A CURIE-WEISS THEORY OF THE CONTINUUM WIDOM-ROWLINSON MODEL

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Abstract. A version of the continuum Widom-Rowlinson model is introduced and studied. It is a two-component gas of point particles placed in $\mathbb{R}^d$ in which like particles do not interact and unlike particles contained in a given vessel of volume $V$ repel each other with intensity $a/V$. This model is thermodynamically equivalent to a one-component gas with multi-particle interaction. For both models, a rigorous theory of a phase transition is presented and the ways of its construction in the framework of the grand canonical formalism are outlined.

1. Introduction

The rigorous theory of phase transitions in continuum particle systems has got much more modest results as compared to its counterpart dealing with lattices, graphs, etc. In fact, there exist only few models with local interactions in which the existence of a liquid-vapor phase transition was rigorously proved. One of them is the model introduced in [1] by B. Widom and J. S. Rowlinson in which the potential energy of $n$ point particles located at $x_1, \ldots, x_n \in \mathbb{R}^d$ is set to be $\theta [W(x_1, \ldots, x_n) - n]$, where $\theta > 0$ is a parameter and $W$ is the volume of the area $\bigcup_{i=1}^n B(x_i)$ covered by the balls of unite volume centered at the corresponding particles. This model is thermodynamically equivalent to a two-component system with binary interactions in which the interaction between unlike particles is a hard-core repulsion and is zero otherwise. In [2], D. Ruelle proved that this model undergoes a phase transition of first order whenever $d \geq 2$. Later on, this result was extended in [3], see also [4] for a review. However, these theories give not too much for understanding the details of the phenomenon. No rigorous results are available on the behavior at the phase-transition threshold. The very existence of such a threshold remains unknown for this model. At the same time, for a number of lattice models the mean field results essentially improve understanding phase transitions in the corresponding models with local interactions [5]. It is then quite natural to develop the mean field theory of phase transitions also in continuum systems. For the Widom-Rowlinson model, the first attempt to do this was undertaken already in [1, Sect. VII]. Assuming that the particles are distributed in a given vessel “at random” the authors heuristically deduced an equation of state (eq. (7.4) in [1]) which manifests a first order phase transition. Unfortunately, such and similar heuristic mean-field theories (sometimes called naive, cf. [6] page 216]) are not free from mathematical inconsistencies and other drawbacks of mathematical nature that diminish the value of their results. The first steps

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in developing a rigorous mean-field theory of a liquid-vapour phase transition were made by J. L. Lebowitz and O. Penrose in [7] where Kac-type interaction potentials were employed to obtain (in the canonical formalism) an explicit form of the pressure-density dependence that describes a first order phase transition. Another way of developing a rigorous mean field theory is to use Curie-Weiss interaction potentials, see [8, 9, 10]. For lattice systems, this way was formulated as a coherent mathematical theory in the framework of which thermodynamic phases are constructed as probability measures on the spin-configuration spaces, see, e.g., [11, Section II]. The aim of this letter is to report the results of a rigorous study of an analog of the Widom-Rowlinson model with Curie-Weiss interaction potentials made in the grand canonical formalism.

2. The Model

We deal with infinite systems of point particles placed in the space $\mathbb{R}^d$. If the particles do not interact, their state of thermal equilibrium (phase) is a Poisson probability measure $P_z$. The only parameter characterizing this measure is activity $z = e^\mu$, where $\mu = (\text{physical chemical potential})/k_B T$, $k_B$ and $T > 0$ being Boltzmann’s constant and absolute temperature, respectively. For a vessel $\Lambda \subset \mathbb{R}^d$ of finite volume $V$ and a nonnegative integer $n$, the measure $P_z$ assigns the probability

$$P_z(\Gamma_{\Lambda,n}) = \frac{(zV)^n}{n!} \exp(-zV)$$

(1)

to the event $\Gamma_{\Lambda,n}$: “$\Lambda$ contains $n$ particles”. Let now point particles of two types, 0 and 1, be placed in the same space $\mathbb{R}^d$. If they do not interact, their state of thermal equilibrium is the Poisson measure $P_{z_0} \otimes P_{z_1}$ such that the event $\Gamma_{\Lambda,n_0} \times \Gamma_{\Lambda,n_1}$ has probability

$$P_{z_{0,1}}(\Gamma_{\Lambda,n_0} \times \Gamma_{\Lambda,n_1}) = P_{z_0}(\Gamma_{\Lambda,n_0}) \cdot P_{z_1}(\Gamma_{\Lambda,n_1}),$$

