Differentiating Dilatons from Axions by their mixing with photons

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Abstract: According to the model (ΛCDM), based on deep cosmological observations, the current universe is constituted of 5% baryonic matter and 25% non-baryonic cold dark matter (of speculative origin). These include quanta of scalar field like dilaton (φ) of scale symmetry origin and quanta of pseudoscalar field of extra standard model symmetry (Peccei-Quinn) origin, like axion (φ′). These fields couple to di-photons through dim-5 operators. In magnetized medium, they in principle can interact with the three degrees of freedom (two transverse (A∥, A⊥) and one longitudinal (A_L)) of photon (γ) as long as the total spin is conserved. Because of intrinsic spin being zero, both φ and φ′ could in principle have interacted with A_L (having s_z = 0). However, out of φ and φ′ only one interacts with A_L. Furthermore, the ambient external magnetic field and media, breaks the intrinsic Lorentz symmetry of the system invoking Charge conjugation, Parity and Time reversal symmetries, we analyse the mixing dynamics of φγ and φ′γ systems and the structural difference of their mixing pattern. The strength of electromagnetic (EM) signals due to φγ and φ′γ mixing as a result would be different. We conclude by commenting on the possibility of detecting this difference—in polarimetric observables the EMS—using the existing space-borne detectors.
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1 Introduction

Scalar or pseudoscalar bosons like dilaton ($\phi(x)$) [1, 2] or axion ($\phi'(x)$) [3–6] are postulated to arise out of symmetry breaking through quantum effects in field theory. That apart, unified theories like string theory [7–12] also predict existence of similar particles, that may appear through moduli compactification. Out of the two bosons mentioned above, dilaton is supposed to restore the scaling symmetry and QCD axion is supposed to cure the $U_A(1)$ anomaly, lastly—string theory axion is supposed to break the shift symmetry of the theory. Both of these particles, though are still illusive to experimental verification, however offer remarkable solutions to the outstanding problems of physics. This makes any effort towards their verification a highly sought after activity. Thus detection of EM signals of their presence has turned out to be a coveted activity all over the globe.

In terms of electromagnetic field strength tensors $f_{\mu\nu}$ (defined in terms of gauge potentials $A_\mu$, as, $f_{\mu\nu}(k) = k_\mu A_\nu - k_\nu A_\mu$) and its dual $\tilde{f}_{\mu\nu} = \frac{1}{2} \varepsilon^{\lambda\sigma\mu\nu} f_{\lambda\sigma}$, the interaction of photons ($\gamma$) with $\phi$ or $\phi'$ is governed by dimension-five anomalous interaction Lagrangian of the form,

$$L_{\text{int},\phi} = g_{\phi\gamma\gamma} f_{\mu\nu} f_{\mu\nu}, \quad \text{is for scalar photon and} \quad L_{\text{int},\phi'} = g_{\phi'\gamma\gamma} f_{\mu\nu} f_{\mu\nu}, \quad (1.1)$$

is for pseudoscalar photon interaction, where symbols $g_{\phi\gamma\gamma}$ or $g_{\phi'\gamma\gamma}$ appearing in eqn. (1.1) are the anomaly induced coupling constants between $\phi$ or $\phi'$ with photons. We denote $\phi$ and $\phi'$ collectively with a subscript $i$ as $\phi_i$ (such that for scalars $\phi_i = \phi$ and for pseudoscalars $\phi_i = \phi'$). As a result of this anomalous coupling, the life time of these particles, having mass $m_{\phi_i}$, against decay into two photons, is given by,

$$\tau_{\phi_i\gamma\gamma} \sim \frac{1}{g_{\phi_i\gamma\gamma}^2 m_{\phi_i}^3}. \quad (1.2)$$

As an aside we note that, except for the axions of quantum chromodynamics (QCD), the mass $m_{\phi'}$ and the coupling constant $g_{\phi'\gamma\gamma}$ for axions of other theories are not related to each other. The same is true for dilatons too. In this work they (scalar and pseudoscalar) are referred, collectively as axion like particles (ALP).

Ever since the cosmological production mechanisms of these particles were realised [13],
the numerical estimates of the parameters \( m_{\phi_i} \) and \( g_{\phi_i\gamma\gamma} \), started being constrained from various astronomical observational facts. One such fact involves the number density of photons \( n_{\gamma} \) at different epochs of cosmological evolution. For instance, cosmological big bang nucleosynthesis puts a very restrictive bound on the number density of photons \( n_{\gamma} \) during primordial deuterium synthesis [14–16]. Therefore processes that lead to a change in \( n_{\gamma} \), e.g., \( \phi_i \rightarrow \gamma\gamma \), have to be fine-tuned – to fulfil this criteria. This in turn puts a bound on parameters \( m_{\phi_i} \) and \( g_{\phi_i\gamma\gamma} \) so that there’s no change in \( n_{\gamma} \) due to (pseudo)scalar decay into diphotons – during cosmic deuterium synthesis.

In the figure below (fig. [1]) an exclusion plot of parameters \( g_{\phi_i\gamma\gamma} \) vs \( m_{\phi_i} \), has been provided. This plot came into existence after considering similar such constraints–those followed from various astrophysical and laboratory based experiments. The relevant point about this plot is, apart from identifying the excluded regions, the same also shows the allowed regions of parameters \( g_{\phi_i\gamma\gamma} \) and \( m_{\phi_i} \), i.e., their possible numerical size.\(^1\)

The emergence of \( \phi_i \)’s as possible candidates of dark matter (DM) followed from the

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\(^1\)We would like to emphasise here that, these bounds are generally insensitive to the nature of the particle(\( \phi \) or \( \phi' \)) and are applicable to any ALP candidate ( particle ) in general.
fig.[1] — there are still regions, for which the life time $\tau_{\phi,\gamma\gamma}$ of (pseudo)scalars can be large enough to be comparable to the age of the universe. That too without disturbing the bound on baryon ($n_B$) to photon ($n_\gamma$) density-ratio observed at the current epoch (i.e., $\frac{n_B}{n_\gamma} \sim 10^{-11}$).

Coming back to the main objective of this investigation, we recall that, the identification of ALP like dark matter candidates— would provide an elegant solution to dual problems; one in the realms of cosmology and the other in the areas of physics beyond the standard model of particle physics.

However this task of ALP identification becomes difficult due to the two fold objectives associated with it, (i) identification of their type; i.e., ALP or not and (ii) identifying their nature, i.e., scalar or pseudoscalar. There are two possible ways to search for them, those are (a) collider based or (b) spectro-polarimetry based. Collider based search is difficult, because of the colliding beam luminosity required to have high signal to noise ratio. But that apart, in some channels it becomes even more difficult, if the final state decay products involve scalar or pseudoscalar interaction—in the initial or intermediate states—have identical interaction parameters. For instance, the life time observations (estimates), in the diphoton or dilepton decay channels—for $\phi$ and $\phi'$ for nearly same numerical strength and size of $m_\phi$ and $g_\phi\gamma\gamma$, would offer very limited scope to distinguish one from the other.

Discussion on collider based search is out of scope for this work and we will focus on the second method. The same was initiated in the seminal work of [17] and then followed by many others — over a considerable period of time.

Depending on the (astrophysical) objects —i.e., their environments and physical situations— detecting— and distinguishing signals due to $\phi$ from $\phi'$, using spectro-polarimetric techniques may also turn out to be equally difficult as their collider based counterparts. However, one can make the same achievable with lesser efforts if one appropriates the symmetry properties of the DM candidates (i.e., $\phi$ and $\phi'$ ) one is looking for and the background media that is under consideration: before analysing the spectro-polarimetric signals from their collective interactions. However to fully appreciate our approach and understand it’s distinction (from that of available ones), a general symmetry based analysis of the system is necessary; we present a deliberation on these issues in the next few sections of this work.

1.1 Motivation : Symmetries and their consequences:

We start by recollecting the obvious fact that, any Lagrangian transforms like a scalar under Lorentz transformation. Hence, the sum of the spin angular momentum assignments of the fields ( background and dynamical) constituting each term of the same must add up to zero. Therefore the interacting Lagrangian in an ambient background EM field ($\tilde{F}^{\mu\nu}$) — for scalar or pseudo-scalar photon — system should also conform accordingly with this observation. That is, (pseudo)scalar-photon interaction Lagrangian in ambient $\tilde{F}^{\mu\nu}$ field should also respect the same. Stated differently, the $\phi\gamma$ interaction Lagrangian (IL) and $\phi'\gamma$ IL are obtained by making the following replacement $f^{\mu\nu} \rightarrow \tilde{F}^{\mu\nu} + f^{\mu\nu}$ in the Lagrangians given in eqn. (1.1) giving the ILs $g_{\phi\gamma\gamma}\phi\tilde{F}^{\mu\nu}$ and $g_{\phi'\gamma\gamma}\phi'\tilde{F}^{\mu\nu}$, should also respect the same.

The characteristics of the scalar-photon ($\phi\gamma$) or pseudoscalar -photon ($\phi'\gamma$) IL, in any ambient background electromagnetic (EM) field depends on the nature of the background
field itself, i.e. whether it is electric or magnetic type and the discrete symmetries of the dynamical fields, i.e., that of the photons and the scalars/pseudoscalars involved.

Furthermore, all the terms present in a Lagrangian endowed with gauge fields do not contribute entirely to the equations of motion, unless gauge fixed, due to the associated gauge ambiguities. Expressing the gauge degrees of freedom of a gauge fixed Lagrangian in terms of a set of orthonormal 4-vectors representing the modes or degrees of freedom, one obtains a transformed Lagrangian – such that each term of the field equations – obtained from this transformed Lagrangian – has a set of definite discrete symmetry assignments. That is each term in every equation of motion, must transform identically either individually–under charge (C) conjugation, parity (P) and time(T) reversal transformations; or identically under the action of a set of their combinations. If the Lagrangian has some additional unitary internal symmetries associated with it then each term in the field equations should also transform identically under these unitary symmetry transformations; thus preserving the covariance of the field equations.

This results in the following consequences: i.e., in a magnetized vacuum (MV)(i.e., in an ambient magnetic field $B_3$, such that, only $B_3 = B_z = F_{12} \neq 0$), only one dof describing a particular polarisation state of the photons out of the two possible states (polarized transversely - along and orthogonal to the ambient (external) magnetic field $B$) interacts with the scalars or pseudoscalars– at this vertex; the other one doesn’t. This interaction depends on the CP symmetry of the polarization state of the photon and that of the interacting fields $\phi$ or $\phi'$. Dynamics of the two states or the transverse degrees of freedom (dof) of the photons can be described in two ways one in terms of gauge fields, two in terms of gauge invariant form-factors, $\Psi$ and $\tilde{\Psi}$, defined as $\Psi = \bar{F}^{\mu\nu}f_{\mu\nu}$ and $\tilde{\Psi} = \bar{F}^{\mu\nu}f_{\mu\nu}$, in the notations of [19]. In this notation the dof of the photons having plane of polarization (POP) orthogonal to the ambient magnetic field $B$ is denoted by $\Psi$. Similarly the same for the photons having POP along $B$, will be described by $\tilde{\Psi}$.

Under the operation of charge conjugation C and parity transformation P, both the tensors $\bar{F}^{\mu\nu}$ and $f^{\mu\nu}$ turn out to be odd and even (see section two for more details)[20]. Hence, the combinations $\bar{F}^{\mu\nu}f_{\mu\nu}$, turns out to be CP even and the other one, i.e., $\tilde{F}^{\mu\nu}f_{\mu\nu}$ turns out to be PT even, therefore CP odd [17, 21]. Since the scalar $\phi(x)$ is always CP even, therefore at the level of the equation of motion (EOM) in a MV, for $\phi \gamma$ interaction the CP even $\Psi$ couples only to CP even $\phi(x)$, remaining CP odd form-factor $\tilde{\Psi}$ propagates freely. In similar situation, when interaction of $\phi'(x)$ with $\gamma$ is considered, the former being (a pseudoscalar i.e., P odd but C even ) CP odd will couple only to the CP odd form-factor $\tilde{\Psi}$ of the photon. The CP even $\Psi$ would propagate freely. This is just the reverse of what happens for $\phi \gamma$ interaction in MV. That is, in MV, the role of $\Psi$ in $\phi' \gamma$ interaction turns out to be similar to that of $\tilde{\Psi}$ in $\phi \gamma$ interaction and vice-versa. Hence one can conclude that– in a MV–as the form of the interaction changes from $\phi \gamma \rightarrow \phi' \gamma$, the role of $\Psi$ and $\tilde{\Psi}$ interchanges with each other. Hence, the mixing dynamics of both systems, $\phi \gamma$ or $\phi' \gamma$, are governed by $2 \times 2$ mixing matrix. This was originally pointed out by Raffelt and Stodolosky in their seminar paper [17].

The propagation modes of the $\phi \gamma$ system in MV, are obtained by diagonalising the the mixing matrix. The CP even propagating modes for the same, in the mass-less limit,
turn out to be \((\Psi \pm \Phi)\) where \(\Phi = (|k\perp|B\phi)\) when \(k\perp = k\sin\theta\) and \(\theta\) is the angle between propagation vector \((\vec{k})\) and the magnetic field \((\vec{B})\). Their corresponding dispersion relations satisfy, \(k^2 \pm g_{\phi\gamma\gamma}|k\perp|B = 0\), [19].

Remarkably, the propagating modes for \(\phi'\gamma\) system, in similar situation, though is CP odd and given by \(\left(\hat{\Psi} \pm \Phi\right)\), (but) their dispersion relation remains the same as \(\phi\gamma\) system; that is, \(k^2 \pm g_{\phi'\gamma\gamma}|k\perp|B = 0\). As can be seen that the dispersion relations (DRs) for both cases are indeed invariant under boost and rotation about the direction of the magnetic field. This happens because, the presence of the external field breaks the Lorentz symmetry of the system. It has been argued in section two that, the action integral \((S = \int Ld^4x)\) remains invariant only under the action of the generators of boost and rotation along and around the 3rd direction of the Lorentz group, so as to keep \(S\) invariant all other generators for the remaining Lorentz symmetry transformations must vanish. The dispersion relation mentioned above is a manifestation of the same. As a result, a MV turns out to be optically active and dichroic in presence of the \(g_{\phi'\gamma\gamma}\phi'f^{\mu\nu}f_{\mu\nu}\) or \(g_{\phi\gamma\gamma}\phi f^{\mu\nu}f_{\mu\nu}\) interactions.\(^2\)

Therefore, for near degenerate magnitudes of the coupling constants and masses of the candidate particles, proper identification of one from the other may become an arduous task, for unfavourably oriented magnetic field, even for some astrophysical situations. One possible way to get out of this impasse and refine the study of mixing physics is through incorporation of matter effects [22–27, 30], by including \(A^\mu(k)\Pi_{\mu\nu}(k,\mu,T)A^\nu(k)\), in the effective Lagrangian \((L_{eff})\) of the system; with \(\Pi_{\mu\nu}(k,\mu,T)\) being the in-medium photon self-energy tensor, that depends on temperature \((T)\), chemical potential \((\mu)\) and 4-momentum \((k^\mu)\). We recall chemical potential \((\mu)\) brings effect of C symmetry into the system.

The introduction of matter effects bring additional subtlety in the mixing dynamics, by introducing additional spin zero dof called longitudinal degree of freedom \((A_L)\) for photons. Both Scalars and pseudoscalars, having spin zero can now mix with \(A_L\) because of spin conservation, thus having a \(4 \times 4\) mixing matrix. And thereby spoiling the cure that one was looking for.

However a PT symmetry analysis of the terms of the field equations for both the systems — expressed – in terms of the photon form-factors, as was introduced in [25], shows that because of PT symmetry, pseudo-scalar photon mixing matrix is \(3 \times 3\), and scalar photon mixing matrix is \(2 \times 2\). This however leaves the question unanswered, that is, if all the elements of the \(3 \times 3\) pseudo-scalar mixing matrix are finite or its just some of them are? The answer is no not all the elements are nonzero. This can be understood from the fact, that all the nonzero elements, those present in the mixing matrix, are due to interaction of the one transverse degree of photon with the pseudo-scalar. Because, since the form factors were expressed in an orthogonal basis, therefore a medium undergoing only EM interaction, the longitudinal one wouldn’t mix with the transverse ones, neither the transverse ones will mix with each other. More about it is detailed in section two.

The refined study, narrated before, accounts only for mixing of two degrees of freedom.

\(^2\)It is noteworthy that an additional possibility of subluminal or superluminal motion of the photons with such interaction, in some energy range, have also been reported in the literature [19, 34–36]
for scalar-photon and three degrees of freedom for pseudo-scalar photon systems. Leaving two degrees of freedom for scalar-photon and one degree of freedom for pseudo-scalar photon system free. In other words, though the mixing pattern for both have changed, but we yet to achieve maximal mixing (for either of the two). This issue can further be cured by incorporation of the parity violating part of photon self-energy tensor in the system, following the steps mentioned before; the necessary details of which, like before, are also relegated in the next section.

2 Interaction Dynamics

2.1 Tree level Lagrangian

The non-minimally coupled \( \phi_i \gamma \gamma \) interacting Lagrangian in an ambient magnetic field (\( \phi_i \) stands for either a scalar (\( \phi \)) or a pseudoscalar (\( \phi' \)) field) can be expressed as a sum over three parts;

\[
L_{\phi_i} = L_{\text{free}, \phi_i} + L_{B \text{int}, \phi_i} + L_{\text{int}, \phi_i}.
\]

(2.1)

When \( L_{\text{free}, \phi_i} \) in eqn. (2.1) stands for free field Lagrangian, given by,

\[
L_{\text{free}, \phi_i} = \frac{1}{2} \phi_i (-k) \left( k^2 - m_{\phi_i}^2 \right) \phi_i (k) - \frac{1}{4} f_{\mu \nu} f^{\mu \nu},
\]

(2.2)

\( L_{B \text{int}, \phi_i} \), in the same equation stands for the interactive part, due to ambient magnetic field and is given by

\[
L_{B \text{int}, \phi_i} = -\frac{1}{4} g_{\phi_i \gamma \gamma} f_{\mu \nu} f^{\mu \nu}, \quad \text{where,} \quad f_{\mu \nu} = \begin{cases} \bar{F}^{\mu \nu} \text{when } \phi_i = \phi \\ \bar{F}^{\mu \nu} \text{when } \phi_i = \phi' \end{cases}
\]

(2.3)

and lastly the Lagrangian for the \( \phi_i \gamma \gamma \) interactive part \( L_{\text{int}, \phi_i} \) – the last term in the same equation is given by,

\[
L_{\text{int}, \phi_i} = -\frac{1}{4} g_{\phi_i \gamma \gamma} f_{\mu \nu} f^{\mu \nu}, \quad \text{where,} \quad f_{\mu \nu} = \begin{cases} f_{\mu \nu} \text{when } \phi_i = \phi \\ \tilde{f}_{\mu \nu} \text{when } \phi_i = \phi' \end{cases}
\]

(2.4)

2.2 Symmetries

2.2.1 Discrete symmetries

The equations of motion for (pseudo)scalar-photon system in MV, following from the Lagrangian (2.1), can be cast in matrix form, in terms of the gauge invariant variables,

\[
\Psi = \bar{F}^{\mu \nu} f_{\mu \nu} \quad \text{and} \quad \bar{\Psi} = \tilde{F}^{\mu \nu} f_{\mu \nu}.
\]

(2.5)

as;

\[
\begin{pmatrix}
  k^2 & 0 & 0 \\
  0 & k^2 & -\frac{g_{\phi \gamma \gamma} \omega B_T}{2} \\
  0 & -\frac{g_{\phi \gamma \gamma} \omega B_T}{2} & k^2 - m_{\phi'}^2
\end{pmatrix}
\begin{pmatrix}
  \Psi \\
  \bar{\Psi} \\
  \Phi'
\end{pmatrix} = 0,
\]

(2.6)
\[
\begin{pmatrix}
k^2 & 0 & 0 \\
0 & k^2 & -g_{\phi\gamma\omega}B_T \\
0 & -g_{\phi\gamma\omega}B_T & k^2 - m^2_\phi
\end{pmatrix}
\begin{pmatrix}
\Psi \\
\Phi
\end{pmatrix} = 0.
\]  

(2.7)

Here \( \Phi = |k_\perp|B\phi, \Phi' = |k_\perp|B\phi' \) and \( B_T = B\sin\theta \). The matrix eqn. \( (2.6) \) stands for the equation of motion (EOM) for photon-pseudoscalar system and the one given by eqn. \( (2.7) \) stands for the EOM of photon-scalar system. Under CP transformation, the bilinear field variables \( \Psi \) and \( \tilde{\Psi} \) and the fields \( \Phi \) and \( \Phi' \) transforms in the following manner,

\[
\begin{align*}
(\text{CP})\tilde{F}_{\mu\nu} & f^{\mu\nu}(\text{CP})^{-1} = \tilde{F}_{\mu\nu}f^{\mu\nu}; (\text{CP})\Phi(x)(\text{CP})^{-1} = \Phi(x) \\
(\text{CP})\tilde{F}_{\mu\nu} & f^{\mu\nu}(\text{CP})^{-1} = -\tilde{F}_{\mu\nu}f^{\mu\nu}; (\text{CP})\Phi'(x)(\text{CP})^{-1} = -\Phi'(x)
\end{align*}
\]  

(2.8) (2.9)

From the equations of motion \( (2.6) \) and \( (2.7) \), it is easy to see that CP even \( \Phi \) (or \( \phi \)) couples to CP even \( \tilde{F}_{\mu\nu}f^{\mu\nu} \) and CP odd \( \Phi' \) (or \( \phi' \)) couples to CP odd \( \tilde{F}_{\mu\nu}f^{\mu\nu} \).

