WAVE PROPAGATION IN PERIODIC LATTICES WITH DEFECTS OF SMALLER DIMENSION

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Abstract. The procedure of evaluating of the spectrum for discrete periodic operators perturbed by operators of smaller dimensions is obtained. This result allows to obtain propagative, guided, localised spectra for different kind of physical operators on graphs with defects.

1. Introduction

There are a lot of papers devoted to discrete periodic operators on graphs, see reference in [BKS]. At the same time there is the strong interest on periodic structures with different kinds of defects, see [MJ], [OA], [CNJMM], [KS], [MS] and [MC]; about the continuous media see e.g. [TMSD], [C], [ZGYZ] and [PVDRDD]. Defects allows us to obtain conductivity of the material for those frequencies (energies) at which it was not in purely periodic structure. For example if we take 2D homogeneous lattice then we have definite band-gap spectral diagram of propagative waves. If we add some line defect we can obtain new waves which propagate only along with this linear defect and decay in perpendicular directions. The main goal of this paper is to provide the general procedure of obtaining spectra appearing after addition of the defects of smaller dimensions than the original periodic lattice. Note that the technique described below was used by the author (see [K1]) to analyse the spectrum of discrete wave equation in 2D lattice with linear defect and with single inclusion to obtain propagative, guided and localised spectra. The difference between this work and [K1] is that in [K1] we provided detailed analysis of special kind of lattices with defects, here we prove the general theorem and provide the overall procedure to find the defect modes (without a detailed study of its behavior) for a wide class of periodic lattices with inclusions.

The work is organized as follows: Section 2 contains known basic definitions and results from the theory of periodic operators on graphs. Section 3 contains the procedure of obtaining the spectrum of periodic operators perturbed by operators of smaller dimensions.

Now, with help of definitions and results from Sections 2, 3 we present briefly the main results of the current work. Consider $N$-periodic lattice $\Gamma$ and $N$-periodic family of operators $A(\omega)$ acting on $\ell^2(\Gamma)$ with the parameter $\omega$. Physically $\omega$ can be frequency, energy and so on, the dependence on $\omega$ is usually smooth or analytic. One of the main problem is to analyse the set

$$\Omega = \{ \omega : 0 \in \text{spec}(A(\omega)) \},$$

(1.1)

which can be called as the set of possible states. The efficient method for the study the spectrum of $N$-periodic operators is to apply Bloch-Floquet transformation $\hat{A}(\omega) := F A(\omega) F^{-1}$. The operators $\hat{A}(\omega)$ acts on $L^2_M([0, \pi]^N)$ and are operators of multiplication by the matrix-function

$$\hat{A}(\omega)f(k) = A(\omega, k)f(k), \quad f(k) \in L^2_M$$

(1.2)
with $M \times M$ matrix $A$ and $k = (k_1, \ldots, k_N) \in [-\pi, \pi]^N$, here $M$ is a number of the nodes in the unite cell of the graph. If $A(\omega, k)$ is a continuous function then the set $\Omega$ can be described as follows

$$\Omega = \{ \omega : \det A(\omega, k) = 0 \text{ for some } k \}. \quad (1.3)$$

The equation

$$\det A(\omega, k) = 0 \quad (1.4)$$

is called the dispersion equation, this defines the Bloch-Floquet dispersion curves

$$\omega = \omega(k), \quad k \in [-\pi, \pi]^N, \quad (1.5)$$

the number of these curves is usually $M$ but can be larger if the dependence on $\omega$ of $A$ is complicated. We have $\Omega = \omega([-\pi, \pi]^N)$, each point of the curves (branches) $\omega(k)$ belongs to $\Omega$ and each point of $\Omega$ belongs at least to one of the dispersion curve $\omega(k)$.

This is regarding to periodic operators, but what happens if we add some periodic perturbations of $A$ of smaller dimension than $N$? Such perturbations leads to study the family of the form

$$\hat{A}(\omega)\hat{f} = A(\omega, k)\hat{f} + A_1(\omega, k)\langle \hat{f} \rangle_1 + \cdots + A_N(\omega, k)\langle \hat{f} \rangle_{1\ldots N}, \quad (1.6)$$

where we denote

$$\langle \hat{f} \rangle_{i_1\ldots i_j} = \frac{1}{(2\pi)^j} \int_{[-\pi, \pi]^j} \hat{f} dk_{i_1}\cdots dk_{i_j}. \quad (1.7)$$

The spectrum of operators $\hat{A}(\omega)$ (1.6) is well described in the Theorem 3.2 below. Using this result we can determine the Bloch-Floquet dispersion curves of smaller dimension than (1.5). The procedure of finding the dispersion relations consists of $N + 1$ steps.