(2)

where $P_{z_i}(\Gamma_{\Lambda,n_i})$, $i = 0, 1$, are as in (1) and the event is “$\Lambda$ contains $n_0$ particles of type 0 and $n_1$ particles of type 1”. In this case, the particle densities are $\varrho_i = z_i = e^{\mu_i}$, $i = 0, 1$, and the equation of state reads

$$p = \varrho_0 + \varrho_1 = e^{\mu_0} + e^{\mu_1},$$

where $p$ is the pressure in the system.

For interacting particles, phases are constructed as limits $\Lambda \to \mathbb{R}^d$ of local Gibbs measures $P^\Lambda_{z,\Phi}$ ($P^\Lambda_{z_0, z_1}$ for two-component systems) describing the portion of the particles contained in the vessel $\Lambda$ and interacting with the energy $\Phi$, see, e.g., [4]. In this work, we introduce two models that – like the Widom-Rowlinson model – can be considered as two versions of the same model. The first one is a two-component gas of point particles in $\mathbb{R}^d$. For a vessel $\Lambda \subset \mathbb{R}^d$ of volume $V$, unlike particles contained in $\Lambda$ repel each other with intensity $a/V > 0$, whereas like particles do not interact. Thus, the potential energy of the collection of $n_0$ particles of type 0 located at $x_{0_i}^0 \in \Lambda$ and of $n_1$ particles of type 1 located at $x_{1_i}^1 \in \Lambda$ is

$$\Phi_V(x_{0_1}^0, \ldots, x_{n_0}^0; x_{1_1}^1, \ldots, x_{n_1}^1) = \sum_{i=1}^{n_0} \sum_{j=1}^{n_1} a \frac{r_{ij}}{V} = \frac{a}{V} n_0 n_1.$$

(3)
The grand canonical partition function of this collection then is
\[ \Xi_V(a, \mu_0, \mu_1) \]
\[ = \sum_{n_0, n_1=0}^{\infty} \frac{1}{n_0! n_1!} \int_{\Lambda^{n_0}} \int_{\Lambda^{n_1}} \exp \left( \mu_0 n_0 + \mu_1 n_1 - \frac{a}{V} n_0 n_1 \right) dx_1^0 \cdots dx_0^{n_0} dx_1^1 \cdots dx_1^{n_1} \]
\[ = \sum_{n_0, n_1=0}^{\infty} \frac{V^{n_0+n_1}}{n_0! n_1!} \exp \left( \mu_0 n_0 + \mu_1 n_1 - \frac{a}{V} n_0 n_1 \right). \]

Here the interaction parameter \( a > 0 \) and the chemical potentials \( \mu_i \in \mathbb{R}, \ i = 1, 2, \) include the reciprocal temperature \( \beta \) and thus are dimensionless. The second our model is a one-component system of point particles interacting as follows. For a vessel \( \Lambda \) of volume \( V \), the potential energy of the collection of \( n \) particles located at \( x_0, \ldots, x_n \in \Lambda \) is set to be
\[ \hat{\Phi}_V(x_1, \ldots, x_n) = V\theta \left[ 1 - \exp \left( -\frac{a}{V} n \right) \right]. \]

