On RIC bounds of Compressed Sensing Matrices for Approximating Sparse Solutions Using \( \ell_q \) Quasi Norms

Yong Hsia

State Key Laboratory of Software Development Environment; LMIB of the Ministry of Education; School of Mathematics and System Sciences, Beihang University, Beijing 100191, P. R. China,

Ruey-Lin Sheu

Department of Mathematics, National Cheng Kung University, Taiwan

Abstract

This paper follows the recent discussion on the sparse solution recovery with quasi-norms \( \ell_q, q \in (0,1) \) when the sensing matrix possesses a Restricted Isometry Constant \( \delta_{2k} \) (RIC). Our key tool is an improvement on a version of “the converse of a generalized Cauchy-Schwarz inequality” extended to the setting of quasi-norm. We show that, if \( \delta_{2k} \leq 1/2 \), any minimizer of the \( \ell_q \) minimization, at least for those \( q \in (0,0.9181] \), is the sparse solution of the corresponding underdetermined linear system. Moreover, if \( \delta_{2k} \leq 0.4931 \), the sparse solution can be recovered by any \( \ell_q, q \in (0,1) \) minimization. The values 0.9181 and 0.4931 improves those reported previously in the literature.

Keywords: compressed sensing, restricted isometry constant, \( \ell_q \) minimization, quasi norm

1 Introduction

Given a \( m \times n \) matrix \( \Phi \) with \( m \ll n \) and a nonzero vector \( b \in \mathbb{R}^m \), one of the recently popular problem in compressed sensing is to find the sparsest solution of the underdetermined linear systems \( \Phi x = b \). Here the sensing matrix \( \Phi \) is assumed to obey a Uniform Uncertainty Principle (UUP) for every \( k \) sparse vector \( x \), or to possess a Restricted Isometry Constant \( \delta_{2k} \) (RIC) defined in \([5]\) as follows:

\*This research was partially supported by Taiwan National Science Council under NSC 101-2115-M-006-005; by Taiwan National Center for Theoretic Studies (South); by National Natural Science Foundation of China under grant 11001006 and 91130019/A011702, and by the fund of State Key Laboratory of Software Development Environment under grant SKLSDE-2011ZX-15.
Definition 1 For $k = 1, 2, \cdots$, the restricted isometry constant is the smallest number $\delta_k$ such that

$$(1 - \delta_k)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1 + \delta_k)\|x\|_2^2$$

holds for all $k$-sparse vector $x \in \mathbb{R}^n$ with $\|x\|_0 \leq k$, where $\|x\|_0$ denotes the number of nonzero elements of $x$.

For simplicity, we only discuss the sparse solution for noiseless recovery, as it can be easily extended to the noisy recovery case. See for example [4, 9].

As the sparsity is fundamental in signal processing, the problem can be mathematically formulated as to find

$$x^* = \arg\min \{\|x\|_0 \mid \Phi x = b, x \in \mathbb{R}^n\}. \quad (1)$$

Unfortunately, the formulation (1) is practically intractable due to its NP-hardness [13]. For more information on the issue and related applications in signal and image processing, see [1] and references therein.

A common alternative is to consider the following convex problem using the $\ell_1$ norm

$$x^{(1)} = \arg\min \{\|x\|_1 \mid \Phi x = b, x \in \mathbb{R}^n\}. \quad (2)$$

Let $x^*$ be the sparse solution to (1). Define

$$T_0 = \{i \in \{1, 2, \cdots, n\} \mid x^*_i \neq 0\}, \quad (3)$$

and define $x_{T_0}$ for any $x \in \mathbb{R}^n$ as

$$(x_{T_0})_i = \begin{cases} x_i, & \text{if } i \in T_0, \\ 0, & \text{otherwise}. \end{cases} \quad (4)$$

Then, a new $\ell_1$ recovery result, which extended many existing ones in literature such as [4, 9, 2, 8], is stated below.

Theorem 1 ([12]) Suppose $\delta_{2k} < \frac{77 - \sqrt{137}}{82} \approx 0.4931$, the solution $x^{(1)}$ of the problem (2) satisfies

$$\|x - x^{(1)}\|_2 \leq C_0 k^{-1/2} \|x - x_{T_0}\|_1, \quad \forall x \in \{x \mid \Phi x = b\},$$

where $C_0$ is a positive constant dependent on $\delta_{2k}$. It follows that if $x$ is $k$-sparse, the recovery is exact.

On the other hand, nonconvex $\ell_q$ quasi-norm minimization with $q \in (0, 1)$ is also considered to recover the sparse solution [13]. It is to solve, for a number $q \in (0, 1)$,

$$x^{(q)} = \arg\min \{\|x\|_q \mid \Phi x = b, x \in \mathbb{R}^n\}. \quad (5)$$

More studies on the nonconvex $\ell_q$ minimization problem can be found in [6, 9, 13, 7]. In particular, the following two theorems will be strengthened in this paper.
Theorem 2 ([9]) Suppose $\delta_{2k} < 2(3 - \sqrt{2}) \approx 0.4531$. Then for any $q \in (0, 1]$

$$\|x - x^{(q)}\|_q \leq C_0 \|x - x_{T_0}\|_q, \quad \forall x \in \{x \mid \Phi x = b\},$$

where $C_0$ is a positive constant dependent on $\delta_{2k}$. In particular, if $x = x^*$ is $k$-sparse, the recovery is exact.

Theorem 3 ([14]) Suppose $\delta_{2k} < 1/2$. There exists a number $q_0 \in (0, 1]$ such that for any $q < q_0$, each minimizer $x^{(q)}$ of the $\ell_q$ minimization ([3]) is the sparse solution of ([5]). Furthermore, there exists a positive constant $C_q$ dependent on $q$ and $\delta_{2k}$ such that

$$\|x - x^{(q)}\|_q \leq C_q \|x - x_{T_0}\|_q, \quad \forall x \in \{x \mid \Phi x = b\}.$$ 

More specifically, the upper bound of $\delta_{2k}$ in Theorem 2 can be improved to 0.4931 in Theorem 11 of the paper. Secondly, the sufficient condition $\delta_{2k} < 1/2$ in Theorem 3 can be extended to $\delta_{2k} \leq 1/2$ and the threshold $q_0$ can be precisely estimated to be at least 0.9181 by Theorem 14. This is indeed a very surprising result since $q_0$, if it would have been computed by the analysis in [14], is only 0.0513. Our main tool to achieve these results is an improvement on a version of “the converse of a generalized Cauchy-Schwarz inequality” extended to the setting of $\ell_q$ quasi-norms. The key inequality is stated and proved in Section 2 below.

