Simplicial sets, Postnikov systems, and bounded cohomology

Nikolai V. Ivanov

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1. Introduction

**Bounded cohomology of topological spaces.** The bounded cohomology groups $\hat{H}^*(X)$ of a topological space $X$ were introduced by Gromov [Gr]. The definition of $\hat{H}^*(X)$ is almost the same as the definition of the singular cohomology $H^*(X, \mathbb{R})$ of $X$ with real coefficients. Namely, in order to define $\hat{H}^*(X)$ one needs only to replace arbitrary singular $n$-cochains by singular $n$-cochains which are bounded as real-valued functions on the set of singular $n$-simplices. The effect of this change is rather dramatic. It turns out that $\hat{H}^*(X)$ depends only on the fundamental group of $X$, or, what is the same, $\hat{H}^*(X)$ does not depend on the higher homotopy groups of $X$. At the same time $\hat{H}^*(X)$ carries an additional structure, a canonical semi-norm, and this semi-norm is the *raison d'être* of the theory.

Gromov’s exposition [Gr] of the bounded cohomology theory is rather cryptic. For more than three decades the only available detailed proofs of the main results of the bounded cohomology theory were author’s proofs [I1], under a technical assumption removed in [I2]. Only recently R. Frigerio and M. Moraschini [FM] reconstructed Gromov’s proofs and provided a detailed exposition of the theory following Gromov’s outline.

**Simplicial sets and Postnikov systems.** The modern singular homology and cohomology theory was created by S. Eilenberg in his paper [E]. Later on Eilenberg, in collaboration with S. MacLane and J. Zilber, undertook a detailed analysis of this theory. In particular, Eilenberg and MacLane [EM1], [EM2] studied the influence of the homotopy groups of spaces on their homology and cohomology groups. The second paper [EM2] in this series relied on the notion of complete semi-simplicial complexes just introduced by Eilenberg and Zilber [EZ]. Nowadays complete semi-simplicial complexes are known as simplicial sets.

The problem of influence of homotopy groups on homology and cohomology groups was addressed by Eilenberg and MacLane only in fairly special cases. Their results were subsumed by the theory of natural systems of M.M. Postnikov [P1], [P2]. Natural systems were quickly renamed into Postnikov systems. The theory of Postnikov systems allows, at least in principle, to determine the homology and cohomology groups of a space starting with its homotopy groups and some additional invariants, known as Postnikov invariants. Simplicial sets served as the natural framework for Postnikov’s theory [P2].

**Simplicial sets and bounded cohomology theory.** The bounded cohomology groups $\hat{H}^*(K)$ of a simplicial set $K$ are defined in an obvious manner. Namely, the simplicial $n$-cochains of $K$ with real coefficients are the real-valued functions on the set $K_n$ of $n$-simplices of $K$. A simplicial $n$-cochain is bounded if it is bounded as a real-valued function on $K_n$, and the coboundary of a bounded cochain is obviously bounded. The bounded cohomology $\hat{H}^*(K)$ are defined as the cohomology of the cochain complex of bounded cochains. Homotopies of simplicial maps lead to cochain homotopies of complexes of bounded cochains and, similarly to the case of spaces, $\hat{H}^*(K)$ depends only on the homotopy type of $K$. 
It is only natural to expect that the theories of simplicial sets and Postnikov systems can be adapted to the bounded version of the cohomology theory and, in particular, used to prove that the bounded cohomology groups $\tilde{H}^*(K)$ do not depend on the higher homotopy groups of $K$. The goal of the present paper is to show that this is indeed the case and, moreover, that the tools provided by the theories of simplicial sets and Postnikov systems are nearly ready for using in the bounded cohomology theory.

**Kan extension property.** It is well known that in the theory of simplicial sets the internal notions of homotopy, homotopy type, and homotopy groups are reasonable only for simplicial sets satisfying an additional condition known as the Kan extension property. Such simplicial sets are called *Kan simplicial sets*, or, more recently, *fibrant simplicial sets*. The need to restrict the class of considered simplicial sets is clear in the bounded cohomology theory. Indeed, the bounded cohomology groups of simplicial sets arising from finite simplicial complexes are the same as the usual real cohomology groups. It turns out the Kan extensions condition is exactly what is needed for the bounded cohomology theory.

**The main theorems.** The first main theorem of this paper is concerned with locally trivial bundles of simplicial sets. The above notwithstanding, the base is allowed to be an arbitrary simplicial set. In this theorem $K(\pi, n)$ denotes the Eilenberg–MacLane simplicial set, as in the classical papers of Eilenberg and MacLane. See Section 5 for the details.

**Theorem A.** Let $E, B$ be simplicial sets, and let $p : E \to B$ be a locally trivial bundle with the Eilenberg–MacLane simplicial set $K(\pi, n)$ as the fiber. If $n > 1$, then the map induced by $p$ in bounded cohomology is an isometric isomorphism.

See Theorem 7.1. This result should be considered as a relative property of $E$ with respect to $B$. Since $K(\pi, n)$ has the Kan extension property, the latter is present, albeit implicitly.

The second main theorem is concerned with the “classifying spaces” $B\pi$ of discrete groups $\pi$, which are actually not spaces, but another incarnation of simplicial sets $K(\pi, 1)$. See Section 4 for a discussion of classifying spaces.

**Theorem B.** Let $\pi$ be a discrete group and $\kappa \subseteq \pi$ be a normal amenable subgroup of $\pi$. Let $p : \pi \to \pi/\kappa$ be the quotient homomorphism. Then $Bp : B\pi \to B(\pi/\kappa)$ induces isometric isomorphism in bounded cohomology.

See Theorem 7.2. Note that the classifying spaces $B\pi$ are Kan simplicial sets.

In addition to the classical theory of simplicial sets, the proofs of these theorems involve the notion of $\Delta$-*sets*, introduced under the name *semi-simplicial complexes* by Eilenberg and Zilber in the same paper [EZ] in which they introduced simplicial sets. The $\Delta$-*sets* differ from simplicial sets by the lack of degeneracy operators. See Section 2 for the details. The
modern term “Δ-set” goes back to C. Rourke and B. Sanderson [RS]. Our main example of a Δ-set is the infinitely dimensional simplex $Δ[∞]$, the union over $n \in \mathbb{N}$ of the standard $n$-dimensional simplices $Δ[n]$ considered as Δ-sets. Again, see Section 2 for the details. The Δ-sets are used to construct unravelings of simplicial sets. The idea of the unraveling goes back to G. Segal [S] and J. Milnor’s construction of classifying spaces [Mi]. The unraveling of a simplicial set $K$ is simply the Δ-set $K × Δ[∞]$. In this product the degeneracy operators of $K$ are ignored and the product is taken dimension-wise, exactly as the products of simplicial sets. The main result about unravelings is the following.

**Theorem C.** Let $K$ be a simplicial set. The projection $p : K × Δ[∞] → K$ induces isometric isomorphisms in the bounded cohomology groups.

See Theorem 6.3. The proof is based on a modification of the method of acyclic models.

**Applications.** When combined with the basic facts of the theory of Postnikov systems, Theorem A easily implies the following theorem.

**Theorem D.** Let $K$ be a connected Kan simplicial set and $f : K → B\pi_1(K, v)$, where $v$ is a vertex of $K$, be a simplicial map inducing isomorphism of fundamental groups. Then $f$ induces an isomorphism in bounded cohomology.

See Theorem 7.4. This theorem together with Theorem B easily implies the following.

**Theorem E.** Let $K, L$ be connected Kan simplicial sets and let $v$ be a vertex of $K$. Let $f : K → L$ be a simplicial map. If $f_* : \pi_1(K, v) → \pi_1(L, f(v))$ is surjective and has amenable kernel, then $f$ induces an isometric isomorphisms in bounded cohomology.

See Theorem 7.6. When applied to the singular simplicial sets of topological spaces, Theorems E and D turn into Gromov’s Mapping theorem and its Corollary (A) respectively. See [Gr], p. 40. In fact, Gromov deduces his Mapping theorem from his Corollary (A). Observing some similarity between this deduction (see [Gr], the top of p. 47) and Segal’s unraveling of categories [S] was the starting point of the present paper.

**The structure of the paper.** Sections 2, 3, and the first half of Section 4 are devoted to the basic definitions and a review of theories used in the paper. The second half of Section 4 introduces some ideas behind the proof of Theorem B and the definition of unravelings. Appendix 1 provides additional motivation, but is not used in the main part of the paper. Section 5 is the technical heart of the paper, laying the groundwork for the proof of Theorem A. Theorem C is proved in Section 6, which does not depends on the rest of the paper. Section 7 is devoted to the proofs of Theorems A and B and deducing Theorems D and E from them. Appendix 2 is devoted to the proofs of several technical lemmas.
2. Simplicial sets and $\Delta$-sets

**The categories $\Delta$ and $\Delta$.** We will include 0 in the set $\mathbb{N}$ of natural numbers. For every $n \in \mathbb{N}$ let $[n]$ be the set $\{0, 1, \ldots, n\}$. The category $\Delta$ has sets $[n]$ as objects and non-decreasing maps $[m] \to [n]$ as morphisms from $[m]$ to $[n]$, with the composition being the composition of maps. The category $\Delta$ is the subcategory of $\Delta$ having the same objects and strictly increasing maps $[m] \to [n]$ as morphisms from $[m]$ to $[n]$.

**Simplicial sets and $\Delta$-sets.** A simplicial set is a contravariant functor from $\Delta$ to the category of sets. Similarly, a $\Delta$-set is a contravariant functor from $\Delta$ to the category of sets. So, a simplicial set $K$ consists of a set $K_n$ for every $n \in \mathbb{N}$ and a map $\theta^*: K_n \to K_m$ for every non-decreasing map $\theta: [m] \to [n]$. For $\Delta$-set $K$ the map $\theta^*$ is defined only if $\theta$ is strictly increasing. If $K$ is a simplicial set, then the restriction of the functor $K$ to the subcategory $\Delta$ of $\Delta$ is a $\Delta$-set, which we will denote by $\Delta K$ or simply by $K$.

The elements of $K_n$ are called $n$-simplices, or simplices of dimension $n$ of $K$, and the maps $\theta^*$ the structure maps of $K$. The 0-simplices are also called vertices and if $\sigma \in K_n$, then the vertices of $\sigma$ are 0-simplices of the form $\theta^*(\sigma)$ with $\theta$ being a map $[0] \to [n]$. If $K, L$ are either simplicial or $\Delta$-sets, then a simplicial map $K \to L$ is a natural transformation of functors, i.e. as a sequence of maps $K_n \to L_n$ such that

$$
\begin{array}{ccc}
K_n & \longrightarrow & L_n \\
\downarrow \theta^* & & \downarrow \theta^* \\
K_m & \longrightarrow & L_m
\end{array}
$$

is a commutative diagram for every morphism $\theta: [m] \to [n]$ of $\Delta$ or $\Delta$ respectively.

**The face and degeneracy operators.** For every $n \in \mathbb{N}$ and $i \in [n]$ there is a unique surjective non-decreasing map $s(i): [n + 1] \to [n]$ taking the value $i$ twice. If $n > 0$, then then there is a unique strictly increasing map $d(i): [n - 1] \to [n]$ not taking the value $i$. The structure maps $s_i = s(i)^*$ and $d_i = d(i)^*$ are called the $i$th degeneracy and face operators respectively. If $\sigma$ is a simplex of $K$, then $d_i \sigma$ is called the $i$th face of $\sigma$.

Clearly, every non-decreasing map $[m] \to [n]$ admits a unique presentation as a composition $[m] \to [k] \to [n]$ of a surjective non-decreasing map $[m] \to [k]$ and a strictly increasing map $[k] \to [n]$. On the other hand, every strictly increasing map is a composition of several maps of the form $d(i)$, and every surjective non-decreasing map is a composition of several maps of the form $s(i)$.
It follows that every structure map is a composition of several face and degeneracy operators. These operators satisfy some simple and well known relations implied by relations between maps \( s(i), d(j) \), which we do not reproduce here. Conversely, the face and degeneracy operators \( \partial_j, s_i \) satisfying these relations can be extended to a contravariant functor from \( \Delta \) to the category of sets, i.e. to a simplicial set. Similarly, face operators \( \partial_j \) satisfying the relations involving only face operators can be extended to a functor from \( \Delta \) to the category of sets, i.e. to a \( \Delta \)-set. If \( K \) is a simplicial set, then the \( \Delta \)-set \( \Delta K \) is the result of ignoring the degeneracy operators of \( K \).

**Non-degenerate simplices.** A simplex \( \sigma \) of a simplicial set \( K \) is said to be *degenerate* if it belongs to the image of some \( s_i \), and *non-degenerate* otherwise. An \( n \)-simplex \( \sigma \) is degenerate if and only if \( \sigma = \theta^*(\tau) \) for an \( m \)-simplex \( \tau \) with \( m < n \) and a surjective non-decreasing \( \theta : [n] \to [m] \). By a lemma of Eilenberg and Zilber [EZ], if \( \tau \) is required to be non-degenerate, then the presentation \( \sigma = \theta^*(\tau) \) is unique. See Lemma A.2.4.

**Simplicial sets from \( \Delta \)-sets.** Let \( D \) be a \( \Delta \)-set. It gives rise to simplicial set \( \Delta D \) defined as follows. The \( n \)-simplices of \( \Delta D \) are the pairs \((\sigma, \rho)\) such that \( \sigma \) is an \( l \)-simplex of \( D \) for some \( l \leq n \) and \( \rho : [n] \to [l] \) is a surjective non-decreasing map. In order to define \( \theta^* \) for a non-decreasing map \( \theta : [m] \to [n] \), we represent \( \theta \) as the composition \( \theta = \tau \circ \varphi \), where \( \tau \) is a strictly increasing map and \( \varphi \) is a surjective non-decreasing map, and set \( \theta^*(\sigma, \rho) = (\tau^*(\sigma), \varphi) \). One can easily check that \( (\theta \circ \eta)^* = \theta^* \circ \eta^* \) and hence \( \Delta K \) is indeed a simplicial set. The correspondence \( D \to \Delta D \) naturally extends to simplicial maps, i.e. leads to a functor from the category of \( \Delta \)-sets to the category of simplicial sets.

