IDENTIFYING THE COVARIATION BETWEEN THE DIFFUSION PARTS AND THE CO-JUMPS GIVEN DISCRETE OBSERVATIONS

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Abstract

In this paper we consider two semimartingales driven by diffusions and jumps. We allow both for finite activity and for infinite activity jump components. Given discrete observations we disentangle the integrated covariation (the covariation between the two diffusion parts, indicated by IC) from the co-jumps. This has important applications to multiple assets price modeling for forecasting, option pricing, risk and credit risk management.

An approach commonly used to estimate IC is to take the sum of the cross products of the two processes increments; however this estimator can be highly

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biased in the presence of jumps, since it approaches the quadratic covariation, which contains also the co-jumps. Our estimator of $IC$ is based on a threshold (or truncation) technique allowing to isolate all the jumps in the finite activity case and the jumps over the threshold in the infinite activity case. We prove that the estimator is consistent in both cases as the number of observations increases to infinity. Further, in presence of only finite activity jumps 1) $\hat{IC}$ is also asymptotically Gaussian; 2) a joint CLT for $\hat{IC}$ and threshold estimators of the integrated variances of the single processes allows to reach consistent and asymptotically Gaussian estimators even of the $\beta$s and of the correlation coefficient among the diffusion parts of the two processes, allowing a better measurement of their dependence; 3) thresholding gives an estimate of $IC$ which is robust to the asynchronicity of the observations.

We conduct a simulation study to check that the application of our technique is in fact informative for values of the step between the observations large enough to avoid the typical problems arising in presence of microstructure noises in the data, and to asses the choice of the threshold parameters.

**Keywords**: co-jumps, integrated covariation, integrated variance, finite activity jumps, infinite activity jumps, threshold estimator.

1 Introduction

We consider two state variables evolving as follows

$$dX_t^{(q)} = a_t^{(q)} dt + \sigma_t^{(q)} dW_t^{(q)} + dJ_t^{(q)}, \quad q = 1, 2$$
for \( t \in [0, T] \), \( T < \infty \) fixed, where \( a \) and \( \sigma \) are cadlag stochastic processes; \( W_t^{(2)} = \rho_t W_t^{(1)} + \sqrt{1 - \rho_t^2} W_t^{(3)} \); \( W \) and \( W^{(3)} \) are independent standard Brownian motions; and \( J^{(1)} \) and \( J^{(2)} \) are possibly correlated pure jump semimartingales. Given discrete observations \( X_{t_j}^{(1)}, X_{\nu_i}^{(2)}, \) with observation times spanned on \([0, T]\), we are interested in the separate identification of the integrated covariation \( \text{IC}_T := \int_0^T \rho_t \sigma_t^{(1)} \sigma_t^{(2)} \, dt \), between the two diffusion parts, and of the co-jumps \( \Delta J_t^{(1)} \Delta J_t^{(2)} \), the simultaneous jumps of \( X^{(1)} \) and \( X^{(2)} \), where, for each \( q = 1, 2 \), \( \Delta J_t^{(q)} \) denotes the size \( J_t^{(q)} - J_{t-}^{(q)} \) of the jump occurred at time \( t \).

The recent empirical interest on co-jumps in financial econometrics is motivated by the problem of a correct assets price model selection. This has important consequences in forecasting, in option pricing, in portfolio risk management, and even in the credit risk management, since a default of a firm is interpretable as a jump in the firm value and contemporaneous defaults give a co-jump, implying default dependence (contagion, \[10\]).

A commonly used approach to estimate \( \int_0^T \rho_t \sigma_t^{(1)} \sigma_t^{(2)} \, dt \) is to take synchronous and evenly-spaced observations \( X_{t_0}^{(1)}, X_{t_1}^{(1)}, ...X_{t_n}^{(1)}, X_{t_0}^{(2)}, X_{t_1}^{(2)}, ...X_{t_n}^{(2)} \), with \( t_n = T \), and to consider the sum of cross products \( \sum_{j=1}^n \Delta_j X^{(1)} \Delta_j X^{(2)} \), where \( \Delta_j X^{(q)} := X_{t_j}^{(q)} - X_{t_{j-1}}^{(q)} \); however this estimate can be highly biased when the processes \( X^{(q)} \) contain jumps; in fact, as \( n \rightarrow \infty \), such a sum approaches the global quadratic covariation

\[
[X^{(1)}, X^{(2)}]_T = \int_0^T \rho_t \sigma_t^{(1)} \sigma_t^{(2)} \, dt + \sum_{0 \leq t \leq T} \Delta J_t^{(1)} \Delta J_t^{(2)},
\]

which contains also the co-jumps. To our aim it is crucial to single out the time intervals where the jumps occurred.

A jump process \( J \) is said to have finite activity (FA) when a.s. only a finite
number of jumps can occur in each finite time interval. On the contrary $J$ is said
to have infinite activity (IA). In the special case where $J$ is Lévy and has IA then
\emph{a.s.} infinitely many jumps occur in each finite time interval.

Our estimator of $IC_T$ is based on a threshold criterion allowing to identify all
the time intervals $[t_{j-1}, t_j]$ where the path of a univariate semimartingale jumped,
if the jump component $J$ has FA, and the intervals where jumps over the threshold
occurred, if the discretely observed realization of $J$ has infinite activity. Extending
the application of the criterion to a bivariate framework allows to derive an asymptotically unbiased estimator of $IC_T$ as well as of the co-jumps occurred up to time $T$. More precisely we construct the following estimator
\[
\hat{IC}_{T,n} := \sum_{j=1}^{n} \Delta_j X^{(1)} 1_{\{(\Delta_j X^{(1)})^2 \leq r(h)\}} \Delta_j X^{(2)} 1_{\{(\Delta_j X^{(2)})^2 \leq r(h)\}}, \quad h = T/n,
\]
where only the variations under a given threshold function $r(h)$ are taken into ac-
count. The first main result of our paper is showing the consistency to $IC_T$, as
the number $n$ of observations tends to infinity. Not equally spaced but synchronous
observations are allowed for such result. The second group of results is given in
presence of only FA jumps. If we dispose of non-synchronous data we still reach
consistency by modifying our estimator in a similar way of [13] and [12]. When
observations are evenly spaced, we prove a joint CLT delivering: 1. that $\hat{IC}_{T,n}$ is
asymptotically Gaussian and converes with speed $\sqrt{h}$, which extends results in [2]
who estimated $IC_T$ in absence of jumps; 2. consistent and asymptotically Gaussian
estimators of the regression coefficients $\beta$s and of the correlation coefficient of the
two continuous parts of processes $X^{(q)}$.

In a further paper ([11]) we explore the speed of convergence of the estimator $\hat{IC}_{T,n}$
proposed here in the presence of, possibly correlated, infinite activity Lévy jump processes \( J^{(1)} \) and \( J^{(2)} \), with dependence structure described by a Lévy copula.

The threshold criterion originated in [21] to separate the diffusion and the jump parts of a univariate parametric Poisson-Gaussian model. The criterion was shown to work even in nonparametric frameworks in [22], [23] and [15]. Potentially the threshold technique can be useful in each context where disentangling the quadratic variation of a signal has some importance to capture the contribution given by the diffusive component of the model and the one given by the jump component.

The literature on non parametric inference for stochastic processes driven by diffusions plus jumps, based on discrete observations, is mainly devoted to univariate cases. As for bivariate processes Barndorff-Nielsen and Shephard ([4]) and Jacod and Torodov ([19]) explore tests for the presence of co-jumps based on estimators constructed basically using cross multipower variations.

We adopt the threshold method here since, at least in the finite activity case, it is a more effective way to identify (asymptotically) the intervals between consecutive observations where jumps occurred. In fact already in the univariate case the threshold estimator of \( IV^{(1)} \) is efficient (in the Cramer-Rao inequality lower bound sense), the asymptotic standard estimation error being \( \sqrt{2 \times IQ^{(1)}} \), where \( IQ^{(1)} := \int_0^T (\sigma_t^{(1)})^4 dt \) (see [23]), while the multipower variation estimators are not efficient (the standard errors are all higher, see [3], [28] and the discussion in [23], section 3.3, the infimum being \( \sqrt{2.609 \times IQ^{(1)}} \)). For the bipower covariation based estimator (BPC) of \( IC_T \), a CLT has been shown to hold only in absence of jumps ([4]), in which case the standard error is a function of \( \rho_t, \sigma_t^{(1)}, \sigma_t^{(2)} \), which for instance equals \( 1.3 \times \int_0^T (1 + \rho_t^2)(\sigma_t^{(1)})^4 dt \) if \( (\sigma_t^{(1)})^2 \equiv (\sigma_t^{(2)})^2 \), while here we show that a CLT
holds for the threshold estimator even in presence of (finite activity) jumps, the asymptotic standard error being
\[
\int_0^T (1 + \rho_t^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt,
\]
for any \(\rho_t, \sigma_t^{(1)}, \sigma_t^{(2)},\) and which is less than the error of the BPC at least when \((\sigma_t^{(1)})^2 \equiv (\sigma_t^{(2)})^2.\) The bipower covariation test of [4] has been discussed by [7], where the Authors show that, when dealing with large portfolios, it is necessary to use a different global cross-variation index to get reliable results.

A CLT using multipowers for a bivariate process and in presence of jumps is given by [19]. More precisely, regarding the co-jumps, they consider the quantity \(\hat{B}_n := \sum_{i=1}^n (\Delta i X^{(1)})^2(\Delta i X^{(2)})^2,\) which is an estimate of \(B := \sum_{s\leq T} (\Delta X_s^{(1)})^2(\Delta X_s^{(2)})^2,\) not directly comparable with an estimate of the sum of the co-jumps \(\sum_{s\leq T} \Delta X_s^{(1)} \Delta X_s^{(2)}\) we give here. Their goal is to give a test for the presence of co-jumps, so they concentrate on
\[
\phi_n^{(j)} = \frac{\sum_i (\tilde{\Delta}_i X^{(1)})^2(\tilde{\Delta}_i X^{(2)})^2}{\sum_i (\Delta_i X^{(1)})^2(\Delta_i X^{(2)})^2},
\]
the quotient of two cross-power variations of the bivariate \(X,\) computed for different lags \(kh\) and \(h: \tilde{\Delta}_i X^{(q)} := X^{(q)}_{i kh} - X^{(q)}_{(i-1)kh},\Delta_i X^{(q)} := X^{(q)}_{ih} - X^{(q)}_{(i-1)h}.\) They reach that \(\phi_n^{(j)} \to 1\) as \(n \to \infty,\) on the space \(\Omega^{(j)}\) where some co-jumps occur, and they prove a CLT for \(\phi_n^{(j)}\) in restriction to \(\Omega^{(j)}\). We remark that to compute an estimator of the conditional asymptotic variance of \(\phi_n^{(j)}\) they in fact use the threshold technique when the volatilities are stochastic and are allowed to co-jump with the respective state variables.

An outline of the paper is as follows. In section 2 we illustrate the framework; in section 3 we deal with the case where each component \(J^{(q)}\) of \(X^{(q)}\) has finite activity of jump. We show that \(\hat{IC}_{T,n}\) is asymptotically Gaussian, so that it is also
consistent. We find a joint CLT allowing to estimate the $\beta$s and the correlation coefficient of the continuous parts of the two processes $X^{(q)}$, and we deal even with the case where we dispose of non-synchronous observations. In section 4 we deal with the more complex case where each $J^{(q)}$ can have an infinite activity semimartingale jump component $J^{(q)}_2$. We show that our estimator is still consistent. Since the given theory asserts that we can asymptotically identify the quantities of our interest, in section 5 we check on simulations that in fact the finite sample performance of $\hat{IC}_{T,n}$ is good even for time step between the observations large enough (five minutes) to avoid considering microstructure effects on the data, at least for commonly used financial models with realistic choices of the parameters. Section 6 concludes and section 7 contains all the proofs and technical details.

2 Framework and notation

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ where $X^{(1)} = (X^{(1)}_t)_{t \in [0,T]}$ and $X^{(2)} = (X^{(2)}_t)_{t \in [0,T]}$ are two real Itô semimartingales defined by

$$X^{(q)}_t = \int_0^t a^{(q)}_s \, ds + \int_0^t \sigma^{(q)}_s \, dW^{(q)}_s + J^{(q)}_t, \quad t \in [0,T], \quad q = 1, 2$$

where

A1. $W^{(1)} = (W^{(1)}_t)_{t \in [0,T]}$ and $W^{(2)} = (W^{(2)}_t)_{t \in [0,T]}$ are two correlated Wiener processes, with quadratic instantaneous covariation given by

$$d < W^{(1)}, W^{(2)} >_t = \rho_t dt, \quad t \in [0,T];$$

we can write

$$W^{(2)}_t = \rho_t W^{(1)}_t + \sqrt{1 - \rho^2_t} W^{(3)}_t,$$
where $W^{(1)}$ and $W^{(3)}$ are independent standard Brownian motions.

