GENERALIZED NUMERICAL RADIUS INEQUALITIES FOR SCHATTEN P-NORMS

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Abstract. In this paper, we present various inequalities for generalized numerical radius of $2 \times 2$ block matrices for Schatten $p$-norm. Moreover, we give a refinement of the triangle inequality for the Schatten $p$-generalized numerical radius.

1. Introduction

Consider the space $B(H)$ of bounded linear operators over a Hilbert space $H$. For $A \in B(H)$, the numerical radius, the usual operator norm, and the Schatten $p$-norm, are denoted by $\omega(A)$, $\|A\|$ and $\|A\|_p$ respectively.

A norm $|||\cdot|||$ on $B(H)$ is said to be unitarily invariant if $|||UAV||| = |||A|||$, where $A \in B(H)$ and $U, V \in B(H)$ being unitary, and weakly unitarily invariant if $|||UAV^*||| = |||A|||$ where $A \in B(H)$ and $U \in B(H)$ being unitary. It is known that $\|A\|_p$ is unitarily invariant.

We should note that if $B(H)$ is the space of $n \times n$ complex matrices, $M_n(\mathbb{C})$, then for $A, B \in M_n(\mathbb{C})$ we have

$$|||A \oplus A^*||| = |||A \oplus A|||$$

(1.1)

and

$$|||A \oplus B||| = \left\| \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right\|$$

(see [11]).

Moreover, we have

$$\|A \oplus B\|_p = (\|A\|^p_p + \|B\|^p_p)^{\frac{1}{p}}.$$
and
\[ \| A \oplus A \|_p = 2^{\frac{p}{2^p}} \| A \|_p. \] (1.4)

The numerical radius \( \omega(\cdot) \) is defined by
\[ \omega(A) = \sup_{\theta \in \mathbb{R}} \| \text{Re}(e^{i\theta}A) \|, \]
where \( A \in B(H) \). Due to its importance, the numerical radius has been generalized several times, and their last was given by Abu-Omar and Kittaneh [2] in 2019, in which they generalized it on \( B(H) \). This generalization is denoted by \( \omega_N(\cdot) \), and is defined by
\[ \omega_N(A) = \sup_{\theta \in \mathbb{R}} N(\text{Re}(e^{i\theta}A)). \]
It was proved by the authors in [2] that \( \omega_N(\cdot) \) generalizes the numerical radius \( \omega(\cdot) \) if \( N \) is the usual operator norm.

In this paper, we are interested in studying the space of \( n \times n \) complex matrices, \( M_n(\mathbb{C}) \). We should note that \( \omega_N(\cdot) \) has the following two important properties, see [1].

**Property 1.1.** The following properties hold:

a) The norm \( \omega_N(\cdot) \) is self adjoint.

b) If the norm \( N(\cdot) \) is weakly unitarily invariant, then so is \( \omega_N(\cdot) \).

In [6], Bhatia and Kittaneh where able to prove the following theorem that relates the Shatten p-norm of an \( n \times n \) block matrix by that of its block entries.

**Theorem 1.2.** Let \( T \in M_n(\mathbb{C}) \) such that \( T = [T_{ij}] \), \( 1 \leq i, j \leq n \) and \( 1 \leq p \leq \infty \) then
\[ n^{2-p} \| T \|_p^p \leq \sum_{i,j=1}^{n} \| T_{ij} \|_p^p \leq \| T \|_p^p, \] (1.5)
for \( 2 \leq p \leq \infty \); and
\[ \| T \|_p^p \leq \sum_{i,j=1}^{n} \| T_{ij} \|_p^p \leq n^{2-p} \| T \|_p^p, \] (1.6)
for \( 1 \leq p \leq 2 \).

Motivated by the results of Bhatia and Kittaneh in [6], and those of Aldalabih and Kittaneh in [3], we aim in this paper to prove some generalized numerical radius inequalities for partitioned general \( 2 \times 2 \) block matrices considering the case when \( N \) is taken to be the Schatten p-norm. We denote this norm by \( \omega_p(\cdot) \) and call it the Schatten p-generalized numerical radius. We emphasize on finding such inequalities
for the off-diagonal part of block matrices. We also provide an application of this norm in which we give a refinement of the triangle inequality for the Schatten p-generalized numerical radius. The following Lemma was proved by the authors in [1], and will be used in our work.

**Lemma 1.3.** Let \( A, B \in M_n(\mathbb{C}) \), then

\[
\omega_p \left( \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \right) = 2^{\frac{p}{p-1}} \omega_p (B)
\]

for all \( p \).

