EXISTENCE OF SCALING LIMIT PHASE TRANSITION IN A TWO-DIMENSIONAL RANDOM POLYMER MODEL

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Abstract. In this paper we prove a scaling limit phase transition for a class of two-dimensional random polymers.

1. Introduction

In this paper we consider two-dimensional random polymers, which are defined as follows. Given \( N \in \mathbb{N} \), a \( N \)-th step polymer \( S \) is an element of \( \mathcal{W}_N \) given by

\[
\mathcal{W}_N := \{ S = (S_0, S_1, \ldots, S_N) : S_i \in \mathbb{Z}^2, S_0 = 0 \text{ and } \|S_{i+1} - S_i\|_1 = 1 \},
\]

where \( \| \cdot \|_1 \) denotes the taxicab norm. Its probability distribution is defined by a Gibbs measure, at the inverse temperature \( \beta > 0 \), given by

\[
P_{N}^{\beta}(\{S\}) = \frac{\exp \left(- \beta \mathcal{H}_N(S) \right)}{Z_N(\beta)},
\]

where \( Z_N(\beta) = \sum_S \exp \left(- \beta \mathcal{H}_N(S) \right) \) is a normalization factor and the Hamiltonian is

\[
\mathcal{H}_N(S) = \sum_{1 \leq i < j \leq N} V_{ij} \cdot \langle X_i, X_j \rangle,
\]

with \( X_i := S_i - S_{i-1} \), \( V_{ij} \geq 0 \) and \( \langle \cdot, \cdot \rangle \) the usual inner product. Here we assume that the interaction \( V_{ij} \) depends only on the distance between \( i, j \in \mathbb{N} \), i.e., \( V_{ij} = V(|i - j|) \) and satisfies the following regularity condition

\[
\sum_{i \in \mathbb{N}} V(i) < +\infty.
\]

Such random polymers have received considerable attention in the literature bringing together physics, chemistry, and more recently biophysics. In [CPP94], the authors considered this type of a random polymer model, where the two body interactions decay as a power law. Their model interpolates between the lattice
Edwards model and an ordinary simple random walk (SRW). Under the assumption of $1 \leq d \leq 4$, the authors proved the existence of an exponent $\gamma = \gamma(d, \alpha)$, for the end-to-end distance, such that $\sum_{S} \|S_N\|^2 P_N^\beta(S) \sim cN^{2\gamma}$ holds. In [BPS05], by considering an appropriate scale, the authors showed that the end-to-end distance of a two-dimensional random polymer with a self repelling interaction of Kac-type undergoes a diffusive-ballistic phase transition. In [PSS08], the authors considered a self repelling model with long-range interactions of the form $V_{ij} = 1/\|i - j\|^\alpha$, $3 < \alpha \leq 4$. They proved that their model is diffusive at sufficiently high temperatures and ballistic at sufficiently low temperatures.

Random polymer models with some similarity to the model proposed in [PSS08] have been studied in [BY91, CPP94, vdHdHS98, MP91]. Recently, in [CDdS14], the authors also proved the existence of a ballistic-diffusive phase transition in terms of the inverse temperature $\beta$, by comparing their long-range polymer model with two independent copies of long-range one-dimensional ferromagnetic Ising model. In addition, they obtained a Central Limit Theorem (CLT) for a similar model with drift.

In the present paper, we prove a phase transition not with respect to the end-to-end distance but for suitable scaling limits. Roughly speaking, we prove the existence of a critical point $\beta_c \in (0, +\infty)$ such that the scaling limit of our random polymer model converges in the Wasserstein distance to the planar standard Brownian motion in the subcritical regime $\beta < \beta_c$. Moreover, we also prove that it does not scales to the Brownian motion when $\beta > \beta_c$. To be more precise, the statements of the main result of this paper are the following. Let $S \in \mathcal{W}_N$ be a random polymer and consider the stochastic process

$$W_n(t) := \frac{1}{\sigma \sqrt{n}} \left\{ S_k + (nt - k)(S_{k+1} - S_k) \right\}, \quad \frac{k}{n} \leq t < \frac{k+1}{n}, \quad t \in [0, 1].$$

**Theorem 1.1** (Phase transition). Consider the random polymer model on $\mathbb{Z}^2$.

Then, for each $0 < p \leq 2$,

1. if $\beta < \beta_c$, then $\lim_{n \to \infty} d_p(W_n(t), B_2(t)) = 0$;
2. if $\beta > \beta_c$, then $\liminf_{n \to \infty} d_p(W_n(t), B_2(t)) > 0$.

where $B_2(t) = (B_1^1(t), B_1^2(t))$ is the planar standard Brownian motion.

We emphasize that the convergence obtained in Theorem 1.1 is stronger than the convergence in distribution. Moreover, by a result of Bickel and Freedman [BF81], it also implies convergence of moments.

This paper is organized as follows. In Section 2 we introduce the Wasserstein distance and recall some of its basics results. Later, in Section 3, we prove an abstract theorem about the scaling limits of positively correlated random variables.
We first consider one-dimensional processes and provide an application on the long-range Ising Model. Next we consider $m$-dimensional processes and use this version to prove our main result in the last section.

