Finding Points in Convex Position in Density-Restricted Sets

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Abstract

For a finite set \( A \subset \mathbb{R}^d \), let \( \Delta(A) \) denote the spread of \( A \), which is the ratio of the maximum pairwise distance to the minimum pairwise distance. For a positive integer \( n \), let \( \gamma_d(n) \) denote the largest integer such that any set \( A \) of \( n \) points in general position in \( \mathbb{R}^d \), satisfying \( \Delta(A) \leq \alpha n^{1/d} \) for a fixed \( \alpha > 0 \), contains at least \( \gamma_d(n) \) points in convex position. About 30 years ago, Valtr proved that \( \gamma_2(n) = \Theta(n^{1/3}) \). Since then no further results have been obtained in higher dimensions. Here we continue this line of research in three dimensions and prove that \( \gamma_3(n) = \Theta(n^{1/2}) \). The lower bound implies the following approximation: Given any \( n \)-element point set \( A \subset \mathbb{R}^3 \) in general position, satisfying \( \Delta(A) \leq \alpha n^{1/3} \) for a fixed \( \alpha \), a \( \Omega(n^{-1/6}) \)-factor approximation of the maximum-size convex subset of points can be computed by a randomized algorithm in \( O(n \log n) \) expected time.

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1 Introduction

A set of points in the \( d \)-dimensional space \( \mathbb{R}^d \) is said to be: (i) in general position if any \( d + 1 \) or fewer points are affinely independent; and (ii) in convex position if none of the points lies in the convex hull of the other points.

In 1935 Erdős and Szekeres proved, as one of the first Ramsey-type results in combinatorial geometry, that for every \( n \in \mathbb{N} \), a sufficiently large point set in the plane in general position contains \( n \) points in convex position [29]. The minimum cardinality of a point set that contains a subset of \( n \) points in convex position is known as the Erdős–Szekeres number, denoted here by \( f(n) \). The resulting upper bound in their paper was \( f(n) \leq \binom{2n-4}{n-2} + 1 = 4^{n(1-o(1))} \). In 1960, the same authors showed a construction that implies a lower bound of \( f(n) \geq 2^{n-2} + 1 \), and conjectured that this lower bound is tight [21]. The current best (asymptotic) upper bound, due to Suk [35], is \( f(n) \leq 2^{n(1+o(1))} \). In other words, every set of \( n \) points in general position in the plane contains \((1 - o(1)) \log n \) points in convex position, and this bound is tight up to lower-order terms.

Let \( A \) be a set of \( n \) points in general position in \( \mathbb{R}^d \). Define

\[
\Delta(A) = \frac{\max \{ \text{dist}(a,b) : a, b \in A, a \neq b \}}{\min \{ \text{dist}(a,b) : a, b \in A, a \neq b \}},
\]

where \( \text{dist}(a,b) \) is the Euclidean distance between points \( a \) and \( b \). The ratio \( \Delta(A) \) is referred to as the aspect ratio or the spread of \( A \); see for instance [12, 17]. We assume without loss of generality that the minimum pairwise distance is 1 and in this case \( \Delta(A) \) is the diameter of \( A \). A standard volume argument shows that if \( A \) has \( n \) points, then \( \Delta(A) \geq c_d n^{1/d} \), where \( c_d > 0 \) is a constant depending only on \( d \); for instance it is known [36] that \( c_2 \geq 21/23^{1/4} \approx 1.05 \). On the other hand, the section of the integer lattice \([n]^d\) shows that this bound is tight up to
Finding Points in Convex Position in Density-Restricted Sets

the aforementioned constant. A point set satisfying the condition $\Delta(A) = \mathcal{O}(n^{1/d})$, is called here density-restricted (or simply dense, see for instance [19, 28, 37]).

In the seminal article of Erdős and Szekeres [20], the constructed point sets with no large subsets in convex position have very large spread. Similarly, in Horton’s seminal article on point sets with no empty convex heptagons, the constructed point sets also have very large spread (Horton sets will be discussed in Section 2). Answering the emerging question of whether such results really require large spreads, Valtr [36] showed the existence of arbitrarily large planar sets $A$ with $\Delta(A) \leq \alpha n^{1/2}$ that have no empty convex heptagons. (Observe that whenever $\Delta(A)$ satisfies this condition, we have $\alpha \geq c_2$.) This property can be achieved for example by a suitable, carefully crafted, small perturbation of the lattice section $[n]^2$.

Moreover, Valtr [36] obtained by probabilistic arguments the following result:

> **Theorem 1.** (Valtr [36]) For every $\alpha > 0$ there exists $\beta = \beta(\alpha) > 0$ such that any set of $n$ points in general position in the plane satisfying $\Delta(A) \leq \alpha n^{1/2}$, contains a subset of $\beta n^{1/3}$ points in convex position.

It is therefore apparent that the size of the largest convex subset contained in an $n$-element point set strongly depends on the spread of the set. In the same paper, Valtr generalized Theorem 1 to a broader range of the spread (by similar arguments):

> **Theorem 2.** (Valtr [36]) For every $\alpha > 0$ and $\tau \in \left[\frac{1}{3}, \frac{2}{3}\right)$, there exists $\beta = \beta(\alpha, \tau) > 0$ such that any set $A$ of $n$ points in general position in the plane satisfying $\Delta(A) \leq \alpha n^\tau$ contains a subset of $\beta n^{1-4\tau/3}$ points in convex position.

On the other hand, as remarked by Valtr, the construction by Erdős and Szekeres [21] can be transformed into one with spread $\Delta < n$ and with at most $\log n + 2$ points in convex position. For the remaining range $\tau \in \left[\frac{2}{3}, 1\right)$, Valtr stated in his PhD Thesis [37] (without proof) that an involved argument shows that any set $A$ of $n$ points in general position in the plane satisfying $\Delta(A) \leq \alpha n^\tau$ (where $\alpha > 0$ is a constant) contains a convex subset of size $\beta n^\tau$ for some $\varepsilon = \varepsilon(\tau) > 0$. We will return to this question in Section 6.

For density-restricted sets, the previously mentioned small perturbation of the lattice section $[n]^2$ provides a matching upper bound:

> **Theorem 3.** (Valtr [36]) For every $n \in \mathbb{N}$ there exists an $n$-element point set $A \subset \mathbb{R}^2$ in general position, satisfying $\Delta(A) \leq \alpha n^{1/2}$, for some constant $\alpha > 0$, in which every subset in convex position has at most $\mathcal{O}(n^{1/3})$ points.

For a positive integer $n$, let $\gamma_d(n)$ denote the largest integer such that any set $A$ of $n$ points in general position in $\mathbb{R}^d$, satisfying $\Delta(A) \leq \alpha n^{1/d}$ for a fixed $\alpha > 0$, contains at least $\gamma_d(n)$ points in convex position. In these terms, Valtr’s result is that $\gamma_2(n) = \Theta(n^{1/3})$. The estimation of $\gamma_d(n)$ for $d \geq 3$ has remained an open problem. Here we continue this line of research in three dimensions and prove that $\gamma_3(n) = \Theta(n^{1/2})$.

Our Results.

> **Theorem 4.** For every $\alpha > 0$ there exists $\beta(\alpha) > 0$ such that any set of $n$ points in $\mathbb{R}^2$ in general position, satisfying $\Delta(A) \leq \alpha n^{1/3}$, contains a subset of at least $\beta n^{1/2}$ points in convex position. In particular, $\gamma_3(n) = \Omega(n^{1/2})$.

> **Theorem 5.** For every $\alpha > 0$ and $\tau \in \left[\frac{1}{3}, \frac{2}{3}\right)$, there exists $\beta = \beta(\alpha, \tau) > 0$ such that any set $A$ of $n$ points in general position in $\mathbb{R}^3$ satisfying $\Delta(A) \leq \alpha n^\tau$ contains a subset of $\beta n^{1-3\tau/2}$ points in convex position.
Theorem 6. For every $n \in \mathbb{N}$ there exists an $n$-element point set $A \subset \mathbb{R}^3$ in general position, satisfying $\Delta(A) \leq an^{1/3}$, for some constant $\alpha > 0$, in which every subset in convex position has at most $O(n^{1/2})$ points. In particular, $\gamma_3(n) = O(n^{1/2})$.

Theorem 7. Given any $n$-element point set $A \subset \mathbb{R}^3$ in general position, satisfying $\Delta(A) \leq an^{1/3}$ for a fixed $\alpha$, a $\Omega((n-1)/\alpha)$-factor approximation of the maximum-size convex subset of points can be computed by a randomized algorithm in $O(n \log n)$ expected time.

In 1960, Erdős and Szekeres [21] constructed for every positive integer $k$, a set of $n := 2^k$ points in general position in the plane, such that the size of the largest convex subset is $k - 1 = \log n - 1$. In 2017, Duque, Fabila-Monroy, and Hidalgo-Toscano [15] showed how to realize that construction on an integer grid of size $O(n^2 \log^3 n)$. In 1978, Erdős asked whether, given any positive integer $k$, every sufficiently large point set in general position in the plane contains $k$ points in convex position such that the respective polygon is empty of other points. In 1983, Horton [25] gave a negative answer by constructing arbitrarily large point sets with no empty convex 7-gon. Such sets are generally called Horton sets. In 2017, Barba, Duque, Fabila-Monroy, and Hidalgo-Toscano [11] showed how to realize a Horton set of size $n$ on an integer grid of size $O(n^{1/2} \log (n/2))$. On the other hand, they proved that any set of $n$ points with integer coordinates combinatorially equivalent to a Horton set contains a point with a coordinate at least $\Omega(n^{1/2} \log (n/2))$.

