Impact of the free-streaming neutrinos to the second order induced gravitational waves

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Abstract The damping effect of the free-streaming neutrinos on the second order gravitational waves is investigated in detail. We solve the Boltzmann equation and give the anisotropic stress induced by neutrinos to second order. The first order tensor and its coupling with scalar perturbations induced gravitational waves are considered. We give the analytic equations of the damping kernel functions and finally obtain the energy density spectrum. The results show that the free-streaming neutrinos suppress the density spectrum significantly for low frequency gravitational waves and enlarge the logarithmic slope \( n \) in the infrared region \( k \ll k_\ast \) of the spectrum. For the spectrum of \( k_\ast \sim 10^{-7} \) Hz, the damping effect in the range of \( k \ll k_\ast \) is significant. The combined effect of the first and second order could reduce the amplitude by 30\% and make \( n \) jump from 1.54 to 1.63 at \( k \sim 10^{-9} \) Hz, which may be probed by the pulsar timing arrays (PTA) in the future.

1 Introduction

Since the detection of gravitational waves by the LIGO and Virgo Collaborations [1,2], gravitational waves have been a popular research object of the cosmology. In particular, the induced gravitational waves have attracted great attention. For recent review please see Ref. [3] and references therein. On large scales, (CMB) gives us information of the primordial fluctuations and inflation [4–6]. In contrast, the induced gravitational waves, generated from the primordial quantum fluctuations, could be used to study the early Universe and new physics beyond the Standard Model on small scales. It contains the information about the physics of the early universe, such as the primordial black holes [7–10] and the primordial non-Gaussianity [11,12]. The induced gravitational waves from the matter dominated [13–15] and the primordial black hole dominated era [16,17] have been studied. More recently, the gauge issue [18–21] and the higher order effects [10,22,23] were analyzed.

It is well known that the neutrinos and the other cosmic particles could have significant impacts on the evolution of the primordial gravitational waves. Considering the anisotropic stress from the free-streaming neutrinos, we know that the square amplitude of the primordial gravitational waves could be reduced by about 35.6\% [24–26]. For the second order scalar induced gravitational waves, the effects of the free-streaming neutrinos have been studied in Refs. [27–29]. At the same time, there are studies [30,31] concerning the tensor induced perturbations independent of neutrinos. In this paper, the first order scalar and tensor perturbations are considered. We give the kernel functions contributed by the scalar–scalar, scalar–tensor and tensor–tensor coupled source terms with impacts of neutrinos for the first time. Besides, we take account of the omitted terms like the integrates of \( \gamma^{i_1} \ldots \gamma^{i_n} \phi^{(1)} \psi^{(1)} \), \( n \neq 0, 2 \) in the second order anisotropic stress \( \Pi_{ij}^{(2)} \) [28]. Finally, giving a monochromatic primordial power spectrum, we obtain the density spectrum of the gravitational waves induced by scalar.

The remaining part of this paper is organized as follows. In Sect. 2, we solve the Boltzmann equation and obtain the anisotropic stress up to the second order induced by neutrinos. In Sect. 3 we give the equation of motion of second order gravitational waves and obtain the kernel functions. The solutions of the equations are shown and analyzed. Finally, in Sect. 4, we calculate and show a comparison of energy density spectrum between no damping and damping gravitational waves. Conclusion and discussions are presented in Sect. 5.
The anisotropic stress induced by neutrinos

The perturbed Friedmann–Robertson–Walker (FRW) metric up to second order in Newtonian gauge is given by

\[
\text{d}x^2 = -(1 + 2\phi^{(1)} + \phi^{(2)})\text{d}t^2 + a(t) V^{(2)} \text{d}r^2 + \text{d}r \text{d}z + \text{d}z \text{d}t,
\]

where \( a(t) \) is the scale factor,

\[
V^{(2)} = V_{,rr} + 2 \frac{V_{,r}}{a} \frac{\dot{a}}{a} + \frac{V_{,t}}{a} \frac{\dot{a}}{a}.
\]

2.1 The first-order Boltzmann equation

In Appendix A, we showed that the zero-order distribution function of neutrinos satisfies the identity \( \left( \frac{\partial F}{\partial \eta} \right)^{(0)} = 0 \). Therefore, the first-order Boltzmann equation can be written as

\[
\left( \frac{\partial F}{\partial \eta} \right)^{(1)} + \left( \frac{\partial F}{\partial x^i} \right)^{(1)} + \left( \frac{\partial F}{\partial q} \right)^{(1)} = 0.
\]

Using Fourier transform, we rearrange the Eq. (10) in the form of

\[
\int_{\eta_{\text{dec}}}^{\eta} d\eta' \left( \frac{\partial F}{\partial \eta'} \right)^{(1)} = 0,
\]

where \( f_k^{(1)}(\eta), \phi_k^{(1)}(\eta), \psi_k^{(1)}(\eta), \) and \( h_{k,ij}^{(1)}(\eta) \) denote the counterparts of \( F(1), \phi(1), \psi(1), \) and \( h_{ij}(1) \) in momentum space, respectively.

We can decompose the stress by helicity

\[
\Pi_{ij}^{(n)} = \sigma_{ij}^{TT(n)} + \frac{1}{2} \left( \frac{\partial \sigma_{ij}^{TT(n)}}{\partial \eta} + \frac{\partial \sigma_{ij}^{TT(n)}}{\partial x^i} + \frac{\partial \sigma_{ij}^{TT(n)}}{\partial q} \right),
\]

where \( \sigma_{ij}^{TT(n)} = 0 \) for \( n = 1, 2 \). Therefore, the first-order Boltzmann equation is given by

\[
\Pi_{ij}^{(n)} = \frac{1}{2} \left( \frac{\partial \sigma_{ij}^{TT(n)}}{\partial \eta} + \frac{\partial \sigma_{ij}^{TT(n)}}{\partial x^i} + \frac{\partial \sigma_{ij}^{TT(n)}}{\partial q} \right),
\]

for \( n = 1, 2 \).
Following the decomposition of the anisotropic stress $\Pi_{ij}$ in Eq. (8), we obtain the explicit expressions of $\sigma^{(1)}_i$ and $\sigma^{(2)}_{\lambda,k}$ in momentum space,

$$\sigma^{(1)}_k = \frac{1}{k^2} \rho_{\nu(0)} \int_{\eta_{\text{dec}}}^\eta d\eta \left[ j_0(k(\eta - \eta')) - 3j_1(k(\eta - \eta')) + 3j_2(k(\eta - \eta')) \right] (\partial_\eta \phi^{(1)}_k(\eta')) + \partial_\eta \psi^{(1)}_k(\eta'),$$

and

$$\sigma^{(2)}_{\lambda,k} = -4 \rho_{\nu(0)}(\eta) \int_{\eta_{\text{dec}}}^\eta d\eta' j_2(k(\eta - \eta')) (k(\eta - \eta')^2 \eta_\lambda^{(1)}(\eta')),$$

where $\rho_{\nu(0)}$ is the unperturbed neutrino energy density, $j_n(\chi)$ is the $n$-order spherical Bessel function.

