NEW EXPLICIT LORENTZIAN EINSTEIN-WEYL STRUCTURES
IN 3-DIMENSIONS

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ABSTRACT. On a 3D manifold, a Weyl geometry consists of pairs \((g, A)\) = (metric, 1-form) modulo gauge \(\hat{g} = e^{2\varphi} g, \hat{A} = A + d\varphi\). In 1943, Cartan showed that every solution to the Einstein-Weyl equations \(R_{(\mu\nu)} - \frac{1}{2} R g_{\mu\nu} = 0\) comes from an appropriate 3D leaf space quotient of a 7D connection bundle associated with a 3rd order ODE \(y''' = H(x, y, y', y'')\) modulo point transformations, provided 2 among 3 primary point invariants vanish:

\[ Wünschmann(H) \equiv 0 \equiv \text{Cartan}(H). \]

We find that point equivalence of a single PDE \(\partial_{\mu} F = F_{\mu} + z_{\mu} F_{\nu} \equiv 0 \) leads to an isomorphic 7D Cartan bundle and connection.

Then magically, the (complicated) equation \(Wünschmann(H) \equiv 0 \equiv \text{Cartan}(H)\) becomes:

\[ 0 \equiv \text{Monge}(F) := 9 F_{pp} F_{pppp} - 45 F_{ppp} F_{ppppp} + 40 F_{ppppp} \quad (p := z_k), \]

whose solutions are just conics in the \((p, F)\)-plane. As an Ansatz, we take:

\[ F(x, y, z, p) := \frac{\alpha(y)(p-2)(y-z)(y-z)(p+2)(y-z)(p+2)(y-z)(p+2)}{\lambda(y)(y+2)(y+2)(y+2)(y+2)} \]

with 9 arbitrary functions \(\alpha, \ldots, \psi\) of \(y\). This \(F\) satisfies \(DF \equiv 0 \equiv \text{Monge}(F)\), and we show that the condition \(\text{Cartan}(H) \equiv 0 \) passes to a certain \(K(F) \equiv 0\) which holds for any choice of \(\alpha(y), \ldots, \psi(y)\). Descending to the leaf space quotient, we gain \(\infty\)-dimensional functionally parametrized and explicit families of Einstein-Weyl structures \(\{g, A\}\) in 3D.

These structures are nontrivial in the sense that \(dA \neq 0\) and \(\text{Cotton}(\{g\}) \neq 0\).

1. INTRODUCTION

On an \(n\)-manifold \(M\), a Weyl geometry is a pair \(\{g, A\}\) of a signature \((k, n-k)\) pseudo-Riemannian metric tensor \(g\) together with a 1-form \(A\) modulo \(\hat{A} = A + d\varphi\), where \(\varphi: M \to \mathbb{R}\) is any function. As in Riemannian geometry, a symmetric Ricci tensor \(R_{(\mu\nu)}\) with scalar curvature \(R\) can be defined (see [2, 11, 10] or Section 2).

The Einstein-Weyl equations in vacuum:

\[ R_{(\mu\nu)} - \frac{1}{n} R g_{\mu\nu} = 0 \quad (1 \leq \mu, \nu \leq n), \]

which depend only on the class \(\{(g, A)\}\), have raised interest, specially in dimension \(n = 3\). We find various functionally parametrized explicit families of solutions. On \(\mathbb{R}^3 \ni (x, y, z)\), take for instance 5 free arbitrary functions \(b, c, k, l, m\) of \(y\) with derivatives \(b', b''\), \(k', k''\).

Theorem 1.1. All pairs \((g, A)\) with \(g\) Lorentzian of signature \((2, 1)\):

\[ g := (k+bz)^2 \ dx^2 + x^2 \ (l^2 - cm) \ dy^2 + x^2 b^2 \ dz^2 + 2x \ (ckz - blz + kl - bm) \ dx \ dy - 2xb \ (k + b) \ dx \ dz - 2x^2 \ (ck - bl) \ dy \ dz, \]

\[ A := \frac{-ck + bl + b' k - bk'}{x \ (ck^2 - 2bkl + b'm)} \ (xb \ dz - (k + b) \ dx) + \frac{bl^2 - cbm - b'kl + bb'm + ckk' - bk'l}{ck^2 - 2bkl + b'm} \ dy, \]

satisfy equations \(1.1\), hence define a Lorentzian Einstein-Weyl structure on \(\mathbb{R}^3\).

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Moreover, all such examples are generically conformally non-flat, and each of the 5 independent components of the Cotton tensor of the underlying conformal structure \((\cal M, [g])\) is not identically zero.

Previous examples ([4] [13] [14] [21] [19] [22] [6] [5] [18] [11] [15] [10]) depended on a finite number of constants and were often conformally flat. We discover in fact even more general explicit families of solutions depending on 9 free arbitrary functions of 1 variable y.

Our main approach is to study point equivalences of a single PDE of the form:

\[ z_y = F(x, y, z, z_x), \]

with unknown \( z = z(x, y) \). From para-CR geometry ([16] [12]), an integrability condition is required, namely:

\[ DF := F_x + z_x F_z \equiv 0. \]

To exclude trivial PDEs, another point invariant condition must be assumed:

\[ F_{pp} \neq 0 \] (abbreviate \( p := z_x \)).

In Theorem 5.1, we construct a 7-dimensional Cartan bundle/connection \( P_7 \to J_4 \ni (x, y, z, p) \) canonically associated to point equivalences of such PDEs \( z_y = F(x, y, z, z_x) \), we determine a canonical coframe \( \{ \theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3 \} \) on \( P_7 \), and we find that its structure equations \([4] [4]\) incorporate exactly 3 primary invariants, named \( A_1, B_1, C_1 \).

