Gap probability in the spectrum of random matrices and asymptotics of polynomials orthogonal on an arc of the unit circle.

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Abstract. We obtain uniform asymptotics for polynomials orthogonal on a fixed and varying arc of the unit circle with a positive analytic weight function. We also complete the proof of the large s asymptotic expansion for the Fredholm determinant with the kernel \( \sin(z/(\pi z)) \) on the interval \([0, s]\), verifying a conjecture of Dyson for the constant term in the expansion. In the Gaussian Unitary Ensemble of random matrices, this determinant describes the probability for an interval of length \( s \) in the bulk scaling limit to be free from the eigenvalues.

1 Introduction

One problem in the random matrix theory is estimation of the probability for a given interval to be free from the eigenvalues. In the Gaussian Unitary Ensemble this probability for any interval of length \( 2s \) in the bulk scaling limit is equal to the following Fredholm determinant:

\[
\Delta(s) = \det[I - K],
\]

where \( K \) is the integral operator on \( L^2(0, 2s) \) given by

\[
(Kg)(x) = \int_0^{2s} \frac{\sin(x - y)}{\pi(x - y)} g(y) dy.
\]

The probability of a gap in the spectrum of random matrices from orthogonal and symplectic ensembles is also expressed in terms of \( \Delta(s) \) (see [1, 3, 18]).
It was shown by Jimbo, Miwa, Môri, and Sato [2] that \((d/ds) \ln \Delta(s)\) satisfies a modified Painlevé V equation (for simpler proofs of this see [3, 4]).

An interesting question is calculation of the asymptotics of \(\Delta(s)\) for large \(s\) (the small \(s\) series are easy to obtain). The first two terms in the expansion of \(\ln \Delta(s)\) were found by des Cloizeaux and Mehta [5] who used a connection with the spheroidal functions. The full asymptotic expansion was obtained by Dyson [6] with the help of the inverse scattering techniques for Schrödinger operators. These calculations were partly conjectural. A rigorous derivation of the main term was given by Widom [7] using continuous analogues of orthogonal polynomials. Finally, Deift, Its, and Zhou [4] (see also that work for a more extensive introduction) calculated, as a particular case of a more general result, the full asymptotics of the derivative \((d/ds) \ln \Delta(s)\) using techniques of matrix Riemann-Hilbert problems. This settled the question up to the constant term in the expansion of \(\ln \Delta(s)\). The first 3 terms in the Dyson expansion are as follows:

\[
\ln \Delta(s) = -\frac{s^2}{2} - \frac{1}{4} \ln s + c_0 + O\left(\frac{1}{s}\right), \quad s \to \infty, \tag{2}
\]

where the constant term \(c_0 = (1/12) \ln 2 + 3\zeta'(-1)\), and \(\zeta'(x)\) is the derivative of Riemann’s zeta function. Thus, justification of \(c_0\) here remained the only problem and it is solved in the present paper. Actually, we obtain the first 3 terms:1

**Theorem 1** The large \(s\) asymptotics of \(\ln \Delta(s)\) are given by [2].

The proof is based on a formula by Deift [9] which connects the determinants of two Toeplitz matrices, a formula by Widom [8] for asymptotics of Toeplitz determinants on a circular arc, and on asymptotics for orthogonal polynomials on a circular arc which are computed here.

Let \(f_\alpha(\theta)\) be a weight function on an arc \(\alpha \leq \theta \leq 2\pi - \alpha, \ z = e^{i\theta}, \ 0 < \alpha < \pi\) of the unit circle \(|z| = 1\), and \(\phi_n(z, \alpha) = \chi_n z^n + \cdots, \ n = 0, 1, \ldots\) the corresponding system of orthonormal polynomials:

\[
\frac{1}{2\pi} \int_\alpha^{2\pi-\alpha} \phi_k(e^{i\theta}, \alpha) \overline{\phi_m(e^{i\theta}, \alpha)} f_\alpha(\theta) \, d\theta = \delta_{km}, \quad k, m = 0, 1, \ldots \tag{3}
\]

Such polynomials in the case of the circle \((\alpha=0)\) were first studied by Szegő (see [10]) who, in particular, found several important asymptotics for them as \(n \to \infty\). Afterwards, asymptotic analysis of such polynomials was carried out by many authors. Specifically for the case of an arc (whose study was initiated by Akhieser [11]), see [12] [13] [14] [15] and references therein. However, the full asymptotic expansion at all points \(z \in \mathbb{C}\) for

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1As this paper was being prepared for publication, an announcement by T. Ehrhardt claiming the same result as Theorem 1 (by a different method) was posted on the internet.

A third solution to the problem by a Riemann-Hilbert approach (related to the present one) is in preparation by P. Deift, A. Its, and X. Zhou.
a wide class of weights became a feasible task only after recent development of Riemann-Hilbert problem methods. It was observed by Fokas, Its, and Kitaev [16] that orthogonal polynomials satisfy certain matrix Riemann-Hilbert problems. An efficient method for their asymptotic solution (steepest descent techniques) was developed by Deift and Zhou [17, 18] and applied for analysis of polynomials orthogonal on the real axis in [19, 20] (see also [21] for a different Riemann-Hilbert approach) and on the unit circle in [22]. The case of polynomials orthogonal on $[-1,1]$ (especially relevant for the present work) was considered by Kuijlaars, McLauphlin, Van Assche, and Vanlessen [23, 24] who found full asymptotics at all points in the case of a positive analytic weight on $[-1,1]$ with power-type singularities at the end-points.

In the present paper, we shall give a procedure to obtain full asymptotics for all $z$ for polynomials $\phi_n(z,\alpha)$ and their leading coefficients as $n \to \infty$ in the case of a positive analytic weight $f_\alpha(\theta)$. The argument will be similar to that of [23]. We consider $2s/n \leq \alpha < \pi$, $n > s$, $s \to \infty$, which includes both the cases of a fixed arc ($\alpha$ is independent of $n$) and a varying arc. The asymptotics for $\phi_n(z,\alpha)$ we obtain are in the inverse powers of $n \sin(\alpha/2)$. The remainder after $k$ terms is uniform in $\alpha$. The general solution is given by equations (61–64).

The first 2 asymptotic terms for any $z$ can be easily written using (58, 59). An example is given in (65, 66).

Our solution can be generalized to the following cases: (1) the weight $f_\alpha(\theta)$ has power-type singularities at the end-points of the arc (this can be done following [23]); (2) the weight $f_\alpha(\theta)$ is not analytic but only smooth enough and positive (one can approximate it then by its Fourier series).

If the weight is symmetric $f_\alpha(\theta) = f_\alpha(2\pi - \theta)$, there exist relatively simple formulas of Szegő type [10, 23, 26] connecting polynomials on an arc with those on an interval. In this case and for a fixed arc one could try to obtain our results from those of [23, 24]. The present argument, however, is more direct.

