On the First Hitting Time Density of an Ornstein-Uhlenbeck Process

Alexander Lipton∗, Vadim Kaushansky†‡

Abstract

In this paper, we study the classical problem of the first passage hitting density of an Ornstein–Uhlenbeck process. We give two complementary (forward and backward) formulations of this problem and provide semi-analytical solutions for both. The corresponding problems are comparable in complexity. By using the method of heat potentials, we show how to reduce these problems to linear Volterra integral equations of the second kind. For small values of $t$, we solve these equations analytically by using Abel equation approximation; for larger $t$ we solve them numerically. We also provide a comparison with other known methods for finding the hitting density of interest, and argue that our method has considerable advantages and provides additional valuable insights.

1 Introduction

Computation of the first hitting time density of an Ornstein-Uhlenbeck process is a longstanding problem, which still remains open. An abstract approach applicable to this problem has been found by Fortet (1943); the Fortet’s equation can be viewed as a variant of the Einstein-Smoluchowski equation (Einstein (1905) and Von Smoluchowski (1906)). A general overview can be found in Borodin and Salminen (2012); Breiman (1967); Horowitz (1985). Attempts to find an analytical result have been made since 1998 when Leblanc and Scaillet (1998) first derived an analytical formula which contained a mistake. Two years later, Leblanc et al. (2000) published a correction on the paper; unfortunately, the correction was erroneous as well. Göing-Jaeschke and Yor (2003) noticed that the authors had incorrectly used a spatial homogeneity property for the three-dimensional Bessel bridge.

Alili et al. (2005), Linetsky (2004), and Ricciardi and Sato (1988) found representations for the hitting density by using the Laplace transform. Alili et al. (2005) gave several representations: a representation in the series of parabolic cylinder functions and its derivatives, an indefinite integral representation via special functions, and a Bessel bridge representation. The first approach is based on inverting the Laplace transform, which is computed analytically. As a result, the authors got the series representation

∗Massachusetts Institute of Technology, Connection Science, Cambridge, MA, USA and Ecole Polytechnique Federale de Lausanne, Switzerland, E-mail: alexlipt@mit.edu
†Mathematical Institute & Oxford-Man Institute, University of Oxford, UK, E-mail: vadim.kaushansky@maths.ox.ac.uk
‡The second author gratefully acknowledges support from the Economic and Social Research Council and Bank of America Merrill Lynch
involving parabolic cylinder functions and its derivatives. The second approach is based on the cosine transform and its inverse. The authors got a representation via an indefinite integral involving some special functions and computed it using the trapezoidal rule. The third approach gives a representation of the density via an expectation of a function of the three-dimensional Bessel bridge, which can be computed using the Monte Carlo method. Linetsky (2004) gave an analytical representation via relevant Sturm–Liouville eigenfunction expansions. The coefficients were found as a solution of a nonlinear equation, which involved nonlinear special (Hermite) functions.

Martin et al. (2015) solved a nonlinear Fokker–Planck equation with steady-state solution by representing it as an infinite product rather than – as usual – an infinite sum. The PDE, which corresponds to the transition probability of Ornstein-Uhlenbeck process, belongs to the class of equations described in Martin et al. (2015). In principle, the results of the paper can be modified to the computation of first time hitting density. An advantage of this method is that it allows quantifying the errors; thereby controlling the number of terms required to reach a given precision.

As we show later, under a suitable change of variables, the hitting problem of an OU process becomes the problem of hitting the square-root boundary of a Brownian motion. Novikov (1971) and Shepp (1967) derived the density and moments expansion for this problem; Novikov et al. (1999), Daniels (1996), and Pötzelberger and Wang (2001) developed numerical methods for general curved boundaries. Hyer et al. (1999) and Avellaneda and Zhu (2001) analyzed the problem with curvilinear boundaries in finance.

The hitting density of an OU process has many applications in applied mathematics, especially in mathematical finance (Martin et al. (2015) for the design of trading strategies; Leblanc and Scaillet (1998) for pricing path-dependent options on yields; Jeanblanc and Rutkowski (2000), Collin-Dufresne and Goldstein (2001), Coculescu et al. (2008), Yi (2010) for credit risk modeling; and Cheridito and Xu (2015) for CoCo bonds pricing). It has also applications in quantitative biology (Smith (1991)), where the hitting time is used for modeling the time between rings of a nerve cell.

The methods described above require substantial numerical computations, and for some of them the convergence rate is unknown. Moreover, most of them are difficult to implement; for example, Cheridito and Xu (2015) preferred the Crank–Nicolson method to other known analytical methods, because it is easier to implement.

In this paper, we develop a fast semi-analytical method to compute the first time hitting density of an Ornstein-Uhlenbeck process, which is easy to implement. After an appropriate change of variables, the corresponding PDE becomes a heat equation with a moving boundary. We solve it using the method of heat potentials (Lipton (2001), Section 12.2.3, pp. 462–467). It leads to a Volterra equation of the second kind, which we solve numerically similar to Lipton et al. (2018). As a result, we got recursive formulas to compute the corresponding hitting density.

The rest of the paper is organized as follows: in Section 2 we formulate the problem and eliminate parameters using a change of variables; in Section 3 we solve the problem analytically for a special case; in Section 4 we derive the solution in terms of a Volterra equation using the method of heat potentials; in Section 5 we consider a numerical solution for the corresponding Volterra equation as well as a solution as an approximation by an Abel equation; in Section 6 we show numerical illustrations and compare the methods; in Section 7 we conclude.
2 Problem formulation and initial transformations

Consider a typical OU process

\[
\begin{align*}
    dX_t &= \lambda (\theta - X_t) \, dt + \sigma dW_t, \\
    X_0 &= z.
\end{align*}
\]

We wish to calculate the stopping time \( s = \inf \{ t : X_t \leq b \} \), where \( z > b \). We introduce new variables

\[
\begin{align*}
    \bar{t} &= \lambda t, \quad \bar{X} = \frac{\sqrt{\lambda}}{\sigma} (X - \theta), \quad \bar{z} = \frac{\sqrt{\lambda}}{\sigma} (z - \theta), \quad \bar{b} = \frac{\sqrt{\lambda}}{\sigma} (b - \theta),
\end{align*}
\]

and rewrite the problem as follows

\[
\begin{align*}
    dX_t &= -X_t \, dt + dW_t, \\
    X_0 &= z, \\
    s &= \inf \{ t : X_t \leq b \},
\end{align*}
\]

where bars are omitted for brevity.

In this formulation we consider the hitting from above problem. But, by symmetry, one can also consider the hitting from below. It can be formulated as the hitting problem from above with the parameters \( \bar{z} = -z \) and \( \bar{b} = -b \).