2. **General 2 × 2 Block Matrices Inequalities**

In this section we give bounds for the generalized Schatten p-numerical radius of general 2 × 2 block matrices. We give emphasis for 2 × 2 block diagonal matrices. Most of the results in this section, generalize those presented in [3].

**Lemma 2.1.** Let \( A, B \in M_n(\mathbb{C}) \) then for all \( p \), we have

\[
\omega_p \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) = 2^{\frac{p}{p-1}} \sup_{\theta \in \mathbb{R}} \| e^{i\theta} A + e^{-i\theta} B^* \|_p .
\]

**Proof.** By equations (1.1), (1.2) and (1.4), we have

\[
\omega_p \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) = \frac{1}{2} \sup_{\theta \in \mathbb{R}} \| \begin{bmatrix} 0 & e^{i\theta} A + e^{-i\theta} B^* \\ e^{i\theta} B + e^{-i\theta} A^* & 0 \end{bmatrix} \|_p \\
= \frac{1}{2} \sup_{\theta \in \mathbb{R}} \| \begin{bmatrix} e^{i\theta} A + e^{-i\theta} B^* & 0 \\ 0 & (e^{i\theta} A + e^{-i\theta} B^*)^* \end{bmatrix} \|_p \\
= \frac{1}{2} \sup_{\theta \in \mathbb{R}} \| \begin{bmatrix} e^{i\theta} A + e^{-i\theta} B^* & 0 \\ 0 & e^{i\theta} A + e^{-i\theta} B^* \end{bmatrix} \|_p \\
= \frac{1}{2} \sup_{\theta \in \mathbb{R}} 2^{\frac{1}{p}} \| e^{i\theta} A + e^{-i\theta} B^* \|_p \\
= 2^{\frac{p}{p-1}} \sup_{\theta \in \mathbb{R}} \| e^{i\theta} A + e^{-i\theta} B^* \|_p .
\]

\[\Box\]

**Proposition 2.2.** Let \( A, B \in M_n(\mathbb{C}) \), then the following inequality holds for all \( p \)

\[
\omega_p \left( \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right) \leq \left( \omega^p_p(A) + \omega^p_p(B) \right)^{\frac{1}{p}}.
\]
Proof. We have

\[
\omega_p \left( \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right) = \sup_{\theta \in \mathbb{R}} \left\| \text{Re} \left( e^{i\theta} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right) \right\|_p,
\]

\[
= \sup_{\theta \in \mathbb{R}} \left\| \begin{bmatrix} \text{Re}(e^{i\theta} A) & 0 \\ 0 & \text{Re}(e^{i\theta} B) \end{bmatrix} \right\|_p
\]

\[
= \sup_{\theta \in \mathbb{R}} \left( \left\| \text{Re}(e^{i\theta} A) \right\|^p_p + \left\| \text{Re}(e^{i\theta} B) \right\|^p_p \right)^{\frac{1}{p}} \quad \text{(by equation (1.3))}
\]

\[
\leq (\omega^p_p(A) + \omega^p_p(B))^{\frac{1}{p}}
\]

as required. \qed

**Theorem 2.3.** Let \( A = [A_{ij}] \) be a \( 2 \times 2 \) block matrix, then

\[
\omega^p_p(A) \leq \frac{1}{2^{p-2}} \sum_{i,j=1}^2 \omega^p_p(a_{ij}) \quad (2.1)
\]

for \( 2 \leq p \leq \infty \), and

\[
\omega^p_p(A) \leq \sum_{i,j=1}^2 \omega^p_p(a_{ij}) \quad (2.2)
\]

for \( 1 \leq p \leq 2 \), where

\[
a_{ij} = \begin{cases} 
A_{ij} & i = j \\
2^{-\frac{1}{p}} \begin{bmatrix} 0 & A_{ij} \\ A_{ji} & 0 \end{bmatrix} & i \neq j.
\end{cases}
\]

