# An Inexact Uzawa Algorithm
for Generalized Saddle-Point Problems and Its Convergence

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## Abstract

We propose an inexact Uzawa algorithm with two variable relaxation parameters for solving the generalized saddle-point system. The saddle-point problems can be found in a wide class of applications, such as the augmented Lagrangian formulation of the constrained minimization, the mixed finite element method, the mortar domain decomposition method and the discretization of elliptic and parabolic interface problems. The two variable parameters can be updated at each iteration, requiring no a priori estimates on the spectrum of two preconditioned subsystems involved. The convergence and convergence rate of the algorithm are analysed. Both symmetric and nonsymmetric saddle-point systems are discussed, and numerical experiments are presented to demonstrate the robustness and effectiveness of the algorithm.

## 1 Introduction

The aim of the current work is to develop an inexact preconditioned Uzawa algorithm for the generalized saddle-point problem of the form

\[
\begin{pmatrix}
A & B \\
B^t & -D
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
=
\begin{pmatrix}
f \\
g
\end{pmatrix},
\]

where \(A\) is an \(n \times n\) symmetric and positive definite matrix, \(B\) is an \(n \times m\) matrix, and \(D\) is an \(m \times m\) symmetric positive semi-definite matrix. We shall assume that the Schur complement matrix

\[
S = B^tA^{-1}B + D
\]

associated with the system (1.1) is an \(m \times m\) symmetric and positive definite matrix, which ensures the unique solvability of system (1.1). System (1.1) arises from many areas of computational sciences and engineering, such as the constrained optimization, the mixed finite element formulation for the second order elliptic equation, the linear elasticity problem, as well as elliptic and parabolic interface problems; see [5] [6] [9] [10] [16] and Section 6 for several such applications.

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Many numerical methods such as Schur complement reduction methods, null space methods, penalty methods, multilevel methods, Krylov subspace methods and preconditioning, are investigated to solve the saddle point problem (1.1), especially for solving the simplest case of the saddle-point system (1.1) when the (2, 2) block \( D \) vanishes; see [1] [4] [3] [10] [18] [19] [2] and the references therein. In particular, the inexact preconditioned Uzawa-type algorithms have attracted wide attention; see [1] [4] [3] [7] [11] [12] [13] [18], and the references therein. These inexact Uzawa-type algorithms have an important feature that they preserve the minimal memory requirement and do not need actions of the inverse matrix \( A^{-1} \). On the contrary, few studies on the convergence analysis of inexact preconditioned Uzawa iterative methods can be found in the literature for the generalized saddle-point system (1.1) where a general block \( D \) is present. This work intends to make some initial efforts to fill in the gap.

Suppose that \( \hat{A} \) and \( \hat{S} \) are two symmetric and positive definite matrices, and act as the preconditioners for \( A \) and \( S \), respectively. We shall be interested in the following inexact preconditioned Uzawa method for solving the system (1.1).

\[
\begin{align*}
x_{i+1} &= x_i + \omega_i \hat{A}^{-1} (f - Ax_i - By_i), \\
y_{i+1} &= y_i + \tau_i \hat{S}^{-1} (B^t x_{i+1} - Dy_i - g),
\end{align*}
\]

(1.2)

where \( \omega_i \) and \( \tau_i \) are two relaxation parameters to be determined at each iteration. Equivalently, the system (1.2) can be written as

\[
\begin{align*}
\hat{A} \frac{x_{i+1} - x_i}{\omega_i} + Ax_i + By_i &= f, \\
\hat{S} \frac{y_{i+1} - y_i}{\tau_i} + B^t x_{i+1} - Dy_i &= g.
\end{align*}
\]

We shall often need the approximate Schur complement of \( S \), namely

\[
H = B^t \hat{A}^{-1} B + D.
\]

The inexact preconditioned Uzawa method (1.2) with two variable relaxation parameters was first proposed and analysed in [11] for the simple case of \( D = 0 \), and different variants of the algorithm were further studied in [12] [13]. The original idea of introducing the variable parameters \( \omega_i \) and \( \tau_i \) was to ensure that the resulting inexact preconditioned Uzawa algorithms always converge for any available symmetric and positive definite preconditioners \( \hat{A} \) and \( \hat{S} \), and converge nicely when effective preconditioners are available. Nearly all other existing preconditioned Uzawa algorithms do not adopt self-updating relaxation parameters, and converge only under some proper scalings of the preconditioners \( \hat{A} \) and \( \hat{S} \).

The choice of the relaxation parameters \( \omega_i \) and \( \tau_i \) in (1.2) is not straightforward. They should be easily updated at each iteration and their evaluations should be less expensive. The usual choices of parameters by minimizing the errors \( x - x_i \) and \( y - y_i \) in certain norms do not work since the evaluation of the resulting parameters always involve the action of \( A^{-1} \); see [11] for details.

Next, we follow [11] to work out an effective way to evaluate the two relaxation parameters \( \omega_i \) and \( \tau_i \) in (1.2). To do so, we consider the two residuals associated with the \( i \)-th iteration:

\[
\begin{align*}
f_i &= f - (Ax_i + By_i), \\
g_i &= B^t x_{i+1} - Dy_i - g.
\end{align*}
\]

(1.3)
Then we may determine the parameter $\omega_i$ by minimizing 

$$|\omega_i \hat{A}^{-1} f_i - A^{-1} f_i|_A^2,$$

which yields

$$\omega_i = \frac{\langle f_i, r_i \rangle}{\langle A r_i, r_i \rangle}, \quad (1.4)$$

where $r_i = \hat{A}^{-1} f_i$, and $\langle \cdot, \cdot \rangle$ stands for the inner product of two vectors in Euclidean space. The parameter $\tau_i$ can be determined by minimizing

$$|\tau_i \hat{S}^{-1} g_i - H^{-1} g_i|_H^2,$$

which gives a prototype choice

$$\tau_i = \frac{\langle g_i, s_i \rangle}{\langle H s_i, s_i \rangle}, \quad (1.5)$$

with $s_i = \hat{S}^{-1} g_i$. But as we shall see, such choice of $\tau_i$ may not guarantee the convergence of Algorithm 1. We need a damping factor $\theta_i$ for the parameter $\hat{\tau}_i$ in (1.5), and will take $\tau_i$ in (1.2) as

$$\tau_i = \theta_i \hat{\tau}_i = \frac{\langle g_i, s_i \rangle}{\langle H s_i, s_i \rangle}, \quad (1.6)$$

The algorithm can be summarized as follows.

**Algorithm 1** Linear inexact Uzawa algorithm with variable relaxation.

1. Compute $f_i = f - (Ax_i + By_i)$, $r_i = \hat{A}^{-1} f_i$, and $\omega_i = \frac{\langle f_i, r_i \rangle}{\langle A r_i, r_i \rangle}$;
2. Update $x_{i+1} = x_i + \omega_i r_i$;
3. Compute $g_i = B^t x_{i+1} - D y_i - g$, $s_i = \hat{S}^{-1} g_i$, and $\tau_i = \theta_i \frac{\langle g_i, s_i \rangle}{\langle H s_i, s_i \rangle}$;
4. Update $y_{i+1} = y_i + \tau_i s_i$.

Algorithm was analyzed in [11] for the simplest case of saddle-point problem (1.1) when the (2,2) block $D$ vanishes. Unfortunately the convergence and convergence rate of Algorithm were established still under some appropriate scaling of preconditioner $\hat{A}$ for $A$, i.e., the smallest eigenvalue of the preconditioned system $\hat{A}^{-1} A$ is larger than one, although no any appropriate scaling of preconditioner $\hat{S}$ for Schur Complement $S$ was needed. In this work we shall extend the analysis in [11] to the more general and challenging indefinite system (1.1), where the block $D$ is present. As it will be seen, such an extension is highly nontrivial for a general block $D$. We need to make essential modifications of the major analysis techniques in [11] and introduce several crucial new techniques in order to succeed in analyzing the convergence and convergence rate of Algorithm for general $D \neq 0$. It is important to remark that for the case of $D = 0$, our subsequent analysis will improve the convergence results, relax the convergence conditions in [11] and provide instructive information on the selection of the damping parameter $\theta_i$ to ensure the convergence. Unlike in [11], we will not
assume appropriate scalings of two preconditioners \(\hat{A}\) and \(\hat{S}\) for the convergence of Algorithm 1.

We will also generalize Algorithm 1 to the cases when the action of preconditioner \(\hat{A}\) or \(\hat{S}\) is replaced by a nonlinear iterative solver. This is more practical and important for some applications where effective preconditioners are not available. The proposed algorithm is also analyzed and tested numerically when \(A\) in (1.1) is nonsymmetric. No such analysis is available in the literature when inexact preconditioners are used.

For the sake of clarity, we list the main notations used later.

### Some notations and definitions

- \(S = B^t A^{-1} B + D\), and \(S = R R^t\) with \(R\) being nonsingular
- \(\hat{S}\): an spd approximation of \(S\)
- \(H = B^t \hat{A}^{-1} B + D\), where \(\hat{A}\) is an approximation of \(A\)
- \(\theta_i\): damping factor for the parameter \(\hat{\tau}_i\)
- \(\alpha = (\kappa_1 - 1)/(\kappa_1 + 1)\), where \(\kappa_1 = \text{cond}(\hat{A}^{-1} A)\)
- \(\beta = (\kappa_2 - 1)/(\kappa_2 + 1)\), where \(\kappa_2 = \text{cond}(\hat{S}^{-1} H)\)
- \(c_0\): the largest eigenvalue of \(D^{-1} B^t \hat{A}^{-1} B\) (see Lemma 2.3)
- \(c_1\): the constant defined in (3.8) or (3.9)
- \(Q_i\): defined by \(Q_i^{-1} = \theta_i G_i^{-1}\) in (3.4), and \(G_i\) is given in Lemma 2.3
- \(\lambda, \lambda_0\): \(\lambda \hat{A} \leq A \leq \lambda_0 \hat{A}\); see (2.2)
- \(\delta_1, \delta_2\): \(\delta_1 = (\lambda_0 + c_0)/[\lambda_0(1 + c_0)]\), \(\delta_2 = (\lambda + c_0)/[\lambda(1 + c_0)]\) (see Lemma 2.3)
- \(\omega, \Omega\): \(\omega |z|_2 \leq |Wz|_2 \leq \Omega |z|_2\), \(\forall z\); see (3.10)

## 2 Basic formulation

We shall often use the condition numbers of the two preconditioned systems

\[
\kappa_1 = \text{cond}(\hat{A}^{-1} A), \quad \kappa_2 = \text{cond}(\hat{S}^{-1} H)
\]

and the following two convergence-rate related constants

\[
\alpha = \frac{\kappa_1 - 1}{\kappa_1 + 1}, \quad \beta = \frac{\kappa_2 - 1}{\kappa_2 + 1}.
\]

For any two symmetric and semi-positive definite matrices \(C_1\) and \(C_2\) of order \(m\) satisfying

\[
\langle C_1 \phi, \phi \rangle \leq \langle C_2 \phi, \phi \rangle, \quad \forall \phi \in \mathbb{R}^m,
\]

we will simply write

\[
C_1 \preceq C_2.
\]