### 2.2 The second-order Boltzmann equation

The second-order Boltzmann equation is given by

$$\left( \frac{\partial F}{\partial \eta} \right)^{(2)} + \left( \frac{dx^i}{d\eta} \right)^{(0)} \left( \frac{\partial F}{\partial x^i} \right)^{(2)} + \left( \frac{dF}{d\eta} \right)^{(0)} \left( \frac{\partial F}{\partial q} \right)^{(0)} + \left( \frac{dq}{d\eta} \right)^{(2)} \left( \frac{\partial F}{\partial q} \right)^{(0)} + \left( \frac{dy'}{d\eta} \right)^{(1)} \left( \frac{\partial F}{\partial y'} \right)^{(1)} = 0.$$  \hspace{1cm} (16)

In momentum space, the equation could be written as

$$\frac{1}{2} \frac{\partial f_k^{(2)}}{\partial \eta} + \frac{1}{2} y' k^i j^2_k = A(q, k, \eta),$$

where $A(q, k, \eta)$ is defined as

$$A(q, k, \eta) \equiv - \left[ \left( \frac{dx^i}{d\eta} \right)^{(1)} \left( \frac{\partial F}{\partial x^i} \right)^{(1)} + \left( \frac{dq}{d\eta} \right)^{(1)} \left( \frac{\partial F}{\partial q} \right)^{(1)} + \left( \frac{dy'}{d\eta} \right)^{(1)} \left( \frac{\partial F}{\partial y'} \right)^{(1)} \right]_k - \left( \frac{dq}{d\eta} \right)^{(2)} \left( \frac{\partial F}{\partial q} \right)^{(0)}.$$  \hspace{1cm} (18)

Then we obtain

$$f_k^{(2)} = 2 \int_{\eta_{\text{dec}}}^\eta d\eta' A(q, k, \eta', \eta') e^{-i\gamma' k_j(\eta - \eta')}.$$  \hspace{1cm} (19)

Finally, the second order anisotropic stress of neutrinos can be written in the form

$$\sigma^{(2)}_{\lambda,k} = 8 \rho_{\nu(0)} \left[ \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \delta(k_1 + k_2 - k) \left[ \tilde{D}_{1,\lambda}(k, k_1, k_2, \eta) - 4 \tilde{D}_{2,\lambda}(k, k_1, k_2, \eta) + \tilde{D}_{3,\lambda}(k, k_1, k_2, \eta) + \tilde{D}_{44,\lambda}(k, \eta) \right] \right].$$  \hspace{1cm} (20)

The functions $\tilde{D}_{i,j,k}$ $(i = 1, 2, 3)$ come from the integrals of the terms $\left( \frac{\partial F}{\partial \eta} \right)^{(1)} \left( \frac{\partial F}{\partial q} \right)^{(1)} \left( \frac{\partial F}{\partial q} \right)^{(0)}$, respectively. $\tilde{D}_{44,\lambda}$ and $\tilde{D}_{444,\lambda}$ are from the integral of $\left( \frac{dq}{d\eta} \right)^{(2)} \left( \frac{\partial F}{\partial q} \right)^{(0)}$. The explicit forms of the functions $\tilde{D}_{i,j,k}$ are defined in Eq. (A.16).

Reference [34] has given a decomposition of the first order distribution function in Fourier space,

$$f_k = f_k^S + \gamma' e_i f_k^T + \gamma' e_j f_{k,ij} f_{k,j}^T.$$  \hspace{1cm} (21)

This decomposition makes sure that $\sigma^{(1)}_k, \sigma^{(1)}_{\lambda,k}$ and $\sigma^{(2)}_{\lambda,k}$ only depend on $f_k^{(1)}, f_k^{(1)} e_i, f_k^{(1)} e_j f_{k,ij} f_{k,j}^T$, respectively.

$$\sigma^{(1)}_{\lambda,k} \propto \varepsilon_{\lambda,ij,k} \int d\Omega_q \gamma' y' j^i f_k.$$  \hspace{1cm} (22)

which is consistent with the first order results shown in Eq. (15). However, the decomposition could not be extended to the second order case. The reason that Eq. (22) holds is that in the $\eta'$-integral of $f_k^{(1)}$, the index of $e$-exponential is $i\gamma' k^i(\eta - \eta')$ and $k^i e_{ij,k} = 0$. For the second order anisotropic stress, in the $\eta'$-integral the index of $e$-exponential is $i\gamma' (k^i \eta - k^i \eta' - k_1 \eta'')$ or $i\gamma' (k^i \eta - k^i \eta' - k_1 \eta'')$ and generally $k^i e_{ij,k} e_{ij,k} = 0$. Therefore, the terms contain only one or more $\gamma'$ all contributed to the anisotropic stress. The terms, like $\gamma' \ldots \gamma' \psi^{(1)}(n \neq 0, 2)$ are not 0 after the angular integration and the effect of the operator $e^i_j$. They are omitted in previous work [28]. Here we have considered all of them. Refer to Sect. 5 and Appendix C for the detailed analyses.

### 3 The transfer and kernel functions

#### 3.1 The transfer functions

Expanding Einstein equation to the first order, we obtain the equations of the scalar and tensor perturbations,

$$-\phi_k^{(1)} + \psi_k^{(1)} = \kappa a^2 \sigma_k^{(1)}.$$  \hspace{1cm} (23)
where $\kappa \equiv 8\pi G$, $\omega$ is the pressure to energy-density ratio, and $c_s$ is the speed of sound.

Usually, we write the first order scalar and tensor perturbations with transfer functions,

$$
\psi^{(1)}_k(\eta) \equiv \Phi_k T_\psi(k\eta), \quad \phi^{(1)}_k(\eta) \equiv \Phi_k T_\phi(k\eta), \quad h^{(1)}_{k,\lambda}(\eta) \equiv h_{k,\lambda}^{\text{inf}}(k\eta),
$$

where $\Phi_k$ is the initial value originated from primordial curvature perturbation, $h_{k,\lambda}^{\text{inf}}$ is the initial value of the primordial gravitational wave mode, $T_\psi(k\eta)$, $T_\phi(k\eta)$ and $\chi(k\eta)$ are transfer functions of $\psi_k$, $\phi_k$ and $h_{k,\lambda}$, respectively.

