Positive co-degree density of hypergraphs

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Abstract

The minimum positive co-degree of a non-empty $r$-graph $H$, denoted $\delta_{r-1}^+(H)$, is the maximum $k$ such that if $S$ is an $(r-1)$-set contained in a hyperedge of $H$, then $S$ is contained in at least $k$ distinct hyperedges of $H$. Given an $r$-graph $F$, we introduce the positive co-degree Turán number $\text{co}^+\text{ex}(n, F)$ as the maximum positive co-degree $\delta_{r-1}^+(H)$ over all $n$-vertex $r$-graphs $H$ that do not contain $F$ as a subhypergraph.

In this paper we concentrate on the behavior of $\text{co}^+\text{ex}(n, F)$ for 3-graphs $F$. In particular, we determine asymptotics and bounds for several well-known concrete 3-graphs $F$ (e.g. $K_4^-$ and the Fano plane). We also show that, for $r$-graphs, the limit

$$\gamma^+(F) := \lim_{n \to \infty} \frac{\text{co}^+\text{ex}(n, F)}{n}$$

exists, and “jumps” from 0 to $1/r$, i.e., it never takes on values in the interval $(0, 1/r)$. Moreover, we characterize which $r$-graphs $F$ have $\gamma^+(F) = 0$. Our motivation comes primarily from the study of (ordinary) co-degree Turán numbers where a number of results have been proved that inspire our results.

1 Introduction

An $r$-graph is a hypergraph whose edges (called $r$-edges) all have size $r$, i.e., an $r$-uniform hypergraph. Given a hypergraph $H$, we use $V(H)$, or simply $V$, to denote the vertex set of $H$ and $E(H)$ or $E$ for the edge set. We often denote a hyperedge by the concatenation of its vertices. For a family of hypergraphs $\mathcal{F}$, a hypergraph is $\mathcal{F}$-free if it does not contain a member of $\mathcal{F}$ as a subhypergraph. The Turán number (or extremal number) $\text{ex}_r(n, \mathcal{F})$ is

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1When the forbidden family $\mathcal{F}$ consists of a single hypergraph $F$ we will write $F$ in place of $\mathcal{F} = \{F\}$. 

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the maximum number of edges in an $n$-vertex $F$-free $r$-graph. The Turán density of $F$ is the limit
\[
\pi(F) := \lim_{n \to \infty} \frac{\text{ex}_r(n, F)}{\binom{n}{r}}
\]
which exists by an argument due to Katona, Nemetz and Simonovits [19]. For graphs (i.e., when $r = 2$), much is known about the Turán density. For example, the Erdős-Stone theorem [12, 13] determines $\pi(F)$. On the other hand, when $r \geq 3$, very few exact results are known. Notoriously, the Turán density of the complete $4$-vertex $3$-graph is unknown.

Instead of maximizing the number of edges, we may examine degree versions of the Turán number. For $r = 2$, this is equivalent to determining the function $\text{ex}(n, F)$, but when $r \geq 3$ there are multiple interpretations of “degree.” In an $r$-graph, the co-degree of an $(r - 1)$-set $S$ is the number of edges containing $S$. The minimum co-degree over all $(r - 1)$-sets in $H$ is denoted $\delta_{r-1}(H)$.

Let $F$ be a family of $r$-graphs. The co-degree Turán number $\text{coex}(n, F)$ is the maximum value of $\delta_{r-1}(H)$ over all $n$-vertex $r$-graphs $H$ that do not contain a member of $F$ as a subhypergraph. Mubayi and Zhao [25] established the existence of the limit (called the co-degree density of $F$)
\[
\gamma(F) := \lim_{n \to \infty} \frac{\text{coex}(n, F)}{n}
\]
and proved several results concerning $\gamma(F)$.

For a summary of these measures of extremality (and others) see the excellent survey of Balogh, Clemen, and Lidický [2].

Motivated by the degree versions of the Erdős-Ko-Rado theorem and co-degree Turán numbers, Balogh, Lemons and Palmer [3] examined the notion of positive co-degree and they determined the maximum size of an intersecting $r$-graph with minimum positive co-degree at least $k$ (see also extensions by Spiro [30]).

**Definition 1.** The minimum positive co-degree of a non-empty $r$-graph $H$, denoted $\delta^+_{r-1}(H)$, is the maximum $k$ such that if $S$ is an $(r - 1)$-set contained in a hyperedge of $H$, then $S$ is contained in at least $k$ distinct hyperedges of $H$.

In light of these results, we propose the problem to maximize the minimum positive co-degree of an $n$-vertex $r$-graph subject to avoiding some forbidden subhypergraph $F$.

**Definition 2.** Let $F$ be a family of $r$-graphs. We define the positive co-degree Turán number $\text{co}^\ast\text{ex}(n, F)$ as the maximum positive co-degree $\delta^+_{r-1}(H)$ over all $n$-vertex $r$-graphs $H$ that do not contain a member of $F$ as a subhypergraph.

We will say that an $n$-vertex, $F$-free $r$-graph $H$ is an extremal $r$-graph for $F$ if $\delta^+_{r-1}(H) = \text{co}^\ast\text{ex}(n, F)$.

Define the positive co-degree density of a forbidden family $F$ as
\[
\gamma^+(F) := \lim_{n \to \infty} \frac{\text{co}^\ast\text{ex}(n, F)}{n}.
\]
It is not immediately clear that \( \lim_{n \to \infty} \text{co}^+ \text{ex}(n, \mathcal{F})/n \) should exist in general; however, the argument demonstrating the existence of \( \gamma(F) \) can be adapted to demonstrate the existence of \( \gamma^+(F) \). Indeed, this was done by Pikhurko \[27\]. We give an alternate proof in Section 3 using the hypergraph removal lemma and a supersaturation result.

In Section 3, we shall also examine the possible values of \( \gamma^+(F) \). In particular, we show that there is no family \( \mathcal{F} \) of forbidden 3-graphs such that \( \gamma^+(\mathcal{F}) \in (0, 1/3) \). This situation is in contrast to the behavior of \( \gamma(\mathcal{F}) \), which is shown not to jump by Mubayi and Zhao in \[25\]. On the other hand, for 3-graphs, there are no Turán densities in the range \((0, 2/9)\) (see \[10, 20\]).

Before deriving these general results, in Section 2 we determine (or bound) the positive co-degree densities of a number of concrete 3-graphs. These results are summarized in the last two columns of the table below with comparisons to the corresponding bounds for Turán density \( \pi \) and co-degree density \( \gamma \). The individual 3-graphs in the table are defined in the respective subsections of Section 2.

Table entries for \( \gamma^+ \) without citations represent original results which are best-known; note that (following initial circulation of these results) several improvements have been reported by other authors for bounds on \( \gamma^+ \). We have also included these bounds in the table below, with citations. For the sake of completeness, even in cases where new bounds on \( \gamma^+ \) have been reported, we will include sections with our originally circulated (sometimes elementary) bounds.

| \( F \) | \( \leq \pi(F) \) | \( \pi(F) \leq \) | \( \leq \gamma(F) \) | \( \gamma(F) \leq \) | \( \leq \gamma^+(F) \) | \( \gamma^+(F) \leq \) |
|-------|----------------|----------------|-----------------|-----------------|-----------------|-----------------|
| \( K_4 \) | 2/7 | 0.28689 | 1/4 | 1/4 | 1/3 | 1/3 |
| \( F_4 \) | 2/9 | 2/9 | 0 | 0 | 1/3 | 1/3 |
| \( F_{3,2} \) | 4/9 | 4/9 | 1/3 | 1/3 | 1/2 | 1/2 |
| \( F_5 \) | 3/4 | 3/4 | 1/2 | 1/2 | 2/3 | 2/3 |
| \( K_4 \) | 3/4 | 3/4 | 1/2 | 1/2 | 2/3 | 2/3 |
| \( F_{3,3} \) | 2\sqrt{3} - 3 | 2\sqrt{3} - 3 | 1/3 | 1/3 | 1/2 | 1/2 |
| \( C_4 \) | 1/4 | 1/4 | 0 | 0 | 1/3 | 1/3 |
| \( J_4 \) | 1/2 | 1/2 | 1/4 | 1/4 | 4/7 | 4/7 |

Note that for all 3-graphs \( F \) which we consider, except for the \( K_4 \), we are able to show that \( \gamma(F) \neq \gamma^+(F) \). The key difference is that in the positive co-degree setting, we are allowed co-degree zero pairs, and indeed our constructions feature large sets of vertices whose co-degrees are pairwise zero. We can often consider such constructions as analogous to blow-ups, which are not possible in the ordinary co-degree setting.

### 2 Bounds for small 3-graphs

In this section, we will consider forbidden 3-graphs which are not 3-partite, i.e., do not appear in a \( K_{n/3, n/3, n/3} \) for any \( n \). We begin with some definitions which will be useful throughout the section.
For $k \geq r$, an $r$-graph $H$ is $k$-partite if there exists a partition $V_1, V_2, \ldots, V_k$ of its vertex set such that each $r$-edge intersects each partition class in at most one vertex. A $k$-partite $r$-graph $H$ is complete if every possible edge is present and is balanced if the class sizes differ by at most 1 (i.e. are as close in size as possible).

A 3-graph $H$ is bipartite if there is a partition $X, Y$ of its vertex set such that any 3-edge of $H$ intersects both $X$ and $Y$. We say that $H$ is one-way bipartite if there exists a partition $X, Y$ such that every 3-edge is of the form $x_1x_2y$, where $x_1, x_2 \in X$ and $y \in Y$, and is complete one-way bipartite if every 3-edge of the form $x_1x_2y$ is present.

Let $H$ be an $r$-graph with vertex set $V$. We say that a subset $S$ of $V$ is independent if every $r$-edge of $H$ includes a vertex in $V \setminus S$. We say that $S$ is strongly independent if every $(r-1)$-subset of $S$ has co-degree zero, i.e., no $r$-edge intersects $S$ in at least $r-1$ vertices. Note that in a one-way bipartite graph one partition class is independent and the other is strongly independent.

Given an 3-graph $H$ and vertices $u, v$ of $H$, the common neighborhood of $u, v$, denoted by $N(u, v)$, is the set of vertices $w$ such that $uvw$ form a 3-edge of $H$.

