Superstring in the \textit{pp}-Wave Background with RR Flux as a Conformal Field Theory

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We provide a concise description of our recent work on the exact conformal field theory (CFT) formulation of the superstring in the \textit{pp}-wave background with Ramond-Ramond (RR) flux, using the Green-Schwarz formalism in the semi-light-cone conformal gauge. Due to the presence of the RR flux, the left- and the right-moving degrees of freedom on the worldsheet are inherently coupled and this makes the canonical analysis intractable even at the classical level. To overcome this difficulty, we develop a phase-space formulation which does not make use of the equations of motion, and construct two independent sets of Virasoro generators classically. Upon quantization, due to the presence of the RR flux, the Virasoro algebra generically develops quantum operator anomalies and only with a judicious choice of normal ordering prescription they cancel between the bosonic and the fermionic contributions. With such properly defined quantum Virasoro generators, one can construct the BRST operators and demonstrate that the BRST cohomology analysis reproduces the physical spectrum previously obtained in the light-cone gauge.

\section{Introduction}

Since its inception about 10 years ago, an impressive collection of “evidence” has been accumulated for the AdS/CFT correspondence,\cite{1,2,3} one of the most profound structures in string theory. On the CFT side, extremely detailed analyses have recently become possible, based on the powerful assumption of “integrability”\cite{4,5,6,7} as well as on the state of the art perturbative techniques.\cite{8,9,10} On the string side, we have also acquired many interesting results, but most of them so far are classical,\cite{11,12,13,14} i.e. based on the classical solutions of either the supergravity or the string sigma model. Understanding of the stringy aspects has been slow due to the difficulty of solving the string theory in the highly curved background with a large RR flux.

In any case, it is fair to say that, despite the tremendous efforts having been made, the understanding of the fundamental mechanism of this remarkable correspondence remains as the central problem.

As we know, although the AdS/CFT correspondence contains some aspects of the conventional open/closed string duality, there are crucial differences:

\begin{itemize}
  \item It is a strong-weak duality, not the usual perturbative duality.
  \item It is a holographic duality. It is not relating the closed and the open strings both in the bulk.
  \item In contrast to the usual open-closed duality, there is a regime where both the open and the closed channel descriptions are dominated by the massless multiplets only, for example by the super-Yang-Mills multiplet and the supergravity multiplet.
\end{itemize}

These aspects must all be related, but let us focus here on the strong/weak
nature of the duality. The familiar fundamental relation expressing this property is

\[ g_{YM}^2 N = 4\pi g_s N = \frac{R^2}{\alpha'}^2. \] (1.1)

The first equality expresses the usual perturbative relation between the open and the closed string couplings. It is the second equality which relates the weak string coupling to the strong sigma-model coupling and vice versa. Let us recall the origin of this relation. The bosonic part of the type IIB supergravity action is of the form

\[ S = \int d^{10}x \sqrt{-g} \left( g_s^{-2} l_s^{-8} R - \frac{2}{5!} F_5^2 \right). \] (1.2)

We know that this admits the D3-brane solution due to the balance between the curvature \( R \) and the RR \( F_5 \) flux. In the near horizon limit, this is expressed as

\[ g_s^{-2} \frac{1}{l_s^8 R^2} \sim \left( \frac{N R^5}{\beta} \right)^2, \] (1.3)

where \( R \) is the common radius of \( AdS_5 \) and \( S^5 \) spaces. Upon rearrangement this immediately gives (1.1). This clearly shows that the presence of the large RR flux is indispensable for the strong/weak duality. It is acting as a kind of “anti-gravity” preventing the spacetime from collapsing.

Unfortunately, the treatment of the RR flux (and the associated curved background) is precisely the largest obstacle which has been hampering the progress on the string side of the AdS/CFT correspondence. To this date, the only such background in which the string theory has been solved, to a certain extent, is the pp-wave background.\(^{15)–17)\) By adopting the light-cone (LC) gauge in the Green-Schwarz formalism,\(^{18)–20)\) the exact LC energy spectrum was found\(^{21)\) to be that of free massive bosons and fermions of the form \( E_n = \sqrt{\mu^2 + \left( n/\alpha' p^+ \right)^2} \), where \( \mu \) is the “mass parameter” characterizing the curvature and the RR flux and \( p^+ \) is the momentum in the light-cone direction. This fact was exploited by Berenstein et al. (BMN)\(^{22)\) to initiate a detailed comparison of the string spectrum and that of the anomalous dimensions of the corresponding gauge-invariant operators in super-Yang-Mills theory (see Refs. 23 and 24 for reviews). The interactions among such stringy modes, however, have been understood only partially. The general form of the 3-point vertex has been studied in the framework of the light-cone string field theory,\(^{25)–34)\) while the higher point functions have not been obtained. Thus, there is still a lot to be learned from the string theory in this simple yet prototypical background in order to unravel, in particular, the role of the large RR flux in the AdS/CFT correspondence.

