SNake Graph Calculus and Cluster Algebras From Surfaces II: Self-Crossing Snake Graphs

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Abstract. Snake graphs appear naturally in the theory of cluster algebras. For cluster algebras from surfaces, each cluster variable is given by a formula which is parametrized by the perfect matchings of a snake graph. In this paper, we continue our study of snake graphs from a combinatorial point of view. We introduce the notions of abstract snake graphs and abstract band graphs, their crossings and self-crossings, as well as the resolutions of these crossings. We show that there is a bijection between the set of perfect matchings of (self-) crossing snake graphs and the set of perfect matchings of the resolution of the crossing. In the situation where the snake and band graphs are coming from arcs and loops in a surface without punctures, we obtain a new proof of skein relations in the corresponding cluster algebra.

1. Introduction

This paper continues the study of abstract snake graphs initiated in [CS]. Our goals are, on the one hand, to establish a computational tool for cluster algebras of surface type, which we call snake graph calculus, and, on the other hand, to introduce a new algebraic structure which is not limited to a particular choice of a surface but rather inspired from the combinatorial structure of all surface type cluster algebras. This theory has already found the following applications. In [CaSc], the authors use snake graphs to study extensions of modules over Jacobian algebras and triangles in the cluster category associated to triangulations of unpunctured surfaces, and in [CLS], the snake graph calculus is used to show that for unpunctured surfaces with exactly one marked point, the upper cluster algebra coincides with the cluster algebra, which was one of the last cases of the mutation finite cluster algebras for which the question was still open.

Cluster algebras were introduced in [FZ1], and further developed in [FZ2, BFZ, FZ4], motivated by combinatorial aspects of canonical bases in Lie theory [L1, L2]. A cluster algebra is a subalgebra of a field of rational functions in several variables, and it is given by constructing a distinguished set of generators, the cluster variables. These cluster variables are constructed recursively and their computation is rather complicated in general. By construction, the cluster variables are rational functions, but Fomin and Zelevinsky showed in [FZ1] that they are Laurent polynomials with integer coefficients. Moreover, these coefficients are known to be non-negative [LS].

An important class of cluster algebras is given by cluster algebras of surface type [GSV, FG1, FG2, FST, FT]. From a classification point of view, this class is very important, since it has been shown in [FeShTu] that almost all (skew-symmetric) mutation finite cluster algebras are of surface type. For generalizations to the skew-symmetrizable case see [FeShTu2, FeShTu3]. The closely related surface skein algebras were studied in [M, T].

If \( \mathcal{A} \) is a cluster algebra of surface type, then there exists a surface with boundary and marked points such that the cluster variables of \( \mathcal{A} \) are in bijection with certain isotopy classes of curves, called arcs, in the surface. Moreover, the relations between the cluster variables are given by the crossing patterns of the arcs in the surface. In [MSW], building on earlier work [S2, ST, S3, MS], the authors gave a combinatorial formula for the cluster variables in cluster algebras of surface type. In the sequel [MSW2], the formula was the key ingredient for the construction of two bases for the cluster algebra, in the case where the surface has no punctures and has at least 2

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marked points. As an application of the computational tools developed in [CS] and in the present paper, it is proved in [CLS] that the basis construction of [MSW2] also applies to surfaces with non-empty boundary and with exactly one marked point.

In order to construct these bases, one associates Laurent polynomials to certain curves in the surface. If the curve is an arc, these Laurent polynomials are cluster variables and are given by the formula of [MSW] in terms of perfect matchings of snake graphs. If the curve is a closed loop, one needs to replace the snake graph by a band graph, and then the Laurent polynomial is still given in terms of perfect matchings of the band graph, see [MSW2].

Perfect matchings of certain graphs have also been used in [MSc] to give expansion formulas in the cluster algebra structure of the homogenous coordinate ring of a Grassmannian.

In our previous work [CS], we have studied the snake graphs from a combinatorial point of view. Instead of constructing snake graphs from arcs in a fixed surface, we gave an abstract definition of a snake graph and studied the properties of these graphs. These abstract snake graphs are not necessarily related to the geometric situation of arcs in a surface. In analogy with the geometric situation, we defined the notion of crossing snake graphs and constructed the resolution of a pair of crossing snake graphs as two pairs of snake graphs. We then constructed a bijection between the set of perfect matchings of the pair of crossing snake graphs and the set of perfect matchings of its resolution. We also showed that, in the case where the snake graphs actually correspond to arcs in a surface, the resolution of crossing snake graphs corresponds precisely to the smoothing of the crossing of the associated arcs. In particular, we obtained a new proof of the skein relations between cluster variables in the cluster algebra.

However, in order to understand the cluster algebra, arcs alone do not provide enough information. One also needs to consider self-crossing curves and closed loops. Self-crossing curves may appear already after smoothing a crossing of two arcs which have more than one crossing point, and closed loops may appear after smoothing a self-crossing. In section 2 of the present paper, we introduce the notion of self-crossing snake graphs, and we construct the resolutions of the self-crossings of section 3. In order to describe these resolutions, the snake graphs alone are no longer sufficient; one also needs to work with band graphs. In section 4, we construct a bijection between the set of perfect matchings of a self-crossing snake graph and the set of perfect matchings of its resolution. We then show in section 6 that, in the case where the self-crossing snake graph is actually coming from a self-crossing arc in a surface, the resolution of the snake graph corresponds exactly to the smoothing of the crossing in the curve. We also show in section 7 that the corresponding skein relation holds in the cluster algebra.

The step from resolving the crossing of a pair of snake graphs to resolving a self-crossing snake graph is surprisingly difficult. The naive approach of cutting one self-crossing snake graph into two crossing snake graphs and then applying the results of [CS] does not work in general, because, in a self-crossing snake graph, the two positions where the crossing occurs may have an intersection and cannot be separated. In the geometric setting, this corresponds to a curve that spirals around a boundary component several times, approaching the boundary, and then running away from the boundary, thereby crossing itself several times.

Even for snake graphs that have a geometric interpretation as curves in a surface, there is a fundamental difference between the smoothing of a crossing of curves and the resolution of a crossing of snake graphs. The definition of smoothing is very simple. It is defined as a local transformation replacing a crossing $\times$ with the pair of segments $\sim \cup$ (resp. $\supset \subset$). But once this local transformation is done, one needs to find representatives inside the isotopy classes of the resulting curves which realize the minimal number of crossings with the fixed triangulation. This means that one needs to deform the obtained curves isotopically, and to ‘unwind’ them if possible, in order to see their actual crossing pattern, which is crucial for the applications to cluster algebras. This can be quite confusing, especially in a higher genus surface.

The situation for the snake and band graphs is exactly opposite. The definition of the resolution is very complicated because one has to consider many different cases. But once all these cases are worked out, one has a complete list of rules in hand, which one can apply very efficiently in actual computations.
In [CS], we gave these rules for pairs of crossing snake graphs, in the present paper, we treat the case of self-crossing snake graphs, and in a forthcoming paper [CS3], we will complete the work by treating crossings of band graphs and self-crossing band graphs.

In the last section, we give an example of an explicit computation of the product of two cluster variables in the cluster algebra of the torus with one boundary component and one marked point.

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2. Abstract snake graphs and abstract band graphs

Abstract snake graphs have been introduced in [CS] motivated by the snake graphs appearing in the combinatorial formulas for cluster variables in cluster algebras of surface type in [PT, MS, MSW]. Here we introduce the notion of abstract band graphs which is motivated by the band graphs used in [MSW2] to construct a bases for cluster algebras of surface type.

The construction of abstract snake graphs and band graphs is completely detached from triangulated surfaces. Our goal is to study these objects in a combinatorial way. We shall simply say snake graphs and band graphs, since we shall always mean abstract snake graphs and abstract band graphs.

In this section, we recall the constructions of [CS] pertaining to the snake graphs and, at the same time, we develop the analogue constructions for the band graphs. Throughout we fix an orthonormal basis of the plane.

2.1. Snake graphs. A tile $G$ is a square of fixed side-length in the plane whose sides are parallel or orthogonal to the fixed basis.

We consider a tile $G$ as a graph with four vertices and four edges in the obvious way. A snake graph $G$ is a connected graph consisting of a finite sequence of tiles $G_1, G_2, \ldots, G_d$ with $d \geq 1$, such that for each $i = 1, \ldots, d - 1$

(i) $G_i$ and $G_{i+1}$ share exactly one edge $e_i$ and this edge is either the north edge of $G_i$ and the south edge of $G_{i+1}$ or the east edge of $G_i$ and the west edge of $G_{i+1}$.

(ii) $G_i$ and $G_j$ have no edge in common whenever $|i - j| \geq 2$.

(ii) $G_i$ and $G_j$ are disjoint whenever $|i - j| \geq 3$.

An example is given in Figure 1. The graph consisting of two vertices and one edge joining them is also considered a snake graph.

We sometimes use the notation $G = (G_1, G_2, \ldots, G_d)$ for the snake graph and $G[i, i + t] = (G_i, G_{i+1}, \ldots, G_{i+t})$ for the subgraph of $G$ consisting of the tiles $G_i, G_{i+1}, \ldots, G_{i+t}$. One may think of this subgraph as a closed interval inside $G$. 

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**Figure 1.** A snake graph with 8 tiles and 7 interior edges (left); a sign function on the same snake graph (right).
The $d - 1$ edges $e_1, e_2, \ldots, e_{d-1}$ which are contained in two tiles are called interior edges of $G$, and the other edges are called boundary edges. Let $\text{Int}(G) = \{e_1, e_2, \ldots, e_{d-1}\}$ be the set of interior edges of $G$. We will always use the natural ordering of the set of interior edges, so that $e_i$ is the edge shared by the tiles $G_t$ and $G_{t+1}$.

If $i \geq 2$ and $i + t \leq d - 1$, the notation $G[i, i + t]$ means $G[i, i + t] \setminus \{e_{i-1}, e_{i+t}\}$. One may think of this subgraph as an open interval inside $G$.

We denote by $SWG$ the 2 element set containing the south and the west edge of the first tile of $G$ and by $G^{NE}$ the 2 element set containing the north and the east edge of the last tile of $G$.

If $G = (G_1, G_2, \ldots, G_d)$ is a snake graph and $e_i$ is the interior edge shared by the tiles $G_i$ and $G_{i+1}$, we define the snake graph $G \setminus \text{pred}(e_i)$ to be the graph obtained from $G$ by removing the vertices and edges that are predecessors of $e_i$, more precisely,

$$G \setminus \text{pred}(e_i) = G[i + 1, d].$$

It will be convenient to extend this construction to the edges $e \in G^{NE}$, thus

$$G \setminus \text{pred}(e) = \{e\} \quad \text{if} \quad e \in G^{NE}.$$ 

Similarly, we define the snake graph $G \setminus \text{succ}(e_i)$ to be the graph obtained from $G$ by removing the vertices and edges that are successors of $e_i$, more precisely,

$$G \setminus \text{succ}(e_i) = G[1, i].$$

It will be convenient to extend this construction to the edges $e \in SWG$, thus

$$G \setminus \text{succ}(e) = \{e\} \quad \text{if} \quad e \in SWG.$$

A snake graph $G$ is called straight if all its tiles lie in one column or one row, and a snake graph is called zigzag if no three consecutive tiles are straight.

2.2. Sign function. A sign function $f$ on a snake graph $G$ is a map $f$ from the set of edges of $G$ to $\{+, -\}$ such that on every tile in $G$ the north and the west edge have the same sign, the south and the east edge have the same sign and the sign on the north edge is opposite to the sign on the south edge. See Figure 1 for an example.

Note that on every snake graph with at least one tile, there are exactly two sign functions. Although the definition of sign function may not seem natural at first sight, it has a geometric meaning which is explained in Remark 5.10.

2.3. Band graphs. Band graphs are obtained from snake graphs by identifying a boundary edge of the first tile with a boundary edge of the last tile, where both edges have the same sign. We use the notation $G^b$ for general band graphs, indicating their circular shape, and we also use the notation $G^b$ if we know that the band graph is constructed by gluing a snake graph $G$ along an edge $b$.

More precisely, to define a band graph $G^b$, we start with an abstract snake graph $G = (G_1, G_2, \ldots, G_d)$ with $d \geq 1$, and fix a sign function on $G$. Denote by $x$ the southwest vertex of $G_1$, let $b \in SWG$ the south edge (respectively the west edge) of $G_1$, and let $y$ denote the other endpoint of $b$, see Figure 2. Let $b'$ be the unique edge in $G^{NE}$ that has the same sign as $b$, and let $y'$ be the northeast vertex of $G_d$ and $x'$ the other endpoint of $b'$.

Let $G^b$ denote the graph obtained from $G$ by identifying the edge $b$ with the edge $b'$ and the vertex $x$ with $x'$ and $y$ with $y'$. The graph $G^b$ is called a band graph or ouroboros.

Note that non-isomorphic snake graphs can give rise to isomorphic band graphs. See Figure 2 for an example.

The interior edges of the band graph $G^b$ are by definition the interior edges of $G$ plus the glueing edge $b = b'$. Given a band graph $G^b$ with an interior edge $e$, we denote by $G^b_e$ the snake graph obtained by cutting $G^b$ along the edge $e$. Note that $(G^b)^c = G^b$, for all band graphs $G^b$ and that $(G^b)_e = G$, for all snake graphs $G$. Moreover, if $G^b$ has $d$ interior edges $e_1, e_2, \ldots, e_d$ then the $d$ snake graphs $G^b_{e_i}$, $i = 1, \ldots, d$, are not necessarily distinct.

\[\text{Ouroboros: a snake devouring its tail.}\]
Definition 2.1. Let $\mathcal{R}$ denote the free abelian group generated by all isomorphism classes of finite disjoint unions of snake graphs and band graphs. If $G$ is a snake graph, we also denote its class in $\mathcal{R}$ by $\mathcal{G}$, and we say that $\mathcal{G} \in \mathcal{R}$ is a positive snake graph and that its inverse $-\mathcal{G} \in \mathcal{R}$ is a negative snake graph.

2.4. Labeled snake and band graphs. A labeled snake graph is a snake graph in which each edge carries a label or weight. For example, for snake graphs from cluster algebras of surface type, these labels are cluster variables.

Formally, a labeled snake graph is a snake graph $\mathcal{G}$ together with two functions

$$\{\text{tiles in } \mathcal{G}\} \rightarrow F \quad \text{and} \quad \{\text{edges in } \mathcal{G}\} \rightarrow F,$$

where $F$ is a set. Labeled band graphs are defined in the same way.

Let $\mathcal{LR}_F$ denote the free abelian group generated by all isomorphism classes of unions of labeled snake graphs and labeled band graphs with labels in $F$.

2.5. Overlaps and self-overlaps. Let $\mathcal{G}_1 = (G_1, G_2, \ldots, G_d)$ and $\mathcal{G}_2 = (G'_1, G'_2, \ldots, G'_{d'})$ be two snake graphs. We say that $\mathcal{G}_1$ and $\mathcal{G}_2$ have an overlap $\mathcal{G}$ if $\mathcal{G}$ is a snake graph consisting of at least one tile and there exist two embeddings $i_1: \mathcal{G} \rightarrow \mathcal{G}_1$, $i_2: \mathcal{G} \rightarrow \mathcal{G}_2$ which are maximal in the following sense.

(i) If $\mathcal{G}$ has at least two tiles and if there exists a snake graph $\mathcal{G}'$ with two embeddings $i'_1: \mathcal{G}' \rightarrow \mathcal{G}_1$, $i'_2: \mathcal{G} \rightarrow \mathcal{G}_2$ such that $i_1(\mathcal{G}) \subseteq i'_1(\mathcal{G}')$ and $i_2(\mathcal{G}) \subseteq i'_2(\mathcal{G}')$ then $i_1(\mathcal{G}) = i'_1(\mathcal{G}')$ and $i_2(\mathcal{G}) = i'_2(\mathcal{G}')$.

(ii) If $\mathcal{G}$ consists of a single tile then using the notation $G_k = i_1(\mathcal{G})$ and $G'_{k'} = i_2(\mathcal{G})$, we have

(a) $k \in \{1, d\}$ or $k' \in \{1, d'\}$ or

(b) $1 < k < d$, $1 < k' < d'$ and the subgraphs $(G_{k-1}, G_k, G_{k+1})$ and $(G'_{k'-1}, G'_{k'}, G'_{k'+1})$ are either both straight or both zigzag subgraphs.

An example of type (i) is shown on the left in Figure 3 and an example of type (ii)(b) on the right in the same figure.

Remark 2.2. Snake graphs may have several overlaps with respect to different snake graphs $\mathcal{G}$.

We say that a snake graph $\mathcal{G}_1 = (G_1, G_2, \ldots, G_d)$ has a self-overlap $\mathcal{G}$ if $\mathcal{G}$ is a snake graph and there exist two embeddings $i_1: \mathcal{G} \rightarrow \mathcal{G}_1$, $i_2: \mathcal{G} \rightarrow \mathcal{G}_1$ which satisfy the conditions (i) and (ii) above. Examples of self-overlaps are shown in Figures 4 and 5.

Let $s, t, s', t'$ be such that $i_1(\mathcal{G}) = \mathcal{G}_1[s, t]$ and $i_2(\mathcal{G}) = \mathcal{G}_1[s', t']$ with $s \leq t, s' \leq t'$ and suppose without loss of generality that $s < s'$. The self-overlap is said to have an intersection
Figure 4. Two snake graphs with self-overlap \((G_s, \ldots, G_t)\) and \((G'_{s'}, \ldots, G'_{t'})\). The self-overlap on the right has an intersection consisting of 3 tiles \(G'_{s'}, G'_{s'+1}\) and \(G'_{t'}\).

