c_\varepsilon$ does exist and is a solution to the LSW model (2.26), (2.30). Closeness of the initial data to equilibrium within the boundary layer is measured in terms of the order of magnitude of the excess free energy $G_\varepsilon(c_\varepsilon(\cdot, 0), 0) - G_\varepsilon(c_{eq, 0}, 0)$ as $\varepsilon \to 0$. With $a(\cdot), V(\cdot), W(\cdot)$ as in (2.7) we have that

$$
e^1 - \alpha G_\varepsilon(c_\varepsilon(\cdot, t), \theta_\varepsilon(t)) = \int_0^\infty \left\{ \frac{(\varepsilon + x)^{1-\alpha} - \varepsilon^{1-\alpha}}{1-\alpha} \right\} c_\varepsilon(x, t) \, dx
$$

$$
+ \varepsilon^{1-\alpha} \left[ \int_0^\infty \log c_\varepsilon(x, t) - 1 \right] \rho_\varepsilon(x, t) \, dx + \int_0^\infty \log \left[ e^{2-\alpha}(\varepsilon + x)^\alpha \right] c_\varepsilon(x, t) \, dx 
$$

$$
+ \frac{1}{2} \varepsilon^\alpha A \tilde{\theta}_\varepsilon(t)^2.
$$

(2.31)

If $\lim_{\varepsilon \to 0} c_\varepsilon = c_0$ exists, then the first term on the RHS of (2.31) converges as $\varepsilon \to 0$ to the LSW energy $(1-\alpha)^{-1} \int_0^\infty x^{1-\alpha} c_0(x, t) \, dx$ corresponding to (2.13). Niethammer’s main condition is that $\varepsilon^{1-\alpha}$ times the excess free energy is bounded independent of $\varepsilon > 0$. It is quite easy to come up with a large class of initial data which satisfy the Niethammer condition. Thus suppose $x \mapsto c_0(x, 0), x \geq 0$, is a continuous non-negative integrable function which satisfies (2.30) with $t = 0$. We define initial data for the system (2.16)–(2.21) as

$$
c_\varepsilon(x, 0) = c_{eq, \varepsilon}(x), \quad \text{if } 0 \leq x < \varepsilon \left[ M \log(1/\varepsilon) \right]^{1/(1-\alpha)},
$$

$$
c_\varepsilon(x, 0) = c_0(x, 0), \quad \text{if } x \geq \varepsilon \left[ M \log(1/\varepsilon) \right]^{1/(1-\alpha)},
$$

(2.32)

where the constant $M$ satisfies $M > 1$. For the initial data (2.32) it follows from (2.22), (2.25), (2.30) at $t = 0$ that $\tilde{\theta}_\varepsilon(0) = O[\varepsilon^{1-\alpha}]$. Thus the final term on the RHS of (2.31) is $O[\varepsilon^{2-\alpha}]$ as $\varepsilon \to 0$ and hence bounded independent of $\varepsilon$. If we subtract from the middle term the corresponding quantity with $c_\varepsilon(\cdot, 0)$ replaced by $c_{eq, \varepsilon}(\cdot)$ we obtain an integral over the interval $\varepsilon \left[ M \log(1/\varepsilon) \right]^{1/(1-\alpha)} < x < \infty$. Evidently this difference is $O[\varepsilon^{1-\alpha}]$, whence again uniformly bounded as $\varepsilon \to 0$. Finally if we subtract from the first term the corresponding quantity with $c_\varepsilon(\cdot, 0)$ replaced by $c_{eq, \varepsilon}(\cdot)$ we obtain the LSW energy for $c_0(\cdot, 0)$ plus $O[\varepsilon^{1-\alpha}]$. Assuming Niethammer’s method applies to the present situation, one expects it to imply that the solution $c_\varepsilon$ to (2.16)–(2.21) with initial data (2.32) converges weakly to the corresponding solution of the LSW model (2.26), (2.30).
So far there are no proofs of strong convergence of solutions to (2.16)–(2.21) with initial data (2.32). However strong convergence for a closely related problem (with $\alpha = 1/3$) was proven in Conlon (2010). The important difference between the system studied in Conlon (2010) and the present one lies in the boundary condition. The diverging Dirichlet condition (2.28) is replaced by the zero one $c_\varepsilon(0, t) = 0$. There are some other minor differences: in Conlon (2010) the conservation law differs from (2.18) in taking $A_\varepsilon = 0$ and replacing $W_\varepsilon$ by $W_\varepsilon \alpha - 1 a_\varepsilon(x) W'_\varepsilon(x) = (\varepsilon + x)^\alpha$ is replaced by $x^\alpha$.

As far as understanding large time behavior of the system (2.1)–(2.5), the replacement of the Dirichlet condition (2.5) by the zero Dirichlet condition should simplify the problem considerably. One also expects that good understanding of the zero Dirichlet case will yield significant insight into the nonzero Dirichlet case. Another important property of the zero Dirichlet boundary condition model is that the Kohn–Otto argument (Kohn and Otto 2002) may be applied to yield a global weak upper bound on the rate of coarsening. This has been shown for the model in Conlon (2010). However the argument is delicate and does not easily extend to the slightly modified model with coefficients (2.7) and conservation law (2.3) considered here.

3 The Whole Line Problem

We consider here the problem of solving the PDE (1.1) on the whole line $\mathbb{R} = (-\infty < x < \infty)$ with initial data which is non-negative on the positive half line $\{x > 0\}$ and zero on the negative half line $\{x < 0\}$. The parameter function $\Lambda_\varepsilon(\cdot)$ is given by (1.3), whence the problem is nonlinear. We have now instead of (1.4) the formula

$$\frac{1}{\Lambda_\varepsilon(t)} \frac{d\Lambda_\varepsilon(t)}{dt} = \left\{ \frac{c_\varepsilon(0, t)}{2} \left[ \frac{\partial c_\varepsilon(0, t)}{\partial x} + \frac{c_\varepsilon(0, t)}{\Lambda_\varepsilon(t)} \right] \right\} \int_0^\infty c_\varepsilon(x, t) dx. \tag{3.1}$$

We shall see below that (3.1) implies $\Lambda_\varepsilon(\cdot)$ is an increasing function, although $\partial c_\varepsilon(0, t)/\partial x$ may be negative.

There is an explicit formula for the whole line Green’s function for (1.1). Let $A : [0, \infty) \to \mathbb{R}$ be a continuous function and define related functions $m_{1,A}, m_{2,A}, \sigma_A^2 : [0, \infty) \to \mathbb{R}^+$ by

$$m_{1,A}(T) = \exp \left[ \int_0^T A(s') ds' \right], \quad m_{2,A}(T) = \int_0^T \exp \left[ \int_s^T A(s') ds' \right] ds,$$

$$\sigma_A^2(T) = \int_0^T \exp \left[ 2 \int_s^T A(s') ds' \right] ds. \tag{3.2}$$

The solution to (1.1) on the whole line with initial data $c_\varepsilon(x, 0)$, $x \in \mathbb{R}$, which is supported on the half line $\mathbb{R}^+$, has the representation

$$c_\varepsilon(x, T) = \int_0^\infty G_\varepsilon(x, y, 0, T) c_\varepsilon(y, 0) dy, \quad x \in \mathbb{R}, \ T > 0. \tag{3.3}$$
The Green’s function \( G_\varepsilon(x, y, 0, T) \) is defined (see eqn. (2.5) of Conlon and Dabkowski (2021)) by the formula

\[
G_\varepsilon(x, y, 0, T) = \frac{1}{\sqrt{2\pi\varepsilon\sigma_A^2(T)}} \exp \left[ -\frac{(x + m_{2,A}(T) - m_{1,A}(T)y)^2}{2\varepsilon\sigma_A^2(T)} \right],
\]

(3.4)

where \( m_{1,A}, m_{2,A}, \sigma_A^2 \) are as in (3.2) and \( A(\cdot) \equiv 1/\Lambda_\varepsilon(\cdot) \). Observe now that

\[
\left[ 1 + \frac{\varepsilon}{2} \frac{\partial}{\partial x} \right] G_\varepsilon(x, y, 0, T) \bigg|_{x=0} = \left[ 1 - \frac{m_{2,A}(T)}{2\sigma_A^2(T)} + \frac{m_{1,A}(T)y}{2\sigma_A^2(T)} \right] G_\varepsilon(0, y, 0, T).
\]

(3.5)

The RHS of (3.5) is non-negative for \( y > 0 \) if \( A(\cdot) \geq 0 \), and consequently from (3.3) the RHS of (3.1) is also non-negative. We have shown that the function \( \Lambda_\varepsilon(\cdot) \) is increasing.

**Lemma 3.1** Assume \( c_\varepsilon(x, 0), \ x \in \mathbb{R}, \) is a non-negative function satisfying \( c_\varepsilon(x, 0) = 0 \) for \( x < 0 \), and the integrability condition

\[
0 < \int_0^\infty (1 + x)c_\varepsilon(x, 0) \, dx < \infty.
\]

(3.6)

Let \( c_\varepsilon(\cdot, t), \ t > 0, \) be the solution to the whole line PDE (1.1) with the constraint (1.3) and initial data \( c_\varepsilon(\cdot, 0) \). Then the function \( \Lambda_\varepsilon(\cdot) \) is increasing and \( \lim_{T \to \infty} \Lambda_\varepsilon(T) = \infty \). If in addition \( c_\varepsilon(\cdot, 0) \) has compact support, then \( \lim_{T \to \infty} m_{1,A}(T) = \infty \), \( \lim_{T \to \infty} \frac{m_{2,A}(T)}{m_{1,A}(T)} = \infty \), \( \lim_{T \to \infty} \frac{\sigma_A^2(T)}{m_{2,A}(T)} = \infty \), and \( \lim_{T \to \infty} \frac{\sigma_A^2(T)}{m_{1,A}(T)^2} < \infty \), where \( A(\cdot) \equiv 1/\Lambda_\varepsilon(\cdot) \).

**Proof** Since \( \Lambda_\varepsilon(\cdot) \) is increasing we have that \( \lim_{T \to \infty} \Lambda_\varepsilon(T) = \Lambda_\infty \leq \infty \). We assume \( \Lambda_\infty < \infty \) and obtain a contradiction. With \( A(\cdot) = 1/\Lambda_\varepsilon(\cdot) \), we see from (3.2) that the functions \( m_{1,A}, m_{2,A}, \sigma_A^2 \) satisfy

\[
\lim_{T \to \infty} m_{1,A}(T) = \infty, \quad \Lambda_\varepsilon(0) \leq \lim_{T \to \infty} \frac{m_{2,A}(T)}{m_{1,A}(T)} \leq \Lambda_\infty, \quad \frac{\Lambda_\varepsilon(0)}{2} \leq \lim_{T \to \infty} \frac{\sigma_A^2(T)}{m_{1,A}(T)^2} \leq \frac{\Lambda_\infty}{2}.
\]

(3.7)

We can obtain a formula for the RHS of (1.3) in terms of expectations of the random variable \( X_{\varepsilon,y,T} \) given by

\[
X_{\varepsilon,y,T} = m_{1,A}(T)y - m_{2,A}(T) + \sqrt{\varepsilon}\sigma_A(T)Z,
\]

(3.8)
where $Z$ is the standard normal variable. This follows by observing that

$$\begin{align*}
\int_0^\infty c_\varepsilon(x, T) \, dx &= \int_0^\infty P(X_{\varepsilon,y,T} > 0)c_\varepsilon(y, 0) \, dy, \\
\int_0^\infty xc_\varepsilon(x, T) \, dx &= \int_0^\infty E[X_{\varepsilon,y,T} \mid X_{\varepsilon,y,T} > 0]P(X_{\varepsilon,y,T} > 0)c_\varepsilon(y, 0) \, dy.
\end{align*}$$

(3.9)

We have now that

$$E[X_{\varepsilon,y,T} \mid X_{\varepsilon,y,T} > 0] = \sqrt{e}\sigma_A(T)E[Z - z_{y,T}/\sqrt{e} \mid Z > z_{y,T}/\sqrt{e}],$$

$$z_{y,T} = \frac{m_{2,A}(T) - m_{1,A}(T)y}{\sigma_A(T)}.$$  

(3.10)

Writing

$$z_{y,T} = a(T) - b(T)y,$$  

(3.11)

we see from (3.7) that

$$0 < \inf_{T \geq 1} a(\cdot) \leq \sup_{T \geq 1} a(\cdot) < \infty, \quad 0 < \inf_{T \geq 1} b(\cdot) \leq \sup_{T \geq 1} b(\cdot) < \infty.$$  

(3.12)

We also have from Lemma A.1 there is a constant $c_\varepsilon > 0$ such that

$$E[Z - z \mid Z > z] \geq \max\{c_\varepsilon, -z\} \quad \text{for } z \leq \sup_{T \geq 1} a(\cdot)/\sqrt{e}.$$  

(3.13)

It follows from (1.3) and (3.9)–(3.13) that

$$\Lambda_\varepsilon(T) \geq c_\varepsilon \sqrt{e}\sigma_A(T).$$  

(3.14)

Since (3.7) implies that $\lim_{T \to \infty} \sigma_A(T) = \infty$, we conclude from (3.14) that $\lim_{T \to \infty} \Lambda_\varepsilon(T) = \infty$, contradicting our assumption that $\Lambda_\infty < \infty$.

Having shown that $\lim_{T \to \infty} \Lambda_\varepsilon(T) = \infty$, we next show that $\lim_{T \to \infty} m_{1,A}(T) = \infty$. This implies that $\Lambda_\varepsilon(T)$ cannot grow too rapidly with $T$. Since $m_{1,A}(\cdot)$ is an increasing function we have that $\lim_{T \to \infty} m_{1,A}(T) = m_{1,A}(\infty) \leq \infty$. Arguing again by contradiction we assume that $m_{1,A}(\infty) < \infty$. Using the identities

$$\frac{m_{2,A}(T)}{m_{1,A}(T)} = \int_0^T \frac{dt}{m_{1,A}(t)}, \quad \frac{\sigma_A^2(T)}{m_{1,A}(T)^2} = \int_0^T \frac{dt}{m_{1,A}(t)^2},$$  

(3.15)

we see that the functions $a(\cdot), b(\cdot)$ of (3.11) satisfy inequalities

$$c_1 \sqrt{T} \leq a(T) \leq C_1 \sqrt{T}, \quad c_1 \leq \sqrt{T} b(T) \leq C_1 \quad \text{for } T > 0,$$  

(3.16)
where $c_1, C_1 > 0$ are constants. Assume now that $c_ε(·, 0)$ has support in the interval $[0, y_∞]$. It follows from (3.10), (3.16) and Lemma A.1 there are constants $C_2, T_2 > 0$ such that

$$E[X_{ε,y,T} | X_{ε,y,T} > 0] ≤ C_2 \text{ for } T ≥ T_2, \ y ∈ [0, y_∞]. \quad (3.17)$$

From (1.3), (3.9) we conclude that $\lim_{T→∞} Λ_{ε}(T) < ∞$, a contradiction.

Next we show that $\lim_{T→∞} m_{2,A}(T)/m_{1,A}(T) = ∞$. Note from (3.15) this implies $m_{1,A}(T)$ cannot approach $∞$ too rapidly, which in turn implies a lower bound on the rate of growth of $Λ_{ε}(T)$. To see this we assume for contradiction that $\lim_{T→∞} m_{2,A}(T)/m_{1,A}(T) = m_∞ < ∞$, whence (3.15) and the inequality $m_{1,A}(·) ≥ 1$ implies that $\lim_{T→∞} σ^2_A(T)/m_{1,A}(T)^2 ≤ m_∞ < ∞$. It follows that the functions $a(·), b(·)$ of (3.11) satisfy the inequality (3.12). Hence $z_{y,T}$ is uniformly bounded for $y ∈ [0, y_∞]$ as $T → ∞$. We conclude from (3.10)–(3.12) there are positive constants $c_3, T_3$ such that

$$E[X_{ε,y,T} | X_{ε,y,T} > 0] ≥ c_3 m_{1,A}(T) \text{ for } T ≥ T_3, \ y ∈ [0, y_∞]. \quad (3.18)$$

It follows from (1.3), (3.9), (3.18) that

$$Λ_{ε}(T) ≥ c_3 m_{1,A}(T) \text{ for } T ≥ T_3. \quad (3.19)$$

From (3.15), (3.19) there is a constant $C_4 > 0$ such that

$$\log m_{1,A}(T) = \int_0^T \frac{dt}{Λ_{ε}(t)} ≤ C_4 \frac{m_{2,A}(T)}{m_{1,A}(T)} \text{ for } T ≥ T_3. \quad (3.20)$$

Since we have already established that $\lim_{T→∞} m_{1,A}(T) = ∞$, we conclude from (3.20) that $m_∞ = ∞$, contradicting our original assumption.

It follows from (3.15), and the fact that $m_{1,A}(·) ≥ 1$, $\lim_{T→∞} m_{2,A}(T)/m_{1,A}(T) = ∞$ that $a(T), b(T) \in (3.11)$ satisfy $\lim_{T→∞} a(T)/b(T) → ∞$ and $\lim_{T→∞} a(T) = ∞$. We see then from (3.10) and Lemma A.1 there are constants $C_3, T_3 > 0$ such that

$$E[X_{ε,y,T} | X_{ε,y,T} > 0] ≤ C_3 \frac{σ^2_A(T)}{m_{2,A}(T)} \text{ for } T ≥ T_3, \ y ∈ [0, y_∞]. \quad (3.21)$$

Now (3.9), (3.21) imply that $Λ_{ε}(T) ≤ C_3 σ^2_A(T)/m_{2,A}(T)$, whence we conclude that $\lim_{T→∞} σ^2_A(T)/m_{2,A}(T) = ∞$. From (3.15) we see that $σ^2_A(T)/m_{2,A}(T) ≤ m_{1,A}(T)$, whence (3.21) implies that $Λ_{ε}(T) ≤ C_3 m_{1,A}(T)$. We have then from (3.15) there is a constant $C_4$ such that

$$\frac{σ^2_A(T)}{m_{1,A}(T)^2} ≤ C_4 \int_0^T \frac{dt}{m_{1,A}(t)Λ_{ε}(t)} = C_4 \left[ 1 - \frac{1}{m_{1,A}(T)} \right]. \quad (3.22)$$

□

Springer
We recall from Conlon et al. (2016) some functions associated with positive random variables $X$ with finite mean $\langle X \rangle < \infty$. Let us assume that $X$ has integrable pdf proportional to a function $c_X : (0, \infty) \to \mathbb{R}^+$. We define functions $w_X, h_X$ with domain $(0, \infty)$ by

$$w_X(x) = \int_x^\infty c_X(x') \, dx', \quad h_X(x) = \int_x^\infty w_X(x') \, dx'. \quad (3.23)$$

The beta function $\beta_X$ associated with $X$ also has domain $(0, \infty)$ and is defined by

$$\beta_X(x) = \frac{c_X(x) h_X(x)}{w_X(x)^2}. \quad (3.24)$$

It is easy to see from (3.23), (3.24) that $E[X - x \mid X > x] = h_X(x)/w_X(x)$ and

$$- \frac{d}{dx} E[X - x \mid X > x] = 1 - \beta_X(x). \quad (3.25)$$

Furthermore, if we define $v_X(x) = E[X - x \mid X > x]^{-1}$ then

$$h_X(x) = h_X(0) \exp \left[ - \int_0^x v_X(x') \, dx' \right]. \quad (3.26)$$

In Proposition 3.1 we show convergence in distribution of $X_{\varepsilon,t}/\langle X_{\varepsilon,t} \rangle$ and also $\beta_{X_{\varepsilon,t}}(\cdot)$ as $t \to \infty$ to the corresponding quantities for the exponential variable. These convergences correspond to convergence of $X_{\varepsilon,t}$ to the exponential variable in various metrics. Consider the set $\mathcal{M}$ of non-negative random variables and a set $\mathcal{H}$ of functions $h : (0, \infty) \to \mathbb{R}$. If $\mathcal{H}$ is a large enough set then we may define an associated metric $d_\mathcal{H}$ on $\mathcal{M}$ by

$$d_\mathcal{H}(X, Y) = \sup_{h \in \mathcal{H}} |E[h(X)] - E[h(Y)]|, \quad X, Y \in \mathcal{M}. \quad (3.27)$$

