Orbifolds, Quantum Cosmology, and Nontrivial Topology

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Abstract

In order to include nontrivial spatial topologies in the problem of quantum creation of a universe, it seems to be necessary to generalize the sum over compact, smooth 4-manifolds to a sum over finite-volume, compact 4-orbifolds. We consider in detail the case of a 4-spherical orbifold with a cone-point singularity. This allows for the inclusion of a nontrivial topology in the semiclassical path integral approach to quantum cosmology, in the context of a Robertson-Walker minisuperspace.

1 Introduction

The question of whether our universe has a finite or infinite spatial extension is still an open question, related to that of the global topology of the

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universe. The Einstein field equations and the demand for homogeneity and isotropy deal only with local geometrical properties, leaving the global topology of the universe undetermined \[1\]. Thus in the standard cosmological framework, the universe is described by a Friedmann-Lamaitre-Robertson-Walker (FLRW) solution, where the spatial sections are usually assumed to be simply connected. These have infinite volumes in the cases of hyperbolic space $H^3$ and Euclidean space $R^3$, and a finite volume in the case of spherical space $S^3$. But each of these geometries can support many topologies with nontrivial topologies and finite volumes, without altering the dynamics or the curvature. Therefore there is no reason, either observational or theoretical, why the universe should not choose a multiply connected topology.

In classical general relativity, a theorem due to Geroch \[2\] and Tipler \[3\] states that changes of cosmic topology are forbidden, in the sense that they would imply the appearance of either singularities or closed timelike curves. Then a natural question appears: where do all kinds of possible cosmic topologies come from? Quantum cosmology aims at getting some insight into this problem, by treating the whole universe quantum mechanically at the Planck era. In fact, it was argued in \[4\] that the global topology of the present universe would be a relic of its quantum era, and it should be statistically predictable by quantum cosmology. Thus, the global topology of cosmic spacetime would not have changed after the Planck era, and the topology of the present universe should be the same as that just after the Planck era.

In the pioneering works on the quantum creation of a closed universe, in both the “tunneling from nothing” \[5\] and “no-boundary” \[6\] proposals, the solution of Einstein’s equations is a Riemannian 4-sphere $S^4$ instanton, joined across an equator to a Lorentzian de Sitter space $S^3$ at its minimal radius. Thus, these and subsequent works on the birth of a closed universe have been limited to the description of an $S^4$ instanton tunneling into a simply connected, Lorentzian de Sitter spacetime $R \otimes S^3$.

Recently many works have appeared, studying the possibility that the universe may possess as spatial sections a compact manifold $M$ with nontrivial topology. $M$ may be represented as a quotient space $R^3/\Gamma$, $S^3/\Gamma$, and $H^3/\Gamma$, where $\Gamma$ is a (nontrivial) discrete group of isometries, which is isomorphic to the fundamental group of $M$. Many methods have been proposed to detect or observationally constrain the spatial topology, using the prediction of topological images, catalogs of discrete sources, the search for circles in the sky, and the cosmic microwave background radiation.
From this analysis, it is worth considering the quantum creation of the universe with a nontrivial topology. The flat universe with 3-torus space topology has been treated in [7], while the hyperbolic case was recently studied by Gibbons, Ratcliffe, and Tschantz [8]. Here we consider the case of a closed universe with \( S^3/\Gamma \) spatial topology. In order to obtain such a universe, it is necessary to begin with a more general instanton, namely \( S^4/\Gamma \). As shown in [9], the action of \( \Gamma \) on \( S^4 \) is not free, so the instanton \( S^4/\Gamma \) is not a manifold, but rather an orbifold with two cone points, which are the poles of \( S^4 \) and are fixed points [10] with respect to the nontrivial elements of \( \Gamma \). Because of this, we generalize Euclidean quantum gravity on smooth, compact 4-manifolds to Euclidean quantum gravity on finite-volume, compact 4-orbifolds. Then we obtain the probability of quantum creation of this universe.

In Sec. 2 we give the general definition of a Riemannian orbifold. In Sec. 3 the Euclidean Einstein-Hilbert action on the orbifold, and the probability amplitude for the transition from an initial 3-manifold \( \Sigma_i \) to a final 3-manifold \( \Sigma_f \) are obtained. In Sec. 4 the probability of creation of a universe with the space topology \( S^3/\Gamma \) is calculated, both with the Hartle-Hawking boundary condition and with the tunneling prescription. And in the last section some final comments are made.

