STRUCTURE THEORY OF GRADED REGULAR GRADED SELF-INJECTIVE RINGS AND APPLICATIONS

ROOZBEH HAZRAT, KULUMANI M. RANGASWAMY, AND ASHISH K. SRIVASTAVA

To S. K. Jain on his 80th birthday

Abstract. In this paper, we develop structure theory for graded regular graded self-injective rings and apply it in the context of Leavitt path algebras. We show that for a finite graph, graded regular graded self-injective Leavitt path algebras are of graded type I and these are precisely graded \(\Sigma\)-YLeavitt path algebras.

1. Introduction

This is a semi-expository article illustrating how the type theory to study the structure of self-injective von Neumann regular rings with identity can be extended to the case of group graded rings. This is accomplished by adapting some of the ideas of Goodearl [2] to the graded situation. As an application, we discuss the type theory of unital Leavitt path algebras over a field.

Kaplansky developed a classification for Baer rings in [7]. Since (von Neumann) regular right self-injective rings are Baer rings, this classification theory of Baer rings applies to them. Let \(R\) be a regular right self-injective ring. Then the ring \(R\) is said to be of Type I provided it contains a faithful abelian idempotent. The ring \(R\) is said to be of Type II provided \(R\) contains a faithful directly finite idempotent but \(R\) contains no nonzero abelian idempotents and the ring \(R\) is said to be of Type III if it contains no nonzero directly finite idempotents.

A regular right self-injective ring \(R\) is said to be of (i) Type \(I_f\) if \(R\) is of Type I and is directly-finite, (ii) Type \(I_{\infty}\) if \(R\) is of Type I and is purely infinite (that is, \(R_R \cong R_R \oplus R_R\)), (iii) Type \(II_f\) if \(R\) is of Type II and is directly-finite, (iv) Type \(II_{\infty}\) if \(R\) is of Type II and is purely infinite.

It is known that [2, Theorem 10.13], if \(R\) is a regular right self-injective ring, then there is a unique decomposition \(R = R_1 \times R_2 \times R_3\), where \(R_1\) is of Type I, \(R_2\) is of Type II, and \(R_3\) is of Type III. This is a special case of Kaplansky’s decomposition theorem for Baer rings [7, Theorem 11] as developed by Goodearl in the setting of regular self-injective rings.

Our motivation here to develop this theory in the graded setting was the result of [3] which shows that all Leavitt path algebras are graded regular and [5] where the graded self-injective Leavitt path algebras were studied. Thus one would like to know how these algebras decompose to different graded types.

In this paper we first obtain a graded version of the decomposition of a regular ring into different types, by suitably modifying the arguments from [2]. One requires to take care of grading along the way and this forces some subtle changes to the details of the proofs from nongraded version to the graded setting (see the introduction to Section \(\S2\)). We then finish the paper by applying these results in the context of

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Leavitt path algebras. In particular, we provide a characterization for when Leavitt path algebra over a finite graph $E$ is of graded Type I (which immediately shows they are in fact of the Type $I_f$).

Unless otherwise stated all the rings that we consider possess a multiplicative identity and all the modules are (unitary) right modules. Also, by a self-injective ring, we mean a right self-injective ring.

2. Structure theory of graded regular graded self-injective rings

The aim of this section is to develop a structure theory for graded von Neumann regular graded self-injective rings. The class of von Neumann regular rings (regular for short) constitutes a large class and from the categorical viewpoint they are a natural generalization of the basic building blocks of ring theory, i.e., simple rings. For an associative ring with identity if all modules are free or projective, then the ring is a division ring or a semi-simple ring, respectively. However if all modules are flat then the ring is regular. There is an element-wise definition for such rings; any element has an “inner inverse”, i.e., for an element $a$, there is $b$ such that $aba = a$. Such rings have very rich structures, and Ken Goodearl has devoted an entire book on them [2]. In the case of regular rings which are self injective, there is a structure theory developed which shows that the ring can be decomposed uniquely into three types based on the behavior of idempotent elements [2, §10].

For the class of graded rings, one can define the notion of graded von Neumann regular rings (graded regular for short) as rings in which any homogeneous element has an inner inverse. There are many interesting class of rings which are not regular but are graded regular. One such class is the Leavitt path algebras [3].

In order to carry over the structure theory of regular self injective rings from nongraded setting to the graded case one needs to carefully analyze the behavior of the homogeneous components throughout the proofs, as in the graded setting, suspensions of the modules would come into consideration. A prototype example of regular self injective rings is the ring of column finite matrices over a division ring. This would not readily generalize to the graded setting; the column finite matrices over a graded division rings is not necessarily graded. Throughout this section our rings have identity element.

