FREE MULTIPLECTIES ON THE MODULI OF $X_3$

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Abstract. In this note we study the freeness of the module of derivations on all moduli of the $X_3$ arrangement with multiplicities. We use homological techniques stemming from work of Yuzvinsky, Brandt, and Terao which have recently been developed for multi-arrangements by the first author, Francisco, Mermin, and Schweig.

1. Introduction

Fix a field $\mathbb{K}$ of characteristic zero and let $S = \mathbb{K}[x, y, z]$ be the polynomial ring on three variables which we associate to the symmetric algebra of a three dimensional vector space $V$. The $X_3$ arrangement is the collection of the hyperplanes $H_1 = \{x = 0\}$, $H_2 = \{y = 0\}$, $H_3 = \{z = 0\}$, $H_4 = \{x + y = 0\}$, $H_5 = \{x + z = 0\}$, and $H_6 = \{y + z = 0\}$ in $V$. The associated matroid is called the rank 3 whirl (see [13] Appendix and pictured in Figure 1) and is the unique relaxation of the braid matroid whose free multiplicities were studied in [9]. A multiplicity on $X_3$ is a function $m : X_3 \to \mathbb{Z}_{>0}$ and let $m = [m_1, \ldots, m_6]$ denote the multiplicity vector on $\mathcal{A}$ where $m(H_i) = m_i$ for all $1 \leq i \leq 6$ respectively. In Figure 1 we show a real projective picture of $X_3$ with the corresponding multiplicities.

The problem of classifying free multiplicities on a free arrangement is usually very difficult (in Section 2 we cover the background material for freeness). At the moment there are only four main results where arrangements (or classes of arrangements) free multiplicities are classified.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{A projective picture of the $X_3$ arrangement labeled with multiplicities and its associated matroid diagram}
\end{figure}
(1) In [6] Abe, Yoshinaga, and Terao show that the only arrangements with the property that all multiplicities are free are products of one or two dimensional arrangements.

(2) In [22] Yoshinaga shows that generic arrangements have no free multiplicities.

(3) In [2] Abe classifies the free multiplicities on the deleted $A_3$ braid arrangement.

(4) In [9] the first author, Francisco, Mermin, and Schweig finish the classification of free multiplicities on the full $A_3$ braid arrangement started in [5] by Abe, Terao, and the second author and progressed in [3] by Abe, Nuida, and Numata.

The idea of this note is to be a positive addition to this list. It turns out that the $X_3$ arrangement has a one dimensional moduli space (the realization space of the rank 3 whirl matroid, see Section 2 for details). The main result, Theorem 1.2, is not only a classification of the free multiplicities on $X_3$ but also a classification of the free multiplicities on all of its moduli. Since the characteristic polynomial of $X_3$ does not factor, no arrangement in the moduli of $X_3$ is free (by Terao’s factorization theorem [16]). This is the first non-generic arrangement to have it’s free multiplicities classified where (1) it’s not free and (2) it has non-trivial moduli. One consequence of this classification is many multiplicities where freeness of the associated multi-arrangement is not combinatorially determined, contrary to Terao’s conjecture for simple arrangements (see Example 2.4).

Our proof of the classification follows along the lines of two recent papers [8, 9] and also draws on work of Brandt and Terao [7] and Schenck and Stillman [15]. We prove that freeness of $D(X_3, m)$ is characterized by the vanishing of a certain homology module of a chain complex. The relevant chain complex is a modification (to the setting of multi-arrangements) of a chain complex appearing in work of Brandt and Terao [7], where it is used to study formality. On the other hand, our analysis of this chain complex closely mirrors techniques in multivariate spline theory [15]. The techniques are not isolated to the $X_3$ arrangement and can be generalized to arbitrary multi-arrangements; this is the intended subject of a forthcoming paper.

Our paper is arranged as follows. In Section 2 we review basic definitions and enumerate the moduli of $X_3$. In section 3 we introduce the homological machinery necessary for our classification and in section 4 we classify all the free multiplicities on $X_3$. In section 5 we discuss free extensions of $X_3$; in particular we show that extensions of $X_3$ and its moduli satisfy Terao’s conjecture.

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2. Set up

In this section we first review the basic definitions of free multiplicities. Then we discuss the moduli on $X_3$. 
2.1. Derivations on multiarrangements. The module of derivations on \( S \) is 
\[ \text{Der}(S) = \{ \theta \in \text{Hom}(S, S) \mid \forall f, g \in S, \theta(fg) = \theta(f)g + f\theta(g) \} \]
and the module of derivations on the multiarrangement \((A, m)\) where \( A = \{ H_i \}\), with defining linear forms \( \alpha_i \), is 
\[ D(A, m) = \{ \theta \in \text{Der}(S) \mid \theta(\alpha_i) \in a_i^{m(H_i)}S \}. \]

Definition 2.1. A multiplicity \( m \) on an arrangement \( A \) is free for \( A \) if \( D(A, m) \) is a free module over the polynomial ring \( S \).

Write \( Q(A, m) = \prod_{H \in A} a_H^{m(H)} \). We may refer to a multi-arrangement by its polynomial \( Q(A, m) \). The following is Saito’s criterion for multi-arrangements.

Proposition 2.2. Suppose \((A, m)\) is a multi-arrangement and \( \theta_1, \ldots, \theta_\ell \in D(A, m) \). Write \( \theta_i = \sum_j \theta_{ij} \partial_j \) in the basis elements \( \partial_j = \partial/\partial x_j \) and let \( M = M(\theta_1, \ldots, \theta_\ell) \) be the \( \ell \times \ell \) matrix with entries \( \theta_{ij} \). The following are equivalent.

1. \( D(A, m) \) is a free \( S \)-module with basis \( \theta_1, \ldots, \theta_\ell \).
2. \( \det(M) = kQ(A, m) \) for some \( k \neq 0 \in \mathbb{K} \).

If \( A \subset V \cong \mathbb{K}^\ell \) is an arrangement and \( H \in A \) a hyperplane, the restriction \( A^H \) is the hyperplane arrangement in \( H \cong \mathbb{K}^{\ell - 1} \) with hyperplanes \( \{ H \cap H' : H' \in A \} \).