Given a point set in general position in $\mathbb{R}^d$, the problem of computing a maximum-size subset in convex position can be solved in polynomial time for $d = 2$ by the dynamic programming algorithm of Chvátal and Klee [16]; their algorithm runs in $O(n^3)$ time. In contrast, the general problem in $\mathbb{R}^d$ was shown to be NP-complete for every $d \geq 3$ by Gianspoulos, Knauer, and Werner [23].

Several problems concerning convex polygons (resp., polytopes) whose vertices lie in a Cartesian product of two (resp., $d$) sets of reals have been recently studied in [15]. See also [13, Sec. 8.2] for additional problems related to Horton sets and the Erdős–Szekeres theorem. A comprehensive survey on the Erdős–Szekeres problem is due to Morris and Soltan [31].

Recently, Bukh and Dong [14] independently proved that $\gamma_d = \Theta_d(n^{(d-1)/(d+1)})$ for all $d \in \mathbb{N}$, using more or less different techniques.

Definitions and notations. For a finite point set $S \subset \mathbb{R}^d$, let $g(S) = g_d(S)$ be the maximum size of a convex subset of $S$; when there is no confusion, the subscript $d$ may be omitted. Let $g(n)$ be defined as

$$g(n) = \min\{g(S) : |S| = n, S \text{ is in general position}\}.$$  

The interior and the boundary of a set $S \subset \mathbb{R}^d$ are denoted by $\text{Int}(S)$ and $\partial S$, respectively. A vector in $\vec{v} = (x_1, \ldots, x_d) \in \mathbb{Z}^d$ is primitive if $\gcd(x_1, \ldots, x_k) = 1$. Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. Unless specified otherwise, all logarithms are in base 2.

Here we use the convention that the approximation ratio of an algorithm is $< 1$ for a maximization problem and $> 1$ for a minimization problem (as in [39]). We frequently write $O_d$, when needed, to indicate that the hidden constant in the $O$ asymptotic notation depends only on $d$.

## 2 Preliminaries

A set of points in the plane $\{p_i = (x_i, y_i), i = 1, \ldots, n\}$ is in strong general position if it is in general position and no two $x$- or $y$-coordinates are the same. More generally, a set of points $P \subset \mathbb{R}^d \{p_i, i = 1, \ldots, n\}$ is said to be in strong general position if $P$ is in general
Finding Points in Convex Position in Density-Restricted Sets

position and no two values of the \( j \)th coordinate are the same, for \( j = 1, \ldots, d \). For any \( f \in \{0, 1, \ldots, d - 1\} \), denote by \( \pi_f \) the orthogonal projection from \( \mathbb{R}^d \) onto \( \mathbb{R}^f \) obtained by retaining the first \( f \) coordinates.

Let \( A, B \subset \mathbb{R}^d \) be finite sets such that \( A \cup B \) is in strong general position in \( \mathbb{R}^d \). We say that \( A \) lies deep below \( B \) (equivalently, \( B \) lies high above \( A \)) if the following holds: every point in \( A \) lies below all hyperplanes determined by \( d \) points in \( B \) and every point in \( B \) lies above all hyperplanes determined by \( d \) points in \( A \) (w.r.t. the \( d \)th coordinate) \cite[Ch. 3.2]{29}. Whenever needed, we extend this relation to apply to any coordinate other than the last one.

Let \( X \subset \mathbb{R}^d \) be a finite set in general position. A \( k \)-element set \( Y \subset X \) is called a \( k \)-hole in \( X \) if \( Y \) is in convex position and \( \text{conv}(Y) \cap X = Y \) \cite[Ch. 3.2]{29}. In the plane \((d = 2)\), \( Y \) is the vertex set of a convex polygon with no point of \( X \) inside. Horton \cite{25}, answering a question by Erdős, constructed arbitrary large finite sets without a 7-hole. On the other hand, every sufficiently large finite set in general position contains six points that determine a 6-hole (i.e., an empty convex hexagon); this was shown about 25 years later by Gerken \cite{22} and respectively Nicolás \cite{32}.

**Horton sets and Valtr grids in the plane.** For a sequence of points sorted by \( x \)-coordinates (with no duplicate \( x \)-coordinates) \( P = \{p_1, p_2, \ldots, p_n\} \), let \( P_0 = \{p_2, p_3, \ldots\} \) denote the subsequence of points with even indexes and \( P_1 = \{p_1, p_3, \ldots\} \) denote the subsequence of points with odd indexes. A finite set \( H \) is a Horton set if either (i) \(|H| = 1\), or (ii) \(|H| \geq 2\), both \( H_0 \) and \( H_1 \) are Horton sets and one of these two sets lies deep below the other one.

The above definition allows one to construct inductively Horton sets in the plane of any size; see \cite[Ch.3.2]{29}. Following \cite{17, 25, 36}, one can construct Horton sets as follows. For a nonnegative integer \( N \), let its binary representation be \( N = \sum_{i \geq 0} a_i 2^i \), where \( a_i \in \{0, 1\} \). For any positive \( 0 < \varepsilon < 1/2 \), denote by \( (N)_{\varepsilon} \) the real number

\[
(N)_{\varepsilon} = \sum_{i \geq 0} a_i \varepsilon^{i+1},
\]

and note that \( 0 < (N)_{\varepsilon} < 2\varepsilon \). Then the set \( S = \{p_i : i \in [n]\} \), where \( p_i = (i, (i)_{\varepsilon}) \), is a Horton set. Indeed, \( S_0 \) lies deep below \( S_1 \) if \( \varepsilon \) is sufficiently small, and further on, this property holds recursively for the smaller sets \( S_0 \) and \( S_1 \).

Interestingly enough, Valtr \cite{36} showed how to use multiple Horton sets to obtain planar sets that are sufficiently small perturbations of a lattice section \([n]^2\) and that preserve the key property of the Horton set, namely that of not containing any 7-hole. The resulting sets (described in Theorem \ref{thm:higher_dimensions}) are not Horton sets per se.

**Higher dimensions.** Recently, Conlon and Lim \cite{17} extended Valtr’s planar construction to higher dimensions such that the resulting sets do not contain large holes (i.e., convex polytopes with a large number of vertices that are empty of points from the set). Their result is summarized in Theorem \ref{thm:topological_results}.

\begin{theorem}
(Conlon and Lim \cite{17}) For any integers \( n \geq 1 \) and \( d \geq 2 \) and any \( \varepsilon > 0 \), there exists an integer \( C_d = d^\Omega(d^2) \) and a set of points \( \{P_{\varepsilon, \overline{x}} : \overline{x} \in [n]^d\} \subset \mathbb{R}^d \) that is \( C_d \)-hole-free and satisfies \( \text{dist}(P_{\varepsilon, \overline{x}}, \overline{x}) \leq \varepsilon \) for all \( \overline{x} \in [n]^d \).
\end{theorem}

Both the Valtr grid \cite{36} and the Conlon-Lim grid \cite{17} can be described in terms of negligible perturbations (or negligible functions). Given two finite sets \( A, B \subset \mathbb{R}^d \), a bijection \( f : A \to B \) is negligible if, whenever \( S \subset A \) and \( x \in A \) with \( x \in \text{Int}(\text{conv}(S)) \), then \( f(x) \in \text{Int}(\text{conv}(f(S))) \).
The planar construction can be described in one paragraph as follows (we mainly follow here the outline of Conlon and Lim [17]). Starting with the lattice section $S_0 = [n]^2$, each column of points is shifted vertically by a small positive amount (i.e., negligible perturbation), resulting in a set $S_1$. The shifting is done in such a way that each row of points is a Horton set. This also implies that any nonvertical line of lattice points yields a subset of $S_1$ that is a Horton set. Finally, each row of points is shifted horizontally by an even smaller positive amount to get $S_2 = S$ so that each column of points is a Horton set. Overall, every line of lattice points in $S_0$ corresponds to a Horton set in $S_2$. In other words, $S$ is the Minkowski sum of two Horton sets; one of them resembling $[n]$ along the $x$-axis and the other along the $y$-axis. It is easy to adjust the two perturbations if needed so that the final set $S$ is in strongly general position. It is worth noting that the final set $S$ is not a Horton set by itself.

**Number of vertices and faces.** The number of vertices of a convex lattice polytope can be bounded from above using a theorem by Andrews [6].

**Lemma 11.** For every finite set $S \subset \mathbb{Z}^d$ and for every integer $t$, $1 \leq t \leq \sqrt{A}$, the lattice polygon $conv(S)$ has $O_d \left( \frac{A}{t^2} \right)$ edges that contain more than $t$ points in $\mathbb{Z}^2$, where $A = Area(conv(S)) > 0$.

