New insight into BRST anomalies in superstring theory

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ABSTRACT

Based on the extended BRST formalism of Batalin, Fradkin and Vilkovisky, we perform a general algebraic analysis of the BRST anomalies in superstring theory of Neveu-Schwarz-Ramond. Consistency conditions on the BRST anomalies are completely solved. The genuine super-Virasoro anomaly is identified with the essentially unique solution to the consistency condition without any reference to a particular gauge for the 2D supergravity fields. In a configuration space where metric and gravitino fields are properly constructed, general form of the super-Weyl anomaly is obtained from the super-Virasoro anomaly as its descendant. We give a novel local action of super-Liouville type, which plays a role of Wess-Zumino-Witten term shifting the super-Virasoro anomaly into the super-Weyl anomaly. These results reveal a hierarchical relationship in the BRST anomalies.

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1. Introduction

The Virasoro anomaly, the Weyl anomaly and the nonvanishing square of BRST charge are known to represent a physically equivalent obstruction in quantization of relativistic strings at subcritical dimensions. In spite of some efforts using the BJL-limit [1] and cohomological techniques [2], however, their internal relationships have been only partially revealed so far. This is mainly because that the considerations on these anomalies including those in the classic references [3-7] have been done only in particular classes of gauges. A general discussion where the gauge dependence of these anomalies is satisfyingly explored is highly desired. This goal is achieved recently in a previous paper [8], hereafter referred to as I, by extensively using the generalized hamiltonian formalism of Batalin, Fradkin and Vilkovisky (BFV) [9]*. It shows up a hierarchical relationship among the anomalies, which can not be clearly seen in the conventional gauge-fixed analyses.

The purpose of this paper is to extend the investigation given in I for bosonic string to superstring of Neveu-Schwarz-Ramond† formulated as 2D supergravity (2D SUGRA) theory coupled with superconformal matter [12]. This extension contains a nontrivial task to find a suitable BRST quantization scheme for 2D SUGRA. The BRST quantization of 2D SUGRA based on the configuration space introduces a lot of redundant variables to maintain off-shell nilpotency of the BRST charge [13,14]. The general scheme given in [15] includes not only the auxiliary field in the supermultiplet but also some pure gauge fields associated with local symmetries. The vanishing-curvature conditions for the gauge fields, however, can not always be solved explicitly, and it would make the general investigation we wish to develop rather complicated‡.

The use of the extended phase space (EPS) of BFV leads to a new BRST quantization scheme as given in sect.2, nicely avoiding such a complexity. The EPS of 2D SUGRA is simplified by taking local Lorentz and Weyl invariant components of spinor fields. The off-shell nilpotency of BRST charge is ensured from the beginning (at the classical level) without introducing the redundant variables appeared in the configuration-space approach[15]. The EPS is yet enough so large that one can easily find out at which stage and how the gauge dependence appears

* See [10] for review articles and references therein.
† See [11] for a review and references therein.
‡ Since we will not assume that supersymmetry is intact upon quantization, superspace formulations [16] are not used here. As for the superfield formulation for anomalies in 2D SUGRA, see refs.[17].
in specification of the BRST anomalies. The quantization scheme given here is of course noncovariant, but the final results of the anomalies turn out to fall into simple covariant expressions.

Once the EPS is fixed, we apply the general method[18,8] for analyzing gauge anomaly as summarized in sect.3. An anomaly can be identified with a cohomologically nontrivial solution of a set of consistency conditions[19], which is obtained in $\hbar$ expansion by imposing the super-Jacobi identities on the anomalous commutators for BRST charge $Q$ and total hamiltonian $H_T$. For theories with reparametrization invariance such as gravity and string theories the $Q^2$ anomaly is of primary importance, because it automatically determines the anomaly in the commutator $[Q, H_T]$. In the BFV formalism, $Q$ is directly constructed from the classically first-class constraints without invoking gauge-fixing conditions, and therefore the $Q^2$ anomaly can be investigated in a completely gauge-independent manner. Only the anomalous $[Q, H_T]$ involves the gauge dependence via the total hamiltonian, which however can be easily tracked in this formalism. It should be stressed that the consistency conditions in our formalism[18,8] may correspond to a hamiltonian version of the descent equations[20,22], but their physical meaning is much more clear than the purely mathematical methods.

In sect.4, we solve algebraically in the full EPS the consistency condition on the BRST anomalies to find the most general form of the $Q^2$ anomaly in 2D SUGRA system. It can be identified with the super-Virasoro anomaly which is generated as anomalous Schwinger terms in the commutator algebra of super-Virasoro constraints defined by the BRST transform of the ghost momenta. The overall factor of the anomaly may be fixed by the explicit calculation using, for instance, the normal ordering prescription. This is the genuine anomaly of the superstring theory in our formalism, being pregeometric and determined without any reference to gauge conditions. In order to identify this $Q^2$ anomaly with the super-reparametrization or the super-Weyl anomaly, one must define metric variables and their superpartners in terms of the BFV basis. A partial gauge-fixing is needed to make this geometrization as described in sect.5.

When the two-dimensional metric is identified, the relation between the BFV ghost basis and the covariant ghost basis is established. If one expresses the genuine $Q^2$ anomaly in terms of the covariant ghosts, this corresponds to a noncovariant anomaly which spoils the super-reparametrization invariance. However, it is geometrized so as to respect that symmetry by adding a suitable coboundary term given in sect.6. This new expression is the supersymmetric extension of curvature
dependent $Q^2$ anomaly found in I. At this stage, there remain three bosonic and four fermionic gauge conditions being left unfixed. They can be used to make conventionally used gauge-fixings in the configuration space such as the superconformal, the light-cone and the harmonic gauges. However, for a wide class of gauge choices including these, the gauge dependence does not appear in the $[Q, H_T]$ anomaly, and its geometrized expression is exactly identified with the super-Weyl anomaly. Therefore, it is easy to see some gauge-fixed forms of the BRST anomalies, for instance, the $Q^2$ anomaly in the super-orthonormal gauge [13,14]. This demonstrates the hierarchical relationship among the anomalies, which can not clearly be recognized in the gauge-fixed approaches to anomalies in superstring or 2D SUGRA [24-26]. We will also give a novel action of super-Liouville type, which naturally emerges from the geometrization of the $Q^2$ anomaly. It converts super-Virasoro anomaly to super-Weyl anomaly, hence plays the role of Wess-Zumino-Witten action. We shall summarize our results in the final section, and give some key formulae in the appendices.

2. BFV formalism of superstring

This section describes a new BFV formulation of 2D SUGRA theory for superstring. We begin by considering the action [11]\textsuperscript{¶}

\[
S_{\text{str}} = \int d^2 \sigma e \left[ -\frac{1}{2} (g_{\alpha\beta} \partial_\alpha X \partial_\beta X - i \bar{\psi} \rho^\alpha \nabla_\alpha \psi) 
- \bar{\chi}_\alpha \rho^\beta \rho^\alpha \psi \partial_\beta X - \frac{1}{4} \bar{\psi} \chi_\alpha \rho^\beta \rho^\alpha \chi_\beta \right],
\]

(2.1)

where $X^\mu$ and $\psi^\mu$ ($\mu = 0, 1, \cdots, D - 1$) are, respectively, the bosonic and the fermionic string variables. This action has reparametrization invariance, local Lorentz invariance and local supersymmetry. Moreover, it is invariant under local Weyl rescalings and local fermionic transformations. Not all of these symmetries can survive in general upon quantization because of anomalies. However, it is legitimate to assume that the local Lorentz invariance remains intact. This is because

\textsuperscript{§} For a harmonic-gauge formulation of 2D gravity, see [23] where some problem in determining the anomaly coefficients is discussed.

\textsuperscript{¶} We shall use the two-dimensional conventions; the flat world-sheet metric is chosen to be $\eta_{ab} = \text{diag}(-1, 1)$. The Dirac matrices $\rho^\alpha$ are $\rho^0 = \sigma_2$ and $\rho^1 = i\sigma_1$ so that $\rho^5 = \rho^0\rho^1 = \sigma_3$ with $\sigma_1$ being Pauli matrices.
that the local Lorentz anomaly is known to be always converted to the Einstein anomaly \[27\]. In what follows, therefore, we shall eliminate the zweibeins from the action by using the local Lorentz invariant variables. We choose the parametrization for the metric variables

\[ \lambda^\pm = \pm \frac{e^0_\pm}{e_1^\pm} = \frac{\sqrt{-g} \pm g_{01}}{g_{11}}, \quad \xi = \ln g_{11}, \]  

(2.2)

where \(e^A_\pm = e^0_\pm \pm e^1_\pm\). For the fermionic fields, we use rescaled upper and lower components defined by

\[ \psi = \begin{pmatrix} (-e_1^-)^{-\frac{1}{2}} \psi_- \\ (e_1^+)^{-\frac{1}{2}} \psi_+ \end{pmatrix}, \quad \chi_\alpha = \begin{pmatrix} (e_1^+)^{\frac{1}{2}} \chi_\alpha^- \\ (-e_1^-)^{\frac{1}{2}} \chi_\alpha^+ \end{pmatrix}, \]  

(2.3)

and parametrize the gravitino fields as

\[ \nu_\pm = (\chi_0 \pm \lambda^\mp \chi_1)_\pm, \quad \Lambda_\pm = 4 \chi_1^\mp. \]  

(2.4)

Note that the local Weyl rescaling \(e_\alpha^a \to e^\sigma e_\alpha^a\) generates a translation \(\xi \to \xi + 2\varphi\) but it leaves \(\lambda^\pm, \phi_\pm, \nu_\pm \) and \(\Lambda_\pm\) unchanged. These rescaled spinor components are local Lorentz and Weyl invariant.* The local fermionic transformation \(\chi_\alpha \to \chi_\alpha + i \rho_\alpha \eta\) with \(\eta\) being an arbitrary Majorana spinor, on the other hand, induces a change \(\Lambda_\pm \to \Lambda_\pm - 4 \eta_\pm\) with \(\eta_\pm\) being the rescaled local Lorentz and Weyl invariant components of \(\eta\), while \(\nu_\pm\) remain invariant. As we shall see in sect.5, the \(\Lambda_\pm\) is the superpartner of the conformal mode \(\xi\). In terms of the variables defined in (2.2) and (2.4), the action (2.1) can be written as**

\[
S_{\text{str}} = \int d^2\sigma \left[ \frac{1}{\lambda^+ + \lambda^-} (\dot{X} - \lambda^+ X') (\dot{X} + \lambda^- X') 
+ \frac{i}{2} \dot{\psi}_+ (\dot{\psi}_+ - \lambda^+ \psi_+') + \frac{i}{2} \dot{\psi}_- (\dot{\psi}_- + \lambda^- \psi_-') 
+ \frac{2}{\lambda^+ + \lambda^-} \left\{ i(\dot{X} - \lambda^+ X') \psi_- \nu_+ - i(\dot{X} + \lambda^- X') \psi_+ \nu_- 
+ \psi_+ \psi_- \nu_+ \nu_- \right\} \right],
\]  

(2.5)

* In ref.[28] similar variables are used to investigate 2D quantum gravity coupled to Majorana field.

