Global Aspects of Doubled Geometry and Pre-rackoid

Noriaki Ikeda and Shin Sasaki

\textsuperscript{a}Department of Mathematical Sciences, Ritsumeikan University
Kusatsu, Shiga 525-8577, Japan,

\textsuperscript{b}Department of Physics, Kitasato University
Sagamihara 252-0373, Japan

Abstract

The integration problem of a C-bracket and a Vaisman (metric, pre-DFT) algebroid which are geometric structures of double field theory (DFT) is analyzed. We introduce a notion of a pre-rackoid as a global group-like object for an infinitesimal algebroid structure. We propose that several realizations of pre-rackoid structures. One realization is that elements of a pre-rackoid are defined by cotangent paths along doubled foliations in a para-Hermitian manifold. Another realization is proposed as a formal exponential map of the algebroid of DFT. We show that the pre-rackoid reduces to a rackoid that is the integration of the Courant algebroid when the strong constraint of DFT is imposed. Finally, for a physical application, we exhibit an implementation of the (pre-)rackoid in a three-dimensional topological sigma model.
1 Introduction

Double field theory (DFT) [1], based on the doubled formalism developed in [2,3], is a gravity theory that inherits T-duality in string theory. DFT is defined in a 2D-dimensional doubled spacetime where the Kaluza-Klein and the string winding modes are realized by the doubled coordinates \( x^M = (x^\mu, \tilde{x}_\mu) \). In this formalism, T-duality is implemented as a global \( O(D,D) \) symmetry in the doubled space. In DFT, the physical \( D \)-dimensional spacetime is defined through the imposition of the strong constraint. Under the strong constraint, the DFT action is reduced to one of supergravity of NSNS sectors in an appropriate frame. Such kind of structure is naturally incorporated in a para-Hermitian manifold \( \mathcal{M} \) [4–7]. It has been shown that the \( D \)-dimensional physical spacetime appears as a leaf in doubled foliations of \( \mathcal{M} \).

The gauge symmetry of DFT originates from the diffeomorphism and the \( U(1) \) gauge symmetry of an NSNS \( B \)-field. The transformations of DFT fields are generated by vector fields in the doubled space and they are governed by a C-bracket. An algebra based on the C-bracket is known to be a Vaisman (metric or pre-DFT) algebroid [4,8,9]. This is a generalization of a Courant algebroid [10] that plays an important role in generalized geometry. Indeed, when the strong constraint is imposed on any vector field and function in \( \mathcal{M} \), the Vaisman algebroid reduces to the Courant algebroid [11]. Both algebroids exhibit local structures of the tangent bundle \( TM \) of the para-Hermitian manifold.
An important aspect of these algebroids involves their doubled structures. For example, it has been shown that the Courant algebroid is composed by a Drinfel’d double of Lie bialgebroids \cite{12}. An analogous result has been obtained even for a Vaisman algebroid. In this picture, the geometric origin of the strong constraint in DFT is traced back to a consistency condition of the Drinfel’d double for Lie bialgebroids \cite{13}. The doubled aspects appearing in the contexts of mathematical and physical sides of DFT shed light on the deep understanding of nature of T-duality. Among other things, the structure of the Drinfel’d double of Lie bialgebras is a key ingredient of the Poisson-Lie T-duality \cite{14,16}. This is a generalization of ordinary T-duality with Abelian isometries and its physical applications have been studied \cite{17,21}. The Poisson-Lie T-duality is interpreted as a freedom of choices for Manin triples in the Drinfel’d double of the Lie bialgebras. If there are several Manin triples, one can generalize the duality to plurality \cite{22}. Recent developments along this line include \cite{23,26}.

It is well known that the Lie bialgebra is the infinitesimal object of the Poisson-Lie group. Conversely, a Lie bialgebra is integrated to a Poisson-Lie group. Obviously, the Poisson-Lie T-duality is named after this group structure. One can imagine that this picture may be generalized to algebroids even in the absence of group structures. Indeed, a Lie algebroid is an infinitesimal object of a Lie groupoid. By the same way, a Lie bialgebroid is an infinitesimal object of a Poisson groupoid \cite{27}. It is known that the Drinfel’d double of Lie bialgebroids gives a Courant algebroid. Manin triples are defined by a Dirac structure of a Courant algebroid \cite{12}. It is therefore conceivable that there is a groupoid-like structure defined by the integration of the Courant algebroid. However, it is a highly non-trivial task to determine a global, group-like object for a given algebroid. This is because there is no analogue of the Lie’s third theorem associated with a Lie algebroid and its global counterpart. Finding a group-like object from a given algebra is known as the \textit{coquecigrue problem} \cite{28}. The coquecigrue problem for the Courant algebroids has been studied intensively in various contexts \cite{29,31}.

The purpose of this paper is to investigate a global structure of the doubled spacetime and examine the geometric meaning of the strong constraint in DFT. To this end, we work on the coquecigrue problem of the Vaisman algebroid which is a natural local structure appearing in a para-Hermitian manifold. We remind ourselves that the standard Courant algebroid is recognized as a Leibniz algebroid \cite{12}. It is known that an integration of a Leibniz algebra is given by a \textit{rack} \cite{33,34}. Racks and associated quandles were first proposed in the contexts of knot theory \cite{36}. They are sets equipped with binary operations satisfying certain axioms. Notably, racks are generalizations of groups. Roughly speaking, racks are group-like objects that are based on the conjugation instead of the product in the sense of ordinary groups. Accordingly, it seems plausible that an integration of a Leibniz algebroid is given by a groupoid-like counterpart of the rack – the \textit{rackoid}. Indeed, the authors in \cite{37} proposed that an integration of the standard Courant algebroid is given by a rackoid. This rackoid structure is defined by cotangent paths associated with underlying paths in the base space. The standard Courant algebroid is shown up as an infinitesimal (or tangent bundle) structure of the cotangent path rackoid. We will generalize this picture to the Vaisman algebroid and look for a groupoid-like structure, which we call the \textit{pre-rackoid}, in the doubled spacetime in DFT.

We also propose another method, a formal exponential map of a Vaisman algebroid. We consider an exponential of the adjoint operation. It is called a formal (pre-)rackoid. This idea
is directly related to one of a large gauge transformations of DFT [38].

As an explicit realization and applications, we discuss the (pre-)rackoid structures in a topological sigma model. A three dimensional topological sigma model with the structure of a Courant algebroid called the Courant sigma model is known [39, 40]. This model is useful to analyze the integration of the Courant algebroid to Lie rackoid. Wilson lines in the model realize a Lie rackoid structure. The doubled geometry version of topological sigma model has been proposed [41]. See also [9, 42]. We will show that the topological sigma model of doubled geometry gives a realization of a (formal) pre-rackoid.

The organization of this paper is as follows. In the next section, we give the definition of the Vaisman algebroid and its relations with the Courant algebroid. In section 3, we introduce the notion of the rack and the rackoid. Mathematical definitions of these structures are presented. In section 4, we first focus on the cotangent path rackoid discussed in [37]. This gives an integration of the standard Courant algebroid. We then generalize this to the doubled cotangent path defined in the doubled foliations of the para-Hermitian manifold $\mathcal{M}$. We show that the integration of the Vaisman algebroid based on the C-bracket is given by a generalization of rackoid, namely, the pre-rackoid. We show that the rack-like product based on the doubled cotangent path defines the pre-rackoid. In this picture, the strong constraint in DFT is an sufficient condition of the self-distributivity of the rack product. We will see that this is re-organized as the quantum Yang-Baxter equation for the rack action. In section 5, we show another way to provide the (pre-)rackoids. We introduce formal exponential maps of the adjoint action to define the (pre-)rack product. This procedure gives a formal integration of the Courant and the Vaisman algebroids. In section 6, we discuss a sigma model implementation of the (pre-)rackoid structures. We show that the (pre-)rackoids associated with the Vaisman and the Courant algebroids are realized as Wilson loops in the sigma models. Section 7 is the conclusion and discussions.

2 Leibniz, Courant and Vaisman algebroids

In this section, we introduce the Courant and the Vaisman algebroids. The latter appears in para-Hermitian manifolds which are natural arenas of the doubled spacetime in DFT. Before the definition of the Courant and the Vaisman algebroids, let us first remind the definition of a Leibniz algebra.

**Definition** ((Left) Leibniz algebra (Loday algebra)). A (left) Leibniz algebra (Loday algebra) $\mathfrak{g}$ is defined as a module over a ring $R$ equipped with a bilinear map $\cdot, \cdot$ (the Leibniz bracket) on $\mathfrak{g}$ satisfying the following left Leibniz identity:

$$[[a, b, c]] = [[a, b], c] + [b, [a, c]], \quad \text{for all } a, b, c \in \mathfrak{g}. \quad (1)$$

A right Leibniz algebra is defined similarly.

Note that the Leibniz bracket $\cdot, \cdot$ is not necessarily skew-symmetric. When the Leibniz bracket $\cdot, \cdot$ is skew-symmetric, it becomes a Lie algebra. The notion of the Leibniz algebra is easily generalized to the ones of algebroids.
Definition ((Left) Leibniz algebroid). A (left) Leibniz algebroid is a triple \((E, [\cdot, \cdot]_D, \rho)\), where \(E \xrightarrow{\pi} M\) is a vector bundle over a smooth manifold \(M\), \([\cdot, \cdot]_D : \Gamma(E) \times \Gamma(E) \to \Gamma(E)\) is a Leibniz bracket satisfying the Leibniz identity \([1]\), \(\rho : E \to TM\) is a bundle map called the anchor map, and \([\cdot, \cdot]_D\) and \(\rho\) satisfy the following relations:

\[
\rho([e_1, e_2]_D) = [\rho(e_1), \rho(e_2)]_{TM},
\]

\[
[e_1, f e_2]_D = f[e_1, e_2]_D + (\rho(e_1) \cdot f)e_2,
\]

for any \(e_i \in \Gamma(E)\) and \(f \in C^\infty(M)\). (2)

Here \([\cdot, \cdot]_{TM}\) is the Lie bracket of vector fields on \(TM\). We note that the first relation can be omitted since it is obtained through the second one and the Leibniz identity \([1]\). In the Leibniz algebroid, the bracket is called the Dorfman bracket.

The Leibniz algebroid is a generalization of the Lie algebroid and the Leibniz algebra. When the bracket \([\cdot, \cdot]_D\) is skew-symmetric, it is a Lie bracket and the triple \((D, [\cdot, \cdot]_D, \rho)\) defines a Lie algebroid. When \(M\) is a point \(M = \{\text{pt}\}\) and \(\rho = 0\), the Leibniz algebroid becomes a Leibniz algebra. Given these definitions, we now introduce the Courant algebroids.

Definition (Courant algebroid). Let \(E \xrightarrow{\pi} M\) be a vector bundle over a manifold \(M\). A Courant algebroid is a quadruple \((E, [\cdot, \cdot]_D, \rho, (\cdot, \cdot))\) where \([\cdot, \cdot]_D\) is a bilinear bracket on \(\Gamma(E)\), \(\rho : E \to TM\) is an anchor map, and \((\cdot, \cdot)\) is a non-degenerate bilinear form on \(\Gamma(E)\). They satisfy the following axioms for any \(e_i \in \Gamma(E)\) and \(f \in C^\infty(M)\):

1. The bracket \([\cdot, \cdot]_D\) satisfies the Leibniz identity \([1]\).

2. \(\rho([e_1, e_2]_D) = [\rho(e_1), \rho(e_2)]_{TM}\).

3. \([e_1, f e_2]_D = f[e_1, e_2]_D + (\rho(e_1) \cdot f)e_2\).

4. \([e, e]_D = \frac{1}{2}D(e, e)\).

5. \(\rho(e_1) \cdot (e_2, e_3) = ([e_1, e_2]_D, e_3) + (e_2, [e_1, e_3]_D)\).

Here \(D\) is a generalized exterior derivative on \(\Gamma(E)\).

The axioms 1,2,3 are just the ones for the Leibniz algebroid. Therefore any Courant algebroids are Leibniz algebroids. An alternative but equivalent definition based on a skew-symmetric – the Courant – bracket is known \([12]\), which is anti-symmetrization of the Dorfman bracket. The Courant algebroids play important roles in generalized geometry \([44]\). This inherits T-duality structure in its Drinfel’d double.

Next we introduce the Vaisman algebroids.