### 2.1 When the \((2, 2)\) block \(D\) vanishes

The convergence of Algorithm 1 was analyzed in [11] for the saddle-point system (1.1) with \(D = 0\) under the condition that preconditioner \(\hat{A}\) for \(A\) is appropriately scaled such that

\[
\hat{A} \leq A \leq \hat{\lambda}_0 \hat{A} \quad (2.1)
\]
for some constant $\lambda_0 \geq 1$. Next, we will demonstrate the convergence of Algorithm 1 without the condition (2.1). In fact, since $\hat{A}$ is a general preconditioner for $A$, there are always two positive constants $\lambda$ and $\lambda_0$ such that $\lambda \leq 1 \leq \lambda_0$ and

$$\lambda \hat{A} \leq A \leq \lambda_0 \hat{A}. \quad (2.2)$$

Noting that (2.2) is not an actual assumption since it is always true. Now we let

$$\tilde{A} = \lambda \hat{A}, \quad \tilde{\lambda}_0 = \lambda_0 / \lambda. \quad (2.3)$$

Then we can rewrite (2.2) as

$$\tilde{A} \leq A \leq \tilde{\lambda}_0 \tilde{A}. \quad (2.4)$$

In terms of this newly introduced $\tilde{A}$, one may express Algorithm 1 as follows (noting that $D = 0$):

$$x_{i+1} = x_i + \tilde{\omega}_i \tilde{A}^{-1}(f - Ax_i - By_i),$$

$$y_{i+1} = y_i + \tilde{\tau}_i \tilde{S}^{-1}(B^t x_{i+1} - g),$$

where $\tilde{S} = \lambda \hat{S}$ and the damping parameters $\tilde{\omega}_i$ and $\tilde{\tau}_i$ are given by

$$\tilde{\omega}_i = \lambda \omega_i, \quad \tilde{\tau}_i = \lambda \tau_i. \quad (2.5)$$

Let us introduce a parameter

$$\alpha_i = \frac{|(I - \tilde{\omega}_i A^{1/2} \hat{A}^{-1} A^{1/2}) A^{-1/2} f_i|}{|A^{-1/2} f_i|} = \frac{|(I - \omega_i A^{1/2} \hat{A}^{-1} A^{1/2}) A^{-1/2} f_i|}{|A^{-1/2} f_i|}, \quad (2.6)$$

where the residual $f_i$ is defined by (1.3). Then we can show

**Lemma 2.1.** For the parameters $\tilde{\lambda}_0$, $\tilde{\omega}_i$ and $\alpha_i$ defined respectively in (2.3), (2.5) and (2.6), it holds that

$$\frac{1}{\tilde{\lambda}_0} \leq \tilde{\omega}_i \leq 1 - \alpha_i^2, \quad 0 \leq \alpha_i \leq \alpha.$$

Proof. First note that if we follow the same way as we get $\omega_i$ in (1.4) when $\hat{A}$ is replaced by $\tilde{A}$, we derive a new parameter $\tilde{\omega}_i$, which is exactly the one given by $\tilde{\omega}_i = \lambda \omega_i$ in (2.5). Then following the proof of Lemma 3.1 in [11] by means of the relations (2.4) and the fact that $\text{cond}(\tilde{A}^{-1} A) = \text{cond}(\hat{A}^{-1} A)$, we can derive the desired estimates. □

Intuitively it is easy to understand that Algorithm 1 may not converge for an arbitrary damping parameter $\theta_i$ in (1.6). Following the convergence analysis in [11] and using Lemma 2.1, we have the following convergence.

**Lemma 2.2.** For any damping parameter $\theta_i$ in (1.6) satisfying

$$\theta_i (1 + \beta) \leq 1 - \alpha_i, \quad (2.7)$$

Algorithm 1 converges.
Lemma 2.3. Schur complement and is essential to the subsequent convergence analysis. Following lemma illustrates this fact in terms of the eigenvalues of the preconditioned $D$ with respect to the convergence of Algorithm 1 depends strongly on the relative scale of $D$. In this subsection we will study the convergence of Algorithm 1 when the block matrix $D$ is present. As we will see, the convergence of Algorithm 1 is much more complicated than the case with $D = 0$, and it is essential to scale preconditioner $\hat{A}$ for $D \neq 0$ since the convergence of Algorithm 1 depends strongly on the relative scale of $B'\hat{A}^{-1}B$ with respect to $D$ in the approximated Schur complement $H = B'\hat{A}^{-1}B + D$. The following lemma illustrates this fact in terms of the eigenvalues of the preconditioned Schur complement and is essential to the subsequent convergence analysis.

**Remark 2.1.** The formula (2.7) gives a range to choose the damping parameter $\theta_i$ for the convergence of Algorithm 1, but it does not provide the best choice of $\theta_i$.

If one can estimate the lower bound $\lambda$ in (2.2), say $\hat{\lambda}$ is such an estimate. That is, $\hat{\lambda} \omega_i \leq \lambda \omega_i = \hat{\omega}_i$. Therefore, $1 - \sqrt{1 - \lambda \omega_i} \leq 1 - \sqrt{1 - \hat{\lambda} \omega_i} \leq 1 - \alpha_i$. Then we can have a more explicit range for $\theta_i$ to ensure the convergence:

$$\theta_i \leq \frac{1 - \sqrt{1 - \lambda \omega_i}}{2}.$$  \hspace{1cm} (2.8)

In fact, it follows from (2.8) that $\theta_i (1 + \beta) \leq 2 \theta_i \leq 1 - \sqrt{1 - \lambda \omega_i} \leq 1 - \alpha_i$, hence Algorithm 1 converges by Lemma 2.2. A lower bound estimate $\hat{\lambda}$ may be obtained by knowing an upper bound $\hat{\kappa}_1$ of the condition number $\kappa_1 = \lambda_0/\lambda$ and a lower bound $\lambda_0$ of $\lambda_0$ and letting

$$\hat{\lambda} = \frac{\lambda_0}{\hat{\kappa}_1}.$$  

The lower bound $\hat{\lambda}_0$ can be easily evaluated, e.g., using the power method for $\hat{A}^{-1}A$.

### 2.2 When the (2, 2) block $D$ is present

In this subsection we will study the convergence of Algorithm 1 when the block matrix $D$ is present. As we will see, the convergence of Algorithm 1 is much more complicated than the case with $D = 0$, and it is essential to scale preconditioner $\hat{A}$ for $D \neq 0$ since the convergence of Algorithm 1 depends strongly on the relative scale of $B'\hat{A}^{-1}B$ with respect to $D$ in the approximated Schur complement $H = B'\hat{A}^{-1}B + D$. The following lemma illustrates this fact in terms of the eigenvalues of the preconditioned Schur complement and is essential to the subsequent convergence analysis.

**Lemma 2.3.** Let

$$\beta_i = \frac{|(I - \hat{\tau}_i \hat{S}^{-1} \hat{H}^\frac{1}{2} H^{-\frac{1}{2}} g_i)|}{|H^{-\frac{1}{2}} g_i|},$$

where $g_i$ is defined in (1.3), then it holds

$$1 - \beta_i^2 = \hat{\tau}_i \frac{\langle g_i, \hat{S}^{-1} g_i \rangle}{\langle g_i, H^{-1} g_i \rangle} \quad \text{and} \quad 0 \leq \beta_i \leq 1.$$

And there exists a symmetric and positive definite matrix $G_i$ such that $G_i^{-1} g_i = \hat{\tau}_i \hat{S}^{-1} g_i$ and all the eigenvalues of the preconditioned matrix $\hat{S}^\frac{1}{2} G_i^{-1} S^\frac{1}{2}$ (or $R_i G_i^{-1} R_i^T$ where $S = RR^T$) lie in the interval

$$[(1 - \beta_i) \delta_1, \ (1 + \beta_i) \delta_2],$$

where $\delta_1 = \frac{\lambda_0 + c_0}{\lambda_0 (1 + c_0)}$, $\delta_2 = \frac{\lambda + c_0}{\lambda (1 + c_0)}$, and $c_0$ is the largest eigenvalue of $D^{-1} B' \hat{A}^{-1} B$.

**Proof.** By the definition of $\hat{\tau}_i$, we can write

$$|\hat{\tau}_i \hat{S}^{-1} g_i - H^{-1} g_i|^2_H = \hat{\tau}_i^2 |\hat{S}^{-1} g_i|^2_H - 2 \hat{\tau}_i < g_i, \hat{S}^{-1} g_i > + |H^{-1} g_i|^2_H$$

$$= \left(1 - \hat{\tau}_i \frac{\langle g_i, \hat{S}^{-1} g_i \rangle}{\langle g_i, H^{-1} g_i \rangle}\right) |H^{-1} g_i|^2_H, \quad (2.9)$$
and
\[
| (I - \hat{\tau}_i H^{1/2} \hat{S}^{-1} H^{1/2} ) H^{-1/2} g_i |^2 = | H^{-1/2} g_i |^2 - 2 \hat{\tau}_i \langle g_i, \hat{S}^{-1} g_i \rangle + \hat{\tau}_i^2 \langle H \hat{S}^{-1} g_i, \hat{S}^{-1} g_i \rangle
\]
\[
= \left( 1 - \frac{\hat{\tau}_i}{\langle g_i, H^{-1} g_i \rangle} \right) | H^{-1} g_i |^2.
\]

Thus it follows from the definition of $\beta_i$ and (2.9) that
\[
\beta_i^2 = 1 - \frac{\hat{\tau}_i \langle g_i, \hat{S}^{-1} g_i \rangle}{\langle g_i, H^{-1} g_i \rangle}
\]
and
\[
| \hat{\tau}_i \hat{S}^{-1} g_i - H^{-1} g_i |_H = \beta_i | H^{-1} g_i |_H.
\]

It is shown in the proof of Lemma 3.2 in [11] by using the Kantorovich inequality that
\[
\hat{\tau}_i \langle g_i, \hat{S}^{-1} g_i \rangle \geq 1 - \beta_i^2
\]
and thus we have $\beta_i \leq \beta$. On the other hand, the estimate (2.10) implies the existence of a symmetric and positive definite matrix $G_i$ such that (cf. [1])
\[
G_i^{-1} g_i = \hat{\tau}_i \hat{S}^{-1} g_i
\]
and
\[
| I - H^{1/2} G_i^{-1} H^{1/2} | \leq \beta_i.
\]
Let $\mu > 0$ be an eigenvalue of $S^{1/2} G_i^{-1} S^{1/2}$, then there exists a vector $\phi$ such that
\[
\langle S \phi, \phi \rangle = \mu \langle G_i \phi, \phi \rangle.
\]
It is easy to see that for any $\phi \in \mathbb{R}^m$,
\[
\langle B^t A^{-1} B \phi, \phi \rangle = \langle (\hat{A}^{1/2} A^{-1} \hat{A}^{1/2}) \hat{A}^{-1/2} B \phi, \hat{A}^{-1/2} B \phi \rangle.
\]
But we know from the assumption (2.2) that
\[
\frac{1}{\lambda_0} I \leq \hat{A}^{1/2} A^{-1} \hat{A}^{1/2} \leq \frac{1}{\lambda} I,
\]
which leads to
\[
\langle \left( \frac{1}{\lambda_0} B^t \hat{A}^{-1} B + D \right) \phi, \phi \rangle \leq \langle (B^t A^{-1} B + D) \phi, \phi \rangle \leq \langle \left( \frac{1}{\lambda} B^t \hat{A}^{-1} B + D \right) \phi, \phi \rangle.
\]