The transfer functions of the first order scalar and tensor perturbations are shown in Figs. 1 and 2. The results of the first order tensor perturbations have been obtained by Weinberg earliest [24]. Later, Ref. [35] gave the fitting formula of the transfer function $\chi(x)$. Here we have set the effective number of neutrinos $N_{\text{eff}} = 3.046$ and the gravitational waves evolve in radiation-dominated era. To study the neutrino effects to the perturbations of different frequencies, we give two damping transfer functions of $x_{\text{dec}} = 1, 2$ in each figure, where $x_{\text{dec}} \equiv k\eta_{\text{dec}}$. The damping of the perturbations would only arise after the decoupling of the neutrinos. From the right panels of Figs. 1 and 2 we obtain that the curves of $x_{\text{dec}} = 2$ are much closer to the no damping ones. It is consistent with Eqs. (14) and (15), in which the anisotropic stress induced by neutrinos are much smaller for high-frequency perturbations than low-frequency perturbations. On the other hand, Fig. 1 shows that for the one order scalar perturbations, neutrino makes $T_\psi$ and $T_\phi$ different with small $x$ (green dot-dashed and red dotted curves). When $x$ becomes larger, $T_\psi \rightarrow T_\phi$. This could be explained by Eq. (23). $\sigma^{(1)}_k$ is higher order infinitesimal compared with $\psi$ and $\phi$ as $x \rightarrow \infty$.

### 3.2 The kernel functions

Considering the anisotropic stress contributed by neutrinos, we present the equation of motion of the second order induced gravitational waves as

$$
\begin{align*}
2 \mathcal{H} & \left( 3(c_s^2 - \omega) \mathcal{H} \phi_0^{(1)} + \phi_0^{(1)r} + (2 + 3c_s^2) \psi_0^{(1)} \right) \\
& + 2 \psi_0^{(1)r} - k^2 \phi_0^{(1)} + k^2 \psi_0^{(1)} + 2c_s^2 k^2 \psi_0^{(1)} \\
& = \frac{\kappa a^2}{3} k^2 \sigma_k^{(1)} , \\
\h^{(1)}_{k,\lambda}'' + 2w k h^{(1)}_{k,\lambda} + h^{(1)}_{k,\lambda} \\
& = 2 \kappa a^2 \sigma_{k,\lambda}^{(1)},
\end{align*}
$$

(24)

where $\h^{(1)}_{k,\lambda}$ is the neutrino effects to the perturbations of different gravitational waves evolve in radiation-dominated era. To study perturbations would only arise after the decoupling of the neutrinos.

On the other hand, Fig. 1 shows that for the one order scalar perturbations, neutrino makes $T_\psi$ and $T_\phi$ different with small $x$ (green dot-dashed and red dotted curves). When $x$ becomes larger, $T_\psi \rightarrow T_\phi$. This could be explained by Eq. (23). $\sigma^{(1)}_k$ is higher order infinitesimal compared with $\psi$ and $\phi$ as $x \rightarrow \infty$.

Define the transfer function $f_i$ \citep[$i = 1, 2I, 2II, 3I, 3II, 3III, 3IV$ of the three sources as

$$
\begin{align*}
S_{1k,i,j} & = \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \delta(k_1 + k_2 - k)k_1 k_2 f_i \Phi_{k_1} \Phi_{k_2}, \\
&S_{2k,i,j} = \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \delta(k_1 + k_2 - k)(\epsilon_{k_1}^{\lambda \mu} k_1^\mu f_2 k_2^\lambda + \epsilon_{k_1}^{\lambda \mu} k_1^\mu f_{2\text{II}} h_{k_1,\lambda}^{\text{inf}} \Phi_{k_1}), \\
&S_{3k,i,j} = \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \delta(k_1 + k_2 - k) \left( \delta^{\mu \nu} k_1^\mu k_2^\nu + \epsilon_{k_1,1}^{\lambda \mu} \epsilon_{k_2,1}^{\nu \lambda} k_1^\mu k_3 f_{2\text{III}} + \epsilon_{k_2,1}^{\lambda \nu} \epsilon_{k_1,1}^{\mu \lambda} k_2 k_3 f_{2\text{IV}} + \epsilon_{k_1,1}^{\nu} \epsilon_{k_2,1}^{\mu \lambda} k_1 k_3 f_{3\text{V}} h_{k_1,\lambda}^{\text{inf}} h_{k_2,\lambda}^{\text{inf}} \right) .
\end{align*}
$$

(31)

(32)

(33)
Equation (20) can be rewritten as

$$\kappa \alpha^2 s_{k,\lambda}^{(2)} = 2 \int \frac{d^3 k_1 d^3 k_2}{(2\pi)^3} \delta(k_1 + k_2 - k)(\epsilon_{k,\lambda}^{ij} k_{1ij}^2 P_{11} + \epsilon_{k,\lambda}^{ij} k_{1ij}^3 P_{111}) \Phi_{k_1} \Phi_{k_2}$$

$$+ (\epsilon_{k,\lambda}^{ij} \epsilon_{k_1,ij}^\prime k_1^2 P_{21} + \epsilon_{k,\lambda}^{ij} \epsilon_{k_1,ij}^\prime k_2^2 P_{31}) \Phi_{k_1} \Phi_{k_2}$$

$$+ (\epsilon_{k,\lambda}^{ij} \epsilon_{k_1,ij}^\prime k_{1ij}^3 P_{311} + \epsilon_{k,\lambda}^{ij} \epsilon_{k_1,ij}^\prime k_{2ij}^3 P_{3111}) \Phi_{k_1} \Phi_{k_2}$$

$$+ \sum_{r} \epsilon_{k,\lambda}^{ij} \epsilon_{k_1,ij}^\prime k_{1r}^3 k^3 P_{3IV} \Phi_{k_1} \Phi_{k_2}$$

$$+ \sum_{r} \epsilon_{k,\lambda}^{ij} \epsilon_{k_1,ij}^\prime k_{1r}^3 k^3 P_{3IV} \Phi_{k_1} \Phi_{k_2}$$

$$- 2\kappa \alpha^2 \rho_0^{(2)} \int x \left( \frac{j_x(x - x')}{(x - x')^2} \right) h_{k,\lambda}^{(2)}. \quad (34)$$

Similarly, $h_{k,\lambda}^{(2)}$ can be written as

$$h_{k,\lambda}^{(2)} = \int \frac{d^3 k_1 d^3 k_2}{(2\pi)^3} \delta(k_1 + k_2 - k)(\epsilon_{k,\lambda}^{ij} k_{1ij}^2 K_{11} + \epsilon_{k,\lambda}^{ij} k_{1ij}^3 K_{111}) \Phi_{k_1} \Phi_{k_2}$$

$$+ (\epsilon_{k,\lambda}^{ij} \epsilon_{k_1,ij}^\prime k_1^2 K_{21} + \epsilon_{k,\lambda}^{ij} \epsilon_{k_1,ij}^\prime k_2^2 K_{21}) \Phi_{k_1} \Phi_{k_2}$$

$$+ (\epsilon_{k,\lambda}^{ij} \epsilon_{k_1,ij}^\prime k_{1ij}^3 K_{211} + \epsilon_{k,\lambda}^{ij} \epsilon_{k_1,ij}^\prime k_{2ij}^3 K_{2111}) \Phi_{k_1} \Phi_{k_2}$$

$$+ \sum_{r} \epsilon_{k,\lambda}^{ij} \epsilon_{k_1,ij}^\prime k_{1r}^3 k^3 K_{2IV} \Phi_{k_1} \Phi_{k_2}$$

$$+ \sum_{r} \epsilon_{k,\lambda}^{ij} \epsilon_{k_1,ij}^\prime k_{1r}^3 k^3 K_{2IV} \Phi_{k_1} \Phi_{k_2}$$

$$- 2\kappa \alpha^2 \rho_0^{(2)} \int x \left( \frac{j_x(x - x')}{(x - x')^2} \right) h_{k,\lambda}^{(2)}. \quad (35)$$

In Eqs. (31)–(35), each second order quantity has been decomposed into three components, scalar–scalar, scalar–tensor and tensor–tensor coupling terms. Besides, according to the contraction form of the polarization tensor $\epsilon_{ij,\lambda}$ and the spatial momentum, each component is divided again, e.g., $K_2 \rightarrow K_{2I} + K_{2II}$. 

