Global Aspects Of
Gauged Wess-Zumino-Witten Models

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A study of the gauged Wess-Zumino-Witten models is given focusing on the effect of topologically non-trivial configurations of gauge fields. A correlation function is expressed as an integral over a moduli space of holomorphic bundles with quasi-parabolic structure. Two actions of the fundamental group of the gauge group is defined: One on the space of gauge invariant local fields and the other on the moduli spaces. Applying these in the integral expression, we obtain a certain identity which relates correlation functions for configurations of different topologies. It gives an important information on the topological sum for the partition and correlation functions.

1. Introduction

The gauged Wess-Zumino-Witten model in two dimensions has two different aspects of interest. On the one hand, it is an exactly soluble quantum gauge theory and is interesting from the point of view of geometry of gauge fields. On the other hand, it is a conformally invariant quantum field theory (CFT): There are observations [1, 2] that a wide class of solved CFTs such as unitary minimal models (bosonic [3] or supersymmetric [4]), parafermionic models [5], etc are realized by gauged WZW models as lagrange field theories. Hence, the model provides a powerful method for the model building and classification of CFTs, important problems for the study of two dimensional statistical systems and string theory.

In this paper, we focus on the former, the geometric aspects of the theory and propose a method to take into account the topologically non-trivial configurations of gauge fields.

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Then, we see that incorporation of non-trivial topology has simple consequences which are of vital importance in the model building of CFTs.

A gauged WZW model is a WZW sigma model with target $G$ a group manifold, coupled to gauge field for a group $H$. We concentrate on the case in which $G$ is compact, connected and simply connected and $H$ is a connected, closed subgroup of the adjoint group $G/Z_G$ where $Z_G$ is the center of $G$. The classical action of the system (the WZW action) at level $k \in \mathbb{N}$ for a closed Riemann surface $\Sigma$, a map $g : \Sigma \to G$ and an $\mathfrak{h} = \text{Lie}(H)$-valued one form $A$ is given by

$$k I_\Sigma(A, g) = \frac{ik}{4\pi} \int_\Sigma \text{tr}(\partial g^{-1} \partial g) - \frac{ik}{12\pi} \int_{B_S} \text{tr}(\tilde{g}^{-1} d\tilde{g})^3$$

(1.1)

$$+ \frac{ik}{2\pi} \int_\Sigma \text{tr}(g \partial g^{-1} A^{01} + A^{10} g^{-1} \tilde{g} g + g A^{10} g^{-1} A^{01} - A^{10} A^{01})$$

where “$\text{tr}$” is a trace in a representation of $G(\mathfrak{h})$ and $A^{10}$ (resp. $A^{01}$) is the $(1,0)$-form (resp. $(0,1)$-form) component of $A$. In the second term, $B_S$ is a compact three manifold having $\Sigma$ as its boundary and $\tilde{g} : B_S \to G$ is an extension of $g$. The value $e^{-k I_\Sigma(A, g)}$ which we call the WZW weight is independent on the choice of $B_S$ and $\tilde{g}$ and hence may be used as the weight for the path integration over $A$ and $g$. This weight is invariant under the gauge transformation $A \to A^h = h^{-1} A h + h^{-1} \partial h$, $g \to h^{-1} g h$ and the resulting system is a quantum gauge theory.

A natural generalization is to consider the topologically non-trivial configurations of $A$ and $g$. Thus, let $\{U_0, U_\infty\}$ be an open covering of $\Sigma$ such that $U_0$ contains a disc $D_0$ and $U_0 \cap U_\infty$ is an annular neighborhood of the boundary circle $\partial D_0$. General configuration is determined by gauge fields $\{A_0, A_\infty\}$ and maps $\{g_0, g_\infty\}$, both defined on $\{U_0, U_\infty\}$ and satisfying the relation

$$A_0 = h^{-1}_{\infty} A_\infty h_{\infty} + h^{-1}_{\infty} \partial h_{\infty} \quad \text{and} \quad g_0 = h^{-1}_{\infty} g_{\infty} h_{\infty}$$

(1.2)

on $U_0 \cap U_\infty$ where $h_{\infty}$ is a map to $H \subset G/Z_G$. In geometric terms, this map $h_{\infty}$, called the transition function, determines a principal $H$-bundle $P$ over $\Sigma$, $\{A_0, A_\infty\}$ determines a connection $A$ of $P$ and $\{g_0, g_\infty\}$ determines a section $g$ of the associated $G$-bundle $P \times_H G$. The homotopy type of the loop $\gamma_{\infty} = h_{\infty}|_{\partial D_0}$ determines the topological type of the $H$-bundle $P$ and hence the fundamental group $\pi_1(H)$ of $H$ classifies the topological types of configurations.\(^1\)

\(^1\)If $G$ is simple, it is normalized by $\text{tr}_g(\text{ad} X \text{ad} Y) = 2 g^Y \text{tr}(XY)$ for $X, Y \in \mathfrak{g} = \text{Lie}(G)$ where $g^Y$ is the dual Coxeter number of $g$. Generalization to the non-simple case is obvious.

\(^2\)Any principal $H$-bundle over $\Sigma$ admits trivialization over $\Sigma_\infty = \Sigma - D_0$ as well as over $D_0$: Take a pants decomposition of $\Sigma_\infty$. Since $H$ is connected, we can choose a gauge over any circle. Now it is enough to observe that given gauges over two boundary circles of a pants can be extended over the whole pants and determines (up to homotopy) a gauge over the third boundary.
One purpose of the paper is to give a method to calculate the correlation function of gauge invariant fields $O_1 \cdots O_s$ restricted to configurations of the topological type determined by $P$:

$$Z_{\Sigma,P}(O_1 \cdots O_s) = \frac{1}{\text{vol} \mathcal{G}_P} \int \mathcal{DA} \mathcal{Dg} e^{-kI_{\Sigma,P}(A,g)}O_1 \cdots O_s,$$

(1.3)

where $\mathcal{G}_P$ is the group of gauge transformations and $kI_{\Sigma,P}(A,g)$ is the WZW action defined in §2 for a general principal $H$-bundle $P$.

Another and the main purpose is to prove certain exact relationships of correlators for configurations of different topologies. Namely, we will see that the group $\pi_1(H)$, which acts on the set of principal $H$-bundles $\gamma : P \mapsto \gamma P$ by multiplication on the transition functions $\gamma_{\infty 0} \mapsto \gamma_{\infty 0} \gamma$, acts on the space of gauge invariant local fields $\gamma : O \mapsto \gamma O$ in such a way that the following holds:

$$Z_{\Sigma,P}(O_1 \cdots O_s \gamma O) = Z_{\Sigma,P\gamma}(O_1 \cdots O_s O).$$

(1.4)

We call this the topological identity. The proof is reduced to the solution of a problem in the geometry of moduli spaces of holomorphic $H_C$-bundles with quasi-parabolic structure. In addition to the case with abelian $H$ in which the problem is trivial, it is solved for the cases $\Sigma = $ sphere with $H$ general and $\Sigma = $ torus with $H = SO(3)$.

The significance of (1.4) can be seen if we take the sum $\sum_P$ over topologies; the fields $O$ and $\gamma O$ are then indistinguishable. For instance, consider the case with $G = SU(2) \times SU(2)$ and $H = SO(3)$ diagonally embedded into $G/Z_G = SO(3) \times SO(3)$. The gauge invariant local fields can be classified by the rectangular grid whose squares are labeled by $\{0, \frac{1}{2}, \cdots, \frac{k}{2}\} \times \{0, \frac{1}{2}, \cdots, \frac{k+1}{2}\}$. The space of fields in the square $(j_1, j)$ is identified with the degenerate representation of the Virasoro algebra of central charge $1 - \frac{6}{(k+2)(k+3)}$ and dimension $\frac{(k+3)j_1 - (k+2)j_1 + 1}{4(k+2)(k+3)}$ as is also the case for the corner $(\frac{k}{2} - j_1, \frac{k+1}{2} - j)$. As we shall see in §4, this transformation $(j_1, j) \leftrightarrow (\frac{k}{2} - j_1, \frac{k+1}{2} - j)$ corresponds precisely to the transformation $O \leftrightarrow \gamma O$ where $\gamma$ is the non-trivial element of $\pi_1(SO(3)) = \mathbb{Z}_2$. Hence, only after the sum over topologies, the set of distinguishable fields coinsides with that of the $k$-th unitary minimal model [3]. The situation is the same for general $G$ and $H$.

The space of local gauge invariant fields, acted on by the infinite conformal symmetry, is identified with the direct sum of Virasoro modules by coset construction [6]. For each element $\gamma \in \pi_1(H)$, there is an isomorphism of coset Virasoro modules, known as the “field identification” [7, 8, 9], that corresponds to our transformation $O \mapsto \gamma O$. Hence, this identification of Virasoro modules leads via the sum over topologies to a genuine identification of quantum fields.

The rest of the paper consists of five sections and four appendices. Sections 2 and 3
are the preparatory parts which follow to some extent the route exploited by Gawędzki and others \[\text{[10, 1]}\]. The main part is section 4 in which a novel expression of the correlator is proposed (see (4.33)) and the topological identity (1.4) is proved at least for the cases mentioned above. An application of (1.4) is made in section 5. The last section includes a remark on alternative choices of the classical action.

2. Wess-Zumino-Witten Model

We start with the study of the WZW model in a general background gauge field with the group \(H = G/Z\). The first material is a construction of the WZW action for topologically non-trivial configurations. It is designed to satisfy the following property of factorization. For a Riemann surface \(\Sigma\), we choose a disc \(D_0\) in \(\Sigma\) and an open covering \(\{U_0, U_\infty\}\) of \(\Sigma\) as in §1 and put \(S = \partial D_0\). Let \(P\) be the principal \(H\)-bundle with the transition function \(h_{\infty0} : U_0 \cap U_\infty \rightarrow H\). For fields \(A = \{A_0, A_\infty\}\) and \(g = \{g_0, g_\infty\}\) satisfying (1.2), the WZW weight on the whole surface \(\Sigma\) is expressed as the product of the weight on \(D_0\) and the weight on \(\Sigma_\infty = \Sigma - D_0\):

\[
e^{-kI_{S,P}(A,g)} = \langle e^{-kI_{S_\infty}(A_\infty,g_\infty)}, \text{ad}_{\gamma_{\infty0}} e^{-kI_{D_0}(A_0,g_0)} \rangle,
\]

(2.1)

Here, the weight on \(D_0\) is not valued in the ordinary number field \(\mathbb{C}\) but in a complex line \(L^k_{WZ}|_{\gamma_0}\) associated to the loop \(\gamma_0 = g_0|S\), and the weight on \(\Sigma_\infty\) is in a line \(L^{*k}_{WZ}|_{\gamma_\infty}\) associated to \(\gamma_\infty = g_\infty|S\). The product is defined through a gauge transformation \(\text{ad}_{\gamma_{\infty0}} : L^k_{WZ}|_{\gamma_0} \rightarrow L^k_{WZ}|_{\gamma_\infty}\) associated to the transition function \(\gamma_{\infty0} = h_{\infty0}|S\).

This factorization property goes over to the quantum theory: A correlation function on the surface \(\Sigma\) is expressed as the pairing of two wave functions at \(S = D_0 \cap \Sigma_\infty\), one coming from \(D_0\) and the other from \(\Sigma_\infty\). Using the infinite dimensional symmetry of the gauge field \(A_0\), we can explicitly determine the wave function coming from the disc \(D_0\) with field insertion at one point, and thus obtain the correspondence of fields and states. If we change the gauge (=reference section) over the boundary \(\partial D_0\), the correspondence effectively changes and we have transformations of states and of fields. For a certain gauge transformation of non-trivial homotopy, the corresponding transformation of states (or of fields) takes a simple form that is known as the spectral flow \[\text{[8]}\]. Consequently, we obtain a relation of correlators of the WZW model that may be considered as a prototype of the equation (1.4) of the gauged WZW model.

2.1 The Line Bundle \(L^k_{WZ}\) And The Adjoint Action Of \(LH\)

We begin with defining the WZW weight on the disc \(D_0 = \{z \in \mathbb{C}; |z| \leq 1\}\) with the parametrized boundary \(\theta \mapsto e^{i\theta} \in \partial D_0\), following the line of argument in \[\text{[10]}\]. In order
to deal with the chiral gauge symmetry, we consider maps to the complexified group $G_C$. For $A \in \Omega^1(D_0, h)$ and $g \in \text{Map}(D_0, G_C)$, by choosing a smooth extension of $g$ to a map $\hat{g}$ defined over the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, we can define the WZW action $I_{\mathbb{P}^1}(\hat{A}, \hat{g})$ by (1.1) where $\hat{\Gamma}$ is the choice of $\hat{g}$, we consider the set of all extensions and put a suitable equivalence relation on $\text{Map}(D_\infty, G_C) \times \mathbb{C}$ so that the class

$$e^{-kD_0(A,g)} = \{(\hat{g}|_{D_\infty}, e^{-k\hat{\Gamma}_{D_\infty}(\hat{A},\hat{g})})\},$$

(2.2)
is independent of the choice. This defines the line bundle $\mathcal{L}_{WZ}^k$ over the group $LG_C$ of loops in $G_C$ so that the WZW weight (2.2) is an element of the line $\mathcal{L}_{WZ}^k|_{\gamma}$ over the boundary loop $\gamma(\theta) = g(e^{i\theta})$.

The group structure of $LG_C$ lifts to a semigroup structure of $\mathcal{L}_{WZ}^k$ by

$$\{(g_1,c_1)\{(g_2,c_2)\} = \{(g_1g_2, c_1c_2e^{-k\Gamma_{D_\infty}(g_1,g_2)})\},$$

(2.3)
where $\Gamma_{\Sigma}(g_1,g_2) = \frac{i}{2\pi} \int_{\Sigma} \text{tr}(g_2 \partial g_2^{-1} g_1^{-1} \bar{\partial} g_1)$. The group $(\mathcal{L}_{WZ})^\times$ of invertible elements for $k = 1$ is isomorphic to the basic central extension $\tilde{LG}_C \mathfrak{L}$ of the loop group $LG_C$ and acts on $\mathcal{L}_{WZ}^k$ on the left and on the right through the homomorphism $\{(g,c)\} \in (\mathcal{L}_{WZ})^\times \mapsto \{(g,c^k)\} \in (\mathcal{L}_{WZ})^\times$. The Polyakov-Wiegmann (PW) identity exhibits the response of the WZW weight to the chiral gauge transformation $A \mapsto A^h$, $g \mapsto g^h$ by $h \in \text{Map}(D_0, G_C)$:

$$(A^h)^{01} = h^{-1} A^{01} h + h^{-1} \bar{\partial} h , \quad (A^h)^{10} = h^* A^{10} h^{*-1} + h^* \bar{\partial} h^{*-1} \quad \text{and} \quad g^h = h^{-1} g h^{*-1} \quad \text{(2.4)}$$

in which $h \mapsto h^*$ is the Cartan involution that corresponds to hermitian conjugation in a unitary representation of $G$. It states that

$$e^{-kD_0(A,g)} = e^{-kD_0(A,h)} e^{-kD_0(A^h,g^h)} e^{-kD_0(A,h^*)} e^{-kD_0(A,h^*)},$$

(2.6)
where $\Gamma_{D_0}$ is given by

$$\Gamma_{D_0}(A, h, h^*) = \frac{i}{2\pi} \int_{D_0} \text{tr}(h^* \partial A h^{*-1} h^{-1} \bar{\partial} A h),$$

(2.7)
in which $h^{-1} \bar{\partial} A h = h^{-1} \bar{\partial} h + h^{-1} A^{01} h - A^{01}$ and similarly for $h^* \partial A h^{*-1}$.

If $h$ is $G$-valued, the above identity can be written as

$$e^{-kD_0(A,g)} = \gamma e^{-kD_0(A^h,g^h)} \gamma^{-1},$$

(2.8)
where $\gamma \in LG$ is the boundary loop of $h$. In this sense, we can say that the WZW weight on $D_0$ is gauge invariant. In (2.8), we have used the fact that the adjoint action
\[ \gamma_1 \mapsto \gamma \gamma_1 \gamma^{-1} \] of \( \gamma \in LG \) on \( LG_C \) lifts to an action on \( L^k_{wz} \) by choosing any element in \((L_{wz})^\times|_{\gamma}\). In fact, the adjoint action of \( LH \) on \( LG_C \), which is apparently well-defined, lifts to an automorphic action on \( L^k_{wz} \) so that the gauge invariance (2.8) holds when \( h^{-1}gh \) is defined on \( D_0 \). The action of a loop \( \gamma^{-1} \in LH \) on the element \( \{(\tilde{g},c)\} \in L^k_{wz} \) for \( \tilde{g} \in Map(D_{\infty},G_C) \) with \( \tilde{g}(\infty) = 1 \) is defined by

\[
\text{ad}\gamma^{-1}\{(\tilde{g},c)\} = \{ (h^{-1}\tilde{g}h, e^{-kC_{D_{\infty}}(h,\tilde{g})}) \},
\]

in which \( h \in Map(D_{\infty} - \{\infty\}, H) \) is any extension of \( \gamma \) and \( C_{D_{\infty}} \) is given by

\[
C_{D_{\infty}}(h,\tilde{g}) = K_{D_{\infty}}(h^{-1}\tilde{g}h) - K_{D_{\infty}}(\tilde{g}) - \frac{i}{4\pi} \int_{D_{\infty}} \text{tr} \left\{ (d\tilde{g}^{-1}d\tilde{g})h\tilde{g}^{-1} + \tilde{h}h^{-1} + \tilde{h}h^{-1} + \tilde{g}h\tilde{g}^{-1} - 1 \right\},
\]

(2.10)

where \( K_{D_{\infty}}(\tilde{g}) = \frac{i}{4\pi} \int_{D_{\infty}} \text{tr} (\partial \tilde{g}^{-1} \tilde{g}) \). In [14], the adjoint action of \( LH \) on \( L\hat{G}_C \) is defined and is shown to be unique. As it should be, it coincides with the action (2.3) for \( k = 1 \) coincides.

Next, we construct the WZW weight on \( \Sigma_{\infty} = \Sigma - D_0 \) where \( D_0 \) is the unit disc in an open subset of \( \Sigma \) with coordinate \( z \). As in the above argument, we put an equivalence relation on \( Map(D_0, G_C) \times C \) defining a line bundle \( L^k_{wz} \) over \( LG_C \) so that the WZW weight \( e^{-kI_{\Sigma_{\infty}}(A,g)} \) for \( A \in \Omega^1(\Sigma_{\infty}, \mathfrak{h}) \) and \( g \in Map(\Sigma_{\infty}, G_C) \) is given as the class \( \{(\tilde{g}|_{D_0}, e^{-kI_{\Sigma}(\tilde{A},\tilde{g}))} \} \) in the line \( L^k_{wz} |_{\gamma} \) over the loop \( \gamma(\theta) = g(e^{i\theta}) \). This bundle has a semigroup structure so that the PW identity holds:

\[
e^{-kI_{\Sigma_{\infty}}(A,g)} = e^{-kI_{\Sigma_{\infty}}(A,h)} e^{-kI_{\Sigma_{\infty}}(A^h,g^h)} e^{-kI_{\Sigma_{\infty}}(A,h^*)} e^{-kI_{\Sigma_{\infty}}(A^h,h^*)},
\]

(2.11)

The line bundles \( L^k_{wz} \) and \( L^k_{wz} \) are dual to each other under the product

\[
\langle \{(g|_{D_0}, c_0)\}, \{(g|_{D_{\infty}}, c_{\infty})\} \rangle = c_0 c_{\infty} e^{kJ_{\mathbb{P}^1}(g)},
\]

(2.12)

where \( g \in \text{Map}(\mathbb{P}^1, G_C) \). The WZW weight for general topology is now defined by (2.1) where \( \text{ad}_{\gamma_{\infty}} \) is the adjoint action (2.14) of the loop \( \theta \mapsto \gamma_{\infty}(e^{i\theta}) \). For the trivial topology, we may take \( \gamma_{\infty} \equiv 1 \) and (2.1) reproduces the action (1.1).

Since the product (2.14) satisfies \( \langle \tilde{\gamma}_1 \tilde{\gamma}_2, \tilde{\gamma}_1 \tilde{\gamma}_2 \rangle = \langle \tilde{\gamma}_1, \tilde{\gamma}_1 \rangle \langle \tilde{\gamma}_2, \tilde{\gamma}_2 \rangle \) for \( \tilde{\gamma}_i \in L^k_{wz} |_{h_i} \) and \( \tilde{\gamma}_i \in L^k_{wz} |_{h_i} \) \((i = 1, 2)\), the PW identities (2.9) and (2.10) lead to the global version of the PW identity:

\[
I_{\Sigma,P}(A^h, g^h) = I_{\Sigma,P}(A, g) - I_{\Sigma,P}(A, h^h^*).
\]

(2.13)

In this expression, \( h \) is a section of the adjoint \( H_C \)-bundle, namely the bundle \( P \times_H H_C \) associated to \( P \) via the adjoint action of \( H \) on \( H_C \). The transformation \( A \mapsto A^h, g \mapsto g^h \)
(the chiral gauge transformation) is locally defined by (2.4) and (2.5). If $h$ is $H$-valued, or precisely if $h$ takes values in the adjoint $H$-bundle $P \times_H H$, we have $hh^* = 1$ and (2.13) is the statement of gauge invariance.

For a section $\epsilon$ of the adjoint bundle $\text{ad}P = P \times H$, the action satisfies

$$\left(\frac{d}{dt}\right)_0 I_{\Sigma,P}(A,e) = \frac{1}{2\pi i} \int_{\Sigma} \text{tr}_p(\epsilon F_A),$$

(2.14)

where $F_A$ is the curvature of $A$ represented in $\text{ad}P$ and $\text{tr}_p$ is the trace of the adjoint bundle normalized by $\text{tr}_{\text{ad}}(X) = 2g^\vee \text{tr}(XY)$ when $G$ is simple. The properties (2.13) and (2.14) are just what we would expect for the chiral anomaly in the massless free fermionic systems. Indeed, the WZW model was first introduced as the non-abelian bosonization of spin-half fermions [12].

Remark. There is another way to construct the WZW action for configurations of general topology. It is to make use of the equivariant version of the Cheeger-Simons differential character (see [13, 14]). A discussion on this is given in §6 and in the future publication [15].

2.2 Space Of States

We proceed next to the quantization of the WZW model. The correlation function of the local fields $O_1 \cdots O_s$ is given by the path-integral

$$Z_{\Sigma,P}(g,A;O_1 \cdots O_s) = \int_{\Gamma(PG)} \mathcal{D}_g g e^{-kI_{\Sigma,P}(A,g)} O_1(g) \cdots O_s(g),$$

(2.15)

where $g$ is a metric on $\Sigma$ and $\mathcal{D}_g g$ is the left-right invariant measure on the configuration space $\Gamma(PG)$ equipped with the metric induced by $g$. In the following, the sign “$g$” will not usually be mentioned for simplicity of notation. Suppose that the fields $O_1 \cdots O_n$ are inserted in $\Sigma_\infty$ whereas the fields $O_{n+1} \cdots O_s$ are in $D_0$. Having in mind the order of integration such that the last is the integration over configurations on the circle $S = D_0 \cap \Sigma_\infty$, we see that the correlation function is expressed as the pairing

$$Z_{\Sigma,P}(A;O_1 \cdots O_s) = \langle Z_{\Sigma_\infty}(A_\infty;O_1 \cdots O_n), \gamma_\infty \rangle = \langle Z_{D_0}(A_0;O_{n+1} \cdots O_s) \rangle$$

(2.16)

of wave functions

$$Z_{\Sigma_\infty}(A_\infty;O_1 \cdots O_n) : \gamma \mapsto \int_{\gamma = g|_S} \mathcal{D}_g g e^{-kI_{\Sigma_\infty}(A_\infty,g)} O_1(g) \cdots O_n(g),$$

(2.17)

$$Z_{D_0}(A_0;O_{n+1} \cdots O_s) : \gamma \mapsto \int_{\gamma = g|_S} \mathcal{D}_g g e^{-kI_{D_0}(A_0,g)} O_{n+1}(g) \cdots O_s(g),$$

(2.18)
The wave functions (2.18) and (2.17) are sections of the line bundles $L^k_{wz}|_{LG}$ and $L^{*k}_{wz}|_{LG}$ over $LG$ respectively, and can be extended to the holomorphic sections over $LG_C$. This observation motivate us to consider the spaces $\Gamma_{hol}(L^k_{wz})$ and $\Gamma_{hol}(L^{*k}_{wz})$ of holomorphic sections of $L^k_{wz}$ and $L^{*k}_{wz}$.

The group $\tilde{LG}_C$ acts on the space $\Gamma_{hol}(L^k_{wz})$ by the left $(J)$ and the right ($\bar{J}$) representations:

$$J(\tilde{\gamma}_1)\bar{J}(\tilde{\gamma}_2)\Phi(\gamma) = \tilde{\gamma}_1\Phi(\gamma_1^{-1}\gamma_2^{-1} \tilde{\gamma}_2^*),$$

where $\{(g,c)\}^* = \{(g^*,c^*)\}$. For any smooth map $h : D_0 \to G_C$, the PW identity (2.20) together with the left-right invariance of the measure leads to

$$J(\tilde{\gamma})\bar{J}(\tilde{\gamma})Z_{D_0}(A_0; O_a O_b \cdots) = Z_{D_0}(A_h^0; h^{-1}O_a h^{-1}O_b \cdots)$$

$$\tilde{\gamma}_1 = e^{-I_{D_0}(A_0,h)}-\frac{1}{2}I_{D_0}(A_0,h),$$

where $h^{-1}O$ is defined by $(h^{-1}O)(g) = O(hg^*)$. Hence, the infinitesimal generators of the representations $J$, $\bar{J}$ can be identified with components of the current that are defined as the responses to infinitesimal variations of the gauge field. The responses to the variations of the metric under infinitesimal conformal transformations can be identified with the Fourier components $\{g^{G,k}\}$ and $\{\bar{g}_{n}^{G,k}\}$ of the Sugawara energy-momentum tensor which is given in (3.7) [12] [14]. These are two copies of representations of Virasoro algebra with central charge $c_{G,k} = \frac{k \dim G}{k+g^2}$.

We now determine the wave function $\Phi_O = Z_{D_0}(0;O)$ for a field insertion $O$ at $z = 0$ in the unit disc $D_0$ with a fixed metric and a gauge field $A_0 = 0$. To describe it explicitly, we choose maximal tori $T_G$ of $G$ and $T = T_G/Z_G$ of $H$ and also a chambre $C$ in it (see Appendix A for notations and basics on the root system and Weyl groups). These choices determine, for a unitary irreducible representation $V$ of $G$, the weight space decomposition and the highest weight $\Lambda$. We shall describe the state $\Phi_\Lambda = \Phi_{O_\Lambda}$ corresponding to the matrix element $O_\Lambda(g) = (v_\Lambda, g(0)^{-1}v_\Lambda)$ for the highest weight vector $v_\Lambda \in V$. Let $g_1$ and $g_2$ be holomorphic maps of $D_0$ to $G_C$ such that the value $g_1(0)$ ($\text{resp. } g_2(0)$) at $z = 0$ belongs to the Borel subgroup $B$ of $G_C$ (resp. the maximal unipotent subgroup $N$ of $B$) that is generated by the Cartan subalgebra $\mathfrak{t}_C$ and the positive root vectors (resp. by only the positive root vectors). Since these preserve the gauge field $A_0 = 0$, the property (2.21) leads to

$$J(e^{-I_{D_0}(g_1)})\bar{J}(e^{-I_{D_0}(g_2)})\Phi_\Lambda = e^\Lambda(g_1(0))\Phi_\Lambda,$$
where e^A is a character of B for the one dimensional representation C\nu_\lambda. It follows that the value of \Phi_\lambda at the loop \gamma_1 \gamma_2^*(\gamma_i = g_i|s) is given by

$$\Phi_\lambda(\gamma_1 \gamma_2^*) = \Phi_\lambda(1)e^{-\lambda}(g_1(0))e^{-kI_D_0(g_1 g_2^*)},$$

(2.24)

where \Phi_\lambda(1) \in L^k_{\gamma_1} is a constant that may be put 1 by a renormalization. Though any loop in G_C is not of the form \gamma_1 \gamma_2^* as above, the set B^+(N^+)\ast of such loops is open and dense in LG_C [1]; by definition, B^+(\text{resp. } N^+) is the subgroup of LG_c consisting of boundary loops of holomorphic maps D_0 \rightarrow G_c such that the values at z = 0 are in B (\text{resp. } N). It is shown in [10] that \Phi_\lambda extends all over LG_C if and only if \Lambda is integrable at level k, namely,

$$0 \leq (\Lambda, \alpha) \leq k$$

for any positive root \alpha.

