Kernel-based Federated Learning with Personalization

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Abstract

We consider federated learning with personalization, where in addition to a global objective, each client is also interested in maximizing a personalized local objective. We consider this problem under a general continuous action space setting where the objective functions belong to a reproducing kernel Hilbert space. We propose algorithms based on surrogate Gaussian process (GP) models that achieve the optimal regret order (up to polylogarithmic factors). Furthermore, we show that the sparse approximations of the GP models significantly reduce the communication cost across clients.

1 Introduction

Federated learning (FL) aims to offer the benefit of data sharing among distributed clients without exposing local data held by each client for privacy considerations. It has seen an increasing application in IoT, edge computing, healthcare and financial sectors. In FL, a central server iteratively aggregates local model updates from distributed clients to learn a global model. When local datasets across clients are statistically homogeneous, the global model, with the benefit of data sharing, offers better performance than locally trained models. When clients are statistically heterogeneous, however, the global model needs to be tailored or personalized at each individual client to strike a balance between data sharing and targeted local performance. In particular, FL with personalization has been recently introduced, where instead of learning a single global model, each client learns a mixture of the global model and its own local model to explicitly tackle heterogeneity across local datasets.

Most existing work on FL focuses on supervised learning with gradient based optimization algorithms such as FedAvg (Fallah et al., 2020; McMahan et al., 2017). Recently, a couple of studies considered zeroth-order (i.e., gradient free) unsupervised learning within the FL framework. For instance, Shi et al. (2021) considered the multi-armed bandit problem (with a finite number of arms) within the framework of FL with personalization. The problem was motivated by applications in cognitive radio and recommender systems. The finite arm assumption, however, can be restrictive.
In this work, we consider kernel-based zeroth-order stochastic optimization within the framework of FL with personalization, where zeroth order pertains to using estimated function values in model updates as compared to using gradient estimates in first-order optimization methods. Zeroth-order stochastic optimization can be equivalently viewed as a continuum-armed bandit problem, where the continuum arms form a high-dimensional set in the Euclidean space (see, e.g., Agarwal et al., 2011a; Vakili et al., 2019).

Our contribution consists of three results. First, we develop a near-optimal learning policy for kernel-based FL with personalization and analyze its regret performance. Referred to as FL with EXploration-exploitation sequencing (FLEX), this policy interleaves exploration and exploitation to not only address the usual tradeoff between learning vs. earning, but also to effectively control the information exchange across clients for learning the global objective (Section 3). Second, we show that the sparse approximation methods can significantly reduce the communication cost without sacrificing the regret performance (Section 4). Third, we derive a regret lower bound for the kernel-based FL with personalization framework (Section 5). The regret lower bound demonstrates that FLEX is order optimal up to a poly-logarithmic factor and a $\gamma^3$ factor, where $\gamma$ is the maximal information gain of the kernel (representing the effective dimension of the kernel, see details in Section 5).

**Related Work:** There is vast literature on FL framework introduced in McMahan et al. (2017). See AbdulRahman et al. (2021); Kairouz et al. (2021) for a survey of recent advances. FL with personalization was considered in Deng et al. (2020); Fallah et al. (2020); Hanzely & Richtárik (2020); Jiang et al. (2019); Kulkarni et al. (2020); Mansour et al. (2020); Smith et al. (2017); Wang et al. (2019), with a focus on supervised learning problems.

Unsupervised learning (e.g., stochastic learning problems) within the FL framework is gaining increasing attention. Shi & Shen (2021) considered the multi-armed bandit problem within the FL framework (without personalization). They derived performance bounds for a UCB based bandit policy. Dubey & Pentland (2020). Li et al. (2020) considered the same problem with distributionally identical clients, with a focus on ensuring privacy. The most relevant work to ours is perhaps Shi et al. (2021) which considered multi-armed bandit problem under FL with personalization framework. All the works mentioned above consider a finite set of arms which may be too restrictive in many settings. The modeling contribution of this work is extending this setting to continuum armed bandit with infinitely many arms, where kernel-based models are used for the objective functions.

The kernel-based bandit problem (in a single client setting) has been considered in several works. Representative examples include Chowdhury & Gopalan (2017), Srinivas et al. (2010); Valko et al. (2013). See Remark 3.3 for a comparison with these works. Other lines of work on continuum armed bandits exist relying on other regularity assumptions such as Lipschitz continuity (Bubeck et al., 2011; Kleinberg et al., 2008), convexity (Agarwal et al., 2011b) and unimodality (Combes et al., 2020), to name a few. We do not compare with these results due to the inherent difference in the regularity assumptions.
2 Problem Formulation

We consider a FL framework consisting of a central server and \( K > 1 \) clients. While clients cannot communicate with each other, they are all connected to the server, and can send and receive information to and from the server. Each client \( i = 1, 2, \ldots, K \) is associated with a local observation function, \( g_i : \mathcal{X} \to \mathbb{R} \), where \( \mathcal{X} \subset \mathbb{R}^d \) is a convex and compact domain. At each time \( t = 1, 2, 3, \ldots \), each client \( i \), can query its own observation function at a point \( x_{i,t} \in \mathcal{X} \) of its choice, and receive a noisy evaluation \( y_{i,t} = g_i(x_{i,t}) + \epsilon_{i,t} \) of the observation function, where \( \epsilon_{i,t} \) is a zero mean noise term.

We next specify our regularity assumptions on the observation functions and noise, which are standard for kernel-based learning.

Assumption 2.1. The observation functions \( g_i \) live in the Reproducing Kernel Hilbert Space (RKHS) associated with a known positive definite kernel \( k : \mathcal{X} \times \mathcal{X} \to \mathbb{R} \). In particular, the RKHS norm of \( g_i \) is bounded by a known constant \( B_i \): \( \|g_i\|_{H_k} \leq B_i \), for each \( i = 1, 2, \ldots, K \).

The RKHS corresponding to \( k \), denoted by \( H_k \), is a Hilbert space equipped with an inner product \( \langle \cdot, \cdot \rangle_{H_k} \), satisfying the reproducing property: \( \langle g, k(\cdot, x) \rangle_{H_k} = g(x) \), for all \( g \in H_k \) and \( x \in \mathcal{X} \). The inner product induces the RKHS norm \( \|g\|_{H_k}^2 = \langle g, g \rangle_{H_k} \), that can be understood as a measure of the smoothness of the function \( g \) with respect to the kernel \( k \). A constructive definition of the RKHS and its norm can be given using Mercer’s theorem [Steinwart & Christmann, 2008 Theorems 4.49, 4.51]. WLOG, we assume that \( k(x, x) \leq 1 \).

We note that with no regularity assumption on the objective functions, the continuum armed bandit problem cannot be effectively solved even for a single client. That being said, the kernel-based modeling is a very general approach to the problem. It is known that the RKHS of typical kernels, such as Matérn family of kernels, can approximate almost all continuous functions on compact subsets of \( \mathbb{R}^d \) [Srinivas et al., 2010].

Assumption 2.2. The noise terms \( \epsilon_{i,t} \) are assumed to be independent across \( i \) and \( t \). In addition \( \epsilon_{i,t} \) are assumed to be \( \mathcal{N} \)-sub-Gaussian for all \( t \). That is \( \mathbb{E} \left[ e^{\epsilon_{i,t}} \right] \leq \exp (\zeta^2 R^2 / 2) \), for all \( i = 1, 2, \ldots, K \), \( t \in \mathbb{N} \) and \( \zeta \in \mathbb{R} \).

Lastly, we assume there exists a fine discretization of the domain for RKHS elements, which is a standard assumption in the literature [Srinivas et al., 2010; Chowdhury & Gopalan, 2017].

Assumption 2.3. For each given \( n \in \mathbb{N} \) and \( f \in H_k \) with \( \|f\|_{H_k} \leq B \), there exists a discretization \( \mathcal{D}_n \) of \( \mathcal{X} \) such that \( |f(x) - f([x]_{\mathcal{D}_n})| \leq 1/\sqrt{n} \), where \( [x]_{\mathcal{D}_n} = \arg \min_{y \in \mathcal{D}_n} \|x - y\|_2 \) is the closest point in \( \mathcal{D}_n \) to \( x \) in terms of \( \ell_2 \) distance and \( |\mathcal{D}_n| \leq C B^d n^{d/2} \), where \( C \) is a constant independent of \( n \) and \( B \).

2.1 FL with Personalization

In the FL framework (without personalization), it is standard to define the global reward\(^1\) function as the average of the local observation functions

\[
f_G(\cdot) = \frac{1}{K} \sum_{i=1}^{K} g_i(\cdot).
\]

\(^1\)It may be referred to as objective or loss (in the case of minimization problems) in other contexts.
For example, Shi & Shen (2021) considered the finite armed bandit problem under this setting. As discussed before, this framework may not effectively capture the heterogeneity between the clients. That is, the global reward may not suitably represent local observations \( g_i \) of each client \( i \). The global reward model can be seen as the most altruistic end of a spectrum, where the clients are only contributing to the global objective. The other end of this spectrum is purely greedy behavior that is when client \( i \) aims at only learning its own \( g_i \), without contributing to learning \( f_G \).

Following Deng et al. (2020); Hanzely & Richtárik (2020); Shi et al. (2021), we consider a FL with personalization model, trading off between these altruistic and greedy objectives. In particular, we consider a personalized reward function

\[
    f_i(\cdot) = \alpha_i g_i(\cdot) + (1 - \alpha_i) f_G(\cdot),
\]

for each client \( i \). It is a linear combination of the global reward and the local observation. Here, the personalization parameter \( \alpha_i \in [0, 1] \) captures the level of personalization for each client \( i \). \( \alpha_i = 0 \) recovers the altruistic behavior of only contributing to the learning of global reward. While, \( \alpha_i = 1 \) recovers the greedy behavior of independently learning the local observation function \( g_i \).

We note that, using the definition of \( f_G \), we can rewrite personalized reward as follows

\[
    f_i(\cdot) = \left( \alpha_i + \frac{1 - \alpha_i}{K} \right) g_i(\cdot) + \sum_{j=1, j\neq i}^{K} \frac{(1 - \alpha_i)}{K} g_j(\cdot).
\]

For clarity of terminology, we refer to \( f_G \) as the global reward, \( g_i \) as the local observation, and \( f_i \) as the personalized reward for client \( i \).