## 2 Orbifold instanton

An orbifold \( O \) under our consideration is a compact, smooth manifold outside of a finite number of singular points in \( O \), and near each of the latter it is locally homeomorphic to the orbit space of group \( \Gamma \). The notion of orbifold was first introduced by Satake in [11], who used for it the term \( V \)-manifold, and was rediscovered by Thurston in [12], where the term orbifold was coined. Roughly speaking, an \( n \)-dimensional manifold is a topological space locally modeled on Euclidean space \( \mathbb{R}^n \), whereas an \( n \)-orbifold generalizes this notion by allowing the space to be modeled on quotients on \( \mathbb{R}^n \) by finite group action. Similarly for a manifold with boundary vs. an orbifold with boundary, whose formal definition is given below.

**Definition 1.** Let \( X \) be a Hausdorff space. For an open set \( U \) in \( X \), an orbifold coordinate chart over \( U \) is a triple \((\tilde{U}, \Gamma, \psi)\) such that: i) \( \tilde{U} \) is a connected open subset of the positive half-space \( \mathbb{R}^n_+ = \{(x^1, x^2, ..., x^n) \in \mathbb{R}^n : x^n \geq 0\} \), ii) \( \Gamma \) is a finite group of diffeomorphisms acting on \( \tilde{U} \) such that the
fixed point set of any $\gamma \in \Gamma$ which does not act trivially has codimension at least 2 in $\tilde{U}$, and $\psi : \tilde{U} \rightarrow U$ is a continuous map such that $\forall \gamma \in \Gamma$, $\psi \circ \gamma = \psi$ induces a homeomorphism $\tilde{\psi} : \tilde{U}/\Gamma \rightarrow U$.

Now suppose that $U$ and $U'$ are two open sets in a Hausdorff space $X$ with $U \subset U'$. Let $(\tilde{U}, \Gamma, \psi)$ and $(\tilde{U'}, \Gamma', \psi')$ be charts over $U$ and $U'$, respectively.

**Definition 2.** An injection $\lambda : (\tilde{U}, \Gamma, \psi) \hookrightarrow (\tilde{U'}, \Gamma', \psi')$ consists of an open embedding $\lambda : \tilde{U} \hookrightarrow \tilde{U'}$ such that $\psi = \psi' \circ \lambda$, and for any $\gamma \in \Gamma$ there exists $\gamma' \in \Gamma'$ for which $\lambda \circ \gamma = \gamma' \circ \lambda$.

An orbifold atlas $U$ on $X$ is a collection $\{(\tilde{U}_i, \Gamma_i, \psi_i)_{i \in \Lambda}\}$, where $\Lambda$ is a countable set of indices, of compatible orbifold charts with boundary on $X$, such that $\cup_{i \in \Lambda} \tilde{U}_i = X$, and boundary defined by $\{\psi_i(\partial \tilde{U}_i) : (\tilde{U}_i, \Gamma_i, \psi_i) \in U\}$.

**Definition 3.** A smooth $C^\infty$-orbifold with boundary $(X, U)$ consists of a Hausdorff space $X$ together with an atlas of orbifold charts $U$ satisfying the following conditions: i) For any pair of charts $(\tilde{U}, \Gamma, \psi)$ and $(\tilde{U'}, \Gamma', \psi')$ in $U$ with $U \subset U'$ there exists an injection $\lambda : (\tilde{U}, \Gamma, \psi) \hookrightarrow (\tilde{U'}, \Gamma', \psi')$. ii) The open sets $U \subset X$ for which there exists a chart $(\tilde{U}, \Gamma, \psi)$ in $U$ form a basis of open sets in $X$. Given an orbifold $(X, U)$, we will call the topological space $X$ the underlying space of the orbifold, and from now on, orbifolds $(X, U)$ will be denoted simply by $O$.

Now, let us take a point $x$ in an orbifold $O$ and let $(\tilde{U}, \Gamma, \psi)$ be a coordinate chart about $x$. Let $\tilde{x}$ be a point in $\tilde{U}$ such that $\psi(\tilde{x}) = x$ and let $\Gamma_x^U$ denote the isotropy group of $\tilde{x}$ under the action of $\Gamma$. As it is known, the group $\Gamma_x^U$ depends only on $x$ and not on a particular choice of $\tilde{x}$ or the chart around $x$. This group $\Gamma_x^U$, denoted by $\Gamma_x$, is the isotropy group of $x$. Thus, the singular set $\mathcal{S}$ of an orbifold consists of those points $x \in O$ whose isotropy group $\Gamma_x$ is nontrivial, and the genuine manifold can be viewed as an orbifold for which all points have trivial isotropy. We say that $\Gamma$ acts freely, if for all $x \in M$, $\gamma x = x$ implies $\gamma = 1$; then the quotient space is another manifold. From this discussion, $O - \mathcal{S}$ is an ordinary manifold with boundary.