**A2.** The diffusion stochastic coefficients $\sigma^{(q)} = (\sigma^{(q)}_t)_{t \in [0,T]}$, $a^{(q)} = (a^{(q)}_t)_{t \in [0,T]}$, $q = 1, 2$, and $\rho = (\rho_t)_{t \in [0,T]}$ are càdlàg adapted processes.

As for the jump components $J^{(q)}$, in the next section we have FA jumps, i.e.

$$J^{(q)}_t = \sum_{k=1}^{N^{(q)}_t} \gamma_{\tau^{(q)}_k}, \quad q = 1, 2,$$

as specified with more detail below, where $N^{(q)} = (N^{(q)}_t)_{t \in [0,T]}$ are counting processes with $E[N^{(q)}_T] < \infty$.

More generally in section 4 each $J^{(q)}$ is allowed to be any pure jump semimartingale with possibly IA.

To begin with we assume to have equally spaced and synchronous observations. The consistency results under not equally spaced but synchronous observations are straightforward using Lemma 2.1 and Theorem 7.1 below. Generalization to not equally spaced and not synchronous observations are dealt with later. Let, for each $n$, $\pi_n = \{0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = T\}$ be a partition of $[0,T]$. For simplicity let us write $t_j$ in place of $t_j^{(n)}$. Define $h := t_j - t_{j-1} = \frac{T}{n}$, for every $j = 1, \ldots, n$ and $n = 1, 2, \ldots$. Note that $h \to 0$ if and only if $n \to \infty$, so when computing the limits of our interest we indifferently indicate one of the two.
A3. We choose a deterministic function, \( h \mapsto r(h) \), satisfying the following properties

\[
\lim_{h \to 0} r(h) = 0, \quad \lim_{h \to 0} \frac{h \log \frac{1}{h}}{r(h)} = 0.
\]

We denote \( r(h) \) by \( r_h \), and, for each \( q = 1, 2 \),

\[
D_{t}^{(q)} = \int_{0}^{t} a_{s}^{(q)} ds + \int_{0}^{t} \sigma_{s}^{(q)} dW_{s}^{(q)},
\]

the Brownian semimartingale part of \( X^{(q)} \).

As a consequence of the Paul Lévy result about the modulus of continuity of the Brownian motion paths, we can control how quickly the increments of the diffusion part of each \( \Delta_j X^{(q)} \) tend to zero. This is the key point to understand when \( \Delta_j X^{(q)} \) is likely to contain some jumps. More precisely, the Paul Lévy law (see e.g. [20], p.114, Theorem 9.25) implies that

\[
\text{a.s. } \lim_{h \to 0} \sup_{j \in \{1, \ldots, n\}} \frac{\left| \Delta_j W^{(q)} \right|}{\sqrt{2h \log \frac{1}{h}}} \leq 1. \tag{2}
\]

However the stochastic integral \( \sigma \cdot W \) is a time changed Brownian motion ([27], Theorems 1.9 and 1.10), i.e. \( \Delta_j (\sigma \cdot W) = B_{IV_{t_j}} - B_{IV_{t_{j-1}}} \), where \( B \) is a Brownian motion and \( IV_t \) is the integrated variance \( \int_{0}^{t} \sigma_{s}^{2} ds \) up to time \( t \). Note that the increments of the drift part of \( X \) tend to zero more quickly than \( \sqrt{2h \log \frac{1}{h}} \) as \( h \to 0 \), so for \( D^{(q)} \) we can reach a result similar to (2), as soon as the boundedness of the paths of \( a \) and \( \sigma \) is guaranteed (which is the case when they are càdlàg). In fact

\[
\sup_{j=1 \ldots n} \frac{\left| \int_{t_{j-1}}^{t_j} a_{s} ds + \int_{t_{j-1}}^{t_j} \sigma_{s} dW_{s} \right|}{\sqrt{2h \log \frac{1}{h}}} \leq \sup_{j} \frac{\left| \int_{t_{j-1}}^{t_j} a_{s} ds \right|}{\sqrt{2h \log \frac{1}{h}}} + \sup_{j} \frac{\left| \int_{t_{j-1}}^{t_j} \sigma_{s} dW_{s} \right|}{\sqrt{2h \log \frac{1}{h}}} \leq \frac{9}{h}. 
\]
\[ C(\omega) \sqrt{\frac{h}{\log \frac{1}{h}}} + \sup_j |B_{IV_{t_j}} - B_{IV_{t_{j-1}}}| \sqrt{\frac{2\Delta IV \log \frac{1}{\Delta IV}}{2M(\omega)h \log \frac{1}{M(\omega)h}}} \sup_j \sqrt{\frac{2M(\omega) \log \frac{1}{M(\omega)h}}{2 \log \frac{1}{h}}}, \]

where \( C(\omega) := \sup_{s \in [0,T]} |a_s(\omega)|, M(\omega) := \sup_{s \in [0,T]} |\sigma^2_s(\omega)| \). By [20] (Theorem 9.25) and the monotonicity of the function \( x \ln \frac{1}{x} \) it follows that as \( h \to 0 \), the right hand side has a limsup which is bounded by \( \sqrt{M(\omega)} \), thus for sufficiently small \( h \), even in the case of not equally spaced observations, the following holds.

**Lemma 2.1.** ([23]) Under \( A2 \) we have that, given an arbitrary partition \( \{t_0 = 0, t_1, \ldots, t_n = T\} \) of \([0, T]\), then for sufficiently small \( h := \sup_{j=1..n} |t_j - t_{j-1}| \) we have a.s.

\[ \sup_{j=1..n} \frac{\sqrt{2h} \log \frac{1}{h}}{h} \leq K_q(\omega), \quad q = 1, 2, \]

where \( K_q(\omega) := \sqrt{M(\omega) + 1} \) are finite random variables.

Last result implies that if \((\Delta_j X^{(q)})^2 > r_h \) and \( r_h \) is, for small \( h \), larger than \( 2K_q^2 h \log \frac{1}{h} \) (as it is, under \( A3 \)), then we have \((\Delta_j X^{(q)})^2 > 2K_q^2 h \log \frac{1}{h} \), and it is not likely that \( \Delta_j X^{(q)} \) coincides with the increment of a Brownian semimartingale, while it is likely that some jumps occurred within \([t_{j-1}, t_j]\) and made \( |\Delta_j X^{(q)}| \) large.

Application of Lemma [2.1] gives us the main tool for the construction of our estimators in the next section.

**Notation.**

- For any semimartingale \( Z \), \( \Delta Z_s = Z_s - Z_{s-} \) denotes the size of the jump of \( Z \) at time \( s \), while \( \Delta_j Z = Z_{t_j} - Z_{t_{j-1}} \) denotes the increment of process \( Z \) in the time interval \([t_{j-1}, t_j]\)

- \( IC_t = \int_0^t \rho_s \sigma_s^{(1)} \sigma_s^{(2)} ds \) denotes the integrated covariation up to time \( t \),

\[ \hat{IC}_{t,n} = \sum_{j=1..n: \, t_j \leq t} \Delta_j X^{(1)} 1_{\{\Delta_j X^{(1)} \leq r_h\}} \Delta_j X^{(2)} 1_{\{\Delta_j X^{(2)} \leq r_h\}}, \quad h = T/n, \]

is its thresh-
old estimator

- $IV_t^{(q)} = \int_0^t (\sigma_s^{(q)})^2 ds$ denotes the integrated variance of process $X^{(q)}$, $q=1,2$, up to time $t$ and $IV_{t,n}^{(q)} = \sum_{j=1..n: t_j \leq t} (\Delta_j X^{(q)})^2 1_{\{ (\Delta_j X^{(q)})^2 \leq r_h \}}$ is its threshold estimator.

- sometimes $\Delta_j X^{(q)} 1_{\{ (\Delta_j X^{(q)})^2 \leq r_h \}}$ is indicated briefly with $\Delta_j X^{(q)} \star$

- sometimes we write Plim to indicate the limit in probability. $\rightarrow$ indicates stable convergence in law of processes. See [18], ch. 8, sec. 5c, for the definition and properties of stable convergence in law, and [16] for further statement of useful properties.

3 Finite activity jumps: consistency and central limit theorem

In this section we assume that $J^{(q)}$ is any FA jump process: for each $q = 1, 2$,

$$J_t^{(q)} = \int_0^t \gamma_s^{(q)} dN_s^{(q)} = \sum_{k=1}^{N_t^{(q)}} \gamma_{\tau_k^{(q)}} ,$$

where $N^{(q)} = (N_t^{(q)})_{t \in [0,T]}$ is a counting process with $E[N_T^{(q)}] < \infty$, $\{\tau_k^{(q)}, k = 1,..., N_T^{(q)}\}$ denote the instants of jump of $J^{(q)}$ and $\gamma_{\tau_k^{(q)}}$ denote the sizes $\Delta J_t^{(q)}$ of the jumps occurred at $\tau_k^{(q)}$. Denote

$$\gamma^{(q)} = \min_{k=1,...,N_T^{(q)}} |\gamma_{\tau_k^{(q)}}| .$$

A4. Assume $E[N_T^{(q)}] < \infty$ and $P(\gamma_{\tau_k^{(q)}} = 0) = 0$, $\forall k = 1,..., N_T^{(q)}, q = 1,2$. 

Remark 3.1. Condition A4 implies that a.s. $\gamma^{(q)} > 0$. 

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Example 3.2. If \( J^{(q)} \) are FA Lévy processes, then they are of compound Poisson type ([9], Proposition 3.3, section 3.2): \( N^{(q)} \) are simple Poisson processes with constant intensities \( \lambda^{(q)} \) and for each \( q \) the random variables \( \gamma_{\tau_k}^{(q)} \) are i.i.d., for \( k = 1, ..., N_T^{(q)} \), are independent on \( N^{(q)} \) and satisfy condition A4.

We remark that the consistency and CLT we reach in this section are valid in presence of general finite activity jump processes, in that we do not need any assumptions on the law of the jump sizes, or of the counting processes \( N^{(q)} \), nor any assumption of independence. We do not even need that \( J^{(q)} \) are FA jumping semimartingales, we only need that A4 holds, which is true if \( J^{(q)} \) are (FA jumping) semimartingales.

Now we construct our threshold estimators.

Definition 3.3. We define for \( r, l \in \mathbb{N} \)

\[
\tilde{v}^{(n)}_{r,l}(X^{(1)}, X^{(2)})_t = h^{1-\frac{r+l}{2}} \sum_{j:t_j \leq t} (\Delta_j X^{(1)})^r (\Delta_j X^{(2)})^l,
\]

\[
w^{(n)}(X^{(1)}, X^{(2)})_t = h^{-1} \sum_{j:t_j+1 \leq t} \prod_{i=0}^1 \Delta_{j+i} X^{(1)} \prod_{i=0}^1 \Delta_{j+i} X^{(2)}.
\]

and their analogous threshold versions

\[
\tilde{v}^{(n)}_{r,l}(X^{(1)}, X^{(2)})_T = h^{1-\frac{r+l}{2}} \sum_{j:t_j \leq t} (\Delta_j X^{(1)})^r 1_{\{ (\Delta_j X^{(1)})^2 \leq r_h \}} (\Delta_j X^{(2)})^l 1_{\{ (\Delta_j X^{(2)})^2 \leq r_h \}},
\]

\[
w^{(n)}(X^{(1)}, X^{(2)})_T = h^{-1} \sum_{j:t_j+1 \leq t} \prod_{i=0}^1 \Delta_{j+i} X^{(1)} 1_{\{ (\Delta_{j+i} X^{(1)})^2 \leq r_h \}} \prod_{i=0}^1 \Delta_{j+i} X^{(2)} 1_{\{ (\Delta_{j+i} X^{(2)})^2 \leq r_h \}}.
\]

\( \tilde{v}^{(n)}_{r,l}(X^{(1)}, X^{(2)})_T \) and \( w^{(n)}(X^{(1)}, X^{(2)})_T \) are used in [2] to estimate IC\(_T\) in the case where \( X^{(q)} \) are diffusion processes. \( \tilde{v}^{(n)}_{r,l}(X^{(1)}, X^{(2)})_T \) and \( \tilde{w}^{(n)}(X^{(1)}, X^{(2)})_T \) are modified versions for the case of jump-diffusion processes: by Theorem 7.1 they exclude
from the sums the terms containing jumps. Note that \(\tilde{v}_{1,1}(X^{(1)}, X^{(2)})_t = \hat{I}C_{t,n}\), for all \(t \in [0, T]\).