In the FL framework, we define a learning policy \( \pi = \{\pi_{i,t} : \mathcal{F}_i^t \to \mathcal{X}\}_{i=1,2,...,K} \) as sequences of mappings, from the information available to each client \( i \) at time \( t \) to an observation point, for each client. The information \( \mathcal{F}_i^t \) consists of the previous observations \( \{(x_{i,t'}, y_{i,t'} = g_i(x_{i,t'} + \epsilon_{i,t'})\}_{t'=1}^{t-1} \) of client \( i \) from its local observation function, as well as some of the local observations from other clients provided to client \( i \) by the server, yet to be specified.

The performance of \( \pi \) is measured in terms of total cumulative regret. Specifically,

\[
    R_{\pi}(T) = \sum_{i=1}^{K} \sum_{t=1}^{T} (f_i(x^*_{i,t}) - f_i(x_{i,t})),
\]

where \( x^*_{i,t} = \arg \max_{x \in \mathcal{X}} f_i(x) \).

The objective is to design a policy \( \pi \) which minimizes the total cumulative regret. We provide high probability regret bounds that hold with probability at least \( 1 - \delta_0 \) for any given \( \delta_0 \in (0, 1) \), a stronger performance guarantee than bounds on expected regret.

### 2.2 Communication Protocol and Cost

Each personalized reward \( f_i \) depends on the observation functions \( g_j \) of all clients \( j \). The clients thus need to share information about their local observations among each other, in order to make learning \( f_i \) feasible. We here describe the communication protocol in our FL framework. The information is shared through the server. At each time \( t \), each client \( i \) is allowed to send some data to the server. The data is then broadcast to all other clients. For simplicity, we assume that all the communication
is synchronized, that is a typical assumption in the FL framework [McMahan et al., 2017], and is the same as [Shi et al., 2021]. The shared data is in the form of the local observations of the clients \((x_{i,t}, y_{i,t})\). We assign a unit cost to each communication of a single observation to other clients through the server.

# The FLEX Policy

In this section, we present the FLEX policy. The policy consists of interleaving exploration and exploitation sequences. In the exploration sequence, all the clients contribute to learning the local observation functions \(g_i\), by choosing observation points distributed across the entire domain. This is crucial for all the clients to learn their own personalized reward functions. In the exploitation sequence, the clients aim at achieving maximum possible personalized reward, based on the information gathered so far. The cardinality of the exploration sequence is carefully designed in a way to optimally trade off the exploration and exploitation, and guarantee low regret. In addition, the separation between exploration and exploitation allows effective control of information exchange across clients for learning the global objective. Both exploration and exploitation in FLEX policy are guided by a fictitious Gaussian process (GP) prior for the elements of the RKHS. We refer to [McMahan et al., 2017, Chowdhury & Gopalan, 2017, Shekhar & Javidi, 2018, Srinivas et al., 2010] for more details.

Before presenting FLEX, we overview the GP models and some useful confidence intervals for the RKHS elements based on GP models, which are central to our policy design.

## 3.1 Preliminaries on GP Models

Following a standard approach in the literature (e.g., see [Chowdhury & Gopalan, 2017, Shekhar & Javidi, 2018, Srinivas et al., 2010]), we assume a fictitious GP prior for a function \(g\) belonging to the RKHS. We use the notation \(G\) to distinguish the fictitious GP model from the fixed function \(g\). Specifically \(G(x), x \in \mathcal{X}\) is a random process indexed on \(\mathcal{X}\), for which all finite subsets \{\(G(x_i)\)\}_{i=1}^{n}, n \in \mathbb{N}\), have a multivariate Gaussian distribution, with mean \(\mu(\cdot) = E[G(\cdot)]\) and covariance \(k(\cdot, \cdot') = E[(G(\cdot) - \mu(\cdot))(G(\cdot') - \mu(\cdot'))]\). We assume \(\mu(x) = 0\), for all \(x \in \mathcal{X}\).

Assuming also a fictitious Gaussian distribution for observation noise, the conjugate property provides closed form expressions for the posterior mean and covariance of the GP model. In particular, provided observations \(\mathcal{H}_t = \{x_t, y_t\}\), where \(x_t = (x_1, x_2, \ldots, x_t)\top, y_t = (y_1, y_2, \ldots, y_t)\top\), \(y_s = g(x_s) + \epsilon_s\), for all \(s\), the following expressions can be derived for the posterior mean and covariance of the GP model

\[
\mu_t(x) = E[G(x)|\mathcal{H}_t] = k_{x_t,x}^\top (K_{x_t,x} + \lambda I_t)^{-1} y_t
\]

(4)

\[
k_t(x, x') = E[(G(x) - \mu_t(x))(G(x') - \mu_t(x'))|\mathcal{H}_t] = k(x, x') - k_{x_t,x}^\top (K_{x_t,x} + \lambda I_t)^{-1} k_{x_t,x'}.
\]

(5)

In the above expressions, \(k_{x_t,x} = [k(x_1, x), \ldots, k(x_t, x)]\top\), \(K_{x_t,x}\) is the \(t \times t\) covariance matrix \([k(x_i, x_j)]_{i,j=1}^t\), \(I_t\) is the \(t \times t\) identity matrix and \(\lambda\) is the variance of the fictitious Gaussian distribution assumed for the noise terms. We use \(\sigma^2_t(\cdot) = k_t(\cdot, \cdot)\) to denote the posterior variance of the GP model.
The reason for introducing this fictitious model is that the posterior mean and variance defined above, respectively, provide powerful tools to predict the values of \( g \), and to quantify the uncertainty in the prediction. We formalize this statement in the following lemma.

**Lemma 3.1** (Theorem 1 in [Vakili et al. (2021a)]). Under Assumptions 2.1 (\( \|g\| \leq B \)) and 2.2 (\( \epsilon_s \) being R-sub-Gaussian, for all \( s \)), provided observations \( \mathcal{H}_t = \{x_t, y_t\} \) as specified above with the query points \( x_t \) chosen independently of the noise sequence, we have, for a fixed \( x \in \mathcal{X} \), with probability at least \( 1 - \delta \),

\[
|g(x) - \mu_t(x)| \leq \beta(\delta)\sigma_t(x),
\]

where \( \beta(\delta) = B + R\sqrt{2/\lambda}\log(2/\delta) \).

This lemma shows that \( \mu_t(\cdot) \) may be used to predict the value of \( g(\cdot) \), where the error in the prediction is bounded by a factor of \( \sigma_t(\cdot) \) with high probability. In FLEX, \( \sigma_t(\cdot) \) is used to guide exploration and \( \mu_t(\cdot) \) is used to guide exploitation.

### 3.2 FLEX

There exist several kernel-based learning algorithms including GP-UCB and GP-TS [Chowdhury & Gopalan (2017)] for a single client setting. We note that even if all the observations of each client were available to all other clients, following the existing algorithms on personalized rewards would result in a trivial regret, linearly growing with \( T \). For example, consider two clients with significantly different optimum points \( x_1^* \neq x_2^* \), \( f_1(x_1^*) \gg f_1(x_2^*) \) and \( f_2(x_2^*) \gg f_2(x_1^*) \). Then, using an algorithm with low regret in the single client setting (e.g., GP-UCB or GP-TS), the observations of client 1 should ideally concentrate around \( x_1^* \), meaning client 1 does not provide informative observations about \( g_1(x_2^*) \) to client 2. Those, however, are crucial for client 2 being able to get closer to \( x_2^* \). The same is true about the crucial information which client 1 requires from client 2. We thus need to design a collaborative algorithm where each client provides altruistic observations for others, which will eventually be beneficial for all the clients, and allow them to learn their own maximum personalized reward.

Inspired by determinstic sequencing of exploration and exploitation (DSEE) algorithm for multi-armed bandits [Vakili et al. (2013)], we propose FLEX that is a kernel-based policy which effectively mixes exploration and exploitation sequences. The selected points in each sequence are guided based on the statistics provided by the GP model.

Let \( \mathcal{A}(t) \) denote the exploration sequence up to time \( t \), meaning if \( s \in \mathcal{A}(t) \), the clients explore at time \( s \). The exploration in FLEX is designed to maximally reduce the uncertainty about the local observation functions \( g_i \), using the variance (uncertainty estimate) provided by the fictitious GP model. In particular, when \( t \in \mathcal{A}(t) \), client \( i \) chooses

\[
x_{i,t} = \arg\max_{x \in \mathcal{X}} \sigma_{i-1}^{(g_i)}(x),
\]

where \( \sigma_{i-1}^{(g_i)}(\cdot) \) is the posterior standard deviation of the GP model of \( g_i \) based on all the previous exploratory observations \( S_{i,t} = \{(x_{i,s}, y_{i,s}) : s \in \mathcal{A}(t-1)\} \) of client \( i \) according to equation \( (\ref{eq:post}) \).
In the exploitation sequence, each client $i$ chooses a point with the highest predicted value based on the GP model. In particular, when $t \notin \mathcal{A}(t)$, client $i$ chooses

$$x_{i,t} = \arg \max_{x \in \mathcal{X}} \mu_{t-1}^{(f_i)}(x),$$

where $\mu_{t-1}^{(f_i)}(\cdot)$ is the posterior mean of the personalized reward function, given as

$$\mu_{t-1}^{(f_i)}(\cdot) = \alpha_i \mu_{t-1}^{(g_i)}(\cdot) + \frac{1 - \alpha_i}{K} \sum_{j=1}^{K} \mu_{t-1}^{(g_j)}(\cdot).$$

In the above expression $\mu_{t-1}^{(g_j)}(\cdot)$ are the posterior mean of the GP model of $g_j$ based on all the previous exploratory observations $\mathcal{S}_{j,t}$ corresponding to client $j = 1, 2, \ldots, K$.

**Communication:** The observations $\mathcal{S}_{i,t}$ of each client $i$ in the exploration sequence are communicated to the server which is then broadcast to all other clients. An advantage of FLEX is that the amount of communication can be easily controlled by the cardinality of the exploration sequence.

**The cardinality of the exploration sequence:** We define the exploration sequence as follows: if $|\mathcal{A}(t)| < N_t$, then $\mathcal{A}(t+1) = \mathcal{A}(t) \cup \{t+1\}$, where $N_t : \mathbb{N} \to \mathbb{R}$ is a positive non-decreasing sequence of real numbers, controlling the cardinality of the exploration sequence.

**Algorithm 1 FLEX**

**Input:** $\{B_j\}_{j=1}^{K}$, the kernel $k(\cdot, \cdot)$.

Set $t \leftarrow 0$, $\mathcal{A}(0) \leftarrow \emptyset$

repeat

if $|\mathcal{A}(t)| < N_t$ then

$\mathcal{A}(t+1) = \mathcal{A}(t) \cup \{t+1\}$

end if

$t \leftarrow t + 1$

if $t \in \mathcal{A}(t)$ then

$x_{i}^{(t)} = \arg \max_{x \in \mathcal{X}} \sigma_{t-1}^{(g_i)}(x)$

else

$x_{i}^{(t)} = \arg \max_{x \in \mathcal{X}} \mu_{t-1}^{(f_i)}(x)$

end if

Query the function $g_i$ at $x_{i,t}$ to obtain $y_{i,t}$

if $t \in \mathcal{A}(t)$ then

Send $(x_{i,t}, y_{i,t})$ to the server which is then broadcast to all clients

end if

until $t = T$

The theorem below establishes an upper bound on the regret performance of FLEX.