Hereafter, the set of weights integrable at level k is denoted by P^+(k).

The state \Phi_\lambda \in \Gamma_{hol}(L^k_{\Lambda}) generates an irreducible LG_C \times \bar{LG}_C module \mathcal{H}_\Lambda^{G,k} \subset \Gamma_{hol}(L^k_{\Lambda}) which is isomorphic to \bar{L}_\Lambda^{G,k} \otimes L_{\Lambda}^{G,k}\ast where \bar{L}_\Lambda^{G,k} (\text{resp. } L_{\Lambda}^{G,k}\ast) is the holomorphic (\text{resp. } anti-holomorphic) irreducible representation of LG_C with highest weight (\Lambda, k).

The subspace

$$\mathcal{H}_\Lambda^{G,k} = \bigoplus_{\Lambda \in P^+(k)} \mathcal{H}_\Lambda^{G,k},$$

(2.25)

of \Gamma_{hol}(L^k_{\Lambda}) is in one to one correspondence under \Phi_\lambda \rightarrow \mathcal{O} with the current descendants of the primary fields \{O_\Lambda; \Lambda \in P^+(k)\}. Though it is not known whether \mathcal{H}_\Lambda^{G,k} is dense in \Gamma_{hol}(L^k_{\Lambda}) with respect to some natural topology, we restrict our attention to this subspace in the rest of the paper.

An advantage of this restriction is that the pairing (2.19) can be given a rigorous definition. It is known [11] that \bar{L}_\Lambda^{G,k} is a unitary representation of the subgroup \bar{LG} = \{\gamma; \gamma^* \gamma^* = 1\} \subset \bar{LG}_C (the basic central extension of the loop group LG), or equivalently, there is a hermitian inner product on the space \mathcal{H}_\Lambda^{G,k} such that

$$(J(\gamma_1)\bar{J}(\gamma_2)\Phi_1, \Phi_2) = (\Phi_1, J(\gamma_2^*)\bar{J}(\gamma_1^*)\Phi_2)$$

(2.26)

In addition, an anti-linear map \Gamma_{hol}(L^s_{\Lambda}) \rightarrow \Gamma_{hol}(L^k_{\Lambda}); \Psi \mapsto \bar{\Psi}(\gamma) = \iota\Psi(\gamma^*) where \iota : L^s_{\Lambda} \rightarrow L^k_{\Lambda} is the map covering \gamma \mapsto \gamma^* defined by

$$\iota\{(g|D_0, c)\} = \{(g^{s-1}|D_0, c^*e^{-kI_D_0(g^{s-1}) + 2kD_0(g^{s-1})|D_0})\}.$$ 

(2.27)

With similar restriction \mathcal{H}_\Lambda^{G,k} \subset \Gamma_{hol}(L^s_{\Lambda}), the pairing \langle \Psi, \Phi \rangle = \int_{\bar{LG}} \mathcal{D}\gamma \langle \Psi(\gamma), \Phi(\gamma) \rangle of \Psi \in \mathcal{H}_\Lambda^{G,k} and \Phi \in \mathcal{H}_\Lambda^{G,k} is now defined by

$$\langle \Psi, \Phi \rangle = \langle \bar{\Psi}, \Phi \rangle.$$ 

(2.28)
This satisfies the property that implies the left-right invariance of the measure $\mathcal{D}_\gamma$.

2.3 The Spectral Flow

Instead of the flat gauge field $A_0 = 0$, we next consider the following configuration. We choose first a real valued smooth function $\varrho : [0, 1 + \epsilon] \to [0, 1]$ such that $\varrho(r) = 0$ for $0 \leq r \leq \epsilon$ and $\varrho(r) = 1$ for $1 - \epsilon \leq r \leq 1 + \epsilon$ where $\epsilon$ is some number in $[0, \frac{1}{2})$. We also choose an element $a$ of $i\mathfrak{t}$ and put

$$A_{\varrho,a} = \varrho(\vert z \rvert) \frac{a}{2} \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) = -\varrho(r) i a d\theta ,$$  \hspace{1cm} (2.29)

where $z = r e^{i\theta}$. If $a$ is in the lattice $\mathbb{P}^+ = \frac{1}{2\pi i} \mathrm{Ker} \{ \exp : t \to T_H \}$, $A_{\varrho,a}$ has trivial holonomy $e^{2\pi ia} = 1$ along the boundary circle $S = \partial D_0$ and one can choose a horizontal gauge $s$ over $S$. It is related to the old standard gauge $s_0$ as $s_0|_S = s\gamma$ by the loop $\gamma(\theta) = g e^{-ia\theta}$ in which $g \in H$ is a constant.

With respect to this horizontal gauge $s$, the state $Z_{D_0}(A_{\varrho,a}; O)$ coming from the disc with field insertion at $z = 0$ looks as the gauge transform $(2.19)$ by $\gamma$ of the state $Z_{D_0}(A_{\varrho,a}; O)$ associated to the standard gauge $s_0|_S$. Let $h_{\varrho,a} : D_0 \to H_C$ be the solution of $A_{\varrho,a}^{01} = h_{\varrho,a} \overline{\partial} h_{\varrho,a}^{-1}$ such that $h_{\varrho,a}(0) = 1$ and $h_{\varrho,a}(z) = c_{\varrho,a}^{-a}|z|^{-a}$ around $S$ with $c_{\varrho,a}$ a real number. Making use of $(2.21)$, one can write $Z_{D_0}(A_{\varrho,a}; O)$ as the transform of $\Phi_O$ by a certain element $\tilde{c}_{\varrho,a} \in \widetilde{LG}_C$ over the constant loop $c_{\varrho,a}^{-a} \in T_C$ and it follows that

$$Z_{D_0}^{(s)}(A_{\varrho,a}; O) = \gamma . J(\tilde{c}_{\varrho,a}) \overline{J}(\tilde{c}_{\varrho,a}) \Phi_O .$$ \hspace{1cm} (2.30)

As we see below, the transformation $\gamma . J(\tilde{c}_{\varrho,a}) \overline{J}(\tilde{c}_{\varrho,a})$ preserves the space $\mathcal{H}^{G,k}$ and permutes the irreducible components $\{ \mathcal{H}^{G,k}_{\Lambda} \}_{\Lambda \in \mathbb{P}_+^{(k)}}$. This is the so-called spectral flow. This line of argument was first suggested in ref. [8].

**Calculation Of $\gamma . \Phi_\Lambda$**

When $O$ is the primary field $O_\Lambda$ with $\Lambda \in \mathbb{P}_+^{(k)}$, the corresponding state $\Phi_\Lambda$ has a definite weight and the new state is given by

$$Z_{D_0}^{(s)}(A_{\varrho,a}; O_\Lambda) = \text{const} \gamma . \Phi_\Lambda ,$$ \hspace{1cm} (2.31)

where the constant is of the form $e^{-ktr(a^2) b_{\varrho} c_{\varrho}^{-2\Lambda(a)}}$ in which $b_{\varrho}$ depends only on $\varrho$.

We calculate $\gamma . \Phi_\Lambda$ when $\gamma \in LH$ represents an element of the group $\Gamma_{\overline{C}}$ (see Appendix A) in which case $\text{ad} \gamma$ preserves the subgroups $B^+$ and $N^+$ of $LG_C$ and the calculation becomes particularly simple. Then, the loop $\gamma$ can be rewritten as

$$\gamma(\theta) = e^{-i\mu \theta} n_w ,$$ \hspace{1cm} (2.32)
where \( n_w = g \) represents an element \( w \) of the Weyl group and \( \mu = wa \) has value 1 or 0 for every positive root. We denote by \( h_\gamma(z) \) the holomorphic extension \( z^{-\mu}n_w \) of \( \gamma \). Note that each connected component of \( \gamma \) contains loops representing a unique element of \( \tilde{\Gamma}_\mathbb{C} \). In §4, we shall make use of such a loop to define a topology changing action of the fundamental group \( \pi_1(H) \) on the set of isomorphism classes of holomorphic \( H\mathbb{C} \)-bundles with parabolic structure.

It suffices to look at the behavior of \( \gamma \cdot \Phi_\Lambda \) over the open dense subset \( B^+(N^+)^* \). Let \( \gamma_1 \in B^+ \) and \( \gamma_2 \in N^+ \) be the boundary loops of holomorphic maps of \( D_0 \) to \( G_\mathbb{C} \), \( g_1 \) and \( g_2 \) respectively with \( g_1(0) \in B \) and \( g_2(0) \in N \). Since \( \text{ad}\gamma \) preserves the subgroups \( B^+ \) and \( N^+ \), holomorphic functions \( h_\gamma^{-1}g_1h_\gamma \) and \( h_\gamma^{-1}g_2h_\gamma \) are defined on \( D_0 \) and satisfy \( (h_\gamma^{-1}g_1h_\gamma)(0) \in B \) and \( (h_\gamma^{-1}g_2h_\gamma)(0) \in N \). Hence we have

\[
\gamma \cdot \Phi_\Lambda(\gamma_1\gamma_2^*) = e^{-\Lambda} \left( (h_\gamma^{-1}g_1h_\gamma)(0) \right) \text{ad}\gamma \left( e^{-kI_{D_0}}( (h_\gamma^{-1}g_1h_\gamma)(h_\gamma^{-1}g_2h_\gamma)^* ) \right). \tag{2.33}
\]

If we put \( g_1(0) \equiv e^{t_0} \in T \text{ mod } N \), we find that \( (h_\gamma^{-1}g_1h_\gamma)(0) \equiv e^{w^{-1}t_0} \) and hence

\[
e^{-\Lambda} \left( (h_\gamma^{-1}g_1h_\gamma)(0) \right) = e^{-w\Lambda(t_0)}. \tag{2.34}
\]

Applying the transformation rule (2.9) of \( \text{ad}\gamma \), we find that

\[
\text{ad}\gamma \left( e^{-kI_{D_0}}(h_\gamma^{-1}g_1h_\gamma) \right) = e^{-k\text{tr}(\mu t_0)}e^{-kI_{D_0}(g_1)}, \tag{2.35}
\]

\[
\text{ad}\gamma \left( e^{-kI_{D_0}}(h_\gamma^{-1}g_2h_\gamma) \right) = e^{-kI_{D_0}(g_2)}. \tag{2.36}
\]

Combining these results, we obtain the expression

\[
\gamma \cdot \Phi_\Lambda(\gamma_1\gamma_2^*) = e^{-w\Lambda(t_0)-k\text{tr}(\mu t_0)}e^{-kI_{D_0}(g_1g_2^*)}. \tag{2.37}
\]

Thus, the result is \( \gamma \cdot \Phi_\Lambda = \Phi_{\gamma\Lambda} \), the vector of highest weight

\[
\gamma \Lambda = w\Lambda + k^\gamma \mu, \tag{2.38}
\]

in which \( ^\gamma \mu \) denotes the weight \( ^v \mu(v) = \text{tr}(\mu v) \). Indeed, if \( \gamma \) represents an element of \( \tilde{\Gamma}_\mathbb{C} \), the transformation \( \Lambda \mapsto \gamma \Lambda \) preserves the set \( \mathbb{P}^{(k)}_+ \) of integrable weights.

This transformation of \( \mathbb{P}^{(k)}_+ \) looks simple with respect to the fundamental affine weights \( \hat{\Lambda}_0, \cdots, \hat{\Lambda}_l \in \hat{\mathbb{V}}^* \) related to the simple affine roots by \( 2(\hat{\Lambda}_i, \hat{\alpha}_j)/||\hat{\alpha}_j||^2 = \delta_{i,j} \). Since \( \Gamma_\mathbb{C} \) is an automorphism group of the extended Dynkin diagram, or an orthogonal group of permutations of simple affine roots; \( \gamma \hat{\alpha}_i = \hat{\alpha}_{\gamma i} \), we find that \( \gamma \in \Gamma_\mathbb{C} \) permutes the
fundamental affine weights modulo $R \times 0 \times 0$. Hence, denoting the highest weight $(\Delta, \Lambda; k)$ by $\hat{\Lambda}$, the transformation is written as

$$\hat{\Lambda} = \sum_{i=0}^{l} n_i \hat{\Lambda}_i \mapsto \gamma \hat{\Lambda} = \sum_{i=0}^{l} n_i \hat{\Lambda}_i \mod R \times 0 \times 0. \quad (2.39)$$

**Remark.** This gauge transformation $\gamma : \mathcal{H}^{G,k} \to \mathcal{H}^{G,k}$ induces the external automorphism of the Virasoro-Kac-Moody algebra. In fact, the spectral flow may be considered as the consequence of such an algebra automorphism.

**Non-Abelian Insertion Theorem**

Let $P$ be the principal $H$-bundle over $\Sigma$ with a connection $A$ which is flat on the unite disc $D_0 \subset \Sigma$ in a coordinatized subset. We choose a horizontal gauge $\sigma_0$ over $D_0$. Gluing $(D_0 \times H, A_{\varphi,a})$ to $(P|_{\Sigma_2}, A|_{\Sigma_2})$ at the boundaries by the identification $(e^{i\theta}, 1) \equiv \sigma_0(e^{i\theta})\gamma(\theta)$, we obtain another $H$-bundle $P_{\gamma}$ with a connection $A_{\gamma}$.

Applying the pairing formula (2.16) to $Z_{\Sigma_2}(A_{\varphi,a}; O_1 \cdots O_s)$ and $\gamma.Z_{D_0}(A_{\varphi,a}; O_\Lambda)$ and using the above result $\gamma.\Phi_\Lambda = \Phi_\Lambda$, we see that

$$Z_{\Sigma_2, P_{\gamma}}(A_{\gamma}; O_1 \cdots O_s O_\Lambda) = \text{const} \cdot Z_{\Sigma_2, P}(A; O_1 \cdots O_s O_\Lambda), \quad (2.40)$$

where the constant is the same as the one in (2.31). This may be considered as the prototype of (1.4). Equation of the same kind is already known in the free fermionic (bosonic) system as the insertion theorem [18].

**3. Integration Over Gauge Fields**

Let $H$ be a connected, closed subgroup of $G' = G/\mathbb{Z}_G$. We denote by $\mathcal{A}_P$ the set of connections of a principal $H$-bundle $P$ over a Riemann surface $\Sigma$ and by $\mathcal{G}_P$ the set of sections of the adjoint $H$-bundle $P \times_H H$ which acts on $\mathcal{A}_P$ as the gauge transformation group. In this section, we turn to the quantization of the gauged WZW model with target group $G$ and gauge group $H$. We develop a method to perform the integration

$$Z_{\Sigma, P}(O_1 \cdots O_s) = \frac{1}{\text{vol} \mathcal{G}_P} \int_{\mathcal{A}_P} \mathcal{D}A Z_{\Sigma, P}^{G,k}(A; O_1 \cdots O_s), \quad (3.1)$$

of the WZW correlator (2.13) of the gauge invariant fields $O_1 \cdots O_s$.

\[\text{superscript "G,k" is introduced for specification since we shall consider several different groups H, H_C, G etc. at the same time. The H-bundle P and its connection A under the superscript are prescribed to mean the extension to G'-bundle and G'-connection.}\]
The method takes advantage of the chiral gauge symmetry
\[ Z_{\Sigma, P}^{G,k}(A^h; O_1 \cdots O_s) = e^{kI_G^P(A, hh^*)} Z_{\Sigma, P}^{G,k}(A; hO_1 \cdots hO_s), \] (3.2)
for a section \( h \) of the adjoint \( H_C \)-bundle \( P \times H H_C \), which is a consequence of the PW identity (2.13). We integrate first over each orbit of the group \( G_{\mathfrak{h}C} = \Gamma(P \times H H_C) \) of chiral gauge transformations, and then over the orbits. One can see that \( \mathcal{A}_P \) contains a submanifold \( \mathcal{A}_{ss} \) with the complement of codimension \( \geq 1 \) such that the orbit space \( \mathcal{A}_{ss}/G_{\mathfrak{h}C} \) is approximately a finite dimensional compact space \( \mathcal{N}_P \) with a preferable structure. Change to the parametrization of \( \mathcal{A}_P \) in terms of \( G_{\mathfrak{h}C} \) and \( \mathcal{N}_P \) induces the Jacobian factor that can be represented by the spin (1, 0) ghost system with values in the adjoint \( \mathfrak{h}_C \)-bundle \( \text{ad}P_C \). The integration over \( G_{\mathfrak{h}C} \) mod \( G_{\mathfrak{h}C} \) leads to a sigma model with the target space \( H_C/H \). Consequently, the correlation function (3.1) is expressed as the integration over \( \mathcal{N}_P \) of a correlation function of the three systems coupled to common representative gauge field — the WZW model with the target \( G \), the sigma model with the target \( H_C/H \) and the ghost system valued in the adjoint bundle.

3.1 The Space Of Gauge Fields

We give a description of the structure of \( G_{\mathfrak{h}C} \)-orbits in \( \mathcal{A}_P \) and argue that we can neglect some orbits in the integration (3.1). To start with, we note that a connection \( A \) of \( P \) determines a unique holomorphic structure \( \overline{\partial}_A \) of the complexified \( H_C \)-bundle \( P_C \): A local section \( \sigma \) of \( P_C \) is holomorphic if \( \overline{\partial}_A \sigma = 0 \) when \( \sigma \) is represented as a local frame of the vector bundle associated to \( P_C \) with a holomorphic representation of \( H_C \). Conversely, any holomorphic structure of \( P_C \) is obtained in this way. Since two connections \( A \) and \( A^h \) related by a chiral gauge transformation \( h \in G_{\mathfrak{h}C} \) correspond to the isomorphic holomorphic structures \( \overline{\partial}_A \) and \( h^{-1}\overline{\partial}_Ah \), we can identify the orbit set \( \mathcal{A}_P/G_{\mathfrak{h}C} \) with the set of isomorphism classes of holomorphic structures of \( P_C \). Such an identification makes easy the explicit description of the orbits for genus zero and makes possible for genus \( \geq 1 \) to use the well-known techniques in analytic and algebraic geometry such as the Riemann-Roch theorem, the Atiyah-Bott stratification and especially the Narasimhan-Seshadri theorem.

It should be noticed that the space \( \mathcal{A}_P \) is given a complex structure \( J_\Sigma \) so that the \( G_{\mathfrak{h}C} \)-action is holomorphic: On each tangent space \( \Omega^1(\Sigma, \text{ad}P) \) which is the set of one forms on \( \Sigma \) valued in the adjoint bundle, \( J_\Sigma \) acts as the Hodge \(*\)-operator; \(*Xd\bar{z} = iXd\bar{z}, \ *\bar{X}dz = -i\bar{X}dz.\) 

On The Sphere

We begin with the case in which \( \Sigma \) is the Riemann sphere \( \mathbb{P}^1 \). It is covered by the
$z$-plane $U_0$ and $w$-plane $U_\infty$ where $z$ and $w$ are related by $zw = 1$. We denote by $D_0$ (resp. $D_\infty$) the unit disc in the $z$-plane (resp. $w$-plane).

For $H = U(1)$: A holomorphic $H_C = C^*$-bundle admits local sections $\sigma_0$ and $\sigma_\infty$ over open neighborhoods of $D_0$ and $D_\infty$ respectively. If they are related by the holomorphic transition function $h_{\infty0}$ on a neighborhood of $S = D_0 \cap D_\infty$

$$\sigma_0(z) = \sigma_\infty(z) h_{\infty0}(z), \quad (3.3)$$

the winding number $a = \frac{1}{2\pi} \oint_S h_{\infty0}^{-1} dh_{\infty0} \in \mathbb{Z}$ determines the topological type. Taking the Laurent expansion of the function $\log\{h_{\infty0}(z)^a\}$, we find that

$$h_{\infty0}(z) = h_{\infty}(z) z^{-a} h_0(z)^{-1}, \quad (3.4)$$

where $h_0$ and $h_\infty$ are $C^*$-valued holomorphic functions on neighborhoods of $D_0$ and $D_\infty$ respectively. Hence, we can always take the transition function of the form $z^{-a}$. In other words, for a $U(1)$-bundle $P$,

$A_P$ is itself a single $G_P$-orbit.

For $H = SU(n)/\mathbb{Z}_n$: We next consider the group $SU(n)/\mathbb{Z}_n$ where $\mathbb{Z}_n$ is the center of $SU(n)$ consisting of identity matrices multiplied by $n$-th roots of unity. The property (3.4) holds also in this case provided that $a$ is an element of $P^\vee$, that is, $a$ is of the form

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \text{ with } a_i + \frac{j}{n} \in \mathbb{Z} \ (i = 1, \cdots, n) \text{ and } \sum_{i=1}^n a_i = 0, \quad (3.5)$$

for some $j \in \hat{J} = \{0, 1, \cdots, n-1\}$. This is due to the Birkhoff factorization theorem [19, 11] which also states that such $a$ is unique up to permutations of $a_1, \cdots, a_n$. Hence, holomorphic $H_C$-bundles over $P^1$ are classified by the countable set $P^\vee/W$ in which $W$ the Weyl group of $H$ acts on diagonal matrices as permutations of diagonal entries. Note that the loop $e^{ia} \mapsto e^{-ia}$ extends to a map on $D_0$ with values in $H$ if and only if all $a_i$ are integers. Thus the topological type of the bundle is determined by the number $j \in \hat{J}$. Stated in another way, for each $j \in \hat{J}$, there is an $H$-bundle $P^{(j)}$ and its complexification admits countably many holomorphic structures classified by $P^\vee_j/W$ in which $P^\vee_j$ is the set of matrices in $P^\vee$ whose diagonal entries differ from $-\frac{j}{n}$ by integers. Since the set $C$ of diagonal matrices $t$ with $t_1^a \geq \cdots \geq t_n^a$ is a fundamental domain of $W$, we see that

$$A_{P^{(j)}} = \bigcup_{a \in P^\vee_j \cap C} A_a, \quad (3.6)$$
where $A_a$ is the $G_{P^{(j)}}$-orbit corresponding to the holomorphic $H_C$-bundle $P_{[a]}$ with the transition function $h_{\infty 0}(z) = z^{-a}$.

Though each $A_a$ is infinite dimensional, one can compare the dimensions of these orbits relative to $G_{P^{(j)}}$. That is to consider the the group $\text{Aut} P_{[a]}$ of holomorphic automorphisms of $P_{[a]}$ that corresponds to the isotropy group of $G_{P^{(j)}}$ at a point of $A_a$. An element $f$ of $\text{Aut} P_{[a]}$ is given by $H_C$-valued holomorphic functions $f_0$ and $f_{\infty}$ on $U_0$ and $U_{\infty}$ respectively such that $f_0(z) = z^a f_{\infty}(z) z^{-a}$ on $U_0 \cap U_{\infty}$. We find that $(f_0(z))^j_j$ is a span of $1, z, \ldots, z^{a_i - a_j}$ if $a_i \geq a_j$ and is zero if $a_i < a_j$. The dimension of $\text{Aut} P_{[a]}$ is thus given by $n - 1 + \sum_{i<j}(\delta_{a_i,a_j} + 1 + |a_i - a_j|)$ and is minimized in $\mathbb{P}^j_\nu \cap \overline{U}$ by $a = \mu_j$ where $(\mu_j)_i = 1 - \frac{j}{n}$ for $i = 1, \ldots, j$ and $(\mu_j)_i = -\frac{j}{n}$ for $i = j + 1, \ldots, n$. Hence $A_{P^{(j)}}$ contains an orbit $A_{\mu_j}$ of maximal dimension and another orbit $A_a$ has codimension $d_a > 0$ given by

$$d_a = \sum_{i<j}(\delta_{a_i,a_j} - 1 + |a_i - a_j|) = \sum_{a_i > a_j} (a_i - a_j - 1).$$

(3.7)

Therefore, in the integration (3.1) for $P = P^{(j)}$, we have only to take into account the single orbit $A_{\mu_j}$.

For general $H$: We follow the preceding argument using the notation of Appendix A. For each $j \in \hat{J}$, there is an $H$-bundle $P^{(j)}$ with the transition function $e^{-i\mu_j \theta}$ and any $H$-bundle is isomorphic to $P^{(j)}$ for some $j \in \hat{J}$. The set of connections of $P^{(j)}$ is decomposed as the disjoint union of the form (3.6) in which $P^j_\nu = \mu_j + Q^\nu$, $\overline{U}$ is the closure of a chambre $C$ in it and $A_a$ is the $G_{P^{(j)}}$-orbit corresponding to a holomorphic bundle with the transition function $z^{-a}$. The orbit $A_{\mu_j}$ is of maximal dimension and the codimension of $A_a$ is

$$d_a = \sum_{\alpha(a) > 0} (\alpha(a) - 1)$$

(3.8)

where $\alpha$ in the sum runs over roots of $H$. Since $d_a \geq 1$ for $a \neq \mu_j$, we may replace $A_{P^{(j)}}$ by $A_{\mu_j}$ in the integration (3.1) for $P = P^{(j)}$.

On A Surface Of Genus $\geq 1$

For a Riemann surface $\Sigma$ of genus $\geq 1$, the set of orbits $A_P/G_{R_k}$ is not in general countable. This can be seen by looking at the index dim $H(1 - g)$ of the operator $\overline{\partial}_A : \Omega^0(\Sigma, \text{ad} R_C) \rightarrow \Omega^{0,1}(\Sigma, \text{ad} R_C)$ which counts the dimension of the symmetry group of $\overline{\partial}_A$ minus the codimension of the $G_{R_k}$-orbit through $A$.

For $H = U(1)$: Let $\mathcal{O}$ (resp. $\mathcal{O}^\times$) be the sheaf of germs of holomorphic functions valued in $\mathbb{C}$ (resp. $\mathbb{C}^\times$). The set of isomorphism classes of holomorphic principal $\mathbb{C}^\times$-bundles is identified with the sheaf cohomology group $H^1(\Sigma, \mathcal{O}^\times)$, the Picard group $\text{Pic}(\Sigma)$. The
long exact sequence induced by the homomorphism $O \to O^\times; f \mapsto e^{2\pi if}$ with kernel $Z$ gives the following description of $\text{Pic}(\Sigma)$:

$$0 \to \text{Jac}(\Sigma) \to \text{Pic}(\Sigma) \xrightarrow{c_1} Z \to 0,$$

(3.9)

where the projection $c_1$ counts the first chern class and $\text{Jac}(\Sigma)$ is the Jacobian variety $H^1(\Sigma, O)/H^1(\Sigma, Z)$ which is a complex torus of dimension $g$.

For each topological type $a \in Z$, a choice $P \in \text{Pic}(\Sigma)$ with $c_1(P) = a$ determines an isomorphism of $\text{Jac}(\Sigma)$ and the set $c_1^{-1}(\{a\})$ of holomorphic $H_C$-bundles of 1-st chern class $a$. Thus, for any $U(1)$-bundle $P$,

$$A_P/G_P \cong \text{Jac}(\Sigma) \text{ (a complex } g\text{-torus)}.$$

(3.10)

In particular, even if $P$ and $P'$ are topologically distinct, there are isomorphisms of $A_P/G_P$ and $A_{P'}/G_{P'}$. In §4, we shall use a certain isomorphism to prove (1.4).

