ON FINITE $p$-GROUPS WITH POWERFUL SUBGROUPS

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ABSTRACT. In this paper we investigate the structure of finite $p$-groups with the property that every subgroup of index $p^i$ is powerful for some $i$. For odd primes $p$, we show that under certain conditions these groups must be potent. Then, motivated by a question of Mann, we investigate in detail the case when all maximal subgroups are powerful. We show that for odd $p$ any finite $p$-group $G$ with all maximal subgroups powerful has a regular power structure - with precisely one exceptional case which is a 3-group of maximal class and order 81. To show this counterexample is unique we use a computational approach. We briefly discuss the case $p = 2$ and some generalisations.

1. Introduction

Loosely speaking, the purpose of this paper is to study $p$-groups which contain large subgroups with some desirable properties and to determine whether these properties are enjoyed by the group itself. In doing so we address questions posed by Mann and Xu.

More precisely, we study finite $p$-groups $G$ such that every subgroup of index $p^i$ is powerful. Powerful $p$-groups share many properties with abelian $p$-groups, in particular products and powers behave well in powerful $p$-groups. We show that under certain conditions these desirable properties of the subgroups of index $p^i$ extend to the group itself.

The study of $p$-groups is important, as they occur in many problems and have connections to many areas of mathematics. However, finite $p$-groups are known to be complicated. The number of $p$-groups grows very quickly and any hope of a strong classification (akin to the classification of finite simple groups) is thought to be hopeless. However, what has been a successful approach is to understand large families of $p$-groups which are in some sense “well behaved”.

To state our results precisely we need to remind the reader of the following terminology. Let $G$ be a finite $p$-group.

$$\Omega_{\{k\}}(G) = \{g \in G \mid g^{p^k} = 1\}$$
$$\Omega_k(G) = \langle \Omega_{\{k\}}(G) \rangle.$$  

$$\bar{\Omega}_{\{k\}}(G) = \{g^{p^k} \mid g \in G\}$$
$$\bar{\Omega}_k(G) = \langle \bar{\Omega}_{\{k\}}(G) \rangle.$$  

We call $\Omega_k(G)$ the $k$th Omega subgroup of $G$. There is a sense in which the subgroups $\bar{\Omega}_k(G)$ and $\Omega_k(G)$ are dual to each other, hence $\bar{\Omega}_k(G)$ is referred to as the $k$th Agemo subgroup of $G$ (Agemo is Omega backwards).

For $p$-groups in general, we cannot guarantee that $\Omega_{\{k\}}(G) = \Omega_k(G)$, nor that $\bar{\Omega}_{\{k\}}(G) = \bar{\Omega}_k(G)$. In words, the product of two elements of order at most $p^k$ can have order greater than $p^k$. Similarly the product of two $p^k$th powers need not be a $p^k$th power.

If we turn our attention to a well understood family of finite $p$-groups - the abelian groups - we notice that they satisfy the following three conditions.

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\[ \Omega_i(G) = \{ g \in G \mid o(g) \leq p^i \} \]  \hspace{1cm} (1.2)

\[ |G : \Omega_i(G)| = |\Omega_i(G)| \]  \hspace{1cm} (1.3)

In [8] Hall showed that a large family of finite p-groups known as regular p-groups also satisfy these properties. Motivated by this, we say that any group which satisfies these three properties enjoys a regular power structure.

The work of Hall in [8] initiated the study of power structure in finite p-groups. Since then, many families of p-groups have been shown to have a regular power structure, and even more have been shown to enjoy some subset of these properties.

One such family of groups known to have a regular power structure for odd primes p are the powerful p-groups. This family of groups was introduced in the paper [13] by Lubotzky and Mann, and has seen widespread applications and generated a huge amount of further research and generalisations (for instance potent p-groups [7]). Therefore it is natural to ask, if all maximal subgroups of a finite p-group G are powerful, does G enjoy any of the same nice properties. This then leads naturally to the generalisation of all subgroups of index \( p^i \) being powerful, and motivates the following definition.

**Definition.** Let G be a finite p-group. We say that G is an \( M_i \) group if all subgroups of index \( p^i \) in G are powerful.

There is a family of finite p-groups called potent p-groups which can be thought of as a generalisation of powerful p-groups. (We recall their definition in Definition 2.5.) In [2] Arganbright began the study of what we now call potent p-groups. Their theory is developed further by González-Sánchez and Jaikin-Zapirain in [7], where among other things it is shown that for odd primes p, potent p-groups have a regular power structure.

For our first main result, we prove the following theorem which says that under certain conditions an \( M_i \) group is a potent p-group.

**Theorem A.** Let p be an odd prime and G be a finite p-group. If \( i \leq p - 3 \) and G is an \( M_i \) group, then G is a potent p-group.

This theorem captures a relationship between the prime p, the index \( p^i \), and the power structure properties of the group. In a similar way, we can capture a relationship with the minimal number of generators of the group. Letting \( d(G) \) be the minimal number of generators of G, we also prove the following result.

**Theorem B.** Let G be a finite p-group. If G is a \( M_i \) group with \( d(G) \geq i + 2 \), then G is powerful.

Turning our attention to the regular power structure properties, as both powerful and potent p-groups have regular power structure, we deduce that for \( p > 3 \) all \( M_1 \) p-groups have a regular power structure.

The main part of the paper focuses on the study of \( M_1 \) groups. We recall a question of Mann from [3, Question 68] which is a motivation for this:

**Question (Mann).** Study the p-groups, \( p > 2 \), all of whose (a) subgroups of index p (of index \( p^2 \)) are powerful (b) subgroups of indices p and \( p^2 \) are powerful.

We remark that to the best of our knowledge there are no published results about p-groups all of whose maximal subgroups are powerful.

For \( p > 3 \) it follows from Theorem A that \( M_1 \) groups are potent and thus have a regular power structure. As often happens, \( p = 3 \) proves to be the challenging case.
We show that for \( p = 3 \), \( M_1 \) groups satisfy condition 1.2 and condition 1.1. We believe our approach to establishing condition 1.1 is unique and may have other uses: to prove it we reduce to a case where we can apply a result of Mann from [15].

Our most surprising result comes when studying condition 1.3. In this case we find that \( M_1 \) 3-groups satisfy the condition, except for one specific 3-group of order 81 and maximal class. Proving the uniqueness of this example is a large part of the paper and our approach makes use of the computer algebra package GAP [1].

Our main theorem is stated below.

**Theorem C.** Let \( G \) be an \( M_1 \) \( p \)-group and \( p \) an odd prime. Then \( G \) has a regular power structure, unless \( G \) is the unique \( M_1 \) group of order 81 and maximal class, in which case condition 1.3 fails to hold.

We briefly discuss some possible generalisations of this theorem in Remark 5.3.

A large motivation for this work is to try to better understand the families of 3-groups with regular power structure. In [21], Xu suggests that the problem of determining all irregular \( p \)-groups with a regular power structure is likely to be very difficult, and so poses some problems with restrictions such as determining the 2-generator 3-groups with regular power structure.

