BIFURCATION IN THE ALMOST PERIODIC 2D RICKER MAP

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Abstract. This paper studies bifurcations in the coupled 2 dimensional almost periodic Ricker map. We establish criteria for stability of an almost periodic solution in terms of the Lyapunov exponents of a corresponding dynamical system and use them to find a bifurcation function. We find that if the almost periodic coefficients of all the maps are identical, then the bifurcation function is the same as the one obtained in the one dimensional case treated earlier, and that this result holds in N dimension under modest coupling constraints. In the general two-dimensional case, we compute the Lyapunov exponents numerically and use them to examine the stability and bifurcations of the almost periodic solutions.

1. Introduction. Much work has been done on the classical Ricker equation

\[ x(t + 1) = F_p(x(t)) \]  

where \( F_p(x) = xe^{p-x} \) since it was first proposed in 1954 by W. E. Ricker [13]. Much of the theoretical work has dealt with global stability in the case of constant and periodic \( p \) while until recently bifurcations have been limited to the study of period doubling and of course chaos. In [20, 21] global stability was proved by a general method that applies to any almost periodic difference or differential equation. Later [19] bifurcations in the almost periodic Ricker equation were studied.

The mapping \( F_p : [0, \infty) \to [0, \infty) \) features a single parameter \( p > 0 \) and two equilibria at \( x = 0 \) and \( x = p \). In the context of population dynamics, these two equilibria correspond to extinction and the carrying capacity respectively. The origin is always unstable, while the other fixed point \( p \) is globally asymptotically stable on \( (0, \infty) \) for \( p \in (0, 2) \), [8]. The stability interval can be extended to \( (0, 2] \) using the Schwarzian derivative when \( p = 2 \) [7, p. 32]. As soon as \( p \) exceeds the value 2, the fixed point turns unstable. Increasing \( p \) further causes a period doubling bifurcation to occur.

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The periodic 1D Ricker Equation was studied in the form
\[ x_{n+1} = x_n e^{p_n - x_n} \]  
where \( \{p_n\} \) is a periodic sequence. It was shown in [18] that for \( p_n \in (0, 2) \) there is a periodic solution having the same period as \( \{p_n\} \) that is globally asymptotically stable with respect to \((0, \infty)\).

More recently, the almost periodic 1D Ricker Equation was studied in [19]. In the case of one period, the equation takes the form
\[ x(t + 1) = x(t) \gamma(t) e^{-x(t)} \]  
where \( \gamma(t) = \gamma_0 + bG(\omega t) \) for some irrational \( \omega \). Here \( G(\theta) \) is a \( C^1 \) function, periodic with period 1 with \( \max |G(\theta)| = 1 \) and with mean value 0 over one period. When \( b = 0 \), equation (3) is identical to equation (1) with \( \gamma_0 = e^p \).

The story for the Ricker Equations in higher dimensions is much more complicated. The 2D Ricker Equations have been studied in the form
\[ x(t + 1) = x(t) e^{p-x(t)-aq(t)} \]  
\[ y(t + 1) = y(t) e^{q-y(t)-bx(t)}, \]  
where the domain is the first quadrant and the parameters \( p, q, a, b \) are all positive. The parameters \( a, b \) serve as coupling parameters; if they were to take the value zero then the equations would be identical to two uncoupled 1D Ricker Equations. For positive coupling and away from the degenerate case of \( a = \frac{p}{q}, b = \frac{q}{p} \), the equation has up to four equilibria. The equilibria \((0, 0), (p, 0), (0, q)\) are always present, while the coexistent equilibrium \( \left( \frac{p-aq}{1-ab}, \frac{q-bp}{1-ab} \right) \) is only present under certain parameter restrictions. For \( ab < 1 \), the stability region for the coexistent fixed point is bounded by three curves [11], see Figure 1. For a given \( a, b \), the boundaries of this region in the \( p-q \) plane are the straight lines \( p = aq, q = bp \), and the hyperbola given by
\[ bp^2 + 2(1 - b)p - (1 + ab)pq + 2(1 - a)q + aq^2 + 4(ab - 1) = 0 \]  
The coexistence fixed point vanishes if either of the two lines are crossed, while it bifurcates into a period 2 orbit as the parameter pair crosses the parabola. Recent studies have deduced conditions under which local asymptotic stability implies global asymptotic stability [2], [16], [17]. For \( ab > 1 \), the interior fixed point is always unstable [11], [15]. The global dynamics in the case where \( ab = 1 \) have also been studied in detail [14].

In this paper, we will study the almost periodic Ricker Equation in higher dimensions. We prefer to view the 2D Ricker Equation as
\[ x_1(t + 1) = \gamma_01x_1(t)e^{-x_1(t)-a_{12}x_2(t)} \]  
\[ x_2(t + 1) = \gamma_02x_2(t)e^{-x_2(t)-a_{21}x_1(t)} \]  
where \( \gamma_01 = e^p, \gamma_02 = e^q \), and we have relabeled \( x, y, a, \) and \( b \) in ways that more easily generalize to higher dimensions.

Our main focus will be to study the almost periodic equation
\[ x_1(t + 1) = \gamma_1(t)x_1(t)e^{-x_1(t)-a_{12}x_2(t)} \]  
\[ x_2(t + 1) = \gamma_2(t)x_2(t)e^{-x_2(t)-a_{21}x_1(t)} \]  
where the constants \( a_{12} \in \left(0, \frac{\log(\gamma_01)}{\log(\gamma_02)}\right) \), \( a_{21} \in \left(0, \frac{\log(\gamma_02)}{\log(\gamma_01)}\right) \). We take \( \gamma_i(t) = \gamma_0i + bG_i(\omega t) \) so that they have the same form as \( \gamma \) from the almost periodic 1D Ricker.
2. The almost periodic 2D ricker equation.

2.1. Preliminaries. In 1925 H. Bohr introduced the notion of almost periodic (AP) functions in his book [3]. The Bohr definition of AP is given as follows:

Definition 2.1. A subset \( E \subset \mathbb{R} \) is relatively dense if there is a number \( \ell > 0 \) such that any interval of \( \mathbb{R} \) of length \( \ell \) contains a point of \( E \). Define

\[
E(f, \varepsilon) = \{ \tau \in \mathbb{R} \mid |f(t) - f(t + \tau)| < \varepsilon, \forall t \in \mathbb{R} \}.
\]

Then \( f : \mathbb{R} \to \mathbb{R} \) is Bohr AP if and only if \( E(f, \varepsilon) \) is relatively dense for each \( \varepsilon > 0 \).
In this paper, we study equation (7) with
\[ \gamma_i(t) = \gamma_{0i} + bG_i(\omega t), \quad i \in \{1, 2\}. \]  
(9)
Here the constants \( \gamma_{01}, \gamma_{02} > 1 \), \( \omega \) is an irrational frequency, and \( b \geq 0 \) is a small parameter. The functions \( G_1, G_2 \) are continuously differentiable with period 1, have mean value 0, and \( \max |G| = 1 \). Clearly, the mean value of \( \gamma_i \) is \( \gamma_{0i} \). Since \( G(\theta) \) is a periodic function with period 1 and \( \omega \) is irrational, each sequence \( \gamma_i : \mathbb{Z}^+ \to \mathbb{R} \) is AP.

