Nonlinear first order PDEs reducible to autonomous form polynomially homogeneous in the derivatives

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Abstract

It is proved a theorem providing necessary and sufficient conditions enabling one to map a nonlinear system of first order partial differential equations, polynomial in the derivatives, to an equivalent autonomous first order system polynomially homogeneous in the derivatives. The result is intimately related to the symmetry properties of the source system, and the proof, involving the use of the canonical variables associated to the admitted Lie point symmetries, is constructive. First order Monge–Ampère systems, either with constant coefficients or with coefficients depending on the field variables, where the theorem can be successfully applied, are considered.

Keywords. Lie symmetries; First order Monge–Ampère systems; Transformation to quasilinear form.

1 Introduction

Lie group analysis [1,2,3,4,5,6,7,8] provides a unified and elegant algorithmic framework to a deep understanding and fruitful handling of differential
equations. It is known that Lie point symmetries admitted by ordinary differential equations allow for their order lowering and possibly reducing them to quadrature, whereas in the case of partial differential equations they can be used for the determination of special (invariant) solutions of initial and boundary value problems. Also, the Lie symmetries are important ingredients in the derivation of conserved quantities, or in the construction of relations between different differential equations that turn out to be equivalent [8, 9, 10, 11, 12, 13, 14, 15, 16]. Lie point symmetries of differential equations, in fact, can be used to construct a mapping from a given (source) system of differential equations to another (target) suitable system; if we consider one-to-one (invertible) point mappings, then a one-to-one correspondence between Lie point symmetries admitted by the source and target system of differential equations arises. In other words, the Lie algebra of infinitesimal operators of the target system of differential equations has to be isomorphic to the Lie algebra of infinitesimal operators of the source system of differential equations. This property has been used to give necessary and sufficient conditions for reducing a system of partial differential equations to autonomous form [12, 13], a system of first order nonlinear partial differential equations to linear form [9, 10, 11, 13], nonautonomous and/or nonhomogeneous quasilinear systems of partial differential equations to autonomous and homogeneous form [14, 16]. In particular, in [16], it has been proved a theorem providing the necessary and sufficient conditions in order to map a general first order quasilinear system of partial differential equations, say

$$\sum_{i=1}^{n} A^i(x, u) \frac{\partial u}{\partial x_i} = B(x, u),$$

where \(x \in \mathbb{R}^n, u \in \mathbb{R}^m, A^i\) are \(m \times m\) matrices with entries depending at most on \(x\) and \(u\), and the source term \(B \in \mathbb{R}^m\) depends at most on \(x\) and \(u\) too, into a first order quasilinear homogeneous and autonomous system. This reduction, when it is possible, is performed by an invertible point transformation like

$$z = Z(x), \quad w = W(x, u),$$

which preserves the quasilinear structure of the system, and whose construction is algorithmically suggested by the Lie symmetries admitted by [11].

In this paper, we consider a general nonlinear system of first order partial differential equations involving the derivatives of the unknown variables in
polynomial (of degree greater than 1) form, and establish a theorem giving necessary and sufficient conditions in order to map it to an autonomous system which is polynomially homogeneous in the derivatives.

In some relevant situations, e.g., Monge–Ampère systems, the target system results to be quasilinear, but there are cases where the system we obtain is polynomially homogeneous in the derivatives but not quasilinear. This means that the conditions of the theorem are only necessary for the reduction of a nonlinear first order system to autonomous and homogeneous quasilinear form [17].

The main difference of the theorem presented in this paper with the similar one proved in [16] (concerned with the transformation of a general first order quasilinear system of partial differential equations into a first order quasilinear homogeneous and autonomous system) consists in the possibility of admitting now an invertible point transformation like

\[ z = Z(x, u), \quad w = W(x, u), \]  

i.e., a mapping where the new independent variables \( z \) are allowed to depend also on the old dependent ones.

The plan of the paper is the following. In Section 2, the theorem giving necessary and sufficient conditions for the existence of an invertible mapping linking a nonlinear system of first order partial differential equations which is polynomial in the derivatives to an autonomous system polynomially homogeneous in the derivatives is proved. In Section 3, the theorem is applied to various general first order Monge–Ampère systems. Finally, Section 4 contains some concluding remarks.

## 2 Main result

Let us consider a general system of first order partial differential equations

\[ \Delta(x, u, u^{(1)}) = 0, \]  

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \) and \( u^{(1)} \in \mathbb{R}^{mn} \) are the independent variables, the dependent variables, and the first order partial derivatives, respectively. In particular, in the following we consider systems [4] composed by equations which are polynomial in the derivatives, with coefficients depending at most
on \( x \) and \( u \), \( i.e. \) systems made by equations of the form

\[
\sum_{|\alpha|,|j|=1}^{N_s} A_{\alpha j}^s(x, u) \prod_{k=1}^{\alpha} \frac{\partial u_{\alpha k}}{\partial x_{j_k}} + B^s(x, u) = 0, \quad s = 1, \ldots, m, \quad (5)
\]

where \( \alpha \) is the multi-index \((\alpha_1, \ldots, \alpha_r)\), \( j \) the multi-index \((j_1, \ldots, j_r)\), \( \alpha_k = 1, \ldots, m \), \( j_k = 1, \ldots, n \), \( N_s \) are integers, and \( A_{\alpha j}^s(x, u) \), \( B^s(x, u) \) smooth functions of their arguments.

The aim is to determine necessary and sufficient conditions for the construction of an invertible point transformation

\[
z = Z(x, u), \quad w = W(x, u), \quad (6)
\]

mapping (5) into an equivalent autonomous system which is homogeneous polynomial in the derivatives \( w^{(1)} \), \( i.e. \), made by equations of the form

\[
\sum_{|\alpha|,|j|=N_s}^{N_s} \tilde{A}_{\alpha j}^s(w) \prod_{k=1}^{\alpha} \frac{\partial w_{\alpha k}}{\partial z_{j_k}} = 0, \quad s = 1, \ldots, m, \quad (7)
\]

for some integers \( N_s \); of course, it may occur that the target system turns out to be linear in the derivatives, \( i.e. \), \( N_s = 1 \) \((s = 1, \ldots, m)\), whereupon we have an autonomous and homogeneous quasilinear system.

The following lemma, guarantees that an invertible point transformation like (6) preserves the polynomial structure in the derivatives.

**Lemma 1** Given a first order system of partial differential equations like (5) which is polynomial in the derivatives, then an invertible point transformation like (6) produces a first order system which is still polynomial in the derivatives.

