NADA-FLD: A general relativistic, multi-dimensional neutrino-hydrodynamics code employing flux-limited diffusion

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ABSTRACT

We present the new code NADA-FLD to solve multi-dimensional neutrino-hydrodynamics in full general relativity (GR) in spherical polar coordinates. The neutrino transport assumes the flux-limited diffusion (FLD) approximation and evolves the neutrino energy densities measured in the frame comoving with the fluid. Operator splitting is used to avoid multidimensional coupling of grid cells in implicit integration steps involving matrix inversions. Terms describing lateral diffusion and advection are integrated explicitly with the Allen-Cheng method, which remains stable even in the optically thin regime. We discuss several toy-model problems in one and two dimensions to test the basic functionality and individual components of the transport scheme. We also perform fully dynamic core-collapse supernova (CCSN) simulations in spherical symmetry. For a Newtonian model we find good agreement with the M1 code ALCAR, and for a GR model we reproduce the main effects of GR in CCSNe already found by previous works.

Key words: hydrodynamics − neutrinos − radiative transfer − methods: numerical − stars: neutron − supernovae: general

1 INTRODUCTION

The transport of neutrinos plays a vital role for core-collapse supernovae (CCSNe) and mergers of neutron stars (NSs). According to the standard explosion mechanism of ordinary CCSNe, the stalled accretion shock is revived due to neutrino heating of material below the shock (Colgate & White 1966; Bethe & Wilson 1985; Janka et al. 2016). In the case of more massive stars undergoing black-hole (BH) formation, neutrino emission has leverage on the time when the central proto-neutron star (PNS) collapses, and subsequently it may regulate the mass-accretion rate onto the BH (e.g. MacFadyen & Woosley 1999; Sekiguchi & Shibata 2011; Mista et al. 2015; Obergaulinger & Aloy 2017). For any type of CCSN, the nucleosynthesis pattern in material ejected from the central regions depends sensitively on the neutron-to-proton ratio, which is determined by the number of neutrinos and antineutrinos emitted and absorbed during the expansion (e.g. Qian & Woosley 1996; Wanajo et al. 2018; Goriely & Janka 2016; Nishimura et al. 2017). In NS mergers, neutrino transport is similarly important for setting the nucleosynthesis conditions in ejected matter and for cooling and heating in the central remnant, which typically consists of an accretion disc surrounding either a NS or a BH (e.g. Metzger & Fernández 2014; Just et al. 2015a; Perego et al. 2014; Siegel & Metzger 2017). Furthermore, in NS−or BH-torus systems a gamma-ray burst jet might be produced or enhanced due to annihilation of neutrinos with their anti-particles (e.g. Paczynski 1986; Eichler et al. 1989; Just et al. 2016; Perego et al. 2017). In order to better understand all of these scenarios, models including a multi-dimensional, general relativistic treatment of neutrino transport are desirable.

During the last several decades, various approaches have been developed for treating neutrino transport in hydrodynamical simulations. The most sophisticated schemes solve the full Boltzmann equation, e.g. by direct discretization using finite differences (see, e.g. Liebendörfer et al. 2004; Livne et al. 2004; Sumiyoshi & Yamada 2012), or by employing a Monte Carlo treatment (see, e.g. Janka & Hillebrandt 1989; Abdikamalov et al. 2012; Richers et al. 2015), or by coupling a somehow simplified Boltzmann solver to an additional system of equations for the lowest angular moments of the Boltzmann equation (see, e.g. Rampp & Janka 2002; Foucart 2018). While these methods have the advantage of providing the full phase-space information of the generally six-dimensional phase-space dependence of the neutrino distribution function, they are still too expensive in terms of computational resources to be used for long-term, high-resolution simulations, or for exhaustive parameter exploration.

A computationally much cheaper alternative of describing neutrino processes comes with so-called neutrino leakage schemes (e.g. Ruffert et al. 1996; Galeazzi et al. 2013; Perego et al. 2016; Ardevol-Pulpillo et al. 2018) that estimate the local matter-neutrino interaction rates just based on the instantaneous fluid configura-
tion without evolving (or evolving only in regions where neutrinos are trapped) conservation equations for neutrino energy and number. Moreover, several additional schemes have been designed for specific application purposes, including light-bulb schemes (e.g. Janka & Müller 1996), the fast-multigroup transport scheme (FMT; Müller & Janka 2015), the M0 scheme (Radice et al. 2016), or the isotropic-diffusion-source approximation (IDSA; Liebendörfer et al. 2009).

In the class of local-closure moment schemes only the lowest angular moments of the distribution function are dynamically evolved, while all higher-order moments are provided by an approximate closure relation as function of the evolved moments. In flux-limited diffusion (FLD) schemes only the zeroth-order moment (i.e. the energy density) is evolved, while in two-moment (M1) schemes additionally the first-order moment (flux density) is integrated. In the recent years a number of M1 schemes have been developed (O’Connor 2015; Just et al. 2015b; Foucart et al. 2015; Müller & Janka 2015), the M0 scheme (Radice et al. 2016), or the fast-multigroup transport scheme (FMT; Janka & Müller 1996), the fast-multigroup transport scheme (FMT; Bowers & Wilber 1982; Bruenn 1985; Burrows & Lattimer 1986; Myra et al. 1987; Baron et al. 1989; Cernohorsky 1990; Burrows et al. 2007; Lentz et al. 2015; Bruenn et al. 2018; Cooperstein & Baron 1992) derived the FLD equation correct to order $L$ (e.g. Lentz et al. 2015; Bruenn et al. 2018) for neutrino transport in CCSNe and PNS cooling (Bowers & Wilber 1982; Bruenn 1985; Burrows & Lattimer 1986; Myra et al. 1987; Baron et al. 1989; Cernohorsky 1990; Burrows et al. 2007; Lentz et al. 2015; Bruenn et al. 2018; Cooperstein & Baron 1992) since then has been used, apart from many applications in the context of photon transport, and Levermore & Pomraning (1981) and Levermore & Pomraning (1981) since then has been used, and has its own shortcomings, for example with respect to the accuracy in beam-crossing regions (see e.g. Foucart et al. 2018). Developing different, complementary algorithms therefore enhances the diversity of applied methods and in the long run might help to discriminate numerical artefacts from physical effects.

Our algorithm employs spherical polar coordinates and integrates the GR equations using the partially implicit Runge-Kutta method (Montero & Cordero-Carron 2012; Baumgarte et al. 2013). The transport equations are solved in the comoving (i.e. fluid-rest) frame. In order to avoid multi-directional coupling of grid cells, and therefore the inversion of bigger matrices spanned over the entire grid, we employ operator splitting. The source terms, the radial- and energy-derivatives, as well as the non-radial derivatives are integrated separately, each using an appropriate discretization scheme. In this way, the scheme can be parallelized in a straightforward manner and remains numerically less complex than an split, fully implicit solver.

In Sect. 2, we outline the basic equations of our general relativistic radiation-hydrodynamics scheme. The discretization scheme for solving the transport equations is described in Sect. 3. In Sect. 4, we discuss the results of various toy-model problems and of fully dynamic neutrino-hydrodynamics simulations of the collapse and post-bounce evolution of a massive star in spherical symmetry. In Appendix A we provide the detailed derivation of the main transport equation used in our code, which evolves the neutrino energy densities measured in the comoving frame.

Throughout most of the paper we assume $c = 1$ for the speed of light, except in some microphysics related cases, in which $c$ appears explicitly. Also gravitational constant, $G$, Planck constant, $\hbar$, and Boltzmann constant, $k_B$, are set to one. We follow the convention that indices or superscripts $a, b, c, \mu, \nu$ run over space-time components $(0, 1, 2, 3)$, while $i, j, k, l$ just run over spatial components $(1, 2, 3)$. We denote quantities defined in the comoving orthonormal frame by an index with a hat (e.g. $\hat{i}$) and quantities defined in the comoving curvilinear frame index with a bar (e.g. $\bar{i}$). We denote electron neutrinos and their anti-neutrinos as $\nu_e$ and $\bar{\nu}_e$, respectively, and we use $\nu_x$ to denote any of the four remaining neutrino types.

2 BASIC EQUATIONS OF GENERAL RELATIVISTIC RADIATION HYDRODYNAMICS

In this section, we outline the basic equations used in our general relativistic radiation-hydrodynamics scheme, namely those describing the evolution of the space-time metric, hydrodynamics, and radiation transport.

2.1 Metric equations

We use a $3 + 1$ decomposition in which the space-time manifold is foliated into space-like hyper-surfaces $\Sigma$ (see, e.g. Baumgarte & Shapiro 2010). We denote the 4-metric as $g_{ab}$. The time-like future pointing normal vector to $\Sigma$ is $n^a$, and the space-like 3-metric on $\Sigma$ is $g_{ij}$. The line element is then given by:

\[
d s^2 = g_{ab} dx^a dx^b = -d\tau^2 + g_{ij}(dx^i + \gamma^i_j dt)(dx^j + \gamma^j_i dt),
\]

where $\alpha, \beta^\mu$ are the lapse function and shift-vector, respectively, and

\[
\gamma_{ab} = g_{ab} + n_a n_b, \quad n^a = (1/\alpha, -\beta^\mu/\alpha), \quad n_a = (-\alpha, 0, 0, 0).
\]

Moreover,

\[
\tilde{\gamma}_{ij} = e^{-\gamma_{ij}} \gamma_{ij}
\]
is the conformal metric, with the conformal factor $\exp(4\phi)$ (see, e.g., chapter 3 of Baumgarte & Shapiro 2010 for a detailed discussion of the conformal transformation). Furthermore, the extrinsic curvature $\mathbf{K}_{ij}$, the conformal traceless extrinsic curvature, $\mathbf{A}_ij$, and the trace of the extrinsic curvature, $K$, are defined as:

$$K_{ij} \equiv -\gamma^{k} \gamma_{j} \nabla_{i} n_{k}$$

$$= -\frac{1}{2a} \partial_{i} \gamma_{ij} + D_{i}(\beta_{j}),$$

$$\mathbf{A}_{ij} \equiv e^{-4\phi}(K_{ij} - \frac{1}{3} K),$$

$$K \equiv K_{i}^{i}. \quad (4)$$

The Minkowski metric in spherical polar coordinates is $\gamma_{ij} = \text{diag}(1, r^{2}, r^{2} \sin^{2} \theta)$. We denote the connection coefficients associated with the metrics $\gamma_{ab}$, $\gamma_{ij}$, and $\mathbf{A}_{ij}$ as $\Gamma_{bc}^{a}$, $\Gamma_{jk}^{i}$, and $\Gamma_{jk}^{i}$, respectively. The covariant derivatives associated with $\gamma_{ij}$, $\gamma_{ij}$, and $\mathbf{A}_{ij}$ are denoted by $D_{i}$, $\bar{D}_{j}$, and $\hat{D}_{k}$, respectively. We define the connection vector $\mathbf{A}^{i}$ as

$$\mathbf{A}^{i} \equiv \gamma^{ik} \Gamma_{kj}^{i}; \quad \text{with} \quad \Delta \Gamma_{jk}^{i} \equiv \Gamma_{jk}^{i} - \hat{\Gamma}_{jk}^{i},$$

and express the Ricci tensor as

$$\bar{R}_{ij} = \frac{1}{2} \gamma^{kl} \bar{D}_{k} \bar{D}_{l} \gamma_{ij} + \gamma_{(i} \bar{D}_{j)} \mathbf{A}^{k} + \Delta \Gamma^{k} \Delta \Gamma_{(ij)k}$$

$$+ \gamma^{kl} \left( 2 \Delta \Gamma_{(ik} \Delta \Gamma_{jm)} + \Delta \Gamma_{ik} \Delta \Gamma_{jm} \right). \quad (7)$$