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**Fig. 1** The transfer functions of the first order scalar perturbations. Left panel shows the no damping $T_\nu$ (blue solid curve), damping $T_\nu$ for $\chi_{dec} = 1$ (orange dashed curve) and $\chi_{dec} = 2$ (green dot-dashed curve), where $\chi_{dec} \equiv k \eta_{dec}$. The horizontal axis represents $x \equiv k \eta$. Right panel gives the differences between the different transfer functions.

**Fig. 2** The transfer functions of the first order tensor perturbation. Left panel shows the no damping $\chi$ (blue solid curve), damping $\chi$ for $\chi_{dec} = 1$ (orange dashed curve) and $\chi_{dec} = 2$ (green dot-dashed curve). Right panel gives the differences between the damping and no damping transfer functions.
Finally, we obtain the equation for the kernel function $K_i$ ($i = 1, 2, I, II, 3I, 3II, 3III, 3IV, 3V$)

$$K_i'' + 2\mathcal{H}K_i' + k^2 K_i = 4 \left[ f_i + Q_i + P_i - 2 \kappa a^2 \rho_\text{v}(0) \int_{x_{\text{dec}}}^x dx' \left( \frac{j_2(x-x')}{(x-x')^2} k K_i' \right) \right],$$

(36)

where $Q_i$ comes from the terms composed of $\Pi_{ij}^{(1)}$ and $\psi^{(1)}$. $P_i$ and the following integral term come from $\Pi_{ij}^{(2)}$. The explicit expressions of $P_i$ and $Q_i$ are shown in Appendix C.

It should be clarified that normally $K_i$ are the functions of $|k_1|, |k_2|$ and $x$. Define $u \equiv |k_1|/|k|$ and $v \equiv |k_2|/|k|$. Then $K_i$ could be written as $K_i(u, v, x)$. Equation (36) shows that $K_i$ also depend on $x_{\text{dec}}$ considering the neutrinos. When $u, v$ have been fixed, we denote the function $K_i(x)$ as $K_i(x_{\text{dec}})(x)$ with no ambiguity. For example, in Fig. 3 we give the kernel function $K_1$ in radiation-dominated era for small $x$. The damping gravitational waves of $x_{\text{dec}} = 1$ and $x_{\text{dec}} = 2$ are shown. For the mode of $x_{\text{dec}} = 2$, as it enter the horizon, neutrinos have not decoupled. $Q_i$, $P_i$ and the integral term in the right hand of Eq. (36) keep 0 until $x = x_{\text{dec}} = 2$. After this moment the neutrinos decouple and affect the propagation of the gravitational wave.

In this paper we study the energy density of the gravitational waves from a monochromatic curvature perturbation. In this case only the behaviors of the kernel functions well inside the horizon ($x \rightarrow \infty$) make sense. In Fig. 4, we give the kernel functions $K_i$ ($i = 1, 2I, 2II, 3I, 3II, 3III, 3IV, 3V$) between $x = 470 \sim 500$ with $u = v = 1$. In each panel of Fig. 4, the no damping ($K_i$) and damping ($K_i(x_{\text{dec}} = 1)$) kernel functions are given. Notice that $K_i = A \frac{\sin(x+\delta)}{x}$ as $x \rightarrow \infty$, which is consistent with the analytic formul-
Fig. 4 The kernel functions $K_i(u, v, x)(i = 1, 2I, 2II, 3I, 3II, 3III, 3IV, 3V)$ of the second order tensor perturbation. We have set $u = v = 1$. For each mode the no damping (blue solid curve) and damping ($x_{dec} = 1$, orange dashed curve) kernel functions are given. The ratios of the amplitudes of two kernel functions $r$ are shown in each panel.
functions of different frequencies are shown. The effect of neutrinos changes this zero point and make the ratio here become infinite.

In the last of this section we show the comparison of the integral term, \( \mathcal{P}_1 \) and \( Q_1 \) in the right hand of Eq. (36). Here we have set \( \mu = v = 1 \)

kernel function [37]. The effect of neutrinos changes this zero point and make the ratio here become infinite.

In the last of this section we show the comparison of the integral term, \( \mathcal{P}_1 \) and \( Q_1 \) in the right hand of Eq. (36) well inside the horizon in Fig. 6. We find that for large \( x \), \(|\text{the integral term}| \gg \mathcal{P}_1 + Q_1 \). Thus, we obtain that the contributions of neutrinos are mainly from the second order anisotropic stress \( \Pi_{ij}^{(2)} \), rather than the couple of the first order scalar perturbation and anisotropic stress \( \psi^{(1)} \Pi_{ij}^{(1)} \) (see Eq. (27)).

4 The energy density spectrum

As we mentioned before, the primordial tensor power spectrum \( \mathcal{P}_h \) is neglected. In this section, we focus on the neutrino effect on the scalar induced gravitational waves, and give the energy density spectrum. The power spectrum \( \mathcal{P}_h^{(2)} \) of the second order induced gravitational wave is defined as

\[
\left( h_k^{(2)}(\eta) h^{(2)*}_k(\eta) \right) = \delta^{(2)}(k + k') \frac{2\pi^2}{k^3} \mathcal{P}_h^{(2)}(\eta, k),
\]

where \( \left( h_k^{(2)}(\eta) h^{(2)*}_k(\eta) \right) \) is the two-point correlation function of \( h_k^{(2)}(\eta) \). Then we obtain [37]

\[
\mathcal{P}_h^{(2)} = \frac{1}{4} \int_0^\infty dv \int_{1-v}^{1+v} du \left\{ k^4 \left( 4v^2 - (1 + v^2 - u^2)^2 \right)^2 \right. \\
\times K_1^2(u, v, x) \mathcal{P}_\Phi(k u) \mathcal{P}_\Phi(k v) \right\}.
\]

Here, only the scalar induced part, i.e., \( K_1 \) in Eq. (35) is considered. \( \mathcal{P}_\Phi \) is the power spectrum of the first order scalar perturbation

\[
\mathcal{P}_\Phi(k) = \delta(k - k_a),
\]

where \( A \) denotes the overall normalization and \( k_a \) is the location where the spectrum has a delta peak.