### 2.1 $K_4^-$

| $H$ | Edges | Figure 1 | Figure 2 |
|-----|-------|----------|----------|
| $K_4^-$ | 123, 124, 134 | ![Figure 1](image1) | ![Figure 2](image2) |

We will begin by determining $\text{co}^+\text{ex}(n, K_4^-)$. The first theorem which we state, due to Frankl and Füredi [17], characterizes a special family of 3-graphs which will provide constructions achieving $\text{co}^+\text{ex}(n, K_4^-)$. Before stating the theorem, we will need to introduce a pair of families of constructions from [17].

**Construction 1.** Let $H$ consist of $n$ vertices placed on the unit circle, with 3-edges those triples $xyz$ such that the triangle with vertices $x, y, z$ contains the origin. (We may assume that no pair of vertices are placed so that the line connecting them passes through the origin.)

**Construction 2.** Let $H_6$ be the unique $(6, 3, 2)$-design, i.e., the 3-graph on vertex set $\{1, 2, 3, 4, 5, 6\}$ with 3-edge set

$$E_6 = \{123, 124, 345, 346, 561, 562, 135, 146, 236, 245\}.$$  

We take an $n$-vertex blow-up of $H_6$, that is, a 3-graph $H$ with vertex set $V$ of size $n$ partitioned as $V = V_1 \cup \cdots \cup V_6$, and 3-edge set

$$E = \{v_1v_2v_3 : i_1 < i_2 < i_3; \text{ for all } i_j, v_{i_j} \in V_{i_j}; i_1i_2i_3 \in E_6\}.$$
**Construction 3.** Let $H_6$ be the unique $(6, 3, 2)$-design, i.e., the 3-graph on vertex set \{1, 2, 3, 4, 5, 6\} with 3-edge set

$$E_6 = \{123, 124, 345, 346, 561, 562, 135, 146, 236, 245\}.$$ 

We take an $n$-vertex blow-up of $H_6$, that is, a 3-graph $H$ with vertex set $V$ of size $n$ partitioned as $V = V_1 \cup \cdots \cup V_6$, and 3-edge set

$$E = \{v_iv_jv_k : \text{for all } v_i \in V_i, v_j \in V_j, v_k \in V_k \text{ such that } ijk \in E_6\}.$$ 

We depict $H_6$ in Figure 1.

![Figure 1: The 3-edges of $H_6$.](image)

It is easy to see that the 3-graphs in Constructions 1 and 2 have the property that any four vertices span 0 or 2 3-edges. In fact, the following theorem says that these are the only 3-graphs with this property.

**Theorem 3** (Frankl-Füredi, [17]). Suppose $H$ is a 3-graph in which any 4 vertices span 0 or 2 3-edges. Then $H$ is isomorphic to one of the 3-graphs in Construction 1 or 2.

We are now ready to state our theorem for $K_4^-$.

**Theorem 4.**

$$\text{co}^+\text{ex}(n, K_4^-) = \left\lfloor \frac{n}{3} \right\rfloor.$$ 

Moreover, when $n \equiv 0 \pmod{6}$, then there are exactly two extremal constructions, namely the complete balanced 3-partite 3-graph and the blow-up of $H_6$ whose class sizes are balanced; when $n \equiv 3 \pmod{6}$, the unique extremal construction is the complete balanced 3-partite 3-graph.

**Proof.** We first show that $\text{co}^+\text{ex}(n, K_4^-) \leq \left\lfloor n/3 \right\rfloor$. Let $H$ be a $K_4^-$-free 3-graph on $n$ vertices. Fix a 3-edge $xyz$ of $H$. It is easy to see that $N(x, y), N(y, z),$ and $N(x, z)$ must be pairwise disjoint; indeed, suppose there is a vertex $w \in N(x, y) \cap N(y, z)$. Then $xyz, xyw, yzw$ are three hyperedges spanning four vertices, i.e., a copy of $K_4^-$, a contradiction. Thus, we must have

$$|N(x, y)| + |N(y, z)| + |N(x, z)| \leq n.$$
This immediately implies that one of $N(x,y), N(y,z), N(x,z)$ has size at most $\lceil n/3 \rceil$.

Note that it is possible to achieve $\delta_2^+(H) = \lfloor n/3 \rfloor$ for any $n$, since the complete balanced 3-partite 3-graph on $n$ vertices has minimum positive co-degree $\lfloor n/3 \rfloor$ and contains no copy of $K_4^-$. Thus, all that is left is to determine whether this extremal construction is unique. We do this when $n$ is divisible by 3.

Suppose $n$ is divisible by 3 and $H$ a is $K_4^-$-free 3-graph on $n$ vertices with $\delta_2^+(H) = n/3$. We claim that any 4 vertices of $H$ span either 0 or 2 3-edges. Indeed, take 4 vertices $x, y, z, w$ of $H$. If these span 0 3-edges, we are done, so suppose not. Thus, $x, y, z, w$ span at least one 3-edge; without loss of generality, $xyz$ is a 3-edge. We have seen that $N(x,y), N(y,z),$ and $N(x,z)$ are pairwise disjoint, and all must have size at least $n/3$ to satisfy the minimum positive co-degree condition on $H$. This implies that all have size exactly $n/3$, and the vertex set is partitioned by $N(x,y), N(y,z), N(x,z)$. In particular, $w$ is in (exactly) one of $N(x,y), N(y,z), N(x,z)$, showing that $x, y, z, w$ span exactly 2 3-edges.

We can now apply Theorem 3 to conclude that $H$ is isomorphic to one of the 3-graphs described in Constructions 1 and 2. Among these 3-graphs, we claim that at most two have minimum positive co-degree $n/3$.

We first consider 3-graphs of the type described in Construction 1. Let $H$ be a 3-graph in the family described by Construction 1, and suppose that $\delta_2^+(H) = n/3$. We claim that $H$ is the complete balanced 3-partite 3-graph.

To prove this claim, let $d(X,Y)$ denote the distance on the circle between vertices $X,Y$. Choose vertices $X,Y$ with positive co-degree such that

$$d(X,Y) = \min\{d(A,B) : A, B \text{ have positive co-degree}\}.$$ 

Denote by $X', Y'$ the antipodes of $X, Y$. Recall that we assume that no two vertices lie on a line through the origin, so the points $X'$ and $Y'$ are not vertices of $H$.

We will write $UW$ to denote the (minor) arc between two points $U, W$ on the circle. For a vertex $V$ of $H$, we write $V \in UW$ if $V$ lies in the arc $UW$.

![Figure 2: The unit circle with three relevant arcs](image)

Observe that $N(X,Y)$ is precisely the set of vertices which lie in $X'Y'$. Thus, at least $n/3$ vertices lie in $X'Y'$.  

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Consider a vertex $Z \in X'Y'$. Since $X', Y'$ are not vertices of $H$, any two vertices which lie in $X'Y'$ have distance strictly smaller than $d(X, Y)$, and therefore have co-degree zero. Thus, $N(X, Z)$ is disjoint from $X'Y'$, so must be contained in $X'Y$. We conclude that at least $n/3$ vertices of $H$ lie in $X'Y$. Analogously, at least $n/3$ vertices of $H$ lie in $XY'$. Thus, exactly $n/3$ vertices lie in each of the three arcs $XY', X'Y'$, and $XY$.

Note that the above argument also shows that $Z$ has positive co-degree with every vertex in $XY'$ and every vertex in $X'Y$. Now, take $U \in XY'$ and consider $N(U, Z)$. Observe that $N(UZ) \subset X'Y$, so in fact to achieve co-degree $n/3$, we must have $V \in N(U, Z)$ for every $V \in X'Y$. Since $Z, U$ are chosen arbitrarily, this implies that $H$ contains $K_{n/3, n/3, n/3}$ as a sub-hypergraph. Thus, $H$ is $K_{n/3, n/3}$, since any 3-edge added to $K_{n/3, n/3}$ yields a $K_3^-$.

Now consider Construction 2. By definition, a 3-graph of this type has six vertex classes $V_1, \ldots, V_6$, and any pair of vertices $x, y$ with positive co-degree have $N(x, y) = V_i \cup V_j$ for some $i \neq j$. Thus, if every class has size exactly $n/6$, then the minimum positive co-degree is exactly $2n/6 = n/3$. This can occur when $n \equiv 0 \pmod{6}$, if we choose to balance the classes. However, when $n \equiv 3 \pmod{6}$, even if we make the construction as balanced as possible, three classes will be of size $\lfloor n/6 \rfloor$. Construction 2 also has the property that for any pair $i, j$ from $\{1, \ldots, 6\}$ (with $i \neq j$), there exists a pair of vertices $x, y$ with $N(x, y) = V_i \cup V_j$. Thus, when $n \equiv 3 \pmod{6}$, we can find a pair of vertices $x, y$ with $|N(x, y)| \leq 2\lfloor n/6 \rfloor < n/3$. Hence, Construction 2 yields an extremal 3-graph when $n \equiv 0 \pmod{6}$, but not when $n \equiv 3 \pmod{6}$.

We do not attempt a uniqueness result for $n \equiv 0 \pmod{3}$; in this case, Theorem 3 may not apply, since given minimum positive co-degree $\lfloor n/3 \rfloor < n/3$, it is possible that some 4-sets of vertices span exactly one edge. When $n \equiv 0 \pmod{3}$, the complete balanced 3-partite 3-graph is extremal, but small perturbations of the construction are also acceptable. For instance, a single vertex can be removed from a class of size $\lfloor n/3 \rfloor$ and isolated without altering the minimum positive co-degree. Construction 2 may or may not yield an extremal 3-graph, depending upon the congruence class of $n \pmod{6}$. Again, when Construction 2 yields an extremal 3-graph, small perturbations may be possible. While it is plausible that only 3-graphs of the types described in Constructions 1 and 2, and small perturbations thereof, yield extremal 3-graphs for $K_4^-$, in general, we also do not rule out the possibility that substantially different constructions are extremal when $n$ is not divisible by 3.

Due to the structure of the $K_4^-$, a supersaturation result also follows quickly. We will first state a lemma of general applicability.

**Lemma 5.** Fix $c > 0$ and suppose $H$ is an $r$-graph with $\delta_{r-1}^+(H) \geq cn$. Then, for $n$ large enough, $|E(H)| \geq \frac{1}{2}c^r n^r$.

