A note on $b$-coloring of Kneser graphs

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Abstract

In this short note, the purpose is to provide an upper bound for the $b$-chromatic number of Kneser graphs. Our bound improves the upper bound that was presented by Balakrishnan and Kavaskar in [b-coloring of Kneser graphs, Discrete Appl. Math. 160 (2012), 9-14].

Keywords: $b$-coloring, $b$-chromatic number, Kneser graph.

Mathematics Subject Classification: 05C15

1 Introduction

In this note, simple graphs whose vertex sets are nonempty and finite are considered. Let $G$ be a graph with vertex set $V(G)$. A coloring of $G$ stands for a function $f : V(G) \rightarrow C$ such that for each $c$ in $C$, the set $f^{-1}(c)$ is independent; in this case, we think of each $c$ in $C$ as a color and call $f^{-1}(c)$ a color class of $f$.

Let $G$ be a graph and $f : V(G) \rightarrow C$ be a coloring of $G$. The vertex $v$ of $G$ is said to be a color-dominating vertex with respect to $f$ if $f(N[v]) = C$, i.e., the vertex $v$ sees all colors on its closed neighborhood. Also, the coloring $f : V(G) \rightarrow C$ is called a $b$-coloring of $G$ whenever each of its color classes contains at least one color-dominating vertex. The $b$-chromatic number of $G$, denoted by $\varphi(G)$, is defined to be the maximum positive integer $k$ for which $G$ admits a $b$-coloring $f : V(G) \rightarrow C$ with $|C| = k$. This concept was introduced by Irving and Manlove in 1999 in [3], and since then there exists an extensive literature on it; see [4] for a survey.

Suppose that $n$ and $m$ are positive integers and $n \geq m$. The Kneser graph $KG(n, m)$ is the graph whose vertex set is the set of all $m$-subsets of $\{1, 2, \ldots, n\}$, in which two vertices $A$ and $B$ are declared to be adjacent iff $A \cap B = \emptyset$. In [1] 2 5 7, $b$-coloring of Kneser graphs has been investigated.

Every $d$-regular graph $G$ satisfies $\varphi(G) \leq d + 1$ [3]. Kratochvíl, Tuza, and Voigt [3] showed that for each $d$ there are only finitely many $d$-regular graphs up to isomorphism whose $b$-chromatic numbers are less than or equal to $d$. So, finding such regular graphs is of interest. In this regard, Balakrishnan and Kavaskar [1] presented some desired Kneser graphs meeting this property.

Theorem 1. [1] Let $n \geq 2$ and $i \geq 0$. Also, let $d$ be the degree-regularity of the Kneser graph $G = KG(2n + k, n)$. If $|V(G)| \leq 2d + 2 - 2i$, then $\varphi(G) \leq d - i$.

The aim of this short note is to provide an improvement of Theorem 1 which is done in the next section.
2 The main result

This section concerns the main result of the note; as follows.

In Theorem 1, the statement $|V(G)| \leq 2d + 2 - 2i$ is equivalent to $\left\lceil \frac{|V(G)|-2}{2} \right\rceil \leq d - i$. Therefore, the upper bound in this Theorem, which is $d - i$, is greater than or equal to $\left\lceil \frac{|V(G)|-2}{2} \right\rceil$. In the next theorem, we provide a sharp upper bound for the $b$-chromatic number of Kneser graphs, which is asymptotic to $\frac{|V(G)|}{3}$.

**Theorem 2.** For fixed $n \geq 2$, the Kneser graph $G_k := KG(2n+k,n)$ satisfies

$$\varphi(G_k) \leq (1 + o(1))\frac{|V(G_k)|}{3},$$

where the $o(1)$ term tends to zero as $k$ tends to infinity.

**Proof.** Let $C$ be the set of color classes of an arbitrary $b$-coloring of $G_k$. For each color class $S$ in $C$, we set $S^0 := \bigcap_{A \in S} A$; and call $S$ an intersecting color class whenever $S^0 \neq \emptyset$. Let us denote by $I$ the set $\{S \mid S \in C, \ S^0 \neq \emptyset\}$.

Consider two distinct color classes $S$ and $T$ in $C$; and let $\hat{S}$ be a color-dominating vertex of $S$. The vertex $\hat{S}$ is adjacent to a vertex of $T$, say $T_1$. So, $\hat{S} \cap T_1 = \emptyset$. Since $S^0 \cap T^0 \subseteq \hat{S} \cap T_1$, we have $S^0 \cap T^0 = \emptyset$. This shows that the function $f : I \rightarrow \{1, 2, \ldots, n + k + 1\}$ that assigns the minimum of $S^0$ to every $S$ in $I$, is an injective mapping. Therefore, $|I| \leq n + k + 1$.

Each non-intersecting color class of $C$ contains at least three vertices of $G_k$. Hence, $|C - I| \leq \frac{|V(G_k)| - |I|}{3}$. Accordingly, $|C| \leq \frac{|V(G_k)| + 2|I|}{3} \leq \frac{|V(G_k)| + 2(n + k + 1)}{3}$. Now, since $\lim_{k \to \infty} \frac{2(n + k + 1)}{2n + k} = 0$, we conclude that $\varphi(G_k) \leq (1 + o(1))\frac{|V(G_k)|}{3}$, which is desired. 

In view of the proof of the Theorem 2, we proved that for any fixed positive integer $n$, the $b$-chromatic number of Kneser graph $G_k := KG(2n+k,n)$ is less than or equal to $U(G_k) := \frac{|V(G_k)|+2(n+k+1)}{3}$. The upper bound $U(G_k)$ is sharp for $n = 1$.

Let us regard an arbitrary integer $n \geq 2$ as fixed. Asymptotically in $k$, the bound $U(G_k)$ is $\frac{|V(G_k)|}{3}$. In [5], Javadi and Omoomei showed that for $n = 2$, the $b$-chromatic number of $G_k$ is asymptotic to $\frac{|V(G_k)|}{3}$. Hence, for $n = 2$, this bound is asymptotically correct; i.e., the ratio $\frac{\varphi(G_k)}{U(G_k)}$ goes to 1 as $k$ tends to infinity.

Since $d + 1$ is an upper bound for the $b$-chromatic number of any $d$-regular graph, it is worth pointing out that for a fixed positive integer $n \geq 2$, if $d_k$ denotes the degree of any vertex of $G_k$, then the upper bound in Theorem 2 is asymptotically $\frac{d_k}{3}$, because the ratio $\frac{d_k}{|V(G_k)|}$ tends to 1 as $k$ tends to infinity.

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