2.1 Forward problem

To calculate the density of the hitting time distribution \( g(t, z) \), we need to solve the following forward problem

\[
\begin{align*}
    p_t (t, x; z) &= p(t, x; z) + xp_x (t, x; z) + \frac{1}{2} p_{xx} (t, x; z), \\
    p(0, x; z) &= \delta (x - z), \\
    p(t, b; z) &= 0.
\end{align*}
\]

This distribution is given by

\[
\begin{align*}
    g(t, z) &= \frac{1}{2} p_x (t, b; z).
\end{align*}
\]

2.2 Backward problem

Alternatively, we can solve the corresponding backward problem for the cumulative hitting probability \( G(t, z) \):

\[
\begin{align*}
    G_t (t, z) &= -z G_z (t, z) + \frac{1}{2} G_{zz} (t, z), \\
    G(0, z) &= 0, \\
    G(t, b) &= 1.
\end{align*}
\]

Then, the first hitting density is

\[
\begin{align*}
    g(t, z) &= G_t (t, z).
\end{align*}
\]

It is clear that problem (3) is easier to solve numerically than problem (1), whilst their analytical solutions are comparable in complexity.
3 Special case \( b = 0 \)

Before solving the general problem via the method of heat potentials, let us consider a special case of \( b = 0 \). It is well known that Green’s function for the OU process in question has the form

\[
H(t, x; 0, z) = \frac{\exp \left( - (e^t x - z)^2 / 2\eta(t) + t \right)}{\sqrt{2\pi\eta(t)}},
\]

where

\[
\eta(t) = \frac{e^{2t} - 1}{2} = e^t \sinh(t).
\]

It is clear that for this case in question, the method of images works. Indeed, \( H(t, x; 0, -z) \) solves the equation in question, and so does the difference

\[
H_0(t, x; 0, 0) = H(t, x; 0, z) - H(t, x; 0, -z).
\]

It is easy to check that

\[
H_0(0, x; 0, z) = \delta(x - z),
\]

and

\[
H_0(0, 0; 0, z) = \frac{\exp \left( - z^2 / 2\eta(t) + t \right)}{\sqrt{2\pi\eta(t)}} - \frac{\exp \left( - z^2 / 2\eta(t) + t \right)}{\sqrt{2\pi\eta(t)}} = 0.
\]

Hence

\[
p(t, x) = H_0(t, x; 0, z).
\]

The corresponding hitting time density \( q(t) \) is given by

\[
q(t) = \frac{1}{2} \left[ \frac{- e^t (e^t x - z)}{\eta(t)} \exp \left( - (e^t x - z)^2 / 2\eta(t) + t \right) \right]_{x=0} + \frac{e^t (e^t x + z) \exp \left( - (e^t x + z)^2 / 2\eta(t) + t \right)}{\eta(t) \sqrt{2\pi\eta(t)}} \bigg|_{x=0}.
\]

\[
q(t) = \frac{e^t z}{\eta(t)} \exp \left( - z^2 / 2\eta(t) + t \right) \frac{1}{\sqrt{2\pi\eta(t)}} = \frac{2 e^t z}{\eta(t) \sqrt{2\pi\eta(t)}}.
\]

4 General case

4.1 Forward problem

Now, we consider the general case. We wish to transform IBVP (1) into the standard IBVP for a heat equation with a moving boundary. To this end, we introduce new independent and dependent variables as follows:

\[
q(\tau, \xi) = e^{-t} p(t, x), \quad \tau = \eta(t), \quad \xi = e^t x.
\]
Given that

\[
\begin{align*}
\frac{\partial}{\partial t} &= (2\tau + 1) \frac{\partial}{\partial \tau} + \xi \frac{\partial}{\partial \xi}, \\
x \frac{\partial}{\partial x} &= x\sqrt{2\tau + 1} \frac{\partial}{\partial \xi} = \xi \frac{\partial}{\partial \xi}, \\
\frac{\partial^2}{\partial x^2} &= (2\tau + 1) \frac{\partial^2}{\partial \xi^2},
\end{align*}
\]

we get

\[
\begin{align*}
q_r (\tau, \xi) &= \frac{1}{2} q_{\xi \xi} (\tau, \xi), \\
q (0, \xi) &= \delta (\xi - z), \\
q (\tau, b (\tau)) &= 0, \\
b (\tau) &= \sqrt{2\tau + 1} b.
\end{align*}
\]

It is clear that \(0 \leq \tau < \infty\) and \(e^t = \sqrt{2\tau + 1}\). We show the moving boundary for different values of \(b\) in Figure 1.

IBVP (5) can be solved via the method of heat potentials (see Tikhonov and Samarskii (1963), pp. 530-535; Lipton (2001), Section 12.2.3, pp. 462-467 for details). As usual, we write

\[
q (\tau, \xi) = H (\tau, \xi, 0, z) + \tilde{q} (\tau, \xi),
\]

where \(H (\tau, \xi, 0, z)\) is the standard heat kernel, while \(\tilde{q} (\tau, \xi)\) solves the IBVP of the form

\[
\begin{align*}
\tilde{q}_r (\tau, \xi) &= \frac{1}{2} \tilde{q}_{\xi \xi} (\tau, \xi), \\
\tilde{q} (0, \xi) &= 0, \\
\tilde{q} (\tau, b (\tau)) &= -H (\tau, b (\tau), 0, z).
\end{align*}
\]
Accordingly,
\[ \tilde{q}(\tau, \xi) = \int_0^\tau \frac{(\xi - b(\tau')) \exp\left(\frac{-(\xi-b(\tau'))^2}{2(\tau-\tau')^3}\right)}{\sqrt{2\pi (\tau-\tau')^3}} \nu^f(\tau') d\tau', \]
where \( \nu^f \) is the solution of the Volterra equation of the second kind,
\[ \nu^f(\tau) + \int_0^\tau \frac{(b(\tau) - b(\tau')) \exp\left(\frac{-(b(\tau)-b(\tau'))^2}{2(\tau-\tau')^3}\right) \nu^f(\tau')}{\sqrt{2\pi (\tau-\tau')^3}} d\tau' + \frac{\exp\left(\frac{-(b(\tau)-z)^2}{2\tau}\right)}{\sqrt{2\pi \tau}} = 0. \tag{6} \]

Now
\[ \frac{b(\tau) - b(\tau')}{\tau - \tau'} = \frac{2b}{\sqrt{2\tau + 1 + \sqrt{2}\tau + 1}}, \]
so that (6) can be written in the form
\[ \nu^f(\tau) + \sqrt{\frac{2}{\pi}} \int_0^\tau \frac{\exp\left(-b^2 \left(\frac{2\tau+1-\sqrt{2}\tau+1}{\sqrt{2}\tau+1+\sqrt{2}\tau+1}\right)\right) \nu^f(\tau')}{(\sqrt{2\tau + 1 + \sqrt{2}\tau + 1}) \sqrt{\tau - \tau'}} d\tau' + \frac{\exp\left(-\frac{(\sqrt{2}\tau+1-b-z)^2}{2\tau}\right)}{\sqrt{2\pi \tau}} = 0. \tag{7} \]

and is easy to solve numerically.

Assuming that \( \nu^f(\tau) \) is found, we can proceed as follows:
\[ p(t, x) = \frac{\exp\left(-\frac{(e^tx - z)^2}{(e^{2t} - 1) + t}\right)}{\sqrt{\pi (e^{2t} - 1)}} + e^t \tilde{q}(\tau, \xi), \tag{8} \]

Then, \( g(t) \) can be computed using (2) by differentiation of (8) and taking its limit at \( b \).