We can also find how individual EM form factors, representing each degree of freedom of the photon couple to \( \phi \) or \( \phi' \), in an ambient external field by expressing the gauge field \( A_\mu(k) \), in a orthogonal 4-vector basis, in terms of the electromagnetic form factors \( (A_\parallel, A_\perp, A_L, \text{and} A_{gf}) \) following \( [25] \), as,

\[
A_\alpha(k) = A_\parallel(k)N_1b^{(1)}_\alpha + A_\perp(k)N_2\bar{I}_\alpha + A_L(k)N_L\bar{u}_\alpha + \frac{k_\alpha}{k^2}A_{gf}(k), \tag{2.10}
\]

and substituting the same in the expression for field strength tensor \( f^{\mu\nu} \) appearing in the field strength bilinear variables \( (\tilde{F}_{\mu\nu}f^{\mu\nu}) \) and \( (\tilde{F}_{\mu\nu}f^{\mu\nu}) \) both. The expressions for the normalization constants \( (N_1, N_2, N_L) \) and form the factors \( (A_\parallel, A_\perp, A_L \text{ and } A_{gf}) \) are to be found in the supplemental document \( [70] \) of this article. Upon doing so we obtain,

\[
\begin{align*}
\tilde{F}_{\mu\nu}f^{\mu\nu} &= A_\parallel N_2b^{(2)}\nu + A_LN_L\bar{u}_\nu b^{(2)}\nu, \tag{2.11} \\
\tilde{F}_{\mu\nu}f^{\mu\nu} &= A_\parallel N_1b^{(1)}\nu\bar{v}_\nu \tag{2.12}
\end{align*}
\]

Using equation \( (2.11) \)– in pseudo-scalar-photon interaction Lagrangian, \( L_{\text{int},\phi'}^B = g_{\phi'\gamma\phi}\tilde{F}^{\mu\nu}f_{\mu\nu} \), we obtain;

\[
L_{\text{int},\phi'}^B = g_{\phi'\gamma\phi'}A_\parallel N_2b^{(2)}\nu + g_{\phi'\gamma\phi'}A_LN_L\bar{u}_\nu b^{(2)}\nu. \tag{2.13}
\]

Equation \( (2.13) \) shows that, in principle, \( A_L \) and \( A_\perp \) can mix with \( \phi' \) in a background magnetic field. But since \( A_L \) comes in to existence only in presence of medium; therefore in magnetized vacuum, \( \phi' \) will mix only with \( A_\perp, A_\parallel \) would remain free. Similar exercise for \( \phi\gamma \) system, with the interaction Lagrangian

\[
L_{\text{int},\phi}^B = g_{\phi\gamma\phi}\tilde{F}^{\mu\nu}f_{\mu\nu}, \tag{2.14}
\]

yields

\[
L_{\text{int},\phi}^B = ig_{\phi\gamma\phi}A_\parallel N_1b^{(1)}\nu\bar{v}_\nu, \tag{2.15}
\]
implying interaction of $A_\parallel$ with $\phi$ only in ambient magnetic field. In other words, mixing is possible only between $A_\parallel$ and $\phi$; $A_\perp$ remaining free. Hence mixing matrix for both systems remain $2 \times 2$ in MV. Furthermore, the mixing matrix for $\phi\gamma$ system, even in a medium would remain $2 \times 2$ because of non-existent $A_L - \phi$ interaction at the Lagrangian level, as was verified in eqn.(2.15). However, since the form factor $A_L$, for longitudinal degree of freedom is physically realised only in a medium, the mixing matrix for $\phi'/\gamma$ system in a medium turn out to be $3 \times 3$. This happens because, $\phi'$ couples directly to the form factors $A_\perp$ and $A_L$ at the level of interaction Lagrangian (eqn. (2.13)), so at the level of EOM they all would drive each other through their coupling with $\phi$. More about the same will be clear when we discuss consequences of magnetized matter effects.

2.2.2 Lorentz symmetry

In the Lagrangian, $L_\phi$, except for $L^{B,\phi_i}$, all other terms respect: Lorentz and gauge symmetry and remain invariant under charge conjugation (C) along with parity (P) and time-inversion (T) transformations. Re-normalizability of the theory however gets compromised due to the presence of dim-5 operators $L^{B,\phi_i}$ and $L^{int,\phi_i}$.

Response of this system to any symmetry transformation can be investigated by subjecting the dynamical fields constituting the action,

$$S = \int d^4x L_\phi = \int d^4x (L_{free,\phi_i} + L^{B,\phi_i} + L^{int,\phi_i}). \quad (2.16)$$

undergo the symmetry transformations - that one intends them to. For instance under an infinitesimal Lorentz transformation ($\Lambda$) around the identity ($I$) – given by:

$$\Lambda_{\mu\nu} = \delta_{\mu\nu} + \omega_{\mu\nu}, \quad (\text{when the parameter, } \omega_{\mu\nu} = -\omega_{\nu\mu}), \quad (2.17)$$

the dynamical fields strength tensor, $f_{\mu\nu}$ would change in the following way,

$$f'_{\mu\nu} = \Lambda^\lambda_{\mu} \Lambda^\rho_{\nu} f_{\lambda\rho}. \quad (2.18)$$

and the background electromagnetic field strength tensor $F^{\mu\nu}$ held constant. It can further be shown that, under an infinitesimal Lorentz transformation given by (2.17) – in a background EM field (such that, $F^{21} = F^{03} \neq 0$ ) – the change in the action (2.16)–turns out to be

$$\delta S = \int d^4x F_{\mu\nu} \omega^{\nu\lambda} f_{\lambda\mu}. \quad (2.19)$$

Having the action (given by equation) (2.16) invariant, under transformation (2.17) i.e., $\delta S = 0$; is possible if two out of the six components of the antisymmetric parameters $\omega_{\mu\nu}$, (only $\omega_{03}$ and $\omega_{12}$) are non-zero [77]. Rest four are identically zero. Implying that the action given by equation (2.16) remains invariant only under the action of two generators of the Lorentz group; (i) the generators of boost ($K_3$) and (ii) rotation ($J_3$) along the third direction. These two generators of the Lorentz symmetry are unbroken, rest four generators are broken. Therefore the only space-time symmetries – those remain preserved
– are boost along third (i.e. z)-direction and rotation in the one-two (xy) plane. Rest of the symmetries are all broken. Hence, the, equations of motion and the dispersion relations, following thereof, should respect these conditions. Therefore it is not surprising that the dispersion relations–obtained from the mixing matrices appearing in eqns. (2.6) and (2.7), when expressed in terms of the four vector \( k \) and the component of the same orthogonal to \( B, k_\perp \text{i.e.,} \)

\[
k^2 = 0, \\
(k^2 \pm g_{\phi\gamma\gamma}k_\perp B_T) = 0; \tag{2.20}
\]
do respect this conclusion. That is, these relations remain invariant under boost and rotation around z-axis, a consequence claimed – on the basis of Lorentz symmetry.

This results in, \( MV \) being birefringent and dichroic. In remaining part of this section, though we may not explicitly discuss the fate of the space-time/Lorentz symmetry of the Lagrangian, upon incorporation of other corrections due to material or magnetized material effects–but the same however can be shown to remain compromised due to the appearance of velocity 4-vector of the centre of mass of the material medium \( u^\mu \), or the external field \( B_z \) or both, in the description of the effective Lagrangian.

### 3 Solutions of field equation

#### 3.1 Solutions in magnetized vacuum

Since the gauge symmetry remains intact, so in the following we’ll study the dynamics of these systems in terms of gauge invariant variables noted earlier in [19]. To find the solutions of the variables appearing in either of the matrix equations (2.6) and (2.7), for \( \phi_\gamma \) or \( \phi'\gamma \) system in \( MV \), its sufficient to solve for first of the two; the solution for the second can be obtained from the first due to the symmetry \( \Psi \leftrightarrow \tilde{\Psi} \). Hence we start with the solns of (2.7). The solutions of the same \([29]\), in terms of constants \( A_0, A_1 \) and \( A_2 \) are given by

\[
\begin{align*}
\tilde{\Psi}(t,x) &= A_0 e^{i(\omega t - k \cdot x)}, \\
\Psi(t,x) &= A_1 \cos(\theta) e^{i(\omega_+ t - k \cdot x)} - A_2 \sin(\theta) e^{i(\omega_- t - k \cdot x)}, \\
\Phi(t,x) &= A_1 \sin(\theta) e^{i(\omega_+ t - k \cdot x)} + A_2 \cos(\theta) e^{i(\omega_- t - k \cdot x)}. \tag{3.1}
\end{align*}
\]

Where \( \omega, \omega_+, \omega_- \) appearing in eqns. (3.1) follows from the dispersion relations and are given by,

\[
\begin{align*}
\omega &= K, \tag{3.2} \\
\omega_+ &= \pm \sqrt{K^2 + \frac{m_{\phi_1}^2}{2} + \left(\frac{m_{\phi_1}^4}{4} + g_{\phi_1\gamma\gamma} B_T^2 \omega^2\right)^{\frac{1}{2}}}, \tag{3.3} \\
\omega_- &= \pm \sqrt{K^2 + \frac{m_{\phi_1}^2}{2} - \left(\frac{m_{\phi_1}^4}{4} + g_{\phi_1\gamma\gamma} B_T^2 \omega^2\right)^{\frac{1}{2}}}. \tag{3.4}
\end{align*}
\]
The absence of (pseudo)scalar at the initial stage, (according to the physics of curvature radiation), favours the following boundary conditions, \( \Phi(0, 0) = 0 \) and \( \Psi(0, 0) = 1 \) and \( \tilde{\Psi}(0, 0) = 1 \). These boundary conditions yield \( A_0 = 1 \), \( A_1 = \cos(\theta) \) and \( A_2 = -\sin(\theta) \). The angle \( \theta \) in equation (3.1) is given by \( \theta = \frac{1}{2} \tan^{-1} \left( \frac{2g_{\phi\gamma}B_\gamma^\omega}{m^2} \right) \). With these conditions the solutions for \( \Psi \) finally turns out to be,

\[
\Psi(t, x) = \cos^2(\theta) e^{i(\omega_+ t - k \cdot x)} + \sin^2(\theta) e^{i(\omega_- t - k \cdot x)},
\]

\( 3.5 \)

\[
\Psi(t, x) = a_2(t) e^{i(\tan^{-1} \frac{\cos^2 \theta \sin \omega + t + \sin^2 \theta \sin \omega - t}{\cos^2 \theta \cos \omega + t + \sin^2 \theta \cos \omega - t})},
\]

\( 3.6 \)

when, \( a_2(t) = 1 + 2 \sin^2 \theta \cos^2 \theta (\cos(\omega_+ - \omega_-) t - 1) \).

4 Observables

4.0.1 Polarimetric observables

The Stokes variables, as obtained from the coherency matrix can be expressed in terms of the solutions, as:

\[
I = \langle \tilde{\Psi}^*(z) \tilde{\Psi}(z) \rangle + \langle \Psi^*(z) \Psi(z) \rangle, \\
Q = \langle \tilde{\Psi}^*(z) \tilde{\Psi}(z) \rangle - \langle \Psi^*(z) \Psi(z) \rangle, \\
U = 2Re \langle \tilde{\Psi}^*(z) \tilde{\Psi}(z) \rangle, \\
V = 2Im \langle \tilde{\Psi}^*(z) \tilde{\Psi}(z) \rangle.
\]

\( 4.1 \)

It may be noted that \( V \) appearing in the equation (4.1) is measure of circular polarisation. Other polarimetric observables like ellipticity angle, polarization angle, degree of linear polarization, follows from the expressions of \( I \), \( U \), \( Q \) and \( V \). Polarization angle (represented by \( \psi_{\phi_i} \)), is defined in terms of \( U \) and \( Q \), as:

\[
tan(2\psi_{\phi_i}) = \frac{U(\omega, z)}{Q(\omega, z)}. 
\]

\( 4.2 \)

The ellipticity angle (denoted by \( \chi_{\phi_i} \)), is defined as:

\[
tan(2\chi_{\phi_i}) = \frac{V(\omega, z)}{\sqrt{Q^2(\omega, z) + U^2(\omega, z)}}. 
\]

\( 4.3 \)

The polarization fraction \( \Pi_{\phi_i}^P \) of the radiation in terms of parameters \( Q \) and \( I \) is given by,

\[
\Pi_{\phi_i}^P = \frac{Q(\omega, z)}{I(\omega, z)}.
\]

\( 4.4 \)

And lastly, the degree of linear polarization (represented as \( P_L \)) is given by,

\[
P_{L\phi_i} = \frac{\sqrt{Q^2(\omega, z) + U^2(\omega, z)}}{I(\omega, z)}.
\]

\( 4.5 \)
4.0.2 Polarometric observables for magnetized vacuum

So using the solutions given in equations (3.1), in equation (4.1) one can obtain the expressions for the Stokes parameters; for scalar photon system. And they are:

\[ I(\omega; z) = \sin^4(\theta) + \cos^4(\theta) + 0.5\sin^2(2\theta)\cos(\omega_+ + \omega_-)z + 1, \]  
\[ Q(\omega; z) = -[\sin^4(\theta) + \cos^4(\theta) + 0.5\sin^2(2\theta)\cos(\omega_+ - \omega_-)z - 1], \]  
\[ U(\omega; z) = 2\sin^2(\theta)\cos(\omega_- - \omega)z + 2\cos^2(\theta)\cos(\omega_+ - \omega)z, \]  
\[ V(\omega; z) = 2\sin^2(\theta)\sin(\omega_- - \omega)z + 2\cos^2(\theta)\sin(\omega_+ - \omega)z. \]  

The same for pseudo-scalar photon system can be obtained by interchanging the solutions of \( \Psi \) with \( \tilde{\Psi} \) and vice-versa in the equations (3.1). As a result, the Stokes parameters for \( \phi\gamma \) (expressed with suffix s) can be related to that of \( \phi'\gamma \) system in the following form:

\[ I_s \to I_{ps}, \quad Q_s \to -Q_{ps} \]  
\[ U_s \to U_{ps}, \quad V_s \to -V_{ps} \]  

As a result, Stokes \( Q, V \), polarisation angle \( \psi_{\phi_s} \) and ellipticity parameter \( \chi_{\phi_s} \) picks up a sign as one moves from \( \phi\gamma \) to \( \phi'\gamma \) system, provided other parameters remain the same. As a result identification of one from the other becomes difficult, when \( \chi_{\phi_s} \) and \( \psi_{\phi_s} \) both tend to zero. Making degree of linear polarisation \( P_L \) work to distinguish \( \phi \) from \( \phi' \) is even more difficult since \( P_L \) remain same.

5 Matter effects

5.1 Matter and magnetized matter effect

Effect of medium on such a system is usually incorporated through the inclusion of \( \Pi_{\mu\nu}(k, \mu, T) \), the photon self energy tensor evaluated in a medium. The magnetic field independent part of the polarization tensor, \( \Pi_{\mu\nu}(k, \mu, T) \) can further be expressed in terms of the scalar form factors times the tensors constructed out of the vectors available in hand for the system under consideration. They are as follows,

\[ \Pi_{\mu\nu}(k, \mu, T) = \Pi_T R_{\mu\nu} + \Pi_L Q_{\mu\nu}, \quad \text{where}, \]
\[ \begin{cases} 
Q_{\mu\nu} = \tilde{u}_\mu \tilde{u}_\nu \\
R_{\mu\nu} = \tilde{g}_{\mu\nu} - Q_{\mu\nu} \\
\tilde{g}_{\mu\nu} = \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right), \\
\tilde{u}_\mu = g_{\mu\nu} u^\nu 
\end{cases} \]  

where, \( \Pi_T \) and \( \Pi_L \) are the transverse and the longitudinal form factors for photon polarization tensor evaluated in an unmagnetised medium. In the long wave length limit, \( \Pi_T \) can be approximated to be equal to the square of the plasma frequency \( \omega_p^2 \). Explicit expression of the same is provided later. The vectors \( u^\mu \) and \( k^\mu \) are the centre of mass velocity of the system and four momentum of the photon. The same for magnetized medium to linear order in magnetic field strength is taken into account by inclusion of the parity violating part.
of photon-self energy tensor \([58]\). Following the same notations of \((2.4)\), the corresponding effective Lagrangian for \(\phi\gamma\) system, are given by
\[
L_{\text{eff},\phi} = L_{\text{eff},\phi_i}^{\text{free}} + L_{\text{eff},\phi_i}^{\text{int}}
\]
when,
\[
L_{\text{eff},\phi_i}^{\text{free}} = \frac{1}{2} \phi_i(-k) \left( k^2 - m_{\phi_i}^2 \right) \phi_i(k) - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \frac{1}{2} A_\mu(-k) \Pi^{\mu\nu}(k, \mu, T) A_\nu(k) + \frac{1}{2} A_\mu(-k) \Pi^{\mu\nu}(k, \mu, T, eB) A_\nu(k)
\]
\[(5.2)\]
\[
L_{\text{eff},\phi_i}^{\text{int}} = -\frac{1}{4} g_{\phi_i\gamma\gamma} \phi_i f_{\mu\nu} f^{\mu\nu},
\]
\[(5.3)\]

The one loop polarization tensor \(\Pi^{\mu\nu}(k, \mu, T, eB)\) in equation \((5.2)\) in general contains the magnetized-medium induced corrections to photon self energy to all orders in \(eB\) \([57–59]\). It can further be decomposed into three pieces, (a) medium induced correction due to temperature \(T\) and chemical potential \(\mu\) only (b) medium plus magnetic field induced correction to all even order terms in \(eB\) and (c) medium and magnetic field induced correction to all odd order terms in \(eB\). In the limit \(\frac{eB}{m^2} < 1\), i.e., weak field and \(\mu \neq 0\), the leading contribution to photon self energy comes from the \(O(eB)\) piece in \(\Pi^{\mu\nu}(k, \mu, T, eB)\). This piece happens to be \(\text{PT}\) symmetric hence \(\text{CP}\) violating or \(\text{P}\) violating. Other pieces being even to all orders in \(eB\), are blind to any \(\text{CP}\) transformations. In most of the astrophysical situations, this is the limit that is realised, hence, in our effective Lagrangian, we have retained terms up-to \(O(eB)\) in \(\Pi^{\mu\nu}(k, \mu, T, eB)\). Henceforth the \(eB\) independent piece is to be denoted by \(\Pi_{\alpha\nu}(k)\) and the \(O(eB)\) parity violating piece by \(\Pi_{\alpha\nu}^{\text{int}}(k)\). The parity violating part can similarly be written as, \(\Pi_{\alpha\nu}^{\text{int}}(k) = \Pi_p(k) P_{\mu\nu}\); where the tensor \(P_{\mu\nu} = i \epsilon_{\mu\nu\beta\gamma} \frac{k^\beta}{|k|} u^{\gamma}\) is the projection operator and \(\Pi_p(k)\) is the associated form factor. Unsubscripted Greek indices in these expressions can have values lying between 0 to 3, however the same with subscript \(\parallel\), (e.g., \(\mu\parallel\)) means that the same can assume only two values, either 0 or 3. The form factor \(\Pi_p(k)\) to order \(eB\), turns out to be
\[
\Pi_p(k) = \frac{\omega B \omega_p^2}{\omega^2 - \omega_p^2} \sim \frac{\omega B \omega_p^2}{\omega},
\]
\[(5.4)\]
and is also called the Faraday term in the literature. We may do the same at places in this work.