Figure 5. Example of a snake graph with crossing self-overlap (shaded) in the opposite direction.

if \(i_1(\mathcal{G}) \cap i_2(\mathcal{G})\) contains at least one edge, that is \(s' \leq t + 1\). In this case, we say \(\mathcal{G}_1\) has an intersecting self-overlap, see the right picture in Figure 4.

We may assume without loss of generality that the embedding \(i_1 : \mathcal{G} \to \mathcal{G}_1[s, t]\) maps the southwest vertex of the first tile of \(\mathcal{G}\) to the southwest vertex of \(G_s\) in \(\mathcal{G}_1[s, t]\). We then say that the self-overlap is in the same direction if the embedding \(i_2 : \mathcal{G} \to \mathcal{G}_1[s', t']\) maps the southwest vertex of the first tile of \(\mathcal{G}\) to the southwest vertex of \(G'_{s'}\) in \(\mathcal{G}_1[s', t']\), and we say that the self-overlap is in the opposite direction if \(i_2\) maps this vertex to the northeast vertex of \(G'_{t'}\) in \(\mathcal{G}_1[s', t']\). The self-overlaps in Figure 4 are in the same direction and the self-overlap in Figure 5 is in the opposite direction.

Remark 2.3. (1) The notion of direction depends on the embeddings \(i_1\) and \(i_2\) and not only on the subgraphs \(\mathcal{G}_1[s, t]\) and \(\mathcal{G}_1[s', t']\). See Figure 12 for an example where \(\mathcal{G}_1[s, t]\) and \(\mathcal{G}_1[s', t']\) can be considered as overlap in either direction.

(2) \(\mathcal{G}_1\) may have an intersecting self-overlap such that the intersection of \(i_1(\mathcal{G})\) and \(i_2(\mathcal{G})\) is a single edge. In this case, we have \(e_t = e_{s'-1}\), see Figure 10 for examples.

For labeled snake graphs, we define overlaps by adding the requirement that the embeddings \(i_1\) and \(i_2\) are label preserving.

2.6. Crossing overlaps. Let \(\mathcal{G}_1 = (G_1, G_2, \ldots, G_d), \mathcal{G}_2 = (G'_1, G'_2, \ldots, G'_d)\) be two snake graphs with \(d, d' \geq 1\), with overlap \(\mathcal{G}\) and embeddings \(i_1(\mathcal{G}) = \mathcal{G}_1[s, t]\) and \(i_2(\mathcal{G}) = \mathcal{G}_2[s', t']\), and suppose without loss of generality that \(s \leq t\) and \(s' \leq t'\). Let \(e_1, \ldots, e_{d-1}\) (respectively \(e'_{1}, \ldots, e'_{d'-1}\)) be the interior edges of \(\mathcal{G}_1\) (respectively \(\mathcal{G}_2\)). Let \(f\) be a sign function on \(\mathcal{G}\). Then \(f\) induces a sign function \(f_1\) on \(\mathcal{G}_1\) and \(f_2\) on \(\mathcal{G}_2\). Moreover, since the overlap \(\mathcal{G}\) is maximal, we have

\[
\begin{align*}
f_1(e_{s-1}) &= -f_2(e'_{s'-1}) & \text{if } s > 1, s' > 1 \\
f_1(e_t) &= -f_2(e'_{t'}) & \text{if } t < d, t' < d'.
\end{align*}
\]

Definition 2.4. We say that \(\mathcal{G}_1\) and \(\mathcal{G}_2\) cross in \(\mathcal{G}\) if one of the following conditions hold.
Remark 2.7. Consider the self-overlaps in Figure 4. In the example on the left side of the figure, the edge \( e_t \) is the edge shared by the first shaded region and the first white region, and the edge \( e_{s'-1} \) is the edge shared by the first white region and the second shaded region. Thus \( f(e_t) = f(e_{s'-1}) \). Moreover, the edge \( e_{t'} \) is the edge shared by the second shaded region and the second white region. Thus \( f(e_{s'-1}) = -f(e_{t'}) \) and the snake graph has a self-crossing in this self-overlap.

The example on the right side of Figure 4, the edge \( e_t \) is the edge shared by the dark shaded region and the second light shaded region, and the edge \( e_{s'-1} \) is the edge shared by the dark shaded region and the first light shaded region. Thus \( f(e_t) = f(e_{s'-1}) \). Moreover, \( e_{t'} \) is the edge shared by the second light shaded region and the second white region. Thus \( f(e_{s'-1}) = -f(e_{t'}) \) and there is a self-crossing in this self-overlap.

Example 2.9. Now consider the example in Figure 5. We have \( f(e_{s-1}) = - \) and \( f(e_t) = + \), and \( f(e_t) = + = f(e_{s'-1}) \) again giving a self-crossing.

Remark 2.10. (1) The definition of self-crossing does not depend on the choice of the sign function \( f \).

(2) The terminology ‘self-cross’ comes from snake graphs that are associated to generalized arcs in a surface. We shall show in Theorem 6.1 that crossings and self-crossings of arcs in a surface correspond precisely to crossings and self-crossings of snake graphs in an overlap.
3. Resolutions

In this section, we define the resolutions of crossings and self-crossings. Given a pair of crossing snake graphs, or a single self-crossing snake graph, the resolution of the crossing consists of a sum of two elements in the group $R$. In a forthcoming paper [CS3], we will introduce a ring structure on the group $R$ and consider the ideal generated by all resolutions. Thus in the resulting quotient ring the crossing pair of snake graphs (or the self-crossing snake graph) is equal to its resolution. This quotient ring is strongly related to cluster algebras from surfaces.

3.1. Resolution of crossing. Let $G_1, G_2$ be two snake graphs crossing in an overlap $G = G_1[s, t] = G_2[s', t']$. Recall that we use the notation $G_k[i, j]$ for the subgraph of $G_k$ given by the tiles with indices $i, i + 1, \ldots, j$. Let $G_k[j, i]$ be the snake graph obtained by reflecting $G_k[i, j]$ such that the order of the tiles is reversed.

We define four connected subgraphs $G_3 - G_6$ below. In the definition of $G_5$ and $G_6$, we shall use the notation $G'_5 = G_1[1, s - 1] \cup G_2[s' - 1, 1]$ and $G'_6 = G_2[d', t' + 1] \cup G_1[t + 1, d]$, where, in the definition of $G'_5$, the two subgraphs are glued along the north edge of $G_{s-1}$ and the east edge of $G'_{s-1}$ if $G_s$ is east of $G_{s-1}$ in $G_1$ and along the east edge of $G_{s-1}$ and the north edge of $G'_{s-1}$ if $G_s$ is north of $G_{s-1}$ in $G_1$ and, in the definition of $G'_6$, the two subgraphs are glued along the west edge of $G_{t+1}$ and the south edge of $G'_{t+1}$ if $G_{t+1}$ is north of $G_1$ in $G_1$ and along the south edge of $G_{t+1}$ and the west edge of $G'_{t+1}$ if $G_{t+1}$ is east of $G_1$ in $G_1$. Let $f_5$ be a sign function on $G'_5$ and $f_6$ a sign function on $G'_6$.

We define four connected subgraphs as follows, see Figure 6 for examples.

$G_3 = G_1[1, t] \cup G_2[t' + 1, d']$ where the adjacency of the two subgraphs is induced by $G_2$.

$G_4 = G_2[1, t'] \cup G_1[t + 1, d]$ where the adjacency of the two subgraphs is induced by $G_1$.

$G_5 = \begin{cases} G'_5 \setminus \text{succ}(e) & \text{if } s > 1, s' > 1 \\ G'_5 \setminus \text{pred}(e) & \text{if } s = 1 \text{ where } e \text{ is the last edge in } \text{Int}(G'_5) \cup SWG'_5 \text{ such that } f_5(e) = f_5(e_{s-1}); \\ G'_5 \setminus \text{pred}(e) & \text{if } s = 1 \text{ where } e \text{ is the first edge in } \text{Int}(G'_5) \cup G'_5^{\text{NE}} \text{ such that } f_5(e) = f_5(e'_{s-1}); \\ G'_6 & \text{if } t < d, t' < d'; \\ G'_6 \setminus \text{succ}(e) & \text{if } t = d, \text{ where } e \text{ is the last edge in } \text{Int}(G'_6) \cup SWG'_6 \text{ such that } f_6(e) = f_6(e_{t'}); \\ G'_6 \setminus \text{pred}(e) & \text{if } t' = d', \text{ where } e \text{ is the first edge in } \text{Int}(G'_6) \cup G'_6^{\text{NE}} \text{ such that } f_6(e) = f_6(e_t). \end{cases}$

Definition 3.1. In the above situation, we say that the element $(G_3 \cup G_4) + (G_5 \cup G_6) \in R$ is the resolution of the crossing of $G_1$ and $G_2$ at the overlap $G$ and we denote it by $\text{Res}_G(G_1, G_2)$.

If $G_1, G_2$ have no crossing in $G$ we let $\text{Res}_G(G_1, G_2) = G_1 \sqcup G_2$.

Remark 3.2. The pair $(G_3, G_4)$ still has an overlap in $G$ but without crossing. The pair $(G_5, G_6)$ can be thought of as a reduced symmetric difference of $G_1$ and $G_2$ with respect to the overlap $G$.

3.2. Resolution of self-crossing. To define the resolution of a self-crossing, we construct two snake graphs and a band graph from the self-crossing snake graph. We consider these snake and band graphs as elements of the group $R$ of Definition 2.1. In particular, we allow them to be negative. In section 9 we show that there is a bijection between the set of perfect matchings of a self-crossing snake graph and the set of perfect matchings of the resolution. In sections 9 and 10 we show that this construction is related to multiplication formulas given by skein relations in cluster algebras.
Let $G_1$ be a self-crossing snake graph with self-overlap $i_1(G) = G_1[s, t] \cong G_1[s', t'] = i_2(G)$. We consider two cases.

Case 1. Overlap in the same direction. We define the following connected graphs.

\begin{align*}
G_3 &= G_1[1, t] \cup G_1[t' + 1, d], \text{ where the adjacency of the subgraphs is induced by } G; \\
G_4 &= G_1[s, s' - 1], \text{ where } b = e_{s' - 1} \text{ is the interior edge shared by } G_{s' - 1} \text{ and } G_{s'}; \\
G_5 &= G_1[t + 1, t'], \text{ where } b' = e_t \text{ is the interior edge shared by } G_t \text{ and } G_{t + 1};
\end{align*}

and $G_{56}$ depends on several cases and is defined below. We illustrate many of these cases in the figures 7–12. These figures also show geometric realizations of the snake graphs in triangulated surfaces; this geometric construction of snake graphs is explained in section 5. Note however that not every self-crossing abstract snake graph has a geometric realization in an unpunctured surface.

1. If $s' \leq t$ (see Figure 7) then

\[ G_{56} = -G_1[1, s - 1] \cup G_1[s', t] \cup G_1[t' + 1, d], \]

where the adjacencies of the subgraphs are induced by $G$;

2. If $s' > t + 1$ (see Figures 8 and 9) then $G_{56}$ is defined to be a subgraph of the following graph

\[ G_{56}' = G_1[1, s - 1] \cup G_1[s' - 1, t + 1] \cup G_1[t' + 1, d], \]

where the first two graphs are glued along the unique boundary edge of $G_{s - 1}$ which is north or east and the unique boundary edge of $G_{s'-1}$ which is north or east, whereas the second two graphs are glued along the unique boundary edge of $G_{t+1}$ which is south or west and the unique boundary edge of $G_{t'+1}$ which is south or west. Let $f_{56}$ be a sign function on $G_{56}'$.

(a) if $s \neq 1$ and $t' \neq d$

\[ G_{56} = G_{56}'. \]
(b) if \( s = 1 \) and \( t' \neq d \) (see Figure 9)

\[ \mathcal{G}_{56} = \mathcal{G}'_{56} \setminus \text{pred}(e), \]

where \( e \) is the first edge in \( \text{Int}(\mathcal{G}'_{56}) \cup \mathcal{G}^{NE}_{56} \) such that \( f_{56}(e) = f_{56}(e_{s'-1}). \)

(c) if \( s \neq 1 \) and \( t' = d \)

\[ \mathcal{G}_{56} = \mathcal{G}'_{56} \setminus \text{succ}(e'), \]

where \( e' \) is the last edge in \( \text{Int}(\mathcal{G}'_{56}) \cup \text{SW}\mathcal{G}'_{56} \) such that \( f_{56}(e') = f_{56}(e_t) \)

(d) if \( s = 1 \) and \( t' = d \) (see Figure 9)

\[ \mathcal{G}_{56} = \mathcal{G}'_{56} \setminus \text{pred}(e) \setminus \text{succ}(e'), \]

where \( e \) is the first edge in \( \text{Int}(\mathcal{G}'_{56}) \cup \mathcal{G}^{NE}_{56} \) such that \( f_{56}(e) = f_{56}(e_{s'-1}) \) and \( e' \) is the last edge in \( \text{Int}(\mathcal{G}'_{56}) \cup \text{SW}\mathcal{G}'_{56} \) such that \( f_{56}(e') = f_{56}(e_t) \), if this set is non-empty. Otherwise, let \( \mathcal{G}_{56} = 0. \)

(3) If \( s' = t + 1 \) (see Figure 10) then

(a) if \( s = 1 \) and \( d \neq t' \) then

\[ \mathcal{G}_{56} = -\mathcal{G}_1 \setminus \text{pred}(e), \] where \( e \) is the first edge in \( \text{Int}(\mathcal{G}_1[t'+1,d]) \cup \mathcal{G}^{NE}_1 \) such that \( f(e) = -f(e_t); \)

(b) if \( s = 1 \) and \( d = t' \) then

\[ \mathcal{G}_{56} = -\{e_t\}; \]

(c) if \( s \neq 1 \) and \( d \neq t' \) (see Figure 10) then we need to consider the local overlap in opposite direction \( \mathcal{G}_1[k+1,s-1] \) and \( \mathcal{G}_1[t'+1,t'+s-k-1] \) consisting of tiles preceding \( G_s \) and tiles succeeding \( G_{t'} \), where \( k \) is given by the maximality condition for overlaps. Thus \( k \) is the least integer such that \( k \geq 0, d \geq t' + s - k - 1, \) and \( \mathcal{G}_1[s-1,k+1] \cong \mathcal{G}_1[t'+1,t'+s-k-1] \). Thus there is a snake graph \( \mathcal{H} \) and embeddings \( j_1(\mathcal{H}) = \mathcal{G}_1[s-1,k+1] \) and \( j_2(\mathcal{H}) = \mathcal{G}_1[t'+1,t'+s-k-1] \). In the examples in Figure 10 we have \( k = 1 \), \( \mathcal{H} \) consists of a single tile, and \( j_1(\mathcal{H}), j_2(\mathcal{H}) \) are the two unlabeled tiles in \( \mathcal{G}_1 \). The four cases in the following definition reflect whether \( j_1(\mathcal{H}) \) contains the first tile and \( j_2(\mathcal{H}) \) contains the last tile of \( A_1 \).

Then let

\[ \mathcal{G}_{56} = \pm \begin{cases} 
\mathcal{G}_1[k+1] \cup \mathcal{G}_1[t'+s-k,d] & \text{if } k > 0 \text{ and } k > s + t' - d - 1, \text{ where the two graphs are glued along the unique boundary edge of } G_k \text{ which is north or east and the unique boundary edge of } G_{t'+s-k} \text{ which is south or west}; \\
\mathcal{G}_1 \setminus \text{pred}(e') & \text{if } k = 0 \text{ and } k > s + t' - d - 1, \text{ where } e' \text{ is the first edge in } \text{Int}(\mathcal{G}_1[s+t',d]) \cup \mathcal{G}^{NE}_1 \text{ such that } f(e') = f(e_{s+t' - d - 1}); \\
\mathcal{G}_1 \setminus \text{succ}(e) & \text{if } k > 0 \text{ and } k = s + t' - d - 1, \text{ where } e \text{ is the last edge in } \text{Int}(\mathcal{G}_1[1,s+t'-d-1]) \cup \text{SW}\mathcal{G}_1 \text{ such that } f(e) = f(e_{s+t'-d-1}); \\
0 & \text{if } k = 0 \text{ and } s + t' - d - 1. 
\end{cases} \]

where the sign is negative if and only if the local overlap \( j_1(\mathcal{H}) \) and \( j_2(\mathcal{H}) \) is crossing.

In the first example in Figure 10 the local overlap \( j_1(\mathcal{H}) \) and \( j_2(\mathcal{H}) \) is crossing, and in the second example it is non-crossing.

