Transition from quintessence to phantom phase in quintom model

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Abstract
Assuming the Hubble parameter is a continuous and differentiable function of comoving time, we investigate necessary conditions for quintessence to phantom phase transition in quintom model. For power-law and exponential potential examples, we study the behavior of dynamical dark energy fields and Hubble parameter near the transition time, and show that the phantom-divide-line $\omega = -1$ is crossed in these models.

1 Introduction
Astronomical data show that the expansion of our universe is accelerated at the present epoch [1]. Assuming that the universe is filled with perfect fluids, the equation of state parameter $\omega = p/\rho$ must satisfy $\omega < -\frac{1}{3}$, indicating a negative pressure. Many theories have been proposed to study the origin of this negative pressure or repulsive gravitational behavior.

One of these theories introduces a smooth energy component with negative pressure dubbed "dark energy". A candidate for dark energy is the cosmological constant [2]: a constant quantum vacuum energy density which fills the space homogeneously, corresponding to a fluid with a constant equation of state parameter $\omega = -1$. Because of conceptual problems associated with the cosmological constant, such as fine-tuning and coincidence problems [3], alternative theories have been proposed where in a class of them, some dynamical scalar fields, with suitably chosen potentials, have been introduced to make the vacuum energy vary with time [2]. For example to describe $\omega > -1$ phase or quintessence regime, a normal scalar field $\phi$, known as quintessence scalar field, can be used [4]. For performing actual

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calculation, it is useful to assume that the quintessence field is slowly rolling in a potential. This approximation (slow-roll), leads to inflationary expansion of the universe \( \Omega < -1 \) phase (or phantom phase) can be related to the presence of a scalar field \( \sigma \), with a wrong sign kinetic term dubbed as phantom scalar field \( \sigma \). Depending on the form of the potential, different solutions such as asymptotic de Sitter, big rip, etc. may be obtained \( \Omega \). The effects of gravitational back-reactions can also counteract that of phantom energy and can become large enough to end the phantom dominated phase before the big rip \( \Omega \).

One of the important issues of the scalar models of dark energy is their stability behaviors in both the classical and quantum mechanical levels. In classical level, one of the main methods of stability studies is achieved by studying the stability of the late-time attractors of the theory by a phase space analysis. This can be done by determining the eigenvalues of the determinate obtained by the set of autonomous equations of motion when perturbed about their critical points. Variety of models have been studied by this method, which some of them can be found in \( \Omega \) and references therein.

The second kind of instabilities, which is more important in phantom models, are those based on the quantum fluctuations. Because the phantom fields have negative kinetic energy, it is possible that a phantom particle decays into arbitrary number of phantoms and ordinary particles, such as gravitons. It can be shown that the decay rates of these interactions are infinite which indicates that the phantom models are dramatically unstable. But if we think of these models as the low-energy effective theories, with the fundamental fields having positive kinetic energy, then we should use a momentum cutoff \( \Lambda \) in calculating the decay rates. In this way it can be shown that for \( \Lambda \sim M_{\text{pl}} \), the lifetimes can become larger than the age of the universe when one chooses suitable phantom-gravity interaction potentials, and this removes the quantum instability of these kinds of phantom models. See \( \Omega \) for a specific example.

Some present data seems to favor an evolving dark energy with \( \omega \) less than \(-1\) at present epoch from \( \omega > -1 \) in the near past \( \Omega \). \( \Omega \) It is therefore instructive to construct physical models in which \( \omega \) can cross \( \omega = -1 \) line (dubbed as phantom-divide-line). Neither the cosmological constant nor the dynamical scalar fields, like quintessence or phantom, can explain the \( \omega = -1 \) crossing, hence some other theories have been introduced to describe this transition \( \Omega \). One of these theories is the quintom model, which based on hybrid models \( \Omega \), describes the transition from \( \omega > -1 \) to \( \omega < -1 \) regime by assuming that the cosmological fluid, besides the ordinary matter and radiation, is consisted of a quintessence and a phantom scalar field \( \Omega \). A brief introduction of quintom model can be found in section two. This model can lead to quintessence domination, i.e. \( \omega > -1 \), at early time and phantom domination, i.e. \( \omega < -1 \), at late time. In \( \Omega \), a
phase-space analysis of the evolution for a spatially flat Friedman-Robertson-Walker (FRW) universe containing a barotropic fluid and phantom-scalar fields with exponential potentials has been presented. It has been shown that the phantom-dominated scaling solution is the stable late-time attractor. In \cite{17} the same calculation has been done by introducing an interaction term between phantom and quintessence fields. Recently, the hessence model, as a new view of quintom dark energy, in which the dark energy is described by a single non-canonical complex scalar field, rather than two independent real scalar fields, has been introduced \cite{18}. The evolution of $\omega$ in this model has been studied in \cite{19} and \cite{20}, via phase-space analysis. Although this model allows $\omega$ to cross $-1$, but it avoids the late time singularity or the "big rip".