**Theorem 3.2.** Consider the kernel-based FL with personalization setting described in Section 2.1. Under Assumptions 2.1 and 2.2 the regret performance of FLEX satisfies, for $\delta_0 \in (0,1)$, with probability at least $1 - \delta_0$,

$$R_{\text{FLEX}}(T) = O \left( KN_T + KT \sqrt{\frac{\gamma T}{N_T} \log \left( \frac{1}{\delta_0} \right)} \right).$$
That implies, when \( N_T = \Theta(T^{\frac{2}{3}}(\gamma_T \log(1/\delta_0))^{\frac{1}{3}}) \),

\[
R_{\text{FLEX}}(T) = O(KT^{\frac{2}{3}}(\gamma_T \log(1/\delta_0))^{\frac{1}{3}}).
\]

In Theorem 3.2, \( \gamma_t \) is the maximal information gain, and is representative of the effective dimension of the kernel. It is defined as \( \gamma_t := \max_{A \subseteq X : |A| = t} I(y_A ; g_A) \) where, \( I(y_A ; g_A) \) denotes the mutual information between \( g_A = [g(x)]_{x \in A} \) and \( y_A = g_A + \epsilon_A \). Bounds on \( \gamma_t \) for several common kernels are known\cite{sridivas2012,vakili2020a} and are increasing sublinear functions of \( t \).

We here provide a sketch of the proof. We bound the regret in the exploration and exploitation sequences separately. The bound on the RKHS norm of \( f_i \) implies a bound on its sup norm. Thus, the regret in the exploration sequence is simply bounded by \( \mathcal{O}(KN_T) \). The regret during the exploitation sequence is bounded using Lemma 3.1 based on the posterior variance of the GP model. Maximal uncertainty reduction sampling and a known bound on the total uncertainty (cumulative conditional variances) of a GP model based on information gain allows us to bound the posterior variance with an \( \mathcal{O}(\frac{2K}{N_T^2}) \) term. Combining these two results, we bound the regret in the exploitation sequence. A detailed proof of the Theorem is provided in Appendix A.

Remark 3.3. The upper bound on the regret of FLEX given in Theorem 3.2 is sublinear in \( T \) as \( \gamma_T = o(T) \) for typical kernels \cite{vakili2020a}. A sublinear regret bound guarantees the convergence to the maximum personalized reward \( f_i(x^*_i) \) across all clients, as \( T \) grows. We would like to emphasize the significance of this guarantee for the performance of FLEX. In comparison, it is not clear whether standard algorithms such as GP-UCB or GP-TS achieve sublinear regret, in the simpler problem of a single client. The existing upper bounds on the regret performance of these algorithms are in the form of \( \tilde{\mathcal{O}}(\sqrt{T}) \), which may be trivial, as \( \gamma_T \) may grow faster than \( \sqrt{T} \). For example, that is the case with a broad range of parameters in the case of Matérn kernel (see, \cite{vakili2021b} for a detailed discussion). The FLEX algorithm thus may be of interest even for a single client setting, in terms of introducing a simple algorithm with sublinear regret. We, however, note that more sophisticated algorithms such as GP-ThreDS \cite{salgia2020} and SupKernelUCB \cite{valko2013} achieve better regret bounds in the case of a single client.

Choosing the exploration sequence: A key design parameter in FLEX is the size \( N_T \) of the exploration sequence. It determines the performance of the algorithm by controlling the trade-off between the cost of exploration and the cost of exploitation. This is seen in the expression of regret given in Theorem 3.2. The first term, \( \mathcal{O}(KN_T) \), corresponds to the regret incurred during the exploration sequence, and increases with \( N_T \). The second term, \( \mathcal{O} \left( KT \sqrt{\frac{7N}{N_T}} \log(\frac{1}{\delta_0}) \right) \), corresponds to the regret incurred during exploitation, and decreases with \( N_T \). That is, a larger exploration sequence enables a better approximation of the arm statistics, which leads to a better performance during the exploitation sequence. The optimal size \( N_T \) of the exploration sequence is one that minimizes the larger value between these two terms. That is how \( N_T = \Theta(T^{\frac{2}{3}}(\gamma_T \log(1/\delta_0))^{\frac{1}{3}}) \) in Theorem 3.2 is selected.

4 Reducing Communication Cost via Sparse Approximation

In Section 3, we introduced FLEX and provided an analysis of its regret performance. Also, as discussed, the amount of communication required by FLEX is \( \mathcal{O}(N_T) \) that is upper bounded by
\[ O(T^{2/3}N^{-1/3}) \] for the optimal choice of \( N_T \). In this section, we build on sparse approximations of the posterior GP models to introduce Sparse FLEX which achieves the same regret order with a significantly reduced communication cost.

### 4.1 Sparse Approximations

The sparse approximations of the GP models are developed under two closely related but nonetheless separate approaches, with different terminologies. One approach is based on variational inference (often referred to as sparse variational Gaussian processes, SVGP), and one approach is based on Nyström method. The representative works are Hensman et al. (2013); Titsias (2009); Vakili et al. (2020b) on the former, and Calandriello et al. (2019, 2017); Seeger et al. (2003) on the latter. A recent overview (Wild et al., 2021) of these methods has shown that they essentially lead to the same result using the same computational steps. In this paper, we use the terminology of Nyström method.

Recall the closed form expressions for the posterior mean and covariance of the GP model based on \( t \) observations \( \mathcal{H}_t = \{(x_i, y_i)\} \), given in equation (1) and (5). Let \( z_m = \{z_1, z_2, \ldots, z_m\} \subset x_t \) be a subset of the observation points, which we aim to use in approximating the posterior distribution of the GP model. These points are often referred to as the *inducing points*. The Nyström approximations are given as follows (Wild et al., 2021):

\[
\hat{\mu}(x) = k_{z_t, x}^T(\lambda K_{z_t, z_t} + K_{z_t, x}K_{x, z_t})^{-1}K_{z_t, x}y_t
\]

\[
\hat{\sigma}^2(x) = \frac{1}{\lambda}(k(x, x) - k_{z_t, x}^TK_{z_t, x}K_{x, z_t})^{-1}k_{z_t, x}^T + K_{z_t, x}K_{x, z_t}K_{x, z_t}^{-1}k_{z_t, x}
\]

where \( K_{z_t, z_t} = [k(z_i, z_j)]_{i,j=1}^{m} \in \mathbb{R}^{m \times m} \), \( K_{z_t, x} = [k(z_i, x_j)]_{i=1,j=1}^{m,t} \in \mathbb{R}^{m \times t} \) and \( K_{x, z_t} = K_{z_t, x}^T \).

### 4.2 Sparse FLEX

Sparse FLEX is similar to FLEX with some minor differences, outlined as follows. Firstly, the exploration sequence, consisting of \( N_T \) points, is entirely concentrated at the beginning of the time horizon by setting \( N_t = N_T \) in FLEX. This design provides the clients with the entire exploration sequence which is required to construct the set of inducing points in Nyström approximation. Then, each client \( i \) constructs its set of inducing points \( z_{i,N_T} = (z_{i,1}, z_{i,2}, \ldots, z_{i,m_i}) \) by selecting each point \( x_{i,j} \in x_{i,N_T} \) to be included in \( z_{i,N_T} \) with a probability equal to \( q_0 \left[ \sigma_{N_T}^{(q)}(x_{i,j}) \right]^2 \). Here \( q_0 \) is a constant that is used to appropriately scale the probabilities of inclusion to ensure a sufficiently large set for a good approximation (Calandriello et al., 2019). Consequently, the set of inducing points for each client \( i \) consists of a random number of points \( m_i \). Each client \( i \), using their set of inducing points, then proceeds to compute the vector \( w_i = (\lambda K_{z_{i,N_T}, z_{i,N_T}} + K_{z_{i,N_T}, x_{i,N_T}}K_{x_{i,N_T}, z_{i,N_T}})^{-1}K_{z_{i,N_T}, z_{i,N_T}}y_{i,N_T} \).

The second difference between FLEX and Sparse FLEX is that, there is a communication sequence of length \( N_T^{(c)} \). At the \( s^{th} \) instant of this sequence, client \( i \) communicates \( (z_{i,s}, w_{i,s}) \) to the server which then broadcasts it to all the clients. Here \( w_{i,s} \) denotes the \( s^{th} \) element of the vector \( w_i \). The length \( N_T^{(c)} \) is chosen such that it is larger than the number of inducing points for all the clients with high probability (See Lemma 4.1). During the communication sequence, the clients may select...
arbitrary points (e.g. the maximizer of the prediction of $g_i$). After the communication sequence, the clients select the maximizer of the Nyström approximate posterior mean of their personalized reward function for the rest of the time horizon. The approximate posterior mean of $f_i$ is given as

$$\tilde{\mu}_{N_T}(f_i) = (1 - \alpha_i) \tilde{\mu}_{N_T}(g_i) + \frac{K}{\alpha_i} \sum_{j=1}^{K} \tilde{\mu}_{N_T}(g_j)$$

with $\tilde{\mu}_{N_T}$'s as defined in (6). A pseudocode is provided in Algorithm 2.

We first establish a bound on the length of the communication sequence. The following lemma, which is a slightly modified version of Theorem 1 in Calandriello et al. (2019), gives us an upper bound on the largest inducing set among all the clients.

**Lemma 4.1.** Fix $\delta \in (0, 1)$, $\varepsilon \in (0, 1)$ and let $\alpha = (1 + \varepsilon)/(1 - \varepsilon)$. If the set of inducing points is constructed as outlined in Section 4.2 with $q_0 \geq 6\alpha \log(4TK/\delta)/\varepsilon^2$, then with probability at least $1 - \delta$, $\max_i m_i \leq 9(1 + 1/\lambda)q_0\gamma N_T$.

Consequently, setting $N_T^{(c)}$ to $9(1 + 1/\lambda)q_0\gamma N_T$, ensures that the set of inducing points for all the clients can be completely transmitted during the communication sequence thereby showing the size of the required communication is at most $O(\gamma N_T)$.

In the following theorem, we formally state that the significant improvement in the communication cost achieved by Sparse FLEX is without sacrificing the regret order.