We begin with some basics of graded rings and graded modules that we will need throughout this paper. Let $\Gamma$ be an additive abelian group and $R$ a unital $\Gamma$-graded ring. Namely, $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$, where $R_\gamma$ are additive subgroups of $R$ and $R_\gamma R_\beta \subseteq R_{\gamma + \beta}$, for $\gamma, \beta \in \Gamma$. We call the elements of components $R_\gamma$, $\gamma \in \Gamma$, homogenous elements. We refer the reader to [4, §1.2.3] for the basics on graded rings. Let $M$ and $N$ be graded right $R$-modules. Consider the hom-group $\text{Hom}_R(M, N)$. One can show that if $M$ is finitely generated or $\Gamma$ is a finite group, then

$$\text{Hom}_R(M, N) = \bigoplus_{\gamma \in \Gamma} \text{Hom}(M, N)_\gamma,$$

where $\text{Hom}(M, N)_\gamma = \{f : M \rightarrow N : f(M_\alpha) \subseteq N_{\alpha + \gamma}, \alpha \in \Gamma\}$ (see [4, §1.2.3]). However if $M$ is not finitely generated, then throughout we will work with the group

$$\text{HOM}_R(M, N) := \bigoplus_{\gamma \in \Gamma} \text{Hom}(M, N)_\gamma,$$

and consequently the $\Gamma$-graded ring

$$\text{END}_R(M) := \text{HOM}_R(M, M). \quad (1)$$

If $f \in \text{END}_R(M)$ then $f = \sum_{\gamma \in \Gamma} f_\gamma$, where $f_\gamma \in \text{Hom}(M, M)_\gamma$ and we write $\text{supp}(f) = \{\gamma \in \Gamma : f_\gamma \neq 0\}$. This is one point of departure from nongraded setting where we should be working with $\text{End}_R(M)$.

Let $R$ be a $\Gamma$-graded ring and $(\delta_1, \cdots, \delta_n)$ an $n$-tuple where $\delta_i \in \Gamma$. Then $M_n(R)$ is a $\Gamma$-graded ring and, for each $\lambda \in \Gamma$, its $\lambda$-homogeneous component consists of $n \times n$ matrices
We say

\[ eR = \sum_{i=0}^{n} e_i R_i \]

where \( e_i \) is a matrix with \( x \) in the \( ij \)-position and with every other entry 0.

Let \( R \) be a \( \Gamma \)-graded ring and \( A \), a graded right \( R \)-module. Recall that for any \( \alpha \in \Gamma \), the \( \alpha \)-suspension of the module \( A \) is defined as \( A(\alpha) = \bigoplus_{\gamma \in \Gamma} A(\alpha)_{\gamma} \), where \( A(\alpha)_{\gamma} = A_{\alpha+\gamma} \).

**Definition 1.** Let \( R \) be a \( \Gamma \)-graded ring and \( A \), a graded right \( R \)-module. We say \( A \) is graded directly-finite if \( A \cong_{gr} A(\alpha) \oplus C \), for some \( \alpha \in \Gamma \), then \( C = 0 \).

A ring \( R \) with identity 1 is said to be directly-finite if for any two elements \( x, y \in R \), \( xy = 1 \) implies \( yx = 1 \).

A \( \Gamma \)-graded ring \( R \) with identity is called graded directly-finite if for any two homogeneous elements \( x, y \in R \) we have that \( xy = 1 \) implies \( yx = 1 \). Clearly here if \( x \in R_\alpha \) then \( y \in R_{-\alpha} \).

**Proposition 2.** [6, Proposition 3.2] Let \( R \) be a \( \Gamma \)-graded ring and \( A \) a graded right \( R \)-module. Then \( A \) is a graded directly-finite \( R \)-module if and only if \( \text{END}_R(A) \) is a graded directly-finite ring.

In the structure theory of self-injective rings various types of idempotents play important roles. We start with giving the graded version of these definitions.

**Definition 3.** Let \( R = \bigoplus_{\gamma \in \Gamma} R_\gamma \) be a \( \Gamma \)-graded ring. Denote by \( I^{gr}(R) \) the set of homogeneous idempotents and by \( B^{gr}(R) \) the set of central homogeneous idempotents.

1. We say \( e \in I^{gr}(R) \) is graded abelian idempotent if \( I^{gr}(eRe) = B^{gr}(eRe) \). We say \( R \) is graded abelian if 1 is a graded abelian idempotent, i.e., \( I^{gr}(R) = B^{gr}(R) \).

2. We say \( e \in I^{gr}(R) \) is graded directly-finite if \( eRe \) is a graded directly-finite ring, i.e., if \( x, y \in (eRe)^h \) such that \( xy = e \) then \( yx = e \). Equivalently, \( eR \) is not graded isomorphic to a proper graded direct summand of \( eR \).

3. We say \( e \in I^{gr}(R) \) is graded faithful if \( y \in B^{gr}(R) \) such that \( ey = 0 \) then \( y = 0 \).

**Definition 4.** Let \( R \) be a \( \Gamma \)-graded von Neumann regular, right self-injective ring.

1. We say \( R \) is of \( gr \)-Type I, if \( R \) contains a graded faithful abelian idempotent.

2. We say \( R \) is of \( gr \)-Type II, if \( R \) contains a graded faithful directly finite idempotent but contains no nonzero graded abelian idempotent.

3. We say \( R \) is of \( gr \)-Type III, if \( R \) contains no nonzero graded directly finite idempotent.

Recall that a \( \Gamma \)-graded ring \( R \) is strongly graded if \( R_\alpha R_\beta = R_{\alpha+\beta} \), for every \( \alpha, \beta \in \Gamma \) (see [4]).

**Lemma 5.** If \( R \) is a \( \Gamma \)-graded regular self-injective ring then \( R_0 \) is a regular self-injective ring. Furthermore if \( R \) is strongly graded, the converse also holds.