The Ziegler multiplicity on \( A^H \) is \( m^H = \# \{ H'' \in A : H'' \cap H = H' \cap H \} \) and the Ziegler multi-restriction of \( A \) to \( H \in A \) is the multi-arrangement \((A^H, m)\) where \( m \) is the Ziegler multiplicity. If \( A \) is a central arrangement, denote by \( L \) its lattice of flats; this is the poset of all intersections of hyperplanes of \( A \) ordered with respect to reverse inclusion. If \( X \in L \) then \( A_X \) is the arrangement consisting of all hyperplanes containing \( X \). The following criterion is due to Yoshinaga [20].

Theorem 2.3. Suppose \( A \) is a central arrangement with maximal flat \( c \) and \( H \in A \). Then \( A \) is free if and only if

1. The Ziegler multi-restriction \((A^H, m)\) is free and
2. \( A_X \) is free for every \( X \neq c \in L \) with \( X > H \).

2.2. Moduli of \( X_3 \). Consider the coefficient matrix \( M = (a_{ij}) \) where \( h_i = \{ a_{ij}x + a_{i2}y + a_{i3}z = 0 \} \) for \( 1 \leq i \leq 6 \). Since hyperplanes \( H_1, H_2, H_5 \) and \( H_6 \) form a generic subarrangement of \( X_3 \) we can fix coordinates so that the coefficient matrix is of the form

\[ M = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
a_{31} & a_{32} & a_{33} \\
a_{41} & a_{42} & a_{43} \\
0 & 0 & 1 \\
1 & 1 & 1
\end{bmatrix} \]

Then we get the following consequences of the three triple points:

1. The triple point \( \{ H_1, H_3, H_5 \} \) gives that \( a_{32} = 0 \).
2. The triple point \( \{ H_2, H_3, H_6 \} \) gives that \( a_{31} = a_{33} \).
3. The triple point \( \{ H_1, H_2, H_4 \} \) gives that \( a_{43} = 0 \).

Since \( a_{31} \neq 0 \) and \( a_{41} \neq 0 \) (otherwise we would have more dependences than are combinatorially allowed for \( X_3 \)) we can scale the rows so that the coefficient matrix
Figure 2. The $X_3$ arrangement with the moduli emphasized.

is

$$M = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 1 \\
1 & a_{42} & 0 \\
0 & 0 & 1 \\
1 & 1 & 1
\end{bmatrix}. $$

Since we do not have any more multiple intersection points this is as far as we can reduce $M$. Hence the dimension of the moduli space of $X_3$ is 1. Moreover, by examining all other possible determinants of $M$ we see that as long as $a_{42} \neq 0, 1$ the corresponding matroid is isomorphic to that of $X_3$. If $a_{42} = 0$ then $H_1 = H_4$. If $a_{42} = 1$ then we get a new dependency which makes the arrangement lattice equivalent to the braid arrangement.

Again changing coordinates and using $\alpha \neq 0, 1$ for the one dimensional moduli we will use the following coefficient matrix for the remainder of the paper

$$M = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & -\alpha & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{bmatrix}$$

which can be viewed as tilting the “diagonal line” in Figure 2.

Example 2.4 (Freeness of multi-arrangements is not combinatorial). Consider the multiplicity $m = [3, 3, 3, 1, 1, 1]$ on the moduli of the $X_3$ arrangement over the complex numbers, with corresponding multi-arrangement defined by $x^3y^3z^3(x + \alpha y)(x + z)(y + z)$. The multi-arrangement defined thus is free if and only if $\alpha \neq -1$, which shows that freeness of multi-arrangements is not a combinatorial property.

We can prove this claim using addition-deletion techniques for multi-arrangements from [5] (if the reader is not familiar with these techniques, feel free to skip the following argument). First, consider the deletion $(\mathcal{A}, m')$ with defining polynomial $x^3y^3z^3(x + z)(y + z)$. Then $\mathcal{A}$ is supersolvable with filtration $\mathcal{A} = \mathcal{A}_3 \supset \mathcal{A}_2 \supset \mathcal{A}_1$, where $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ have defining polynomials $x, xz(x + z), \text{and } xyz(x + z)(y + z)$, respectively. We can check that $(\mathcal{A}, m')$ is free with exponents $(3, 4, 4)$ by [5].
Theorem 5.10. The Euler restriction (see [3]) of \((X_3, \mathbf{m})\) to \(V(x) = H_1\) is the multi-arrangement \(x^3z^3(x + z)(x + \alpha z)\) (some care must be taken to see that 3 is the proper exponent on \(x\), but this follows from [5, Proposition 4.1]). Ziegler considered the multi-arrangement \(x^3z^3(x + z)(x + \alpha z)\) to show that exponents of multi-arrangements are not combinatorial [24] Proposition 10; he showed that the exponents are \((3, 5)\) if \(\alpha = -1\) and \((4, 4)\) if \(\alpha \neq -1\). It then follows from the addition theorem [5, Theorem 0.7] that the multi-arrangement \(x^3y^3z^3(x - \alpha y)(x + z)(y + z)\) is free if \(\alpha \neq -1\) and from the restriction theorem [5, Theorem 0.6] that it is not free when \(\alpha = -1\).

**Remark 2.5.** A significant obstruction to using addition-deletion techniques from [5] to classify all free multiplicities on the moduli of \(X_3\) is that exponents for multi-arrangements of four points in \(\mathbb{P}^1\) are largely unknown. If the multiplicity vector \(\mathbf{m}\) of an arrangement \(A\) of points in \(\mathbb{P}^1\) satisfies certain inequalities and the points in \(A\) are ‘generic’ then the second author and Yuzvinsky show that the exponents of \((A, \mathbf{m})\) are \((\lfloor \mathbf{m}/2 \rfloor, \lfloor \mathbf{m}/2 \rfloor, \lfloor \mathbf{m}/2 \rfloor)\) [19]. However it is not easy to determine what makes the arrangement of points in \(\mathbb{P}^1\) ‘generic’: it depends in particular on the multiplicity vector \(\mathbf{m}\).