**Theorem 4.2.** Consider the kernel-based FL with personalization setting described in Section 2.1. Consider Sparse FLEX based on Nyström approximation as described in Algorithm 2. For $\delta_0 \in (0, 1)$, with probability at least $1 - \delta_0$, Sparse FLEX Satisfies

$$R_{Sparse \text{ FLEX}}(T) = O \left( KN_T + KT \sqrt{\frac{\gamma N_T}{N_T} \log \left( \frac{1}{\delta_0} \right)} \right)$$

In addition, the communication cost of Sparse FLEX is bounded by $N_T^{(c)} = O(K\gamma N_T)$.

**Remark 4.3.** The communication cost of Sparse FLEX is significantly lower than that of FLEX. For example in the case of SE kernel, with $N_T = O(KT^{3/2} (\gamma T \log(1/\delta_0))^{3/2})$, the communication cost of Sparse FLEX is bounded by $O(K \log^{d+1}(T))$, while the communication cost of FLEX can be as large as $O(KT^{3/2} (\log(T))^{d+1})$.

The proof of the theorem is similar to that of Theorem 3.2. We bound the regret in exploration, communication and the exploitation sequences separately. The exploration regret is bounded by $O(KN_T)$. The regret in communication regret is similarly bounded based on the length of the communication sequence: $O(K\gamma T)$. The exploitation regret is bounded following similar steps as in the proof of 3.2 except that we bound the instantaneous regret based on approximate posterior standard deviation in contrast to the exact one. To do this, we use the following lemma which is a counterpart of Lemma 3.1 in the case of sparse approximation of the posterior mean and variance.
Lemma 4.4. Under Assumptions 2.1 and 2.2, provided a set of observations $\mathcal{H}_t = \{x_t, y_t\}$, which are chosen independently of the associated noise sequence, and corresponding subset of inducing points $\mathbf{z}_t$ obtained as outlined in Sec. 4.3, we have, for a fixed $x \in \mathcal{X}$, with probability at least $1 - \delta$,

$$|g(x) - \hat{\mu}_t(x)| \leq \beta(\delta)\tilde{\sigma}_t(x),$$

where $\hat{\mu}(x)$ and $\hat{\sigma}(x)$ are as defined in (7) and $\tilde{\beta}(\delta) = B\sqrt{\frac{2\lambda}{1-\varepsilon}} + R\sqrt{2\log\left(\frac{1}{\delta}\right)}$.

Please refer to Appendix B for a detailed proof of Theorem 4.2 and the above lemma.

Algorithm 2 Sparse FLEX

Input: $N_T$, the kernel $k(\cdot, \cdot)$, $\varepsilon \in (0, 1)$, $\delta_0 \in (0, 1)$, $\lambda$.

$z_{i,N_T} \leftarrow \emptyset$, $q_0 \leftarrow 6(1 + \varepsilon) \log(8TK/\delta_0)/\varepsilon^2(1-\varepsilon)$, $N_T^{(c)} \leftarrow 9(1 + 1/\lambda)q_0\gamma N_T$

for $t = 1, 2, \ldots, N_T$ do

Choose $x_{i,t} = \arg \max_{x \in \mathcal{X}} \sigma_{N_T}^{(g)}(x)$

Query the function $g_i$ at $x_{i,t}$ to obtain $y_{i,t}$

end for

for $j = 1, 2, \ldots, N_T$ do

Set $p_j \leftarrow q_0 \left[ \sigma_{N_T}^{(g)}(x_{i,j}) \right]^2$

Draw $W_j \sim \text{Bernoulli}(p_j)$

if $W_j == 1$ then

$z_{i,N_T} \leftarrow z_{i,N_T} \cup \{x_{i,j}\}$

end if

end for

Evaluate the vector $w_i = (\lambda K_{z_{i,N_T}, z_{i,N_T}} + K_{z_{i,N_T}, x_{i,N_T}} K_{x_{i,N_T}, z_{i,N_T}})^{-1}K_{z_{i,N_T}, x_{i,N_T}}y_{i,N_T}$

for $s = 1, 2, \ldots, N_T^{(c)}$ do

$t \leftarrow t + 1$

$x_{i,t} \leftarrow \arg \max_{x \in \mathcal{X}} \hat{\mu}_{N_T}^{(g)}(x)$

if $s \leq m_i$ then

Send $(z_{i,s}, w_{i,s})$ to the server which is then broadcast to all clients

end if

end for

repeat

$t \leftarrow t + 1$

$x_{i,t} \leftarrow \arg \max_{x \in \mathcal{X}} \hat{\mu}_{N_T}^{(f)}(x)$

until $t = T$

5 Regret Lower Bound

In this section, we establish a lower bound on regret for the kernel-based FL with personalization problem. In particular, we prove an $\Omega(T^{2/3})$ lower bound on total regret defined in (3), when there is no communication constraint.
Theorem 5.1. Consider the FL with personalization setting given in Sec. 2.1 with no constraint on communication. There exists a choice of observation functions \( \{g_i\}_{i=1}^{K} \), for which, the cumulative regret of any policy \( \pi \), satisfies
\[
E[R_\pi(T)] = \Omega(T^{\frac{2}{3}}).
\] (8)

Please refer to Appendix C for a detailed proof.

Remark 5.2. Shi et al. (2021) also conjectured a lower bound for the regret in the finite arm setting. Their lower bound is however different in the sense that it is a distribution-dependent lower bound. The distribution-dependent lower bounds typically are given in terms of the gaps (in the mean and KL-divergence) between the best arms and suboptimal arms. These bounds do not apply to our setting, in the sense that such distribution-dependent gaps are always zero in the case of infinite arms (modeled as continuous functions on compact domains). We thus use a different method to prove a minimax bound on the regret.

Figure 1: (a) Cumulative regret of FLEX over \( T = 1000 \) steps when run with different lengths of exploration sequence averaged over 10 Monte Carlo runs. (b) Cumulative regret of SparseFLEX after \( T = 1000 \) steps relative to FLEX plotted against the reduction in communication offered relative to FLEX when SparseFLEX is run with different sizes of inducing sets. (c) Cumulative regret incurred by different algorithms over \( T = 1000 \) steps averaged over 10 Monte Carlo runs.

Remark 5.3. Theorem 5.1 shows that the regret performance of (Sparse) FLEX is optimal up to logarithmic factors as well as an \( \gamma_T \) factor. In the case of SE kernel, the regret performance of (Sparse) FLEX is optimal up to logarithmic factors.

6 Empirical Studies

In this section, we corroborate our theoretical findings with empirical evidence. We perform three sets of experiments. In the first one, we provide numerical results on the performance of FLEX for various choices of the length of exploration sequence. We empirically confirm our theoretically obtained optimal choice of \( N_t \). In the second one, we study the effect of reduced communication in
Sparse FLEX on its regret performance. We show that Sparse FLEX can offer a 15-fold reduction in the communication cost without a significant increase in regret. Lastly, we compare the performance of FLEX against baseline algorithms: IGP-UCB [Chowdhury & Gopalan 2017], GP-EI (Expected Improvement) [Wang & de Freitas 2014], and GP-PI (Probability of Improvement) [Wang et al. 2018].

In all experiments, we consider a setting of $K = 4$ agents with $a_i = 0.1i$, for $i = 1, 2, 3, 4$. For each observation function, we consider a variation of the commonly used benchmark function $Branin$, defined on $[0,1]^2$, denoted by $B(x_1, x_2)$. An analytical expression of the $Branin$ function is given in Azimi et al. (2012); Picheny et al. (2013). The observation functions are chosen as $g_1 = B(x_1, x_2)$, $g_2(x_1, x_2) = B(x_1, x_2^2)$, $g_3(x_1, x_2) = B(x_1^2, x_2)$ and $g_4(x_1, x_2) = B(x_1^2, x_2^2)$. All the algorithms are implemented using a uniform discretization of 1600 points over the domain, $[0,1]^2$. We use a SE kernel with lengthscale parameter $l = 0.2$. The observations are corrupted with Gaussian noise with variance 0.01, and $\lambda$ is also set to 0.01.

For the first experiment, we run the FLEX algorithm for $T = 2000$ steps, with $N_t = \lceil 0.4t^\eta \rceil$ for the value of $\eta \in \{1/6, 1/4, 1/3, 1/2, 2/3, 3/4, 5/6\}$. In Fig. 1a, we plot the average regret over 10 Monte Carlo runs for each choice of exploration sequence. Fig. 1a shows that the choice of $N_T = \mathcal{O}(T^{2/3})$ results in the smallest regret, which is consistent with our theoretical findings.

In the second experiment, we evaluate the cumulative regret of Sparse FLEX at the end of $T = 1000$ steps for varying sizes of the set of inducing points. We set $N_T = 300$. The value of $q_0$ is varied to obtain various number of inducing points. Recall that the number of inducing points determines the communication cost. The vertical axis shows the ratio between the regret performance of Sparse FLEX to FLEX. The horizontal axis shows the ratio between the communication cost of FLEX to Sparse FLEX. All experiments are averaged over 10 Monte Carlo runs. Fig. 1b shows that Sparse FLEX may offer a reduction in communication cost by a factor of 15; while, sacrificing regret with a factor less than 1.5. This shows the efficiency of sparse approximation methods in practice.

In the third experiment, we compare the regret performance of FLEX with standard baseline GP bandit algorithms, namely, IGP-UCB, GP-EI and GP-PI. For these baselines, all agents optimize their own objective functions $f_i$. For IGP-UCB, we set $B = 15$ and $R^2 = \lambda = 0.01$. For GP-EI and GP-PI, the margin terms are set to 0.01. We compare these algorithms with FLEX which is run with $N_t = 40$, a constant value.

Fig 1c shows the cumulative regret of each algorithm over $T = 1000$ averaged over 10 Monte Carlo runs. We see that although the regret of SparseFLEX initially grows faster—corresponds to the exploration sequence—eventually it achieves the lowest regret. This experiment corresponds to our discussion in Section 3.2, that baseline GP bandit algorithms do not perform well in the federated setting, even with communication at each round. The observations obtained during the exploration sequence in FLEX helps all the agents to perform better.

