FINITE-TIME SYNCHRONIZATION OF COMPETITIVE NEURAL NETWORKS WITH MIXED DELAYS

TINGTING SU AND XINSONG YANG*

Department of Mathematics
Chongqing Normal University
Chongqing 401331, China

Abstract. In this paper, finite-time synchronization of competitive neural networks (CNNs) with bounded time-varying discrete and distributed delays (mixed delays) is investigated. A simple controller is added to response (slave) system such that it can be synchronized with the driving (master) CNN in a setting time. By introducing a suitable Lyapunov-Krasovskii’s functional and utilizing some inequalities, several sufficient conditions are obtained to ensure the control object. Moreover, the setting time is explicitly given. Different from previous results, the setting is related to both the initial value of error system and the time delays. Finally, numerical examples are given to show the effectiveness of the theoretical results.

1. Introduction. In the past decades, much attention has been attracted to chaos control and synchronization due to their important applications in image processing and secure communication [15], [11]. Neural networks, as a special kind of nonlinear dynamical systems, can also exhibit chaotic behavior if their parameters are appropriately chosen [22]. Therefore, much effort has been devoted to the research of synchronization of neural networks [22], [13], [9], [14], [6], [18]. For instance, exponential stochastic synchronization of coupled memristor-based neural networks was studied in [22], and synchronization induced by temporal delays in pulse-coupled neural networks was investigated in [9]. Recently, another kind of neural networks, called competitive neural networks (CNNs), has received increasing attention of researchers [10], [23]. In the models of [10] and [23], there are two types of state variables: the short-term memory variable describing the fast neural activity, and the long memory variable describing the slow unsupervised synaptic modifications. In the literature, there are many results concerning synchronization of CNNs [17], [4], [8], [5], [20]. For instance, authors in [17] and [4] studied asymptotic synchronization of CNNs with mixed delays and perturbations. Authors in [20] further investigated the problem of exponential synchronization of switched stochastic CNNs with both interval time-varying delays and distributed delays. It is worth reminding

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*Corresponding author: Xinsong Yang.
that most of existing results concerning synchronization of CNNs with delays are asymptotic, i.e., CNNs with delays can achieve synchronization when time goes to infinity. To the best of our knowledge, few published papers consider finite-time synchronization of CNNs with delays.

Compared with asymptotic synchronization, finite-time synchronization is optimal [19]. Finite-time synchronization means that the synchronization is achieved in a setting time. Moreover, it is well known that finite-time synchronization technique has better robustness and disturbance rejection properties than asymptotic technique. Hence, the study of finite-time synchronization problem is of great importance [1], [21], [2], [16]. Authors in [1] and [21] investigated finite-time synchronization of some chaotic systems. By using finite-time stability theorem, inequality techniques, the properties of Weiner process and adding suitable controllers, the authors [2] studied finite-time synchronization of complex networks with noises without delay. The authors in [16] studied the finite-time stochastic synchronization problem for complex networks with stochastic noise perturbations. It should be noted that most of existing results concerning finite-time synchronization are obtained by using the finite-time stability theorem in [12]. However, it is reported in [3] that the finite-time stability theorem in [12] cannot be applied to time-delay systems. Recently, the authors of [7] tried to study finite-time stability and stabilization of functional differential equations with delays by using Lyapunov functionals. Unfortunately, the theoretical result in [7] is not usable in practice for studying the finite-time stabilization problem because it is extremely difficult to find a Lyapunov functional satisfying the assumptions in [7]. Since CNN has both two types of state variables (LTM and STM), it is more difficult to investigate the finite-time of CNNs with mixed delays. This paper aims to fill this challenging gap.

Motivated by the above discussions, in this paper, we investigate finite-time synchronization of drive-response CNNs with mixed delays by using a simple discontinuous controller. By introducing a suitable Lyapunov-Krasovskii’s functional and utilizing some inequalities, several sufficient conditions are obtained to ensure the finite-time synchronization. Moreover, the setting time is explicitly given. Different from previous results, the setting time is related to both the initial value of error system and the time delays. Finally, numerical examples demonstrate the effectiveness of the theoretical results.

The rest of this paper is organized as follows. In Section II, CNNs with mixed delays is proposed. Some necessary assumptions are also given in this section. In Section III, finite-time synchronization of the considered system is studied. Then, in Section IV, numerical simulations are given to show the effectiveness of the theoretical results. Finally, conclusions and future research interests are given in Section V.