Quite unexpectedly, we realize that these structure equations are exactly the same as the structure equations of the canonical 7-dimensional Cartan bundle/connection associated with point equivalences of 3\(^{rd}\) order ODEs \( y''' = H(x, y, y', y'') \). Furthermore, it is known that quite similarly, 3 primary differential invariants govern such geometries. Two among them are: the \textit{Wünschmann invariant} \( W(H) \) and the \textit{Cartan invariant} \( C(H) \). Since Cartan 1943, it is also known ([4] [13] [9] [11] [10]) that all solutions to the Einstein-Weyl structure equations \([1.1]\) can be obtained from ODEs satisfying \( W(H) \equiv 0 \equiv C(H) \). Translating what is known for ODEs or performing computations from scratch, we will set up and state Cartan’s construction \textit{from the PDE side, see Theorem 5.2.}

But from the ODE side unfortunately, it is quite difficult to solve Wünschmann’s nonlinear equation incorporating 25 differential monomials:

\[ 0 \equiv W(H) := -18 q H q H y q + 9 p H y q + 18 q H H y q q + 9 q H p H y q - 18 p q H y H y q - 18 p H H y q q - 18 H H y q q - 18 H q H y q - 18 H H H y q q - 18 H y q H y q - 18 H H H y H y q - 18 H q H y q - 18 H H H y H H y q - 27 H H y + 4 H^4 q + 9 p^2 H y q q + 27 H y p + 9 q H y q + q^2 H p p q - 27 H q p - 18 H H p q + 9 H^2 H y q q + 54 H y. \]

This inspired us to try to work on the PDE side \( z_y = F(x, y, z, z_x) \), instead of the ODE side. Then \textit{magically,} \( W(H) \equiv 0 \) transforms into the much simpler classical invariant equation:

\[ 0 \equiv \text{Monge}(F) := 9 F^2_{pp} F_{ppppp} - 45 F_{pp} F_{ppp} F_{pppp} + 40 F^3_{ppp}, \]

When \( F_{pp} \neq 0 \), it is known that \( M(F) \equiv 0 \) holds if and only if there exist functions \( A, B, C, K, L, M \) of \( x, y, z \) such that:

\[ 0 \equiv A F^2 + 2 B F p + C p^2 + 2 K F + 2 L p + M. \]

Assuming \( A := 0 \), we then solve the problem completely.
Proposition 1.2. The general solution \( F = F(x, y, z, p) \) to:
\[
0 \equiv F_x + p F_z, \\
0 \equiv 0 + 2B F_p + c p^2 + 2k F + 2\tau p + M,
\]
is:
\[
F = \frac{\alpha(y)(z - xp)^2 + \beta(y)(z - xp)p + \gamma(y)(z - xp) + \delta(y)p^2 + \epsilon(y)p + \zeta(y)}{\lambda(y)(z - xp) + \mu(y)p + \nu(y)},
\]
with 9 arbitrary functions \( \alpha, \beta, \gamma, \delta, \epsilon, \zeta, \lambda, \mu, \nu \) of \( y \).

Of course, to the Cartan invariant \( C(H) \) from the ODE side there corresponds from the PDE side a certain invariant we name \( K(F) \): its expression appears in Theorem 5.1. Miraculously, then, a direct calculation shows that no further constraint is imposed.

Proposition 1.3. For any choice of \( \alpha(y), \beta(y), \gamma(y), \delta(y), \epsilon(y), \zeta(y), \lambda(y), \mu(y), \nu(y) \), the second condition:
\[
K(F_{\alpha,\ldots,\nu}) = 0
\]
for obtaining Weyl pairs \( [(g, A)] \) satisfying the Einstein-Weyl field equations (1.1) holds automatically.

We then get — quite a bit long — formulas for pairs \( [(g, A)] \) expressed explicitly in terms of \( \alpha, \beta, \gamma, \delta, \epsilon, \zeta, \lambda, \mu, \nu \). The subfamily for which \( \beta = 0, \delta = 0, \epsilon = 0, \mu = 0 \) corresponds (with different notations) to Theorem 1.1.

Theorem 1.4. Same conclusion as in Theorem 1.1 with:
\[
g := \tau^1 \otimes \tau^2 + \tau^2 \otimes \tau^1 + \tau^3 \otimes \tau^3,
\]
\[
A := \tau^3 \frac{1}{2} \Pi \left( \gamma \lambda x - \gamma \mu + \chi \lambda \nu' + \beta \lambda z + \lambda \mu' z - 2 \alpha \mu z - \lambda' \mu z - \mu' \nu' - \chi \lambda' \nu - 2 \chi \alpha \nu + \beta \nu + \mu' \nu \right),
\]
with the coframe:
\[
\tau^1 := dx + \frac{dy}{\chi \lambda - \mu} (x \beta - \gamma - x^2 \alpha),
\]
\[
\tau^2 := \frac{2}{\chi \lambda - \mu} dy, \\
\tau^3 := (- \lambda z - \nu) dx + \frac{1}{\chi \lambda - \mu} dy \left( - \epsilon \mu + 2 \chi^2 \alpha \nu + \chi \gamma \mu - 2 \chi \beta \nu - \beta \mu z + 2 \delta \lambda z + 2 \chi \alpha \mu z + \chi \epsilon \lambda + 2 \chi \delta - x^2 \gamma \lambda - \chi \beta \lambda \nu \right),
\]
and the function:
\[
\Pi := x^2 \zeta \lambda^2 + \alpha \mu z^2 + 2x \alpha \mu \nu z + x^2 \alpha \nu^2 - \beta \lambda \mu z^2 - x \beta \lambda \nu z + \delta \lambda^2 z^2 + x \epsilon \lambda^2 z - 2x \zeta \lambda \mu - \beta \mu \nu z - x \beta \nu^2 + 2 \delta \lambda \nu z - \epsilon \lambda \mu z + \chi \epsilon \lambda z - x^2 \gamma \lambda \nu + \zeta \mu z + \delta \nu^2 - \epsilon \mu \nu + \gamma \mu^2 z + \chi \gamma \nu,
\]
again with \( dA \neq 0 \) and Cotton([g]) \( \neq 0 \).

At the end, we also present other families of functionally parametrized solutions, when \( A \neq 0 \).
2. WEYL GEOMETRY: A SUMMARY

In Einstein’s theory, gravity is described in terms of a (pseudo-)riemannian metric \( g \) called the gravitational potential. In Maxwell’s theory, the electromagnetic field is described in terms of a 1-form \( A \) called the Maxwell potential.