For the proof of Theorem 1, we shall only need asymptotics of polynomials with the weight $f_\alpha(\theta) = 1$ and only at the point $e^{i\alpha}$. What we need is summarized in the following theorem proved in Section 2 (after the argument in the general case is given):

**Theorem 2** Let $0 < \alpha < \pi$, $\gamma = \cos(\alpha/2)$,

\[
\begin{align*}
r_+^1 &= \frac{e^{-i\alpha/2}}{3 \cdot 2i} (1 + e^{-i\alpha} - 2e^{i\alpha}), \\
r_+^2 &= \frac{1}{3 \cdot 29} (16 - 9e^{i\alpha} + 43e^{-i\alpha} - 2e^{-2i\alpha}), \\
r_-^1 &= \frac{1}{2} (-6 + 7e^{i\alpha} - 17e^{-i\alpha}), \\
r_-^2 &= \frac{e^{i\alpha/2}}{4 \cos^2(\alpha/2)}, \\
\tau &= \frac{1 + 2\cos\alpha}{6i}, \\
\rho &= n \sin(\alpha/2), \\
\varepsilon &= 0.
\end{align*}
\]

Let $f_\alpha(\theta) = 1$. Then the polynomial $\phi_n(z,\alpha)$ admits an asymptotic expansion for large $\rho$ in
the inverse powers of \( \rho \). We have for \( z = e^{i\alpha} \)

\[
\phi_n(e^{i\alpha}, \alpha) = \chi_n \gamma^n e^{i\alpha(n/2-1/4)} \sqrt{\pi i \rho} \left[ 1 + \frac{r_1}{\rho} + \frac{r_2}{\rho^2} + \frac{r_3}{\rho^3} + O\left(\frac{1}{\rho^4}\right) \right],
\]

where \( \chi_n \) is the leading coefficient of \( \phi_n(z, \alpha) = \chi_n z^n + \cdots \) for which we have

\[
\chi_n^{2} = \gamma^{-2n+1} \left[ 1 + \frac{1}{4n} + \frac{5}{2^{5}n^{2}} + O\left(\frac{1}{n^{3}}\right) \right],
\]

and \( r_3 \) is a bounded function of \( \alpha \). The derivative of the polynomial \( \phi'_n(z, \alpha) = (d/dz)\phi_n(z, \alpha) \) at \( e^{i\alpha} \) can be written as

\[
\phi'_n(e^{i\alpha}, \alpha) = \frac{n}{2} \phi_n(e^{i\alpha}, \alpha) e^{-i\alpha} + \chi_n \gamma^n e^{i\alpha(n/2-5/4)} \frac{\sqrt{\pi i \rho}}{2 \sin \alpha} \left[ i \rho^2 + e^{i\alpha/2} \rho + \tau + \frac{1}{\rho} (r_1 (i \rho^2 + \tau) + r_1 (i \rho^2 + \tau)) \right].
\]

There exists \( s_0 > 0 \) such that all the remainder terms are valid and uniform in \( \alpha, s, \) and \( n \) for \( \alpha \in [2s/n, \pi - \varepsilon], s > s_0, \) and \( n > s \).

Remark The uniformity of the remainders here is crucial for the proof of Theorem 1.

After constructing asymptotics for polynomials in Section 2, we give a proof of Theorem 1 in Section 3. Note that the present method could also be used to obtain the full asymptotic expansion of \( \Delta(s) \).

## 2 Asymptotics of polynomials on an arc

In the present section we construct asymptotics for polynomials \( \phi_n(z) = \chi_n z^n + \cdots \) orthonormal with a weight \( f(z) = f_\alpha(\theta) \) on an arc \( \alpha \leq \theta \leq 2\pi - \alpha, z = e^{i\theta}, \) for \( 2s/n \leq \alpha \leq \pi - \varepsilon, \varepsilon > 0, n > s, s \to \infty \). This includes both the cases of a fixed arc \( 0 < \alpha < \pi \) and the varying arc \( \alpha = 2s/n \). The function \( f(z) \) is assumed positive and analytic on the arc for a fixed arc case, and on the whole circle in the general case. In the general case, we obtain \( \alpha \)-uniform asymptotics in the inverse powers of \( n \sin(\alpha/2) \).

Consider the following \( 2 \times 2 \) matrix

\[
Y(z) = \begin{pmatrix}
\chi_n^{-1} \phi_n(z) & \chi_n^{-1} \int_\Sigma \frac{\phi_n(\xi)}{\xi - z} f(\xi) d\xi \\
\chi_n^{-1} \phi_n^{*}(z) & \chi_n^{-1} \int_\Sigma \frac{\phi_n^{*}(\xi)}{\xi - z} f(\xi) d\xi
\end{pmatrix},
\]

where \( \Sigma \) is the arc \( \alpha \leq \theta \leq 2\pi - \alpha \) of the unit circle traversed in the direction from \( 2\pi - \alpha \) to \( \alpha \), and \( \phi_n^{*}(z) = z^n \phi_n(1/z) \). As is easy to verify (see [22, 23]), \( Y(z) \) is the unique solution of the following Riemann-Hilbert problem:
(a) $Y(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma$.

(b) For $\theta \in (\alpha, 2\pi - \alpha)$, $Y$ has continuous boundary values $Y_+(x)$ as $z$ approaches $x = e^{i\theta}$ from the outside of the circle, and $Y_-(x)$, from the inside. They are related by the jump condition

$$Y_+(x) = Y_-(x) \left( \begin{array}{cc} 1 & x^{-n} f(x) \\ 0 & 1 \end{array} \right), \quad x = e^{i\theta}, \quad \theta \in (\alpha, 2\pi - \alpha).$$ (9)

(c) $Y(z)$ has the following asymptotic behavior at infinity:

$$Y(z) = \left( I + O \left( \frac{1}{z} \right) \right) \left( \begin{array}{cc} z^n & 0 \\ 0 & z^{-n} \end{array} \right), \quad \text{as } z \to \infty.$$ (10)

(d) Near the end-points of the arc $e^{\pm i\alpha}$

$$Y(z) = O \left( \frac{1}{\log |z - e^{\pm i\alpha}|} \right),$$ (11)

as $z \to e^{\pm i\alpha}, z \in \mathbb{C} \setminus \Sigma$.

The function

$$\psi(z) = \frac{1}{2\gamma} \left( z + 1 + \sqrt{(z - e^{i\alpha})(z - e^{-i\alpha})} \right), \quad \gamma = \cos(\alpha/2),$$ (12)

which conformally maps the outside of the arc $\Sigma$ into the outside of the unit circle, will have an important role in what follows (cf. (6.18) of [4]). Here we take the branch of the square root which is positive for positive arguments. Note that the boundary values of $\psi(z), \psi_+(x)$ as $z$ approaches $x \in \Sigma$ from the outside of the circle, and $\psi_-(x)$, from the inside, are related as:

$$\psi_+(x) \psi_-(x) = x.$$ (13)

Let $\mu(z)$ be defined by the equation (12) but with the minus sign in front of the square root. Then $\mu(z)$ is the mapping of the outside of the arc into the inside of the unit circle. Hence $|\mu(z)| < 1$, whereas $|\psi(z)| > 1$ for $z \in \mathbb{C} \setminus \Sigma$. Therefore we have that first, for $|z| < 1$ $|z/\psi(z)^2| < 1$, and second, for $|z| > 1$

$$\left| \frac{z}{\psi(z)^2} \right| = \left| \frac{z\mu(z)^2}{\psi(z)^2 \mu(z)} \right| = \left| \frac{z\mu(z)^2}{z^2} \right| = \left| \frac{\mu(z)^2}{z} \right| < 1.$$ (14)

Thus,

$$\left| \frac{z}{\psi(z)^2} \right| < 1 \quad \text{for} \quad |z| \neq 1.$$ (14)

This inequality will be useful later on.
We now replace the original Riemann-Hilbert problem with an equivalent one which is normalized to unity at infinity and has oscillating elements of the jump matrix. Namely, set

\[ T(z) = \gamma^{-n\sigma_3} Y(z)\psi(z)^{-n\sigma_3}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  

(15)

Then, as is easy to verify, \( T(z) \) satisfies the same problem as \( Y(z) \) but with the changed conditions (b) and (c):

(b) \[
T_+(x) = T_-(x) \begin{pmatrix} x^n\psi_+(x)^{-2n} & f(x) \\ 0 & x^n\psi_-(x)^{-2n} \end{pmatrix}, \quad x \in \Sigma,
\]

(16)

(c) \[
T(z) = I + O\left(\frac{1}{z}\right), \quad \text{as } z \to \infty.
\]

