CONIC S-PROCEDURE AND CONSTRAINED DISSIPATIVITY

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1 Introduction

Recently a number of new tools for systems analysis and design related to frequency domain inequalities (FDI) over a finite frequency range (so called Generalized KYP-lemma) have been developed [2–4]. It follows from the results of [2–4] that fullfillness of a standard FDI in a finite frequency range is equivalent to validity of some nonclassical linear matrix inequalities (LMI) for a pair of matrices $P, Q$ replacing inequalities for a single matrix $P$ appearing in the classical KYP-lemma. It was shown in [1] that FDI, in turn, are equivalent to some time-domain inequality (TDI, dissipation inequality [5]), valid only over a part of the system trajectories, determined by an additional integral matrix inequality (restricted or constrained dissipativity [1]). Thus, a complete extension of the classical KYP-results on equivalence between FDI, TDI and LMI to the "finite-frequency" case was obtained. Note that the proof of equivalence between FDI and TDI in [1] goes along the lines of the necessity proof for the frequency-domain absolute stability criterion [6, 7].

In this paper a new proof of the result of [1] is provided based on the losslessness result for a new version of the classical S-procedure [8]. A new version of the S-procedure, also included in the paper, deals with constraints in LMI form, or more generally, conic inequalities in linear spaces.

In the next section a new S-procedure results are presented. In Section 3 they are applied to the proof of equivalence between TDI and LMI.

We use the following notation. The set of square integrable functions on $[0, \infty)$ is denoted by $L^2_0(0, \infty), M^\dagger$, where $M$ is a matrix, stands for its transposition and complex conjugate of all elements. For a square matrix $M$, its Hermitian part is defined by $He(M) := (M + M^\dagger)/2$. The interior of a set $\Omega$ is denoted by $\text{Int} \Omega$.

2 Conic S-procedure

Let $X, Y_1, \ldots, Y_m$ be linear topological spaces, $G_j : X \to Y_j, j = 1, \ldots, m$ be continuous mappings.

Let for any $j = 1, \ldots, m$ a convex cone $K_j \subset Y_j$ be given defining inequality $G_j(x) \preceq 0$ for $x \in X$ as inclusion $G_j(x) \in K_j$. Let $Y_j^*$ denote an dual space to $Y_j$, i. e. a linear space of linear continuous functionals $y_j^*$ on $Y_j$ and $K_j^* \subset Y_j^*$ denote an dual cone to $K_j$, i. e. $K_j^* = \{ y_j^* \in Y_j^* : \langle y_j^*, y_j \rangle \geq 0 \forall y_j \in K_j \}$, where $\langle y_j^*, y_j \rangle$ is the value of the functional $y_j^*$ at the element $y_j$.

Obviously, if $Y = \mathbb{R}^1 \times Y_1 \times \cdots \times Y_m$, then $Y^* = \mathbb{R}^1 \times Y_1^* \times \cdots \times Y_m^*$ is the set of all correges $(y_0^*, y_1^*, \ldots, y_m^*)$, where $y_0^* \in \mathbb{R}^1, y_j^*$ a linear functional from $Y_j^*$.

Consider the following two relations for the mappings $F_0, G_1, \ldots, G_m$.

(A) $F(x) \succeq 0$ for $x \in X, G_j(x) \in K_j, j = 1, \ldots, m$;

(B) $\exists \tau_0 \geq 0, \tau_j \in K_j^*: \tau_0 F(x) - \sum_{j=1}^m \langle \tau_j, G_j(x) \rangle \geq 0 \forall x \in X$.

Obviously, validity of (B) with $\tau_0 > 0$ implies (A). Indeed, if $x \in X$ satisfies inequalities $G_j(x) \in K_j, j = 1, \ldots, m$, then it follows from (B) that $\tau_0 F(x) \succeq 0$, since $\langle \tau_j, G_j(x) \rangle \succeq 0$ for $j = 1, \ldots, m$. The opposite statement is not true even in the case of scalar constraints $Y_j = \mathbb{R}^1, j = 1, \ldots, m$, corresponding to the classical $S$-procedure [8].

Similarly to the classical case we will say that $S$-procedure with conic constraints $G_j(x) \succeq 0$ is lossless, if (B) with $\tau_0 > 0$ implies (A).
It is well known [9] that losslessness of the classical S-procedure is equivalent to the duality theorem in the corresponding optimization problem. However, the problem is, in general, nonconvex and only a few classes of functionals \( F, G_1, \ldots, G_m \) are known to possess the losslessness property.

