Numerical scheme for semilinear Stochastic PDEs via Backward Doubly Stochastic Differential Equations

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Abstract: In this paper, we investigate a numerical probabilistic method for the solution of a class of semilinear stochastic partial differential equations (SPDEs in short). Our numerical scheme is based on discrete time approximation for solutions of systems of a decoupled forward-backward doubly stochastic differential equations. Under standard assumptions on the parameters, we prove the convergence and the rate of convergence of our numerical scheme. The proof is based on a generalization of the result on the path regularity of the backward equation.

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1. Introduction

Stochastic partial differential equations (SPDEs) combine the features of partial differential equations and Itô equations. Such equations play important roles in many applied fields such as the filtering of partially observable diffusion processes, genetic population and other areas. We study the following stochastic partial differential equation (in short SPDE) for a system-valued of predictable random field \( u_t(x) = u(t,x), \) satisfying the following equation:

\[
du_t(x) + (Lu_t(x) + f(t,x,u_t(x),\nabla u_t(x)))\,dt + g(t,x,u_t(x),\nabla u_t(x))\cdot dB_t = 0, \tag{1.1}
\]

over the time interval \([0,T]\), with a given final condition \( u_T = \Phi \) and non-linear deterministic coefficients \( f \) and \( g \). \( Lu = (Lu_1,\cdots,Lu_k) \) is a second order differential operator and \( \sigma \) is the diffusion coefficient. The differential term with \( dB_t \) refers to the backward stochastic integral with respect to a \( L \)-dimensional brownian motion on \((\Omega,F,P,(B_t)_{t\geq 0})\). We use the backward stochastic integral in the SPDE because we will employ the framework of Backward Doubly Stochastic Differential Equation (BDSDE) introduced first by Pardoux and Peng [29]. They gave a probabilistic representation for the classical solution \( u_t(x) \) of the SPDE (1.1) (written in the integral form) in term of the following class of BDSDE’s:

\[
Y_t^{t,x} = \Phi(X_T^{t,x}) + \int_t^T f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\,dr + \int_t^T g(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\cdot dB_r - \int_t^T Z_r^{t,x}\,dW_r, \tag{1.2}
\]

where \((X_t^{t,x})_{t\leq s\leq T}\) is a diffusion process starting from \( x \) at time \( t \) driven by the finite dimensional brownian motion \((W_t)_{t\geq 0}\) and with infinitesimal generator \( L \). More precisely, under some regularity assumptions on the final condition \( \Phi \) and coefficients \( f \) and \( g \), they have proved that \( u_t(x) = Y_t^{t,x} \) and \( \nabla u_t(x) = Z_t^{t,x}, \forall (t,x) \in [0,T] \times \mathbb{R}^d \). Then, Bally and Matoussi [5] (see also [25]) showed that the same representation remains true in the case when the final condition (respectively the coefficients \( f \) and \( g \)) is only measurable in \( x \) (resp. are jointly measurable in \((t,x)\) and Lipschitz in \( u \) and \( \nabla u \)). In this paper, weak Sobolev solution of the equation (1.1) has been considered, and the approach was based on stochastic flow technics (see also [19, 20]). Moreover their results were generalized in [25] in the case of a larger class of SPDE’s (1.1) driven by a Kunita-Ito non-linear noise (see [19, 20, 21] for more details). In particular, the Kunita-Ito non-linear noise covers a class of infinite dimensional space-time colored-white noise (see [13], [31], [16]). Generally, the explicit resolution of semi-linear SPDEs is not possible, so it is then necessary to resort to numerical methods.

The first approach used to solve numerically nonlinear SPDEs is analytic methods, based on time-space discretization of the SPDEs. The discretization on space can be achieved either by finite differences, or finite elements and spectral Galerkin methods. But most numerical works on SPDEs have concentrated on the Euler finite-difference scheme. Gyongi and Nualart [14] have proved that these schemes converge, and Gyongy [15] determined the order of convergence. J.B. Walsh [32] investigated schemes based on the finite elements methods. He studied the rate of convergence of these schemes for parabolic SPDEs, including the Forward and Backward Euler and the Crank-Nicholson schemes. He found substantially similar rate of convergence to those found for finite difference schemes. The spectral Galerkin approximation was used by Jentzen and Kloeden [16]. They based their method on Taylor expansions derived for the solution of the SPDE, under some
regularity conditions.
Lototsky, Mikulevicius and Rozovskii in 1997 [22] used the spectral approach for the numerical estimation of the conditional distribution solution of a linear SPDE known as the Zakai equation. Further developments on spectral methods can be found in Lototsky [23]. Milstein and Tretyakov [24] solved a linear Stochastic Partial Differential Equation by using the method of characteristics (the averaging over the characteristic formula). They proposed a numerical scheme based on Monte Carlo technique. Layer methods for linear and semilinear SPDEs are constructed. Picard [30] considered a filtering problem where the observation is a function of a diffusion corrupted by an independent white noise. He estimated the error caused by a discretization of the time interval. He obtained some approximations of the optimal filter which can be computed with Monte-Carlo methods. Crisan [8] studied a particle approximation for a class of nonlinear stochastic partial differential equations. Very interesting results have been obtained by Gyongy and Krylov [13] where they considered a symmetric finite difference scheme for a class of linear SPDE driven by infinite dimensional brownian motion. They have proved that the approximation error is proportional to $k^2$ where $k$ is the discretization step in space and by the Richardson acceleration method they have even got the error proportional to $k^4$.

The other alternative for resolving numerically SPDEs is the probabilistic approach by using Monte Carlo methods. These latter methods are tractable especially when the dimension of the state process is very large unlike the finite difference method. Furthermore, their parallel nature provides another advantage to the probabilistic approach: each processor of a parallel computer can be assigned the task of making a random trial and doing the calculus independently. The probabilistic approach requires weaker assumptions on the SPDE’s coefficients. In the deterministic PDE’s case i.e. $g \equiv 0$, the numerical approximation of the BSDE has already been studied in the literature by Bally [3], Zhang [33], Bouchard and Touzi [6], Gobet, Lemor and Warin [12] and Bouchard and Elie [7]. Zhang [33] proposed a discrete-time numerical approximation, by step processes, for a class of decoupled FBSDEs with possible path-dependent terminal values. He proved an $L^2$-type regularity of the BSDE’s solution, the convergence of his scheme and he derived its rate of convergence. Bouchard and Touzi [6] suggested a similar numerical scheme for decoupled FBSDEs. The conditional expectations involved in their discretization scheme were computed by using the kernel regression estimation. Therefore, they used the Malliavin approach and the Monte carlo method for its computation. Crisan, Manolarakis and Touzi [9] proposed an improvement on the Malliavin weights. Gobet, Lemor and Warin in [12] proposed an explicit numerical scheme. In the case when $g \neq 0$ and it does not depend on the control variable $z$, Aman [1] proposed a numerical scheme following the idea used by Bouchard and Touzi [6] and obtained a convergence of order $h$ of the square of the $L^2$- error ($h$ is the discretization step in time). Aboura [2] studied the same numerical scheme under the same kind of hypothesis, but following Gobet et al. [11]. He obtained a convergence of order $h$ in time and used the regression Monte Carlo method to implement his scheme, following always [11].
In our work, we extend the approach of Bouchard-Touzi-Zhang in the general case when $g$ depends also on the control variable $z$. We wish to emphasize that this generalization is not obvious because of the strong impact of the backward stochastic integral term on the numerical approximation scheme. It is known that in the associated Stochastic PDE’s (1.1), the term $g(u, \nabla u)$ leads to a second order perturbation type which explains the contraction condition assumed on $g$ with
respect to the variable \( z \) (see [29], [27]). Our scheme is implicit in \( Y \) and explicit in \( Z \). We prove
the convergence of our numerical scheme and we give the rate of convergence. The square of the \( L^2 \)- error has an upper bound of order the discretization step in time. As a consequence, we get
a numerical scheme for the weak solution of the associated semi-linear SPDE. We give also a rate
of convergence result for the later weak solution. Then, we propose a numerical scheme based on
iterative regression functions which are approximated by projections on vector space of functions
with coefficients evaluated using Monte Carlo simulations. Finally, we present some numerical
tests. Compared to the deterministic numerical method developed by Gyongy and Krylov [13],
the probabilistic approach could tackle the semilinear SPDE which could be degenerate and needs
less regularity conditions on the coefficients than the finite difference scheme. However, the rate
of convergence obtained (as the classical Monte Carlo method) is clearly slower than the results
obtained by finite difference and finite element schemes, but of course more available in higher
dimension.

This paper is organized as follows. In section 2 we introduce preliminaries and assumptions and we
describe the approximation scheme for the BDSDE. In section 3 we show an upper bound result for
the time discretization error. In section 4 we give a Malliavin regularity result for the solution of
our Forward-Backward Doubly SDE’s. Then, we show a \( L^2 \)-regularity result for the \( Z \)-component of
the solution of the BDSDE (1.2) which is crucial to obtain the rate of convergence of our numerical
scheme. Section 5 is devoted to the numerical scheme of the SPDE’s weak solution. In section 6,
we test statistically the convergence of this scheme by using a path dependent algorithm based on
the regression Monte Carlo Method. Finally, we give some technical results in the Appendix.

2. Preliminaries and notations

2.1. Forward Backward Doubly Stochastic Differential Equation

Let \( \{W_s, 0 \leq s \leq T\} \) and \( \{B_s, 0 \leq s \leq T\} \) be two mutually independent standard brownian motion
processes, with values respectively in \( \mathbb{R}^d \) and in \( \mathbb{R}^l \) where \( T > 0 \) is a fixed horizon time, on the
probability space \((\Omega, \mathcal{F}, P)\).

We shall work on the product space \( \Omega := \Omega_W \times \Omega_B \), where \( \Omega_W \) is the set of continuous functions
from \([0, T]\) into \( \mathbb{R}^d \) and \( \Omega_B \) is the set of continuous functions from \([0, T]\) into \( \mathbb{R}^l \).

We fix \( t \in [0, T] \). For each \( s \in [t, T] \), we define

\[ \mathcal{F}_s^t := \mathcal{F}_{t,s}^W \cup \mathcal{F}_{s,T}^B, \]

where \( \mathcal{F}_{t,s}^W = \sigma\{W_r - W_t, t \leq r \leq s\} \), and \( \mathcal{F}_{s,T}^B = \sigma\{B_r - B_s, s \leq r \leq T\} \). We take \( \mathcal{F}_W = \mathcal{F}_{0,T}^W \),
\( \mathcal{F}_B = \mathcal{F}_{0,T}^B \) and \( \mathcal{F} = \mathcal{F}_W \cup \mathcal{F}_B \).

We define the probability measures \( P_W \) on \((\Omega_W, \mathcal{F}_W)\) and \( P_B \) on \((\Omega_B, \mathcal{F}_B)\). We then define the probability measure \( P := P_W \otimes P_B \) on \((\Omega, \mathcal{F}_W \times \mathcal{F}_B)\). Without loss of generality, we assume that \( \mathcal{F}_W \) and \( \mathcal{F}_B \) are complete.

Note that the collection \( \{\mathcal{F}_s^t, s \in [t, T]\} \) is neither increasing nor decreasing, and it does not constitute a filtration.

Afterthat, we introduce the following spaces:

- \( C_b^k(\mathbb{R}^p, \mathbb{R}^q) \) (respectively \( C_b^{\infty}(\mathbb{R}^p, \mathbb{R}^q) \)) denotes the set of functions of class \( C^k \) from \( \mathbb{R}^p \) to \( \mathbb{R}^q \) whose partial derivatives of order less or equal to \( k \) are bounded (respectively the set of functions
of class $C^\infty$ from $\mathbb{R}^p$ to $\mathbb{R}^q$ whose partial derivatives are bounded).

For any $m \in \mathbb{N}$, we introduce the following notations:

- $\mathbb{H}_m^2([0, T])$ denotes the set of (classes of $dP \times dt$ a.e. equal) $\mathbb{R}^m$-valued jointly measurable processes
- $\psi_u u \in [0, T]$ satisfying:
- (i) $\|\psi\|^2_{\mathbb{H}_m^2([0, T])} := E[\int_0^T |\psi_u|^2 du] < \infty$,
- (ii) $\psi_u$ is $\mathcal{F}_u$-measurable, for a.e. $u \in [0, T]$.

- $\mathbb{S}_m^2([0, T])$ denotes similarly the set of $\mathbb{R}^m$-valued continuous processes satisfying:
- (i) $\|\psi\|^2_{\mathbb{S}_m^2([0, T])} := \sup_{0 \leq u \leq T} |\psi_u|^2 < \infty$,
- (ii) $\psi_u$ is $\mathcal{F}_u$-measurable, for any $u \in [0, T]$.

- $\mathbb{S}$ the set of random variables $F$ of the form: $F = \hat{f}(W(h_1), \ldots, W(h_{m_1}), B(k_1), \ldots, B(k_{m_2}))$

For any random variable $F \in \mathbb{S}$, we define its Malliavin derivative $(D_s F)_s$ with respect to the brownian motion $W$ by

$$D_s F := \sum_{i=1}^{m_1} \nabla_i \hat{f} \left( W(h_1), \ldots, W(h_{m_1}); B(k_1), \ldots, B(k_{m_2}) \right) h_i(s),$$

where $\nabla_i \hat{f}$ is the derivative of $\hat{f}$ with respect to its i-th argument.

We define a norm on $\mathbb{S}$ by:

$$\|F\|_{1,2} := \left\{ E[F^2] + E\left[ \int_0^T |D_s F|^2 ds \right] \right\}^{\frac{1}{2}}.$$

- $\mathbb{D}^{1,2} := \mathbb{S}^{1,2}$ is then a Sobolev space.

- $\mathbb{S}_k^2([0, T], \mathbb{D}^{1,2})$ is the set of processes $Y = (Y_u, 0 \leq u \leq T)$ such that $Y \in \mathbb{S}_k^2([0, T]), Y_u^i \in \mathbb{D}^{1,2}, 1 \leq i \leq k, 0 \leq u \leq T$ and

$$\|Y\|_{1,2} := \left\{ E\left[ \int_0^T |Y_u|^2 du \right] + E\left[ \int_0^T \int_0^T \|D_\theta Y_u\|^2 d\theta du \right] \right\}^{\frac{1}{2}} < \infty.$$

- $\mathcal{M}_k^{2 \times d}([0, T], \mathbb{D}^{1,2})$ is the set of processes $Z = (Z_u, 0 \leq u \leq T)$ such that $Z \in \mathcal{M}_k^{2 \times d}([0, T]), Z_u^{i,j} \in \mathbb{D}^{1,2}, 1 \leq i \leq k, 1 \leq j \leq d, 0 \leq u \leq T$ and

$$\|Z\|_{1,2} := \left\{ E\left[ \int_0^T \|Z_u\|^2 du \right] + E\left[ \int_0^T \int_0^T \|D_\theta Z_u\|^2 d\theta du \right] \right\}^{\frac{1}{2}} < \infty.$$

- $\mathbb{B}^2([0, T], \mathbb{D}^{1,2}) := \mathbb{S}_k^2([0, T], \mathbb{D}^{1,2}) \times \mathcal{M}_k^{2 \times d}([0, T], \mathbb{D}^{1,2}).$

We define also for a given $t \in [0, T]$:

- $\mathbb{L}^2([t, T], \mathbb{D}^{1,2})$ is the set of progressively measurable processes $(v_s)_{t \leq s \leq T}$ such that:
  - (i) $v(s, .) \in \mathbb{D}^{1,2}$, for a.e. $s \in [t, T]$.
  - (ii) $(s, w) \rightarrow Duv(s, w) \in L^2([t, T] \times \Omega),$
  - (iii) $E\left[ \int_t^T |v_s|^2 ds \right] + E\left[ \int_t^T \int_t^T |D_u v_s|^2 duds \right] < \infty.$
- $\mathbb{L}^2([t, T], \mathbb{D}^{1,2} \times \mathbb{D}^{1,2}) := \mathbb{L}^2([t, T], \mathbb{D}^{1,2}) \times \mathbb{L}^2([t, T], \mathbb{D}^{1,2}).$
For all \((t, x) \in [0, T] \times \mathbb{R}^d\), let \((X^{t,x}_s)\) be the unique strong solution of the following stochastic differential equation:

\[
dx^{t,x}_s = b(X^{t,x}_s)ds + \sigma(X^{t,x}_s)dW_s, \quad s \in [t, T], \quad X^{t,x}_t = x, \quad 0 \leq s \leq t, \tag{2.1}\]

where \(b\) and \(\sigma\) are two functions on \(\mathbb{R}^d\) with values respectively in \(\mathbb{R}^d\) and \(\mathbb{R}^{d \times d}\). We will omit the dependence of the forward process \(X\) in the initial condition if it starts at time \(t = 0\).

We consider the following BDSDE: For all \(t \leq s \leq T\),

\[
\begin{cases}
    dY^{t,x}_s = -f(s, X^{t,x}_s, Y^{t,x}_s, Z^{t,x}_s)ds - g(s, X^{t,x}_s, Y^{t,x}_s, Z^{t,x}_s)dB_s + Z^{t,x}_s dW_s, \\
    Y^{t,x}_T = \Phi(X^{t,x}_T),
\end{cases} \tag{2.2}
\]

where \(f\) and \(\Phi\) are two functions respectively on \([0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}\) and \(\mathbb{R}^d\) with values in \(\mathbb{R}^k\) and \(g\) is a function on \([0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}\) with values in \(\mathbb{R}^{k \times l}\).

We note that the integral with respect to \((B_s, t \leq s \leq T)\) is a "backward Itô integral" (see Kunita [21] and Nualart and Pardoux [27] for the definition) and the integral with respect to \((W_s, t \leq s \leq T)\) is a standard forward Itô integral.

Finally, for each real matrix \(A\), we denote by \(\|A\|\) its Frobenius norm defined by \(\|A\| = (\sum_{i,j} a_{i,j}^2)^{1/2}\).