Notations. The notations are quite standard. Throughout this paper, $\mathbb{R}^+$ and $\mathbb{R}^n$ denote, respectively, the set of nonnegative real numbers and the $n$-dimensional Euclidean space. The superscript $T$ denotes matrix or vector transposition. For a vector $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$, if $|x_i|, i = 1, 2, ..., n$ is bounded, then we say $x$ is bounded. The notion $X \preceq Y$ (respectively, $X < Y$), where $X$ and $Y$ are symmetric matrices, means that $X - Y$ is negative semidefinite (respectively, negative definite). The shorthand $\text{diag}(\omega_1, \omega_2, ..., \omega_n)$ denotes a diagonal matrix with the diagonal elements $\omega_1, \omega_2, ..., \omega_n$. Sometimes, the arguments of a function or a matrix will be omitted in the analysis when no confusion can arise.
2. **Model description and some preliminaries.** Consider a model of master and slave systems of \( N \) nonidentical nodes with time-varying discrete delay and unbounded delay is described as follows:

\[
\begin{align*}
\varepsilon \dot{x}_i(t) &= -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} f_j(x_j(t) - h_{ij}(t)) \\
&+ \sum_{j=1}^{n} d_{ij} \int_{-\tau_{ij}}^{0} f_j(x_j(s)) \, ds + E_i \sum_{l=1}^{p} m_{il}(t) F_l,
\end{align*}
\]

where the first equation denotes the STM, the second equation denotes the LTM, \( i = 1, 2, \ldots, n \) denotes the number of neurons, \( l = 1, 2, \ldots, p \) denotes the number of the constant external stimulus, \( x_i(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^N \) is the neuron current activity level, \( \varepsilon \) is the time scale of STM state, \( f_j(x_j(t)) = (f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t)))^T \) is output of neurons, \( m_{ij}(t) = (m_{i1}(t), m_{i2}(t), \ldots, m_{ip}(t))^T \) is the synaptic feedback, \( E_i = (E_1, E_2, \ldots, E_p)^T \) is the constant external stimulus, \( b_{ij} \) is diagonal matrix \( b_{ij} = \text{diag}(a_2, a_3, \ldots, a_n) \) with \( a_i > 0 \) is the time constant of neurons. \( b_{ij} \) and \( d_{ij} \) are the connection weight matrices of delay feedback, \( E_i = \text{diag}(E_1, E_2, \ldots, E_p) \) is the strength of the external stimulus. \( a_i = \text{diag}(a_2, a_3, \ldots, a_n) \) and \( \beta_i = \text{diag}(\beta_1, \beta_2, \ldots, \beta_n) \) denote observable scaling constants with \( a_i > 0 \). The bounded function \( h_{ij} > 0 \) represents the time-varying discrete delay of the \( j \) th unit from the \( i \) th unit. \( S_i(t) = \sum_{j=1}^{p} m_{ij} F_j = m_{i}(t) F_i, S_i(t) = (S_1(t), S_2(t), \ldots, S_n(t))^T \in \mathbb{R}^N \).

From (1), we obtain the following CNN:

\[
\begin{align*}
\varepsilon \dot{x}_i(t) &= -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} f_j(x_j(t) - h_{ij}(t)) \\
&+ \sum_{j=1}^{n} d_{ij} \int_{-\tau_{ij}}^{0} f_j(x_j(s)) \, ds + E_i S_i(t), \\
\dot{S}_i(t) &= - \alpha_i S_i(t) + \beta_i |F|^2 f_i(x_i(t)), \quad i = 1, 2, \ldots, n,
\end{align*}
\]

where \( |F|^2 = F_1^2 + F_2^2 + \cdots + F_p^2 \) is a constant. Without loss of generality, the input stimulus vector \( F \) is assumed to be normalized with unit magnitude \( |F|^2 = 1 \). Then (2) can be written as the following following:

\[
\begin{align*}
\dot{x}_i(t) &= -\frac{1}{\varepsilon} a_i x_i(t) + \frac{1}{\varepsilon} \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \frac{1}{\varepsilon} \sum_{j=1}^{n} c_{ij} f_j(x_j(t) - h_{ij}(t)) \\
&+ \frac{1}{\varepsilon} \sum_{j=1}^{n} d_{ij} \int_{-\tau_{ij}}^{0} f_j(x_j(s)) \, ds + \frac{1}{\varepsilon} E_i S_i(t), \\
\dot{S}_i(t) &= - \alpha_i S_i(t) + \beta_i f_i(x_i(t)), \quad i = 1, 2, \ldots, n.
\end{align*}
\]

The initial conditions of (3) is given as \( x(t) = \phi^x(s), \phi^x(s) = (\phi_1^x(s), \phi_2^x(s), \ldots, \phi_n^x(s))^T \in C([-\kappa, 0], \mathbb{R}^n), S(t) = \phi^s(s), \phi^s(s) = (\phi_1^s(s), \phi_2^s(s), \ldots, \phi_n^s(s))^T \in C([-\kappa, 0], \mathbb{R}^n), \) where \( \kappa = \max\{h, \tau_{ij}\} \). \( x(t) \) can be any desired state: equilibrium point, a nontrivial periodic orbit, or even a chaotic orbit.