Now one of the major factors preventing further developments of this theory is the lack of conformal invariance in the light-cone gauge treatment. As it is a string theory, it should be possible to formulate it as a conformal field theory (CFT), which was so powerful in the case of the flat background. More explicitly, let us list some of the concrete motivations for developing CFT description:

- It would be extremely interesting to understand how a (left-right coupled) “massive theory” can be understood as a CFT, especially since the string in the \( AdS_5 \times S^5 \) also has such an inherent left-right coupling.
• Even at the classical level, the Virasoro algebra structure for the superstring in the \( pp \)-wave background with RR flux has not been discussed.\(^{a})\n• One wishes to be able to compute the correlation functions using a worldsheet description. Once one can do this exactly in \( \alpha' \), we will obtain the corresponding information in the gauge theory to all loop orders.
• It should be useful in understanding the modular invariance property. In the LC gauge, the modular \( S \) transformation \( (\tau \leftrightarrow \sigma) \) alters the gauge condition itself and the modular invariance is not seen directly).\(^{36,37})\n• It should serve as a step towards developing a fully covariant pure spinor formalism\(^{38–40})\) for the \( pp \)-wave background in operator formulation.

In the rest of this article, we will give a concise description of our recent work\(^{41})\) on such a CFT description of the plane-wave string theory. In §2, we will begin with the attempt at classical canonical analysis in (conformally-invariant) semi-light-cone gauge\(^{42–44})\) and describe the difficulty one encounters in such an analysis. To overcome this difficulty, we develop, in §3, a phase space formulation without the use of equations of motion. In §4, we will perform the quantization, construct the quantum Virasoro operators carefully and show that the physical spectrum agrees with the one computed in the light-cone gauge. In §5 we will discuss remaining issues.

§2. Attempt at canonical analysis of classical superstring in \( pp \)-wave background in conformal gauge

2.1. Green-Schwarz Lagrangian in the semi-light-cone gauge

We will employ the Green-Schwarz formulation developed by Metsaev\(^{21})\) (see also Ref. 45)). The basic fields are the 10 string coordinates\(^{**})\) \( X^\mu = (X^+, X^-, X^I) \), with \( I = 1 \sim 8 \), and the two sets of 16-component Majorana spinors \( \theta^A_a = (\theta^A_a, \dot{\theta}^A_a) \), with \( A = 1, 2 \), where \( a \) and \( \dot{a} \) respectively denote the \( SO(8) \) chiral and anti-chiral indices. The local \( \kappa \)-symmetry is fixed by the semi-light-cone (SLC) gauge condition \( \gamma^+ \theta^A = 0 \), where \( \gamma^+ \equiv \frac{1}{2} (\gamma^9 + \gamma^0) \). This is equivalent to \( \theta^A_a = 0 \) and hence only the \( SO(8) \) chiral components \( \theta^A_a \) will be present.

With some slight redefinitions and simplifications, the Lagrangian in the SLC gauge constructed by Metsaev\(^{21})\) can be cast into the form

\[
\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{WZ}},
\]

\[
\mathcal{L}_{\text{kin}} = -\frac{T}{2} \sqrt{-g} g^{ij} \left( 2 \partial_i X^+ \partial_j X^- + \partial_i X^I \partial_j X^I - \mu^2 X_i^2 \partial_i X^+ \partial_j X^+ \right) - i \sqrt{2} T \epsilon^{ij} \left( \partial_i X^+(\theta^1 \partial_j \theta^1 + \theta^2 \partial_j \theta^2) + 2 \mu \partial_i X^+ \partial_j X^+ \theta^1 \theta^2 \right),
\]

where \( \mathcal{L}_{\text{kin}} \) and \( \mathcal{L}_{\text{WZ}} \) are, respectively, the kinetic and the Wess-Zumino part, \( T = 1/2\pi\alpha' \) is the string tension and \( \mu \) represents the strength of the RR flux. The world-

\(^{a})\) With NSNS flux in RNS formalism, see Ref. 35).
\(^{**}) We use the convention \( X^\pm = \frac{1}{\sqrt{2}} (X^9 \pm X^0) \).
sheet coordinates are denoted by $\xi^i = (\xi^0, \xi^1) = (t, \sigma)$. After generating the Virasoro constraints by varying $g_{ij}$, we take the conformal gauge $g_{ij} = \eta_{ij} = \text{diag}(-1,1)$. Note that couplings to the curved geometry and to the RR flux give quartic interactions in addition to the cubic interactions, which are present even for $\mu = 0$. In contrast, in the light-cone gauge these non-linearities disappear as one can set $\partial_i X^+ \partial^i X^+ = 0$.

2.2. Equations of motion and their solutions

Despite the non-linearities of the Lagrangian mentioned above, it is possible to obtain the general solutions of the equations of motion exactly.

First, the equation of motion for $X^+$ is easily seen to be

$$\partial_i \partial^i X^+ = 0.$$  \hspace{2cm} (2.8)

This can be solved by introducing new variables $\rho_{\pm}$ (effectively adopting the light-cone frame)

$$\xi_{\pm} \equiv (\rho_{\pm} + \rho_{\mp}), \hspace{2cm} \rho_{\pm} \equiv \frac{\partial}{\partial \rho_{\pm}} = \left(\partial_{\rho_{\pm}}\right)^{-1} \partial_{\rho_{\pm}}.$$  \hspace{2cm} (2.9)

Then the equation (2.8) simplifies to

$$\partial_+ \partial_- X^I + \mu^2 (\partial_+ \mathcal{X}^+_L \partial_- \mathcal{X}^+_R) X^I = 0.$$  \hspace{2cm} (2.10)