For a vector \(x\), \(|x|\) stands for its Euclidean norm defined by \(|x| = (\sum_i |x_i|^2)^{1/2}\).

The following assumptions will be needed in our work:

**Assumption (H1)** There exist a positive constant \(K\) such that

\[|b(x) - b(x')| + \|\sigma(x) - \sigma(x')\| \leq K|x - x'|, \forall x, x' \in \mathbb{R}^d.\]

**Assumption (H2)** There exist two constants \(K > 0\) and \(0 \leq \alpha < 1\) such that for any \((t_1, x_1, y_1, z_1), (t_2, x_2, y_2, z_2) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}\),

(i) \(|f(t_1, x_1, y_1, z_1) - f(t_2, x_2, y_2, z_2)| \leq K(\sqrt{|t_1 - t_2| + |x_1 - x_2| + |y_1 - y_2| + \|z_1 - z_2\|}),\)

(ii) \(\|g(t_1, x_1, y_1, z_1) - g(t_2, x_2, y_2, z_2)\|^2 \leq K(|t_1 - t_2| + |x_1 - x_2|^2 + |y_1 - y_2|^2 + \alpha^2 \|z_1 - z_2\|^2),\)

(iii) \(\sup_{0 \leq t \leq T} (|f(t, 0, 0, 0)| + \|g(t, 0, 0, 0)\|) \leq K.\)

**Assumption (H3)**

(i) \(b \in C^2_b(\mathbb{R}^d, \mathbb{R}^d)\) and \(\sigma \in C^2_b(\mathbb{R}^d, \mathbb{R}^{d \times d})\)

(ii) \(\Phi \in C^2_b(\mathbb{R}^d, \mathbb{R}^k), f \in C^2_b([0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{d \times k}, \mathbb{R}^k)\)

and \(g \in C^2_b([0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{d \times k}, \mathbb{R}^{k \times l})\).

Pardoux and Peng [29] proved that there exists a unique solution \((Y, Z) \in S^2_b([t, T]) \times H^2_{k \times d}([t, T])\) to the BDSDE (2.2).

**Remark 2.1** Pardoux and Peng [29] assumed the contraction condition \(0 \leq \alpha < 1\) to prove the existence and the uniqueness results for the BDSDE’s solution.
From [10], [29] and [18], the standard estimates for the solution of the Forward-Backward Doubly SDE (2.1)-(2.2) hold and we remind the following theorem:

**Theorem 2.1** Under Assumptions (H1) and (H2), there exist, for any \( p \geq 2 \), two positive constants \( C \) and \( C_p \) and an integer \( q \) such that:

\[
E \left[ \sup_{t \leq s \leq T} |X_s^{t,x}|^2 \right] \leq C(1 + |x|^2), \tag{2.3}
\]

\[
E \left[ \sup_{t \leq s \leq T} |Y_s^{t,x}|^p + \left( \int_t^T |Z_s^{t,x}|^2 ds \right)^{p/2} \right] \leq C_p(1 + |x|^q), \tag{2.4}
\]

**2.2. Numerical Scheme for decoupled Forward-BDSDE**

In order to approximate the solution of the BDSDE (2.2), we introduce the following discretized version. Let

\[ \pi : t_0 = 0 < t_1 < \ldots < t_N = T, \tag{2.5} \]

be a partition of the time interval \([0, T]\). For simplicity we take an equidistant partition of \([0, T]\) i.e. \( h = \frac{T}{N} \) and \( t_n = nh \), \( 0 \leq n \leq N \). Through the rest, we will use the notations \( \Delta W_n = \left(W_{t_{n+1}} - W_{t_n}\right) \) and \( \Delta B_n = \left(B_{t_{n+1}} - B_{t_n}\right) \), for \( n = 1, \ldots, N \).

The forward component \( X \) will be approximated by the classical Euler scheme:

\[
\begin{align*}
X_{t_0}^N &= X_{t_0}, \\
X_{t_n}^N &= X_{t_{n-1}}^N + b(X_{t_{n-1}}^N)(t_n - t_{n-1}) + \sigma(X_{t_{n-1}}^N)(W_{t_n} - W_{t_{n-1}}), \quad \text{for } n = 1, \ldots, N. \tag{2.6}
\end{align*}
\]

It is known that as \( N \) goes to infinity, one has \( \sup_{0 \leq n \leq N} E |X_{t_n}^N - X_{t_n}^N|^2 \to 0 \).

Quite naturally, the solution \((Y, Z)\) of (2.2) is approximated by \((Y^N, Z^N)\) defined by:

\[
Y_{t_N}^N = \Phi(X_T^N) \quad \text{and} \quad Z_{t_N}^N = 0, \tag{2.7}
\]

and for \( n = N - 1, \ldots, 0 \), we set

\[
\begin{align*}
Y_{t_n}^N &= E_{t_n}[Y_{t_{n+1}}^N] + hE_{t_n}[f(t_n, \Theta_n^N)] + E_{t_n}[g(t_{n+1}, \Theta_{n+1}^N)\Delta B_n], \tag{2.8}
\end{align*}
\]

\[
\begin{align*}
hZ_{t_n}^N &= E_{t_n} \left[ Y_{t_{n+1}}^N \Delta W_n^* + g(t_{n+1}, \Theta_{n+1}^N)\Delta B_n \Delta W_n^* \right], \tag{2.9}
\end{align*}
\]

where \( \Theta_n^N := (X_{t_n}^N, Y_{t_n}^N, Z_{t_n}^N), \forall n = 0, \ldots, N \).

* denotes the transposition operator and \( E_{t_n} \) denotes the conditional expectation over the \( \sigma \)-algebra \( \mathcal{F}_{t_n} \).

For all \( n = 0, \ldots, N - 1 \), we define the pair of processes \((Y_t^N, Z_t^N)_{t_n \leq t < t_{n+1}}\) as the solution of the following BDSDE:

\[
Y^N_t = Y^N_{t_{n+1}} + \int_t^{t_{n+1}} f(t, \Theta_t^N)ds + \int_t^{t_{n+1}} g(t_{n+1}, \Theta_{n+1}^N)dB_s - \int_t^{t_{n+1}} Z_s^N dW_s, \quad t_n \leq t < t_{n+1}. \tag{2.10}
\]
Remark 2.2 Equation (2.10) is the continuous approximation of the solution of BDSDE (2.2). The sequences \((Y_{t_n}^N)_{0 \leq n \leq N}\) given by (2.8) and (2.10) coincide. In Lemma 3.1 we will give the relation between \((Z_{t_n}^N)_{0 \leq n \leq N-1}\) and \((Z_{t_n}^N)_{n \leq s < t_n+1}^s\).

Remark 2.3 For the approximation of \(Y_{t_n}^N\), (2.8) is well-defined, indeed if we set

\[
\begin{cases}
Y_{t_n}^{N,0} = 0 \\
Y_{t_n}^{N,i} = E_{t_n}[f(t_n, X_{t_n}^N, Y_{t_n}^{N,i-1}, Z_{t_n}^N)] + E_{t_n}[g(t_n+1, X_{t_n+1}^N, Y_{t_n+1}^N, Z_{t_n+1}^N)dB_n], & i \geq 1
\end{cases}
\]

By using the Isometry property for the Itô integral, we have

\[
E[|Y_{t_n}^{N,i} - Y_{t_n}^{N,i-1}|^2] = h^2E\left[\left|E_{t_n}[f(t_n, X_{t_n}^N, Y_{t_n}^{N,i-1}, Z_{t_n}^N)] - f(t_n, X_{t_n}^N, Y_{t_n}^{N,i-2}, Z_{t_n}^N)\right|^2\right] \\
\leq h^2E\left[\left|f(t_n, X_{t_n}^N, Y_{t_n}^{N,i-1}, Z_{t_n}^N) - f(t_n, X_{t_n}^N, Y_{t_n}^{N,i-2}, Z_{t_n}^N)\right|^2\right]
\]

By Assumption (H2), we obtain for \(i \geq 2\)

\[
E[|Y_{t_n}^{N,i} - Y_{t_n}^{N,i-1}|^2] \leq h^2K^2E[|Y_{t_n}^{N,i-1} - Y_{t_n}^{N,i-2}|^2]
\]

This shows that, for \(h\) small enough, \((Y_{t_n}^{N,i})_{i \geq 0}\) is a Cauchy sequence and \(Y_{t_n}^{N,i}\) in \(L^2(\Omega, \mathcal{F}_t^0)\) where \(L^2(\Omega, \mathcal{F}_t^0)\) denotes the set of square integrable random variables which are \(\mathcal{F}_t^0\)-measurable.

Remark 2.4 The superscript \((t, x)\) indicates the dependence of the solution \((X, Y, Z)\) on the initial date \((t, x)\). To alleviate notations, we omit the dependence on \((t, x)\) of \((Y^{t,x}, Z^{t,x})\) and \((Y^{N,t,x}, Z^{N,t,x})\) when the context is clear.

We note also that in the next computations, the constant \(C\) denotes a generic constant that may change from line to line. It depends on \(K, T, \alpha, |b(0)|, ||\sigma(0)||, |f(t, 0, 0, 0)|\) and \(||g(t, 0, 0, 0)||\).

3. Upper bound for the discrete time approximation error

First, we define the process \(\hat{Z}\) by

\[
\begin{cases}
\hat{Z}_t = \frac{1}{h}E_{t_n}[\int_{t_n}^{t_{n+1}} Z_s ds], & \forall t \in [t_n, t_{n+1}), \forall n \in \{0, \ldots, N-1\}, \\
\hat{Z}_{t_N} = 0.
\end{cases}
\]

Then we give the following property of the continuous approximation \(Z^N\) which shows that \(Z_{t_n}^N\) is the best \(L^2(\mathcal{F}_t)\)-estimate of \((Z_{t_n}^N)_s\).

Lemma 3.1 For all \(n = 0, \ldots, N-1\), we have

\[
Z_{t_n}^N = \frac{1}{h}E_{t_n}[\int_{t_n}^{t_{n+1}} Z_s ds].
\]

Proof. From (2.10) we have

\[
\int_{t_n}^{t_{n+1}} Z_s^N dW_s = Y_{t_n+1}^N \Delta W_n + \int_{t_n}^{t_{n+1}} f(t_n, \Theta_{t_n}^N) ds \Delta W_n \\
+ \int_{t_n}^{t_{n+1}} g(t_n+1, \Theta_{t_n+1}^N) dB_s \Delta W_n - Y_{t_n}^N \Delta W_n.
\]
then
\[
E_t^n \left[ \int_{t_n}^{t_{n+1}} Z_s^N dW_s \Delta W_n \right] = E_t^n [Y_{t_{n+1}}^N - Y_{t_{n+1}}^N] + \mathbb{E}_t^n \left[ \int_{t_n}^{t_{n+1}} f(t, \Theta_n^N) dt \right] \Delta W_n \\
+ \mathbb{E}_t^n \left[ \int_{t_n}^{t_{n+1}} g(t, \Theta_n^N, \Theta_{n+1}^N) dB_s \Delta W_n \right] - \mathbb{E}_t^n [Y_{t_{n}}^N - Y_{t_{n}}^N] \\
= \mathbb{E}_t^n [Y_{t_{n+1}}^N - Y_{t_{n+1}}^N] + h \mathbb{E}_t^n [f(t, \Theta_n^N) \Delta W_n] \\
+ \mathbb{E}_t^n [g(t, \Theta_n^N, \Theta_{n+1}^N) \Delta B_n \Delta W_n] - \mathbb{E}_t^n [Y_{t_{n}}^N - Y_{t_{n}}^N] \\
= \mathbb{E}_t^n [Y_{t_{n+1}}^N - Y_{t_{n+1}}^N] + hZ_{t_{n+1}}^N - hZ_{t_{n}}^N. \\
\tag{3.3}
\]

Here we used the fact that $Y_{t_n}^N$ and $f(t, \Theta_n^N)$ are $\mathcal{F}_{t_n}$-measurable and then we have
\[
E_t^n [f(t, \Theta_n^N) \Delta W_n] = E_t^n [Y_{t_{n}}^N \Delta W_n] = 0.
\]

Now by using the integration by parts formula we have
\[
E_t^n \left[ \int_{t_n}^{t_{n+1}} Z_s^N dW_s \Delta W_n \right] = E_t^n \left[ \int_{t_n}^{t_{n+1}} \int_{t_n}^{t_{n+1}} Z_s^N dW_s dW_s \right] \\
+ E_t^n \left[ \int_{t_n}^{t_{n+1}} \int_{t_n}^{t_{n+1}} Z_s^N dW_s dW_s \right] + E_t^n \left[ \int_{t_n}^{t_{n+1}} Z_s^N ds \right].
\]

Then
\[
E_t^n \left[ \int_{t_n}^{t_{n+1}} Z_s^N dW_s \Delta W_n \right] = E_t^n \left[ \int_{t_n}^{t_{n+1}} Z_s^N ds \right]. \\
\tag{3.4}
\]

Equations (3.3) and (3.4) give that
\[
Z_{t_n}^N = \frac{1}{h} E_t^n \left[ \int_{t_n}^{t_{n+1}} Z_s^N ds \right].
\]

The next theorem states an upper bound result regarding the time discretization error.

**Theorem 3.1** Assume that Assumptions (H1) and (H2) hold, define the error
\[
\text{Error}_N(Y, Z) := \sup_{0 \leq s \leq T} E[|Y_s - Y_s^N|^2] + \sum_{n=0}^{N-1} E[\int_{t_n}^{t_{n+1}} |Z_s - Z_{t_n}^N|^2 ds], \\
\tag{3.5}
\]

where $Y^N$ and $Z^N$ are given by (2.10). Then
\[
\text{Error}_N(Y, Z) \leq Ch(1 + |x|^2) + C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[|Z_s - \bar{Z}_{t_n}|^2] ds \\
+ C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[|Z_s - \bar{Z}_{t_{n+1}}|^2] ds. \\
\tag{3.6}
\]
Proof. For all \( t \in [t_n, t_{n+1}) \), \( n = 0, \ldots, N - 1 \) we define the following quantities:

\[
\begin{cases}
\theta_t := (X_t, Y_t, Z_t), \quad \delta Y_t := Y_t - Y_t^N, \quad \delta Z_t := Z_t - Z_t^N, \\
\delta f_t := f(t, \theta_t) - f(t_n, \Theta_n^N), \\
\delta g_t := g(t, \theta_t) - g(t_{n+1}, \Theta_n^N+1).
\end{cases}
\]

We have:

\[
\delta Y_t^N = \delta Y_t^N + \int_t^{t_{n+1}} \delta f_s \, ds + \int_t^{t_{n+1}} \delta g_s \, dB_s - \int_t^{t_{n+1}} \delta Z_s \, dW_s, \forall t \in [t_n, t_{n+1}).
\]

Using the Generalized Itô’s Lemma (see Lemma 1.3, [29]), we obtain

\[
|\delta Y_t|^2 + \int_t^{t_{n+1}} |\delta Z_s|^2 ds - |\delta Y_t^N|^2 = 2 \int_t^{t_{n+1}} (\delta Y_s, \delta f_s) ds + 2 \int_t^{t_{n+1}} (\delta Y_s, \delta g_s) \bar{dB}_s + \int_t^{t_{n+1}} |\delta g_s|^2 ds - 2 \int_t^{t_{n+1}} (\delta Y_s, \delta Z_s) dW_s, \forall t \in [t_n, t_{n+1}).
\]

Then taking the expectation, we have

\[
A_t^n := E[|\delta Y_t|^2] + \int_t^{t_{n+1}} E[|\delta Z_s|^2] ds - E[|\delta Y_t^N|^2] = 2 \int_t^{t_{n+1}} E[(\delta Y_s, \delta f_s)] ds + \int_t^{t_{n+1}} E[|\delta g_s|^2] ds.
\]

From Assumption (H2)-(ii),

\[
\int_t^{t_{n+1}} E[|\delta g_s|^2] ds \leq K h^2 + K \int_t^{t_{n+1}} E[|X_s - X_{t_{n+1}}|^2] ds + K \int_t^{t_{n+1}} E[|Y_s - Y_{t_{n+1}}|^2] ds + \alpha^2 E[\int_t^{t_{n+1}} |Z_s - Z_{t_{n+1}}|^2 ds].
\]

Using the Young’s inequality, for a positive constant \( \epsilon \), we obtain for all \( n = 0, \ldots, N - 2 \),

\[
E[\int_t^{t_{n+1}} |Z_s - Z_{t_{n+1}}|^2 ds] \leq (1 + \frac{1}{\epsilon}) E[\int_t^{t_{n+1}} |Z_s - \bar{Z}_{t_{n+1}}|^2 ds] + (1 + \epsilon) E[\int_t^{t_{n+1}} |\bar{Z}_{t_{n+1}} - Z_{t_{n+1}}|^2 ds].
\]

Let’s note that the last inequality is not needed for \( n = N - 1 \), since \( Z_{t_{N-1}}^N = 0 \) and then

\[
E[\int_t^{t_{N-1}} |Z_s - Z_{t_{N-1}}|^2 ds] = E[\int_t^{t_{N-1}} |Z_s|^2 ds], \forall t \in [t_{N-1}, t_N).
\]

For all \( n = 0, \ldots, N - 2 \), we use Lemma 3.1, the definition of \( \bar{Z} \) and the Jensen’s inequality to get

\[
E[|\bar{Z}_{t_{n+1}} - Z_{t_{n+1}}|^2] = E\left[\frac{1}{h} E_{t_{n+1}} \left[\int_{t_{n+1}}^{t_{n+2}} \delta Z_r^N \, dr\right]^2\right] \leq \frac{1}{h^2} E\left[ E_{t_{n+1}} \left[\int_{t_{n+1}}^{t_{n+2}} |\delta Z_r^N|^2 \, dr\right]^2\right].
\]

By using Cauchy Schwartz inequality, we obtain for all \( n = 0, \ldots, N - 2 \)

\[
E[|\bar{Z}_{t_{n+1}} - Z_{t_{n+1}}|^2] \leq \frac{1}{h} E\left[ \int_{t_{n+1}}^{t_{n+2}} |\delta Z_r^N|^2 \, dr\right].
\]
Plugging (3.11) in (3.10) then (3.10) in (3.9), we get for all \( n = 0, \ldots, N - 2 \)
\[
\int_t^{t_{n+1}} E[\|\delta g_s\|^2]ds \leq Kh^2 + K \int_t^{t_{n+1}} E[\|X_s - X_{t_{n+1}}^N\|^2]ds + K \int_t^{t_{n+1}} E[\|Y_s - Y_{t_{n+1}}^N\|^2]ds \\
+ \left( 1 + \frac{1}{\epsilon} \right) \alpha^2 \int_t^{t_{n+1}} E[\|Z_s - \tilde{Z}_{t_{n+1}}\|^2]ds + (1 + \epsilon) \alpha^2 \int_{t_{n+1}}^{t_{n+2}} E[\|\delta Z_N^N\|^2]ds. \tag{3.12}
\]

The previous inequality becomes trivially for \( n = N - 1 \),
\[
\int_t^{t_N} E[\|\delta g_s\|^2]ds \leq Kh^2 + K \int_t^{t_N} E[\|X_s - X_{t_N}^N\|^2]ds + K \int_t^{t_N} E[\|Y_s - Y_{t_N}^N\|^2]ds \\
+ \alpha^2 E\left[ \int_{t_{N-1}}^{t_N} \|Z_s\|^2 ds \right], \forall t \in [t_{N-1}, t_N). \tag{3.13}
\]

We mention that in the rest of the proof, we will omit to treat the case \( n = N - 1 \). This case remains only to replace inequality (3.12) by inequality (3.13) in the following estimations, which is simpler to handle.