2 A Key Inequality

According to Cauchy-Schwarz inequality, we have the standard inequality

$$\|x\|_2 \geq \frac{\|x\|_1}{\sqrt{n}}.$$ 

The following converse of the above inequality is very recent:

Lemma 1 ([3]) For any $x \in \mathbb{R}^n$,

$$\|x\|_2 \leq \frac{\|x\|_1}{\sqrt{n}} + \frac{\sqrt{n}}{4} \left( \max_{1 \leq i \leq n} |x_i| - \min_{1 \leq i \leq n} |x_i| \right).$$

On the other hand, Cauchy-Schwarz inequality can be extended to the setting of quasi-norm $\|x\|_q$, $q \in (0, 1)$ with

$$\|x\|_2 \geq \frac{\|x\|_q}{n^{1/q - 1/2}}$$

(6)

by using Hölder’s inequality. The first converse of (6) was proposed in ([14]).

Lemma 2 ([14]) Fix $0 < q < 1$. For any $x \in \mathbb{R}^n$,

$$\|x\|_2 \leq \frac{\|x\|_q}{n^{1/q - 1/2}} + \sqrt{n} \left( \max_{1 \leq i \leq n} |x_i| - \min_{1 \leq i \leq n} |x_i| \right).$$

(7)
Our key result, Lemma 3 below, gives a sharpened estimation on the right hand side of (7). When \(p\) in (9) is replaced by 1 for any \(q \in (0, 1)\), Lemma 3 reduces to Lemma 2.

**Lemma 3** For \(q \in (0, 1)\) and \(x \in \mathbb{R}^n\), there is

\[
\|x\|_2 \leq \frac{\|x\|_q}{n^{1/q-1/2}} + p_q \sqrt{n} \left( \max_{1 \leq i \leq n} |x_i| - \min_{1 \leq i \leq n} |x_i| \right),
\]

where

\[
p_q := \left( \frac{q}{2} \right)^{\frac{1}{1-q}} - \left( \frac{q}{2} \right)^{\frac{1}{1-q}}.
\]

Moreover, \(p_q\) is a decreasing convex function of \(q \in (0, 1)\) with \(\lim_{q \to 0} p_q = 1\) and \(\lim_{q \to 1} p_q = 1/4\).

**Proof.** Due to the symmetry of the inequality (8) in the components \(|x_1|, |x_2|, \ldots, |x_n|\), we only have to prove the case for \(x \in S = \{(x_1, x_2, \ldots, x_n) \neq 0 \mid x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}\) (notice that \(x = 0\) is a trivial case). Furthermore, suppose the inequality (8) is true for \(x \in S\), \(x_1 = 1\). By substituting \(\frac{x_i}{x_1}, x \in S\) into (8) and canceling the common factor \(\frac{1}{x_1}\), we immediately generalize the result to all \(x \in S\). In other words, our goal is to show

\[
\|x\|_2 \leq \frac{\|x\|_q}{n^{1/q-1/2}} + p_q \sqrt{n} (1 - x_n), \quad x \in S_1 = \{x \in S \mid x_1 = 1\}
\]

where \(p_q\) is a function of \(q\) specified in (9).

Following the approach in [14], we define for any fixed \(q \in (0, 1)\)

\[
f(x) = \|x\|_2 - \frac{\|x\|_q}{n^{1/q-1/2}}
\]

and compute the first order partial derivatives as

\[
\frac{\partial f(x)}{\partial x_i} = \frac{x_i}{\|x\|_2} - \frac{\|x\|_q^{1-q} x_i^{q-1}}{n^{1/q-1/2}}.
\]

Note that, when \(x_i\) increases with all other components fixed, the following two terms

\[
\|x\|_2^{1/q} x_i^{q-1} = \frac{1}{n^{1/q-1/2}} \left( \sum_{j=1}^{n} \left( \frac{x_j}{x_i} \right)^q \right)^{\frac{1-q}{q}}
\]

are both decreasing. As the result, \(\frac{\partial f(x)}{\partial x_i}\) is increasing and \(f(x)\) is convex in each of the components \(x_i\) for \(i = 1, 2, \cdots, n\). Analogously from (11) and (12), we can show that the composite function \(g : \mathbb{R}^{n-1} \to \mathbb{R}\) such that

\[
g(x_1, x_3, x_4, \ldots, x_n) = f(x_1, x_3, x_4, \ldots, x_n)
\]
is also convex in the variable $x_3$ while all other components $x_1, x_4, \ldots, x_n$ remaining fixed. Likewise, we can conclude that $f(1, \ldots, 1, x_n, \ldots, x_n)$ is convex in the variable $x_n$ where $x_n$ is repeated for a couple of times.