Let us further form the following field-dependent “light-cone coordinates”:

$$\tilde{t} \equiv \frac{1}{2 \ell_s^2 \rho^+} (\rho_+ + \rho_-), \hspace{2cm} \tilde{\sigma} \equiv \frac{1}{2 \ell_s^2 \rho^+} (\rho_+ - \rho_-).$$  \hspace{2cm} (2.11)

Then, the equation becomes that of a free massive field

$$\left(\frac{\partial^2}{\partial \tilde{t}^2} - \frac{\partial^2}{\partial \tilde{\sigma}^2}\right) X^I + M^2 X^I = 0.$$  \hspace{2cm} (2.12)
where the dimensionless “mass” $M$ is defined as $M \equiv 2\ell_s^2 p^+\mu$. The general solution $2\pi$ periodic in $\sigma$ can be written as\(^*)

$$\mathcal{X}^I = \sum_n (a_n^I u_n + \tilde{a}_n^I \tilde{u}_n),$$

$$u_n = e^{-i(\omega_n t + n\sigma)} = e^{-i(\lambda_n^+ X_R^+ + \lambda_n^- X_L^-)},$$

$$\tilde{u}_n = e^{-i(\tilde{\omega}_n t - n\sigma)} = e^{-i(\tilde{\lambda}_n^+ \tilde{X}_R^+ + \tilde{\lambda}_n^- \tilde{X}_L^-)}.$$  

Here $a_n^I$ and $\tilde{a}_n^I$ are constant coefficients and $\lambda_n^\pm$ etc. are given by

$$\lambda_n^\pm = \frac{1}{2\ell_s^2 p^+}(\omega_n \pm n), \quad \tilde{\lambda}_n^\pm = \frac{1}{2\ell_s^2 p^+}(\tilde{\omega}_n \pm n),$$

$$\omega_n = \tilde{\omega}_n = \frac{n}{|n|}\sqrt{n^2 + M^2} \quad \text{for} \ n \neq 0,$$

$$\omega_0 = -\tilde{\omega}_0 = M.$$

$u_n$ and $\tilde{u}_n$ consist of product of left- and right-going functions.

Next, let us consider the equations of motion for the fermions $\theta^A$. In terms of the LC-frame coordinates $\rho^\pm$, they read

$$\tilde{\partial}_+ \theta^1 = -\mu \theta^2, \quad \tilde{\partial}_- \theta^2 = \mu \theta^1.$$  

Combining them we get $\tilde{\partial}_+ \tilde{\partial}_- \theta^A + \mu^2 \theta^A = 0$, which is the same equation satisfied by $\mathcal{X}^I$. Therefore the general solution can be written in terms of the functions $u_n$ and $\tilde{u}_n$ as

$$\theta^A = \sum_n (b_n^A u_n + \tilde{b}_n^A \tilde{u}_n)$$

with $\mu b_n^2 = i\lambda_n^+ b_n^1$, $\mu \tilde{b}_n^2 = i\tilde{\lambda}_n^- \tilde{b}_n^1$.  

Finally, look at the equation of motion for $X^-$. It takes the form

$$\tilde{\partial}_+ \tilde{\partial}_- X^- = \mu^2 \mathcal{X}^I (\tilde{\partial}_+ + \tilde{\partial}_-) \mathcal{X}^I + i\sqrt{2}\mu(\vartheta^1 \tilde{\partial}_+ \vartheta^2 - \vartheta^2 \tilde{\partial}_- \vartheta^1).$$

Since the RHS consists of known functions, it can be easily solved for $X^-$ by inverting the Laplacian $\tilde{\partial}_+ \tilde{\partial}_-$ with some appropriate boundary condition.

In this way one can obtain the general classical solutions. But now we have an apparent puzzle: How can we construct purely left- and right-going energy-momentum tensors out of the basis functions $u_n(\sigma_+, \sigma_-)$ and $\tilde{u}_n(\sigma_+, \sigma_-)$?

2.3. Energy-momentum tensor

To solve this puzzle, let us look at the energy-momentum tensors $T^\pm$ obtained through standard procedure. In the $\rho^\pm$ basis they read

$$T^\pm_T = (\partial_\pm \rho^\pm)^2 \left[\frac{1}{2}(\tilde{\partial}_-^+ X^- \tilde{\partial}_-^+ X^-) - \frac{i}{\sqrt{2}} \tilde{\partial}_-^+ \vartheta^1 (\vartheta^1 \tilde{\partial}_-^+ \theta^2 + \theta^2 \tilde{\partial}_-^+ \vartheta^1) \right.$$  

$$- \frac{1}{4}(\tilde{\partial}_-^+ X^2)^2 (\mu^2 X^2_T + 4\sqrt{2}\mu \theta^1 \theta^2) \right].$$

\(^*)\) This solution was obtained in the appendix of Ref. 46.
We want to see if $T_{\pm}$ are functions of $\sigma_{\pm}$, with the use of equations of motion. Focus on $T_{+}$. Using $\tilde{\partial}_{+}\mathcal{X}^{+} = 1$ and $\mu \vartheta^{2} = -\tilde{\partial}_{+}\vartheta^{1}$, it reduces to
\[
T_{+} = \frac{T}{2}(\partial_{+}\rho_{+})^{2} \left[ \tilde{\partial}_{+}\mathcal{X}^{-} + \frac{1}{2} \left( (\tilde{\partial}_{+}\mathcal{X}^{I})^{2} - \mu^{2}\mathcal{X}^{2}_{I} \right) - i\sqrt{2}(\partial^{2}\tilde{\partial}_{+}\vartheta^{2} - \vartheta^{1}\tilde{\partial}_{+}\vartheta^{1}) \right].
\] (2.21)