In view of the practical application of our estimator we are now interested in the speed of convergence of \(\hat{I}C_{T,n}\). We in fact reach even more. The first main result of this section is a joint central limit theorem for the threshold estimators

\[
\begin{pmatrix}
\hat{I}V^{(1)} & \hat{I}C \\
\hat{I}C & \hat{I}V^{(2)}
\end{pmatrix}
\]

which implies that in presence of finite activity jumps \(\hat{I}C_{T,n}\) converges to \(IC\) at speed \(\sqrt{h}, h = T/n\), and it allows to give estimators of standard dependence measures between the diffusion parts \(D^{(q)}\) of our processes \(X^{(q)}\), such as the realized diffusion regression coefficients up to time \(t\)

\[
\beta_{t}^{(1,2)} := \frac{IC_{t}}{IV_{t}^{(2)}}, \quad \beta_{t}^{(2,1)} := \frac{IC_{t}}{IV_{t}^{(1)}},
\]

and the realized diffusion correlation

\[
\rho_{t}^{(1,2)} := \frac{IC_{t}}{\sqrt{IV_{t}^{(1)}IV_{t}^{(2)}}}.
\]

**Theorem 3.4** (Joint CLT, FA jumps). Under assumptions from \(A1\) to \(A4\), with \(h = T/n\), we have, as \(h \to 0\),

\[
h^{-1/2} \begin{pmatrix}
\hat{I}V_{n}^{(1)} - IV_{n}^{(1)} & \hat{I}C_{n} - IC \\
\hat{I}C_{n} - IC & \hat{I}V_{n}^{(2)} - IV_{n}^{(2)}
\end{pmatrix} \overset{st}{\to} \frac{1}{\sqrt{2}} \begin{pmatrix}
2Z_{11} & Z_{12} + Z_{21} \\
Z_{12} + Z_{21} & 2Z_{22}
\end{pmatrix},
\]

where \(Z\) is the \(2 \times 2\) process with components

\[
Z_{11,t} := \int_{0}^{t} (\sigma_{s}^{(1)})^2 dB_{11s}
\]
\begin{align}
Z_{12,t} &:= \int_0^t \rho_s \sigma_s^{(1)} \sigma_s^{(2)} dB_{11s} + \int_0^t \sqrt{1 - \rho_s^2} \sigma_s^{(1)} dB_{12s} \\
Z_{21,t} &:= \int_0^t \rho_s \sigma_s^{(1)} \sigma_s^{(2)} dB_{11s} + \int_0^t \sqrt{1 - \rho_s^2} \sigma_s^{(1)} dB_{21s} \\
Z_{22,t} &:= \int_0^t \rho_s^2 (\sigma_s^{(2)})^2 dB_{11s} + \int_0^t \rho_s \sqrt{1 - \rho_s^2} (\sigma_s^{(2)})^2 (dB_{12s} + dB_{21s}) + \int_0^t (1 - \rho_s^2)(\sigma_s^{(2)})^2 dB_{21s}
\end{align}

and \( B \) is a \( 2 \times 2 \)-dimensional standard Brownian motion independent on the filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P) \) where our model is defined.

Note that the result for \( \hat{\beta}_{j}^{(q)} \) is consistent with [23], since

\[
\text{Var}(\sqrt{2} Z_{qq,t}) = 2 \int_0^T \rho^4_s (\sigma_s^{(q)})^4 ds + 4 \int_0^T \rho^2_s (1 - \rho^2_s) (\sigma_s^{(q)})^4 ds + 2 \int_0^T (1 - \rho^2_s)(\sigma_s^{(q)})^4 ds = 2 \int_0^T (\sigma_s^{(q)})^4 ds.
\]

**Corollary 3.5** (Consistency, FA jumps). *Under A1 to A4, as \( n \to \infty \), for all \( t \in [0,T] \)

\[
\hat{IC}_{t,n} \overset{P}{\to} IC_t,
\]

if a.s \( IV_j^j \neq 0 \) then

\[
\hat{\beta}_{i,n}^{(i,j)} := \frac{\hat{IC}_{t,n}}{IV_{t,n}^{(j)}} \overset{P}{\to} \beta_{i}^{(i,j)}, \quad (i, j) = (1, 2), (2, 1)
\]

if a.s \( IV_1^1 IV_2^2 \neq 0 \) then

\[
\hat{\rho}_{t,n}^{(1,2)} := \frac{\hat{IC}_{t,n}}{\sqrt{IV_{t,n}^{(1)}} \sqrt{IV_{t,n}^{(2)}}} \overset{P}{\to} \rho_{t}^{(1,2)}.
\]

**Corollary 3.6** (Speed of convergence of \( \beta \)s and \( \rho \), FA jumps). *If a.s. \( IV_{t}^{(j)} \neq 0 \) for all \( t \in [0,T] \) we have, for \( (i, j) = (1, 2) \) or \( (2, 1) \),

\[
h^{-1/2} \left( \hat{\beta}_{n}^{(i,j)} - \beta_{i}^{(i,j)} \right) \Rightarrow Z_{12} + Z_{21} \sqrt{\frac{2}{IV_{t}^{(j)}}} + \sqrt{2} Z_{j, j} \frac{IC}{(IV_{t}^{(j)})^2}.
\]
If a.s. $IV_t^{(1)} IV_t^{(2)} \neq 0$ for all $t \in [0, T]$ we have

$$h^{-1/2} \left( \hat{\rho}_n^{(1,2)} - \frac{IC}{\sqrt{IV^{(1)} IV^{(2)}}} \right) \xrightarrow{st} Z_{12} + Z_{21} \sqrt{2 IV^{(1)} IV^{(2)}} - Z_{22} IC \sqrt{2 IV^{(1)} (IV^{(2)})^{3/2}} - Z_{11} IC \sqrt{2 IV^{(2)} (IV^{(1)})^{3/2}}.$$ 

The following proposition allows us to give a CLT for the standardized version of the estimation error $\hat{IC}_{T,n} - IC_T$. Note that the asymptotic variance of $h^{-1/2}(\hat{IC}_{T,n} - IC_T)$, by Theorem 3.4, is given by

$$\left( \text{Var}(Z_{12,T} + Z_{21,T}) \right) / 2 = \int_0^T (1 + \rho_s^2)(\sigma_s^{(1)})^2(\sigma_s^{(2)})^2 ds.$$ 

**Proposition 3.7** (Estimate of the standard error for $\hat{IC}_{t,n}$, FA jumps). Under assumptions A1 to A4 we have, for all $t \in [0, T]$,

$$\hat{v}^{(n)}_{2,2}(X^{(1)}, X^{(2)})_t - \tilde{w}^{(n)}(X^{(1)}, X^{(2)})_t \xrightarrow{P} \int_0^t (1 + \rho_s^2)(\sigma_s^{(1)})^2(\sigma_s^{(2)})^2 ds.$$ 

We now are ready to present the central limit theorem for the standardized estimation error.

**Corollary 3.8** (CLT for the standardized version of $\hat{IC}_{t,n} - IC_t$, FA jumps). Under A1 to A4, if a.s. $\int_0^t (1 + \rho_s^2)(\sigma_s^{(1)})^2(\sigma_s^{(2)})^2 ds \neq 0$ we have

$$\frac{\hat{IC}_{t,n} - IC_t}{\sqrt{h\sqrt{\hat{v}^{(n)}_{2,2}(X^{(1)}, X^{(2)})_t - \tilde{w}^{(n)}(X^{(1)}, X^{(2)})_t}}} \xrightarrow{d} \mathcal{N},$$

where $\mathcal{N}$ denotes a standard Gaussian random variable.

**Remark 3.9** (Estimate of the co-jumps). By Corollary 3.5 clearly we have an estimate of the sum of the co-jumps up to $T$ simply subtracting $\hat{IC}_{T,n}$ from the quadratic covariation estimator:

$$\sum_{j=1}^n \Delta_j X^{(1)} \Delta_j X^{(2)} - \hat{IC}_{T,n} \xrightarrow{P} \sum_{0 \leq s \leq T} \Delta J_s^{(1)} \Delta J_s^{(2)},$$
as \( n \to \infty \). Analogously we can obtain an estimator of the sum of the co-jumps up to each time \( t \in [0, T] \).

An estimate of each \( \Delta J_s^{(1)} \Delta J_s^{(2)} \), with \( s \in [0, T] \), is obtained using

\[
\Delta_j X^{(1)} \Delta_j X^{(2)} - \Delta_j X^{(1)} 1_{\{ (\Delta_j X^{(1)})^2 \leq r_h \}} \Delta_j X^{(2)} 1_{\{ (\Delta_j X^{(2)})^2 \leq r_h \}},
\]

with \( j \) such that \( s \in ]t_{j-1}, t_j] \). Alternatively, as we consider one single term, and not the sum of \( n \) terms, even

\[
\Delta_j X^{(1)} 1_{\{ (\Delta_j X^{(1)})^2 > r_h \}} \Delta_j X^{(2)} 1_{\{ (\Delta_j X^{(2)})^2 > r_h \}}
\]

or

\[
\Delta_j X^{(1)} \Delta_j X^{(2)}
\]

estimate the co-jump \( \Delta J_s^{(1)} \Delta J_s^{(2)} \), with \( s \in ]t_{j-1}, t_j] \), since \( |\Delta_j X^{(q)} 1_{\{ (\Delta_j X^{(q)})^2 \leq r_h \}} \Delta_j X^{(\ell)}| \leq 2\sqrt{r_h} \sup_{s \in [0, T]} |X^{(\ell)}|, q = 1, 2, \ell = 3 - q \), and \( |\Delta_j X^{(1)} 1_{\{ (\Delta_j X^{(1)})^2 \leq r_h \}} | \times 1_{\{ (\Delta_j X^{(2)})^2 \leq r_h \}} \leq r_h \) tend to zero in probability as \( h \to 0 \), by the pathwise boundedness of each \( X^{(\ell)} \) on \([0, T]\). However, as we show in section 5, estimator (5) has the best finite sample properties in the simulations of Model 1 having FA jumps. 

Remark 3.10. Finite sample performance and microstructure noises. Our theoretic results allow to estimate \( IC_T \) and the co-jumps asymptotically for \( h \to 0 \), while in practice for very small values of \( h \) financial time series are affected by microstructure noises which introduce a bias which is larger as \( h \) is smaller. In section 5 we implement our estimators of the integrated covariance and of the co-jumps on simulations of realistic financial time series and we find that they have good performance already with temporal mesh \( h \) corresponding to five minutes, a time lag at which prices are not usually affected by microstructure noises ([5]). However
we remark that when iid microstructure noises contaminate the observations of each asset price \( X^{(q)} \), the threshold estimator rules out even the noises, similarly as it rules out the contribution of the jumps ([24]).

**Remark 3.11. Asynchronous observations.** It is known that the problem of the estimation of the covariation among two assets undergoes the so called Epps effect, i.e. in the empirical applications the estimator tends to zero as the step of observation \( h \) tends to zero. The asynchronicity among the observations of \( X^{(1)} \) and \( X^{(2)} \) is considered one of the possible causes ([26]; [3], section 2.10.2). In fact some Authors have tackled the problem of reaching a consistent estimator of the covariation even when data are asynchronous and \( h \to 0 \), under the assumption of Brownian semimartingale models (in [13] the estimator is introduced, however we refer to [12] where the observation times are allowed to be dependent on \( X^{(q)} \)).

At the time scale of five minutes the Epps effect probably does not affect our estimate of \( IC_T \). However even in presence of this microstructure-type noise (for smaller \( h \)) it is possible to make our estimator correctly converge to the integrated covariation, as detailed below.

Assume we dispose of two records \( \{ D^{(1)}_{\tau_0^{(n)}}, D^{(1)}_{\tau_1^{(n)}}, ... D^{(1)}_{\tau_{m(n)}^{(n)}} \}, \{ D^{(2)}_{\nu_0^{(n)}}, D^{(2)}_{\nu_1^{(n)}}, ... D^{(2)}_{\nu_{k(n)}^{(n)}} \} \), of observations of two Brownian semimartingales \( D^{(1)} \) and \( D^{(2)} \), with the two stochastic partitions \( 0 = \tau_0^{(n)} < \tau_1^{(n)} < ... \tau_{m(n)}^{(n)} \) and \( 0 = \nu_0^{(n)} < \nu_1^{(n)} < ... \nu_{k(n)}^{(n)} \) spanned on \([0, T]\). For simplicity let us write \( \nu_i \) and \( \tau_j \) in place of \( \nu_i^{(n)} \) and \( \tau_j^{(n)} \). The idea of Hayashi and Kusuoka is to select only some of the cross variations \((D^{(1)}_{\tau_j} - D^{(1)}_{\tau_{j-1}})(D^{(2)}_{\nu_i} - D^{(2)}_{\nu_{i-1}})\), in order to estimate the covariation, and precisely the ones for which there is an intersection between the time intervals \([\tau_{j-1}, \tau_j]\) and \([\nu_{i-1}, \nu_i]\).
We show here that using their result ([12], Corollary 2.2) we in fact reach the same kind of consistency in the case of asynchronous observations even in presence of finite activity jumps. The idea is very simple: we first eliminate the jumps, using threshold technique, and then apply the Hayashi and Kusuoka estimator to the estimated continuous components $\hat{D}(q)$. Recall that

$$(X_a^{(q)} - X_b^{(q)})_* = (X_a^{(q)} - X_b^{(q)})1_{\{(X_a^{(q)} - X_b^{(q)})^2 \leq r_h\}},$$

for any two time instants $a$ and $b$, $h := \sup_{j=1..m(n)}(\tau_j - \tau_{j-1}) \vee \sup_{i=1..k(n)}(\nu_i - \nu_{i-1})$.

