RENORMALIZATION AND EXISTENCE OF FINITE-TIME BLOW UP SOLUTIONS FOR A ONE-DIMENSIONAL ANALOGUE OF THE NAVIER-STOKES EQUATIONS

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ABSTRACT. The one-dimensional quasi-geostrophic equation is the one-dimensional Fourier-space analogue of the famous Navier-Stokes equations. In [9], Li and Sinai have proposed a renormalization approach to the problem of existence of finite-time blow up solutions of this equation. In this paper we revisit the renormalization problem for the quasi-geostrophic blow ups, prove existence of a family of renormalization fixed points, and deduce existence of real \( C^\infty([0,T), C^\infty(\mathbb{R}) \cap L^2(\mathbb{R})) \) solutions to the quasi-geostrophic equation whose energy and enstrophy become unbounded in finite time, different from those found in [9].

1. Introduction

Bakhtin, Dinaburg and Sinai [2] proposed a novel approach to the question of existence of the blow-up solutions in the Navier-Stokes equations. In their approach, a solution of an initial value problem with a certain self-similar initial conditions is “shadowed” by a solution of a fixed point problem for an integral nonlinear “renormalization” operator in an appropriate functional space. The self-similarity of the solution is of the type

\[
    u(x, t) = (T - t)^{-\frac{1}{2}} U \left( (T - t)^{-\frac{1}{2}} x \right).
\]

The question of existence of blow-up solutions of this form has been first addressed by J. Leray in [7], conditions for the triviality of such solutions have been studied in [16], [14]. In this regard, Bakhtin-Dinaburg-Sinai renormalization can be viewed as a technique of proving existence of Leray self-similar solutions through a fixed point theorem.

Later this renormalization approach was applied to several hydrodynamics models by Li and Sinai [8–11]. The method of these publications is a development of that of [2], with the addition that the authors derive an exact, and not an approximate, renormalization fixed point equation.

In this paper we study existence of Leray solutions in the 1D quasi-geostrophic equation

\[
    \partial_t u - (Hu \cdot u)_x = u_{xx},
\]

where \( Hu \) is the usual Hilbert transform:

\[
    Hu(y) := \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{u(x)}{y - x} \, dx.
\]

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The importance of this equation is that in the Fourier space it serves as a 1D analogue of the Navier-Stokes equations.

We will derive an equation for the Fourier transform of certain self-similar solutions of (2) and demonstrate that this equation reduces to a fixed point problem for a renormalization operator. We then prove existence of a family of even and exponentially decaying renormalization fixed points, which correspond to real $C^\infty$-solutions of (2) which blow up in finite time $T$.

Our approach to the renormalization is different from that of Li and Sinai: rather than looking for solutions in the form of a power series and deriving a fixed point equation for the series' coefficients, we use considerations of compactness borrowed from the renormalization theory in dynamics to prove existence of a class of real solutions different from [9].

1.1. Outline of the proof. As usual, a solution to a Picard-Lindelöf integral equation associated to a PDE, will be called a mild solution.

In Section 2 we will demonstrate that the problem of existence of Leray mild self-similar blow ups is equivalent to a problem of existence of a common $L^2$-fixed point of a family of integral operators $R_\beta$ acting on functions of one variable, with $\beta \in (0,1)$ playing the role of “time”: $\beta^2 = (T-t)/T$.

In Section 3 this fixed point problem will be reduced to a construction of relatively compact sets $\mathcal{N}_p$ in $L^p$, $p \geq 1$, invariant by all $R_\beta$ with $\beta \in (\beta_0,1)$ for some $\beta_0 \in (0,1)$. The sets $\mathcal{N}_p$ will be constructed as the non-empty intersections of two convex sets $\mathcal{E}_{a,k,K,A,\beta_0}^p$ and $\mathcal{M}_{\mu,\sigma}$. The first, $\mathcal{E}_{a,k,K,A,\beta_0}^p$, is the set of all $\psi \in L^p(\mathbb{R})$ satisfying the following convex conditions:

1) $|\psi(\eta)| \leq ke^{-a|\eta|}$;
2) $|\psi(\eta) - \psi(\eta - \delta)| \leq \omega(\eta)|\delta|^\alpha$ with $\omega > 0$, $\omega \in L^p(\mathbb{R})$, $\|\omega\|_p \leq K$ and $\omega(\eta) \leq A|\eta|e^{-a|\eta|}$ whenever $|\eta| > 1$.

The second set $\mathcal{M}_{\mu,\sigma}$ is the set of all $\psi$ in $L^1_u(\mathbb{R})$, where $u$ is the weight $u = |\eta|^{\sigma}e^{-\eta^2}$, such that the following integral is larger than some $\mu > 0$:

$$
\mathcal{I}[\psi] = \int_{\mathbb{R}} \psi(\eta)|\eta|^{\sigma}e^{-\eta^2} \, d\eta.
$$

The invariance of the compact closure of the convex set $\mathcal{N}_p = \mathcal{E}_{a,k,K,A,\beta_0}^p \cap \mathcal{M}_{\mu,\sigma}$ under $R_\beta$ results, via Tikhonov fixed point theorem, in existence of fixed points $\psi_{\beta,p} \in \overline{\mathcal{N}_p}$ of $R_\beta$ for every $\beta \in (\beta_0,1)$. At the same time, as already mentioned, existence of mild blow ups requires existence of one and the same fixed point $\psi_p$ for all $R_\beta$ with $\beta \in (0,1)$. We show in Section 4 that any limit along a subsequence of the fixed points $\psi_{\beta_i,p}$ with $\beta_i \in (\beta_0,1)$ is a fixed point of $R_\beta$ for all $\beta \in (0,1)$. Together with the fact that $\overline{\mathcal{N}_p} \subset L^2(\mathbb{R})$, for all $p \geq 1$, this completes the proof of existence of Leray blow ups.

Additionally, in Section 5 we compute the common point $\psi_p$ as the inverse Fourier transform of the inverse Laplace transform of an explicit expression.

1.2. Statement of results. The main result of this paper reads as follows:

Main theorem. There exist $\nu \in (0,1)$ such that for any $T > 0$ there is a solution of equation (2) in the class $C^\infty\left([0,T), C^\infty(\mathbb{R}) \cap \bigcap_{m \geq 2} L^m(\mathbb{R})\right)$,
given by
\[ u_\nu(x,t) = (T-t)^{-\frac{1}{2}}F^{-1} \circ L^{-1}[\hat{\nu}] \left( x(T-t)^{-\frac{1}{2}} \right), \]
where \( F^{-1} \) denotes the inverse Fourier transform, and \( L^{-1} \) the inverse Laplace transform, and \( \hat{\nu} \) is a solution of the equation
\[ \hat{\nu}'(s) = -\hat{\nu}(s)^2 - \frac{s}{2} \hat{\nu}(s) + \frac{\nu^2}{2}, \]
such that the energy \( E[u_\nu](t) := \|u_\nu(\cdot,t)\|_2^2 \) and the enstropy \( \Omega[u_\nu](t) := \|w(\cdot,t)\|_2^2 \) where \( w(x,t) := xu(x,t) \), become unbounded as \( t \to T \).

1.3. The Navier-Stokes equations. Recall the Navier-Stokes equations:
\[ \partial_t u + (u \cdot \nabla)u - \Delta u + \nabla p = 0, \quad \nabla \cdot u = 0, \quad u(\cdot,0) = u_0, \]
where \( u_0 \) is the initial datum, and \( u \) and \( p \) are the unknowns.

We apply the Fourier transform \( v(y) = F[u](y) \), so that the Navier-Stokes PDE becomes the following system of integro-differential equations:
\[ \partial_t v(y,t) = -|y|^2v(y,t) + i \int_{\mathbb{R}^d} (y \cdot v(y-z,t))P_y v(z,t) dz, \]
where \( P_y \) is the Leray projection to the subspace orthogonal to \( y \), i.e.
\[ P_y v = v - \frac{v \cdot y}{|y|^2} y, \]
indeed, incompressibility condition \( \nabla \cdot u = 0 \) reads \( v(y,t) \cdot y = 0 \) in Fourier space. Integrating (6) with respect to \( t \), we obtain the integral equation:
\[ v(y,t) = e^{-|y|^2t}v_0(y) + i \int_0^t \int_{\mathbb{R}^d} (y \cdot v(y-z,\tau))P_y v(z,\tau)e^{-|y|^2(t-\tau)} dz d\tau, \]
where \( v_0 = v(y,0) \) stands for the initial datum at \( t = 0 \).

1.4. The quasi-geostrophic equation. Our primary focus will be equation (2), which according to the discussion of [3] is a 1D version of the 2D quasi-geostrophic initial value problem:
\[ \theta_t + (u \cdot \nabla)\theta = 0, \]
\[ u = -R^1 \theta := (-R_2 \theta, R_1 \theta), \]
\[ \theta(x,0) = \theta_0(x), \]
where \( R_j \) is the Riesz transform:
\[ R_j(\theta)(x,t) = \frac{1}{2\pi} PV \int_{\mathbb{R}} \frac{(x_j - y_j) \theta(y,t)}{|x-y|^3} dy. \]
This system of equations models the evolution of the temperature in a mixture of cold and hot air fronts.