We do the necessary computations in Appendix A and write the final formula below:
\[ g(t) = \frac{1}{2} p_x(t, b) \tag{9} \]
\[ = -\frac{(e^t b - z) \exp\left(-\frac{(e^t b - z)^2}{(e^{2t} - 1)} + 2t\right)}{\sqrt{\pi (e^{2t} - 1)^3}} \left( e^t b + \frac{e^{2t}}{\sqrt{\pi (e^{2t} - 1)}} \right) \nu^f(t) \]
\[ + \frac{e^{2t}}{\sqrt{8\pi}} \int_0^\tau \frac{1 - 2b^2 \left(\frac{\sqrt{2\tau+1-\sqrt{2}\tau+1}}{\sqrt{2}\tau+1+\sqrt{2}\tau+1}\right) \exp\left(-b^2 \left(\frac{2\tau+1-\sqrt{2}\tau+1}{\sqrt{2}\tau+1+\sqrt{2}\tau+1}\right)\right) \nu^f(\tau') - \nu^f(\tau)}{\sqrt{(\tau - \tau')^3}} d\tau'. \]

We can rewrite (7) in an alternative way. Let \( \theta = \sqrt{2\tau + 1 - 1}, \theta' = \sqrt{2\tau' + 1 - 1}, 0 \leq \theta' \leq \theta < \infty \). Then
\[ \nu^f(\theta) + \frac{2b}{\sqrt{\pi}} \int_0^\theta \frac{\exp\left(-b^2 \left(\frac{\theta - \theta'}{2 + \theta + \theta'}\right) (1 + \theta') \nu^f(\theta')}{\sqrt{(2 + \theta + \theta')^3 (\theta - \theta')}} d\theta' + \frac{\exp\left(-\frac{(1 + \theta b - z)^2}{(1 + \theta)^2 - 1}\right)}{\sqrt{\pi ((1 + \theta)^2 - 1)}} = 0. \tag{10} \]
Symbolically,
\[ \nu^f (\theta) + \int_0^\theta \frac{\Phi'_b (\theta, \theta') \nu^f (\theta')}{\sqrt{\theta - \theta'}} d\theta' + \frac{\exp \left( - \frac{((1+\theta)b - z)^2}{((1+\theta)^2-1)} \right)}{\sqrt{\pi} ((1+\theta)^2 - 1)} = 0, \]

where
\[ \Phi'_b (\theta, \theta') = \frac{2b \exp \left( -b^2 \frac{\theta - \theta'}{2(1+\theta)} \right)}{\sqrt{\pi}} \left( 1 + \theta' \right). \]

Accordingly, (9) can be written in the form
\[
g(t) = - \frac{(e^tb - z) \exp \left( - \frac{(e^tb - z)^2}{(e^t - 1)} + 2t \right)}{\sqrt{\pi (e^{2t} - 1)^2}} - \left( e^tb + \frac{e^{2t}}{\sqrt{\pi (e^{2t} - 1)}} \right) \nu^f (t) \\
+ \frac{1}{\sqrt{\pi}} e^{2t} \int_0^\theta \left( 1 - 2b^2 \frac{(\theta - \theta')}{(2+\theta + \theta')} \right) \exp \left( -b^2 \frac{(\theta - \theta')}{(2+\theta + \theta')} \right) \nu^f (\theta') - \nu^f (\theta) \left( 1 + \theta' \right) \left( 1 + \theta' \right) \left( 1 + \theta' \right) d\theta'.
\]

### 4.2 Backward problem

#### 4.2.1 General case

By the same token as before, we introduce
\[ \lambda = \varpi (t) = \frac{1 - e^{-2t}}{2} = e^{-t} \sinh (t), \quad 0 \leq \lambda < \frac{1}{2}, \quad \mu = e^{-t}z, \quad (11) \]

notice that
\[
\begin{align*}
\frac{\partial}{\partial t} &= (1 - 2\lambda) \frac{\partial}{\partial \lambda} - \mu \frac{\partial}{\partial \mu}, \\
\frac{\partial}{\partial z} &= z \sqrt{1 - 2\lambda} \frac{\partial}{\partial \mu} = \mu \frac{\partial}{\partial \mu}, \\
\frac{\partial^2}{\partial z^2} &= (1 - 2\lambda) \frac{\partial^2}{\partial \mu^2},
\end{align*}
\]

and rewrite IBVP (3) as follows
\[
G_{\lambda} (\lambda, \mu) = \frac{1}{2} G_{\mu\mu} (\lambda, \mu), \\
G (0, \mu) = 0, \\
G (\lambda, b (\lambda)) = 1, \\
b (\lambda) = \sqrt{1 - 2\lambda}.
\]

It is clear that \( 0 \leq \lambda < 1/2 \), and \( e^{-t} = \sqrt{1 - 2\lambda} \). It is worth noting that for the backward problem the computational domain is compactified in the \( \lambda \) direction. We show the moving boundary for different values of \( b \) in Figure 2.
Accordingly,

\[ G(\lambda, \mu) = \int_{0}^{\lambda} \frac{(\mu - b(\lambda')) \exp \left( -\frac{(\mu - b(\lambda'))^2}{2(\lambda - \lambda')^2} \right) \nu^b(\lambda')}{\sqrt{2\pi (\lambda - \lambda')^3}} \, d\lambda', \quad (13) \]

\[ g(\lambda, \mu) = \left( (1 - 2\lambda) \frac{\partial}{\partial \lambda} - \mu \frac{\partial}{\partial \mu} \right) \int_{0}^{\lambda} \frac{(\mu - b(\lambda')) \exp \left( -\frac{(\mu - b(\lambda'))^2}{2(\lambda - \lambda')^2} \right) \nu^b(\lambda')}{\sqrt{2\pi (\lambda - \lambda')^3}} \, d\lambda' \quad (14) \]

\[ = \frac{(1 - 2\lambda)}{2} \int_{0}^{\lambda} \left( -3(\lambda - \lambda') + (\mu - b(\lambda'))^2 \right) (\mu - b(\lambda')) \]

\[ \times \frac{\exp \left( -\frac{(\mu - b(\lambda'))^2}{2(\lambda - \lambda')^2} \right)}{\sqrt{2\pi (\lambda - \lambda')^3}} \nu^b(\lambda') \, d\lambda' \]

\[ + \mu \int_{0}^{\lambda} \left( - (\lambda - \lambda') + (\mu - b(\lambda'))^2 \right) \frac{\exp \left( -\frac{(\mu - b(\lambda'))^2}{2(\lambda - \lambda')^2} \right)}{\sqrt{2\pi (\lambda - \lambda')^3}} \nu^b(\lambda') \, d\lambda'. \]

where \( \nu^b \) is the solution of the Volterra equation of the second kind,

\[ \nu^b(\lambda) + \int_{0}^{\lambda} \frac{(b(\lambda) - b(\lambda')) \exp \left( -\frac{(b(\lambda) - b(\lambda'))^2}{2(\lambda - \lambda')^2} \right) \nu^b(\lambda')}{\sqrt{2\pi (\lambda - \lambda')^3}} \, d\lambda' - 1 = 0. \]
Since
\[
\frac{b(\lambda) - b(\lambda')}{\lambda - \lambda'} = \frac{b\left(\sqrt{1-2\lambda} - \sqrt{1-2\lambda'}\right)}{\lambda - \lambda'} = -\frac{2b}{\sqrt{1-2\lambda} + \sqrt{1-2\lambda'}},
\]
so that (15) can be written in the form
\[
\nu^b(\lambda) = \sqrt{\frac{2b}{\pi}} \int_0^\lambda \frac{\exp\left(-b^2\left(\frac{1}{\sqrt{1-2\lambda'}} - \frac{1}{\sqrt{1-2\lambda}}\right)\nu^b(\lambda')\right)}{\sqrt{1-2\lambda'} + \sqrt{1-2\lambda}} d\lambda' - 1 = 0. \tag{16}
\]

Once (16) is solved, \(g(\lambda, \mu)\) and \(g(t, z)\) can be calculated by virtue of (13) and (11) in a straightforward fashion.