**Step 1.** The matrix

$$B_0(\omega, k) := A(\omega, k) \quad (1.8)$$

defines the dispersion curves of the dimension $N$

$$\det B_0(\omega, k) = 0 \Rightarrow \omega = \omega_0(k), \quad k = (k_1, \ldots, k_N) \in [-\pi, \pi]^N. \quad (1.9)$$

(Note that $\omega_0$ coincides with (1.5))

**Step 2.** The matrix

$$B_1 := I + \langle B_0^{-1}A_1 \rangle_1 \quad (1.10)$$

defines the dispersion curves of the dimension $N - 1$

$$\det B_1(\omega, k_1) = 0 \Rightarrow \omega = \omega_1(k_1), \quad k_1 = (k_2, \ldots, k_N) \in [-\pi, \pi]^{N-1}. \quad (1.11)$$

The equation (1.11) is not valid in the sets

$$I_1(k_1) = \omega_0([-\pi, \pi], k_2, \ldots, k_N) \quad (1.12)$$

because $B_0^{-1}$ does not exists here. The sets $I_1$ are the projections of the Bloch-Floquet dispersion (spectral) curves on the plane $(\omega, k_1)$.

Continuing the process $N$ times we come to the last step

**Step N+1.** The matrix

$$B_N := I + \langle B_{N-1}^{-1}\cdots B_0^{-1}A_N \rangle_{1\ldots N} \quad (1.13)$$

defines the dispersion curves of the dimension 0 (these are points, i.e. discrete spectrum)

$$\det B_N(\omega) = 0 \Rightarrow \omega_N. \quad (1.14)$$
The equation (1.11) is not valid in the sets
\[ I_N = I_{N-1}([−π, π]) \cup \omega_{N-1}([−π, π]), \] (1.15)
which are the projections of the Bloch-Floquet dispersion (spectral) curves of the dimension greater or equal than 1 on the axis \( ω \).

The set \( Ω \) (1.1) is union of all Bloch-Floquet branches
\[ Ω = \bigcup_{j=0}^{N} \omega_j([−π, π]^{N-j}). \] (1.16)

Note that the restriction to consider the dispersion curves \( \omega_j(k_j) \) outside the projection \( I_j(k_j) \) is physically natural because the sets \( I_j \) already consist of spectral points \( ω \) of higher dimension.

2. Periodic lattices

Def. 1. Periodic lattice. We call the set \( Γ \) is the \( N \)-periodic lattice with \( M \)-point unit cell if
\[ Γ = \bigcup_{n∈\mathbb{Z}^N} L_n \] (2.1)
where components \( L_n \) are disjoint
\[ L_{n_1} \cap L_{n_2} = \emptyset, \quad n_1 \neq n_2, \] (2.2)
and for each \( n \) there is the bijection
\[ \varphi_n : L_n \rightarrow [1, .., M]. \] (2.3)

Def. 2. Group of translations. For the lattice \( Γ \) (2.1) define the translation \( ψ_n \) by
\[ ψ_n : Γ \rightarrow Γ, \] (2.4)
\[ ∀m : ψ_n|L_m = \varphi_{n+m}^{-1} \varphi_m. \] (2.5)
The translations satisfy
\[ ψ_{n_1} \circ ψ_{n_2} = ψ_{n_1+n_2} \] (2.6)
and the set of all translations
\[ T(Γ) = \{ ψ_n : n ∈ \mathbb{Z}^N \} \] (2.7)
is a group isomorphic to \( \mathbb{Z}^N \).

Def. 3. Group of unitary translations. For the lattice \( Γ \) (2.1) define the unitary operators acting on the Hilbert space of quadratic-summable functions \( ℓ^2(Γ) \)
\[ S_n : ℓ^2(Γ) \rightarrow ℓ^2(Γ), \] (2.8)
\[ S_nh = h \circ ψ_n. \] (2.9)
The set of all such operators
\[ U(Γ) = \{ S_n : n ∈ \mathbb{Z}^N \} \] (2.10)
is a group isomorphic to \( \mathbb{Z}^N \), since
\[ S_{n_1}S_{n_2} = S_{n_1+n_2}. \] (2.11)

Def. 4. \( N \)-periodic operators. The operator
\[ A : ℓ^2(Γ) \rightarrow ℓ^2(Γ) \] (2.12)
is called $N$-periodic iff
\[ AS = SA, \quad \forall S \in U(\Gamma). \quad (2.13) \]

**Remark.** It is sufficient to check the condition (2.13) only for basis translations $S_j$, $j = 1, \ldots, N$, where
\[ e_j = (\delta_{ij})_{i=1}^N \quad (2.14) \]
with Kronecker symbol $\delta$.

