PALINDROMIC WIDTH OF FREE NILPOTENT GROUPS

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Abstract. In this paper we consider the palindromic width of free nilpotent groups. In particular, we prove that the palindromic width of a finitely generated free nilpotent group is finite. We also prove that the palindromic width of a free abelian-by-nilpotent group is finite.

1. Introduction

Let \( G \) be a group with a set of generators \( A \). A reduced word in the alphabet \( A^{\pm 1} \) is a palindrome if it reads the same forwards and backwards. The palindromic length \( l_P(g) \) of an element \( g \) in \( G \) is the minimum number \( k \) such that \( g \) can be expressed as a product of \( k \) palindromes. The palindromic width of \( G \) with respect to \( A \) is defined to be \( \text{pw}(G) = \sup_{g \in G} l_P(g) \). In analogy with commutator width of groups (for example see [2, 3, 4, 5]), it is a problem of potential interest to study palindromic width of groups. Palindromes of free groups have already proved useful in studying various aspects of combinatorial group theory, for example see [8, 9, 12]. In [6], it was proved that the palindromic width of a non-abelian free group is infinite. This result was generalized in [7] where the authors proved that almost all free products have infinite palindromic width; the only exception is given by the free product of two cyclic groups of order two, when the palindromic width is two. Piggot [13] studied the relationship between primitive words and palindromes in free groups of rank two. It follows from [6, 13] that up to conjugacy, a primitive word can always be written as either a palindrome or a product of two palindromes and that certain pairs of palindromes will generate the group. Recently Gilman and Keen [10, 11] have used tools from hyperbolic geometry to reprove this result and further have obtained discreteness criteria for two generator subgroups in \( \text{PSL}(2, \mathbb{C}) \) using the geometry of palindromes. The work of Gilman and Keen indicates a deep connection between palindromic width of groups and geometry.

Let \( N_{n,r} \) be the free nilpotent group of rank \( n \) and of step \( r \). In this paper we consider the palindromic width of free nilpotent groups. We prove that the palindromic width of a finitely generated free nilpotent group is finite. In fact, we prove that the palindromic width of an arbitrary rank \( n \) free nilpotent group is bounded by \( 3n \). For the 2-step free nilpotent groups, we improve this bound. For the groups, \( N_{n,1} \) and \( N_{2,2} \) we get the exact values of the palindromic width. Our main theorem is the following.

**Theorem 1.1.** Let \( N_{n,r} \) be the free \( r \)-step nilpotent group of rank \( n \geq 2 \). Then the following holds:

\[ \text{pw}(N_{n,r}) \leq 3n \]

\[ \text{pw}(N_{2,2}) = 3 \]

\[ \text{pw}(N_{n,1}) \text{ is exact for } n \geq 2 \]

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(-1) The palindromic width \( \text{pw}(N_{n,1}) \) of a free abelian group of rank \( n \) is equal to \( n \).
(2) For \( r \geq 2 \), \( n \leq \text{pw}(N_{n,r}) \leq 3n \).
(3) \( \text{pw}(N_{n,2}) \leq 3(n-1) \).

We prove the theorem in Section 3. Along the way, we also prove \( \text{pw}(N_{2,2}) = 3 \). In Section 2 after recalling some basic notions and related basic results, we prove Lemma 2.5 which is a key ingredient in the proof of Theorem 1.1. As a consequence of Lemma 2.5 we also prove that the palindromic width of a free abelian-by-nilpotent group of rank \( n \) is bounded by \( 5n \), see Proposition 3.7. For the group \( N_{3,2} \) it is possible to improve the bound given in (2) of the above theorem. In fact, \( 4 \leq \text{pw}(N_{3,2}) \leq 6 \). A detailed proof of this fact will appear elsewhere. It would be interesting to obtain solutions to the following problems.

