\textbf{Q-SOB as an epireflective hull in $Q\text{-TOP}_0$}

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\textbf{Abstract}

We show that the epireflective hull of the $Q$-Sierpinski space in the category $Q\text{-TOP}_0$ of $Q$-$T_0$-topological spaces is the category $Q\text{-SOB}$ of $Q$-sober topological spaces.

\textit{Keywords:} $Q$-topological space, $Q$-sober topological space, $Q$-$T_0$-topological space, epireflective hull.

\section{Introduction}

For a given (but fixed) variety $A$ of $\Omega$-algebras and a fixed member $Q$ of $A$, S.A. Solovyov \cite{Solovyov94} introduced the notion of a $Q$-topological space (and $Q$-continuous maps between them), providing thereby the category $Q\text{-TOP}_0$ of such spaces. He also introduced the notions of $Q$-$T_0$-topological spaces, $Q$-sober topological spaces and $Q$-Sierpinski space. If $Q\text{-SOB}$ denotes the category of $Q$-sober topological spaces, then Solovyov also showed implicitly that $Q\text{-SOB}$ is reflective in $Q\text{-TOP}_0$ (cf. Lemma 19 of \cite{Solovyov94}).

In this note, (motivated by results in \cite{Solovyov94} \cite{SinghSrivastava13}) we have shown that $Q\text{-SOB}$ is the epireflective hull of the $Q$-Sierpinski space in the category $Q\text{-TOP}_0$ of $Q$-$T_0$-topological spaces.

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1recently, a characterization of the category $Q\text{-TOP}$ has been given in \cite{SinghSrivastava13}
2 Preliminaries

For all undefined category theoretic notions used in this paper, [8] may be referred. All subcategories used here are assumed to be full.

We begin by recalling the notions of $\Omega$-algebras and their homomorphisms; for details, cf. [2], [3].

**Definition 2.1.** Let $\Omega = (n_\lambda)_{\lambda \in I}$ be a class of cardinal numbers.

- An $\Omega$-algebra is a pair $(A, (\omega^A_\lambda)_{\lambda \in I})$ consisting of a set $A$ and a family of maps $\omega^A_\lambda : A^{n_\lambda} \to A$. $B \subseteq A$ is called a subalgebra of $(A, (\omega^A_\lambda)_{\lambda \in I})$ if $\omega^B_\lambda((b_i)_{i \in n_\lambda}) \in B$, for every $\lambda \in I$ and every $(b_i)_{i \in n_\lambda} \in B^{n_\lambda}$. Given $S \subseteq A$, $(S)$ denotes the subalgebra of $(A, (\omega^A_\lambda)_{\lambda \in I})$ generated by $S$, i.e., $(S)$ is the intersection of all subalgebras of $(A, (\omega^A_\lambda)_{\lambda \in I})$ containing $S$. (In fact, $(S) = \{\omega^A_\lambda((s_i)_{i \in n_\lambda}) \mid s_i \in S \text{ and } \lambda \in I\}$).
- Given $\Omega$-algebras $(A, (\omega^A_\lambda)_{\lambda \in I})$ and $(B, (\omega^B_\lambda)_{\lambda \in I})$, a map $f : A \to B$ is called an $\Omega$-algebra homomorphism provided that for every $\lambda \in I$, the following diagram

\[
\begin{array}{ccc}
A^{n_\lambda} & \xrightarrow{f^{n_\lambda}} & B^{n_\lambda} \\
\omega^A_\lambda \downarrow & & \downarrow \omega^B_\lambda \\
A & \xrightarrow{f} & B
\end{array}
\]

commutes.

Let $\text{Alg}(\Omega)$ denote the category of $\Omega$-algebras and $\Omega$-algebra homomorphisms (this category has products).

- A variety of $\Omega$-algebras is a full subcategory of $\text{Alg}(\Omega)$ which is closed under the formation of products, subalgebras, and homomorphic images.

Throughout this paper, $\Omega = (n_\lambda)_{\lambda \in I}$ denotes a fixed class of cardinal numbers, $A$ denotes a fixed variety of $\Omega$-algebras and $Q$ denotes a fixed member of $A$.