Finally, we obtain

\[
\Omega = \lim_{x \to \infty} \frac{\pi^2}{96} \frac{x^2}{k^2} \left( 1 - \frac{k^2}{4} \right)^2 k^4 \mathcal{P}_h^{(2)} \left( \frac{1}{k}, \frac{1}{k'}, x \right) \theta(2 - k) \theta(k),
\]

where \( \tilde{k} = k/k_a \), and \( \theta \) denotes the Heaviside step function.

Considering the damping effect of the free-streaming neutrinos, we show the modified energy density spectra of the second order gravitational waves in Fig. 7. A significant difference between the modified spectra and the no damping spectra is that the shapes of the former depend on the choices of the frequency \( k_a \). Shown in Fig. 7, the two spectra of \( k_a \eta_{dec} = 1 \) and \( k_a \eta_{dec} = 100 \) have an obvious difference. The right panel of Fig. 7 gives the ratio between the damping spectra \( \Omega(x_{dec}) \) to the no damping spectrum \( \Omega_0 \). The curve of \( \Omega(k_a \eta_{dec} = 1)/\Omega_0 \) is below the curve of \( \Omega(k_a \eta_{dec} = 100)/\Omega_0 \), which means that the damping effect is more significant for former. There are peaks at \( \tilde{k} = 2/\sqrt{6} \), which corresponds with the infinities at \( u = \sqrt{6}/2 \) in Fig. 5. We find that for the density spectrum of \( k_a \eta = 100 \), the neutrino could suppress it to 70% at \( k = 0.01 \). The damping
effect gradually weakens with the increase of $\tilde{k}$ and nearly vanishes at $\tilde{k} = 0.1$ and above. As a result, the logarithmic slope $n$ in the infrared region, which is defined as $\Omega \propto \tilde{k}^n$ at small $\tilde{k}$ [39], jumps from 1.54 to 1.63 at $\tilde{k} = 0.01$.

5 Conclusion and discussion

Notice for the density spectrum of $k_s \eta_{\text{dec}} = 100$, i.e., $k_s \sim 100$ nHz, the modified energy spectrum have the possibility of being probed by the pulsar timing arrays (PTA) in the future. In Fig. 8, we show the current density spectra $\Omega_{\text{current}} = \Omega R \Omega [10,40]$ taking into account of the effect of neutrinos and the sensitivity curve of the Square Kilometer Array (SKA) [41]. Here we have set $A = 10^{-8}$. $\Omega R$ is the density parameter of radiation at present. $f = k/2\pi$ is the frequency of the gravitational waves. Although the free-streaming neutrinos suppress $\Omega(k_s \eta_{\text{dec}} = 1)$ more significantly, unfortunately, it locates far below the frequency of nHz and cannot be detected by SKA. For $\Omega(k_s \eta_{\text{dec}} = 100)$, SKA may have a precise detection and figure out the neutrino effect on the gravitational waves.

In Sect. 2.2 we have mentioned that we considered the integrations of the terms like $\gamma^{ij} \cdots \gamma^{k\ell} \phi^{(1)} \psi^{(1)}$, $(n \neq 0, 2)$, which make a difference with Ref. [28] in the second order anisotropic stress $\Pi_{ij}^{(2)}$. More precisely, this difference finally appears on $P_1$ in Eq. (36). Fig. 9 gives us the comparison between our $P_1$ and Ref. [28]. It shows that the two results have obvious difference at small $x$, which could affect $K_1$ at small $x$. As $x \to \infty$, since the two $P_1$'s are of the same order and $|\text{the integral term}| \gg P_1 + Q_1$, the kernel function $K_1$ well inside the horizon are the same.

In this paper, we analyzed the effect of the free-streaming neutrinos to the second order induced gravitational waves. We give the first order transfer function and the second order kernel functions considering the neutrinos. For the reason that the gravitational waves with lower frequencies influenced by the neutrinos for much longer times, they are damped more significantly than the higher frequency modes. Finally, we give the energy density spectrum of $k_s \sim 100$ nHz. It has
been damped to 70% at $\tilde{k} = 0.01$. As $\tilde{k}$ increases, the damping effect gradually vanishes, which enlarges the logarithmic slope $n$ in the infrared region. These effects may be examined by PTA in the future.

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Appendix A: The second order quantities

The temporal component of the geodesic equation is written as

$$\frac{dP^0}{d\lambda} = P^0 \frac{dp^0}{d\tau} = - P_\alpha^0 P^\alpha P^\beta.$$  \hspace{1cm} (A.1)

Using Eqs. (1) and (A.1), we obtain

$$\frac{dp^0}{d\tau} = P^0 \left( - \frac{\dot{a}}{a} - \partial_i \phi^{(1)} + \frac{\dot{\psi}(1)}{a} + \partial_\psi \psi^{(1)} \ight.$$  
$$\left. + \frac{1}{2} \dot{\psi} \psi^{(1)} + \frac{1}{2} \frac{\dot{\psi}(1)}{a} + \frac{\psi(1)}{a} \right) - \frac{1}{2} \gamma^{ij} \partial_{ij} h^{(1)}$$  
$$\left. - \frac{4}{a} \psi^{(1)} \gamma^{ij} \partial_i \phi^{(1)} + \frac{2}{a} \psi^{(1)} \gamma^{ij} \partial_i \phi^{(1)} + 4 \frac{\psi^{(1)}}{a} \partial_i \psi^{(1)} \ight.$$  
$$\left. - 2 \psi^{(1)} \gamma^{ij} \partial_i h^{(1)} - \gamma^{ij} \dot{h}^{(1)} \partial_i \psi^{(1)} \ight.$$  
$$\left. + \gamma^{ij} \gamma^{kl} \partial_{ij} h^{(1)} - \gamma^{ij} \dot{h}^{(1)} \partial_i \psi^{(1)} - \frac{2}{a} \gamma^{ij} \partial_i \phi^{(1)} + \frac{1}{a} \gamma^{ij} \psi \partial_i \psi^{(1)} - \frac{2 \dot{a}}{a} \gamma^{ij} \psi \partial_i \psi^{(1)} \right) \right).$$  \hspace{1cm} (A.2)

Considering that in the Boltzmann equation Eq. (5), the comoving momentum and comoving time $\eta$ are used. By using Eqs. (3) and (4) we can rewrite Eq. (A.2) as

$$\frac{dq}{d\eta} = q \left( - \gamma^{ij} \partial_i \phi^{(1)} + \partial_\eta \psi^{(1)} + \frac{1}{2} \gamma^{ij} \gamma^{kl} \partial_{ij} h^{(1)} \ight.$$  
$$\left. + \frac{1}{2} \gamma^{ij} \gamma^{kl} \partial_{ij} h^{(1)} + \frac{1}{2} \gamma^{ij} \gamma^{kl} \partial_{ij} h^{(1)} - \gamma^{ij} \partial_i \psi^{(1)} \right)$$  
$$\left. + \frac{1}{2} \gamma^{ij} \gamma^{kl} \partial_{ij} h^{(1)} + \frac{1}{2} \gamma^{ij} \gamma^{kl} \partial_{ij} h^{(1)} - \gamma^{ij} \partial_i \psi^{(1)} \right) \right).$$  \hspace{1cm} (A.3)