**Problem 1.** (1) For \( n \geq 3 \), \( r \geq 2 \), find \( \text{pw}(N_{n,r}) \).
(2) Construct an algorithm that determines \( l_p(g) \) for arbitrary \( g \in N_{n,r} \).

The above problem can be asked for any other groups as well. In general, the palindromic width of an arbitrary group depends on the generating set of the group. However, the advantage of working with free nilpotent groups is that, in this case we have a basis and hence, the palindromic width is the same regardless of the choice of a basis as a generating set.

For \( g, h \) in \( G \), the commutator of \( g \) and \( h \) is defined as \([g, h] = g^{-1}h^{-1}gh\). If \( C \) is the set of commutators in some group \( G \) then the commutator subgroup \( G' \) is generated by \( C \). The length \( l_C(g) \) of an element \( g \in G' \) is called the commutator length. The width \( \text{wid}(G',C) \) is called the commutator width of \( G \) and is denoted by \( \text{cw}(G) \). It is well known [1] that the commutator width of a free non-abelian group is infinite, but the commutator width of a finitely generated nilpotent group is finite (see [3, 4]). An algorithm of the computation of the commutator length in free non-abelian groups can be found in [5]. The following problem is natural to ask.

**Problem 2.** Is it true that for a finitely generated group \( G = \langle A \rangle \), the palindromic width \( \text{pw}(G) \) is finite if and only if the commutator width \( \text{cw}(G) \) is finite?

## 2. Background and Preliminary Results

### 2.1. Background.

2.1.1. **Widths of groups.** Let \( G \) be a group and \( A \subseteq G \) a subset that generates \( G \). For each \( g \in G \) define the length \( l_A(g) \) of \( g \) with respect to \( A \) to be the minimal \( k \) such that \( g \) is a product of \( k \) elements of \( A^{\pm 1} \). The supremum of the values \( l_A(g), g \in G \), is called the width of \( G \) with respect to \( A \) and is denoted by \( \text{wid}(G,A) \). In particular, \( \text{wid}(G,A) \) is either a natural number or \( \infty \). If \( \text{wid}(G,A) \) is a natural number, then every element of \( G \) is a product of at most \( \text{wid}(G,A) \) elements of \( A \).

Let \( A \) be a set of generators of a group \( G \). A reduced word \( w \) in the alphabet \( A^{\pm 1} \) is called a palindrome if \( w \) reads the same left-to-right and right-to-left. An element \( g \) of \( G \) is called a palindrome if \( g \) can be represented by some word \( w \) that is a palindrome in the alphabet \( A^{\pm 1} \). We denote the set of all palindromes in \( G \) by \( \mathcal{P} = \mathcal{P}(G) \). Evidently, the set \( \mathcal{P} \) generates \( G \). Then any element \( g \in G \) is a product of palindromes

\[ g = p_1p_2 \ldots p_k. \]
The minimal \( k \) with this property is called the \textit{palindromic length} of \( g \) and is denoted by \( l_{\mathcal{P}}(g) \). The \textit{palindromic width} of \( G \) is then given by

\[
\text{pw}(G) = \text{wid}(G, \mathcal{P}) = \sup_{g \in G} l_{\mathcal{P}}(g).
\]

2.1.2. \textit{Free Nilpotent Groups}. Let \( N_{n,r} \) be the free \( r \)-step nilpotent group of rank \( n \) with a basis \( x_1, \ldots, x_n \). For example, when \( r = 1 \), \( N_{n,1} \) is simply the free abelian group generated by \( x_1, \ldots, x_n \), so every element of \( N_{n,1} \) can be presented uniquely as

\[
g = x_1^{\alpha_1} \cdots x_n^{\alpha_n}
\]

for some integers \( \alpha_1, \ldots, \alpha_n \). For \( r = 2 \), every element \( g \in N_{n,2} \) has the form

\[
g = \prod_{i=1}^{n} x_i^{\alpha_i} \cdot \prod_{1 \leq i < j \leq n} [x_i, x_j]^{\beta_{ij}}
\]

for some integers \( \alpha_i \) and \( \beta_{ij} \), where \( [x_i, x_j] = x_i^{-1}x_j^{-1}x_ix_j \) are basic commutators (see [14, Chapter 5]).