2. Wasserstein Distance

The Wasserstein distance \cite{Vas69} is also known as Monge-Kantorovich-Rubinstein distance \cite{KeRsn58}, Mallows distance \cite{Mal72} or optimal transport distance in optimization \cite{Amb03}. It is a useful tool in order to derive CLT type results including the case of heavy-tailed stable distributions \cite{JS05, DO14}. To define it, let $(\mathcal{X}, d)$ be a complete metric space and let $\mathcal{P}(\mathcal{X})$ be the set of all probability measures $\mu$ on the Borel $\sigma$-field of $\mathcal{X}$. The Wasserstein distance of order $p > 0$ between two probability measures $\mu_1, \mu_2 \in \mathcal{P}(\mathcal{X})$ is defined as

$$d_p(\mu_1, \mu_2) = \left\{ \inf_{\nu \in \Pi(\mu_1, \mu_2)} \int_{\mathcal{X} \times \mathcal{X}} d^p(x, y) \nu(dx, dy) \right\}^{1/p},$$

where $\Pi(\mu_1, \mu_2)$ is the set of all Borel probability measures on $\mathcal{X} \times \mathcal{X}$ with marginals $\mu_1$ and $\mu_2$, respectively.

Note that in the case where $\mathcal{X} = \mathcal{B}$ is a finite dimensional Euclidean space, with its standard norm $\| \cdot \|$, the Wasserstein distance of order $p > 0$ between Borel probability measures $\mu_1$ and $\mu_2$ is alternatively given by

$$d_p(\mu_1, \mu_2) = \inf_{(X,Y)} \left\{ \mathbb{E}(\|X - Y\|^p) \right\}^{1/p}, \quad (2.1)$$

where the infimum is taken over all $\mathcal{B}$-valued random variables (r.v.'s) $X$ and $Y$, where $X$ has law $\mu_1$ and $Y$ has law $\mu_2$. When convenient, we write $d_p(X, Y)$ instead of $d_p(\mu_1, \mu_2)$ and we shall remark that this particular case is enough for the purpose of this work.

For $p \geq 1$, the Wasserstein distance defines a metric on a subspace of $\mathcal{P}(\mathcal{B})$ and bears a close connection with weak convergence. Let $\Gamma_p(\mathcal{B})$ be the set of all probability measures $\mu \in \mathcal{P}(\mathcal{B})$ such that $\int_\mathcal{B} \|x\|^p d\mu(x) < +\infty$.

**Theorem 2.1** (Bickel and Freedman \cite{BF81}). Let $p \geq 1$, and $\mu$ and $\{\mu_n\}_{n \geq 1}$ in $\Gamma_p(\mathcal{B})$. Then, $\lim_{n \to \infty} d_p(\mu_n, \mu) = 0$ if and only if, for every bounded continuous function $g : \mathcal{B} \to \mathbb{R}$, we have,

$$\lim_{n \to \infty} \int_\mathcal{B} g(x) d\mu_n(x) = \int_\mathcal{B} g(x) d\mu(x) \quad \text{and} \quad \lim_{n \to \infty} \int_\mathcal{B} \|x\|^p d\mu_n(x) = \int_\mathcal{B} \|x\|^p d\mu(x).$$

Assume $X \overset{d}{=} F$, $Y \overset{d}{=} G$ and $(X, Y) \overset{d}{=} H$, where $H(x, y) = \min\{F(x), G(y)\}$. Then the following representation result, known as the representation Theorem, whose proof can be found in \cite{DF12}, will be helpful to evaluate $d_p(F, G)$ when $\mathcal{B}$ is the real line.
Theorem 2.2. For \( p \geq 1 \) we have
\[
 d_p(F,G) = \left\{ \int_{\mathbb{R}^2} |x-y|^p dH(x,y) \right\}^{1/p} =: \{ \mathbb{E}_H(|X-Y|^p) \}^{1/p}.
\]

This theorem will be used in our one-dimensional setting. When moving the discussion to the \( m \)-dimensional setting, the following generalization will be required, see [BF81].

Theorem 2.3. Let \( X \) and \( Y \) be the coordinate functions on \( \mathcal{B} \times \mathcal{B} \). The infimum of \( d_p(\mu_1, \mu_2) \) is attained by
\[
 \int_{\mathcal{B} \times \mathcal{B}} \|x-y\|^p d\pi(x,y),
\]
for some probability \( \pi \) on \( \mathcal{B} \times \mathcal{B} \) such that \( \pi X^{-1} = \mu_1 \) and \( \pi Y^{-1} = \mu_2 \).