**Proof.** For brevity, let $P = conv(S)$. The boundary of $P$ can be decomposed into two $x$-monotone polygonal chains, a lower arc $P_1$ and an upper arc $P_2$. It suffices to show that $P_1$ and $P_2$ each have $O((A/t^2)^{1/3})$ edges that contain more than $t$ grid points.

Consider $P_1$ (the argument for $P_2$ is analogous and omitted). Suppose that $P_1$ has $L_1$ edges that contain more than $t$ grid points. Denote the vertex set of $P_1$ as $S_1 = \{v_0, v_1, \ldots, v_k\}$, labeled in left-to-right order, and assume that $v_0$ is the origin. Note that $S_1 \subset S$, and so $Area(conv(S_1)) \leq Area(conv(S)) = A$. For $i = 1, \ldots, k$, if the edge $v_{i-1}v_i$ contains $m_i$ grid points, then $v_i - v_{i-1} = (m_i - 1)\vec{e}_i$. For every $i = 1, \ldots, k$, let $m_i'$ be the largest multiple of $t$ such that $0 \leq m_i' \leq m_i - 1$. Note that if $m_i \leq t$ then $m_i' = 0$.

Let $S_1''$ be the vertex set of the polygonal arc obtained by concatenating the vectors $m_1'\vec{e}_1, \ldots, m_k'\vec{e}_k$. Since $m_i' \leq m_i$ for all $i$, then $Area(conv(S_1'')) \leq Area(conv(S_1)) \leq A$. By construction, we have $S_1'' \subset t\mathbb{Z}^2$, hence $S_1'' := \frac{1}{t} S_1'' \subset \mathbb{Z}^2$. Thus $conv(S_1'')$ is a lattice polygon of area at most $A/t^2$. If $conv(S_1'')$ has positive area, then Theorem [3] applies, otherwise $L_1 \leq 2$. In both cases, $conv(S_1'')$ has $L_1 = O((A/t^2)^{1/3})$ vertices, as required.

## Lower bound: Proof of Theorem 4

Let $A$ be a set of $n$ points in $\mathbb{R}^3$ in general position, satisfying $\Delta(A) \leq \alpha n^{1/3}$. We may assume that $A \subset B$, where $B$ is a ball of radius $R = \alpha n^{1/3}$ centered at $O$; see Fig. 2(left).

Consider any maximal packing $\mathcal{P}$ of spherical caps of height $h = \alpha n^{-1/6}$ and radius $r = (R^2 - (R - h)^2)^{1/2} = \Theta(\alpha n^{1/12})$. For instance, $\mathcal{P}$ may be constructed greedily from an empty packing by successively adding spherical caps that do not intersect previous caps.
Lemma 12. The spherical caps in $\mathcal{P}$ cover at least one quarter of the surface area of $B$.

Proof. Consider the set $\mathcal{P}'$ of enlarged caps with the same centers as in $\mathcal{P}$ but with double polar angle. Due to the maximality of $\mathcal{P}$, the caps in $\mathcal{P}'$ cover the surface $\partial B$ of $B$. Indeed, if there is an uncovered point $p \in \partial B$, then the cap of spherical radius $r$ centered at $p$ would not intersect any cap in $\mathcal{P}$, contradicting the maximality of $\mathcal{P}$. Doubling the polar angle of a spherical cap increases its area by a factor of at most 4. Consequently, the surface area of the union of spherical caps in $\mathcal{P}'$ is less than four times the surface area covered by $\mathcal{P}$. Since $\mathcal{P}'$ covers $\partial B$, then the caps in $\mathcal{P}$ cover at least one quarter of the surface area of $B$.

Since the area of $\partial B$ is $4\pi R^2$ and the area of each spherical cap in $\mathcal{P}$ is $2\pi Rh$, the cardinality of $\mathcal{P}$ is bounded from below (by Lemma 12) as follows:

$$|\mathcal{P}| \geq \frac{4\pi R^2}{4 \cdot 2\pi Rh} = \frac{R}{2h} = \frac{n^{1/2}}{2}.$$  \hspace{1cm} (2)

Label these caps as $\Psi_i, i = 1, \ldots, |\mathcal{P}|$, in an arbitrary fashion. The following is analogous with Lemma 2.2 in [36], however, due to the specifics of our construction, its proof is obvious.

Lemma 13. If we choose at most one point from each of the spherical caps $\Psi_i, i = 1, \ldots, |\mathcal{P}|$, then the chosen points are in convex position.

We wish to find a suitable placement of a ball $C$ congruent to $B$ (in a rotated position) so that a constant fraction of its spherical caps in $\mathcal{P}$ contains some points from $A$. This would prove Theorem 4. As in Valtr’s work, we proceed by probabilistic methods; the resulting algorithm is randomized.

Let $S$ be the (infinite) set of all spherical caps of radius $r$ and height $h$ corresponding to all balls $C$ congruent to $B$ contained in a ball $B''$ of radius $2R$ centered at $O$; note that any such ball $C$ is centered in a ball $B'$ of radius $2R$ centered at $O$. Refer to Fig. 1(right). A spherical cap in $S$ is determined by two parameters: (1) the direction, which is a unit vector in $S^2$ parallel to the vector from the center of the corresponding ball $C$ to the center of the cap; and (2) the center of the corresponding ball $C$, which is in $B'$. Consequently, $S$ is parameterized by the Cartesian product $S^2 \times B'$; and the Lebesgue measures on $B'$ and $S^2$, resp., induce a product measure $\mu$ on $S$. Since we have Vol($B'$) = $\frac{4\pi}{3}(2R)^3 = \frac{32\pi}{3}R^3$ and Area($S^2$) = $4\pi^2$, then $\mu(S)$ has a simple formula:

$$\mu(S) = \frac{128}{3} \pi^3 R^3.$$  \hspace{1cm} (3)

A key lemma (analogous to Lemma 2.3 in [36]) is the following.
Lemma 14. There exists a positive constant $c = c(\alpha)$ such that
\[ \mu(\{S \in \mathcal{S} : S \cap A \neq \emptyset\}) \geq c \mu(S). \]

Its proof relies on two lemmas (analogous to Lemmas 2.4 and 2.5 in [36]).

Lemma 15. Let $a \in A$ and for every $j \in \mathbb{N}$ let $A(j)$ be the set of points $a' \in A$ such that $j \leq \text{dist}(a, a') \leq j + 1$. Then $|A(j)| \leq c_1 j^2$, for some constant $c_1 > 0$.

Proof. Draw $O(j^2)$ planes incident to $a$ that divide the surface of the ball of radius $j + 1$ centered at $a$ into $O(j^2)$ surface patches of diameter at most $1/2$. The three concentric spheres of radii $j$, $j + 1/2$, and $j + 1$ together with the $O(j^2)$ planes partition the set $A(j)$ into $O(j^2)$ 3-dimensional cells of diameter less than 1. As such, each cell contains at most one point from $A$ and the lemma follows. ▶

For every point $a \in A$, and every integer $i = 0, 1, \ldots, n$, we introduce the notation
\[ S(a) = \{S \in \mathcal{S} : a \in S\}, \]
\[ S_i = \{S \in \mathcal{S} : |S \cap A| = i\}, \]
\[ m_i = \mu(S_i). \]

Lemma 16. The locus of directions $\vec{v} \in S^2$ corresponding to spherical caps in $S(a) \cap S(a')$ is contained in a spherical ring of polar angle $\theta$, where $\theta/2 = \arcsin \frac{h}{\text{dist}(a, a')}$; see Fig. 2.

Proof. Let $\Psi$ be a spherical cap in $S(a) \cap S(a')$ and let $p$ denote its center. We distinguish between two cases.

Case 1. $1 \leq \text{dist}(a, a') \leq \sqrt{r^2 + h^2}$. In an extremal position, $a' = p$ and $a$ is on the base of the cap. Refer to Fig. 3(left). The locus of directions $\vec{v} \in S^2$ corresponding to spherical caps in $S(a) \cap S(a')$ is a spherical ring of polar angle $\theta$, where $\theta/2 = \arcsin \frac{h}{\text{dist}(a, a')}$. ▶

Case 2. $\sqrt{r^2 + h^2} < \text{dist}(a, a') \leq 2r$. In an extremal position, $a$ is on the circle at the base of the cap, $a'$ is on the surface of the cap, and the great circle incident to $a$ and $a'$ passes
Finding Points in Convex Position in Density-Restricted Sets

through $p$; see Fig. 3(right). The locus of directions $\vec{v} \in S^2$ corresponding to spherical caps in $\mathcal{S}(a) \cap \mathcal{S}(a')$ is a spherical ring of polar angle $\theta'$, where $\theta'/2 = \arcsin \frac{y}{\text{dist}(a,a')} \leq \arcsin \frac{h}{\text{dist}(a,a')}$, and $y$ is the distance between $a'$ and the base of $\Psi$. Consequently, this locus is contained in a spherical ring of polar angle $\theta$, where $\theta/2 = \arcsin \frac{h}{\text{dist}(a,a')}$. In particular, observe that when $\text{dist}(a,a') = 2r$, then $y = \theta' = 0$.