Each function $f : X \to Y$ between sets $X$ and $Y$ gives rise to two functions $f^\lor : 2^Y \to 2^X$ and $f^\lnot : 2^X \to 2^Y$, given by $f^\lor (B) = \{x \in X \mid f(x) \in B\}$ and $f^\lnot (A) = \{f(x) \mid x \in A\}$, and also a function $f^Q_{\lnot} : Q^X \to Q^X$, given by $f^Q_{\lnot} (\alpha) = \alpha \circ f$.

- Given a set $X$, a subset $\tau$ of $Q^X$ is called a $Q$-topology on $X$ if $\tau$ is a subalgebra of $Q^X$, in which case, the pair $(X, \tau)$ is called a $Q$-topological space.
- Given two $Q$-topological spaces $(X, \tau)$ and $(Y, \eta)$, a $Q$-continuous function from $(X, \tau)$ to $(Y, \eta)$ is a function $f : X \to Y$ such that $f^Q_{\lnot} (\alpha) \in \tau$, for every $\alpha \in \eta$.

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2 Most of the definitions in the preliminaries are given in [3] also, we recall these here for the sake of completeness.
• Given a Q-topological space \((X, \tau)\) and \(Y \subseteq X\), \((i_Y)^{-1}(\tau) (= \{ p \circ i \mid p \in \tau \})\) is called the Q-subspace topology on \(Y\), where \(i : Y \to X\) is the inclusion map. We shall denote the Q-subspace topology on \(Y\) as \(\tau_Y\).

• A Q-topological space \((X, \tau)\) is called Q-T0 if for every distinct \(x, y \in X\), there exists \(p \in \tau\) such that \(p(x) \neq p(y)\).

The meanings of homeomorphisms, embeddings, and products etc. for Q-topological spaces, are on expected lines.

Let Q-TOP denote the category of all Q-topological spaces and Q-continuous maps between them.

Remark 2.1. In [10], it has been noted that, Q-TOP, like TOP, has products. One can go on further and verify that Q-TOP is initially complete; in fact Q-TOP turns out to be a topological category over SET. As a consequence of the above, Q-TOP is complete; in particular, it has equalizers which are constructed, at the set-theoretical level, in the same way as in SET.

Let \(\mathcal{C}\) be a category, \(\mathcal{H} \subseteq \text{mor}\mathcal{C}\), \(\text{epi}\mathcal{C}\) be the class of all \(\mathcal{C}\)-epimorphisms and \(\mathcal{R}\) be a subcategory of \(\mathcal{C}\).

Definition 2.2. [1][2] \(\mathcal{R}\) is said to be epireflective in \(\mathcal{C}\) if for each \(\mathcal{C}\)-object \(X\), there exists an epimorphism \(r_X : X \to RX\), with \(RX \in \text{ob}\mathcal{R}\), such that for each \(\mathcal{C}\)-morphism \(f : X \to Y\), with \(Y \in \text{ob}\mathcal{R}\), there exists a unique \(\mathcal{R}\)-morphism \(f^* : RX \to Y\), such that \(f^* \circ r_X = f\). If moreover, each \(r_X \in \mathcal{H}\) and \(f^*\) is a \(\mathcal{C}\)-isomorphism, whenever \(f \in \text{epi}\mathcal{C} \cap \mathcal{H}\), then \(\mathcal{R}\) is said to be an \(\mathcal{H}\)-firm epireflective subcategory of \(\mathcal{C}\) (or that the epireflectivity of \(\mathcal{R}\) in \(\mathcal{C}\) is \(\mathcal{H}\)-firm).

3 The Q-sober space

Consider the identity function \(id : Q \to Q\) and let \(\nu = \langle id \rangle\) be the subalgebra of \(Q^Q\), generated by \(id\).

Definition 3.1. [9] The Q-topological space \((Q, \nu)\) is called the Q-Sierpinski space.

We shall denote the Q-Sierpinski space \((Q, \nu)\) as \(Q_S\).

The next result is from [8] (which is also the same as Lemma 57 in [9]).

Theorem 3.1. For every Q-topological space \((X, \tau)\), \(p \in \tau\) if and only if \(p : (X, \tau) \to Q_S\) is Q-continuous.