3. Homological characterization of freeness

In this section we prove that freeness of \((D(X_3), \mathbf{m})\) can be characterized by the vanishing of a certain homology of a chain complex. We devote this section to a careful description of the chain complex and a proof that the vanishing of this particular homology is both necessary and sufficient for freeness of \(D(X_3, \mathbf{m})\). The techniques are close in spirit to [15], where it is shown that vanishing homologies of a chain complex determine freeness of the module of splines.

We will denote by \(L = L_{X_3}\) the intersection lattice of \(X_3\), and we denote by \(L_c\) intersections of codimension (or rank) \(c\). The \(X_3\) arrangement has three triple points \(Y_1, Y_2, Y_3\) given as intersections of \(\{H_1, H_2, H_4\}, \{H_1, H_3, H_5\}\), and \(\{H_2, H_3, H_6\}\), respectively. Let \(L^{\text{trip}} = \{Y_1, Y_2, Y_3\}\) be the set of triple points. For every hyperplane \(H_i\) in \(X_3\) set \(J(H_i) = \alpha_i^{m(H_i)}S\). Furthermore, for a flat \(Y \in L\) of rank two let \(J(Y) = \sum_{Y \subset H} J(H)\).

3.1. Hilbert-Burch resolutions. For later use, we will need to have a good understanding of the syzygies of the ideals \(J(Y_i)\), which we write explicitly below:

\[
\begin{align*}
J(Y_1) &= \langle x^{m_1}, y^{m_2}, (x - \alpha y)^{m_3}\rangle \\
J(Y_2) &= \langle x^{m_1}, z^{m_3}, (x + z)^{m_2}\rangle \\
J(Y_3) &= \langle y^{m_2}, z^{m_3}, (y + z)^{m_2}\rangle.
\end{align*}
\]

Each of these ideals is visibly codimension two (being supported at the relevant triple point in \(\mathbb{P}^2\)) and is Cohen-Macaulay. As such, each ideal \(J(Y_i)\) has a Hilbert-Burch resolution (see [11, Theorem 3.2]) of the form

\[
0 \rightarrow S^2 \xrightarrow{\phi_i} S^3 \rightarrow J(Y_i) \rightarrow 0,
\]

where \(\phi_i\) is a \(2 \times 3\) matrix of forms whose \(2 \times 2\) minors are (up to multiplication by a non-zero constant) the given generators of \(J(Y_i)\). The columns of \(\phi_i\) are syzygies
The chain complex.

whose homologies will tell us about freeness of the multi-arrangement \((X_3, \mathbf{m})\) where

\[
\begin{bmatrix}
A_1 & B_1 \\
A_2 & B_2 \\
A_3 & B_3 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
C_1 & D_1 \\
C_2 & D_2 \\
C_3 & D_3 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
E_1 & F_1 \\
E_2 & F_2 \\
E_3 & F_3 \\
\end{bmatrix}
\]

To make the meaning of these matrices clearer, consider the ideal \(J(Y_1) = \langle x^{m_1}, y^{m_2}, (x - \alpha y)^{m_4} \rangle\). Then the columns of \(\phi_1\) express the relations \(A_i x^{m_1} + B_i y^{m_2} + C_i (x - \alpha y)^{m_4} = 0\) for \(i = 1, 2\) (notice the relative order of the generators is important). Since these are matrices for Hilbert-Burch resolutions, we also have (up to multiplication by a non-zero constant)

\[
A_1 B_2 - B_1 A_2 = (x - \alpha y)^{m_4}, A_1 B_3 - B_1 A_3 = y^{m_2}, A_2 B_3 - B_2 A_3 = x^{m_1}, \text{etc.}
\]

It is standard practice in commutative algebra to omit redundant generators from the generating set. However, it is important for our analysis that we fix the given generators for the ideals \(J(Y_i)\). We illustrate with two simple examples.

**Example 3.1.** Suppose \(m_1 = m_2 = 2\) and \(m_4 = 1\), so \(J(Y_1) = \langle x^2, y^2, (x - \alpha y) \rangle\). Clearly either \(x^2\) or \(y^2\) is an extraneous generator. There are several possible choices for what \(\phi_1\) will look like, depending on what syzygies are chosen on \(J(Y_1)\). We make the following choice:

\[
\phi_1 = \begin{bmatrix}
1 \\
-\alpha^2 \\
-(x + \alpha y) \\
\end{bmatrix}
\begin{bmatrix}
0 \\
-(x - \alpha y) \\
y^2 \\
\end{bmatrix}
\]

which expresses the fact that syzygies on \(J(Y_1)\) can be generated by the Koszul syzygy between \(y^2\) and \((x + y)\), followed by the expression of \(x^2\) in terms of \(y^2\) and \((x - \alpha y)\). Notice that the 2 \times 2 minors are \(-(x - \alpha y), y^2, -x^2\). More generally, suppose \(J(Y_1)\) is minimally generated by two of the three forms \(x^{m_1}, y^{m_2}, (x - \alpha y)^{m_4}\); without loss suppose \(y^{m_2}\) and \((x - \alpha y)^{m_4}\) generate \(J(Y_1)\). Then \(\phi_1\) has the form

\[
\phi_1 = \begin{bmatrix}
1 \\
-F \\
-G \\
\end{bmatrix}
\begin{bmatrix}
0 \\
-(x - \alpha y)^{m_4} \\
y^{m_2} \\
\end{bmatrix}
\]

where \(F, G\) are polynomials so that \(F y^{m_2} + G (x - \alpha y)^{m_4} = x^{m_1}\).

Now suppose instead that \(m_1 = m_2 = m_4 = 2\), so \(J(Y_1) = \langle x^2, y^2, (x - \alpha y)^2 \rangle\). This time none of the generators are redundant. Then one choice for \(\phi_1\) is:

\[
\phi_1 = \begin{bmatrix}
x - 2\alpha y \\
\alpha^2 x \\
-x \\
\end{bmatrix}
\begin{bmatrix}
y \\
-2\alpha x + \alpha^2 y \\
-y \\
\end{bmatrix}
\]

Notice that the 2 \times 2 minors are \(-2\alpha(x + y)^2, 2\alpha y^2, -2\alpha x^2\).