For general $H$: If $H$ is non abelian, the situation is a little different. For simplicity of the discussion, we assume that $H$ is simple. We make use of the following stratification (decomposition into submanifolds) of the space of connections of a principal $H$-bundle $P$ over $\Sigma$ which is due to Atiyah and Bott [20]:

$$A_P = \bigcup_{\mu} A_{\mu}.$$

(3.11)

This generalizes the disjoint union (3.6) for genus zero. Here, $\mu$ runs over a discrete subset of $\mathbb{C}$ and $A_{\mu}$ is a $G_P$-invariant submanifold of $A_P$ of codimension

$$d_\mu = \sum_{\alpha(\mu) > 0} (\alpha(\mu) + g - 1).$$

(3.12)

The unique solution to $d_\mu = 0$ is $\mu = 0$ for genus $\geq 1$. It is known [20] that $A \in A_0$ if and only if the adjoint bundle $\text{ad}P_C$ with the holomorphic structure $\partial_A$ is semi-stable, namely, any holomorphic subbundle has non-positive first chern class. In view of this characterization, we hence-forth denote $A_0$ by $A_{ss}$ ($ss$ means “semi-stable”). The space $A_{ss}$ contains a $G_P$-invariant, open and dense submanifold $A_{ss}^\circ$ such that the quotient $A_{ss}^\circ/G_P$ is a non-empty complex manifold whose dimension $d_N = \dim H(g - 1)$ for genus $\geq 2$ and is between 0 and rank $H$ for genus one. Hence, we may replace $A_P$ by $A_{ss}^\circ$ in the integration (3.1). A compactification of $A_{ss}^\circ/G_P$ is given by the quotient $N_P = A_{ss}/G_P$ of $A_{ss}$ under a certain equivalence relation. The theorem of Narasimhan and Seshadri [21] [23] [22] essentially states that the set $A_F$ of flat connections is included in $A_{ss}$ and the inclusion map induces the identification of the moduli space $A_F/G_P$ of flat connections and the moduli space $N_P = A_{ss}/G_P$ of semi-stable $H_C$-bundles.
Example — Flat SO(3)-Connections over the Torus

We explicitly describe the moduli spaces of flat connections of the trivial and the non-trivial \( H = SO(3) \)-bundles on the torus \( \Sigma = C/(Z + \tau Z) \) of period 1 and \( \tau \) where \( \tau_2 = \text{Im} \tau > 0 \). We denote by \( \zeta \) the coordinate of this plane \( C \). The homology base \( A, B : [0, 1] \to \Sigma \) defined by \( \zeta(A_t) = t \) and \( \zeta(B_t) = t\tau \) provides a set of generators of the fundamental group \( \pi_1 \Sigma = Z^2 \). A flat connection of an \( H \)-bundle \( P \) defines (up to conjugation) a holonomy representation \( \rho : \pi_1 \Sigma \to H \). It is determined by the commuting elements \( a = \rho(A) \) and \( b = \rho(B) \) of \( H \).

If \( P \) is trivial, \( a \) and \( b \) are represented by commuting elements \( \tilde{a} \) and \( \tilde{b} \) of \( \tilde{H} = SU(2) \). By conjugation, we can bring them to diagonal matrices
\[
\tilde{a} = \begin{pmatrix} e^{2\pi i \phi} & 0 \\ 0 & e^{-2\pi i \phi} \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} e^{2\pi i \psi} & 0 \\ 0 & e^{-2\pi i \psi} \end{pmatrix}.
\]
Such holonomy is provided by the gauge field of the following form:
\[
A_u = \left( \frac{\pi}{\tau_2} u d\bar{\zeta} - \frac{\pi}{\tau_2} \bar{u} d\zeta \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
where \( u = \psi - \tau \phi \). This \( u \) can be considered as a holomorphic parameter. \( A_u' \) is gauge equivalent to \( A_u \) if and only if \( u' = \pm u - \frac{m}{2} + \frac{n\tau}{2} \) for some \( n, m \in Z \). Hence, the moduli space is given by
\[
\mathcal{N}_{\text{triv}} = C/\{ (\frac{1}{2}Z + \frac{\tau}{2}Z) \tilde{\times} \{ \pm 1 \} \}.
\]
It is an orbifold with four singularities \( u \equiv 0, \frac{1}{4}, \frac{1}{4} + \frac{\tau}{4}, \frac{1}{4} + \frac{3\tau}{4} \) of order 2. The manifold \( \mathcal{A}_{ss}/G_{\tilde{c}} \) in this case is \( \mathcal{N}_{\text{triv}} \) with these singular points deleted.

If \( P \) is non-trivial, \( a \) and \( b \) are represented by elements \( \tilde{a}, \tilde{b} \) of \( \tilde{H} = SU(2) \) that do not commute but satisfy
\[
\tilde{a} \tilde{b} \tilde{a}^{-1} \tilde{b}^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
There is only one such pair \( (\tilde{a}, \tilde{b}) \) modulo conjugation:
\[
\tilde{a} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]
Hence, on the torus,
\[
\mathcal{N}_{\text{non-triv}} = \{ \text{one point} \}.
\]
In contrast with the abelian case, \( \mathcal{N}_{\text{triv}} \) is not isomorphic to \( \mathcal{N}_{\text{non-triv}} \) and even the dimensions are different. For a general semi-simple group \( H \), the moduli space of topologically trivial semi-stable \( HC \)-bundles over the torus \( \Sigma \) is
\[
\mathcal{N}_{\text{triv}} = t_c/C/(P^\vee + \tau P^\vee) \tilde{\times} W,
\]
where \( t_c \) and \( W \) are associated with the fundamental group \( \pi_1 \Sigma \) and the gauge group \( G \).
and hence of dimension rank \( H \). But for each \( j \in \mathcal{J} \) in the terminology of Appendix A, we have a non-trivial \( H \)-bundle \( P^{(j)} \) and we can see that \( \dim \mathcal{N}_{P^{(j)}} = \dim \ker(w_jw_0 - 1) \) which is strictly less than the rank of \( H \).

### 3.2 The Path Integration

To define the measure for the integration (3.1), we introduce metrics on the spaces \( \mathcal{A}_P \) and \( \mathcal{G}_{R^c} \). We identify the tangent spaces at \( A \in \mathcal{A}_P \) and \( h \in \mathcal{G}_{R^c} \) as

\[
T^0_A^0 \mathcal{A}_P \cong \Omega^{0,1}(\Sigma, \text{ad}P_C), \quad T^1_h \mathcal{G}_{R^c} \cong \Omega^0(\Sigma, \text{ad}P_C),
\]

where \( a \in \Omega^{0,1}(\Sigma, \text{ad}P_C) \) is tangent to the curve \( \partial_A = \partial_A + t a \) at \( t = 0 \) and \( \epsilon \in \Omega^0(\Sigma, \text{ad}P_C) \) is tangent to the curve \( h_t = he^{t\epsilon}e^\ast \) at \( t = 0 \). We define inner products on those spaces by \( (a_1, a_2) = \frac{1}{2\pi} \int_{\Sigma} tr_p(a_1^\ast a_2) \) and by \( (\epsilon_1, \epsilon_2) = \frac{1}{2\pi} \int_{\Sigma} \epsilon_1^\ast \epsilon_2 \). Then, \( \mathcal{A}_P \) becomes a \( \mathcal{G}_P \) invariant Kähler manifold and \( \mathcal{G}_{R^c} \) becomes a Hermitian manifold invariant under the left action of \( \mathcal{G}_{R^c} \) and the right action of \( \mathcal{G}_P \).

**Local Parametrization of Gauge Fields**

As is noticed above, we may replace \( \mathcal{A}_P \) in the integration (3.1) by a submanifold \( \mathcal{A}^0_P \) whose \( \mathcal{G}_{R^c} \)-quotient \( \mathcal{N}^0_P \) is a complex manifold of dimension \( d_N \). For \( H = U(1) \), \( \mathcal{A}^0_P \) is \( \mathcal{A}_P \) itself and \( \mathcal{N}^0_P \) is a complex \( d_N = g \)-torus. In general, we put

\[
\begin{align*}
(g = 0) & \quad \mathcal{A}^0_P = \mathcal{A}_{P^+} \text{ for } P = P^{(j)}, \quad \mathcal{N}^0_P = \text{one point}, \quad d_N = 0, \\
(g = 1) & \quad \mathcal{A}^0_P = \mathcal{A}_{s,s}^0, \quad \mathcal{N}^0_P = \mathcal{A}_{s,s}^0 / \mathcal{G}_{R^c} \subset \mathcal{N}_P, \quad 0 \leq d_N \leq \text{rank} H, \\
(g \geq 2) & \quad \mathcal{A}^0_P = \mathcal{A}_{s,s}^0, \quad \mathcal{N}^0_P = \mathcal{A}_{s,s}^0 / \mathcal{G}_{R^c} \subset \mathcal{N}_P, \quad d_N = \text{dim} H(g - 1).
\end{align*}
\]

For every point \( u_0 \in \mathcal{N}^0_P \), we can take a neighborhood \( U \) of \( u_0 \) in \( \mathcal{N}^0_P \) with a holomorphic family \( \{A_u \}_{u \in U} \) of representatives, that is, a holomorphic map \( U \rightarrow \mathcal{A}^0_P; \ u \mapsto A_u \) such that the \( \mathcal{G}_{R^c} \)-orbit through \( A_u \) is \( u \). We denote by \( \mathcal{A}_U \) the inverse image of \( U \) by the quotient map \( \mathcal{A}^0_P \rightarrow \mathcal{N}^0_P \) and define a surjective map

\[
f : U \times \mathcal{G}_{R^c} \longrightarrow \mathcal{A}_U \quad \text{by} \quad f(u, h) = A^h_u.
\]

This is not injective if the symmetry group \( S_u = \text{Aut} \partial_A \) is non-trivial, in particular if its dimension \( d_S = d_N + \text{dim} H(1 - g) \) is non zero which is the generic situation for \( g = 0, 1 \).

Let \( (u^1, \ldots, u^{d_N}) \) be a complex coordinate system on \( U \). The differentials

\[
\nu_u(u) = (\partial / \partial u^a) A_u^0, \quad a = 1, \ldots, d_N,
\]

provide a base of the tangent space \( T^1_u^0 U = H^{0,1}_{\partial A_u}(\Sigma, \text{ad}P_C) \). We also choose a base \( \{a^b(u)\}_{b=1}^{d_N} \) of the cotangent space \( (T^1_u^0 U)^* = H^{1,0}_{\partial A_u}(\Sigma, \text{ad}P_C) \) and a base \( \{\epsilon_i(u)\}_{i=1}^{d_S} \) of the space \( \text{Lie}(S_u) = H^0_{\partial A_u}(\Sigma, \text{ad}P_C) \) of infinitesimal symmetries of \( \partial A_u \).
At the point \( f(u, h) = A_h^u \), we choose an orthonormal base \( \{a_n(u, h)\}_{n=1}^\infty \) of the tangent space \( \tilde{\partial}_{A_h^u} \Omega^0(\Sigma, \text{ad} P) \) of the \( G_R \)-orbit through \( A_h^u \) and an orthonormal base \( \{e_n(u, h)\}_{n=1}^\infty \) of the orthogonal complement of \( h^{-1} \text{Lie}(S_u) h \) in \( \Omega^0(\Sigma, \text{ad} P) \). Putting \( a_{b-d_N}(u, h) = h^* a^b(u) h^{-1} \) and \( e_{i-d_S}(u, h) = h^{-1} e_i(u) h \), we have a base \( \{a_n(u, h)\}_{n=1-d_N}^\infty \) of the tangent space of \( \mathcal{A}_P \) at \( A_h^u \) and a base \( \{e_n(u, h)\}_{n=1-d_S}^\infty \) of the tangent space of \( G_R \) at \( h \).

Let \( x = (x^{1-d_N}, x^0, x^1, \ldots) \) and \( t = (t^{1-d_S}, t^0, t^1, \ldots) \) be the complex coordinate systems on neighborhoods of \( A_h^u \) in \( \mathcal{A}_U \) and of \( h \) in \( G_R \) defined by

\[
A(x)^{01} = (A_h^u)^{01} + \sum_{n=1-d_N}^\infty x^n a_n(u, h),
\]

and

\[
h(t) = \exp \left\{ \sum_{i=1}^{d_S} t_i^{d_S} e_i(u) \right\} h \exp \left\{ \sum_{n=1}^{\infty} t^n e_n(u, h) \right\}.
\]

Then, the pull backs of differentials \( dx^n \) by the map \( f \) are expressed as

\[
f^* dx^{a-d_N} = \sum_{b,c=1}^{d_N} M^{a\bar{b}}(u, h) \langle a^\bar{b}(u), \nu_c(u) \rangle du^c,
\]

\[
f^* dx^n = \sum_{m=1}^{\infty} (a_n(u, h), \tilde{\partial}_{A_h^u} e_m(u, h)) dt^m + \sum_{c=1}^{d_N} (a_n(u, h), h^{-1} \nu_c(u) h) du^c,
\]

where \( M^{a\bar{b}}(u, h) \) is the inverse matrix of \( M_{a\bar{b}}(u, h) = (a_{a-d_N}(u, h), a_{b-d_N}(u, h)) \) and \( \langle \ , \ \rangle \) is the natural pairing given by \( \langle a, \nu \rangle = \frac{1}{2\pi i} \int_\Sigma \text{tr}(a \nu) \).

**The Measure \( DA \)**

The pull back by \( f \) of the volume element \( DA = \det M_{a\bar{b}}(u, h) \prod_{n=1-d_N}^\infty d^2 x^n \) of \( \mathcal{A}_P \) (where \( d^2 x^n = id x^n \wedge d\bar{x}^n \)) is then given by

\[
f^* DA(u, h) = \frac{\left| \det \langle a^\bar{b}(u), \nu_c(u) \rangle \right|^2}{\det M_{a\bar{b}}(u, h)} \prod_{c=1}^{d_N} d^2 u^c \det \left( \tilde{\partial}_{A_h^u} \tilde{\partial}_{A_h^u} \right) \prod_{n=1}^\infty d^2 t^n
\]

\[
\times \det S(u, h) \prod_{n=1}^\infty d^2 t^n e^{\frac{1}{2\pi i} \int_{\Sigma_R \nu_{\mathcal{A}_U, h}}^\dagger D^* D} (A_{A_h^u, h})
\]

where \( S_{ij}(u, h) = (\epsilon_{i-d_S}(u, h), \epsilon_{j-d_S}(u, h)) \) and \( \det'(D^* D) \) denotes the regularized determinant of \( D^* D \) restricted to its positive eigen-spaces. The factor \( e^{\frac{1}{2\pi i} \int_{\Sigma_R \nu_{\mathcal{A}_U, h}}^\dagger D^* D} (A_{A_h^u, h}) \) is the chiral anomaly of the infinite determinant which shall be written down shortly.
If the dimension $d_S \neq 0$, $f^* \mathcal{DA}$ has ‘lower’ degree compared to the volume element $\prod d^2 u^c \mathcal{D}h = \prod d^2 u^c \det S(u, h) \prod_{n=1-d_S} \infty d^2 v$ of $U \times \mathcal{G}_\mathbb{R}$. In order to deal with such a case, we assume that there is a function $F_u : \mathcal{G}_\mathbb{R} \to \mathbb{C}^{d_S}$ with the following property: On each $S_u$-orbit in $\mathcal{G}_\mathbb{R}$, $F_u$ takes the value zero at one and only one point and that, at each zero point $h$, the differential $F_{u,h} : \text{Lie}(S_u) \to \mathbb{C}^{d_S}$ defined by $F_{u,h}(\epsilon) = \frac{d}{dh} \bigg|_0 F_u(\epsilon^t h)$ is a linear isomorphism. Then, the last factor $\det S(u, h) \prod \cdots$ in (3.27) can be identified with the volume form

$$\int_{S_u} \mathcal{D}h \, \delta^{(2d_S)}(F_u(h)) \left| \det F_u (\epsilon_j(u)) \right|^2 \delta_{\Sigma,\mathbb{P}}^{gh}(A_u, hh^*)$$

(3.28)
on $S_u \setminus \mathcal{G}_\mathbb{R}$. Since the role of $F_u$ is to fix the ‘gauge degrees of freedom’ $S_u$, we call it the residual gauge fixing function.

Remark. The assumption of the existence of the residual gauge fixing function on the whole space $\mathcal{G}_\mathbb{R}$ may fail. However, we can find at least a function $F_u$ defined on a neighborhood of a (local) section of the fibre bundle $\mathcal{G}_\mathbb{R} \to S_u \setminus \mathcal{G}_\mathbb{R}$ and the factor $\delta^{(2d_S)}(F_u(h)) \left| \det F_u (\epsilon_j(u)) \right|^2$ makes sense by localizing the integration (3.28) in that neighborhood.

The infinite determinant and some other factors can neatly be expressed in terms of the system of free fermions called the adjoint ghosts — spin $(1, 0)$ (resp. $(0, 0)$) anti-commuting field $b$ (resp. $c$) valued in $(\text{ad} \mathbb{P}_\mathbb{C})^*$ (resp. $\text{ad} \mathbb{P}_\mathbb{C}$) and its anti-holomorphic partner $\bar{b}$ (resp. $\bar{c}$) — with the classical action

$$I_{\Sigma,\mathbb{P}}^{gh}(A, b, c, \bar{b}, \bar{c}) = \frac{i}{2\pi} \int_{\Sigma} \bar{b} \partial_A c + \bar{c} \partial_A \bar{b}.$$  

(3.29)

That is, we have

$$\det' \left( \partial_{A_u} \bar{\partial}_{A_u} \right) \frac{\det F_u (\epsilon_j(u)) \left| \det \langle a^b(u), \nu_c(u) \rangle \right|^2}{\det \langle \epsilon_i(u), \epsilon_j(u) \rangle \det \langle a^a(u), a^b(u) \rangle}$$

$$= Z_{\Sigma,\mathbb{P}}^{gh}(A_u; \prod_{i=1}^{d_S} F_{u,h}(c) \bar{F}_{u,h}(\bar{c}) \prod_{a=1}^{d_{\Sigma}} \langle b, \nu_a(u) \rangle \langle \bar{b}, \bar{\nu}_a(u) \rangle),$$

(3.30)

where $Z_{\Sigma,\mathbb{P}}^{gh}(A; \cdots)$ is the correlation function of the adjoint ghost system.

These results (3.27), (3.28) and (3.30) together with the chiral gauge symmetry (3.2) lead to the expression

$$Z_{\Sigma,\mathbb{P}}(O_1 \cdots O_s) = \int_{\mathbb{P}_\Sigma} \Omega_{\Sigma,\mathbb{P}}^{\text{tot}}(O_1 \cdots O_s),$$

(3.31)

where the integrand is as follows: On $U \subset \mathbb{P}_\Sigma$ with representatives $\{A_u\}_{u \in U}$,

$$\Omega_{\Sigma,\mathbb{P}}^{\text{tot}}(O_1 \cdots O_s)$$

(3.32)
= \prod_{a=1}^{d_N} d^2 u^a Z^\text{tot}_{\Sigma,P} \left( A_u; \delta \left( F_u(h) \right) \prod_{i=1}^{d_S} F_{u,b_i}(c) \prod_{a=1}^{d_N} \langle b_i, v_a(u) \rangle h O_1 \cdots h O_s \right),

in which \( Z^\text{tot}_{\Sigma,P}(A; \cdots) \) is given by

\[
\int_{\mathcal{G}_R/\mathcal{G}_P} \mathcal{D}h \ e^{ikI_{\Sigma,P}(A,hh^*) + I_{\Sigma,P}^{ad}(A,hh^*)} Z^{G,k}_{\Sigma,P}(A; \cdots) Z^{gh}_{\Sigma,P}(A; \cdots). (3.33)
\]

Note that the factor \( \frac{1}{\text{vol} \mathcal{G}_P} \) in (3.1) corresponds to the \( \mathcal{G}_P \)-quotient of \( \mathcal{G}_R \) here, though we need some care if \( d_S > 0 \) since the residual gauge fixing term may be gauge variant. In such a case, “integration over \( \mathcal{G}_P \) with the factor \( \frac{1}{\text{vol} \mathcal{G}_P} \)” leads to a new expression of the residual gauge fixing term that depends only on the \( \mathcal{G}_P \)-invariant combination \( hh^* \). The issue is illustrated in Appendix C for \( \Sigma = \mathbb{P}^1 \).

### 3.3 A Few Remarks On The Induced Systems

Two new systems of quantum fields are introduced above. One is the adjoint ghost system and the other is the system of field \([h] \in \mathcal{G}_R/\mathcal{G}_P \) with the classical action \( I_{\text{chg}} = -kI_{\Sigma,P}(A, hh^*) - I_{\Sigma,P}^{ad}(A, hh^*) \). The standard calculation of anomaly [24] shows (for simple \( H \)) that \( I_{\text{chg}} = -(\tilde{k} + 2h^*)I_{\Sigma,P}^H(A, hh^*) \) where \( \tilde{H} \) is the universal covering of \( H \) and \( \tilde{k} \) is the induced level such that \( \kappa e_{\mathfrak{g}}(XY) = \tilde{k} e_{\mathfrak{g}}(XY) \) for \( X, Y \in \mathfrak{h} \subset \mathfrak{g} \). Since \([h] \in \mathcal{G}_R/\mathcal{G}_P \) is a section of \( P \times_H \mathcal{H}_C/H \), we call this system the WZW model with the target \( \mathcal{H}_C/H \). Both systems are conformally invariant up to the anomalies \( c_{\text{gh}} = -2 \dim H \) and \( c_{\text{chg}} = \frac{\tilde{k} + 2h^*}{k + h^*} \dim H \) (if \( H \) is simple) respectively. Below, we shall give a brief description of the spaces of states.

#### Fermion Fock Space

Theory of free fermions on a Riemann surface has been studied by many authors. So, we only give a minimal account on the particular system of adjoint ghosts, referring for detail and generality to the references [25, 26, 27, 28, 29, 21].

The ghost Fock space \( \mathcal{F}_{gh}^{\text{ad}} \) is the Hilbert space spanned by states at the parametrized boundary \( S \) of a unit disc \( D_0 \) with several ghost insertions. The disc with flat gauge field and no insertion corresponds with respect to a horizontal gauge to the vacuum state \([0] \in \mathcal{F}_{gh}^{\text{ad}} \). Ghost fields at \( S \) act on \( \mathcal{F}_{gh}^{\text{ad}} \) in the standard way satisfying the anticommutation relations:

\[
\{ c^a_n, c^b_m \} = \{ b^a_n, b^b_m \} = 0, \quad \{ c^a_n, b^b_m \} = \delta_{n+m,0} \delta^a_b, \quad (3.34)
\]

and similarly for \( \bar{b}^a_n \) and \( \bar{c}^a_n \). Here, \( b^a_n \) and \( c^a_n \) are Fourier components of \( b_\Sigma(s_\Sigma(z)) \) and \( s^a(c(z)) \) respectively where \( \{ s_\Sigma \} \) is a horizontal frame of the adjoint bundle over \( S \) and
\{s^a\} is its dual. Hilbert space structure of \(\mathcal{F}_{gh}^{ad}\) are stated by \(b_{ma}^\dagger = \eta_{ab}c_{-m}^a\) where \(\eta_{ab} = \text{tr}_p(s_a s_b)\). A central extension \(\tilde{L}H_C^{ad}\) of \(LH_C\) acts on \(\mathcal{F}_{gh}^{ad}\) by the holomorphic (\(J\)) and the anti-holomorphic (\(\bar{J}\)) representations whose infinitesimal version could be read by looking at the expression (B.3) of the currents \(J\) and \(\bar{J}\) in terms of the ghost fields. The Cartan involution \(\gamma \mapsto \gamma^*\) of \(LH_C\) lifts to \(\tilde{L}H_C^{ad}\) so that \(J(\gamma^*) = \bar{J}(\gamma)^\dagger\) and \(\bar{J}(\gamma^*) = \bar{J}(\gamma)^\dagger\). The gauge transformation \(\gamma\) by the loop \(\gamma \in LH\) is provided by the diagonal action \(J(\tilde{\gamma}).\bar{J}(\tilde{\gamma})\) of any element \(\tilde{\gamma} \in \tilde{L}H_C^{ad}\) over \(\gamma\) with \(\tilde{\gamma}\tilde{\gamma}^* = 1\). The energy momentum tensor provides Virasoro generators \(\{L_n^g\}\) and \(\{\bar{L}_n^g\}\) acting on \(\mathcal{F}_{gh}^{ad}\) with central charge \(c_{gh}\) (see (B.4)).

**WZW model with the target \(H_C/H\)**

Let \(L^{ad}\) denote the line bundle \(\tilde{L}H_C^{ad} \times C \subset LH\) over \(LH\) on which the group \(\tilde{L}H_C^{ad}\) acts on the left and on the right. We pull back the line bundle \(L^{\text{wz}}_k \otimes L^{\text{ad}-1}_{\text{wz}}\) over \(LG_C \times LH\) by the map \(L(H_C/H) \rightarrow LG_C \times LH\); \([\gamma] \mapsto (\gamma^*, \gamma^*)\) where \(\gamma\gamma^*\) in the first factor is considered as a loop in \(\tilde{H}_C \subset GC\), the covering group of \(H_C \subset GC/Z_G\). Then, we obtain a line bundle denoted by \(L_{\text{chg}}\) over the loop space \(L(H_C/H)\). The state space of the system is the space \(\Gamma(L_{\text{chg}})\) of sections.

The left action of \(\tilde{L}H_C\) on \(L(H_C/H)\) induces a projective representation \(\mathcal{J}\) of \(\tilde{L}H_C\) on \(\Gamma(L_{\text{chg}})\). Unlike in the case of \(\Gamma_{\text{had}}(L^{k}_{\text{wz}})\), the group \(\tilde{L}H_C\) has only one projective representation on \(\Gamma(L_{\text{chg}})\). However, the infinitesimal version splits into two copies of a representation of the Kac-Moody algebra for \(H\). This is obtained by decomposing the complexification of the Lie algebra of \(LH\) into holomorphic and anti-holomorphic subspaces. These infinitesimal generators are identified with the currents. The Sugawara forms \(\{L_n^{gh}\}\) and \(\{\bar{L}_n^{gh}\}\) constructed by these act on (a subspace of) \(\Gamma(L_{\text{chg}})\) as Virasoro generators with central charge \(c_{gh}\).

To make things explicit, we consider a simple group \(H\) with the universal covering \(\tilde{H}\). Then, \(L_{\text{chg}}\) is isomorphic to the pull back of the line bundle \(L^{\text{wz}}_{\text{wz}} \otimes L^{\text{ad}}_{\text{ad}}\) over \(\tilde{L}H_C\) by \([\gamma] \mapsto \gamma\gamma^*\). The representation \(\mathcal{J}\) of \(\tilde{L}H_C\) on \(\Gamma(L_{\text{chg}})\) is defined by

\[
\mathcal{J}(\tilde{\gamma})\Phi([\gamma_0]) = \tilde{\gamma}\Phi([\gamma^{-1}\gamma_0])\tilde{\gamma}^*.
\] (3.36)

For use in the next section, choosing a maximal torus \(T\) and a chambre \(C\), we look at the state coming from the disc \(D_0\) with an insertion of the field of the form

\[
\left| e^{\lambda + 2\rho(b(h))}\right|^2 \quad \lambda \in P_+^{(k)},
\] (3.37)

where \(\rho\) is half the sum of positive roots of \(H\) and \(b(h)\) is the ‘Borel part’ of the Iwasawa decomposition \(h = b(h)U(h); b(h) \in B, U(h) \in \tilde{H}\). (The Iwasawa decomposition is
associated to the decomposition $\mathfrak{h}_c = \mathfrak{n} \oplus i\mathfrak{t} \oplus \mathfrak{h}$ of the Lie algebra where $\mathfrak{n}$ is spanned by positive root vectors. In this paper, we call the factor corresponding to $\mathfrak{n} \oplus i\mathfrak{t}$ the ‘Borel-part’. The state at the boundary $S = \partial D_0$ is given by

$$\Phi_{-\lambda-2\rho}(\gamma) = e^{(\tilde{k}+2n^\vee)I_D_0(b)b^\vee} e^{\lambda+2\rho(b(0))} |2, (3.38)$$

where $\gamma$ is bounded by a holomorphic function $b$ on $D_0$ with $b(0) \in B$. A calculation as in §2 shows that $\gamma\Phi_{-\lambda-2\rho} = \Phi_{-\gamma\lambda-2\rho}$ for $\gamma \in \Gamma_C$. The argument is easily generalized to the case in which $H$ is not simple.

