January 12, 2024

THE DIFFERENTIAL TOPOLOGY OF THE THURSTON SPINE

INGRID IRMER

Abstract. In [28], a short, simple and elegant construction of a mapping class group-equivariant deformation retraction of Teichmüller space of a closed compact surface was given. The preprint [28], which unfortunately is not online, has not been broadly accepted. The purpose of this paper is to go through the construction in detail, resolve any questions that have arisen in the literature and in personal communications. An explicit example is given to show that one of the claims needs to be modified, and details of how to do this are given. A mapping class group-equivariant deformation retraction of the Thurston spine onto a complex of dimension equal to the virtual cohomological dimension of the mapping class group is then constructed.

Contents

1. Introduction 1
   Acknowledgements 3
2. Definitions and conventions 3
3. Thurston’s deformation retraction 6
4. The modified claim 11
5. Flowing into a regular neighbourhood 12
6. A deformation retraction of the Thurston spine 19
References 25

1. Introduction

One reason for studying the Teichmüller space of a surface is because it is a contractible space on which the mapping class group acts properly discontinuously. On the other hand, it is clear that Teichmüller space is not the lowest dimensional space on which the mapping class group acts properly discontinuously; it is known that there exist nontrivial mapping class group-equivariant deformation retractions of Teichmüller space, for example Theorem 2.7 of [10].

When the surface has at least one puncture, an explicit mapping class group-equivariant deformation retraction of Teichmüller space was given in [13] and [25]. The dimension of the image of this deformation retraction was shown to be equal to the virtual cohomological dimension of mapping class group; this is a homological invariant that provides a lower bound on the dimension of the image of such a deformation retraction. The existence of a puncture was a crucial ingredient in all these constructions; informally speaking, a puncture or some form of marked point is needed, relative to which coordinates defining a cell decomposition are defined. In this case, determining whether a deformation retraction onto a CW-complex of dimension equal to the virtual cohomological dimension actually exists is listed as the first...
open question in [8]. The interested reader is referred to [8], [4] and Chapter 3.3 of [14] for a survey of the background and applications of this question.

Thurston’s construction in [28] resolved the problem of a missing basepoint by using curve lengths to parametrise Teichmüller space, constructing a mapping class group-equivariant deformation retraction of the Teichmüller space of a closed, compact surface. The image of this deformation retraction is the so-called Thurston spine $\mathcal{P}_g$. This is a CW-complex contained in $\mathcal{T}_g$ consisting of the set of points representing hyperbolic surfaces that are cut into polygons by the set of shortest geodesics (also known as the systoles).

**Theorem 1.** There is a mapping class group-equivariant deformation retraction of the Thurston spine of a closed orientable surface of genus $g$ onto a CW-complex of dimension equal to $4g - 5$.

For large genus, the reduction in dimension achieved by the deformation retraction from Theorem 1 can be significant, as there are examples known for which the codimension of the Thurston spine is small relative to the genus of the surface. This follows from the construction in [6] of small index critical points of the topological Morse function (these are defined in Section 2) $f_{\text{sys}} : \mathcal{T}_g \to \mathbb{R}$, whose value at a point $x$ of $\mathcal{T}_g$ is given by the length of the systoles. Any spine constructed as in [28] must contain the “unstable manifolds” of critical points of $f_{\text{sys}}$, so the index of a critical point gives an upper bound on the codimension.

The Teichmüller space $\mathcal{T}_g$ is contractible, and by Fricke’s theorem, the mapping class group acts properly discontinuously on it. One characterisation of moduli space is as the quotient of $\mathcal{T}_g$ by the action of the mapping class group. Studying mapping class group-equivariant deformation retractions of Teichmüller space is therefore intimately connected with questions about the virtual cohomological dimension of the mapping class group and about the problem of finding a space of the lowest possible dimension on which the mapping class group acts properly discontinuously. The virtual cohomological dimension gives a lower bound on this dimension. As the Thurston spine is the image of a mapping class group-equivariant deformation retraction of Teichmüller space, Theorem 1 together with the construction in [28] shows that this lower bound is achieved.

**The Steinberg module and the Thurston Spine** The “thick” part, $\mathcal{T}_g^{\epsilon_M}$, of $\mathcal{T}_g$ is defined to be the set of all points of $\mathcal{T}_g$ corresponding to surfaces with injectivity radius greater than or equal to a specific constant called the Margulis constant $\epsilon_M$. The Margulis constant has many important geometric and algebraic properties; for example, when $f_{\text{sys}}$ is less than or equal to $\epsilon_M$, it follows from Margulis’s Lemma that the systoles are pairwise disjoint. It was shown in [13] that Harvey’s complex of curves $\mathcal{C}_g$ is a simplicial complex homotopy equivalent to an infinite wedge of spheres of dimension $2g - 2$. It is known, [18], that $\partial \mathcal{T}_g^{\epsilon_M}$ is $\Gamma_g$-equivariantly homotopy equivalent to $\mathcal{C}_g$. 

The Steinberg module of the mapping class group is defined to be the reduced homology group \( \tilde{H}_{2g-2}(C_g; \mathbb{Z}) \). As there is a simplicial action of the mapping class group on the complex of curves, the Steinberg module inherits the structure of a mapping class group-module.

Theorem 1 is proven by using the homology of \( \partial T^\epsilon_M \) to show that certain cells dual to the spine — analogous to the cells defined by Schmutz — must be homotopic into \( \partial T^\epsilon_M \) relative to their boundary on \( \partial T^\epsilon_M \). When the dimension of \( P_g \) is greater than \( 4g - 5 \), this shows the existence of a boundary of the Thurston spine, used as a starting point for a deformation retraction. This deformation retraction can be done \( \Gamma_g \)-equivariantly, because \( \Gamma_g \) preserves the level sets of \( f_{\text{sys}} \), and the intersection of the fixed point sets of \( \Gamma_g \) with \( P_g \) behave like unstable manifolds of critical points.

Section 3 discusses the construction from [28] in detail. A number of questions about this construction have been raised, for example [19]. Section 3 answers all such questions of which the authors are aware. In Section 4, an explicit counterexample is given to a claim made in [28]. As shown in Section 5 this could be resolved by replacing certain “\( \epsilon \)-neighbourhoods” of the spine by slightly more general “tubular” neighbourhoods.

The basic intuition behind the construction would appear to be based on the classic Morse-theoretic construction of a deformation retraction onto a Morse-Smale complex. One aim of this paper is to preserve the elegant simplicity of the construction from [28], while providing the technical details.