The work in this paper is then a contribution towards better understanding the 3-groups with regular power structure. In particular for 3-groups of order strictly larger than 81 we have found a new family of 3-groups with regular power structure.

**Notation.** Our notation is standard. For a group \( G \), we denote the Frattini Subgroup by \( \Phi(G) \). Our commutators are left normed. We define \( \gamma_1(G) = G \), and then for \( i > 1 \)

\[ \gamma_i(G) = [\gamma_{i-1}(G), G]. \]

We use bar notation to denote images under a quotient map and in all cases we shall make it explicit what we are quotienting by. We denote the minimal number of generators of \( G \) by \( d(G) \) and the exponent of \( G \) by \( \exp G \).

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2. **Preliminaries**

For the convenience of the reader we collect here the basic definitions and results that will be used in this paper. A reader familiar with powerful and potent \( p \)-groups may skip this section.

Powerful \( p \)-groups were introduced by Lubotzky and Mann in [13]. We recall their definition below.

**Definition 2.1.** For odd primes \( p \), a \( p \)-group \( G \) is **powerful** if \( [G, G] \leq \Phi_1(G) \). A 2-group \( G \) is **powerful** if \( [G, G] \leq \Phi_2(G) \).

The property of being powerful is preserved by quotients (see [5, Lemma 2.2(i)]), but not necessarily when taking subgroups.

In this paper we make use of several properties of powerful \( p \)-groups, often without explicit mention.

**Proposition 2.2.** Let \( G \) be a powerful \( p \)-group with \( p \) odd and \( k, j \) non-negative integers. Then the following statements hold:

(i) \( \Phi_k(G), G] \leq \Phi_{k+1}(G) \).
(ii) \( \gamma_k(G) \leq \Phi_{k-1}(G) \).
(iii) \( \Phi_j(\Phi_k(G)) = \Phi_{j+k}(G) \).

For a proof of the above see [12, Theorem 11.10].
Proposition 2.3. Let $G = \langle a_1, \ldots, a_r \rangle$ be a powerful $p$-group. Then $\mathcal{U}_1(G) = \langle a_1^p, \ldots, a_r^p \rangle$.

For a proof of this see [12, Theorem 11.11]. For an introduction to the theory of powerful $p$-groups, we recommend Chapter 11 of [12].

Theorem 2.4. For odd primes $p$, powerful $p$-groups have a regular power structure.

The first power structure condition was established in [13, Proposition 1.7]. The second and third power structure conditions were first established in [20, Theorem 3.1], where the proof of the third condition relied on a result of Héthelyi and Lévai [9, Corollary 2]. A shorter proof of conditions 1.2 and 1.3 were given by Fernández-Alcober in [6], see Theorem 1(iii) and Theorem 4. We note also that an independent proof of the fact that $|\Omega_{\{k\}}(G)| = |G : \mathcal{U}_k(G)|$ for a powerful $p$-group $G$ was given by Mazur in [16, Theorem 1].

A generalisation of powerful $p$-groups is the notion of a potent $p$-group.

Definition 2.5. For an odd prime $p$, a finite $p$-group $G$ is said to be potent if $\gamma_{p-1}(G) \leq \mathcal{U}_1(G)$. A finite 2-group $G$ is said to be potent if $[G, G] \leq \mathcal{U}_2(G)$.

Notice that for $p = 2$ and $p = 3$ the notions of potent and powerful coincide.

In [7] it is proved that for odd primes potent $p$-groups also have regular power structure.

Theorem 2.6. For odd primes $p$, potent $p$-groups have a regular power structure.

In [8] P. Hall introduced regular $p$-groups.

Definition 2.7. We say that a $p$-group $G$ is regular if for any two elements $a, b \in G$ and any $n$ we have that:

$$(ab)^{p^n} = a^{p^n}b^{p^n}S_1^{p^n} \cdots S_t^{p^n},$$

with $S_1, \ldots, S_t$ appropriate elements from the commutator subgroup of the group generated by $a$ and $b$.

In [8] P. Hall showed, amongst other things, that regular $p$-groups satisfy the regular power structure properties. For a textbook exposition of this see [3, Theorem 7.2 (b), (c), (d)].

There are many known conditions which imply that a $p$-group $G$ is regular. It is clear from the definition that $p$-groups of exponent $p$ are regular. It is also the case that $p$-groups whose nilpotency class $c$ is strictly less than $p$ are regular ([3, Theorem 7.1(b)]).

We will use these facts throughout this paper, often without explicit mention.

At one point in this paper, we will make use of Fitting’s Lemma to bound the nilpotency class of a group.

Lemma 2.8 (Fitting). Let $G$ be a group and $M$ and $N$ normal nilpotent subgroups of $G$ with nilpotency classes $a$ and $b$ respectively. Then $MN$ is also a nilpotent normal subgroup of $G$, with nilpotency class at most $a + b$.

For a proof of Lemma 2.8 see [11, Theorem 10.3.2].

In Section 4.3 we will attempt to bound the order of some groups. In doing so, we end up considering the order of a 2-generator group of exponent 3. We make use of the fact that the largest 2-generator group of exponent 3 has order 27 (See Chapter 18 of [11] for an introduction to the Burnside problem and proof of a more general version of this fact).

Finally, we would like to remind the reader that the group $\mathcal{U}_n(G)$ is a verbal subgroup and behaves nicely when we take quotients. In particular we have that

$$\mathcal{U}_n \left( \frac{G}{N} \right) \cong \frac{\mathcal{U}_n(G)N}{N}.$$
2.1. **A note on GAP code.** In Section 4.3 we will reduce problems about 3-groups, to problems about 3-groups of specific small orders (at most $3^7$). As the groups of order $3^n$ for $n \in \{1, \ldots, 7\}$ have been classified and are readily available in computer algebra software such as GAP [1] through the SmallGroupsLibrary [4], we are then able to concretely deal with these problems.

Although some of these cases and problems could surely be dealt with directly through various ad hoc arguments, we believe there is a significant aesthetic benefit to reducing to a finite problem and then having the computer deal with it uniformly in one clean sweep. For the benefit of the reader, we include alongside this manuscript our GAP code to verify the claims made in Section 4.3.

3. **On $M_i$ groups**

One of our motivations for this work is to study $p$-groups all of whose maximal subgroups are powerful, these are the $M_1$ groups. However to begin we study the more general case of $M_i$ groups; these are the finite $p$-groups for which all subgroups of index $p^i$ are powerful.

In this paper we will often use the fact that if $G$ is an $M_1$ group, then so are its quotients. This is easily seen, recalling that the property of being powerful is preserved under quotients.

3.1. **$M_i$ groups and potency.** Fundamentally, the behaviour of $p$-groups is determined by the interaction of powers and commutators. For an $M_i$ group, we are ensuring that all subgroups of index $p^i$ have a good behaviour with regards to the interactions of powers and commutators. It is natural to ask, what does this mean for the group itself.