To evolve the space-time metric we solve the covariant BSSN equations (Baumgarte et al. 2013), which are given by:

$$\partial_{\perp} \tilde{\gamma}_{ij} = -\frac{2}{3} \tilde{\gamma}_{ij} \bar{D}_{k} \beta^{k} - 2a A_{ij},$$

$$\partial_{\perp} A_{ij} = -\frac{2}{3} A_{ij} \bar{D}_{k} \beta^{k} - 2a A_{ik} \hat{A}^{k} + a A_{ij} K + e^{-4\phi} \left[ -2a \bar{D}_{i} \bar{D}_{j} \phi + 4a \bar{D}_{i} \phi \bar{D}_{j} \phi 

+ 4 \bar{D}_{i} \alpha \bar{D}_{j} \phi + \bar{D}_{i} \bar{D}_{j} \alpha + a (\bar{R}_{ij} - 8 \pi S_{ij}) \right]_{\text{TF}},$$

$$\partial_{\perp} \phi = \frac{1}{6} \bar{D}_{k} \beta^{k} - \alpha K,$$

$$\partial_{\perp} K = \frac{\alpha}{3} K^{2} + a A_{ij} A^{ij}$$

$$- e^{-4\phi} \left( \bar{D}^{2} \alpha + 2 \bar{D}^{i} \alpha \bar{D}_{i} \phi \right) + 4 \pi a (\varphi + S),$$

$$\partial_{\perp} \bar{A}^{i} \equiv \bar{\gamma}^{ij} \bar{D}_{j} \beta^{i} = \frac{2}{3} \Delta \Gamma^{i} \Delta \Gamma_{ij} + \frac{1}{3} \beta^{i} \bar{D}_{j} \beta^{j}$$

$$- 2 \bar{A}^{i} \left( \delta^{i} \alpha \partial_{i} \phi - 6 \alpha \delta^{i} \partial_{i} \phi - a \Delta \Gamma_{ik} \right)$$

$$- \frac{4}{3} \alpha \gamma^{ij} \partial_{i} K - 16 \pi a \gamma^{ij} S_{ij}. \quad (8)$$

Here, \n
$$\partial_{\perp} \equiv \partial_{t} - \mathcal{L}_{\beta},$$

where $\mathcal{L}_{\beta}$ is the Lie derivative along the shift vector $\beta^{i}$. The superscript TF denotes the trace-free part of a tensor. The matter-radiation

1 We use parenthesis () to denote the symmetric part $A_{ij(i)} \equiv \frac{1}{2}(A_{ij} + A_{ji})$ of any tensor $A_{ij}$.  

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### 2.2 Hydrodynamics

The general relativistic hydrodynamics equations expressing the local conservation of baryonic mass (with current density $J^{a}$), baryon-momentum (with energy-momentum tensor $T_{h}^{ab}$), and electron lepton number (with current density $J_{e}^{a}$) read (e.g. Font 2008):

$$\nabla_{a} J^{a} = 0,$$

$$\nabla_{a} T_{h}^{ab} = s^{b},$$

$$\nabla_{a} J_{e}^{a} = S_{N}, \quad (12)$$

where

$$J^{a} = \rho u^{a},$$

$$T_{h}^{ab} = \rho w^{a} u^{b} + pg^{ab},$$

$$J_{e}^{a} = \rho a^{e} Y_{e}, \quad (13)$$

and

$$h = 1 + e + p/r,$$

$$u^{0} = W/a,$$

$$u^{i} = W(v^{i} - \beta^{i} a^{-1}). \quad (14)$$

The symbols $\rho$, $e$, $v^{i}$, $W$, $p$, $h$, and $Y_{e}$ denote the baryonic mass density, specific internal energy, 3-velocity, Lorentz factor, gas pressure, specific enthalpy, and electron fraction (equal to the number of protons per nucleon), respectively. In order to obtain explicit expressions of equations (12), we use the flux-conservative Valencia formulation generalized to curvilinear coordinates, as described in Montero et al. (2014). In this formulation, singular terms proportional to $1/r$ and $\cot \theta$ are scaled out by using the reference metric $\gamma_{ij}$. The conservative variables $D$, $S_{i}$, $\tau$, and $D_{e}$ that are evolved in time are defined in terms of the primitive variables $\rho$, $e$, $v^{i}$, $p$, and

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The continuity, Euler, energy, and lepton-number equations in the generalized Valencia formulation read:

\[
\begin{align*}
\partial_t (\sqrt{g} D) + \partial_i (f S) j &= 0, \\
\partial_t (\sqrt{g} S_i) + \partial_j (f S) j &= \partial_t (s S) i + (f S) k (F^k j - (f S) k F^j) \\
&+ \alpha \sqrt{g/\gamma} (S M) i, \\
\partial_t (\sqrt{g} \tau) + \partial_j (f H) j &= s \tau - (f H) k F^j + \alpha \sqrt{g} S_E, \\
\partial_t (\sqrt{g} D) + \partial_j (f H) j &= \alpha \sqrt{g} S_N,
\end{align*}
\]

(15)

where \( \gamma \) is the determinant of the metric \( \gamma_{ij} \) and the flux functions are given by:

\[
\begin{align*}
(f p) j &= \alpha \sqrt{g} D (\psi - \beta^i a^{-1}), \\
(f s) j &= \alpha e^\phi \sqrt{g/\gamma} (W^2 \phi v^i - \beta^i a^{-1} + p^i a^i), \\
(f H) j &= \alpha \sqrt{g} (\tau (\psi - \beta^i a^{-1} + p v^i) + T^i j), \\
(f D) j &= \alpha \sqrt{g} D (\psi - \beta^i a^{-1}),
\end{align*}
\]

(16)

and the source functions are defined by:

\[
\begin{align*}
(s S) i &= \alpha e^\phi \sqrt{g/\gamma} (T^0 0 \alpha \beta^j - T^0 k \partial_t \beta^j) \\
&+ \frac{1}{2} \left( T^0 0 \beta^j \beta^k + 2 T^0 j \beta^k + T^k j \right) \partial_t \gamma_{jk}, \\
(s \tau) &= \alpha e^\phi \sqrt{g/\gamma} (T^0 0 (\beta^i \beta^j K_{ij} - \beta^j \partial_i \alpha)) \\
&+ T^0 i (2 \beta^j K_{ij} - \partial_i \alpha) + T^i j K_{ij}.
\end{align*}
\]

(18)

The source terms \( S_E, \) \( S_M \) and \( S_N \) express the change of gas energy, momentum, and lepton number, respectively, due to neutrino-matter interactions and will be quantified in Section 2.3.2. To close the system of equations, an equation of state is required that provides the pressure, temperature and composition as functions of the primitive variables. The hydrodynamic equations are solved using a finite difference Godunov-type High-Resolution-Shock-Capturing Method (HRSC) \[\text{[Toro, 2009]}.\] For the reconstruction of primitive variables at the cell interfaces, the PPM \[\text{[Colella & Woodward, 1984]}, \ CEN0 \text{[Liu \\& Osher, 1998]} \text{ and MP5 [Suresh \\& Huynh, 1997]} \text{ methods are implemented. The fluxes at cell interfaces are calculated from primitive variables using the HLL Riemann solver [Harten et al., 1983]. The time integration is done with a second-order Runge-Kutta method, where the time step is the same as that used for integrating the BSSN equations (cf. equation 11). The numerical implementation and test of the hydrodynamics part of the code is discussed in [Montero et al., 2014]. The hydrodynamics equations are integrated using the same timestep (given by equation 11) as used for integrating the GR equations.

2.3 Neutrino transport

In this section we present the evolution equations used in our FLD neutrino transport scheme and their coupling to the evolution of the metric, eqs. \[\text{[5]}\] and of the hydrodynamic quantities, eqs. \[\text{[16]}\]. The formalism of fully general relativistic truncated-moment schemes has been developed and extensively discussed in [Shibata et al., 2011; Endeve et al., 2012; Cardall et al., 2013], from whom we adopt a great share of our notation. Like in the aforementioned works, all (comoving-frame and lab-frame) angular moments as well as the neutrino stress-energy tensor are expressed as functions of Eulerian (i.e. lab-frame) space-time coordinates, \( \alpha^\mu \), and of the neutrino energy measured by a comoving observer, \( \epsilon \). One difference of our scheme compared to those of the aforementioned papers is, however, that we evolve the neutrino moments (i.e. the energy densities) as measured in the orthonormal comoving frame, instead of those measured in the lab frame. In this respect, our scheme is similar to that of \[\text{[Müller et al., 2010]}\].

2.3.1 Basic definitions

In terms of the neutrino distribution function, \( f \), the comoving-frame 0th-, 1st-, and 2nd-order moments are given by

\[
\begin{align*}
\mathcal{J}(\alpha^\mu, \epsilon) &= \epsilon^3 \int f(\alpha^\mu, \epsilon^l) \, d\Omega, \\
\mathcal{H}^l(\alpha^\mu, \epsilon) &= \epsilon^3 \int l^l f(\alpha^\mu, \epsilon^l) \, d\Omega, \\
\mathcal{K}^{ij}(\alpha^\mu, \epsilon) &= \epsilon^3 \int l^i l^j f(\alpha^\mu, \epsilon^l) \, d\Omega,
\end{align*}
\]

(19)

where \( \epsilon^l = (1, l^i) \) denotes the neutrino momentum-space coordinates, with unit momentum three-vector \( l^i \), and the angular integration is performed in the comoving-frame momentum space.

The comoving-frame moments in eqs. \[\text{[19]}\] are related to the monochromatic lab-frame neutrino stress-energy tensor, \( T^{ab}_{\epsilon} \), by

\[
T^{ab}_{\epsilon}(\alpha^\mu, \epsilon) = \int T^{ab} \, d\epsilon.
\]

(20)

In eq. \[\text{[20]}\], the matrices \( L^a_{\epsilon} \equiv e^a \epsilon A^\alpha_{\epsilon} \) are responsible for transforming tensors from the orthonormal comoving frame to the global coordinate (i.e. lab) frame. Here, the Lorentz transformation, \( A^\alpha_{\epsilon} \), converts orthonormal comoving-frame quantities into an orthonormal (i.e. locally Minkowskian) tetrad basis in the lab frame, and the tetrad transformation \( e^a \epsilon \) converts from the orthonormal lab-frame tetrad basis to the basis of global coordinates (which are generally not orthonormal in curved space-time).

The lab-frame moments of 0th-, 1st-, and 2nd-order are respectively given in terms of the comoving-frame moments by (cp. 2011; Endeve et al., 2012; Cardall et al., 2013).
The neutrino contributions to the source terms for the Einstein equations (cf. eqs. (5)) are obtained from the lab-frame neutrino angular moments (cf. eq. (22)) using eq. (25) as:

\[ \varrho_t = \sum_i \int dE \mathcal{E}^i, \]

\[ S_t^i = \sum_i \int dE \mathcal{F}^i, \]

\[ S_t^{ij} = \sum_i \int dE \mathcal{S}^{ij}. \]  

**2.3.3 Energy equation and flux-limited diffusion approximation**

The evolution equation for the comoving-frame neutrino energies, \( \mathcal{F} \), can be derived from the evolution equations for the lab-frame moments, \( \mathcal{E} \) and \( \mathcal{F} \), which are discussed in Shibata et al. (2011); En leve et al. (2012); Cardall et al. (2013). We refer to Appendix A for a detailed derivation. The resulting evolution equation reads:

\[ \frac{1}{\alpha} \frac{\partial}{\partial t} \left[ \alpha W(v - \beta') \hat{\mathcal{F}} \right] + \frac{1}{\alpha} \frac{\partial}{\partial x} \left[ \alpha W(v - \beta') \hat{\mathcal{F}} \right] + \hat{\mathcal{K}} = \kappa_{\alpha} (\mathcal{J} - \mathcal{F}) \]  

where we use the notation \( \hat{\mathcal{F}} \), \( \hat{\mathcal{K}} \), \( \hat{\mathcal{J}} \), and \( \hat{\mathcal{H}} \), are functions of the comoving-frame moments by virtue of eqs. (A14), (A15), and (22). The specific shape of the neutrino source terms on the right-hand side of eq. (28) takes account of the fact that the current implementation is restricted to absorption and emission (or formally equivalent) reactions, and iso-energetic scattering processes, i.e. scattering processes without exchange of energy between neutrino and target particle.