In his attempt Raum, Zeit, Materie \[23\] of unifying gravitation and electromagnetism, Weyl was inspired to introduce the synthetic geometric structure on any \( n \)-dimensional manifold \( M \) which consists of classes of such pairs \( [(g, A)] \) under the equivalence relation:

\[
(g, A) \sim (\tilde{g}, \tilde{A})
\]

holding by definition if and only if there exists a function \( \varphi : M \rightarrow \mathbb{R} \) such that:

1. \( \tilde{g} = e^{2\varphi} g \);
2. \( \tilde{A} = A + d\varphi \).

Clearly, the electromagnetic field \( F := dA \) depends only on the class. The signature \( (k, n-k) \) of \( g \) with \( p+q = n \) can be arbitrary. Einstein-conformal structures are a special class of Weyl structures, corresponding to the choice of a closed — hence locally exact — 1-form \( A \).

Inspired by Levi-Civita, Weyl established that to such a Weyl structure \( (M, [(g, A)]) \) is associated a unique connection \( D \) on \( TM \) satisfying:

(A) \( D \) has no torsion;

(B) \( Dg = 2A \, g \) for any representative \( (g, A) \) of the class \( [(g, A)] \).

In any (local) coframe \( \omega^\mu, \mu = 1, \ldots, n \), for the cotangent bundle \( T^*M \) in which \( g = g^{\mu\nu} \omega^\mu \otimes \omega^\nu \), the connection 1-forms \( \Gamma^\mu_\nu \) of \( D \), or equivalently the \( \Gamma^\mu_\nu := g^{\mu\rho} \Gamma^\rho_\nu \), are indeed uniquely defined from the more explicit conditions:

(A') \( d\omega^\mu + \Gamma^\mu_\nu \wedge \omega^\nu = 0 \);

(B') \( Dg^{\mu\nu} := dg^{\mu\nu} - \Gamma^\mu_\nu - \Gamma^\nu_\mu = 2A \, g^{\mu\nu} \).

Then the curvature of this Weyl connection identifies with the collection of \( n^2 \) curvature 2-forms:

\[ \Omega^\mu_\nu := d\Gamma^\mu_\nu + \Gamma^\mu_\rho \wedge \Gamma^\rho_\nu, \]

which produce the curvature tensor \( R^\mu_{\nu\rho\sigma} \) by expanding in the given coframe \( \omega^\mu \):

\[ \Omega^\mu_\nu = \frac{1}{2} R^\mu_{\nu\rho\sigma} \, \omega^\rho \wedge \omega^\sigma. \]

It turns out that \( R^\mu_{\nu\rho\sigma} \) is a tensor density, which means in particular that its vanishing is independent of the choice of a representative \( (g, A) \), and hence as such, serves as a starting point for all invariants of a Weyl geometry \( (M, [(g, A)]) \), produced by covariant differentiation.

Other invariant objects are:

- the (Weyl-)Ricci tensor \( R_{\mu\nu} := R^\rho_{\mu\rho\nu} \);
- its symmetric part \( R_{[\mu\nu]} := \frac{1}{2} (R_{\mu\nu} + R_{\nu\mu}) \);
- its antisymmetric part \( R_{\{\mu\nu\}} := \frac{1}{2} (R_{\mu\nu} - R_{\nu\mu}) \).

In particular, an appropriately contracted Bianchi identity shows that in 3-dimensions:

\[ R_{[\mu\nu]} = -\frac{3}{2} F_{\mu\nu}, \]

where \( F = dA = \frac{1}{2} F_{\mu\nu} \omega^\mu \wedge \omega^\nu. \)
In [4], Élie Cartan proposed dynamical Einstein equations for a Weyl geometry \((M, ([g, A]])\) postulating that the trace-free part of the symmetric Ricci tensor vanishes:

\[ R_{(\mu \nu)} - \frac{1}{n} R g_{\mu \nu} = 0, \]

where \(R := g^{\mu \nu} R_{\mu \nu}\), with \(g^{\mu \rho} g_{\rho \nu} = \delta_{\mu \nu}\) and \(n = \dim M\).

These equations (2.1) are called Einstein-Weyl equations, and a Weyl geometry satisfying (2.1) is called an Einstein-Weyl structure. The reason for this name is as follows. Since a Weyl structure \((M, [g, A])\) with vanishing \(F = dA \equiv 0\) is equivalent to a plain (pseudo-)conformal structure \((M, [g])\) and since the Weyl connection then reduces to the Levi-Civita connection, these equations (2.1) are a natural generalization of Einstein’s field equations. According to Weyl’s approach, a gravity potential \(g\) is thereby coupled with an electromagnetic field \(F = dA\).

3. CARTAN’S SOLUTION TO THE EINSTEIN-WEYL VACUUM EQUATIONS

In [3], Cartan gave a geometric description of all solutions to the Einstein-Weyl equations (2.1) in 3-dimensions. In particular, he showed that there is a one-to-one correspondence between 3rd-order ODEs \(y''' = H(x, y, y', y'')\) considered modulo point transformations of variables which satisfy certain two point-invariant conditions:

(Wünschmann) \( W(H) \equiv 0 \)

(Cartan) \( C(H) \equiv 0 \)

and 3-dimensional Einstein-Weyl structures with Lorentzian metrics \(g\) of signature \((2, 1)\). Abbreviating \(p := y', q := y''\), in terms of the total differentiation operator:

\[ D := \partial_x + p \partial_y + q \partial_p + H \partial_q, \]

their explicit expressions are:

\[ W := 9 D D H_q - 27 D H_p - 18 H_q D H_q + 18 H_q H_p + 4 H^3_q + 54 H_q, \]

(3.1)

\[ C := 18 H_{qq} D H_q - 12 H_{qq} H^2_q - 54 H_{qq} H_p + 36 H_{pq} H_q - 108 H_{yq} + 54 H_{pp}. \]

(3.2)

Although Cartan’s geometric arguments (4) offer, in the Lorentzian setting, a complete — but abstract — understanding of the space of all solutions of the Einstein-Weyl equations (2.1), it is quite difficult to find explicit solutions to the Wünschmann-Cartan equations \(0 \equiv W(H) \equiv C(H)\), which would provide workable formulas for such Einstein-Weyl structures.