**Acknowledgements.** The author would like to thank Stavros Garoufalidis, Olivier Mathieu, Scott Wolpert and Don Zagier for helpful discussions and comments.

## 2. Definitions and conventions

The purpose of this section is to supply detailed definitions and background for the rest of the paper.

Let \( S_g \) be a closed, compact, connected, orientable surface of genus \( g \).

A *marking* of \( S_g \) is a diffeomorphism \( f : S_g \to M \), where \( M \) is a closed, orientable, hyperbolic surface with genus \( g \geq 2 \), and \( S_g \) is a closed, orientable, topological surface of genus \( g \). The Teichmüller space \( T_g \) is the set of pairs \((M, f)\) modulo the equivalence relation \((M, f) \sim (N, h)\) if \( f \circ h^{-1} \) is isotopic to an isometry. A topology on \( T_g \) is usually assumed that makes it homeomorphic to \( \mathbb{R}^{6g-6} \). More information can be found in Section 10.6 of [12].

The group of isotopy classes of orientation preserving diffeomorphisms from \( S_g \to S_g \) is known as the *mapping class group* \( \Gamma_g \) of \( S_g \). The mapping class group acts on \( T_g \) by changing the marking, namely \( \Gamma_g \times T_g \to T_g \) is given by \( \gamma \times (M, f) \mapsto (M, f \circ \gamma^{-1}) \). The quotient of \( T_g \) by this action is called the *moduli space* of \( S_g \) and will be denoted by \( \mathcal{M}_g \).
It will be assumed that the genus \( g \geq 2 \) in order to ensure that all surfaces admit a hyperbolic structure. Once a point in Teichmüller space is chosen, by an abuse of notation, \( S_g \) will be used to denote the surface \( S_g \) endowed with the corresponding hyperbolic structure.

A curve on \( S_g \) is assumed to be a closed, embedded, nonoriented, nontrivial isotopy class of maps of \( S^1 \) to \( S_g \). The length of the curve will be defined to be the length of its geodesic representative. When there is no possibility of confusion, the image of a particular representative of the isotopy class, such as a geodesic, will also be referred to as a curve. An individual curve will be denoted by a lowercase \( c \), sometimes with a subscript, whereas a finite set of curves will be denoted by an uppercase \( C \).

A set of curves on a surface is said to fill the surface if the complement of the geodesic representatives is a union of polygons. The Thurston spine, \( P_g \), is the set of points at which the set of shortest curves (these are called systoles) fill the surface.

The curve complex, \( C_g \), defined by Harvey in [15], is the flag complex with \( n \)-simplices in 1-1 correspondence with sets of \( n + 1 \) homotopically nontrivial, pairwise disjoint, closed curves on \( S_g \). Since the action of the mapping class group on curves preserves intersection properties, the mapping class group acts simplicially on \( C_g \).

Whenever a metric is required, the Weil-Petersson metric will be assumed.

Definition 2 (Systole stratum \( \text{Sys}(C) \)). For a fixed set of curves \( C \), the systole stratum \( \text{Sys}(C) \) is the set of points of \( T_g \) on which the set of systoles is exactly \( C \).

Following Thurston, the term “stratum” is used to mean a decomposition into locally closed subsets, with each point contained in a neighbourhood that intersects only finitely many strata. This stratification can also be extended to the metric completion of \( T_g \) with respect to the Weil-Petersson metric, where a stratum of noded surfaces is labelled by the multicurve that has been pinched to obtain the noded surface.

A systole stratum is a closed semi-analytic subset of \( T_g \). It is the solution to a system of analytic equations, stating that certain geodesics (the systoles) have the same length in addition to a locally finite set of inequalities ensuring that these geodesics are shorter than all others. The local finiteness is well-known, and follows for example from the collar lemma. As a consequence of Lojasiewicz’s theorem, [22], a neighbourhood of \( P_g \) in \( T_g \) admits a triangulation compatible with the stratification. \( P_g \) is therefore a simplicial complex. Any reference to the boundary of a stratum assumes such a triangulation and is defined by applying the usual boundary operator to the subcomplex consisting of the simplices with interior in \( \text{Sys}(C) \).

Tangent cones and cone of increase. The piecewise-smooth structure behind many of the arguments in this paper make it convenient to define tangent cones, analogously to
definitions given for polyhedra in [9] or [2].

The tangent cone to a simplex $T$ at point $p$ of $T$ such that $v = \dot{\gamma}(0)$ for a smooth oriented path $\gamma(t)$ with $\gamma(0) = p$ and $\gamma(\epsilon)$ in $T$ for sufficiently small $\epsilon > 0$. In other words, it is the set of 1-sided limits of tangent vectors to the simplex. When $p \in \mathcal{P}$ is on the boundary of more than one simplex of $\mathcal{P}$, the tangent cone to $\mathcal{P}$ at $p$ is the union of the tangent cones of the simplices with $p$ on the boundary. The unit tangent cone is the set of unit vectors in the tangent cone. Tangent cones of strata are defined similarly. For a triangulation compatible with the stratification, the tangent cone to $\text{Sys}(C)$ at $p$ is the union of the tangent cones of the simplices with $p$ on the boundary. Lojasiewicz’s theorem also applies to level sets within strata, which can therefore also be triangulated, and for which tangent cones can be defined.

As $f_{\text{sys}}$ is only piecewise-smooth, the notion of gradient is replaced by the cone of increase. By local finiteness, at a point $p \in \text{Sys}(C)$, $f_{\text{sys}}$ can only be increasing or stationary in a direction in which the lengths of all the curves in $C$ are increasing or stationary. The cone of increase of $f_{\text{sys}}$ at $p$ is given by the tangent cone at $p$ of the intersection $I(C, p) := \{x \in T \mid L(c)(x) \geq f_{\text{sys}}(p) \forall c \in C\}$.

A length function is an analytic map $T \to \mathbb{R}^+$, the simplest example of which is the map whose value at the point $x \in T$ is given by the length of $c$ at $x$. More general length functions are positive linear combinations of lengths of curves. Length functions were shown to be convex along earthquake paths in [20] and strictly convex on Weil-Petersson geodesics in [29]. In [5] it was shown that Fenchel-Nielsen coordinates can be chosen such that length functions are convex functions of these coordinates.

As $f_{\text{sys}}$ is not smooth, it cannot be a Morse function. There is however a sense in which it behaves just like a Morse function.