Let $\gamma \in (0, 1)$ be a constant to be determined such that for any $\phi \in \mathbb{R}^m$,
\[
\langle (B^t \hat{A}^{-1} B + \lambda D) \phi, \phi \rangle \leq \gamma \langle H \phi, \phi \rangle = \gamma \langle B^t \hat{A}^{-1} B \phi, \phi \rangle + \gamma \langle D \phi, \phi \rangle,
\]
which implies
\[
\langle B^t \hat{A}^{-1} B \phi, \phi \rangle \leq \frac{\gamma - \lambda}{1 - \gamma} \langle D \phi, \phi \rangle.
\]
As \( c_0 \) is the largest eigenvalue of \( D^{-1}B^{t}\hat{A}^{-1}B \), we can choose \( \gamma \) such that \((\gamma - \lambda)/(1 - \gamma) = c_0\), that gives \( \gamma = (\lambda + c_0)/(1 + c_0) \). Hence we know
\[
\langle(B^{t}\hat{A}^{-1}B + \lambda D)\phi, \phi\rangle \leq \frac{\lambda + c_0}{1 + c_0}(H\phi, \phi).
\]
Similarly we can derive
\[
\langle(B^{t}\hat{A}^{-1}B + \lambda_0 D)\phi, \phi\rangle \geq \frac{\lambda_0 + c_0}{1 + c_0}(H\phi, \phi).
\]
Using the above two estimates we deduce from (2.11) that
\[
\frac{1}{\lambda_0} \frac{\lambda_0 + c_0}{1 + c_0}(H\phi, \phi) \leq \mu \langle G_i\phi, \phi\rangle \leq \frac{1}{\lambda} \frac{\lambda + c_0}{1 + c_0}(H\phi, \phi).
\]
Since \(|I - H^{t}G_i^{-1}H^{t}| \leq \beta_i\),
\[
\frac{1 - \beta_i}{\lambda_0} \frac{\lambda_0 + c_0}{1 + c_0}(G_i\phi, \phi) \leq \mu \langle G_i\phi, \phi\rangle \leq \frac{1 + \beta_i}{\lambda} \frac{\lambda + c_0}{1 + c_0}(G_i\phi, \phi).
\]
which implies the claimed eigenvalue bounds. \(\square\)

**Remark 2.2.** We give some comments on the constant \( c_0 \) introduced in Lemma 2.3. If \( D = 0 \), then \( c_0 = \infty \) and the estimate in Lemma 2.3 coincides with the one in [11]. Noting that \( \lambda A^{-1} \leq \hat{A}^{-1} \leq \lambda_0 A^{-1} \), \( c_0 \) is bounded above and below by the eigenvalues of \( D^{-1}B^{t}A^{-1}B \). Thus if \( D \) dominates, then \( c_0 \) is very small.

**Remark 2.3.** Lemma 2.3 shows that unless \( D \) dominates \( B^{t}\hat{A}^{-1}B \) (i.e., \( c_0 \) is small) the eigenvalues of the preconditioned matrix \( \theta_iS^\frac{1}{2}G_i^{-1}S^\frac{1}{2} \) are sensitive to the scaling of preconditioner \( \hat{A} \). Thus, we may assume \( \lambda = 1 \) as in (2.4), i.e.,
\[
\hat{A} \leq A \leq \lambda_0 \hat{A},
\]
which may be achieved by scaling \( \hat{A} \) by \( \lambda \); see [2,3]. Then we may simply choose the damping parameter \( \theta_i \) in (1.6) such that
\[
\theta_i \leq \frac{1 - \sqrt{1 - \omega_i}}{2}
\]
for the convergence of Algorithm [7]. In general we may choose the damping parameter \( \theta_i = M/\kappa_i \), where the constant \( M \) should be selected for guaranteeing the convergence of Algorithm [7] and achieving an appropriate convergence rate; we refer to the further discussions in the next section.

### 3 Convergence analysis

Now, we are ready to analyze the convergence of Algorithm [11]. Let us introduce the errors
\[
e^x_i = x - x_i, \quad e^y_i = y - y_i.
\]
Then the residuals $f_i$ and $g_i$ can be expressed as

$$f_i = Ae_i^x + Be_i^y, \quad g_i = -B^te_{i+1} + De_i^y. \quad (3.1)$$

Using the definition of $f_i$ and the iteration (1.2) for updating $x_i$, we can write

$$A^\frac{1}{2}e_{i+1}^x = A^\frac{1}{2}(e_i^x - \omega_i\hat{A}^{-1}f_i) = (I - \omega_i A^\frac{1}{2}\hat{A}^{-1}A^\frac{1}{2})A^\frac{1}{2}f_i - A^\frac{1}{2}Be_i^y. \quad (3.2)$$

On the other hand, using the iteration (1.2) for updating $y_i$, the definition of $g_i$, the formula (3.2) and the matrix $G_i$ introduced in Lemma 2.3 we derive

$$e_{i+1}^y = e_i^y - \tau_i\tilde{S}^{-1}g_i = e_i^y - Q_i^{-1}g_i = e_i^y + Q_i^{-1}(B^te_{i+1} - De_i^y)$$

$$= e_i^y + Q_i^{-1}B^tA^{-\frac{1}{2}}(I - \omega_i A^\frac{1}{2}\hat{A}^{-1}A^\frac{1}{2})A^{-\frac{1}{2}}f_i - A^{-\frac{1}{2}}Be_i^y - Q_i^{-1}De_i^y \quad (3.3)$$

$$= Q_i^{-1}B^tA^{-\frac{1}{2}}(I - \omega_i A^\frac{1}{2}\hat{A}^{-1}A^\frac{1}{2})A^{-\frac{1}{2}}f_i + (I - Q_i^{-1}S)e_i^y,$$

where

$$Q_i^{-1} \equiv \tau_i\tilde{S}^{-1} = \theta_i\tilde{S}^{-1} = \theta_iG_i^{-1}. \quad (4.4)$$

Now it follows from (3.1), (3.2) and (3.3) that

$$A^{-\frac{1}{2}}f_{i+1} = A^\frac{1}{2}e_{i+1}^x + A^{-\frac{1}{2}}B^te_{i+1}^y$$

$$= (I + A^{-\frac{1}{2}}BQ_i^{-1}B^tA^{-\frac{1}{2}})(I - \omega_i A^\frac{1}{2}\hat{A}^{-1}A^\frac{1}{2})A^{-\frac{1}{2}}f_i - A^{-\frac{1}{2}}BQ_i^{-1}SE_i^y. \quad (3.5)$$

Consider the singular value decomposition of the matrix $B^tA^{-\frac{1}{2}}$,

$$B^tA^{-\frac{1}{2}} = U\Sigma V, \quad \Sigma = [\Sigma_0, 0],$$

where $U$ is an $m \times m$ orthogonal matrix, $V$ is an $n \times n$ orthogonal matrix, and $\Sigma_0$ is an $m \times m$ diagonal matrix with its diagonal entries being the singular values of $B^tA^{-\frac{1}{2}}$, and $S = RR^t$, where $R$ is a non-singular $m \times m$ matrix. Set

$$E_i^{(1)} = \sqrt{\alpha}V^tA^{-\frac{1}{2}}f_i, \quad E_i^{(2)} = R^te_i^y,$$

then we can write by using (3.3) and (3.5) that

$$
\begin{pmatrix}
E_{i+1}^{(1)} \\
E_{i+1}^{(2)}
\end{pmatrix}
= 
\begin{pmatrix}
\alpha(I + \Sigma U^tQ_i^{-1}U\Sigma) & \sqrt{\alpha}\Sigma U^tQ_i^{-1}R \\
\sqrt{\alpha}R^tQ_i^{-1}U\Sigma & -(I - R^tQ_i^{-1}R)
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\alpha}V^t(I - \omega_i A^\frac{1}{2}\hat{A}^{-1}A^\frac{1}{2})VE_i^{(1)} \\
0
\end{pmatrix}.
$$

One can easily verify by noting $\alpha_i \leq \alpha$ that

$$\frac{1}{\alpha}V^t(I - \omega_i A^\frac{1}{2}\hat{A}^{-1}A^\frac{1}{2})VE_i^{(1)}|E_i^{(1)}|^2 = \frac{\alpha_i^2}{\alpha^2}|E_i^{(1)}|^2 \leq |E_i^{(1)}|^2,$$

which, along with the relations

$$(I + \Sigma U^tQ_i^{-1}U\Sigma) = 
\begin{pmatrix}
I + \Sigma_0 U^tQ_i^{-1}U\Sigma_0 & 0 \\
0 & I
\end{pmatrix}, \quad \Sigma U^tQ_i^{-1}R = 
\begin{pmatrix}
\Sigma_0 U^tQ_i^{-1}R \\
0
\end{pmatrix},$$

"
enables us to reduce the estimate of components $E_{i+1}^{(1)}$ and $E_{i+1}^{(2)}$ to the spectral estimate of the following symmetric matrix

$$F_i = \begin{pmatrix} \alpha (I + \Sigma_0 U^i Q_i^{-1} U \Sigma_0) & \sqrt{\alpha} \Sigma_0 U^i Q_i^{-1} R \\ \sqrt{\alpha} R^i Q_i^{-1} U \Sigma_0 & -(I - R^i Q_i^{-1} R) \end{pmatrix}.$$ \hfill (3.6)

So if all the eigenvalues of $F_i$ are bounded by $\rho = ||F_i|| < 1$ in their magnitude, then

$$|E_{i+1}^{(1)}|^2 + |E_{i+1}^{(2)}|^2 \leq \rho^2 \left( \frac{\alpha^2}{\alpha^2} |E_i^{(1)}|^2 + |E_i^{(2)}|^2 \right),$$ \hfill (3.7)

and the convergence of Algorithm [1] can be ensured. In following we will have an estimate on $\rho$. We first examine the convergence the algorithm, and then further estimate the convergence rate.

For the spectral estimate of $F_i$, we introduce a parameter $c_1$ satisfying

$$c_1 R^i Q_i^{-1} R \geq \Sigma_0 U^i Q_i^{-1} U \Sigma_0,$$ \hfill (3.8)

where $c_1$ measures the magnitude of $B^t A^{-1} B$ relatively to the one of $D$ in an appropriately weighted sense. If we let

$$W = Q_i^{-\frac{t}{2}} R, \quad T = Q_i^{-\frac{t}{2}} U \Sigma_0,$$

then (3.8) is equivalent to the following inequality

$$|Tu|^2 \leq c_1 |Wu|^2 \quad \forall u \in \mathbb{R}^m.$$ \hfill (3.9)

Using the parameter $c_1$, we have the following result.

**Theorem 3.1.** If the damping parameter $\theta_i$ satisfies

$$\theta_i (1 + \beta) \delta_2 < \frac{2(1 - \alpha)}{1 - \alpha + 2c_1 \alpha},$$

then $\rho = ||F_i|| < 1$, so Algorithm [1] converges.

**Proof.** We estimate the upper and lower bounds of all the eigenvalues of matrix $F_i$.

To see the lower bound of $F_i$, we observe that for any $u, v \in \mathbb{R}^m$ with one of them being non-zero,

$$\left\langle (F_i + I) \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle$$

$$= (\alpha + 1)|u|^2 + \alpha \langle Tu, Tu \rangle + \sqrt{\alpha} (\langle Wv, Tu \rangle + \langle Tu, Wv \rangle) + \langle Wv, Wv \rangle$$

$$= (\alpha + 1)|u|^2 + |\sqrt{\alpha} Tu + Wv|^2 > 0,$$

thus all the eigenvalues of $F_i$ are bounded below by $-1$.