We in fact have the following

**Theorem 3.12** (Asynchronous observations). Let A1 to A4 hold, $0 = \tau_0 < \tau_1 < ... < \tau_{m(n)}$, $0 = \nu_0 < \nu_1 < ... < \nu_{k(n)}$ be two sequences of stopping times such that $\tau_{m(n)} \uparrow T$, $\nu_{k(n)} \uparrow T$ a.s., as $n \to \infty$ then

$$\sum_{j=1..m(n), i=1..k(n)} (X_{\tau_j}^{(1)} - X_{\tau_{j-1}}^{(1)})_* (X_{\nu_i}^{(2)} - X_{\nu_{i-1}}^{(2)})_* 1_{\{[\tau_{j-1}, \tau_j] \cap [\nu_{i-1}, \nu_i] \neq \emptyset\}} \overset{P}{\to} IC_T,$$

as $h := \sup_{j=1..m(n)}(\tau_j - \tau_{j-1}) \vee \sup_{i=1..k(n)}(\nu_i - \nu_{i-1}) \overset{P}{\to} 0$.

**4 Infinite activity jumps: consistency**

In this section we allow the jump components of processes $X^{(q)}$ to have infinite activity, so we are here in the case where $X^{(q)}$ are general Itô semimartingales.

Any unidimensional Itô semimartingale has a representation as in (1) with each $J^{(q)}$ decomposed as

$$J^{(q)} = J_1^{(q)} + \tilde{J}_2^{(q)},$$

$$J_1^{(q)}(\omega) = \int_0^t \int_{|\gamma^{(q)}(\omega, t, x)| > 1} \gamma^{(q)}(\omega, t, x) \mu^{(q)}(\omega, dx, dt),$$

$$\tilde{J}_2^{(q)}(\omega) = \int_0^t \int_{|\gamma^{(q)}(\omega, t, x)| \leq 1} \gamma^{(q)}(\omega, t, x) \tilde{\mu}^{(q)}(\omega, dx, ds),$$

(7)
where $\mu^{(q)}$ is the Poisson random measure of the jumps of $J^{(q)}$, $\tilde{\mu}^{(q)}(\omega, dx, ds) = \mu^{(q)}(\omega, dx, ds) - \nu^{(q)}(\omega, dx, ds)$ is its compensated measure, $\nu^{(q)}(\omega, dx, ds) = dx \times ds$, the coefficients $a^{(q)}, \sigma^{(q)}, \gamma^{(q)}$ are predictable and $\int 1 \wedge (\gamma^{(q)})^2(\omega, t, x) dx$ is a.s. finite (see [16], pp.3,4; [15], (2.11)).

Conditions A2 and A4’ below guarantee local boundedness properties of such coefficients.

A4’. $\int 1 \wedge (\gamma^{(q)})^2(\omega, t, x) dx$ is locally bounded.

For each $q = 1, 2$, $J_1^{(q)}$ is a finite activity jump process of type $J_1^{(q)} = \sum_{k=1}^{N^{(q)}_T} \gamma^{(q)}_k$, as in section 3, where $E[N^{(q)}_T] < \infty$ is equivalent to $E[\int_{|\gamma^{(q)}| > 1} dx] < \infty$ and now the sizes $|\gamma_k^{(q)}|$ are all larger than 1; on the contrary $\tilde{J}_2^{(q)}$ accounts for the infinite activity jumps of $J^{(q)}$, since generally $\int_{|\gamma^{(q)}| \leq 1} dx = +\infty$. $\tilde{J}_2^{(q)}$ is a compensated sum of jumps, where each jump is bounded in absolute value by 1. Therefore, for each $q = 1, 2$, $J_1^{(q)}$ accounts for the ”large” and rare jumps of $X^{(q)}$, while $\tilde{J}_2^{(q)}$ accounts for the frequent and small jumps.

Example 4.1. If one of the two processes $J^{(q)}$ is a pure jump Lévy process, it is always possible to decompose it as in (7) with $\gamma^{(q)}(\omega, t, x) \equiv x$ but $\nu^{(q)}(\omega, dx, ds) \equiv \nu^{(q)}(dx) \times ds$, where $\nu^{(q)}$ is the Lévy measure of $J^{(q)}$ and is a deterministic $\sigma$-finite measure such that $\int_{\mathbb{R}} 1 \wedge x^2 \nu^{(q)}(dx) < \infty$ but generally such that $\int_{|x| \leq 1} \nu^{(q)}(dx) = +\infty$.

We prove that $\hat{IC}_{t,n}$ is still a consistent estimator of $IC_t$, for all $t \in [0, T]$. For ease of notation we only consider $IC$ up to time $T$ and evenly spaced synchronous observations. Not evenly spaced but synchronous observations (with $h = \sup_j |t_j - t_{j-1}|$) and arbitrary $t \in [0, T]$ are straightforward. As a consequence the same
estimators of the co-jumps, presented in the previous section, are consistent even in the present framework.

As for the speed of convergence of $\hat{IC}_{T,n}$, in the presence of infinite activity jump components, things are more complicated in that such a speed is determined both by the dependence structure between $\tilde{J}_2^{(1)}$, $\tilde{J}_2^{(2)}$ and by the amount of jump activity of each $\tilde{J}_2^{(q)}$. In [11] we consider two Lévy infinite activity jump components $\tilde{J}_2^{(1)}$ and $\tilde{J}_2^{(2)}$ with a dependence structure described by a Lévy copula. We find that, when $\tilde{J}_2^{(1)}$ and $\tilde{J}_2^{(2)}$ do depend, the speed is still $\sqrt{h}$ only when the activity of jump of at least one process is moderate (Blumenthal-Getoor index smaller than 1), otherwise the speed is less than $\sqrt{h}$.

We now state the main result in presence of infinite activity jumps.

**Theorem 4.2** (Consistency in presence of IA jumps, synchronous observations).

Let $(X_t^{(1)})_{t \in [0,T]}$ and $(X_t^{(2)})_{t \in [0,T]}$ be two processes of the form (7). Assume $A1$, $A2$, $A3$ and $A4'$. Then

$$\hat{IC}_{T,n} \xrightarrow{P} IC_T,$$

as $n \to \infty$.

**Remark 4.3** (Estimate of the co-jumps). Even in this framework of infinite activity jumps, as a consequence of Theorem 4.2, the sum of the co-jumps up to $T$ is estimated by

$$\sum_{j=1}^{n} \Delta_j X^{(1)} \Delta_j X^{(2)} - \sum_{j=1}^{n} \Delta_j X^{(1)} \mathbf{1}_{\{\left(\Delta_j X^{(1)}\right)^2 \leq r_h\}} \Delta_j X^{(2)} \mathbf{1}_{\{\left(\Delta_j X^{(2)}\right)^2 \leq r_h\}}.$$

Estimates of a single co-jump $\Delta J_s^{(1)} \Delta J_s^{(2)}$, with $s \in [t_{j-1}, t_j]$, exactly as in section 3, are given by (4), (5) or (6). Simulations in section 5.3 show that for Model
2, with IA jumps, in fact estimator (5) is a little bit more biased than (6) but is still acceptable. Note that since in sec. 5 each $J^{(q)}$ has infinite activity and $J^{(2)} = \rho J^{(1)} + \sqrt{1 - \rho^2} J^{(3)}$, each $]t_{j-1}, t_j]$ contains an infinite number of co-jump instants.

5 Implementation

5.1 Choice of the threshold

Our estimators depend on the threshold function $r_h$. In this section we check on simulations how the results are sensitive to the choice of $r_h$ in a given class. This is only an informal and necessarily limited investigation. Formal study of methods for optimal threshold selection in a given model is object of further research.

In principle there are many functions $r_h$ satisfying conditions $A3$. However on simulations we find that the choice of $r_h$ within the family of powers of $h$, $r_h = ch^\beta$, with $c$ a constant and $\beta$ a power in $]0, 1[$, seems to be sufficiently good.

We simulate two kind of models: Model 1, proposed in [14], where each $X^{(q)}$ has stochastic volatility and a FA Compound Poisson jump part and Model 2, proposed in [8], where each $X^{(q)}$ has constant volatility and IA jumps, as described in Table A. For Model 1 the parameters of the univariate $X^{(q)}$ are taken from [14]. A path of each $\sigma$ varies most between 0.013 and 0.019 in a day. For Model 2 the parameters of the univariate $X^{(q)}$ are taken from Table 2 of [8] for GE and HWP stocks. Note that the parameter $Y$ is not significantly different from zero for the two considered stocks, so that the CGMY process can be reduced to the VG process.

The VG process is characterized by three parameters $\kappa$, $\theta$ and $\zeta$. It is obtained
by evaluating a Brownian motion with drift, $\theta t + \zeta B_t$, at a random time $G_t$ given by a gamma process, a Lévy process whose lag $h$ increments $G_{t+h} - G_t$ are distributed as Gamma r.v.s with mean $h$ and variance $h\kappa$. It turns out that the VG process is pure jump and has infinite, but moderate, activity (it is a process with finite variation).

To effectively introduce non zero co-jumps, in each model the jump component $J^{(2)}$ of $X^{(2)}$ is correlated with $J^{(1)}$ of $X^{(1)}$ in the following way: we generate $J^{(1)}$ and an independent $J^{(3)}$ with parameters as in Table A, then $J^{(2)} = \rho J^{(1)} + \sqrt{1-\rho^2} J^{(3)}$. The simulation of the model paths has been made using the Euler scheme with increments of 1 second, then we have taken the five minutes synchronous returns and constructed our daily threshold estimator $\hat{IC}_{T,n}$. We simulated 3000 bivariate paths.

For each model we implement the estimator of $IC$ as $r_h$ varies. Figures 1-2 show how the mean relative bias in percentage form

$$100 \frac{(IC_{T,n} - IC)}{IC}$$

varies as $\beta$ varies in $]0, 1[$ for $c = 0.1, \ldots, 5.6$ with step 0.5 in Models 1 and 2, for $h$ fixed equal to five minutes ($n = 84$ observations per day, time unit of measure $T=1$ day, $h = 1/84$). It is evident that the choice $c = 0.1$ is the best one since in presence of FA jumps ($\lambda^{(q)} = 0.118$) it allows to decrease the bias as $\beta$ increases. In fact in the case of IA jumps the bias is much larger but $c = 0.1$ allows to reach, for high $\beta$, the lowest possible error. Figures 3-4 show the empirical densities and the QQ-plots of the normalized bias

$$\frac{\hat{IC}_{T,n} - IC_T}{\sqrt{h} \sqrt{\mathbb{E}^{(n)}_{2,2}(X^{(1)}, X^{(2)})_T - \bar{w}^{(n)}(X^{(1)}, X^{(2)})_T}}$$

(8)
when \( r_h \) varies as before, for fixed \( h \) equal to five minutes, for Model 1 with \( \lambda^{(q)} = 0.014 \). The same plots for Model 1 with \( \lambda^{(q)} = 0.118 \) and Model 2 are shown in Figures 5-6 and 7-8 respectively. We conclude that the best choice is \( r_h = 0.1 h^{0.99} \).

As a further check in Figures 9-10 we made the same plots for Model 1 with \( \lambda^{(q)} = 0 \) and we found that the choice of \( r_h \) gives good results as well.

### 5.2 Estimates of \( IC \) and \( \sum_{0 \leq t \leq T} \Delta J_t^{(1)} \Delta J_t^{(2)} \) on simulations

We report here the performance of the estimators of \( IC_T \) and of the sum \( \sum_{0 \leq t \leq T} \Delta J_t^{(1)} \Delta J_t^{(2)} \) of the co-jumps up to time \( T \), where the threshold is the one selected in the previous subsection. \( T \) is kept fixed to one day, \( h \) equals five minutes. Figures 11-12-13 show the histograms of \( 100 \frac{(\hat{IC}_{T,n} - IC)}{IC} \) to check the efficiency of \( \hat{IC}_{T,n} \) for Model 1, \( \lambda^{(q)} = 0.014 \), Model 1, \( \lambda^{(q)} = 0.118 \) and Model 2 respectively. The relative summary statistics are shown in Tables 2 and 3. \( \hat{IC}_{T,n} \) has an acceptable performance in Model 1 and it is biased in Model 2 but note that the estimation errors for \( IC_T \) and \( \int_0^T (1 + \rho_t^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt \) compensate and give good empirical densities of the normalized bias in Figure 7 for \( c = 0.1 \) and \( \beta = 0.99 \) and Table 1. Figures 14-15-16 show the histograms of the following relative bias in percentage form

\[
100 \frac{(\sum_{j=1}^n \Delta_j X_t^{(1)} \Delta_j X_t^{(2)} - \hat{IC}_{T,n}) - \sum_{0 \leq t \leq T} \Delta X_t^{(1)} \Delta X_t^{(2)}}{\sum_{0 \leq t \leq T} \Delta X_t^{(1)} \Delta X_t^{(2)}}
\]

for the sum of the co-jumps in Model 1, \( \lambda^{(q)} = 0.014 \), Model 1, \( \lambda^{(q)} = 0.118 \) and Model 2. Note that since in Model 2 each \( J^{(q)} \) is a pure jump process with IA in fact each movement of \( J^{(q)} \) is a jump, and each time that \( J^{(1)} \) jumps even \( J^{(2)} \) does by the way we correlated them, therefore the best we can do to reach the true \( \sum_{0 \leq t \leq T} \Delta X_t^{(1)} \Delta X_t^{(2)} \) is to take the sum of the cross-products of the one-second
differences of processes $J^{(q)}$. Tables 4 and 5 show the relative summary statistics. The performance of the estimator of the sum of co-jumps is very good under Model 1 and a bit worse under Model 2. Under model 2 the estimate of $\sum_{0 \leq t \leq T} \Delta J^{(1)}_t \Delta J^{(2)}_t$ is much better than the one of $IC_T$.