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A Proof of Theorem 3.2

To bound the regret of FLEX, we consider the regret incurred during the exploration and exploitation sequences separately. The regret corresponding to the exploration sequence is bounded with a constant factor of the cardinality of the exploration sequence, yielding the $KN_T$ term. The bound on the regret corresponding to the exploitation sequence in obtained in two steps. In the first step, we use the choice of $x_{i,t}$ and Lemma 3.1 to bound $|f_i(x_i^*) - f_i(x_{i,t})|$ in terms of the posterior standard deviation at $x_i^*$ and $x_{i,t}$. It is notable that, in this analysis, since $x_{i,t}$ in the exploration sequence of FLEX are selected in a purely exploratory way, we can use Lemma 3.1 which provides tighter confidence intervals compared to the ones used in the analysis of GP-UCB and GP-TS (Chowdhury & Gopalan, 2017). That is the key in proving an always sublinear regret bound for FLEX. See also Vakili et al. (2021a) for further discussions on the confidence intervals. In the second step, we show that the point selection rule in the exploration sequence, based on maximally reducing the uncertainty, allows us to bound both of the posterior standard deviation terms mentioned above by $\sqrt{\gamma N_t/N_t}$, up to an absolute constant factor. We arrive at the theorem by summing the instantaneous regret over $i$ and $t$.

We first consider the regret incurred during the exploration sequence. Let $R_1$ denote the regret incurred during the exploration sequence. We then have

$$R_1 = \sum_{i=1}^{K} \sum_{t \in \mathcal{A}(T)} f_i(x_i^*) - f_i(x_{i,t})$$

$$\leq \sum_{i=1}^{K} \sum_{t \in \mathcal{A}(T)} 2B_i$$

$$\leq \sum_{i=1}^{K} 2B_i|\mathcal{A}(T)|$$

$$\leq 2(\max_i B_i)KN_T.$$

In the second step, we have used $\sup_{x \in \mathcal{X}} f(x) \leq \|f\|_{H_k}$, which can be shown as follows

$$\sup_{x \in \mathcal{X}} f(x) = \sup_{x \in \mathcal{X}} \langle f, k(\cdot, x) \rangle$$

$$\leq \sup_{x \in \mathcal{X}} \|f\|_{H_k} \|k(\cdot, x)\|_{H_k}$$

$$\leq (\sup_{x \in \mathcal{X}} k(x, x)) \|f\|_{H_k}.$$

Using $k(x, x) \leq 1$ for all $x \in \mathcal{X}$ gives us the required bound.

We now consider the regret incurred during the exploitation sequence, which we denote by $R_2$. We first consider the regret incurred by a single client $i$. Based on Assumption 2.3, we consider a discretization $\mathcal{D}_T$ with $|\mathcal{D}_T| = O(T^d)$ such that for all $x \in \mathcal{X}$, $i \in \{1, 2, \ldots, K\}$ and $t \leq T$, we have, $|f_i(x) - f_i([x]_{\mathcal{D}_T})| \leq N_T^{-1/2}$ and $|\mu_i^{(f_i)}(x) - \mu_i^{(f_i)}([x]_{\mathcal{D}_T})| \leq N_T^{-1/2}$, where $[x]_{\mathcal{D}_T}$ denotes the point closest to $x$ on the discretization $\mathcal{D}_T$. Note that we do not need to explicitly construct such a discretization for the actual algorithm. It is considered only for the purpose of analysis.
Using Lemma 3.1 and a union bound over all the points in the discretization $\mathcal{D}_T$, we can conclude that with probability at least $1 - \delta_0/K$, $[f_i(x') - \mu_t^{(f_i)}(x')] \leq \beta_t(\delta_0/2K|\mathcal{D}_T|T)\sigma_t^{(f_i)}(x')$ for all $x' \in \mathcal{D}_T$, $t \leq T$ and given client $i \in \{1, 2, \ldots, K\}$. With this relation, we move onto bound the regret incurred by any client $i$. With a probability greater than $1 - \delta_0/K$, we have,

$$
\sum_{t \notin \mathcal{A}(T)} f_i(x^*_t) - f_i(x_{i,t}) = \sum_{t \notin \mathcal{A}(T)} f_i(x^*_t) - f_i([x_{i,t}]_{\mathcal{D}_T}) + f_i([x_{i,t}]_{\mathcal{D}_T}) - f_i(x_{i,t}) \\
\leq \sum_{t \notin \mathcal{A}(T)} f_i(x^*_t) - f_i([x_{i,t}]_{\mathcal{D}_T}) + N_T^{-1/2} \\
\leq \sum_{t \notin \mathcal{A}(T)} f_i([x_{i,t}]_{\mathcal{D}_T}) - \mu_t^{(f_i)}(x^*_t) + \mu_t^{(f_i)}(x_{i,t}) - f_i([x_{i,t}]_{\mathcal{D}_T}) + N_T^{-1/2} \\
\leq \sum_{t \notin \mathcal{A}(T)} f_i([x_{i,t}]_{\mathcal{D}_T}) - \mu_t^{(f_i)}(x^*_t) + \mu_t^{(f_i)}(x_{i,t}) - \mu_t^{(f_i)}([x_{i,t}]_{\mathcal{D}_T}) + \mu_t^{(f_i)}([x_{i,t}]_{\mathcal{D}_T}) - f_i([x_{i,t}]_{\mathcal{D}_T}) + N_T^{-1/2} \\
\leq \sum_{t \notin \mathcal{A}(T)} f_i([x_{i,t}]_{\mathcal{D}_T}) - \mu_t^{(f_i)}(x^*_t) + \mu_t^{(f_i)}([x_{i,t}]_{\mathcal{D}_T}) - f_i([x_{i,t}]_{\mathcal{D}_T}) + 2N_T^{-1/2} \\
\leq \sum_{t \notin \mathcal{A}(T)} \beta_t(\delta_0/2K|\mathcal{D}_T|T)\sigma_t^{(f_i)}(x^*_t) + \beta_t(\delta_0/2K|\mathcal{D}_T|T)\sigma_t^{(f_i)}([x_{i,t}]_{\mathcal{D}_T}) + 2N_T^{-1/2}.
$$

In the third step, we used the fact that $x_{i,t} = \arg\max_{x \in \mathcal{X}} \mu_t^{(f_i)}(x)$. The expression on the RHS in the above equation is bounded using the following lemma.

**Lemma A.1.** Consider the sequence of points $\{x_1, x_2, \ldots, \}$ generated using the maximum posterior variance sampling scheme, that is, $x_t = \arg\max_{x \in \mathcal{X}} \sigma_t(x)$, where $\sigma_t^2$ is the posterior variance computed from the history $\{x_1, x_2, \ldots, x_{t-1}\}$. Then, at any time $t$, the following inequality is true.

$$
\sup_{x \in \mathcal{X}} \sigma_t(x) \leq \sqrt{\frac{12\gamma_t}{t}},
$$

where $\gamma_t$ is the information gain after $t$ steps.

The lemma provides an upper bound on the posterior standard deviation under the maximum posterior variance sampling scheme, as adopted in the exploration sequence of FLEX. The proof of this lemma is provided at the end of this section.

Note that $t \notin \mathcal{A}(T) \implies t \notin \mathcal{A}(t)$ which in turn implies that $|\mathcal{A}(t)| > N_t$. Thus, at any time instant $t$ in the exploitation sequence, the posterior standard deviation is computed based on a history of at least $N_t$ observations taken according to the maximum posterior variance sampling scheme. Hence, using Lemma A.1 we can conclude that both $\sigma_t^{(f_i)}(x^*_t)$ and $\sigma_t^{(f_i)}([x_{i,t}]_{\mathcal{D}_T})$ can be upper bounded by
a factor of $\sqrt{\gamma_t/N_t}$. Using this bound along with a union bound over all the clients, we have that

$$R_2 = \sum_{i=1}^{K} \sum_{t \in \mathcal{A}(T)} f_i(x_i^*) - f_i(x_{i,t})$$

$$\leq \sum_{i=1}^{K} \sum_{t \in \mathcal{A}(T)} 2\beta_i(\delta_0/2K|D_T|T)\sqrt{12\gamma_t/N_t} + 2N_T^{-1/2}$$

$$\leq 2KT \left( \max_i B_i + R \sqrt{\frac{2}{\lambda} \log \left( \frac{4KT}{\delta_0} \right)} \right) \sqrt{\frac{12\gamma_T}{N_T}} + \frac{2}{\sqrt{N_T}} ,$$

holds with a probability of at least $1 - \delta_0$. Adding the bounds on $R_1$ and $R_2$, we arrive at the theorem.

A.1 Proof of Lemma A.1

The proof is based on Lemma 4 of Chowdhury & Gopalan (2017) which bounds the sum of sequential posterior standard deviations of a GP model as follows

$$\sum_{s=1}^{t} \sigma_{s-1}(x_s) \leq \sqrt{12t\gamma_t}.$$ 

This is true for all $t \geq 1$ and for any sampling scheme that sequentially generates the points $\{x_1, x_2, \ldots\}$. In the case of maximally reducing the variance, for any choices of $s' \leq s$, we have

$$\sigma_{s'-1}(x_{s'}) \geq \sigma_{s'-1}(x_s) \geq \sigma_{s-1}(x_s).$$

The first inequality comes from the selection rule. The second inequality comes from the fact that conditioning a GP on a larger set of points reduces the variance (that follows from the positive definiteness of the covariance matrix).

This implies that $\sigma_{t-1}(x_t)$ is the smallest term in the sum $\sum_{s=1}^{t} \sigma_{s-1}(x_s)$ and hence has to be smaller than the mean. Consequently, we have,

$$\sup_{x \in \mathcal{X}} \sigma_{t-1}(x) = \sigma_{t-1}(x_t) \leq \frac{1}{t} \sum_{s=1}^{t} \sigma_{s-1}(x_s) \leq \sqrt{\frac{12\gamma_t}{t}} ,$$

as required.

B Proof of Theorem 4.2

We first introduce some additional notation corresponding the feature representation of functions in a RKHS, which is used in a part of the proof. In particular, the kernel function $k(\cdot, \cdot)$ is associated with a nonlinear feature map $\phi(\cdot) : \mathbb{R}^d \to \mathcal{H}_k$ such that $k(x, x') = \phi(x')^\top \phi(x)$. Consequently, the reproducing property can also be written as $f(x) = \phi(x)^\top f$. Corresponding to a client $i$, at each
time instant $t$, we define the matrix $\Phi_{i,t} = [\phi(x_{i,1}), \phi(x_{i,2}), \ldots, \phi(x_{i,t})]^T$ based on the points sampled by client $i$, i.e., $x_{i,t}$.