The system (9) – (11) can be alternatively written as
\[ \theta_t + \text{div}(R^1 \theta)\theta = 0. \]
One obtains the 1D quasi-geostrophic equation by assuming that \( \theta(x,t) \equiv \theta(x_1,t) \): integration of the Riesz transform with respect to the variable \( y_2 \) results in a Hilbert transform of \( \theta(x_1,t) \).
The model (2) has been considered in the classical references [4, 5, 12, 15].

and consider the periodic case, and proof existence of finite-time blow-up $C\infty(T \times [0, T])$-solutions for this equation. Li and Sinai show in [9] that (2) admits complex-valued solutions that blow up in finite time.

The integral equation for this model, analogous to (8), is given by:

$$\psi(x,t) = \psi(x,0) + \int_0^t \int_R y\psi(y,z)\psi(z,t) \, dz \, dt.$$ (12)

The operator $P_y \psi(z) = -i \psi(y - z)\psi(z)$, appearing in (12) is also a projection. In Section 2 we will derive a renormalization operator for a generalized version of (12) with an exponent $\gamma > 1$:

$$\psi(y, t) = e^{-|y|^\gamma t} \psi(y, 0) + \int_0^t \int_R y\psi(y, z)\psi(z, t) \, dz \, dt.$$ (14)

2. Renormalization Problem for a Self-Similar Solution

In this section we will consider an initial value problem for (14) with a Leray initial condition, and deduce that existence of a Leray solution to (14) is equivalent to existence of a fixed point for a certain operator.

We will make an assumption that

$$\psi(y, t) = \tau(t)^{-c} \psi(y\tau(t)),$$ (15)

where $\psi$ is some bounded function, and $\tau(t) = (T - t)^{1/\gamma}$. The following is equation (14) for the initial value problem with the initial condition $v_0(y) = \tau_0^{-c} \psi(y\tau_0) = T^{-\frac{\gamma}{\gamma - 1}} \psi(y\tau_0^{\frac{1}{\gamma}})$, $\tau_0 \equiv \tau(0)$ under the assumption that $\psi(y, t)$ remains of the form (15) for $t > 0$:

$$\frac{\psi(y\tau(t))}{\tau(t)^c} = e^{-|y|^\gamma \frac{\tau(y\tau_0)}{\tau_0^c}} + \int_0^t \int_R e^{-|y|^\gamma (y - z)^\tau(s)} \frac{\psi(z, \tau(s)) \psi((y - z)\tau(s))}{\tau(s)^{2c}} \, dz \, ds,$$

or, denoting the convolution $\int_0^y \psi(x)\psi(y - x) \, dx$ as $\psi \ast \psi(y)$,

$$\frac{\psi(y\tau(t))}{\tau(t)^c} = e^{-|y|^\gamma \frac{\tau(y\tau_0)}{\tau_0^c} - \gamma} \int_0^{\tau(t)} e^{-|y|^\gamma \tau(s)^{2c+2-\gamma}} \frac{(\psi \ast \psi)(y\tau(s))}{\tau(s)^{2c+2-\gamma}} \, d\tau(s).$$

Set $y = y\tau_0$, $\xi = y\tau(t)$, $\zeta = y\tau(s)$, then

$$\frac{\psi(\xi)}{\xi^c} = e^{\xi^{-|y|^\gamma} \frac{\psi(\eta)}{\eta^c}} + \gamma \int_\xi^\eta e^{\xi^{-|y|^\gamma} \frac{\psi(\zeta)}{\zeta^c}} d\zeta.$$

We can now make the choice of $c = \gamma - 2$, after which we get

$$\psi(\xi) = e^{\xi^{-|y|^\gamma} \frac{\xi^{\gamma - 2}}{\eta^{\gamma - 2}} \psi(\eta)} + \gamma \int_\xi^\eta e^{-\xi^{-\gamma} \frac{\xi^{\gamma - 2}}{\eta^{\gamma - 2}} \zeta^{\gamma - 2} \psi(\zeta)} d\zeta.$$ (16)

A function $\psi$ solving this equation for all real $\eta$ and $\xi$ such that $|\xi| \leq |\eta|$, provides a solution

$$v(y, t) = \frac{2 - \gamma}{\gamma} \psi \left( y(T - t)^{\frac{1}{\gamma}} \right).$$
to the initial value problem (14) with the initial data

\[ v_0(y) = T^{\frac{2-\gamma}{\gamma}} \psi \left( y T^{\frac{1}{\gamma}} \right). \]

By introducing a new time variable,

\[ \beta = T^{-\frac{1}{\gamma}} (T - t)^{\frac{1}{\gamma}}, \]

equation (16) can be written as

\[ \psi(\eta \beta) = \beta^{\gamma - 2} \left( e^{\beta \eta^{\gamma - \eta^{\gamma}}} \psi(\eta) + \gamma \eta^{\gamma - 2} e^{\beta \eta^{\gamma}} \int_{\beta \eta}^{\eta} e^{-\zeta^{\gamma}} \zeta^{2-\gamma} (\psi \cdot \psi)(\zeta) \, d\zeta \right), \]

while the problem of existence of solution of (19) can be stated as a fixed point problem for a family of operators

\[ R_\beta[\psi](\eta) = \beta^{\gamma - 2} e^{-\left( \frac{\pi}{\beta} - 1 \right) \eta^{\gamma}} \psi \left( \frac{\eta}{\beta} \right) + \gamma \eta^{\gamma - 2} e^{\eta^{\gamma}} \int_{\eta}^{\eta} e^{-\zeta^{\gamma}} \zeta^{2-\gamma} (\psi \cdot \psi)(\zeta) \, d\zeta. \]

In the next sections we will demonstrate that in the most interesting case \( \gamma = 2 \), the operator (20) has a fixed point \( \psi_{\beta, p} \) for every \( p \geq 1 \) and for each \( \beta \in (0, 1) \) for some \( \beta_0 \in (0, 1) \), in a certain subset \( N_p \subset L^p(\mathbb{R}) \cap L^2(\mathbb{R}) \) of measurable exponentially decaying functions. In particular, the inverse Fourier transform, \( F^{-1}[\psi](x) = F[\psi](-x) \) is a well defined operator from \( N_p \) to \( L^2(\mathbb{R}) \), in fact, to \( C^\infty(\mathbb{R}) \). We will eventually show that the function

\[ u(x, t) = (T - t)^{-\frac{1}{2}} F^{-1}[\psi_p] \left( x (T - t)^{-\frac{1}{2}} \right) \]

is \( C^\infty(\mathbb{R} \times [0, T]) \) solution of (2). For such solution both the energy and the enstrophy become unbounded in finite time. By Plancherel’s theorem:

\[ E[u](t) = \int_\mathbb{R} |v(y, t)|^2 \, dy = \frac{1}{(T - t)^{\frac{1}{2}}} \int_\mathbb{R} \psi(\zeta)^2 \, d\zeta; \]

\[ \Omega[u](t) = \int_{\mathbb{R}^d} |y|^2 |v(y)|^2 \, dy = \frac{1}{(T - t)^{\frac{3}{2}}} \int_\mathbb{R} |\zeta|^2 \psi(\zeta)^2 \, d\zeta. \]

The exponential decay of \( \psi \) implies boundedness of the integrals above.

The operator (20) will be referred to as a renormalization operator: the equation

\[ \psi_p(\eta) = R_\beta[\psi_p](\eta) \]

says informally that the time evolution of the initial data at a later time looked at at a larger scale is equivalent to the initial data itself.

3. A-priori bounds renormalization fixed point

In this section we will consider the family \( R_\beta \) given in (20) for \( \gamma = 2 \).

The operator \( R_\beta \) has two “simple” fixed points: \( \psi = 0 \) and \( \psi = 1 \), corresponding to the trivial and the distributional solutions of (2), \( u(x) = 0 \) and \( u(x) = \delta(x) \). Additionally, \( R_\beta \) preserve the set of even functions.

In what follows we will construct a convex \( R_\beta \)-invariant, equicontinuous family \( N_p \) of exponentially decaying even functions in \( L^p(\mathbb{R}) \), \( p \geq 1 \). The key result that allows us to claim precompactness of \( N_p \), and, consequently, existence of fixed points, is the following (see [6]):
Frechet-Kolmogorov-Riesz-Weil Compactness Theorem. Let $N_p$ be a subset of $L^p(\mathbb{R})$ with $p \in [1, \infty)$, and let $\tau_\delta f$ denote the translation by $\delta$, $(\tau_\delta f)(x) := f(x - \delta)$. The subset $N_p$ is relatively compact iff the following properties hold:

1) Equicontinuity: \( \lim_{|\delta| \to 0} \| \tau f - f \|_p = 0 \) uniformly on $N_p$;
2) Equitightness: \( \lim_{r \to \infty} \int_{|x| > r} |f|^p = 0 \) uniformly on $N_p$.

By analogy with the renormalization theory in dynamics, existence of a renormalization-invariant precompact set will be called a-priori bounds.

We will denote for brevity:

\begin{equation}
J(\eta) = e^{\beta \eta^2} \int_{\eta \eta} e^{-\zeta^2} (\psi \bullet \psi)(\zeta) \, d\zeta.
\end{equation}

We remark, that by Young’s convolution inequality

\begin{equation}
|J(\eta)| \leq C e^{\beta \eta^2} \| \psi \|_s \| \psi \|_r \left( \text{erf}(\sqrt{q} \eta) - \text{erf}(\sqrt{q} \beta \eta) \right)^{\frac{1}{q}},
\end{equation}

whenever $\psi \in L^r(\mathbb{R}) \cap L^s(\mathbb{R})$, $1/r + 1/s + 1/q = 2$. Here, and below, $C$ serves as a stand in for a constant whose value is irrelevant to the proof.