We can rewrite (16) in an alternative way. Let \(\vartheta = 1 - \sqrt{1-2\lambda}, \vartheta' = 1 - \sqrt{1-2\lambda'},\)
\(0 \leq \vartheta' \leq \vartheta < 1.\) Then
\[
\nu^b(\vartheta) = \frac{2b}{\sqrt{\pi}} \int_0^\vartheta \frac{\exp\left(-b^2\frac{\vartheta(\vartheta - \vartheta')}{2(\vartheta - \vartheta')}\right)(1 - \vartheta') \nu^b(\vartheta')}{\sqrt{(2 - \vartheta - \vartheta')^3(2 - \vartheta - \vartheta')}} d\vartheta' - 1 = 0. \tag{17}
\]

Symbolically,
\[
\nu^b(\vartheta) = \int_0^\vartheta \frac{\Phi^b_b(\vartheta, \vartheta') \nu^b(\vartheta')}{\sqrt{\vartheta - \vartheta'}} d\vartheta' - 1 = 0,
\]
where
\[
\Phi^b_b(\vartheta, \vartheta') = \frac{2b}{\sqrt{\pi}} \exp\left(-b^2\frac{(\vartheta - \vartheta')}{2(\vartheta - \vartheta')}\right)(1 - \vartheta').
\]

As one would expect,
\[
\Phi^b_b(\vartheta, \vartheta') = -i\Phi^f_{\vartheta, \vartheta'}(-\vartheta, -\vartheta').
\]

As a result, (13), (14) become
\[
G(t, z) = 2\int_0^\vartheta \frac{\left(z(1 - \vartheta) - b(1 - \vartheta')\right) \exp\left(-\frac{(z(1 - \vartheta) - b(1 - \vartheta'))^2}{(\vartheta - \vartheta')(2 - \vartheta - \vartheta')}\right)(1 - \vartheta') \nu^b(\vartheta')}{\sqrt{\pi(\vartheta - \vartheta')^3(2 - \vartheta - \vartheta')^3}} d\vartheta',
\]
\[
g(t, z) = 4e^{-2t} \int_0^\vartheta \left(-3(\vartheta - \vartheta')(2 - \vartheta - \vartheta') + (z(1 - \vartheta) - b(1 - \vartheta'))^2\right)
\times (z(1 - \vartheta) - b(1 - \vartheta')) \exp\left(-\frac{(z(1 - \vartheta) - b(1 - \vartheta'))^2}{(\vartheta - \vartheta')(2 - \vartheta - \vartheta')}\right)(1 - \vartheta') \nu^b(\vartheta')
\times \frac{\sqrt{\pi(\vartheta - \vartheta')^7(2 - \vartheta - \vartheta')^7}}{\sqrt{\pi(\vartheta - \vartheta')^3(2 - \vartheta - \vartheta')^3}} d\vartheta',
\]
\[
\exp\left(-\frac{(z(1 - \vartheta) - b(1 - \vartheta'))^2}{(\vartheta - \vartheta')(2 - \vartheta - \vartheta')}\right)(1 - \vartheta') \nu^b(\vartheta')
\times \frac{\sqrt{\pi(\vartheta - \vartheta')^7(2 - \vartheta - \vartheta')^7}}{\sqrt{\pi(\vartheta - \vartheta')^3(2 - \vartheta - \vartheta')^3}} d\vartheta',
\]
where \(\vartheta = 1 - e^{-t}.\)
4.2.2 Special case

When \( b = 0 \), we have

\[ \nu^b (\lambda) = 1, \]

\[
G (\lambda, \mu) = \mu \int_0^\lambda \frac{\exp \left( -\frac{\mu^2}{2(\lambda - \lambda')} \right)}{\sqrt{2\pi (\lambda - \lambda')^3}} d\lambda' \\
= \mu \int_0^\lambda \frac{\exp \left( -\frac{\mu^2}{2\lambda} \right)}{\sqrt{2\pi \lambda^3}} d\lambda' \\
= 2\mu \int_0^\infty \frac{\exp \left( -\frac{\mu^2 u^2}{2} \right)}{\sqrt{2\pi}} du \\
= 2 \int_0^\infty \frac{\exp \left( -\frac{v^2}{2} \right)}{\sqrt{2\pi}} dv \\
= 2N \left( -\frac{\mu}{\sqrt{\lambda}} \right).
\]

Thus,

\[
g (\lambda, \mu) = \left( (1 - 2\lambda) \frac{\partial}{\partial \lambda} - \mu \frac{\partial}{\partial \mu} \right) G (\lambda, \mu) \\
= 2 \left( (1 - 2\lambda) \frac{\mu}{2\sqrt{\lambda}^3} + \frac{\mu}{\sqrt{\lambda}} \right) \exp \left( -\frac{\mu^2}{2\lambda} \right) \\
= \frac{\mu}{\sqrt{\lambda}^3} (1 - 2\lambda + 2\lambda) \exp \left( -\frac{\mu^2}{2\lambda} \right) \\
= \frac{\mu}{\sqrt{\lambda}^3} \exp \left( -\frac{\mu^2}{2\lambda} \right),
\]

\[
g (t, z) = \frac{z \exp \left( -\frac{e^{-t/2} z^2}{2 \sinh (t)} + \frac{1}{2} \right)}{\sqrt{2\pi (\sinh (t))^3}}.
\]

Alternatively,

\[
G (t, z) = 2N \left( -\frac{e^{-t/2} z}{\sqrt{\sinh (t)}} \right), \quad (18)
\]
and
\[
g(t, z) = G_t(t, z) = 2 \left( \frac{e^{-t/2}z + \cosh(t) e^{-t/2}z}{2} \right) \frac{\exp\left(-\frac{e^{-t/2}z}{2\sinh(t)}\right)}{\sqrt{2\pi}}
\]
\[
= \frac{e^{-t/2}z (\sinh(t) + \cosh(t)) \exp\left(-\frac{e^{-t/2}z}{2\sinh(t)}\right)}{\sqrt{(\sinh(t))^3}}
\]
\[
= \frac{z \exp\left(-\frac{e^{-t/2}z}{2\sinh(t)} + \frac{1}{2}\right)}{\sqrt{2\pi (\sinh(t))^3}},
\]
as before.