If we consider \( \Gamma = (\gamma_1, \gamma_2) \) as a point in \( X = C(\mathbb{R}, \mathbb{R}^2) \) with flow \( \sigma(\Gamma, \tau) = \Gamma_{\tau} \), where \( \Gamma_{\tau}(t) = \Gamma(t + \tau) \), then the hull of \( \Gamma \),
\[ H(\Gamma) = \bigcup_{\tau \in \mathbb{R}} \Gamma_{\tau}, \]  
(10)
is a closed curve \( \mathcal{P} \) in \( X \). Since \( \omega \) is irrational the points \( \{\Gamma(t) : t \in \mathbb{Z}^+ \} \) are dense on \( \mathcal{P} \). Thus, (7) generates a skew-product semi-flow on the space \( \mathbb{R}_+^2 \times \mathcal{P} \) with base space \( \mathcal{P} \),
\[ \pi_1 : \mathbb{R}_+^2 \times \mathcal{P} \times \mathbb{Z}^+ \longrightarrow \mathbb{R}_+^2 \times \mathcal{P}, \]  
(11)
defined by
\[ (x, \Gamma, 1) \xrightarrow{\pi} (\phi(x, \Gamma, 1), \Gamma_1), \]
where \( \phi(x, \Gamma, 1) \) is the value of the left hand side of (7), \( \Gamma_1(t) = \Gamma(t + 1) \) and
\[ \pi_n = \pi \circ \pi \circ \cdots \circ \pi(x, \theta, t), \quad n \text{-times.} \]

See also [20, Lemma 3.3] for a further discussion of skew-product spaces.

Our initial aim in this work is to determine when equation (7) has an asymptotically stable almost periodic solution. By an almost periodic solution, we mean a solution \((x_1(t), x_2(t))\) which satisfies equation (7) such that \( x_1, x_2 \) are AP. We will then show how to find, for given coupling constants \( a_{12}, a_{21} \), a function \( B(\gamma_{01}, \gamma_{02}, b) \) so that \( B < 0 \) guarantees this asymptotic stability. Then \( B = 0 \) (the bifurcation equation) describes a loss of stability so that \( B > 0 \) implies a bifurcation into two AP solutions.

The approach is a generalization of the approach used in one-dimension [19]. First, an AP solution with one irrational frequency generates an invariant curve \( \Lambda \) and it is this curve that bifurcates. Then we derive the recurrence relation that follows from the invariance (given below). The following theorem sums up the results in the case of one dimension. Note that \( \Lambda \) lies in the product \( \mathbb{R}_+^2 \times \mathcal{P} \) and the projection \( \pi|_\Lambda : \Lambda \to \mathcal{P} \) is a diffeomorphism, [19, p. 606].

**Theorem 2.2.** [19, p. 606]

For \( k \in \mathbb{Z} \), define the sequence \( \theta_k = \theta + k\omega \) (mod 1) and \( v_k(\theta) = v(\theta_k) \). Let \( v \) satisfy the recurrence
\[ v_n(\theta) = \mu s(b, \theta_{n-1}) v_{n-1}(\theta) + R_1(b, \theta_{n-1}, v_{n-1}(\theta)), \]  
(12)
where
\[ s(\theta) = 1 + \frac{b}{\gamma_0} G(\theta), \quad R_1(b, \theta, v) = \frac{f_0}{\gamma_0} G(\theta) + h(b, \theta, v) \quad \text{and} \quad \mu = 1 - \log \gamma_0. \]  
(13)
Then $v_n$ satisfies the recurrence

$$v_n(\theta) = \left( \prod_{j=1}^{n} \mu_s(b, \theta_{j-1}) \right) v_0(\theta) + \sum_{j=1}^{n-1} \left( \prod_{k=1}^{n-j} \mu_s(b, \theta_{n-k}) \right) R_1(b, \theta_{j-1}, v_{j-1}(\theta)) .$$  
(14)

Moreover, if $b$ is sufficiently small such that $|\mu_s(\theta_{j-1})| \leq \sigma < 1$ the sequence (14) will converge to a unique exponentially asymptotically stable solution of (12) if and only if

$$L(\theta_{j-1}) = \limsup_{n \to \infty} \left| \prod_{j=1}^{n} \mu_s(\theta_{j-1}) \right|^{1/n} < 1 .$$  
(15)

After taking the logarithm and using the definition of $\mu$,

$$B(\gamma_0, b) = \log(\log \gamma_0 - 1) + \int_{0}^{1} \log \left( 1 + \frac{b}{\gamma_0} G(\theta) \right) d\theta < 0.$$  
(16)

2.2. Invariance equations. A set of points $S$ in the domain of a discrete dynamical system $D : x_1 = g(x)$ is invariant under the action of $D$ if $x \in S$ implies $x_1 \in S$. In the present case we assume the invariant curve $A$ has the form

$$f(\theta) = f_0 + u(\theta), \quad f_0 = \begin{pmatrix} f_{01} \\ f_{02} \end{pmatrix}, \quad u(\theta) = \begin{pmatrix} u_1(\theta) \\ u_2(\theta) \end{pmatrix} .$$

For $b = 0$ the equations (7) reduces to the autonomous equation (6) having an asymptotically stable fixed point under certain conditions.

The invariance equations are

$$f_{01} + u_1(\theta_1) = \gamma_1(\theta) f_{01} + u_1(\theta) e^{-f_{01} - u_1(\theta) - a_{12}(f_{02} + u_2(\theta))} ,$$

$$f_{02} + u_2(\theta_1) = \gamma_2(\theta) f_{02} + u_2(\theta) e^{-f_{02} - u_2(\theta) - a_{21}(f_{01} + u_1(\theta))} ,$$

where $\gamma_j = \gamma_{0j} + b G_j(\theta)$, $j = 1, 2$ and $\theta_k = \theta + k \omega \ (\text{mod} \ 1)$. Rewriting (17),

$$f_{01} + u_1(\theta_1) = (\gamma_{01} + b G_1(\theta)) (f_{01} + u_1(\theta)) e^{-f_{01} - a_{12} f_{02} - u_1(\theta) - a_{12} u_2(\theta)} ,$$

$$f_{02} + u_2(\theta_1) = (\gamma_{02} + b G_2(\theta)) (f_{02} + u_2(\theta)) e^{-f_{02} - a_{21} f_{01} - u_2(\theta) - a_{21} u_1(\theta)} .$$

At $b = 0$ one has $u_1 = u_2 = 0$ and the equations reduce to

$$f_{01} = \gamma_{01} f_{01} e^{-a_{12} f_{02}} ,$$

$$f_{02} = \gamma_{02} f_{02} e^{-a_{21} f_{01}} ,$$

or equivalently

$$e^{f_{01} + a_{12} f_{02}} = \gamma_{01} \quad \text{and} \quad e^{f_{02} + a_{21} f_{01}} = \gamma_{02} .$$  
(19)

These lowest order relations (19) hold even for $b > 0$. Then,

$$f_{01} + u_1(\theta_1) = \left( 1 + \frac{b}{\gamma_{01}} G_1(\theta) \right) (f_{01} + u_1(\theta)) e^{-u_1(\theta) - a_{12} u_2(\theta)} ,$$

$$f_{02} + u_2(\theta_1) = \left( 1 + \frac{b}{\gamma_{02}} G_2(\theta) \right) (f_{02} + u_2(\theta)) e^{-u_2(\theta) - a_{21} u_1(\theta)} .$$