Convergence in distribution corresponds to convergence in the metric (3.27) for $\mathcal{H}$ the space of characteristic functions of intervals $[0, x), \ x > 0$. If $\mathcal{H}$ is the set of differentiable functions satisfying $\|h'\|_\infty \leq 1$ then (3.27) is a Wasserstein metric. In view of (3.26) we may also define a metric $d_\beta$ associated to beta functions by

$$d_\beta(X, Y) = \sup_{0 \leq x < \infty} |\beta_X(x) - \beta_Y(x)|, \quad X, Y \in \mathcal{M}. \quad (3.28)$$

for $X, Y \in \mathcal{M}$ with infinite support so $\|X\|_\infty = \|Y\|_\infty = \infty$. The metric (3.28) is dilation invariant in the sense that $d_\beta(\lambda X, \lambda Y) = d_\beta(X, Y)$ for $\lambda > 0$. In contrast, the metrics (3.27) are dilation invariant only if the set $\mathcal{H}$ is dilation invariant in the sense that $h \in \mathcal{H}, \ \lambda > 0$ implies that $h_\lambda \in \mathcal{H}$ where $h_\lambda(x) = h(\lambda x), \ x > 0$. Proposition 3.1 shows that $\beta_{X_{\varepsilon,t}}(\cdot)$ converges in the metric (3.28) as $t \to \infty$ to the exponential function. A corresponding result for the LSW model has been proven in Conlon and Dabkowski (2020).
Proposition 3.1 Assume \( c_\varepsilon(x, 0), \ x \in \mathbb{R}, \) has compact support, satisfies the conditions of Lemma 2.1 and \( c_\varepsilon(\cdot, t), \ t > 0, \) is the solution to (1.1), (1.3). Let \( X_{\varepsilon, t} \) be the non-negative random variable with pdf proportional to \( c_\varepsilon(x, t), \ x > 0, \) and \( \mathcal{X} \) be the exponential variable with mean 1. Then

\[
\frac{X_{\varepsilon, t}}{\langle X_{\varepsilon, t} \rangle} \xrightarrow{D} \mathcal{X}, \quad \text{as } t \to \infty.
\] (3.29)

In addition one has

\[
\lim_{t \to \infty} \|\beta_{X_{\varepsilon, t}}(\cdot) - 1\|_\infty = 0.
\] (3.30)

Proof We have from Lemma 3.1 that the functions \( a(\cdot), b(\cdot) \) of (3.11) satisfy \( \lim_{T \to \infty} a(T) = \infty, \lim_{T \to \infty} b(T)/a(T) = 0. \) Hence \( z_{y, T} \) given by (3.10) satisfies \( \lim_{T \to \infty} z_{y, T} = +\infty, \) and the limit is uniform for \( y \in [0, y_\infty]. \) Let \( \tilde{X}_{\varepsilon, y, T} \) be the random variable \( X_{\varepsilon, y, T} \) of (3.8) conditioned on \( X_{\varepsilon, y, T} > 0. \) It follows from Lemma A.2 that

\[
\frac{\tilde{X}_{\varepsilon, y, T}}{\langle \tilde{X}_{\varepsilon, y, T} \rangle} \xrightarrow{D} \mathcal{X} \quad \text{as } T \to \infty.
\] (3.31)

The limit in (3.30) is uniform for \( y \in [0, y_\infty]. \) We also have from Lemma A.1 that

\[
\frac{\langle \tilde{X}_{\varepsilon, y, T} \rangle}{\langle \tilde{X}_{\varepsilon, 0, T} \rangle} \to 1 \quad \text{as } T \to \infty,
\] (3.32)

and the limit is uniform for \( y \in [0, y_\infty]. \) We conclude from (3.9), (3.32) that

\[
\frac{\langle \tilde{X}_{\varepsilon, y, T} \rangle}{\langle X_{\varepsilon, T} \rangle} \to 1 \quad \text{as } T \to \infty,
\] (3.33)

and the limit is uniform for \( y \in [0, y_\infty]. \) We have from (3.3), (3.8) that

\[
P \left( \frac{X_{\varepsilon, T}}{\langle X_{\varepsilon, T} \rangle} > x \right) = \int_0^{y_\infty} P \left( \frac{X_{\varepsilon, y, T}}{\langle X_{\varepsilon, y, T} \rangle} > x \right) c_\varepsilon(y, 0) \ dy \int_0^{y_\infty} P(X_{\varepsilon, y, T} > 0) c_\varepsilon(y, 0) \ dy
\]
\[
= \int_0^{y_\infty} P \left( \frac{\tilde{X}_{\varepsilon, y, T}}{\langle \tilde{X}_{\varepsilon, y, T} \rangle} > x \right) \left[ \int_0^{y_\infty} P(X_{\varepsilon, y, T} > 0) c_\varepsilon(y, 0) \ dy \right] \ dy.
\] (3.34)

The convergence in distribution (3.29) then follows from (3.31)–(3.34).
The limit of the beta function in (3.30) already follows from Lemma 2.1 of Conlon et al. (2016) in the case when $\delta(\cdot, 0)$ is a point distribution. For $\delta(\cdot, 0)$ supported in an interval $[0, \gamma_\infty]$ we write $\beta(x, T) = A_\varepsilon(x, T)C_\varepsilon(x, T)/B_\varepsilon(x, T)^2$. The function $A_\varepsilon$ is given by the formula

$$
A_\varepsilon(x, T) = \int_0^{\gamma_\infty} \exp \left[ \frac{b(T)xy}{\varepsilon \sigma_A(T)} \right] \tilde{\varepsilon}_\varepsilon(y, 0) \, dy ,
$$

where $\tilde{\varepsilon}_\varepsilon(y, 0) = \exp \left[ \frac{a(T)b(T)y - b(T)^2y^2}{2\varepsilon} \right] \varepsilon(y, 0) , \tag{3.35}

and $a(\cdot), b(\cdot)$ are given by (3.10), (3.11). The functions $B_\varepsilon, C_\varepsilon$ are given by the formulae

$$
B_\varepsilon(x, T) = \int_0^{\gamma_\infty} dy \int_0^{\gamma_\infty} dx' \exp \left[ \frac{b(T)(x + x')y}{\varepsilon \sigma_A(T)} - \frac{a(T)x'}{\varepsilon \sigma_A(T)} - \frac{x'(2x + x')}{2\varepsilon A(T)} \right] \tilde{\varepsilon}_\varepsilon(y, 0) ,

C_\varepsilon(x, T) = \int_0^{\gamma_\infty} dy \int_0^{\gamma_\infty} dx' x' \exp \left[ \frac{b(T)(x + x')y}{\varepsilon \sigma_A(T)} - \frac{a(T)x'}{\varepsilon \sigma_A(T)} - \frac{x'(2x + x')}{2\varepsilon A(T)} \right] \tilde{\varepsilon}_\varepsilon(y, 0). \tag{3.36}

$$

Observe now that for $\delta > 0$ small one has

$$
\int_0^{\gamma_\infty} e^{-z - \delta z^2/2} \, dz = 1 - \delta + O(\delta^2) ,

\int_0^{\gamma_\infty} ze^{-z - \delta z^2/2} \, dz = 1 - 3\delta + O(\delta^2). \tag{3.37}
$$

From (3.36), (3.37) we see there exists $T_0 \geq 1$ sufficiently large such that for $T \geq T_0$,

$$
B_\varepsilon(x, T) = [1 - \delta_\varepsilon(x, T)] \int_0^{\gamma_\infty} \left[ \frac{a(T) - b(T)y}{\varepsilon \sigma_A(T)} + \frac{x}{\varepsilon A^2(T)} \right]^{-1} \exp \left[ \frac{b(T)xy}{\varepsilon \sigma_A(T)} \right] \tilde{\varepsilon}_\varepsilon(y, 0) \, dy ,

C_\varepsilon(x, T) = [1 - 3\delta_\varepsilon(x, T)] \int_0^{\gamma_\infty} \left[ \frac{a(T) - b(T)y}{\varepsilon \sigma_A(T)} + \frac{x}{\varepsilon A^2(T)} \right]^{-2} \exp \left[ \frac{b(T)xy}{\varepsilon \sigma_A(T)} \right] \tilde{\varepsilon}_\varepsilon(y, 0) \, dy. \tag{3.38}
$$

with $\|\delta(\cdot, T)\|_\infty \leq 2\varepsilon/a(T)^2$. Using the fact that $\lim_{T \to \infty} a(T) = \infty, \lim_{T \to \infty} b(T)/a(T) = 0$, we conclude from (3.35), (3.38) that $\beta_\varepsilon(\cdot, T)$ converges uniformly to 1 as $T \to \infty$. The limit (3.30) follows. \qed

Springer
Proposition 3.2 Assume \( c_\varepsilon(x, 0) \), \( x \in \mathbb{R} \), has compact support, satisfies the conditions of Lemma 3.1 and \( c_\varepsilon(\cdot, t) \), \( t > 0 \), is the solution to (1.1), (1.3). Then \( \lim_{t \to \infty} \frac{d \Lambda_\varepsilon(t)}{dt} = 1 \).

Proof We see from (3.5) and Lemma 3.1 that for any \( \delta > 0 \) there exists \( T_\delta \geq 1 \) such that

\[
\varepsilon \left| \frac{\partial c_\varepsilon(0, t)}{\partial x} \right| \leq \delta c_\varepsilon(0, t) \quad \text{for } t \geq T_\delta.
\]

(3.39)

If we also choose \( T_\delta \) sufficiently large so that \( \varepsilon/2 \Lambda_\varepsilon(t) \leq \delta \) for \( t \geq T_\delta \), then we conclude from (3.1), (3.24), (3.39) that

\[
[1 - 2\delta]\beta_{X_{\varepsilon,t}}(0) \leq \frac{d \Lambda_\varepsilon(t)}{dt} \leq [1 + 2\delta]\beta_{X_{\varepsilon,t}}(0) \quad t \geq T_\delta.
\]

(3.40)

The result follows from (3.30) of Proposition 3.1 and (3.40).

Remark 1 Observe that the rate of coarsening for the \( \varepsilon = 0 \) CP model is given by

\[
\frac{d}{dt} \langle X_{0,t} \rangle = \beta_{X_{0,t}}(0).
\]

(3.41)

Hence the formula (3.1) for the whole line \( \varepsilon > 0 \) model can be considered as a linear combination of the \( \varepsilon = 0 \) formula (3.41) and the \( \varepsilon > 0 \) formula (1.4) for the Dirichlet boundary condition case.

Remark 2 It follows from (3.38) that

\[
\lim_{T \to \infty} \frac{\Lambda_\varepsilon(T) a(T)}{\varepsilon \sigma_A(T)} = \lim_{T \to \infty} \frac{\Lambda_\varepsilon(T) m_{2,A}(T)}{\varepsilon \sigma_A^2(T)} = 1, \quad \text{where } A(\cdot) \equiv \frac{1}{\Lambda_\varepsilon(\cdot)}.
\]

(3.42)

Equation (3.42) suggests that \( \Lambda_\varepsilon(T)/T \) converges to 1 at a logarithmic rate. To see this note from Lemma 2.1 that

\[
\lim_{T \to \infty} \frac{\sigma_A^2(T)}{m_{1,A}(T)^2} = C_\varepsilon < \infty.
\]

(3.43)

We conclude from (3.15), (3.42), (3.43) and Proposition 3.2 that

\[
\lim_{T \to \infty} \frac{T}{m_{1,A}(T)} \int_0^T \frac{ds}{m_{1,A}(s)} = \varepsilon C_\varepsilon.
\]

(3.44)

If we choose \( m_{1,A}(T) \approx T (\log T)^{1/2} \) for large \( T \) then the limit on the LHS of (3.44) is 2 and

\[
\frac{dm_{1,A}(T)}{dT} \approx \frac{m_{1,A}(T)}{T} \left[ 1 + \frac{1}{2 \log T} \right].
\]

(3.45)
From (3.2) we have that
\[
\frac{dm_{1,A}(T)}{dT} = A(T)m_{1,A}(T) = \frac{1}{\Lambda_\epsilon(T)}m_{1,A}(T). \tag{3.46}
\]
Comparing (3.45), (3.46) we conclude that
\[
\frac{\Lambda_\epsilon(T)}{T} \simeq 1 - \frac{1}{2 \log T} \quad \text{as } T \to \infty. \tag{3.47}
\]

4 The Half Line Problem

We consider now the half line problem (1.1), (1.2) with zero Dirichlet boundary condition. Our approach will be to regard this problem as a perturbation from the whole line problem studied in Sect. 2. Hence it is helpful to regard the half line problem as (1.1), (1.3) rather than the equivalent (1.1), (1.2). In comparing the half line problem to the full line problem, the main difficulty is in finding estimates on the ratio of the half line Dirichlet Green’s function \(G_{\epsilon,D}(x,y,t,T)\) for (1.1) to the whole line Green’s function (3.4). We write the ratio as
\[
K_{\epsilon,D}(x, y, t, T) = \frac{G_{\epsilon,D}(x, y, t, T)}{G_{\epsilon}(x, y, t, T)}, \quad x > 0, \ 0 < t < T. \tag{4.1}
\]

In Conlon and Dabkowski (2021) we proved a log concavity property for this ratio (see Theorem 1.1 of Conlon and Dabkowski (2021)). We shall use this fact to show that the conclusions of Lemma 3.1 extend to the half line problem.

**Lemma 4.1** Let \(c_\epsilon(x, t), \ x, t > 0\), be the solution to (1.1), (1.2) with zero Dirichlet boundary condition and non-negative initial data \(c_\epsilon(\cdot, 0)\) satisfying (3.6). Then the function \(\Lambda_\epsilon(\cdot)\) of (1.3) is increasing and \(\lim_{T \to \infty} \Lambda_\epsilon(T) = \infty\). If in addition \(c_\epsilon(\cdot, 0)\) has compact support, then \(\lim_{T \to \infty} m_{1,A}(T) = \infty\), \(\lim_{T \to \infty} \frac{m_{2,A}(T)}{m_{1,A}(T)} = \infty\), \(\lim_{T \to \infty} \frac{\sigma_\epsilon^2(T)}{m_{2,A}(T)} = \infty\) and \(\lim_{T \to \infty} \frac{\sigma_\epsilon^2(T)}{m_{1,A}(T)^2} < \infty\), where \(A(\cdot) = 1/\Lambda_\epsilon(\cdot)\).

**Proof** For \(T > 0\) we define the function \(u_\epsilon\) by
\[
u_\epsilon(y, t, T) = \int_0^\infty G_{\epsilon,D}(x, y, t, T) \, dx, \quad y > 0, \ t < T. \tag{4.2}
\]

In place of (3.9) we have the formulas
\[
\int_0^\infty c_\epsilon(x, T) \, dx = \int_0^\infty u_\epsilon(y, 0, T) \, c_\epsilon(y, 0) \, dy, \tag{4.3}
\]
and
\[
\int_0^\infty xc_\epsilon(x, T) \, dx = \int_0^\infty E[X_{\epsilon,y,T}]u_\epsilon(y, 0, T)c_\epsilon(y, 0) \, dy. \tag{4.4}
\]
where $X_{\varepsilon,y,T}$ is the positive random variable with pdf proportional to the function $x \to G_{\varepsilon}(x,y,0,T)$. Since the function $x \to G_{\varepsilon}(x,y,0,T)$ of (3.4) is the pdf of the random variable (3.8) we have that

$$E[X_{\varepsilon,y,T}] = \sqrt{\varepsilon} \sigma_A(T)$$

$$E[K_{\varepsilon,D}(\sqrt{\varepsilon} \sigma_A(T)[Z - z_{y,T}/\sqrt{\varepsilon}], y, 0, T)(Z - z_{y,T}/\sqrt{\varepsilon}) | Z > z_{y,T}/\sqrt{\varepsilon}]$$

$$E[K_{\varepsilon,D}(\sqrt{\varepsilon} \sigma_A(T)[Z - z_{y,T}/\sqrt{\varepsilon}], y, 0, T) | Z > z_{y,T}/\sqrt{\varepsilon}]$$

(4.5)

where $Z$ is the standard normal variable and $K_{\varepsilon,D}$ is given by (4.1). Observe that the function $f(z) = K_{\varepsilon,D}(\sqrt{\varepsilon} \sigma_A(T)z, y, 0, T)$, $z \geq 0$, is continuous increasing with $f(0) = 0$ and $\lim_{z \to \infty} f(z) = 1$ (see Proposition 3.2 of Conlon and Dabkowski, 2021).

It follows from (1.4) that $\Lambda_{\varepsilon}(\cdot)$ is increasing. As in Lemma 3.1 we assume for contradiction that $\lim_{T \to \infty} \Lambda_{\varepsilon}(T) = \Lambda_{\infty} < \infty$. Then (3.12) holds, and in place of (3.13) we have from (A.17) of Lemma A.3 that $E[X_{\varepsilon,y,T}] \geq c \sqrt{\varepsilon} \sigma_A(T)$, $y > 0$, $T \geq 1$, for some constant $c > 0$. The inequality (3.14) follows now from (4.3), (4.4), whence we conclude $\lim_{T \to \infty} \Lambda_{\varepsilon}(T) = \infty$. To prove that $\lim_{T \to \infty} \frac{m_{1,A}(T)}{m_{1,A}(T) - \sigma_A^2(T)} = \infty$ we need to use the concavity of the function $z \to -\log(1 - f(z))$ (see (1.17) of Conlon and Dabkowski, 2021). In particular, we need to show that if the functions $a(\cdot), b(\cdot)$ satisfy (3.16) then the analogue of (3.17) given by

$$E[X_{\varepsilon,y,T}] \leq C_2 \text{ for } T \geq T_2, \ y \in (0, y_\infty)$$

(4.6)

holds. Using the representation (4.5), we see that (4.6) follows from (A.1) of Lemma A.1 and (A.18) of Lemma A.3.

The argument that $\lim_{T \to \infty} \frac{m_{2,A}(T)}{m_{1,A}(T)} = \infty$ follows as in Lemma 2.1 from (3.14). To prove that $\lim_{T \to \infty} \frac{\sigma_A^2(T)}{m_{2,A}(T)} = \infty$ we again need to use the concavity of the function $z \to -\log[1 - f(z)]$. Thus (A.1) and (A.18) imply the analogue of (3.21),

$$E[X_{\varepsilon,y,T}] \leq C_3 \frac{\sigma_A^2(T)}{m_{2,A}(T)} \text{ for } T \geq T_3, \ y \in (0, y_\infty)$$

(4.7)

whence we conclude from (4.3), (4.4) that $\Lambda_{\varepsilon}(T) \leq C_3 \frac{\sigma_A^2(T)}{m_{2,A}(T)}$ for $T \geq T_3$. The remainder of the argument proceeds as in Lemma 3.1. □

**Remark 3** Lemma 7.1 of Conlon et al. (2016) gives a proof that $\lim_{T \to \infty} \Lambda_{\varepsilon}(T) = \infty$ by using some simple inequalities for solutions to (1.1), (1.2) with zero Dirichlet condition. Note that the proof in Lemma 4.1 of $\lim_{T \to \infty} \Lambda_{\varepsilon}(T) = \infty$ uses only the elementary property (proved using the maximum principle) that the function $x \to K_{\varepsilon,D}(x,y,0,T)$ is increasing. The proof in Lemma 4.1 of $\lim_{T \to \infty} m_{1,A}(T) = \infty$ uses the more subtle log concavity property of the function $x \to K_{\varepsilon,D}(x,y,0,T)$. Evidently the result that $\lim_{T \to \infty} m_{1,A}(T) = \infty$ is a type of upper bound on the rate of coarsening, i.e., the rate at which $\Lambda_{\varepsilon}(\cdot)$ can increase. Theorem 1.2 of Conlon et al.
(2016) yields a bound $\Lambda_\varepsilon(T) \leq CT + \Lambda_\varepsilon(0)$, which implies that $\lim_{T \to \infty} m_{1,A}(T) = \infty$. The proof of Theorem 1.2 of Conlon et al. (2016) also proceeds by establishing a log concavity condition on solutions to (1.1) with zero Dirichlet boundary condition.

In Theorem 1.2 of Conlon et al. (2016) we obtained an upper bound on the rate of coarsening provided the initial data for (1.1), (1.2) satisfies a log concavity condition on solutions to (1.1) with zero Dirichlet boundary condition.

**Lemma 4.2** Let $c_\varepsilon(x,t)$, $x,t > 0$, be the solution to (1.1), (1.2) with zero Dirichlet boundary condition and non-negative initial data $c_\varepsilon(\cdot,0)$ satisfying (3.6). Assume $c_\varepsilon(\cdot,0)$ has compact support and for $t > 0$ denote by $X_{\varepsilon,t}$ the random variable with pdf proportional to $c_\varepsilon(\cdot,t)$. Then for any $T_0 > 0$ there is a constant $C$ such that $\beta_{X_{\varepsilon,t}}(\cdot) \leq C$ for $t \geq T_0$.