We set \( \alpha' := (1 + \epsilon) \alpha^2 \). We choose \( \epsilon \) such that \( \alpha' \in (0,1) \). This is possible since \( \alpha^2 \in (0,1) \). Then, we use the inequality \( 2ab \leq \frac{1}{4a'x^2} a^2 + \frac{4K}{\alpha'x^2} b^2 \) and equation (3.12) to obtain for all \( n = 0, \ldots, N - 2 \)
\[
A_n^1 \leq \frac{4K}{1 - \alpha'} \int_t^{t_{n+1}} E[\|\delta Y_N^N\|^2]ds + \frac{1 - \alpha'}{4K} \int_t^{t_{n+1}} E[\|\delta f_s\|^2]ds + Kh^2 \\
+ K \int_t^{t_{n+1}} E[\|X_s - X_{t_{n+1}}^N\|^2]ds + K \int_t^{t_{n+1}} E[\|Y_s - Y_{t_{n+1}}^N\|^2]ds \\
+ \left( 1 + \frac{1}{\epsilon} \right) \alpha^2 \int_t^{t_{n+1}} E[\|Z_s - \tilde{Z}_{t_{n+1}}\|^2]ds + \alpha' \int_{t_{n+1}}^{t_{n+2}} E[\|\delta Z_N^N\|^2]ds.
\]

Now using Assumption (H2)-(i) in the last inequality, we get
\[
A_n^1 \leq \frac{4K}{1 - \alpha'} \int_t^{t_{n+1}} E[\|\delta Y_N^N\|^2]ds + \frac{1 - \alpha'}{4K} \int_t^{t_{n+1}} E[\|X_s - X_{t_{n+1}}^N\|^2]ds + \int_t^{t_{n+1}} E[\|Y_s - Y_{t_{n+1}}^N\|^2]ds \\
+ \int_t^{t_{n+1}} E[\|Z_s - \tilde{Z}_{t_{n+1}}\|^2]ds + Kh^2 + K \int_t^{t_{n+1}} E[\|X_s - X_{t_{n+1}}^N\|^2]ds + K \int_t^{t_{n+1}} E[\|Y_s - Y_{t_{n+1}}^N\|^2]ds \\
+ \left( 1 + \frac{1}{\epsilon} \right) \alpha^2 \int_t^{t_{n+1}} E[\|Z_s - \tilde{Z}_{t_{n+1}}\|^2]ds + \alpha' \int_{t_{n+1}}^{t_{n+2}} E[\|\delta Z_N^N\|^2]ds.
\]

Then, by plugging \( \tilde{Z}_{t_n} \) in the last inequality
\[
A_n^1 \leq \frac{4K}{1 - \alpha'} \int_t^{t_{n+1}} E[\|\delta Y_N^N\|^2]ds + \frac{1 - \alpha'}{4K} \int_t^{t_{n+1}} E[\|X_s - X_{t_{n+1}}^N\|^2]ds + \int_t^{t_{n+1}} E[\|Y_s - Y_{t_{n+1}}^N\|^2]ds \\
+ 2 \int_t^{t_{n+1}} E[\|Z_s - \tilde{Z}_{t_n}\|^2]ds + 2 \int_{t_{n+1}}^{t_{n+2}} E[\|\delta Z_N^N\|^2]ds + Kh^2 + K \int_t^{t_{n+1}} E[\|X_s - X_{t_{n+1}}^N\|^2] \\
+ K \int_t^{t_{n+1}} E[\|Y_s - Y_{t_{n+1}}^N\|^2]ds + (1 + \frac{1}{\epsilon}) \alpha^2 \int_t^{t_{n+1}} E[\|Z_s - \tilde{Z}_{t_{n+1}}\|^2]ds \\
+ \alpha' \int_{t_{n+1}}^{t_{n+2}} E[\|\delta Z_N^N\|^2]ds.
\]

It is well known from Kloeden and Platen [17] that for all \( s \in [t_n, t_{n+1}) \) and for all \( n = 0, \ldots, N - 1 \)
\[
E[\|X_s - X_{t_n}^N\|^2] \leq Ch \quad \text{and} \quad E[\|X_s - X_{t_{n+1}}^N\|^2] \leq Ch, \tag{3.14}
\]
where $C$ is a positive constant independent of $x$ and depending on $K, T, |b(0)|$ and $\|\sigma(0)\|$. On the other hand, it is easy to check that 
\[ \sup_{t_n \leq s \leq t_{n+1}} (|Y_s - Y_{t_{n+1}}|^2 + |Y_s - Y_{t_n}|^2) \leq Ch(1 + |x|^2), \]
This implies that
\[
E[|Y_s - Y_{t_{n+1}}|^2] \leq C[E[|Y_s - Y_{t_{n+1}}|^2] + E[|Y_{t_{n+1}} - Y_{t_n}|^2]]
\leq C\{h(1 + |x|^2) + E[|\delta Y_{t_{n+1}}|^2]\} \tag{3.15}
\]
and similarly we have
\[
E[|Y_s - Y_{t_n}|^2] \leq C\{h(1 + |x|^2) + E[|\delta Y_{t_n}|^2]\}, \tag{3.16}
\]
where $C$ is a positive constant independent of $x$.

From (3.14), (3.15) and (3.16), we obtain
\[
A^n_t \leq C \int_t^{t_{n+1}} E[|\delta Y_s|^2]ds + ChE[|\delta Y_{t_{n+1}}|^2] + ChE[|\delta Y_{t_n}|^2] + Ch^2(1 + |x|^2)
+ C \int_t^{t_{n+1}} E[|Z_s - \bar{Z}_{t_n}|^2]ds + \left(\frac{1 - \alpha'}{2}\right) \int_t^{t_{n+1}} E[|\delta Z_s|^2]ds
+ (1 + \frac{1}{\epsilon})\alpha^2 \int_t^{t_{n+1}} E[|Z_s - \bar{Z}_{t_{n+1}}|^2]ds + \alpha' \int_{t_{n+1}}^{t_{n+2}} E[|\delta Z_s|^2]ds. \tag{3.17}
\]
where $C$ is a generic positive constant depending on $\alpha'$ and independent of $x$.

From (3.8) and (3.17), we get
\[
E[|\delta Y_t|^2] \leq A^n_t + E[|\delta Y_{t_{n+1}}|^2]
= E[|\delta Y_t|^2] + \int_t^{t_{n+1}} E[|\delta Z_s|^2]ds
\leq C \int_t^{t_{n+1}} E[|\delta Y_s|^2]ds + B_n, \ \forall t \in [t_n, t_{n+1}), \tag{3.18}
\]
where we set for all $n = 0, \ldots, N - 2$:
\[
B_n := E[|\delta Y_{t_{n+1}}|^2] + ChE[|\delta Y_{t_{n+1}}|^2] + ChE[|\delta Y_{t_n}|^2] + Ch^2(1 + |x|^2)
+ C \int_{t_n}^{t_{n+1}} E[|Z_s - \bar{Z}_{t_n}|^2]ds + \left(\frac{1 - \alpha'}{2}\right) \int_{t_n}^{t_{n+1}} E[|\delta Z_s|^2]ds
+ (1 + \frac{1}{\epsilon})\alpha^2 \int_{t_n}^{t_{n+1}} E[|Z_s - \bar{Z}_{t_{n+1}}|^2]ds + \alpha' \int_{t_{n+1}}^{t_{n+2}} E[|\delta Z_s|^2]ds. \tag{3.19}
\]
Using Gronwall Lemma, we have
\[
E[|\delta Y_t|^2] \leq B_n e^{Ch}, \ \forall t \in [t_n, t_{n+1}). \tag{3.20}
\]
From inequalities (3.20) and (3.18), we get for $h$ small enough
\[
E[|\delta Y_t|^2] + \int_t^{t_{n+1}} E[|\delta Z_s|^2]ds \leq (1 + Ch)B_n
\leq (1 + Ch)B_n, \ \forall t \in [t_n, t_{n+1}). \tag{3.21}
\]
By taking \( t = t_n \) in the last inequality, we obtain

\[
E[\|\delta Y_{t_n}^N\|^2] + \int_{t_n}^{t_{n+1}} E[\|\delta Z_{s}^N\|^2]ds \leq (1 + Ch) \left\{ E[\|\delta Y_{t_{n+1}}^N\|^2] + ChE[\|\delta Y_{t_n}^N\|^2] \right\} + Ch^2 (1 + |x|^2) + C \int_{t_n}^{t_{n+1}} E[\|Z_s - \bar{Z}_{t_n}\|^2]ds + \frac{1}{2} \alpha' \int_{t_n}^{t_{n+1}} E[\|\delta Z_{s}^N\|^2]ds
\]

\[
+ (1 + \frac{1}{\alpha}) \alpha^2 \int_{t_n}^{t_{n+1}} E[\|Z_s - \bar{Z}_{t_n}\|^2]ds + \alpha' \int_{t_n}^{t_{n+1}} E[\|\delta Z_{s}^N\|^2]ds \right\}.
\]

Then

\[
(1 - Ch)E[\|\delta Y_{t_n}^N\|^2] + [1 - (1 + Ch) \frac{1 - \alpha'}{2} \int_{t_n}^{t_{n+1}} E[\|\delta Z_{s}^N\|^2]ds \leq (1 + Ch) \left\{ E[\|\delta Y_{t_{n+1}}^N\|^2] + \alpha' \int_{t_{n+1}}^{t_{n+2}} E[\|\delta Z_{s}^N\|^2]ds \right\] + Ch^2 (1 + |x|^2) + C \int_{t_n}^{t_{n+1}} E[\|Z_s - \bar{Z}_{t_n}\|^2]ds + (1 + \frac{1}{\alpha}) \alpha^2 \int_{t_n}^{t_{n+1}} E[\|Z_s - \bar{Z}_{t_n+1}\|^2]ds
\]

\[
+ \alpha' \int_{t_{n+1}}^{t_{n+2}} E[\|\delta Z_{s}^N\|^2]ds \right\}.
\]

For \( h \) small enough, we get

\[
E[\|\delta Y_{t_n}^N\|^2] + \frac{1 + \alpha'}{2} \int_{t_n}^{t_{n+1}} E[\|\delta Z_{s}^N\|^2]ds \leq (1 + Ch) \left\{ E[\|\delta Y_{t_{n+1}}^N\|^2] + \alpha' \int_{t_{n+1}}^{t_{n+2}} E[\|\delta Z_{s}^N\|^2]ds \right\] + Ch^2 (1 + |x|^2) + C \int_{t_n}^{t_{n+1}} E[\|Z_s - \bar{Z}_{t_n}\|^2]ds + (1 + \frac{1}{\alpha}) \alpha^2 \int_{t_n}^{t_{n+1}} E[\|Z_s - \bar{Z}_{t_n+1}\|^2]ds
\]

\[
+ \alpha' \int_{t_{n+1}}^{t_{n+2}} E[\|\delta Z_{s}^N\|^2]ds \right\}.
\]

Iterating the last inequality, we obtain for all \( n = 0, \ldots, N - 1 \)

\[
E[\|\delta Y_{t_n}^N\|^2] + \frac{1 + \alpha'}{2} \int_{t_n}^{t_{n+1}} E[\|\delta Z_{s}^N\|^2]ds \leq (1 + Ch)^{N-1} \left\{ E[\|\delta Y_T^N\|^2] + Ch(1 + |x|^2) \right\] + C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[\|Z_s - \bar{Z}_{t_n}\|^2]ds + C \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E[\|Z_s - \bar{Z}_{t_{n+1}}\|^2]ds
\]

\[
+ \alpha^2 \int_{t_{N-1}}^{t_N} E[\|Z_s\|^2]ds \right\}.
\]

Using the Assumption \((\text{H2})-(\text{iii})\), we get

\[
E[\|\delta Y_{t_n}^N\|^2] + \frac{1 + \alpha'}{2} \int_{t_n}^{t_{n+1}} E[\|\delta Z_{s}^N\|^2]ds \leq (1 + Ch)^{N-1} \left\{ Ch(1 + |x|^2) \right\] + C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[\|Z_s - \bar{Z}_{t_n}\|^2]ds + C \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E[\|Z_s - \bar{Z}_{t_{n+1}}\|^2]ds
\]

\[
+ \alpha^2 \int_{t_{N-1}}^{t_N} E[\|Z_s\|^2]ds \right\}.
\]

(3.23)
Now we sum up inequality (3.22) over \( n \), we get
\[
\sum_{n=0}^{N-1} E[|\delta Y_{t_n}^N|^2] + \frac{1 + \alpha'}{2} \int_0^T E[|\delta Z_s^N|^2]ds \leq (1 + Ch) \left\{ \sum_{n=0}^{N-1} E[|\delta Y_{t_{n+1}}^N|^2] \right\}
\]
\[+ \sum_{n=0}^{N-1} Ch^2 (1 + |x|^2) + C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds + C \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_{n+1}}||^2]ds
\]
\[+ \alpha^2 E\left[ \int_{t_{N-1}}^{t_N} ||Z_s||^2 ds \right] + (1 + Ch) \alpha' \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E[|\delta Z_s^N|^2]ds.
\]
Using that \( Nh = T \) and \( \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E[|\delta Z_s^N|^2]ds = \int_t^T E[|\delta Z_s^N|^2]ds \), we get
\[
\sum_{n=0}^{N-1} E[|\delta Y_{t_n}^N|^2] + \frac{1 + \alpha'}{2} - (1 + Ch) \alpha' \int_0^T E[|\delta Z_s^N|^2]ds \leq (1 + Ch) \left\{ \sum_{n=0}^{N-1} E[|\delta Y_{t_{n+1}}^N|^2] \right\}
\]
\[+ Ch(1 + |x|^2) + C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds + C \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_{n+1}}||^2]ds
\]
\[+ \alpha^2 E\left[ \int_{t_{N-1}}^{t_N} ||Z_s||^2 ds \right].
\]
Then
\[
\sum_{n=0}^{N-1} E[|\delta Y_{t_n}^N|^2] + \frac{1 + \alpha'}{2} - (1 + Ch) \alpha' \int_0^T E[|\delta Z_s^N|^2]ds \leq (1 + Ch) \left\{ \sum_{n=0}^{N-1} E[|\delta Y_{t_{n+1}}^N|^2] \right\}
\]
\[+ Ch(1 + |x|^2) + C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds + C \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_{n+1}}||^2]ds
\]
\[+ \alpha^2 E\left[ \int_{t_{N-1}}^{t_N} ||Z_s||^2 ds \right].
\]
We obtain for \( h \) small enough
\[
\sum_{n=0}^{N-1} E[|\delta Y_{t_n}^N|^2] + \frac{1 + \alpha'}{2} - (1 + Ch) \alpha' \int_0^T E[|\delta Z_s^N|^2]ds \leq (1 + Ch) \left\{ \sum_{n=0}^{N-1} E[|\delta Y_{t_{n+1}}^N|^2] \right\}
\]
\[+ Ch(1 + |x|^2) + C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds + C \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_{n+1}}||^2]ds
\]
\[+ \alpha^2 E\left[ \int_{t_{N-1}}^{t_N} ||Z_s||^2 ds \right].
\]
Hence, we get
\[
\frac{1 - \alpha'}{2} \int_0^T E[|\delta Z_s^N|^2]ds \leq (1 + Ch) E[|\delta Y_{t_N}^N|^2] + (1 + Ch) - 1 \sum_{n=1}^{N-1} E[|\delta Y_{t_n}^N|^2]
\]
\[+ C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds + C \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_{n+1}}||^2]ds
\]
\[+ \alpha^2 E\left[ \int_{t_{N-1}}^{t_N} ||Z_s||^2 ds \right].
\]
Using Assumption (H2)-(iii) on Φ, we get

\[
\frac{1 - \alpha'}{2} \int_0^T E[\|\delta Z_s^N \|^2]ds \leq Ch(1 + |x|^2) + Ch \sum_{n=1}^{N-1} E[\|\delta Y_{t_n}^N \|^2] + C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds \\
+ C \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds + \alpha^2 E[\int_{t_{N-1}}^{t_N} ||Z_s||^2]ds.
\]