For the upper bound, we consider

$$J = \left\langle (I - F_i) \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle = (1 - \alpha) |u|^2 - |\sqrt{\alpha} Tu + Wv|^2 + 2|v|^2.$$
Let \( \omega \) and \( \gamma \) be the spectral bounds of \( W \) given by
\[
\omega |z|^2 \leq |Wz|^2 \leq \gamma |z|^2 \quad \forall z \in \mathbb{R}^m, \tag{3.10}
\]
then using Young’s inequality we know for all \( \delta > 0, \)
\[
|\sqrt{\alpha}Tu + Wv|^2 \leq (1 + \delta)\alpha |Tu|^2 + (1 + \frac{1}{\delta})|Wv|^2,
\]
hence it follows from (3.9) and (3.10) that
\[
J \geq (1 - \alpha)|u|^2 - c_1\alpha(1 + \delta)\gamma |u|^2 + 2|v|^2 - (1 + \frac{1}{\delta})\gamma|v|^2. \tag{3.11}
\]
For \( J > 0 \), we need the existence of a \( \delta > 0 \) such that
\[
(1 - \alpha) - c_1\alpha(1 + \delta)\gamma > 0 \text{ and } 2 - (1 + \frac{1}{\delta})\gamma > 0,
\]
which is equivalent to
\[
\frac{\gamma}{2 - \gamma} < \delta < \frac{1 - \alpha}{c_1\alpha\gamma} - 1,
\]
and hence
\[
2(1 - \alpha) - (1 - \alpha + 2c_1\alpha)\gamma > 0,
\]
which requires \( \gamma \) to satisfy
\[
0 < \gamma < \frac{2(1 - \alpha)}{1 - \alpha + 2c_1\alpha}. \tag{3.12}
\]
Clearly if this condition holds, then it follows from (3.11) that \( I - F_i > 0. \) Thus all the eigenvalues of \( F_i \) have the upper bound 1. But by Lemma 2.3, we know
\[
|Wy|^2 \leq \theta_i (1 + \beta)\delta_2 |y|^2
\]
which leads to the desired result of Theorem 3.1 by taking \( \gamma = \theta_i (1 + \beta)\delta_2. \)

**Remark 3.1.** We may comment on some direct consequences of Theorem 3.1 at the extreme cases of \( c_1 \) close to 0 or 1. It is easy to see that for \( 0 < c_1 \leq 1, 2(1 - \alpha)/(1 - \alpha + 2c_1\alpha) \) is monotonically decreasing with respect to \( c_1, \) implying that
\[
\frac{2(1 - \alpha)}{1 + \alpha} \leq \frac{2(1 - \alpha)}{1 - \alpha + 2c_1\alpha} < 2.
\]
Then Theorem 3.1 ensures the convergence of Algorithm 1 for any \( \theta_i \) satisfying
\[
\theta_i(1 + \beta)\delta_2 < \frac{2(1 - \alpha)}{1 + \alpha} = \frac{2}{\kappa_1}
\]
in the case that \( c_1 \) is close to 1, i.e., \( D \) is relatively small compared to \( B^t A^{-1} B \) in the sense of (3.8). In the case that \( c_0 \) and \( c_1 \) are close to 0, i.e., \( D \) dominates \( B^t A^{-1} B, \) we take \( \theta_i \) satisfying
\[
\theta_i(1 + \beta) < 2
\]
to guarantee the convergence according to Theorem 3.1, or roughly we take \( \theta_i \leq 1. \)
**Estimate of convergence rate.** The following of this section is devoted to estimating the convergence rate of Algorithm 1. That is, the more precise size of $\rho = ||F_i||$ in (3.7). Following exactly the same arguments as the one for the upper bound of $F_i$ in the proof of Theorem 3.1, we can show that $F_i \leq \mu I$ for some $\mu \in (\alpha, 1)$ provided that there exists a $\delta > 0$ such that

$$(\mu - \alpha) - c_1 \alpha (1 + \delta) \gamma \geq 0 \quad \text{and} \quad \mu + 1 - (1 + \frac{1}{\delta}) \gamma \geq 0,$$

where $\gamma$ is the spectral bound of $W$ in (3.10). This implies

$$\frac{\gamma}{\mu + 1 - \gamma} \leq \delta \leq \frac{\mu - \alpha}{\alpha c_1 \gamma} - 1,$$

or equivalently

$$0 < \gamma \leq \frac{(\mu + 1)(\mu - \alpha)}{\alpha c_1 (\mu + 1) + \mu - \alpha} \equiv \gamma(\mu, \alpha, c_1). \quad (3.13)$$

That is, if $\gamma \leq \gamma(\mu, \alpha, c_1)$ then all the eigenvalues of $F_i$ are bounded above by $\mu$.

To estimate the lower bound of $F_i$, for any $\tilde{\mu} \in (0, 1)$ and $0 < \delta < 1$ we can derive

$$J = \langle (F_i + \tilde{\mu}I) \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \rangle = (\alpha + \tilde{\mu})|u|^2 + |\sqrt{\alpha}Tu + Wv|^2 + (\tilde{\mu} - 1)|v|^2$$

$$\geq (\tilde{\mu} + \alpha)|u|^2 + (1 - \frac{1}{\delta})\alpha|Tu|^2 + (\tilde{\mu} - 1)|v|^2 + (1 - \delta)|Wv|^2.$$

Using (3.10) and (3.9), we get $|Tu|^2 \leq c_1 \gamma |u|^2$, thus

$$J \geq (\tilde{\mu} + \alpha + (1 - \frac{1}{\delta})c_1 \alpha \gamma)|u|^2 + (\tilde{\mu} - 1 + (1 - \delta)\omega)|v|^2.$$

This implies $F_i \geq -\tilde{\mu}I$ if there exists a $\delta > 0$ such that

$$(\tilde{\mu} + \alpha + (1 - \frac{1}{\delta})c_1 \alpha \gamma) \geq 0, \quad \tilde{\mu} - 1 + (1 - \delta)\omega \geq 0,$$

or equivalently

$$\frac{c_1 \alpha \gamma}{\tilde{\mu} + \alpha + \alpha c_1 \gamma} \leq \delta \leq \frac{\omega + \tilde{\mu} - 1}{\omega},$$

which is equivalent to requiring that

$$\omega(\tilde{\mu}, \alpha, c_1, \gamma) \equiv (1 - \tilde{\mu})(1 + \frac{c_1 \alpha \gamma}{\tilde{\mu} + \alpha}) \leq \omega, \quad (3.14)$$

then all the eigenvalues of $F_i$ are bounded below by $-\tilde{\mu}$. By Lemma 2.3 we know that

$$\theta_i(1 - \beta)\delta_1 |y|^2 \leq |Wy|^2,$$

so we can take $\omega = \theta_i(1 - \beta)\delta_1$.

Note that in (3.14) $\omega(\tilde{\mu}, \alpha, c_1, \gamma) \leq (1 - \tilde{\mu})(1 + c_1 \frac{\alpha}{\alpha + \gamma \gamma(\mu, \alpha, c_1)})$ for $\tilde{\mu} \in [\beta, 1)$.

Using Lemma 2.3 we derive immediately from (3.13)–(3.14) the following results.
Theorem 3.2. For any \( \mu \in (\alpha, 1) \), if \( \theta_i \) satisfies
\[
\theta_i \leq \frac{1}{\delta_2(1 + \beta)} \gamma(\mu, \alpha, c_1),
\]
and \( \tilde{\mu} \in [\beta, 1) \) satisfies
\[
1 - \tilde{\mu} \leq \frac{\delta_1 (1 - \beta)}{1 + c_1 \frac{\alpha}{\alpha + \beta} \gamma(\mu, \alpha, c_1)} \theta_i,
\]
then we have
\[-\tilde{\mu} I \leq F_i \leq \mu I,\]
and the convergence rate \( \rho = \max\{\mu, \tilde{\mu}\} \).

Rate estimates at extreme cases. We are now trying to provide more detailed conditions for the convergence rates at some extreme cases. It is easy to see that \( \gamma(\mu, \alpha, c_1) \) is monotonically decreasing with respect to \( c_1 \in (0, 1] \), which implies
\[
\frac{(1 + \mu)(\mu - \alpha)}{\mu(1 + \alpha)} \leq \gamma(\mu, \alpha, c_1) < \mu + 1 < 2.
\]
When \( c_1 \) is close to 1, i.e., \( D \) is relatively small, we have
\[
\gamma(\mu, \alpha, c_1) \approx \frac{(\mu + 1)(\mu - \alpha)}{\mu(1 + \alpha)}.
\]
Hence for any \( \theta_i \) satisfying
\[
\delta_2(1 + \beta) \theta_i \leq \frac{(\mu + 1)(\mu - \alpha)}{\mu(1 + \alpha)} < 2 \frac{1 - \alpha}{1 + \alpha} < \frac{2}{\kappa_1},
\]
we know \( F_i \leq \mu I \), while for \( \tilde{\mu} \) in the following range
\[
1 - \tilde{\mu} \leq \frac{\delta_1 (1 - \beta)}{1 + \frac{\alpha}{\beta + \alpha} \gamma(\mu, \alpha, c_1)} \theta_i,
\]
we know \( F_i \geq -\tilde{\mu} I \).

In the case that \( c_0 \) and \( c_1 \) are both close to 0, i.e., \( D \) dominates \( B^t A^{-1} B \), we see
\[
\gamma(\mu, \alpha, c_1) \approx \mu + 1, \quad \delta_1 \approx 1, \quad \delta_2 \approx 1.
\]
Thus for \( \theta_i \) satisfying
\[
\theta_i < \frac{\mu + 1}{1 + \beta} \quad (3.15)
\]
then \( F_i \leq \mu I \). On the other hand, it follows from (3.14) that if
\[
1 - \tilde{\mu} \leq \theta_i (1 - \beta) \quad (3.16)
\]
then \( F_i \geq -\tilde{\mu} I \).

From above we can see that the convergence rate \( \rho = \max\{\mu, \tilde{\mu}\} \) can be estimated using Theorem 3.2 and (3.15)–(3.16) when \( D \) is dominant in the approximate Schur complement \( H = B^T \tilde{A}^{-1} B + D \).
Corollary 3.1. Let \( D = 0 \) in (1.1). Then for any damping parameter \( \theta_i \) satisfying
\[
\theta_i \leq \frac{\lambda}{\kappa_1},
\]
Algorithm \( \mathcal{I} \) converges; and if the eigenvalues of \( S_{1/2}Q_{i}^{-1}S_{1/2} \) is clustered around \((1 - \alpha)/(1 + \alpha)\), the algorithm achieves approximately the optimal rate \( \sqrt{\alpha} \).