**Proof.** Straightforward, by using the chain rule. \( \square \)

**Theorem 1** The nonlinear first order system of partial differential equations polynomial in the derivatives

\[
\sum_{|\alpha|,|j|=1}^{N_s} A_{\alpha j}^s(x, u) \prod_{k=1}^{\alpha} \frac{\partial u_{\alpha k}}{\partial x_{j_k}} + B^s(x, u) = 0, \quad s = 1, \ldots, m, \quad (8)
\]
is mapped by an invertible point transformation, say
\[ z = Z(x, u), \quad w = W(x, u), \]
(9)
to the equivalent nonlinear first order autonomous system having homogeneous polynomial form, say
\[ \sum_{|\alpha|,|\beta| = N_s} \tilde{A}_{\alpha\beta}(w) \prod_{k=1}^{N_s} \frac{\partial w_{\alpha_k}}{\partial z_{j_k}} = 0, \quad s = 1, \ldots, m, \]
(10)
for some integers \( N_s \), if and only if there exists an \( (n+1) \)-dimensional subalgebra of the Lie algebra of point symmetries, admitted by system (8), spanned by the vector fields
\[ \Xi_i = \sum_{i=1}^{n} \xi_i^j(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{m} \eta_\alpha^i(x, u) \frac{\partial}{\partial u_\alpha}, \quad i = 1, \ldots, n+1, \]
(11)
generating a distribution of rank \( (n+1) \), and such that
\[ [\Xi_i, \Xi_j] = 0, \quad i = 1, \ldots, n - 1, \quad i < j \leq n, \]
\[ [\Xi_i, \Xi_{n+1}] = \Xi_i, \quad i = 1, \ldots, n. \]
(12)

Moreover, the variables \( w \), which by construction are invariants of \( \Xi_1, \ldots, \Xi_n \), have to be invariant with respect to \( \Xi_{n+1} \) too.

**Proof.** Suppose the conditions of Theorem 1 are satisfied, and so the system (8) admits an \( (n+1) \)-dimensional algebra as subalgebra of the algebra of its Lie point symmetries generating a distribution of rank \( (n+1) \) and verifying the structure conditions (12). Let us introduce a set of canonical variables for the vector field \( \Xi_1 \), say
\[ y_i^1 \quad (i = 1, \ldots, n), \quad \nu_\alpha^1 \quad (\alpha = 1, \ldots, m), \]
such that
\[ \Xi_1 y_i^1 = 1, \quad \Xi_1 y_{i_1}^1 = 0, \quad \Xi_1 \nu_\alpha^1 = 0 \]
\((i_1 = 2, \ldots, n); \) as a consequence, \( \Xi_1 \) takes the form
\[ \Xi_1 = \frac{\partial}{\partial y_1^1}. \]
\( i.e. \), it corresponds to a translation in the variable \( y^1_i \).

Since \([\Xi_1; \Xi_2] = 0\), it is

\[
\Xi_1(\Xi_2y^1_i) = \Xi_2(\Xi_1y^1_i) = 0, \quad \Xi_1(\Xi_2v^1_\alpha) = \Xi_2(\Xi_1v^1_\alpha) = 0, \quad (13)
\]

for \( i = 1, \ldots, n \) and \( \alpha = 1, \ldots, m \). Thus, the infinitesimals of \( \Xi_2 \), represented in terms of the canonical variables of \( \Xi_1 \), will depend upon the invariants of \( \Xi_1 \) only, \( i.e. \), \( \Xi_2 \) writes as

\[
\Xi_2 = \sum_{i=1}^n \Theta^2_i(y^1_j, v^1_\beta) \frac{\partial}{\partial y^1_i} + \sum_{\alpha=1}^m \Lambda^2_\alpha(y^1_j, v^1_\beta) \frac{\partial}{\partial v^1_\alpha} \quad (14)
\]

\((j = 2, \ldots, n, \beta = 1, \ldots, m)\).

If \( \Theta^2_i \neq 0 \) we need to replace \( y^1_i \) with

\[
y^1_i + \varphi^1_i(y^1_j, v^1_\beta),
\]

where the function \( \varphi^1_i \) satisfies

\[
\Theta^2_i(y^1_j, v^1_\beta) + \sum_{i_1=2}^n \Theta^2_{i_1}(y^1_j, v^1_\beta) \frac{\partial \varphi^1_i}{\partial y^1_{i_1}} + \sum_{\alpha=1}^m \Lambda^2_\alpha(y^1_j, v^1_\beta) \frac{\partial \varphi^1_i}{\partial v^1_\alpha} = 0.
\]

That enables us to write \( \Xi_1 \) and \( \Xi_2 \) as follows

\[
\Xi_1 = \frac{\partial}{\partial y^1_i}, \quad \Xi_2 = \sum_{i_1=2}^n \Theta^2_{i_1}(y^1_j, v^1_\beta) \frac{\partial}{\partial y^1_{i_1}} + \sum_{\alpha=1}^m \Lambda^2_\alpha(y^1_j, v^1_\beta) \frac{\partial}{\partial v^1_\alpha}, \quad (15)
\]

where \( j = 2, \ldots, n \).

Introducing the canonical variables

\[
y^2_1 = y^1_1, \quad y^2_2, \quad y^2_{i_2} \quad (i_2 = 3, \ldots, n), \quad v^2_\alpha \quad (\alpha = 1, \ldots, m),
\]

such that

\[
\Xi_2y^2_2 = 1, \quad \Xi_2y^2_{i_2} = 0, \quad \Xi_2v^2_\alpha = 0
\]

\((i = 2, \ldots, n)\), it is obtained

\[
\Xi_2 = \frac{\partial}{\partial y^2_2}.
\]
Continuing inductively for \( k = 2, \ldots, n - 1 \), since \( \Xi_{k+1} \) commutes with \( \Xi_1, \ldots, \Xi_k \), in terms of the canonical variables

\[
y^k_i \quad (i = 1, \ldots, n), \quad v^k_\alpha \quad (\alpha = 1, \ldots, m),
\]

we have

\[
\begin{align*}
\Xi_1 &= \frac{\partial}{\partial y^1_k}, \\
\Xi_2 &= \frac{\partial}{\partial y^2_k}, \\
\ldots & \ldots \\
\Xi_k &= \frac{\partial}{\partial y^k_k}, \\
\Xi_{k+1} &= \sum_{i=1}^{n} \Theta^{k+1}_{i}(y^k_{j_k}, v^k_\beta) \frac{\partial}{\partial y^k_i} + \sum_{\alpha=1}^{m} \Lambda^{k+1}_{k}(y^k_{j_k}, v^k_\beta) \frac{\partial}{\partial v^k_\alpha},
\end{align*}
\]