Expanding the exponentials, we obtain

$$f_{01} + u_1(\theta_1) = \left( 1 + \frac{b}{\gamma_{01}} G_1(\theta) \right) (f_{01} + u_1(\theta))(1 - u_1(\theta) - a_{12} u_2(\theta)) + O(\|u\|^2) ,$$

$$f_{02} + u_2(\theta_1) = \left( 1 + \frac{b}{\gamma_{02}} G_2(\theta) \right) (f_{02} + u_2(\theta))(1 - u_2(\theta) - a_{21} u_1(\theta)) + O(\|u\|^2) .$$
where \( u = (u_1, u_2)^T \). After some manipulation, we obtain

\[
\begin{align*}
  f_{o1} + u_1(\theta_1) &= \left(1 + \frac{b}{\gamma_{o1}} G_1(\theta)\right) \left[f_{o1} + u_1(\theta) - f_{o1}(u_1(\theta) - a_{12} u_2(\theta))\right] + \mathcal{O}(\|u\|^2), \\
  f_{o2} + u_2(\theta_1) &= \left(1 + \frac{b}{\gamma_{o2}} G_2(\theta)\right) \left[f_{o2} + u_2(\theta) - f_{o2}(u_2(\theta) - a_{21} u_1(\theta))\right] + \mathcal{O}(\|u\|^2),
\end{align*}
\]

and finally

\[
\begin{align*}
  f_{o1} + u_1(\theta_1) &= f_{o1} + \left(1 + \frac{b}{\gamma_{o1}} G_1(\theta)\right) \left[(1 - f_{o1}) u_1(\theta) - a_{12} f_{o1} u_2(\theta)\right] \\
  &+ \frac{b f_{o1}}{\gamma_{o1}} G_1(\theta) + \mathcal{O}(\|u\|^2), \\
  f_{o2} + u_2(\theta_1) &= f_{o2} + \left(1 + \frac{b}{\gamma_{o2}} G_2(\theta)\right) \left[(1 - f_{o2}) u_2(\theta) - a_{21} f_{o2} u_1(\theta)\right] \\
  &+ \frac{b f_{o2}}{\gamma_{o2}} G_2(\theta) + \mathcal{O}(\|u\|^2).
\end{align*}
\]

Cancelling the \( f_{oj} \) from each equation, we arrive at the system

\[
u(\theta_1) = \begin{bmatrix}
1 + \frac{b}{\gamma_{o1}} G_1(\theta) & 0 \\
0 & 1 + \frac{b}{\gamma_{o2}} G_2(\theta)
\end{bmatrix} \begin{bmatrix}
1 - f_{o1} & -a_{12} f_{o1} \\
-a_{21} f_{o2} & 1 - f_{o2}
\end{bmatrix} u(\theta) + R(b, \theta, u), \quad (21)
\]

where

\[
R(b, \theta, u) = b \begin{bmatrix}
\frac{f_{o1}}{\gamma_{o1}} G_1(\theta) \\
\frac{f_{o2}}{\gamma_{o2}} G_2(\theta)
\end{bmatrix} + \mathcal{O}(\|u\|^2).
\]

We set \( u(\theta) = b v(\theta) \) in order to remove the degeneracy from this equation at \( b = 0 \). Then (21) reduces to

\[
v(\theta_1) = \begin{bmatrix}
1 + \frac{b}{\gamma_{o1}} G_1(\theta) & 0 \\
0 & 1 + \frac{b}{\gamma_{o2}} G_2(\theta)
\end{bmatrix} \begin{bmatrix}
1 - f_{o1} & -a_{12} f_{o1} \\
-a_{21} f_{o2} & 1 - f_{o2}
\end{bmatrix} v(\theta) + R_1(b, \theta, v), \quad (22)
\]

where

\[
R_1(b, \theta, v) = \begin{bmatrix}
\frac{f_{o1}}{\gamma_{o1}} G_1(\theta) \\
\frac{f_{o2}}{\gamma_{o2}} G_2(\theta)
\end{bmatrix} + h(b, \theta, v),
\]

and for some \( M_2 > 0 \)

\[
\|h(b, \theta, v)\| \leq b M_2 \|v\|^2 \text{ for } \|bv\| < 1. \quad (23)
\]

Now we denote

\[
v_n(\theta) = v(\theta_n), \quad S(b, \theta) = I + b \begin{bmatrix}
G_1(\theta) & 0 \\
0 & G_2(\theta)
\end{bmatrix}, \quad M = \begin{bmatrix}
1 - f_{o1} & -a_{12} f_{o1} \\
-a_{21} f_{o2} & 1 - f_{o2}
\end{bmatrix}. \quad (24)
\]

Then equation (22) implies

\[
v_n(\theta) = S(b, \theta_{n-1}) M v_{n-1}(\theta) + R_1(b, \theta_{n-1}, v_{n-1}(\theta)). \quad (25)
\]

Note that equation (25) is similar to equation (12). Instead of a constant \( \mu = 1 - \log(\gamma_0) \), we have a \( 2 \times 2 \) constant matrix \( M \). Also, \( S \) is now a \( 2 \times 2 \) diagonal
matrix with diagonal entries of the same form as $s$ from equation (12). Nonetheless, by writing $v_n$ in terms of $v_0$, we obtain the recurrence

$$v_n(\theta) = \left( \prod_{j=1}^{n} S(b, \theta_{n-j})M \right) v_0(\theta)$$

$$+ \sum_{j=1}^{n} \left( \prod_{k=1}^{n-j} S(b, \theta_{n-k})M \right) R_1(b, \theta_{j-1}, v_{j-1}(\theta)).$$

which should be compared to the very similar recurrence from equation (14) in Theorem 2.2. To discuss convergence of (14), it was necessary to compute the magnitude of the coefficients $v_0(\theta)$. In equation (26), we will instead investigate the norm of the coefficient of $v_0(\theta)$. This is tied directly to the calculation of Lyapunov exponents.

2.3. Linear cocycles and Lyapunov exponents. The Lyapunov exponent associated with an orbit simply measures the rate of divergence of nearby orbits to a given orbits. This is accomplished by studying the solutions of the linearized variational equation. The equation we wish to study is the non-homogeneous equation (26) and the variational equation is obtained by replacing $v_n$ by $v_n + \eta_n$,

$$\eta_n = \left( \prod_{j=1}^{n} S(b, \theta_{n-j})M \right) \eta_0(\theta) + bR'_1 \eta_0.$$

For $b$ on a small neighborhood of zero, it is the first expression that determines stability,

$$\eta_n = \left( \prod_{j=1}^{n} S(b, \theta_{n-j})M \right) \eta_0(\theta).$$

Then the evolution operator or cocycle is given by

$$A^{(n)}(\theta) = \prod_{j=1}^{n} A(\theta_{n-j}),$$

where $A(\theta) = S(b, \theta)M$ where these quantities are defined in (24). Then the cocycle condition [1] is

$$A^{(m+n)}(\theta) = A^{(m)}(\theta_n) \cdot A^{(n)}(\theta), \quad \forall n, m \in \mathbb{Z}^+, \: \theta \in S.$$  

It follows that $A^{(n)}(\theta)$ is a linear cocycle. We define the Lyapunov exponents [22], [23] at the point $\theta$ and a nonzero $v \in \mathbb{R}^d$ as

$$\chi(\theta, v) = \limsup_{n \to \infty} \frac{1}{n} \log \| A^{(n)}(\theta)v \|.$$  

As $\omega$ is irrational, the $\theta_n$ are uniformly distributed mod 1 and by Weyl’s Criterion [4], [5], [19, p. 609] the limits in (31) exist. In this setting, the limits are also independent of $\theta$ [12], [22]. Moreover, over all nonzero $v$, there are at most $d = 2$ distinct Lyapunov exponents $\chi_1, \chi_2$.

