Quantum Gravity as a quantum field theory of simplicial geometry

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Abstract. This is an introduction to the group field theory approach to quantum gravity, with emphasis on motivations and basic formalism, more than on recent results; we elaborate on the various ingredients, both conceptual and formal, of the approach, giving some examples, and we discuss some perspectives of future developments.

1. Introduction: ingredients and motivations for the group field theory approach

Our aim in this paper is to give an introduction to the group field theory (GFT) approach to non-perturbative quantum gravity. We want especially to emphasize the motivations for this type of approach, the ideas involved in its construction, and the links with other approaches to quantum gravity, more than reviewing the results that have been obtained up to now in this area. For other introductory papers on group field theory, see [1], but especially [2], and for a review of the state of the art see [3]. No need to say, the perspective on the group field theory approach we provide is a personal one and we do not pretend it to be shared or fully agreed upon by other researchers in the field, although of course we hope this is the case. First of all what do we mean by 'quantizing gravity'in the GFT approach? What kind of theory are we after? The GFT approach seeks to construct a theory of quantum gravity that is non-perturbative and background independent. By this we mean that we seek to describe at the quantum level all the degrees of freedom of the gravitational field and thus obtain a quantum description of the full spacetime geometry; in other words no perturbative expansion around any given gravitational background metric is involved in the definition of the theory, so on the one hand states and observables of the theory will not carry any dependence on such background structure, on the other hand the theory will not include only the gravitational configurations that are obtainable perturbatively
starting from a given geometry. Also, let us add a (maybe not necessary) note: we are not after unification of fundamental forces; it cannot be excluded that a group field theory formulation of quantum gravity would be best phrased in terms of unified structures, be it the group manifold used or the field, but it is not a necessary condition of the formalism nor among the initial aims of the approach. So what are group field theories? In a word: group field theories are particular field theories on group manifolds that (aim to) provide a background independent third quantized formalism for simplicial gravity in any dimension and signature, in which both geometry and topology are thus dynamical, and described in purely algebraic and combinatorial terms. The Feynman diagrams of such theories have the interpretation of simplicial spacetimes and the theory provides quantum amplitudes for them, in turn interpreted as discrete, algebraic realisation of a path integral description of gravity. Let us now motivate further the various ingredients entering the formalism (for a similar but a bit more extensive discussion, see [20]), and at the same time discuss briefly other related approaches to quantum gravity in which the same ingredients are implemented.

1.1. Why path integrals? The continuum sum-over-histories approach

Why to use a description of quantum gravity on a given manifold in terms of path integrals, or sum-over-histories? The main reason is its generality: the path integral formulation of quantum mechanics, let alone quantum gravity, is more general than the canonical one in terms of states and Hamiltonians, and both problems of interpretation and of recovering of classicality (via decoherence) benefit from such a generalisation [4]. Coming to quantum gravity in particular, the main advantages follow from its greater generality: one does not need a canonical formulation or a definition of the space of states of the theory to work with a gravity path integral, the boundary data one fixes in writing it down do not necessarily correspond to canonical states nor have to be of spacelike nature (one is free to consider timelike boundaries), nor the topology of the manifold is fixed to be of direct product type with a space manifold times a time direction (no global hyperbolicity is required). On top of this, one can maintain manifest diffeomorphism invariance, i.e. general covariance, and does not need any \((n - 1) + 1\) splitting, nor the associated enlargement of spacetime diffeomorphism symmetry to the symmetry group of the canonical theory [5]. Finally, the most powerful non-perturbative techniques of quantum field theory are based on path integrals and one can hope for an application of some of them to gravity. So how would a path integral for continuum gravity look like? Consider a compact four manifold (spacetime) with trivial topology \(\mathcal{M}\) and all the possible geometries (spacetime metrics up to diffeomorphisms) that are compatible with it. The partition function of the theory would then be defined [6] by an integral over all possible 4-geometries with a diffeomorphism invariant measure and weighted by a quantum amplitude given by the exponential of \(\iota \)times the action of the classical theory one wants to quantize, General Relativity. For computing transition amplitudes for given boundary configurations of the field, one would instead consider a manifold \(\mathcal{M}\) again, of trivial topology,
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with two disjoint boundary components $S$ and $S'$ and given boundary data, i.e. 3-geometries, on them: $h(S')$ and $h'(S')$, and define the transition amplitude by:

$$Z_{QG}(h(S), h'(S')) = \int_{g(M|h(S), h'(S'))} Dg e^{iS_{GR}(g, M)}$$

i.e. by summing over all 4-geometries inducing the given 3-geometries on the boundary, with the amplitude possibly modified by boundary terms if needed. The expression above is purely formal: first of all we lack a rigorous definition of a suitable measure in the space of 4-geometries, second the expectation is that the oscillatory nature of the integrand will make the integral badly divergent. To ameliorate the situation somehow, a ‘Wick rotated’ of the above expression was advocated with the definition of a “Euclidean quantum gravity” where the sum would be only over Riemannian metrics with a minus sign in front of the action in the definition of the integral [6]. This however was not enough to make rigorous sense of the theory and most of the related results were obtained in semiclassical approximations [6]. Also, the physical interpretation of the above quantities presents several challenges, given that the formalism seems to be bound to a cosmological setting, where our usual interpretations of quantum mechanics are not applicable. We do not discuss this here, but it is worth keeping this issue in mind, given that a good point about group field theory is that it seems to provide a rigorous version of the above formulas (and much more than that) which is also local in a sense to be clarified below.

1.2. Why topology change? Continuum 3rd quantization of gravity

In spite of the difficulties in making sense of a path integral quantization of gravity on a fixed spacetime, one can think of doing even more and treat not only geometry but also topology as a dynamical variable in the theory. One would therefore try to implement a sort of “sum over topologies” alongside a sum over geometries, thus extending this last sum to run over all possible spacetime geometries and not only those that can live on a given topology. Again therefore the main aim in doing this is to gain in generality: there is no reason to assume that the spacetime topology is fixed to be trivial, so it is good not to assume it. Of course this has consequences on the type of geometries one can consider, in the Lorentzian case, given that a non-trivial spacetime topology implies spatial topology change [7] and this in turn forces the metric to allow either for closed timelike loops or for isolated degeneracies (i.e. the geometry may be degenerate, have zero volume element, at isolated points).

While in a first order or tetrad formulation of gravity one can thus avoid the first possibility by allowing for the second, in the second order metric formulation one is bound to include metrics with causality violations. This argument was made stronger by Horowitz [8] to the point of concluding that if degenerate metrics are included in the (quantum) theory, then topology change is not only possible but unavoidable and non-trivial topologies therefore must be included in the quantum theory. However, apart from greater generality, there are various results that hint to the need for topology change in quantum gravity. Work on topological geons
topological configurations with particle-like properties, suggest that spatial topology change (the equivalent of pair creation for geons) is needed in order for them to satisfy a generalisation of the spin-statistics theorem. Work in string theory indicates that different spacetime topology can be equivalent with respect to stringy probes. Wormholes, i.e. spatial topology changing spacetime configurations, have been advocated as a possible mechanism that turn off the cosmological constant decreasing its value toward zero, and the possibility has been raised that all constants of nature can be seen as vacuum parameters, thus in principle computed, in a theory in which topology is allowed to fluctuate. This last idea, together with the analogy with string perturbation theory and the aim to solve some problems of the canonical formulation of quantum gravity, prompted the proposal of a “third quantization” formalism for quantum gravity. The idea is to define a (scalar) field in superspace $\mathcal{H}$ for a given choice of basic spatial manifold topology, i.e. in the space of all possible 3-geometries (3-metrics $h_{ij}$ up to diffeos) on, say, the 3-sphere, essentially turning the wave function of the canonical theory into an operator: $\phi(h)$, whose dynamics is defined by an action of the type:

$$S(\phi) = \int_{\mathcal{H}} D^3 h \phi(h) \Delta \phi(h) + \lambda \int_{\mathcal{H}} D^3 h \mathcal{V}(\phi(h))$$

with $\Delta$ being the Wheeler-DeWitt operator of canonical gravity here defining the kinetic term (free propagation) of the theory, while $\mathcal{V}(\phi)$ is a generic, e.g. cubic, and generically non-local (in superspace) interaction term for the field, governing the topology changing processes. Notice that because of the choice of basic spatial topology needed to define the 3rd quantized field, the topology changing processes described here are those turning $X$ copies of the 3-sphere into $Y$ copies of the same.