**Definition 3 (Topological Morse function).** Let $M$ be an $n$-dimensional topological manifold. A continuous function $f : M \to \mathbb{R}^+$ is a topological Morse function if the points of $M$ consist of regular points and critical points. When $p \in M$ is a regular point, there is an open neighbourhood $U$ containing $p$, where $U$ admits a homeomorphic parametrisation by $n$ parameters, one of which is $f$. When $p$ is a critical point, there exists a $k \in \mathbb{Z}$, $0 \leq k \leq n$, called the index of $p$, and a homeomorphic parametrisation of $U$ by parameters $\{x_1, \ldots, x_n\}$, such that everywhere on $U$, $f$ satisfies

$$f(x) - f(p) = \sum_{i=1}^{n-k} x_i^2 - \sum_{n-k+1}^{n} x_i^2$$

Topological Morse functions were first defined in [24], where it was shown that, when they exist, they can be used in most of the same ways as their smooth analogues for constructing cell decompositions of manifolds and computing homology.
The cohomological dimension of a group $G$ is
\[ \sup \{ n \in \mathbb{N} \mid H^n(G, M) \neq 0 \text{ for some module } M \}. \]

The mapping class group is known to contain finite index torsion-free subgroups; a discussion is given in Chapter 6 of [12]. By Serre’s theorem, [11] Chapter VIII, any finite index torsion-free subgroup of a group has the same cohomological dimension. The cohomological dimension of a (and hence any) finite index torsion free subgroup is then called the virtual cohomological dimension.

3. Thurston’s deformation retraction

This section describes Thurston’s deformation retraction onto the Thurston spine in detail. References are [28] and Chapter 3 of [14].

Thurston constructed a $\Gamma_g$-equivariant isotopy $\phi_t$ from $T_g$ into a regular neighbourhood of $P_g$. The map $\phi_t$ is defined for all $t > 0$. This has led to some confusion, as $\phi_t$ was referred to as an isotopy in [28]. Although $\phi_t$ can be defined for all $t > 0$, it will be shown that $\phi_t$ is the identity map for all sufficiently large $t$, and it is only the restriction of $\phi_t$ to a compact interval $[0, T]$ that will be needed. The construction relies on the next proposition.

**Proposition 4** (Proposition 0.1 of [28]). Let $C$ be any collection of curves on a surface that do not fill. Then at any point of $T_g$, there are tangent vectors that simultaneously increase the lengths of all the geodesics representing curves in $C$.

**Remark 5.** It is important to note that Proposition 4 implies that all critical points of $f_{sys}$ are contained in $P_g$.

The proof of Proposition 4 given in [28] uses Lipschitz maps, and is quite intuitive. A different proof will be given here, illustrating how the convexity of length functions constrains the differential topology of $M_g$. There is no claim to originality here. Results similar to Proposition 4 have been proven using a variety of techniques; the first instance of which the author is aware can be found in Lemma 4 of [3]. Wolpert has also pointed out that it follows from Riera’s formula, [26].

**Proof.** Let $C = \{ c_1, \ldots, c_n \}$. The length of a curve $c$ will be denoted by $L(c)$. Let $L(c)_x$ be the level set of $L(c)$ passing through a point $x$ of $T_g$.

Since the curves in $C$ do not fill, the intersection $N(x) := \cap_{j=1,\ldots,n} L(c_j)_x$ is not compact. This is because the intersection must be invariant under the action of a subgroup of $\Gamma_g$ generated by Dehn twists around curves disjoint from the curves in $C$.

A length function $\sum_{i=1}^n a_i L(c_i)$ with each $a_i \in \mathbb{R}^+ \cup \{0\}$ and not uniformly zero cannot have a minimum in $T_g$. This is because such a minimum must be a unique point by strict convexity, but $N(x)$ is not compact for any $x \in T_g$.

It is always possible to find a point $w \in T_g$ at which the lemma holds. This can be done by finding a point $q$ in the metric completion of $T_g$ with respect to the Weil-Petersson metric.
with the property that a curve $c$ is pinched at $q$, where $c$ has nonzero geometric intersection number with each of the curves in $C$. Choosing $w$ sufficiently close to $q$ will ensure that the lemma holds at $w$.

Suppose the proposition breaks down at $y \in T_g$. Along a path $\gamma$ from $w$ to $y$, there must be a point $z \in T_g$ at which the lemma first breaks down. At $z$, there exists therefore a nontrivial subset $G_z$ of $\{\nabla L(c_i) \mid c_i \in C\}$ that spans a proper subspace of $T_z T_g$, and whose elements are not contained in a halfspace of this subspace.

The existence of $G_z$ implies that it is possible to find $a_1, \ldots, a_n \in \mathbb{R}^+ \cup \{0\}$ not all zero such that the sum

$$\sum_{i=1}^n a_i \nabla L(c_i)(z)$$

is zero. By strict convexity of length functions along Weil-Petersson geodesics, this implies that the length function

$$L = \sum_{i=1}^n a_i L(c_i)$$

has a local—and hence global—minimum at $z$. The proposition follows by contradiction. □

For any $\epsilon > 0$, an open subset $P_{g,\epsilon}$ of $T_g$ is defined to be the subset of $T_g$ consisting of hyperbolic structures such that the set of geodesics whose length is within $\epsilon$ of the shortest length fill the surface. It is not hard to see that each $P_{g,\epsilon}$ is open, its projection to $M_g$ has compact closure, and the intersection of $P_{g,\epsilon}$ over all positive $\epsilon$ is the subcomplex $P_g$. It follows that for any regular neighbourhood $N$ of $P_g$, there is an $\epsilon$ such that $P_{g,\epsilon} \subset N$.

Recall that the Weil-Petersson metric used here is invariant under the action of the mapping class group. At a point $x$ of $T_g \setminus P_g$, let $C(x)$ be a set of shortest geodesics. The notion “shortest” will be made precise later; $C(x)$ can contain more curves than just the systoles at $x$. If the geodesics in $C(x)$ do not fill the surface, by Proposition 4, it is possible to define a $\Gamma_g$-equivariant vector field $X_C$ with the property that the length of every curve in $C$ is increasing in the direction of $X_C$. Thurston gave as an example the vector field $X_C$ with the property that at any point $x$, $X_C(x)$ has unit length and points in the direction that maximises the sum of the (real) logarithms of the derivatives of the lengths of the curves in $C$. This is a shorthand way of saying that $X_C(x)$ points in a smooth choice of direction in which the length of each curve in $C$ is increasing, because if one of the derivatives were negative or zero, the log would be imaginary or $-\infty$. This vector field is discontinuous only at places where the set of shortest geodesics changes.