**Proposition 3.1.** Let $G$ be an $M_i$ group of exponent $p$. Then the nilpotency class of $G$ is at most $i + 1$.

**Proof.** If the group is abelian the result is clear, hence we assume that $G$ is not abelian. All subgroups in $G$ of index $p^i$ are powerful of exponent $p$, and so they are abelian. Let $j$ be the smallest integer such that all subgroups of index $p^j$ in $G$ are abelian, and note that $j \geq i$. Then there is some subgroup $H$ in $G$ with index $p^{j-1}$ and $H$ is not abelian. However every maximal subgroup of $H$ is abelian, and so $H$ is a minimal non-abelian $p$-group of exponent $p$. As $H$ is minimal non-abelian it follows must be 2-generator and have nilpotency class 2 with $|[H,H]| = p$. Thus $|H| \leq p^3$. (See [3, Exercise 8a] for a classification of minimal non-abelian $p$-groups).

Therefore the order of $G$ is at most $p^3 \cdot p^{j-1} = p^{j+2}$ and so the nilpotency class of $G$ is at most $j + 1$ as required. □

Notice that this means an $M_i$ $p$-group of exponent $p$ is a potent $p$-group.

**Corollary 3.2.** Let $p > 2$. If $i \leq p - 3$ and $G$ is an $M_i$ group, then $G$ is a potent $p$-group.

**Proof.** We prove this by induction on the order of $G$. The result is clearly true for groups of order $p$. Now consider some $M_i$ group $G$ and suppose that the claim holds for all groups of smaller order. If $G$ has exponent $p$, the result follows from Proposition 3.1. Now suppose that the exponent of $G$ is greater than $p$. Then $\overline{G} = G/\overline{U}_1(G)$ is an $M_i$ group of order strictly less than $G$ and so by induction $\gamma_{p-1}(\overline{G}) \leq \overline{U}_1(\overline{G})$, and it then follows that $\gamma_{p-1}(G) \leq \overline{U}_1(G)$ as required. □

This establishes Theorem A from the introduction. We remark that when $p = 2$, the definition of potent and powerful coincide. We can readily find examples of 2-groups for which all maximal subgroups are powerful (even abelian) and yet the group itself is not powerful - for example the dihedral and quaternion groups of order 8.
3.2. $\mathcal{M}_i$ groups and minimal number of generators. In this section we ask ourselves, for a given $\mathcal{M}_i$ group $G$, is there some condition on the number of generators of $G$ that will ensure desirable properties for our group $G$.

**Proposition 3.3.** Let $G$ be a finite $p$-group. If $G$ is a $\mathcal{M}_i$ group with $d(G) \geq i + 2$, then $G$ is powerful.

*Proof.* Let $\{a_1, \ldots, a_r\}$ be a minimal generating set for $G$. Then $H = \langle a_i, a_j, \Phi(G) \rangle$, for $i, j \in \{1, \ldots, r\}$ and $i \neq j$, is a subgroup of index $p^{d(G) - 2} \geq p^i$. In particular $H$ is contained in some powerful subgroup $M$ of index $p^i$ in $G$. If $p$ is an odd prime, then $[a_i, a_j] \in \mathfrak{U}_1(M) \leq \mathfrak{U}_1(G)$. It follows that $[G, G] \leq \mathfrak{U}_1(G)$ and that $G$ is powerful. If $p = 2$ then $[a_i, a_j] \in \mathfrak{U}_2(M) \leq \mathfrak{U}_2(G)$ and again it follows $G$ is powerful.

This is Theorem 13 from the introduction.

In the next section we focus our attention on $\mathcal{M}_1$ groups and their power structure properties. Corollary 3.2 and Proposition 3.3 will allow us to reduce to studying 2-generator 3-groups.

4. On power structure of $\mathcal{M}_1$ groups

In this section we investigate the power structure properties of $\mathcal{M}_1$ groups. As all maximal subgroups are powerful, and hence have a regular power structure by Theorem 2.4, it seems reasonable to expect a $\mathcal{M}_1$ group $G$ to have good power structure properties. We recall the definition of regular power structure from the introduction.

**Definition.** A finite $p$-group $G$ has a regular power structure if the following three conditions hold for all positive integers $i$:

\begin{align}
\mathfrak{U}_i(G) &= \{g^{p^i} \mid g \in G\}.
\end{align}

\begin{align}
\Omega_i(G) &= \{g \in G \mid o(g) \leq p^i\}.
\end{align}

\begin{align}
|G : \mathfrak{U}_i(G)| &= |\Omega_i(G)|.
\end{align}

It is our goal in this section to show that all $\mathcal{M}_1$ groups satisfy conditions (1.1) and (1.2). Remarkably we show that they also satisfy (1.3), except for one specific exception – a 3-group of maximal class and order $3^4$.

To begin, we use the results from Section 3.1 to show that we can reduce the problem of whether $\mathcal{M}_1$ groups have a regular power structure, to the study of 2-generator 3-groups.

**Reduction to $p = 3$:** By Corollary 3.2 for primes $p \geq 5$ we have that an $\mathcal{M}_1$ group $G$ is a potent $p$-group, and by Theorem 2.6 such groups have a regular power structure. Thus we need only focus on $p = 3$.

**Reduction to 2-generator groups:** Furthermore by Proposition 3.3 if the number of generators of an $\mathcal{M}_1$ 3-group were greater than 2, then the group is powerful and also has a regular power structure by Theorem 2.4. Thus in what follows we only need to consider the case of 2-generator 3-groups. However we note that many of our arguments work for any odd prime $p$ and so may be of independent interest. We will not need to rely on this reduction until Section 4.6.

4.1. Our first goal is to show that condition (1.3) is satisfied. That is, we want to show that the product of two elements of order at most $p^i$, has order at most $p^i$.

Note that for an $\mathcal{M}_1$ group the maximal subgroups are powerful, and so have a regular power structure and in particular satisfy condition (1.2). This condition is inherited by subgroups and so any proper subgroup of $G$ must satisfy this condition. We use this fact freely in what follows to conclude that proper subgroups which are generated by elements of exponent $p$ must have exponent $p$.

**Proposition 4.1.** Let $G = \langle a, b \rangle$ be an $\mathcal{M}_1$ group, with $o(a) = p, o(b) = p$, then $\text{exp} G = p$. 

Thus in particular nilpotency class at most 2, and so will also satisfy all three power structure conditions.

4.2. In this section we move to showing that condition 1.1 is satisfied. That is, we wish condition 1.2.

Observe that if some subgroup \( H \) of \( G \) then can be no counterexample.

The method of proof in this part is interesting, as we apply a result of Mann from [15].

Let \( \exp(\Omega_G) \leq p \). Then the maximal subgroups \( M_i = \langle a, \Phi(G) \rangle \) and \( M_2 = \langle b, \Phi(G) \rangle \) are both of exponent 1, and as they are powerful by hypothesis, they must be abelian since \( [M_i, M_j] \leq \Omega(G) = 1 \) for all 

Thus the product \( \Phi(G) = [G, G] \Omega(G) \) must also have exponent \( p \), for the same reason (as \( [G, G] \) and \( \Omega(G) \) are both contained in a powerful maximal subgroup, and so condition 1.2 holds). Then the maximal subgroups \( i = 1, \) and \( i = \Omega(G) \) it follows that the exponent of \( G \) is 1. Hence there can be no counterexample.