Thus, a smooth $n$-dimensional orbifold $O$ is a topological space which locally has the structure of a quotient space of $\mathbb{R}^n$ by a smooth finite group action, and its singular set $\mathcal{S}$ corresponds to the fixed points of these local actions. For our purpose here, the boundary $\partial O$ is required to be an $n - 1$ compact, smooth manifold without boundary, which implies that the singular sets are in the interior of $O$. An orbifold is compact if its underlying topological space is compact.

The following proposition, due to Thurston, leads to an important classification of orbifolds into two types:
**Proposition.** If $M$ is a smooth manifold with boundary and $\Gamma$ a group that acts properly discontinuously on $M$ such that the fixed point set of each element of $\Gamma$ has codimension greater than or equal to two and fixes the boundary, then the quotient space $M/\Gamma$ is an orbifold with boundary. An orbifold is called **good** or **global** if it arises as a global quotient of a manifold by a properly discontinuous group action. Otherwise the orbifold is called **bad**.

Let $O = M/\Gamma$ be a good orbifold covered by $\{(\tilde{U}_i, \Gamma_i, \psi_i)_{i \in \Lambda}\}$. $O$ is orientable if, for any overlapping charts $(\tilde{U}_i, \Gamma_i, \psi_i)$ and $(\tilde{U}_j, \Gamma_j, \psi_j)$, there exist local manifold coordinate charts $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ on $\tilde{U}_i$ and $(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)$ on $\tilde{U}_j$, such that for any injection $\lambda$ we have $J = \det(\partial \tilde{y}_j / \partial \tilde{x}_i) > 0$. Thus an orbifold is orientable if it has an atlas such that $\tilde{U}_i$ can all be oriented consistently with $\Gamma_i$ and the overlap maps are orientation preserving.

Let us introduce a Riemannian structure on orbifolds. The construction of a Riemannian metric on a orbifold proceeds, as in the manifold case, with the metric being locally defined via coordinate charts, and then patched together through a partition of unity. The main difference in the orbifold case is that the structure involved must be invariant under the local group actions. Let $(X, \mathcal{U})$ be a $C^\infty$-Riemannian orbifold with a $\Gamma$-invariant metric. By definition, to give a Riemannian metric $(,)$ on $(X, \mathcal{U})$ is to give a $\Gamma$-invariant Riemannian metric $(,)$ on each $(\tilde{U}_i, \Gamma_i, \psi_i)$, such that for any injection $\lambda$ of $(\tilde{U}, \Gamma, \psi)$ into $(\tilde{U}', \Gamma', \psi')$, $(\tilde{X}, \tilde{Y}) = \left(\lambda(\tilde{X}), \lambda(\tilde{Y})\right)$, where $\tilde{X}, \tilde{Y}$ are vector fields on $\tilde{U}$, and $\lambda(\tilde{X}), \lambda(\tilde{Y})$ are the corresponding vector fields on $\lambda(\tilde{U}) \subset \tilde{U}'$. The $C^\infty$-Riemannian orbifold $(X, \mathcal{U})$ has the Levi-Civita connection $\nabla$ defined by a $\Gamma$-invariant Riemannian connection $\nabla^{\tilde{U}}$ on each $(\tilde{U}_i, \Gamma_i, \psi_i)$, such that for any injection $\lambda$ of $(\tilde{U}, \Gamma, \psi)$ into $(\tilde{U}', \Gamma', \psi')$, $\lambda(\nabla^{\tilde{U}}_{\tilde{X}} \tilde{Y}) = \nabla^{\tilde{U}'}_{\lambda(\tilde{X})} \lambda(\tilde{Y})$. Then the curvature tensor of $\nabla$ is defined by

$$ (R^{\tilde{U}})(\tilde{X}, \tilde{Y}) \tilde{Z} = \nabla^{\tilde{U}}_{\tilde{U} \tilde{X}} \nabla^{\tilde{U}}_{\tilde{U} \tilde{Y}} \tilde{Z} - \nabla^{\tilde{U}}_{\tilde{U} \tilde{Y}} \nabla^{\tilde{U}}_{\tilde{U} \tilde{X}} \tilde{Z} - \nabla^{\tilde{U}}_{[\tilde{X}, \tilde{Y}]} \tilde{Z}, $$

for vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}$ on each $\tilde{U}$. Now, patching the local metrics together using a partition of unity, we obtain a global Riemannian metric on $O$. A compact orbifold together with a Riemannian metric is called a **Riemannian orbifold**. From the curvature tensor the scalar curvature $R[g(x)]$ is obtained as usual, by a process of contraction.