For every \(A \in \text{ob}\mathcal{A}\), let \(ptA = \text{hom}_A(A, Q)\). Define a map \(\phi : A \to Q^{ptA} \) as \(\phi(a)(p) = p(a), \forall a \in A \) and \(\forall p \in ptA\). Then \(\phi\) turns out to be an \(\Omega\)-algebra homomorphism (cf. [9]). Hence \(\phi(A)\) is a subalgebra of \(Q^{ptA}\), whereby \(\phi(A)\) is a Q-topology on \(ptA\).

For each Q-topological space \((X, \tau)\), let \(X \xrightarrow{\eta_X} pt\tau\) be the function defined by \(\eta_X(x)(p) = p(x), \forall x \in X \) and \(\forall p \in \tau\).

Definition 3.2. [9] A Q-topological space \((X, \tau)\) is called Q-sober if \(\eta_X\) is bijective.
Let $Q$-$SOB$ denote the subcategory of $Q$-$TOP$ whose objects are $Q$-sober topological spaces.

The following fact is easily verified.

**Proposition 3.1.** The $Q$-Sierpinski space $Q_S$ is $Q$-sober.

**Proposition 3.2.** [9] For every $A \in \text{ob} A$, $(pA, \phi(A))$ is $Q$-sober.

**Proposition 3.3.** [9] Let $(X, \tau) \in \text{ob} Q$-$TOP$. Then

1. the $Q$-topological space $(pt\tau, \phi(\tau))$ is $Q$-sober,
2. $\eta_X : (X, \tau) \to (pt\tau, \phi(\tau))$ is $Q$-continuous,
3. $(X, \tau)$ is $Q$-$TOP$ if and only if $\eta_X$ is injective,
4. $(X, \tau)$ is $Q$-sober if and only if $\eta_X : (X, \tau) \to (pt\tau, \phi(\tau))$ is $Q$-homeomorphism.

**Proposition 3.4.** $Q$-$SOB$ is reflective in $Q$-$TOP$.

**Proof:** It follows from Lemma 19 of [9].

**Proposition 3.5.** If $(X, \tau) \in \text{ob} Q$-$TOP_0$, then $\eta_X : (X, \tau) \to (pt\tau, \phi(\tau))$ is a $Q$-$TOP_0$-embedding.

**Proof:** Clearly, $\eta_X : (X, \tau) \to (pt\tau, \phi(\tau))$ is injective. Let $f : (X, \tau) \to (\eta_X(X), \phi(\tau)_{\eta_X(X)})$ be the ‘corestriction’ of $\eta_X$ onto $\eta_X(X)$. It is enough to show that $f^{-1} : (\eta_X(X), \phi(\tau)_{\eta_X(X)}) \to (X, \tau)$ is $Q$-continuous, i.e., to show that $p \circ f^{-1} \in \phi(\tau)_{\eta_X(X)}$, $\forall p \in \tau$, where $\phi(\tau)_{\eta_X(X)}$ is the $Q$-subspace topology on $\eta_X(X)$. Note that $\phi(\tau)_{\eta_X(X)} = \{\phi(q) \circ i \mid q \in \tau\}$.

where $i : \eta_X(X) \to pt\tau$ be the inclusion map. For every given $p \in \tau$, as $(p \circ f^{-1})(\eta_X(x)) = p(f^{-1}(\eta_X(x))) = p(x)$ and also, $(\phi(p) \circ i)(\eta_X(x)) = \phi(p)(\eta_X(x)) = \eta_X(x)(p) = p(x)$, $\forall x \in X$, so, $p \circ f^{-1} = \phi(p) \circ i$, $\forall p \in \tau$.

Hence $f^{-1}$ is $Q$-continuous.

### 3.1 Another description of $Q$-soberity

This section is motivated by [7] (Section 6; Prop. 28(c), page 107) wherein, sobriety in $TOP$ was shown to have some link with an adjoint situation between $TOP$ and $SET^{op}$, arising out of a use of the two-point Sierpinski topological space. We show here that an analogous link exists for $Q$-soberity also.