The vector field $X_C$ is arbitrarily defined to be zero on $P_g$. For a point $x$ very close to $P_g$ the curves in $C(x)$ might also fill, depending on how the notion of “set of shortest curves” is defined. For simplicity, $X_C$ will also be defined to be zero when the curves in $C$ fill. The
Consequently, it is always possible to flow for a time $\epsilon$ by an example. Theorem 9.16 of [21]. In particular, first recall that the standard result that a smooth, compactly supported vector field is complete; see for example [23]. For $x \in T_g$, the cardinality of a set $C(x)$ of curves of length less than $f_{sys}(x) + |C|\epsilon$ is either uniformly bounded depending on $\epsilon$, or infinite, in which case $U_C$ is not in $U$.

This unusual construction of a cover was presumably made because it has the property that if two sets $U_{C_1}$ and $U_{C_2}$ intersect, either $C_1 \subset C_2$ or $C_2 \subset C_1$. This choice is not in any way canonical, and different choices are clearly also allowable here.

Note that for every point $x$ not on $P_g$, there is an $\epsilon$ such that for some set $U_{C_i}$ containing $x$, the curves in $C_i$ do not fill.

Let $\{\lambda_{C_i}\}$ be a partition of unity subordinate to the covering $\{U_{C_i}\}$. The partition of unity is chosen in such a way as to be invariant under the action of $\Gamma_g$ on the sets of geodesics $\{C\}$. For example, it could be defined as a function of geodesic lengths. The vector field $X_\epsilon$ is constructed by using the partition of unity $\{\lambda_{C_i}\}$ to average over the vector fields $\{X_{C_i}\}$. Note that this averaging process does not create zeros. For a point $x$ in the intersection of the open sets $U_{C_i}$, $i = 1, \ldots, k$, there is at least one shortest or equal shortest curve $c$ in the intersection of the sets $C_i$. Any vector field $X_{C_i}$, $i = 1, \ldots, k$ evaluated at $x$ has the property that if it is nonzero, it increases the length of $c$ at $x$. It follows that $X_\epsilon$ can only be zero at $x$ if every vector field being averaged over at $x$ is zero.

Denote by $K$ a subset of $T_g$ that is compact modulo the action of the mapping class group. The goal is now to construct an isotopy $\phi_t$ of $T_g$ with the property that for any $\epsilon$ there is a $T(\epsilon)$ such that taking $t > T(\epsilon)$ ensures that for any $K$, $\phi_t(K)$ is contained within $P_{g,\epsilon}$. This is done by using the flow generated by $X_{\epsilon'(t)}$ where $\epsilon'(t) > 0$ is small and is decreased further as time goes on.

The proof that $X_{\epsilon'(t)}$ generates a flow defined for all $t \in [0, \infty)$ is the same as the proof of the standard result that a smooth, compactly supported vector field is complete; see for example Theorem 9.16 of [21]. In particular, first recall that the $\alpha$-thick part of $T_g$, call it $T_g^\alpha$, defined as $T_g^\alpha = \{x \in T_g \mid f_{sys}(x) \geq \alpha\}$ is invariant under the flow and compact modulo the action of $\Gamma_g$. This means that by the existence and uniqueness theorem of ODEs, around each point $p$ in $T_g^\alpha$ there is a neighbourhood $U_p$ and an $\epsilon > 0$ such that the flow is defined on $U_p \times [0, \epsilon(p))$. By compactness, there is a nonzero uniform lower bound $\epsilon$ of $\epsilon(p)$ on $T_g^\alpha$. Consequently, it is always possible to flow for a time $\epsilon$ longer. This concludes the proof that
$X_{\epsilon(t)}$ generates a flow defined for all $t \in [0, \infty)$.

The next lemma gives control over the rate at which $f_{sys}$ increases along the flowlines of $X_{\epsilon}$ outside of the $\delta$-thick part of $T_g$. It was not explicitly contained in Thurston’s preprint, but has been included here by request. Recall that $C_i(x)$ is the set of curves with length at $x$ less than $f_{sys}(x) + |C_i|\epsilon$ defined above.

**Claim 7.** For any $\epsilon > 0$ there is a $T(\epsilon)$ such that flowing for $t > T(\epsilon)$ ensures that any $K$ is carried inside - and remains within - $P_{g, \epsilon}$.

For the moment Claim 7 will be assumed. It will be discussed in detail in Sections 4 and modified slightly in Section 5.

Back to the construction of $\phi_t$. Denote by $I_t$ the closed set in $T_g$ containing $P_g$ whose boundary is the image of $\partial T_g^{\epsilon M}$ after it has been flowed for time $t$. For any $t \in [0, \infty)$, the isotopy $\phi_t$ takes a point to its image at time $t$ under the flow. The set $T_g^{\epsilon M}$ is therefore mapped to $I_t$ by $\phi_t$.

Suppose $\epsilon'(t)$ has been chosen small enough to ensure that $X_{\epsilon'(t)}$ is nonzero on a neighbourhood of $\partial \phi_t(T_g^{\epsilon M})$. The boundary of $\phi_t(T_g^{\epsilon M})$ is similar to the boundary of a level set of $f_{sys}$ such as $T_g^{\epsilon M}$ in the sense that $X_{\epsilon'(t)}$ points inward at every point of $\phi_t(T_g^{\epsilon M})$. One
way of proving this is to use that \( \phi_t \) gives a flowline-preserving diffeomorphism of a regular neighbourhood of \( \partial T^x_g \) onto a regular neighbourhood of \( \partial \phi_t(T^x_g) \). A point on a flowline outside \( T^x_g \) is mapped to a point on a flowline outside of \( \partial \phi_t(T^x_g) \), and vice versa. Since the flowlines determine a foliation of the regular neighbourhood of \( \partial T^x_g \), this is also the case for \( \partial \phi_t(T^x_g) \). This implies that there are no places where a flowline is tangent to \( \partial \phi_t(T^x_g) \), which would need to exist for some value of \( t \) if \( X_{e'}(t) \) were to transition from pointing inwards to pointing outwards.

Choose \( t^* \) such that the isotopy \( \phi_{t^*} \) maps any \( K \) into \( P_{g,\epsilon} \) for some \( \epsilon \) small enough to ensure that \( P_{g,\epsilon} \) is contained in a regular neighbourhood \( \mathcal{N} \) of \( P_g \). Existence of such a \( t^* \) is guaranteed by Claim 7. A deformation retraction of \( K \) onto \( P_g \) is obtained by taking a composition of \( \phi_{t^*} \) with a deformation retraction that arises from the deformation retraction of \( \mathcal{N} \) onto \( P_g \).