Proof. We have

\[
\left\| \text{Re}(e^{i\theta} A) \right\|_p = \left\| \begin{bmatrix} \text{Re}(e^{i\theta} A_{11}) & \frac{1}{2}(e^{i\theta} A_{12} + e^{-i\theta} A_{21}^*) \\ \frac{1}{2}(e^{i\theta} A_{21} + e^{-i\theta} A_{12}^*) & \text{Re}(e^{i\theta} A_{22}) \end{bmatrix} \right\|_p,
\]
then by inequality (1.5) and Lemma 2.1, we get

\[
\|\text{Re}(e^{i\theta} A)\|_p^p \leq \frac{1}{2p-2} \sum_{i,j=1}^{2} \|\text{Re}(e^{i\theta} A)_{ij}\|_p^p
\]

\[
= \frac{1}{2p-2} \left( \sum_{i=j} \|\text{Re}(e^{i\theta} A_{ij})\|_p^p + \sum_{i \neq j} \left( \frac{1}{2} \|e^{i\theta} A_{ij} + e^{-i\theta} A_{ji}^*\|_p^p \right) \right)
\]

\[
\leq \frac{1}{2p-2} \left( \sum_{i=j} \omega_p^p(A_{ij}) + \sum_{i \neq j} \left( 2^{-\frac{p}{2}} \omega_p \left( \begin{bmatrix} 0 & A_{ij} \\ A_{ji} & 0 \end{bmatrix} \right) \right) \right)
\]

\[
= \frac{1}{2p-2} \sum_{i=j} \omega_p^p(a_{ij}).
\]

Then

\[
\omega_p^p(A) = \sup_{\theta \in \mathbb{R}} \|\text{Re}(e^{i\theta} A)\|_p^p
\]

\[
\leq \frac{1}{2p-2} \sum_{i=j} \omega_p^p(a_{ij}).
\]

for \(2 \leq p \leq \infty\). The second inequality is proved in a similar manner, using Lemma 2.1 and inequality (1.6). \(\square\)

### 3. Off-Diagonal 2 × 2 Block Matrices Inequalities

In this section, our interest was finding inequalities for \(\omega_p\) of the off-diagonal 2 × 2 block matrices. The following lemma is useful in our work.

**Lemma 3.1.** Let \(A, B \in M_n(\mathbb{C})\), then

a) \(\omega_p \left( \begin{bmatrix} 0 & A \\ e^{i\theta} B & 0 \end{bmatrix} \right) = \omega_p \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right)\) for all \(\theta \in \mathbb{R}\).

b) \(\omega_p \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) = \omega_p \left( \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix} \right)\).

for all \(p\).
Proof. Let \( U = \begin{bmatrix} I & 0 \\ 0 & e^{i\theta}I \end{bmatrix} \), then \( U \) is unitary. Then by Property 1.1 we have

\[
\omega_p \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) = \omega_p \left( U \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} U^* \right) 
\]

\[
= \omega_p \left( e^{-i\theta} \begin{bmatrix} 0 & A \\ e^{i\theta}B & 0 \end{bmatrix} \right) 
\]

\[
= \omega_p \left( \begin{bmatrix} 0 & A \\ e^{i\theta}B & 0 \end{bmatrix} \right) 
\]

which ends the proof of (a). Now to prove the equality (b), consider \( U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \), then \( U \) is unitary.

Then by Property 1.1 we have

\[
\omega_p \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) = \omega_p \left( U \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} U^* \right) 
\]

\[
= \omega_p \left( \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix} \right) 
\]

which ends the proof. \( \square \)

The next theorem gives upper and lower bounds for \( \omega_p \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \) in terms of \( \omega_p(A + B) \) and \( \omega_p(A - B) \).

**Theorem 3.2.** Let \( A, B \in M_n(\mathbb{C}) \), then

\[
\frac{\max(\omega_p(A + B), \omega_p(A - B))}{2^{1 - \frac{1}{p}}} \leq \omega_p \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leq \frac{\omega_p(A + B) + \omega_p(A - B)}{2^{1 - \frac{1}{p}}}
\]

for all \( p \).
Proof. By Lemma 1.3 we have

\[ 2^{\frac{1}{p}} \omega_p(A + B) = \omega_p \left( \begin{bmatrix} 0 & A + B \\ A + B & 0 \end{bmatrix} \right) \]

\[ = \omega_p \left( \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix} + \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \right) \]

\[ \leq \omega_p \left( \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix} \right) + \omega_p \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \] (by triangle inequality).