Based on the concept of drive-response synchronization, we take (1) as the drive system. The corresponding response system is constructed as follows:

\[
\begin{align*}
\dot{y}_i(t) &= -\frac{1}{\varepsilon} a_i y_i(t) + \frac{1}{\varepsilon} \sum_{j=1}^{n} b_{ij} f_j(y_j(t)) + \frac{1}{\varepsilon} \sum_{j=1}^{n} c_{ij} f_j(y_j(t) - h_{ij}(t)) \\
&+ \frac{1}{\varepsilon} \sum_{j=1}^{n} d_{ij} \int_{-\tau_{ij}}^{0} f_j(y_j(s)) \, ds + \frac{1}{\varepsilon} E_i R_i(t) + U_i, \\
\dot{R}_i(t) &= - \alpha_i R_i(t) + \beta_i f_i(y_i(t)), \quad i = 1, 2, \ldots, n,
\end{align*}
\]

where \( y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T \in \mathbb{R}^n \) is the state vector of the response system at time \( t \). \( U_i(t) \) is the controller to be designed.

The initial conditions of (4) is given as \( \phi^y(s) = (\phi_1^y(s), \phi_2^y(s), \ldots, \phi_n^y(s))^T \in C([-\kappa, 0], \mathbb{R}^n), \phi^r(s) = (\phi_1^r(s), \phi_2^r(s), \ldots, \phi_n^r(s))^T \in C([-\kappa, 0], \mathbb{R}^n), \) where \( \kappa = \max\{h, \tau_{ij}\} \).
In order to investigate the problem of finite-time synchronization between (3) and (4), we define the synchronization error signal \( e_i(t) = y_i(t) - x_i(t) \). Subtracting (3) from (4) yields the following error system:

\[
\begin{aligned}
\dot{e}_i(t) &= -\frac{1}{\varepsilon}a_i e_i(t) + \frac{1}{\varepsilon} \sum_{j=1}^{n} b_{ij} g_j(e_j(t)) + \frac{1}{\varepsilon} \sum_{j=1}^{n} c_{ij} \dot{g}_j(e_j(t - h_{ij}(t))) \\
\dot{Z}_i(t) &= -\alpha_i Z_i(t) + \beta_i g_i(e_i(t)), \quad i = 1, 2, \ldots, n,
\end{aligned}
\]

where \( g_j(e_j(.)) = f_j(u_j(.)) - f_j(x_j(.)), e_i(t) = (e_1(t), e_2(t), \ldots, e_n(t)) \).

The initial conditions of (5) is given as

\[
\begin{aligned}
Z_i(0) &= \sum_{j=1}^{n} c_{ij} |e_j(0)| + \frac{1}{1 - \mu_{ij}} |d_{ij}| \gamma_j, \\
\end{aligned}
\]

\[
\dot{Z}_i(t) = \frac{1}{\varepsilon} \sum_{s} |e_i(s)|ds - \frac{1}{\varepsilon} \sum_{s} |d_{ij} \gamma_j \int_{t-\tau_j}^{t} |e_j(s)|ds|ds\]

\[
\dot{t}^* \leq \frac{1}{\rho} \left[ \sum_{i=1}^{n} \sum_{s} |e_i(0)| + \sum_{i=1}^{n} \sum_{j=1}^{n} |Z_i(s)| + \frac{1}{\varepsilon} \sum_{i=1}^{n} \sum_{j=1}^{n} |d_{ij} \gamma_j \int_{t-\tau_j}^{t} |e_j(s)|ds|ds\right] - \hat{h},
\]

where \( \hat{h} \) is a tunable constant.
where $\dot{\rho} = \min\{\rho_i = 1, 2, \ldots, n\}$, $\dot{h} = \max\{\dot{h}_{ij}, \tau_{ij}, i, j = 1, 2, \ldots, n\}$.