For example, classical S-procedure is lossless, if \( m = 1 \) and \( F, G_1 \) are quadratic forms on real or complex linear space \( X \). It is also lossless, if \( m = 2 \) and \( F, G_1, G_2 \) are quadratic (Hermitian) forms on the complex linear space \( X \). However, classical S-procedure for quadratic forms is, in general, lossy for \( m \geq 2 \) in real case and for \( m \geq 3 \) in complex case [9]. A. Megretski and S. Treil proved in 1990 [11] that the classical S-procedure is lossless for all \( m \geq 1 \), if \( F, G_1, \ldots, G_m \) are integral quadratic forms on \( L_2(0, \infty) \). V. Yakubovich extended this result to a more broad class of quadratic functionals, forming the so-called S-system [10].

Below an extension of the results of [10] to the case of the S-procedure with conic constraints is formulated. Note that the general formulation of the S-procedure with conic constraints was presented, e.g. in [12, 13].

**Theorem 1.** Let \( n_0 = 1 \) and \( K \) is the closure of the cone \( K \) generated by the set
\[
F(X) = \{(F_0(x), F_1(x), \ldots, F_m(x)) : x \in X\}.
\]
If the cone \( K \) is convex, then the S-procedure with conic constraints is lossless.

If, in addition, constraints \( G_j(x) \in K_j \) are regular, namely
\[
\exists x_0 : G_j(x_0) \in \text{Int} K_j, \text{ then one can choose } \tau_0 = 1 \text{ in (B)}.
\]

**Proof.** Condition (A) implies that \( F(x) \geq 0 \) for \( G_j(x) \in \text{Int} K_j \), i.e. intersection of the set \( F(X) \) and the open cone
\[
D = \{(y_0, y_1, \ldots, y_m) : y_0 > 0, y_j \in \text{Int} K_j, j = 1, \ldots, m\}
\]
is empty: \( D \cap F(X) = \emptyset \). Therefore, \( D \cap K = \emptyset \). Applying separation theorem for cones, we obtain that there exists vector \( \tau^* = (\tau_0^*, \tau_1^*, \ldots, \tau_m^*) \in Y^* \) such that \( \langle \tau_0^*, F(x) \rangle + \sum_{j=1}^m \langle \tau_j^*, G_j(x) \rangle > 0 \) for all \( x \in X \) and \( \tau_j^* \), \( y_j > 0 \) for all \( y_j \in D \), i.e. \( \langle \tau_0^*, y_0 \rangle > 0 \) + \sum_{j=1}^m \langle \tau_j^*, y_j \rangle > 0 \). For any \( j = 1, \ldots, m \) pick up \( y_j \in \text{Int} K_j \) and choose sequences \( y_k \rightarrow 0 \), \( y_{sk} \rightarrow 0 \) as \( k \rightarrow \infty \), such that \( y_{0j} > 0, y_{sk} \in K_j, s \neq j \). If \( k \rightarrow \infty \), then we obtain \( \langle \tau_j^*, y_j \rangle > 0 \), i.e. \( \tau_j^* \in K_j \). The first part of the theorem is proved.

Taking \( x = x_0 \) from regularity condition and \( y_0 \neq 0 \) yields \( \tau_0^*, y_0 > 0 \), i.e. \( \tau_0^* > 0 \). Dividing the inequality (B) by \( \tau_0 \), we arrive at the second statement of the theorem. End of the proof.

Our next step is to extend the definition of S-system [10] to the case of conic constraints.

**Definition 1.** Let \( F_j, j = 0, 1, \ldots, m \) be mappings from a Hilbert space \( Z \) to spaces of self-adjoint operators over corresponding Euclidean space \( \mathbb{R}^{n_j} \), such that \( F_j : Z \rightarrow SR(n_j \times n_j) \).

We say that \( F_0, F_1, \ldots, F_m \) form a S-system if there exists a subspace \( Z_0 \) and a sequence of linear bounded operators \( T_k : Z \rightarrow Z_0, k = 1, 2, \ldots \) such that
\[
(i) < T_k z_1, z_2 > \rightarrow 0 \text{ as } k \rightarrow \infty \text{ for all } z_1, z_2 \in Z;
\]
\[
(ii) Z_0 \text{ is invariant for } T_k \text{ for all } k = 1, 2, \ldots;
\]
\[
(iii) F_j(T_k z) \rightarrow F_j(z) \text{ as } k \rightarrow \infty \text{ for all } j = 0, 1, \ldots, m, z \in Z_0.
\]

**Lemma 1** Let \( F_j, j = 0, 1, \ldots, m \) form S-system. Define the map \( F : Z \rightarrow \prod_{j=0}^m \mathbb{R}^{n_j \times n_j} \) by means of the relation
\[
F(z) = F_0(z) \otimes F_1(z) \otimes \cdots \otimes F_m(z)
\]
\[
\in \mathbb{R}^{n_0 \times n_0} \otimes \mathbb{R}^{n_1 \times n_1} \otimes \cdots \otimes \mathbb{R}^{n_m \times n_m}.
\]

Then the closure of the image \( F(Z) \) is a convex set in \( \mathbb{R}^{n_0^2 + n_1^2 + \cdots + n_m^2} \).