3. Convergence in Wasserstein distance

In this section, we consider one and \( m \)-dimensional random processes defined on some underlying complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\). In what follows, \( \mathbb{N} \) denotes the set of positive integers and for \( x \in \mathbb{R} \), \( \lfloor x \rfloor \) denotes the greatest integer less than or equal to \( x \). Let \( X \) be a \( m \)-dimensional stochastic process,
\[
 S_n := \sum_{i=1}^n X_i \quad \text{and} \quad Y_{j,n} := \sum_{k=\lfloor(j-1)\ell_n+1\rfloor}^{\lfloor j\ell_n \rfloor} X_k, \quad j = 1, \ldots, m_n, \tag{3.1}
\]
where the first \( m_n = \lfloor n/\ell_n \rfloor \) blocks have size \( \ell_n \) large enough such that
\[
 \lim_{n \to \infty} \ell_n = \infty, \quad \lim_{n \to \infty} \frac{n}{\ell_n} = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{\ell_n^3}{m_n} = 0. \tag{3.2}
\]
The condition (3.2) is satisfied, for example, if we take \( \ell_n = n^\delta \) with \( \delta < 1/4 \).

Given \( t > 0 \), we denote by \( Z_t(n) \) the stabilized partial sum of the r.v.’s \( X_i \)’s. That is,
\[
 Z_t(n) := \frac{S_{\lfloor nt \rfloor}}{\sigma \sqrt{n}}, \tag{3.3}
\]
where \( \sigma^2 \in (0, +\infty) \). When \( t = 1 \), we simply write \( Z_n \) instead of \( Z_1(n) \).

3.1. Dimension one. The stochastic process \( X \) is said positively associated [EPW67] if, given two coordinate-wise non-decreasing functions \( f, g : \mathbb{R}^n \to \mathbb{R} \) and \( i_1, \ldots, i_n \in \mathbb{N} \), we have
\[
 \text{Cov} (f(X_{i_1}, \ldots, X_{i_n}), g(X_{i_1}, \ldots, X_{i_n})) \geq 0,
\]
provided the covariance exists, and for coordinate-wise non-decreasing function we mean \( f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n) \) whenever \( x_j \leq y_j \) for all \( j = 1, \ldots, n \).
Lemma 3.1 (Newman and Wright [NW81]). Let $X$ be positively associated. If all $X_j$’s have finite second moment, then the characteristic functions $\varphi_j(r_j) = E(\exp\{ir_jX_j\})$ and $\varphi(r_1, \cdots, r_n) = E(\exp\{i \sum_{j=1}^n r_jX_j\})$ satisfy
\[
|\varphi(r_1, \cdots, r_n) - \prod_{j=1}^n \varphi_j(r_j)| \leq \frac{1}{2} \sum_{1 \leq j \neq k \leq n} |r_j r_k| \text{Cov}(X_j, X_k).
\]

Now we state the main result of this section and prove it using the previous lemma.

Theorem 3.2. Let $X$ be a centered, one-dimensional positively associated and stationary stochastic process. Assume that the following conditions are satisfied:
\[
\lim_{n \to \infty} \frac{1}{n} \text{Var}(S_n) = \sigma^2 \in (0, +\infty),
\]
\[
\lim_{n \to \infty} \frac{1}{m_n \ell_n} \sum_{j=1}^{m_n} \text{Var}(Y_{j,n}) = \sigma^2
\text{ and }
\]
\[
E(|X_j|^3) < C_* < +\infty, \quad \forall j \in \mathbb{N}.
\]
Then for each $0 < p \leq 2$,
\[
\lim_{n \to \infty} d_p(Z_n, Z) = 0 \quad \text{and} \quad \lim_{n \to \infty} E(|Z_n|^p) = E(|Z|^p),
\]
where $Z \sim N(0,1)$.

Proof. By Minkowsky’s inequality and by (2.1),
\[
dl(Z_n, Z) \leq \left\{ \frac{E\left(\left( S_n - \frac{1}{\sqrt{m_n \ell_n}} \sum_{i=1}^{m_n} X_i \right)^2 \right)}{\sigma m_n \ell_n} \right\}^{1/2} + \left\{ \frac{E\left(\left( \frac{1}{\sqrt{m_n \ell_n}} \sum_{i=1}^{m_n} X_i - Z \right)^2 \right)}{\sigma m_n \ell_n} \right\}^{1/2}
= A_n^{1/2} + B_n^{1/2}.
\]
Then
\[
\lim_{n \to \infty} d_2(Z_n, Z) = 0, \quad \text{whenever} \quad \lim_{n \to \infty} A_n = 0 \quad \text{and} \quad \lim_{n \to \infty} B_n = 0.
\]
Consider the blocks (3.1) and assume that the block size $\ell_n$ satisfies (3.2). We will first prove that $\lim_{n \to \infty} A_n = 0$. For this, using properties of variance and the positivity of covariances, we have
\[
A_n = \frac{1}{\sigma^2} \text{Var} \left( \sum_{j=1}^{m_n \ell_n} \left( \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m_n \ell_n}} \right) X_j + \sum_{j=m_n \ell_n+1}^{n} \frac{1}{\sqrt{n}} X_j \right)
\leq \frac{2}{\sigma^2} \left( \frac{1}{\sqrt{m_n \ell_n}} \right)^2 \sum_{i,j=1}^{m_n \ell_n} \text{Cov}(X_i, X_j) + \frac{2}{n \sigma^2} \sum_{i,j=m_n \ell_n+1}^{n} \text{Cov}(X_i, X_j).
By inequality \( \left( \frac{1}{\sqrt{m_n \ell_n}} \pm \frac{1}{\sqrt{n}} \right)^2 \leq \frac{\ell_n}{m_n (\sqrt{m_n \ell_n} + \sqrt{n})} \), the rhs of above inequality is
\[
\leq \frac{2}{n \sigma^2} \left( \frac{\ell_n}{\sqrt{m_n \ell_n} + \sqrt{n}} \right)^2 \frac{1}{m_n \ell_n} \sum_{i,j=1}^{m_n \ell_n} \text{Cov}(X_i, X_j)
\]
\[+ \frac{2}{\sigma^2} \left( \frac{n - m_n \ell_n}{n} \right) \frac{1}{n - m_n \ell_n} \sum_{i,j=m_n \ell_n+1}^{n} \text{Cov}(X_i, X_j). \]