** The world-sheet coordinates \(\sigma^\alpha (\alpha = 0, 1)\) are denoted by \((\tau, \sigma)\), and take \(-\infty < \sigma < \infty\). It is straightforward to make the analysis on a finite interval of \(\sigma\) so as to impose the Neveu-Schwarz or Ramond boundary conditions. We also use \(\dot{F} = \partial_\tau F\) and \(F' (= \partial F) = \partial_\sigma F\) for any variable \(F\).
Let us denote the canonical momenta for $X$, $\lambda^\pm$, $\xi$, $\nu^\pm$ and $\Lambda^\pm$ by $P$, $\pi^\pm_\lambda$, $\pi_\xi$, $\pi^\pm_\nu$ and $\pi^\pm_\Lambda$, respectively. Then the canonical theory of this system involves the following set of primary constraints

$$
\varphi_A \equiv \pi_A \approx 0 \quad \text{for } A = \lambda^\pm, \xi, \tag{2.6}
$$
$$
\mathcal{J}^z \equiv \pi^z \approx 0 \quad \text{for } z = \nu^\pm, \Lambda^\pm,
$$

and the secondary constraints, the super-Virasoro constraints

$$
\varphi_\pm = \frac{1}{4}(P \pm X')^2 \pm \frac{i}{2}\psi_\pm\psi'_\pm \approx 0
$$
$$
\mathcal{J}_\pm = \psi_\pm(P \pm X') \approx 0. \tag{2.7}
$$

They satisfy the classical super-Virasoro algebra

$$
\{\varphi_\pm(\sigma), \varphi_\pm(\sigma')\} = \pm(\varphi(\sigma) + \varphi(\sigma'))\partial_\sigma\delta(\sigma - \sigma'),
$$
$$
\{\mathcal{J}_\pm(\sigma), \varphi_\pm(\sigma')\} = \pm\frac{3}{2}\mathcal{J}_\pm(\sigma)\partial_\sigma\delta(\sigma - \sigma') \pm \mathcal{J}'_\pm\delta(\sigma - \sigma'),
$$
$$
\{\mathcal{J}_\pm(\sigma), \mathcal{J}_\pm(\sigma')\} = -4i\varphi_\pm(\sigma)\delta(\sigma - \sigma'),
$$
all other super-Poisson brackets vanish,

where $\{ , \}$ denotes super-Poisson bracket[10]. All the constraints (2.6) and (2.7) are classically first-class and generate full set of the local symmetries of the classical action (2.1).

The BFV algorithm then defines the EPS by introducing canonical pairs of the ghost, anti-ghost and auxiliary fields to each constraint as

$$
\varphi_A : (C^A, P_A), \quad (P^A, \overline{C}_A), \quad (N^A, B_A) \quad \text{for } A = \lambda^\pm, \xi, \pm,
$$
$$
\mathcal{J}^z : (\gamma^z, \beta_z), \quad (\beta^z, \gamma_z), \quad (M^z, A_z) \quad \text{for } z = \nu^\pm, \Lambda^\pm, \pm. \tag{2.9}
$$

Here $A = \lambda^\pm, \xi, \pm$ and $z = \nu^\pm, \Lambda^\pm, \pm$, respectively, label the bosonic and fermionic first-class constraints given in (2.6) and (2.7). The grassmannian parities of the ghost variables for the bosonic (fermionic) constraints are chosen to be odd (even), while those of auxiliary fields are even (odd).
In the BFV formalism the BRST charge is constructed without fixing a gauge. It is solely determined by the Poisson algebra among the constraints and the requirement of nilpotency $\{Q, Q\} = 0$. In the case at hand it takes of the form

$$Q = \int d\sigma \left[ C^A \varphi_A + \gamma^z J_z + \mathcal{P}^A B_A + \beta^z A_z \right.$$ 

$$+ C^+ (\mathcal{P}^+_+ C^+ + \beta^+_+ \gamma^+ + \gamma^+ \left(2i \mathcal{P}^+_+ \gamma^+ - \frac{1}{2} \beta^+_+ C^+\right)$$ 

$$- C^- (\mathcal{P}^-_- C^- + \beta^-_- \gamma^- + \gamma^- \left(2i \mathcal{P}^-_- \gamma^- + \frac{1}{2} \beta^-_- C^-\right) \right],$$

where $A$ and $z$ run over all constraint labels. It generates the BRST transformations of the fundamental variables:

$$\delta X = \frac{1}{2} \{(C^+ - C^-) X' + (C^+ + C^-) P\} + \gamma^+ \psi_+ + \gamma^- \psi_-,$$

$$\delta P = \frac{1}{2} \{(C^+ - C^-) P + (C^+ + C^-) X'\} + \gamma^+ \psi_+ - \gamma^- \psi_-,'$$

$$\delta \psi_\pm = \pm \frac{1}{2} C^\pm X' \pm C^\pm \psi'_\pm + i \gamma^\pm (P \pm X'),$$

$$\delta \lambda_\pm = C^\pm_\lambda, \quad \delta \xi = C^\xi, \quad \delta C^\pm_\lambda = 0, \quad \delta C^\xi = 0,$$

$$\delta \nu_\pm = -\gamma^\nu, \quad \delta \Lambda_\pm = -\gamma^\Lambda, \quad \delta \gamma^\nu_+ = 0, \quad \delta \gamma^\Lambda_+ = 0,$$

$$\delta N^A = \mathcal{P}^A, \quad \delta M^z = -\beta^z, \quad \delta \mathcal{P}^A = 0, \quad \delta \beta^z = 0,$$

$$\delta N^\pm = \mathcal{P}^\pm, \quad \delta M^\pm = -\beta^\pm, \quad \delta \mathcal{P}^\pm = 0, \quad \delta \beta^\pm = 0,$$

$$\delta \mathcal{C}_A = -\mathcal{B}_A, \quad \delta \varphi_z = -A_z, \quad \delta \mathcal{B}_A = 0, \quad \delta A_z = 0,$$

$$\delta \mathcal{C}^\pm = -\mathcal{B}_\pm, \quad \delta \gamma^\pm = -A^\pm, \quad \delta \mathcal{B}^\pm = 0, \quad \delta A^\pm = 0,$$

$$\delta \varphi_A = -\mathcal{C}_A, \quad \delta \beta_z = -J_z, \quad \delta \mathcal{C}_A = -\mathcal{B}_A, \quad \delta \mathcal{C}^\pm = -\mathcal{B}^\pm, \quad \delta \mathcal{B}^\pm = -\Phi_\pm,$$

$$\delta \gamma^\pm = \mp \left(\frac{1}{2} C^\pm \gamma^\pm + \frac{1}{2} \gamma^\pm C^\pm\right), \quad \delta \beta^\pm = -I_\pm,$$

with $A = \lambda^\pm, \xi$ and $z = \nu^\pm, \Lambda^\pm$,

where $\delta F = -\{Q, F\}$ for any $F$. We have introduced the generalized super-Virasoro operators

$$\Phi^\pm = \varphi^\pm \pm 2 \mathcal{P}^\pm C^\pm + \mathcal{P}^\pm C^\pm + \frac{3}{2} \gamma^\pm C^\pm + \frac{1}{2} \beta^\pm \gamma^\pm,$$

$$I^\pm = J^\pm \mp \frac{3}{2} \mathcal{C}^\pm C^\pm + \mathcal{P}^\pm C^\pm + 4i \mathcal{P}^\pm C^\pm + 2i \mathcal{B}^\pm \gamma^\pm.$$

The BRST charge (2.10) and the transformation (2.11) have a reflection symmetry

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in the EPS which we will use below. All the EPS variables are divided into those with (+) or (−) indices and the remainings carrying no ± indices. The symmetry* is an invariance under replacements, ± → ∓, and ∂σ → −∂σ.

The canonical structure of the classical theory is now described in the EPS. In the next section we shall discuss the BRST anomalies associated with quantization of the theory.

3. BRST anomalies and consistency conditions

In this section we will review the general description of the BRST anomalies in the BFV formalism[8]. When one applies the BFV formalism to theories with local gauge symmetries, the basic gauge algebras reflecting its classical gauge invariance are formulated in terms of the BRST charge Q by the condition

\[ \{Q , Q\} = 0. \]  \hspace{1cm} (3.1)

The gauge symmetries must be consistent with the time development of the system. This can be expressed as the conservation of Q, i.e.,

\[ \dot{Q} \equiv \frac{dQ}{dt} = \{Q , H_T\} = 0, \]  \hspace{1cm} (3.2)

where \( H_T \) is the total Hamiltonian.

Quantization can be achieved formally by replacing super-Poisson brackets with supercommutators. At the quantum level, however, these operators must be suitably regularized to become well-defined. An anomaly arises if (3.1) and (3.2) cannot be maintained upon quantization. The anomalous terms may be expanded in \( \hbar \) as†

\[ [Q , Q] \equiv i\hbar^2\Omega + O(\hbar^3) \]
\[ [Q , H_T] \equiv \frac{i}{2}\hbar^2\Gamma + O(\hbar^3), \]  \hspace{1cm} (3.3)

It is convenient to distinguish a supercommutator from a naive one \([ , ]_0\) which is

* In the superconformal gauge, it reduces to a holomorphic (anti-holomorphic) decomposition of conformal fields and at the same time a chiral projection on the world-sheet spinors.
† We will explicitly write \( \hbar \) in dealing with \( \hbar \) expansion. Otherwise, we use \( \hbar = 1 \).
defined via classical super-Poisson bracket \{ , \}, i.e.,

\[ [A , B]_0 \equiv i\hbar \{A , B\}. \] (3.4)

Our basic assumption is that the supercommutation relations between \( Q \) and \( H_T \) obey the commutation law, the distribution law and especially the super-Jacobi identity. They read in the present case

\[ [Q , [Q , Q]] = 0 , \] (3.5)

\[ 2 [Q , [Q , H_T]] + [H_T , [Q , Q]] = 0 . \] (3.6)

To the lowest order, \( \hbar^3 \), the outer commutators in these super-Jacobi identities can be truncated by the naive commutators, yielding two consistency conditions:

\[ \delta \Omega = 0 , \] (3.7)

\[ \delta \Gamma = \{ H_T , \Omega \} = -\frac{d\Omega}{dt} , \] (3.8)

where \( \delta \) is the classical BRST transformation defined by (2.11).