The quantum theory is defined by the partition function $Z = \int D\phi e^{-S(\phi)}$, that produces the sum over histories outlined above, including a sum over topologies with definite weights, as a dynamical process, in its perturbative expansion in Feynman graphs:

$$\text{---}$$

The quantum gravity path integral for each topology will represent the Feynman amplitude for each ‘graph’, with the one for trivial topology representing a sort of one particle propagator, thus a Green function for the Wheeler-DeWitt equation. Some more features of this (very) formal setting are worth mentioning:

1) the full classical equations of motions for the scalar field will be a non-linear extension of the Wheeler-DeWitt equation of canonical gravity, due to the interaction term in the action, i.e. of the inclusion of topology change;
2) the perturbative 3rd quantized vacuum of the theory will be the “no spacetime” state, and not any state with a semiclassical geometric interpretation in terms of a smooth geometry,
say a Minkowski state. We will see shortly how these ideas are implemented in the group field theory approach.

1.3. Why going discrete? Matrix models and simplicial quantum gravity

However good the idea of a path integral for gravity and its extension to a third quantized formalism may be, there has been no definite success in the attempt to realise them rigorously, nor in developing the formalism to the point of being able to do calculations and then obtaining solid predictions from the theory. A commonly held opinion is that the main reason for the difficulties encountered is the use of a *continuum* for describing spacetime, both at the topological and at the geometrical level. One can indeed advocate the use of *discrete structures* as a way to regularize and make computable the above expressions, to provide a more rigorous definition of the theory, with the continuum expressions and results emerging only in a continuum *limit* of the corresponding discrete quantities. This was in fact among the motivations for discrete approaches to quantum gravity as matrix models, or dynamical triangulations or quantum Regge calculus. At the same time, various arguments can be and have been put forward for the point of view that discrete structures instead provide a more *fundamental* description of spacetime. These arguments come from various quarters. On the one hand there is the possibility, suggested by various approaches to quantum gravity such as string theory or loop quantum gravity, that in a more complete description of space and time there should be a fundamental length scale that sets a least bound for measurable distances and thus makes the notion of a continuum loose its physical meaning, at least as a fundamental entity. Also, one can argue on both philosophical and mathematical grounds [13] that the very notion of “point” can correspond at most to an idealization of the nature of spacetime due to its lack of truly operational meaning, i.e. due to the impossibility of determining with absolute precision the location in space and time of any event (which, by the way, is implemented mathematically very precisely in non-commutative models of quantum gravity, see the contribution by Majid in this volume). Spacetime points are indeed to be replaced, from this point of view, by small but finite regions corresponding to our finite abilities in localising events, and a more fundamental (even if maybe not ultimate [13]) model of spacetime should take these local regions as basic building blocks. Also, the results of black hole thermodynamics seem to suggest that there should be a discrete number of fundamental spacetime degrees of freedom associated to any region of spacetime, the apparent continuum being the result of the microscopic (Planckian) nature of them. This means that the continuum description of spacetime will replace a more fundamental discrete one as an *approximation* only, as the result of a *coarse graining* procedure. In other words, a finitary topological space [14] would constitute a better model of spacetime than a smooth manifold. All these arguments against the continuum and in favor of a finitary substitute of it can be naturally seen as arguments in favor of a simplicial description of spacetime, with the simplices playing indeed the role of a finitary substitute of the concept of a point or fundamental event, or of a minimal
spacetime region approximating it. Simplicial approaches to quantum gravity are matrix models, dynamical triangulations and quantum Regge calculus. The last one \cite{15} is the straightforward translation of the path integral idea in a simplicial context. One starts from the definition of a discrete version of the Einstein-Hilbert action for General Relativity on a simplicial complex $\Delta$, given by the Regge action $S_R$ in which the basic geometric variables are the lengths of the edges of $\Delta$, and then defines the quantum theory usually via Euclidean path integral methods, i.e. by:

$$Z(\Delta) = \int \mathcal{D}l \ e^{-S_R(l)}.$$  \hspace{1cm} (3)

The main issue is the definition of the integration measure for the edge lengths, since it has to satisfy the discrete analogue of the diffeomorphism invariance of the continuum theory (the most used choices are the $dl$ and the $dl/l$ measures) and then the proof that the theory admits a good continuum limit in which continuum general relativity is recovered, indeed the task that has proven to be the most difficult. Matrix models \cite{23} can instead be seen as a surprisingly powerful implementation of the third quantization idea in a simplicial context, but in an admittedly simplified framework: 2d Riemannian quantum gravity. Indeed group field theories are a generalisation of matrix models to higher dimension and to Lorentzian signature. Consider the action

$$S(M) = \frac{1}{2} \text{tr} M^2 - \frac{\lambda}{3! \sqrt{N}} \text{tr} M^3$$  \hspace{1cm} (4)

for an $N \times N$ hermitian matrix $M_{ij}$, and the associated partition function $Z = \int dM e^{-S(M)}$. This in turn is expanded in perturbative expansion in Feynman diagrams; propagators and vertices of the theory can be expressed diagrammatically and the corresponding Feynman diagrams, obtained as usual by gluing vertices with propagators, are given by fat graphs of all topologies. Moreover, propagators can be understood as topologically dual to edges and vertices to triangles of a 2-dimensional simplicial complex that is dual to the whole fat graph in which they are combined; this means that one can define a model for quantum gravity in 2d, via the perturbative expansion for the matrix model above, as sum over all 2d triangulations $T$ of all topologies. Indeed the amplitude of each Feynman diagram for the above theory is related to the Regge action for classical simplicial gravity in 2dm for fixed edge lengths equal to $N$ and positive cosmological constant, and
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more specifically, the partition function is:

$$Z = \int dM e^{-S(M)} = \sum_T \frac{1}{\text{sym}(T)} \lambda^{n_2(T)} N \chi(T)$$

where \(\text{sym}(T)\) is the order of symmetries of the triangulation \(T\), \(n_2\) is the number of triangles in it, and \(\chi\) is the Euler characteristic of the same triangulation. Many results have been obtained over the years for this class of models, for which we refer to the literature [23]. Closely related to matrix models is the dynamical triangulations approach [24], that extends the idea and results of defining a path integral for gravity as a sum over equilateral triangulations of a given topology to higher dimensions, weighted by the (exponential of the) Regge action for gravity:

$$Z(G, \lambda, a) = \sum_T \frac{1}{\text{sym}(T)} e^{iS_R(T, G, \Lambda, a)}$$

where \(G\) is the gravitational constant and \(\Lambda\) is a cosmological constant. In the Lorentzian case one also distinguishes between spacelike edges (length square \(a^2\)) and timelike ones (length square \(-a^2\)), and imposes some additional restrictions on the topology considered and on the way the triangulations are constructed via the gluing of d-simplices. In particular, one may then look for a continuum limit of the theory, corresponding to the limit \(a \to 0\) accompanied by a suitable renormalisation of the constants of the theory \(\Lambda\) and \(G\), and check whether in this limit the structures expected from a continuum quantum gravity theory are indeed recovered, i.e. the presence of a smooth phase with the correct macroscopic dimensionality of spacetime. And indeed, the exciting recent results obtained in this approach seem to indicate that, in the Lorentzian context and for trivial topology, a smooth phase with the correct dimensionality is obtained even in 4 dimensions, which makes the confidence in the correctness of the strategy adopted to define the theory grow stronger.