**Simplicial sets and \( \Delta \)-sets from simplicial complexes.** Recall that a *simplicial complex* \( S \) is a set of *vertices* \( V = V_S \) together with a collection of finite subsets of \( V \), called *simplices* of \( S \), subject to the condition that a subset of a simplex is also a simplex. Elements of a simplex \( \sigma \subset V \) are called the *vertices of \( \sigma \). A simplex \( \tau \) is said to be a *face* of a simplex \( \sigma \) if \( \tau \subset \sigma \). A *local order* on a simplicial complex \( S \) is an assignment of a linear order \( <_\sigma \) on \( \sigma \) for each simplex \( \sigma \). These orders are required to agree in the sense that \( <_\tau \) is the restriction of \( <_\sigma \) if \( \tau \) is a face of \( \sigma \). For example, if \( < \) is a linear order on \( V_S \), then the restrictions of \( < \) to simplices form a local order on \( S \). The simplest examples of simplicial sets and \( \Delta \)-sets are provided by the following construction.

A locally ordered simplicial complex \( S \) gives rise to a \( \Delta \)-set \( \Delta S \) and a simplicial set \( \Delta S \). The \( n \)-simplices of \( \Delta S \) are injective maps \( \sigma : [n] \to V \) such that the image \( \sigma([n]) \) is a simplex of \( S \) and \( \sigma \) is order-preserving. Of course, the sets \( [n] \) are considered with their natural order induced from \( N \). The \( n \)-simplices of \( \Delta S \) are maps \( \sigma : [n] \to V \) such that \( \sigma([n]) \) is a simplex of \( S \) and \( \sigma \) in non-decreasing with respect to the orders on \([n]\) and this simplex. In both cases the structure maps \( \theta^* \) are defined by \( \theta^*(\sigma) = \sigma \circ \theta \). An easy check shows that \( \Delta S = \Delta \Delta S \). The local order involved in this construction almost never matters and is rarely mentioned.
Basic examples. For \( n \in \mathbb{N} \) the set \([n]\) can be considered as a simplicial complex having \([n]\) as its set of vertices and all subsets of \([n]\) as simplices. The usual order on \( \mathbb{N} \) turns \([n]\) into a locally ordered simplicial complex. The \( \Delta \)-set \( \Delta[n] \) has as \( k \)-simplices strictly increasing maps \([k] \to [n]\), and the simplicial set \( \Delta[n] \) has as \( k \)-simplices non-decreasing maps \([k] \to [n]\). Clearly, \( \Delta[n] = \Delta\Delta[n] \). We will need also the simplicial complex \([\infty]\) having \( \mathbb{N} \) as its set of vertices and all finite subsets of \( \mathbb{N} \) as simplices, as also the \( \Delta \)-set \( \Delta[\infty] \) and the simplicial set \( \Delta[\infty] = \Delta\Delta[\infty] \).

Every non-decreasing map \( \theta: [m] \to [n] \) defines a simplicial map \( \theta_*: \Delta[m] \to \Delta[n] \) by the rule \( \theta_*(\sigma) = \theta \circ \sigma \). Clearly, \( (\theta \circ \eta)_* = \theta_* \circ \eta_* \) and the assignments \( n \mapsto \Delta[n] \) and \( \theta \mapsto \theta_* \) define a covariant functor from \( \Delta \) to the category of simplicial sets. Similarly, strictly increasing maps \( \theta: [m] \to [n] \) define simplicial maps \( \theta_*: \Delta[m] \to \Delta[n] \) and lead to a functor from \( \Delta \) to \( \Delta \)-sets.

Kan extension and lifting properties. Let \( n \in \mathbb{N} \) and \( k \in [n] \). The \( k \)-horn of \( [n] \) is the simplicial complex \( [n]_k \) having \([n]\) as the set of vertices and subsets of \([n]\) not containing \([n] \sim \{k\}\) as simplices. Equivalently, \([n]_k \) is obtained from \([n]\) by removing simplices \([n]\) and \([n] \sim \{k\}\). The \( k \)-horn \( [n]_k \) of \([n]\) leads to the \( k \)-horn \( \Lambda_k[n] = \Delta[n]_k \) of \( \Delta[n] \).

A simplicial set \( K \) is said to have the Kan extension property, or to be a Kan simplicial set, if every simplicial map \( \Lambda_k[n] \to K \) can be extended to a simplicial map \( \Delta[n] \to K \). Let \( E, B \) be simplicial sets. A simplicial map \( p: E \to B \) is said to have Kan lifting property, or to be a Kan fibration if every commutative diagram of solid arrows of the form

\[
\begin{array}{ccc}
\Lambda_k[n] & \longrightarrow & E \\
\downarrow \scriptstyle{i} & \quad & \downarrow \scriptstyle{p} \\
\Delta[n] & \longrightarrow & B,
\end{array}
\]

where \( i \) is the inclusion, can be completed by a dashed arrow to a commutative diagram. A simplicial set \( K \) is Kan if and only if the unique map \( K \to \Delta[0] \) is a Kan fibration.

Simplices and simplicial maps. Let \( \iota_n \) be the identity map \([n] \to [n] \) considered as an \( n \)-simplex of \( \Delta[n] \). Then every \( m \)-simplex of \( \Delta[n] \) is equal to \( \theta^*(\iota_n) \) for a unique non-decreasing map \( \theta: [m] \to [n] \). It follows that for every simplicial set \( K \) simplicial maps \( f: \Delta[n] \to K \) are uniquely determined by the images \( f(\iota_n) \). Conversely, if \( \sigma \in K_n \), then there is a unique simplicial map

\[
i_\sigma: \Delta[n] \to K
\]

such that \( \sigma = i_\sigma(\iota_n) \).
Skeletons. Let $n \in \mathbb{N}$. If $D$ is a $\Delta$-set, then the $n$th skeleton $\text{sk}_n D$ of $D$ is the $\Delta$-subset of $D$ consisting of all $k$-simplices with $k \leq n$. If $K$ is a simplicial set, then $\text{sk}_n K$ consists of all simplices of the form $\theta^*(\sigma)$, where $\sigma$ is a $k$-simplex for some $k \leq n$. The boundary $\partial \Delta[n]$ of $\Delta[n]$ is defined as the skeleton $\text{sk}_{n-1} \Delta[n]$. The simplicial set $\partial \Delta[n]$ has as simplices non-decreasing maps $[m] \to [n]$ with the image $\neq [n]$.

Products. The product $K \times L$ of simplicial sets and $\Delta$-sets $K, L$ is defined dimension-wise. In more details, $(K \times L)_n = K_n \times L_n$ and the structure maps of $K \times L$ are the products of the structure maps of $K$ and $L$. This dimension-wise product is hardly natural for finite $\Delta$-sets, but products with $\Delta[\infty]$ play a key role in our theory.

Homotopies. The maps $d(0), d(1) : [0] \to [1]$ take 0 to 1 and 0 respectively. Let $i(0) = d(1)_*$ and $i(1) = d(0)_*$ be the corresponding maps $\Delta[0] \to \Delta[1]$. Suppose that $K, L$ are simplicial sets and $f, g : K \to L$ are simplicial maps. A homotopy between $f$ and $g$ is a simplicial map $h : K \times \Delta[1] \to L$ such that

$$f = k \circ (\text{id}_K \times i(0)) \quad \text{and} \quad g = k \circ (\text{id}_K \times i(1)),$$

where $K$ is identified with $K \times \Delta[0]$. Homotopy equivalences are defined in terms of homotopies in the usual manner. A simplicial set $K$ is said to be contractible if $K$ is homotopy equivalent to $\Delta[0]$. In this case $\text{id}_K$ is homotopic to the composition $K \to \Delta[0] \to K$ of the unique map $K \to \Delta[0]$ with $i_v : \Delta[n] \to K$ for some $v \in K_0$. If $K$ is contractible, then the projection $K \times L \to L$ is a homotopy equivalence for every $L$.

Local systems of coefficients. Let $K$ be either a simplicial or a $\Delta$-set. Let $\varepsilon$ be a 1-simplex of $K$ and $v, w$ are vertices of $K$. We say that $\varepsilon$ connects $v$ with $w$ if $\partial_1 \varepsilon = v$ and $\partial_0 \varepsilon = w$. A local system of coefficients, or simply a local system $\pi$ on $K$ is an assignment of a group $\pi_v$ to every vertex $v$ of $K$ and an isomorphism $\varepsilon^* : \pi_w \to \pi_v$ to every 1-simplex $\varepsilon$ connecting $v$ with $w$. These groups and isomorphisms are subject to the following condition: if $\omega \in K_2$ and $\rho = \partial_2 \omega$, $\sigma = \partial_0 \omega$, $\tau = \partial_1 \omega$, then $\tau^* = \rho^* \circ \sigma^*$. A local system of coefficients $\pi$ is said to be abelian if all groups $\pi_v$ are abelian.

The leading vertex of an $n$-simplex $\sigma$ is the vertex $v_{\sigma} = \theta^*(\sigma)$, where $\theta : [0] \to [n]$ is the inclusion, and the leading edge of an $n$-simplex $\sigma$ is the 1-simplex $\varepsilon_{\sigma} = \eta^*(\sigma)$, where $\eta : [1] \to [n]$ is the inclusion. An $n$-cochain of $K$ with coefficients in the local system $\pi$ is a map $c$ assigning every $\sigma \in K_n$ an element $c(v) \in \pi_v$, where $v = v_{\sigma}$ is the leading vertex of $\sigma$. The group of such cochains is denoted by $C^n(K, \pi)$. For abelian $\pi$ the coboundary operators $\partial^* : C^n(K, \pi) \to C^{n+1}(K, \pi)$ are defined by the formula

$$\partial^* c(\sigma) = \varepsilon_{\sigma}^* (c(\partial_0 \sigma)) + \sum_{i=1}^{n+1} (-1)^i c(\partial_i \sigma) \in \pi_{v_{\sigma}}.$$

The cocycles and coboundaries are defined in terms $\partial^*$ in the usual manner.
3. Postnikov systems and minimality

Comparing simplices. Let $K$ be a simplicial set and $q \in \mathbb{N}$. Recall that for every $\sigma \in K_q$ there exists a unique simplicial map $i_\sigma : \Delta[q] \to K$ such that $i_\sigma (\iota_q) = \sigma$. Two simplices $\sigma, \tau \in K_q$ are said to be $n$-equivalent if the restrictions of the maps $i_\sigma, i_\tau : \Delta[q] \to K$ to $sk_n \Delta[q]$ are equal, or, equivalently, if $\theta^* (\sigma) = \theta^* (\tau)$ for every non-decreasing map $\theta : [n] \to [q]$. We write $\sigma \sim_n \tau$ if $\sigma, \tau$ are $n$-equivalent. Obviously, if $q \leq n$, then $\sigma \sim_n \tau$ if and only if $\sigma = \tau$.

Clearly, $\sim_n$ is an equivalence relation on the set of simplices. If $\sigma, \tau \in K_q$ and $\sigma \sim_n \tau$, then $\theta^* (\sigma) \sim_n \theta^* (\tau)$ for every non-decreasing map $\theta : [m] \to [q]$. Therefore the structural maps $\theta^* \circ K$ induce maps between sets of equivalence classes of $\sim_n$. These induced maps are the structure maps of a canonical structure of a simplicial set on the set $K(n)$ of equivalence classes with respect to $\sim_n$. Clearly, there is a canonical simplicial map

$$p_n : K \to K(n).$$

Also, if $n \leq m$ and $\sigma \sim_m \tau$, then $\sigma \sim_n \tau$, and hence there is a canonical simplicial map

$$p_{m,n} : K(m) \to K(n).$$

Clearly, the maps $p_n$ and $p_{m,n}$ induce isomorphisms of $n$th skeletons. When there is no danger of confusion, we denote maps $p_n$ and $p_{m,n}$ simply by $p$.

Let $M$ be a simplicial subset of $K$. Clearly, for simplices $\sigma, \tau$ of $M$ the relation $\sigma \sim_n \tau$ holds in $M$ if and only if it holds in $K$. Therefore there is a canonical injective map $M(n) \to K(n)$. In such a situation we will identify $M(n)$ with its image in $K(n)$.

Two simplices $\sigma, \tau \in K_q$ are said to be homotopic if the maps $i_\sigma$ and $i_\tau$ are homotopic relatively to $\partial \Delta[q]$. We write $\sigma \sim \tau$ if $\sigma, \tau$ are homotopic. If $K$ is a Kan simplicial set, then $\sim$ is an equivalence relation on the set of simplices.

Postnikov systems. The sequence of simplicial sets $K(0), K(1), \ldots, K(n), \ldots$ together with the maps $p_n$ and $p_{m,n}$ is called Postnikov or Moore-Postnikov system of $K$. More precisely, this definition is the version of the original construction of Postnikov [P1], [P2] due to Moore [Mo]. This construction and the notion of homotopic simplices are useful only for Kan simplicial sets. If $K$ is Kan, then every term $K(n)$ of the Postnikov system is also Kan, and all maps $p_n$ and $p_{m,n}$ are Kan fibrations. See [Ma], Proposition 8.2.
Minimality. Postnikov systems are especially powerful when $K$ is minimal in the following sense. A Kan simplicial set $K$ is said to be minimal if every two homotopic simplices of $K$ are equal, i.e. that $\sigma \sim \tau$ implies $\sigma = \tau$ for every two simplices $\sigma, \tau$ of $M$. This notion is going back to Eilenberg and Zilber [EZ] and Postnikov [P1]. Every Kan simplicial set $K$ contains a minimal Kan simplicial subset $M$ as a strong deformation retract. Moreover, every two such simplicial subsets $M$ are isomorphic. See [Ma], Theorems 9.5 and 9.8.

Let $E, B$ be simplicial sets and $p: E \to B$ be a Kan fibration. Two simplices $\sigma, \tau \in E_q$ are fiberwise homotopic if $p(\sigma) = p(\tau)$ and there exists a relative to $\partial \Delta[q]$ homotopy $h: \Delta[q] \times \Delta[1] \to E$ between the maps $i_\sigma$ and $i_\tau$ such that the diagram

$$\begin{array}{ccc}
\Delta[q] \times \Delta[1] & \xrightarrow{h} & E \\
\downarrow \mathrm{pr} & & \downarrow p \\
\Delta[q] & \xrightarrow{i_\rho} & B,
\end{array}$$

where $\rho = p(\sigma) = p(\tau)$ and $\mathrm{pr}$ is the projection, is commutative. In particular, if $\sigma, \tau$ are fiberwise homotopic, then $\sigma, \tau$ are homotopic. A Kan fibration $p: E \to B$ is said to be minimal if every two fiberwise homotopic simplices are equal. For every Kan fibration $p: E \to B$ there exists a simplicial subset $M$ of $E$ such that $p|M: E \to B$ is a minimal Kan fibration and $M$ is a fiberwise strong deformation retract of $E$ in a natural sense. See [GJ], Chapter I, Proposition 10.3, or [Ma], Theorem 10.9.