6 Systems with dim-5 interactions
6.1 Scalar-photon system in magnetized medium

The equations of motion for the coupled \(\phi\gamma\) interacting system, including Faraday term, can be cast in a compact matrix form, as:
\[
\left[ k^2 \mathbf{I} - \mathbf{M} \right] \begin{pmatrix} A_\parallel(k) \\ A_\perp(k) \\ A_L(k) \\ \phi(k) \end{pmatrix} = 0,
\]
\[(6.1)\]
where \( \mathbf{I} \) is a \( 4 \times 4 \) identity matrix, and \( \mathbf{M} \) is a \( 4 \times 4 \) mixing matrix given by,

\[
\mathbf{M} = \begin{bmatrix}
\Pi_T & -N_1 N_2 \Pi^p(k) P^{(1)\mu} I^\nu & 0 & -i g_{\phi \gamma \gamma} N_2 b_0^{(2)} I^\mu \\
N_1 N_2 \Pi^p(k) P^{(1)\mu} I^\nu & \Pi_T & 0 & 0 \\
0 & 0 & \Pi_L & 0 \\
i g_{\phi \gamma \gamma} N_2 b_0^{(2)} I^\mu & 0 & 0 & m_\phi^2
\end{bmatrix}
\] (6.2)

It is necessary to note that, \( A_L \) are not getting mixed up with other degrees of freedom in magnetized medium. Therefore we would not be considering the longitudinal degree of freedom of photon in further calculations.

Solutions for the field equations can be obtained by diagonalising the matrix provided in eqn. (6.2). Performing the same and considering the initial condition that at the origin there are no scalars (i.e., \( \phi(\omega,0) = 0 \)); the final solutions for the electromagnetic form factors can be written (in terms of the orthonormal eigenvectors of the mixing matrix) as:

\[
\mathbf{A}_\parallel(\omega, z) = (e^{-i\Omega_\parallel z} \bar{u}_1 \bar{u}_1^* + e^{-i\Omega_\parallel z} \bar{u}_2 \bar{u}_2^* + e^{-i\Omega_\parallel z} \bar{u}_3 \bar{u}_3^*) \mathbf{A}_\parallel(\omega, 0) \\
+ (e^{-i\Omega_\parallel z} \bar{v}_1 \bar{v}_1^* + e^{-i\Omega_\parallel z} \bar{v}_2 \bar{v}_2^* + e^{-i\Omega_\parallel z} \bar{v}_3 \bar{v}_3^*) \mathbf{A}_\parallel(\omega, 0).
\] (6.3)

Similarly the perpendicular \( A_\perp(\omega, z) \) component turns out to be,

\[
\mathbf{A}_\perp(\omega, z) = (e^{-i\Omega_\parallel z} \bar{v}_1 \bar{v}_1^* + e^{-i\Omega_\parallel z} \bar{v}_2 \bar{v}_2^* + e^{-i\Omega_\parallel z} \bar{v}_3 \bar{v}_3^*) \mathbf{A}_\parallel(\omega, 0) \\
+ (e^{-i\Omega_\parallel z} \bar{v}_1 \bar{v}_1^* + e^{-i\Omega_\parallel z} \bar{v}_2 \bar{v}_2^* + e^{-i\Omega_\parallel z} \bar{v}_3 \bar{v}_3^*) \mathbf{A}_\parallel(\omega, 0).
\] (6.4)

The variables \( \Omega_\parallel \), \( \Omega_\perp \) and \( \Omega_\phi \) introduced in eqns. (6.4) and (6.3), are functions of the roots of the \( 3 \times 3 \) matrix \( \mathbf{M} \). They are given by:

\[
\Omega_\parallel = \left( \omega - \frac{\lambda_1}{2\omega} \right), \quad \Omega_\perp = \left( \omega - \frac{\lambda_2}{2\omega} \right) \quad \text{and} \quad \Omega_\phi = \left( \omega - \frac{\lambda_3}{2\omega} \right).
\] (6.5)

The energy of the photon in equation (6.5) is given by \( \omega \). In the final solutions, i.e., in eqns. (6.3) and (6.4), the initial conditions for the two form factors of the photons at the origin are denoted by \( \mathbf{A}_\perp(\omega, 0) \) and \( \mathbf{A}_\parallel(\omega, 0) \). Their magnitude can be estimated according to the process under consideration. The expressions for the Stokes parameter turns out to be:

\[
\mathbf{I}(\omega, z) = I_\parallel |\mathbf{A}_\parallel(\omega, 0)|^2 + I_\perp |\mathbf{A}_\perp(\omega, 0)|^2 - 2I_\parallel |\mathbf{A}_\parallel(\omega, 0)\mathbf{A}_\perp(\omega, 0)|,
\] (6.6)

\[
\mathbf{Q}(\omega, z) = Q_\parallel |\mathbf{A}_\parallel(\omega, 0)|^2 - Q_\perp |\mathbf{A}_\perp(\omega, 0)|^2 + 2Q_\parallel |\mathbf{A}_\parallel(\omega, 0)\mathbf{A}_\perp(\omega, 0)|,
\] (6.7)

\[
\mathbf{U}(\omega, z) = 2U_\parallel |\mathbf{A}_\parallel(\omega, 0)|^2 + 2U_\perp |\mathbf{A}_\perp(\omega, 0)|^2 + 2U_\parallel |\mathbf{A}_\parallel(\omega, 0)\mathbf{A}_\perp(\omega, 0)|,
\] (6.8)

\[
\mathbf{V}(\omega, z) = 2V_\parallel |\mathbf{A}_\parallel(\omega, 0)|^2 + 2V_\perp |\mathbf{A}_\perp(\omega, 0)|^2 + 2V_\parallel |\mathbf{A}_\parallel(\omega, 0)\mathbf{A}_\perp(\omega, 0)|.
\] (6.9)
6.2 Pseudoscalar-photon system in magnetized medium

In this section we study $\phi'\gamma$ mixing in a magnetized medium when the parity violating part of photon self-energy or polarization tensor $\Pi_{\mu\nu}^p(\mu, T, e B)$, is included in $\phi'\gamma$ effective Lagrangian, $L_{\text{eff},\phi'}$.

In this work, we first point out the distinct analytic features of the mixing matrix that makes the $\gamma\phi'$ mixing dynamics different from the $\gamma\phi$ case. Following that, we obtain the estimates of polarimetric observables arising out this mixing— for compact astrophysical objects.

Next we evaluate the same observables, for $\phi\gamma$ system, for same physical situation, in the same physical parameter range and point out the difference in their magnitude from that for the $\phi'\gamma$ system, arising due to the difference in their respective mixing dynamics.

The equations of motion for $\phi'\gamma$ system obtained from Lagrangian $-L_{\text{eff},\phi'}$, as before, can be expressed in compact matrix notations as:

$$[k^2\mathbf{I} - \mathbf{M}'] \begin{pmatrix} A_{\parallel}(k) \\ A_{\perp}(k) \\ A_L(k) \\ \phi'(k) \end{pmatrix} = 0,$$

where $\mathbf{I}$ is an identity matrix and matrix $\mathbf{M}'$ is the $4 \times 4$ mixing matrix. The same, in terms of its elements is given by,

$$\mathbf{M}' = \begin{pmatrix} \Pi_T & -\Pi_p N_1 N_2 P_{\mu\nu} b_1^{(1)\mu} I^\nu & 0 & 0 \\ \Pi_p N_1 N_2 P_{\mu\nu} b_1^{(1)\mu} I^\nu & \Pi_T & 0 & -i g_{\phi'\gamma\gamma} N_2 b_1^{(2)\mu} I^\mu \\ 0 & 0 & \Pi_L & -i g_{\phi'\gamma\gamma} N_L b_1^{(2)\mu} \bar{u}^\mu \\ 0 & i g_{\phi'\gamma\gamma} N_2 b_1^{(2)\mu} I^\mu & i g_{\phi'\gamma\gamma} N_L b_1^{(2)\mu} \bar{u}^\mu & m_\phi^2 \end{pmatrix}. \tag{6.11}$$

Note that, projection operator $P_{\mu\nu}$, appearing in the $M'_{12}$ and $M'_21$ elements of the mixing matrix $\mathbf{M}'$ is a complex one, that makes the matrix, $\mathbf{M}'$, a hermitian matrix, that is expected even otherwise on general grounds.

In order to obtain the solutions in terms of form factors, we apply the boundary conditions that at origin there is no pseudoscalar field i.e., $\phi'(0) = 0$ Necessary steps to arrive at the solution has been provided in [70], i.e,

$$A_{\parallel}(\omega, z) = \left(e^{i \Omega_{L} z \tilde{u}_1 \tilde{u}_1^*} + e^{i \Omega_{L} z \tilde{u}_2 \tilde{u}_2^*} + e^{i \Omega_{L} z \tilde{u}_3 \tilde{u}_3^*} + e^{i \Omega_{\phi'} z \tilde{u}_4 \tilde{u}_4^*}\right) A_{\parallel}(\omega, 0) + \left(e^{i \Omega_{L} z \tilde{u}_1 \tilde{v}_1^*} + e^{i \Omega_{L} z \tilde{u}_2 \tilde{v}_2^*} + e^{i \Omega_{L} z \tilde{u}_3 \tilde{v}_3^*} + e^{i \Omega_{\phi'} z \tilde{u}_4 \tilde{v}_4^*}\right) A_{\perp}(\omega, 0) + \left(e^{i \Omega_{L} z \tilde{u}_1 \tilde{w}_1^*} + e^{i \Omega_{L} z \tilde{u}_2 \tilde{w}_2^*} + e^{i \Omega_{L} z \tilde{u}_3 \tilde{w}_3^*} + e^{i \Omega_{\phi'} z \tilde{u}_4 \tilde{w}_4^*}\right) A_L(\omega, 0). \tag{6.12}$$

The component $A_{\perp}(\omega, z)$ is given by,

$$A_{\perp}(\omega, z) = \left(e^{i \Omega_{L} z \tilde{v}_1 \tilde{v}_1^*} + e^{i \Omega_{L} z \tilde{v}_2 \tilde{v}_2^*} + e^{i \Omega_{L} z \tilde{v}_3 \tilde{v}_3^*} + e^{i \Omega_{\phi'} z \tilde{v}_4 \tilde{v}_4^*}\right) A_{\parallel}(\omega, 0) + \left(e^{i \Omega_{L} z \tilde{v}_1 \tilde{w}_1^*} + e^{i \Omega_{L} z \tilde{v}_2 \tilde{w}_2^*} + e^{i \Omega_{L} z \tilde{v}_3 \tilde{w}_3^*} + e^{i \Omega_{\phi'} z \tilde{v}_4 \tilde{w}_4^*}\right) A_{\perp}(\omega, 0) + \left(e^{i \Omega_{L} z \tilde{v}_1 \tilde{w}_1^*} + e^{i \Omega_{L} z \tilde{v}_2 \tilde{w}_2^*} + e^{i \Omega_{L} z \tilde{v}_3 \tilde{w}_3^*} + e^{i \Omega_{\phi'} z \tilde{v}_4 \tilde{w}_4^*}\right) A_L(\omega, 0). \tag{6.13}$$

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The complete expressions for the coefficients of $A_{\parallel}$ explicitly defined in $\left(\omega - \frac{\lambda g}{2K}\right)$, $\Omega_{\perp} = \left(\omega - \frac{\lambda \phi}{2K}\right)$, $\Omega_{L} = \left(\omega - \frac{\lambda}{2K}\right)$, and $\Omega_{\phi'} = \left(\omega - \frac{\lambda \phi'}{2K}\right)$. The difference between $\phi'\gamma$ and this system, is the appearance of contribution due to $A_{L}$ or longitudinal component of photon. Using the solutions obtained in (6.12) and (6.13) for $\phi'\gamma$ interaction, we provide the polarimetric variables for pseudoscalar-photon interaction taking magnetized matter effect into account, as follows: 

$$ I(\omega, z) = |A_{\parallel}(\omega, 0)|^2 + I_{\perp}|A_{\perp}(\omega, 0)|^2 - I_{L}|A_{L}(\omega, 0)|^2 $$

$$ I_{\perp}|A_{\parallel}(\omega, 0)|A_{\perp}(\omega, 0) + I_{L}|A_{\parallel}(\omega, 0)|A_{L}(\omega, 0) $$

$$ I_{L}|A_{\perp}(\omega, 0)|A_{L}(\omega, 0)|, \quad (6.14) $$

$$ Q(\omega, z) = Q_{\parallel}|A_{\parallel}(\omega, 0)|^2 - Q_{\perp}|A_{\perp}(\omega, 0)|^2 - Q_{L}|A_{L}(\omega, 0)|^2 $$

$$ Q_{\perp}|A_{\parallel}(\omega, 0)|A_{\perp}(\omega, 0) + Q_{L}|A_{\parallel}(\omega, 0)|A_{L}(\omega, 0) $$

$$ Q_{L}|A_{\perp}(\omega, 0)|A_{L}(\omega, 0)|, \quad (6.15) $$

$$ U(\omega, z) = U_{\parallel}|A_{\parallel}(\omega, 0)|^2 + U_{\perp}|A_{\perp}(\omega, 0)|^2 + U_{L}|A_{L}(\omega, 0)|^2 $$

$$ U_{\perp}|A_{\parallel}(\omega, 0)|A_{\perp}(\omega, 0) + U_{L}|A_{\parallel}(\omega, 0)|A_{L}(\omega, 0) $$

$$ U_{L}|A_{\perp}(\omega, 0)|A_{L}(\omega, 0)|, \quad (6.16) $$

$$ V(\omega, z) = V_{\parallel}|A_{\parallel}(\omega, 0)|^2 + V_{\perp}|A_{\perp}(\omega, 0)|^2 + V_{L}|A_{L}(\omega, 0)|^2 $$

$$ V_{\perp}|A_{\parallel}(\omega, 0)|A_{\perp}(\omega, 0) + V_{L}|A_{\parallel}(\omega, 0)|A_{L}(\omega, 0) $$

$$ V_{L}|A_{\perp}(\omega, 0)|A_{L}(\omega, 0)|. \quad (6.17) $$

The complete expressions for the coefficients of $A_{\parallel}, A_{\perp}, A_{L}, A_{\parallel}A_{\perp}, A_{\parallel}A_{L}$ and $A_{\perp}A_{L}$ are explicitly defined in [70].

7 Mixing

7.1 Analysis of mixing.

In this section, we deliberate on the origin of the couplings between various degrees of freedom – on the mixing dynamics – from the point of view of symmetry. We begin our discussion first on the consequences of including just the matter effects, followed by, the same upon including the matter plus the magnetized matter effects – on the mixing dynamics.

We begin with the demonstration of the claim made in sec. (1.1), that, with the incorporation of matter effects by including $A_{\mu}(k)\Pi_{\mu\nu}(k, \mu, T)A_{\nu}(k)$ in the effective Lagrangian $L_{\vee}k, \phi_{i}$ – the mixing matrix of scalar photon system turns out to be $2 \times 2$ but the same for pseudoscalar photon system turns out to be $3 \times 3$.

This feature can easily be verified by setting the parity violating part $\Pi_{\mu} = 0$ in eqns. (6.2) and (6.11) for $\phi'\gamma$ and $\phi'\gamma$ system. This is due to the fact that eqns. (6.2) has both the effects incorporated in it, so by setting $\Pi_{\mu} = 0$, one retains only the matter effects.

Next we address the question of the coupling between different degrees of freedom and
their discrete symmetry assignments. The emerging 2 × 2 mixing matrix structure for \( \phi \gamma \) system is due to mixing between \( A_\parallel \) and \( \phi \). This follows from the expression of the Lagrangian \( \mathcal{L}_{\text{int,}\phi}^B \) given by eqn. (2.15), denoting \( \phi \gamma \) interaction in an external magnetic field. It follows from there that the only form factor that has nonzero interaction with the scalar \( \phi \), is \( A_\parallel \). No other form factor has any interaction with \( \phi \) at the level of \( \mathcal{L}_{\text{int,}\phi}^B \). Hence the mixing matrix is 2 × 2.

The other interesting issue in this mixing is: although naively one would have expected the longitudinal degree of freedom of the photon and the scalar \( \phi \) to mix with each other for having same spin assignments \( s_z = 0 \), but that remains forbidden because at the level of the Lagrangian \( \mathcal{L}_{\text{eff,}\phi}^B \), they have no interaction. So the mixing matrix for \( \phi \gamma \) system turns out to be 2 × 2.

Lastly, we examine another subtlety in this mixing dynamics, following from the discrete symmetry arguments. We begin with the observation that, the discrete symmetries of the associated degrees of freedom of electromagnetic form factors, appearing in the description of the gauge potential, given by (2.10) – including \( A_\parallel \) – are odd under Time (T) reversal and even under Parity (P) inversion symmetry transformations (for details consult table [1]. And \( \phi \), being a scalar remains even under PT transformation. So at the level of field equations coupling of PT odd \( A_\parallel \) with PT even scalar \( \phi \), is a bit bizarre.

However, a careful inspection of the fields-equations reveal that, these two degrees of freedom with uneven PT symmetry, appears in the field -equations after being multiplied by specific PT dependent factors. It so happens that upon PT transformation the products in field equations have retained same PT symmetry. Stated differently, if we consider the IL of equation (2.15) i.e.,

\[
\mathcal{L}_{\text{int,}\phi}^B = ig_{\phi\gamma} \phi A_\parallel N_1 b_1^{(1)\nu} b_\nu^{(1)}
\] (7.1)

the product, \( b_1^{(1)\nu} b_\nu^{(1)} \) is even under Time reversal symmetry transformation (T) and so are the transformation properties of \( N_1 \) and \( \phi \), but \( A_\parallel \) is odd under T. Therefore, the eqn. (7.1) wouldn’t have been T symmetric unless the multiplicative factor \( i \) was absent. Therefore, here the factor \( i \) is the T dependent multiplicative factor, that was discussed in preceding paragraph.

Next we consider the issue of the effect of magnetized medium on \( \phi \gamma \) mixing dynamics. The effect of the same is obtained through the incorporation of the parity violating part \( \Pi_p \), in the effective Lagrangian. Since the same makes the two transverse degrees of freedom of the photon \( A_\parallel \) with \( A_\perp \) couple directly with each other and \( \phi \) too couples directly with \( A_\parallel \), thus the appearance of \( \Pi_p \) makes \( A_\perp \) coupled indirectly to \( \phi \). The longitudinal degree of freedom of photon however remains decoupled. Thus the mixing matrix for \( \phi \gamma \) system turns out to be 3 × 3 on incorporation of the same.

Next we turn our attention to pseudoscalars (axion) photon system. The same under similar situation, has non-zero interaction with \( A_\perp \) and \( A_L \) at the level of the interaction Lagrangian, as can be checked from eqn. (2.13). Therefore, the inclusion of matter effects, makes direct mixing possible between \( \phi' \), \( A_L \) and \( A_\perp \), thus turning the mixing matrix a 3 × 3 one.

On inclusion of magnetized-matter effects along with unmagnetized matter-effects, the
mixing matrix for the $\phi'\gamma$ becomes $4 \times 4$, due to the following reason: because of matter effects alone, $\phi'$ couples directly to $A_L$ and $A_\perp$, at the level of $L^{\mu\nu}_{\text{int},\phi'}$. As we already have pointed out, upon inclusion of the parity violating $\Pi_\mu$ part, $A_\perp$ further couples directly with $A_\parallel$, hence there is an indirect mixing between $i\phi'$ and photon EM form-factor for transverse polarization ($A_\parallel$); thus, making all the degrees of freedom for pseudoscalar photon system interact with each other. The essence of the same is captured through the elements of the $4 \times 4$ mixing matrix.

Coming to question of the discrete symmetry assignments of the degrees of freedom and their coupling in the field equations for the pseudoscalar photon system, we note the existence coupling between degrees of freedom with uneven PT symmetry assignments. This can be understood as follows. Due to odd PT transformation assignment, $A_L$ (see table [1] in [70]) can couple only to the product $i \times \phi'$ in any equation; because, as noted before, though $\phi'$ is even under PT but $i$ (i.e., $\sqrt{-1}$) being odd under T, and even under P makes $i \times \phi'$ odd under PT; thus favouring the coupling between the two at the level of field equations. This can be verified from the last equation of the equations presented at (S3) in the supplementary section [70]. Similar argument follows for other equations too. We conclude this discussion with the observation that, the difference in the mixing pattern between $\phi\gamma$ and $\phi'\gamma$ system in an environment having magnetized-matter can have far reaching consequences on the polarimetric signals produced by each. Hence, the same can serve as an ideal astrophysical laboratory, to distinguish one type of interaction from the other. A detailed discussion on the possible nonzero elements of the mixing matrix, is however, left for a future communication [71].