Let $z_{i,t} = \{z_{i,1}, z_{i,2}, \ldots, z_{i,m_i}\}$ denote the set of inducing points chosen from $x_{i,t}$ according to the strategy outlined in Section 4. Let $S_{i,t}$ be the diagonal $\mathbb{R}^{t \times t}$ matrix with $W_{i,j}/\sqrt{p_{i,j}}$ as the $j^{th}$ element on the diagonal. Recall that $p_{i,j} = q_0 \left[ \sigma(g_i)(x_{i,j}) \right]^2$ was the probability of choosing point $x_{i,j}$ for $j = 1, 2, \ldots, t$ and $W_{i,j}$ was the corresponding $\{0, 1\}$ random variable that determined whether that point was a part of the inducing set $z_{i,t}$ or not. Consequently, we have the following relation -

$$\sum_{z \in z_{i,t}} \frac{1}{p_{i,j}} \phi(z)\phi(z)^T = \sum_{j=1}^{t} W_{i,j} \phi(x_{i,j})\phi(x_{i,j})^T = \Phi_{i,t}S_{i,t}S_{i,t}^T\Phi_{i,t}^T.$$

Furthermore, if $\tilde{H}_k$ denotes the subspace of $H_k$ spanned by the inducing points, then using $S_{i,t}$, we can also define the projection operator onto this subspace, which we denote by $P_{i,t}$. In particular, we have,

$$P_{i,t} = \Phi_{i,t}S_{i,t}(S_{i,t}^T\Phi_{i,t}^T\Phi_{i,t}S_{i,t})^{-1}S_{i,t}^T\Phi_{i,t}^T,$$

where $A^+$ denotes the pseudo-inverse of the matrix $A$. Using $P_{i,t}$, we define an approximate feature map $\tilde{\phi}(\cdot) : P_{i,t}\phi(\cdot)$ that corresponds to the subspace $\tilde{H}_k$. Similar to $\Phi_{i,t}$, we define $\tilde{\Phi}_{i,t} = [\tilde{\phi}(x_{i,1}), \tilde{\phi}(x_{i,2}), \ldots, \tilde{\phi}(x_{i,t})]^T = \Phi_{i,t}P_{i,t}$. Lastly, we also define two more matrices, $A_{i,t} = \Phi_{i,t}^T\Phi_{i,t} + \lambda I$ and $\tilde{A}_{i,t} = \tilde{\Phi}_{i,t}^T\tilde{\Phi}_{i,t} + \lambda I$.

With the notations set up, we move on to proving Theorem 4.2. The basic proof follows the same structure as that of Theorem 3.2. We consider the regret separately in the exploration, communication and the exploitation sequence. For the exploration and the communication sequence, the regret is bounded with a constant factor of their lengths. Since the algorithm queries the same point, $\hat{x}^* = \arg \max_{x \in X} \tilde{\mu}(x)$, throughout the exploitation sequence, an upper bound on the regret incurred during the exploitation sequence is simply obtained by bounding the error $f(x^*) - f(\hat{x}^*)$ and multiplying with $T$, the length of the time horizon. The bound on $f(x^*) - f(\hat{x}^*)$ is obtained using Lemma 4.4 using argument similar to the ones used in the proof of Theorem 3.2. The overall regret is then obtained by summing the contributions of these two different terms.

We first consider the regret incurred during the exploration and communication sequence. Let $R_1$
denote the regret incurred during this period. We then have

\[ R_1 = \sum_{i=1}^{K} \sum_{t=1}^{N_T+N_{T_i}^c} f_i(x_t^*) - f_i(x_{t,i,t}) \]

\[ \leq \sum_{i=1}^{K} \sum_{t=1}^{N_T+N_{T_i}^c} 2B_i \]

\[ \leq \sum_{i=1}^{K} 2B_i(N_T+N_{T_i}^c) \]

\[ \leq 2(\max_i B_i)K(N_T + 9(1 + 1/\lambda)q_0\gamma N_T) \]

\[ \leq 2(1 + 9(1 + 1/\lambda)q_0)(\max_i B_i)K N_T. \]

In the second step, we have once again used \( \sup_{x \in \mathcal{X}} f(x) \leq \|f\|_{\mu_t}. \)

To bound the regret incurred during the exploitation sequence, denoted by \( R_2, \) we consider a discretization \( D_{N_T} \) with \( D_{N_T} = O(T^d) \) of the domain \( \mathcal{X}. \) Based on Assumption 2.3, we have that for all \( x \in \mathcal{X}, i \in \{1,2,\ldots,K\}, |f_i(x) - f_i([x]_{D_{N_T}})| \leq N_T^{-1/2} \) and \( |\tilde{\mu}_i(f_i(x^*) - \tilde{\mu}_i(f_i)(x^*)| \leq N_T^{-1/2}, \) where \([x]_{D_{N_T}} \) denotes the point closest to \( x \) on the discretization \( D_{N_T}. \) As before, we do not need to explicitly construct such a discretization for the actual algorithm.

Using Lemma 4.1 and a union bound over all the points in the discretization \( D_{N_T}, \) we can conclude that with probability at least \( 1 - \delta_0/K, |f_i(x^*) - f_i([x^*]_{D_{N_T}})| \leq \tilde{\beta}_i(\delta_0/2K|D_{N_T}|)|\tilde{\sigma}_i(f_i)(x^*)| \) for all \( x^* \in D_{N_T} \) and given client \( i \in \{1,2,\ldots,K\}. \) With this relation and using a similar series of arguments as used in Appendix A, we have,

\[ f_i(x_t^*) - f_i(\tilde{x}_{i,N_T}^*) \leq \tilde{\beta}_i(\delta_0/2K|D_T|T)|\tilde{\sigma}_i(f_i)(x_t^*)| + \tilde{\beta}_i(\delta_0/2K|D_T|T)|\tilde{\sigma}_i(f_i)([\tilde{x}_{N_T}^*]_{D_T})| + 2N_T^{-1/2}, \]

where \( \tilde{x}_{i,N_T}^* = \arg \max_{x \in \mathcal{X}} \tilde{\mu}_i(f_i)(x). \)

For the particular choice of \( q_0 \) as used in Lemma 4.1 (Calandriello et al. 2019) shows that the dictionary \( D_i = \{(\phi(x_{i,j}), W_{i,j}/\sqrt{\tilde{\sigma}_i})(|W_{i,j}, i = 1,j = 1,2,\ldots,N_T\} \) for client \( i \) is \( \varepsilon \)-accurate with probability at least \( 1 - \delta_0/3 \) simultaneously for all the clients. Consequently, they establish the following relation between \( \tilde{\sigma}(x) \) and \( \sigma(x): \sigma(x)/\alpha \leq \tilde{\sigma}(x) \leq \alpha \sigma(x), \) where \( \alpha = (1 + \varepsilon)/(1 - \varepsilon). \) This implies that approximate posterior variance and the actual posterior variance are within a constant factor of each other. Using Lemma A.1, we have,

\[ f_i(x_t^*) - f_i(\tilde{x}_{i,N_T}^*) \leq \alpha \tilde{\beta}_i(\delta_0/2K|D_{N_T}|T)|\sigma_i(f_i)(x_t^*)| + \alpha \tilde{\beta}_i(\delta_0/2K|D_{N_T}|T)|\sigma_i(f_i)([\tilde{x}_{N_T}^*]_{D_{N_T}})| + 2N_T^{-1/2} \]

\[ \leq \alpha \tilde{\beta}_i(\delta_0/2K|D_{N_T}|T)\sqrt{\frac{12\gamma N_T}{N_T}} + \alpha \tilde{\beta}_i(\delta_0/2K|D_{N_T}|T)\sqrt{\frac{12\gamma N_T}{N_T}} + 2\frac{1}{N_T}. \]
Consequently, $R_2$ can be bounded as

$$R_2 = \sum_{i=1}^{K} \sum_{t=N_T + N_T'} f_i(x_t^*) - f_i(\tilde{x}_{i,N_T})$$

$$\leq T \sum_{i=1}^{K} (f_i(x_t^*) - f_i(\tilde{x}_{i,N_T}))$$

$$\leq 2KT \left[ \alpha \beta_t \left( \frac{\delta_0}{2K|D_{N_T}|} \right) \sqrt{\frac{12\gamma_{N_T}}{N_T}} + \frac{1}{\sqrt{N_T}} \right].$$

Adding the bound on $R_1$ and $R_2$ yields the required result.

### B.1 Proof of Lemma 4.4

Throughout the proof, we use the same notation as established earlier with the subscript corresponding to the client dropped for simplicity. We begin considering the definition of $\tilde{\mu}_t$. We have,

$$g(x) - \tilde{\mu}_t(x) = g(x) - k_{z_t,x}^T (\lambda K_{z_t,x} + K_{z_t,x} K_{x_t,z_t})^{-1} K_{z_t,x} y_t$$

$$= g(x) - k_{z_t,x}^T (\lambda K_{z_t,x} + K_{z_t,x} K_{x_t,z_t})^{-1} K_{z_t,x} g_{1:t} - k_{z_t,x}^T (\lambda K_{z_t,x} + K_{z_t,x} K_{x_t,z_t})^{-1} K_{z_t,x} \epsilon_{1:t},$$

where $g_{1:t} = [g(x_1), g(x_2), \ldots, g(x_t)]^\top$ and $\epsilon_{1:t} = [\epsilon_1, \epsilon_2, \ldots, \epsilon_t]^\top$.