**Proof.** Let $f(H)$ be the number of $(r-1)$-sets in $H$ with positive co-degree. Observe that

$$f(H)cn \leq f(H)\delta_{r-1}^-(H) \leq r|E(H)|,$$

so it suffices to estimate $f(H)$. Since $\delta_{r-1}^+(H) \geq cn > 0$, there is an $(r-1)$-set $T$ in $V(H)$ with positive co-degree. Consider the following process for building $(r-1)$-sets with positive
co-degree, starting from \( T \). Set \( T_1 = T \). At step \( i \), choose a vertex \( t_i \) in \( T_i \cap T \) and a vertex \( v_i \in V(H) \setminus T \) such that \( T_i \cup \{ v_i \} \) span an \( r \)-edge of \( H \). Set \( T_{i+1} = (T_i \setminus \{ t_i \}) \cup \{ v_i \} \). If \( T_i \cap T = \emptyset \), we end the process and return \( T_i \). Observe that, by construction, all \( T_i \)'s have positive co-degree. Thus, at each step, we will be able to find a vertex \( v_i \) to add to \( T_i \). Observe also that the process will terminate at a set \( T_r \) such that \( T_r \cap T = \emptyset \). Now, we count the number of ways to build \( T_r \).

At step \( i \), there are at least \( cn - i + 1 \) choices for \( v_i \), since \( d_{r-1}(T_i) \geq cn \) and \( T \) contains \( i - 1 \) vertices which are not in \( T_i \). A set \( T_r \) may be generated up to \((r-1)!\) times by this process. Therefore, the number of distinct \((r-1)\)-sets \( T_r \) of positive co-degree is at least

\[
\frac{1}{(r-1)!} \prod_{i=1}^{r-1} (cn - i + 1) \geq \frac{c^{r-1}}{2(r-1)!} n^{r-1}
\]

for large enough \( n \). Thus, \(|E(H)| \geq \frac{c^{r-1}}{2(r-1)!} n^{r-1}\) for large enough \( n \).

\[ \square \]

We can now prove a supersaturation result for \( K_4^- \). We will later prove a more general positive co-degree supersaturation result. However, the following theorem has several nice properties: it gives an explicit value of \( \delta \) which is relatively large in terms of \( \varepsilon \), and it directly leverages the structure of \( K_4^- \) for an elementary proof.

**Theorem 6.** Fix \( \varepsilon > 0 \). If \( H \) has \( \delta^2_3(H) = (1/3 + \varepsilon)n \), then there exists \( \delta > 0 \) such that \( H \) contains at least \( \delta n^4 \) copies of \( K_4^- \).

**Proof.** We claim that \( \delta = \frac{\varepsilon}{324} \) is sufficient.

We first show that each \( 3 \)-edge of \( H \) is contained in at least \( 3 \varepsilon n \) copies of \( K_4^- \). Indeed, suppose \( abc \) is a \( 3 \)-edge, and consider \( N(ab), N(bc), N(ac) \). The number of copies of \( K_4^- \) which use the \( 3 \)-edge \( abc \) is precisely \(|N(ab) \cap N(bc)| + |N(ab) \cap N(ac)| + |N(bc) \cap N(ac)|\), where the first term in the sum counts the number of copies of \( K_4^- \) of the form \( abc, abd, bcd \), and so on. Observe,

\[
n \geq |N(ab) \cup N(bc) \cup N(ac)| \\
\geq |N(ab)| + |N(bc)| + |N(ac)| - \left(|N(ab) \cap N(bc)| + |N(ab) \cap N(ac)| + |N(bc) \cap N(ac)|\right) \\
\geq 3 \left( \frac{n}{3} + \varepsilon n \right) - \left(|N(ab) \cap N(bc)| + |N(ab) \cap N(ac)| + |N(bc) \cap N(ac)|\right) \\
= n + 3 \varepsilon n - \left(|N(ab) \cap N(bc)| + |N(ab) \cap N(ac)| + |N(bc) \cap N(ac)|\right),
\]

which implies that

\[
|N(ab) \cap N(bc)| + |N(ab) \cap N(ac)| + |N(bc) \cap N(ac)| \geq 3 \varepsilon n.
\]

We now count copies of \( K_4^- \) in \( H \) by \( 3 \)-edge. Since each \( 3 \)-edge in \( H \) is contained in at least \( 3 \varepsilon n \) copies of \( K_4^- \), and each copy of \( K_4^- \) will be counted exactly \( 3 \) times, we know we have at least \( \varepsilon n \cdot |E(H)| \) copies of \( K_4^- \). By Lemma 5, with \( c = 1/3 + \varepsilon \), we have

\[
|E(H)| \geq \frac{(1/3 + \varepsilon)^3}{12} n^3 > \frac{1}{324} n^3.
\]

So \( H \) contains at least \( \frac{\varepsilon}{324} n^4 \) copies of \( K_4^- \).

\[ \square \]
The following theorem gives an exact result for graphs on at least 6 vertices. Note that the statement does not hold for values smaller than 6; for \( n = 4 \), the \( K_4 \) has minimum positive co-degree \( 2 > \lceil 4/3 \rceil \) and no \( F_5 \), while for \( n = 5 \), the graph obtained by adding a single isolated vertex to a \( K_4 \) has minimum positive co-degree \( 2 > \lceil 5/3 \rceil \) and no \( F_5 \).

**Theorem 7.** For \( n \geq 6 \),

\[
\text{co}^+ \text{ex}(n, F_5) = \left\lfloor \frac{n}{3} \right\rfloor.
\]

*Proof.* The complete balanced 3-partite 3-graph is easily checked to be \( F_5 \)-free, and has minimum positive co-degree \( \lceil n/3 \rceil \). So, we need only supply a matching upper bound.

Suppose \( H \) has at least 6 vertices and \( \delta^+_2(H) > \lfloor n/3 \rfloor \). We claim that \( H \) must contain an \( F_5 \). Observe that, since \( n \geq 6 \), we have \( \lfloor n/3 \rfloor \geq 2 \), so \( \delta^+_2(H) > 2 \). Consider a 3-edge \( abc \) of \( H \). Since \( \delta^+(H) > \lfloor n/3 \rfloor \), it follows that at least two of \( N(ab), N(bc), N(ac) \) have non-empty intersection. Without loss of generality, there exists \( d \) in \( N(ab) \cap N(bc) \). So the pair \( c, d \) has positive co-degree. Since \( \delta^+(H) \geq 3 \), this means that there is a new vertex \( e \) such that \( cde \) is a 3-edge. But now \( abc, abd, cde \) form an \( F_5 \)-copy. \( \square \)

When \( n \) is divisible by 3, we can also easily show that the complete balanced 3-partite 3-graph is the unique extremal 3-graph for \( F_5 \). Indeed, suppose \( H \) is a 3-graph on \( n \geq 6 \) vertices, where 3 divides \( n \), with minimum positive co-degree \( n/3 \). Let \( abc \) be a 3-edge of \( H \). As noted above, \( N(a,b), N(a,c), \) and \( N(b,c) \) are pairwise disjoint, so form a balanced tripartition of the vertex set. Thus, \( H \) is a 3-partite 3-graph with parts \( N(a,b), N(a,c), \) and \( N(b,c) \). It is clear that in order to achieve the co-degree condition, \( H \) must be complete.

### 2.3 \( F_{3,2} \)

The following theorem gives an exact result for graphs on at least 6 vertices. Note that the statement does not hold for values smaller than 6; for \( n = 4 \), the \( K_4 \) has minimum positive co-degree \( 2 > \lceil 4/3 \rceil \) and no \( F_{3,2} \), while for \( n = 5 \), the graph obtained by adding a single isolated vertex to a \( K_4 \) has minimum positive co-degree \( 2 > \lceil 5/3 \rceil \) and no \( F_{3,2} \).

**Theorem 8.** For \( n \geq 6 \),

\[
\text{co}^+ \text{ex}(n, F_{3,2}) = \left\lfloor \frac{n}{3} \right\rfloor.
\]

*Proof.* The complete balanced 3-partite 3-graph is easily checked to be \( F_{3,2} \)-free, and has minimum positive co-degree \( \lceil n/3 \rceil \). So, we need only supply a matching upper bound.

Suppose \( H \) has at least 6 vertices and \( \delta^+_2(H) > \lfloor n/3 \rfloor \). We claim that \( H \) must contain an \( F_{3,2} \). Observe that, since \( n \geq 6 \), we have \( \lfloor n/3 \rfloor \geq 2 \), so \( \delta^+_2(H) > 2 \). Consider a 3-edge \( abc \) of \( H \). Since \( \delta^+(H) > \lfloor n/3 \rfloor \), it follows that at least two of \( N(ab), N(bc), N(ac) \) have non-empty intersection. Without loss of generality, there exists \( d \) in \( N(ab) \cap N(bc) \). So the pair \( c, d \) has positive co-degree. Since \( \delta^+(H) \geq 3 \), this means that there is a new vertex \( e \) such that \( cde \) is a 3-edge. But now \( abc, abd, cde \) form an \( F_{3,2} \)-copy. \( \square \)
Theorem 8.

\[ \text{co}^+ \text{ex}(n, F_{3,2}) \leq \frac{n}{2}. \]

Moreover, if \( H \) is an \( F_{3,2} \)-free \( n \)-vertex 3-graph with \( \delta_2^+(H) = n/2 \), then \( n \) is divisible by 4 and \( H \) is the complete balanced 4-partite 3-graph.

Proof. Let \( H \) be an \( n \)-vertex 3-graph with minimum positive co-degree \( \delta_2^+(H) \geq n/2 \). Let \( a, b \) be a pair of vertices with positive co-degree. Then \( |N(a, b)| \geq n/2 \). Let \( c \in N(a, b) \) and consider \( N(a, c) \). The common neighborhood \( N(a, c) \) does not include \( a \) and has size at least \( n/2 \), so \( N(a, b) \cap N(a, c) \neq \emptyset \). So let \( d \) be a vertex in \( N(a, b) \cap N(a, c) \). There is no 3-edge in \( N(a, b) \) as \( H \) is \( F_{3,2} \)-free. Therefore, \( N(c, d) \cap N(a, b) = \emptyset \). Thus, as \( |N(a, b)| + |N(c, d)| \leq n \) and \( |N(c, d)| \geq n/2 \), we have \( |N(a, b)| \leq n/2 \).

Now, if \( \delta_2^+(H) = n/2 \), then we can find \( a, b, c, d \) as above, so that \( N(a, b) \) and \( N(c, d) \) partition the vertex set of \( H \). Note that \( N(a, c) \) and \( N(b, d) \) also partition the vertex set of \( H \), so we have a partition into four classes, \( N(a, b) \cap N(a, c) \), \( N(a, b) \cap N(b, d) \), \( N(c, d) \cap N(a, c) \), and \( N(c, d) \cap N(b, d) \). Since the common neighborhood of any pair of vertices with positive co-degree must be independent to avoid an \( F_{3,2} \), each of these four parts is independent. We claim, moreover, that each part is strongly independent.

