G-graded irreducibility and the index of reducibility

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ABSTRACT
Let \( R \) be a commutative Noetherian ring graded by a torsion-free abelian group \( G \). We introduce the notion of \( G \)-graded irreducibility and prove that \( G \)-graded irreducibility is equivalent to irreducibility in the usual sense. This is a generalization of Chen and Kim’s result in the \( \mathbb{Z} \)-graded case. We also discuss the concept of the index of reducibility and give an inequality for the indices of reducibility between any radical non-graded ideal and its largest graded subideal.

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1. Introduction

Let \( R \) be a commutative Noetherian ring, \( M \) a finitely generated \( R \)-module, and \( N \) an \( R \)-submodule of \( M \). It is known that \( N \) has an irreducible decomposition, that is, \( N \) is an intersection of irreducible submodules in \( M \). When \( R, M, N \) are also graded with respect to a torsion-free abelian group \( G \), we can talk about \( G \)-graded irreducible submodules of \( M \) and irreducible decomposition of \( N \) in \( M \) in the category of \( G \)-graded modules. It is natural to ask whether these two irreducibilities are the same. More precisely, we want to know whether graded irreducibility implies irreducibility in the nongraded sense. It is well known that irreducibility implies being primary; in [1, IV.3.3.5] we know being graded primary is the same as being primary. Chen and Kim proved in [3] that the two irreducibilities are the same in the \( \mathbb{Z} \)-graded case. In this paper, we extend this result to the case of any \( G \)-grading where \( G \) is a torsion-free abelian group. Here \( G \) is not necessarily finitely generated as an abelian group. In particular, as a consequence, a \( G \)-graded irreducible decomposition is an irreducible decomposition in the usual sense, and both indices of reducibility, defined for \( G \)-grading and in the usual sense, will be the same. Finally we estimate the indices of reducibility of a nongraded ideal and its largest graded subideal. We prove one inequality in the radical case and show by example that it fails in the general case.

We always assume \( R \) is a commutative ring and \( G \) is an abelian group in the following sections. We also make the following assumptions if necessary:

Assumption 1.1. \( R \) is a commutative Noetherian ring, \( M \) is a finitely generated \( R \)-module. When we say \( R \) and \( M \) are \( G \)-graded, we assume that \( G \) is torsion-free abelian, but not necessarily finitely generated. The identity element of \( G \) is denoted by \( 0 \).

The reason for these assumptions are as follows.
When $R$ is Noetherian and $N \subseteq M$ are Noetherian modules, we have a finite irredundant irreducible decomposition for $N$. So the index of reducibility defined below will make sense.

The torsion-free property is essential. In fact, in the $G$-graded case where $G$ has torsion, the definition of prime ideals, primary ideals, associated primes will be different. An example is the group algebra $k[\mathbb{Z}_2] = k[x]/(x^2 - 1)$. The ideal 0 is not a prime ideal; however, it is a graded prime ideal in the sense that if two homogeneous elements multiply to get 0 then one of them is 0. Also the associated primes $(x + 1), (x - 1)$ are all nongraded, so here we need a different definition for graded associated primes. Such definitions can be found in [6]. In the torsion-free case, a graded prime ideal is just a prime ideal that is graded; and the same holds for graded primary submodules and graded associated prime ideals.

2. Preliminaries

We recall the following standard definitions.

**Definition 2.1.** Let $(G, +)$ be an abelian group. A ring $R$ is said to be $G$-graded if there is a family of additive subgroups $R_g$ such that $R = \bigoplus_{g \in G} R_g$ and $R_gR_h \subseteq R_{g+h}$ for any $g, h \in G$. For a $G$-graded ring $R$, an $R$-module $M$ is $G$-graded if there is a family of additive subgroups $M_g$ such that $M = \bigoplus_{g \in G} M_g$ and $R_gM_h \subseteq M_{g+h}$ for any $g, h \in G$.

**Definition 2.2.** Let $R$ be a Noetherian ring, $M$ an $R$-module, $N$ an $R$-submodule of $M$, $G$ an abelian group. We assume (1.1). Then

1. The submodule $N$ is called irreducible if whenever $N_1, N_2$ are two submodules of $M$ satisfying $N_1 \cap N_2 = N$, we have $N_1 = N$ or $N_2 = N$.
2. Suppose moreover that $R, M, N$ are $G$-graded. Then $N \subseteq M$ is called $G$-graded-irreducible, or simply graded-irreducible when $G$ is clear, if whenever $N_1, N_2$ are two $G$-graded submodules of $M$ satisfying $N_1 \cap N_2 = N$ we have $N_1 = N$ or $N_2 = N$.
3. The submodule $N$ is called primary if $M/N$ has only one associated prime. If this prime is $\mathfrak{p}$ we say $N$ is $\mathfrak{p}$-primary. The set of associated primes of a module $M$ is denoted by $\text{Ass}(M)$.