The largest of these two exponents is given by $\lim_{n \to \infty} \frac{1}{n} \log \| A^{(n)}(\theta) \|$. Note, $A^{(n)}(\theta)$ is the coefficient of $v_0(\theta)$ in equation (26). Consequently, an identical calculation to the proof of Theorem (2.2) given in [19] also gives the following theorem.
Theorem 2.3. Let \( b \geq 0, \omega \) be irrational, and let \( G_1, G_2 \) be \( C^1 \). Assume further that \( \max_{\theta} |G_i(\theta)| = 1, G_i(\theta + 1) = G_i(\theta) \), and each \( G_i \) has mean value 0 over one period. Let \( a_{12} \in \left( 0, \frac{\log(\gamma_0)}{\log(\gamma_2)} \right), a_{21} \in \left( 0, \frac{\log(\gamma_2)}{\log(\gamma_0)} \right) \) Define \( S \) and \( M \) as in (24).

Let \( A(\theta) = S(b, \theta)M \) and \( \theta_k = \theta + k\omega \pmod{1} \). Then the AP 2D Ricker Equation (7) with \( \gamma_i(t) = \gamma_0 + bG_i(\omega t) \) has a locally asymptotically stable AP solution if and only if the largest Lyapunov exponent of the cocycle (29) of \( A \) is negative.

Theorem 2.3 implies that there is a bifurcation from a stable AP solution precisely when the largest Lyapunov exponent is 0. Except in special cases, Lyapunov exponents are difficult to compute analytically. We first consider \( \gamma \) explicitly when the largest Lyapunov exponent is 0. Throughout this subsection, we assume \( \omega \) is an irrational number, \( G \in C^1 \) with \( \max |G| = 1 \), \( G(\theta + 1) = G(\theta) \), and \( G \) has mean value 0 over one period. The values of \( f_{01}, f_{02} \) are given by the system of equations

\[
\begin{align*}
  f_{01} + a_{12} f_{02} &= f_{02} + a_{21} f_{01} = \log(\gamma_0) \quad (32)
\end{align*}
\]

from (19). We will prove the following Theorem.

Theorem 2.4. Define

\[
S(b, \theta) = I + b \begin{bmatrix}
    \frac{G(\theta)}{\gamma_0} & 0 \\
    0 & \frac{G(\theta)}{\gamma_0}
\end{bmatrix}, \quad M = \begin{bmatrix}
1 - f_{01} & -a_{12} f_{01} \\
-a_{21} f_{02} & 1 - f_{02}
\end{bmatrix}.
\]

Let \( \lambda_1, \lambda_2 \) denote the eigenvalues of \( M \). Let \( A(\theta) = S(b, \theta)M \) and \( \theta_k = \theta + k\omega \pmod{1} \), \( k \in \mathbb{Z}^+ \). If \( M \) is invertible, then the Lyapunov exponents of the cocycle

\[
A^{(n)}(\theta) = \prod_{j=1}^{n} A(\theta_{n-j})
\]

are

\[
\chi_1 = \log |\lambda_1| + \int_{0}^{1} \log \left( 1 + \frac{b}{\gamma_0} G(\theta) \right) \, d\theta
\]

and

\[
\chi_2 = \log |\lambda_2| + \int_{0}^{1} \log \left( 1 + \frac{b}{\gamma_0} G(\theta) \right) \, d\theta.
\]

Proof of Theorem 2.4. Let \( \lambda \) be an eigenvalue for the constant matrix \( M \) and let \( v \) be a corresponding eigenvector. Recall that the Lyapunov exponent for \( v \) is given by

\[
\chi(v) = \lim_{n \to \infty} \frac{1}{n} \log \|A^{(n)}(\theta)v\|.
\]

\[
(33)
\]
where

\[
A^{(n)}(\theta) = \prod_{j=1}^{n} S(b, \theta_{n-j}) M
\]

\[
= \prod_{j=1}^{n} \left( 1 + \frac{bG(\theta_{n-j})}{\gamma_0} \right) M
\]

\[
= \left( \prod_{j=1}^{n} \left( 1 + \frac{bG(\theta_{n-j})}{\gamma_0} \right) \right) M^n .
\]

Then

\[
\frac{1}{n} \log \| A^{(n)}(\theta) v \| = \frac{1}{n} \log \left\| \prod_{j=1}^{n} \left( 1 + \frac{bG(\theta_{n-j})}{\gamma_0} \right) M^n v \right\|
\]

\[
= \frac{1}{n} \log \left( \prod_{j=1}^{n} \left( 1 + \frac{bG(\theta_{n-j})}{\gamma_0} \right) \right) \| M^n v \|
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} \log \left( 1 + \frac{bG(\theta_{n-j})}{\gamma_0} \right) + \frac{1}{n} \log \| \lambda^n v \|
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} \log \left( 1 + \frac{bG(\theta_{n-j})}{\gamma_0} \right) + \frac{1}{n} \log | \lambda^n | \| v \|
\]

\[
= \log | \lambda | + \frac{1}{n} \log \| v \| + \frac{1}{n} \sum_{j=1}^{n} \log \left( 1 + \frac{bG(\theta_{n-j})}{\gamma_0} \right) .
\]

Now we take limits as \( n \to \infty \) so that \( \frac{1}{n} \log \| v \| \) vanishes. Further, since the sequence \( \{ \theta_{n-j} \} \) is uniformly distributed modulo 1, Weyl’s Criterion [4], [5], [19, p. 609] implies

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \log \left( 1 + \frac{b}{\gamma_0} G(\theta_{j-1}) \right) = \int_{0}^{1} \log \left( 1 + \frac{b}{\gamma_0} G(\theta) \right) d\theta .
\]

Finally, since there are at most two Lyapunov exponents, the result follows by repeating the calculation for the other eigenvalue. If the eigenvalue has multiplicity two, then the above analysis works for all \( v \in \mathbb{R}^2, \ v \neq (0, 0) \).

Now let us discuss the implications of Theorem 2.4. A combination of Theorem 2.3 and Theorem 2.4 gives us the following corollary.

**Corollary 1.** Let \( M \) denote the matrix as in \((24)\). Let \( \rho(M) \) denote the spectral radius of \( M \). Then the AP 2D Ricker Equation \((7)\)

\[
\begin{align*}
    x_1(t+1) &= \gamma_1(t) x_1(t) e^{-x_1(t-a_{12}x_2(t)} \\
    x_2(t+1) &= \gamma_2(t) x_2(t) e^{-x_2(t-a_{21}x_1(t)}
\end{align*}
\]

with \( \gamma_i(t) = \gamma_0 + bG(\omega t), \ i \in \{1, 2\} \) has a locally asymptotically stable AP solution if and only if

\[
\log | \rho(M) | + \int_{0}^{1} \log \left( 1 + \frac{b}{\gamma_0} G(\theta) \right) d\theta < 0 .
\]
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Figure 2. Plot of the Bifurcation Equation $B(\gamma, \gamma, b) = 0$. A bifurcation takes place as the parameter pair $(b, \log \gamma)$ crosses from the stability region ($B < 0$) to the instability region ($B > 0$).

Note that Corollary 1 implies the bifurcation equation is

$$B(\gamma, \gamma, b) \doteq \log |\rho(M)| + \int_0^1 \log \left(1 + \frac{b}{\gamma_0} G(\theta)\right) d\theta = 0,$$  

(34)

where $B(\gamma, \gamma, b)$ the bifurcation function. The graph of $B = 0$ is shown in Figure 2. In [19] it is proved that the graph is increasing and convex with slope zero at $b = 0$.