Proof. When \( D = 0 \), the error propagating matrix \( F_i \) in (3.6) becomes
\[
\tilde{F}_i = \begin{pmatrix}
\alpha (I + \Sigma_0 U^t Q_i^{-1} U \Sigma_0) & \sqrt{\alpha} \Sigma_0 U^t Q_i^{-1} U \Sigma_0 \\
\sqrt{\alpha} \Sigma_0 U^t Q_i^{-1} U \Sigma_0 & -(I - \Sigma_0 U^t Q_i^{-1} U \Sigma_0)
\end{pmatrix}.
\]
Then we have
\[
\begin{pmatrix}
E_{i+1}^{(1)} \\
E_{i+1}^{(2)}
\end{pmatrix} = \tilde{F}_i \begin{pmatrix}
\frac{1}{\alpha} V^t (I - \omega_i A_2 \hat{A}^{-1} A_1^2)V E_i^{(1)} \\
E_i^{(2)}
\end{pmatrix}.
\]
Clearly, \( \tilde{F}_i \) is a function of the single matrix \( R_t^t Q_{i}^{-1} = \Sigma_0 U^t Q_{i}^{-1} U \Sigma_0 \). Let \( M := R_t^t Q_{i}^{-1} R \), then
\[
\tilde{F}_i - \mu I = \begin{pmatrix}
(\alpha - \mu)I + \alpha M & \sqrt{\alpha} M \\
\sqrt{\alpha} M & -(\mu + 1)I + M
\end{pmatrix}.
\]
Using the factorization
\[
\begin{pmatrix}
\sqrt{\alpha} M \\
(\alpha - \mu)I + \alpha M
\end{pmatrix}
\begin{pmatrix}
-(\mu + 1)I + M \\
\sqrt{\alpha} M
\end{pmatrix} = \begin{pmatrix}
\sqrt{\alpha} M & 0 \\
(\alpha - \mu)I + \alpha M & X
\end{pmatrix}
\begin{pmatrix}
I & Y \\
0 & I
\end{pmatrix},
\]
where \( X = \sqrt{\alpha} M - [(\alpha - \mu)I + \alpha M] \alpha^{-\frac{3}{2}} M^{-1} [-(\mu + 1)I + M], Y = \alpha^{-\frac{3}{2}} M^{-1} [-(\mu + 1)I + M] \), we know that \( \det(\tilde{F}_i - \mu I) = 0 \) is equivalent to
\[
\det(\alpha M^2 - [(\mu - \alpha)I - \alpha M][(\mu + 1)I - M]) = 0.
\]
Let \( z \) be an eigenvalue of \( R_t^t Q_{i}^{-1} R \), then the corresponding eigenvalue \( \mu \) of \( \tilde{F}_i \) satisfies
\[
f(\mu) = (\mu - \alpha(1 + z))(\mu + 1 - z) - \alpha z^2 = \mu^2 + (1 - \alpha - (1 + \alpha)z)\mu - \alpha = 0.
\]
It is easy to see that \( f(0) = -\alpha < 0 \), and \( f(-1) = (1 + \alpha)z > 0 \). If
\[
f(1) = 2(1 - \alpha) - (1 + \alpha)z > 0,
\]
then we know \( \mu \in (-1, 1) \). This is equivalent to
\[
z < \frac{2(1 - \alpha)}{1 + \alpha} = \frac{2}{\lambda_0} = \frac{2}{\kappa_1}.
\]
Noting that \( S_{1/2}Q_{i}^{-1}S_{1/2} \) has the same eigenvalues as \( R_t^t Q_{i}^{-1} R \), we know from Lemma 2.3 that
\[
\theta_i \frac{1 - \beta}{\lambda_0} \leq z \leq \theta_i \frac{1 + \beta}{\lambda},
\]
which indicates that condition (3.17) holds if \( \theta_i \leq \lambda/\kappa_1 \). This proves the first part of Corollary 3.1.

To see the second part, we know if \( z \) is clustered around \( 1−\alpha = 1/\kappa_1 \), \( f(\mu) \) approaches \( \mu^2 − \alpha \), indicating that Algorithm 1 achieves approximately the optimal convergence rate \( \sqrt{\alpha} \). □

Remark 3.2. A few remarks are in order.

1. The convergence of Algorithm 1 was analyzed in [11] when \( D = 0 \) under Assumption (2.1), and the convergence was established by evaluating the maximum eigenvalues of \( \hat{F}_i^T \hat{F}_i \) directly, where \( \hat{F}_i \) is a non-symmetric matrix given by

\[
\hat{F}_i = \begin{pmatrix}
\alpha_i (I + \Sigma_0^T U_i^T Q_i^{-1} U \Sigma_0) & -\sqrt{\alpha} \Sigma_0^T U_i^T Q_i^{-1} U \Sigma_0 \\
\alpha_i \Sigma_0^T U_i^T Q_i^{-1} U \Sigma_0 & (I - \Sigma_0^T U_i^T Q_i^{-1} U \Sigma_0)
\end{pmatrix}.
\]

Note that \( \hat{F}_i \) is a function of the single matrix \( \Sigma_0^T U_i^T Q_i^{-1} U \Sigma_0 \). Our estimate (3.7) is different from the one in [11] since it contains the additional decay factor \( \alpha_i^2/\alpha_2 \). Moreover, a direct extension of the analysis in [11] for the general case of \( D \neq 0 \) is considerably difficult since the analysis in [11] depends on the fact that \( \hat{F}_i \) is a function of a single matrix, but the corresponding matrix \( F_i \) for the case \( D \neq 0 \) involves two different matrices.

2. For the estimate of eigenvalues of \( S_1^{1/2} Q_1^{-1} S_2^{1/2} \) in Lemma 2.3 the estimate (2.11) may be very conservative and can be replaced by the specific conditioning of \( \hat{A}^{-1} \) on \( \text{Range}(B) \), i.e.,

\[
\gamma_1 B^i \hat{A}^{-1} B \leq B^i A^{-1} B \leq \gamma_2 B^i \hat{A}^{-1} B.
\] (3.18)

As a consequence the estimate of the range of eigenvalues of \( S_1^{1/2} Q_1^{-1} S_2^{1/2} \) is sharper and the convergence rate can be improved.

3. In all the above estimates \( \beta \) can be replaced by \( \beta_i (\beta_i \leq \beta) \) since eigenvalues of \( S_1^{1/2} Q_1^{-1} S_2^{1/2} \) is bounded in terms of \( \beta_i \) in Lemma 2.3. In practice \( \beta_i \) may be much smaller than \( \beta \), thus it may result in much sharper estimate for the lower bound of the eigenvalues of \( F_i \).

4 Nonlinear Preconditioners

Our analysis in the previous sections still applies when the preconditioner \( \hat{A}^{-1} \) for \( A \) in (1.1) is replaced by a more general one. A general preconditioner is a nonlinear mapping \( \Psi_A : \mathbb{R}^n \rightarrow \mathbb{R}^n \) for the linear system

\[
Ax = \xi
\]
such that $\Psi_A(\xi)$ gives an approximation of the solution $x$ with certain accuracy. We assume that $\Psi_A$ satisfies

$$
|\Psi_A(\xi) - A^{-1}\xi|_A \leq \delta |A^{-1}\xi|_A \quad \forall \xi \in \mathbb{R}^n, 
$$

(4.1)

$$
|\Psi_A(Bd) - A^{-1}Bd|_A \leq \delta_0 |A^{-1}Bd|_A \quad \forall d \in \mathbb{R}^n
$$

(4.2)

for some $\delta, \delta_0 \in (0, 1)$. General preconditioners of this type can be realized, for example, by the approximate inverse generated via the preconditioned conjugate gradient (PCG) iteration, or by one sweep of a multigrid method with conjugate gradient smoothing. With the help of this general preconditioner $\Psi_A$ we consider the following iterative method for solving the generalized saddle-point system (1.1).

**Algorithm 2** Nonlinear inexact Uzawa algorithm when good approximate Schur complement available.

1. Compute $f_i = f - Ax_i - By_i$, $r_i = \Psi_A(f_i)$, and the relaxation parameter $\omega_i = \frac{\langle f_i, r_i \rangle}{\langle Ar_i, r_i \rangle}$ for $f_i \neq 0$ ($\omega_i = 1$, otherwise).

2. Update $x_{i+1} = x_i + \omega_i \Psi_A(f - Ax_i - By_i) = x_i + \omega_i r_i$;

3. Compute $g_i = B^t x_{i+1} - Dy_i - g$, $s_i = \hat{S}^{-1}g_i$, and $\tau_i = \theta_i \frac{\langle g_i, s_i \rangle}{\langle \Psi_A(Bs_i), Bs_i \rangle + \langle Ds_i, s_i \rangle}$ for $s_i \neq 0$ ($\tau_i = 1$, otherwise);

4. Update $y_{i+1} = y_i + \tau_i \hat{S}^{-1}(B^t x_{i+1} - Dy_i - g) = y_i + \tau_i s_i$.

Using condition (4.1), one can find a symmetric and positive definite matrix $Q_{i,A}$ such that (cf. [1])

$$
Q_{i,A} \Psi_A(f_i) = f_i
$$

and

$$
|I - A^{\frac{1}{2}}Q_{i,A}^{-1}A^{\frac{1}{2}}| \leq \delta.
$$

Similarly, there exists a symmetric and positive definite matrix $Q_{i,B}$ such that

$$
Q_{i,B} \Psi_A(Bs_i) = Bs_i
$$

and

$$
|B^t A^{-1}B - B^t Q_{i,B}^{-1}B| \leq \delta_0.
$$

Using the same arguments as in steps (3.1)–(3.4), we can obtain

$$
e_{i+1} = Q_i^{-1}B^t A^{-\frac{1}{2}}(I - \omega_i A^{\frac{1}{2}}Q_{i,A}^{-1}A^{\frac{1}{2}})A^{-\frac{1}{2}}f_i + (I - Q_i^{-1}S)e_i,
$$

$$
A^{-\frac{1}{2}}f_{i+1} = (I + A^{-\frac{1}{2}}BQ_i^{-1}B^t A^{-\frac{1}{2}})(I - \omega_i A^{\frac{1}{2}}Q_{i,A}^{-1}A^{\frac{1}{2}})A^{-\frac{1}{2}}f_i - A^{-\frac{1}{2}}BQ_i^{-1}S e_i.
$$
Let \( H_i = B^t Q_{i,A}^{-1} B + D \), and 
\[
\beta_i = \frac{|(I - \tau_i H_i^{-\frac{1}{2}} \hat{S}^{-1} H_i^{-\frac{1}{2}}) H_i^{-\frac{1}{2}} g_i|}{|H_i^{-\frac{1}{2}} g_i|}, \quad \kappa = \text{cond}(\hat{S}^{-1}(B^t A^{-1} B + D)).
\]

Then one can prove that there exists \( \beta = \beta(\delta_0, \kappa) \) such that \( \beta_i \leq \beta \leq 1 \) as it was done in the proof of Lemma 2.3. Consequently, one can prove Lemma 2.3 and (3.15) in Remark 3.2 with \( \gamma_1 = 1 - \delta_0, \quad \gamma_2 = 1 + \delta_0 \), thus we can carry out exactly the same convergence analysis as we did in the previous sections for the nonlinear inexact Uzawa algorithm above.

When there is no good preconditioner for the Schur complement system, especially when \( \text{cond}(\hat{S}^{-1} S) \gg \text{cond}(\hat{A}^{-1} A) \), we use a nonlinear solver, for example, CG, to solve \( H z = \zeta \), where \( H = B^t A^{-1} B + D \), and get the approximate solution \( \psi_H(\zeta) \).

**Algorithm 3** Nonlinear inexact Uzawa algorithm when no good approximate Schur complement available.

1. Compute \( f_i = f - (A x_i + B y_i) \), \( r_i = \hat{A}^{-1} f_i \), and the relaxation parameter \( \omega_i = \frac{\langle f_i, r_i \rangle}{\langle A r_i, r_i \rangle} \) for \( f_i \neq 0 \) (\( \omega_i = 1 \), otherwise).