Here \( \theta > 0 \) is a parameter, similar to that in \([1]\) mentioned above. Then the corresponding grand canonical partition function is
\[ \hat{\Xi}_V(a, \mu, \theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \exp \left( \mu n - V\theta \left[ 1 - \exp \left( -\frac{a}{V} n \right) \right] \right) dx_1 \cdots dx_n \]
\[ = \sum_{n=0}^{\infty} \frac{V^n}{n!} \exp \left( \mu n - V\theta \left[ 1 - \exp \left( -\frac{a}{V} n \right) \right] \right) \]
\[ = \exp \left( -V\theta \right) \Xi_V(a, \mu, \ln \theta). \]

The latter equality can readily be derived by summing out in (4) over \( n_1 \). The dependence of the pressure \( p \) in the two-component system (resp. \( \hat{p} \) in the one-component system) on \( a \) and \( \mu_i, \ i = 0, 1 \) (resp. on \( a, \theta \) and \( \mu \)) is then obtained in the thermodynamic limit
\[ p = p(a, \mu_0, \mu_1) = \lim_{V \to +\infty} \frac{1}{V} \ln \Xi_V(a, \mu_0, \mu_1), \]
\[ \hat{p} = \hat{p}(a, \theta, \mu) = \lim_{V \to +\infty} \frac{1}{V} \ln \hat{\Xi}_V(a, \theta, \mu), \]
which by \([9]\) yields \( \hat{p} = p - \theta \). Thus, the particle density \( \varrho \) in the one-component system and the density \( \varrho_0 \) of the particles of type 0 in the two-component system are related to each other by
\[ \varrho = \frac{\partial \hat{p}}{\partial \mu} = \frac{\partial p}{\partial \mu_0} \bigg|_{\mu_0=\mu, \mu_1=\ln \theta} = \varrho_0. \]

Therefore, the study of the one-component system amounts to studying its two-component counterpart by employing \([1]\) and \([7]\).  

### 3. The Results

In view of \([1]\), we will deal with three thermodynamic variables \( a, \mu_0, \mu_1 \), and hence with the phase space
\[ \mathcal{F} = \{(a, \mu_0, \mu_1) : a \geq 0, \mu_0, \mu_1 \in \mathbb{R} \}. \]
Our main result is the statement that $\mathcal{F}$ can be divided into three disjoint subsets, i.e., presented as

$$\mathcal{F} = \mathcal{R} \cup \mathcal{C} \cup \mathcal{M}. \quad (8)$$

A point $(a, \mu_0, \mu_1)$ belongs to one of these subsets according to the number of global maxima of the function

$$E(y) = f(a, \mu_0 + y) + f(a, \mu_1 - y) - \frac{y^2}{2a}, \quad (9)$$

where

$$f(a, x) = \frac{a}{2} [u(a, x)]^2 + u(a, x), \quad x \in \mathbb{R}. \quad (10)$$

Here $u$ is a special function, which can be expressed through Lambert’s $W$-function\(^\text{[12]}\) as follows

$$u(a, x) = \frac{1}{a} W(a e^x), \quad x \in \mathbb{R}. \quad (11)$$

For a fixed $a > 0$, the function $\mathbb{R} \ni x \mapsto u(a, x)$ can be obtained as the inverse to $(0, +\infty) \ni u \mapsto x(u) = au + \ln u$, by which one gets that

$$u(a, x) \exp [au(a, x)] = e^x, \quad (12)$$

$$\frac{\partial}{\partial x} u(a, x) = \frac{u(a, x)}{1 + au(a, x)}. \quad (13)$$

By (10) and (12) it follows that the points of global maximum of $E$ are also its local maximum points and hence can be obtained from the equation

$$y = au(a, \mu_0 + y) - au(a, \mu_1 - y), \quad y \in \mathbb{R}. \quad (14)$$

The single-phase domain $\mathcal{R} = \mathcal{F} \setminus (\mathcal{C} \cup \mathcal{M})$, cf. (8), consists of all those $(a, \mu_0, \mu_1)$ for each of which there exists a unique global maximum of $E$ (at some $y_\ast$). Then the unique phase existing at this point $(a, \mu_0, \mu_1)$ is the Poisson state $P_{\tilde{z}_0, \tilde{z}_1}$, see (1) and (2), where

$$\tilde{z}_0 = u(a, \mu_0 + y_\ast), \quad \tilde{z}_1 = u(a, \mu_1 - y_\ast). \quad (15)$$