Since the maximum of a convex function always happens on the boundary, we have

$$\max_{1 \geq x_2 \geq x_3 \geq \cdots \geq x_n} f(1, x_2, \ldots, x_n)$$

$$= \max_{1 \geq x_3 \geq \cdots \geq x_n} \left\{ \max_{x_2 \in [x_3, 1]} f(1, x_2, \ldots, x_n) \right\}$$

$$= \max_{1 \geq x_3 \geq \cdots \geq x_n} \left\{ \max \{f(1, 1, x_3, \ldots, x_n), f(1, x_3, x_3, \ldots, x_n)\} \right\}$$

$$= \max \left\{ \max_{1 \geq x_3 \geq \cdots \geq x_n} f(1, 1, x_3, \ldots, x_n), \max_{1 \geq x_3 \geq \cdots \geq x_n} f(1, x_3, x_3, x_4, \ldots, x_n) \right\}. \quad (13)$$

In (13), since $f(1, x_3, x_3, x_4, \ldots, x_n)$ is convex in $x_3$, it follows that

$$\max_{x_3 \in [x_4, 1]} f(1, x_3, x_3, x_4, \ldots, x_n) = \max \{f(1, 1, 1, x_4, \ldots, x_n), f(1, x_4, x_4, x_4, x_5, \ldots, x_n)\}.$$

Repeating the arguments iteratively, we can thus express the maximum of $f$ only in terms of 1 and $x_n$ as follows:

$$h(x_n) = \max_{1 \geq x_2 \geq x_3 \geq \cdots \geq x_n} f(1, x_2, \ldots, x_n) = f(1, \ldots, 1, x_n, \ldots, x_n), \quad x_n \in [0, 1].$$

Suppose the distribution of $x_1 = 1$ appears for $r$ times ($1 \leq r \leq n$) in the maximum solution of $f$, we have

$$h(x_n) = \sqrt{r(1 - x_n^2) + nx_n^2} - \frac{(r(1 - x_n^2) + nx_n^2)^{1/2}}{n^{1/q - 1/2}}.$$

Since $h(x_n)$ is convex and $h(1) = 0$, we have

$$h(x_n) \leq (1 - x_n) h(0) + x_n h(1) = (1 - x_n) h(0).$$

Then it holds that

$$f(x) \leq h(x_n)$$

$$\leq (1 - x_n) h(0)$$

$$\leq (1 - x_n) \max_{r \in \{1, 2, \ldots, n\}} \left\{ \sqrt{r(1^2 - 0^2) + n0^2} - \frac{(r(1^q - 0^q) + n0^q)^{1/2}}{n^{1/q - 1/2}} \right\} \quad (14)$$

$$\leq (1 - x_n) \max_{r \in [1, n]} \left\{ \sqrt{r - \frac{r^{1/q}}{n^{1/q - 1/2}}} \right\} \quad (15)$$

$$= (1 - x_n) \left( \left( \frac{q}{2} \right)^{\frac{1}{2-q}} - \left( \frac{q}{2} \right)^{\frac{1}{2-q}} \right) \sqrt{n}.$$
Finally, it is easy to verify that
\[
\lim_{q \to 0} p_q = 1, \quad \lim_{q \to 1} p_q = 1/4,
\]
and \(p_q\) is a decreasing function of \(q \in (0, 1)\) since
\[
\frac{d}{dq} p_q = \frac{(q/2)^\frac{q}{2-q}}{(2-q)^2} (2 \ln(q/2) + 2 - q) - \frac{(q/2)^\frac{q}{2-q}}{(2-q)^2} (2 \ln(q/2) + \frac{4}{q} - 2)
\]
\[
= \frac{2 \ln(q/2)}{(2-q)^2} p_q < 0.
\] (16)

Moreover, the convexity of \(p_q\) over \(q \in (0, 1)\) can be verified by
\[
\frac{d^2}{dq^2} p_q = \frac{(2 \ln(q/2) + 2 - q)^2 + (2 - q)^3/q}{(2-q)^4} p_q > 0.
\]
and the proof is thus completed. 

In the following, we give an upper estimate of \(p_q\) for small \(q\), which will be used later.

**Lemma 4** Denote Euler’s number by \(e = 2.718 \cdots\). It holds that
\[
p_q < 1 + \frac{q \ln(q/2)}{2 - q}, \quad \forall q \in (0, 0.4797] \subset (0, 1 - \sqrt{2}/e].
\]
Proof. Since \(e^x < 1 + x + x^2/2\) for \(x < 0\), we have
\[
p_q = (q/2)^\frac{q}{2-q} - (q/2)^\frac{q}{2-q} = e^{\frac{q \ln(q/2)}{2 - q}} - (q/2)^\frac{q}{2-q} < 1 + \frac{q \ln(q/2)}{2 - q} + \frac{q^2 \ln^2(q/2)}{2(2 - q)^2} - (q/2)^\frac{q}{2-q}.
\]
To prove the lemma, it is sufficient to show that
\[
\frac{-q \ln(q/2)}{\sqrt{2(2 - q)}} \leq (q/2)^\frac{1}{1 + \sqrt{2}/e}, \quad \forall q \in (0, 1 - \sqrt{2}/e].
\] (17)

Since \(\frac{1}{2-q} \leq \frac{1}{1+\sqrt{2}/e}\) and \((q/2)^\frac{1}{1 + \sqrt{2}/e} \geq (q/2)^\frac{1}{1 + \sqrt{2}/e}\), the inequality (17) can be confirmed by verifying
\[
\sqrt{2} \frac{-(q/2) \ln(q/2)}{(1 + \sqrt{2}/e)} \leq (q/2)^\frac{1}{1 + \sqrt{2}/e}, \quad \forall q \in (0, 1 - \sqrt{2}/e],
\]
or equivalently, by verifying
\[
-(q/2)^\frac{1}{1 + \sqrt{2}/e} \ln(q/2) \leq \frac{1 + \sqrt{2}/e}{\sqrt{2}}, \quad \forall q \in (0, 1 - \sqrt{2}/e].
\]

Let \(y = (q/2)^\frac{1}{1 + \sqrt{2}/e} \in (0, 1)\). We then have the desired result by
\[
-(q/2)^\frac{1}{1 + \sqrt{2}/e} \ln(q/2) = -\frac{1 + \sqrt{2}/e}{\sqrt{2}} y \ln(y) \leq \frac{1 + \sqrt{2}/e}{\sqrt{2}}
\]
because the negative entropy function \(-y \ln(y)\) attains the maximum value of \(1/e\). 

\[\Box\]
3 Main Results

Let \( \text{Null}(\Phi) \) be the null space of \( \Phi \); \( x^*, x^{(q)} \) be the solutions to (1) and (5), respectively; \( T_0 \) be defined in (3). Suppose \( \|x^*\|_0 = k \) and define

\[ T_0^c = \{1, 2, \cdots, n\} \setminus T_0. \]

The following null space property is essential. However, a refined version is immediately stated in Lemma 6.