The existence of this second deformation retraction will now be shown. For ease of notation, it will be shown that \( I_{t^*} = \phi_{t^*}(T^x_g) \) deformation retracts onto \( P_g \). An identical argument works with the \( \alpha \)-thick part of \( T_g \) in place of \( T^x_g \), for any \( \alpha \) small enough such that the \( \alpha \)-thick part of \( T_g \) contains \( P_g \). As the \( \alpha \)-thick subsets are an exhaustion of \( T_g \) by sets compact modulo the action of \( \Gamma_g \), this will then give the required deformation retraction of \( T_g \).

First note that the boundary of \( I_{t^*} \) is connected. This is because, as shown in Proposition 12.10 of [12], \( \partial T^x_g \) is connected and by Proposition 4 there are no critical points of \( f_{sys} \) between \( \partial I_{t^*} \) and \( \partial T^x_g \). By construction, the set \( I_{t^*} \) has \( P_g \) in the interior, because a flowline is prevented from actually reaching \( P_g \) by the fact that for any \( \epsilon \), \( X_\epsilon \) is zero at points sufficiently close to \( P_g \). Consequently, \( I_{t^*} \) is a connected subset of \( \mathcal{N} \) with a connected boundary that separates \( \partial \mathcal{N} \) from \( P_g \). The deformation retraction of the regular neighbourhood \( \mathcal{N} \) onto \( P_g \) then gives the required deformation retraction of \( I_{t^*} \) onto \( P_g \).

Questions raised in the literature. In the last two pages of [19], a list of questions about the construction in [28] was made. All but one of these questions were answered in the exposition above. The final (and main) objection, given on page 14 is as follows. Point 4 on page 13 of [19] states “Use the flow defined by the vector field \( X \) to deform points of \( T_g \) into a neighbourhood \( P_{B_\epsilon} \) of \( P \) [the Thurston spine].” Below is stated “Certainly, there is no problem to deform any compact subset \( K \) of \( T_g - P_{B_\epsilon} \) into \( P_{B_\epsilon} \) in a fixed time, but we need to deform the whole space....If it can be shown that points in \( P_{B_\epsilon} \) cannot be flowed out of \( P_{B_\epsilon} \), then it is fine, and Step (4) is valid. In summary, for this method in [28] to succeed, we need vector fields whose flows increase the number of geodesics whose lengths are close to the systole of the surface, rather than only increasing their lengths simultaneously.”

The number of curves at \( x \) of length close to \( f_{sys}(x) \) is not the right measure of complexity to use in this context. Consider for example a stratum on which the systoles intersect. At any point on the boundary of this stratum the number of systoles is larger than in the interior. A vector field whose flow increases the number of curves with length close to \( f_{sys} \) would have a zero inside the stratum.
The construction in [28] was founded on the intuition that $f_{\text{sys}}$ is increasing towards the spine. As $f_{\text{sys}}|_{\mathcal{P}_g}$ is not constant, the preprint [17] clarifies exactly what is meant here. In Lemma 12 it will be shown that near $\mathcal{P}_g$, the flow can be chosen to decrease the value of the $\epsilon$ in the definition of $\mathcal{P}_{g,\epsilon}$. By formula 3.3 of [30], the vector field suggested in [28] presumably also does this.

In an arXiv paper [7], it was claimed that there is “reason to be skeptical” of Thurston’s construction of a deformation retraction because “the systole is a topological Morse function on $\mathcal{T}_g$, so one can also define an invariant vector field that vanishes only at critical points, but $\mathcal{T}_g$ does not deformation retract into this infinite discrete set”.

This objection is based on the misconception that a deformation retraction analogous to Thurston’s flow, obtained by taking the flow of a smooth $\Gamma_g$-equivariant vector field with isolated zeros and support compact modulo the action of $\Gamma_g$ gives a deformation retraction onto the set of points at which the vector field vanishes. This is not the case, as there can be flowlines of the vector field from one critical point to another.

4. The modified claim

Claim 7 states that for any $\epsilon > 0$ there is a $T(\epsilon)$ such that flowing for $t > T(\epsilon)$ ensures that any $K$ is carried inside — and remains within — $\mathcal{P}_{g,\epsilon}$. This section constructs a smooth vector field $X'_\epsilon$ satisfying the following properties

(1) The systole function is increasing in the direction of $X'_\epsilon$.
(2) The zeros of $X'_\epsilon$ are contained in a neighbourhood of $\mathcal{P}_g$ that shrinks onto $\mathcal{P}_g$ as $\epsilon$ approaches zero.
(3) $X'_\epsilon$ is mapping class group-equivariant.

for which the flow of $X'_\epsilon$ contradicts Claim 7.

Example 8. In the final section of [17] an example of a 3-dimensional stratum $\text{Sys}({c_1, c_2, c_3, c_4})$ in $\mathcal{P}_2$ was given, where the set of curves $C = \{c_1, c_2, c_3, c_4\}$ is shown in Figure 1.

![Figure 1](image-url)
The set $C$ is minimal in the sense that removing any curve gives a set that does not fill. Consequently, $\text{Sys}(C)$ is not on the boundary of any larger dimensional stratum of $P_g$. It was shown in Section 3 of [17] that, away from isolated critical points, for any point of $\text{Sys}(C)$ there is an open cone of directions in which $f_{\text{sys}}$ is increasing and the gradients of the lengths of curves in $C$ are linearly independent. It follows that each proper subset of $C$ realises a stratum adjacent to $\text{Sys}(C)$.

Let $c_1^*$ be a curve that intersects $c_1$ but is disjoint from every curve in the set $C\setminus \{c_1\}$.

For $c_1 \in C$, denote by $V$ the vector field of unit length in the direction in which the twist parameter around $c_1^*$ is increasing, and suppose $X_\epsilon$ is any vector field constructed as in the previous section. Also recalling the partition of unity in the previous section, define $\lambda$ to be $\sum \lambda_{C'}$ where the sum is taken over all subsets $C'$ of $C\setminus \{c_1\}$. As $U_{\{c_2,c_3,c_4\}}$ is open in $T_g$, there are points in $U_{\{c_2,c_3,c_4\}}$ at which $\nabla L(c_1)$ has nonzero projection onto $V$. If there is a point $x$ at which $L(c_1)$ is increasing in the direction of $V$, define $X'_\epsilon = X_\epsilon + r\lambda V$, where $r \in \mathbb{R}^+$ is chosen large enough to ensure that in a neighbourhood of $x$, $L(c_1)$ is increasing in the direction of $X'_\epsilon$ faster than the lengths of any of the other curves in $C$. If there is no point $x$ in $U_{\{c_2,c_3,c_4\}}$ at which $L(c_1)$ is increasing in the direction of $V$, define $X'_\epsilon = X_\epsilon - r\lambda V$, where $r \in \mathbb{R}^+$ is chosen large enough to ensure that in a neighbourhood of $x$, $L(c_1)$ is increasing in the direction of $X'_\epsilon$ faster than the lengths of any of the other curves in $C$.