Similarly, using the spatial component of the geodesic equation, we could obtain $\frac{dy^i}{d\eta}$ to the first order

$$\frac{dy^i}{d\eta} = - \partial_i \phi^{(1)} + \gamma^{ij} \partial_j \phi^{(1)} + \gamma^{ij} \partial_j \psi^{(1)}$$  
$$\left. - 2 \gamma^{ij} \gamma^{kl} \partial_{ij} h^{(1)} - \gamma^{ij} \gamma^{kl} \partial_{ij} h^{(1)} - \gamma^{ij} \gamma^{kl} \partial_{ij} h^{(1)} \ight.$$  
$$\left. + \frac{1}{2} \gamma^{ij} \gamma^{kl} \partial_{ij} h^{(1)} + 2 \gamma^{ij} \gamma^{kl} \partial_{ij} h^{(1)} + \gamma^{ij} \gamma^{kl} \partial_{ij} h^{(1)} \right).$$  \hspace{1cm} (A.4)

Following Eq. (1) and the constraint condition $g_{\mu \nu} P^\mu P^\nu = 0$, we obtain

$$\frac{dx^i}{d\eta} = a \frac{dx^i}{d\tau} = a \frac{P^i}{P^0}$$  
$$= \left( 1 + \phi^{(1)} + \psi^{(1)} - \frac{1}{2} \gamma^{ij} \gamma^{kl} \partial_{ij} h^{(1)} \right) y^i.$$  \hspace{1cm} (A.5)

Substituting Eqs. (A.3)–(A.5) into Eq. (5), the explicit form of the Boltzmann equation to the second order is obtained.
The components of the right hand of Eq. (18) can be written as

\[
\frac{dx_{i}^{(1)}}{d\eta_{k_{1}}} \left( i k_{2} f_{k_{2}}^{(1)} \right) = \left( \frac{\partial F}{\partial q} \right)^{(0)} \times \int \frac{d^{3}k_{1}d^{3}k_{2}}{(2\pi)^{3}} \delta(k_{1} + k_{2} - k) D_{1}(k, k_{1}, k_{2}, \gamma, \eta),
\]

\[
\frac{dq}{d\eta_{k_{1}}} \left( \frac{\partial f_{k_{2}}^{(1)}}{\partial q} \right) = \left( \frac{\partial F}{\partial q} \right)^{(0)} \times + q^{2} \left( \frac{\partial^{2} F}{\partial q^{2}} \right)^{(0)} \int \frac{d^{3}k_{1}d^{3}k_{2}}{(2\pi)^{3}} \delta(k_{1} + k_{2} - k) D_{2}(k, k_{1}, k_{2}, \gamma, \eta),
\]

\[
\frac{dq}{d\eta_{k_{1}}} \left( \frac{\partial F}{\partial q} \right)^{(0)} \left( \frac{\partial f_{k_{2}}^{(1)}}{\partial q} \right) = \left( \frac{\partial F}{\partial q} \right)^{(0)} \times \int \frac{d^{3}k_{1}d^{3}k_{2}}{(2\pi)^{3}} \delta(k_{1} + k_{2} - k) D_{3}(k, k_{1}, k_{2}, \gamma, \eta) + D_{44}(k, \gamma, \eta),
\]

where

\[
D_{1}(k, k_{1}, k_{2}, \gamma, \eta) = \left( \phi_{k_{1}}^{(1)}(\eta') + \psi_{k_{1}}^{(1)}(\eta') \right) - \frac{1}{2} \gamma_{j}^{(1)} \left( \frac{\partial F}{\partial q} \right)^{(0)} \left( \frac{\partial f_{k_{2}}^{(1)}}{\partial q} \right)
\]

\[
+ \frac{1}{4} \gamma_{j}^{(1)} \left( \frac{\partial F}{\partial q} \right)^{(0)} \left( \frac{\partial f_{k_{2}}^{(1)}}{\partial q} \right)^{(0)} \int_{\text{dec}} d\eta'' \left( \frac{1}{2} \gamma_{j}^{(1)} \left( \frac{\partial F}{\partial q} \right)^{(0)} \left( \frac{\partial f_{k_{2}}^{(1)}}{\partial q} \right)^{(0)} \right) \left( \frac{\partial f_{k_{2}}^{(1)}}{\partial q} \right)^{(0)}
\]

\[
D_{2}(k, k_{1}, k_{2}, \gamma, \eta) = \left( -i \gamma_{j}^{(1)} k_{1} \phi_{k_{1}}^{(1)}(\eta') + \psi_{k_{1}}^{(1)}(\eta') \right) - \frac{1}{2} \gamma_{j}^{(1)} \left( \frac{\partial F}{\partial q} \right)^{(0)} \left( \frac{\partial f_{k_{2}}^{(1)}}{\partial q} \right)
\]

\[
+ \frac{1}{4} \gamma_{j}^{(1)} \left( \frac{\partial F}{\partial q} \right)^{(0)} \left( \frac{\partial f_{k_{2}}^{(1)}}{\partial q} \right)^{(0)} \int_{\text{dec}} d\eta'' \left( \frac{1}{2} \gamma_{j}^{(1)} \left( \frac{\partial F}{\partial q} \right)^{(0)} \left( \frac{\partial f_{k_{2}}^{(1)}}{\partial q} \right)^{(0)} \right) \left( \frac{\partial f_{k_{2}}^{(1)}}{\partial q} \right)^{(0)}
\]

\[
D_{3}(k, k_{1}, k_{2}, \gamma, \eta) = \left[ -i k_{1} \phi_{k_{1}}^{(1)}(\eta') + \psi_{k_{1}}^{(1)}(\eta') \right] + i k_{1} \gamma_{j}^{(1)} \phi_{k_{1}}^{(1)}(\eta') + \psi_{k_{1}}^{(1)}(\eta')
\]

\[
- \delta_{j}^{(1)} \gamma_{j}^{(1)} \left( \frac{\partial F}{\partial q} \right)^{(0)} \left( \frac{\partial f_{k_{2}}^{(1)}}{\partial q} \right)
\]

\[
+ \frac{1}{2} i k_{1} \gamma_{j}^{(1)} \left( \frac{\partial F}{\partial q} \right)^{(0)} \left( \frac{\partial f_{k_{2}}^{(1)}}{\partial q} \right)^{(0)} \int_{\text{dec}} d\eta'' \left( \frac{1}{2} \gamma_{j}^{(1)} \left( \frac{\partial F}{\partial q} \right)^{(0)} \left( \frac{\partial f_{k_{2}}^{(1)}}{\partial q} \right)^{(0)} \right) \left( \frac{\partial f_{k_{2}}^{(1)}}{\partial q} \right)^{(0)}
\]