Then, by (3.4) \( A_n \to 0 \) as \( n \to \infty \).

The next step is to prove that \( \lim_{n \to \infty} B_n = 0 \). In fact, if
\[
I_n(t) := \left| \mathbb{E} \left( \exp \left\{ i \frac{t}{\sqrt{m_n \ell_n}} \sum_{j=1}^{m_n} Y_{j,n} \right\} \right) - \prod_{j=1}^{m_n} \mathbb{E} \left( \exp \left\{ i \frac{t}{\sqrt{m_n \ell_n}} Y_{j,n} \right\} \right) \right|, \quad t \in \mathbb{R},
\]
by Lemma 3.1,
\[
I_n(t) \leq \frac{|t|}{2 m_n \ell_n} \sum_{i,j=1}^{m_n} \text{Cov}(Y_{i,n}, Y_{j,n})
\]
\[= \frac{|t|}{2} \left( \frac{1}{m_n \ell_n} \text{Var} \left( \sum_{j=1}^{m_n \ell_n} X_j \right) - \frac{1}{m_n \ell_n} \sum_{i=1}^{m_n} \text{Var}(Y_{i,n}) \right). \]

Taking \( n \to \infty \) in the above inequality, by (3.4)-(3.5) follows that \( \lim_{n \to \infty} I_n(t) = 0 \). That is, the blocks \( Y_{j,n}/\sqrt{m_n \ell_n}, j = 1, \ldots, m_n \) are asymptotically independent.

On the other hand, denoting \( \mathcal{M}_n := \sum_{j=1}^{m_n} Y_{j,n}/\sigma \sqrt{m_n \ell_n} \) and using the independence of the blocks, it follows from (3.5) that
\[
\lim_{n \to \infty} \text{Var}(\mathcal{M}_n) = \lim_{n \to \infty} \frac{1}{\sigma^2 m_n \ell_n} \sum_{j=1}^{m_n} \text{Var}(Y_{j,n}) = 1. \quad (3.9)
\]

For each fixed \( n \), by using Minkowsky’s inequality and (3.6) we get \( \mathbb{E}[|Y_{j,n}|^3] \leq \ell_n^3 C_* \).

Since the \( Y_{j,n} \)’s are zero-mean and independent r.v.’s with finite third moment, it follows from Berry-Essen’s Theorem (see, e.g., [Fel71]) that
\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left( \frac{\mathcal{M}_n}{\sqrt{\text{Var}(\mathcal{M}_n)}} \leq x \right) - \Phi(x) \right| \leq \frac{6\ell_n^3 C_*}{(\sigma^2 m_n \ell_n)^{3/2} \{\text{Var}(\mathcal{M}_n)\}^{3/2}}, \quad \Phi \overset{d}{=} N(0, 1).
\]

By (3.2) and (3.9) the rhs of the above inequality tends to zero, when \( n \to \infty \). Therefore, \( \mathcal{M}_n/\sqrt{\text{Var}(\mathcal{M}_n)} \overset{d}{\to} Z \), where \( \overset{d}{\to} \) denotes convergence in distribution.

Then, by (3.9) and by Slutsky’s Theorem, \( S_n \overset{d}{\to} Z \) and \( \lim_{n \to \infty} \text{Var}(S_n) = 1 \) or equivalently (see Theorem 2.1) \( \lim_{n \to \infty} d_2(\mathcal{M}_n, Z) = 0 \).
From Theorem 2.2 follows that there exists a r.v. $Z^* \overset{d}{=} \Phi$ such that the joint distribution of $(\mathcal{M}_n, Z^*)$ is given by $H(x, y) = \min\{F_{\mathcal{M}_n}(x), \Phi(y)\}$ and

$$d_2(\mathcal{M}_n, Z) = \mathbb{E}((\mathcal{M}_n - Z^*)^2) \to 0 \quad \text{as} \ n \to \infty \Leftrightarrow B_n \to 0 \quad \text{as} \ n \to \infty.$$  

Hence, by (3.8) $\lim_{n \to \infty} d_2(Z_n, Z) = 0$. Therefore, by Theorem 2.2 there exists $\tilde{Z}^* \overset{d}{=} \Phi$ such that

$$d_2(Z_n, Z^*) = \mathbb{E}((Z_n - \tilde{Z}^*)^2) \to 0 \quad \text{as} \ n \to \infty.$$  