The above definitions hold for $N \subseteq M$ if and only if they hold for $0 \subseteq M/N$.

The following proposition is well known.

**Proposition 2.3.** An abelian group is torsion free if and only if it can be given a total order compatible with the group operation.

**Proof.** The if part is trivial. Now for the converse, if $G$ is torsion free, we can embed $G$ into some $\mathbb{Q}$-vector space, order the basis element, and give the lexicographic order on the vector space and restrict this order to $G$.

So now we can equip each torsion-free abelian group with a total order. We have the following proposition.

**Proposition 2.4.** Let $R$ be a graded ring and $G$ an abelian group. Then

1. Assume $G$ is torsion free. Let $\mathfrak{p}$ be a graded proper ideal in $R$ such that if $f, g$ are homogeneous elements in $R$, $fg \in \mathfrak{p}$, then $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Then $\mathfrak{p}$ is a prime ideal.
2. Let $M$ be a graded $R$-module, $N$ be a graded submodule of $M$. Assume (1.1) for $R, M, N, G$. Then every associated prime of $M/N$ is graded and is the annihilator of a homogeneous
element. In particular, \( N \subseteq M \) is primary if and only if \( M/N \) has only one associated \( G \)-graded prime.

(3) If \( N,M \) are as in (2), then there is a graded primary decomposition.

(4) If \( N,M \) are as in (2) and \( N \subseteq M \) is graded irreducible, then \( M/N \) is graded primary, hence primary with a unique graded associated prime.

Proof. For (1), See [7, A.II.1.4]. For (2), See [7, A.II.7.3] or [5, Prop 3.12]. For (3), See [7, A.II.7.11], or [5, Ex 3.5]. (4) is a corollary of (3).

The following definition comes from [7, A.I.4], [2, definition 1.5.13], and [3] in the \( \mathbb{Z} \)-graded case. In [2, 3], “\( G \)-graded local” is called “\( \ast \)local”.

Definition 2.5. Let \( G \) be a group. A \( G \)-graded maximal ideal of \( R \) is a \( G \)-graded ideal \( m \) which is maximal with respect to inclusion in all \( G \)-graded ideals properly contained in \( R \). A \( G \)-graded ring \( R \) is called \( G \)-graded local if it has a unique \( G \)-graded maximal ideal. A \( G \)-graded field is a \( G \)-graded ring \( k \) such that all the nonzero homogeneous elements in \( k \) are invertible.

Remark 2.6. A \( G \)-graded ideal \( m \) is a \( G \)-graded maximal ideal if and only if \( R = m \) is a \( G \)-graded field. In particular, if \( k \neq 0 \) is a \( G \)-graded ring, then it is a graded field if and only if it has only two \( G \)-graded ideals, namely 0 and \( k \), if and only if 0 is a \( G \)-graded maximal ideal of \( k \).

Definition 2.7. Let \( R \) be a \( G \)-graded ring for some group \( G \). For an ideal \( I \) which is not necessarily \( G \)-graded, as in [2, 3], we define \( I^g \) to be the ideal of \( R \) generated by all the homogeneous elements in \( I \).

Remark 2.8. Assume \( G \) is torsion free. Proposition 2.4(1) yields that \( \mathfrak{p}^* \) is a graded prime ideal contained in \( \mathfrak{p} \) when \( \mathfrak{p} \) is a prime ideal of \( R \). In particular, every \( G \)-graded maximal ideal \( m \) in \( R \) is a prime ideal of \( R \), because \( m \) is contained in some (not necessarily graded) maximal ideal \( \mathfrak{n} \), and it follows that \( m = n^* \) by definition. So a graded field \( k \) must be a domain. Therefore, it makes sense to talk about the rank of a \( k \)-module if \( k \) is a graded field.

Definition 2.9. Let \( G \) be a group, \( R \) be \( G \)-graded, \( M \) a \( G \)-graded \( R \)-module, and \( \mathfrak{p} \) a \( G \)-graded prime of \( R \). The homogeneous localization of \( M \) at \( \mathfrak{p} \), denoted by \( M_{(\mathfrak{p})} \), is \( W^{-1}M \) where \( W \) is the multiplicative set of all homogeneous elements not in \( \mathfrak{p} \).