5.3 Estimate of the single co-jumps

Using the threshold function selected in subsection 5.1 both for Model 1 and for Model 2 we implement (4), (5) and (6) to estimate each single co-jump, in order to check which is the most informative estimator for the single co-jumps. We consider 1 day time horizon and $h$ equal to five minutes. Figures 17-18 show the histograms of the 3000 values of $100 \frac{\hat{J}J_t - \Delta J^{(1)}_t \Delta J^{(2)}_t}{\Delta J^{(1)}_t \Delta J^{(2)}_t}$ for each estimator for both Model 1 ($\lambda_1 = \lambda_2 = 0.118$) and Model 2, where we define by $\hat{J}J_t$ (joint jumps) the estimate of $\Delta J^{(1)}_t \Delta J^{(2)}_t$. Tables 6-7 report the relative summary statistics. We conclude that the most informative estimate of $\Delta J^{(1)}_t \Delta J^{(2)}_t$ is (5) for Model 1 and (6) for Model 2, however, we find that (5) for Model 2 is still well acceptable. We find that, anyway all three estimators show a good performance, since the mean percentage estimation error in the worst case (estimator (4) Model 2) is 1% with low standard deviation.

6 Conclusions

In this paper we introduce a new estimator of the diffusion part $IC$ and of the co-jumps in the quadratic covariation of two semimartingales $X^{(q)}$. To capture the separate contributions to the quadratic covariation has important applications in finance (forecasting, option pricing, risk and credit risk management).
The estimator \( IC_{T,n} \) is constructed using a threshold criterion introduced in \[21\], and consists in summing properly selected cross products of increments of the two processes. Our estimator is consistent, and when the two jump parts have only finite activity a joint CLT for \( IC_{T,n} \) and the estimators \( \hat{IV}_T^{(q)} \) of the integrated variances is proved and delivers the following important consequences.

1. \( IC_{T,n} \) is also asymptotically Gaussian with speed of convergence \( \sqrt{h} \). A central limit theorem in presence of infinite activity jump parts is studied in a further paper (\[11\]) where we find that the speed of convergence of \( IC_{T,n} \) is determined both by the dependence structure between the two processes \( X^{(q)} \) and by the amount of jump activity of each \( J^{(q)} \).

2. Consistent estimators both of the sum of the co-jumps occurred within \([0, T]\) and of each single co-jump are obtained.

3. We construct asymptotically Gaussian estimators of the regression coefficients \( \beta \)'s and of the correlation coefficient between the two processes \( X^{(q)} \).

Further we find that in presence of FA jumps a slight modification of \( IC_{T,n} \) is consistent even when only non-synchronous observations are available.

We assess the choice of the threshold and check the performance of our estimators on two different kind of simulated models which are common in the financial literature. Model 1 has components with stochastic volatilities and FA jumps, while Model 2 has components with constant volatilities and IA jumps. We find that even with five minutes observations the performances of the estimators of \( \sum_{0 \leq t \leq T} \Delta J^{(1)}_t \Delta J^{(2)}_t \) and of the single co-jumps are satisfactory. \( IC_{T,n} \) is satisfactory in Model 1 while is biased in Model 2 but the corresponding normalized bias has still Gaussian behavior.
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7 Appendix

The following theorem is the key result, in the finite jump activity case, validating the idea that if \((\Delta_j X^{(q)})^2\) is larger than \(r_h\) then some jumps occurred in \([t_{j-1}, t_j]\) (and vice-versa). It is stated in the general case of not equally spaced observations.

**Theorem 7.1.** (23, FA jumps) Under the assumptions from A1 to A4, given an arbitrary partition \(\{t_0 = 0, t_1, ..., t_n = T\}\) of \([0, T]\), then for sufficiently small, but
strictly positive, \( h := \sup_{j=1..n} |t_j - t_{j-1}| \) (depending on \( \omega \)) we have a.s.

\[
1_{\{(\Delta_j X^{(q)})^2 \leq h \}} = 1_{\{\Delta_j N^{(q)} = 0\}}, \quad j = 1, 2, \ldots, n, \quad q = 1, 2.
\]

**Proof of Theorem 3.4 [Joint CLT]** By Theorem 7.1 we have, for all \( t \in [0, T] \),

\[
h^{-1/2} \left[ \sum_{t_j \leq t} \Delta_j X^{(1)} \Delta_j X^{(2)} - IC_t \right] = h^{-1/2} \left[ \sum_{t_j \leq t} \Delta_j X^{(1)} \Delta_j X^{(2)} I_{\{\Delta_j N^{(1)} = 0, \Delta_j N^{(2)} = 0\}} - IC_t \right]
\]

\[
= h^{-1/2} \left[ \sum_{t_j \leq t} \Delta_j D^{(1)} \Delta_j D^{(2)} - \int_0^t \rho_s \sigma_s^{(1)} \sigma_s^{(2)} ds \right] - h^{-1/2} \sum_{t_j \leq t} \Delta_j D^{(1)} \Delta_j D^{(2)} I_{\{\Delta_j N^{(2)} \neq 0\}} - h^{-1/2} \sum_{t_j \leq t} \Delta_j D^{(1)} 1_{\{\Delta_j N^{(1)} \neq 0\}} \Delta_j D^{(2)} I_{\{\Delta_j N^{(2)} \neq 0\}}.
\]

Each one of last three sums tends a.s. to zero as \( h \to 0 \), since it contains at least one \( I_{\{\Delta_j N^{(q)} \neq 0\}} \) and for any \( q = 1, 2 \) we have

\[
\text{Plim}_{n \to \infty} h^{-1/2} \sum_{t_j \leq t} \Delta_j D^{(1)} \Delta_j D^{(2)} I_{\{\Delta_j N^{(q)} \neq 0\}} \leq \text{Plim}_{n \to \infty} K_1(\omega) K_2(\omega) \sqrt{h} \log \frac{1}{h} N_{T}^{(q)} = 0.
\]

Moreover, analogously as in [23], for each \( q = 1, 2 \) we reach that

\[
h^{-1/2} \left( \sum_{t_j \leq t} (\Delta_j X^{(q)})^2 - IV_t^{(q)} \right) = h^{-1/2} \left( \sum_{t_j \leq t} (\Delta_j X^{(q)})^2 I_{\{\Delta_j N^{(q)} = 0\}} - IV_t^{(q)} \right) =
\]

\[
h^{-1/2} \left( \sum_{t_j \leq t} (\Delta_j D^{(q)})^2 - IV_t^{(q)} \right) - h^{-1/2} \sum_{t_j \leq t} (\Delta_j D^{(q)})^2 I_{\{\Delta_j N^{(q)} \neq 0\}},
\]

where the last term tends a.s. to zero as \( h \to 0 \). Therefore we have that

\[
h^{-1/2} \begin{pmatrix}
IV_n^{(1)} - IV^{(1)} & \hat{IC}_{T,n} - IC \\
\hat{IC}_n - IC & \hat{IV}_n^{(2)} - IV^{(2)}
\end{pmatrix}
\]

(9)
has the same limit in distribution as
\[
h_{t}^{-1/2} \left( \sum_{t_j \leq t} (\Delta_j D^{(1)})^2 - IV^{(1)} \quad \sum_{t_j \leq t} \Delta_j D^{(1)} \Delta_j D^{(2)} - IC \quad \sum_{t_j \leq t} \Delta_j D^{(2)} - IV^{(2)} \right).
\]
Note that
\[
\sum_{t_j \leq t} \Delta_j D^{(1)} \Delta_j D^{(2)} - IC_t = \sum_{t_j \leq t} \left( \Delta_j D^{(1)} \Delta_j D^{(2)} - \Delta_j < D^{(1)}, D^{(2)} > \right)
\]
and, along the lines of [5] (proof of Theorem 1, sec. 3.1), using Itô formula we know that
\[
d(D^{(1)}D^{(2)}) = D^{(1)}_1 dD^{(2)} + D^{(2)}_1 dD^{(1)} + d < D^{(1)}, D^{(2)} >,
\]
so
\[
\Delta_j(D^{(1)}D^{(2)}) = \int_{t_{j-1}}^{t_j} D^{(1)}_s dD^{(2)} + \int_{t_{j-1}}^{t_j} D^{(2)}_s dD^{(1)} + \Delta_j < D^{(1)}, D^{(2)} >.
\]
Therefore
\[
\Delta_j D^{(1)} \Delta_j D^{(2)} = \Delta_j(D^{(1)}D^{(2)}) - D^{(1)}_{t_{j-1}} \Delta_j D^{(2)} - D^{(2)}_{t_{j-1}} \Delta_j D^{(1)}
\]
\[
= \int_{t_{j-1}}^{t_j} D^{(1)}_s dD^{(2)} + \int_{t_{j-1}}^{t_j} D^{(2)}_s dD^{(1)} + \Delta_j < D^{(1)}, D^{(2)} > - D^{(1)}_{t_{j-1}} \Delta_j D^{(2)} - D^{(2)}_{t_{j-1}} \Delta_j D^{(1)},
\]
so that (10) equals
\[
\int_0^t \left( D^{(1)}_s - \sum_{t_j \leq t} D^{(1)}_{t_j-1} I_{\{s \in [t_j-1, t_j]\}} (s) \right) dD^{(2)}_s + \int_0^t \left( D^{(2)}_s - \sum_{t_j \leq t} D^{(2)}_{t_j-1} I_{\{s \in [t_j-1, t_j]\}} (s) \right) dD^{(1)}_s
\]
\[
= A^{(n)}_{12,t} + A^{(n)}_{21,t},
\]
where
\[
A^{(n)} = \begin{pmatrix}
\int_0^t \left( D^{(1)}_s - D^{(1)}_{\lfloor s \rfloor / n} \right) dD^{(1)}_s & \int_0^t \left( D^{(1)}_s - D^{(1)}_{\lfloor s \rfloor / n} \right) dD^{(2)}_s \\
\int_0^t \left( D^{(2)}_s - D^{(2)}_{\lfloor s \rfloor / n} \right) dD^{(1)}_s & \int_0^t \left( D^{(2)}_s - D^{(2)}_{\lfloor s \rfloor / n} \right) dD^{(2)}_s
\end{pmatrix}.
\]
As special cases, for each $q = 1, 2$

$$\sum_{t_j \leq t} (\Delta_j D^{(q)}_t)^2 - IV_t^{(q)} = \sum_{t_j \leq t} \left((\Delta_j D^{(q)}_t)^2 - \Delta_j < D^{(q)}, D^{(q)} > \right) =$$

$$2 \int_0^t \left(D_s^{(q)} - \sum_{j=1}^n D^{(q)}_{t_j} I_{\{s \in [t_{j-1}, t_j]\}}(s)\right) dD_s^{(q)} = 2 A^{(n)}_{qq,t}.$$

By Theorem 5.5 in [17] we have that

$$h^{-1/2} A^{(n)} \overset{st}{\to} \frac{Z}{\sqrt{2}},$$

with $Z$ as in (3). It follows that, as $n \to \infty$, (9) converges stably in law to

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 2Z_{11} & Z_{12} + Z_{21} \\ Z_{12} + Z_{21} & 2Z_{22} \end{pmatrix}.$$

**Proof of Corollary 3.6 [Speed of convergence of $\beta$'s and $\rho$, FA jumps]**

For all $t \in [0, T]$ we have

$$h^{-1/2} \left( \hat{\beta}_n^{(i,j)} - \frac{IC_t}{IV_t^{(j)}} \right) = h^{-1/2} \frac{IC_{t,n} - IC_t}{IV_t^{(j)}} + h^{-1/2} IC_t \frac{IV_{t,n}^{(j)} - \hat{IV}_t^{(j)}}{IV_{t,n}^{(j)} IV_t^{(j)}},$$

therefore

$$h^{-1/2} \left( \hat{\beta}_n^{(i,j)} - \frac{IC}{IV^{(j)}} \right) \overset{st}{\to} \frac{Z_{12} + Z_{21}}{\sqrt{2} IV^{(j)}} - IC \frac{\sqrt{2} Z_{jj}}{IV^{(j)}},$$