The Total System

Space of states of the total system — the combined system of the WZW model and the two new systems is given by

$$\mathcal{H}^{\text{tot}} = \mathcal{H}^{G,k} \otimes \Gamma(L_{\text{chg}}) \otimes \mathcal{F}_{\text{ad}}.$$ (3.39)

The left and the right representations of the Kac-Moody algebra $\text{Lie}(L H_C)$ on the three spaces determine the representations $J^{\text{tot}}$ and $\tilde{J}^{\text{tot}}$ of the loop algebra $\text{Lie}(L H_C)$ on $\mathcal{H}^{\text{tot}}$. In the similar way, the representations $\{L_n^{\text{tot}}\}$ and $\{\tilde{L}_n^{\text{tot}}\}$ of the Virasoro algebra is defined with central charge $c_{\text{tot}} = c_{G,k} + c_{\text{chg}} + c_{\text{gh}}$.

Another ingredient is the BRST operator which is the zero mode of the meromorphic and gauge invariant fermionic current $J_{WZW} + J_{\text{chg}} + \frac{1}{2}J_{\text{gh}}$ where $J_{WZW}$, $J_{\text{chg}}$ and $J_{\text{gh}}$ are the $H$-currents of the three sectors. It is nilpotent and may be used to specify the physical states or fields by determining the cohomology group under suitable equivariant condition. In this paper, however, we do not use it but argue in the following way.

3.4 Gauge Invariant Local Fields

We specify the set of $G_P$-invariant fields $O$ in the WZW model and describe the dressed fields $hO$. Recall that the fields and states in $\mathcal{H}^{G,k}$ are in one to one correspondence under $O \leftrightarrow \Phi_O = Z_{D_0}(0; O)$. The gauge invariance condition on $O$ is equivalent to the following conditions on $\Phi_O$:

$$\left(J_0(v) + \tilde{J}_0(v)\right) \Phi_O = 0 \quad \text{for} \quad v \in \mathfrak{h}, \quad (3.40)$$

$$J_n(v)\Phi_O = \tilde{J}_n(v)\Phi_O = 0 \quad \text{for} \quad v \in \mathfrak{h}_C \quad \text{and} \quad n = 1, 2, \ldots, \quad (3.41)$$

where $J_n(v)$ and $\tilde{J}_n(v)$ are infinitesimal generators of $J$ and $\tilde{J}$ corresponding to the tangent vector to the curve $t \mapsto \exp\{-I_{D_0}(e^{iz^\vee}v)\}$ in $\tilde{L}G_C$.

To distinguish the space $\mathcal{H}_{\text{inv}}$ of states satisfying these conditions, we choose maximal tori and chambrés $(T_G, C_G)$ for $G$ and $(T, C)$ for $H$ and consider the decomposition of
the integrable representations $L_{\Lambda}^{G,k}$ of $\tilde{L}G$ à la Goddard-Kent-Olive [3]. The restriction of $\tilde{L}G$ to $L\tilde{H} \subset LG$ is a central extension $\tilde{L}H$ of $L\tilde{H}$ where $\tilde{H} \subset G$ is the covering group of $H \subset G/Z_G$. We decompose $L_{\Lambda}^{G,k}$ into irreducible representations of the subgroup $L\tilde{H}$:

$$L_{\Lambda}^{G,k} = \bigoplus_{\lambda} B_{\lambda} \otimes L_{\lambda}^{\tilde{H},k}, \quad (3.42)$$

in which $B_{\lambda}$ is the subspace of $L_{\Lambda}^{G,k}$ consisting of highest weight vectors of weight $(\lambda, \tilde{k})$ with respect to $L\tilde{H}$ where $\tilde{k}$ is the induced level. We denote by $\mathcal{H}_\lambda^\Lambda$ the subspace of $L_{\Lambda}^{G,k}$ corresponding to the subspace $B_{\lambda} \otimes 
\overline{B_{\lambda}}$ of $L_{\Lambda}^{G,k} \otimes \overline{L_{\Lambda}^{G,k}}$. Each state $\Phi \in \mathcal{H}_\lambda^\Lambda$ generates an irreducible $J_0(\tilde{H}) \times \tilde{J}_0(\tilde{H})$-module $E_\lambda(\Phi) \subset \mathcal{H}_\lambda^\Lambda$ which is isomorphic to the tensor product $V_\lambda \otimes \overline{V_\lambda}$ of the irreducible $\tilde{H}$-module $V_\lambda$ of highest weight $\lambda$ and its dual $\overline{V_\lambda}$. Choosing a base $\{e_m\}$ of $V_\lambda$ and the dual base $\{e^m\}$ of $V_\lambda^*$, we denote by $\Phi^m \in E_\lambda(\Phi)$ the vector corresponding to $e_m \otimes e^m \in V_\lambda \otimes \overline{V_\lambda}$. Then, the $\tilde{H}$-invariant element $\sum_m \Phi^m \in E_\lambda(\Phi)$ satisfies (3.40) and (3.41). In this way, the space $\mathcal{H}_{\text{inv}}$ can be identified with the subspace

$$\mathcal{H}_{\text{inv}} = \bigoplus_{\Lambda,\lambda} \mathcal{H}_\lambda^\Lambda \quad (3.43)$$

of $\mathcal{H}_{\Lambda}^{G,k}$ spanned by highest weight states of left-right equal weights with respect to $J(L\tilde{H}_C) \times \tilde{J}(L\tilde{H}_C)$.

Let $O_{\Phi}^\phi_m$ denote the field corresponding to the state $\Phi^m$ and we consider it as a matrix element of a field $O_\Phi$ valued in $\text{End}(V_\lambda)$. Then, the gauge invariant field $O_\Phi$ corresponding to the state $\frac{1}{\dim V_\lambda} \sum_m \Phi^m$ is expressed as

$$O_\Phi = \frac{1}{\dim V_\lambda} \text{tr}_{V_\lambda}(O_\Phi). \quad (3.44)$$

Since $J_0(\tilde{h})\tilde{J}_0(\tilde{h})\Phi^m_m$ with $\tilde{h} \in \tilde{H}_C$ is expanded as $\sum \tilde{h}^{n_m'} \Phi^m_{m'} \tilde{h}^{n_m'}$, the dressed field for $O_\Phi$ is given by

$$hO_\Phi = \frac{1}{\dim V_\lambda} \text{tr}_{V_\lambda}(O_\Phi hh^*). \quad (3.45)$$

**Remark.** One can construct the Sugawara forms $\{L_n^{G,k}\}$ from the current algebra $\text{Lie}(L\tilde{G}_C)$ and also from the subalgebra $\text{Lie}(\tilde{L}\tilde{H}_C)$: $\{L_n^{\tilde{H},k}\}$. The difference $L_n^{\text{GKO}} = L_n^{G,k} - L_n^{\tilde{H},k}$ commutes with $\text{Lie}(L\tilde{H}_C)$ and hence acts on $B_{\lambda}^\Lambda$ [3]. Thus we have a Virasoro actions $\{L_n^{\text{GKO}}\}$ and $\{\tilde{L}_n^{\text{GKO}}\}$ on $\mathcal{H}_{\text{inv}}$. As is shown in [3], these generators coincides with the generators $\{L_n^{\text{tot}}\}$ and $\{\tilde{L}_n^{\text{tot}}\}$ up to BRST-exact terms.
4. Actions Of The Fundamental Group

A prototype (2.40) of the topological identity (1.4) is obtained in §2.3, but we recognize two kinds of gaps to be filled. One is that the field $O_\Lambda$ in (2.40) is not gauge invariant (for $H$ non-abelian) but corresponds to a highest weight state. The other is that (2.40) holds for certain gauge fields of special configurations over a neighborhood of the insertion point, while (1.4) is an equation in the quantum gauge theory. In §3, we have developed a method to integrate over gauge fields and obtained a formula (3.31) that expresses a correlation function as an integral over the moduli space of semi-stable $H_C$-bundles. If we are to use this formula to prove (1.4), we must find some relation of the moduli spaces $N_P$ and $N_{P,\gamma}$ of semi-stable bundles of different topology.

In this section we shall fill these gaps by taking into account the variety of choices of highest weight conditions — the flag manifold: We express a correlator as an integral over a certain moduli space of holomorphic bundles with a flag at the insertion point. This leads us to define two actions of the fundamental group $\pi_1(H)$; one on the set of gauge invariant local fields and the other on the moduli spaces of bundles with flags.

4.1 The Flag Partner

As a step to the new expression of correlators, a dressed gauge invariant local field $hO$ is expressed as an integral over the flag manifold of $H$. When it is inserted into a correlation function of the total system, the integrand is given by contour integrals of ghost currents encircling a field $\hat{O}$ of ghost number $|\Delta|$ (the number of roots of $H$) which is referred to as the flag partner of $O$.

Flag Manifold and the Borel-Weil Theorem

We recall the representation theory of compact groups due to Borel and Weil. Let $Fl(H)$ be the ensemble of choices of maximal tori and chambres:

$$Fl(H) = \left\{ (T, C) ; \begin{array}{l} T \text{ is a maximal torus of } H \\ \text{and } C \text{ is a chambre in } \mathfrak{t} \end{array} \right\} . \quad (4.1)$$

A choice $(T, C) \in Fl(H)$ determines an identification $Fl(H) = H/T$ which makes $Fl(H)$ a compact manifold called the flag manifold of $H$. Furthermore, $Fl(H)$ becomes a homogeneous complex manifold since the embedding $H \hookrightarrow H_C$ induces the isomorphism $H/T \cong H_C/B$ where $B$ is the Borel subgroup of $H_C$ determined by $(T, C)$.

A weight $\lambda \in P$ gives a character $e^\lambda : T \rightarrow U(1)$ by $e^{2\pi iv} \mapsto e^{2\pi i\lambda(v)}$ and its extension $e^\lambda : B \rightarrow \mathbb{C}^*$ defines a homogeneous holomorphic line bundle

$$L_{-\lambda} = H_C \times_B \mathbb{C} \longrightarrow Fl(H), \quad (4.2)$$
by the equivalence relation \((hb, c) \sim (h, e^{-\lambda}(b)c)\) where \(h \in H_C, b \in B\) and \(c \in C\). We denote by \(h\cdot c \in L_{-\lambda}\) the equivalence class represented by \((h, c) \in H_C \times C\). The Borel-Weil theorem states that the space \(H^0(Fl(H), L_{-\lambda})\) of holomorphic sections is an irreducible \(H_C\)-module \(V_{\lambda}\) of highest weight \(\lambda^* = -\omega_\lambda\), which is non-zero if and only if \(\lambda\) takes positive values on \(C\) (see [31] and also [32, 33]).

The line bundle \(L_{-\lambda}\) is equipped with an \(H\)-invariant hermitian metric \((., .)_{-\lambda}\) such that an element \(h\) of \(H\) determines a unitary frame \(h \cdot 1; (h \cdot c_1, h \cdot c_2)_{-\lambda} = c_1c_2\). There also exists an \(H\)-invariant volume form \(\Omega\) on \(Fl(H) = H/T\). These induces the following \(H\)-invariant hermitian inner product on the space \(H^0(Fl(H), L_{-\lambda})\):

\[
(\psi_1, \psi_2)_{Fl(H)} = \frac{1}{\text{vol}Fl(H)} \int_{Fl(H)} (\psi_1, \psi_2)_{-\lambda}\Omega.
\]

(4.3)

Let \(\{e_m; m \in \tilde{P}_\lambda\}\) be an orthonormal base of \(V_\lambda\) consisting of weight vectors where \(\tilde{P}_\lambda\) is an indexing set. We always take the weight \(\lambda\) itself as the index for the highest weight vector. Denoting the matrix element \((e_{m_1}, he_{m_2})\) by \((h)_m^{m_2}\), we put

\[
\psi^m(hB) = h \cdot (h)_m^m,
\]

(4.4)

for \(m \in \tilde{P}_\lambda\). Then, \(\{\psi^m; m \in \tilde{P}_\lambda\}\) forms an orthogonal base of \(H^0(Fl(H), L_{-\lambda})\):

\[
(\psi^{m_1}, \psi^{m_2})_{Fl(H)} = \frac{1}{\text{vol}Fl(H)} \int_{Fl(H)} (\psi^{m_1}, \psi^{m_2})_{-\lambda}\Omega
= \frac{1}{\text{vol}H} \int_H (\psi^{m_1}(hB), \psi^{m_2}(hB))_{-\lambda} dh
= \frac{1}{\text{vol}H} \int_H \delta^{m_1,m_2}(h)^{m_2} dh = \frac{1}{\dim V_\lambda} \delta^{m_1,m_2},
\]

(4.5)

where \(dh\) is the Haar measure of \(H\) and the Peter-Weyl theorem is used.

**Integral Expression of Gauge Invariant Fields**

Recall that (see §3.4) to each \(\Phi \in \mathcal{H}_\lambda\) is associated a gauge invariant field \(O_\Phi = \frac{1}{\dim V_\lambda} \text{tr}_{V_\lambda}(O_\Phi)\) or the dressed field \(hO_\Phi = \frac{1}{\dim V_\lambda} \text{tr}_{V_\lambda}(O_\Phi hh^*)\). In the following argument, \(\Phi\) is fixed all through and will not usually be mentioned.

We now express \(hO\) as an integral over the flag manifold \(Fl(H)\). We introduce a field \(\Omega(hh^*)\) with values in differential forms on \(Fl(H)\) of top degree. At the point \(h_1B \in Fl(H)\) represented by \(h_1 \in H\), it is expressed as

\[
\Omega(hh^*)|_{h_1B} = h_1O_\lambda^\overline{\lambda} \left| e^{\lambda + 2\rho}(b(h_1^{-1}h))\right|^2 \Omega|_{h_1B},
\]

(4.6)

where \(b(h_1^{-1}h) \in B\) is the ‘Borel-part’ of the Iwasawa decomposition of \(h_1^{-1}h \in H_C\). Let \(L_{h_1^{-1}}: Fl(H) \to Fl(H)\) be the left translation by \(h^{-1}\). The relation

\[
L_{h_1^{-1}}\Omega|_{h_1B} = \left| e^{2\rho}(b(h_1^{-1}h))\right|^2 \Omega|_{h_1B},
\]

(4.7)
which shall be proved shortly shows
\[
L^*_h \Omega(hh^*)|_{h^{-1}h_1B} = h_1 O^\lambda_\chi l \left| e^\lambda (b(h_1^{-1}h)) \right|^2 \Omega|_{h^{-1}h_1B}.
\] (4.8)

The definition of \(b(h_1^{-1}h)^{-1}\) says that there is a representative \(U \in H\) of \(h^{-1}h_1B\) such that
\[
h^{-1}h_1 O^\lambda_\chi = U O^\lambda_\chi e^{-\lambda}(b(h_1^{-1}h)) \right|^2
\]
and hence (4.8) gives
\[
L^*_h \Omega(hh^*|_{UB} = h U O^\lambda_\chi \Omega|_{UB} = \sum_{m,m} h O^\lambda_m (U^{-1})^\lambda_m (U)^m \Omega|_{UB}.
\] (4.9)

This amounts to the following identity of top differential forms:
\[
L^*_h \Omega(hh^*) = \sum_{m,m} h O^\lambda_m (\psi^m, \psi^m)_{-\lambda} \Omega,
\] (4.10)
where \(\psi^m\) is given in (4.4). Due to the orthogonality (4.5), it follows that
\[
\frac{1}{\text{vol} Fl(H)} \int_{Fl(H)} \Omega(hh^*) = \frac{1}{\dim V_\lambda} \text{tr}_{v_\lambda} (O hh^*).
\] (4.11)

**Proof of the relation (4.7)**. It is enough to prove \(L^*_h \Omega|_B = |e^{-2\rho}(b)|^2 \Omega|_B\) for \(b \in B\). Since the holomorphic tangent space of \(Fl(H)\) at \(B\) is isomorphic to \(\mathfrak{h}_\mathbb{C}/\mathfrak{b}\), we have only to show that \(e^{-2\rho}(b)\) is the determinant of \(\text{ad}_{-}(b) : \mathfrak{h}_\mathbb{C}/\mathfrak{b} \rightarrow \mathfrak{h}_\mathbb{C}/\mathfrak{b}\). In view of \(2\rho = \sum_{\alpha > 0} \alpha\), the proof is now trivial since we can order the base of \(\mathfrak{h}_\mathbb{C}/\mathfrak{b}\) consisting of negative root vectors so that \(\text{ad}_{-}(b)\) is represented by an upper triangular matrix.

**The Flag Partner**

Suppose that \(hO\) is inserted at \(x \in \Sigma\) in a correlator \(Z^\text{tot}_{\Sigma,P}(A; \cdots)\) of the total system (3.3), where we assume that the background gauge field \(A\) is chosen to be flat over a disc \(D_0 \subset \Sigma\) centered by \(x\). Then, (1.11) leads to an integral expression of \(hO(x)\) over the flag manifold \(Fl(P_x) = P_x/T \cong RC_x/B\) of the fibre \(P_x\) of \(P\) over \(x\): The integrand \(\Omega_x(hh^*)\) is expressed at the flag \(f \in Fl(P_x)\) by
\[
\Omega_x(hh^*)|_f = O^\lambda_\chi(f) \left| e^{\lambda + 2\rho}(b_f(h)) \right|^2 \Omega|_f.
\] (4.12)

Here, \(O^\lambda_\chi(f) = O_{\Phi^\lambda}(f)\) is the field corresponding to \(\Phi \in \mathcal{H}^\lambda_\chi\) and \(b_f(h)\) is the ‘Borel-part’ of the Iwasawa decomposition of \(h(x)\), both with respect to the horizontal section \(s\) of \(P|_{D_0}\) with \(s(x)B = f\). \(\Omega\) is the invariant volume form on \(Fl(P_x)\).

This measure can be rewritten using ghost fields. Let \(U_F\) be an open subset of \(Fl(P_x)\) with complex coordinates \(f^1, \cdots, f^{|\Delta|}\) and a family \(\{\sigma_f\}_{f \in U_F}\) of holomorphic sections of \((P_C, \partial_A)|_{D_0}\) such that \(\sigma_f(x)B = f\). Then, the symbol \((\partial \sigma_f/\partial f^a)\sigma_f^{-1}\) determines a
holomorphic section \( \nu_a(f) \) of \( (\text{ad}P_C, \partial_A)|_{D_0} \). Using the singular behavior of the product of ghosts, we obtain the following expression:

\[
\Omega_x(hh^*)|_f = \prod_{\beta=1}^{\Delta_x} d^2 f^\beta \oint_x b^{\nu_\beta}(f) \oint_x b^{\bar{\nu}_\beta}(f) \hat{O}(f),
\]

(4.13)

where

\[
\hat{O}(f) = O^{\lambda}_x(f) \left| e^{\lambda+2\rho}(b(f)) \right|^2 \prod_{-\alpha<0} c^{-\alpha}(x) c^{-\alpha}(x).
\]

(4.14)

Here and henceforth, we denote the normalized contour integral \( \frac{1}{2\pi i} \oint \) by \( \oint \). In the expression, \( c^{-\alpha}(x) \) is the coefficient of the ghost; \( c(x) = \sum s_n(x) e^x \) where \( \{s_n(x)\} \) is the frame associated to any \( s(x) \in P_x \) representing \( f \) and to the base of \( h_C \) including root vectors \( \{e_a\}_{\alpha \in \Lambda} \) normalized by \( \text{tr}(e_a e_{\beta}) = \delta_{\alpha+\beta,0} \). We call this field \( \hat{O}(f) \) the flag partner of \( O \) associated to \( f \in Fl(P_x) \).

**Remark.** Construction/determination of BRST complex/cohomology is a standard method of determining the space of physical states of a theory with gauge symmetry. In the literature (see and references therein), there are several constructions which seem to come from the gauged WZW model. The cohomology groups include as a non-trivial element, the state of the form

\[
|\lambda\rangle^{G,k} \otimes | -\lambda - 2\rho \rangle^{H_k/H} \otimes \prod_{-\alpha<0} c_0^{-\alpha} |0\rangle^{gh},
\]

(4.15)

where \( |\lambda\rangle^{G,k} \) is a state in \( B^\Lambda_A \subset L^{G,k}_\Lambda \) for some \( \Lambda \); \( | -\lambda - 2\rho \rangle^{H_k/H} \) is a high weight state with weight \( -\lambda - 2\rho, -k^e \) in a suitably chosen Lie \( (LH_C) \)-module and \( |0\rangle^{gh} \) is the natural vacuum of the ghost Fock space. This state seems to correspond to (the left moving part) of our flag partner \( \hat{O} \).

4.2 A New Integral Expression

We consider the correlation function \( Z_{\Sigma,P}(O_1 \cdots O_s O(x)) \) of gauge invariant fields. As is seen in §3.2, it is expressed as an integral of \( \Omega^{\text{tot}}_{\Sigma,P}(\cdots O(x)) \) on an open dense subset \( \mathcal{N}_P^\Sigma \) of the moduli space of semi-stable \( H_C \)-bundles. Let \( U \subset \mathcal{N}_P^\Sigma \) be provided with a holomorphic family \( \{A_u\}_{u \in U} \) of representative gauge fields. The result of §4.1 shows that the following measure on \( U \times Fl(P_x) \) reproduces \( \Omega^{\text{tot}}_{\Sigma,P}(\cdots O(x))|_U \) after the integration along each \( Fl(P_x) \):

\[
\tilde{\Omega}^{\text{tot}}_{\Sigma,P,x}(\cdots \hat{O})_U = \prod_{a=1}^{d_N} d^2 u^a \prod_{\alpha=1}^{\Delta_u} d^2 f^\alpha Z^{\text{tot}}_{\Sigma,P}(A_u; |\text{rgf}_{A_u}(c, h)|^2 h(\cdots)
\]

(4.16)

\[\times \prod_{a=1}^{\Delta_u} \langle b, \nu_a(u) \rangle \langle \bar{b}, \bar{\nu}_a(u) \rangle \prod_{\alpha=1}^{\Delta_u} \oint_x b^{\nu_\alpha}(f) \oint_x b^{\bar{\nu}_\alpha}(f) \hat{O}(f) \rangle.
\]
where $\tilde{O}$ is the flag partner of $O$, $h(\cdots)$ is the dressed insertion $hO_1 \cdots hO_s$ and $|\text{rgf}_{A_u}(c, h)|^2$ denote the residual gauge fixing term:

$$|\text{rgf}_{A_u}(c, h)|^2 = \delta^{(2dS)} \left( F_u(h) \prod_{i=1}^{dS} F_{u,h}^i(c) \tilde{F}_{u,h}(\tilde{c}) \right).$$  

(4.17)

We ask whether this form on $U \times Fl(P_x)$ extends to a globally defined form on some well-defined flag manifold bundle over $N^\circ_P$. Below, we shall see that the answer is generally no but $\tilde{\Omega}_{\Sigma,P,x}^{\text{tot}}(\cdots \tilde{O})_U$ determines a new form $\Omega_{\Sigma,P,x}^{\text{tot}}(\cdots \tilde{O})$ which is globally defined on a geometric object $N^\circ_{P,x}$ associated to $P$ and $x$.

**Transformation Properties of $\tilde{\Omega}_{\Sigma,P,x}^{\text{tot}}$**

Let $\{A_{1u}\}_{u \in U}$ and $\{A_{2u}\}_{u \in U}$ be families of representatives that are related by

$$A_{1u} = A_{2u}^{h_{21,u}},$$

(4.18)

by a family $\{h_{21,u}\}_{u \in U}$ of chiral gauge transformations. The groups $S_{1u} = \text{Aut} \tilde{O}_{A_{1u}}$ of symmetries are then related by $S_{1u} = h_{21,u}^{-1} S_{2u} h_{21,u}$. Hence, if $\{F_{1u}\}$ is a family of residual gauge-fixing functions for $\{S_{1u}\}$, $F_{2u}(h) = F_{1u}(h^{-1} h)$ determines a family $\{F_{2u}\}$ of residual gauge-fixing functions for the symmetries $\{S_{2u}\}$. Since the chiral anomaly is absent in the total system, it follows that

$$Z_{\Sigma,P}^{\text{tot}}(A_{1u} ; |\text{rgf}_{A_{1u}}(c, h)|^2 h(\cdots) \prod (b; \tilde{b}) \tilde{O}(f))
= Z_{\Sigma,P}^{\text{tot}}(A_{2u} ; |\text{rgf}_{A_{2u}}(c, h)|^2 h(\cdots) \prod (h^{-1} h; h_{21,u}^* \tilde{b}) h_{21,u} \tilde{O}(f)),$$

(4.19)

where $\prod (b; \tilde{b})$ is any functional of $b-\tilde{b}$ and $h^{-1} h$ is the coadjoint action of $h^{-1} \in G_R$ on the field with values in $(\text{ad}P_C)^*$. Making use of the Iwasawa decomposition of $h_{21,u}(x) \in P_x \times H \text{H}_C$ with respect to the flag $f \in Fl(P_x)$, we can see that

$$h_{21,u} \tilde{O}(f) = \tilde{O}(h_{21,u} f),$$

(4.20)

where the action of $G_R$ on $Fl(P_x) = P_C / B$ is induced by the action on $P_C / B$.

Now, it is enough to note the relation

$$\delta A_{1u}^{01} = h_{21,u}^{-1} \delta A_{2u}^{01} h_{21,u} + \tilde{O}_{A_{1u}}(h_{21,u}^{-1} \delta h_{21,u}) ,$$

(4.21)

to see that the form $\tilde{\Omega}_{\Sigma,P,x}^{\text{tot}}(\cdots \tilde{O})$ on $\{1\} \times U \times Fl(P_x)$ with the backgrounds $\{A_{1u}\}_{u \in U}$ coincides with the one on $\{2\} \times U \times Fl(P_x)$ with the backgrounds $\{A_{2u}\}_{u \in U}$, under the following identification of the two spaces:

$$ (1, u, f) \longleftrightarrow (2, u, h_{21,u} f) .$$

(4.22)
The Space $\mathcal{N}_{P,x}^\circ$

Let $\{U_i\}$ be an open covering of $\mathcal{N}_{P}^\circ$ such that each $U_i$ is provided with a holomorphic family $\{A_{i,u}\}_{u \in U_i}$ of representatives. If $U_i$ and $U_j$ intersect, we can choose a family $\{h_{i,u}\}_{u \in U_i \cap U_j}$ of chiral gauge transformations such that $A_{j,u} = A_{i,u}^h$.

If the symmetry group $S_{i,u} = \text{Aut} \bar{\Lambda}_{i,u}$ is trivial everywhere, the families $\{h_{i,u}\}_{i,j}$ necessarily satisfy the triangle identities:

$$h_{i,j}h_{j,k} = h_{i,k}, \quad \text{for } u \in U_i \cap U_j \cap U_k.$$  \hspace{1cm} (4.23)

Then, the identification rules as $[1.22]$ glue the spaces $\{i\} \times U_i \times \text{Fl}(P_x)$ and forms $\tilde{\Omega}^{\text{tot}}_{\Sigma,P,x}(\cdots \tilde{\Omega})_{U_i}$ together and define an $\text{Fl}(P_x)$-bundle $\mathcal{N}_{P,x}^\circ$ over $\mathcal{N}_{P}^\circ$ and a measure $\Omega^{\text{tot}}_{\Sigma,P,x}(\cdots \tilde{\Omega})$ of it.