If \( R \) is graded, \( \mathfrak{p} \) is a graded prime of \( R \), then \( R_{(\mathfrak{p})} \) is graded local.

Lemma 2.10. Let \( R \) be a Noetherian ring, \( \mathfrak{p} \) a prime ideal of \( R \), \( M \) an \( R \)-module, \( N \) an \( R \)-submodule of \( M \) which is \( \mathfrak{p} \)-primary in \( M \), and \( G \) be an abelian group. We assume (1.1). Then \( N \) is irreducible in \( M \) if and only if \( N_{\mathfrak{p}} \) is irreducible in \( M_{\mathfrak{p}} \). Moreover, if \( R, M, N, \mathfrak{p} \) are all \( G \)-graded, then \( N \) is graded-irreducible in \( M \) if and only if \( N_{(\mathfrak{p})} \) is graded-irreducible in \( M_{(\mathfrak{p})} \).

Proof. See [3, Lemma 2]. The proof is for the \( \mathbb{Z} \)-graded case but can be applied to the \( G \)-graded case.

3. The structure of modules over graded fields

It is well known that if \( k \) is a field, then every vector space over \( k \) is free. Here we prove a similar result when \( k \) is a graded field.

Definition 3.1. Let \( R \) be a \( G \)-graded ring. We define the support of \( R \), denoted by \( \text{Supp}(R) \), to be \( \{ g \in G : R_g \neq 0 \} \).
If $R$ is a domain, then $\text{Supp}(R)$ is a subsemigroup of $G$. If $k$ is a graded field, then $\text{Supp}(k)$ is a subgroup of $G$.

The following two theorems have more general versions using the notions of strongly graded rings and graded division rings, see [7, A.1.3 and A.1.4]. We present explicit proofs in our particular case.

**Theorem 3.2.** Let $G$ be a group, and $k$ a $G$-graded field. Then $k_0$ is a field, and $k_g \cong k_0$ as a $k_0$-vector space for any $g \in \text{Supp}(k)$.

**Proof.** Every nonzero element in $k_0$ has an inverse in $k_0$, so $k_0$ is a field. Take any nonzero $u \in k_g$. Then the multiplication by $u$ is a $k_0$ isomorphism of $k_0 \to k_g$ with an inverse which is the multiplication by $u^{-1}$.

**Theorem 3.3.** Let $G$ be a group, and $k$ a $G$-graded field. Then any $G$-graded $k$-module $M$ is free over $k$.

**Proof.** Let $G' = \text{Supp}(k)$. Let $S$ be a set of representatives in $G$ of all the cosets in $G/G'$. Then $M = \bigoplus_{g \in G} M_g = \bigoplus_{i \in S} (\bigoplus_{h \in G} M_{gh})$ as a $k$-module. So it suffices to prove $\bigoplus_{h \in G} M_{gh}$ is a free $k$-module for any $s$ and any graded $k$-module $M$. Now $M_s$ is a $k_0$-vector space. Let $\{e_i, i \in I\}$ be a basis for $M_s$ and choose a basis $u_h$ in $k_h$ for each degree $g \in G$. Then for every $h$, $u_h \ast e_i$ is a $k_0$-basis for $M_{gh}$. This means that $\bigoplus_{h \in G} M_{gh} = \bigoplus_{h \in G, i \in I} k_0 u_h \ast e_i = \bigoplus_{i \in I} k_0 e_i$, hence it is free as a $k$-module, so $M$ is a free $k$-module.

This theorem shows the simplicity of graded modules over a graded field, which is very important in the proof of Lemma 4.2 and Theorem 3.3. We want to restrict to the case where $G$ is finitely generated using the Noetherian condition. We pose the following question.

**Question 3.4.** Let $R$ be a Noetherian ring, $G$ be a torsion-free abelian group. Suppose $R$ is $G$-graded with $\text{Supp}(R) = G$. Do we have that $G$ is finitely generated?

We have the following two theorems.

**Theorem 3.5.** Let $G$ be a finite generated torsion-free abelian group, say $\mathbb{Z}^n$. Then every $G$-graded field $k$ is isomorphic to $k_0[G']$ as a graded field, where $G'$ is the support of $k$.