Proposition 3.1. For all $\beta \in (0, 1)$ and for all $p > 3/2$ the operator $R_\beta$ is a well-defined continuous operator of $L^p(\mathbb{R})$ into itself.

Proof. Let $s = r = p$ in (24), i.e. $q = p/(2p - 2)$. Notice, that

\[
\text{erf}(\sqrt{sb}) - \text{erf}(\sqrt{sa}) = \int_{\sqrt{sa}}^{\sqrt{sb}} e^{-t^2} \, dt = \int_{\sqrt{sa}}^{\sqrt{sb}} e^{-\sqrt{st}} \, dt = \frac{1}{\sqrt{sa}} \left( e^{-sa} - e^{-sab} \right).
\]

Therefore, we get, using (24):

\[
\|J\|_p^p \leq C \| \psi \|_s^2 \int_{\mathbb{R}} \left( e^{q^2 \eta^2} \left( \text{erf}(\sqrt{q} \eta) - \text{erf}(\sqrt{q} \beta \eta) \right) \right)^{\frac{2}{q}} \, d\eta
\]

\[
\leq C \| \psi \|_s^2 \int_{\mathbb{R}} \left( 1 - e^{-q(1-\beta)\eta^2} \right)^{\frac{2}{q}} \, d\eta
\]

\[
\leq C (1 - \beta)^{\frac{2}{q}} \| \psi \|_s^2 \int_{\mathbb{R}} \left( 1 - e^{-x^2} \right)^{\frac{2}{q}} \, dx.
\]

The last integral converges for all $p > q = p/(2p - 2)$. The claim follows. \(\square\)

We will now construct a renormalization invariant subset in $L^p(\mathbb{R})$.

Proposition 3.2. For any $a > 0$ the subset $E_{a,k} \subset \bigcap_{p > 0} L^p(\mathbb{R})$ of exponentially decaying even functions,

\begin{equation}
E_{a,k} = \left\{ f \text{ - measurable and even on } \mathbb{R} : |f(x)| \leq ke^{-a|x|} \right\},
\end{equation}

is invariant under $R_\beta$ for all $\beta \in (0, 1)$.

Proof. It is straightforward to check that $R_\beta$ maps even functions to even functions. Set

\[
\phi(\eta) = ke^{-\eta^2} \psi(\eta).
\]
The operator $R_\beta$ acts on $\phi$ as follows:

\begin{equation}
\hat{R}_\beta[\phi](\eta) = \phi\left(\frac{\eta}{\beta}\right) + 2 \int_\eta^t \int_0^\zeta \phi(z)\phi(\zeta - z)e^{-2\zeta(\zeta - z)}dz \, d\zeta.
\end{equation}

Consider an upper bound for non-negative $\eta$:

\begin{align*}
\hat{R}_\beta[\phi](\eta) &\leq ke^{-\frac{\eta^2}{\beta^2} - \frac{a^2}{2}} + 2k^2 \int_\eta^t \int_0^\zeta e^{-\zeta^2 - az}e^{-\eta(\zeta - z)}e^{-2\zeta(\zeta - z)}dz \, d\zeta \\
&= ke^{-\frac{\eta^2}{\beta^2} - \frac{a^2}{2}} + 2k^2 \int_\eta^t e^{-\zeta^2 - a\zeta} \, d\zeta,
\end{align*}

and $\hat{R}_\beta[\phi](\eta) \leq ke^{-\eta^2 - \frac{a^2}{2}}$ for non-negative $\eta$ if

\[\hat{R}_\beta[\phi](\eta) - ke^{-\eta^2 - \frac{a^2}{2}} = k \int_\eta^t (-2\zeta - a)e^{-\zeta^2 - a\zeta} + 2k\zeta e^{-\zeta^2 - a\zeta} \, d\zeta\]

is less or equal to 0. A sufficient condition for this to be non-positive for all $\beta \in (0, 1)$ is the non-positivity of the integrand, i.e. $k \leq 1$.

The lower bound $f(x) \geq -ke^{-|x|}$ is proved in a similar way. \hfill \Box

The set $E_{a,k}$ contains the trivial function. We will now introduce an extra condition which will define a convex subset of $E_{a,k}$ that does not contain 0.

Consider the weighted space $L^1_u(\mathbb{R})$ where $u$ is some weight.

**Lemma 3.3.** For every $\sigma > -3$ and $\beta \in (0, 1)$, the space $L^1_u(\mathbb{R})$ with $u(\eta) = |\eta|\sigma e^{-|\eta|^2}$ is $R_\beta$ invariant. Additionally, for every $0 < \beta_0 < 1$ and $-3 < \sigma < -1$ there exists $\mu_0 > 0$,

\begin{equation}
\mu_0 = \frac{k^2}{\beta_0^2} \left(1 - \beta_0^2\right)^{1/2} \Gamma(s_1) \Gamma(s_2) - a\Gamma(s_2) \Gamma(s_1) \left(s_1, 1 + a^2/4\right)
\end{equation}

where $s_1 = (\sigma + 3)/2$ and $s_2 = (\sigma + 4)/2$, such that for all $\beta \in (0,1)$ and all $\mu > \mu_0$ the convex set $E_{a,k} \cap M_{\mu,\sigma}$,

\begin{equation}
M_{\mu,\sigma} : = \left\{ \psi \in L^1_u(\mathbb{R}) : I[\psi] \geq \mu \right\}, \text{ where}
I[\psi] = \int_\mathbb{R} \psi(\eta)e^{-|\eta|^2|\eta|\sigma} \, d\eta,
\end{equation}

is $R_\beta$ invariant.

**Proof.** Consider an upper bound on the $L^1_u$-norm of $R_\beta[\psi](\eta)$:

\[
\|R_\beta[\psi]\|_u \leq \int_\mathbb{R} e^{-\frac{\eta^2}{\beta^2}} \psi\left(\frac{\eta}{\beta}\right) |\eta|\sigma \, d\eta + 2 \int_\mathbb{R} \int_\eta^t e^{-\zeta^2} \psi(\zeta) \, d\zeta \, d\eta \\
\leq \beta^{\sigma+1} \|\psi\|_u + 2k^2 \int_\mathbb{R} \int_\eta^t e^{-\zeta^2 - a\zeta} \, d\zeta \, d\eta \\
\leq \beta^{\sigma+1} \|\psi\|_u + 2k^2 \left(\frac{1}{\beta^2 - 1}\right) \int_0^\infty e^{-\eta^2 - an} \eta^{2+\sigma} \, d\eta.
\]
A sufficient condition for the last integral to converge is $\sigma > -3$. Moreover,

$$I[\mathcal{R}_\beta[\psi]] \geq \beta^{\sigma+1}I[\psi] - \frac{2Ak^2}{\beta^2} (1 - \beta^2),$$

where

$$A = \frac{1}{2} \left( \Gamma \left( \frac{\sigma + 3}{2} \right) \right) \left( \frac{\sigma + 3}{2} \cdot \frac{a^2}{4} \right) - a\Gamma \left( \frac{\sigma + 4}{2} \right) \left( \frac{\sigma + 4}{2} \cdot \frac{3a^2}{4} \right).$$

A sufficient condition for $I[\mathcal{R}_\beta[\psi]] \geq \mu$ whenever $I[\psi] \geq \mu$ is $\sigma < -1$ and

$$\beta^{\sigma+1} \mu - \frac{2Ak^2}{\beta^2} (1 - \beta^2) \geq \mu \iff \mu \geq \frac{2A}{\beta^{\sigma+3}} \frac{1 - \beta^2}{1 - \beta^{1-\sigma}}.$$  \hspace{1cm} (29)

Since the function $\beta^{\nu-2}(1 - \beta^2)/(1 - \beta^\nu)$ is decreasing on $(\beta_0, 1)$ for $0 < \nu < 2$, we get that $\mu \geq \mu_0$, with $\mu_0$ as in (27), is a sufficient condition for (29). \hspace{1cm} $\Box$

We would now like to show that for any fixed $0 < k < 1$ there is a choice of $a > 0$, $0 < \beta_0 < 1$ and $\sigma$ such that for any $a > a_0$ the convex sets $\mathcal{E}_{a,k}$ and $\mathcal{M}_{\mu,\sigma}$, defined in (26) and (28), have a non-empty intersection.

**Lemma 3.4.** For every $-3 < \sigma < -1$, $0 < k < 1$ and $\beta_0 \in (0, 1)$ there exists $a_0 > 0$ such that for any $a > a_0$ the sets $\mathcal{E}_{a,k}$ and $\mathcal{M}_{\mu,\sigma}$ have a non-empty intersection.