(d) if \( s \neq 1 \) and \( d = t' \)

\[ \mathcal{G}_{56} = -\mathcal{G}_1 \setminus \text{pred}(e) \] where \( e \) is the last edge in \( \text{Int}(\mathcal{G}_1[1,s-1]) \cup \text{SW}\mathcal{G}_1 \) such that \( f(e) = -f(e_{s-1}). \)

**Definition 3.3.** In the above situation, we say that the element \( (\mathcal{G}_1 \cup \mathcal{G}^2_1) + \mathcal{G}_{56} \in \mathcal{R} \) is the resolution of the self-crossing of \( \mathcal{G}_1 \) at the overlap \( \mathcal{G} \) and we denote it by \( \text{Res}_\mathcal{G}(\mathcal{G}_1) \).
Case 2. Overlap in the opposite direction. As before, let $\mathcal{G}_1$ be a self-crossing snake graph with self-overlap $i_1(\mathcal{G}) = \mathcal{G}_1[s,t] \cong \mathcal{G}_1[s',t'] = i_2(\mathcal{G})$. We suppose now that the overlap is in the opposite direction, thus the first tile of $\mathcal{G}$ is mapped to the first tile of $\mathcal{G}_1[s,t]$ under $i_1$ and to the last tile of $\mathcal{G}_1[s',t']$ under $i_2$. 

\[
\begin{align*}
\mathcal{G}_1 & = \mathcal{G}_3 + \mathcal{G}_4 + \mathcal{G}_56 \\
& = \mathcal{G}_1[1,s-1] \cup \mathcal{G}_1[s',t] \cup \mathcal{G}_1[t'+1,d]
\end{align*}
\]
We define the following connected graphs, see Figure 11 for an example. In the definition of $G_5$ we shall use the notation

$$G'_5 = G_1[1,s-1] \cup G_1[t'+1,d],$$

where the two subgraphs are glued along the unique boundary edges of $G'^{NE}_{s-1}$ and $SWG_{t'+1}$.

Let $f_5$ be a sign function on $G'_5$. With this notation, we define

$$G_{34} = \begin{cases} 
G'_5 & \text{if } s \neq 1, t' \neq d; \\
G'_5 \setminus \text{pred}(e) & \text{if } s = 1, t' \neq d, \text{ where } e \in \text{Int}(G'_5) \cup G'^{NE}_{s-1} \text{ is the first edge such that } f_5(e) = f_5(e'); \\
G'_5 \setminus \text{succ}(e) & \text{if } s \neq 1, t' = d, \text{ where } e \in \text{Int}(G'_5) \cup SEG'_5 \text{ is the last edge such that } f_5(e) = f_5(e_{s-1}); \\
0 & \text{if } s = 1, t' = d.
\end{cases}$$

$$G'_6 = G'_1[t+1,s'-1] \text{ where } b \text{ is the unique boundary edge in } SWG_{t+1} \text{ and } b' \text{ is the unique boundary edge in } G'^{NE}_{s'-1}.$$ 

**Definition 3.4.** In the above situation, we say that the element $G_{34} + (G_5 \cup G'_6) \in \mathcal{R}$ is the resolution of the self-crossing of $G_1$ at the overlap $G$ and we denote it by $\text{Res}_G(G_1)$. 

**Figure 9.** Example of resolution of selfcrossing when $s' > t + 1$, $s = 1$ and $t' = d$, together with geometric realization on the annulus. Here $G_{56}$ is the empty set and the corresponding arc in the surface is a contractible loop at a boundary point.
Figure 10. Examples of selfcrossing when $s' = t + 1$.

Remark 3.5. For the two possible directions of overlap, our choice of notation for indices is consistent in the sense that indices 3, 4 always refer to the part of the resolution that contains the overlaps, and the indices 5, 6 always refer to the part that does not.

3.3. Grafting. In this subsection, we define another operation which to two snake graphs associates two pairs of snake graphs. This operation does not involve the notion of overlaps.

Let $G_1 = (G_1, G_2, \ldots, G_d)$, $G_2 = (G'_1, G'_2, \ldots, G'_d)$ be two snake graphs and let $f_1$ be a sign function on $G_1$.

Case 1. Let $s$ be such that $1 \leq s < d$.

If $G_{s+1}$ is north of $G_s$ in $G_1$, then let $\delta_s$ denote the east edge of $G_s$, $\delta_5$ the west edge of $G_{s+1}$ and $\delta'_5$ the west edge of $G'_1$.

If $G_{s+1}$ is east of $G_s$ in $G_1$, then let $\delta_s$ denote the north edge of $G_s$, $\delta_5$ the south edge of $G_{s+1}$ and $\delta'_5$ the south edge of $G'_1$. Thus we have one of the following two situations.
Figure 11. Resolution of a self-crossing in opposite direction.

Define four snake graphs as follows; see Figure 13 for examples.

\[ G_3 = G_1[1,s] \cup G_2, \] where the two subgraphs are glued along the edges \( \delta_3 \) and \( \delta'_3 \);

\[ G_4 = G_1 \setminus \text{pred}(e), \] where \( e \in \text{Int}(G_1[s+1,d]) \cup G_1^{NE} \) is the first edge such that \( f_1(e) = f_1(\delta_3) \);

\[ G_5 = G_1 \setminus \text{succ}(e), \] where \( e \in \text{SW}G_1 \cup \text{Int}(G_1[1,s]) \) is the last edge such that \( f_1(e) = f_1(\delta_3) \);

\[ G_6 = G_2[d',1] \cup G_1[s+1,d], \] where the two subgraphs are glued along the edges \( \delta_5 \) and \( \delta'_5 \).

**Case 2.** Now let \( s = d \). Choose a pair of edges \((\delta_3, \delta'_3)\) such that either \( \delta_3 \) is the north edge in \( G_s \) and \( \delta'_3 \) is the south edge in \( G'_s \) or \( \delta_3 \) is the east edge in \( G_s \) and \( \delta'_3 \) is the west edge in \( G'_s \). Let \( f_2 \) be a sign function on \( G_2 \) such that \( f_2(\delta'_3) = f_1(\delta_3) \). Then define four snake graphs as follows.

\[ G_3 = G_1[1,s] \cup G_2, \] where the two subgraphs are glued along the edges \( \delta_3 \) and \( \delta'_3 \);

\[ G_4 = \{ \delta_3 \} \]

\[ G_5 = G_1 \setminus \text{succ}(e), \] where \( e \in \text{SW}G_1 \cup \text{Int}(G_1[1,s]) \) is the last edge such that \( f_1(e) = f_1(\delta_3) \);

\[ G_6 = G_2 \setminus \text{pred}(e), \] where \( e \in \text{Int}(G_2) \cup G_2^{NE} \) is the first edge such that \( f_2(e) = f_2(\delta'_3) \)

**Definition 3.6.** In the above situation, we say that the element \((G_3 \cup G_4) + (G_5 \cup G_6) \in R\) is the resolution of the grafting of \( G_4 \) on \( G_1 \) in \( G_s \) and we denote it by \( \text{Graft}_{s,\delta_3}(G_1,G_2) \).

**Case 3.** Grafting with a single edge. In this case let \( G_1 = (G_1,\ldots,G_d) \) and let the snake graph \( G_2 \) consist of a single edge. We are grafting this edge at a position \( s \), where \( 1 \leq s \leq d \), as follows. We define
Figure 12. Example of a snake graph with self-crossing in a single tile which can be considered as in either direction.

\[ \mathcal{G}_3 = \mathcal{G}_1 \setminus \text{succ}(e), \text{ where } e \in \text{Int}(\mathcal{G}_1[1, s]) \cup \text{SW} \mathcal{G}_1 \text{ is the last edge such that } f(e) = -; \]

\[ \mathcal{G}_4 = \mathcal{G}_1 \setminus \text{pred}(e), \text{ where } e \in \text{Int}(\mathcal{G}_1[s, d]) \cup \mathcal{G}_1^{NE} \text{ is the first edge such that } f(e) = +; \]

\[ \mathcal{G}_5 = \mathcal{G}_1 \setminus \text{succ}(e), \text{ where } e \in \text{Int}(\mathcal{G}_1[1, s]) \cup \text{SW} \mathcal{G}_1 \text{ is the last edge such that } f(e) = +; \]

\[ \mathcal{G}_6 = \mathcal{G}_1 \setminus \text{pred}(e), \text{ where } e \in \text{Int}(\mathcal{G}_1[s, d]) \cup \mathcal{G}_1^{NE} \text{ is the first edge such that } f(e) = -. \]

In particular, if \( \mathcal{G}_1 \) consists of a single tile, then \( s = d = 1 \) and one of \( \mathcal{G}_3 \cup \mathcal{G}_4 \), \( \mathcal{G}_5 \cup \mathcal{G}_6 \) is the east and the west edge of \( \mathcal{G}_1 \) and the other is the north and the south edge of \( \mathcal{G}_1 \).

3.4. Self-grafting. In this subsection, we define the grafting operation on a single snake graph. Let \( \mathcal{G}_1 = (G_1, G_2, \ldots, G_d) \), be a snake graph and let \( f \) be a sign function on \( \mathcal{G}_1 \). We will define the resolution of self-grafting on \( \mathcal{G}_1 \) in \( \mathcal{G}_s \).

Case 1. Let \( s \) be such that \( 1 \leq s < d \). If \( G_{s+1} \) is north of \( G_s \) in \( \mathcal{G}_1 \) then let \( \delta_3 \) denote the east edge of \( G_s \), \( \delta_5 \) the west edge of \( G_{s+1} \) and if \( G_{s+1} \) is east of \( G_s \) in \( \mathcal{G}_1 \), then let \( \delta_3 \) denote the north edge of \( G_s \).
edge of $G_s$, $\delta_5$ the south edge of $G_{s+1}$. In either case we have $f(\delta_3) = f(\delta_5)$. Let $\delta'_3, \delta'_5 \in SW G_1$ be such that $f(\delta'_3) = f(\delta_3) = f(\delta'_5)$ and $f(\delta'_5) = -f(\delta_3) = -f(\delta_5)$.

Thus we have one of the following two situations.

Define two snake graphs and a band graph as follows, see Figure 14 for an example.

$\mathcal{G}_3 = \mathcal{G}_1 \setminus \text{pred}(e)$, where $e \in \text{Int}(\mathcal{G}_1[s+1,d]) \cup \mathcal{G}_1^{NE}$ is the first edge such that $f(e) = f(\delta_3)$;

$\mathcal{G}_4 = (\mathcal{G}_1[s,1])^{\delta_3}$, the band graph obtained by identifying $\delta_3$ and $\delta'_3$;

Let $\mathcal{G}'_{56} = \mathcal{G}_1'[s,1] \cup \mathcal{G}_1'[s+1,d]$ glued along $\delta_5$ and $\delta'_5$, and let $f_{56}$ be a sign function on $\mathcal{G}'_{56}$. Then define

$\mathcal{G}_{56} = \mathcal{G}'_{56} \setminus \text{pred}(e)$, where $e \in \text{Int}(\mathcal{G}'_{56}) \cup \mathcal{G}'_{56}^{NE}$ is the first edge such that $f_{56}(e) = f_{56}(\delta_3)$.
Case 2. Now let $s = d$ (see Figure 15). Choose an edge $\delta_3 \in G_1^{NE}$, and let $\delta'_3 \in SW G_1$ be the unique edge such that $f(\delta_3) = f(\delta'_3)$. Then define

$$
G_3 = \{\delta_3\}; \\
G_4^{\delta_3} = \text{the band graph obtained by identifying } \delta_3 \text{ and } \delta'_3; \\
G_{56} = \begin{cases} \\
G_1 \setminus \text{succ}(e) \setminus \text{pred}(e') & \text{where } e \text{ is the last edge in } \text{Int}(G_1) \text{ such that } f(e) = f(\delta_3) \\
& \text{and } e' \text{ is the first edge in } \text{Int}(G_1 \setminus \text{succ}(e)) \text{ such that } f(e') = f(\delta'_3), \text{ if such an } e' \text{ exists}; \\
0 & \text{otherwise.}
\end{cases}
$$

Remark 3.7. This last case is particularly important in actual computations, since it computes the difference between a band graph and any of its underlying snake graphs.

Definition 3.8. In the above situation, we say that the element $(G_3 \sqcup G_4^{\delta_3}) + G_{56} \in \mathcal{R}$ is the resolution of the self-grafting of $G_1$ in $G_s$ and we denote it by $\text{Graft}_{s,\delta_3}(G_1)$.

4. Perfect Matchings

In this section, we show that there is a bijection between the set of perfect matchings of (self-)crossing snake graphs and the set of perfect matchings of the resolution. In sections 6 and 7 we will show that for labeled snake graphs coming from an unpunctured surface, this bijection is weight preserving and induces an identity in the corresponding cluster algebra.

Recall that a perfect matching $P$ of a graph $G$ is a subset of the set of edges of $G$ such that each vertex of $G$ is incident to exactly one edge in $P$.

For band graphs, we need the notion of good perfect matchings, which was introduced in [MSW2, Definition 3.8]. Let $G$ be a snake graph and let $G^b$ be the band graph obtained from $G$ by gluing along the edge $b$ as defined in section 2.3. If $P$ is a perfect matching of $G$ containing
the edge \( b \), then \( P \setminus \{ b \} \) is a perfect matching of \( G^b \). In the following definition of good perfect matchings for arbitrary band graphs, we start with the band graph and cut it at an interior edge.

**Definition 4.1.** Let \( G^o \) be a band graph. A perfect matching \( P \) of \( G^o \) is called a good perfect matching if there exists an interior edge \( e \) in \( G^o \) such that \( P \cup \{ e \} \) is a perfect matching of the snake graph \( (G^o)_e \) obtained by cutting \( G^o \) along \( e \).

**Definition 4.2.**

1. If \( G \) is a snake graph, let \( \text{Match} \ G \) denote the set of all perfect matchings of \( G \).
2. If \( G^o \) is a band graph, let \( \text{Match} \ G^o \) denote the set of all good perfect matchings of \( G^o \).
3. If \( R = (G_1 \sqcup G_2 + G_3 \sqcup G_4) \in R \), we let \( \text{Match} \ R = \text{Match} \ G_1 \times \text{Match} \ G_2 \sqcup \text{Match} \ G_3 \times \text{Match} \ G_4 \).

The following Lemma will be useful later on.

**Lemma 4.3.** Let \( G \) be a snake graph with sign function \( f \) and let \( P \) be a matching of \( G \) which consists of boundary edges only. Let \( NE \) be the set of all north and all east edges of the boundary of \( G \) and let \( SW \) be the set of all south and west edges of the boundary. Then

1. \( f(a) = f(b) \) if \( a \) and \( b \) are both in \( P \cap NE \) or both in \( P \cap SW \).
2. \( f(a) = -f(b) \) if one of \( a \), \( b \) is in \( P \cap NE \) and the other in \( P \cap SW \).
3. If \( a \in P \cap NE \), or if \( a \in SW \) but \( a \notin P \), then \( P = \{ b \in NE \mid f(b) = f(a) \} \cup \{ b \in SW \mid f(b) = -f(a) \} \).
4. If \( a \in P \cap SW \), or if \( a \in NE \) but \( a \notin P \), then \( P = \{ b \in NE \mid f(b) = -f(a) \} \cup \{ b \in SW \mid f(b) = f(a) \} \).

**Proof.** Statements (1) and (2) are Lemma 7.2 in [CS]. The statements (3) and (4) clearly hold for straight snake graphs. For general snake graphs it follows from the following observation. Suppose that three consecutive tiles \( G_{i-1}, G_i, G_{i+1} \) form a zigzag, and suppose without loss of generality that \( G_i \) is east of \( G_{i-1} \) and \( G_{i+1} \) is north of \( G_i \). Then the north edge \( n \) of \( G_{i-1} \), the south edge \( s \) and the east edge \( e \) of \( G_i \), and the west edge \( w \) of \( G_{i+1} \) are boundary edges. Moreover all four edges have the same sign \( f(n) = f(s) = f(e) = f(w) \). Since \( P \) consists of boundary edges only there are two possibilities, either \( n, e \in P \) and \( s, w \notin P \) or \( n, e \notin P \) and \( s, w \in P \).

In [CS], we constructed bijections between the set of perfect matchings of two crossing snake graphs and the set of perfect matchings of the resolution of the crossing. One of the main results of [CS] is the following.

**Theorem 4.4.** [CS] Theorem 3.1 Let \( G_1, G_2 \) be two snake graphs. Then there are bijections

1. \( \text{Match} \ (G_1 \sqcup G_2) \rightarrow \text{Match} \ (\text{Res}_G(G_1 \sqcup G_2)) \);
2. \( \text{Match} \ (G_1 \sqcup G_2) \rightarrow \text{Match} \ (\text{Graft}_{x,e_3}(G_1 \sqcup G_2)) \).

We now extend this construction to give bijections between sets of perfect matchings of a self-crossing snake graph and of the resolution of the self-crossing.

### 4.1. Switching operation.

The following construction was introduced in [CS]. Let \( G_1, G_2 \) be two snake graphs with a crossing local overlap \( G \) and embeddings \( i_1 : G \to G_1 \) and \( i_2 : G \to G_2 \). Let \( G_3, G_4 \) be the pair of snake graphs in the resolution of the crossing which contain the overlap. Thus \( G_3 \) contains the initial part of \( G_1 \), the overlap, and the terminal part of \( G_2 \), whereas \( G_4 \) contains the initial part of \( G_2 \), the overlap, and the terminal part of \( G_1 \).