### 3.2. The chain complex.

We will now assemble the promised chain complex whose homologies will tell us about freeness of the multi-arrangement \((X_3, \mathbf{m})\). First consider the chain complex \(\mathcal{S} = \mathcal{S}_0 \xrightarrow{\delta^0} \mathcal{S}_1 \xrightarrow{\delta^1} \mathcal{S}_2\) with modules

\[
\mathcal{S}_0 \cong S^3 \quad \mathcal{S}_1 \cong S^6 \quad \mathcal{S}_2 \cong S^3
\]

We will consider taking homologies beginning at the left-hand side, so will denote homologies as cohomologies. The free module \(\mathcal{S}_0\) has a basis in correspondence with the variables \(x, y, z\), \(\mathcal{S}_1\) has basis in correspondence with hyperplanes (codimension
one flats), and $S_2$ has basis in correspondence with the triple points of $X_3$. The maps $\delta^i : S_i \to S_{i+1}$ are given by the matrices

$$\delta^0 = \begin{pmatrix} x & y & z \\ H_1 & 1 & 0 & 0 \\ H_2 & 0 & 1 & 0 \\ H_3 & 0 & 0 & 1 \\ H_4 & 1 & -\alpha & 0 \\ H_5 & 1 & 0 & 1 \\ H_6 & 0 & 1 & 1 \end{pmatrix} \quad \delta^1 = \begin{pmatrix} H_1 & H_2 & H_3 & H_4 & H_5 & H_6 \\ Y_1 & 1 & -\alpha & 0 & -1 & 0 \\ Y_2 & 1 & 0 & 1 & 0 & -1 \\ Y_3 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Note that $\delta^0$ is the coefficient matrix of forms defining $X_3$ and $\delta^1$ is the matrix of relations around triple points. It is readily checked that the chain complex $S$ is exact. In fact, it can be checked that the homologies of $S$ govern when $X_3$ is 2-formal in the sense of Brandt and Terao [7] (see also [17]); as a consequence we see that $X_3$ is 2-formal.

Now define the sub-complex $J = J_1 \xrightarrow{\delta^1} J_2$ of $S$ with modules

$$J_1 = \bigoplus_{H \in L_1} J(H) \subset S_1 \quad J_2 = \bigoplus_{Y \in L_2^\text{trip}} J(Y) \subset S_2.$$  

The quotient complex $S/J$ has the form

$$S^3 \xrightarrow{\delta^0} \bigoplus_{H \in L_1} S_{J(H)} \xrightarrow{\delta^1} \bigoplus_{Y \in L_2^\text{trip}} S_{J(Y)},$$

where $\delta^0, \delta^1$ are the quotient maps. Notice that $D(X_3, m)$ is the kernel of the map $\delta^0$, in other words $D(X_3, m) \cong H^0(S/J)$.

**Remark 3.2.** The modules of the chain complex $S/J$ are a natural extension to multi-arrangements of the modules $\bigoplus_{Y \in L_c} D_c(A_Y)$ which appear in the paper of Brandt and Terao [7].

The main point of the chain complex $S/J$ is that its homologies control freeness of $D(X_3, m)$. First, we notice that we may consider instead homologies of the complex $J$.

**Proposition 3.3.** With the chain complexes $J, S, S/J$ as above, $D(X_3, m) \cong H^0(S/J) \cong H^1(J), H^1(S/J) \cong H^2(J)$, and $H^2(S/J) = 0$.

**Proof.** This follows directly from the long exact sequence in homology derived from the short exact sequence of complexes $0 \to J \to S \to S/J \to 0$, the fact that the homologies of $S$ vanish, and the fact that $D(X_3, m) \cong H^0(S/J)$. 

The main result of this section is the following proposition.

**Proposition 3.4.** $D(X_3, m)$ is a free $S$-module if and only if $H^2(J) = 0$.

**Proof.** Let $m = \langle x, y, z \rangle$ be the maximal ideal of $S$. First we claim that if $H^2(J)$ is nonzero, then it is only supported at $m$ (hence has finite length). It is easiest to show this using $H^1(S/J)$; by Proposition 3.3 $H^1(S/J) \cong H^2(J)$. We will show that the localization $H^1(S/J)_p$ at any homogeneous non-maximal prime is
zero, using the fact that localization commutes with taking homology. To this end, localize the complex $S/\mathcal{J}$:

$$0 \to S^3_p \xrightarrow{(\delta^3)_p} \bigoplus_{H \in L_1} \frac{S}{J(H)_p} \xrightarrow{(\delta^1)_p} \bigoplus_{Y \in L_2^{\text{trip}}} \frac{S}{J(Y)_p} \to 0.$$

If $J(H) \not\subset P$ or $J(Y) \not\subset P$, then the corresponding summand in the above chain complex vanishes. Suppose $P$ is codimension one. Then $P$ is principal, hence $P$ can contain $J(H)$ for at most one $H \in L_1$. If $P$ contains none of the ideals $J(H)$, for $H \in L_1$, then clearly $H^1(S/\mathcal{J})_P = 0$. So suppose there is some $H \in L_1$ so that $J(H) \subset P$ (since $P$ is codimension one, this must be the only such $J(H)$). Clearly in this case $(\delta_0)_p : S^3_p \to S/J(H)_p$ is surjective, so again $H^1(S/\mathcal{J})_P = 0$.

Now suppose $P$ is codimension two (so $P$ is the ideal of a point in projective two-space). If $P$ contains only one $H \in L_1$, then the same argument as above shows that $H^1(S/\mathcal{J})_P = 0$. If $P$ contains two or more hyperplanes of $A$, then $P$ must be the ideal of the point of intersection of whatever hyperplanes $H$ satisfy $J(H) \subset P$.