The set

$$\mathcal{M} := \{(a, \mu, \mu) : a > 0, \mu > 1 - \ln a\}$$

consists of phase coexistence points, and

$$\mathcal{C} := \{(a, 1 - \ln a, 1 - \ln a) : a > 0\} \quad (16)$$

is the line of critical points. For $(a, \mu_0, \mu_1) \in \mathcal{M}$ (i.e., for $\mu_0 = \mu_1 = \mu > 1 - \ln a$), the function in (9) has two equal maxima located at $\pm \bar{y}(a, \mu)$, where the order parameter $\bar{y}(a, \mu) > 0$ is the unique solution of the equation $\psi(y) = \mu - (1 - \ln a)$ with

$$\psi(y) = y + \frac{y}{e^y - 1} - 1 + \ln \frac{y}{e^y - 1}, \quad y > 0.$$

For small $y > 0$, we have that $\psi(y) = y^2/24 + o(y^2)$. Thus,

$$\bar{y}(a, \mu) = \sqrt{24(\mu - (1 - \ln a))} + o(\mu - (1 - \ln a))$$

for small positive $\mu - (1 - \ln a)$. For $\bar{y}(a, \mu) > 0$, there exist two phases $P_{\tilde{z}^+, \tilde{z}^-}$ and $P_{\tilde{z}^-, \tilde{z}^+}$, where

$$\tilde{z}^\pm = u(a, \mu \pm \bar{y}(a, \mu)).$$
The equation of state of the two-component system has the following form, cf. (16),

\[ p = a \varrho_0 \varrho_1 + \varrho_0 + \varrho_1, \tag{17} \]

where the densities \( \varrho_i \) are, cf. (14),

\[ \varrho_0 = u(a, \mu_0 + y_\ast), \quad \varrho_1 = u(a, \mu_1 - y_\ast). \tag{18} \]

Note that each \( \varrho_i \) depends on both \( \mu_0, \mu_1 \) and \( \varrho_i = \partial \varrho / \partial \mu_i, \ i = 0, 1, \) cf. (7). Note also that the densities satisfy

\[ \varrho_0 = \exp (\mu_0 - a \varrho_1), \quad \varrho_1 = \exp (\mu_1 - a \varrho_0). \tag{19} \]

That is, due to the repulsion both densities are smaller than in the free case \( a = 0 \).

Let us turn now to the ground states of the two-component model which one obtains by passing to the limit \( a \to +\infty \). To this end, we consider \( \varrho_i, \ i = 0, 1, \) as differentiable functions of \( a \) defined in (12). Let \( \varrho_i, \ i = 0, 1, \) stand for the corresponding \( a \)-derivatives. Differentiating both sides of each equality in (19) after some calculations we get

\[ \dot{\varrho}_0 - \dot{\varrho}_1 = \frac{a \varrho_0 \varrho_1}{1 - a^2 \varrho_0 \varrho_1} (\varrho_0 - \varrho_1). \tag{20} \]

The denominator here is positive by the fact that \( y_\ast \) used in (18) is the point of local maximum of \( E \) given in (9). For \( \mu_0 > \mu_1 \), we have that \( \varrho_0 > \varrho_1 \) for all \( a > 0 \).