**Proof.** Consider the integral $I[\phi]$ for $\phi(\eta) = k \min \{1, |\eta|^\nu\} e^{-a|\eta|} \in \mathcal{E}_{a,k}$, $\nu > -1 - \sigma > 0$:

$$I[\phi] = 2k \int_0^1 e^{-a\eta} e^{-\eta^2 \eta^{\nu+\sigma}} d\eta + 2k \int_1^\infty e^{-a\eta} e^{-\eta^2 \eta^{\sigma}} d\eta$$

$$\geq 2ke^{-1} \int_0^1 e^{-a\eta} e^{-\eta^2 \eta^{\nu+\sigma}} d\eta + 2k \int_1^a e^{-2a\eta} e^{-\eta^2 \eta^{\sigma}} d\eta + 2k \int_a^\infty e^{-2a\eta} e^{-\eta^2 \eta^{\sigma}} d\eta$$

$$\geq 2k \frac{\gamma(\sigma + \nu + 1, a)}{e^{a\sigma + \nu + 1}} + \frac{k}{2^2 a^{2+\sigma}} \frac{(1 + \sigma, 2\eta)^{a^2}}{\gamma(1 + \sigma, 2a^2)} + \frac{k}{2a^{2+\sigma}} \frac{1}{\Gamma(1 + \frac{1}{2}, 2a^2)}.$$  \hspace{1cm} (30)

The asymptotics of this expression as $a \to \infty$ is

$$Ck \left( \frac{1}{a^{\sigma+\nu+1}} + \frac{e^{-2a}}{2^2 a^{\sigma}} (1 - e^{-2a(a-1)\sigma}) + \frac{e^{-2a^2 \sigma-1}}{2^{2+\sigma}} \right) \sim Ck \frac{1}{a^{\sigma+\nu+1}}.$$  \hspace{1cm} (31)

We would like to compare (31) to the asymptote of (27). The confluent hypergeometric function has the following expansion for $x \to \infty$:

$$1F_1(u, v, x) = \frac{\Gamma(v)}{\Gamma(v - u)} \frac{1 + O \left( \frac{1}{x} \right)}{(-x)^u} + \frac{\Gamma(v)}{\Gamma(u)} e^x \sum_{k=0}^\infty \frac{\Gamma(v - u + k) \Gamma(1 - u + k)}{\Gamma(v - u) \Gamma(1 - u) k!} x^k.$$

Using this expansion, we obtain that the parts of the terms in (27) proportional to $\exp(a^2/4)$ cancel, and the lower bound $\mu_0$ becomes

$$\mu_0 \leq \frac{k^2}{a^{\sigma+3}}.$$  \hspace{1cm} (32)

Therefore, if $\sigma + 3 > \nu + \sigma + 1 \iff \nu < 2$ then for any fixed $k$ the bound (32) decreases faster as $a \to \infty$ than the lower bound (31) on $I[\psi]$, and the conclusion follows. \hspace{1cm} $\Box$
We continue with the second and the third ingredient of our \textit{a-priori} bounds: equitightness and equicontinuity in the sense of $L^p(\mathbb{R})$ spaces.

Since $\mathcal{E}_{a,k} \subset L^p(\mathbb{R})$ for all $p > 0$, we see that the exponentially decaying family $\mathcal{E}_{a,k}$ is equitight:

\textbf{Corollary 3.5.} The set $\mathcal{E}_{a,k}$ is equitight for all $p > 0$, i.e., for all $f \in \mathcal{E}_{a,k}$

\begin{equation}
\lim_{r \to \infty} \int_{|x| > r} |f|^p = 0
\end{equation}

uniformly on $\mathcal{E}_{a,k}$.

Let $\mathcal{E}^p_{a,k,K,\alpha}$ be the subset of all $\alpha$-Hölder uniformly continuous functions with constant $K$ in $\mathcal{E}_{a,k}$, i.e., all functions $f \in \mathcal{E}_{a,k}$ such that for every such $f$ there exists a non-negative valued $\omega_\alpha \in L^p(\mathbb{R})$, such that

\begin{equation}
|\tau_\delta f(\eta) - f(\eta)| \leq \omega_\alpha(\eta)|\delta|^{\alpha}, \quad \|\omega_\alpha\|_p \leq K,
\end{equation}

where $(\tau_\delta f)(x) = f(x - \delta)$. We will also denote the subset of $\alpha$-Hölder locally uniformly continuous functions as $\mathcal{E}^p_{a,k,K,\alpha,\delta_0}$, i.e., the subset for which (34) holds for all $|\delta| < \delta_0$. Both of these families are equicontinuous in $L^p(\mathbb{R})$, i.e.

\begin{equation}
\lim_{|\delta| \to 0} \|\tau_\delta f - f\|_p = 0
\end{equation}

uniformly on both $\mathcal{E}^p_{a,k,K,\alpha}$ and $\mathcal{E}^p_{a,k,K,\alpha,\delta_0}$. By Frechet-Kolmogorov-Riesz-Weil Compactness Theorem (see [6]), both $\mathcal{E}^p_{a,k,K,\alpha}$ and $\mathcal{E}^p_{a,k,K,\alpha,\delta_0}$ are relatively compact in $L^p(\mathbb{R})$ for all $p \geq 1$.

We will require a straightforward technical lemma before we proceed to the \textit{a-priori} bounds.

\textbf{Lemma 3.6.} Fix $\beta_0 \in (0, 1)$. There exist a constant $C = C(\beta_0) > 0$ such that for all $\beta \in (\beta_0, 1)$

\begin{equation}
E(\eta, \delta, \beta) := \left| e^{-(1-\beta^2)\eta^2/|\delta|^2} - e^{-(1-\beta^2)\eta^2/|\delta|^2} \right| \leq C \left( \frac{1}{\beta^2} - 1 \right)^{\frac{1}{2}} |\delta|.
\end{equation}

\textbf{Proof.} Using the shorthanded notation $E$ for the difference of exponentials as in (36), we have in the case $|\eta| > |\eta - \delta|$ and $|\eta| > |\delta|:

\begin{equation}
E \leq e^{-\frac{1-\beta^2}{\beta^2}|\eta|\delta^2/|\delta|^2} \left| 1 - e^{-\frac{1-\beta^2}{\beta^2}(|\eta - \delta|^2 - |\eta|^2)} \right| \leq e^{-\frac{1-\beta^2}{\beta^2}(|\delta|^2 - |\eta|^2)} \left( 1 - e^{-\frac{1-\beta^2}{\beta^2}D|\eta||\delta|} \right)
\end{equation}

for some constant $D > 0$. To get a bound on this expression, proportional to a power of $|\delta|$, we first notice that for non-negative $x$,

\begin{equation}
e^{-x^2} \left( 1 - e^{-ax} \right) \leq \frac{2ax}{(1 + ax)(1 + x^2)} \leq \frac{2ax}{(1 + x^2)} \leq a.
\end{equation}

We can now use this estimate on (37) with

\[ x = \frac{1}{\sqrt{2}} \left( \frac{1}{\beta^2} - 1 \right)^{\frac{1}{2}} |\eta|, \quad a = \sqrt{2}D|\delta| \left( \frac{1}{\beta^2} - 1 \right)^{\frac{1}{2}}, \]

to get

\begin{equation}
E \leq C e^{-\frac{1-\beta^2}{\beta^2}|\delta|^2} \left( \frac{1}{\beta^2} - 1 \right)^{\frac{1}{2}} |\delta|.\n\end{equation}
In the case $|\eta| > |\eta - \delta|$ and $|\eta| < |\delta|$,\n\begin{equation}
E \leq 1 - e^{-\frac{1-\beta^2}{\beta^2}D|\eta|}\leq C \left( \frac{1}{\beta^2} - 1 \right)|\delta|^2. \tag{40}
\end{equation}
In the case $|\eta - \delta| > |\eta| > |\delta|$, we can again use estimate (38):
\begin{equation}
E \leq e^{-\frac{1-\beta^2}{\beta^2}|\eta|^2} \left( 1 - e^{-\frac{1-\beta^2}{\beta^2}D|\eta|} \right) \leq C \left( \frac{1}{\beta^2} - 1 \right)^\frac{\beta}{2}|\delta|. \tag{41}
\end{equation}
In the case $|\eta - \delta| > |\eta|$ and $|\delta| > |\eta|$, we can crudely bound as follows:
\begin{equation}
E \leq e^{-\frac{1-\beta^2}{\beta^2}|\eta|^2} \left( 1 - e^{-\frac{1-\beta^2}{\beta^2}D|\eta|} \right) \leq 1 - e^{-\frac{1-\beta^2}{\beta^2}D|\delta|^2} \leq C \left( \frac{1}{\beta^2} - 1 \right)|\delta|^2. \tag{42}
\end{equation}
The claim follows after one combines bounds (39), (40), (41) and (42).\[\square\]

We continue with the proof of the a-priori bounds. Per our convention, $C$ denotes a constant whose size has no bearing on the arguments in the proofs (they may, however depend on $\beta_0$, $\delta_0$ and $a_0$).

**Proposition 3.7.** (A-priori bounds: equicontinuity.) For every $p \geq 1$ there are constants $0 < \beta_0 < 1$, $A > 0$, $K > 0$, $a > 0$, $\delta_0 > 0$, $k > 0$ and $0 < \alpha < 1/p$, such that for every $\beta \in (\beta_0, 1)$ the convex subset
\begin{equation}
\mathcal{E}^p_{a,k,K,A,\alpha,\delta_0} = \{ \psi \in \mathcal{E}^p_{a,k,K,A,\alpha,\delta_0} : \omega_\psi(\eta) \leq A|\eta| e^{-a|\eta|} \text{ for } |\eta| > 1 \}
\end{equation}
of $\mathcal{E}^p_{a,k,K,A,\alpha,\delta_0}$, is renormalization invariant under $R_\beta$.