In this paper we study the transition from quintessence to phantom phase in the quintom model. Instead of considering late time behavior of the model we try to investigate the behavior of cosmological parameters and scalar fields in a neighborhood of transition time. In section one, by assuming that the Hubble parameter is continuous and differentiable function at transition time, we seek the necessary conditions that must be satisfied in order that the phase transition occurs. In section two, instead of a phase-space analysis, we try to obtain solution of Einstein and Friedmann equations in the vicinity of transition time for exponential and power law potentials for slowly varying fields. We show that the dynamical equations are consistent with the conditions needed for quintessence to phantom phase transition. It is worth noting that these two specific examples have stable late time attractor solutions \cite{16,17}.

Through this paper we use $\bar{\hbar} = c = G = 1$ units.

## 2 Phase transition in quintom model: general results

We consider a spatially flat FRW space-time in comoving coordinates $(t, x, y, z)$

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2),$$

(1)

where $a(t)$ is the scale factor. We assume that the universe is filled with two kinds of fluid: (dark) matter and quintom dark energy. The equation of state of matter is

$$P_m = (\gamma_m - 1)\rho_m, \quad 1 < \gamma_m < 2,$$

(2)

where $\rho_m$, and $P_m$ are the matter density and pressure, respectively, and $\gamma_m = 1 + \omega_m$. The evolution equation of $\rho_m$ is

$$\dot{\rho}_m + 3H\gamma_m\rho_m = 0.$$

(3)
\(H(t) = \dot{a}(t)/a\) is the Hubble parameter and "dot" denotes time derivative. The quintom dark energy consists of a negative kinetic energy scalar field \(\sigma\) and a normal scalar field \(\phi\), described by the Lagrangian density [11, 16]:
\[
L_D = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma + V(\phi, \sigma),
\]
(4)
where \(V(\phi, \sigma)\) is the quintom potential. Restricting ourselves to homogeneous fields, the energy density \(\rho_D\) and pressure \(P_D\) of the homogenous quintom dark energy is then
\[
\rho_D = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \dot{\sigma}^2 + V(\phi, \sigma),
\]
\[P_D = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \dot{\sigma}^2 - V(\phi, \sigma).\]
(5)
The evolution equations of the fields are
\[
\ddot{\phi} + 3H \dot{\phi} + \frac{\partial V(\phi, \sigma)}{\partial \phi} = 0,
\]
\[
\ddot{\sigma} + 3H \dot{\sigma} - \frac{\partial V(\phi, \sigma)}{\partial \sigma} = 0.
\]
(6)
Using Einstein equation, one can show that the Hubble parameter satisfies the Friedmann equations
\[
H^2 = \frac{8\pi}{3} (\rho_D + \rho_m)
\]
\[= \frac{4\pi}{3} [\dot{\phi}^2 - \dot{\sigma}^2 + 2V(\phi, \sigma) + 2\gamma_m \rho_m],
\]
(7)
and
\[
\dot{H} = -4\pi (\rho_D + \rho_m + P_D + P_m)
\]
\[= -4\pi (\dot{\phi}^2 - \dot{\sigma}^2 + \gamma_m \rho_m).
\]
(8)
Note that eqs. (6), (7) and (8) are not independent. The equation of state parameter \(\omega\) is defined through
\[
\omega = \frac{P_D + P_m}{\rho_D + \rho_m} = \frac{\dot{\phi}^2 - \dot{\sigma}^2 - 2V(\phi, \sigma) + 2(\gamma_m - 1) \rho_m}{\dot{\phi}^2 - \dot{\sigma}^2 + 2V(\phi, \sigma) + 2\rho_m}.
\]
(9)
In terms of Hubble parameter we have
\[\omega = -1 - \frac{2}{3} \frac{\dot{H}}{H^2}.
\]
(10)
For an accelerating universe, \(\ddot{a}(t) > 0\), we have \(\dot{H}(t) + H^2(t) > 0\) and \(\omega < -\frac{1}{3}\). Through this paper we will restrict ourselves to \(H(t) > 0\). In the
quintessence phase, $\dot{H} < 0$ and therefore $\omega > -1$. In the phantom phase we have $\dot{H} > 0$ so $\omega < -1$. In contrast to cosmological models with only one scalar field, transition from quintessence to phantom era, in principle, is possible in quintom model. If $H(t)$ has a local minimum at $t = t_0$, i.e. $\dot{H}(t_0) = 0$, then at $t < t_0$, $\dot{H}(t) < 0$ and $\omega > -1$ and at $t > t_0$, $\dot{H}(t) > 0$ and $\omega < -1$. This behavior is in agreement with present data of equation of state parameter $\omega$. So the transition from quintessence to phantom phase, or crossing the phantom-divide-line, can be studied by investigating the behavior of the Hubble parameter near $t = t_0$ point.