If the point is a simple point of $A$, then there are only two hyperplanes $H, H' \in A$ so that $J(H) \subset P$ and $J(H') \subset P$. So $(S/\mathcal{J})_P$ looks like

$$0 \to S^3_p \xrightarrow{(\delta^3)_p} \frac{S}{J(H)_p} \oplus \frac{S}{J(H')_p} \xrightarrow{(\delta^1)_p} \frac{S}{J(H'')_p} \to 0,$$

which is clearly exact on the right, so $H^1(S/\mathcal{J})_P = 0$. Finally, suppose $P$ is the ideal of a triple point $Y \in L_2^{\text{trip}}$ (intersection of $H, H'$, and $H'' \in A$). Then $(S/\mathcal{J})_P$ looks like

$$0 \to S^3_p \xrightarrow{(\delta^3)_p} \frac{S}{J(H)_p} \oplus \frac{S}{J(H')_p} \oplus \frac{S}{J(H'')_p} \xrightarrow{(\delta^1)_p} \frac{S}{J(Y)_p} \to 0.$$

A check yields that $\text{coker}(\delta_0)_p = (S/J(Y)_p)_P$, so again $H^1(S/\mathcal{J})_P = 0$. It follows that $H^1(S/\mathcal{J})$ is only supported at $m$, the homogeneous maximal ideal.

Now to show that freeness of $D(X_3, m)$ is equivalent to vanishing of $H^2(\mathcal{J})$, consider the following four-term exact sequence:

$$(3.1) \quad 0 \to D(X_3, m) \cong H^1(\mathcal{J}) \to \bigoplus_{H \in L_1} J(H) \xrightarrow{\delta^1} \bigoplus_{Y \in L_2^{\text{trip}}} J(Y) \to H^2(\mathcal{J}) \to 0.$$ 

Write $\mathcal{J}_1$ for $\bigoplus_{H \in L_1} J(H)$ and $\mathcal{J}_2$ for $J(Y_1) \oplus J(Y_2) \oplus J(Y_3)$. Then break this four term sequence into the two short exact sequences

$$0 \to K \xrightarrow{\delta^1} H^2(\mathcal{J}) \to 0 \quad \text{and} \quad 0 \to D(X_3, m) \xrightarrow{\delta^1} J^1 \to K \to 0,$$

where $K = \text{im}(\delta^1)$. By the long exact sequence in Ext applied to the first short exact sequence, coupled with the fact that $\text{Ext}^1_S(J(Y), S)$ vanishes when $j > 1$ (hence $\text{Ext}^2_S(J_2, S)$ vanishes when $j > 1$), we obtain $\text{Ext}^3_S(H^2(\mathcal{J}), S) \cong \text{Ext}^2_S(K, S)$. Similarly, applying the long exact sequence in Ext to the second short exact sequence (and using that $\text{Ext}^1(J_1, S) = 0$ for $j > 0$ since $\mathcal{J}_1$ is a sum of principal ideals) yields that $\text{Ext}^2_S(K, S) \cong \text{Ext}^1(D(X_3, m), S)$. All in all, we see that $\text{Ext}^3_S(H^2(\mathcal{J}), S) \cong \text{Ext}^3_S(D(X_3, m), S)$. Since $D(X_3, m)$ is a second syzygy, freeness of $D(X_3, m)$ is equivalent to vanishing of $\text{Ext}^3_S(D(X_3, m), S)$, which is equivalent to vanishing of $\text{Ext}^3_S(H^2(\mathcal{J}), S)$. Since $H^2(\mathcal{J}) \cong H^1(S/\mathcal{J})$ is only supported at $m$, $\text{Ext}^3_S(H^2(\mathcal{J}), S)$ vanishes if and only if $H^2(\mathcal{J}) = 0$, and we are done. \hfill \Box
4. FREE MULTIPLICITIES

In this section we derive the full characterization of free multiplicities on \((X_3, m)\). The characterization hinges on the use of Proposition 3.4 together with a precise description of the homology module \(H^2(J)\). We begin by giving this description in terms of syzygies.

Denote by \(e_H\) a generator for \(J(H)\) (of degree \(\text{m}(H)\)) and given a triple point \(Y\) and a hyperplane \(H\) passing through \(Y\) we will denote by \(e_{H,Y}\) (of degree \(\text{m}(H)\)) a generator for \(J(H)\) viewed as a sub-ideal of \(J(Y)\). We will use the following commutative diagram to get a presentation for \(H^2(J)\) using syzygies of the ideal \(J(Y)\).

\[
\begin{array}{ccc}
0 & \to & \bigoplus_{Y \in L^\text{trip}_2} \text{syz}(J(Y)) \\
\uparrow & & \uparrow \\
\bigoplus_{H \in L_1} J(H) & \xrightarrow{\delta^1} & \bigoplus_{Y \in L^\text{trip}_2} J(Y) \\
\uparrow & & \uparrow \\
\bigoplus_{H \in L_1} S[e_H] & \xrightarrow{\delta^1} & \bigoplus_{H \in L_1} S[e_{H,Y}] \\
\uparrow & & \uparrow \\
0 & \to & \bigoplus_{Y \in L^\text{trip}_2} \text{syz}(J(Y)) \\
\end{array}
\]

The snake lemma yields an exact sequence

\[
\bigoplus_{Y \in L^\text{trip}_2} \text{syz}(J(Y)) \to \text{coker}(\delta^1) \to \text{coker}(\delta^1) = H^2(J) \to 0.
\]

We now explicitly identify the image of the syzygy modules inside of \(\text{coker}(\delta^1)\), which we will show is isomorphic to \(S^3\). As noted in §3.4 each of the ideals \(J(Y_i)\) is Cohen-Macaulay of codimension two, with Hilbert-Burch resolution of the form

\[
0 \to S^2 \xrightarrow{\varphi_1} S^3 \to J(Y_i) \to 0,
\]

where, as in §3.4, we will write

\[
\varphi_1 = \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \\ A_3 & B_3 \end{bmatrix} \quad \varphi_2 = \begin{bmatrix} C_1 & D_1 \\ C_2 & D_2 \\ C_3 & D_3 \end{bmatrix} \quad \varphi_3 = \begin{bmatrix} E_1 & F_1 \\ E_2 & F_2 \\ E_3 & F_3 \end{bmatrix}.
\]