(17)

Following the idea of the steepest descent method of Deift and Zhou, we now replace the Riemann-Hilbert problem for \( T(z) \) with an equivalent one on a system of 3 contours where some of the jump matrix elements are exponentially small. Divide the complex plane into 3 regions as shown on Figure 1 (the contours \( \Sigma_{1,3} \) lie sufficiently close to \( \Sigma_2 \equiv \Sigma \) for \( f(z) \) to remain nonzero and analytic in regions 1 and 2) and define the matrix-valued function \( S(z) \) by the formulas:

1) in region 1

\[
S(z) = T(z) \begin{pmatrix} 1 & 0 \\ \frac{1}{f(z)\psi(z)^{2n}} & 1 \end{pmatrix},
\]

(18)
2) in region 2
\[ S(z) = T(z) \left( \frac{1}{z^n} \begin{pmatrix} \frac{1}{f(x)\psi(x)} & 0 \\ 0 & 1 \end{pmatrix} \right), \quad (19) \]

3) in region 3
\[ S(z) = T(z). \quad (20) \]

The condition (c) in the problem for \( S(z) \) is the same as for \( T(z) \), the conditions (a), (b) and (d) are different. Namely,

(a,b) \( S(z) \) is analytic in \( \mathbb{C} \setminus (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3) \) with the following jump conditions on the contours:

\[ S_+(x) = S_-(x) \left( \frac{1}{x^n} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right), \quad x \in \Sigma_1 \cup \Sigma_3, \quad (21) \]

\[ S_+(x) = S_-(x) \left( \begin{pmatrix} f(x) \\ 0 \end{pmatrix} \right), \quad x \in \Sigma_2 \equiv \Sigma, \quad (22) \]

(c) as \( z \to \infty \)
\[ S(z) = I + O \left( \frac{1}{z} \right), \quad (23) \]

(d) near the end-points of the arc
\[ S(z) = O \left( \begin{pmatrix} \log |z - e^{\pm i\alpha}| & \log |z - e^{\pm i\alpha}| \\ \log |z - e^{\pm i\alpha}| & \log |z - e^{\pm i\alpha}| \end{pmatrix} \right), \quad (24) \]

as \( z \to e^{\pm i\alpha}, \quad z \in \mathbb{C} \setminus (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3) \).

Recalling (14), we see that, for \( n \) large, for \( x \) outside some neighborhoods of the endpoints of the arc, the jump matrix on \( \Sigma_1 \cup \Sigma_3 \) is uniformly exponentially close to the identity. We therefore approximate the function \( S(z) \) with parametrices inside the mentioned neighborhoods and in the outside region where we neglect the jumps on \( \Sigma_1 \cup \Sigma_3 \). The Riemann-Hilbert problems for the parametrices can be solved and the solution closely resemble that in case of polynomials on an interval (23).

The parametrix for the outside region is defined as the solution of the following Riemann-Hilbert problem:

(a) \( N(z) \) is analytic for \( z \in \mathbb{C} \setminus \Sigma \equiv \Sigma_2 \),

(b) with the jump condition on \( \Sigma \)
\[ N_+(x) = N_-(x) \left( \begin{pmatrix} 0 \\ -f(x)^{-1} \end{pmatrix} \right), \quad x \in \Sigma, \quad (25) \]
(c) and the following behavior at infinity

\[ N(z) = I + O \left( \frac{1}{z} \right), \quad \text{as } z \to \infty. \]  

(26)

The solution \( N(z) \) is found in the same way as in \cite{23}. Consider the Szegő function:

\[ D(z) = \exp \left[ \frac{\sqrt{(z - e^{i\alpha})(z - e^{-i\alpha})}}{2\pi i} \int_{\Sigma} \frac{\ln f(\xi)}{(\xi - e^{i\alpha})(\xi - e^{-i\alpha})} \xi - z \right]. \]  

(27)

This function is analytic outside the arc, and its boundary values satisfy

\[ D_+(x)D_-(x) = f(x), \quad x \in \Sigma. \]  

(28)

Denote

\[ D_\infty = \lim_{z \to \infty} D(z) = \exp \left[ -\frac{1}{2\pi i} \int_{\Sigma} \frac{\ln f(\xi)d\xi}{(\xi - e^{i\alpha})(\xi - e^{-i\alpha})} \right]. \]  

(29)

Then the solution of the above Riemann-Hilbert problem is as follows:

\[ N(z) = \frac{1}{2}(D_\infty)^{\sigma_3} \begin{pmatrix} a + a^{-1} & -i(a - a^{-1}) \\ i(a - a^{-1}) & a + a^{-1} \end{pmatrix} D(z)^{-\sigma_3}, \quad a(z) = \left( \frac{z - e^{i\alpha}}{z - e^{-i\alpha}} \right)^{1/4}, \]  

(30)

where the value of the root satisfies the condition \( a(z) \to 1 \) as \( z \to \infty \). Note that \( \det N(z) = 1 \), which allows, in particular, to write a simple expression for the inverse \( N(z)^{-1} \).

Now consider a \( \delta \)-neighborhood \( U_\delta \) of the point \( e^{i\alpha} \), small enough so that \( f(z) \) is analytic and nonzero there. The jump matrices on \( \Sigma_1, \Sigma_3 \) are not close to the identity in this region, so we need to construct a separate local parametrix. We look for a matrix-valued function \( P(z) \) which is analytic in \( U_\delta \), satisfies the same jump relations on \( (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3) \cap U_\delta \) as \( S(z) \), has the same behavior at \( z = e^{i\alpha} \) as \( S(z) \), and matches \( N(z) \) at the boundary:

\[ P(z)N(z)^{-1} = I + O \left( \frac{1}{n \sin(\alpha/2)} \right), \quad z \in \partial U_\delta \setminus (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3), \]  

(31)

where \( \rho = n \sin(\alpha/2) \to \infty \). We seek \( P(z) \) in the form

\[ P(z) = E(z)\hat{P}(z) \left( \frac{\psi(z)}{\sqrt{z}} \right)^{-n\sigma_3} f(z)^{-\sigma_3/2}, \]  

(32)

where \( E(z) \) is invertible and analytic in a neighborhood of \( U_\delta \). The function \( E(z) \) does not affect jump relations and will be chosen later so that \( P(z) \) satisfies (31). Using the boundary-value property (13), we obtain (cf. \cite{23}):

\[ \hat{P}(x)_+ = \hat{P}(x)_- \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad x \in (\Sigma_1 \cup \Sigma_3) \cap U_\delta, \]  

\[ \hat{P}(x)_+ = \hat{P}(x)_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad x \in \Sigma_2 \cap U_\delta. \]  

(33)
Consider the function \( \omega(z) \) defined by the equation:

\[
e^{\sqrt{\omega(z)}} = \frac{\psi(z)}{\sqrt{z}}
\]

Using (13), we have for the boundary values of \( \omega \) on the arc:

\[
\sqrt{\omega(x)}_+ = \ln \frac{\psi(x)}{\sqrt{x}} = -\sqrt{\omega(x)}_-, \quad (35)
\]

therefore \( \omega(z) \) is analytic in \( U_\delta \). For \( z \) near \( e^{i\alpha} \), we obtain uniformly for all \( \alpha \)

\[
\frac{\psi(z)}{\sqrt{z}} = 1 + \left( \sqrt{\frac{i \sin(\alpha/2)}{\cos(\alpha/2)}} e^{-i\alpha/2} \sqrt{1 + \frac{u}{2i \sin \alpha}} \sqrt{u + (e^{i\alpha/2} \cos(\alpha/2)^{-1} - 1)e^{-i\alpha} u^2} \right) (1 + O(u)),
\]

\( u = z - e^{i\alpha} \).