For any reparametrization invariant theory, the specification of the BRST anomalies can be considerably simplified. Such a theory has vanishing canonical hamiltonian, and its total hamiltonian takes the form, \( H_T = \frac{1}{i\hbar}[Q , \Psi] \), where \( \Psi \) is the gauge fermion [9] needed to fix the gauge degrees of the system. Using (3.3) and the super-Jacobi identity, we obtain

\[ \Gamma = \{ \Omega , \Psi \} . \] (3.9)

One finds that \( \Gamma \) can be calculated from \( \Omega \) without solving (3.8). In this sense \( \Omega \) is of primary importance for any theory being reparametrization invariant at the classical level. This fact is used below to analyse the BRST anomalies in 2D SUGRA of superstring theory. It is also noted that the consistency condition (3.7)
shows up cohomological nature of the anomalies. If $\Omega$ is a solution of (3.7), then $\tilde{\Omega}$ defined by

$$\tilde{\Omega} = \Omega + \delta \Xi$$

(3.10)

also solves (3.7) for any $\Xi$. This is nonvanishing square of a suitably redefined BRST charge, $[\bar{Q}, \bar{Q}] = i\hbar^2 \tilde{\Omega}$, where

$$\bar{Q} = Q - \frac{\hbar}{2} \Xi .$$

(3.11)

The shift of the BRST charge corresponds to a redefinition of the total hamiltonian as

$$\bar{H}_T = H_T - \frac{\hbar}{2} \{ \Xi, \Psi \} .$$

(3.12)

It generates an anomalous commutator $[\bar{Q}, \bar{H}_T]$, which defines

$$\bar{\Gamma} = \{ \tilde{\Omega}, \Psi \} = \Gamma - \frac{d\Xi}{dt} + \delta \{ \Xi, \Psi \} .$$

(3.13)

One finds from (3.10) and (3.12) that adding a coboundary term to a given solution $\Omega$ is related with introducing a counteraction

$$S_{\text{count}} = \frac{\hbar}{2} \int d\tau \{ \Xi, \Psi \} ,$$

(3.14)

which reflects the difference of underlying regularization schemes used to obtain $\Omega$ and $\tilde{\Omega}$.

In summary, a BRST anomaly is determined by a cohomology class of the non-trivial solutions of (3.7), totally independent of gauge fixings and regularization schemes. On the other hand, $\Gamma$, the descendant of $\Omega$, is gauge dependent. The gauge dependence can be yet easily tracked, and is shown to disappear eventually for the conventionally used gauge choices.

With these discussions in mind, we shall solve the consistency condition (3.7) for 2D SUGRA theory considered in the previous section.

\* We regard $\Gamma$ as the descendant of $\Omega$ because $\Gamma$ is basically determined by $\Omega$ via (3.9).
4. Solution in the extended phase space:
The genuine super-Virasoro anomaly

In order to solve the consistency condition (3.7) in the EPS of 2D SUGRA, we
will basically follow the method developed in I. Let us first summarize as signments
of ghost number and canonical dimension for the ghost and auxiliary fields in
(2.9). The BFV ghosts $C^A$, $P^A$, $\gamma^z$ and $\beta^z$ carry one unite of the ghost number,
$\text{gh}(C^A) = \text{gh}(P^A) = \text{gh}(\gamma^z) = \text{gh}(\beta^z) = 1$, while $\text{gh}(\overline{P}_A) = \text{gh}(\overline{C}_A) = \text{gh}(\overline{\gamma}_z) = \text{gh}(\overline{\beta}_z) = -1$ for their canonical momenta, $\overline{P}_A$, $\overline{C}_A$, $\overline{\gamma}_z$ and $\overline{\beta}_z$. Canonical pairs
of the auxiliary fields $(N^A, B_A)$ and $(M_A, A_{\pm})$ have no ghost number. We assign
0 to the canonical dimension of $X_{\mu,\lambda}^\pm$ and $\xi$, and correspondingly $+1$ to $P_{\mu}^\lambda$ and $\pi_\xi$. Their superpartners, $\psi_{\pm}$, $\nu_{\pm}$, $\Lambda_{\pm}$, and their conjugate momenta have the
canonical dimension $\frac{1}{2}$. The canonical dimensions of ghosts and anti-ghosts are
fixed only relative to that of $C^\pm$. Putting $c \equiv \text{dim}(C^\pm)$, we find

$$\begin{align*}
\text{dim}(C^\pm_\lambda) &= \text{dim}(C^\pm_\xi) = 1 + c, \\
\text{dim}(\overline{P}_\pm) &= 1 - c, \\
\text{dim}(\overline{P}_\pm^\lambda) &= \text{dim}(\overline{P}_\xi) = -c, \\
\text{dim}(\gamma^z) &= c + \frac{1}{2}, \\
\text{dim}(\beta^z) &= -c + \frac{1}{2}.
\end{align*}$$

Note that all the bosonic (anti-)ghosts have the same canonical dimensions.†

We are now ready to solve the consistency condition (3.7) and seek the solution
in the form

$$\Omega = \int d\sigma \omega ,$$

(4.2)

where we assume that $\omega$ is a polynomial of local operators with $\text{gh}(\omega) = 2$ and
$\text{dim}(\omega) = 3 + 2c$. According to the general structure of the BFV formalism, the
total phase space can be divided, with respect to the action of $\delta$, into two sectors;

$$\begin{align*}
S_1 : \text{consisting of } (X^\mu, P_\mu, \psi_{\pm}), (C^\pm, \overline{C}_\pm) \text{ and } \\
S_2 : \text{consisting of all the other fields.}
\end{align*}$$

(4.3)

It is easy to see that on each sector the $\delta$ operation closes:

$$\delta_1^2 = \delta_2^2 = 0 , \quad \delta_1 \delta_2 + \delta_2 \delta_1 = 0 ,$$

(4.4)

where $\delta = \delta_1 + \delta_2$, and $\delta_1$ acts only on $S_1$ ($S_2$) variables. Since the variables
in the $S_2$-sector form pairs $(U, V)$ with properties $\delta_2 U = \pm V$, it can be shown

† We do not need to know canonical dimensions of other fields as we will see later.
that variables belonging to $S_2$ can be removed from $\Omega$ as coboundary terms [18]. Therefore, $\omega$ can be chosen to be independent of the $S_2$-variables.

To reduce the number of possibilities for $\omega$, we shall use here the following global symmetries compatible with the BRST transformation (2.11). The BRST charge (2.10) is invariant under the space-time Poincaré transformation $X^\mu \rightarrow \Lambda^\mu_{\nu} X^\nu + a^\mu$. It has also the reflection symmetry, as described in sect.2, of interchanging the indices of the variables $(\pm) \leftrightarrow (\mp)$ and reversing the sign of $\partial_\sigma$. For simplicity we will denote this symmetry by $\{ \pm \leftrightarrow \mp \}$. We may assume therefore, without loss of generality, that $\Omega$ also respects these symmetries. The translational invariance forbids $X^\mu$ to appear in $\omega$ without derivatives. It is convenient thus to introduce the variables

$$Y^\mu_\pm \equiv (P \pm X')^\mu \quad \text{with} \quad \delta Y^\mu_\pm = \pm \partial_\sigma (C^\pm Y_\pm + 2\gamma^\pm \psi_\pm)^\mu .$$

(4.5)

With these symmetries imposed, there are still great many of the operators composed of $S_1$ variables with ghost number 2 and canonical dimension $2c + 3$. We can, however, reduce further the number of operators contributing to $\Omega$ by noting that the ghost momenta $\overline{P}_\pm$ and $\overline{\beta}_\pm$ can be removed from nontrivial solutions to (3.7) as coboundary terms. We will give the proof of this lemma in Appendix B. This implies that we have only to consider the operators composed of ghosts variables ($C^\pm, \gamma^\pm$) and the string variables ($Y_\pm, \psi_\pm$). These operators can be classified into the following six groups; (1) operators constructed only from ($C^+, \gamma^+$), and those obtained by applying the reflection symmetry $\{ \pm \leftrightarrow - \}$, (2) operators bilinear in ($C^+, \gamma^+$) and ($C^-, \gamma^-$), and containing no string variables, (3) operators constructed from ($C^+, \gamma^+$) and quadratic in ($Y_+, \psi_+$), and those obtained by $\{ \pm \leftrightarrow \mp \}$, (4) operators bilinear in ($C^+, \gamma^+$) and ($C^-, \gamma^-$) and quadratic in ($Y_+, \phi_+$), and those obtained by $\{ \pm \leftrightarrow \mp \}$, (5) operators constructed from ($C^\pm, \gamma^\pm$) and bilinear in ($Y_+, \psi_+$) and ($Y_-, \psi_-$), (6) operators quartic in the string variables ($Y_\pm, \psi_\pm$). Since the BRST transformation does not mix operators belonging to different groups, we can investigate solutions to (3.7) group by group. We first assume $\Omega$ to be the most general linear combinations of operators belonging to each group and then determine the unknown coefficients appearing in $\Omega$ to satisfy the consistency condition (3.7). In spite of a great many unknown coefficients, it can be shown that there are no nontrivial solutions in the cases (2),(3),(4) and (6), and the operators (1) and (5), respectively, possess only one nontrivial solution.
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\( \Omega_1 \) and \( \Omega_5 \) given by

\[
\begin{align*}
\Omega_1 &= \int d\sigma \left( C^+ \partial_n^3 C^+ - 8i \gamma^+ \partial_n^2 \gamma^+ \right) + \{ + \leftrightarrow - \}, \\
\Omega_5 &= \int d\sigma \left[ (C^+ \partial_n C^+ - C^+ \partial_n C^- - 2i(\gamma^+)^2)Y_+ Y_- - 4\psi_+ \psi_- \gamma^+ \partial_n \gamma^- \\
&\quad + 2C^+ (\partial_n Y_+ \psi_+ + 2Y_+ \partial_n \psi_+) \gamma^- + 2C^+ \psi_+(\partial_n Y_+ \gamma^+ + 2Y_- \partial_n \gamma^+) \right] \\
&\quad + \{ + \leftrightarrow - \}. 
\end{align*}
\] (4.6)

We thus obtain the general solution to the consistency condition (3.7)

\[ \Omega = k\Omega_1 + k'\Omega_5. \] (4.7)

The coefficients \( k \) and \( k' \) can not be determined in this algebraic approach. The two nontrivial solutions (4.6) are exactly supersymmetric extension of those in the bosonic string theory found in 1. The first solution \( \Omega_1 \) can be related to the super-Virasoro anomaly as we will see below, while \( \Omega_5 \) have never been noted before as far as we know. It depends on string coordinates and is algebraically allowed anomaly, whose physical implication is not yet unkown. In what follow we will simply assume

\[ k' = 0. \] (4.8)

The coefficient \( k \) can be computed within a specific regularization scheme. Since \( \Omega \) is bilinear in the ghost fields, it may originate from the anomalous terms in the gauge algebra for the generalized super-Virasoro constraints (2.7). Such anomalous Schwinger terms in the constraint algebra can be calculated by using the normal ordering prescription in the Schrödinger picture.* We find

\[
\begin{align*}
[\Phi_\pm(\sigma), \Phi_\pm(\sigma')] &= \pm i(\Phi(\sigma) + \Phi(\sigma')) \partial_\sigma (\sigma - \sigma') + i \frac{D - 10}{16\pi} \partial_\sigma^3 \delta(\sigma - \sigma'), \\
[I_\pm(\sigma), \Phi_\pm(\sigma')] &= \pm i \frac{3}{2} I_\pm(\sigma) \partial_\sigma \delta(\sigma - \sigma') \pm i I_\pm'(\sigma) \delta(\sigma - \sigma'), \\
[I_\pm(\sigma), I_\pm(\sigma')] &= 4\Phi_\pm(\sigma) \delta(\sigma - \sigma') - \frac{D - 10}{2\pi} \partial_\sigma^3 \delta(\sigma - \sigma'),
\end{align*}
\] (4.9)

all other supercommutators vanish.