In the special case \( n_j = 1, j = 1, \ldots, m \), Lemma 1 coincides with Lemma 1 of the paper [10] and it is proved similarly to the Lemma 1 of [10].

**Example 1.** An important series of examples for S-systems is provided by finite family of integral quadratic operators on the Hilbert space \( L_2(0, \infty) \) of square integrable functions with values \( z(t) \in \mathbb{R}^{n_j} \). The mappings are defined for any \( z \in L_2(0, \infty) \) as follows:
\[
F_j(z) = \frac{1}{\tau_j} \int_0^{\infty} F_j'(z(t)) z(t) z(t) dt,
\]
where \( F_j', F_j'' \) are \( n_j \times n_j \) symmetric matrices. In this case the family of the operators \( T_k \) can be chosen as time shifts: \( T_k(z)(t) = z(t + k) \), while the subspace \( Z_0 \) can be chosen as the set of functions with zero initial conditions:
\[
Z_0 = \{ z(\cdot) : z(\cdot) \in L_2(0, \infty), z_2(0) = 0 \}.
\]

The proof of the S-system property for Example 1 is again similar to [10]. Note that the cone of positive semidefinite matrices is selfdual. Therefore S-procedure with conic constraints determined by functions (1) deals with positive semidefinite matrix Lagrange multipliers.

Properties of the S-procedure in general case are given by the following theorem.

**Theorem 2.** S-procedure with conic constraints is lossless for any family of self-adjoint operators \( F_0, F_1, \ldots, F_m \) forming an S-system.

Proof follows immediately from Theorem 1 and Lemma 1. The result can be extended to the case of equality constraints and to the case of the so called generalized S-procedure introduced in [2].
3 Constrained dissipativity

In this section, we first present a special case of the generalized KYP lemma [4], characterizing FDIIs in the continuous-time setting. Let complex matrices \( A, B, \Pi, \) and real scalars \( \varpi_1, \varpi_2 \) be given. Define

\[
\Omega := \{ \omega \in \mathbb{R} \mid (\omega - \varpi_1)(\omega - \varpi_2) \leq 0 \}. \tag{2}
\]

(We may assume \( \varpi_2 > 0 \) without loss of generality).

**Theorem 3** [4]. Suppose \( \Pi \) is Hermitian matrix, pair \((A, B)\) is controllable, and \( \Omega \) has a nonempty interior. Then the following statements are equivalent.

(i) The frequency domain inequality

\[
\left( j\omega I - A \right)^{-1} B \Pi \left( j\omega I - A \right)^{-1} B^* \leq 0
\]

holds for all \( \omega \in \Omega \) such that \( \det(j\omega I - A) \neq 0 \).

(ii) There exist Hermitian matrices \( P \) and \( Q \) such that \( Q \geq 0 \) and the linear matrix inequality

\[
\begin{bmatrix}
A & B \\
B^* & 0
\end{bmatrix} \Pi \begin{bmatrix}
A & B^* \\
B & 0
\end{bmatrix} \leq P + j\omega_o Q - \omega_o \Pi
\]

holds, where \( \omega_o := (\varpi_1 + \varpi_2)/2 \).

Choosing the parameters \( \varpi_1 = \varpi_2 = 0 \) and \( \tau = -1 \), the set \( \Omega \) becomes the entire real numbers, and thus statement (i) becomes the FDI for all frequencies. In this case, the term associated with \( Q \) in the LMI (4) becomes positive semidefinite, and hence the best choice of \( Q \) for satisfaction of (4) is \( Q = 0 \). The resulting LMI with variable \( P \) is exactly the same as the one in the standard KYP lemma.

The following result extends the result of [1]. It provides an equivalence between FDI and time domain dissipation inequality over a restricted class of input signals.

**Theorem 4**. Let complex matrices \( A, B, \Pi, \) and real scalars \( \varpi_1, \varpi_2 \) be given and \( \Omega \) be defined by (2). Consider the system

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad t \in [0, \infty), \tag{5}
\]

where \( x(t) \in \mathbb{C}^n \) is the state and \( u(t) \in \mathbb{C}^m \) is the input. Assume that \((A, B)\) is controllable, \( \Pi \) is Hermitian, and \( \Omega \) has a nonempty interior. Then the following statements are equivalent.