If also \( |u| < 2 \sin \alpha \), we get for the function \( \omega(z) \) the following (nonuniform) expansion at \( z = e^{i\alpha} \):

\[
\omega(z) = \frac{i \sin(\alpha/2)}{\cos(\alpha/2)} e^{-i\alpha} u \left( 1 - \frac{1 - 2e^{-i\alpha} - 2e^{-2i\alpha}}{6i \sin \alpha} u + O(u^2) \right).
\]

Denote

\[
\hat{P}(z) = Q(\zeta), \quad \zeta = n^2 \omega(z).
\]

Now we reached a crucial moment. The circle \( \partial U_\delta \) is transformed in the \( \zeta \) variable into a curve \( \partial \hat{U}_\delta \) whose minimal distance from zero is \( n^2 \min_{0 \leq t \leq 2\pi} |\omega(e^{i\alpha} + \delta e^{it})| \). In order to construct a solution, we need this distance to be large. This is so for large \( n \) if \( \delta \) and \( \alpha \) are independent of \( n \) (see (37)). In the general case of \( 2s/n \leq \alpha \leq \pi - \varepsilon \), there exists some small \( \alpha_0 \) (depending on \( f(z), \varepsilon \)) such that we can assume

\[
\delta = \begin{cases} 
\sin(\alpha_0/2), & \text{for } \alpha_0 \leq \alpha \leq \pi - \varepsilon \\
\sin(\alpha/2), & \text{for } 2s/n \leq \alpha \leq \alpha_0
\end{cases}.
\]

Putting \( u = \delta e^{it} \) in (36) and choosing \( \alpha_0 \) sufficiently small, we obtain after simple analysis:

\[
n^2 \min_{0 \leq t \leq 2\pi} |\omega(e^{i\alpha} + \delta e^{it})| \geq C(n \sin(\alpha/2))^2
\]

for some constant \( C \) which is larger than zero. We see that for large \( s \) this distance remains uniformly large (not less than of order \( s^2 \)) for any \( \alpha \in [2s/n, \pi - \varepsilon] \), where \( n > s \).

In \( \hat{U}_\delta \) the image \( \hat{\Sigma}_{1,2,3} \) of the cuts can be considered as 3 direct lines emanating from zero. (The image of \( \Sigma \) is a line, and the exact form of \( \Sigma_1 \) and \( \Sigma_3 \) can be chosen at will.) The analytic matrix-valued function \( Q(\zeta) \) which satisfies the jump conditions (33) on \( \hat{\Sigma}_{1,2,3} \) and singularity conditions (24) at \( \zeta = 0 \) was constructed in [23]. Namely, we have in \( \hat{U}_\delta \) (the regions 1, 2, and 3 correspond to the \( \zeta \)-variable images of the regions in Figure 1) the following expressions in terms of modified Bessel and Hankel functions (see, e.g., [27]):
1) region 1

\[ Q(\zeta) = \frac{1}{2} \left( \pi \zeta^{1/2} \left( H_0^{(1)}(e^{-i\pi/2} \zeta^{1/2}) \right)'(e^{-i\pi/2} \zeta^{1/2}) - \pi \zeta^{1/2} \left( H_0^{(2)}(e^{-i\pi/2} \zeta^{1/2}) \right)'(e^{-i\pi/2} \zeta^{1/2}) \right), \]  

\[ (41) \]

2) region 2

\[ Q(\zeta) = \frac{1}{2} \left( -\pi \zeta^{1/2} \left( H_0^{(2)}(e^{i\pi/2} \zeta^{1/2}) \right)'(e^{i\pi/2} \zeta^{1/2}) - \pi \zeta^{1/2} \left( H_0^{(1)}(e^{i\pi/2} \zeta^{1/2}) \right)'(e^{i\pi/2} \zeta^{1/2}) \right), \]  

\[ (42) \]

3) region 3

\[ Q(\zeta) = \left( \begin{array}{cc} I_0(\zeta^{1/2}) & \frac{1}{\zeta} K_0(\zeta^{1/2}) \\ \pi i \zeta^{1/2} I_0'(\zeta^{1/2}) & -\zeta^{1/2} K_0'(\zeta^{1/2}) \end{array} \right), \]  

\[ (43) \]

where \(-\pi < \arg(\zeta) < \pi\).

We now have to choose \(E(z)\) so that the matching condition (31) is satisfied. For that we can use the first term in the asymptotic expansion of Bessel and Hankel functions for large \(\zeta\). The expansion for \(Q(\zeta)\) is the same in all the 3 regions. We can write down an arbitrary number of terms. Below we shall make use only of the first three. We have:

\[ Q(\zeta) = \frac{1}{\sqrt{2}}(\pi \sqrt{\zeta})^{-\sigma_3/2} \left( \begin{array}{cc} 1 & i \\ i & 1 \end{array} \right) \left[ I + \frac{1}{8\sqrt{\zeta}} \left( \begin{array}{cc} -1 & -2i \\ -2i & 1 \end{array} \right) \right] - \frac{3}{2\sqrt{\zeta}} \left( \begin{array}{cc} 1 & -4i \\ 4i & 1 \end{array} \right) + O(\zeta^{-3/2}) e^{\sqrt{\zeta}\sigma_3} \]  

\[ (44) \]

uniformly on the boundary of \(\tilde{U}_\delta\). We now define \(E(z)\) as follows:

\[ E(z) = \frac{1}{\sqrt{2}} N(z) f(z)^{\sigma_3/2} \left( \begin{array}{cc} 1 & -i \\ -i & 1 \end{array} \right) (\pi n \sqrt{\omega(z)})^{\sigma_3/2}. \]  

\[ (45) \]

We verify exactly as in (32) that it is an analytic function in \(U_\delta\). Using (32), (34), and (40) (and estimating \(N(z)\) as below) we see that the matching condition (31) is now satisfied. Thus we have

\[ P(z) = E(z) Q(n^2 \omega(z)) \left( \frac{\psi(z)}{\sqrt{z}} \right)^{-n\sigma_3} f(z)^{-\sigma_3/2}. \]  

\[ (46) \]

Solution in the neighborhood \(\tilde{U}_\delta\) of \(e^{i(2\pi - \alpha)}\) is similar (but note the reversed direction of the contours). We have there:

\[ \tilde{P}(z) = \tilde{E}(z) \sigma_3 Q(n^2 \omega(z)) \left( \frac{\psi(z)}{\sqrt{z}} \right)^{-n\sigma_3} f(z)^{-\sigma_3/2}, \]  

\[ (47) \]

\[ \tilde{E}(z) = \frac{1}{\sqrt{2}} N(z) f(z)^{\sigma_3/2} \left( \begin{array}{cc} 1 & i \\ i & 1 \end{array} \right) (\pi n \sqrt{\omega(z)})^{\sigma_3/2}. \]
We are now ready for the last transformation of the Riemann-Hilbert problem. Let

\[ R(z) = S(z)N^{-1}(z), \quad z \in \mathbb{C} \setminus (U_\delta \cup \bar{U}_\delta \cup \Sigma_{1,2,3}), \]
\[ R(z) = S(z)P^{-1}(z), \quad z \in U_\delta \setminus \Sigma_{1,2,3}, \]
\[ R(z) = S(z)\bar{P}^{-1}(z), \quad z \in \bar{U}_\delta \setminus \Sigma_{1,2,3}. \]  

(48)