* See, for example,[29,2].
which implies
\begin{equation}
  k = -\frac{D - 10}{16\pi}.
\end{equation}

Thus one finds that \( \Omega \) is essentially unique and determined without referring to specific gauge choices.

Let us turn to consider \( \Gamma \), which is given by a naive commutator between \( \Omega \) and \( \Psi \) as in (3.9). \( \Gamma \) can not be calculated without any assumption on the gauge fermion \( \Psi \). First, we restrict ourselves to the standard form of the gauge fermion\[10\]
\begin{equation}
  \Psi = \int d\sigma [ C_A \chi^A + \gamma^z \zeta_z + \overline{P}_A N^A + \overline{\beta}_z M_z ],
\end{equation}
where \( \chi \)'s and \( \zeta \)'s are the gauge-fixing functions. In order for \( N^A \) and \( M_z \) to be identified with the multiplier fields, these gauge-fixing functions are assumed not to depend on the ghost momenta \( \overline{P}_A \) and \( \overline{\beta}_z \). They are otherwise arbitrary. These are all the assumptions we need to compute unambiguously the naive commutator \( \{ \Omega, \Psi \} \) given in (3.9):
\begin{equation}
  \Gamma = 2k \int d\sigma \left[ \partial_\sigma N^+ \partial_\sigma^2 C^+ + 8i\partial_\sigma \gamma^+ \partial_\sigma M^+ \right] + \{ + \leftrightarrow - \}.
\end{equation}
This result is independent of the gauge-fixing functions \( \chi \)'s and \( \zeta \)'s as long as the above assumptions, which are about the weakest ones imposed on \( \Psi \), are satisfied.

Although \( \Omega \equiv k\Omega(1) \) has exactly the same form as the \( Q^2 \) anomaly of ref.[13,14], its theoretical content is much richer. The \( \Omega \), along with its descendant \( \Gamma \), exhibits namely the most general form of anomaly in the extended phase space, suitably called as the genuine super-Virasoro anomaly. It is a pregeometric result because it has been obtained without any reference to a two dimensional super-metric. The geometrical meaning of \( \lambda^\pm \) and \( \xi \), which is given in (2.2), has disappeared in the EPS, as one can see from their BRST transformations (2.11); they are no longer related to some metric variables, since the associated ghosts, \( C^+_\lambda \) and \( C^\xi \), are by no means the reparametrization ghosts and the Weyl ghost. The same thing happens in the sector of their superpartner. So at the present stage, the genuine super-Virasoro anomaly can not be identified with the super-reparametrization or super-Weyl anomaly. To distinguish these anomalies from each other we certainly need a metric.
5. Geometrization

In the previous section we have obtained the BRST anomalies Ω and Γ in terms of EPS variables, which do not possess direct geometrical meaning as they are. To endow them with a geometrical interpretation, we must specify the gauge conditions to relate the EPS variables with those of configuration space. This provides us with the basis needed to geometrize the genuine super-Virasoro anomaly into the super-Weyl anomaly.

We begin by constructing the BRST gauge-fixed action. Using the BRST invariant total hamiltonian

\[
H_T = \frac{1}{\hbar}[Q, \Psi].
\]

(5.1)

For the standard form of the gauge fermion (4.11), we obtain the BRST gauge-fixed action

\[
S = S_{cl} + S_{FP} + S_{gf}
\]

(5.2)

where

\[
S_{cl} = \int d^2\sigma [p\dot{X} + \frac{i}{2}\psi_+\dot{\psi}_+ + \frac{i}{2}\psi_-\dot{\psi}_- + \pi^+_\lambda^+ + \pi^-_\lambda^- + \pi_\xi\dot{\xi} + \pi^+_\lambda - \pi^-_\lambda - N^\Lambda\varphi_\Lambda - M_\zeta\mathcal{J}^z],
\]

\[
S_{FP} = \int d^2\sigma [\overline{p}_A\dot{\chi}_A + \overline{z}_\gamma\dot{\zeta}_z - \overline{A}_\zeta\delta\lambda^A + \overline{\gamma}_\delta\zeta_z - \overline{p}_A\mathcal{P}^A - \overline{\gamma}_z\beta_z
\]

(5.3)

\[
- N_+(2\overline{p}_+c^{+t} + \overline{z}_+\gamma^+ + \frac{3}{2}\beta^+_\gamma^t + \frac{1}{2}\beta^+_\gamma^t) + N_-(2\overline{p}_-c^{-t} + \overline{z}_-\gamma^- + \frac{3}{2}\beta^-\gamma^{-t} + \frac{1}{2}\beta^-\gamma^{-t})
\]

- \[M^+(\frac{3}{2}\beta^+c^{+t} + \beta^+\gamma^+ - 4i\overline{p}_+\gamma^+)
\]

+ \[M^- (\frac{3}{2}\beta^-c^{-t} + \beta^-\gamma^- + 4i\overline{p}_-\gamma^-),
\]

\[
S_{gf} = \int d^2\sigma [-B_\lambda\chi^A - A^z\zeta_z].
\]

Because of the presence of the primary constraints (2.6), the EPS variables \(\lambda^\pm\) and \(\nu^\pm\) have lost their original geometrical meaning as metric variables and

* The ghost sector can be simplified by shifting the gauge fermion (4.11) by \(\Psi \to \Psi + \int d\sigma [\overline{c}_A\lambda^A + \overline{z}_\gamma\zeta_z].\) This just cancels the Legendre terms \(\int d^2\sigma [\overline{c}_A\mathcal{P}^A + \overline{z}_\gamma\beta_z + B_\lambda\chi^A + A^z\zeta_z]\) in constructing the effective action.
gravitino fields, as discussed in the previous section. Instead the auxiliary fields \(N^\pm\) and \(M^\pm\) play that role as is seen from (5.3). For the geometrization, these two sets of variables must be identified by imposing the gauge conditions \[8\]

\[
\chi^{\pm} = \lambda^{\pm} - N^{\pm}, \quad \zeta^{\pm} = \nu^{\pm} \mp iM^{\mp}.
\]

There still remain the gauge conditions \(\chi^{\xi}, \chi^{\pm}, \zeta^{\Lambda^{\pm}}\) and \(\zeta^{\pm}\) left unfixed. These are for the gauge degrees of freedom, which we would like to identify with two reparametrization symmetries, one Weyl symmetry and their supersymmetric counterparts. In order to construct the covariant ghost fields from the BFV ghosts by using some equations of motion, one needs in general to fix these gauge symmetries. However, it suffices for us to assume that the gauge-fixing functions \(\chi^{\xi}, \chi^{\pm}, \zeta^{\Lambda^{\pm}}\) and \(\zeta^{\pm}\) and their BRST transforms do not depend on \(P, \pi_{A}, \pi^{z}, \overline{P}_{A}\) and \(\overline{\beta}^{z}\). This is the only assumption we have to make for the general analysis. The gauge conditions can be arbitrary otherwise.

From the action (5.2) we obtain

\[
\dot{X} = \frac{1}{2} \left\{ (N^{+} + N^{-})P + (N^{+} - N^{-})X' \right\} + M^{+} \psi_{+} + M^{-} \psi_{-} = 0,
\]

\[
\dot{\psi}_{\pm} = \frac{1}{2} N^{\pm'} \psi_{\pm} + iM^{\pm}(P \pm X') = 0,
\]

\[
\dot{\lambda}^{\pm} = N_{\lambda}^{\pm}, \quad \dot{\xi} = N^{\xi}, \quad \dot{\nu}_{\pm} = M_{\pm}^{\nu'}, \quad \dot{\Lambda}_{\pm} = M_{\pm}^{\Lambda},
\]

\[
\dot{C}_{\lambda}^{\pm} = \mathcal{P}_{\lambda}^{\pm}, \quad \dot{C}^{\xi} = \mathcal{P}^{\xi}, \quad \dot{\gamma}_{\pm}^{\nu'} = \beta_{\pm}^{\nu'}, \quad \dot{\gamma}_{\pm}^{\Lambda} = \beta_{\pm}^{\Lambda},
\]

\[
\mathcal{P}^{\pm} = C^{\pm} \pm C^{\pm'} N^{\pm'} \mp C^{\pm'} N^{\pm} - 4i\gamma^{\pm} M^{\pm},
\]

\[
\beta_{\pm} = \dot{\gamma}^{\pm} \pm \frac{1}{2} \gamma^{\pm} N^{\pm'} \mp \gamma^{\pm'} N^{\pm} \mp C^{\pm} M^{\pm'} \pm \frac{1}{2} C^{\pm'} M^{\pm}.
\]

To identify the covariant ghost variables we consider the BRST transformation of \(X\) given in (2.11). Using the equation of motion for \(P\) in (5.5), we find on the EPS basis

\[
\delta X = \frac{C^{+} + C^{-}}{N^{+} + N^{-}} \dot{X} + \frac{N^{-}C^{+} - N^{+}C^{-}}{N^{+} + N^{-}} X' + \left(\gamma^{+} + \frac{C^{+} + C^{-}}{N^{+} + N^{-}} M^{+}\right) \psi_{+} + \left(\gamma^{-} + \frac{C^{+} + C^{-}}{N^{+} + N^{-}} M^{-}\right) \psi_{-}.
\]

On the other hand the BRST transformation in terms of covariant ghosts is given
by
\[ \delta X = C^\alpha \partial_\alpha X + \bar{\omega} \psi = C^\alpha \partial_\alpha X - i(\omega_- \psi_+ - \omega_+ \psi_-), \quad (5.7) \]

where \( C^\alpha (\alpha = 0, 1) \) and \( \omega_\pm \) are the ghost variables for the reparametrization and local supersymmetry. The covariant bosonic ghost is defined by
\[
\omega = \begin{pmatrix}
(e_1^+)\frac{i}{2} \omega_-

(-e_1^-)\frac{i}{2} \omega_+
\end{pmatrix}.
\quad (5.8)
\]

By comparing (5.6) with (5.7) we obtain
\[
C^0 = \frac{C^+ + C^-}{N^+ + N^-}, \quad C^1 = \frac{N^-C^+ - N^+C^-}{N^+ + N^-}, \quad \omega_\pm = \mp i \left( \gamma^\mp + \frac{C^+ + C^-}{N^+ + N^-} M^\mp \right) \quad (5.9)
\]

The Weyl ghost \( C_w \) can be found by considering the BRST transformation of \( \xi = \ln g_{11} \). In the configuration space it is given by
\[
\delta \xi = C_w + C^\alpha \partial_\alpha \xi + 2C^1 + C^0(\lambda^+ - \lambda^-) - i(\omega_- \Lambda_+ - \omega_+ \Lambda_-). \quad (5.10)
\]