Definition (Vaisman algebroid). Let \(E \xrightarrow{\pi} M\) be a vector bundle over a manifold \(M\). A Vaisman algebroid is a quadruple \((E, [\cdot, \cdot]_D, \rho, (\cdot, \cdot))\) where \([\cdot, \cdot]_D\) is a bracket on \(\Gamma(E)\), \(\rho : E \to TM\) is an anchor map, and \((\cdot, \cdot)\) is a non-degenerate bilinear form on \(\Gamma(E)\). They satisfy the following axioms for any \(e_i \in \Gamma(E)\) and \(f \in C^\infty(M)\):

1. \([e_1, f e_2]_D = f[e_1, e_2]_D + (\rho(e_1) \cdot f)e_2\).
2. \( \rho(e_1) \cdot (e_2, e_3) = ([e_1, e_2]_D, e_3) + (e_2, [e_1, e_3]_D) \)

The bracket \([\cdot, \cdot]_D\) is called a D-bracket. One can find an alternative definition based on a skew-symmetric bracket which satisfies the same axioms above. The skew symmetrization of a D-bracket is called a C-bracket. Obviously, any Courant algebroids are Vaisman algebroids. From the viewpoint of DFT, a Vaisman algebroid appears on the tangent bundle of a 2D-dimensional para-Hermitian manifold \([4, 5, 8]\). In this case, the bracket is given by the D-bracket in DFT. A remarkable property of the Vaisman algebroid is that its defining bracket is composed of a double of Lie algebroids \([13]\). The skew-symmetric bracket of the Vaisman algebroid on a para-Hermitian manifold is nothing but the C-bracket that governs the gauge symmetry of DFT. One can switch the C- and D-brackets by the standard procedures \([43]\). In order that the algebra of the gauge transformation of DFT closes, a constraint, known as the strong constraint, should be imposed on all the fields and gauge parameters \([11]\). In this case, the D-bracket reduces to the Dorfman bracket of generalized geometry and the Vaisman algebroid reduces to the Courant algebroid.

In the next section, we introduce racks and rackoids which are integrations of the Leibniz algebras and algebroids.

### 3 Racks and rackoids

In this section, we introduce the notion of racks and rackoids. The rack has been proposed as a global group-like object associated with its infinitesimal counterpart – the Leibniz algebra \([33, 35]\). This observation can be generalized to those for Leibniz algebroids. The corresponding global, groupoid-like structure is known as rackoids. In the following, we first define racks and then generalize the notion to rackoids. This section is based on \([45]\). More details can be found there.

#### 3.1 Racks

The definition of racks is the following.

**Definition (Rack).** The set \( S \) together with a binary operation \((x, y) \mapsto x \triangleright y\) for any \( x, y \in S \) is called a rack if the map \( y \mapsto x \triangleright y \) is bijective and the operation \( \triangleright \) satisfies the following left self-distributivity:

\[
x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z),
\]

for any \( x, y, z \in S \). \( x \triangleright y \) and the map \( y \mapsto x \triangleright y \) are called the rack product and the rack action of \( x \) on \( y \), respectively.

We note that in general, the rack product \( \triangleright \) is not associative \( x \triangleright (y \triangleright z) \neq (x \triangleright y) \triangleright z \). Since the rack action \( x \triangleright \cdot \) is a bijection, there exists the unique element \( y \in S \) such that \( x \triangleright y = z \) for any \( z \in S \). This implies the left invertibility of the rack product \( \triangleright \). A right rack is defined similarly for a right rack product \( \triangleleft \). In the following, we employ left racks and never consider the right ones.
An example of the rack product is the conjugation \( g \triangleright h = ghg^{-1} \) on a group \( G \). This example trivially satisfies the self-distributivity. In this sense, racks are defined over wrack of group structures. There remains only the conjugation operation out of the group multiplication.

There does not necessarily exist unit element in racks. We introduce pointed racks which are racks with the unit element 1 with respect to the rack product.

**Definition (Pointed rack).** Let \((S, \triangleright)\) be a rack. When there is an element \( 1 \in S \) such that the following relation holds for any \( x \in S \),

\[
1 \triangleright x = x, \quad x \triangleright 1 = 1,
\]

then \((S, \triangleright)\) is called a **pointed rack**.

One can show that a group \( G \) with the rack product defined by the conjugation \( g \triangleright h = ghg^{-1} \) is an example of a pointed rack \((G, \triangleright)\). The element 1 is obviously the unit element of the group \( 1 \in G \). We next define a Lie rack.

**Definition (Lie rack).** Let \( S \) be a manifold. When all the structures in a pointed rack \((S, \triangleright)\) are smooth and the rack action \( y \mapsto x \triangleright y \) for any \( x, y \in S \) is a diffeomorphism, then \((S, \triangleright)\) is called a **Lie rack**.

We can then consider a tangent space of \((S, \triangleright)\) at the unit element. Indeed, it is shown that an infinitesimal algebra defined on the tangent space at a unit element of a Lie rack is nothing but a Leibniz algebra [33]. This is quite analogous to the fact that the algebra on the tangent space at the unit element of a Lie group \( G \) is isomorphic to the Lie algebra \( g \) of \( G \).

### 3.2 Rackoids

We next generalize the notion of the rack to that of rackoids. One finds that this leads to an integration of the Leibniz algebroids. The notion of rackoids has been introduced as a base ground of an integration of the Courant algebroids [45]. Before going to the rackoids, let us begin with precategories.

**Definition (Precategory).** Let \((\mathcal{G}, M)\) be a pair composed of sets \( \mathcal{G} \) and \( M \). When there exist bijection maps \( s, t : \mathcal{G} \to M \), then \((\mathcal{G}, M)\) is called a semi-precategory. Here \( s, t \) are called the source and the target maps. When there is a unit map \( \epsilon : M \to \mathcal{G} \) satisfying \( s \circ \epsilon = t \circ \epsilon = \text{id}_M \), then \((\mathcal{G}, M)\) is called a **precategory**. For each \( x \in M \), we express \( \epsilon(x) = 1_x \in \mathcal{G} \).

An element \( g \) of \( \mathcal{G} \) is regarded a morphism from \( s(g) \) to \( t(g) \). In the following, \( \mathcal{G} \ni M \) denote a precategory. We also note that \( \epsilon(x) \) satisfying \( s \circ \epsilon = t \circ \epsilon = \text{id}_M \) corresponds to a unit morphism at \( x \in M \) if there is a composition map of morphisms. When there exists an associative composition of morphisms, then \( \mathcal{G} \ni M \) becomes a category. In addition, when there exists an inverse for all the morphisms, \( \mathcal{G} \ni M \) is a groupoid.

Since there are not necessarily compositions of morphisms in (semi-)precategories, we next define bisections which enable us to find an appropriate action of \( \mathcal{G} \) on \( M \).
Definition (Bisection). Let $\mathcal{G} \rightrightarrows M$ be a semi-precategory. A bisection of $\mathcal{G}$ is defined by the following equivalent data:

1. A subset $\Sigma \subset \mathcal{G}$ such that the restricted source and the target maps $s, t : \Sigma \to M$ are bijection.

2. A map $\sigma = t \circ \sigma : M \to M$ that is bijection. Here $\sigma : M \to \mathcal{G}$ is a right inverse of $s$, namely, it is defined by $s \circ \sigma = \text{id}_M$.

In the following, maps associated with bisections $\Sigma, T, \cdots$ are denoted by $\sigma, \tau, \cdots$. Now, a set of all the morphisms $g \in \mathcal{G}$ that satisfy $s(g) = x, t(g) = y$ for all $x, y \in M$ is denoted by $\mathcal{G}^y_x$.

Then rackoids are defined as follows.

Definition (Rackoid). For a semi-precategory $\mathcal{G} \rightrightarrows M$, a bisection $\Sigma \subset \mathcal{G}$ and $g \in \mathcal{G}^y_x$, one defines an action of $\Sigma$ on $g$

\[ \triangleright : (\Sigma, g) \mapsto \Sigma \triangleright g \in \mathcal{G}^y_x, \]

For bisections $\Sigma, T \subset \mathcal{G}$, we define $\Sigma \triangleright T$ as the image of an assignment $\Sigma \triangleright (\cdot)$ on $T$. When the action $\triangleright$ satisfies the following properties,

1. For any bisections $\Sigma$, an assignment $\Sigma \triangleright (\cdot) : \mathcal{G} \to \mathcal{G}$ is bijective.

2. For any bisections $\Sigma, T$ and any $g \in \mathcal{G}$, the action $\triangleright$ satisfies the following self-distributivity,

\[ \Sigma \triangleright (T \triangleright g) = (\Sigma \triangleright T) \triangleright (\Sigma \triangleright g). \]

then, this becomes a rack action and $(\mathcal{G} \rightrightarrows M, \triangleright)$ is called a non-unital rackoid. In addition, for any $x \in M$, $g \in \mathcal{G}$, when there exists $\epsilon(x) = 1_x \in \mathcal{G}$ such that

\[ 1_M \triangleright g = g, \quad \sigma(x) \triangleright 1_x = 1_{\sigma(x)}, \]

then $(\mathcal{G} \rightrightarrows M, \triangleright)$ is called a unital (or pointed) rackoid. Here $1_M$ stands for the bisection $\epsilon(M)$, namely, the collection of $1_x$ for all $x \in M$. When all the structures defined above are smooth, $(\mathcal{G} \rightrightarrows M, \triangleright)$ is called a Lie rackoid.

Note that since the map is bijective, the image of the map $\Sigma \triangleright (\cdot)$ on a bisection $T$ is a bisection. The geometrical meaning of the rack product defined above is obvious. The rack action by $\Sigma$ shifts the initial (source) and the end (target) points of $g \in \mathcal{G}$ along the $\Sigma$-direction.

This is understood by the following relation of the associated diffeomorphism on $M$:

\[ \sigma \triangleright \tau = \sigma \circ \tau \circ \sigma^{-1}. \]

Therefore the rack action is essentially a conjugation. The rackoid defined in this way is a groupoid-like generalization of the rack. Indeed, when $M$ is a point $M = \{\text{pt}\}$, namely, there is only one point in $M$, then the rackoid becomes a rack. This corresponds to the fact that a groupoid over a point $M = \{\text{pt}\}$ becomes a group. In the following, all the collections
bisections in a smooth precategory \( \mathcal{G} \rightrightarrows M \) is denoted by \( \text{Bis(}\mathcal{G}) \). Since all the structures in a Lie rackoid is smooth, we can now discuss its infinitesimal algebra on the tangent space. Parallel to the fact that a Lie algebroid is an infinitesimal object of a Lie groupoid, one can define a Leibniz algebroid as an infinitesimal counterpart of a Lie rackoid. Let us sketch this procedure in the following.

Let \( \mathcal{G} \rightrightarrows M \) be a unital (pointed) Lie rackoid. Through the unit map \( \epsilon : M \rightarrow \mathcal{G} \), we identify \( M \) to a subset of \( \mathcal{G} \). Namely, \( x \in M \) is identified with \( 1_x \in \mathcal{G} \). Fibers of the pullback bundle \( \epsilon^*T\mathcal{G} \rightrightarrows M \) by \( \epsilon \) are given by \( T_1\mathcal{G} \). We consider differentials of source and the target maps of \( \mathcal{G} \rightrightarrows M, T_s, T_t : T\mathcal{G} \rightarrow TM \). By definition, since the fibers of the pullback bundle \( \epsilon^*T\mathcal{G} \) are just copies of the \( T\mathcal{G} \) fibers, diffeomorphisms \( \epsilon^*T\mathcal{G} \rightarrow TM \) are naturally defined. Explicitly, we have the induced maps,

\[
T_1s : T_1\mathcal{G} \rightarrow T_xM, \quad T_1t : T_1\mathcal{G} \rightarrow T_xM.
\]  (9)

Using two maps, we define an infinitesimal algebroid \( \mathcal{A} \) of a unital Lie rackoid.

**Definition (Infinitesimal algebroids of rackoid).** Given a unital (pointed) Lie rackoid \( \mathcal{G} \rightrightarrows M \), we have a vector bundle \( \mathcal{A} \) over \( M \) defined by

\[
\mathcal{A} = \text{Ker}(Ts) = \bigoplus_{x \in M} \text{Ker}(T_1x, s) \in \bigoplus_{x \in M} T_1x\mathcal{G}.
\]  (10)

This is called an infinitesimal algebroid of \( \mathcal{G} \). The anchor \( \rho : \mathcal{A} \rightarrow TM \) is defined by

\[
\rho = -Tt|_\mathcal{A}.
\]  (11)

Since all the structures discussed above are well-defined, we consider the adjoint action on \( \mathcal{A} \) induced by the rack action \( \Sigma \triangleright \) of a bisection \( \Sigma \):

\[
\text{Ad}_\Sigma = T_1x, \Sigma \triangleright |_\mathcal{A} : A_x \rightarrow A_{\Sigma(x)},
\]  (12)

The rack action satisfies, by definition, the self-distributivity. Namely, for any bisections \( \Sigma, T \subset \mathcal{G} \) and \( g \in \mathcal{G} \), we have \( \Sigma \triangleright T \overset{\rho}{=} (\Sigma \triangleright T) \overset{\rho}{=} \Sigma \triangleright \). Since \( T_1x, \Sigma \triangleright \) induces the adjoint action, the self-distributive relation of the rack action results in the composition of the adjoint action:

\[
\text{Ad}_\Sigma \circ \text{Ad}_T = \text{Ad}_{\Sigma \triangleright T} \circ \text{Ad}_\Sigma.
\]  (13)

The following lemma enable us to make contact with explicit derivation of the bracket in the algebroid \( \mathcal{A} \):

**Lemma (Family of bisections).** Let \( (\mathcal{G} \rightrightarrows M, \triangleright) \) and \( \mathcal{A} \) be a Lie rackoid and its infinitesimal algebroid. Let us consider the section \( \Gamma : M \rightarrow \epsilon^*\mathcal{G} \). Namely, \( \Gamma : x \mapsto b_x \in A_x \subset T_1x\mathcal{G} \). Then, there is a family of bisection, \( (\Sigma_u)_{u \in I}, I = (-1, 1) \) of \( \mathcal{G} \) such that \( \Sigma_0 = \epsilon(M) \) and \( \frac{\partial}{\partial u}\sigma_u(x)|_{u=0} \) coincides with \( b_x \) for all \( x \in M \). Here \( \sigma_u \) is a map \( \sigma_u : M \rightarrow \mathcal{G} \) associated with the bisection \( \Sigma_u \) and it is parametrized by \( u \). This satisfies \( s \circ \sigma_u = \text{id}_M, \sigma_u(x) \in \Sigma_u \subset \mathcal{G} \). In this case, the assignment \( x \mapsto \frac{\partial}{\partial u}\sigma_u(x)|_{u=0} \) is a smooth section of \( \mathcal{A} \). Since \( \Sigma_{u=0} = \Sigma_0 = \epsilon(M) = \bigoplus_{x \in M} 1_x \), we have \( \sigma_{u=0}(x) = 1_x \).
The proof is found in [45]. With this fact at hand, we find that the tangent space of Bis($\mathcal{G}$) at $g = \epsilon(M)$ is $\Gamma(\mathcal{A})$. Using the adjoint map for families of bisections, we now define the bracket structure on $\Gamma(\mathcal{A})$.