**Def. 5.** Finite operators. The operator $A : \ell^2(\Gamma) \to \ell^2(\Gamma)$ is finite iff for any $h$ with finite support $A h$ has finite support too.

**Remark 1.** For $N$-periodic operator $A$ we need to check finiteness only for the functions $A h_m$, $m = 1, \ldots, M$, where $h_m$ is a function with the single support at the point $\varphi^{-1}_0(m)$.

**Remark 2.** Finite $N$-periodic operator is always bounded.

**Def. 6.** Bloch-Floquet transformation. Define the unitary operator $F$
\[ F : \ell^2(\Gamma) \to L^2_M := \oplus_{m=1}^M L^2([\pi, \pi]^N), \quad (2.15) \]
\[ F h = ( \hat{h}_m(k))_{m=1}^M, \quad (2.16) \]
\[ \hat{h}_m(k) = \frac{1}{(2\pi)^N} \sum_{n \in \mathbb{Z}^N} h(\varphi^{-1}_n(m)) e^{in \cdot k}, \quad (2.17) \]
where $k = (k_j)_1^N \in [\pi, \pi]^N$ and $\cdot$ is a scalar product.

Floquet-Bloch transformation allows us to study $N$-periodic operators efficiently, because

**Proposition 2.1.** i) For $N$-periodic bounded operator $A$ the operator $\hat{A} := F A F^{-1}$ is an operator of multiplication by the matrix
\[ \forall \hat{f} \in L^2_M : \hat{A} \hat{f} = A(k) \hat{f}, \quad (2.18) \]
where $M \times M$ matrix-function $A$ is defined as
\[ A(k) := (\hat{A} \hat{e}_1 \ldots \hat{A} \hat{e}_N) \quad (2.19) \]
with constant functions $\hat{e}_j(k) = e_j \quad (2.14)$. ii) If $A$ is a finite $N$-periodic operator then the matrix $A(k)$ is a finite sum
\[ A(k) = \sum_n e^{in \cdot k} A^{(n)} \quad (2.20) \]
with constant matrices $A^{(n)}$, $n \in \mathbb{Z}^N$.

**Proof.** i) It is not difficult to show that the operator
\[ \hat{S}_n := F^{-1} S_n F = e^{-in \cdot k}, \quad (2.21) \]
is the operator of multiplication by $e^{-in \cdot k}$. So if $A$ is $N$-periodic then $\hat{A}$ commutes with any $\hat{S}_n$, i.e.
\[ \hat{A} e^{-in \cdot k} \hat{f} = e^{-in \cdot k} \hat{A} \hat{f} \quad (2.22) \]
for any $\hat{f} \in L^2_M$. Using linearity of $\hat{A}$ we deduce that
\[ \hat{A} r(k) \hat{f} = r(k) \hat{A} \hat{f} \quad (2.23) \]
for any $f \in L^2_M$ and for any trigonometric polynomial $r(k)$. Then (2.23) is fulfilled for any function $r(k) \in L^2$, since $\hat{A}$ is bounded and trigonometric polynomials are dense in $L^2$. The identity (2.23) yields (2.18) and (2.19). The statement ii) immediately follows from (2.19) and Remark 1 after Def. 6. □
3. Periodic operators of smaller dimensions

Def. 7. Sublattices of smaller dimensions. For $N$-periodic lattice $\Gamma$ introduce the sublattices:

$$\Gamma_1 = \bigcup_{n \in \mathbb{Z}^{N-1}} \mathcal{L}_n$$

(3.1)

corresponding to the hyperplane

$$\tilde{Z}^{N-1} = \{n \in \mathbb{Z}^N : n \cdot e_1 = 0\}$$

(3.2)

and so on

$$\Gamma_j = \bigcup_{n \in \mathbb{Z}^{N-j}} \mathcal{L}_n$$

(3.3)

corresponding to the hyperplane

$$\tilde{Z}^{N-j} = \{n \in \mathbb{Z}^{N-j+1} : n \cdot e_j = 0\}.$$  

(3.4)

For $j = N$ we have $\Gamma_N = \mathcal{L}_0$ is a finite set, for $1 \leq j < N$ the set $\Gamma_j$ is a $(N-j)$-periodic lattice. Note that

$$\Gamma_N \subset \Gamma_{N-1} \subset \ldots \subset \Gamma_1 \subset \Gamma.$$  

(3.5)

Def. 8. Projectors on the sublattice. Define the natural projectors

$$P_j : \ell^2(\Gamma) \to \ell^2(\Gamma_j) \subset \ell^2(\Gamma).$$

(3.6)