Since \( \delta \xi = C^\xi \) on the EPS basis and we are working with the gauge conditions (5.4), we can solve (5.10) for \( C_w \) to obtain
\[
C_w = C^\xi - V^+_C + V^-_C \quad (5.11)
\]

where \( V^\pm_C \) and their companions \( V^\pm_N \) are defined by
\[
V^\pm_N = \frac{1}{2} G^\pm N^\pm \pm \Lambda^\pm M^\pm + N'^\pm, \quad (5.12)
\]

\[
V^\pm_C = \frac{1}{2} G^\pm C^\pm \pm \Lambda^\pm \gamma^\pm + C'^\pm
\]

with
\[
G^\pm = \frac{2}{N^+ + N^-} [\pm N^\xi + N^\xi' \mp (N^+ - N^-)' \mp (\Lambda^+ M^+ + \Lambda^- M^-)]. \quad (5.13)
\]

We have replaced the time derivative \( \dot{\xi} \) with \( N^\xi \) by way of (5.5).
The covariant ghost for the fermionic symmetry can also be obtained from the BRST transformation of $\Lambda_{\pm}$ in the configuration space

$$
\delta \Lambda_{\pm} = -4\eta_{w\pm} + C^\alpha \partial_\alpha \Lambda_{\pm} - 4C^0\nu_{\pm} + \frac{1}{2}(C^{1\prime} \pm \lambda^\pm C^{0\prime})\Lambda_{\pm}
$$

$$
\pm \frac{4g_{11}}{\sqrt{-g}} \omega_\mp \{ \dot{\xi} \pm \lambda^\mp \xi' - (\lambda^+ - \lambda^-)' - i(\nu_+\Lambda_- - \nu_-\Lambda_+) \} + 4\omega'_\mp,
$$

(5.14)

where $\eta_{w\pm}$ stand for the ghost for the fermionic symmetry. In spinor notation the covariant bosonic ghost for the fermionic symmetry is given by

$$
\eta_w = \begin{pmatrix}
(-e_1^-)^{-\frac{1}{2}}\eta_{w^-} \\
(e_1^+)^{-\frac{1}{2}}\eta_{w^+}
\end{pmatrix}.
$$

(5.15)

Comparing (5.14) with the relation $\delta \Lambda_{\pm} = -\gamma^A_{\pm}$ in the EPS, we obtain

$$
\eta_{w\pm} = \frac{1}{4} \left( W_{\mathcal{C}}^\pm + \frac{C^+ + C^-}{N^+ + N^-} W_N^\pm \right),
$$

(5.16)

where $W_{\mathcal{C}}^\pm$ and $W_N^\pm$ are defined by

$$
W_{\mathcal{C}}^\pm = M_{\pm}^A \mp \Lambda_{\pm}^t N_{\pm} \mp \frac{1}{2} \Lambda_{\pm}^t N_{\pm}^t \mp i(G_{\pm} M_{\pm}^t + 4 M_{\pm}^t'),
$$

$$
W_N^\pm = \gamma_{\pm} \mp \Lambda_{\pm}^t C_{\pm} \mp \frac{1}{2} \Lambda_{\pm} C_{\pm}^t \mp i(G_{\pm} \gamma_{\pm} + 4 \gamma_{\pm}').
$$

(5.17)

Here the time derivatives $\dot{\Lambda}_{\pm}$ have been replaced by $M_{\pm}^A$.

The equations (5.9), (5.11) and (5.16) completely fix the ghost relations between the BFV basis and the covariant one. It is straightforward to obtain the BRST transformation for the configuration space variables. We give a complete list for the BRST transformations in covariant form in Appendix A. However, in the next section we shall use the original form (2.11) written in terms of the BFV basis. This is because that the transformation rule in the EPS, which clearly satisfies the nilpotency condition, is particularly simple compared with those for the configuration space variables. This makes the general analysis much easier. The ghost relations found above are used only after performing the BRST transformation to obtain the manifestly covariant expression of the super-Weyl anomaly.
6. Derivation of the super-Weyl anomaly

This section describes the central issue of the present paper, derivation of the super-Weyl anomaly from the genuine super-Virasoro anomaly found in section 4.

In the previous section we have shown that the original EPS variables are related to covariant geometrical ones after the partial gauge fixing specified by (5.4). When expressed in terms of these covariant ghosts via (5.9), the genuine Virasoro anomaly Ω and Γ might be interpreted as an anomaly associated with super-reparametrizations. One expects, however, that this anomaly may be shifted into the one which respects the super-reparametrization symmetry and is endowed with a geometrical interpretation. As we shall show in this section, this can, indeed, be achieved and the geometrized anomaly turns out to be the super-Weyl anomaly.

Let us denote the geometrized $Q^2$ anomaly and its descendant by $Ω_g$ and $Γ_g$, respectively. They must be expressed in terms of covariant variables and have well-defined transformation properties under the covariant BRST transformations given in Appendix A. As we have argued in sect.4, these two different expressions for $Q^2$ anomaly, $Ω$ given in (4.7) with $k' = 0$ and $Ω_g$, should necessarily belong to the same cohomology class defined by the BRST transformation (2.11) in the extended phase space. Therefore, the difference between $Ω$ and $Ω_g$ is proportional to a coboundary term. Such coboundary term, however, can be traced back to the redefinition of the BRST charge as discussed in sect.3. Let $Q_g$ stand for the BRST charge producing the geometrized anomalies $Ω_g$ and $Γ_g$. Then it is related with the original BRST charge $Q$ by

$$Q_g = Q - \frac{ℏ}{2} Ξ.$$  \hspace{1cm} (6.1)

The construction of $Ξ$ proceeds with some guess work. It must be a suitable supersymmetric generalization of the corresponding quantities obtained in I. Such a $Ξ$ exists and is given by

$$Ξ = k \int dσ \left[ \frac{1}{2} C^c \left( G_+ - G_- \right) + U_+^C + U_-^C \right],$$  \hspace{1cm} (6.2)
where $U_\pm^C$ and their companions $U_\pm^N$ are defined by

$$
U_\pm^C = - C^\pm \left\{ \left( \frac{1}{4} G_\pm^2 - G'_\pm \right) \pm \frac{i}{2} \Lambda_\pm \Lambda'_\pm \right\} + \frac{i}{2} \gamma_\pm \Lambda_\pm \mp \gamma_\pm (G_\pm - 4\Lambda'_\pm),
$$

$$
U_\pm^N = - N^\pm \left\{ \left( \frac{1}{4} G_\pm^2 - G'_\pm \right) \pm \frac{i}{2} \Lambda_\pm \Lambda'_\pm \right\} - \frac{i}{2} M_\pm \Lambda_\pm \pm M_\pm (G_\pm - 4\Lambda'_\pm). \tag{6.3}
$$

In deriving (6.2) we have used the assumption on the gauge-fixing functions mentioned in sect.5. It is now straightforward to obtain $\Omega_g$ and $\Gamma_g$. On the BFV basis, the $\Omega_g$ is given by

$$
\Omega_g = k \int d\sigma \left[ \frac{C^+ + C^-}{N^+ + N^-} \left\{ \frac{1}{2} \partial_\tau (G_+ - G_-) - \partial_\sigma (V^+_{N} + V^-_{N}) \right\} (C^\xi - V^+_{C} + V^-_{C})
+ \left\{ \frac{2}{N^+ + N^-} \partial_\tau (C^\xi - V^+_{C} + V^-_{C}) - \frac{N^+ - N^-}{N^+ + N^-} \partial_\sigma (C^\xi - V^+_{C} + V^-_{C})
- \frac{2}{N^+ + N^-} (\gamma^+ W^+_{N} + \gamma^- W^-_{N} + M^+ W^+_{C} + M^- W^-_{C}) \right\}
\times (C^\xi - V^+_{C} + V^-_{C}) - \frac{i}{2} \left\{ (W^+_{C})^2 + (W^-_{C})^2 \right\} \right], \tag{6.4}
$$

where $V^\pm_{N}$ are given in (5.12). To show that $\Omega_g$ indeed be equivalent to $\Omega$, we briefly describe how to get $\Omega_g$ in Appendix C. Using the relations

$$
eR + 4ie^{\alpha\beta} \partial_\alpha (\chi_\beta \rho_5 \rho^\gamma \chi_\gamma) = \frac{1}{2} \partial_\tau (G_+ - G_-) - \partial_\sigma (V^+_{N} + V^-_{N})
$$

$$
2e^{\alpha\beta} \overline{\rho}_0^0 \rho^\delta \nabla_\alpha \chi_\beta = \frac{1}{N^+ + N^-} (\omega_- W^+_{N} - \omega_+ W^-_{N}),
$$

$$
e\overline{\eta}_w \rho^0 \rho^0 \chi_\alpha = \frac{2}{N^+ + N^-} (M^+ \eta_{w+} + M^- \eta_{w-}), \tag{6.5}
e\overline{\eta}_w \rho^0 \eta_w = \eta^2_{w+} + \eta^2_{w-},
$$

$$
16e^{\alpha\beta} \overline{\eta}_w \rho_5 \nabla_\alpha \chi_\beta = i (W^+_{N} W^+_{C} + W^-_{N} W^-_{C}),
$$

and, (5.9) and (5.16), one obtains the covariant expression

$$
\Omega_g = k \int d\sigma \left[ \left\{ eR + 4ie^{\alpha\beta} \partial_\alpha (\chi_\beta \rho_5 \rho^\gamma \chi_\gamma) \right\} C^0 C_w + e g^{0\alpha} C_w \partial_\alpha C_w
+ (4ie^{\alpha\beta} \overline{\rho}_0^0 \rho^\delta \nabla_\alpha \chi_\beta - 4e \overline{\eta}_w \rho^0 \rho^0 \chi_\alpha) C_w
- 8ie \overline{\eta}_w \rho^0 \eta_w - 16C^0 e^{\alpha\beta} \overline{\eta}_w \rho_5 \nabla_\alpha \chi_\beta \right]. \tag{6.6}
$$
where $R$ is the scalar curvature for the metric $g_{\alpha\beta}$ and $e \equiv \sqrt{-g}$. Once $\Omega_g$ is given on the BFV basis, it is straightforward to calculate $\Gamma_g = \{\Omega_g, \Psi\}$. In this naive commutator, the ghost fields are simply replaced by the relevant multiplier field due to our assumption. We thus obtain by using the identity,

$$V_N^+ - V_N^- = N^\xi,$$

This is nothing but the super-Weyl anomaly. One may also confirm that this expression is obtained by adding to (4.12) the contribution from the coboundary term $\Xi$ as

$$\Gamma_g = \Gamma - \dot{\Xi} + \delta(\{\Xi, \Psi\}). \quad (6.8)$$

The results (6.6) and (6.7) are just the supersymmetric generalization of the geometrized BRST anomalies obtained in I.