\[ = 2 \omega_p \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \] (by Lemma 5.1),

then

\[ \omega_p \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \geq \frac{1}{2^{1 - \frac{1}{p}}} \omega_p(A + B), \] (3.1)

replacing \( B \) by \(-B\) in inequality (3.1), we get

\[ \omega_p \left( \begin{bmatrix} 0 & A \\ -B & 0 \end{bmatrix} \right) \geq \frac{1}{2^{1 - \frac{1}{p}}} \omega_p(A - B), \]

taking \( \theta = \pi \) in Lemma 3.1 we get

\[ \omega_p \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) = \omega_p \left( \begin{bmatrix} 0 & A \\ -B & 0 \end{bmatrix} \right) \]

\[ \geq \frac{1}{2^{1 - \frac{1}{p}}} \omega_p(A - B), \] (3.2)

therefore, by the estimations (3.1) and (3.2), we get

\[ \omega_p \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \geq \frac{\max(\omega_p(A + B), \omega_p(A - B))}{2^{1 - \frac{1}{p}}}. \]
Now for the second inequality, consider \( U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -I \\ I & I \end{bmatrix} \), where \( I \) is the \( n \times n \) identity matrix, then \( U \) is unitary, and thus by Property 1.1 we get

\[
\omega_p \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) = \omega_p \left( U \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} U^* \right)
\]

\[
= \frac{1}{2} \omega_p \left( \begin{bmatrix} -(A + B) & A - B \\ -(A - B) & A + B \end{bmatrix} \right)
\]

\[
\leq \frac{1}{2} \left( \omega_p \left( \begin{bmatrix} -(A + B) & 0 \\ 0 & A + B \end{bmatrix} \right) + \omega_p \left( \begin{bmatrix} 0 & A - B \\ -(A - B) & 0 \end{bmatrix} \right) \right)
\]

(by triangle inequality)

\[
\leq \frac{1}{2} \left( 2^{\frac{p}{2}} \omega_p(A + B) + 2^{\frac{p}{2}} \omega_p(A - B) \right)
\]

(by Proposition 2.2, Lemma 3.1 and Lemma 1.3).

as required. \( \square \)

**Corollary 3.3.** Let \( T \in M_n(\mathbb{C}) \) such that \( T = A + iB \), where \( A = \text{Re}(T) \) and \( B = \text{Im}(T) \), then

\[
\frac{\omega_p(T)}{2} \leq \frac{1}{2^{\frac{p}{2}}} \omega_p \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leq \omega_p(T).
\]

for all \( p \).

**Proof.** Replacing \( B \) by \( iB \) in theorem 3.2 we get

\[
\max \left( \omega_p(A + iB), \omega_p(A - iB) \right) \leq \omega_p \left( \begin{bmatrix} 0 & A \\ iB & 0 \end{bmatrix} \right) \leq \frac{\omega_p(A + iB) + \omega_p(A - iB)}{2^{1 - \frac{1}{p}}},
\]

then,

\[
\max \left( \omega_p(T), \omega_p(T^*) \right) \leq \omega_p \left( \begin{bmatrix} 0 & A \\ iB & 0 \end{bmatrix} \right) \leq \frac{\omega_p(T) + \omega_p(T^*)}{2^{1 - \frac{1}{p}}},
\]

However, \( \omega_p(T) = \omega_p(T^*) \), then

\[
\frac{\omega_p(T)}{2^{1 - \frac{1}{p}}} \leq \omega_p \left( \begin{bmatrix} 0 & A \\ iB & 0 \end{bmatrix} \right) \leq \frac{\omega_p(T)}{2^{-\frac{1}{p}}}.
\]
Take $\theta = \frac{\pi}{2}$ in Lemma 3.1 then

$$\frac{\omega_p(T)}{2^{1-\frac{1}{p}}} \leq \omega_p\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{\omega_p(T)}{2^{1-\frac{1}{p}}}. \tag{3.3}$$

Multiply (3.3) by $2^{-\frac{1}{p}}$, then

$$\frac{\omega_p(T)}{2} \leq \frac{1}{2^{\frac{1}{p}}} \omega_p\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \omega_p(T).$$

□

**Remark 3.1.** Let $A \in M_n(\mathbb{C})$, then we have

$$e^{i(\theta - \frac{\pi}{2})} = -ie^{i\theta}, \text{ and } e^{-i(\theta - \frac{\pi}{2})} = ie^{-i\theta}.$$