2. Update \( x_{i+1} = x_i + \omega_i r_i \).

3. Compute \( g_i = B^t x_{i+1} - D y_i - g \), \( s_i = \Psi_H(g_i) \), and the parameter \( \tau_i = \theta_i \frac{\langle g_i, s_i \rangle}{\langle H s_i, s_i \rangle} \) for \( s_i \neq 0 \) (\( \tau_i = 1 \), otherwise).

4. Update \( y_{i+1} = y_i + \tau_i s_i \).

Assume that \( |\Psi_H(\zeta) - H^{-1} \zeta|_H \leq \delta_H \|H^{-1} \zeta\|_H, \forall \zeta \in \mathbb{R}^m \). There is a symmetric positive definite matrix \( \hat{Q}_i \) (see Lemma 9 in [1]) such that \( \hat{Q}_i^{-1} g_i = \Psi_H(g_i) \) and all eigenvalues of the matrix \( \hat{Q}_i^{-1} H \) are in the interval \([1 - \delta_H, 1 + \delta_H]\). That is,

\[
(1 - \delta_H) \langle \hat{Q}_i \psi, \psi \rangle \leq \langle H \psi, \psi \rangle \leq (1 + \delta_H) \langle \hat{Q}_i \psi, \psi \rangle, \quad \forall \psi \neq 0.
\]

Suppose that \( \langle S \phi, \phi \rangle = \lambda \langle \hat{Q}_i \phi, \phi \rangle \), where \( \lambda \) is the eigenvalue of \( \hat{Q}_i^{-1} S \). We can verify that

\[
\langle S \phi, \phi \rangle = \langle A^{-1} B \phi, B \phi \rangle + \langle D \phi, \phi \rangle = \langle \hat{A}^{-\frac{1}{2}} A^{-1} \hat{A}^{\frac{1}{2}} \hat{A}^{-\frac{1}{2}} B \phi, \hat{A}^{-\frac{1}{2}} B \phi \rangle + \langle D \phi, \phi \rangle.
\]

Let \( \mu_1 \) and \( \mu_2 \) be the minimal and maximal eigenvalues of \( \hat{A}^{-1} A \), and we have

\[
\mu_1 \langle \hat{A}^{-\frac{1}{2}} B \phi, \hat{A}^{-\frac{1}{2}} B \phi \rangle + \langle D \phi, \phi \rangle \leq \langle S \phi, \phi \rangle \leq \mu_2 \langle \hat{A}^{-\frac{1}{2}} B \phi, \hat{A}^{-\frac{1}{2}} B \phi \rangle + \langle D \phi, \phi \rangle.
\]

Assuming that the spectra of \( \hat{A}^{-1} A \) are around 1 and \( \mu_1 \leq 1 \leq \mu_2 \), we obtain

\[
\mu_1 \langle B^t \hat{A}^{-1} B \phi, \phi \rangle + \mu_1 \langle D \phi, \phi \rangle \leq \langle S \phi, \phi \rangle \leq \mu_2 \langle B^t \hat{A}^{-1} B \phi, \phi \rangle + \mu_2 \langle D \phi, \phi \rangle.
\]
That is,
\[ \mu_1 \langle H\phi, \phi \rangle \leq \lambda \langle \hat{Q}_i \phi, \phi \rangle \leq \mu_2 \langle H\phi, \phi \rangle. \]
Hence,
\[ \mu_1(1 - \delta_H)\langle \hat{Q}_i \phi, \phi \rangle \leq \lambda \langle \hat{Q}_i \phi, \phi \rangle \leq \mu_2(1 + \delta_H)\langle \hat{Q}_i \phi, \phi \rangle. \]
One can directly check that
\[ \text{cond}(\hat{Q}_i^{-1}S) \leq \frac{1 + \delta_H \mu_2}{1 - \delta_H \mu_1} = \frac{1 + \delta_H}{1 - \delta_H} \text{cond}(A^{-1}A). \]

Therefore, the nonlinear solver \( \Psi_H(g_i) = \hat{Q}_i^{-1}g_i \) corresponds to a new preconditioner \( \hat{Q}_i \) such that \( \text{cond}(\hat{Q}_i^{-1}S) \) is much more improved than \( \text{cond}(S^{-1}S) \) and has about the same order as \( \text{cond}(A^{-1}A) \). Algorithm \( I \) can be recovered if we replace \( \hat{Q}_i \) by \( \hat{S} \) in Algorithm \( 3 \). Obviously Algorithm \( 3 \) can be regarded as a variant of the previous Algorithm \( I \) and similar convergence analysis can be performed for Algorithm \( 3 \).

5 Nonsymmetric case

In this section we consider the convergence of Algorithm \( I \) for the case when \( A \) in (1.1) is nonsymmetric. This study seems to be new, and still no such investigations are available in the literature. Let \( A_0 \) be the symmetric part of \( A \), with \( A_0 \) being positive definite. Let
\[ J = A_0^{\frac{1}{2}}A^{-1}A_0^{\frac{1}{2}}. \]
First, we note that the relaxation parameter \( \omega_i \) in Algorithm \( I \) is now replaced by
\[ \omega_i = \frac{(f_i, r_i)}{(A_0 r_i, r_i)}. \]
Using (3.1) and the iteration (1.2) for updating \( x_i \), we can write
\[ A_0^{\frac{1}{2}}e_{i+1} = A_0^{\frac{1}{2}}(e_i - \omega_i \hat{A}^{-1}f_i) = (J - \omega_i A_0^{\frac{1}{2}}\hat{A}^{-1}A_0^{\frac{1}{2}})A_0^{\frac{1}{2}}f_i - JA_0^{-\frac{1}{2}}Be_i^y. \quad (5.1) \]
On the other hand, using the iteration (1.2) for updating \( y_i \), the definition of \( g_i \), (5.1) and matrix \( G_i \) introduced in Lemma 2.3, we derive
\[ e_{i+1}^y = e_i^y - Q_i^{-1}g_i = e_i^y + Q_i^{-1}(B^t e_{i+1}^x - De_i^y) \]
\[ = e_i^y + Q_i^{-1}B^t A_0^{-\frac{1}{2}}((J - \omega_i A_0^{\frac{1}{2}}\hat{A}^{-1}A_0^{\frac{1}{2}})A_0^{\frac{1}{2}}f_i - JA_0^{-\frac{1}{2}}Be_i^y) - Q_i^{-1}De_i^y \]
\[ = Q_i^{-1}B^t A_0^{-\frac{1}{2}}(J - \omega_i A_0^{\frac{1}{2}}\hat{A}^{-1}A_0^{\frac{1}{2}})A_0^{\frac{1}{2}}f_i + (I - Q_i^{-1}S)e_i^y, \quad (5.2) \]
where \( Q_i^{-1} = \theta_i G_i^{-1} \). Now, it follows from (3.1), (5.1) and (5.2) that
\[ A_0^{-\frac{1}{2}}f_{i+1} = J^{-1}A_0^{\frac{1}{2}}e_{i+1} + A_0^{-\frac{1}{2}}Be_{i+1}^y \]
\[ = (J^{-1} + A_0^{\frac{1}{2}}BQ_i^{-1}B^t A_0^{\frac{1}{2}})(J - \omega_i A_0^{\frac{1}{2}}\hat{A}^{-1}A_0^{\frac{1}{2}})A_0^{\frac{1}{2}}f_i - A_0^{\frac{1}{2}}BQ_i^{-1}Se_i^y. \quad (5.3) \]
Consider the singular value decomposition of matrix $B^tA_0^{-\frac{1}{2}}$,

$$B^tA_0^{-\frac{1}{2}} = U\Sigma V^t, \quad \Sigma = [\Sigma_0, \ 0]$$

where $U$ is an orthogonal $m \times m$ matrix, $V$ is an orthogonal $n \times n$ matrix, and $\Sigma_0$ is a $m \times m$ diagonal matrix with its diagonal entries being the singular values of $B^tA_0^{-\frac{1}{2}}$. Let $S_0 = B^tA_0^{-1}B + D$ and $S_0 = RR^t$. Set

$$E_i^{(1)} = \sqrt{\alpha}V^tA_{0}^{-\frac{1}{2}}f_i, \quad E_i^{(2)} = R^te_i^y.$$ 

Using (5.3), we have

$$\sqrt{\alpha}V^tA_{0}^{-\frac{1}{2}}f_{i+1} = [V^t(J^{-1} + XV^t)(J - \omega_iY)]V(\sqrt{\alpha}V^tA_{0}^{-\frac{1}{2}}f_i)$$

$$+ \sqrt{\alpha}V^t\Sigma U^tQ_i^{-1}SS_0^{-1}R(R^te_i^y),$$

where $X = \Sigma^tU^tQ_i^{-1}U\Sigma$ and $Y = A_0^{-\frac{1}{2}}A_0^{-\frac{1}{2}}$. Noticing that $(V^tJ^{-1} + XV^t)(J - \omega_iY) = (I + X)V^t(I - \omega_iY) + XV^t(J - I) - \omega_iV^t(J^{-1} - I)Y$ and $SS_0^{-1}R = (S - S_0 + S_0)S_0^{-1}R = R - (S_0 - S)S_0^{-1}R = R - (S_0 - S)R^{-t}$, we rewrite the formula above as follows,

$$E_{i+1}^{(1)} = [(I + X)V^t(I - \omega_iY) + XV^t(J - I) - \omega_iV^t(J^{-1} - I)Y]\alpha(I - \omega_iY)^{-1}V$$

$$+ \frac{1}{\alpha}V^t(I - \omega_iY)VE_i^{(1)} - \sqrt{\alpha}V^t\Sigma U^tQ_i^{-1}[R - (S_0 - S)R^{-t}]E_i^{(2)}$$

$$= \alpha\{(I + X) + [XV^t(J - I) - \omega_iV^t(J^{-1} - I)Y](I - \omega_iY)^{-1}V\}$$

$$+ \frac{1}{\alpha}V^t(I - \omega_iY)VE_i^{(1)} - \sqrt{\alpha}U^tQ_i^{-1}[R - (S_0 - S)R^{-t}]E_i^{(2)}.$$ 

Using (5.2), we obtain

$$R^te_i^y_{i+1} = \sqrt{\alpha}R^tQ_i^{-1}USV^t(I - \omega_iY)\frac{1}{\alpha}V(\sqrt{\alpha}V^tA_{0}^{-\frac{1}{2}}f_i)$$

$$+ R^t(I - Q_i^{-1}S)R^{-t}(R^te_i^y).$$

Noticing that $(J - \omega_iY)V = (I - \omega_iY + J - I)(I - \omega_iY)^{-1}VV^t(I - \omega_iY)V = [V + (J - I)(I - \omega_iY)^{-1}V]V^t(I - \omega_iY)V$, and $R^t(I - Q_i^{-1}S)R^{-t} = R^t[I - Q_i^{-1}(S_0 + S - S_0)]R^{-t} = I - R^tQ_i^{-1}R - R^tQ_i^{-1}(S - S_0)R^{-t}$, we rewrite the formula above as follows,

$$E_{i+1}^{(2)} = \sqrt{\alpha}[R^tQ_i^{-1}US + R^tQ_i^{-1}USV^t(J - I)(I - \omega_iY)^{-1}V]\frac{1}{\alpha}V^t(I - \omega_iY)VE_i^{(1)}$$

$$- (I - R^tQ_i^{-1}R) + R^tQ_i^{-1}(S - S_0)R^{-t}]E_i^{(2)}.$$ 

The error propagation can be reformulated as

$$\begin{pmatrix} E_{i+1}^{(1)} \\ E_{i+1}^{(2)} \end{pmatrix} = \begin{pmatrix} \alpha(I + \Sigma^tU^tQ_i^{-1}US) & \sqrt{\alpha}\Sigma^tU^tQ_i^{-1}R \\ \sqrt{\alpha}R^tQ_i^{-1}US & -(I - R^tQ_i^{-1}R) \end{pmatrix} + \Delta$$

$$\cdot \begin{pmatrix} \frac{1}{\alpha}V^t(I - \omega_iA_0^{-\frac{1}{2}}A_0^{-\frac{1}{2}})VE_i^{(1)} \\ -E_i^{(2)} \end{pmatrix}.$$ 