Some particular solutions are known, e.g.:

\[ H = \frac{3 q^2}{2 q}, \quad H = \frac{3 q^2 p}{p^2 + 1}, \quad H = q^{3/2}, \quad H = \alpha \left(\frac{2 q y - p^2}{y^2}\right)^{3/2} \quad (\alpha \in \mathbb{R}), \]

or the ‘horrible’:

\[ H = \frac{pq (-12 + 3pq - 8\sqrt{T-pq}) + 8(1 + \sqrt{T-pq})}{p^3}. \]

They were all obtained by rather \textit{ad hoc} methods.

In fact, the main difficulty in getting a systematic approach to finding the solutions is an annoying nonlinearity of the Wünschmann condition \(W \equiv 0\).
4. Third-Order ODEs Modulo Point Transformations of Variables

It was Cartan ([3]) who solved the equivalence problem for 3rd order ODEs considered modulo point transformations. Nowadays, the result may be stated more elegantly in terms of a certain Cartan connection ([11, 10]), as follows.

To any 3rd order ODE:

\[
y''' = H(x, y, y', y'')
\]  

(4.1)

one associates a contact-like coframe on the space \( J_4 \ni (x, y, p, q) \) of 2-jets of graphs \( x \mapsto y(x) \):

\[
\begin{align*}
\omega^1 & := dy - p \, dx, \\
\omega^2 & := dx, \\
\omega^3 & := dp - q \, dx, \\
\omega^4 & := dq - H(x, y, p, q) \, dx.
\end{align*}
\]  

(4.2)

It follows that if a 3rd order ODE (4.1) undergoes a point transformation of variables:

\[
(x, y) \mapsto (\bar{x}, \bar{y}) = (\bar{x}(x, y), \bar{y}(x, y)),
\]

then the 1-forms \((\omega^1, \omega^2, \omega^3, \omega^4)\) transform as:

\[
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4
\end{pmatrix}
\mapsto
\begin{pmatrix}
u_1 & 0 & 0 & 0 \\
u_2 & u_3 & 0 & 0 \\
u_4 & 0 & u_5 & 0 \\
u_6 & 0 & u_7 & u_8
\end{pmatrix}
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4
\end{pmatrix} =:
\begin{pmatrix}
\theta^1 \\
\theta^2 \\
\theta^3 \\
\theta^4
\end{pmatrix},
\]  

(4.3)

where the \(u_i\) are certain functions on \(J_4\).

Actually, Cartan assures us that the entire equivalence problem for 3rd order ODEs considered modulo point transformations of variables is the same as the equivalence problem for 1-forms (4.2), considered modulo transformations (4.3). There is a unique way of reducing these eight group parameters \(u_i\) to only three \(u_3, u_5, u_7\), the other ones being expressed in terms of them. This is achieved by forcing the exterior differentials of the \(\theta^\mu\)'s to satisfy the EDS (4.4) below.

**Theorem 4.1.** [3, 11, 10] A 3rd order ODE \(y''' = H(x, y, y', y'')\) with its associated 1-forms:

\[
\begin{align*}
\omega^1 & := dy - p \, dx, & \omega^2 & := dx, & \omega^3 & := dp - q \, dx, & \omega^4 & := dq - H(x, y, p, q) \, dx,
\end{align*}
\]

uniquely defines a 7-dimensional fiber bundle \(P_7 \rightarrow J_4\) over the space of second jets \(J_4 \ni (x, y, p, q)\) and a unique coframe \(\{\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3\}\) on \(P_7\) enjoying structure
three primary invariants vanish identically:

\[ M = \Omega_1 \wedge \Omega_2 \wedge \Omega_3. \]

between the corresponding bundles \( P_7 \rightarrow J_4 \) and \( \overline{P}_7 \rightarrow J_4 \) satisfying:

\[ \Phi^* \overline{\Omega}^\mu = \Omega_i \quad (\mu = 1, 2, 3, 4; \ i = 1, 2, 3). \]

Exactly 3 (boxed) invariants are primary: \( A_1, B_1, C_1 \), while others express in terms of them and their covariant derivatives. Point equivalence to \( \overline{y}'''' = 0 \) is characterized by \( 0 = A_1 = B_1 = C_1 \). Two relevant explicit expressions are:

(W in (3.1)) \[ A_1 = \frac{1}{3^4} \frac{u^1}{u_1} W, \]

(C in (3.2)) \[ C_1 = \frac{1}{3^4} \frac{u^2}{u_1} \left( C + \frac{1}{27} W^q \right). \]

The seven 1-forms \( (\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3) \) set up a Cartan connection \( \tilde{\omega} \) on \( P_7 \) via:

\[
\tilde{\omega} := \begin{pmatrix}
\frac{1}{2} \Omega_1 & \frac{1}{2} \Omega_2 & 0 & 0 \\
-\theta^2 & \Omega_3 - \frac{1}{2} \Omega_1 & 0 & 0 \\
\theta^3 & -\theta^4 & \frac{1}{2} \Omega_1 - \frac{1}{2} \Omega_3 & -\frac{1}{2} \Omega_2 \\
2 \theta^1 & \theta^3 & & \\
\end{pmatrix},
\]

and the structure equations (4.4) are just the equations for the curvature \( \tilde{K} \) of this connection:

\[ d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega} =: \tilde{K}. \]

Now, the structure equations (4.4) guarantee that the bundle \( P_7 \) is foliated by a 4-dimensional distribution annihilating the three 1-forms \( (\theta^1, \theta^2, \theta^3) \), and that the leaf space \( M_3 \) of this foliation is equipped with a natural Weyl geometry, if and only if two among three primary invariants vanish identically:

\[ 0 = A_1(H) = C_1(H). \]
A representative \((g, A)\) of the concerned Weyl class \([[(g, A)]]\) on \(M_3\) has then the signature \((2, 1)\) symmetric bilinear form:

\[ g := \theta^3 \otimes \theta^3 + \theta^1 \otimes \theta^4 + \theta^4 \otimes \theta^1, \]

which is obtained as the determinant of the lower-left \(2 \times 2\) submatrix of the connection matrix \(\hat{\omega}\), while the 1-form is defined as:

\[ A := \Omega^3. \]

It is thanks to the hypothesis \(A_1 \equiv 0 \equiv C_1\) that \(g\) and \(A\), originally defined on \(P_7\), descend on \(M_3\).