7.2 Searches so far:

Based on the physics principles mentioned above, systematic studies to find ALP, through Laboratory or astrophysics based experiments has been going on, for some time now. The ongoing and proposed laboratory based searches for ALP are [47, 49–56]. Among those, the CAST collaboration of CERN [47] is a prominent one. This collaboration is engaged in detecting ALP, produced at the interior of the Sun via primakoff process [45, 46], based on energy loss arguments from stellar interior. Earlier they had reported the following bounds on ALP parameters, $m_{\phi'} \leq 0.02$ eV and $g_{\phi'\gamma\gamma} < 1.16 \times 10^{-10}$ GeV$^{-1}$ [47]; and their last improvement to this result reads, $g_{\phi'\gamma\gamma} < 0.66 \times 10^{-10}$ GeV$^{-1}$ in the same mass range [48].

In the astrophysical front, bound on ALP parameters also obtained by study the cooling rate of stellar objects, e.g., super-giants, red giants, helium core burning stars, white dwarfs, and neutron stars [38–41] etc., due to the same. An up to date introduction to the relevant issues can be found in [44]. The interesting part is that the astrophysical bound on axion photon coupling, based on measuring the ratio of the number of horizontal branch stars to that of red giant branch stars in the globular cluster, also provide $g_{\phi'\gamma\gamma} < 0.66 \times 10^{-10}$ GeV$^{-1}$ at 95% C.L. [42] as that of [48]. Most of these bounds were obtained by taking just the matter effects into account in the axion-photon effective Lagrangian. That is mixing occurs between only three degrees of freedom ($\phi'$, $A_\perp$ and $A_L$), leaving $A_\parallel$ free.

However according to current observations, the data gives a hint of additional cooling at
various stages of their evolution, termed as cooling anomaly. This has renewed the interest in a model dependent axion induced cooling [43]. However a possible of way to achieve this is through **mixing of all the degrees of freedom of photon with ALPs** that will open **additional channels of energy transport** via conversion of photons into axions.\(^3\) Same can be achieved by incorporation of the parity violating part of photon self energy tensor \(\Pi_{\mu\nu}^P\) that emerges from magnetized-matter effects, in the pseudoscalar photon effective-Lagrangian \(L_{eff}\). This happens to changes the mixing dynamics, hence, the **mixing matrix** for \(\phi'\gamma\) system. The mixing matrix for \(\phi'\gamma\) system, turns out to be \(4\times4\), implying, complete mixing between all the three degrees of freedom of (in-medium) photon with the single degree of freedom of \(\phi'\).

For \(\phi\gamma\) system, the same procedure changes the mixing matrix **from** \(2\times2\) **to a** \(3\times3\) **one** [61]. That is, the mixing is only between the two transverse polarization states of the photon and \(\phi\). The longitudinal degree of freedom of the photon gets decoupled and propagates freely. The spectro-polarimetric signatures of these features, are the ones investigated in the subsequent sections of this work.

### 8 Astrophysical application.

In this section we would focus on a possible astrophysical scenario, that is likely to provide the distinct signatures that would distinguish scalar photon mixing from pseudoscalar photon mixing using magnetized media effects. The choice of the astrophysical system considered here for getting a reasonably strong signal is: an eclipsing compact binary system. This choice is due to the strong surface magnetic field (\(10^9 \text{ – } 10^{14}\) Gauss) associated with them. We expect that the \(\phi_i\gamma\) interaction in such strong field would generate strong enough signal to be detected by the upcoming space-borne X-ray observatories.

Since polarization angle \((\psi_{\phi_i})\) and ellipticity angle \((\chi_{\phi_i})\) have emerged as a reliable observable for polarimetric signatures, therefore we would consider estimating their contribution for signals, for the kind of situation discussed here.

The binary system considered here, is assumed to be composed of a neutron star (primary) and its “companion”, that may be a white dwarf or a neutron star. The primary is imagined to be an axisymmetric rotor and the companion to be an orthogonal rotor, having their individual dipole field direction nearly orthogonal to each other. At the onset of the eclipsing phase, electromagnetic radiation beam from the primary grazes past the polar cap of the companion – to the observer. This happens during a finite fraction of their orbital cycle [fig. 2]. For rest of the cycle, the primary would beam its radiation directly to the observer. In this scenario, any difference in the observed values of \((\Psi_{\phi_i})\) and \((\chi)\) between the **direct beaming phase** to the **grazing phase** can be attributed to propagation in the magnetized environment of the companion. If scalars or pseudo-scalars do exist in nature, their interaction in the magnetized environment of the companion would also contribute to the difference in the measured values of \(\psi_{\phi_i}\) and \(\chi_{\phi_i}\); the study of which happens to be the main goal of this investigation, as mentioned before.

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\(^3\)Incorporation of \(\Pi\) does not do that
Since the EM beam, according to our physical picture, grazes past the polar cap of the companion, hence the magnetized path length that photons travel would be $\sim$ twice the polar cap radius ($R_{PC}$) of the companion. Assuming the magnetic field to be dipolar, the polar cap opening angle for the last open field lines at the surface of the compact star can be estimated from dipole field equations; the same in terms of light-cylinder radius $R_{LC}$ turns out to be,

$$\theta_{pc} = \sin^{-1} \left[ \left( \frac{R}{R_{LC}} \right)^{\frac{1}{2}} \right] \sim \left( \frac{R}{R_{LC}} \right)^{\frac{1}{2}}, \text{ for } \left( \frac{R}{R_{LC}} \right) << 1. \quad (8.1)$$

Therefore, the polar-cap radius of the companion, in terms of its radius $R$ and period $P$ turns out to be,

$$R_{pc} = R \left( \frac{2\pi R}{P} \right)^{\frac{1}{2}}, \quad (8.2)$$

while writing equation (8.2) we have used the relation $R_{pc} = \frac{P}{2\pi}$. Assuming period $P \sim 100$ sec, radius $R = 10$ km, for the companion, the polar cap radius turns out to be $R_{pc} = 6.5$ meter. We further assume the primary and the companion to be separated enough, not to pass through each others light cylinder during their orbital motion. The companion considered here is cold enough to have no charge particles in its magnetosphere. Hence, the availability of plasma would be around the polar-cap where the plasma frequency will be given by $\omega_p = \frac{n_{GJ}}{m_e}$. Goldreich Julien number density [37] given by $n_{GJ} = \frac{\Omega B}{2\pi}$ for an orthogonal rotor can in principle be very small, depending on the angle between $\Omega$ and $B$; So we consider the plasma frequency to be of the order of $10^{-9}$ eV and surface magnetic field, $B_S = 1.25 \times 10^{12}$G. The choice of these fiducial parameters ensure that, the particle density in the stellar magnetosphere will be negligible and the narrow EM beam will just pass through the plasma close to the surface of the companion. This scenario implies that, the interaction with the magnetized media is confined only in the polar cap region for the

![Figure 2. Schematic diagram of the binary configuration, as considered in the text.](image)
propagating EM beam.

9 Results

In this section we would like to discuss the signatures of $\gamma\phi$ and $\gamma\phi'$ interactions on the spectro-polarimetric observables $I, Q, U, V, \chi_{\phi_i}, \psi_{\phi_i}$, $P_{L\phi_i}$ and polarization fraction $\Pi_{\phi_i}^P$ of the EM radiations coming from the astrophysical sources. We would also like to establish whether the contributions to the polarimetric variables are same or different for these two types of interactions. Further an attempt would be made to understand the origin of the difference in the size of their contributions based on the number of participating dof in mixing.

The current and proposed space based polarimetric experiments to detect polarimetric signals from astrophysical sources include, APEX [66], IXPE [67], e-ASTOGRAM [69], ASTRO-2020 [68] to name a few. They are expected to cover the soft X-ray band of the EM spectra. Furthermore since the energy distribution of the observed cosmic $\gamma$-ray spectra is seen to peak in the 1-10 KeV range [69], we have confined our analysis in a slightly broader energy domain of 1-100 KeV that covers the peak emission range.

Assuming the physical scenario mentioned already, we have estimated the Stokes parameters, polarization and ellipticity angles for each type of interaction (i.e., $g_{\phi'\gamma\gamma} F_{\mu\nu} F^{\mu\nu}$) keeping $\omega_p$, path length $z$, magnetic field $B$, $g_{\phi,\gamma\gamma}$ and $m_{\phi_i}$ fixed, using the same parameters. We further have estimated the difference in the polarization angle, $\Delta \psi = (\psi_{\phi_i}' - \psi_{\phi_i})$ and the ellipticity angle $\Delta \chi = (\chi_{\phi_i}' - \chi_{\phi_i})$ acquired by a light beam, due to pseudoscalar $g_{\phi'\gamma\gamma} F_{\mu\nu} F^{\mu\nu}$ and scalar $g_{\phi\gamma\gamma} F_{\mu\nu} F^{\mu\nu}$ interactions, as the EM beam passes through the magnetosphere of the companion star of a binary system.

The coupling constant $g_{\phi'\gamma\gamma}$ is considered to be $g_{\phi\gamma\gamma} = 1.0 \times 10^{-11} \text{ GeV}^{-1}$ for the scalar ($\phi$) and the pseudoscalar ($\phi'$) interactions and their mass $m_{\phi_i} = 1.0 \times 10^{-11} \text{ GeV}$. We have considered the path length of the photon to be twelve meter, keeping magnetic field $(B) \sim 1.0 \times 10^{12} \text{ Gauss}$ and plasma frequency $\omega_p = 1.0 \times 10^{-12} \text{ GeV}$. The energy band of investigation has been kept at soft X-ray region $(1-100 \text{ KeV})$. The evolution of Stokes parameters with energy of the photon $\omega$ for both the interactions has been plotted in fig.[3] and in fig.[4] respectively.

The individual plots of the ellipticity angle $(\chi_{\phi_i})$ for scalar photon and pseudoscalar photon interaction can be found in left panel of fig.[5] plotted as a function of $\omega$ and the same for polarization angle $(\psi_{\phi_i})$ have been plotted in right panel of the fig.[5]. The magnitudes of the two initial amplitudes of the orthogonal modes of the photon beam are considered to be identical, which is not too drastic an assumption if the system is isotropic and homogeneous. In a more realistic situation, same can be extracted from the direct beaming phase of the radiation.

The plots are obtained by estimating the respective quantities numerically maintaining certain identities (e.g. $I^2 = Q^2 + U^2 + V^2$) to accuracies of one part in $10^{-7}$ to $10^{-5}$, at least. The angles in the plots are in radians.

The important features in the plots for $\Delta \psi$ vs $\omega$ and $\Delta \chi$ vs $\omega$, are (a) their spectral
Figure 3. Stokes parameters I, Q, U and V (in ordinates) for $\phi\gamma$ interaction vs photon energy $\omega$ (in abscissas) plotted in panel A, B, C and D respectively. The parameter V in D is scaled by the factor of $10^6$. The parameters chosen for this estimations are: Mass of scalar and pseudo scalar $m_{\phi,\phi'} = 1.0 \times 10^{-11}$ GeV, coupling constant $g_{\phi\gamma\gamma} = 1.0 \times 10^{-11}$ GeV, magnetic field $\sim 1.0 \times 10^{12}$ Gauss and plasma frequency $= 1.0 \times 10^{-12}$ GeV, photon pathlength $= 12$ meter.

dependence (b) the variations in magnitudes of the difference of the ellipticity angle $\Delta \chi$ and polarization angle difference $\Delta \psi$ – that is the variation in angle-difference (AD) with $\omega$. It is note worthy that, although $\Delta \psi$ may get affected due to geometrical effects like rotation of the coordinate axis of the observer frame etc.; however, $\Delta \chi$ is free from that problem – because – $\chi_{\phi_i}$ remains invariant under rotation about the propagation direction. The variation of AD, particularly $\Delta \chi$ with $\omega$ may be used for making compact-star models and DM identification through multi-wavelength spectro-polarimetric studies, This is because of the distinct variational pattern in AD with energy as is evident from figure [6].

It may be noted in the passing that an identity involving ellipticity angle $\chi_{\phi_i}$, the polarization angle $\psi_{\phi_i}$ and polarization fraction $\Pi^P_{\phi_i}$, can be derived from their constitutive

\textsuperscript{4}The technology to detect circular polarization for high energy photons in the KeV-MeV region needs to be developed. The space observatories – currently existing and the planned ones are capable of detecting only linear polarization [68].
relations. The same turns out to be,

\[
\cos^2 2\chi_{\phi_i} \cos^2 2\psi_{\phi_i} = \Pi_{\phi_i}^P, \tag{9.1}
\]
\[
\cos 2\chi_{\phi_i} = P_{L\phi_i}. \tag{9.2}
\]

When the expressions of the \(\chi_{\phi_i}, \psi_{\phi_i}, \Pi_{\phi_i}^P\) and \(P_{L\phi_i}\) can be found in eqns. (4.2) to (4.5). One can in principle identify the interaction type by using the observational data in models of compact stars and verifying eqns. (9.1) and (9.2). Though this identity in principle should be satisfied for each \(\omega\), but the observed data seems to fall short of that expectation.

One of the reason behind this may be use of energy band spectra by the detectors, when this relation is supposed to be satisfied for line spectrum. This brings us to explore the use of numerical data in statical form and use the same for drawing conclusions.

However for correct identification of the nature of the interaction, the magnitude of the variations in \(\Delta \psi_P\) and \(\Delta \chi\) should be greater than the half power diameter (HPD) or the angular resolution of the detectors in a given energy band.

If the angular resolution of the detectors in the soft X-ray region (1 to 100 KeV) following [73, 74] falls at 4', they could in principle be detected with the current avail-
Figure 5. The ellipticity angle, polarization angle of dilaton are in the top panel and the same for axion are in bottom panel in the same parameter space as of fig. [3]. The x-axes are in unit of KeV.

Figure 6. Plot of $\Delta \chi$ vs $\omega$ in the left panel and $\Delta \psi$ vs $\omega$ in the right panel. The x-axes are in unit of KeV. The parameters for these plots are same as used in fig.[5].

able resolution of onboard detectors. Furthermore, the difference in polarization angles $\Delta \psi = (\psi_{\phi'} - \psi_{\phi})$ and in ellipticity angles $\Delta \chi = (\chi_{\phi'} - \chi_{\phi})$, due to two different interactions, also seem to come in the observable range in some energy intervals as may be noticed from the plot in fig.[6].

On the face of the discrepancy between observed data and the identities presented in
eqns. (9.1) and (9.2), we have tried to estimate the confidence level (C.L) on the numerical size of the observables obtained from the exact numerical solutions. This analysis is relegated to the end of the supplementary section [70].

10 Conclusions

In this paper we have compared the mixing dynamics of $\frac{1}{M}\phi FF$ with $\frac{1}{M}\phi' F\tilde{F}$ with the inclusion of the photon-self-energy-correction (PSEC) evaluated in a weakly-magnetized-finite density-medium evaluated at linear order in field strength ($eB$). Inclusion of the same modifies their mixing-dynamics of the available degrees of freedom among themselves; as a result, the mixing matrix for $\frac{1}{M}\phi FF$ becomes $3 \times 3$ and the same for $\frac{1}{M}\phi' F\tilde{F}$ becomes $4 \times 4$ from $2 \times 2$, that is realised in MV. That is, for the $\phi\gamma$ system, there is a mixing between only three degrees of freedom but for $\phi'\gamma$ system there is a mixing among all four available degree of freedom; i.e., the pseudoscalar, the longitudinal and the two transverse degrees of freedom of the photon. Hence the spectro-polarimetric signals, due to each of them, once PSEC is considered – undergoes sizeable amount of modification.

Here we have also estimated difference in the polarimetric observables due to change in the mixing pattern for both cases i.e., $\phi\gamma$ and $\phi'\gamma$ system numerically and have plotted in figures [6]. Our results indicate that the differences in the observables are within the detector resolution limit.

Our findings indicate that, magnetized environment of compact astrophysical systems may be a good place to look for the DM signatures, coming from scalars as well as pseudoscalars. Such activity complimentary to the searches going on else-where [27, 28].

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11 Introduction

The particles (beyond the standard model) associated with the breaking of chiral symmetry (pseudoscalars) \( \phi'(x) \) \([3-6]\), through quantum effects and the Goldstone boson of a spontaneously broken scale symmetry (dilaton) \( \phi(x) \) \([1, 2]\), have remained possible candidates of Dark matter for some times now.

According to our current understanding, about 27% of the total matter-energy density is in the form of Dark matter. If \( \phi \) or \( \phi' \) constitutes the dark-matter that is being talked about, then their cosmological relic density \( \rho(t) \) (as \( \phi \) or \( \phi' \)) at some epoch \( t \), in terms of their lifetime \( \tau_{\phi_i} \) in \( \phi \rightarrow \gamma\gamma \) channel, would be given by \([1]\),

\[
\rho(t) = \rho_d \left( \frac{\eta_d}{\eta} \right)^3 e^{-\frac{t}{\tau_{\phi_i}}},
= \frac{\zeta(3)}{\pi^2} \frac{g_\ast(t)}{g_{\ast d}(t)} T^3(t) e^{-\frac{t}{\tau_{\phi_i}}}. \tag{S1}
\]

In equation (S1) \( \rho_d \) corresponds to the density and \( \eta_d \) the magnitude of the scale factor of Friedman-Robertson-Walker metric. And \( g_{\ast d} \) along with \( g_\ast \) are the number of degrees of freedom available at the time of decoupling and the same at the epoch \( t \), respectively. The last line in equation (S1) has been obtained demanding entropy conservation in the co-moving volume of the universe, from the time of decoupling to the epoch \( t \).

Their life-time \( \tau_{\phi_i} \), in turn depends on the parameters of the theory, i.e., scalar-photon coupling constant \( g_{\phi\gamma\gamma} \) and scalar mass \( m_\phi \) or pseudoscalar-photon coupling constant \( g_{\phi'\gamma\gamma} \) and pseudoscalar’s \( m_{\phi'} \); those regulate the interaction dynamics of these exotic particles with photon (\( \gamma \)). Hence estimations of these are intimately related to the identifications of these particles.

The Dim-5 operator \( g_{\gamma\phi\gamma\phi} F^{\mu
u} F_{\mu
u} \) that regulates the massive scalar - photon interaction also induces optical activity to the vacuum in a quasi-static or static magnetic field \([17, 18]\). The same is also true for pseudo-scalar- photon system. These interactions open up the possibility of determination of \( g_{\phi\gamma\gamma}, m_\phi \) or \( g_{\phi'\gamma\gamma}, m_{\phi'} \) by studying their polarimetric signatures. However, if there are degeneracies in their masses \( (m_\phi \text{ or } m_{\phi'}) \) as well as in coupling constants \( (g_{\phi\gamma\gamma} \text{ or } g_{\phi'\gamma\gamma}) \), identification would not be so easy.

In the following sections, we have proposed a model to resolve the issue of indistinguishable nature of these particles (scalars and pseudoscalars) by studying their mixing dynamics in presence of magnetised plasma. The mixing matrix for \( \phi F_{\mu\nu} F^{\mu\nu} \) interaction , turns out to be \( 3 \times 3 \) instead of \( 2 \times 2 \), usually encountered in \( \text{MV} \), or magnetized plasma. For \( \phi' F_{\mu\nu} F^{\mu\nu} \) interaction the situation turns out to be more interesting. In this situation, mixing of all four degrees of freedom –three degrees of freedom of a photon and one degree of freedom of pseudoscalar – takes place . This results in he mixing matrix being \( 4 \times 4 \).
11.1 Electromagnetic form-factors for $A_\mu(k)$

In contrast to vacuum – the gauge fields $A_\mu(k)$ – for in medium photons, can be written in momentum space, in terms of three electromagnetic form factors: $A_{||}(k)$, $A_{\perp}(k)$, $A_L(k)$ along with the gauge fixing term $A_{gf}(k)$ and four orthonormal four vectors constructed out of the available 4-vectors and tensors for the system in hand:

$$A_\alpha(k) = A_{||}(k)N_1b_\alpha^{(1)} + A_{\perp}(k)N_2I_\alpha + A_L(k)N_L\tilde{u}_\alpha + \frac{k_\alpha}{k^2}A_{gf}(k). \quad (S2)$$

Here $N_i$s are the normalization constants and $A_i$s are the form factors already defined above. We next set $A_{gf} = 0$, to comply with the Lorentz gauge condition $k_\mu A^\mu(k) = 0$. These vectors, introduced in equation (S2), are defined as,

$$\hat{b}^{(1)\nu} = N_1k_\mu F^{\mu\nu}, \hat{b}^{(2)\nu} = k_\mu \tilde{F}^{\mu\nu}, \hat{\tilde{u}}^\nu = N_L \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) u_\mu, \hat{\tilde{u}}^\nu = N_2 \left( b^{(2)\nu} - \frac{(\tilde{u}^{\mu}b^{(2)\mu})}{u^2} \tilde{u}^\nu \right) \quad (S3)$$

The normalization constants, $N_1$, $N_2$ and $N_L$ in equation (S3) are given by,

$$N_1 = \frac{1}{\sqrt{-b^{(1)\mu}b^{(1)\mu}}}, \quad N_2 = \frac{1}{\sqrt{-I_{\mu}I^\mu}} = \frac{K}{\omega K_{\perp}B}, \quad \text{and} \quad N_L = \frac{1}{\sqrt{-\tilde{u}_\mu\tilde{u}^\mu}} = \frac{\sqrt{(k^\mu k^\nu)}}{|k|} \quad (S4)$$

where $K_\perp = (k_1^2 + k_2^2)^{\frac{1}{2}}$.