The flux-limited diffusion (FLD) approximation is implemented as follows. The flux density as measured in the orthonormal comoving frame, \( \mathcal{H} \), is in the diffusion limit approximately given by \( \mathcal{H}^{\text{diff}} = e^{k_1} \delta_{k} J / (3 \lambda_1) \). This expression can be obtained from the evolution equation of the neutrino flux densities (shown in eq. (A9)), by neglecting time derivatives, velocity terms, and general relativistic corrections, which all should be subdominant. Going towards lower optical depths, radiation approaches the causality limit, i.e. \( \mathcal{H}^{\text{free}} \approx J \). In FLD, a smooth interpolation between these two regimes is accomplished by the use of a scalar flux-limiter, \( \lambda \in [0, 1/3] \), in terms of which the flux is expressed as:

\[ \mathcal{H} \rightarrow -D e^{k_1} \delta_{k} J, \]  

where

\[ D = \frac{1}{\lambda_1}, \]  

is the (scalar) diffusion coefficient. In doing so, it is implicitly assumed that the partial time derivative of the flux vanishes, i.e. \( \partial_t \mathcal{H} = 0 \). In this work, we use the Levermore-Pomraning (LP) limiter [Pomraning (1981), Levermore & Pomraning (1981)] and the Wilson limiter [Bowers & Wilson (1982)], which are computed as:

\[ \lambda_{\text{LP}} = \frac{2 + R}{6 + 3R + R^2}, \]

\[ \lambda_{\text{Wilson}} = \frac{1}{3 + R}. \]
where

\[ R = \frac{|\alpha^i_\phi \partial_\phi J|}{\kappa_\phi J} \]  

is the Knudsen number, and we use the flat metric tetrad \( e^i_\phi = \text{diag}(1, 1/r, 1/\sin \theta) \) in eq. \( \{32\} \). To calculate the Knudsen number, diffusion coefficient and the Eddington scalar and tensor, we follow the procedure described in the Appendix H.4 of Swesty & Myra (2009). The Knudsen number along a direction \( i \) is calculated using the absolute value of the diffusive flux, \( \mathcal{H}^{i, \text{diff}}_i \), in \( i \) direction. This procedure ensures that causality is not violated for individual flux components, i.e. \( |\mathcal{H}_i| \leq J \), but the total flux, \( \mathcal{H} = \sqrt{(\mathcal{H}_1^2 + \mathcal{H}_2^2 + \mathcal{H}_3^2)} \), might violate causality. We will come back to this unsatisfactory point when discussing our results for 2D test problems.

The Eddington tensor, \( \chi^{ij} \), which is related to the second moment tensor, \( \mathcal{K}^{ij} \), by

\[ \chi^{ij} = \frac{\mathcal{K}^{ij}}{J}, \]  

is in the FLD approximation given by (see, e.g. Pomraning 1981; Levermore & Pomraning 1981; Swesty & Myra 2009):

\[ \chi^{ij} = \frac{1}{2} \left( (1 - \chi) \delta^{ij} + (3 \chi - 1) h^i h^j \right), \]  

where \( h^i \) is the unit vector along \( \mathcal{H}^i \) and the (scalar) Eddington factor, \( \chi \), is given by

\[ \chi = \lambda + (\Lambda R)^2. \]  

For future reference, we also define the flux factor as

\[ f^i = \frac{\mathcal{H}^i}{J}. \]

The final FLD equation solved in our code reads:

\[
\begin{align*}
\frac{1}{\alpha} \frac{\partial}{\partial t} (W J) + \frac{1}{\alpha} \frac{\partial}{\partial x i} \left[ a W (v^j - \beta^j a^{-1}) J \right] - \frac{1}{\alpha^2} \frac{\partial}{\partial x i} \left[ \alpha \sqrt{\gamma} \left[ \gamma^{jk} + W \left( \frac{W}{W + 1} \gamma^{jk} - \beta^j a^{-1} \right) \right] D \partial_\phi J \right] - \frac{1}{\alpha} \frac{\partial}{\partial x i} (W \sqrt{\gamma} \gamma) D^i_\phi \partial_\phi J + R_e - \frac{\partial}{\partial \epsilon} (\epsilon R_e) \\
= \kappa_\phi (f^q - J).
\end{align*}
\]

The second, third and fourth terms in the above equation describe advection, diffusion, and aberration due to fluid acceleration, respectively. We simplify the equation by neglecting all spatial cross derivatives, which appear due to off-diagonal metric components \( \gamma^r_\phi, \gamma^\phi_\phi \) and \( \gamma^\phi_\phi \). Since these off-diagonal components are typically strongly subdominant compared to the diagonal components, the corresponding error should remain small.

### 3 NUMERICAL TREATMENT OF THE TRANSPORT

In this section, we describe the numerical method used to solve the neutrino transport equations together with the Einstein and hydrodynamics equations. The neutrino energy space is discretized into energy groups, and for each of these and for each neutrino species we solve the evolution equation for \( J \), eq. \( \{37\} \), which generally depends on three spatial dimensions. We use finite-difference methods for the spatial discretization on the same spatial grid as for the GR and hydrodynamics steps.

![Flow chart illustrating the steps performed in the evolution scheme.](figure1)

The flow chart of our evolution algorithm is depicted in Fig.1. After advancing the GR and hydrodynamics equations by one integration step, we calculate the opacity using updated hydrodynamics quantities as well as transport quantities from the previous time step. Next, we evolve the neutrino energy densities. During the transport steps, all hydrodynamics and GR quantities are kept fixed. Since the FLD equations are generally parabolic and the propagation speed of information is in principle infinity, many existing FLD codes employ a fully implicit time integration. However, with the computational cost roughly increasing with the number of grid points to the third power, unsplit, fully implicit integration schemes become particularly expensive in multi-dimensional applications, and they tend to scale poorly on large numbers of computational cores. In the present scheme we avoid this inconvenience by using operator splitting and treating parts of the equation explicitly. In the following subsections, we first estimate the relevant timescales to motivate the time-integration steps, and then we present the detailed discretization procedure employed at each step.

For the calculation of the diffusion coefficient and the Eddington scalar and tensor, we follow Swesty & Myra (2009) (see their Appendix H.4). In particular, we compute a flux-limiter, \( \Lambda(R) \), and diffusion coefficient, \( D \), separately for each coordinate direction. In

\[ \text{Iterative implicit methods on domains decomposed using the Message Passing Interface (MPI) require several collective MPI communications per time step, whereas explicit methods only require a single point-to-point communication per time step and domain boundary.} \]
what follows $D_1$, $D_2$, $D_3$ will denote the diffusion coefficients in the radial, polar, and azimuthal coordinate direction, respectively.

### 3.1 Relevant timescales and motivation of the integration scheme

Using simple dimensional estimates, we first identify the characteristic timescales on which the different terms in the FLD equation induce a change of $J$. We denote the grid spacing for simplicity by $\Delta x$, keeping in mind that this quantity generally depends on the grid location. For clarity, in this section we explicitly include the speed of light, c.

Ignoring the energy derivatives, the FLD equation, eq. (37), is an advection-diffusion-reaction equation (e.g. Anderson 2011). The velocity-dependent terms of eq. (37) are in this sense advection terms, the characteristic timescale of which is bounded from below by the light-crossing time of a grid cell,

$$t_{\text{light}} = \frac{\Delta x}{c}.$$  \tag{38}

The reaction (i.e. neutrino source) terms are associated with timescales

$$t_{\text{source}} = \frac{1}{c\kappa_a},$$  \tag{39}

that are typically much shorter than $t_{\text{light}}$ inside a hot PNS and practically infinity far away from any neutrino sources. Finally, the characteristic timescale of the FLD-related terms can be estimated by

$$t_{\text{diff}} = \frac{\Delta x^2}{D}.$$  \tag{40}

The time step for an explicit treatment of the advection terms, $\Delta t$, must always be less than or equal to the light-crossing timescale of a grid cell, i.e. $\Delta t \leq t_{\text{light}}$. Now, a useful quantity to assess the performance of any method used to integrate the diffusion terms is

$$r_{\text{diff}} \equiv \frac{\Delta t}{t_{\text{diff}}}.$$  \tag{41}

For conventional explicit integration schemes the condition for numerical stability is $r_{\text{diff}} \leq 0.5 \sim 1$. In order to get some idea about typical values of $r_{\text{diff}}$ encountered in post-bounce configurations, we can use (assuming $\Delta t \sim t_{\text{light}}$ and recalling that $\lambda$ and $\kappa_a\Delta x$ denote the flux-limiter and the optical depth per grid cell, respectively)

$$r_{\text{diff}} \sim \frac{\lambda}{\kappa_a\Delta x},$$  \tag{42}

and consider the (simplified) case of constant grid width of $\Delta x \sim O(100)$: Inside the hot PNS we have $\lambda \sim 1/3$ and $\kappa_a\Delta x \gg 1$, and therefore we expect $r_{\text{diff}} \ll 1$. Far away from any neutrino source the Knudsen number roughly scales as $R \sim (\kappa_a\Delta x)^{-1}$, giving $\lambda \sim R^{-1} \sim \kappa_a\Delta x$ and hence $r_{\text{diff}} \sim O(1)$. In other words, both the PNS center and the region far away from the PNS do not necessarily demand an implicit integration. Nevertheless, $r_{\text{diff}}$ may still attain high values in the intermediate, semi-transparent region. However, estimates based on our 1D simulations indicate (cf. Fig. 8) that $r_{\text{diff}}$ may reach values far greater than unity only close to shock. At such large distances lateral neutrino fluxes are strongly subdominant compared to radial fluxes.

Backed by these considerations, we decompose eqs. (37) into three parts and integrate each part in its own operator-split step: In the first step, we integrate the neutrino source terms implicitly (because $t_{\text{source}}$ may be $\ll t_{\text{light}}$) using a Newton-Raphson scheme.

Then we solve for the contributions from the radial derivatives and the spectral-shift terms (i.e. $R_e - \partial_e(eR_e)$) using an implicit Crank-Nicolson scheme. Finally, we obtain the contribution from the non-radial derivatives using an explicit method, namely the Allen-Cheng method [Allen (1970)]. Although this method is explicit, it has the appealing property that it remains stable for any value of $r_{\text{diff}}$; see Sect. 3 for exemplary tests and, e.g., Anderson (2011) for detailed comparisons with other methods. By using an explicit compared to an implicit scheme for the non-radial terms, not only the single-core efficiency is improved but, even more importantly, the scheme can be parallelized very efficiently using MPI decomposition in the polar and azimuthal directions. The trade-off for using an explicit scheme is some loss of accuracy at high values of $r_{\text{diff}} \gtrsim 1$. For this reason, we apply the Allen-Cheng method only to the non-radial fluxes. Since the non-radial fluxes tend to be subdominant compared to the radial fluxes in near-shock regions where $r_{\text{diff}}$ peaks, the error introduced by this integration method should remain manageable.