In general, the triangle identity $(4.23)$ does not hold but modulo actions of the symmetry groups. In such a situation, it is a natural idea to consider the quotient of $\text{Fl}(P_x)$ by the symmetry group $S_u$. Then, we expect that the integration of $\tilde{\Omega}^{\text{tot}}_{\Sigma,P,x}$ along the $S_u$-orbits determine a globally defined form on some fibre bundle over $\mathcal{N}_{P}^\circ$ having the quotient $S_u \backslash \text{Fl}(P_x)$ as the fibre over $u$. However, $S_u$ is generically non-compact and the quotient space $S_u \backslash \text{Fl}(P_x)$ is not even Hausdorff. At this stage, we assume that we can find an open dense subset $\text{Fl}_{S_u}(P_x)$ of $\text{Fl}(P_x)$ consisting of $S_u$-orbits of maximum dimension such that the family $\cup_{u \in U} \{u\} \times S_u \backslash \text{Fl}_{S_u}(P_x)$ of quotients is given a good geometric structure (such as manifold or orbifold). Under this assumption, the spaces $\{i\} \times \cup_{u \in U_i} \{u\} \times S_{i,u} \backslash \text{Fl}_{S_{i,u}}(P_x)$ are glued together by the identification rules as $[1.22]$ and result in a space denoted by $\mathcal{N}_{P,x}^\circ$.

Note that the space $\mathcal{N}_{P,x}^\circ$ can be considered as a subset of the quotient of $\mathcal{A}_{P,x} = \mathcal{A}_P \times \text{Fl}(P_x)$ by $\mathcal{G}_K$ (where $\mathcal{G}_K$ acts on $\mathcal{A}_{P,x}$ by $h : (A, f) \mapsto (Ah, h^{-1}f)$). In §4.3, we shall identify elements of $\mathcal{A}_{P,x}/\mathcal{G}_K$ with certain holomorphic objects over the Riemann surface and make sure in simple cases that the assumptions involved in this argument hold true.

Local Coordinatization of $\mathcal{N}_{P,x}^\circ$

Recall that $d_N$ and $d_S$ denote the dimensions of the moduli space $\mathcal{N}_{P}^\circ$ and the symmetry group $S_u$ for $u \in \mathcal{N}_{P}^\circ$ respectively. We denote by $\hat{d}_s$ the dimension of the group $S_{u,f}$ of symmetries of $A_u$ that fix the flag $f$. Then, the dimension $\hat{d}_N$ of the space $\mathcal{N}_{P,x}^\circ$ is given by $\hat{d}_N = d_N + |\Delta_+| - d_S + \hat{d}_s$.

We assume without proof that the following holds: For a generic point $v_0 \in \mathcal{N}_{P,x}^\circ$, we can find a coordinatized neighborhood $\mathcal{V}$ of $v_0$ in $\mathcal{N}_{P,x}^\circ$ so that there is a family $\{(A_v, f_v)\}_{v \in \mathcal{V}} \subset \mathcal{A}_{P,x}$ of representatives depending holomorphically on the coordinates.
\( v_1, \ldots, v_v \). We choose families \( \{ \sigma_0(v) \}, \{ \sigma_\infty(v) \} \) of holomorphic trivializations on \( U_0, U_\infty \) such that \( \sigma_0(v)B = f_v \), where \( U_0 \) is a neighborhood of \( x \) and \( U_\infty \) is a neighborhood of \( \Sigma - U_0 \). We assume that the transition function \( h_{\infty 0}(v) \) of \( \sigma_0(v) \) and \( \sigma_\infty(v) \) depends holomorphically on \( v \).

We choose the coordinate system \( v_1, \ldots, v_v \) in such a way that \( A_v \) and \( \sigma_\infty(v) \) depend only on the first \( d_\Sigma \)-tuples \( v = (v_\nu, \ldots, v_v) \). The symmetry group and the residual gauge fixing function for \( A_v \) are then denoted as \( S_\Sigma \) and \( F_\Sigma \) respectively. We also choose a family \( \{ h_{v,t} \}_{v,t} \subset G_k \) parametrized by \( v \in V \) and \( t = (t_1, \ldots, t_v) \) such that the family \( \{ h_{v,t} \}_{v,t} \) for a fixed \( v \) is in \( S_\Sigma \) and is transversal to \( S_\Sigma \), \( f \)-orbits. Then, the holomorphic sections

\[
\nu_{d_\Sigma + \alpha}(v) = \sigma_0(v) \cdot h_{\infty 0}(v)^{-1} \frac{\partial}{\partial v_{d_\Sigma + \alpha}} h_{\infty 0}(v) \quad \alpha = 1, \ldots, d_\Sigma - d_\nu,
\]

\[
\nu_i(v, t) = (\frac{\partial}{\partial t_i}) h_{v,t}^{-1} \quad i = 1, \ldots, d_\nu - d_S,
\]

of \( \text{ad} P \big|_{U_v} \) provide a base of the tangent space of \( Fl(P_x) \) at \( h_v, t \).

Let \( s_0 \) and \( s_\infty \) be fixed sections of \( P|_{U_0} \) and \( P|_{U_\infty} \) and we put \( s_0 = \sigma_0(v) h_0(v) \) and \( s_\infty = \sigma_\infty(v) h_{\infty}(v) \). Since the connection \( A_v \) is represented by \( \tilde{\partial}_A \sigma_i = s_i \cdot h_1(v)^{-1} \tilde{\partial} h_1(v) \) on \( U_I \) \( (I = 0, \infty) \), the variation of \( A_v \) is expressed as \( \delta A_v^0 = \tilde{\partial}_A(s_i \cdot h_1(v)^{-1} \partial h_1(v)) \). For a holomorphic differential \( b \) valued in \( \text{ad} P \), we thus have

\[
\frac{1}{2\pi i} \int_{\Sigma} b \delta A_v^0 \frac{\partial}{\partial \nu} = \frac{1}{2\pi i} \int_{\Sigma} b \left( s_0 \cdot h_0(v)^{-1} \delta h_0(v) - s_\infty \cdot h_{\infty}(v)^{-1} \delta h_{\infty}(v) \right)
\]

where the contour encircles the point \( x \).

The measure \( \tilde{\Omega}_{\Sigma, P, x}^\nu (O) \) is then expressed as

\[
\tilde{\Omega}_{\Sigma, P, x}^\nu (O) = \prod_{\alpha=1}^{d_\nu} d^2 v^\alpha \prod_{i=1}^{d_S - d_S} Z_{\Sigma, P}^\nu \left( A_v \; | \; \text{rgf}_{A_v}^\nu (c, h) \right)^2 h(\cdots)
\]

\[
\times \prod_{\alpha=1}^{d_\nu} \phi_b \nu_\alpha(v) \phi_b \tilde{\nu}_\alpha(v) \prod_{i=1}^{d_S - d_S} \phi_b h_i(v, t) \phi_b \tilde{h}_i(v, t) \phi_b \tilde{f}_v(v, t)
\]

where \( \nu_\alpha(v) \) are the holomorphic sections of \( \text{ad} P \big|_{U_v \cap U_\infty} \)

\[
\nu_\alpha(v) = \sigma_0(v) \cdot h_{\infty 0}(v)^{-1} \frac{\partial}{\partial v_\alpha} h_{\infty 0}(v) \quad \alpha = 1, \ldots, d_\nu.
\]

Due to the absence of chiral anomaly, we can rewrite the above measure by

\[
\prod_{\alpha=1}^{d_\nu} d^2 v^\alpha \prod_{i=1}^{d_S - d_S} Z_{\Sigma, P}^\nu \left( A_v \; | \; \text{rgf}_{A_v}^\nu (h_{v,t} c, h_{v,t} h) \right)^2 h(\cdots)
\]

\[
\times \prod_{\alpha=1}^{d_\nu} \phi_b \nu_\alpha(v) \phi_b \tilde{\nu}_\alpha(v) \prod_{i=1}^{d_S - d_S} \phi_h h_i(v, t) \phi_h \tilde{h}_i(v, t) \phi_h \tilde{f}_v(v, t)
\].
The New Expression

Now we integrate over each $S_v$-orbit in $Fl_{\Sigma}(P_x)$. We exchange the order of integration; we perform $\int dt_i$’s before $\int Dh$. Then, the delta function $\delta^{(2d_S)}(F_v(h_{v,i}, h))$ in the residual gauge fixing term serves another delta function of lower dimension $2d_S$ multiplied by a certain determinant factor. On the other hand, deforming the contours of the integrals $\oint h_{v,t}b_{\nu}i$ so that they encircle the $2d_S$ $c\bar{c}$-insertions in the residual gauge fixing term, we get another $2d_S$ $c\bar{c}$-insertions multiplied by the determinant which is reciprocal to the one from the delta function. Thus, the integration over $t_i$’s serves the following residual gauge fixing term for $(A_v, f_v)$:

$$|\text{rgf}_{A_v, f_v}(c, h)|^2 = \delta^{(2d_S)}(F_v(h)) \prod_{i=1}^{d_S} F_{v, h_i}(c) \bar{F}_{v, h}(\bar{c})$$

(4.31)

where $F_v : G_{\mathbb{R}} \to C^{d_S}$ is a gauge fixing function for $S_{\Sigma,v} \subset G_{\mathbb{R}}$.

Finally, we have reached to the following measure on $\mathcal{V}$:

$$\Omega^\text{tot}_{\Sigma, P, x}(O_1 \cdots O_s \hat{O}) = \prod_{\Lambda=1}^{d_N} d^{2v_{\Lambda}} Z^\text{tot}_{\Sigma, P}(A_v ; |\text{rgf}_{A_v, f_v}(c, h)|^2 hO_1 \cdots hO_s \prod_{\Lambda=1}^{d_N} \phi_{\Lambda}(v) \phi_{\bar{\Lambda}}(\bar{v}) \hat{O}(f_v))$$

(4.32)

We can check that this expression is independent on the choice of the representatives $\{(A_v, f_v)\}_{v \in \mathcal{V}}$. This shows that the form $\Omega^\text{tot}_{\Sigma, P, x}(\cdots \hat{O})$ extends to a well-defined measure on the space $\mathcal{N}_{P, x}^\circ$. We have thus obtained the new integral expression for the correlation function:

$$Z_{\Sigma, P}(O_1 \cdots O_s O(x)) = \frac{1}{\text{vol Fl}(H)} \int_{\mathcal{N}_{P, x}^\circ} \Omega^\text{tot}_{\Sigma, P, x}(O_1 \cdots O_s \hat{O}) .$$

(4.33)

4.3 The Moduli Space Of Holomorphic Principal Bundles With Flag Structure — Examples

In the preceding subsection, we have introduced the space $\mathcal{N}_{P, x}^\circ$ which can be considered as a subset of the quotient of $\mathcal{A}_{P, x}$ by $G_{\mathbb{R}}$. This quotient $\mathcal{A}_{P, x}/G_{\mathbb{R}}$ can naturally be identified with the set of isomorphism classes of certain holomorphic objects — holomorphic $H_{\mathbb{C}}$-bundles with quasi-parabolic structure at $x$. Using this fact, we give an explicit description of the space $\mathcal{N}_{P, x}^\circ$ for some simple cases.

Holomorphic $H_{\mathbb{C}}$-Bundles With Quasi-Parabolic Structure

We fix a maximal torus $T$ and a chambre $C$ of $H$ and denote by $B$ the corresponding Borel subgroup of $H_{\mathbb{C}}$. For a holomorphic $H_{\mathbb{C}}$-bundle $\mathcal{P}$ over $\Sigma$, a choice of flag $f \in \mathcal{P}/B$
at \( x \in \Sigma \) is called a \textit{quasi-parabolic structure} of \( \mathcal{P} \) at \( x \). In this paper, we shall simply call it a \textit{flag structure} instead. Two holomorphic \( H_{\mathbb{C}} \)-bundles with flag structure at \( x \), \((\mathcal{P}_1, f_1)\) and \((\mathcal{P}_2, f_2)\), are said to be \textit{isomorphic} when there is an isomorphism \( \mathcal{P}_1 \rightarrow \mathcal{P}_2 \) which sends \( f_1 \) to \( f_2 \). Notice that the set of isomorphism classes of flag structures at \( x \) of a holomorphic \( H_{\mathbb{C}} \)-bundle \( \mathcal{P} \) is given by \( \text{Aut}(\mathcal{P})/\mathcal{P}_x/B \). As in the case without flags, for a principal \( H \)-bundle \( \mathcal{P} \), the set \( \mathcal{A}_{\mathcal{P},x}/\mathcal{G}_K \) can naturally be identified with the set of isomorphism classes of holomorphic \( H_{\mathbb{C}} \)-bundles of topological type \( P_{\mathbb{C}} \) with flag structure at \( x \).

For a holomorphic \( H_{\mathbb{C}} \)-bundle \( \mathcal{P} \) with flag structure \( f \) at \( x \), we denote by \( \text{Aut}(\mathcal{P}, f) \) the group of automorphisms of \( \mathcal{P} \) that preserve the flag \( f \). Then, \((\mathcal{P}, f)\) represents an element of \( N^0_{\mathcal{P},x} \) if and only if \( \mathcal{P} \) represents an element of \( N^0_{\mathcal{P}} \) and \( \dim \text{Aut}(\mathcal{P}, f) \leq \dim \text{Aut}(\mathcal{P}, f') \) for other choices \( f' \) of flags.

\textit{On the Sphere}

We classify the holomorphic principal bundles over the Riemann sphere \( \mathbb{P}^1 \) with flag structure at one point. We follow the notation of section 3.1.

We start with the case \( H = SU(n)/\mathbb{Z}_n \). The Borel subgroup \( B \) we choose is represented by the set of upper triangular matrices. Let \((\mathcal{P}, f)\) be a holomorphic \( H_{\mathbb{C}} \)-bundle with a flag at \( z = 0 \). We choose a section \( \sigma_0 \) on the \( z \)-plane \( U_0 \) with \( \sigma_0(0)B = f \) and a section \( \sigma_\infty \) on the \( z^{-1} \)-plane \( U_{\infty} \), and let \( h_{\infty 0} : U_0 \cap U_{\infty} \rightarrow H_{\mathbb{C}} \) be the holomorphic transition function. The Birkhoff theorem states [1] that there is a unique element \( a \in \mathbb{P}^\vee \) such that \( h_{\infty 0}(z) = h_{\infty}(z)z^{-a}h_0(z)^{-1} \) for some holomorphic maps \( h_I : U_I \rightarrow H_{\mathbb{C}} \ (I = 0, \infty) \) with \( h_0(0) \in B \). Thus, the set of isomorphism classes of holomorphic \( H_{\mathbb{C}} \)-bundles with flag structure at \( z = 0 \) is represented by \( \{\mathcal{P}_a ; a \in \mathbb{P}^\vee \} \) where \( \mathcal{P}_a = (\mathcal{P}_{[a]}, f_a) \) is an \( H_{\mathbb{C}} \)-bundle \( \mathcal{P}_{[a]} \) described by the transition relation \( \sigma_0 = \sigma_\infty z^{-a} \) with the flag \( f_a = \sigma_0(0)B \).

The automorphism group \( \text{Aut}\mathcal{P}_a \) of \((\mathcal{P}_{[a]}, f_a)\) is a subgroup of \( \text{Aut}\mathcal{P}_{[a]} \). Recall that an element \( h \) of \( \text{Aut}\mathcal{P}_{[a]} \) is represented with respect to \( \sigma_0 \) by an \( SL(n, \mathbb{C}) \)-valued function whose \( i \)-\( j \)-th entry \( (h_0)^j_i(z) \) is a span of \( 1, z, \cdots, z^{a_i-a_j} \) if \( a_i \geq a_j \) and zero if \( a_i < a_j \). It belongs to \( \text{Aut}\mathcal{P}_a \) if \( (h_0)^j_i(0) = 0 \) for \( i > j \). Thus, we see

\[
\dim \text{Aut}\mathcal{P}_a = \dim \text{Aut}\mathcal{P}_{[a]} - \sum_{i,j} \sum_{a_i \geq a_j} 1 = n - 1 + \sum_{i<j} (a_i - a_j + \theta_{a_i,a_j}),
\]

(4.34)

where \( \theta_{x,y} = 0 \) if \( x < y \) and \( \theta_{x,y} = 1 \) if \( x \geq y \). An element \( a \in \mathbb{P}^\vee \) minimizes this value in its permutation class if and only if the entries satisfy \( a_1 \leq a_2 \leq \cdots \leq a_n \).

Remember that for each \( j \in \tilde{J} = \{0, \cdots, n-1\} \) there is a smooth \( SU(n)/\mathbb{Z}_n \)-bundle \( P^{(j)} \) such that \( N^0_{P^{(j)}} = \{\mathcal{P}_{[\mu]}\} \). By the above argument, the set of distinct flag structures
on $\mathcal{P}_{[\mu_j]}$ is identified with the Weyl orbit $W\mu_j$. Let $n_{w_jw_0}$ denote the matrix
\[
\begin{pmatrix}
0 & 1_j \\
1_{n-j} & 0
\end{pmatrix} (-1)^{\frac{j(j-n)}{2}}, \tag{4.35}
\]
where $1_j$ is the $j \times j$ identity matrix. Since $a = \text{ad}_{w_jw_0}^{-1} \mu_j$ satisfies $a_1 \leq \cdots \leq a_n$, it is
the unique element in $W\mu_j$ that minimizes the dimension of the symmetry group. Thus, $\mathcal{N}_{\mathcal{P}(j),x}^\circ$ consists
of one point represented by $\mathcal{P}_j = \mathcal{P}_{(w_jw_0)^{-1}\mu_j}$.

For a general compact group $H$, the story is essentially the same. Each $a \in P^\vee$ indexes an isomorphism class represented by $\mathcal{P}_a = (\mathcal{P}_{[a]}, f_a)$, an $H_C$-bundle $\mathcal{P}_{[a]}$ described by
the transition rule $\sigma_0^{(a)} = \sigma_{\infty}^{(a)} z^{-a}$ with the flag $f_a = \sigma_0^{(a)}(0) B$. The group $\text{Aut}\mathcal{P}_a$ of automorphisms has
\[
\dim \text{Aut}\mathcal{P}_a = l + \sum_{\alpha > 0} \left( |\alpha(a)| + \theta_{\alpha(a),0} \right), \tag{4.36}
\]
which is minimized by $a = (w_jw_0)^{-1} \mu_j$ ($j \in \mathcal{J}$). Thus, $\mathcal{N}_{\mathcal{P}(j),x}^\circ$ consists of one point represented by $\mathcal{P}_j$, an $H_C$-bundle described by the transition rule
\[
\sigma_0(z) = \sigma_{\infty}(z) z^{-\mu_j} n_{w_jw_0}, \tag{4.37}
\]
with the flag $\sigma_0(0) B$, where $n_{w_jw_0}$ is an element of $N_T$ that represents $w_jw_0 \in W$.

On Torus with $H = SO(3)$

Next, we describe $\mathcal{N}_{\mathcal{P},x}^\circ$ for the trivial or the non-trivial principal $SO(3)$-bundle over the torus $\Sigma_x = C/Z + \tau Z$. This time, we realize $\Sigma_x$ as $C^*/qZ$ where $qZ$ is the subgroup of $C^*$ generated by $q = e^{2\pi i \tau}$. We take $z(x) \equiv 1 \mod qZ$.

Below, we list up some holomorphic $H_C = PSL(2, C)$-bundles over $\Sigma_x$ that are relevant to our story. Every bundle $\mathcal{P}$ is obtained by putting the relation
\[
\sigma(qz) = \sigma(z) h(q;z) \tag{4.38}
\]
on the section $\sigma(z)$ of the trivial bundle $C^* \times H_C$. In the following list, bundles and transition functions are exhibited as $\mathcal{P} : h(q;z)$ (where we put $t_u = e^{-2\pi i u}$).

[Some Holomorphic Principal $PSL(2, C)$-Bundles]

| trivial | non-trivial |
|---------|-------------|
| $\mathcal{P}_u^{(0)} : \begin{pmatrix} t_u & 0 \\ 0 & t_u^{-1} \end{pmatrix}$ \quad $u \sim \pm u + \frac{m}{2} + \frac{n}{2} \tau$ | $\mathcal{P}_F^{(1)} : \begin{pmatrix} 0 & q^{-\frac{1}{2}}z^{-\frac{1}{2}} \\ -q^\frac{1}{2}z^{-\frac{1}{2}} & 0 \end{pmatrix}$ |
| $\mathcal{P}_{00}^{(0)} : \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ | $\mathcal{P}_u^{(1)} : \begin{pmatrix} it_u^{-1}z^{-\frac{1}{2}} & 0 \\ 0 & -it_uz^{\frac{1}{2}} \end{pmatrix}$ \quad $u \equiv u + \frac{m}{2} + \frac{n}{2} \tau$ |
Remark that \( \mathcal{P}_u^{(0)} \cong \mathcal{P}_{u'}^{(0)} \iff u \equiv \pm u' \mod \frac{1}{2} \mathbb{Z} + \frac{3}{2} \mathbb{Z} \) and also that \( \mathcal{P}_u^{(1)} \cong \mathcal{P}_{u'}^{(1)} \iff u \equiv u' \mod \frac{1}{2} \mathbb{Z} + \frac{3}{2} \mathbb{Z} \). \( \{ \mathcal{P}_u^{(0)}, \mathcal{P}_{00}^{(0)} \) and \( \mathcal{P}_F^{(1)} \) are all the semi-stable \( H_C \)-bundles over \( \Sigma_\tau \). This is implicitly seen in \([36]\) but here we content ourselves by stating that \( \mathcal{P}_u^{(0)} \) and \( \mathcal{P}_F^{(1)} \) come from the flat \( SO(3) \) connections whose holonomies are \((3.13)\) and \((3.17)\) respectively (under \( z = e^{-2\pi i \zeta} \)). For use in \( \S 4.4 \), some unstable (i.e. non semi-stable) bundles \( \mathcal{P}_u^{(1)} \) are also included in the list above.

To determine isomorphism classes of flag structures at \( z = 1 \) of these bundles, we list below the groups of holomorphic automorphisms. An automorphism \( \mathcal{P} \to \mathcal{P} \) is described by

\[
h : \sigma(z) \mapsto \sigma(z) h(z),
\]

where \( h : \mathbb{C}^* \to PSL(2, \mathbb{C}) \) satisfies \( h(q; z) h(zq) = h(z) h(q; z) \). Typical elements \( h(z) \) are exhibited in the list below:

**[Automorphism Groups]**

\[
\begin{align*}
\text{Aut} \mathcal{P}_u^{(0)} & \cong \begin{cases} 
\mathbb{C}^* & \text{if } u \neq 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \\
PSL(2, \mathbb{C}) & \text{if } u \sim 0 \\
\mathbb{C}^* \times \mathbb{Z}_2 & \text{if } u \sim \frac{1}{3} \\
\mathbb{C}^* \times \mathbb{Z}_2 & \text{if } u \sim \frac{5}{3}, \frac{1}{3} 
\end{cases}, \\
\text{Aut} \mathcal{P}_{00}^{(0)} & \cong \mathbb{C} \\
\text{Aut} \mathcal{P}_F^{(1)} & \cong \mathbb{Z}_2 \times \mathbb{Z}_2 = \{ \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right), \left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \} \\
\text{Aut} \mathcal{P}_u^{(1)} & \cong B_0^0 \left( \begin{array}{cc} c & 0 \\ x \vartheta_{\tau,u}(z) & c^{-1} \end{array} \right)
\end{align*}
\]

In the above expression, \( \vartheta_{\tau,u} \) is given by \( \vartheta_{\tau,u}(z) = \vartheta(\tau, \zeta + 2u + \frac{1+i}{2}) \) where \( \vartheta(\tau, \zeta) \) is the Riemann’s theta function \( \sum_{n \in \mathbb{Z}} q^{\frac{1}{4} n^2 z^{-n}} \); \( q = e^{2\pi i \tau}, z = e^{-2\pi i \zeta} \). Note that \( \vartheta_{\tau,u}(1) = 0 \) if and only if \( u \equiv 0 \mod \frac{1}{2} \mathbb{Z} + \frac{3}{2} \mathbb{Z} \).

The flag manifold over \( z = 1 \) is identified with the Riemann sphere \( \mathbb{C} \cup \{ \infty \} \) by

\[
y \in \mathbb{C} \cup \{ \infty \} \mapsto \sigma(1) \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) B \in \mathcal{P}_{z=1}/B \quad ; \quad y = c/a.
\]

Looking at the action of \( \text{Aut} \mathcal{P} \) on \( \mathcal{P}_{z=1}/B \), we see that the flag structures over our holomorphic bundles are classified as follows:
Holomorphic principal \( U \)

respectively. Namely, \( N \) is presented by \( P \)

Generically there is no non-trivial symmetry, but there are three points with \( \text{Aut} \) derivation of (1.4).

we see that the compactified moduli space \( A \)

\[ \begin{align*}
\{ (P_u^{(0)}, 1) : u \neq 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} \} & \quad \text{trivial} \\
\{ (P_u^{(0)}, 0) : u \neq 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} \} & \quad \text{non-trivial} \\
\{ (P_u^{(0)}, \infty) \} & \quad \text{trivial}, \ C^* \\
\{ (P_0^{(1)}, 0) : y \neq 0, 1, i \} & \quad \text{non-trivial} \\
\{ (P_0^{(1)}, 1) \} & \quad \text{C} \\
\{ (P_0^{(1)}, 0), (P_0^{(1)}, \infty) \} & \quad \text{non-trivial, B}
\end{align*} \]

Note that \( (P_F^{(1)}, y) \cong (P_F^{(1)}, y') \) if and only if \( y' = y, -y, y^{-1} \) or \( -y^{-1} \). The group \( \text{Aut}(P, f) \) is presented in the list on the right of \( (P, f) \).

Recall that the moduli space \( N_{\text{triv}}^\infty \) for the trivial topology is represented by the family \( \{ (P_u^{(0)}, 1) \}_{u \neq 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}} \) whereas for the non-trivial topology \( N_{\text{non-triv}}^\infty \) is the one point set represented by \( P_F^{(1)} \). Counting the dimension of the automorphism groups, we see that \( N_{\text{triv}, x}^\infty \) and \( N_{\text{non-triv}, x}^\infty \) are represented by the families \( \{ (P_u^{(0)}, 1) \}_{u \neq 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}} \) and \( \{ (P_F^{(1)}, y) \}_{y \in \mathbb{C}} \) respectively. Namely,

\[
N_{\text{triv}, x}^\infty \cong \mathbb{C} \bigg/ \left( \left\{ \left( \frac{1}{2} \mathbb{Z} + \zeta \mathbb{Z} \right) \right\} \mathbb{Z}_2 \right) \cong \{ [0], [\zeta], [\zeta + 1] \}, \quad (4.45)
\]

\[
N_{\text{non-triv}, x}^\infty \cong \left( \mathbb{Z}_2 \times \mathbb{Z}_2 \right) \setminus \mathbb{P}^1. \quad (4.46)
\]

If \( N_{\text{triv}, x}^\infty \) is compactified by attaching the points \( (P_u^{(0)}, 1); u = \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} \) and \( (P_0^{(0)}, \infty) \), then we see that the compactified moduli space \( N_{\text{triv}, x}^\infty \) coincides topologically with \( N_{\text{non-triv}, x}^\infty \cong S^2 \). Moreover, it seems that the families of automorphism groups coincide with each other: Generically there is no non-trivial symmetry, but there are three points with \( \text{Aut} \cong \mathbb{Z}_2 \).

In §4.4, we shall construct a bijection between \( A_{\text{triv}, x}/G_{\text{triv}}^\infty \) and \( A_{\text{non-triv}, x}/G_{\text{non-triv}}^\infty \) which induces an isomorphism \( N_{\text{triv}, x}^\infty \cong N_{\text{non-triv}, x}^\infty \). In fact, this is an essential step in the derivation of \( \Box \).

4.4 Action Of \( \pi_1(H) \) On The Moduli Spaces

Let \( (\Sigma, x) \) be a closed Riemann surface with a point in it. We choose a neighborhood \( U_0 \) of \( x \) with a coordinate \( z \) such that \( z(x) = 0 \) and that \( z(U_0) \) is an open disc. A holomorphic principal \( \mathbb{C}^* \)-bundle admits trivializations over \( U_0 \) and \( U_\infty = \Sigma - x \) that are
related by a holomorphic transition function \( h_{\infty 0} : U_0 \cap U_\infty \to \mathbb{C}^* \). For each \( a \in \mathbb{Z} \), the transformation

\[
h_{\infty 0}(z) \mapsto h_{\infty 0}(z)z^{-a}
\]

of transition functions induces the translation of the group \( \text{Pic}(\Sigma) \) by an element of first chern class \( a \) (see §3.1). This defines an action of \( \pi_1(U(1)) \cong \mathbb{Z} \) on \( \text{Pic}(\Sigma) \) that covers the natural action on the set \( H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z} \) of topological types of \( U(1) \)-bundles. This action depends on \( x \) but not on the choice of coordinate \( z \).