Since the lengths of the curves $C\setminus \{c_1\}$ are stationary along $V$, the vector field $X'_\epsilon$ increases both $f_{\text{sys}}$ and the difference in length between the curves $\{c_2,c_3,c_4\}$ and $c_1$, increasing $\epsilon$ in the definition of $P_{g,\epsilon}$. Moreover, the zeros of $X'_\epsilon$ coincide with the zeros of $X_\epsilon$. The vector field $X'_\epsilon$ is mapping class group-equivariant whenever $r \in \mathbb{R}^+$ is chosen consistently.

5. Flowing into a regular neighbourhood

This section shows that for small enough $\epsilon'$, any vector field $X_{\epsilon'}$ constructed in Section 3 generates a flow that maps $T_g$ into a regular neighbourhood of $P_g$. In Claim 7, $P_{g,\epsilon}$ is replaced by a tubular neighbourhood of $P_g$ (defined below), which is sufficient for the construction of the deformation retraction.

The gap function $g_{\text{sys}} : T_g \rightarrow \mathbb{R}_+$ is the piecewise smooth function whose value at the point $x$ is given by the smallest real number $r$ such that the set of curves with lengths within $r$ of $f_{\text{sys}}$ fill $S_g$. Recall the definitions of tangent cone and cone of increase from Section 2.

**Theorem 9** (Theorem 1 of [17]). Suppose a point $p \in P_g$ has the property that the cone of increase of $f_{\text{sys}}$ at $p$ has dimension equal to that of $T_g$. Then the intersection of this cone of increase with the unit tangent cone of $P_g$ at $p$ is nonempty and stably contractible (Defined below).

Due to the fact that tangent cones are only first order approximations, when there are strata at $p$ that agree to first order on a subspace of $T_p T_g$, in the statement of Theorem 9 contractibility of the intersection of the cone of increase with the unit tangent cone of $P_g$
at \( p \) is not enough. This leads to the following definition, which is explained in more detail in [17]. Let \( C \) be the set of systoles at a point \( p \in \mathcal{P}_g \) and \( I(C, x) := \{ x \in \mathcal{T}_g \mid L(c)(x) \geq f_{\text{sys}}(p) \ \forall c \in C \} \). Recall that the cone of increase of \( f_{\text{sys}} \) is defined to be the tangent cone to \( I(C, x) \). Denote by \( I(C, x) \cap \partial B_\epsilon(p) \) the set of all points in \( I(C, x) \) at distance \( \epsilon \) from \( p \). Then the intersection of the cone of increase with the \( \epsilon \)-tangent cone of \( \mathcal{P}_g \) at \( p \) is \textit{stably contractible} if the intersection of \( I(C, x) \cap \partial B_\epsilon(p) \) with \( \mathcal{P}_g \) is contractible.

\textbf{Remark 10.} In the language of topological Morse functions, Corollary 9 implies that restricting \( f_{\text{sys}} \) to \( \mathcal{P}_g \) does not create critical points.

\textbf{Remark 11.} The condition that the cone of increase of \( f_{\text{sys}} \) has dimension less than the dimension of \( \mathcal{T}_g \) is not necessarily the same as being a critical point in the context of topological Morse functions. Boundary points of \( f_{\text{sys}} \), defined in [27], are regular points and satisfy this condition. The function \( f_{\text{sys}} \) can only increase away from a boundary point to second order. As for critical points, boundary points are isolated and all contained in \( \mathcal{P}_g \). The local structure of \( \mathcal{T}_g \) around a boundary point is discussed in the last section of [17], where it is shown that all directions in which \( f_{\text{sys}} \) is increasing at a boundary point are in the tangent cone to \( \mathcal{P}_g \).

\textbf{Lemma 12.} For all sufficiently small \( \delta \), there is a neighbourhood \( N_{g, \delta} \) of \( \mathcal{P}_g \) and a smooth vector field \( X_g \) satisfying the following properties
\begin{itemize}
  \item The vector field \( X_g \) is nonzero on \( N_{g, \delta} \setminus \mathcal{P}_g \),
  \item The gap function \( g_{\text{sys}} \) is strictly decreasing in the direction of \( X_g \) whenever \( X_g \) is nonzero,
  \item The lengths of the systoles are increasing in the direction of \( X_g \) wherever \( X_g \) is nonzero, and
  \item \( X_g \) is mapping class group-equivariant.
\end{itemize}
In this theorem, \( \delta \) is a parameter measuring the size of the neighbourhood, i.e. \( N_{g, \delta_1} \subset N_{g, \delta_2} \) for \( \delta_1 < \delta_2 \).

\textbf{Proof.} The basic idea here is that when passing from a locally top dimensional stratum \( \text{Sys}(C) \) into a stratum of nonfilling curves \( \text{Sys}(C'), C' \subset C \), the set of curves whose lengths determine \( g_{\text{sys}} \) consist of the curves \( C' \) in addition to some curves in \( C \setminus C' \), for which the gradients of the lengths are linearly independent near \( \text{Sys}(C) \) from the gradients of the lengths of the curves in \( C' \). This makes it possible to locally construct the required vector field near \( \text{Sys}(C) \), and properties of the level sets that foliate the systole strata are then used to show the definition can be extended to the desired domain without creating zeros.

Critical points and boundary points are isolated points of \( \mathcal{P}_g \). It will first be explained how to construct \( X_g \) near a point \( x \) of \( \text{Sys}(C) \) in the interior of a locally top dimension simplex, where \( x \) is not a critical point or boundary point of \( f_{\text{sys}} \). Recall that the systole function is smooth when restricted to the interior of a simplex and that a triangulation compatible with the stratification is assumed when referring to the boundary of a simplex or its tangent cone. This ensures that the boundary of a stratum can also be triangulated.
Figure 2. Theorem 9 implies that on a neighbourhood of \( x \in P_g \) that is not a critical or boundary point of \( f_{sys} \), the level sets of \( f_{sys} \) in \( \text{Sys}(C) \) extend to a foliation of \( \text{Sys}(C_1) \), some leaves of which are shown in blue on the left. The vectors indicate a direction in which the lengths of curves in \( C \setminus C_1 \) are increasing. Theorem 9 is used to rule out examples such as on the right of this figure.