### 3.2 Neutrino source terms

In the first step, we compute the contribution from the neutrino source terms in an implicit manner. We solve the following equations:

$$W = \frac{\alpha}{\partial_t} J_{\nu,\xi} = \left[ \kappa_a (J^\text{eq} - J) \right]_{\nu,\xi},$$  \tag{43}

$$W = \frac{\rho_0}{\partial_t} e(T, Y_e) = -\sum_{\nu,\xi} \left[ \kappa_a (J^\text{eq} - J) \Delta \xi \right]_{\nu,\xi},$$  \tag{43}

$$W = \frac{\rho_0}{\partial_t} Y_e = -m_{\nu} \sum_{\xi} \left[ \kappa_a (J^\text{eq} - J) \Delta \xi \right]_{\nu,\xi},$$  \tag{43}

$$\left( J^\text{eq} - J \right)_{\nu,\xi}.$$  \tag{43}

The subscripts $\nu$ and $\xi$ indicate the neutrino species and energy bin, respectively, and $\Delta \xi$ is the width of the energy bin centered at $e$. We discretize eq. (43) in time employing a backward Euler scheme and solve the resulting system of equations for the neutrino energy densities, $J_{\nu,\xi}$, temperature, $T$ and electron fraction, $Y_e$, using the Newton-Raphson method. We keep $\alpha$, $W$, $\rho_0$, and $\kappa_a$ constant during this step at values obtained after the GR-hydro step. The Jacobian of eq. (43) is determined numerically, and a direct matrix solver from the LAPACK library (Anderson et al. 1999) is used for inverting the Jacobian. The values of neutrino energy densities, $J_{\nu,\xi}$, obtained in this step are used as initial values in the next step.

### 3.3 Radial derivatives and spectral-shift terms

In the next operator-split step, the following equation is solved:

$$W \partial_r \tilde{J} + R_r = 0,$$  \tag{44}

where

$$R_r = \partial_r (W)\tilde{J} + \partial_r \left[ \alpha W (v^r - \beta^r \alpha^{-1} \tilde{J}) \right] - \partial_r \left[ \alpha \sqrt{\gamma_f} v^r + W \left( \frac{W}{W + 1} v^r - \beta^r \alpha^{-1} \right) \right] D_1 \partial_r \tilde{J} - \partial_r (W \sqrt{\gamma_f}) D_1 e^{r_0} \partial_r \tilde{J} + \alpha \left( \tilde{R}_e - \frac{\partial}{\partial e} (e\tilde{R}_e) \right).$$  \tag{45}
is integrated by using the implicit Crank-Nicolson method. The old time is denoted as \( t^n \) and the new time as \( t^{n+1} \). The time indices for all GR and hydrodynamical quantities are omitted as they are kept fixed in all transport steps. Using superscripts \( n \) and \( n+1 \) to label quantities defined before and after this partial integration step, respectively, the discretized equation reads:

\[
(W \sqrt{\gamma})(\frac{J_i^{n+1} - J_i^n}{\Delta \tau}) = \frac{1}{2}(R_{r,i}^{n+1} + R_{r,i}^n) .
\]

(46)

Here, \( \Delta \tau \equiv t^{n+1} - t^n \) and \( i \) denotes quantities measured at the cell center in the radial direction. In the following, we provide the constituents of \( R_{r,i}^{n+1} \), while the corresponding expressions for \( R_{r,i}^n \) are obtained by replacing \( n+1 \) with \( n \). For simplicity, we assume a uniform radial grid with constant cell size \( \Delta \tau \); the generalization to non-uniform grids is straightforward.

The diffusion term is spatially discretized as:

\[
\frac{\partial}{\partial \tau} (\Delta' D_1 \partial_r J_i) + (\frac{J_i^{n+1} - J_i^n}{\Delta \tau}) = -\Lambda_1 \frac{J_i^{n+1} - J_i^n}{\Delta \tau} ,
\]

where

\[
\Delta' \equiv \alpha \sqrt{\gamma} \left( \omega r \right) + W \left( \frac{W}{W+1} \right) r .
\]

(47)

Indices \( i+1/2 \) and \( i-1/2 \) denote the right and left cell interface of the \( i \)-th cell, respectively. If not mentioned otherwise, all cell interface values of hydrodynamic quantities and metric terms (contained in \( \Delta' \) and in other terms below) are calculated by linear interpolation of the cell centered values.

The fluid-acceleration term (fourth term in eq. [50]) is computed as:

\[
\frac{\partial}{\partial \tau} (\Delta' D_1 \partial_r J_i) + (\frac{J_i^{n+1} - J_i^n}{\Delta \tau}) = \frac{B'}{2} \left[ D_{r,i}^{n+1/2} - \frac{J_i^{n+1}}{\Delta \tau} + D_{r,i}^{n-1/2} - \frac{J_i^n}{\Delta \tau} \right] ,
\]

where

\[
B' \equiv \frac{\partial}{\partial \tau} (W \sqrt{\gamma} \omega r) .
\]

(50)

The time derivative in eq. [56] is calculated using values of the hydrodynamic and metric quantities before and after the initial GR-hydro step.

The advection term is discretized using an upwind-type method (see, e.g., [Dorfi 1998; Rampf & Janka 2002]) as:

\[
\frac{\partial}{\partial \tau} (C' \partial_r J_i) + (\frac{J_i^{n+1} - J_i^n}{\Delta \tau}) = \frac{1}{\Delta \tau} \left[ C_{r,i}^{n+1/2} - C_{r,i}^{n-1/2} \right] ,
\]

where

\[
C' \equiv \alpha \sqrt{\gamma} W (\omega r - \beta r \alpha) .
\]

(51)

\[
(i + 1/2) \equiv \begin{cases} i, & \text{if } v_{i+1/2} > 0, \\ i + 1, & \text{otherwise} . \end{cases}
\]

(53)

The spectral-shift term, \( R_e = \partial_r (\epsilon \epsilon r) \), is discretized using the number-conservative scheme developed in [Müller et al., 2010]. The terms with \( H^j \) and \( K^j \) that appear in \( R_e \) are replaced by \( J^j \) and \( J^j \), respectively, and the flux factor, \( F \), and Eddington tensor, \( \hat{J} \), are defined at instance \( t^n \), while only \( J \) is defined at \( t^{n+1} \).

The Crank-Nicolson method requires to solve a linear system of equations. Direct methods for solving linear systems are relatively expensive, therefore we use the iterative “Generalized Minimal Residual Method with Restart” (GMRES) along with the incomplete LU decomposition as a preconditioner from the NAG library [4] for this purpose. The values of neutrino energy densities, \( \nu \), obtained in this step are used as initial values in the next step.

### 3.4 Non-radial derivatives

Finally, we include the contribution from the remaining lateral advection and diffusion terms by integrating the equation

\[
\partial_t (W \hat{J}^*) + \partial_\omega [\alpha W (v^\phi - \beta^\phi \alpha^{-1}) \hat{J}^*] + \partial_{\bar{w}} [\alpha W (v^\bar{w} - \beta^\bar{w} \alpha^{-1}) \hat{J}^*] +
\]

\[
\partial_\nu \left[ \alpha \sqrt{\gamma} (\nu^{\phi} + W ) \left( \frac{W}{W+1} \right) (\nu^{\phi} - \beta^\phi \alpha^{-1}) \right] D_2 \partial_\nu \hat{J}^* = 0
\]

(54)

using the explicit Allen-Cheng method ([Allen 1970]), where \( D_2 \) and \( D_3 \) are the diffusion coefficients in polar and azimuthal direction, respectively.

The discretized version of eq. (54) is presented below exemplarily for a single dimension (representative of the \( \theta \) or \( \phi \)-direction) and a uniform grid, whose points are labeled by \( k \) and spaced apart by \( \Delta \tau \). The method consists of two steps, a predictor step and a corrector step. We again use \( n \) and \( n+1 \) to label quantities before and after the two substeps. The value of \( \hat{J}^* \) obtained after the predictor step, \( \hat{J}^*_n \), is used in the corrector step to determine \( \hat{J}^*_n+1 \).

The predictor step is given by

\[
\frac{(W \sqrt{\gamma})}{\Delta \tau} (J^*_k - J^*_n) = -\frac{1}{2 \Delta \tau} (J^*_k - J^*_n) -\frac{1}{2 \Delta \tau} (J^*_k - J^*_n) -\frac{1}{2 \Delta \tau} (J^*_k - J^*_n)
\]

(55)

and the corrector step by

\[
\frac{(W \sqrt{\gamma})}{\Delta \tau} (J^*_n + J^*_k) = -\frac{1}{2 \Delta \tau} (J^*_k - J^*_n) +\frac{1}{2 \Delta \tau} (J^*_k - J^*_n) +\frac{1}{2 \Delta \tau} (J^*_k - J^*_n)
\]

(56)

where we used

\[
F = \alpha \sqrt{\gamma} W (v^j - \beta^j \alpha^{-1} ,
\]

\[
E = \alpha \sqrt{\gamma} \left( \nu^{\phi} + W \left( \frac{W}{W+1} \right) (\nu^{\phi} - \beta^\phi \alpha^{-1}) \right) D .
\]

(57)

with \( j \) denoting the considered direction, \( \theta \) or \( \phi \). The values \( J^* \) obtained in this step are the final values at the new time \( t^{n+1} \). These values are used to calculate the neutrino source terms for the hydrodynamics equations (cf. eqs. [49] and for the metric equations [27], which are used in the next GR-Hydro step.

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3.5 Boundary conditions

For our spherical polar coordinate system, we use the standard boundary conditions in angular directions, namely reflecting boundary conditions at the polar axis and periodic boundary conditions in azimuthal direction. For the outer radial boundary, we typically use the “free” boundary condition, meaning that the flux is set according to free-streaming conditions, $D\partial_r J = J$. For the inner radial boundary, the user may choose a “flat” boundary condition, given by $D\partial_r J = 0$ (adequate, e.g., at the coordinate center for symmetry reasons), or a “fixed” boundary condition, for which $J$ is set to some predefined value (e.g., if the inner boundary is placed at a nonzero radius). We set the lower boundary of the neutrino energy grid at $e = 0$ and, therefore, $eR_e = 0$. At the boundary of the highest energy bin, we exponentially extrapolate the neutrino energy density, $J$.

4 TEST PROBLEMS

In this section, we discuss various setups for testing and validating the transport scheme. In Sects. 4.1 and 4.2, we will consider 1D and 2D tests with simplified radiation-matter interactions, and in Sect. 4.3 we examine fully dynamic 1D core-collapse supernova simulations with a microphysical equation of state.

4.1 1D test problems

We first consider 1D toy-model problems, namely the diffusion of a Gaussian pulse and a differentially expanding isothermal atmosphere.