Remark 1. It is not difficult to show that

$$\hat{P}_j := F P_j F^{-1}$$

(3.7)

acts on any $\hat{f} \in L^2_M$ as

$$\hat{P}_j \hat{f} = (\hat{f})_{1\ldots j},$$

(3.8)

where we denote

$$(\hat{f})_{i_1 \ldots i_j} = \frac{1}{(2\pi)^{j}} \int_{[-\pi,\pi]^j} \hat{f} dk_{i_1} \ldots dk_{i_j}.$$  

(3.9)

Def. 9. Periodic operators on the sublattice. The operator $A_j : \ell^2(\Gamma) \to \ell^2(\Gamma)$ is called $(N-j)$-periodic on the sublattice $\Gamma_j$ iff it acts on the sublattice

$$P_j A_j P_j = A_j$$

(3.10)

and $P_j A_j P_j$ restricted to the subspace $\ell^2(\Gamma_j)$ is $(N-j)$-periodic operator (see Def. 4).

Using properties of $P_j$ (3.6)-(3.8) it is not difficult to show the analog of the Proposition 2.1 for the operators on sublattices

Proposition 3.1. i) For $(N-j)$-periodic bounded operator $A_j$ (see Def. 9) the operator $\hat{A}_j := F A_j F^{-1}$ is an operator of multiplication by the matrix

$$\forall f \in L^2_M : \hat{A}_j \hat{f} = A_j(k) \hat{f}_{1\ldots j},$$

(3.11)

where $M \times M$ matrix-function $A_j$ is defined as

$$A_j(k) := \hat{A}_j \hat{e}_1 \ldots \hat{A}_j \hat{e}_N$$

(3.12)

with constant functions $\hat{e}_i(k) = e_i (2.14)$.
ii) If, in addition, \( A_j \) is a finite operator then the matrix \( A_j(k) \) is a finite sum
\[
A_j(k) = \sum_n e^{in\cdot k} A_j^{(n)}
\]
(3.13)
with constant matrices \( A_j^{(n)} \) and \( n \in \mathbb{Z}^{N-j} \).

Now we will study our main object: the spectrum of the \( N \)-periodic operator \( A \) on the lattice \( \Gamma \) perturbed by the \((N-j)\)-periodic operators \( A_j \) on the sublattices \( \Gamma_j \). Thus consider the operator
\[
C = A + A_1 + \ldots + A_N.
\]
(3.14)
The spectrum of \( C \) is the same as the spectrum of \( \hat{C} = \mathcal{F}CF^{-1} \)
\[
\hat{C} = \hat{A} + \hat{A}_1 + \ldots + \hat{A}_N.
\]
(3.15)
Due to the Propositions \[2.1, 3.1\] the operator \( \hat{C} : L^2_M \to L^2_M \) has the following form:
\[
\hat{C}\hat{f} = A\hat{f} + A_1\langle \hat{f}\rangle_1 + \ldots + A_N\langle \hat{f}\rangle_{1\ldots N}
\]
(3.16)
for any \( \hat{f} \in L^2_M \). The matrices \( A_j \) depends on \( k = (k_1, \ldots, k_N) \), but precisely does not depend on \( k_1, \ldots, k_j \). The following Theorem provide the procedure of verification \( \lambda \in \text{spec}(\hat{C}) \) or not.

**Theorem 3.2.** Let \( \hat{C} \) be defined in (3.16) with continuous matrix-functions \( A_j \). For given \( \lambda \in \mathbb{C} \) the condition \( \lambda \in \text{spec}(\hat{C}) \) can be verified as follows: denote \( B_0 := A - \lambda I \) with identical matrix \( I \).

*Step 1.* If
\[
\det B_0(k) = 0 \quad \text{for some} \quad k
\]
(3.17)
then \( \lambda \in \text{spec}(\hat{C}) \) else define the matrix
\[
B_1 := I + \langle B_0^{-1} A_1 \rangle_1.
\]
(3.18)

*Step 2.* If
\[
\det B_1(k) = 0 \quad \text{for some} \quad k
\]
(3.19)
then \( \lambda \in \text{spec}(\hat{C}) \) else define the matrix
\[
B_2 := I + \langle B_1^{-1} B_0^{-1} A_2 \rangle_{12}.
\]
(3.20)

*Step N.* If
\[
\det B_{N-1}(k) = 0 \quad \text{for some} \quad k
\]
(3.21)
then \( \lambda \in \text{spec}(\hat{C}) \) else define the matrix
\[
B_N := I + \langle B_{N-1}^{-1} \ldots B_1^{-1} B_0^{-1} A_N \rangle_{12\ldots N}.
\]
(3.22)