(16)

where \( j_k = k + 1, \ldots, n \). If \( \Theta^{k+1}_{\ell} \neq 0 \), for \( \ell = 1, \ldots, k \), we need to replace the variable \( y^k_\ell \) with

\[
y^k_\ell + \varphi^k_\ell(y^k_{j_k}, v^k_\beta),
\]

where the function \( \varphi^k_\ell \) satisfies

\[
\Theta^{k+1}_{\ell}(y^k_{j_k}, v^k_\beta) + \sum_{i_k=k+1}^{n} \Theta^{k+1}_{i_k}(y^k_{j_k}, v^k_\beta) \frac{\partial \varphi^k_\ell}{\partial y^k_{i_k}} + \sum_{\alpha=1}^{m} \Lambda^{k+1}_{k}(y^k_{j_k}, v^k_\beta) \frac{\partial \varphi^k_\ell}{\partial v^k_\alpha} = 0,
\]

so that \( \Xi_{k+1} \) writes as

\[
\Xi_{k+1} = \sum_{i_k=k+1}^{n} \Theta^{k+1}_{i_k}(y^k_{j_k}, v^k_\beta) \frac{\partial}{\partial y^k_{i_k}} + \sum_{\alpha=1}^{m} \Lambda^{k+1}_{k}(y^k_{j_k}, v^k_\beta) \frac{\partial}{\partial v^k_\alpha};
\]

(17)

hence, we may construct the canonical variables

\[
y^k_1 = y^k_1, \ldots, y^k_{k+1} = y^k_{k+1} \quad (i_k = k+1, \ldots, n), \quad v^k_{\alpha+1} \quad (\alpha = 1, \ldots, m),
\]

related to the operator \( \Xi_{k+1} \), such that the latter writes as

\[
\Xi_{k+1} = \frac{\partial}{\partial y^k_{k+1}}.
\]

The complete application of the described algorithm enables us to write each operator \( \Xi_i \) in the form

\[
\Xi_i = \frac{\partial}{\partial z_i}, \quad i = 1, \ldots, n,
\]
and the new independent and dependent variables are \( z_i = y_i^n \) \((i = 1, \ldots, n)\), \( w_\alpha = v_\alpha^n \) \((\alpha = 1, \ldots, m)\), respectively.

Therefore, what we have obtained is a variable transformation like \(9\) allowing to write the system \(8\) in autonomous form.

Finally, since \([\Xi_i, \Xi_{n+1}] = \Xi_i \ (i = 1, \ldots, n)\), it is
\[
\Xi_i(\Xi_{n+1}z_j) = \Xi_{n+1}(\Xi_i z_j) + \Xi_i z_j = \delta_{ij},
\Xi_i(\Xi_{n+1}w_\alpha) = \Xi_{n+1}(\Xi_i w_\alpha) + \Xi_i w_\alpha = 0;
\]
(18)

where \(\delta_{ij}\) is the Kronecker symbol; these relations, together with the hypothesis that the variables \(w_\alpha \ (\alpha = 1, \ldots, m)\) are invariant with respect to \(\Xi_{n+1}\), allow the vector field \(\Xi_{n+1}\) to gain the representation
\[
\Xi_{n+1} = \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}.
\]
(19)

As a consequence, since the resulting system, written in the variables \(z\) and \(w\), is autonomous and polynomial in the derivatives, and is invariant with respect to a uniform scaling of all independent variables, then it necessarily must be polynomially homogeneous in the derivatives, \(i.e.,\) it has the form \(10\).

The condition that the symmetries generate a distribution of rank \((n+1)\), whence the vector fields spanning the \(n\)–dimensional Abelian Lie subalgebra generate a distribution of rank \(n\), ensures that we may construct the complete set of the new independent variables \(z\).

Conversely, if the nonautonomous and/or nonhomogeneous system \(8\) can be mapped by the invertible point transformation \(9\) to the autonomous system polynomially homogeneous in the derivatives \(10\), then, since the latter admits the \(n\) vector fields \(\frac{\partial}{\partial z_i}\), spanning an \(n\)–dimensional Abelian Lie algebra, and the vector field \(\sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}\), then it follows that also the system \(8\) must admit \((n+1)\) Lie point symmetries with the requested algebraic structure. \(\Box\)
Example 1 Let us consider the first order system made by the equations

$$\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} = 0,$$

$$κ_1 \left( \frac{\partial u_1}{\partial x_2} \right)^4 + \left( κ_2 \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} + κ_3 \left( \frac{\partial u_1}{\partial x_2} \right)^2 \right) \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2}$$

$$+ \left( κ_4 \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} + κ_5 \left( \frac{\partial u_1}{\partial x_2} \right)^2 \right) \frac{\partial u_1}{\partial x_1} + \left( κ_6 \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} + κ_7 \left( \frac{\partial u_1}{\partial x_2} \right)^2 \right) \frac{\partial u_1}{\partial x_2}$$

$$+ \left( κ_8 \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} + κ_9 \left( \frac{\partial u_1}{\partial x_2} \right)^2 \right) \frac{\partial u_2}{\partial x_1} + κ_{10} \left( \frac{\partial u_1}{\partial x_1} \right)^2$$

$$+ κ_{11} \frac{\partial u_1}{\partial x_1} \frac{\partial u_1}{\partial x_2} + κ_{12} \left( \frac{\partial u_1}{\partial x_2} \right)^2 + κ_{13} \left( \frac{\partial u_1}{\partial x_2} \right)^2$$

$$+ κ_{14} \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_2} + κ_{15} \left( \frac{\partial u_2}{\partial x_2} \right)^2 = 0,$$ 

(20)

with $u_1(x_1, x_2), u_2(x_1, x_2)$ scalar functions, and $κ_i(u_1, u_2)$ ($i = 1, \ldots, 15$) arbitrary smooth functions of the indicated arguments.

It can be easily ascertained that system (20) admits the Lie point symmetries spanned by the operators

$$Ξ_1 = \frac{\partial}{\partial x_1}, \quad Ξ_2 = \frac{\partial}{\partial x_2},$$

$$Ξ_3 = (x_1 - au_1 - bu_2) \frac{\partial}{\partial x_1} + (x_2 - bu_1 - cu_2) \frac{\partial}{\partial x_2},$$

(21)

$a, b, c$ being constants, provided that the conditions

$$κ_1 - c^2 κ_{10} + bck_{11} - acκ_{12} - b^2 κ_{13} + abκ_{14} - a^2 κ_{15} = 0,$$

$$κ_2 - c^2 κ_{10} + bck_{11} - acκ_{12} - b^2 κ_{13} + abκ_{14} - a^2 κ_{15} = 0,$$

$$κ_3 + 2(c^2 κ_{10} - bck_{11} + acκ_{12} + b^2 κ_{13} - abκ_{14} + a^2 κ_{15}) = 0,$$

$$κ_4 + 2cκ_{10} - bκ_{11} + ak_{12} = 0,$$

$$κ_5 - 2cκ_{10} + bκ_{11} - ak_{12} = 0,$$

$$κ_6 + ck_{11} - 2bκ_{13} + ak_{14} = 0,$$

$$κ_7 - ck_{11} + 2bκ_{13} - ak_{14} = 0,$$

$$κ_8 + ck_{12} - bk_{14} + 2ak_{15} = 0,$$

$$κ_9 - ck_{12} + bk_{14} - 2ak_{15} = 0$$

(22)

are satisfied. Since

$$[Ξ_1, Ξ_2] = 0, \quad [Ξ_1, Ξ_3] = Ξ_1, \quad [Ξ_2, Ξ_3] = Ξ_2,$$

(23)