**Proof.** If \( R \) is graded regular, clearly \( R_0 \) is regular and consequently \( R \) is flat over \( R_0 \). Let \( I \) be a right ideal of \( R_0 \). Tensoring any map \( f : I \to R_0 \) with \( R \), thanks to \( R \) being graded self-injective, \( f \otimes 1 \) lifts to a graded homomorphism \( \tilde{f} : R \to R \) and its zero component \( \tilde{f}_0 \in \text{Hom}(R_0, R_0) \) extends \( f \). Thus \( R_0 \) is self-injective as well.

If \( R \) is strongly graded, by Dade’s theorem \( Gr-R \) is Morita equivalent to \( \text{Mod-}R_0 \) ([4, §1.5]). Thus any diagram can be completed via going to \( \text{Mod-}R_0 \).
Lemma 6. If a graded regular self-injective ring $R$ has graded type I (type II), then $R_0$ also has type I (type II).

Proof. Suppose $R$ is a $\Gamma$-graded regular self-injective ring. There is a bijection between the right (left) ideals of $R_0$ and graded right (left) ideals of $R$ which sends any right ideal $A$ of $R_0$ to the graded right ideal $AR$. Moreover $(AR)_0 = A$. From this and [2, Propositions 10.4 (b) and 10.8 (b)], we can conclude that if $R$ has graded type I (type II), then $R_0$ also has type I (type II). $\square$

Theorem 7. Let $R$ be a $\Gamma$-graded regular ring and $A$ be a graded projective right $R$-module. Then any graded finitely generated submodule $B$ of $A$ is a direct summand of $A$. In particular $B$ is graded projective.

Proof. Since $A$ is graded projective, there is a graded module $C$ such that $A \oplus C \cong_{gr} F$, where $F \cong_{gr} \bigoplus_{i \in I} R(\alpha_i)$, is a graded free $R$-module. Consider $A$ as a graded submodule of $F$. Since $B \leq_{gr} A$ is finitely generated, there is a graded finitely generated free $R$-module $G$ which is a direct summand of $F$ and contains $B$.

Suppose $B$ is generated by $n$ elements $b_i$ of degree $\alpha_i$, $1 \leq i \leq n$, respectively, and consider a graded free module $H$ containing $G$ which has a graded basis of at least $n$ elements $h_i$ of degrees $\alpha_i$’s. Sending $h_i \mapsto b_i$, $1 \leq i \leq n$, and the rest of the basis to zero, will define a graded homomorphism of degree zero $f : H \to H$ such that $f(H) = B$. Since $\text{End}_R(H)$ is graded von Neumann regular, there is $g \in \text{End}_R(H)_0$ such that $fgf = f$. Thus $fg$ is a homogeneous idempotent and $fgf(H) = f(H) = B$. Thus $H = f(g(H) \bigoplus (1 - fg)(H)) = B \bigoplus (1 - fg)(H)$.

and so $B$ is a direct summand of $H$. (Note that no suspension of $B$ is required as $f$ and $g$ are of degree zero.) To finish the proof, we use the following fact twice: if $N \leq M \leq F$ and $N$ and $M$ are direct summand of $F$, then $N$ is a direct summand of $M$. $\square$

Recall that a graded $R$-module $A$ is called graded quasi-injective if for any graded $R$-submodule $M$, and graded homomorphism $f$, there is a graded homomorphism $g$ such that one can complete the following diagram.

\[
\begin{array}{ccc}
M & \xrightarrow{i} & A \\
\downarrow{f} & & \downarrow{g} \\
A & & \\
\end{array}
\]

Our next theorem extends [2, Theorem 1.22] to the graded setting. Since in the following Theorem, the $R$-module $A$ is not necessarily finitely generated, we need to work with the graded ring $\text{END}_R(A)$ (see (1)) as apposed to the nongraded case which we consider the ring $\text{End}_R(A)$. Throughout the note, we denote the graded Jacobson radical of a $\Gamma$-graded ring $R$ by $J^{gr}(R)$. Recall also that for $r \in R$, writing $r = \sum_{\alpha \in \Gamma} r_{\alpha}$, we denote by supp$(r) := \{r_{\alpha} \mid r_{\alpha} \neq 0\}$.

Theorem 8. Let $R$ be a $\Gamma$-graded ring, and $A$ be a graded quasi-injective right $R$-module. Let $Q = \text{END}_R(A)$. Then

(a) $J^{gr}(Q) = \{f \in \text{END}_R(A) \mid \ker f_{\alpha} \leq_{gr} A, \text{ for all } \alpha \in \text{supp}(f)\}$.
(b) $Q/J^{gr}(Q)$ is graded regular.
(c) If $J^{gr}(Q) = 0$ then $Q$ is graded right self-injective.