Let $H(t)$ be differentiable on an open set containing $t_0$, then it can be expanded at $t = t_0$

$$H(t) = \sum_{n=0}^{\infty} \frac{H^{(n)}(t_0)}{n!}(t - t_0)^n, \quad \dot{H}(t_0) = 0,$$

(11)

where $H^{(n)}(t_0)$ is the $n$-th derivative of $H(t)$ at $t = t_0$. If $\alpha \geq 2$ is the order of the first non-zero derivative of $H(t)$ at $t = t_0$, then

$$H(t) \simeq h_0 + h_1(t - t_0)^\alpha + O((t - t_0)^{\alpha+1}).$$

(12)

where $h_0 = H(t_0)$ and $h_1 = \frac{1}{\alpha}H^{(\alpha)}(t_0)$. A transition from quintessence to phantom phase occurs when $\alpha$ is an even integer and $h_1 > 0$. $h_1 > 0$ follows from the fact that $H(t_0)$ must be a local minimum of $H(t)$. If we considered the transition from phantom to quintessence universe, we should take $h_1 < 0$, instead. Using eqs. (11), (12), one finds

$$H^2 = \frac{8\pi}{3}V - \frac{1}{3}\dot{H},$$

(13)

where $V(t) = V(\phi, \sigma) + (1 - \gamma_m^2)\rho_m$. By expanding both sides of eq. (13) near $t = t_0$, and noting $H^{(\beta<\alpha)}(t_0) = 0$, one finds

$$V^{(\beta<\alpha-1)}(t_0) = 0,$$

(14)

and

$$h_0^2 = \frac{8\pi}{3}V_0,$$

(15)

$$h_1 = \frac{8\pi}{\alpha}V_1,$$

(16)

in which $V_0 = V(t_0)$ and $V_1 = \frac{1}{(\alpha-1)!}V^{(\alpha-1)}(t_0)$. For matter density $\rho_m$, one finds, using eq. (14)

$$\rho_m^{(\alpha-1)}(t_0) = (-3H(t_0)\gamma_m)^{\alpha-1}\rho_m(t_0),$$

(17)

so

$$h_1 = \frac{8\pi}{\alpha!} \left( V^{(\alpha-1)}(\phi(t_0), \sigma(t_0)) + (1 - \gamma_m^2)(-3\gamma_m h_0)^{\alpha-1}\rho_m(t_0) \right), \quad \alpha \geq 2.$$
Assuming $\rho_m(t) > 0$ and noting $\gamma_m \leq 2$, the second term in the right-hand-side of eq. (15) is negative for even $\alpha$’s. So to have $h_1 > 0$, it is necessary

$$V^{(\alpha-1)}(\phi(t_0), \sigma(t_0)) > 0. \quad (19)$$

It is also interesting to note that eq. (15) implies

$$V(t_0) > 0. \quad (20)$$

But $V(t) > 0$ is a necessary condition for acceleration of the universe at any time $t$. This can be verified by considering that when $\omega < -\frac{1}{3}$, eq. (9) yields

$$\dot{\phi}^2 - \dot{\sigma}^2 < V(\phi, \sigma) + \left(1 - \frac{3}{2} \gamma_m\right) \rho_m. \quad (21)$$

Also for $H(t) \neq 0$, eq. (7) results

$$\dot{\phi}^2 - \dot{\sigma}^2 > -2V(\phi, \sigma) - 2\rho_m. \quad (22)$$

$V(t) > 0$ follows from eqs. (21) and (22).

Let us for a moment relax our even-$\alpha$ condition and look at the models with odd $\alpha$. Assume that at some time $\tilde{t}$ we have $\dot{H}(t = \tilde{t}) = 0$. If the system is in quintessence phase in $t < \tilde{t}$ times, i.e. $\dot{H}(t < \tilde{t}) < 0$, expanding $H(t)$ near $t = t_0$ results

$$H(t) = h_0 + h_1(t - \tilde{t})^{2n+1} + O((t - \tilde{t})^{2n+2}), n \geq 1. \quad (23)$$

We assume $h_1 < 0$ to guarantee the existence of quintessence phase in $t < \tilde{t}$ times (for $h_1 > 0$, the system describes a phantom universe). In this case $\tilde{t}$ is an inflection point and $\dot{H}(t) < 0$ for both $t > \tilde{t}$ and $t < \tilde{t}$, and therefore no transition to phantom phase happens. Assuming the potential has a lower positive bound $V(\phi, \sigma) \geq v_0 > 0$, and using the fact that $\dot{H}(t) < 0$ for all $t$’s, eqs. (7) and (8) result

$$H^2(t) > \frac{4\pi}{3}((2 - \gamma_m)\rho_m + 2V(\phi, \sigma))$$

$$> \frac{8\pi}{3}v_0. \quad (24)$$

Since $H(t)$ is a decreasing function, the above relation shows that $H^2(t)$ achieves its minimum at infinity (provided we assume that the system remains in quintessence phase for all $t$). Hence $\lim_{t \to \infty} H^2(t) = (8\pi/3)v_0$, while $\lim_{t \to \infty} V(\phi, \sigma) = v_0$ and $\lim_{t \to \infty} \rho_m = 0$, and the system tends to a de Sitter space time at late time, i.e $\lim_{t \to \infty} \dot{H}(t) = 0$ [21].