1.4. Why groups and representations? Loop quantum gravity and spin foams

We will see many of the previous ideas at work in the group field theory context. There, however, a crucial role is played by the Lorentz group and its representations, as it is in terms of them that geometry is described. Another way to see group field theories in fact is as a re-phrasing (in addition to a generalisation) of the matrix model and simplicial quantum gravity formalism in an algebraic language. Why would one want to do this? One reason is the physical meaning and central role that the Lorentz group plays in gravity and in our description of spacetime; another is that by doing this, one can bring in close contact with the others yet another approach to quantum gravity: loop quantum gravity, through spin foam models. But let us discuss one thing at the time. The Lorentz group enters immediately into play and immediately in a crucial role as soon as one passes from a description of gravity in terms of a metric field to a first order description in terms of tetrads and connections. Gravity becomes not too dissimilar from a gauge
theory, and as such its basic observables (intended as correlations of partial observables [16]) are given by parallel transports of the connection itself along closed paths, i.e. holonomies, contracted in such a way as to be gauge invariant. Indeed these have a clear operational meaning [16]. The connection field is a $so(3, 1)$ valued 1-form (in 4d) and therefore its parallel transports define elements of the Lorentz group, so that the above observables (in turn determining the data necessary to specify the states of a canonical formulation of a theory based on this variables) are basically given by collections of group elements associated to possible paths in spacetime organized in the form of networks. They are classical spin networks. In a simplicial spacetime, the valence of these networks will be constrained but they will remain the basic observables of the theory. A straightforward quantization of them would be obtained by the choice of a representation of the Lorentz group for each of the links of the network to which group elements are associated. Indeed, the resulting quantum structures are spin networks, graphs labeled by representations of the Lorentz group associated to their links, of the type characterizing states and observables of loop quantum gravity [16], the canonical quantization of gravity based on a connection formulation. A covariant path integral quantization of a theory based on spin networks will have as histories (playing the role of a 4-dimensional spacetime geometries) a higher-dimensional analogue of them: a spin foam [17, 18, 20], i.e. a 2-complex (collection of faces bounded by links joining at vertices) with representations of the Lorentz group attached to its faces, in such a way that any slice or any boundary of it, corresponding to a spatial hypersurface, will be indeed given by a spin network. Spin foam models [17, 18, 20] are intended to give a path integral quantization of gravity based on these purely algebraic and combinatorial structures.

In most of the current models the combinatorial structure of the spin foam is restricted to be topologically dual to a simplicial complex of appropriate dimension, so that to each spin foam 2-complex it corresponds a simplicial spacetime, with the representations attached to the 2-complex providing geometric information to the simplicial complex; in fact they are interpreted as volumes of the (n-2)-simplices topologically dual to the faces of the 2-complex. The models are then defined by an assignment of a quantum probability amplitude (here factorised in terms of face, edge, and vertex contributions) to each spin foam $\sigma$ summed over, depending on the representations $\rho$ labeling it, also being summed over, i.e. by the transition amplitudes for given boundary spin networks $\Psi, \Psi'$ (which may include the empty spin network as well):

$$Z = \sum_{\sigma|\Psi, \Psi'} w(\sigma) \sum_{\{\rho\}} \prod_f A_f(\rho_f) \prod_e A_e(\rho_{f|e}) \prod_v A_v(\rho_{f|v});$$

one can either restrict the sum over spin foams to those corresponding to a given fixed topology or try to implement a sum over topologies as well; the crucial point is in any case to come up with a well-motivated choice of quantum amplitudes, either coming from some sort of discretization of a classical action for gravity or from some other route. Whatever the starting point, one would then have an
implementation of a sum-over-histories for gravity in a combinatorial-algebraic context, and the key issue would then be to prove that one can both analyse fully the quantum domain, including the coupling of matter fields, and at the same recover classical and semi-classical results in some appropriate limit. A multitude of results have been already obtained in the spin foam approach, for which we refer to [17, 18, 20]. We will see shortly that this version of the path integral idea is the one coming out naturally from group field theories.

2. Group field theory: what is it? The basic GFT formalism

Group field theories, as anticipated, are a new realization of the third quantization idea that we have outlined above, in a simplicial setting, and in which the geometry of spacetime as well as superspace itself are described in an algebraic language. As such, they bring together most of the ingredients entering the other approaches we have briefly discussed, thus providing hopefully a general encompassing framework for developing them, as we will try to clarify in the following. We describe the basic framework of group field theories and the rationale for its construction first, and then we will give an explicit (and classic) example of it so to clarify the details of the general picture.

2.1. A discrete superspace

The first ingredient in the construction of a third quantization theory of gravity in \( n \) dimensions is a definition of superspace, i.e. the space of \((n-1)\)-geometries. In a simplicial setting, spacetime is discretized to a simplicial complex and thus it is built out of fundamental blocks represented by \( n \)-simplices; in the same way, an \((n-1)\)-space, i.e. an hypersurface (not necessarily spacelike) embedded in it, is obtained gluing together along shared \((n-2)\)-simplices a number of \((n-1)\)-simplices in such a way as to reproduce through their mutual relations the topology of the hypersurface. In other words, a \((n-1)\)-space is given by a \((n-1)\)-dimensional triangulation and its geometry is given not by a metric field (thanks to which one can compute volumes, areas, lengths and so on) but by the geometric data assigned...
to the various elements of the complex: volumes, areas, lengths etc. There is some freedom in the choice of variables to use as basic ones for describing geometry and from which to compute the various geometric quantities. In Regge calculus, as we have seen, the basic variables are chosen to be the edge lengths of the complex; in group field theories \[2, 25\], as currently formulated, the starting assumption is that one can use as basic variables the volumes of \((n-2)\)-simplices (edge lengths in 3d, areas of triangles in 4d, etc). The consequences and possible problems following from this assumption have not been fully investigated yet. These \((n-2)\)-volumes are determined by unitary irreducible representations \(\rho\) of the Lorentz group, one for each \((n - 2)\) face of the simplicial complex. Equivalently, one can take as basic variables appropriate Lorentz group elements \(g\) corresponding to the parallel transports of a Lorentz connection along dual paths (paths along the cell complex dual to the triangulation), one for each \((n-2)\)-face of the complex. The equivalence between these two sets of variables is given by harmonic analysis on the group, i.e. by a Fourier-type relation between the representations \(\rho\) and the group elements \(g\), so that they are interpreted as conjugate variables, as momenta and position of a particle in quantum mechanics \[25\]. Therefore, if we are given a collection of \((n-1)\)-simplices together with their geometry in terms of associated representations \(\rho\) or group elements \(g\), we have the full set of data we need to characterize our superspace. Now one more assumption enters the group field theory approach: that one can exploit the discreteness of this superspace in one additional way, i.e. by adopting a local point of view and considering as the fundamental superspace a single \((n-1)\)-simplex: this means that one considers each \((n-1)\)-simplex as a “one-particle state”, and the whole \((n-1)\)-d space as a “multiparticle” state, but with the peculiarity that these many “particles” (many \((n-1)\)-simplices) can be glued together to form a collective extended structure, i.e. the whole of space. The truly fundamental superspace structure will then be given by a single \((n-1)\)-simplex geometry, characterized by \(n\) Lorentz group elements or \(n\) representations of the Lorentz group, all the rest being reconstructed from it, either by composition of the fundamental superspace building blocks (extended space configurations) or by interactions of them as a dynamical process (spacetime configurations), as we will see. In the generalised group field theory formalism of \[26\], one uses an extended or parametrised formalism in which additional variables characterize the geometry of the fundamental \((n-1)\)-simplices, so that the details of the geometric description are different, but the overall picture is similar, in particular the local nature of the description of superspace is preserved.