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where $\Delta = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix}$ with the blocks defined by

$\Delta_{11} = \alpha [\Sigma^t U_i Q^{-1} U \Sigma V^t (J - I) - \omega_i V^t (J^{-1} - I) A_0^{\frac{1}{2}} \hat{A}^{-1} \hat{A}_0^{\frac{1}{2}} (I - \omega_i A_0^{\frac{1}{2}} \hat{A}^{-1} \hat{A}_0^{\frac{1}{2}})]^{-1} V,$

$\Delta_{12} = -\sqrt{\alpha} \Sigma^t U_i Q^{-1} (S_0 - S) R^{-t},$

$\Delta_{21} = \sqrt{\alpha} R^t Q_i^{-1} U \Sigma V^t (J - I) (I - \omega_i A_0^{\frac{1}{2}} \hat{A}^{-1} \hat{A}_0^{\frac{1}{2}})^{-1} V,$

$\Delta_{22} = R^t Q_i^{-1} (S - S_0) R^{-t}.$

Now it follows from Theorem 3.1 that Algorithm 1 will converge when

$|J - I|, \quad |J^{-1} - I|, \quad |S - S_0|$

are sufficiently small.

### 6 Applications

The saddle-point system (1.1) arises from many applications. We present a few such examples in this section.

The first example arises naturally from the standard quadratic constrained programming with linear constraints:

$$\min_{x \in \mathbb{R}^n} J(x) = \frac{1}{2} (Ax,x) - (f,x) \quad \text{subject to} \quad Bx = g.$$  \hspace{1cm} (6.1)

If we apply the Lagrangian multiplier approach with penalty for the minimization problem (6.1), we come to solve system (1.1) for the primal variable $x$ and the Lagrange multiplier $y$, with $D = \epsilon \hat{D}$, where $\epsilon > 0$ is usually a small parameter and $\hat{D}$ is an appropriately selected symmetric and positive definite matrix. If we apply the above approach iteratively with respect to $\epsilon$, then the parameter $\epsilon$ needs not be too small.

The second example is related to the mixed formulation for the second order elliptic equation, $-\nabla \cdot (\mu \nabla u) + cu = f$. In some applications the flux $p = \mu \nabla u$ is an important quality to know. For the purpose, we may introduce the new variable $p = \mu \nabla u$, then the elliptic equation can be written as the system

$$\frac{1}{\mu} p - \nabla u = 0, \quad -\nabla \cdot p + cu = f.$$  \hspace{1cm} (6.2)

When we apply the mixed finite element formulation to the above system, we obtain a discrete system of form (1.1).

The third example comes from the linear elasticity equation

$$-\mu \Delta u - \nabla ((\lambda + \mu) \nabla \cdot u) = f$$  \hspace{1cm} (6.2)

where $\mu, \lambda$ are Lame coefficients. If one needs to follow the compressiveness of the displacement more closely, one may introduce a new variable $p = (\lambda + \mu) \nabla \cdot u$, then (6.2) can be equivalently written as

$$-\mu \Delta u - \nabla p = f, \quad \nabla \cdot u - \frac{1}{\lambda + \mu} p = 0.$$  \hspace{1cm} (6.3)
This formulation allows us to develop some stable numerical methods for the nearly incompressible case, \( \lambda \gg 1 \). Now the application of the mixed finite element formulation to the above system results in a discrete system of form (1.1).

The next example arises from the following elliptic interface problem

\[
-\nabla \cdot (\mu \nabla u) = f \quad \text{in} \quad \Omega;
\]

\[
[\mu \frac{\partial u}{\partial \nu}] + \alpha u = g \quad \text{on} \quad \Gamma,
\]

where \( \Omega \) is occupied by, e.g., two different fluids or materials \( \Omega_1 \) and \( \Omega_2 \), with different physical property \( \mu \) and a common interface \( \Gamma = \partial \Omega_1 \cap \partial \Omega_2 \). \([\mu \frac{\partial u}{\partial \nu}]\) stands for the jump of the flux \( \mu \frac{\partial u}{\partial \nu} \) across the interface. In some applications, the jump of the flux \([\mu \frac{\partial u}{\partial \nu}]\) can be an important physical quantity to know. For this purpose, we may introduce a new variable \( p = -[\mu \frac{\partial u}{\partial \nu}] \), then the above interface system can be written as

\[
-\nabla \cdot (\mu \nabla u) + \gamma^* p = f \quad \text{in} \quad \Omega;
\]

\[
\gamma u - \frac{1}{\alpha} p = g \quad \text{on} \quad \Gamma,
\]

where \( \gamma \) is the trace operator from \( H^1(\Omega) \) to \( L^2(\Gamma) \) and \( \gamma^* p \in H^1(\Omega)^* \) is defined by

\[
\langle \gamma^* p, \phi \rangle = (p, \gamma \phi)_{L^2(\Gamma)}, \quad \forall \ p \in L^2(\Gamma), \phi \in H^1(\Omega).
\]

The advantage of this formulation is that it can be easily utilized in the domain decomposition approach for a wide class of interface problems, e.g., one uses a subdomain solver, given the boundary value \( g \) and solves the Schur complement system (Neumann-to-Dirichlet map) that equates the continuity of the solution at \( \Gamma \).

In addition, (1.1) can be regarded as a regularization of the simplified saddle-point problem where the (2,2) diagonal block vanishes, with \( D \) arising from the regularization on \( y \). This regularization is often used to remedy the lack of the inf-sup condition and prevent the locking phenomena; see [5, 9, 10], for example, the stabilized Q1-P0 finite element method on the steady-state Stokes problem:

\[
-\nu \Delta u + \nabla p = 0, \quad -\nabla \cdot u = 0 \quad \text{in} \quad \Omega \quad (6.4)
\]

with Dirichlet boundary conditions on \( \partial \Omega \), where \( u \) stands for the velocity field and \( p \) denotes the pressure.

## 7 Numerical experiments

In the following we present some numerical experiments to show the performance of Algorithm [1] with parameters \( \omega_i \) and \( \tau_i \) selected by (1.4) and (1.6). As our first testing example, we consider the two-dimensional elasticity problem (6.2) and its mixed formulation (6.3) in the domain \( \Omega = (0,1) \times (0,1) \). For convenience we use \( (u, v) \) and \( (f, g) \) below to stand respectively for the displacement vector \( u \) and forcing.
vector \(-f\) in \([6.3]\). The system \([6.2]\) is complemented by the following boundary conditions

\[
\begin{align*}
  u &= 0, \quad v_x = 0 \quad \text{on} \quad x = 0, \ 1. \\
  u_y &= 0, \quad v = 0 \quad \text{on} \quad y = 0, \ 1.
\end{align*}
\]

\(7.1\)

\(7.2\)

We partition the domain \(\Omega\) into \(n^2\) equal rectangular elements, and the displacement components \(u\) and \(v\) and the pressure \(p\) are approximated respectively at the staggered grids as follows:

\[
\begin{align*}
  p^{i,j} &\approx p((i - \frac{1}{2})h, (j - \frac{1}{2})h) \quad \text{for} \ 1 \leq i \leq n, \ 1 \leq j \leq n, \\
  u^{i,j-\frac{1}{2}} &\approx u(ih, (j - \frac{1}{2})h) \quad \text{for} \ 0 \leq i \leq n, \ 1 \leq j \leq n, \\
  v^{i-\frac{1}{2},j} &\approx v((i - \frac{1}{2})h, jh) \quad \text{for} \ 1 \leq i \leq n, \ 0 \leq j \leq n,
\end{align*}
\]

\(7.3\)

\(7.4\)

\(7.5\)

with the meshsize \(h = 1/n\). Applying the central difference approximation to \([6.3]\) results in the following scheme:

\[
\begin{align*}
  \mu &\frac{u^{i+1,j-\frac{1}{2}} - 2u^{i,j-\frac{1}{2}} + u^{i-1,j-\frac{1}{2}}}{h^2} + \frac{v^{i,j+\frac{1}{2}} - 2v^{i,j-\frac{1}{2}} + v^{i,j-\frac{1}{2}}}{h^2} \\
  &+ \frac{p^{i+\frac{1}{2},j-\frac{1}{2}} - p^{i-\frac{1}{2},j-\frac{1}{2}}}{h} = f^{i,j-\frac{1}{2}}, \\
  \mu &\frac{v^{i+\frac{1}{2},j} - 2v^{i,j} + v^{i-\frac{1}{2},j}}{h^2} + \frac{u^{i-\frac{1}{2},j+1} - 2u^{i,\frac{j+1}{2}} + u^{i,\frac{j+1}{2}}}{h^2} \\
  &+ \frac{p^{i-\frac{1}{2},j+\frac{1}{2}} - p^{i-\frac{1}{2},j-\frac{1}{2}}}{h} = g^{i-\frac{1}{2},j}, \\
  \mu &\frac{v^{i-\frac{1}{2},\frac{j-1}{2}} - 2v^{i,j-\frac{1}{2}} + v^{i-\frac{1}{2},\frac{j-1}{2}}}{h} + \frac{u^{i,j-1} - 2u^{i,j-\frac{1}{2}} + u^{i,j-\frac{1}{2}}}{h} \\
  &+ \frac{1}{\mu + \lambda^{i-\frac{1}{2},j-\frac{1}{2}}}p^{i-\frac{1}{2},j-\frac{1}{2}} = 0.
\end{align*}
\]

Equivalently the matrices \(A, B\) and \(D\) in \([11]\) can be written as

\[
A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad D = \text{diag}\left(\frac{1}{\mu + \lambda^{i-\frac{1}{2},j-\frac{1}{2}}}\right),
\]

where

\[
A_1 = I \otimes H_1 + H_2 \otimes I, \quad A_2 = I \otimes H_2 + H_1 \otimes I,
\]

\[
B_1 = I \otimes D, \quad B_2 = D \otimes I,
\]

and the tridiagonal matrices \(H_1 \in \mathbb{R}^{(n-1)\times(n-1)}\) and \(H_2 \in \mathbb{R}^{n\times n}\) are given by

\[
H_1 = \begin{pmatrix} 2 & -1 \\ -1 & 2 & -1 \\ & \ddots & \ddots & \ddots \\ -1 & 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & -1 & \ & \ & \ \\ -1 & 2 & -1 & \ & \ \\ & \ddots & \ddots & \ddots \ \\ & -1 & 2 & -1 \\ & -1 & 2 \end{pmatrix}.
\]
computation domain with $Q$ the pressure is constant in the center of each element, and the cell width is $u$.