5 Numerical solution

In this section, we show how to solve the Volterra equations (10) and (17) numerically, and also derive an analytical approximation for small values of \( t \).

5.1 Numerical method

In this section, we briefly discuss two methods to solve the corresponding Volterra equations (10) and (17). We start with a simple trapezoidal method, and then consider a more advanced method based on a quadratic interpolation, which gives a better convergence rate. Both methods can be found in Linz (1985) (Section 8.2 and Section 8.4).

In this section we solve
\[
f(t) = g(t) + \int_0^t K(t, s) \sqrt{t - s} f(s) ds,
\]
where \( K(t, s) \) is a non-singular part of the kernel.

Both (10) and (17) can be formulated as (19) with an appropriate choice of \( K(t, s) \) and \( g(t) \).

For the forward equation we take (in \((\theta, \theta')\) variables)
\[
g(\theta) = -\frac{\exp\left(-\frac{(1+\theta)b-z^2}{2(1+\theta)^2-1}\right)}{\sqrt{\pi (1+\theta)^2 - 1}}, \quad K(\theta, \theta') = -\frac{2b}{\sqrt{\pi}} \frac{\exp\left(-b^2(\theta-\theta')\right) (1 + \theta')}{\sqrt{(2 + \theta + \theta')^3}},
\]
and for the backward equation we take (in \((\vartheta, \vartheta')\) variables)
\[
g(\vartheta) = 1, \quad K(\vartheta, \vartheta') = \frac{2b}{\sqrt{\pi}} \frac{\exp\left(-b^2(\vartheta-\vartheta')\right) (1 - \vartheta')}{\sqrt{(2 - \vartheta - \vartheta')^3}}.
\]
5.1.1 Simple trapezoidal method

Consider the integral in (19) separately

$$\int_0^t \frac{K(t, s) \nu(s)}{\sqrt{t-s}} \, ds = -2 \int_0^t K(t, s) \nu(s) \, d\sqrt{t-s}. \tag{20}$$

We consider a grid $0 = t_0 < t_1 < \ldots < t_N = T$, and denote $F_k$ for the approximated value of $f(t_k)$ and $\Delta_{k,i} = t_k - t_i$. Then, by trapezoidal rule of the Stieltjes integral (20), (19) can be approximated as

$$F_k = g(t_k) + \sum_{i=1}^k \left( K(t_k, t_i) F_i + K(t_k, t_{i-1}) F_{i-1} \right) \left( \sqrt{\Delta_{k,i-1}} - \sqrt{\Delta_{k,i}} \right) = 0.$$  

From the last equation we can express $F_k$

$$F_k = \left( 1 - K(t_k, t_k) \sqrt{\Delta_{k,k-1}} \right)^{-1} \times \left( g(t_k) + K(t_k, 0) \left( \sqrt{\Delta_{k,0}} - \Delta_{k,1} \right) + \sum_{i=1}^{k-1} K(t_k, t_i) \left( \sqrt{\Delta_{k,i-1}} - \Delta_{k,i+1} \right) F_i \right).$$

Taking $F_0 = g(0)$, we can recursively compute $F_m$ using the previous values $F_0, \ldots, F_{m-1}$.

The approximation error of the integrals is of order $O(\Delta^2)$, where $\Delta = \max_{k,i} \sqrt{\Delta_{k,i-1}} - \sqrt{\Delta_{k,i+1}}$ is the step size. Hence, on the uniform grid $t_i = i \Delta$, the convergence rate is of order $O(h)$.

5.1.2 Block-by-block method based on quadratic interpolation

Now we consider a block-by-block method based on quadratic interpolation from [Linz 1985] (Section 8.4).

Using piece-wise quadratic interpolation, Linz (1985) derived

$$F_{2m+1} = g(t_{2m+1}) + (1 - \delta_{m0}) \sum_{i=0}^{2m} w_{2m+1,i} K(t_{2m+1}, t_i) F_i$$

$$+ \alpha \left( t_{2m+1}, t_{2m}, \frac{h}{2} \right) K(t_{2m+1}, t_{2m}) F_{2m}$$

$$+ \beta \left( t_{2m+1}, t_{2m}, \frac{h}{2} \right) K(t_{2m+1}, t_{2m+1/2}) \left( \frac{3}{8} F_{2m} + \frac{3}{4} F_{2m+1} - \frac{1}{8} F_{2m+2} \right)$$

$$+ \gamma \left( t_{2m+1}, t_{2m}, \frac{h}{2} \right) K(t_{2m+1}, t_{2m+1}) F_{2m+1}, \tag{21}$$

and

$$F_{2m+2} = g(t_{2m+2}) + (1 - \delta_{m0}) \sum_{i=0}^{2m} w_{2m+2,i} K(t_{2m+2}, t_i) F_i$$

$$+ \alpha \left( t_{2m+2}, t_{2m}, h \right) K(t_{2m+2}, t_{2m}) F_{2m}$$

$$+ \beta \left( t_{2m+2}, t_{2m}, h \right) K(t_{2m+2}, t_{2m+1}) F_{2m+1}$$

$$+ \gamma \left( t_{2m+2}, t_{2m}, h \right) K(t_{2m+2}, t_{2m+2}) F_{2m+2}, \tag{22}$$
where
\[ w_{n,i} = (1 - \delta_{i,n-1})\alpha(t_n, t_i, h) + (1 - \delta_{i,0})\gamma(t_n, t_i - 2h, h), \quad i \text{ is even,} \]
\[ w_{n,i} = \beta(t_n, t_i - h, h), \quad i \text{ is odd,} \]
and \( \delta_{ij} \) is a Kronecker delta.

The functions \( \alpha, \beta, \) and \( \gamma \) are defined by
\[
\alpha(x, y, z) = \frac{z}{2} \int_0^2 \frac{(1 - s)(2 - s)}{\sqrt{x - y - sz}} ds,
\]
\[
\beta(x, y, z) = \frac{z}{2} \int_0^2 \frac{s(2 - s)}{\sqrt{x - y - sz}} ds,
\]
\[
\gamma(x, y, z) = \frac{z}{2} \int_0^2 \frac{s(s - 1)}{\sqrt{x - y - sz}} ds.
\]

We note that \( \alpha, \beta, \) and \( \gamma \) depend only on \( x - y, z \), and can be written as a function of two variables. Moreover, these integrals can be computed analytically.

Then, one can find \([F_{2m+1}, F_{2m+2}]\) by solving the system of two linear equations (21) and (22). We present the numerical algorithm in Algorithm 1.

**Algorithm 1** Block-by-block method based on quadratic interpolation

**Require:** \( N \) — number of time steps: \( 0 = t_0 < t_1 < t_2 < \ldots < t_N = T \)
1: \( F_0 = f(0) \)
2: for \( m = 0 : N/2 \) do
3: \hspace{1cm} Compute \( w_{2m+1,i} \) for \( i = 0, \ldots, 2m \)
4: \hspace{1cm} Compute \( w_{2m+2,i} \) for \( i = 0, \ldots, 2m \)
5: \hspace{1cm} Get \( [F_{2m+1}, F_{2m+2}] \) by solving (21) and (22)
6: end for

Under some assumptions on regularity, the convergence rate of this method is 3.5. In our case, for the backward equation, these assumptions are not satisfied, and we empirically confirm the convergence of order 1.5.