(3.24)

Summing up (3.23) over n, we have

\[
h \sum_{n=0}^{N-1} E[\|\delta Y_{t_n}^N \|^2] \leq Ch(1 + |x|^2) + C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds \\
+ C \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds + C E[\int_{t_{N-1}}^{t_N} ||Z_s||^2]ds.
\]

Plugging the last inequality in (3.24), we obtain

\[
\frac{1 - \alpha'}{2} \int_0^T E[\|\delta Z_s^N \|^2]ds \leq Ch(1 + |x|^2) + C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds \\
+ C \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds + C E[\int_{t_{N-1}}^{t_N} ||Z_s||^2]ds.
\]

(3.25)

Now, turning Back to equation (3.21), we have for all \(n = 0, \ldots, N - 2\)

\[
E[\|\delta Y_t^N \|^2] \leq (1 + Ch)B_n \\
\leq (1 + Ch) \left\{ E[\|\delta Y_{t_{n+1}}^N \|^2] + \alpha' \int_{t_n}^{t_{n+1}} E[\|\delta Z_s^N \|^2]ds \right\} \\
+ Ch E[\|\delta Y_{t_n}^N \|^2] + \left( \frac{1 - \alpha'}{2} \right) \int_{t_n}^{t_{n+1}} E[\|\delta Z_s^N \|^2]ds \\
+ Ch^2 (1 + |x|^2) + C \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds \\
+ \left( 1 + \frac{1}{\epsilon} \right) \alpha^2 \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds, \forall t \in [t_n, t_{n+1}].
\]

Using inequality (3.23), we get

\[
E[\|\delta Y_t^N \|^2] \leq Ch(1 + |x|^2) + C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds \\
+ C \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds + C E[\int_{t_{N-1}}^{t_N} ||Z_s||^2]ds.
\]

Then by taking the supremum over \(t\) in the last inequality, we obtain

\[
\sup_{0 \leq t \leq T} E[\|\delta Y_t^N \|^2] \leq Ch(1 + |x|^2) + C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds \\
+ C \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E[||Z_s - \bar{Z}_{t_n}||^2]ds + C E[\int_{t_{N-1}}^{t_N} ||Z_s||^2]ds.
\]

(3.26)
Equations (3.26) and (3.25) give together
\[
\sup_{0 \leq t \leq T} E[|\delta Y_t^N|^2] + \int_0^T E[|\delta Z_t^N|^2] ds \leq C h(1 + |x|^2) + C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[|Z_s - \bar{Z}_{t_n}|^2] ds \\
+ C \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E[|Z_s - \bar{Z}_{t_{n+1}}|^2] ds + CE \int_{t_{N-1}}^{t_N} E[|Z_s|^2] ds. \tag{3.27}
\]

Plugging $\bar{Z}_{t_n}$, we deduce from Lemma 3.1 that
\[
E \left[ \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} |Z_s - Z_{t_n}^N|^2 ds \right] \leq C E \left[ \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} |Z_s - \bar{Z}_{t_n}|^2 ds \right] \\
+ C E \left[ \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} |\delta Z_{t_n}^N|^2 ds \right].
\]

Using the last inequality in (3.27), we get
\[
\sup_{0 \leq t \leq T} E[|\delta Y_t^N|^2] + E \left[ \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} |Z_s - Z_{t_n}^N|^2 ds \right] \leq C h(1 + |x|^2) + C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[|Z_s - \bar{Z}_{t_n}|^2] ds \\
+ C \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E[|Z_s - \bar{Z}_{t_{n+1}}|^2] ds + CE \int_{t_{N-1}}^{t_N} E[|Z_s|^2] ds
\]

which can be written, if we set $\bar{Z}_{t_N} := 0$
\[
\sup_{0 \leq t \leq T} E[|\delta Y_t^N|^2] + E \left[ \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} |Z_s - Z_{t_n}^N|^2 ds \right] \leq C h(1 + |x|^2) \\
+ C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[|Z_s - \bar{Z}_{t_n}|^2] ds + C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[|Z_s - \bar{Z}_{t_{n+1}}|^2] ds. \tag{3.28}
\]

\[\square\]

4. Path regularity of the process $Z$

The purpose of this section is to prove $L^2$-regularity of the $Z$ component of the solution of the BDSDE (1.2). Such result is crucial to obtain the rate of convergence of our numerical scheme. For this end, we need to introduce the Malliavin derivatives of the solution. This will allow us to provide a representation and a regularity results for $Y$ and $Z$ that will immediately imply the rate of our scheme.

We recall the tools on the Malliavin calculus in the context of BDSDEs introduced in Pardoux and Peng [29]. Pardoux and Peng have skipped details of this part considering that it is just a natural extension of the work on standard BSDEs [28]. For the sake of completeness, we give some details which are crucial to obtain regularity result of the process $Z$ and we give some technical proofs in the Appendix.
4.1. Malliavin calculus on the Forward SDE’s

In this section, we recall some properties on the differentiability in the Malliavin sense of the forward process \((X^{t,x}_r)\). Under (H3(i)), Nualart [26] stated that \(X^{t,x}_s \in \mathbb{D}^{1,2}\) for any \(s \in [t, T]\) and for \(l \leq k\) the derivative \(D^i_sX^{t,x}_s\) is given by:

(i) \(D^i_sX^{t,x}_s = 0\) for \(s < r \leq T\),

(ii) For any \(t < r \leq T\), a version of \(\{D^i_sX^{t,x}_s, r \leq s \leq T\}\) is the unique solution of the linear SDE

\[
D^i_sX^{t,x}_s = \sigma^i(X^{t,x}_r) + \int_r^s \nabla b(X^{t,x}_u)D^i_rX^{t,x}_u \, du + \sum_{i=1}^d \int_r^s \nabla \sigma^i(X^{t,x}_u)D^i_rX^{t,x}_u \, dW^i_u,
\]

where \((\sigma^i)_{i=1,...,k}\) denotes the \(i\)-th column of the matrix \(\sigma\).

Moreover, \(D^i_sX^{t,x}_s \in \mathbb{D}^{1,2}\) for all \(r, s \leq T\). For all \(v \leq T\) and \(l' \leq k\), we have

\[
D^i_vD^j_sX^{t,x}_s = 0 \text{ if } s < v \wedge r,
\]

and for all \(s \geq v \vee r\) a version of \(D^i_vD^j_sX^{t,x}_s\) is the unique solution of the SDE:

\[
D^i_vD^j_sX^{t,x}_s = \nabla \sigma^i(X^{t,x}_r)D^j_vX^{t,x}_r + \sum_{i=1}^d \nabla \sigma^i(X^{t,x}_u)D^j_rX^{t,x}_u \mathbf{1}_{\{t \leq v \leq s\}} + \int_r^s \left[ \sum_{j=1}^k \nabla((\nabla b)^j(X^{t,x}_u))D^i_vX^{t,x}_u(D^j_rX^{t,x}_u)^j + \nabla b(X^{t,x}_u)D^i_vD^j_rX^{t,x}_u \right] \, du
\]

\[
+ \sum_{i=1}^d \int_r^s \left[ \sum_{j=1}^k \nabla \sigma^i(X^{t,x}_u)^j D^i_vX^{t,x}_u(D^j_rX^{t,x}_u)^j + \nabla \sigma^i(X^{t,x}_u)D^i_vD^j_rX^{t,x}_u \right] \, dW^i_u,
\]

where \((\nabla b)^j\) denotes the \(j\)-th column of the matrix \((\nabla b)\) and \((\nabla \sigma^i(X^{t,x}_u)^j)\) denotes the \(j\)-th component of the vector \((D^i_vX^{t,x}_u)\).

The following inequalities will be useful later. For the proofs, we refer to Nualart [26] for example. From Lemma 2.7 in [26] applied to \(X\) and \(D_sX\) and any \(0 \leq r \leq s \leq T\), we have the following inequalities

\[
E\left[ \sup_{0 \leq u \leq T} \|D_sX_u\|^p \right] \leq C(1 + |x|^p),
\]

(4.1)

\[
E\left[ \sup_{s \leq u \leq T} \|D_sX_u - D_rX_u\|^p \right] \leq C|s - r|(1 + |x|^p),
\]

(4.2)

and the similar argument for \(D_rD_sX\) shows that there exists \(C > 0\) such that

\[
E\left[ \sup_{0 \leq u \leq T} \|D_rD_sX_u\|^p \right] \leq C(1 + |x|^{2p}).
\]

(4.3)

4.2. Malliavin calculus for the solution of BDSDE’s

Now, our aim is to study the differentiability in the Malliavin sense of the solution of the BDSDE (2.2). We start with the following lemma which shows that a backward Itô integral is differentiable in the Malliavin sense if and only if its integrand is so. We recall that Pardoux and Peng [28] proved that the result holds for the classical Itô integral.
Lemma 4.1 Let \( U \in \mathbb{H}^2_\mathbb{F}([t, T]) \) and \( I_i(U) = \int_t^T U_r dW_r^i \), \( i = 1, \ldots, d \). Then, for each \( \theta \in [0, T] \) we have \( U_\theta \in \mathbb{D}^{1,2} \) if and only if \( I_i(U) \in \mathbb{D}^{1,2} \), \( i = 1, \ldots, d \) and for all \( \theta \in [0, T] \), we have

\[
D_\theta I_i(U) = \int_\theta^T D_\theta U_r dW_r^i, \quad \theta > t, \\
D_\theta I_i(U) = \int_t^\theta D_\theta U_r dW_r^i, \quad \theta \leq t.
\]

For backward Itô integral, and since the Malliavin derivative is with respect to the brownian motion \( W \), we have the following result:

Lemma 4.2 Let \( U \in \mathbb{H}^2_\mathbb{F}([t, T]) \) and \( I_i(U) = \int_t^T U_r dB_r^i \), \( i = 1, \ldots, l \). Then, for each \( \theta \in [0, T] \) we have \( U_\theta \in \mathbb{D}^{1,2} \) if and only if \( I_i(U) \in \mathbb{D}^{1,2} \), \( i = 1, \ldots, l \) and for all \( \theta \in [0, T] \), we have

\[
D_\theta I_i(U) = \int_\theta^T D_\theta U_r dB_r^i, \quad \theta > t, \\
D_\theta I_i(U) = \int_t^\theta D_\theta U_r dB_r^i, \quad \theta \leq t.
\]

For later use, we need to prove the a priori estimates for the solution of the BDSDE (see [10] for similar estimates for a standard BSDE).

Proposition 4.1 Let \( (\phi^1, f^1, g^1) \) and \( (\phi^2, f^2, g^2) \) be two standard parameters of the BDSDE (2.2) and \( (Y^1, Z^1) \) and \( (Y^2, Z^2) \) the associated solutions. Assume that Assumption (H2) holds. For \( s \in [t, T] \), set \( \delta Y_s := Y_s^1 - Y_s^2 \) and \( \delta Z_s := f^1(s, X_s, Y_s^2, Z_s^2) - f^2(s, X_s, Y_s^1, Z_s^1) \). Then, we have

\[
\|\delta Y_t\|_{\mathbb{D}^{2}(t, T)}^2 + \|\delta Z_t\|_{\mathbb{D}^{2}(t, T)}^2 \leq C \int_t^T |\delta_2 f_s|^2 \, ds + \int_t^T |\delta_2 g_s|^2 \, ds, \tag{4.4}
\]

where \( C \) is a positive constant depending only on \( K, T \) and \( \alpha \).

Proof. The proof of this result is classical in the BSDE’s theory (see El Karoui et al.[10]) so we omit it. \( \square \)

Now, we study the differentiability in the Malliavin sense of the solution of the BDSDE which is technical. To our knowledge, it does not exist in the literature. We have to precise that Pardoux and Peng [29] have skipped details considering that it was just an easy extension of the work on standard BSDEs [28]. We show that the derivative is a solution of a linear BDSDE as Peng and Pardoux [28] have given for the standard BSDE’s, see also El Karoui Peng and Quenez ([10], Proposition 5.3)).

Proposition 4.2 Assume that (H1)-(H3) hold. For any \( t \in [0, T] \) and \( x \in \mathbb{R}^d \), let \( \{Y_s, Z_s, t \leq s \leq T\} \) denotes the unique solution of the BDSDE:

\[
Y_s = \Phi(X_{T-s}^x) + \int_s^T f(r, X_{T-r}^x, Y_r, Z_r)dr + \int_s^T g(r, X_{T-r}^x, Y_r, Z_r)dB_r - \int_s^T Z_r dW_r, \quad t \leq s \leq T.
\]

Then, \( (Y, Z) \in \mathcal{B}^2([t, T], \mathbb{D}^{1,2}) \) and \( \{D_\theta Y_s, D_\theta Z_s; t \leq s, \theta \leq T\} \) is given by:

(i) \( D_\theta Y_s = 0, D_\theta Z_s = 0 \) for all \( t \leq s < \theta \leq T \)
(ii) for any fixed $\theta \in [t,T]$, $\theta \leq s \leq T$ and $1 \leq i \leq d$, a version of $(D_0^i Y_s, D_0^i Z_s)$ is the unique solution of the BDSDE:

$$
D_0^i Y_s = \nabla \Phi(X_T^t, X_T^t)D_0^i X_T^t + \int_s^T \left( \nabla_x f(r, X_r^t, Y_r, Z_r)D_0^i X_r^t \right)dr \\
+ \int_s^T \left( \nabla_y f(r, X_r^t, Y_r, Z_r)D_0^i Y_r + \sum_{j=1}^d \nabla_{z,j} f(r, X_r^t, Y_r, Z_r)D_0^i Z_r^j \right)dr \\
+ \sum_{n=1}^l \int_s^T \left( \nabla_x g^n(r, X_r^t, Y_r, Z_r)D_0^i X_r^t + \nabla_y g^n(r, X_r^t, Y_r, Z_r)D_0^i Y_r \right)dB_r^n \\
+ \sum_{n=1}^l \int_s^T \sum_{j=1}^d \left( \nabla_{z,j} g^n(r, X_r^t, Y_r, Z_r)D_0^i Z_r^j \right)dB_r^n - \int_s^T \sum_{j=1}^d D_0^i Z_r^j dB_r^j,
$$

(4.5)

where $(z^j)_{1 \leq j \leq d}$ denotes the $j$-th column of the matrix $z$, $(g^n)_{1 \leq n \leq l}$ denotes the $n$-th column of the matrix $g$ and $B = (B^1, \ldots, B^l)$.

**Proof.** See Appendix. The second order differentiability in the Malliavin sense of the solution of the BDSDE will be given in Appendix.

### 4.3. Representation results for BDSDEs

In this subsection, we will prove a representation result of $(Z, DZ)$ which will be useful to prove the rate of convergence of our numerical scheme.

**Proposition 4.3** Assume that (H1)-(H3) hold. Then : For $t \leq s \leq T$, we have

$$
D_s Y_s = Z_s,
$$

(4.6)

and

$$
||Z||_{L_{x,d}^2([t,T])}^2 \leq C(1 + ||x||^2).
$$

(4.7)

For $l_1, l_2 \leq d$, $t \leq s \leq T$, we have

$$
D^{l_1}_s D^{l_2}_s Y_s = D^{l_2}_s Z^{l_1}_s,
$$

(4.8)

and

$$
||D^{l_1}_s Z||_{L_{x,d}^2([t,T])}^2 \leq C(1 + ||x||^4).
$$

(4.9)

**Proof.** To simplify the notations, we restrict ourselves to the case $k = d = 1$.

1. Notice that for $t \leq s$

$$
Y_s = Y_t - \int_t^s f(r, \Sigma_r)dr - \int_t^s g(r, \Sigma_r)dB_r + \int_t^s Z_r dW_r,
$$

where $\Sigma_r := (X_r^t, Y_r, Z_r)$.

It follows from Lemma 4.1 and Lemma 4.2 that, for $t < \theta \leq s$

$$
D_\theta Y_s = Z_\theta - \int_\theta^s \left( \nabla_x f(r, \Sigma_r)D_\theta X_r + \nabla_y f(r, \Sigma_r)D_\theta Y_r + \nabla_z f(r, \Sigma_r)D_\theta Z_r \right)dr \\
- \int_\theta^s \left( \nabla_x g(r, \Sigma_r)D_\theta X_r + \nabla_y g(r, \Sigma_r)D_\theta Y_r + \nabla_z g(r, \Sigma_r)D_\theta Z_r \right)dB_r + \int_\theta^s D_\theta Z_r dW_r.
$$
Then by taking \( \theta = s \), it follows that equality (4.6) holds. From (7.1), we deduce that (4.7) holds.