In particular, this implies that for $B(\gamma, \gamma, b) < 0$ there is an asymptotically stable almost periodic orbit that is dense on $O_0$ a closed curve. As $B(\gamma, \gamma, b)$ increases through zero, the orbit on $O_0$ branches off continuously and inherits the asymptotic stability. This new orbit jumps back and forth between two new closed curves $O_1$ and $O_2$ under the action of (7) and densely fills out the two curves.

This should be compared to the bifurcation equation for the one-dimensional Ricker equation [19], given by

$$\log \left(\log(\gamma_0) - 1\right) + \int_0^1 \log \left(1 + \frac{b}{\gamma_0} G(\theta)\right) d\theta = 0.$$  

(35)

For $b = 0$, we have that the bifurcation in the 1D case occurs at $\gamma_0 = e^2$. The following lemmas prove in a neighborhood of $\gamma_0 = e^2$, that the bifurcation curve (34) is identical to that of (35).

Lemma 2.5. Let $M$ denote the matrix as in (24). The two eigenvalues of $M$ are $\lambda_i = 1 - \beta_i \log(\gamma_0)$ where $\beta_i$ are independent of $\gamma_0$ and $\beta_1 = 1$, $\beta_2 \in (0, 1)$.

Lemma 2.6. Let $M$ denote the matrix as in (24). There exists an $\epsilon > 0$ such that $\rho(M) = \log(\gamma_0) - 1$ for all $\log(\gamma_0) \geq 2 - \epsilon$. 


The remainder of this subsection gives the proofs of these two lemmas. In the next subsection, we will return to the general 2D case without the assumptions of $\gamma_0_1 = \gamma_0_2$ and $G_1 = G_2$.

**Proof of Lemma 2.5.** The values of $f_{01}$ and $f_{02}$ that appear in the matrix $M$ are obtained from solving (19),

$$(1 - a_{21})f_{01} = (1 - a_{12})f_{02} = \alpha, \quad (36)$$

where $\alpha$ is a constant to be determined. From equations (20),

$$f_{01} + a_{12}f_{02} = \log \gamma_0 = a_{21}f_{01} + f_{02}. \quad (37)$$

Solving (36) for $f_{01}$ and $f_{02}$ and inserting them into the first equality in (37) we obtain,

$$\frac{\alpha}{1 - a_{21}} + a_{12} \frac{\alpha}{1 - a_{12}} = \log \gamma_0.$$

Solving for $\alpha$ we obtain,

$$\alpha = \frac{(1 - a_{12})(1 - a_{21})}{1 - a_{12}a_{21}} \log \gamma_0.$$

Finally,

$$f_{01} = \frac{1 - a_{12}}{1 - a_{12}a_{21}} \log \gamma_0 \quad \text{and} \quad f_{02} = \frac{1 - a_{21}}{1 - a_{12}a_{21}} \log \gamma_0. \quad (38)$$

Notice that we may rewrite the matrix $M$ as

$$I - \begin{bmatrix} f_{01} & 0 \\ 0 & f_{02} \end{bmatrix} \begin{bmatrix} 1 & a_{12} \\ a_{21} & 1 \end{bmatrix} \quad (39)$$

Hence, it suffices to find the eigenvalues $\mu_i$ of

$$\begin{bmatrix} f_{01} & 0 \\ 0 & f_{02} \end{bmatrix} \begin{bmatrix} 1 & a_{12} \\ a_{21} & 1 \end{bmatrix} \quad (40)$$

as the eigenvalues $\lambda_i$ of $M$ are given by $\lambda_i = 1 - \mu_i$. However, these are the same eigenvalues as that of

$$\begin{bmatrix} 1 & a_{12} \\ a_{21} & 1 \end{bmatrix} \begin{bmatrix} f_{01} & 0 \\ 0 & f_{02} \end{bmatrix} = \begin{bmatrix} f_{01} & a_{12}f_{02} \\ a_{21}f_{01} & f_{02} \end{bmatrix}. \quad (41)$$

Notice from (37) that both rows sum to $\log(\gamma_0)$, so one of the eigenvalues of $M$ is $\lambda_1 = 1 - \log(\gamma_0)$. The other eigenvalue may be obtained from the trace of $M$, yielding

$$\lambda_1 + \lambda_2 = 2 - f_{01} - f_{02}$$

so that

$$\lambda_2 = 1 + \log(\gamma_0) - f_{01} - f_{02} = 1 + \log(\gamma_0) \left(1 - \frac{1 - a_{12}}{1 - a_{12}a_{21}} - \frac{1 - a_{21}}{1 - a_{12}a_{21}}\right). \quad (42)$$

This expression is equivalent to $\lambda_2 = 1 - \beta_2 \log(\gamma_0)$ where

$$\beta_2 = \frac{(1 - a_{12})(1 - a_{21})}{1 - a_{12}a_{21}}.$$

It is clear that $\beta_2 > 0$ since $a_{12}, a_{21} \in (0, 1)$. Also, $\beta_2 < \frac{1 - a_{12}}{1 - a_{12}a_{21}} < 1$, proving the claim. \qed
Proof of Lemma 2.6. We claim Lemma 2.6 holds for $\epsilon < \frac{2\beta_2}{1 + \beta_2}$. The result is clear from Lemma 2.5 if both eigenvalues are negative. If instead $\lambda_2 > 0$, then it suffices to show that $\log(\gamma_0) - 1 > 1 - \beta_2 \log(\gamma_0)$. This is equivalent to showing

$$
\log(\gamma_0) > \frac{2}{1 + \beta_2} = 2 - \frac{2\beta_2}{1 + \beta_2}
$$

and the claim follows.

2.5. The general 2D case. In this subsection, we investigate the Lyapunov exponents for a general $G_1, G_2, \gamma_{01}, \gamma_{02}$ numerically. We follow the numerical scheme of [6] using QR decompositions, where we decompose a matrix as the product of an orthogonal matrix $Q$ and an upper triangular matrix $R$. Recall that

$$
A^{(n)}(\theta) = \prod_{j=1}^{n} A(\theta_{n-j})
$$

Set $Q^{(1)} R^{(1)} = A^{(1)}(\theta) = A(\theta)$ where $Q^{(1)} R^{(1)}$ is the QR decomposition of $A^{(1)}(\theta)$ into a product of an orthogonal matrix $Q^{(1)}$ and an upper triangular matrix $R^{(1)}$. Then recursively define $Q^{(n)} R^{(n)} = A(\theta_{n-1})Q^{(n-1)}$ where $Q^{(n)} R^{(n)}$ is the QR decomposition of $A(\theta_{n-1})Q^{(n-1)}$. Then since $Q^{(n)}$ is orthogonal

$$
\chi(v) = \lim_{n \to \infty} \frac{1}{n} \log \left\| A^{(n)}(\theta)v \right\| = \lim_{n \to \infty} \frac{1}{n} \log \left\| Q^{(n)} \prod_{j=0}^{n-1} R^{(n-j)}v \right\|
$$

$$
= \lim_{n \to \infty} \frac{1}{n} \log \left\| \prod_{j=0}^{n-1} R^{(n-j)}v \right\|
$$

We now compute the two Lyapunov exponents. Taking $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ gives

$$
\chi\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \lim_{n \to \infty} \frac{1}{n} \log \left\| \prod_{j=0}^{n-1} R^{(n-j)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|
$$

$$
= \lim_{n \to \infty} \frac{1}{n} \log \left\| \prod_{j=0}^{n-1} R_{11}^{(n-j)} \right\|
$$

$$
= \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left| R_{11}^{n-j} \right|
$$
Lyapunov Exponents $\chi_1$ and $\chi_2$ as a function of the parameter $b$