4.1.1 Diffusion of Gaussian pulse with Crank-Nicolson

We set up a well-known test problem consisting of a Gaussian-shaped pulse of radiation that diffuses through a medium with constant scattering opacity, $\kappa_s$. This problem is chosen to test the basic working capability of the code, in particular the correct implementation of the implicit Crank-Nicolson method used for the radial diffusion terms. Diffusion of a Gaussian-shaped pulse with constant scattering opacity has the analytical solution (e.g., Swesty & Myra 2009; Kuroda et al. 2016):

$$ J^{\text{an}}(\hat{r}) = \frac{\kappa_s}{\hat{r}} \exp \left(-\frac{3\kappa_s \hat{r}^2}{4d}\right) $$

(59)

in $d = 1, 2, 3$ dimensions, where $\hat{r}$ is the distance to the center of the pulse. In the present 1D case, a constant scattering opacity of $\kappa_s = 10^3$ is used. The pulse is initialized at time $t = 10^{-9}$ such that its peak coincides with the center of our computational domain, which has a total length of 2. In our spherical polar coordinate system, we mimic the 1D Cartesian grid (plane geometry) by locating the computational domain at some very large radius $r \sim 10^4$. The domain is divided into $N = 128, 256, 512$ cells and a single radiation energy bin is evolved. We employ a “flat” boundary condition for the inner boundary and a “free” boundary condition for the outer boundary, following Swesty & Myra (2009). We consider two choices for the CFL value (cp. eq. (11)), 1 and 10. The problem is stopped at $t = 2 \times 10^{-9}$. In Table 1, the L2-error and the ratio of the L2-error to that resulting with twice the value of $\Delta t$, is given together with the ratio of the current L2-error to that resulting with half the resolution. The L2-error decreases quadratically with the number of grid points, consistent with the 2nd-order accuracy of the Crank-Nicolson method.

| CFL resolution, $N$ | L2 error | error ratio |
|---------------------|----------|-------------|
| 1.0 128             | 0.258    |             |
| 2.0 256             | 0.054    | 4.751       |
| 4.0 512             | 0.013    | 4.023       |
| 10.0 128            | 0.259    |             |
| 20.0 256            | 0.055    | 4.711       |
| 40.0 512            | 0.013    | 4.062       |

Table 2. Gaussian pulse test using the explicit Allen-Cheng method (cf. Sect. 4.1.2). For each value of $\Delta t$, the L2-error is given together with the ratio of the current L2-error to that resulting with twice the value of $\Delta t$. The L2-error decreases linearly with $\Delta t$, consistent with the 1st-order temporal accuracy of the Allen-Cheng method.

| $\Delta t$ resolution, $N$ | L2 error | error ratio |
|---------------------------|----------|-------------|
| 1.6 200                    | 0.0583   |             |
| 0.8 200                    | 0.0359   | 1.623       |
| 0.4 200                    | 0.0206   | 1.742       |
| 0.2 200                    | 0.0089   | 2.314       |
| 0.1 200                    | 0.0037   | 2.405       |
demonstrates that for $r_{\text{diff}} > 0.5$ the Allen-Cheng method indeed remains stable, and that, as expected, the accuracy decreases for higher values of $r_{\text{diff}}$. In Sect. 4.2.2, we will consider a similar test in two dimensions.

4.1.3 Differentially expanding atmosphere

Next, we consider a differentially expanding, isothermal atmosphere in spherical symmetry having a temperature of $T = 1$ [Mihalas 1980; Ramp et al. 2002; Just et al. 2015b] in order to check the correct implementation of the energy-bin coupling and velocity-dependent terms in our code. The velocity profile is given by

$$v_r(r) = v_{\max} \frac{r - r_{\min}}{r_{\max} - r_{\min}}.$$

In the region $[r_{\min}, r_{\max}]$ and by $v_r = 0$ elsewhere. We consider three cases with $v_{\max} = \{0.0, 0.1, 0.3\}$. The radius- and energy-dependent absorption opacity is given by:

$$\kappa_a = \begin{cases} \left( \frac{10 \alpha_s}{r^2} \exp \left( - \frac{(\epsilon - \epsilon_0)^2}{\Delta^2} \right) + \frac{\alpha_s}{r^2} \left( 1 - \exp \left( - \frac{(\epsilon - \epsilon_0)^2}{\Delta^2} \right) \right) \right), & \epsilon \leq \epsilon_0, \\ \left( \frac{10 \alpha_s}{r^2} \exp \left( - \frac{(\epsilon - \epsilon_0)^2}{\Delta^2} \right) \right), & \epsilon > \epsilon_0, \end{cases}$$

and the equilibrium distribution by:

$$J^{\text{eq}} = \frac{8 \pi \epsilon^3}{\exp(\epsilon/T) - 1}.$$

The parameters in the aforementioned prescriptions are given by $(\{r_{\min}, r_{\max}, \epsilon_0, A\}) = \{1.0, 1.1, 0.3, 0.7, 0.2, 7, 10.9989\}$. We use 400 grid points to discretize the simulation domain within $[0.1, 15]$ and employ 40 energy bins to cover the radiation energy range $[0, 11.8]$. At $r = 0.1$ the “flat” boundary condition is applied and at $r = 15$ the free-streaming boundary condition. Each simulation is performed with the Crank-Nicolson scheme using a CFL value of 0.5 and is stopped once stationarity is reached. We run a simulation for each of the three values of $v_{\max}$ as well as for both the LP and the Wilson limiters (cf. eqs. (31)).

In the left plot of Fig. 2, we show radial profiles of the energy-integrated radiation energy density in the comoving frame, $E(r) \equiv \int J(r, \epsilon) d\epsilon$, normalized by $E_0 = \int J^{\text{eq}}(r = 0, \epsilon) d\epsilon$. In agreement with the reference solution (taken from Mihalas 1980 and indicated by markers), $E$ shows a gradual decrease with growing expansion velocities at each given radius $r \leq 10$, which is because of Doppler redshifting in the comoving frame. At higher radii, $r \geq 10$, cases with higher velocities show, again in agreement with the reference solution, higher values of $E$, mainly because of the cumulative effect of reduced absorption rates in the underlying layers where $E$ is reduced.

We notice that radiation in the FLD solutions departs from equilibrium and transitions into free-streaming conditions at somewhat lower radii than radiation in the reference solution. However, the L1-error of the FLD solution with respect to the reference solution is still rather small, namely 4% for the LP limiter and 3% for the Wilson limiter. In this test, the Wilson limiter reproduces the reference solution slightly better than the LP limiter.

In the right plot of Fig. 2, the radiation energy density spectra, normalized by the maximum of equilibrium distribution function, $J_{\max}^{\text{eq}}$, are shown at radii $r = 5.5$ and 11.0, representative of optically thick and thin conditions, respectively, along with the equilibrium spectrum at $r = 5.5$ (see, Fig. 2 of Just et al. 2015b for comparison). The jump in the spectra is associated with the jump in the absorption opacity at energy $\epsilon = \epsilon_0$. Due to radiation being redshifted (in the frame comoving with the background fluid) on its way to the surface, the jump in the spectra around $\epsilon_0$ is smeared out, all the more for higher values of $v_{\max}$.

The overall satisfactory results of this test prove that our FLD code can handle the transition of radiation from diffusion to free-streaming, and they indicate that the velocity-dependent terms describing Doppler effects are implemented properly.

4.2 2D test problems

In this section, we have a look at two-dimensional (2D) toy-model problems in order to check basic multi-dimensional features of our transport solver.

4.2.1 Hemispheric difference test

We first discuss a simple configuration to test the basic ability of the code to deal with multiple dimensions without becoming unstable or producing numerical artefacts. We consider radiation diffusing out of a static scattering atmosphere. The absorption opacity vanishes everywhere.

In the first of two versions of this test, the scattering opacity, $\kappa_s$, has a spherically symmetric profile, given by

$$\kappa_s(r) = \begin{cases} \frac{1}{r}, & r \leq r_{\max}, \\ 10^{-10}, & r > r_{\max}, \end{cases}$$

with $r_{\max} = 10$, while in the second version we consider a dipole-shaped opacity profile by multiplying the opacity with the factor $(1 + 0.5 \cos \theta)$. We use 600 grid points to cover the radial domain of $r \in [0, 11]$, with 200 grid points uniformly distributed between 0 and 1 (optically thick region) and 400 grid points uniformly distributed between 1 and 11 (optically thin region). We use 64 uniformly spaced grid points in polar direction with $\theta \in [0, \pi]$. A single energy group is used and the CFL value is set to 0.5. At $r = 0.01$ the “fixed” boundary condition is applied with $J(r = 0.01, t = 1)$. The problem is initialized with a constant value of $J(r, t = 0) = 10^{-10}$.

In panel (a) of Fig. 3 the two top plots show the scattering opacity, while the two bottom plots depict, at an exemplary time of $t = 10.8$ (after 16000 iterations) the radiation energy density. The case of spherically symmetric (dipole-shaped) opacity is shown left (right). We see that for the spherically symmetric opacity configuration the solution remains spherically symmetric, i.e. our mixed-type integration scheme combines the Crank-Nicolson and Allen-Cheng methods does not lead to spurious asphericities. The relative pole-to-equator and pole-to-pole differences of $J$ are $< 0.1\%$.

In the case of the dipole-shaped opacity, in which the southern hemisphere has lower scattering opacity than the northern hemisphere, we observe, as expected, also a hemispheric difference in the radiation energy density. A greater amount of radiation is able to escape out of the southern hemisphere compared to the northern hemisphere. In panel (b) of Fig. 3, radial profiles of the radiation energy density, $J$, and radial flux density, $H^{\text{r}}$, both multiplied by $r^2$, are shown along the $\theta = 0, \pi/2, \pi$ directions. For higher $\theta$, we observe enhanced fluxes and energies, as well as a transition to free-streaming (i.e. $H^{\text{r}}/J \to 1$) at smaller radii. We checked the causality violation of the total flux (see the comments after eq. {32}). We obtained a maximum total flux factor of 1.2 in this test. Further work is necessary to develop a limiter which ensures causality for both the individual flux components and the total flux. Nevertheless, the stability of the conducted simulation and the plausible physics
results demonstrate the basic functionality of the multidimensional version of our transport solver.

4.2.2 Diffusion of Gaussian pulse

We now investigate two-dimensional diffusion of a Gaussian pulse, which has been considered already in 1D in Sects. 4.1.1 and 4.1.2. In contrast to the test in Sect. 4.2.1 the diffusion test allows to compare with an analytical solution and, hence, we are now able to check also on a quantitative level the proper functionality of the multi-dimensional transport, with a particular focus on the impact of the dimensional splitting with mixed explicit-implicit treatments.

We use Cartesian coordinates in a domain of size $1 \times 1$. A uniform grid with 100 points is employed in each direction, and one energy bin is used. The diffusion coefficient is set to $D = 10^{-3}$ and the problem is initialized at $t = 1$ using equation (59) with $d = 2$ and $r^2 = (x - 0.5)^2 + (y - 0.5)^2$. We again define the characteristic time-step parameter $r_{\text{diff}} = D\Delta t/\Delta x^2$, where $\Delta t$ is the integration time step and $\Delta x$ the grid spacing. The values of $r_{\text{diff}}$ are varied between $\{0.1, 0.5, 1.0\}$, corresponding to CFL values of $\{1, 5, 10\}$, respectively. The Allen-Cheng scheme is applied along the $x$-direction and Crank-Nicolson scheme is applied along the $y$-direction. The simulation is stopped at $t = 1.995$.

The left panels in Fig. 4 show contour plots of the radiation energy density with $r_{\text{diff}} = 0.5$ (top) and $r_{\text{diff}} = 1$ (bottom) at the end of the simulation at time $t = 1.995$. The right column compares profiles along the lines at $y = 0.5$ (top) and $x = 0.5$ (bottom) of the numerical solution with that of the analytical solution, which is given by eq. (62). We first note that the integration remains well-behaved and numerically stable, which is indicated by the absence of spurious numerical features in the plotted data. Moreover, as one can see, the Gaussian pulse retains a circular shape up to a good degree, even for $r_{\text{diff}} = 0.5$, although a non-circular deformation is visible and becomes stronger for values of $r_{\text{diff}} \geq 0.5$. The deformation is a result of the fact that for high values of $r_{\text{diff}}$ the diffusion rates are somewhat reduced in $x$-direction, along which the explicit Allen-Cheng method is used. The error for higher values of $r_{\text{diff}}$ increases much more strongly in $x$-direction than in $y$-direction. This is expected, because the Allen-Cheng method is only first-order accurate while the Crank-Nicolson method is second-order accurate. However, large relative errors only appear at energy densities that are orders of magnitude smaller than the peak energy, for which reason the global error is still small. The test confirms that the dimensional splitting of our algorithm works well and that the Allen-Cheng method remains stable and reasonably accurate even for values $r_{\text{diff}} \sim 0.5 - 1$.