**Lemma 5** ([10]) \( x^{(q)} \) is the unique sparse solution \( x^* \) if and only if

\[ \|h_{T_0}\|_q < \|h_{T_0^c}\|_q, \quad \forall h \in \text{Null}(\Phi), \ h \neq 0, \]  \hspace{1cm} (18)

where \( h_{T_0} \) and \( h_{T_0^c} \) are similarly defined as in (4).

**Lemma 6** \( x^{(q)} \) is the unique sparse solution \( x^* \) if and only if

\[ \|h_{T_0}\|_q < \|h_{T_0^c}\|_q, \quad \forall h \in \text{Null}(\Phi), \ h_{T_0^c} \neq 0. \]  \hspace{1cm} (19)

Proof. It is sufficient to study the difference between (18) and (19). Suppose \( 0 \neq h \in \text{Null}(\Phi) \) and \( h_{T_0^c} = 0 \). It follows that \( h_{T_0} \neq 0 \) and \( \Phi h_{T_0} = 0 \). Therefore,

\[ \Phi(x^* + th_{T_0}) = b, \quad \forall t \in \mathbb{R}, \]

and

\[ \min_{t \in \mathbb{R}} \|x^* + th_{T_0}\|_0 \leq k - 1, \]

which contradicts the optimality of \( x^* \). \( \square \)

The purpose of this research is to establish sufficient conditions for (19) with the help of Lemmas 3 and 4 so that

\[ \tau(h, q) := \frac{\|h_{T_0}\|_q}{\|h_{T_0^c}\|_q} < 1, \quad \forall h \in \text{Null}(\Phi), \ h_{T_0^c} \neq 0. \]

To this end, let

\[ h = h_{T_0} + h_{T_1} + h_{T_2} + \cdots, \]

where \( T_1 \) corresponds to the locations of the \( k \) largest entries of \( h_{T_0} \), \( T_2 \) the locations of the next \( k \) largest entries of \( h_{T_0} \) and so on. Without loss of generality, we assume

\[ h = (h_{T_0}, h_{T_1}, h_{T_2}, \cdots)^T \]

with the cardinality of \( T_i \) being equal to \( k \) for \( i = 0, 1, 2, \cdots \). Define a ratio

\[ t := t(h, q) \in (0, 1] \]
such that
\[ \|h_T_1\|_q^q = t \sum_{i \geq 1} \|h_T_i\|_q^q. \]

According to Lemma 6, we only focus on nonzero \( h_{T_0} \), which immediately implies that \( h_{T_1} \neq 0 \), i.e., \( t > 0 \). Several technique lemmas are needed.

**Lemma 7** ([14]) For \( q \in (0, 1) \), we have
\[
\sum_{i \geq 2} \|h_{T_i}\|_2^2 \leq \frac{1}{k^{(2-q)/q}} (1 - t)^{(2-q)/q} \left( \sum_{i \geq 1} \|h_{T_i}\|_q^q \right)^{2/q}
\]

**Lemma 8** For \( q \in (0, 1) \), \( h \in \text{Null} (\Phi) \), \( h_{T_0} \neq 0 \), we have
\[
\sum_{i \geq 2} \|h_{T_i}\|_2 \leq \frac{1 + (p_q - 1)t^{1/q}}{k^{1/q - 1/2}} \left( \sum_{i \geq 1} \|h_{T_i}\|_q^q \right)^{1/q}
\]

**Proof.** We first apply Lemma 3 to each \( h_{T_i} \) to get
\[
k^{1/q - 1/2} \sum_{i \geq 2} \|h_{T_i}\|_2 \leq \|h_{T_i}\|_q + p_k^{1/q} (|h_{ik+1}| - |h_{ik+k}|), \quad i = 2, 3, \ldots \tag{20}
\]
and sum up (20) over all \( i \geq 2 \). Then,
\[
k^{1/q - 1/2} \sum_{i \geq 2} \|h_{T_i}\|_2 \leq \sum_{i \geq 2} \|h_{T_i}\|_q + p_k^{1/q} (|h_{2k+1}| - (|h_{2k+k}| - |h_{3k+1}|) - \cdots
\]
\[
\leq \sum_{i \geq 2} \|h_{T_i}\|_q + p_k^{1/q} |h_{2k+1}|
\]
\[
= \sum_{i \geq 2} \|h_{T_i}\|_q + p_k^{1/q} (|h_{2k+1}|^q)^{1/q}
\]
\[
\leq \sum_{i \geq 2} \|h_{T_i}\|_q + p_k^{1/q} (\|h_{T_i}\|_q^q/k)^{1/q}
\]
\[
= \sum_{i \geq 1} \|h_{T_i}\|_q + (p_q - 1) (\|h_{T_i}\|_q^q)^{1/q}
\]
\[
\leq \left( \sum_{i \geq 1} \|h_{T_i}\|_q^q \right)^{1/q} + (p_q - 1) \left( t \sum_{i \geq 1} \|h_{T_i}\|_q^q \right)^{1/q} \quad \text{(since } q < 1 \text{)}
\]
\[
= (1 + (p_q - 1)t^{1/q}) \left( \sum_{i \geq 1} \|h_{T_i}\|_q^q \right)^{1/q}
\]

\[ \square \]
Lemma 9 For $q \in (0, 1)$, $h \in \text{Null}(\Phi)$, $h_{T_0} \neq 0$, we have

$$\|\Phi(h_{T_0} + h_{T_1})\|_2^2 = \|\Phi(\sum_{j \geq 2} h_{T_j})\|_2^2 \leq \left(\frac{(1-t)t^{(2-q)/q}}{k^{2/q}-1} + \frac{\delta_{2k}(1 + (p_q-1)t^{1/q})^2}{k^{2/q}-1}\right)\left(\sum_{i \geq 1} \|h_{T_i}\|_q^q\right)^{2/q}.$$ 