We can now prove the more general result, showing that \( M_i \) groups satisfy condition 1.2 that \( \exp(\Omega_i(G)) \leq p^i \) for any positive integer \( i \).

Theorem 4.3. Let \( G \) be an \( M_i \) group, then \( \exp(\Omega_i(G)) \leq p^i \).

Proof. The case \( i = 1 \) is Corollary 1.2 thus we may assume that \( i > 1 \).

Let \( N = \Omega_1(\Phi(G)) \). The group \( \bar{G} = G/N \) is an \( M_i \) group. Then \( \Omega_{i-1}(\bar{G}) = H/N \) for some subgroup \( H \) of \( G \) with \( N \leq H \). By induction we know that \( \exp(\Omega_{i-1}(\bar{G})) \leq p^{i-1} \). Observe that if \( g \in G \) has order \( p^j \) with \( j > 1 \), then \( \bar{g} \in \Omega_{i-1}(\bar{G}) \). This is because \( g^{p^{j-1}} \) is a \( j \)-th power, and is of order \( p \), and so \( g^{p^{j-1}} \in N \).

Hence \( \Omega_i(G) \leq H \), and for any \( h \in H \), we have that \( h^{p^{j-1}} \in N \), and the exponent of \( N \) is \( p \) since \( N \leq \Omega_1(M) \) for a maximal (and hence powerful) subgroup \( M \). Thus \( \exp(\Omega_i(G)) \leq \exp H \leq p^i \).

4.2. In this section we move to showing that condition 1.4 is satisfied. That is, we wish to show that for a \( M_i \) group, the product of two \( p \)-th powers is equal to a \( p \)-th power. The method of proof in this part is interesting, as we apply a result of Mann from [15].

We recall some definitions from [15]:

Definition 4.4. A \( p \)-group \( G \) is a \( P_i \) group if \( G \), as well as all sections of \( G \), satisfy condition 1.1. A \( p \)-group \( G \) is a \( P_2 \) group if \( G \), as well as all sections of \( G \), satisfy condition 1.2.

For example if \( G \) is a \( p \)-group (for \( p \) odd) of nilpotency class 2, then \( G \) is a regular \( p \)-group and so satisfies all three power structure conditions. Every section of \( G \) will have nilpotency class at most 2, and so will also satisfy all three power structure conditions. Thus in particular \( G \) is a \( P_2 \) group. We use this fact in the next proof.

As in the previous section, we begin with a reduced 2-generator case.
Proposition 4.5. Let $G$ be an $\mathcal{M}_1$ group with $d(G) \leq 2$ and $\exp G \leq p^2$. Then $G$ is a $P_2$ group.

Proof. The proof is by induction on the order of the group. For orders $p$ and $p^2$ the groups are abelian and so the claim holds.

For any subgroup $Z \leq Z(G)$ we have that $G/Z$ satisfies the hypothesis.

For any normal subgroup $N$ of $G$, we have a nontrivial $z \in N \cap Z(G)$. Then by the isomorphism theorems we know that $G/N \cong G/(z)$, and notice that $G/(z)$ is a $P_2$ group by induction, hence $G/N$ is a $P_2$ group. Hence all proper quotients of $G$ are $P_2$ groups. We also know that $G$ itself satisfies condition (1.2) by Theorem 4.3. Thus all that remains is to prove that proper subgroups of $G$ are $P_2$.

For any maximal subgroup $M$ of $G$, we have that $|M, M, M| \leq [U_1(M), M] \leq U_2(M) = 1$, since $M$ is powerful. Then in particular all proper subgroups have class at most 2. For $p$ odd, groups of class 2 are $P_2$ groups.

Thus we have established that $G$ is a $P_2$ group. □

We now quote a result from [15].

Theorem 4.6 ([15, Corollary 4]). A $P_2$ group is a $P_1$ group.

Combining Proposition 4.5 and Theorem 4.6 we obtain the following corollary.

Corollary 4.7. Let $G$ be an $\mathcal{M}_1$ group with $d(G) \leq 2$ and $\exp G \leq p^2$. Then $\bar{U}_1(G) = \{g^p \mid g \in G\}$.

The results so far can now be used to prove that for an $\mathcal{M}_1$ group, the product of $p$th powers is equal to a $p$th power.

Proposition 4.8. Let $G$ be an $\mathcal{M}_1$ group, then $\bar{U}_1(G) = \{g^p \mid g \in G\}$.

Proof. As usual, we may assume that $d(G) = 2$, by Proposition 4.3. Next we show that we may assume that the exponent of the group is at most $p^2$.

If $\exp G > p^2$, then $\bar{U}_1(\bar{U}_1(G)) \neq 1$. Then $\bar{U}_1(\bar{U}_1(G))$ contains some minimal normal subgroup $Z$. Then by induction, it follows that for any $a, b \in G$ we have that $a^{p^2}b^p = c^pZ$ for some $c \in G$ and $z \in Z$. The element $c$ is contained in some maximal subgroup $M$. Notice that $z \in \bar{U}_1(\bar{U}_1(G)) \leq \bar{U}_1(M)$, and so $z = m^p$ for some $m \in M$ (here we use that $M$ is powerful and so satisfies condition (1.1)).

But now, $a^{p^2}b^p = c^pZ = c^p m^p = x^p$ for some $x \in M$, where we have used again that $c, m \in M$ and $M$ is powerful. Thus it follows that $\bar{U}_1(G) = \{g^p \mid g \in G\}$.

Hence we may assume that the exponent of $G$ is at most $p^2$. This is now the case of Corollary 4.7 above and the result follows. □

We next prove that for an $\mathcal{M}_1$ group, the first Agemo subgroup, $\bar{U}_1(G)$ is powerful. This will help us generalise Proposition 4.8 and will also be used in the proof Proposition 4.24. We also note that this is interesting in it’s own right, in light of a question of Wilson, see [20, Question 4.8].

Question 4.9 ([20, Question 4.8]). If $G$ has the property that $\bar{U}_k(\bar{U}_k(G))$ is the set of $p^k$th powers of elements of $G$ for all $k$, then must $\bar{U}_1(G)$ be powerful.

This property has been verified for many families of groups, and we are pleased to be able to add $\mathcal{M}_1$ groups to this list of families.

Proposition 4.10. Let $G$ be an $\mathcal{M}_1$ group, then $\bar{U}_1(G)$ is powerful.

Proof. First, observe that by Proposition 4.8 we know that $\bar{U}_1(G) = \{g^p \mid g \in G\}$. Thus $\bar{U}_1(\bar{U}_1(G)) = \{g^{p^2} \mid g \in G\} = \bar{U}_2(G)$. We wish to show that $\bar{U}_1(G)$ is powerful. To do this, we may quotient by $\bar{U}_2(G) = \bar{U}_1(\bar{U}_1(G))$ and show that the resulting group
\[ H = G/\mathcal{U}_2(G), \] which is of exponent at most \( p^2 \), has \( \mathcal{U}_1(H) \) being abelian. (Recall that the Agemo subgroups behave well under taking quotients so \( \mathcal{U}_1(H/\mathcal{U}_2(G)) = \mathcal{U}_1(G/\mathcal{U}_2(G)) \).