From the definition of Wasserstein distance (2.1) and Liapounov’s inequality we have for $0 < p \leq 2$

$$d_p(Z_n, Z) \leq \left\{ \mathbb{E}(|Z_n - \tilde{Z}^*|^p) \right\}^{1/p} \to 0 \quad \text{as} \ n \to \infty.$$  

□

**Corollary 3.3.** Under conditions of Theorem 3.2, for each $0 < p \leq 2$ we have

$$\lim_{n \to \infty} d_p(Z_t(n), B_t) = 0 \quad \text{and} \quad \lim_{n \to \infty} \mathbb{E}(|Z_t(n)|^p) = \mathbb{E}(|B_t|^p), \quad t \in [0, 1], \quad (3.10)$$

where $B_t$ is the standard one-dimensional Brownian motion.

**Proof.** Assume first that $p = 2$. By Theorem 3.2 we have $\lim_{n \to \infty} d_2(Z_n, Z) = 0$ or equivalently

$$Z_n \overset{d}{=} Z, \quad Z \sim N(0, 1), \quad \text{and} \quad \lim_{n \to \infty} \text{Var}(Z_n) = 1. \quad (3.11)$$

Since,

$$\frac{1}{\sigma \sqrt{n}} S_{[nt]} = \sqrt{\frac{|nt|}{n}} \frac{1}{\sigma \sqrt{[nt]}} S_{[nt]} \quad \text{and} \quad \lim_{n \to \infty} \frac{|nt|}{n} = t,$$

by (3.11) and Slutsky’s Theorem we obtain $Z_t(n) \overset{d}{=} B_t$ and $\lim_{n \to \infty} \text{Var}(Z_t(n)) = t$. By Theorem 2.1, it follows that $\lim_{n \to \infty} d_2(Z_t(n), B_t) = 0$.

As in the proof of Theorem 3.2, the Liapounov’s inequality completes the proof for $0 < p < 2$. □

**Applications of Theorem 3.2.** The examples and discussion presented in this section are inspired by the ones in preprint [CDV17].

For the volume $\Lambda_N = \{1, 2, \ldots, N\}$, the Gibbs measure of the one-dimensional Ising model with free boundary conditions, at inverse temperature $\beta > 0$ on $\Lambda_N$, is given by

$$\mu_N^\beta(\sigma) = \frac{\exp \left( \beta \sum_{1 \leq i < j \leq N} V_{ij} \sigma_i \sigma_j \right)}{Z_N(\beta)}, \quad (3.12)$$
where \( \sigma = (\sigma_1, \ldots, \sigma_N) \in \{-1, 1\}^N \), the interactions \( V_{ij} \) being the same we used to define our two-dimensional random polymer model (1.2) and

\[
Z_N(\beta) = \sum_{\sigma} \exp \left( \beta \sum_{1 \leq i < j \leq N} V_{ij} \sigma_i \sigma_j \right)
\]

is the normalization factor.

Let \( \mu^\beta \) be the thermodynamical limit of \( \mu^\beta_\Lambda \) when \( \Lambda \to \mathbb{N} \). The susceptibility \( \chi(\beta) \) of the Ising model at the inverse temperature \( \beta \) is defined by

\[
\chi(\beta) := \sum_{n=1}^\infty \left\{ \mu^\beta(\sigma_1 \sigma_n) - \mu^\beta(\sigma_1) \mu^\beta(\sigma_n) \right\}.
\]  (3.13)

Let \( X = \{X_i : i \in \mathbb{N}\} \) be a stochastic process defined on \( \Omega = \{-1, 1\}^N \), where the variables \( X_i \) are projections, i.e., for \( \sigma = (\sigma_1, \sigma_2, \ldots) \in \{-1, 1\}^\mathbb{N} \) we have \( X_i(\sigma) = \sigma_i, \forall i \).

**Example 3.4.** For fixed \( L > 0 \), define \( V_{ij} \) as:

\[
V_{ij} = V, \quad i, j \in \mathbb{N} \text{ s.t. } 0 < |i - j| \leq L,
\]

where \( V > 0 \) is a constant. In this case, it is well-known that the set of the Gibbs measures \( \mathcal{G}(\beta) \) is a singleton. It can be verified that \( X \) on \( (\Omega, \mathcal{F}, \mu) \) is not a sequence of independent r.v.’s by applying the GKS-II inequality. Moreover, one can verify that \( X \) on \( (\Omega, \mathcal{F}, \mu) \) is stationary and positively associated by using the FKG inequality [FKG71]. From the Lieb-Simon inequality (cf. [Lie80] and [Sim80]) follows that the susceptibility \( \chi(\beta) < +\infty \) (for a more recent version see [DCT16]). By a straightforward calculation, we have

\[
\lim_{n \to \infty} \frac{1}{n} \text{Var}(S_n) = \text{Var}(\sigma_1) + 2 \lim_{n \to \infty} \frac{1}{n} \sum_{j=2}^{n-1} \sum_{i=1}^{j-1} \text{Cov}(\sigma_1, \sigma_j) = \sum_{j=1}^{\infty} \text{Cov}(\sigma_1, \sigma_j) = \chi(\beta)
\]

and similarly \( \lim_{n \to \infty} \sum_{j=1}^{m_n} \text{Var}(Y_{j,n})/m_n \ell_n = \chi(\beta) \). Therefore, all the hypothesis of Theorem 3.2 hold. Moreover, the convergences (3.7) and (3.10) also hold for \( 0 < p \leq 2 \).

