Methods of constructing superposition measures

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The resource theory of quantum superposition is an extension of the quantum coherent theory, in which linear independence relaxes the requirement of orthogonality. It can be used to quantify the nonclassical in superposition of finite number of optical coherent states. Based on convex roof extended, state transformation and weight, we give three methods of constructing superposition measures of quantum states, respectively. We also generalize the superposition resource theory from two perspectives.

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I. INTRODUCTION

Quantum information is a comprehensive subject which combines quantum mechanics and information. In quantum information, based on the basic principles of quantum mechanics, quantum states are used to encode information and the storage, processing and transmission of information are implemented by using quantum system. Quantum communication [1–3], quantum cryptography [4–7], and quantum secret sharing [8] are the main content of quantum information.

Quantum entanglement, first proposed by Einstein, Podolsky and Rosen [9] in 1935, is a quantum correlation which can be regarded as a kind of resource. Because of the great success of entanglement theory [10–16], researchers naturally think whether the framework of entanglement theory can be applied to other fields. In the quantum coherence, Åberg proposed a method of quantifying the superposition of orthogonal quantum states in 2006 [17]. In 2014, Baumgratz et al. established a rigorous framework of resource theory for quantifying quantum coherence [18]. The frameworks of resource theory in various fields have been presented shortly [19–30].

Quantum resource theory provides a structural framework in quantum information [31–39]. Resource theory is determined by imposed constraints that may be due to fundamental conservation laws, such as super-selection rules and conservation of energy, or by practical difficulties in performing certain operations on the quantum states of system. The operations allowed by the constraints are called free operation. In addition to free operation, there are another two elements in resource theory. One element is to measure the resources contained in the quantum states, such as entanglement and coherence. The other element is to define the so-called free state which does not contain resources. The free operation maps the free state to the free state.

In coherent resource theory, all the coherent resources used are considered with the condition that the base of the free states are orthogonal. In 2017, Theurer et al. proposed a generalized coherence theory called superposition resource theory, in which the requirement of orthogonality of the base of free states is extended to linear independence [40]. This is of great significance for the extension of coherence theory. First, although linear independence relaxes the requirement for orthogonality, it still has a limitation requirement, which requires the basis to form a linearly independent set. So from a conceptual point of view, superposition resource theory helps clarify the difference between orthogonality and linear independence. Many of the results of coherence theory are special cases of results obtained in non-orthogonal environments. This suggests that linear independence between free states, rather than orthogonality, is the main underlying cause of these results. Second, superposition resource theory can quantify the nonclassical superposition of finite number of optical coherent states. However, the coherent theoretical framework cannot be used in this case because the optical coherent states are not orthogonal. Therefore, superposition resource theory can be regarded as a new starting point and a general resource theory.

In this paper we mainly study the methods of constructing superposition measures of quantum states from different aspects. We organize this paper as follows. In Sec.II, we give an overview of the superposition resource theory. In Sec.III, we present two methods for constructing superposition measure and obtain some new results. In Sec.IV,
we provide the method of constructing superposition measure by weight. In Sec.V, we generalize the superposition resource theory. A brief summary is given in Sec.VI.

II. OVERVIEW OF THE SUPERPOSITION RESOURCE THEORY

A. Linearly independent bases and Gram matrix

In superposition resource theory, one uses \[ \{ |c_i \rangle \}^d_{i=1} \] to denote a set of linearly independent bases on Hilbert space, where the bases are not required to be orthogonal to each other. The Gram matrix is a useful tool for determining whether a given set of vectors is linearly independent [41, 42]. In the inner product space, given a set of finite vectors \[ \{ v_1, v_2, \ldots, v_m \} \], the Gram matrix is expressed as

\[
G = \begin{pmatrix}
\langle v_1 | v_1 \rangle & \langle v_1 | v_2 \rangle & \cdots & \langle v_1 | v_m \rangle \\
\langle v_2 | v_1 \rangle & \langle v_2 | v_2 \rangle & \cdots & \langle v_2 | v_m \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle v_m | v_1 \rangle & \langle v_m | v_2 \rangle & \cdots & \langle v_m | v_m \rangle
\end{pmatrix}.
\]

A set of vectors is linearly independent if and only if the determinant of its Gram matrix is positive [43]. For a quantum state set \[ \{ |c_1 \rangle, |c_2 \rangle, \ldots, |c_d \rangle \} \], if \[ \langle c_i | c_j \rangle = \mu_{ij} \], its Gram matrix becomes

\[
G = \begin{pmatrix}
1 & \mu_{12} & \cdots & \mu_{1d} \\
\mu_{12} & 1 & \cdots & \mu_{2d} \\
\vdots & \vdots & \ddots & \vdots \\
\mu_{1d} & \mu_{2d} & \cdots & 1
\end{pmatrix}.
\]

If \[ \{ |c_1 \rangle, |c_2 \rangle, \ldots, |c_d \rangle \} \] is linearly independent, then the determinant of the Gram matrix is positive. In special cases, in which \[ \langle c_i | c_j \rangle = \mu \] and \[ \mu \] be real numbers for arbitrary \( i \neq j \), the positive determinant of the Gram matrix corresponds \[ \mu \in (\frac{1}{d}, 1) \] for \( d \)-dimensional Hilbert space.

B. Free state and free operation

The quantum state whose density matrix being \[ \rho = \sum_{i=1}^{d} \rho_i |c_i \rangle \langle c_i | \] is called a free state, where \( \{ \rho_i \} \) is the probability distribution. The set of free states is denoted by \( \mathcal{F} \). All quantum states except free states, are called resource states.

If the quantum operation \( \Phi \) is trace-preserving and has the following form when it acts on quantum state \( \rho \)

\[
\Phi(\rho) = \sum_n K_n \rho K_n^\dagger,
\]

then the quantum operation \( \Phi \) is called free operation and the Kraus operator \( K_n \) is called superposition-free, where \( K_n = \sum_k c_{k,n} |f_n(k)\rangle \langle c_k^\dagger |, \ c_{k,n} \) is a complex number, \( f_n(k) \) is an index function, the quantum states \( |c_i^\dagger \rangle \) satisfy \( \langle c_i^\dagger | c_j \rangle = \xi_i \delta_{ij} \) for arbitrary \( i, j = 1, 2, \cdots, d \), \( \xi_i \) is a real number. The set of free operations is denoted by \( \mathcal{FO} \).

C. Superposition measure

A function \( \mathcal{M} \) mapping all quantum states to the non-negative real numbers is called a superposition measure of quantum states if \( \mathcal{M} \) satisfies the following conditions:

(S1) Faithful: \( \mathcal{M}(\rho) = 0 \) if and only if \( \rho \in \mathcal{F} \).
(S2) Monotonic under \( \mathcal{FO} \): For any \( \Lambda \in \mathcal{FO} \), \( \mathcal{M}(\Lambda(\rho)) \leq \mathcal{M}(\rho) \).
(S3) Monotonic under superposition-free selective measurements on average: \( \sum_n p_n \mathcal{M}(\rho_n) \leq \mathcal{M}(\rho) \), where \( p_n = \text{Tr}(K_n \rho K_n^\dagger) \), \( \rho_n = \frac{K_n \rho K_n^\dagger}{p_n} \) for all \( \{ K_n \} \): \( \sum_n K_n^\dagger K_n = 1 \), \( K_n \mathcal{F} K_n^\dagger \subset \mathcal{F} \).
(S4) Convex: \( \mathcal{M}(\sum_i p_i \rho_i) \leq \sum_i p_i \mathcal{M}(\rho_i) \), where the probability distribution \( \{ p_i \} \) satisfies \( p_i \geq 0 \) and \( \sum_i p_i = 1 \).

A few valid superposition measures of quantum states are listed in the following [40].
(1) The $l_1$ measure: $\mathcal{M}_{l_1}(\rho) = \sum_{i\neq j} |\rho_{ij}|$, for $\rho = \sum_{i,j} \rho_{ij} |c_i\rangle \langle c_j|$. 