As for $\hat{\rho}_n^{(1,2)}$, note preliminarily that Theorem 3.4 implies that $h^{-1/2} \left( \sqrt{IV_n^{(j)}} - \sqrt{\hat{IV}_n^{(j)}} \right)$ converges stably, since, $t$ by $t$,

$$h^{-1/2} \left( \sqrt{IV_n^{(j)}} - \sqrt{\hat{IV}_n^{(j)}} \right) = h^{-1/2} \left( IV_n^{(j)} - \hat{IV}_n^{(j)} \right) \overset{st}{\to} - \frac{Z_{jj}}{\sqrt{2} IV^{(j)}}.$$

As a consequence

$$h^{-1/2} \left( \hat{\rho}_n^{(1,2)} - \frac{IC}{\sqrt{IV^{(1)}_n} IV^{(2)}_n} \right) =$$
\[ h^{-1/2} \frac{\hat{IC}_n - IC}{\sqrt{IV_n^{(1)}IV_n^{(2)}}} + h^{-1/2}IC \left( \frac{1}{\sqrt{IV_n^{(1)}}} - \frac{1}{\sqrt{IV_n^{(1)}IV_n^{(2)}}} \right). \]

The first term converges stably to \( \frac{Z_{12} + Z_{21}}{\sqrt{2 IV_n^{(1)}IV_n^{(2)}}} \), while the second term equals

\[
\frac{h^{-1/2}IC}{\sqrt{IV_n^{(1)}}} \left( \frac{1}{\sqrt{IV_n^{(2)}}} - \frac{1}{\sqrt{IV_n^{(1)}IV_n^{(2)}}} \right) =
\frac{h^{-1/2}IC}{\sqrt{IV_n^{(1)}}} \left( \frac{\sqrt{IV_n^{(2)}} - \sqrt{IV_n^{(2)}}}{\sqrt{IV_n^{(2)}IV_n^{(1)}}} \right) + \frac{h^{-1/2}IC}{\sqrt{IV_n^{(1)}IV_n^{(2)}}} \left( \sqrt{IV_n^{(1)}} - \sqrt{IV_n^{(1)}} \right)
\]

\[ \overset{st}{\rightarrow} - \frac{Z_{22}IC}{\sqrt{2 IV_n^{(1)}IV_n^{(2)}}^{3/2}} - \frac{Z_{11}IC}{\sqrt{2 IV_n^{(1)}IV_n^{(2)}}^{3/2}}. \]

\( \square \)

**Proof of Proposition 3.7** [Estimate of the standard error for \( \hat{IC}_n \), FA jumps] For \( t = T \) it is sufficient to show that as \( n \to \infty \)

\[ \hat{v}_{2,2}^{(n)}(X^{(1)}, X^{(2)}) \overset{P}{\rightarrow} \int_0^T (2\rho_t^2 + 1)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt, \]

and

\[ \hat{w}^{(n)}(X^{(1)}, X^{(2)}) \overset{P}{\rightarrow} \int_0^T \rho_t^2(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt. \]

For \( t < T \) the proof is analogous with \( \sum_{j=1}^n \) replaced by \( \sum_{j:t_j \leq t} \). By Theorem 7.1

we can write

\[
\text{Plim}_{n \to \infty} \hat{v}_{2,2}^{(n)}(X^{(1)}, X^{(2)})_T = \text{Plim}_{n \to \infty} h^{-1} \sum_{j=1}^n (\Delta_j D^{(1)})^2 1_{\{\Delta_j N^{(1)} = 0\}} (\Delta_j D^{(2)})^2 1_{\{\Delta_j N^{(2)} = 0\}}
\]

\[ = \text{Plim}_{n \to \infty} v_{2,2}^{(n)}(D^{(1)}, D^{(2)})_T - \text{Plim}_{n \to \infty} h^{-1} \sum_{j=1}^n (\Delta_j D^{(1)})^2 (\Delta_j D^{(2)})^2 1_{\{\Delta_j N^{(1)} \neq 0\}}. \]
\[-\operatorname{Plim}_{n \to \infty} h^{-1} \sum_{j=1}^{n} (\Delta_j D^{(1)})^2 (\Delta_j D^{(2)})^2 1_{\{\Delta_j N^{(1)} \neq 0\}} \]
\[+ \operatorname{Plim}_{n \to \infty} h^{-1} \sum_{j=1}^{n} (\Delta_j D^{(1)})^2 1_{\{\Delta_j N^{(1)} \neq 0\}} (\Delta_j D^{(2)})^2 1_{\{\Delta_j N^{(2)} \neq 0\}}.\]

By Theorem 2.1 in [6],
\[\operatorname{Plim}_{n \to \infty} n^{-1} \sum_{j=1}^{n} (\Delta_j D^{(1)})^2 (\Delta_j D^{(2)})^2 1_{\{\Delta_j N^{(1)} \neq 0\}} (\Delta_j D^{(2)})^2 1_{\{\Delta_j N^{(2)} \neq 0\}}.\]

whereas the other terms are all zero. In fact for any \(q = 1, 2\)
\[\operatorname{Plim}_{n \to \infty} n^{-1} \sum_{j=1}^{n} (\Delta_j D^{(1)})^2 (\Delta_j D^{(2)})^2 1_{\{\Delta_j N^{(q)} \neq 0\}} \leq \operatorname{Plim}_{n \to \infty} K^2_1(\omega) K^2_2(\omega) h \left( \log \frac{1}{h} \right)^2 N_T^{(q)} = 0.\]  

(11)

Now we deal with \(\bar{w}^{(n)}(X^{(1)}, X^{(2)})_T\). Analogously as before
\[\operatorname{Plim}_{n \to \infty} \bar{w}^{(n)}(X^{(1)}, X^{(2)})_T\]
\[= \operatorname{Plim}_{n \to \infty} n^{-1} \sum_{j=1}^{n-1} \left[ \prod_{i=0}^{1} \Delta_{j+i} D^{(1)} (1 - 1_{\{\Delta_{j+i} N^{(1)} \neq 0\}}) \prod_{i=0}^{1} \Delta_{j+i} D^{(2)} (1 - 1_{\{\Delta_{j+i} N^{(2)} \neq 0\}}) \right],\]
which coincides with the sum of \(\operatorname{Plim}_{n \to \infty} w^{(n)}(D^{(1)}, D^{(2)})_T\) with a finite number of terms which are shown to be negligible. By Theorem 2.1 in [6], \(\operatorname{Plim}_{n \to \infty} w^{(n)}(D^{(1)}, D^{(2)})_T = \int_0^T \rho^2_s (\sigma^{(1)}_s)^2 (\sigma^{(2)}_s)^2 dt\), while the other terms are given by the product of \(\prod_{i=0}^{1} \Delta_{j+i} D^{(1)} \prod_{i=0}^{1} \Delta_{j+i} D^{(2)}\) with at least one of the indicators \(1_{\{\Delta_{j+i} N^{(q)} \neq 0\}}\), for an \(s \in \{0, 1\}\).

Therefore the limit in probability of each such term is zero as in (11).

Proof of Corollary 3.8 [CLT for the standardized version of \(\hat{IC}_{t,n} - IC_t, FA\ jumps\)]. By Theorem 3.4 we have
\[h^{-1/2} \left( \hat{IC}_n - IC \right) \overset{st}{\to} \frac{1}{\sqrt{2}} \left( \int_0^T 2\sigma^{(1)}_s \sigma^{(2)}_s \rho_s dB^{11}_s + \int_0^T \sigma^{(1)}_s \sigma^{(2)}_s \sqrt{1 - \rho^2_s} \left[ dB^{12}_s + dB^{21}_s \right] \right).\]  

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The variance of the last term at time \( t \) is \( \int_0^t (1 + \rho_s^2)(\sigma_t^s)^2(\sigma_t^2)^2 ds \). By Proposition 3.7, we then obtain that

\[
\sqrt{h} \frac{\hat{IC}_{t,n} - IC_t}{\sqrt{\sum_{j=1}^m \tilde{L}_j(X^{(1)}, X^{(2)})_t - \tilde{L}_n(X^{(1)}, X^{(2)})_t}} \xrightarrow{d} \mathcal{N},
\]

where \( \mathcal{N} \) is a standard Gaussian r.v.. \( \square \)

**Proof of Theorem 3.12 [Asynchronous observations]** Note that we can assume that \( a \) and \( \sigma \) are bounded on \([0, T]\) (15), so that the Brownian semimartingale parts \( D^{(q)} \) of \( X^{(q)} \) belong to \( L^8 \). Using Theorem 7.1 in the not evenly-spaced observations case (23) with \( h := \sup_{j=1..m^{(n)}} (\tau_j - \tau_{j-1}) \vee \sup_{i=1..k^{(n)}} (\nu_i - \nu_{i-1}) \), a.s. for sufficiently small \( h \) we can write

\[
\sum_{j=1..m^{(n)}, i=1..k^{(n)}} (X^{(1)}_{\tau_j} - X^{(1)}_{\tau_{j-1}}) (X^{(2)}_{\nu_i} - X^{(2)}_{\nu_{i-1}}) \mathbf{1}_{\{\tau_{j-1}, \nu_i, \nu_{i-1} \neq \emptyset\}} = \\
\sum_{j=1..m^{(n)}, i=1..k^{(n)}} (D^{(1)}_{\tau_j} - D^{(1)}_{\tau_{j-1}}) \mathbf{1}_{\{N^{(1)}_{j} - N^{(1)}_{j-1} = 0\}} (D^{(2)}_{\nu_i} - D^{(2)}_{\nu_{i-1}}) \mathbf{1}_{\{N^{(2)}_{\nu_i} - N^{(2)}_{\nu_{i-1}} = 0\}} \mathbf{1}_{\{\tau_{j-1}, \nu_i, \nu_{i-1} \neq \emptyset\}}
\]

\[
= \sum_{j=1..m^{(n)}, i=1..k^{(n)}} (D^{(1)}_{\tau_j} - D^{(1)}_{\tau_{j-1}}) (D^{(2)}_{\nu_i} - D^{(2)}_{\nu_{i-1}}) \mathbf{1}_{\{\tau_{j-1}, \nu_i, \nu_{i-1} \neq \emptyset\}}
\]

\[
- \sum_{j=1..m^{(n)}, i=1..k^{(n)}} (D^{(1)}_{\tau_j} - D^{(1)}_{\tau_{j-1}}) (D^{(2)}_{\nu_i} - D^{(2)}_{\nu_{i-1}}) \left[ \mathbf{1}_{\{N^{(1)}_{\nu_i} - N^{(1)}_{\nu_{i-1}} \neq 0\}} + \mathbf{1}_{\{N^{(2)}_{\nu_i} - N^{(2)}_{\nu_{i-1}} \neq 0\}} \right] \mathbf{1}_{\{\tau_{j-1}, \nu_i, \nu_{i-1} \neq \emptyset\}},
\]

The first sum of the r.h.s. tends to \( IC_T \) in probability by Corollary 2.2 in (12), with \( f \equiv g \equiv 1 \), while each sum in the second term is dominated in absolute value, for a suitable \( q \), by

\[
\sup_j |D^{(1)}_{\tau_j} - D^{(1)}_{\tau_{j-1}}| \sup_i |D^{(2)}_{\nu_i} - D^{(2)}_{\nu_{i-1}}| N^{(q)}_T,
\]

which tends a.s. to zero as \( h \to 0 \), by Lemma 2.1 \( \square \)
The following facts are used within the proof of Theorem 4.2.

Without loss of generality (as in [15], Lemma 4.6) we can assume that

\[ A5. \int_{x \in \mathbb{R}} 1 \wedge (\gamma_{(q)})^2(\omega, t, x)dx \text{ is bounded.} \]

**Lemma 7.2.** For each \( q = 1, 2 \) we have the following.

1. If processes \( a \) and \( \sigma \) are càdlàg then, under \( A3 \), a.s., for small \( h \), \( 1_{\{(\Delta_j D^{(q)})^2 > rh\}} = 0 \), uniformly in \( j \).

2. Under \( A5 \) we have that, for each \( j = 1, \ldots, n \), \( E[(\Delta_j \tilde{J}^{(q)})^2] \leq Kh \), for a positive constant \( K \).

**Proof.** Part 1. is a consequence of Lemma 2.1.

Part 2.

\[
E[(\Delta_j \tilde{J}^{(q)})^2] = E\left[ \int_{t_{j-1}}^{t_j} \int_{|\gamma_{(q)}| \leq 1} (\gamma_{(q)})^2 \nu_{(q)}(dx, ds) \right] = E\left[ \int_{t_{j-1}}^{t_j} \int_{|\gamma_{(q)}| \leq 1} (\gamma_{(q)})^2 dxds \right]
\]

since, by assumption \( A5 \), \( \int_{|\gamma_{(q)}| \leq 1} (\gamma_{(q)})^2 dx \) is bounded, the last term is dominated by \( Kh \) for some positive constant \( K \). \( \square \)

The following lemma generalizes analogous results given in [23] from the framework of Lévy jumps to the one of Itô semimartingale jumps.