![Graph](image)

**Figure 3.** Plots of the values of the two Lyapunov Exponents $\chi_1$ (left plot) and $\chi_2$ (right plot) as functions of the parameter $0 \leq b \leq 0.99$. Values used were $\gamma_{01} = e^{1.2}$, $\gamma_{02} = e^{1.6}$, $a_{12} = 0.6$, $a_{21} = 0.8$, $G_1(\theta) = \sin(2\pi \theta)$, $G_2(\theta) = \sin(2\pi \theta) \cos(2\pi \theta)$, and $\omega = \frac{e^{\pi/25}}{25} \approx 0.8984$. Both are decreasing functions with respect to $b$. The $b$ values used were $0.01m$ for $m = 0, 1, \cdots, 99$.

while taking $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ gives

$$
\chi \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \lim_{n \to \infty} \frac{1}{n} \log \left\| R_{22}^{(n-j)} \right\| = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left( \left| R_{22}^{n-j} \right| \right).
$$

That is, we can approximate the numerical values of the Lyapunov exponents by averaging the sum of the logarithms of the absolute values of the diagonal entries of the $R^{(j)}$. As an example, we show numerically computed Lyapunov exponents for $\gamma_{01} = e^{1.2}$, $\gamma_{02} = e^{1.6}$, $a_{12} = 0.6$, $a_{21} = 0.8$ in Figure 3. Here we used $G_1(\theta) = \sin(2\pi \theta)$, $G_2(\theta) = \sin(2\pi \theta) \cos(2\pi \theta)$, and $\omega = \frac{e^{\pi/25}}{25} \approx 0.8984$. We observe that as $b$ increases, the Lyapunov exponents decrease as a function of $b$. This type of plot is typical and we summarize our main numerical finding as a conjecture.
Conjecture 1. Let \( \gamma_{01} > 1, \gamma_{02} > 1, a_{12} \in \left( 0, \frac{\log(\gamma_{01})}{\log(\gamma_{02})} \right), a_{21} \in \left( 0, \frac{\log(\gamma_{02})}{\log(\gamma_{01})} \right) \). Let \( f_{01}, f_{02} \) satisfy \( f_{01} + a_{12} f_{02} = \log(\gamma_{01}), f_{02} + a_{21} f_{01} = \log(\gamma_{02}) \). Define

\[
S(b, \theta) = I + b \begin{bmatrix}
G_1(\theta)
\frac{\gamma_{01}}{\gamma_{02}} & 0
0 & G_2(\theta)
\end{bmatrix},
M = \begin{bmatrix}
1 - f_{01} & -a_{12} f_{01}
-a_{21} f_{02} & 1 - f_{02}
\end{bmatrix}.
\]

Let \( A(\theta) = S(b, \theta)M \) and \( \theta_k = \theta + k \omega \pmod{1} \). If \( M \) is invertible, then the Lyapunov exponents of the cocycle

\[
A^{(n)}(\theta) = \prod_{j=1}^{n} A(\theta_{n-j})
\]

are decreasing functions of \( b \).

Observe that at \( b = 0 \), the Lyapunov exponents are given by \( \log |\lambda_1|, \log |\lambda_2| \) where \( \lambda_i \) are the eigenvalues of \( M \). As \( M \) is also the Jacobian matrix of the 2D Ricker Equation (6) at the interior fixed point, these Lyapunov exponents are negative whenever the interior fixed point of (6) is locally asymptotically stable. Thus, if the conjecture is true, Theorem 2.3 implies that anywhere the 2D Ricker Equation (7) has a locally asymptotically stable interior fixed point (that is, the region in Figure 1) the AP 2D Ricker Equation (7) has a locally asymptotically stable AP solution. Moreover, this stability region grows as \( b \) grows. For a given \( b \), the border of this stability region consists of the values \( \gamma_{01}, \gamma_{02} \) where the maximum Lyapunov exponent is exactly 0. A zoom in of this border near the hyperbola in Figure 1 for the values \( b = 0, b = 0.2, b = 0.4, \) and \( b = 0.6 \) is shown in Figure 4, where the border moves further northeast as \( b \) grows.

To illustrate the change in behavior outside of the stability region, we take \( \gamma_{01} = e^{2.0015}, \gamma_{02} = e^{1.998} \), which lies inside between the \( b = 0.2 \) and \( b = 0.4 \) borders shown in Figure 4, so that for \( b = 0.4 \) there is a locally stable AP solution, while for \( b = 0.2 \) the same values yield an unstable solution. We plot the long term behavior of \( (\gamma_1(t), x_1(t), x_2(t)) \) and \( (\gamma_2(t), x_1(t), x_2(t)) \) for \( b = 0.4 \) in the top row of Figure 5. As expected, they each converge to a closed curve. In contrast, for \( b = 0.2 \) in the bottom row of Figure 5, they converge to a 2-1 lifting of closed curves, an AP generalization of period 2 orbits.

3. Higher dimensions. The almost periodic Ricker Equation in \( N \) dimensions has the form

\[
x_i(t + 1) = \gamma_i(t)x_i(t) \exp \left( -\sum_{j=1}^{N} a_{ij}x_j(t) \right), \quad i = 1, \cdots, N
\]

where

\[
\gamma_i(t) = \gamma_{0i} + bG_i(\omega t).
\]

We do not explore this equation in detail in this work and only consider the special case \( \gamma_{0i} = \gamma_0 > 1, G_i = G \) for all \( i \). Throughout this section, \( G \) is a \( C^1 \) function, periodic with period 1 with \( \max |G| = 1 \) and mean value 0 over one period, while \( \omega \) is an irrational number. In two-dimensions, this special case gave rise to the same bifurcation curve as the one-dimensional case. Our goal here is to show
that this also applies to the $N$ dimensional case under certain coupling constraints. In particular, we assume that

$$a_{ij} \geq 0, \quad a_{jj} = 1, \quad \text{and} \quad \sum_{j=1, j \neq i}^{N} a_{ij} < 1 . \quad (45)$$

These conditions include the two-dimensional case already studied where the coupling parameters $a_{12}, a_{21} \in (0, 1)$ and the coefficients of $x_1(t), x_2(t)$ in the exponential were 1. Mimicking the case of two dimensions, we assume there is an invariant curve $f(\theta) = f_0 + u(\theta)$ where

$$f(\theta) = f_0 + u(\theta), \quad f_0 = \begin{pmatrix} f_{01} \\ \vdots \\ f_{0N} \end{pmatrix}, \quad u(\theta) = \begin{pmatrix} u_1(\theta) \\ \vdots \\ u_N(\theta) \end{pmatrix} .$$

Then the invariance equations have the form

$$f_{0j} + u_j(\theta_1) = \gamma(t) (f_{0j} + u_j(\theta)) e^{-\sum_{k=1}^{N} a_{jk}(f_{0k} + u_k(\theta))}, \quad j = 1, \cdots, N . \quad (46)$$

At $b = 0$ where $u(\theta) \equiv 0$ the equations become

$$f_{0j} = \gamma_0 f_{0j} e^{-\sum_{k=1}^{N} a_{jk} f_{0k}}, \quad j = 1, \cdots, N . \quad (47)$$
or equivalently that
\[ \sum_{k=1}^{n} a_{jk} f_{0k} = \log(\gamma_0), \ j = 1, \cdots, N. \] (48)

Since the matrix \( A \) with coefficients \( a_{ij} \) is diagonally dominant, it is invertible and the linear system (48) can be solved for the \( f_{0j} \). In the context of population dynamics, the \( f_{0j} \) should be positive. We will show that this is the case for the given set of coupling parameters.