(2) The relative entropy: $\mathcal{M}_{\text{rel. ent.}}(\rho) = \min_{\sigma \in \mathcal{F}} S(\rho \| \sigma)$, where $S(\rho \| \sigma) = \text{Tr}(\rho \log \rho) - \text{Tr}(\rho \log \sigma)$ denotes the quantum relative entropy.

(3) The rank-measure: $\mathcal{M}_{\text{rank}}(\rho) = \log r(\rho), \mathcal{M}_{\text{rank}}(\rho) = \min_{\{p_n,|\varphi_n\rangle\}} \sum_n p_n \mathcal{M}_{\text{rank}}(|\varphi_n\rangle)$, where $\{p_n,|\varphi_n\rangle\}$ is the ensemble of decomposition of $\rho$, $r(\rho)$ is the rank of the quantum state $|\psi\rangle$ with respect to the basis $\{c_i\}$.

(4) The robustness measure: $\mathcal{M}_{\text{robust}}(\rho) = \min_{\tau \in \text{density matrix}} \{s \geq 0 | \frac{\rho + s \tau}{1 + s} \in \mathcal{F} \}$. 

III. SUPERPOSITION MEASURE BASED ON KNOWN FUNCTION

In this section we will provide two methods of constructing superposition measure based on functions satisfying (S1) and (S3).

A. Convex roof extended

**Theorem 1.** If a function $\mathcal{M}$ satisfies (S1) and (S3) for pure states, then the superposition measure can be obtained by the convex roof extended

$$\mathcal{M}'(\rho) = \min_{\{p_n,|\varphi_n\rangle\}} \sum_n p_n \mathcal{M}(|\varphi_n\rangle),$$

where $\{p_n,|\varphi_n\rangle\}$ is the ensemble of decomposition of $\rho$.

In order to simplify the expression, we use $\mathcal{M}(|\varphi\rangle)$ to represent $\mathcal{M}(|\varphi\rangle,\langle\varphi|)$. Here the function $\mathcal{M}$ satisfies (S1) for pure states means that for any pure state $|\psi\rangle$, there is $\mathcal{M}(|\psi\rangle) \geq 0$, the equal sign holds if and only if $|\psi\rangle \in \mathcal{F}$, the set of free states; the function $\mathcal{M}$ satisfies (S3) for pure states means that the Kraus operator $\{K_n\}$ must satisfy

$$\mathcal{M}(|\varphi\rangle) \geq \sum_n \text{Tr}(K_n|\varphi\rangle \langle \varphi| K_n^\dagger) \mathcal{M}(\frac{K_n|\varphi\rangle \langle \varphi| K_n^\dagger}{\text{Tr}(K_n|\varphi\rangle \langle \varphi| K_n^\dagger)}).$$

**Proof.** First we prove $\mathcal{M}'(\rho)$ satisfies (S1). Because the function $\mathcal{M}$ satisfies (S1) for pure states, so

$$\mathcal{M}'(\rho) = \min_{\{p_n,|\varphi_n\rangle\}} \sum_n p_n \mathcal{M}(|\varphi_n\rangle) \geq 0. \quad (6)$$

If $\rho$ is a free state, then $\rho$ can be expressed as $\rho = \sum_{i=1}^d \rho_i |c_i\rangle \langle c_i|$, so $\mathcal{M}'(\rho) \leq \sum_i \rho_i \mathcal{M}(|c_i\rangle) = 0$, hence $\mathcal{M}'(\rho) = 0$.

If $\mathcal{M}'(\rho) = 0$, there is an ensemble decomposition of $\rho$ which is $\{p_i,|\psi_i\rangle\}$ makes $\sum_i p_i \mathcal{M}(|\psi_i\rangle) = 0$. At this point, we have $\mathcal{M}(|\psi_i\rangle) = 0$ for every $i$, so $\rho$ is free state.

Second we prove $\mathcal{M}'(\rho)$ satisfies (S4). We want to prove that, for $p_1$ and $p_2$, there are $\mathcal{M}'(p_1 \rho_1 + p_2 \rho_2) \leq p_1 \mathcal{M}'(\rho_1) + p_2 \mathcal{M}'(\rho_2)$, in which $p_1 + p_2 = 1$. Assume that the set $\{p_1^1,|\psi_1^1\rangle\}$ is the optimal ensemble decomposition of $\rho_1$ and the set $\{p_2^2,|\psi_2^2\rangle\}$ is the optimal ensemble decomposition of $\rho_2$. That is to say $\mathcal{M}'(\rho_1) = \sum_i p_1^i \mathcal{M}(|\psi_1^i\rangle)$ and $\mathcal{M}'(\rho_2) = \sum_j p_2^j \mathcal{M}(|\psi_2^j\rangle)$. Then one has

$$p_1 \mathcal{M}'(\rho_1) + p_2 \mathcal{M}'(\rho_2) = \sum_i p_1^i p_1^1 \mathcal{M}(|\psi_1^i\rangle) + \sum_j p_2^j p_2^2 \mathcal{M}(|\psi_2^j\rangle). \quad (7)$$

Because $p_1 \rho_1 + p_2 \rho_2 = \sum_i p_1^i |\psi_1^i\rangle \langle \psi_1^i| + \sum_j p_2^j |\psi_2^j\rangle \langle \psi_2^j|$, therefore

$$\mathcal{M}'(p_1 \rho_1 + p_2 \rho_2) \leq \sum_i p_1^i \mathcal{M}(|\psi_1^i\rangle) + \sum_j p_2^j \mathcal{M}(|\psi_2^j\rangle) = p_1 \mathcal{M}'(\rho_1) + p_2 \mathcal{M}'(\rho_2). \quad (8)$$
After that we prove $\mathcal{M}'(\rho)$ satisfies (S3). Suppose the set \{\$q_m, |\$\varphi_m\$\} is the optimal ensemble decomposition of \$\rho\$. We have

$$
\mathcal{M}'(\rho) = \sum_m q_m \mathcal{M}(|\varphi_m\rangle)
\geq \sum_m q_m \sum_n \Tr(K_n|\varphi_m\rangle\langle\varphi_m|K_n^\dagger) \mathcal{M}(\frac{K_n|\varphi_m\rangle\langle\varphi_m|K_n^\dagger}{\Tr(K_n|\varphi_m\rangle\langle\varphi_m|K_n^\dagger)})
= \sum_n \Tr(K_n\rho K_n^\dagger) \sum_m q_m \Tr(K_n|\varphi_m\rangle\langle\varphi_m|K_n^\dagger) \mathcal{M}(\frac{K_n|\varphi_m\rangle\langle\varphi_m|K_n^\dagger}{\Tr(K_n|\varphi_m\rangle\langle\varphi_m|K_n^\dagger)})
\geq \sum_n \Tr(K_n\rho K_n^\dagger) \mathcal{M}'(\frac{K_n\rho K_n^\dagger}{\Tr(K_n\rho K_n^\dagger)}).
$$
(9)

The first inequality in Eq.(9) is based on the fact that \$\mathcal{M}\$ satisfies (S3) for pure states. The second inequality in Eq.(9) comes from the fact that \$\mathcal{M}'(\rho)\$ satisfies (S4).

Finally, we prove \$\mathcal{M}'\$ satisfies (S2). Evidently

$$
\mathcal{M}'(\Lambda(\rho)) \leq \sum_n \Tr(K_n\rho K_n^\dagger) \mathcal{M}'(\frac{K_n\rho K_n^\dagger}{\Tr(K_n\rho K_n^\dagger)}) \leq \mathcal{M}'(\rho).
$$
(10)

The first inequality in Eq.(10) takes the fact that \$\mathcal{M}'(\rho)\$ satisfies (S4). The second inequality in Eq.(10) is the result of the fact that \$\mathcal{M}'(\rho)\$ satisfies (S3).

Thus we have proved the above theorem.