It will suffice to show that

\[ N(b, c) = [N(c, d) \cap N(b, d)] \cup [N(a, b) \cap N(a, c)] \]

and

\[ N(a, d) = [N(c, d) \cap N(a, c)] \cup [N(a, b) \cap N(b, d)], \]

for then we will have established that the union of any two parts is the common neighborhood of a pair of vertices, so is independent, and therefore every 3-edge in \( H \) must use vertices from three parts.

Suppose, for a contradiction, that \( x \in N(b, c) \) is contained in \( N(a, b) \cap N(b, d) \). Then \( c \) and \( x \) have positive co-degree. Both \( c \) and \( x \) are in \( N(a, b) \), so \( N(c, x) \cap N(a, b) = \emptyset \), and thus \( N(c, x) = N(c, d) \). In particular, \( acc \) is a 3-edge, so \( x \) is in \( N(a, c) \), as are \( b \) and \( d \). But by assumption, \( x \in N(b, d) \), a contradiction, as \( N(a, c) \) must be independent to avoid an \( F_{3,2} \). Thus, \( N(b, c) \) does not intersect \( N(a, b) \cap N(b, d) \).

Similarly, \( N(b, c) \) does not intersect \( N(c, d) \cap N(a, c) \). We conclude that

\[ N(b, c) \subseteq [N(c, d) \cap N(b, d)] \cup [N(a, b) \cap N(a, c)]. \]

An analogous argument shows that

\[ N(a, d) \subseteq [N(c, d) \cap N(a, c)] \cup [N(a, b) \cap N(b, d)]. \]

Together, these containments imply that

\[ N(b, c) = [N(c, d) \cap N(b, d)] \cup [N(a, b) \cap N(a, c)] \]

and

\[ N(a, d) = [N(c, d) \cap N(a, c)] \cup [N(a, b) \cap N(b, d)], \]
since each of \([N(c, d) \cap N(b, d)] \cup [N(a, b) \cap N(a, c)]\), \([N(c, d) \cap N(b, d)] \cup [N(a, b) \cap N(a, c)]\) must have size exactly \(n/2\) to allow both \(|N(b, c)| \geq n/2\) and \(|N(a, d)| \geq n/2\).

We conclude that \(H\) is 4-partite, so it only remains to show that \(H\) is complete and balanced. Firstly, from the fact that the union of any two parts is the common neighborhood of a pair of vertices, it follows that the union of any two parts has size exactly \(n/2\). From this, it is simple to deduce that all class sizes are equal, so \(H\) is balanced (and \(n\) must be divisible by 4). Now, to show that \(H\) is complete, it suffices to show that any two vertices \(x, y\) which lie in different parts have positive co-degree. Consider a vertex \(x\) of \(H\). Without loss of generality, \(x\) is in \([N(c, d)] \cap [N(b, d)]\). So \(x\) has positive co-degree with \(b\) and \(c\). Since \(N(x, b)\) does not intersect \([N(c, d)]\), we must have \(N(x, b) = N(a, b)\), so \(x\) has positive co-degree with every vertex in \([N(a, b)]\). Similarly, \(N(x, c)\) must not intersect \([N(b, d)]\), so \(N(x, c) = N(a, c)\), and \(x\) has positive co-degree with every vertex in \([N(a, c)]\). This shows that \(x\) has positive co-degree with every vertex which is not in \([N(c, d)] \cap [N(b, d)]\). We conclude that \(H\) is complete. \(\square\)

We can now quickly characterize the exact value of \(\text{co}^{+}\text{ex}(n, F_{3,2})\) for all \(n\).

**Corollary 9.** If \(4 \nmid n\), then

\[\text{co}^{+}\text{ex}(n, F_{3,2}) = \left\lfloor \frac{n-1}{2} \right\rfloor.\]

**Proof.** By Theorem 8, we know that if \(4 \nmid n\), then \(\text{co}^{+}\text{ex}(n, F_{3,2}) < \frac{n}{2}\). Note that when \(n\) is odd, \(\left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor\), and when \(n \equiv 2 \mod 4\), \(\left\lfloor \frac{n-1}{2} \right\rfloor = \frac{n}{2} - 1\). So in both cases, \(\left\lfloor \frac{n-1}{2} \right\rfloor\) is the largest integer which is strictly less than \(\frac{n}{2}\). Thus, Theorem 8 tells us that when \(4 \nmid n\), we have \(\text{co}^{+}\text{ex}(n, F_{3,2}) \leq \left\lfloor \frac{n-1}{2} \right\rfloor\).

For a matching lower bound, observe that the complete 4-partite 3-graph whose class sizes are as balanced as possible has minimum positive co-degree \(\left\lfloor \frac{n-1}{2} \right\rfloor\) when \(4 \nmid n\). \(\square\)

While we have fully characterized the values of \(\text{co}^{+}\text{ex}(n, F_{3,2})\), we may still ask about uniqueness of extremal constructions when \(4\) does not divide \(n\). In these cases, multiple constructions achieve a minimum positive co-degree of \(\left\lfloor \frac{n-1}{2} \right\rfloor\); moreover, the situation seems to vary based on the congruence class of \(n\) modulo 4. We do not fully characterize the families of extremal 3-graphs in these cases, but mention some alternate extremal constructions. For all \(n\) which are not congruent to 0 modulo 4, the complete one-way bipartite graph with balanced classes is also extremal (it is easily checked to be \(F_{3,2}\)-free). Depending on the congruence class of \(n\) modulo 4, small perturbations of the complete 4-partite 3-graph and the complete one-way bipartite 3-graph may also be extremal. For example, a vertex can in some cases be removed from one of the classes and left as an isolated vertex without lowering the minimum positive co-degree. While we do not claim to describe all constructions, we believe that it would be possible to do so.

### 2.4 Fano plane \(\mathcal{F}\)
Theorem 10.

\[ \text{co}^+ \text{ex}(n, \mathcal{F}) \leq \frac{2}{3} n. \]

Moreover, this bound is sharp when \( n \) is divisible by 6.

Proof. Let \( H \) be an \( n \)-vertex 3-graph with minimum positive co-degree \( \delta_2^+(H) > \frac{2}{3} n \). Observe that the positive co-degree condition implies that a vertex \( x \) with positive degree (i.e., \( x \) is contained in a 3-edge) necessarily has degree greater than \( \frac{2}{3} n \).

Let \( v_1v_2v_3 \) be a 3-edge. Then \( N(v_1, v_2) \cap N(v_3) \) is non-empty, so there is a vertex \( v_4 \in N(v_1, v_2) \cap N(v_3) \). Observe that \( N(v_1) \cap N(v_2) \cap N(v_3, v_4) \) is non-empty, so there is a vertex \( v_5 \) such that \( v_3v_4v_5 \) is a 3-edge and the pair \( v_5, v_i \) has positive co-degree for \( i = 1, 2 \). Now \( N(v_1, v_5) \cap N(v_2, v_4) \cap N(v_3) \) is non-empty, so there is a vertex \( v_6 \) such that \( v_1v_5v_6 \) and \( v_2v_4v_6 \) are 3-edges and the pair \( v_6, v_3 \) has positive co-degree. Finally, \( N(v_1, v_4) \cap N(v_2, v_5) \cap N(v_3, v_6) \) is non-empty, so there is a vertex \( v_7 \) such that \( v_1v_4v_7, v_2v_5v_7 \) and \( v_3v_6v_7 \) are 3-edges. These seven 3-edges form an \( \mathcal{F} \).

When \( n \) is divisible by 6, we obtain a matching lower bound by considering the complete balanced 6-partite 3-graph. \( \square \)

We do not attempt to prove that the complete balanced 6-partite 3-graph is uniquely extremal when \( n \) is divisible by 6. When \( n \) is not divisible by 6, the complete balanced 6-partite 3-graph still gives a lower bound of at least \( 4 \left\lceil \frac{n}{6} \right\rceil \), which shows that \( \text{co}^+ \text{ex}(n, \mathcal{F}) \) is asymptotic to \( \frac{2n}{3} \). However, we do not attempt a more precise result when 6 does not divide \( n \).

2.5 \( K_4 \)

The extremal co-degree for 3-graphs excluding a \( K_4 \) has previously been studied. It was shown by Czygrinow and Nagle [7] that the co-degree density \( \gamma(K_4) \) is at least \( 1/2 - o(1) \), and they conjecture that \( 1/2 \) is the correct upper bound. The construction achieving this
bound is a nice application of the probabilistic method. We provide a different (deterministic) construction of a 3-graph with minimum positive co-degree $n/2 - 1$. For each $n$, our construction gives a larger minimum positive co-degree than the random construction of [7], but asymptotically the resulting densities are the same: $1/2$. This is the one case considered here where we could not improve (at least asymptotically) on the co-degree density.

Finally, while we do not consider larger complete graphs, we note that these have also been studied in the literature. In particular, Lo and Markström [21] proved that for all $k$ the limit

$$\lim_{n \to \infty} \frac{1}{n} \text{coex}(n, K_k)$$

exists.

Later, Sidorenko [28] showed that

$$\lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{n} \text{coex}(n, K_k) \geq 1 - o(k^{-1.084}).$$

Lo and Zhao [22] gave a construction showing that

$$\lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{n} \text{coex}(n, K_k) = 1 - \Theta\left(\frac{\ln k}{k^2}\right).$$

The authors of these last three publications in fact prove more general statements for general $r$-graphs. We give the following simple bounds.

**Theorem 11.**

$$\frac{n}{2} - 1 \leq \text{co}^+\text{ex}(n, K_4) \leq \frac{2n}{3}.$$

**Proof.** For the lower bound, we take an $n$-vertex complete one-way bipartite 3-graph with classes $X, Y$. When $n$ is even, put $|X| = |Y| = n/2$. Observe that two vertices in $X$ have co-degree $|Y| = n/2$, two vertices in $Y$ have co-degree zero, and a pair $x \in X, y \in Y$ have co-degree $|X| - 1 = \frac{n}{2} - 1$. When $n$ is odd, we can take $|X| = \lceil n/2 \rceil$ and $|Y| = \lfloor n/2 \rfloor$ to achieve minimum positive co-degree $\lceil n/2 \rceil - 1 = \lfloor n/2 \rfloor > n/2 - 1$.