**Proof.** Since $G'$ is a subgroup of $G$, it is still a finitely generated torsion-free abelian group, say $\bigoplus_{i=1}^n \mathbb{Z} e_i$. For each $i$, take a nonzero element $a_i$ in $k_0$. Then for any $h = n_1 e_1 + n_2 e_2 + \cdots + n_m e_m \in G'$, $a_1^{n_1} a_2^{n_2} \cdots a_m^{n_m}$ is a nonzero element in $k_h$, thus $k_h = k_0 \ast a_1^{n_1} a_2^{n_2} \cdots a_m^{n_m}$. This means that $k = k_0[a_1, a_2, ..., a_m] \cong k_0[G']$.

**Remark 3.6.** The conclusion of the theorem above is not true in general for torsion-free abelian groups which are not finitely generated. In fact, we have to find a basis for all the nonzero components of the graded field or graded module; and there is no guarantee that one can find a collection of bases, labeled by the group, that is closed under multiplication if the group is not finitely generated. There are different isomorphic classes of such graded fields corresponding to the cohomology classes in $H^2(G, k_0)$; see [8, Ex 1.5.10].

**Theorem 3.7.** Let $k = k_0[G]$ be a group algebra over a field $k_0$. $G$ any abelian group, not necessarily torsion free. Then $k$ is Noetherian if and only if $G$ is finitely generated.

**Proof.** The if direction is obvious since in the finitely generated case the group algebra is the localization of a quotient of a polynomial ring over a field. Now suppose $G$ is not finitely
generated. Consider any finitely generated ideal $I$. The generators live in finitely many degrees. Let $H$ be the subgroup generated by these degrees, then $I$ must be $G/H$-graded for some finitely generated $H$. Now consider the map $\pi : k \rightarrow k_0, \Sigma a_0 g_i \rightarrow \Sigma a_i$. The kernel $J$ is an ideal in $k[g]$ generated by $\langle e_g - 1 \rangle_{g \in G}$. If $H$ is a subgroup such that $J$ is $G/H$-graded then we must have $G = H$. Thus $J$ cannot be finitely generated.

**Remark 3.8.** The last two theorems tell us that under the Noetherian hypothesis on the ring and the torsion-free hypothesis on the group, a graded field is graded over a finitely generated group if and only if it is a group algebra over a field. But we cannot conclude anything for a general graded field. So in order to generalize the graded irreducibility, it is necessary to consider a torsion-free group which is not finitely generated. In this case, we cannot use induction on the support of the ring, or use the graded injective envelope. So we use the graded scale in the next section for the main result.

### 4. G-graded irreducibility implies irreducibility

In this section, we prove our main result, that is, a graded irreducible submodule of a graded module is irreducible.

**Definition 4.1.** Let $(R, m, k)$ be a local ring or a graded-local ring. Let $M$ be an $R$-module. The socle of $M$, denoted by $soc(M)$, is $(0: m^\infty)$. When $M$ is graded, $soc(M)$ is also graded. In both cases, $soc(M)$ is a free $k$-module by Theorem 3.3.

**Lemma 4.2.** Let $R$ be a Noetherian ring, $M$ a finitely generated $R$-module, and $N \subseteq M$ a submodule. Suppose: (1) $(R, m, k)$ is local and $M/N$ is Artinian or (2) $R$, $M$ and $N$ are all $G$-graded for a torsion-free abelian group $G$, $(R, m, k)$ is graded-local, $M/N$ is $m$-primary. Then $N \subseteq M$ is irreducible (resp. graded-irreducible) if and only if $soc(M/N)$ has rank 1.

**Proof.** We may assume $N = 0$ after replacing $M$ with $M/N$. We know $soc(M)$ is a free $k$-module. Now suppose the rank of $soc(M)$ is at least two. Then there exist a $k$-basis of $soc(M)$. Take the first two basis element: $e_1, e_2 \in soc(M) \subset M$ then $Re_1 \cap Re_2 = 0$. So 0 is not irreducible. Now suppose the rank of $soc(M)$ is one, say, $soc(M)$ is $Re \cong k$ as an $R$-module. Then for any $N_1, N_2 \subseteq M, Ass(N_1) = Ass(N_2) = Ass(M) = \{m\}$ because $Ass(M)$ consists of one prime and $Ass(N_1), Ass(N_2)$ are nonempty subsets of $Ass(M)$. Now $soc(N_1) \neq 0, soc(N_2) \neq 0$ so we can take nonzero $e_1 \in soc(N_1), e_2 \in soc(N_2)$. They must all lie in $soc(M) = Re$. Now $k$ is a domain, hence 0 is an irreducible $k$-submodule of $k$, hence 0 is an irreducible $R$-submodule of the $R$-module $k$. So $Re_1 \cap Re_2 \neq 0$ in $Re \cong k$. So 0 is irreducible. In the graded case, just take all the modules to be graded and elements to be homogeneous.