It is easy to see that this function has jumps only on \( \partial U_\delta, \partial \bar{U}_\delta, \) and parts of \( \Sigma_1, \) and \( \Sigma_3 \) lying outside of the neighborhoods \( U_\delta, \bar{U}_\delta \) (we denote these parts \( \Sigma_{\text{out}} \)). Namely,

\[ R_+(x) = R_-(x)N_-(x) \begin{pmatrix} \frac{1}{x^n} & 0 \\ f(x) & 1 \end{pmatrix} N_-(x)^{-1}, \quad x \in \Sigma_{\text{out}}, \]
\[ R_+(x) = R_-(x)P(x)N(x)^{-1}, \quad x \in \partial U_\delta, \]
\[ R_+(x) = R_-(x)\bar{P}(x)N(x)^{-1}, \quad x \in \partial \bar{U}_\delta. \]  

(49)

As is easy to verify, the matrix elements of \( N(x) \) and \( N(z)^{-1} \) remain bounded for \( |x - e^{\pm i\alpha}| \geq \delta \geq \sin(s/n) \) and all \( \alpha \). For example,

\[ |a^4| = (1 + 8\gamma \sin t + 16\gamma^2)^{-1/2} \leq (1 - 4\gamma)^{-1} \quad \text{for} \quad z = e^{i\alpha + \sin(s/n)e^{it}} \in \partial U_{\sin(s/n)}. \]  

(50)

(For evaluation of \( D(z) \) near \( e^{\pm \alpha} \) see a similar calculation in Lemma 6.4 of [23].) Therefore the jump matrix on \( \Sigma_{\text{out}} \) can be uniformly (both in \( z \) and \( \alpha \)) estimated by (34,40) as \( I + O(\exp(-C_1 n \sin(\alpha/2))) \), where \( C_1 \) is a positive constant. Note that in the case of a fixed arc, it is sufficient to use \( [4] \).

The jump matrices on \( \partial U_\delta \cup \partial \bar{U}_\delta \) admit an asymptotic expansion in powers of \( 1/\sqrt{\zeta} \) (which turns into expansion in powers of \( 1/n \) for a fixed arc or \( 1/s \) for a varying arc case). First, \([4], [15], \) and \([40] \) yield

\[ P(z)N(z)^{-1} = I + N(z)f(z)^{\sigma_3/2} \left[ \frac{1}{8\sqrt{\zeta}} \begin{pmatrix} -1 & -2i \\ -2i & 1 \end{pmatrix} - \frac{3}{27\zeta} \begin{pmatrix} 1 & -4i \\ 4i & 1 \end{pmatrix} \right] + O(\zeta^{-3/2}) \]
\[ f(z)^{-\sigma_3/2}N(z)^{-1} = I + \Delta_1 + \Delta_2 + O(\zeta^{-3/2}), \quad z \in \partial U_\delta, \]  

(51)

where \( \Delta_1 \) and \( \Delta_2 \) denote the terms with \( \sqrt{\zeta} \) and \( \zeta \), respectively (the remainder term will be justified below). We write them down explicitly for the case of \( f(z) = 1 \) needed later on:

\[ \Delta_1 = \frac{1}{24\sqrt{\zeta}} \begin{pmatrix} -(3a^2 - a^{-2}) & -i(3a^2 + a^{-2}) \\ -i(3a^2 + a^{-2}) & 3a^2 - a^{-2} \end{pmatrix}, \quad \Delta_2 = -\frac{3}{27\zeta} \begin{pmatrix} 1 & -4i \\ 4i & 1 \end{pmatrix}, \]  

(52)

where \( f = 1 \). The functions \( \Delta_1(z) \) and \( \Delta_2(z) \) for \( z \in \partial \bar{U}_\delta \) are given by the same expressions with \( a \) exchanged with \( 1/a \) and \( i \) replaced by \(-i\).

Using the expansion for Bessel functions, we can write a general term \( \Delta_j \) in \([51] \) which is of order \( 1/\zeta^{j/2} \). Indeed, apart from the prefactor with \( \zeta^{j/2} \), the matrix elements of \( \Delta_j \) are obviously \( O(1) \) as \( \zeta \to \infty \) for a fixed arc. It is also true in the general case, because of the remark after equation \([49] \). Now it is clear that \([51] \) is an asymptotic expansion in \( \zeta \).
Since for \( z \in \partial U \) by \((37,40)\)

\[
\sqrt{\zeta} = n \sqrt{\omega(z)} = O(n \sin(\alpha/2)) = \begin{cases} O(n), & \alpha \text{ fixed} \\ O(s), & \alpha = 2s/n, \end{cases}
\] 

the component \( \Delta_j \) on this boundary is of order \( 1/n^j \) (fixed arc) and \( 1/s^j \) (varying arc), and the remainder term in \((51)\) is uniform for all \( \alpha \) between a fixed positive value and \( 2s/n \), all \( n > s \), and all \( s \) larger than some \( s_0 \). (As we shall see this uniformity persists for the remainder in the asymptotics of our polynomials.) We show as in \[23\] that \( \Delta_j(z) \) is an analytic function in \( U_\delta \) with a pole at \( e^{i\alpha} \) of order less than or equal to \((j+1)/2\). The same reasoning also holds for the neighborhood \( \tilde{U}_\delta \). We shall denote the components of the jump matrix there by the same symbols \( \Delta_j(z) \).

If \( R(z) \) is known, we can trace the sequence \( Y \mapsto T \mapsto S \mapsto R \) backwards, and obtain an expression for the polynomials.

We look for \( R(z) \) asymptotically in the form

\[
R(z) \sim R_0(z) + R_1(z) + R_2(z) + \cdots,
\]

where \( R_j(z) \) is of the same order as \( \Delta_j \) (in our case, of order \((n \sin(\alpha/2))^{-j})\). For more discussion and justification of this expansion see \[19, 23\]. More precisely, it can be shown as in Theorems 7.8–7.10 of \[19\] that for any \( k \geq 1 \)

\[
R(z) = I + \sum_{j=1}^{k-1} R_j(z) + O((n \sin(\alpha/2))^{-k})
\]

uniformly for all \( z \) if the arc is fixed, and for \( z \) outside a neighborhood of \( z = 1 \) if the arc is varying (\( \alpha = 2s/n \)). The proof of this expansion in the general case \( 2s/n \leq \alpha < \pi \) for \( z \) close to \( 1 \) requires a special argument since the contours then are close to \( z \) and can not be trivially deformed. Such a proof will be given later on for the case of \( f = 1, z \in U_\delta \), we need below. The argument in the general case is similar. Moreover, it follows directly from the proofs that the remainder term in \((51)\) has the same uniformity property as that in \((53)\).

Substituting this asymptotic expansion into \((48)\) and collecting the terms of the same order, we obtain:

\[
\begin{align*}
R_{0+}(x) + R_{1+}(x) + \cdots & \sim (R_{0-}(x) + R_{1-}(x) + \cdots)(I + \Delta_1(x) + \cdots), \quad x \in \partial U_\delta \cup \partial \tilde{U}_\delta. \\
R_{0+}(x) = R_{0-}(x) & \Rightarrow R_0(z) = I, \\
R_{1+}(x) - R_{1-}(x) & = \Delta_1(x), \\
R_{2+}(x) - R_{2-}(x) & = R_{1-}(x)\Delta_1(x) + \Delta_2(x), \\
R_{k+}(x) - R_{k-}(x) & = \sum_{j=1}^{k} R_{k-j-}(x)\Delta_j(x), \quad k = 1, 2, \ldots
\end{align*}
\]

The main term in the asymptotics of polynomials is given therefore by the parametrices at the appropriate points \( z \). The expressions for \( R_k(z) \) follow from the Sokhotsky-Plemelj
formulas:

\[ R_1(z) = \frac{1}{2\pi i} \int_{\partial U_\delta \cup \partial \tilde{U}_\delta} \frac{\Delta_1(x)dx}{x - z}, \quad R_2(z) = \frac{1}{2\pi i} \int_{\partial U_\delta \cup \partial \tilde{U}_\delta} \frac{R_1(x)\Delta_1(x) + \Delta_2(x)}{x - z}dx, \ldots \]

Note that the contours are traversed in the negative direction (see Figure 1).