Our covariant expressions (6.6) and (6.7) are invariant under supersymmetry transformation while the Weyl invariance and the fermionic symmetry are necessarily broken upon quantization unless $D = 10$. We stress that derivation of the super-Weyl anomaly given here is by construction nonperturbative and practically gauge independent. At first sight the gauge independence of $\Gamma_g$ might be considered to be odd since it is directly related with the gauge fermion as in (3.9). As we have noted in sect.4, this peculiar property can be understood if one notice that the gauge conditions $\chi$’s and $\zeta$’s do not contribute to the rhs of (3.9) so far as the assumptions on the gauge conditions mentioned in sect.5 are satisfied.

To make it clear the implication of geometrization, it is interesting to note that the genuine super-Virasoro anomaly (4.7) implies the breakdown of the reparametrization invariance and the local supersymmetry of the classical action (2.1) with the superghost sector included, while the Weyl rescaling and the fermionic symmetry remain intact. As we have shown in this section, it can be converted to the super-Weyl anomaly by a suitable redefinition of the BRST charge given by (6.1). Then the geometrized BRST charge $Q_g$ necessarily respects the reparametrization invariance and the local supersymmetry. This can most easily be seen from the
counteraction associated with the transition from $Q$ to $Q_g$. Let us denote the counteraction by $S_g$, then it is given by

$$S_g = \frac{k}{2} \int d\sigma \left[ \frac{1}{2} N\xi(G_+ - G_-) + U^N_+ + U^N_- \right].$$

In terms of configuration space variables, (6.9) can be written as

$$S_g = -\frac{k}{2} \int d^2\sigma \left[ e \left\{ -\frac{1}{2} \left( g^{\alpha\beta} \partial_\alpha \xi \partial_\beta \xi - i\mathcal{N}_\alpha \partial_\alpha \Lambda \partial_\beta \xi - \frac{1}{4} \mathcal{N}_\Lambda \chi_\alpha \rho^\alpha \rho^\alpha \chi_\beta \right) \right. \right]$$

$$+ e R\xi + 4i\epsilon^{\alpha\beta} \mathcal{N}_\alpha \rho_5 \gamma_\gamma \partial_\beta \xi + 4\epsilon^{\alpha\beta} \mathcal{N}_\alpha \rho_5 \nabla_\beta \Lambda - \frac{2g_{11}}{\sqrt{-g}} \left( \left( \frac{g_{11}}{g^1} \right)' \right)^2,$$

where $\Lambda$ is defined by

$$\Lambda = \begin{pmatrix} (-e^{-1})^{-\frac{1}{2}} \Lambda_- \\ (e^1)^{-\frac{1}{2}} \Lambda_+ \end{pmatrix}.$$

Since both $\xi$ and $\Lambda$ do not possess simple transformation properties like scalars and spinors under reparametrization and local supersymmetry, $S_g$ is not invariant under these symmetries. Instead, it exactly cancels the super-Virasoro anomaly of the action $S_{cd} + S_{FP}$ given in (5.3) [30]. Under the Weyl rescaling $\delta\varphi e_\alpha^a = \frac{\varphi}{2} e_\alpha^a$ and $\delta\varphi \chi_\alpha = \frac{\varphi}{4} \chi_\alpha$, and the fermionic symmetry $\delta\eta e_\alpha^a = 0$ and $\delta\eta \chi_\alpha = i\rho_\alpha \eta$ with $\eta$ being an arbitrary Majorana field, $\xi$ and $\Lambda$ behave as super-Liouville mode. Furthermore, $S_g$ correctly reproduces the super-Weyl anomaly relations

$$\delta\varphi S_g = -\frac{k}{2} \int d^2\sigma \left[ eR + 4i\epsilon^{\alpha\beta} \partial_\alpha (\mathcal{N}_\beta \rho_5 \gamma_\gamma \chi_\gamma) \right] \varphi,$$

$$\delta\eta S_g = -8k \int d^2\sigma \epsilon^{\alpha\beta} \eta \rho_5 \nabla_\alpha \chi_\beta.$$

In this sense $S_g$ is nothing but the super-Liouville action with $\xi$ and $\Lambda$ as the super-Liouville fields. It is a local functional of 2D supergravity fields in sharp contrast to the nonlocal super-Liouville action [16,17,31]. This is only possible by sacrificing the reparametrization invariance and the local supersymmetry. Since the super-Virasoro anomaly is converted to the super-Weyl anomaly by introducing $S_g$ as a counteraction, it can be regarded as a Wess-Zumino-Witten term [19,27].
Another point to be noted in connection with (6.10) is that the super-Liouville mode of the 2D supergravity becomes propagating if one include (6.10) to (2.1) as was discussed in ref.[3]. Some of the classical constraints generating the super-Weyl symmetry will be lost from (2.6), and the contributions from the 2D supergravity sector should be included in (2.7). The issues related to this has been discussed in [32] for the bosonic string. The extension to fermionic string will be discussed in [33].

With these observations in mind it is instructive to discuss some gauge-fixed versions of the BRST anomalies. We examine the superconformal gauge[12,5,14] and the supersymmetric extension of the light-cone gauge [34,35,31,36].

(i) Superconformal gauge

In addition to (5.4) this gauge is realized by the following choice of the gauge conditions which are the simple extension of the bosonic string case[8]

\[
\begin{align*}
\chi^\pm &\equiv N^\pm - 1, \\
\chi^\xi &\equiv \xi - \hat{\xi}, \\
\zeta^\pm &\equiv iM^\pm, \\
\zeta^\Lambda &\equiv \Lambda^\pm - \hat{\Lambda}^\pm,
\end{align*}
\]

where \(\hat{\xi}\) and \(\hat{\Lambda}^\pm\) are fixed functions. This corresponds to the choice, \(g_{\alpha\beta} = \eta_{\alpha\beta}\exp(\hat{\xi})\) and \(\chi_\alpha = \frac{i}{4}\rho_\alpha\hat{\Lambda}\). The Weyl ghost [37] and its superpartner can be then eliminated via the equations of motion

\[
\begin{align*}
C_w &= -\frac{1}{2}[(C^+\partial_+ + C^-\partial_-)\hat{\xi} + (\partial_+C^+ + \partial_-C^-) + i(\Lambda_+\omega_- - \Lambda_-\omega_+)], \\
\eta_{\nu\pm} &= \frac{1}{8}(C^+\partial_+ + C^-\partial_-)\hat{\Lambda}^\pm \pm \frac{1}{16}\partial_\pm C^\pm\hat{\Lambda}^\pm \pm \omega_\mp \partial_\pm\hat{\xi} \pm \frac{1}{2}\partial_\pm\omega_\mp,
\end{align*}
\]

where the reparametrization ghost, \(C^\pm = C^0 + C^1\), and its partner, \(\omega_\pm\), satisfy \(\partial_\pm C^\mp = 0\) and \(\partial_\pm\omega_\mp = 0\). It is easy to compute gauge-fixed forms of \(\Omega_g\) and \(\Gamma_g\), which depend on \(\hat{\xi}\) and \(\hat{\Lambda}^\pm\) in (6.13). Especially in the super-orthonormal gauge with \(\hat{\xi} = \hat{\Lambda}^\pm = 0\) we obtain

\[
\begin{align*}
\Omega_g &= k \int d\sigma [C^+\partial^3C^+ + 8i\omega_+\partial^2\omega_+] + \{+ \leftrightarrow -\}, \\
\Gamma_g &= 0.
\end{align*}
\]

This is the supersymmetric extension[13,14] of the result of Kato-Ogawa[5], and corresponds to the BRST gauge-fixed version of the genuine super-Virasoro anomaly.*

---

* The light-cone coordinates are defined by \(\sigma^\pm = \tau \pm \sigma\). For derivatives we employ the convention \(\partial_\pm = \partial_\tau \pm \partial_\sigma\).
The super-Weyl anomaly can not directly be seen in this special gauge. One should not conclude from this result the absence of super-Weyl anomaly. As noted in I, it is not legitimate to fix all the classical gauge degrees if some of the local symmetries become anomalous. In the present case we can not gauge-fix the superconformal mode of 2D supergravity due to the super-Weyl anomaly. It can be shown that one can recover the super-Weyl anomaly if the contributions of the superconformal mode to $\Gamma_g$ are properly taken into account.

(ii) Light-cone gauge

The supersymmetric extension of the light-cone gauge fixing for the bosonic string can be defined by

$$(e^a_\alpha) = \begin{pmatrix} e^{++} & e^{+-} \\ e^{-+} & e^{--} \end{pmatrix}, \quad \chi^- = \begin{pmatrix} \chi^{--} \\ \chi^{-+} \end{pmatrix} = 0.$$ (6.16)

In terms of BFV variables this is equivalent to the following set of gauge conditions

$$\chi^+ \equiv N^+ - 1, \quad \chi^- \equiv \epsilon^\xi (N^- + 1) - 2,$$

$$\zeta^+ \equiv iM^+, \quad \zeta^- \equiv iM^- - \frac{1}{4} (N^- + 1) \Lambda_-$$

(6.17)

together with (5.4). As in the conformal gauge fixing we should specify $\chi^\xi$ and $\zeta^A_\pm$ to completely fix the classical gauge symmetries corresponding to the Weyl rescaling and fermionic symmetry. Since they are broken by anomalies, we will leave $\chi^\xi$ and $\zeta^A_\pm$ unspecified.

The equations of motion for spinor components become simple in terms of rescaled variables: $(\pm e_1^\pm)\frac{1}{2} \omega_\mp \rightarrow \omega_\mp$, $\mp e_1^{\mp} \rightarrow \eta_{w\mp}$ and $(\pm e_1^\pm)\frac{1}{2} \chi^{\mp} \rightarrow \chi^{\mp}$. Then the equations of motion for ghosts are given by

$$\partial_- C^+ = 0, \quad C_w = -\frac{1}{2} (\partial_+ C^+ + \partial_- C^-) + 4i \omega_- \chi_{+-},$$

$$\partial_- \omega_- = 0, \quad \eta_{w+} = -\frac{1}{2} \partial_- \omega_+,$$

(6.18)

while the equations for $C^-$ and $\eta_{w+}$ can not be derived from the gauge conditions (6.17). Using (6.18), we can easily find the BRST anomalies in the light-cone
gauge. In particular, $\Gamma_g$ is given by

$$
\Gamma_g = \frac{k}{2} \int d\sigma (\partial_+ C^+ + \partial_- C^-) \partial^2 g_{++} \\
+ 8ik \int d\sigma (\partial_- \omega_+ \partial_- \chi_{++} + \frac{1}{2} \omega_- \chi_+ - \partial^2 g_{++}) - 16ik \int d\sigma \eta_{w+} \partial_- \chi_{+-} \\
= - \frac{k}{2} \int d\sigma C^- \partial_+^3 g_{++} - 8ik \int d\sigma \omega_+ \partial_-^2 \chi_{++} \\
- 16ik \int d\sigma (\eta_{w+} + \frac{1}{4} \omega_- \partial_- g_{++}) \partial_- \chi_{+-} + \cdots.
$$

(6.19)

In the second equality we have omitted the total time derivatives.