Given two perfect matchings \( P_1 \in \text{Match} \ G_1 \) and \( P_2 \in \text{Match} \ G_2 \), there may or may not exist a switching position, that is, a position in the overlap such that using the edges of \( P_1 \) up to that position and the edges of \( P_2 \) after that position yield a perfect matching on \( G_3 \). In this case, using the edges of \( P_2 \) up to the switching position and those of \( P_1 \) after the switching position also yields a matching of \( G_4 \). The possible local configurations of the perfect matchings at the switching position are explicitly listed in Figures 6-11 in [CS] section 3].
If a switching position exists, we define the \textit{switching operation} to be the map \((P_1, P_2) \mapsto (P_3, P_4)\), where \(P_3\), respectively \(P_4\), is the matching of \(G_3\), respectively \(G_4\), obtained by applying the method above at the first switching position. In the same way, we define the switching operation sending matchings of \(G_3 \sqcup G_4\) to matchings of \(G_1 \sqcup G_2\). If no switching position exists, then the restrictions \((P_1 \cup P_2)|_{G_3}\) and \((P_1 \cup P_2)|_{G_4}\) are perfect matchings of \(G_3\) and \(G_4\), respectively.

The switching operation generalizes in a straightforward way when we consider a single snake graph \(G_1\) with a crossing self-overlap instead of the pair \((G_1, G_2)\), as long as the self-overlap does not have an intersection.

By Definition 3.3, \(G_3\) and \(G_3^2\) are still two distinct graphs if the overlaps have the same orientation, and if the orientations are opposite, then, by Definition 3.4, we obtain a single snake graph \(G_{34}\).

Throughout the rest of this section let \(G_1 = (G_1, \ldots, G_d)\) be a snake graph with self-crossing in the overlap \(i_1(G) = G_1[s, t] \cong G_1[s', t'] = i_2(G)\), with \(s < s'\), and let \(f\) be a sign function on \(G_1\).

4.2. \textbf{Self-crossing case 1: Overlap in the same direction and \(s' > t + 1\).} Let \(P_1 \in \text{Match } G_1\). If \(P_1\) has a switching position in \(i_1(G)\) and \(i_2(G)\), we define \(P_3 \in \text{Match } G_3\) by

\[
\begin{align*}
P_3 &= P_1 \\
P_3 &= \begin{cases} P_1|_{i_1(G)} & \text{before the first switching position;} \\
P_1|_{i_2(G)} & \text{after the first switching position;} \end{cases} \\
&\quad \text{on } G_1[1, s-1] \cup G_1[t'+1, d],
\end{align*}
\]

And we define \(P_4 \in \text{Match } G_4^2\) by

\[
\begin{align*}
P_4 &= P_1 \\
P_4 &= \begin{cases} P_1|_{i_2(G)} & \text{before the first switching position;} \\
P_1|_{i_1(G)} & \text{after the first switching position;} \end{cases} \\
&\quad \text{on } G_1[t+1, s'-1],
\end{align*}
\]

With this notation, we define

\[
\varphi(P_1) = \begin{cases} (P_3, P_4) & \text{if } P_1 \text{ has a switching position in } i_1(G) \text{ and } i_2(G); \\
P_3 = P_1|_{G_{56}} & \text{if } P_1 \text{ has no switching position in } i_1(G) \text{ and } i_2(G). \end{cases}
\]

To show that \(\varphi\) is a bijection, we define its inverse function \(\psi\). Let \(i_3 : G \rightarrow G_3\) and \(i_4 : G \rightarrow G_4^2\) be the embeddings of the overlap, and let \((P_3, P_4) \in \text{Match } G_3 \times G_4^2\). It follows from [CS, Lemma 7.3] that \((P_3, P_4)\) has a switching position in \(i_3(G)\) and \(i_4(G)\). We define \(\psi(P_3, P_4) = P_1\) to be

\[
\begin{align*}
P_1 &= P_3 \\
P_1 &= \begin{cases} P_3|_{i_3(G)} & \text{before the first switching position;} \\
P_4|_{i_4(G)} & \text{after the first switching position;} \end{cases} \\
&\quad \text{on } G_1[1, s-1] \cup G_1[t'+1, d] \\
P_1 &= P_4 \\
P_1 &= \begin{cases} P_4|_{i_4(G)} & \text{before the first switching position;} \\
P_3|_{i_3(G)} & \text{after the first switching position;} \end{cases} \\
&\quad \text{on } G_1[t+1, s'-1].
\end{align*}
\]

Now let \(P_5 \in \text{Match } G_{56}\). Define \(\psi(P_5) \in \text{Match } G_1\) as follows. If \(G_{56} \neq G_{56}'\) then first complete \(P_5\) to a matching \(P_5'\) of \(G_{56}'\) using only boundary edges of \(G_{56}\) and then complete \(P_5'\) to a matching of \(G_1\) using only boundary edges of \(G_1\). This unique completion does not depend on \(P_5\) and it is determined by the fact that it does not contain the edges \(e\) and \(e'\), where \(e\) is the unique edge in \(SW G_s\) such that \(f(e) = f(e_{s-1})\) and \(e'\) is the unique edge in \(G_{56}^{NE}\) such that \(f(e') = f(e_t)\). Indeed, if \(s \neq 1\), then \(SW G_s = \{e, e_{s-1}\}\) and the endpoints of \(e_{s-1}\) are matched in \(\psi(P_5)\) by edges of \(G_{s-1}\), so \(e \notin \psi(P_5)\). If \(s = 1\), then the endpoints of \(e_1\) are matched in \(\psi(P_5)\) by edges of \(G_{1+1}\), because \(G_{1+1} \subset G_{56}\). Since \(e_i\) is in \(G_{56}^{NE}\), it follows from Lemma 4.3 that all north and east edges of \(G_1[s, t] \cap \psi(P_5)\) have the same sign as \(e_i\) and all south and west edges have the opposite sign. Since \(e \in SW G_s\) and \(f(e) = f(e_{s-1}) = f(e_t)\), it follows that \(e \notin \psi(P_5)\). The proof for \(e'\) is similar.

It is clear that if \(P_1\) has a switching position in \(i_1(G)\) and \(i_2(G)\) then the resulting pair \(\varphi(P_1) = (P_3, P_4)\) has the same switching position in \(i_3(G)\) and \(i_4(G)\), and the resulting matching \(\psi(P_3, P_4) = P_1\). Similarly, if \((P_3, P_4)\) has a switching position in \(i_3(G)\) and \(i_4(G)\) then the
resulting matching \( \psi(P_3, P_4) = P_1 \) has the same switching position in \( i_1(G) \) and \( i_2(G) \) and the resulting pair is \( \varphi(P_1) = (P_3, P_4) \). Since we are always using the first switching position, it follows that \( \varphi \) and \( \psi \) are mutually inverse bijections between the sets

\[
\{ P_1 \in \text{Match } G_1 \mid P_1 \text{ has a switching position in } i_1(G) \text{ and } i_2(G) \}
\]

and \( \text{Match } G_3 \times \text{Match } G_2^s \). On the other hand, it follows from [CS] section 3] that \( \psi(P_{56}) \) does not have a switching position and therefore \( \varphi \) and \( \psi \) are mutually inverse bijections between the sets

\[
\{ P_1 \in \text{Match } G_1 \mid P_1 \text{ has no switching position in } i_1(G) \text{ and } i_2(G) \}
\]

and \( \text{Match } G_{56} \). We have proved the following theorem.

**Theorem 4.5.** In the case \( s' > t + 1 \), the map

\[
\varphi : \text{Match } G_1 \rightarrow \text{Match } (\text{Res } G(G_1))
\]

is a bijection with inverse \( \psi \).

Let us also point out the following useful fact.

**Proposition 4.6.** In the case \( s' > t + 1 \), let \( P_{56} \in \text{Match } G_{56} \) and let \( P_1 = \psi(P_{56}) \). Then \( P_1|_{i_1(G)} \) and \( P_1|_{i_2(G)} \) consist of boundary edges of \( G \) only and are complementary as matchings of \( G \).

**Proof.** By definition the restriction of \( P_1 \) to both subgraphs consists of boundary edges only. So we need to show complementarity.

\( P_1 \) does not contain the boundary edge \( e \) in \( G_{s'} \) whose sign is \( f(e) = f(e_{s'-1}) \). Then by Lemma [4.3] the matching \( P_1|_{i_1(G)} \) consists precisely of all south and west boundary edges of \( i_1(G) \) whose sign is \( -f(e) \) and all north and east boundary edges of \( i_1(G) \) whose sign is \( f(e) \). Similarly \( P_1 \) does not contain the boundary edge \( e' \) in \( G_{t'} \) whose sign is \( f(e') = f(e_t) \). Then again by Lemma [4.3] the matching \( P_1|_{i_2(G)} \) consists precisely of all south and west boundary edges of \( i_1(G) \) whose sign is \( f(e') \) and all north and east boundary edges of \( i_2(G) \) whose sign is \( -f(e') \).

Since \( i_1(G) \) and \( i_2(G) \) form a crossing overlap, we have \( f(e_i) = f(e_{s'-1}) \) and thus \( f(e) = f(e') \). This shows that \( P_1|_{i_1(G)} \) and \( P_1|_{i_2(G)} \) consist of boundary edges of \( G \) only and are complementary as matchings of \( G \). \( \square \)

### 4.3. Self-crossing case 2: Overlap in the opposite direction.

Recall that

\[
G_{34} = G_1[1, t] \cup \overline{G}_1[s' - 1, t + 1] \cup G_1[s', d],
\]

where \( i_1(G) = G_1[s, t] \) and \( i_2(G) = G_1[s', t'] \) are the self-overlaps.

Let \( P_1 \in \text{Match } G_1 \). If \( P_1 \) has a switching position in \( i_1(G) \) and \( i_2(G) \), choose the first switching position and define a matching \( P_{34} \) by

\[
P_{34} =
\begin{cases}
  P_1|_{i_1(G)} & \text{before the first switching position;} \\
  P_1|_{i_2(G)} & \text{after the first switching position;}
\end{cases}
\text{ on } i_1(G).
\]

\[
P_{34} =
\begin{cases}
  P_1|_{i_2(G)} & \text{before the first switching position;} \\
  P_1|_{i_1(G)} & \text{after the first switching position;}
\end{cases}
\text{ on } i_2(G).
\]

If \( P_1 \) has no switching position in \( i_1(G) \) and \( i_2(G) \), we define \( P_3 \in \text{Match } G_5 \) by \( P_3 = P_1|_{G_5} \) and \( P_6 \in \text{Match } G_6^b \) by

\[
P_6 =
\begin{cases}
  P_1|_{G_6 \setminus \{b, b'\}} & \text{if both } b, b' \in P_1; \\
  P_1|_{G_6} & \text{otherwise.}
\end{cases}
\]

With this notation, we define

\[
\varphi(P_1) =
\begin{cases}
  P_{34} & \text{if } P_1 \text{ has a switching position in } i_1(G) \text{ and } i_2(G); \\
  (P_3, P_6) & \text{otherwise.}
\end{cases}
\]
To show that \( \varphi \) is a bijection, we define its inverse function \( \psi \). Let \( P_{34} \in \text{Match} \ G_{34}^0 \) and \( (P_5, P_6) \in \text{Match} \ G_5 \times G_6^0 \). Define \( \psi(P_{34}) \) to be the matching of \( G_1 \) obtained from \( P_{34} \) by switching at the first possible switching position. Thus

\[
\psi(P_{34}) = \begin{cases} 
\text{on } G_1[1, s - 1] \cup G_1[1, t + 1] \cup G_1[t + 1, d] 
\end{cases}
\]

where \( j_1 : G \to G_{34} \) and \( j_2 : G \to G_{34} \) are the embeddings of the overlap such that \( j_1(G) \subset G_{34} \) corresponds to \( i_1(G) \subset G_1 \).

On the other hand, \( (P_5, P_6) \) is the unique matching of \( G_1 \) whose restriction to \( G_5 \cup G_6^0 \) is the pair \( (P_5, P_6) \) and whose restriction to the complement \( G_1 \setminus (G_5 \cup G_6^0) \) consists of boundary edges only.

**Lemma 4.7.** \( P_6 \in \text{Match} \ G_6^0 \).

**Proof.** Suppose first that \( G_6^0 \) consists of at least two tiles. Let \( G_6^0 = (G_6^0)_{\epsilon t+1} \) be the snake graph obtained from \( G_6^0 \) by cutting along the interior edge \( \epsilon t+1 \) between the tiles \( G_{t+1} \) and \( G_{t+2} \). We cut \( G_1 \) along the same edge into the two snake graphs

\[ G_1' = G_1[1, t + 1], \quad G_1'' = G_1[t + 2, d], \]

with \( t + 2 < s' \) and obtain two perfect matchings \( P_1' \) and \( P_1'' \) of \( G_1' \) and \( G_1'' \) respectively, one by restricting \( P_1 \) and the other by restricting \( P_1 \) and adding the cut edge. Say \( P_1' = P_1|_{G_1'} \) and \( P_1'' = P_2|_{G_1''} \cup \{ \text{cut edge} \} \). Observe that \( G_1' \) and \( G_1'' \) have a crossing overlap \( i_1(G) \) and \( i_2(G) \) and \( (P_1', P_1'') \) is a pair of matchings without switching position in the overlap. Theorem 3.1 of [CS] implies that \( (P_1', P_1'') \) is a perfect matching of the band graph \( G_6^0 \). Observe that, since \( P_1 \) originally was a matching of the snake graph \( G_1 \), its restriction to \( G_6^0 \) is a good matching of the band graph. This completes the proof in this case.

Suppose now that \( G_6^0 \) consists of a single tile. Since there is no switching position at the tiles \( G_1, G_6^0 \), we see from [CS, Figure 7] that the local configuration is one of the two shown in Figure 16. Note that the tiles \( (G_t, G_{t+1}, G_{s'}) \) must form a zigzag, since the crossing overlap condition implies that \( f(\epsilon t) = f(\epsilon s'-1) \).

In both cases, \( G_6^0 \) is glued along the boundary edges \( b, b' \) of the tile \( G_{t+1} \), and thus \( P_6 \in \text{Match} \ G_6^0 \).

The following lemma shows that \( \psi \) is well-defined.

**Lemma 4.8.**

(a) \( P_{34} \) always has a switching position in \( j_1(G) \) and \( j_2(G) \).

(b) \( P_5 \cup P_6 \) can be completed to a matching of \( G_1 \) using only boundary edges.

(c) For every pair of matchings \( (P_5, P_6) \), the completion in (b) is unique and complementary on the overlaps \( i_1(G) \) and \( i_2(G) \).

**Proof.** Part (a) follows from [CS, Lemma 7.3] and parts (b) and (c) from [CS, Lemma 7.4].

**Theorem 4.9.** If the overlap is in the opposite direction then the map

\[ \varphi : \text{Match} \ G_1 \to \text{Match} (\text{Res} \varphi(G_1)) \]

is a bijection with inverse \( \psi \).

**Proof.** The proof is analogous to the proof of Theorem 4.5.
4.4. Self-crossing case 3: Overlap in the same direction and $s' < t$. Let $G_1$ be a self-crossing snake graph with self-overlap $i_1(G) = G_1[s, t] \cong G_1[s', t'] = i_2(G)$ with $s' < t$, and let $G_3, G_4', G_56$ be the resolution of the self-crossing as defined in section 3.2. We will define a map

$$\varphi : \text{Match } G_1 \cup \text{Match } G_{56} \longrightarrow \text{Match } G_3 \times \text{Match } G_4' \times \text{Match } G_56$$

Recall that in this case $G_{56}$ is a negative snake graph in $R$. This is why Match $G_{56}$ is now part of the domain of $\varphi$. Let $P_1 \in \text{Match } G_1$. See the top row of Figure 17 for an example. If the matching $P_1$ contains an edge of $G_1[s, 2s' - s - 1]$ which is an interior edge in $G_4'$, then let $e$ denote the first such edge. Then the snake graph $(G_4')_e$ obtained from $G_4'$ by cutting along the edge $e$ is isomorphic to the subsnake graph $G_1$ consisting of the last $(s' - s)$ tiles preceding the edge $e$ in $G_1$ or the first $(s' - s)$ tiles following $e$ in $G_1$, depending whether the edge $e$ comes after or before the tile $G_{s'}$ in $G_1$. On the other hand, the band graph $G_4'$ can be recovered from $G_4'$ by gluing the edge $e$ to an edge $e'$ at the opposite end of $G_4'$. Thus $G_4' = (G_4')^e$. The restriction of $P_1$ to this subgraph $G_4'$ induces a perfect matching $\varphi_4(P_1)$ on $G_4' = (G_4')^e$, since $P_1$ contains the gluing edge $e$. Indeed, if the vertices incident to $e'$ are matched in $P_1$ by the edges in $G_4'$, then the induced matching $\varphi_4(P_1)$ on $G_4'$ is the restriction of $P_1$ to $G_4' \setminus \{e\}$ (as in Figure 17), and if the vertices incident to $e'$ are matched in $P_1$ by edges in $G_1 \setminus G_4'$, then $\varphi_4(P_1)$ is the restriction of $P_1$ to $G_4$.

On the other hand, if $P_1$ does not contain an edge from $G_1[s, 2s' - s - 1]$ which is an interior edge in $G_4'$, then the first $(s' - s)$ edges in $P_1$ on the set $G_1[s, s' - 1] \setminus \{e\}$, where $e$ is the boundary edge of $G_s$ that becomes interior in $G_4$, induce a matching $\varphi_4(P_1)$ on $G_4' \setminus \text{boundary edges only}$.