Proof. (a) and (b) Let

\[
K = \{f \in \text{END}_R(A) \mid \ker f_{\alpha} \leq_{gr} A, \text{ for all } \alpha \in \text{supp}(f)\}.
\]

One checks that $K$ is a graded two sided ideal of $A$. We first show that $K \subseteq J^{gr}(Q)$. It is enough to show that $K^h \subseteq J^{gr}(Q)$. Let $f \in K^h$, i.e., $f \in \text{Hom}(A, A)_{\alpha}$, for some $\alpha \in \Gamma$ and $\ker f \leq_{gr} A$. We show that for any $r \in \text{Hom}(A, A)_{-\alpha}$, $1 - rf$ is invertible. We have $\ker(1 - rf) \cap \ker f = 0$. Since $\ker f$ is essential, and
ker(1 − rf) is a graded right ideal of A, it follows that ker(1 − rf) = 0. Thus θ := 1 − rf : A → (1 − rf)A is graded isomorphism. Since A is graded quasi-injective, there is a graded homomorphism g which completes the following diagram.

\[
\begin{array}{ccc}
(1 − rf)A & \xrightarrow{\theta^{-1}} & A \\
\downarrow{g} & & \downarrow{g} \\
A & & A
\end{array}
\]

It follows that g(1 − rf) = 1. So f ∈ J^gr(Q) and consequently K ⊆ J^gr(Q). Next we show that Q/K is a graded regular ring. Let f ∈ Q^h, i.e., f ∈ Hom(A, A) for some α ∈ Γ. Set S = \{N ≤^gr A | N ∩ ker f = 0\}. This is a poset in which each chain has an upper bound. Thus by Zorn’s lemma S has a maximal element B. It follows that B ⊕ ker f ≤^gr A. Since θ = f : B → f(B) is graded isomorphism, and A is graded quasi-injective, there is graded homomorphism g such that

\[
\begin{array}{ccc}
f(B) & \xrightarrow{\theta^{-1}} & A \\
\downarrow{g} & & \downarrow{g} \\
A & & A
\end{array}
\]

It follows that gf = 1 on B. Thus (fgf − f)B = 0. So B ⊕ ker f ≤ ker(fgf − f) and therefore fgf − f ∈ Hom(A, A) and ker(fgf − f) ≤^gr A. So fgf − f ∈ K. Thus in Q/K we have \(\overline{fgf} = \overline{f}\), i.e., Q/K is graded regular. So J^gr(Q/K) = 0, which gives that J^gr(Q) = K. So Q/J^gr(Q) is graded regular. This gives (a) and (b).

(c) Suppose J^gr(Q) = 0. Then Q is graded regular and A is the graded left Q-module. So A is (graded) flat over Q. Suppose J is a graded right ideal of Q and f : J → Q a graded Q-module homomorphism. We will show that there is a h ∈ Hom(A, A) such that one can complete the following diagram and thus Q is self-injective.

\[
\begin{array}{ccc}
J & \xrightarrow{f} & Q \\
\downarrow{h} & & \downarrow{h} \\
Q & & Q
\end{array}
\]

(4)

We have the following commutative diagram of graded maps.

\[
\begin{array}{ccc}
J ⊗_Q A & \xrightarrow{f ⊗ 1} & Q ⊗_Q A \\
\cong & & \cong \\
J A & \xrightarrow{g} & A
\end{array}
\]

Since A is quasi-injective the graded homomorphism q extends to a h ∈ Hom(A, A) such that h(xy) = g(x)y = f(x)y for any x ∈ J and y ∈ A. Thus we have f(x) = h(x) for any x ∈ J. This completes Diagram 4 and thus the proof.

**Corollary 9.** Let R be a graded ring, A a graded right R-module and Q = END_R(A). If A is graded semi-simple or graded non-singular quasi-injective, then Q is graded regular self-injective ring.

**Proof.** We show that J^gr(Q) = 0 and the corollary follows from Theorem 8. If f ∈ J^gr(Q) then ker f ≤^gr A for all α ∈ supp(f). If A is graded semi-simple, then f is homogeneous and ker f = A and so f = 0. If A is graded non-singular, then f_α : A → A(−α) is a graded homomorphism. Thus A/ker f ∼ A(−α). If A is non-singular then A(−α) is non-singular as well and so f = 0. Thus J^gr(Q) = 0.

The corollary below follows immediately from Corollary 9 and it gives a prototype example of graded regular self-injective rings.
Corollary 10. Let $R$ be a graded division ring and $A$ be a graded right $R$-module. Then $\text{END}_R(A)$ is a graded regular self-injective ring.

Corollary 11. Let $R$ be a graded ring such that $R$ is graded right nonsingular. Then its maximal graded right quotient ring $Q^\gr(R)$ is graded regular and graded right self-injective.

Since the Leavitt path algebra $L_K(E)$ associated to an arbitrary graph $E$ is graded regular, it is graded nonsingular. Consequently, its maximal graded right quotient ring is graded regular and graded right self-injective.

Since for any central homogeneous idempotent $e$, $eR(1-e)R = 0$, if $R$ is graded prime, it follows that $B^\gr(R) = \{0,1\}$. If $R$ is, in addition, graded regular and abelian, i.e., $I^\gr(R) = B^\gr(R)$, then for any $0 \neq x \in R^h$, there is $y \in R^h$ such that $xyx = x$, so $xy \in B^\gr(R)$, and thus $xy = 1$. It follows that $R$ is a graded division ring. The following proposition shows that if a corner of $R$ is abelian, then $R$ is graded isomorphic to matrices over a graded division ring.

Theorem 12. Let $R$ be a $\Gamma$-graded ring. Then $R \cong_{gr} \text{END}_D(A)$, where $D$ is a graded division ring and $A$ is a graded $D$-module if and only if $R$ is graded prime, graded regular, self-injective ring with $\text{soc}^\gr(R) \neq 0$.