**Proof** We bound $\beta_{X_{\varepsilon,t}}(\cdot)$ from above by a constant times the beta function of the corresponding random variable in the whole line problem. The result then follows from the argument in the proof of Proposition 3.1. To obtain the bound we note that

$$c_\varepsilon(x,T) = \int_0^{\infty} K_{\varepsilon,D}(x,y,0,T)G_\varepsilon(x,y,0,T)c_\varepsilon(y,0) \, dy ,$$

and write

$$w_\varepsilon(x,T) = \int_x^\infty c_\varepsilon(x',T) \, dx'$$

$$= \int_x^\infty dx' \int_0^{\infty} K_{\varepsilon,D}(x',y,0,T)G_\varepsilon(x',y,0,T)c_\varepsilon(y,0) \, dy$$

$$\geq \int_0^{\infty} K_{\varepsilon,D}(x,y,0,T) \left[ \int_x^\infty dx' G_\varepsilon(x',y,0,T) \right] c_\varepsilon(y,0) \, dy ,$$

since the function $x \to K_{\varepsilon,D}(x,y,0,T)$ is increasing. Observe that

$$\frac{1}{G_\varepsilon(x,y,0,T)} \int_x^\infty dx' G_\varepsilon(x',y,0,T) = \frac{B_\varepsilon(x,y,T)}{A_\varepsilon(x,y,T)} ,$$

where $A_\varepsilon(x,y,T)$, $B_\varepsilon(x,y,T)$ are the functions $A_\varepsilon(x,T)$, $B_\varepsilon(x,T)$ of (3.35), (3.36) with $\tilde{c}_\varepsilon(\cdot,0)$ given by the Dirac delta function concentrated at $y$. Since Lemma 4.1 implies that $\lim_{T \to \infty} a(T) = \infty$, $\lim_{T \to \infty} b(T)/a(T) = 0$, we have from (3.35), (3.38) there exists $T_1 \geq T_0$ such that
\[
\frac{1}{G_\varepsilon(x, y, 0, T)} \int_x^\infty dx' G_\varepsilon(x', y, 0, T) \geq \frac{1}{2} \left[ \frac{a(T)}{\varepsilon\sigma_A(T)} + \frac{x}{\varepsilon\sigma_A^2(T)} \right]^{-1} x \geq 0, \ T \geq T_1, \ 0 < y < y_\infty. \quad (4.11)
\]

Arguing as in the proof of Lemma 4.1 we also see that the variable \( X_{\varepsilon, y, T} \) defined there satisfies the inequality
\[
E[X_{\varepsilon, y, T} - x | X_{\varepsilon, y, T} > x] \leq C_1 \left[ \frac{a(T)}{\varepsilon\sigma_A(T)} + \frac{x}{\varepsilon\sigma_A^2(T)} \right]^{-1} x \geq 0, \ T \geq T_1, \ 0 < y < y_\infty, \quad (4.12)
\]
for some constant \( C_1 > 0 \). It follows from (4.11), (4.12) that \( \beta_{X_{\varepsilon, T}}(\cdot) \leq 2C_1 \) if \( T \geq T_1 \). The upper bound on \( \beta_{X_{\varepsilon, T}}(\cdot) \) in the region \( \theta_0 < T < T_1 \) is straightforward. \( \square \)

**Proposition 4.1** Assume \( c_\varepsilon(\cdot, 0) \) satisfies the conditions of Lemma 3.2. Then for any \( T_0 > 0 \) there is a constant \( C_0 \) such that the function \( \Lambda_\varepsilon(\cdot) \) defined by (1.3) satisfies the inequality \( \frac{d\Lambda_\varepsilon(t)}{dt} \leq C_0 \) when \( t \geq T_0 \).

**Proof** We follow the arguments of Lemma 7.2 and Lemma 7.3 of Conlon et al. (2016), using Lemma 4.2 to substitute for the argument of Lemma 7.2. Thus by rescaling and Lemma 4.2 we can assume that
\[
\langle X_{\varepsilon, 0} \rangle = 1, \ \beta_{X_{\varepsilon, 0}}(\cdot) \leq C_1, \quad (4.13)
\]
where \( C_1 \) is the bound obtained in Lemma 4.2. We wish to show there is a second constant \( C_2 \), depending only on \( C_1 \), such that
\[
\frac{\varepsilon}{2} \frac{\partial c_\varepsilon(0, 1)}{\partial x} \leq C_2 c_\varepsilon(\varepsilon, 1) \ \text{if} \ \varepsilon \leq 1. \quad (4.14)
\]
Expressing the function \( K_{\varepsilon, D} \) of (4.1) as
\[
K_{\varepsilon, D}(x, y, 0, T) = \left\{ 1 - \exp \left[ -\frac{q_\varepsilon(x, y, T)}{\varepsilon} \right] \right\}, \quad (4.15)
\]
we have that
\[
\frac{\varepsilon}{2} \frac{\partial c_\varepsilon(0, 1)}{\partial x} = \frac{1}{2} \int_0^\infty \frac{\partial q_\varepsilon(0, y, 1)}{\partial x} G_\varepsilon(0, y, 0, 1)c_\varepsilon(y, 0) \, dy. \quad (4.16)
\]
We have then from (4.16) and Proposition 5.1 of Conlon and Dabkowski (2021) the inequality
\[
\frac{\varepsilon}{2} \frac{\partial c_\varepsilon(0, 1)}{\partial x} \leq \int_0^\infty \left[ 1 + \frac{m_{1, A}(1)y}{\sigma_A^2(1)} \right] G_\varepsilon(0, y, 0, 1)c_\varepsilon(y, 0) \, dy, \quad (4.17)
\]
with \( m_{1, A}(1) \) the variance of the noise process.
where \( A(\cdot) \equiv 1/\Lambda_\varepsilon(\cdot) \). Using the lower bound in Proposition 3.3 of Conlon and Dabkowski (2021) we have that
\[
c_\varepsilon(\varepsilon, 1) \geq \frac{2m_{1, A}(1)}{\sigma_A^2(1)} \int_0^\infty y \exp \left[ -\frac{2m_{1, A}(1)y}{\sigma_A^2(1)} \right] G_\varepsilon(\varepsilon, y, 0, 1) c_\varepsilon(y, 0) \, dy. \tag{4.18}
\]

Observe that the function \( A(\cdot) \) is decreasing and satisfies \( A(0) = 1 \). The inequality (4.14) follows from (4.17), (4.18) by using the argument in the proof of Lemma 7.2 of Conlon et al. (2016). The key point is that the inequality (7.6) of Conlon et al. (2016) continues to hold. That is if \( X_{\varepsilon, 0} \) satisfies (4.13) then for any \( \delta \) with \( 0 < \delta < 1 \) there exists a constant \( \nu(\delta) > 0 \), depending only on \( C_1 \), such that
\[
P(X_{\varepsilon, 0} < \nu(\delta)) \leq \delta. \tag{4.19}
\]

The inequality (4.19) follows from (3.25), (3.26). Once (4.14) is established the proof of the lemma follows as in the proof of Lemma 7.3 of Conlon et al. (2016). \( \square \)

Proposition 4.1 gives an upper bound on the rate of coarsening. We can also obtain a lower bound by using the log concavity property of the function \( K_{\varepsilon, D} \), which was crucial to the proof of Lemma 4.1. We illustrate this first for the classical \((\varepsilon = 0)\) CP model.

**Lemma 4.3** Assume \( 0 < \varepsilon \leq 1 \) and \( c_\varepsilon(x, 0) = f(x)e^{-x}, \ x > 0, \) is the initial data for the diffusive CP model (1.1), (1.2), where \( f : [0, \infty) \to \infty \) is continuous non-negative increasing, satisfying \( \lim_{x \to \infty} f(x) = 1 \) and with the property that the function \(-\log[1 - f(\cdot)]\) is concave. Then there is a universal constant \( \delta_0 > 0 \) such that \( \Lambda_\varepsilon(1) \geq [1 + \delta_0]\Lambda_\varepsilon(0) \).

**Proof** We first prove the result in the case \( \varepsilon = 0 \), so for the classical model of Carr and Penrose (1998). Let \( X_{0, t}, \ t \geq 0, \) be the random variables with pdf proportional to \( c_0(\cdot, t), \ t \geq 0, \) where \( c_0(x, t), \ x, t \geq 0, \) is the solution of the CP model (1.1), (1.2) with \( \varepsilon = 0 \). Then one has that
\[
\frac{d\Lambda_0(t)}{dt} = \beta_{X_0, t}(0), \quad t > 0. \tag{4.20}
\]

Furthermore \( \beta_{X_0, t}(x) = \beta_{X_0, 0}(F_A(x, t)), \ x \geq 0, \) where \( A(\cdot) \equiv 1/\Lambda_0(\cdot) \) and \( F_A \) is defined by
\[
F_A(x, t) = \frac{x + m_{2, A}(t)}{m_{1, A}(t)}, \quad x, t \geq 0. \tag{4.21}
\]

We have from (3.24) that
\[
\beta_X(x) = \frac{c_X(x)}{w_X(x)} E[X - x \mid X > x]. \tag{4.22}
\]
Arguing as in the proof of Lemma A.3, we see there is a universal constant $c_1 > 0$ such that $E[X_{0,0} - x \mid X_{0,0} > x] \geq c_1$, $x > 0$. We consider for $X = X_{0,0}$,

$$\frac{w_X(x)}{c_X(x)} = \int_x^\infty \frac{f(x')}{f(x)} e^{x-x'} \, dx'. \quad (4.23)$$

We write $f(x) = 1 - e^{-q(x)}$ where $q(\cdot)$ is positive increasing and concave. If $q(x) \geq 1$, then the RHS of (4.23) is bounded above by $[1 - e^{-1}]^{-1}$. If $q(x) \leq 1$, then we have from (4.23) that

$$\frac{w_X(x)}{c_X(x)} \leq e \int_x^\infty \frac{q(x')}{q(x)} e^{x-x'} \, dx'. \quad (4.24)$$

From the concavity of $q(\cdot)$ and the fact that $q(0) \geq 0$ we obtain the inequality

$$q(x') \leq q(x) + \frac{q(x)}{x}(x' - x) \quad \text{for} \quad x' > x, \quad (4.25)$$

whence the RHS of (4.24) is bounded above by $e[1 + x]/x$. We conclude from (4.22)–(4.25) that

$$\beta_{X_{0,0}}(x) \geq \frac{c_1 x}{e[1 + x]}, \quad x > 0. \quad (4.26)$$

Since the function $A(\cdot) = 1/\Lambda_0(\cdot)$ is decreasing we see that $m_{1,A}(s) \leq \exp[A(0)s]$, $s > 0$, whence we have that

$$\frac{m_{2,A}(t)}{m_{1,A}(t)} \geq \frac{1}{A(0)} \left[ 1 - e^{-A(0)t} \right], \quad t > 0. \quad (4.27)$$

By the argument of Lemma A.3 there are universal constants $C_2, c_2 > 0$ such that $c_2 \leq \Lambda_0(0) \leq C_2$. The result follows from this and (4.26), (4.27) upon using the identity,

$$\Lambda_0(1) - \Lambda_0(0) = \int_0^1 \beta_{X_{0,0}}(F_{A}(0, t)) \, dt. \quad (4.28)$$

To extend the argument to $\varepsilon > 0$ we use (4.16) and Proposition 5.1 of Conlon and Dabkowski (2021), whence we obtain the inequality

$$\frac{\varepsilon}{2} \frac{\partial c_\varepsilon(0, t)}{\partial x} \geq \int_0^\infty \frac{m_{1,A}(t) y}{\sigma^2_A(t)} G_\varepsilon(0, y, 0, t) c_\varepsilon(y, 0) \, dy. \quad (4.29)$$

Then (1.2), (1.4), (3.24), (4.26), (4.29) yield the inequality

$$\frac{d\Lambda_\varepsilon(t)}{dt} \geq \frac{c_1 m_{1,A}(t)}{e\sigma^2_A(t)} \int_0^\infty \frac{y^2}{1 + y} G_\varepsilon(0, y, 0, t) \frac{w_X(y)^2}{h_X(0)h_X(y)} \, dy, \quad (4.30)$$
where $X = X_{0,0}$. As in the proof of Lemma 7.3 of Conlon et al. (2016), we use an inequality

$$h_X(y) \geq \frac{1}{12} h_X(0), \quad \text{if } 0 < y \leq \Lambda_0(0)/2.$$  \hfill (4.31)

We combine (4.31) with the inequality $h_X(y)/w_X(y) = E[X - y \mid X > y] \leq C_1$ for some constant $C_1$, as in the proof of Lemma A.3. Then we integrate (4.30) as before to obtain the result for $0 < \varepsilon \leq 1$.

**Proposition 4.2** Assume $c_\varepsilon(\cdot, 0)$ satisfies the conditions of Lemma 3.2. Then for any $T_0 > 0$ there is a constant $\delta_0 > 0$ such that the function $\Lambda_\varepsilon(\cdot)$ defined by (1.3) satisfies the inequality $\Lambda_\varepsilon(T + \Lambda_\varepsilon(T)) \geq [1 + \delta_0]\Lambda_\varepsilon(T)$ when $T \geq T_0$.

**Proof** We may assume that $T_0$ is sufficiently large so that we are in the asymptotic regime established in Lemma 4.1. Noting that Lemma A.3 also yields a lower bound comparable to the upper bound (4.12), we conclude that

$$E[X_{\varepsilon,T} - x \mid X_{\varepsilon,T} > x] \geq c_1 \left[ \frac{a(T)}{\varepsilon\sigma_A(T)} + \frac{x}{\varepsilon\sigma_A^2(T)} \right]^{-1} x \geq 0, \; T \geq T_0,$$ \hfill (4.32)

for some constant $c_1 > 0$. We wish to obtain a lower bound on $\beta_{X_{\varepsilon,T}}(\cdot)$ similar to the one established in the proof of Lemma 4.3. This will follow from (4.32) and an upper bound on the ratio $w_\varepsilon(x, T)/c_\varepsilon(x, T)$, where $c_\varepsilon(x, T), \; w_\varepsilon(x, T)$ are given by (4.8), (4.9).

In analogy to (3.35), (3.36) we define functions $A_{\varepsilon,D}, \; B_{\varepsilon,D}$ by

$$A_{\varepsilon,D}(x, T) = \int_0^{y_\infty} K_{\varepsilon,D}(x, y, 0, T) \exp \left[ \frac{b(T)yx}{\varepsilon\sigma_A(T)} \right] \tilde{c}_\varepsilon(y, 0) \, dy,$$ \hfill (4.33)

$$B_{\varepsilon,D}(x, T) = \int_0^{y_\infty} dy \int_0^{\infty} dx' K_{\varepsilon,D}(x + x', y, 0, T) \times \exp \left[ \frac{b(T)(x + x')y}{\varepsilon\sigma_A(T)} - \frac{a(T)x'}{\varepsilon\sigma_A(T)} - \frac{x'(2x + x')}{2\varepsilon\sigma_A^2(T)} \right] \tilde{c}_\varepsilon(y, 0).$$ \hfill (4.34)

Then one has for $X = X_{\varepsilon,T}$ the formula

$$\frac{w_X(x)}{c_X(x)} = \frac{B_{\varepsilon,D}(x, T)}{A_{\varepsilon,D}(x, T)},$$ \hfill (4.35)

Using the representation (4.15) we define $y_\infty(x, T)$ by

$$y_\infty(x, T) = \sup\{y : 0 < y < y_\infty, \; q_\varepsilon(x, y, T) < \varepsilon\}.$$ \hfill (4.36)

Since $q_\varepsilon(x, 0, T) = 0$ and the function $y \to q_\varepsilon(x, y, T)$ is increasing, it follows from (4.36) that $0 < y_\infty(x, T) \leq y_\infty$ and $q_\varepsilon(x, y, T) \leq \varepsilon$ if $y < y_\infty(x, T)$, with
\( q_\varepsilon(x, y, T) \geq \varepsilon \) if \( y_\infty(x, T) < y_\infty \) and \( y > y_\infty(x, T) \). It follows then from (4.33) that

\[
A_{\varepsilon, D}(x, T) \geq \frac{1}{\varepsilon} \int_0^{y_\infty(x, T)} q_\varepsilon(x, y, T) \exp \left[ \frac{b(T)xy}{\varepsilon \sigma_A(T)} \right] \tilde{c}_\varepsilon(y, 0) \, dy \\
+ \left[ 1 - e^{-1} \right] \int_{y_\infty(x, T)}^{y_\infty} \exp \left[ \frac{b(T)xy}{\varepsilon \sigma_A(T)} \right] \tilde{c}_\varepsilon(y, 0) \, dy. \tag{4.37}
\]

Similarly we have that

\[
B_{\varepsilon, D}(x, T) \leq \frac{1}{\varepsilon} \int_0^{y_\infty(x, T)} dy \int_0^\infty dx' q_\varepsilon(x+x', y, T) \\
\times \exp \left[ \frac{b(T)(x + x')y}{\varepsilon \sigma_A(T)} - \frac{a(T)x'}{\varepsilon \sigma_A(T)} - \frac{x'(2x + x')}{2\varepsilon \sigma_A^2(T)} \right] \tilde{c}_\varepsilon(y, 0) \\
+ \int_{y_\infty(x, T)}^{y_\infty} dy \int_0^{\infty} dx' \exp \left[ \frac{b(T)(x + x')y}{\varepsilon \sigma_A(T)} - \frac{a(T)x'}{\varepsilon \sigma_A(T)} - \frac{x'(2x + x')}{2\varepsilon \sigma_A^2(T)} \right] \tilde{c}_\varepsilon(y, 0). \tag{4.38}
\]

Applying (4.25) to the functions \( x \rightarrow q_\varepsilon(x, y, T) \) and using (3.37) we have from (4.37), (4.38) the inequality

\[
\frac{B_{\varepsilon, D}(x, T)}{A_{\varepsilon, D}(x, T)} \leq C_1 \left[ \frac{a(T)}{\varepsilon \sigma_A(T)} + \frac{x}{\varepsilon \sigma_A^2(T)} \right]^{-1} \left\{ 1 + \frac{1}{x} \left[ \frac{a(T)}{\varepsilon \sigma_A(T)} + \frac{x}{\varepsilon \sigma_A^2(T)} \right]^{-1} \right\}, \tag{4.39}
\]

for some constant \( C_1 \).

It follows from (3.42) and Lemma A.3 that \( T_0 \) can be chosen sufficiently large so that

\[
c_2 \frac{\varepsilon \sigma_A(T)}{a(T)} \leq \Lambda_\varepsilon(T) \leq C_2 \frac{\varepsilon \sigma_A(T)}{a(T)}, \quad \text{for } T \geq T_0, \tag{4.40}
\]

for some constants \( C_2, c_2 > 0 \). We conclude from (4.32)–(4.40) there is a constant \( c_3 > 0 \) such that

\[
\beta_{X_{\varepsilon, T}}(x) \geq c_3 \frac{x}{\Lambda_\varepsilon(T) + x}, \quad \text{for } x > 0, \ T \geq T_0. \tag{4.41}
\]

The result follows now from (4.41) and the argument of Lemma 4.3 by scaling \( \Lambda_\varepsilon(T) \) to 1, whence the diffusion coefficient in (1.1) becomes \( \varepsilon / \Lambda_\varepsilon(T) < < 1 \). \( \square \)
5 Convergence of the Function \( x \rightarrow q_\varepsilon(x, y, T) \) as \( T \rightarrow \infty \)

In Proposition 3.1 we proved convergence in distribution to the exponential variable for solutions to the whole line diffusive CP problem. This follows from the corresponding convergence in distribution of the positive random variable \( \tilde{X}_{\varepsilon,y,T} \), with density proportional to the function \( x \rightarrow G_\varepsilon(x, y, 0, T) \), \( x > 0 \). To prove that we used certain properties of functions associated with \( A(\cdot) \equiv 1/\Lambda_\varepsilon(\cdot) \), established in Lemma 3.1. These are as follows:

\[
(a) \lim_{T \rightarrow \infty} A(T) = 0, \quad (b) \lim_{T \rightarrow \infty} m_{1,A}(T) = \infty, \quad (c) \lim_{T \rightarrow \infty} \frac{m_{2,A}(T)}{m_{1,A}(T)} = \infty, \quad (d) \lim_{T \rightarrow \infty} \frac{\sigma_A^2(T)}{m_{2,A}(T)} = \infty, \quad (e) \lim_{T \rightarrow \infty} \frac{\sigma_A^2(T)}{m_{1,A}(T)^2} < \infty. \tag{5.1}
\]

We wish to follow a similar strategy for the half line problem. In Lemma 4.1 we showed that (5.1) with \( A(\cdot) \equiv 1/\Lambda_\varepsilon(\cdot) \) holds for the half line problem. The random variable \( X_{\varepsilon,y,T} \) defined in the proof of Lemma 4.1 has pdf proportional to \( x \rightarrow G_{\varepsilon,D}(x, y, 0, T) \), \( x > 0 \). The variables \( \tilde{X}_{\varepsilon,y,T} / \langle \tilde{X}_{\varepsilon,y,T} \rangle \) and \( X_{\varepsilon,y,T} / \langle X_{\varepsilon,y,T} \rangle \) have therefore the same distributional limit as \( T \rightarrow \infty \) if we can show the ratio of their pdfs, given by the function \( x \rightarrow K_{\varepsilon,D}(x, y, 0, T) \) of (4.1), converges to 1 as \( T \rightarrow \infty \) for \( x \) larger than any small constant times \( \min \{ \langle \tilde{X}_{\varepsilon,y,T} \rangle, \langle X_{\varepsilon,y,T} \rangle \} \). In view of (4.15), and the fact that the function \( x \rightarrow q_\varepsilon(x, y, T) \) is increasing, this is equivalent to obtaining lower bounds on the function \( q_\varepsilon(\cdot, y, T) \).