**Example 3.5.** For all \( i \in \mathbb{N} \) we define \( V_{ii} = 0 \) and

\[
V_{ij} = \beta |i - j|^{-\alpha}, \quad i, j \in \mathbb{N} \text{ and } i \neq j
\]

where \( \beta > 0 \) and \( \alpha > 1 \). The analysis in terms of the parameter \( \alpha \) is twofold. The first one is \( 1 < \alpha \leq 2 \) and the second is \( \alpha > 2 \).

Suppose that \( 1 < \alpha \leq 2 \). In this case there is a real number \( \beta_c(\alpha) \in (0, +\infty) \) called critical point such that, for all \( \beta < \beta_c(\alpha) \) the set of the Gibbs measures \( \mathcal{G}(\beta) \) is a singleton [Dys69, FSS82] and the unique probability measure \( \mu_{\beta,0} \) has the FKG property and the stochastic process \( X = \{X_i : i \in \mathbb{N}\} \) on \( (\Omega, \mathcal{F}, \mu_{\beta,0}) \) is associated and stationary. It was obtained in [AN86] a polynomial decay for \( \text{Cov}_{\mu_{\beta,0}}(X_0, X_i) \),
up to the critical point, and since $\alpha > 1$, the susceptibility $\chi (\mu_{\beta, \alpha}) < +\infty$. The conditions (3.4)-(3.6) are checked in analogy to that made in Example 3.4. In this case, the convergences (3.7) and (3.10) hold for $0 < p \leq 2$. On the other hand, for all $\beta > \beta_c (\alpha)$, the set $\mathcal{Y} (\beta)$ has infinitely many elements. Then, we can not ensure that the stochastic process $X$ on $(\Omega, \mathcal{F}, \mu)$ is stationary for any $\mu \in \mathcal{Y} (\beta)$. Moreover, the susceptibility is not finite anymore.

The case $\alpha > 2$ is similar to the case $1 < \alpha \leq 2$ and $\beta < \beta_c (\alpha)$, but no restriction on the parameter $\beta$ is needed to ensure the uniqueness of the Gibbs measures and the other used properties.

3.2. Higher dimensions. Given $m \in \mathbb{N}$, let $\{(X_i^1, \ldots, X_i^m) : i \in \mathbb{N}\}$ be a $m$-dimensional stationary random process with independent coordinates. We define

$$S^j_n := \sum_{i=1}^{n} X_i^j, \quad Z^j_i (n) := \frac{S^j_i (nt)}{\sqrt{n}}, \quad j = 1, \ldots, m, \quad (3.14)$$

$$S^m_n := (S^1_n, \ldots, S^m_n)$$

and $Z^m_i (n) := (Z^1_i (n), \ldots, Z^m_i (n))$, where $\sigma^2 \in (0, +\infty)$.

**Theorem 3.6.** Let $\{X_i^j : i \in \mathbb{N}\}, j = 1, \ldots, m$ be a centered, one-dimensional positively associated and stationary random process satisfying the hypothesis of Theorem 3.2. Then, for each $0 < p \leq 2$ we have

$$\lim_{n \to \infty} d_p (Z^m_n (t), B^m_m (t)) = 0 \quad \text{and} \quad \lim_{n \to \infty} \mathbb{E} (\|Z^m_n (t)\|^p) = \mathbb{E} (\|B^m_m (t)\|^p),$$

where $B^m_m (t) = (B^1_t, \ldots, B^m_t)$ is the $m$-dimensional standard Brownian motion.

**Proof.** In view of Theorem 2.2, there exists an r.v. $B^j_{t}^{i,*} \overset{d}{=} B^j_{t}^i$ such that the joint distribution of $(Z^j_i (n), B^j_{t}^{i,*})$ is given by $H (x, y) = \min \{F_{Z^j_i (n)} (x), F_{B^j_{t}^{i,*}} (y)\}$ and

$$d_2 (Z^j_i (n), B^j_{t}^{i,*}) = \mathbb{E} ((Z^j_i (n) - B^j_{t}^{i,*})^2), \quad j = 1, \ldots, m.$$

By above identity and by Minkowski’s inequality,

$$d_2 (Z^m_n (t), B^m_m (t)) \leq \left\{ \mathbb{E} \left( \|Z^m_n (t) - B^m_m (t)\|^2 \right) \right\}^{1/2} \leq \sum_{j=1}^{m} \left\{ \mathbb{E} \left( (Z^j_i (n) - B^j_{t}^i)^2 \right) \right\}^{1/2} = \sum_{j=1}^{m} d_2 (Z^j_i (n), B^j_{t}^i). \quad (3.15)$$

Combining this inequality with (3.10), we obtain the convergence of order $p = 2$.