Using Eqs. (18) and (A.6)–(A.9), we obtain

\[
A(q, k, \gamma, \eta) = \left( \frac{\partial F}{\partial q} \right)^{(0)} \left[ \int \frac{d^{3}k_{1}d^{3}k_{2}}{(2\pi)^{3}} \delta(k_{1} + k_{2} - k) D_{1} + D_{2} + D_{3} + D_{44} \right]
\]

\[
- q^{2} \left( \frac{\partial^{2} F}{\partial q^{2}} \right)^{(0)} \int \frac{d^{3}k_{1}d^{3}k_{2}}{(2\pi)^{3}} \delta(k_{1} + k_{2} - k) D_{2}.
\]

Define

\[
D_{n, \lambda}(k, \ldots, \eta) \equiv \int_{\text{dec}} d\eta'
\]

\[
+ \frac{1}{4\pi} \int \frac{d\Omega_{\gamma} \gamma^{(1)} \epsilon_{k_{1}, i, j} D_{n}(k, \ldots, \gamma, \eta') e^{-ik_{1} \gamma^{(1)}(\eta' - \eta)}}{1 + \left( \frac{\partial F}{\partial q} \right)^{(0)} \left( \frac{\partial f_{k_{2}}^{(1)}}{\partial q} \right)}.
\]
where $D_n$ denotes the functions $D_1$, $D_2$, $D_3$, $D_{4I}$ and $D_{4II}$. Following Eqs. (9) and (19), we obtain

$$\sigma_{k,\lambda} = 2\epsilon_{ij}^k a^{-4} \int \frac{d^3q}{(2\pi)^3} q_j \gamma_j \int_{0_{\text{uc}}}^{\eta} \frac{d\eta}{\epsilon_{ij}(q, k, \lambda)} e^{-i\epsilon_{ij}^k(\eta - \eta')}$$

$$= -2a^{-4} \int \frac{d^3q}{(2\pi)^3} \left[ q_j \left( \frac{\partial F}{\partial q} \right)^{(0)} \left( \int \frac{dk_1 dk_2}{(2\pi)^3} \delta(k_1 + k_2 - k) \right) + q_j \left( \frac{\partial^2 F}{\partial q^2} \right)^{(0)} \left( \int \frac{dk_1 dk_2}{(2\pi)^3} \delta(k_1 + k_2 - k) \right) \right]$$

$$= 8\epsilon_{ij}^k \left[ \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \delta(k_1 + k_2) + k (\delta_{1, \lambda} - 4 \delta_{2, \lambda} + \delta_{3, \lambda}) + \delta_{4, \lambda} \right]. \quad (A.17)$$

### Appendix B: Bessel function and the angular integration

Following Rayleigh’s formulas, one can write the spherical Bessel functions $j_n(x)$, $n = 0, 1, 2 \ldots$ as

$$j_n(x) = (-x)^n \left( \frac{1}{x} \frac{d}{dx} \right)^n \frac{\sin x}{x}. \quad (B.18)$$

Then we obtain the recurrence formula

$$\frac{1}{x} \frac{d}{dx} \left( \frac{j_n(x)}{(-x)^n} \right) = \frac{j_{n+1}(x)}{(-x)^{n+1}}. \quad (B.19)$$

Define the integrals

$$I_{l_1 l_2 \ldots l_N}^{(i)}(x) \equiv \frac{1}{4\pi} \int d\Omega_q y_1^{i_1} y_2^{i_2} \ldots y_N^{i_N} e^{-i\gamma^{i}}. \quad (B.20)$$

We find that

$$I_{l_1 l_2 \ldots l_{N+1}}^{(i)}(x) = i \frac{\partial}{\partial l_{N+1}} I_{l_1 l_2 \ldots l_N}^{(i)}, \quad (B.21)$$

and

$$I_0(x) = \frac{1}{4\pi} \int d\Omega_q e^{-i\gamma^{i}} = \frac{\sin x}{x} = j_0(x), \quad (B.22)$$

where $x = |x|$. Using the recurrence formulas Eqs. (B.19), (B.21) and the initial condition Eq. (B.22), we get the general term formula of $I_{l_1 l_2 \ldots l_N}^{(i)}(x)$,

$$I_{2n+1}^{(i)}(x) = \sum_{l=0}^{n} \delta^{(n-l)} X_{2l+1}(x) j_{n+l+1}(x) \frac{x}{(n+l+1)!}, \quad (B.23)$$

$$I_{2n}^{(i)}(x) = \sum_{l=0}^{n} \delta^{(n-l)} X_{2l}(x) j_{n+l}(x) \frac{x}{(n+l)!}, \quad (B.24)$$

where $n = 0, 1, 2 \ldots$, $\delta^{(n-l)}$ denotes $n - l$ Kronecker $\delta$’s with $2(n - l)$ distinct indexes, $x^{[2l]}$ denotes $2l$ $x$’s with $2l$ distinct indexes. These $2n + 1$ ($2n$) indexes are circulant symmetric. The explicit forms of Eqs. (B.23) and (B.24) for $n = 1, 2$ have been shown in Ref. [42]. From these equations, we obtain

$$I_{2i}^{(i)}(x) e_{k,ij}^\lambda = \frac{j_i(x)}{(-x)^2} \epsilon_i x_i x_j \epsilon_{k,ij}^\lambda, \quad (B.25)$$

and

$$I_{3i}^{(i)}(x) e_{k,ij}^\lambda = -i \frac{j_3(x)}{(-x)^4} \delta^{(i)} x_i x_j \epsilon_{k,ij}^\lambda, \quad (B.26)$$

and

$$I_{4i}^{(i)}(x) e_{k,ij}^\lambda x_i x_j \epsilon_{k,ij}^\lambda = \frac{j_4(x)}{(-x)^6} \delta^{(i)} x_i x_j \epsilon_{k,ij}^\lambda. \quad (B.27)$$

For $x \to \infty$, it should be noticed that

$$\delta_{n-l} X^{[2l+1]}(x) j_{n+l+1}(x) \frac{x}{(n+l+1)!} \ll \delta_{n-l} X^{[2l]}(x) j_{n+l}(x) \frac{x}{(n+l)!} \quad \text{as} \quad x \to \infty. \quad (B.29)$$

Thus we have

$$I_{N}^{(i)}(x) \simeq i^N X^{[N]}(x) j_N(x) \frac{x}{(-x)^N} \quad (B.31)$$

for $x \to \infty$.