7. Summary

Applying the generalized hamiltonian formalism of Batalin, Fradkin and Vilkovisky, we have quantized superstring theory of Ramond-Neveu-Schwarz to perform an exhaustive algebraic analysis on anomalies in the EPS. To make the analysis most general, we have presented a new canonical formulation of 2D SUGRA theory. On the basis of this BRST scheme, the genuine super-Virasoro anomaly expressed by $\Omega$ is identified with the essentially unique solution of the consistency condition, $\delta \Omega = 0$, without invoking any particular gauge for the metrics and gravitinos on the world-sheet. The absolute normalization for the $\Omega$ can be fixed by using the canonical normal ordering prescription. Our analysis shows up the primary importance of the genuine super-Virasoro anomaly; it is totally gauge independent, and the super-Weyl anomaly in the configuration space is obtained from it as a descendant. We have derived the most general form of the super-Weyl anomaly by making a partial gauge-fixing and explicitly finding a local counterterm needed for the covariantization. The conditions under which this expression can be independent of the remaining gauge choices are clarified. Our results are obtained in a nonperturbative way without assuming the weak gravitational field as in ref. [6, 24], and valid in any space-time dimensions. It is straightforward to give gauge-fixed forms of these BRST anomalies. We have examined superconformal gauge and supersymmetric light-cone gauge as particularly interesting cases.

The above results are summarized as a hierarchial relationship among the anomalies in fermionic string theory. In the unconstrained EPS, the genuine super-Virasoro anomaly is sitting on the top of the hierarchy of anomalies. In its subspace,
where the two dimensional metric variables and their superpartner can be identified, this pregeometrical anomaly obtains its geometrical meaning and appears as the super-Weyl anomaly. The $Q^2$ anomaly being supersymmetric extension[13,14] of the one considered by Kato and Ogawa[5] for bosonic string is obtained as a complete gauge-fixed form of the anomaly in the super-orthonormal gauge. The relationship clarified here between the super-Virasoro and the super-Weyl anomaly should be compared with that discussed in the algebraic approaches [2, 25, 26] using descent equations for cocycles[27-22]. There, the super-Virasoro anomaly has been calculated as a 2-cocyle from 1-cocyle, the super-Weyl anomaly. The hierarchial relationships discussed here can not be revealed in the gauge-fixed analyses.

Our result of the genuine super-Virasoro anomaly is the starting point for quantizing subcritical fermionic string theory or 2D SUGRA as an anomalous gauge theory[32,33]. A systematic construction of the super-Liouville action for the theory will be discussed in a forthcoming paper[30].
APPENDIX A

BRST transformations in the configuration space

In order to establish the ghost relations between the BFV and the covariant basis, we give the BRST transformation in the configuration space:

\[
\delta X = C^a \partial_a X - i(\omega_- \psi_+ - \omega_+ \psi_-),
\]

\[
\delta \psi_\pm = C^a \partial_a \psi_\pm + \frac{1}{2}(C^{1\prime} \pm \lambda^\pm C^{0\prime}) \psi_\pm
\]

\[
\pm \frac{2}{\lambda^+ + \lambda^-} \omega_\mp \{ \dot{X} \pm \lambda^\mp X' - i(\nu_- \psi_+ - \nu_+ \psi_-) \},
\]

\[
\delta \lambda^\pm = C^a \partial_a \lambda^\pm \pm (\dot{\lambda}^1 \pm \lambda^\pm \dot{\lambda}^0) - \lambda^\pm (C^{1\prime} \pm \lambda^\pm C^{0\prime}) - 4i \omega_\mp \nu_\mp,
\]

\[
\delta \xi = C_w + C^a \partial_a \xi + 2C^{1\prime} + C^{0\prime}(\lambda^+ - \lambda^-) - i(\omega_- \Lambda_+ - \omega_+ \Lambda_-),
\]

\[
\delta \nu_\pm = C^a \partial_a \nu_\pm + (\dot{\nu}^0 \pm \lambda^\mp C^{0\prime}) \nu_\pm - \frac{1}{2}(C^{1\prime} \mp \lambda^\mp C^{0\prime}) \nu_\pm + \dot{\omega}_\pm \pm \lambda^\mp \omega_\pm \mp \frac{1}{2} \lambda^\mp \omega_\pm,
\]

\[
\delta \Lambda_\pm = -4\eta_{w\pm} + C^a \partial_a \Lambda_\pm - 4C^{0\prime} \nu_\pm + \frac{1}{2}(C^{1\prime} \pm \lambda^\mp C^{0\prime}) \lambda_\pm
\]

\[
\pm \frac{8}{\lambda^+ + \lambda^-} \omega_\mp \{ \dot{\xi} \pm \lambda^\mp \xi' - (\lambda^+ - \lambda^-)' - i(\nu_+ \Lambda_+ - \nu_- \Lambda_-) \} + 4\omega'_\mp,
\]

\[
\delta C^0 = C^a \partial_a C^0 + \frac{2i}{\lambda^+ + \lambda^-}(\omega^2_+ + \omega^2_-),
\]

\[
\delta C^1 = C^a \partial_a C^1 - \frac{2i}{\lambda^+ + \lambda^-}(\lambda^+ \omega^2_+ - \lambda^- \omega^2_-),
\]

\[
\delta \omega_\pm = C^a \partial_a \omega_\pm - \frac{1}{2}(C^{1\prime} \mp \lambda^\pm C^{0\prime}) \omega_\mp - \frac{2i}{\lambda^+ + \lambda^-}(\omega^2_+ + \omega^2_-) \nu_\pm,
\]

\[
\delta C_w = C^a \partial_a C_w - 4i(\omega_- \eta_{w+} - \omega_+ \eta_{w-}),
\]

\[
\delta \eta_{w\pm} = C^a \partial_a \eta_{w\pm} + \frac{1}{2}(C^{1\prime} \pm \lambda^\pm C^{0\prime}) \eta_{w\pm}
\]

\[
- \frac{1}{\lambda^+ + \lambda^-} \omega_\mp \{ \dot{\xi} \pm \lambda^\mp \xi' - (\lambda^+ - \lambda^-)' - i(\nu_+ \Lambda_+ - \nu_- \Lambda_-) \}
\]

\[
+ \frac{i}{2(\lambda^+ + \lambda^-)} \omega_\mp \{ (\dot{\Lambda}_+ - \lambda^+ \Lambda'_+ + \frac{1}{2} \lambda^+ \Lambda_- - 4\nu'_+) \omega_- 
\]

\[
+ (\dot{\Lambda}_- + \lambda^- \Lambda'_- + \frac{1}{2} \lambda^- \Lambda_+ - 4\nu'_-) \omega_- 
\]

\[
+ \frac{i}{\lambda^+ + \lambda^-} \omega_\mp \{ (\dot{\xi} - \lambda^+ \xi' - (\lambda^+ - \lambda^-)' - i\lambda^+ \nu_- ) \nu_\mp \omega_- 
\]

\[
- (\dot{\xi} + \lambda^- \xi' - (\lambda^+ - \lambda^-)' + i\lambda^- \nu_+ ) \nu_\pm \omega_+ \} [A.1]
\]

These expressions are not manifestly covariant. It is the price of not introducing
the pure gauge and auxiliary fields introduced in the configuration space approach [15]. The manifest covariance is, however, retained in our final expressions of the super-Weyl anomaly.

APPENDIX B

**Nontrivial Ω contains no ghost momenta \( \overline{p}_\pm \) and \( \overline{\beta}_\pm \)**

We shall show here the lemma in sect.4 that ghost momenta (\( \overline{p}_\pm \) and \( \overline{\beta}_\pm \)) dependence can be removed in \( \Omega = \int d\sigma \omega \) up to a coboundary term, where \( \omega \) denotes the density throughout Appendix B.

We begin by noting that any operator \( \omega \) with \( \text{gh}(\omega) = 2, \dim(\omega) = 2c + 3 \) can be expanded to linear in \( \overline{p}_+ \) and to cubic in \( \overline{\beta}_+ \):

\[
\omega = \overline{p}_+ f_1 + \overline{\beta}_+ f_2 + \overline{p}_+ \overline{\beta}_+ f_3 + \overline{\beta}_+^2 f_4 + \overline{\beta}_+^3 f_5 + \omega_1, \quad (B.1)
\]

where \( f_i \) (\( i = 1, \ldots, 5 \)) and \( \omega_1 \) are operators containing no \( \overline{p}_+ \) and \( \overline{\beta}_+ \). Since \( \delta(\overline{\beta}_+^3 f_5) \) contains a term \( \overline{p}_+ \overline{\beta}_+^2 \) which can not be cancelled with other terms unless \( f_5 = 0 \). Hence,

\[
f_5 = 0. \quad (B.2)
\]

Requiring \( \delta \omega = \partial \chi \) for some operator \( \chi \), we obtain

\[
\begin{align*}
\delta \omega_1 &= \varphi_+ f_1 + \mathcal{J}_+ f_2 + \partial \chi_1, \\
\delta f_1 &= C^+ f_1' - C^+ f_1 - 4i\gamma^+ f_2 + \mathcal{J} f_3, \\
\delta f_2 &= C^+ f_2' - \frac{1}{2} C^+ f_2 - \frac{1}{2} \gamma^+ f_1' + \gamma^+ f_1 + \varphi_+ f_3 + 2 \mathcal{J} f_4, \\
\delta f_3 &= C^+ f_3' - \frac{5}{2} C^+ f_3 - 8i\gamma^+ f_4, \\
\delta f_4 &= C^+ f_4' - 2 C^+ f_4 - \frac{1}{4} \gamma^+ f_3' + \frac{5}{4} \gamma^+ f_3,
\end{align*} \quad (B.3)
\]

where

\[
\chi_1 = \chi + (\overline{p}_+ C^+ + \frac{1}{2} \overline{\beta}_+ \gamma^+) f_1 - \overline{\beta}_+ C^+ f_2 \\
+ (\overline{p}_+ \overline{\beta}_+ C^+ + \frac{1}{4} \overline{\beta}_+ \gamma^+) f_3 - \overline{\beta}_+^2 C^+ f_4.
\]