**Definition** (Bracket in $\mathcal{A}$). For any $a, b \in \Gamma(\mathcal{A})$, a bracket $[\cdot, \cdot] : (b, a) \mapsto [b, a]$ is defined by

$$[b, a] = \frac{\partial}{\partial u} \text{Ad}_{\Sigma_u} a |_{u=0}. \quad (14)$$

Here $\Sigma_0 = \epsilon(M)$ and $\frac{\partial}{\partial u} \sigma_u(x)|_{u=0} = b_x$. $\Sigma_u, u \in (-1, 1)$ is a family of smooth bisection of $\mathcal{G}$.

One can check the bracket $[\cdot, \cdot]$ defined above satisfies the Leibniz identity (1) as follows. By the self-distributivity of the rack product for families of bisections $\Sigma_u, T_v$ acting on any $a \in \Gamma(\mathcal{A})$, we have

$$\text{Ad}_{\Sigma_u} \circ \text{Ad}_{T_v} a = \text{Ad}_{\Sigma_u \circ T_v} \circ \text{Ad}_{\Sigma_u} a. \quad (15)$$

By differentiating both sides with respect to $v$ and evaluating at $v = 0$, we find

$$\text{Ad}_{\Sigma_u} \circ [b, a] = \frac{\partial}{\partial v} \text{Ad}_{\Sigma_u \circ T_v} |_{v=0} \circ \text{Ad}_{\Sigma_u} a. \quad (16)$$

Here we have used the relation $[b, a] = \frac{\partial}{\partial v} \text{Ad}_{T_v} a |_{v=0}, b_x = \frac{\partial}{\partial v} \tau_v(x)|_{u=0}$. Since the rack action on $\Gamma(\mathcal{A})$ induces the adjoint action, we have

$$\frac{\partial}{\partial v} \Sigma_u \triangleright \tau_u(x)|_{v=0} = \Sigma_u \triangleright \frac{\partial}{\partial v} \tau_v(x)|_{v=0}$$

$$= \text{Ad}_{\Sigma_u} b_x. \quad (17)$$

Then the right hand side of (19) becomes $[\text{Ad}_{\Sigma_u} b, \text{Ad}_{\Sigma_u} a]$. By differentiating both sides again with respect to $u$ and setting $u = 0$, we obtain

$$\frac{\partial}{\partial u} \text{Ad}_{\Sigma_u}([b, a])|_{u=0} = \frac{\partial}{\partial u} \text{Ad}_{\Sigma_u} b|_{u=0}, \text{Ad}_{\Sigma_u} a|_{u=0} |_{u=0} + [\text{Ad}_{\Sigma_u} b|_{u=0}, \frac{\partial}{\partial u} \text{Ad}_{\Sigma_u} a|_{u=0}]. \quad (18)$$

The left hand side gives $[c, [b, a]]$ while the first and the second terms in the right hand side are $[[c, b], b]$ and $[b, [c, a]]$. Here we have used the facts $\frac{\partial}{\partial u} \text{Ad}_{\Sigma_u} b|_{u=0} = [c, b]$, $\text{Ad}_{\Sigma_u} a|_{u=0} = a$. Then the Leibniz identity follows:

$$[c, [b, a]] = [[c, b], a] + [b, [c, a]]. \quad (19)$$

One also finds that due to the definition of the anchor map $\rho = -T t|_{\mathcal{A}}$, the Leibniz rule for the bracket holds. We then end up with the following theorem.

**Theorem** (Tangent Leibniz algebroid of Lie rackoid). Let $(\mathcal{G} \rightrightarrows M, \triangleright)$ be a unital Lie rackoid over a manifold $M$. There is a vector bundle $\mathcal{A} = \text{Ker}(\tau s) \to M$ in the induced bundle $\epsilon^* T \mathcal{G}$. An anchor $\rho = -T t|_{\mathcal{A}}$ and a bracket $[b, a] = \frac{\partial}{\partial u} \text{Ad}_{\Sigma_u} a|_{u=0}$ is defined for a family of bisections $\Sigma_u$. Then the triple $(\mathcal{A}, [\cdot, \cdot], \rho)$ defines a Leibniz algebroid over $M$. This is called a tangent Leibniz algebroid of $\mathcal{G}$.

We stress that the Leibniz identity (19) derived by the adjoint map (13) originates from the self-distributive relation of the rack product. This fact will be an important clue for the integration of the Vaisman algebroid.
4 Pre-rackoids and doubled cotangent paths

In this section, we exhibit an explicit example of the Lie rackoid through the cotangent paths discussed in [37]. This is a basic ground for the group-like, global structures of the standard Courant algebroid. We first introduce the standard Courant algebroid and discuss the rackoid structure associated with it. We then proceed to the rackoid-like structure for the Vaisman algebroid. A key ingredient is the notion of the pre-rackoid based on the doubled foliations of a para-Hermitian manifold.

4.1 Cotangent path rackoids and Courant algebroid

The most familiar example of the Courant algebroid is the standard Courant algebroid. The vector bundle of the standard Courant algebroid is given by the generalized tangent bundle $T^*M = TM \oplus T^*M$ over a manifold $M$. The Dorfman bracket of the standard Courant algebroid is given by

$$[e_1, e_2]_D = [X_1, X_2]_{TM} + \mathcal{L}_{X_1} \xi_2 - \iota_{X_2} d \xi_1. \quad (20)$$

Here $e_i = X_i + \xi_i$, $(i = 1, 2)$ and $X_i \in \Gamma(TM)$ are vector fields, $\xi_i \in \Gamma(T^*M)$ are 1-forms. The bracket $[\cdot, \cdot]_{TM}$ in the right hand side is the ordinary Lie bracket of the vector fields, $\mathcal{L}_X$, $\iota_X$ are the Lie derivative and the interior product associated with $X$ and $d$ is the exterior derivative operator. In [37], a rackoid $(\mathcal{G} \rightrightarrows M, \triangleright)$ that results in a tangent Leibniz algebroid equipped with the bracket (20) is constructed. Here $\mathcal{G} = PT^*M$ is a set of paths $PT^*M = C^\infty([0, 1], T^*M)$ on a compact manifold $M$ and the rack product $\triangleright$ is defined by automorphisms on the Dorfman bracket.

In the following, we briefly introduce the discussions in [37] and then generalize the construction to the one in the doubled geometry. Before examining the space $PT^*M$, we first clarify a precategory structure for paths on $M$.

**Definition (Precategory by paths).** Let $M$ be a compact manifold of finite dimensions. Let $PM$ be a set of smooth paths $\gamma : [0, 1] \to M$. The source and the target maps are defined by

$$s(\gamma) = \gamma_0, \quad t(\gamma) = \gamma_1, \quad (21)$$

where the path $\gamma_t$ is parametrized by $t \in [0, 1]$. The unit map $\epsilon : M \to PM$ is defined by a map to a constant $c$, $\epsilon(x) = c$ for any $x \in M$. Then $PM \rightrightarrows M$ becomes a smooth precategory.

Since the path can be seen as a map $\gamma : [0, 1] \to C^\infty(M, M)$, bisections of the precategory $PM \rightrightarrows M$ is defined as follows.

**Definition (Bis$(PM)$ of $PM \rightrightarrows M$).** A set of smooth bisections Bis$(PM)$ in the precategory $PM \rightrightarrows M$ is defined by

$$\text{Bis}(PM) = \{\gamma_0 = \text{id}_M, \gamma_1 \text{ is a diffeomorphism of } M\}. \quad (22)$$

Namely, there is one to one correspondence between a pair of fixed points $x = \gamma_0(x)$, $y = \gamma_1(x)$ and a path in a bisection of $PM \rightrightarrows M$. We then define the path rackoids.
Definition (Path rackoid). Given a precategory \( PM \rightrightarrows M \) defined by paths \( \gamma : [0,1] \rightarrow M \), we introduce the rack product \( \triangleright \) for any elements of bisections \( \psi, \varphi \in \text{Bis}(PM) \) and for any \( t \in [0,1] \) as

\[
(\psi \triangleright \varphi)_t = \psi_1 \circ \varphi_t \circ \psi_1^{-1}.
\]

Further, we define the rack action of a bisection \( \psi \) on a path \( \gamma \in PM \) as

\[
(\psi \triangleright \gamma)_t = \psi_1 \circ \gamma_t.
\]

Then it is obvious that the product \( \triangleright \) satisfies the self-distributivity and \( (PM \rightrightarrows M, \triangleright) \) becomes an infinite-dimensional unital Lie rackoid. We call this a path rackoid.

As we have discussed in the previous section, the rack action of \( \psi \) on \( \phi \) shifts the initial and the end points of \( \phi \) along the \( \psi \) direction. Since the path in the base space \( M \) induces the path in the cotangent bundle \( T^*M \), the notion of the path rackoid is generalized to that of the cotangent path.

Definition (Prcategory by cotangent paths). Let \( M \) be a finite-dimensional compact manifold and \( T^*M \rightrightarrows \) be the cotangent bundle over \( M \). We define a pair of smooth paths on \( M \) as

\[
PT^*M = \{ (\gamma, \eta) : [0,1] \rightarrow T^*M \mid \eta : \text{smooth} \}.\]

Here \( \eta \) is the actual cotangent path on \( T^*M \) and it is related to the path \( \gamma \) in the base space \( M \) through the projection \( \gamma = \pi \circ \eta \). The path \( \eta \) is recognized as a morphism whose source and target maps \( s, t \) are defined by \( s = \pi \circ \text{ev}_0 \), \( t = \pi \circ \text{ev}_1 \). Here \( \text{ev}_{0,1} \) is the evaluation of \( \eta \) at \( t = 0,1 \) and takes values in \( T^*M \). One notices that the projection results in the initial and the end points of the path \( \gamma \) on \( M \). Again, the unit map \( \epsilon \) is defined by the constant path. These structures make \( PT^*M \rightrightarrows M \) be a smooth precategory.

We next define bisections of the precategory \( PT^*M \rightrightarrows M \).

Definition (Bis(\( PT^*M \)) of \( PT^*M \rightrightarrows M \)). Let \( PT^*M \rightrightarrows M \) be the precategory defined by the cotangent paths. Bisections of \( PT^*M \rightrightarrows M \) are pairs of paths \( \Sigma = (\phi, \eta) \subset PT^*M \) where each path is defined as follows. First, the paths \( \phi \) in the base space \( M \) are bisections of the precategory \( PM \rightrightarrows M \). Second, the cotangent paths \( \eta \) are the section of the pullback bundle of \( \phi_t : M \rightarrow M \) defined by \([0,1] \ni t \mapsto \phi_t^*\eta_t \). Namely, they are 1-forms on \( M \). Explicitly, the path of the 1-form is defined by

\[
t \in [0,1] \mapsto \phi_t^*\eta_t(X_x) = \eta_t(T_x\phi_t(X_x)) \quad \text{for} \quad X_x \in T_xM, \quad x \in M.
\]

Given these definitions, we introduce a rack structure in bisections \( \Sigma \) of the precategory \( PT^*M \rightrightarrows M \).