Proof. Suppose $R \cong_{gr} \text{END}_D(A)$, where $D$ is a graded division ring and $A$ is a graded $D$-module. Then by Corollary 9, $R$ is graded regular, self-injective. For $0 \neq x, y \in \text{END}_D(A)^h$, there are homogeneous elements $a, b \in A$ such that $x(a) \neq 0$ and $y(b) \neq 0$. Since $y(b)$ is homogeneous, one can extend $\{y(b)\}$ to a graded basis for $A$ [4, §1.4]. Thus there is a $z \in \text{END}_D(A)^h$ such that $z(y(b)) = a$. It follows that $x(z(y(b))) = x(a) \neq 0$. Thus $xy \neq 0$. This shows that $R$ is a graded prime ring. Furthermore, one can choose a homogeneous element $e \in R = \text{END}_D(A)$ such that $eRe \cong_{gr} D$. Thus $\text{End}_{eRe}(eRe) \cong_{gr} eRe \cong_{gr} D$ and the fact that $R$ is graded prime, implies that $eR$ is graded simple. Thus $\text{soc}^\gr(R) \neq 0$.

For the converse of the theorem, since $\text{soc}^\gr(R) \neq 0$, one can choose a homogeneous idempotent $e \in R$ such that $eR$ is graded right simple. Thus $eRe \cong_{gr} \text{End}_R(eRe)$ is a graded division ring. Since $R$ is graded prime, $B^\gr(R) = \{0,1\}$. Thus by the graded version of [2, Theorem 9.8], the graded ring homomorphism $R \rightarrow \text{END}_{eRe}(Re)$ is an isomorphism.

Proposition 13. Let $R$ be a $\Gamma$-graded prime, regular and right self-injective ring. Then $R$ is $gr$-Type I if and only if $R$ is graded isomorphic to $\text{END}_D(A)$, where $D$ is a graded division ring and $A$ is a graded right $D$-module.

Proof. Suppose $R \cong_{gr} \text{END}_D(A)$, where $D$ is a graded division ring and $A$ is a graded $D$-module. Then $\text{soc}^\gr(R) \neq 0$ by Theorem 12. Thus there is a homogeneous idempotent $e \in R$ such that $eR$ is graded right simple $R$-module. Thus $\text{End}_R(eRe, eR) \cong eRe$ is a graded division ring. Thus $e$ is graded abelian idempotent. Since $R$ is graded prime, $B^\gr(R) = \{0,1\}$. Hence $e$ is faithful as well, and so $R$ is $gr$-Type I.

For the converse of the theorem, suppose $R$ is $gr$-Type I. Thus there is homogeneous idempotent $e \in R$ which is faithful and abelian. Hence, $eRe$ is graded prime, regular and abelian. It follows that $R$ is a graded division ring (see the argument before Theorem 12) and consequently, we have that $eR$ is graded simple. Thus $\text{soc}^\gr(R) \neq 0$. Now Theorem 12 implies that $R \cong_{gr} \text{END}_D(A)$.

Proposition 14. Let $R$ be a graded regular self-injective ring and $A$ be a graded right ideal of $R$. Then there is a unique graded direct summand $B$ of $R$ such that $A \leq_{gr} B$.

Proof. Let $B$ be the graded injective envelope of $A$. Then clearly $B$ is a graded direct summand of $R$ and $A \leq_{gr} B$. Note that $B/A$ is graded singular and $R/B$ is graded nonsingular and hence $B/A$ equals the graded singular submodule of $R/A$. If $B'$ is any graded direct summand of $R$ with $A \leq_{gr} B'$, then $B'/A$ also equals the graded singular submodule of $R/A$. This shows $B = B'$.

Proposition 15. Let $J$ be a graded two-sided ideal of $R$.

(a) There is a unique homogeneous idempotent $e \in B^\gr(R)$ such that $J_R \leq_{gr} eR_R$.  


(b) If $R/J$ is graded non-singular, then $J = eR$.

Proof. The following proof is essentially Goodearl’s proof in [2] with minor modifications.

(a) By Proposition 14, $J \leq_{gr} D$, a graded direct summand of $R_R$ which is unique. Now $D = eR$ where $e$ is a homogeneous idempotent of degree 0. (Note that $e = \pi(1)$, where $\pi : R \to D$ is the graded coordinate projection. So $\pi(1)$ is homogeneous, as 1 is homogeneous of degree 0. Since $\pi(1)$ is also an idempotent, it is of degree 0). Given any homogeneous element $x \in R$, we have $xJ \leq_{gr} J_R \leq_{gr} eR$ and so $(1 - e)xJ = 0$. Then the left multiplication by the homogeneous element $(1 - e)x$ induces a graded morphism from the graded singular module $eR/J$ to $(1 - e)xeR$. So $(1 - e)xeR$ is a graded singular submodule of $(1 - e)R$. As $(1 - e)R$ is graded non-singular, $(1 - e)xeR = 0$. This holds for all homogeneous elements $x$ and, consequently, $(1 - e)ReR = 0$ and so $(1 - e)Re = 0$. This implies $(eR(1 - e))^2 = 0$ and so $(eR(1 - e))R = 0$ from which we obtain $eR(1 - e) = 0$. So the Pierce decomposition of $R$ corresponding to $e$ becomes $R = eRe + (1 - e)R(1 - e)$. Writing each element $a \in R$ in terms of the last equation, it is clear that $e$ commutes with $a$ and so $e \in B^{gr}(R)$ and $J \leq_{gr} eR$, as desired. If there is another homogeneous idempotent $f \in B(R)$ such that $J \leq_{gr} fR$, then by the uniqueness of $D$, $eR = D = fR$ which implies $e = f$. Hence $e$ is unique.