**Proof.** Assume that $\psi \in \mathcal{E}^p_{a,k,K,A,\alpha,\delta_0}$. We consider linear and non-linear terms separately for $|\delta| < \beta\delta_0$. Clearly, it is sufficient to prove the result for $0 \leq \delta < \beta\delta_0$.

**1) Linear terms, case $0 < \delta < \beta\delta_0$.** Denote,
\begin{equation}
L(\eta, \delta, \beta) = \left| e^{-(1-\beta^2)\frac{1}{\beta^2}|\eta|^2} \psi \left( \frac{\eta}{\beta} \right) - e^{-(1-\beta^2)\frac{1}{\beta^2}|\eta-\delta|^2} \psi \left( \frac{\eta-\delta}{\beta} \right) \right|. \tag{44}
\end{equation}
We have for all $|\delta| < \beta\delta_0$, using the notation of (36)
\[
L(\eta, \delta, \beta) \leq E(\eta, \delta, \beta) \left| \psi \left( \frac{\eta-\delta}{\beta} \right) \right| + e^{-\frac{1-\beta^2}{\beta^2}|\eta|^2} \left| \psi \left( \frac{\eta}{\beta} \right) - \psi \left( \frac{\eta-\delta}{\beta} \right) \right|
\]
\[
\leq E(\eta, \delta, \beta) \left| \psi \left( \frac{\eta-\delta}{\beta} \right) \right| + \beta^{-\alpha} e^{-\frac{1-\beta^2}{\beta^2}|\eta|^2} \omega_\psi \left( \frac{\eta}{\beta} \right) |\delta|^\alpha
\]
\[
= L_1(\eta, \delta, \beta) + L_2(\eta, \delta, \beta).
\]
We have
\[
E(\eta, \delta, \beta) \leq \begin{cases}
1 - e^{-\frac{1}{\beta^2}(|\eta|^2-|\eta-\delta|^2)} & \text{if } |\eta| > |\eta-\delta|, \\
1 - e^{-\frac{1}{\beta^2}(|\eta-\delta|^2-|\eta|^2)} & \text{if } |\eta| \leq |\eta-\delta|.
\end{cases}
\]
In both cases, if $|\eta| > 2|\delta|$, the following very conservative estimate holds:
\[
E(\eta, \delta, \beta) \leq 1 - e^{-\frac{1-\beta^2}{\beta^2}D|\eta|}\leq \frac{1-\beta^2}{\beta^2}D|\eta||\delta|
\]
for some constant $D > 0$. Therefore,

$$\|L_1\|_p^p \leq \int_{|\eta| \leq 2|\delta|} E(\eta, \delta, \beta)^p \left| \psi \left( \frac{\eta - \delta}{\beta} \right) \right|^p d\eta + \int_{|\eta| > 2|\delta|} E(\eta, \delta, \beta)^p \left| \psi \left( \frac{\eta - \delta}{\beta} \right) \right|^p d\eta$$

$$\leq k \left( 1 - e^{-\frac{1-\alpha}{\beta^2} |\delta|^2} \right)^p \int |\eta| e^{-p(\eta - \delta)} d\eta + C k (1 - \beta)^p |\delta|^p \int \left| |\eta|^p e^{-p(\eta - \delta)} d\eta \right|$$

$$\leq C k (1 - \beta)^p \left( |\delta|^3 \right)$$

We have, therefore,

$$\|L\|_p \leq C k (1 - \beta)|\delta| + \beta^{\frac{1}{2} - \alpha} K |\delta|^\alpha.$$  \hfill (45)

Additionally, in the case $\eta > 1$ and $\delta > 0$, we have the following bound on the linear terms themselves (and not the norm):

$$L(\eta, \delta) \leq C k (1 - \beta) |\eta| e^{-a|\eta - \delta|} + \beta^{-\alpha} e^{-\frac{1-\beta^2}{\beta^2} |\eta|^2} \omega_\beta \left( \frac{\eta}{\beta} \right) \delta^\alpha$$

$$\leq C k (1 - \beta) |\eta| e^{-a|\eta - \delta|} + A \beta^{-\alpha - 1} e^{-\frac{1-\beta^2}{\beta^2} |\eta|^2} |\eta| e^{-a |\eta| \delta} \delta^\alpha$$  \hfill (46)

2) Non-linear terms, case $0 < \delta < \beta \delta_0$. We consider $\|\hat{J} - \tau_\delta \hat{J}\|_p$, $\hat{J} = 2 J \circ \beta^{-1}$ where $J$ is defined in [23]. We have

$$\hat{J} = 2 \eta \int_1^t e^{2 |\eta|^2 (t - t') \int_0^{\eta t} |\psi(x)| \psi(\eta t - x) \, dx \, dt,$$

and

$$|\hat{J}(\eta) - \hat{J}(\eta - \delta)| \leq J_1(\eta, \delta) + J_2(\eta, \delta) + J_3(\eta, \delta) + J_4(\eta, \delta),$$

where

$$J_1(\eta, \delta) \leq C |\eta| \int_1^t \left| e^{2 |\eta|^2 (t - t')} - e^{2 |\eta - \delta|^2 (t - t')} \right| \left| \int_0^{\eta t} |\psi(x)| \psi(\eta t - x) \, dx \right| \, dt,$$

$$J_2(\eta, \delta) \leq C \delta \int_1^t e^{2 |\eta - \delta|^2 (t - t')} \left| \int_0^{\eta t} |\psi(x)| \psi(\eta t - x) \, dx \right| \, dt,$$

$$J_3(\eta, \delta) \leq C |\eta - \delta| \int_1^t e^{2 |\eta - \delta|^2 (t - t')} \left| \int_0^{\eta t} |\psi(x)| \psi(\eta t - x) - \psi(\eta t - x + (\eta - \delta) t - x) \right| \, dx \, dt,$$

$$J_4(\eta, \delta) \leq C |\eta - \delta| \int_1^t e^{2 |\eta - \delta|^2 (t - t')} \left| \int_{(\eta - \delta) t}^{\eta t} |\psi(x)| \psi((\eta - \delta) t - x) \right| \, dx \, dt.$$

We consider each $J_i$ separately.

a) Calculation of $J_1$. Since $|(|\psi \cdot \psi(x)| \leq e^{-a|x|} |x|$, we have

$$J_1(\eta, \delta) \leq C k^2 |\eta| \int_1^t \left| e^{2 |\eta|^2 (t - t')} - e^{2 |\eta - \delta|^2 (t - t')} \right| \, dt.$$
Therefore, in the case \( \eta > 1 \), we have \( \eta \geq \eta - \delta \) and \( \eta - \delta > (1 - \delta)\eta \), and

\[
J_1(\eta, \delta) \leq Ck^2 \int_1^{\beta} e^{-(\eta-\delta)^2(t^2-1)} \left( 1 - e^{-\delta(2\eta-\delta)(t^2-1)} \right) e^{-a\eta t} \, dt \, d\eta
\]

\[
\leq Ck^2 \int_1^{\beta} e^{-(\eta)^2(t^2-1)} \left( 1 - e^{-C(1-\beta^2)\eta^2} \right) e^{-a\eta t} \, dt \, d\eta
\]

\[
\leq Ck^2 \delta \eta (1 - \beta^2) e^{(1-\delta)^2\eta^2} \int_0^{(1-\delta)\eta} e^{-y^2} e^{-\frac{\eta}{\beta^2} y} \, dy
\]

\[
\leq Ck^2 \delta \eta (1 - \beta^2) e^{-\delta} \left( 1 - e^{-\left(\frac{1}{\beta^2} - 1\right)\eta^2 - \left(\frac{\eta}{\beta} - 1\right)\eta} \right),
\]

while the norm itself can be bounded, using the bound (46) on the difference of exponentials, as

\[
\|J_1\|_p \leq Ck^2 (1 - \beta)^{\frac{3}{2}} \delta.
\]

b) \textit{Calculation of } \( J_2 \). For \( \eta > 1 \) this bound follows closely that on \( J_1 \):

\[
J_2(\eta, \delta) \leq Ck^2 \delta \int_1^{\beta} e^{(\eta-\delta)^2(1-t^2)} e^{-a\eta t} \, dt \, d\eta
\]

\[
\leq Ck^2 \delta \eta^{-1} e^{-\delta} \left( 1 - e^{-\left(\frac{1}{\beta^2} - 1\right)\eta^2 - \left(\frac{1}{\beta} - 1\right)\eta} \right),
\]

while the norm itself is bounded as

\[
\|J_2\|_p \leq Ck^2 \delta \left( \int e^{-ap\eta} \left( \frac{1}{\beta} - 1 \right)^p \eta^p \, d\eta \right)^{\frac{1}{p}} \leq Ck^2 (1 - \beta)\delta.
\]

c) \textit{Calculation of } \( J_3 \). We first consider the case of \( |\eta| \leq 1 \).