Theorem 2.1 of Conlon and Dabkowski (2021) states that the function \( q_0(x, y, T) = \lim_{\varepsilon \rightarrow 0} q_\varepsilon(x, y, T) \) is the solution to the variational problem (see (2.28) of Conlon and Dabkowski (2021))

\[
q_0(x, y, T) = \min \left\{ \frac{1}{2} \int_{\tau}^{T} \left[ \frac{dx(s)}{ds} - \lambda(x(s), y, s) \right]^2 ds \mid 0 < \tau < T, \ x(T) = x, \ x(\tau) = 0 \right\}, \tag{5.2}
\]

where \( \lambda(\cdot, \cdot, \cdot) \) is defined by

\[
\lambda(x, y, s) = A(s) + \frac{1}{\sigma_A^2(s)} x - 1 + \frac{m_{2,A}(s)}{\sigma_A^2(s)} y - \frac{m_{1,A}(s)y}{\sigma_A^2(s)}, \ x, y, s > 0. \tag{5.3}
\]

We see from (5.3) that if (5.1) holds then \( \lim_{s \rightarrow \infty} \lambda(x, y, s) = -1 \). The solution to (5.2) when \( \lambda(\cdot, \cdot, \cdot) \equiv -1 \) is given by \( q_0(x, y, T) = 2x \) and the optimal exit time is \( \tau = \tau(x, y, T) = T - x \). We therefore expect that \( \lim_{T \rightarrow \infty} q_0(x, y, T) = 2x \) for all \( x, y > 0 \). The situation is however more subtle than we just described because it is possible that the minimizing trajectory \( x_{\min}(s), \tau < s < T \), for (5.2) could have
$T - \tau$ large. In that case $x_{\text{min}}(\cdot)$ is approximately the integral curve for the vector field $\lambda(\cdot, \cdot, \cdot)$ with terminal condition $x_{\text{min}}(T) = x$ for a long time, but then close to time $\tau$ the trajectory $x_{\text{min}}(\cdot)$ exits the positive half line with small cost. We shall show that the conditions (5.1) rule out this possibility.

**Proposition 5.1** Assume the function $A : [0, \infty) \to \mathbb{R}$ is continuous positive decreasing and (5.1) holds. Then for all $x, y > 0$ one has $\lim_{T \to \infty} q_0(x, y, T) = 1$. In addition the limit is uniform in any region $0 < x < M$, $y_0 < y < y_\infty$, where $M > 0$ and $0 < y_0 < y_\infty < \infty$.

**Proof** It follows from the upper bound $q_\varepsilon(x, y, T) \leq -2\lambda(0, y, T)x$, proved in Proposition 3.3 of Conlon and Dabkowski (2021), and (5.1) that

$$\limsup_{T \to \infty} [q_\varepsilon(x, y, T) - 2x] / x \leq 0, \quad \text{uniformly for } 0 < x < M, \ 0 < y < y_\infty.$$  \hfill (5.4)

From Theorem 2.1 of Conlon and Dabkowski (2021) we have that $q_0(x, y, T) = \lim_{\varepsilon \to 0} q_\varepsilon(x, y, T)$, whence (5.4) also holds when $\varepsilon = 0$. To obtain a lower bound on $q_0(x, y, T)$ we consider solutions to the equation

$$\frac{dx(s)}{ds} = \lambda(x(s), y, s), \quad s < T, \ x(T) = x,$$  \hfill (5.5)

which we denote by $x_{\text{class}}(s, T), \ s < T$. We have from (2.12) of Conlon and Dabkowski (2021) the explicit formula

$$\sigma_A^2(T)x_{\text{class}}(s, T) = xm_{1, A}(s, T)\sigma_A^2(s) + ym_{1, A}(s)\sigma_A^2(s, T) + m_{1, A}(s, T)m_{2, A}(s, T)\sigma_A^2(s) - m_{2, A}(s)\sigma_A^2(s, T),$$  \hfill (5.6)

where the functions $(s, T) \to m_{1, A}(s, T), \ m_{2, A}(s, T), \ \sigma_A^2(s, T), \ s < T$, are defined for the interval $[s, T]$ similarly to the corresponding functions $T \to m_{1, A}(T) = m_{1, A}(0, T), \ m_{2, A}(T) = m_{2, A}(0, T), \ \sigma_A^2(T) = \sigma_A^2(0, T)$ of (3.2) defined for the interval $[0, T]$. Evidently one has $\lim_{s \to T} x_{\text{class}}(s, T) = x$ and $\lim_{s \to 0} x_{\text{class}}(s, T) = y$. We can estimate $x_{\text{class}}(s, T)$ in the interval $0 < s < T$ using the properties (5.1) by observing from (5.6) that

$$x_{\text{class}}(s, T) \geq \frac{\sigma_A^2(s)}{m_{1, A}(s)^2} \left( \frac{\sigma_A^2(T)}{m_{1, A}(T)^2} \right)^{-1} \frac{m_{2, A}(s, T)}{m_{1, A}(s, T)} - \frac{1}{m_{1, A}(s)} \frac{\sigma_A^2(s, T)}{m_{1, A}(s, T)^2} \left( \frac{\sigma_A^2(T)}{m_{1, A}(T)^2} \right)^{-1} \frac{m_{2, A}(s)}{m_{1, A}(s)}, \ 0 < s < T.$$  \hfill (5.7)
From (5.1) (c), (e) the first term on the RHS of (5.7) goes to $\infty$ as $T \to \infty$, whereas from (5.1) (e) the second term converges to a finite number. We conclude that $\lim_{T \to \infty} x_{\text{class}}(s, T) = \infty$.

We can use the method of characteristics to construct a solution to the Hamilton–Jacobi equation (2.29) of Conlon and Dabkowski (2021), corresponding to the variational problem (5.2), in a neighborhood of the line $\{(x, T) \in \mathbb{R}^2 : x = 0, \ T > 0\}$. The characteristic equation, given by (4.28) of Conlon and Dabkowski (2021), is

$$\frac{dx(s)}{ds} = \left[A(s) + \frac{1}{\sigma^2_A(s)}\right] x(s) + \frac{m_{1,A}(s)}{\sigma^2_A(s)} \left[y + 2g_{2,A}(\tau, \tau) - g_{2,A}(s, s)\right], \ s > \tau, \ x(\tau) = 0,$$

(5.8)

where the function $s \to g_{2,A}(s, s)$ is given by the formula

$$g_{2,A}(s, s) = \frac{\sigma^2_A(s) - m_{2,A}(s)}{m_{1,A}(s)}, \ s > 0.$$

(5.9)

Differentiating (5.9) we find that

$$\frac{d}{ds} g_{2,A}(s, s) = \frac{A(s)\sigma^2_A(s)}{m_{1,A}(s)}.$$

(5.10)

Evidently the function $s \to g_{2,A}(s, s)$, $s > 0$, is non-negative increasing, and from (5.1) (c), (d) we have $\lim_{s \to \infty} g_{2,A}(s, s) = \infty$. Choosing $T_1 > 0$ so that $\sigma^2_A(s) \geq 2m_{2,A}(s)$ for $s \geq T_1$, we see from (5.9), (5.10) that

$$\frac{d}{ds} g_{2,A}(s, s) \leq 2A(s)g_{2,A}(s, s), \ s > T_1.$$

(5.11)

Integrating (5.11) we conclude that

$$g_{2,A}(s, s) \leq \exp \left[2 \int_T^s A(s') \ ds'\right] g_{2,A}(T, T), \text{ for } s > T \geq T_1.$$

(5.12)

Since the function $A(\cdot)$ is decreasing it follows from (5.8), (5.12) that the characteristic curve $s \to x_{\text{char}}(T, s)$, $s > T$, $x_{\text{char}}(T, T) = 0$, is increasing provided $T > T_1$ and $T < s < T + \log 2/2A(T)$. We may also obtain a lower bound on $x_{\text{char}}(T, s)$ by observing from (5.8), (5.12) that

$$\frac{d}{ds} x_{\text{char}}(T, s) \geq \frac{m_{1,A}(s)}{3\sigma^2_A(s)} g_{2,A}(s, s) \geq \frac{1}{6}, \text{ for } T < s < T + \log(3/2)/2A(T), \ T \geq T_1.$$

(5.13)
whence we conclude that

\[ x_{\text{char}}(T, s) \geq \frac{s - T}{6}, \quad \text{if } T < s < T + \log(3/2)/2A(T), \ T \geq T_1. \tag{5.14} \]

The first variation equation for the characteristics is obtained by differentiating (5.8) with respect to \( \tau \). This yields the equation

\[
\frac{d}{ds} DT x_{\text{char}}(T, s) = \left[ A(s) + \frac{1}{\sigma_A^2(s)} \right] DT x_{\text{char}}(T, s) + \frac{2m_{1,A}(s) A(T)\sigma_A^2(T)}{\sigma_A^2(s)} \frac{m_{1,A}(T)}{m_{1,A}(s)} , \ s > T ,
\]

\[
DT x_{\text{char}}(T, s) \bigg|_{s=T} = -\frac{m_{1,A}(T)}{\sigma_A^2(T)} [y + g_{2,A}(T, T)]. \tag{5.15}
\]

Using the fact that the function \( s \to m_{1,A}(s)/\sigma_A^2(s), \ s > 0 \), is decreasing we have from (5.15) that

\[
\frac{d}{ds} DT x_{\text{char}}(T, s) \leq 2A(T) \quad \text{if } s > T \quad \text{and } DT x_{\text{char}}(T, s) \leq 0,
\]

\[
DT x_{\text{char}}(T, s) \bigg|_{s=T} \leq -\frac{1}{2}, \quad \text{if } T \geq T_1. \tag{5.16}
\]

It follows from (5.16) that \( DT x_{\text{char}}(T, s) < 0 \) if \( T < s < T + 1/4A(T), \ T > T_1 \).

We define a domain \( D_y(T_1) \) by

\[
D_y(T_1) = \{ [x, s] : x = x_{\text{char}}(T, s), \ T < s < T + 5A(T), \ T > T_1 \}. \tag{5.17}
\]

Letting \( U(T_1) = \{ [s, T] : T < s < T + 5A(T), \ T > T_1 \} \), we have shown that the mapping \( U(T_1) \to D_y(T_1) \) defined by \([s, T] \to [x_{\text{char}}(T, s), s]\) is a diffeomorphism.

One can use the method of characteristics to construct a \( C^1 \) solution \( q_{\text{char}}(x, y, T), [x, T] \in D_y(T_1) \), of the Hamilton–Jacobi equation in the domain \( D_y(T_1) \). From (4.26) of Conlon and Dabkowski (2021) we have the formula

\[
\frac{\partial q_{\text{char}}(x_{\text{char}}(T, s), y, s)}{\partial x} = \frac{2m_{1,A}(s) [y + g_{2,A}(T, T)]}{\sigma_A^2(s)}, \quad s > T. \tag{5.18}
\]

In view of (5.1) (a) and (5.14), we see that for any \( M > 0 \) there exists \( T_M \geq T_1 \) such that \( D_y(T_1) \) contains the infinite rectangle \( \{ [x, T] : 0 < x < M, \ T > T_M \} \). Furthermore, (5.1) (a), (b), (d), (e) and (5.14), (5.18) imply for any \( y_\infty > 0 \) that

\[
\lim_{T \to \infty} \frac{\partial q_{\text{char}}(x, y, T)}{\partial x} = 2, \quad \text{uniformly for } 0 < x < M, \ 0 < y < y_\infty. \tag{5.19}
\]
As a consequence of (5.19) we have that \( \lim_{T \to \infty} q_{\text{char}}(x, y, T)/2x = 1 \) for all \( x, y > 0 \). Let \( x(s), \tau < s \leq T, \) be a path in \( \mathbb{R}^+ \) such that \( x(T) = x \) and with first exit time \( \tau > 0 \) from \( \mathbb{R}^+ \). The associated Lagrangian \( L \) and action integral \( A \) are given by the expressions

\[
L(x(\cdot), x, y, s, T) = \frac{1}{2} \left[ \frac{dx(s)}{ds} - \lambda(x(s), y, s) \right]^2 , \quad \tau < s < T ,
\]
\[
A(x(\cdot), x, y, T) = \int_{\tau}^{T} L(x(\cdot), x, y, s, T) \, ds . \quad (5.20)
\]

The usual verification theorem (see Proposition 4.2 of Conlon and Dabkowski (2021)) implies that if the path \( s \to [x(s), s], \tau < s \leq T, \) lies in \( D_y(T_1) \), then \( A(x(\cdot), x, y, T) \geq q_{\text{char}}(x, y, T) \). Suppose now that \( T > T_M \) and the path exits \( D_y(T_1) \), but then reenters \( D_y(T_1) \) at time \( \tau^* > T_M \) and remains in \( D_y(T_1) \), until it exits \( \mathbb{R}^+ \). In that case we have the inequality \( A(x(\cdot), x, y, T) \geq q_{\text{char}}(M, y, \tau^*) \).

Next we wish to obtain a lower bound on the action in the case \( 0 < \tau < T_1 \). We observe that the path \( x(s), \tau \leq s \leq T, \) is the solution to the terminal value problem

\[
\frac{dx(s)}{ds} = \lambda(x(s), y, s) - f(s) , \quad \tau < s \leq T ,
\]
\[
x(T) = x , \quad \tau < s \leq T , \quad \frac{1}{2} f(s)^2 = L(x(\cdot), x, y, s, T) \quad (5.21)
\]

It follows from (5.5) that \( \tilde{x}(s) = x(s) - x_{\text{class}}(s, T) \) is a solution to the terminal value problem

\[
\frac{d\tilde{x}(s)}{ds} = \left[ A(s) + \frac{1}{\sigma_A^2(s)} \right] \tilde{x}(s) - f(s) , \quad \tau < s < T , \quad \tilde{x}(T) = 0 . \quad (5.22)
\]

Integrating (5.22) we obtain the integral formula

\[
\tilde{x}(s) = \frac{\sigma_A^2(s)}{m_{1,A}(s)} \int_{s}^{T} \frac{m_{1,A}(s')}{\sigma_A^2(s')} f(s') \, ds' . \quad (5.23)
\]

Applying the Schwarz inequality in (5.23) we have from (5.1) (e) and (5.21) that

\[
\tilde{x}(s)^2 \leq C \left[ \int_{s}^{T} \frac{|f(s)|}{m_{1,A}(s,s')} \, ds' \right]^2 
\]
\[
\leq 2C \frac{\sigma_A^2(s, T)}{m_{1,A}(s, T)^2} \int_{s}^{T} L(x(\cdot), x, y, s', T) \, ds' , \quad (5.24)
\]
for some constant $C$. It follows from (5.2), (5.20), (5.24) that

\[
\mathcal{A}(x(\cdot), x, y, T) \geq \frac{m_{1, A}(s, T)^2}{2 C \sigma_A^2(s, T)} [x(s) - x_{\text{class}}(s, T)]^2 + q_0(x(s), y, s), \quad \tau < s < T. \tag{5.25}
\]

We wish to show that

\[
\liminf_{T \to \infty} q_0(x, y, T) \geq 1, \quad \text{uniformly for } 0 < x < M, \quad y_0 < y < y_\infty. \tag{5.26}
\]

To do this we first note the lower bound from Proposition 4.1 of Conlon and Dabkowski (2021),

\[
q_0(x, y, T) \geq \frac{2m_{1, A}(T)xy}{\sigma_A^2(T)}, \quad x, y, T > 0. \tag{5.27}
\]

Consider a path $x(s)$, $\tau < s < T$, with $x(T) = x$ and first exit time $\tau$, where $\tau < T_{2M} < T$. Taking $s = T_{2M}$ in (5.25) and recalling that $\lim_{T \to \infty} x_{\text{class}}(T_{2M}, T) = \infty$, it follows from (5.1) (e) and (5.25), (5.27) that there exists $T_{2M}^* > T_{2M}$ such that

\[
\mathcal{A}(x(\cdot), x, y, T) \geq 2x \quad \text{for } T > T_{2M}^*, \quad 0 < x < M, \quad y > y_0. \tag{5.28}
\]

We have already seen that the method of characteristics yields a lower bound on the action in the case $\tau > T_{2M}$. Combining that with (5.28) implies (5.26). We have already established the upper bound $\limsup_{T \to \infty} q_0(x, y, T)/2x \leq 1$ in (5.4). It also follows from the inequality $q_0(x, y, T) \leq q_{\text{char}}(x, y, T)$ and (5.19). \hfill \Box

Next we show that the result of Proposition 5.1 extends to the case $\epsilon > 0$. We have already obtained an upper bound in (5.4), so we just need to obtain a lower bound on $q_\epsilon(x, y, T)$. We consider solutions $X_\epsilon(\cdot)$ to the SDE

\[
dX_\epsilon(s) = \mu_\epsilon(X_\epsilon(s), y, s) \, ds + \sqrt{\epsilon} dB(s), \tag{5.29}
\]

run backward in time with controller $\mu_\epsilon$ and given terminal data. The optimal controller for the stochastic control problem corresponding to the function $[x, T] \to q_\epsilon(x, y, T)$ is

\[
\mu_\epsilon^*(x, y, T) = \lambda(x, y, T) + \frac{\partial q_\epsilon(x, y, T)}{\partial x}, \quad x, T > 0. \tag{5.30}
\]

Letting $X^*_\epsilon(\cdot)$ be solutions to (5.29) with $\mu_\epsilon = \mu_\epsilon^*$, we have from Lemma 2.1 and Lemma 2.3 of Conlon and Dabkowski (2021) the identity

\[
q_\epsilon(x, y, T) = E \left[ \frac{1}{2} \int_{x_\epsilon^*}^{T} \left[ \mu^*(X^*_\epsilon(s), y, s) - \lambda(X^*_\epsilon(s), y, s) \right]^2 \, ds \mid X^*_\epsilon(T) = x \right], \tag{5.31}
\]
where \( \tau^*_{\varepsilon,x,T} \) is the first exit time of \( X_\varepsilon^*(s), \ s < T \), with \( X_\varepsilon^*(T) = x \) from the half line \((0, \infty)\). Lemma 2.3 of Conlon and Dabkowski (2021) also establishes that \( \tau^*_{\varepsilon,x,T} > 0 \) with probability 1.