\vspace{1em}

\textbf{Lemma 17.} There exist positive constants $c_2$, $c_3$, and $c_4$ such that:

(i) $\forall a \in A : c_2 \leq \mu(\mathcal{S}(a)) \leq c_3$, and

(ii) $\forall a \in A : \sum_{a' \in A} \mu(\mathcal{S}(a) \cap \mathcal{S}(a')) \leq c_4$.

\textbf{Proof.} (i) The centers of spherical caps $S \in \mathcal{S}(a)$ with a fixed direction $\vec{v} \in S^2$ form a congruent spherical cap of the same direction $\vec{v}$. Hence $\mu(\mathcal{S}(a)) = v_0 \cdot \text{Area}(S^2)$, where $v_0 = \frac{\pi}{2} (3h - h) = \pi \alpha(1 - n^{-1/2}/3)$ is the volume of a spherical cap. Since $1 - n^{-1/2}/3 \geq 5/6$ for $n \geq 4$, and $\text{Area}(S^2) = 4\pi^2$, we deduce that $c_2 \leq \mu(\mathcal{S}(a)) \leq c_3$, for constants $c_2 = \pi \alpha^3 \cdot \frac{5}{6} \cdot 4\pi^2$ and $c_3 = \pi \alpha^3 \cdot 4\pi^2$.

(ii) For $a, a' \in A$, let $W_{a,a'}$ be the locus of directions $\vec{v} \in S^2$ corresponding to spherical caps in $\mathcal{S}(a) \cap \mathcal{S}(a')$. Equivalently, for a fixed spherical cap $S \in \mathcal{S}$ with direction vector $\vec{a}' / \text{dist}(a,a')$, the set $W_{a,a'}$ is the locus of directions $\vec{v} \in S^2$ corresponding to the line segments of length $\text{dist}(a,a')$ contained in $S$; see Fig. 3. Note that $\text{dist}(a,a') \geq 1$, since the pairwise minimum distance is equal to 1; and if $\text{dist}(a,a') > 2r$, then $W_{a,a'} = \emptyset$, of measure 0. We therefore have $1 \leq \text{dist}(a,a') \leq 2r$. By Lemma 16, $W_{a,a'}$ is contained in a spherical ring of polar angle $\theta$, where $\theta/2 = \arcsin \frac{h}{\text{dist}(a,a')}$, and so its measure in $S^2$ is $O(\arcsin(h/\text{dist}(a,a'))) = O(h/\text{dist}(a,a'))$. Since the spherical caps from $\mathcal{S}(a) \cap \mathcal{S}(a')$ are in $\mathcal{S}(a)$, we have

$\mu(\mathcal{S}(a) \cap \mathcal{S}(a')) \leq c_5 \frac{h}{\text{dist}(a,a')}$, \hspace{1em} (4)

for some constant $c_5 > 0$. Recall that $A(j) = \{a' \in A : j \leq \text{dist}(a,a') \leq j + 1\}$. Hence

$\sum_{a' \in A} \mu(\mathcal{S}(a) \cap \mathcal{S}(a')) = \mu(\mathcal{S}(a)) + \sum_{j=1}^{2r} \sum_{a' \in A(j)} \mu(\mathcal{S}(a) \cap \mathcal{S}(a'))$

$\leq c_3 + \sum_{j=1}^{2r} c_5 j \sum_{a' \in A(j)} h j^{-1} \leq c_3 + \sum_{j=1}^{2r} c_5 j^2 c_5 h j^{-1}$

$\leq c_3 + c_1 c_5 h \sum_{j=1}^{2r} j \leq c_3 + 3c_1 c_5 r^2 h \leq c_3 + 6c_1 c_5 \alpha^3 = c_4,$

for some constant $c_4 > 0$. Here we used part (i), Inequality 4, Lemma 15 and $r^2 h < 2\alpha^3$. 

\textbf{Proof of Lemma 14} We wish to bound $\sum_{i=1}^{n} m_i$ from below. Consider the sums $\sum_{i=1}^{n} m_i$ and $\sum_{i=1}^{n} i^2 m_i$ for which we apply the Cauchy-Schwarz inequality in the following form:

$\left( \sum_{i=1}^{n} m_i \right) \left( \sum_{i=1}^{n} i^2 m_i \right) \geq \left( \sum_{i=1}^{n} i m_i \right)^2$. 
We can bound $\sum_{i=1}^{n} im_i$ from below and $\sum_{i=1}^{n} i^2 m_i$ from above as follows:

\[
\sum_{i=1}^{n} im_i = \sum_{a \in A} \mu(S(a)) \geq \sum_{a \in A} c_2 = c_2 n, \quad \text{[by Lemma 17(i)]}
\]

\[
\sum_{i=1}^{n} i^2 m_i = \sum_{a \in A} \sum_{a' \in A} \mu(S(a) \cap S(a')) \leq \sum_{a \in A} c_4 = c_4 n. \quad \text{[by Lemma 17(ii)]}
\]

Applying these estimates yields the desired lower bound

\[
\sum_{i=1}^{n} m_i \geq \left( \sum_{i=1}^{n} i^2 m_i \right) \geq \frac{(c_2 n)^2}{c_4 n} = \frac{c_2}{c_4} n.
\]

Consequently,

\[
\frac{\mu(\{S \in S : S \cap A \neq \emptyset\})}{\mu(S)} = \frac{\sum_{i=1}^{n} m_i}{\mu(S)} \geq \frac{(c_2/c_4)n}{128\pi^3 \alpha^3 n} = c > 0,
\]

for some positive constant $c$.

**Proof of Theorem 4**. We randomly place a ball $C$ congruent to $B$ inside $B'$. Specifically, recall that $B$, $B'$, and $B''$ are concentric balls of radii $R$, $2R$, and $3R$, respectively, where $A \subset B \subset B' \subset B''$. We construct a random congruence as follows: Let $g$ be rotation in $\text{SO}(3)$, the group of rotation in $\mathbb{R}^3$, chosen uniformly at random; and let $\tau$ be a translation that maps $O$ to point in $B'$ chosen uniformly at random. Put $\Phi := \tau \circ g$, and $C := \Phi(B)$. Since the center of $C$ is in $B'$, then $C \subset B''$. For any spherical cap $S_0 \in \mathcal{P}$ of direction $\vec{v} \in \mathbb{S}^2$, the direction $g(\vec{v})$ is distributed uniformly on $\mathbb{S}^2$. Consequently, the probability distribution of the indicator variable $I[\Phi(S_0) = S]$ over $S \in \mathcal{S}$ is a scalar multiple of $\mu$. By Lemma 17, $\Phi(S)$ contains some point in $A$ with a probability at least

\[
\frac{\mu(\{S \in S : S \cap A \neq \emptyset\})}{\mu(S)} \geq c,
\]

for some constant $c > 0$. By linearity of expectation and [3], the expected number of nonempty spherical caps in $\Phi(\mathcal{P})$ is at least $\frac{1}{2} \alpha n^{1/2}$. Setting $\beta = \frac{c}{2}$ completes the proof of Theorem 4.

**Proof of Theorem 5** (sketch). Set $R = \alpha n^w$, $h = \alpha n^{-w/2}$, $r = (R^2 - (R - h)^2)^{1/2} = \Theta(\alpha n^{w/4})$, and proceed as in the proof of Theorem 4. With this setting we have $r^2 h = \Theta(\alpha^3)$. The resulting lower bound is of the form

\[
\Omega \left( n^{1-3\tau} \frac{R^2}{r^2} \right) = \Omega \left( n^{1-3\tau/2} \right),
\]

as required.

**4 Upper bound: Proof of Theorem 6**

In this section, we describe and analyze a 3-dimensional construction, similar to the Valtr and the Conlon-Lim grids. It suffices to prove Theorem 6 for every $n$ of the form $n = 2^{3k}$, where $k \in \mathbb{N}$. Our point set is a suitable perturbation of the 3-dimensional Cartesian grid $G = \{0, 1, \ldots, 2^k - 1\}^3$, where each point lies within a ball of radius $O(\varepsilon) > 0$ centered at an integer point in $G$. Here $\varepsilon < 10^{-2}2^{-k} < 10^{-2}$ is a sufficiently small positive real that
Finding Points in Convex Position in Density-Restricted Sets

Lemma 18.

Lemma 20.