Therefore,

$$\text{Re}(e^{i(\theta - \frac{\pi}{2})} A) = \frac{e^{i(\theta - \frac{\pi}{2})} A + e^{-i(\theta - \frac{\pi}{2})} A^*}{2}$$

$$= \frac{-ie^{i\theta} A + ie^{-i\theta} A^*}{2}$$

$$= \frac{e^{i\theta} A - e^{-i\theta} A^*}{2i}$$

$$= \text{Im}(e^{i\theta} A).$$

Therefore,

$$\omega_p(A) = \sup_{\alpha \in \mathbb{R}} \left\| \text{Re}(e^{i\alpha} A) \right\|_p$$

$$= \sup_{\theta \in \mathbb{R}} \left\| \text{Re}(e^{i(\theta - \frac{\pi}{2})} A) \right\|_p$$

$$= \sup_{\theta \in \mathbb{R}} \left\| \text{Im}(e^{i\theta} A) \right\|_p.$$

See [8].
**Remark 3.2.** Let \( X \in M_n(\mathbb{C}) \), and \( 2 \leq p < \infty \) then

\[
\omega_p \left( \begin{bmatrix} X & X \\ -X & -X \end{bmatrix} \right) \leq \sup_{\theta \in \mathbb{R}} \left\| \begin{bmatrix} \text{Re}(e^{i\theta}X) & \text{Im}(e^{i\theta}X) \\ -\text{Im}(e^{i\theta}X) & -\text{Re}(e^{i\theta}X) \end{bmatrix} \right\|_p \\
\leq \frac{1}{2^{\frac{1}{p}-1}} \sup_{\theta \in \mathbb{R}} \left( 2 \left\| \text{Re}(e^{i\theta}X) \right\|_p^p + 2 \left\| \text{Im}(e^{i\theta}X) \right\|_p^p \right)^{\frac{1}{p}} \\
= \frac{1}{2^{\frac{1}{p}-1}} 2^\frac{p}{2} \omega_p(X) \\
= 2 \omega_p(X).
\]

**Theorem 3.4.** Let \( A, B \in M_n(\mathbb{C}) \), then

\[
\omega_p \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leq 2^\frac{p}{2} \min(\omega_p(A), \omega_p(B)) + \min(\omega_p(A + B), \omega_p(A - B)).
\]

for \( 2 \leq p < \infty \).

*Proof.*** Consider \( U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix} \), where \( I \) is the \( n \times n \) identity matrix, then \( U \) is unitary, we have

\[
\omega_p \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) = \omega_p \left( U \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} U^* \right) \\
= \frac{1}{2} \omega_p \left( \begin{bmatrix} A + B & A - B \\ -(A - B) & -(A + B) \end{bmatrix} \right) \\
= \frac{1}{2} \omega_p \left( \begin{bmatrix} A + B & A + B \\ -(A + B) & -(A + B) \end{bmatrix} + \begin{bmatrix} 0 & -2B \\ 2B & 0 \end{bmatrix} \right) \\
\leq \frac{1}{2} \left( \omega_p \left( \begin{bmatrix} A + B & A + B \\ -(A + B) & -(A + B) \end{bmatrix} \right) + \omega_p \left( \begin{bmatrix} 0 & -2B \\ 2B & 0 \end{bmatrix} \right) \right).
\]
(by triangle inequality)
\[ \leq \frac{1}{2} \left( 2\omega_p(A + B) + 2\bar{\tau}\omega_p(2B) \right) \] (by Remark 3.2, Lemma 3.1 and Lemma 1.3)
\[ = \omega_p(A + B) + 2\bar{\tau}\omega_p(B). \] \hspace{1cm} (3.4)

Replacing \( B \) by \(-B\) in (3.4), we get
\[ \omega_p \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) = \omega_p \left( \begin{bmatrix} 0 & A \\ -B & 0 \end{bmatrix} \right) \] \hspace{1cm} (taking \( \theta = \pi \) in Lemma 3.1)
\[ \leq \omega_p(A - B) + 2\bar{\tau}\omega_p(B). \] \hspace{1cm} (3.5)

Then by (3.4) and (3.5), we get
\[ \omega_p \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leq 2\bar{\tau}\omega_p(B) + \min(\omega_p(A + B), \omega_p(A - B)). \] \hspace{1cm} (3.6)

Interchanging \( A \) and \( B \) in (3.6), we get
\[ \omega_p \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) = \omega_p \left( \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix} \right) \] \hspace{1cm} (by Lemma 3.1)
\[ \leq 2\bar{\tau}\omega_p(A) + \min(\omega_p(A + B), \omega_p(A - B)). \] \hspace{1cm} (3.7)

Therefore, by (3.6) and (3.7), we have
\[ \omega_p \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leq 2\bar{\tau} \min(\omega_p(A), \omega_p(B)) + \min(\omega_p(A + B), \omega_p(A - B)) \]
\[ \text{as required.} \]

\[ \square \]

4. An Application

In this section we present an application, which is a refinement of the triangle inequality for the generalized Schatten p-numerical radius. The following remark is presented by "Yamazaki" in [14].