For the free nilpotent group \( N_{n,r} \), let \( N'_{n,r} \) be its commutator subgroup. We note the following lemmas that will be used later.

\begin{lemma}
Let \( N'_{n,r} \) be the commutator subgroup of \( N_{n,r} \). Any element \( g \) in the commutator subgroup \( N'_{n,r} \) can be represented in the form

\[
g = [u_1, x_1][u_2, x_2] \cdots [u_n, x_n], \ u_i \in N_{n,r}.
\]

In fact, Allambergenov and Roman’kov [4] proved the following.

(i) Any element of the commutator subgroup \( N'_{n,2} \) is a product of no more than \( \lfloor n/2 \rfloor \) commutators.

(ii) Any element of the commutator subgroup \( N'_{n,r} \) in all other cases (\( r \geq 3, \ n \geq 4 \) or \( r > 3, \ n = 2 \)) is a product of no more than \( n \) commutators.

\end{lemma}

\begin{lemma}
Let \( A \) be a normal subgroup of \( N_{n,r} \). If \( A \) is abelian or \( A \) lies in the second center of \( N_{n,r} \), then every element of \([A, N_{n,r}]\) has the form \([u_1, x_1][u_2, x_2] \cdots [u_n, x_n]\) for some \( u_i \in A \).

\end{lemma}

2.2. \textit{Preliminary Results}. Let \( G = \langle A \rangle \) be a group and \( \mathcal{P} = \mathcal{P}(A) \) be the set of palindromes in \( G \). Evidently, any palindrome \( p \in \mathcal{P} \) can be represented in the form

\[
p = ua^\alpha \overline{u}, \text{ for some } a \in A, \alpha \in \mathbb{Z},
\]

where

\[
u = a_1^{\alpha_1}a_2^{\alpha_2} \cdots a_k^{\alpha_k}, \ a_i \in A, \alpha_i \in \mathbb{Z}
\]

is a word and

\[
\overline{u} = a_k^{\alpha_k}a_{k-1}^{\alpha_{k-1}} \cdots a_1^{\alpha_1}
\]

is its reverse word. Clearly, \( \overline{u}^{-1} = \overline{u} \).

The following lemma is easy to prove.

\begin{lemma}
Let \( G = \langle A \rangle \) and \( H = \langle B \rangle \) be two groups, \( \mathcal{P}(A) \) is the set of palindromes in the alphabet \( A^{\pm1} \), \( \mathcal{P}(B) \) is the set of palindromes in the alphabet \( B^{\pm1} \). If \( \varphi : G \to H \) is an epimorphism such that \( \varphi(A) = B \) then

\[
\text{pw}(H) \leq \text{pw}(G).
\]

\end{lemma}
For free nilpotent groups of rank $n$ we have the following set of epimorphisms

$$N_{n,1} \leftarrow N_{n,2} \leftarrow N_{n,3} \leftarrow \ldots$$

where

$$N_{n,1} = N_{n,2}/\gamma_2(N_{n,2}), \ N_{n,2} = N_{n,3}/\gamma_3(N_{n,3}), \ldots$$

Applying the above lemma we have:

**Corollary 2.4.** The following inequalities hold:

$$\text{pw}(N_{n,1}) \leq \text{pw}(N_{n,2}) \leq \text{pw}(N_{n,3}) \leq \ldots$$

**Lemma 2.5.** Let $G = (A)$ be a group generated by a set $A$. Then the following hold.

1) If $p$ is a palindrome, then for $m$ in $\mathbb{Z}$, $p^m$ is also a palindrome.

2) Any element in $G$ which is conjugate to a product of $n$ palindromes, $n \geq 1$, is a product of $n$ palindromes if $n$ is even, and of $n+1$ palindromes if $n$ is odd.