Now let us come back to even-$\alpha$ case and study the behavior of the equation of state parameter $\omega$ near $t = t_0$, where $\dot{H}(t_0) = 0$. For non-zero
differentiable \( H(t) \), \( \omega \) can be Taylor expanded as (to see a discussion about Taylor expansion of \( \omega \) in terms of scale parameter see [22])

\[
\omega(t) = \sum_{n=0}^{\infty} \frac{\omega^{(n)}(t_0)}{n!} (t - t_0)^n. \tag{25}
\]

From eq.(10), it is found that \( \omega(t_0) = -1 \) and that the first non-zero derivative of \( \omega(t) \) in its Taylor expansion is

\[
\omega^{(\alpha-1)}(t_0) = \frac{-2 H^{(\alpha)}(t_0)}{3 H^2(t_0)}, \tag{26}
\]

where as before we have denoted the order of the first non zero derivative of \( H(t) \) at \( t = t_0 \) by \( \alpha \). Therefore

\[
\omega(t) = -1 - \frac{2\alpha h_1}{3 h_0^2} (t - t_0)^{\alpha-1} + O((t - t_0)^\alpha), \tag{27}
\]
or in terms of \( V \),

\[
\omega(t) = -1 - \frac{2V_1}{V_0} (t - t_0)^{\alpha-1} + O((t - t_0)^\alpha). \tag{28}
\]

The above equation shows that (i) to cross the phantom-divide-line \( w = -1 \) \( \alpha \) must be an even integer, (ii) to go from quintessence phase, \( \omega > -1 \), to phantom phase, \( \omega < -1 \), we must have \( h_1 > 0 \), or equivalently \( V_1 > 0 \). Note that if initially the universe is in phantom phase, the condition (ii) for phase transition to quintessence era becomes \( h_1 < 0 \). These results are consistent with those we have obtained from studying the behavior of \( H(t) \) near \( t = t_0 \).

In brief, to have an accelerating universe, a specific combination of the quintom potential and matter density, denoted by \( V(t) \), must be positive. Also a transition from quintessence to phantom phase occurs provided the order of the first non-vanishing derivative of \( H(t) \) at \( t = t_0 \), in which \( \dot{H}(t_0) = 0 \), is even and the parameter \( h_1 \) obtained in eq.(18) is positive. We will illustrate these general results and discuss the phase transition via two important examples: exponential and power like quintom potentials, in the next section.

3 Examples

3.1 Exponential potential

As a first example we consider the quintom model with potential

\[
V = V_\phi + V_\sigma \\
= v_1 e^{\lambda_1 \phi} + v_2 e^{\lambda_2 \sigma}, \quad v_1 > 0, \quad v_2 > 0. \tag{29}
\]
To study the possible occurrence of transition from $\omega > -1$ to $\omega < -1$ for this potential, we try to obtain the solutions of eqs. (6), (7), and (8), when the Hubble parameter behaves as

$$H(t) = h_0 + h_1 t^\alpha + O(t^{\alpha+1}), \ \alpha \geq 2, \ \ h_1 \neq 0,$$

(30)

near $t = 0$, in which $H(0) = 0$. Eq. (6) cannot be solved exactly with the potential (29), but for slowly varying fields, where $\ddot{\phi} \ll H \dot{\phi}$ and $\ddot{\sigma} \ll H \dot{\sigma}$, these equations become

$$3(h_0 + h_1 t^\alpha) \dot{\phi}(t) = -\lambda_1 v_1 e^{\lambda_1 \phi(t)}$$

(31)

$$3(h_0 + h_1 t^\alpha) \dot{\sigma}(t) = \lambda_2 v_2 e^{\lambda_2 \sigma(t)},$$

with solutions

$$\phi(t) = \frac{1}{\lambda_1} \ln \left\{ \frac{3h_0 \alpha}{v_1 \lambda_1^2 \Phi\left(\frac{v_0 h_1}{h_0}, 1, \frac{1}{\alpha} \right) + c_1 h_0 \alpha} \right\},$$

(32)

$$\sigma(t) = \frac{1}{\lambda_2} \ln \left\{ -\frac{3h_0 \alpha}{v_2 \lambda_2^2 \Phi\left(\frac{v_0 h_1}{h_0}, 1, \frac{1}{\alpha} \right) + c_2 h_0 \alpha} \right\},$$

in which $\Phi(z, a, b)$ is the Lerchphi function. The constants $c_1$ and $c_2$, in terms of the initial conditions at $t = 0$, are defined through

$$\phi(0) = \frac{1}{\lambda_1} \ln \left( \frac{3}{v_1 \lambda_1^2 c_1} \right)$$

(33)

$$\sigma(0) = \frac{1}{\lambda_2} \ln \left( -\frac{3}{v_2 \lambda_2^2 c_2} \right).$$

Hence $c_1 > 0$ and $c_2 < 0$. Near $t = 0$, up to order $O(t)$, we have

$$\ddot{\phi}(0) = \frac{1}{h_0^2 c_1^2 \lambda_1},$$

$$3h_0 \dot{\phi}(0) = -\frac{3}{\lambda_1 c_1},$$

(34)

and

$$\ddot{\sigma}(0) = \frac{1}{h_0^2 c_2^2 \lambda_2},$$

$$3h_0 \dot{\sigma}(0) = -\frac{3}{\lambda_2 c_2}.$$  

(35)

Therefore the slowly varying condition is equivalent to

$$|c_1 h_0^2| \gg 1, \quad |c_2 h_0^2| \gg 1.$$  

(36)
These inequalities can be expressed in terms of the potentials as following: 
\[ |\frac{1}{\alpha} (\frac{dV}{d\alpha})^2(0)| < h_0^2 \] and 
\[ |\frac{1}{\alpha} (\frac{d\phi}{d\alpha})^2(0)| < h_0^2, \] respectively.