**Corollary 1.** $\mathcal{M}'_{l_1}(\rho) = \min_{\{p_n, |\varphi_n\rangle\}} \sum_n p_n \mathcal{M}_{l_1}(|\varphi_n\rangle)$ is a superposition measure, where \{\$p_n, |\$\varphi_n\$\}$ expresses an ensemble decomposition of \$\rho\$, \$|\varphi_n\rangle = \sum_i \varphi_n,|c_i\rangle\$, \$\mathcal{M}_{l_1}(|\varphi_n\rangle) = \sum_{i,j} |\varphi_n,|\varphi_{n,j}\rangle|\$ stands for the \$l_1\$ measure.

**Proof.** Since \$\mathcal{M}_{l_1}(\rho)$ is a superposition measure, hence \$\mathcal{M}_{l_1}(\rho)$ satisfies (S1), (S2), (S3) and (S4) for any states. Thus \$\mathcal{M}_{l_1}(|\varphi\rangle\langle\varphi|)$ satisfies (S1) and (S3) for all pure states. According to Theorem 1, Corollary 1 is true.

**Corollary 2.** For any state \$\rho\$, \$\mathcal{M}'_{l_1}(\rho) \geq \mathcal{M}_{l_1}(\rho)$.

**Proof.** Suppose \{\$q_m, |\$\varphi_m\$\}$ is the optimal ensemble decomposition of \$\rho\$ for which \$\mathcal{M}'_{l_1}(\rho)$ is minimized. Let $|\varphi_m\rangle = \sum_i \varphi_m,|c_i\rangle$, $\rho = \sum_{i,j} \rho_{ij}|c_i\rangle\langle c_j|$. Then we have $\rho = \sum_m q_m |\varphi_m\rangle\langle\varphi_m| = \sum_m q_m \sum_{i,j} \varphi_m,|\varphi_{m,j}|\langle c_i|\langle c_j|$.

Therefore $\rho_{ij} = \sum_m q_m \varphi_m,|\varphi_{m,j}|$. One can easily derive

$$
\mathcal{M}_{l_1}(\rho) = \sum_{i,j} |\rho_{ij}| = \sum_{i,j} \sum_m q_m \varphi_i,|\varphi_{j,m}| = \sum_{i,j} q_m \sum_m |\varphi_i,|\varphi_{j,m}| = \sum_m q_m \mathcal{M}_{l_1}(|\varphi_m\rangle\langle\varphi_m|) = \mathcal{M}'_{l_1}(\rho).
$$
(11)

**Corollary 3.** $\mathcal{M}'_{rel,net}(\rho) = \min_{\{p_n, |\varphi_n\rangle\}} \sum_n p_n \mathcal{M}_{rel,net}(|\varphi_n\rangle)$ is a superposition measure, where \{\$p_n, |\$\varphi_n\$\}$ is an ensemble decomposition of \$\rho\$, \$\mathcal{M}_{rel,net}(|\varphi_n\rangle)$ is the relative entropy.

**Proof.** As \$\mathcal{M}_{rel,net}(\rho)$ is a superposition measure, therefore \$\mathcal{M}_{rel,net}(\rho)$ satisfies (S1), (S2), (S3) and (S4) for any states. Hence \$\mathcal{M}_{rel,net}(|\varphi\rangle\langle\varphi|)$ satisfies (S1) and (S3) for all pure states. By using Theorem 1, it is not difficult to confirm Corollary 3.

**Theorem 2.** Assume that some pure states form the set $\mathcal{S}$, and the convex combinations of states within $\mathcal{S}$ form the set $\mathcal{S}'$. If a function $\mathcal{M}$ satisfies (S1) and (S3) for states in $\mathcal{S}$, then the superposition measure of all states in $\mathcal{S}'$ can be obtained by the convex roof extended

$$
\mathcal{M}'(\rho) = \min_{\{p_n, |\varphi_n\rangle\}} \sum_n p_n \mathcal{M}(|\varphi_n\rangle),
$$
(12)

where \{\$p_n, |\$\varphi_n\$\}$ is the ensemble decomposition of \$\rho\$, and meets $|\varphi_n\rangle \in \mathcal{S}$. 

The function $\mathcal{M}$ satisfies (S1) for states in $\mathcal{S}$ implies that for any $|\psi\rangle \in \mathcal{S}$, $\mathcal{M}(|\psi\rangle) \geq 0$ is true if and only if $|\psi\rangle \in \mathcal{F}$ and $|\psi\rangle \in \mathcal{S}$; the function $\mathcal{M}$ satisfies (S3) for states in $\mathcal{S}$ means that the Kraus operator $\{K_n\}$ must satisfy
\[
\frac{K_n|\psi\rangle\langle\psi|K_n^\dagger}{\text{Tr}(K_n|\psi\rangle\langle\psi|K_n)} \in \mathcal{S} \text{ for any } |\psi\rangle \in \mathcal{S}; \text{ for } |\psi\rangle \in \mathcal{S} \text{ and } |\phi\rangle \in \mathcal{F}, \frac{K_n|\psi\rangle\langle\psi|K_n^\dagger}{\text{Tr}(K_n|\psi\rangle\langle\psi|K_n^\dagger)} \in \mathcal{S} \text{ and } \frac{K_n|\phi\rangle\langle\phi|K_n^\dagger}{\text{Tr}(K_n|\phi\rangle\langle\phi|K_n^\dagger)} \in \mathcal{F}; \text{ for } |\psi\rangle \in \mathcal{S},
\]
\[
\mathcal{M}(|\psi\rangle) \geq \sum_n \text{Tr}(K_n|\psi\rangle\langle\psi|K_n^\dagger)\mathcal{M}\left(\frac{K_n|\psi\rangle\langle\psi|K_n^\dagger}{\text{Tr}(K_n|\psi\rangle\langle\psi|K_n^\dagger)}\right).
\tag{13}
\]

**Proof.** First we prove $\mathcal{M}'(\rho)$ satisfies (S1) for states in $\mathcal{S}'$. It is equivalent to prove that $\mathcal{M}'(\rho) = 0$ if and only if $\rho \in \mathcal{F}$ and $\rho \in \mathcal{S}'$. For $|\varphi_n\rangle \in \mathcal{S}$, because the function $\mathcal{M}$ satisfies (S1) for states in $\mathcal{S}$, so
\[
\mathcal{M}'(\rho) = \min_{(p_{\varphi_n}, \varphi_n)} \sum_n p_n \mathcal{M}(|\varphi_n\rangle) \geq 0.
\tag{14}
\]
If $\rho \in \mathcal{F}$, then $\rho$ can be expressed as $\rho = \sum_{i=1}^d p_i\rho_i(c_i|c_i)$, so $\mathcal{M}'(\rho) \leq \sum_i p_i \mathcal{M}(|\rho_i\rangle) = 0$, therefore $\mathcal{M}'(\rho) = 0$.

If $\mathcal{M}'(\rho) = 0$, there is an ensemble decomposition of $\rho$, $\{p_i, |\psi_i\rangle\}$ which makes $\sum_i p_i \mathcal{M}(|\psi_i\rangle) = 0$. Since $\mathcal{M}(|\psi_i\rangle) \geq 0$, so we have $\mathcal{M}(|\psi_i\rangle) = 0$ for every $i$, therefore $\rho \in \mathcal{F}$.

Second we prove $\mathcal{M}'(\rho)$ satisfies (S4) for states in $\mathcal{S}'$. For $\rho_1 \in \mathcal{S}'$ and $\rho_2 \in \mathcal{S}'$, evidently there is $\rho = \rho_1 \rho_2 + \rho_2 \rho_1 \in \mathcal{S}'$, where $\rho_1 + \rho_2 = 1$. We want to prove that, for $\rho_1 \in \mathcal{S}'$ and $\rho_2 \in \mathcal{S}'$, then $\mathcal{M}'(\rho_1 \rho_2 + \rho_2 \rho_1) \leq \rho_1 \mathcal{M}'(\rho_2) + \rho_2 \mathcal{M}'(\rho_2)$, in which $\rho_1 + \rho_2 = 1$. Assume that the set $\{p_1|\psi_1\rangle\}$ is the optimal ensemble decomposition of $\rho_1$ and the set $\{p_2|\psi_2\rangle\}$ is the optimal ensemble decomposition of $\rho_2$. It implies that $\mathcal{M}'(\rho_1) = \sum_i p_1|\psi_1\rangle\langle\psi_1|\rangle$ and $\mathcal{M}'(\rho_2) = \sum_j p_2|\psi_2\rangle\langle\psi_2|\rangle$.