Let \( a, b \in \mathcal{U}_1(H) \); as \( H \) is still an \( \mathcal{M}_1 \) group, then by Proposition 4.8 we can write \( a = x^p \) and \( b = y^p \) for some \( x, y \in H \). Let \( M \) be any maximal subgroup of \( H \) which contains \( x \). Note that \( b \) is contained in all maximal subgroups of \( H \) (as it is a \( p \)th power).

We observe that the nilpotency class of \( M \) is at most 2, since \( [M, M, M] \leq [\mathcal{U}_1(M), M] \leq M^{p^2} = 1 \). Then as \( x \) and \( b \) are both contained in \( M \), we have that \( [a, b] = [x^p, b] = [x, b]^p \in \mathcal{U}_1(\mathcal{U}_1(H)) = 1 \). In particular \( \mathcal{U}_1(H) \) is abelian. The result now follows.

\[ \square \]

We can now prove that for \( \mathcal{M}_1 \) groups, the product of any two \( p^i \)th powers is always a \( p^i \)th power. Indeed, by Proposition 4.8 we know that \( \mathcal{U}_1(G) = \{ g^p \mid g \in G \} \). Then for \( i > 1 \), we have that
\[ a^{p^i} b^{p^i} = (a^p)^{p^{i-1}} (b^p)^{p^{i-1}} = c^{p^{i-1}} \]
for some \( c \in \mathcal{U}_1(G) \), since \( \mathcal{U}_1(G) \) is powerful by Proposition 4.10. However then \( c = dp \) for some \( d \in G \) by Proposition 4.8. Hence \( a^{p^i} b^{p^i} = d^{p^i} \). Thus we have proved the following theorem.

**Theorem 4.11.** Let \( G \) be an \( \mathcal{M}_1 \) group. Then \( \mathcal{U}_i(G) = \{ g^{p^i} \mid g \in G \} \) for \( i \geq 1 \).

4.3. We now address the remaining power structure property, condition 1.3. We recall the condition below.

\[ |G : \mathcal{U}_i(G)| = |\Omega_i(G)|. \tag{1.3} \]

This condition may seem the least natural of the three. We offer here one possible way to motivate this condition: For an abelian \( p \)-group \( G \), the \( p^i \)th power map on \( G \) is a homomorphism with image \( \mathcal{U}_i(G) \) and kernel \( \Omega_i(G) \), and thus condition 1.3 follows by the isomorphism theorems. For \( p \)-groups in general (and even for groups with a regular power structure) this map need not be a homomorphism, but perhaps this property could be thought of as trying to capture some essence of that behaviour.

We believe that status of condition 1.3 for \( \mathcal{M}_1 \) groups is surprising. The first two properties have now been established for \( \mathcal{M}_1 \) groups and any odd prime \( p \). Furthermore by Proposition 5.3 we know that the final condition holds for primes \( p \geq 5 \), and by Proposition 6.3 all regular power structure properties hold for \( p = 3 \) if \( d(G) > 2 \). Thus given all the cases where condition 1.3 holds, it is surprising that in the remaining case of \( \mathcal{M}_1 \) 3-groups \( G \) with \( d(G) = 2 \), we will see that the condition 1.3 need not hold. Even more intriguing is that there is only one example where this conditions fails - and in this case the group is of a very small order, \( |G| = 3^4 \).

**Example 4.12.** The following group can be easily constructed in GAP as \texttt{SmallGroup(81,10)}. For completeness we list below a power commutator presentation for the group.

\[ J = \langle a_1, a_2, a_3, a_4 \mid a_1^3 = a_4, a_2^3 = (a_4)^2, a_3^3 = 1, a_4^3 = 1, [a_2, a_1] = a_3, [a_3, a_1] = a_4 \rangle, \]

the four remaining commutator relations not listed are trivial.

We state some properties of this group - these can be readily verified in GAP, or with more effort, by hand.

This group has nilpotency class 3 and order \( 3^4 \) and so is a group of maximal class. It has four maximal subgroups, one of which is abelian (isomorphic to \( C_3 \times C_9 \)), and the other three all isomorphic to

\[ \langle x, y \mid x^3, y^9, [x, y] = y^3 \rangle, \]

a semidirect product of the form \( C_9 \rtimes C_3 \).

Then it is clear that our group is an \( \mathcal{M}_1 \) group.
In this group we have that $\Omega_1(J) = \langle a_4 \rangle$ and $\Omega_1(J) = \langle a_3, a_4 \rangle$. Hence we have that $|\Omega_1(J)||\Omega_1(J)| = 27 \neq |J| = 81$, and so the group does not satisfy condition [1.3]. We also remark here that this is the only $M_1$ group of order $3^4$ and of maximal class (nilpotency class 3), which can be readily verified in GAP.

For $p$ odd, this is the only example of an $M_1$ $p$-group which does satisfy condition [1.3].

Proving this, and the method we use to do so, is one of the main contributions of the paper.

By the reductions outlined at the start of Section 4 we only need to be concerned with 2-generator 3-groups. Unlike previous sections where we still offered arguments that worked for any odd prime, here we will take advantage of these reductions and focus only on $p = 3$. In contrast to the very theoretical arguments in the previous section, we now move to more concrete arguments making use of the classification of 3-groups of small order.

As all groups of exponent 3 are regular, it is clear that the smallest exponent for which a counterexample to condition [1.3] can occur is exponent $3^2$. We exhibited such a group in Example 4.12. To begin, we show that it is the only counterexample of exponent $3^2$.

**Lemma 4.13.** Let $G$ be an $M_1$ group of exponent $p^2$. Then the nilpotency class of $G$ is at most 3.

**Proof.** Every maximal subgroup of $G$ is powerful and of exponent at most $p^2$. For a powerful $p$-group $H$ we have that $[H, H, H] \leq H^{p^2}$ and so it follows that every proper subgroup of $G$ must have nilpotency class at most 2. Now, by a remarkable theorem of MacDonald [14, Corollary 1 to Theorem 1], it is known that if all proper subgroups of $G$ have class at most 2 then $G$ has class at most 3. □

**Lemma 4.14.** There is exactly one 2-generator, $M_1$ 3-group of exponent 9 which does not satisfy $|G| = |\Omega_1(G)||\Omega_1(G)|$.

**Proof.** Let $G = \langle a, b \rangle$ be a 2-generator $M_1$ group of exponent 9. By Lemma 4.13 the class of $G$ is at most 3. The largest (most free) 2-generator 3-group of exponent at most 9 and nilpotency class at most 3 has order $3^8$. This group can be constructed using the nilpotent quotient algorithm from the GAP package "nq" [10]. This group is not a $M_1$ group, but any 2-generator $M_1$ group with exponent at most 9 and class at most 3 will be a quotient of this group. In particular their order will be at most $3^7$.