### 5.2 Approximation by an Abel integral equation

For small values of \( \theta \), (7) can be approximated by an Abel integral equation of the second kind.
\[
\nu^f(\theta) + \frac{b}{\sqrt{2\pi}} \int_0^\theta \frac{1}{\sqrt{\theta - \theta'}} \nu^f(\theta') d\theta' + \frac{\exp\left(-\frac{(b-z)^2}{2\theta}\right)}{\sqrt{2\pi\theta}} = 0. \tag{23}
\]

The last equation is an Abel equation of the second kind and can be solved analytically using direct and inverse Laplace transforms (Abramowitz and Stegun (1964)).

The Laplace transform yields
\[
\tilde{\nu}^f(\Lambda) + b\tilde{\nu}^f(\Lambda) + \frac{e^{-\sqrt{2\Lambda}(z-b)}}{\sqrt{2\Lambda}} = 0.
\]

Then, \( \tilde{\nu}^f(\Lambda) \) can be expressed as
\[
\tilde{\nu}^f(\Lambda) = -\frac{e^{-\sqrt{2\Lambda}(z-b)}}{\sqrt{2\Lambda} + b}.
\]
Taking inverse Laplace transform, we get the final expression for \( \nu^f(\theta) \)

\[
\nu^f(\theta) = be^{\frac{b^2}{2} \theta} + b(z-b)N\left(-\frac{b\theta + z - b}{\sqrt{\theta}}\right) - \exp\left(-\frac{(b-z)^2}{2\theta}\right)\sqrt{2\pi\theta},
\]

(24)

where \( N(x) \) is the CDF of the standard normal distribution.

Now consider the backward equation. For small values of \( \vartheta \), (17) can be approximated by

\[
\nu^b(\vartheta) - \frac{b}{\sqrt{2\pi}} \int_{0}^{\vartheta} \frac{1}{\sqrt{\vartheta - \vartheta'}} \nu^b(\vartheta') d\vartheta' - 1 = 0.
\]

(25)

Similar to the forward equation, we solve it by taking direct and inverse Laplace transforms. The direct Laplace transform yields

\[
\bar{\nu}^b(\Lambda) - b\frac{\bar{\nu}^b(\Lambda)}{\sqrt{2\Lambda}} - \frac{1}{\Lambda} = 0.
\]

Hence,

\[
\nu^b(\Lambda) = \frac{1}{\sqrt{\Lambda}(\sqrt{\Lambda} - b/\sqrt{2})}.
\]

Taking the inverse Laplace transform, we get

\[
\nu^b(\vartheta) = 2e^{\frac{b^2}{2} \vartheta} N(b\sqrt{\vartheta}).
\]

(26)

Alternatively, one can find an analytical solution using the following results (Polyanin and Manzhirov (1998)). The solution of Abel equation

\[
y(t) + \xi \int_{0}^{t} \frac{y(s)ds}{\sqrt{t-s}} = f(t).
\]

has the form

\[
y(t) = F(t) + \pi\xi^2 \int_{0}^{t} \exp[\pi\xi^2(t-s)]F(s) ds,
\]

where

\[
F(t) = f(t) - \xi \int_{0}^{t} \frac{f(s) ds}{\sqrt{t-s}}.
\]

6 Numerical examples

6.1 Numerical tests

Consider \( T = 2 \) and \( z = 2 \). In Figure 3 we examine the density and cumulative density functions for different values of \( b \) and compare them with the analytical solution (4) for \( b = 0 \). In Figure 4 we show \( G(t, z) \) as a function of both \( t \) and \( z \) for \( b = 1 \). In Figure 5 we present \( \nu^f(\tau) \) and in Figure 6 we present \( \nu^b(\vartheta) \) for various values of \( b \).

To test how good the Abel equation approximates the corresponding Volterra equation, we plot them together for small values of \( t \). In Figure 7 we compare the corresponding forward equations and in Figure 8 we compare the corresponding backward equations. We can see that up to \( t = 0.02 \) the solutions are visually indistinguishable.
Figure 3: (a) Density function $g(t)$ (b) Cumulative density function $G(t)$.

Figure 4: $G(t, z)$ as a function of both $t$ and $z$ for $b = 1$.

Figure 5: $\nu^f(\theta)$, the solution of (10) with $z = 2$ (a) as a function of $\theta$ and $b$ (b) as a function of $\theta$ for different values of $b$. 
Figure 6: \( \nu^b(\vartheta) \), the solution of (17) (a) as a function of \( \vartheta \) and \( b \) (b) as a function of \( \vartheta \) for different values of \( b \).

Figure 7: Comparison between the approximation by the Abel equation and the numerical solution of the forward equation \( \nu^f(t) \) in \( t \)-coordinates (a) \( b = -0.5, z = -0.25 \) (b) \( b = 0.5, z = 1 \).
We also empirically test the convergence rate of our numerical method by implementing it for the forward and backward Abel equations (23) and (25) and comparing them with the analytical solutions (24) and (26). In Figure 9a we get the order of convergence 3.2 for the quadratic interpolation method and order 1 for the trapezoidal method for the forward equation. It confirms our estimate in Section 5.1. In Figure 9b we get the order 1.5 for the quadratic interpolation method and the order 1 for the trapezoidal method for the backward equation. The convergence order of the quadratic interpolation method for the backward equation is smaller than the theoretical estimate, because the assumptions on the regularity of the solution are not satisfied. Potentially, a non-uniform grid improves the convergence order.

6.2 Comparison with other methods

6.2.1 Description of other methods

We compare our method with other available alternatives.

1. Leblanc et al. (2000) method. We used (3) in Leblanc et al. (2000) to compute the hitting density. In our notation it is

\[ g(t) = \left( z - b \right) \exp \left( \frac{z^2 - b^2 + t - (z-b)^2 \coth t}{2} \right) \]

\[ \quad \cdot \frac{1}{\sqrt{2\pi (\sinh t)^3}} \]

\[ = \left( z - b \right) \exp \left( b(z-b) - \frac{e^{-t}(z-b)^2}{2 \sinh(t)} + \frac{t}{2} \right) \]

\[ \cdot \frac{1}{\sqrt{2\pi (\sinh t)^3}}. \]

Integrating \( g(t) \), we get

\[ G(t) = 2e^{b(z-b)} N \left( -\frac{(z-b)e^{-t/2}}{\sqrt{\sinh t}} \right). \]
2. Alili et al. (2005), Linetsky (2004), and Ricciardi and Sato (1988) method. The method is based on the inversion of the Laplace transform of the hitting density. In our notation, the Laplace transform $u(\Lambda)$ has the form

$$u(\Lambda) = e^{z^2 - b^2 / 2} \frac{D_n(z \sqrt{2})}{D_{-n}(b \sqrt{2})},$$

(27)

where $D_{\nu}(x)$ is the parabolic cylinder function. Alili et al. (2005) found a representation of the inverse Laplace transform as a series of parabolic cylinder function and its derivatives

$$g(t) = e^{(z^2 - b^2)/2} \sum_{j=1}^{\infty} \frac{D_{\nu_j,b\sqrt{2}}(z \sqrt{2})}{D'_{\nu_j,b\sqrt{2}}(b \sqrt{2})} \exp(\nu_j,b\sqrt{2}),$$

where $D'_{\nu}(x) = \frac{\partial D_{\nu}(x)}{\partial \nu}$ and $\nu_{j,b}$ the ordered sequence of positive zeros of $\nu \to D_{\nu}(x)$. However, we used Gaver-Stehfest algorithm (Abate et al. (2000)) for the inversion, as it gives more stable and robust results. Moreover, the representation from Alili et al. (2005) works only for $t > t_0$ for some $t_0$, while Gaver-Stehfest algorithm works for all positive $t$.