This can be simplified drastically upon expressing $\tilde{\partial}_{+}\mathcal{X}^{-}$ in terms of the other fields. Using various equations of motion, the once-integrated equation for $X^{-}$ can be written as
\[
\tilde{\partial}_{+}\mathcal{X}^{-} = -\frac{1}{2} \left( (\tilde{\partial}_{+}\mathcal{X}^{I})^{2} - \mu^{2}\mathcal{X}^{2}_{I} \right) + i\sqrt{2}(\partial^{2}\tilde{\partial}_{+}\vartheta^{2} - \vartheta^{1}\tilde{\partial}_{+}\vartheta^{1}) + f_{+}(\sigma_{+}).
\] (2.22)

where $f_{+}(\sigma_{+})$ is an arbitrary function of $\sigma_{+}$ produced through integration process. Substituting this into (2.21), we see that all the terms containing physical fields $\mathcal{X}^{I}$ and $\vartheta^{A}$ cancel, except for $f_{+}(\sigma_{+})$, and $T_{+}$ collapses to an exceedingly compact expression
\[
T_{+} = \frac{T}{2}(\partial_{+}\mathcal{X}_{L}^{+})^{2}f_{+}(\sigma_{+}).
\] (2.23)

So we have a very peculiar situation. Although $T_{+}$ is indeed a function only of $\sigma_{+}$, $f_{+}(\sigma_{+})$ cannot be made out of local products of physical fields $\mathcal{X}^{I}(\sigma_{+}, \sigma_{-}), \vartheta^{A}(\sigma_{+}, \sigma_{-})$. Dependence on these fields must be through their integrals, i.e. through $\sigma_{+}$-independent modes $a_{n}^{I}, \tilde{a}_{n}^{I}, b_{n}^{A},$ and $\tilde{b}_{n}^{A}$. Another notable feature is that obviously the above form is not smoothly connected to $\mu = 0$ flat space case: No matter how small $\mu$ is, as long as it is non-zero the classical solutions are connected through this quantity and hence the energy-momentum tensors take non-flat forms.

How should $f_{\pm}(\sigma_{\pm})$ be fixed? The requirement is that it must be determined so that the correct canonical equal time commutation relations are realized among the fields. To examine this, we now turn to the Poisson-Dirac brackets for the fields and the modes.

2.4. Poisson-Dirac brackets for the fields and the modes

The bosonic momenta are given by
\[
P^{+} = T\partial_{0}X^{+},
\] (2.24)
\[
P^{-} = T[\partial_{0}X^{-} - \partial_{0}X^{+}(\mu^{2}X^{2}_{I} + 4\sqrt{2}i\mu\theta^{1}\theta^{2}) - 2\sqrt{2}i(\theta^{1}\tilde{\partial}_{+}\theta^{1} + \theta^{2}\tilde{\partial}_{+}\theta^{2})],
\] (2.25)
\[
P^{I} = T\partial_{0}X^{I},
\] (2.26)

while the fermionic momenta take the form
\[
p^{1} = i\sqrt{2}T(\partial_{0}X^{+} - \partial_{1}X^{+})\theta^{1} = i\pi^{+1}\theta^{1},
\] (2.27)
\[
p^{2} = i\sqrt{2}T(\partial_{0}X^{+} + \partial_{1}X^{+})\theta^{2} = i\pi^{+2}\theta^{2}.
\] (2.28)

Here the quantity $\pi^{\pm}$ are defined by
\[
\pi^{+1} \equiv \sqrt{2}(P^{+} - T\partial_{1}X^{+}), \quad \pi^{+2} \equiv \sqrt{2}(P^{+} + T\partial_{1}X^{+}).
\] (2.29)
Of course the above definitions of the fermionic momenta should be regarded as primary constraints
\[ d^A = p^A - i\pi^+_A \theta^A = 0. \] (2.30)

We define the Poisson brackets as
\[ \{ X^I(\sigma, t), P^J(\sigma', t) \}_P = \delta^{IJ}\delta(\sigma - \sigma') , \] (2.31)
\[ \{ X^\pm(\sigma, t), P^\mp(\sigma', t) \}_P = \delta(\sigma - \sigma') , \] (2.32)
\[ \{ \theta^A_a(\sigma, t), p^B_b(\sigma', t) \}_P = -\delta^{AB}\delta_{ab}\delta(\sigma - \sigma') , \] (2.33)
rest = 0. \] (2.34)

Under this bracket, the fermionic constraints \( d^A_a \) form the second class algebra
\[ \{ d^A_a(\sigma, t), d^B_b(\sigma', t) \}_P = 2i\delta^{AB}\delta_{ab}\pi^+ A(\sigma, t)\delta(\sigma - \sigma') . \] (2.35)

Thus we introduce the Dirac bracket in the standard way. Then, \( \theta^A \) become self-conjugate:
\[ \{ \theta^A_a(\sigma, t), \theta^B_b(\sigma', t) \}_D = \frac{i\delta^{AB}\delta_{ab}}{2\pi^+ A(\sigma, t)}\delta(\sigma - \sigma') . \]

It is convenient to define the new field \( \Theta^A_a \) by
\[ \Theta^A_a \equiv \sqrt{2\pi^+ A} \theta^A_a . \] (2.36)

It enjoys the canonical bracket relation with itself of the form
\[ \{ \Theta^A_a(\sigma, t), \Theta^B_b(\sigma', t) \}_D = i\delta^{AB}\delta_{ab}\delta(\sigma - \sigma') . \] (2.37)

In fact it is easy to check that the set \( \{ X^\mu, P^\mu, \Theta^A_a \} \) satisfy the canonical (anti)-commutation relations under the Dirac bracket.