We generalize the lower bound (5.25) on the action. To do this we denote by \( X_{\varepsilon, \text{class}}(s, T), \ s < T \), the solution to (5.29) with \( \mu_\varepsilon = \lambda \) and terminal condition \( X_{\varepsilon, \text{class}}(T, T) = x \). The SDE (5.29) is then linear and may be explicitly solved, whence we have that

\[
X_{\varepsilon, \text{class}}(s, T) = x_{\text{class}}(s, T) - \sqrt{\varepsilon} \frac{\sigma^2_A(s)}{m_{1,A}(s)} Z(s),
\]

with

\[
Z(s) = \int_s^T \frac{m_{1,A}(s') dB(s')}{\sigma^2_A(s')},
\]

(5.32)

where \( x_{\text{class}}(s, T) \) is given by (5.6). Since the function \( x \rightarrow q_\varepsilon(x, y, T) \) is increasing it follows from (5.30) that \( \mu_\varepsilon(x, y, s) \geq \lambda(x, y, s), \ x, y, s > 0 \), whence \( X_{\varepsilon, \text{class}}(s, T) > 0 \) for \( \tau^*_{\varepsilon,x,T} < s < T \). Integrating (5.29) over an interval \([s, T]\) we have similarly to (5.22), (5.23) the identity

\[
X_{\varepsilon, \text{class}}(s, T) - X_{\varepsilon}^*(s) = \frac{\sigma^2_A(s)}{m_{1,A}(s)} \int_s^T \frac{m_{1,A}(s')}{\sigma^2_A(s')} \left[ \mu_\varepsilon^*(X_{\varepsilon}^*(s'), y, s') - \lambda(X_{\varepsilon}^*(s'), y, s') \right] ds', \quad \tau^*_{\varepsilon,x,T} < s < T.
\]

(5.33)

Let \( \tau_T \) be a stopping time for the diffusion (5.29), run backward in time with terminal time \( T \), such that \( \tau_T > \tau^*_{\varepsilon,x,T} \). Applying the Schwarz inequality in (5.33), we conclude from (5.31) that

\[
q_\varepsilon(x, y, T) \geq E \left[ q_\varepsilon(X_{\varepsilon}^*(\tau_T), y, \tau_T) \right| X_{\varepsilon}^*(T) = x \right] + \frac{\sigma^2_A(T)}{2} E \left[ \frac{X_{\varepsilon, \text{class}}(\tau_T, T) - X_{\varepsilon}^*(\tau_T)}{\sigma^2_A(\tau_T)\sigma^2_A(T)} \right] \left| X_{\varepsilon}^*(T) = x \right|
\]

(5.34)

where we have used the identity

\[
\int_s^T \frac{m_{1,A}(s')^2 ds'}{\sigma^2_A(s')} = \frac{m_{1,A}(s)^2 \sigma^2_A(s)}{\sigma^2_A(T)\sigma^2_A(s)}.
\]

(5.35)

In order to proceed further we first consider a simpler problem in which the function \( \lambda \) is replaced by a constant \(-k\) with \( k > 0 \). We see from (4.15) and (2.18), (2.19) of Conlon and Dabkowski (2021) that the function \( v_\varepsilon \), defined by

\[
v_\varepsilon(x, y, T) = 1 - K_{\varepsilon,D}(x, y, 0, T) = \exp \left[ -\frac{q_\varepsilon(x, y, T)}{\varepsilon} \right],
\]

(5.36)
is a solution to the PDE
\[
\frac{\partial v_\varepsilon(x, y, T)}{\partial T} = -\lambda(x, y, T) \frac{\partial v_\varepsilon(x, y, T)}{\partial x} + \frac{\varepsilon}{2} \frac{\partial^2 v_\varepsilon(x, y, T)}{\partial x^2},
\]
(5.37)
with boundary condition \( \lim_{x \to 0} v_\varepsilon(x, y, T) = 1 \). Replacing the function \( \lambda \) in (5.37) by the constant \(-k\), we are interested in solutions \((x, T) \to v_\varepsilon(x, T)\) to the Dirichlet boundary value problem
\[
\frac{\partial v_\varepsilon(x, T)}{\partial T} = k \frac{\partial v_\varepsilon(x, T)}{\partial x} + \frac{\varepsilon}{2} \frac{\partial^2 v_\varepsilon(x, T)}{\partial x^2}, \quad x, T > 0, \quad v_\varepsilon(0, T) = 1, \quad T > 0.
\]
(5.38)
As in (5.36) we define a function \((x, T) \to q_\varepsilon(x, T)\) by
\[
v_\varepsilon(x, T) = \exp\left[-\frac{q_\varepsilon(x, T)}{\varepsilon}\right], \quad x, T > 0.
\]
(5.39)
Observe that if we set \(q_\varepsilon(x, T) = 2kx\) in (5.39), then \((x, T) \to v_\varepsilon(x, T)\) is a solution to (5.38).

**Proposition 5.2** Let the function \((x, T) \to v_\varepsilon(x, T)\) be the solution to the boundary value problem (5.38) with initial data \(v_\varepsilon(x, 0) = \exp[-q_\varepsilon(x, 0)/\varepsilon], \ x > 0, \) and assume there is a non-negative function \(f : \mathbb{R}^+ \to \mathbb{R}\), independent of \(\varepsilon\), with the property that \(\lim_{x \to \infty} f(x) = \infty\) and \(q_\varepsilon(x, 0) \geq f(x), \ x \geq 0\). Then the function \(q_\varepsilon\) defined by (5.39) has the property \(\lim_{T \to \infty} [q_\varepsilon(x, T) - 2kx]/x = 0\) if \(x > 0\). In addition for any \(M, \varepsilon_0 > 0\), the limit is uniform in the region \(0 < x \leq M, \ 0 < \varepsilon \leq \varepsilon_0\).

**Proof** The solution to (5.38) with initial data \(v_\varepsilon(\cdot, 0)\) has the representation
\[
v_\varepsilon(x, T) = \frac{\varepsilon}{2} \int_0^T \frac{\partial G_{\varepsilon, D}(x, 0, t)}{\partial x'} \, dt + \int_0^\infty G_{\varepsilon, D}(x, x', T) v_\varepsilon(x', 0) \, dx', \quad x, T > 0,
\]
(5.40)
where \(G_{\varepsilon, D}\) is the Dirichlet Green’s function for the PDE (5.38). In the case of (5.38) there is an explicit formula for \(G_{\varepsilon, D}\),
\[
G_{\varepsilon, D}(x, x', t) = \frac{1}{\sqrt{2\pi \varepsilon t}} \exp\left[-\frac{(x - x' + kt)^2}{2\varepsilon t}\right] \left\{ 1 - \exp\left[-\frac{2xx'}{\varepsilon t}\right]\right\}.
\]
(5.41)
We first observe that
\[
\frac{\varepsilon}{2} \int_0^\infty \frac{\partial G_{\varepsilon, D}(x, 0, t)}{\partial x'} \, dt = \exp\left[-\frac{2kx}{\varepsilon}\right].
\]
(5.42)
In fact the LHS of (5.42) is given by
\[
\int_0^\infty \frac{x}{t^{3/2} \sqrt{2\pi \varepsilon}} \exp \left[ -\frac{(x + kt)^2}{2\varepsilon t} \right] \, dt
\]
\[
= \exp \left[ -\frac{kx}{\varepsilon} \right] \int_0^\infty \frac{x}{t^{3/2} \sqrt{2\pi \varepsilon}} \exp \left[ -\frac{x^2}{2\varepsilon t} - \frac{k^2 t}{2\varepsilon} \right] \, dt
\]
\[
= \exp \left[ -\frac{kx}{\varepsilon} \right] 2\sqrt{\varepsilon} \int_0^\infty s g(s) e^{-zs} \, ds,
\]
(5.43)
where \( z, g(\cdot) \) are given by the formulae
\[
z = \frac{k^2 x^2}{\varepsilon^2}, \quad g(s) = \frac{1}{2\sqrt{\pi} s^{3/2}} \exp \left[ -\frac{1}{4s} \right].
\]
(5.44)
Now (5.42) follows since the Laplace transform \( \mathcal{L} g(z) = \exp[-\sqrt{z}] \).

To obtain an upper bound on \( \limsup_{T \to \infty} q_\varepsilon(x, T)/x \) we show that for any \( \delta > 0 \) there exists \( T_\delta > 0 \) such that
\[
v_\varepsilon(x, T) \geq \exp \left[ -\frac{2kx(1+\delta)}{\varepsilon} \right] \quad \text{if} \quad 0 < x \leq M, \quad 0 < \varepsilon \leq \varepsilon_0, \quad T \geq T_\delta.
\]
(5.45)
The inequality (5.45) follows from (5.42) and the inequality
\[
\varepsilon \int_T^\infty \partial G_{\varepsilon, D}(x, 0, t) \, dt \leq \left( \frac{2\varepsilon}{\pi} \right)^{1/2} \frac{x}{k^2 T^{3/2}} \exp \left[ -\frac{k^2 T}{2\varepsilon} \right].
\]
(5.46)
To obtain a lower bound on \( \liminf_{T \to \infty} q_\varepsilon(x, T)/x \), we show that for any \( \delta > 0 \) there exists \( T_\delta > 0 \) such that
\[
\int_0^\infty G_{\varepsilon, D}(x, x', T) v_\varepsilon(x', 0) \, dx' \leq e^{-2kx/\varepsilon} \left( e^{\delta x/\varepsilon} - 1 \right) = e^{-(2k-\delta)x/\varepsilon} \left[ 1 - e^{-\delta x/\varepsilon} \right]
\]
\[
\leq \frac{\delta x}{\varepsilon} e^{-(2k-\delta)x/\varepsilon} \quad \text{if} \quad T \geq T_\delta, \quad 0 < x \leq M, \quad 0 < \varepsilon \leq \varepsilon_0.
\]
(5.47)
To prove (5.47) we first observe there exists \( T_0 > 0 \) such that
\[
\inf_{x' > 0} \left[ \frac{(x - x' + kT)^2}{2T} + f(x') \right] \geq 2kM \quad \text{for all} \quad x > 0, \quad \text{if} \quad T \geq T_0.
\]
(5.48)
Evidently \( T_0 \) must satisfy the inequalities
\[
T_0 \geq \frac{4M}{k}, \quad f(kT_0) \geq 2kM.
\]
(5.49)
From (5.41), (5.48) we see that the LHS of (5.47) is bounded above by
\[ e^{-(2k-\delta)M/\varepsilon} \frac{1}{\sqrt{2\pi \varepsilon T}} \int_0^\infty dx' \frac{2xx'}{\varepsilon T} \exp \left[ -\frac{\delta}{2k\varepsilon} \left\{ \frac{(x-x'+kT)^2}{2T} + f(x') \right\} \right] , \]
(5.50)
provided \( T \geq T_0 \). It follows from (5.47), (5.50) it is sufficient to choose \( T_\delta \geq T_0 \) such that
\[ \frac{1}{\sqrt{2\pi \varepsilon T}} \int_0^\infty dx' \frac{2xx'}{T} \exp \left[ -\frac{\delta}{2k\varepsilon} \left\{ \frac{(x-x'+kT)^2}{2T} + f(x') \right\} \right] \leq \delta \]
(5.51)
if \( T \geq T_\delta \). Evidently \( T_\delta \) may be chosen uniformly for \( 0 < x \leq M, \ 0 < \varepsilon \leq \varepsilon_0 \).

The proof of Proposition 5.2 depends heavily on using the explicit formula (5.41) for the Dirichlet Green’s function. No such formula exists in the general case of the drift (5.3). We therefore give an alternative proof of Proposition 5.2, using the representation of \( q_\varepsilon(x, T) \) as the cost function of a stochastic control problem. We will then generalize this approach to the case of the drift \( \lambda \) of (5.3).

**Proof of Proposition 5.2 upper bound** As in the proof of Lemma 2.1 of Conlon and Dabkowski (2021), we have the inequality
\[ q_\varepsilon(x, T) \leq E \left[ \frac{1}{2} \int_{\tau_{e,x,T} \wedge 0}^T \left[ \mu_\varepsilon(X_\varepsilon(s), s) + k \right]^2 ds \left| X_\varepsilon(T) = x \right. \right. \]
\[ + E \left[ q_\varepsilon(X_\varepsilon(\tau_{e,x,T} \wedge 0), \tau_{e,x,T} \wedge 0) \left| X_\varepsilon(T) = x \right. \right] , \]
(5.52)
where \( X_\varepsilon(s), s \leq T, \) is the solution to the SDE (5.29), and \( \tau_{e,x,T} \) is the first exit time of \( X_\varepsilon(\cdot) \) from the half line \((0, \infty)\). We define the drift \( \mu_\varepsilon \) by
\[ \mu_\varepsilon(x, s) = k \text{ if } 1 < s < T, \ \mu_\varepsilon(x, s) = \frac{x}{s} \text{ if } 0 < s < 1. \]
(5.53)
Arguing as in the proof of Lemma 2.1 of Conlon and Dabkowski (2021), we see that \( \tau_{e,x,T} > 0 \) with probability 1. Hence the second term on the RHS of (5.52) is zero. It follows then from (5.52) that
\[ q_\varepsilon(x, T) \leq 2k^2E[T - \tau_{e,x,T}^a] \]
\[ + E \left[ \frac{1}{2} \int_{\tau_{e,x,T} \wedge 1}^1 \left[ \mu_\varepsilon(X_\varepsilon(s), s) + k \right]^2 ds \left| X_\varepsilon(T) = x \right. \right] , \]
(5.54)
where \( \tau_{e,x,T}^a \) is the first exit time from the half line \((0, \infty)\) for the diffusion with the constant drift \( k \). It is easy to see that \( E[T - \tau_{e,x,T}^a] = x/k \). To estimate the second term on the RHS of (5.54) we consider a diffusion run backward in time from \( s = 1 \).
conditioned on $X_{\varepsilon}(1) = x$, which satisfies the SDE (5.29) with $\mu_{\varepsilon}(x, s) = x/s$. The solution to (5.29) is then given by the formula

$$X_{\varepsilon}(s) = s \left[ x - \sqrt{\varepsilon} Z(s) \right], \quad Z(s) = \int_s^1 \frac{dB(s')}{s'}.$$  \hspace{1cm} (5.55)

Let $\tau_{\varepsilon,x} < 1$ be the first exit time from $(0, \infty)$ for $X_{\varepsilon}(\cdot)$ with $X_{\varepsilon}(1) = x$. Then using (5.55) and arguing as in the proof of Lemma 2.2 of Conlon and Dabkowski (2021), we obtain the identity

$$E \left[ \frac{1}{2} \int_{\tau_{\varepsilon,x}}^1 \frac{[X_{\varepsilon}(s)/s + k]^2}{s} \, ds \mid X_{\varepsilon}(1) = x \right] = \frac{(x + k)^2}{2} E[1 - \tau_{\varepsilon,x}] + x(x + k)E[\tau_{\varepsilon,x}] + \frac{\varepsilon}{2} E \left[ \int_{\tau_{\varepsilon,x}}^1 Z(s)^2 \, ds \right].$$  \hspace{1cm} (5.56)

Following the argument of Conlon and Dabkowski (2021), we see that

$$E \left[ \int_{\tau_{\varepsilon,x}}^1 Z(s)^2 \, ds \right] \leq C \left[ 1 + |\log \varepsilon| + x \right],$$  \hspace{1cm} (5.57)

with constant $C$ independent of $\varepsilon, x$ for $x > 0, 0 < \varepsilon \leq \varepsilon_0$. Since $0 < \tau_{\varepsilon,x} < 1$ one obtains from (5.56), (5.57) an upper bound on the LHS of (5.56), which we denote by $F_{\varepsilon}(x)$. It follows then from (5.54), noting that (5.41) with $k$ replaced by $-k$ and $t = T - 1$ gives the distribution of $X_{\varepsilon}(1)$ on paths $X_{\varepsilon}(s)$ with $\tau_{\varepsilon,x,T} < 1$, that

$$q_{\varepsilon}(x, T) \leq 2k + \frac{1}{\sqrt{2\pi \varepsilon(T - 1)}} \int_0^\infty \exp \left[ -\frac{(x - x' - k(T - 1))^2}{2\varepsilon(T - 1)} \right] \frac{1}{x} \left\{ 1 - \exp \left[ -\frac{2xx'}{\varepsilon(T - 1)} \right] \right\} F_{\varepsilon}(x') \, dx'.$$  \hspace{1cm} (5.58)

The integral in (5.58) converges to 0 as $T \to \infty$, uniformly for $0 < \varepsilon \leq \varepsilon_0, 0 < x \leq M$. We conclude that $\limsup_{T \to \infty} q_{\varepsilon}(x, T)/x \leq 2k$ and the limit is uniform for $0 < \varepsilon \leq \varepsilon_0, 0 < x \leq M$. \hfill \square

**Proof of Proposition 5.2 lower bound** Let $\mu_{\varepsilon}^*$ be the optimal controller

$$\mu_{\varepsilon}^*(x, T) = -k + \frac{\partial q_{\varepsilon}(x, T)}{\partial x}, \quad x, T > 0,$$  \hspace{1cm} (5.59)

and $X_{\varepsilon}^*(\cdot)$ be the solution to the SDE (5.29) with $\mu_{\varepsilon} = \mu_{\varepsilon}^*$. Denoting by $\tau_{\varepsilon,x,T}^*$ the first exit time from the half line $(0, \infty)$ for the diffusion $X_{\varepsilon}^*(s), s \leq T$, with
\(X_\varepsilon^*(T) = x > 0\), we have an identity similar to (5.31),

\[
q_\varepsilon(x, T) = E \left[ \frac{1}{2} \int_{\tau^*_{x,T} > 0} \left[ \mu_\varepsilon(X_\varepsilon(s), s) + k \right]^2 \, ds \mid X_\varepsilon^*(T) = x \right] + E \left[ q_\varepsilon(X_\varepsilon(\tau^*_{\varepsilon,x,T} \lor 0), \tau^*_{\varepsilon,x,T} \lor 0) \mid X_\varepsilon^*(T) = x \right].
\]  

(5.60)

Let \(X_\varepsilon, \text{class}(s), s \leq T\), be the solution to (5.29) with \(\mu_\varepsilon = -k\). Evidently, conditioned on \(X_\varepsilon, \text{class}(T) = x\), one has the formula

\[
X_\varepsilon, \text{class}(s) = x + k(T - s) + \sqrt{\varepsilon}B(s), \quad s \leq T.
\]  

(5.61)

Arguing as in (5.33), (5.34) we obtain from (5.60) the lower bound

\[
q_\varepsilon(x, T) \geq E \left[ \frac{1}{2} \left( X_\varepsilon, \text{class}(\tau) - X_\varepsilon^*(\tau) \right)^2 \right] + q_\varepsilon(X_\varepsilon^*(\tau), 0) \mid X_\varepsilon^*(T) = x \right]
\]  

(5.62)

where \(\tau\) is any stopping time satisfying \(\tau \geq \tau^*_{\varepsilon,x,T} \lor 0\). We use the fact that

\[
\inf_{\tau < T} \frac{X_0, \text{class}(\tau)^2}{2(T - \tau)} = \inf_{\tau < T} \frac{(x + k(T - \tau))^2}{2(T - \tau)} = 2kx,
\]  

(5.63)

where the minimizing \(\tau\) satisfies \(T - \tau = x/k\). We can obtain a lower bound on \(q_\varepsilon(x, T)\) from (5.62), (5.63) if we show for any \(\delta > 0\) that with high probability \(X_\varepsilon, \text{class}(s) \geq (1 - \delta)X_0, \text{class}(s), s \leq T\), provided \(X_\varepsilon, \text{class}(T) = X_0, \text{class}(T) = x\) is sufficiently large. Thus we need to compute the probability that

\[
X_\varepsilon, \text{class}(s) - (1 - \delta)X_0, \text{class}(s) = \delta x + \delta k(T - s) + \sqrt{\varepsilon}B(s) > 0 \quad \text{for} \ s < T.
\]  

(5.64)

This probability is given by 1 minus the RHS of (5.42) with \(\delta x\) replacing \(x\) and \(\delta k\) replacing \(k\), whence the probability of the event (5.64) is \(1 - \exp[-2\delta^2 k x / \varepsilon]\). Choosing \(\tau = \tau^*_{\varepsilon,x,T} \lor 0\) in (5.62), we conclude that

\[
q_\varepsilon(x, T) \geq \left\{ 1 - \exp \left[ -\frac{2\delta^2 k x}{\varepsilon} \right] \right\} \min \left[ 2k(1 - \delta)^2 x, \inf_{x > (1 - \delta)(x + k T), x' > 0} \left\{ \frac{X - x'}{2T} + q_\varepsilon(x', 0) \right\} \right].
\]  

(5.65)

Since we are assuming that \(q_\varepsilon(x', 0) \geq f(x')\) and \(\lim_{x' \to \infty} f(x') = \infty\), we obtain on taking the limit \(T \to \infty\) in (5.65) the inequality,

\[
\liminf_{T \to \infty} \frac{q_\varepsilon(x, T)}{x} \geq 2k(1 - \delta)^2 \left\{ 1 - \exp[-2\delta^2 k x / \varepsilon] \right\}.
\]  

(5.66)
The limit in (5.66) is uniform for $0 < x \leq M$, $0 < \varepsilon \leq \varepsilon_0$. Evidently (5.66) yields a lower bound close to $2k$ if $x/\varepsilon >> 1$.

We wish to obtain a lower bound, which also holds as $x \to 0$. Let $\tau_{\varepsilon, \text{class}, x, T}$, $s < T$, be the first exit time from the half line $(0, \infty)$ for the diffusion $X_{\varepsilon, \text{class}}(s)$, $s < T$, with $X_{\varepsilon, \text{class}}(T) = x$. Since the function $x \to q_\varepsilon(x, T)$, $x > 0$, is increasing, it follows from (5.59) that $\tau_{\varepsilon, x, T}^* > \tau_{\varepsilon, \text{class}, x, T}$. We also have that

$$P(\tau_{\varepsilon, \text{class}, x, T} < T - t) = \int_0^\infty G_{\varepsilon, D}(x, x', t) \, dx', \quad (5.67)$$

where $G_{\varepsilon, D}$ is given by (5.41). Evidently the function $t \to P(\tau_{\varepsilon, \text{class}, x, T} < T - t)$ is decreasing, and from (5.41), (5.67) we see that

$$\lim_{t \to \infty} P(\tau_{\varepsilon, \text{class}, x, T} < T - t) = 1 - \exp \left[ -\frac{2kx}{\varepsilon} \right], \quad t > 0. \quad (5.68)$$

Note that (5.42) is the integral of the pdf of $\tau_{\varepsilon, \text{class}, x, T}$, whence the RHS of (5.42) and the RHS of (5.68) add up to 1. If $x/\varepsilon \leq K$ for some constant $K$, the asymptotic limit on the RHS of (5.68) is closely approximated, uniformly for $0 < \varepsilon \leq \varepsilon_0$, when $t \geq N\varepsilon$ where $N >> 1$ is independent of $\varepsilon$. The inequality (5.46) gives a quantitative estimate of this.