As in the proof of Theorem 3.2, the Liapounov’s inequality completes the proof for $0 < p < 2$. \hfill \square

**Theorem 3.7.** Under conditions of Theorem 3.6, consider the random process

$$W_{n,m} (t) := \frac{1}{\sigma \sqrt{2n}} \left\{ S^m_k + (nt - k) (S^m_{k+1} - S^m_k) \right\}, \quad \frac{k}{n} \leq t < \frac{k+1}{n}, \quad t \in [0, 1].$$
Then, for each $0 < p \leq 2$ we have
\[
\lim_{n \to \infty} d_p(W_{n,m}(t), B_m(t)) = 0 \quad \text{and} \quad \lim_{n \to \infty} \mathbb{E}(\|W_{n,m}(t)\|^p) = \mathbb{E}(\|B_m(t)\|^p),
\]
where $B_m(t) = (B^1_t, \ldots, B^m_t)$ is the $m$-dimensional standard Brownian motion.

Proof. The inequality $\frac{k}{n} t < \frac{k+1}{n} t$ implies that $k = \lfloor nt \rfloor$. Then,
\[
W_{n,m}(t) = \frac{1}{\sqrt{2}} \left\{ Z^m_n(t) + \frac{(nt - |nt|)}{\sigma \sqrt{n}} (S^m_{\lfloor nt \rfloor+1} - S^m_{\lfloor nt \rfloor}) \right\} =: \frac{1}{\sqrt{2}} \left\{ Z^m_n(t) + \psi^m_n(t) \right\}.
\]
Note that
\[
\|\psi^m_n(t)\|^2 = \frac{(nt - |nt|)}{\sigma \sqrt{n}} \sum_{j=1}^m (X^j_{k+1})^2.
\]
By above decomposition of $W_{n,m}(t)$ and Minkowski’s inequality,
\[
d_2(W_{n,m}(t), B_m(t)) \leq \frac{1}{\sqrt{2}} \left\{ \mathbb{E}(\|W_{n,m}(t) - B_m(t)\|^2) \right\}^{1/2}
\leq \frac{1}{\sqrt{2}} \left\{ \mathbb{E}(\|Z^m_n(t) - B_m(t)\|^2) \right\}^{1/2} + \frac{1}{\sqrt{2}} \left\{ \mathbb{E}(\|\psi^m_n(t)\|^2) \right\}^{1/2}
\leq \frac{1}{\sqrt{2}} \sum_{j=1}^m d_2(Z^j_n(n), B^j_t) + \frac{1}{\sqrt{2}} \left\{ \frac{(nt - |nt|)}{\sigma \sqrt{n}} \sum_{j=1}^m \text{Var}(X^j_{k+1}) \right\}^{1/2},
\]
where in the last inequality we used (3.15) and (3.16). Taking $n \to \infty$ in the above inequality and using Corollary 3.3, we get
\[
\lim_{n \to \infty} d_2(W_{n,m}(t), B_m(t)) = 0.
\]

The main result of this paper is an application of the results of this section for a two-dimensional random polymer model presented in the next section.

4. Proof of Theorem 1.1

In this section, we will prove that below the critical temperature $\beta_c$ (respectively, above the critical temperature), the random polymer model, after a rescaling, converges (respectively, not converge) in Wasserstein distance to the planar standard Brownian motion, leaving open the problem of asymptotic behavior of the random polymers in the critical phase.

4.1. Proof of Item 1.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation through an angle $\pi/4$. If $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n) \in \mathcal{W}_n$ and $\mathcal{X}_i = \mathcal{S}_{i+1} - \mathcal{S}_i$, then for each step $\mathcal{X}_i$ there are $\sigma^{(1)}_i, \sigma^{(2)}_i \in \{-1, 1\}$ so that
\[
T \mathcal{X}_i = \sigma^{(1)}_i \frac{\epsilon_1}{\sqrt{2}} + \sigma^{(2)}_i \frac{\epsilon_2}{\sqrt{2}}.
\]
Let $\mu^\beta$ be the thermodynamical limit of $\mu^\beta_\Lambda$ when $\Lambda \to \mathbb{N}$, and $\chi(\beta)$ the susceptibility of the Ising model at the inverse temperature $\beta$ defined in (3.13). Once we are assuming $V_{ij} \geq 0$, it follows from the FKG inequality [FKG71] that $\chi(\beta) \geq 0$ and that the sequence of r.v.'s $\{\sigma_i^{(j)} : i \in \mathbb{N}\}, j = 1, 2$, are associated with respect to the infinite volume Gibbs measure $\mu^\beta$.