In the table below, we provide the transformation properties of different vectors and the associated form factors under the charge conjugation (C), Time reversal (T) and Parity inversion (P) transformations. Although P and T transformation are a part of Lorentz transformation but they are not orthogonal proper Lorentz transformation (OPLT). Therefore the variables transforming under discrete transformations may remain inert under the action of the generators of symmetries of the OPLT.

|        | $F_{\mu\nu}$ | $k_\mu$ | $u_\mu$ | $\tilde{u}_\mu$ | $b^{(1)}_\mu$ | $b^{(2)}_\mu$ | $I_\mu$ | $A_{||}$ | $A_{\perp}$ | $A_L$ | $f_{\mu\nu}$ | $i$ | $\epsilon_{\mu\nu\rho\sigma}$ |
|--------|---------------|---------|---------|-----------------|---------------|---------------|--------|---------|----------|-------|-------------|----|-----------------|
| C      | $-F_{\mu\nu}$ | $k_\mu$ | $-u_\mu$ | $-\tilde{u}_\mu$ | $-b^{(1)}_\mu$ | $-b^{(2)}_\mu$ | $-I_\mu$ | $A_{||}$ | $A_{\perp}$ | $A_L$ | $-f_{\mu\nu}$ | $i$ | $\epsilon_{\mu\nu\rho\sigma}$ |
| P      | $F^{\mu\nu}$  | $k_\mu$ | $u_\mu$  | $\tilde{u}_\mu$ | $b^{(1)}_\mu$  | $b^{(2)}_\mu$  | $I_\mu$  | $A_{||}$ | $A_{\perp}$ | $A_L$ | $f_{\mu\nu}$  | $i$ | $-\epsilon_{\mu\nu\rho\sigma}$ |
| T      | $-F^{\mu\nu}$ | $k_\mu$ | $-u_\mu$ | $-\tilde{u}_\mu$ | $b^{(1)}_\mu$  | $-b^{(2)}_\mu$ | $-I_\mu$ | $-A_{||}$ | $-A_{\perp}$ | $-A_L$ | $-f_{\mu\nu}$ | $-i$| $-\epsilon_{\mu\nu\rho\sigma}$ |

Table 1. Transformation properties for the vectors, tensors and the electromagnetic form factors used to describe $A_\mu(k)$ in equation (2.10), under C, P and T.

11.2 Gauge field degrees of freedom

In a material medium the effective Lagrangian in momentum space will be given by,

$$L_{eff(m)} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}A_\mu(-k)\Pi^{\mu\nu}(k, \mu, T)A_\mu(k). \quad (S5)$$
Where, $\Pi_{\mu\nu}(k, \mu, T)$ is the photon self energy tensor including medium corrections. The number of effective dof for the system can be estimated by estimating the number of first class constraints (FCC) and second class constraints (SCC), that the Lagrangian $L_{\text{eff}(m)}$ have. Due to complicated structure of $\Pi_{\mu\nu}(k, \mu, T)$ in momentum space, the evaluation of the analytically without any approximation is bit difficult. However, in the infrared limit the same takes the form,

$$L_{\text{eff}(m)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \omega_p^2 (u.A)(u.A).$$  \hspace{1cm} (S6)$$

This form of the effective Lagrangian, when $u^\mu = (1, 0, 0, 0)$ turns out to be very close to the Proca Lagrangian. So following the analysis of Proca Lagrangian, this one also would have two SCC and no FCC. Hence the number of degrees $N$ of freedom would be,

$$N = \left( N_{PSV} - n_{SCC} \right) / 2 = 3,$$ \hspace{1cm} (S7)

where $N_{PSV}$ is number of phase space variables. Therefore, one can now set $A_{gf}$ equal to zero in equation (S2). As a consequence the $U(1)$ gauge field is seen to obey the condition,

$$k_\mu A^\mu = 0.$$ \hspace{1cm} (S8)

That resembles the Lorentz gauge condition. Introduction of the contribution from magnetized medium does not alter this conclusion.

12 Incorporation of matter effects

Matter effects are incorporated through the inclusion of a term of the following form, $A^\mu \Pi_{\mu\nu}(k, \mu, T) A^\nu$ in the effective Lagrangian ($L_{\text{eff}}$) of the system [22–27, 30]. This is a scalar made up by contracting in-medium photon self-energy tensor $\Pi_{\mu\nu}(k, \mu, T)$ with gauge fields. Parameters $T$ and $\mu$ stand for temperature and chemical potential as arguments of $\Pi_{\mu\nu}(k, \mu, T)$.

Such a system can best be described in terms of a set of photon form-factors introduced in [25]. Unlike [19] these form factors [25], turn out to be even or odd under Time (T) reversal or parity (P) symmetry transformations (for details consult table [1] in [70]. The $\phi\gamma\gamma$ Lagrangian in an external magnetic field, generates couplings between $PT$ violating $A_\parallel$ and $PT$ symmetric $\phi$ through a multiplicative $T$ violating factor $i$. The other two $PT$ violating electromagnetic form factors $A_\perp$ and $A_L$ (see table [1]) have no coupling with the scalar $\phi$. However, the pseudoscalars (axion) under similar situation, couples to $A_\perp$ and $A_L$. Therefore, with the inclusion of $\Pi_{\mu\nu}(k, \mu, T)$, when $A_L$ becomes non-zero, the mixing matrix for $\phi\gamma$ remains $2 \times 2$ but the same for $\phi'\gamma$, turns out to be $3 \times 3$.

13 Mixing dynamics of $\phi\gamma$ interaction

In magnetised media the effective Lagrangian for $\phi\gamma$ interaction including the faraday effect is given as:

$$L_{\text{eff}, \phi} = \frac{1}{2} \phi [k^2 - m_\phi^2] - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \frac{1}{2} A_\mu \Pi^{\mu\nu}(k, \mu, T, eB) A_\nu - \frac{1}{4} g_{\phi\gamma\gamma} \phi F^{\mu\nu} f_{\mu\nu}. \hspace{1cm} (S1)$$
The equations of motion for the electromagnetic form factors for the photon, in the notation of [25], turn out to be,

\[
(k^2 - \Pi_T)A_{\parallel}(k) + i\Pi^p(k)N_1N_2 \left[ \epsilon_{\mu\nu\delta\beta} \frac{k^\beta}{|k|} u^{(1)\mu} I^\nu \right] A_{\perp}(k) = \frac{ig \phi\gamma A_{\parallel}(k)}{N_1}, \tag{S2}
\]

\[
(k^2 - \Pi_T)A_{\perp}(k) - i\Pi^p(k)N_1N_2 \left[ \epsilon_{\mu\nu\delta\beta} \frac{k^\beta}{|k|} u^{(1)\mu} I^\nu \right] A_{\parallel}(k) = 0, \tag{S3}
\]

\[
(k^2 - \Pi_L)A_L(k) = 0. \tag{S4}
\]

The three equations (i.e., (S2)-(S4)), describe the dynamics of the three degrees of freedom of the photon. And the equation of motion for \(\phi\) is given by,

\[
(k^2 - m^2)\phi(k) = -ig \phi\gamma A_{\parallel}(k) N_1. \tag{S5}
\]

For notational convenience, we, introduce further the new variables \(F\) and \(G\), defined as,

\[
F = N_1 N_2 \Pi^p(k) \left[ \epsilon_{\mu\nu\delta\beta} \frac{k^\beta}{|k|} u^{(1)\mu} I^\nu \right] \quad \text{and} \quad G = \frac{g \phi\gamma}{N_1}. \tag{S6}
\]

As stated already, in the long wavelength limit, we consider \(\Pi_T = \omega_p^2\), where \(\omega_p = \sqrt{4\pi\alpha n_e} \) is the plasma frequency, \(\alpha\) is electromagnetic coupling constant and \(n_e\) is the density of electrons. Other terms, \(F\) and \(G\) introduced in equation (S6) can be simplified to yield \(F = \frac{\omega_p^2}{\omega} \frac{eB\cos\theta}{m_e}\) and \(G = -g \phi\gamma B\sin\theta\omega\), where \(\theta\) is angle between the photon propagation vector \(\vec{k}\) and the magnetic field \(B\). Here \(m_e\) is the mass of electron and \(e\) is the electronic charge. The same can be cast in a compact matrix form, as:

\[
\begin{bmatrix}
 k^2I - M
\end{bmatrix}
\begin{pmatrix}
 A_{\parallel}(k) \\
 A_{\perp}(k) \\
 \phi(k)
\end{pmatrix}
= 0. \tag{S7}
\]

As we have already mentioned in section two that for \(\phi\gamma\) in presence of magnetized media, the longitudinal degree of freedom of photon does not mix with rest others, therefore we will exclude this term from the mixing matrix. Hence, the above \(4\times4\) mixing matrix \(M\), can be reduce in \(3\times3\) matrix. For the sake of brevity, using shorthand notations the equations of motions can be written as follows:

\[
\begin{bmatrix}
 k^2I - M
\end{bmatrix}
\begin{pmatrix}
 A_{\parallel}(k) \\
 A_{\perp}(k) \\
 \phi(k)
\end{pmatrix}
= 0, \tag{S8}
\]

where the \(3\times3\) mixing matrix for scalar-photon interaction is given as;

\[
M = \begin{pmatrix}
 \omega_p^2 & IF & -iG \\
 -iF & \omega_p^2 & 0 \\
 iG & 0 & m_\phi^2
\end{pmatrix}. \tag{S9}
\]

In order to get the stoke parameters \((I(\omega,z), Q(\omega,z), U(\omega,z), V(\omega,z))\) of the radiations, we need to get the solutions of \(M\). This can be achieved by diagonalizing the same.
13.1 Diagonalizing the $3 \times 3$ mixing matrix

Our objective here is to obtain the analytical expression for the diagonalizing (unitary) matrix $U$, for the hermitian matrix $M$. We express the elements of the same ($U$), in terms of algebraic expressions for maintaining the desired numerical accuracy ($O(10^{-15})$). In order to achieve that, we need to solve for the characteristic equation, obtained from $\text{Det} (M - \lambda I) = 0$ to find the eigen values(roots) of $M$; then use the same to find the corresponding eigen vectors. Finally, using the eigen vectors, construct the unitary matrix $U$ the diagonalizes $M$. The characteristic equation for this $3 \times 3$ hermitian matrix $M$, for obvious reasons turns out to be a cubic equation, having real roots.

The cubic equation, that follows from the characteristic equation can be written, in terms of parameters $b$, $c$ and $d$ as,

$$\lambda^3 + b\lambda^2 + c\lambda + d = 0,$$  \hspace{1cm} (S10)

where the parameters $b$, $c$ and $d$ are functions of the elements of mixing matrix $M$, denoted by:

$$b = -(2\omega^2_p + m^2_\phi),$$  \hspace{1cm} (S11)

$$c = \omega^4_p + 2\omega^2_p m^2_\phi - \left(\frac{eB\parallel \omega^2_p}{m_e}\right)^2 - (g_{\gamma\phi} B_\perp \omega^2)^2,$$  \hspace{1cm} (S12)

$$d = -\left[\omega^4_p m^2_\phi - \left(\frac{eB\parallel \omega^2_p}{m_e}\right)^2 m^2_\phi - (g_{\gamma\phi} B_\perp \omega^2)^2 \omega^2_p\right].$$  \hspace{1cm} (S13)

Next we introduce the variables $p$ and $Q$, when $p = \left(\frac{3c - b^2}{9}\right)$ and $2Q = \left(\frac{2b^3}{27} - \frac{bc}{9} + d\right)$; in terms of them, the roots turn out to be,

$$\lambda_1 = R \cos \alpha + \sqrt{3} R \sin \alpha - b/3,$$

$$\lambda_2 = R \cos \alpha - \sqrt{3} R \sin \alpha - b/3, \quad \text{With} \quad \left\{ \begin{array}{l} \alpha = \frac{1}{3} \cos^{-1}\left(\frac{Q}{R}\right) \\ R = \sqrt{(-p) \text{sgn}(Q)} \end{array} \right.$$  \hspace{1cm} (S14)

$$\lambda_3 = -2R \cos \alpha - b/3.$$

The orthonormal eigenvectors $X_j$ of $M$ are to be found from the matrix relation, $[M - \lambda_j] [X_j] = 0$. In terms of its elements, the normalised column vector $[X_j]$, can be denoted as,

$$[X_j] = \begin{bmatrix} \bar{u}_j \\ \bar{v}_j \\ \bar{w}_j \end{bmatrix}.$$

Following standard methods, one can evaluate these elements, in terms of the roots $\lambda_j$ and elements of the mixing matrix $M$. The same, once evaluated turns out to be,

$$\bar{u}_j = (\omega^2_p - \lambda_j)(m^2_\phi - \lambda_j) \times N^{(j)}_{\text{vn}},$$

$$\bar{v}_j = i eB\parallel \omega^2_p (m^2_\phi - \lambda_j) \times N^{(j)}_{\text{vn}}, \quad \text{when} \quad \left\{ N^{(j)}_{\text{vn}} = \frac{1}{\sqrt{|\bar{u}_j|^2 + |\bar{v}_j|^2 + |\bar{w}_j|^2}} \right\}$$  \hspace{1cm} (S15)

$$\bar{w}_j = i g_{\gamma\phi} B_\perp \omega (\omega^2_p - \lambda_j) \times N^{(j)}_{\text{vn}}.$$
Here \( \Lambda_{vn}^{(j)} \) is normalisation constant and \( j \) can take values from 1 to 3. Using these eigenvectors, the unitary matrix \( U \) turns out to be:

\[
U = \begin{pmatrix}
(\omega_p^2 - \lambda_1)(m_\phi^2 - \lambda_1)\Lambda_{vn}^{(1)} & (\omega_p^2 - \lambda_2)(m_\phi^2 - \lambda_2)\Lambda_{vn}^{(2)} & (\omega_p^2 - \lambda_3)(m_\phi^2 - \lambda_3)\Lambda_{vn}^{(3)} \\
\frac{\e^{iB_1}}{m_e} \frac{\omega_p^2}{2} (m_\phi^2 - \lambda_1)\Lambda_{vn}^{(1)} & \frac{\e^{iB_1}}{m_e} \frac{\omega_p^2}{2} (m_\phi^2 - \lambda_2)\Lambda_{vn}^{(2)} & \frac{\e^{iB_1}}{m_e} \frac{\omega_p^2}{2} (m_\phi^2 - \lambda_3)\Lambda_{vn}^{(3)} \\
ig_{\gamma\gamma} B_\perp \omega (\omega_p^2 - \lambda_1)\Lambda_{vn}^{(1)} & ig_{\gamma\gamma} B_\perp \omega (\omega_p^2 - \lambda_2)\Lambda_{vn}^{(2)} & ig_{\gamma\gamma} B_\perp \omega (\omega_p^2 - \lambda_3)\Lambda_{vn}^{(3)}
\end{pmatrix}
\]

(S16)

The obtained Unitary matrix (S16) diagonalises the 3\(\times\)3 mixing matrix \( M \). We used the diagonalized mixing matrix \( M_D \) to produce the solution of the field equation. The necessary steps have been stated in succeeding sections.

### 13.2 Field equation : Solutions

In order to solve the coupled equation (6.1) we multiply the same by \( U^{-1} \) to get,

\[
U^{-1} \left[ k^2 I - M \right] U U^{-1} \begin{pmatrix}
A_{||}(k) \\
A_{\perp}(k) \\
\phi(k)
\end{pmatrix} = \left[ k^2 I - M_D \right] \begin{pmatrix}
A'_{||}(k) \\
A'_{\perp}(k) \\
\phi'(k)
\end{pmatrix} = 0, \tag{S17}
\]

where matrix \( U \) is given in equation (S16) and \( U^{-1} \) is the inverse of the same and \( M_D \) is the diagonal matrix having eigen values as diagonal elements. Here we have denoted

\[
\begin{pmatrix}
A'_{||}(k) \\
A'_{\perp}(k) \\
\phi'(k)
\end{pmatrix} = U^{-1} \begin{pmatrix}
A_{||}(k) \\
A_{\perp}(k) \\
\phi(k)
\end{pmatrix}. \tag{S18}
\]

For a photon beam, propagating in the \( z \) direction, one can Fourier transform the momentum, \( k_3 \) back to \( z \) and write \( k^2 \approx 2\omega (\omega - i\partial_z) \). Recalling that \( U^{-1}MU = M_D \), equation (S18) can further be cast in the form,

\[
(\omega - i\partial_z)I - \begin{pmatrix}
\frac{\lambda_2}{2} & 0 & 0 \\
0 & \frac{\lambda_2}{2} & 0 \\
0 & 0 & \frac{\lambda_3}{2}
\end{pmatrix} \begin{pmatrix}
A'_{||}(z) \\
A'_{\perp}(z) \\
\phi'(z)
\end{pmatrix} = 0. \tag{S19}
\]

The matrix equation (S19) is easy to solve now. Introducing the variables, \( \Omega_{||} = \left( \omega - \frac{\lambda_2}{2} \right) \), \( \Omega_{\perp} = \left( \omega - \frac{\lambda_3}{2} \right) \) and \( \Omega_\phi = \left( \omega - \frac{\lambda_1}{2} \right) \), we can now directly write down the solutions for the column vector \([A(z)]\); in matrix form, they are:

\[
\begin{pmatrix}
A_{||}(z) \\
A_{\perp}(z) \\
\phi(z)
\end{pmatrix} = U \begin{pmatrix}
\e^{-i\Omega_{||}z} & 0 & 0 \\
0 & \e^{-i\Omega_{\perp}z} & 0 \\
0 & 0 & \e^{-i\Omega_\phi z}
\end{pmatrix} U^{-1} \begin{pmatrix}
A_{||}(0) \\
A_{\perp}(0) \\
\phi(0)
\end{pmatrix}. \tag{S20}
\]

The magnitudes of the elements of column vector \([A(0)]\) in equation (S20), are subject to the physical situations/conditions prevailing at the origin. Using the initial conditions
stated in section (2.1), one can write down the solution of equation (S20) for $\mathbf{A}_\parallel (\omega, z)$, and it is as follows:

$$
\mathbf{A}_\parallel (\omega, z) = (e^{-i\Omega_\parallel z} \tilde{u}_1 \tilde{u}_1^* + e^{-i\Omega_\perp z} \tilde{u}_2 \tilde{u}_2^* + e^{-i\Omega_\phi z} \tilde{u}_3 \tilde{u}_3^*) \mathbf{A}_\parallel (\omega, 0) \\
+ (e^{-i\Omega_\parallel z} \tilde{v}_1 \tilde{v}_1^* + e^{-i\Omega_\perp z} \tilde{v}_2 \tilde{v}_2^* + e^{-i\Omega_\phi z} \tilde{v}_3 \tilde{v}_3^*) \mathbf{A}_\perp (\omega, 0). \tag{S21}
$$

Similarly the perpendicular component $\mathbf{A}_\perp (\omega, z)$, turns out to be,

$$
\mathbf{A}_\perp (\omega, z) = (e^{-i\Omega_\parallel z} \tilde{v}_1 \tilde{v}_1^* + e^{-i\Omega_\perp z} \tilde{v}_2 \tilde{v}_2^* + e^{-i\Omega_\phi z} \tilde{v}_3 \tilde{v}_3^*) \mathbf{A}_\parallel (\omega, 0) \\
+ (e^{-i\Omega_\parallel z} \tilde{u}_1 \tilde{u}_1^* + e^{-i\Omega_\perp z} \tilde{u}_2 \tilde{u}_2^* + e^{-i\Omega_\phi z} \tilde{u}_3 \tilde{u}_3^*) \mathbf{A}_\perp (\omega, 0). \tag{S22}
$$