\[
J_3(\eta, \delta) \leq C\eta \int_1^{\beta} e^{(\eta-\delta)^2(1-t^2)} \int_0^{\eta t} |\psi(x)| |\omega_{\eta}(\eta t - x)(\delta t)^{\alpha}| \, dx \, dt
\]

\[
\leq C\delta^\alpha \eta e^{(\eta-\delta)^2} \int_1^{\beta} e^{-\eta^2 t^2} t^{\alpha} \, dt \|\psi\|_q \|\omega_{\eta}\|_p
\]

\[
\leq CK k (1 - \beta)^{\delta^\alpha} \eta,
\]

by Young’s inequality with \( 1/p + 1/q + 1/r = 2 \) and \( r = 1 \). Next, for \( \eta > 1 \),

\[
\int_0^{\eta t} |\psi(\eta t - x)| |\omega_{\eta}(x)| \, dx \leq \int_0^1 |\psi(\eta t - x)| |\omega_{\eta}(x)| \, dx + \int_1^{\eta t} |\psi(\eta t - x)| |\omega_{\eta}(x)| \, dx
\]

\[
\leq k \int_0^1 e^{-a|\eta t - x|} \omega_{\eta}(x) \, dx + Ak \int_1^{\eta t} e^{-a(\eta t - x)} e^{-ax} \, dx
\]

\[
\leq CK ke^{-\alpha t} + CAk e^{-\alpha t} \eta^2 t^2.
\]
Therefore, in the case $\eta > 1$, 
\[
J_3(\eta, \delta) \leq C k^\delta e^{-\eta^2 - \alpha t} \int_1^\infty e^{-(\eta^2 - \alpha t)} \left( K + A(\eta^2 t^2) \right) \, d\eta 
\]
\[
\leq C k \delta e^{-\eta^2} \left( 1 - e^{-\left(\frac{1}{\beta} - 1\right)\eta^2} \right) + C A k \delta e^{-\eta^2/2} \left( 1 - e^{-\left(\frac{1}{\beta} - 1\right)\eta^2} \right) \cdot g(\eta)
\]
(52)
\[
\leq C k \delta e^{-\eta^2} \left( 1 - e^{-\left(\frac{1}{\beta} - 1\right)\eta^2} \right) + A \eta \left( 1 - e^{-\left(\frac{1}{\beta} - 1\right)\eta^2} \right).
\]

\textit{d) Calculation of } J_4. \textit{ For } \eta > 1,
\[
J_4(\eta, \delta) \leq C k^2 (\eta - \delta) \int_1^\infty e^{-(\eta^2 - \alpha t)} \int_1^\infty e^{-a|t|} e^{-a(\eta^2 t - |t|)} \, dx \, dt 
\]
\[
\leq C (\eta - \delta) k^2 \int_1^\infty e^{-(\eta^2 - \alpha t)} e^{-a(\eta^2 t - |t|)} \, dt 
\]
\[
\leq C k^2 (\eta - \delta)^{-1} e^{-a(\eta - \delta)} \left( 1 - e^{-\left(\frac{1}{\beta} - 1\right)(\eta - \delta)^2} \right) \delta, 
\]
(53)
\[
\leq C k^2 \eta^{-1} e^{-\eta^2} \left( 1 - e^{-\left(\frac{1}{\beta} - 1\right)\eta^2} \right) \delta,
\]

and
\[
(54) \quad \|J_4\|_p \leq C k^2 (1 - \beta) \delta.
\]

3) \textit{Bound on the constant } A. \textit{ Denote } f(\eta) = \left( \frac{1}{\beta^2} - 1 \right) \eta^2 + \left( \frac{1}{\beta} - 1 \right) \alpha \eta 
\textit{ and } g(\eta) = \left( \frac{1}{\beta} - 1 \right) \alpha |\eta|. \textit{ We can now combine } (45), (47), (49), (51), (52) \textit{ and } (53) \textit{ to get the following bound on } \omega_\mathcal{R}_\beta[|\cdot|] \textit{ for } \eta > 1:
\[
\omega_\mathcal{R}_\beta[|\cdot|] \leq A e^{-\frac{1}{\beta^2} \eta^2} \eta e^{-a\eta^2} + C k (1 - \beta) \delta_0^{1 - \alpha} e^{-\eta} + C k^2 (1 - \beta) \delta_0^{1 - \alpha} \eta e^{-\eta + f(\eta)} + C k^2 \delta_0^{1 - \alpha} |\eta|^{-1} e^{-a|\eta|} \left( 1 - e^{-f(\eta)} \right) + C k e^{-a\eta} \left( K (1 - \beta) + A \eta \left( 1 - e^{-f(\eta)} \right) \right).
\]
(55)

A sufficient condition for \(\omega_\mathcal{R}_\beta[|\cdot|] \leq A \eta e^{-\eta^2}\) is therefore
\[
A \geq A e^{-\frac{1}{\beta^2} \eta^2} + C k (1 - \beta) \delta_0^{1 - \alpha} e^{-g(\eta)} + C k \delta_0^{1 - \alpha} + A \left( 1 - e^{-f(\eta)} \right) + C k (1 - \beta),
\]
which is implied by
\[
A \frac{\beta^{1 + \alpha} - e^{-f(\eta)}}{1 - e^{-f(\eta)}} - A C k \geq C k \delta_0^{1 - \alpha} e^{-g(\eta)} + C k^2 \delta_0^{1 - \alpha} + C k \delta_0^{1 - \alpha} + C k (1 - \beta).
\]
As $\beta \to 1$, this condition is tangent to

$$A \left(1 - Ck - \frac{1 + \alpha}{2 + a}\right) \geq Ck \left(e^{-g(1)} + k\delta_0^{1-\alpha} + K\right).$$

Suppose that there is a choice of constants such that (56) holds, then the condition on $\omega_\psi$ in (43) for $\eta > 1$ and $\delta > 0$, is renormalization invariant. This implies that for $\eta > 1 - \delta$ and $\delta > 0$,

$$|\psi(\eta) - \psi(\eta + \delta)| \leq A(\eta + \delta)e^{-\eta(\eta + \delta)}\delta^\alpha < A(1 + \delta)e^{-\eta\delta} < A\eta^{-\eta\delta},$$

if $a > 1$ and $\delta_0 = \delta_0(a)$ is sufficiently small. Using the fact that $\psi$ is an even function, it follows that

$$|\psi(\eta) - \psi(\eta - \delta)| \leq A\eta^{-\eta\delta}$$

for $\eta < -(1 - \delta)$ and $\delta > 0$, and

$$|\psi(\eta) - \psi(\eta + \delta)| \leq A\eta^{-\eta\delta}$$

for $\eta < -1$ and $\delta > 0$. We conclude that the condition on $\omega_\psi$ in the definition of the set (45) is invariant for all $0 < \delta < \delta_0$ and $|\eta| > 1$. A similar argument gives the invariance condition for all $|\delta| < \delta_0$.

**4) Bound on the constant $K$.** We have computed bounds on the norms of the terms $L$, $J_1$, $J_2$ and $J_4$. The only remaining bound is that on $J_3$.

First, we remark, that one gets a bound identical to (52) in the case of $\eta < -1$. Then, for $|\eta| > 1$, we can bound the factor $1 - e^{-g(\eta)}$ in (52) very crudely by $C(1 - \beta)|\eta|$, and the factor $1 - e^{-f(\eta)}$ as $C(1 - \beta)|\eta|^2$.

$$||J_3||_p \leq C(1 - \beta)\delta^\alpha \left(K^p \int_{|\eta| \leq 1} |\eta|^p d\eta + K^p \int_{|\eta| > 1} |\eta|^p e^{-a|\eta|} d\eta + A^p \int_{|\eta| > 1} |\eta|^{3p} e^{-a|\eta|} d\eta\right)^{\frac{1}{p}}$$

(57)

$$\leq Ck(1 - \beta) \delta^\alpha (K + A).$$

We can now collect (45), (48), (50), (57) and (54) to get that the condition $||\omega_\psi||_p$ is renormalization invariant if

$$K \geq Ck(1 - \beta)\delta_0^{1-\alpha} + \beta^\frac{1}{p} - \alpha K + Ck^2(1 - \beta)\delta_0^{1-\alpha} + Ck(1 - \beta) (K + A).$$

As $\beta \to 1$, this condition is tangent to

$$K \left(\frac{1}{p} - \alpha\right) - Ck \geq Ck\delta_0^{1-\alpha} + Ck^2\delta_0^{1-\alpha} + CAk.$$

For large $K$ and $A$, conditions (50) and (58) are asymptotic to

$$A \left(1 - Ck - \frac{1 + \alpha}{2 + a}\right) \geq CkK \quad \text{and} \quad K \left(\frac{1}{p} - \alpha\right) - Ck \geq CkA.$$

These two conditions have a solution if

$$\frac{\frac{1}{p} - \alpha - Ck}{Ck} \geq \frac{Ck}{1 - Ck - \frac{1 + \alpha}{2 + a}},$$

where $C$ is a general stand in for a constant independent of $A$, $K$ or $k$, not necessarily the same in all instances. Condition (59) can be satisfied by a choice of $k$. 

5) Non-linear terms, case $\delta_0 > |\delta| > \beta \delta_0$. In this case
\[ \| \psi - \tau_\delta \psi \|_p \leq \| \psi \|_p + \| \tau_\delta \psi \|_p \leq \left( \int_{\mathbb{R}} e^{-ap|\eta|} \, d\eta \right)^{\frac{1}{p}} \frac{2k}{\beta_0 \delta_0} |\delta|, \]
and $K$ can be chosen to be the maximum of $K$ from Part 4) and
\[ \left( \int_{\mathbb{R}} e^{-ap|\eta|} \, d\eta \right)^{\frac{1}{p}} \frac{2k}{\beta_0 \delta_0}. \]

We will now demonstrate that the intersection $\mathcal{N}_p = \mathcal{E}_{a,k,K,A,\alpha,\delta_0}^p \cap \mathcal{M}_{\mu,\sigma}$ is non-empty. We have already shown in Lemma 3.8 that the function $\phi(\eta) = k \min\{1, |\eta|^{\nu} \} e^{-a|\eta|}$ is in $\mathcal{E}_{a,k} \cap \mathcal{M}_{\mu,\sigma}$ for $\nu > -1 - \sigma$. The following Lemma shows that the same function belongs to $\mathcal{N}_p$.