*Step N+1.* If
\[
\det B_N = 0
\]
(3.23)
then \( \lambda \in \text{spec}(\hat{C}) \) else \( \lambda \notin \text{spec}(\hat{C}) \).
Proof. Denoting \( \hat{C}_0 := \hat{C} - \lambda \) we can rewrite the condition \( \lambda \in \text{spec}(\hat{C}) \) as \( 0 \in \text{spec}(\hat{C}_0) \). At the same time the condition \( 0 \in \text{spec}(\hat{C}_0) \) is equivalent to that there is no \( \hat{C}_0^{-1} \) or by the Banach theorem that \( \hat{C}_0 \) is not a bijection, since \( \hat{C}_0 \) is bounded.

Step 1. Suppose that \( \det B_0(k_0) = 0 \) for some \( k_0 \). This means that there exists the vector \( f_0 \) with quadratic norm \( \|f_0\| = 1 \) and with

\[
B_0(k_0)f_0 = 0. \tag{3.24}
\]

We take sufficiently small \( \delta > 0 \) and take the function

\[
\hat{f}_0(k) = \begin{cases} \frac{1}{\sqrt{\pi}} f_0, & k - \tilde{k}_0 \in [0, \delta]^N \\ 0, & \text{otherwise}, \end{cases} \tag{3.25}
\]

where \( \tilde{k}_0 \) is close enough to \( k_0 \) with the condition \( \tilde{k}_0 + [0, \delta]^N \subset [-\pi, \pi]^N \). Note that if \( k_0 \in (-\pi, \pi)^N \) then we can take \( \tilde{k}_0 = k_0 \). We have that \( L_2^\delta \)-norm of the function \( \hat{f}_0 \) and its integrals are

\[
\|\hat{f}_0\| = 1, \quad \|\langle \hat{f}_0 \rangle_1\| = \delta^\frac{1}{2}, \quad \ldots \quad \|\langle \hat{f}_0 \rangle_{1...N}\| = \delta^\frac{N}{2}. \tag{3.26}
\]

Using (3.24), (3.26) with the definition (3.16) we obtain

\[
\|\hat{C}_0\hat{f}_0\| \leq \max_{k - k_0 \in [0, \delta]^N} \|B_0(k) - B_0(k_0)\| + \delta^\frac{1}{2} \max_{1 \leq j, k} \|A_j(k)\| =: \varepsilon(\delta). \tag{3.27}
\]

The continuity of \( B_0(k) \), \( A_j(k) \) leads to \( \varepsilon(\delta) \to 0 \) for \( \delta \to 0 \). This means that \( 0 \in \text{spec}(\hat{C}_0) \) because we found \( \hat{f}_0 \) with \( \|\hat{f}_0\| = 1 \) and with arbitrary small norm of \( \hat{C}_0\hat{f}_0 \).

Now suppose that \( \det B_0(k) \neq 0 \) for all \( k \). Consider the equation

\[
\hat{C}_0\hat{f} = B_0\hat{f} + A_1\langle \hat{f} \rangle_1 + \ldots + A_N\langle \hat{f} \rangle_{1...N} = \hat{g} \tag{3.28}
\]

with some \( \hat{g} \). After multiplying (3.28) on \( B_0^{-1} \)

\[
\hat{f} + B_0^{-1}A_1\langle \hat{f} \rangle_1 + \ldots + B_0^{-1}A_N\langle \hat{f} \rangle_{1...N} = B_0^{-1}\hat{g} \tag{3.29}
\]

and taking \( \langle \cdot \rangle_1 \) we obtain

\[
\hat{C}_1\langle \hat{f} \rangle_1 := B_1\langle \hat{f} \rangle_1 + \ldots + B_0^{-1}A_N\langle \hat{f} \rangle_{1...N} = \langle B_0^{-1}\hat{g} \rangle_1. \tag{3.30}
\]

Note that the operator \( \hat{C}_1 \) acts on \( L_2^\delta([-\pi, \pi]^{N-1}) \). If \( \hat{C}_0 \) is a bijection then \( \hat{C}_1 \) is a bijection too. The inverse statement is true also, because we can uniquely reconstruct \( \hat{f} \) from \( \langle \hat{f} \rangle_1 \) by using (3.29). So we conclude that

\[
0 \in \text{spec}(\hat{C}_0) \iff 0 \in \text{spec}(\hat{C}_1). \tag{3.31}
\]

Now applying the Step 1 to the operator \( \hat{C}_1 \) we finish the Proof by induction.

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