**Proof.** Define the following Lyapunov-Krasovskii functional candidate:

$$ V(t) = \sum_{i=1}^{3} V_i(t), \quad (10) $$

where

$$ V_1(t) = \sum_{i=1}^{n} |e_i(t)| + \sum_{i=1}^{n} |z_i(t)|, $$

$$ V_2(t) = \frac{1}{\varepsilon} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{1 - \mu_{ij}} |c_{ij}| \gamma_j \int_{t-h(t)}^{t} |e_j(s)| ds, $$

$$ V_3(t) = \frac{1}{\varepsilon} \sum_{i=1}^{n} \sum_{j=1}^{n} |d_{ij}| \gamma_j \int_{t-\tau_{ij}}^{t} \int_{s}^{t} |e_j(u)| du ds. $$

Calculating the time derivative of $V_1(t)$ along the trajectories of the error system (5), it follows from $(H_1)$ that:

$$ \dot{V}_1(t) \leq \sum_{i=1}^{n} \left\{ \left( -\frac{\alpha_i}{\varepsilon} |e_i(t)| + \text{sgn}(e_i(t)) \left[ \frac{1}{\varepsilon} \sum_{j=1}^{n} h_{ij} g_j(e_j(t)) + \frac{1}{\varepsilon} \sum_{j=1}^{n} c_{ij} g_j(e_j(t-h_{ij}(t))) \right] \right) \\
+ \frac{1}{\varepsilon} \sum_{j=1}^{n} d_{ij} \int_{t-\tau_{ij}}^{t} g_j(e_j(s)) ds + \frac{1}{\varepsilon} E_i z_i(t) - l_i e_i(t) - \rho_i \text{sgn}(e_i(t)) \right\} \\
+ \sum_{i=1}^{n} \left\{ -\alpha_i |z_i(t)| + |\beta_i g_i(e_i(t))| \right\} $$

$$ \leq \sum_{i=1}^{n} \left\{ -\left( \frac{\alpha_i}{\varepsilon} + l_i \right) |e_i(t)| + \beta_i \gamma_j |e_i(t)| + \frac{1}{\varepsilon} \sum_{j=1}^{n} \left[ h_{ij} \gamma_j |e_j(t)| \\
+ c_{ij} \gamma_j |(e_j(t-h_{ij}(t)))| + d_{ij} \gamma_j \int_{t-\tau_{ij}}^{t} |(e_j(s))| ds \right] \\
+ \left( \frac{1}{\varepsilon} \sum_{i=1}^{n} E_i - \sum_{i=1}^{n} \alpha_i \right) |z_i(t)| - \rho_i \lambda_i \right\}, \quad (12) $$

where $|e_i(t)| \neq 0$, $\lambda_i = 1$, otherwise, $\lambda_i = 0$.

In view of derivative of time and our assumptions $(H_1)$-$(H_2)$, it is obtained from $V_2(t)$ and $V_3(t)$ that

$$ \dot{V}_2(t) = \frac{1}{\varepsilon} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{1 - \mu_{ij}} |c_{ij}| \gamma_j |e_j(t)| $$

$$ - \frac{1}{\varepsilon} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{1 - \mu_{ij}} (1 - \dot{h}_{ij}(t)) |c_{ij}| \gamma_j |e_j(t-h_{ij}(t))| $$

$$ \leq \frac{1}{\varepsilon} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{1 - \mu_{ij}} |c_{ij}| \gamma_j |e_j(t)| $$
By (12), (13), and (14), we obtain that:

$$-\frac{1}{\varepsilon} \sum_{i=1}^{n} \sum_{j=1}^{n} |e_{ij}| \gamma_j |e_j(t - h_{ij}(t))|,$$

and

$$\dot{V}_3(t) = \frac{1}{\varepsilon} \sum_{i=1}^{n} \sum_{j=1}^{n} |d_{ij}| \gamma_j \int_{t - \tau_{ij}}^{t} |e_j(s)|ds - \frac{1}{\varepsilon} \sum_{i=1}^{n} \sum_{j=1}^{n} |d_{ij}| \gamma_j \int_{t - \tau_{ij}}^{t} |e_j(s)|ds$$

$$= \frac{1}{\varepsilon} \sum_{i=1}^{n} \sum_{j=1}^{n} \tau_{ij} |d_{ij}| \gamma_j |e_j(t)| - \frac{1}{\varepsilon} \sum_{i=1}^{n} \sum_{j=1}^{n} |d_{ij}| \gamma_j \int_{t - \tau_{ij}}^{t} |e_j(s)|ds. \quad (14)$$

By (12), (13), and (14), we obtain that:

$$\dot{V}(t) \leq \sum_{i=1}^{n} \left[ -\left( \frac{\alpha_i}{\varepsilon} + i_i \right) + \beta_i \gamma_i + \frac{1}{\varepsilon} \sum_{j=1}^{n} (|b_{j|i}| \gamma_i + \frac{1}{1 - \mu_{ji}} |c_{ji}| \gamma_i$$

$$+ \tau_{ij} |d_{ij}| \gamma_i) |e_i(t)| + \left( \frac{1}{\varepsilon} \sum_{i=1}^{n} E_i - \sum_{i=1}^{n} \alpha_i \right) |z_i(t)| - \sum_{i=1}^{n} \rho_i \lambda_i. \quad (15)$$