2.2. The field and its symmetries

Accordingly to the above description of superspace, the fundamental field of GFTs, as in the continuum a scalar field living on it, corresponds to the 2nd quantization of a \((n-1)\)-simplex. The 1st quantization of a 3-simplex in 4d was studied in detail in \[27\] in terms of the algebraic set of variables motivated above, and the idea is that the field of the GFT is obtained promoting to an operator the wave function arising from the 1st quantization of the fundamental superspace building block.
We consider then a complex scalar field over the tensor product of $n$ copies of the Lorentz group in $n$ dimensions and either Riemannian or Lorentzian signature, $\phi(g_1, g_2, ..., g_n) : G^\otimes n \to \mathbb{C}$.

The order of the arguments in the field, each labeling one of its $n$ boundary faces ($(n-2)$-simplices), corresponds to a choice of orientation for the geometric $(n-1)$-simplex it represents; therefore it is natural to impose the field to be invariant under even permutations of its arguments (that do not change the orientation) and to turn into its own complex conjugate under odd permutations. This ensures \textsuperscript{[28]} that the Feynman graphs of the resulting field theory are given by orientable 2-complexes, while the use of a real field, with invariance under any permutation of its arguments, has as a result Feynman graphs including non-orientable 2-complexes as well. If the field has to correspond to an $(n-1)$-simplex, with its $n$ arguments corresponding to an $(n-2)$-simplex each, one extra condition is necessary: a global gauge invariance condition under Lorentz transformations \textsuperscript{[27]}. We thus require the field to be invariant under the global action of the Lorentz group, i.e. under the simultaneous shift of each of its $n$ arguments by an element of the Lorentz group, and we impose this invariance through a projector operator: $P_g \phi(g_1; g_2; ...; g_n) = \int_G dq \phi(g_1 q; g_2 q; ...; g_n q)^\dagger$. Geometrically, this imposes that the $n(n-2)$-simplices on the boundary of the $(n-1)$-simplex indeed close to form it \textsuperscript{[27]}; algebraically, this causes the field to be expanded in modes into a linear combination of Lorentz group invariant tensors (intertwiners). The mode expansion of the field takes in fact the form:

$$\phi^n (g_i, s_i) = \sum_{J, \Lambda, k_i} \phi^{J \Lambda}_{k_i} \prod_i D_{k_i}^{J_i} (g_i) C_{J_1; J_4 \Lambda}^{J_i}$$. 

with the $J$'s being the representations of the Lorentz group, the $k$'s vector indices in the representation spaces, and the $C$'s are intertwiners labeled by an extra representation index $\Lambda$. In the generalised formalism of \textsuperscript{[28]}, the Lorentz group is extended to $(G \times \mathbb{R})^n$ with consequent extension of the gauge invariance one imposes and modification of the mode expansion. Note also that the timelike or spacelike nature of the $(n-2)$-simplices corresponding to the arguments of the field depends on the group elements or equivalently to the representations associated to them, and nothing in the formalism prevents us to consider timelike $(n-1)$-simplices thus a superspace given by a timelike $(n-1)$-geometry.

2.3. The space of states or a third quantized simplicial space

The space of states resulting from this algebraic third quantization is to have a structure of a Fock space, with $N$-particle states created out of a Fock vacuum, \textsuperscript{1}The Lorentzian case, with the use of the non-compact Lorentz group as symmetry group, will clearly involve, in the definition of the symmetries of the field as well as in the definition of the action and of the Feynman amplitudes, integrals over a non-compact domain; this produces trivial divergences in the resulting expressions and care has to be taken in making them well-defined. However, this can be done quite easily in most cases with appropriate gauge fixing. We do not discuss issues of convergence here in order to simplify the presentation.
corresponding as in the continuum to the “no-spacetime” state, the absolute vacuum, not possessing any spacetime structure at all. Each field being an invariant tensor under the Lorentz group (in momentum space), labeled by \( n \) representations of the Lorentz group, it can be described by a \( n \)-valent spin network vertex with \( n \) links incident to it labeled by the representations. One would like to distinguish a ‘creation’ and an ‘annihilation’ part in the mode expansion of the field, as \( \phi_{J_i \Lambda}^{k_i} = \varphi_{k_i}^{J_i \Lambda} + \left( \varphi_{k_i}^{J_i \Lambda} \right)^\dagger \), and then one would write something like: 

\[
\phi_{J_i \Lambda}^{k_i} | 0 \rangle
\]

for a one particle state, 

\[
\varphi_{k_i}^{J_i \Lambda} \varphi_{k_i}^{\tilde{J}_i \tilde{\Lambda}} | 0 \rangle
\]

for a disjoint 2-particle state (two disjoint \((n-1)\)-simplices), or 

\[
\varphi_{k_1 k_2 \ldots k_n}^{J_1 \ldots J_n \Lambda} \varphi_{k_1 k_2 \ldots k_n}^{\tilde{J}_1 \ldots \tilde{J}_n \tilde{\Lambda}} | 0 \rangle
\]

for a composite 2-particle state, made out of two \((n-1)\)-simplices glued along one of their boundary \((n-2)\)-simplices (the one labeled by \( J_2 \)), and so on. Clearly the composite states will have the structure of a spin network of the Lorentz group. This way one would have a Fock space structure for a third quantized simplicial space of the same type as that of usual field theories, albeit with the additional possibility of creating or destroying at once composite structures made with more than one fundamental ‘quanta’ of space. At present this has been only formally realised [29] and a more complete and rigorous description of such a third quantized simplicial space is needed.

### 2.4. Quantum histories or a third quantized simplicial spacetime

In agreement with the above picture of (possibly composite) quanta of a simplicial space being created or annihilated, group field theories describe the evolution of these states in perturbation theory as a scattering process in which an initial quantum state (that can be either a collection of disjoint \((n-1)\)-simplices, or spin network vertices, or a composite structure formed by the contraction of several such vertices, i.e. an extended \((n-1)\)-dimensional triangulation) is transformed into another one through a process involving the creation or annihilation of a number of quanta. Being these quanta \((n-1)\)-simplices, their interaction and evolution is described in terms of \( n \)-simplices, as fundamental interaction processes, in which \( D \) \((n-1)\)-simplices are turned into \( n + 1 - D \) ones (in each \( n \)-simplex there are \( n + 1 \) \((n-1)\)-simplices). Each of these fundamental interaction processes corresponds to a possible \( n \)-dimensional Pachner move, a sequence of which is known to allow the transformation of any given \((n-1)\)-dimensional triangulation into any other.