9
applying the theorem, we introduce the new independent and dependent variables
\[ z_1 = x_1 - au_1 - bu_2, \quad z_2 = x_2 - bu_1 - cu_2, \]
\[ w_1 = u_1, \quad w_2 = u_2, \]  
and the nonlinear system (20) reduces to
\[ \frac{\partial w_1}{\partial z_2} - \frac{\partial w_2}{\partial z_1} = 0, \]
\[ \kappa_{10} \left( \frac{\partial w_1}{\partial z_1} \right)^2 + \kappa_{11} \frac{\partial w_1}{\partial z_1} \frac{\partial w_1}{\partial z_2} + \kappa_{12} \frac{\partial w_1}{\partial z_1} \frac{\partial w_2}{\partial z_2} + \kappa_{13} \left( \frac{\partial w_1}{\partial z_2} \right)^2 + \kappa_{14} \frac{\partial w_1}{\partial z_2} \frac{\partial w_2}{\partial z_2} + \kappa_{15} \left( \frac{\partial w_2}{\partial z_2} \right)^2 = 0, \]  
i.e., reads as an autonomous system polynomially homogeneous in the derivatives.

We notice that system (25), by specializing the functions \( \kappa_{10}, \ldots, \kappa_{15} \) as follows,
\[ \kappa_{10} = -\kappa (1 + w_2^2)^2, \]
\[ \kappa_{11} = 4 \kappa w_1 w_2 (1 + w_2^2), \]
\[ \kappa_{12} = 2 ((2 - \kappa) (1 + w_1^2 + w_2^2) - \kappa w_1^2 w_2^2), \]
\[ \kappa_{13} = -4 (1 + w_1^2 + w_2^2 + \kappa w_1^2 w_2^2), \]
\[ \kappa_{14} = 4 \kappa w_1 w_2 (1 + w_2^2), \]
\[ \kappa_{15} = -\kappa (1 + w_1^2)^2, \]  
is equivalent to the second order partial differential equation
\[ \kappa \left( 1 + w_2^2 \right)^2 w_{z_1 z_1}^2 - 4 \kappa w_{z_1} w_{z_2} \left( 1 + w_2^2 \right) w_{z_1 z_1}^2 w_{z_2 z_2}
- 2 \left( (2 - \kappa) (1 + w_1^2 + w_2^2) - \kappa w_1^2 w_2^2 \right) w_{z_1 z_1}^2 w_{z_2 z_2}
+ 4 (1 + w_1^2 + w_2^2 + \kappa w_1^2 w_2^2) w_{z_1 z_2}^2 - 4 \kappa w_{z_1} w_{z_2} \left( 1 + w_1^2 \right) w_{z_1 z_2} w_{z_2 z_2}
+ \kappa \left( 1 + w_2^2 \right)^2 w_{z_2 z_2}^2 = 0, \]  
where \( w_{z_1} = \frac{\partial w}{\partial z_1} = w_1, \quad w_{z_2} = \frac{\partial w}{\partial z_2} = w_2, \) and \( \kappa \) is an arbitrary function of \( w_{z_1} \) and \( w_{z_2} \).

Considering a smooth surface in \( \mathbb{R}^3 \) with the metric \( ds^2 = dz_1^2 + dz_2^2 + dw^2, \)
and its Gaussian and mean curvature,

\[
G = \frac{w_{z_1}w_{z_2} - w_{z_1z_2}}{(1 + w_{z_1}^2 + w_{z_2}^2)^2},
\]

\[
H = \frac{1}{2} \frac{(1 + w_{z_2}^2)w_{z_1z_2} - 2w_{z_1}w_{z_2}w_{z_1z_2} + (1 + w_{z_1}^2)w_{z_2z_2}}{(1 + w_{z_1}^2 + w_{z_2}^2)^{3/2}},
\]

respectively, equation (27) can be written as

\[
G = \kappa H^2, \quad (29)
\]

whereupon it should be \( \kappa(w_{z_1}, w_{z_2}) \leq 1 \). In the limit case \( \kappa \equiv 1 \), Eq. (29) characterizes a surface with all its points umbilic; it is known that a surface with all its point umbilic is a (open) domain of a plane or a sphere [18]. It is worth of being remarked that Eq. (27) with \( \kappa \equiv 1 \) is strongly Lie remarkable [19], since it is the unique second order partial differential equation uniquely characterized by the conformal Lie algebra in \( \mathbb{R}^3 \) [20].

**Remark 1** Notice that the Example 1 provides a system polynomial in the derivatives which is transformed into a system polynomially homogeneous of degree 2 in the derivatives. In Section 3, we will analyze various first order Monge–Ampère systems, and show that they can be transformed to quasilinear form.

### 3 Applications

In this section, we provide some examples of first order nonlinear systems polynomial in the derivatives whose Lie symmetries satisfy the conditions of Theorem 1 and prove that they can be transformed under suitable conditions to autonomous first order systems having homogeneous polynomial form; the systems that will be considered are of Monge–Ampère type, and, remarkably, they are reduced to quasilinear (or linear) form.

In particular, we are concerned with the nonlinear first order systems of Monge–Ampère equations for the unknowns \( u_{\alpha}(x_i) \) (\( \alpha = 1, \ldots, m, i = 1, \ldots, n \)). These systems have been characterized by Boillat in 1997 [21] by looking for the nonlinear first order systems possessing, as the quasilinear systems, the property of the linearity of the Cauchy problem. These systems are also completely exceptional [22, 23], and are made by equations which
are expressed as linear combinations (with coefficients depending at most on the independent and the dependent variables) of all minors extracted from the gradient matrix of $u_\alpha = u_\alpha(x_i)$.

Hereafter, to shorten the formulas, we denote with $u_{\alpha,i}$ the first order partial derivative of $u_\alpha(x_i)$ with respect to $x_i$, and with $w_{\alpha,i}$ the first order partial derivative of $w_\alpha(z_i)$ with respect to $z_i$; moreover, we denote with $f_{i;\alpha}$ the first order partial derivative of the function $f_i$ with respect to $u_\alpha$ (or $w_\alpha$). In the following we limit ourselves to consider the coefficients of the Monge–Ampère systems at most functions of the field variables.

### 3.1 Case $m = n = 2$

Let us consider the nonlinear first order system of Monge–Ampère made by the equations

$$
\kappa^i_0 (u_{1,2} - u_{2,1}) + \kappa^i_1 u_{1,1} + \kappa^i_2 u_{1,2} + \kappa^i_3 u_{2,1} + \kappa^i_4 u_{2,2} + \kappa^i_5 = 0
$$

(i = 1, 2), with $u_1(x_1, x_2)$, $u_2(x_1, x_2)$ scalar functions, and $\kappa^i_j (u_1, u_2)$ (i = 1, 2; j = 0, ..., 5) arbitrary smooth functions of the indicated arguments.