2. Notice that for \( \theta \leq t \leq s \)

\[
D_y Y_s = D_y Y_t - \int_t^s \left( \nabla_x f(r, \Sigma_r) D_y X_r + \nabla_y f(r, \Sigma_r) D_y Y_r + \nabla_z f(r, \Sigma_r) D_y Z_r \right) dr
- \int_t^s \left( \nabla_x g(r, \Sigma_r) D_y X_r + \nabla_y g(r, \Sigma_r) D_y Y_r + \nabla_z g(r, \Sigma_r) D_y Z_r \right) \overline{dB}_r + \int_t^s D_y Z_r dW_r.
\]

It follows from Lemma 4.1 and Lemma 4.2 that, for \( \theta \leq t < v \leq s \)

\[
D_v D_y Y_s = D_v D_y Z_v - \int_v^s D_v (\Sigma_r)^* [H f](r, \Sigma_r) D_y (\Sigma_r) dr - \int_v^s \nabla f(r, \Sigma_r) D_v D_y (\Sigma_r) dr
- \int_v^s D_v (\Sigma_r)^* [H g](r, \Sigma_r) D_y (\Sigma_r) \overline{dB}_r - \int_v^s \nabla g(r, \Sigma_r) D_v D_y (\Sigma_r) \overline{dB}_r
+ \int_v^s D_v D_y Z_r dW_r.
\]

Then by taking \( v = s \) and \( t = \theta \), it follows that equality (4.8) holds. We have from estimate (2.4) and inequality (4.3), that for each \( v \leq T \) and \( \theta \leq T \)

\[
E \left[ \sup_{t \leq s \leq T} |D_v D_y Y_s|^2 \right] + E \int_t^T |D_v D_y Z_s|^2 ds \leq C(1 + |x|^2).
\]

(4.10)

and then by taking \( v = s \) and \( t = \theta \) we deduce that (4.9) holds.

\[\Box\]

### 4.4. Path regularity

In this subsection, we extend the result of Zhang [33] which concerns the \( L^2 \)-regularity of the martingale integrand \( Z \). Such result is crucial to derive the rate of convergence of our numerical scheme. We start with the following proposition which gives an upper bound for

\[
E \left[ \sup_{r \in [s, u]} |Y_r - Y_s|^2 \right] \quad \text{and} \quad E \left[ ||Z_u - Z_s||^2 \right], \quad t \leq s \leq u \leq T.
\]

**Proposition 4.4** Assume that (H1)-(H3) hold. Then for \( t \leq s \leq u \leq T \), we have

\[
E \left[ \sup_{r \in [s, u]} |Y_r - Y_s|^2 \right] \leq C(1 + |x|^2)|u - s|, \quad (4.11)
\]

\[
E \left[ ||Z_u - Z_s||^2 \right] \leq C(1 + |x|^2)|u - s|. \quad (4.12)
\]

**Proof.** To simplify the notations, we restrict ourselves to the case \( k = d = l = 1 \).

(i) Plugging inequality (4.7) in the estimate (7.12), the result (4.11) holds.

(ii) From Proposition 4.3, we have

\[
E \left[ |Z_u - Z_s|^2 \right] \leq CE||D_u Y_u - D_s Y_s|^2] + CE|D_s Y_u - D_s Y_s|^2]. \quad (4.13)
\]
From the definition of the BDSDE (4.5), we have
\[
D_u Y_u - D_s Y_u \triangleq \nabla \Phi(X_T)(D_u X_T - D_s X_T) + \int_u^T \left( \nabla_x f(r, \Sigma_r)(D_u X_r - D_s X_r) \right) \, dr
\]
\[
+ \int_u^T \left( \nabla_y f(r, \Sigma_r)(D_u Y_r - D_s Y_r) + \nabla_z f(r, \Sigma_r)(D_u Z_r - D_s Z_r) \right) \, dr
\]
\[
+ \int_u^T \left( \nabla_x g(r, \Sigma_r)(D_u X_r - D_s X_r) + \nabla_y g(r, \Sigma_r)(D_u Y_r - D_s Y_r) \right) \, dB_r
\]
\[
+ \int_u^T \left( \nabla_z g(r, \Sigma_r)(D_u Z_r - D_s Z_r) \right) \, dB_r - \int_u^T (D_u Z_r - D_s Z_r) \, dW_r.
\]
Applying the generalized Itô’s formula (see [29], Lemma 1.3), we obtain
\[
\left| D_u Y_T - D_s Y_T \right|^2 - \left| D_u Y_u - D_s Y_u \right|^2 =
\]
\[
- 2 \int_u^T \nabla_x f(r, \Sigma_r)(D_u X_r - D_s X_r)(D_u Y_r - D_s Y_r) \, dr - 2 \int_u^T \nabla_y f(r, \Sigma_r)(D_u Y_r - D_s Y_r)^2 \, dr
\]
\[
- 2 \int_u^T \nabla_z f(r, \Sigma_r)(D_u Z_r - D_s Z_r)(D_u Y_r - D_s Y_r) \, dr
\]
\[
- 2 \int_u^T \nabla_x g(r, \Sigma_r)(D_u X_r - D_s X_r)(D_u Y_r - D_s Y_r)^2 \, dB_r
\]
\[
- 2 \int_u^T \nabla_y g(r, \Sigma_r)(D_u Y_r - D_s Y_r)^2 \, dB_r
\]
\[
- 2 \int_u^T \nabla_z g(r, \Sigma_r)(D_u Z_r - D_s Z_r)(D_u Y_r - D_s Y_r)^2 \, dB_r
\]
\[
+ 2 \int_u^T (D_u Z_r - D_s Z_r)(D_u Y_r - D_s Y_r) \, dW_r
\]
\[
- \int_u^T \left| \nabla_x g(r, \Sigma_r)(D_u X_r - D_s X_r) + \nabla_y g(r, \Sigma_r)(D_u Y_r - D_s Y_r) + \nabla_z g(r, \Sigma_r)(D_u Z_r - D_s Z_r) \right|^2 \, dr
\]
\[
+ \int_u^T \left| D_u Z_r - D_s Z_r \right|^2 \, dr.
\]
From inequalities (7.1) and (4.1), using the Burkholder-Davis-Gundy’s inequality and Assumption (H2), the stochastic integrals which appear in the last equation disappear when we take the
expectation. By Young inequality, we obtain, for $\epsilon' > 0$

\[
E[|D_u Y_u - D_s Y_u|^2] + E\left[ \int_u^T |D_u Z_r - D_s Z_r|^2 \, dr \right] \leq E[|\nabla \Phi(X_T) (D_u X_T - D_s X_T)|^2]
\]

\[+
2E\left[ \int_u^T \nabla_x f(r, \Sigma_r) (D_u X_r - D_s X_r) (D_u Y_r - D_s Y_r) \, dr \right]
\]

\[+
2E\left[ \int_u^T \nabla_y f(r, \Sigma_r) (D_u Y_r - D_s Y_r)^2 \, dr \right]
\]

\[+
2E\left[ \int_u^T \nabla_z f(r, \Sigma_r) (D_u Z_r - D_s Z_r) (D_u Y_r - D_s Y_r) \, dr \right]
\]

\[+
C(1 + \frac{1}{\epsilon'}) E\left[ \int_u^T \nabla_x g(r, \Sigma_r)^2 |D_u X_r - D_s X_r|^2 \, dr \right]
\]

\[+
C(1 + \frac{1}{\epsilon'}) E\left[ \int_u^T \nabla_y g(r, \Sigma_r)^2 |D_u Y_r - D_s Y_r|^2 \, dr \right]
\]

\[+
(1 + \epsilon') E\left[ \int_u^T \nabla_z g(r, \Sigma_r)^2 |D_u Z_r - D_s Z_r|^2 \, dr \right].
\]

Hence by using Assumption (H2) and Young inequality, we have for $\epsilon, \epsilon' > 0$ and $C > 0$,

\[
E[|D_u Y_u - D_s Y_u|^2] + E\left[ \int_u^T |D_u Z_r - D_s Z_r|^2 \, dr \right] \leq K^2 E[|D_u X_T - D_s X_T|^2]
\]

\[+
2KE\left[ \int_u^T |D_u X_r - D_s X_r|^2 \, dr \right] + 4KE\left[ \int_u^T |D_u Y_r - D_s Y_r|^2 \, dr \right]
\]

\[+
K\epsilon E\left[ \int_u^T |D_u Y_r - D_s Y_r|^2 \, dr \right] + \frac{K}{\epsilon} E\left[ \int_u^T |D_u Z_r - D_s Z_r|^2 \, dr \right]
\]

\[+
CK^2(1 + \frac{1}{\epsilon'}) E\left[ \int_u^T |D_u X_r - D_s X_r|^2 \, dr \right] + CK^2(1 + \frac{1}{\epsilon'}) E\left[ \int_u^T |D_u Y_r - D_s Y_r|^2 \, dr \right]
\]

\[+
(1 + \epsilon')\alpha^2 E\left[ \int_u^T |D_u Z_r - D_s Z_r|^2 \, dr \right].
\]

Then, we obtain

\[
E[|D_u Y_u - D_s Y_u|^2] + E\left[ \int_u^T |D_u Z_r - D_s Z_r|^2 \, dr \right] \leq K^2 E[|D_u X_T - D_s X_T|^2]
\]

\[+
K(2 + KC(1 + \frac{1}{\epsilon'})) E\left[ \int_u^T |D_u X_r - D_s X_r|^2 \, dr \right]
\]

\[+
(K^2 C(1 + \frac{1}{\epsilon'}) + (4 + \epsilon)K) E\left[ \int_u^T |D_u Y_r - D_s Y_r|^2 \, dr \right]
\]

\[+
((1 + \epsilon')\alpha^2 + \frac{K}{\epsilon}) E\left[ \int_u^T |D_u Z_r - D_s Z_r|^2 \, dr \right].
\]

For $\epsilon$ large enough and $\epsilon'$ small enough, we have $(1 + \epsilon')\alpha^2 + \frac{K}{\epsilon} < 1$. From inequality (4.2), we deduce that

\[
E[|D_u Y_u - D_s Y_u|^2] \leq C \left( (1 + |x|^2)|u - s| + E\left[ \int_u^T |D_u Y_r - D_s Y_r|^2 \, dr \right] \right),
\]

where $C$ is a positive constant. From Gronwall’s lemma we have

\[
E[|D_u Y_u - D_s Y_u|^2] \leq C(1 + |x|^2)|u - s|. \tag{4.14}
\]
Since \((D,Y_u)_{s\leq u\leq T}\) satisfies the BDSDE (4.5), inequalities (7.12)-(4.7) hold for \((D,Y,Du)_{s\leq u\leq T}\) and yield
\[
E[|D_sY_u - D_sY_s|^2] \leq C(1 + |x|^2)|u - s|. \tag{4.15}
\]
Plugging (4.14) and (4.15) into (4.13), we obtain (4.12).

The following theorem states the rate of convergence of our numerical scheme.

**Theorem 4.1** Under Assumptions (H1)-(H3), there exists a positive constant \(C\) (depending only on \(T\), \(K\), \(\alpha\), \(|b(0)|\), \(|\sigma(0)|\), \(|f(t,0,0,0)|\) and \(|g(t,0,0,0)|\)) such that
\[
\text{Error}_N(Y,Z) \leq C(1 + |x|^2). \tag{4.16}
\]

**Proof.** From the definition (3.1), \(\bar{Z}_{t_n}\) is the best approximation of \((Z_t)_{t_n \leq t < t_{n+1}}\) by \(F_{t_n}\)-measurable random variable in the following sense
\[
E\left[\int_{t_n}^{t_{n+1}} \|Z_s - \bar{Z}_{t_n}\|^2 \, ds\right] = \inf_{Z_n \in L^2(\Omega,F_{t_n})} E\left[\int_{t_n}^{t_{n+1}} \|Z_s - Z_n\|^2 \, ds\right]
\]
which implies
\[
E[\|Z_s - \bar{Z}_{t_n}\|^2] \leq E[\|Z_s - Z_{t_n}\|^2].
\]
From Proposition 4.4, we have
\[
E[\|Z_s - Z_{t_n}\|^2] \leq C(1 + |x|^2)|s - t_n| \leq Ch(1 + |x|^2),
\]
for all \(s \in [t_n,t_{n+1}]\) and \(0 \leq n \leq N - 1\) where \(C\) depends only on \(T\), \(K\), \(b(0)\), \(\sigma(0)\), \(f(t,0,0,0)\) and \(g(t,0,0,0)\). Then
\[
\sum_{n=0}^{N-1} E\left[\int_{t_n}^{t_{n+1}} \|Z_s - \bar{Z}_{t_n}\|^2 \, ds\right] \leq Ch(1 + |x|^2).
\]
Similarly, from Proposition 4.4 we get
\[
\sum_{n=0}^{N-2} E\left[\int_{t_n}^{t_{n+1}} \|Z_s - \bar{Z}_{t_{n+1}}\|^2 \, ds\right] \leq Ch(1 + |x|^2).
\]
Finally, using the same argument again, we obtain
\[
E\left[\int_{t_{N-1}}^{t_N} \|Z_s\|^2 \, ds\right] \leq Ch(1 + |x|^2).
\]
Then, from Theorem 3.1
\[
\text{Error}_N(Y,Z) \leq Ch(1 + |x|^2).
\]

**Remark 4.1** One could define the error as follows
\[
\tilde{\text{Error}}_N(Y,Z) := \sup_{0 \leq s \leq T} E[|Y_s - \bar{Y}_n|^2] + \sum_{n=0}^{N-1} E\left[\int_{t_n}^{t_{n+1}} \|Z_s - Z_s^n\|^2 \, ds\right]. \tag{4.17}
\]
Then, we have
\[
\tilde{\text{Error}}_N(Y,Z) \leq Ch(1 + |x|^2). \tag{4.18}
\]
5. Numerical scheme for the weak solution of the SPDE

Most numerical work on SPDEs has concentrated on the Euler finite-difference scheme (see [14], [15], [13]), on finite element method (see [32]) and also on spectral Galerkin methods (see [16] and the references therein). Here, we follow a probabilistic method based on the Feynman-Kac’s formula for the weak solution of the semilinear SPDE’s (1.1) based on BSDE’s approach (see [5], [25]). We consider a weak Sobolev solution of such SPDE in the sense that \( u \) shall be considered as a predictable process in some first order Sobolev space. Therefore, we shall improve the convergence and the rate of convergence of the \( L^2 \)-norm error of such solution by using the convergence results on BDSDEs proved in section 4 and an equivalence norm result given in Barles and Lesigne [4] and Bally and Matoussi [5].

5.1. Weak solution for SPDE

Since we work on the whole space \( \mathbb{R}^d \), we introduce a weight function \( \rho \) satisfying the following conditions: \( \rho \) is a positive locally integrable function, \( \frac{1}{\rho} \) are locally integrable and \( \int_{\mathbb{R}^d}(1+|x|^2)\rho(x)dx < \infty \). For example, we can take \( \rho(x) = e^{-|x|^2} \) or \( \rho(x) = e^{-|x|} \). As a consequence of (H3), we have
\[
\int_{\mathbb{R}^d} |\Phi(x)|^2 \rho(x)dx < \infty, \quad \int_0^T \int_{\mathbb{R}^d} |f(t,x,0,0)|^2 \rho(x)dxdt < \infty \quad \text{and} \quad \int_0^T \int_{\mathbb{R}^d} |g(t,x,0,0)|^2 \rho(x)dxdt < \infty.
\]

We denote by \( L^2(\mathbb{R}^d, \rho(x)dx) \) the weighted Hilbert space and we employ the following notation for its scalar product and its norm: \( (u,v)_\rho = \int_{\mathbb{R}^d} u(x)v(x)\rho(x)dx \) and \( \|u\|_\rho = (u,u)_\rho^{\frac{1}{2}} \). Then, we define by \( H^1_\rho(\mathbb{R}^d) \) the associated weighted first order Dirichlet space and its norm \( \|u\|_{H^1_\rho(\mathbb{R}^d)} = (\|u\|_\rho^2 + \|\nabla u\|_\rho^2)^{\frac{1}{2}} \). Finally, \( (.,.) \) denotes the usual scalar product in \( L^2(\mathbb{R}^d, dx) \).

We define also \( \mathcal{D} := C^\infty_c([0,T]) \otimes C^2_c(\mathbb{R}^d) \) the space of test functions where \( C^\infty_c([0,T]) \) denotes the space of all real valued infinite differentiable functions with compact support in \([0,T]\) and \( C^2_c(\mathbb{R}^d) \) the set of \( C^2 \)-functions with compact support in \( \mathbb{R}^d \).

We introduce \( \mathcal{H}_T \) the space of predictable processes \( (u_t)_{t \geq 0} \) with values in \( H^1_\rho(\mathbb{R}^d) \) such that
\[
\|u\|_T = \left( E\left[ \sup_{0 \leq t \leq T} \|u_t\|_\rho^2 \right] + E\left[ \int_0^T \|\nabla u_t\|_\rho^2 dt \right] \right)^{\frac{1}{2}} < \infty.
\]

Définition 5.1 We say that \( u \in \mathcal{H}_T \) is a weak solution of the equation (1.1) associated with the terminal condition \( \Phi \) and the coefficients \( (f,g) \), if the following relation holds almost surely, for each \( \varphi \in \mathcal{D} \)
\[
\int_0^T (u(s,.), \partial_s \varphi(s,.))ds + \int_0^T \mathcal{E}(u(s,.), \varphi(s,.))ds + (u(t,.), \varphi(t,.)) - (\Phi(.), \varphi(T,.)) \quad (5.1)
\]
\[
= \int_0^T (f(s,.u(s,.), (\nabla u\sigma) (s,.)), \varphi(s, .))ds + \sum_{i=1}^l \int_0^T (g(s,.u(s,.), (\nabla u\sigma) (s,.)), \varphi(s,.))dB^i_s,
\]
where \( \mathcal{E}(u, \varphi) = \langle Lu, \varphi \rangle = \int_{\mathbb{R}^d}((\nabla u\sigma)(\nabla \varphi) + \varphi \nabla((\frac{1}{2}\sigma^* \sigma + b)u)))(x)dx \) is the energy associated to the diffusion operator..

From Bally and Matoussi [5], we have the following result:
From the Markov property of \( H_T \) of the SPDE (1.1). Moreover, \( u(t,x) = Y_t^{t,x} \) and \( Z_t^{t,x} = \nabla u_t \sigma, dt \otimes dx \otimes dP \) a.e. where \((Y_s^{t,x}, Z_s^{t,x})_{t \leq s \leq T} \) is the solution of the BDSDE (1.2). Furthermore, we have for all \( s \in [t,T] \), \( u(s, X_s^{t,x}) = Y_s^{t,x} \) and \((\nabla u)(s, X_s^{t,x}) = Z_s^{t,x} \) \( dt \otimes dx \otimes dP \) a.e.