We see that the image of \(\bigoplus_{Y \in L^\text{trip}_2} \text{syz}(J(Y))\) inside of \(\bigoplus_{Y \in L^\text{trip}_2} \bigoplus_{H \in L_1} S[e_{H,Y}]\) is generated by the columns of the following matrix (we label the rows by pairs \((H_i, Y_i)\)
corresponding to \( e_{H_i,Y_i} \):

\[
\begin{pmatrix}
H_1, Y_1 \\
H_2, Y_1 \\
H_3, Y_1 \\
H_4, Y_2 \\
H_5, Y_2 \\
H_6, Y_3 \\
\end{pmatrix}
\begin{pmatrix}
A_1 & B_1 & 0 & 0 & 0 & 0 \\
A_2 & B_2 & 0 & 0 & 0 & 0 \\
A_3 & B_3 & 0 & 0 & 0 & 0 \\
0 & 0 & C_1 & D_1 & 0 & 0 \\
0 & 0 & C_2 & D_2 & 0 & 0 \\
0 & 0 & 0 & 0 & E_1 & F_1 \\
0 & 0 & 0 & 0 & E_2 & F_2 \\
0 & 0 & 0 & 0 & E_3 & F_3 \\
\end{pmatrix}
\]

The map \( \delta^1 \) is lifted from \( \delta^1 \) and has the following matrix:

\[
\begin{pmatrix}
H_1 & H_2 & H_3 & H_4 & H_5 & H_6 \\
[1, Y_1] & 1 & 0 & 0 & 0 & 0 \\
[1, Y_2] & 0 & -\alpha & 0 & 0 & 0 \\
[1, Y_3] & 0 & 0 & 1 & 0 & 0 \\
[1, Y_4] & 0 & 0 & 1 & 0 & 0 \\
[1, Y_5] & 0 & 1 & 0 & 0 & 0 \\
[1, Y_6] & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix}
\]

This matrix clearly has full rank, so \( \text{coker}(\delta^1) \cong S^3 \), generated for instance by the images of \( e_{H_1,Y_1} \), \( e_{H_2,Y_1} \), and \( e_{H_3,Y_2} \); call these \( [e_{H_1,Y_1}], [e_{H_2,Y_1}], \) and \( [e_{H_3,Y_2}] \).

With respect to this basis we may represent \( \text{coker}(\delta^1) \) as the image of \( S^9 \) under the projection with matrix

\[
\begin{pmatrix}
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
\end{pmatrix}
\]

With this choice of projection, the image of the syzygies inside of \( S^3 \) is generated by the columns of the following matrix (rows are labeled by \( [H_i,Y_i] \) corresponding to \( [e_{H_i,Y_i}] \) for \( i = 1, 2, 4 \)):

\[
M = \begin{pmatrix}
H_1, Y_1 \\
H_2, Y_1 \\
H_3, Y_2 \\
\end{pmatrix}
\begin{pmatrix}
A_1 & B_1 & -C_1 & -D_1 & 0 & 0 \\
A_2 & B_2 & 0 & 0 & \alpha E_1 & \alpha F_1 \\
0 & 0 & C_2 & D_2 & -E_2 & -F_2 \\
\end{pmatrix}
\]

Together with Proposition 3.4 and the presentation (4.1), we have proved the following result.

**Proposition 4.1.** With all notation as above, \( D(X_3, \mathfrak{m}) \) is free if and only if the columns of \( M \) generate the free module \( S^3 = \text{coker}(\delta^1) \).

Now we classify free multiplicities on \( X_3 \) (and all of its moduli).
Theorem 4.2. For any \( \alpha \neq 0, 1 \), freeness of \((X_3, m)\) implies \( m_4 = m_5 = m_6 = 1 \) and \( m_1 = m_2 = m_3 = n \) for some positive integer \( n \). So free multiplicities are of the form \([n, n, n, 1, 1, 1]\). Furthermore,

- if \( \alpha \) is not a root of unity then \([n, n, n, 1, 1, 1]\) is a free multiplicity for any \( n > 1 \), and
- if \( \alpha \) is a root of unity with order \( m \), then \([n, n, n, 1, 1, 1]\) is a free multiplicity if and only if \( n \equiv 1 \mod m \). (In particular, if \( \alpha = -1 \), then \([n, n, n, 1, 1, 1]\) is a free multiplicity if and only if \( n \) is even.)

Moreover, when \((X_3, m)\) is free the exponents are \((n + 1, n + 1, n + 1)\).

Remark 4.3. In Theorem 4.2, if we fix a multiplicity \( m = [n, n, n, 1, 1, 1] \) then \((X_3, m)\) is free as long as \( \alpha \) avoids the roots of \( t^{n-1} - 1 \). Moreover \((X_3, m)\) is not free for any \( \alpha \) if \( m \) does not have the form \([n, n, n, 1, 1, 1]\). So among the moduli of arrangements with the \( X_3 \) lattice and a fixed multiplicity \( m \), the multi-arrangements which are free form a (possibly empty) Zariski-open set. We do not know if this is true in general. Yuzvinsky \( [23] \) has shown that this is the case for simple arrangements.

Proof. We use Proposition 4.1. The columns of \( M \) generate \( S^3 \) if and only if there is a \( 3 \times 3 \) minor \( M' \) of \( M \) so that \( \det(M') \) is a non-zero constant. Let us see what constraints this assumption places on the entries of \( M' \).

Suppose first that \( M' \) contains two columns corresponding to the same ideal; without loss of generality assume that

\[
M' = \begin{pmatrix}
A_1 & B_1 & F \\
A_2 & B_2 & G \\
0 & 0 & H
\end{pmatrix},
\]

where the third column is some other column of \( M \). Then \( \det(M') = H(A_1B_2 - B_1A_2) \). However, by the structure of the Hilbert-Burch matrix (see §3.1), \( A_1B_2 - B_1A_2 \) is a (non-zero) constant multiple of a generator of the ideal \( J(Y_1) \), hence this implies \( \det(M') \) is either zero or has positive degree, so cannot be non-vanishing constant. It follows that if \( \det(M') \) is a non-zero constant, it must have a column corresponding to a single syzygy from each of the three ideals \( J(Y_1) \), \( J(Y_2) \), and \( J(Y_3) \). Without loss of generality, we may assume that \( M' \) has the form