We have now reduced to a finite problem, and as all the 3-groups of orders $3^n$, $n \in \{1, \ldots, 7\}$ are classified we can check the classification and see that there is indeed only one counterexample. The group of Example 4.12. □

In what follows we use the well known commutator expansion formula (see [17], Exercise 1.2). If $G$ is a group, $x, y \in G$, and $n \in \mathbb{N}$ then

$$(xy)^{p^n} \equiv x^{p^n}y^{p^n} \mod \Omega_n(\gamma_2(T))\Omega_{n-1}(\gamma_3(T)) \cdots \gamma_p(T))$$

where $T = \langle x, y \rangle$.

**Lemma 4.15.** Let $G$ be a $M_1$ 3-group and let $M$ be a maximal subgroup of $G$. Then either $|\Omega_1(G)| = |\Omega_1(M)|$ or $|\Omega_1(G)| = 3 \cdot |\Omega_1(M)|$.

**Proof.** We may write $G = \langle M, a \rangle$ for some element $a \in G$. Then any element of the group can be written in the form $ma^i$ for some $m \in M$ and $i \in \{0, 1, 2\}$.

Now we know the form of a generic element, we can consider the form of a generic 3rd power.

Using formula 4.1 above, we obtain:

$$(ma^i)^3 = (m)^3(a^i)^3\Omega_1(\gamma_2(T))\gamma_3(T),$$
where \( T = \langle m, a^i \rangle \).

Notice that \( \gamma_2(T)^3 \leq \mathcal{U}_1(M) \).

We next consider \( \gamma_3(T) = \gamma_3(\langle m, a^i \rangle) \). Notice that for some collection of elements \( m_i \in M \) there is a maximal subgroup \( N = \langle a, m_1, \ldots, m_n \rangle \) in \( G \). As it is maximal, it contains \( \Phi(G) \) and so \( \gamma_2(T) \leq N \). Then \( [a, m, a] \in N^3 = \langle a^3, m_1^3, \ldots, m_n^3 \rangle \) by Proposition 2.3. Also notice that \( [a, m, m] \in M^3 \). In particular it follows that \( \gamma_3(T) \leq \langle a^3, \mathcal{U}_1(M) \rangle \).

Then it follows that \( (ma^i)^3 = \tilde{m}^3 a^j \) for some \( m \in M \) and \( j \in \{0, 1, 2\} \) (notice \( a^3 \in M \)).

Then we see we have 3 choices for \( j \), and \( \tilde{m}^3 \) can take \( |\mathcal{U}_1(M)| \) values (where we note that \( \mathcal{U}_1(M) \) consists solely of 3rd powers because it is powerful). Then there are at most \( 3 \cdot |\mathcal{U}_1(M)| \) possible 3rd powers in \( G \).

As we proved in Theorem 4.11 \( \mathcal{U}_1(G) \) consists solely of 3rd powers and so \( |\mathcal{U}_1(G)| \leq 3 \cdot |\mathcal{U}_1(M)| \).

Clearly \( \mathcal{U}_1(G) \geq \mathcal{U}_1(M) \) and so either \( |\mathcal{U}_1(G)| = |\mathcal{U}_1(M)| \) or \( |\mathcal{U}_1(G)| = 3 \cdot |\mathcal{U}_1(M)| \) as required.

\[ \square \]

**Lemma 4.16.** Let \( G \) be an \( M_1 \)-3-group of exponent at least \( p^2 \), and let \( M \) be a maximal subgroup of \( G \) such that \( \Omega_1(G) \leq M \). If \( \mathcal{U}_1(G) \neq \mathcal{U}_1(M) \) then \( |G| = |\mathcal{U}_1(G)||\Omega_1(G)| \).

**Proof.** \( \Omega_1(G) \) consists of all elements of order at most \( p \) in \( G \) by Theorem 2.3. Similarly, \( \Omega_1(M) \) consists of all elements of order \( p \) in \( M \), because \( M \) is powerful. Then because \( \Omega_1(G) \leq M \) we must have that

\[
\Omega_1(G) = \Omega_1(M). \tag{4.2}
\]

Let \( |G| = p^n \), then for the maximal subgroup \( M \) we have \( |M| = p^{n-1} \). As \( M \) is powerful, we have that condition 1.3 holds for \( M \):

\[
p^{n-1} = |M| = |\mathcal{U}_1(M)||\Omega_1(M)|. \tag{4.3}
\]

By hypothesis \( \mathcal{U}_1(G) \neq \mathcal{U}_1(M) \) and so we have \( |\mathcal{U}_1(G)| = p|\mathcal{U}_1(M)| \) by Lemma 4.15.

Hence

\[
|\mathcal{U}_1(G)||\Omega_1(G)| = p|\mathcal{U}_1(M)||\Omega_1(G)| \tag{Lemma 4.14}
\]

\[
= p|\mathcal{U}_1(M)||\Omega_1(M)| \tag{by 4.2}
\]

\[
= p \cdot p^{n-1} \tag{by 4.3}
\]

\[
= |G|. \tag{4.4}
\]

\[ \square \]

We have seen that there is only one counterexample of exponent at most 9. We now wish to show there are no counterexamples of exponent \( 3^i \) with \( i \geq 3 \). In this case, by Theorem 4.13 we know that \( \exp \Omega_1(G) = 3 \) and so we must have that \( \Omega_1(G) \) is contained in a maximal subgroup \( M \) of \( G \). Then by Lemma 4.16 in the cases that follow we assume we have that \( \mathcal{U}_1(G) = \mathcal{U}_1(M) \).

The point of the following lemma is to say if a counterexample \( G \) exists, then we can bound the size of \( \Omega_1(G) \). This will allow us to reduce our search for counterexamples to checking a finite number of groups later on.

**Lemma 4.17.** Let \( G \) be an \( M_1 \)-2-generator 3-group, such that \( \Omega_1(G) \leq M \) for some maximal subgroup \( M \) and additionally that \( |\mathcal{U}_1(G)| = |\mathcal{U}_1(M)| \). Then \( |\Omega_1(G)| \leq 3^2 \).

**Proof.** As \( M \) is powerful we have \( |\mathcal{U}_1(M)||\Omega_1(M)| = |M| \). As \( \Omega_1(G) \leq M \) we have \( \Omega_1(G) = \Omega_1(M) \), and by the hypothesis we have \( |\mathcal{U}_1(G)| = |\mathcal{U}_1(M)| \). Hence

\[
|\mathcal{U}_1(G)||\Omega_1(G)| = |\mathcal{U}_1(M)||\Omega_1(M)| = |M| = |G|/3. \tag{4.5}
\]

Rearranging yields:

\[
3 \cdot |\Omega_1(G)| = |G|/|\mathcal{U}_1(G)| = |G/\mathcal{U}_1(G)|. \tag{4.6}
\]
The group $G/\Omega_1(G)$ is 2 generator of exponent 3. It is well known the largest such 3-group has order $3^3$. Hence $3 \cdot |\Omega_1(G)| \leq 3^3$ and so $|\Omega_1(G)| \leq 3^2$ as required.

In the proof of the following lemma we use a result due to L. Wilson from [20]. We recall some notation from [20].