Substituting conditions (7) and (8) into (15) yields the following inequality:

$$\dot{V}(t) \leq -\sum_{i=1}^{n} \rho_i \lambda_i. \quad (16)$$

Because $V(t)$ is positive definite and strictly monotone decreasing known from (14), there exists nonnegative constant $V'$ such that:

$$\lim_{t \to +\infty} V(t) = V' \text{ and } V(t) \geq V', \forall t \geq 0. \quad (17)$$

On the other hand, integrating both side of the inequality (16) from 0 to t gets the following inequality:

$$V(t) - V(0) \leq -\sum_{i=1}^{n} \rho_i \lambda_i t, \quad (18)$$

If $|e(T)| + |Z(T)| = 0$ at a instant $T \in [0, +\infty)$ for $i = 1, 2, \ldots, n$, then we can get that $\|e(T)\|_1 + \|Z(T)\|_1 = 0$ at a instant $T \in [0, +\infty]$ and jump to the step two.

If $\|e(t)\|_1 + \|Z(T)\|_1 > 0$ for all $t \in [0, +\infty)$, when $\|e(t)\|_1 > 0$, $\|Z(t)\|_1 = 0$ then $\sum_{i=1}^{n} \rho_i \lambda_i < 0$, which imply that $\lim_{t \to +\infty} V(t) = -\infty$. This contradicts (16), so the inequality (18) does not hold when $t \to +\infty$. otherwise, when $\|Z(t)\|_1 > 0$, we can get that $\|e(t)\|_1 > 0$, by the analysis above it is This contradicts (16). So, there exists $T \in (0, +\infty)$ such that:

$$\lim_{t \to T} V(t) = V' \text{ and } V(t) = V', \forall t \geq T. \quad (19)$$

According to Definition 1, we should prove that

$$\|e(T)\|_1 + \|Z(T)\|_1 = 0 \text{ and } \|e(t)\|_1 + \|Z(t)\|_1 \equiv 0 \text{ for } \forall t \geq T. \quad (20)$$

We divide the prove by two steps.

**Step 1.** We prove that $\|e(T)\|_1 + \|Z(T)\|_1 = 0$.

Suppose that there is a small constant $\tau > 0$ such that $\|e(t)\|_1 + \|Z(t)\|_1 > 0$ for all $t \in (T, T + \tau]$. For one thing we let $\|Z(t)\|_1 = 0$ and $\|e(t)\|_1 > 0$. So there exists at least one $i \in \{1, 2, \ldots, n\}$ such that $\|e_{i}(t)\|_1 > 0$ for any instant $t \in (T, T + \tau]$. According to the definition of the controllers in (6), that the controller $U_{i}$ is active at the instant $t \in (T, T + \tau]$. Based on the previous analysis, it is easy to know
that \( \dot{V}(t) \leq \rho \| t \) holds for the instant \( t \in (T, T + \tau] \). This contradicts (19). For the other thing we let \( \| Z(t) \| > 0 \), According to the model of (5), When the \( \| Z(T) \| > 0 \), we can get \( \| e(T) \| > 0 \). This contradicts (19), too.

**Step 2.** Now we prove \( \| e(t) \| + \| Z(t) \| \equiv 0 \) for \( \forall t \geq T \).

Suppose that there exists \( t_3 > T \) such that \( \| e(t_3) \| \equiv 0 \). Let \( t_s = \sup \{ t \in [T, t_3] : \| e_1(t) \| = 0 \} \). We have \( t_s < t_1, \| e(t) \| > 0 \) for all \( t \in (t_s, t_1] \). By \( \| e(t) \| > 0 \), for all \( t \in (t_s, t_1] \), there exists at least one \( i_0 \in 1, 2, \ldots, n \) such that \( |e_{i_0}(t)| > 0 \) at any instant \( t \in (t_s, t_1] \). By the argument as above, it follows that \( \dot{V}(t) \leq -\rho < 0 \) holds for the instant \( t \in (t_s, t_1] \), where \( -\rho = \min \rho_i, i = 1, 2, \ldots, n \). Hence \( 0 \leq V(t_1) = V(t_s) + \int_{t_s}^{t_1} \dot{V}(t) dt < -\rho (t_1 - t_s) < 0 \), which is a contradiction. When the \( \| e(t) \| \equiv 0 \), we can also obtain that \( \| Z(t) \| \equiv 0 \). Therefore, the conditions in (20) hold. According to Definition 1, the neural network (4) is synchronized with (3) in a finite time under the controller (6).