Following [23], we can also obtain the expressions for \( R_j(z) \) in a different way. As mentioned, \( \Delta_1(z) \) is analytic in \( U_\delta \cup \tilde{U}_\delta \) with simple poles at the end-points of the arc. Thus,

\[ \Delta_1(z) = \frac{A^{(1)}}{z - e^{i\alpha}} + O(1), \quad \text{as} \quad z \to e^{i\alpha}, \quad \Delta_1(z) = \frac{B^{(1)}}{z - e^{-i\alpha}} + O(1), \quad \text{as} \quad z \to e^{-i\alpha}, \quad (57) \]

where the constant matrices \( A^{(1)} \) and \( B^{(1)} \) are obtained by expanding various functions in (51) at \( z = e^{i\alpha} \) and \( z = e^{-i\alpha} \). It is easy to verify directly that the Riemann-Hilbert problem for \( R_1(z) \) has the solution:

\[ R_1(z) = \begin{cases} \frac{A^{(1)}}{z - e^{i\alpha}} + \frac{B^{(1)}}{z - e^{-i\alpha}}, & \text{for} \ z \in \mathbb{C} \setminus (U_\delta \cup \tilde{U}_\delta) \\ \frac{A^{(1)}}{z - e^{i\alpha}} + \frac{B^{(1)}}{z - e^{-i\alpha}} - \Delta_1(z), & \text{for} \ z \in U_\delta \cup \tilde{U}_\delta. \end{cases} \quad (58) \]

Expanding the functions in (51), we obtain:

\[ A^{(1)} = \frac{\cos(\alpha/2)}{8n} \left( \frac{1}{-iD_{-1}^2} \right) e^{i\alpha/2}, \quad B^{(1)} = \overline{A^{(1)}}, \quad (59) \]

where \( \overline{M} \) means complex conjugation applied to every matrix element of \( M \). The general term \( R_k(z) \) is obtained similarly provided we have the expressions for \( R_j(z) \), \( j = 1, 2, \ldots, k - 1 \). Since \( R_j(z) \) are analytic in \( U_\delta \cup \tilde{U}_\delta \) and \( \Delta_j(z) \) have poles at \( e^{\pm i\alpha} \) of order at most \( [(j+1)/2] \), we see that

\[ \sum_{j=1}^{k} R_{k-j,-}(z)\Delta_j(z) = \frac{A_p^{(k)}}{(z - e^{i\alpha})^p} + \frac{A_{p-1}^{(k)}}{(z - e^{i\alpha})^{p-1}} + \cdots + \frac{A_1^{(k)}}{z - e^{i\alpha}} + O(1), \quad \text{as} \quad z \to e^{i\alpha}, \quad (60) \]

where \( p = [(k+1)/2] \), and similar expressions hold with matrices \( A \) replaced with some matrices \( B \) in a neighborhood of \( e^{-i\alpha} \). Then

\[ R_k(z) = \begin{cases} \sum_{j=1}^{p} \left( \frac{A_j^{(k)}}{(z - e^{i\alpha})^j} + \frac{B_j^{(k)}}{(z - e^{-i\alpha})^j} \right), & \text{for} \ z \in \mathbb{C} \setminus (U_\delta \cup \tilde{U}_\delta) \\ \sum_{j=1}^{p} \left( \frac{A_j^{(k)}}{(z - e^{i\alpha})^j} + \frac{B_j^{(k)}}{(z - e^{-i\alpha})^j} \right) - \sum_{j=1}^{k} R_{k-j,-}(z)\Delta_j(z), & \text{for} \ z \in U_\delta \cup \tilde{U}_\delta. \end{cases} \quad (61) \]

Recalling the definitions of \( R, S, \) and \( T \), we finally get

\[ Y(z) = \gamma^{n\sigma_3} R(z) M(z) \psi(z)^{n\sigma_3}, \quad R(z) \sim I + R_1(z) + R_2(z) + \cdots, \quad (62) \]
where for $z$ restricted to region 3 $M(z) = N(z)$, $P(z)$, or $\tilde{P}(z)$ if $z \in \mathbb{C} \setminus (\overline{U_\delta} \cup \overline{\widetilde{U}_\delta})$, $U_\delta$, or $\overline{U_\delta}$, respectively (the expressions in regions 1 and 2 can also be readily written). Therefore, by (53),

$$
\chi_n^{-1} \phi_n(z) = Y_{11}(z) = \gamma^n \psi^n(z) [R_{11}(z)M_{11}(z) + R_{12}(z)M_{21}(z)].
$$

(63)

Furthermore,

$$
\chi_{n-1}^2 = Y_{21}(0) = \gamma^{-2n}[R_{21}(0)N_{11}(0) + R_{22}(0)N_{21}(0)].
$$

(64)

Since we know the expressions for $M(z)$ and can obtain $R(z)$ with an arbitrary precision, the last 2 equations give an implicit solution for asymptotics of polynomials $\phi_n(z)$ and their leading coefficients. These are the asymptotic series in the inverse powers of $n \sin(\alpha/2)$. The error after $k$ terms is $O(n \sin(\alpha/2))^{-k-1}$ and remains uniform for all $\alpha$, $s$, and $n$, provided $\alpha \in [2s/n, \pi - \varepsilon]$, $n > s$, $s > s_0$.

As an example, we give below the first two terms in the asymptotics of $\phi_n(z)$ valid for $z$ outside a fixed arbitrary small $\varepsilon$-neighborhood of the arc $\Sigma$:

$$
\chi_n^{-1} \phi_n(z) \sim \gamma^n \psi^n(z) \frac{D_{\infty}}{D(z)} \left[ \frac{a(z) + a(z)^{-1}}{2} + \gamma \left( \frac{a(z)e^{i\alpha/2}}{z - e^{i\alpha}} + \frac{a(z)^{-1}e^{-i\alpha/2}}{z - e^{-i\alpha}} \right) \right],
$$

(65)

$$
\chi_{n-1}^2 \sim \frac{\gamma^{-2n+1}}{D(0)D_{\infty}} \left[ 1 + \frac{1}{4n} \right],
$$

(66)

where all the quantities are defined as above (see (12) (27) (29) (30)).

We now give an explicit solution for the first 3 terms in the asymptotics of $\phi_n(z)$ and its first derivative at an end-point of the arc for the weight $f(z) = 1$.

Recall that we still have to prove the remainder term in the expansion (54) for $z \in U_\delta$, $2s/n \leq \alpha < \alpha_0$. Since $f(z) = 1$ we see that $D(z) = 1$. Each matrix element of $\Delta_k(z)$ can be written as $(\alpha_k a^2 + \beta_k a^{-2} + \gamma_k)\zeta^{-k/2}$, where $\alpha_k$, $\beta_k$, and $\gamma_k$ are independent of $\alpha$. In a neighborhood of $e^{i\alpha}$, we have the series:

$$
\zeta^{-k/2} = (n^2 u \sin \alpha)^{-k/2} \sum_{j=0}^{\infty} b_j(k, \alpha) \left( \frac{u}{\sin \alpha} \right)^j,
$$

$$
a(z)^2 = \sqrt{\frac{u}{\sin \alpha}} \sum_{j=0}^{\infty} c_j \left( \frac{u}{\sin \alpha} \right)^j,
$$