It can be shown that an operator whose BRST transformation is linear in the constraints \( \varphi_+ \) and \( \mathcal{J}_+ \) is given, up to a coboundary term and a total divergence,
by a linear combination of the constraints and of the terms proportional to \(Y_+^2\) and \(Y_+\psi'_+\). Hence, \(\omega_1\) takes the form

\[
\omega_1 = \varphi_+ g_1 + \mathcal{J}_+ g_2 + Y_+^2 g_3 + Y_+ \psi'_+ g_4 + \omega_2, \tag{B.4}
\]

where operators \(g_i\) \((i = 1, \ldots, 4)\) do not depend on \(\overrightarrow{P}_+\), \(\overrightarrow{\beta}_+\), \(Y_+\) and \(\psi_+\), and \(\omega_2\) satisfies \(\delta \omega_2 = \partial \chi_2\) for some \(\chi_2\). Calculating \(\delta \omega_1\) of (B.4) and comparing this with the one in (B.3), one finds that

\[
\delta \omega_2 = \partial \chi_2, \\
f_1 = \delta g_1 - C^+ g_1' + C^{++'} g_1 + 4i\gamma^+ g_2 - 4i\gamma^{'+} g_4 + \mathcal{J}_+ g_5, \\
f_2 = -[\delta g_2 - C^+ g_2' + \frac{\gamma^+}{2} g_2' - \frac{\gamma^{'+}}{2} (g_1 + 4g_3) + \frac{1}{2} C^{''} g_4] \\
- \varphi_+ g_5 + \mathcal{J}_+ g_6, \\
\delta g_3 = C^+ g_3' - C^{++'} g_3 + i\frac{\gamma^+}{2} g_4' - \frac{3}{2} i\gamma^{'+} g_4, \\
\delta g_4 = C^+ g_4' - \frac{3}{2} C^{++'} g_4 + 4\gamma^+ g_3,
\]

where

\[
\chi_2 = \chi_1 - (C^+ \varphi_+ + \frac{1}{2} \mathcal{J}_+ \gamma_+) g_1 - C^+ (\mathcal{J}_+ g_2 + Y_+^2 g_3) \\
- (\frac{i}{2} \gamma^+ Y_+ + C^{'} \psi'_+) Y_+ g_4. \tag{B.6}
\]

In (B.5) \(g_5\) and \(g_6\) are any operators containing no ghost momenta \(\overrightarrow{P}_+\) and \(\overrightarrow{\beta}_+\). They carry \(\dim(g_5) = 3c + 1/2\), \(\dim(g_6) = 3c + 1\) and \(\text{gh}(g_5) = \text{gh}(g_6) = 3\). Therefore, \(g_5\) and \(g_6\) cannot contain \(Y_+\) and \(\psi_+\). Furthermore, (B.3) and (B.5) imply that

\[
\mathcal{J}_+ f_3 = \delta f_1 - C^+ f_1' + C^{++'} f_1 + 4i\gamma^+ f_2 \\
= -\mathcal{J}_+ (\delta g_5 - C^+ g_5' + \frac{5}{2} C^{++'} g_5 - 4i\gamma^+ g_6), \\
\mathcal{J}_+ f_4 = \frac{1}{2} (\delta f_2 - C^+ f_2' + \frac{\gamma^+}{2} C^{++'} f_2 + \frac{1}{2} \gamma^+ f_1' - \gamma^{'+} f_1 - \varphi_+ f_3) \\
= -\frac{1}{2} \mathcal{J}_+ (\delta g_6 - C^+ g_6' + 2C^{++'} g_6 - \frac{1}{2} \gamma^+ g_5' + \frac{5}{2} \gamma^{'+} g_5), \tag{B.7}
\]

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which leads to

\[
f_3 = - (\delta g_5 - C^+ g'_5 + \frac{5}{2} \gamma^+ g_5 - 4i \gamma^+ g_6),
\]

\[
f_4 = - \frac{1}{2} (\delta g_6 - C^+ g'_6 + 2 \gamma^+ g_6 - \frac{1}{2} \gamma^+ g'_5 + \frac{5}{2} \gamma^+ g_5).
\]

Substituting expressions of \( f_i \) \((i = 1, \ldots, 4)\) into (B.5) and (B.8) and \( \omega_1 \) in (B.4) into (B.1), we finally obtain

\[
\omega = \omega_3 + \delta \eta + \partial \zeta
\]

where

\[
\begin{align*}
\omega_3 &= \omega_2 - \frac{8}{3} \varphi + g_3 + \frac{1}{3} \mathcal{J}_+ g'_4 + Y^2 g_3 + Y_+ \psi'_4 g_4 \\
\eta &= - \overline{P}_+(g_1 + \frac{8}{3} g_3) - \overline{\beta}_+(g_2 - \frac{1}{3} g'_4) + \overline{P}_+ \overline{\beta}_+ g_5 - \frac{1}{2} \overline{\beta}_+^2 g_6 \\
\zeta &= - (\overline{P}_+ C^+ + \frac{1}{2} \overline{\beta}_+ \gamma^+)(g_1 + \frac{8}{3} g_3) + \overline{\beta}_+ C^+(g_2 - \frac{1}{3} g'_4) \\
&\quad + (\overline{P}_+ \overline{\beta}_+ C^+ + \frac{1}{4} \overline{\beta}_+^2 \gamma^+ g_5 + \frac{1}{2} \overline{\beta}_+^2 C^+ g_6.
\end{align*}
\]

It is easy to show

\[
\delta \omega_3 = \partial \chi_3
\]

for some \( \chi_3 \). The ghost momenta \( \overline{P}_+ \) and \( \overline{\beta}_+ \) are shown to appear only in the coboundary term \( \delta \eta \) or in the total derivative term \( \partial \zeta \). Since the same argument obviously applies to remove the dependence on \( \overline{P}_- \) and \( \overline{\beta}_- \) from the nontrivial solution, we have proved the lemma.
APPENDIX C

Derivation of the covariant form $\Omega_g$ of $Q^2$ anomaly

We summarize here the derivation of $\Omega_g = \Omega + \delta \Xi$, where $\Xi$ is given in (6.2). The calculation will be simplified if one uses the reflection symmetry under $\{ + \leftrightarrow - \}$. Note that $\partial_\sigma, G_\pm, V_\mp C \to -\partial_\sigma, -G_\mp, -V_\mp C$ while the variables with $\xi$-index are unchanged under the reflection. Using the relations

$$G_+ + G_- = 2\xi', \quad V_N^+ - V_N^- = N^\xi$$

(C.1)

and

$$\delta G_\pm = \frac{2}{N^+ + N^-}[ -P^\xi + N^\mp C^\xi \mp J^+ \mp J^- ]$$

(C.2)

with

$$J^\pm \equiv \pm \frac{1}{2} G_\pm P^\pm \pm P^{\pm'} - \gamma^A \Lambda^{\pm'} + \Lambda_{\pm} \beta_{\pm},$$

(C.3)

we find that $\delta \Xi$ is given by

$$\delta \Xi = k \int d\sigma \left[ \left( \frac{1}{2} C^\xi + V_c^+ \right) \delta G_+ - \left( \frac{1}{4} C^{\pm'} G_+^2 + C^{\pm''} G_+ \right) + L^+ \right] + \{ + \leftrightarrow - \},$$

(C.4)

where we have introduced $L^\pm$ by

$$L^\pm = -i \left( \gamma_{\pm} \right)^2 \pm i \left[ \{ \mp C^\pm C^{\pm'} + 2i(\gamma^\pm)^2 \} \Lambda_{\pm} \Lambda_{\pm'} - C^\pm (\gamma^A \Lambda_{\pm} - \Lambda_\pm \gamma_{\pm'}) \right]
+ \left( \frac{1}{2} C^{\pm'} \gamma^\pm - C^\pm \gamma^\pm' \right) (G_{\pm} \Lambda_{\pm} + 4 \Lambda_{\pm}) \pm \gamma^\pm (G_{\pm} \gamma^A - 4 \gamma^A_{\pm})
+ 2i(\gamma^\pm)^2 \left( \frac{1}{4} G^2_{\pm} - G_{\pm}' \right).$$

(C.5)

Using (5.11) and (C.2), we obtain

$$\delta \Xi = k \int d\sigma \left[ \frac{1}{N^+ + N^-} C_w \left\{ -P^\xi + 2J^+ + N^+ C^\xi \right\} + V_c^+ C^\xi
- (V_c^+ - C^{\pm'}) (V_c^{\pm'} + C^{\pm''}) + K^+ + L^+ \right] + \{ + \leftrightarrow - \},$$

(C.6)
where \(K^{\pm}\) are given by

\[
K^{\pm} = (V^{\pm}_{c} - C^{\pm})(\Lambda^{\pm}\gamma^{\pm})' - (V^{\pm}_{c} + C^{\pm})'\Lambda^{\pm}\gamma' \pm \Lambda^{\pm}\Lambda^{\prime}_{\pm}(\gamma^{\pm})^{2}. \tag{C.7}
\]

In the integrand of the rhs of (C.6) the terms proportional to \(C_{w}\) can be written as

\[-\mathcal{P}^{\xi} + 2J^{+} + N^{+}C^{\xi'} + \{+ \leftrightarrow -\} = -2\dot{C}_{w} + (N^{+} + N^{-})C'_{w} \]

\[
- \left[ \frac{1}{2} \partial_{r}(G_{+} - G_{-}) - \partial_{\sigma}(V^{+}_{N} + V^{-}_{N}) \right] (C^{+} + C^{-}) \]

\[- (N^{+} - N^{-})(V^{+}_{c} + V^{-}_{c})' + 2(Z^{+} + Z^{-}) \tag{C.8}
\]

with

\[
Z^{\pm} = \pm [N^{\pm}(A^{\pm}\gamma^{\pm})' - (A^{\pm}M^{\pm})'C^{\pm}] \mp [G^{\pm}\gamma^{\pm}M^{\pm} + 2(\gamma^{\pm}M^{\pm})'] \]

\[- \partial_{r}(A^{\pm}\gamma^{\pm}) - (\gamma^{\pm}M^{\pm} - \Lambda^{\pm}\beta^{\pm}) \]

\[- = -2(\gamma^{\pm}W^{\pm}_{N} + M^{\pm}W^{\pm}_{c}), \tag{C.9}
\]

where \(W^{\pm}_{N}\) and \(W^{\pm}_{c}\) are defined by (5.17). By utilizing the identities

\[
K^{\pm} + L^{\pm} = -8i(\gamma^{\pm})^{2} - \frac{i}{2}(W^{\pm}_{c})^{2}, \tag{C.10}
\]

we find that the coboundary term is finally given by

\[
\delta \Xi = k \int d\sigma \left[ \frac{C^{+} + C^{-}}{N^{+} + N^{-}} \left\{ \frac{1}{2} \partial_{r}(G_{+} - G_{-}) - \partial_{\sigma}(V^{+}_{N} + V^{-}_{N}) \right\} C_{w} \right.
\]

\[
+ \left\{ \frac{2}{N^{+} + N^{-}} \partial_{r}C_{w} - \frac{N^{+} - N^{-}}{N^{+} + N^{-}} \partial_{\sigma}C_{w} \right\} C_{w} \]

\[
- \frac{2}{N^{+} + N^{-}}(\gamma^{+}W^{+}_{N} + \gamma^{-}W^{-}_{N} + M^{+}W^{+}_{c} + M^{-}W^{-}_{c})C_{w} \tag{C.11}
\]

\[- \frac{i}{2} \left\{ (W^{+}_{c})^{2} + (W^{-}_{c})^{2} \right\} \]

\[- C^{+}C^{+''} + C^{-}C^{-''} - 8i(\gamma^{+})^{2} - 8i(\gamma^{-})^{2} \right].
\]

One thus obtains the result (6.4) of the geometrized \(\Omega_{g}\) in terms of EPS variables.
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