We assume $x$ satisfies $0 < x \leq Nk\varepsilon/2$ with $N \geq 1$ and take $\tau = \tau_{\varepsilon, x, T}^* \vee (T - N\varepsilon)$ in (5.62). Using the quadratic inequality in (5.62) and the formula (5.61), we obtain a lower bound

$$q_\varepsilon(x, T) \geq \rho \left[ 2kx P(\tau_{\varepsilon, x, T}^* > T - N\varepsilon) - 2k\sqrt{\varepsilon}E \left[ B(T - N\varepsilon); \tau_{\varepsilon, x, T}^* < T - N\varepsilon \right] \right]$$

$$+ E \left[ \frac{X_{\varepsilon, \text{class}}(T - N\varepsilon) - X_{\varepsilon}(T - N\varepsilon)}{2N\varepsilon} \right]^2$$

$$+ q_\varepsilon(X_{\varepsilon}(T - N\varepsilon), 0); \tau_{\varepsilon, x, T}^* < T - N\varepsilon \right], \quad (5.69)$$

for any $\rho$ satisfying $0 \leq \rho \leq 1$. In (5.69) we have used the identity $E[B(\tau)] = 0$. Observe from (5.66) that for any $M > 0$ there exists $N_0 \geq 1$, $T_0 > 0$ such that

$$\inf_{X > Nk\varepsilon/2, \ v > 0} \left\{ \frac{(X - x')^2}{2N\varepsilon} + q_\varepsilon(x', T - N\varepsilon) \right\}$$

$$\geq \frac{Nk^2\varepsilon}{32} \quad \text{if } N \geq N_0, \ T \geq T_0, \ Nk\varepsilon \leq M. \quad (5.70)$$

We also have as in (5.67) that

$$P\left( \tau_{\varepsilon, \text{class}, x, T} < T - N\varepsilon, X_{\varepsilon, \text{class}}(T - N\varepsilon) < Nk\varepsilon/2 \right)$$

$$= \int_0^{Nk\varepsilon/2} G_{\varepsilon, D}(x, x', N\varepsilon) \, dx' \leq \frac{C \varepsilon}{\varepsilon} \exp \left[ -\frac{Nk^2}{16} \right], \quad (5.71)$$
for some constant $C$, independent of $x, \varepsilon, N$ in the range $x > 0$, $0 < \varepsilon \leq \varepsilon_0$, $N \geq 1$. Suppose now that

$$P \left( \tau_{\varepsilon,x,T}^* < T - N\varepsilon \right) \geq \frac{2Cx^2}{\varepsilon} \exp \left[ -\frac{Nk^2}{16} \varepsilon \right]. \quad (5.72)$$

Setting $\rho = 0$ in (5.69) and using the fact that $\tau_{\varepsilon,x,T}^* > \tau_{\varepsilon,\text{class},x,T}$, we conclude from (5.69)–(5.71) that if (5.72) holds then

$$q_{\varepsilon}(x, T) \geq \frac{Nk^2\varepsilon}{64} P \left( \tau_{\varepsilon,x,T}^* < T - N\varepsilon \right) \quad \text{if} \quad N \geq N_0, \ T \geq T_0, \ Nk\varepsilon \leq M. \quad (5.73)$$

Evidently (5.73) yields the lower bound on $q_{\varepsilon}(x, T)$ unless

$$P \left( \tau_{\varepsilon,x,T}^* < T - N\varepsilon \right) \leq \frac{128x}{Nk\varepsilon}. \quad (5.74)$$

We assume now that $0 < x \leq Nk\varepsilon/2$ and that (5.74) holds. We apply the inequality (5.69) with $\rho = 1$ and divide the second term on the RHS of (5.69) into the expectation over $B(T - N\varepsilon) \leq N^{2/3}/\varepsilon$ and $B(T - N\varepsilon) \geq N^{2/3}/\varepsilon$. Using (5.61) we then have that

$$q_{\varepsilon}(x, T) \geq 2kx \left[ 1 - \frac{128x}{Nk\varepsilon} - \frac{128}{N^{1/3}k} \right] - 2kE[X_{\varepsilon,\text{class}}(T - N\varepsilon) - X_{0,\text{class}}(T - N\varepsilon) ; \ \tau_{\varepsilon,\text{class},x,T} < T - N\varepsilon, \ X_{\varepsilon,\text{class}}(T - N\varepsilon) - X_{0,\text{class}}(T - N\varepsilon) > N^{2/3}\varepsilon]. \quad (5.75)$$

The expectation in (5.75) is given by

$$\int_{x+Nk\varepsilon+N^{2/3}\varepsilon}^{\infty} G_{\varepsilon,D}(x, x', N\varepsilon)[x' - (x + Nk\varepsilon)] \, dx' \leq Cx \exp \left[ -N^{1/6} \right] \quad (5.76)$$

where $C$ is a constant which can be chosen uniformly for $0 < x \leq Nk\varepsilon/2$, $0 < \varepsilon \leq \varepsilon_0$, $N \geq N_0$.

To obtain the lower bound on $q_{\varepsilon}(x, T)$ we first choose $N \geq N_0$ large and $M$ satisfying $Nk\varepsilon_0 = M$. Then (5.66) holds and hence the lower bound up to an $O(1/N^{1/3})$ correction, when $Nk\varepsilon/2 \leq x \leq M$. In the case $0 < x \leq Nk\varepsilon/2$ the argument has just been given. \hfill \Box

**Proposition 5.3** Assume the function $A(\cdot)$ satisfies the conditions of Proposition 5.1. Assume also there exists $\delta_0, T_0 > 0$ such that

$$A(T + 1/A(T)) \leq \frac{A(T)}{1 + \delta_0} \quad \text{if} \quad T \geq T_0. \quad (5.77)$$
Letting $q_\varepsilon(x, y, T)$ be the function defined by (4.1), (4.15), then for all $x$, $y > 0$ one has $\lim_{T \to \infty} \frac{q_\varepsilon(x, y, T)}{2x} = 1$. In addition the limit is uniform in any region $0 < x \leq M$, $0 < \varepsilon \leq \varepsilon_0$, $y_0 < y < y_\infty$, where $\varepsilon_0$, $M > 0$ and $0 < y_0 < y_\infty < \infty$.

**Proof** In view of (5.4) it is sufficient to establish a lower bound. Since $\lim_{T \to \infty} \lambda(x, y, T) = -1$ we may use a similar argument to the one just given of the lower bound in Proposition 5.2. We replace the assumption in Proposition 5.2 on the initial data by the following:

For any $\varepsilon, \delta, y_0 > 0$, there exists a function $f : [1, \infty) \to \mathbb{R}$ such that

$$\lim_{T \to \infty} f(T) = \infty, \quad q_\varepsilon(\delta/A(T), y, T) \geq f(T) \quad \text{for } 0 < \varepsilon \leq \varepsilon_0, \quad y \geq y_0.$$  

(5.78)

The extra assumption (5.77) on $A(\cdot)$ is needed for the proof of (5.78).

To begin the proof of (5.78) we consider any $T_0 \geq 1$ and define the stopping time $\tau_T$ by $\tau_T = \tau_{\varepsilon,x,T} \vee T_0$. We use the formula (5.34), whence we have the inequality

$$q_\varepsilon(x, y, T) \geq E\left[q_\varepsilon(X_\varepsilon^*(T_0), y, T_0); \quad \tau_{\varepsilon,x,T}^* \leq T_0\right] + \frac{\sigma_A^2(T)}{2} E\left[\frac{(X_\varepsilon,\text{class}(T_0, T) - X_\varepsilon^*(T_0))^2}{\sigma_A^2(T_0)\sigma_A^2(T, T)}; \quad \tau_{\varepsilon,x,T}^* \leq T_0\right]. \quad (5.79)$$

We see from (5.32), (5.35) that

$$\text{Var}\left[\sqrt{\varepsilon} \frac{\sigma_A^2(s)}{m_{1,A}(s)} Z(s)\right] = \varepsilon \frac{\sigma_A^2(s)\sigma_A^2(s, T)}{\sigma_A^2(T)} \leq \varepsilon \sigma_A^2(s), \quad 0 < s < T. \quad (5.80)$$

From Proposition 3.3 of Conlon and Dabkowski (2021) one has that the inequality (5.27) extends to $\varepsilon > 0$, so

$$q_\varepsilon(x, y, T) \geq \frac{2m_{1,A}(T)xy}{\sigma_A^2(T)}, \quad x, y, T > 0. \quad (5.81)$$

It follows then from (5.7), (5.32), (5.79)–(5.81) that for any $\rho, K > 0$ there exists $T_{\rho,K} > T_0$ such that

$$q_\varepsilon(x, y, T) \geq K \quad \text{for } y \geq y_0, \quad 0 < \varepsilon \leq \varepsilon_0, \quad T \geq T_{\rho,K}, \quad \text{if } P(\tau_{\varepsilon,x,T}^* \leq T_0) \geq \rho. \quad (5.82)$$

From (5.82) we may assume in the remainder of the argument that $\tau_{\varepsilon,x,T}^* > T_0$ with probability at least $1 - \rho$, where $\rho > 0$ may be taken arbitrarily small.

Next we consider $T > 1/A(0)$. Since the function $s \mapsto s + 1/A(s)$, $s > 0$, is strictly increasing, there exists unique $\tilde{T} < T$ such that $\tilde{T} + 1/A(\tilde{T}) = T$. It follows
from (5.77) that \( A(T) \leq A(\tilde{T})/(1 + \delta_0) \). We have from (5.6), (5.7) that

\[
x_{\text{class}}(s, T) \geq \frac{\sigma_A^2(s)}{m_{1,A}(s)^2} \left( \frac{\sigma_A^2(T)}{m_{1,A}(T)^2} \right)^{-1} \frac{[x + m_{2,A}(s, T)]}{m_{1,A}(s, T)} - \frac{1}{m_{1,A}(s)} \frac{\sigma_A^2(s, T)}{m_{1,A}(s, T)^2} \left( \frac{\sigma_A^2(T)}{m_{1,A}(T)^2} \right)^{-1} \frac{m_{2,A}(s)}{m_{1,A}(s)}, \quad 0 < s < T.
\]

(5.83)

It follows from (5.1) (d),(e) and (5.83) that for any \( \nu > 0 \) there exists \( T_\nu > 1/A(0) \) such that

\[
x_{\text{class}}(\tilde{T}, T) \geq \frac{1}{e(1 + \nu)} \left[ x + \frac{1}{A(\tilde{T})} \right] \text{ if } x > 0, \ T \geq T_\nu.
\]

(5.84)

From (5.32) we have that

\[
X_{\varepsilon, \text{class}}(s, T) = x_{\text{class}}(s, T) - \sqrt{e} \Phi(s, T) \tilde{Z}(s, T),
\]

where \( \Phi(s, T) = \frac{\sigma_A^2(s)m_{1,A}(s, T)}{\sigma_A^2(T)}, \quad \tilde{Z}(s, T) = \int_s^T \frac{dB(s')}{\Phi(s', T)}. \) (5.85)

The function \( s \rightarrow \Phi(s, T) \) is positive increasing and \( \Phi(T, T) = 1 \). We also have that

\[
\Phi(\tilde{T}, T) = \frac{\sigma_A^2(\tilde{T})m_{1,A}(\tilde{T}, T)}{\sigma_A^2(T)} = \frac{\sigma_A^2(\tilde{T})m_{1,A}(\tilde{T}, T)}{m_{1,A}(\tilde{T}, T)^2 \sigma_A^2(\tilde{T}) + \sigma_A^2(\tilde{T}, T)}.
\]

(5.86)

Since \( \sigma_A^2(\tilde{T}, T) \leq m_{1,A}(\tilde{T}, T)^2/A(\tilde{T}) \) and \( 1 \leq m_{1,A}(\tilde{T}, T) \leq e \), we can obtain a lower bound on \( \Phi(\tilde{T}, T) \), independent of \( T \) as \( T \to \infty \), if we can bound above \( 1/\sigma_A^2(\tilde{T}) \) by a constant times \( A(\tilde{T}) \). Note this is tantamount to the linear term in the formula (5.3) for the drift \( x \rightarrow \lambda(x, y, T) \) being dominated by \( A(T)x \). From (5.1) (e) we see that \( \lim_{T \to \infty} m_{1,A}(\tilde{T}, T) \Phi(\tilde{T}, T) = 1 \), whence the lower bound follows. From (5.35) we see that

\[
\text{Var}[\tilde{Z}(s, T)] = \frac{\sigma_A^2(s, T)}{m_{1,A}(s, T)\Phi(s, T)}.
\]

(5.87)

We also have by the reflection principle that

\[
P \left( \sup_{s < s' < T} \tilde{Z}(s', T) > a \right) = 2P \left( \tilde{Z}(s, T) > a \right), \quad a > 0.
\]

(5.88)

The identities (5.85)–(5.88) enable us to bound below the diffusive path \( X_{\varepsilon, \text{class}}(\cdot, T) \) with high probability.
Observe that the RHS of (5.6) is non-negative if \( x, y \geq 0 \) and \( A(\cdot) \) is non-negative. Hence similarly to (5.84), we may choose \( T_v \) large enough so that

\[
x_{\text{class}}(s, T) \geq \frac{x}{e(1 + \nu)} \quad \text{if } x > 0, \; T \geq T_v, \; \tilde{T} \leq s \leq T.
\]

From (5.87), (5.88) and (A.9) we have that

\[
\begin{align*}
\mathbb{P}
\left(
\sup_{\tilde{T} < s' < T} \tilde{Z}(s', T) > a
\right)
&\leq \left(\frac{2}{\pi}\right)^{1/2}\frac{\sigma_T}{a} \exp\left[-\frac{a^2}{2\sigma_T^2}\right], \\
\sigma_T^2 &= \frac{\sigma_A^2(\tilde{T}, T)}{m_{1, A}(\tilde{T}, T) \Phi(\tilde{T}, T)}.
\end{align*}
\]

(5.90)

We choose \( T_v \) large enough so that

\[
\sigma_T^2 \leq \frac{e^2(1 + \nu)}{A(\tilde{T})} \quad \text{if } T \geq T_v,
\]

(5.91)

and define the event \( \mathcal{E}_T \) by

\[
\mathcal{E}_T = \left\{ \sup_{\tilde{T} < s < T} \sqrt{\varepsilon_0} \; \Phi(s, T) \tilde{Z}(s, T) < \frac{1}{A(\tilde{T})^{2/3}} \right\}.
\]

(5.92)

Choosing \( a \approx 1/A(\tilde{T})^{2/3} \) in (5.90) we see from (5.91) and (5.1)(a) there exists \( T_1 > 1/A(0) \) such that

\[
\mathbb{P}(\mathcal{E}_T) \geq 1 - \exp\left[-\frac{1}{A(\tilde{T})^{1/4}}\right], \; T \geq T_1.
\]

(5.93)

We conclude from (5.84), (5.85), (5.89), (5.92) that for any \( \delta \) satisfying \( 0 < \delta < 1/3 \), there exists \( T_\delta \geq T_1 \) such that

\[
\inf_{\tilde{T} < s < T} X_{\varepsilon, \text{class}}(s, T) \geq \frac{\delta}{3A(T)} \quad \text{and} \quad X_{\varepsilon, \text{class}}(\tilde{T}, T) \geq \frac{1}{3} \left[ \frac{\delta}{A(T)} + \frac{1}{A(\tilde{T})} \right]
\]

on the event \( \mathcal{E}_T \) when \( x = \frac{\delta}{A(T)} \), \( T \geq T_\delta \).

(5.94)

It follows from (5.94) that

\[
\frac{\sigma_A^2(T)}{2} \frac{X_{\varepsilon, \text{class}}(s, T)}{\sigma_A^2(s, T)} \geq \frac{\delta^2}{20A(T)}, \quad \frac{\sigma_A^2(T)}{2} \frac{[X_{\varepsilon, \text{class}}(\tilde{T}, T) - x']^2}{\sigma_A^2(\tilde{T}) \sigma_A^2(\tilde{T}, T)} \geq \frac{\delta^2}{20A(T)}
\]

on the event \( \mathcal{E}_T \) when \( x = \frac{\delta}{A(T)}, \; T \geq T_\delta, \) if \( \tilde{T} < s \leq T, \; 0 < x' < \frac{\delta}{A(T)} \).

(5.95)
Let $\Omega_T$ be the event

$$\Omega_T = \left\{ x = \frac{\delta}{A(T)}, \tilde{T} < \tau_{e,x,T}^* < T \right\} \cup \left\{ x = \frac{\delta}{A(T)}, \tau_{e,x,T}^* \leq \tilde{T}, 0 < X_{\tilde{T}}^* < \frac{\delta}{A(\tilde{T})} \right\}. \quad \text{(5.96)}$$

We conclude from (5.34), (5.95), (5.96) that

$$q_{\epsilon}\left(\frac{\delta}{A(T)}, y, T\right) \geq P(\Omega_T^c \cap \mathcal{E}_T) q_{\epsilon}\left(\frac{\delta}{A(\tilde{T})}, y, \tilde{T}\right) + P(\Omega_T \cap \mathcal{E}_T) \frac{\delta^2}{20A(T)}. \quad \text{(5.97)}$$

where $\Omega_T^c$ is the complement of $\Omega_T$.

We choose any $\delta$ with $0 < \delta < 1/3$ and $T_\delta$ as in (5.94). For $T > T_\delta + 1/A(T_\delta)$ we may define a sequence of times $T_n, n = 1, 2, ..., N$, with $T_\delta \leq T_1 < T_\delta + 1/A(T_\delta), T_N = T$, and $T_{n+1} = T_n + 1/A(T_n), n = 1, 2, ..., N - 1$. We define events $\Omega_n, n = 1, 2, ..., N - 1$, by

$$\Omega_n = \{ T_{n+1} \geq \tau_{e,x,T}^* > T_n, X_{\tilde{T}}^*(T_m) \geq \frac{\delta}{A(T_m)}, m = n + 1, \ldots, N \}$$

$$\cup \{ \tau_{e,x,T}^* \leq T_n, X_{\tilde{T}}^*(T_n) < \frac{\delta}{A(T_n)} \} \quad \text{(5.98)}$$

where $x = \frac{\delta}{A(T)}$. The events $\Omega_n$ are disjoint and

$$\sum_{n=1}^{N-1} P(\Omega_n) + P\left( \tau_{e,x,T}^* \leq T_1, X_{\tilde{T}}^*(T_m) \geq \frac{\delta}{A(T_m)}, m = 1, \ldots, N \right) = 1. \quad \text{(5.99)}$$

We define the probabilities

$$p_{j-1} = \frac{P(\Omega_{T_j} \cap \mathcal{E}_{T_j})}{P(\mathcal{E}_{T_j})}, \quad \lambda_{j-1} = P(\mathcal{E}_{T_j}), \quad j = 2, \ldots, N. \quad \text{(5.100)}$$

Then (5.97) implies that

$$q_{\epsilon}\left(\frac{\delta}{A(T_n)}, y, T_n\right) \geq \lambda_{n-1} \left[ (1 - p_{n-1}) q_{\epsilon}\left(\frac{\delta}{A(T_{n-1})}, y, T_{n-1}\right) + p_{n-1} \frac{\delta^2}{20A(T_n)} \right], \quad n = 2, \ldots, N. \quad \text{(5.101)}$$

Iterating the inequality (5.101) starting with $n = N$ down to $n = 2$, we conclude that

$$q_{\epsilon}\left(\frac{\delta}{A(T)}, y, T\right) \geq \prod_{n=1}^{N-1} \lambda_n \left[ 1 - \prod_{n=1}^{N-1} (1 - p_j) \right] \frac{\delta^2}{20A(T_\delta)}. \quad \text{(5.102)}$$
We also have that
\[ P(\Omega_{T_n}) \geq P(\Omega_{n-1}), \quad n = 2, \ldots, N, \]
whence we see that
\[
\sum_{n=1}^{N-1} p_n \geq \sum_{n=2}^{N} P(\Omega_{T_n}) - \sum_{n=2}^{N} P(\mathcal{E}^c_{T_n}) \geq \sum_{n=1}^{N-1} P(\Omega_n) - \sum_{n=2}^{N} P(\mathcal{E}^c_{T_n}). \tag{5.103}
\]

It follows from (5.77), (5.93) that \( T_1 \) in (5.93) may be chosen so that
\[
N \sum_{n=1}^{N-1} \lambda_n \geq \frac{1}{4} \text{ if } T_\delta \geq T_1. \tag{5.104}
\]

Suppose now that
\[
N \sum_{n=1}^{N-1} p_n \geq \frac{1}{2}. \tag{5.105}
\]

It follows then from (5.103)–(5.105) that
\[
\prod_{n=1}^{N-1} (1 - p_j) \leq \exp \left[ - \sum_{n=1}^{N-1} p_j \right] \leq \exp \left[ - \frac{1}{4} \right]. \tag{5.106}
\]

We conclude from (5.102), (5.104), (5.106) that if (5.105) holds then
\[
q_\varepsilon(\delta/A(T), y, T) \geq \frac{1}{2} \left[ 1 - e^{-1/4} \right] \frac{\delta^2}{20A(T_\delta)}. \tag{5.107}
\]

To prove (5.78) we observe first that if (5.105) holds then \( q_\varepsilon(\delta/A(T), y, T) \) is bounded below by the RHS of (5.107) for \( T > T_\delta + 1/A(T_\delta) \). This bound is uniform for all \( \varepsilon, y \) satisfying \( 0 < \varepsilon \leq \varepsilon_0, \ y > 0 \). In the case when (5.105) does not hold we have from (5.99) that
\[
P \left( \tau_{\varepsilon,x,T}^* \leq T_1 \right) \geq \frac{1}{2}, \quad x = \frac{\delta}{A(T)}. \tag{5.108}
\]

Then (5.82), (5.108) imply a lower bound on \( q_\varepsilon(\delta/A(T), y, T) \) for large \( T \) provided \( 0 < \varepsilon \leq \varepsilon_0, \ y \geq y_0 \). Since \( K \) in (5.82) may be chosen arbitrarily large, and \( A(T_\delta) \) in (5.107) arbitrarily small, the lower bound (5.78) follows.