The next step is to introduce the appropriate notion of integration on an orbifold. Suppose that $O$ is a compact and orientable Riemannian orbifold.
Let $w$ be an $n$-form on $O$ such that the support of $w$ is contained in the chart $(\tilde{U}, \Gamma, \psi)$. Then the integral of $w$ on $O$ is defined as follows [11]:

$$\int_{O} w = \frac{1}{|\Gamma|} \int_{\tilde{U}} \tilde{w},$$

(2)

where $\tilde{w}$ is the pullback of $w$ via the projection $\psi$: $\tilde{w} = w \circ \psi$. This definition is independent on the choice of coordinate chart and the integral of a globally defined differential form is defined using a partition of unity, as in the manifold case. In general the integration on a orbifold is defined using the measure $d\mu(g) = \sqrt{g(x)} d^n x$ associated with the Riemannian metric $g(x)$ on the orbifold: for an integrable function $f$, $\int_{O} f \equiv \int_{O} d\mu[g(x)] f(x)$. Thus we have defined the metric, the curvature, the connection, and the integration on an $n$-dimensional compact, Riemannian, good orbifold $O = M/\Gamma$ with boundary $\partial O$. The Einstein-Hilbert functional (which appears in quantum gravity) on this compact orbifold may be written as

$$S[g] = \int_{O} d\mu(g) R[g] + \int_{\partial O} d\mu(h) K[h],$$

(3)

where $R$ is the scalar curvature of $g$ and $K$ is the trace of the extrinsic curvature of the boundary hypersurface $(\Sigma, h)$ in $(O, g)$. If $\Gamma$ is a discrete subgroup of the isometries of either a spherical, a hyperbolic, or a Euclidean $n$-manifold, then the quotient spaces $S^n/\Gamma$, $H^n/\Gamma$, and $R^n/\Gamma$ will be respectively called a spherical, a hyperbolic and a Euclidean orbifold. The induced metric and the curvature will have singularities at the fixed points of the orbifold. However, as observed by Schleich and Witt [13], the character of the curvature singularity at these points is dimension dependent; and in dimensions greater than two the integral leaves out only a set of zero measure, so the Einstein-Hilbert functional is finite.

### 3 Euclidean quantum gravity and orbifolds

Following Wheeler’s and Hawking’s seminal ideas, we formulate a Euclidean approach to quantum gravity on orbifolds. Now the basic quantity of interest is the transition probability from an initial 3-orbifold to a final 3-orbifold, and for $n = 4$ the theory of cobordism guarantees that every compact, oriented 3-orbifold bounds an oriented, compact 4-orbifold. But we are interested in explaining the origin of our universe, which we assume to be
described by a de Sitter model during the inflationary era and later by a Friedmann-Robertson-Walker model. Since these models are based on manifolds the final 3-orbifold \( \Sigma_f \) is a manifold. The transition probability amplitude from an initial 3-manifold \( \Sigma_i \) with a metric \( h_i \) to a final 3-manifold \( \Sigma_f \) with a metric \( h_f \) is then given by

\[
K(\Sigma_f, h_f; \Sigma_i, h_i) = \sum_{(O,g)} \int D(g_{\mu\nu}) \exp[-S_E(O, g)],
\]

where the sum includes any 4-dimensional compact orbifold \( O \) with metric \( g \); the integral is over all Lorentz signature 4-metrics on \( O \) which interpolate between \( (\Sigma_i, h_i) \) and \( (\Sigma_f, h_f) \); and \( S_E \) is the Euclidean orbifold action with cosmological constant,

\[
S_E[O, g] = -\frac{1}{16\pi G} \int_{O_E} d^4x \, g^{1/2} (R - 2\Lambda) - \frac{1}{8\pi G} \int_{\partial O} d^3x \, h^{1/2} K,
\]

where \( \partial O \) is the union of the disjoint pair of boundaries: \( \partial O = \Sigma_i \cup \Sigma_f \). By definition, the orbifolds are 4-dimensional, compact, Riemannian, good orbifolds \( O = M/\Gamma \) - cf. Sec. 2; but the important point is that these quotients of \( M \) by \( \Gamma \) will induce a nontrivial topology on the boundary \( \partial O \). So the transition probability is from an initial three-orbifold \( \Sigma_i \) with nontrivial topology to a final three-manifold \( \Sigma_f \) also with a nontrivial topology.