5.2. Numerical Scheme for SPDE

Let us first recall that \((X^N, Y^N, Z^N)\) denotes the numerical Euler scheme of the FBDSDE’s (1.2) given in (2.6)-(2.7)-(2.8)-(2.9). The numerical approximation of the SPDE (1.1) will be presented in the following lemma:

Lemma 5.1 Let \( x \in \mathbb{R}^d \) and \( t, t_n \in \pi \) such that \( t \leq t_n \). Define

\[
u_{t_n}^N(x) := Y_{t_n}^{N,t_n,x} \quad \text{and} \quad v_{t_n}^N(x) := Z_{t_n}^{N,t_n,x}
\]

(5.2)

Then \( u_{t_n}^N \) (resp. \( v_{t_n}^N \)) is \( F_{t_n}^{B,T} \)-measurable and we have

\[
u_{t_n}^N(X_{t_n}^{t,x}) = Y_{t_n}^{N,t_n,x} \quad \text{resp.} \quad v_{t_n}^N(X_{t_n}^{t,x}) = Z_{t_n}^{N,t_n,x}.
\]

Proof. From the Markov property of \( Y^N \) and \( Z^N \), the random variables \( u_{t_n}^N \) and \( v_{t_n}^N \) are \( F_{t_n}^{B,T} \)-measurable. From the definition of \( u_{t_n}^N \) and \( v_{t_n}^N \), we have

\[
u_{t_n}^N(X_{t_n}^{t,x}) = Y_{t_n}^{N,t_n,x} \quad \text{and} \quad v_{t_n}^N(X_{t_n}^{t,x}) = Z_{t_n}^{N,t_n,x}.
\]

From (2.8), (2.9) and by taking \((t,x) = (t_n, X_{t_n}^{t,x})\), we obtain:

\[
u_{t_n}^N(X_{t_n}^{t,x}) = E_{t_n}[Y_{t_n+1}^{N,t_n,x}X_{t_n+1}^{t,x}] + hE_{t_n}[f(t_n, X_{t_n}^{N,t_n,x}, Y_{t_n}^{N,t_n,x}, Z_{t_n}^{N,t_n,x})]
\]

\[+ E_{t_n}[g(t_{n+1}, X_{t_{n+1}}^{N,t_{n+1},x}, Y_{t_{n+1}}^{N,t_{n+1},x}, Z_{t_{n+1}}^{N,t_{n+1},x})dB_n] \]

\[hZ_{t_n}^{N,t_n,x} = E_{t_n}[Y_{t_n+1}^{N,t_n,x}X_{t_n+1}^{t,x} \Delta W_n^* + g(t_{n+1}, X_{t_{n+1}}^{N,t_{n+1},x}, Y_{t_{n+1}}^{N,t_{n+1},x}, Z_{t_{n+1}}^{N,t_{n+1},x})dB_n \Delta W_n^*].
\]

From the flow property, we have \( X_{t_n}^{N,t_n,x} = X_{t_n,x}^{N,x} \), then we obtain

\[
u_{t_n}^N(X_{t_n}^{t,x}) = E_{t_n}[Y_{t_n+1}^{N,t_n,x}] + hE_{t_n}[f(t_n, X_{t_n}^{N,t_n,x}, Y_{t_n}^{N,t_n,x}, Z_{t_n}^{N,t_n,x})]
\]

\[+ E_{t_n}[g(t_{n+1}, X_{t_{n+1}}^{N,t_{n+1},x}, Y_{t_{n+1}}^{N,t_{n+1},x}, Z_{t_{n+1}}^{N,t_{n+1},x})dB_n] \]

\[hZ_{t_n}^{N,t_n,x} = E_{t_n}[Y_{t_n+1}^{N,t_n,x} \Delta W_n^* + g(t_{n+1}, X_{t_{n+1}}^{N,t_{n+1},x}, Y_{t_{n+1}}^{N,t_{n+1},x}, Z_{t_{n+1}}^{N,t_{n+1},x})dB_n \Delta W_n^*].
\]

Then from the uniqueness of the solution of (2.8)-(2.9) we obtain the result.

5.3. Rate of convergence for the weak solution of SPDEs

We give a norm equivalence result which was already proved by Barles and Lesigne [4] and Bally and Matoussi [5] when \( b \in C^1_{b}(\mathbb{R}^d, \mathbb{R}^d) \) and \( \sigma \in C^2(\mathbb{R}^d, \mathbb{R}^{d \times d}) \). We note that \( \sigma \) can be degenerated and so we do not assume ellipticity condition.
Proposition 5.1 Under Assumptions (H1) – (H3), there exist two positive constants $C_1$ and $C_2$ such that for every $t \leq s \leq T$ and $\phi \in L^1(\mathbb{R}^d \times \Omega_B, \rho(x)dx \otimes dP)$, we have
\[ C_1 \int_{\mathbb{R}^d} E_B[|\phi(x)||\rho(x)dx] \leq \int_{\mathbb{R}^d} E[|\phi(X_{s,t}^{t,x})||\rho(x)dx] \leq C_2 \int_{\mathbb{R}^d} E_B[|\phi(x)||\rho(x)dx], \tag{5.3} \]
where $E_B$ is the expectation under $P_B$. Moreover, for every $\Psi \in L^1(\mathbb{R}^d \times (0,T) \times \Omega_B, \rho(x)dx \otimes dt \otimes dP_B)$
\[ C_1 \int_{\mathbb{R}^d} \int_t^T E_B[|\Psi(s,x)||d\rho(x)dx] \leq \int_{\mathbb{R}^d} \int_t^T E[|\Psi(s,X_{s,t}^{t,x})||d\rho(x)dx] \leq C_2 \int_{\mathbb{R}^d} \int_t^T E_B[|\Psi(x)||d\rho(x)dx]. \tag{5.4} \]

We recall that $u(t,x) = Y_t^{t,x}$ and $v(t,x) = Z_t^{t,x}$ $dt \otimes dx \otimes dP$ a.e. We define the process $(u_N^s, v_N^s)$ as follows:
\[ u_N^s(x) := Y_s^{N,s,x} \text{ and } v_N^s(x) := Z_s^{N,s,x}, \forall s \in [t_n, t_{n+1}). \tag{5.5} \]

Using (2.10) and following the proof of Lemma 5.1, we obtain
\[ u_N^s(X_{s,t}^{t,x}) = Y_s^{N,t,x} \text{ and } v_N^s(X_{s,t}^{t,x}) = Z_s^{N,t,x}, \forall t \leq s, s \in [t_n, t_{n+1}). \tag{5.6} \]
As in Gyongy and Krylov [13], we define the error between the solution of the SPDE and the numerical scheme as follows:
\begin{align*}
\text{Error}_N(u,v) &:= \sup_{0 \leq s \leq T} E_B[\int_{\mathbb{R}^d} |u_N^s(x) - u(s,x)|^2 \rho(x)dx] \\
&\quad + \sum_{n=0}^{N-1} E_B[\int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} |v_N^s(x) - v(s,x)|^2 d\rho(x)dx]. \tag{5.7}
\end{align*}
The following theorem shows the convergence of the numerical scheme (5.2) of the solution of the SPDE (1.1).

Theorem 5.2 Assume that (H1)-(H3) hold. Then, the error $\text{Error}_N(u,v)$ converges to 0 as $N \rightarrow \infty$ and there exists a positive constant $C$ (depending only on $T$, $K$, $\alpha$, $|b(0)|$, $||\sigma||$, $|f(t,0,0,0)|$ and $||g(t,0,0,0)||$) such that
\[ \text{Error}_N(u,v) \leq Ch. \tag{5.8} \]

Proof. We take $t = t_0$. From the norm equivalence result (see inequality (5.3)), for all $s \in [t_n, t_{n+1})$ such that $s \geq t$, we have
\[ E_B[\int_{\mathbb{R}^d} |u_N^s(x) - u(s,x)|^2 \rho(x)dx] \leq CE[\int_{\mathbb{R}^d} |u_N^s(X_{s,t}^{t,x}) - u(s,X_{s,t}^{t,x})|^2 \rho(x)dx], \]
where $C$ is positive generic constant. From equation (5.6), we get
\[ E_B[\int_{\mathbb{R}^d} |u_N^s(x) - u(s,x)|^2 \rho(x)dx] \leq C \int_{\mathbb{R}^d} E[|Y_s^{N,t,x} - Y_s^{t,x}|^2] \rho(x)dx. \]
Therefore Remark 4.1 implies that
\[
\sup_{0 \leq s \leq T} EB \int_{\mathbb{R}^d} |u^N_s(x) - u(s,x)|^2 \rho(x) dx \leq Ch \int_{\mathbb{R}^d} (1 + |x|^2) \rho(x) dx \leq Ch. \tag{5.9}
\]
From the norm equivalence result (see inequality (5.4)), we have
\[
\sum_{n=0}^{N-1} E_B \left[ \int_{t_n}^{t_{n+1}} \int_{\mathbb{R}^d} \|u^N_s(x) - v(s,x)\|^2 \rho(x) dx ds \right] 
\leq C \sum_{n=0}^{N-1} E \left[ \int_{t_n}^{t_{n+1}} \|u^N_s(X^t,x) - v(s,X^t_s,x)\|^2 \rho(x) dx ds \right].
\]
From equation (5.6), we get
\[
\sum_{n=0}^{N-1} E \left[ \int_{t_n}^{t_{n+1}} \|v^N_s(X^t,x) - v(s,X^t_s,x)\|^2 \rho(x) dx ds \right] 
= \sum_{n=0}^{N-1} E \left[ \int_{t_n}^{t_{n+1}} \|Z^N_s - Z_s\|^2 \rho(x) dx ds \right],
\]
and so from Remark 4.1 we deduce that
\[
\sum_{n=0}^{N-1} E_B \left[ \int_{t_n}^{t_{n+1}} \int_{\mathbb{R}^d} \|v^N_s(x) - v(s,x)\|^2 \rho(x) dx ds \right] \leq Ch \int_{\mathbb{R}^d} (1 + |x|^2) \rho(x) dx \leq Ch. \tag{5.10}
\]
From inequalities (5.9) and (5.10), we deduce that (5.8) holds. \hfill \Box

**Remark 5.1** Gyongy and Krylov [13] considered the following linear SPDE on $[0, T] \times \mathbb{R}^d$,
\[
\begin{align*}
    du(t, x) &= (L_1 u(t, x) + f(t, x)) dt + \sum_{i=1}^{\infty} (L_{2,i} u(t, x) + g(t, x)_i) dw^i_t, \\ u(0, x) &= u_0 \in L^2(\Omega, P),
\end{align*}
\]
where $L_1 u(t, x) = \sum_{q,l=1}^{d} a(t, x)_{il} \frac{\partial^2 u(t, x)}{\partial x_l \partial x_q}$, $L_{2,i} u(t, x) = \sum_{q=1}^{d} b_{iq}(t, x) \frac{\partial u(t, x)}{\partial x_q}, 1 \leq i \leq \infty$ and $(b(t, x)_i)_i \in \ell^2$, $(t, x) \in [0, T] \times \mathbb{R}^d, 1 \leq q \leq d$. They approximate the SPDE by
\[
\begin{align*}
    du^h(t, x) &= (L^h_1 u^h(t, x) + f(t, x)) dt + \sum_{i=1}^{\infty} (L^h_{2,i} u^h(t, x) + g(t, x)_i) dw^i_t, \\
\end{align*}
\]
\(L^h_1, L^h_{2,i}\) are the approximation of \(L_1, L_{2,i}\) by using finite difference scheme on the space grid \(G_h\). Their results revolve to prove the existence of the random process \( u^{(j)}(t, x), j = 1, \ldots, k \) for some \( k \geq 0 \) s.t.
\[
    u^h(t, x) = u^{(0)}(t, x) + \sum_{j=1}^{k} \frac{h^j}{j!} u^{(j)}(t, x) + R^h(t, x),
\]
where \( u^{(0)} \) is the solution of the SPDE. They assumed that the SPDE is non degenerate and for \( m > k + 1 + \frac{d}{2} \), the coefficients are \( m \)-times continuously differentiable in \( x \). When they used a
symmetric finite difference scheme and \(d = 2\), the \(L^2\)-error is proportional to \(h^2\) where \(h\) is the discretization step in space and by the Richardson acceleration, the error is proportional to \(h^4\).

Compared to their work, our scheme is more general. It converges in the non linear case. Our convergence is of order \(\sqrt{N}\) where \(h\) is the discretization step in time. However, in our work, the rate of convergence does not depend on the space dimension \(d\).

**Remark 5.2** If we assume more regularity conditions on the coefficients and the final condition as in Pardoux and Peng [29], namely, \(\Phi \in C^3_b(\mathbb{R}^d, \mathbb{R}^k)\), \(f \in C^3_b([0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{d \times k}, \mathbb{R}^k)\) and \(g \in C^3([0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{d \times k}, \mathbb{R}^{k \times 1})\). If \((Y^{t,x}_s, Z^{t,x}_s)_{t \leq s \leq T}\) is the solution the BDSDE (1.2). Then, \(u_t(x) = Y^{t,x}_t\), \(\forall (t, x) \in [0, T] \times \mathbb{R}^d\) is the unique classical solution of the SPDE (1.1) in the integral sense (see [29]). Therefore, we can obtain a stronger result. In fact, the estimation on the error (5.7) obtained in the previous theorem can be replaced by:

\[
E[\sup_{0 \leq t \leq T} |u_t^N(x) - u(t, x)|^2] + E[\int_0^T ||u_t^N(x) - v(t, x)||^2 dt] \leq Ch.
\]

This last equation gives an estimation which holds for all \(x \in \mathbb{R}^d\) and which is not only almost sure anymore. For the Monte Carlo method, we estimate the solution for one point \(x\) at time \(t\), and by varying \(x\) and \(t\) we obtain the solution \(u(t, x)\) on the whole domain.

6. Implementation and numerical tests

In this part, we are interested in implementing our numerical scheme. Our aim is only to test statically its convergence. Further analysis of the convergence of the used method and of the error bounds will be accomplished in a future work.

6.1. Notations and algorithm

We use a path-dependent algorithm, for every fixed path of the brownian motion \(B\), we approximate by a regression method the solution of the associated PDE. Then, we replace the conditional expectations which appear in (6.1) and (6.2) by \(L^2(\Omega, \mathcal{F})\) projections on the function basis approximating \(L^2(\Omega, \mathcal{F})\). We compute \(Y^N_{t_n}\) in an explicit manner and we use I Picard iterations to compute \(Y^N_{t_n}\) in a implicit way. Actually, we proceed as in [12], except that in our case the solutions \(Y^N_{t_n}\) and \(Z^N_{t_n}\) are measurable functions of \((X^N_{t_n}, (\Delta B_i)_{n \leq i \leq N-1})\). So, each solution given by our algorithm depends on the fixed path of \(B\).

6.1.1. Numerical scheme

For each fixed path of \(B\), the solution of (2.1)-(2.2) is approximated by \((Y^N, Z^N)\) defined by the following algorithm, given in the multidimensional case.

For \(0 \leq n \leq N - 1\):

\[
\forall j_1 \in \{1, \ldots, k\}, \quad Y^N_{t_n,j_1} = E_{t_n} \left[ Y^N_{t_{n+1},j_1} + h f_{j_1}(X^N_{t_n}, Y^N_{t_n}, Z^N_{t_n}) + \sum_{j=1}^{l} g_{j, j_1}(X^N_{t_{n+1}}, Y^N_{t_{n+1}}, Z^N_{t_{n+1}}) \Delta B_{n,j} \right], \quad (6.1)
\]
\( \forall j_1 \in \{1, \ldots, k\} \text{ and } \forall j_2 \in \{1, \ldots, d\} \)

\[ hZ^N_{t_n,j_1,j_2} = E t_n \left[ Y^N_{t_{n+1},j_1} \Delta W_{n,j_2} + \sum_{j=1}^l g_{j_1,j}(X^N_{t_{n+1}}, Y^N_{t_{n+1}}, Z^N_{t_{n+1}}) \Delta B_{n,j} \Delta W_{n,j_2} \right]. \]  

(6.2)

We stress that at each discretization time, the solution of the algorithm depends on the fixed path of the brownian motion \( B \).

### 6.1.2. Vector spaces of functions

At every \( t_n \), we select \( k(d+1) \) deterministic functions bases \((p_{i,n}(\cdot))_{1 \leq i \leq k(d+1)}\) and we look for approximations of \( Y^N_{t_n} \) and \( Z^N_{t_n} \) which will be denoted respectively by \( y^N_{t_n} \) and \( z^N_{t_n} \), in the vector space spanned by the basis \((p_{j_1,j_2,n}(\cdot))_{1 \leq j_1 \leq k} \text{ respectively } (p_{j_1,j_2,n}(\cdot))_{1 \leq j_1 \leq k, 1 \leq j_2 \leq d}\). Each basis \( p_{i,n}(\cdot) \) is considered as a vector of functions of dimension \( L_{t,n} \). In other words, \( P_{t,n}(\cdot) = \{\alpha.p_{i,n}(\cdot), \alpha \in \mathbb{R}^{L_{t,n}}\} \).

As an example, we cite the hypercube basis \((HC)\) used in [12]. In this case, \( p_{i,n}(\cdot) \) does not depend nor on \( i \) neither on \( n \) and its dimension is simply denoted by \( L \). A domain \( D \subset \mathbb{R}^d \) centered on \( X_0 = x \), that is \( D = \prod_{i=1}^d (x_i - a, x_i + a] \), can be partitioned on small hypercubes of edge \( \delta \). Then, \( D = \bigcup D_{i_1 \ldots i_d} \) where \( D_{i_1 \ldots i_d} = (x_i - a + i_1 \delta, x_i - a + i_1 \delta) \times \ldots \times (x_i - a + i_d \delta, x_i - a + i_d \delta) \).

Finally we define \( p_{i,n}(\cdot) \) as the indicator functions of this set of hypercubes.

### 6.1.3. Monte Carlo simulations

To compute the projection coefficients \( \alpha \), we will use \( M \) independent Monte Carlo simulations of \( X^N_{t_n} \) and \( \Delta W_n \) which will be respectively denoted by \( X^N_{t_n} \) and \( \Delta W^m_n, m = 1, \ldots, M \).