(i) The frequency domain inequality (4) holds for \( \omega \in \Omega \).

(ii) The time domain inequality

\[
\int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^* \Pi \begin{bmatrix} x \\ u \end{bmatrix} dt \leq 0 \tag{6}
\]

holds for all solutions of (5) with \( u \in L_2[0, \infty) \) such that \( x(0) = 0, \ x \in L_2[0, \infty) \) and

\[
\text{He} \int_0^\infty (\varpi_1 x + j\dot{x})(\varpi_2 x + j\dot{x})^* dt \leq 0. \tag{7}
\]

Note that the corresponding result of [1] was obtained under additional condition of asymptotic stability for the system (5) which is not required in the current statement.

In Theorem 4 a general frequency interval \( \Omega \) is considered for the FDI, and this has translated to the input constraint described by (4). Though the physical meaning of this constraint may be not clear in general, it becomes clear for the following special case.

**Corollary 1**. Let real matrices \( A, B, \Pi, \) and a positive scalar \( \varpi \) be given. Suppose \( \Pi \) is symmetric and consider the system (5) where \( x(t) \in \mathbb{R}^n \) is the state and \( u(t) \in \mathbb{R}^m \) is the input. Assume that \((A, B)\) is controllable. Then the following statements are equivalent.

(i) The frequency domain inequality (4) holds for all \( \omega \) such that \( |\omega| \leq \varpi \).

(ii) The time domain inequality holds for all \( u \in L_2[0, \infty) \) such that \( x \in L_2[0, \infty) \) and

\[
\int_0^\infty \dot{x}^\dagger \dot{x} dt \leq \varpi^2 \int_0^\infty xx^\dagger dt. \tag{8}
\]

Moreover, the above two statements are equivalent when the two inequalities “\( \leq \)” are replaced by “\( \geq \).”

Loosely speaking, the first part of Corollary 1 states that the FDI in the low finite frequency range means that the system possesses the property (4) for the input signals \( u \) that drive the states not too fast (slowly). The bound on the “slowness” is given by \( \varpi \) in the sense of (4). The second part of Corollary 1 makes a similar statement for the FDI in the high frequency range. To derive Corollary 1 from Theorem 4 one needs to put \( \omega_1 = -\omega_2 = \varpi \).

**Proof of Theorem 4**. In view of Theorem 3 it is sufficient to prove equivalence of (ii) to the condition (ii) of Theorem 4 (solvability of matrix inequality (4)). The result follows from Theorem 2 of the previous section with \( m = 2 \) (one matrix constraint). Denote

\[
z = \begin{bmatrix} x \\ u \end{bmatrix}, \quad F(z) = -\int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^\dagger \Pi \begin{bmatrix} x \\ u \end{bmatrix} dt,
\]

\[
G_1(z) = -\text{He} \int_0^\infty (\varpi_1 x + j\dot{x})(\varpi_2 x + j\dot{x})^\dagger dt. \tag{9}
\]
Obviously, TDI (6), (7) correspond to the statement (A) of the conic S-procedure. At the same time the statement (B) means existence of an $n \times n$-matrix $\tau^*$ from the dual cone $K^*$ to the cone of positive semidefinite matrices satisfying inequality $F(z) - \langle \tau^*, G_1(z) \rangle \geq 0 \ \forall z$ or

$$\text{He} \int_0^\infty \left( -\begin{bmatrix} x \\ u \end{bmatrix}^T \Pi \begin{bmatrix} x \\ u \end{bmatrix} - (\varpi_1 x + j\dot{x})^* \tau(\varpi_2 x + j\dot{x}) \right) dt \geq 0.$$  

(10)

Replacement of $\tau$ by $Q$ and substitution of $\dot{x}$ from (5) transforms (10) into LMI (4). To verify regularity condition of Theorem 2 take $x_0 = (\bar{x}, \bar{u})$, where $\bar{x}(t) = -(A+\mu I)^{-1}B \exp(-\mu t)$, $\bar{u}(t) = \exp(-\mu t)$, $\mu > 0$ and $\mu \in \text{Int} \Omega$. Application of Theorem 2 ends the proof.

Remark. Similar results hold for discrete-time case.

4 Conclusions

The property of the system defined by the item (ii) of Theorem 4 and the Corollary can be called constrained dissipativity or restricted dissipativity. It is weaker than standard passivity or dissipativity conditions and may better reflect specifications for real systems. At the same time the property of “slowness” described by the inequality (8) leaves enough flexibility to be useful for robustness analysis of systems.

The results of the paper shed new light on the intimate interrelations between S-procedure and KYP-lemma. They allow to extend classical S-procedure tool to allow for analysis and design of robust systems with matrix inequalities constraints.

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