Now we can estimate synchronization time for the neural network with bounded mixed delays. From the above discussion we know that there exists nonnegative constant \( V \) such that:

\[
\lim_{t \to T} V(t) = V' \quad \text{and} \quad V(t) = V' \forall t \geq T. \tag{21}
\]

Now we prove that \( V' = 0 \). If \( V' > 0 \), then it is obtained from (9) that there exists \( t_2 \) satisfying \( T - \max \{ \sum_{i,j} r_{ij}, \tau_{ij}, i, j = 1, 2, \ldots, n \} \leq t_3 < t_2 < T \) such that \( \| e(t) \| > 0 \) for all \( t \in [t_3, t_2] \). Note that \( \| e(T) \| = 0 \) and \( \| e(t) \| \equiv 0, \forall t \geq T \). It is obtained from the integrations in \( V(t) \) that, for any instant \( t_4 > T \), there has \( V' = V(T) > V(t_4) = V' \), which is a contradiction. Therefore, \( V' = 0 \).

From the above discussion we also get that \( \dot{V}(t) \leq -\hat{\rho} \) when \( \| e(t) \| \neq 0 \), where \( \hat{\rho} = \min \{ \rho_i - M_i, i = 1, 2, \ldots, n \} \). Integrating both sides of the inequality from 0 to \( T \) obtains that:

\[
T \leq \frac{V(0)}{\hat{\rho}}.
\]

At the same time \( \| e(t^*) \| = 0 \) and \( \| e(t) \| \equiv 0 \) for \( \forall t \geq t^* \), where \( t^* = T - \hat{\rho} \). This completes the proof. \( \square \)

**Remark 1.** In the proof of Theorem 1, the finite-time stability theorem in [12] is not used. The key step in the proof of Theorem 1 is to obtain the inequality (16), which implies that the derivative of the Lyapunov function is less than a negative constant before the realization of synchronization. However, the finite-time stability theorem in [12] is based on the inequality \( \dot{V}(x) \leq -\alpha V^\alpha(x) \), where \( \alpha > 0 \) and \( 0 < \eta < 1 \) are constants, \( \alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \) with class-\( \mathcal{K} \) functions \( \alpha_1(\cdot) \) and \( \alpha_2(\cdot) \). The finite-time synchronization criteria in [1], [21], [2], and [16] are obtained by using the finite-time stability theorems in [12] and their Lyapunov functionals are quadratic form (2-norm). If 2-norm based Lyapunov functional are used, the inequality (16) cannot be derived.

**Remark 2.** The controller (6) has two terms \( -l_i e_i(t) \) and \( -\rho_i \sgn(e_i(t)) \), while the controllers in [17], [4], [8], [5], and [20] do not include the part \( -\rho_i \sgn(e_i(t)) \). From the inequality (15) one can see that the control gains \( l_i, i = 1, 2, \ldots, n \) are used to keep the error system stable (not necessarily to be asymptotic), while, by the inequality (9), the role of control gains \( \rho_i, i = 1, 2, \ldots, n \) is to tune the setting time. Numerical simulations in Fig. 5 demonstrates the role of control gains \( \rho_i, i = 1, 2, \ldots, n \).
When neural networks without delay are concerned, we can consider $V_1(t)$ as the Lyapunov function. Since $V_1(t) = 0$ for all $t$, we get the following Corollary 1 from Theorem 1, we omit its proof here.

**Corollary 1.** Let $c_{ij} = d_{ij} = 0, i, j = 1, 2, \ldots, n$. Suppose that (H2) is satisfied. Then the CNN (4) can be synchronized with (3) in finite time under the controller (6) if the inequalities (8) and

$$l_i \geq \frac{a_i}{\varepsilon} + |\beta_i| \gamma_i + \gamma_i \sum_{j=1}^{n} \frac{1}{\varepsilon} |b_{ji}|,$$

hold. Moreover, the setting time is estimated as $T \leq \frac{1}{\hat{\rho}} \sum_{i=1}^{n} [ |e_i(0)| + |Z_i(0)| ]$, where $\hat{\rho} = \min \{ \rho_i = 1, 2, \ldots, n \}$.  

4. **Examples.** In this section, we will give two examples to illustrate the effectiveness of the theoretical results above.