**Lemma 3.1.** Assume \( \gamma_0 > 1 \) and the \( a_{ij} \) satisfy
\[ a_{ij} \geq 0, \ a_{jj} = 1, \text{ and } \sum_{j=1 \atop j \neq i}^{N} a_{ij} < 1. \]

Then
\[ \sum_{j=1}^{n} a_{ij} f_{0j} = \log(\gamma_0), \ i = 1, \cdots, N. \]
implies \( f_{0j} > 0 \) for all \( j \).
Lemma 3.1 follows from a more general Theorem [10] that gives sufficient conditions on existence of positive solutions to a linear system. However, we provide a simpler proof to Lemma 3.1 specific to the conditions on $a_{ij}$ that uses some related ideas.

Proof. Let $A$ be the matrix with entries $A_{ij} = a_{ij}$. The $f_{0j}$ are given by the linear system

$$
A \begin{bmatrix} f_{01} \\ \vdots \\ f_{0n} \end{bmatrix} = \begin{bmatrix} \log(\gamma_0) \\ \vdots \\ \log(\gamma_0) \end{bmatrix}.
$$

(49)

The conditions imply we may write $A = I + B$ with $\rho(B) < 1$. Since $A$ is diagonally dominant, it is invertible, and we have that

$$
\begin{bmatrix} f_{01} \\ \vdots \\ f_{0n} \end{bmatrix} = (I + B)^{-1} \begin{bmatrix} \log(\gamma_0) \\ \vdots \\ \log(\gamma_0) \end{bmatrix}.
$$

(50)

We may rewrite the right hand side as

$$(I + B)^{-1}(I - B)^{-1}(I - B) \begin{bmatrix} \log(\gamma_0) \\ \vdots \\ \log(\gamma_0) \end{bmatrix} = (I - B^2)^{-1}(I - B) \begin{bmatrix} \log(\gamma_0) \\ \vdots \\ \log(\gamma_0) \end{bmatrix}.
$$

(51)

We will argue separately that the entries of

$$(I - B^2)^{-1}
$$

are nonnegative while

$$(I - B) \begin{bmatrix} \log(\gamma_0) \\ \vdots \\ \log(\gamma_0) \end{bmatrix}
$$

consist of only positive entries.

For the former, since $\rho(B) < 1$, the sum $\sum_{j=0}^{\infty} B^{2j}$ converges and we may write

$$(I - B^2)^{-1} = \sum_{j=0}^{\infty} (B^2)^j.
$$

(52)

As all of the entries of $B$ are nonnegative, the entries of the infinite sum are also nonnegative. For the latter, since $A = I + B$ is diagonally dominant and $B$ has no diagonal entries, we have that $I - B$ is also diagonally dominant. This implies that

$$(I - B) \begin{bmatrix} \log(\gamma_0) \\ \vdots \\ \log(\gamma_0) \end{bmatrix}
$$

has all positive entries since $\log(\gamma_0) > 0$. The product of the two expressions is positive since the diagonal entries of the infinite sum (52) are bounded below by 1 due to the sum containing $I$. $\square$
Now we impose conditions (48) for \( b > 0 \) and repeat the calculation from the two-dimensional case. After an omitted calculation, this yields the recurrence

\[
v(\theta_1) = S(b, \theta) M v(\theta) + R_1(b, \theta, v).
\]

where \( S \) is the diagonal matrix with \( S_{ii} = 1 + b \gamma_0 G(\theta) \) and \( M \) is the matrix with entries \( M_{ii} = 1 - f_{0i} \) and \( M_{ij} = -a_{ij} f_{0i} \) for \( i \neq j \). The remainder \( R_1 \) is of the form

\[
R_1(b, \theta, v) = \begin{bmatrix}
\frac{f_{01}}{\gamma_0} G(\theta) \\
\vdots \\
\frac{f_{0N}}{\gamma_0} G(\theta)
\end{bmatrix} + h(b, \theta, v),
\]

and for some \( M_2 > 0 \)

\[
\|h(b, \theta, v)\| \leq bM_2 \|v\|^2 \text{ for } \|bv\| < 1.
\]

Since \( S \) is a scalar multiple of the identity, we can compute the \( N \) Lyapunov exponents explicitly.

**Theorem 3.2.** Let \( N \geq 2 \) be a positive integer and suppose \( a_{ij} \) satisfy (45) and \( f_{0j} \) satisfy (48). Let \( S(b, \theta) \) be the \( N \times N \) diagonal matrix with \( S_{ii} = 1 + \frac{b}{\gamma_0} G(\theta) \) and \( M \) the matrix with entries \( M_{ii} = 1 - f_{0i} \) and \( M_{ij} = -a_{ij} f_{0i} \) for \( i \neq j \). Let \( \lambda_i \) denote the eigenvalues of \( M \). Let \( A(\theta) = S(b, \theta) M \) and \( \theta_k = \theta + k\omega \mod 1 \). If \( M \) is invertible, then the Lyapunov exponents of the cocycle

\[
A^{(n)}(\theta) = \prod_{j=1}^{n} A(\theta_{n-j})
\]

are

\[
\chi_i = \log |\lambda_i| + \int_{0}^{1} \log \left( 1 + \frac{b}{\gamma_0} G(\theta) \right) d\theta.
\]

The proof of Theorem 3.2 is nearly identical to that of Theorem 2.4 and is omitted. As a consequence, we have the following result.

**Corollary 2.** Let \( a_{ij} \) satisfy (45) and let \( f_{0j} \) satisfy (48). Let \( M \) be the matrix with entries \( M_{ii} = 1 - f_{0i} \) and \( M_{ij} = -a_{ij} f_{0i} \) for \( i \neq j \). Let \( \rho(M) \) denote the spectral radius of \( M \). Then the \( N \)-dimensional AP Ricker Equation (8)

\[
x_i(t+1) = \gamma_i(t) x_i(t) \exp \left( -\sum_{j=1}^{N} a_{ij} x_j(t) \right), \quad i = 1, \ldots, N
\]

with \( \gamma_i(t) = \gamma_0 + bG(\omega t) \), \( i \in \{1, \ldots, N\} \) has a locally asymptotically stable AP solution if and only

\[
\log |\rho(M)| + \int_{0}^{1} \log \left( 1 + \frac{b}{\gamma} G(\theta) \right) d\theta < 0.
\]

This leads to the same bifurcation equation as in the one-dimensional case. In two-dimensions, we showed for sufficiently large \( \gamma_0 \) that \( \rho(M) = \log(\gamma_0) - 1 \). We will conclude this section by showing a similar result for \( N \)-dimensions. The proof is a bit more delicate since now the eigenvalues of \( M \) may be complex. We will make use of Gershgorin’s circle theorem [9].
Theorem 3.3. \[9\] Let $U$ be an $n \times n$ complex matrix with entries $u_{ij}$. Let

$$R_i = \sum_{j=1}^{n} |u_{ij}|.$$  \hspace{1cm} (55)

Then every eigenvalue lies inside one of the disks

$$\{z : |z - u_{ii}| \leq R_i \}.$$  \hspace{1cm} (56)

Theorem 3.4. Let $a_{ij}$ satisfy (45) and let $f_{0j}$ satisfy (48). Let $M$ be the matrix with entries $M_{ii} = 1 - f_{0i}$ and $M_{ij} = -a_{ij}f_{0i}$ for $i \neq j$. All eigenvalues of $M$ are of the form

$$\lambda_k = 1 - \beta_k \log(\gamma_0)$$

where $\beta_k$ is independent of $\gamma_0$ and only depends on the coupling constants $a_{ij}$. Further, $\beta_1 = 1$ and

$$|\beta_k - c| \leq 1 - c, \; k \geq 2$$

for some $c \in (0, 1)$ where $c$ only depends on $a_{ij}$.

