Conservation, impermeability and potential vorticity in relativistic magnetohydrodynamics

S J Fletcher
Trinity College, University of Melbourne, Parkville, Victoria 3052, Australia
E-mail: stephefl@trinity.unimelb.edu.au

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Abstract
The conservation and impermeability conditions are reformulated utilising differential forms and generalised to spacetime. The thermodynamic and electromagnetic potential vorticity scalar fields are defined for relativistic magnetohydrodynamics and their evolution equations are derived.

1. Introduction
Conserved quantities play a key role in physical theory. In geophysical fluid dynamics perhaps the most important and useful conserved quantity is potential vorticity (PV), (Ertel 1942). PV is a scalar field which combines information on the mass distribution, rotation and stability of a fluid:

$$\rho \ q := \zeta \cdot \nabla \psi := \Gamma$$  \hspace{1cm} (1)

$\rho$ denotes fluid density, $q$ denotes PV, $\zeta$ denotes fluid absolute vorticity and $\psi$ denotes specific entropy. When specific entropy depends only on fluid density and pressure the Ertel theorem states:

PV is conserved along the trajectories of a fluid when the flow is isentropic and the only forces acting on the fluid are the pressure gradient force and conservative body forces.

With $\Gamma$ defined in (1), Haynes and McIntyre (1987, 1990) (HM) introduced the terminology potential vorticity substance (PVS) for the integral quantity:

$$\Sigma := \int_{V} \Gamma \ dV$$  \hspace{1cm} (2)

HM demonstrated that PVS is conserved along the trajectories of a vector field which is, in general, different from the velocity of the fluid. Along the trajectories of this vector field PVS is conserved even in the presence of diabatic heating and frictional forces, a result known as the HM conservation theorem. Furthermore, a hypothetical parcel of conserved PVS flowing along the trajectories of this vector field will remain within the same surface of constant $\psi$, a result known as the HM impermeability theorem.

Palmer (1988) discussed PVS and mathematically analogous electromagnetic integral quantities in a non-relativistic context, formulating expressions for their conservation laws utilising the Lie derivative. Webb and Mace (2015) have derived the conservation law for PV in non-relativistic ideal MHD. Friedman (1978) and Katz (1984) have constructed relativistic expressions for PV and demonstrated the Ertel theorem in spacetime for a particle conserving perfect fluid. Sakuma et al (2022) recently rederived the Ertel theorem in spacetime for a particle conserving perfect fluid.

In this paper key results from HM, Webb and Mace (2015), Friedman (1978) and Katz (1984) are extended to relativistic inviscid magnetohydrodynamics (MHD) and integrated into a single coherent mathematical framework: differential forms in spacetime. This framework produces derivations which are economical and results which are extremely general. Furthermore, this framework does not introduce extraneous mathematical...
structures such as the Riemann curvature tensor, utilised in Katz (1984), nor does it explicitly\(^1\) apply covariant derivatives, utilised in Friedman (1978).

Bekenstein and Oron (2001) applied a variational method to derive the governing equations of relativistic MHD and demonstrated the Kelvin circulation theorem in the case of ideal relativistic MHD.

Markakis et al (2017) (MU) utilised a tensorial formulation of Hamiltonian mechanics to discuss conserved quantities in relativistic MHD. In particular, MU discussed conservation of circulation and helicity in both perfect fluids and magnetoflows and discussed conservation of PV in perfect fluids. For a barotropic perfect fluid MU placed particular emphasis on the fact that the Kelvin circulation theorem applies in any spacetime. In this paper results of the same generality are derived: only the most general properties of spacetime are assumed and the specific form of the covariant metric tensor field is not required. In particular, it is not necessary to assume or apply the field equations of general relativity.

Cheviakov (2014) applied the multiplier method for systems of partial differential equations to derive non-relativistic results which are closely related to those presented in this paper.

Publications discussing the practical application of PV concepts to non-relativistic electromagnetic fluids seems to have emerged only recently. Madsen et al (2015) have derived an evolution equation for PV in magnetised plasmas, hypothesising that magnetically confined fusion plasmas may be partitioned into discrete regions of homogenised PV separated by interfaces at which there is strong zonal flow. Chen and Diamond (2020) have developed a theory of PV mixing and zonal flow generation in tangled magnetic fields suitable for modelling the solar tachocline.

Astrophysical scenarios for which relativistic MHD may be applied as a first order model include black hole accretion disks (Mościobrodzka et al (2009)), jets in active galactic nuclei (Meliani et al (2010)), pulsars (Beskin (2009)) and magnetars (Sotani et al (2008)). It does not appear that PV concepts have, to date, been applied in relativistic MHD models of these astrophysical scenarios.

In section 2 standard techniques of vector calculus are utilised to review the construction of the HM conservation vector field. In section 3 the same construction is presented utilising differential forms, firstly in 3-dimensional space and then in spacetime. Sections 2 and 3 provide a clear pathway from the 3-dimensional vector calculus framework to the more general framework of differential forms in spacetime, making explicit the inductive process.

The core definitions and results of this paper are presented in sections 4 and 5. Starting from simple but general mathematical definitions in section 4, a deductive process is applied. A reader principally interested in the central argument of this paper may thus choose to start with section 4 in which the resolution identity is derived and the double conservation and alignment properties are defined. In section 5 integral and scalar thermodynamic and electromagnetic PV fields are defined and their evolution equations derived. Thermodynamic PV follows from the double conservation property, electromagnetic PV follows from the alignment property.

Section 6 concludes with a summary and some brief remarks. Appendix A details the general identities and conservation laws utilised in this paper. Appendix B presents a special case of a useful kinematic 3-form field.

2. Conservation and impermeability utilising vector calculus

In this section the construction of a vector field satisfying the conservation and impermeability conditions utilising vector calculus is presented. This vector calculus construction has already been presented in the literature (HM, Webb and Mace (2015), Bretherton and Schär (1993)).