3) Any commutator of the type $[u, p]$, where $p$ is a palindrome is a product of 3 palindromes. Any element $[u, a^\alpha]a^\beta$, $a \in A$, $\alpha, \beta \in \mathbb{Z}$, is a product of 3 palindromes.

4) In $G$ any commutator of the type $[u, pq]$, where $p, q$ are palindromes is a product of 4 palindromes. Any element $[u, pa^\alpha]a^\beta$, $a \in A$, $\alpha, \beta \in \mathbb{Z}$, is a product of 4 palindromes.

**Proof.**

1) Let $p = ua^\alpha\overline{u}$, where $u$ is as above. Then $p^2 = ua^\alpha\overline{u}ua^\alpha\overline{u}$, $p^3 = ua^\alpha\overline{u}ua^\alpha\overline{u}ua^\alpha\overline{u}$ are palindromes. The result now follows by induction.

2) Let $v = u^{-1}pu$ be conjugate to a palindrome $p$. If $\overline{u}$ is the reverse word of $u$, then

$$v = (u^{-1}pu^{-1}) \cdot \overline{u}u$$

and we see that $u^{-1}pu^{-1}$ and $\overline{u}u$ are palindromes.

If $v$ is the conjugate to the product of $2m$ palindromes $p_1, \ldots, p_{2m}$, for some $u \in G$ let $v = u^{-1}p_1p_2 \ldots p_{2m}u$. Then

$$v = (u^{-1}p_1 u^{-1})(\overline{u}p_2u)(u^{-1}p_3 u^{-1}) \ldots \overline{u}p_{2m}u$$

is the product of $2m$ palindromes.

If $v = u^{-1}p_1p_2 \ldots p_{2m}p_{2m+1}u$, then

$$v = u^{-1}p_1 \ldots p_{2m}u^{-1}\overline{u}p_{2m+1}u = u^{-1}p_1 \ldots p_{2m}u \cdot (u^{-1}u^{-1}) \cdot \overline{u}p_{2m+1}u.$$

By part (2) above, $u^{-1}p_1 \ldots p_{2m}u$ is a product of $2m$ palindromes, $u^{-1}u^{-1}$ and $\overline{u}p_{2m+1}u$ are palindromes. Hence, $v$ is a product of $2m + 2$ palindromes.

3) We can check that

$$[u, p] = u^{-1}p^{-1}up = u^{-1}p^{-1}u^{-1} \cdot \overline{u} \cdot p$$

and $u^{-1}p^{-1}u^{-1}$ and $\overline{u}u$ are palindromes. If we take $p = a^\alpha$ then it is clear that $[u, a^\alpha]a^\beta$ is the product of three palindromes.

4) Observe that $[u, pq] = (u^{-1}(q^{-1}p^{-1})u)pq$. By (2), $u^{-1}(pq)^{-1}u$ is a product of two palindromes, hence the result follows.

**Proposition 2.6.** Let $G$ be a group and let an element $g$ in the center of $G$ be a product of 2 palindromes. Then for any integer $m$ the power $g^m$ is a product of 2 palindromes.
Proof. Let \( m > 0 \). Use induction on \( m \). If \( g = p_1p_2 \) is a product of 2 palindromes then
\[
g^2 = p_1p_2 \cdot g = p_1gp_2 = p_1^2 \cdot p_2^2
\]
and by Lemma 2.5(1) \( p_1^2 \) and \( p_2^2 \) are palindromes. Assume the result for some \( m \). Then
\[
g^{m+1} = (p_1p_2)^m \cdot g = p_1^m p_2^m = p_1^{m+1} \cdot p_2^{m+1}
\]
and by Lemma 2.5(1) \( p_1^{m+1} \) and \( p_2^{m+1} \) are palindromes.