We let $O_p$ be the class of all $p$-groups for which $\Omega_k(G)$ is the set of elements of order dividing $p^k$ for all $k$. Theorem 4.13 tells us that for odd primes $p$, $M_1$-groups are in $O_p$.

**Lemma 4.18 (20 Lemma 2.1).** Let $G$ be in $O_p$. Then for all $m$ and $k$
\[ \Omega_k(G/\Omega_m(G)) = \Omega_{m+k}(G)/\Omega_m(G). \]

**Proof.** As the exponent of $G$ is at least $3^3$, and the group satisfies condition 1.2 by Theorem 4.3, then $\exp \Omega_2(G) \leq 9$ and so $\Omega_1(G)$ is a proper subgroup of $G$ and thus is contained in some maximal subgroup $M$. Then clearly $\Omega_1(G)$ is also contained in $M$ and so by Lemma 4.10 we only need to consider the case where
\[ \Omega_1(G) = \Omega_1(M). \] (4.4)

We will show that this situation never occurs.

Let $|G| = 3^n$ and $|\Omega_1(G)| = 3^a$, clearly $a > 0$. Note that $\Omega_1(M) = \Omega_1(G)$ and $\Omega_2(M) = \Omega_2(G)$.

We now wish to consider what happens when we quotient by $\Omega_1(G)$, we shall denote the image under this quotient with bar notation.
\[ p^{n-1-a} = |\bar{M}| = |\Omega_1(\bar{M})| \cdot |\bar{U}_1(\bar{M})| \]

since $\bar{M}$ is powerful and so $\bar{M}$ satisfies 1.3.

By hypothesis we know that $|\bar{G}| = |\Omega_1(\bar{G})||\bar{U}_1(\bar{G})|$. We also know that $|\bar{G}| = |G|/|\Omega_1(G)| = 3^{n-a}$. Hence
\[ |\Omega_1(\bar{G})||\bar{U}_1(\bar{G})| = 3^{n-a}. \] (4.5)

However we also have that
\[ \Omega_1(\bar{M}) = \Omega_1\left(\frac{M}{\Omega_1(M)}\right) \text{ Lemma 4.18} = \frac{\Omega_2(M)}{\Omega_1(M)} = \frac{\Omega_2(G)}{\Omega_1(G)}. \]

and also
\[ \Omega_1(\bar{G}) = \Omega_1\left(\frac{G}{\Omega_1(G)}\right) \text{ Lemma 4.18} = \frac{\Omega_2(G)}{\Omega_1(G)}. \]

Hence
\[ \Omega_1(\bar{M}) = \Omega_1(\bar{G}). \] (4.6)

Also
\[ \bar{U}_1(\bar{M}) = \bar{U}_1\left(\frac{M}{\Omega_1(G)}\right) = \frac{\bar{U}_1(M)}{\Omega_1(G)} = \frac{\bar{U}_1(G)}{\Omega_1(G)} = \bar{U}_1(\bar{G}). \] (4.7)

Hence
\[ 3^{n-1-a} = |\Omega_1(\bar{M})||\bar{U}_1(\bar{M})| \]
\[ = |\Omega_1(\bar{G})||\bar{U}_1(\bar{G})| \quad \text{ (by 4.6)} \]
\[ = |\Omega_1(\bar{G})||\bar{U}_1(\bar{G})| \quad \text{ (by 4.7)} \]
\[ = 3^{n-a}. \] (4.5)

Thus we obtain a contradiction, and so the case when $\Omega_1(G) = \Omega_1(H)$ cannot occur. The proof is complete. □
We can now prove that there are no counterexamples of exponent $p^3$.

**Lemma 4.20.** There are no $\mathcal{M}_1$ 2-generator 3-groups of exponent 27 such that $|G| \neq |\Omega_1(G)||\Omega_1(G)|$.

**Proof.** Suppose that $G$ is a minimal counterexample with respect to its order. By Lemma 4.19 we must have that $H = G/\Omega_1(G)$ is such that $|H| \neq |\Omega_1(H)||\Omega_1(H)|$. Notice that $H$ is 2-generator 3-group with $|H| = |\Omega_1(H)||\Omega_1(H)|$ and smaller order than $G$. As $G$ is the smallest counterexample of exponent $p^3$, the group $H$ must have smaller exponent to avoid contradicting this minimality. Thus $H$ must have exponent 9 (since $\exp \Omega_1(G) = 3$).

By Lemma 4.14 there is only one possibility for $H$, and we see that $|H| = 3^4 = 81$. By Lemma 4.16 we can assume that $|\Omega_1(G)| = |\Omega_1(M)|$ for some maximal subgroup $M$, with $\Omega_1(G) \leq M$. Now by Lemma 4.17 we can assume that $|\Omega_1(G)| \leq 9$. Then if $G/\Omega_1(G) \cong H$ we must have that $|G| \leq 3^2 \cdot 3^4$.

By checking the classification of the 3-groups of order at most $3^6$, for example by using GAP, we see that all $\mathcal{M}_1$ groups $G$ of exponent 27 and $|G| \leq 3^6$ satisfy $|G| = |\Omega_1(G)||\Omega_1(G)|$.

**□**

**Proposition 4.21.** Let $G$ be a $\mathcal{M}_1$ 3-group with 2-generators and exponent at least $3^3$. Then $|G| = |\Omega_1(G)||\Omega_1(G)|$.

**Proof.** Let the exponent of the group be $3^e$, with $e \geq 3$. The proof is by induction on $e$. The base case $e = 3$ is established by Lemma 4.20. Now consider some $e > 3$ and suppose the claim holds for smaller values.

As usual we may assume that $G$ contains a maximal subgroup $M$ containing $\Omega_1(G)$. The group $G/\Omega_1(G)$ is of exponent $p^{e-1}$, because if $g$ had order $p^e$, then $p^{e-1} \in \Omega_1(G)$.

As $e - 1 \geq 3$ it follows inductively that $H = G/\Omega_1(G)$ has the property that $|H| = |\Omega_1(H)||\Omega_1(H)|$. Then by Lemma 4.19 $G$ also has the property and the proof follows by induction.

We now seek to prove the more general result, condition 1.3. First we prove the following lemma.

**Lemma 4.22.** There are no $\mathcal{M}_1$ 2-generator 3-groups $G$ such that $G/\Omega_1(G)$ is isomorphic to the group from Example 4.12 for any $i \geq 1$.

**Proof.** We let $J$ denote the group from Example 4.12.

The proof is by induction on $i$. First consider the base case $i = 1$. Suppose that $G$ is such a group and so $G/\Omega_1(G) \cong J$. Then $\exp G \geq 27$ (Lemma 4.18). By Proposition 4.21 we know that $|G| = |\Omega_1(G)||\Omega_1(G)|$. Then $|\Omega_1(G)| = |G/\Omega_1(G)|$. As $G/\Omega_1(G)$ is at most 2-generator of exponent 3, its order is at most 27. Therefore the order of $G$ is at most $3^4 \cdot 3^3$. The groups of order $3^7$ have been classified. From this classification one can verify that there are 3-groups $G$ of order at most $3^7$ with $G/\Omega_1(G) \cong J$, where $J$ is the group from Example 4.12.