For the upper bound, suppose $\delta^*_2(H) > \frac{2n}{3}$ and let $abc$ be a 3-edge of $H$. Observe that $|N(a, b) \cap N(a, c) \cap N(b, c)| > 0$, so $H$ contains a $K_4$. \hfill \Box

Since initial circulation of this manuscript on arXiv, Volec [33] reported that a bound of $\gamma^+(K_4) \leq 0.54296$ can be obtained via Flag Algebras. An intriguing question raised by Theorem [11] is whether or not the co-degree and positive co-degree densities are equal when forbidding a $K_4$, i.e., does $\gamma(K_4) = \gamma^+(K_4)$?
We begin with a lemma.

**Lemma 12.** Let $H = (V, E)$ be a 3-graph with an independent set of size $cn$. Then $\delta^+_2(H) \leq (1 - c)n$.

**Proof.** Let $A$ be an independent set of $H$ of size $cn$ and $B = V \setminus A$. If there is no pair $x \in A, y \in B$ such that $x, y$ have positive co-degree, then every vertex in $A$ is isolated, and it is immediate that $H$ has $\delta^+_2(H) \leq (1 - c)n - 2 < (1 - c)n$. So we may assume that there exists $x \in A, y \in B$ such that $x, y$ have positive co-degree. If $N(x, y) \subseteq B$, then the co-degree of $x, y$ is at most $|B| - 1 = (1 - c)n - 1 < (1 - c)n$, so $\delta^+_2(H) < (1 - c)n$. On the other hand, if there is a 3-edge $xyz$ with $z \in A$, then $x, z$ have positive co-degree. Since $A$ contains no 3-edge, we must have $N(x, z) \subseteq B$, so the co-degree of $x, z$ is at most $|B| = (1 - c)n$. Thus, $\delta^+_2(H)$ must be at most $(1 - c)n$. \hfill $\square$

**Theorem 13.**

$$3 \left\lfloor \frac{n}{5} \right\rfloor \leq \text{co}^+\text{ex}(n, F_{3,3}) \leq \frac{3}{4}n.$$  

**Proof.** For the lower bound, observe that the complete balanced 5-partite 3-graph on $n$ vertices contains no $F_{3,3}$ copy and has minimum positive co-degree at least $3\left\lfloor \frac{n}{5} \right\rfloor$.

For the upper bound, let $\varepsilon > 0$ and suppose $H = (V, E)$ is a 3-graph with minimum positive co-degree $\left(\frac{2}{3} + \varepsilon\right)n$. Let $abc$ be a 3-edge of $H$. We wish to obtain a lower bound on $|N(a, b) \cap N(b, c) \cap N(a, c)|$. For $x \in V$, we define $f(x)$ to be the number of common neighborhoods in $\{N(a, b), N(b, c), N(a, c)\}$ containing $x$. So $f(x) \in \{0, 1, 2, 3\}$ for all $x$. Notice that

$$|N(a, b)| + |N(b, c)| + |N(a, c)| = \sum_{x \in V} f(x)$$

and

$$|N(a, b) \cap N(b, c) \cap N(a, c)| = |\{x \in V : f(x) = 3\}|.$$

From the first equation, we know that

$$\sum_{x \in V} f(x) \geq 3\delta^+_2(H) = 3\left(\frac{2}{3} + \varepsilon\right)n = (2 + 3\varepsilon)n.$$
Observe, if $|\{x \in V : f(x) = 3\}| < 3\varepsilon n$, then strictly more than $(1 - 3\varepsilon)n$ vertices $x \in V$ have $f(x) \leq 2$. Therefore,

$$\sum_{x \in V} f(x) < 3(3\varepsilon n) + 2((1 - 3\varepsilon)n) = (2 + 3\varepsilon)n.$$ 

So we must have $|\{x \in V : f(x) = 3\}| \geq 3\varepsilon n$. Thus, $|N(a, b) \cap N(b, c) \cap N(a, c)| \geq 3\varepsilon n$.

Now, suppose further that $H$ is $F_{3,3}$-free. For any 3-edge $abc$ of $H$, we must have that $N(a, b) \cap N(b, c) \cap N(a, c)$ is independent in order to avoid an $F_{3,3}$. We also know that $|N(a, b) \cap N(b, c) \cap N(a, c)| \geq 3\varepsilon n$. Thus, $H$ has an independent set of size at least $3\varepsilon n$, so by Lemma 12 we have that $\delta^+_2(H) \leq (1 - 3\varepsilon)n$. Now, since $\delta^+_2(H) = \left(\frac{2}{3} + \varepsilon\right)n$, we must have

$$\frac{2}{3} + \varepsilon \leq 1 - 3\varepsilon,$$

which implies $\varepsilon \leq \frac{1}{12}$. So $\delta^+_2(H) \leq \left(\frac{2}{3} + \frac{1}{12}\right)n = \frac{3}{4}n$. \hfill \Box

Since posting of this manuscript, Balogh and Lidický reported an improved upper bound of $\gamma^+(F_{3,3}) \leq 0.616$ via flag algebras.

### 2.7 $C_5$ and $C_5^-$

We give elementary bounds for both $C_5$ and $C_5^-$. 

**Theorem 14.**

$$2 \left\lfloor \frac{n}{4} \right\rfloor \leq \co^+(n, C_5) \leq \frac{2}{3}n.$$

**Proof.** For the lower bound, it is easy to see that $C_5$ is not 4-partite so it is not contained in the balanced complete 4-partite 3-graph. For the upper bound, let $H$ be an $n$-vertex 3-graph with $\delta^+_2(H) > \frac{2}{3}n$. It is easy to find four vertices $v_1, v_2, v_3, v_4$ such that $v_1v_2v_3, v_2v_3v_4 \in E(H)$ and $v_1, v_4$ have positive co-degree. Then observe that $N(v_1, v_2) \cap N(v_3, v_4) \cap N(v_1, v_4) \neq \emptyset$. Therefore, there exists a $v_5$ in all three neighborhoods. Thus, $v_4v_5v_1$, $v_5v_1v_2$ and $v_3v_4v_5$ are all 3-edges in $H$. Together with $v_1v_2v_3$ and $v_2v_3v_4$ they form a $C_5$. \hfill \Box

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Theorem 15.

\[ \left\lfloor \frac{n}{3} \right\rfloor \leq \text{co}^+ \text{ex}(n, C_5^-) \leq \frac{1}{2} n. \]

Proof. For the lower bound, consider a triangle blowup, i.e., a balanced complete 3-partite 3-graph. For the upper bound, let \( H \) be an \( n \)-vertex 3-graph with \( \delta^+_2(H) > \frac{1}{2} n \). It is easy to find four vertices \( v_1, v_2, v_3, v_4 \) such that \( v_1v_2v_3, v_2v_3v_4 \in E(H) \). Observe that \( N(v_1, v_2) \cap N(v_3, v_4) \neq \emptyset \). Therefore, there exists a vertex \( v_5 \) in both neighborhoods. Thus, \( v_5v_1v_2 \) and \( v_3v_4v_5 \) are also both 3-edges in \( H \). Together with \( v_1v_2v_3 \) and \( v_2v_3v_4 \) they form a \( C_5^- \). \( \square \)

Since our manuscript was posted to arXiv, Wu [34] has determined the values \( \text{co}^+ \text{ex}(n, C_5) \) and \( \text{co}^+ \text{ex}(n, C_5^-) \) exactly.

2.8 \( J_k \)

\( J_k \) is the 3-graph on \( k + 1 \) vertices with \( \left( \begin{array}{c} k \\ 2 \end{array} \right) \) 3-edges such that there is a distinguished vertex contained in every 3-edge, and all other vertex pairs are contained in a 3-edge.

\[ J_4 \]

\begin{align*}
J_4 & \quad 123, 124, 125, 134, 135, 145 \\
& \quad \bullet \bullet \bullet \bullet \bullet \bullet
\end{align*}

Theorem 16.

\[ (k - 2) \left\lfloor \frac{n}{k} \right\rfloor \leq \text{co}^+ \text{ex}(n, J_k) \leq \frac{k - 2}{k - 1} n. \]

Proof. For \( k \leq 2 \), the bounds trivially hold. For the lower bound when \( k > 2 \), consider a complete \( k \)-partite graph. We prove the upper bound by induction on \( k \). Note that the base case, \( k = 2 \) trivially holds. Assume then that an \( n \)-vertex hypergraph \( H \) has minimum positive co-degree greater than \( \frac{k - 2}{k - 1} n \). By the induction hypothesis, \( H \) contains a \( J_{k-1} \). Let \( x \) denote the distinguished vertex of the \( J_{k-1} \). We need to find a vertex \( y \) such that \( xyz \) form an edge of the graph for each of the \( k - 1 \) vertices \( z \) distinct from \( x \) in the \( J_{k-1} \). As the co-degree of each pair \( z \) and \( x \) is greater than \( \frac{k - 2}{k - 1} n \) such a vertex \( y \) must exist by the pigeonhole principle. \( \square \)

For the \( J_4 \), the lower bound can be improved to \( 4 \left\lfloor \frac{n}{7} \right\rfloor \) with the following construction.

**Construction 4.** Let \( F \) be the Fano plane (see Section 2.4). We take an \( n \)-vertex balanced blow-up of the complement of \( F \), that is, a 3-graph \( H_7 \) with vertex set \( V \) of size \( n \) partitioned as \( V = V_1 \cup \cdots \cup V_7 \), and 3-edge set

\[ E = \{ v_iv_jv_k : \text{ for all } v_i \in V_i, v_j \in V_j, v_k \in V_k \text{ such that } i < j < k \text{ and } ijk \notin F \}. \]
It is easy to check that $J_4$ is not a subhypergraph of the complement of $\mathbb{F}$. Moreover, every pair of vertices in $J_4$ has non-zero co-degree. Therefore, the blow-up $H_7$ is $J_4$-free. A pair of vertices in the same class $V_i$ have co-degree 0. Now let $x, y$ be vertices in distinct classes $V_i, V_j$. There is a unique edge $ijk$ in the Fano plane corresponding to classes $V_i$ and $V_j$. Therefore, there is an edge $xyz$ for each vertex $z$ in a class other than $V_i, V_j, V_k$. Thus, $x, y$ has co-degree at least $4[\frac{n}{7}]$.

Therefore, $\gamma^+(J_4) \geq 4/7 > 0.57$. Since initial circulation of these results, Balogh and Lidický [4] reported an improved upper bound of $\gamma^+(J_4) \leq 0.58$ via flag algebras.