Definition (Rack action of cotangent paths). Let \( PT^*M \rightrightarrows M \) be a precategory defined by the cotangent paths. For any bisections \( \Sigma = (\phi, \eta), T = (\psi, \zeta) \) of \( PT^*M \rightrightarrows M \), the rack action is defined by

\[
(\Sigma \triangleright T)_t = (\phi, \eta) \triangleright (\psi, \zeta)_t = (\phi_1 \circ \psi_t \circ \phi_1^{-1}, (\phi_t^*)^{-1} (\zeta_t - t\dot{\phi}_t \circ \phi_1^*d\beta_\Sigma))
\]
where $\dot{\psi}_t = \frac{d}{dt}\psi_t$ and

$$\beta_\Sigma = \int_0^1 ds \, \dot{\phi}_s^* \eta_s$$  \hspace{1cm} (28)

is a 1-form associated with $\Sigma$. Similarly, for a cotangent path $a \in PT^*M$ composed of a cotangent vector $\theta_t(x), x \in M$ at $\gamma_t(x)$, where the path $[0, 1] \ni t \mapsto \gamma_t$ is given by the projection $\gamma = \pi \circ a$, we define a rack action of a bisection $\Sigma = (\phi, \eta) \subset PT^*M$ on the cotangent path $a = (\gamma, \theta)$ as

$$(\Sigma \triangleright a)_t(x) = (\phi_1 \circ \gamma_t(x), (\phi_1^{-1})^* \circ \{\theta_t(x) - \iota_{\dot{\gamma}_t} \circ \phi_1^* \circ d\beta_\Sigma(x)\}) \hspace{1cm} (29)$$

Note that for a path $(\gamma, \theta) \subset PT^*M$, $(\gamma, \theta)$ is a pair of a tangent vector and a 1-form and $(\dot{\gamma}_t, \theta_t)$ is regarded as $C^\infty([0, 1], TM)$. Here $TM = TM \oplus T^*M \xrightarrow{\pi} M$ is the generalized tangent bundle over $M$. The rack product by $\Sigma = (\phi, \eta)$ shifts the initial and the end points of the path on the base space $M$. On the other hand, the rack product in (27), (29) induces the pull-back by $\phi$ and the gauge transformation by $\eta$ in the cotangent space. It is easy to check that the first component of the rack action (27) satisfies the self-distributivity. On the other hand, the self-distributivity of the rack action in the 1-forms is little bit non-trivial. A careful analysis revealed that the second component of the cotangent path also satisfies the self-distributivity. One finds the detailed proof in [37]. The proof is based on the explicit form of the exterior derivative of the 1-form associated with the bisection $\Sigma \triangleright T$. This is given by the direct calculations. The result is

$$d\beta_\Sigma T = (\phi_1^*)^{-1} d\beta T - (\phi_1^*)^{-1} \psi_1^* \phi_1^* d\beta_\Sigma + d\beta_\Sigma.$$  \hspace{1cm} (30)

Here bisections are expressed as $\Sigma = (\phi, \eta), T = (\psi, \zeta)$. This is a key expression to prove the self-distributivity of the rack product [37]. We will see in the next subsection that this specific expression, that is necessary to prove the self-distributivity, does not hold for the pre-rackoid.

Since the structure discussed above are all smooth and the rack action is well-defined, $(PT^*M \rightrightarrows M, \triangleright)$ becomes a unital Lie rackoid. Now we are in a position to discuss the infinitesimal algebroid $A$ of the rackoid $(PT^*M \rightrightarrows M, \triangleright)$. Consider the $s^{-1}$-fiber over $M$. The infinitesimal algebroid $A = \text{Ker}Ts \xrightarrow{\pi} M$ for the Lie rackoid $PT^*M \rightrightarrows M$ is well-defined. Recall that for the bisections, we have a tangent bundle $T(PT^*M) = \{\dot{\gamma} : [0, 1] \rightarrow TM \oplus T^*M\}$. Then given by the section $\Gamma : M \rightarrow PT^*M$, we define the bracket on $\Gamma(A) = \Gamma_s([0, 1], X(M) \oplus \Omega^1(M))$ as clarified in the general discussion. From the definition of the rack product of $(PT^*M \rightrightarrows M, \triangleright)$, the bracket is calculated as the adjoint action associated with the rack action for families of bisections. We define the following quantities,

$$\frac{\partial}{\partial u} \phi_t^u|_{u=0} = X_{1,t}, \quad \frac{\partial}{\partial u} \psi_t^u|_{u=0} = X_{2,t},$$

$$\frac{\partial}{\partial u} \beta_{\Sigma t}^u|_{u=0} = \alpha_{1,t}, \quad \frac{\partial}{\partial u} \delta_t^u|_{u=0} = -\alpha_{2,t}, $$  \hspace{1cm} (31)

where $\phi^u, \psi^u$ are families of bisections parametrized by $u$, $X_i, \alpha_i$ $(i = 1, 2)$ are vector fields and
1-forms on $M$. Then we find
\[ \frac{\partial}{\partial u} \frac{\partial}{\partial v} \phi^{u} \circ \psi^{u} \circ (\phi^{v})^{-1}|_{u=v=0} = [X_{1,t=1}, X_{2,t}], \]
\[ \frac{\partial}{\partial u} \frac{\partial}{\partial v} (\phi^{v})^{-1} (\zeta^{v})|_{u=v=0} = \mathcal{L}_{X_{1,t=1}} \alpha_{2,t}. \] (32)

Therefore, the adjoint action associated with the rack action $\Sigma \triangleright (\cdot)$ results in the following bracket:
\[ [X_{1,t} + \xi_{1,t}, X_{2,t} + \xi_{2,t}]_{D} = [X_{1,t=1}, X_{2,t}]_{TM} + \mathcal{L}_{X_{1,t=1}} \alpha_{2,t} - \iota_{X_{2,t}} d \int_{0}^{1} ds \alpha_{1,s}. \] (33)

This by definition satisfies the left Leibniz identity. The bracket is almost the Dorfman bracket of the standard Courant algebroid. We then define the following subbundle of $\mathcal{A}$:
\[ I = \{ (X_{t}, \xi_{t}) | X_{1} = 0, \int_{0}^{1} ds \alpha_{s} = 0 \} \] (34)

This defines an isomorphism between $\mathcal{A}/I$ and the standard Courant algebroid via the map $\varphi = (ev_{1}, \int_{0}^{1} dt) : \mathcal{A} \to TM$. Indeed, we have
\[ \varphi : (X_{t}, \alpha_{t}) \mapsto (X_{1}, \int_{0}^{1} dt \alpha_{t}). \] (35)

Using these facts, by defining $X_{i} = X_{i,t=1}, \xi_{i} = \int_{0}^{1} \alpha_{i,s} ds$, we finally obtain
\[ \varphi[X_{1,t} + \alpha_{1,t}, X_{2,t} + \alpha_{2,t}]_{D} = [X_{1}, X_{2}]_{TM} + \mathcal{L}_{X_{1}} \xi_{2} - \iota_{X_{2}} d \xi_{1}. \] (36)

This is nothing but the Dorfman bracket (20) of the standard Courant algebroid.

### 4.2 Pre-rackoids and doubled cotangent paths

Exploiting the discussions in the previous subsections, we next explore the group-like, global structure associated with the Vaisman algebroid. To this end, we look for a rackoid-like structure whose bracket in the infinitesimal algebroid results in the bracket of the Vaisman algebroids. An explicit example of the Vaisman algebroid appears in the tangent bundle of a para-Hermitian manifold $\mathcal{M}$ [4, 8]. The bracket is given by the D-bracket:
\[ [e_{1}, e_{2}]_{D} = [X_{1}, X_{2}]_{T\mathcal{M}^{+}} + \mathcal{L}_{\xi}, X_{2} - \iota_{\xi} d^* X_{1} \]
\[ + [\xi_{1}, \xi_{2}]_{T\mathcal{M}^{-}} + \mathcal{L}_{X_{1}} \xi_{2} - \iota_{X_{2}} d \xi_{1}. \] (37)

Here $e_{i} = X_{i} + \xi_{i}$ and $X_{i}, \xi_{i}$ are vectors and dual vectors, $d^*, d$ are exterior derivatives on the vector and its dual vector spaces. $[\cdot, \cdot]_{T\mathcal{M}^{+}}, [\cdot, \cdot]_{T\mathcal{M}^{-}}$ are Lie brackets on vector and dual vector spaces and $\mathcal{L}_{X}, \mathcal{L}_{\xi}$ are Lie derivatives associated with the vectors and their duals. There are two independent parts in the D-bracket. The first term in the first line and the second, third terms in the second line define the Dorfman bracket (20) for the standard Courant algebroid.
The other terms are necessary for the Vaisman algebroid. As we have observed, brackets in the Vaisman algebroids do not necessarily satisfy the Leibniz identity. Indeed, the Leibniz identity of the D-bracket never holds without the strong constraint. If one solves the strong constraint in DFT by the para-holomorphic quantities in the para-Hermitian manifold, the latter all vanish [13].

A skew-symmetric bracket is defined by the anti-symmetrization of the D-bracket.

\[
[e_1, e_2]_C = [X_1, X_2]_{T\mathcal{M}_+} + \mathcal{L}_{\xi_1} X_2 - \mathcal{L}_{\xi_2} X_1 - \frac{1}{2} d(\iota_{X_1} \xi_2 - \iota_{X_2} \xi_1) + [\xi_1, \xi_2]_{T\mathcal{M}_-} + \mathcal{L}_{X_1} \xi_2 - \mathcal{L}_{X_2} \xi_1 - \frac{1}{2} d^*(\iota_{\xi_1} X_2 - \iota_{\xi_2} X_1)
\] (38)

This is nothing but the C-bracket which governs the gauge symmetry of DFT.

As we have seen in the previous sections, the origin of the Leibniz identity is the self-distributivity of the rack action. Therefore it is natural that a global counterpart of the Vaisman algebroid is given by a rackoid without self-distributive rack actions. To elucidate such a structure, we first define the notion of pre-rackoids.

**Definition (Pre-rackoid).** Let \( G \rightrightarrows M \) be a semi-precategory. Bisections of \( G \) are defined as in the definition 3.2. For any bisection \( \Sigma \) and \( g \in G_y \), we define an action of \( \Sigma \) on \( g \in G_y \).

\[
\triangleright : (\Sigma, g) \mapsto \Sigma \triangleright g \in G_{\sigma \cdot y}.
\] (39)

Here \( \sigma \cdot x \) stands for a smooth action of \( \sigma \) on \( x \in M \). When the assignment \( \Sigma \triangleright (\cdot) : G \to G \) is bijective, we call this the pre-rack action (product). We then call \( (G \rightrightarrows M, \triangleright) \) the pre-rackoid. We can introduce an additional unital structure by \( \epsilon \) by which we define the unital pre-rackoid.

Similar to the rack action, the pre-rack action of \( \Sigma \) on \( g \in G \) is defined by the shift of the initial and the end points of the morphism \( g \) by the \( \Sigma \) action. However, this action is not given by \( g = t \circ \sigma \) in general. Due to this property, we stress that \( \triangleright \) does not necessarily satisfy the self-distributivity. We will see the explicit example of this pre-rack action in the following. We note that the smooth pre-rack action \( \triangleright \) still induces a bracket via the adjoint action.

In order to find an explicit example of the pre-rackoid, we once again write down the necessary conditions. (i) The pre-rack product \( \triangleright \) does not satisfy the self-distributivity in general. (ii) Under the imposition of the strong constraint in DFT, the pre-rack product \( \triangleright \) reduces to the rank product \( \triangleright \) that satisfies the self-distributivity. (iii) A bracket obtained by the induced adjoint map of the pre-rack action \( \triangleright \) is given by the D-bracket (37).

A key ingredient is the doubled structure of the D-bracket. To incorporate this structure, we begin with a 2\( D \)-dimensional para-Hermitian manifold \( \mathcal{M} \) as the base space of the pre-rackoid. The local coordinates \( x^\mathcal{M} = (x^\mu, \tilde{x}_\mu) \) of doubled geometry in DFT naturally appears in a para-Hermitian manifold [6,7]. Due to the para-complex structure \( K^2 = 1 \) of \( \mathcal{M} \), the tangent bundle \( T\mathcal{M} \) is decomposed into two parts \( T\mathcal{M} = T\mathcal{M}_+ \oplus T\mathcal{M}_- \). Here each part is determined by the eigenbundles of the para-complex structure \( K^2 = 1 \). There are doubled foliations \( \mathcal{F}, \tilde{\mathcal{F}} \) of \( \mathcal{M} \) associated with the integrability of \( T\mathcal{M}_+ \) and \( T\mathcal{M}_- \). The leaves associated with \( T\mathcal{M}_+ \) are characterized by spaces where \( \tilde{x} = \text{const} \). We express a leaf defined by a locus for fixed \( \tilde{x} \) by \( \mathcal{F}_{x,[\tilde{x}]} \). The coordinate along the space \( \mathcal{F}_{x,[\tilde{x}]} \) is \( x^\mu \). The same is true for \( T\mathcal{M}_- \), namely, spaces...
defined by \( x = \text{const} \) is denoted by \( \tilde{\mathcal{F}}_{[\vec{x}]} \). In this picture, solving the strong constraint and defining the physical space is equivalent to choose a leaf in the foliations of \( \mathcal{M} \). For example, the strong constraint in DFT is trivially solved by the para-holomorphic quantities, i.e., those that depend only on \( x^\mu \) coordinates. This implies that we choose a \( \mathcal{F}_{x,[\vec{x}]} \) space as a physical spacetime.