Following the Hartle and Hawking’s (HH) proposal [6] of no-boundary boundary condition for the creation of a universe, in which the initial \( \Sigma_i \) is absent, Eq. (4) is the wave function for the metric on the 3-manifold boundary \( \Sigma = \Sigma_f \). Then the wave function for the universe is defined by a Euclidean sum-over-histories of the form

\[
\Psi(\Sigma, h_{ij}) = \sum_{(O,g)} \int D(g_{\mu\nu}) \exp[-S_E(O, g)].
\]

The path integral (6) has no precise definition, but its qualitative behaviour can be obtained by using semiclassical techniques in simple cosmological models, or, equivalently, by restricting the summation to minisuperspace. The boundary condition in the the semiclassical approximation to the wave function is of the form

\[
\Psi(h_{ij}) = N_0 \sum A_i \exp(-B_i),
\]
where $N_0$ is a normalization constant and $B_i$ are the actions of the Euclidean classical solutions. The prefactors $A_i$ denote fluctuations about the latter. Thus the wave function for the orbifold instanton in the semiclassical approximation has the same form as in the case of manifold instantons. In order to consider the quantum creation of a closed universe with nontrivial topology, let us outline the main ideas about $S^4/\Gamma$ obtained in [9]. First, let us use the embedding of the Euclidean four-sphere $S^4$ with unit radius in 5-dimensional Euclidean space $\mathbb{R}^5$: 

$$S^4 = \{X_\alpha, \alpha = 0, 4; X^\alpha X_\alpha = 1\}.$$ 

The action of the discrete, finite group of isometries $\Gamma$ on $S^4$ is obtained by extending its action on the unit radius 3-sphere $S^3$ to all infinite “parallel” $S^3$ on the four-sphere $S^4$, that is, for $|X_0| \leq 1$, $S^3_{X_0} = \{X_0, X_i, i = 1, 4; X^i X_i = 1 - X_0^2\}$. Thus, the action is already defined on the equator $S^3_0$, which is isometric to $S^3$. Let $(X_0, X_i) \in S^3_{X_0}$ and $\gamma \in \Gamma$. If $|X_0| < 1$, then $(0, X'_i = X_i/\sqrt{1 - X_0^2}) \in S^3_0$, so that $\gamma(0, X'_i) = (0, X''_i) \in S^3_0$; we define $\gamma(0, X_i) \equiv (X_0, X''_i/\sqrt{1 - X_0^2}) \in S^3_0$. If $|X_0| = 1$, then $\gamma S^3_{\pm 1} = S^3_{\pm 1}$, which are the poles of $S^4$. Thus the action of $\Gamma$ on $S^4$ is not free, and the quotient space $S^4/\Gamma$ is an orbifold (cf. Sec. 2 above) with two cone-like points corresponding to the poles of the 4-sphere - see Fig. 1.

In the quantum creation of a universe, in both the tunneling from nothing and the no-boundary proposals, only the lower half $(X_0 \leq 0)$ of the orbifold instanton takes part in the solution. The full spacetime solution is $M = O \cup_{\Sigma} M_L$, where $O$ and $M_L = R \otimes S^3/\Gamma$ are attached smoothly by $\Sigma = S^3_0/\Gamma = \partial O$ (see Fig. 2). Thus Gibbons’s condition [5] is satisfied: $O$ is a compact orbifold with $\Sigma$ as sole boundary; $\Sigma$ is a possible Cauchy surface for $M_L$, and has a vanishing second fundamental form with respect to both $O$ and $M_L$ - this is true for the $S^3$ covering, and the action of $\Gamma$ does not interfere with the local metrics.