(b) Now $eR/J$ is graded singular and so if $R/J$ is graded non-singular, then $eR/J = 0$ Hence $J = eR$, where $e \in B^{gr}(R)$.

Definition 16. A graded right $R$-module $M$ is said to be graded faithful if for each homogeneous element $r \in R$, there is a homogeneous element $a \in M$ such that $ar \neq 0$. If a graded right $R$-module is graded faithful, it is also faithful.

We give below the description of graded faithful idempotents.

Lemma 17. Let $R$ be a graded regular self-injective ring. The following properties are equivalent for a homogeneous idempotent $e \in R$.

(a) $e$ is graded faithful;
(b) $(Re)_R \leq_{gr} R_R$;
(c) $Re$ is a graded left faithful ideal;
(d) $eR$ is a graded right faithful ideal;
(e) $\text{HOM}(eR, J) \neq 0$ for all non-zero graded right ideals $J$ of $R$.

Proof. The proof similar to the one in [2], after minor modification to handle the graded version, works here. For the sake of completeness, we outline the proof.

Assume (a). By Proposition 15, $(Re)_R \leq_{gr} (fR)_R$ for some homogeneous idempotent $f \in B^{gr}(R)$. Now $(1 - f)e = 0$ and so $1 - f = 0$. Consequently, $(Re)_R \leq_{gr} R_R$, thus proving (b).

Assume (b). Now $R$ is graded non-singular. Since $(Re)_R$ is graded essential right ideal, it is then clear that for any homogeneous element $x \in R$, $x(ReR) \neq 0$. In particular, $xRe \neq 0$ and it is then clear there is a homogeneous element $a \in Re$ such that $xa \neq 0$.

Assume (c). Let

$$K = \{x \in R \mid x = x_{i_1} + \cdots + x_{i_n} \text{ is the homogeneous decomposition of } x \text{ and } eRx_{i_k} = 0 \text{ for all } k = 1, \cdots, n\}.$$  \hspace{1cm} (5)

It is easy to see that $K$ is a graded left ideal of $R$ and $eK = 0$. So $(KR)e = KR(eK)Re = 0$. Consequently, $KR = 0$. By (c), $K = 0$. Hence $eR$ is faithful, thus proving (d).

Assume (d). Let $J$ be any graded right ideal and let $0 \neq x \in J$ be a homogeneous element. Since $eR$ is graded faithful, there is a homogeneous element $y \in eR$ such that $yx \neq 0$. Thus $yxR$ is a non-zero graded projective (principal) right ideal of $R$. Then the graded epimorphism $xR \to yxR$ given by $xa \mapsto yxa$ splits.
with the split map \( \phi : yxR \to xR \). Since \( yxR \) is a graded direct summand, \( \phi \) extends to a non-zero graded morphism \( \theta : eR \to J \). Hence \( \text{HOM}(eR, J) \neq 0 \), thus proving (e).

Assume (e). Suppose \( f \in B(R) \) is a homogeneous central idempotent such that \( ef = 0 \). Then for any graded morphism \( \theta : eR \to fR \), we have \( \theta(ef) = \theta(e)a = fba = bfe = 0 \). This contradicts (d), as \( \text{HOM}(eR, fR) \neq 0 \) for non-zero \( fR \). Hence \( f = 0 \) and we conclude that \( e \) is a graded faithful idempotent. This proves (a).

**Theorem 18.** For a graded regular graded right self-injective ring, the following are equivalent.

(a) \( R \) has type I;

(b) Every non-zero graded right ideal of \( R \) contains a non-zero graded abelian idempotent;

(c) The graded two-sided ideal generated by all the graded abelian idempotents in \( R \) is graded essential as a graded right ideal.

**Proof.** Assume (a), so \( R \) contains a non-zero graded faithful abelian idempotent \( e \). Let \( J \) be a non-zero graded right ideal of \( R \). By Lemma 17, there is a non-zero graded morphism \( \theta : eR \to J \).

Now \( J' = \theta(eR) = \theta(e)R \subseteq J \) is a graded principal right ideal of \( R \) and so \( J' = fR \) for some homogeneous idempotent \( f \). Since \( fR \) is graded projective, there is a graded morphism \( \phi : fR \to eR \) such that \( \theta \phi = 1_{fR} \). Then \( g = \phi \theta \in \text{End}(eR) = eRe \) and \( g = ege \) is a homogeneous idempotent such that \( gR \cong_{gr} fR \). In particular, \( eR = (\ker \theta) \oplus gR \). Consequently, \( fR \neq 0 \) and we conclude that \( e \) is graded faithful. Hence \( \phi = 1 \).

Assume (b). Let \( X = \{ \text{homogeneous idempotents } e \in B(R) \text{ such that } eR \text{ is type I} \} \).