**Example 1.** Consider the following CNN introduced by [17] which incorporates both discrete and distributed delays.

\[
\begin{align*}
\dot{x}(t) &= -\frac{1}{2}a_{ii}x_i(t) + \frac{1}{\varepsilon} \sum_{j=1}^{n} b_{ij}f_j(x_j(t)) \\
&\quad + \frac{1}{\varepsilon} \sum_{j=1}^{n} c_{ij}f_j(x_j(t - h_{ij}(t))) \\
&\quad + \frac{1}{\varepsilon} \sum_{j=1}^{n} d_{ij} \int_{t-\tau_{ij}}^{t} f_j(x_j(s))ds + \frac{1}{\varepsilon} E_i S_i(t), \\
\dot{S}_i(t) &= -\alpha_i S_i(t) + \beta_i f_i(x_i(t)), \quad i = 1, 2, \ldots, n,
\end{align*}
\]

where $x(t) = (x_1(t), x_2(t))^T$, $f(x(t)) = (\tanh x_1(t), \tanh x_2(t))^T$, $h_{ij}(t) = 1$, $\tau_{ij} = 0.3$, $\varepsilon = 2.5$, 

\[
A = \begin{pmatrix} 1.2 & 0 \\ 0 & 1.0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -0.3 \\ 6 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} -1.4 & 0.1 \\ 0.3 & -8 \end{pmatrix},
\]
Taking the initial conditions as $x(t) = (0.4, 0.6)^T$ and $S(t) = (0.1, 0.6)^T$ for $\forall t \in [-1, 0]$, we have the trajectories of $x(t)$ and $S(t)$ shown in Figs. 3 and 4.

Consider (23) as the master system, the slave CNN is presented as follows:

$$
\begin{align*}
\dot{y}_i(t) &= -\frac{1}{\tau} a_i y_i(t) + \frac{1}{\tau} \sum_{j=1}^{n} b_{ij} f_j(y_j(t)) + \frac{1}{\tau} \sum_{j=1}^{n} c_{ij} f_j(y_j(t - h_{ij}(t))) \\
+ &\frac{1}{\tau} \sum_{j=1}^{n} d_{ij} \int_{t-\tau_{ij}}^{t} f_j(y_j(s))ds + \frac{1}{\tau} E_i S_i(t), \\
\dot{S}_i(t) &= -\alpha_i S_i(t) + \beta_i f_i(y_i(t)), \quad i = 1, 2, \ldots, n.
\end{align*}
$$

The trajectories of (24) with initial values $y(t) = (-1, -0.5)^T$ and $R(t) = (0.5, 1.5)^T$ for $\forall t \in [-1, 0]$ are shown in Figs. 3 and 4.

In this example, one observes that $h_{ij} = 1$ and $\bar{\mu}_{ij} = 0$, $\bar{\tau}_{ij} = 0.3$, $\gamma_1 = \gamma_2 = 1$. Then we can get that $l_1 = -\frac{a_1}{\tau} + |\beta_1| \gamma_1 + \gamma_1 \sum_{j=1}^{n} \frac{1}{\tau} |b_{ij}| + \frac{1}{1-\mu_1} |c_{ij}| + |d_{ij}| \tau_{ij} = 4.88$, $l_2 = -\frac{a_2}{\tau} + |\beta_2| \gamma_2 + \gamma_2 \sum_{j=1}^{n} \frac{1}{\tau} |b_{ij}| + \frac{1}{1-\mu_2} |c_{ij}| + |d_{ij}| \tau_{ij} = 5.38$. According to Theorem 1, the systems can realize finite-time synchronization under the controller (6) for any positive constants $\rho_1$ and $\rho_2$.

In the simulations we choose the initial value of driver and response system as $x(t) = (0.4, 0.6)^T$, $S(t) = (0.1, 0.6)^T$, $y(t) = (-1, -0.5)^T$, $R(t) = (0.5, 1.5)^T$. Fig. 5 describes the time evolution of $\|e(t)\|_1 + \|Z(t)\|_1$ with different values of $\rho_1 = \rho_2$, we can obtain that when $\rho_1 = \rho_2 = 0.5$, $\|e(t)\|_1 + \|Z(t)\|_1$ reaches 0 before 15.8732, when $\rho_1 = \rho_2 = 1$, while $\|e(t)\|_1 + \|Z(t)\|_1$ reaches 0 before 7.4366. Fig. 5 demonstrates that the larger $\rho_1 = \rho_2$ can shorten the synchronization time and the effectiveness of our Theorem 1.
Example 2. Consider the following CNN without delay

\[
\begin{cases}
\dot{x}_i(t) &= -\frac{1}{\varepsilon} a_i x_i(t) + \frac{1}{\varepsilon} \sum_{j=1}^{n} \frac{1}{\varepsilon} b_{ij} f_j(x_j(t)) + \frac{1}{\varepsilon} E_i S_i(t), \\
\dot{S}_i(t) &= -\alpha_i S_i(t) + \beta_i f_i(x_i(t)), \quad i = 1, 2, \ldots, n,
\end{cases}
\]