Let us consider a path \((\phi, \eta) \subset PT^*\mathcal{M}\) in a leaf \( \mathcal{F}_{x,[\vec{x}]} \) given by \( \vec{x} = \text{const} \). As we have discussed in the previous subsection, one can define a cotangent path rackoid \((PT^*\mathcal{M} \rightrightarrows \mathcal{F}_{x,[\vec{x}]}, \triangleright)\) whose base space is \( \mathcal{F}_{x,[\vec{x}]} \). We also define another cotangent path rackoid \((\tilde{PT}^*\mathcal{M} \rightrightarrows \tilde{\mathcal{F}}_{[x],\vec{x}}, \triangleright)\) based on the cotangent path \((\tilde{\phi}, \tilde{\eta}) \subset \tilde{PT}^*\mathcal{M}\) on the leaf \( \tilde{\mathcal{F}}_{[x],\vec{x}} \) at \( x = \text{const} \). These rackoids \((PT^*\mathcal{M}, \tilde{PT}^*\mathcal{M})\) are defined independently. Now we introduce a new path based on a pair of paths \((\phi, \eta) \subset PT^*\mathcal{M}\), \((\tilde{\phi}, \tilde{\eta}) \subset \tilde{PT}^*\mathcal{M}\) in the doubled foliations on \( \mathcal{M} \) and the cotangent bundle \( T^*\mathcal{M} \). The new path on the base space \( \mathcal{M} \) is defined by the concatenation of the paths \( \phi : [0, 1] \to \mathcal{F}_{x,[\vec{x}]} \) and \( \tilde{\phi} : [0, 1] \to \tilde{\mathcal{F}}_{\tilde{\phi}(\vec{x})}, \vec{x} \) along the leaves \( \mathcal{F}_{x,[\vec{x}]} \) and \( \tilde{\mathcal{F}}_{\tilde{\phi}(\vec{x}),\vec{x}} \), respectively (see Fig 1). Here \( t \in [0, 1] \) is the parameter of the paths. More explicitly, we have the path acting on the point \((x, \vec{x}) \in \mathcal{M}\) as

\[
(\phi, \tilde{\phi})_t(x, \vec{x}) = (\phi_t(x), \tilde{\phi}_t(\vec{x}))
\]  

(40)

The paths in the cotangent space are defined similarly by the concatenation of \( \eta \) and \( \tilde{\eta} \) on \((T\mathcal{M}_+)^* \) and \((T\mathcal{M}_-)\)\(^*\). We call this the doubled cotangent path and denote it \( PT^*\mathcal{M} \odot \tilde{PT}^*\mathcal{M} \equiv PT^*\mathcal{M} \). We define the source and the target maps \( PT^*\mathcal{M} \to \mathcal{M} \) as \( s \) and \( \tilde{t} \). Here \( s, \tilde{t} \) are the source and the target maps of \( PT^*\mathcal{M} \rightrightarrows \mathcal{F}_{x,[\vec{x}]} \) and \( \tilde{PT}^*\mathcal{M} \rightrightarrows \tilde{\mathcal{F}}_{\tilde{\phi}(\vec{x})}, \vec{x} \). Then, \( PT^*\mathcal{M} \rightrightarrows \mathcal{M} \) becomes a semi-precategory. If we employ the pair of the unit maps \((\epsilon, \tilde{\epsilon})\) of \( PT^*\mathcal{M} \) and \( \tilde{PT}^*\mathcal{M} \) as the unit map of \( PT^*\mathcal{M} \rightrightarrows \mathcal{M} \), it becomes a smooth precategory. Bisections \((PT^*\mathcal{M})\) are defined similarly through the ones in the precategories \( PT^*\mathcal{M}, \tilde{PT}^*\mathcal{M} \), namely, they are diffeomorphisms associated with paths in \( \mathcal{M} \) and \( T\mathcal{M} \). Explicitly, for bisections \( \Sigma = (\phi, \eta) \),

\[\text{Figure 1: Leaves for the doubled foliations of } \mathcal{M} \text{ (thin lines). The doubled path is the concatenation of the paths along the leaves } \mathcal{F}_{x,[\vec{x}]} \text{ and } \tilde{\mathcal{F}}_{\tilde{\phi}(\vec{x})}, \vec{x} \text{ (bold lines).}\]
\( \tilde{\Sigma} = (\tilde{\phi}, \tilde{\eta}) \) of \( PT^* M \) and \( \hat{PT}^* M \), a bisection \( \Sigma \) of \( PT^* M \) is given by \( \Sigma = \Sigma \circ \tilde{\Sigma} \).

We then define a pre-rack product \( \triangleright \) in the precategory \( PT^* M \cong M \). We propose a product of bisections between \( \Sigma = \tilde{\Sigma} \circ \Sigma \) and \( T = \tilde{T} \circ T \) of \( PT^* M \cong M \) as

\[
\Sigma \triangleright T = \left( \phi_1 \circ \psi_t \circ \phi_1^{-1} \circ \tilde{\phi}_1 \circ \tilde{\psi}_t \circ \tilde{\phi}_1^{-1}, \right.
\]

\[
\left. \left( (\phi_1^{-1})^*(\zeta_t) - (\phi_1^{-1})^* \iota_\phi \phi_1^* d\beta_\Sigma \right) + \left( (\tilde{\phi}_1^{-1})^*(\tilde{\zeta}_t) - (\tilde{\phi}_1^{-1})^* \iota_\psi \tilde{\phi}_1^* d\tilde{\beta}_\Sigma \right) \right). \tag{41}
\]

Here

\[
\beta_\Sigma = \int_0^1 ds \phi_1^* \eta_s, \quad \tilde{\beta}_\Sigma = \int_0^1 ds \tilde{\phi}_1^* \tilde{\eta}_s \tag{42}
\]

are 1-forms associated with the bisections \( \Sigma, \tilde{\Sigma} \). We note that the product defined by (41) is quite different from the rack product in \( \Sigma \). Remarkably, in the cotangent \( \phi \) in \( \Sigma \), the pull-back \( \phi^* \) induced by the path \( \phi \) in the base space acts on the 1-form by the rack product. In (41), it is not true that the pull-back \((\phi \circ \phi)^{-1}\mathfrak{r}^*\) by the path \( \phi \circ \phi \) in \( M \) acts on the doubled 1-form, given in the form of \( \beta^* dx^\mu + \xi_\mu d\tilde{x}^\mu \), like that way. More explicitly, the pre-rack product of \( \Sigma \) and \( T \) at \((x, \tilde{x}) \in M \) is given by

\[
(\Sigma \triangleright T)_t(x, \tilde{x}) = \left( (\phi_1 \circ \psi_t \circ \phi_1^{-1}(x), \tilde{\phi}_1 \circ \tilde{\psi}_t \circ \tilde{\phi}_1^{-1}(\tilde{x})), \right)
\]

\[
\left( (\phi_1^{-1})^*(\zeta_t) - (\phi_1^{-1})^* \iota_\phi \phi_1^* d\beta_\Sigma)(x, \tilde{x}) + ((\tilde{\phi}_1^{-1})^*(\tilde{\zeta}_t) - (\tilde{\phi}_1^{-1})^* \iota_\psi \tilde{\phi}_1^* d\tilde{\beta}_\Sigma)(x, \tilde{x}) \right) . \tag{43}
\]

One notices that there are no terms such as \((\phi_1^{-1})^* \tilde{\zeta}_t\). Here the first line provides the components of path in \( M \). The path is represented by a pair of local coordinates \((x, \tilde{x})\). The second line gives the components of the doubled cotangent vectors in the form of \( X^\mu dx^\mu + \xi_\mu d\tilde{x}^\mu \) on \( T^* M \). The maps \( \phi, \psi \) controls the translation along \( x \)-direction while \( \phi, \tilde{\psi} \) gives the path in the \( \tilde{x} \)-direction. With these structures, we have the following proposition.

**Proposition** (Pre-rackoid by doubled cotangent path). The pre-rack product \( \triangleright \) defined in (41) does not satisfy the self-distributivity in general. Therefore \((PT^* M \cong M, \triangleright)\) is a pre-rackoid. We call this the doubled cotangent path pre-rackoid.

The non-self-distributivity of the pre-rack product (41) is shown by direct calculations. However we present a concise reason in the following. The path in the base space is nothing but the one in the path rackoid. Since the rack action along the \( x(\tilde{x}) \)-direction is given by the adjoint action

\[
(\phi \triangleright \psi)_t = \phi_1 \circ \psi_t \circ \phi_1^{-1}, \tag{44}
\]

then, it is obvious that this satisfies the self-distributivity. The cotangent part seems to be less trivial. For example, the 1-form \( \beta_{\Sigma \triangleright T} \) associated with \( \Sigma \triangleright T \) is given by

\[
\beta_{\Sigma \triangleright T} = \int_0^1 ds \left[ \phi_1 \circ \psi_s \circ \phi_1^{-1} \circ \tilde{\phi}_1 \circ \tilde{\psi}_t \circ \tilde{\phi}_1^{-1} \right]^* \times \left[ ((\phi_1^{-1})^*(\zeta_t) - (\phi_1^{-1})^* \iota_\phi \phi_1^* d\beta_\Sigma) + ((\tilde{\phi}_1^{-1})^*(\tilde{\zeta}_t) - (\tilde{\phi}_1^{-1})^* \iota_\psi \tilde{\phi}_1^* d\tilde{\beta}_\Sigma) \right]. \tag{45}
\]
Here one finds that \( \left[ \phi_1 \circ \psi_s \circ \phi_1^{-1} \circ \phi_1 \circ \psi_t \circ \phi_1^{-1} \right]^* \), composed of the pull-back of \( \phi_1 \) along \( x \)-direction, cancels \( (\phi_1^{-1})^* \) in front of \( \zeta_t \), but that for the \( \bar{x} \)-direction picks up shifts of the path by \( \phi_1^* \). Moreover, the \( d \) operation not only acts on the cotangent path in \( \mathcal{F}_{x,[\bar{x}]} \) but also on that in \( \tilde{\mathcal{F}}_{[x],[\bar{x}]} \). The same is true for \( d^* \). Due to these properties, the expression (33) does not hold anymore and the proof in [37] is not applied to the product (41). These properties trigger the breaking of the self-distributivity.

The remaining discussion is completely parallel to the ones in [37]. Since the pre-rack action is smooth, the adjoint map induced by the \( \ominus \) results in a bracket on the infinitesimal algebroid \( \mathcal{A} \) of \( (\mathbf{PT}^* \mathcal{M} \rightrightarrows \mathcal{M}, \geq) \). Again, by differentiating the families of bisections in the base space, we obtain a pair of vectors:

\[
\frac{\partial}{\partial u} \phi^a|_{u=0} = X = X^\mu \partial_\mu, \quad \frac{\partial}{\partial u} \phi^a|_{u=0} = \xi = \xi_\mu \partial_\mu.
\] (46)

Similarly, for families of bisections in the cotangent bundle, we obtain a pair of 1-forms:

\[
\frac{\partial}{\partial u} \beta_{\Sigma^a}|_{u=0} = \alpha = \alpha_\mu dx^\mu, \quad \frac{\partial}{\partial u} \beta_{\Sigma^a}|_{u=0} = \Lambda = \Lambda^\mu dx^\mu.
\] (47)

We here clarify the relation of the doubled geometry and generalized geometry. The vectors \( \xi_\mu(x, \bar{x}) \partial \mu^\mu \) and 1-forms \( X^\mu(x, \bar{x}) d\bar{x}_\mu \) on \( T \mathcal{M}_-,(T \mathcal{M}_-)^* \) in the doubled geometry are identified with the 1-forms \( \xi_\mu(x, \bar{x}) dx^\mu \) and vectors \( X^\mu(x, \bar{x}) \partial_\mu \) on \( T \mathcal{M}_+,(T \mathcal{M}_+)^* \) through the following natural isomorphism [6]:

\[
\Phi^+ : \xi_\mu \partial \mu^\mu \sim \xi_\mu dx^\mu, \quad X^\mu d\bar{x}_\mu \sim X^\mu \partial_\mu.
\] (48)

Then, the pair of paths \( (\phi, \bar{\eta}) \) on \( \mathcal{M} \) and pair of vectors on \( T \mathcal{M}_+ \) defined by the derivatives of the paths, and the pair of 1-forms \( (\eta, \partial) \) on the dual bundle \( (T \mathcal{M}_+)^* \) are obtained. Given these identifications, it is now straightforward to obtain the D-bracket structure from the the pre-rack action (11). One finds that the Dorfman bracket of the standard Courant algebroid comes from the \( \mathbf{PT}^* \mathcal{M} \rightrightarrows \mathcal{F}_{x,[\bar{x}]} \) part in (11). The extra terms needed for the D-bracket of the Vaisman algebroid is obtained from the \( \mathbf{PT}^* \mathcal{M} \rightrightarrows \tilde{\mathcal{F}}_{[x],[\bar{x}]} \) part. As we have seen in the case of the standard Courant algebroid, the map \( \varphi = (\text{ev}_1, \int_0^1 dt) \) finally provides the complete D-bracket

\[
[e_1, e_2]_D = [X_1, X_2]_{T \mathcal{M}_+} + \mathcal{L}_{\xi_1} X_2 - \iota_{\xi_2} d^* X_1 \\
+ [\xi_1, \xi_2]_{(T \mathcal{M}_+)^{*}} + \mathcal{L}_{X_1} \xi_2 - \iota_{X_2} d \xi_1.
\] (49)

By definition, the infinitesimal algebroid \( \mathcal{A} \) of the pre-rackoid \( \mathbf{PT}^* \mathcal{M} \rightrightarrows \mathcal{M} \) equipped with the bracket (49) is the Vaisman algebroid.