In both cases, we have constructed a perfect matching $\varphi_4(P_1)$ on $G_4'$, and moreover $\varphi_4(P_1)$ is a subset of $P_1$. Let $\varphi_3(P_1) = P_1 \setminus \varphi_4(P_1)$ be the complement. Then it follows from the construction that $\varphi_3(P_1)$ is a perfect matching of $G_3$.

Now let $P_{56} \in \text{Match } G_{56}$. See the bottom row of Figure 17 for an example. We define a pair $(\varphi_3(P_{56}), \varphi_4(P_{56})) \in \text{Match } G_3 \times \text{Match } G_4'$. Suppose without loss of generality that in $G_{56}$ the tile $G_{s-1}$ is west of the tile $G_{s'}$, and denote by $a$ the interior edge between these tiles. Let $x$ be the northern endpoint of $a$, and let $y$ be the southern endpoint. Let $e(x) \in P_{56}$ (respectively $e(y) \in P_{56}$) be the unique edge incident to $x$ (respectively $y$). Then there are three cases.
case 2: \(e(x) = e(y) = a\), the interior edge shared by \(G_{s-1}\) and \(G_{s'}\). In this case, define \(\varphi_3(P_{56})\) to be the matching \(P_{56}\) on \(G_{56} \subset G_3\), where we agree that the edge \(a\) is in the tile \(G_{s-1}\), and complete \(\varphi_3(P_{56})\) with boundary edges of \(G_3\).

case 3: \(e(x)\) is not an edge of \(G_{s-1}\). This implies that \(e(y)\) is the south edge of \(G_{s'}\). In this case, define \(\varphi_3(P_{56})\) to be the matching \(P_{56}\) on \(G_{56} \subset G_3\) together with the edge \(a\) on the tile \(G_{s-1}\) and then completed by boundary edges of \(G_3\). This is the case shown in Figure 17.

In each case, define \(\varphi_4(P_{56})\) to be the unique matching on \(G_3^o\) consisting of those boundary edges that were not used in the completion of \(\varphi_3(P_{56})\). In other words, the matchings \(\varphi_3(P_{56}) \setminus P_{56}\) and \(\varphi_4(P_{56})\) are complementary and consist of boundary edges only.

**Remark 4.10.** In each case, the vertex \(x\) of the tile \(G_{s-1}\) in \(G_3\) is matched in \(\varphi_3(P_{56})\) by an edge in \(G_{s-1}\) (namely \(e(x)\) in case 1 and the edge \(a\) in the other cases). This implies that

1. the completed part \(\varphi_3(P_{56}) \cap G_3(s, s'-1)\) is the same in each case,
2. \(\varphi_4(P_{56})\) does not depend on \(P_{56}\),
3. the matching \(\varphi_4(P_{56})\) does not contain the edge \(a\).

In order to show that the map \(\varphi\) is a bijection, we construct its inverse

\[\psi : \text{Match } G_3 \times \text{Match } G_4^o \rightarrow \text{Match } G_1 \cup \text{Match } G_{56}^o.\]

Let \((P_3, P_4) \in \text{Match } G_3 \times \text{Match } G_4^o\). We define \(\psi(P_3, P_4)\) by

\[\psi(P_3, P_4)|_{G_1 \setminus G_4^o} = P_3\quad \text{and}\quad \psi(P_3, P_4)|_{G_4^o} = P_4\]

where \(G_4'' \subset G_1[s, 2s' - s - 1]\) is the first subgraph such that this definition yields a matching on \(G_1\) and \(G_4'' = (G_4')_*\) for some interior edge \(e\), if such a subgraph \(G_4''\) exists. If such a subgraph \(G_4''\) does not exist, we define

\[\psi(P_3, P_4) = P_{56} = P_3|_{G_{56}^o}.\]

**Lemma 4.11.** The subgraph \(G_4''\) in the definition above exists if and only if the pair \((P_3, P_4)\) is not of the form \((\varphi_3(P_{56}), \varphi_4(P_{56}))\) with \(P_{56} \in \text{Match } G_{56}^o\).

**Proof.** (\(\Rightarrow\)) Suppose that \((P_3, P_4) = (\varphi_3(P_{56}), \varphi_4(P_{56}))\) for some \(P_{56} \in \text{Match } G_{56}^o\). By definition of \(\varphi\), the matchings \(P_4\) and \(P_3|_{G_1[s, s'-1]}\) are complementary and consist of boundary edges only. It follows that \(G_4''\) could only be \(G_1[s, s'-1]\) or \(G_1[s', 2s' - s - 1]\). On the other hand, Remark 4.10 implies that the northeast vertex \(x\) of \(G_{s-1}\) is matched in \(P_3\) by the edges of \(G_{s-1}\) and the northwest vertex of \(G_s\), which is also the point \(x\), is matched in \(P_4\) by a boundary edge. This implies that \(G_4''\) cannot be \(G_1[s, s'-1]\). A similar argument shows that \(G_4''\) cannot be \(G_1[s', 2s'-s - 1]\) either. It follows that \(G_4''\) does not exist.

(\(\Leftarrow\)) Suppose that the pair \((P_3, P_4)\) is not of the form \((\varphi_3(P_{56}), \varphi_4(P_{56}))\) with \(P_{56} \in \text{Match } G_{56}^o\). It has been shown in [CS Section 3] that if one of \(P_4\) or \(P_3|_{G_1 \setminus G_{56}^o}\) contains an interior edge of \(G_4^o\), or if both \(P_4\) and \(P_3|_{G_1 \setminus G_{56}^o}\) have an edge in common, then \(G_4''\) always exists.

Thus we only need to consider the pairs \((P_3, P_4)\) where \(P_4\) and \(P_3|_{G_1 \setminus G_{56}^o}\) are complementary and consist of boundary edges only. Moreover \((P_3, P_4)\) is not of the form \((\varphi_3(P_{56}), \varphi_4(P_{56}))\) with \(P_{56} \in \text{Match } G_{56}^o\), and from the construction of \(\varphi\) and Remark 4.10 (3), it follows that the edge \(a\) in the definition of \(\varphi\) belongs to \(P_4\). Therefore \(G_4'' = G_1[s, s'-1]\) satisfies the required properties.

**Theorem 4.12.** In the case \(s' < t\), the map

\[\varphi : \text{Match } G_1 \cup \text{Match } G_{56} \rightarrow \text{Match } G_3 \times \text{Match } G_4^o\]

is a bijection with inverse \(\psi\).

**Proof.** Let \(P_1 \in \text{Match } G_1\). If the interior edge \(e\) in the definition of \(\varphi\) exists, we have \(\psi \varphi(P_1) = \psi(\varphi(P_1), \varphi_4(P_1)) = P_1\), because in this case the subgraph \(G_4''\) in the definition of \(\psi\) is equal to the subgraph \(G_4^o\) in the definition of \(\varphi\). On the other hand, if no such \(e\) exists then \(G_4''\) is the
subgraph $G'[s', s' - 1] \setminus \{e\}$, where $e$ is the boundary edge of $G_s$ that becomes interior in $G_4$, and again $\psi_\varphi(P_1) = \psi_\varphi(P_2) = \psi_\varphi(P_3) = P_1$.

Next, let $P_{56} \in \text{Match } G_{56}$. Then $\psi_\varphi(P_{56}) = \psi_\varphi(P_{56}) = \psi_\varphi(P_{56}) = P_{56}$, where the second identity holds because of Lemma 4.11 and the last identity by definition of $\varphi_3$.

This shows that $\psi_\varphi$ is the identity.

Now let $(P_3, P_4) \in \text{Match } G_3 \times G_4$. If $G_3'$ in the definition of $\psi$ exists then $\psi_\varphi(P_3, P_4) = \varphi_3(P_3 \cup P_4) = (P_3, P_4)$, by construction. On the other hand, if $G_3''$ does not exist, then Lemma 4.11 implies that there exists $P_{56} \in \text{Match } G_{56}$ such that $(P_3, P_4) = \varphi_3(P_{56})$. In this case $\psi_\varphi(P_3, P_4) = \varphi_3(P_{56}) = P_{56}$, and thus $\psi_\varphi(P_3, P_4) = \varphi_3(P_{56}) = (P_3, P_4)$. This shows that $\varphi_\varphi$ is the identity.

\hfill \Box

4.5. Self-crossing case 4: Overlap in the same direction and $s' = t + 1$. The bijections $\varphi, \psi$ are almost exactly as in the case $s' > t + 1$ except for one particular type of matchings which we describe now. Let $k$ be as in section 3.2 case (3)(c). Recall that in this case there is a second overlap $j_1(\mathcal{H}) = G_1[s - 1, k + 1]$ and $j_2(\mathcal{H}) = G_1[t + 1, t' + s - k - 1]$ just before and after the overlap $i_1(\mathcal{G}) = G_1[s, t]$ and $i_2(\mathcal{G}) = G_1[s', t']$, which determines the sign of $G_{56}$ in the resolution of the self-crossing.

We consider $G_1$ as a union of 6 subgraphs

$$G_1 = G_1[1, k] \cup G_1[k + 1, s - 1] \cup G_1[s, t] \cup G_1[s', t'] \cup G_1[t' + 1, t' + s - k - 1] \cup G_1[t' + s - k, d]$$

Recall that $G_{56} = G_1[1, k] \cup G_1[t' + s - k, d]$ glued along the edge $e$ which is the boundary edge in $G_{56}^{NE}$ and the edge $e'$ which is the boundary edge in $SW G_{t' + s - k - k}$.

Suppose $P_1 \in \text{Match } G_1$ is such that $e_1 \in P_1, P_1|_{G_1(k+1,t)}$ consists of boundary edges only, $P_1|_{G_1(s,t'+s-k-1)}$ consists of boundary edges only, and is complementary on both pieces.

Without loss of generality, suppose $e_1$ is north of $G_1$ and $f(e_1) = -$. Then $f(e_{s-1}) = + = f(e')$, because we have a crossing overlap. Moreover, Lemma 4.3 implies that the east and the north edges in $P_1|_{G_1(k+1,t)}$ have sign $-$ (thus the east and the north edges in $P_1|_{G_1(s',t'+s-k-1)}$ have sign $+$) and the south and the west edges in $P_1|_{G_1(k+1,t)}$ have sign $+$ (thus the south and the west edges in $P_1|_{G_1(s',t'+s-k-1)}$ have sign $-$). Moreover, since the overlaps $j_1(\mathcal{H}), j_2(\mathcal{H})$ have opposite direction, we have $f(e_{s-\ell-1}) = -f(e_{\ell+t})$ for $\ell = 1, 2, \ldots, s - k - 2, f(e_k) = f(e_{t'+s-k-1})$ and $f(e) = f(e')$.

There are two cases: the overlaps $j_1(\mathcal{H})$ and $j_2(\mathcal{H})$ cross or not. Let us suppose first that they do not cross. Then $f(e_k) = f(e_{s-1}) = f(e_{t'+s-k-1}) = f(e') = +$, because it is a non-crossing overlap in the opposite direction. Then $f(e) = -f(e_k) = -$.

If $G_{k+1}$ is east of $G_k$ then $e$ is the north edge of $G_k$ and this shows that the south edge of $G_{k+1}$ is not in $P_1$ because it has sign $-$; and if $G_{k+1}$ is north of $G_k$ then $e$ is the east edge of $G_k$ and this shows that the west edge of $G_{k+1}$ is not in $P_1$ because it has sign $-$. Therefore the two vertices of the interior edge $e_k$ shared by the tiles $G_k$ and $G_{k+1}$ are matched by edges from $G_k$ or $G_{k-1}$. Thus either $e_k \in P_1$ or $e \in P_1$ and similarly, either $e_{t'+s-k-1} \in P_1$ or $e' \in P_1$. We define

$$\varphi(P_1) = P_1|_{G_1 \setminus \{e\}} \text{ if } e \in P_1 \text{ or } e, e' \in P_1;$$

and

$$\varphi(P_1) = P_1|_{G_1 \setminus \{e'\}} \text{ if } e' \in P_1 \text{ and } e \notin P_1.$$
On the other hand, if $P$ both pieces, then the glueing edge $e$ is not in $P$.

This completes the case $G$ using only boundary edges, together with the unique complementary matching on $P$. Theorem 4.13. There is a bijection

\[ \varphi : \text{Match } G_1 \to \text{Match (Graft}_{s,t}(G_1)). \]

Proof. The bijection $\varphi$ is the same as in the grafting case in Theorem 4.4 (2). It is given explicitly by the operation described in [CS, Figure 11].

5. Labeled snake and band graphs arising from cluster algebras of unpunctured surfaces

In this section we recall how snake graphs and band graphs arise naturally in the theory of cluster algebras. We follow the exposition in [MSW2].

5.1. Cluster algebras from unpunctured surfaces. Let $S$ be a connected oriented 2-dimensional Riemann surface with nonempty boundary, and let $M$ be a nonempty finite subset of the boundary of $S$, such that each boundary component of $S$ contains at least one point of $M$. The elements of $M$ are called marked points. The pair $(S, M)$ is called a bordered surface with marked points.

For technical reasons, we require that $(S, M)$ is not a disk with 1, 2 or 3 marked points.

Definition 5.1. A generalized arc in $(S, M)$ is a curve $\gamma$ in $S$, considered up to isotopy, such that:

(a) the endpoints of $\gamma$ are in $M$;
(b) except for the endpoints, $\gamma$ is disjoint from the boundary of $S$; and
(c) $\gamma$ does not cut out a monogon or a bigon.
A generalized arc $\gamma$ is called an arc if in addition $\gamma$ does not cross itself, except that its endpoints may coincide;

Thus a generalized arc is allowed to cross itself a finite number of times.

Curves that connect two marked points and lie entirely on the boundary of $S$ without passing through a third marked point are boundary segments. Note that boundary segments are not arcs.

For any two arcs $\gamma, \gamma'$ in $S$, let $e(\gamma, \gamma')$ be the minimal number of crossings of arcs $\alpha$ and $\alpha'$, where $\alpha$ and $\alpha'$ range over all arcs isotopic to $\gamma$ and $\gamma'$, respectively. We say that arcs $\gamma$ and $\gamma'$ are compatible if $e(\gamma, \gamma') = 0$.

A triangulation is a maximal collection of pairwise compatible arcs (together with all boundary segments).

Triangulations are connected to each other by sequences of flips. Each flip replaces a single arc $\gamma$ in a triangulation $T$ by a (unique) arc $\gamma' \neq \gamma$ that, together with the remaining arcs in $T$, forms a new triangulation.

**Definition 5.2.** Choose any triangulation $T$ of $(S, M)$, and let $\tau_1, \tau_2, \ldots, \tau_n$ be the $n$ arcs of $T$. For any triangle $\Delta$ in $T$, we define a matrix $B_{\Delta} = (b_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}$ as follows.

- $b_{ij}^\Delta = 1$ and $b_{ji}^\Delta = -1$ if $\tau_i$ and $\tau_j$ are sides of $\Delta$ with $\tau_j$ following $\tau_i$ in the clockwise order.
- $b_{ij}^\Delta = 0$ otherwise.

Then define the matrix $B_T = (b_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}$ by $b_{ij} = \sum_{\Delta} b_{ij}^\Delta$, where the sum is taken over all triangles in $T$.

Note that $B_T$ is skew-symmetric and each entry $b_{ij}$ is either 0, ±1, or ±2, since every arc $\tau$ is in at most two triangles.

**Theorem 5.3.** [FST Theorem 7.11] and [FT Theorem 5.1] Fix a bordered surface $(S, M)$ and let $A$ be the cluster algebra associated to the signed adjacency matrix of a triangulation. Then the (unlabeled) seeds $\Sigma_T$ of $A$ are in bijection with the triangulations $T$ of $(S, M)$, and the cluster variables are in bijection with the arcs of $(S, M)$ (so we can denote each by $x_\gamma$, where $\gamma$ is an arc). Moreover, each seed in $A$ is uniquely determined by its cluster. Furthermore, if a triangulation $T'$ is obtained from another triangulation $T$ by flipping an arc $\gamma \in T$ and obtaining $\gamma'$, then $\Sigma_T'$ is obtained from $\Sigma_T$ by the seed mutation replacing $x_\gamma$ by $x_{\gamma'}$.

From now on suppose that $A$ has principal coefficients in the initial seed $\Sigma_T = (x_T, y_T, B_T)$.

**Definition 5.4.** A closed loop in $(S, M)$ is a closed curve $\gamma$ in $S$ which is disjoint from the boundary of $S$. We allow a closed loop to have a finite number of self-crossings. As in Definition 5.1, we consider closed loops up to isotopy. A closed loop in $(S, M)$ is called essential if it is not contractible and it does not have self-crossings.

**Definition 5.5.** A multicurve is a finite multiset of generalized arcs and closed loops such that there are only a finite number of pairwise crossings among the collection. We say that a multicurve is simple if there are no pairwise crossings among the collection and no self-crossings.

If a multicurve is not simple, then there are two ways to resolve a crossing to obtain a multicurve that no longer contains this crossing and has no additional crossings. This process is known as smoothing.

**Definition 5.6.** (Smoothing) Let $\gamma, \gamma_1$ and $\gamma_2$ be generalized arcs or closed loops such that we have one of the following two cases:

1. $\gamma_1$ crosses $\gamma_2$ at a point $x$.
2. $\gamma$ has a self-crossing at a point $x$.

Then we let $C$ be the multicurve $\{\gamma_1, \gamma_2\}$ or $\{\gamma\}$ depending on which of the two cases we are in. We define the smoothing of $C$ at the point $x$ to be the pair of multicurves $C_+ = \{\alpha_1, \alpha_2\}$ (resp. $\{\alpha\}$) and $C_- = \{\beta_1, \beta_2\}$ (resp. $\{\beta\}$).