Proof. We first let $A$ be the matrix with entries $A_{ij} = a_{ij}$. We note that the $f_{0j}$ can be found by inverting the linear system given in equation (48) which has $A$ as its coefficient matrix. Cramer’s rule then gives that

$$f_{0j} = \log(\gamma_0) \frac{\det(A_j)}{\det(A)}$$

where $A_j$ is the matrix $A$ with the $j$-th column replaced by all 1s. We will denote

$$c_j = \frac{\det(A_j)}{\det(A)}$$

so that $f_{0j} = \log(\gamma_0)c_j$. Note that equation (48) then implies that for all $i$ we have

$$\sum_{j=1}^{n} c_j a_{ij} = 1.$$  \hspace{1cm} (58)

Further, since by Lemma 3.1 all $f_{0j} > 0$, we also have $c_j > 0$. This then implies from equation (58) that all $c_j < 1$ since $a_{jj} = 1$. Now we observe that we may write

$$M = I - \log(\gamma_0)CA$$

where $C$ is the $n \times n$ diagonal matrix with diagonal entries $c_j$. It is then clear that the eigenvalues $\lambda_k$ of $M$ are given by $1 - \beta_k \log(\gamma)$ where $\beta_k$ are the eigenvalues of $CA$, which is independent of $\gamma_0$ and only depends on $a_{ij}$.

Note that the eigenvalues of $CA$ are the same as those of $AC$, and that the entries of $AC$ are given by $c_j a_{ij}$. In particular, $AC$ is a right stochastic matrix since all rows sum to 1, so we can take $\beta_1 = 1$. Now we apply Theorem 3.3 to our matrix $AC$. This yields that every eigenvalue lies in one of the disks

$$G_i = \{ z \in \mathbb{C} : |z - c_i| \leq 1 - c_i \}, \; i = 1, \ldots, N.$$  \hspace{1cm} (59)

We set $c$ to be the minimum of all $c_j$. We know for each $k$ that $\beta_k \in G_i$ for some $i$. But then

$$|\beta_k - c| = |\beta_k - c_i + c_i - c|$$

$$\leq |\beta_k - c_i| + |c_i - c|$$

$$= |\beta_k - c_i| + c_i - c$$

$$\leq 1 - c_i + c_i - c$$

$$= 1 - c$$
Lemma 3.5. Let \( a_{ij} \) satisfy (45) and let \( f_{0j} \) satisfy (48). Let \( M \) be the matrix with entries \( M_{ii} = 1 - f_{0i} \) and \( M_{ij} = -a_{ij} f_{0i} \) for \( i \neq j \). Then there exists a value \( C \) such that the spectral radius \( \rho(M) = \log(\gamma_0) - 1 \) for all \( \gamma_0 > C \).

Proof. From Theorem 3.4, we know the eigenvalues are of the form
\[
\lambda_k = 1 - \beta_k \log(\gamma_0)
\]
with \( \beta_1 = 1 \) and
\[
|\beta_k - c| \leq 1 - c, \; k \geq 2 .
\]
Let \( C = e^{1/c} \) where \( c \) is the value from Theorem 3.4. We will show that for all \( \gamma_0 > C \) that
\[
|1 - \beta_k \log(\gamma_0)| \leq \log(\gamma_0) - 1 .
\]
The stated inequality is equivalent to showing
\[
|\beta_k - \frac{1}{\log(\gamma_0)}| \leq 1 - \frac{1}{\log(\gamma_0)} .
\]
From Theorem 3.4, we know
\[
|\beta_k - c| \leq 1 - c .
\]
Thus,
\[
|\beta_k - \frac{1}{\log(\gamma_0)}| = |\beta_k - c + c - \frac{1}{\log(\gamma_0)}| \\
\leq |\beta_k - c| + \left| c - \frac{1}{\log(\gamma_0)} \right| \\
\leq 1 - c + c \frac{1}{\log(\gamma_0)} \\
= 1 - \frac{1}{\log(\gamma_0)} .
\]

For slightly more restrictive conditions for the \( a_{ij} \) than those given in (45), we have that the bound \( C \) in Lemma 3.5 is identical to that of the two-dimensional case. These conditions and the bound are given in the following lemma.

Lemma 3.6. Let \( a_{ij} \) satisfy
\[
a_{ij} \geq 0, \; a_{jj} = 1, \quad \text{and} \quad \sum_{j=1}^{N} a_{ij} < \frac{1}{2}
\]
and let \( f_{0j} \) satisfy (48). Let \( M \) be the matrix with entries \( M_{ii} = 1 - f_{0i} \) and \( M_{ij} = -a_{ij} f_{0i} \) for \( i \neq j \). Then there exists an \( \epsilon > 0 \) such that the spectral radius \( \rho(M) = \log(\gamma_0) - 1 \) for all \( \log(\gamma_0) > 2 - \epsilon \).

Proof. Denote \( c_j \) as in the proof of Theorem 3.4, so that equation (58) is satisfied and each \( c_j \in (0, 1) \). Let \( c \) denote the minimum of all \( c_j \). From the proof of Lemma
3.5, we have that \( \rho(M) = \log(\gamma_0) - 1 \) for all \( \gamma_0 > C \) where \( C = e^{1/e} \). It thus suffices to prove that \( c_j > \frac{1}{2} \) for all \( j \). From equation (58), we have that

\[
1 = \sum_{j=1}^{n} c_j a_{ij} = c_i + \sum_{j=1}^{n} c_j a_{ij} < c_i + \sum_{j=1}^{n} a_{ij} < c_i + \frac{1}{2}
\]

from which the result follows.

As a corollary, in a neighborhood of \( \gamma_0 = e^2 \), under the restrictions of Lemma 3.6 we have that the bifurcation equation is given by

\[
\log (\log(\gamma_0) - 1) + \int_0^1 \log \left( 1 + \frac{b}{\gamma_0} G(\theta) \right) d\theta = 0.
\]

4. Conclusion. We have investigated the almost periodic (AP) Ricker Equation and established a connection between AP solutions and Lyapunov exponents of a linear cocycle. In the case of identical AP sequences for each variable, we obtained a closed form for the Lyapunov exponents and show that under certain coupling constraints, the bifurcation equation is identical to that of the one dimensional case. This result is independent of the dimension of the system. We also examined the general two-dimensional AP Ricker Equation and observed numerically that the Lyapunov Exponents are decreasing functions of the parameter \( b \) in the AP functions. We discussed implications of this observation, in particular concerning the size of the stability region for AP solutions. We also give a numerical example to confirm our results and illustrate the 2-1 lifting that occurs when the AP solutions turn unstable.

Finally, we remark that although our study has been focused specifically on the Ricker map, we suspect our results can be carried over to other couplings of unimodal mappings. In particular, the results should hold for two-dimensional mappings where the recurrence relation for the invariant curve has a form similar to that of equation (25). We defer this generalization to further work.

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