### 6.1.4. Description of the algorithm

→ Initialization: For \( n = N \), take \((y^N_{N,m,I}, (\Phi(X^N_{t_n})) \text{ and } (z^N_{N,m}) = 0 . \)

→ Iteration: For \( n = N - 1, \ldots, 0 \):

- We approximate (6.2) by computing for all \( j_1 \in \{1, \ldots, k\} \text{ and } j_2 \in \{1, \ldots, d\} \)

\[
\alpha^{M}_{j_1,j_2,n} = \arg \min_{\alpha} \frac{1}{M} \sum_{m=1}^{M} y^N_{n+1,j_1}(X^N_{t_{n+1}}, \frac{\Delta W^m_{n,j_2}}{h}) + \sum_{j=1}^{l} g_{j_1,j}(X^N_{t_{n+1}}, Y^N_{t_{n+1}}, Z^N_{t_{n+1}}) \Delta B_{n,j} \Delta W^m_{n,j_2} = \alpha.p^{m}_{j_1,j_2,n} \ 
\]

Then we set \( z^N_{n,j_1,j_2,\cdot} = (\alpha^{M}_{j_1,j_2,n}, p_{j_1,j_2,n}(\cdot)) \), \( j_1 \in \{1, \ldots, k\} \), \( j_2 \in \{1, \ldots, d\} \).

- We use \( I \) Picard iterations to obtain an approximation of \( Y_{t_n} \) in (6.1):

  - For \( i = 0 \): \( \forall j_1 \in \{1, \ldots, k\} \), \( \alpha^{M,0}_{j_1,n} = 0. \)

  - For \( i = 1, \ldots, I \): We approximate (6.1) by calculating \( \alpha^{M,i}_{j_1,n}, \forall j_1 \in \{1, \ldots, k\} \), as the minimizer

...
We take at each time $t$

In this case, the solutions of our least squares problems are given by:

\[ \frac{1}{M} \sum_{m=1}^{M} y_{n+1,j_1} g_{n} N, M, i, -1 \left( X_{t_{n+1}}^{N, m}, z_{n+1}^{M} \right) \]

\[ + \sum_{j=1}^{l} g_{j, l, i} \left( X_{t_{n+1}}^{N, m}, z_{n+1}^{M} \right) \Delta B_{n, j} - \alpha p_{j, l}^m \left| \frac{\Delta B_{n, j}^m}{\Delta B_{n, j}^m} \right|^2. \]

Finally, we define $y_{n, M, i}^n$ as:

\[ y_{n, M, i}^n = (\alpha_{p_{j_1, i}, n} M, i), \forall j_1 \in \{1, \ldots, k\}. \]

6.2. One-dimensional case (Case when $d = k = l = 1$)

6.2.1. Function bases

We use the basis (HC) defined above. So we set:

\[ d_1 = \min_n x_{n, m}^m, \quad d_2 = \max_n x_{n, m}^m \quad \text{and} \quad L = \frac{d_2 - d_1}{\delta} \]

where $\delta$ is the edge of the hypercubes $(D_j)_{1 \leq j \leq L}$ defined by $D_j = \left[ d + (j - 1)\delta, d + j\delta \right), \forall j$.

We take at each time $t_n$

\[ 1_{D_j}(X_{t_n}^{N, m}) = 1_{[d + (j - 1)\delta, d + j\delta]}(X_{t_n}^{N, m}), j = 1, \ldots, L \]

and

\[ \psi_{n, i}^m = \left\{ \sqrt{\frac{M}{\text{card}(D_j)}} 1_{D_j}(X_{t_n}^{N, m}), 1 \leq j \leq L \right\}, i = 0, 1. \]

\text{Card}(D_j) \text{ denotes the number of simulations of } X_{t_n}^N \text{ which are in our cube } D_j. \]

This system is orthonormal with respect to the empirical scalar product defined by

\[ \langle \psi_1, \psi_2 \rangle_{n, M} = \frac{1}{M} \sum_{m=1}^{M} \psi_1(X_{t_n}^{N, m}) \psi_2(X_{t_n}^{N, m}). \]

In this case, the solutions of our least squares problems are given by:

\[ \alpha_{n, M} = \frac{1}{M} \sum_{m=1}^{M} p_{n, m} (X_{t_n}^{N, m}) \left\{ y_{n+1}^{N, M, i} (X_{t_{n+1}}^{N, m}) \Delta W_n^m \right\}, \]

\[ + g \left( X_{t_{n+1}}^{N, m}, z_{n+1}^{N, M} \right) \Delta B_n^m \Delta W_n^m \]

\[ \alpha_{n, M} = \frac{1}{M} \sum_{m=1}^{M} p_{0, n} (X_{t_n}^{N, m}) \left\{ y_{n+1}^{N, M, i} (X_{t_{n+1}}^{N, m}) \Delta W_n^m \right\}, \]

\[ + g \left( X_{t_{n+1}}^{N, m}, z_{n+1}^{N, M} \right) \Delta B_n^m \Delta W_n^m \].
Remark 6.1 We note that for each value of \( M, N \) and \( \delta \), we launch the algorithm 50 times and we denote by \((Y_{0,m'}^{0,x,N,M,I})_{1 \leq m' \leq 50}\) the set of collected values. Then we calculate the empirical mean \( \overline{Y}_{0,x,N,M,I} \) and the empirical standard deviation \( \sigma_{N,M,I} \) defined by:

\[
\overline{Y}_{0,x,N,M,I} = \frac{1}{50} \sum_{m'=1}^{50} Y_{0,m'}^{0,x,N,M,I}, \quad \sigma_{N,M,I} = \sqrt{\frac{1}{49} \sum_{m'=1}^{50} |Y_{0,m'}^{0,x,N,M,I} - \overline{Y}_{0,x,N,M,I}|^2}.
\]

We also note before starting the numerical examples that our algorithm converges after at most three Picard iterations. Finally, we stress that \((6.3)\) gives us an approximation of \( u(0,x) \) the solution of the SPDE \((1.1)\) at time \( t = 0 \).

6.2.2. Case when \( f \) and \( g \) are linear in \( y \) and independent of \( z \)

\[
\left\{ \begin{array}{l}
  dX_t = X_t(\mu dt + \sigma dW_t), \\
  \Phi(x) = -x + K, \quad f(y) = a_0 y, \quad g(y) = b_0 y
\end{array} \right.
\]

and we set \( K = 115, r = 0.01, R = 0.06, X_0 = 100, \mu = 0.05, \sigma = 0.2, T = 0.25, d_1 = 60, d_2 = 200, a_0 \) and \( b_0 \) are fixed constants.

Let \( Y_{\text{explicit}} \) be the solution of our BDSDE in this particular case. By the integration by parts formula, we get

\[
Y_{t,\text{explicit}} = \mathbb{E}[\Phi(X_{T}^{t,x})e^{a_0(T-t)+b_0(B_T-B_t)-\frac{b_0^2}{2}(T-t)}/\mathcal{F}_t].
\]

At \( t=0 \), we have

\[
Y_{0,\text{explicit}} = \mathbb{E}[\Phi(X_{T}^{0,x})e^{a_0-\frac{b_0^2}{2}(T)}+b_0B_T]/\mathcal{F}_0] = e^{(a_0-\frac{b_0^2}{2})T+b_0B_T} \mathbb{E}[\Phi(X_{T}^{0,x})].
\]

Then, we define \( \overline{Y}_{0,x,N,M,I} \) as the numerical approximation of the solution of the BDSDE in this case (computed by our algorithm) and \( \sigma_{N,M,I} \) as its standard deviation. In the other hand, we compute the solution \( Y_{0,\text{explicit}} \) in this linear case by using the explicit formula of the expectation of \( \Phi(X_{T}^{0,x}) \), as follows

\[
Y_{\text{explicit}} = e^{(a_0-\frac{b_0^2}{2})T+b_0B_T} \mathbb{E}[\Phi(X_{T}^{0,x})] = e^{(a_0-\frac{b_0^2}{2})T+b_0B_T}(K - x e^\mu T).
\]

For \( a_0 = 0.5, b_0 = 0.5 \) and \( \delta = 1 \)

\[
\begin{array}{|c|c|c|}
\hline
N=20, Y_{\text{explicit}}^{0,x} & 13.724 \\
\hline
M & \overline{Y}_{0,x,N,M,I}^{0,x,N,M,I} & (\sigma_{N,M,I}^{0,x,N,M,I}) \\
\hline
100 & 13.910(1.178) & 0.013 \\
1000 & 13.792(0.309) & 0.004 \\
5000 & 13.847(0.117) & 0.008 \\
\hline
\end{array}
\]
For $a_0 = 0.5$, $b_0 = 0.5$ and $\delta = 0.5$

\[
N=30, \; Y_{\text{explicit}}^{0,x} = 14.115
\]

| $M$ | $Y_{0}^{0,x,N,M,I}(\sigma^{N,M,I})$ | $\frac{|Y_{\text{explicit}}^{0,x,N,M,I} - Y_{0}^{0,x,N,M,I}|}{Y_{\text{explicit}}^{0,x,N,M,I}}$ |
|-----|----------------------------------|------------------------------------------|
| 100 | 14.246(1.045)                    | 0.009                                    |
| 1000| 14.195(0.337)                    | 0.005                                    |
| 5000| 14.236(0.129)                    | 0.008                                    |

6.2.3. **Comparison of numerical approximations of the solutions of the FBDSDE and the FBSDE: the general case**

Now we take

\[
\Phi(x) = -x + K,
\]

\[
f(t, x, y, z) = -\theta z - ry + (y - \frac{z}{2})^-(R - r),
\]

\[
g_1(t, x, y, z) = 0.1z + 0.5y + log(x)
\]

and we set $\theta = (\mu - r)/\sigma$, $K = 115$, $X_0 = 100$, $\mu = 0.05$, $\sigma = 0.2$, $r = 0.01$, $R = 0.06$, $\delta = 1$, $N = 20$, $T = 0.25$ and we fix $d_1 = 60$ and $d_2 = 200$ as in [11]. The functions $g_1, g_2$ and $g_3$ taken in the following are examples of the function g. They are sufficiently regular and Lipschitz on $[60, 200] \times \mathbb{R} \times \mathbb{R}$ and could be extended to regular Lipschitz functions on $\mathbb{R}^3$. In this case, Assumptions (H1)-(H3) are satisfied.

We compare the numerical solution of our BDSDE (noted again $Y_{t,x,N,M,I}$) and the BSDE’s one (noted here by $Y_{t,BSDE}$), without $g$ and $B$.

When $t$ is close to maturity

\[
\begin{array}{c|c|c}
M & Y_{t_{19},BSDE}^{0,x,N,M}(\sigma^{N,M}) & Y_{t_{19}}^{0,x,N,M,I}(\sigma^{N,M,I}) \\
\hline
128 & 13.748(0.879) & 15.452(0.948) \\
512 & 13.827(0.384) & 15.534(0.409) \\
2048 & 13.762(0.223) & 15.464(0.240) \\
8192 & 13.781(0.091) & 15.484(0.097) \\
32768 & 13.796(0.054) & 15.501(0.058) \\
\end{array}
\]

\[
\begin{array}{c|c|c}
M & Y_{t_{15},BSDE}^{0,x,N,M}(\sigma^{N,M}) & Y_{t_{15}}^{0,x,N,M,I}(\sigma^{N,M,I}) \\
\hline
128 & 14.168(0.905) & 17.894(1.096) \\
512 & 14.113(0.388) & 17.774(0.429) \\
2048 & 13.988(0.226) & 17.607(0.270) \\
8192 & 13.985(0.093) & 17.623(0.104) \\
32768 & 13.994(0.055) & 17.627(0.064) \\
\end{array}
\]

When $t = 0$
For \( g_2(y, z) = 0.1z + 0.5y \):

| \( M \) | \( Y_{0,BSDE}^{0,x,N,M}(\sigma_{N,M}) \) | \( Y_{0,i}^{0,x,N,M,I}(\sigma_{N,M,I}) \) |
|---|---|---|
| 128 | 15.431(1.005) | 13.571(1.146) |
| 512 | 15.029(0.428) | 13.173(0.500) |
| 2048 | 14.763(0.243) | 12.885(0.280) |
| 8192 | 14.718(0.098) | 12.825(0.106) |
| 32768 | 14.715(0.060) | 12.804(0.064) |

When \( t = 0 \):

| \( M \) | \( Y_{0,i}^{0,x,N,M,I}(\sigma_{N,M,I}) \) |
|---|---|
| 128 | 14.767(0.949) |
| 512 | 14.850(0.410) |
| 2048 | 14.781(0.240) |
| 8192 | 14.801(0.097) |
| 32768 | 14.818(0.058) |

For \( g_3(x, y) = \log x + 0.5y \):

When \( t \) is close to maturity:

| \( M \) | \( Y_{1,10}^{0,x,N,M,I}(\sigma_{N,M,I}) \) |
|---|---|
| 128 | 15.452(0.948) |
| 512 | 15.534(0.409) |
| 2048 | 15.464(0.240) |
| 8192 | 15.484(0.097) |
| 32768 | 15.501(0.058) |

| \( M \) | \( Y_{1,15}^{0,x,N,M,I}(\sigma_{N,M,I}) \) |
|---|---|
| 128 | 18.253(1.068) |
| 512 | 18.166(0.453) |
| 2048 | 18.010(0.266) |
| 8192 | 18.006(0.109) |
| 32768 | 18.017(0.065) |
When $t = 0$

| $M$  | $Y_0^{N,N,M,I}(\sigma^{N,M,I})$ |
|------|---------------------------------|
| 128  | 12.071(0.054)                   |
| 512  | 12.075(0.088)                   |
| 2048 | 12.122(0.218)                   |
| 8192 | 12.384(0.381)                   |
| 32768| 12.791(0.903)                   |

In the previous tables, we test our algorithm for different examples of the function $g$ ($g_1$ and $g_2$ are dependent in $z$, $g_3$ is independent of $z$). We see the convergence of the BDSDE’s solution when we increase the number of simulations $M$.

In figure 1, we study statically the main result of this paper. So, we fix all the parameters ($\delta = 1$, and $M = 2000$ ) and we draw the map of the BDSDE’s solution, for the function $g_1$, with respect to the number of time discretization steps $N$. The solution is computed for five different paths of the brownian motion $B$. We can examine there the convergence of our scheme.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{image.png}
\caption{The BDSDE’s solution with respect to the number of time discretization steps for five different paths of $B$. The figure is obtained for $M = 2000$ and $\delta = 1$.}
\end{figure}

We see on Figure 2 the impact of the function $g$ on the solution; we variate $N$, $M$ and $\delta$ as in [12], by taking these quantities as follows: First we fix $d_1 = 40$ and $d_2 = 180$ (which means that $x \in [d_1, d_2] = [40, 180]$ and in this case our assumptions (H1)-(H3) are satisfied). Let $j \in \mathbb{N}$, we take $\alpha_M = 3$, $\beta = 1$, $N = 2(\sqrt{2})^{(j-1)}$, $M = 2(\sqrt{2})^{\alpha_M(j-1)}$ and $\delta = 50/(\sqrt{2})^{(j-1)(\beta+1)/2}$. Then, we draw the map of each solution at $t = 0$ with respect to $j$. 
The parameter $j$

The approximation of the solution $Y$ at time $t=0$ BSDE

BDSDE for $g_1$

BDSDE for $g_2$

Figure 2. Comparison of the BSDE’s solution and the BDSDE’s one: The solution of the BSDE is with circle markers, the solution of the BDSDE for $g_1(x,y,z) = 0.1z + 0.5y + \log(x)$ is with star markers and the one for $g_2(y,z) = 0.1z + 0.5y$ is with cross markers. Confidence intervals are with dotted lines.

7. Appendix

7.1. Proof of Proposition 4.2.

To simplify the notations, we restrict ourselves to the case $k = d = l = 1$. $(D_0Y, D_0Z)$ is well defined and from inequalities (2.4) and (4.1), we deduce that for each $\theta \leq T$

$$E[ \sup_{t \leq s \leq T} |D_0Y_s|^2] + E[\int_t^T |D_0Z_s|^2 ds] \leq C(1 + |x|^2). \quad (7.1)$$

We define recursively the sequence $(Y^m, Z^m)$ as follows. First we set $(Y^0, Z^0) = (0, 0)$. Then, given $(Y^{m-1}, Z^{m-1})$, we define $(Y^m, Z^m)$ as the unique solution in $S^{2k}([t,T]) \times H^2_{\mathbb{F}}([t,T])$ of

$$Y^m_s = \Phi(X_T^{t,x}) + \int_t^s f(r, X^{t,x}_r, Y^{m-1}_r, Z^{m-1}_r) dr + \int_t^s g(r, X^{t,x}_r, Y^{m-1}_r, Z^{m-1}_r) dB_r - \int_t^s Z^m_r dW_r.$$

We recursively show that $(Y^m, Z^m) \in B^2([t,T], \mathbb{D}^{1,2})$. Suppose that $(Y^m, Z^m) \in B^2([t,T], \mathbb{D}^{1,2})$ and let us show that $(Y^{m+1}, Z^{m+1}) \in B^2([t,T], \mathbb{D}^{1,2})$.

From the induction assumption, we have $\Phi(X_T) + \int_s^T f(r, \Sigma^m_r) dr \in \mathbb{D}^{1,2}$. We have $g(r, \Sigma^m_r) \in \mathbb{D}^{1,2}$ for all $r \in [t,T]$. From Lemma 4.2, we have $\int_t^T g(r, \Sigma^m_r) dB_r \in \mathbb{D}^{1,2}$, then

$$Y^{m+1}_s = E[\Phi(X_T^{t,x}) + \int_t^s f(r, \Sigma^m_r) dr + \int_t^s g(r, \Sigma^m_r) dB_r | \mathcal{F}_t \lor \mathcal{V}_{t,s} \lor \mathcal{F}_{t,T}] \in \mathbb{D}^{1,2},$$

where $\Sigma^m_r := (X^{t,x}_r, Y^m_r, Z^m_r)$.