\[
M' = \begin{pmatrix}
A_1 & -C_1 & 0 \\
A_2 & 0 & \alpha E_1 \\
0 & C_2 & -E_2
\end{pmatrix},
\]

with \( \det(M') = -\alpha A_1E_1C_2 - A_2C_1E_2 \). Suppose that one of the entries off of the anti-diagonal of \( M' \) vanishes: without loss suppose \( A_1 = 0 \). Then it follows that \( A_2y^{m_2} + A_3(x + y)^{m_4} \) is a Koszul syzygy on the forms \( y^{m_2} \) and \( (x + y)^{m_4} \) (see Example 3.1). However, this implies that the second term \( A_2C_1E_2 \) either has positive degree or vanishes. Both cases contradict that \( \det(M') \) is a non-zero constant. It follows that all entries other than the anti-diagonal entries of \( M' \) are non-zero. Therefore, since both terms of \( \det(M') \) are homogeneous, the only way that \( \det(M') \) is a non-zero constant is if all entries of \( M' \) other than the anti-diagonal entries are non-zero constants.

Now recall the analysis in Example 3.1. The presence of any non-zero constant in the matrix \( \phi_1 \) expresses the fact that \( J(Y_1) \) is minimally generated by two of the three given generators. For simplicity let us consider \( J(Y_1) \). The fact that \( A_1 \) and
Remark 4.5. It is possible to recover portions of Theorem 4.2 using other techniques for multi-arrangements. We list a few here.

1. If \( \alpha = -1 \) and we assume \( m_4 = m_5 = m_6 = 1 \), then local and global mixed products [4] can be used to show that \((X_3, \mathbf{m})\) is free if and only if \( m_1 = m_2 = m_3 = 2k \) for some natural number \( k \).

2. If \((X_3, \mathbf{m})\) satisfies that all sub-\(A_2\) multi-arrangements are 'balanced,' then local and global mixed products show that \( \mathbf{m} \) is not a free multiplicity if one of \( m_4, m_5, \) or \( m_6 \) is greater than one.

3. If \( m_i = 1 \) for some \( i = 1, \ldots, 6 \), then the deletion of \((X_3, \mathbf{m})\) with respect to \( H_i \) has the deleted \( A_3 \) arrangement as the base arrangement. The classification of all free multiplicities on the deleted \( A_3 \) arrangement by Abe [2], along with addition/deletion/restriction theorems for multi-arrangements [5], yields some constraints on the \( m_i \) (however, this method runs into the problem of Remark 2.5).
Proposition 4.6. The derivations
\[ x^{2k+1} \partial_1 + y^{2k+1} \partial_2 + z^{2k+1} \partial_3 \]
\[ (y + z)(x^{2k} \partial_1 - y^{2k} \partial_2 - z^{2k} \partial_3) \]
\[ x^{2k} z \partial_1 - (x + y + z)y^{2k} \partial_2 + x z^{2k} \partial_3 \]
form a basis for the S-module \( D(X_3, \mathbf{m}) \) for \( \alpha = -1 \) where \( \mathbf{m} = [2k, 2k, 1, 1, 1] \).

Proof. This is a routine check with Saito’s criterion [14].

5. Free extensions

Given an arrangement \( \mathcal{A}' \) of rank \( r \), an extension of \( \mathcal{A}' \) is an arrangement \( \mathcal{A} \) of rank \( r + 1 \) so that there exists a hyperplane \( H_0 \in \mathcal{A} \) with \( \mathcal{A}' = \mathcal{A} \). In our case, where \( \mathcal{A}' \) is defined by linear forms in \( S = \mathbb{K}[x, y, z] \), we will assume \( \mathcal{A} \) is defined by forms in \( S = \mathbb{K}[x, y, z, w] \) and \( H_0 \) is defined by \( w = 0 \). A free extension of a multi-arrangement \((\mathcal{A}', \mathbf{m})\) is a free arrangement \( \mathcal{A} \) with a hyperplane \( H_0 \in \mathcal{A} \) so that the Ziegler multi-restriction satisfies \( \mathcal{A}' = (\mathcal{A}', \mathbf{m}) \). The free multiplicities of the \( X_3 \) arrangement do admit free extensions; we write these down in Proposition 5.2 but first we need a preliminary lemma.

Lemma 5.1. Let \( n > 0 \) be an integer and let \( \mathcal{A} \) be the central arrangement given by the vanishing of forms
\[
\begin{align*}
x - a_1 z, & \ldots , x - a_n z, \\
y - b_1 z, & \ldots , y - b_n z, \\
Ax + By + Cz, & \ldots, z,
\end{align*}
\]
where \( a_1, \ldots , a_n, b_1, \ldots , b_n, A, B, C \) are all constants with \( A, B \neq 0 \). Then \( \mathcal{A} \) is free if and only if there is a re-ordering of the \( a_i \) so that \( (a_1, b_1, 1), \ldots , (a_n, b_n, 1) \) all lie on the line \( Ax + By + Cz = 0 \) (in \( \mathbb{P}^2(\mathbb{K}) \)).

Proof. We use Yoshinaga’s criterion for freeness of 3-arrangements [21]. Namely, a three-arrangement is free if and only if
\[
\begin{align*}
(1) \quad & \chi(\mathcal{A}, t) = (t - 1)(t - d_1)(t - d_2) \\
(2) \quad & \text{The Ziegler restriction } (\mathcal{A}'', \mathbf{m}) \text{ is free with exponents } (d_2, d_2).
\end{align*}
\]
Restricting to the hyperplane \( H = V(z) \), we get the multi-arrangement \( \mathcal{A}'' = V(x) \cup V(y) \cup V(Ax + By) \) with multiplicities \( \mathbf{m} = [n, n, 1] \). By a characterization due to Wakamiko [18], \( (\mathcal{A}''', \mathbf{m}) \) is free with exponents \( (n, n + 1) \). Suppose that the line \( Ax + By + Cz = 0 \) passes through \( q \) of the \( n^2 \) intersection points of the grid defined by \( x_1 = a_1 z, \ldots , x_n = a_n z, y_1 = b_1 z, \ldots , y_n = b_n z \). Then we compute
\[
\chi(\mathcal{A}, t) = (t - 1)(t^2 - (2n + 1)t + \ldots + n^2 + 2n - q),
\]
which factors as \( (t - 1)(t - n)(t - (n + 1)) \) if and only if \( q = n \). Since \( A, B \neq 0 \), the re-ordering in the statement of the lemma is possible.