Let $G : SET^{op} \to Q$-$TOP$ and $F : Q$-$TOP \to SET^{op}$ be the functors, described as follows:

$G$ sends an object $X$ to $Q_S^{X}$ (the $X$-fold product of $Q_S$) and a morphism $f : X \to Y$ to $G(f) : Q_S^{X} \to Q_S^{Y}$, given by $G(f)(g) = g \circ f$, while $F$ sends an object $X = (X, \tau)$ to the set $C(X, Q_S)$ of all $Q$-continuous functions from $(X, \tau)$ to $Q_S$ (which is equal to $\tau$) and a morphism $f : X \to Y$ to $F(f) : C(X, Q_S) \to C(Y, Q_S)$, given by $F(f)(\alpha) = \alpha \circ f$.

It can be easily verified that $G$ is right adjoint to $F$ and that the unit $\psi : Id_{Q$-$TOP} \to GF$ of this adjunction is given as follows:

for every $X = (X, \tau) \in \text{ob} Q$-$TOP$, $\psi_X : X \to GF(X) = Q_S^{X}$, is defined as $\psi_X(x)(p) = p(x), \forall x \in X, \forall p \in \tau$.

Let $T = GF$ and let $e : EX \to TX$ be the equalizer (in $Q$-$TOP$) of $T\psi_X$ and $\psi_T X, e$ being the inclusion map. Then $EX = \{f \in TX \mid T\psi_X(f) = \psi_T X(f)\}$. As $\psi$ is a natural transformation, we have $T\psi_X \circ
\[ \psi_X = \psi_{TX} \circ \psi_X, \text{ so there exists a unique morphism } k_X : X \to EX \text{ in } Q\text{-TOP} \text{ such that } \psi_X = e \circ k_X \text{ (see the following diagram).} \]

\[
\begin{array}{ccc}
  X & \xrightarrow{k_X} & EX \\
  \downarrow{\psi_X} & & \downarrow{e} \\
  TX & \xrightarrow{\tau_{TX}} & T^2X
\end{array}
\]

**Proposition 3.6.** Let \((X, \tau) \in obQ\text{-TOP}.\) Then \((X, \tau)\) is Q-sober iff \(k_X : X \to EX\) is a Q-homeomorphism.

**Proof:** In view of Prop. 3.4, it will suffice to show that (i) the \(Q\)-topological space \(EX\) is the same as \((pt\tau, \phi(\tau))\) and (ii) \(k_X = \eta_X.\)

For \((X, \tau) \in obQ\text{-TOP},\) note that \(T\psi_X, \psi_{TX} : Q_S \to Q^C(Q_S, QS)\) are given by \((T\psi_X)(\alpha) = f(\alpha \circ \psi_X)\) and \((\psi_{TX})(\alpha) = \alpha(f), \forall f \in Q_S, \forall \alpha \in C(Q_S, QS).\) So, \(EX = \{ f \in TX \mid f(\alpha \circ \psi_X) = \alpha(f), \forall \alpha \in C(Q_S, QS) \}.\) As \(Q_S\) has the \(Q\)-product \(Q\text{-topology,}\) for every \(\alpha \in C(Q_S, QS),\) there is some \(\lambda \in I\) such that \(\alpha = \omega^Q_\alpha((\pi_{\alpha n})_{\lambda \in n}),\) where \(\pi_{\alpha n} : Q_S^\tau \to Q_S\) is the \(i^{th}\) projection map, for \(\alpha \in \tau\) and \(i \in n.\) So, \(f(\alpha \circ \psi_X) = \omega^Q_\alpha((\pi_{\alpha n})_{\lambda \in n}) \circ \psi_X = \omega^Q_\alpha((\pi_{\alpha n} \circ \psi_X)_{\lambda \in n}) = \omega^Q_\alpha((\alpha_{\lambda n})_{\lambda \in n}).\) Hence, \(f(\alpha \circ \psi_X) = f(\omega^Q_\alpha((\alpha_{\lambda n})_{\lambda \in n})).\) Also, \(\alpha(f) = (\omega^Q_\alpha((\pi_{\alpha n})_{\lambda \in n}))(f) = \omega^Q_\alpha((\pi_{\alpha n}(f))_{\lambda \in n}) = \omega^Q_\alpha((f(\alpha))_{\lambda \in n}).\) Consequently, \(EX = \{ f \in TX \mid f(\omega^Q_\alpha((\alpha_{\lambda n})_{\lambda \in n})) = \omega^Q_\alpha((f(\alpha))_{\lambda \in n}) \forall \lambda \in I \}, i.e., EX = \{ f \mid f : \tau \to Q \text{ is an Q-algebra homomorphism} \} = pt\tau.\)