Lemma 24 below. We give an explicit formula for the perturbation \( \Phi : G \rightarrow \mathbb{R}^3 \) in terms of \( \varepsilon > 0 \). Let \( e_1, e_2, \) and \( e_3 \) denote the three standard basis vectors in \( \mathbb{R}^3 \). For a grid point \( p \in G \), let \( p = (p_1, p_2, p_3) \) denote the three coordinates of \( p \). For all \( i, j \in \{1, 2, 3\} \), let \( \varphi_{i,j} : G \rightarrow \mathbb{R}^3 \), \( \varphi_{i,j}(p) = (p_1) \varepsilon \cdot e_j \). For every \( p \in \mathbb{Z}^3 \), let \( u(p) \in \mathbb{R}^3 \) be a random unit vector. We can now define the perturbation \( \Phi : G \rightarrow \mathbb{R}^3 \) as

\[
\Phi(p) = p + \left( \sum_{i \neq j} \varepsilon^{(3i+j-5)} \cdot \varphi_{i,j}(p) \right) + \varepsilon^{7k} \cdot u(p)
\]

and let \( A = \Phi(G) \). The last term, \( \varepsilon^{7k} \cdot u(p) \), ensures that \( A \) is in strongly general position. It is convenient to think of \( \Phi \) as a concatenation of seven successive perturbations, corresponding to the terms in \( (5) \). Terms with different powers of \( \varepsilon \) ensure that each successive perturbation is negligible with respect to previous perturbations if \( \varepsilon > 0 \) is sufficiently small. We introduce notation for the result of the first perturbation: let \( \Phi_1 : G \rightarrow \mathbb{R}^3 \), \( \Phi_1(p) = p + \varphi_{1,2}(p) = (p_1, p_2 + (p_1) \varepsilon, p_3) \).

We next analyze this construction and show that it contains no large convex subsets as quantified in Theorem 19. Let \( B \subset A \) be a set in convex position, and let \( C = \Phi^{-1}(B) \), i.e., \( C \subset G \) with \( B = \Phi(C) \). If \( \varepsilon > 0 \) is sufficiently small, then for every vertex \( p \) of \( \text{conv}(C) \), the point \( \Phi(p) \) is a vertex of \( \text{conv}(B) \). However, for every vertex \( q \) of \( \text{conv}(B) \), the point \( \Phi^{-1}(q) \) is either a vertex of \( \text{conv}(C) \) or lies in the interior of an edge or a face of \( \text{conv}(C) \). We have

\[
\text{conv}(C) \subset \text{conv}(G), \quad \text{and} \quad \text{Vol}(\text{conv}(C)) \leq \text{Vol}(\text{conv}(G)) = n.
\]

By Theorem 19. \( \text{conv}(C) \) has \( O(n^{3-1}/(3+1)) = O(n^{1/2}) \) vertices. By Euler’s polyhedral formula, \( \text{conv}(C) \) has \( O(n^{1/2}) \) edges and facets. We will show that if an edge of \( \text{conv}(C) \) contains \( m \) lattice points in \( G \), then only \( O(\log^2 m) \) of these points are in \( C \) (Lemma 20); and if a face \( F \) of \( \text{conv}(C) \) contains \( m \) lattice points in \( G \), then \( O(m^{1/3}) \) of these points are in \( C \) (Lemma 24).

We use three lemmas (Lemmas 18, 20, and 24). Lemma 18 describes the subadditivity of the function \( g(\cdot) \) (defined at the end of Section 11). Its easy proof is left to the reader.

**Lemma 18.** Let \( S = \bigcup_{i=1}^{k} S_i \) be an arbitrary partition of \( S \). Then \( g(S) \leq \sum_{i=1}^{k} g(S_i) \).

**Lemma 20** below states that in a perturbation of a set of collinear points in the grid \( G \), there are only \( O(\log^2 n) \) points in convex position. A possible proof for Lemma 20 would establish that the perturbation of collinear points in \( G \) is a Horton set in \( \mathbb{R}^3 \), and then apply the following result due to Károlyi and Valtr 26.

**Lemma 19.** (Károlyi and Valtr 26) Let \( S \subset \mathbb{R}^d \) be a Horton set of size \( n \). Then \( g_d(S) = O(\log^{d-1} n) \), where the constant hidden in the \( O \) notation depends only on \( d \).

Instead, we give a direct proof for Lemma 20 and then generalize it to handle a perturbation of a set of coplanar points in Lemma 24 below.

**Lemma 20.** Let \( S \subset G \) be a set of collinear points. Then \( \text{conv}(\Phi(S)) \) has \( O(\log^2 m) \) vertices, where \( m = |G \cap \text{conv}(S)| \).

**Proof.** We may assume that \( |S| \geq 2 \). Let \( L \) be the line spanned by \( S \), and let \( m = |G \cap \text{conv}(S)| \). Consider the first coordinate axis (\( x \)-, \( y \)-, or \( z \)-axis) that is not orthogonal
to $L$. We give a detailed proof for the case that the $x$-axis is not orthogonal to $L$. The other two cases are analogous (and are omitted): If $L$ is orthogonal to the $x$-axis, then the first two iterations of the perturbation (which depend on the first coordinate) translate all points $S$ uniformly, hence they have no impact on the convex hull of $\Phi(S)$. Similarly, if $L$ is orthogonal to both $x$- and $y$-axes, then we can ignore the components of the perturbation that depend on the first and second coordinates (and use the third coordinate instead).

Assume that $L$ is not orthogonal to the $x$-axis. Label the points in $S$ as $S = \{s_1, \ldots, s_t\}$ sorted by increasing $x$-coordinates. Let the binary representation of the $x$-coordinate of $s \in S$ be $x(s) = \sum_{j=1}^{t-1} x_j(s) 2^j$. We recursively define the sets

$$S_1 \supset S_2 \supset \ldots \supset S_b,$$

for some suitable $b \leq O(\log m)$ as follows; see Fig. 4. Let $S_1 = S$. Given a set $S_a \subset S$, for $a \geq 1$, with $|S_a| \geq 2$, we define $S_{a+1}$ as follows. Let $j(a) \geq 0$ be the smallest integer such that $\{x_{j(a)}(s) : s \in S_a\} = \{0, 1\}$; and let $S_{a+1} = \{s \in S_a : x_{j(a)}(s) = 1\}$. Importantly, this implies that the translation vector $\varphi_{1,2}(s)$ has a term $e^{j(a)}e_2$ for all $s \in S_{a+1}$; but this term is missing for all $s \in S_a \setminus S_{a+1}$. Note that for all $s \in S_a$, the last (i.e., the least significant) $j(a)$ bits in the binary expansion of $x(s)$ are the same. Consequently, $j(a+1) > j(a)$ for all $a \geq 1$; hence $a \leq O(\log m)$, and the recursion terminates with $b \leq O(\log m)$, as claimed. Note also that $|S_0| = 1$ by definition.

---

**Figure 4** Perturbation for a set of collinear grid points. Left: Solid dots indicate $S = \{s_1, \ldots, s_9\}$; all other grid points are marked by empty dots. The binary expansion of the $x$-coordinates of all grid points.

We show that $\text{conv}(\Phi(S))$ has $O(\log^2 m)$ vertices in two steps: First we consider the convex hull of a projection of $\Phi(S)$ to a plane, and then extend the argument to 3-space.

**Convex hull of the orthogonal projection to the $xy$-plane.** We consider the impact of the first step of the perturbation, $\Phi_1(S)$. Let $\Phi^{xy}(S)$ and $\Phi^{xy}_1(S)$, resp., be the orthogonal projection of $\Phi(S)$ and $\Phi_1(S)$ to the $xy$-plane.

▷ **Claim 21.** The convex polygon $\text{conv}(\Phi^{xy}(S))$ has $O(\log m)$ vertices.

If a point $\Phi^{xy}(p)$ is a vertex of $\text{conv}(\Phi^{xy}(S))$, then $\Phi^{xy}_1(p)$ is on the boundary of $\text{conv}(\Phi^{xy}_1(S))$, since subsequent perturbations are negligible. Note that both $\Phi^{xy}_1(s_1)$ and $\Phi^{xy}_1(s_t)$ are vertices of $\text{conv}(\Phi^{xy}_1(S))$. These two points decompose the boundary of $\text{conv}(\Phi^{xy}_1(S))$ into two Jordan arcs: an upper arc and a lower arc in the $xy$-plane. It suffices to show that each arc contains $O(\log n)$ points of $\Phi^{xy}_1(S)$. Without loss of generality, consider the upper arc.
Finding Points in Convex Position in Density-Restricted Sets

\(\triangleright\) Claim 22. Let \(s_\ell \in S\). If \(\Phi_1^{\tau_1}(s_\ell)\) lies in the upper arc of \(\text{conv}(\Phi_1^{\tau_1}(S))\), then \(s_\ell\) is the first or last point in \(S_a\) for some \(a \in \{1, \ldots, b\}\).

We prove the contrapositive of Claim 22. Suppose that \(s_\ell\) is neither the first nor the last point in \(S_a\) for any \(a \in \{1, \ldots, b\}\). Due to \(O\) and \(|S_b| = 1\), there exists an \(a \in \{1, \ldots, b-1\}\) such that \(s_\ell \in S_a\) but \(s_\ell \not\in S_{a+1}\). Denote the first and last points in \(S_a\) and \(S_{a+1}\), resp., by \(s_{\sigma(1)}, s_{\sigma(2)}\) and \(s_{\tau(1)}, s_{\tau(2)}\). Then \(\text{conv}(S_a)\) and \(\text{conv}(S_{a+1})\), resp., are the line segments \(s_{\sigma(1)}s_{\sigma(2)}\) and \(s_{\tau(1)}s_{\tau(2)}\); see Fig. 3. Note that \(\sigma(1) \leq \tau(1) \leq \tau(2) \leq \sigma(2)\) and \(\sigma(1) < \sigma(2)\). Then we have \(\sigma(1) < \ell < \tau(2)\) or \(\tau(1) < \ell < \sigma(2)\). Assume w.l.o.g. that \(\sigma(1) < \ell < \tau(2)\).