In this section, and for the remainder of this paper, all fields will be assumed to be sufficiently differentiable for the applied operations to be valid. In this section it will be assumed that \( \partial_t \) commutes with \( \nabla \), \( \nabla \), and \( \nabla \times \).

Let \( \mathbf{v} \) denote the fluid velocity vector field, define \( \zeta := \nabla \times \mathbf{v} \), let \( \mathbf{G} \) denote the vector field representing the sum of all forces per unit mass acting on the fluid and let \( \mathbf{0} \) denote the zero vector field. The momentum equation is:

\[
0 = \partial_t \mathbf{v} + \frac{1}{2} \nabla(\mathbf{v} \cdot \mathbf{v}) + \zeta \times \mathbf{v} - \mathbf{G}
\]  

(3)

\( \partial_t \) denotes partial differentiation with respect to \( t \). Let \( \psi \) denote a scalar field and let \( \mathbf{B} \) denote a vector field. The following is an identity:

\( \psi \) The Riemann–Christoffel connection is utilised when the energy and momentum equations are derived from the spacetime stress–energy tensor field. If the energy and momentum equations are simply imposed, rather than derived, then the results in this paper may be obtained without requiring the manifold to have a connection.
\[(\nabla \times \mathbf{B}) \cdot \nabla \psi = \nabla \cdot (\mathbf{B} \times \nabla \psi)\]  

(4)

Combining equation (3) with identity (4) and applying the vector triple product\(^2\):

\[0 = (\partial_t \zeta) \cdot \nabla \psi + \nabla \cdot [(\zeta \cdot \nabla \psi) \mathbf{v} - \mathbf{G} \times \nabla \psi - (\mathbf{v} \cdot \nabla \psi) \zeta]\]  

(5)

Recall that \( \Gamma \) is defined in (1). In this paper only the cases with \( \Gamma = 0 \) will be considered. Define:

\[\Gamma \mathbf{x} := \Gamma \mathbf{v} - \mathbf{G} \times \nabla \psi - (\partial_t \psi + \mathbf{v} \cdot \nabla \psi) \zeta\]  

(6)

With these definitions equation (5) becomes\(^3\):

\[0 = \partial_t \Gamma + \nabla \cdot (\Gamma \mathbf{x})\]  

(7)

With integral quantity \( \Sigma \) defined in (2) equation (7) implies:

\[0 = \frac{d \Sigma}{d \tau}\]  

(8)

\(d/d\tau\) denotes the derivative along the trajectories of \( \mathbf{x} \). Hence, equation (7) is the conservation condition utilising vector calculus.

Definition (6) implies:

\[0 = \Gamma (\partial_t \psi + \mathbf{x} \cdot \nabla \psi)\]  

(9)

Equation (9) is the impermeability condition utilising vector calculus.

For the remainder of this paper the following terminology will be applied. Equation (7) and its generalisations will be referred to as the conservation condition. Equation (9) and its generalisations will be referred to as the impermeability condition.

A vector field satisfying both equations (7) and (9) or their generalisations will be referred to as a double conservation vector field. Hence, in the terminology of this paper \( \mathbf{x} \), as defined by (1) and (6), is a double conservation vector field.

For a vector field \( \mathbf{x} \) satisfying equation (7), \( \mathbf{q} = \Gamma \mathbf{x} \) will be referred to as the PVS flux vector field. In 3 dimensions \( \mathbf{x} \) satisfies the impermeability condition, given by equation (9), but \( \mathbf{q} \) does not.

The HM impermeability and conservation theorems are the demonstration that a double conservation vector field exists and a second requirement that the double conservation vector field is constructed solely from terms specified or calculated locally from fields with an explicitly stated physical definition (c.f. Bretherton and Schär 1993).

We are now in a position to examine conservation and impermeability in the more general mathematical framework of differential forms

### 3. Conservation and impermeability utilising differential forms

#### 3.1. Notation

Adopt the notation of Frankel (2012). Differential p-forms are denoted using a superscript indicating the value of p. The interior product of a vector, denoted \( \mathbf{Z} \), and a p-form, denoted \( \mathcal{E}^p \), is denoted \( i_\mathbf{Z} \mathcal{E}^p \). \(^4\) \( \wedge \) denotes the exterior product, \( d \) denotes the exterior derivative and \( \mathcal{L}_\mathbf{Z} \) denotes the Lie derivative with respect to \( \mathbf{Z} \).

Require the manifold under consideration to be endowed with a covariant metric tensor field, denoted \( \mathbf{g} \), and a contravariant metric tensor field, denoted \( \mathbf{g}^{-1} \). For any vector field, \( \mathbf{Z} \), its corresponding 1-form field will be denoted:

\[\mathcal{Z} := \mathbf{g}(\mathbf{Z}, \cdot) \implies \mathbf{g}^{-1}(\mathcal{Z}, \cdot) = \mathbf{Z}\]  

(10)

In this paper the same letter will be utilised to indicate vectors and 1-forms related to one another by definition (10). Finally, symbol \( * \) denotes the Hodge star operator which, in an n-dimensional Riemannian manifold, defines a mapping from p-forms to (n-p)-forms in their respective spaces at each point (Flanders 1989).

General identities are detailed in appendix A.

#### 3.2. Conservation and impermeability in 3-dimensional space

In this section utilise a 3-dimensional manifold endowed with a covariant metric tensor field, \( \mathbf{g} \), with components \( \text{diag}(1, 1, 1) \) in an orthonormal basis so that the signature takes the Euclidean value +3. Require \( \mathbf{g} \), \( \mathbf{H} = \nabla \phi - (\nabla \phi)/\rho \)

\(^2\) Define \( \mathbf{H} = \nabla \phi - (\nabla \phi)/\rho \). When \( \psi = \psi(\rho, \mathbf{p}) \), equation (5) allows us to replace \( \mathbf{G} \rightarrow \mathbf{G}' = \mathbf{G} - \mathbf{H} \). \( \mathbf{G}' \) was the force per unit mass term utilised by HM in definition (6). See also Cheviakov (2014).