Furthermore, it is the result of Cartan in \([4]\) that any such Weyl geometry \([[(g, A)]]\) defined on such a leaf space \(M_3\) is automatically Einstein-Weyl!

We stress that given \(H = H(x, y, p, q)\) satisfying \(A_1 \equiv 0 \equiv C_1\), or equivalently:

\[ W(H) \equiv 0 \equiv C(H), \]

one can in principle set up explicit formulae for the corresponding forms \(\theta^1, \theta^3, \theta^4, \Omega_3\) on \(P_7\), and this in turn can provide explicit formulae for \((g, A)\) on \(M_3\). However, one substantial obstacle is the:

**Question 4.2.** How to solve \(W(H) \equiv 0 \equiv C(H)\)?

5. PDE ON THE PLANE \(z_y = F(x, y, z, z_x)\) MODULO POINT TRANSFORMATIONS

In \([12]\), it was shown that the equivalence problem for 3rd-order ODEs considered modulo point transformations of variables can be embedded into an equivalence problem for 4-dimensional para-CR structures of type \((1, 1, 2)\), cf. also \([17]\). This thus suggests us a new approach here to constructing Lorentzian Einstein-Weyl structures via para-CR structures of type \((1, 1, 2)\).

Let us therefore associate a para-CR structure with PDEs on the plane of the form:

\[ z_y = F(x, y, z, z_x), \]

for an unknown function \(z = z(x, y)\). Using the abbreviation \(z_x =: p\), we will consider such PDEs modulo point transformations of variables:

\[ (x, y, z) \mapsto (\bar{x}, \bar{y}, \bar{z}) = (\bar{x}(x, y, z), \bar{y}(x, y, z), \bar{z}(x, y, z)). \]

This conducts to an equivalence problem for the four 1-forms:

\[
\begin{cases}
\omega_1^0 := dz - p \, dx - F(x, y, z, p) \, dy,
\omega_2^0 := dp,
\omega_3^0 := dx,
\omega_4^0 := dy,
\end{cases}
\]

given up to transformations:

\[
\begin{pmatrix}
\omega_1^0 \\
\omega_2^0 \\
\omega_3^0 \\
\omega_4^0
\end{pmatrix} \mapsto \begin{pmatrix}
u_1 & 0 & 0 & 0 \\
u_2 & u_3 & 0 & 0 \\
u_4 & 0 & u_6 & 0 \\
u_7 & u_8 & u_9 & 0
\end{pmatrix}
\begin{pmatrix}
\omega_1^0 \\
\omega_2^0 \\
\omega_3^0 \\
\omega_4^0
\end{pmatrix}.
\]

Within this coframe \(\{\omega_1^0, \omega_2^0, \omega_3^0, \omega_4^0\}\), in terms of the two operators:

\[ D := \partial_x + p \, \partial_z \quad \text{and} \quad \Delta := \partial_y + F \partial_z, \]
The exterior differential of any function $F = F(x, y, z, p)$ rewrites as:

$$\text{d}F = F_z \omega^1_0 + F_p \omega^2_0 + \text{DF} \omega^3_0 + \Delta F \omega^4_0.$$ 

The only nontrivial integrability condition for such an equivalence problem to constitute a true para-CR structure of type $(1, 1, 2)$ comes from:

$$0 = \text{d} \omega^1_0 \wedge \omega^1_0 \wedge \omega^3_0 = -\text{DF} \omega^1_0 \wedge \omega^2_0 \wedge \omega^3_0 \wedge \omega^4_0,$$

hence is:

$$0 \equiv \text{DF} = F_x + p F_z.$$ 

From now on, we will only consider PDEs $z_y = F(x, y, z, z_x)$ satisfying $\text{DF} \equiv 0$. Furthermore, we will also assume that another point-invariant condition holds:

$$0 \neq F_{pp} \quad \text{(everywhere)}.$$ 

Cartan’s process conducts to choose more convenient representatives of these forms:

$$\omega^1 := \omega^1_0,$$

$$\omega^2 := \omega^2_0 - \frac{\Delta F_{ppp} F_{pp} - \Delta F_{pp} F_{ppp} + 3 F_p F_{pp} F_{zpp} - 3 F_{pp}^2 F_{zp} - 2 F_p F_{ppp} F_{zp}}{6 F_{pp}^3} \omega^1_0,$$

$$\omega^3 := \omega^3_0 + F_p \omega^4_0 - \frac{1}{3} \frac{F_{ppp}}{F_{pp}} \omega^1_0,$$

$$\omega^4 := F_{pp} \omega^4_0 + \frac{4 F_{pp}^2 - 3 F_{pp} F_{pppp}}{18 F_{pp}^2} \omega^1_0,$$

and we will use this choice in the sequel. Using Cartan’s method, it is then straightforward to solve the equivalence problem for point equivalence classes of such PDEs $z_y = F(x, y, z, z_x)$. The solution is summarized in the following

**Theorem 5.1.** A PDE system $z_y = F(x, y, z, z_x)$ satisfying the two point-invariant conditions:

$$\text{DF} \equiv 0 \neq F_{z_x z_x},$$

with its associated 1-forms $\omega^1$, $\omega^2$, $\omega^3$, $\omega^4$ as above, uniquely defines a 7-dimensional principal bundle $H_3 \to P_7 \to J_4 \ni (x, y, z, p)$ with the (reduced) structure group $H_3$ consisting of matrices:

$$\begin{pmatrix} u_3 u_5 & 0 & 0 & 0 \\ 0 & u_3 & 0 & 0 \\ -u_3 u_8 & 0 & u_5 & 0 \\ -u_3 u_8^2 & 0 & u_8 & u_3 & \end{pmatrix} \quad (u_3 \in \mathbb{R}^+, u_5 \in \mathbb{R}^+, u_8 \in \mathbb{R}),$$

together with a unique coframe \{\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3\} on $P_7$ where:

$$\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \\ \theta^4 \end{pmatrix} := \begin{pmatrix} u_3 u_5 & 0 & 0 & 0 \\ 0 & u_3 & 0 & 0 \\ -u_3 u_8 & 0 & u_5 & 0 \\ -u_3 u_8^2 & 0 & u_8 & u_3 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix},$$

such that the coframe enjoys precisely the structure equations \{4.4\}. This time however, the curvature invariants $A_1, A_2, A_3, B_1, B_2, B_3, B_4, C_1, C_2, C_3, E_1, E_2$ depend on $F = F(x, y, z, p)$ and its derivatives up to order 6.
Two relevant explicit expressions are:

\[
A_1 = -\frac{1}{\Xi F_{pp}^2} M F_{pp}^2,
\]

\[
C_1 = \frac{1}{3 \Xi F_{pp}^2} K F_{pp}^3,
\]

where:

\[
M := 9 F_{ppppp} F_{pp}^2 - 45 F_{ppppp} F_{pp} F_{ppp} + 40 F_{ppppp},
\]

\[
K := 4 \Delta F_{ppppp} F_{pp}^2 - 5 \Delta F_{ppppp} F_{pp}^2 F_{ppp} + 12 \Delta F_{ppppp} F_{pp} F_{ppp}^2 - 12 \Delta F_{ppp} F_{pp}^3 - 4 \Delta F_{ppp} F_{ppp} F_{ppppp} F_{ppp} - 5 \Delta F_{ppp} F_{ppp} F_{ppppp} F_{ppppp} + 6 \Delta F_{ppp} F_{ppppp} - 20 \Delta F_{ppp} F_{ppp} F_{ppppp} F_{ppppp} - 12 \Delta F_{ppp} F_{ppp} F_{ppppp} F_{ppppp} + 36 \Delta F_{ppp} F_{ppp} F_{ppppp} F_{ppppp} + 8 \Delta F_{ppp} F_{ppp} F_{ppppp} + 24 F_{ppp} F_{ppp} F_{ppp} - 3 F_{ppp} F_{ppp} F_{ppppp} F_{ppppp} - 2 F_{ppp} F_{ppp} F_{ppppp} F_{ppppp}.
\]

Two equations \( z_y = F(x, y, z, z_\chi) \) and \( \bar{z}_{\bar{y}} = F_\bar{y}(\bar{x}, \bar{y}, \bar{z}, \bar{z}_\chi) \) satisfying \( D F = 0 \neq F_{zzzz} \) and \( D F \neq 0 \neq F_{zzzz} \) are locally point equivalent if and only if there exists a bundle isomorphism \( \Phi: \mathcal{P}_7 \rightarrow \mathcal{P}_7 \) between the corresponding principal bundles \( \mathcal{H}_3 \rightarrow \mathcal{P}_7 \rightarrow \mathcal{J}_4 \) and \( \mathcal{H}_3 \rightarrow \mathcal{P}_7 \rightarrow \mathcal{J}_4 \) satisfying:

\[
\Phi^* \bar{\Omega}^i = \Omega_i \quad (\mu = 1, 2, 3, 4; i = 1, 2, 3).
\]

This theorem shows that the geometry of \( 3^{rd} \) order ODEs \( y'''' = H(x, y, y', y'') \) considered modulo point transformations of variables is the same as the geometry of PDEs \( z_y = F(x, y, z, z_\chi) \) with \( D F \equiv 0 \neq F_{zzzz} \), also considered modulo point transformations. Thus provided that \( M(|F|) \equiv 0 \), there should exist a conformal Lorentzian metric on the leaf space of the integrable distribution in \( \mathcal{P}_7 \) annihilated by \( \{ \theta^1, \theta^3, \theta^4 \} \), and when moreover \( K(|F|) \equiv 0 \), all this should produce (new) Einstein-Weyl geometries. Actually, we gain the following

**Theorem 5.2.** A PDE \( z_y = F(x, y, z, z_\chi) \) defines a bilinear form \( \bar{g} \) of signature \( (+, +, -, 0, 0, 0, 0) \) on the bundle \( \mathcal{P}_7 \ni (x, y, z, p, u_3, u_5, u_8) \):

\[
\bar{g} = \theta^3 \otimes \theta^3 + \theta^1 \otimes \theta^4 + \theta^4 \otimes \theta^1
\]

\[
= \frac{u_3^2}{F_{pp}^2} \left\{ \left( 3 F_{pp} [dx + F_p dy] - F_{ppppp} [dz - p dx - F dy] \right)^2 + (dz - p dx - F dy) \left( 18 F_{pp}^2 dy + [4 F_{ppppp} - 3 F_{ppp} F_{ppppp}] [dz - p dx - F dy] \right) \right\},
\]

degenerate along the rank 4 integrable distribution \( D_4 \) which is the annihilator of \( \theta^1, \theta^3, \theta^4) \). The PDE \( z_y = F(x, y, z, z_\chi) \) also defines the 1-form