Using this method, the density can be expressed as

$$g(t) \approx \frac{\ln 2}{t} \sum_{k=1}^{2m} \omega_k u \left( \frac{k \ln 2}{t} \right),$$

where

$$\omega_k = (-1)^{m+k} \sum_{j=\lfloor(k+1)/2\rfloor}^{\min(k,m)+1} \frac{j^{m+1}}{m} C_m^j C_2^j C_{k-j}^j.$$

As we can see, the method only requires the values of $u$ on the positive real semi-axis, and from (27) one can observe that $u(\Lambda)$ is non-singular on $\mathbb{R}_+$. As an example, in Figure 10 we plot $u(\Lambda)$ for $\Lambda > 0$ with $z = 2$ and various values of $b$. 

Figure 9: Empirical estimation of the numerical methods from Section 5.1: (a) forward equation, (b) backward equation.
As an example of using Gaver-Stehfest algorithm in finance, one can refer to Lipton and Sepp (2011), where the authors used it for the calibration of a local volatility surface.

3. Crank–Nicolson method. We solved (3) using Crank–Nicolson numerical scheme (Duffy (2013)).

6.2.2 Comparison

We start with a comparison with Leblanc et al. (2000) method to show that it gives wrong results. In Figure 11 we compare two methods for different parameters. We also give the analytical solution, when it is available for $b = 0$. We can observe that only for $b = 0$ the Leblanc et al. (2000) method gives correct results, while for other parameters it totally differs from our method. Moreover in Figure 11c we can see that it gives $G(t) > 1$.

A closed-form solution is only available when $b = 0$ and is given by (4). We take other parameters as before $T = 2$ and $z = 2$. We compare our method (forward and backward), the Crank-Nicolson method, and the method based on the inversion of the Laplace transform in Figure 12 and Figure 13. We note that our method has an advantage for this case, because Volterra equations (7) and (17) become trivial for $b = 0$. Nevertheless, the comparison is still very useful.

We take $N = 500$ for both forward and backward methods, $h = k = 0.005$ for the Crank-Nicolson method, and $m = 24$ for the Gaver-Stehfest algorithm. We used mpmath library in Python (Johansson et al. (2013)) for the implementation of the Gaver-Stehfest algorithm and parabolic cylinder functions with arbitrary precision arithmetics. In our computations we used 100 digits precision.

From these graphs we observe that the methods developed in this paper and the method based on the inversion of the Laplace transform (Alili et al. (2005)) significantly outperform the Crank-Nicolson scheme.

In order to compare the solutions for non-trivial parameters, we take the Laplace transform method with a maximum possible precision, and our method for $N = 100$ and $N = 10000$. The first example shows how our results can be comparable to the Laplace
Figure 11: Comparison with Leblanc et al. (2000) method: (a) $z = 2, b = -1$, (b) $z = 2, b = 0$, (c) $z = 2, b = -1$.

Figure 12: The difference between numerical and analytical solutions for $b = 0$ (a) Forward method (b) Backward method.
transform method for a relatively small value of $N$, when the computations can be done very fast; the second example demonstrates that our method can obtain a very good precision for a large $N$. The results are presented in Figure 14.

6.3 Asymptotic behavior when $t \to \infty$

It is clear that for $b = 0$, and hence all $b \geq 0$, $G(t, z) \to 1$, when $t \to \infty$. One can easily verify it by taking the limit in (18). But it is unclear what happens for a negative $b$. We empirically investigate it by plotting $G(t, z)$ as a function of $b$ for a fixed large value of $T$ and fixed $z$. In Figure 15a we take $z = 2$ and $T = 500$ and compute $G(T, z)$ for $b \in [-5, 2]$. We can see that it still remains close to 1 up to $b = -2$, and then rapidly approaches zero. In further research we want to explore the asymptotic behavior of $G(t)$ in more details. In Figure 15b we show the expected hitting time; this quantity is very important for the design of mean-reverting trading strategies since it allows a trader to decide when to go in and out of the trade.

7 Conclusion

We developed a semi-analytical approach to finding the first time hitting density of an Ornstein-Uhlenbeck process. We transformed our problem to a free-boundary value problem; using the method of heat potentials, we derived the corresponding expression for the density via a weight function, which is a solution of a Volterra equation of the second kind. For small values of $t$ we solved this equation analytically by using Abel equation approximation, while for $t$ which are not small, we developed a numerical recursive scheme. We showed the third order of convergence for the forward scheme and the order 1.5 for the backward scheme and confirmed it numerically.

We compared our method to several other methods known in the literature. We showed that our method significantly outperforms the Crank-Nicolson scheme and is at least as good as the Laplace transform method and is markedly more stable.
Figure 14: The difference between our method and the Laplace transform method (a) \( N = 100 \) (b) \( N = 10000 \). 

Figure 15: (a) \( G(T, z) \) as a function of \( b \) with \( T = 500 \) and \( z = 2 \) (b) \( T(z, b) = \mathbb{E}[s] \), where \( s \) is the hitting time.
As we pointed out, the problem has many practical applications, especially for con-
struction of mean-reverting trading strategies. Using the surface Figure [15b], one can
estimate the expected time for the reversion to the mean, and construct either a long or
a short strategy.

This paper leaves several important questions open as to the asymptotic behavior of
the hitting time density, which we are going to address in further research.
A Derivation of (9)

\[
g(t) = \frac{1}{2} p_x(t, b) = \frac{1}{2} \lim_{x \to b} p_x(t, x) = \\
(\epsilon t b - z) \exp \left( -\frac{(\epsilon t b - z)^2}{(e^{2t} - 1)^3} + 2t \right) \\
\]

\[
= - \frac{1}{\sqrt{\pi} (e^{2t} - 1)^3} + 2t \\
+ \frac{1}{2} e^{2t} \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \int_0^\tau \frac{(\epsilon + b(\tau) - b(\tau')) \exp \left( -\frac{(\epsilon + b(\tau) - b(\tau'))^2}{2(\tau - \tau')} \right) \nu(\tau') d\tau'}{\sqrt{2\pi (\tau - \tau')^3}} \\
= \frac{1}{2} e^{2t} \lim_{\epsilon \to 0} \int_0^\tau \left(1 - \frac{\epsilon + b(\tau) - b(\tau')}{(\tau - \tau')} \right) \exp \left( -\frac{(\epsilon + b(\tau) - b(\tau'))^2}{2(\tau - \tau')} \right) \nu(\tau') d\tau' \\
= \mathbb{L}_1 + \mathbb{L}_2 - 2\mathbb{L}_3 - \mathbb{L}_4,
\]