\( \rho_m(t) \) satisfies the equation

\[ \dot{\rho}_m(t) + 3\gamma_m(h_0 + h_1t)\rho_m(t) = 0. \] (37)

The solution of this equation is

\[ \rho_m(t) = c_3 e^{-3\gamma_m(h_0 t + \frac{h_1}{h_0} t^{\alpha+1})}, \] (38)

where \( c_3 = \rho_m(t = 0) \).

In terms of dimensionless variables \( C_1 = 1/(c_1 h_0^2) \), \( C_2 = 1/(c_2 h_0^2) \), \( C_3 = c_3/h_0^2 \), \( \tau = h_0 t \), and \( H_1 = h_1/h_0^{\alpha+1} \), eq.(7) becomes

\[ \frac{H^2(\tau)}{h_0^2} = \frac{4\pi}{3} \left\{ \frac{\alpha^2}{\tau \Phi(-\tau^\alpha H_1, 1, \frac{1}{\alpha}) + \frac{\alpha}{C_1}}^2 (1 + H_1\tau^\alpha)^2 \lambda_1^2 - \right\} \] (39)

\[ \frac{\alpha^2}{\tau \Phi(-\tau^\alpha H_1, 1, \frac{1}{\alpha}) + \frac{\alpha}{C_2}}^2 (1 + H_1\tau^\alpha)^2 \lambda_2^2 + \frac{6\alpha}{\lambda_1^2 \tau \Phi(-\tau^\alpha H_1, 1, \frac{1}{\alpha}) + \frac{\alpha}{C_1}} + \frac{6\alpha}{\lambda_2^2 \tau \Phi(-\tau^\alpha H_1, 1, \frac{1}{\alpha}) + \frac{\alpha}{C_2}} + 2C_3 e^{-3\gamma_m(\tau + \frac{H_1}{h_0^{\alpha+1}} \tau^{\alpha+1})} \}

and eq.(8) reduces to

\[ \frac{1}{h_0} \frac{dH(\tau)}{d\tau} = -4\pi \left\{ \frac{\alpha^2}{\tau \Phi(-\tau^\alpha H_1, 1, \frac{1}{\alpha}) + \frac{\alpha}{C_1}}^2 (1 + H_1\tau^\alpha)^2 \lambda_1^2 - \right\} \] (40)

\[ \frac{\alpha^2}{\tau \Phi(-\tau^\alpha H_1, 1, \frac{1}{\alpha}) + \frac{\alpha}{C_2}}^2 (1 + H_1\tau^\alpha)^2 \lambda_2^2 + \gamma_m C_3 e^{-3\gamma_m(\tau + \frac{H_1}{h_0^{\alpha+1}} \tau^{\alpha+1})} \}

By expanding the right hand sides of eqs. (39) and (40) near \( \tau = 0 \) and using eq. (39) for their left hand sides, one finds:

\[ 1 + 2H_1\tau^\alpha + O(\tau^{\alpha+1}) = \frac{4\pi}{3} \left[ \frac{C_1(C_1 + 6)}{\lambda_1^2} - \frac{C_2(C_2 + 6)}{\lambda_2^2} + 2C_3 \right] \]

\[ -8\pi \left[ \frac{C_1^2(C_1 + 1)}{\lambda_1^4} - \frac{C_2^2(C_2 + 1)}{\lambda_2^4} + \gamma_m C_3 \right] \tau \]

\[ + \frac{4\pi}{3} \left[ 3\tau^2 C_3 + 2 \frac{C_1^3}{\lambda_1^4} - 2 \frac{C_2^3}{\lambda_2^4} \right] + H_1 \]

\[ + 3 \left( \frac{C_1^4}{\lambda_1^4} - \frac{C_2^4}{\lambda_2^4} \right) \tau^2 + O(\tau^3), \] (41)
and
\[ \alpha H_1 \tau^{\alpha - 1} + O(\tau^\alpha) = -4\pi \left( \gamma_m C_3 + \frac{C_1^2}{\lambda_1^2} - \frac{C_2^2}{\lambda_2^2} \right) \]
\[ + 4\pi \left( 3\gamma_m^2 C_3 + \frac{2C_1^3}{\lambda_1^2} - \frac{2C_2^3}{\lambda_2^2} \right) \tau + O(\tau^2), \]  
(42)
respectively. To have a consistent set of equations for \( \alpha \geq 2 \), one finds the equalities
\[ \frac{4\pi}{3} \left( \frac{C_1(C_1 + 6)}{\lambda_1^2} - \frac{C_2(C_2 + 6)}{\lambda_2^2} + 2C_3 \right) = 1, \]
(43)
and
\[ \frac{C_1^2(C_1 + 1)}{\lambda_1^2} - \frac{C_2^2(C_2 + 1)}{\lambda_2^2} + \gamma_m C_3 = 0, \]
(44)
from the first two terms of eq.(41) and
\[ \gamma_m C_3 + \frac{C_1^2}{\lambda_1^2} - \frac{C_2^2}{\lambda_2^2} = 0, \]
(45)
from the first term of eq.(42), respectively. In slow-roll approximation, which we are working in, the parameters satisfy eq.(36), which imply \( |C_1| << 1 \) and \( |C_2| << 1 \). In this approximation, the eqs.(44) and (45) are equivalent.