\(^3\) Definition (6) with equation (7) and \( \partial_t \psi + \mathbf{v} \cdot \nabla \psi = 0 \) is Webb and Mace (2015) general conservation equation (4.37) from which their specific conservation equation (1.3) follows.

\(^4\) In index notation this corresponds to contraction of \( \mathbf{Z} \) on the first index of \( \mathcal{E}^p \): \( i_\mathbf{Z} \mathcal{E}^p = \mathbf{Z}^a \mathcal{E}_{abc} \ldots \)
\( g^{-1} \) and the orthonormal basis vector fields to be time independent. Assume that \( \partial_t \) commutes with the exterior derivative, \( d \).

For 3-dimensional Riemannian manifolds with Euclidean signature:

\[
\begin{align*}
\star 1 &= \text{vol}^3 \\
\star \text{vol}^3 &= 1
\end{align*}
\] (11)

\( \text{vol}^3 \) denotes the volume 3-form. For \( p \in \{1, 2, 3\} \):

\[
\text{vol}^3 = \star (\star \text{vol}^p \wedge Z^3)
\] (13)

Equation (13) is true for the specific case being considered in this section, that is a manifold of dimension 3 and covariant metric tensor field of signature +3. Equation (13) is not true for general Riemannian manifolds.

For the remainder of this section require only general properties (11)–(13) to apply. Other than their assumed existence, and properties (11)–(13), the covariant and contravariant metric tensor fields are completely general and may be flat or curved.

Let \( V \) denote the fluid velocity vector field, let \( V^i \) denote the corresponding fluid velocity 1-form field, let \( G^i \) denote the 1-form field representing the sum of all forces per unit mass acting on the fluid and let \( O^i \) denote the zero 1-form field in 3-dimensional space. The momentum equation is:

\[
O^i = \partial_t V^i + \frac{1}{2} d(\nabla V^i) + \nabla V^i - G^i
\] (14)

Let \( \psi \) denote a scalar field and let \( B^i \) denote a 1-form field. Consider also identity:

\[
dB^i \wedge d\psi = d(B^i \wedge d\psi)
\] (15)

Combining identities (15), (A.4) and (A.5) with equation (14):

\[
0 \text{vol}^3 = \partial_t (dV^i) \wedge d\psi + d(\nabla V^i \wedge d\psi) - G^i \wedge d\psi - (\nabla V^i) dV^i
\] (16)

Define:

\[
\Gamma_3 := \star (dV^i \wedge d\psi)
\] (17)

Consider only the cases with \( \Gamma_3 \neq 0 \). Equations (11)–(13), identity (A.8) and definition (17) imply:

\[
\text{vol}^3 = \star \Gamma_3 = \star (\text{vol}^2)
\] (18)

Define:

\[
\Gamma_3 \star X^i = \Gamma_3 \star V^i - \left[G^i \wedge d\psi + \left(\partial_t \psi + \nabla V^i \right) dV^i\right]
\] (19)

Definitions (17) and (19), equation (18) and identity (A.5) applied to equation (16) imply:

\[
0 \text{vol}^3 = \partial_t (\Gamma_3 \text{vol}^3) + \nabla_X (\Gamma_3 \text{vol}^3)
\] (20)

Equation (20) implies:

\[
0 = \frac{d}{d\tau} \int V^i \Gamma_3 \text{vol}^3
\] (21)

d/d\tau denotes the derivative along the trajectories of \( X \) and \( V^3 \) denotes an oriented compact 3-dimensional submanifold transported along the trajectories of \( X \). Hence, equation (20) is the conservation condition utilising differential forms in a 3-dimensional space.

Definition (19) implies:

\[
0 = \Gamma_3 (\partial_t \psi + \nabla_X \psi)
\] (22)

Equation (22) is the impermeability condition utilising differential forms in a 3-dimensional space. Since both equations (20) and (22) are satisfied, \( X \) is a double conservation vector field.

The separation of the time and space derivatives in this section is inelegant. We are now in a position to examine conservation and impermeability in the more general setting of spacetime where no such separation is required. The spacetime covariant metric tensor field may vary in both time and space.

### 3.3. Conservation and impermeability in spacetime

In this section, and for the remainder of this paper, utilise a 4-dimensional manifold endowed with a covariant metric tensor field with components \( \text{diag}(-1, 1, 1, 1) \) in an pseudo-orthonormal basis so that the signature takes the spacetime value +2. Choose the orientation with:

\[
\begin{align*}
\star 1 &= \text{vol}^4 \\
\star \text{vol}^4 &= -1
\end{align*}
\] (23)

Equations (23) and (24) hold if and only if the manifold is simply connected.

(S3) Consider only the cases with the coordinate \( \tau \) which varies in \( [0, \infty) \) only. Otherwise, the manifold is simply connected.

(S4) Assume that the manifold is simply connected. The coordinate \( \tau \) then varies in \( [0, \infty) \) only.
vol$^4$ denotes the volume 4-form. The spacetime quantities under consideration in this paper are most naturally expressed in terms of 3-form fields. Hence, introduce the following spacetime notation:

$$Z^3 := * Z^4 \implies * Z^3 = Z^4$$  \hspace{1cm} \text{(25)}

In this paper the same letter will be utilised to indicate 1-forms and 3-forms related to one another by definition (25).

For $p \in \{1, 2, 3, 4\}$ in spacetime:

$$i_Z \mathcal{E}^p = - * (Z^3 \wedge * \mathcal{E}^p)$$  \hspace{1cm} \text{(26)}

Equation (26) is true for the specific case being considered in this section and for the remainder of this paper, that is a manifold of dimension 4 and covariant metric tensor field of signature $+2$. Equation (26) is not true for general Riemannian manifolds.