We show that the point \(\Phi_1^{\tau_1}(s_\ell)\) lies below the line segment \(\Phi_1^{\tau_1}(s_{\sigma(1)})\Phi_1^{\tau_1}(s_{\tau(2)})\), and so it cannot be on the upper arc of \(\text{conv}(\Phi_1^{\tau_1}(S))\). Since the points \(s_{\sigma(1)}, s_\ell, s_{\tau(2)} \in \mathcal{L}\) are collinear, and \(\Phi_1(p) = p + \varphi_{1,2}(p)\), it is enough to compare the perturbations incurred by \(\varphi_{1,2}\), which depend only on the \(x\)-coordinates. By the choice of \(a\), we have \(s_{\sigma(1)}, s_\ell, s_{\tau(2)} \in S_a\). This means that in the binary expansion of their \(x\)-coordinates, the last \(j(a)\) bits are the same. Regarding the bit \(x_{j(a)}(\ell)\), we know that \(x_{j(a)}(s_\ell) = 0\) and \(x_{j(a)}(s_{\tau(2)}) = 1\). However, \(s_{\sigma(1)}\) may or may not be in \(S_{a+1}\), and so we do not know \(x_{j(a)}(s_{\sigma(1)})\). Consequently, by using the inequality \((N_x) < 2\varepsilon\) in (9), we have

\[
\|\varphi_{1,2}(s_{\sigma(1)})\| \geq \left( \sum_{j=0}^{(a)-1} x_j(s_\ell) \cdot \varepsilon^{j+1} \right) + X = X
\]

\(1\)

\[
\|\varphi_{1,2}(s_\ell)\| < \left( \sum_{j=0}^{(a)-1} x_j(s_\ell) \cdot \varepsilon^{j+1} \right) + 2\varepsilon^{(a)+2} = X + 2\varepsilon^{(a)+2}
\]

\(2\)

\[
\|\varphi_{1,2}(s_{\tau(2)})\| \geq \left( \sum_{j=0}^{(a)-1} x_j(s_\ell) \cdot \varepsilon^{j+1} \right) + \varepsilon^{(a)+1} = X + \varepsilon^{(a)+1}.
\]

\(3\)

Since \(s_{\sigma(1)}, s_\ell, s_{\tau(2)}\) are collinear grid points, we can express \(s_\ell\) as a convex combination:

\[
s_\ell = \alpha \cdot s_{\sigma(1)} + (1 - \alpha) \cdot s_{\tau(2)}
\]

\(10\)

for some coefficient \(\frac{1}{n - 1} \leq \alpha \leq 1 - \frac{1}{n - 1}\). Denote by \(q\) the point in the line segment \(\Phi_1^{\tau_1}(s_{\sigma(1)})\Phi_1^{\tau_1}(s_{\tau(2)})\) above \(s_\ell\). Then, substituting (7)-(9) into (10), we obtain

\[
\|q - s_\ell\| = \alpha \cdot \|\varphi_{1,2}(s_{\sigma(1)})\| + (1 - \alpha) \cdot \|\varphi_{1,2}(s_{\tau(2)})\|
\]

\[
\geq \left( \sum_{j=0}^{(a)-1} x_j(s_\ell) \cdot \varepsilon^{j+1} \right) + (1 - \alpha) \cdot \varepsilon^{(a)+1}
\]

\[
\geq X + \frac{\varepsilon^{(a)+1}}{n^{1/4}} > X + 2\varepsilon^{(a)+2} > \|\varphi_{1,2}(s_\ell)\|,
\]

if \(\varepsilon < \frac{1}{2n^{1/4}}\). This confirms that \(\Phi_1^{\tau_1}(s_\ell)\) lies below the line segment \(\Phi_1^{\tau_1}(s_{\sigma(1)})\Phi_1^{\tau_1}(s_{\tau(2)})\); and completes the proof of Claim 22.

Since \(b = \mathcal{O}(\log m)\), there are at most \(2b = \mathcal{O}(\log m)\) points that are first or last in \(S_1, \ldots, S_b\). Consequently, Claim 22 implies that the upper arc of \(\text{conv}(\Phi_1^{\tau_1}(S))\) has \(\mathcal{O}(\log m)\) vertices. Similarly, one can show that the lower arc of \(\text{conv}(\Phi_1^{\tau_1}(S))\) has \(\mathcal{O}(\log m)\) vertices (the key difference for handling the lower arc is that in the recursive definition of the sets \(S_1 \supset S_2 \supset \ldots \supset S_b\), we would put \(S_{a+1} = \{s \in S_a : x_{j(a)}(s) = 0\}\)). Overall, \(\text{conv}(\Phi_1^{\tau_1}(S))\) has \(\mathcal{O}(\log m)\) vertices; completing the proof of Claim 21.
Convex hull in 3-space. Consider $\text{conv}(\Phi(S))$. The orthogonal projection of $\text{conv}(\Phi(S))$ to the $xy$-plane is $\text{conv}(\Phi^x(S))$. The boundary of $\text{conv}(\Phi^x(S))$ is the projection of a closed curve $\gamma$ on the boundary of $\text{conv}(\Phi(S))$. We have shown that the projection of $\gamma$, hence $\gamma$ itself, has $O(\log n)$ vertices. The Jordan curve $\gamma$ partitions the boundary of $\text{conv}(\Phi(S))$ into two components: An upper surface and a lower surface. It suffices to show that each has $O(\log^2 m)$ vertices.

$\triangleright$ Claim 23. Let $s_t \in S$. If $\Phi(s_t)$ is a vertex of the upper surface of $\text{conv}(\Phi(S))$, then $\Phi^x(s_t)$ is a vertex of $\text{conv}(\Phi^x(S_a))$ for some $a \in \{1, 2, \ldots, b\}$.

We prove the contrapositive of Claim 23. Suppose that $s_t \in S$ but $\Phi(s_t)$ is not a vertex of $\text{conv}(\Phi(S_a))$ for any $a \in \{1, \ldots, b\}$. Note that (6) implies

$$\text{conv}(\Phi^x(S)) = \text{conv}(\Phi^x(S_1)) \supset \text{conv}(\Phi^x(S_2)) \supset \ldots \supset \text{conv}(\Phi^x(S_b)).$$

There exists some $a \in \{1, \ldots, b\}$ such that $s_t \in S_a$ but $s_t \notin S_{a+1}$. First triangulate the convex polygon $\text{conv}(\Phi^x(S_{a+1}))$, and then triangulate the nonconvex polygon $\text{conv}(\Phi^x(S_a)) \setminus \text{conv}(\Phi^x(S_{a+1}))$ such that each triangle is spanned by some vertices of $\text{conv}(\Phi^x(S_a))$ and some vertices of $\text{conv}(\Phi^x(S_{a+1}))$. The point $\Phi^x(s_t)$ lies in some triangle $\Phi^x(p)\Phi^x(q)\Phi^x(r)$ in which at least one corner is a vertex of $\text{conv}(\Phi^x(S_{a+1}))$. Considering the perturbation $\Phi_1(s_t) = s + \varphi_{1,2}(s)$, the triangle $\Phi(p)\Phi(q)\Phi(r)$ lies above $\Phi(s_t)$ if $\varepsilon > 0$ is sufficiently small, analogously to (7) and (10), except that $s_t$ is now the convex combination of three grid points $p$, $q$, and $r$. Consequently, $\Phi(s_t)$ cannot be on the upper surface. This proves Claim 23.

By Claim 22 $\text{conv}(\Phi^x(S_a))$ has $O(\log m)$ vertices for all $a \in \{1, 2, \ldots, b\}$. Summation over all $a = 1, \ldots, b$ shows that there are $\sum_{a=1}^{b} O(\log m) = O(\log^2 m)$ vertices on the upper surface of $\text{conv}(\Phi(S))$. Analogously, the lower surface of $\text{conv}(\Phi(S))$ also has $O(\log^2 m)$ vertices. Overall, $\text{conv}(\Phi(S))$ has $O(\log^2 m)$ vertices. This completes the proof of Lemma 20.

$\triangleright$ Lemma 24. Let $S \subset G$ be a set of coplanar points. Then $\text{conv}(\Phi(S))$ has $O(m^{1/3})$ vertices, where $m = |G \cap \text{conv}(S)|$.

Proof. We proceed similarly to the proof of Lemma 20. Consider the first coordinate axis (the $x$- or the $y$-axis) that is not orthogonal to $F$, and another coordinate axis that is not parallel to $F$. Assume that $F$ is not orthogonal to the $x$-axis (or else we would use the $y$-axis); and it is not parallel to the $y$-axis (or else we would use the $z$-axis). Label the points in $S \cap F$ as $\{s_1, \ldots, s_t\}$ sorted by increasing $x$-coordinates (ties are broken arbitrarily). Let the
Finding Points in Convex Position in Density-Restricted Sets

We prove the contrapositive of Claim 25. Suppose that for some suitable \( b \leq O(\log m) \) the same way as in the proof of Lemma 20. Let \( S_1 = S \). Given a set \( S_a \subset S \), for \( a \geq 1 \), where the points in \( S_a \) do not all have the same \( x \)-coordinates, we define \( S_{a+1} \) as follows. Let \( j(a) \geq 1 \) be the smallest integer such that \( \{ x_{j(a)}(s) : s \in S_a \} = \{0, 1\} \); and let \( S_{a+1} = \{ s \in S_a : x_{j(a)}(s) = 1 \} \).