Note that \( X \subseteq B^{gr}(R) \cap R_0 \subseteq B(R_0) \) and, by Lemma 5 and [2, Proposition 9.9], \( B(R_0) \) is a complete lattice. Let \( \text{sup}(X) = f \in B(B_0) \). We claim \( f = 1 \). Suppose \( f \neq 1 \). Then \( 1 - f \in R_0 \) is a homogeneous idempotent and, by (b), \((1 - f)R \) contains a homogeneous abelian idempotent \( g \). By Proposition 15, \( RgR \leq_{gr} eR \) for a unique homogeneous \( e \in B^{gr}(R) \). We then appeal to Lemma 17 to conclude that \( g \) is faithful in the ring \( eR \). Hence \( eR \) is of type I. Thus \( e \in X \) and so \( e \leq f \), that is \( e = fe \). Hence \( g \in fR \). But \( g \in (1 - f)R \) and so \( g = 0 \), a contradiction. Hence \( f = 1 \).

Let \( Y \subseteq X \subseteq B(R_0) \) be a maximal family of mutually orthogonal homogeneous idempotents. Let \( \epsilon = \text{sup}(Y) \). Now \( \epsilon = 1 \), since otherwise \( Y \cup \{1 - \epsilon\} \) will contradict the maximality of \( X \). Write \( Y = \{ e_i : i \in I \} \). Since \( e_i \in X \), \( e_iR \) contains a homogeneous faithful abelian idempotent \( h_i \). Then \( h = (\cdots, h_i, \cdots) \in S = \prod_{i \in I} e_iR \) is an idempotent which is clearly faithful. Also if \( \epsilon = (\cdots, e_i, \cdots) \) is a homogeneous idempotent in \( h(\prod_{i \in I} e_iR)h = \prod_{i \in I} h_i(e_iR)h_i \), then since for each \( i \), \( e_i \) is central in \( h_i(e_iR)h_i \), \( \epsilon \) is central in \( h(\prod_{i \in I} e_iR)h \). Thus \( h \) is abelian and so \( \prod_{i \in I} e_iR \) is type I. Since \( \text{sup}\{e_i : i \in I\} = 1 \), we appeal to the graded version of [2, Proposition 9.10] to conclude that the natural ring map \( R \to \prod_{i \in I} e_iR \) is a graded isomorphism. Hence \( R \) is of type I, thus proving (a).

Assume (c). Then the graded two-sided ideal \( T \) generated by all the graded abelian idempotents in \( R \) is graded essential as a graded right ideal in \( R \). Let \( x \) be a non-zero graded homogeneous element in \( R \). Since \( R \) is graded non-singular, \( XT \neq 0 \). This implies that \( xre \neq 0 \) for some graded abelian idempotent and some homogeneous element \( r \in R \). (because, \( xre = 0 \) for all such \( e \) and for all homogeneous \( r \in R \) will imply that \( xa = 0 \) for all \( a \in T \)). Then the graded cyclic right ideal \( xreR = fR \) for some homogeneous idempotent \( f \in xR \). Now the map \( \theta : ey \mapsto xrey \) is a graded non-zero morphism from \( eR \) to \( xR \) with image \( xreR = fR \) and since \( fR \) is graded projective, there is a graded morphism \( \phi : fR \to eR \) such that \( \theta \phi = 1_{fR} \). Then proceeding as in the proof of (a) \( \Rightarrow \) (b), \( g = \phi \theta \) is a homogeneous idempotent in \( eRe \) with
by using the fact that if $fRf \cong_{gr} gRg \subseteq eRe$. This shows that $f \in xR$ is a homogeneous abelian idempotent. This proves (b).

Now (b) obviously implies (c).

**Theorem 19.** Let $R$ be a graded regular graded right self-injective ring. Then the following properties are equivalent:

(a) $R$ has type II;
(b) Every non-zero graded right ideal of $R$ contains a non-zero graded directly finite idempotent;
(c) The graded two-sided ideal generated by all the graded directly finite idempotents of $R$ is graded essential as a right ideal of $R$.

**Proof.** The proof follows the same line of the proof of Theorem 18 by using the fact that if $e$ is a graded directly finite idempotent, then any homogeneous idempotent $f \in eRe$ is also graded directly finite.

**Theorem 20.** Let $R$ be a $\Gamma$-graded regular graded right self-injective ring. Then $R \cong_{gr} R_1 \times R_2 \times R_3$, where $R_1, R_2, R_3$ are graded rings of gr-Type I, II and III, respectively. Furthermore, this decomposition is unique.

**Proof.** Let $X$ be the set of all graded abelian idempotents in $R$. By Proposition 15, there is a unique homogeneous idempotent $e_1 \in B^{gr}(R)$ such that $(RXR)_R \leq_{gr} e_1 R$. Theorem 18 implies that the graded regular graded right self-injective ring $e_1 R$ is of type I. Now the ring $(1-e_1)R$ has no non-zero abelian idempotents since $(1-e_1)R \cap X = \emptyset$. Let $Y$ denote the set of all graded directly finite idempotents in $(1-e_1)R$. Again, by Proposition 15, there exists a unique homogeneous idempotent $e_2 \in B^{gr}((1-e_1)R)$ such that $(RYR)_R \leq_{gr} e_2 R$. Theorem 19 implies that the ring $e_2 R$ is of type II. Let $e_3 = 1-e_1-e_2$. Since $e_3 R \cap Y = \emptyset$, the ring $e_3 R$ contains no non-zero directly finite graded idempotents and consequently, $e_3 R$ is a graded regular graded right self-injective ring of type III. Clearly $e_1, e_2, e_3$ are homogeneous orthogonal central idempotents with $e_1 + e_2 + e_3 = 1$. Consequently, $R = e_1 R \oplus e_2 R \oplus e_3 R$.