Spacetime identity (26) is both very useful and facilitates index-free derivations. For example, utilising spacetime identity (26) with identity (24), definition (25) and Cartan’s formula (A.5), together with the definition of vector divergence:

$$\text{div}(\mu Z \text{vol}^4) = \mathcal{E}_{\mu Z} \text{vol}^4 = d(\mu Z^3)$$  \hspace{1cm} \text{(27)}

$\mu$ denotes a scalar field and $Z$ denotes a vector field with corresponding 3-form field $Z^3$. The zero divergence case of (27) forms the basis of our later work on conservation and the right hand differential forms expression makes this particularly simple to work with.

In what follows, require only general properties (23)–(26) to apply. Other than their assumed existence the spacetime covariant and contravariant metric tensor fields are completely general. The geometry may be flat or curved, stationary or non-stationary.

Let $V$ denote the fluid velocity (4-)vector field and let $V^3$ denote the corresponding fluid velocity 1-form field. Utilise geometrised units and normalise the fluid velocity:

$$1 = - iv V^3$$  \hspace{1cm} \text{(28)}

Let $G^1$ denote the 1-form field representing the sum of all forces per unit mass acting on the fluid. In this section, and for the remainder of this paper, let $\mathcal{O}^p$ denote the zero $p$-form field in spacetime. With normalisation condition (28) the momentum equation is:

$$\mathcal{O}^1 = i_v d V^3 - G^3$$  \hspace{1cm} \text{(29)}

A simpler equation than its 3-dimensional counterpart, (14). Let $\psi$ denote a scalar field. Equation (29) allows us to choose:

$$0 \text{vol}^4 = d[V^3 \wedge i_v d V^3 \wedge d\psi - V^3 \wedge G^1 \wedge d\psi]$$  \hspace{1cm} \text{(30)}

Combining normalisation condition (28), equations (29) and (30), and identities (A.4) and (A.5):

$$0 \text{vol}^4 = d[- i_v (V^3 \wedge d V^3 \wedge d\psi) - V^3 \wedge [G^2 \wedge d\psi + (\mathcal{E}_V \psi) d V^3]]$$  \hspace{1cm} \text{(31)}

Define:

$$\Gamma_4 := * (V^3 \wedge d V^3 \wedge d\psi)$$  \hspace{1cm} \text{(32)}

Consider only the cases with $\Gamma_4 = 0$. Equations (23)–(26), identity (A.8) and definition (32) imply:

$$- i_Z (V^3 \wedge d V^3 \wedge d\psi) = i_Z * \Gamma_4 = i_Z (\Gamma_4 \text{vol}^4) = \Gamma_4 Z^3$$  \hspace{1cm} \text{(33)}

Define:

$$\Gamma_4 X^3 := \Gamma_4 V^3 - V^3 \wedge [G^2 \wedge d\psi + (\mathcal{E}_V \psi) d V^3]$$  \hspace{1cm} \text{(34)}

Equations (27) and (31)–(34) imply:

$$0 \text{vol}^4 = d(\Gamma_4 X^3) = \mathcal{E}_{\Gamma_4 X} \text{vol}^4 = \text{div}(\Gamma_4 X) \text{vol}^4$$  \hspace{1cm} \text{(35)}

Equation (35) is the conservation condition utilising differential forms in spacetime. To connect spacetime conservation condition (35) to the conservation of a volume integral note that the left hand equation in (35), identity (26) and Cartan’s formula (A.5) imply:

$$\mathcal{O}^3 = \mathcal{E}_X (\Gamma_4 X^3)$$  \hspace{1cm} \text{(36)}

Equation (36) is of form (A.9) and is sufficient to guarantee:

$$0 = d / d\tau \int_{V^3} \Gamma_4 X^3$$  \hspace{1cm} \text{(37)}

d/d\tau denotes the derivative along the trajectories of $X$ and $V^3$ denotes an oriented compact 3-dimensional submanifold transported along the trajectories of $X$. 

Equation (26), definition (34) and identity (A.7) imply:
\[ 0 = \Gamma_\nu \mathcal{F}_\nu \psi \]  
(38)
Equation (38) is the impermeability condition utilising differential forms in spacetime. Since both equations (35) and (38) are satisfied, \( \mathbf{X} \) is a double conservation vector field.

The key definitions and results of this paper now follow.

4. Resolution, double conservation and alignment

Starting with spacetime impose the additional mathematical structure of a 1-form field, \( \mathcal{V}^1 \), satisfying normalisation condition (28) and a p-form field, \( \mathcal{R}^p \). These fields have no pre-existing identification with physical quantities.

Identity (26) implies that \( \mathcal{R}^p \) may be decomposed:
\[ \mathcal{R}^p = \mathcal{T}^p + \mathcal{K}^p \]  
(39)
\[ \mathcal{T}^p := \star [\mathcal{V}^1 \wedge \star (\mathcal{V}^1 \wedge \mathcal{R}^p)] \]  
(40)
\[ \mathcal{K}^p := - \mathcal{V}^1 \wedge \mathbf{i}_\nu \mathcal{R}^p = \mathcal{V}^1 \wedge \star (\mathcal{V}^1 \wedge \star \mathcal{R}^p) \]  
(41)
Identity (26) also implies that:
\[ \mathbf{i}_\nu \mathcal{T}^p = \mathcal{O}^{p-1} \]  
(42)
\[ \mathbf{i}_\nu \mathcal{K}^p = \mathcal{O}^{n-p-1} \]  
(43)
Specific cases of general p-form identity (39) were applied in MU, most notably to decompose the Faraday tensor.