Denote by \( M = \mathbb{Z}^3 \cap F \) the set of grid points in \( F \). Based on the recursion above, we define a sequence of nested sets

\[
M_1 \supset M_2 \supset \ldots \supset M_b
\]

as follows. Let \( M_1 = M \), and \( M_{a+1} = \{ s \in M_a : x_{j(a)}(s) = 1 \} \) for \( a \in \{1, \ldots, b-1\} \). Then \( |M_a| = O(m/2^a) \) for all \( a \in \{1, \ldots, b\} \).

![Figure 6](image-url) The nested sequence \( F \supset \text{conv}(S_1) \supset \ldots \supset \text{conv}(S_4) \). The point \( s_\ell \) lies in \( \text{conv}(S_2) \setminus \text{conv}(S_1) \).

We interpret the above-below relationship with respect to the \( y \)-axis. Assume w.l.o.g. that \( \text{conv}(S) \) lies below the face \( F \).

\( \triangleright \) Claim 25. Let \( s_\ell \in S \cap F \). If \( \Phi(s_\ell) \) is a vertex of \( \text{conv}(\Phi(S \cap F)) \), then \( s_\ell \) lies on the boundary of \( \text{conv}(S_a) \) for some \( a \in \{1, \ldots, b\} \).

We prove the contrapositive of Claim 25. Suppose that \( s_\ell \) is not on the boundary of \( \text{conv}(S_a) \) for any \( a \in \{1, \ldots, b\} \). Due to (4), there exists some \( a \in \{1, \ldots, b\} \) such that \( s_\ell \in S_a \) but \( s_\ell \notin S_{a+1} \); see Fig. 6. First triangulate the convex polygon \( \text{conv}(S_{a+1}) \); and then triangulate the nonconvex polygon \( \text{conv}(S_a) \setminus \text{conv}(S_{a+1}) \) such that each triangle is spanned by some vertices of \( \text{conv}(S_a) \) and some vertices of \( \text{conv}(S_{a+1}) \). Then point \( s_\ell \) lies in some triangle \( pqr \), and at least one corner is a vertex of \( \text{conv}(S_{a+1}) \). Considering the perturbation \( \varphi_{1,2}, \) the triangle \( \Phi(p)\Phi(q)\Phi(r) \) lies above \( \Phi(s_\ell) \) if \( \varepsilon > 0 \) is sufficiently small. Consequently, \( \Phi(s_\ell) \) cannot be a vertex of \( \text{conv}(\Phi(S \cap F)) \). This completes the proof of Claim 25.

For every \( a \in \{1, \ldots, b\} \), the convex hull \( \text{conv}(S_a \cap M_a) \) has \( O(m_a^{1/3}) \) vertices and edges, where \( m_a = |M_a| \leq O(m/2^a) \), by Theorem 9. Further, by Lemma 11, it has \( O((m_a/t^2)^{1/3}) \) edges that contain more than \( t \) points in \( M_a \) for any \( t \geq 1 \). By Lemma 20 if an edge \( e \)
of \( \text{conv}(S_a \cap M_a) \) contains more than \( t \) grid points, then \( \text{conv}(\Phi(S \cap e)) \) contains \( \mathcal{O}(\log^2 t) \) vertices of \( B \). Let \( E_j \) be the set of edges \( e \) with \( |e \cap \mathbb{Z}^3| \in (2^{j-1}, 2^j] \); and \( E = \bigcup_{j \in \mathbb{N}} E_j \).

Summation over all edges of \( S_a \cap F \) yields

\[
\sum_{e \in E} \mathcal{O}(\log^2 |e \cap \mathbb{Z}^3|) = \sum_{j \in \mathbb{N}} |E_j| \cdot \mathcal{O}(\log^2 2^j)
\]

\[
= \sum_{j \in \mathbb{N}} \mathcal{O} \left( m_a / 2^{3j} \right)^{1/3} \cdot j^2
\]

\[
= \mathcal{O}(m_a^{1/3}) \sum_{j \in \mathbb{N}} j^2 / (2^{2j/3}) = \mathcal{O}(m_a^{1/3}).
\]

Finally, summation for all \( a \in \{1, \ldots, b\} \) yields

\[
\sum_{a=1}^b \mathcal{O}(m_a^{1/3}) = \sum_{a=1}^b \mathcal{O} \left( m_a / 2^{3a} \right)^{1/3} = \mathcal{O} \left( m^{1/3} \sum_{a \geq 1} 2^{-a/3} \right) = \mathcal{O}(m^{1/3}),
\]

as claimed.

**Lemma 26.** Let \( S \subset G \) and \( P = \text{conv}(S) \). Suppose that \( P \) has \( f \) faces (of any dimension), which contain \( m_1, \ldots, m_f \) lattice points in their interior. Then \( \sum_{i=1}^f m_i^{1/3} = \mathcal{O}(n^{1/2}) \).

**Proof.** The surface area of the lattice polytope \( P = \text{conv}(S) \) is bounded above by that of the cube \( \text{conv}(G) \), which is \( 6n^{2/3} \). Let \( F \) be the set of all facets of \( P \). Then summation of the area over all facets yields an upper bound \( \sum_{F \in F} \mathcal{O}(\text{Area}(F)) = \mathcal{O}(n^{2/3}) \).

For every face \( F \in F \), let \( n_f = (a, b, c) \) be the integer normal vector of the plane spanned by \( F \), where \( a, b, c \in \mathbb{Z} \), and \( \gcd(a, b, c) = 1 \). It is known that \( F \) contains \( \mathcal{O}(\text{Area}(F) / \|n_f\|_2) \) lattice points [3].

For every integer \( j \in \mathbb{N} \), let \( F_j \) be the set of facets \( F \in F \) such that \( \|n_f\|_2 \in (2^{j-1}, 2^j] \). In particular, a face \( F \in F_j \) contains \( \mathcal{O}(\text{Area}(F) / 2^j) \) lattice points. For an integer \( N \in \mathbb{N} \), let \( r_3(N) \) denote the number of representations of \( N \) as a sum of squares of three integers (where signs and the order of terms matters). It is known [24, Theorem 340] that

\[
\sum_{N=1}^M r_3(N) = \frac{4}{3} \pi M^{3/2} + \mathcal{O}(M).
\]

In particular, for \( M = 2^{2j} \), we have \( \sum_{N=1}^M r_3(N) = \Theta(2^{3j}) \), and so \( |F_j| = \Theta(2^{3j}) \).

The total area of all facets in \( F_j \) is \( \mathcal{O}(n^{2/3}) \) for each \( j \in \mathbb{N} \). Thus the facets in \( F_j \) contain at most \( \sum_{F \in F_j} \mathcal{O}(\text{Area}(F) / 2^j) = \mathcal{O}(n^{2/3} / 2^j) \) lattice points. Jensen’s inequality gives

\[
\sum_{F \in F_j} m_i^{1/3} \log m_i \leq |F_j| \cdot \left( \frac{n_i^{3/2} / 2^j}{|F_j|} \right)^{1/3}
\]

\[
= |F_j|^{2/3} n_i^{2/9} 2^{-i/3} = \mathcal{O}(n^{2/9} 2^{(2/3)j - i/3}) = \mathcal{O}(n^{2/9} 2^{5j/3}). \tag{14}
\]

Since the number of facets is \( \mathcal{O}(n^{1/2}) \), then \( |F_j| = \mathcal{O}(n^{1/2}) \) for all \( j \in \mathbb{N} \); and (14) becomes:

\[
\sum_{F \in F_j} m_i^{1/3} = |F_j| \cdot \left( \frac{n_i^{3/2} / 2^j}{|F_j|} \right)^{1/3}
\]

\[
\leq |F_j|^{2/3} n_i^{2/9} 2^{-i/3} = (\mathcal{O}(n^{1/2}))^{2/3} n_i^{2/9} 2^{-i/3} = \mathcal{O}(n^{5/9} 2^{-i/3}). \tag{15}
\]
The bounds in \([14]\) and \([15]\) are equal, up to constant factors, when \(2^j = n^{1/6}\), or equivalently, \(j = \frac{1}{6} \log n\). Summation over all \(j \in \mathbb{N}\) yields

\[
\sum_{i=1}^{f} m_i^{1/3} = \sum_{j \in \mathbb{N}} \sum_{P \in \mathcal{F}_j} m_i^{1/3} = \sum_{j=1}^{\log n^{1/6}} \sum_{P \in \mathcal{F}_j} m_i^{1/3} + \sum_{j > \log n^{1/6}} \sum_{P \in \mathcal{F}_j} m_i^{1/3} \leq \sum_{j=1}^{\log n^{1/6}} O(n^{2/9}2^{5j/3}) + \sum_{j > \log n^{1/6}} O(n^{5/9}/2^{j/3}) \leq O(n^{2/9+5/18}) + O(n^{5/9−1/18}) = O(n^{1/2}).
\]

We can now complete the proof of Theorem 6. Recall that \(G = \{0, \ldots, 2^k − 1\}^3\) is a section of the integer lattice \(\mathbb{Z}^3\) with \(n = 2^{3k}\) points. We have \(A = \Phi(G)\); and \(C \subset G\) such that \(B = \Phi(C)\) is in convex position. As noted above, every point in \(C\) lies on the boundary of the lattice polytope \(P = \text{conv}(C)\).