Here, the multicurve $C_+$ (resp. $C_-$) is the same as $C$ except for the local change that replaces the crossing $\times$ with the pair of segments $\subset$ (resp. $\supset$).
Figure 19. On the left, a triangle with two vertices; on the right the tile $G_j$ where $i_j = 2$.

Since a multicurve may contain only a finite number of crossings, by repeatedly applying smoothings, we can associate to any multicurve a collection of simple multicurves. We call this resulting multiset of multicurves the smooth resolution of the multicurve $C$.

**Remark 5.7.** This smoothing operation can be rather complicated, since the multicurves are considered up to isotopy. Thus after performing the local operation of smoothing described above, one needs to find representatives of the isotopy classes of $C_+$ and $C_-$ which have a minimal number of crossings with the triangulation. In practice, this can be quite difficult especially if one needs to smooth several crossings. This difficulty was one of the original motivations to develop the snake graph calculus. The isotopy is already contained in the definition of the resolutions of the (self-)crossing snake graphs.

**Theorem 5.8.** (Skein relations) [MW, Propositions 6.4,6.5,6.6] Let $C, C_+, C_-$ be as in Definition 5.6. Then we have the following identity in $A$, $x_C = \pm Y_1 x_{C_+} \pm Y_2 x_{C_-}$, where $Y_1$ and $Y_2$ are monomials in the variables $y_{\tau_{ij}}$. The monomials $Y_1$ and $Y_2$ can be expressed using the intersection numbers of the elementary laminations (associated to triangulation $T$) with the curves in $C, C_+$ and $C_-$. 

5.2. **Labeled snake graphs from surfaces.** Let $\gamma$ be an arc in $(S,M)$ which is not in $T$. Choose an orientation on $\gamma$, let $s \in M$ be its starting point, and let $t \in M$ be its endpoint. We denote by $s = p_0, p_1, p_2, \ldots, p_{d+1} = t$ the points of intersection of $\gamma$ and $T$ in order. Let $\tau_{ij}$ be the arc of $T$ containing $p_j$, and let $\Delta_{j-1}$ and $\Delta_j$ be the two triangles in $T$ on either side of $\tau_{ij}$. Note that each of these triangles has three distinct sides, but not necessarily three distinct vertices, see Figure 19. Let $G_j$ be the graph with 4 vertices and 5 edges, having the shape of a square with a diagonal, such that there is a bijection between the edges of $G_j$ and the 5 arcs in the two triangles $\Delta_{j-1}$ and $\Delta_j$, which preserves the signed adjacency of the arcs up to sign and such that the diagonal in $G_j$ corresponds to the arc $\tau_{ij}$ containing the crossing point $p_j$. Thus $G_j$ is given by the quadrilateral in the triangulation $T$ whose diagonal is $\tau_{ij}$.

Given a planar embedding $\tilde{G}_j$ of $G_j$, we define the relative orientation $\text{rel}(\tilde{G}_j,T)$ of $\tilde{G}_j$ with respect to $T$ to be $\pm 1$, based on whether its triangles agree or disagree in orientation with those of $T$. For example, in Figure 19 $\tilde{G}_j$ has relative orientation $+1$.

Using the notation above, the arcs $\tau_{ij}$ and $\tau_{ij+1}$ form two edges of a triangle $\Delta_j$ in $T$. Define $\tau_{e_j}$ to be the third arc in this triangle.

We now recursively glue together the tiles $G_1, \ldots, G_d$ in order from 1 to $d$, so that for two adjacent tiles, we glue $G_{j+1}$ to $\tilde{G}_j$ along the edge labeled $\tau_{e_j}$, choosing a planar embedding $\tilde{G}_{j+1}$ for $G_{j+1}$ so that $\text{rel}(\tilde{G}_{j+1},T) \neq \text{rel}(\tilde{G}_j,T)$. See Figure 20.
After gluing together the $d$ tiles, we obtain a graph (embedded in the plane), which we denote by $G^\Delta_\gamma$.

**Definition 5.9.** The (labeled) snake graph $G_\gamma$ associated to $\gamma$ is obtained from $G^\Delta_\gamma$ by removing the diagonal in each tile.

In Figure 21, we give an example of an arc $\gamma$ and the corresponding snake graph $G_\gamma$. Since $\gamma$ intersects $T$ five times, $G_\gamma$ has five tiles.

**Remark 5.10.** Let $f$ be a sign function on $G_\gamma$ as in section 2. The interior edges $e_1, \ldots, e_{d-1}$ are corresponding to the sides of the triangles $\Delta_1, \ldots, \Delta_{d-1}$ that are not crossed by $\gamma$. Two interior edges $e_j, e_k$ have the same sign $f(e_j) = f(e_k)$ if and only if the sides $\tau_{e_j}, \tau_{e_k}$ lie on the same side of the segments of $\gamma$ in $\Delta_j$ and $\Delta_k$, respectively.

**Definition 5.11.** If $\tau \in T$ then we define its (labeled) snake graph $G_\tau$ to be the graph consisting of one single edge with weight $x_\tau$ and two distinct endpoints (regardless whether the endpoints of $\tau$ are distinct).

Now we associate a similar graph to closed loops. Let $\zeta$ be a closed loop in $(S, M)$, which may or may not have self-intersections, but which is not contractible and has no contractible kinks. Choose an orientation for $\zeta$, and a triangle $\Delta$ which is crossed by $\gamma$. Let $p$ be a point in the interior of $\Delta$ which lies on $\gamma$, and let $b$ and $c$ be the two sides of the triangle crossed by $\gamma$ immediately before and following its travel through point $p$. Let $a$ be the third side of $\Delta$. We let $\tilde{\gamma}$ denote the arc from $p$ back to itself that exactly follows closed loop $\gamma$.

We start by building the snake graph $G_{\tilde{\gamma}}$ as defined above. In the first tile of $G_{\tilde{\gamma}}$, let $x$ denote the vertex at the corner of the edge labeled $a$ and the edge labeled $b$, and let $y$ denote the vertex at the other end of the edge labeled $a$. Similarly, in the last tile of $G_{\tilde{\gamma}}$, let $x'$ denote the vertex at the corner of the edge labeled $a$ and the edge labeled $b$, and let $y'$ denote the vertex at the other end of the edge labeled $a$. See the right of Figure 22. Our convention for $x'$ and $y'$ are exactly opposite to those in [MSW2].
Definition 5.12. The (labeled) band graph \( G_{\zeta} \) associated to the loop \( \zeta \) is the graph obtained from \( \tilde{G}_{\zeta} \) by identifying the edges labeled \( a \) in the first and last tiles so that the vertices \( x \) and \( x' \) and the vertices \( y \) and \( y' \) are glued together.

5.3. Snake graph formula for cluster variables. Recall that if \( \tau \) is a boundary segment then \( x_{\tau} = 1 \).

If \( G \) is a (labeled) snake graph and the edges of a perfect matching \( P \) of \( G \) are labeled \( \tau_{j_1}, \ldots, \tau_{j_r} \), then the weight \( x(P) \) of \( P \) is \( x_{\tau_{j_1}} \ldots x_{\tau_{j_r}} \).

Let \( \gamma \) be a generalized arc and \( \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_d} \) be the sequence of arcs in \( T \) which \( \gamma \) crosses. The crossing monomial of \( \gamma \) with respect to \( T \) is defined as

\[
\text{cross}(T, \gamma) = \prod_{j=1}^{d} x_{\tau_{i_j}}.
\]

By induction on the number of tiles it is easy to see that the snake graph \( G_{\gamma} \) has precisely two perfect matchings which we call the minimal matching \( P_- = P_-(G_{\gamma}) \) and the maximal matching \( P_+ = P_+(G_{\gamma}) \), which contain only boundary edges. To distinguish them, if \( \text{rel}(\tilde{G}_1, T) = 1 \) (respectively, \(-1\)), we define \( e_1 \) and \( e_2 \) to be the two edges of \( G_{\gamma} \) which lie in the counterclockwise (respectively, clockwise) direction from the diagonal of \( \tilde{G}_1 \). Then \( P_- \) is defined as the unique matching which contains only boundary edges and does not contain edges \( e_1 \) or \( e_2 \). \( P_+ \) is the other matching with only boundary edges. In the example of Figure 21, the minimal matching \( P_- \) contains the bottom edge of the first tile labeled 4.

Lemma 5.13. [MS, Theorem 5.1] The symmetric difference \( P_- \ominus P \) is the set of boundary edges of a (possibly disconnected) subgraph \( G_P \) of \( G_{\gamma} \), which is a union of cycles. These cycles enclose a set of tiles \( \cup_{j \in J} G_j \), where \( J \) is a finite index set.

Definition 5.14. With the notation of Lemma 5.13, we define the height monomial \( y(P) \) of a perfect matching \( P \) of a snake graph \( G_{\gamma} \) by

\[
y(P) = \prod_{j \in J} y_{\tau_{i_j}}.
\]

Following [MSW2], for each generalized arc \( \gamma \), we now define a Laurent polynomial \( x_\gamma \), as well as a polynomial \( F_T^\gamma \) obtained from \( x_\gamma \) by specialization.

Definition 5.15. Let \( \gamma \) be a generalized arc and let \( G_{\gamma} \), be its snake graph.

1. If \( \gamma \) has a contractible kink, let \( \tau \) denote the corresponding generalized arc with this kink removed, and define \( x_\gamma = (-1)x_{\tau} \).
2. Otherwise, define

\[
x_\gamma = \frac{1}{\text{cross}(T, \gamma)} \sum_{P} x(P)y(P),
\]

where the sum is over all perfect matchings \( P \) of \( G_{\gamma} \).
Define $F^T_\gamma$ to be the polynomial obtained from $x_\gamma$ by specializing all the $x_{\tau_i}$ to 1.

If $\gamma$ is a curve that cuts out a contractible monogon, then we define $\gamma = 0$.

Theorem 5.16. [MSW, Thm 4.9] If $\gamma$ is an arc, then $x_\gamma$ is a the cluster variable in $A$, written as a Laurent expansion with respect to the seed $\Sigma_T$, and $F^T_\gamma$ is its F-polynomial.

Again following [MSW2], we define for every closed loop $\zeta$, a Laurent polynomial $x_\zeta$, as well as a polynomial $F^T_\zeta$ obtained from $x_\zeta$ by specialization.

Definition 5.17. Let $\zeta$ be a closed loop.

1. If $\zeta$ is a contractible loop, then let $x_\zeta = -2$.
2. If $\zeta$ has a contractible kink, let $\zeta'$ denote the corresponding closed loop with this kink removed, and define $x_\zeta = (-1)x_{\zeta'}$.
3. Otherwise, let
   
   $x_\zeta = \frac{1}{\text{cross}(T, \gamma)} \sum_P x(P)y(P),$  

   where the sum is over all good matchings $P$ of the band graph $G_\zeta$.

Define $F^T_\zeta$ to be the Laurent polynomial obtained from $x_\zeta$ by specializing all the $x_{\tau_i}$ to 1.

5.4. Bases of the cluster algebra. We recall the construction of the two bases given in [MSW2] in terms of bangles and bracelets.

Definition 5.18. Let $\zeta$ be an essential loop in $(S,M)$. The bangle $\text{Bang}_k \zeta$ is the union of $k$ loops isotopic to $\zeta$. (Note that $\text{Bang}_1 \zeta$ has no self-crossings.) And the bracelet $\text{Brac}_k \zeta$ is the closed loop obtained by concatenating $\zeta$ exactly $k$ times, see Figure 23. (Note that it will have $k-1$ self-crossings.)

Note that $\text{Bang}_1 \zeta = \text{Brac}_1 \zeta = \zeta$.

Definition 5.19. A collection $C$ of arcs and essential loops is called $C^o$-compatible if no two elements of $C$ cross each other. Let $C^o(S,M)$ be the set of all $C^o$-compatible collections in $(S,M)$.

Definition 5.20. A collection $C$ of arcs and bracelets is called $C$-compatible if:

- no two elements of $C$ cross each other except for the self-crossings of a bracelet; and
- given an essential loop $\zeta$ in $(S,M)$, there is at most one $k \geq 1$ such that the $k$-th bracelet $\text{Brac}_k \zeta$ lies in $C$, and, moreover, there is at most one copy of this bracelet $\text{Brac}_k \zeta$ in $C$.

Let $C(S,M)$ be the set of all $C$-compatible collections in $(S,M)$.

Note that a $C^o$-compatible collection may contain bangles $\text{Bang}_k \zeta$ for $k \geq 1$, but it will not contain bracelets $\text{Brac}_k \zeta$ except when $k = 1$. And a $C$-compatible collection may contain bracelets, but will never contain a bangle $\text{Bang}_k \zeta$ except when $k = 1$.

Definition 5.21. Given an arc or closed loop $c$, let $x_c$ denote the corresponding Laurent polynomial defined in Section 5.3. Let $B^o$ be the set of all cluster algebra elements corresponding to the set $C^o(S,M)$,

$$B^o = \left\{ \prod_{c \in C} x_c \mid C \in C^o(S,M) \right\}.$$
Similarly, let

\[ \mathcal{B} = \left\{ \prod_{c \in C} x_c \mid C \in \mathcal{C}(S, M) \right\}. \]

Remark 5.22. Both \( \mathcal{B}^\circ \) and \( \mathcal{B} \) contain the cluster monomials of \( \mathcal{A} \).

We are now ready to state the main result of [MSW2].

**Theorem 5.23.** [MSW2] Theorem 4.1] If the surface has no punctures and at least two marked points then the sets \( \mathcal{B}^\circ \) and \( \mathcal{B} \) are bases of the cluster algebra \( \mathcal{A} \).

Remark 5.24. This result has been extended to surfaces with only one marked point in [CLS].

6. Relation to cluster algebras

In this section we show how our results on abstract snake graphs are related to computations in cluster algebras from unpunctured surfaces.

6.1. **Homotopy after removing a puncture and its effect on the snake graph.** We start by studying the technique of introducing and removing a puncture and its effect on snake graphs. Let \( T \) be a triangulation of an unpunctured surface \( (S, M) \) and let \( \Delta \) be a triangle in \( T \) and label the sides of \( \Delta \) by 1, 2, 3.

Let \( (\hat{S}, \hat{M}) \) be the surface obtained from \( (S, M) \) by introducing a puncture in the interior of the triangle \( \Delta \), and let \( T' \) be the triangulation obtained from \( T \) by adding three arcs \( a, b, c \) connecting the puncture to the vertices of \( \Delta \).

Let \( \hat{\gamma} \) be an arc in \( (\hat{S}, \hat{M}) \) and let \( \gamma \) be the corresponding arc in \( (S, M) \). Consider the three local configurations illustrated in Figure 24.

In the first case, the arc \( \hat{\gamma} \) crosses one of the arcs \( a, b, c \) and then the arc \( \gamma \) crosses the same arcs as \( \hat{\gamma} \) (locally) except for the arc \( a, b \) or \( c \) crossed by \( \hat{\gamma} \).

In the second case, the arc \( \hat{\gamma} \) crosses two of the arcs \( a, b, c \) and we have two subcases. If \( \gamma \) crosses a side of \( \Delta \) immediately before and after crossing two of the arcs \( a, b, c \) then \( \gamma \) crosses the same arcs as \( \hat{\gamma} \) except for the two arcs of \( a, b, c \). On the other hand, if \( \gamma \) starts at a vertex \( x \) of \( \Delta \) and then crosses two of the arcs \( a, b, c \), then the difference in the crossings of \( \hat{\gamma} \) and \( \gamma \) is the two arcs among \( a, b, c \) plus a whole fan of arcs with common vertex \( x \). In the example in Figure 24 these arcs are \( b, c, 3, 4, \ldots, k \).

In the third case, the arc \( \hat{\gamma} \) crosses all three arcs \( a, b, c \) and the difference in the crossings of \( \hat{\gamma} \) and \( \gamma \) is the 3 arcs \( a, b, c \) plus the local overlap of arcs before and after the arcs \( a, b, c \). In the example in Figure 24 these arcs are \( \ell, \ldots, k+1, k, \ldots, 4, 3, a, b, c, 3, 4, \ldots, k, k+1, \ldots, \ell \).
The corresponding snake graphs $G_\gamma, G_\hat{\gamma}$ are related by deleting the corresponding tiles and glueing the remaining pieces.

Thus in the first case:

In the second case:

Delete tiles $b, c$ and everything up to the next sign change

In the third case:

Delete tiles $a, b, c$ and then cancel tiles with the same label

### 6.2. Crossing arcs and crossing snake graphs.

In this subsection, we show that the notions of crossings for arcs and snake graphs coincide.