4.3 Spherically symmetric core collapse

In this section, we discuss spherically symmetric simulations with more realistic microphysics of the collapse and post-bounce evolution of a $20M_\odot$ stellar progenitor with solar metallicity (Woosley).
at which the optical depth in Sect. 4.3. "Table 3. With CFL of 0.6). We apply neutrino transport every 50 hydrodynamics time steps. The energy grid is logarithmic, with 15 points covering energies from 0 to 400 MeV, where 400 MeV is the upper boundary of the last energy bin. We evolve electron neutrinos ($\nu_e$), electron anti-neutrinos ($\bar{\nu}_e$), and $\nu_x$ neutrinos that are representative of the four heavy-lepton neutrinos. The neutrino reactions taken into account are listed in Table 3. Their formulation is mostly based on Bruenn (1985) and Rampp & Janka (2002), but additionally includes corrections due to weak magnetism and recoil (Horowitz 2002). We also take into account nucleon-nucleon bremsstrahlung. Following the recipe suggested by O’Connor (2015), we neglect pair-processes for electron-type neutrinos and treat pair-processes for $\nu_x$ neutrinos with a prescription that is formally equivalent to emission/absorption.

We perform simulations with fully general relativistic hydrodynamics and transport, denoted by NADA GR, using each of the two flux-limiters, LP and Wilson (cf. eqs. [31]). However, in order to compare our code with a reference solution, we first discuss a simulation, called NADA NEWT, that is identical to NADA GR with the LP limiter, but that is conducted with a Newtonian treatment of gravity and special relativistic hydrodynamics. We compare this model to model ALCAR NEWT which is performed with the ALCAR code using the Minerbo closure (Just et al. 2015b, 2018). Model ALCAR NEWT contains exactly the same input physics, but it employs the M1 approximation for the neutrino transport and assumes non-relativistic hydrodynamics.

In the left column of Fig. 5 we compare characteristic prop-

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**Table 3. Neutrino opacities used for the 1D CCSN simulations discussed in Sect. 4.3.** "N" denotes nucleons and "A" and "$A'$" denote nuclei. The $\nu\bar{\nu}$ pair processes are taken into account only for $\nu_e$ (for $\nu_e$, $\bar{\nu}_e$, the $\beta$-processes are by far dominant).

| Reaction                  | Neutrino |
|---------------------------|----------|
| $\nu + A \leftrightarrow \nu + A$ | $\nu_e$, $\bar{\nu}_e$ |
| $\nu + N \leftrightarrow \nu + N$ | $\nu_e$, $\bar{\nu}_e$ |
| $\nu_e + n \leftrightarrow e^- + p$ | $\nu_e$, $\bar{\nu}_e$ |
| $\nu_e + A \leftrightarrow e^- + A'$ | $\nu_e$ |
| $\bar{\nu}_e + p \leftrightarrow e^+ + n$ | $\nu_e$, $\bar{\nu}_e$ |
| $\nu + \bar{\nu} + N + N \leftrightarrow N + N + N$ | $\nu_x$ |

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properties of the collapse between models NADA NEWT and ALCAR NEWT, namely the electron fraction, $Y_e$, lepton fraction, $Y_{lep}$, and entropy per baryon at the stellar center as function of the central density, $\rho$. Neutrino trapping sets in once the central density reaches $\sim 2 \times 10^{12} \text{g cm}^{-3}$. After the onset of neutrino trapping, the lepton fraction remains constant with a value around 0.37 for both codes. The electron fraction roughly asymptotes at a central density of $\sim 2 \times 10^{13} \text{g cm}^{-3}$. The deleptonization slows down around $5 \times 10^{10} - 10^{11} \text{g cm}^{-3}$ in both models due to neutron shell blocking and a low abundance of free protons (e.g. Bruenn 1985). After the onset of trapping, the entropy per baryon of the gas increases to $\approx 1.15 k_b/\text{baryon}$ because of a growing number of free nucleons and $\alpha$-particles. Overall, both simulations agree very well in their deleptonization behavior.

In the right column of Fig. 5, we show the neutrino luminosity as well as the shock- and PNS-radii as functions of time until 20 ms post bounce. For the present spherically symmetric case we define the comoving-frame luminosity as

$$L_\nu(r) \equiv 4\pi c^4 \rho^2 \int \mathcal{H}_\nu(r, e) \, de,$$  

(64)

where $\phi = 0$ for the case of Newtonian gravity. The luminosity obtained with the NADA code agrees well with that of the ALCAR code. The integrated energy loss is ~5% higher in the NADA model than that in the ALCAR model within the first 20 ms of post-bounce evolution. The peak in the neutrino luminosities around 10-15 ms post-bounce time is due to early expansion and subsequent compression of matter behind the shock as shown in the bottom-right plot of Fig. 5. We also see that the luminosity of $\nu_e$ rises earlier than $\nu_x$, which is different from existing earlier work where the $\nu_x$ luminosity rises earlier than the $\nu_e$ luminosity (see, e.g. Thompson et al. 2003, Kachelrieß et al. 2005). This difference, which is shared with the comparative ALCAR model, might be a consequence of the use of an analytic closure relation in the FLD scheme, or it might be linked to different sets of neutrino reactions employed by our scheme and previous models (e.g. the current NADA version ignores neutrino-electron scattering). Indeed, the rapid rise and first peak of the $\nu_e$ luminosity disappears in ALCAR simulations when neutrino electron scattering is taken into account.

The left column of Fig. 6 provides various quantities as functions of time for the NADA NEWT and ALCAR NEWT simulations, namely the neutrino luminosities, $L_\nu$, the neutrino mean...
energies,\[\langle \epsilon_\nu(r) \rangle \equiv \frac{\int J(r, \epsilon) \, d\epsilon}{\int J(r, \epsilon) \, \epsilon^{-1} \, d\epsilon},\]
the mass-accretion rate at 500 km, \( M \), the mass of the PNS, \( M_{\text{ns}} \),
the total mass in the gain layer, \( M_{\text{gain}} \), and the total neutrino-heating
rates, \( Q_{\text{gain}} \). The luminosities and mean energies, as well as almost
all other quantities agree remarkably well between both codes. The
integrated energy loss is \(-3\%\) higher in the NADA model than in
the ALCAR model during the post-bounce evolution until the end of
the simulations. We also notice a secular drift towards higher mean
energies in the NADA NEWT model, particularly at late times. We
speculate that this difference might be related to what we see in
Fig. 7 where radial profiles of the mean flux factor,
\[\langle f_\nu(r) \rangle \equiv \frac{\int H_\nu(r, \epsilon) \, \epsilon^{-1} \, d\epsilon}{\int H_\nu(r, \epsilon) \, \epsilon^{-1} \, d\epsilon},\]
are plotted: In the NADA simulation, the flux factors rise at slightly
smaller radii than in the ALCAR simulation, which means that
neutrinos are effectively released from deeper within the PNS and
therefore at higher temperatures. As a result, the neutrino mean
energies, \( \langle \epsilon_\nu \rangle \), have higher values in the NADA simulation com-
pared to the ALCAR simulation (see below for further discussion of Fig. 7). The higher neutrino luminosities in model NADA NEWT
might be linked to slightly higher mass accretion rates compared to
model ALCAR NEWT, especially during the phase of high mass
accretion \( (t_{\text{ph}} \lesssim 0.2s) \); see Fig. 6 left column). We also notice that
model NADA NEWT produces larger PNS radii (by \(-2\sim 7\)km) and
shock radii (by \(-5\sim 20\)km) compared to model ALCAR NEWT.

The most natural explanation for this disagreement (and for the
disagreement in some other properties) seems to be that NADA
and ALCAR use different approximate closures. However, at this
point we cannot exclude that also other, more subtle discrepancies
between the two codes exist in the numerical treatment of the trans-
port or hydrodynamics sector (e.g., the detailed implementation of
neutrino rates, the description of the gravity term in the Newtonian
version of NADA), which could contribute to the observed differ-
ences. Identifying these remaining inconsistencies is challenging
and can be vastly time consuming, and is therefore out of the scope
of the present paper. It might be worth noting that hardly any code

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**Figure 5.** Properties obtained during the collapse (left) and shortly after bounce (right) in the NADA and ALCAR CCSN simulations of a 20 solar mass progenitor, both with Newtonian gravity (see Sect. 4.3). NADA (ALCAR) results are marked by thick (thin) lines. The top left plot shows the central electron fraction (red) and total lepton fraction (yellow) and the bottom left plot shows the central entropy per baryon as functions of central density. The top right plot depicts as functions of time the neutrino luminosities of \( \nu_e \) (red), \( \bar{\nu}_e \) (yellow), and an individual species of \( \nu_\mu \) (blue), measured in the comoving frame at a radius of 500 km. The bottom right plot shows the shock radii (red) and proto-neutron star radii (blue) together with contours of constant enclosed mass in units of the solar mass (gray). The mass shells are depicted for the NADA NEWT model.
Figure 6. Comparison of global properties as functions of time for several CCSN simulations. Shown are results in the left column for the NADA (thick lines) and ALCAR (thin lines) simulations that both use Newtonian gravity, in the middle column for NADA simulations with GR (thick lines) and Newtonian (thin lines) treatment of gravity, and in the right column for NADA simulations with GR gravity using different flux-limiters (cf. Sect. 2.3.3), namely the LP flux-limiter (thick lines) and the Wilson limiter (thin lines). From top to bottom the panels display the neutrino luminosities, the neutrino mean energies, the shock-, PNS-, and gain radii, the mass accretion rate measured at 500 km, the mass in the gain layer, and the total neutrino-heating rate in the gain layer.

The maximum difference in PNS masses between models NADA NEWT and ALCAR NEWT is $4 \times 10^{-3} M_{\odot}$, and therefore, the differences between the corresponding lines are hardly visible.
Figure 7. Comparison of the energy-averaged flux factors, eq. (66), as functions of radius between the NADA (thick lines) and ALCAR (thin lines) CCSN simulations using Newtonian gravity for different neutrino species $\nu_e$ (red), $\bar{\nu}_e$ (yellow), and $\nu_x$ (blue). The left panel (only for $\nu_e$) shows the mean flux factors at a time during collapse when the central density reaches $2 \times 10^{12} \text{g/cm}^3$, and the right plot at a post-bounce time of 500 ms. For all cases the transition to free-streaming (i.e. to high flux factors, $\langle f_\nu \rangle \gtrsim 0.5$), takes place at somewhat smaller radii and higher densities for the NADA simulations, which employ the FLD approximation, compared to the ALCAR simulations, which make use of the M1 approximation.

In the middle column of Fig. 6, we compare the fully relativistic NADA GR simulation with the NADA NEWT model, using for both cases the LP limiter. The main impact of GR is to produce an effectively steeper gravitational potential. Hence, the core bounces $\approx 40$ ms earlier in the GR case compared to the Newtonian case. Subsequently, the GR treatment produces a considerably more compact PNS and post-shock configuration. As a consequence of the higher compactness, the temperatures at the PNS surface are increased, which results in significantly enhanced neutrino luminosities and mean energies. The enhancement is even strong enough to overcompensate for the lower masses in the gain layer and to yield considerably higher total neutrino-heating rates compared to the Newtonian model. The qualitative differences found here between Newtonian and general relativistic CCSN models are in good agreement with previous studies (e.g. Bruenn et al. 2001; Marek et al. 2006; Müller et al. 2012; O’Connor & Couch 2018). We conclude that the coupling of the neutrino-hydrodynamics components of the code to the Einstein solver is working well, at least in spherical symmetry.