3.1. Theorem. Every minimal Kan fibration with connected base is a locally trivial bundle.

Proof. See [GZ], Section VI.5.4, or [Ma], Theorem 11.11. □

Postnikov systems and locally trivial bundles. If $K$ is a minimal Kan simplicial set, then all maps $p_n$ and $p_{m,n}$ are minimal Kan fibrations. See [Ma], Lemma 12.1. Therefore, Theorem 3.1 implies that these maps are locally trivial bundles. In particular,

$$p = p_{n,n-1}: K(n) \to K(n-1)$$

is a locally trivial bundle. Its fiber is the Eilenberg–MacLane simplicial set $K(\pi, n)$, where $\pi = \pi_n(K, \nu)$ is the $n$th homotopy groups of $K$. See [Ma], the beginning of Section 25. For our purposes it is sufficient to know that for $n > 1$ the fiber is an Eilenberg–MacLane simplicial set $K(\pi, n)$ with an abelian group $\pi$, and to identify the simplicial set $K(1)$. See Lemma 7.3 for the latter.
4. Classifying spaces of categories and groups

**Classifying spaces of categories.** This section is devoted to some classical constructions of Milnor [Mi] and Segal [S] (who attributed some of the ideas of [S] to Grothendieck).

A set $S$ with a partial order $\leq$ defines a category having $S$ as its set of objects. For $a, b \in S$ there is exactly one morphism $a \to b$ if $a \leq b$, an none otherwise. In particular, sets $[n]$ together with their natural order can be considered as categories. From this point of view non-decreasing maps $\theta: [m] \to [n]$ are nothing else but functors $[m] \to [n]$.

Every small category $\mathcal{C}$ defines a simplicial set $B\mathcal{C}$, its nerve in the sense of G. Segal [S], often called also the classifying space of $\mathcal{C}$. The vertices of $B\mathcal{C}$ are the objects of $\mathcal{C}$, and the $n$-simplices are functors $\sigma: [n] \to \mathcal{C}$. As usual, the structure maps are defined as compositions. Namely, if $\theta: [m] \to [n]$ is a non-decreasing map, then $\theta^*(\sigma) = \sigma \circ \theta$, where in the right hand side $\theta$ is considered as a functor. Clearly, a functor $[n] \to \mathcal{C}$ is determined by its values on objects and on morphisms $i \to i+1$, where $i \in [n-1]$. Therefore $n$-simplices of $B\mathcal{C}$ correspond to sequences of morphisms of the form

\begin{equation}
(4.1) \quad v_0 \xrightarrow{p_1} v_1 \xrightarrow{p_2} \ldots \xrightarrow{p_n} v_n,
\end{equation}

where each $v_i$ is an object of $\mathcal{C}$ and each $p_i$ is a morphism $v_{i-1} \to v_i$. Of course, the objects $v_i$ are determined by the morphisms $p_k$ and hence $n$-simplices correspond to sequences $(p_1, p_2, \ldots, p_n)$ of morphisms such that the composition $p_{i+1} \circ p_i$ is defined for each $i$ between 1 and $n-1$. For $0 < i < n$ the boundary operators $\partial_i$ acts by replacing $v_i$ and morphisms $p_i, p_{i+1}$ by the composition $p_{i+1} \circ p_i$. The boundary operators $\partial_0$ and $\partial_n$ act by simply removing $v_0, p_1$ and $p_{n-1}, v_n$ respectively. The degeneracy operator $s_i$ acts by inserting the identity morphism $v_i \to v_i$. Cf. [GJ], Example I.1.4.

Let $\mathcal{C}, \mathcal{D}$ be two categories. A functor $f: \mathcal{C} \to \mathcal{D}$ defines, in an obvious way, a simplicial map $Bf: B\mathcal{C} \to B\mathcal{D}$. Given two functors $f, g: \mathcal{C} \to \mathcal{D}$, a natural transformation $f \to g$ defines a homotopy between $Bf$ and $Bg$. Indeed, a natural transformation $t: f \to g$ can be considered as a functor $\mathcal{C} \times [1] \to \mathcal{D}$, where $[1]$ is considered as a category. One can easily see that the operation $\mathcal{C} \to B\mathcal{C}$ commutes with the products. Since, obviously, $B[1] = \Delta[1]$, the natural transformation $t$ defines a simplicial map $Bt: B\mathcal{C} \times \Delta[1] \to B\mathcal{D}$, i.e. a homotopy. We leave to the reader the verification that this is a homotopy between $Bf$ and $Bg$.

A discrete group $\pi$ can be considered as a category with a single object and $\pi$ being the set of morphisms from this object to itself, with the composition being the group multiplication. The classifying space $B\pi$ is a Kan simplicial set. See [GJ], Lemma I.3.5. Comparing the definitions shows that the usual and the bounded cohomology of the group $\pi$ are, in fact, cohomology of the classifying space $B\pi$. 
Milnor's classifying spaces. Another classical construction of classifying spaces of groups is due to Milnor [Mi]. While we will not use it directly, it serves as a motivation for the definitions of unravelings of classifying spaces and simplicial sets below and in Section 6.

For a discrete group $\pi$ let $E\pi$ be the simplicial complex having the product $\pi \times \mathbb{N}$ as the set of vertices and as simplices finite subsets $\sigma \subset \pi \times \mathbb{N}$ such that the projection $\pi \times \mathbb{N} \to \mathbb{N}$ is injective on $\sigma$. There is a left action $\pi \times E\pi \to E\pi$ of the group $\pi$ on $E\pi$ by the rule

$$h \cdot (g, k) = (h \cdot g, k),$$

where $h \in \pi$ and $(g, k) \in \pi \times \mathbb{N}$ is a vertex of $E\pi$. The quotient $B\pi = \pi \backslash E\pi$ of the simplicial complex $E\pi$ by this action is a well-defined simplicial complex. We will call $B\pi$ the Milnor classifying space of $\pi$. Milnor defined directly the geometric realization $|B\pi|$ for arbitrary topological group $\pi$. By this reason his construction is different from ours one.

It is convenient to enhance the structure of Milnor's classifying space to a $\Delta$-set. The natural order on $\mathbb{N}$ defines local orders on the simplicial set $E\pi$ and $B\pi$ and allows to turn them into $\Delta$-sets, which we will still denote by $E\pi$ and $B\pi$. The local order on $E\pi$ is invariant under the left action of $\pi$, and $B\pi = \pi \backslash E\pi$ as $\Delta$-sets also.

One of advantages of $B\pi$ is the existence of many automorphisms. Let $C^0(\mathbb{N}, \pi)$ be the group of all maps $\mathbb{N} \to \pi$. Let us define a right action of $C^0(\mathbb{N}, \pi)$ on $E\pi$ by the rule

$$(g, k) \mapsto (g \cdot c(k), k),$$

where $c \in C^0(\mathbb{N}, \pi)$ and $(g, k) \in \pi \times \mathbb{N}$ is a vertex of $E\pi$. Clearly, the right action of $C^0(\mathbb{N}, \pi)$ on $E\pi$ preserves the order of vertices and commutes with the left action of $\pi$. Hence this action leads to a right action of $C^0(\mathbb{N}, \pi)$ on $B\pi$. If $\kappa \subset \pi$ is a subgroup of $\pi$, then $C^0(\mathbb{N}, \kappa)$ is a subgroup of $C^0(\mathbb{N}, \pi)$, and if $\kappa$ is a normal subgroup, then

$$(4.2) \quad B(\pi/\kappa) = B\pi/C^0(\mathbb{N}, \kappa).$$

This obvious property is the main reason of our interest in $B\pi$. This property strongly contracts with the properties of the classifying spaces $B\pi$. Namely, the classifying space $B(\pi/\kappa)$ is not a quotient of $B\pi$, at least not in any natural way.

Unravelings of classifying spaces of groups. The $\Delta$-set $B\pi$ is isomorphic to $B\pi \times \Delta[\infty]$. See Lemma A.1.1. While we are not going to use this result, it motivates our interest to the $\Delta$-set $B\pi \times \Delta[\infty]$, which we will call the unraveling of $B\pi$.

As we will see in a moment, the action of $C^0(\mathbb{N}, \pi)$ can be defined directly for $B\pi \times \Delta[\infty]$. In fact, it is easier and more useful to define an action of $C^0(\mathbb{N}, \pi)$ on the simplicial set $B\pi \times \Delta[\infty]$ first. The latter is the classifying space of a category. Indeed, the simplicial set
\(\Delta[\infty]\) has non-decreasing maps \([n] \rightarrow \mathbb{N}\) as \(n\)-simplices, with the usual structure maps.

Let \(\mathbf{n}\) be the category having \(\mathbb{N}\) as its set of objects, exactly one morphism \(n \rightarrow m\) when \(n \leq m\), and no morphisms \(n \rightarrow m\) when \(n > m\). Clearly, \(\Delta[\infty] = B\mathbf{n}\). It follows that

\[
B\pi \times \Delta[\infty] = B\pi \times B\mathbf{n} = B(\pi \times \mathbf{n}) .
\]

The category \(\pi \times \mathbf{n}\) has \(\mathbb{N}\) as the set of objects. The set of morphisms \(n \rightarrow m\) is a copy of \(\pi\) if \(n \leq m\), and is empty if \(n > m\). Given \(c \in C^0(\mathbb{N}, \pi)\), let \(a(c): \pi \times \mathbf{n} \rightarrow \pi \times \mathbf{n}\) be the functor equal to the identity on objects and acting on morphisms \(n \rightarrow m\) identified with elements of the group \(\pi\) by the rule

\[
g \mapsto c(n)^{-1} \cdot g \cdot c(m) .
\]

Clearly, \(a(c)\) is an automorphism of \(\pi \times \mathbf{n}\) and even an automorphism over \(\mathbf{n}\), in the sense that \(pr \circ a(c) = pr\), where \(pr: \pi \times \mathbf{n} \rightarrow \mathbf{n}\) is the projection. Also, all diagrams

\[
\begin{array}{ccc}
n & \xrightarrow{g} & n \\
n \downarrow & & \downarrow \\
c(n) & \xrightarrow{a(c)} & c(m) \\
\end{array}
\]

are commutative and hence morphisms \(c(n): n \rightarrow n\) form a natural transformation from the identity functor to \(a(c)\). It follows that the simplicial map

\[
Ba(c): B(\pi \times \mathbf{n}) \rightarrow B(\pi \times \mathbf{n})
\]

is an automorphism of \(B(\pi \times \mathbf{n})\) homotopic to the identity. The map \(c \mapsto Ba(c)\) defines an action of \(C^0(\mathbb{N}, \pi)\) on \(B(\pi \times \mathbf{n}) = B\pi \times \Delta[\infty]\). Since the functors \(a(c)\) are automorphisms over \(\mathbf{n}\), the simplicial maps \(Ba(c)\) are automorphisms of \(B\pi \times \Delta[\infty]\) over \(\Delta[\infty]\).

It follows that maps \(Ba(c)\) leave the \(\Delta\)-subset \(B\pi \times \Delta[\infty]\) of \(B\pi \times \Delta[\infty]\) invariant. By restricting these maps to \(B\pi \times \Delta[\infty]\) we get an action of \(C^0(\mathbb{N}, \pi)\) on \(B\pi \times \Delta[\infty]\). If \(\kappa \subseteq \pi\) is a normal subgroup of \(\pi\), then a direct verification shows that

\[
B(\pi/\kappa) \times \Delta[\infty] = B\pi \times \Delta[\infty]/C^0(\mathbb{N}, \kappa) .
\]

Of course, this is simply another form of the property (4.2). We will need a slightly stronger, but still obvious, form of this property. Let \(1 \in \pi\) be the unit of \(\pi\), and let \(C_0(\mathbb{N}, \pi)\) be the group of maps \(c: \mathbb{N} \rightarrow \pi\) such that \(c(n) = 1\) for almost every \(n\). Then

\[
(4.3) \quad B(\pi/\kappa) \times \Delta[\infty] = B\pi \times \Delta[\infty]/C_0(\mathbb{N}, \kappa) .
\]
5. Bundles with Eilenberg–MacLane fibers

Locally trivial bundles. Let \( p : E \to B \) be a simplicial map thought as a bundle, and let \( i : A \to B \) a simplicial map. Let \( i^*E \subset E \times A \) be the simplicial subset of \( E \times A \) having as \( n \)-simplices pairs \((\sigma, \tau)\) such that \( \sigma \in E_n, \tau \in A_n \), and \( p(\sigma) = i(\tau) \). In other terms,

\[
(i^*E)_n = \{ (\sigma, \tau) \in E_n \times A_n \mid p(\sigma) = i(\tau) \}.
\]

The restriction \( i^*p : i^*E \to A \) of the projection \( E \times A \to A \) to \( i^*E \) is called the pull-back of \( p \) by \( i \), or the bundle induced from the bundle \( p : E \to B \) by \( i \). The bundle \( i^*p \) has the usual universal properties of pull-backs. A simplicial map \( p : E \to B \) is said to be a trivial bundle with the fiber \( F \) if there exists a commutative diagram

\[
\begin{array}{ccc}
B \times F & \xrightarrow{t} & E \\
\downarrow{\text{pr}} & & \downarrow{p} \\
B & \xrightarrow{=} & B,
\end{array}
\]

such that \( t : B \times F \to E \) is an isomorphism. Such \( t \) is called a trivialization of \( p \). A map \( p : E \to B \) is a locally trivial bundle with the fiber \( F \) if for every simplex \( \sigma \) of \( B \) the pull-back \( i^*_\sigma p \) is a trivial bundle with the fiber \( F \). In this case \( E \) is called the total space and \( B \) the base of \( p \). Clearly, if \( p \) is a locally trivial bundle, then \( p \) is surjective.

Normalized and non-abelian cochains. Let \( n \in \mathbb{N}, \ n > 1 \), and let \( \pi \) be an abelian group. A cochain of a simplicial set \( K \) is said to be normalized if it is equal to 0 on degenerate simplices. We will denote by \( \mathcal{C}^n(K, \pi) \) the group of normalized \( n \)-cochains of \( K \) with coefficients in \( \pi \) and by \( \mathcal{Z}^n(K, \pi) \) the subgroup of normalized cocycles.