The tangent space \( T_x T_g \) is subdivided into segments corresponding to tangent cones of strata. On a neighbourhood of \( x \), by local finiteness the systoles are all contained in \( C \). The lowest dimensional segments not in the tangent cone to \( P_g \) are tangent cones to strata of maximal nonfilling subsets \( C_1, \ldots, C_k \) of \( C \). For each maximal nonfilling subset \( C_i \) of \( C \), the curves in \( C_i \) are all disjoint from a multicurve \( m(C_i) \).

Every point in the systole stratum \( \text{Sys}(C_i) \) satisfies the system of constraints that the lengths of the curves in \( C_i \) are all equal. The dimension of \( \text{Sys}(C_i) \) at any point is one more than the dimension of the level sets in \( \text{Sys}(C_i) \) of the curves in \( C_i \). This is a consequence of the fact that \( f_{sys}|_{P_g} \) is not constant at \( x \), so by continuity \( f_{sys}|_{\text{Sys}(C_i)} \) cannot be constant on a neighbourhood of \( x \). Theorem 9 implies that the level sets of the curves \( C \) in \( \text{Sys}(C) \) are on the boundary of level sets of curves in \( C_i \) in \( \text{Sys}(C_i) \). This is illustrated schematically in Figure 2. As \( f_{sys} \) is a topological Morse function and there are no critical points outside of \( P_g \), in a neighbourhood of any point of \( \text{Sys}(C_i) \), the level sets are homeomorphic to hyperplanes.

For \( x \in \text{Sys}(C) \cap \partial \text{Sys}(C_i) \), the tangent cone to the level sets of \( C_i \) in \( \text{Sys}(C_i) \) determines a set of directions in which the lengths of all the curves in \( C \setminus C_i \) are increasing. There exists an intersection \( N(C_i) \) of an open set in \( T_g \) with \( \text{Sys}(C_i) \), with \( \text{Sys}(C) \subset \partial N(C_i) \), on which one obtains a smooth vector field \( X(C_i)^+ \) tangent to the level sets of the curves in \( C_i \) specifying a direction in which the lengths of all the curves in \( C \setminus C_i \) are increasing. On \( N(C_i) \), but perhaps not on \( \text{Sys}(C) \), this increase is first order.

Proposition 4 implies that at every point of \( T_g \) there is an open cone of directions in which the lengths of all the curves in \( C_i \) are simultaneously increasing. Let \( X(C_i) \) be a smooth vector field defined on \( N(C_i) \) with this property. It follows that nonvanishing functions \( \eta_{1,i} \) and \( \eta_{2,i} \) on \( N(C_i) \) can be chosen such that \( \eta_{1,i} X(C_i) + \eta_{2,i} X(C_i)^+ \) is a smooth vector field in which \( g_{sys} \) is decreasing and \( f_{sys} \) increasing. The functions \( \eta_{1,i} \) and \( \eta_{2,i} \) approach zero near
Sys($C$) on $\partial N(C_i)$. Note that level sets are preserved under the action of the mapping class group, so any choices can be made in a mapping class group-equivariant way. This gives a vector field on a $\Gamma_g$-orbit of $N(C_i)$ satisfying the conditions of the lemma.

**Extending the vector field to higher dimensional systole strata.** Let $\text{Sys}(C_{i,j})$ be a systole stratum with dimension larger than $\text{Sys}(C_i)$, with boundary along $\text{Sys}(C_i)$, $\text{Sys}(C)$ and $\text{Sys}(C_j)$. The set $C_{i,j}$ is therefore contained in the intersection of $C_i$ and $C_j$.

The level sets in $\text{Sys}(C_{i,j})$ have the level sets of $\text{Sys}(C)$, $\text{Sys}(C_i)$ and $\text{Sys}(C_j)$ on the boundary. As before, there is an intersection $N(C_{i,j})$ of an open set in $T_g$ with $\text{Sys}(C_{i,j})$, with $\text{Sys}(C)$, $\text{Sys}(C_i)$ and $\text{Sys}(C_j)$ on $\partial N(C_{i,j})$, on which the smooth vector field $X(C_{i,j})^\perp$ tangent to the level sets of the curves in $\text{Sys}(C_{i,j})$ is defined. Here the vector field $X(C_{i,j})^\perp$ has the property that the lengths of all the curves in $C \setminus C_{i,j}$ are increasing in the direction of $X(C_{i,j})^\perp$. Moreover, since the dimension of the level sets of $\text{Sys}(C_{i,j})$ is at least one more than the level sets of $\text{Sys}(C_i)$ and $\text{Sys}(C_j)$, this can be extended to a smooth vector field on $N(C_i) \cup N(C_{i,j}) \cup N(C_j)$ that agrees with $X(C_i)^\perp$ on $N(C_i)$ and with $X(C_j)^\perp$ on $N(C_j)$ and approaches zero near $\text{Sys}(C)$. A vector field $\eta_{i,j}X(C_{i,j}) + \eta_{i,j}X(C_{i,j})^\perp$ is defined on $N(C_i) \cup N(C_{i,j}) \cup N(C_j)$ as above, and extended to the $\Gamma_g$-orbit of $N(C_{i,j})$.

This construction can be extended to higher dimensional systole strata on a neighbourhood of $\text{Sys}(C)$ analogously.

**Extending the vector field beyond $\text{Sys}(C)$.** Suppose $\text{Sys}(C_L)$ meets $\text{Sys}(C_R)$ along $\text{Sys}(C_M)$, where $\text{Sys}(C_L)$ and $\text{Sys}(C_R)$ are locally top dimensional systole strata on $\mathcal{P}_g$, and $\text{Sys}(C_M)$ has lower dimension. Suppose also that $\text{Sys}(C_{L,1})$ has $\text{Sys}(C_L)$ on the boundary, and $C_{L,1}$ is a maximal nonfilling subset of $C_L$. Then $\text{Sys}(C_{L,1})$ either has $\text{Sys}(C_R)$ on the boundary or not. Suppose $\text{Sys}(C_R)$ is on the boundary of $\text{Sys}(C_{L,1})$.

A curve $c_{L,1} \in C_L$ intersecting $m(C_{L,1})$ that is shortest near $\text{Sys}(C_L)$ might not be shorter than a shortest curve $c_{R,1} \in C_R$ intersecting $m(C_{L,1})$ near $\text{Sys}(C_R)$.