We ask whether such an action exists for a general compact connected group \( H \): Does the natural action of \( \pi_1(H) \) on the set of topological types of principal \( H \)-bundles lift to an action on the set \( \text{Pic}^{H_C}(\Sigma) \) of isomorphism classes of holomorphic principal \( H_C \)-bundles? As an answer, we shall find that, instead of on \( \text{Pic}^{H_C}(\Sigma) \), \( \pi_1(H) \) acts on the set \( \text{Pic}^{H_C}(\Sigma, x) \) of isomorphism classes of holomorphic principal \( H_C \)-bundles with flag structure at \( x \). We conjecture that the action permutes the moduli spaces \( \mathcal{N}^2_{P,x} \). It is verified on the sphere for a general group and on torus for \( H = SO(3) \).

**Action Of \( \pi_1(H) \) On \( \text{Pic}^{H_C}(\Sigma, x) \)**

We first describe \( \text{Pic}^{H_C}(\Sigma, x) \) in terms of the loop group \( LH_C \). For a holomorphic principal \( H_C \)-bundle \( \mathcal{P} \) with a flag \( f \in \mathcal{P}_x/B \), we say that a section of \( \mathcal{P}|_{U_0} \) is admissible with respect to \( f \) when it represents \( f \) over \( x \). Let \( h : (\mathcal{P}, f) \to (\mathcal{P}', f') \) be an isomorphism. Under a choice of trivializations \( \{\sigma_0, \sigma_\infty\} \) and \( \{\sigma'_0, \sigma'_\infty\} \) over \( \{U_0, U_\infty\} \) of \( \mathcal{P} \) and \( \mathcal{P}' \) respectively such that \( \sigma_0 \) and \( \sigma'_0 \) are admissible with respect to \( f \) and \( f' \), \( h \) is represented by \( H_C \)-valued holomorphic functions \( \{h_0, h_\infty\} \) on \( \{U_0, U_\infty\} \) with \( h_0(x) \in B : \sigma_I \mapsto \sigma'_I h_I \) (\( I = 0, \infty \)). Then, the transition functions \( h_{\infty 0} \) and \( h'_{\infty 0} \) are subject to the relation

\[
h'_{\infty 0}(z) = h_\infty(z)h_{\infty 0}(z)h_0(z)^{-1} \quad z \in U_0 \cap U_\infty.
\]

For an open Riemann surface \( U \), we denote by \( L^U H_C \) the group of holomorphic maps \( U \to H_C \). Pulling back by inclusions \( U_\infty = U_0 \cap U_\infty \hookrightarrow U_0, U_\infty \), the groups \( L^{U_0} H_C \) and \( L^{U_\infty} H_C \) may be considered as subgroups of \( L^{U_\infty} H_C \).

\[^4\text{With a choice } S^1 \hookrightarrow U_\infty \text{ of parametrized circle, the group } L^{U_\infty} H_C \text{ is identified with a dense open subgroup of the loop group } LH_C. \text{ This is the origin of the notation.}\]

The fundamental group \( \pi_1(H) \) is isomorphic to the subgroup \( \Gamma_C \) of the affine Weyl group \( W_{\text{aff}} \) consisting of elements that preserve the above \( \hat{C} \) (see Appendix A). For each
\( \gamma \in \Gamma_{\mathbb{C}} \), there is a holomorphic extension \( h_{\gamma} : \mathbb{C}^* \to H_{\mathbb{C}} \). For example, \( h_{\gamma_j}(z) = z^{-\mu_j} n_{w_j w_0} \).

Using the coordinate \( z : U_{\infty, 0} \to \mathbb{C}^* \), we identify \( h_{\gamma} \) as an element of \( L^{U_{\infty, 0}} H_{\mathbb{C}} \). Since the adjoint action of \( h_{\gamma} \) on \( L^{U_{\infty, 0}} H_{\mathbb{C}} \) preserves the subgroup \( B^{U_0} \subset L^{U_{\infty, 0}} H_{\mathbb{C}} \), we find in view of (4.43) that the transformation

\[
h_{\infty 0}(z) \mapsto h_{\infty 0}(z) h_{\gamma}(z)
\]

of transition functions induces the transformation

\[
\gamma_x : \text{Pic}^{H_{\mathbb{C}}}(\Sigma, x) \longrightarrow \text{Pic}^{H_{\mathbb{C}}}(\Sigma, x).
\]

This transformation changes the homotopy type of the transition function by \( \gamma \in \pi_1(H) \) and hence permutes the subsets \( \{ A_{P, x} / G_{P_{\gamma}} \}_{P} \):

\[
\gamma_x : A_{P, x} / G_{P_{\gamma}} \longrightarrow A_{P_{\gamma}, x} / G_{P_{\gamma} C}.
\]

Thus \( \gamma \mapsto \gamma_x \) is the desired action of \( \pi_1(H) \) on \( \text{Pic}^{H_{\mathbb{C}}}(\Sigma, x) \).

**The Conjecture**

One important thing to notice is that this action preserves the automorphism groups. Namely, if the class of \((P, f)\) is mapped by \( \gamma_x \) to a class represented by \((P\gamma, f\gamma)\), we have

\[
\text{Aut}(P, f) \cong \text{Aut}(P\gamma, f\gamma).
\]

This can be seen by multiplying \( h_{\gamma}(z) \) on the right to both sides of (4.48) in which we put \( h'_{\infty 0} = h_{\infty 0} \).

Recall that an element of the moduli space \( \mathcal{N}_{P, x}^{0} \subset \text{Pic}^{H_{\mathbb{C}}}(\Sigma, x) \) is represented by \((P, f)\) that satisfies the following conditions: \( P \) represents an element of \( \mathcal{N}_{P}^{0} \) and \( \dim \text{Aut}(P, f) \leq \dim \text{Aut}(P, \tilde{f}) \) for every flag \( \tilde{f} \) at \( x \).

Having these in mind, we conjecture that the following holds: *There is a method to compactify the moduli space \( \mathcal{N}_{P, x}^{0} \) in such a way that for each \( \gamma \in \Gamma_{\mathbb{C}} \), \( \gamma_x \) maps the compactified moduli space \( \overline{\mathcal{N}_{P, x}^{0}} \) isomorphically onto another space \( \overline{\mathcal{N}_{P_{\gamma}, x}} \).*

If, furthermore, we can compactify \( \mathcal{N}_{P}^{0} \) so that the natural projection \( \mathcal{N}_{P, x}^{0} \to \mathcal{N}_{P}^{0} \) (forgetting the flags) extends to a surjective map \( \overline{\mathcal{N}_{P, x}^{0}} \to \overline{\mathcal{N}_{P}^{0}} \), we have the following double fibration:

\[
\begin{array}{ccc}
\text{\( \mathcal{N}_{P, x}^{0} \)}} \cong \overline{\mathcal{N}_{P_{\gamma}, x}} & \xrightarrow{\gamma_x} & \overline{\mathcal{N}_{P}^{0}} \\
\downarrow & & \downarrow \\
\overline{\mathcal{N}_{P}^{0}} & \xrightarrow{\gamma} & \overline{\mathcal{N}_{P_{\gamma}}^{0}}
\end{array}
\]

This seems to be what mathematicians call the *Hecke correspondence*.
Verification On The Sphere

As seen in §4.3, the moduli space $N_{P^{(1)},x}^\circ$ for $z(x) = 0$ is the one point represented by $P_i$ that is described by the transition relation $\sigma_0(z) = \sigma_\infty(z) h_{\gamma_i}(z)$ where $\sigma_0$ is admissible. The $\gamma_{jx}$-transform of $P_i$ is then described by $\sigma_0^{\gamma_j}(z) = \sigma_\infty^{\gamma_j}(z) h_{\gamma_i}(z)$ where $\sigma_0^{\gamma_j}$ is now admissible. Hence we see that

$$\gamma_{jx} : N_{P^{(1)},x}^\circ \longrightarrow N_{P^{(1)},x}^\circ,$$  \hspace{1cm} (4.55)

where $\gamma_{i0j} = \gamma_i \gamma_j$. The conjecture is thus verified on the sphere.

Verification On Torus With $H = SO(3)$

For $H = SO(3)$, the non-trivial element $\gamma$ of $\Gamma_C \cong \mathbb{Z}_2$ is represented by a path

$$\gamma(\theta) = \begin{pmatrix} 0 & -e^{\frac{i\theta}{2}} \\ e^{\frac{i\theta}{2}} & 0 \end{pmatrix}, \hspace{1cm} 0 \leq \theta \leq 2\pi$$  \hspace{1cm} (4.56)

in $SU(2)$. We apply $\gamma_x$ to the topologically trivial semi-stable bundles with flag structure at $z(x) = 1$.

A $PSL(2, \mathbb{C})$-bundle $P$ we consider is described by the transition relation $\sigma(zq) = \sigma(z) h(q; z)$ and a flag is parametrized by $y \in \mathbb{C} \cup \{\infty\}$. If we choose a matrix $h_f \in SL(2, \mathbb{C})$ such that $(h_f)_1^0/(h_f)_1^1 = y$, then, $\sigma_0(z) = \sigma(z) h_f$ is an admissible section on a small neighborhood $U_0$ of $z = 1$. Hence, the $\gamma_x$-transform of $(P, y)$ is represented by a bundle $P^\gamma$ with an admissible section $\sigma_0^\gamma$ on $U_0$ and a section $\sigma'$ on $\mathbb{C}^* - q^\mathbb{Z}$ that are related by

$$\sigma_0^\gamma(z) = \sigma'(z) h_f \gamma(z - 1), \hspace{1cm} z \in U_0 - \{1\},$$  \hspace{1cm} (4.57)

$$\sigma'(zq) = \sigma'(z) h(q; z), \hspace{1cm} z \not\equiv 1 \mod q^\mathbb{Z}.$$  \hspace{1cm} (4.58)

The conservation $\text{Aut}(P, f) \cong \text{Aut}(P^\gamma, f^\gamma)$ of automorphism groups enables us to guess how $\gamma_x$ transforms the bundles listed in §4.3. After a calculation, we find the following solution (see Appendix D for the proof):

[The Transformation $\gamma_x$]

\begin{align*}
(P_{u0}^{(0)}, 1) & \rightarrow (P_{F}^{(1)}, y_u) \hspace{1cm} u \not\sim 0 \\
(P_{00}^{(0)}, \infty) & \rightarrow (P_{F}^{(1)}, y_0) \\
(P_{u0}^{(0)}, 0) & \rightarrow (P_{u}^{(1)}, 0) \\
(P_{u\infty}^{(0)}, \infty) & \rightarrow (P_{-u}^{(1)}, 0) \\
(P_{00}^{(0)}, 0) & \rightarrow (P_{0}^{(1)}, 1)
\end{align*}

(4.59)

where

$$y_u = iq^\frac{1}{2} e^{2\pi i u} \frac{\vartheta(2\tau, 2u + \tau)}{\vartheta(2\tau, 2u)}.$$  \hspace{1cm} (4.60)
The most important point to notice is that the compactified moduli space $\overline{N}_{\text{triv}, x}^\circ$ represented by $\{(P_u, 1)_{u \neq 0}\}$ and $(P_{00}, \infty)$ is mapped bijectively to the (compact) moduli space $N_{\text{non-triv}, x}^\circ$ of flag structures on the semi-stable bundle $P^{(1)}_F$:

$$\gamma_x : \overline{N}_{\text{triv}, x}^\circ \longrightarrow N_{\text{non-triv}, x}^\circ.$$  \hfill (4.61)

In terms of the coordinates $u$ and $y$, this map is given by $u \mapsto y$ where $y$ is given in $(4.60)$ and satisfies $y - u = y_u$, $y_u + \frac{1}{2} = -y_u$ and $y_u + \frac{3}{2} = -y_u^{-1}$. The orbifold points $u \sim \frac{1}{4}, \frac{3}{4}, \frac{7}{4}$ are mapped to the orbifold points $y \sim 0, i, 1$ respectively, whereas $(P_{00}, \infty)$ is mapped to the smooth point $y_0$. If we decide to take $u^2$ as the complex coordinate around the point $(P_{00}, \infty)$, the bijection $(4.61)$ becomes an isomorphism and the conjecture is verified also in this case.

**Remark.** Even though bundles $\{P_u\}$ are semi-stable, the points $\{(P_u, 0)\}$ which do not lie in $\overline{N}_{\text{triv}, x}^\circ$ are mapped by $\gamma_x$ to $\{(P_u^{(1)}, 0)\}$ that are projected to unstable bundles by the 'flag forgetful map'.

### 4.5 The Topological Identity

We have just seen that the holomorphic function $h_\gamma(z)$ for $\gamma \in \Gamma_{\hat{C}} \cong \pi_1(H)$ induces the bijection $\gamma_x : A_{P, x}/G_{\hat{C}} \rightarrow A_{P_\gamma, x}/G_{P_\gamma C}$. In §2.3, we have seen (for the case $H = G/Z_G$) that the gauge transformation by the same function $h_\gamma(z)$ induces the spectral flow which maps the highest weight state $\Phi_\Lambda$ to another $\Phi_{\gamma \Lambda}$. Applying such spectral flow to the flag partners, we define the action of $\pi_1(H)$ on the space of gauge invariant local fields. Under the assumption that the conjecture is verified, we shall observe that the double role of $h_\gamma$ results in the identity (1.4).

**States Corresponding To The Flag Partners**

From the state space $\mathcal{H}^\text{tot} = \mathcal{H}^{G,k} \otimes \Gamma(\mathcal{L}_{\text{chg}}) \otimes \mathcal{F}_{\text{gh}}^{\text{ad}}$ of the total system, we shall select out the subspace $\hat{\mathcal{H}}_{\text{inv}}$ corresponding to the flag partner fields. Let $\Phi \in \mathcal{H}_\Lambda^\text{inv}$. The expression (4.14) shows that the following is the state corresponding to the flag partner of $\hat{\mathcal{O}}_\Phi$ associated to the standard flag $f_0 = s_0(0)B$:

$$Z^{\text{tot}}_{D_0}(0; \hat{\mathcal{O}}_\Phi(f_0)) = \Phi \otimes \Phi_{-\lambda - 2\rho} \otimes |\Omega\rangle.$$ \hfill (4.62)

Here, $\Phi_{-\lambda - 2\rho} \in \Gamma(\mathcal{L}_{\text{chg}})$ is the highest weight state $(3.38)$ and $|\Omega\rangle \in \mathcal{F}_{\text{gh}}^{\text{ad}}$ is defined by $\prod_{\alpha < 0} c_0^{-\alpha} c_{0}^{-\alpha} |0\rangle$ and satisfies

$$b_n(v)|\Omega\rangle = 0 \quad \text{for} \ n \geq 1, \ v \in \mathfrak{h}_C \ \text{or} \ n = 0, \ v \in \mathfrak{h},$$

$$c_n(v^*)|\Omega\rangle = 0 \quad \text{for} \ n \geq 1, \ v^* \in \mathfrak{h}_C^* \ \text{and} \ c_{0}^{-\alpha} |\Omega\rangle = 0 \quad \alpha \in \Delta_+.$$ \hfill (4.64)

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Thus we see that
\[ \hat{\mathcal{H}}_{\text{inv}} = \bigoplus_{\lambda,\lambda} \mathcal{H}_\lambda^\Lambda \otimes \Phi_{-\lambda-2\rho} \otimes |\Omega\rangle. \] (4.65)

**Action Of \( \pi_1(H) \) On Gauge Invariant Local Fields**

Let \( \gamma \in \Gamma_\mathcal{C} \cong \pi_1(H) \) be represented by the loop \( \gamma(\theta) = e^{-i\theta}n_w \). We consider the configuration \( A_{\theta,w^{-1}\mu} \) of gauge field which is introduced in §2.3. For a gauge invariant local field \( O \), we shall define another field \( \gamma O \) by
\[ Z_{D_0}^{\text{tot}}(0; \gamma \hat{O}(f_0)) = \gamma \cdot Z_{D_0}^{\text{tot}}(A_{\theta,w^{-1}\mu}; \hat{O}(f_0)) , \] (4.66)
where \( \gamma \cdot \) is the gauge transformation on \( \mathcal{H}_{\text{inv}}^{\text{tot}} \) induced by the loop \( \gamma \in LH \). Below, we show that the right hand side belongs to \( \hat{\mathcal{H}}_{\text{inv}} \) and is independent on the choices involved, such as the representative \( n_w \) of \( w \) or as the function \( g \) etc.

As we have seen in §2.3, \( A_{\theta,w^{-1}\mu} \) can be made flat by a chiral gauge transformation and the right hand side is expressed as \( \gamma \cdot \mathcal{J}(\hat{c})Z_{D_0}^{\text{tot}}(0; \hat{O}(f_0)) \) where \( \mathcal{J}(\hat{c}) \) is the action of an element \( \hat{c} \) of \( LH_\mathcal{C} \) over a constant loop in \( T_\mathcal{C} \). Since the total central charge vanishes and the state \( Z_{D_0}^{\text{tot}}(0; \hat{O}(f_0)) \) has weight zero, we see that the right hand side for \( O = O_\Phi \) is given by \( \gamma \cdot \Phi \otimes \gamma \cdot \Phi_{-\lambda-2\rho} \otimes \gamma \cdot |\Omega\rangle \).

Since \( \gamma \cdot \) preserves the highest weight condition for \( LH_\mathcal{C} \), it sends \( \mathcal{H}_\Lambda^\Lambda \) to \( \mathcal{H}_{\gamma G\Lambda}^{\gamma\Lambda} \) where \( \gamma G \) is the image of \( \gamma \) under the natural map \( \pi_1(H) \to \pi_1(G/Z_G) \). Also, it preserves the ray \( \mathcal{C}|\Omega\rangle \) characterized by (4.63) and (4.64), and a calculation shows that \( \gamma \cdot |\Omega\rangle = |\Omega\rangle \). Finally, we recall that \( \gamma \cdot \Phi_{-\lambda-2\rho} = \Phi_{-\gamma \lambda-2\rho} \). Combining these all, we have
\[ Z_{D_0}^{\text{tot}}(0; \gamma \hat{O}_\Phi(f_0)) = \gamma \cdot \Phi \otimes \Phi_{-\gamma \lambda-2\rho} \otimes |\Omega\rangle , \] (4.67)
for \( \Phi \in \mathcal{H}_\Lambda^\Lambda \). Thus, the action of \( \pi_1(H) \) on the gauge invariant local fields is defined and is identified with the action of \( \Gamma_\mathcal{C}(H) \) on \( \bigoplus \mathcal{H}_\Lambda^\Lambda \).

**Proof Of (1.4)**

We are now in a position to prove (1.4). We make use of the new integral expression (4.33). Let \( \mathcal{V} \) be an open subset of \( \mathcal{N}_{P,x}^0 \) with a holomorphic family \( \{(A_v,f_v)\}_{v \in \mathcal{V}} \) of representatives. The non-anomalous chiral gauge symmetry of the total system enables us to take the representatives so that there is a family \( \{\sigma_0(v)\}_{v \in \mathcal{V}} \) of horizontal and admissible sections on a neighborhood \( U_0 \) of \( x \).

We choose a complex coordinate \( z \) on \( U_0 \) such that \( z(x) = 0 \) and \( z(U_0) \) contains the unit disc \( D_0 \). Let \( \gamma \in LH \) be a representative loop of \( \gamma \in \Gamma_\mathcal{C} \cong \pi_1(H) \). We put \( \Sigma_\infty = \Sigma - D_0 \). Gluing \( (P|_{\Sigma_\infty},A_v|_{\Sigma_\infty}) \) and \( (D_0 \times H, A_{\theta,w^{-1}\mu}) \) by the identification of \( \sigma_0(v,e^{i\theta}) \gamma(\theta) \) and
\[ s_0(e^{i\theta}) := (e^{i\theta}, 1), \]
we obtain an \( H \)-bundle \( P\gamma \) over \( \Sigma \) with a smooth connection \( A^\gamma_0 \). We denote by \( \sigma(v) \) and \( s_0^\gamma(v) \) the sections of \( P\gamma \) over \( \Sigma_\infty \cap U_0 \) and \( D_0 \) respectively which had been \( \sigma_0(v)|_{\Sigma_\infty \cap U_0} \) and \( s_0 \) before the gluing. If we put \( f^\gamma_0 = s_0^\gamma(v, x)B \), we can find a holomorphic and admissible section \( \sigma_0^\gamma(v) \) of \( (A^\gamma_0, f^\gamma_0) \) over \( U_0 \) such that \( \sigma_0^\gamma(v, z) = \sigma(v, z)h_\gamma(z) \) on a neighborhood of \( S \). This shows that the transformation

\[ (A_v, f_v) \mapsto (A^\gamma_0, f^\gamma_0), \] (4.68)

represents \( \gamma_x : \mathcal{V} \to \gamma_x(\mathcal{V}) \). If the conjecture is verified, \( \{(A^\gamma_0, f^\gamma_0)\}_{v \in \mathcal{V}} \) is a family of representatives over \( \gamma_x(\mathcal{V}) \subset N^\gamma_{P,\gamma,x} \).

By construction of the \( \pi_1(H) \) action on gauge invariant local fields, we have

\[ Z_{D_0}^{\text{tot}(\sigma_0(v))}(A_v; \gamma \tilde{O}(f_v)) = Z_{D_0}^{\text{tot}(\sigma(v))}(A^\gamma_0; \tilde{O}(f^\gamma_0)). \] (4.69)

If we choose the gauge fixing function \( F_v \) for \( \text{Aut}(A_v, f_v) \) so that \( F_v(h) \) is independent on \( h|_{U_0} \), then it defines the gauge fixing function \( F^\gamma_0 \) for \( \text{Aut}(A^\gamma_0, f^\gamma_0) \) by the identification \( P|_{\Sigma_\infty} \cong P\gamma|_{\Sigma_\infty} \).

Now we see that \( (4.69) \) leads to the equality

\[ Z_{\Sigma, P}^{\text{tot}}(A_v; \text{rgf}_{A_v, f_v}(c, h), \gamma \tilde{O}(f_v)) = Z_{\Sigma, P, \gamma}^{\text{tot}}(A^\gamma_0; \text{rgf}_{A^\gamma_0, f^\gamma_0}(c, h), \gamma \tilde{O}(f^\gamma_0)), \] (4.70)

where \( \nu^\gamma_\lambda(v) \) is given by

\[ \nu^\gamma_\lambda(v) = \sigma(v) \cdot h^\gamma_0(v)^{-1} \frac{\partial}{\partial u^\lambda} h^\gamma_0(v) = \sigma_0^\gamma(v) \cdot h^\gamma_0(v)^{-1} \frac{\partial}{\partial u^\lambda} h^\gamma_0(v), \] (4.71)
in which \( h^\gamma_0(v) = h_\infty(v)h_\gamma \). This amounts to

\[ \Omega^{\text{tot}}_{\Sigma, P, x}(O_1 \cdots O_s \gamma \tilde{O}) = \gamma_x^* \Omega^{\text{tot}}_{\Sigma, P, \gamma, x}(O_1 \cdots O_s \hat{O}), \] (4.72)

which shows \( (1.4) \).
5. Sum Over Topologies

The full correlation function of the gauge invariant fields $O_1 \cdots O_s$ is given by

$$Z_\Sigma(O_1 \cdots O_s) = \sum_P Z_{\Sigma,P}(O_1 \cdots O_s),$$

where the sum is over all topological types of principal $H$-bundles over $\Sigma$. If we use the topological identity (1.4), we have

$$Z_\Sigma(O_1 \cdots O_s Q) = Z_{\Sigma,P}(O_1 \cdots O_s Q),$$

in which $P$ is any principal $H$-bundle and $Q := \sum_{\gamma \in \pi_1(H)} \gamma O$. We say that two gauge invariant local fields are equivalent when they are indistinguishable in any full correlator, and we denote by $\dot{H}$ the set of equivalence classes of gauge invariant local fields. The equation (5.2) shows that $O$ and $O'$ are equivalent if and only if $O = O'$. In particular, $O$ and $\gamma O$ are equivalent and so are $O$ and $\frac{1}{2}(O + \gamma O)$. Thus, $\hat{H}$ is the quotient of the space $\mathcal{H}$ of gauge invariant local fields by the kernel of the operator $\sum_{\gamma \in \pi_1(H)} \gamma$. Since the $\pi_1(H)$-action on $\mathcal{H}$ is identified with the $\hat{\Gamma}_C$-action on $\mathcal{H}_{hw}$, $\hat{H}$ is in one to one correspondence with the quotient $\hat{H}_{hw}$ of $\mathcal{H}_{hw}$ by the kernel of $\sum_{\gamma \in \Gamma_C} \gamma$. By the general principle of CFT, we expect that the torus partition function $Z_{\Sigma_T}(1)$ satisfies

$$Z_{\Sigma_T}(1) = \text{tr}_{\mathcal{H}_{hw}}(q^{L_0} - \frac{c}{24} q^{- \frac{c}{24}}),$$

in which $c = c_{\text{tot}} = c_{G,k} - c_{\hat{H},k}$ and $L_0$ and $\bar{L}_0$ are dilatation generators $L_0^{GKO}$ and $\bar{L}_0^{GKO}$ by GKO construction; the generators $L_n^{GKO}$ and $\bar{L}_n^{GKO}$ commute with the operators $\gamma$, for $\gamma \in \Gamma_C$ and hence can act on the quotient space $\mathcal{H}_{hw}$.

In this section, we calculate the full partition function on the torus $\Sigma_T$ and see whether (5.3) holds. For simplicity of the argument, we consider only the case in which $H$ is semi-simple. In this case, $\pi_1(H)$ is a finite group and the quotient $\mathcal{H}_{hw}$ is mapped by $\frac{1}{|\pi_1(H)|} \sum_{\gamma} \gamma$ isomorphically (as Virasoro module) onto the subspace $\mathcal{H}_{hw}^{\Gamma_C}$ of $\Gamma_C$-invariant elements.

5.1 Torus Partition Function For The Trivial Topology

We start with the calculation of the partition function for the trivial bundle $P_{\text{triv}} = \Sigma_T \times H$. We recall that the moduli space $\mathcal{N}_H = \mathcal{N}_{P_{\text{triv}}}$ is parametrized by $u \in t_C$ with the representative family

$$u \mapsto A_u = \frac{\pi}{\tau_2} u d\zeta - \frac{\pi}{\tau_2} \bar{u} d\bar{\zeta}$$
of flat gauge fields. $A'_u$ is gauge equivalent to $A_u$ if and only if $u' = wu + n + \tau m$ for some $w \in W$ and $n, m \in \mathbb{P}^\vee$.