(25)

where \( x(t) = (x_1(t), x_2(t))^T \), \( f(x(t)) = (\tanh x_1(t), \tanh x_2(t))^T \), \( h_{ij}(t) = 1 \), \( \tau_{ij} = 0.3 \), \( \varepsilon = 2.5 \),

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2.2 & -1.2 \\ 1.2 & 2.2 \end{pmatrix}, \quad E = \begin{pmatrix} -2 & 0 \\ 0 & -10 \end{pmatrix}.
\]
The trajectories of (25) with initial value \( x(t) = (0.4, 0.6)^T \) and \( S(t) = (0.1, 0.4)^T \) for \( \forall t \in [-1, 0] \) are described in Figs. 6 and 7.

We design the following slave system

\[
\begin{align*}
\dot{y}_i(t) &= -\frac{1}{\tau} \alpha_i y_i(t) + \frac{1}{\tau} \sum_{j=1}^{n} b_{ij} f_j(y_j(t)) + \frac{1}{\tau} E_i R_i(t), \\
\dot{R}_i(t) &= -\alpha_i R_i(t) + \beta_i f_i(y_i(t)), \quad i = 1, 2, \ldots, n.
\end{align*}
\]

(26)

Trajectory of (26) with initial value \( y(t) = (-1, -0.1)^T \), \( R(t) = (0.5, 1)^T \) is shown in Figs. 8 and 9.

---

\( \alpha = \begin{pmatrix} 1.5 & 0 \\ 0 & 3 \end{pmatrix} \), \( \beta = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.5 \end{pmatrix} \).

The trajectories of (26) with initial values for \( y(t) = (-1, -0.1)^T \) and \( R(t) = (0.5, 1)^T \) for \( \forall t \in [-1, 0] \) are shown in Figs. 8 and 9.

---

**Figure 8.** Trajectory of \( y(t) \) of (26) with initial conditions \( y(t) = (-1, -0.1)^T \) and \( R(t) = (0.5, 1)^T \), \( \forall t \in [-1, 0] \).

**Figure 9.** Trajectory of \( R(t) \) of (26) with initial conditions \( y(t) = (-1, -0.1)^T \) and \( R(t) = (0.5, 1)^T \), \( \forall t \in [-1, 0] \).

**Figure 10.** Time evolution of \( \|e(t)\|_1 + \|Z(t)\|_1 \) with \( \rho_1 = \rho_2 = 0.5 \) and \( \rho_1 = \rho_2 = 1 \).
According to Corollary 1 if we take $l_1 = -\frac{a_1}{2} + |\beta_1|\gamma_1 + \gamma_1 \sum_{j=1}^{n_l}(|b_{1j}| = 1.56,$ $l_2 = -\frac{a_2}{2} + |\beta_2|\gamma_2 + \frac{a_2}{2} \sum_{j=1}^{n_l}(|b_{2j}| = 1.46,$ the response system (26) can synchronize the derive system (25) in the setting time. Fig. 10 describes the time evolution of $||e(t)||_1 + ||Z(t)||_1$ with different values of $\rho_1 = \rho_2,$ we can obtain that when $\rho_1 = \rho_2 = 0.5,$ $||e(t)||_1 + ||Z(t)||_1$ reaches 0 before 6.2, when $\rho_1 = \rho_2 = 1,$ while $||e(t)||_1 + ||Z(t)||_1$ reaches 0 before 3.1. Fig. 5 demonstrates effectiveness of our Corollary 1. Moreover, it can be seen from Fig. 5 that the larger $\rho_1 = \rho_2$ can shorten the synchronization time.

5. Conclusion and future research issues. In this paper, finite-time synchronization of CNNS with mixed delays has been studied without using existing finite-time stability theorem. By utilizing Lyapunov stability theory and new analytical techniques, several sufficient conditions have been obtained to ensure that the synchronization of CNNS with heterogeneous mixed delays can be achieved in a setting time. Moreover, the setting time has also been explicitly estimated. Results of this paper are general and can be easily extended to other nonlinear time-delay systems. Two numerical simulations verified the effectiveness of our theoretical results.

It is reported in [9] that neurons is an important class of oscillators characterized by a short pulse from an oscillator to its partners. In [9], asymptotic synchronization of pulse-coupled neurons with inhibitory coupling and temporal delays has been studied. Inspired by [9], our next research is to extend the results in [9] to finite-time synchronization of CNNs with inhibitory coupling and temporal delays.

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