The next step is to find the commutation relations among the modes so that the fields satisfy the canonical Poisson-Dirac bracket relations at equal \( t \). Here we encounter a grave difficulty. Let us recall the solution for the transverse coordinate \( \mathcal{X}^I \). It is given in terms of the functions \( u_n \) and \( \tilde{u}_n \) as
\[ \mathcal{X}^I = \sum_n (a^I_n u_n + \tilde{a}^I_n \tilde{u}_n) , \] (2.38)
\[ u_n = e^{-i(\omega_n \bar{t} + n\bar{s})} = e^{-i(\lambda^+_n \mathcal{X}^+_R(\sigma_-) + \lambda^-_n \mathcal{X}^+_L(\sigma_+))} , \] (2.39)
\[ \tilde{u}_n = e^{-i(\bar{\omega}_n \bar{t} - n\bar{s})} = e^{-i(\bar{\lambda}^+_n \mathcal{X}^+_R(\sigma_-) + \bar{\lambda}^-_n \mathcal{X}^+_L(\sigma_+))} . \] (2.40)

To extract \( a^I_n \) and \( \tilde{a}^I_n \), one needs completeness relations for the functions \( u_n \) and \( \tilde{u}_n \) at “equal time”. Evidently, it is easy for the equal \( \bar{t} \) slice, just as in the LC gauge, but extremely hard for the equal \( t \) slice of our interest. Formal expression can be derived but it depends on the modes of \( \mathcal{X}^+_R \) and \( \mathcal{X}^+_L \) in an intractably complicated way. (The reason for this is that a non-trivial field-dependent conformal transformation is involved between the symplectic structures in the canonical \( (t, \sigma) \) basis and the \( (\bar{t}, \bar{s}) \) basis.) So at this point the canonical analysis has to be abandoned.
§3. Phase space formulation without the use of equations of motion

3.1. Basic observation

Fortunately, the difficulty encountered in the canonical analysis described above can be overcome by the use of the phase-space formulation. The basic observation is as follows.

In ordinary field theories, the knowledge of the Poisson(-Dirac) brackets at equal \(t\) is not enough to describe the dynamics which relates different \(t\). This is precisely the reason why we follow the canonical procedure. Namely, one first try to find the brackets for \(t\)-independent modes and then compute the brackets for fields at arbitrary (unequal) times.

But the situation is different for a string theory in the conformally invariant gauge. It is a type of theory in which the Hamiltonian \(H\) is a member of the generators of a large symmetry algebra, called the “spectrum generating algebra”. In such a case, the representation theory of the algebra should know about the spectrum and the dynamics: The analysis of the Virasoro constraints gives the information of the spectrum and by constructing the primary fields one should be able to compute the correlation functions which carry the dynamical information. In such a case, one may use the phase space formulation, where the equations of motion are not needed explicitly and only the equal-time brackets should be sufficient to develop the representation theory.

3.2. Classical Virasoro algebra in the phase space formulation

Let us now construct the Virasoro generators in terms of the phase space variables at the classical level. To simplify the description, let us introduce dimensionless fields \(A, B, S, \tilde{\Pi}, \text{ and } \Pi\), and a constant \(\hat{\mu}\) as

\[
A = \sqrt{2\pi T}X, \quad B = \sqrt{\frac{2\pi}{T}}P, \quad S = \sqrt{2\pi\Theta},
\]

\[
\tilde{\Pi} = \frac{1}{\sqrt{2}}(B + \partial_1 A), \quad \Pi = \frac{1}{\sqrt{2}}(B - \partial_1 A), \quad \hat{\mu} = \frac{\mu}{\sqrt{2\pi T}}.
\]

It is useful to remember that the fields \((\{\tilde{\Pi}^*\}, S^2)\) are “left-moving”, while \((\{\Pi^*\}, S^1)\) are “right-moving”. Then, the two sets of energy-momentum tensors can be written as

\[
\mathcal{T}_+ = \frac{1}{2}(\mathcal{H} + \mathcal{P}) = \frac{1}{2\pi} \left( \tilde{\Pi}^* + \tilde{\Pi} - \frac{i}{2} S^2 \partial_1 S^2 + \frac{\hat{\mu}^2}{2} A^2 \tilde{\Pi}^* + \frac{i\hat{\mu}}{\sqrt{2}} \sqrt{\tilde{\Pi}^* + \tilde{\Pi} + S^1 S^2} \right),
\]

\[
\mathcal{T}_- = \frac{1}{2}(\mathcal{H} - \mathcal{P}) = \frac{1}{2\pi} \left( \Pi^* + \Pi - \frac{i}{2} S^2 \partial_1 S^2 + \frac{\hat{\mu}^2}{2} A^2 \tilde{\Pi}^* + \frac{i\hat{\mu}}{\sqrt{2}} \sqrt{\tilde{\Pi}^* + \tilde{\Pi} + S^1 S^2} \right),
\]
where $\mathcal{H}$ is the Hamiltonian density and $\mathcal{P}$ is the momentum density. Although $T_+$ and $T_-$ do not manifestly commute with each other, after some careful computations, we verify the two independent sets of closed algebras