Indeed, assuming \( \varrho_0 = \varrho_1 \) for some \( a > 0 \), we then get by (19) that \( e^{\mu_0} = e^{\mu_1} \), which contradicts the assumed inequality \( \mu_0 > \mu_1 \). Thus, by (20) \( \varrho_0 - \varrho_1 \) is an increasing function of \( a \), which yields \( \varrho_0 - \varrho_1 \geq \varrho_0 - \varrho_1 |_{a=0} = e^{\mu_0} - e^{\mu_1} \). By (19) and the latter estimate we then get

\[ \varrho_1 = \varrho_0 \exp (- (\mu_0 - \mu_1) - a (e^{\mu_0} - e^{\mu_1})), \tag{21} \]

Since \( \varrho_0 \leq e^{\mu_0} \), see (19), by (21) we get that \( a \varrho_1 \to 0 \), and hence \( \varrho_1 \to 0 \), as \( a \to +\infty \). At the same time, \( \varrho_0 \geq \varrho_1 + (e^{\mu_0} - e^{\mu_1}) \), which by (19) yields that \( \varrho_0 \to e^{\mu_0} \) as \( a \to +\infty \). By (14) and (18) we thus conclude that the two-component system has two ground states: \( P_{z_0,0} \) and \( P_{0, z_1} \). In each of them, there is only one free component.

Turn now to the one-component system. Its equation of state reads

\[ \hat{p} = a \varrho e^{-a \varrho} + \varrho - \theta (1 - e^{-a \varrho}), \tag{22} \]

that can be obtained by (19) and the formula \( \hat{p} = p - \theta \). Here \( \varrho \) is a function of \( \mu \in \mathbb{R} \) obtained from (18), i.e., \( \varrho = u(a, \mu + y_\ast) \). It is increasing and continuous whenever \( \theta \leq e/a \). For \( \theta > e/a \), \( \varrho \) makes a jump at \( \mu = \ln \varrho \) with one-sided limits \( \lim_{\mu \to \ln \varrho \pm 0} \varrho = u(a, \mu \pm y_\ast(a, \mu)) \). That is, the system undergoes a first-order phase transition with the increment of the density \( \Delta \varrho = \bar{y}(a, \mu)/a \). The expression in (22) can also be used to define \( \hat{p} \) as a function of \( \varrho \). Namely, \( \hat{p} \) is as in (22) for \( \varrho \leq \tilde{z}_0^- \) and \( \varrho \geq \tilde{z}_0^+ \), and \( \hat{p} \equiv \hat{p}_s := a \tilde{z}_0^+ + \tilde{z}_0^- + \tilde{z}_0^- - \theta \) for \( \varrho \in [\tilde{z}_0^-, \tilde{z}_0^+] \). Note that the equation of state in (22) with \( a = 1 \) formally coincides with that found heuristically in (1), in which, however, the horizontal part \( \hat{p} \equiv \hat{p}_s \) should be found from the Maxwell rule, see (13) for more detail.

The part of the phase diagram of the two-component system in the plane in \( \mathcal{F} \) with constant \( a > 0 \) is presented in Fig (1). Points from the grayed area correspond to the existence of three solutions of (13), one of which is \( y_\ast \). Note that \( y_\ast > 0 \).
Figure 1. Phase diagram at fixed $a$

for $\mu_0 > \mu_1$. At the boundaries of this area (symmetric under $\mu_0 \leftrightarrow \mu_1$), (13) has only two solutions. The upper branch of the boundary is described by the equation

$$\eta = \sqrt{\xi^2 - 1} + \ln \left(\xi - \sqrt{\xi^2 - 1}\right),$$

where $\xi = (\mu_0 + \mu_1)/2 + \ln a$ and $\eta = (\mu_1 - \mu_0)/2$. For all points from the complement to the grayed area, (13) has only one solution. Note that $y^* = 0$ for $\mu_0 = \mu_1 < 1 - \ln a$.

4. Deriving the Results

Here we outline the way of deriving the results presented in the preceding section. In [13], these results are formulated and proved as mathematical statements.