Proof. According to Lemma 1.2 in [5], we have

$$\langle \Phi(h_{T_i}), \Phi(h_{T_j}) \rangle \leq \delta_{2k}\|h_{T_i}\|_2\|h_{T_j}\|_2.$$ 

Therefore,

$$\|\Phi(\sum_{j \geq 2} h_{T_j})\|_2^2 = \sum_{i,j \geq 2} \langle \Phi(h_{T_i}), \Phi(h_{T_j}) \rangle$$

$$= \sum_{j \geq 2} \langle \Phi(h_{T_j}), \Phi(h_{T_j}) \rangle + 2 \sum_{2 \leq i < j} \langle \Phi(h_{T_i}), \Phi(h_{T_j}) \rangle$$

$$\leq (1 + \delta_{k}) \sum_{i \geq 2} \|h_{T_i}\|_2^2 + 2\delta_{2k} \sum_{i > j \geq 2} \|h_{T_i}\|_2\|h_{T_j}\|_2$$

$$\leq \sum_{i \geq 2} \|h_{T_i}\|_2^2 + \delta_{2k}\left(\sum_{i \geq 2} \|h_{T_i}\|_2^2\right)^2$$

$$\leq \left(\frac{(1-t)t^{(2-q)/q}}{k^{2/q}-1} + \frac{\delta_{2k}(1 + (p_q-1)t^{1/q})^2}{k^{2/q}-1}\right)\left(\sum_{i \geq 1} \|h_{T_i}\|_q^q\right)^{2/q}.$$ 

where the last inequality follows from Lemma 7 and Lemma 8. □

Lemma 10 ([14]) For $q \in (0, 1)$, we have

$$\|\Phi(h_{T_0} + h_{T_1})\|_2^2 \geq \frac{1 - \delta_{2k}}{k^{2/q}-1}(r(h, q)^2 + t^{2/q})\left(\sum_{i \geq 1} \|h_{T_i}\|_q^q\right)^{2/q}.$$ 

(21)

Combining Lemma 9 with Lemma 10 we obtain

$$(1 - \delta_{2k})(r(h, q)^2 + t^{2/q}) \leq (1-t)t^{(2-q)/q} + \delta_{2k}(1 + (p_q-1)t^{1/q})^2.$$ 

After rearrangement of the terms, it implies that

$$r(h, q)^2 \leq \delta_{2k}(1 + (p_q-1)t^{1/q})^2 + t^{(2-q)/q} - (2 - \delta_{2k})t^{2/q}/(1 - \delta_{2k}).$$ 

(22)

Then, an immediate sufficient condition for $x^q$ being the sparse solution of (1) is to require the right hand side of (22) being less than 1 for all $t \in (0, 1]$. If we focus on $\delta_{2k} < 1$, the sufficient condition that the right hand side of (22) being less than 1 can be expressed as

$$r(t, q, \delta_{2k}) := 2\delta_{2k} + (p_q-1)t^{1/q}(2 + (p_q-1)t^{1/q})\delta_{2k} + t^{(2-q)/q} - (2 - \delta_{2k})t^{2/q} < 1, \forall t \in (0, 1].$$ 

(23)
The problem then becomes to estimate the range of \( q \in (0, 1) \) and \( \delta_{2k} \in (0, 1) \) for which (23) is true.

Notice that by setting \( p_q = 1, \forall q \in (0, 1) \) and by the fact (shown in [14]) that when \( q \to 0^+ \),

\[
\sup_{t \in (0,1)} \left\{ t^{(2-q)/q} - (2 - \delta_{2k})t^{2/q} \right\} \leq q/e \to 0,
\]

one can immediately obtain Theorem 3 In general, as \( p_q < 1 \) for \( q \in (0, 1) \), we expect an improvement over Theorem 3.

To begin with, we rewrite

\[
r(t, q, \delta_{2k}) = r_1(t, q, \delta_{2k}) + r_2(t, q, \delta_{2k}) + r_3(t, q, \delta_{2k}).
\]

where

\[
\begin{align*}
r_1(t, q, \delta_{2k}) &= \delta_{2k} + \delta_{2k}(1 - (1 - p_q)t^{1/q})^2; \\
r_2(t, q, \delta_{2k}) &= t^{(2-q)/q}; \\
r_3(t, q, \delta_{2k}) &= -(2 - \delta_{2k})t^{2/q}
\end{align*}
\]

with the ranges \( \delta_{2k}, q \in (0, 1) \) and \( t \in (0, 1] \).

We first discuss the monotonicity of functions \( r, r_1, r_2, r_3 \). Here is the summary:

(a) \( r_1(t, q, \delta_{2k}) \) and \( r_3(t, q, \delta_{2k}) \) are decreasing functions of \( t \in (0, 1] \) whereas \( r_2(t, q, \delta_{2k}) \) is increasing.

(b) \( r_1(t, q, \delta_{2k}) \) is decreasing in terms of \( q \) since both \( 1 - p_q \) and \( t^{1/q} \) are increasing functions of \( q \).

(c) The sum of the latter two functions \( (r_2 + r_3)(t, q, \delta_{2k}) \) is an increasing function of \( q \) for \( t \leq \frac{1}{2-\delta_{2k}} \) since

\[
\frac{\partial(r_2 + r_3)(t, q, \delta_{2k})}{\partial q} = -2/q^2(\ln t)t^{2/q - 1}(1 - (2 - \delta_{2k})t) \geq 0.
\]

(d) \( r(t, q, \delta_{2k}) \) is an increasing function of \( \delta_{2k} \) since we can rewrite (23) as

\[
r(t, q, \delta_{2k}) = \delta_{2k}(2 + 2(p_q - 1)t^{1/q} + (p_q - 1)^2t^{2/q} + t^{2/q} - 2t^{2/q} - 2t^{2/q})
\]

\[
= \delta_{2k}(1 + (1 + (p_q - 1)t^{1/q})^2 + t^{(2-q)/q} - 2t^{2/q} - 2t^{2/q})
\]

with \( 1 + (1 + (p_q - 1)t^{1/q})^2 > 0. \)

Moreover, we have

**Lemma 11** Suppose \( \delta_{2k} \leq 1/2 \), it holds that

\[
r(t, q, \delta_{2k}) < 1, \forall t \in \left(\frac{1}{2-\delta_{2k}}, 1\right], \forall q \in (0, 1).
\]

10
Proof. Suppose \( t > \frac{1}{2 - \delta_{2k}} \), we always have

\[
\begin{align*}
    r(t, q, \delta_{2k}) &= 2 \delta_{2k} + (p_q - 1) t^{1/q} (2 + (p_q - 1) t^{1/q}) \delta_{2k} + (1/t - 2 + \delta_{2k}) t^{2/q} \\
    &\leq 2 \delta_{2k} + (p_q - 1) t^{1/q} (2 + (p_q - 1) t^{1/q}) \delta_{2k} \\
    &< 2 \delta_{2k} \leq 1.
\end{align*}
\]

Now we present our main result.