Note also that \(EX\) is the \(Q\)-subspace of \(Q^S(TX)\). It can be easily verified that the \(Q\)-subspace topology on \(EX\) is the same as the \(Q\)-topology \(\phi(\tau)\) on \(pt\tau\). Thus the \(Q\)-topological spaces \(EX\) and \((pt\tau, \phi(\tau))\) are the same. This establishes (i).

From the definition of \(\psi_X,\) it is clear that \(\eta_X\) is the ‘corestriction’ of \(\psi_X\) to \(pt\tau.\) Also, from the diagram above, \(\psi_X = e \circ k_X.\) Hence, \(\forall x \in X, \eta_X(x) = \psi_X(x) = e(k_X(x)) = k_X(x).\) Thus \(\eta_X = k_X,\) which establishes (ii). \(\square\)

### 3.2 \(Q\text{-SOB as the epireflective hull of } Q_S\)

For \((X, \tau) \in obQ\text{-TOP} \text{ and } M \subseteq X, \text{ put } [M] = \cap \{ Eq(f, g) \mid f, g \in \tau \text{ and } f|M = g|M \}, \text{ where } Eq(f, g) = \{ x \in X \mid f(x) = g(x) \}. \text{ It turns out that } [[M]] = [M]. \text{ Also, if } [M] = M, \text{ then we say that } M \text{ is a closed Q-subobject.}\)

For showing that \(Q\text{-SOB}\) is the epireflective hull of \(Q_S\) in \(Q\text{-TOP}_0,\) we shall need to identify (i) the epimorphisms in \(Q\text{-TOP}_0\) and (ii) the extremal subobjects in \(Q\text{-TOP}_0.\)

**Proposition 3.7.** A morphism \(e : (X, \tau) \to (Y, \delta)\) in \(Q\text{-TOP}_0\) is an epimorphism if and only if \(e_Q\) is injective.

**Proof:** Suppose \(e\) is an epimorphism and for \(q_1, q_2 \in \delta, e_Q(q_1) = e_Q(q_2).\) Then \(q_1 \circ e = q_2 \circ e,\) implying that \(q_1 = q_2.\)

Conversely, suppose the given condition is satisfied. Now, consider any distinct pair \(f, g : (Y, \delta) \to (Z, \sigma)\) of morphisms in \(Q\text{-TOP}_0.\) Then for
some \( y \in Y \), \( f(y) \neq g(y) \). Since \( Z \) is \( Q\)-\( T_0 \), \( \exists \ p \in \sigma \) such that \( p(f(y)) \neq p(g(y)) \), i.e., \( f_Q^p (p) \neq g_Q^p (p) \). This gives \( c_Q (f_Q^p (p)) \neq c_Q (g_Q^p (p)) \) i.e., \( p \circ f \circ e \neq p \circ g \circ e \), implying that \( f \circ e \neq g \circ e \). Thus \( e \) is an epimorphism.

\[ \blacksquare \]

**Proposition 3.8.** A morphism \( f : (X, \tau) \to (Y, \delta) \) in \( Q\)-\( TOP_0 \) is an epimorphism if and only if \( [f(X)] = Y \).

**Proof.** First, let \( f : (X, \tau) \to (Y, \delta) \) be an epimorphism in \( Q\)-\( TOP_0 \). Let \( [f(X)] \neq Y \). Then \( \exists \ y \in Y \) such that \( y \notin [f(X)] \), and so \( \exists \) morphisms \( g, h : (Y, \delta) \to Q_S \) in \( Q\)-\( TOP_0 \) with \( g|_{f(X)} = h|_{f(X)} \) and \( g(y) \neq h(y) \).

Since \( g|_{f(X)} = h|_{f(X)} \), \( g \circ f = h \circ f \), which is a contradiction. Thus \( [f(X)] = Y \).