(67)

$$
|u| < \sin(\alpha/2), \quad u = z - e^{i\alpha}, \quad 0 < |c_j| \leq 1, \quad j \geq 0,
$$

where $b_j(k, \alpha)$ are bounded functions of $\alpha$. Since $\Delta_k(z)$ is single-valued in $U_\delta$, its matrix elements do not contain terms with $\sqrt{u}$. Hence, $\alpha_k, \beta_k = 0$ if $k$ is even, and $\gamma_k = 0$ if $k$ is odd. We have in the same neighborhood:

$$
\Delta_k(z) = \frac{1}{(n \sin \alpha)^k} \sum_{j=-(k+1)/2}^{\infty} C_j(k, \alpha) \frac{u^j}{\sin^j \alpha}.
$$

(68)

Now using (60) and the second formula in (61), it is easy to show by induction that

$$
R_k(z) = \frac{1}{(n \sin \alpha)^k} \sum_{j=0}^{\infty} \hat{C}_j(k, \alpha) \frac{u^j}{\sin^j \alpha}.
$$

(69)
The matrix elements of $C_j$ and $\hat{C}_j$ are bounded functions of $\alpha$. Considering the remainder term in the asymptotic expansion of Bessel functions, we see that $R(z) - I - \sum_{j=1}^{k-1} R_j(z)$ is given by the same series with different matrices $\hat{C}_j(k, \alpha)$ (but also bounded in $\alpha$). By analyticity, these series and their derivative w.r.t. $u$ converge in $U_\delta$. This is also true for $\alpha > \alpha_0$. Thus,

$$R(z) = I + \sum_{j=1}^{k-1} R_j(z) + O((n \sin(\alpha/2))^{-k}),$$

$$\frac{d}{dz} R(z) = \sum_{j=1}^{k-1} \frac{d}{dz} R_j(z) + O(n^{-k} \sin(\alpha/2))^{-1}), \quad z \in U_\delta,$$

where the remainder terms are uniform for $z \in U_\delta$, $2s/n < \alpha \leq \pi - \varepsilon$, $n > s$, $s > s_0$.

Using (63, 46, 45, 43), we obtain for $z$ in the intersection of $U_\delta$ and region 3 an asymptotic equivalence:

$$\chi^{-1}_n \phi_n(z) \sim n^{1/2} \sqrt{\omega(z)} \left[ a^{-1} I_0 + a I'_0 + \sum_{j=1}^{\infty} \{(R_{j, 11}(z) - iR_{j, 12}(z))a^{-1} I_0 + (R_{j, 11}(z) + iR_{j, 12}(z))a I'_0]\right],$$

where the Bessel functions $I_0$ and $I'_0 = I_1$ are taken at $n \sqrt{\omega(z)}$. We now estimate the “$R$” terms and some of their derivatives at the point $e^{i\alpha}$. From (56) and (52) we get:

$$R_{1, 11}(e^{i\alpha}) - iR_{1, 12}(e^{i\alpha}) = \frac{1}{2\pi i} \int_{\partial U_\delta \cup \partial \bar{U}_\delta} \frac{\Delta_{1, 11}(x) - i\Delta_{1, 12}(x)}{x - e^{i\alpha}} dx =$$

$$\frac{1}{2\pi i} \int \frac{3a^2}{8\sqrt{\zeta}} \frac{du}{u} + \frac{1}{2i \sin \alpha} \text{Res}_{x=e^{-i\alpha}} \frac{a^2}{8\sqrt{\zeta}},$$

where $u = z - e^{i\alpha} = e^{it}$ describes a circle of a small radius $\epsilon$ in the positive direction. The expansion of $\zeta$ near $e^{i\alpha}$ is given by (37), and that near $e^{-i\alpha}$ is obtained from it by changing the sign of $\alpha$. Expanding also $a^2(z)$ near $e^{\pm i\alpha}$ and calculating residues, we obtain

$$R_{1, 11}(e^{i\alpha}) - iR_{1, 12}(e^{i\alpha}) = \frac{3e^{i\alpha/2} + e^{-i\alpha/2}}{16i \sin(\alpha/2)}.$$  

Similarly, we calculate

$$R_{1, 11}(e^{i\alpha}) + iR_{1, 12}(e^{i\alpha}) = e^{-i\alpha/2} \frac{1 + e^{-i\alpha} - 2e^{i\alpha}}{3 \cdot 16i \sin(\alpha/2)}.$$  

Now differentiating $u$ w.r.t. $z$, we get

$$R'_1(z) = \frac{1}{2\pi i} \int_{\partial U_\delta \cup \partial \bar{U}_\delta} \frac{\Delta_1(x)}{(x - z)^2} dx.$$
From here we obtain as above

\[ R'_{1,11}(e^{i\alpha}) - iR'_{1,12}(e^{i\alpha}) = \frac{e^{-i\alpha/2}\cos(\alpha/2)}{16n\sin^2(\alpha/2)}; \quad (74) \]

To estimate \( R_2(z) \) using \( (56) \), we need to know \( L(z) = R_1(z)\Delta_1(z) + \Delta_2(z) \) in the neighborhoods of \( e^\pm i\alpha \). Here \( \Delta_1(z), \Delta_2(z) \) are given by \( (52) \) and for \( R_1(z) \) we use the second formula in \( (58) \). Then we obtain in the same way as above:

\[
\begin{align*}
R_{2,11}(e^{i\alpha}) - iR_{2,12}(e^{i\alpha}) &= \frac{-6 + 7e^{i\alpha} - 17e^{-i\alpha}}{2^9n^2\sin^2(\alpha/2)}; \\
R_{2,11}(e^{i\alpha}) + iR_{2,12}(e^{i\alpha}) &= \frac{16 - 9e^{i\alpha} + 43e^{-i\alpha} - 2e^{-2i\alpha}}{3 \cdot 2^9n^2\sin^2(\alpha/2)}.
\end{align*}
\]  

(75)

Substituting these expressions into \( (71) \), expanding \( a(z) \), Bessel functions, and \( \omega(z) \) at \( e^{i\alpha} \) (see \( (37) \)), and recalling \( (70) \), we obtain \( (5) \). A simple calculation of \( (64) \) yields \( (6) \).

Taking the derivative of \( (71) \) at \( z = e^{i\alpha} \) and using the expressions for \( R(e^{i\alpha}) \) and \( R'(e^{i\alpha}) \) we complete the proof of Theorem 2. \( \square \)

### 3 Proof of Theorem 1

Consider the following weight function on the unit circle:

\[ f_\alpha(\theta) = \begin{cases} 1, & 0 \leq \theta < 2\pi - \alpha, \\ 0, & \text{otherwise}. \end{cases} \quad 0 < \alpha < \pi \]

and the corresponding orthonormal polynomials \( \phi_k(z, \alpha) = \chi_kz^k + b_{k-1}z^{k-1} + \ldots + b_0 \) satisfying \( (3) \). Since \( f_\alpha(\theta) = f_\alpha(2\pi - \theta) \), the coefficients of the polynomials \( \phi_k(z, \alpha) \) are real.

Associated with \( f_\alpha(\theta) \) is an \( (n+1) \times (n+1) \) Toeplitz matrix \( T_n(\alpha) \) whose matrix elements are as follows:

\[
(T_n(\alpha))_{jk} = \frac{1}{2\pi} \int_0^{2\pi} e^{-i(j-k)\theta} f_\alpha(\theta) d\theta = \begin{cases} 1 - \frac{\alpha}{\pi}, & j = k, \\ -\frac{\sin(\alpha(j-k))}{\pi(j-k)}, & j \neq k, \end{cases} \quad j, k = 0, 1, \ldots, n.
\]

Putting \( \alpha = 2s/n \) and taking the limit \( n \to \infty \), we easily obtain

\[
\Delta(s) = \lim_{n \to \infty} \det T_{n-1}(\frac{2s}{n}). \quad (76)
\]

If \( \alpha \) is fixed, the large \( n \) asymptotics of \( \det T_n(\alpha) \) were obtained by Widom \( \[8\] \) (see also \( \[28\] \) for an alternative derivation), namely,

\[
\det T_{n-1}(\alpha) = \cos^2(\alpha/2) \left(n \sin \frac{\alpha}{2}\right)^{-1/4} 2^{1/12} e^{3\zeta'(1)}(1 + o(1)), \quad 0 < \alpha < \pi. \quad (77)
\]
An idea of Dyson [6] was to put $\alpha = 2s/n$ in these asymptotics. Then, in view of (6), one formally obtains the first 3 terms of (2). However, as noted in [6], since the remainder term in (7) is unknown, we cannot say if this expansion is uniform in $\alpha$ and therefore cannot justify such a limit.