**Theorem 4** Suppose \( \delta_{2k} \leq 1/2 \). For any \( q \in (0, 0.9181] \), each minimizer \( x^q \) of the \( \ell_q \) minimization (5) is the sparse solution of (7).

Proof. Since \( r(t, q, \delta_{2k}) \) is an increasing function of \( \delta_{2k} \) by monotonicity (d), for any \( \delta_{2k}' < \delta_{2k} \), we have

\[ r(t, q, \delta_{2k}') < 1, \forall t \in (0, 1] \implies r(t, q, \delta_{2k}) < 1, \forall t \in (0, 1]. \]

Hence, it is sufficient to assume that \( \delta_{2k} = 1/2 \). Then we have

\[ r(t, q, 1/2) = 1 + (p_q - 1) t^{1/q} (2 + (p_q - 1) t^{1/q}) / 2 + t^{2-q}/q - 3/2 t^{2/q}, \]

which is less than 1, by Lemma [11] for all \( t \in (\frac{1}{q}, 1] \) and \( q \in (0, 1) \). The rest of the proof is to show that \( r(t, q, 1/2) < 1 \) on \( t \in (0, 2/3] \) and \( q \in (0, 0.9181] \) by incorporating monotonicity (a) - (c) on sufficiently fine meshes.

First, for any \( q \in (0, 1) \),

\[
\begin{align*}
    \frac{\partial r(t, q, 1/2)}{\partial t} &= \frac{t^{(2/q)-2}}{q} \left( \frac{p_q - 1}{t^{(1/q)-1}} + 2 - q - (3 - (p_q - 1)^2) t \right) \\
    &\leq \frac{p_q - 1}{q} \frac{t^{(1/q)-1} + 2 - q}{t^{(2/q)-2}} \\
    &= (p_q - 1 + (2 - q) t^{(1/q)-1}) t^{(1/q)-1} \\
    &< 0, \quad \forall 0 < t^{(1/q)-1} < \frac{1 - p_q}{2 - q}.
\end{align*}
\]

In other words, \( r(t, q, 1/2) \) is strictly decreasing on \( t \in (0, (1-p_q)/(2-q)) \). Moreover, \( r(0, q, 1/2) = 1 \). Hence, for any \( q \in (0, 1) \), we have

\[
r(t, q, 1/2) < 1, \forall 0 < t < \left( \frac{1 - p_q}{2 - q} \right)^{\frac{1}{1-q}}, \tag{24}
\]

which specifies the range on which \( r(t, q, 1/2) < 1 \) by a function of \( q \in (0, 1) \).

Secondly, to analyze the function \( \left( \frac{1 - p_q}{2 - q} \right)^{\frac{1}{1-q}} \), we can compute to get

\[
\lim_{q \to 0^+} \left( \frac{1 - p_q}{2 - q} \right)^{\frac{1}{1-q}} = 1; \quad \lim_{q \to 1^-} \left( \frac{1 - p_q}{2 - q} \right)^{\frac{1}{1-q}} = 0.
\]
and its first derivative
\[
\frac{d}{dq} \left( \frac{1 - p_q}{2 - q} \right)^{\frac{q}{1-q}} = \left( \frac{1 - p_q}{2 - q} \right)^{\frac{q}{1-q}} \frac{q(1 - p_q - p_q'(2 - q))}{(1 - q)(1 - p_q)(2 - q)}
\]

\[
\frac{1}{(1 - q)^2} \ln \frac{1 - p_q}{2 - q} + q \frac{1 - p_q - 2\ln(2 - q) p_q}{(1 - q)(1 - p_q)(2 - q)}
\]

where \(16\) is used for \(p_q'\). We consider the following two cases.

(i) Let \(q \in (0, 0.3]\). In this case, \(p_q \geq p_{0.3} > 0.6081\). According to Lemma \(14\) we have

\[
-\frac{q \ln(q/2)}{(1 - p_q)(2 - q)} < 1.
\]

Therefore,

\[
\frac{1}{(1 - q)^2} \ln \frac{1 - p_q}{2 - q} + q \frac{1 - p_q - 2\ln(q/2) p_q}{(1 - q)(1 - p_q)(2 - q)} < \frac{1}{1 - q} \left\{ \frac{\ln(\frac{1 - p_q}{2 - q})}{1 - q} + \frac{q}{2 - q} + \frac{2p_q}{2 - q} \right\}
\]

\[
\leq \frac{1}{1 - q} \left\{ \ln \left( \frac{1 - 0.6081}{2 - 0.3} \right) + \frac{0.3}{2 - 0.3} + \frac{2}{2 - 0.3} \right\}
\]

\[
< 0,
\]

implying that \(\left( \frac{1 - p_q}{2 - q} \right)^{\frac{q}{1-q}}\) is strictly decreasing for \(q \in (0, 0.3]\).