## 4 Calculation of the wave function

In this section we will consider the probability of creation of a universe with the space topology $S^3/\Gamma$ in both the Hartle-Hawking no-boundary condition and the tunneling prescription. In order to calculate the minisuperspace path integral the semiclassical approximation given by Eq. (7) is used.
The line element of a Euclidean Robertson-Walker (RW) universe is
\[
d s^2 = N(\tau)^2 d\tau^2 + a^2(\tau) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega_2^2 \right].
\] (8)

The (comoving) volume of the space section \( \Sigma = S^3/\Gamma \) is given by
\[
V(S^3/\Gamma) = \frac{2\pi^2}{|\Gamma|},
\] (9)

where \(|\Gamma|\) is the order of \(\Gamma\), so the Euclidean Einstein-Hilbert action (5) is
\[
S_E = \frac{3V(S^3/\Gamma)}{8\pi G} \int_{\tau_0}^{\tau_1} d\tau \left[ -\frac{\dot{a}^2 a}{N} - Na + \frac{Na^3 \Lambda}{3} \right].
\] (10)
Figure 2: Global structure of the orbifold instanton and its continuation. The orbifold instanton $S^4/\Gamma$ is joined across an equator to a Lorentzian de Sitter space with nontrivial topology $S^3/\Gamma$ at its radius of maximum contraction.
Thus an information about the topology of the 4-orbifold is encoded in the action.

The field equation is

\[ \left( \frac{\dot{a}}{N} \right)^2 - 1 + \frac{\Lambda}{3} a^2 = 0, \]

whose solution

\[ a(\tau) = \frac{1}{H} \cos(HN\tau), \]

with \( H = \sqrt{\frac{\Lambda}{3}} \), satisfies the boundary conditions \( a(\tau) = 0 \) at the south pole of \( O_E \) and \( a(\tau) = a_0 \) = radius both of the Lorentzian de Sitter space with topology \( S^3/\Gamma \) and of \( O_E \).

Using this boundary condition, the final Euclidean action is

\[ S_E = -\frac{V(S^3/\Gamma)}{6\pi GH^2} \left[ 1 - (1 - H^2 a_0^2)^{3/2} \right]. \]

The semiclassical approximation of the HH wave function (7) for \( a < H^{-1} \) is therefore

\[ \Psi(S^3/\Gamma) = N_0 \exp \left( \frac{\pi}{3|\Gamma|GH^2} \left[ 1 - (1 - H^2 a^2)^{3/2} \right] \right), \]

and the unnormalized probability of creation of this multiply connected universe is

\[ |\Psi_{HH}|^2 = N_1 \exp \left( \frac{2\pi}{3|\Gamma|GH^2} \left[ 1 - (1 - H^2 a^2)^{3/2} \right] \right). \]

On the other hand, Vilenkin [14] observed that the full Wheeler-DeWitt equation is invariant under the transformation

\[ h_{ij} \rightarrow \exp(i\pi)h_{ij}, \quad V(\phi) \rightarrow \exp(-i\pi)V(\phi) . \]

This means that there is a relation between the tunneling and the HH wave functions: \( \Psi_{HH} \) and \( \Psi_T \) are related by an analytical continuation.

Using these relations for our case, we obtain the tunneling wave function for \( a < H^{-1} \)

\[ \Psi_T(S^3/\Gamma) = N_0 \exp \left( -\frac{\pi}{3|\Gamma|GH^2} \left[ 1 - (1 - H^2 a^2)^{3/2} \right] \right), \]
and the unnormalized probability of creation of a universe in the tunneling approach, for $a < H^{-1}$ is

$$|\Psi_T|^2 = N_1 \exp \left( -\frac{2\pi}{3|\Gamma|GH^2} \left[ 1 - \left( 1 - H^2a^2 \right)^{3/2} \right] \right).$$

(17)

Thus we obtain the probability of quantum creation of a universe with an $S^3/\Gamma$ spatial topology, in both the Hartle-Hawking (HH) boundary condition and Vilenkin’s tunneling prescription.

## 5 Final remarks

We have considered the generalization of Euclidean quantum gravity on compact, smooth 4-manifolds to a Euclidean quantum gravity on finite-volume, compact 4-orbifolds, and as an application we obtained the probability of quantum creation of a closed universe with an $S^3/\Gamma$ spatial topology. Using this Euclidean functional integral prescription as applied to quantum cosmology, we calculated the wave function of such a universe with a positive cosmological constant and without matter. The minisuperspace path integral is calculated in the semiclassical approximation and thus we obtained the probability of quantum creation of a closed universe with nontrivial topology, and comparing Eqs. (14) and (17), it appears that in HH approach the probability of creation is maximum for minimum order of $\Gamma$, while in the tunneling approach it increases with $|\Gamma|$.

Further generalizations, including a cosmological constant and a scalar field of matter, both in the above spherical case and in that of a compact, hyperbolic orbifold instanton $H^4/\Gamma$ going over to a compact, hyperbolic FLRW universe, are under study.

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