4.1. Resolution

Consider definitions (40) and (41) and p-form decomposition identity (39) in the case \( p = 3 \). Again, 3-form field \( \mathcal{R}^3 \) has no pre-existing identification with a physical quantity. Define:
\[ \mathbf{m} := \star (\mathcal{V}^1 \wedge \mathcal{R}^3) \]  
(44)
In this case:
\[ \mathcal{R}^3 = \mathcal{T}^3 + \mathcal{K}^3 \]  
(45)
\[ \mathcal{T}^3 := \mathbf{m} \mathcal{V}^3 \]  
(46)
\[ \mathcal{K}^3 := - \mathcal{V}^1 \wedge \mathbf{i}_\nu \mathcal{R}^3 = \mathcal{V}^1 \wedge \star (\mathcal{V}^1 \wedge \star \mathcal{R}^3) \]  
(47)
Let \( \mathbf{R}, \mathbf{I}, \mathbf{K} \) and \( \mathbf{V} \) denote the vectors corresponding to the 3-forms above via definitions (10) and (25). Definition (46) implies that vector \( \mathbf{I} \) is parallel to vector \( \mathbf{V} \). Also, from equation (43):
\[ \mathbf{i}_\nu \mathcal{K}^3 = 0 \]  
(48)
Vector \( \mathbf{K} \) is perpendicular to vector \( \mathbf{V} \). Hence, when written in terms of their corresponding vectors, equation (45) represents the resolution of \( \mathbf{R} \) into a sum of vector components parallel to \( \mathbf{V} \) and perpendicular to \( \mathbf{V} \).

3-form field \( \mathcal{R}^3 \) may now be specialised by imposing additional mathematical properties. The two, independently imposed, mathematical properties utilised in this paper are double conservation and alignment.

4.2. Property 1: Double conservation

Applying identities (26) and (A.5) to equations (35) and (38) with \( Q := \Gamma_\nu \mathbf{X} \) the conservation and impermeability conditions in spacetime are:
\[ 0 \mathbf{vol}^1 = dQ^3 \]  
(49)
\[ 0 \mathbf{vol}^1 = Q^3 \wedge d\psi \]  
(50)
A remarkably simple system. Although motivated by earlier physical considerations, equations (49) and (50) will be imposed as mathematical statements. Fields \( Q^3 \) and \( \psi \) will not be assumed to have a pre-existing identification with physical quantities. Equations (49) and (50) imply the local existence of a 1-form field, \( \mathcal{Q}^1 \), such that:
\[ Q^3 = d\mathcal{Q}^1 \wedge d\psi \]  
(51)
Double conservation has been reduced to the simple mathematical statement that \( Q^3 \) specified in equation (51) satisfies equations (49) and (50). \( Q \), the vector field associated with \( Q^3 \), is a double conservation vector field.

\( Q^3 \), as specified in equation (51), is sufficiently general to include the conserved current of MU definition (2.34).
Now set $\mathcal{R}^3 \rightarrow Q^3 = \mathcal{D}S^1 \wedge d\psi$. Definitions and equations (44)–(47) become:

$$m = \ast (\mathcal{V} \wedge dS^1 \wedge d\psi)$$

$$Q^3 = dS^1 \wedge d\psi = m\mathcal{V}^3 - \mathcal{V}^1 \wedge iv(dS^1 \wedge d\psi)$$

The right hand equation in (53) represents resolved double conservation.

The specific double conservation vector field of section 3.3 may be recovered by setting $\mathcal{S}^3 \rightarrow \mathcal{V}^3$, identifying $\mathcal{V}^3$ with the (4-)velocity 1-form field of the fluid, identifying $\psi$ with specific entropy and reintroducing momentum equation (29):

$$\Gamma^i \mathcal{V}^3 = m\mathcal{V}^3 = Q^3 = d\mathcal{V}^3 \wedge d\psi = m\mathcal{V}^3 - \mathcal{V}^1 \wedge [\mathcal{G}^{\mathcal{V}^1} \wedge d\psi + (\xi_{\mathcal{V}} \psi) d\mathcal{V}^1]$$

Each term on the RHS of equation (54) can be specified or calculated locally from fields with an explicit physical definition and the second requirement for the HM conservation and impermeability theorems is satisfied.

4.3. Property 2: Alignment

Vector field $\mathcal{P}$ is aligned with vector field $\mathcal{V}$ when:

$$\mathcal{O}^2 = \mathcal{V}^1 \wedge \mathcal{P}^1$$

Equivalently, utilising (26):

$$\mathcal{O}^2 = i_{\mathcal{V}} \mathcal{P}^3$$

Now set $\mathcal{R}^3 \rightarrow \mathcal{P}^3$ where $\mathcal{P}^3$ satisfies equation (56). Definitions and equations (44)–(47) become:

$$m = \ast (\mathcal{V}^1 \wedge \mathcal{P}^3)$$

$$\mathcal{P}^3 = m\mathcal{V}^3$$

5. Potential vorticity evolution in spacetime

5.1. Thermodynamic potential vorticity evolution

In this section the double conservation property will be imposed utilising equations (52) and (53).

Consider the laws of physics. Utilise the spacetime stress-energy tensor field of an electromagnetic perfect fluid, require conservation of particle number and impose the Maxwell equations. The energy and momentum equations may be obtained by requiring non-divergence of the spacetime stress-energy tensor field and applying particle conservation and the Maxwell equations. Together, these physical conditions will be taken to represent a general electromagnetic fluid.

Let $\mathcal{F}^2$ denote the Faraday 2-form field, let $n \neq 0$ denote particle number density and let $\mathcal{J}^3$ denote the (4-) current per particle 3-form field. The Maxwell equations are:

$$\mathcal{O}^3 = d\mathcal{F}^2$$

$$n \mathcal{J}^3 = d \ast \mathcal{F}^2$$

Let $\mathcal{J}$ denote the (4-)current per particle vector field and let $\mathcal{L}^0$ denote the Lorentz force per particle:

$$\mathcal{L}^0 = -i_{\mathcal{V}} \mathcal{F}^2$$

Let $\mathcal{V}$ denote the fluid velocity vector field. The particle number conservation condition is:

$$\mathcal{O}^3 = \xi_{\mathcal{V}} (n \mathcal{V}^3)$$

Let $\psi$ denote specific\textsuperscript{5} entropy, let $h$ denote relativistic specific enthalpy, and let $T$ denote temperature. Utilising equations (59)–(62) and identity (A.2) the combined energy-momentum equation for a general electromagnetic fluid may be written:

$$\mathcal{O}^1 = iv(dh^1) - \mathcal{L}^0 - T d\psi$$

Equation (63) is by far the most compact expression for the energy-momentum equation in a general electromagnetic fluid.