When the strong constraint is imposed, the Leibniz identity of the Vaisman algebroid is recovered and it becomes the Courant algebroid. This is obvious if one employs a solution as the one that \( \partial \mu^\mu = 0 \), namely, the para-holomorphic quantities. In this case, the physical spacetime is defined by a leaf \( \mathcal{F}_{x,[\bar{x}]} \) for a fixed \( \bar{x} \) and all the physical quantities depend only on \( x \). The directions along the leaf \( \tilde{\mathcal{F}}_{[x],[\bar{x}]} \) is frozen in the doubled space. Then, the \( \mathbf{PT}^* \mathcal{M} \)
component of the pre-rackoid $PT^*M \supseteq \mathcal{M}$ becomes trivial and it reduces to the rackoid $PT^*M \supseteq \mathcal{F}_{x,[2]} \subset \mathcal{M}$. The tangent space of $\mathcal{M}$ is identified with the generalized tangent space $TM$ through the natural isomorphism \([\mathcal{E}]\). At the level of the algebroid, the condition $\tilde{\partial}^* = 0$ implies $[\cdot, \cdot]^* = L_{\zeta}^* = d^* = 0$ and the D-bracket reduces to the Dorfman bracket \([13]\).

Now we revisit the geometrical meaning of the strong constraint in DFT. As we have shown above, an infinitesimal algebroid of the pre-rackoid is isomorphic to the Vaisman algebroid equipped with the D-bracket, which is equivalent to the C-bracket through the antisymmetrization. An essential difference between the Vaisman and the Courant algebroids is the Leibniz (Jacobi) identity of the D-bracket (C-bracket). As we have mentioned, the Leibniz identity is nothing but the self-distributivity of the rack action. From the viewpoint of the doubled geometry, the strong constraint can be seen as a condition for the recovery of the self-distributivity of the pre-rackoid.

This fact is rephrased in the following suggestive form. We define the operator $R(g \otimes h) = g \otimes g \triangleright h$ on $\text{Bis}(PT^*M) \otimes \text{Bis}(PT^*M)$ for any $g, h \in \text{Bis}(Y)$. Then the action of $R$ on the tensor products $\text{Bis}(PT^*M) \otimes \text{Bis}(PT^*M) \otimes \text{Bis}(PT^*M)$ results in

$$R_{12}R_{13}R_{23}(g, h, i) = g \otimes g \triangleright h \otimes g \triangleright (h \triangleright i),$$

$$R_{23}R_{13}R_{12}(g, h, i) = g \otimes g \triangleright h \otimes (g \triangleright h) \triangleright (g \triangleright i).$$

Therefore the self-distributivity of the doubled cotangent path in $\mathcal{M}$ is recast in the following quantum Yang-Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$  \hspace{1cm} (51)

In other words, we can say that the strong constraint in DFT is a sufficient condition of the quantum Yang-Baxter equation for the rack action.

### 5 Formal rackoids and pre-rackoids

In the previous section, we work on the integration of the Vaisman algebroid by a heuristic approach based on the doubled geometry. In this section, we propose a formal (pre)-rackoids which enable one to find a formal integration of the Courant and the Vaisman algebroid. The prescription discussed here is useful for perturbative treatment of the (pre)-rackoids.

#### 5.1 Formal rackoids

Let $g$ be a (left) Leibniz algebra with a Leibniz bracket $[\cdot, \cdot]$. Let $\text{ad}(X)Y = [X, Y]$. We define an operation \([33]\),

$$X \triangleright Y := \exp \text{ad}(X)Y.$$  \hspace{1cm} (52)

Then, the operation satisfies

$$X \triangleright (Y \triangleright Z) = (X \triangleright Y) \triangleright (X \triangleright Z).$$  \hspace{1cm} (53)
since
\[
X \triangleright (Y \triangleright Z) = \exp \text{ad}(X)(\exp \text{ad}(Y)Z) = \exp(\exp \text{ad}(X)\text{ad}(Y))\exp \text{ad}(X)Z = \exp(\text{ad}(\exp \text{ad}(X)Y))\exp \text{ad}(X)Z = (X \triangleright Y) \triangleright (X \triangleright Z).
\]
(54)

We generalize this construction to a Leibniz algebroid, Courant algebroid and a Vaisman algebroid.

We introduce a formal integration of a Leibniz algebroid.

**Definition.** Let \(E\) be a vector bundle over \(M\). For \(e_i \in \Gamma(E)\) and \(f \in C^\infty(M)\), we consider a product \(\cdot: \Gamma(E) \times C^\infty(M) \to C^\infty(M)\), and operations, a rack operation, \(\triangleright: \Gamma(E) \times \Gamma(E) \to \Gamma(E)\) and a rack action, \(\triangleright: \Gamma(E) \times C^\infty(M) \to C^\infty(M)\). If they satisfy
\[
e_1 \triangleright (e_2 \triangleright e_3) = (e_1 \triangleright e_2) \triangleright (e_1 \triangleright e_3),
\]
(55)
\[
e_1 \triangleright (e_2 \triangleright f) = (e_1 \triangleright e_2) \triangleright (e_1 \triangleright f),
\]
(56)
\[
e_1 \triangleright fe_2 = (e_1 \triangleright f) \cdot (e_1 \triangleright e_2),
\]
(57)

\((E, \triangleright, \cdot)\) is called a bundle (Lie) rackoid.

In fact a bundle rackoid is constructed by the formal exponential of operations of the Leibniz algebroid. Let \((E, [-, -]_D, \rho)\) be a Leibniz algebroid over a smooth manifold \(M\). Then, for sections \(e_i \in \Gamma(E)\) and a function \(f \in C^\infty(M)\), we define a rack operation \(\triangleright: \Gamma(E) \times \Gamma(E) \to \Gamma(E)\) and a rack action on \(C^\infty(M)\) as
\[
e_1 \triangleright e_2 := \exp \text{ad}(e_1)e_2,
\]
(58)
\[
e_1 \triangleright f := (\exp \rho(e_1))f.
\]
(59)

Here \(\text{ad}(e_1)e_2 = [e_1, e_2]_D\). Then, we obtain the rack identities, (55)–(57) from the following three identities of a Leibniz algebroid,
\[
[e_1, e_2, e_3]_D = [[e_1, e_2]_D, e_3]_D + [e_2, [e_1, e_3]_D]_D, \\
\rho([e_1, e_2]_D)f = [\rho(e_1), \rho(e_2)]f, \\
[e_1, fe_2]_D = f[e_1, e_2] + \rho(e_1)f \cdot e_2.
\]

We call a formal rackoid a rackoid defined by formal exponentials (58) and (59) of a tangent algebroid.

**Theorem.** If \(E\) is a Leibniz algebroid with \((\rho, [-, -]_D)\), \(E\) is a formal rackoid, if we define
\[
e_1 \triangleright e_2 := \exp \text{ad}(e_1)e_2,
\]
(60)
\[
e_1 \triangleright f := (\exp \rho(e_1))f.
\]
(61)

Next, we consider a bundle rackoid corresponding to a Courant algebroid.
Definition. Let \((E, \triangleright, \cdot)\) be a bundle rackoid. Let \((\cdot, \cdot)\) be a pseudo-Euclidean inner product on \(E\). If a bundle rackoid satisfies
\[
e_1 \triangleright (e_2, e_3) = ((e_1 \triangleright e_2), (e_1 \triangleright e_3)),
\]
\(E\) is called a metric bundle rackoid.

Let \((E, [-, -]_D, \rho, (\cdot, \cdot))\) be a Courant algebroid. We consider a formal rackoid defined by operations (58) and (59). Then, the condition (62) is proved from the identity of the Courant algebroid,
\[
\rho(e_1)(e_2, e_3) = ([e_1, e_2]_D, e_3) + (e_2, [e_1, e_3]_D).
\]
This identity with identities as Leibniz algebroid is enough to obtain other identities of the Courant algebroid. Then, we obtain the following theorem.

Theorem. If \(E\) is a Courant algebroid with \(([-, -]_D, \rho, (\cdot, \cdot))\), If we define a formal rackoid with operations,
\[
e_1 \triangleright e_2 := \exp \text{ad}(e_1)e_2, \quad (64)
\]
\[
e_1 \triangleright f := \exp \rho(e_1)f, \quad (65)
\]
\(E\) is a metric bundle rackoid.

5.2 Formal pre-rackoids

We consider a pre-rackoid version of a formal exponential.

Definition. Let \(E\) be a vector bundle over a smooth manifold \(M\) with a pseudo-Euclidean inner product \((\cdot, \cdot)\). For \(e_i \in \Gamma(E)\) and \(f \in C^\infty(M)\), we consider a product \(\cdot : \Gamma(E) \times C^\infty(M) \to C^\infty(M)\), and operations, a rack operation, \(\triangleright : \Gamma(E) \times \Gamma(E) \to \Gamma(E)\) and a rack action, \(\triangleright : \Gamma(E) \times C^\infty(M) \to C^\infty(M)\).

If an operation \(\triangleright\) satisfies
\[
e_1 \triangleright fe_2 = (e_1 \triangleright f) \cdot (e_1 \triangleright e_2), \quad (66)
\]
\[
e_1 \triangleright (e_2, e_3) = (((e_1 \triangleright e_2), (e_1 \triangleright e_3)), \quad (67)
\]
\(E\) is called a metric (bundle) (Lie) pre-rackoid.

Let \((E, [-, -]_D, \rho, (\cdot, \cdot))\) be a Vaisman algebroid. As in the case of a Leibniz algebroid, we define the following operation \(\triangleright\): \(\Gamma(E) \times \Gamma(E) \to \Gamma(E)\) and the action on \(C^\infty(M)\) \(\triangleright\): \(\Gamma(E) \times C^\infty(M) \to C^\infty(M)\) for \(e_i \in \Gamma(E)\) and \(f \in C^\infty(M)\),
\[
e_1 \triangleright e_2 := \exp \text{ad}(e_1)e_2, \quad (68)
\]
\[
e_1 \triangleright f := \exp \rho(e_1)f. \quad (69)
\]
Here \(\text{ad}(e_1)e_2 = [e_1, e_2]_D\) is a D-bracket. The exponential of a D-bracket or a generalized Lie derivative \(\exp L_v\) has appeared as a large gauge transformation of DFT [38]. We can prove
that operations (68) and (69) satisfy (66) and (67). A Vaisman algebroid gives a formal metric pre-rackoid. In general, the rack identities are not satisfied,

\[ e_1 \bowtie (e_2 \bowtie e_3) \neq (e_1 \bowtie e_2) \bowtie (e_1 \bowtie e_3), \]  
\[ e_1 \bowtie (e_2 \bowtie f) \neq (e_1 \bowtie e_2) \bowtie (e_1 \bowtie f), \]  

The closure condition is expressed by

\[ e_1 \bowtie (e_2 \bowtie e_3) - (e_1 \bowtie e_2) \bowtie (e_1 \bowtie e_3) = 0, \]  
\[ e_1 \bowtie (e_2 \bowtie f) - (e_1 \bowtie e_2) \bowtie (e_1 \bowtie f) = 0, \]  

6 Rackoids and pre-rackoids from sigma models

In this section, we consider a sigma model description of a cotangent path rackoid and a doubled cotangent path pre-rackoid. We discuss description of a Lie rackoid and pre-rackoid using topological sigma models.

6.1 Courant sigma model

We can naturally construct a three dimensional topological sigma model with a Courant algebroid structure called a Courant sigma model. [39, 40]

Let \((E, \rho, [-,-]_D, H)\) be a Courant algebroid over a smooth manifold \(M\). Let \(N\) be a three dimensional manifold with local coordinates \(\sigma^\mu, \phi^i: N \to M\) is a smooth map, \(A \in \Omega^1(N, \varphi^*E)\) is a 1-form and \(B \in \Omega^2(N, \varphi^*T^*M)\) is a 2-form.

The action of the Courant sigma model is

\[ S = \int_N \left( -B_i \wedge d\varphi^i + \frac{1}{2} k_{ab} A^a \wedge dA^b + \rho^i_a(\varphi) B_i \wedge A^a + \frac{1}{3!} H_{abc}(\varphi) A^a \wedge A^b \wedge A^c \right), \]  

where \(k_{ab}\) is defined from a fiber metric as \(k_{ab}(e_a, e_b)\), \(\rho(e_a) = \rho_a^b(x) \partial_i = \rho^i_a(x) \frac{\partial}{\partial x^i}\) and \(\frac{1}{3!} H_{abc}(x) = H(e_a, e_b, e_c)\). Here \(e_a\) is a basis of the fiber of \(E\).

The gauge transformation is

\[ \delta \varphi^i = \rho^i_a(\varphi) t^a, \]  
\[ \delta A^a = dt^a + k_{ab} \rho^i_b(\varphi) u_i + k_{ab} H_{bcd}(\varphi) A^c t^d, \]  
\[ \delta B_i = du_i + \partial_i \rho^a_A(A^a u_j - t^a B_j) + \frac{1}{2} \partial_i H_{abc}(\varphi) A^a A^b t^c, \]  

where \(t^a\) is a 0-form gauge parameter and \(u_i\) is a 1-form gauge parameter. The action (74) is gauge invariant under the gauge transformation (75)–(77) if and only if the target space \(E\) with structures \((k, \rho, H)\) is a Courant algebroid. Thus this topological sigma mode is called a Courant sigma model.