### 13.3 Stokes parameters

To find out the Stokes parameters for Scalar-photon mixing, we need to evaluate $|\mathbf{A}_\parallel (\omega, z)|^2$ and $|\mathbf{A}_\perp (\omega, z)|^2$, using eqns. [S21] and [S22]. Introducing the new variables, $P = |\tilde{u}_1||\tilde{v}_1|$, $Q = |\tilde{u}_2||\tilde{v}_2|$ and $R = |\tilde{u}_3||\tilde{v}_3|$, the expression for $|\mathbf{A}_\parallel (\omega, z)|^2$ in terms of them, turns out to be,

$$
|\mathbf{A}_\parallel (\omega, z)|^2 = \left[ 1 - 4|\tilde{u}_1|^2|\tilde{u}_2|^2 \sin^2 \left( \frac{(\Omega_\parallel - \Omega_\perp)z}{2} \right) - 4|\tilde{u}_2|^2|\tilde{u}_3|^2 \sin^2 \left( \frac{(\Omega_\perp - \Omega_\phi)z}{2} \right) \\
- 4|\tilde{u}_3|^2|\tilde{u}_1|^2 \sin^2 \left( \frac{(\Omega_\phi - \Omega_\parallel)z}{2} \right) \right] \\
\times |\mathbf{A}_\parallel (\omega, 0)|^2 - 4 \left[ P Q \sin^2 \left( \frac{(\Omega_\parallel - \Omega_\perp)z}{2} \right) + Q R \sin^2 \left( \frac{(\Omega_\perp - \Omega_\phi)z}{2} \right) \\
+ R P \sin^2 \left( \frac{(\Omega_\phi - \Omega_\parallel)z}{2} \right) \right] \\
\times |\mathbf{A}_\perp (\omega, 0)|^2 + \left[ |\tilde{u}_1 \tilde{u}_2|(|\tilde{u}_1 \tilde{v}_1| - |\tilde{v}_2 \tilde{v}_1|) \sin \left( (\Omega_\parallel - \Omega_\perp)z \right) \\
+ |\tilde{u}_2 \tilde{u}_3|(|\tilde{v}_2 \tilde{v}_3| - |\tilde{u}_3 \tilde{v}_2|) \sin \left( (\Omega_\perp - \Omega_\phi)z \right) \\
+ |\tilde{u}_3 \tilde{u}_1|(|\tilde{u}_3 \tilde{v}_1| - |\tilde{v}_1 \tilde{v}_3|) \sin \left( (\Omega_\phi - \Omega_\parallel)z \right) \right] \\
\times 2 |\mathbf{A}_\parallel (\omega, 0)||\mathbf{A}_\perp (\omega, 0)|. \tag{S23}
$$
Similarly we can find $|A_{\perp}(\omega, z)|^2$. The expression for the same, in terms of the variables introduced earlier, turns out to be,

$$
|A_{\perp}(\omega, z)|^2 = -4 \left[ PQ \sin^2 \left( \frac{(\Omega_{\parallel} - \Omega_{\perp}) z}{2} \right) + QR \sin^2 \left( \frac{(\Omega_{\perp} - \Omega_{\phi}) z}{2} \right) 
+ \mathbb{R}P \sin^2 \left( \frac{(\Omega_{\phi} - \Omega_{\parallel}) z}{2} \right) \right] 
\times |A_{\parallel}(\omega, 0)|^2 
+ \left[ 1 - 4|\bar{v}_1|^2|\bar{v}_2|^2 \sin^2 \left( \frac{(\Omega_{\parallel} - \Omega_{\perp}) z}{2} \right) - 4|\bar{v}_2|^2|\bar{v}_3|^2 \sin^2 \left( \frac{(\Omega_{\perp} - \Omega_{\phi}) z}{2} \right) 
- 4|\bar{v}_3|^2|\bar{v}_1|^2 \sin^2 \left( \frac{(\Omega_{\phi} - \Omega_{\parallel}) z}{2} \right) \right] 
\times |A_{\perp}(\omega, 0)|^2 
+ \left[ |\bar{v}_1| |\bar{u}_2|-|\bar{u}_2| \bar{v}_1| \sin \left( (\Omega_{\parallel} - \Omega_{\perp}) z \right) \right] 
+ \left[ |\bar{v}_2| |\bar{u}_3|-|\bar{u}_3| \bar{v}_2| \sin \left( (\Omega_{\perp} - \Omega_{\phi}) z \right) \right] 
+ \left[ |\bar{v}_3| |\bar{u}_1|-|\bar{u}_1| \bar{v}_3| \sin \left( (\Omega_{\phi} - \Omega_{\parallel}) z \right) \right] 
\times 2|A_{\parallel}(\omega, 0)||A_{\perp}(\omega, 0)|. 
$$

(S24)

To express the Stokes parameters given by equations (6.6-6.9), obtained from (S23) and (S24) we had introduced variables $I_{\parallel}$, $I_{\perp}$ and $I_{\parallel \perp}$ to express $I$. They are given by:

$$
I_{\parallel} = 1 - 4|\bar{u}_1|^2|\bar{u}_2|^2 + PQ \sin^2 \left( \frac{(\Omega_{\parallel} - \Omega_{\perp}) z}{2} \right) - 4|\bar{u}_2|^2|\bar{u}_3|^2 + QR \sin^2 \left( \frac{(\Omega_{\perp} - \Omega_{\phi}) z}{2} \right) 
- 4|\bar{u}_3|^2|\bar{u}_1|^2 + \mathbb{R}P \sin^2 \left( \frac{(\Omega_{\phi} - \Omega_{\parallel}) z}{2} \right) . 
$$

(S25)

$$
I_{\perp} = 1 - 4|\bar{v}_1|^2|\bar{v}_2|^2 + PQ \sin^2 \left( \frac{(\Omega_{\parallel} - \Omega_{\perp}) z}{2} \right) - 4|\bar{v}_2|^2|\bar{v}_3|^2 + QR \sin^2 \left( \frac{(\Omega_{\perp} - \Omega_{\phi}) z}{2} \right) 
- 4|\bar{v}_3|^2|\bar{v}_1|^2 + \mathbb{R}P \sin^2 \left( \frac{(\Omega_{\phi} - \Omega_{\parallel}) z}{2} \right) . 
$$

(S26)

$$
I_{\parallel \perp} = (|\bar{u}_1|^2 - |\bar{u}_2|^2)|\bar{w}_1|\bar{w}_2| \sin \left( (\Omega_{\parallel} - \Omega_{\perp}) z \right) 
+ (|\bar{u}_2|^2 - |\bar{u}_3|^2)|\bar{w}_2|\bar{w}_3| \sin \left( (\Omega_{\perp} - \Omega_{\phi}) z \right) 
+ (|\bar{u}_3|^2 - |\bar{u}_1|^2)|\bar{w}_3|\bar{w}_1| \sin \left( (\Omega_{\phi} - \Omega_{\parallel}) z \right) .
$$

(S27)

Similarly, to express Stokes parameter $Q$, we had introduced the variables $Q_{\parallel}$, $Q_{\perp}$ and $Q_{\parallel \perp}$. Their actual forms are,

$$
Q_{\parallel} = 1 - 4|\bar{u}_1|^2|\bar{u}_2|^2 - PQ \sin^2 \left( \frac{(\Omega_{\parallel} - \Omega_{\perp}) z}{2} \right) - 4|\bar{u}_2|^2|\bar{u}_3|^2 - QR \sin^2 \left( \frac{(\Omega_{\perp} - \Omega_{\phi}) z}{2} \right) 
- 4|\bar{u}_3|^2|\bar{u}_1|^2 - \mathbb{R}P \sin^2 \left( \frac{(\Omega_{\phi} - \Omega_{\parallel}) z}{2} \right) . 
$$

(S28)
\[
Q_\perp = 1 - 4(|v_1v_2|^2 - FTQ)\sin^2 \left( \frac{(\Omega || - \Omega_\perp)z}{2} \right) - 4(|v_2v_3|^2 - QR)\sin^2 \left( \frac{(\Omega_\perp - \Omega_\phi)z}{2} \right) - 4(|v_3v_1|^2 - RP)\sin^2 \left( \frac{(\Omega_\phi - \Omega_\perp)z}{2} \right).
\] (S29)

\[
Q_{||} = (|\bar{u}_1\bar{v}_2| - |\bar{u}_2\bar{v}_1|)(|\bar{u}_1\bar{v}_2| - |\bar{v}_1\bar{v}_2|)\sin \left((\Omega || - \Omega_\perp)z\right) + (|\bar{u}_2\bar{v}_3| - |\bar{u}_3\bar{v}_2|)(|\bar{u}_2\bar{v}_3| - |\bar{v}_2\bar{v}_3|)\sin \left((\Omega_\perp - \Omega_\phi)z\right) + (|\bar{v}_3\bar{v}_1| - |\bar{u}_1\bar{v}_3|)(|\bar{v}_3\bar{v}_1| - |\bar{v}_3\bar{v}_1|)\sin \left((\Omega_\phi - \Omega_\perp)z\right).
\] (S30)

In the same spirit, the expanded expressions for the variables \(U_||, U_\perp\) and \(U_{||\perp}\), already introduced in the text to express Stokes parameter \(U\), are given by:

\[
U_|| = |\bar{u}_1\bar{u}_2||\bar{v}_1\bar{u}_2| - |\bar{u}_1\bar{v}_2|)\sin \left((\Omega || - \Omega_\perp)z\right) + |\bar{u}_2\bar{u}_3|(|\bar{v}_2\bar{v}_3| - |\bar{u}_2\bar{v}_3|)\sin \left((\Omega_\perp - \Omega_\phi)z\right) + |\bar{u}_3\bar{u}_1|(|\bar{v}_3\bar{u}_1| - |\bar{u}_3\bar{v}_1|)\sin \left((\Omega_\phi - \Omega_\perp)z\right).
\] (S31)

\[
U_\perp = |\bar{v}_1\bar{v}_2|(|\bar{v}_1\bar{u}_2| - |\bar{v}_1\bar{v}_2|)\sin \left((\Omega || - \Omega_\perp)z\right) + |\bar{v}_2\bar{v}_3|(|\bar{v}_2\bar{u}_3| - |\bar{v}_2\bar{v}_3|)\sin \left((\Omega_\perp - \Omega_\phi)z\right) + |\bar{v}_3\bar{v}_1|(|\bar{v}_3\bar{u}_1| - |\bar{v}_3\bar{v}_1|)\sin \left((\Omega_\phi - \Omega_\perp)z\right).
\] (S32)

\[
U_{||\perp} = (|\bar{v}_1\bar{u}_2| - |\bar{u}_1\bar{v}_2|)^2 \cos \left((\Omega || - \Omega_\perp)z\right) + (|\bar{v}_2\bar{v}_3| - |\bar{u}_2\bar{v}_3|)^2 \cos \left((\Omega_\perp - \Omega_\phi)z\right) + (|\bar{v}_3\bar{u}_1| - |\bar{v}_3\bar{v}_1|)^2 \cos \left((\Omega_\phi - \Omega_\perp)z\right).
\] (S33)

And lastly, the expressions for \(V_||, V_\perp\) and \(V_{||\perp}\), introduced to express the measure of circular polarisation \(V\) are:

\[
V_|| = (|\bar{u}_1\bar{v}_1| |\bar{u}_1|^2 + |\bar{u}_2\bar{v}_2|^2 + |\bar{u}_3\bar{v}_3|^2) + |\bar{u}_1\bar{u}_2|(|\bar{v}_1\bar{u}_2| + |\bar{v}_1\bar{v}_2|) \cos \left((\Omega || - \Omega_\perp)z\right) + |\bar{u}_2\bar{u}_3|(|\bar{v}_2\bar{u}_3| + |\bar{v}_2\bar{v}_3|) \cos \left((\Omega_\perp - \Omega_\phi)z\right) + |\bar{v}_3\bar{u}_1|(|\bar{v}_3\bar{u}_1| + |\bar{v}_3\bar{v}_1|) \cos \left((\Omega_\phi - \Omega_\perp)z\right).
\] (S34)

\[
V_\perp = (|\bar{u}_1\bar{v}_1|^2 + |\bar{u}_2\bar{v}_2|^2 + |\bar{u}_3\bar{v}_3|^2) + |\bar{v}_1\bar{v}_2|(|\bar{v}_1\bar{v}_2| + |\bar{v}_1\bar{v}_2|) \cos \left((\Omega || - \Omega_\perp)z\right) + |\bar{v}_2\bar{v}_3|(|\bar{v}_2\bar{v}_3| + |\bar{v}_2\bar{v}_3|) \cos \left((\Omega_\perp - \Omega_\phi)z\right) + |\bar{v}_3\bar{v}_1|(|\bar{v}_3\bar{v}_1| + |\bar{v}_3\bar{v}_1|) \cos \left((\Omega_\phi - \Omega_\perp)z\right).
\] (S35)

\[
V_{||\perp} = (|\bar{v}_1|^2 |\bar{v}_2|^2 + |\bar{v}_2|^2 |\bar{v}_1|^2) \sin \left((\Omega || - \Omega_\perp)z\right) + (|\bar{u}_2|^2 |\bar{v}_3|^2 + |\bar{u}_3|^2 |\bar{v}_2|^2) \sin \left((\Omega_\perp - \Omega_\phi)z\right) + (|\bar{u}_3|^2 |\bar{v}_1|^2 - |\bar{u}_1|^2 |\bar{v}_3|^2) \sin \left((\Omega_\phi - \Omega_\perp)z\right).
\] (S36)

14 Mixing dynamics of \(\phi'\gamma\) interaction in magnetized medium

The effective Lagrangian for photon-pseudoscalar interaction including Faraday term is given as:

\[
L_{eff,\phi'} = \frac{1}{2} \phi' [k^2 - m_\phi^2] - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \frac{1}{2} A_\mu \Pi^{\mu\nu}(k, \mu, T, eB) A_\nu - \frac{1}{4} g_{\phi'\gamma\phi'} \tilde{F}^{\mu\nu} f_{\mu\nu}.
\] (S1)
The dynamics of the gauge potential in terms, for coupled pseudoscalar-photon system, following [25] turns out to be,

\[
[k^2 g_{\mu\nu} - \Pi_{\mu\nu}(k) - \Pi_p(k) P_{\mu\nu}] A^{\nu}(k) = ig_{\phi'\gamma\phi} \tilde{F}_{\mu\nu} k^\nu \varphi'(k),
\]

(S2)

where the terms have their usual meaning. The resulting equations of motion for the form factors are given by:

\[
(k^2 - \Pi_T(k))A_\perp(k) - \Pi_pN_1N_2 \left[ P_{\mu\nu}b^{(1)\mu} I^\nu \right] A_{||}(k) + \left( ig_{\phi'\gamma\phi}N_2 b^{(2)\mu} I^\mu \right) \varphi'(k) = 0,
\]

\[
(k^2 - \Pi_T(k))A_{||}(k) + \Pi_pN_1N_2 \left[ P_{\mu\nu}b^{(1)\mu} I^\nu \right] A_\perp(k) = 0,
\]

\[
(k^2 - \Pi_L) A_L(k) + ig_{\phi'\gamma\phi} N_L \left( b^{(2)\mu} \tilde{u}^\mu \right) \varphi'(k) = 0. \quad \text{(S3)}
\]

And the equation of motion (EOM) for the pseudoscalar field is,

\[
(k^2 - m^2) \varphi'(k) = \left[ \left( ig_{\phi'\gamma\phi} b^{(2)\mu} I^\mu \right) N_2 A_\perp(k) + \left( ig_{\phi'\gamma\phi} b^{(2)\mu} \tilde{u}^\mu \right) N_L A_L(k) \right]. \quad \text{(S4)}
\]

Introducing \( \Pi_T = \frac{\omega^2_{\phi'}}{m^2_{\phi'}} \) and the following notations for the elements of \( \mathbf{M}' \), i.e., \( \mathbf{M}'_{12} = -\mathbf{M}'_{21} = iF = \frac{\omega^2_{\phi'}}{m^2_{\phi'}} \cos \theta \) followed by \( \mathbf{M}'_{24} = -\mathbf{M}'_{42} = iG = ig_{\phi'\gamma\phi} B \) and \( \mathbf{M}'_{34} = -\mathbf{M}'_{43} = -iL = -ig_{\phi'\gamma\phi} \omega_p B \) sin \( \theta \), the EOMs i.e. eqns. (S3) and equation. (S4), can be combined to be expressed in a compact matrix notation as,

\[
[k^2 \mathbf{I} - \mathbf{M}'] \begin{pmatrix} A_{||}(k) \\ A_\perp(k) \\ A_L(k) \\ \varphi'(k) \end{pmatrix} = 0, \quad \text{(S5)}
\]

where \( \mathbf{I} \) is an identity matrix and matrix \( \mathbf{M}' \) is the \( 4 \times 4 \) mixing matrix, given as follows:

\[
\mathbf{M}' = \begin{pmatrix}
\omega^2_{\phi'} & iF & 0 & 0 \\
-iF & \omega^2_{\phi'} & 0 & iG \\
0 & 0 & \Pi_L & -iL \\
0 & -iG & iL & m^2_{\phi'}
\end{pmatrix}. \quad \text{(S6)}
\]

The angle \( \theta \), in (S6) is the angle between vector \( \vec{k} \) and \( \vec{B} \).

### 14.1 Diagonalizing the \( 4 \times 4 \) mixing matrix

Diagonalising the matrix \( \mathbf{M}' \) given in (6.11), with its full generality is a complex task, involving finding the roots of a fourth order polynomial. Therefore the same was performed perturbatively in [25]. In the following we discuss that procedure briefly.

In order to find the approximate eigen values and eigen vectors we can write \( \mathbf{M}' \) as a sum of two parts,

\[
\mathbf{M}' = \begin{pmatrix}
\omega^2_{\phi'} & iF & 0 & 0 \\
-iF & \omega^2_{\phi'} & 0 & iG \\
0 & 0 & \Pi_L & 0 \\
0 & -iG & 0 & m^2_{\phi'}
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -iL \\
0 & 0 & iL & 0
\end{pmatrix}. \quad \text{(S7)}
\]
The first matrix on the rhs of equation (S7) is $M^0$ and the second one is $M^6$, constructed with the $M'_{44}$ and $M_{43}$. In the limit $m_{\phi'} \rightarrow 0$, the eigenvalues of the $M^0$ matrix can be expressed in terms of the variables $X$ and $Y$ (defined as: $X = \frac{F}{(m_{\phi'} - \omega_p)}$ and $Y = \frac{G}{(m_{\phi'} - \omega_p)}$), in the following way,

$$\lambda_0^0 = \frac{G^2}{2 \omega_p} + \omega_p^2 - \sqrt{F^2 + \left( \frac{G}{2 \omega_p} \right)^2},$$

$$\lambda_0^0 = \frac{G^2}{2 \omega_p} + \omega_p^2 + \sqrt{F^2 + \left( \frac{G}{2 \omega_p} \right)^2},$$

$$\lambda_0^0 = \Pi_L,$$

$$\lambda_0^0 = - \frac{G^2}{2 \omega_p}. \tag{S8}$$

The eigenvectors of $M^0$, defined in terms of $\lambda_1 = \sqrt{X^2 + \frac{Y^2}{4} - \frac{Y^2}{2}}, \lambda_2 = - \sqrt{X^2 + \frac{Y^2}{4} - \frac{Y^2}{2}}$ and $\lambda_3 = 1 + Y^2$ turn out to be,

$$|\lambda_1^0\rangle = \frac{1}{d_1} \begin{pmatrix} -1 \\ i \frac{\lambda_1}{\lambda_2} \\ 0 \\ Y \frac{\lambda_1}{\lambda_2} \end{pmatrix}, \quad |\lambda_2^0\rangle = \frac{1}{d_2} \begin{pmatrix} -1 \\ i \frac{\lambda_1}{\lambda_2} \\ 0 \\ Y \frac{\lambda_1}{\lambda_2} \end{pmatrix},$$

$$|\lambda_3^0\rangle = \frac{1}{d_3} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\lambda_4^0\rangle = \frac{1}{d_4} \begin{pmatrix} 0 \\ -i Y \\ 0 \\ 1 \end{pmatrix}. \tag{S9}$$

In eqns. of (S9) the variables $d_1 = \sqrt{1 + \frac{\lambda_1}{X^2}}, d_2 = \sqrt{1 + \frac{\lambda_2}{X^2}}$ and $d_3 = d_4 = 1$.