In order to test the sensitivity with respect to the chosen flux-limiter, we also compare the NADA GR simulation that uses the LP limiter against a similar simulation that employs the Wilson limiter; see the right column of Fig. 6 for the corresponding quantities as functions of time. Using the Wilson limiter instead of the LP limiter results in an overall less compact configuration, i.e. in higher values of the shock, PNS-, and gain-radii, particularly at earlier times, $t_{pb} \gtrsim 0.3$ s, while later on the differences become smaller. The most likely reason is found when comparing the luminosities, which for electron-type neutrinos are significantly reduced during the first $\sim 0.2$–$0.3$ s of post-bounce evolution in the case of using the WILSON limiter. The lower neutrino-cooling rates explain the larger PNS radii, and those also cause (see e.g. Janka 2012) larger gain- and shock-radii in the case of using the WILSON limiter. The more powerful neutrino heating in the gain layer with the WILSON limiter is thus mainly a result of the increased mass in the gain layer compared to the case with the LP limiter.

It is interesting that the differences between NADA GR (LP) and NADA GR (WILSON) are bigger than those between NADA...
Figure 8. Estimate of the time-step parameter $r_{\text{diff}}$ characterizing the accuracy of the explicit integration of lateral fluxes in the future case of a 2D axisymmetric simulation. We use data from the one-dimensional, Newtonian NADA simulation with LP limiter, employ eqs. (67) and (68), and assume an angular resolution of 1.4 degrees. The resulting $r_{\text{diff}}$ is shown at post-bounce times of 100 ms (left) and 300 ms (right) for species $\nu_e$ (red), $\bar{\nu}_e$ (yellow), and $\nu_x$ (blue).

NEWT (LP) and ALCAR NEWT. In order to understand this, we point out that ALCAR uses an M1 scheme with Minerbo closure (Just et al. 2015b). From Janka (1992) we know that the LP limiter shows a better agreement with the limiter belonging to the Minerbo closure than with the Wilson limiter in the optically thick and semi-transparent regimes. We suspect that this fact explains why there is good agreement between models NADA NEWT (LP) and ALCAR NEWT but comparatively large deviations between model NADA GR (LP) and NADA GR (WILSON).

In Fig. 7 we show the radial profile of the mean flux factor, eq. (65), for models NADA NEWT (with LP limiter) and ALCAR NEWT, at a time when the central density is $2 \times 10^{12}$ g cm$^{-3}$ (left plot) and at 300 ms post bounce (right plot). Although the M1 scheme used in ALCAR is not a fully accurate solution of the Boltzmann equation either, it is likely somewhat more reliable than the FLD solution (see, Just et al. 2015b for a comparison of FLD and M1 with a Boltzmann solver for static CCSN-related configurations). In both cases, we see that the FLD solution makes the transition to free-streaming conditions at smaller radii compared to the M1-based ALCAR solution. Furthermore, in the FLD scheme, the flux factor jumps to high values artificially strongly near sharp drops in the transport opacity (see, Janka 1992 for a detailed discussion). As a result, the mean flux factor abruptly becomes $\approx 1$ already close to the PNS surface, i.e. well behind the shock, which lies at $r \approx 80 - 90$ km in the right panel of Fig. 7. The results concerning the flux factor are consistent with previous investigations of the FLD scheme; see, Dgani & Janka (1992) who identify a “missing opacity” problem of FLD that can be solved, only in 1D however, by introducing an “artificial opacity”. However, the otherwise good agreement between NADA NEWT and ALCAR NEWT suggests that the aforementioned deficiencies are small enough to affect the 1D dynamics at most on the few-percent level.

In order to assess the energy conservation error of our code we show in Fig. 9 different components of the energy for our NADA NEWT model with neutrino transport (left plot) and without neutrino transport (right plot). We also evaluate the magnitude of energy violation (brown lines), $\Delta E = E_{\text{tot}} - E_{\text{tot},0}$ (where $E_{\text{tot},0}$ is the total energy at the beginning of the simulation), in our models. For the calculation of the total energy, $E_{\text{tot}}$, we have taken into account the energy loss due to neutrino escape from the computational grid and energy gain due to the mass inflow through the outer boundary (not shown in Fig. 9 because it is tiny). With neutrino transport included we find a total energy violation of about $-2 \times 10^{51}$ erg at bounce and $(6 - 7) \times 10^{51}$ erg at 600 ms after bounce (left plot of Fig. 9). However, referring this to the relevant energy scale of the problem, this is $\sim 2\%$ energy violation with respect to the gravitational energy or internal energy at bounce for the NADA NEWT model with neutrino transport and $\sim 2\%$ ($\sim 3.5\%$) energy violation at $\sim 600$ ms post-bounce time relative to the gravitational (internal) energy (or again $\sim 2\%$ when compared to the sum of internal energy and en-
ergies stored and escaping in neutrinos). For the NADA NEWT model without neutrino transport, we obtained an energy violation with respect to the gravitational energy (internal energy) of \(-1.3\%\) \((-1.3\%\) at bounce and \(-7\%\) \((-7\%) \) at \(-550\) ms post-bounce time. By comparison, the Fornax code is mentioned to conserve the total energy on an excellent level of \(0.05\times10^{51}\) erg until 1 s after bounce for a pure hydrodynamical Newtonian simulation with 608 radial zones (see section 8.9 of Skinner et al. 2018). To a large extent the energy violation is likely to be due to the non-conservative implementation of the gravitational potential in the Newtonian version of NADA, which could be improved, e.g., by implementing the gravity treatment of Müller et al. (2010).

As a final point we discuss the time-integration accuracy of a (future) multi-dimensional CCSN simulation based on our 1D simulation data. As we recall from Sect. 3, the time integration of the transport equations is done implicitly for the source terms as well as the radial fluxes and energy derivatives, and explicitly for the lateral fluxes. We consider for the case of an axisymmetric simulation the resulting characteristic time-step parameter,

\[
r_{\text{diff}} = \frac{\langle D_{\nu} \rangle}{\Delta t / (r \Delta \theta)^2},
\]

i.e. the ratio of the integration time step employed for all explicit terms, \(\Delta t\), and the characteristic timescale associated with the lateral diffusion terms, \((r \Delta \theta)^2 / \langle D_{\nu} \rangle\). We assign \(\Delta t\) the value of the hydrodynamics time step employed in the 1D simulation, given by equation (11), and assume a suitable value of 1.4 degrees for \(\Delta \theta\).

The energy-averaged diffusion coefficient, \(\langle D_{\nu} \rangle\), is estimated as

\[
\langle D_{\nu} \rangle(r) = \frac{\int J_{\nu}(r, \epsilon) \epsilon^{-1} \, \text{d} \epsilon}{\int J_{\nu}(r, \epsilon) \, \text{d} \epsilon}.
\]

For this setup, the estimates of \(r_{\text{diff}}\), shown in Fig. 8 for an early and a late post-bounce time, allow us to identify regions, \(r_{\text{diff}} \gtrsim 1\), in which the explicit Allen-Cheng method is potentially less accurate in describing the lateral neutrino propagation. We find however, that high values, \(r_{\text{diff}} \gtrsim 1\), are reached only near the very center of the PNS and close to the shock. This is reassuring, because deep inside the PNS \((r \lesssim 2\) km), neutrinos are trapped and neutrino fluxes are strongly dominated by radial advection fluxes, while at large radii in the vicinity of the shock lateral neutrino fluxes are anyway small compared to radial fluxes. Hence, our estimate indicates that the explicit treatment of lateral terms in multi-dimensional simulations will only have minor consequences on the dynamical evolution. In future 2D simulations, we will apply the lateral transport sweep at every hydrodynamics time step, but the radial transport sweep will be applied only at every few, say 10–50, hydrodynamics time steps.

5 SUMMARY

In this paper, we presented a new code to solve multi-dimensional neutrino transport in spherical polar coordinates coupled to the GR-hydro code NADA (Baumgarte et al. 2013; Montero et al. 2014).
The transport solver assumes the flux-limited diffusion approximation and evolves the neutrino energy densities as measured in the frame comoving with the fluid. In order to improve the computational efficiency and parallel scalability compared to a scheme that solves the multi-dimensional FLD equations in a single, unsplit step, we employ operator splitting such that different parts of the equations (and different coordinate directions) are dealt with in separate, consecutive steps. The source terms as well as the radial- and energy-derivatives are integrated implicitly, while the non-radial derivatives are integrated explicitly using the Allen-Cheng method (Allen1970).

We tested the algorithm and its implementation by conducting several problems in 1D and 2D and comparing to reference solutions. The tests demonstrate that the code runs stably and it robustly handles diffusion, transition to free-streaming, energy-bin coupling, multi-dimensional transport, microphysical neutrino interactions, and the coupling to GR-hydro. We confirmed that the Allen-Cheng method is, in contrast to conventional explicit schemes, unconditionally stable even if the diffusion timescale of a grid cell is shorter than the time step used for integration. However, estimates indicate that in multidimensional CCSN simulations, the diffusion timescale is typically longer than the integration time step except close to the coordinate center and the shock locations where lateral neutrino fluxes are strongly subdominant.

In terms of physics ingredients the most sophisticated tests performed here consider the core collapse and post-bounce evolution of a massive star in spherical symmetry. We compared a Newtonian version of this configuration with the results of the M1 code ALCAR (Just et al.2015b,2018) and found that most global properties agree remarkably well, namely within 5 – 10 %. We also compared the Newtonian simulation with its GR counterpart and were able to confirm the tendency of GR (e.g. Bruenn et al.2001, Marek et al.2006, Müller et al.2012, O’Connor & Couch2018) to lead to an overall more compact post-bounce configuration along with higher neutrino luminosities and mean energies. A final comparison of the GR simulation using the Levermore-Pomraning (LP) flux-limiter with another GR simulation using the Wilson limiter revealed notable differences, which, given the good agreement of the LP simulation with the ALCAR simulation, suggests that the LP limiter may be a better choice for CCSN simulations than the Wilson limiter.

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APPENDIX A: DERIVATION OF ENERGY EQUATION IN COMOVING FRAME

The transport equation used in our code evolves the energy density, \( \mathcal{F} \), measured in an orthonormal comoving frame. This Appendix shows how this evolution equation can be obtained from corresponding equations evolving the lab-frame moments, \( E \) and \( F^i \). The lab-frame equations are derived and discussed in Shibata et al. (2011); Endeve et al. (2012); Cardall et al. (2013). In Table A1 we summarize the meaning of various quantities used here and where to find more information about them. The lab-frame equations contain the quantities, \( G, I^i, P^{ij}, Q^{ijk} \), which are related to the third-moment tensor,

\[
\mathcal{U}^{abc} \equiv e^3 \int d^3\mathbf{x} \, f (\mathbf{u}, \rho) \, d\Omega ,
\]

by

\[
\mathcal{U}^{abc} = G n^a n^b n^c + I^a n^b n^c + I^b n^a n^c + F^{abc} n^d + P^{ab} n^c + P^{bc} n^a + P^{ac} n^b + Q^{abc}.
\]

We start from the lab-frame neutrino-energy equation as given in a conservative form by equations (171), (91-93), (146), (147), and (173) of Cardall et al. (2013):

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial t} \left( \sqrt{\gamma} \mathcal{E} \right) + \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^j} \left( \sqrt{\gamma} (\alpha F^j - \beta^j \mathcal{E}) \right) + F^j \frac{\partial \ln a}{\partial x^j} - \mathcal{S}^{ijk} K_{jk} - \frac{1}{e^2} \frac{\partial}{\partial e} (e^2 F^c) = -n_\mu \frac{\epsilon}{\epsilon} \int p^\mu C d\Omega .
\]