To prove the uniqueness of the decomposition, suppose $f_1, f_2, f_3$ are central orthogonal idempotents such that $f_1 + f_2 + f_3 = 1$ and that $f_1 R, f_2 R, f_3 R$ are of type I, II, III respectively. Now $e_1 = e_1 f_1 + e_1 f_2 + e_1 f_3$. Here $e_1 f_2 \in e_1 R e_1 \cap f_2 R f_2 = 0$ and $e_1 f_3 \in e_1 R e_1 \cap f_3 R f_3 = 0$. Consequently, $e_1 = e_1 f_1 \in f_1 R$ and so $e_1 R \subseteq f_1 R$. Similarly, $f_1 R \subseteq e_1 R$ and so $e_1 R = f_1 R$. In the same way, $e_2 R = f_2 R$ and $e_3 R = f_3 R$. This finishes the proof.

## 3. Applications to Leavitt path algebras

In this section we apply the results from the previous section to the case of Leavitt path algebras. We first recall the basics of Leavitt path algebras. Let $K$ be a field and $E$ be an arbitrary directed graph. Let $E^0$ be the set of vertices, and $E^1$ be the set of edges of directed graph $E$. Consider two maps $r: E^1 \to E^0$ and $s: E^1 \to E^0$. For any edge $e$ in $E^1$, $s(e)$ is called the source of $e$ and $r(e)$ is called the range of $e$. If $e$ is an edge starting from vertex $v$ and pointing toward vertex $w$, then we imagine an edge starting from vertex $w$ and pointing toward vertex $v$ and call it the ghost edge of $e$ and denote it by $e^*$. We denote by $(E^1)^*$, the set of all ghost edges of directed graph $E$. If $v \in E^0$ does not emit any edges, i.e. $s^{-1}(v) = \emptyset$, then $v$ is called a sink and if $v$ emits an infinite number of edges, i.e. $|s^{-1}(v)| = \infty$, then $v$ is called an infinite emitter. If a vertex $v$ is neither a sink nor an infinite emitter, then $v$ is called a regular vertex.

The **Leavitt path algebra** of $E$ with coefficients in $K$, denoted by $L_K(E)$, is the $K$-algebra generated by the sets $E^0$, $E^1$, and $(E^1)^*$, subject to the following conditions:

(A1) $v_i v_j = \delta_{ij} v_i$ for all $v_i, v_j \in E^0$.
(A2) $s(e)e = e = er(e)$ and $r(e)e^* = e^* = e^* s(e)$ for all $e$ in $E^1$.
(CK1) $e^* e_j = \delta_{ij} r(e_i)$ for all $e_i, e_j \in E^1$.
(CK2) If $v \in E^0$ is any regular vertex, then $v = \sum_{e \in E^1: s(e) = v} ee^*$.
Conditions (CK1) and (CK2) are known as the Cuntz-Krieger relations. If $E^0$ is finite, then $\sum_{v_i \in E^0} v_i$ is an identity for $L_K(E)$ and if $E^0$ is infinite, then $E^0$ generates a set of local units for $L_K(E)$. We refer the reader to [1] for the basics on the theory of Leavitt path algebras.

It is known that the Leavitt path algebra $L_K(E)$ over any graph $E$ is a graded regular ring [3]. As an application of the structure theory of graded regular graded self-injective rings, we have the following in case of unital Leavitt path algebras.

**Theorem 21.** Let $L_K(E)$ be a Leavitt path algebra associated to a finite graph $E$, where $K$ is a field. Then the following are equivalent.

(a) $L_K(E)$ is an algebra of gr-Type I;
(b) $L_K(E)$ is graded self-injective;
(c) No cycle in $E$ has an exit;
(d) $L_K(E)$ is a graded $\Sigma$-V ring;
(e) There is a graded isomorphism

$$L_K(E) \cong_{gr} \bigoplus_{v_i \in X} M_{\Lambda_i}(K)((\mathbb{P}^\mathbb{N})) \oplus \bigoplus_{w_j \in Y} M_{\Upsilon_j}(K[x^{t_j}, x^{-t_j}])((\mathbb{Q}^\mathbb{N}))$$

where $\Lambda_i, \Upsilon_j$ are suitable index sets, the $t_j$ are positive integers, $X$ is the set of sinks in $E$ and $Y$ is the set of all distinct cycles (without exits) in $E$.

**Proof.** The equivalence of (b), (c), (d) and (e) is established in [6, Theorem 4.16]. Now the equivalence of (a) with these conditions follows from Proposition 13. $\square$

In fact the above theorem shows that self-injective unital Leavitt path algebras are of gr-Type $I_f$, i.e., they are of graded Type I and are graded directly finite (see [6]).

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Centre for Research in Mathematics, Western Sydney University, AUSTRALIA

E-mail address: r.hazrat@westernsydney.edu.au

Department of Mathematics, University of Colorado at Colorado Springs, Colorado-80918, USA

E-mail address: kragasw@uccs.edu

Department of Mathematics and Statistics, St. Louis University, St. Louis, MO-63103, USA

E-mail address: ashish.srivastava@slu.edu