The substitutions

$$
u_1 \rightarrow u_1 + \alpha_{11} x_1 + \alpha_{12} x_2,

\nu_2 \rightarrow u_2 + \alpha_{21} x_1 + \alpha_{22} x_2,$$

where $\alpha_{ij}$ are arbitrary constants, produce a system with $\kappa^i_5 = 0$ (i = 1, 2) provided that

$$
\kappa^i_0 (\alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21}) + \kappa^i_1 \alpha_{11} + \kappa^i_2 \alpha_{12} + \kappa^i_3 \alpha_{21} + \kappa^i_4 \alpha_{22} + \kappa^i_5 = 0
$$

(i = 1, 2). Conditions (32) provide two constraints on the functional form of the coefficients so that not all systems can be written in a form where $\kappa^i_5 = 0$; however, if the coefficients $\kappa^i_j$ are constant, due to the arbitrariness of the constants $\alpha_{ij}$, then (32) can always be satisfied whatever the values of the coefficients are.

It is easily recognized that system (30), now taken with $\kappa^i_5 = 0$, admits the Lie point symmetries spanned by the operators

$$
\Xi_1 = \frac{\partial}{\partial x_1},

\Xi_2 = \frac{\partial}{\partial x_2},

\Xi_3 = (x_1 - f_1) \frac{\partial}{\partial x_1} + (x_2 - f_2) \frac{\partial}{\partial x_2},
$$

where $f_i(u_1, u_2)$ (i = 1, 2) are arbitrary smooth functions of their arguments, provided that

$$
\kappa^i_0 + \kappa^i_1 f_{2;2} - \kappa^i_2 f_{1;2} - \kappa^i_3 f_{2;1} + \kappa^i_4 f_{1;1} = 0, \quad i = 1, 2.
$$
The constraints (34), once we assign the 10 functions \( k_{ij}(u_1, u_2) \) \((i = 1, 2, j = 0, \ldots, 4)\), are the differential equations providing us the functional form of \( f_1(u_1, u_2) \) and \( f_2(u_1, u_2) \).

In the case where all the coefficients \( \kappa_{ij} \) are constant, then the functions \( f_1 \) and \( f_2 \) are forced to be linear, i.e.,

\[
  f_1 = \beta_{11} u_1 + \beta_{12} u_2, \quad f_2 = \beta_{21} u_1 + \beta_{22} u_2,
\]

\( \beta_{ij} \) being constants whose value is determined by the coefficients \( \kappa_{ij} \).

Since

\[
  [\Xi_1, \Xi_2] = 0, \quad [\Xi_1, \Xi_3] = \Xi_1, \quad [\Xi_2, \Xi_3] = \Xi_2,
\]

we introduce the new variables

\[
  z_1 = x_1 - f_1, \quad z_2 = x_2 - f_2, \quad w_1 = u_1, \quad w_2 = u_2,
\]

and the generators of the point symmetries write as

\[
  \Xi_1 = \frac{\partial}{\partial z_1}, \quad \Xi_2 = \frac{\partial}{\partial z_2}, \quad \Xi_3 = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}.
\]

In terms of the new variables (37), the nonlinear system (30) becomes

\[
  \kappa_1^i w_{1,1} + \kappa_2^i w_{1,2} + \kappa_3^i w_{2,1} + \kappa_4^i w_{2,2} = 0,
\]

i.e., reads as an autonomous and homogeneous quasilinear system. This system is linear if all the coefficients \( \kappa_{ij} \) are constant; nevertheless, since it is a \( 2 \times 2 \) homogeneous and autonomous quasilinear system, it can be written in linear form by means of the hodograph transformation also when the coefficients \( \kappa_{ij} \) depend on \( u_1 \) and \( u_2 \).

In conclusion, all Monge–Ampère systems with \( m = n = 2 \) can be reduced to a linear system when the coefficients \( \kappa_{ij} \) are constant; on the contrary, when the coefficients depend upon \( u_1 \) and \( u_2 \) the reduction to the linear form is possible provided that the constraints (32) are satisfied.

### 3.2 Case \( m = 2, n = 3 \)

By considering the gradient matrix of \( u_\alpha(x_i) \) \((\alpha = 1, 2, i = 1, \ldots, 3)\)

\[
  H = \begin{pmatrix}
    u_{1,1} & u_{1,2} & u_{1,3} \\
    u_{2,1} & u_{2,2} & u_{2,3}
  \end{pmatrix}
\]

(40)
and its extracted minors
\[ H^1 = \begin{bmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{bmatrix}, \quad H^2 = \begin{bmatrix} u_{1,1} & u_{1,3} \\ u_{2,1} & u_{2,3} \end{bmatrix}, \quad H^3 = \begin{bmatrix} u_{1,2} & u_{1,3} \\ u_{2,2} & u_{2,3} \end{bmatrix}, \] (41)
the nonlinear first order system of Monge–Ampère is made by equations like
\[ \kappa_i^1 H^1 + \kappa_i^2 H^2 + \kappa_i^3 H^3 + \kappa_i^4 u_{1,1} + \kappa_i^5 u_{1,2} + \kappa_i^6 u_{1,3} + \kappa_i^7 u_{2,1} + \kappa_i^8 u_{2,2} + \kappa_i^9 u_{2,3} + \kappa_i^{10} = 0, \]
(42)
\(\kappa_j^i(u_\alpha)\) \((i = 1, 2, j = 1, \ldots, 10)\) arbitrary smooth functions of the indicated arguments.

The substitutions
\[ u_1 \to u_1 + \alpha_{11} x_1 + \alpha_{12} x_2 + \alpha_{13} x_3, \quad u_2 \to u_2 + \alpha_{21} x_1 + \alpha_{22} x_2 + \alpha_{23} x_3, \] (43)
where \(\alpha_{ij}\) are arbitrary constants, produce a system with \(\kappa_{10}^i = 0\) \((i = 1, 2)\) provided that
\[ \kappa_1^i(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}) + \kappa_2^i(\alpha_{11}\alpha_{23} - \alpha_{13}\alpha_{21}) + \kappa_3^i(\alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{22}) + \kappa_4^i \alpha_{11} + \kappa_5^i \alpha_{12} + \kappa_6^i \alpha_{13} + \kappa_7^i \alpha_{21} + \kappa_8^i \alpha_{22} + \kappa_9^i \alpha_{23} + \kappa_{10}^i = 0 \]
\((i = 1, 2)\). Actually, conditions (44) can always be satisfied when the coefficients \(\kappa_j^i\) are constant because of the arbitrariness of the constants \(\alpha_{ij}\).