Thus we have
\[
\mathcal{M}'(\rho_1) + \mathcal{M}'(\rho_2) = \sum_i p_1|\psi_1\rangle\langle\psi_1|\rangle + \sum_j p_2|\psi_2\rangle\langle\psi_2|\rangle.
\tag{15}
\]
Because $p_1 \rho_1 + p_2 \rho_2 = \sum_i p_1|\psi_1\rangle\langle\psi_1|\rangle + \sum_j p_2|\psi_2\rangle\langle\psi_2|\rangle$, so
\[
\mathcal{M}'(\rho_1 \rho_2 + \rho_2 \rho_1) \leq \sum_i p_1|\psi_1\rangle\langle\psi_1|\rangle + \sum_j p_2|\psi_2\rangle\langle\psi_2|\rangle = p_1 \mathcal{M}'(\rho_2) + p_2 \mathcal{M}'(\rho_2).
\tag{16}
\]

Next we prove $\mathcal{M}'(\rho)$ satisfies (S3) for states in $\mathcal{S}'$. Suppose set $\{q_m, |\varphi_m\rangle\}$ is the optimal ensemble decomposition of $\rho$, therefore
\[
\mathcal{M}'(\rho) = \sum_m q_m \mathcal{M}(|\varphi_m\rangle)
\]
\[
\geq \sum_m q_m \sum_n \text{Tr}(K_n|\varphi_m\rangle\langle\varphi_m|K_n^\dagger)\mathcal{M}\left(\frac{K_n|\varphi_m\rangle\langle\varphi_m|K_n^\dagger}{\text{Tr}(K_n|\varphi_m\rangle\langle\varphi_m|K_n)}\right)
\]
\[
= \sum_n \text{Tr}(K_n\rho K_n^\dagger) \sum_m q_m \text{Tr}(K_n|\varphi_m\rangle\langle\varphi_m|K_n^\dagger)\mathcal{M}\left(\frac{K_n|\varphi_m\rangle\langle\varphi_m|K_n^\dagger}{\text{Tr}(K_n|\varphi_m\rangle\langle\varphi_m|K_n)}\right)
\tag{17}
\]
\[
\geq \sum_n \text{Tr}(K_n\rho K_n^\dagger) \mathcal{M}'(\frac{K_n\rho K_n^\dagger}{\text{Tr}(K_n \rho K_n^\dagger)}).
\]

The first inequality in Eq.(17) holds because $\mathcal{M}$ satisfies (S3) for states in $\mathcal{S}$. The second inequality in Eq.(17) is true since $\mathcal{M}'(\rho)$ satisfies (S4) for states in $\mathcal{S}'$.

Finally, we show $\mathcal{M}'$ satisfies (S2) for states in $\mathcal{S}'$. Evidently
\[
\mathcal{M}'(\Lambda(\rho)) \leq \sum_n \text{Tr}(K_n \rho K_n^\dagger) \mathcal{M}'(\frac{K_n \rho K_n^\dagger}{\text{Tr}(K_n \rho K_n^\dagger)}) \leq \mathcal{M}'(\rho).
\tag{18}
\]

The first inequality in Eq.(18) comes from the fact that $\mathcal{M}'(\rho)$ satisfies (S4) for states in $\mathcal{S}'$. The second inequality in Eq.(18) holds since $\mathcal{M}'(\rho)$ satisfies (S3) for states in $\mathcal{S}'$.

The proof of Theorem 2 has been completed.
In this section we will give another method of constructing superposition measure called the state transformation method, which is stated as the following theorem.

**Theorem 3.** For \( i \neq j \), let \( \langle c_i | c_j \rangle = \mu \) and \( \mu \) be real numbers. If a function \( M \) satisfies (S1) and (S3) for pure states and \( M \) satisfies \( M(|\varphi_0\rangle) \leq \sum_i p_i M(|\psi_i\rangle) \), the superposition measure can be obtained by

\[
\Gamma(\rho) = \inf_{|\phi\rangle \in R(\rho)} M(|\phi\rangle). \tag{19}
\]

Here \( R(\rho) \) is the set of all pure states that can be converted to \( \rho \) by free operation, \( |\varphi_0\rangle \in R(\rho) \), \( \{p_i, |\psi_i\rangle\} \) is any ensemble decomposition of \( \rho \) [44].

**Proof.** First we prove \( \Gamma(\rho) \) satisfies (S1) for any state.

Obviously, \( \Gamma(\rho) \geq 0 \). Evidently, an arbitrary free state can be expressed as \( \rho' = \sum_{i=1}^{d} \rho_i |c_i\rangle \langle c_i| \), where \( \rho_i \geq 0 \) and \( \sum_{i=1}^{d} \rho_i = 1 \). Choose Kraus operators as

\[
K_1 = \sqrt{\rho_1} \frac{1}{\xi_1} |c_1\rangle \langle c_1| + \frac{1}{\xi_2} |c_2\rangle \langle c_2| + \cdots + \frac{1}{\xi_{d-1}} |c_{d-1}\rangle \langle c_{d-1}| + \frac{1}{\xi_d} |c_d\rangle \langle c_d|,
\]

\[
K_2 = \sqrt{\rho_2} \frac{1}{\xi_1} |c_2\rangle \langle c_2| + \frac{1}{\xi_2} |c_3\rangle \langle c_3| + \cdots + \frac{1}{\xi_{d-1}} |c_{d-1}\rangle \langle c_{d-1}| + \frac{1}{\xi_d} |c_1\rangle \langle c_1|,
\]

\[
K_3 = \sqrt{\rho_3} \frac{1}{\xi_1} |c_3\rangle \langle c_3| + \frac{1}{\xi_2} |c_4\rangle \langle c_4| + \cdots + \frac{1}{\xi_{d-1}} |c_{d-1}\rangle \langle c_{d-1}| + \frac{1}{\xi_d} |c_2\rangle \langle c_2|,
\]

\[
K_d = \sqrt{\rho_d} \frac{1}{\xi_1} |c_d\rangle \langle c_d| + \frac{1}{\xi_2} |c_1\rangle \langle c_1| + \cdots + \frac{1}{\xi_{d-1}} |c_{d-1}\rangle \langle c_{d-1}| + \frac{1}{\xi_d} |c_1\rangle \langle c_1|,
\]

where \( \xi_i = \langle c_i| c_i \rangle \). It is easy to prove that \( \{K_1, K_2, \cdots, K_d\} \) is a free operation, \( \sum_{i=1}^{d} K_i^\dagger K_i = I \) and \( \rho' = \sum_{i=1}^{d} \rho_i |c_i\rangle \langle c_i| = \sum_{i=1}^{d} K_i |c_1\rangle \langle c_1| K_i^\dagger \). Thus we have

\[
0 \leq \Gamma(\rho') \leq \Gamma(|c_1\rangle) = 0. \tag{21}
\]

It means \( \Gamma(\rho') = 0 \).

Suppose that \( \Gamma(\rho) = 0 \). According to the definition, there must be one pure state \( |\phi\rangle \) and free operation \( \Lambda \), such that \( M(|\phi\rangle) = 0 \) and \( \rho = \Lambda(|\phi\rangle \langle \phi|) \). It implies that both \( |\phi\rangle \) and \( \rho \) are free states. Therefore, the superposition measure satisfies (S1).

Next we prove \( \Gamma(\rho) \) satisfies (S2).