If a facet \(F\) of \(P\) contains \(m\) lattice points, then \(F\) contains \(O(m^{1/3})\) points of \(C\) by Lemma 24. Summation over all facets yields \(O(n^{1/2})\) by Lemma 26. Consequently, we have \(|B| = |C| = O(n^{1/2})\), as required.

5 Approximation algorithm: Proof of Theorem 7

In this section we analyze the randomized algorithm described in Section 3 and show that its approximation factor is \(\Omega(n^{−1/6})\). We also make some small twists that allow for an efficient implementation. We are given an \(n\)-element set \(A \subset \mathbb{R}^3\) with \(\Delta(A) \leq O(n^{1/3})\). As in the proof of Theorem 4, we may assume that \(A \subset B\), where \(B\) is a ball of radius \(R \leq O(n^{1/3})\) centered at \(O\); e.g., a smallest enclosing ball of \(A\). First, deterministically construct a (single) spherical packing \(\mathcal{P}\) on a sphere \(C\) congruent to \(B\) of prescribed minimum size \(\Theta(n^{1/2})\); \(\mathcal{P}\) does not have to be maximal. For example, slice the sphere by latitudes, and choose equally spaced spherical caps between consecutive latitudes. Second, the randomized phase proceeds as follows. Guess a center for a ball \(C\) and apply a random rotation in \(SO(3)\) (applying the same rotation to all caps in \(\mathcal{P}\)), and then test whether a constant fraction of the caps are nonempty; if not, repeat the process. The expected number of repetitions is bounded by a constant (the sum of a geometric series).

Approximation ratio. As in the proof of Theorem 4 we may assume that \(A \subset B\), where \(B\) is a ball of radius \(R = \alpha n^{1/3}\) centered at \(O\). Let \(A_{\text{OPT}} \subset A\) be a maximum-size subset in convex position, i.e., \(\text{OPT} = |A_{\text{OPT}}|\). Since \(A_{\text{OPT}} \subset B\), we have

\[
\text{Area}(\partial(\text{conv}(A_{\text{OPT}}))) \leq \text{Area}(\partial B) = 4\pi R^2 = O(n^{2/3}).
\]

Since \(A\) is density-restricted, then \(\text{OPT} = O(n^{2/3})\). (The above argument is the 3-dimensional variant of \([14]\) Theorem 3.2.) On the other hand, Theorem 4 yields \(\text{ALG} = \Omega(n^{1/2})\). Consequently, the approximation ratio is

\[
\frac{\text{ALG}}{\text{OPT}} = \Omega \left( \frac{n^{1/2}}{n^{2/3}} \right) = \Omega(n^{-1/6}),
\]
as claimed.
Running time analysis. A smallest enclosing ball of \( n \) points in \( \mathbb{R}^3 \) can be computed in \( \mathcal{O}(n) \) expected time \( \text{[35]} \), and a random rotation and translation in \( \mathcal{O}(1) \) time \( \text{[31]} \). The packing \( \mathcal{P} \) can be constructed in \( \mathcal{O}(n^{1/2}) \) time (i.e., in time linear in \( |\mathcal{P}| \)). We next consider the time complexity of \( \Theta(\Omega(1/2)) \) range-emptiness queries for spherical cap ranges. After \( \mathcal{O}(n) \) preprocessing, points outside the chosen ball are excluded from further consideration. Then each range-emptiness query for a spherical cap is equivalent to (and answered by) a halfspace emptiness query determined by the plane containing the base of the cap. (Here we take advantage of the fact that all spherical cap ranges pertain to the same ball.) After \( \mathcal{O}(n \log n) \) expected preprocessing time, such queries in 3-space can be answered in \( \mathcal{O}(\log n) \) time per query using the algorithm by Afshani and Chan \( \text{[1]} \); but also by other algorithms, see \( \text{[2, 3]} \). Consequently, the range-emptiness queries can be answered in \( \mathcal{O}(n^{1/2} \log n) \) time. Adding up the running times of the steps we have \( \mathcal{O}(n^{1/2}) + \mathcal{O}(n) + \mathcal{O}(n \log n) + \mathcal{O}(n^{1/2} \log n) = \mathcal{O}(n \log n) \). Overall, the randomized algorithm runs in \( \mathcal{O}(n \log n) \) expected time.

Generalization to higher dimensions. The machinery developed here generalizes to \( \mathbb{R}^d \).

Theorem 27. Given any \( n \)-element point set in \( \mathbb{R}^d \) in general position, satisfying \( \Delta(A) \leq an^{(d-1)/(d+1)} \) for a fixed \( a \), a \( \Omega \left( n^{-\frac{(d-1)}{d+1}} \right) \)-factor approximation of the maximum size convex subset of points can be computed by a randomized algorithm in \( \mathcal{O}(n \log n) \) expected time.

The proof of Theorem \( \text{[27]} \) is analogous to the proof of Theorem \( \text{[7]} \). The approximation ratio is

\[
\frac{\text{ALG}}{\text{OPT}} = \Omega \left( \frac{n^{(d-1)/(d+1)}}{\text{n}^{(1-\frac{1}{d+1})}} \right) = \Omega \left( n^{-\frac{(d-1)}{d+1}} \right).
\]

As the exponent tends to zero when \( d \to \infty \), the approximation ratio improves with the dimension (if \( n \) is sufficiently large). As such, our algorithm enjoys the ‘blessing of dimensionality’ rather than the usual ‘curse of dimensionality’.

6 Concluding remarks

Conlon and Lim \( \text{[17]} \) raised the question of whether the extension of the Valtr grid to higher dimensions presented in their paper bears any influence on the problem of constructing density-restricted sets with no large convex subsets in higher dimensions. Here we gave a positive answer and a tight asymptotic bound for \( d = 3 \). We also obtained the first approximation algorithm for the problem of finding a maximum-size subset of points in convex position in a density-restricted set in \( \mathbb{R}^3 \). Next, we list a few open questions regarding the remaining gaps and the quality of approximation.

1. Is the problem of finding a maximum-cardinality subset in convex position, in given finite set in \( \mathbb{R}^3 \), still NP-complete for density-restricted sets?

2. Is there a constant-ratio approximation algorithm for finding a maximum-size subset in convex position for a given finite set in \( \mathbb{R}^3 \)? Is there one for density-restricted sets?

Next are several open questions regarding the size of the largest convex subset in point sets where the density constraints are relaxed. Let \( A \) be a set of \( n \) points in general position in \( \mathbb{R}^d \) satisfying \( \Delta(A) \leq an^{\tau} \), where \( d \geq 2 \) and \( \alpha, \tau > 0 \) are constants. Note that for \( \tau \geq 1 \), only poly-logarithmic bounds are in effect \( \text{[26]} \).

3. Let \( d = 2 \). What upper bounds on the size of the largest convex subset can be derived when \( \tau \in \left( \frac{1}{2}, 1 \right) \)? What lower bounds can be derived when \( \tau \in \left[ \frac{1}{2}, 1 \right] \)?
4. Let $d = 3$. What upper bounds on the size of the largest convex subset can be derived when $\tau \in \left(\frac{1}{3}, 1\right)$? What lower bounds can be derived when $\tau \in \left[\frac{2}{3}, 1\right)$?

A natural candidate for a lower bound in the third question is a suitable perturbation—in the form of the Valtr grid—of a rectangular section of the integer lattice. Indeed, if $A$ is a $n^\tau \times n^{1-\tau}$ section of this lattice and $\tau > \frac{1}{2}$, then $\Delta(A) = \mathcal{O}(n^\tau)$.

We conclude with the following conjecture that generalizes Lemma 11:

**Conjecture 28.** For every finite set $S \subset \mathbb{Z}^d$, $d \geq 2$, and for every integer $t$, $1 \leq t \leq V^{1/d}$, the lattice polytope $\text{conv}(S)$ has $\mathcal{O}\left((V/t^d)^{\frac{d-1}{d}}\right)$ faces (of any dimension) that contain more than $t$ points in $\mathbb{Z}^d$, where $V = \text{Vol}(\text{conv}(S)) > 0$.

Using this technical tool, our upper bound $\gamma_3(n) = \mathcal{O}(n^{1/2})$ in Section 4 would generalize to higher dimensions. Together with the direct generalization of our lower bound $\gamma_3(n) = \Omega(n^{1/2})$ in Section 3, it would yield an alternative proof of the bound $\gamma_d(n) = \Theta_d\left(n^{\frac{d-1}{d}}\right)$ for $d \geq 4$, obtained by Bukh and Dong [13] independently.

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