If \( m < 0 \) then \( g^{-m} = (g^{-m})^{-1} \) and the result follows from the previous case and the fact that the inverse of a palindrome is a palindrome. \( \square \)

3. Proof of Theorem 1.1

3.1. The palindromic widths of free nilpotent groups. Let \( N_{n,r} \) be a free \( r \)-step nilpotent group of rank \( n \geq 2 \). Let \( x_1, x_2, \ldots, x_n \) be a basis for \( N_{n,r} \). Let \( \mathcal{P} \) be the set of all palindromes in \( N_{n,r} \). Note that an element \( p \in N_{n,r} \) is a palindrome if it can be be represented in the form
\[
p = x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_k^{\alpha_k}x_{k+1}^{\alpha_k+1}x_k^{\alpha_k} \cdots x_2^{\alpha_2}x_1^{\alpha_1}
\]
where
\[
i_j \in \{1, 2, \ldots, n\}, \quad \alpha_j \in \mathbb{Z} \setminus \{0\}.
\]

Lemma 3.1. \( \text{pw}(N_{n,1}) = n \).

Proof. In this case \( N_{n,1} \) is a free abelian group of rank \( n \). Since any element \( g \in N_{n,1} \) has the form
\[
g = x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \quad \alpha_i \in \mathbb{Z},
\]
then \( g \) is a product of \( n \) palindromes \( x_i^{\alpha_i} \) and hence
\[
\text{pw}(N_{n,1}) \leq n.
\]

To prove the equality we shall show that \( l_{\mathcal{P}}(x_1x_2 \cdots x_n) = n \). For this define a map
\[
\hat{\cdot}: N_{n,1} \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2,
\]
where \( \mathbb{Z}_2 \) is a cyclic group of order 2 by the rule
\[
\hat{g} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n),
\]
where
\[
\varepsilon_i = \begin{cases} 0 & \text{if } \alpha_i \text{ in (3.2) is even}, \\ 1 & \text{if } \alpha_i \text{ in (3.2) is odd}. \end{cases}
\]

Evidently, that for any \( g, h \in G \) we have \( \hat{gh} = \hat{g} + \hat{h} \).

If a palindrome \( p \) has the form (3.2) then
\[
\hat{p} = (\nu_1, \nu_2, \ldots, \nu_n)
\]
contains no more than one non-zero component. On the other hand, for \( w = x_1x_2 \cdots x_n \),
\[
\hat{w} = (1, 1, \ldots, 1).
\]

Thus \( x_1x_2 \cdots x_n \) is a product of at least \( n \) palindromes. \( \square \)

Lemma 3.2. For \( r \geq 2 \), \( n \leq \text{pw}(N_{n,r}) \leq 3n \).
Lemma 3.3. For some integers $a$ and $b$, any palindrome in $N_{2,2}$ has one of the following forms:

$$p(2a, b) = x^{2a} y^b z^{ab}, \quad p(a, 2b) = x^a y^{2b} z^{ab}, \quad \text{where } z = [y, x].$$
Lemma 3.4. If a product of two palindromes lies in $N'_{2,2}$ then this product is trivial. More generally, we have the following.

Proof. We know that any palindrome has the form $p = x^a$ or $p = y^b$. If it is equal to 1 then $p = x^a$ or $p = y^b$. If the syllable length is 3 then,

$$p = x^ay^bx^a = x^{2a}y^b[y, x]^\alpha$$

or

$$p = y^bx^ay^b = x^b y^{2a}[y, x]^\alpha.$$

Using the note before the lemma, we see that all other possibilities are reduced to these two cases. 

We see that if a palindrome lies in the commutator subgroup $N'_{2,2}$ then the palindrome is trivial. More generally, we have the following.

Lemma 3.4. If a product of two palindromes lies in $N'_{2,2}$ then this product is trivial. 