In ideal MHD it is assumed that the fluid has infinite electrical conductivity and that the electric field is zero in the momentarily comoving rest frame of the fluid (MCRF). In ideal MHD (c.f. MU):

\textsuperscript{5} Per particle.
\* \mathcal{F}^2 = B^i \wedge V^i \tag{64}

$B^i$ denotes the magnetic 1-form field in the MCRF. In ideal MHD, Maxwell equations (59) and (60) become:

\[ \mathcal{O}^i = \epsilon \ast (B^i \wedge V^i) \tag{65} \]

\[ \eta \ast J^i = d(B^i \wedge V^i) \tag{66} \]

In ideal MHD, the Lorentz force per particle 3-form field, obtained by taking $\ast \, (61)$ and applying (26) and (66) is:

\[ n \, L^i = B^i \wedge V^i \wedge \ast (d(B^i \wedge V^i)) \tag{67} \]

Comparison of equations (53) and (63) suggests identifying $\mathcal{S}^i \Rightarrow h \, V^i$. Equation (52) becomes:

\[ m_T \ast (\psi \wedge d(\lambda \psi) \wedge d(V^i)) = h \ast (\, \wedge V^i \wedge d(V^i) \, d(\psi \, V^i)) \tag{68} \]

Equation (53) with energy-momentum equation (63) becomes:

\[ d(h V^i \wedge d(\psi) + V^i \wedge \xi \wedge d(\xi) + h(\psi V) V^i \wedge d(V^i)) = m_T V^i \tag{69} \]

For the remainder of this paper the 3-form field $V^i \wedge dV^i$ will be referred to as the kinematic 3-form field. Define:

\[ \lambda := - \ast (V^i \wedge \xi \wedge d(\psi)) \tag{70} \]

\[ \eta := - \ast (h(\psi V) V^i \wedge d(V^i)) \tag{71} \]

Each term in the definitions of $\lambda$ and $\eta$ may be specified or calculated locally from known physical fields. For the remainder of this paper scalar field $\lambda$ will be referred to as the Lorentz term and scalar field $\eta$ will be referred to as the diabatic term.

Equation (69) and definitions (70) and (71), together with identity (26) and Cartan’s formula (A.5) imply:

\[ (\lambda + \eta) \, V^i = \mathcal{E}_V (m_T \, V^i) \tag{72} \]

Equation (72) represents evolution of the volume integral of $m_T \, V^i$. The Lorentz and diabatic terms indicate that this integral quantity is not conserved along the trajectories of $V$ in general.

Turn attention now to the definition and evolution of the related scalar field. Following Friedman (1978) and Katz (1984) define the thermodynamic PV (TPV) scalar field, denoted $q_T$:

\[ q_T := \frac{m_T}{n} \tag{73} \]

$m_T$ is defined in (68) and $n$ denotes particle number density.

Identities (26), (A.3) and (A.7) and Cartan’s formula (A.5) combined with particle number conservation condition (62) and equation (72) imply:

\[ \frac{\lambda + \eta}{n} = \mathcal{E}_V q_T \tag{74} \]

For the remainder of this paper equation (74) will be referred to as the TPV evolution equation. Electromagnetism, via the Lorentz and diabatic terms, acts as a source in TPV evolution equation (74).

For particle conserving perfect fluids all electromagnetic terms are zero and (74) simplifies to the PV conservation result first obtained by Friedman (1978) and Katz (1984).

As an illustrative example consider TPV when all physical variables, including the metric tensor field, are time independent and have an additional spatial symmetry. Utilise the definitions and terminology of Appendix B. Introduce coordinates $\{t, r, \theta, \phi\}$ which are adapted to the symmetry so that $\psi = \psi(r, \theta)$ and similarly for the metric functions and the boost and angle parameters.

Make the restriction $\mathcal{L}^i = H(r, \theta) \, dr + I(r, \theta) \, d\theta$. The Lorentz term in equation (74) is zero. Restrict the fluid flow to be isentropic so that the diabatic term in equation (74) is zero. Since both the Lorentz and diabatic terms are zero, TPV is conserved along the trajectories of the fluid. Equations (B.1) and (B.3) and conservation of specific entropy along the fluid trajectories imply:

\[ 0 = \sinh(\gamma) \left[ f(\partial_r \psi) \sin(\epsilon) + c(\partial_{\theta \psi}) \sin(\delta) \cos(\epsilon) \right] \tag{75} \]

c and $f$ denote metric functions, $\gamma$ denotes the boost parameter, $\delta$ and $\epsilon$ denote angle parameters. In the case with non-zero boost, $\gamma \neq 0$, and a non-zero ’radial’ gradient of specific entropy, $\partial_r \psi \neq 0$, we may solve equation (75) for $\epsilon$ as a function of the other variables.

Apply $q_T$ to combined energy-momentum equation (63). With the restricted form of $\mathcal{L}^i$ specified above and equation (75) it may be demonstrated that: $\mathcal{L}_V = \kappa(t, \theta) \, d\psi$.