**Lemma 3.8.** For any $0 < k < 1$, any $-1 > \sigma > -2$ and any $1 > \nu > -1 - \sigma$, there exist $a > 0$, $A > 0$, $K > 0$, $\delta_0 > 0$ and $\mu > 0$, such that the function $\phi(\eta) = k \min\{1, |\eta|^{\nu} \} e^{-a|\eta|}$ is in $\mathcal{N}_p = \mathcal{E}_{a,k,K,A,\alpha,\delta_0}^p \cap \mathcal{M}_{\mu,\sigma}$.

**Proof.** Let $\delta_0 < 1$ and consider first $\eta > 1$. We can combine the following inequalities
\[ k|e^{-a \eta} - e^{-a(\eta-\delta)}| \leq Ck \delta e^{-a \eta}, \]
\[ k|\eta^\nu e^{-a \eta} - \eta^\nu e^{-a (\eta-\delta)}| \leq Ck \delta \eta^\nu e^{-a \eta}, \]
\[ k|\eta^\nu e^{-a \eta} - (\eta-\delta)^\nu e^{-a \eta}| \leq Ck \delta^\nu e^{-a \eta}, \]
to get that there exists some $A > 0$, such that
\[ |\phi(\eta) - \phi(\eta-\delta)| \leq A|\eta|^{\nu}|\delta|^\nu e^{-a \eta}. \]
A similar bound holds for $\delta < 0$ and $\eta < -1$. It is also straightforward that there is $K > 0$ such that for all $|\delta| < \delta_0$
\[ \| \phi - \tau_\delta \phi \|_p \leq K \delta^\nu. \]

We have demonstrated the following result.

**Proposition 3.9.** For any $p \geq 1$ there exist $A > 0$, $K > 0$, $\alpha$, $a$, $k$, $\mu$, $\beta_0$, $\sigma$ and $\delta_0$ such that for every $\beta \in (\beta_0, 1)$ the operator $\mathcal{R}_\beta$ has a fixed point $\psi_{\beta,p}$ in the $L^p(\mathbb{R})$-closure $\overline{\mathcal{N}_p}$ of $\mathcal{N}_p = \mathcal{E}_{a,k,K,A,\alpha,\delta_0}^p \cap \mathcal{M}_{\mu,\sigma}$.

Additionally, every $\overline{\mathcal{N}_p}$, together with $\psi_{\beta,p}$, is in $\cap_{n \geq 1} L^n(\mathbb{R})$.

**Proof.** Choose $A$ and $K$ to be the maximum of those provided by Proposition 3.7 and Lemma 3.8 and take the rest of the constants as those specified in Proposition 3.7 and Lemma 3.8. Then $\overline{\mathcal{N}_p}$ is non-empty and renormalization-invariant. Minkowski inequality for the $L^p(\mathbb{R})$-norm together with the convexity of the set $\mathcal{E}_{a,k}$ and $\mathcal{M}_{\mu,\sigma}$ also implies that $\overline{\mathcal{N}_p}$ is convex. Since all functions in $\mathcal{N}_p$ are exponentially bounded, $\overline{\mathcal{N}_p} \subset \cap_{n \geq 1} L^n(\mathbb{R})$.

The claim follows by Tikhonov fixed point theorem for the continuous operator $\mathcal{R}_\beta$ on a convex compact $\mathcal{R}_\beta$-invariant set $\overline{\mathcal{N}_p}$. \(\blacksquare\)
4. Limits of renormalization fixed points

We emphasize, that existence of fixed points for the family $\mathcal{R}_\beta$ does not yet imply existence of a solution to the equation (16). Rather, one needs to show, that there is one and the same fixed point $\psi_p$ for all $\beta \in (0, 1)$. The next Lemma provides a step in this direction.

**Lemma 4.1.** Any fixed point $\phi_\beta$ of the operator $\mathcal{R}_\beta$ is also fixed by $\mathcal{R}_{\beta^n}$ for all $n \in \mathbb{N}$.

**Proof.** We have at the base of induction.

$$e^{-\beta^2 n^2} \phi_\beta(\beta^2 n) = e^{-\beta^2 n^2} \phi_\beta(\beta n) + 2 \int_{\beta^2 n}^{\beta n} e^{-\xi^2} (\phi_\beta \bullet \phi_\beta)(\xi) \, d\xi$$

$$= e^{-n^2} \phi_\beta(n) + 2 \int_{\beta n}^{n} e^{-\xi^2} (\phi_\beta \bullet \phi_\beta)(\xi) d\xi + 2 \int_{\beta^2 n}^{\beta n} e^{-\xi^2} (\phi_\beta \bullet \phi_\beta)(\xi) d\xi$$

$$= e^{-n^2} \phi_\beta(n) + 2 \int_{\beta n}^{n} e^{-\xi^2} (\phi_\beta \bullet \phi_\beta)(\xi) d\xi + 2 \int_{\beta^2 n}^{\beta n} e^{-\xi^2} (\phi_\beta \bullet \phi_\beta)(\xi) d\xi$$

Assume the result for $n = k$. Then

$$e^{-\beta^{2k+1} n^2} \phi_\beta(\beta^{k+1} n) = e^{-\beta^{2k} n^2} \phi_\beta(\beta^k n) + 2 \int_{\beta^{k+1} n}^{\beta^k n} e^{-\xi^2} (\phi_\beta \bullet \phi_\beta)(\xi) \, d\xi$$

$$= e^{-n^2} \phi_\beta(n) + 2 \int_{\beta^k n}^{n} e^{-\xi^2} (\phi_\beta \bullet \phi_\beta)(\xi) d\xi + 2 \int_{\beta^{k+1} n}^{\beta^k n} e^{-\xi^2} (\phi_\beta \bullet \phi_\beta)(\xi) d\xi$$

$$= e^{-n^2} \phi_\beta(n) + 2 \int_{\beta^k n}^{n} e^{-\xi^2} (\phi_\beta \bullet \phi_\beta)(\xi) d\xi + 2 \int_{\beta^{k+1} n}^{\beta^k n} e^{-\xi^2} (\phi_\beta \bullet \phi_\beta)(\xi) d\xi$$

$\square$

We would like now to address the question of what happens to the functions $\psi_{\beta, p}$, described in Proposition 3.9, as $\beta$ approaches 1.

Consider a sequence of $\{\psi_{\beta_i, p}\}_{i=0}^\infty$, $\beta_i \in (\beta_0, 1)$ and $\beta_i \to 1$ with $\psi_{\beta_i, p} \in N_p$ where $N_p$ is as in Proposition 3.9. By compactness of the set $N_p$, we can pass to a converging subsequence, which we will also denote $\{\psi_{\beta_i, p}\}_{i=0}^\infty$. Its limit $\psi_p$ is non-trivial. Since $\psi_p$ is exponentially bounded, we have that $\psi_p \in \cap_{n \geq 1} L^n(\mathbb{R})$, and

$$F^{-1}[\psi_p] \in C^\infty(\mathbb{R}) \cap \bigcap_{m \geq 2} L^m(\mathbb{R}),$$

(recall that for any $p \in (1, 2]$, the Fourier transform is a bounded operator from $L^p$ to $L^{p'}$, where $p'$ is the Hölder conjugate of $p$).

Consider this sequence $\{\psi_{\beta_i, p}\}_{i=0}^\infty$, $L^p$-converging to $\psi_p$. For any $\beta \in (0, 1)$ there exists a diverging sequence of integers $n_i$, such that $\beta_i^{n_i} \to \beta$. Indeed, for any $\varepsilon > 0$, a sufficient condition for $\beta_i^{n_i} \in (\beta - \varepsilon, \beta + \varepsilon)$ is

$$n_i \in \left(\frac{\ln(\beta + \varepsilon)}{\ln \beta_i}, \frac{\ln(\beta - \varepsilon)}{\ln \beta_i}\right).$$

The length of this interval is of the order $-\varepsilon / \beta \ln \beta_i$. Therefore, for any $\varepsilon$ there exists $I \in \mathbb{N}$, such that this length is larger than 1 for all $i \geq I$, and
the interval contains some positive integer \(n_i\). Therefore,
\[
\|\mathcal{R}_\beta[\psi_{\beta_i,p}] - \psi_{\beta_i,p}\|_p \leq \left\| \mathcal{R}_{\beta_i}\psi_{\beta_i,p} - \psi_{\beta_i,p} + \left( e^{(\beta^2-1)\eta^2} - e^{\left(\beta_i^2\eta^2\right)} \right) \psi_{\beta_i,p} \right\|_p + \\
+ \left\| 2e\eta^2 \int_{\frac{\eta}{\beta_i}}^{\eta} e^{-\zeta^2} (\psi_{\beta_i,p} \cdot \psi_{\beta_i,p})(\zeta) \, d\zeta \right\|_p.
\]

According to Lemma \((4,1)\), we have \(\mathcal{R}_{\beta_i}\psi_{\beta_i,p} = \psi_{\beta_i,p}\), therefore,
\[
\|\mathcal{R}_\beta[\psi_{\beta_i,p}] - \psi_{\beta_i,p}\|_p \leq C|\beta_i^{n_i} - \beta| \left( \left( \frac{\int_{\mathbb{R}} \eta^{2\nu} e^{-a\eta|\eta|} \, d\eta}{\int_{\mathbb{R}} \eta^{2\nu} e^{-a\eta|\eta|} \, d\eta} \right)^\frac{1}{p} + \left( \frac{\int_{\mathbb{R}} \eta^{2\nu} e^{-\eta|\eta|} \, d\eta}{\int_{\mathbb{R}} \eta^{2\nu} e^{-\eta|\eta|} \, d\eta} \right)^\frac{1}{p} \right).
\]

We conclude with the following

**Proposition 4.2.** For every \(p \geq 1\) there exists \(\psi_p \in \mathcal{N}_p \subset \cap_{n \geq 1} L^n(\mathbb{R})\), which is a fixed point of \(\mathcal{R}_\beta\) for all \(\beta \in (0,1)\).