Eilenberg–MacLane simplicial sets \( K(\pi, n) \). For every \( q \in \mathbb{N} \) let us consider the groups \( \mathcal{C}^n(\Delta[q], \pi) \) and \( \mathcal{Z}^n(\Delta[q], \pi) \). Every non-decreasing map \( \theta : [r] \to [q] \) induces a simplicial map \( \theta^* : \Delta[r] \to \Delta[q] \), which, in turn, induces homomorphisms

\[
\theta^* : \mathcal{C}^n(\Delta[q], \pi) \to \mathcal{C}^n(\Delta[r], \pi) \quad \text{and} \quad \theta^* : \mathcal{Z}^n(\Delta[q], \pi) \to \mathcal{Z}^n(\Delta[r], \pi).
\]

Eilenberg–MacLane simplicial set \( K(\pi, n) \) is defined as follows. Its set of \( q \)-simplices is

\[
K(\pi, n)_q = \mathcal{Z}^n(\Delta[q], \pi),
\]
and the structural maps $\theta^*: K(\pi, n) \rightarrow K(\pi, n)_r$ are the above induced homomorphisms $\theta^*$. Let $0_q \in \mathcal{Z}^n(\Delta[q], \pi) = K(\pi, n)_q$ be the zero cocycle.

The $n$-cocycles of $K(\pi, n)$. Every normalized $n$-cochain of $\Delta[n]$ is a cocycle, i.e.

$$\mathcal{Z}^n(\Delta[n], \pi) = \mathcal{C}^n(\Delta[n], \pi).$$

Clearly, a normalized $n$-cochain $c$ of $\Delta[n]$ is determined by its value $c(t_n)$ on the unique non-degenerate $n$-simplex $t_n$ of $\Delta[n]$. Therefore we can identity $K(\pi, n)_n$ with $\pi$.

A $n$-cochain of $K(\pi, n)$ with coefficients in $\pi$ is a map $K(\pi, n)_n \rightarrow \pi$, and hence can be thought as a map $c: \pi \rightarrow \pi$. Clearly, the zero cocycle $0_n \in \mathcal{Z}^n(\Delta[n], \pi)$ is the only degenerate $n$-simplex of $K(\pi, n)$. Therefore we can identify normalized $n$-cochains $c$ of $K(\pi, n)$ with maps $c: \pi \rightarrow \pi$ subject only to the condition $c(0) = 0$. It turns out that $c$ is a cocycle if and only if $c$ is a homomorphisms $\pi \rightarrow \pi$. See Lemma A.2.2 for a proof. In particular, the identity map $\text{id}_\pi: \pi \rightarrow \pi$ is an $n$-cocycle of $K(\pi, n)$.

Simplicial maps to $K(\pi, n)$. Let $K$ be a simplicial set. Let us assign to every simplicial map $f: K \rightarrow K(\pi, n)$ the $n$-cocycle

$$z(f) = f^*(\text{id}_\pi) \in \mathcal{Z}^n(K, \pi)$$

(the cochain $z(f)$ is a normalized cocycle because $\text{id}_\pi$ is). Unraveling the definitions shows that $f$ is uniquely determined by $z(f)$ and for every $z \in \mathcal{Z}^n(K, \pi)$ there is a map $f: K \rightarrow K(\pi, n)$ such that $z(f) = z$. One can say that the definition of $K(\pi, n)$ is dictated by this property. If $\pi$ is abelian, then two maps $f, g: K \rightarrow K(\pi, n)$ are homotopic if and only if $z(f) - z(g)$ is a coboundary of a normalized $(n-1)$-cochain. See, for example, [Ma], Lemma 24.3 and Theorem 24.4, or [EM3], Theorems 5.1 and 5.2.

In the case of $K = K(\pi', n)$, we see that every homomorphism $h: \pi' \rightarrow \pi$ defines a map $K(\pi', n) \rightarrow K(\pi, n)$. We will denote this map by $s(h)$. There are no other simplicial maps $K(\pi', n) \rightarrow K(\pi, n)$, and $s(h)$ is an isomorphisms if and only if $h$ is.

Maps over the base. Let $p: E \rightarrow B$ be a simplicial map thought of as a bundle. A simplicial map $f: E \rightarrow E$ is said to be a map over the base, or a map over $B$, if $p \circ f = f$. For example, the identity map $\text{id}_E: E \rightarrow E$ is a map over the base.

Simplicial groups. A simplicial group is a contravariant functor from $\Delta$ to the category of groups. In other words, a simplicial group is a simplicial set $G$ together with group structures on sets $G_n$, $n \in \mathbb{N}$, such that the structural maps $\theta^*$ are homomorphisms. Let $G$ be a simplicial group. Then for every $q \in \mathbb{N}$ there is a natural action of the group $G_q$ on $\Delta[q] \times G$ defined as follows. Let $g \in G_q$. The $m$-simplices of $\Delta[q] \times G$ are the pairs

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\((\theta, \tau), \) where \(\tau \in G_m\) and \(\theta: [m] \rightarrow [q]\) is a \(q\)-simplex of \(\Delta[q]\). Let
\[
g \cdot (\theta, \tau) = \left( \theta, \theta^* (g) \cdot \tau \right),
\]
where the product in the right hand side is taken in \(G_q\). A routine check shows that the map \( (\theta, \tau) \mapsto g \cdot (\theta, \tau) \) is a simplicial map \(\Delta[q] \times G \rightarrow \Delta[q] \times G\). We will denote this map by \(t(g)\). Clearly, \(t(g)\) is an automorphism of the bundle \(pr: \Delta[q] \times G \rightarrow \Delta[q]\) over the base. Another routine check shows that \(g \mapsto t(g)\) is an action.

The addition of normalized cocycles turns \(K(\pi, n)\) into a simplicial group. In the case of \(G = K(\pi, n)\) we get an automorphism
\[
(5.1) \quad t(c): \Delta[q] \times K(\pi, n) \rightarrow \Delta[q] \times K(\pi, n)
\]
of the bundle
\[
(5.2) \quad pr: \Delta[q] \times K(\pi, n) \rightarrow \Delta[q].
\]
over the base for every normalized cocycle \(c \in \mathcal{Z}^n(\Delta[q], \pi)\).

**5.1. Theorem.** Every automorphism of (5.2) over the base is equal to the composition
\[
(\text{id}_{\Delta[q]} \times s(h)) \circ t(c),
\]
where \(h: \pi \rightarrow \pi\) is an automorphism and \(c \in \mathcal{Z}^n(\Delta[q], \pi)\). Both \(h\) and \(c\) are uniquely determined by the automorphism.

**Proof.** See [Ma], Propositions 25.2 and 25.3. ■

**Translations of trivial bundles.** Let \(p: E \rightarrow \Delta[q]\) be a trivial bundle with the fiber \(K(\pi, n), n > 1\), and let \(f: E \rightarrow E\) be an automorphism over the base. We will say that \(f\) is a translation if \(f = t \circ t(c) \circ t^{-1}\) for some trivialization \(t: \Delta[q] \times K(\pi, n) \rightarrow E\) and some normalized cocycle \(c \in \mathcal{Z}^n(\Delta[q], \pi)\).

Theorem 5.1 implies that if \(f\) has the form \(t \circ t(c) \circ t^{-1}\) for some trivialization \(t\), then \(f\) has such form for every trivialization. But the cocycle \(c\) depends on the choice of \(t\) because (5.2) has automorphisms of the form \(\text{id}_{\Delta[q]} \times s(h)\). Still, there is a way to make \(c\) to be uniquely determined by \(f\).

Let \(i_0: \Delta[0] \rightarrow \Delta[q]\) be the map defined by the inclusion \([0] \rightarrow [q]\). The total space \(F\) of the pull-back bundle \(i_0^* p\) is isomorphic to \(K(\pi, n)\). Any two isomorphisms differ by an automorphism of the form \(s(h)\), where \(h\) is an automorphism of \(\pi\). Therefore F
is a simplicial group isomorphic to $K(\pi, n)$. In particular, the set $F_n$ of $n$-simplices of $F$ is a group isomorphic to $\pi$. Let us denote this group by $\pi_0$. Then $p$ is also a trivial bundle with the fiber canonically isomorphic to $K(\pi_0, n)$. Let us call a trivialization $t: \Delta[q] \times K(\pi_0, n) \to E$ special if the induced map $K(\pi_0, n) \to F$ is the canonical isomorphism. By Theorem 5.1 two special trivializations differ by a translation. But, if

$$g: \Delta[q] \times K(\pi_0, n) \to \Delta[q] \times K(\pi_0, n)$$

is a translation, then $g \circ t(c) \circ g^{-1} = t(c)$. It follows that for every translation $f$ of the bundle $p$ there is a well defined cocycle

$$d(f) \in \mathcal{Z}^n(\Delta[q], \pi_0)$$

such that $t^{-1} \circ f \circ t = t(d(f))$ for every special trivialization $t$.

**The canonical local system.** Let $p: E \to B$ be a locally trivial bundle with the fiber $K(\pi, n)$ and $n \geq 1$. The above discussion of translations suggests to associate with each vertex $v \in B_0$ a group $\pi_v$ isomorphic to $\pi$. Namely, the total space $F_v$ of the pull-back bundle $i_v^* p$ is a simplicial group isomorphic to $K(\pi, n)$. Let $\pi_v$ be the group of $n$-simplices of $F_v$. Then $F_v$ is canonically isomorphic to $K(\pi_v, n)$.

Suppose that $v, w \in B_0$ and $\epsilon$ is a 1-simplex of $B$ such that $\partial_1 \epsilon = v$, $\partial_0 \epsilon = w$. Let $E_\epsilon$ be the total space of the pull-back bundle $i_\epsilon^* p$, and let $t: \Delta[1] \times K(\pi_v, n) \to E_\epsilon$ be a special trivialization. Recall that two special trivializations differ by a translation. Since $n > 1$, every normalized $n$-chain of $\Delta[1]$ is equal to 0. It follows that every translation is equal to the identity and hence $t$ is uniquely determined. Therefore, the isomorphism

$$K(\pi_v, n) \to F_w = K(\pi_w, n)$$

induced by $t$ depends only on $\epsilon$. Let $\epsilon(p)$ be its inverse, and let $\epsilon^*: \pi_w \to \pi_v$ be the unique isomorphism such that $\epsilon(p) = s(\epsilon^*)$. By using trivializations of the pull-back bundles $i_\sigma^* p$ for 2-simplices $\sigma$ of $B$ one can easily check that the groups $\pi_v$ together with isomorphisms $\epsilon^*$ form a local system of coefficients on $B$, which we will denote by $\pi(p)$.

**Translations of locally trivial bundles.** Let $p: E \to B$ be a locally trivial bundle with the fiber $K(\pi, n)$, $n \geq 1$. Let $f: E \to E$ be an automorphism over the base. For every $q$-simplex $\sigma$ of $B$ the automorphism $f$ induces an automorphism $f_\sigma$ of the pull-back bundle $i_\sigma^* p$ over the base. Since the bundle $i_\sigma^* p$ is trivial, it make sense to ask if $f_\sigma$ is a translation and to call $f$ a translation if $f_\sigma$ is a translation for every simplex $\sigma$ of $B$. Suppose that $f: E \to E$ is a translation. Let $\sigma$ be an $n$-simplex of $B$, and let $\pi_\sigma = \pi_v$, where $v = v_\sigma$ is the leading vertex of $\sigma$. Then

$$d(f_\sigma) \in \mathcal{Z}^n(\Delta[n], \pi_\sigma).$$

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The group $\mathcal{Z}^n(\Delta[n], \pi_\sigma)$ is canonically isomorphic to $\pi_\sigma$ and hence we can consider $d(f_\sigma)$ as an elements of $\pi_\sigma$. The map

$$D_f : \sigma \mapsto d(f_\sigma)$$

is an $n$-cochain of $B$ with coefficients in the local system $\pi(p)$.

5.2. Lemma. The $n$-cochain $D_f$ is a normalized cocycle.

Proof. If $\sigma$ is a degenerate $n$-simplex of $B$, then $\sigma = \theta^*(\tau)$ for an $m$-simplex $\tau$ such that $m < n$ and a non-decreasing map $\theta : [n] \rightarrow [m]$. Therefore $i_\sigma = i_\tau \circ \theta_*$, where $\theta_* : \Delta[n] \rightarrow \Delta[m]$ is induced by $\theta$, and hence

$$i_\sigma^* = \theta_*^*(i_\tau^*)$$

If $t$ be a special trivialization of $i_\tau^* p$, then $t^{-1} \circ f_\tau \circ t = t(d(f_\tau))$. But $d(f_\tau)$ is a normalized $m$-cochain of $\Delta[m]$. Therefore $m < n$ implies that $d(f_\tau) = 0$ and hence $f_\tau$ is equal to the identity. In view of (5.3) this implies that $f_\sigma$ is equal to the identity and hence $d(f_\sigma) = 0$. It follows that $D_f$ is normalized.

Let $\rho$ be an $(n+1)$-simplex of $B$, and let $\epsilon = \theta^*(\rho)$, where $\theta : [1] \rightarrow [n]$ is the inclusion. Then $v = \partial_1 \epsilon$ is the leading vertex of $\rho$ and each face $\partial_i \rho$ with $i > 0$, and $w = \partial_0 \epsilon$ is the leading vertex of $\tau = \partial_0 \rho$. Let $t$ be a special trivialization of $i_\tau^* p$. Then $t$ induces a trivialization of $i_\sigma^* p$ for every simplex $\tau = \partial_i \rho$. If $i > 0$, then the induced trivialization is special. If $i = 0$, it differs from a special trivialization by the isomorphism

$$\epsilon(p) : K(\pi_w, n) \rightarrow K(\pi_\nu, n)$$

corresponding to the isomorphism $\epsilon^* : \pi_w \rightarrow \pi_\nu$. It follows that

$$d(f_\rho)(\partial_i 1_{n+1}) = D_f(\partial_i \rho) \quad \text{if} \quad i > 0 \quad \text{and}$$
$$d(f_\rho)(\partial_0 1_{n+1}) = \epsilon^*(D_f(\tau)).$$

Since $d(f_\rho)$ is a cocycle with coefficients in $\pi_\rho = \pi_\nu$, this implies that $D_f$ is a cocycle with coefficients in the local system $\pi(p)$. ■

5.3. Lemma. For every normalized $n$-cocycle $c \in \mathcal{Z}^n(B, \pi(p))$ there exists a unique translation $f = f(c) : E \rightarrow E$ such that $D_f = c$.