This nonlinear system (42), now taken with \(\kappa_{10}^i = 0\), admits the Lie point symmetries spanned by the operators
\[ \Xi_1 = \frac{\partial}{\partial x_1}, \quad \Xi_2 = \frac{\partial}{\partial x_2}, \quad \Xi_3 = \frac{\partial}{\partial x_3}, \quad \Xi_4 = (x_1 - f_1) \frac{\partial}{\partial x_1} + (x_2 - f_2) \frac{\partial}{\partial x_2} + (x_3 - f_3) \frac{\partial}{\partial x_3}, \]
(45)
where \(f_i(u_1, u_2)\) \((i = 1, \ldots, 3)\) are arbitrary smooth functions of their arguments, provided that
\[ \kappa_1^i + \kappa_4^i f_{2;2} - \kappa_5^i f_{1;2} - \kappa_7^i f_{2;1} + \kappa_8^i f_{1;1} = 0, \]
\[ \kappa_2^i + \kappa_4^i f_{3;2} - \kappa_6^i f_{1;2} - \kappa_7^i f_{3;1} + \kappa_9^i f_{1;1} = 0, \]
\[ \kappa_3^i + \kappa_5^i f_{3;2} - \kappa_6^i f_{2;2} - \kappa_8^i f_{3;1} + \kappa_9^i f_{2;1} = 0. \]
(46)
The six conditions (46) cannot be fulfilled for an arbitrary choice of the coefficients \(\kappa^i_j\). In the simplest case, where the coefficients \(\kappa^i_j\) are constant, they can be always satisfied and the functions \(f_i\) must be linear:

\[
\begin{align*}
f_1 &= \beta_{11}u_1 + \beta_{12}u_2, \\
f_2 &= \beta_{21}u_1 + \beta_{22}u_2, \\
f_3 &= \beta_{31}u_1 + \beta_{32}u_2,
\end{align*}
\]

(47) \(\beta_{ij}\) being arbitrary constants.

The symmetries (45) generate a 4–dimensional solvable Lie algebra,

\[
[\Xi_i, \Xi_j] = 0, \quad [\Xi_i, \Xi_4] = \Xi_i, \quad (i, j = 1, 2, 3),
\]

(48) whereupon we may introduce the new variables

\[
\begin{align*}
z_1 &= x_1 - f_1, \\
z_2 &= x_2 - f_2, \\
z_3 &= x_3 - f_3,
\end{align*}
\]

and the generators of the point symmetries write as

\[
\Xi_1 = \frac{\partial}{\partial z_1}, \quad \Xi_2 = \frac{\partial}{\partial z_2}, \quad \Xi_3 = \frac{\partial}{\partial z_3}, \quad \Xi_4 = z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3}.
\]

(50)

In terms of the new variables (49), the nonlinear system (42) reduces to

\[
\kappa^i_4w_{1,1} + \kappa^i_5w_{1,2} + \kappa^i_6w_{1,3} + \kappa^i_7w_{2,1} + \kappa^i_8w_{2,2} + \kappa^i_9w_{2,3} = 0,
\]

(51) i.e., reads as an autonomous and homogeneous quasilinear system.

### 3.3 Case \(m = 3, n = 2\)

By considering the gradient matrix of \(u_\alpha(x_i)\) \((\alpha = 1, \ldots, 3, \ i = 1, 2)\)

\[
H = \left(\begin{array}{cc}
u_{1,1} & u_{1,2} \\
u_{2,1} & u_{2,2} \\
u_{3,1} & u_{3,2}
\end{array}\right)
\]

(52) and its extracted minors

\[
H^1 = \left|\begin{array}{cc}
u_{1,1} & u_{1,2} \\
u_{2,1} & u_{2,2}
\end{array}\right|, \quad H^2 = \left|\begin{array}{cc}
u_{1,1} & u_{1,2} \\
u_{3,1} & u_{3,2}
\end{array}\right|, \quad H^3 = \left|\begin{array}{cc}
u_{2,1} & u_{2,2} \\
u_{3,1} & u_{3,2}
\end{array}\right|
\]

(53) the nonlinear first order system of Monge–Ampère is made by equations like

\[
\kappa^i_1H^1 + \kappa^i_2H^2 + \kappa^i_3H^3 + \kappa^i_4u_{1,1} + \kappa^i_5u_{1,2} + \kappa^i_6u_{2,1} + \kappa^i_7u_{2,2} + \kappa^i_8u_{3,1} + \kappa^i_9u_{3,2} + \kappa^i_{10} = 0,
\]

(54)
\( \kappa^i_j (u_\alpha) \) \( (i = 1, \ldots, 3, \ j = 1, \ldots, 10) \) arbitrary smooth functions of the indicated arguments.

The substitutions

\begin{align*}
u_1 &\to u_1 + \alpha_{11}x_1 + \alpha_{12}x_2, \\
u_2 &\to u_2 + \alpha_{21}x_1 + \alpha_{22}x_2, \\
u_3 &\to u_3 + \alpha_{31}x_1 + \alpha_{32}x_2,
\end{align*}

where \( \alpha_{ij} \) are arbitrary constants, produce a system with \( \kappa^i_{10} = 0 \) \( (i = 1, \ldots, 3) \) provided that

\begin{align*}
\kappa^i_1(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}) + \kappa^i_2(\alpha_{11}\alpha_{32} - \alpha_{12}\alpha_{31}) + \kappa^i_3(\alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31}) \\
+ \kappa^i_4\alpha_{11} + \kappa^i_5\alpha_{12} + \kappa^i_6\alpha_{21} + \kappa^i_7\alpha_{22} + \kappa^i_8\alpha_{31} + \kappa^i_9\alpha_{32} + \kappa^i_{10} = 0
\end{align*}

\( (i = 1, \ldots, 3) \). Also in this case, conditions \( (56) \) can always be satisfied when the coefficients \( \kappa^i_j \) are constant because of the arbitrariness of the constants \( \alpha_{ij} \).

This nonlinear system \( (54) \), now taken with \( \kappa^i_{10} = 0 \), admits the Lie point symmetries spanned by the operators

\begin{align*}
\Xi_1 &= \frac{\partial}{\partial x_1}, \\
\Xi_2 &= \frac{\partial}{\partial x_2}, \\
\Xi_3 &= (x_1 - f_1) \frac{\partial}{\partial x_1} + (x_2 - f_2) \frac{\partial}{\partial x_2},
\end{align*}

where \( f_i(u_1, u_2, u_3) \) \( (i = 1, 2) \) are arbitrary smooth functions of their arguments, provided that

\begin{align*}
\kappa^i_1 + \kappa^i_4f_2;2 - \kappa^i_5f_{1;2} - \kappa^i_6f_{2;1} + \kappa^i_7f_{1;1} &= 0, \\
\kappa^i_2 + \kappa^i_4f_2;3 - \kappa^i_5f_{1;3} - \kappa^i_8f_{2;1} + \kappa^i_9f_{1;1} &= 0, \\
\kappa^i_3 + \kappa^i_6f_{2;3} - \kappa^i_7f_{1;3} - \kappa^i_8f_{2;2} + \kappa^i_9f_{1;2} &= 0.
\end{align*}

The six conditions \( (58) \) cannot be fulfilled for an arbitrary choice of the coefficients \( \kappa^i_j \). In the simplest case, where the coefficients \( \kappa^i_j \) are constant, they can be always satisfied and the functions \( f_i \) must be linear:

\begin{align*}
f_1 &= \beta_{11}u_1 + \beta_{12}u_2 + \beta_{13}u_3, \\
f_2 &= \beta_{21}u_1 + \beta_{22}u_2 + \beta_{23}u_3,
\end{align*}

\( \beta_{ij} \) being arbitrary constants.