A generic scattering process involves however an arbitrary number of these fundamental interactions, with given boundary data, and each of these represents a possible quantum history of simplicial geometry, so our theory will appropriately sum over all these histories with certain amplitudes. The states being collections of suitably contracted spin network vertices, thus spin networks themselves labeled with representations of the Lorentz group (or equivalently by Lorentz group elements), dual to triangulations of a \((n-1)\)-dimensional space, their evolution history will be given by \( 2 \)-complexes labeled again by representations of the Lorentz group, dual to \( n \)-dimensional simplicial complexes. Spacetime is thus purely virtual in this context, as in the continuum third quantized formalism and as it should be in a
sum over histories formulation of quantum gravity, here realised as a sum over labeled simplicial complexes or equivalently their dual labeled complexes, i.e. spin foams. We see immediately that we have here a formalism with the ingredients of the other discrete and algebraic approaches to quantum gravity we have outlined above.

2.5. The third quantized simplicial gravity action

The action of group field theories [2, 17, 18, 26] is defined so to implement the above ideas, and it is given by:

$$S_n(\phi, \lambda) = \frac{1}{2} \prod_{i=1, \ldots, n} \int dg_i d\tilde{g}_i \phi(g_i) K(g_i \tilde{g}_i^{-1}) \phi(\tilde{g}_i) + \frac{\lambda}{n+1} \prod_{i \neq j=1}^{n+1} \int dg_{ij} \phi(g_{ij}) \ldots \phi(g_{n+1j}) V(g_{ij} g_{ji}^{-1}),$$

where of course the exact choice of the kinetic and interaction operators is what defines the model. We see that indeed the interaction term in the action has the symmetries and the combinatorial structure of a $n$-simplex made out of $n+1$ (n-1)-simplices glued pairwise along common (n-2)-simplices, represented by their arguments, while the kinetic term represent the gluing of two $n$-simplices along a common (n-1)-simplex, i.e. the free propagation of the (n-1)-simplex between two interactions. $\lambda$ is a coupling constant governing the strength of the interactions, and the kinetic and vertex operators satisfy the invariance property $K(g_i \tilde{g}_i^{-1}) = K(gg_i \tilde{g}_i^{-1}g')$ and $V(g_{ij} g_{ji}^{-1}) = V(g_{ij} g_{ji}^{-1}g_j^{-1})$ as a consequence of the gauge invariance of the field itself. A complete analysis of the symmetries of the various group field theory actions has not been carried out yet, and in 3d for example it is known that there exist symmetries of the Feynman amplitudes (i.e. of the histories) of the theory that are not yet identified at the level of the GFT action. In the generalised models [26], the structure of the action is exactly the same, with the group extended to $G \times \mathbb{R}$. The simplest choice of action is given by $K = \int dg \prod_{i=1}^{n} \delta(g_i \tilde{g}_i^{-1})$ and $V = \prod_{i+1}^{n+1} \int dg_i \prod_{i<j} \delta(g_{ij} g_{ji}^{-1} g_{ji}^{-1})$, that corresponds to a GFT formulation of topological BF theories in $n$ dimensions, that gives gravity in 1st order formalism in 3d, as we will see shortly, while in dimension $n=2$ gives a sum over matrix models of increasing matrix dimension if one choses $SU(2)$ as group manifold [2]. Less trivial actions can be constructed [20], while a simple modification of the BF action gives much studied models of 4d quantum gravity [17, 18]. Unfortunately, we do not understand much at present of the classical theory described by these actions, and paradoxically we understand better the (perturbative) quantum theory, thanks to the work done in the context of spin foam models [17, 18], and we turn now to this.

2.6. The partition function and its perturbative expansion

The partition function of the theory is then given by an integral over the field of the exponential of (minus) the GFT action. Our current understanding of the non-perturbative properties of the partition function, i.e. of the quantum theory, is quite poor, even if some work is currently in progress on instantonic calculations in GFTs [30]. More is known about perturbative dynamics in terms of Feynman
graphs, thanks to work on spin foam models. The perturbation expansion of the partition function is as usual given by the Schwinger-Dyson expansion in Feynman graphs:

\[ Z = \int \mathcal{D}\phi \, e^{-S[\phi]} = \sum_{\Gamma} \frac{\lambda^N}{\text{sym}[\Gamma]} Z(\Gamma), \]

where \( N \) is the number of interaction vertices in the Feynman graph \( \Gamma \), \( \text{sym}[\Gamma] \) is a symmetry factor for the graph and \( Z(\Gamma) \) the corresponding Feynman amplitude. The Feynman amplitudes can be constructed easily after identification of the propagator, given by the inverse of the kinetic term in the action, and the vertex amplitude of the theory; each edge of the Feynman graph is made of \( n \) strands running parallel to each other, one for each argument of the field, and each is then re-routed at the interaction vertex, with the combinatorial structure of an \( n \)-simplex. Diagrammatically:

Each strand in an edge of the Feynman graph goes through several vertices and then comes back where it started, for closed Feynman graphs, and therefore identifies a 2-cell. The collection of 2-cells (faces), edges and vertices of the Feynman graph then characterizes a 2-complex that, because of the chosen combinatorics for the arguments of the field in the action, is dual to a \( n \)-dimensional simplicial complex. Each strand carries a field variable, i.e. a group element in configuration space or a representation label in momentum space. Therefore in momentum space each Feynman graph is given by a spin foam, and each Feynman amplitude (a complex function of the representation labels) of the GFT by a spin foam model. Indeed, one can show that the inverse is also true: any local spin foam model can be obtained from a GFT perturbative expansion \[25, 2\]. The sum over Feynman graphs for the partition function gives then a sum over spin foams (histories of the spin networks on the boundary in any scattering process considered), and equivalently a sum over triangulations, augmented by a sum over algebraic data (group elements or representations) with a geometric interpretation. This is true of course also for the generalised GFT models of \[26\]. This perturbative expansion of the partition function also allows (in principle) the explicit evaluation of expectation values of GFT observables; these are given \[2\] by gauge invariant combinations of the field operators that can be constructed using spin networks. In particular, the transition amplitude (probability amplitude for a certain scattering process) between certain boundary data represented by two spin networks can be
expressed as the expectation value of field operators contracted as to reflect the combinatorics of the two spin networks [2].

2.7. GFT definition of the canonical inner product

Even though, as mentioned in the beginning, this is not meant to be a review of the results obtained so far in the GFT approach, for which we refer to [3], we would like to mention just one important recent result, because it shows clearly how useful this new formalism can be in addressing long-standing open problems of quantum gravity research. Also, it proves that the overall picture of GFTs that we have outline above (and will summarize in the next subsection) is consistent and fertile. In ordinary QFT, the classical equations of motion for the (free) 2nd quantized field are also the quantum equation of motion for the 1st quantized relativistic particle wave function; in other words, the full dynamical content of the 1st quantized (non-interacting) theory is contained in the classical level of the 2nd quantized theory. If we buy the picture of GFTs as representing a 3rd quantization formalism of (simplicial) gravity, we expect a similar situation in which the classical GFT level encodes all the dynamical information about canonical quantum gravity (with fixed topology). We will mention in the final part of this paper some caveats to this perspective and some open issues. Given the limits in our understanding of the classical structure of GFT and of the non-perturbative level of the theory, at present we may hope to see these ideas realized explicitly only in the perturbative expansion of the partition function. Indeed, the hopes are fulfilled. It has been shown [2] that the restriction of the sum over Feynman graphs outlined above to tree level, thus neglecting all quantum correction and encoding the classical information only, gives a definition of the ‘2-point function’, for fixed boundary spin networks, that is positive semidefinite and finite, including a sum over all triangulations and thus fully triangulation independent; the triangulations involved are of fixed trivial topology, as appropriate for dealing with canonical quantum gravity, and the sum over geometric data (representations) is well-defined. This prompts the interpretation of the resulting quantity as a well-posed and computable definition of the canonical inner product between quantum gravity states, represented by spin networks, the object that in canonical loop quantum gravity encodes the whole dynamical content of the theory (the action of the Hamiltonian constraint operator on spin network states). Even if more work is certainly needed to build up on this result, we see that the use of GFT techniques and ideas has an immediate and important usefulness even from the point of view of canonical quantum gravity: the GFT definition of the physical inner product for canonical spin network states provides finally a solution to the long standing issues of canonical quantum gravity on the definition of the Hamiltonian constraint operator, its action on kinematical states, the definition of the physical inner product through some kind of projection operator formalism, the computation of physical observables, etc. Now it is time to put this concrete proposal to test. Finally, let us stress that this results also highlights the richness of the GFT formalism, and suggests that we have barely started to scratch the surface, with
much more lying underneath. If the classical level of GFTs encodes already the full content of canonical quantum gravity, it is clear that a complete analysis of the quantum level of the GFTs will lead us even much further, in understanding a quantum spacetime, than we have hoped to do by studying canonical quantum gravity. This includes the physics of topology change, of course, and the use of quantum gravity sum over histories for other purposes than defining the canonical inner product, but probably much more than that.