A state undergoing two free operations can be regarded as undergoing one free operation. For \( \rho_0 = \Lambda(\rho) = \Lambda(\varepsilon(|\psi\rangle \langle \psi|)) \), where \( \Lambda \) and \( \varepsilon \) are free operations, \( \rho_0 \) and \( \rho \) can be obtained from \( |\psi\rangle \) by free operations. If \( |\psi\rangle \) is optimal for \( \rho \), then \( \Gamma(\rho) = M(|\psi\rangle) \). Because \( |\psi\rangle \) is not necessarily optimal for \( \rho_0 \), we know that \( \rho \) can be transformed into \( \rho' \) by free operations, then \( \mathcal{C}(\rho) = \mathcal{C}(\rho') \) is given by the monotonicity of \( \mathcal{C} \). Second we show that when \( \mathcal{M} \) satisfies \( \mathcal{M}(|\psi_0\rangle) \leq \sum_i p_i \mathcal{M}(|\psi_i\rangle) \), we have \( \Gamma(\rho) = \mathcal{M}(\rho') \).

Obviously, \( \mathcal{M}'(\rho) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i \mathcal{M}(|\psi_i\rangle) \) is one of superposition measure \( \mathcal{C} \). So we have \( \Gamma(\rho) \geq \mathcal{M}'(\rho) \). By \( \mathcal{M}(|\varphi_0\rangle) \leq \sum_i p_i \mathcal{M}(|\psi_i\rangle) \), we know that \( \Gamma(\rho) \leq \mathcal{M}(|\varphi_0\rangle) \leq \mathcal{M}'(\rho) \). Therefore when \( \mathcal{M} \) satisfies \( \mathcal{M}(|\varphi_0\rangle) \leq \sum_i p_i \mathcal{M}(|\psi_i\rangle) \), we have \( \Gamma(\rho) = \mathcal{M}'(\rho) \). Because \( \mathcal{M}'(\rho) \) satisfies (S4), thus \( \Gamma(\rho) \) meets (S4).
Finally, we prove $\Gamma(\rho)$ satisfies (S3). Suppose $|\psi\rangle$ is converted to $\rho = \sum_i t_i |\varphi_i\rangle \langle \varphi_i |$ by free operation, and then converted to $\{p_i, \rho_i = \sum_l \frac{t_{il} q_{il}}{p_i} |\phi_{il}\rangle \langle \phi_{il} | \}$ by free operation. Suppose $|\psi\rangle$ is optimal for $\rho$, where $p_i = \sum_i t_i q_{il}$, $\rho_i = \sum_l \frac{t_{il} q_{il}}{p_i} |\phi_{il}\rangle \langle \phi_{il} |$, there is

$$\Gamma(\rho) = \mathcal{M}(|\psi\rangle) \geq \sum_{i,l} t_{il} q_{il} \mathcal{M}(|\phi_{il}\rangle) = \sum_i p_i \sum_l t_{il} q_{il} \mathcal{M}(|\phi_{il}\rangle) = \sum_i p_i \sum_l t_{il} q_{il} \Gamma(|\phi_{il}\rangle) \geq \sum_i p_i \Gamma(\rho_i). \quad (22)$$

The first inequality in Eq.(22) is based on the fact that $\mathcal{M}$ satisfies (S3) for pure states. Another inequality in above equation holds by the fact that $\Gamma(\rho)$ meets (S4).

Thus we have proved Theorem 3.

**Example 1.** For qubit, let us consider a class of states $\rho(x) = \frac{1}{1+2\mu^2} (\frac{1}{2} |c_0\rangle \langle c_0 | + x |c_0\rangle \langle c_1 | + |c_0\rangle \langle c_0 | + \frac{1}{2} |c_1\rangle \langle c_1 |)$, where $-1 < \mu = \langle c_0 | c_1 \rangle < 1$ and $-\frac{1}{2} \leq x \leq \frac{1}{2}$. Choose $\mathcal{M}_i(\rho)$ as the function that satisfies (S1) and (S3) for pure states. The quantum state $|\varphi_0\rangle$ for the state $\rho(x)$ will be given as follows [45].

Apparently, quantum state $\rho(x)$ can be decomposed as

$$\rho(x) = p_1(x, \alpha) |\psi_1(x, \alpha)\rangle \langle \psi_1(x, \alpha) | + p_2(x, \alpha) |\psi_2(x, \alpha)\rangle \langle \psi_2(x, \alpha) |, \quad (23)$$

and each $\alpha$ corresponds a decomposition of $\rho(x)$, where

$$\sqrt{p_1(x, \alpha)} |\psi_1(x, \alpha)\rangle = \cos \alpha \sqrt{\lambda_1(x)} |c_+\rangle + \sin \alpha \sqrt{\lambda_2(x)} |c_-\rangle,$$

$$\sqrt{p_2(x, \alpha)} |\psi_2(x, \alpha)\rangle = -\sin \alpha \sqrt{\lambda_1(x)} |c_+\rangle + \cos \alpha \sqrt{\lambda_2(x)} |c_-\rangle,$$

$$p_1(x, \alpha) = \cos^2 \alpha \lambda_1(x) + \sin^2 \alpha \lambda_2(x),$$

$$p_2(x, \alpha) = \sin^2 \alpha \lambda_1(x) + \cos^2 \alpha \lambda_2(x),$$

$$\lambda_1(x) = \frac{(1 + \mu)(1 + 2x)}{2 + 4\mu x},$$

$$\lambda_2(x) = \frac{(1 - \mu)(1 - 2x)}{2 + 4\mu x},$$

$$|c_+\rangle = \frac{1}{\sqrt{2\mu + 2}} (|c_0\rangle + |c_1\rangle), \quad |c_-\rangle = \frac{1}{\sqrt{2 - 2\mu}} (|c_0\rangle - |c_1\rangle).$$

So we have

$$p_1(x, \alpha) \mathcal{M}_i(|\psi_1(x, \alpha)\rangle) + p_2(x, \alpha) \mathcal{M}_i(|\psi_2(x, \alpha)\rangle) = \frac{1}{2 + 4\mu x} (\sqrt{2x + \cos 2\alpha} + |2x - \cos 2\alpha|) \geq \frac{|2x|}{1 + 2\mu x}. \quad (25)$$

Obviously $\sum_{i=1}^2 p_i(x, \alpha) \mathcal{M}_i(|\psi_i(x, \alpha)\rangle)$ is a function of $\alpha$. When $\alpha = \frac{\pi}{4}$, the equal sign holds in Eq.(25). Therefore, we find that the optical ensemble decomposition of $\rho$ which minimizes $\sum_{i=1}^2 p_i(x, \alpha) \mathcal{M}_i(|\psi_i(x, \alpha)\rangle)$, corresponds to $\alpha = \frac{\pi}{4}$ and yields

$$p_1(x, \frac{\pi}{4}) = \frac{1}{2},$$

$$|\psi_1(x, \frac{\pi}{4})\rangle = \sqrt{\frac{\lambda_1(x)}{4\mu + 4}} |c_0\rangle + \sqrt{\frac{\lambda_2(x)}{4\mu + 4}} |c_1\rangle,$$

$$|\psi_2(x, \frac{\pi}{4})\rangle = -\sqrt{\frac{\lambda_1(x)}{4\mu + 4}} |c_0\rangle + \sqrt{\frac{\lambda_2(x)}{4\mu + 4}} |c_1\rangle. \quad (26)$$

Thus if we choose $|\varphi_0\rangle = |\psi_1(x, \frac{\pi}{4})\rangle$, then we have that $|\varphi_0\rangle$ satisfies

$$\Gamma(\rho(x)) = \mathcal{M}_i(|\varphi_0\rangle) \leq \sum_{i=1}^2 p_i(x, \alpha) \mathcal{M}_i(|\psi_i(x, \alpha)\rangle). \quad (27)$$
We choose the free operation \( \{ K_1 = \sqrt{\frac{1}{2}} (|00\rangle \langle 00| + |11\rangle \langle 11|), K_2 = \sqrt{\frac{1}{2}} (|c_1\rangle \langle c_2| + |c_1\rangle \langle c_2|) \} \). It is easy to deduce that
\[
\rho(x) = \sum_{i=1}^{2} K_i |\varphi_0\rangle \langle \varphi_0| K_i^\dagger.
\]
Therefore, the \(|\varphi_0\rangle\) which meets the requirements has been found.