As mentioned before, we estimated the approximate eigen-values and eigen-vectors of matrix $M^0$, by treating $M^6$ as perturbation over $M^0$. The eigenvectors, under this approximation turn out to be,

$$|\lambda_1\rangle = \begin{pmatrix} - \frac{1}{d_1} \left( 1 - \frac{L^2 w_1^2}{d_1^2 (\lambda_1^3 - \lambda_2^3)} \right)^{\frac{1}{2}} \\ \frac{w_1}{d_1} \left( 1 - \frac{L^2 w_1^2}{d_1^2 (\lambda_1^3 - \lambda_2^3)} \right)^{\frac{1}{2}} \\ -i \frac{L w_1}{d_1} \left( 1 - \frac{L^2 w_1^2}{d_1^2 (\lambda_1^3 - \lambda_2^3)} \right)^{\frac{1}{2}} \\ \frac{w_2}{d_1} \left( 1 - \frac{L^2 w_1^2}{d_1^2 (\lambda_1^3 - \lambda_2^3)} \right)^{\frac{1}{2}} \end{pmatrix}, \quad |\lambda_2\rangle = \begin{pmatrix} - \frac{1}{d_2} \left( 1 - \frac{L^2 w_2^2}{d_2^2 (\lambda_1^3 - \lambda_2^3)} \right)^{\frac{1}{2}} \\ \frac{w_2}{d_2} \left( 1 - \frac{L^2 w_2^2}{d_2^2 (\lambda_1^3 - \lambda_2^3)} \right)^{\frac{1}{2}} \\ -i \frac{L w_2}{d_2} \left( 1 - \frac{L^2 w_2^2}{d_2^2 (\lambda_1^3 - \lambda_2^3)} \right)^{\frac{1}{2}} \\ \frac{w_3}{d_2} \left( 1 - \frac{L^2 w_2^2}{d_2^2 (\lambda_1^3 - \lambda_2^3)} \right)^{\frac{1}{2}} \end{pmatrix},$$

$$|\lambda_3\rangle = \begin{pmatrix} -i \left[ \frac{L w_1}{d_1^2 (\lambda_1^3 - \lambda_2^3)} + \frac{L w_2}{d_2^2 (\lambda_1^3 - \lambda_2^3)} \right] \\ -i \left[ \frac{L^2 w_1^2}{d_1^2 (\lambda_1^3 - \lambda_2^3)} + \frac{L^2 w_2^2}{d_2^2 (\lambda_1^3 - \lambda_2^3)} \right] \\ 1 - \frac{L^2 w_1^2}{d_1^2 (\lambda_1^3 - \lambda_2^3)} - \frac{L^2 w_2^2}{d_2^2 (\lambda_1^3 - \lambda_2^3)} \end{pmatrix}^{\frac{1}{2}} \\ -i \left[ \frac{L w_1}{d_1^2 (\lambda_1^3 - \lambda_2^3)} + \frac{L w_2}{d_2^2 (\lambda_1^3 - \lambda_2^3)} \right] \\ -i \left[ \frac{L^2 w_1^2}{d_1^2 (\lambda_1^3 - \lambda_2^3)} + \frac{L^2 w_2^2}{d_2^2 (\lambda_1^3 - \lambda_2^3)} \right] \\ 1 - \frac{L^2 w_1^2}{d_1^2 (\lambda_1^3 - \lambda_2^3)} - \frac{L^2 w_2^2}{d_2^2 (\lambda_1^3 - \lambda_2^3)} \end{pmatrix}^{\frac{1}{2}} \end{pmatrix}, \quad |\lambda_4\rangle = \begin{pmatrix} 0 \\ \frac{w_3}{d_3} \left( 1 - \frac{L^2 w_3^2}{d_3^2 (\lambda_1^3 - \lambda_2^3)} \right)^{\frac{1}{2}} \\ -i \frac{L w_3}{d_3} \left( 1 - \frac{L^2 w_3^2}{d_3^2 (\lambda_1^3 - \lambda_2^3)} \right)^{\frac{1}{2}} \\ \frac{w_4}{d_3} \left( 1 - \frac{L^2 w_3^2}{d_3^2 (\lambda_1^3 - \lambda_2^3)} \right)^{\frac{1}{2}} \end{pmatrix}. \tag{S10}$$

In the above equations, $v_1 = i \frac{\lambda_1}{\lambda_2}, v_2 = i \frac{\lambda_2}{\lambda_1}, v_4 = -i Y$, and $w_1 = Y \frac{\lambda_1}{\lambda_2}, w_2 = Y \frac{\lambda_2}{\lambda_1}, w_4 = 1$. 

- 39 -
We use these eigenvectors to construct the Unitary matrix $U$ to diagonalize the mixing matrix $M'$ by unitary transformation. The constructed Unitary matrix is given as follows:

$$
U = \begin{pmatrix}
-\frac{1}{\sqrt{N_v}} \sqrt{1 - \xi_1 N_v^{(1)}} & -\frac{1}{\sqrt{N_v}} \sqrt{1 - \xi_2 N_v^{(2)}} & \cdots \\
\frac{1}{\sqrt{N_v}} \sqrt{1 - \xi_1 N_v^{(1)}} & \frac{1}{\sqrt{N_v}} \sqrt{1 - \xi_2 N_v^{(2)}} & \cdots \\
\cdots & \cdots & \cdots \\
\frac{1}{\sqrt{N_v}} \sqrt{1 - \xi_1 N_v^{(1)}} & \frac{1}{\sqrt{N_v}} \sqrt{1 - \xi_2 N_v^{(2)}} & \cdots
\end{pmatrix} 
$$

(S11)

Variables introduced in (S11) are defined as; $\xi_i = \frac{E_{W_i}}{d_i(\lambda_i^2 - \lambda_f^2)}$ ($i$ can take values 1,2,4). Here, $u_j, v_j, w_j$ and $x_j$ are the components of the $j^{th}$ column vectors of $U$, and

$$
N_v^{(j)} = \frac{1}{\sqrt{|u_j|^2 + |v_j|^2 + |w_j|^2 + |x_j|^2}}
$$
is the corresponding normalisation constant ($j = 1, 2, 3, 4$).

14.2 Field equation : Solutions

To get the solution of the coupled equation (S5), we multiply the same by $U^{-1}$, hence:

$$
U^{-1} \left[ k^2 I - M' \right] UU^{-1} \begin{pmatrix} A_{||}(k) \\ A_{\perp}(k) \\ A_{L}(k) \\ \phi'(k) \end{pmatrix} = \begin{pmatrix} k^2 I - M_D \end{pmatrix} \begin{pmatrix} A'_{||}(k) \\ A'_{\perp}(k) \\ A'_L(k) \\ \phi''(k) \end{pmatrix} = 0,
$$

(S12)

$$
[C_{A',\phi'}] = U \begin{pmatrix} A_{||}(k) \\ A_{\perp}(k) \\ A_L(k) \\ \phi'(k) \end{pmatrix} = U \begin{pmatrix} A'_{||}(k) \\ A'_{\perp}(k) \\ A'_L(k) \\ \phi''(k) \end{pmatrix}.
$$

(S13)

Following [17], one can write partial of $k_3$ (Fourier transform) for a beam propagating in z direction, equation (S13) can further be written explicitly in the form,

$$
(\omega - i\partial_z)I - \begin{pmatrix} \frac{\lambda_i}{2\omega} & 0 & 0 & 0 \\ 0 & \frac{\lambda_\perp}{2\omega} & 0 & 0 \\ 0 & 0 & \frac{\lambda_L}{2\omega} & 0 \\ 0 & 0 & 0 & \frac{\lambda_\phi}{2\omega} \end{pmatrix} \begin{pmatrix} A'_{||}(z) \\ A'_{\perp}(z) \\ A'_L(z) \\ \phi''(z) \end{pmatrix} = 0.
$$

(S14)

The solutions for the column vector $[C_{A',\phi'}]$, can be written in matrix form as was performed in the scalar-photon case with the introduction of the variables, $\Omega_{||} = (\omega - \frac{\lambda_i}{2\omega})$, $\Omega_{\perp} = (\omega - \frac{\lambda_\perp}{2\omega})$ and $\Omega_{L} = (\omega - \frac{\lambda_L}{2\omega})$ and $\Omega_{\phi'} = (\omega - \frac{\lambda_\phi}{2\omega})$:

$$
\begin{pmatrix} A_{||}(z) \\ A_{\perp}(z) \\ A_L(z) \\ \phi'(z) \end{pmatrix} = U \begin{pmatrix} e^{-\Omega_{||}z} & 0 & 0 & 0 \\ 0 & e^{-\Omega_{\perp}z} & 0 & 0 \\ 0 & 0 & e^{-\Omega_Lz} & 0 \\ 0 & 0 & 0 & e^{-\Omega_{\phi'}z} \end{pmatrix} U^{-1} \begin{pmatrix} A_{||}(0) \\ A_{\perp}(0) \\ A_L(0) \\ \phi'(0) \end{pmatrix}.
$$

(S15)
The elements of the column vector, \( A(0) \) appearing in equation (S15), are the initial values whose magnitudes are subject to the physical conditions considered at \( z = 0 \). For the boundary conditions for pseudoscalar field, the solution of equation (S15) for \( A_\parallel(\omega, z) \), turns out to be:

\[
A_\parallel(\omega, z) = \left( e^{i\Omega z} \hat{u}_1 \hat{u}_1^* + e^{i\Omega_\perp z} \hat{u}_2 \hat{u}_2^* + e^{i\Omega L z} \hat{u}_3 \hat{u}_3^* + e^{i\Omega \phi z} \hat{u}_4 \hat{u}_4^* \right) A_\parallel(\omega, 0) + \\
\left( e^{i\Omega_\parallel z} \hat{v}_1 \hat{v}_1^* + e^{i\Omega_\perp z} \hat{v}_2 \hat{v}_2^* + e^{i\Omega L z} \hat{v}_3 \hat{v}_3^* + e^{i\Omega \phi z} \hat{v}_4 \hat{v}_4^* \right) A_\perp(\omega, 0) + \\
\left( e^{i\Omega_\parallel z} \hat{w}_1 \hat{w}_1^* + e^{i\Omega_\perp z} \hat{w}_2 \hat{w}_2^* + e^{i\Omega L z} \hat{w}_3 \hat{w}_3^* + e^{i\Omega \phi z} \hat{w}_4 \hat{w}_4^* \right) A_L(\omega, 0),
\]

(S16)

and the component \( A_\perp(\omega, z) \) turns out to be,

\[
A_\perp(\omega, z) = \left( e^{i\Omega_\parallel z} \hat{v}_1 \hat{v}_1^* + e^{i\Omega_\perp z} \hat{v}_2 \hat{v}_2^* + e^{i\Omega L z} \hat{v}_3 \hat{v}_3^* + e^{i\Omega \phi z} \hat{v}_4 \hat{v}_4^* \right) A_\parallel(\omega, 0) + \\
\left( e^{i\Omega_\parallel z} \hat{w}_1 \hat{w}_1^* + e^{i\Omega_\perp z} \hat{w}_2 \hat{w}_2^* + e^{i\Omega L z} \hat{w}_3 \hat{w}_3^* + e^{i\Omega \phi z} \hat{w}_4 \hat{w}_4^* \right) A_L(\omega, 0),
\]

(S17)

### 14.3 Stokes parameters

For \( \phi' \gamma \) mixing, the Stokes parameter \( (I, Q, U \text{ and } V) \) is determined by the set of equations appearing in (6.14 - 6.17) by using (S16) and (S17). The variables introduced in (6.14) for the expression for Stokes parameter \( I \) can be expressed as:

\[
I_\parallel = \left[ 1 - 4|\hat{u}_1 \hat{u}_2| V_{12} \sin^2 \left( \frac{(\Omega_\parallel - \Omega_\perp)z}{2} \right) - 4|\hat{u}_2 \hat{u}_3| V_{23} \sin^2 \left( \frac{(\Omega_\perp - \Omega_L)z}{2} \right) - 4|\hat{u}_3 \hat{u}_4| V_{34} \sin^2 \left( \frac{(\Omega_L - \Omega_\parallel)z}{2} \right) \right].
\]

(S18)

\[
I_\perp = \left[ 1 - 4|\hat{v}_1 \hat{v}_2| V_{12} \sin^2 \left( \frac{(\Omega_\parallel - \Omega_\perp)z}{2} \right) - 4|\hat{v}_2 \hat{v}_3| V_{23} \sin^2 \left( \frac{(\Omega_\perp - \Omega_L)z}{2} \right) - 4|\hat{v}_3 \hat{v}_4| V_{34} \sin^2 \left( \frac{(\Omega_L - \Omega_\parallel)z}{2} \right) \right].
\]

(S19)

\[
I_L = \left[ 4|\hat{w}_1 \hat{w}_2| V_{12} \sin^2 \left( \frac{(\Omega_\parallel - \Omega_\perp)z}{2} \right) + 4|\hat{w}_2 \hat{w}_3| V_{23} \cos^2 \left( \frac{(\Omega_\perp - \Omega_L)z}{2} \right) + 4|\hat{w}_3 \hat{w}_4| V_{34} \cos^2 \left( \frac{(\Omega_L - \Omega_\parallel)z}{2} \right) \right].
\]

(S20)
\[ I_{\|L} = \left[ 2(\hat{u}_2^2|\hat{u}_1\hat{v}_1| - |\hat{v}_2|^2|\hat{u}_1\hat{v}_1| - |\hat{u}_1|^2|\hat{u}_2\hat{v}_2| + |\hat{v}_1|^2|\hat{u}_2\hat{v}_2|) \sin((\Omega_\| - \Omega_\bot)z) \\
+ 2(|\hat{u}_3|^2|\hat{u}_2\hat{v}_2| - |\hat{v}_3|^2|\hat{u}_2\hat{v}_2| - |\hat{u}_2|^2|\hat{u}_3\hat{v}_3| + |\hat{v}_2|^2|\hat{u}_3\hat{v}_3|) \sin((\Omega_\| - \Omega_L)z) \\
+ 2(|\hat{u}_3|^2|\hat{u}_1\hat{v}_1| - |\hat{v}_3|^2|\hat{u}_1\hat{v}_1| - |\hat{u}_1|^2|\hat{u}_3\hat{v}_3| + |\hat{v}_1|^2|\hat{u}_3\hat{v}_3|) \sin((\Omega_\| - \Omega_L)z) \\
+ 2|\hat{u}_3|^2|\hat{u}_1\hat{v}_1|) \sin((\Omega_\| - \Omega_\bot)z) - 2|\hat{u}_4|^2|\hat{u}_2\hat{v}_2|) \sin((\Omega_\| - \Omega_\bot)z) \right]. \] (S21)

\[ I_{\|L} = \left[ 2(|\hat{u}_1|^2|\hat{u}_2\hat{w}_2| - |\hat{v}_1|^2|\hat{u}_2\hat{w}_2| + |\hat{u}_3|^2|\hat{u}_1\hat{w}_1| - |\hat{v}_3|^2|\hat{u}_1\hat{w}_1|) \sin((\Omega_\| - \Omega_\bot)z) \\
- 2(|\hat{u}_2|^2|\hat{u}_3\hat{w}_3| + |\hat{v}_2|^2|\hat{u}_3\hat{w}_3| + |\hat{u}_3|^2|\hat{u}_2\hat{w}_2| + |\hat{v}_3|^2|\hat{u}_2\hat{w}_2|) \sin((\Omega_\| - \Omega_L)z) \\
- 2(|\hat{u}_3|^2|\hat{u}_1\hat{w}_1| + |\hat{v}_3|^2|\hat{u}_1\hat{w}_1| + |\hat{u}_1|^2|\hat{u}_3\hat{w}_3| + |\hat{v}_1|^2|\hat{u}_3\hat{w}_3|) \sin((\Omega_\| - \Omega_L)z) \\
+ 2|\hat{u}_3|^2|\hat{u}_1\hat{w}_1|) \sin((\Omega_\| - \Omega_\bot)z) - 2|\hat{u}_1|^2|\hat{u}_4\hat{w}_4|) \sin((\Omega_\bot - \Omega_\|)z) \\
+ 2|\hat{u}_2|^2|\hat{u}_4\hat{w}_4|) \sin((\Omega_\bot - \Omega_\|)z) \right]. \] (S22)

\[ I_{\bot L} = \left[ 2(|\hat{u}_3|^2|\hat{u}_3\hat{w}_3| - |\hat{u}_1|^2|\hat{u}_3\hat{w}_3| - |\hat{u}_2|^2|\hat{u}_3\hat{w}_3| - |\hat{v}_2|^2|\hat{u}_3\hat{w}_3|) \sin((\Omega_\bot - \Omega_\|)z) \\
+ \sin((\Omega_\bot - \Omega_\|)z) \\
+ 2(|\hat{v}_3|^2|\hat{v}_3\hat{w}_3|) \cos((\Omega_\bot - \Omega_\|)z) \\
- 2(|\hat{v}_1|^2|\hat{v}_4\hat{w}_4|) \cos((\Omega_\| - \Omega_\bot)z) \\
- 2(|\hat{v}_1|^2|\hat{v}_4\hat{w}_4|) \cos((\Omega_\| - \Omega_\bot)z) \\
- 2(|\hat{v}_2|^2|\hat{v}_2\hat{w}_2| + |\hat{v}_2|^2|\hat{v}_2\hat{w}_2| + |\hat{v}_1|^2|\hat{v}_2\hat{w}_2|) \cos((\Omega_\| - \Omega_\bot)z) \\
+ 2(|\hat{v}_2|^2|\hat{v}_3\hat{w}_3| - |\hat{v}_3|^2|\hat{v}_3\hat{w}_3| - |\hat{v}_1|^2|\hat{v}_2\hat{w}_2|) \cos((\Omega_\| - \Omega_\bot)z) \\
+ 2(|\hat{v}_1|^2|\hat{v}_3\hat{w}_3| - |\hat{v}_3|^2|\hat{v}_3\hat{w}_3| - |\hat{v}_1|^2|\hat{v}_2\hat{w}_2|) \cos((\Omega_\| - \Omega_\bot)z) \right]. \] (S23)

Also, the variables introduced in expressing the Stokes parameter \( Q \), are expressed as follows,

\[ Q_\| = \left[ 1 - 4|\hat{u}_1\hat{u}_2|\vec{V}_{12}\sin^2\left(\frac{(\Omega_\| - \Omega_\bot)z}{2}\right) - 4|\hat{u}_2\hat{u}_3|\vec{V}_{23}\sin^2\left(\frac{(\Omega_\| - \Omega_L)z}{2}\right) \\
- 4|\hat{u}_3\hat{u}_1|\vec{V}_{31}\sin^2\left(\frac{(\Omega_\| - \Omega_\bot)z}{2}\right) \right]. \] (S24)

\[ Q_\bot = \left[ 1 + 4|\hat{v}_1\hat{v}_2|\vec{V}_{12}\sin^2\left(\frac{(\Omega_\| - \Omega_\bot)z}{2}\right) + 4|\hat{v}_2\hat{v}_3|\vec{V}_{23}\sin^2\left(\frac{(\Omega_\| - \Omega_L)z}{2}\right) \\
+ 4|\vec{V}_{31}\sin^2\left(\frac{(\Omega_\| - \Omega_\|)z}{2}\right) - 4|\vec{V}_{31}\sin^2\left(\frac{(\Omega_\| - \Omega_\|)z}{2}\right) \\
- 4|\vec{V}_{12}\sin^2\left(\frac{(\Omega_\| - \Omega_\|)z}{2}\right) - 4|\vec{V}_{12}\sin^2\left(\frac{(\Omega_\| - \Omega_\|)z}{2}\right) \right]. \] (S25)
\[ Q_L = \left[ 4|\dot{w}_1\dot{w}_2|\bar{V}_{12}\sin^2\left(\frac{(\Omega_\| - \Omega_\perp)z}{2}\right) + 4|\dot{w}_2\dot{w}_3|\bar{V}_{23}\cos^2\left(\frac{(\Omega_\perp - \Omega_L)z}{2}\right) \right. \]
\[ \left. + 4|\dot{w}_3\dot{w}_1|\bar{V}_{31}\cos^2\left(\frac{(\Omega_L - \Omega_\|)z}{2}\right) - 4|\dot{v}_3\dot{v}_4|\bar{w}_3\dot{w}_4\cos^2\left(\frac{(\Omega_\perp - \Omega_\|)z}{2}\right) \right] \]
\[ - 4|\dot{v}_4\dot{v}_1||\dot{w}_4\dot{w}_1|\sin^2\left(\frac{(\Omega_\perp - \Omega_\|)z}{2}\right) - 4|\dot{v}_2\dot{v}_4||\dot{w}_2\dot{w}_4|\sin^2\left(\frac{(\Omega_\perp - \Omega_\|)z}{2}\right) \right]. \quad (S26) \]

\[ Q_{\|\perp} = \left[ 2(|\dot{u}_2|^2|\dot{u}_1\dot{v}_1| + |\dot{v}_2|^2|\dot{u}_2\dot{v}_2| - |\dot{u}_1|^2|\dot{u}_2\dot{v}_2| - |\dot{v}_1|^2|\dot{u}_2\dot{v}_2| \right. \]
\[ \left. \left(\Omega_\| - \Omega_\perp\right)z \right] \]
\[ + 2(|\dot{u}_3|^2|\dot{u}_2\dot{v}_3| + |\dot{v}_3|^2|\dot{u}_2\dot{v}_3| - |\dot{u}_2|^2|\dot{u}_3\dot{v}_3| - |\dot{v}_2|^2|\dot{u}_3\dot{v}_3| \right) \sin((\Omega_\| - \Omega_L)z) \]
\[ + 2(|\dot{u}_3|^2|\dot{u}_1\dot{v}_3| + |\dot{v}_3|^2|\dot{u}_1\dot{v}_3| - |\dot{u}_1|^2|\dot{u}_3\dot{v}_3| - |\dot{v}_1|^2|\dot{u}_3\dot{v}_3| \right) \sin((\Omega_\perp - \Omega_L)z) \]
\[ + 2|\dot{v}_3|^2|\dot{u}_1\dot{v}_1| |\dot{v}_1\dot{v}_1| \sin((\Omega_\| - \Omega_\perp)z) \]
\[ - 2|\dot{v}_3|^2|\dot{u}_1\dot{v}_3| \sin((\Omega_\| - \Omega_\perp)z) \]. \quad (S27) \]