We proceed now as in the proof of the lower bound in Proposition 5.2 beginning at (5.59), replacing the condition on the initial data in Proposition 5.2 by (5.78). Thus we consider \( q_\varepsilon(x, y, T) \) for large \( T \) and use (5.78) at time \( \tilde{T} \). We take \( \tau_T = \max \{ \tau_\varepsilon^{*,x,T}, \tilde{T} \} \) in (5.34). It follows from (5.1) (d),(e), (5.6), (5.7) that for any \( \nu > 0 \) there exists \( T_v > 1/A(0) \) such that if \( T \geq T_v \) then
\[
x_{\text{class}}(s, T) \geq \frac{1}{(1 + \nu)m_{1,A}(s, T)} \left[ x + m_{2,A}(s, T) \right] \quad \text{for } \tilde{T} \leq s \leq T. \tag{5.109}
\]
We wish next to estimate from below the probability that $\tau^*_{\epsilon, x, T} < \tilde{T}$. To see this we first observe from (5.32) that

$$X_{\epsilon, \text{class}}(s, T) > 0 \quad \text{if} \quad \frac{\sigma_A^2(T)}{m_{1, A}(T)^2} \left( \frac{\sigma_A^2(s)}{m_{1, A}(s)^2} \right)^{-1} m_{1, A}(s, T) x_{\text{class}}(s, T)$$

$$- \sqrt{\epsilon} \frac{\sigma_A^2(T)}{m_{1, A}(T)} Z(s) > 0.$$

(5.110)

From (5.35) we have that

$$\text{Var}\left[ \frac{\sigma_A^2(T)}{m_{1, A}(T)} Z(s) \right] = \frac{\sigma_A^2(T)}{m_{1, A}(T)^2} \left( \frac{\sigma_A^2(s)}{m_{1, A}(s)^2} \right)^{-1} \sigma_A^2(s, T), \quad 0 < s < T.$$

(5.111)

Since the function $s \rightarrow \sigma_A^2(s)/m_{1, A}(s)^2$ is increasing, we have from (5.109), (5.111) the inequality

$$\frac{\sigma_A^2(T)}{m_{1, A}(T)^2} \left( \frac{\sigma_A^2(s)}{m_{1, A}(s)^2} \right)^{-1} m_{1, A}(s, T) x_{\text{class}}(s, T)$$

$$\geq \frac{x}{1 + \nu} + \frac{1}{(1 + \nu)m_{1, A}(s, T)} \text{Var}\left[ \frac{\sigma_A^2(T)}{m_{1, A}(T)} Z(s) \right], \quad s < T. \quad (5.112)$$

Using the fact that the martingale $s \rightarrow Z(s)$ is a rescaled Brownian motion, and the inequality $m_{1, A}(s, T) \leq e$ for $\tilde{T} \leq s \leq T$, we conclude from (5.42) with $x$ replaced by $x/(1 + \nu)$ and $k = 1/(1 + \nu)e$ and (5.109)–(5.111) that

$$P\left( \tau_{\epsilon, \text{class}, x, T} < \tilde{T} \right) \geq 1 - \exp\left[ -\frac{2x}{\epsilon(1 + \nu)^2 e} \right] \quad \text{if} \quad T \geq T_v, \quad (5.113)$$

where $\tau_{\epsilon, \text{class}, x, T}$ is the first exit time from the half line $(0, \infty)$ for $X_{\epsilon, \text{class}}(s, T), s < T$.

We use (5.113) to prove the analogue of (5.65). Thus from (5.113) we have for $0 < \delta < 1$ that

$$X_{\epsilon, \text{class}}(s, T) > (1 - \delta)x_{\text{class}}(s, T), \quad \tilde{T} \leq s \leq T,$$

with probability at least $1 - \exp\left[ -\frac{2\delta^2 x}{\epsilon(1 + \nu)^2 e} \right]$. \quad (5.114)
We also have from (5.109) that for any \( M > 0 \) then
\[
\frac{\sigma_A^2(T)}{2} \inf_{\tilde{T} < \tau < T} \frac{x_{\text{class}}(\tau, T)^2}{\sigma_A^2(\tau)\sigma_A^2(\tilde{T}, T)} \geq \frac{2x}{(1 + \nu)^2}, \quad 0 < x \leq M, \ T \geq T_{v,M},
\]
(5.115)
where \( T_{v,M} \geq T_v \) depends also on \( M \). From (5.109) we see there are constants \( C_1, C_2 > 0 \) such that
\[
\inf_{X > (1 - \delta)x_{\text{class}}(\tilde{T}, T)} \left[ \frac{\sigma_A^2(T)}{2} \frac{(X - x')^2}{\sigma_A^2(\tilde{T})\sigma_A^2(\tilde{T}, T)} + q_\epsilon(x', y, \tilde{T}) \right] \geq \min \left[ \frac{C_1 A(\tilde{T})}{A(T)}, q_\epsilon \left( \frac{C_2 A(\tilde{T})}{A(T)}, y, \tilde{T} \right) \right] \quad \text{if} \ 0 < \delta < \frac{1}{2}, \ x' > 0, \ T \geq T_v.
\]
(5.116)
It follows from (5.34), (5.78) and (5.114)–(5.116) that
\[
\liminf_{T \to \infty} q_\epsilon(x, y, T) \geq \left( 1 - \delta \right)^2 \left( 1 + \nu \right)^2 \left\{ 1 - \exp \left[ - \frac{2\delta^2 x}{\epsilon(1 + \nu)^2} \right] \right\}.
\]
(5.117)
The limit in (5.117) is uniform for \( 0 < x \leq M, \ 0 < \epsilon \leq \epsilon_0, \ y_0 \leq y \leq y_\infty \).

The remainder of the proof follows the same lines as the proof of the lower bound in Proposition 5.2, beginning after (5.66). \( \square \)

6 Convergence to the Exponential Distribution

Here we complete the proof of Theorem 1.1. First we extend the method used in proving Proposition 4.1 to prove an analogous result for the half line problem.

**Proposition 6.1** Assume \( c_\epsilon(x, t), \ x, t > 0, \) and \( X_{\epsilon,t}, \ t > 0, \) are as in Lemma 4.2. Then
\[
\frac{X_{\epsilon,t}}{\langle X_{\epsilon,t} \rangle} \overset{D}{\to} \mathcal{X}, \quad \text{as} \ t \to \infty.
\]
(6.1)
Let \( g : [0, \infty) \to \mathbb{R}^+ \) be a continuous function which satisfies \( \lim_{t \to \infty} g(t) = \infty \). Then one has
\[
\lim_{t \to \infty} \sup_{x > g(t)} |\beta X_{\epsilon,t}(x) - 1| = 0.
\]
(6.2)
Proof We first show that (3.42) holds for the half line problem. To do this we write \(\Lambda_1\varepsilon(T)\) as the ratio of (4.4) to (4.3). Note that Lemma 4.1 implies that (5.1) holds for \(A(\cdot) = 1/\Lambda_1\varepsilon(\cdot)\), and Proposition 4.2 implies that (5.77) also holds when \(A(\cdot) = 1/\Lambda_1\varepsilon(\cdot)\). Hence the conclusion of Proposition 5.3 holds when \(A(\cdot) = 1/\Lambda_1\varepsilon(\cdot)\). It follows then from (4.5), Lemma 4.1 and Proposition 5.3 that

\[
\lim_{T \to \infty} \frac{m_2, A(T) E[X_{\varepsilon, T}, T]}{\varepsilon \sigma^2_A(T)} = 1 ,
\]

with the limit in (6.3) being uniform in \(y\) for \(y\) in any interval \(0 < y_0 < y < \infty\). The function \((y, t) \to u_{\varepsilon}(y, t, T)\), \(y > 0, t < T\), defined by (4.2) is the solution to the terminal value problem

\[
\frac{\partial u_{\varepsilon}(y, t)}{\partial t} + [A(t)y - 1] \frac{\partial u_{\varepsilon}(y, t)}{\partial y} + \varepsilon \frac{\partial^2 u_{\varepsilon}(y, t)}{\partial y^2} = 0, \quad y > 0, t < T, \tag{6.4}
\]

with zero Dirichlet condition \(u_{\varepsilon}(0, t) = 0, t < T\), and terminal condition \(u_{\varepsilon}(\cdot, T) \equiv 1\). By the maximum principle (Protter and Weinberger 1984) we see that for any \(t < T\) the function \(y \to u_{\varepsilon}(y, t, T)\) is increasing. Now (3.42) follows from (4.7), (6.3).

To prove (6.1) we use a similar identity to (3.34),

\[
P \left( \frac{X_{\varepsilon, T}}{\langle X_{\varepsilon, T} \rangle} > x \right) = \int_0^{\infty} P \left( \frac{X_{\varepsilon, T}}{\langle X_{\varepsilon, T} \rangle} > x \right) u_{\varepsilon}(y, 0, T) c_{\varepsilon}(y, 0) \, dy \int_0^{\infty} u_{\varepsilon}(y, 0, T) c_{\varepsilon}(y, 0) \, dy. \tag{6.6}
\]

We have for \(x > 0\) that

\[
P \left( \frac{X_{\varepsilon, T}}{\langle X_{\varepsilon, T} \rangle} > x \right) = \frac{E[K_{\varepsilon, D}(\sqrt{\varepsilon} \sigma_A(T)\{Z - z_{y, T}/\sqrt{\varepsilon}\}, y, 0, T) H(\sqrt{\varepsilon} \sigma_A(T)\{Z - z_{y, T}/\sqrt{\varepsilon}\} - x | Z > z_{y, T}/\sqrt{\varepsilon})]}{E[K_{\varepsilon, D}(\sqrt{\varepsilon} \sigma_A(T)\{Z - z_{y, T}/\sqrt{\varepsilon}\}, y, 0, T) | Z > z_{y, T}/\sqrt{\varepsilon}]} . \tag{6.7}
\]

where \(H : \mathbb{R} \to \mathbb{R}\) is the Heaviside function. As with proving the limit (6.3), we conclude from (3.42), (6.7) and Proposition 5.3 that

\[
\lim_{T \to \infty} P \left( \frac{X_{\varepsilon, T}}{\langle X_{\varepsilon, T} \rangle} > x \right) = e^{-x} \quad \text{for } x > 0 , \tag{6.8}
\]
and the limit is uniform in any interval $0 < y_0 < y < y_\infty$. The convergence in distribution (6.1) follows from (6.6), (6.8) upon using the monotonicity of the function $y \to u_\varepsilon(y, 0, T)$ again.

To begin the proof of (6.2) we first note we cannot set $g(\cdot) \equiv 0$ as in Proposition 2.1 since the zero Dirichlet boundary condition implies that $\beta_{X_{\varepsilon,T}}(0) = 0$ for $T > 0$. Similarly to (3.35), (3.36) we observe that $\beta_{X_{\varepsilon,T}}(x) = A_{\varepsilon,D}(x, T)C_{\varepsilon,D}(x, T)/B_{\varepsilon,D}(x, T)^2$. The functions $A_{\varepsilon,D}, B_{\varepsilon,D}$ are as in (3.33), (3.34), while $C_{\varepsilon,D}$ is given by the formula

$$C_{\varepsilon,D}(x, T) = \int_0^\infty dy \int_0^\infty dx' K_{\varepsilon,D}(x + x', y, 0, T) \times x' \exp\left[ \frac{b(T)(x + x')y}{\varepsilon\sigma_A(T)} - \frac{a(T)x'}{\varepsilon\sigma_A(T)} - \frac{x'(2x + x')}{2\varepsilon\sigma_A^2(T)} \right] \tilde{c}_\varepsilon(y, 0). \quad (6.9)$$

Comparing (4.33), (4.34), (6.9) to (3.35), (3.36) we see from Proposition 5.3 and the argument of Proposition 3.1 that for any $\delta > 0$ there exists $x_\delta, T_\delta > 0$ such that $\sup_{x > x_\delta} |\beta_{X_{\varepsilon,T}}(x) - 1| < \delta$ for $T \geq T_\delta$. Note that to conclude this we use the fact that the function $y \to \exp[b(T)xy/\varepsilon\sigma_A(T)]$ is increasing for all $x > 0$. The limit (6.2) evidently follows.

**Proposition 6.2** Assume $c_\varepsilon(x, t), x, t > 0$, and $X_{\varepsilon,t}, t > 0$, are as in Lemma 4.2. Then

$$\lim_{t \to \infty} \frac{d}{dt}(X_{\varepsilon,t}) = 1. \quad (6.10)$$

**Proof** We proceed similarly to the proof of Proposition 4.1, whence we may assume that (4.13) holds. Also from (6.2) of Proposition 6.1 we may additionally assume that

$$\varepsilon < \nu \quad \text{and} \quad \sup_{x > \delta} |\beta_{X_{\varepsilon,0}}(x) - 1| \leq \rho, \quad (6.11)$$

where $\rho, \nu, \delta > 0$ may be chosen arbitrarily small. As in (3.9) we have that

$$\int_0^\infty dy \int_0^\infty dx G_\varepsilon(x, y, 0, T) c_\varepsilon(y, 0) = \int_0^\infty dy P\left( Z > \frac{m_{2,A}(T) - m_{1,A}(T)y}{\sqrt{\varepsilon\sigma_A(T)}} \right) c_\varepsilon(y, 0) = I_\varepsilon(T), \quad (6.12)$$

where $A(\cdot) = 1/\Lambda_\varepsilon(\cdot)$ is decreasing and $A(0) = 1$. We define the function $w_\varepsilon(\cdot)$ by

$$w_\varepsilon(y) = \int_y^\infty c_\varepsilon(y', 0) \, dy' \quad \text{for} \quad y \geq 0. \quad (6.13)$$

The integral on the RHS of (6.12) is bounded above as

$$I_\varepsilon(T) \leq P\left( Z > \frac{1}{\varepsilon^{1/4}} \right) w_\varepsilon(0) + w_\varepsilon \left( \alpha(T) - \varepsilon^{1/4}\beta(T) \right), \quad (6.14)$$

\[\text{Quantity}\]
\[
\alpha(T) = \frac{m_{2,A}(T)}{m_{1,A}(T)}, \quad \beta(T) = \frac{\sigma_A(T)}{m_{1,A}(T)}.
\] (6.15)

Similarly we have a lower bound
\[
I_\varepsilon(T) \geq \left[ 1 - P \left( Z > \frac{1}{\varepsilon^{1/4}} \right) \right] w_\varepsilon \left( \alpha(T) + \varepsilon^{1/4} \beta(T) \right). \tag{6.16}
\]

We obtain using (4.13) bounds for \( I_\varepsilon(T) \) in terms of \( w_\varepsilon(\alpha(T)) \). To see this first observe from (3.23)–(3.26) the identity
\[
w_\varepsilon(x) = w_\varepsilon(0) v_\varepsilon(x) v_\varepsilon(0) \exp \left[ - \int_0^x v_\varepsilon(x') \, dx' \right], \quad x > 0. \tag{6.17}
\]

From (3.25), (4.13) we see that \( v_\varepsilon(\cdot) \) has the properties
\[
1 - C_1 \leq \frac{1}{v_\varepsilon(x)^2} \frac{dv_\varepsilon(x)}{dx} \leq 1, \quad x > 0, \quad v_\varepsilon(0) = 1, \tag{6.18}
\]

where \( C_1 \) is the constant in (4.13). It follows from (6.18) that
\[
v_\varepsilon(x) \leq 2, \quad \left| \frac{dv_\varepsilon(x)}{dx} \right| \leq 4(C_1 + 1), \quad \text{for } 0 < x < 1/2. \tag{6.19}
\]

Choosing \( x_0 = 1/8(C_1 + 1) \), we have from (6.18), (6.19) that \( 1/2 \leq v_\varepsilon(x) \leq 3/2 \) for \( 0 \leq x \leq x_0 \). Applying (6.17), (6.19) to (6.14), (6.16) we have that
\[
\left[ 1 - P \left( Z > \frac{1}{\varepsilon^{1/4}} \right) \right] \left[ 1 - 8(C_1 + 1)\varepsilon^{1/4} \beta(T) \right] \exp \left[ -3\varepsilon^{1/4} \beta(T)/2 \right]
\leq \frac{I_\varepsilon(T)}{w_\varepsilon(\alpha(T))} \leq 2e^{3x_0/2} P \left( Z > \frac{1}{\varepsilon^{1/4}} \right)
+ \left[ 1 + 8(C_1 + 1)\varepsilon^{1/4} \beta(T) \right] \exp \left[ 3\varepsilon^{1/4} \beta(T)/2 \right], \tag{6.20}
\]

provided \( \alpha(T) + \varepsilon^{1/4} \beta(T) \leq x_0 \).