Now suppose $i = k$ and the claim has been established for smaller values. Using Lemma 4.18 notice that:

$$\Omega_1(G) = \Omega_1(G) \cdot \frac{G}{\Omega_1(G)} = \frac{\Omega_i(G)}{\Omega_{i-1}(G)}. \tag{4.8}$$

Then

$$\frac{(G/\Omega_{i-1}(G))}{\Omega_1((G/\Omega_{i-1}(G)))} = \frac{G/\Omega_{i-1}(G)}{\Omega_i(G)/\Omega_{i-1}(G)} = \frac{G}{\Omega_i(G)} \cong J. \tag{4.9}$$

The result then follows by the base case, as there can be no group $G/\Omega_{i-1}(G)$.

**□**

**Remark 4.23.** In the proof of the next proposition, our approach very closely follows the approach of Wilson in [20]. We note that we also followed the same method in [19] to prove that quasi-powerful $p$-groups satisfy property 1.3 for odd primes $p$. We think that the approach in [20] is a very nice method for establishing the final power structure property.
Proposition 4.24. Let $G$ be a $\mathcal{M}_1$ 2-generator 3-group such that $G$ is not isomorphic to the group from Example 4.12. Then $|G| = |\Omega_i(G)\Omega_i(G)|$ for any $i \geq 1$.

Proof. We use induction on $i$. The base case $i = 1$ is established by Proposition 4.21 (exp $G \geq 27$) and Lemma 4.14 (exp $G = 9$) and the fact that groups of exponent 3 are regular.

Thus we may assume the result holds for $i$. We wish to find the order of $\Omega_{i+1}(G)$. By Theorem 1.11 we have that $\Omega_{i+1}(G) = \Omega_i((\Omega_i(G)))$. As $\Omega_i(G)$ is powerful (by Proposition 4.10), we know that $\Omega_i(G)$ satisfies condition 1.3. In particular we can conclude that

$$|\Omega_{i+1}(G)| = |\Omega_i(G) : \Omega_i((\Omega_i(G)))|. \quad (4.10)$$

By Theorem 1.3 we know that the exponent of $\Omega_i(G)$ is at most $p^i$ and so we have $\Omega_i((\Omega_i(G))) = \Omega_i(G) \cap \Omega_i(G)$. Then

$$\Omega_i(G)/\Omega_i((\Omega_i(G))) \cong \Omega_i(G)\Omega_i(G)/\Omega_i(G) = (G/\Omega_i(G))^p.$$  

Thus

$$|\Omega_i(G) : \Omega_i((\Omega_i(G)))| = |(G/\Omega_i(G))^p|. \quad (4.11)$$

Since quotients of $\mathcal{M}_1$ groups are $\mathcal{M}_1$ groups, we have that $G/\Omega_i(G)$ is an $\mathcal{M}_1$ group. If $G/\Omega_i(G)$ is 2-generator then by Lemma 1.12 we can apply the base case, to find that

$$|(G/\Omega_i(G))^p| = |G/\Omega_i(G) : \Omega_i(G/\Omega_i(G))| = |G : \Omega_{i+1}(G)|. \quad (\text{Lemma 4.18})$$

If $G/\Omega_i(G)$ is cyclic then (4.12) clearly holds. In either case we see that

$$|\Omega_{i+1}(G)| = |\Omega_i(G) : \Omega_i((\Omega_i(G)))| = |G : \Omega_{i+1}(G)|. \quad (4.12)$$

4.4. Putting together the results of the previous sections we obtain the following Theorem. This is Theorem C from the Introduction.

Theorem 4.25. Let $p$ be an odd prime. Let $G$ be an $\mathcal{M}_1$ $p$-group. Then $G$ has a regular power structure, unless $G$ is isomorphic to the group of Example 4.12 in which case condition 1.3 fails to hold.

Proof. If $p > 3$ the group is potent by Corollary 3.2 and so has a regular power structure. If $d(G) > 2$ the group is powerful by Proposition 3.3. Thus we only need to consider 2-generator 3-groups.

The second power structure property 1.2 is established by Theorem 1.3. The first power structure property is established by Theorem 1.11.

Proposition 4.24 establishes the third and final power structure property for all 2-generator $\mathcal{M}_1$ 3-groups with exactly one exception, the group from Example 4.12.

5. Further Remarks

5.1. In the theory of $p$-groups it is well known that small primes can behave differently to large primes - often being more difficult to deal with. Newman refers to this as the tyranny of the small in [13]. When studying $\mathcal{M}_1$ groups our focus has been on odd primes, with a large part of the paper spent dealing with the difficult case $p = 3$. We briefly address here the case $p = 2$. We should not expect $p = 2$ to behave as well as the other primes. It often happens that some condition which guarantees regular power structure for odd primes fails when considered for $p = 2$. For example powerful 2-groups need not have a regular power structure (see [9] page 2) for an example of a powerful 2-group where condition 1.2
fails to hold). In [19] we introduced the notion of a quasi-powerful $p$-group for odd primes $p$. We showed that for odd primes these groups have a regular power structure, but if the definition was used unmodified for $p = 2$, the resulting groups need not have regular power structure.

We can readily find examples of $M_1$ 2-groups which do not have a regular power structure. For example, there is a group of order 64 (which can be created in GAP as SmallGroup(64,31)) which fails all three power structure properties. It would be interesting to explore if the theory in this paper could be adapted for 2-groups.

5.2. A natural generalisation to consider is what happens if we replace powerful with potent. For $p = 3$ the notions are the same, and so we focus now on the case $p \geq 5$. We could then ask for $p \geq 5$ if all maximal subgroups of $G$ are potent, must $G$ satisfy conditions [1.1] or [1.2] or [1.3]

Using GAP, we can find examples of 5-groups with all maximal subgroups potent, such that condition [1.2] fails, and also examples where condition [1.3] fails (for example SmallGroup(78125, 784) fails both). However in all the examples we have found condition [1.1] does hold. Thus we ask the following question.

**Question 5.1.** Does there exist a $p$-group ($p \geq 5$) such that all maximal subgroups are potent, for which condition [1.1] fails to hold.

5.3. We may also ask, how are the power structure conditions affected if $i$ is increased from 1. Corollary 3.2 gives us a result in this direction, for instance for $p \geq 5$ we have that if all subgroups of index $p^2$ are powerful then $G$ is potent. We might then ask how sharp is this result, for instance what can be said when $p = 3$ about $p$-groups with all subgroups of index $p^2$ powerful? We have examples of $M_2$ 3-groups to demonstrate each of the properties can fail. For instance the groups with the following SmallGroup Ids (in GAP or MAGMA) all fail condition [1.1] and condition [1.3]

\[ \text{SmallGroup}(2187,83), \text{SmallGroup}(2187,84), \text{SmallGroup}(2187,85), \text{SmallGroup}(2187,90), \text{SmallGroup}(2187,91), \text{SmallGroup}(2187,92) \]

SmallGroup(81,7) is an example of an $M_2$ 3-group for which condition [1.2] fails; interestingly it is the only example we could find.

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