Proof. We know that any palindrome has the form $p(2a, b)$ or $p(a, 2b)$. Consider the product of two palindromes. If both palindromes have the type the $p(2a, b)$ then their product

$$p(2a_1, b_1) \cdot p(2a_2, b_2) = x^{2a_1}y^{b_1}z^{a_1b_1} \cdot x^{2a_2}y^{b_2}z^{a_2b_2} = x^{2(a_1+a_2)}y^{b_1+b_2}z^{b_1(a_1+2a_2)+a_2b_2}$$

lies in the commutator subgroup if and only if

$$\begin{cases} a_1 + a_2 = 0, \\ b_1 + b_2 = 0, \end{cases}$$

or

$$\begin{cases} a_1 = -a_2, \\ b_1 = -b_2. \end{cases}$$

But this means that

$$p(2a_1, b_1) \cdot p(2a_2, b_2) = z^{-b_2a_2+a_2b_2} = z^0 = 1.$$ 

The case of a product $p(a_1, 2b_1) \cdot p(2a_2, b_2)$ is similar.

Consider a product of palindromes of different types:

$$p(2a_1, b_1) \cdot p(a_2, 2b_2) = x^{2a_1}y^{b_1}z^{a_1b_1} \cdot x^{a_2}y^{2b_2}z^{a_2b_2} = x^{2a_1+a_2}y^{b_1+2b_2}z^{a_1b_1+a_2b_2+b_1a_2}.$$ 

We see that this product lies in the commutator subgroup if and only if

$$\begin{cases} 2a_1 + a_2 = 0, \\ b_1 + 2b_2 = 0, \end{cases}$$

or

$$\begin{cases} a_2 = -2a_1, \\ b_1 = -2b_2. \end{cases}$$

But this means that

$$p(2a_1, b_1) \cdot p(a_2, 2b_2) = z^{b_1(a_1+a_2)+a_2b_2} = z^0 = 1.$$ 

The case of the product $p(a_2, 2b_2) \cdot p(2a_1, b_1)$ is similar. 

Proposition 3.5. $\text{pw}(N_{2,2}) = 3$.

Proof. Note that $[y, x]$ is an element in the center of $N_{2,2}$. Note that

$$x^\alpha y^\beta[y, x] = x^\alpha y^{\beta - \gamma}y^{\gamma}[y, x] = x^\alpha y^{\beta - \gamma}(x^{-1}yx)^\gamma.$$
It follows that

\[ x^\alpha y^\beta z^\gamma = x^\alpha y^{\beta-\gamma} y^{x^\alpha} \cdot x^{-\alpha-2} \cdot xy^\gamma x. \]

Hence, \( \text{pw}(N_{2,2}) \leq 3 \). On the other hand, it follows from Lemma 2.5 that \( z \) is not a product of 2 palindromes. Hence \( \text{pw}(N_{2,2}) \geq 3 \).

In the general case we can prove

**Lemma 3.6.** Any element in \( N_{n,2} \), \( n \geq 2 \) is a product of at most \( 3(n-1) \) palindromes.

**Proof.** Let \( g \in N_{n,2} \). Then \( g \) has the form

\[ g = x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \cdots x_{n}^{\alpha_{n}} \prod_{1 \leq j < i \leq n} [x_{i}, x_{j}]^{\gamma_{ij}} \]

for some integers \( \alpha_{i} \) and \( \gamma_{ij} \). Using the commutator identities (see e.g. [14]) we have

\[ \prod_{1 \leq j < i \leq n} [x_{i}, x_{j}]^{\gamma_{ij}} = [x_{n}^{\gamma_{n-1}}, x_{n-1}^{\gamma_{n-2}}, \ldots, x_{2}^{\gamma_{2}}, x_{1}] [x_{n}^{\gamma_{n-2}}, x_{n-1}^{\gamma_{n-3}}, \ldots, x_{3}^{\gamma_{3}}, x_{2}] \cdots \]

\[ [x_{n}^{\gamma_{n-2}}, x_{n-1}^{\gamma_{n-3}}, \ldots, x_{2}^{\gamma_{2}}, x_{1}]^{\gamma_{n}} \]