□
Proposition 5.2. Let $n > 1$ be an positive integer, $m_n = [n,n,n,1,1,1]$, and $\alpha \neq 0,1 \in K$. If $(X_3, m_n)$ has a free extension then $\alpha$ is a root of unity and $n = mt$ where $m$ is the order of $\alpha$ and $t$ is a positive integer. Suppose $A_1, \ldots, A_t \in K^*$ are nonzero constants. Then the arrangement defined by the vanishing of the forms

$$x - A_1 w, x - \alpha A_1 w, \ldots, x - \alpha^{m-1} A_1 w,$$

$$y - A_1 w, y - \alpha A_1 w, \ldots, y - \alpha^{m-1} A_1 w,$$

$$z + A_1 w, z + \alpha A_1 w, \ldots, z + \alpha^{m-1} A_1 w$$

is a free extension of $(X_3, m_n)$. Moreover, up to translations, every free extension has this form.

Proof. Freeness of the arrangement $A$ given by the indicated forms can be verified by applying Yoshinaga’s criterion [20]; since $(X_3, m_n)$ is free by Theorem 4.2, we check that $A$ is locally free along $w = 0$. This reduces to checking freeness of the rank three closed sub-arrangements of $A$ along $w = 0$; all of these are free by Lemma 5.1.

Now we show that any free extension must have this form. By Theorem 2.3, a free extension $A$ of $(X_3, m)$ exists only if $m = [n,n,n,1,1,1]$ for some integer $n > 1$. In this case $A$ has the form

$$x - a_1 w, \ldots, x - a_n w$$

$$y - b_1 w, \ldots, y - b_n w$$

$$z - c_1 w, \ldots, z - c_n w$$

$$x - \alpha y + Aw, x + z + Bw, y + z + Cw, w,$$

for some constants $a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n, A, B,$ and $C$. Translating in the $x,y,$ and $z$ directions we may assume that $A = B = C = 0$. We now check local freeness along $w = 0$. Let $A_{xyw}, A_{xzw}, A_{yzw}$ be the closed rank three sub-arrangements consisting of hyperplanes which contain the line $x = y = w = 0$, $x = z = w = 0$, and $y = z = w = 0$, respectively. Projectively, these are all arrangements of the form considered in Lemma 5.1. Assuming that $A_{xzw}$ and $A_{yzw}$ are both free, Lemma 5.1 allows a re-indexing of the $a_i, b_i$ so that the points $(a_i,0,c_i,1)$ for $i = 1, \ldots, n$ lie on the line determined by $x + z$ and the points $(0, b_i, c_i, 1)$ for $i = 1, \ldots, n$ lie along the line determined by $y + z$. It follows that $a_i = b_i = -c_i$ for $i = 1, \ldots, n$.

Now, by Lemma 5.1, $A_{xyw}$ is free if and only if there are $n$ points among the $n^2$ points $(a_i, b_j, 0, 1) = (a_i, a_j, 0, 1)$ which lie along the line $x - \alpha y$. Set $X = \{a_1, \ldots, a_n\}$ and consider the action of $\alpha$ on $X$ by multiplication. We see that this action must be a permutation action. Since $|X| = n > 1$, $\alpha$ must be a root of unity; we denote the order of $\alpha$ by $m$. Suppose $0 \in X$. Then the multiplication action of $\alpha$ on $X - \{0\}$ is faithful, so $|X - \{0\}| = tm$ (for some integer $t$) and $n = |X| = tm + 1$. But then $m_n = [n,n,n,1,1,1]$ is not a free multiplicity by
Theorem 4.2 since \( n \equiv 1 \mod m \). It follows that \( 0 \notin X \), so \( \alpha \) acts faithfully on \( X \) by multiplication, and \( n = |X| = mt \) for some integer \( t \). Choosing representatives \( A_1, \ldots, A_t \) in each \( \alpha \)-orbit gives the form in the statement of the proposition. \( \square \)

Remark 5.3. Every free arrangement \( \mathcal{A} \) in Proposition 5.2 has the property that \( \mathcal{A}^{w=0} = X_3 \) is not free. For instance, taking \( \alpha = -1 \) and \( t = 1 \) provides a one-parameter family of free arrangements of rank four with ten hyperplanes whose restriction is not free. The first such example was a free arrangement of rank five with 21 hyperplanes provided by Edelman and Reiner [10].

Remark 5.4. Yoshinaga’s extendability criterion [22] does not apply; the \( X_3 \) arrangement is locally \( A_2 \) but only has a positive system when \( \alpha = -1 \). Furthermore, even if \( \alpha = -1 \), the (•) criterion of [22, Theorem 2.5] does not hold for the multiplicities \([n, n, 1, 1, 1]\) when \( n \) is even.

Corollary 5.5. If \( \mathcal{A} \) is an extension of \( xyz(x - \alpha y)(x + z)(y + z) \) (for \( \alpha \neq 0, 1 \)), then it satisfies Terao’s conjecture (see [12, Conjecture 4.138]). That is, freeness of \( \mathcal{A} \) is determined from its intersection lattice.

Proof. This is implicit in the proof of Proposition 5.2. Explicitly, all of the following steps are combinatorial:

\begin{itemize}
  \item determining that the Ziegler multi-restriction has the multiplicity \([n, n, n, 1, 1, 1]\) for \( n > 1 \) (by Theorem 4.2)
  \item determining local freeness of \( \mathcal{A} \) along \( w = 0 \) (by Lemma 5.1), which implies \( \alpha \) is a root of unity
  \item determining that \( n = mt \) for some \( t \), where \( m \) is the order of \( \alpha \) (by Theorem 4.2)
  \item Finally, determining that \( \mathcal{A} \) is free by Yoshinaga’s criterion [20] and Theorem 4.2
\end{itemize}

\( \square \)

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