The important observation is that since \( C_1 \) and \( C_3 \) are positive and \( C_2 \) is negative real numbers, the coefficient of \( \tau \) in eq.(42) is a non-zero positive number, which results the following for exponential potential:
\[ \alpha_{\text{exp.}} = 2, \]
\[ (h_1)_{\text{exp.}} > 0. \]
(46)
In this way it is proved that in the quintom model with exponential potential, the system has a phase transition from quintessence to phantom phase, if any other remaining relations, obtained from eqs.(41) and (42), will be satisfied consistently up to the lowest order.

To check the remaining relations, we first note that since \( \alpha = 2 \), eq.(42) results another equation, as follows,
\[ H_1 = 2\pi \left( 3\gamma_m^2 C_3 + \frac{2C_1^3}{\lambda_1^2} - \frac{2C_2^3}{\lambda_2^2} \right). \]
(47)
But eqs.(44) and (45) imply: \( C_1^3/\lambda_1^2 - C_2^3/\lambda_2^2 << |C_1^2/\lambda_1^2 - C_2^2/\lambda_2^2| \approx C_3 \), so eq.(47) reduces to:
\[ H_1 \simeq 6\pi \gamma_m^2 C_3. \]
(48)
This shows that the rate of the phase transition depends on the parameter of state \( \gamma_m \) and the dark matter density at transition time.
In terms of dark energy and matter densities, $\rho_D, \rho_m$, eq. (43) can be written as

$$\frac{\rho_m}{\rho_c} + \frac{2D}{\rho_c} = 1,$$

(49)

where $\rho_c = 3h_0^2/8\pi$, $\rho_m(0)/\rho_c = (8\pi/3)C_3$ and $\rho_D(0)/\rho_c = (4\pi/3)[C_1(C_1 + 6)/\lambda_1^2 - C_2(2C_2 + 6)/\lambda_2^2]$. The dark energy density is composed of two parts: the density of kinetic dark energy, $(4\pi/3)(C_1^2/\lambda_1^2 - C_2^2/\lambda_2^2)\rho_c$, which is negative according to eq. (45), and the density of potential dark energy, $(4\pi/3)(6C_1/\lambda_1^2 - 6C_2/\lambda_2^2)\rho_c$. Now since $C_1/\lambda_1^2 - C_2/\lambda_2^2 >> C_1^2/\lambda_1^2 - C_2^2/\lambda_2^2 \simeq C_3$, it is obvious that the main part of the energy density at the transition time is coming from the potential of dark energy.

The remaining equation that can be extracted from eq.(41) is

$$H_1 = \frac{2\pi}{3} \left[ 3\left( 3\gamma_m C_3 + 2\frac{C_3^2}{\lambda_1^2} - 2\frac{C_2^2}{\lambda_2^2} \right) + 2\left( \frac{C_2^2}{\lambda_2^2} - \frac{C_1^2}{\lambda_1^2} \right) \right] H_1$$

$$+ 3\left( \frac{C_1^4}{\lambda_1^2} - \frac{C_2^4}{\lambda_2^2} \right)$$

$$= H_1 \left[ 1 + 2\left( \frac{C_2^2}{\lambda_2^2} - \frac{C_1^2}{\lambda_1^2} \right) \right] + O(C^3),$$

(50)

which can be verified using the fact that $C_1^2/\lambda_1^2 - C_2^2/\lambda_2^2 << C_1/\lambda_1^2 - C_2/\lambda_2^2 \simeq \rho_{\text{total}}/\rho_c = 1$. This completes our desired checking of all relations in the lowest order.

Expanding $V(t)/h_0^2 = (V + (1 - \frac{2\pi}{3})\rho_m)/h_0^2$ at $t = 0$ results

$$\frac{V(t)}{h_0^2} = \frac{(2 - \gamma_m)C_3}{2} \left[ \frac{3C_1}{\lambda_1^2} - \frac{3C_2}{\lambda_2^2} \right]$$

$$+ \frac{3}{2} \left[ C_3 \left( \gamma_m^2 - 2\gamma_m \right) \right] + O(\tau^2)$$

$$= \frac{3}{8\pi} \left[ 1 + \frac{1}{4\pi}H_1\tau \right] + O(\tau^2),$$

(51)

where we have used eqs. (43), (45), and (48) in the second equality. The above result is in complete agreement with our previous assertions eqs.(43) and (48).

To test, in another way, the validity of our approximations let us choose $C_1 = 10^{-4}$, $C_2 = -1.1 \times 10^{-4}$ (since it must be small quantities), $\gamma_m = 1$, and $\lambda_1 = \lambda_2$ (for simplicity), at $t = 0$. Eqs. (43), (45) and (47) then result $C_3 = 4.0 \times 10^{-7}$, $\lambda_1 = \lambda_2 = 0.073$, and $H_1 = 7.5 \times 10^{-6}$. Fig. (11) shows the plot of $(1 + H_1\tau^2)^2$ and the same plot obtained from eq.(49). Fig. (2) shows the comparison of $\omega$ in these two approaches.