Proof. Let $\sigma$ be a $q$-simplex of $B$, and let $t$ be a special trivialization of $i_\sigma^* p$. Then $t$ induces a trivialization of $i_\tau^* p$ for every simplex $\tau$ of the form $\tau = \theta^*(\sigma)$. In general,
the induced trivialization is not special, but differs from a special one by the isomorphism

\[ \varepsilon(p) : K(\pi_w, n) \rightarrow K(\pi_v, n) \]

where \( v, w \) are the leading vertices of \( \sigma, \tau \) respectively, and \( \varepsilon \) is the unique 1-simplex of the form \( \varepsilon = \eta^*(p) \) such that \( \partial_1 \varepsilon = v, \partial_0 \varepsilon = w \). It follows that

\[ \theta^*(d(f_\sigma)) = \varepsilon^*(d(f_\tau)) , \]

where the isomorphism \( \varepsilon^* : \pi_w \rightarrow \pi_v \) is applied to the coefficients of \( d(f_\tau) \). By applying this observation to \( n \)-simplices \( \tau \) we see that \( d(f_\sigma) \) is determined by \( D_f \), and hence

\[ f_\sigma : i_\sigma^* p \rightarrow i_\sigma^* p \]

is also determined by \( D_f \). Since this is true for every simplex \( \sigma \) of \( B \), the translation \( f \) is determined by \( D_f \). This proves the uniqueness. To prove the existence, suppose that \( c \in Z^n(B, \pi(p)) \) is given. Let \( \sigma \) be a \( q \)-simplex of \( B \). The isomorphisms \( \varepsilon^* \) from the first part of the proof establish an isomorphism between the induced local system \( i_\sigma^* \pi(p) \) and the constant coefficients system \( \pi_v \). This isomorphism turns the \( n \)-cochain \( i_\sigma^*(c) \) of \( \Delta[q] \) with coefficients in \( i_\sigma^* \pi(p) \) into an \( n \)-cochain

\[ c(\sigma) \in Z^n(\Delta[q], \pi_v) . \]

There is a unique translation \( f_\sigma : i_\sigma^* p \rightarrow i_\sigma^* p \) such that \( d(f_\sigma) = c(v) \). Since the cochains \( c(\sigma) \) result from a single cochain \( c \), the translations \( f_\sigma \) agree with each other in the sense that if \( \tau = \theta^*(\sigma) \), then the diagram

\[
\begin{array}{ccc}
  i_\tau^* E & \xrightarrow{\theta^*} & i_\sigma^* E \\
  f_\tau \downarrow & & \downarrow f_\sigma \\
  i_\tau^* E & \xrightarrow{\theta^*} & i_\sigma^* E \\
\end{array}
\]

is commutative. It follows that the maps \( f_\sigma \) together define a translation \( f : E \rightarrow E \). By the construction, \( D_f = c \). This proves the existence. ■

5.4. Lemma. Let \( f, g : E \rightarrow E \) be two translations. Then

\[ D_{f \circ g} = D_f + D_g . \]

If \( c, d \in Z^n(B, \pi(p)) \), then \( f(c + d) = f(c) \circ f(d) \).
Proof. The first part of the lemma follows directly from the definitions. In view of Lemma 5.3 the second part follows from the first one.

5.5. Lemma. Let \( c, d \in \mathcal{Z}^n(B, \pi(p)) \). If \( c - d \) is equal to the coboundary of a normalized cochain, then the maps \( f(c) \) and \( f(d) \) are homotopic.

Proof. In view of Lemma 5.4 it is sufficient to consider the case when \( d = 0 \). Suppose that \( b \in \mathcal{C}^{n-1}(B, \pi(p)) \) and \( c = \partial^* b \). In this case we need to prove that \( f(c) \) is homotopic to the identity. Let us consider the bundle

\[
p_1 = p \times \text{id}_{\Delta[1]} : E \times \Delta[1] \to B \times \Delta[1].
\]

Equivalently, \( p_1 \) is induced from \( p \) by the projection \( \text{pr}_B : B \times \Delta[1] \to B \). Clearly, the local system \( \pi(p_1) \) is induced from \( \pi(p) \) by the same projection. Recall the simplicial maps \( i(e) : \Delta[0] \to \Delta[1] \), where \( e = 0 \) or \( 1 \), from the definition of homotopies. These maps lead to the maps

\[
\text{id}_B \times i(e) : B \times \Delta[0] \to B \times \Delta[1].
\]

Let \( B_e \) be the image of \( \text{id}_B \times i(e) \). Similarly, let \( E_e \) be the image of \( \text{id}_E \times i(e) \). Let us identify \( B_0 \) with \( B \) and consider the \( (n - 1) \)-cochain

\[
b_0 \in \mathcal{C}^{n-1}(B \times \Delta[1], \pi(p_1))
\]

equal to \( b \) on \( B_0 = B \) and to \( 0 \) on all \( (n - 1) \)-simplices of \( B \times \Delta[1] \) not in \( B_0 \). Let

\[
c_0 = \partial^* b_0 \quad \text{and}
\]

\[
h = f(c_0) : E \times \Delta[1] \to E \times \Delta[1].
\]

Then \( h \) is a translation of \( p_1 \). Clearly, the map \( E_0 \to E_0 \) induced by \( h \) can be identified with \( f(c) \), and the map \( E_1 \to E_1 \) induced by \( h \) is equal to the identity. Therefore, the composition of \( h \) with the projection \( E \times \Delta[1] \to E \) is a homotopy between \( f(c) \) and the identity. The lemma follows.

Remark. Since \( \mathcal{Z}^n(\Delta[q], \pi_0) = 0 \) if \( q < n \), every translation \( f : E \to E \) is equal to the identity over \( \text{sk}_{n-1} B \). By the same reason the homotopy constructed in Lemma 5.5 is constant over \( \text{sk}_{n-2} B \).

Remark. If \( p \) is the trivial bundle \( B \times K(\pi, n) \to B \), then the translations of \( p \) correspond to maps \( B \to K(\pi, n) \). So, Lemmas 5.3 and 5.5 provide a “twisted” version of the classification of maps \( f : K \to K(\pi, n) \) in terms of cocycles \( z(f) \).
A group acting on $E$. Let $G = \mathcal{C}^{n-1}(B, \pi(p))$ be the group of normalized $(n-1)$-cochains of $B$ with coefficients in the local system $\pi(p)$. By Lemma 5.3 for every $g \in G$ there exists a unique automorphism

$$a(c) = f(\partial^*c) : E \rightarrow E$$

over $B$ such that $D_a(c) = \partial^*c$. By Lemma 5.4 the map $c \rightarrow a(c)$ is a homomorphism. Hence this map defines an action of $G$ on $E$. By Lemma 5.5 every automorphism $a(c)$ is homotopic to the identity. Moreover, by the remark after the proof of Lemma 5.5 the homotopy can be chosen to be constant over $sk_{n-2} B$. Therefore, the group $G$ acts on $E$ by automorphisms homotopic to the identity by homotopies constant over $sk_{n-2} B$.

Free simplices. Let us say that a $q$-simplex $\sigma$ is free in dimension $m$ if the restriction of the simplicial map $i_\sigma : \Delta[q] \rightarrow K$ to $sk_m \Delta[q]$ is an isomorphism onto its image.

5.6. Lemma. Suppose that $\tau, \tau'$ are $q$-simplices of $E$ such that $p(\tau) = p(\tau')$. If $p(\tau)$ is free in dimension $n-1$, then there exists $c \in G$ such that $a(c)(\tau) = \tau'$.

Proof. Let $\sigma = p(\tau)$ and $v$ be the leading vertex of $\sigma$. Let

$$t : \Delta[q] \times K(\pi_v, n) \rightarrow i^*_\sigma E$$

be a special trivialization of the pull-back bundle $i^*_\sigma p$. Then

$$t^{-1}(\tau) = (\iota_q, z) \quad \text{and} \quad t^{-1}(\tau') = (\iota_q, z').$$

for some $q$-simplices $z, z'$ of $K(\pi_v, n)$, i.e. for some $z, z' \in Z^n(\Delta[q], \pi_v)$. Clearly,

$$t(z' - z)(\iota_q, z) = (\iota_q, z').$$

Since the cohomology of $\Delta[q]$ vanish, the cocycle $z' - z$ is the coboundary of some normalized $(n-1)$-cochain $d$. Since $\sigma$ is free in dimension $n-1$, there exists a normalized $(n-1)$-cochain $c$ of $B$ such that $d = i^*_\sigma(c)$ (the values of $c$ on non-degenerate $(n-1)$-simplices not belonging to the image of $i_\sigma$ are arbitrary) and hence

$$z' - z = \delta^*d = i^*_\sigma(\delta^*c).$$

Let $f = a(c) = f(\delta^*c)$. Then $D_f = \delta^*c$ and hence $d(f_\sigma) = z' - z$. It follows that

$$a(c)(\tau) = f(\tau) = \tau'.$$

This completes the proof. ■
6. Unraveling simplicial sets

The unraveling. Let $\Gamma = \Delta[\infty]$. As usual, we will denote by $\Gamma_n$ the set of $n$-simplices of $\Gamma$. The unraveling of a simplicial set $K$ is the dimension-wise product $\Delta K \times \Gamma$. We will denote this $\Delta$-set simply by $K \times \Gamma$. The goal of this section is to prove that the projection $p : K \times \Gamma \to K$ induces isomorphisms in bounded cohomology.

Averaging operators. An averaging operator on $\Gamma_n$ is a bounded linear functional $m_n : B(\Gamma_n) \to \mathbb{R}$, of the norm 1 equal to the identity on constant functions. More precisely, if $f(n) = a$ for all $n \in \mathbb{N}$, then it is required that $m_n(f) = a$. A family of averaging operators $m_n$, where $n \in \mathbb{N}$, is said to be coherent if the operators $m_n$ commute with the adjoints of the face operators $\partial_i : \Gamma_n \to \Gamma_{n-1}$, i.e. if

$$m_n \circ \partial_i^* = m_{n-1}$$

for every $n \geq 1$ and $i \in \mathbb{N}$. Such a family defines a graded map of degree 0 $m_* : B^*(K \times \Gamma) \to B^*(K)$ by averaging cochains over preimages in $K \times \Gamma$ of simplices of $K$. In fact, $m_*$ is a cochain map. See Lemma A.2.1. Clearly, $m_* \circ p^* = \text{id}$.

Banach limits. Given a function $f : \mathbb{N} \to \mathbb{R}$, let $sf$ be the function $\mathbb{N} \to \mathbb{R}$ defined by $sf(n) = f(n+1)$. A Banach limit is a linear functional $l : B(\mathbb{N}) \to \mathbb{R}$ such that its norm is equal to 1, $l(f) = a$ if $f$ is the constant function with the value $a \in \mathbb{R}$, and $l(sf) = l(f)$ for all $f$. It is well known that Banach limits exist. See, for example, [R], Exercise 4 to Chapter 3. Let us fix a Banach limit and denote it by $\text{lim}$. Suppose now that $f(n)$ is a bounded real-valued function of the natural argument $n$ defined only for sufficiently large numbers $n$. If $N$ is sufficiently large, then the function $f_N(n) = f(n+N)$ is defined for all $n \in \mathbb{N}$. Clearly, $f_{N+1} = sf_N$. Therefore we can define $\text{lim} f$ as the common values of $\text{lim} f_N$ for sufficiently large natural numbers $N$.

Suppose now that $f(a, b, ..., z)$ is a bounded real-valued function of several natural variable $a, b, ..., z$ and that $k$ is one of these variables. By fixing values of other variables and applying $\text{lim}$ to the resulting function of $k$ we will get a bounded function of the other variables $a, ..., \hat{k} ..., z$, which we will denote by $\text{lim}_k f(a, ..., \hat{k} ..., z)$. As above, this operation applies even if $f(a, b, ..., z)$ is defined only for sufficiently large $k$. 
6.1. Lemma. Coherent families of averaging operators exist.

Proof. The $n$-simplices of $\Gamma$ can be identified with the sequences $(k_0, k_1, \ldots, k_n) \in \mathbb{N}^{n+1}$ such that $k_0 < k_1 < \ldots < k_n$. Given a bounded function $f : \Gamma_n \to \mathbb{R}$, let $f^{(1)}$ be the function $\Gamma_{n-1} \to \mathbb{R}$ defined by

$$f^{(1)}(k_0, k_1, \ldots, k_{n-1}) = \lim_{k_n} f(k_0, k_1, \ldots, k_n).$$

For $0 \leq m \leq n + 1$ let us define $f^{(m)}$ recursively by $f^{(0)} = f$ and

$$f^{(m+1)} = \left( f^{(m)} \right)^{(1)}.$$

Then $f^{(m)}$ is a function of $n + 1 - m$ natural variables. In particular, $f^{(n+1)}$ is a function of zero variables, i.e. is a constant. Let $m_n(f)$ be this constant. Clearly, each $m_n$ is an averaging operator. We claim that the family of these operators is coherent, i.e. that

$$(\partial^* f)^{(n+2)} = f^{(n+1)}$$

for every $i \in [n + 1]$. By the definition,

$$\partial^*_i f(k_0, k_1, \ldots, k_{n+1}) = f(k_0, \ldots, \widehat{k_i} \ldots, k_{n+1})$$

is a function independent of $k_i$. By consecutively taking limits we see that

$$\left( \partial^*_i f \right)^{(m)}(k_0, k_1, \ldots, k_{n+1-m}) = f^{(m)}(k_0, \ldots, \widehat{k_i} \ldots, k_{n+1-m})$$

for $n + 1 - m \geq i$, i.e. for $m \leq n + 1 - i$. In particular,

$$\left( \partial^*_i f \right)^{(n+1-i)}(k_0, k_1, \ldots, k_i) = f^{(n+1-i)}(k_0, \ldots, k_{i-1}).$$

By taking the limit of the left hand side, which is independent of $k_i$, we see that

$$\left( \partial^*_i f \right)^{(n+2-i)}(k_0, k_1, \ldots, k_{i-1}) = f^{(n+1-i)}(k_0, \ldots, k_{i-1}).$$

Taking the limits $i$ more times shows that the equality (6.1) holds. Therefore our family of averaging operators is indeed coherent. ■

Acyclicity of $\Delta[n] \times \Gamma$. Recall that an $m$-chain of a $\Delta$-set $D$ is a finite formal sum of $m$-simplices of $D$ with coefficients in some abelian group. A vertex of a chain is defined as a vertex of some simplex entering into this sum with non-zero coefficient. If an $m$-chain $c$ of $\Delta[n] \times \Gamma$ is a cycle, then $c$ is a boundary in $\Delta[n] \times \Gamma$. Indeed, since $c$ is a finite sum,
there exists \( m \in \mathbb{N} \) such that for every vertex \((v, k)\) of \( c \) the inequality \( k < m \) holds. Let \( w \) be a vertex of \( \Delta[n] \), and let us consider the cone \( b \) over \( c \) with the apex \((w, m)\). In order to ensure that this cone is indeed a chain of \( \Delta[n] \times \Gamma \) one needs to build the cone by adding the apex as the last vertex of every simplex of \( c \). Then

\[
c = (-1)^{m+1} \partial b.
\]

The sign is caused by adding the apex as the last vertex.