\[ Q_{\perp\|} = \left[ 2(|\dot{u}_1|^2|\dot{u}_2\dot{v}_2| - |\dot{u}_2|^2|\dot{u}_1\dot{v}_1| - |\dot{u}_1|^2|\dot{v}_2\dot{v}_2| + |\dot{u}_2|^2|\dot{v}_1\dot{v}_1| \right) \sin((\Omega_\| - \Omega_\perp)z) \]
\[ - 2(|\dot{u}_3|^2|\dot{u}_2\dot{v}_3| + |\dot{v}_3|^2|\dot{u}_2\dot{v}_3| - |\dot{u}_2|^2|\dot{u}_3\dot{v}_3| - |\dot{v}_2|^2|\dot{u}_3\dot{v}_3| \right) \sin((\Omega_\| - \Omega_L)z) \]
\[ - 2(|\dot{u}_3|^2|\dot{u}_1\dot{v}_3| + |\dot{v}_3|^2|\dot{u}_1\dot{v}_3| - |\dot{u}_1|^2|\dot{u}_3\dot{v}_3| - |\dot{v}_1|^2|\dot{u}_3\dot{v}_3| \right) \sin((\Omega_\perp - \Omega_L)z) \]
\[ - 2|\dot{v}_3|^2|\dot{u}_1\dot{v}_1| \sin((\Omega_\perp - \Omega_\|)z) + 2|\dot{u}_1|^2|\dot{v}_1\dot{v}_1| \sin((\Omega_\perp - \Omega_\|)z) \]
\[ - 2|\dot{u}_2|^2|\dot{v}_1\dot{v}_1| \sin((\Omega_\| - \Omega_\perp)z) \]. \quad (S28) \]

\[ Q_{\perp\perp} = \left[ 2(|\dot{u}_3|^2|\dot{u}_3\dot{v}_3| - |\dot{u}_1|^2|\dot{u}_1\dot{v}_1| - |\dot{u}_2|^2|\dot{u}_2\dot{v}_2| - |\dot{v}_1|^2|\dot{v}_1\dot{v}_1| \right) \sin((\Omega_\| - \Omega_\perp)z) \]
\[ - |\dot{v}_3|^2|\dot{v}_3\dot{v}_3| + |\dot{v}_3|^2|\dot{u}_4\dot{v}_4|) - 2(|\dot{u}_4|^2|\dot{v}_3\dot{v}_3| - |\dot{v}_3|^2|\dot{u}_4\dot{v}_4|) \cos((\Omega_\| - \Omega_\perp)z) \]
\[ + 2(|\dot{v}_4|^2|\dot{u}_1\dot{v}_1| + |\dot{v}_4|^2|\dot{u}_1\dot{v}_1|) \cos((\Omega_\perp - \Omega_\|)z) \]
\[ + 2(|\dot{v}_4|^2|\dot{u}_4\dot{v}_4| + |\dot{v}_4|^2|\dot{u}_4\dot{v}_4|) \cos((\Omega_\| - \Omega_\perp)z) \]
\[ - 2(|\dot{u}_1|^2|\dot{v}_1\dot{v}_1| - |\dot{v}_1|^2|\dot{v}_1\dot{v}_1| - |\dot{v}_1|^2|\dot{v}_1\dot{v}_1|) \cos((\Omega_\| - \Omega_\perp)z) \]
\[ + 2(|\dot{u}_2|^2|\dot{v}_3\dot{v}_3| + |\dot{v}_3|^2|\dot{u}_2\dot{v}_2| - |\dot{v}_2|^2|\dot{v}_3\dot{v}_3| + |\dot{v}_3|^2|\dot{u}_2\dot{v}_2|) \cos((\Omega_\| - \Omega_\perp)z) \]
\[ + 2(|\dot{u}_3|^2|\dot{u}_3\dot{v}_3| - |\dot{u}_3|^2|\dot{u}_3\dot{v}_3| + |\dot{v}_3|^2|\dot{u}_3\dot{v}_3|) \cos((\Omega_\perp - \Omega_L)z) \]. \quad (S29) \]

where \( \mathbf{V}_{ij} = |\dot{u}_i\dot{u}_j| + |\dot{v}_i\dot{v}_j| \) and \( \bar{\mathbf{V}}_{ij} = |\dot{u}_i\dot{u}_j| - |\dot{v}_i\dot{v}_j| \) (i and j may take value from 1 to 3). In similar way, variables used in the expression of Stokes parameters U and V, written as:

\[ U_\| = 2\left[ |\dot{u}_1\dot{u}_2|(|\dot{u}_1\dot{v}_2| - |\dot{u}_2\dot{v}_1|) \sin((\Omega_\| - \Omega_\perp)z) + |\dot{u}_2\dot{v}_3|(|\dot{u}_2\dot{v}_3| - |\dot{u}_3\dot{v}_2|) \sin((\Omega_\perp - \Omega_L)z) \right. \]
\[ + |\dot{u}_3\dot{v}_1|(|\dot{u}_3\dot{v}_1| - |\dot{u}_1\dot{v}_3|) \sin((\Omega_\| - \Omega_\perp)z) \]
\[ \left. + |\dot{u}_3\dot{v}_3| |\dot{u}_4|^2 \sin((\Omega_\perp - \Omega_\|)z) \right] \]. \quad (B.13) \]

\[ U_\perp = 2\left[ |\dot{v}_1\dot{v}_2|(|\dot{u}_1\dot{v}_2| - |\dot{u}_2\dot{v}_1|) \sin((\Omega_\| - \Omega_\perp)z) + |\dot{v}_2\dot{v}_3|(|\dot{u}_2\dot{v}_3| - |\dot{u}_3\dot{v}_2|) \sin((\Omega_\perp - \Omega_L)z) \right. \]
\[ + |\dot{v}_3\dot{v}_1|(|\dot{u}_3\dot{v}_1| - |\dot{u}_1\dot{v}_3|) \sin((\Omega_\| - \Omega_\perp)z) - |\dot{u}_1\dot{v}_1| |\dot{v}_4|^2 \sin((\Omega_\perp - \Omega_\|)z) \]
\[ + |\dot{u}_3\dot{v}_3| |\dot{u}_4|^2 \sin((\Omega_\perp - \Omega_\|)z) \right]. \quad (S30) \]
\[
U_L = 2\left[|\dot{w}_1|\dot{w}_2|(|\dot{u}_1| \dot{v}_2| - |\dot{u}_2\dot{v}_1|)\sin((\Omega二是 - \Omega_{\perp})z) - |\dot{w}_2\dot{w}_3|(|\dot{u}_2\dot{v}_3| - |\dot{u}_3\dot{v}_3|)\sin((\Omega_{\perp} - \Omega_{\perp})z) \right.
\]
\[
+ |\dot{u}_3\dot{w}_3|(|\dot{u}_3\dot{v}_1| - |\dot{u}_3\dot{v}_3|)\sin((\Omega_{\perp} - \Omega_{\perp})z) + |\dot{u}_1\dot{w}_4|\sin((\Omega_{\perp} - \Omega_{\perp})z) + |\dot{u}_3\dot{w}_4|\sin((\Omega_{\perp} - \Omega_{\perp})z) - |\dot{u}_3\dot{w}_3|\sin((\Omega_{\perp} - \Omega_{\perp})z) \bigg]. 
\]

(S31)

\[
U_{\parallel\perp} = 2\left[|\dot{u}_1|\dot{u}_2|(|\dot{w}_1| \dot{v}_2| + |\dot{w}_2|\dot{v}_1|)^2 \cos((\Omega\parallel - \Omega_{\perp})z) + (|\dot{u}_1|^2|\dot{v}_2|^2 + |\dot{u}_2|\dot{v}_1|)^2 \cos((\Omega_{\perp} - \Omega_{\perp})z) \right.
\]
\[
+ (|\dot{u}_3|^2|\dot{v}_1|^2 + |\dot{u}_1|\dot{v}_3|)^2 \cos((\Omega - \Omega_{\perp})z) + |\dot{u}_1|^2|\dot{v}_4|^2 \cos((\Omega_{\perp} - \Omega_{\perp})z) + |\dot{u}_2|\dot{v}_4|^2 \cos((\Omega_{\perp} - \Omega_{\perp})z) + |\dot{u}_3|^2|\dot{v}_4|^2 \cos((\Omega_{\perp} - \Omega_{\perp})z) - 2|\dot{u}_1\dot{u}_2||\dot{v}_1\dot{v}_2|\cos((\Omega_{\perp} - \Omega_{\perp})z)
\]
\[
- 2|\dot{u}_2\dot{u}_3|\sin((\Omega_{\perp} - \Omega_{\perp})z) - 2|\dot{u}_3\dot{u}_1||\dot{v}_3\dot{v}_1|\cos((\Omega_{\perp} - \Omega_{\perp})z) \bigg].
\]

(S32)

\[
U_{\perp\perp} = 2\left[|\dot{v}_1|\dot{v}_2|(|\dot{u}_1| \dot{v}_2| - |\dot{u}_2\dot{v}_1|)^2 \sin((\Omega_{\perp} - \Omega_{\perp})z) - (|\dot{v}_1|^2|\dot{v}_2|^2 + |\dot{v}_2|\dot{v}_1|)^2 \sin((\Omega_{\perp} - \Omega_{\perp})z) \right.
\]
\[
- |\dot{v}_1|\dot{v}_2||\dot{v}_3\dot{v}_1|\sin((\Omega_{\perp} - \Omega_{\perp})z) - (|\dot{v}_1|^2|\dot{v}_3|^2 + |\dot{v}_3|\dot{v}_1|)^2 \sin((\Omega_{\perp} - \Omega_{\perp})z)
\]
\[
- |\dot{v}_2|\dot{v}_2||\dot{v}_4\dot{v}_3|\sin((\Omega_{\perp} - \Omega_{\perp})z) - |\dot{v}_3|\dot{v}_3|\dot{v}_4^2 \sin((\Omega_{\perp} - \Omega_{\perp})z) \bigg].
\]

(S33)

\[
V_{\perp} = 2\left[|\dot{u}_1|\dot{u}_2|(|\dot{w}_1| \dot{v}_2| + |\dot{w}_2|\dot{v}_1|)^2 + |\dot{u}_3|\dot{u}_2|(|\dot{w}_3| \dot{v}_2| - |\dot{w}_2|\dot{v}_3|)\cos((\Omega_{\perp} - \Omega_{\perp})z) + |\dot{u}_1|\dot{u}_2|(|\dot{w}_1| \dot{v}_2| + |\dot{w}_2|\dot{v}_1|)\cos((\Omega_{\perp} - \Omega_{\perp})z) \right.
\]
\[
+ |\dot{w}_2|\dot{w}_3|(|\dot{w}_3| \dot{v}_3| + |\dot{w}_3|\dot{v}_3|)\cos((\Omega_{\perp} - \Omega_{\perp})z) + |\dot{u}_3|\dot{u}_2|(|\dot{u}_3| \dot{v}_2| + |\dot{u}_2|\dot{v}_3|)\cos((\Omega_{\perp} - \Omega_{\perp})z)
\]
\[
+ |\dot{w}_2|\dot{w}_3|(|\dot{w}_2| \dot{v}_2| + |\dot{w}_2|\dot{v}_2|)\cos((\Omega_{\perp} - \Omega_{\perp})z) + |\dot{u}_3|\dot{u}_2|(|\dot{u}_3| \dot{v}_3| + |\dot{u}_3|\dot{v}_3|)\cos((\Omega_{\perp} - \Omega_{\perp})z) + |\dot{w}_2|\dot{w}_3|\cos((\Omega_{\perp} - \Omega_{\perp})z) + |\dot{w}_2|\dot{w}_3|\cos((\Omega_{\perp} - \Omega_{\perp})z) \bigg].
\]

(S34)

\[
V_{\perp} = 2\left[|\dot{u}_1|\dot{u}_2|(|\dot{w}_1| \dot{v}_2| + |\dot{w}_2|\dot{v}_1|)^2 + |\dot{v}_3|\dot{v}_2|(|\dot{v}_3| \dot{v}_3| + |\dot{v}_3|\dot{v}_3|)\cos((\Omega_{\perp} - \Omega_{\perp})z)
\]
\[
- |\dot{v}_2|\dot{v}_3|(|\dot{v}_2| \dot{v}_3| + |\dot{v}_2|\dot{v}_3|)\cos((\Omega_{\perp} - \Omega_{\perp})z) - |\dot{v}_3|\dot{v}_3|(|\dot{v}_3| \dot{v}_3| + |\dot{v}_3|\dot{v}_3|)\cos((\Omega_{\perp} - \Omega_{\perp})z) + |\dot{v}_2|\dot{v}_4|(|\dot{v}_1| \dot{v}_1| + |\dot{v}_1|\dot{v}_1|)\cos((\Omega_{\perp} - \Omega_{\perp})z) - \dot{v}_3\dot{v}_3|\dot{v}_4^2 \cos((\Omega_{\perp} - \Omega_{\perp})z) \bigg].
\]
\[ V_{\perp} = 2 \left[ (|\hat{u}_2|^2|\hat{v}_1|^2 - |\hat{u}_1|^2|\hat{v}_2|^2) \sin((\Omega_\parallel - \Omega_\perp)z) + (|\hat{u}_3|^2|\hat{v}_2|^2 - |\hat{u}_2|^2|\hat{v}_3|^2) \sin((\Omega_\perp - \Omega_L)z) + |\hat{u}_1|^2|\hat{v}_1|^2 \sin((\Omega_{\parallel} - \Omega_{\perp})z) \right]. \]

\[ V_{\parallel} = 2 \left[ |\hat{u}_1| |\hat{v}_2| (|\hat{w}_2\hat{v}_1| - |\hat{w}_1\hat{v}_2|) \sin((\Omega_\parallel - \Omega_\perp)z) - |\hat{u}_2| |\hat{v}_1| (|\hat{w}_3\hat{v}_2| + |\hat{w}_2\hat{v}_3|) \sin((\Omega_\perp - \Omega_L)z) \right] - (|\hat{u}_3|^2|\hat{v}_1^2| + |\hat{u}_1|^2|\hat{v}_3^2|) \sin((\Omega_\parallel - \Omega_\perp)z). \]

\[ V_{\perp L} = 2 \left[ |\hat{u}_2|^2|\hat{v}_3| - 2|\hat{v}_2|^2|\hat{u}_2\hat{w}_2| - 2|\hat{v}_1|^2|\hat{u}_1\hat{w}_1| - (|\hat{v}_2|^2|\hat{u}_1\hat{w}_1| + |\hat{v}_1|^2|\hat{u}_2\hat{w}_2|) \cos((\Omega_\parallel - \Omega_\perp)z) \right] - (|\hat{v}_3|^2|\hat{u}_2\hat{w}_2| - |\hat{v}_2|^2|\hat{u}_3^2|) \cos((\Omega_{\parallel} - \Omega_\perp)z) + (|\hat{v}_1|^2|\hat{u}_3^2| - |\hat{v}_1|^2|\hat{u}_2\hat{w}_2|) \cos((\Omega_\perp - \Omega_L)z) \right] - (|\hat{v}_1|^2|\hat{u}_2\hat{w}_2| + |\hat{v}_2|^2|\hat{u}_3^2|) \cos((\Omega_{\parallel} - \Omega_\perp)z) + (|\hat{v}_2|^2|\hat{u}_3^2| - |\hat{v}_3|^2|\hat{u}_2\hat{w}_2|) \cos((\Omega_\perp - \Omega_L)z) \right] - (|\hat{v}_3|^2|\hat{u}_2\hat{w}_2| - |\hat{v}_2|^2|\hat{u}_3^2|) \cos((\Omega_{\parallel} - \Omega_\perp)z) + (|\hat{v}_2|^2|\hat{u}_3^2| - |\hat{v}_3|^2|\hat{u}_2\hat{w}_2|) \cos((\Omega_\perp - \Omega_L)z) \right] - (|\hat{v}_2|^2|\hat{u}_3^2| - |\hat{v}_3|^2|\hat{u}_2\hat{w}_2|) \cos((\Omega_{\parallel} - \Omega_\perp)z) + (|\hat{v}_3|^2|\hat{u}_2\hat{w}_2| - |\hat{v}_2|^2|\hat{u}_3^2|) \cos((\Omega_\perp - \Omega_L)z) \right]. \]
The oscillating behaviour of the Stokes parameters and the observables ($\chi_{\phi_i}$ and $\psi_{\phi_i}$) as a function of energy is evident from the plots displayed in figures [S7-S12]. None of the current space based observations confirm the same and we believe the proposed observations would not be able to detect the same. The chief reason behind this being the space based detectors are mostly sensitive over an energy band not a line.

So, in the spirit of the analysis of the observational X-ray polarization data, we divide initially the window of the energy band under consideration i.e., $1 - 100$ KeV into 10000 intervals and evaluated the average of the parameters $I, Q$ and $U$ to get $\bar{I}, \bar{Q}$ and $\bar{U}$ and hence the average of degree of linear polarization $P_{L\phi_i}$ to get the first set of samples. This procedure was followed by increasing $N$ to $N'$ such that $N' = N + \Delta N$, when $\Delta N = 100$. We continued this process until $N' = N + n\Delta N$ when $n = 400$. Thus we generated a

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5 The word symmetries here mean the discrete and continuous symmetries, internal or space time (those following from the actions of the generators of the symmetry group: constituting the set of orthogonal proper Lorentz transformations (OPLT)) that the associated dynamical and background-fields respect
samples of 401 data points for $P_{L\phi_i}$.

These samples were further distributed on frequency distribution plots for both the systems ($\gamma\phi$, $\gamma\phi'$) by choosing the appropriate bin size along the x-axis. These frequency distributions were further plotted in fig.[S13] for scalar photon (in the left panel) as well as pseudoscalar photon (in the right panel) systems.

Assuming the obtained frequency distributions of the samples as normal distributions, we have estimated the (mean) average value and the variance of the of the $P_{L\phi_i}$ obtained in the energy interval of 1 – 100 KeV. The mean of $P_{L\phi_i}$ for $\gamma\phi$ and $\gamma\phi'$ system turns out to be $\mu_\phi = 0.414$ and $\mu_\phi' = 0.215$ and the corresponding variance turns out to be $\sigma^2_\phi = 0.275 \times 10^{-6}$ and $\sigma^2_\phi' = 0.264 \times 10^{-5}$ respectively.

Statistically, it means in the one sigma level for $\gamma\phi$ system the range of $P_{L\phi_i}$ lies between $0.414 - 0.275 \times 10^{-6}$ to $0.414 + 0.275 \times 10^{-6}$ and for $\gamma\phi'$ system it is $0.215 + 0.264 \times 10^{-5}$ to $0.215 + 0.264 \times 10^{-5}$. As can be seen from the range that with increase in the numbers of dof the range for the confidence level increases by the factor of 10. The C.L analysis
for other interaction parameters should be performed for designing the detectors for space borne experiments.

15.2 MDP

Many of the astrophysical polarimeters use minimum detectable polarization (MDP) to define detector efficiency. The probability amplitude of polarization that has 1% chance of being detected is called MDP. Usually the expression of the same is given by,

\[
MDP = \frac{4.29}{\sqrt{N\mu}}
\]  

(S1)

when \( \mu \) is the modulation factor that can vary between 0 and 1 and \( N \) is the sample size. Instruments like APEX has MDP is \( \sim 1\% \) at 5.2 KeV [66]. The MDP in terms of mean \( \mu_{\phi_i} \) and \( \sigma_{\phi_i} \) of the observed data of \( P_{\phi_i} \) when background is zero and modulation factor
is one, can be cast as:

$$ MDP_{\phi_i} = \mu_{\phi_i} + \frac{4.29}{\sqrt{2}} \sigma_{\phi_i}. $$

In our case of investigation, we have found that $MDP_{\phi} > MDP_{\phi'}$ for the parameters range under consideration in this work. Thus in the same parameter space the increase in the number of degrees of freedom of the system, the detectability of $P_{L\phi}$ seems to be reduced. This information should be considered while searching for ALPs in compact star models using observed data.