We focus on the first term which can be rewritten as

$$|g(x) - k_{z_t,x}^T (\lambda K_{z_t,x} + K_{z_t,x} K_{x_t,z_t})^{-1} K_{z_t,x} g_{1:t}|$$

$$= |\phi^T(x) g - \phi^T(x) A_t^{-1} \Phi_t g_{1:t}|$$

$$= |\phi^T(x) g - \phi^T(x) A_t^{-1} \Phi_t \Phi_t g|$$

$$\leq \left\| \phi^T(x) \left( I - A_t^{-1} \Phi_t \Phi_t \right) \right\| H_k \|g\| H_k.$$

Consider,

$$A_t^{-1} \Phi_t \Phi_t - I = A_t^{-1} \left( \Phi_t \Phi_t - A_t \right)$$

$$= A_t^{-1} \left( \Phi_t \Phi_t - (\Phi_t \Phi_t + \lambda I) \right)$$

$$= A_t^{-1} \left( \Phi_t \Phi_t - \Phi_t \Phi_t P_t - \lambda I \right)$$

$$= A_t^{-1} \Phi_t (I - P_t) - \lambda A_t^{-1}.$$
Thus,

\[
\|\phi^T(x) \left( I - \hat{A}_t^{-1} \tilde{\Phi}_t^T \Phi_t \right) \|_{H_k} = \|\phi^T(x) \left( \tilde{\Phi}_t^T \Phi_t (I - P_t) - \lambda I \right) \|_{H_k} \\
\leq \|\phi^T(x) \tilde{\Phi}_t^T \Phi_t (I - P_t)\|_{H_k} + \lambda \|\phi^T(x) \tilde{A}_t^{-1}\|_{H_k} \\
\leq \sqrt{\phi^T(x) \tilde{\Phi}_t^T \Phi_t (I - P_t)^2 \Phi_t^T \tilde{\Phi}_t \tilde{A}_t^{-1}\phi(x)} + \lambda \sqrt{\phi^T(x) \tilde{A}_t^{-2} \phi(x)} \\
\leq \sqrt{\frac{\lambda}{1 - \varepsilon} \phi^T(x) \tilde{A}_t^{-1} \tilde{\Phi}_t \Phi_t \tilde{A}_t^{-1}\phi(x)} + \lambda \sqrt{\phi^T(x) \tilde{A}_t^{-2} \phi(x)} \\
\leq \frac{\lambda}{1 - \varepsilon} \phi^T(x) \tilde{A}_t^{-1} \phi(x) + \lambda^2 \phi^T(x) \tilde{A}_t^{-2} \phi(x) \left(1 - \frac{1}{1 - \varepsilon}\right) \\
\leq 2 \left\{ \frac{2\lambda}{1 - \varepsilon} \phi^T(x) \tilde{A}_t^{-1} \phi(x) \right\} \\
\leq \sqrt{\frac{2\lambda}{1 - \varepsilon} \tilde{\sigma}_t(x)}.
\]

The fifth line follows from Lemma 1 in [Calandriello & Rosasco, 2018] and by noting that for the given choice of \( q_0 \), the dictionaries are \( \varepsilon \)-accurate with a probability of at least \( 1 - \delta_0/3 \). On combining everything, we obtain,

\[
|g(x) - k_{z_1,x}^T (\lambda K_{z_1,z_1} + K_{z_1,x} K_{x_1,z_1})^{-1} K_{z_1,x} g_{1:t}| \leq \|g\|_{H_k} \sqrt{\frac{2\lambda}{1 - \varepsilon} \tilde{\sigma}_t(x)}.
\]

We now consider the other term given by \( k_{z_1,x}^T (\lambda K_{z_1,z_1} + K_{z_1,x} K_{x_1,z_1})^{-1} K_{z_1,x} \varepsilon_{1:t} \). For simplicity, we denote \( k_{z_1,x}^T (\lambda K_{z_1,z_1} + K_{z_1,x} K_{x_1,z_1})^{-1} K_{z_1,x} : = \zeta^T(x) \). Using the fact that the components of \( \varepsilon_{1:t} \) are independent \( \tilde{R} \)-sub-Gaussian random variables, we have,

\[
E \left[ \exp \left( \zeta^T(x) \varepsilon_{1:t} \right) \right] = \exp \left( \frac{R^2}{2} \|\zeta(x)\|^2 \right).
\]

To bound the RHS, we focus on bounding the norm of the vector \( \zeta(x) \). We have,

\[
\|\zeta(x)\|^2 = \zeta^T(x) \zeta(x) \\
\leq k_{z_1,x}^T (\lambda K_{z_1,z_1} + K_{z_1,x} K_{x_1,z_1})^{-1} k_{z_1,x} \\
\leq \frac{1}{\lambda} \left( k(x, x) - k_{z_1,x}^T k_{z_1,x}^{-1} k_{z_1,x} \right) \\
\leq \tilde{\sigma}_t^2(x).
\]
Thus, by using Chernoff-Hoeffding inequality, we can conclude that

\[ |k_{z_t,x_t}^\top (\lambda K_{z_t,x_t} + K_{z_t,x_t} K_{x_t,x_t})^{-1} K_{z_t,x_t}, x_t \epsilon_{1:t}) - \frac{1}{K} k_{z_t,x_t}(x_t)\epsilon_{1:t}| \leq R_{\tilde{r}}(x) \sqrt{2 \log \left( \frac{1}{\delta} \right)} \]

holds probability at least \(1 - \delta\). On adding the two terms, we have the required answer.

C Proof of Theorem 5.1

We establish the lower bound on regret of any policy by explicitly constructing a choice of observations functions that make learning under the Federated setup a difficult problem. To this effect, we first outline the function that we would use as a building block in our proof. Similar to Scarlett et al. (2017), consider the following multi-dimensional bump function

\[
H(\xi) = \begin{cases} 
\exp \left( -\frac{1}{1-\|\xi\|^2} \right) & \text{if } \|\xi\|^2 \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

Let \( h(x) \) be inverse Fourier transform of \( H(\xi) \). As shown in Scarlett et al. (2017), there exists a \( \zeta > 0 \) such that for \( \|x\|_\infty > \zeta, h(x) \leq 0.5 h(0) \). Using \( h(x) \), we define \( g_\alpha(x) \) as the following function for any \( \alpha > 0 \):

\[
g(x; \alpha) = \frac{\alpha}{h(0)} h \left( \frac{x \zeta}{w} \right),
\]

where \( w \) is a parameter. Using the results in Scarlett et al. (2017), it is straightforward to show that for a given RKHS norm \( B \), there exists an \( \alpha_0(B) \), such that for all \( \alpha < \alpha_0 \), \( \|g(x; \alpha)\|_{H_2} \leq B \) for \( w \leq 0.5 \). Thus, for a given constraint of RKHS norm, \( B \), we fix an \( \alpha < \alpha_0 \).

We consider a federated bandit setup on the unit cube \([0,1]^d\) with a central server and two agents. Let \( V \) denote the set of all the \( 2^d \) vertices of the cube and \( \mathbf{1} = (1, \ldots, 1)^\top \) be the vector of all ones. We define \( V' = V \setminus \mathbf{1} \). Given this notation, the observation functions of the two agents \( g_1 \) and \( g_2 \) are given as follows:

\[
g_1(x) = \sum_{v \in V'} g(x - v; 2\alpha) + g(x - \mathbf{1}; \alpha)
\]

\[
g_2(x) = \sum_{v \in V \setminus V'} g(x - v; \alpha) + g(x - \mathbf{1}; 2\alpha) + g(x - V; \alpha + \epsilon).
\]

In the above expressions, \( V \) is a chosen uniformly at random from \( V' \) and \( \epsilon > 0 \) is a small constant whose value will be specified later. Thus, each of the observation function is effectively a sum of function centered at the vertices of the cube. The reward function for agent 1 is set to \( f_1 = g_1 + g_2 \) and that of agent 2 is set to \( f_2 = g_2 \). Thus, effectively, agent 1 is interested in optimizing the global function, while agent 2 is just interested in maximizing its local function. For the observation, we assume that the noise is Gaussian with unit variance.

We introduce some additional notation for the analysis of the lower bound. Throughout the proof, a subscript \( i \) will be used to refer to quantities related to client \( i = 1, 2 \). We divide the cube in \( 2^d \)
equal regions by equally splitting all the sides. We denote these regions \( \{ R_v \}_{v \in V} \), where \( R_v \) denotes the region that contains the vertex \( v \in V \). Let \((x_{i,t}, y_{i,t})\) denote the input observation pair at time \( t \) and consequently we define \( j_{i,t} = \{ v : x_{i,t} \in R_v \} \). Using this, we can define \( N_{i,v} = \sum_{t=1}^T \mathbb{1} \{ j_{i,t} = v \} \) for \( v \in V \).

We use \( P_v(\cdot) \) to denote the distribution of the rewards when \( V = v \) and \( P_s(\cdot) \) to denote the joint distribution of the rewards and the random variable \( V \). Also let \( P_0 \) denote the reward distribution for the case when \( g_2(x) = \sum_{v \in V \setminus V} g(x - v; \alpha) + g(x - 1; 2\alpha) \). Lastly, let \( Q_V(\cdot) \) denote the reward distribution for the case when \( g_2(x) = \sum_{v \in V \setminus V} g(x - v; \alpha) + g(x - 1; 2\alpha) + g(x - V; \alpha + \varepsilon) \mathbb{1} \{ x \in R_1 \} \), where \( V \) is the random variable defined earlier. We denote the corresponding expectations as \( E_{P_v}[\cdot], E_s[\cdot], E_0[\cdot] \) and \( Q_V[\cdot] \). We would like to point out that the choice of \( g_2(x) \) for defining \( P_0 \) and \( Q_V \) is solely for analysis purposes and the actual observation function is as described earlier.

Let \( \pi = (\pi_1, \pi_2) \) denote the combined strategy for the two agents. For simplicity, we assume that the policy is deterministic. The arguments can be extended for the case of randomized strategies in a straightforward manner. Furthermore, since there is no constraint on communication, we assume that all the observations are exchanged between the two clients. This assumption is without loss of generality as any policy that communicates only a subset of observations can not do any better than the policy that communicates all the observations. Let \( I_t = (y_{1,t}, y_{2,t}) \) denote information gained at time \( t \) and \( I^{(t)} = (I_1, I_2, \ldots, I_t) \) denote information state at time \( t \). Since the policy is deterministic, \( x_{1,t} \) and \( x_{2,t} \) are deterministic functions of \( I^{(t-1)} \).

We state the following lemma that will be useful in the analysis.

**Lemma C.1.** Let \( a \) be any function defined on the information state vector \( I^{(T)} \) whose range is in the bounded interval \([0, A]\). Then for any \( v \in V' \), we have,

\[
E_{P_v}[a(I^{(T)})] \leq E_{Q_v}[a(I^{(T)})] + A \sqrt{\sum_{u \in V'} E_{P_v}[N_{2,u}] \frac{\varepsilon^2}{2}}.
\]

The lemma is largely similar to Lemma B.1 in [Auer et al., 1995] and has been modified for the setup considered in this work. The proof of the lemma is provided at the end of the section.