**Theorem 6.1.** Let $\gamma_1, \gamma_2$ be generalized arcs and $G_1, G_2$ their corresponding labeled snake graphs (might have self-crossings.)

a) $\gamma_1, \gamma_2$ cross with a nonempty local overlap $(\tau_{i_1}, \ldots, \tau_{i_t}) = (\tau'_{i_1}, \ldots, \tau'_{i_t})$ if and only if $G_1, G_2$ cross in $G_1[s, t] \cong G_2[s', t'].$

b) $\gamma$ has a self-crossing with a nonempty local overlap $(\tau_{i_1}, \ldots, \tau_{i_t}) = (\tau'_{i_1}, \ldots, \tau'_{i_t})$ (with or without self-intersection) if and only if $G_\gamma$ has a crossing self-overlap on $G_\gamma[s, t] \cong G_\gamma[s', t'].$

**Proof.** a) This follows directly from Theorem 5.3 of [CS].

b) Let $\gamma$ be a self-crossing arc. Choose a parametrization $\gamma = \gamma(t)$, and say the self-crossing occurs at the times $t_1$ and $t_2$. Take now two copies $\gamma_1, \gamma_2$ of $\gamma$ and consider their crossing at $\gamma_1(t_1) = \gamma_2(t_2)$. The local overlap of $\gamma_1$ and $\gamma_2$ at this crossing is the same as the local overlap of the self-crossing, and the local overlap in the corresponding snake graphs $G_1, G_2$ of $\gamma_1, \gamma_2$ is the same as the local self-overlap of the snake graph $G_\gamma$. Now the result follows from part a).

### 6.3. Smoothing crossings and resolving snake graphs.

In this subsection, we show that the smoothing operation for arcs corresponds to the resolution of crossings for snake graphs. The following result has been shown in [CS].

**Theorem 6.2.** [CS, Theorem 5.4] Let $\gamma_1$ and $\gamma_2$ be two (generalized) arcs which cross with a non-empty local overlap, and let $G_1$ and $G_2$ be the corresponding labeled snake graphs with overlap $G_1[s, t] \cong G_2[s', t'].$
\[ \mathcal{G}. \] Then the labeled snake graphs of the four arcs obtained by smoothing the crossing of \( \gamma_1 \) and \( \gamma_2 \) in the overlap are given by the resolution \( \text{Res}_G(\mathcal{G}_1, \mathcal{G}_2) \) of the crossing of the labeled snake graphs \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) at the overlap \( \mathcal{G} \).

We now show the analogous statement for self-crossings.

**Theorem 6.3.** Let \( \gamma_1 \) be a self-crossing arc with nonempty local overlap and let \( \mathcal{G}_1 \) be the corresponding labeled snake graph with crossing overlap \( i_1(\mathcal{G}) = \mathcal{G}_1[s, t] \) and \( i_2(\mathcal{G}) = \mathcal{G}_1[s', t'] \). Then the labeled snake graphs of the two arcs and the labeled band graph of the loop obtained by smoothing the crossing of \( \gamma_1 \) on the overlap are given by the resolution \( \text{Res}_G(\mathcal{G}_1) \) of the self-crossing of the labeled snake graph \( \mathcal{G}_1 \) at the overlap \( \mathcal{G} \).

**Proof.** We start with the case where the overlap is in the same direction and \( s' \geq t + 1 \). Let \( \Delta \) be the triangle in the surface which contains the segment of \( \gamma_1 \) between the \( t \)-th and the \( (t+1) \)-st crossing point. We introduce a puncture \( p \) on this segment of \( \gamma_1 \) and in the interior of \( \Delta \), see Figure 25 where the triangulation arcs are black and the arc \( \gamma \) is red. We complete \( T \) to a triangulation \( \tilde{T} \) by adding three arcs \( a, b, c \) from the puncture \( p \) to the vertices of \( \Delta \). Let \( \gamma_{11} \) be the segment of \( \gamma_1 \) up to the puncture \( p \), and let \( \gamma_{12} \) be the segment of \( \gamma_1 \) after \( p \).

On the other hand, consider the snake graphs \( \tilde{\mathcal{G}}_{11} = \mathcal{G}_{11}[1, t] \) and \( \tilde{\mathcal{G}}_{12} = \mathcal{G}_{12}[t + 1, d] \). Observe that these snake graphs do not necessarily correspond to snake graphs of \( \gamma_{11} \) and \( \gamma_{12} \), since the arc \( \gamma \) might run through the triangle \( \Delta \) several times, and introducing a puncture in the surface might create crossings with the new arcs in \( \tilde{T} \setminus T \). Then the snake graphs \( \tilde{\mathcal{G}}_{11}, \tilde{\mathcal{G}}_{12} \) corresponding to \( \gamma_{11} \) and \( \gamma_{12} \) will be obtained from \( \mathcal{G}_{11} \) and \( \mathcal{G}_{12} \), respectively, by inserting single tiles which correspond to these new crossings.

Suppose first that \( t' < d \). Then the triangle \( \Delta \) has sides \( \tau_i = \tau_{i'}, \tau_{i+1}, \tau_{i'+1} \), and \( i_{t+1} \neq i_{t'+1} \), since we have an overlap. Smoothing the self-crossing of \( \gamma_1 \) is the same as smoothing the corresponding crossing of the two curves \( \gamma_{11} \) and \( \gamma_{12} \) and then removing the puncture again. This will produce a pair \( (\gamma_3, \gamma_4) \), where \( \gamma_3 \) is an arc and \( \gamma_4 \) is a loop, and an arc \( \gamma_{56} \) which crosses \( \tau_{t+1} \) and \( \tau_{t'+1} \). In Figure 25, the arc \( \gamma_{56} \) is the blue one. In terms of snake graphs, \( \text{Res}_G(\mathcal{G}_1) \) is obtained from \( \text{Res}_G(\tilde{\mathcal{G}}_{11}, \tilde{\mathcal{G}}_{12}) \) by removing the tiles corresponding to the crossings with the arcs at the puncture and glueing. The graphs \( \mathcal{G}_3 \) and \( \mathcal{G}_4 \) are glued along the edge labeled \( \tau_{t'+1} \) and correspond to \( \gamma_3 \) and \( \gamma_4 \). The graph \( \gamma_{56} \) is glued along the edge labeled \( \tau_i \) and corresponds to \( \gamma_{56} \).

Now suppose that \( t' = d \). Then the triangle \( \Delta \) has sides \( \tau_i = \tau_{i'}, \tau_{i+1}, \tau_{i'+1}, \tau_{i+t} \), see the right hand side of Figure 25. For \( \gamma_3 \) and \( \gamma_4 \) the proof is exactly as above. For \( \gamma_{56} \), there is a slight difference. Now smoothing the crossing of the two arcs \( \gamma_{11} \) and \( \gamma_{12} \) will produce an arc \( \gamma_5 \) starting at \( p \) and then crossing \( \tau_{i+1} \), as well as an arc \( \gamma_6 = c \) of the triangulation \( \tilde{T} \setminus T \) from the puncture \( p \) to the common endpoint \( v \) of \( \tau_{i+1} \) and \( \tau_{i+t} \). Removing the puncture will then produce an arc \( \gamma_{56} \) by glueing the two arcs \( \gamma_5 \) and \( \gamma_6 \). Thus \( \gamma_{56} \) starts at \( v \), follows \( c \) and then follows \( \gamma_{12} \), but because of isotopy, this arc will not cross the arcs in the fan of arcs \( \tau_{i+1}, \tau_{i+2}, \ldots, \tau_{i+t} \) that are crossed.
by γ_{12} and incident to v. This situation is exactly reflected in the definition of the snake graph \( G_{56} = G_{56} \setminus \text{succ}(e') \), since \( e' \) is the last interior edge such that \( f_{56}(e') = f_{56}(e_t) \).

When the overlap is in the opposite direction, the proof is similar.

It remains the case where the overlap is in the same direction and \( s' \leq t \). Thus the self-overlap has an intersection \( G_{1}[s', t] \).

Let \( \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_r} \) be the sequence of arcs of the triangulation crossed by \( \gamma_1 \) in order. By the definition of overlap, we have that the sequences \( \tau_{s_i}, \tau_{s_{i+1}}, \ldots, \tau_{t_i} \) and \( \tau_{i'_s}, \tau_{i'_s+1}, \ldots, \tau_{i'_t} \) are equal or opposite to each other. Since \( s' < t \) it follows that they have to be equal, since otherwise the segment of \( \gamma \) crossing \( \tau_{i'_s+1}, \tau_{i'_s+2}, \ldots, \tau_{i'_t} \) would be isotopic to a curve not crossing these arcs at all, see Figure 26.

Let \( \gamma[s, t] \) and \( \gamma[s', t'] \) be the segments of \( \gamma \) corresponding to the overlaps and let \( \gamma[s', t'] \) be their common subsegment corresponding to the intersection of the overlaps.

Since the sequences \( \tau_{i_s}, \tau_{i_{s+1}}, \ldots, \tau_{i_t} \) and \( \tau_{i'_s}, \tau_{i'_s+1}, \ldots, \tau_{i'_t} \) are equal, it follows that the curves \( \gamma[s, t] \) and \( \gamma[s', t'] \) run parallel before and after their crossing at \( p \). Moreover, since \( s' < t \), the following sequences are equal as well:

\[
\tau_{i_s}, \tau_{i_{s+1}}, \ldots, \tau_{i_{t-1}} \\
\tau_{i'_s}, \tau_{i'_{s+1}}, \ldots, \tau_{i'_{t-1}} \\
\tau_{2s'-s}, \tau_{2s'-s+1}, \ldots, \tau_{2s'-s+1} \\
\tau_{t_s}, \tau_{t_{s+1}}, \ldots, \tau_{t_{t-1}}
\]

Thus the curve \( \gamma[s, s'] \) after identifying its endpoints is a closed and non-contractible curve. This implies that the curve \( \gamma[s, t] \) is of the form as in Figure 27 where points with equal labels \( a, b, c, d, e \) are identified. The crossing point \( p \) can be any of the points labeled 1,2,3,4,5. For example, if \( p \) is the point labeled 5,4,3,2,1 respectively, then the crossing point \( s' \) at the beginning of the second overlap must be the point on \( \tau_{i_s} \) crossed by \( \gamma \) after the point \( e, d, c, b, a \) respectively, and the crossing point \( t \) at the end of the first overlap must be the point on \( \tau_{i'_t} \) first crossed by \( \gamma \) after passing through the point \( a, b, c, d, e \) respectively.

Therefore the condition \( s' \leq t \) implies that in the example in Figure 27 the point \( p \) must be the point 2 or 1. We now study the smoothing of these self-crossings.

If \( p = 1 \) then the smoothing at \( p \) will produce the two multicurves

\( \{\zeta, \text{generalized arc with 4 self-crossings at 2,3,4,5}\} \)

and

\( \{\text{generalized arc with self-crossings at 3,4,5 and a kink at 2}\} \).

If \( p = 2 \) then the smoothing at \( p \) will produce the two multicurves

\( \{\text{Brac}_1 \zeta, \text{generalized arc with 3 self-crossings at 3,4,5}\} \)

and

\( \{\text{generalized arc with self-crossings at 4,5 and a kink at 3}\} \).

Again we conclude that the snake graphs of the arcs (and bands) obtained by smoothing the self-crossing of \( \gamma \) are given by \( \text{Res}_\varphi(G_1) \). \( \square \)
Figure 27. Proof of Theorem 6.3. This situation can arise in any non-simply connected surface. The curve $\gamma$ between the points 5 and 5 is the concatenation of an essential loop $\zeta$ with itself 5 times, thus $\text{Brac}_5 \zeta$.

So far, we have considered two arcs which cross with a non-empty local overlap. Now we study two arcs which cross with an empty local overlap. The following result has been shown in [CS].

**Theorem 6.4.** [CS, Theorem 5.7] Let $\gamma_1$ and $\gamma_2$ be two arcs which cross in a triangle $\Delta$ with an empty local overlap, and let $G_1$ and $G_2$ be the corresponding snake graphs. Assume $\Delta = \Delta_0$ is the first triangle $\gamma_2$ meets. Then the snake graphs of the four arcs obtained by smoothing the crossing of $\gamma_1$ and $\gamma_2$ in $\Delta$ are given by the resolution $\text{Graft}_{s, \delta_3}(G_1, G_2)$ of the grafting of $G_2$ on $G_1$ in $G_s$, where $0 \leq s \leq d$ is such that $\Delta = \Delta_s$, and if $s = 0$ or $s = d$ then $\delta_3$ is the unique side of $\Delta$ that is not crossed by neither $\gamma_1$ nor $\gamma_2$.

For self-crossing arcs with empty local overlap, we have the following result, see Figure 28.

**Theorem 6.5.** Let $\gamma_1$ be a generalized arc which has a self-crossing in a triangle $\Delta$ with an empty local overlap, and let $G_1$ be the corresponding snake graph. Thus $\Delta = \Delta_0$ is the first triangle $\gamma_1$ meets and $\Delta = \Delta_s$ is met again after $s$ crossings. Then the snake graphs of the two arcs and the band graph of the loop obtained by smoothing the self-crossing of $\gamma_1$ in $\Delta$ are given by the resolution $\text{Graft}_{s, \delta_3}(G_1)$ of the self-grafting of $G_1$ in $G_s$, and if $s = d$ then $\delta_3$ is the unique side of $\Delta$ that is not crossed by $\gamma_1$.

**Proof.** As in the proof of Theorem 6.3, we can introduce a puncture on the segment of $\gamma_1$ between the two crossing points and complete to a triangulation. Then the two segments of $\gamma_1$ before and after the puncture still have the same crossing. We can use Theorem 6.4 to resolve that crossing and then remove the puncture to get the desired resolution.

7. SNAKE GRAPH CALCULUS FOR CLUSTER ALGEBRAS

In this section, we show that we can use snake graph calculus to make explicit computations in the cluster algebras from unpunctured surfaces. In particular, we give a new proof of the skein relations.

7.1. Non-empty overlaps. If $\mathcal{G}$ is a snake graph associated to an arc $\gamma$ in a triangulated surface $(S, M, T)$ then each tile of $\mathcal{G}$ corresponds to a quadrilateral in the triangulation $T$, and we denote by $\tau_{i(G)} \in T$ the diagonal of that quadrilateral. With this notation we define

$$x(\mathcal{G}) = \prod_{\mathcal{G} \text{ tile in } \mathcal{G}} x_{i(G)} \quad y(\mathcal{G}) = \prod_{\mathcal{G} \text{ tile in } \mathcal{G}} y_{i(G)}$$
If \( \mathcal{G} = \{ \tau \} \) consists of a single edge, we let \( x(\mathcal{G}) = 1 \) and \( y(\mathcal{G}) = 1 \).

Let \( \gamma_1 \) and \( \gamma_2 \) be two arcs which cross with a non-empty overlap. Let \( x_{\gamma_1} \) and \( x_{\gamma_2} \) be the corresponding cluster variables and \( G_1 \) and \( G_2 \) the snake graphs with corresponding overlap \( \mathcal{G} \).

Recall that \( \text{Res}_G(G_1, G_2) \) consists of two pairs \((G_3, G_4)\) and \((G_5, G_6)\) of snake graphs. The number of tiles in \((G_3, G_4)\) is equal to the number of tiles in \((G_1, G_2)\), whereas the number of tiles in \((G_5, G_6)\) is strictly smaller, since this pair does not contain the overlaps.

Define \( \tilde{\mathcal{G}} \) to be the union of all tiles in \( G_1 \cup G_2 \) which are not in \( G_5 \cup G_6 \).

Similarly, if \( \gamma_1 \) is a self-crossing arc with non-empty local overlap, let \( x_{\gamma_1} \) be the corresponding Laurent polynomial and \( G_1 \) be the snake graph with corresponding self-overlap \( \mathcal{G} \). In this situation, if the self-overlap is in the same direction, then the resolution of the crossing consists of a pair \((G_3, G_4)\) of a snake and a band graph and a snake graph \( G_{56} \); whereas if the self-overlap is in the opposite direction, then the resolution of the crossing consists of a snake graph \( G_{34} \) and a pair \((G_5, G_6)\) of a snake and a band graph.

If the self-overlap is in the same direction, the number of tiles in \((G_3, G_4)\) is equal to the number of tiles in \((G_1, G_2)\), whereas the number of tiles in \( G_{56} \) is strictly smaller. On the other hand, if the overlap is in the opposite direction, then the number of tiles in \( G_{34} \) is equal to the number of tiles in \( G_1 \), whereas the number of tiles in \( G_5 \cup G_6 \) is strictly smaller.

Also in this case, define \( \tilde{\mathcal{G}} \) to be the union of all tiles in \( G_1 \) which are not in \( G_{56} \) or \( G_5 \cup G_6 \).

In all cases, under the bijections of section 4 the matchings \( P_{56} \) of \( G_5 \cup G_6 \) (respectively \( G_{56} \) or \( G_5 \cup G_6 \)) are completed to matchings of \( G_1 \cup G_2 \) (respectively \( G_1 \) or \( G_5 \cup G_6 \)) in a unique way which does not depend on \( P_{56} \). Moreover, the \( y \)-monomial of the completion is maximal on a connected subgraph of \( \tilde{\mathcal{G}} \) and trivial on its complement. We denote by \( \tilde{G}_{\text{max}} \) the component on which the \( y \) monomial is maximal.