A similar argument may be made to estimate the integral
\[
J_\varepsilon(T) = \int_0^\infty dy \int_0^\infty dx \, G_\varepsilon(x, y, 0, T) c_\varepsilon(y, 0) \tag{6.21}
\]
in terms of the function \( h_\varepsilon(\cdot) \) defined by
\[
h_\varepsilon(y) = \int_y^\infty w_\varepsilon(y') \, dy', \quad y \geq 0. \tag{6.22}
\]
To do this we first observe that
\[
\int_{0}^{\infty} dx \ x G_\varepsilon(x, y, 0, T) = m_{1,A}(T) [y - \alpha(T)] P \left( \frac{Z - m_{2,A}(T) - m_{1,A}(T)y}{\sqrt{\varepsilon \sigma_A(T)}} \right) + \left( \frac{\varepsilon \sigma_A^2(T)}{2\pi} \right)^{1/2} \exp \left[ - \frac{(m_{2,A}(T) - m_{1,A}(T)y)^2}{2\varepsilon \sigma_A^2(T)} \right].
\]

(6.23)

We conclude from (6.21)–(6.23) that
\[
J_\varepsilon(T) \leq \left( \frac{\varepsilon \sigma_A^2(T)}{2\pi} \right)^{1/2} \left( w_\varepsilon(0) + m_{1,A}(T)h_\varepsilon(\alpha(T)) \right).
\]

(6.24)

Similarly to (6.16) we also have from (6.23) a lower bound
\[
J_\varepsilon(T) \geq 1 - P \left( \frac{Z - \alpha(T)}{\varepsilon^{1/4}} \right) m_{1,A}(T)h_\varepsilon \left( \alpha(T) + \varepsilon^{1/4} \beta(T) \right).
\]

(6.25)

Assuming again that \( \alpha(T) + \varepsilon^{1/4} \beta(T) \leq x_0 \), we bound \( J_\varepsilon(T) \) using (3.26), (6.19) as
\[
\left[ 1 - P \left( \frac{Z - \alpha(T)}{\varepsilon^{1/4}} \right) \right] \exp \left[ -3\varepsilon^{1/4} \beta(T)/2 \right] \leq \frac{J_\varepsilon(T)}{m_{1,A}(T)h_\varepsilon(\alpha(T))} \leq 3e^{3x_0/2} \left( \frac{\varepsilon \sigma_A^2(T)}{2\pi m_{1,A}(T)^2} \right)^{1/2} + 1.
\]

(6.26)

Next as in (4.16) we use the formula
\[
\frac{\varepsilon}{2} \frac{\partial c_\varepsilon(0, T)}{\partial x} = \frac{1}{2} \int_{0}^{\infty} \frac{\partial q_\varepsilon(0, y, T)}{\partial x} G_\varepsilon(0, y, 0, T)c_\varepsilon(y, 0) \, dy
\]

(6.27)

to estimate the LHS of (6.27) in terms of \( c_\varepsilon(\alpha(T), 0) \). We write the RHS of (6.27) as a sum of the integral over the interval \( \alpha(T) - \varepsilon^{1/4} \beta(T) < y < \alpha(T) + \varepsilon^{1/4} \beta(T) \) and the integral over the complement of this interval in \( \mathbb{R}^+ \). From Proposition 5.1 of Conlon and Dabkowski (2021) this latter integral is bounded above by
\[
\frac{1}{\sqrt{2\pi \varepsilon \sigma_A^2(T)}} \exp \left[ - \frac{1}{2\varepsilon^{1/2}} \right] \left[ w_\varepsilon(0) + \frac{m_{1,A}(T)}{\sigma_A^2(T)}h_\varepsilon(0) \right].
\]

(6.28)

Again using Proposition 5.1. of Conlon and Dabkowski (2021), the former integral is bounded above by
\[
\frac{1}{m_{1,A}(T)} \left[ 1 + \frac{\varepsilon^{1/4}}{\sigma_A(T)} \right] \sup_{|y - \alpha(T)| < \varepsilon^{1/4} \beta(T)} c_\varepsilon(y, 0).
\]

(6.29)
To obtain a lower bound on the RHS of (6.27) we use Proposition 6.1 of Conlon and Dabkowski (2021). Thus since \( \sup A(\cdot) \leq 1 \), there exist universal constants \( C_1, C_2 > 0 \) such that

\[
\frac{1}{2} \frac{\partial q_{\varepsilon}(0, y, T)}{\partial x} \geq 1 + \frac{m_{1,A}(T)[y - \alpha(T)]}{\sigma_A^2(T)} - \frac{C_1\varepsilon T^2}{y^2},
\]

provided \( 0 < T \leq 1, y \geq C_2 T^2 \), \( \varepsilon \leq T^3 \). Similarly to (6.29) one obtains from (6.30) the lower bound

\[
\frac{1}{m_{1,A}(T)} \left[ 1 - P \left( |Z| > \frac{\varepsilon^{1/4}}{\sigma_A(T)} \right) - \frac{\varepsilon^{1/4}}{\sigma_A(T)} \right] - \frac{C_1\varepsilon T^2}{\{\alpha(T) - \varepsilon^{1/4} \beta(T)\}^2} \inf_{|y-\alpha(T)|<\varepsilon^{1/4} \beta(T)} c_{\varepsilon}(y, 0).
\]

We may bound \( c_{\varepsilon}(\cdot, 0) \) in terms of the beta function \( \beta_{X_{\varepsilon,0}}(\cdot) = c_{\varepsilon}(\cdot, 0) h_{\varepsilon}(\cdot)/w_{\varepsilon}(\cdot)^2 \) of (3.24). Thus from (3.26), (1.17), (1.19) we have

\[
\sup_{|y-\alpha(T)|<\varepsilon^{1/4} \beta(T)} c_{\varepsilon}(y, 0) \leq \left\{ 1 + 8(C_1 + 1)\varepsilon^{1/4} \beta(T) \right\}^2 \times \exp \left[ 9e^{1/4} \beta(T)/2 \right] \frac{w_{\varepsilon}(\alpha(T))^2}{h_{\varepsilon}(\alpha(T))} \sup_{|y-\alpha(T)|<\varepsilon^{1/4} \beta(T)} \beta_{\varepsilon}(y, 0),
\]

provided \( \alpha(T) + \varepsilon^{1/4} \beta(T) \leq x_0 \). Similarly one obtains a lower bound

\[
\inf_{|y-\alpha(T)|<\varepsilon^{1/4} \beta(T)} c_{\varepsilon}(y, 0) \geq \left\{ 1 - 8(C_1 + 1)\varepsilon^{1/4} \beta(T) \right\}^2 \times \exp \left[ -9e^{1/4} \beta(T)/2 \right] \frac{w_{\varepsilon}(\alpha(T))^2}{h_{\varepsilon}(\alpha(T))} \inf_{|y-\alpha(T)|<\varepsilon^{1/4} \beta(T)} \beta_{\varepsilon}(y, 0).
\]

We choose now \( T_0, \nu > 0 \) such that

\[
0 < T_0 \leq 1, \quad 2C_2 T_0^2 \leq \alpha(T_0) \leq \frac{x_0}{2}, \quad \nu^{1/4} \beta(T_0) \leq \frac{\alpha(T_0)}{2}.
\]

Note that the choice in (6.34) is possible since \( \lim_{T \to 0} \alpha(T)/T = 1 \). The inequalities (6.32), (6.33) both hold for \( T = T_0 \) and \( \varepsilon \) satisfying (6.11). From (1.2), (1.4) we have that

\[
\frac{d}{dt}(X_{\varepsilon,t}) \bigg|_{t=T_0} = \frac{\varepsilon}{2} \frac{\partial c_{\varepsilon}(0, T_0)}{\partial x} \frac{J_{\varepsilon,D}(T_0)}{I_{\varepsilon,D}(T_0)^2},
\]
where $I_{\varepsilon,D}(T)$, $J_{\varepsilon,D}(T)$ are defined as in (6.12), (6.21), but with the half line Dirichlet Green’s function $G_{\varepsilon,D}$ replacing the whole line Green’s function $G_{\varepsilon}$. Evidently we have that $I_{\varepsilon,D}(T) \leq I_{\varepsilon}(T)$, $J_{\varepsilon,D}(T) \leq J_{\varepsilon}(T)$, whence (6.20), (6.26) yield upper bounds on $I_{\varepsilon,D}(T)$, $J_{\varepsilon,D}(T)$. We can also see from the lower bound on $q_{\varepsilon}(x, y, T)$ in Proposition 3.3 of Conlon and Dabkowski (2021) and (4.15) that the lower bounds in (6.20), (6.26) also hold with $T = T_0$ and $\nu$ small, up to a multiplicative factor close to 1. We conclude that the LHS of (6.35) is equal to 1 modulo terms in the parameters $\nu, \rho$ of (6.11) which converge to zero as $\nu, \rho \to 0$. Now (6.10) follows by arguing as in the proof of Lemma 7.3 of Conlon et al. (2016).

\[ \square \]

### Appendix A. Properties of Gaussian Conditional Variables

Let $Z$ be the standard normal variable and for $z \in \mathbb{R}$ let $X_z$ be the random variable $Z - z$ conditioned on $Z > z$. Here we derive some properties of the variables $X_z$.

**Lemma A.1** Let $m : \mathbb{R} \to \mathbb{R}$ be the function $m(z) = \langle X_z \rangle$, $z \in \mathbb{R}$. Then $m(\cdot)$ is a continuous positive decreasing function satisfying the inequalities,

\[
\begin{align*}
\frac{1}{z} - \frac{2}{z^3} < m(z) < \frac{1}{z} & \text{ for } z > 1, \\
\max\{|z|, \sqrt{2/\pi}\} < m(z) < \sqrt{2/\pi} + |z| & \text{ for } z < 0.
\end{align*}
\]

**Proof** We have that

\[
m(z) = \int_{z}^{\infty} (z' - z)e^{-z'^2/2} \, dz' / \int_{z}^{\infty} e^{-z'^2/2} \, dz'.
\]

Thus

\[
\frac{dm(z)}{dz} = \int_{0}^{\infty} z'e^{-z'^2/2-zz'} \, dz' / \left[\int_{0}^{\infty} e^{-z'^2/2-zz'} \, dz'\right]^2 - 1.
\]

Using the fact that

\[
m(z) = \int_{0}^{\infty} z'e^{-z'^2/2-zz'} \, dz' / \int_{0}^{\infty} e^{-z'^2/2-zz'} \, dz',
\]

we conclude that $m(\cdot)$ is the solution to the Riccati equation

\[
\frac{dm(z)}{dz} = m(z)^2 + zm(z) - 1,
\]

which satisfies $m(0) = \sqrt{2/\pi}$. Observe that $m(z) = -z$ is a solution to (A.6).
We show that the function (A.3) is decreasing. To see this we use the identity
\[
\int_0^\infty z'(z'+z)e^{-z'^2/2-zz'} \, dz' = \int_0^\infty z' \left( -\frac{d}{dz'} \right) e^{-z'^2/2-zz'} \, dz' \\
= \int_0^\infty e^{-z'^2/2-zz'} \, dz'. \tag{A.7}
\]
Evidently (A.7) implies that
\[
\langle X_z^2 \rangle + z \langle X_z \rangle = 1. \tag{A.8}
\]
Since \( \langle X_z \rangle^2 < \langle X_z^2 \rangle \), we see from (A.8) that the RHS of (A.6) is strictly negative, whence \( m(\cdot) \) decreases. Note also that (A.8) implies the upper bound in (A.1). To obtain the lower bound we recall the well-known inequality
\[
\left[ \frac{1}{z} - \frac{1}{z^3} \right] < e^{z^2/2} \int_z^\infty e^{-z'^2/2} \, dz' < \frac{1}{z} \quad \text{for } z > 0. \tag{A.9}
\]
Considering (A.6) to be a linear equation with inhomogeneous term \( m(z)^2 - 1 \), the solution \( m(z) \) has the representation
\[
m(z) = e^{z^2/2} \int_z^\infty [1 - m(z')^2]e^{-z'^2/2} \, dz'. \tag{A.10}
\]
It follows from the upper bound in (A.1) and (A.10) that
\[
m(z) \geq \left( 1 - \frac{1}{z^2} \right) e^{z^2/2} \int_z^\infty e^{-z'^2/2} \, dz'. \tag{A.11}
\]
We then obtain the lower bound in (A.1) from the lower bound in (A.9) and (A.11).

To obtain the lower bound in (A.2) we use the fact that trajectories of the non-autonomous differential Eq. (A.6) do not intersect in \( \mathbb{R}^2 \), in particular the trajectories \( z \to [m(z), z] \) and \( z \to [-z, z] \). To obtain the upper bound we observe that the function \( w(z) = \alpha - z - m(z) \) is a solution to the initial value problem
\[
\frac{dw(z)}{dz} = a(z)w(z) - b(z) , \quad w(0) = \alpha - \sqrt{2/\pi} , \tag{A.12}
\]
where the functions \( a(\cdot) \), \( b(\cdot) \) are given by
\[
a(z) = \alpha + m(z) , \quad b(z) = \alpha [\alpha - z] . \tag{A.13}
\]
Evidently \( w(z) \geq 0 \) for \( z < 0 \) if \( w(0) \geq 0 \) and \( b(z) \geq 0 \) for \( z < 0 \). This is the case if \( \alpha = \sqrt{2/\pi} \). \( \square \)
Lemma A.2 For any \( \delta \) satisfying \( 0 < \delta < 1 \) there exists \( c(\delta) > 0 \) depending on \( \delta \) such that for all \( z \in \mathbb{R} \),

\[
P(m(z) < X_z < (1 + \delta)m(z)) \geq c(\delta) , \quad P((1 - \delta)m(z) < X_z < m(z)) \geq c(\delta) .
\]  \hspace{1cm} (A.14)

Furthermore there exist constants \( C, c > 0 \) such that for all \( z \in \mathbb{R} \),

\[
P(X_z > km(z)) \leq Ce^{-ck} , \quad k = 1, 2, ...
\]  \hspace{1cm} (A.15)

The variables \( X_z/m(z) \) converge in distribution as \( z \to \infty \) to the exponential variable \( X' \) with mean 1.

Proof Let \( \rho_z(z') \), \( z' > 0 \), be the pdf of the variable \( X_z \). Then

\[
\rho_z(z') = A(z) \exp \left[-\frac{(z' + z)^2}{2} \right] = B(z) \exp \left[-z'z - \frac{z'^2}{2} \right] , \quad z' > 0 ,
\]  \hspace{1cm} (A.16)

where \( A(z), B(z) \) depend only on \( z \). For \( z < 1 \) the variable \( X_z \) is approximately Gaussian with mean \( m(z) \). Hence using the first representation on the RHS of (A.16) and Lemma A.1, we conclude that (A.14), (A.15) hold. For \( z \geq 1 \) the variable \( X_z \) is approximately exponential with mean \( m(z) \). Using the second representation on the RHS of (A.16) we see that (A.14), (A.15) hold. We similarly see that \( X_z/m(z) \) converges in distribution to the exponential variable as \( z \to \infty \). \( \square \)

Lemma A.3 Let \( f : [0, \infty) \to \mathbb{R} \) be a continuous non-negative increasing function. Then there is a universal constant \( c > 0 \) such that

\[
\frac{E[X_z f(X_z)]}{E[f(X_z)]} \geq cm(z) \quad \text{for} \quad z \in \mathbb{R} .
\]  \hspace{1cm} (A.17)

If in addition \( \lim_{z \to \infty} f(z) = 1 \) and \( -\log[1 - f(\cdot)] \) is a concave function, then there is a universal constant \( C \) such that

\[
\frac{E[X_z f(X_z)]}{E[f(X_z)]} \leq Cm(z) \quad \text{for} \quad z \in \mathbb{R} .
\]  \hspace{1cm} (A.18)

Proof The ratio of expectations in (A.17) is given by

\[
\int_{0}^{\infty} z' f(z') \rho_z(z') \, dz' / \int_{0}^{\infty} f(z') \rho_z(z') \, dz' ,
\]  \hspace{1cm} (A.19)

where \( \rho_z(\cdot) \) is the pdf of \( X_z \). From Lemma A.2 we see there is a constant \( C_1 > 0 \) such that

\[
\int_{0}^{m(z)} \rho_z(z') \, dz' \leq C_1 \int_{m(z)}^{\infty} \rho_z(z') \, dz' , \quad z \in \mathbb{R} .
\]  \hspace{1cm} (A.20)
It follows from (A.20) that the LHS of (A.19) is bounded below by \( m(z)/(C_1 + 1) \), whence (A.17) follows.

To obtain the upper bound (A.18) we first observe that the expression on the LHS of (A.18) is bounded above by

\[
m(z) + \int_{m(z)}^{\infty} z' f(z') \rho_z(z') \, dz' / \int_{0}^{\infty} f(z') \rho_z(z') \, dz'. \tag{A.21}
\]

Suppose now that \( f(z') \geq 1 - e^{-1} \) for \( z' \geq m(z) \). Then since \( \sup f(\cdot) = 1 \), it follows from (A.14) that the expression (A.21) is bounded above by \( C_2 m(z) \) for some constant \( C_2 \). Hence to complete the proof of (A.18) we may assume that \( f(m(z)) \leq 1 - e^{-1} \).

In that case \( f(z') = 1 - e^{-q(z')} \), where \( q(\cdot) \) is a non-negative increasing concave function and \( q(m(z)) \leq 1 \). From the concavity of \( q(\cdot) \) we have that \( q(z') \leq \tilde{q}(z') \) for \( z' \geq m(z) \), and \( q(z') \geq \tilde{q}(z') \) for \( m(z)/2 < z' < m(z) \), where \( \tilde{q}(\cdot) \) is the secant line function

\[
\tilde{q}(z') = \frac{2}{m(z)} [(m(z) - z') q(m(z)/2) + (z' - m(z)/2) q(m(z))]. \tag{A.22}
\]

Hence the second term in (A.21) is bounded above by

\[
\int_{m(z)}^{\infty} z'[1 - e^{-\tilde{q}(z')} ] \rho_z(z') \, dz' / \int_{m(z)/2}^{m(z)} [1 - e^{-\tilde{q}(z')} ] \rho_z(z') \, dz' \\
\leq e \int_{m(z)}^{\infty} z' \tilde{q}(z') \rho_z(z') \, dz' / \int_{m(z)/2}^{m(z)} \tilde{q}(z') \rho_z(z') \, dz'. \tag{A.23}
\]

We see from (A.22) that

\[
\tilde{q}(z') \geq q(m(z))/2 \quad \text{for } 3m(z)/4 \leq z' \leq m(z), \\
\tilde{q}(z') \leq (2n + 1) q(m(z)) \quad \text{for } z' \leq (n + 1)m(z), \quad n = 1, 2, \ldots \tag{A.24}
\]

It follows from (A.23), (A.24) that the second term in (A.21) is bounded above by

\[
2e \sum_{n=1}^{\infty} (2n + 1) \int_{nm(z)}^{(n+1)m(z)} z' \rho_z(z') \, dz' / \int_{3m(z)/4}^{m(z)} \rho_z(z') \, dz'. \tag{A.25}
\]

We conclude from (A.14), (A.15) that the expression (A.25) is bounded above by \( Cm(z), \ z \in \mathbb{R} \), for some constant \( C \).

\[\square\]

References

Ball, J., Carr, J., Penrose, O.: The Becker–Döring cluster equations: basic properties and asymptotic behavior of solutions. Commun. Math. Phys. 104, 657–692 (1986)

Becker, R., Döring, W.: Kinetische Behandlung der Keimbildung in übersättigten Dämpfen. Ann. Phys. (Leipzig) 24, 719–752 (1935)
Carr, J., Penrose, R.: Asymptotic behavior of solutions to a simplified Lifshitz–Slyozov equation. Physica D 124, 166–176 (1998)

Conlon, J.: On a diffusive version of the Lifschitz–Slyozov–Wagner equation. J. Nonlinear Sci. 20, 463–521 (2010)

Conlon, J., Dabkowski, M.: On global asymptotic stability for the LSW model with subcritical initial data. J. Stat. Phys. 178, 117–177 (2020)

Conlon, J., Dabkowski, M.: On properties of the Dirichlet Green’s function for linear diffusions on a half line (2021). arXiv:2103.03929

Conlon, J., Dabkowski, M., Wu, J.: On Large time behavior and selection principle for a diffusive Carr–Penrose model. J. Nonlinear Sci. 26, 453–518 (2016)

Conlon, J., Schlichting, André: A non-local problem for the Fokker–Planck equation related to the Becker–Döring model. Discrete Contin. Dyn. Syst. 39, 1821–1889 (2019)

Goudon, T., Monasse, L.: Fokker–Planck approach of Ostwald ripening: simulation of a modified Lifschitz–Slyozov–Wagner system with a diffusive correction. SIAM J. Sci. Comput. 42(1), B157–B184 (2020)

Kohn, R., Otto, F.: Upper bounds on coarsening rates. Commun. Math. Phys. 229, 375–395 (2002)

Lifshitz, I.M., Slyozov, V.V.: Kinetics of precipitation from supersaturated solid solutions. J. Phys. Chem. Sol. 19, 35–50 (1961)

Meerson, B.: Fluctuations provide strong selection in Ostwald ripening. Phys. Rev. E 60, 3072–3075 (1999)

Niethammer, B.: On the evolution of large clusters in the Becker–Döring model. J. Nonlinear Sci. 13, 115–155 (2003)

Niethammer, B., Pego, Robert L.: Well-posedness for measure transport in a family of nonlocal domain coarsening models. Indiana Univ. Math. J. 54, 499–530 (2005)

Pego, R.L.: Lectures on dynamics in models of coarsening and coagulation, 1–61. In: Dynamics in models of coarsening, coagulation, condensation and quantization. Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore, vol. 9. World Scientific Publication (2007)

Penrose, O.: Metastable states for the Becker–Döring cluster equations. Commun. Math. Phys. 124, 515–541 (1989)

Penrose, O.: The Becker–Döring equations at large times and their connection with the LSW theory of coarsening. J. Stat. Phys. 89, 305–320 (1997)

Protter, M., Weinberger, H.: Maximum Principles in Differential Equations. Springer, New York (1984)

Rubinstein, I., Zaltzman, B.: Diffusional mechanism of strong selection in Ostwald ripening. Phys. Rev. E 61, 709–717 (2000)

Schlichting, A.: Macroscopic limit of the Becker–Döring equation via gradient flows. ESAIM Control Optim. Calc. Var. 25, 22 (2019)

Velázquez, J.J.L.: The Becker–Döring equations and the Lifshitz–Slyozov theory of coarsening. J. Stat. Phys. 92, 195–236 (1998)

Wagner, C.: Theorie der alterung von niederschlägen durch umlösen. Z. Elektrochem. 65, 581–591 (1961)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.