The demonstration of TPV conservation argued above provides no useful information if TPV is uniformly zero throughout the region under consideration. To see that TPV is, in general, non-zero combine definition (68) with definitions (B.6) and (B.7) and equation (B.8):
\[
q_T = \frac{m_T}{n} = \frac{h}{n} [ (\partial_t \psi)(\partial_t G) - (\partial_x \psi)(\partial_x G)] \cosh^2(\gamma) \star ( - dt \wedge dr \wedge d\theta \wedge d\phi )
\]  

(76)

G is defined in (B.7). In general, expression (76) for TPV is non-zero.\(^6\)

At this point it is worth emphasising that there is nothing to prevent application of these definitions and results to non-relativistic inviscid MHD. Simply require boost parameter \(\gamma\) to satisfy \(|\gamma| \ll 1\) and impose Minkowski spacetime. One potential application is to non-relativistic, magnetically confined plasmas for which toroidal coordinates in Minkowski spacetime would seem to be an appropriate choice.

Electromagnetism plays a role in determining the evolution of TPV, but there is no electromagnetic term within the definition of TPV itself. We are now in a position to include an electromagnetic term within a redefined PV.

### 5.2. Electromagnetic potential vorticity evolution

In this section the alignment property will be imposed utilising equations (57) and (58).

Again let \(\mathcal{L}\) denote the Lorentz force per particle, let \(\psi\) denote specific entropy, let \(h\) denote relativistic specific enthalpy and let \(T\) denote temperature. Define:

\[
\beta T := 1
\]

\[
\mathcal{M}^4 := \beta \mathcal{L}^4 + d\psi
\]

\[
\mathcal{N}^2 := d(h\psi^2) + \mathcal{V}^3 \wedge T \mathcal{M}^4
\]

(77)

(78)

(79)

In a general electromagnetic fluid the energy and momentum equations may be written:

\[
0 = i_{\mathcal{V}} \mathcal{M}^4
\]

\[
\mathcal{O}_1 = i_{\mathcal{V}} \mathcal{N}^2
\]

(80)

(81)

Equations (80) and (81) and identity (A.4) imply that setting \(\mathcal{D} \rightarrow \mathcal{N}^2 \wedge \mathcal{M}^4\) is sufficient to achieve alignment and equations (57) and (58) may be assumed to apply. Equation (57) becomes:

\[
m_E := * ( \mathcal{V}^3 \wedge \mathcal{N}^2 \wedge \mathcal{M}^4 ) = h \star ( \mathcal{V}^3 \wedge d \mathcal{V}^3 \wedge \mathcal{M}^4 )
\]

(82)

Equations (58), (80) and (81) imply:

\[
d(h\psi^2) + d(h\psi^2) \wedge \beta \mathcal{L}^4 = m_E \mathcal{V}^3
\]

(83)

Define:

\[
\alpha := \star [d(h\psi^2) \wedge d(\beta \mathcal{L}^4)]
\]

(84)

Each term in the definition of \(\alpha\) may be specified or calculated locally from known physical fields.

Equation (83) and definition (84), together with identity (26) and Cartan’s formula (A.5) imply:

\[
\alpha \mathcal{V}^3 = \mathcal{V} \cdot (m_E \mathcal{V}^3)
\]

(85)

Equation (85) represents evolution of the volume integral of \(m_E \mathcal{V}^3\). The \(\alpha\) term indicates that this integral quantity is not conserved along the trajectories of \(\mathcal{V}\) in general.

Define the electromagnetic PV (EPV) scalar field, denoted \(q_E\):

\[
q_E := \frac{m_E}{n}
\]

(86)

\(m_E\) is defined in (82) and \(n\) denotes particle number density.

Identities (26), (A.3) and (A.7) and Cartan’s formula (A.5) combined with particle number conservation condition (62) and equation (85) imply:

\[
\frac{\alpha}{n} = \mathcal{V} \cdot q_E
\]

(87)

For the remainder of this paper equation (87) will be referred to as the EPV evolution equation. Electromagnetism, via the \(\alpha\) term, acts as a source in EPV evolution equation (87).

For particle conserving perfect fluids all electromagnetic terms are zero and \(m_E = m_T\). Particle conserving perfect fluids satisfy both the double conservation and alignment conditions.

Interestingly, the Sakuma et al (2022) definition of PV in particle conserving perfect fluids utilises scalar field:

\[
m_S := \frac{1}{2T} \star [d(h\psi^2) \wedge d(h\psi^3)]
\]

(88)

In a general particle conserving perfect fluid: \(m_S \equiv m_E = m_T\). After application of the energy-momentum equation, however:

\(^6\) In the general case TPV is non-zero in ‘most’ of the fluid, however submanifolds of lower dimension may exist within which TPV is zero. In special cases such as cylindrical symmetry TPV may be zero throughout the entire fluid.
Conservation of the volume integral of $m_3 V^3$ in a particle conserving perfect fluid follows:

$$\mathcal{O}^3 = \mathcal{E}_V(m_3 V^3)$$  \hspace{1cm} (90)

### 6. Summary and concluding remarks

In this paper it has been implicitly assumed that the spacetime region under consideration is non-singular and has no physical boundaries.

The general Einsteinian position has been taken: the momentum equation contains no gravitational force term. Instead, the effects of gravitation occur within the acceleration term in the momentum equation. The results presented in this paper apply to any theory of gravitation compatible with this general Einsteinian position, including to general relativity.

The perfect fluid subcase of the results presented in this paper reproduces the PV conservation result of Friedman (1978) and Katz (1984). The demonstration of this result is much simpler utilising the framework of differential forms in spacetime and does not introduce extraneous mathematical structures.

The new physical results presented in this paper are the addition of the Lorentz and diabatic source terms to the TPV evolution equation and the definition of scalar field EPV together with its evolution equation. The new mathematical results presented in this paper consist of the simple formulations of the resolution identity and the double conservation and alignment properties. As far as the author is aware, very useful spacetime identity (26) is also new.

The central argument of this paper is that there is a surprisingly simple process to define ‘PV-type’ integral quantities and scalar fields in spacetime and obtain their evolution equations. Construct scalar field $\psi$ and the resolution identity for $\mathcal{R}^3$. Impose a mathematical property on $\mathcal{R}^3$. In this paper the double conservation and alignment mathematical properties were imposed independently. Motivated by the energy and momentum equations assign physical meaning to the mathematical fields. Apply the energy and momentum equations and derive an expression for $\mathcal{E}_V(m^3)$. Define the associated PV scalar field and derive its evolution equation.