### Appendix C: The Expressions of $f_i$, $\mathcal{P}_i$ and $\mathcal{Q}_i$

In the radiation dominated epoch, the explicit expression of $f_i$ are as follows,

$$f_1 = -2\tau_{\phi}(ux) T_{\phi}(ux) - 2\tau_{\psi}(ux) T_{\psi}(ux) + T_{\phi}(ux) T_{\psi}(ux)$$

$$- \tau_{ux} \frac{dT_{\psi}(ux)}{d(ux)} - \tau_{ux} \frac{dT_{\phi}(ux)}{d(ux)} \quad (C.32)$$

and

$$f_{21} = \frac{a^2 X^{[u]}(ux)}{d(ux)^2} T_{\phi}(ux) + \frac{1}{2} \frac{dX^{[u]}(ux)}{d(ux)} T_{\phi}(ux)$$

$$+ \frac{1}{2} \frac{dX^{[u]}(ux)}{d(ux)} T_{\phi}(ux) \quad (C.32)$$

$$- \frac{2}{x} \frac{dX^{[u]}(ux)}{d(ux)} T_{\phi}(ux)$$
In the radiation dominated epoch, define \( f \) as the fraction of the total energy density in neutrinos. The explicit expression of \( \mathcal{P}_1 \) is as follows,

\[
\mathcal{P}_1 = \frac{12 f_v}{x^2} \int_{x_{dec}}^{x} dx' \int_{x_{dec}}^{x'} dx'' \left( 2 \frac{j_2(\tilde{x}_1)}{(-\tilde{x}_1)^2} (x' - x'') + \frac{j_3(\tilde{x}_1)}{(-\tilde{x}_1)^3} (x' - x'')^2 \right)
+ \frac{1}{x^2} \int_{x_{dec}}^{x} dx' \int_{x_{dec}}^{x'} dx'' \left( \frac{d\mathcal{T}_\Phi(ux')}{d(ux'')} + \frac{d\mathcal{T}_\Phi(ux'')}{d(ux''')} \right)
- \frac{48 f_v}{x^2} \int_{x_{dec}}^{x} dx' \int_{x_{dec}}^{x'} dx'' \left[ \frac{j_2(\tilde{x}_1)}{(-\tilde{x}_1)^2} (x' - x'') \right]
+ \frac{1}{x^2} \int_{x_{dec}}^{x} dx' \int_{x_{dec}}^{x'} dx'' \left( \frac{d\mathcal{T}_\Phi(ux')}{d(ux'')} + \frac{d\mathcal{T}_\Phi(ux'')}{d(ux''')} \right)
+ \frac{12 f_v}{x^2} \int_{x_{dec}}^{x} dx' \int_{x_{dec}}^{x'} dx'' \left( \frac{d\mathcal{T}_\Phi(ux')}{d(ux'')} + \frac{d\mathcal{T}_\Phi(ux'')}{d(ux''')} \right)
+ \frac{1}{x^2} \int_{x_{dec}}^{x} dx' \int_{x_{dec}}^{x'} dx'' \left( \frac{d\mathcal{T}_\Phi(ux')}{d(ux'')} + \frac{d\mathcal{T}_\Phi(ux'')}{d(ux''')} \right)
+ \frac{1}{x^2} \int_{x_{dec}}^{x} dx' \int_{x_{dec}}^{x'} dx'' 
\]
\begin{align*}
&\times \chi(u'x)v \left( \frac{dT_\Phi(u'x'v)}{d(u'x')} + \frac{dT_\Psi(u'x'v)}{d(u'x')} \right) \\
&- \frac{12 f_v}{x^2} \int_{\Delta_{\text{dc}}}^x dx' \int_{\Delta_{\text{dc}}}^{x'} dx'' \left( \frac{f_j(\tilde{x}_2)}{(-\tilde{x}_2)^2} (x-x') \right) \\
&\times \frac{d\chi(u'x'v)}{d(u'x')} \left( T_\Phi(u'x'v) + T_\Psi(u'x'v) \right) \\
&- \frac{48 f_v}{x^2} \int_{\Delta_{\text{dc}}}^x dx' \int_{\Delta_{\text{dc}}}^{x'} dx'' \left( \frac{f_j(\tilde{x}_2)}{(-\tilde{x}_2)^2} (x-x') \right) (x-x') (x''-x'') u v \\
&\times \frac{d\chi(u'x'v)}{d(u'x')} \left( T_\Phi(u'x'v) + T_\Psi(u'x'v) \right) \\
&- \frac{48 f_v}{x^2} \int_{\Delta_{\text{dc}}}^x dx' \int_{\Delta_{\text{dc}}}^{x'} dx'' \left( \frac{f_j(\tilde{x}_2)}{(-\tilde{x}_2)^2} (x-x') \right) (x-x') (x''-x'') u v \\
&\times \frac{d\chi(u'x'v)}{d(u'x')} \left( T_\Phi(u'x'v) + T_\Psi(u'x'v) \right) \\
&- \frac{12 f_v}{x^2} \int_{\Delta_{\text{dc}}}^x dx' \int_{\Delta_{\text{dc}}}^{x'} dx'' \left( \frac{f_j(\tilde{x}_1)}{(-\tilde{x}_1)^2} (x-x') \right) (x-x') (x''-x'') u v \\
&\times \frac{d\chi(u'x'v)}{d(u'x')} \left( T_\Phi(u'x'v) + T_\Psi(u'x'v) \right) \\
&- \frac{12 f_v}{x^2} \int_{\Delta_{\text{dc}}}^x dx' \int_{\Delta_{\text{dc}}}^{x'} dx'' \left( \frac{f_j(\tilde{x}_1)}{(-\tilde{x}_1)^2} (x-x') \right) (x-x') (x''-x'') u v \\
&\times \frac{d\chi(u'x'v)}{d(u'x')} \left( T_\Phi(u'x'v) + T_\Psi(u'x'v) \right) \\
&\times \frac{d\chi(u'x'v)}{d(u'x')} \left( T_\Phi(u'x'v) + T_\Psi(u'x'v) \right) \\
&\times \frac{d\chi(u'x'v)}{d(u'x')} \left( T_\Phi(u'x'v) + T_\Psi(u'x'v) \right)
\end{align*}

In the radiation dominated epoch, the explicit expression of $Q_i$ is expressed as,

\begin{align}
Q_1 &= - \frac{48 f_v}{x^2} \int_{\Delta_{\text{dc}}}^x dx' \left( \frac{1}{u} \right) j_2(u(x-x')) \\
&\times \left( \frac{dT_\Phi(u'x'v)}{d(u'x')} + \frac{dT_\Psi(u'x'v)}{d(u'x')} \right) T_\Psi(u'x'), \\
Q_{2i} &= \frac{24 f_v}{x^2} \int_{\Delta_{\text{dc}}}^x dx' \left( \frac{1}{u(x-x')^2} \right) u v \frac{d\chi(u'x'v)}{d(u'x')} T_\Psi(u'x'),
\end{align}

\begin{align}
Q_{3j} &= 0. \quad (j = I, II, III, IV, V),
\end{align}

where $x = k\eta$, $x' = k\eta'$, $x'' = k\eta''$, $\tilde{x}_1 = x - u'x' - v''x''$, $\tilde{x}_2 = x - u'x' - v''x''$, $\tilde{x}_1 = |\tilde{x}_1|$, $\tilde{x}_2 = |\tilde{x}_2|$. It should be clarified that during the calculation of $P_{3i}$, $P_{3m}$ ($l = I, II, m = I \sim V$) shown above, the integrals $F_{N-l}^{m}$ have used the approximation for $N \geq 5$ shown in Eq. (B.31). Thus, Eqs. (C.41)–(C.45) only established as $x \rightarrow \infty$.

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