**Theorem 4.3.** Let $G$ be an abelian group which is torsion free but not necessarily finitely generated. Let $R$ be $G$-graded Noetherian ring, $M$ be a finitely generated graded $R$-module, $N$ be a graded primary submodule of $M$, Then $N$ is graded irreducible if and only if $N$ is irreducible.

**Proof.** We know that $M/N$ has a unique associated prime, denoted by $p$, and it is graded under assumption (2). Also, we may assume $N = 0$ after replacing $M$ with $M/N$. The “if” direction is trivial. Now let 0 be graded irreducible in $M$. Then 0 is an $R_{(p)}$-submodule in $M_{(p)}$ which is graded irreducible by Lemma 2.10. Then by Lemma 4.2 $soc(M_{(p)}) \cong R_{(p)}/pR_{(p)}$. Since $(0: m^\infty)_{(p)} = (0: m^\infty)_{p}$ because $m$ is finitely generated. So $soc(M_p) \cong R_p/pR_p$. Still by Lemma 4.2, 0 is irreducible in $M_p$. So 0 is irreducible in $M$ by Lemma 2.10.
We have proved that graded-irreducibility is the same as being graded and irreducible. Now we give the following definitions from [3] and [4], generalized to the G-graded case. In [3], they are called the index of irreducibility and denoted by \( r_M(N) \) (resp. \( \overline{r}_M(N) \)).

**Definition 4.4.** Let \( N \subset M \) be \( R \)-modules. We assume (1.1). Then

1. The index of reducibility of \( N \) in \( M \) is \( ir_M(N) = \min \{ r : N = \bigcap_{i=1}^r N_i N_i \text{ irreducible } R\text{-submodules of } M \} \).
2. When \( M \) and \( N \) are graded, the graded index of reducibility of \( N \) in \( M \) is \( ir_M^g(N) = \min \{ r : N = \bigcap_{i=1}^r N_i N_i \text{ graded-irreducible } R\text{-submodules of } M \} \).

When \( M \) is clearly understood we simply denote them by \( ir(N) \) (resp. \( ir^g(N) \)). Here is the G-graded version of [3, Theorem 7]. The proof is identical.

**Theorem 4.5.** Let \( R \) be a G-graded ring, \( M \) a G-graded module. We assume (1.1). Then we have the following:

1. Every graded-irreducible submodule of \( M \) is irreducible.
2. For every graded submodule \( N \) of \( M \), \( ir_M(N) = ir_M^g(N) \).
3. Every graded submodule \( N \) of \( M \) is a finite intersection of irreducible graded submodules of \( M \).

**Theorem 4.6.** We assume (1.1). Let \((R, m, k)\) be a G-graded local ring, \( N \subset M \) are graded R-modules such that \( M/N \) is \( m \)-primary. Then \( ir_M(N) = \text{rank}_k \text{soc}(M/N) \).

**Proof.** Localizing at \( m \). We have \( ir_M(N) = \text{rank}_k \text{soc}(M/N)_m \). Notice that the rank of \( \text{soc}(M/N) \) will not change after localizing. \( \square \)

## 5. The relation between the index of reducibility of \( I \) and \( I^* \)

Let \( R \) be a graded ring and \( I \) be an ideal of \( R \) which is not necessarily graded. We want to compare \( ir_R(I) \) and \( ir_R(I^*) \). Let’s consider a special case: \( G = \mathbb{Z} \) and \( R \) be the coordinate ring of a cone \( C \) in an affine variety \( \mathbb{A}^n \), then \( R \) is G-graded. In this case, the operation \( I \to I^* \) has a geometric interpretation. Suppose \( I \) is a radical ideal corresponding to a closed subset \( X \) in \( \mathbb{C} \), and \( X \) is not supported at the origin. That is, \( X = V(I) \) is the vanishing set of \( I \). There is a natural projection \( \pi : \mathbb{A}^n - \{0\} \to \mathbb{P}^{n-1} \). \( \pi \) restricts to two maps: \( \mathbb{C} - \{0\} \to \mathbb{P}^{n-1} \) and \( X - \{0\} \to \mathbb{P}^{n-1} \). Then \( I^* \) is a radical ideal, and its vanishing set is \( \pi(X - \{0\}) \in \mathbb{P}^{n-1} \), because the maximal homogeneous ideal in \( I \) corresponds to the minimal closed subset containing \( \pi(X - \{0\}) \).