(ii) Let \(q \in [0.3, 0.99]\). Define a partition of \(q \in (0, 0.3]\) by \(q_s = 0.3 + 0.01s\) for \(s = 0, 1, \ldots, 69\) and try to estimate the right hand side of \((25)\) on each mesh \(q \in [q_s, q_{s+1}]\) where \(q_s, q_{s+1}\) are points in the partition. Then,

\[
\ln \frac{1 - p_q}{2 - q} + q \frac{1 - p_q - 2\ln(q/2) p_q}{(1 - q)(1 - p_q)(2 - q)}
\]

\[
= \ln \frac{1 - p_q}{2 - q} + q \frac{1}{(1 - q)(2 - q)} + \frac{2\ln(q/2) p_q}{(1 - q)(1 - p_q)(2 - q)^2}
\]

\[
\leq \ln \frac{1 - p_{q_{s+1}}}{2 - q_{s+1}} + q_{s+1} \frac{1 - p_{q_{s+1}}}{(1 - q_{s+1})(2 - q_{s+1})} + \frac{2q_{s+1} \ln(q_{s+1}/2) p_{q_{s+1}}}{(1 - q_{s+1})(1 - p_{q_{s+1}})(2 - q_{s+1})^2}.
\]

With the aid of computer, the evaluation of \((26)\) shows that they are negative on all mesh points. That is, \(\left( \frac{1 - p_q}{2 - q} \right)^{\frac{q}{1-q}}\) is strictly decreasing for \(q \in [0.3, 0.99]\).

Together with (i) and (ii), we conclude that \(\left( \frac{1 - p_q}{2 - q} \right)^{\frac{q}{1-q}}\) is strictly decreasing for \(q \in (0, 0.99]\), which leads to the following estimation:

\[
\left( \frac{1 - p_q}{2 - q} \right)^{\frac{q}{1-q}} \geq \left( \frac{1 - p_{q_{9.181}}}{2 - 0.9181} \right)^{\frac{q_{9.181}}{0.9181}} > 0.0105, \quad \forall q \in (0, 0.9181];
\]

\[
\left( \frac{1 - p_q}{2 - q} \right)^{\frac{q}{1-q}} \geq \left( \frac{1 - p_{q_{0.17}}}{2 - 0.17} \right)^{\frac{q_{0.17}}{0.17}} > 0.6821, \quad \forall q \in (0, 0.17].
\]
It follows from (27) and (24) that

$$r(t, q, 1/2) < 1, \quad \forall t \in (0, 0.0105], \forall q \in (0, 0.9181];$$

from (28) and Lemma 11 that

$$r(t, q, 1/2) < 1, \quad \forall t \in (0, 1], \forall q \in (0, 0.17].$$

That is, to prove (23), it is left to verify that

$$r(t, q, 1/2) < 1 \quad \text{on} \quad (t, q) \in [0.0105, 2/3] \times [0.17, 0.9181].$$

Our idea is to subdivide the region $[0.0105, 2/3] \times [0.17, 0.9181]$ into the union of small squares with the type $[t_r, t_{r+1}] \times [q_s, q_{s+1}]$. Then, by the monotonicity (a), (b) and (c), we can estimate $r(t, q, 1/2)$ on this square by the corner points as follows:

$$\max_{t \in [t_r, t_{r+1}]} \max_{q \in [q_s, q_{s+1}]} r(t, q, 1/2) \leq r_1(t_r, q_s, 1/2) + r_2(t_{r+1}, q_{s+1}, 1/2) + r_3(t_r, q_{s+1}, 1/2).$$

If the evaluation by computer shows that (29) is less than one, we are done with the square. Otherwise, the estimation might not be tight enough so that we have to subdivide the square into finer meshes.

Our calculation shows that, by covering $[0.0105, 2/3] \times [0.17, 0.9181]$ with

$$[0.0105, 0.6667] \times [0.17, 0.9172] \bigcup [0.0105, 0.66667] \times [0.9172, 0.91809] \bigcup [0.0105, 0.66667] \times [0.91809, 0.9181],$$

we can get the desired result as

$$\max_{r \in \{105, 106, \ldots, 6666\}} \max_{s \in \{1700, 1701, \ldots, 9171\}} \{r_1(t_r, q_s, 1/2) + r_2(t_{r+1}, q_{s+1}, 1/2) + r_3(t_r, q_{s+1}, 1/2)\} < 1,$$

where $t_r = 0.0001r$ and $q_s = 0.0001s$;

$$\max_{r \in \{1050, 1051, \ldots, 6666\}} \max_{s \in \{91720, 91721, \ldots, 91808\}} \{r_1(t_r, q_s, 1/2) + r_2(t_{r+1}, q_{s+1}, 1/2) + r_3(t_r, q_{s+1}, 1/2)\} < 1,$$

where $t_r = 0.00001r$ and $q_s = 0.00001s$ in this partition; and

$$\max_{r \in \{10500, 10501, \ldots, 66666\}} \{r_1(t_l, q_{91809}, 1/2) + r_2(t_{r+1}, q_{91810}, 1/2) + r_3(t_r, q_{91810}, 1/2)\} < 1,$$

where $t_r = 0.000001r$.

All the above calculations were carried out by computer and the proof is thus complete. $\square$
**Remark 1** Let $q \to 0^+$. Then $p_q \to 1$ and it follows from (23) that $\delta_{2k} \leq \frac{1}{2}$. In other words, $\delta_{2k} \leq \frac{1}{2}$ can not be further improved based on the sufficient condition (22). Furthermore, one can verify that

$$r(0.064, 0.9182, 1/2) > 1.0000002,$$

implying that the condition $q \in (0, 0.9181]$ in Theorem 4 is tight up to the fourth decimal digit.

**Remark 2** Based on the analysis in [14], the threshold $q_0$ in Theorem 3 can be estimated to be around 0.0513, far less than the number 0.9181 reported in our Theorem 4.

From the above analysis, if the sparse recovery is to be exact for any $q \in (0, 1)$, we may require a tighter restricted isometric constant than $\delta_{2k} = 0.5$. This is the spirit of Theorem 2 which we shall show immediately an improvement on their result.