At the end, it is worth noting that all the above approximations are valid until $\tau << 1$, which is equivalent to $t << h_0^{-1}$. But this is reasonable period of time since $h_0^{-1}$ is of order of the age of our universe.
Figure 1: $H^2/h_0^2$ as a function of $\tau$, obtained from eq.(39) (points), and $(1 + H_1 \tau^2)^2$ (line).

Figure 2: $\omega$ versus $\tau$, using approximation (39) (points) and $H = h_0(1 + H_1 \tau^2)$ (line) in eq.(10).
3.2 Power-law potential

As the second example consider the potential

\[ V = V_\phi + V_\sigma = v_1 \phi^{b_1} + v_2 \sigma^{b_2}, \quad (52) \]

Using the approximation (30), the solutions of the eq.(6) are

\[ \phi(t) = \left( \frac{v_1 b_1 (b_1 - 2) t \Phi(-\frac{\alpha h_0}{h_0}, 1, \frac{1}{\alpha})}{3h_0 \alpha} + c_1 \right)^{\frac{1}{2-b_1}} \]

\[ \sigma(t) = \left( \frac{v_2 b_2 (-b_2 + 2) t \Phi(-\frac{\alpha h_0}{h_0}, 1, \frac{1}{\alpha})}{3h_0 \alpha} + c_2 \right)^{\frac{1}{2-b_2}} \quad (53) \]

where \( c_1 = \phi^{2-b_1}(0) \) and \( c_2 = \sigma^{2-b_2}(0) \). We assume \( \phi(0) > 0 \) and \( \sigma(0) > 0 \).

Using these solutions one can show that neglecting second derivatives in eq.(6), is allowed when

\[ |b_1 (b_1 - 1) \frac{v_1}{h_0^2} \phi^{b_1-2}(0) | \ll 1, \]

\[ |b_2 (b_2 - 1) \frac{v_2}{h_0^2} \sigma^{b_2-2}(0) | \ll 1. \quad (54) \]

In term of potentials, these inequalities are \( \frac{d^2 V_\phi}{d\phi^2} |(0) \ll h_0^2 \) and \( \frac{d^2 V_\sigma}{d\sigma^2} |(0) \ll h_0^2 \). For \( b_1 = b_2 = 2; v_1 = m_\phi^2 / 2 \) and \( v_2 = m_\sigma^2 / 2 \), eq.(53) reduces to

\[ \phi(t) = \phi(0) e^{-m_\phi^2 t^2 \Phi(-\frac{\alpha h_0}{h_0}, 1, \frac{1}{\alpha})}, \]

\[ \sigma(t) = \sigma(0) e^{-m_\sigma^2 t^2 \Phi(-\frac{\alpha h_0}{h_0}, 1, \frac{1}{\alpha})}. \quad (55) \]

In this case, the approximation (54) is reduced to small mass limits for phantom and quintessence fields: \( m_\phi \ll h_0 \) and \( m_\sigma \ll h_0 \).

\( \rho_m \) is given by eq.(38). By putting eqs.(38) and (39) into eqs.(7) and (9), and using dimensionless variables \( V_1 = v_1 h_0^2, V_2 = v_2 h_0^2, C_3 = c_3 / h_0^2, \)

\( \tau = \theta_0, \) and \( H_1 = h_1 / h_0^{\alpha+1}, \) near \( \tau = 0, \) we obtain

\[ 1 + 2H_1 \tau^\alpha + O(\tau^{\alpha+1}) = \frac{4\pi}{3} \left[ \phi^{2b_1-2}(0) b_1^2 V_1^2 - \sigma^{2b_2-2}(0) b_2^2 V_2^2 \right] + 2C_3 + 2V_1 \phi^{b_1}(0) + \]

\[ 2V_2 \sigma^{b_2}(0) + 4\pi \left[ \frac{1}{27} - \frac{2 b_1^3 (b_1 - 1) V_1^3 \phi^{3b_1-4}(0)}{27} - \frac{2 b_2^3 (b_2 - 1) V_2^3 \sigma^{3b_2-4}(0)}{27} - 67 \right] \tau \]
and

\[
\begin{align*}
\alpha H_1 \tau^{\alpha - 1} + O(\tau^\alpha) &= -4\pi \left[ \frac{\phi^{2(b_1 - 1)}(0)b_1^2 V_1^2}{9} - \frac{\sigma^{2(b_2 - 1)}(0)b_2^2 V_2^2}{9} + \gamma_m C_3 \right] \\
&\quad - 4\pi \left[ -\frac{2b_1^3(b_1 - 1)\phi^{2(b_1 - 1)}(0)}{27} \left( 1 + \frac{1}{9} b_1 (b_1 - 1) V_1 \phi^{b_1 - 2}(0) \right) - \right. \\
&\quad \left. V_2^2 b_2^2 \sigma^{2(b_2 - 1)}(0) \left( 1 - \frac{1}{9} b_2 (b_2 - 1) V_2 \phi^{b_2 - 2}(0) \right) \right] = 0,
\end{align*}
\]

(58)

and

\[
\begin{align*}
\frac{4\pi}{3} \left[ \frac{\phi^{2(b_1 - 2)}(0)b_1^2 V_1^2}{9} - \frac{\sigma^{2b_2 - 2}(0)b_2^2 V_2^2}{9} + 2C_3 + 2V_1 \phi^{b_1}(0) + 2V_2 \sigma^{b_2}(0) \right] &= 1.
\end{align*}
\]

