Moufang Theorem for a variety of local non-Moufang loops

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Abstract

An open problem in theory of loops is to find the variety of non-Moufang loops satisfying the Moufang Theorem. In this note, we present a variety of local smooth diassociative loops with such property.

Key words: Binary-Lie algebras, Malcev algebras, diassociative loops, Moufang loops, Steiner loops.

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1 Introduction

At the conference “Loops ’11” in Trest, Czech Republic, Andrew Rajah proposed the following question:

(Moufang theorem in non-Moufang loops). We say that a variety $V$ of loops satisfies the Moufang theorem if for every loop $Q$ in $V$ the following implication holds: if $x(yz) = (xy)z$ for every $x, y, z \in Q$ then the subloop generated by $x, y, z$ is a group. Is every variety satisfying the Moufang theorem contained in the variety of Moufang loops?

Recently this question was discussed in several articles, as for example [ColbSt], [MGColb], and [St].
In all cited articles there were found some cases or classes of Steiner loops satisfying the statement of Moufang Theorem. None of those examples present the variety of loops. Our example of a variety in the context of local smooth loops doesn’t give the complete answer to the question of A. Rajah, but it sheds some additional light on his question.

R. Moufang, in 1931, started studying algebraic structures which today are called Moufang loops. Recall that a loop is a set \( Q \) with a binary operation \( \cdot \) such that the equations \( x \cdot a = b \) \( a \cdot y = b \) have a unique solution \( \forall a, b \in Q \) and \( x \cdot 1 = x = 1 \cdot x \) \( \forall x \in Q \). And in the same way, a Moufang loop is a loop in which the identity \((xy)xz = x(yxz))\) hold for every three elements of the loop. A loop \( Q \) is called diassociative if every two elements generate a subgroup of \( Q \). In addition to Moufang loops another example of diassociative loops are Steiner loops. Steiner loops are diassociative commutative loops of exponent 2. (See [ColbSt] and [MGColb]).

2 Main Theorem

In 1935 Ruth Moufang showed her famous theorem [RM]: Let \( M \) be a Moufang loop. If \((a,b,c) = ((ab)c)(a(bc))^{-1} = 1\) for some elements \( a, b, c \in M \) then \( a, b, c \) generate a subgroup of \( M \). It is easy to see that the Moufang Theorem implies the diassociativity of loops: to verify this fact, consider the identity \((a,b,1) = 1\) in the loop, which obviously always has a place.

In 1955 A.I. Malcev applying the Campbell-Hausdorff formula to the varieties of smooth local loops introduced the Binary-Lie algebras as tangent algebras of smooth diassociative loops and Moufang-Lie algebras (now called Malcev algebras) as tangent algebras of smooth local Moufang loops [Ma]. The identities \( x^2 = 0, J(x,y,xy) = 0 \), (where \( J(x,y,z) = (xy)z + (yz)x + (zx)y \)) defining the variety of Binary-Lie algebras were found in [Ga]. On the other hand, the identities \( x^2 = 0, J(x,y,xz) = J(x,y,z)x \), which define the variety of Malcev algebras were stated in [Sa]. The identities \( x^2 = 0, J(x,y,zt) = J(x,y,z)t = 0 \) of the variety of Malcev algebras which are tangent algebras of smooth local left-automorphic Moufang loops are discussed in [CS1] and the identities \( x^2 = 0, J(x,y,xz) = J(x,y,z)x = 0 \) of the variety of Malcev algebras which are tangent algebras of smooth local almost left automorphic Moufang loops were found in [CS2].

The analog of Moufang Theorem in the context of Malcev algebras has the following form: Let \( M \) be a Malcev algebra. If for given three elements \( \{x_1, x_2, x_3\} \) of \( M \) the equality \( J(x_1, x_2, x_3) = 0 \) is satisfied, the subalgebra of \( M \) generated by these elements, \( \{x_1, x_2, x_3\} \), is a Lie algebra.

Thus, Rajah’s question in this sense should be: Is there a variety of Binary-Lie algebras satisfying the analog of Moufang theorem which does not belong to the variety of Malcev algebras?
In the following, all the algebras will be considered as algebras over a field $k$.