**Remark 4.1.** Let \( T \in M_n(\mathbb{C}) \), we have
\[ \omega_p(T) = \sup_{\alpha^2 + \beta^2 = 1} \| \alpha Re(T) + \beta Im(T) \|_p, \]
for all \( p \). Then
\[ \| T + T^* \|_p \leq 2\omega_p(T). \]
Proof. We have

\[ \omega_p(T) = \sup_{\theta \in \mathbb{R}} \left\| \operatorname{Re}(e^{i\theta}T) \right\|_p \]

\[ = \frac{1}{2} \sup_{\theta \in \mathbb{R}} \left\| e^{i\theta}T + e^{-i\theta}T^* \right\|_p \]

\[ = \sup_{\theta \in \mathbb{R}} \left\| \cos(\theta)T + i\sin(\theta)T + \cos(\theta)T^* - i\sin(\theta)T^* \right\|_p \]

\[ = \sup_{\theta \in \mathbb{R}} \left\| \cos(\theta) \operatorname{Re}(T) - \sin(\theta) \operatorname{Im}(T) \right\|_p \]

\[ = \sup_{\alpha^2 + \beta^2 = 1} \left\| \alpha \operatorname{Re}(T) + \beta \operatorname{Im}(T) \right\|_p. \]

Take \( \theta = 2\pi \), then

\[ \omega_p(T) \geq \left\| \cos(2\pi) \operatorname{Re}(T) - \sin(2\pi) \operatorname{Im}(T) \right\|_p \]

\[ = \frac{1}{2} \left\| T + T^* \right\|_p \]

so, \( \left\| T + T^* \right\|_p \leq 2\omega_p(T) \). \( \square \)

The next theorem is a refinement of the triangle inequality for the generalized Schatten p-numerical radius.

**Theorem 4.1.** Let \( A, B \in M_n(\mathbb{C}) \), then

\[ \| A + B \|_p \leq 2^{1 - \frac{1}{p}} \omega_p \left( \begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix} \right) \leq \| A \|_p + \| B \|_p. \]

for all \( p \).

Proof. Let \( T = \begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix} \). We have

\[ \left\| \begin{bmatrix} 0 & A + B \\ A^* + B^* & 0 \end{bmatrix} \right\|_p^p = \|(A + B) \oplus (A^* + B^*)\|_p^p \quad \text{(by equation 1.2)} \]

\[ = \|(A + B) \oplus (A + B)\|_p^p \quad \text{(by equation 1.6)} \]

\[ = 2 \left\| A + B \right\|_p^p \quad \text{(by equation 1.4)} \] (4.1)
Then,

\[
2 \|A + B\|_p^p = \left\| \begin{bmatrix} 0 & A + B \\ A^* + B^* & 0 \end{bmatrix} \right\|_p^p
= \|T + T^*\|_p^p
\leq 2^p \omega_p^p(T) \quad \text{(by Remark 4.1)}
= 2^p \omega_p^p \left( \begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix} \right).
\]

Thus,

\[
\|A + B\|_p \leq 2^{1 - \frac{1}{p}} \omega_p \left( \begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix} \right).
\]

For the second inequality, we have

\[
\omega_p(T) = \sup_{\theta \in \mathbb{R}} \| \text{Re}(e^{i\theta} T) \|_p
= \frac{1}{2} \sup_{\theta \in \mathbb{R}} \left\| \begin{bmatrix} 0 & e^{i\theta} A + e^{-i\theta} B \\ e^{-i\theta} A^* + e^{i\theta} B^* & 0 \end{bmatrix} \right\|_p
= \frac{1}{2^{1 - \frac{1}{p}}} \sup_{\theta \in \mathbb{R}} \| e^{i\theta} A + e^{-i\theta} B \|_p \quad \text{(by same argument as (4.1))}
\leq \frac{1}{2^{1 - \frac{1}{p}}} \left( \|A\|_p + \|B\|_p \right). \quad \text{(by triangle inequality)}
\]

Therefore,

\[
\|A + B\|_p \leq 2^{1 - \frac{1}{p}} \omega_p \left( \begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix} \right) \leq \|A\|_p + \|B\|_p.
\]

□

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