The partition function of the WZW model with the target $G$ is given by

$$Z_{\Sigma^r, P_{triv}}^{G,k}(A_u; 1) = e^{\frac{i}{\sqrt{2}} \kappa_{tr}^G(u-\bar{u})^2} \sum_{\Lambda \in \mathbb{P}_+^k(G)} |\chi^G_\Lambda(\tau, u)|^2, \quad (5.5)$$

(see [38, 39, 1]) in which $\chi^G_\Lambda$ is the character of the representation $L^G_{\Lambda}$ of $LG$:

$$\chi^G_{\Lambda}(\tau, u) = \text{tr} L^G_{\Lambda}(q^{L_0 - \frac{c_G}{24} \theta} e^{2\pi i J_0(u)}), \quad (5.6)$$

where $u \in \mathfrak{t}C$ is considered as an element of $\mathfrak{g}_C$. As it should be, (5.5) is invariant under the gauge transformation $u \mapsto wu + n + \tau m$. This can be seen by looking at the transformation rule (5.2) in which $(x, t, y)$ is identified with $xL_0 + J_0(t) + ky$ and by noting (i) since $T \subset T_G$, $\mathbb{P}^\vee$ is a sublattice of $\mathbb{P}^G$ and (ii) for any $w \in W$, there is an element $w' \in W_G$ such that $w = w'$ on $\mathfrak{t}C$ [41]. It is also invariant under the modular transformations generated by $T$ and $S$: $(\tau, u) \mapsto (\tau + 1, u)$ and $(-\frac{1}{\tau}, \frac{u}{\tau})$. This is due to unitarity [42] of the modular transformation matrices for the characters $\chi^G_{\Lambda}$. Now we calculate

$$Z_{\Sigma^r, P_{triv}}(1) = \int_{\mathcal{N}^l} \prod_{i=1}^l d^2u_i Z^{\text{vol}}_{\Sigma^r, P_{triv}}\bigl( A_u ; |\text{rgf}_{A_u}(c, h)|^2 \bigr) \prod_{j=1}^l \frac{i}{2\pi} \int_{\Sigma^r} \bar{b} \frac{\partial A^0_{u_1}}{\partial u^j} \frac{i}{2\pi} \int_{\Sigma^r} \bar{b} \frac{\partial A^{10}_{u_1}}{\partial \bar{u}^j} \bigl). \quad (5.7)$$

The symmetry group of $A_u$ for generic $u$ is the group $T_C = T \times \mathbb{C}^k$ of constant gauge transformations. Parametrizing $hh^*$ as $n+e^\varphi n_+^*$ where $n_+$ is $N$-valued and $\varphi$ is $it$-valued, the residual gauge fixing term can be chosen as

$$|\text{rgf}_{A_u}(c, h)|^2 = \frac{\delta(t)(\varphi(x_0))}{\text{vol}(T)} \prod_{i=1}^l c^i(x_0)\bar{c}^i(x_0), \quad (5.8)$$

where $x_0$ is any point of $\Sigma_r$. As calculated essentially in [1], we have

$$Z^{gh}_{\Sigma^r, P_{triv}}\bigl( A_u ; \prod_{i=1}^l c^i(x_0)\bar{c}^i(x_0) \prod_{j=1}^l \frac{i}{2\pi} \int_{\Sigma^r} \bar{b} \frac{\partial A^0_{u_1}}{\partial u^j} \frac{i}{2\pi} \int_{\Sigma^r} \bar{b} \frac{\partial A^{10}_{u_1}}{\partial \bar{u}^j} \bigr) \quad (5.9)$$

$$= \left( \frac{\pi}{\tau_2} \right)^2 \det_{ad} \bigl( \bar{\bar{a}} A \bar{a} \bar{A}_u \bigr),$$

$$Z^{R_{h/|H, -\bar{h}^{\vee} |}}_{\Sigma^r, P_{triv}}\bigl( A_u, \frac{\delta(t)(\varphi(x_0))}{\text{vol}(T)} \bigr) = \left( \frac{2\pi \tau_2 (\bar{h} + h^{\vee})}{(2\pi)^4 \text{vol}(T)} \right)^{\frac{1}{2}} \left( \det_{ad} \bigl( \bar{\bar{a}} A \bar{a} \bar{A}_u \bigr) \right)^{-\frac{1}{2}}. \quad (5.10)$$
where $\det'_\text{ad}(\partial^\dagger_{A_u} \partial_{A_u})$ is the $\zeta$-regularized determinant of the Laplace operator $\partial^\dagger_{A_u} \partial_{A_u}$ acting on sections of the adjoint bundle. Calculation of the determinant is done in [13] and the result is

$$\det'_\text{ad}(\partial^\dagger_{A_u} \partial_{A_u}) = (2\tau_2)^2 e^{\frac{\pi}{\tau_2} \zeta_{\text{ad}}(u) - \frac{1}{2} \zeta_{\text{ad}}(u)^2} |\Pi_{\tilde{H}}(\tau, u)|^4,$$  

(5.11)
in which $\Pi_{\tilde{H}}(\tau, u)$ is the Weyl-Kac denominator defined by

$$\Pi_{\tilde{H}}(\tau, u) = q^{\frac{\text{dim} H}{24}} \prod_{\alpha \in \Delta_+} (e^{\pi i \alpha(u)} - e^{-\pi i \alpha(u)}) \prod_{n=1}^\infty \left\{ (1 - q^n)^l \prod_{\alpha \in \Delta} (1 - q^n e^{-2\pi i \alpha(u)}) \right\}. \quad (5.12)$$

Thus, $Z_{\Sigma_r, \text{triv}}(1)$ is equal to

$$\left( \frac{k + h'}{2\tau_2} \right)^{\frac{1}{2}} (2\pi)^l \frac{1}{\text{vol}(T)} \int \prod_{i=1}^l d^2 u_i e^{\frac{\pi}{\tau_2} (k + h') \zeta_{\text{ad}}(u_i)^2} \sum_{\lambda} \left| \chi_{\lambda}^{G, k}(\tau, u) \Pi_{\tilde{H}}(\tau, u) \right|^2. \quad (5.13)$$

The branching rule (3.42) leads to the expansion

$$\chi_{\lambda}^{G, k}(\tau, u) = \sum_{\lambda'} b_{\lambda'}^{\lambda}(\tau) \chi_{\lambda'}^{\tilde{H}, \tilde{k}}(\tau, u), \quad (5.14)$$
in which the branching function $b_{\lambda'}^{\lambda}$ for $\Lambda \in \text{P}_+(k)(G)$ and $\lambda \in \text{P}_+(\tilde{k})(\tilde{H})$ is defined by $b_{\lambda}^{\lambda'}(\tau) = \text{tr}_{B_{\lambda}}(q^{2\tau_2 k})$ where $c = c_{G, k} - c_{\tilde{H}, \tilde{k}}$ and $L_0 = L_0^{\text{GKO}}$. Since the Virasoro generators by the GKO construction commute with the spectral flow, we have $b_{\gamma \lambda'}^{\gamma \lambda} = b_{\gamma \lambda}^{\gamma \lambda}$. This enables us to replace the integration $\int_{N_{\tilde{H}}}$ in (5.13) by $\frac{1}{|\text{P}^\vee/\text{Q}^\vee|^2} \int_{N_{\tilde{H}}}$. Using the obvious identity $\text{vol}(T) = (2\pi)^l \text{vol}(it/P^\vee)$ and the orthogonality

$$\int_{N_{\tilde{H}}} \prod_{i=1}^l d^2 u_i e^{\frac{\pi}{\tau_2} (k + h') \zeta_{\text{ad}}(u_i)^2} \left| \chi_{\lambda}^{G, k}(\tau, u) \chi_{\lambda'}^{G, k}(\tau, u) \right|^2 = \text{vol}(it/Q^\vee) \left( \frac{2\tau_2}{k + h'} \right)^{\frac{1}{2}} \delta_{\lambda, \lambda'}, \quad (5.15)$$
of characters for $\tilde{LH}$ at level $\tilde{k}$, we finally have

$$Z_{\Sigma_r, \text{triv}}(1) = \frac{1}{|\pi_1(H)|} \sum_{\lambda, \lambda'} \left| b_{\lambda'}^{\lambda}(\tau) \right|^2, \quad (5.16)$$
in which $(\Lambda, \lambda)$ runs over $\text{P}_+(k)(G) \times \text{P}_+(\tilde{k})(\tilde{H})$. Due to the invariance $b_{\gamma \lambda'}^{\gamma \lambda} = b_{\lambda'}^{\lambda}$, it can also be expressed as

$$Z_{\Sigma_r, \text{triv}}(1) = \sum_{[\Lambda, \lambda]} \frac{1}{|\mathcal{S}_{\lambda}|} \left| b_{\lambda}^{\lambda}(\tau) \right|^2, \quad \quad (5.17)$$

where the sum is over the quotient $(\text{P}_+(k)(G) \times \text{P}_+(\tilde{k})(\tilde{H}))/\Gamma_{\tilde{C}}$ and $\mathcal{S}_{\lambda}$ is the isotropy subgroup of $\Gamma_{\tilde{C}}$ at $(\Lambda, \lambda)$.  

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If $S^\Lambda_\lambda = 1$ for every $(\Lambda, \lambda)$, obviously we have
\[
Z_{\Sigma_r, P_{\text{triv}}}(1) = \text{tr}_{L_{\text{hw}}} (q^{L_0} - \bar{q}^{\bar{L}_0} \bar{q}^{\bar{L}_0} - \bar{q}) .
\]
(5.18)
As we shall see shortly, in this case, topologically non-trivial bundles do not contribute to the partition function and hence $Z_{\Sigma_r, P_{\text{triv}}}(1)$ is itself the full partition function. Thus, (5.3) holds if $\pi_1(H)$ acts freely on $P^{(k)}_+(G) \times P^{(\tilde{k})}_+(H)$.

5.2 Field Identification Fixed Points

To an element $\gamma \in \pi_1(H)$ is associated a principal $H$-bundle $P_\gamma = P_{\text{triv}} \gamma$ over $\Sigma_r$. Due to the topological identity $([4.4])$, the partition function for $P_\gamma$ is the one point function for the trivial bundle:
\[
Z_{\Sigma_r, P_\gamma}(1) = Z_{\Sigma_r, P_{\text{triv}}}( \gamma(1)) ,
\]
(5.19)
where $\gamma(1)$ is associated to the state $\gamma \Phi_0$ in $\mathcal{H}^0_{\gamma \Phi_0}$. It is expressed as an integral over $\mathcal{N}_H$ whose integrand contains a factor $Z^{G, k}_{\Sigma_r, P_{\text{triv}}} (A_{ui}, \tilde{O}_r \Phi_0 m_i)$. For this to be non-vanishing, the fusion rule $[38, 44, 45]$ requires
\[
\sum_{\Lambda_0 \in P^{(k)}_+(G)} N^\Lambda_{0 \Lambda_0} \neq 0 \quad \text{and} \quad \sum_{\lambda_0 \in P^{(\tilde{k})}_+(H)} N^\Lambda_{\gamma \lambda_0} \neq 0 ,
\]
(5.20)
where $N^\Lambda_{0 \Lambda'}$ (resp. $N^\Lambda_{\lambda \lambda'}$) is the fusion coefficient of the WZW model with target $G$ and level $k$ (resp. target $\tilde{H}$ and level $\tilde{k}$). From the Gepner’s observation $[4]$ $S^\gamma_{\Lambda \lambda} = (-1)^{l(w)} e^{-2 \pi i (\lambda + \rho)(\mu)} S^\Lambda_\lambda$ for $\gamma(\theta) = e^{-i \mu \theta} w$ on the modular transformation matrix and from the Verlinde formula $N^\Lambda_{0 \Lambda_2} = \sum_\lambda S^\Lambda_{0 \lambda} S\lambda^{\Lambda_2 \lambda} / S^\Lambda_0$ $[44]$, it follows that $N^\gamma_{\gamma_1 \lambda \gamma_2 \lambda'} = N^\Lambda_{\lambda \lambda'}$. Since $N^\Lambda_{0 \lambda} = \delta^\Lambda_\lambda$, (5.20) is equivalent to the condition that there exist $\Lambda \in P^{(k)}_+(G)$ and $\lambda \in P^{(\tilde{k})}_+(H)$ such that $\gamma_0 \Lambda = \Lambda$ and $\gamma \lambda = \lambda$.

This observation is desirable in the following sense. If there is a pair $(\Lambda, \lambda)$ at which the isotropy $S^\Lambda_\lambda \subset \pi_1(H)$ is not $\{1\}$ (such a pair is called the fixed point in the literature), the partition function for the trivial topology has fractional coefficients in the $q, \bar{q}$-expansion (see (5.17)) and we can hardly expect that this function is expressed as a trace of $q^{L_0} - \frac{q^{\bar{L}_0}}{\bar{q}} q^{\bar{L}_0} - \frac{q^{L_0}}{q}$ in any Virasoro module. In algebraic treatments of coset models $[3, 46]$, this was recognized as the field identification problem in the presence of fixed points. We expect that a natural resolution is provided by the sum over topologies: If $S^\Lambda_\lambda \neq \{1\}$, the contribution $Z_{\Sigma_r, P_\gamma}(1)$ for $\gamma \in S^\Lambda_\lambda - \{1\}$ may be non-vanishing and the integrality of the coefficients may be restored for the full partition function. (3) In the next subsection, choosing a concrete example, we examine whether this happens.

---

5In [46], a method for “fixed point resolution” is presented. Characters of the “fixed point CFTs” in that reference may be related to the partition functions for non-trivial topologies.
The partition function (5.16) for the trivial topology is manifestly modular invariant, and so is expected for any topology $P$ since $P$ and $f^*P$ are topologically isomorphic for any diffeomorphism $f$ of $\Sigma_r$. Hence, the modular invariance may still hold for the full partition function. This is also examined below.

5.3 Models With $G = SU(2) \times SU(2)$ and $H = SO(3)$

We consider the case in which $G = SU(2) \times SU(2)$ and $H$ is the subgroup $\{(g, g); g \in SO(3)\}$ of the adjoint group $G/Z_G = SO(3) \times SO(3)$. For the level $k = (k_1, k_2)$, the induced level is $\tilde{k} = k_1 + k_2$. Since a highest weight representation of $SU(2)$ is convensionally labeled by the spin $j \in \frac{1}{2} \mathbb{Z}$, we identify $P^{(k)}_+ = P^{(k)}_+(SU(2))$ with the set $\{0, \frac{1}{2}, 1, \ldots, \frac{k_1}{k_2}\}$ of “integrable spins”. The non-trivial element of $\pi_1(H) = \mathbb{Z}_2$ induces the involution $((j_1, j_2), j) \mapsto ((\frac{k_1}{2} - j_1, \frac{k_2}{2} - j_2), \frac{k}{2} - j)$ in $P^{(k)}_+(G) \times P^{(k)}_+$. If $k_1$ or $k_2$ is an odd integer, there is no fixed point and the full partition function is given by $\frac{1}{2} \sum_{j_1, j_2, j} |b^{(j)}_{j_1, j_2}(\tau)|^2$. For the case $k_2 = 1$, it is the diagonal modular invariant partition function of the $k_1$-th unitary minimal model.

**Partition Function For The Non-trivial Topology**

In the following, we assume that $k_1$ and $k_2$ are both even integers. Then, there is a unique fixed point $((\frac{k_1}{4}, \frac{k_2}{4}), \frac{k}{4})$ and the topologically non-trivial configurations contribute to the partition function. Recall that the moduli space $N_{\text{non-triv}}$ of semi-stable $H_{\mathbb{C}}$-bundles of non-trivial topology consists of one point represented by $\mathcal{P}^{(1)}_F$, which is obtained by the identification

$$\sigma(zq) = \sigma(z) \left( \begin{array}{cc} 0 & q^{-\frac{1}{4}}z^{-\frac{1}{4}} \\ -q^{\frac{1}{4}}z^{\frac{1}{4}} & 0 \end{array} \right)$$

of a holomorphic section $z \mapsto \sigma(z)$ of the bundle $\mathbb{C}^* \times H_{\mathbb{C}}$ over $\mathbb{C}^*$. Denoting by $A_F$ the flat $SO(3)$-connection corresponding to the holomorphic bundle $\mathcal{P}^{(1)}_F$, we have

$$Z_{\Sigma_r, \text{non-triv}}(1) = Z^{\text{tot}}_{\Sigma_r, \text{non-triv}}(A_F; \frac{1}{4}),$$

where $\frac{1}{4}$ is the residual gauge fixing term for $\text{Aut}\mathcal{P}^{(1)}_F = \mathbb{Z}_2 \times \mathbb{Z}_2$. This factorises into the product of the partition functions for the three (or four) constituents:

$$\frac{1}{4} \prod_{i=1}^2 Z^{SU(2), k_i}_{\Sigma_r, \text{non-triv}}(A_F; 1) Z^{H_{\mathbb{C}}/H_{\mathbb{C}} - \frac{k}{4}}_{\Sigma_r, \text{non-triv}}(A_F; 1) Z^{gh}_{\Sigma_r, \text{non-triv}}(A_F; 1).$$

(5.23)

We shall show that each is constant, that is, independent on $\tau$. For this, we introduce the Green’s function of the operator $\delta_{A_F}$ for the adjoint bundle. Let $\sigma_{ad}(z) : g_{\mathbb{C}} \cong \text{ad} |_{\mathbb{C}}$ be the frame associated to $\sigma(z) \in R_{\mathbb{C}}|_{z}$. The Green function is then expressed as $G_w(z) =$
\[ \sigma_{ad}(w)g(w, z)\sigma_{ad}(z)^{-1} \otimes dz \text{ where } g(w, z) \in \text{End}(h_C) \text{ is represented by the matrix} \]
\[
g(w, z) = \left( \begin{array}{ccc} \sum z^{-q_n^2 - q_{n-1}^2} w & 0 & -\sum z^{-1} q_{n-1}^2 z^{-q_n^2} w \\ 0 & f(w, z) & 0 \\ -\sum w q_{n-1}^2 w & 0 & \sum w z^{-1} q_n^2 w \end{array} \right), \tag{5.24} \]

with respect to the base \((\sigma_+, \sigma_3, \sigma_-)\) of \(h_C = \mathfrak{sl}(2, \mathbb{C})\) in which \(\sigma_i\)'s are Pauli matrices. The sums \(\sum_n\) in the four entries are over all integers and \(f(w, z)\) is expressed by the theta function \(\vartheta\) and its derivative \(\vartheta' = \partial_{\zeta} \vartheta\) as
\[
f(e^{-2\pi i \xi}, e^{-2\pi i \zeta}) = \frac{1}{2\pi iz} \frac{\vartheta(\tau, \xi - \zeta + \frac{z+1}{2})}{\vartheta(\tau, \zeta - \frac{z}{2})}. \tag{5.25} \]

The partition function \(Z_{\Sigma, \text{non-triv}}^{SU(2), k}(A_F; 1)\) for the \(SU(2)\)-WZW model satisfies the following Ward identities for the chiral gauge symmetry:
\[
Z_{\Sigma, \text{non-triv}}^{SU(2), k}(A_F; J) = 0 \tag{5.26}
\]
\[
Z_{\Sigma, \text{non-triv}}^{SU(2), k}(A_F; J \cdot \epsilon(z) J \cdot \epsilon'(w)) = k \text{tr}_\tau (\partial^{(w)}_{A_F} G_w \epsilon(z) \epsilon'(w)) Z_{\Sigma, \text{non-triv}}^{SU(2), k}(A_F; 1). \tag{5.27} \]

Putting these into the expression \((B.7)\) for the energy momentum tensor, we find
\[
\frac{\partial}{\partial \tau} Z_{\Sigma, \text{non-triv}}^{SU(2), k}(A_F; 1) = 0. \tag{5.28} \]

This also holds for the WZW model with the target \(H_C/H\). As for the ghost system, putting the identity
\[
Z_{\Sigma, \text{non-triv}}^{gh}(A_F; c(w)b(z)) = G_w(z) Z_{\Sigma, \text{non-triv}}^{gh}(A_F; 1), \tag{5.29} \]
into the expression \((B.4)\) of the energy momentum tensor, we find
\[
\frac{\partial}{\partial \tau} Z_{\Sigma, \text{non-triv}}^{gh}(A_F; 1) = 0. \tag{5.30} \]

Thus, the partition function is a constant:
\[
Z_{\Sigma, \text{non-triv}}(1) = C_{\text{non-triv}}. \tag{5.31} \]

The Full Partition Function

The partition function for topologically trivial configurations is given by
\[
Z_{\Sigma, \text{triv}}(1) = \sum_{(j_1, j_2), \omega} |b_{(j_1, j_2)}^{(}\omega(\tau)|^2 + \frac{1}{2} |b_{(k_1/4, k_2/4)}^{(}\omega(\tau)|^2, \tag{5.32} \]

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where the sum $\sum^{\circ}$ is over the $\mathbb{Z}_2$-quotient of $P^{(k)}_+(G) \times P^{(k)}_+$ $- \{( (\frac{k_1}{4}, \frac{k_2}{4}), \frac{k}{2} ) \}$. For the non-trivial topology, we have (5.30). Since we have no way to determine $C_{\text{non-triv}}$ now, we change the question to the following form: Can we tune $C_{\text{non-triv}}$ so that (5.3) holds?

The term $\sum^{\circ}$ in (5.31) is the trace of $q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}}$ on the space $\mathcal{H}_hw$ where $\mathcal{H}^{\circ}_{hw} = \mathcal{H}_{hw} \ominus \mathcal{H}^f$ ; $\mathcal{H}^f := \mathcal{H}^{\frac{k}{4}}_{(k_1/4,k_2/4)}$.

Hence, the question is whether there is a constant $C_{\text{non-triv}}$ such that

$$\frac{1}{2} \left| b^{\frac{k}{4}}_{(k_1/4,k_2/4)}(\tau) \right|^2 + C_{\text{non-triv}} = \text{tr}_{\mathcal{H}^f}(q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}}).$$

We answer this in the case $k_2 = 2$. The Virasoro modules by the GKO construction $SU(2) \times SU(2) / SU(2)$ at level $(k_1,2)$ are known [6] to be the ones appearing in the $k_1$-th $N = 1$ superconformal minimal model. Among others, $B^f := B^{\frac{k}{4}}_{(k_1/4,k_2/4)}$ is in the Ramond sector and contains a unique ground state with $L_0 = \frac{c}{24}$. In particular, there is a supercharge $G_0 : B^f \to B^f$ such that $G_0^2 = L_0 - \frac{c}{24}$. One can show that $\gamma : \mathcal{H}^f \to \mathcal{H}^f$ induces an involution $U_\gamma$ of the Virasoro module $B^f$ such that

$$G_0 U_\gamma + U_\gamma G_0 = 0.$$  

If $B^f$ is decomposed as $B^f = \oplus_{n=0}^{\infty} B_n$ in which $B_n$ is the $L_0$-eigen space with $G_0^2 = n$, we may put $U_\gamma = 1$ on $B_0 \cong \mathbb{C}$ and the anti-commuting relation (5.34) shows that

$$B_n = B_n^{(+)} \oplus B_n^{(-)}, \quad B_n^{(+)} \xrightarrow{G_0} B_n^{(-)} \quad \text{(isomorphic)} \quad (5.35)$$

for $n \geq 1$, where $B_n^{(\pm)}$ is the subspace of $B_n$ on which $U_\gamma = \pm 1$. Thus, we have $\mathcal{H}^f = \mathcal{H}^{(+)} \oplus \mathcal{H}^{(-)}$ where

$$\mathcal{H}^{(+)} \cong \bigoplus_{n,m=0}^{\infty} B_n^{(+)} \otimes B_m^{(+)} \oplus \bigoplus_{n,m=1}^{\infty} B_n^{(-)} \otimes B_m^{(-)}$$

$$\mathcal{H}^{(-)} \cong \bigoplus_{n \geq 0, m \geq 1} \left\{ B_n^{(+)} \otimes B_m^{(-)} \oplus B_n^{(-)} \otimes B_m^{(+)} \right\}$$

are subspaces on which $\gamma = 1$ and $\gamma = -1$ respectively. Since $\mathcal{H}^f$ is isomorphic to $\mathcal{H}^{(+)}$, we see that (5.33) and hence (5.3) hold if we tune $C_{\text{non-triv}} = \frac{1}{2}$. 

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6. Concluding Remarks

So far, we have been considering the gauged WZW model whose classical action is defined by (2.1). However, we could have started with another choice of an action generalizing (1.1). One familiar way to modify the action is to add the “theta term”

$$\int_{\Sigma} \frac{i}{2\pi} \theta(F_A),$$  \hfill (6.1)

where \(\theta\) is some adjoint invariant linear form \(\mathfrak{h} \rightarrow \mathbb{R}\). Then, the equivalence relation of gauge invariant local fields is modified by a phase factor. If \(H\) contains a \(U(1)\)-factor, we thus have a continuous series of quantum field theories having one common partition function.

If \(H\) is semi-simple, the term (6.1) vanishes. However, there is another way to modify the action. It arises from the variety of WZW actions constructed via the equivariant differential characters. Construction of an action in terms of Cheeger-Simons differential character was initiated by Dijkgraaf and Witten in Chern-Simons gauge theory [13] and the method was elaborated in ref. [14] (see also [47]). It provides a way to define topological lagrangians satisfying suitable physical conditions such as locality, unitarity, gluing property, etc. According to it, WZW actions with the target \(G\) and the gauge group \(H\) are classified by the equivariant cohomology (Borel cohomology) \(H^3_{H}(G; \mathbb{Z}) := H^3(EH \times_H G; \mathbb{Z})\) in which \(EH\) is the universal \(H\)-bundle and \(H\) acts on \(G\) via adjoint transformations. Importantly, for a semi-simple group \(H\), we have

$$H^3_{H}(G; \mathbb{Z}) = H^3(G; \mathbb{Z}) \oplus \text{Hom}(\pi_1(H), \mathbb{R}/\mathbb{Z}).$$ \hfill (6.2)

The levels are classified by \(H^3(G; \mathbb{Z})\) and presumably the torsion part \(\text{Hom}(\pi_1(H), \mathbb{R}/\mathbb{Z})\) classifies the “theta terms”. In the quantum theory, such a theta term would modify the equivalence relation of gauge invariant local fields. In a theory with fixed points, it would modify the partition function as well. For example, when \(G = SU(2) \times SU(2)\) and \(H = SO(3)\), the theory corresponding to \((k_1, k_2, \pm 1) \in \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \cong H^3_{SO(3)}(SU(2)^2; \mathbb{Z})\) with even \(k_1, k_2\) would have the full partition function

$$Z_{\Sigma, \text{triv}}(1) \pm C_{\text{non-triv}} = \frac{1}{2} |\eta^{k_1/4,k_2/4}(\tau)|^2 \pm C_{\text{non-triv}} + \cdots.$$ \hfill (6.3)

For \(k_2 = 2\), both have positive integral coefficients in the \(q, \bar{q}\)-expansions if and only if \(C_{\text{non-triv}} = \pm \frac{1}{2}\). If \(C_{\text{non-triv}} = \frac{1}{2}\), (5.3) holds in each theory. Due to the relation (5.34), the involution \(\gamma\), can be identified with the mod two fermion number \((-1)^F\) and the theory for \((k_1, 2, \pm 1)\) is the spin model [4, 48] with the projection \((-1)^F = \pm 1\) on the Ramond
sector. We expect in a general model that adding a torsion (theta term) has such a simple and significant consequence in physics. This will be elaborated in a future work [15].

In this paper, we have been concentrated on the model whose matter theory is the WZW model with a compact simply connected target group. However, our argument is readily applicable to other models such as (i) the model whose target group is compact and connected but non simply connected, (ii) the model whose matter theory is a free fermionic system of arbitrary spin and (iii) a combined system of free fermions and WZW models. Study of models of the type (i) may be important for the classification of rational CFTs. An interesting class of theories of the type (iii) is the (twisted) $N = 2$ coset conformal field theory (Kazama-Suzuki model) [19, 54]. Algebraic structure of the spectral flows of such a model has been studied by many authors [4, 50, 51, 52]. In the ref. [52], a geometric interpretation of field identification is attempted along the line similar to ours. However, the argument in that reference uses the counterpart (in the twisted $N = 2$ system) of our old integral expression (3.31) and hence is applicable only for abelian gauge groups. Our method completes this. The fixed point resolution in these systems (see [51, 53] for algebraic approaches) by the topological sum with theta terms will be interesting and perhaps of some importance in superstring theory.

Appendix A

We describe some basic facts on root systems and Weyl groups [17]. Let $H$ be a compact connected Lie group and let $\pi : \tilde{H} \to H$ be the universal covering with kernel $\pi^{-1}(1) \cong \pi_1(H)$. We choose a maximal torus $T$ of $H$ and put $\tilde{T} = \pi^{-1}(T)$. The Lie algebras of $T$ and $\tilde{T}$ are identified and the imaginary part $i\mathfrak{t}$ of its complexification is denoted by $V$. We introduce lattices $Q^\vee \subset P^\vee$ in $V$ so that the exponential maps induce isomorphisms $t/2\pi iP^\vee \cong P^\vee$ and $t/2\pi iQ^\vee \cong \tilde{T}$. Then we have $P^\vee/Q^\vee \cong \pi_1(H)$.