We have tested the algorithm with the damping factor $\theta$ and observed the convergence of Algorithm 1 for this symmetric case, where ‘Iter’ stands for the iteration numbers. The first 4 columns are for the poor Jacobi preconditioner and show numbers of iterates and CPU time (seconds) to achieve the error $\| (f_i, g_i) \| < 10^{-4}$ for $n = 20$. The next 4 columns are for the more reasonable preconditioner generated by the incomplete Cholesky factorization with no fill-in for the cases $n = 20$ and $n = 50$. The next 4 columns are for the good preconditioner by incomplete Cholesky factorization with drop tolerance of $10^{-3}$ for the cases $n = 20, 50, 100, 200$, with a total number of degrees of freedom being 120,000 for $n = 200$. The last column is for the case when the exact preconditioner for $A$ is used. From our experiments and observations, the number of iterations is insensitive to mesh refinements if good preconditioners are used. With the poor Jacobi preconditioner, Algorithm 1 always converges. We have tested the algorithm with the damping factor $\theta$ selected from the range $[0.01, 1.0]$, and observed the convergence of the algorithm for all the cases. But for the well-conditioned case for $A$, $\theta_i = 1$ produces the best results. For the very ill-conditioned preconditioner $\hat{A}$, $\theta_i$ may need to be small.

Next we consider the Stokes flow in a rectangular domain $\Omega = (0, 1) \times (0, 1)$. Here Dirichlet boundary conditions are used: $u = 1$, $v = 0$ on the top ($y = 1$); $u = v = 0$ on the other three sides (i.e., $x = 0$, $x = 1$, and $y = 0$). We discretize the computation domain with $Q_1 - P_0$ element, where the velocity is located on the node, the pressure is constant in the center of each element, and the cell width is $h = 1/n$. After discretization of (6.4), we obtain

$$
\begin{bmatrix}
A_0 & 0 & B_1^T \\
0 & A_0 & B_2^T \\
B_1 & B_2 & -D
\end{bmatrix}
\begin{bmatrix}
u/6 (M \otimes K + K \otimes M) & 0 & h/2 (H_n^T \otimes H_n^T) \\
0 & \nu/6 (M \otimes K + K \otimes M) & h/2 (H_o^T \otimes H_o^T) \\
h/2 (H_n \otimes H_o) & h/2 (H_o \otimes H_n) & -\beta k^2 (I \otimes T_N + T_N \otimes I)
\end{bmatrix}
\begin{bmatrix}
u/6 (M \otimes K + K \otimes M) & 0 & h/2 (H_n^T \otimes H_n^T) \\
0 & \nu/6 (M \otimes K + K \otimes M) & h/2 (H_o^T \otimes H_o^T) \\
h/2 (H_n \otimes H_o) & h/2 (H_o \otimes H_n) & -\beta k^2 (I \otimes T_N + T_N \otimes I)
\end{bmatrix}.
$$

\begin{table}[h]
\centering
\begin{tabular}{l|cccc|cccc|c}
\hline
$\theta_i$ & .03 & .1 & .5 & 1 & 1 & .1 & .05 & 1 & 1 \\
\hline
Iter & 659 & 737 & 906 & 1074 & 95 & 752 & 463 & 434 & 11 & 17 & 61 & 152 & 5 \\
CPU & .52 & .56 & .66 & .74 & .22 & 9.57 & 5.70 & 5.37 & .04 & 1.86 & 4.91 & 56.9 & 5.62 \\
$n$ & 20 & 20 & 20 & 20 & 20 & 50 & 50 & 50 & 20 & 50 & 100 & 200 & 200 \\
\hline
\end{tabular}
\caption{Number of iterates with different $\theta_i$'s and preconditioners for linear elasticity problem.}
\end{table}

We will choose the following set of parameters in our test: $f = 0$, $g = 1$ and $\mu = 1$. The parameter $\lambda$ is taken to be discontinuous: $\lambda = 1000$ in $(0.25, 0.75) \times (0.25, 0.75)$, and $\lambda = 0$ otherwise.

After discretization of (6.4), we obtain
Here we define \( A_0 = \nu/6 (M \otimes K + K \otimes M) \), \( B_1 = h/2 (H_n \otimes H_o), B_2 = h/2 (H_o \otimes H_n) \), \( D = \beta h^2 (I \otimes T_N + T_N \otimes I) \), where \( M = \text{tridiag}(1, 4, 1) \in \mathbb{R}^{(n-1) \times (n-1)} \), \( K = \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{(n-1) \times (n-1)} \), \( T_N = \text{tridiag}(-1, 2, -1) - e_1 e_1^T - e_n e_n^T \in \mathbb{R}^{n \times n} \), and \( H_o, H_n \) are bidiagonal matrices with \( H_o = \text{sparse}(1 : n - 1, 1 : n - 1, -\text{ones}(1, n - 1), n, n - 1) + \text{sparse}(2 : n, 1 : n - 1, \text{ones}(1, n - 1), n, n - 1) \in \mathbb{R}^{n \times (n-1)} \), \( H_n = \text{sparse}(1 : n - 1, 1 : n - 1, \text{ones}(1, n - 1), n, n - 1) + \text{sparse}(2 : n, 1 : n - 1, \text{ones}(1, n - 1), n, n - 1) \in \mathbb{R}^{n \times (n-1)}. \) Here \text{sparse} and \text{ones} are MATLAB notations, \( e_1 \) and \( e_n \) are the first and \( n \)-th column vector of unit matrix \( I_n \). For the right hand side, \( f_1 = (6 \times \frac{9}{2}) (\epsilon_{n-1} \otimes \epsilon) \in \mathbb{R}^{(n-1) \times 1} \), where \( \epsilon_{n-1} \) is the \( (n - 1) \)-th column vector of unit matrix \( I_{n-1} \), and \( \epsilon = [1, \ldots, 1]^T \in \mathbb{R}^{(n-1) \times 1}. \) The choice of \( \beta \) represents a trade-off between stability and accuracy. We use \( \beta = 0.25 \) for the local stabilization and \( \beta = 1 \) for the global stabilization. The iteration stops when the residual \( \max \{|f_r|, |g_r|\} < 10^{-6} \). The iteration numbers and computation times are listed in Table 7.2. We compare the iteration numbers for using different preconditioners. The preconditioner for \( A \) include Jacobi iteration, the incomplete Cholesky decomposition with no fill-in or with tolerance \( 10^{-3} \), and the exact solver as well. The preconditioner for Schur complement is the pressure mass matrix for all cases. The CPU times (in seconds) are given correspondingly.

| \( \theta = 0.5 \) | \( \theta = 0.3 \) | \( \theta = 0.1 \) | \( \theta = 0.05 \) |
|-----------------|-----------------|-----------------|-----------------|
| Iter | CPU | Iter | CPU | Iter | CPU | Iter | CPU |
| Jacobi | 2006 | 0.57 | 891 | 0.26 | 725 | 0.21 | 749 | 0.21 |
| cholinc('0') | 192 | 0.081 | 164 | 0.069 | 139 | 0.061 | 156 | 0.065 |
| cholinc(10^{-3}) | 37 | 0.022 | 47 | 0.028 | 93 | 0.056 | 175 | 0.11 |
| Exact | 37 | 0.27 | 45 | 0.32 | 98 | 0.71 | 184 | 1.31 |
| Jacobi | 16823 | 17.4 | 14518 | 15.1 | 3329 | 3.50 | 2845 | 3.02 |
| cholinc('0') | 873 | 1.51 | 779 | 1.37 | 494 | 0.87 | 343 | 0.62 |
| cholinc(10^{-3}) | 38 | 0.12 | 55 | 0.17 | 80 | 0.25 | 147 | 0.46 |
| Exact | 36 | 1.70 | 48 | 2.25 | 94 | 4.52 | 177 | 8.34 |
| Jacobi | 4103 | 1.19 | 1318 | 0.38 | 1278 | 0.37 | 1300 | 0.38 |
| cholinc('0') | 295 | 0.13 | 203 | 0.094 | 235 | 0.097 | 291 | 0.12 |
| cholinc(10^{-3}) | 101 | 0.061 | 117 | 0.071 | 169 | 0.10 | 271 | 0.16 |
| Exact | 80 | 0.57 | 115 | 0.85 | 169 | 1.21 | 269 | 1.96 |
| Jacobi | 22026 | 23.1 | 3884 | 4.06 | 2777 | 2.91 | 3756 | 3.92 |
| cholinc('0') | 1385 | 2.37 | 755 | 1.30 | 391 | 0.67 | 386 | 0.67 |
| cholinc(10^{-3}) | 143 | 0.45 | 117 | 0.37 | 160 | 0.50 | 242 | 0.75 |
| Exact | 77 | 3.60 | 95 | 4.47 | 151 | 7.14 | 247 | 11.5 |

Table 7.2: Stokes problem.

The third testing case is a purely algebraic example from [17]. Consider the linear system \([\mathbf{A}]\) with \( A = (a_{ij})_{n \times n}, B = [T; 0] \in \mathbb{R}^{n \times m}, \) and \( D = I, \) where

\[
a_{ij} = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-y)^2}{2\sigma^2}}, \quad T = \frac{1}{1000} \text{tridiag}(1, 4, 1) \in \mathbb{R}^{m \times m}.
\]

We set \( \sigma = 1.5. \) The right hand side is chosen such that the exact solution is a vector of all ones. Note that \( A \) is an ill-conditioned Toeplitz matrix. Fortunately, the
Schur complement $S$ is well-conditioned for $n = 800$ and $m = 600$, or $n = 1600$ and $m = 1200$. We set $\hat{S} = 2I$ as the preconditioner.

| $\theta_i$ | $n = 800, m = 600$ | $n = 1600, m = 1200$ |
|-----------|-----------------|-----------------|
|          | Iter | CPU | Iter | CPU | Iter | CPU |
| Jacobi   |      |     | Jacobi |      |     | Jacobi |      |     |
| Exact    |      |     | Exact |      |     | Exact |      |     |
| 0.05     | 263  | 263 | 1.10  | 30.2 | 263  | 263 | 3.69  | 129.0 |
| 0.1      | 206  | 129 | 0.87  | 14.9 | 129  | 129 | 1.86  | 63.1  |
| 0.5      | 171  | 21  | 0.72  | 2.53 | 150  | 21  | 2.14  | 10.3  |
| 0.9      | 183  | 7   | 0.82  | 0.83 | 143  | 7   | 2.07  | 3.44  |

Table 7.3: The purely algebraic example.

As our last testing example, we consider the nonsymmetric saddle-point system (1.1) arising from the discretization of the mixed formulation of the following system

$$-\mu \Delta u + b \begin{pmatrix} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_2} \end{pmatrix} + \nabla p = f,$$

which is a compressible linearized Navier-Stokes system. Numerical results are summarized in Table 7.4.

The first five columns are for the preconditioner generated by the incomplete Cholesky factorization with no fill-in for the case $n = 50$. The next three columns are for the preconditioner by the incomplete Cholesky factorization with drop tolerance of $10^{-3}$ for $n = 50$. The last three columns are for the case with exact preconditioner for $A$ with $n = 50$. The number of iterations depends significantly on $b$ (the magnitude of the convection term). The algorithm may fail to converge when $|b|$ is very large, which is consistent with the convergence analysis in Section 5 as the symmetric part of block $A$ is not dominant.

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| $\theta_i$ | no fill-in Cholesky | cholinc($A, 10^{-3}$) | Exact |
|-----------|---------------------|-----------------------|-------|
| Iter      | 343                 | 438                   | 660   |
| CPU       | 4.23                | 5.45                  | 796.2 |
| $b$       | 40                  | 2                     | 10    |

Table 7.4: Nonsymmetric case with $n = 50$ and different $b$’s.
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