The simple process just described reverses the order of earlier relativistic PV derivations by Friedman (1978) and Katz (1984) which start with the equations of motion. By imposing mathematical proto-structures first, into which physical content is subsequently injected, this simple process facilitates interpretation of the HM conservation and impermeability theorems by clearly separating the mathematical and physical content. In this paper we have confirmed that the conservation and impermeability conditions are essentially mathematical in nature. This statement does not imply that the HM conservation and impermeability theorems are devoid of physical content. Identify $\psi$, $V^3$ and $\mathcal{O}^3$ with physical fields. Each term in $\mathcal{O}^3$, given by equation (54), may then be specified or calculated locally from these physical fields.

Utilising differential forms in spacetime produces derivations which are economical and results which are extremely general. The TPV and EPV evolution equations apply in any non-singular spacetime region with no physical boundaries. The geometry may be flat or curved, stationary or non-stationary.

**Data availability statement**

No new data were created or analysed in this study.

**Appendix A. General identities and conservation laws**

Let $\mathcal{O}^p$ denote the zero p-form field. The following identities may be found in Frankel (2012):

$$d \mathcal{E}^p = \mathcal{O}^{p+2}$$  \hspace{1cm} (A.1)

$$i_2 i_V(\mathcal{E}^p) = i_V i_2(\mathcal{E}^p)$$  \hspace{1cm} (A.2)

$$d(\mathcal{E}^p \wedge \mathcal{H}_t) = (d \mathcal{E}^p) \wedge \mathcal{H}_t + (-1)^p \mathcal{E}^p \wedge (d \mathcal{H}_t)$$  \hspace{1cm} (A.3)

$$i_2(\mathcal{E}^p \wedge \mathcal{H}_t) = (i_2 \mathcal{E}^p) \wedge \mathcal{H}_t + (-1)^p \mathcal{E}^p \wedge (i_2 \mathcal{H}_t)$$  \hspace{1cm} (A.4)

Cartan’s formula is:

$$\mathcal{E}_Z \mathcal{E}^p = i_2 d \mathcal{E}^p + d i_2 \mathcal{E}^p$$  \hspace{1cm} (A.5)
Two useful identities are (Flanders 1989):
\[ E^p \land \ast H^p = H^p \land \ast E^p \] (A.7)
\[ \ast \ast E^p = (-1)^{p(n-p)+(n-1)/2} E^p \] (A.8)

n is the dimension of the manifold and t the signature of the metric.

A differential p-form, \( E^p \), satisfies a differential conservation law, when there exists a vector field, \( Z \), such that:
\[ \varepsilon_E E^p = O^p \] (A.9)

Frankel (2012) equation (4.33) implies that (A.9) is sufficient to guarantee integral conservation law:
\[ \frac{d}{d\lambda} \int_{\Sigma} E^p = 0 \] (A.10)

\( d/d\lambda \) denotes the derivative along the trajectories of \( Z \) and \( V^p \) denotes an oriented compact p-dimensional submanifold transported along the trajectories of \( Z \).

Appendix B. A special case of the kinematic 3-form field

Let a (pseudo-)orthonormal vector basis \( \{ \hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3 \} \) and dual 1-form basis \( \{ e^0, e^1, e^2, e^3 \} \) be specified. Let the fluid velocity be obtained from a boost in some combination of the \( \hat{e}_1 \), \( \hat{e}_2 \) and \( \hat{e}_3 \) directions:
\[ V = \cosh(\gamma) \hat{e}_0 + \sinh(\gamma) [\sin(\epsilon) \hat{e}_1 + \sin(\delta) \cos(\epsilon) \hat{e}_2 + \cos(\delta) \cos(\epsilon) \hat{e}_3] \] (B.1)
\[ V^i = -\cosh(\gamma) e^0 + \sinh(\gamma) [\sin(\epsilon) e^1 + \sin(\delta) \cos(\epsilon) e^2 + \cos(\delta) \cos(\epsilon) e^3] \] (B.2)

\( \gamma \) denotes the boost parameter. Angle parameters \( \delta \) and \( \epsilon \) apportion the components of \( V \) into each of the \( \hat{e}_1 \), \( \hat{e}_2 \) and \( \hat{e}_3 \) directions. Introduce coordinates \( \{ t, r, \theta, \phi \} \) and restrict consideration to spacetimes with 1-form orthonormal and coordinate bases related by:
\[ e^0 = a (dt + b d\phi); \quad e^1 = a_c dr; \quad e^2 = a_f d\theta; \quad e^3 = a_g d\phi \] (B.3)

\( a, b, c, f \) and \( g \) will be referred to non-rigorously as the metric functions. Each metric function may depend on all four coordinates.

In the coordinate basis:
\[ V^i = a \cosh(\gamma) [ -dt + C \, dr + F \, d\theta + G \, d\phi ] \] (B.4)
\[ C := c \tanh(\gamma) \sin(\epsilon) \] (B.5)
\[ F := f \tanh(\gamma) \sin(\delta) \cos(\epsilon) \] (B.6)
\[ G := g \tanh(\gamma) \cos(\delta) \cos(\epsilon) - b \] (B.7)

The kinematic 3-form field is given by:
\[ V^i \wedge d V^i = a^2 \cosh^2(\gamma) [dt \wedge dr \wedge dC + dt \wedge d\theta \wedge dF + dt \wedge d\phi \wedge dG + d\theta \wedge d\phi \wedge (G \, dF - F \, dG) + d\phi \wedge dr \wedge (C \, dG - G \, dC) + dr \wedge d\theta \wedge (F \, dC - C \, dF) ] \] (B.8)
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