A.1 For A Simple Centerless Group

We first consider the case in which $H$ is simple and centerless. Then, $\tilde{H}$ is compact, $\tilde{T}$ is its maximal torus and $\pi_1(H)$ is the center of $\tilde{H}$. $P^\vee$ and $Q^\vee$ are dual to the root lattice $Q$ of $\tilde{H}$ and the weight lattice $P$ of $\tilde{T}$ respectively:

$$V^* \supset P \cdots Q^\vee \tag{A.1}$$

where $A \cdots B$ means that $A$ is the dual of $B$. 

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Weyl Group

The Weyl group $W$ of $(H, T)$ is defined by $W = N_T / T$ where $N_T$ is the normalizer of $T$ in $H$. The adjoint action of $W$ on $T$ induces its linear actions on $V^*$ and $V$ leaving invariant the four lattices and the set $\Delta$ of roots. For each root $\alpha \in \Delta$, we introduce a hyperplane $H_\alpha = \{ x \in V; \alpha(x) = 0 \}$ and we denote by $s_\alpha \in W$ the reflection with respect to $H_\alpha$. Since $W$ preserves $\Delta$, the family $\{ H_\alpha \}_{\alpha \in \Delta}$ of hyperplanes is invariant by $W$. A chambre of $\Delta$ is, by definition, a connected component of $V = \bigcup_{\alpha \in \Delta} H_\alpha$. We now have the

Theorem 1  
(1) $W$ acts simply transitively on the set of chambers.
(2) If $H_1, \cdots, H_l$ are walls of a chambre $C$, for each $i$ there exist a unique root $\alpha_i$ such that $H_{\alpha_i} = H_i$ and that $\alpha_i$ takes positive values on $C$.
(3) The set $B(C) = \{ \alpha_1, \cdots, \alpha_l \}$ forms a base of the free abelian group $Q$.
(4) The set $S(C) = \{ s_{\alpha_1}, \cdots, s_{\alpha_l} \}$ generates $W$.
(5) Any root $\alpha \in \Delta$ is expressed as $\alpha = \sum_{i=1}^l n_i \alpha_i$ where $n_i$ are all non-negative integers or all non-positive integers.

We fix a chambre $C$. By (3) and (5), we can choose a base $\{ \mu_1, \cdots, \mu_l \}$ of $P^\vee$ such that $\alpha_i(\mu_j) = \delta_{i,j}$. (5) of the theorem shows that $\Delta$ is decomposed as a disjoint union of the set $\Delta_+$ of positive roots and the set $\Delta_- = -\Delta_+$ of negative roots where a root is positive if it takes positive values on $C$. We see from (1) that there exists a unique element $w_0 \in W$ such that $w_0 \Delta_+ = \Delta_-$. It is the longest element where the length $l(w)$ of $w \in W$ is the minimum length $n$ of such sequence $s_{i_1}, \cdots, s_{i_n}$ in $S(C)$ that $w = s_{i_1} \cdots s_{i_n}$. The highest weight of the adjoint representation is called highest root and is denoted by $\hat{\alpha}$. It can be shown that, for any $\alpha \in \Delta$, $\hat{\alpha} - \alpha$ is a span of $\alpha_1, \cdots, \alpha_l$ with non-negative integral coefficients. In particular, $\hat{\alpha}$ is expressed as $\hat{\alpha} = \sum_{i=1}^l n_i \alpha_i$ for $n_i \geq 1$. We define $J \subset \{ 1, \cdots, l \}$ by $j \in J \iff n_j = 1$. We put $\mu_0 = 0$, $\hat{J} = \{ 0 \} \cup J$ and $M_C = \{ \mu_j; j \in \hat{J} \}$. Then, we have the following

Proposition 2  $a \in P^\vee$ satisfies $\alpha(a) = 0$ or 1 for any $\alpha \in \Delta_+$ if and only if $a \in M_C$. Moreover, any $Q^\vee$ orbit in $P^\vee$ contains one and only one element of $M_C$.

The latter part can be understood after we introduce the group $\Gamma_C$.

Affine Weyl Groups

The affine Weyl groups of $\tilde{H}$ and $H$ are defined by

$W_\text{aff} = \text{Hom}(U(1), \tilde{T}) \rtimes W \cong Q^\vee \times W$ and $W'_\text{aff} = \text{Hom}(U(1), T) \rtimes W \cong P^\vee \times W$ respectively. Since $Q^\vee \subset P^\vee$, $W_\text{aff}$ is considered as a subgroup of $W'_\text{aff}$. $W_{\text{aff}}$ can also be defined as the Weyl group of $U(1) \tilde{\times} \tilde{L}H$ with
respect to the torus $U(1) \times L\tilde{H}|_T$ where $U(1)$ acts on $L\tilde{H}$ covering the rotation action $e^{ix} : \gamma(\theta) \rightarrow \gamma(\theta - x)$ on $L\tilde{H}$. Hence comes the linear action of $W_{aff}$ (and also of $W_{aff}'$) on $\text{Lie}(U(1) \times L\tilde{H}|_T)$ = $i\hat{V}$ where $\hat{V}$ = $R_r \oplus V \oplus R_c$

\[ e^{-i\theta} w : (x, t, y) \in \hat{V} \mapsto (x, wt - xa, y - tr(awt) + \frac{1}{2}\text{tr}(a^2)) \in \hat{V}, \quad (A.2) \]
\[ e^{-i\theta} w : (n, \lambda, k) \in \hat{V}^* \mapsto (n + w\lambda(a) + \frac{1}{2}\text{tr}(a^2), w\lambda + k\text{tr}a, k) \in \hat{V}^*. \quad (A.3) \]

where “$tr$” is a normalized trace in $\mathfrak{h}$ that induce the inner product on $V^*$ with $(\alpha, \alpha) = 2$ for a long root $\alpha$. The dual action $(A.3)$ preserves the following inner product on $\hat{V}^*$:

\[ \left( (n_1, \lambda_1, k_1), (n_2, \lambda_2, k_2) \right) = (\lambda_1, \lambda_2) - n_1k_2 - k_1n_2. \quad (A.4) \]

An affine root of $LG$ is, by definition, a weight of the adjoint action of $U(1)_r \times L\tilde{H}|_{\tilde{H}}$ on $\text{Lie}(LH_C)$. The set $\Delta_{aff} \subset \hat{V}^*$ of non-zero affine roots is invariant under the action of $W_{aff}'$ and is given by $\Delta_{aff} = Z_{\neq 0} \times \{0\} \cup Z \times \Delta \times \{0\}$.

$W_{aff}'$ acts linearly on the quotient $V_{aff} = \hat{V}/R_c$ and affinely on each hyperplane $V_x = \{x\} \times V \subset V_{aff}$ (see $(A.2)$). If we put $\tilde{H}_{\tilde{\alpha}} = \{v \in V_{aff} : \tilde{\alpha}(v) = 0\}$, the family $\{\tilde{H}_{\tilde{\alpha}}\}_{\tilde{\alpha} \in \Delta_{aff}}$ of hyperplanes in $V_{aff}$ is $W_{aff}'$-invariant. Hence, if we put $H_{\tilde{\alpha}} = \tilde{H}_{\tilde{\alpha}} \cap V_{-1}$, the family $\{H_{\tilde{\alpha}}\}_{\tilde{\alpha} \in Z \times \Delta \times \{0\}}$ of hyperplanes in $V_{-1}$ is also $W_{aff}'$-invariant. We denote by $s_{\tilde{\alpha}} \in W_{aff}$ the reflection with respect to $H_{\tilde{\alpha}} \neq \emptyset$. An alcôve is, by definition, a connected component of $V_{-1} - \bigcup_{\tilde{\alpha} \in \Delta_{aff}} H_{\tilde{\alpha}}$. Then, we have the

**Theorem 3**  
1. $W_{aff}$ acts simply transitively on the set of alcôves.
2. If $H_0, H_1, \ldots, H_l$ are walls of an alcôve $\tilde{C}$, for each $i$ there exists a unique affine root $\hat{\alpha}_i$ such that $H_{\hat{\alpha}_i} = H_i$ and that $\hat{\alpha}_i$ takes positive values on $\tilde{C}$.
3. The set $B(\tilde{C}) = \{\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_l\}$ forms a base of $Z \oplus Q \oplus \{0\}$.
4. The set $S(\tilde{C}) = \{s_{\hat{\alpha}_0}, s_{\hat{\alpha}_1}, \ldots, s_{\hat{\alpha}_l}\}$ generates $W_{aff}$.
5. Any affine root $\tilde{\alpha} \in \Delta_{aff}$ is expressed as $\tilde{\alpha} = \sum_{i=0}^l n_i \hat{\alpha}_i$ where $n_i$ are all non-negative integers or all non-positive integers.

A chambre $C$ determines an alcôve $\tilde{C} = \{(-1, t) \in V_{-1} : t \in C, \tilde{\alpha}(t) < 1\}$ which gives $B(\tilde{C}) = \{\tilde{\alpha}_0, \ldots, \tilde{\alpha}_l\}$ where $\tilde{\alpha}_0 = (-1, -\tilde{\alpha}, 0)$ and $\tilde{\alpha}_i = (0, \alpha_i, 0)$ for $i = 1, \ldots, l$. (5) of the theorem shows that the set $\Delta_{aff}$ is decomposed as a disjoint union of the set $\Delta_{aff}^+$ of positive affine roots and the set $\Delta_{aff}^- = -\Delta_{aff}^+$ of negative affine roots where an affine root is positive if it takes positive values on the alcôve $\tilde{C}$.

$W_{aff}'$ acts on the set of alcôves and we denote by $\Gamma_{\tilde{C}}$ the isotropy subgroup at $\tilde{C}$. Then we see that $W_{aff}'$ decomposes into semi-direct product of $W_{aff}$ and $\Gamma_{\tilde{C}}$:

\[ W_{aff}' \cong W_{aff} \rtimes \Gamma_{\tilde{C}}. \quad (A.5) \]
The subgroup $\Gamma_{\hat{C}}$ preserves the decomposition $\Delta_{\text{aff}} = \Delta_{\text{aff}+} \cup \Delta_{\text{aff}-}$ which shows with the aid of (5) of the theorem that $\Gamma_{\hat{C}}$ permutes the elements $\hat{\alpha}_0, \cdots, \hat{\alpha}_l$ of $B(\hat{C})$. Looking at the transformation rule (A.3), we see that the homogeneous part of $\Gamma_{\hat{C}}$ permutes the distinct elements $\alpha_0 = -\tilde{\alpha}, \alpha_1, \cdots, \alpha_l$ of $\Delta$. Since the relative disposition of these $l + 1$ roots is used to construct the extended Dynkin diagram, $\Gamma_{\hat{C}}$ can be identified with a group of Dynkin diagram automorphisms.

We explicitly describe the group $\Gamma_{\hat{C}}$. For each $j \in \hat{J}$, the set $S_j = S(C) - \{s_{\alpha_j}\}$ generates a subgroup $W_j$ of $W$ and determines a length in $W_j$. Let $w_j$ be the longest element in $W_j$ and we put $\gamma_j(\theta) = e^{-i\mu_j\theta}w_jw_0$. Then we can show the

**Proposition 4** The group $\Gamma_{\hat{C}}$ is given by $\Gamma_{\hat{C}} = \{\gamma_j; j \in \hat{J}\}$.

The embedding $P^\vee \hookrightarrow W'_\text{aff}$ induces the isomorphism $P^\vee / Q^\vee \cong W'_\text{aff} / W_{\text{aff}} \cong \Gamma_{\hat{C}}$. It then follows the latter part of proposition 2.

### A.2 For A General Compact Connected Group

In general, $\tilde{H}$ is isomorphic to $\mathbb{R}^M \times \prod_{n=1}^N G_n$ for some $M = 0, 1, 2, \cdots$ and some sequence $G_1, \cdots, G_N$ of simple simply connected groups. The root lattice $Q$ is no longer dual to $P^\vee$. If $H$ is not semi-simple ($M \neq 0$), $P^\vee / Q^\vee$ is an infinite group.

The definition of the Weyl group $W$, hyperplanes, chambrés are the same as in A.1. Theorem 1 holds with the modification that $l$ is replaced by the rank $l^{ss}$ of the semi-simple part $\prod_n G_n$. (The actual rank is dim $C = M + l^{ss}$.) A choice of chambre $C$ determines the decomposition $\Delta = \Delta_+ \cup \Delta_-$ and we put

$$M_C = \{\mu \in P^\vee; \alpha(\mu) = 0 \text{ or } 1 \text{ for any } \alpha \in \Delta_+\}.$$  \hspace{1cm} (A.6)

From the proposition 2 of A.1, one can see that $M_C \subset P^\vee$ is a section of the projection $P^\vee \rightarrow P^\vee / Q^\vee$.

The definition of the affine Weyl groups $W_{\text{aff}}, W'_\text{aff}$ are the same as in A.1. However, $W'_\text{aff}$ action on $\hat{V} = \mathbb{R}^{M+N}_+ \oplus V \oplus \mathbb{R}^{M+N}_c$ needs some modification if $H$ is not semi-simple: It depends on choice of the physical model which determines the lift of the rotation action on $LH$ to $\tilde{LH}$. (This could be read by looking at the energy momentum tensor given in Appendix B.) Hence, we shall consider only the action on $V_{\text{aff}} = \hat{V} / \mathbb{R}^{M+N}_c$ which is the same as in A.1. Affine root set $\Delta_{\text{aff}}$ is defined as the subset of $V_{\text{aff}}^* \cong \mathbb{R}_+^{M+N} \oplus V^*$. A system of hyperplanes in $\{(-1, \cdots, -1)\} \times V \subset V_{\text{aff}}$ is defined by using the affine roots and it leads to the definition of an alcôve. Theorem 3 holds under the modification that the number of walls of an alcôve is $l^{ss} + N$ and that $B(\hat{C})$ forms a base of $\mathbb{Z}^N \oplus Q$. 54
A choice of chambre $C$ determines an alcove $\hat{C}$ which in turn determines the decompositions $\Delta_{\text{aff}} = \Delta_{\text{aff}+} \cup \Delta_{\text{aff}-}$ and $W'_{\text{aff}} \cong W_{\text{aff}} \times \Gamma_{\hat{C}}$ where $\Gamma_{\hat{C}}$ is the isotropy subgroup of $W'_{\text{aff}}$ at $\hat{C}$. For an indexing set $\hat{J}$ of $M_C = \{ \mu_j ; j \in \hat{J} \}$, we can give a map $j \in \hat{J} \mapsto w_j \in W$ so that $\Gamma_{\hat{C}}$ is given by $\Gamma_{\hat{C}} = \{ \gamma_j ; j \in \hat{J} \}$ where $\gamma_j(\theta) = e^{-iw_j \theta} w_j w_0$. ($w_0$ is the longest element of $W$)

In any case, the groups

$$\pi_1(H), \: P^{\vee}/Q^{\vee}, \: W'_{\text{aff}}/W_{\text{aff}} \text{ and } \Gamma_{\hat{C}}$$

are all isomorphic.

**Appendix B.**

In a two dimensional quantum field theory coupled to background metric $g$ and gauge field $A$ for a Lie group $H$, the energy momentum tensor $T$ and the current $J$ is defined as the response to variation of $g$ and $A$: 

$$\delta Z_S(g, A; O_1 O_2 \cdots) = Z_S(g, A; \frac{1}{2\pi i} \int_{\Sigma} \{ \frac{i}{2} \sqrt{|g|} \delta g^{ab} T_{ab} + J \cdot \delta A \} O_1 O_2 \cdots). \quad (B.1)$$

The theory is said to be conformally invariant up to anomaly $c$ when $T_{zz} = -\frac{c}{12} R_{zz}$ on an insertionless region on which $A$ is flat, where $R_{zz}$ is the curvature of $g$. In this appendix, we give expressions of $T_{zz}$ and $J_z$ of the adjoint ghost system ($c = -2 \dim H$) and give a description of the Sugawara energy momentum tensor of the level $k$ WZW model with the target $H$ ($c = \frac{k}{k+h^c} \dim H$).

We first assume that $H$ is simple and compact. Choose a local complex coordinate $z$ ($g_{zz} = 0$) and a local holomorphic section $\sigma$ with respect to $A$. To a base $\{ e_a \}$ of $\mathfrak{h}_C$, $\sigma$ associates a local holomorphic frame $\{ \sigma_a \}$ of the adjoint bundle and the dual frame $\{ \sigma^a \}$ of the coadjoint bundle. We denote by $\omega$ and $A^a$ the Levi-Chivita connection and the connection $A$ represented via the holomorphic sections $\frac{\partial}{\partial z}$ and $\sigma$ respectively.

**Ghost System**

We put $c^a(z) = \sum_a e_a \sigma^a \cdot c(z) \in \mathfrak{h}_C$ and $b^a_z(z) = \sum_a e^a b_z \cdot \sigma_a(z) \in \mathfrak{h}_C^*$. Defining the regularized product $:b^a_\infty(z) c^a(w) :$ by 

$$b^a_z(z) \otimes c^a(w) = \frac{\sum_a e^a \otimes e_a}{z - w} + : b^a_z(z) \otimes c^a(w) :,$$  \quad (B.2)

we have

$$J_z \cdot \sigma X = : b^a_z \cdot [X, c^a] : - 2h^c \text{tr}(A^a_z X), \quad (B.3)$$

$$T_{zz} = : \partial_z b^a_z \cdot c^a : - : b^a_z \cdot [A^a_z, c^a] : + h^c \text{tr}(A^a_z A^a_z) - \frac{c}{12} S_{zz}, \quad (B.4)$$

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where \( \sigma X = \sum_a \sigma_a X^a \), \( \text{tr}(XY) = \frac{1}{2h} \text{tr}_\mathfrak{h} (\text{ad}X \text{ad}Y) \) and \( S_{zz} = \partial_z \omega_z - \frac{1}{2} \omega_z^2 \).

**Group H WZW Model At Level \( k \)**

To the current, we associate an \( \mathfrak{h}_C \)-valued holomorphic differential \( J^\sigma_z \) defined by

\[
J^\sigma_z \cdot \sigma X = J^\sigma_z \cdot X - k \text{tr}(A^\sigma_z X).
\]  
(B.5)

Defining the regularized product : \( J^\sigma_z(z) J^\sigma_z(w) : \) by

\[
J^\sigma_z(z) \cdot X J^\sigma_z(w) \cdot Y = \frac{k \text{tr}(XY)}{(z-w)^2} + J^\sigma_z(w) \cdot [X,Y] \frac{z-w}{z-w} + : J^\sigma_z(z) \cdot X J^\sigma_z(w) \cdot Y :,
\]  
(B.6)

the Sugawara energy momentum tensor is expressed as

\[
T_{zz} = \frac{\eta^{ab}}{2(k + h^\vee)} : J^\sigma_z \cdot e_a J^\sigma_z \cdot e_b : - J^\sigma_z \cdot A^\sigma_z + \frac{k}{2} \text{tr}(A^\sigma_z A^\sigma_z) - \frac{c}{12} S_{zz} ,
\]  
(B.7)

where \( \eta^{ab} \text{tr}(e_b e_c) = \delta^c_e \). This leads to differential equations of correlation functions \([16, 33, 40]\).

If \( H \) is an arbitrary compact group, \( \mathfrak{h} \) is decomposed as the sum \( \mathfrak{R}^M \oplus \oplus_{n=1}^N \mathfrak{g}_n \) of abelian and simple components. Since one can always find a local holomorphic section \( \sigma \) such that \( A^\sigma \) is of block diagonal form, generalization of (B.3), (B.4) and (B.7) to this case is obvious under the prescription that \( h^\vee = 0 \) for \( H = U(1) \).

**Appendix C.**

In this appendix, we derive residual gauge fixing term on the Riemann sphere \( \mathbb{P}^1 \). We assume that the gauge group \( H \) is simple and use notations introduced in Appendix A. Let \( \mu \in \mathfrak{C} \) be one of \( \{ \mu_j; j \in \mathfrak{J} \} \) and let \( \mathcal{P}_[\mu] \) be the holomorphic \( HC \)-bundle with transition function \( h_{\infty o}(z) = z^{-\mu} \).

**Description Of Symmetries.**

We introduce subsets of the set \( \Delta \) of roots:

\[
\Delta^{\mu,0} = \{ \alpha \in \Delta ; \alpha(\mu) = 0 \} , \quad \Delta^{\mu,0}_\pm = \Delta^{\mu,0} \cap \Delta_\pm ,
\]  
(C.1)

\[
\Delta^{\mu,1} = \{ \alpha \in \Delta ; |\alpha(\mu)| = 1 \} , \quad \Delta^{\mu,1}_\pm = \Delta^{\mu,1} \cap \Delta_\pm .
\]  
(C.2)

A holomorphic automorphism of \( \mathcal{P}_[\mu] \) is given by \( HC \)-valued holomorphic functions \( f_0(z) \) and \( f_\infty(z) \) of \( z \) and \( z^{-1} \) respectively such that \( f_0(z) = z^a f_\infty(z) z^{-a} \). An infinitesimal automorphism is thus of the form

\[
\delta f_0(z) = v + \sum_{\alpha \in \Delta^{\mu,0}} a_\alpha e_\alpha + \sum_{\alpha \in \Delta^{\mu,1}_\pm} (b_\alpha + z c_\alpha) e_\alpha ,
\]  
(C.3)
where \( v \in \mathfrak{t}_c \) and \( e_{\alpha} \in \mathfrak{h}_C \) is a root vector corresponding to \( \alpha \in \Delta \). Putting
\[ n^\mu_+ = \text{the abelian subalgebra of } \mathfrak{h}_C \text{ spanned by } \{ e_{\alpha}; \alpha \in \Delta^\mu_+ \}, \]
\( H^{\mu,0} \) is the subgroup of \( H \) of maximal rank with root system \( \Delta^{\mu,0} \), we can describe the automorphism group by
\[ f(h) \]
leads to the following expression depending only on the combination \( H \rightarrow \]
\[ \sigma \text{ flag} \]
by positive root vectors. We also take the Iwasawa decomposition
\[ \gamma \in \mathfrak{h}_C \]
to fix \( g \in \mathfrak{h}_C \) and similarly for \( h_\infty(\infty) \) where \( g_0 \in H \), \( \phi_0 \in \mathfrak{i}t \) and \( v_0 \) is spanned by root vectors. We also take the Iwasawa decomposition
\[ f^0 = g^0 e^{\frac{i\pi}{2}} v^0 \]
where \( g^0 \in H^{\mu,0}, \phi^0 \in \mathfrak{i}t \) and \( v^0 \) is spanned by root vectors for \( \Delta^\mu_+ \).

The condition \( h_\infty(\infty) \in H \) is necessary and enough to fix \( n_\infty, v^0 \) and \( \phi^0 \). In order to fix \( g^0 \), we decompose \( H^{\mu,0} \backslash H \) into pieces \( \{ U_\sigma \} \) so that we can find a section \( s_\sigma \) of \( H \rightarrow H^{\mu,0} \backslash H \) over each piece \( U_\sigma \). If we further require \( h_\infty(\infty) \) to be in some \( s_\sigma U_\sigma \), then, \( g^0 \) is fixed. The rest, \( n_0 \in n^\mu_+ \) is fixed by the condition \( v^0_0 = 0 \) for \( \alpha \in \Delta^\mu_+ \) where \( v^0_0 \) is the coefficient of \( v_0 \).

These conditions determine the residual gauge fixing term. The complexity arising from the gauge fixing of \( g^0 \) disappears if we integrate \( g_0 \) over \( H \) with the factor \( \frac{1}{\text{vol} H} \). This leads to the following expression depending only on the combination \( hh^* \) which is gauge invariant in the usual sense:
\[ \prod_{\alpha > 0} \delta^{(2)}(v^0_\alpha)e^\alpha e^\alpha \prod_{i=1}^l \delta(\phi^i)e^x e^x \frac{1}{\text{vol} H_{u,0}} \prod_{-\beta \in \Delta^{\mu,0}} c^\beta c^\beta \prod_{\alpha \in \Delta^\mu_+} \delta^{(2)}(v^0_\alpha)e^\alpha e^\alpha . \]
(C.5)

Appendix D.

This appendix gives an outline of the proof of the transformation rule \( (1.59) \) of \( \gamma_x \). The \( \gamma_x \)-transform \( (P^\gamma, f^\gamma) \) of an \( \mathcal{H}_C \)-bundle \( \mathcal{P} \) described by \( \sigma(qz) = \sigma(z)h(q; z) \) with a flag \( \sigma(1)h_f \) is defined by the relations \( (1.57) \) and \( (1.58) \) of an admissible section \( \sigma^*_0 \) around \( z = 1 \) and a section \( \sigma' \) over \( C^* - q^\mathbb{Z} \).

We shall find an everywhere regular (but multivalued) section \( \sigma^\gamma \). We put \( \sigma^\gamma(z) = \sigma'(z)\chi(z) \) for \( z \neq 1 \) and require the relation \( \sigma^\gamma(qz) = \sigma^\gamma(z)\chi(q; z) \) to hold. The task is
then to find such \( \bar{\chi}(z) \) that
\[
\begin{align*}
\bar{\chi}(zq) &= h(q; z)^{-1}\bar{\chi}(z)h'(q; z) \\
\chi(z) &= \sigma(\gamma(z-1))^{-1}h^{-1}_f \bar{\chi}(z)
\end{align*}
\]
is regular as \( z \to 1 \).

The latter condition arises by the requirement that \( \sigma_0^\gamma(z) = \sigma(\gamma(z))^{-1} \) is an admissible section around \( z = 1 \). In the following, the solution is exhibited as \((P, f) \to (P^\gamma f^\gamma) : \bar{\chi}(z)\).

\[
\begin{align*}
(P_{u0}^{(0)}, 1) &\to (P_{F1}^{(1)}, y_u) : \begin{pmatrix} r_uR_u(z) & ie^{-2\pi iu}q^{-\frac{1}{2}}r_uR_u(z) \\
r_uR_u(z) & -ie^{-2\pi iu}q^{\frac{1}{2}}r_uR_u(z) \end{pmatrix} ; u \neq 0 \\
(P_{00}^{(0)}, \infty) &\to (P_{F1}^{(1)}, y_0) : \begin{pmatrix} R_0(z)F(z) & iq^{-\frac{1}{4}}R_0(z)G(z) \\
R_0(z) & iq^{-\frac{1}{4}}R_0(z) \end{pmatrix} \\
(P_u^{(0)}, 0) &\to (P_{u1}^{(1)}, 0) : \begin{pmatrix} 0 & -c(z)^{-1} \\
c(z) & 0 \end{pmatrix} \\
(P_u^{(0)}, \infty) &\to (P_{u-1}^{(1)}, 0) : \begin{pmatrix} c(z) & 0 \\
0 & c(z)^{-1} \end{pmatrix} \\
(P_0^{(0)}, 0) &\to (P_{01}^{(1)}, 1) : \begin{pmatrix} c(z)H(z) & -c(z)^{-1} \\
c(z) & 0 \end{pmatrix}
\end{align*}
\]
where
\[
c(z) = \left( \vartheta(\tau, \zeta + \frac{\tau + 1}{2}) \right)^{\frac{1}{2}}, \quad z = e^{-2\pi i\zeta}
\]
\[
R_u(z) = \vartheta(2\tau, \zeta + 2u + \tau)/c(z), \quad r_u = c_\tau \cdot (z-1)^\frac{1}{2}R_u(z)|_{z=1},
\]
\[
F(z) = 2z \frac{\partial}{\partial z} \log \vartheta(2\tau, \zeta + \tau) - 1,
\]
\[
G(z) = 2z \frac{\partial}{\partial z} \log \vartheta(2\tau, \zeta),
\]
\[
H(z) = 2z \frac{\partial}{\partial z} \log c(z).
\]
in which \( c_\tau \) is a constant and \( \vartheta \) is the theta function \( \vartheta(\tau, \zeta) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2} z^{-n} \).

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