Conversely, let \( [f(X)] = Y \). Consider any two morphisms \( g, h : (Y, \delta) \to (Z, \sigma) \) in \( Q\)-\( TOP_0 \) such that \( g \circ f = h \circ f \). If possible, let \( g \neq h \). Then \( \exists \ y \in Y \) such that \( g(y) \neq h(y) \). Since \( g \circ f = h \circ f \), \( g|_{f(X)} = h|_{f(X)} \). But then \( y \notin [f(X)] \), a contradiction. Thus \( f \) is an epimorphism. \( \blacksquare \)

We say that an embedding \( e : (X, \tau) \to (Y, \delta) \) in \( Q\)-\( TOP_0 \) is \([\_]\)-closed if \( [e(X)] = e(X) \).

**Proposition 3.9.** The extremal monomorphisms in \( Q\)-\( TOP_0 \) are precisely the \([\_]\)-closed embeddings (in \( Q\)-\( TOP_0 \)).

**Proof.** Let \( m : (X, \tau) \to (Y, \delta) \) be an extremal monomorphism in \( Q\)-\( TOP_0 \) and \( Z = [m(X)] \) (with the \( Q\)-subspace topology \( \delta_Z \)). Define a map \( e : (X, \tau) \to (Z, \delta_Z) \) as \( e(x) = m(x), \forall x \in X \). Then \( e \) is an epimorphism in \( Q\)-\( TOP_0 \) and \( m = i \circ e \), where \( i : (Z, \delta_Z) \to (Y, \delta) \) is the inclusion map. But then \( e \) is a \( Q\)-homeomorphism. Thus \( m \) is a \([\_]\)-closed embedding.

Conversely, let \( m : (X, \tau) \to (Y, \delta) \) be a \([\_]\)-closed embedding. Let the elements of the set \( \{ (f, g) \in \tau \times \tau \mid f|_{m(X)} = g|_{m(X)} \} \) be indexed by an index set \( J \). Then \( [m(X)] = \bigcap \{ Eq(f_j, g_j) \mid f_j, g_j \in \tau \text{ and } j \in J \} \). For every \( j \in J \), let \( \pi_j : Q_S \to Q_S \) be the \( j^\text{th} \) projection map. Then by the property of the product, there exists unique \( Q\)-continuous maps \( f^*, g^* : (X, \tau) \to Q_S \) such that \( \pi_j \circ f^* = f_j \) and \( \pi_j \circ g^* = g_j \), \( \forall j \in J \). Now it can be easily verified that \( [m(X)] = Eq(f^*, g^*) \), whereby \( m(X) \) (in fact, the inclusion map from \( (m(X), \delta_{m(X)}) \)) to \( (Y, \delta) \) is an equalizer in \( Q\)-\( TOP_0 \) (this also follows from Proposition 1.6 of \( [3] \), which, however, is stated in a more general set-up). But as equalizers are extremal monomorphisms, \( m \) is an extremal monomorphism. \( \blacksquare \)

**Corollary 3.1.** The extremal subobjects in \( Q\)-\( TOP_0 \) are precisely the \([\_]\)-closed subspaces of \( Q\)-\( T_0 \)-topological spaces.

The next result is analogous to the corresponding results in \([5] \) and \([10] \).

**Theorem 3.2.** (i) \( Q\)-\( SOB \) is epireflective in \( Q\)-\( TOP_0 \) and (ii) this epireflectivity is \( \mathcal{H} \)-firm, where \( \mathcal{H} \) is the class of all \( Q\)-\( TOP_0 \)-embeddings.

\(^3\)This result has been proved in \([3] \) (Theorem 1.11) in a more general set-up. The proof being given here is somewhat more direct.
Proof: (i) Let \((X, \tau) \in \text{ob-Q-TOP}_0\). We show that \(\eta_X : (X, \tau) \rightarrow (\text{ptr}, \varphi(\tau))\) is the desired epireflection of \((X, \tau)\) in \(\text{Q-SOB}\). We use Proposition 3.6 to show first that \(\eta_X\) is an epimorphism. Let \(\eta_X^{-1}(\varphi(p_1)) = \eta_X^{-1}(\varphi(p_2))\), where \(p_1, p_2 \in \tau\). Then \(\eta_X^{-1}(\varphi(p_1))(x) = \eta_X^{-1}(\varphi(p_2))(x)\), \(\forall x \in X\), implying that \(\varphi(p_1)(\eta_X(x)) = \varphi(p_2)(\eta_X(x))\), i.e., \(\eta_X(x)(p_1) = \eta_X(x)(p_2)\), which gives \(p_1(x) = p_2(x)\). Hence, \(p_1 = p_2\), whereby \(\varphi(p_1) = \varphi(p_2)\).