We consider the case of the standard Courant algebroid on \(E = TM \oplus T^*M\). We take the \(O(D, D)\) metric as

\[ k = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]  

21
and the anchor map $\rho : X + \alpha \mapsto X$, where $X + \alpha \in \Gamma(TM \oplus T^*M)$,

$$\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

(79)

and $H = 0$. The 1-form field $A^a$ is decomposed to components of $TM$ and $T^*M$ as $A^a = (A^i, C_i)$, where $A \in \Omega^1(N, \varphi^*T^*M)$ and $C \in \Omega^1(N, \varphi^*TM)$. The action of the Courant sigma model (74) becomes a simple form,

$$S = \int_N -B_i \land d\varphi^i + C_i \land dA^i + B_i \land A^i.$$  

(80)

Let $N = \Sigma \times \mathbb{R}$, where $\Sigma$ is a two dimensional manifold and $\mathbb{R}$ is the time direction. Then, we can compute the symplectic form,

$$\omega = \int_\Sigma \delta X^i \land \delta B_i + \delta A^i \land \delta C_i.$$  

(81)

Nonzero Poisson brackets of fields are obtained from Poisson brackets of canonical conjugates,

$$\{X^i(\sigma), B_{abj}(\sigma')\} = -\epsilon_{ab} \delta_j^2 (\sigma - \sigma'),$$  

(82)

$$\{A_a^i(\sigma), C_{bj}(\sigma')\} = \epsilon_{ab} \delta_j^2 (\sigma - \sigma'),$$  

(83)

where $a, b = 1, 2$ are indices on $\Sigma$. The Hamiltonian

$$H = \int_\Sigma (B_{0i} \land G^i - C_{0i} \land F^i - A_{0i}^i K_{21i}),$$  

(84)

is purely written by terms with constraints, where $B_{0i} = B_{0ai} d\sigma^a$ where $a = 1, 2$. Here constraints are

$$G^i = d\varphi^i - A^i,$$  

(85)

$$F_i = dC_i + B_i,$$  

(86)

$$K_i = dA^i.$$  

(87)

Poisson brackets of constraints $G^i$, $F_i$ and $K_i$ show that they consist of the first class constraints.

### 6.2 Path rackoids from Courant sigma models

We analyze correspondence of the standard Courant sigma model and a path Lie rackoid.

We consider a path $I_t = [0, t] \subset \Sigma$ and a map from $I_t$ to the target space $M$, $\gamma(t) : I_t \rightarrow M$, where $t \in \mathbb{R}$. $a = (\gamma, \alpha) \in PT^*M$ consists of a cotangent path on $M$.

We can easily identify a path to a map $\varphi$ on $I_t$, $\gamma = \varphi|_{I_t}$. A section of a generalized tangent bundle $TM \oplus T^*M$, $X + \alpha = X^i(x) \partial_i + \alpha_i(x) dx^i$, is mapped as follows,

$$(\tilde{X} + \tilde{\alpha}) = \int_0^1 (X^i(\varphi(\sigma))C_i(\sigma) + \alpha_i(\varphi(\sigma))A^i(\sigma)),$$  

(88)
using fields $A^i(\sigma)$ and $C_i(\sigma)$ in the Courant sigma model. This map is denoted by $j$,

$$j : X + \alpha \mapsto \tilde{X} + \tilde{\alpha},$$

(89)

Then the bilinear form $(\cdot, \cdot)$ is mapped to a Poisson bracket since $j$ is a homomorphism,

$$j : (X + \alpha, Y + \beta) \mapsto -\{\tilde{X} + \tilde{\alpha}, \tilde{Y} + \tilde{\beta}\},$$

(90)

for $X + \alpha, Y + \beta \in \Gamma(TM \oplus T^*M)$. The anchor map $\rho(X + \alpha) = X$ is mapped to

$$j : \rho(X + \alpha) f(x) \mapsto -\{\tilde{X} + \tilde{\alpha}, H\}, f(\phi(\sigma)),$$

(91)

where $f \in C^\infty(M)$. The Dorfman bracket is mapped to the derived bracket of Poisson brackets,

$$j : [X + \alpha, Y + \beta]_D \mapsto -\{\tilde{X} + \tilde{\alpha}, H\}, \tilde{Y} + \tilde{\beta}\},$$

(92)

where $f \in C^\infty(M)$. Such a derived bracket construction has been formulated in [43]. We can prove that $j$ is a homomorphism of a Courant algebroid. Therefore a Courant algebroid structure on $E$ is mapped to the mapping space $\text{Map}(\mathcal{N}, E)$ with a symplectic form $\omega$, and operations are calculated by (90)–(92).

From the above correspondence, a cotangent path is mapped to

$$j : (\gamma, \alpha) \mapsto (\phi(t), \tilde{\alpha}(t)).$$

(93)

Quantities $\phi, \psi, \beta, \zeta$ in a rackoid which satisfy differential equations (31) are formally described by exponential maps, which are Wilson lines

$$j : \phi \mapsto \text{P exp } \tilde{X} = \text{P exp } \int_{\gamma} X^i(\phi(\sigma)) C_i(\sigma),$$

(94)

$$j : \beta \mapsto \text{P exp } \tilde{\alpha} = \text{P exp } \int_{\gamma} \alpha_i(\phi(\sigma)) A^i(\sigma),$$

(95)

Here $\gamma = \gamma(1)$. We obtain a formal rackoid structure in section 5 from Wilson lines (95) and (95). Wilson lines provide formal exponential maps, but they are useful for concrete calculations of the integration using the quantization of the sigma model.

We can easily generalize the above construction of the standard Courant algebroid to a general Courant algebroid. The equivalent but more familiar construction for quantization of a Courant algebroid using a topological sigma model is a so-called AKSZ sigma model using supergeometry. [46, 47]

### 6.3 Topological double sigma models and pre-rackoids

In this section, we consider a construction of a pre-rackoid using a topological sigma model of doubled target spacetime with a Vaisman algebroid structure. A topological sigma model with a structure of DFT geometry is proposed in [41].

Let $\mathcal{M}$ is $2D$-dimensional manifold corresponding a doubled spacetime. Typically, it is a direct product of a physical spacetime and a T-dual spacetime, $\mathcal{M} = M \times \tilde{M}$. Suppose an
$O(D, D)$ invariant metric $\eta_{IJ}$ on $\mathcal{M}$, where $I, J = 1, \cdots, 2D$ are indices of local coordinate on $\mathcal{M}$. 

Let $N$ be a 3 dimensional manifold with local coordinates $\sigma^\mu$. $\varphi : N \to \mathcal{M}$ is a smooth map from $N$ to the target double spacetime. $A \in \Omega^1(N, \varphi^*T^*\mathcal{M})$ is a 1-form taking a value on the pullback of $T^*\mathcal{M}$. $B \in \Omega^2(N, \varphi^*T^*\mathcal{M})$ is a 2-form taking a value on the pullback of $T^*\mathcal{M}$.

An action of a topological double sigma model is
\begin{equation}
S = \int_N \left(-B_I d\varphi^I + \frac{1}{2} \eta^{IJ} A_I dA_J + \frac{1}{3!} F^{IJK}(\varphi) A_I A_J A_K \right),
\end{equation}
where $F(x) \in \Omega^3(\mathcal{M})$ is a generalized flux which is a 3-form on $\mathcal{M}$. Now we consider a simplest $F = 0$ case for a genuine Vaisman algebroid. Then the action becomes
\begin{equation}
S = \int_N (-B_I d\varphi^I + \frac{1}{2} \eta^{IJ} A_I dA_J).
\end{equation}

Nonzero Poisson brackets of fields are computed as
\begin{align}
\{X^I(\sigma), B_{ab}J(\sigma')\} &= -\epsilon_{ab} \delta^I_2(\sigma - \sigma'), \\
\{A_aI(\sigma), A_bJ(\sigma')\} &= \epsilon_{ab} \eta_{IJ} \delta^2(\sigma - \sigma'),
\end{align}
where $a, b = 1, 2$ are indices on $\Sigma$.

The section condition to reduce the 2D-dimensional double spacetime $\mathcal{M}$ to a D-dimensional spacetime $\mathcal{M}$ drop functions of $\phi^I$. If we use equations of motion, half degrees of $A_I$ drop and we obtain the first class constraint $G^I_{\mid M}$. On the other hand, we can impose $G^I$ consist of the first class constraints. The condition gives the section condition on fields on the target space.

It is natural to consider the following gauge transformations analogous to gauge transformations of the CSM, (75)–(77),
\begin{align}
\delta \varphi^I &= \eta^{IJ} t_J, \\
\delta A_I &= dt_I + u_I, \\
\delta B_I &= du_I,
\end{align}
where $t_I$ is a zero-form gauge parameter and $u_I$ is a 1-form gauge parameter. However, the action $S$ is not gauge invariant under this transformation, $\delta S \neq 0$.

In the Hamiltonian analysis, we obtain a Hamiltonian,
\begin{equation}
H = \int_\Sigma (B_{0I} \wedge G^I + A^I_0 F_I),
\end{equation}
where constraints are
\begin{align}
G^I &= d\varphi^I - \eta^{IJ} A_J, \\
F_I &= dA_I + B_I.
\end{align}
In this case, the constraints are not the first class, since
\begin{equation}
\{G_a^I(\sigma), G_b^J(\sigma)\} = \eta^{IJ} \epsilon_{ab} \delta^2(\sigma - \sigma).
\end{equation}
where \( a, b = 1, 2 \) are indices on \( \Sigma \).

Operations in the Vaisman algebroid are constructed similar to one of the Courant algebroid discussed in sections 6.1 and 6.2.

A section of tangent bundle \( T\mathcal{M} \), \( X = X^I(x)\partial_I \) is mapped to a field (a pullback 1-form to the mapping space),

\[
\tilde{X}(t) = \int_{I_t} X^I(\varphi(\sigma))A_I(\sigma).
\]

(107)

The map is denoted by \( j \),

\[
j : X \mapsto \tilde{X},
\]

(108)

The bilinear form \( (\cdot, \cdot) \) is mapped to a Poisson bracket since \( j \) is a homomorphism,

\[
j : (X, Y) \mapsto -\{\tilde{X}, \tilde{Y}\}.
\]

(109)

for \( X + \alpha, Y + \beta \in \Gamma(TM \oplus T^*M) \). The D-bracket is mapped to the derived bracket of Poisson brackets,

\[
j : [X, Y]_D \mapsto -\{\{\tilde{X}, H\}, \tilde{Y}\}.
\]

(110)

where \( X, Y \in \mathfrak{x}(\mathcal{M}) \). A derived bracket construction of a D-bracket has been analyzed in [48–50]. Therefore a Vaisman algebroid structure on \( T\mathcal{M} \) is mapped to the mapping space \( \text{Map}(N, T\mathcal{M}) \) with a symplectic form \( \omega \), and operations are calculated by (109) and (110).

From the above correspondence, a cotangent path is mapped to

\[
j : (\gamma, \alpha) \mapsto (\varphi(t), \tilde{\alpha}(t)).
\]

(111)

where a metric \( \eta \) naturally identify \( T\mathcal{M} \) to \( T^*\mathcal{M} \). Quantities \( \phi, \psi, \beta, \zeta \) and tilde quantities in a pre-rackoid which satisfy differential equations (31) are formally described by exponential maps, which are Wilson lines

\[
j : (\phi, \beta) \mapsto \text{P exp } \tilde{X} = \text{P exp } \int_{I_t} X^I(\varphi(\sigma))A_I(\sigma).
\]

(112)

Similar to the Courant algebroid case, the equivalent but more familiar construction for quantization of a Courant algebroid using a topological sigma model is an AKSZ type sigma model formulation.

7 Conclusion and discussions

In this paper, we studied a global aspect of the doubled geometry in DFT through the coquecigrue problem of the Vaisman algebroid.

We first focus on the global structure associated with the Courant algebroid. A global, group-like structure corresponding to the Leibniz algebroid is a rackoid, which is a groupoid-like generalization of a rack. A Leibniz algebroid appears in the tangent bundle of a rackoid
as an infinitesimal counterpart of the global structure. The cotangent path rackoid proposed in [37] provides the standard Courant algebroid as a tangent Leibniz algebroid.

With these results at hand, we studied a generalization of the cotangent path rackoid that gives rise to the Vaisman algebroid as an infinitesimal algebroid. It is obvious that the Vaisman algebroid fail to satisfy the Leibniz identity in the Courant algebroid. Since the Leibniz identity is encoded into the self-distributivity of the rack product, an integration of the Vaisman algebroid is given by a rackoid type structure without the self-distributivity. We called this structure the pre-rackoid. A crucial ingredient for the construction of a pre-rackoid is the doubled cotangent path that is defined by the doubled foliations of the para-Hermitian manifold. With this structure, we defined the pre-rack product by (41). Due to the intermediate shift between different leaves of the doubled foliations, the self-distributivity of the pre-rack product is explicitly broken.