(60)

For slowly varying fields (eq. 541), eq. 59 reduces to eq. 58. The coefficient of \( \tau \) in eq. 57 is

\[
\begin{align*}
4\pi \left[ \frac{2b_1^3(b_1 - 1)\phi^{3b_1 - 4}(0)V_1^3}{27} + \frac{2b_2^3(b_2 - 1)\sigma^{3b_2 - 4}(0)V_2^3}{27} \right] + 3\gamma_m^2 C_3.
\end{align*}
\]

(61)

Comparing eqs. (58) and (59) shows \( \phi^{3b_1 - 4}(0)V_1^3 + \sigma^{3b_2 - 4}(0)V_2^3 \ll \phi^{2b_1 - 2}(0)V_1^2 - \sigma^{2b_2 - 2}(0)V_2^2 \approx C_3 \), which reduces eq. (61) to

\[
H_1 \approx 6\pi \gamma_m^2 C_3.
\]

(62)
This is a non-zero positive number. So for quintom model with power-law potential, we show

\[ \alpha_{\text{power-law}} = 2, \]
\[ (h_1)_{\text{power-law}} > 0, \]

which proves the existence of quintessence to phantom phase transition in this model. Of course the remaining relations must be also checked. Note that the value of \( H_1 \) at transition point does not depend on the potential, compare eqs. (15) and (52), a fact that we guess is true for any other potential.

To check the consistency of the remaining equations, first we note that by the same arguments used in the preceding example, it can be shown that the main contribution in eq. (60), which can be written as \( \rho_{\text{total}} / \rho_c = 1 \), comes from the quintom potential, i.e. \( V_1 \phi^{b_1}(0) + V_2 \sigma^{b_2}(0) \). Now the coefficient of \( \tau^2 \) in the right-hand-side of eq. (56), using eq. (61), can be written as

\[ 2H_1 \left[ 1 + \frac{4\pi}{3} \left( -\frac{\phi^{2(b_1-1)}(0)b_1^2V_1^2}{9} + \frac{\sigma^{2(b_2-1)}(0)b_2^2V_2^2}{9} \right) + \cdots \right], \]

which is equal to \( 2H_1 \), using the fact that \( -V_1^2 \phi^{2(b_1-1)}(0) + V_2^2 \sigma^{2(b_2-1)}(0) << V_1 \phi^{b_1}(0) + V_2 \sigma^{b_2}(0) \approx \rho_{\text{total}} / \rho_c = 1 \). Finally it can be shown that eq. (51) is also verified for this potential.

As an illustration of the transition behavior of quintom model with potential (52), see Fig. (3), which shows the plot of \( H_2 / h_0^2 \) obtained from eq. (7), for \( b_1 = b_2 = 2, \gamma_m = 1, v_1 = m_\phi^2 / 2, v_2 = m_\sigma^2 / 2, \) with \( m_\phi / h_0 = m_\sigma / h_0 = 0.01 \) (since they must be small quantities) and \( \sigma^2(0) = 2\sigma^2(0) \). Eqs. (55), (60) and (62) then result \( \phi^2(0) = 2500.0 / \pi, C_3 = 0.28 \times 10^{-5} / \pi, \) and \( H_1 = 0.17 \times 10^{-4} \). See also Fig. (4) for \( \omega \).

4 Conclusion

In this paper the phase transition from quintessence to phantom era in quintom model has been discussed. By the assumption that the Hubble parameter has a Taylor expansion in terms of comoving time, the behavior of the Hubble parameter and the potential near the transition time, and the relation between them, have been studied. See eqs. (14)-(16). The conditions need to satisfy to have a quintessence to phantom phase transition have been calculated, i.e. evenness of \( \alpha \) and eq. (18). The same results obtained by studying the relations between equation of state parameter \( \omega \), the quintom potential and matter density, see eq. (28).

To be specific, we have considered special cases by determining the quintom potential. For slowly varying exponential quintom potential, we have shown that the equations are consistent when \( \alpha = 2 \) and \( h_1 > 0 \). In this way it has been proved that the phase transition occurs with a rate depending
Figure 3: $H^2/h_0^2$ as a function of $\tau$, obtained from eq. (7) (with initial conditions mentioned in the text) (points), and $(1 + H_1 \tau^2)^2$ (line).

Figure 4: $\omega$ versus $\tau$, using approximation (56) (points) and $H = h_0(1 + H_1 \tau^2)$ (line) in eq.(10) for power-law potential.
on the density of matter at transition time, see the discussion after eq. (48). The relation of matter and dark energy fields and their densities have been obtained at transition time. See eqs. (43), (45) and discussion after eq. (49). For slowly varying power law potentials, similar results have been deduced.

In these examples we restricted ourselves to slow roll quintom model near the transition time. For $\gamma_m \simeq 1$, we expect at the transition time $\dot{H} = 0$, the kinetic energy of quintom field be of the same order of magnitude as dark matter density, see eq. (8). Using this conclusion in eq. (7) results that the main part of the energy density is provided by the potential energy of scalar fields. Therefore the dark matter density is very small with respect to the density of dark energy, hence the coincidence problem still unsolved in these examples.

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