Let \( \text{Res}_G(G_1, G_2) \) be the resolution of the crossing of \( G_1 \) and \( G_2 \) at \( \mathcal{G} \) and \( \text{Res}_G(G_1) \) the resolution of the self-crossing of \( G_1 \) at \( \mathcal{G} \). Define the Laurent polynomial of the resolutions by

\[
(7.1) \quad \mathcal{L}(\text{Res}_G(G_1, G_2)) = \mathcal{L}(G_3 \cup G_4) + y(\tilde{G}_{\text{max}}) \mathcal{L}(G_5 \cup G_6),
\]

and

\[
(7.2) \quad \mathcal{L}(\text{Res}_G(G_1)) = \begin{cases} 
\mathcal{L}(G_3 \cup G_4) + y(\tilde{G}_{\text{max}}) \mathcal{L}(G_{56}), & \text{if the self-overlap is in the same direction;} \\
\mathcal{L}(G_{34}) + y(\tilde{G}_{\text{max}}) \mathcal{L}(G_5 \cup G_6) & \text{if the self-overlap is in the opposite direction},
\end{cases}
\]

where

\[
\mathcal{L}(G_k) = \frac{1}{x(G_k)} \sum_{P \in \text{Match}(G_k)} x(P)y(P), \quad \text{if } G_k \text{ is positive};
\]
and

\[ \mathcal{L}(G_k) = -\mathcal{L}(-G_k), \quad \text{if } G_k \text{ is negative.} \]

**Theorem 7.1.** (1) Let \( \gamma_1 \) and \( \gamma_2 \) be two arcs which cross with a non-empty local overlap and let \( G_1 \) and \( G_2 \) be the corresponding snake graphs with local overlap \( G \). Then

\[ \mathcal{L}(G_1 \cup G_2) = \mathcal{L}(\text{Res}_G(G_1, G_2)) \]

(2) Let \( \gamma_1 \) be a self-crossing generalized arc a non-empty local overlap and let \( G_1 \) be the corresponding snake graph with local overlap \( G \). Then

\[ \mathcal{L}(G_1) = \mathcal{L}(\text{Res}_G(G_1)) \]

**Proof.** (1) This is [CS] Theorem 6.1. The essential step of the proof is to show that the switching operation of section 4.1 is weight preserving. That is, if \( G \) is a (union of) labeled snake and band graphs coming from an unpunctured surface, \( P \in \text{Match} \, G \), and \( P' \) is obtained from \( P \) by a switching operation, then \( x(P) = x(P') \) and \( y(P) = y(P') \). Then, since the bijection \( \varphi \) of section 4.2 is defined using switching and restriction, it is also weight preserving. To finish the proof one needs to take care of the missing tiles in \( G_5 \) and \( G_6 \), and show that the \( y(\tilde{G}_{\text{max}}) \) is absorbing this discrepancy.

(2) As usual we use the notation \( G_1[s, t] \equiv G_1[s', t'] \) for the overlap. In the case where \( s' > t + 1 \), the proof is the exact analogue of the proof of (1).

If \( s' = t + 1 \), we have two cases. Either the second overlaps \( j_1(H) \) and \( j_2(H) \) from the definition of the resolution in section 3.2 cross or not. If they cross then the proof is the same as in the case \( s' = t + 1 \). If they do not cross, then the proof is exactly analogue to the proof of (1), except that we need to check that the bijection is weight preserving for the one special type of matching \( P_\circ \) defined in section 4.5. Observe that on each piece of \( G_3 \), the definition of \( \varphi(P_\circ) \) is given by restricting \( P_\circ \) to that piece. Moreover, on \( G_1^s \), \( \varphi(P_\circ) \) is also given by restricting \( P_\circ \) except for one edge, namely the boundary edge of \( G_{s-1}^{\text{NE}} \) is mapped to the edge of \( G_1 \) that corresponds to the edge \( e_\tau \) of \( G_1 \).

To show that \( x(\varphi(P_\circ)) = x(P_\circ) \) it suffices to show that the boundary edge of \( G_{s-1}^{\text{NE}} \) has the same weight as the edge \( e_\tau \). Recall that \( s_i = i \), \( i = i \), and \( s_{i-1} \neq i \), \( i+1 \neq i\), since we have an overlap. Moreover, since we are in the case \( s' = t + 1 \), it follows that \( s_{i-1} \neq i \) and \( s_{i} \neq i_{i+1} \), and that \( \tau_{i-1} \) and \( \tau_{i} \) are two distinct sides of a triangle \( \Delta \) in the triangulation. Therefore \( \tau_{i-1} = \tau_{i+1} \) is the third side of \( \Delta \). Now the edge \( e_\tau \) is the edge shared by the tiles \( G_{i+1} \) and \( G_{i+1} \), and its weight is given by the side of \( \Delta \) different from \( \tau_{i-1} \) and \( \tau_{i+1} \), thus the weight of \( e_\tau \) is equal to \( s_i \). On the other hand, the weight of the boundary edge of \( G_{s-1}^{\text{NE}} \) is also equal to \( s_i \) since \( G_s \) is following \( G_{s-1} \) in \( G_1 \).

To show that \( y(\varphi(P_\circ)) = y(P_\circ) \), observe that \( y(\varphi(P_\circ)|_{\tilde{G}_s}) = y(\varphi(P_\circ)|_{\tilde{G}_s[s', t']}) \) by definition and that \( y(\varphi(P_\circ)|_{\tilde{G}_s}) = y(P_\circ|_{\tilde{G}_s[s', t']}) \) because even if the edges on the parts that come from \( j_1(H) \) and \( j_2(H) \) are swapped, together they still create the same contribution to the \( y \)-monomial.

Suppose now that \( s' \leq t \). Let

\[ \varphi : \text{Match} \, G_1 \cup \text{Match} \, G_{56} \mapsto \text{Match} \, G_3 \times \text{Match} \, G_{4} \]

be the bijection defined in section 4.4.

First we note that

\[ x(G_1) = x(G_3 \cup G_{56}^i) = x(G_{56}^i) x(G), \]

where the first identity holds because \( G_1 \) and \( G_{4} \cup G_{56}^i \) have the same set of tiles, and the second identity holds because \( G \) consists of the tiles of \( G_1 \setminus G_{56} \). Using the equation (7.3) on the definition
of $L(G_1)$ as well as the bijection $\varphi$, we obtain

$$L(G_3 \sqcup \tilde{G}_1^s) = \frac{1}{x(G_1)} \sum_{P \in \text{Match } G_1} x(\varphi(P))y(\varphi(P)) + \frac{1}{x(\tilde{G})} \sum_{P \in \text{Match } \tilde{G}_{56}} x(\varphi(P))y(\varphi(P)),$$

and therefore it suffices to show the following lemma.

Lemma 7.2. Let $s' \leq t$.

(a) $x(\varphi(P))y(\varphi(P)) = x(P)y(P)$ if $P \in \text{Match}(G_1)$.

(b) $x(\varphi(P))y(\varphi(P)) = x(\tilde{G})y(\tilde{G}_{\text{max}})x(P)y(P)$ if $P \in \text{Match}(\tilde{G}_{56})$.

Proof. (a) This is [CS, Lemma 6.2].

(b) Since $s' < t$, we see that $\tilde{G}$ consists of two copies of each tile in $G_2$. By definition of $\varphi$ in this case, we have

$$\varphi(P) = P \cup P_- \cup P_+$$

where $P_-$ is the minimal matching of $G_1^s$ and $P_+$ is its maximal matching, since $\varphi(P)$ is extended to a matching on $G_1$ using boundary edges that are complementary on $G_1[s, s' - 1]$ and $G_1[t + 1, t']$. Now $x(P_-)x(P_+) = x(G_1^s)x(\tilde{G}_1^s)$ and thus

$$x(\varphi(P)) = x(P)x(P_-)x(P_+) = x(P)(x(G_1^s))^2 = x(P)x(\tilde{G}).$$

Moreover, $\tilde{G}_{\text{max}} = G_4^s$ and thus

$$y(\varphi(P)) = y(P)y(P_+) = y(P)y(G_3^s) = y(P)y(\tilde{G}_{\text{max}}).$$

$\square$

7.2. Empty overlaps.

7.2.1. Two arcs crossing. Now let $\gamma_1$ and $\gamma_2$ be two arcs which cross in a triangle $\Delta$ with an empty overlap. We may assume without loss of generality that $\Delta$ is the first triangle $\gamma_2$ meets. Let $x_{\gamma_1}$ and $x_{\gamma_2}$ be the corresponding cluster variables and $G_1$ and $G_2$ be their associated snake graphs, respectively. We know from Theorem 6.4 that the snake graphs of the arcs obtained by smoothing the crossing of $\gamma_1$ and $\gamma_2$ are given by the resolution $\text{Graft}_{s, \delta_3}(G_1, G_2)$ of the grafting of $G_2$ on $G_1$ in $G_s$, where $s$ is such that $\Delta = \Delta_s$ is the triangle $\gamma_1$ meets after its $s$-th crossing point, and, if $s = 0$, then $\delta_3$ is the unique side of $\Delta$ which is not crossed neither by $\gamma_1$ nor $\gamma_2$.

The edge of $G_s$ which is the glueing edge for the grafting is called the grafting edge. We say that the grafting edge is minimal in $G_1$ if it belongs to the minimal matching on $G_1$.

Recall that $\text{Graft}_{s, \delta_3}(G_1, G_2)$ is a pair $(G_3 \sqcup G_4), \tilde{G}_{56})$. Let $G_{34}$ be the union of all tiles in $G_1 \sqcup G_2$ that are not in $G_3 \sqcup G_4$ and let $\tilde{G}_{56}$ be the union of all tiles in $G_1 \sqcup G_2$ that are not in $G_5 \sqcup G_6$. Define

$$L(\text{Graft}_{s, \delta_3}(G_1, G_2)) = y_{34}L(G_3 \sqcup G_4) + y_{56}L(G_5 \sqcup G_6),$$

where

$$y_{34} = \begin{cases} 1 & \text{if the grafting edge is minimal in } G_1; \\ y_{34} & \text{otherwise.} \end{cases}$$

The following result has been shown in [CS].

Theorem 7.3. [CS, Theorem 6.3] With the notation above, we have

$$L(G_1 \sqcup G_2) = L(\text{Graft}_{s, \delta_3}(G_1, G_2)).$$
7.2.2. **Self-crossing.** Similarly, if \( \gamma_1 \) is a generalized arc which self-crosses in a triangle \( \Delta \) with an empty overlap, let \( \mathcal{G}_1 \) be the associated snake graph and \( x_{\gamma_1} \) be the corresponding Laurent polynomial. We know from Theorem 6.3 that the snake graphs of the arcs obtained by smoothing the self-crossing of \( \gamma_1 \) are given by the resolution \( \text{Graft}_{s,\delta_3}(\mathcal{G}_1) \) of the self-grafting of \( \mathcal{G}_1 \) in \( G_s \), where \( s \) is such that \( \Delta = \Delta_s \) and, if \( s = d \), then \( \delta_3 \) is the unique side of \( \Delta \) which is not crossed by \( \gamma_1 \).

Let \( \tilde{\mathcal{G}}_{34} \) be the union of tiles in \( \mathcal{G}_1 \) that are not in \( \mathcal{G}_3 \sqcup \mathcal{G}_4^\circ \) and \( \tilde{\mathcal{G}}_{56} \) be the union of tiles in \( \mathcal{G}_1 \) that are not in \( \mathcal{G}_{56} \).

If \( s < d \), then \( \delta_3 \) is the north or the east edge in \( G_s \), and we let

\[
y_{34} = \begin{cases} y(\tilde{\mathcal{G}}_{34}) & \text{if } \delta_3 \text{ is maximal in } \mathcal{G}_1; \\ 1 & \text{otherwise;}
\end{cases}
y_{56} = \begin{cases} 1 & \text{if } \delta_3 \text{ is maximal in } \mathcal{G}_1; \\ y(\tilde{\mathcal{G}}_{56}) & \text{otherwise;}
\end{cases}
\]

(7.6)  

If \( s = d \), then \( \mathcal{G}_1 \) and \( \mathcal{G}_1 \sqcup \mathcal{G}_4^\circ \) have the same tiles, so \( y_{34} = 1 \). On the other hand, \( \mathcal{G}_1 \setminus \mathcal{G}_{56} \) has two components \( \mathcal{G}_1[1,k'-1] \) containing the glueing edge \( \delta'_3 \) and \( \mathcal{G}_1[k+1,d] \) containing the glueing edge \( \delta_3 \). We let

\[
y_{56} = \begin{cases} y(\mathcal{G}_1[1,k'-1]) & \text{if } \delta'_3 \text{ is minimal in } \mathcal{G}_1; \\ 1 & \text{otherwise;}
\end{cases}
y_{\prime 56} = \begin{cases} 1 & \text{if } \delta_3 \text{ is minimal in } \mathcal{G}_1; \\ y(\mathcal{G}_1[k+1,d]) & \text{otherwise.}
\end{cases}
\]

(7.7)  

where

\[
y_{\prime 56} = \begin{cases} y(\mathcal{G}_1[1,k'-1]) & \text{if } \delta'_3 \text{ is minimal in } \mathcal{G}_1; \\ 1 & \text{otherwise;}
\end{cases}
y_{\prime \prime 56} = \begin{cases} 1 & \text{if } \delta_3 \text{ is minimal in } \mathcal{G}_1; \\ y(\mathcal{G}_1[k+1,d]) & \text{otherwise.}
\end{cases}
\]

With this notation define

\[
\mathcal{L}(\text{Graft}_{s,\delta_3}(\mathcal{G}_1)) = y_{34} \mathcal{L}(\mathcal{G}_3 \sqcup \mathcal{G}_4^\circ) + y_{56} \mathcal{L}(\mathcal{G}_{56}).
\]

**Theorem 7.4.** With the notation above, we have

\[
\mathcal{L}(\mathcal{G}_1) = \mathcal{L}(\text{Graft}_{s,\delta_3}(\mathcal{G}_1))
\]

**Proof.** As in the proof of Theorem 6.3 we can introduce a puncture on the segment of \( \gamma_1 \) between the two crossing points and complete to a triangulation. Then the two segments of \( \gamma_1 \) before and after the puncture still have the same crossing with empty overlap. Applying Theorem 7.3 to this situation and then removing the puncture will complete the proof. \( \Box \)

7.3. **Skein relations.** As a corollary we obtain a new proof of the skein relations.

**Corollary 7.5.** Let \( C, C_+ \), and \( C_- \) be as in Definition 5.6 but not including any closed loops. Then we have the following identity in the cluster algebra \( \mathcal{A} \),

\[
x_C = \pm Y_1 x_{C_+} \pm Y_2 x_{C_-}.
\]

Moreover the coefficients \( Y_1 \) and \( Y_2 \) are given by (7.1), (7.2), (7.5), (7.6) and (7.7). \( \Box \)

8. **An example**

Let \( (S,M) \) be the torus with one boundary and one marked point, and consider the two arcs \( \gamma_1 \) (in black) and \( \gamma_2 \) (in red) shown in Figure 29. In the left picture, the arcs are drawn directly on the torus and in the right picture, the arcs are drawn in the standard covering of the torus. In the picture on the right, we also have fixed a triangulation whose arcs carry the labels 1,2,3, and 4.

We want to compute the product of the two corresponding cluster variables and express this product in the basis \( \mathcal{B} \). This could be done using the smoothing operation 4 times to smooth the 4 crossings of the two arcs. Instead of using the smoothing operation, we do the computation with snake graphs, see Figure 30.

The first equation in the figure is the resolution of the second crossing overlap given by the tiles with labels 3,4. The graph \( \mathcal{G}_6 \) consists of the single edge with label 3, coming from the east edge of the last tile in the snake graph \( \mathcal{G}_1 \). The second equation uses self-grafting with \( s = d \).
replace the first snake graph by a product of the band graph with tiles 1,3,4,1,3,4 and the single edge 2 plus the self-crossing snake graph with tiles 4,1,3, and to replace the second snake graph with the product of the band graph with tiles 3,4,1 and the single edge 2 plus zero. The third equation uses self-grafting with $s = d$ to replace the self-crossing snake graph with tiles 3,4,1 by the product of the corresponding band graph and the single edge $b$ (for boundary).

For the computation of the coefficients, we choose the orientation such that the south edge of the first snake graph is minimal, see Figure 30. Thus the term $G_5 G_6$ in the first row of the figure carries the coefficient $y(\bar{G}_{\text{max}}) = y_1^2 y_3 y_4$. Recall that the second row is obtained from the first by two self-graftings with $s = d$, thus the two terms $G_3 G_4$ do not carry a coefficient. In the first parenthesis in the second row, the snake graph with tiles 4,1,3 is the term $G_{56}$ of the self-grafting. The grafting edge $\delta_3$ is minimal, whereas the grafting edge $\delta'_3$ is maximal, and consequently, the coefficient $y_4'_{56} = y_1 y_3$ corresponds to the initial segment of the self-crossing snake graph. In the second parenthesis of the second row, the term $G_{56}$ is zero and does not carry a coefficient. Recall that the third row is obtained from the second by a self-grafting with $s = d$ on the graph with tiles 4,1,3. The single edge with label 2 and coefficient $y_4$ is the term $G_{56}$ of this self-grafting.

**Figure 29.** Two arcs on the torus

**Figure 30.** Snake graph calculus computing the product of two cluster variables
The grafting edge $\delta_3$ is minimal, whereas the grafting edge $\delta_3'$ is maximal, and consequently, the coefficient $y_{56} = y_4$ corresponds to the initial segment of the self-crossing snake graph.

Finally, in the last row we rewrite the expression as
\[(\text{Brac}_2 \zeta x_2 + y_1 y_3 x_\zeta + y_1 y_3 y_4 x_2) x_\zeta x_2 + y_1^2 y_3^2 y_2^2 x'_1 x_3,\]
where $\zeta$ is the loop that crosses 1,3,4 and $x'_1$ is the cluster variable obtained from the initial cluster by mutating in direction 1.

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