\[
\Omega_3 := r_x dx + r_y dy + r_z dz + \frac{1}{3} d \left[ \log \left( u_3^2 F_{pp} \right) \right],
\]
where:
\[
\begin{align*}
 r_x & := \frac{1}{3F_p} \left\{ \Delta F_{ppp} F_p^2 - \Delta F_{pp} F_p F_{ppp} + 3 F_p F_{pp} F_{ppp} - F_{pp}^3 F_p - 2 F_{pp} F_{ppp} F_p z \\
& \quad - \Delta F_{pppp} F_p^2 p + 3 \Delta F_{ppp} F_p F_{ppp} p - 3 \Delta F_{pp} F_p F_{ppp} p + \Delta F_{pp} F_p F_{pppp} p \\
& \quad - 4 F_p F_{pp} F_{ppp} p - 2 F_{pp}^3 p + 9 F_p F_{pp} F_{ppp} F_{ppp} p + F_{pp}^2 p \right. \\
& \quad \left. - 6 F_p F_{ppp} F_{ppp} p + 2 F_{pp} F_{pppp} F_p p \right\} \\
 r_y & = \frac{1}{3F_p} \left\{ - \Delta F_{pppp} F_p^2 F_p + \Delta F_{ppp} F_p F_p F_{ppp} + \Delta F_{pp} F_p F_{ppp} - 3 \Delta F_{pp} F_p F_{ppp} - 3 \Delta F_{pp} F_p F_{pppp} \\
& \quad + 3 F_p F_{ppp} F_p - 2 F_p F_{ppp} F_{ppp} - 2 F_{pp} F_{ppp} F_p - 9 F_p F_{pp} F_{ppp} F_p - 3 F_{pp}^3 F_p \\
& \quad - 2 F_p F_{ppp} F_{ppp} F_p + F_p F_{ppp} F_{pppp} F_p - 6 F_p F_{ppp} F_{pppp} F_p + 2 F_{pp} F_{ppp} F_{pppp} F_p \right. \\
& \quad \left. + 3 F_{pp} F_p \right\}, \\
 r_z & = \frac{1}{3F_p} \left\{ \Delta F_{pppp} F_p^2 - 3 \Delta F_{ppp} F_p F_{ppp} - \Delta F_{pp} F_p F_{ppp} F_p - 3 \Delta F_{pp} F_p F_{pppp} F_p \\
& \quad + 4 F_p F_{ppp} F_p F_{ppp} - 2 F_{pp} F_{ppp} F_p - 9 F_p F_{pp} F_{ppp} F_{ppp} - F_{pp}^2 F_p - F_p F_{ppp} F_p \right. \\
& \quad \left. + 6 F_p F_{ppp} F_{ppp} F_p - 2 F_p F_{ppp} F_{pppp} F_p \right\}.
\end{align*}
\]

The degenerate bilinear form \( \tilde{g} \) descends to a Lorentzian conformal class \([g]\) on the leaf space \( M_3 \) of the distribution \( D_4 \), if and only if the Monge invariant \( M(F) = 0 \) vanishes identically.

When \( M(F) = 0 \), the local coordinates on \( M_3 \) are \((x, y, z)\) with the projection:
\[
\begin{align*}
P_7 & \rightarrow M_3 \\
(x, y, z, p, u_3, u_5, u_8) & \mapsto (x, y, z),
\end{align*}
\]
and the conformal class \([g]\) has a representative which is explicitly expressed in terms of \( dx, dy, dz \), with coefficients depending only on \((x, y, z)\).

Next, \( \Omega_3 \) descends to a 1-form denoted \( \tilde{A} \) given up to the differential of a function on \( M_3 \ni (x, y, z) \), if and only if \( K(F) = 0 \).

Moreover, the pair \((\tilde{g}, \Omega_3)\) descends to a representative of a Einstein-Weyl structure \(([g, A])\) on \( M_3 \), if and only if both \( M(F) = 0 \) and \( K(F) = 0 \).

Finally, this Weyl structure is actually Einstein-Weyl, namely it satisfies \((2.1)\), and all Einstein-Weyl structures in 3-dimensions emerge from this construction.

6. Transformation of the Wünschmann Invariant
Into the Monge Invariant

In particular, PDEs with \( A_1 = 0 \equiv C_1 \) always define an Einstein-Weyl geometry on the leaf space \( M_3 \) of the integrable distribution in \( P_7 \) annihilated by \( \{ \theta^1, \theta^3, \theta^4 \} \).

The advantage of looking at this Weyl geometry from the PDE point of view \( z_9 = F(x, y, z, z_z) \) rather than from the ODE side \( y'' = H(x, y, y', y'') \), is that now the Wünschmann invariant of the ODE becomes the much simpler and classical Monge invariant:
\[
A_1(H) \sim M(F) = 9 F_{pp}^2 F_{ppppp} - 45 F_{pp} F_{pppp} F_{ppp} + 40 F_{ppp}^3.
\]
Serendipitously, the identical vanishing $M(F) \equiv 0$ is well known to be equivalent to the condition that the graph of $p \mapsto F(p)$ is contained in a conic of the $(p, F)$-plane, with parameters $(x, y, z)$. More precisely:

\[
\begin{align*}
0 \equiv M(F) & \iff A F^2 + 2 B F p + C p^2 + 2 k F + 2 l p + m \equiv 0,
\end{align*}
\]

for some functions $A, B, C, k, l, m$ depending only on $(x, y, z)$.

Thus, passing from the formulation of Einstein-Weyl’s equations in terms of a 3rd order ODE $y''' = H(x, y, y', y'')$ to the — equivalent! — formulation in terms of a PDE $z_\mu = F(x, y, z, z_\nu)$, we are able to find the general solution to the equation:

\[
W(H) \equiv 0!
\]

By replacing $W(H) \rightsquigarrow M(F)$, the general solution \eqref{eq:6.1} is just conical!

7. How to Construct New Explicit Lorentzian Einstein-Weyl Metrics

But remember we also have to assure that:

\[
0 \equiv DF = \partial_x F + p \partial_z F.
\]

The simultaneous conditions $DF \equiv 0 \equiv M(F)$ can be achieved for instance by taking $F$ satisfying:

\[
a F^2 + 2b F (z - px) + c (z - px)^2 + 2k F + 2l (z - px) + m \equiv 0,
\]

with $a, b, c, k, l, m$ being now functions of $y$ only!

From now on, we will analyze this special solution for $M(F) \equiv 0 \equiv DF$. The simplest case occurs when avoiding square root by choosing:

\[
a := 0,
\]

so that:

\[
F := -\frac{c(z-xp)^2 - 2l(z-xp) - m}{2b(z-xp)+2k}.
\]

Here:

\[
b = b(y), \quad c = c(y), \quad k = k(y), \quad l = l(y), \quad m = m(y)
\]

are free arbitrary differentiable functions of one variable $y$.

A direct check shows that magically this solution \eqref{eq:7.1} also satisfies $K(F) \equiv 0$!

**Proposition 7.1.** All such $F := -\frac{c(z-xp)^2 - 2l(z-xp) - m}{2b(z-xp)+2k}$ with any functions $b, c, k, l, m$ of $y$, lead to Einstein-Weyl structures in 3-dimensions.