3 General results

In this section, we collect results of a more general flavor. Our first goal is to establish the existence of the limit

$$\gamma^+(F) = \lim_{n \to \infty} \frac{\co^+ \ex(n, F)}{n}.$$ 

As previously mentioned, the existence of this limit can be proved by adapting the argument of Mubayi and Zhao [25] which establishes the existence of the limit

$$\gamma(F) = \lim_{n \to \infty} \frac{\coex(n, F)}{n}.$$ 

We refer the reader to [27] for a rigorous treatment. We shall give a different proof of the existence of $\lim_{n \to \infty} \frac{\co^+ \ex(n, F)}{n}$, which highlights some of the nice aspects of working with positive co-degree; the key argument which allows our approach will also yield simple proofs of general supersaturation and invariance of co-degree density under the blow-up operation. In fact, we shall prove supersaturation and blow-up invariance before proving the existence of $\gamma^+(F)$.

The key step in these proofs will be an application of the hypergraph removal lemma, which we state below (see, for example, [9]).

Lemma 17. Let $F$ be an $r$-graph and fix $\varepsilon > 0$. There exists $\delta > 0$ such that if $H$ is an $n$-vertex $r$-graph containing at most $\delta n^{V(F)}$ copies of $F$, then there exists $E' \subset E(H)$ such that $|E'| \leq \varepsilon n^r$ and $H - E'$ is $F$-free.

Roughly speaking, given an $r$-graph $H$ with few copies of $F$, we can remove these copies via few $r$-edge deletions to obtain an $F$-free subhypergraph $H_1$ of $H$. However, it is unclear that $\delta^+_{r-1}(H_1)$ should be close to $\delta^+_{r-1}(H)$. In fact, it may be the case that $\delta^+_{r-1}(H_1)$ is much smaller than $\delta^+_{r-1}(H)$. The following lemma, from which a number of general theorems will quickly follow, shows that we can “clean up” $H_1$ to obtain a subhypergraph $H_2$ with $\delta^+_{r-1}(H_2)$ near to $\delta^+_{r-1}(H)$. Note that such a “clean-up” process would not work in the ordinary co-degree setting, since the procedure relies upon deleting $r$-edges so that sets whose co-degree is small in $H_1$ will have co-degree 0 in $H_2$.

Throughout this section we will use $d_{r-1}(S)$ to denote the co-degree of an $(r-1)$-set $S$, i.e., the number of $r$-edges containing $S$. 

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Lemma 18. Let $H$ be an $n$-vertex $r$-graph and fix $0 < \varepsilon < 1$ small enough that

$$(r + 1)! \sqrt[r]{\varepsilon n}^r < |E(H)|.$$ 

Let $H_1$ be a subhypergraph of $H$ obtained by the deletion of at most $\varepsilon n^r$ $r$-edges. Then $H_1$ has a subhypergraph $H_2$ with $\delta_{r-1}^+(H_2) \geq \delta_{r-1}^+(H) - 2r! \varepsilon^{r-1} n^r$.

Proof. Let $E'$ be the set of $r$-edges which we delete from $H$ to obtain $H_1$. For $S \subseteq \binom{V(H)}{r-1}$, let $f_{r-1}(S)$ be the amount by which $d_{r-1}(S)$ drops after the deletion of $E'$. Each deleted $r$-edge drops $d_{r-1}(S)$ by 1 for $r$ different $(r-1)$-sets $S$, so

$$\sum_S f_{r-1}(S) \leq r \varepsilon n^r.$$

Thus, at most $r \sqrt[r]{\varepsilon n^{r-1}}$ sets $S$ have $f(S) \geq \sqrt[r]{\varepsilon n}$. We shall ultimately delete all edges of $H_1$ which contain an $(r-1)$-set $S$ with $f_{r-1}(S) \geq \sqrt[r]{\varepsilon n}$. However, this deletion alone may be insufficient to guarantee the desired minimum positive co-degree. We make the following definition. We say that an $(r-1)$-set $S_{r-1}$ is $(r-1)$-bad if $S_{r-1}$ is contained in at least $\sqrt[r]{\varepsilon n}$ edges of $E'$, i.e., if $f_{r-1}(S) \geq \sqrt[r]{\varepsilon n}$. For $1 \leq i \leq r - 2$, we say that an $i$-set $S_i$ is $i$-bad if $S_i$ is contained in at least $\sqrt[r]{\varepsilon n^{r-i}}$ sets which are $(i+1)$-bad. We shall in fact delete all $r$-edges of $H_1$ which contain any $i$-bad set, for any $i \in \{1, \ldots, r-1\}$.

We bound the number of $i$-bad sets in $H_1$. First, we consider $(r-2)$-bad sets. For a set $S_{r-2} \subseteq \binom{V(H)}{r-2}$, let $f_{r-2}(S_{r-2})$ be the number of $(r-1)$-bad sets containing $S_{r-2}$. Since $H_1$ contains at most $r \sqrt[r]{\varepsilon n^{r-1}}$ $(r-1)$-bad sets, and since each $(r-1)$-bad set contains $r-1$ sets of size $r-2$, we have

$$\sum_{S_{r-2}} f_{r-2}(S_{r-2}) \leq (r-1) \cdot r \sqrt[r]{\varepsilon n^{r-1}}.$$ 

Thus, at most $r(r-1) \sqrt[r]{\varepsilon n^{r-2}}$ sets $S_{r-2}$ have $f_{r-2}(S_{r-2}) \geq \sqrt[r]{\varepsilon n}$, i.e., at most $\frac{r!}{(r-2)!} \sqrt[r]{\varepsilon n^{r-2}}$ sets of size $r-2$ are $(r-2)$-bad. Proceeding inductively, at most $\frac{r!}{i!} \sqrt[r]{\varepsilon n^{r-i}}$ $i$-sets are $i$-bad.

Thus, for any $i$, there are few $i$-bad sets in $V(H_1)$. Now, we delete all $r$-edges of $H_1$ which contain any $i$-bad set. Let $H_2$ be the resulting subhypergraph of $H_1$. We claim that $H_2$ has the desired minimum positive co-degree. Indeed, suppose $T$ is an $(r-1)$-set of vertices with $d_{r-1}(T) > 0$ in $H_2$. Thus, no subset of $T$ is bad. We now estimate the number of $r$-edges containing $T$ which have been deleted.

Suppose $T \subseteq e \subseteq E(H) \setminus E(H_2)$. Thus, either $e \subseteq E'$ or there is some bad $i$-set $S_i \subseteq e$. Since $T$ itself is not $(r-1)$-bad, $T$ is contained in at most $\sqrt[r]{\varepsilon n}$ $r$-edges of $E'$. If $e$ is not in $E'$, but contains an $i$-bad set $S_i$, then $|S_i \cap T| = i-1$. Let $T_{i-1} = S_i \cap T$ be the set of $i-1$ vertices of $T$ which are distinguished by $e$. If $T_{i-1} = \emptyset$, then $S_i$ is a $1$-bad set. There are at most $\frac{i!}{r-i} \sqrt[r]{\varepsilon n}$ $i$-bad sets. Now, suppose $T_{i-1} \neq \emptyset$. Since $T_{i-1}$ is not $(i-1)$-bad, $T_{i-1}$ is contained in fewer than $\sqrt[r]{\varepsilon n^{r-i}}$ $i$-bad sets. Thus, each subset of $T$ of size $i-1$ corresponds to fewer than $\sqrt[r]{\varepsilon n}$ $r$-edges containing $T$ which are deleted from $H_1$. Therefore, the described deletions drop the co-degree of $T$ by at most

$$\sqrt[r]{\varepsilon n} + r! \sqrt[r]{\varepsilon n} + \sum_{i=1}^{r-1} \binom{r-1}{i} \sqrt[r]{\varepsilon n} \leq 2r! \sqrt[r]{\varepsilon n}.$$
Thus, any set $T$ which maintains positive co-degree in $H_2$ satisfies the desired minimum positive co-degree condition. It remains to show that $H_2$ is not empty. We estimate the total number of $r$-edges deleted from $H$ to obtain $H_2$. At most $\varepsilon n$ $r$-edges are deleted to obtain $H_1$. From $H_1$, we delete all $r$-edges which contain any $i$-bad set, for any $1 \leq i \leq r - 1$. A fixed $i$-bad set is in at most $n^{r-i}$ $r$-edges, so, using previous bounds on the number of $i$-bad sets, we delete at most

$$\sum_{i=1}^{r-1} n^{r-i} r! \frac{r-1}{i!} \varepsilon^i n^i$$

$r$-edges from $H_1$ to obtain $H_2$. Thus, the total number of $r$-edge deletions required to obtain $H_2$ is at most

$$\varepsilon n^r + \sum_{i=1}^{r-1} n^{r-i} r! \frac{r-1}{i!} \varepsilon^i n^i < (r+1)! \varepsilon n^r$$

By hypothesis, $|E(H)| > (r+1)! \varepsilon n^r$, so $H_2$ is not empty. \hfill \Box

Note that in general, Lemma 18 will only be useful for $r$-graphs with order $n^r$ $r$-edges, since we must be able to pick $\varepsilon$ with $(r+1)! \varepsilon n^r < |E(H)|$. However, recall that Lemma 5 tells us that if $\delta^+_{r-1}(H)$ is linear, then $|E(H)| = \Theta(n^r)$. Thus, in situations where we cannot apply Lemma 18, we will have $\delta^+_{r-1}(H) = o(n)$, which will be enough for us to work with.

Our first application of Lemma 18 is a general supersaturation result.

**Theorem 19.** Let $F$ be an $r$ graph and fix $\varepsilon > 0$. There exists $\delta > 0$ such that, if $H$ is an $n$-vertex $r$-graph with

$$\delta^+_{r-1}(H) > \co^+ \ex(F) + \varepsilon n,$$

then $H$ contains at least $\delta n^{|V(F)|}$ copies of $F$.