Since, the commutator subgroup \( N_{n,2}' \) is equal to the center of \( N_{n,2} \), then

\[ g = [x_{n}^{\gamma_{n-1}}, x_{n-1}^{\gamma_{n-2}}, \ldots, x_{2}^{\gamma_{2}}, x_{1}] [x_{n}^{\gamma_{n-2}}, x_{n-1}^{\gamma_{n-3}}, \ldots, x_{3}^{\gamma_{3}}, x_{2}] [x_{n}^{\gamma_{n-3}}, x_{n-1}^{\gamma_{n-4}}, \ldots, x_{4}^{\gamma_{4}}, x_{3}] \cdots \]

\[ [x_{n}^{\gamma_{n-2}}, x_{n-1}^{\gamma_{n-3}}, \ldots, x_{2}^{\gamma_{2}}, x_{1}]^{\gamma_{n}} \]

By Lemma 2.5(2) any element

\[ [x_{n}^{\gamma_{n-1}}, x_{n-1}^{\gamma_{n-2}}, \ldots, x_{2}^{\gamma_{2}}, x_{1}] [x_{n}^{\gamma_{n-2}}, x_{n-1}^{\gamma_{n-3}}, \ldots, x_{3}^{\gamma_{3}}, x_{2}] [x_{n}^{\gamma_{n-3}}, x_{n-1}^{\gamma_{n-4}}, \ldots, x_{4}^{\gamma_{4}}, x_{3}] \cdots [x_{n}^{\gamma_{n-2}}, x_{n-1}^{\gamma_{n-3}}, \ldots, x_{2}^{\gamma_{2}}, x_{1}]^{\gamma_{n}} \]

is a product of 3 palindromes. Elements \( x_{n-1} \) and \( x_{n} \) generate a group which is isomorphic to \( N_{2,2} \) and by Proposition 3.5 the element \( x_{n-1}^{\gamma_{n-1}} [x_{n}^{\gamma_{n-1}}, x_{n-1}] x_{n}^{\alpha_{n}} \) is a product of 3 palindromes. Hence, \( g \) is a product of \( 3(n-1) \) palindromes. \( \square \)

### 3.3. Proof of Theorem 1.1

Theorem 1.1 is obtained by combining Lemma 3.1, Lemma 3.2, and Lemma 3.6.

### 3.4. The palindromic widths of free abelian-by-nilpotent groups.

**Proposition 3.7.** Let \( G \) be a non-abelian free abelian-by-nilpotent group of rank \( n \). Then \( \text{pw}(G) \leq 5n \).

**Proof.** Let \( G \) be a non-abelian free abelian-by-nilpotent group with a basis \( x_{1}, x_{2}, \ldots, x_{n} \). Let \( A \) be an abelian normal subgroup of \( G \) such that \( G/A \) is nilpotent. It follows from [2] Theorem 2) that any element \( g \in G \) has the form

\[ g = x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \cdots x_{n}^{\alpha_{n}} [u_{1}, x_{1}]^{\alpha_{1}} [u_{2}, x_{2}]^{\alpha_{2}} \cdots [u_{n}, x_{n}]^{\alpha_{n}}, \alpha_{i} \in \mathbb{Z}, u_{i} \in G, \alpha_{i} \in A. \]

By (3) of Lemma 2.3 any commutator \([u_{i}, x_{i}]\) is a product of 3 palindromes, thus by (2) of Lemma 2.6 any commutator \([u_{i}, x_{i}]^{\alpha_{i}}\) is a product of 4 palindromes. Hence, \( g \) is a product of \( n + 4n = 5n \) palindromes. \( \square \)

Since free metabelian groups are free abelian-by-abelian groups, hence we have the following.

**Corollary 3.8.** The palindromic width of a finitely generated free metabelian group is finite.
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