The symmetries \( (57) \) generate a 3–dimensional solvable Lie algebra,

\begin{align*}
[\Xi_i, \Xi_j] &= 0, \\
[\Xi_i, \Xi_3] &= \Xi_i, \\
(i, j &= 1, 2),
\end{align*}

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whereupon we may introduce the new variables
\[ z_1 = x_1 - f_1, \quad z_2 = x_2 - f_2, \]
\[ w_1 = u_1, \quad w_2 = u_2, \quad w_3 = u_3, \]  
(61)
and the generators of the point symmetries write as
\[ \Xi_1 = \frac{\partial}{\partial z_1}, \quad \Xi_2 = \frac{\partial}{\partial z_2}, \quad \Xi_3 = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}. \]  
(62)
In terms of the new variables (61), the nonlinear system (54) reduces to
\[ \kappa_i^4 w_{1,1} + \kappa_i^3 w_{1,2} + \kappa_i^2 w_{2,1} + \kappa_i^1 w_{2,2} + \kappa_i^0 w_{3,1} + \kappa_i^0 w_{3,2} = 0, \]  
(63)
i.e., reads as an autonomous and homogeneous quasilinear system.

### 3.4 Case \( m = n = 3 \)

By considering the gradient matrix of \( u_\alpha(x_i) \) \( (i = 1, \ldots, 3, \alpha = 1, \ldots, 3) \)
\[ H = \begin{pmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{pmatrix} \]  
(64)
and its extracted minors of order 2
\[
\begin{align*}
H^1 &= \begin{vmatrix} u_{2,2} & u_{2,3} \\ u_{3,2} & u_{3,3} \end{vmatrix}, \\
H^2 &= \begin{vmatrix} u_{2,1} & u_{2,3} \\ u_{3,1} & u_{3,3} \end{vmatrix}, \\
H^3 &= \begin{vmatrix} u_{2,1} & u_{2,2} \\ u_{3,1} & u_{3,2} \end{vmatrix}, \\
H^4 &= \begin{vmatrix} u_{1,2} & u_{1,3} \\ u_{3,2} & u_{3,3} \end{vmatrix}, \\
H^5 &= \begin{vmatrix} u_{1,1} & u_{1,3} \\ u_{3,1} & u_{3,3} \end{vmatrix}, \\
H^6 &= \begin{vmatrix} u_{1,1} & u_{1,2} \\ u_{3,1} & u_{3,2} \end{vmatrix}, \\
H^7 &= \begin{vmatrix} u_{1,2} & u_{1,3} \\ u_{2,2} & u_{2,3} \end{vmatrix}, \\
H^8 &= \begin{vmatrix} u_{1,1} & u_{1,3} \\ u_{2,1} & u_{2,3} \end{vmatrix}, \\
H^9 &= \begin{vmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{vmatrix},
\end{align*}
\]
the nonlinear first order system of Monge–Ampère results composed by equations like
\[
\begin{align*}
\kappa_i^0 \det(H) + \kappa_i^1 H^1 + \kappa_i^2 H^2 + \kappa_i^3 H^3 + \kappa_i^4 H^4 + \kappa_i^5 H^5 + \kappa_i^6 H^6 \\
+ \kappa_i^7 H^7 + \kappa_i^8 H^8 + \kappa_i^9 H^9 + \kappa_{10}^1 u_{1,1} + \kappa_{11}^1 u_{1,2} + \kappa_{12}^1 u_{1,3} + \kappa_{13}^1 u_{2,1} \\
+ \kappa_{14}^1 u_{2,2} + \kappa_{15}^i u_{2,3} + \kappa_{16}^i u_{3,1} + \kappa_{17}^i u_{3,2} + \kappa_{18}^i u_{3,3} + \kappa_{19}^i &= 0,
\end{align*}
\]
(65)
\( \kappa^i_j(u_{\alpha}) \) \( (i = 1, \ldots, 3, \ j = 0, \ldots, 19, \ \alpha = 1, \ldots, 3) \) arbitrary smooth functions of the indicated arguments.

Also in this case, the substitutions

\[
\begin{align*}
    u_1 &\to u_1 + \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3, \\
    u_2 &\to u_2 + \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3, \\
    u_3 &\to u_3 + \alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}x_3,
\end{align*}
\]

(66)

where \( \alpha_{ij} \) are arbitrary constants, allow us to obtain a system with \( \kappa^i_{19} = 0 \) provided that \( u_{i,j} = \alpha_{ij} \) is a solution of equations (65). This requirement implies some constraints on the coefficients \( \kappa^i_j \) in the general case; on the contrary, no limitation to the values of the coefficients exists if they are assumed to be constant.

The system (65), with \( \kappa^i_{19} = 0 \), admits the Lie point symmetries spanned by the operators

\[
\Xi_1 = \frac{\partial}{\partial x_1}, \quad \Xi_2 = \frac{\partial}{\partial x_2}, \quad \Xi_3 = \frac{\partial}{\partial x_3}, \quad \Xi_4 = (x_1 - f_1) \frac{\partial}{\partial x_1} + (x_2 - f_2) \frac{\partial}{\partial x_2} + (x_3 - f_3) \frac{\partial}{\partial x_3},
\]