### IV. SUPERPOSITION MEASURE BASED ON WEIGHT

In superposition resource theory, we can divide each state into a free part and a resource part [46], one can construct the following superposition measure based on weight. For quantum state \( \rho \), we define
\[
\mathcal{M}_w(\rho) = \min_{\text{density matrix}} \{1 - \lambda |\rho = \lambda \delta + (1 - \lambda) \tau, \delta \in \mathcal{F}\}.
\]
Evidently, \( 0 \leq \mathcal{M}_w(\rho) \leq 1 \). Then we arrive at the following result.

**Theorem 4.** \( \mathcal{M}_w(\rho) \) is a superposition measure.

**Proof.** Clearly, \( \mathcal{M}_w(\rho) \) satisfies (S1).

Next we prove \( \mathcal{M}_w(\rho) \) meets (S4). It implies that we want to prove \( \mathcal{M}_w(\rho) \leq p \mathcal{M}_w(\rho_1) + (1 - p) \mathcal{M}_w(\rho_2) \), where \( \rho = p(p_1 + (1 - p)\rho_2) \). Suppose \( \delta_1 \) and \( \tau_1 \) are the optimal states for \( \rho_1 \) to obtain \( \mathcal{M}_w(\rho_1) \), and suppose \( \delta_2 \) and \( \tau_2 \) are the optimal states for \( \rho_2 \) to obtain \( \mathcal{M}_w(\rho_2) \), then \( \delta_1 \) and \( \delta_2 \) are free states, \( \tau_1 \) and \( \tau_2 \) are resource states. We have
\[
\mathcal{M}_w(\rho_1) = \max_{\text{ensemble}} \{1 - \lambda |\rho = \lambda \delta + (1 - \lambda) \tau, \delta \in \mathcal{F}\}.
\]
Without loss of generality, we assume \( \mathcal{M}_w(\rho_1) \neq 0 \) and \( \mathcal{M}_w(\rho_2) \neq 0 \). Hence
\[
\rho = p\rho_1 + (1 - p)\rho_2 = p(1 - \mathcal{M}_w(\rho_1))\delta_1 + \mathcal{M}_w(\rho_1)\tau_1 + (1 - p)\mathcal{M}_w(\rho_2)(1 - \mathcal{M}_w(\rho_2))\delta_2 + \mathcal{M}_w(\rho_2)\tau_2.
\]
where \( M = p\mathcal{M}_w(\rho_1) + (1 - p)\mathcal{M}_w(\rho_2) \).

Note that \( \delta_1 \) and \( \delta_2 \) are free states, \( \tau_1 \) and \( \tau_2 \) are resource states, \( \mathcal{M}_w(\rho) \) satisfies (S3).

Later on we will show \( \mathcal{M}_w(\rho) \) satisfies (S3). That is one wants to prove \( \sum_n \text{Tr}(K_n \rho K_n^\dagger) \mathcal{M}_w(\rho K_n(K_n^\dagger)) \leq \mathcal{M}_w(\rho) \), where \( K_n \) is superposition-free. Suppose \( \delta \) and \( \tau \) are the optimal states for \( \rho \) to minimize \( \mathcal{M}_w(\rho) \), where \( \delta \) is a free state and \( \tau \) is a resource state. Then we obtain \( \rho = [1 - \mathcal{M}_w(\rho)]\delta + \mathcal{M}_w(\rho)\tau \).

Furthermore
\[
K_n \rho K_n^\dagger = (1 - \mathcal{M}_w(\rho))K_n \delta K_n^\dagger + \mathcal{M}_w(\rho)K_n \tau K_n^\dagger.
\]
Therefore, we get
\[
\frac{K_n \rho K_n^\dagger}{\text{Tr}(K_n \rho K_n^\dagger)} = \frac{[1 - \mathcal{M}_w(\rho)]K_n \delta K_n^\dagger}{\text{Tr}(K_n \rho K_n^\dagger)} + \frac{\mathcal{M}_w(\rho)K_n \tau K_n^\dagger}{\text{Tr}(K_n \rho K_n^\dagger)}.
\]
So
\[
\mathcal{M}_w\left( \frac{K_n \rho K_n^\dagger}{\text{Tr}(K_n \rho K_n^\dagger)} \right) = \mathcal{M}_w\left( \frac{[1 - \mathcal{M}_w(\rho)]K_n \delta K_n^\dagger}{\text{Tr}(K_n \rho K_n^\dagger)} + \frac{\mathcal{M}_w(\rho)K_n \tau K_n^\dagger}{\text{Tr}(K_n \rho K_n^\dagger)} \right) \leq \mathcal{M}_w\left( \frac{[1 - \mathcal{M}_w(\rho)]K_n \delta K_n^\dagger}{\text{Tr}(K_n \rho K_n^\dagger)} \right) + \mathcal{M}_w\left( \frac{\mathcal{M}_w(\rho)K_n \tau K_n^\dagger}{\text{Tr}(K_n \rho K_n^\dagger)} \right) \leq \mathcal{M}_w(\rho)\text{Tr}(K_n \tau K_n^\dagger) \leq \mathcal{M}_w(\rho)\text{Tr}(\tau K_n K_n^\dagger) \leq \mathcal{M}_w(\rho)\text{Tr}(\tau K_n K_n^\dagger).\]
In Eq.(33), the first inequality is based on that $\mathcal{M}_w(\rho)$ satisfies (S4); the second inequality comes from the fact $0 \leq \mathcal{M}_w(\frac{K_n \rho K_n^\dagger}{\text{Tr}(K_n \rho K_n)}) \leq 1$; the second equality holds because $\frac{K_n \rho K_n^\dagger}{\text{Tr}(K_n \rho K_n)}$ is a free state.

So, one gets

$$\text{Tr}(K_n \rho K_n^\dagger) \mathcal{M}_w(\frac{K_n \rho K_n^\dagger}{\text{Tr}(K_n \rho K_n)}) \leq \mathcal{M}_w(\rho) \text{Tr}(K_n \tilde{\tau} K_n^\dagger).$$

(34)

By summing both sides over $n$, we have

$$\sum_n \text{Tr}(K_n \rho K_n^\dagger) \mathcal{M}_w(\frac{K_n \rho K_n^\dagger}{\text{Tr}(K_n \rho K_n)}) \leq \sum_n \mathcal{M}_w(\rho) \text{Tr}(K_n \tilde{\tau} K_n^\dagger) = \mathcal{M}_w(\rho) \sum_n \text{Tr}(K_n \tilde{\tau} K_n^\dagger)
= \mathcal{M}_w(\rho) \text{Tr}(\tilde{\tau}) = \mathcal{M}_w(\rho).$$

(35)

Here the result that the free operation is trace-preserving has been used.

Finally, we prove $\mathcal{M}_w(\rho)$ satisfies (S2). Let $\Lambda$ be a free operation, then

$$\mathcal{M}_w(\Lambda(\rho)) = \mathcal{M}_w(\sum_n \text{Tr}(K_n \rho K_n^\dagger) \frac{K_n \rho K_n^\dagger}{\text{Tr}(K_n \rho K_n)}) \leq \sum_n \text{Tr}(K_n \rho K_n^\dagger) \mathcal{M}_w(\frac{K_n \rho K_n^\dagger}{\text{Tr}(K_n \rho K_n)}) \leq \mathcal{M}_w(\rho).$$

(36)

In Eq.(36), the first inequality takes the fact that $\mathcal{M}_w(\rho)$ satisfies (S4), the second inequality is true, since $\mathcal{M}_w(\rho)$ satisfies (S3). Eq.(36) implies that $\mathcal{M}_w(\rho)$ meets (S2).

Thus we have proved the above theorem.