Recently, Deift [9] found a formula which connects the determinants of two different Toeplitz matrices. A proof of it we need below was noticed by Simon. We shall use this formula together with equation (77) for $\det T_{n-1}(\alpha)$ ($\alpha$ fixed) to obtain an expression for $\det T_{n-1}(2s/n)$. The variant of Deift’s formula we shall use is the following:

$$
\frac{d}{d\alpha} \ln \det T_{n-1}(\alpha) = -\sum_{k=0}^{n-1} \frac{1}{2\pi} \int_{\alpha}^{2\pi-\alpha} \frac{d}{d\alpha} |\phi_k(e^{i\theta}, \alpha)|^2 d\theta, \quad n = 1, 2, \ldots
$$

(78)

Indeed, it is well-known (e.g., [10]) that the Toeplitz determinant has the following representation in terms of the leading coefficients of $\phi_n(z, \alpha)$:

$$
\det T_{n-1}(\alpha) = \prod_{k=0}^{n-1} \chi_k^{-2}, \quad n = 1, 2, \ldots
$$

(79)

Therefore

$$
\frac{d}{d\alpha} \ln \det T_{n-1}(\alpha) = -2 \sum_{k=0}^{n-1} \frac{\chi_k'(\alpha)}{\chi_k(\alpha)}.
$$

On the other hand, the orthogonality of our polynomials implies

$$
-\sum_{k=0}^{n-1} \frac{1}{2\pi} \int_{\alpha}^{2\pi-\alpha} \frac{d}{d\alpha} |\phi_k(e^{i\theta}, \alpha)|^2 d\theta =
$$

$$
-\sum_{k=0}^{n-1} \frac{1}{2\pi} \int_{\alpha}^{2\pi-\alpha} (\phi_k(e^{i\theta}, \alpha)(\chi_k'(\alpha)e^{-ik\theta} + b_{k-1}(\alpha)e^{-i(k-1)\theta} + \ldots) + \text{c.c.}) d\theta = -2 \sum_{k=0}^{n-1} \frac{\chi_k'(\alpha)}{\chi_k(\alpha)},
$$

and (78) is obtained. A corollary of it is the following

**Lemma 3** Let $\phi_n(z, \alpha)$ and $T_{n-1}(\alpha)$ be defined as at the beginning of the section, and $\phi'_n(z, \alpha) = (d/dz)\phi_n(z, \alpha)$. Then for any $n = 1, 2, \ldots$

$$
\frac{d}{d\alpha} \ln \det T_{n-1}(\alpha) = -\pi \frac{n}{n} |\phi_n(e^{i\alpha}, \alpha)|^2 - \frac{1}{\pi} \left\{ \phi_n(e^{-i\alpha}, \alpha)e^{i\alpha}\phi'_n(e^{i\alpha}, \alpha) + \text{c.c.} \right\}.
$$

(80)

**Proof** From the identity

$$
\frac{d}{d\alpha} \left( \sum_{k=0}^{n-1} \frac{1}{2\pi} \int_{\alpha}^{2\pi-\alpha} |\phi_k(e^{i\theta}, \alpha)|^2 d\theta \right) = \frac{d}{d\alpha} n = 0
$$

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we obtain using (78),
\[ \frac{d}{d\alpha} \ln \det T_{n-1}(\alpha) = -\frac{1}{\pi} \sum_{k=0}^{n-1} |\phi_k(e^{i\alpha}, \alpha)|^2. \]

As is known (e.g., [10]), any system of orthonormal polynomials on the circle satisfies an analogue of the Christoffel formula which we write for \( x \) and \( y \) on the unit circle in the form:
\[ \sum_{k=0}^{n-1} \phi_k(x)\overline{\phi_k(y)} = \frac{\phi_n^*(x)\overline{\phi_n^*(y)} - \phi_n(x)\overline{\phi_n(y)}}{1 - xy} = \frac{\phi_n(x)(\phi_n(x) - \phi_n(y)) - \phi_n^*(x)(\phi_n^*(x) - \phi_n^*(y))}{x(x - y)}, \]
where \( \phi_n^*(x) = x^n\overline{\phi_n(x)} \). Letting \( y \to x \) along the unit circle and noting that \( (d/dx)\phi_n^*(x) = nx^{-n+1}\overline{\phi_n(x)} - x^{-n+2}\overline{\phi_n'(x)} \), we obtain
\[ \sum_{k=0}^{n-1} |\phi_k(x)|^2 = x\phi_n(x)\overline{\phi_n'(x)} + \frac{1}{x} \phi_n(x)\overline{\phi_n'(x)} - n|\phi_n(x)|^2, \quad |x| = 1, \quad (81) \]
which, after substituting \( x = e^{i\alpha} \) and recalling that the coefficients of our polynomials are real, completes the proof. \( \square \)

Note [21, 29] that the logarithmic derivative (80) satisfies a \( \tau \)-function version of Painlevé VI equation.

Now we want to integrate (80) over \( \alpha \) between \( \alpha_1 = 2s/n \) and any fixed \( \alpha_2 < \pi \) for large \( n \) (for \( \alpha_2 \) asymptotics (77) are valid). We therefore need to know large \( n \) asymptotics of \( \phi_n(z, \alpha) \) and its derivative at the endpoint \( z = e^{i\alpha} \) of the orthogonality arc. These are given by Theorem 2. Substituting them into (80), we see that the term \( \frac{1}{n} |\phi_n(e^{i\alpha}, \alpha)|^2 \) cancels at once, and purely imaginary terms in the rest of the expression also disappear (in particular, the terms with \( r_3^k \) drop out). As a result we have:
\[ \frac{d}{d\alpha} \ln \det T_{n-1}(\alpha) = -n^2 \frac{\sin(\alpha/2)}{2\cos(\alpha/2)} - \frac{\cos(\alpha/2)}{8\sin(\alpha/2)} + O \left( \frac{1}{n\sin^2(\alpha/2)} \right), \quad (82) \]
where the remainder term is uniform for \( 2s/n \leq \alpha \leq \pi - \varepsilon, s > s_0, n > s, \varepsilon > 0 \). Integrating this expression over \( \alpha \) from \( 2s/n \) to \( \alpha_2 \) and using Widom’s asymptotics (77) for \( \det T_{n-1}(\alpha_2) \), we get
\[ \ln \det T_{n-1}(2s/n) = n^2 \ln \cos \frac{s}{n} - \frac{1}{4} \ln n \sin \frac{s}{n} + c_0 + O \left( \frac{1}{n \sin(s/n)} \right) + o(1) \quad (83) \]
as \( n \to \infty \) with the first remainder term turning into \( O(1/s) \) valid for all \( s > s_0 \). By (76), this equation yields (2). \( \square \)

**Remark** The Riemann-Hilbert problem methods allow us to calculate asymptotics of orthogonal polynomials to arbitrary precision. Because of the integral identity (Theorem 2b and equation (4) of [30], see also [31]) that expresses Toeplitz determinants in terms of orthogonal polynomials, an arbitrary number of terms in asymptotics of Toeplitz determinants could also be computed. Thus one should be able to estimate the remainder term in (77) and therefore give another proof of Theorem 1.
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