By means of the identity

$$-a V_n^0 n_1^1 = -a^2 V_n^0 n_1^1 + a^2 (n_0 - n_1)^2,$$

and by a standard Gaussian formula we rewrite (4) as

$$\Xi_V(a, \mu_0, \mu_1) = \sqrt{\frac{V}{2\pi a}} \int_{-\infty}^{+\infty} \exp \left( V E_V(y) \right) dy,$$

with

$$E_V(y) = f_V(a, \mu_0 + y) + f_V(a, \mu_1 - y) - \frac{y^2}{2a}.$$

Here $f_V$ is defined by the following relation

$$\exp \left( V f_V(a, x) \right) = \sum_{n=0}^{\infty} \frac{V^n}{n!} \exp \left( x n - \frac{a}{2V} n^2 \right),$$

and thus is an infinitely differentiable function of $x \in \mathbb{R}$ for each fixed $a > 0$ and $V > 0$. Then so is $E_V$ as a function of $y \in \mathbb{R}$. In view of (6) we have to find the large $V$ asymptotic in (23). To this end we employ a more advanced version of Laplace’s method as $E_V$ depends on $V$. According to [13, Theorem 2.2], this amounts to proving that the third $x$-derivative of $f_V$ is bounded as $V \to +\infty$,
which is the most challenging aspect of the theory. Taking the \( x \)-derivative of both sides of (24) we obtain that 
\[ u_V'(a, x) \exp \left( \int_0^1 u_V \left( a, x - \frac{a}{V} t \right) \, dt \right) = \exp \left( x - \frac{a}{2V} \right). \]

On the other hand, also by (24) we get that
\[ u_V(a, x) = \langle n \rangle / V = \frac{1}{V} \sum_{n=1}^{\infty} n \pi_n, \]

with
\[ \pi_n = \frac{V^n}{n!} \exp \left( nx - \frac{an^2}{2V} \right) / \sum_{n=0}^{\infty} \frac{V^n}{n!} \exp \left( nx - \frac{an^2}{2V} \right). \]

Then the consecutive \( x \)-derivatives of \( u_V \) are
\[ u_V'(a, x) = \frac{1}{V} \langle (n - \langle n \rangle)^2 \rangle, \quad u_V''(a, x) = \frac{1}{V} \langle (n - \langle n \rangle)^3 \rangle. \]

These formulas allow for obtaining uniform in \( V \) upper bounds for \( u_V'(a, x) \) and \( |u_V''(a, x)| \). Thereafter, we apply the mentioned version of Laplace’s method by which the problem of calculating \( p \) in (9) is reduced to finding the global maxima of \( E \) given in (9). This yields, in particular, that \( p \) is given by (17). We also prove that \( f_V \rightarrow f \), \( u_V \rightarrow u \) and \( u_V' \rightarrow u' \) as \( V \rightarrow \infty \), where \( f, u \) and \( u' \) are as in (10), (11) and (12), respectively. The proof that the limiting states are \( P_{\tilde{z}_i, \tilde{z}_i} \) is done by showing that the correlation functions of the local Gibbs measures, see (10), converge as \( V \rightarrow +\infty \) to those of \( P_{\tilde{z}_i, \tilde{z}_i} \). Here we also employ the result of (11) mentioned above, as well as the convergence of the derivatives of \( f_V \) just mentioned. Then the representation of \( \mathcal{F} \) in (5) with \( M \) and \( C \) as in (15) and (16), respectively, are obtained by studying the global maxima of \( E \).

5. Concluding Remarks

We recall that the parameters \( a, V, \mu_0 \) and \( \mu_1 \) in the expression for \( \Xi_V \) in (4) are dimensionless. By chosen them we, in fact, fix some metric of the habitat space \( \mathbb{R}^d \). The change of this metric (scale) can be done by passing to \( V^\alpha := \alpha V \) for some scale parameter \( \alpha > 0 \). This, of course, leads to the change \( \xi_i \rightarrow \tilde{\xi}_i = \xi_i / \alpha \), \( i = 0, 1 \), and also to \( a \rightarrow a^\alpha = \alpha a \) and \( p \rightarrow p^\alpha = p / \alpha \), cf. (3) and (17), respectively. Then by (19) we obtain that the rescaled densities and the interaction parameter \( a^\alpha \) satisfy
\[ \tilde{\rho}_0^\alpha = \exp \left( \mu_0^\alpha - a^\alpha \tilde{\rho}_1^\alpha \right), \quad \tilde{\rho}_1^\alpha = \exp \left( \mu_1^\alpha - a^\alpha \tilde{\rho}_0^\alpha \right), \]