The aim of this note is to show the following:

**Theorem 1.** Let $\mathfrak{w}$ be a variety of Binary-Lie algebras defined by the identities

$$x^2 = 0, \quad J(x, y, zu) = 0,$$

where $J(x, y, z) = (xy)z + (yz)x + (zx)y$.

Then

1. $\mathfrak{w}$ is not a variety of Malcev algebras.
2. Any algebra of the variety $\mathfrak{w}$ satisfies the statement of an analog of Moufang Theorem.

**Proof.**

1. Consider a non-nilpotent solvable 4-dimensional algebra $\mathcal{L}$ from the variety $\mathfrak{w}$ generated by the elements $\{a, b, c\}$ with the following relations:

$$ab = ac = 0, \quad bc = d, \quad da = d, \quad bd = cd = 0$$

we have $J(a, b, c) = d$, and therefore by direct computation we get:

$$J(\mathcal{L}) = \mathcal{L}^2 = kd = Lie(\mathcal{L}),$$

where $k \in k$, $J(\mathcal{L})$ is an ideal generated by all jacobians $J(x, y, z) \forall x, y, z \in \mathcal{L}$ and $Lie(\mathcal{L}) = \{x \in \mathcal{L} \mid J(x, y, z) = 0, \forall y, z \in \mathcal{L}\}$ is a Lie center of an algebra $\mathcal{L}$. We have $J(a, b, ac) - J(a, b, c)a = d \neq 0$, hence $\mathcal{L}$ is not a Malcev algebra.

2. In order to prove the second statement of the Theorem let us consider the algebra $\mathcal{C}$ from the variety $\mathfrak{w}$, generated by the elements $\{a, b\}$, such that the following condition holds:

$$J(a, b, c) = 0.$$

Let us note that in this case

$$J(w_1, w_2, w_3) = 0,$$

for all $w_i \in \mathcal{C}$, Indeed, we have $w_i = v_i + u_i$ where $v_i$ is an element from the vector space $V$ with the base $\{a, b, c\}$ and $u_i \in \mathcal{C}^2$. Therefore by definition of $\mathfrak{w}$ we have

$$J(w_1, w_2, w_3) = J(v_1, v_2, v_3) = \alpha J(a, b, c) = 0, \quad \alpha \in k.$$

Using the Malcev Theorem on the correspondence between local diassociative loops and their tangent Binary-Lie algebras [Ma] we get the following...
Corollary 1. There exists the non-Moufang variety of local dissociative loops \( \mathfrak{W} \) such that the statement of Moufang Theorem holds for every loop in \( \mathfrak{W} \).

Now we give the general construction for all algebras from the variety \( \mathfrak{w} \).

Let \( L \) be a Lie algebra, \( \text{Der} L \) the Lie algebra of derivations of \( L \) and \( P \) an arbitrary vector space. Let \( P_0 \subseteq \text{Der} L \) be a subspace of \( \text{Der} L \) such that \([P_0, P_0] \subseteq \text{Inn} L\), where \( \text{Inn} L \) is the ideal of all inner derivations of \( L \), i.e. 
\[
\text{Inn} L = \{ad x \mid x \in L\} \quad (ad x : a \rightarrow [ax]).
\]
Let \( \psi : P \to P_0 \) be some epimorphism.
Consider \( L = L_0 \oplus Z(L) \), where \( Z(L) = \{x \in L \mid [x, L] = 0\} \). It is possible to identify:
\[
L_0 \cong L/Z(L) \cong \text{Inn} L.
\]

Proposition 1. The algebra \( B \) with the operation (1) belongs to the variety \( \mathfrak{w} \).
Any algebra from the variety \( \mathfrak{w} \) can be obtained by the construction described above.

Proof. 1. An algebra \( B \) belongs to the variety \( \mathfrak{w} \) if and only if \( B^2 \) belongs to the Lie center \( Z(B) \).
Since \( B \cdot B \subseteq L \) it is enough to show that \( L \subseteq \text{Lie}(B) \). By construction 
\[
J(p_1, a, b) = 0,
\]
since \( p_1 \) acts on \( L \) as a derivation. In the same way, \( J(p_1, p_2, a) = 0 \), since \( \lambda(p_1, p_2) \in Z(L) \) for all \( p_1, p_2 \in P, a, b \in L \).