**Lemma 7.3.** The following facts hold.

1. Let us consider any sequence \( \pi_n \) of partitions \( \{0, t_1, \ldots, t_n = T\} \) of \([0, T]\), \( n \in \mathbb{N} \), such that \( \max_{j=1,..,n} |t_j - t_{j-1}| \to 0 \) as \( n \to \infty \). For each \( q = 1, 2 \), as long as \( \tilde{J}^{(q)}_2 \) is a semimartingale, we can find a subsequence \( n_k \) for which a.s., for
any \( \delta > 0 \) there exists a sufficiently large \( k \) such that for all \( j = 1, \ldots, n_k \) on \( \{(\Delta_j \tilde{J}_2^{(q)})^2 \leq 4r_{hk}\} \) we have

\[
(\Delta_j \tilde{J}_2^{(q)})^2 \leq 4r_{hk} + \delta, \quad \forall s \in [t_{j-1}, t_j].
\]

2. Under A3 and A5, for each \( q = 1, 2 \), we have \( \sum_{j=1}^{n} P\{\Delta_j N^{(q)} \neq 0, (\Delta_j \tilde{J}_2^{(q)})^2 > 4r_h\} \to 0 \) as \( h \to 0 \).

Proof. Statement 1 is a consequence of the fact that (25, Theorem 25.1) there is a subsequence \( n_k \) such that, defined \( h_k = T/n_k \), \( \sum_{j=1}^{n} (\Delta_j \tilde{J}_2^{(q)})^2 \) tends to \( \sum_{s \in [0,t]} (\Delta_j \tilde{J}_2^{(q)})^2 \) a.s. uniformly w.r.t. \( t \in [0, T] \), as \( k \to \infty \), where \([x]\) denotes the integer part of \( x \). Since a.s.

\[
\sup_{j=1\ldots n_k} |(\Delta_j \tilde{J}_2^{(q)})^2 - \sum_{s \in [t_{j-1}, t_j]} (\Delta_j \tilde{J}_2^{(q)})^2| = \sup_{j=1\ldots n_k} \left\{ \left[ \sum_{\ell=1}^{[t_j/h_k]} (\Delta_j \tilde{J}_2)^2 - \sum_{s \in [0,t_j]} (\Delta_j \tilde{J}_2^{(q)})^2 \right] - \left[ \sum_{\ell=1}^{[t_{j-1}/h_k]} (\Delta_j \tilde{J}_2)^2 - \sum_{s \in [0,t_{j-1}]} (\Delta_j \tilde{J}_2^{(q)})^2 \right] \right\} \\
\leq 2 \sup_{t \in [0,T]} \left[ \sum_{j=1}^{n} (\Delta_j \tilde{J}_2^{(q)})^2 - \sum_{s \in [0,t]} (\Delta_j \tilde{J}_2^{(q)})^2 \right] \to 0,
\]

we in fact have that a.s. for all \( j = 1, \ldots, n_k \) each squared increment \((\Delta_j \tilde{J}_2^{(q)})^2\) is uniformly, on \( j \), arbitrarily close to \( \sum_{s \in [t_{j-1}, t_j]} (\Delta_j \tilde{J}_2^{(q)})^2 \). More precisely, a.s. for all \( \delta > 0 \) we can find a sufficiently large \( k \) such that

\[
\sup_{j=1\ldots n_k} \left| (\Delta_j \tilde{J}_2^{(q)})^2 - \sum_{s \in [t_{j-1}, t_j]} (\Delta_j \tilde{J}_2^{(q)})^2 \right| < \delta,
\]

so, for all \( j \) such that \((\Delta_j \tilde{J}_2^{(q)})^2 \leq 4r_h\) we have

\[
\sum_{s \in [t_{j-1}, t_j]} (\Delta_j \tilde{J}_2^{(q)})^2 \leq \sup_{j=1\ldots n_k} \left| (\Delta_j \tilde{J}_2^{(q)})^2 - \sum_{s \in [t_{j-1}, t_j]} (\Delta_j \tilde{J}_2^{(q)})^2 \right| + (\Delta_j \tilde{J}_2^{(q)})^2 \leq 4r_h + \delta.
\]
In particular for any $s \in ]t_{j-1}, t_j]$ with $j$ such that $(\Delta_j \tilde{J}^{(q)}_2)^2 \leq 4r_h$, each squared jump size $(\Delta_j \tilde{J}^{(q)}_2)^2$ is bounded by $4r_h + \delta$.

Statement 2. The predictable compensator of $N_t^{(q)} = \sum_{s \leq t} I_{|\Delta J_s^{(q)}| > 1}$ and the predictable quadratic variation of $\tilde{J}^{(q)}_2$ are of the form $\Lambda_t^{(q)} = \int_0^t \lambda_t^{(q)} ds$ and $\Lambda_t'^{(q)} = \int_0^t \lambda_t'^{(q)} ds$ respectively. Assumption A5 guarantees that both $\int |\gamma_t^{(q)}| > 1$ and $\int |\gamma_t^{(q)}| dx$ are bounded, and therefore that $\lambda^{(q)}$ and $\lambda'^{(q)}$ are bounded processes. Using exactly the same argument as in [1], eq. (60), with $\delta = 1$ and $\zeta = 3\sqrt{r_h}$ and replacing $M(\delta) = \int \int_{|x| \leq \delta} x \tilde{\mu}(dx, dt)$ with our $\tilde{J}^{(q)}_2$, we conclude that

$$\sum_{j=1}^n P\{\Delta_j N^{(q)} \neq 0, (\Delta_j \tilde{J}^{(q)}_2)^2 > 4r_h\} = O(nh \frac{h}{r_h}).$$

For any $\delta > 0$ denote by $Z^{(q),\delta}_{hk}$ the following pure jump plus drift semimartingales having only jumps bounded in absolute value by $\sqrt{4r_h k + \delta}$, $q = 1, 2$:

$$Z^{(q),\delta}_{hk,t} := \int_0^t \int_{|\gamma| \leq \sqrt{4r_h k + \delta}} \gamma^{(q)} \mu^{(q)}(dx, ds) - \int_0^t \int_{\sqrt{4r_h k + \delta} < |\gamma| \leq 1} \gamma^{(q)} dx dt, \quad t \geq 0.$$  

By Lemma 7.3 we have that for any $\delta > 0$, for sufficiently large $k$ the indexes $j$ for which $(\Delta_j \tilde{J}^{(q)}_2)^2 \leq 4r_h$ are such that the increment $(\Delta_j \tilde{J}^{(q)}_2)^2$ coincides with the increment $(\Delta_j Z^{(q),\delta}_{hk})^2$ of $Z^{(q),\delta}_{hk}$, since $(\Delta_j \tilde{J}^{(q)}_2)^2$ does not contain jumps larger than $\sqrt{4r_h k + \delta}$.

Lemma 7.4. For each $q = 1, 2$

$$\text{Plim}_{n \to \infty} \sum_{j=1}^n (\Delta_j \tilde{J}^{(q)}_2)^2 \mathbf{1}_{\{(\Delta_j \tilde{J}^{(q)}_2)^2 \leq 4r_h\}} = 0$$

Proof Consider the sequence of partitions $\pi_n = \{0, T/n, 2T/n, \ldots, T\}$. Take any subsequence $\pi_{nk}$. By Lemma 7.3 point 1, a.s. there exists a sub-subsequence
Let $n_{\ell_k}$ such that for any $\delta > 0$ and $k$ sufficiently large then for all $j = 1, \ldots, n_{\ell_k}$ on $(\Delta_j \tilde{J}_2^{(q)})^2 \leq 4r_{h_{\ell_k}}$ we have $(\Delta_j \tilde{J}_2^{(q)})^2 = (\Delta_j Z_{h_{\ell_k}}^{(q),\delta})^2$. Denote 

$$S_n^{(q)} := \sum_{j=1}^n (\Delta_j \tilde{J}_2^{(q)})^2 1_{\{(\Delta_j \tilde{J}_2^{(q)})^2 \leq 4r_{h_{\ell_k}}\}}.$$ 

Therefore

$$0 \leq \text{Plim}_{k \to \infty} S_n^{(q)} \leq \text{Plim}_{k \to \infty} \sum_{j=1}^{n_{\ell_k}} (\Delta_j Z_{h_{\ell_k}}^{(q),\delta})^2$$

$$= \text{Plim}_{k \to \infty} \int_0^T \int_{|\gamma(q)| \leq \sqrt{4r_{h_{\ell_k}} + \delta}} (\gamma(q))^2 \nu(q)(dx, ds) = \int_0^T \int_{|\gamma(q)| \leq \sqrt{\delta}} (\gamma(q))^2 dxds.$$

Since a.s. $\int_{|\gamma(q)| \leq 1} (\gamma(q))^2 dx < \infty$, the last term above tends a.s. to zero as $\delta \to 0$, which implies that $\text{Plim}_{k \to \infty} S_n^{(q)} = 0$.

Since then from any subsequence of $S_n$ we can extract a sub-subsequence tending to zero in probability, we in fact have that the whole sequence $S_n^{(q)} \to 0$ in probability, as we need. \hfill $\square$

**Proof of Theorem 4.2** We decompose $\hat{IC}_{T,n} - IC_T$ into the sum of five terms and we show that each term tends a.s. to zero, as $n \to \infty$. We need some further notation. Recall that for each $q = 1, 2$

$$D_t^{(q)} = \int_0^t a_s^{(q)} ds + \int_0^t \sigma_s^{(q)} dW_s^{(q)},$$

and denote

$$Y_t^{(q)} := D_t^{(q)} + J_t^{(q)}$$

so that we have $X_t^{(q)} = Y_t^{(q)} + J_t^{(q)}$, $q = 1, 2$.

Adding and subtracting $\sum_{j=1}^n \Delta_j Y^{(1)} 1_{\{(\Delta_j Y^{(1)})^2 \leq 9r_{h_k}\}} \Delta_j Y^{(2)} 1_{\{(\Delta_j Y^{(2)})^2 \leq 9r_{h_k}\}}$ from $\hat{IC}_{T,n} - IC_T$, we reach

$$|\hat{IC}_{T,n} - IC_T|$$
\[
\sum_{j=1}^{n} (\Delta_j Y^{(1)} + \Delta_j \tilde{J}_2^{(1)}) 1_{\{ (\Delta_j X^{(1)})^2 \leq r_h \}} (\Delta_j Y^{(2)} + \Delta_j \tilde{J}_2^{(2)}) 1_{\{ (\Delta_j X^{(2)})^2 \leq r_h \}} - IC_T
\]
\[
\leq \sum_{j=1}^{n} \Delta_j Y^{(1)} 1_{\{ (\Delta_j Y^{(1)})^2 \leq 9r_h \}} \Delta_j Y^{(2)} 1_{\{ (\Delta_j Y^{(2)})^2 \leq 9r_h \}} - IC_T
\]
\[
+ \sum_{j=1}^{n} \Delta_j Y^{(1)} \Delta_j \tilde{J}_2^{(2)} \left( 1_{\{ (\Delta_j X^{(1)})^2 \leq r_h \}} 1_{\{ (\Delta_j X^{(2)})^2 \leq r_h \}} - 1_{\{ (\Delta_j Y^{(1)})^2 \leq 9r_h \}} 1_{\{ (\Delta_j Y^{(2)})^2 \leq 9r_h \}} \right)
\]
\[
+ \sum_{j=1}^{n} \Delta_j \tilde{J}_2^{(1)} \Delta_j Y^{(2)} 1_{\{ (\Delta_j X^{(1)})^2 \leq r_h \}} 1_{\{ (\Delta_j X^{(2)})^2 \leq r_h \}}
\]
\[
+ \sum_{j=1}^{n} \Delta_j \tilde{J}_2^{(1)} \tilde{J}_2^{(2)} \left( 1_{\{ (\Delta_j X^{(1)})^2 \leq r_h \}} 1_{\{ (\Delta_j X^{(2)})^2 \leq r_h \}} \right)
\]
\[
= \left[ \sum_{j=1}^{n} \Delta_j Y^{(1)} \Delta_j Y^{(2)} \left[ 1_{\{ (\Delta_j X^{(1)})^2 \leq r_h, (\Delta_j X^{(2)})^2 \leq r_h, (\Delta_j Y^{(1)})^2 > 9r_h \}}
\right.
\]
\[
+ 1_{\{ (\Delta_j X^{(1)})^2 \leq r_h, (\Delta_j X^{(2)})^2 \leq r_h, (\Delta_j Y^{(2)})^2 > 9r_h \}}
\]
\[
- 1_{\{ (\Delta_j X^{(1)})^2 \leq r_h, (\Delta_j X^{(2)})^2 \leq r_h, (\Delta_j Y^{(1)})^2 > 9r_h \}}
\]
\[
- 1_{\{ (\Delta_j X^{(1)})^2 > r_h, (\Delta_j Y^{(1)})^2 \leq 9r_h \}} - 1_{\{ (\Delta_j Y^{(2)})^2 > r_h, (\Delta_j Y^{(1)})^2 \leq 9r_h \}} - 1_{\{ (\Delta_j Y^{(2)})^2 \leq 9r_h \}}
\]
\]
\[
= \left[ \sum_{j=1}^{n} \Delta_j Y^{(1)} \Delta_j Y^{(2)} \left[ 1_{\{ (\Delta_j X^{(1)})^2 \leq r_h, (\Delta_j X^{(2)})^2 \leq r_h, (\Delta_j Y^{(1)})^2 > 9r_h \}}
\right.
\]
\[
+ 1_{\{ (\Delta_j X^{(1)})^2 \leq r_h, (\Delta_j X^{(2)})^2 \leq r_h, (\Delta_j Y^{(2)})^2 > 9r_h \}}
\]
\[
- 1_{\{ (\Delta_j X^{(1)})^2 \leq r_h, (\Delta_j X^{(2)})^2 \leq r_h, (\Delta_j Y^{(1)})^2 > 9r_h \}}
\]
\[
- 1_{\{ (\Delta_j X^{(1)})^2 > r_h, (\Delta_j Y^{(1)})^2 \leq 9r_h \}} - 1_{\{ (\Delta_j Y^{(2)})^2 > r_h, (\Delta_j Y^{(1)})^2 \leq 9r_h \}} - 1_{\{ (\Delta_j Y^{(2)})^2 \leq 9r_h \}}
\]
\]
\[
+ 1_{\{ (\Delta_j X^{(1)})^2 > r_h, (\Delta_j X^{(2)})^2 > r_h, (\Delta_j Y^{(1)})^2 \leq 9r_h, (\Delta_j Y^{(2)})^2 \leq 9r_h \}}
\]
\].