We first focus on bounding the regret of the first client under the policy \( \pi \) denoted by \( R_{1,\pi} \). We fix the value of \( V = v \). Under this case, the maximizer of \( f_1(x) \) clearly lies in \( R_v \). Using the construction of \( g(x; \alpha) \), we can conclude that the maximum value in any region other than \( R_v \) is at least \( \varepsilon/2 \) smaller than the maximum value. Thus, everytime the algorithm queries a point in a region other that \( R_v \), it incurs a regret of at least \( \varepsilon/2 \). The regret incurred for points queried in \( R_v \) can be trivially lower bounded by 0. Thus, we have,

\[
E_{P_v}[R_{1,\pi}] \geq \sum_{u \in V \setminus v} E_{P_v}[N_{1,u}] \frac{\varepsilon}{2} \geq \frac{\varepsilon}{2}(T - E_{P_v}[N_{1,v}]).
\]
To bound $\mathbb{E}_{P_v}[N_{1,v}]$, we use Lemma C.1 to obtain,

$$\mathbb{E}_{P_v}[N_{1,v}] \leq \mathbb{E}_{Q_v}[N_{1,v}] + A \sqrt{\sum_{u \in V'} \mathbb{E}_{P_v}[N_{2,u}] \frac{\epsilon^2}{2}}.$$

Taking a sum over $v \in V'$, we obtain,

$$\sum_{v \in V'} \mathbb{E}_{P_v}[N_{1,v}] \leq \sum_{v \in V'} \mathbb{E}_{Q_v}[N_{1,v}] + \sum T \sqrt{\sum_{u \in V'} \mathbb{E}_{P_v}[N_{2,u}] \frac{\epsilon^2}{2}}$$

$$\leq \sum_{v \in V'} \mathbb{E}_0[N_{1,v}] + T \sqrt{|V'|} \sum_{v \in V'} \sum_{u \in V'} \mathbb{E}_{P_v}[N_{2,u}] \frac{\epsilon^2}{2}$$

$$\leq T + T \sqrt{|V'|} \sum_{v \in V'} \sum_{u \in V'} \mathbb{E}_{P_v}[N_{2,u}] \frac{\epsilon^2}{2}.$$

Thus, we have,

$$\mathbb{E}_* [R_{1,\pi}] \geq \frac{\epsilon}{2} \left( T - \frac{1}{|V'|} \sum_{v \in V'} \mathbb{E}_{P_v}[N_{1,v}] \right)$$

$$\geq \frac{\epsilon}{2} \left( T - \frac{T}{|V'|} - T \sqrt{\frac{1}{|V'|} \sum_{v \in V'} \sum_{u \in V'} \mathbb{E}_{P_v}[N_{2,u}] \frac{\epsilon^2}{2}} \right)$$

$$\geq \frac{T \epsilon}{2} \left( 1 - \frac{1}{|V'|} - \sqrt{\sum_{u \in V'} \mathbb{E}_*[N_{2,u}] \frac{\epsilon^2}{2}} \right).$$

We similarly consider the regret incurred by client 2 denoted by $R_{2,\pi}$. Clearly, the maximum lies in $R_1$ and every time the policy queries a point in the domain that does not lie in $R_1$, it incurs regret of $c\alpha$ for some universal constant $c > 0$. The regret of points that lie in $R_1$ can be trivially lower bounded by 0. Thus, for any fixed $v \in V'$, we have,

$$\mathbb{E}_{P_v}[R_{2,\pi}] \geq \sum_{u \in V \setminus 1} \mathbb{E}_{P_v}[N_{2,u}] \alpha$$

$$\geq \alpha \sum_{u \in V'} \mathbb{E}_*[N_{2,u}].$$

Taking expectation over $V$ gives us

$$\mathbb{E}_* [R_{2,\pi}] \geq \alpha \sum_{u \in V'} \mathbb{E}_*[N_{2,u}].$$

If we denote $\sum_{u \in V'} \mathbb{E}_*[N_{2,u}]$ with $N_0$, then we have the following relations:

$$\mathbb{E}_* [R_{1,\pi}] \geq \frac{T \epsilon}{2} \left( 1 - \sqrt{\frac{N_0 \epsilon^2}{2}} \right) \quad \text{and} \quad \mathbb{E}_* [R_{2,\pi}] \geq \alpha N_0.$$
On setting \( \epsilon^2 = (8N_0)^{-1} \), we obtain that \( \mathbb{E}_v[R_{1,v}] \geq T/16\sqrt{2N_0} \) and \( \mathbb{E}_v[R_{2,v}] \geq c\alpha N_0 \). In order to ensure both that regret of both the agents are simultaneously minimized, the policy has to ensure that \( N_0 = \Theta(T^{2/3}) \), yielding a lower bound of \( \Omega(T^{2/3}) \) on the regret on any agent, as required. Intuitively, \( N_0 \) essentially captures the number of times client 2 has to query points that are suboptimal for them. They need to query it in order to ensure that client 1 can learn about their objective function and achieve a sublinear regret.

\[ \text{C.1 Proof of Lemma C.1} \]

The proof of this lemma is similar to the lemma obtained for the lower bound in adversarial bandits in Auer et al. (1995). We have,

\[
\mathbb{E}_{P_v}[a(I^T)] - \mathbb{E}_{Q_v}[a(I^T)] = \sum_{T} a(I^T) \left( \mathbb{P}_v(I^T) - \mathbb{Q}_v(I^T) \right)
\leq \sum_{T} a(I^T) \left( \mathbb{P}_v(I^T) - \mathbb{Q}_v(I^T) \right)
\leq A \sum_{T} \mathbb{P}_v(I^T) - \mathbb{Q}_v(I^T)
\leq A \frac{1}{2} \mathbb{P}_v(I^T) - \mathbb{Q}_v(I^T)
\leq A \sqrt{D(P_v(I^T)\|Q_v(I^T))}.
\]

In the last step, we have used Pinsker’s inequality and \( D(P\|Q) \) denotes the Kullback Leibler Divergence between two distributions \( P \) and \( Q \).

We can compute the KL divergence between \( P_v \) and \( Q_v \) using the chain rule. We have,

\[
D(P_v\|Q_v) = \sum_{t=1}^{T} \mathbb{E}_{P_v} \left[ D(P_v(I_t|I^{t-1})\|Q_v(I_t|I^{t-1})) \right]
= \sum_{t=1}^{T} \mathbb{E}_{P_v} \left[ D(P_v((y_{1,t}, y_{2,t})|I^{t-1})\|Q_v((y_{1,t}, y_{2,t})|I^{t-1})) \right]
= \sum_{t=1}^{T} \mathbb{E}_{P_v} \left[ D(P_v(y_{1,t}|I^{t-1})\|P_v(y_{1,t}|I^{t-1})) + D(Q_v(y_{2,t}|I^{t-1})\|P_v(y_{2,t}|I^{t-1})) \right]
= \sum_{t=1}^{T} \mathbb{E}_{P_v} \left[ D(P_v(y_{2,t}|I^{t-1})\|Q_v(y_{2,t}|I^{t-1})) \right]
= \sum_{t=1}^{T} \mathbb{E}_{P_v} \left[ \sum_{x_{2,t} \in \mathcal{R}_u} I \{ x_{2,t} \in \mathcal{R}_u \} D(P_v(y_{2,t}|x_{2,t})\|Q_v(y_{2,t}|x_{2,t})) \right].
\]

Since \( P_v \) and \( Q_v \) are Gaussian distributions with unit variance, their KL divergence is half the square of difference of their means. Since the mean function is same for over \( \mathcal{R}_1 \) for both \( P_v \) and \( Q_v \),
and at most $\varepsilon$ apart at any other point in the domain, the expression can be bounded as

$$
D(\mathbb{P}_v \| \mathbb{Q}_v) = \sum_{t=1}^{T} \mathbb{E}_{\mathbb{P}_v} \left[ \sum_{u \in V} \mathbb{1}_{\{ x_{2,t} \in R_u \}} D(\mathbb{P}_v(x_{2,t} \mid x_t) \| \mathbb{Q}_v(x_{2,t} \mid x_t)) \right]
$$

$$
\leq \sum_{t=1}^{T} \mathbb{E}_{\mathbb{P}_v} \left[ \sum_{u \in V \setminus \{1\}} \mathbb{1}_{\{ x_{2,t} \in R_u \}} \varepsilon^2 \right]
$$

$$
\leq \sum_{u \in V'} \mathbb{E}_{\mathbb{P}_v} \left[ \sum_{t=1}^{T} \mathbb{1}_{\{ x_{2,t} \in R_u \}} \varepsilon^2 \right] \frac{\varepsilon^2}{2}
$$

On plugging this back in the original equation, we obtain the required result.

D Supplemental Material on Experiments

In this section, we provide further details on the experiments and additional plots.

D.1 Additional Details about the Empirical Study

We have used the function Branin, a standard benchmark function for Bayesian Optimization, in our experiments. Its analytical expression is given below [Azimi et al. (2012); Picheny et al. (2013)]

$$
B(x, y) = -\frac{1}{51.95} \left( \left( v - 5.1u^2 + \frac{5u}{\pi} - 6 \right)^2 + \left( 10 - \frac{10}{8\pi} \right) \cos(u) - 44.81 \right)
$$

, where $u = 15x - 5$ and $v = 15y$.

The domain is set to $\mathcal{X} = [0, 1]^2$. The details of IGP-UCB, PI and EI are provided below.

1. IGP-UCB: The algorithm is implemented exactly as outlined in [Chowdhury & Gopalan (2017)] with $B$ (in scaling parameter $\beta_t$) set to 15. The parameters $R$ and $\delta_0$ are set to $10^{-2}$ and $10^{-3}$. $\gamma_t$ was set to $\log t$.

2. Expected Improvement (EI) / Probability of Improvement (PI): Similar to IGP-UCB, EI and PI select the observation points based on maximizing an index often referred to as an acquisition function. The acquisition function of EI is $(\mu(x) - f^* - \varepsilon)\Phi(z) + \sigma(x)\phi(z)$, where $z = \frac{\mu(x) - f^* - \varepsilon}{\sigma(x)}$. Here, $\Phi(\cdot)$ and $\phi(\cdot)$ denote the CDF and PDF of a standard normal random variable. $f^*$ is set to be the maximum value of $\mu(x)$ among the current observation points. The parameters $\varepsilon$ and $\xi$ are used to balance the exploration-exploitation trade-off. We follow [Hoffman et al. (2011)] that showed the best choice of these parameters are small non-zero values. In particular, $\varepsilon$ and $\xi$ are both set to 0.01.

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For the first study, we run the FLEX algorithm for $T = 2000$ steps, with $N_t = \lceil 0.4t^{\eta} \rceil$ for different value of $\eta \in \{1/6, 1/4, 1/3, 1/2, 2/3, 3/4, 5/6\}$. In the second study, we set $N_T = 300$ and decrease $q_0$ to reduce the number of inducing points and consequently the communication cost. In particular, we set $q_0$ to values in $\{300, 200, 100, 70, 50, 30\}$ which gives the average maximum size of the inducing set (rounded to the nearest integer) over the clients as $\{145, 96, 51, 38, 28, 16\}$ respectively. We also run FLEX with $N_T = 300$ to get the baseline with all the points. For the last study, we run FLEX with $N_t = 40$, a constant value.

D.2 Additional Plots

We plots the curves obtained in the first and the third studies with error bars corresponding to 1 standard deviation.

![Regret vs Time](image)

Figure 2: Cumulative regret of FLEX for different values of $N_T$. 

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Figure 3: Cumulative regret for different algorithms