Let \((Y, \delta) \in \text{ob-Q-SOB}\) and \(f : (X, \tau) \rightarrow (Y, \delta)\) be \(Q\)-continuous. We need to find a \(Q\)-TOP-morphism \(f^* : (\text{ptr}, \varphi(\tau)) \rightarrow (Y, \delta)\) such that \(f^* \circ \eta_X = f\). For any \(\alpha \in \text{ptr}\), define \(\alpha' : \delta \rightarrow Q\) by \(\alpha'(q) = \alpha(q \circ f)\).

It can be verified that \(\alpha'\) is an \(\Omega\)-algebra homomorphism, i.e., \(\alpha' \circ \delta\). Since \((Y, \delta)\) is \(Q\)-sober, there is a unique \(y \in Y\) with \(\eta_Y(y) = \alpha'\). Put \(f'(\alpha) = y\). This gives us a map \(f^* : \text{ptr} \rightarrow Y\). Now, given \(q \in \delta\) and \(\alpha \in \text{ptr}, f^* \circ \varphi \circ (\alpha) = (q \circ f^*) \circ (\alpha) = q(y) = \eta_Y(y)(q) = \alpha'(q) = \alpha(q \circ f) = \alpha(f^{\varphi}(q)) = \alpha(f^{\varphi}_\alpha(q)), \) whereby \(f^{\varphi}_\alpha(q) \in \varphi(\tau)\), showing the \(Q\)-continuity of \(f^*\). Now, \(\forall q \in \delta\) and \(\forall x \in X\), \(\eta_Y(f(x))(q) = q(f(x)) = (q \circ f)(x) = \eta_Y(x)(q \circ f) = \eta_Y(x)(q)\). Hence, \(\eta_Y(f(x)) = (\eta_Y(x))'\), \(\forall x \in X\). So \(f^*(\eta_X(x)) = f(x)\), \(\forall x \in X\). Hence, \(f^* \circ \eta_X = f\). Finally, as \(\eta_X\) is an epimorphism, \(f^*\) is unique. This proves (i).

We now prove (ii). Let \((X, \tau) \in \text{ob-Q-TOP}_0\). Then the epireflection \(\eta_X : (X, \tau) \rightarrow (\text{ptr}, \varphi(\tau))\) is injective and it is an embedding in \(\text{Q-TOP}_0\). Let \((Y, \delta) \in \text{ob-Q-SOB}\), \(f : (X, \tau) \rightarrow (Y, \delta)\) be an epimorphic-embedding in \(\text{Q-TOP}_0\) and \(f^* : (\text{ptr}, \varphi(\tau)) \rightarrow (Y, \delta)\) the unique \(Q\)-TOP-morphism such that \(f^* \circ \eta_X = f\). Let \(\hat{f} : X \rightarrow f(X)\) be the ‘corestriction’ of \(f\) to \(f(X)\). Then \(\hat{f}^{-1} : (f(X), \delta_{f(X)}) \rightarrow (X, \tau)\) is clearly \(Q\)-continuous. So, \(\forall p \in \tau \exists p_f \in \delta\) such that \((\hat{f}^{-1})_{\epsilon}(p) = p_f \circ i\), where \(i : f(X) \rightarrow Y\) is the inclusion map. Hence \(f^{\varphi}_\alpha(p_f) = p\). As \(f\) is an epimorphism, this \(p_f\) is unique such that \(f^{\varphi}_\alpha(p_f) = p\). Now, define \(g : (Y, \delta) \rightarrow (\text{ptr}, \varphi(\tau))\) by \(g(y)(p) = p_f(y), \forall y \in Y\) and \(\forall p \in \tau\). It can be easily verified that \(g(y) : \tau \rightarrow Q\) is an \(\Omega\)-algebra homomorphism. Now, \(\forall y \in Y\) and \(\forall p \in \tau\) and \(\forall y \in Y, g^\alpha_q(\varphi(p)(y)) = \varphi(p)(g(y)) = g(y)(p) = p_f(y)\). So, \(g^\alpha_q(\varphi(p)) = p_f\). Thus \(g^\alpha_q(\varphi(p)) \in \delta\), showing that \(g\) is \(Q\)-continuous. Since \(\forall q \in \delta\), \(f^{\varphi}_\alpha(q) \in \tau\) and so \(f^{\varphi}_\alpha((f^{\varphi}_\alpha(q))) = f^{\varphi}_\alpha(q)\), whereby \((f^{\varphi}_\alpha(q))' = q\). Hence, \(\forall q \in \delta\) and \(\forall y \in Y, (f^{\varphi}_\alpha(q)) = g(y), \) implying that \(g(y)(f^{\varphi}_\alpha(q)) = \eta_Y(y)(q), \) i.e., \(g(y)(q \circ f) = \eta_Y(y)(q)\). So, \(f^*(g(y)) = y, \forall y \in Y\). Thus \(f^* \circ g = id_Y\).