The definition of different symbols can be found in Table A1. Here,

\[
F^\epsilon = W \{ I_j \frac{\partial v^j}{\partial t} + P^{jk} \frac{\partial v^j}{\partial x^k} + \frac{1}{2} P^{jk} v^l \frac{\partial v_l}{\partial x^k} - (I^j - G^j) \frac{\partial \ln a}{\partial x^j} - P^{jk} K_{jk} + v^j I_k \frac{1}{\alpha} \frac{\partial \beta^k}{\partial x^j} \} + \frac{1}{2} \left( I_j \frac{\partial v^j}{\partial t} - G^j \frac{\partial W}{\partial t} + (P^{jk} + I^j - I) \frac{\partial W}{\partial x^j} \right),
\]

with \( \frac{\partial}{\partial t} \equiv n^a \frac{\partial}{\partial \alpha}. \) Equations (A3) and (A4) are copied directly from Cardall et al. (2013), who employs slightly different definitions than us concerning the power of \( \epsilon \) in the prefactor of the angular moments and third-moment projections. We now switch to our notation by doing the replacements \( e^2 \{ \mathcal{E}, F^j, S^{ij}, L^{ijk} \} \rightarrow \left\{ \mathcal{E}, F^j, S^{ij}, L^{ijk} \right\} \) and \( e \{ G^i, I^j, P^{ij}, Q^{ijk} \} \rightarrow \{ G^i, I^j, P^{ij}, Q^{ijk} \}. \) Moreover, we multiply eq. (A3) by \( e^2 \sqrt{\gamma} \) and introduce the notation \( X \equiv \sqrt{\gamma} X \) for any quantity \( X \) to obtain:

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial t} (\sqrt{\gamma} \mathcal{E}) + \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^j} \left( \sqrt{\gamma} (\alpha F^j - \beta^j \mathcal{E}) \right) + F^j \frac{\partial \ln a}{\partial x^j} - \mathcal{S}^{ijk} K_{jk} - \frac{1}{e^2} \frac{\partial}{\partial e} (e^2 F^c) = -n_\mu \sqrt{\gamma} \int p^\mu C d\Omega ,
\]

where the formal definition of \( F^\epsilon \) is still given by eq. (A4). Multiplying eq. (A5) by \( W \) and using the product rule (i.e. \( W \frac{\partial f}{\partial x} = \frac{\partial (W f)}{\partial x} - f \frac{\partial W}{\partial x} \)) one finds

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial t} (W \mathcal{E}) + \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^j} \{ W(\alpha F^j - \beta^j \mathcal{E}) \} + W F^j \frac{\partial \ln a}{\partial x^j} - W \mathcal{S}^{ijk} K_{jk} - \frac{1}{e^2} \frac{\partial}{\partial e} (W e F^c) = -n_\mu W \sqrt{\gamma} \int p^\mu C d\Omega .
\]

We now extract the lab-frame energy-momentum equation from eqs. (172), (95-97), (149), (150), and (174) of Cardall et al. (2013), again keeping their notation first:

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial t} (\sqrt{\gamma} \mathcal{E}_i) + \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^j} \{ \sqrt{\gamma} (\alpha S^j_i - \beta^j \mathcal{E}_i) \} + \mathcal{E} \frac{\partial \ln a}{\partial x^j} - \mathcal{F} \frac{1}{\alpha} \frac{\partial \beta^j}{\partial x^j} - \frac{1}{2} \frac{\partial S^{ijk}}{\partial x^j} = \gamma_\mu \frac{1}{\epsilon} \int p^\mu C d\Omega .
\]
where

\[
S_{1}^{\varepsilon} = W (P_{ij} \frac{\partial v_{j}}{\partial \tau} + Q_{ijk} \frac{\partial v_{j}}{\partial x_{k}} + \frac{1}{2} Q_{ijk} \frac{\partial \gamma_{jk}}{\partial x_{i}} + (P_{ij} - I_{ij} \nu) \frac{\partial \ln \alpha}{\partial x_{j}} - Q_{ijk} K_{jk} + \nu P_{ik} \frac{1}{\alpha} \frac{\partial \beta_{k}}{\partial x_{i}}) + (P_{ik} \nu - I_{ik}) \frac{\partial W}{\partial \tau} + (Q_{ikj} \nu - P_{ij}) \frac{\partial W}{\partial x_{j}}.
\]

(A8)

Switching to our notation by doing the same replacements as for the energy equation above, we obtain:

\[
\frac{1}{\alpha} \frac{\partial}{\partial t} (\tilde{S}_{1}^{\varepsilon}) + \frac{1}{\alpha} \frac{\partial}{\partial x_{i}} \left((\alpha \tilde{S}_{i}^{\varepsilon} - \beta \tilde{F}_{i}^{\varepsilon}) + \tilde{E} \frac{\partial \ln \alpha}{\partial x_{i}} - \tilde{F}_{i} \frac{1}{\alpha} \frac{\partial \beta_{i}}{\partial x_{i}} - \frac{1}{2} \tilde{S}_{ij} \frac{\partial \gamma_{jk}}{\partial x_{i}} - \frac{1}{\alpha} (\varepsilon \tilde{S}_{i}^{\varepsilon}) \right) = \gamma \mu \nu \sqrt{\varepsilon} \int p^{\#} C d\Omega.
\]

(A9)

where the formal definition of \( S_{i}^{\varepsilon} \) is still given by eq. (A8). We multiply eq. (A9) by \( W \) and contract with \( v_{j} \), to end up with

\[
\frac{1}{\alpha} W \frac{\partial}{\partial t} W (v_{i} \tilde{F}_{i}) + \frac{1}{\alpha} \frac{\partial}{\partial x_{i}} \left[W (\alpha \tilde{S}_{i}^{\varepsilon} - \beta \tilde{F}_{i}) + W (\tilde{E} \frac{\partial \ln \alpha}{\partial x_{i}} - W \tilde{F}_{i} \frac{1}{\alpha} \frac{\partial \beta_{i}}{\partial x_{i}} - \frac{1}{2} W \tilde{S}_{ij} \frac{\partial \gamma_{jk}}{\partial x_{i}} - W \frac{1}{\alpha} \tilde{S}_{i}^{\varepsilon}) \right] = \gamma \mu \nu \sqrt{\varepsilon} \int p^{\#} C d\Omega.
\]

(A10)

Subtracting eq. (A10) from eq. (A6) we get

\[
\frac{1}{\alpha} W (\tilde{E} - v_{i} \tilde{F}_{i}) + \frac{1}{\alpha} \frac{\partial}{\partial x_{i}} \left[W (\alpha \tilde{S}_{i}^{\varepsilon} - \beta \tilde{F}_{i}) + W (\tilde{E} \frac{\partial \ln \alpha}{\partial x_{i}} - W \tilde{F}_{i} \frac{1}{\alpha} \frac{\partial \beta_{i}}{\partial x_{i}} - \frac{1}{2} W \tilde{S}_{ij} \frac{\partial \gamma_{jk}}{\partial x_{i}} - W \frac{1}{\alpha} \tilde{S}_{i}^{\varepsilon}) \right] = -\gamma \mu \nu \sqrt{\varepsilon} (n_{\mu} + \mu \nu v_{i} \sqrt{\varepsilon}) \int p^{\#} C d\Omega.
\]

(A11)

We further rewrite \( n^{\#} \frac{\partial}{\partial x_{i}} = \frac{\partial}{\partial x_{i}} \) with \( n^{\#} = (1/\alpha - \beta / \alpha) \), \( W (n_{\mu} \gamma \mu \nu v_{i} \sqrt{\varepsilon}) = u_{\mu} \), and redefine \( e^{2} C \rightarrow C \):

\[
\frac{1}{\alpha} W (\tilde{E} - v_{i} \tilde{F}_{i}) + \frac{1}{\alpha} \frac{\partial}{\partial x_{i}} \left[W (\alpha \tilde{S}_{i}^{\varepsilon} - \beta \tilde{F}_{i}) + W (\tilde{E} \frac{\partial \ln \alpha}{\partial x_{i}} - W \tilde{F}_{i} \frac{1}{\alpha} \frac{\partial \beta_{i}}{\partial x_{i}} - \frac{1}{2} W \tilde{S}_{ij} \frac{\partial \gamma_{jk}}{\partial x_{i}} - W \frac{1}{\alpha} \tilde{S}_{i}^{\varepsilon}) \right] = -\gamma \mu \nu \sqrt{\varepsilon} \int p^{\#} C d\Omega.
\]

(A12)

The term inside the energy derivative of eq. (A12) is given by:

\[
e W (F^{\varepsilon} - v_{i} S_{i}^{\varepsilon}) = e W \left[ (I_{ij} \frac{\partial v_{j}}{\partial \tau} + P_{ij} \frac{\partial v_{j}}{\partial x_{i}} + \frac{1}{2} Q_{ijk} \frac{\partial \gamma_{jk}}{\partial x_{i}} + (I_{ij} - G_{ij}) \frac{\partial \ln \alpha}{\partial x_{j}} - P_{ijk} K_{jk} + \nu P_{ik} \frac{1}{\alpha} \frac{\partial \beta_{k}}{\partial x_{i}}) \right]
\]

(A13)

In order to express \( \{G, F^{\varepsilon}, P^{\varepsilon}, Q^{\varepsilon}, S_{i}^{\varepsilon}\} \) in terms of the lab-frame moments, \( \{E, F^{\varepsilon}, S_{i}^{\varepsilon}\} \), we use eqs. (74)-(80) of Endeve et al. [2012], however with \( \varepsilon \) set to 1 in their equations to account for the different definitions:

\[
e W (F^{\varepsilon} - v_{i} S_{i}^{\varepsilon}) = e W \left[ \tilde{F}_{i} \frac{\partial v_{j}}{\partial \tau} + S_{i}^{\varepsilon} \frac{\partial v_{j}}{\partial x_{i}} + (F_{ij} - E_{ij} v^{\varepsilon}) \frac{\partial \ln \alpha}{\partial x_{i}} + v^{\varepsilon} F_{ij} \frac{1}{\alpha} \frac{\partial \beta_{k}}{\partial x_{i}} + S_{ij}^{\varepsilon} \frac{1}{2} \frac{\partial \gamma_{jk}}{\partial x_{i}} - K_{jk}) \right]
\]

(A14)

where in the last line we defined \( R_{e} \) used in the main text of this paper. While \( R_{e} \) is expressed in eq. (A14) in terms of the lab-frame moments,
it can be re-expressed in terms of the comoving-frame moments using
\begin{align}
E - v^i F_i &= J + \bar{v}_i \mathcal{H}^i, \\
F^i - S^i_{\ j} v^j &= \frac{1}{W} e^i_j \mathcal{H}^j + v^i J + W v^i \bar{v}_i \mathcal{H}^i, \\
F^i - E v^i &= W \left( e^i_j - \frac{W}{W + 1} v^i \bar{v}_j \mathcal{H}^j \right)
\end{align}
(A15)
as well as eqs. (22). Using the same transformations also for the remaining terms of eq. (A12), we finally obtain the neutrino energy equation in terms of the comoving-frame neutrino moments as:
\begin{align}
\frac{1}{\alpha} \frac{\partial}{\partial t} [W (J + \bar{v}_i \mathcal{H}^i)] + \frac{1}{\alpha} \frac{\partial}{\partial x^j} \left[ \alpha W (v^j - \beta^j / \alpha) J^j \right] + \frac{1}{\alpha} \frac{\partial}{\partial x^j} \left[ \alpha e^j_i \mathcal{H}^i \right] \\
+ \frac{1}{\alpha} \frac{\partial}{\partial x^j} \left[ \alpha W \left( \frac{W}{W + 1} v^j - \beta^j / \alpha \right) \bar{v}_i \mathcal{H}^i \right] + \dot{\mathcal{R}}^j_{\ e} = \frac{\partial}{\partial e} (e \dot{\mathcal{R}}^j_{\ e}) = \sqrt{\gamma} \int C d\Omega .
\end{align}
(A16)

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