$$\{T_+(\sigma, t), T_+(\sigma', t)\}_D = \pm 2T_+(\sigma, t)\delta'(\sigma - \sigma') \pm \partial_1 T_+(\sigma, t)\delta(\sigma - \sigma'), \quad (3.5)$$

$$\{T_+(\sigma, t), T_-(\sigma', t)\}_D = 0. \quad (3.6)$$

These relations contain the information of the time-development. In particular,

$$\partial_0 \mathcal{H} = \{\mathcal{H}, \mathcal{H}\}_P = \partial_1 \mathcal{P}, \quad \partial_0 \mathcal{P} = \{\mathcal{P}, \mathcal{H}\}_P = \partial_1 \mathcal{H}, \quad (3.7)$$

where $\mathcal{H} \equiv \int d\sigma \mathcal{H}$ is the Hamiltonian. By adding and subtracting these equations, we readily get $\partial_\pm T_\pm = 0$, which means

$$T_\pm = T_\pm(\sigma_\pm). \quad (3.8)$$

From (3.5) and (3.8) we learn that the modes of $T_\pm$ form the (classical) Virasoro algebra:

$$T_\pm = \frac{1}{2\pi} \sum_n T^{\pm}_n e^{-in\sigma_\pm}, \quad (3.9)$$

$$\{T^\pm_m, T^\pm_n\}_D = \frac{1}{i}(m - n)T^\pm_{m+n}, \quad \{T^\pm_m, T^\mp_n\}_D = 0. \quad (3.10)$$

Note that to extract the modes we only need the information at $t = 0$, namely

$$T^n_\pm = \int_0^{2\pi} d\sigma e^{\pm in\sigma} T_\pm(\sigma, t = 0). \quad (3.11)$$

Thus, despite the left-right coupling, we have two independent sets of classical Virasoro algebras for any value of $\mu$. It is interesting to observe that viewed as exactly marginal deformations from the flat space case they are somewhat unusual since the $\mu$-dependent terms are not primary with respect to the $\mu = 0$ (flat space) theory.

§4. Quantization, quantum Virasoro algebra, and the physical spectrum

4.1. Quantization

The quantization of the basic fields is done by replacing the Poisson-Dirac brackets by quantum brackets in the usual way at $t = 0$. For example,

$$\{\tilde{\Pi}^+(\sigma), \tilde{\Pi}^-(\sigma')\}_D = 2\pi \delta'(\sigma - \sigma') \Rightarrow \left[\tilde{\Pi}^+(\sigma), \tilde{\Pi}^-(\sigma')\right] = 2\pi i \delta'(\sigma - \sigma'). \quad (4.1)$$

As for the mode expansion at $t = 0$, we will adopt the following convention:

$$\phi(\sigma) = \sum_n \phi_n e^{-in\sigma}. \quad (4.2)$$

Then, the commutator of the modes takes the form such as

$$\tilde{\Pi}^\pm(\sigma) = \sum_m \tilde{\Pi}^\pm_m e^{-im\sigma}, \quad \left[\tilde{\Pi}^\pm_m, \tilde{\Pi}^\pm_n\right] = m\delta_{m+n,0}, \quad \text{etc.} \quad (4.3)$$
4.2. Quantum Virasoro algebra

Now we come to the construction of the quantum Virasoro operators. For this purpose, we must (i) find an appropriate normal-ordering, (ii) make sure that the central charges add up to 26 and (iii) add quantum corrections, if necessary.

Finding the correct normal-ordering turned out to be quite non-trivial. To describe and check the appropriate scheme we found, it is convenient to introduce the operators \( L_\pm(\sigma) \), which satisfy the same form of the Virasoro algebra (as opposed to \( T_\pm \) which satisfied the relations (3.5) with different signs)

\[
L_\pm(\sigma) \equiv \pm T_\pm(\sigma) = \frac{1}{2\pi} \sum_n L_n^\pm e^{-i n \sigma}.
\] (4.4)

Now to define \( L_n^\pm \) quantum-mechanically, we adopt the “phase-space normal-ordering” where \( B^*_n(n \geq 0), A^*_n(n \geq 1), S^{A^*_n}(n \geq 1) \) are regarded as annihilation operators.

The crucial feature of this normal-ordering is that with such a prescription the quantum operator anomalies produced by the double contractions between the non-linear terms cancel exactly between the bosonic and fermionic contributions.

These anomalous contributions are of the form (the subscript \( B(F) \) stands for bosonic (fermionic))

\[
C_B = \frac{1}{(2\pi)^2} \left( \left[ \frac{1}{2} \tilde{\Pi}^2(\sigma), \frac{\bar{\mu}^2}{2} \tilde{\Pi}^+ A_1^2(\sigma') \right] - (\sigma \leftrightarrow \sigma') \right),
\] (4.5)

\[
C_F = \frac{1}{(2\pi)^2} \left[ -i \frac{\bar{\mu}}{\sqrt{2}} \sqrt{\tilde{\Pi}^+ \tilde{\Pi}^+ S^1 S^2(\sigma), -i \frac{\bar{\mu}}{\sqrt{2}} \sqrt{\tilde{\Pi}^+ \tilde{\Pi}^+ S^1 S^2(\sigma')} \right].
\] (4.6)