**Corollary 4.** There is an upper bound $C_d \mathcal{M}_w(\rho)$ for any superposition measure $\mathcal{C}(\rho)$. Here $C_d$ is the value of $\mathcal{C}(\rho)$ for the maximum superposition state, and $\mathcal{M}_w(\rho)$ stands for the superposition measure based on weight.

**Proof.** Suppose free state $\tilde{\delta}$ and resource state $\tilde{\tau}$ are the optimal states for $\rho$ to minimize $\mathcal{M}_w(\rho)$. So $\rho = [1 - \mathcal{M}_w(\rho)] \tilde{\delta} + \mathcal{M}_w(\rho) \tilde{\tau}$. Because of the convexity of $\mathcal{C}(\rho)$, then

$$\mathcal{C}(\rho) \leq [1 - \mathcal{M}_w(\rho)] \mathcal{C}(\tilde{\delta}) + \mathcal{M}_w(\rho) \mathcal{C}(\tilde{\tau}) = \mathcal{M}_w(\rho) \mathcal{C}(\tilde{\tau}) \leq \mathcal{M}_w(\rho) C_d.$$

(37)

So Corollary 4 is true.

**V. THE GENERALIZATION OF SUPERPOSITION RESOURCE THEORY**

In this section we will generalize the superposition resource theory from the following two perspectives.

**A. Convex function**

Assume that $\mathcal{F}$ is a subset of quantum state space, and there is a convex function $\mathcal{M}$ which maps every state in the set $\mathcal{F}$ to zero and maps the other state to a positive real number. The operation $\Lambda$ constituted by Kraus operators $\{K_n\}$ is called free operation if it satisfies the condition

$$\sum_n \text{Tr}(K_n \rho K_n^\dagger) \mathcal{M}(\frac{K_n \rho K_n^\dagger}{\text{Tr}(K_n \rho K_n)}) \leq \mathcal{M}(\rho)$$

(38)

for every quantum state $\rho$. Then one has the following conclusion.

**Theorem 5.** The convex function $\mathcal{M}$ is the superposition measure with free operation set $\{\Lambda\}$ [47].

**Proof.** Let’s first prove that free operation $\Lambda$ maps a free state to a free state. Since $\mathcal{M}$ is a convex function, then

$$\mathcal{M}(\Lambda(\rho)) \leq \sum_n \text{Tr}(K_n \rho K_n^\dagger) \mathcal{M}(\frac{K_n \rho K_n^\dagger}{\text{Tr}(K_n \rho K_n)}) \leq \mathcal{M}(\rho).$$

(39)
If \( \rho \) is free state, then \( \mathcal{M}(\rho) = 0 \). Thus \( \mathcal{M}(\Lambda(\rho)) = 0 \). It implies that \( \Lambda(\rho) \) is also a free state.

In the case where \( \{\Lambda\} \) is a free operation set, obviously the convex function \( \mathcal{M} \) satisfies (S1), (S2), (S3), (S4), so \( \mathcal{M} \) is a valid measure with free operation set \( \{\Lambda\} \).

**Example 2.** Suppose \( \{|i\rangle\}_{i=1}^d \) is an orthonormal basis and \( V \) is a full rank \( d \times d \) real matrix. We define \( |c_i\rangle = V|i\rangle \). For any state \( \rho = \sum_{ij} \rho_{ij} |c_i\rangle \langle c_j| \) if every \( \rho_{ij} \) is real, one defines the state \( \rho \) as free state \([29, 30]\). It is well known that there is a convex function \( \mathcal{M}(\rho) = S(\rho||\Delta(\rho)) \), where \( \Delta(\rho) = \frac{1}{2}(\rho + \rho') \), \( \rho' = \sum_{ij} \rho_{ij} |c_i\rangle \langle c_i| \). Next we will show that the Kraus operators of the free operation \( \Lambda \) are \( \{K_n = \sum_{ij} c_{ij}^* |c_i^\perp\rangle \langle c_j^\perp| \} \), where \( c_{ij}^* \) is a real number for any \( i, j, n, \) and \( \sum_{n} K_n^\dagger K_n = I \). Furthermore we will also demonstrate that the convex function \( \mathcal{M}(\rho) \) is a superposition measure.

First we show that \( S(\rho||\Delta(\rho)) = 0 \) holds only for the state \( \rho \) whose \( \rho_{ij} \) are real number. Let us show that \( \Delta(\rho) \) is a trace-preserving map. Note that \( |c_i\rangle = V_i |1\rangle + V_{i2} |2\rangle + \ldots + V_{id} |d\rangle \), and the matrix elements of \( V_i \) are all real numbers for arbitrary \( i, j = 1, 2, \ldots, d \), one obtains the transposition matrix of quantum state \( \rho \),

\[
\rho^T = \left( \sum_{ij} \rho_{ij} |c_i\rangle \langle c_j| \right)^T = \sum_{ij} \rho_{ij} (|c_j\rangle \langle c_i|)^* = \sum_{ij} \rho_{ij} |c_j\rangle \langle c_i| = \rho'.
\]

(40)

Therefore, \( \text{Tr}(\Delta(\rho)) = \frac{1}{2}(\text{Tr}\rho + \text{Tr}\rho') = \frac{1}{2}(\text{Tr}\rho + \text{Tr}\rho^T) = \text{Tr}\rho \). It means \( \Delta(\rho) \) is a trace-preserving map. Later on, we will show that \( \rho = \Delta(\rho) \) only if \( \rho_{ij} \) is real. Evidently, \( \rho^T = \sum_{ij} \rho_{ij}^* |c_j\rangle \langle c_i| = \sum_{ij} \rho_{ij}^* |c_i\rangle \langle c_j| \). As \( \rho^T = \rho \), so \( \rho_{ij}^* = \rho_{ij} \).

Therefore

\[
\rho + \rho' = \sum_{ij} (\rho_{ij} + \rho_{ij}^*) |c_i\rangle \langle c_j| = \sum_{ij} (\rho_{ij} + \rho_{ij}^*) |c_i\rangle \langle c_i|.
\]

(41)

Hence, if \( \rho = \Delta(\rho) \), we have \( \rho_{ij} = \rho_{ij}^* \), that means \( \rho_{ij} \) are real. Thus \( S(\rho||\Delta(\rho)) \) can map just the state whose \( \rho_{ij} \) are real to 0. Therefore we have demonstrated that \( S(\rho||\Delta(\rho)) = 0 \) holds only for the state \( \rho \) whose \( \rho_{ij} \) are real number.

Next we will show that \( S(\rho||\Delta(\rho)) \) is convex. By the joint convexity of relative entropy \([48]\),

\[
S(\rho||\sigma) \leq \sum_i p_i S(\rho_i||\sigma_i),
\]

(42)

where \( \rho = \sum_i p_i \rho_i \), \( \sigma = \sum_i p_i \sigma_i \). So

\[
S(\rho||\Delta(\rho)) = S\left( \sum_i p_i |\psi_i\rangle \langle \psi_i| || \Delta\left( \sum_i p_i |\psi_i\rangle \langle \psi_i| \right) \right) = S\left( \sum_i p_i |\psi_i\rangle \langle \psi_i| \right) \sum_i p_i \Delta(|\psi_i\rangle \langle \psi_i|) \leq \sum_i p_i S(|\psi_i\rangle \langle \psi_i| || \Delta(|\psi_i\rangle \langle \psi_i|)),
\]

(43)

where \( \rho = \sum_i p_i |\psi_i\rangle \langle \psi_i| \). It implies \( S(\rho||\Delta(\rho)) \) is convex.