Next, let \(\alpha \in \text{ptr}\) and \(f^*(\alpha) = y\). Then \(\eta_Y(y)(q) = \alpha(q \circ f), \forall q \in \delta\).

For \(p \in \tau\), \(g(y)(p) = p_f(y) = \eta_Y(y)(p_f) = \alpha(p_f \circ f) = \alpha(f^{\varphi}_\alpha(p_f)) = \alpha(p)\) implying that \(g(y) = \alpha\). Hence \(g \circ f^* = id_{\text{ptr}}\). Thus \(f^*\) is a \(Q\)-TOP-isomorphism. \(\square\)

Using Theorem 1 of [3], we get the following corollary:

Corollary 3.2. \(\text{Q-SOB}\) is closed under forming products and extremal subobjects in \(\text{Q-TOP}_0\).

Proposition 3.10. If \((X, \tau) \in \text{ob-Q-TOP}_0\), then \((\text{ptr}, \varphi(\tau))\) is a \([\ ]\)-closed subspace of \(Q^\tau_S\).

Proof: From [3] (Theorem 58), it follows that the map \(e : X \rightarrow Q^\tau_S\) defined by \(e(x)(\mu) = \mu(x), \forall x \in X, \forall \mu \in \tau\), is a \(Q\)-TOP\(_0\)-embedding.
Hence $f : X \to [e(X)]$, the ‘corestriction’ of $e$ to $[e(X)]$, is an epimorphic-embedding in $Q\text{-}\text{TOP}_0$ (by Proposition 3.7). $[e(X)]$, being a $[\ ]$-closed subspace of $Q_S^\tau$ (which is $Q$-sober), is, therefore, an extremal subobject of $Q_S^\tau$. Hence $[e(X)]$ is $Q$-sober. Theorem 3.2(ii), now provides a $Q$-sober isomorphism $f^* : pt \tau \to [e(X)]$ (such that $f^* \circ \eta_X = f$). Hence $(pt \tau, \phi(\tau))$ is a $[\ ]$-closed subspace of $Q_S^\tau$. □

**Proposition 3.11.** $(X, \tau) \in \text{ob} \ Q\text{-}\text{SOB}$ if and only if it is $Q$-homeomorphic to a $[\ ]$-closed subspace of $Q_S^\tau$.

**Proof:** Let $(X, \tau) \in \text{ob} \ Q\text{-}\text{SOB}$. Then, via $\eta_X$, $(X, \tau)$ is $Q$-homeomorphic to $(pt \tau, \phi(\tau))$, which is a $[\ ]$-closed subspace of $Q_S^\tau$. The converse follows from Proposition 3.1, Corollary 3.1 and Corollary 3.2. □

Using Theorem 2 of [5], together with Corollary 3.1 and Proposition 3.10 above, we now obtain the following result:

**Theorem 3.3.** $Q\text{-}\text{SOB}$ is the epireflective hull of $Q_S$ in $Q\text{-}\text{TOP}_0$.

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