Upon careful computation with appropriate regularization one finds

\[
C_B = -C_F = -\frac{i \bar{\mu}^2}{\pi} (2 \tilde{\Pi}^+ \tilde{\Pi}^+ \delta'(\sigma - \sigma') + \partial_\sigma (\tilde{\Pi}^+ \tilde{\Pi}^+) \delta(\sigma - \sigma')).
\] (4.7)

With this cancellation, one verifies that \( L_n^\pm \) form two sets of quantum Virasoro algebra:

\[
[L_\pm(\sigma), L_\pm(\sigma')] = i \left( 2L_\pm(\sigma) \delta'(\sigma - \sigma') + \partial_\sigma L_\pm(\sigma) \delta(\sigma - \sigma') - \frac{1}{24\pi} (14 \delta''(\sigma - \sigma') - 2 \delta''(\sigma - \sigma')) \right),
\] (4.8)

\[
[L_+(\sigma), L_-(\sigma')] = 0.
\] (4.9)

As is evident from the form of (4.8), the central charge is only 14, 10 from \( X^\mu \) and 4 from \( S^a \). This is the same as in the flat space case in the SLC gauge and the cure is known.\(^{47}\) One needs to add the quantum corrections \( \Delta L_\pm \) of the following form:

\[
L_\pm \rightarrow L_\pm + \Delta L_\pm,
\] (4.10)

\[
\Delta L_+ = -\frac{1}{2\pi} \partial_\sigma^2 \ln \tilde{\Pi}^+, \quad \Delta L_- = \frac{1}{2\pi} \partial_\sigma^2 \ln \tilde{\Pi}^+.
\] (4.11)

\(^{47}\) This in particular means that in the plane-wave background with RR flux, the Virasoro operators made out of bosonic fields alone cannot close quantum-mechanically in a consistent way.
Δℓ± almost behave as primary operators of dimension 2, except that they provide the wanted 12 units of central charge. One can indeed verify

\[
\left[ \mathcal{L}_\pm(\sigma), \Delta \mathcal{L}_\pm(\sigma') \right] + \left[ \Delta \mathcal{L}_\pm(\sigma), \mathcal{L}_\pm(\sigma') \right] = i \left( 2 \Delta \mathcal{L}_\pm(\sigma) \delta'(\sigma - \sigma') + \partial_\sigma(\Delta \mathcal{L}_\pm(\sigma)) \delta(\sigma - \sigma') - \frac{1}{24\pi} 12 \delta''''(\sigma - \sigma') \right),
\]

so that with this addition we have the desired quantum Virasoro operators with central charge 26.

4.3. BRST formulation and the physical spectrum

Having constructed the quantum Virasoro operators, it is now straightforward to construct the nilpotent BRST operators \( Q \) and \( \tilde{Q} \) for the right- and the left-sector. \( Q \), for instance, takes the familiar form

\[
Q = \sum_n c_{-n} L_n^- - \frac{1}{2} \sum_{m,n} (m - n) : c_{-m} c_{-n} b_{m+n} :.
\]

Although the Virasoro generators \( L_n^- \) in this formula contain non-linear terms, the decoupling of the unphysical degrees of freedom, namely the non-zero modes of \( \tilde{H}^+ \), \( \Pi^\pm \), \( b, c, \tilde{b} \) and \( \tilde{c} \), works in a simple way. This is because one can easily prove the equivalence \( Q \)-cohomology \( \cong Q_{-1} \) cohomology, where \( Q_{-1} \equiv -\Pi_0^+ \sum_{n \neq 0} \Pi_{-n} c_n \) is exactly the same as in the free bosonic string. Consequently, the physical states are the ones in the transverse space \( \mathcal{H}_T \) (i.e. without the non-zero modes above), satisfying the constraints \( H = L_0^+ + L_0^- = 0 \), \( P = L_0^+ - L_0^- = 0 \).

After dropping the non-zero modes of \( \tilde{H}^\pm \) and \( \Pi^\pm \), our Hamiltonian becomes (with phase-space normal-ordering understood)

\[
H = H_B + H_F, \quad (4.14)
\]

\[
H_B = \alpha' p^+ p^- + \frac{1}{2} \sum (B_n^I B_n^I + \omega_n^2 A_n^I A_n^I), \quad (4.15)
\]

\[
H_F = \frac{1}{2} \sum (-n S_{-n}^1 S_{-n}^1 + n S_{n}^2 S_{n}^2 - i M S_{-n}^1 S_{n}^2 + i M S_{n}^2 S_{-n}^1), \quad (4.16)
\]

where \( \omega_n \equiv \frac{n}{|n|} \sqrt{n^2 + M^2} \). Evidently, \( H_B \) describes free massive bosonic excitations. As for \( H_F \), we need to perform diagonalization to see that it describes the corresponding free massive fermionic excitations. To this end, construct massive oscillators in terms of massless oscillators in the following way:

\[
\alpha_n^I \equiv \frac{1}{\sqrt{2}} (B_n^I - i n \omega_n A_{-n}^I), \quad \alpha_n^I \equiv \frac{1}{\sqrt{2}} (B_{-n}^I - i n \omega_n A_{n}^I), \quad (n \neq 0) \quad (4.17)
\]