To investigate the issue, we look into the case when $q = 1, p_q = 1/4$. The sufficient condition (23) becomes

$$r(t, 1, \delta_{2k}) = 2\delta_{2k} + (1 - 3\delta_{2k}/2)t - (2 - 25\delta_{2k}/16)t^2 < 1, \forall t \in (0, 1].$$

Since $2 - 25\delta_{2k}/16 > 0$, $r(t, 1, \delta_{2k})$ is a concave parabola of $t \in (0, 1]$. The sufficient condition in (30) holds if and only if the equation $r(t, 1, \delta_{2k}) = 1$ has no solution. Namely, we need

$$(1 - 3\delta_{2k}/2)^2 + 4(2 - 25\delta_{2k}/16)(2\delta_{2k} - 1) < 0.$$ 

It follows that

$$\delta_{2k} < \frac{77 - \sqrt{1337}}{82} (> 0.493109).$$

We therefore have the following theorem.

**Theorem 5** Suppose $\delta_{2k} \leq 0.4931$. Then for any $q \in (0, 1)$ each minimizer $x_q^*$ of the $\ell_q$ minimization (5) is the sparse solution of (1).

Proof. Since $r(t, q, \delta_{2k})$ is an increasing function of $\delta_{2k}$, we fix $\delta_{2k}$ at 0.4931. According to Lemma 11 $r(t, q, 0.4931) < 1$ for $t \in (0.663614, 1]$. Therefore, it is sufficient to check the maximum of $r(t, q, 0.4931)$ over the region $(t, q) \in [0, 0.6637] \times [0.9181, 1]$. This is done by dividing the region into two parts

$$[0, 0.6637] \times [0.9181, 1] = [0, 0.6637] \times [0.9181, 0.9992] \cup [0, 0.6637] \times [0.9992, 1].$$

14
the first of which with a mesh size 0.0001 whereas the latter with 0.00001. The computation has been verified by computer as follows

\[
\max_{t \in [0.0001, 0.6637]} \max_{q \in [0.9181, 1]} r(t, q, 0.4931) \\
\leq \max\{ \max_{r=0,1,2,\ldots,6636} \max_{s=0,1,2,\ldots,9999} \{ r_1(t_r, q_s, 0.4931) \\
\quad + r_2(t_{r+1}, q_{s+1}, 0.4931) + r_3(t_r, q_{s+1}, 0.4931) \}, \}
\]

\[
\max_{u=0,1,2,\ldots,6639} \max_{v=0,1,2,\ldots,9999} \{ r_1(t_u, q_v, 0.4931) \\
\quad + r_2(t_{u+1}, q_{v+1}, 0.4931) + r_3(t_u, q_{v+1}, 0.4931) \} \}
\]

< 1

where \( t_r = 0.0001r, q_s = 0.0001s, t_u = 0.00001u, q_v = 0.00001v \) and the proof is thus complete. \( \square \)

4 Conclusion

In this paper, based on the extended “converse of a generalized Cauchy-Schwarz inequality” of quasi-norm, we establish new sufficient conditions under which the minimizer of the \( \ell_q \) minimization is the sparse solution of the corresponding underdetermined linear system. More precisely, we show that if the restricted isometry constant \( \delta_{2k} \leq 1/2 \), then for any \( q \leq 0.9181 \) the solution of \( \ell_q \) minimization also solves the \( \ell_0 \) minimization. Furthermore, if \( \delta_{2k} \leq 0.4931 \), then for any \( q \in (0, 1] \), the minimizer of \( \ell_q \) minimization remains optimal in \( \ell_0 \) minimization. Our results strongly improves those reported previously in the literature. We believe the value 0.4931 can be improved to \( \frac{77 - \sqrt{1337}}{54} \) with a new proof, but now it is still open. The other future research is to extend the presented results to the rank minimization problem.

Acknowledgments

This research was undertaken while Y. Hsia visited National Cheng Kung University, Tainan, Taiwan.

References

[1] A. Bruckstein, D. Donoho and M. Elad, From sparse solutions of systems of equations to sparse modeling of signals and images, *SIAM Rev.*, **51**, 34–81 (2009)

[2] T. Cai, L. Wang, G. Xu, Shifting inequality and recovery of sparse signals, *IEEE Trans. Signal Process.*, **58**, 1300–1308 (2010)

[3] T. T. Cai, L. Wang, and G. Xu, New bounds for restricted isometry constants, *IEEE Transactions Information Theory*, **56(9)**, 4388–4394 (2010)
[4] E. Candès, The restricted isometry property and its implications for compressed sensing, *C. R. Acad. Sci. Ser. I*, 346, 589–592 (2008)

[5] E. Candès and T. Tao, Decoding by linear programing, *IEEE Trans. Inform. Theory*, 51, 4203–4215 (2005)

[6] R. Chartrand, Exact reconstruction of sparse signals via nonconvex minimization, *IEEE Signal Process. Lett.*, 14, 707–710 (2007)

[7] M. Davies and R. Gribonval, Restricted isometry constants where q sparse recovery can fail for $0 < p < 1$, *IEEE Trans. Inform. Theory*, 55(5), 2203–2214 (2009)

[8] S. Foucart, A note on guaranteed sparse recovery via $\ell_1$-minimization, *Appl. Comput. Harmon. Anal.*, 29, 97–103 (2010)

[9] S. Foucart and M.J. Lai, Sparsest solutions of underdetermined linear systems via $\ell_q$-minimization for $0 < q < 1$, *Appl. Comput. Harmon. Anal.*, 26, 395–407 (2009)

[10] R. Gribonval and M. Nielsen, Sparse decompositions in unions of bases, *IEEE Trans. Info. Theory*, 49(12), 3320–3325 (2003)

[11] M.J. Lai and L.Y. Liu, A New Estimate of Restricted Isometry Constants for Sparse Solutions, 2011

[12] S. Li and Q. Mo, New bounds on the restricted isometry constant $\delta_{2k}$, *Appl. Comput. Harmon. Anal.*, 31(3), 460–468 (2011)

[13] B. Natarajan, Sparse approximate solutions to linear systems, *SIAM J. Comput.*, 24, 227–234 (1995)

[14] M. J. Lai and J. Wang, An unconstrained $lq$ minimization for sparse solution of under determined linear systems, *SIAM J. Optimization*, 21(1), 82–101, 2011