Hence

$$\int_t^T Z^{m+1}_r dW_r = \Phi(X_T^{t,x}) + \int_t^T f(r, \Sigma^m_r) dr + \int_t^T g(r, \Sigma^m_r) dB_r - Y^{m+1}_t \in \mathbb{D}^{1,2}.$$
It follows from Lemma 4.1 that $Z^{m+1} \in \mathcal{M}^2_{\text{loc}}([t,T], \mathbb{D}^{1,2})$ and we have $D\theta Y^{m+1}_s = D\theta Z^{m+1}_s = 0$ for $t \leq s \leq \theta$ and for $\theta \leq s \leq T$

$$D\theta Y^{m+1}_s = \nabla \Phi(X^{t,x}_T)D\theta X^{t,x}_T$$

(7.2)

$$+ \int_s^T \left( \nabla_x f(r, \Sigma_r)D\theta X_r + \nabla_y f(r, \Sigma_r)D\theta Y^{m}_r + \nabla_z f(r, \Sigma_r)D\theta Z^{m}_r \right) dr$$

$$+ \int_s^T \left( \nabla_x g(r, \Sigma_r)D\theta X_r + \nabla_y g(r, \Sigma_r)D\theta Y^{m}_r + \nabla_z g(r, \Sigma_r)D\theta Z^{m}_r \right) dB_r$$

$$- \int_s^T D\theta Z^{m+1}_r dW_r.$$ From inequality (2.4), we deduce that for each $\theta \leq T$

$$E[ \sup_{t \leq s \leq t} |D\theta Y^{m+1}_s|^2 ] + E[\int_t^T |D\theta Z^{m+1}_s|^2 ds] \leq C(1 + |x|^2).$$

It is known that inequality (2.4) holds for $(Y^{m+1}, Z^{m+1})$ and so we deduce that

$$\|Y^{m+1}\|_{1,2} + \|Z^{m+1}\|_{1,2} < \infty,$$

which shows that $(Y^{m+1}, Z^{m+1}) \in \mathcal{B}^2([t,T], \mathbb{D}^{1,2})$. Using the contraction mapping argument as in El Karoui Peng and Quenez [10], we deduce that $(Y^{m+1}, Z^{m+1})$ converges to $(Y, Z)$ in $\mathcal{S}^2([t,T]) \times \mathbb{H}^2([t,T])$. We will show that $(D\theta Y^{m}, D\theta Z^{m})$ converges to $(Y^\theta, Z^\theta)$ in $L^2(\Omega \times [t,T] \times [t,T], dP \otimes dt \otimes dt)$, where $Y^\theta_s = Z^\theta_s = 0$ for all $t \leq s \leq \theta$ and $(Y^\theta_s, Z^\theta_s, \theta \leq s \leq T)$ is the solution of the BDSDE.

$$Y^\theta_s = \nabla \Phi(X^{t,x}_T)D\theta X^{t,x}_T$$

(7.3)

$$+ \int_s^T \left( \nabla_x f(r, \Sigma_r)D\theta X_r + \nabla_y f(r, \Sigma_r)Y^\theta_r + \nabla_z f(r, \Sigma_r)Z^\theta_r \right) dr$$

$$+ \int_s^T \left( \nabla_x g(r, \Sigma_r)D\theta X_r + \nabla_y g(r, \Sigma_r)Y^\theta_r + \nabla_z g(r, \Sigma_r)Z^\theta_r \right) dB_r$$

$$- \int_s^T Z^\theta_r dW_r.$$ From equations (7.2) and (7.3), we have

$$D\theta Y^{m+1}_s - Y^\theta_s = \int_s^T \left( \nabla_x f(r, \Sigma_r) - \nabla_x f(r, \Sigma_r) \right) D\theta X^{t,x}_T$$

$$+ \nabla_y f(r, \Sigma_r)D\theta Y^{m}_r - \nabla_y f(r, \Sigma_r)Y^\theta_r + \nabla_z f(r, \Sigma_r)D\theta Z^{m}_r - \nabla_z f(r, \Sigma_r)Z^\theta_r dr$$

$$+ \int_s^T \left( \nabla_x g(r, \Sigma_r) - \nabla_x g(r, \Sigma_r) \right) D\theta X^{t,x}_T + \nabla_y g(r, \Sigma_r)D\theta Y^{m}_r - \nabla_y g(r, \Sigma_r)Y^\theta_r \right) dB_r$$

$$+ \int_s^T \left( \nabla_z g(r, \Sigma_r) - \nabla_z g(r, \Sigma_r) \right) D\theta Z^{m}_r - \nabla_z g(r, \Sigma_r)Z^\theta_r \right) dB_r$$

$$- \int_s^T (D\theta Z^{m+1}_r - Z^\theta_r) dW_r.$$
From Proposition 4.1, we have

\[
E[\sup_{0 \leq s \leq T} |D_\theta Y_s^{m+1} - Y_s^\theta|^2] + E[\int_t^T |D_\theta Z_s^{m+1} - Z_s^\theta|^2 \, dr]
\leq CE\left[\int_t^T \left(\nabla_x f(r, \Sigma_r^m) - \nabla_x f(r, \Sigma_r)\right) D_\theta X_r^t \, dr + \nabla_y f(r, \Sigma_r^m) Y_r^\theta - \nabla_y f(r, \Sigma_r) Y_r^\theta + \nabla_z f(r, \Sigma_r^m) Z_r^\theta - \nabla_z f(r, \Sigma_r) Z_r^\theta\right] \, dr.
\]

Therefore, we obtain

\[
E[\int_t^T \int_t^T |D_\theta Y_s^{m+1} - Y_s^\theta|^2 \, ds \, d\theta] + E[\int_t^T \int_t^T |D_\theta Z_s^{m+1} - Z_s^\theta|^2 \, ds \, d\theta] 
\leq CE\left[\int_t^T \int_t^T |\delta_{r,\theta}^m|^2 \, dr \, d\theta\right] + CE\left[\int_t^T \int_t^T |\rho_{r,\theta}^m|^2 \, dr \, d\theta\right],
\]

where

\[
\delta_{r,\theta}^m = (\nabla_x f(r, \Sigma_r^m) - \nabla_x f(r, \Sigma_r)) D_\theta X_r^t \, dr + \nabla_y f(r, \Sigma_r^m) Y_r^\theta - \nabla_y f(r, \Sigma_r) Y_r^\theta + \nabla_z f(r, \Sigma_r^m) Z_r^\theta - \nabla_z f(r, \Sigma_r) Z_r^\theta,
\]

and

\[
\rho_{r,\theta}^m = (\nabla_x f(r, \Sigma_r^m) - \nabla_x f(r, \Sigma_r)) D_\theta X_r^t \, dr + \nabla_y g(r, \Sigma_r^m) Y_r^\theta - \nabla_y g(r, \Sigma_r) Y_r^\theta + \nabla_z g(r, \Sigma_r^m) Z_r^\theta - \nabla_z g(r, \Sigma_r) Z_r^\theta.
\]

From the definition of \((\delta_{r,\theta}^m)_{t \leq r, \theta \leq T}\), we have

\[
E[\int_t^T \int_t^T |\delta_{r,\theta}^m|^2 \, dr \, d\theta] \leq C \int_t^T (A_m(\theta, t, T) + B_m(\theta, t, T)) d\theta,
\]

where

\[
A_m(\theta, t, T) = E\left[\int_t^T |(\nabla_x f(r, \Sigma_r^m) - \nabla_x f(r, \Sigma_r)) D_\theta X_r^t|^2 \, dr\right],
\]

\[
B_m(\theta, t, T) = E\left[\int_t^T |(\nabla_y f(r, \Sigma_r^m) - \nabla_y f(r, \Sigma_r^m)) Y_r^\theta|^2 \, dr\right] + E\left[\int_t^T |(\nabla_z f(r, \Sigma_r^m) - \nabla_z f(r, \Sigma_r^m)) Z_r^\theta|^2 \, dr\right].
\]

Moreover, since \(\nabla_x f\) is bounded and continuous with respect to \((x, y, z)\), it follows by the dominated convergence theorem and inequality (2.3) that

\[
\lim_{m \to \infty} \int_t^T A_m(\theta, t, T) d\theta = 0.
\]

Furthermore, since \(\nabla_y f\) and \(\nabla_z f\) are bounded and continuous with respect to \((x, y, z)\), it follows, also, by the dominated convergence theorem and inequality (2.4) that

\[
\lim_{m \to \infty} \int_t^T B_m(\theta, t, T) d\theta = 0.
\]
From the definition of \((ρ_r^m)_{s≤r≤T}\), we have

\[
E\left[\int_t^T \left|\rho_r^m\right|^2 dr\right] \leq C \int_t^T (A_m^r(θ, t, T) + B_m^r(θ, t, T)) dθ
\]

with

\[
A_m^r(θ, t, T) = E\left[\int_t^T \left| (∇zg(θ, Σ_{r}^m) - ∇zg(r, Σ_{r})D_{θ}X_{r}^1)^2 \right| dr\right]
\]

\[
B_m^r(θ, t, T) = E\left[\int_t^T \left| (∇yg(θ, Σ_{r}) - ∇yg(r, Σ_{r}^m))Y_{r}^θ\right|^2 dr\right] + E\left[\int_t^T \left| (∇zg(θ, Σ_{r}) - ∇zg(r, Σ_{r}^m))Z_r^θ\right|^2 dr\right].
\]

Similarly as shown above, since \(∇y\) and \(∇z\) are bounded and continuous with respect to \((x, y, z)\) we can show that:

\[
\lim_{m \to \infty} \int_t^T A_m^r(θ, t, T) dθ = \lim_{m \to \infty} \int_t^T B_m^r(θ, t, T) dθ = 0. \quad (7.10)
\]

Plugging (7.8), (7.9) and (7.10) into inequality (7.5), we deduce that

\[
\lim_{m \to \infty} E\left[\int_t^T \int_t^T |D_{θ}Y_{s}^{m+1} - Y_θ^s|^2 ds dθ\right] + E\left[\int_t^T \int_t^T |D_{θ}Z_{s}^{m+1} - Z_θ^s|^2 ds dθ\right] = 0.
\]

It follows that \((Y^m, Z^m)\) converges to \((Y, Z)\) in \(L^2([t, T], \mathbb{D}^{1,2})\) and a version of \((D_{θ}Y, D_{θ}Z)\) is given by \((Y^θ, Z^θ)\) which is the desired result.

\[\square\]

7.2. Second order Malliavin derivative of the solution of BDSDE’s

We apply know similar computation to get the second order Malliavin derivatives representation of the solution of BDSDE’s, so we will omit the proof.

**Proposition 7.1** Assume that assumptions (H2) and (H3) hold. We fix \(t \in [0, T]\). Then for each \(i ≤ θ ≤ T\), \((D_{θ}Y, D_{θ}Z)\) belongs to \(B^2([t, T], \mathbb{D}^{1,2})\). For each \(t ≤ v ≤ T\) and \(1 ≤ i, j ≤ d\),

\[
D_{v}^i D_{θ}^j Y_s = D_{v}^i D_{θ}^j Z_s = 0, \quad 1 ≤ n ≤ d, \quad \text{if } s < θ \lor v,
\]

and a version of \((D_{θ}^i D_{θ}^j Y_s, D_{θ}^i D_{θ}^j Z_s)_{v \lor θ ≤ s ≤ T}\) is the unique solution of the equation:

\[
D_{v}^i D_{θ}^j Y_s = T_1(Φ) + T_2(f) + T_3(g) + T_4(W),
\]

where

\[
T_1(Φ) = \sum_{n_1=1}^{k} ∇((∇Φ)^{n_1}(X_{T}^{1,x})) D_{v}^i X_{T}^{1,x}(D_{θ}^j D_{θ}^j X_{T}^{1,x})^{n_1} + ∇Φ(X_{T}^{1,x})D_{v}^i D_{θ}^j X_{T}^{1,x},
\]
\[ T_2(f) = \int_s^T \sum_{n_1=1}^k (\nabla_x ((\nabla_x f)^n_1 (r, X^{t,x}_r, Y_r, Z_r))) D^i_t X^{t,x}_r (D^j_{\theta} Y^{t,x}_r)^n_1 dr + \nabla_x f(r, X^{t,x}_r, Y_r, Z_r) D^j_{\theta} Y^{t,x}_r \] 
\[ + \int_s^T \left( \sum_{n_1=1}^k \nabla_y ((\nabla_y f)^n_1 (r, X^{t,x}_r, Y_r, Z_r)) D^j_t Y_t (D^j_{\theta} Y_t)^n_1 \right) dr + \nabla_y f(r, X^{t,x}_r, Y_r, Z_r) D^j_{\theta} Y_t dr \] 
\[ + \int_{n_2=1}^{d} \int_s^T \sum_{n_1=1}^k \nabla_{z^{n_2}} ((\nabla_z f)^n_1 (r, X^{t,x}_r, Y_r, Z_r)) D^j_t Z^{n_2}_r (D^j_{\theta} Z^{n_2}_r)^n_1 dr \] 
\[ + \int_{n_2=1}^{d} \int_s^T \nabla_{z^{n_2}} f(r, X^{t,x}_r, Y_r, Z_r) D^j_{\theta} Z^{n_2}_r dr, \]

\[ T_3(g) = \sum_{n_3=1}^l \int_s^T \sum_{n_1=1}^k \nabla_x ((\nabla_x g^{n_3})^n_1 (r, X^{t,x}_r, Y_r, Z_r)) D^i_t X^{t,x}_r (D^j_{\theta} X^{t,x}_r)^n_1 \frac{dB^{n_3}_t}{\theta^n_i} \] 
\[ + \int_{n_3=1}^l \int_s^T \nabla_x g^{n_3}(r, X^{t,x}_r, Y_r, Z_r) D^j_{\theta} X^{t,x}_r dB^{n_3}_t \] 
\[ + \int_{n_3=1}^l \int_s^T \sum_{n_1=1}^k \nabla_y ((\nabla_y g^{n_3})^n_1 (r, X^{t,x}_r, Y_r, Z_r)) D^j_t Y_t (D^j_{\theta} Y_t)^n_1 \frac{dB^{n_3}_t}{\theta^n_i} \] 
\[ + \int_{n_3=1}^l \int_{n_2=1}^{d} \sum_{n_1=1}^k \nabla_{z^{n_2}} ((\nabla_z g^{n_3})^n_1 (r, X^{t,x}_r, Y_r, Z_r)) D^j_t Z^{n_2}_r (D^j_{\theta} Z^{n_2}_r)^n_1 \frac{dB^{n_3}_t}{\theta^n_i} \] 
\[ + \int_{n_3=1}^l \int_{n_2=1}^{d} \sum_{n_1=1}^k \nabla_{z^{n_2}} g^{n_3}(r, X^{t,x}_r, Y_r, Z_r) D^j_{\theta} Z^{n_2}_r dB^{n_3}_t, \]

\[ (z^i)_1 \leq j \leq d \text{ denotes the } j\text{-th column of the matrix } z, \quad (g^{n_3})_1 \leq n_3 \leq l \text{ denotes the } n_3\text{-th column of the matrix } g, \quad B = (B^1, \ldots, B^l), \quad (D^j_{\theta} X^{t,x}_r)^n_1 \text{ is the } n_1\text{-th component of the vector } (D^j_{\theta} X^{t,x}_r), \quad (D^j_{\theta} Y_t)^n_1 \text{ is the } n_1\text{-th component of the vector } (D^j_{\theta} Y_t) \] 
\[ \text{ and } (D^j_{\theta} Z^{n_2}_r)^n_1 \text{ is the } n_1\text{-th component of the vector } (D^j_{\theta} Z^{n_2}_r). \]

7.3. Some estimates on the solution of the FBDSDE

Lemma 7.1 Let \((b^1, \sigma^1)\) and \((b^2, \sigma^2)\) be the standard parameters of the SDE (2.1) with initial condition \(x^1\) (resp. \(x^2\)). We assume that (H1) holds. Put \(\delta X_s = X^1_s - X^2_s\), \(\delta b_s = (b^1 - b^2)(X^1_s)\) and \(\delta \sigma_s = (\sigma^1 - \sigma^2)(X^1_s)\). Then

\[ ||X^1||_{L^2} \leq C(1 + |x|^2). \]
For all \( s_1, s_2 \in [0, T] \), we have
\[
E \left[ \sup_{s_1 \leq u \leq s_2} |X^1_u - X^1_{s_1}| \right] \leq C(1 + |x|^2)|s_2 - s_1|,
\]
and for all \( s_1 \leq s \leq s_2 \), we have
\[
||\delta X||_{S_{[s_1,s_2]}}(s) \leq C \left( |x^1 - x^2|^2 + |s_2 - s_1| + E\left[ \int_{s_1}^{s_2} |\delta b_s|^2 + |\delta \sigma_s|^2 ds \right] \right),
\]
where \( C \) is a generic constant depending only on \( K, T, (b^1(0), \sigma^1(0)) \) and \( (b^2(0), \sigma^2(0)) \).

**Lemma 7.2** Let \((X^{t,x}, Y^{t,x}, Z^{t,x})\) be the solution of the FBDSDE (2.1)-(2.2). We assume that Assumptions (H1) and (H2) hold. Then, we have
\[
||Y^{t,x}||_{S_t} + ||Z^{t,x}||_{M_{[s_1,s_2]}} \leq C(1 + |x|^2),
\]
and for all \( s' \leq s \in [t, T] \), we have
\[
E \left[ \sup_{s' \leq u \leq s} |Y^t_{u} - Y^{t,s'}_{s}|^2 \right] \leq C \left( (1 + |x|^2)|s - s'| + ||Z^{t,x}||_{M_{[s_1,s_2]}} \right).\]

**Proof.** The technics used to prove these estimates are classical in the BSDE’s theory (see El Karoui et al. [10]) so we omit it. \( \square \)

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