5. More on the common fixed point

We can, however, go one step further and derive an exact form of the limit function \(\psi_p\). First,
\[
e^{-\beta_i^2\eta^2}\psi_{\beta_i,p}(\beta_i\eta) - e^{-\eta^2}\psi_{\beta_i,p}(\eta) = 2 \int_{\beta_i \eta}^{\eta} e^{-\zeta^2} (\psi_{\beta_i,p} \cdot \psi_{\beta_i,p})(\zeta) \, d\zeta
\]
\[
(61) = (1 - \beta_i) \left( 2e^{-\eta^2}(\psi_{\beta_i,p} \cdot \psi_{\beta_i,p})(\eta) + \eta I(\eta, \beta_i) \right),
\]

where, according to Lebesgue differentiation theorem, the remainder \(I\)
\[
|I(\eta, \beta_i)| \leq \frac{1}{(1 - \beta_i)|\eta|} \left| \int_{\beta_i \eta}^{\eta} e^{-\zeta^2} \psi_{\beta_i,p} \cdot \psi_{\beta_i,p}(\zeta) - e^{-\eta^2}\psi_{\beta_i,p} \cdot \psi_{\beta_i,p}(\eta) \, d\zeta \right|
\]
converges to zero a.e. as \(\beta_i \to 1\). In fact, by the Lebesgue dominated convergence theorem, \(I(\eta, \beta_i)\) converges to zero in \(L^p\). Indeed, \(|I(\eta, \beta_i)|\) can be bounded by the function \(2e^{-\beta_i^2\eta^2 - \alpha|\eta|}\) \(|\eta| \in L^p\), with \(a\) as in Proposition \(4.2\). We, therefore, have that the following functions are in \(L^p(\mathbb{R})\):
\[
F_p(\eta, \beta_i) = \frac{e^{-\beta_i^2\eta^2}\psi_{\beta_i,p}(\beta_i\eta) - e^{-\eta^2}\psi_{\beta_i,p}(\eta)}{(1 - \beta_i)|\eta|},
\]
and \(F_p(\eta, \beta_i) - 2e^{-\eta^2}(\psi_{\beta_i,p} \cdot \psi_{\beta_i,p})(\eta)\) converges to 0 in \(L^p(\mathbb{R})\) as \(\beta_i \to 1\).

We claim that the sequence \(F_p(\eta, \beta_i)\) converges to the weak derivative of the sequence \(\phi_{\beta_i,p}(\eta) = -e^{-\eta^2}\psi_{\beta_i,p}(\eta)\). Indeed, given \(v \in C_0^\infty(\mathbb{R})\)
\[
\int_{\mathbb{R}} F_p(\eta, \beta) v(\eta) \, d\eta = \int_{\mathbb{R}} \phi_{\beta,p}(\eta) \frac{v(\eta) - v\left(\frac{\eta}{\beta}\right)}{(1 - \beta)|\eta|} \, d\eta - \int_{\mathbb{R}} \phi_{\beta,p}(\eta) \frac{v(\eta)}{(1 - \beta)|\eta|} \, d\eta + \\
+ \int_{\mathbb{R}} \phi_{\beta,p}(\eta) \frac{v\left(\frac{\eta}{\beta}\right)}{(1 - \beta)|\eta|} \, d\eta.
\]
The last two integrals cancel after a change of variables. Therefore,
\[ \int_{\mathbb{R}} F_p(\eta, \beta) \nu(\eta) \, d\eta = - \int_{\mathbb{R}} \phi_{\beta,p}(\eta) \nu'(\eta) \, d\eta + \int_{\mathbb{R}} \phi_{\beta,p}(\eta) O((1 - \beta)\eta) \, d\eta. \]

Since \( \phi_{\beta,p} \) is exponentially bounded, the last integral is \( O(1 - \beta) \). Therefore, for any \( p \geq 1 \) the points \( \phi_{\beta,p} \) \( L^p \)-converge to a weak \( L^p \)-solution of (62)
\[ \hat{\nu}'(\eta) = 2\eta \hat{\nu}(\eta) - 2\partial \bullet \hat{\nu}(\eta). \]

Since any weak \( L^p \)-solution of equation (62) is necessarily differentiable, it is enough to consider only classical solutions of (62). This is a Riccati equation after the Laplace transform \( \hat{\nu}(s) = \mathcal{L}[\nu](s) \):
\[ \hat{\nu}'(s) = -\hat{\nu}(s)^2 - \frac{s}{2} \hat{\nu}(s) + \frac{\nu}{2}, \]
where \( \nu := \nu(0) = \psi_p(0) \). Equation (63) has a trivial solution when \( \nu = 0 \).
When \( \nu = 1 \), it has a solution \( \hat{\nu}(s) = 1/s \), corresponding to the solution \( \nu(\eta) = 1 \) of (62). The substitution \( \hat{\nu}(s) = v'(s)/v(s) \) reduces this to a second order linear differential equation, and a further change of variables \( v(s) = e^{-\frac{s^2}{4}} g(s) \), reduces it to a Kummer’s equation for the function \( u(s^2/4) = g(s) \):
\[ zu''(z) + \left( \frac{1}{2} - z \right) u'(z) - \frac{1 + \nu}{2} u(z) = 0, \]
with the general solution
\[ u(z) = c_1 M \left( s_1, \frac{1}{2}, z \right) + c_2 U \left( s_1, \frac{1}{2}, z \right), \]
where \( s_1 = (1 + \nu)/2, \) \( M \) is the confluent hypergeometric function, and \( U \) is Tricomi’s confluent hypergeometric function:
\[ U \left( s_1, \frac{1}{2}, z \right) = \frac{\sqrt{\pi}}{\Gamma(s_2)} M \left( s_1, \frac{1}{2}, z \right) - \frac{2\sqrt{\pi}}{\Gamma(s_1)} \sqrt{z} M \left( s_2, \frac{3}{2}, z \right), \]
where \( s_2 = (2 + \nu)/2 \). In particular,
\[ \hat{\nu}(s) = \frac{d}{ds} \ln \left( c_1 e^{-\frac{s^2}{4}} M \left( s_1, \frac{1}{2}, \frac{s^2}{4} \right) + c_2 e^{-\frac{s^2}{4}} U \left( s_1, \frac{1}{2}, \frac{s^2}{4} \right) \right) \]
is a one-parameter family of solutions of (63), the parameter being \( (c_1, c_2) \in \mathbb{R}P^1 \). Taking the inverse Laplace transform of (64) and equating its value at 0 with \( \nu(0) \), one obtains the the parameter \( (c_1, c_2) \).

We consider solutions for \( \nu < 1 \).
\[ \hat{\nu}(s)(s) = \frac{d}{ds} \ln \left( c_1 e^{-\frac{s^2}{4}} M \left( s_1, \frac{1}{2}, \frac{s^2}{2} \right) + c_2 e^{-\frac{s^2}{4}} U \left( s_1, \frac{1}{2}, \frac{s^2}{2} \right) \right). \]
Since the asymptotics of function \( M \) is
\[ M(a, b, z) \sim \Gamma(b) \left( \frac{e^{\frac{a-b}{2}}}{\Gamma(a)} + \frac{(-z)^{-a}}{\Gamma(b-a)} \right), \]
we get that for large positive \( s \),
\[ e^{-\frac{s^2}{4}} M \left( \frac{1 + \nu}{2}, \frac{1}{2}, \frac{s^2}{2} \right) \sim \text{const} \ s^\nu, \]
and similarly for $U$. Therefore, the asymptotics of $\hat{\vartheta}(s)$ is $\nu/s$, which is bounded for large $s$ from above by $1/(s + a)$ for some positive $a$, and $\hat{\vartheta}$ is indeed a Laplace transform of an exponentially bounded function.

We conclude that

$$\vartheta_\nu(\eta) = \mathcal{L}^{-1}[\hat{\vartheta}](\eta),$$

where $\mathcal{L}^{-1}$ is the inverse Laplace transform, is a family of solutions of equation (62) such that $\vartheta_\nu \in \mathcal{X}_p$ for some $\nu \in (0, 1)$.

This concludes the proof of the Main Theorem.

We emphasize, however, that the computation of the inverse Laplace transform for the function (65) is a non-trivial undertaking which cannot be performed in quadratures.

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