with \( \mu_i^\alpha = \mu_i - \ln \alpha, \ i = 0, 1 \). By this we conclude that the description of the phases corresponding to the points \( (a, \mu_0, \mu_1) \in \mathcal{F} \) is scale-invariant. Thus, one can choose \( \alpha = 1/a \) and hence consider the plane in \( \mathcal{F} \) with \( a = 1 \). Then the results corresponding to a general point \( (a, \mu_0, \mu_1) \in \mathcal{F} \) can be obtained from those obtained for \( a = 1 \) by the rescaling as just described. Note, however, that in view of taking the thermodynamic limit \( V \rightarrow +\infty \), the role of the scale of \( V \) may get be lost. Then considering \( a \) as an interaction parameter helps to reveal it. Moreover, from our analysis it follows that the phase transition occurs for an arbitrarily small interaction \( a \). Also for the following reasons it might be worth to deal with general values of \( a \): (i) to be able to pass to the free case \( a = 0 \); (ii) to obtain the ground states in the limit \( a \rightarrow +\infty \) as described above; (iii) to get clues on the phase transition in the original Widom-Rowlinson model in which the
metric is rather fixed by the choosing the radius of the hard-core repulsion. In this model the interaction energy of the collection of \( n_0 \) particles of type zero located at \( x_0^1, \ldots, x_{n_0}^1 \) with \( n_1 \) particles of type one located at \( x_1^1, \ldots, x_{n_1}^1 \) is written in the form, cf. [4, Section 10.2],

\[
\Phi(x_0^1, \ldots, x_{n_0}^1, x_1^1, \ldots, x_{n_1}^1) = \sum_{k=1}^{n_0} \sum_{l=1}^{n_1} \infty \mathbb{1}_{|x_k^0 - x_l^1| \leq r_d}(x_k^0, x_l^1),
\]

where \( \mathbb{1}_{|x_k^0 - x_l^1| \leq r_d} \) is the corresponding characteristic function and \( r_d \) is the radius of the ball in \( \mathbb{R}^d \) of unit volume. In contrast to our \( \Phi_V \) given in [3] \( \Phi \) takes values either zero or infinity, and hence no parameter like our \( a \) can be associated with this model. Regarding its thermodynamic phases the following is known. There exist \( \mu_*, \mu^* \in \mathbb{R} \) such that \( \mu_* < \mu^* \) and: (a) for \( \mu_0 = \mu_1 = \mu < \mu_* \), there exists only one thermodynamic phase; (b) for \( \mu_0 = \mu_1 = \mu > \mu^* \), there exist at least two thermodynamic phases. In contrast to our description, nothing is known about the threshold that might exist on the interval \([\mu_*, \mu^*]\). Having in mind that for spin systems on \( \mathbb{Z}^d \) with local (properly normed) interactions the mean-field theory of a phase transition becomes exact in the limit \( d \to +\infty \), see [6, Theorem II.14.1, page 228], one might speculate that also for the model in (25) there exists a threshold value \( \mu_c = 1 - \ln a_{\text{WR}}(d) \in [\mu_*, \mu^*] \) with \( a_{\text{WR}}(d) \) satisfying \( a_{\text{WR}}(d) \to a_{\text{WR}} \) as \( d \to +\infty \) for some \( a_{\text{WR}} > 0 \). If this is true, then our model with a particular value \( a = a_{\text{WR}} \) can be considered as a mean-field limit of the original Widom-Rowlinson model.

\[
1_{|x_k^0 - x_l^1| \leq r_d}(x_k^0, x_l^1),
\]

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