2.8. Summary: GFT as a general framework for quantum gravity

Let us briefly summarise the nature of GFTs, before giving an explicit example of it, so to clarify the details of the formalism. We have a field theory over a group manifold that makes no reference to a physical spacetime (except implicitly in the combinatorial structure that one chooses for the GFT action), in which the field (thus the fundamental “particle” it describes) has a geometric interpretation of a quantized (n-1)-simplex. The states of the theory (given in momentum space by spin networks) are correspondingly interpreted as triangulations of (n-1)-dimensional (pseudo)manifolds [28]. The theory can be dealt with perturbatively through an expansion in Feynman graphs that describe the possible interactions of the field quanta, that can be created and annihilated as well as change intrinsic geometry (configuration variables and associated momenta), as a scattering process. Geometrically each possible interaction process for given boundary states is a possible simplicial spacetime with assigned geometry, and it is given by a spin foam, to which the theory assigns a precise quantum amplitude. The simplicial spacetimes summed over have arbitrary topology, as are constructed by all possible gluings of fundamental interaction vertices each corresponding to an n-simplex. GFTs are therefore a third quantized formulation of simplicial geometry. Interestingly, group field theories also have all the ingredients that enter other approaches to quantum gravity: boundary states given by spin networks, as in loop quantum gravity, a simplicial description of spacetime and a sum over geometric data, as in Quantum Regge calculus, a sum over triangulations dual to 2-complexes, as in dynamical triangulations, a sum over topologies like in matrix models, of which GFTs are indeed higher-dimensional analogues, as we said, an ordering of fundamental events (vertices of Feynman diagrams), given by the orientation of the 2-complex, which has similarities to that defining causal sets [19], and quantum amplitudes for histories that are given by spin foam models. Therefore one can envisage GFTs as a general framework for non-perturbative quantum gravity, that encompasses most of the current approaches. This is at present only a vague and quite optimistic point of view, not yet established nor strongly supported by rigorous results, but we feel that it can be a fruitful point of view both for the development of GFTs themselves and of these other approaches as well, by offering a new perspective on them and possibly new techniques that can be used to address the various open issues they still face.
3. An example: 3d Riemannian Quantum Gravity

For simplicity we consider explicitly in more detail only the 3d Riemannian quantum gravity case, whose group field theory formulation was first given by Boulatov [32]. Consider the real field: \( \phi(g_1, g_2, g_3) : (SU(2))^3 \to \mathbb{R} \), with the symmetry: \( \phi(g_1 g, g_2 g, g_3 g) = \phi(g_1, g_2, g_3) \), imposed through the projector: \( P_g \phi(g_1, g_2, g_3) = \int dg \phi(g_1 g, g_2 g, g_3 g) \) and the symmetry: \( \phi(g_1, g_2, g_3) = \phi(g_{\pi(1)}, g_{\pi(2)}, g_{\pi(3)}) \), with \( \pi \) an arbitrary permutation of its arguments, that one can realise through an explicit sum over permutations: \( \phi(g_1, g_2, g_3) = \sum_\pi \phi(g_{\pi(1)}, g_{\pi(2)}, g_{\pi(3)}) \). In this specific case, the interpretation is that of a 2nd quantized triangle with its 3 edges corresponding to the 3 arguments of the field; the irreps of \( SU(2) \) labeling these edges in the mode expansion of the field have the interpretations of edge lengths.

The classical theory is defined by the action:

\[
S[\phi] = \frac{1}{2} \int dg_1 dg_3 [P_g \phi(g_1, g_2, g_3)]^2 + \frac{\lambda}{4} \int dg_1 dg_6 [P_{g_1} \phi(g_1, g_2, g_3)] [P_{g_2} \phi(g_3, g_5, g_4)] [P_{g_3} \phi(g_4, g_2, g_6)] [P_{g_4} \phi(g_6, g_5, g_1)].
\]

As we have discussed in the general case, we see that the structure of the action is chosen so to reflect the combinatorics of a 3d triangulation, with four triangles (fields) glued along their edges (arguments of the field) pairwise, to form a tetrahedron (vertex term) and two tetrahedra being glued along their common triangles (kinetic term \( \to \) propagator). The quantum theory is given by the partition function, in turn again defined in terms of perturbative expansion in Feynman graphs:

\[
Z = \int d\phi e^{-S[\phi]} = \sum \frac{\lambda^N}{\text{sym} |\Gamma|} Z(\Gamma).
\]

Therefore, in order to construct explicitly the quantum amplitudes of the theory for each of its Feynman graphs, we need to identify their building blocks, i.e. propagator and vertex amplitude. These are to be read out from the action:

\[
S[\phi] = \frac{1}{2} \int dg_1 dg_3 \phi(g_1) K(g_1, g_3, \phi(g_3)) + \frac{\lambda}{4} \int dg_1 dg_2 \phi(g_1) \phi(g_2)
\]

For the propagator, starting from the kinetic term \( P_g \phi(g_1, g_2, g_3) P_g \phi(g_1, g_2, g_3) \), and considering the permutation symmetry, one gets immediately:

\[
P = K^{-1} = K = \sum_\pi \int dg_1 g_2 \delta(g_1 g_2^{-1} g_3^{\pi(1)}) \delta(g_2 g_3^{\pi(2)}) \delta(g_3^{\pi(3)}) \]

while the vertex is given by:

\[
\mathcal{V} = \int dh_1 \delta(h_1 h_3^{\pi(1)}) \delta(h_2 h_4^{\pi(1)}) \delta(h_3 h_1 h_2^{\pi(2)}) \delta(h_4 h_5 h_6^{\pi(3)}).
\]

We see that, as for BF theories in any dimensions, the vertex and propagator are given simply by products of delta functions over the group, represented simply by lines in the diagrams below, with boxes representing the integration over the
group (following from the requirement of gauge invariance. The Feynman graphs are obtained as usual by gluing vertices with propagators. Let us see how these look like. As we explained for the general case, each line in a propagator goes through several vertices and for closed graphs it comes back to the original point, thus identifying a 2-cell, and these 2-cells, together with the set of lines in each propagator, and the set of vertices of the graph, identify a 2-complex. Each of these 2-complexes is dual to a 3d triangulation, with each vertex corresponding to a tetrahedron, each link to a triangle and each 2-cell to an edge of the triangulation.