\[
\alpha_0^I \equiv \frac{1}{\sqrt{2}} (B_0^I - i M A_0^I), \quad \alpha_0^I \equiv \frac{1}{\sqrt{2}} (B_{-0}^I + i M A_{0}^I), \quad (4.18)
\]

\[
\tilde{S}_{-n} \equiv N(n) \left( S_{-n}^2 + i \frac{M}{\Omega_{-n}^+} S_{-n}^1 \right), \quad S_n \equiv N(n) \left( S_{-n}^1 - i \frac{M}{\Omega_{n}^+} S_{-n}^2 \right), \quad (n \neq 0) \quad (4.19)
\]

\[
S_0 \equiv \frac{1}{\sqrt{2}} (S_0^1 - i S_0^2), \quad S_0^\dagger \equiv \frac{1}{\sqrt{2}} (S_0^1 + i S_0^2), \quad (4.20)
\]
\( \Omega_n^+ \equiv \omega_n + n, \quad N(n) \equiv \sqrt{\frac{\Omega_n^+}{2\omega_n}}. \) (4.21)

They satisfy the (anti-)commutation relations
\[
\left[ \tilde{\alpha}^I_m, \tilde{\alpha}^J_n \right] = [\alpha^I_m, \alpha^J_n] = \omega_n \delta^{IJ} \delta_{m+n,0}, \quad [\tilde{\alpha}^I_m, \alpha^J_n] = 0, \quad (4.22)
\]
\[
\left\{ \tilde{S}_m, \tilde{S}_n \right\} = \{ S_m, S_n \} = \delta_{m+n,0}, \quad \{ S_0, S_0^\dagger \} = 1, \quad \{ \tilde{S}_m, S_n \} = 0. \quad (4.24)
\]

Now re-express \( H \) in terms of these oscillators and re-normal-order appropriately for the new oscillators. Then one finds that, due to supersymmetry, the constants produced in this process cancel exactly and the Hamiltonian simplifies to
\[
H = \alpha' p^+ p^- + \alpha_0^I \alpha_0^J + \sum_{n \geq 1} (\alpha_{-n}^I \alpha_n^I + \tilde{\alpha}_{-n}^I \tilde{\alpha}_n^I) \\
+ MS_0^\dagger S_0 + \sum_{n \geq 1} \omega_n (S_n^\dagger S_n + \tilde{S}_n^\dagger \tilde{S}_n). \quad (4.25)
\]

Setting \( H = 0 \) and solving for \(-p^- = H_{lc}\), the light-cone Hamiltonian, we get
\[
H_{lc} = \frac{1}{\alpha' \omega^+} \left( \alpha_0^I \alpha_0^J + \sum_{n \geq 1} (\alpha_{-n}^I \alpha_n^I + \tilde{\alpha}_{-n}^I \tilde{\alpha}_n^I) \\
+ MS_0^\dagger S_0 + \sum_{n \geq 1} \omega_n (S_n^\dagger S_n + \tilde{S}_n^\dagger \tilde{S}_n) \right). \quad (4.26)
\]

This coincides with the well-known light-cone Hamiltonian in the LC gauge. Further, \( P = 0 \) yields the level-matching condition
\[
P = \sum_{n \geq 1} \left( \frac{n}{\omega_n} \tilde{\alpha}_{-n}^I \tilde{\alpha}_n^I + n S_n^\dagger \tilde{S}_n \right) - \sum_{n \geq 1} \left( \frac{n}{\omega_n} \alpha_{-n}^I \alpha_n^I + n S_n^\dagger S_n \right) = 0. \quad (4.27)
\]

This shows that the physical spectrum of our conformal field theory is precisely the same as in the light-cone gauge.

§5. Summary and remaining issues

We have initiated the study of the superstring in the \( pp \)-wave background with RR-flux as an exact conformal field theory in an operator formulation. Despite non-linearity, the equations of motion can be solved exactly. However, the canonical analysis based on these solutions meets difficulties: Transverse fields are functions of both \( \sigma_+ \) and \( \sigma_- \). Only their modes can appear in the Virasoro generators and it turned out to be extremely hard to find the commutation relations for these modes.

We pointed out that to overcome this difficulty an alternative phase space formulation without the use of equations of motion can be utilized. Despite the coupling
between left- and right-going fields, two independent sets of classical Virasoro generators are constructed. Fields can be quantized in a straightforward manner and in terms of them the quantum Virasoro generators are defined with appropriate normal-ordering and a quantum modification. They are checked to form correct Virasoro algebra. Also we have shown that they reproduce, via BRST formulation, the correct physical spectrum previously obtained in the light-cone gauge.

Clearly, there are many remaining problems to be investigated. We must clarify how the global symmetries are realized, including the supersymmetry. The most important and challenging task is the construction of the primary fields, at least for the low lying excitations. Once this is achieved, we should be able to compute the correlation functions exactly in \( \alpha' \). It would also be of interest to construct the DDF operators for all the excitations. Study of the modular invariance, which is awkward in the light-cone gauge, is another important problem. It is intriguing to see how the presence of the large RR flux affects the nature of the open/closed duality. The CFT we constructed in the SLC gauge may be a starting point of the operator formulation of the fully covariant pure-spinor formalism, through the double spinor extension\(^{49}\) of the Green-Schwarz superstring. Finally, we should study if our phase-space formalism could be applied to the case of superstring in the \( AdS_5 \times S^5 \) background in a useful way. We hope to be able to report progress on these issues in the near future.

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