**Proof.** Since $\delta^+_{r-1}(H) \geq \varepsilon n$, we know $|E(H)| = \Theta(n^r)$ by Lemma 5. Fix $\alpha > 0$ such that $(r+1)! \varepsilon^r \alpha n^r < |E(H)|$ and $2^r r! \varepsilon^r \alpha < \varepsilon$, and apply Lemma 17 with $\delta = \delta(\alpha)$. Thus, if $H$ contains at most $\delta n^{|V(F)|}$ copies of $F$, then $H$ can be made $F$-free with at most $\alpha n^r$ $r$-edge deletions. Let $H_1$ be the resulting subhypergraph of $H$. By Lemma 18 and the choice of $\alpha$, $H_1$ contains a subhypergraph $H_2$ with $\delta^+_{r-1}(H_2) > \delta^+_{r-1}(H) - \varepsilon n > \co^+ \ex(n, F)$, a contradiction, as $H_2$ is $F$-free. We conclude that $H$ contains more than $\delta n^{|V(F)|}$ copies of $F$. \hfill \Box

As a standard consequence of supersaturation (see, e.g. [20]), we have blow-up invariance for $\co^+ \ex(n,F)$. First we set some additional notation. Given a $r$-graph $H$ and an integer $t$, the $t$-blow-up $H[t]$ is the $r$-graph obtained by replacing each vertex $v_i \in V(H)$ with a strongly independent class $V_i$ of $t$ vertices. A set of $r$ vertices in $H[t]$ spans an $r$-edge if and only if the vertices’ classes correspond to the $r$ vertices of an $r$-edge in $H$. We will also sometimes wish to consider blow-ups whose classes are not exactly equal in size. We shall denote by $H[n/v(H)]$ a blow-up of $H$ with $n$ vertices and vertex classes of size $[n/v(H)]$ or $\lceil n/v(H) \rceil$, called a balanced blow-up of $H$. Note that there is not always a unique such blow-up. However, in the one instance where we work with balanced blow-ups which are not of
the form $H[t]$, our argument will apply to any balanced blow-up of appropriate size; thus, we are happy to leave the notation somewhat ambiguous. Note that when $v(H)$ divides $n$, there is a unique balanced blow-up of $H$ with vertex classes of size $n/v(H)$, i.e., if $n/v(H) = t$, then $H[n/v(H)] = H[t]$.

**Corollary 20.** Let $F$ be an $r$-graph and $t$ a positive integer. Then

$$\alpha^+ \text{ex}(n, F) \leq \alpha^+ \text{ex}(n, F[t]) \leq \alpha^+ \text{ex}(n, F) + o(n).$$

Now, we are nearly ready to prove that $\gamma^+(F) = \lim_{n \to \infty} \frac{\alpha^+ \text{ex}(n, F)}{n}$ exists. The broad strategy will be to show that $c_n := \frac{\alpha^+ \text{ex}(n, F)}{n}$ becomes arbitrarily close to the least upper bound $\ell$ on $\{c_n\}$ by constructing (for $n$ large enough) $n$-vertex, $F$-free $r$-graphs with minimum positive co-degree nearly $\ell n$. We obtain these $r$-graphs by finding one “good” construction and taking balanced blow-ups. However, for general $H, F$, and $t$, it is possible that $F \not\subset H$ but $F \subset H[t]$, so we shall have to be somewhat careful.

Given two $r$-graphs, $H$ and $F$, we shall say that $H$ blowup-contains $F$ if there is some $t$ for which $F \subseteq H[t]$. For a fixed $r$-graph $F$ on $f$ vertices, we define $\mathcal{B}(F)$ to be the family of $r$-graphs on at most $f$ vertices which blowup-contain $F$. Note that $\mathcal{B}(F)$ always contains $F$; depending upon the structure of $F$, $\mathcal{B}(F)$ may contain other $r$-graphs as well. However, since its members have bounded size, $\mathcal{B}(F)$ will always be a finite set. The standard proof of Corollary 20 can be easily adapted to the following result.

**Corollary 21.** For any $r$-graph $F$,

$$\alpha^+ \text{ex}(n, \mathcal{B}(F)) \leq \alpha^+ \text{ex}(n, F) \leq \alpha^+ \text{ex}(n, \mathcal{B}(F)) + o(n).$$

**Proof.** Fix $\varepsilon > 0$ and let $F$ be an $r$-graph on $f$ vertices. We wish to show that, for $n$ large enough, any $n$-vertex $r$-graph $H$ with $\delta^+_{r-1}(H) \geq \frac{\alpha^+ \text{ex}(n, \mathcal{B}(F))}{n} + \varepsilon n$ contains a copy of $F$. Put $k := |\mathcal{B}(F)|$, and label the members of $\mathcal{B}(F)$ as $F_1, \ldots, F_k$. Observe, for any $F_i \in \mathcal{B}(F)$, we have $F \subset F_i[f]$, so it suffices to show that $H$ contains a large enough blow-up of some $F_i$. This fact follows by essentially the same proof as Corollary 20 fixing $\alpha$ as in the proof of Theorem 19, we apply the Hypergraph Removal Lemma $k$ times to find $\delta_1, \ldots, \delta_k$ such that if $H$ contains at most $\delta_i n^{|V(F_i)|}$ copies of $F_i$, then all copies of $F_i$ can be destroyed by the deletion of at most $\frac{2\alpha}{k} n^r$ $r$-edges. If $H$ in fact contains at most $\delta_i n^{|V(F_i)|}$ copies of $F_i$ for all $i$, then $H$ can be made $\mathcal{B}(F)$-free by deleting at most $\alpha n^r$ $r$-edges. Thus we can apply Lemma 18 obtain a $\mathcal{B}(F)$-free subhypergraph $H'$ of $H$ with $\delta^+_{r-1}(H') > \delta^+_{r-1}(H) - \varepsilon n \geq \alpha^+ \text{ex}(n, \mathcal{B}(H))$ for a contradiction. Thus, for some $i$, $H$ contains at least $\delta_i n^{|V(F_i)|}$ copies of $F_i$. By a standard hypergraph Ramsey argument, this implies that $H$ contains a copy of $F_i[f]$ for $n$ large enough.

Now, observe that if $H$ is $\mathcal{B}(F)$-free, then any blow-up of $H$ must be $\mathcal{B}(F)$-free. By Corollary 21 in order to prove that the limit

$$\gamma^+(F) = \lim_{n \to \infty} \frac{\alpha^+ \text{ex}(n, F)}{n}$$

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exists, it suffices to show the existence of the limit
\[ \gamma^+(B(F)) = \lim_{n \to \infty} \frac{\coex{n}{B(F)}}{n}. \]

We do so now.

**Theorem 22.** For any \( r \)-graph \( F \), the limit
\[ \gamma^+(F) = \lim_{n \to \infty} \frac{\coex{n}{F}}{n} \]
exists. Moreover, for any \( \varepsilon > 0 \), there exists an \( F \)-free \( r \)-graph \( H = H(F, \varepsilon) \) such that every blow-up of \( H \) is \( F \)-free, and for all \( n \) sufficiently large,
\[ \frac{\delta^+_r(H[n/v(H)])}{n} > \gamma^+(F) - \varepsilon. \]

**Proof.** As described above, we instead will show the existence of
\[ \lim_{n \to \infty} \frac{\coex{n}{B(F)}}{n}. \]

Let \( c_n := \frac{\coex{n}{B(F)}}{n} \). Note that adding an isolated vertex to an \( F \)-free \( r \)-graph does not change the minimum positive co-degree, so \( \coex{n}{B(F)} \leq \coex{n+1}{B(F)} \). On the other hand, removing any vertex from a \( B(F) \)-free \( r \)-graph drops the minimum positive co-degree by at most 1, so \( \coex{n+1}{B(F)} \leq \coex{n}{B(F)} + 1 \).

From here we get
\[ \frac{\coex{n}{B(F)}}{n+1} \leq c_{n+1} \leq \frac{\coex{n}{B(F)}+1}{n+1}. \]

Observe that
\[ \frac{\coex{n}{B(F)}}{n+1} = \frac{\coex{n}{B(F)}}{n+1} \cdot \frac{n}{n+1} = \frac{\coex{n}{B(F)}}{n+1} \cdot \frac{n}{n+1} = c_n \frac{1}{n+1} = \frac{c_n}{n+1}. \]

Similarly,
\[ \frac{\coex{n}{B(F)}+1}{n+1} = \frac{\coex{n}{B(F)}+1}{n+1} \cdot \frac{n}{n+1} = \frac{\coex{n}{B(F)}+1}{n+1} \cdot \frac{n}{n+1} = \left( c_n + \frac{1}{n} \right) \frac{n}{n+1} = \left( c_n + \frac{1}{n} \right) \left( 1 - \frac{1}{n+1} \right). \]

Therefore,
\[ c_n \left( 1 - \frac{1}{n+1} \right) \leq c_{n+1} \leq \left( c_n + \frac{1}{n} \right) \left( 1 - \frac{1}{n+1} \right). \]

Since \( c_n \leq 1 \) for all \( n \), observe that
\[ c_n - \frac{1}{n} \leq c_n \left( 1 - \frac{1}{n+1} \right) \]

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and clearly \[
\left( c_n + \frac{1}{n} \right) \left( 1 - \frac{1}{n+1} \right) \leq c_n + \frac{1}{n}.
\]

So we conclude that \[
c_n - \frac{1}{n} \leq c_{n+1} \leq c_n + \frac{1}{n}.
\]

Thus, \[
|c_{n+1} - c_n| \leq \frac{1}{n},
\]

and it follows that for \( k \geq 0, \)
\[
|c_{n+k} - c_n| \leq \sum_{i=0}^{k-1} \frac{1}{n+i}.
\]

As \( 0 \leq c_n \leq 1 \) for all \( n, \) there is a least upper bound \( \ell \) on the sequence \( \{c_n\}. \) We shall prove that \( \lim_{n \to \infty} c_n = \ell. \) Fix \( \varepsilon > 0. \) Since \( \ell \) is the least upper bound for \( \{c_n\}, \) there exists some \( k \) such that \( c_k > \ell - \frac{\varepsilon}{2}. \) Let \( H_k \) be a \( k \)-vertex extremal \( r \)-graph for \( B(F). \)

Now, observe that if \( H \) is a \( B(F) \)-free \( r \)-graph, then any blow-up of \( H \) is also \( B(F) \)-free. Indeed, suppose that some blow-up \( H[t] \) of \( H \) contains a copy of some \( F_1 \in B(F), \) and let \( V_1, \ldots, V_k \) be the classes of \( H[t] \) which intersect this copy. These classes \( V_1, \ldots, V_k \) correspond to vertices \( v_1, \ldots, v_k \in V(H). \) Let \( F_2 = H[v_1, \ldots, v_k], \) the subhypergraph of \( H \) induced on \( v_1, \ldots, v_k. \) We claim that \( F_2 \in B(F), \) a contradiction to the assumption that \( H \) is \( B(F) \)-free. Indeed, we have \( v(F_2) \leq v(F_1) \leq v(F), \) so we will have \( F_2 \in B(F) \) as long as \( F \) is contained in some blow-up of \( F_2. \) We know that \( F_1 \subseteq F_2[t], \) and since \( F_1 \in B(F), \) there exists \( t' \) such that \( F \subseteq F_1[t']. \) Thus, \( F \subseteq F_1[t'] \subseteq F_2[tt'], \) so indeed, \( F_2 \in B(F). \)