2. Now we show that every algebra from the variety \( \mathfrak{w} \) may be obtained using the general construction.

Consider \( B \in \mathfrak{w} \) and denote \( L = \text{Lie}(B) \), then \( B = L \oplus P \), where \( P \) is some vector space. By definition of \( \mathfrak{w} \) \( B^2 \subseteq L \), in particular \([P, P] \subseteq L\). For any \( p \in P \) denote
\[
p^\psi \in \text{Der} L : p^\psi(a) = [p, a].
\]

Notice that \( \psi \) is defined correctly, since \( L \) is a Lie center. Moreover \( P_0 = P^\psi \subseteq \text{Der} L \) satisfies the condition
\[
[P_0, P_0] \subseteq \text{Inn} L \cong L/Z(L)
\]
In the case \( L = Z(L) \oplus L_0 \), where \( L_0 \) is a suitable vector space, we have a map
\[
\phi : P_0 \land P_0 \to L_0, \quad \phi(p_1^\psi \land p_2^\psi) = l_0
\]
where \([p_1, p_2] = l_0 + z\). \(l_0 \in L_0, z \in Z(L)\).

Finally, define \(\lambda : P \wedge P \to Z(L)\) \(\lambda(p_1 \wedge p_2) = z\). Under our notations \([Z(L), p] \subseteq Z(L), p \in P\) implies the correctness of the definition of the map \(\phi\). This way one can construct the algebra \(B\) using the Lie algebra \(L\) and two maps defined above: \(\psi : P \to P_0\) and \(\lambda : P \wedge P \to Z(L)\). \(\Box\)

**Example.**

Let \(L = \kappa c, \kappa \in k\) be a one-dimensional Lie algebra, let \(P\) be a vector space, generated by elements \(\{t, a, b\}\). Consider \(P_0 = \text{Der}_L\).

\(P_0 = \kappa t_0, \kappa \in k, \ ct_0 = c, \ \psi(t) = t_0, \ \psi(a) = \psi(b) = 0, \)

\(\lambda(t, a) = \alpha_1 c, \ \lambda(t, b) = \alpha_2 c, \ \lambda(a, b) = \alpha_3 c\)

Let us denote \(B(\alpha_1, \alpha_2, \alpha_3)\) the corresponding Binary-Lie algebra. It is easy to see that if \(\alpha_3 \neq 0\), then every such an algebra is isomorphic to \(B(0, 0, 1)\) with base \(\{t, a, b, c\}\) and operation:

\[at = bt = 0 \quad ab = c, \quad ac = bc = 0, \quad ct = c.\]

Consider the variety \(v\) of algebras defined by the identities

\[x^2 = 0, \quad J(x, y, xz) = 0.\]

The variety \(v\) is the variety of Binary-Lie algebras, which are not Malcev algebras, because \(w\) is contained in \(v\). Now, lets consider a free algebra of the variety \(v\) generated by the elements \(\{a, b, ac\}\). We know that \(J(a, b, (ac)) = 0\). Then, one can conjecture that \(J(a, b, (ab)(ac)) \neq 0\). If this conjecture is true then it will shown that the variety \(v\) does not obey the statement of Moufang Theorem.

Some natural questions arise:

1. Is it possible to find a maximal subvariety of the variety of Binary-Lie algebras for which the analog of Moufang Theorem holds?

2. Analysing our example of the variety \(M\) of local loops one can consider the variety \(\Omega\) of (discrete) diassociative loops with the additional identity

\[(x, y, [z, t]) = 1.\]

It is known that the variety of Steiner loops is a subvariety of \(\Omega\). It is also known that not all Steiner loops obey the statement of Moufang Theorem. In the case of the variety of Steiner loops as we mentioned above the identity \(x^2 = 1\) holds.

In this context, we conjecture that we can find a positive answer of A.Rajah’s question for the variety of diassociative loops with the additional identities

\[(x, y, [z, t]) = 1, \quad x^m = 1\]

for some odd \(m\).
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