(67)

where \( f_i(u_1, u_2, u_3) \) \( (i = 1, \ldots, 3) \) are arbitrary smooth functions of their
arguments, provided that
\begin{align}
\kappa_i^0 - (f_{2,2}f_{3,3} - f_{2,3}f_{3,2})\kappa_{10} - (f_{1,3}f_{3,2} - f_{1,2}f_{3,3})\kappa_{11} \\
- (f_{1,2}f_{2,3} - f_{1,3}f_{2,2})\kappa_{12} - (f_{2,3}f_{3,1} - f_{2,1}f_{3,3})\kappa_{13} \\
- (f_{1,1}f_{3,3} - f_{1,3}f_{3,1})\kappa_{14} - (f_{1,3}f_{2,1} - f_{1,1}f_{2,3})\kappa_{15} \\
- (f_{2,1}f_{3,2} - f_{2,2}f_{3,1})\kappa_{16} - (f_{2,3}f_{3,1} - f_{1,1}f_{3,2})\kappa_{17} \\
- (f_{1,1}f_{2,2} - f_{1,2}f_{2,1})\kappa_{18} = 0, \\
\kappa_i^1 + \kappa_{14} f_{3,3} - \kappa_{15} f_{2,3} - \kappa_{17} f_{3,2} + \kappa_{18} f_{2,2} = 0, \\
\kappa_2 + \kappa_{13} f_{3,3} - \kappa_{15} f_{1,3} - \kappa_{16} f_{3,2} + \kappa_{18} f_{1,2} = 0, \\
\kappa_3 + \kappa_{13} f_{2,3} - \kappa_{14} f_{1,3} - \kappa_{16} f_{2,2} + \kappa_{17} f_{1,2} = 0, \\
\kappa_4 + \kappa_{11} f_{3,3} - \kappa_{12} f_{2,3} - \kappa_{17} f_{3,1} + \kappa_{18} f_{2,1} = 0, \\
\kappa_5 + \kappa_{10} f_{3,3} - \kappa_{12} f_{1,3} - \kappa_{16} f_{3,1} + \kappa_{18} f_{1,1} = 0, \\
\kappa_6 + \kappa_{10} f_{2,3} - \kappa_{11} f_{1,3} - \kappa_{16} f_{2,1} + \kappa_{17} f_{1,1} = 0, \\
\kappa_7 + \kappa_{11} f_{3,3} - \kappa_{12} f_{2,3} - \kappa_{14} f_{3,1} + \kappa_{15} f_{2,1} = 0, \\
\kappa_8 + \kappa_{10} f_{3,1} - \kappa_{12} f_{1,3} - \kappa_{15} f_{3,1} + \kappa_{14} f_{1,1} = 0, \\
\kappa_9 + \kappa_{10} f_{2,3} - \kappa_{11} f_{1,3} - \kappa_{15} f_{2,1} + \kappa_{14} f_{1,1} = 0. \\
\end{align}

(68)

The vector fields (67) span a 4-dimensional solvable Lie algebra,
\begin{align}
[\Xi_i, \Xi_j] = 0, \quad [\Xi_i, \Xi_4] = \Xi_i, \quad (i, j = 1, \ldots, 3),
\end{align}

(69)

whereupon we may introduce the new variables
\begin{align}
z_1 = x_1 - f_1, \quad z_2 = x_2 - f_2, \quad z_3 = x_3 - f_3, \\
w_1 = u_1, \quad w_2 = u_2, \quad w_3 = u_3,
\end{align}

(70)

and the generators of the point symmetries write as
\begin{align}
\Xi_1 = \frac{\partial}{\partial z_1}, \quad \Xi_2 = \frac{\partial}{\partial z_2}, \quad \Xi_3 = \frac{\partial}{\partial z_3}, \\
\Xi_4 = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_3}.
\end{align}

(71)

In terms of the new variables (70), Eqs. (65) write as
\begin{align}
\kappa_{10} w_{1,1} + \kappa_{11} w_{1,2} + \kappa_{12} w_{1,3} + \kappa_{13} w_{2,1} + \kappa_{14} w_{2,2} + \kappa_{15} w_{2,3} \\
+ \kappa_{16} w_{3,1} + \kappa_{17} w_{3,2} + \kappa_{18} w_{3,3} = 0,
\end{align}

(72)
i.e., they are in autonomous and homogeneous quasilinear (linear, if the coefficients are constant) form.

Conditions (68) play severe restrictions to the expression of the coefficients $\kappa^i_j$. When these coefficients are assumed to be constant, we are forced to take

\[
\begin{align*}
  f_1 &= \beta_{11}u_1 + \beta_{12}u_2 + \beta_{13}u_3, \\
  f_2 &= \beta_{21}u_1 + \beta_{22}u_2 + \beta_{23}u_3, \\
  f_3 &= \beta_{31}u_1 + \beta_{32}u_2 + \beta_{33}u_3,
\end{align*}
\]

(73)

where $\beta_{ij}$ are arbitrary constants; also in such simple case, the reduction to linear form is not always possible due to (68).

### 3.5 Case $m$ and $n$ arbitrary

It is easily recognized that, a general Monge–Ampère system with $m$ dependent variables and $n$ independent variables, provided that some suitable conditions on the coefficients (at most depending on the field variables) are satisfied, is invariant with respect to the Lie groups generated by the vector fields

\[
\Xi_i = \frac{\partial}{\partial x_i} (i = 1, \ldots, n), \quad \Xi_{n+1} = \sum_{i=1}^{n} (x_i - f_i(u_\alpha)) \frac{\partial}{\partial x_i},
\]

(74)

where $f_i(u_\alpha)$ are smooth functions of $(u_1, \ldots, u_m)$ which have to be linear in their arguments when the coefficients of the Monge–Ampère system are constant.

As one expects, for $m > 3$ or $n > 3$, we have a situation similar to the case $m = n = 3$, i.e., even in the case of constant coefficients, not all Monge–Ampère systems can be reduced to (quasi)linear form.

### 4 Conclusions

In this paper, we proved a theorem giving necessary and sufficient conditions for transforming a nonlinear first order system of partial differential equations involving the derivatives in polynomial form to an equivalent autonomous system polynomially homogeneous in the derivatives. The theorem is based on the Lie point symmetries admitted by the nonlinear system, and the proof
is constructive, in the sense that it leads to the algorithmic construction of the invertible mapping performing the task.

The theorem is applied to a class of first order nonlinear systems belonging to the family of Monge–Ampère systems that have been characterized by Boillat in 1997. These systems share with the quasilinear systems the property of the linearity of the Cauchy problem. These systems are also completely exceptional [22, 23], and are made by equations which are expressed as linear combinations (with coefficients depending at most on the independent and the dependent variables) of all minors extracted from the gradient matrix of $u_\alpha = u_\alpha(x_i)$ ($\alpha = 1, \ldots, m, i = 1, \ldots, n$). We considered explicitly either the case of constant coefficients or the case of coefficients depending on the field variables, for $m = 2, 3$ and $n = 2, 3$. If $m = 2$ and $n = 2, 3$, or $n = 2$ and $m = 2, 3$, and the coefficients are assumed to be constant, we proved that the Monge–Ampère systems can always be transformed to linear form.

Nevertheless, for arbitrary $m$ and $n$, Monge–Ampère systems, provided that the coefficients entering their equations satisfy some constraints, can be mapped to first order quasilinear autonomous and homogeneous systems. This, in some sense, casts new light on the fact, underlined by Boillat [21], that Monge–Ampère systems, because of the linearity of the Cauchy problem, are the closest to quasilinear systems, which are Monge systems.

Moreover, an example of a first order system polynomial in the derivatives that can be reduced to a system polynomially homogeneous in the derivatives (equivalent to a second order partial differential equation for a surface in $\mathbb{R}^3$ such that its Gaussian curvature is proportional to the square of its mean curvature), is provided.

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