Now in particular, for any integer \( m \geq 1, \) the blow-up \( H_k[m] \) is \( F \)-free. Observe that
\[
\delta^+(H_k[m]) = m \cdot \delta^+(H_k),
\]

which shows that \( c_{mk} \geq c_k > \ell - \frac{\varepsilon}{2} \) for all integers \( m \geq 1. \)

Now, choose \( N \) such that \( k \leq N \) and \( \sum_{i=0}^{k-2} \frac{1}{N+i} < \frac{\varepsilon}{2}. \) Let \( n \geq N, \) and choose an integer \( m \) so that \( mk \) is the smallest multiple of \( k \) with \( n \leq mk. \) Since \( |mk - n| \leq k - 1, \) we know that
\[
|c_{mk} - c_n| \leq \sum_{i=0}^{k-2} \frac{1}{N+i} < \sum_{i=0}^{k-2} \frac{1}{N+i} < \frac{\varepsilon}{2}.
\]

Since \( \ell - \frac{\varepsilon}{2} < c_{mk}, \) we must have \( \ell - \varepsilon < c_n. \) Thus, \( \{c_n\} \) converges to \( \ell. \)

For the second part of the theorem statement, we must show that for any \( \varepsilon > 0, \) there exists some \( H \) such that all sufficiently large balanced blow-ups \( H[n/v(H)] \) are \( F \)-free and have \( \frac{\delta^+(H[n/v(H)])}{n} > \gamma^+(F) - \varepsilon. \) As above, we choose \( k \) such that \( c_k > \gamma^+(F) - \frac{\varepsilon}{2} \) and let \( H_k \) be a \( k \)-vertex extremal \( r \)-graph for \( B(F). \) We know that \( \frac{\delta^+(H_k[q])}{qk} = \frac{\delta^+(H_k)}{k} > \gamma^+(F) - \varepsilon \) for all positive integers \( q, \) so we simply must address blow-ups \( H_k[n/k] \) where \( k \) does not
divide $n$. Let $n = qk + r$ for integers $q, r$ with $0 \leq r < k$. We have

$$\frac{\delta^+_{r-1}(H_k[n/k])}{qk + r} \geq \frac{\delta^+_{r-1}(H_k[q])}{qk} \geq \frac{r}{qk + r} \frac{\delta^+_{r-1}(H_k[q])}{qk} > \left(1 - \frac{1}{q}\right) \frac{\delta^+_{r-1}(H_k[q])}{qk} = \left(1 - \frac{1}{q}\right) \frac{\delta^+_{r-1}(H_k)}{k} > \left(1 - \frac{1}{q}\right) \left(\gamma^+(F) - \frac{\varepsilon}{2}\right).$$

If $n$ is large enough that $q \geq \frac{2}{\varepsilon}$, we have $\left(1 - \frac{1}{q}\right) \left(\gamma^+(F) - \frac{\varepsilon}{2}\right) > \gamma^+(F) - \varepsilon$. Thus, for $n$ sufficiently large, the balanced blow-up $H_k[n/k]$ has $\delta^+_{r-1}(H_k[n/k])$ above the desired threshold. □

We now turn our attention to 3-partite 3-graphs. We start by considering $K_{2,2,2}$.

**Theorem 23.** $\operatorname{co^+ex}(n, K_{2,2,2}) = O\left(n^{5/6}\right)$.

**Proof.** Let $H$ be an $n$-vertex 3-graph with maximal minimum positive co-degree such that $H$ contains no $K_{2,2,2}$. Note that we may assume that fewer than $n/2$ vertices of $H$ are isolated (if not, we can replace a subset of the isolated vertices of $H$ with a copy of the non-trivial components of the positive co-degree). Let $\delta^+_2 = \delta^+_2(H)$ be the minimum positive co-degree of $H$.

Let $T$ denote the number of ordered pairs $\{(x,y), \{z_1, z_2\}\}$ such that $x, y, z_i$ form a 3-edge of $H$ for $i = 1, 2$. On the one hand if $z_1$ and $z_2$ are fixed, then the admissible pairs $\{x, y\}$ form a $C_4$-free 2-graph. This gives the upper bound

$$T \leq \binom{n}{2} cn^{3/2}.$$

On the other hand, let $E^+$ be the set of (unordered) pairs $\{x,y\}$ with positive co-degree. Then

$$|E^+| = \binom{\delta^+_2}{2} \leq \sum_{\{x,y\} \in E^+} \binom{d(x,y)}{2} = T.$$

Since at most $n/2$ vertices of $H$ are isolated, we have that $|E^+| \geq \frac{n}{4} \delta^+_2$. Combining these estimates for $T$ gives

$$\left(\delta^+_2\right)^3 \leq O\left(n^{5/2}\right).$$

Thus, unlike previous examples, $\operatorname{co^+ex}(n, K_{2,2,2})$ is sub-linear. The next construction shows that, at least, $\operatorname{co^+ex}(n, K_{2,2,2})$ is not constant.

**Theorem 24.** $\operatorname{co^+ex}(n, K_{2,2,2}) = \Omega(n^{1/2})$.

**Proof.** Partition $n$ vertices into three classes, $X,Y,Z$, so that $|Y| = |Z| \geq cn$ for some $c > 0$ and $|X| = \Omega(n^{1/2})$. Let $G$ be a $C_4$-free bipartite graph with classes $Y,Z$, with
minimum degree as large as possible. Now, let $H$ be the 3-graph with 3-edges $xyz$ such that $x \in X, y \in Y, z \in Z,$ and $yz$ is an edge in the graph $G$.

The fact that $H$ is $K_{2,2,2}$-free follows directly from the condition that $G$ is $C_4$-free. Indeed, since $H$ is 3-partite, if it contains a $K_{2,2,2}$, then this subgraph has two vertices in each of $X, Y, Z$. Let $x_1, x_2 \in X, y_1, y_2 \in X$ and $z_1, z_2 \in Z$. If these six vertices induce a $K_{2,2,2}$, then $x_1, y_1, z_1$ is a 3-edge for all choices of $(i, j, k)$. But this means that $y_1 z_1$ is an edge in $G$ for all choices of $(i, j, k)$, i.e., a $G$ contains a $C_4$, a contradiction.

Now, we consider co-degrees in $H$. Let $x \in X, y \in Y,$ and $z \in Z$. Observe that $y, z$ have co-degree either 0 (if $yz$ is not an edge in $G$) or $|X|$, which we assume to be $\Omega(n^{1/2})$. Next, $x, y$ have co-degree equal to the degree of $y$ in $G$. We chose $G$ so that $\delta(G)$ is maximized, so we know that $\delta(G) = \Omega(n^{1/2})$ by a construction of Erdős, Rényi and Sós [11]. So the co-degree of $x, y$ (and, analogously, $x, z$) is also $\Omega(n^{1/2})$.

We do not make a further effort to determine the order of $\coex^+(n, K_{2,2,2})$, but a resolution of this question remains interesting.

We next consider 3-partite 3-graphs more broadly. Unlike hypergraphs considered in the previous section, we have seen that $\coex^+(n, K_{2,2,2}) = o(n)$, i.e., $\gamma^+(K_{2,2,2}) = 0$. A natural goal is to characterize those 3-graphs $F$ for which $\gamma^+(F) = 0$.

**Theorem 25.** Let $F$ be a 3-graph. If $F$ is 3-partite, then $\coex^+(n, F) \leq O(n^{1-\varepsilon})$ for $\varepsilon = \varepsilon(F)$. Otherwise, $\coex^+(n, F) = \Theta(n)$.

**Proof.** If $F$ is not 3-partite, then it is not contained in a complete balanced 3-partite $n$-vertex 3-graph, so $\coex^+(n, F) = \Theta(n)$.

Now suppose that $F$ is 3-partite and choose $j \leq k \leq l$ minimal such that $F \subseteq K_{j,k,l}$. We have $\coex^+(n, F) \leq \coex^+(n, K_{j,k,l})$. Let $H$ be an $n$-vertex 3-graph with maximal minimum positive co-degree such that $H$ is $K_{j,k,l}$-free.

To bound $\coex^+(n, K_{j,k,l})$, we count pairs $(\{x, y\}, \{z_1, z_2, \ldots, z_l\})$ such that $x,y,z_i$ form a 3-edge of $H$ for each $i \leq l$. Denote the total number of such pairs by $T$. The proof follows by a similar counting argument used in the proof of Theorem 23.

For a short proof of a slightly weaker result, it was shown in [10] that for 3-partite $F$, the Turán number $\ex_3(n, F)$ is $O(n^{3-c})$ for $c = c(F)$. Thus, by the contrapositive of Lemma 5, it is immediate that any $n$-vertex $F$-free 3-graph has minimum positive co-degree $o(n)$. Another short proof of this slightly weaker statement follows from Corollary 24. Note that if $F$ is 3-partite, then $\mathcal{B}(F)$ contains the 3-graph consisting of a single 3-edge. Thus, $\coex^+(n, \mathcal{B}(F)) = 0$, so $\coex^+(n, F) \leq 0 + o(n) = o(n)$. In fact, these arguments generalize to all uniformities:

**Corollary 26.** Let $F$ be an $r$-graph. If $F$ is $r$-partite, then $\gamma^+(F) = 0$.

In light of Corollary 26, we can now quickly show that the positive co-degree density of $r$-graphs jumps from 0 to $1/r$.

**Corollary 27.** Let $F$ be an $r$-graph such that $\gamma^+(F) > 0$. Then $\gamma^+(F) \geq 1/r$. 24
Proof. If $F$ is such an $r$-graph, then by Corollary 26, $F$ cannot be $r$-partite. Therefore for all $n$, $F$ is not a subgraph of the complete balanced $r$-partite $r$-graph on $n$ vertices. This implies that $\text{co}^+\text{ex}(n,F) \geq \lfloor n/r \rfloor$ for all $n$, which completes the proof. □

Note that it also follows immediately that for any family $\mathcal{F}$ of forbidden $r$-graphs either has $\gamma^+(\mathcal{F}) = 0$ or $\gamma^+(\mathcal{F}) \geq 1/r$. Thus, we say that positive co-degree density “jumps” from 0 to $1/r$. In a forthcoming manuscript, we develop additional techniques which demonstrate at least one more jump in positive co-degree density for 3-graphs; thus, the situation described in Corollary 27 is not an isolated phenomenon.

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