The sum over Feynman graphs is thus equivalent to a **sum over 3d triangulations of any topology**, as anticipated in the general case. Let us now identify the quantum amplitudes for these Feynman graphs. These are obtained the usual way using the above propagators and vertex amplitudes. In configuration space, where the variables being integrated over are group elements, the amplitude for each 2-complex is:

$$Z(\Gamma) = \left( \prod_{e \in \Gamma} \int dg_e \right) \prod_f \delta \left( \prod_{e \in \partial f} g_e \right)$$

which has the form of a lattice gauge theory partition function with simple delta function weights for each plaquette (face of the 2-complex) and one connection variable for each edge; the delta functions constraint the the curvature on any face to be zero, as we expect from 3d quantum gravity. To have the corresponding
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expression in momentum space, one expands the field in modes \( \phi(x_1, x_2, x_3) = \sum_{j_1, j_2, j_3} \phi^{j_1, j_2, j_3}_{m_1 n_1 m_2 n_2 m_3 n_3} D^{j_1}_{m_1 n_1} D^{j_2}_{m_2 n_2} (g_1) D^{j_3}_{m_3 n_3} (g_3) \), where the \( j \)'s are irreps of the group \( SU(2) \) (the Lorentz group, local gauge group of gravity, for 3d and Riemannian signature) obtaining, for the propagator, vertex and amplitude:

\[
\begin{align*}
\mathcal{P} &= \delta_{j_1 j_4} \delta_{m_1 m_4} \delta_{j_2 j_5} \delta_{m_2 m_5} \delta_{j_3 j_6} \delta_{m_3 m_6} \\
\mathcal{V} &= \delta_{j_1 j_4} \delta_{m_1 m_4} \delta_{j_2 j_5} \delta_{m_2 m_5} \delta_{j_3 j_6} \delta_{m_3 m_6} \delta_{j_4 j_5} \delta_{m_4 m_5} \delta_{j_5 j_6} \delta_{m_5 m_6} \\
Z(\Gamma) &= \left( \prod_f \sum_{j_f} \right) \prod_f \Delta_{j_f} \prod_v \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} 
\end{align*}
\]

where \( \Delta_j \) is the dimension of the representation \( j \) and for each vertex of the 2-complex we have a so-called \( 6j \)-symbol, i.e. a scalar function of the 6 representations meeting at that vertex. The amplitude for each 2-complex is given then by a spin foam model, the Ponzano-Regge model for 3d gravity without cosmological constant, about which a lot more is known [31]. This amplitude, after gauge fixing, gives a well-defined topological invariant of 3-manifolds, as one expects from 3d quantum gravity, and as such it is invariant under choice of triangulation. This means that it evaluates to the same number for any triangulation (2-complex) for given topology, so that the only dynamical degrees of freedom in the theory are indeed the topological. The full theory is then defined as we said by the sum over all Feynman graphs weighted by the above amplitudes:

\[
Z = \sum_{\Gamma} \frac{\chi^N_{\text{sym}[\Gamma]}}{\prod_{f} \sum_{j_f}} \prod_{f} \Delta_{j_f} \prod_v \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} .
\]

This gives a rigorous (after gauge-fixing of the symmetries of the theory [7] the above quantity is well-defined, even in the Lorentzian case, in spite of the non-compactness of the Lorentz group used and of the infinite dimensionality of the irreps used) and un-ambiguous (in the sense that every single element in the above formula has a known closed expression and therefore it can be computed exactly at least in principle) realisation, in purely algebraic and combinatorial terms, of the sum over both geometries and topologies, i.e. of the third quantization idea, in the 3-dimensional case. Issues about interpretation and about the convergence of the sum over complexes (but see [38]) of course remain, but we see a definite progress at least from what concerns the definition of the amplitude and measure for given 2-complex (i.e. given spacetime) with respect to the continuum path integral.

Strikingly, group field theory models for quantum gravity in 4 spacetime dimensions, that seem to have many of the right properties we seek, can be obtained by a very simple modification of the 3-dimensional model [17, 18]. Motivated by the classical formulation of gravity as a constrained BF theory [17, 18], one first generalises the above field to a 4-valent one with arguments living in the 4-dimensional
Lorentz group $SO(3,1)$, modifying the combinatorial structure of the above action to mimic the combinatorics of a 4-simplex in the interaction term, with 5 tetrahedra (fields) glued along triangles, and then simply imposes a restriction on the arguments of the field to live in the homogeneous space $SO(3,1)/SO(3) \simeq H^3$, i.e. on the upper sheet of the timelike hyperboloid in Minkowski space. The resulting Feynman graphs expansion produces quantum amplitudes for them given by the Barrett-Crane model [33], that has been recently the focus of much work for which we refer to the literature [17,18].

4. Assorted questions for the present, but especially for the future

We have outlined the general formalism of group field theories and the picture of a third quantized simplicial spacetime they suggest. We have organised this outline and presented these models from the perspective that sees GFTs as a candidate to and a proposal for a fundamental formulation of a theory of quantum gravity, as opposed to just a tool to be used to produce for example spin foam models, that is the way they have been used up to now. We have tried to highlight the features of the formalism and of the resulting picture that we find most appealing and fascinating. However, it is probably transparent that there is and there should be much more in the theory than what we have described. As a matter of fact, there is certainly in the GFTs much more than it is presently known, which is basically only their action, their perturbative expansion, and a few properties of their Feynman amplitudes, i.e. spin foam models. To tell the whole truth, even many of the details of the general picture we have presented are only tentative and there remains lots to be understood about them. Therefore we want to conclude by posing a (limited) set of assorted questions regarding GFTs and that future work should answer, if GFTs are to be taken seriously as candidates for a fundamental formulation of third quantized simplicial gravity, and thus of non-perturbative quantum gravity.

- **What about the classical theory?** The description of the classical simplicial superspace that we have briefly described is to be investigated further as the validity of the variables that GFTs use to describe classical geometry is not solidly established yet. Also, even given this for granted, not much is known about the classical theory behind the GFT action. Some work is in progress [30], but it is fair to say that we do not have a good understanding of the physical meaning of the classical equations of motion of the theory nor we know enough solutions of them. As for the physical meaning, we stress again that the classical GFT dynamics should already encode the canonical quantum gravity dynamics in full, as it happens in ordinary quantum field theories (where the classical field equations are the quantum Schroedinger equation for the one-particle theory). Also, in analogy to the continuum third quantization setting, one would expect them to describe a modification of the Wheeler-DeWitt equation, in a simplicial setting, due to the presence of topology change; however, unlike the continuum case it is not easy, and maybe not
even possible, to distinguish (the simplicial analogue of) a Wheeler-DeWitt operator and a topology changing term as distinct contributions to the equation, due to the local nature of the description of superspace that we have in GFTs. As for obtaining solutions of the classical equations of motion, the non-local nature of the equations of motion (coming from that of the action) makes solving them highly non-trivial even in lower dimensions. The form of the action in the generalised GFT formalism \[26\] may probably facilitate this task somehow, due to a greater similarity with usual scalar quantum field theories. Similarly for the Hamiltonian analysis of the theory, not yet performed and that is quite non-trivial in the usual formalism, but may be easier to carry out in a covariant fashion thanks to the proper time variables in the generalised formalism. In this regard, the fact established in \[20\] that usual GFTs are the Static Ultra-local (SUL), and un-oriented, limit of well-defined generalised GFTs may help in that one may first tackle with standard methods the Hamiltonian analysis of such generalised models and then study the SUL limit of the corresponding Hamiltonian structures.