All these terms tend a.s. to zero. In fact for the first three ones notice that on
\{(\Delta_j X^{(q)})^2 \leq r_h, (\Delta_j Y^{(q)})^2 > 9r_h\} \text{ we have } \sqrt{r_h} \geq |\Delta_j X^{(q)}| \geq |\Delta_j Y^{(q)}| - |\Delta_j \tilde{J}_2^{(q)}|

and thus \(|\Delta_j \tilde{J}_2^{(q)}| \geq |\Delta_j Y^{(q)}| - \sqrt{r_h} > 3\sqrt{r_h} - \sqrt{r_h} = 2\sqrt{r_h}\), so that \(\{(\Delta_j X^{(q)})^2 \leq r_h, (\Delta_j Y^{(q)})^2 > 9r_h\} \subset \{(\Delta_j X^{(q)})^2 \leq r_h, (\Delta_j \tilde{J}_2^{(q)})^2 > 4r_h\}, q = 1, 2\), and thus the probability that each one of the first three terms of (13) is non zero is bounded by

\[
P\left\{\sum_{j=1}^n 1_{\{\{(\Delta_j X^{(q)})^2 \leq r_h, (\Delta_j \tilde{J}_2^{(q)})^2 > 4r_h\}\}} \neq 0\right\} \tag{14}
\]

for a suitable \(q\). Now on \(\{(\Delta_j X^{(q)})^2 \leq r_h, (\Delta_j \tilde{J}_2^{(q)})^2 > 4r_h\}\) we in fact have that \(\Delta_j N^{(q)} \neq 0\). Actually, since

\[2\sqrt{r_h} - |\Delta_j Y^{(q)}| < |(\Delta_j \tilde{J}_2^{(q)})^2| - |\Delta_j Y^{(q)}| \leq |\Delta_j X^{(q)}| \leq \sqrt{r_h}\]

then

\[K_q \sqrt{2h \log \frac{1}{h} + |\Delta_j J_1^{(q)}|} \geq |\Delta_j D^{(q)}| + |\Delta_j J_1^{(q)}| \geq |\Delta_j Y^{(q)}| > \sqrt{r_h},\]

so

\[|\Delta_j \tilde{J}_1^{(q)}| > \sqrt{r_h} \left(1 - K_q \sqrt{\frac{2h \log \frac{1}{h}}{r_h}}\right)\]

since a.s. for sufficiently small \(h\) the quantity \(1 - K_q \sqrt{2h \log \frac{1}{h}}/r_h\) is positive, then in fact \(|\Delta_j \tilde{J}_1^{(q)}| > 0\), so that \(\Delta_j N^{(q)} \neq 0\).

So (14) is dominated by \(\sum_{j=1}^n P\{\Delta_j N^{(q)} \neq 0, (\Delta_j \tilde{J}_2^{(q)})^2 > 4r_h\}\) which tends to zero as \(h \to 0\) by Lemma 7.3 part 2).

As for the last three terms of (13) note that on \(\{(\Delta_j Y^{(q)})^2 \leq 9r_h\}\) we have a.s., for \(h\) small such that \(\Delta_j N^{(q)} \in \{0, 1\}\),

\[\Delta_j N^{(q)} \leq |\Delta_j J_1^{(q)}| = |\Delta_j Y^{(q)} - \Delta_j D^{(q)}| < |\Delta_j D^{(q)}| + |\Delta_j Y^{(q)}|\]

\[\leq |\Delta_j D^{(q)}| + 3\sqrt{r_h} \leq K_q \sqrt{2h \log \frac{1}{h} + 3\sqrt{r_h}} \to 0, \quad q = 1, 2,\]
hence, for small $h$ on $\{(\Delta_j Y^{(q)})^2 \leq 9r_h\}$ we have $\Delta_j N^{(q)} = 0$, $j = 1, \ldots, n$. Therefore $\{(\Delta_j X^{(q)})^2 > r_h, (\Delta_j Y^{(q)})^2 \leq 9r_h\} \subset \{(\Delta_j D^{(q)} + \Delta_j \tilde{J}_2^{(q)})^2 > r_h\} \subset \{(\Delta_j D^{(q)})^2 > \frac{r_h}{4}\} \cup \{(\Delta_j \tilde{J}_2^{(q)})^2 > \frac{r_h}{4}\}$, $q = 1, 2$; however, by Lemma 7.2 part 1), a.s., for small $h$,

$$1_{\{(\Delta_j D^{(q)})^2 > \frac{r_h}{4}\}} = 0,$$

thus the last three terms of (13) are dominated by

$$\sum_{j=1}^n |\Delta_j D^{(1)} \Delta_j D^{(2)}|1_{\{(\Delta_j \tilde{J}_2^{(q)})^2 > \frac{r_h}{4}\}}$$

for a suitable $q$. However this last term tends to zero in probability, since

$$\sum_{j=1}^n |\Delta_j D^{(1)} \Delta_j D^{(2)}|1_{\{(\Delta_j \tilde{J}_2^{(q)})^2 > \frac{r_h}{4}\}} \leq K_1 K_2 2h \log \frac{1}{h} \sum_{j=1}^n 1_{\{(\Delta_j \tilde{J}_2^{(q)})^2 > \frac{r_h}{4}\}}$$

and $E[2h \log \frac{1}{h} \sum_{j=1}^n 1_{\{(\Delta_j \tilde{J}_2^{(q)})^2 > \frac{r_h}{4}\}}] = 2h \log \frac{1}{h} \sum_{j=1}^n 1_{\{(\Delta_j \tilde{J}_2^{(q)})^2 > \frac{r_h}{4}\}} = O\left(\frac{2h \log \frac{1}{h}}{r_h} nh\right)$.

We now show that the third and fourth terms of the right hand side of (12), which are similar, tend to zero in probability. We have

$$\sum_{j=1}^n \Delta_j Y^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{(\Delta_j X^{(1)})^2 \leq r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \leq r_h\}}$$

$$= \sum_{j=1}^n \Delta_j Y^{(1)} \Delta_j \tilde{J}_2^{(2)} \left[1_{\{\Delta_j X^{(1)} \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(1)}| \leq 2 \sqrt{r_h}\}} 1_{\{\Delta_j X^{(2)} \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(2)}| \leq 2 \sqrt{r_h}\}}ight.$$

$$+ 1_{\{\Delta_j X^{(1)} \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(1)}| > 2 \sqrt{r_h}\}} 1_{\{\Delta_j X^{(2)} \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(2)}| > 2 \sqrt{r_h}\}}$$

$$+ 1_{\{\Delta_j X^{(1)} \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(1)}| > 2 \sqrt{r_h}\}} 1_{\{\Delta_j X^{(2)} \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(2)}| \leq 2 \sqrt{r_h}\}}$$

$$+ 1_{\{\Delta_j X^{(1)} \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(1)}| \leq 2 \sqrt{r_h}\}} 1_{\{\Delta_j X^{(2)} \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(2)}| > 2 \sqrt{r_h}\}} \right].$$

As before on $\{(\Delta_j X^{(q)})^2 \leq r_h, (\Delta_j \tilde{J}_2^{(q)})^2 > 4r_h\}$ we have that $\Delta_j N^{(q)} \neq 0$, so, for each one of the last three terms of (15), the probability it is different from zero is
dominated by
\[ \sum_{j=1}^{n} P\{\Delta_j N^{(q)} \neq 0, (\Delta_j \tilde{J}_2^{(q)})^2 > 4r_h\} \to 0.\]

Now we show that the first term of (15) is asymptotically negligible. Notice that on
\{\Delta_j X^{(q)} \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(q)}| \leq 2 \sqrt{r_h}\} a.s. for small h we have \(\Delta N_j^{(q)} = 0\); in fact a.s.,
for small h we have \(\Delta_j N^{(q)} \in \{0, 1\}\), and
\[ \Delta_j N^{(q)} \leq |\Delta_j \tilde{J}_1^{(q)}| = |\Delta_j X^{(q)} - \Delta_j D^{(q)} - \Delta_j \tilde{J}_2^{(q)}| \]
\[ \leq \sqrt{r_h} + \sup_j |\Delta_j D^{(q)}| + 2 \sqrt{r_h} \to 0, \]
for all \(j = 1, \ldots, n\), for \(q = 1, 2\). So we have
\[
\operatorname{Plim}_{n \to \infty} \left| \sum_{j=1}^{n} \Delta_j Y^{(1)} \Delta_j \tilde{J}_2^{(2)} \mathbb{1}_{\{|\Delta_j X^{(1)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(1)}| \leq 2 \sqrt{r_h}\}} \mathbb{1}_{\{|\Delta_j X^{(2)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(2)}| \leq 2 \sqrt{r_h}\}} \right|
\]
\[ \leq \operatorname{Plim}_{n \to \infty} \sum_{j=1}^{n} |\Delta_j D^{(1)} \Delta_j \tilde{J}_2^{(2)}| \mathbb{1}_{\{|\Delta_j \tilde{J}_2^{(2)}| \leq 2 \sqrt{r_h}\}}. \]

By the Cauchy-Schwarz inequality, last term is dominated by
\[
\operatorname{Plim}_{n \to \infty} \sqrt{\sum_{j=1}^{n} (\Delta_j D^{(1)})^2} \sqrt{\sum_{j=1}^{n} (\Delta_j \tilde{J}_2^{(2)})^2} \mathbb{1}_{\{|\Delta_j \tilde{J}_2^{(2)}| \leq 2 \sqrt{r_h}\}} \leq \sqrt{\int_0^T (\sigma_s^{(1)})^2 \, ds} \operatorname{Plim}_{n \to \infty} \sqrt{S_n^{(2)}} = 0,
\]
by Lemma 7.4.

It remains to consider the last term of (12), which is rewritten as in (15) with \(\Delta_j \tilde{J}_2^{(1)} \Delta_j \tilde{J}_2^{(2)}\) in place of \(\Delta_j Y^{(1)} \Delta_j \tilde{J}_2^{(2)}\), so that last three terms converge to zero in probability as before. As for the first term
\[
\operatorname{Plim}_{n \to \infty} \sum_{j=1}^{n} \Delta_j \tilde{J}_2^{(1)} \Delta_j \tilde{J}_2^{(2)} \mathbb{1}_{\{(\Delta_j X^{(1)})^2 \leq r_h, (\Delta_j \tilde{J}_2^{(1)})^2 \leq 4r_h\}} \mathbb{1}_{\{(\Delta_j X^{(2)})^2 \leq r_h, (\Delta_j \tilde{J}_2^{(2)})^2 \leq 4r_h\}}, \quad (16)
\]
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we remark that it is bounded in absolute value by

\[
\text{Plim}_{n \to \infty} \sum_{j=1}^{n} |\Delta_j \tilde{J}_2^{(1)}|_1 \{ (\Delta_j \tilde{J}_2^{(1)})^2 \leq 4r_h \} |\Delta_j \tilde{J}_2^{(2)}|_1 \{ (\Delta_j \tilde{J}_2^{(2)})^2 \leq 4r_h \}
\leq \text{Plim}_{n \to \infty} \sqrt{S_n^{(1)}} \sqrt{S_n^{(2)}} = 0,
\]

by Lemma [7.4] \hfill \Box