For a morphism \( f : Y \to Y' \) between varieties, if \( Z \subset Y \) is an irreducible closed subset, then \( f(Z) \) is also irreducible. So if all the irreducible components are reflected in the variety as a set, \( ir_R(I) \) should be greater or equal to \( ir_R(I^*) \), because every irreducible component of \( I \) or \( V(I) \) map to an irreducible subset contained in \( V(I^*) \). The equality holds if and only if different irreducible components do not collapse to contain each other. Let \( \text{Min}(I) \) denote the set of minimal primes over an ideal \( I \), and \( |S| \) denote the cardinality of a set \( S \). We have the following theorem:

**Theorem 5.1.** Let \( R \) be a G-graded Noetherian ring where \( G \) is torsion-free abelian. Let \( I \) be an \( R \)-ideal which is radical but not G-graded. Then \( ir(I) \geq ir(I^*) \). The equality holds if and only if \( p \to p^* \) gives a bijection between \( \text{Min}(I) \) and \( \text{Min}(I^*) \).

**Proof.** Since \( I \) is radical, \( I^* \) is also radical. The irredundant irreducible decompositions of \( I \) and \( I^* \) are \( I = \bigcap_{p \in \text{Min}(I)} p \) and \( I^* = \bigcap_{q \in \text{Min}(I^*)} q \). By definition \( ir(I) = |\text{Min}(I)|, ir(I^*) = |\text{Min}(I^*)| \).
For any set of ideals \( \{I_i, i \in S\} \), we have \( (\cap_{i \in S} I_i)^s = \{r \in I_i \text{ for any } i \in S \text{ and } r \text{ is homogeneous}\} = \cap_{i \in S} (I_i)^s \). So \( I^* = \cap (p_i)^s_{Min(I)} \). Comparing this decomposition with the irredundant irreducible decomposition of \( I^* \) we know that for any \( q \in Min(I^*) \) there exist a \( p \) such that \( p^* = q \). So \( ir(I) \geq ir(I^*) \). The equality holds if and only if the second decomposition is also irredundant. That is, for each \( q \in Min(I^*) \), there exists exactly one \( p \in Min(I) \) with \( q = p^* \); and there does not exist \( p \in Min(I) \) such that \( p^* \notin Min(I^*) \). This means that \( p \rightarrow p^* \) gives a bijection between \( Min(I) \) and \( Min(I^*) \).

Example 5.2. Let \( R = k[x, y] \) which is \( \mathbb{Z} \)-graded. Let \( I = (x, (y-a_1)(y-a_2)\cdots(y-a_r)) \) where \( a_1, a_2, \ldots, a_r \) are pairwise distinct and all nonzero elements in \( k \). Then \( I^* = (x) \). In this case, we see that \( I \) has \( r \) components which collapse to become one component of \( I^* \).

In general, there is no fixed inequality between \( ir(I) \) and \( ir(I^*) \). Here are two examples where \( I \) and \( I^* \) are both \( m \)-primary for a graded prime ideal \( m \).

Example 5.3. Let \( R = k[x, y] \) where \( k \) is a field. \( R \) is \( \mathbb{Z} \)-graded-local with maximal ideal \( m = (x, y) \) and \( R/m = k \). Let \( I = (x^2, xy, y^3, x-y^2) \), then the largest graded subideal \( I^* = (x^2, xy, y^3) \). \( I \) and \( I^* \) are both \( m \)-primary. Now we have \( soc(I) = kx, soc(I^*) = kx + ky^2 \), so \( ir(I) = 1 < 2 = ir(I^*) \).

Example 5.4. Let \( R, m, k, G \) be as above, \( I = (x^4, x^2y^2, y^4, x^3y-y^3x) \). Then we have \( I^* = (x^4, x^2y^2, y^4) \), and \( I \) and \( I^* \) are still \( m \)-primary. Now we have \( soc(I^*) = kx^3y + kxy^3 \), \( soc(I) = kx^3y+k(x^3-xy^3) + k(x^3y-y^3) \), so \( ir(I) = 3 > 2 = ir(I^*) \).

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