Denoting $S_i^j = \sum_{k=1}^k X_i^j$ with $X_i^j := \sigma_i^{(j)}$, $j = 1, 2$, by (4.1) note that $TW_n(t)$ has the following expression

$$TW_n(t) = \frac{1}{\sqrt{2n\chi(\beta)}} \sum_{j=1}^2 \left( S_{\lfloor nt\rfloor}^j + (nt - \lfloor nt\rfloor)(S_{\lfloor nt\rfloor+1}^j - S_{\lfloor nt\rfloor}) \right) e_j = W_{n,2}(t),$$

where $W_{n,2}(t)$ is the random process defined in Theorem 3.7 with $m = 2$ and $e_1, e_2$ are elements of the canonical basis of $\mathbb{R}^2$. By the Aizenman-Barsky-Fernández’s Theorem (see [ABF87] and for a more recent version see [DCT16]), we have that the susceptibility is finite as long as $\beta < \beta_c$. If $\mathcal{G}(\beta)$ denotes the set of all infinite volume Gibbs measures, it is well know that the set $\mathcal{G}(\beta) = \{\mu^\beta\}$ is a singleton for any $\beta < \beta_c$. In this case, this unique measure $\mu^\beta$ is translation invariant and this fact implies that the r.v.'s $\{X_i^j : i \in \mathbb{N}\}, j = 1, 2$, form a stationary sequence. Once these r.v.'s are bounded, they have finite third moment and (3.6) is valid. The conditions (3.4) and (3.5) of Theorem 3.2 are proved analogously to Example 3.4, so all the hypotheses of Theorem 3.7 hold. Therefore,

$$\lim_{n \to \infty} d_p(W_{n,2}(t), B_2(t)) = 0. \quad (4.2)$$

On the other hand, since $T$ is invertible and its inverse is a linear transformation, denoted by $T^{-1}$, we have $W_n(t) = T^{-1}W_{n,2}(t)$. Combining this equality with Item (8.3) of reference [Mal72] and with the fact that the distribution of the Brownian motion is invariant under rotations in the plane, we obtain

$$d_2(W_n(t), B_2(t)) = d_2(T^{-1}W_{n,2}(t), B_2(t)) \leq ||T^{-1}|| \cdot d_2(W_{n,2}(t), B_2(t)).$$

Taking $n \to \infty$ in the above inequality, by (4.2) we get $\lim_{n \to \infty} d_2(W_n(t), B_2(t)) = 0$.

Again, the Liapounov’s inequality completes the proof for $0 < p < 2$. \hfill \square

4.2. Proof of Item 2.

By Theorem 2.1 it is sufficient to prove that $W_n(t) \xrightarrow{p} B_2(t)$ for each $\beta > \beta_c$. The proof is by contradiction. Suppose that $W_n(t) \xrightarrow{p} B_2(t)$. So for each $t \in \mathbb{R}$, we have

$$\lim_{n \to \infty} \mathbb{E}(||W_n(t)||_1^2) = \mathbb{E}(||B_2(t)||_1^2) = 2t. \quad (4.3)$$
From now, to simplify the notation we write $W_n(t)$ as

$$W_n(t) = \left\{ \frac{1}{\sigma \sqrt{n}} S_{\lfloor nt \rfloor} + \frac{(nt - \lfloor nt \rfloor)}{\sigma \sqrt{n}} (S_{\lfloor nt \rfloor + 1} - S_{\lfloor nt \rfloor}) \right\} =: \frac{1}{\sigma \sqrt{n}} S_{\lfloor nt \rfloor} + \psi_n(t).$$

For any $t > 0$, the mean square of $W_n(t)$ is given by

$$E(\|W_n(t)\|^2) = \frac{1}{n \sigma^2} E(\|S_{\lfloor nt \rfloor}\|^2_1) + \frac{1}{\sigma \sqrt{n}} E(\langle S_{\lfloor nt \rfloor}, \psi_n(t) \rangle) + E(\|\psi_n(t)\|^2_1)$$

$$= a_n(t) + b_n(t) + c_n(t).$$

It is simple to check that $\psi_n(t) = \frac{(nt - \lfloor nt \rfloor)}{\sigma \sqrt{n}} (S_{\lfloor nt \rfloor + 1} - S_{\lfloor nt \rfloor}) \overset{\mathcal{D}}{\to} 0$ and $\lim_{n \to \infty} c_n(t) = 0$. We also have $\lim_{n \to \infty} b_n(t) = 0$, since

$$\frac{S_{\lfloor nt \rfloor}}{\sqrt{\lfloor nt \rfloor}} \overset{\mathcal{D}}{\to} B_2(t) \quad \text{and} \quad \frac{\sqrt{nt - \lfloor nt \rfloor}}{\sigma \sqrt{n}} \to 0.$$ 

Finally, applying the main theorem of [CDdS14] on the Thermodynamical limit, we get

$$\frac{1}{2} m_w(\beta)^2 \leq \frac{1}{\lfloor nt \rfloor^2} E(\|S_{\lfloor nt \rfloor}\|^2_1) \leq 1.$$ 

This shows that for any $\epsilon > 0$ there exists $N_0$ so that, if $n \geq N_0$, then

$$\frac{\lfloor nt \rfloor^2}{2n \sigma^2} m_w(\beta)^2 \leq a_n(t) \leq \frac{\lfloor nt \rfloor^2}{n \sigma^2} + \epsilon,$$

which implies $\lim_{n \to \infty} a_n(t) = +\infty$. Hence, for each $t > 0$,

$$\lim_{n \to \infty} E(\|W_n(t)\|^2_1) = +\infty,$$

which contradicts (4.3). The proof is complete. \hfill $\square$

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