The method of acyclic models. For a simplicial or \( \Delta \)-set \( K \) let \( C_*(K) \) be the complex of chains in \( K \) with coefficients in some abelian group. The method of acyclic models applied to the functors \( K \mapsto C_*(K) \) and \( K \mapsto C_*(K \times \Gamma) \) from simplicial sets to chain complexes implies that \( p_* : C_*(K \times \Gamma) \to C_*(K) \) is a chain homotopy equivalence.

We will adapt the method of acyclic models to prove that \( p^* : B^*(K) \to B^*(K \times \Gamma) \) is a cochain homotopy equivalence. Let \( m_n \) be a coherent family of averaging operators. Since \( m_* \circ p^* = \text{id} \), it is sufficient to prove that \( p^* \circ m_* : B^*(K \times \Gamma) \to B^*(K \times \Gamma) \) is cochain homotopic to the identity.

Some special chains. Recall that \( d(i) : [n-1] \to [n] \) is the unique strictly increasing map not having \( i \) as a value. Let \( \delta_i = d(i)_* : \Delta[n-1] \to \Delta[n] \) be the simplicial map induced by \( d(i) \). We will use the following abbreviated notation for sums:

\[
\sum_i' \cdot = \sum_i (-1)^i \cdot.
\]

For every two simplices \( \tau, \tau' \in \Gamma_n \) we are going to define an \((n+1)\)-chain \( c_n(\tau, \tau') \) of \( \Delta[n] \times \Gamma \) with integer coefficients in such a way that

\[
(6.2) \quad \partial c_n(\tau, \tau') = (t_n, \tau) - (t_n, \tau') - \sum_i' (\delta_i \times \text{id}_\Gamma)_* \left( c_{n-1}(\partial_i \tau, \partial_i \tau') \right).
\]

In addition, we will require that the \( l_1 \)-norm of \( c_n(\tau, \tau') \) (i.e. the sum of the absolute values of the coefficients) can be bounded by constants depending only on \( n \). For \( n = 0 \) the condition (6.2) simplifies to

\[
\partial c_0(\tau, \tau') = (t_0, \tau) - (t_0, \tau').
\]

We will construct such chains using a recursion by \( n \). The chain \((t_0, \tau) - (t_0, \tau')\) has the augmentation 0 and is a boundary in \( \Delta[0] \times \Gamma \) if \( N \geq 2 \). Assuming that \( N \geq 2 \), let us choose a vertex \( v \) of \( \Gamma \) strictly larger that \( \tau, \tau' \) in the natural order (recall that \( \Gamma_0 = \mathbb{N} \)) and take as \( c_0(\tau, \tau') \) the cone with the apex \((t_0, v)\) over the cycle \((t_0, \tau) - (t_0, \tau')\). Then the \( l_1 \)-norm of \( c_n(\tau, \tau') \) is \( \leq 2 \).
Suppose that the chains \( c_m(\tau, \tau') \) are already defined for \( m \leq n - 1 \), the condition (6.2) holds for them, and there are required bounds on the \( l_1 \)-norms. In order to define the chains \( c_n(\tau, \tau') \) we need to verify that the right hand side of (6.2) is a cycle. The boundary of the right hand side is

\[
\partial(t_n, \tau) - \partial(t_n, \tau') - \sum_i \partial(\delta_i \times \text{id}_\Gamma)_* (c_{n-1}(\partial_i \tau, \partial_i \tau')) \\
= \sum_i' (\delta_i \times \text{id}_\Gamma)_* (\delta_i \times \text{id}_\Gamma)_* (c_{n-1}(\partial_i \tau, \partial_i \tau')) \\
= \sum_i' (\delta_i \times \text{id}_\Gamma)_* \left( (t_{n-1}, \partial_i \tau) - (t_{n-1}, \partial_i \tau') - \partial c_{n-1}(\partial_i \tau, \partial_i \tau') \right).
\]

By applying (6.2) with \( n - 1 \) in the role of \( n \) and cancelling two occurrences of

\[
(t_{n-1}, \partial_i \tau) - (t_{n-1}, \partial_i \tau')
\]

we conclude that the boundary of the right hand side of (6.2) is equal to

\[
\sum_i' (\delta_i \times \text{id}_\Gamma)_* \left( \sum_k' (\delta_k \times \text{id}_\Gamma)_* (c_{n-1}(\partial_k \partial_i \tau, \partial_k \partial_i \tau')) \right) \\
= \sum_i' \sum_k' (\delta_i \times \text{id}_\Gamma)_* \circ (\delta_k \times \text{id}_\Gamma)_* (c_{n-1}(\partial_k \partial_i \tau, \partial_k \partial_i \tau')) \\
= \sum_i' \sum_k' (\delta_i \circ \delta_k \times \text{id}_\Gamma)_* (c_{n-1}(\partial_k \partial_i \tau, \partial_k \partial_i \tau')).
\]

As in the proof of the identity \( \partial \circ \partial = 0 \), all summands in the last double sum cancel (recall that \( \sum' \) denotes an alternating sum). It follows that the right hand side of (6.2) is a cycle. Clearly, the \( l_1 \)-norm of the right hand side can be bounded in terms of \( n \) and the \( l_1 \)-norm of \( c_{n-1}(\rho, \rho') \). By the inductive assumption this implies that these norms can bounded in terms of \( n \) only. Hence one can take as \( c_n(\tau, \tau') \) the cone over the right hand side with an appropriate apex. Then the \( l_1 \)-norms of \( c_n(\tau, \tau') \) and of the right hand side are equal and can be bounded in terms of \( n \). This completes the construction of the chains \( c_n(\tau, \tau') \).

**Partial averaging.** We will deal with the functions of several variables such as bounded cochains \( \sigma, \tau, \tau' \rightharpoonup g(\sigma, \tau, \tau') \) and apply the averaging operators to only one of the variables. If, say, \( \tau' \) runs over \( \Gamma_n \), we will denote by

\[
(\sigma, \tau) \rightharpoonup m_n(\tau') g(\sigma, \tau, \tau')
\]

the function of variables \( \sigma, \tau \) resulting from applying \( m_n \) to functions

\[
\tau' \rightharpoonup g(\sigma, \tau, \tau').
\]
With these notations the coherence condition takes the form

\[ m_n(t') g(\sigma, \tau, \partial_i t') = m_{n-1}(\rho) g(\sigma, \tau, \rho), \]

where \( t' \) runs over \( \Gamma_n \) and \( \rho \) runs over \( \Gamma_{n-1} \).

**Constructing cochain homotopies.** Let \( K \) be a simplicial set. Every \( n \)-simplex of \( K \times \Gamma \) has the form \((\sigma, \tau, \tau')\), where \( \sigma \in K_n \) and \( \tau, \tau' \in \Gamma_n \). Clearly,

\[ (\sigma, \tau, \tau') = (i_\sigma \times \text{id}_{\Gamma \times \Gamma})_* (t_n, \tau, \tau'). \]

Let \( k_n : C_n(K \times \Gamma) \to C_n(K \times \Gamma) \) be the unique homomorphism such that

\[ k_n(\sigma, \tau, \tau') = (i_\sigma \times \text{id}_{\Gamma \times \Gamma})_* (c_n(\tau, \tau')). \]

for every \( n \)-simplex \((\sigma, \tau, \tau')\). The condition (6.2) implies that

\[ (6.3) \quad \partial k_n(\sigma, \tau, \tau') = (\sigma, \tau) - (\sigma, \tau') - \sum_i' k_{n-1}(\partial_i \sigma, \partial_i \tau, \partial_i \tau'). \]

The next step is to apply the averaging operators \( m_n \). If \( f \in B^{n+1}(K \times \Gamma) \), then

\[ (\sigma, \tau, \tau') \mapsto f\left(k_n(\sigma, \tau, \tau')\right) \]

is a bounded function because, together with the \( l_1 \)-norm of \( c_n(\tau, \tau') \), the \( l_1 \)-norm of \( k_n(\sigma, \tau, \tau') \) can be bounded in terms of \( n \) only. Let

\[ h_{n+1}(f)(\sigma, \tau) = m_n(t') f\left(k_n(\sigma, \tau, \tau')\right). \]

Then \( h_{n+1}(f) \in B^n(K \times \Gamma) \) and, moreover,

\[ h_{n+1} : B^{n+1}(K \times \Gamma) \to B^n(K \times \Gamma) \]

is a bounded operator.

**6.2. Lemma.** The operators \( h_n \) form a cochain homotopy between \( p^* \circ m^* \) and the identity.

**Proof.** Let \( f \in B^n(K \times \Gamma) \). By applying \( f \) to (6.3) we see that

\[ f\left(\partial k_n(\sigma, \tau, \tau')\right) = f(\sigma, \tau) - f(\sigma, \tau') - \sum_i' f\left(k_{n-1}(\partial_i \sigma, \partial_i \tau, \partial_i \tau')\right), \]
or, equivalently,

$$\partial^* f\left( k_n(\sigma, \tau, \tau') \right) = f(\sigma, \tau) - f(\sigma, \tau') - \sum_i f\left( k_{n-1}(\partial_i \sigma, \partial_i \tau, \partial_i \tau') \right).$$

Next, let us apply $m_n(\tau')$ to the terms of this equality. By the definition of $h_{n+1}$,

$$m_n(\tau') \partial^* f\left( k_n(\sigma, \tau, \tau') \right) = h_{n+1}(\partial^* f)(\sigma, \tau).$$

Since $f(\sigma, \tau)$ does not depend on $\tau'$,

$$m_n(\tau') f(\sigma, \tau) = f(\sigma, \tau).$$

By the definition of $m^*(f)$,

$$m_n(\tau') f(\sigma, \tau') = m^*(f)(\sigma) = p^* \circ m^*(f)(\sigma, \tau).$$

Finally, the coherence of the family $m_n$ implies that

$$m_n(\tau') \sum_i f\left( k_{n-1}(\partial_i \sigma, \partial_i \tau, \partial_i \tau') \right) = \sum_i m_n(\tau') f\left( k_{n-1}(\partial_i \sigma, \partial_i \tau, \partial_i \tau') \right)$$

$$= \sum_i m_{n-1}(\rho) f\left( k_{n-1}(\partial_i \sigma, \partial_i \tau, \rho) \right)$$

$$= \sum_i h_n(f)(\partial_i \sigma, \partial_i \tau) = \partial^*(h_n(f))(\sigma, \tau).$$

By collecting all these observations together, we see that

$$h_{n+1}(\partial^* f) = f - p^* \circ m^*(f) - \partial^*(h_n(f))$$

for every $f \in B^n(K \times \Gamma)$ and hence

$$h_{n+1} \circ \partial^* = \text{id} - p^* \circ m^* - \partial^* \circ h_n,$$

or, equivalently, $\text{id} - p^* \circ m^* = h_{n+1} \circ \partial^* + \partial^* \circ h_n$. ■

6.3. Theorem. The projection $p: K \times \Gamma \rightarrow K$ induces isometric isomorphisms in the bounded cohomology groups.

Proof. Lemma 6.2 together with $m^* \circ p^* = \text{id}$ implies that the induced homomorphisms are isomorphisms. Since the norms of $m^*$ and $p^*$ are $\leq 1$, these induced homomorphisms are isometries. ■
7. Isometric isomorphisms in bounded cohomology

7.1. Theorem. Let $p : E \to B$ be a locally trivial bundle with the fiber $K(\pi, n)$. If $n > 1$, then the map induced by $p$ in bounded cohomology is an isometric isomorphism.

Proof. Let $\Gamma = \Delta[\infty]$. Let us consider the diagram

$$
\begin{array}{ccc}
E & \xrightarrow{p} & E \times \Gamma & \xrightarrow{p \times \text{id}} & E \times \Gamma \\
\downarrow & & \downarrow & & \downarrow \\
B & \leftarrow & B \times \Gamma & \xrightarrow{p \times \text{id}} & B \times \Gamma,
\end{array}
$$

where the left horizontal arrows are projections, and the right horizontal arrows are inclusions. The unravellings $E \times \Gamma$ and $B \times \Gamma$ are only $\Delta$-sets, and the arrows of this diagram are simplicial maps of $\Delta$-sets, except of $p : E \to B$ and $p \times \text{id} : E \times \Gamma \to B \times \Gamma$, which are maps of simplicial sets. Clearly, this diagram is commutative. By Theorem 6.3 the left horizontal arrows induce isometric isomorphisms in bounded cohomology. Therefore, it is sufficient to prove that the simplicial map $q = p \times \text{id} : E \times \Gamma \to B \times \Gamma$ induces isometric isomorphisms in bounded cohomology. Since $\Gamma$ is contractible, the projections $E \times \Gamma \to E$ and $B \times \Gamma \to B$ are homotopy equivalences. In particular, these projections induce isometric isomorphisms in bounded cohomology. It follows that the right horizontal arrows of the diagram induce isometric isomorphisms in bounded cohomology.

Clearly, the simplicial map $q = p \times \text{id} : E \times \Gamma \to B \times \Gamma$ is a locally trivial bundle. The main part of the proof is an application of the theory developed in Section 5 to $q$ in the role of $p$. Let $G = \mathcal{C}^{n-1}(B \times \Gamma, \pi(q))$ be the group of normalized $(n-1)$-cochains of $B \times \Gamma$ with coefficients in the local system $\pi(q)$. The group $G$ acts on $E \times \Gamma$ by homotopic to the identity automorphisms over $B \times \Gamma$. The group $G$ is abelian and hence is amenable.