Performing the Cartan procedure to determine the coframe $\{\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3\}$, projecting both $\theta^3 \otimes \theta^3 + \theta^1 \otimes \theta^4 + \theta^4 \otimes \theta^1$ and $\Omega_3$ to the leaf space of the annihilator $M_3$ of $\{\theta^1, \theta^3, \theta^4\}$, equipping $M_3 \equiv \mathbb{R}^3$ with coordinates $(x, y, z)$, we therefore obtain functionally parameterized Einstein-Weyl structures $(g, A)$ on $\mathbb{R}^3 \ni (x, y, z)$ represented by the signature $(2, 1)$ Lorentzian metric:

\[
g := (k + b z)^2 dx^2 + x^2 (l^2 - cm) dy^2 + x^2 b^2 dz^2
\]

\[
+ 2x (ckz - blz + kl - bm) dx dy - 2xb(k + b z) dx dz - 2x^2 (ck - bl) dy dz,
\]
together with the differential 1-form:
\[ \Lambda := \frac{-ck + bl + b'k - bk'}{x(ck^2 - 2bkl + b'm)} (xb\ dz - (k + b\ z)\ dx) + \frac{bl^2 - cbm - b'kl + bb'm + ckk' - bk'l}{ek^2 - 2bkl + b'm} \ dy. \]

An independent direct check confirms that equations (1.1) are indeed identically fulfilled!

About the Cotton tensor, we compute their 5 components, and realize that they are not identically zero. Hence the obtained Einstein-Weyl structures are generically conformally non-flat. Thus, Theorem 1.1 is established. The story for Theorem 1.4 is quite similar. □

Next, without assuming \( \Lambda \equiv 0 \) in (6.1), let us now make the Ansatz that:
\[ (7.2) \quad a\ F^2 + 2b\ F(z - xp) + c\ (z - xp)^2 + 2k\ F + 2l\ (z - xp) + m \equiv 0, \]
for some arbitrary functions \( a, b, c, k, l, m \) of \( y \). The (two) solutions \( F \) automatically satisfy \( DF \equiv 0 \equiv M(F) \).

Since the solutions to Monge’s equation are conics in the \((p, F)\)-plane, we can rewrite in a hyperbolic setting:
\[ (a\ F + b\ (z - xp) + c\ y)^2 - (k\ F + l\ (z - xp) + m)^2 \equiv 1, \]
with changed functions \( a, b, c, k, l, m \) of \( y \). To avoid transcendental functions in computations, we parametrize \( \cosh t = \frac{1 + a^2}{2q} \) and \( \sinh t = \frac{1 - a^2}{2q} \), and then, solving for \( F \) and for \( z - xp \), we may start from:
\[ F = a(y) \frac{1 + a^2}{2q} + b(y) \frac{1 - a^2}{2q} + c(y), \]
\[ z - xp = k(y) \frac{1 + a^2}{2q} + l(y) \frac{1 - a^2}{2q} + m(y), \]
again with (changed) free functions \( a, b, c, k, l, m \) of \( y \). Taking:
\[ \omega_0 := d(z - xp) + x\ dp - F\ dy, \quad \omega_0^2 := dx, \quad \omega_0^3 := dy, \quad \omega_0^4 := dp, \]
and performing para-CR Cartan reduction to an \{e\}-structure/connection, we obtain the

**Proposition 7.2.** The second invariant condition \( K(F) \equiv 0 \) holds precisely in the following two cases:

1. \( k = l; \)
2. \( c = m' \) and \( a = \frac{bl + kk' - ll'}{k}. \)

In case (1), we obtain Einstein-Weyl structures for all free functions \( a, b, c, l, m \) of \( y \) given by:
\[ g := 2\tau^1\ \tau^2 + (\tau^3)^2, \]
\[ \Lambda := -\frac{2(a + b)}{x(a - b)l}\ \tau^2 - \frac{c - m'}{x(a - b)}\ \tau^3, \]
where:
\[ \tau^1 := x(a + b)\ dy - 2l\ dx, \]
\[ \tau^2 := -\frac{1}{x} x(a - b)\ dy, \]
\[ \tau^3 := xc\ dy - x\ dz + (z - m)\ dx. \]

We verify that these Einstein-Weyl structures have nontrivial \( F = dA \neq 0 \) and nontrivial \( \text{Cotton}([g]) \neq 0 \).

In case (2), we obtain Einstein-Weyl structures given by:
\[ g := 2\tau^1\ \tau^2 + (\tau^3)^2, \quad A := d\left[\log(x^2\ e)\right], \]
where:
\[ \tau_1 := (k + l)k \, dx + x(bk - bl + kk' - ll') \, dy, \]
\[ \tau_2 := \frac{1}{2} (k - l)k \, dx + \frac{1}{2} x(bk - bl + kk' - ll') \, dy, \]
\[ \tau_3 := -(z - m)k \, dx - xkm \, dy + xk \, dz. \]

But this structure, which depends on 3 functions \( b, k, l \) of \( y \), is flat:
\[ dA \equiv 0 \equiv \text{Cotton}(\langle g \rangle). \]

Finally, without replacing \( p \) by \( z - xp \), let us make the Ansatz that:
\[ (7.3) \]
\[ a F^2 + 2b F p + c p^2 + 2k F + 2l p + m \equiv 0. \]

Dealing similarly with the hyperbolic case,
\[ F = a(y) \frac{1+q^2}{2 q} + b(y) \frac{1-q^2}{2 q} + c(y), \]
\[ p = k(y) \frac{1+q^2}{2 q} + l(y) \frac{1-q^2}{2 q} + m(y), \]

we obtain nontrivial Einstein-Weyl structures. For instance, when \( k = l \) as in (1) above:
\[ g := 2 \tau_1 \tau_2 + (\tau_3)^2, \quad A := -\frac{m'}{[a-b]l} \tau_3, \]
where:
\[ \tau_1 := 2l \, dx + (a + b) \, dy, \]
\[ \tau_2 := -\frac{1}{2} (a - b) \, dy, \]
\[ \tau_3 := dz - m \, dx + (a + b) \, dy. \]

Note that this is again nontrivial:
\[ dA \not\equiv 0 \not\equiv \text{Cotton}(\langle g \rangle). \]

and note that we do not have \( x, z \) dependence here.

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