Finally, we’re going to show how to obtain \( \{K_n\} \). According to Ref.\([40]\), for any completely positive trace-preserving map formed by Kraus operator \( \{L_n\} \), we have

\[
\sum_n \text{Tr}(L_n \rho L_n^\dagger) S\left( \frac{L_n \rho L_n^\dagger}{\text{Tr}(L_n \rho L_n^\dagger)} || \frac{L_n \delta L_n^\dagger}{\text{Tr}(L_n \delta L_n^\dagger)} \right) \leq \sum_n S(L_n \rho L_n^\dagger || L_n \delta L_n^\dagger) \leq S(\rho||\delta).
\]

(44)

Therefore, if the Kraus operators \( \{K_n\} \) which constitute \( \Lambda \) satisfy \( K_n \Delta(\rho) K_n^\dagger = \Delta(\rho_\Lambda K_n^\dagger) \), then the following result
holds

\[
\begin{align*}
\sum_n \text{Tr}(K_n \rho K_n^\dagger) S\left( \frac{K_n \rho K_n^\dagger}{\text{Tr}(K_n \rho K_n)} \right) & \leq \sum_n \text{Tr}(K_n \rho K_n^\dagger) S\left( \frac{K_n \rho K_n^\dagger}{\text{Tr}(K_n \rho K_n)} \right) \\
& = \sum_n \text{Tr}(K_n \rho K_n^\dagger) S\left( \frac{K_n \rho K_n^\dagger}{\text{Tr}(K_n \rho K_n)} \right) \\
& = \sum_n \text{Tr}(K_n \rho K_n^\dagger) S\left( \frac{K_n \rho K_n^\dagger}{\text{Tr}(K_n \rho K_n)} \right) \\
& \leq S(\rho \| \Delta(\rho))
\end{align*}
\]

Next we give a concrete form of the Kraus operator \( \{ K_n \} \) of \( \Lambda \) satisfying \( K_n \Delta(\rho) K_n^\dagger = \Delta(\rho) K_n^\dagger \). Choose \( K_n = \sum_{ij} c_{ij}^n |c_i^+ \rangle \langle c_j^+| \). Evidently,

\[
\begin{align*}
\Delta(K_n \rho K_n^\dagger) &= \Delta(\sum_{ij} c_{ij}^n |c_i^+ \rangle \langle c_j^+| \sum_{\alpha \beta} \rho_{\alpha \beta} |c_\alpha \rangle \langle c_\beta| \sum_{ml} (c_{ml}^n)^* (c_i^+)^* \langle c_m^+|) \\
& = \Delta(\sum_{ijml} c_{ij}^n \rho_{jl} (c_{ml}^n)^* |c_i^+ \rangle \langle c_m^+|) \\
& = \text{Re}(\sum_{ij} c_{ij}^n \rho_{jl} (c_{ml}^n)^* |c_i^+ \rangle \langle c_m^+|),
\end{align*}
\]

(45)

\[
K_n \Delta(\rho) K_n^\dagger = \sum_{ij} c_{ij}^n |c_i^+ \rangle \langle c_j^+| \sum_{\alpha \beta} \rho_{\alpha \beta} |c_\alpha \rangle \langle c_\beta| \sum_{ml} (c_{ml}^n)^* (c_i^+)^* \langle c_m^+| \\
& = \sum_{ij} \sum_{ml} c_{ij}^n \text{Re}(\rho_{jl}) (c_{ml}^n)^* |c_i^+ \rangle \langle c_m^+|.
\]

So when \( c_{ij}^n \) is a real number for any \( i, j, n \), we have \( \text{Re}(c_{ij}^n \rho_{jl} (c_{ml}^n)^*) = c_{ij}^n \text{Re}(\rho_{jl})(c_{ml}^n)^* \) for any \( j, l \). Therefore \( \Delta(K_n \rho K_n^\dagger) = K_n \Delta(\rho) K_n^\dagger \).

Thus when the Kraus operators of the free operation \( \{ \Lambda \} \) are \( \{ K_n = \sum_{ij} c_{ij}^n |c_i^+ \rangle \langle c_j^+| \}, \) where \( c_{ij}^n \) is a real number for any \( i, j, n \), and \( \sum_n K_n K_n = \mathbb{I} \), the convex function \( M(\rho) = S(\rho \| \Delta(\rho)) \) is a superposition measure.

**B. Operators**

Suppose that there are operators \( \{ E_i \}_{i=1}^n \) acting on the Hilbert space, where \( E_i E_j = E_i \delta_{ij} \). It is easy to see that the following operators

\[
E_1 = \sum_{i=1}^{d_1} |c_i \rangle \langle c_i^+|, \quad E_2 = \sum_{i=d_1+1}^{d_2} |c_i \rangle \langle c_i^+|, \quad E_3 = \sum_{i=d_2+1}^{d_3} |c_i \rangle \langle c_i^+|, \quad \ldots, \quad E_n = \sum_{i=d_{n-1}+1}^{d_n} |c_i \rangle \langle c_i^+|,
\]

satisfy the requirement, where \( d > d_{n-1} > d_{n-2} > \cdots > d_2 > d_1 \geq 1 \).

Then we define \( \sigma = \frac{\sum_i E_i \rho E_i^\dagger}{\text{Tr} \left( \sum_i E_i \rho E_i^\dagger \right)} \) as the free states, where \( \rho \) is an arbitrary state in the Hilbert space. The set of free states is denoted by \( \mathcal{F} \). Let \( K_n = \sum_i E_{f_n(i)} c_{n,i} E_i \) [49], where \( f_n(i) \) is index permutation function and \( c_{n,i} \) is
complex matrix. Obviously,
\[
K_n\sigma K_n^\dagger = \sum_i E_{f_n(i)} c_{n,i} E_i \sum_{\alpha=1} E_{\alpha} \rho E_{\alpha}^\dagger \sum_j E_j^\dagger \sigma_{n,j} E_{f_n(i)}^\dagger / \text{Tr}(\sum_i E_i \rho E_i^\dagger) 
= \sum_i E_{f_n(i)} c_{n,i} E_i \rho E_i^\dagger c_{n,i} E_{f_n(i)}^\dagger / \text{Tr}(\sum_i E_i \rho E_i^\dagger).
\] (48)

Define the free operation \( \Lambda = \{ K_n \mid \sum_n K_n K_n^\dagger = I\} \). Clearly \( \Lambda \) maps a free state to a free state. So with the free operation set \( \{ \Lambda \} \) one has the following two superposition measures.

1. Generalized superposition measure based on weight:
\[
\mathcal{M}_w(\rho) = \min_{\tau \text{ density matrix}} \{ 1 - \lambda \geq 0 | \rho = \lambda \delta + (1 - \lambda) \tau, \delta \in \mathcal{F} \}.
\] (49)

2. Generalized superposition measure based on robustness:
\[
\mathcal{M}_R(\rho) = \min_{\tau \text{ density matrix}} \{ s \geq 0 | \frac{\rho + \tau}{1 + s} \in \mathcal{F} \}.
\] (50)

It is easy to see from the Sec.IV, with the good definition of free states and free operations, generalized superposition measure \( \mathcal{M}_w(\rho) \) satisfies (S1), (S2), (S3) and (S4). By Ref.[50], we can obtain that generalized superposition measure \( \mathcal{M}_R(\rho) \) satisfies (S1), (S2), (S3) and (S4) with free states and free operations well defined.

VI. CONCLUSION

In this paper, we present three methods of constructing superposition measure, which are methods based on convex roof extended, state transformation and weight, respectively. We demonstrate that \( \mathcal{M}_{\text{rel,net}}(\rho) \) and \( \mathcal{M}_{\text{net}}(\rho) \) are superposition measures, and that \( \mathcal{M}_{\text{net}}(\rho) \geq \mathcal{M}_{\text{net}}(\rho) \). We prove that \( \mathcal{C}_d M_w(\rho) \) is an upper bound of any superposition measure \( \mathcal{C}(\rho) \), where \( \mathcal{C} \) is the value of \( \mathcal{C}(\rho) \) for the maximum superposition state. Our approach provides new insights on better understanding of superposition measures. We also generalize the superposition resource theory with convex function \( \mathcal{M} \), and give a method to find free operation and new superposition measure. Finally, we generalize the superposition resource theory based on operators, and obtain two good superposition measures with well-defined free operation. We hope that these works can help us to better understand the resource theory of quantum superposition.

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