This picture is consistent with DFT. One remembers that the strong constraint in DFT picks up a leaf of the foliations as a physical spacetime. In this case, the pre-rack product is restricted only on a leaf and the self-distributivity is trivially recovered.

We next focused on a direct approach on the integration of the Courant and the Vaisman algebroid. We introduced the formal exponential map of the adjoint action or the bracket in algebroids. The resulting structures lead to notions of formal rackoid and pre-rackoids. A formal rackoid together with the metric bundle rackoid structure defines an integration of the Courant algebroid. We showed that these notions are generalized to the formal metric pre-rackoid. The D-bracket in the Vaisman algebroid is exponentiated, providing an example of the pre-rack product. Compare to the heuristic approach by the explicit examples of the (pre-)rack product, the approach based on the a formal integration of the algebroid will help to understand an intuitive feature of the (pre-)rackoid. The idea is familiar to a formal deformation of an algebra and a perturbative calculation of a quantum theory.

In the end of the discussion, we exhibited another realization of a (pre)-rackoid. We introduced a three-dimensional topological sigma model, the so-called Courant sigma model. This provides a natural arena for a realization of the Courant algebroid. We showed that the structure of the formal rackoid is explicitly implemented by an exponential map of fields in the Courant sigma model. This is physically interpreted as Wilson lines associated with gauge fields. The construction is generalized to a topological double sigma model of the Vaisman algebroid and the pre-rackoid. These examples give physical applications of the (pre-)rackoid.

In this paper, we studied a global structure associated with the Vaisman algebroid from the several viewpoints. The underlying geometry is the doubled geometry in DFT.

A pre-rackoid structure is related to a global gauge structure of DFT. Quantizations of topological sigma models will provide quantum version of doubled geometry, which is important for analysis of quantum T-duality.

It is known that DFT contains various solutions that are not the ones in supergravity [51–55]. They are called the non-geometric solutions. The global aspects of doubled geometry discussed here will be helpful to understand these non-geometric nature of spacetimes in string theory. We noted that the strong constraint in DFT is represented by the quantum Yang-Baxter equation for the rack action. It is well-known that the quantum and the classical Yang-Baxter relations have deep connections with the integrable systems. Indeed, the classical Yang-Baxter
deformation is a key ingredient of the integrability of string theory [20, 21, 56]. We will come back to these issues in future studies.

Acknowledgments

The authors would like to thank H. Mori and K. Shiozawa for useful discussions. The work of S. S. was supported by Grant-in-Aid for Scientific Research (C), JSPS KAKENHI Grant Number JP20K03952.

References

[1] C. Hull and B. Zwiebach, “Double Field Theory,” JHEP 0909 (2009) 099 [arXiv:0904.4664 [hep-th]].
[2] W. Siegel, “Superspace duality in low-energy superstrings,” Phys. Rev. D 48 (1993), 2826-2837 [arXiv:hep-th/9305073 [hep-th]].
[3] W. Siegel, “Manifest duality in low-energy superstrings,” [arXiv:hep-th/9308133 [hep-th]].
[4] I. Vaisman, “On the geometry of double field theory,” J. Math. Phys. 53 (2012) 033509 [arXiv:1203.0836 [math.DG]].
[5] I. Vaisman, “Towards a double field theory on para-Hermitian manifolds,” J. Math. Phys. 54 (2013) 123507 [arXiv:1209.0152 [math.DG]].
[6] L. Freidel, F. J. Rudolph and D. Svoboda, “Generalised Kinematics for Double Field Theory,” JHEP 1711 (2017) 175 [arXiv:1706.07089 [hep-th]].
[7] L. Freidel, F. J. Rudolph and D. Svoboda, “A Unique Connection for Born Geometry,” Commun. Math. Phys. 372 (2019) no.1, 119 [arXiv:1806.05992 [hep-th]].
[8] D. Svoboda, “Algebroid Structures on Para-Hermitian Manifolds,” J. Math. Phys. 59 (2018) no.12, 122302 [arXiv:1802.08180 [math.DG]].
[9] A. Chatzistavrakidis, L. Jonke, F. S. Khoo and R. J. Szabo, “Double Field Theory and Membrane Sigma-Models,” JHEP 1807 (2018) 015 [arXiv:1802.07003 [hep-th]].
[10] T. J. Courant, “Dirac Manifolds,” Transactions of the American Mathematical Society, vol. 319 (1990) 631.
[11] C. Hull and B. Zwiebach, “The Gauge algebra of double field theory and Courant brackets,” JHEP 09 (2009), 090 [arXiv:0908.1792 [hep-th]].
[12] Z-J. Liu, A. Weinstein, P. Xu, “Manin Triples for Lie Bialgebroids,” J. Differential Geom. 45 (1997) 547 [dg-ga/9508013].
[13] H. Mori, S. Sasaki and K. Shiozawa, “Doubled Aspects of Vaisman Algebroid and Gauge Symmetry in Double Field Theory,” J. Math. Phys. 61 (2020) no.1, 013505 [arXiv:1901.04777 [hep-th]].

[14] C. Klimcik and P. Severa, “Dual nonAbelian duality and the Drinfeld double,” Phys. Lett. B 351 (1995) 455 [hep-th/9502122].

[15] C. Klimcik, “Poisson-Lie T duality,” Nucl. Phys. Proc. Suppl. 46 (1996) 116 [hep-th/9509095].

[16] C. Klimcik and P. Severa, “Poisson-Lie T duality and loop groups of Drinfeld doubles,” Phys. Lett. B 372 (1996), 65-71 [arXiv:hep-th/9512040 [hep-th]].

[17] F. Hassler, “Poisson-Lie T-Duality in Double Field Theory,” [arXiv:1707.08624 [hep-th]].

[18] V. E. Marotta and R. J. Szabo, “ParaHermitian Geometry, Dualities and Generalized Flux Backgrounds,” Fortsch. Phys. 67 (2019) no.3, 1800093 [arXiv:1810.03953 [hep-th]].

[19] Y. Sakatani, “Type II DFT solutions from Poisson-Lie T-duality/plurality,” PTEP (2019) 073B04 [arXiv:1903.12175 [hep-th]].

[20] C. Klimcik, “Yang-Baxter sigma models and dS/AdS T duality,” JHEP 12 (2002), 051 [arXiv:hep-th/0210095 [hep-th]].

[21] C. Klimcik, “On integrability of the Yang-Baxter sigma-model,” J. Math. Phys. 50 (2009), 043508 [arXiv:0802.3518 [hep-th]].

[22] R. Von Unge, “Poisson Lie T plurality,” JHEP 0207 (2002) 014 [hep-th/0205245].

[23] Y. Sakatani, “U-duality extension of Drinfel’d double,” PTEP 2020 (2020) no.2, 023B08 [arXiv:1911.06320 [hep-th]].

[24] E. Malek and D. C. Thompson, “Poisson-Lie U-duality in Exceptional Field Theory,” JHEP 04 (2020), 058 [arXiv:1911.07833 [hep-th]].

[25] Y. Sakatani and S. Uehara, “Non-Abelian U-duality for membrane,” [arXiv:2001.09983 [hep-th]].

[26] L. Hlavaty, “Classification of six-dimensional Leibniz algebras E_3,” [arXiv:2003.06164 [hep-th]].

[27] K. C. H. Mackenzie and P. Xu, “Lie bialgebroids and Poisson groupoids,” Duke Math. J. 73 (1994) 415.

[28] J. L. Loday, “Une version non commutative des algébres de Lie: les algébres de Leibniz,” Enseign. Math. 39 (1993) 269-293.

[29] P. Severa, “Some title containing the words “homotopy” and “symplectic”, e.g. this one,” [arXiv:math/0105080 [math.SG]].
[30] Y. Sheng and C. Zhu, “Higher Extensions of Lie Algebroids,” Commun. Contemp. Math. 19, 1650034 (2017), [arXiv:1103.5920 [math-ph]]

[31] D. Li-Bland and P. Severa, “Integration of Exact Courant Algebroids,” Electronic Research Announcements in Mathematical Sciences (ERA-MS), 19 (2012) 58-76, [arXiv:1101.3996 [math.DG]].

[32] R. A. Mehta, X. Tang, “Symplectic structures on the integration of exact Courant algebroids,” Journal of Geometry and Physics 127 (2018), 68-83, [arXiv:1310.6587 [math.DG]].

[33] M. K. Kinyon, “Leibniz algebras, Lie racks, and digroups,” J. Lie Theory 17 (2007) 99-114, [arXiv:math/0403509 [math.RA]]

[34] S. Covez, “The local integration of Leibniz algebras,” [arXiv:1011.4112 [math.RA]].

[35] M. Bordemann and F. Wagemann, “A dirty integration of Leibniz algebras,” [arXiv:1606.08214 [math.DG]].

[36] J. S. Carter, “A Survey of Quandle Ideas,” [arXiv:1002.4429 [math.GT]]

[37] C. Laurent-Gengoux and F. Wagemann, “Lie rackoids integrating Courant algebroids,” [arXiv:1807.05891 [math.AT]].

[38] O. Hohm and B. Zwiebach, “Large Gauge Transformations in Double Field Theory,” JHEP 02 (2013), 075 doi:10.1007/JHEP02(2013)075 [arXiv:1207.4198 [hep-th]].

[39] N. Ikeda, “Chern-Simons gauge theory coupled with BF theory,” Int. J. Mod. Phys. A 18 (2003), 2689-2702 [arXiv:hep-th/0203043 [hep-th]].

[40] D. Roytenberg, “AKSZ-BV Formalism and Courant Algebroid-induced Topological Field Theories,” Lett. Math. Phys. 79 (2007), 143-159 [arXiv:hep-th/0608150 [hep-th]].

[41] Z. Kokenyesi, A. Sinkovics and R. J. Szabo, “Double Field Theory for the A/B-Models and Topological S-Duality in Generalized Geometry,” Fortsch. Phys. 66 (2018) no.11-12, 1800069 [arXiv:1805.11485 [hep-th]].

[42] A. Chatzistavrakidis, C. J. Grewcoe, L. Jonke, F. S. Khoo and R. J. Szabo, “BRST symmetry of doubled membrane sigma-models,” PoS CORFU2018 (2019), 147 [arXiv:1904.04857 [hep-th]].

[43] D. Roytenberg, “Courant algebroids, derived brackets and even symplectic supermanifolds,” [arXiv:math/9910078 [math.DG]].

[44] M. Gualtieri, “Generalized complex geometry,” Oxford University DPhil thesis. [arXiv:math/0401221 [math.DG]].

[45] C. Laurent-Gengoux and F. Wagemann, “Lie rackoids,” [arXiv:1511.03018 [math.DG]].
[46] M. Alexandrov, M. Kontsevich, A. Schwartz and O. Zaboronsky, *The Geometry of the master equation and topological quantum field theory*, Int. J. Mod. Phys. A 12 (1997) 1405.

[47] N. Ikeda, *Lectures on AKSZ Sigma Models for Physicists*, Noncommutative Geometry and Physics 4, Workshop on Strings, Membranes and Topological Field Theory: 79-169, 2017, World scientific, Singapore, p.79-169.

[48] A. Deser and J. Stasheff, “Even symplectic supermanifolds and double field theory,” Commun. Math. Phys. 339 (2015) no.3, 1003-1020 doi:10.1007/s00220-015-2443-4 [arXiv:1406.3601 [math-ph]].

[49] A. Deser and C. Smann, “Extended Riemannian Geometry I: Local Double Field Theory,” doi:10.1007/s00023-018-0694-2 [arXiv:1611.02772 [hep-th]].

[50] M. A. Heller, N. Ikeda and S. Watamura, “Unified picture of non-geometric fluxes and T-duality in double field theory via graded symplectic manifolds,” JHEP 02 (2017), 078 doi:10.1007/JHEP02(2017)078 [arXiv:1611.08346 [hep-th]].

[51] J. Berkeley, D. S. Berman and F. J. Rudolph, “Strings and Branes are Waves,” JHEP 1406 (2014) 006 [arXiv:1403.7198 [hep-th]].

[52] D. S. Berman and F. J. Rudolph, “Branes are Waves and Monopoles,” JHEP 1505 (2015) 015 [arXiv:1409.6314 [hep-th]].

[53] I. Bakhmatov, A. Kleinschmidt and E. T. Musaev, “Non-geometric branes are DFT monopoles,” JHEP 1610 (2016) 076 [arXiv:1607.05450 [hep-th]].

[54] D. Lst, E. Plauschinn and V. Vall Camell, “Unwinding strings in semi-flatland,” JHEP 07 (2017), 027 [arXiv:1706.00835 [hep-th]].

[55] T. Kimura, S. Sasaki and K. Shiozawa, “Worldsheet Instanton Corrections to Five-branes and Waves in Double Field Theory,” JHEP 1807 (2018) 001 [arXiv:1803.11087 [hep-th]].

[56] F. Delduc, M. Magro and B. Vicedo, “An integrable deformation of the $AdS_5 \times S^5$ superstring action,” Phys. Rev. Lett. 112 (2014) no.5, 051601 [arXiv:1309.5850 [hep-th]].