Clearly, a simplex of $E \times \Gamma$ belongs to $E \times \Gamma$ if and only if its image in $B \times \Gamma$ belongs to $B \times \Gamma$. It follows that the action of $G$ leaves the $\Delta$-subset $E \times \Gamma$ invariant. Obviously, every simplex of $\Gamma$ is free in every dimension. It follows that every simplex of $B \times \Gamma$ is free in every dimension. Hence Lemma 5.6 implies that $E \times \Gamma/G = B \times \Gamma$ and

$$
q^* : B^*(B \times \Gamma) \to B^*(E \times \Gamma)
$$

induces isomorphism from $B^*(B \times \Gamma)$ to the space $G$-invariant cochains in $B^*(E \times \Gamma)$. Let

$$
q^{**} : \widehat{H}^*(B \times \Gamma) \to \widehat{H}^*(E \times \Gamma)
$$
be the map induced by \( q^* \). Let us prove that \( q^{**} \) is surjective. Let \( \gamma \in B^m(E \times \Gamma) \) be a cocycle. Since \( E \times \Gamma \to E \times \Gamma \) induces isomorphisms in bounded cohomology, there exists a cocycle \( c \in B^m(E \times \Gamma) \) such that the restriction of \( c \) to \( E \times \Gamma \) is cohomologous to \( \gamma \). Since \( G \) is amenable and acts on \( E \) by automorphisms homotopic to the identity, the cocycle \( c \) is cohomologous to a \( G \)-invariant bounded cocycle \( b \). See Lemma A.2.3. Let \( \beta \) be the restriction of \( b \) to \( E \times \Gamma \). Then \( \beta \) is cohomologous to \( \gamma \) and is \( G \)-invariant. Since \( \beta \) is \( G \)-invariant, \( \beta = q^*(\alpha) \) for some \( \alpha \in B^m(B \times \Gamma) \). It follows that \( q^{**} \) is surjective.

Let us prove now that \( q^{**} \) is injective. Since \( G \) is amenable, there exists a \( G \)-invariant mean \( \mu : B(G) \to \mathbb{R} \). For each \( m \in \mathbb{N} \) let us define a map

\[
\mu_* : B^m(E \times \Gamma) \to B^m(B \times \Gamma)
\]

as follows. Let \( c \in B^m(E \times \Gamma) \) and \( \sigma \) is an \( m \)-simplex of \( B \times \Gamma \). Let us choose an \( m \)-simplex \( \sigma' \) of \( E \times \Gamma \) such that \( p(\sigma') = \sigma \) and consider the function \( g \mapsto c(g \cdot \sigma') \) on \( G \). Let the value of the cochain \( \mu_*(c) \) on \( \sigma \) be equal to the value of \( \mu \) on this function. Since \( \mu \) is \( G \)-invariant, this value is independent on the choice of \( \sigma' \). Therefore \( \mu_* \) is well-defined. Clearly, the composition \( \mu_* \circ q^* \) is equal to the identity. Since \( G \) acts by automorphisms, \( \mu_* \) commutes with the duals \( \partial_i^* \) of the face operators \( \partial_i \) and hence with the coboundary operator \( \partial^* \). Hence \( \mu_* \) leads to maps

\[
\mu_{**} : \hat{H}^m(E \times \Gamma) \to \hat{H}^m(B \times \Gamma).
\]

Since the composition \( \mu_* \circ q^* \) is equal to the identity, the composition \( \mu_{**} \circ q^{**} \) is also equal to the identity. It follows that \( q^{**} \) is injective.

We see that \( q^{**} \) is an isomorphism. It remains to prove that \( q^{**} \) is an isometric isomorphism. Since \( q^{**} \) is induced by a simplicial map, the norm of \( q^{**} \) is \( \leq 1 \). On the other hand, the norm of \( \mu \) is \( \leq 1 \) and hence the norms of \( \mu_* \) and \( \mu_{**} \) are also \( \leq 1 \). Since \( q^{**} \) is an isomorphism and \( \mu_{**} \circ q^{**} \) is the identity, \( \mu_{**} \) is the inverse of \( q^{**} \). So, the norms of \( q^{**} \) and of its inverse are \( \leq 1 \). It follows that \( q^{**} \) is an isometry. \( \blacksquare \)

**Remark.** One can prove that \( q^{**} \) is an isometry in a different way. Let us return to the proof of the surjectivity of \( q^{**} \). Since \( E \times \Gamma \to E \times \Gamma \) induces isometric isomorphisms in bounded cohomology, one can choose \( c \) in such a way that \( \| \gamma \| \geq \| c \| \). By Lemma A.2.3 one can choose \( b \) in such a way that \( \| c \| \geq \| b \| \). Clearly, \( \| b \| \geq \| \hat{\beta} \| \) and \( \| \hat{\beta} \| = \| \alpha \| \). It follows that \( \| \gamma \| \geq \| \alpha \| \). Therefore for every cohomology class \( \gamma \in \hat{H}^m(E \times \Gamma) \) there exists a cohomology class \( \alpha \in \hat{H}^m(B \times \Gamma) \) such that \( \gamma = q^{**}(\alpha) \) and \( \| \alpha \| \leq \| \gamma \| \). Since, at the same time, \( \| q^{**}(\alpha) \| \leq \| \alpha \| \), the isomorphism \( q^{**} \) is an isometric isomorphism.

**7.2. Theorem.** Let \( \pi \) be a discrete group and \( \kappa \subset \pi \) be a normal amenable subgroup of \( \pi \). Let \( p : \pi \to \pi/\kappa \) be the quotient homomorphism. Then \( Bp : B\pi \to B(\pi/\kappa) \) induces isometric isomorphism in bounded cohomology.
Proof. The proof is completely similar to the proof of Theorem 7.1, with the map

\[ Bp : B\pi \rightarrow B(\pi/\kappa) \]

playing the role of \( p : E \rightarrow B \). As we saw in Section 4, the group \( G = C_0(N, \kappa) \) acts on \( B\pi \times \Gamma \) and \( B\pi \times \Gamma/G = B(\pi/\kappa) \times \Gamma \). See (4.3). While, to the best of author's knowledge, the group \( C_0(V, \kappa) \) is not known to be amenable, the group \( C_0(V, \kappa) \) is a direct sum of copies of \( \kappa \) and hence is amenable. Hence one can argue as in the proof of Theorem 7.1 and conclude that \( Bp \) induces isometric isomorphism in bounded cohomology. ■

The fundamental group. Let \( K \) be a connected Kan simplicial set. Suppose that \( K \) has only one vertex, which we will denote by \( v \). Let us interpret a 1-simplex \( \sigma \in K_1 \) as a loop based at \( v \). The Kan extension property implies that for every two 1-simplices \( \rho, \sigma \) there exists a 2-simplex \( \omega \) such that \( \rho = \partial_2 \omega \) and \( \sigma = \partial_0 \omega \). One can easily check that up to homotopy \( \tau = \partial_1 \omega \) does not depend on the choice of \( \omega \), and, moreover, up to homotopy \( \tau \) depends only on the homotopy classes of \( \sigma, \tau \). One can take the homotopy class of \( \tau \) as the product \( r \cdot s \) of the homotopy classes \( r, s \) of \( \rho, \sigma \) respectively. The set of homotopy classes of 1-simplices together with this product is the fundamental group \( \pi_1(K, v) \) of \( K \). If, in addition, \( K \) is minimal, then every two homotopic 1-simplices are equal. In this case \( \pi_1(K, v) \) can be identified with \( K_1 \) as a set.

7.3. Lemma. Suppose that \( K \) is a connected minimal Kan simplicial set. Then \( K(1) \) is canonically isomorphic to \( B\pi_1(K, v) \), where \( v \) is the unique vertex of \( K \).

Proof. For \( i, j, n \in N \) such that \( 0 \leq i < j \leq n \) let \( \theta_{i,j} : [1] \rightarrow [n] \) be the map

\[ \theta_{i,j} : 0 \rightarrow i, \quad 1 \rightarrow j. \]

Suppose that \( \rho_1, \rho_2, ..., \rho_n \) are 1-simplices of \( K \). If

\[ \theta_{i-1,i}^* (\sigma) = \rho_i \]

for some \( n \)-simplex \( \sigma \) of \( K \) and every \( i \) between 1 and \( n \), then

\[ \theta_{i,j}^* (\sigma) = \rho_i \cdot ... \cdot \rho_j \]

for every \( i < j \). This follows from the definition of the product together with an induction by \( j - i \). In turn, this implies that the restriction of \( i_\sigma \) to \( \text{sk}_1 \Delta[n] \) is uniquely determined by \( \rho_1, \rho_2, ..., \rho_n \). On the other hand, Kan extension property implies that such a simplex \( \sigma \) exists for every \( n \)-tuple \( \rho_1, \rho_2, ..., \rho_n \). It follows that one can identify \( n \)-simplices of \( K(1) \) with sequences \( (\rho_1, \rho_2, ..., \rho_n) \) of elements of \( \pi_1(K, v) \), i.e. with \( n \)-simplices of \( B\pi_1(K, v) \). A direct check shows that this identification respects the boundary and degeneracy operators. The lemma follows. ■
7.4. Theorem. Let $K$ be a connected Kan simplicial set and $f : K \to B\pi_1(K, v)$, where $v$ is a vertex of $K$, be a simplicial map inducing isomorphism of fundamental groups. Then $f$ induces an isomorphism in bounded cohomology.

Proof. The proof is based on the theory of Postnikov systems. See Section 3 for a review of the definitions and the theorems used in this proof.

Let $\pi_1$ be the fundamental group of $K$. Since $B\pi_1$ is a Kan simplicial set, every two map $K \to B\pi_1$ inducing isomorphism of the fundamental groups are homotopic. Hence it is sufficient to prove to prove the theorem for one such map. Let $M$ be a minimal Kan simplicial subset of $K$ which is a strong deformation retract of $K$. Since $M$ is minimal and connected, $M$ has only one vertex, which we denote by $v$. Let $M(0), M(1), \ldots, M(n), \ldots$ and the maps $p_n$ and $p_{m,n}$ be the Postnikov system of $M$. Then every map

$$p_{n,n-1} : M(n) \to M(n-1)$$

is a locally trivial bundle with the fiber $K(\pi_n, n)$, where $\pi_n = \pi_n(M, v)$ is the $n$th homotopy group of $M$. If $n > 1$, then $p_{n,n-1}$ induces isometric isomorphism in bounded cohomology. It follows that for every $n > 1$ the map

$$p_{n,1} : M(n) \to M(1)$$

induces isometric isomorphism in bounded cohomology. On the other hand, for $n \geq m$ the $m$th skeletons of $M$ and $M(n)$ are the same by the very definition of $M(n)$. By the definition, the bounded cohomology group $\hat{H}^m(M)$ depends only on the $(m+1)$th skeleton $\text{sk}_{m+1}M$ of $M$. It follows that the map

$$p_1 : M \to M(1)$$

induces isometric isomorphism in bounded cohomology. By Lemma 7.3 the simplicial set $M(1)$ is canonically isomorphic to $B\pi_1$. Moreover, the description of fundamental groups preceding Lemma 7.3 shows that $p_1 : M \to M(1)$ induces isomorphism of fundamental groups. If $r : K \to M$ is a strong deformation retraction, then $r$ induces isomorphism of fundamental groups and isometric isomorphism in bounded cohomology. It follows that

$$p_1 \circ r : K \to M(1) = B\pi_1$$

also has this property. This proves the theorem for $f = p_1 \circ r$. As was pointed out above, any special case of the theorem implies the general one. This completes the proof. ■

7.5. Corollary. Let $K, L$ be connected Kan simplicial sets. If $f : K \to L$ is a simplicial map inducing isomorphism of fundamental groups, then $f$ induces isomorphism in bounded cohomology. ■
7.6. Theorem. Let $K, L$ be connected Kan simplicial sets and let $v$ be a vertex of $K$. Let $f : K \to L$ be a simplicial map. If $f_* : \pi_1(K, v) \to \pi_1(L, f(v))$ is surjective and has amenable kernel, then $f$ induces an isometric isomorphism in bounded cohomology.

Proof. In order not to clutter the notations, we will not mention the base points $v, f(v)$ anymore. Let us consider the diagram

$$
\begin{array}{ccc}
K & \xrightarrow{f} & L \\
\downarrow^{p_K} & & \downarrow^{p_L} \\
B\pi_1(K) & \xrightarrow{Bf_*} & B\pi_1(L),
\end{array}
$$

where $p_K, p_L$ are some maps inducing isomorphisms of fundamental groups. By Theorem 7.2 the map $Bf_*$ induces isometric isomorphism in bounded cohomology. By Theorem 7.4 the maps $p_K, p_L$ induce isometric isomorphisms in bounded cohomology. Since the above diagram is commutative up to homotopy, it follows that $f$ induces isometric isomorphism in bounded cohomology. ■
A.1. The constructions of Milnor and Segal

A.1.1. Lemma. The $\Delta$-set $\mathcal{B} \pi$ is canonically isomorphic to the product $B \pi \times \Delta[\infty]$.

Proof. To begin with, we will give an explicit description of simplices of $\mathcal{B} \pi$. The $n$-simplices of $\mathcal{E} \pi$ can be identified with pairs of sequences $$(g_0, g_1, \ldots, g_n) \in \pi^{n+1}, \quad (k_0, k_1, \ldots, k_n) \in \mathbb{N}^{n+1},$$ such that $k_0 < k_1 < \ldots < k_n$, and $g \in \pi$ acts by the rules

$$g \cdot (g_0, g_1, \ldots, g_n) = (gg_0, gg_1, \ldots, gg_n) \quad \text{and}$$

$$g \cdot (k_0, k_1, \ldots, k_n) = (k_0, k_1, \ldots, k_n).$$

In order to give a direct description of simplices of $\mathcal{B} \pi$, let us set use the bar notations $g_0 [g_1 | g_2 | \ldots | g_n] = (g_0, g_0 g_1, \ldots, g_0 g_1 \ldots g_n)$.

In these notations the action of $\pi$ takes the form

$$g \cdot (g_0 [g_1 | g_2 | \ldots | g_n]) = gg_0 [g_1 | g_2 | \ldots | g_n].$$

Therefore $n$-simplices of $\mathcal{B} \pi$ can be identified with pairs of sequences $$(g_1 | g_2 | \ldots | g_n) \in \pi^{n}, \quad (k_0, k_1, \ldots, k_n) \in \mathbb{N}^{n+1},$$ such that $k_0 < k_1 < \ldots < k_n$.