where

\[
\mathbb{L}_1 = \nu(\tau) \lim_{\epsilon \to 0} \int_0^\tau \left(1 - \frac{\epsilon^2}{(\tau - \tau')} \right) \exp \left( -\frac{\epsilon^2}{2(\tau - \tau')} + \frac{\epsilon(b(\tau) - b(\tau'))}{\tau - \tau'} \right) \Xi(\tau, \tau') \nu(\tau') - \nu(\tau) d\tau', \\
\mathbb{L}_2 = \lim_{\epsilon \to 0} \int_0^\tau \left(1 - \frac{\epsilon^2}{(\tau - \tau')} \right) \exp \left( -\frac{\epsilon^2}{2(\tau - \tau')} + \frac{\epsilon(b(\tau) - b(\tau'))}{\tau - \tau'} \right) \Xi(\tau, \tau') \nu(\tau') - \nu(\tau) d\tau', \\
\mathbb{L}_3 = \lim_{\epsilon \to 0} \int_0^\tau \frac{\epsilon(b(\tau) - b(\tau'))}{\tau - \tau'} \exp \left( -\frac{\epsilon^2}{2(\tau - \tau')} + \frac{\epsilon(b(\tau) - b(\tau'))}{\tau - \tau'} \right) \Xi(\tau, \tau') \nu(\tau') d\tau', \\
\mathbb{L}_4 = \lim_{\epsilon \to 0} \int_0^\tau \frac{b(\tau) - b(\tau^2)}{(\tau - \tau')} \exp \left( -\frac{\epsilon^2}{2(\tau - \tau')} + \frac{\epsilon(b(\tau) - b(\tau'))}{\tau - \tau'} \right) \Xi(\tau, \tau') \nu(\tau') d\tau',
\]

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and
\[ \Xi(\tau, \tau') = \exp \left(-\frac{(b(\tau) - b(\tau'))^2}{2(\tau - \tau')} \right). \]

We have
\[
L_1 = \nu(\tau) \lim_{\varepsilon \to 0} \int_0^\tau \left( 1 - \frac{\varepsilon^2}{\tau - \tau'} \right) \frac{\exp \left(-\frac{\varepsilon^2}{2(\tau - \tau')} + \frac{\varepsilon(b(\tau) - b(\tau'))}{\tau - \tau'} \right)}{\sqrt{2\pi (\tau - \tau')^3}} \, d\tau'
\]
\[ = \nu(\tau) \lim_{\varepsilon \to 0} \int_0^{\tau/\varepsilon^2} \left( 1 - \frac{1}{u} \right) \frac{\exp \left(-\frac{1}{u} \right)}{\sqrt{2\pi}} \, du
\]
\[ = 2\nu(\tau) \lim_{\varepsilon \to 0} \int_{\varepsilon/\sqrt{\pi}}^{\infty} (1 - v^2) \frac{\exp \left(-\frac{v^2}{2} \right)}{\sqrt{2\pi}} \, dv
\]
\[ = -2\nu(\tau) \lim_{\varepsilon \to 0} \int_0^{\varepsilon/\sqrt{\pi}} (1 - v^2) \frac{\exp \left(-\frac{v^2}{2} \right)}{\sqrt{2\pi}} \, dv
\]
\[ = -\frac{2}{\sqrt{2\pi\tau}} \nu(\tau) = -\frac{2}{\sqrt{\pi(e^{2\tau} - 1)}} \nu(t),
\]
where \((\tau - \tau') = u, u = 1/v^2\) and we have used the fact that
\[ \int_0^\infty (1 - v^2) \frac{\exp \left(-\frac{v^2}{2} \right)}{\sqrt{2\pi}} \, dv = 0.
\]

Further,
\[
L_2 = \lim_{\varepsilon \to 0} \int_0^\tau \left( 1 - \frac{\varepsilon^2}{\tau - \tau'} \right) \exp \left(-\frac{\varepsilon^2}{2(\tau - \tau')} + \frac{\varepsilon(b(\tau) - b(\tau'))}{\tau - \tau'} \right) (\Xi(\tau, \tau')\nu(\tau') - \nu(\tau)) \, d\tau'
\]
\[ = \lim_{\varepsilon \to 0} \int_0^\tau \exp \left(-\frac{\varepsilon^2}{2(\tau - \tau')} + \frac{\varepsilon(b(\tau) - b(\tau'))}{\tau - \tau'} \right) (\Xi(\tau, \tau')\nu(\tau') - \nu(\tau)) \, d\tau'
\]
\[ = \int_0^\tau \left( \Xi(\tau, \tau')\nu(\tau') - \nu(\tau) \right) \, d\tau'
\]
where we have dropped the higher order \(\varepsilon^2\) term in the integral in the second line. \(L_3\) is computed as in \cite{Tikhonov_and_Samarskii_1963}, pp. 530-535
\[
L_3 = \lim_{\varepsilon \to 0} \int_0^\tau \frac{\varepsilon(b(\tau) - b(\tau'))}{\tau - \tau'} \exp \left(-\frac{\varepsilon^2}{2(\tau - \tau')} + \frac{\varepsilon(b(\tau) - b(\tau'))}{\tau - \tau'} \right) \Xi(\tau, \tau') \, d\tau'
\]
\[ = b'(\tau) \Xi(\tau, \tau') \nu(\tau) = b e^{-t} \nu(t),
\]
and
\[
L_4 = \lim_{\varepsilon \to 0} \int_0^\tau \frac{(b(\tau) - b(\tau'))^2}{\tau - \tau'} \exp \left(-\frac{\varepsilon^2}{2(\tau - \tau')} + \frac{\varepsilon(b(\tau) - b(\tau'))}{\tau - \tau'} \right) \Xi(\tau, \tau') \, d\tau'
\]
\[ = \int_0^\tau \left( \frac{b(\tau) - b(\tau')}{\tau - \tau'} \right)^2 \Xi(\tau, \tau') \, d\tau'.
\]
Combining all terms, we finally have

\[
g(t) = -\frac{(e^t b - z) \exp \left( -\frac{(e^t b - z)^2}{e^{2t} - 1} + 2t \right)}{\sqrt{\pi (e^{2t} - 1)^3}} - \frac{e^t + \frac{e^{2t}}{\sqrt{\pi (e^{2t} - 1)}}}{\nu(t)}
\]

\[
+ \frac{1}{2} e^{2t} \int_0^\tau \left( 1 - 2b^2 \frac{\sqrt{\tau + 1} - \sqrt{\tau' + 1}}{\sqrt{\tau + 1} + \sqrt{\tau' + 1}} \right) \exp \left( -b^2 \frac{\sqrt{\tau + 1} - \sqrt{\tau' + 1}}{\sqrt{\tau + 1} + \sqrt{\tau' + 1}} \right) \nu(\tau') - \nu(\tau) \frac{d\tau'}{\sqrt{2\pi (\tau - \tau')^3}}.
\]

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