- **Fock space and +ve/-ve energy states?** The picture of creation and annihilation of quanta of simplicial geometry, the construction of quantum states of the field theory, and the Fock structure of the corresponding space of states built out of the 3rd quantized vacuum may be appealing and geometrically fascinating, but it is at present mostly based on intuition. A rigorous construction needs to be carried out, starting from a proper hamiltonian analysis, and a convincing identification of creation and annihilation operators from the mode expansion of the field. Having at hand the hamiltonian form and the harmonic decomposition of the field itself in modes (resulting from the harmonic analysis on group manifolds), one should investigate the quantum gravity analogue of the notion of positive and negative energy (particle/anti-particle) states of QFT. This has probably to do with the opposite orientations one can assign to the fundamental building blocks of our quantum simplicial spacetime, so maybe it is again best investigated in the generalised GFT formalism \[26\], in which the orientation data play a crucial role.

- **What is \( \lambda \)?** In \[2\], one consistent and convincing interpretation of the GFT coupling constant has been given: it was shown that a suitable power of the coupling constant can be interpreted as the parameter governing the sum over topologies in the perturbative expansion in Feynman graphs. This result was based on the analysis of the Schwinger-Dyson equations of the n-dimensional GFT. More work is now needed to expand on these result and elucidate all its implications. Another interpretation of the GFT coupling constant, although not as well established is suggested by the work \[28\]. If one considers only one fixed representation of the Lorentz group, thus reducing GFTs to simple tensor extensions of matrix models, then the coupling constant enters in the resulting amplitudes as the exponential of a cosmological constant, and the perturbative expansion is interpreted as an expansion in the “size” of spacetime \[28\]; this is beautiful, but one must check whether a similar picture
applies for the full GFT with no restriction on the representations. Assuming
this can be established, more work is then needed to prove the compatibility
of the two interpretations. This last point has important consequences, in that
it may lead to a simplicial and exact realisation of the old idea, put forward
in the context of continuum formulations of 3rd quantised gravity [11], of
a connection between cosmological constant and wormholes, i.e. topology
change, in quantum gravity.

- **Where are the diffeos? actually, where are all the continuum symmetries?**
  Another crucial point that needs to be addressed is a deeper understanding
  of GFT symmetries; in fact, already in the 3-dimensional case it is known [31]
  that the amplitudes generated by the GFT possess symmetries that are not
  immediately identified as symmetries of the GFT action. The most important
  symmetry of a gravity theory is diffeomorphism symmetry. One could argue
  that GFTs are manifestly diffeomorphism invariant in the sense that there is
  no structure in the theory corresponding to a continuum spacetime, but one
  should identify a discrete symmetry corresponding to diffeos in the appro-
  priate continuum approximation of the theory. Since any continuum metric
  theory that is invariant under diffeomorphisms reproduces Einstein gravity
  (plus higher derivative terms), the identification of diffeomorphism symme-
  try in GFTs is crucial, it that it would support the idea that they possess a
  continuum approximation given by General Relativity. It was argued in [2]
  that diffeomorphisms are the origin of loop divergences in spin foam models,
  that in turn are just Feynman amplitudes for the GFT. This needs to be
  investigated. Another possibility to be investigated is that diffeomorphisms
  originate from a non-trivial renormalisation group acting on the parameter
  space of GFTs.

- **A GFT 2-point functions zoo? where is causality?** In ordinary quantum field
  theory, one can define different types of N-point functions or transition am-
  plitudes, with different uses and meanings; is this the case also for GFTs, and
  more generally for Quantum Gravity? if so, what is their respective use and
  interpretation? The difference between various N-point functions in QFT is
  in their different causal properties, so this question is related to the more
  general issue of causality in GFTs. Where is causality? How to implement
  causality restrictions? Recent work [26] seems to suggest that these issues
  can be dealt with satisfactorily, but much more work is certainly needed.

- **What is the exact relation with the canonical theory?** A canonical theory
  based on Lorentz spin networks, adapted to a simplicial spacetime, and thus
  a kind of covariant discretization of loop quantum gravity [16], has to be
  given by a subsector of the GFT formalism. The reduction to this subsector
  as well as the properties of the resulting theory are not yet fully understood.
  A canonical theory of quantum gravity needs a spacetime topology of product
  type, i.e. $\Sigma \times \mathbb{R}$, so that the only non-trivial topology can be in the topology
  of space $\Sigma$, and a positive definite inner product between quantum states.
  A precise and well-posed definition of such an inner product and a way to
reduce to trivial topology in the perturbative expansion of the GFT was proposed in [2], as we discussed: it is given by the perturbative evaluation of the expectation value of appropriate spin network observables in a tree level truncation, that indeed generates only 2-complexes with trivial topology. The consequences of this proposal, that would amount to a complete definition of the canonical theory corresponding to the GTFs, need to be investigated in detail. It may be possible for example to extract a definition of an Hamiltonian constraint operator from the so-defined inner product, to study its properties, and to compare it with those proposed in the loop quantum gravity approach. Even when this has been done, it would remain to investigate the relation between covariant spin network structures based on the full Lorentz group, and the loop quantum gravity ones based on $SU(2)$, and to check how much of the many mathematical results obtained on the kinematical Hilbert space of $SU(2)$ spin networks can be reproduced for the covariant ones.

- **How to include matter?** The inclusion and the correct description of matter fields at the group field theory level is of course of crucial importance. Work on this has started only recently [34, 35, 36] for the 3-dimensional case, with very interesting results. The idea pursued there was that one could perform a 3rd quantization of gravity and a 2nd quantization of matter fields in one stroke, thus writing down a coupled GFT action for both gravity and matter fields that would produce, in perturbative expansion, a sum over simplicial complexes with dynamical geometry (quantum gravity histories) together with Feynman graphs for the matter fields living on the simplicial complexes (histories for the matter fields). The whole description of the coupled system would thus be purely algebraic and combinatorial. Indeed, this can be realised consistently for any type of matter field [36]. However, in 3d life is made easier by the topological nature of gravity and by the fact that one can describe matter as a topological defect. The difficult task that lies ahead is to extend these results to 4 dimensions. In this much more difficult context, some work is currently in progress regarding the coupling of gauge fields to quantum gravity at the GFT level [37].

- **Does the GFT perturbation theory make sense?** Even if the only thing we know about GFTs is basically their perturbative expansion in Feynman graphs, strangely enough we do not know for sure if this perturbative expansion makes sense. Most likely the perturbative series is not convergent, but this is not too bad, as it is what happens in ordinary field theories. One would expect (better, hope) it to be an asymptotic series to a non-perturbatively defined function, but this has been realised up to now only in a specific model in 3d [38], and more work is needed for what concerns other models especially in 4d. Let us recall that the perturbative GFT expansion entails a sum over topologies, so that gaining control over it is a mathematically highly non-trivial issue with very important physical consequences.
• **How to relate to the other approaches to quantum gravity?** Even if one is optimistic and buys the picture of GFTs as a general framework for quantum gravity encompassing other approaches, or at least the main ingredients of other approaches, the links with these approaches need to be investigated in detail to start really believing the picture. For example, to obtain a clear link with simplicial approaches to quantum gravity, one needs first to construct a GFT that has Feynman amplitudes given by the exponential of the Regge action for the corresponding simplicial complex, probably building on the results of [26]. Then one would be left to investigate the properties of the measure in front of the exponential, to be compared with those used in Regge calculus, and to find a nice procedure for reducing the model to involve only equilateral triangulations and to admit a slicing structure, so to compare it with dynamical triangulations models. And this would be just a start.

• **What about doing some physics?** The ultimate aim is of course to have a consistent framework for describing quantum gravity effects and obtain predictions that can be compared to experiments. This may be seen as far-fetched at present, and maybe it is, but a consistent coupling of matter fields at the GFT level, a better understanding of its semiclassical states and of perturbations around them, and a better control over the continuum approximation of the GFT structures, all achievable targets for current and near future studies, may bring even this ultimate aim within our reach not too far from now.

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