The boundary operators act independently on these sequences. Namely, the action of the boundary operator $\partial_i$ on the sequences $(k_0, k_1, \ldots, k_n)$ is the same as in $\Delta[\infty]$, and the action on the sequences $[g_1 | g_2 | \ldots | g_n]$ is given by the rules

$$\partial_0 [g_1 | g_2 | \ldots | g_n] = [g_2 | g_3 | \ldots | g_n],$$

$$\partial_n [g_1 | g_2 | \ldots | g_n] = [g_1 | g_2 | \ldots | g_{n-1}], \quad \text{and}$$

$$\partial_i [g_1 | g_2 | \ldots | g_n] = [g_1 | \ldots | g_i g_{i+1} | \ldots | g_n] \quad \text{for} \quad 0 < i < n.$$
Unravelings of classifying spaces of categories. For a category $\mathcal{C}$ let $\mathcal{C}_n$ be the subcategory of $\mathcal{C} \times n$ obtained by deleting all morphisms of the form $(c, n) \rightarrow (c', n)$ where $c, c'$ are objects of $\mathcal{C}$ and $n \in \mathbb{N}$, except identity morphisms. This construction is due to Segal [S], who called $\mathcal{C}_n$ the unravelling of $\mathcal{C}$ over the ordered set $\mathbb{N}$ and pointed out that for a group $\pi$ the geometric realizations of $B\pi$ and $B\pi_n$ are homeomorphic.

This result can be interpreted in terms of simplicial sets and extended to arbitrary categories. Namely, Lemma A.1.1 suggests that the $\Delta$-set $BC = BC \times \Delta[\infty]$ is an analogue of $B\pi$. In contrast with the case groups, in general $BC$ is not arising from a simplicial complex. It turns out that the simplicial set $\Delta BC$ is isomorphic to $BC_n$. Before proving this, it is convenient to introduce the notion of the core of a simplicial set.

The core of a simplicial set. Following Rourke and Sanderson [RS], let us define the core of a simplicial set $K$ as the $\Delta$-subset core$(K)$ of $K$ consisting of simplices of the form $\Theta^*(\sigma)$ with non-degenerate $\sigma$ and strictly increasing $\Theta$. The simplicial set $K$ is said to have non-degenerate core if non-degenerate simplices of $K$ form a $\Delta$-subset of $K$. Clearly, this $\Delta$-subset is equal to core$(K)$. There is a canonical simplicial map $\Theta: \Delta core(K) \rightarrow K$ defined by $\Theta(\sigma, \rho) = \rho^*(\sigma)$.

A.1.2. Lemma. If $K$ has non-degenerate core, then $\Theta$ is an isomorphism.

Proof. Every simplex $\sigma$ of a simplicial set admits a unique presentation $\sigma = \Theta^*(\tau)$ with non-degenerate $\tau$ and surjective $\Theta$. See Lemma A.2.4. This implies that $\Theta$ is surjective. If $K$ has non-degenerate core and $\Theta(\sigma_1, \rho_1) = \Theta(\sigma_2, \rho_2)$, then $\rho_1^*(\sigma_1) = \rho_2^*(\sigma_2)$ and $\sigma_1, \sigma_2$ are non-degenerate. Therefore the uniqueness part of Lemma A.2.4 implies that $(\sigma_1, \rho_1) = (\sigma_2, \rho_2)$. ■

A.1.3. Theorem. The simplicial set $\Delta BC$ is isomorphic to $BC_n$.

Proof. By restricting the projection $\mathcal{C} \times n \rightarrow n$ to the subcategory $\mathcal{C}_n$ we get a functor $p: \mathcal{C}_n \rightarrow n$. This functor induces a simplicial map

$$Bp: B\mathcal{C}_n \rightarrow Bn = \Delta[\infty].$$

By the definition of the category $\mathcal{C}_n$ a morphism $f$ of this category is an identity morphism if and only if $p(f)$ is an identity morphism. It follows that a simplex $\sigma$ of $B\mathcal{C}_n$ is non-degenerate if and only if $Bp(\sigma)$ is non-degenerate. This implies that

$$\text{core}(B\mathcal{C}_n) = B\mathcal{C} \times \text{core} \Delta[\infty] = B\mathcal{C} \times \Delta[\infty] = \beta\mathcal{C},$$

where $B\mathcal{C}$ is considered as a $\Delta$-set. Also, since $\Delta[\infty]$ has non-degenerate core, this implies that $B\mathcal{C}_n$ has non-degenerate core. Therefore $B\mathcal{C}_n = \Delta\text{core}(B\mathcal{C}_n) = \Delta\beta\mathcal{C}$. ■
A.2. Few technical lemmas

A.2.1. Lemma. If the averaging maps \( m_n \), \( n \in \mathbb{N} \) form a coherent family, then \( m_* \) is a cochain map.

Proof. For an \( n \)-cochain \( f \in B^n(K \times \Gamma) \) and an \( n \)-simplex \( \sigma \in K_n \) let \( f_\sigma : \Gamma_n \rightarrow \mathbb{R} \) be defined by \( f_\sigma(\tau) = f(\sigma, \tau) \). Then \( m_*(f)(\sigma) = m_n(f_\sigma) \).

Suppose that \( f \in B^n(K \times \Gamma) \) and \( \rho \in K_{n+1} \). Then

\[
\partial^*(m_*(f))(\rho) = \sum_{i=0}^{n+1} (-1)^i m_*(f)(\partial_i \rho)
\]

\[
= \sum_{i=0}^{n+1} (-1)^i m_n(f_\partial_i \rho)
\]

\[
= \sum_{i=0}^{n+1} (-1)^i m_{n+1}(\partial^*_i(f_\partial_i \rho))
\]

because \( m_n = m_{n+1} \circ \partial_i \). If \( \tau \in \Gamma_{n+1} \), then

\[
\partial^*_i(f_\partial_i \rho)(\tau) = f_\partial_i(\rho, \partial_i \tau)
\]

\[
= f_\partial_i(\rho, \tau)
\]

\[
= \partial^*_i(f)(\rho, \tau) = \partial^*_i(f)_\rho(\tau)
\]

It follows that \( \partial^*_i(f_\partial_i \rho) = \partial^*_i(f)_\rho \) and hence

\[
\partial^*(m_*(f))(\rho) = \sum_{i=0}^{n} (-1)^i m_{n+1}(\partial^*_i(f)_\rho)
\]

Since the maps \( m_{n+1} \) and \( h \mapsto h_\rho \) are linear, it follows that

\[
\partial^*(m_*(f))(\rho) = m_{n+1}\left( \sum_{i=0}^{n+1} (-1)^i \partial^*_i(f)_\rho \right)
\]

\[
= m_{n+1}(\partial^* f)(\rho)
\]

and hence \( \partial^*(m_*(f)) = m_*(\partial^* f) \). ■
A.2.2. Lemma. Suppose that $n \geq 1$. A normalized $n$-cochain $c : \pi \rightarrow \pi$ of $K(\pi, n)$ is a cocycle if and only if $c$ is a homomorphism $\pi \rightarrow \pi$.

Proof. An $n$-cochain $u$ of $\Delta^{n+1}$ is determined by its values $u_i = u(\delta_i \iota_{n+1})$ on the non-degenerate $n$-simplices of $\Delta^{n+1}$. Therefore, one can identify $u$ with the $(n+2)$-tuple $(u_0, u_1, \ldots, u_{n+1})$ of elements of $\pi$. The boundary operators $\delta_i$ are given by the restrictions to faces, i.e.

$$\delta_i (u_0, u_1, \ldots, u_{n+1}) = u_i$$

Suppose that $\pi$ is abelian. Then $u$ is cocycle, i.e. belongs to $K(\pi, n)_{n+1}$, if and only if

$$\sum_{i=0}^{n+1} (-1)^i u_i = 0.$$ 

An $n$-cochain $c : \pi \rightarrow \pi$ is a cocycle if and only if $\delta^* c(u) = 0$ for every simplex $u \in K(\pi, n)_{n+1}$, i.e. if and only if the last equality implies

$$\sum_{i=0}^{n+1} (-1)^i c(u_i) = 0$$

for every $(n+2)$-tuple $u$. Clearly, this is the case when $c$ is a homomorphism. Conversely, if $c$ is a cocycle, then the last equality for the $(n+2)$-tuples

$$(u_0, u_1, \ldots, u_{n+1}) = (v, v+w, w, 0, \ldots, 0),$$

where $v, w \in \pi$, together with the fact that $c(0) = 0$ implies that

$$c(v) - c(v+w) + c(w) = 0$$

for every $v, w \in \pi$. This proves that $c$ is a homomorphism when $\pi$ is abelian.

If $\pi$ is not abelian, then $n = 1$ and a triple $u = (u_0, u_1, u_2)$ is a cocycle if and only if $u_1 = u_2 \cdot u_0$. It follows that $c$ is a cocycle if and only if $c(u_2 \cdot u_0) = c(u_2) \cdot c(u_0)$ for every pair $u_2, u_0 \in \pi$, i.e. if and only if $c$ is a homomorphism. In addition, we see that when $n = 1$, every cocycle is automatically normalized. ■

A.2.3. Lemma. Suppose that $K$ is a simplicial set and $G$ is an amenable group acting on $K$ on the left by automorphisms homotopic to the identity. Then every bounded cocycle $c$ of $K$ is boundedly cohomologous to a $G$-invariant bounded cocycle with the norm $\leq \| c \|$.
Proof. We will denote the action by \((g, \sigma) \mapsto g \cdot \sigma\), where \(g \in G\) and \(\sigma\) is a simplex of \(K\). Let \(B(G)\) be the space of bounded real-valued functions on \(G\). For \(g \in G\) and \(f \in B(G)\) let \(g \cdot f\) be the function \(h \mapsto f(hg)\). This defines an action of \(G\) on \(B(G)\).

Since \(G\) is amenable, there exists a \(G\)-invariant mean on \(B(G)\), i.e. a linear functional \(\mu : B(G) \to \mathbb{R}\) such that the norm of \(\mu\) is \(\leq 1\), \(\mu\) takes a constant function to its value, and \(\mu(g \cdot f) = \mu(f)\) for every \(g \in G\) and \(f \in B(G)\).

For \(g \in G\) let \(a(g) : K \to K\) be the automorphism defined by \(g\). Since \(a(g)\) is homotopic to the identity, there exists a cochain homotopy between \(a(g)^* : B^*(K) \to B^*(K)\) and the identity. In other words, for each \(m > 0\) a homomorphism

\[
k_m(g) : B^m(K) \to B^{m-1}(K)
\]

is defined, and

\[
a(g)^*(c) - c = k_{m+1}(g) \circ \partial^*(c) + \partial^* \circ k_m(g)(c)
\]

for every \(c \in B^m(K)\). Suppose that \(c\) is a cocycle. Then this identity simplifies to

\[
a(g)^*(c) - c = \partial^* \circ k_m(g)(c).
\]

By applying this equality to an \(m\)-simplex \(\sigma\) of \(K\) using the definition of \(\partial^*\), we get

\[
(7.1) \quad a(g)^*(c)(\sigma) - c(\sigma) = k_m(g)(c)(\partial \sigma).
\]

We would like to consider all terms of this equality as functions of \(g\) and apply \(\mu\) to them.

To begin with, let \(\gamma(\sigma)\) be the result of applying \(\mu\) to the function

\[
g \mapsto a(g)^*(c)(\sigma) = c(g \cdot \sigma).
\]

The map \(\gamma : \sigma \mapsto \gamma(\sigma)\) is a bounded \(m\)-cochain of \(K\) and \(\|\gamma\| \leq \|c\|\). Since \(\mu\) is \(G\)-invariant, \(\gamma\) is also \(G\)-invariant. Next, the result of applying \(\mu\) to the constant map \(g \mapsto c(\sigma)\) is \(c(\sigma)\). Let \(\tau\) be an \((m-1)\)-simplex of \(K\) and consider the function

\[
g \mapsto k_m(g)(c)(\tau).
\]

Let \(\kappa_m(c)(\tau) \in \mathbb{R}\) be the result of applying \(\mu\) to this function. The map

\[
\kappa_m(c) : \tau \mapsto \kappa_m(c)(\tau)
\]

is a bounded \((m-1)\)-cochain of \(K\). i.e. \(\kappa_m(c) \in B^{m-1}(K)\). In terms of \(\gamma\) and \(\kappa_m(c)\)
the result of applying $\mu$ to (7.1) can be written as follows:

$$\gamma(\sigma) - c(\sigma) = \kappa_m(c)(\partial \sigma).$$

Therefore $\gamma - c = \partial^* \kappa_m(c)$. The lemma follows. ■

A.2.4. Lemma. Every $n$-simplex $\sigma$ of a simplicial set $K$ admits a unique presentation of the form $\sigma = \theta^*(\tau)$ with a surjective non-decreasing map $\theta$ and a non-degenerate simplex $\tau$.

Proof. This is a well known lemma of Eilenberg and Zilber [EZ]. See [EZ], (8.3).

Let us choose among all presentations $\sigma = \theta^*(\tau)$ with surjective $\theta: [n] \rightarrow [m]$ and an $m$-simplex $\tau$ some presentation with minimal possible $m$. Clearly, the minimality of $m$ implies that $\tau$ is non-degenerate. This proves the existence. Suppose that also $\sigma = \eta^*(\rho)$, where $\eta: [n] \rightarrow [k]$ is surjective and $\rho$ is a $k$-simplex. Since $\theta, \eta$ are surjective non-decreasing maps, there exist strictly increasing maps $\alpha: [m] \rightarrow [n]$ and $\beta: [k] \rightarrow [n]$ such that $\theta \circ \alpha$ and $\eta \circ \beta$ are the identity maps. Then

$$(\eta \circ \alpha)^*(\rho) = \alpha^*(\eta^*(\rho)) = \alpha^*(\theta^*(\tau)) = (\theta \circ \alpha)^*(\tau) = \tau.$$ 

Similarly, $(\theta \circ \beta)^*(\tau) = \rho$. Since $\tau$ and $\rho$ are both non-degenerate, both $\eta \circ \alpha$ and $\theta \circ \beta$ are strictly injective. It follows that $m = k$ and both $\eta \circ \alpha$ and $\theta \circ \beta$ are equal to the identity. In turn, this implies that $\tau = \rho$. Suppose that $\theta(i) \neq \eta(i)$ for some $i \in [n]$. One can choose the map $\alpha$ in such a way that $\alpha(\theta(i)) = i$. Then

$$(\eta \circ \alpha)(\theta(i)) = \eta \circ \alpha \circ \theta(i) = \eta(i) \neq \theta(i),$$

contrary to $\eta \circ \alpha$ being equal to the identity. Hence $\theta = \eta$. The uniqueness follows. ■
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https://nikolaivivanov.com

E-mail: nikolai.v.ivanov@icloud.com