Claim - either the entire tangent cone to $\text{Sys}(C_{L,1})$ at a point $x$ on $\text{Sys}(C_M)$ consists of directions in which $f_{\text{sys}}$ is decreasing and $g_{\text{sys}}$ is increasing, or in the intersection of $\text{Sys}(C_{L,1})$ with a neighbourhood of $\text{Sys}(C_m)$ the projections of $\nabla L(c_{L,1})$ and $\nabla L(c_{R,1})$ to $\left\{\nabla L(c) \mid c \in C_{L,1}\right\}$ have positive inner product.

To prove the claim, first consider the case in which the intersection of the cone of increase of $f_{\text{sys}}$ with the tangent cone of $\text{Sys}(C_m)$ at $x \in \text{Sys}(C_m)$ is nonempty. Away from critical points and boundary points of $f_{\text{sys}}$, this implies that the level sets in $\text{Sys}(C_{L,1})$ have the level sets of $\text{Sys}(C_L)$ on their boundary, similarly for $C_R$. Since the level sets of $\text{Sys}(C_L)$ have the level sets of $\text{Sys}(C_M)$ on their boundary, so do the level sets of $\text{Sys}(C_{L,1})$.

If the claim were not true, the tangent cone to a level set of $\text{Sys}(C_{L,1})$ from a point in $\text{Sys}(C_M)$ would contain a direction in which one of $L(c_{L,1})$ or $L(c_{R,1})$ is decreasing. This
gives a contradiction, because the decreasing curve would then be shorter than the systoles. The claim follows in this case from the observation that if the projection of the gradients to the subspace in question have positive inner product near \( \text{Sys}(C_M) \), this holds on a neighbourhood.

If the cone of increase of \( f_{\text{sys}} \) at \( x \) is disjoint from the tangent cone to \( \text{Sys}(C_m) \) at \( x \), by contractibility in Theorem 9, the cone of increase of \( f_{\text{sys}} \) cannot intersect both the tangent cone of \( \text{Sys}(C_L) \) and \( \text{Sys}(C_R) \) at \( x \). If it intersects exactly one of the tangent cones of \( \text{Sys}(C_L) \) and \( \text{Sys}(C_R) \), there will be a level set of \( \text{Sys}(C_{L,1}) \) with a level set of \( \text{Sys}(C_m) \) on the boundary, which suffices to prove the claim.

If the cone of increase of \( f_{\text{sys}} \) at \( x \in \text{Sys}(C_m) \) intersects neither the tangent cone at \( x \) of \( \text{Sys}(C_L) \), \( \text{Sys}(C_R) \) nor \( \text{Sys}(C_m) \), then by Theorem 9 it cannot intersect the tangent cone to \( \text{Sys}(C_{L,1}) \) either. It follows that the entire tangent cone to \( \text{Sys}(C_{L,1}) \) at \( x \in \text{Sys}(C_M) \) consists of directions in which \( f_{\text{sys}} \) is decreasing and \( g_{\text{sys}} \) is increasing. In this case, there is a lot a freedom in constructing the vector field; any smooth \( \Gamma_g \)-equivariant choice of vector field in the tangent space to \( \text{Sys}(C_{L,1}) \) near \( x \), and pointing towards \( \mathcal{P}_g \) will satisfy the conditions of the Lemma. This concludes the proof of the claim.

As a result of the claim, it is possible to extend the construction of the vector field on \( N(C_{L,1}) \) to a neighbourhood in \( \text{Sys}(C_{L,1}) \) of the boundary \( \text{Sys}(C_L) \cup \text{Sys}(C_M) \cup \text{Sys}(C_R) \).

Now consider the possibility that \( \text{Sys}(C_{L,1}) \) does not have \( \text{Sys}(C_R) \) on the boundary for any possible choice of \( \text{Sys}(C_R) \). (There might be more than 2 locally top dimensional systole strata in \( \mathcal{P}_g \) meeting along \( \text{Sys}(C_M) \).) Then \( \text{Sys}(C_{L,1}) \) has boundary along \( \text{Sys}(C_{M,1}) \). Suppose the two systole strata \( \text{Sys}(C_{L,1}) \) and \( \text{Sys}(C_{R,1}) \) meet along \( \text{Sys}(C_{M,1}) \).

As before, either the tangent cone at \( x \) to one or more of \( \text{Sys}(C_{M,1}) \), \( \text{Sys}(C_{L,1}) \) and \( \text{Sys}(C_{R,1}) \) consists of directions in which \( f_{\text{sys}} \) is decreasing and \( g_{\text{sys}} \) is increasing, or the vector field near \( x \) on \( \text{Sys}(C_{L,1}) \cup \text{Sys}(C_{M,1}) \cup \text{Sys}(C_{R,1}) \) can be constructed as a linear combination. Suppose the vector field is constructed as a linear combination. Then \( m(C_{M,1}) \subset m(C_{R,1}) \) and \( m(C_{R,1}) \subset m(C_{L,1}) \). This implies it is possible to extend the vector field \( X(C_{L,1}) \) out over \( \text{Sys}(C_{L,1}) \cup \text{Sys}(C_{M,1}) \cup \text{Sys}(C_{R,1}) \). For example, by Riera’s formula, \( X(C_{L,1}) \) could be taken to be the Weil-Petersson gradient of the length of \( m(C_{M,1}) \). Moreover, it follows from the same arguments as in the previous case that \( X(C_{L,1}) \) can be extended out along \( \text{Sys}(C_{L,1}) \cup \text{Sys}(C_{M,1}) \cup \text{Sys}(C_{R,1}) \) in a neighbourhood of the boundary \( \text{Sys}(C_L) \cup \text{Sys}(C_M) \cup \text{Sys}(C_R) \).

The construction relies on the same ideas when \( \text{Sys}(C_{L,1}) \) and \( \text{Sys}(C_{R,1}) \) do not meet in \( \text{Sys}(C_{M,1}) \) but when there are more strata in between. Extending the vector fields defined over higher dimensional systole strata past \( \text{Sys}(C_M) \) is also analogous.

The structure of \( \mathcal{P}_g \) near critical and boundary points of \( f_{\text{sys}} \) is well understood; a reference for this is the last section of [17]. It was shown in [17] that at a critical point \( p \in \text{Sys}(C) \), the
orthogonal complement of the span of \( \{ \nabla L(c) \mid c \in C \} \) is tangent to \( \mathcal{P}_g \). At critical points, \( f_{sys} \) is known to be “eutactic”. This implies that any direction in \( T_p \mathcal{T}_g \) is either a direction tangent to \( P_g \) in which \( f_{sys} \) is stationary, or a direction in which \( f_{sys} \) is increasing towards \( p \) and \( g_{sys} \) is either zero or decreasing. For almost any choice of any vector field \( X(C_{1,j,k,...}) \) on a neighbourhood of \( p \), \( X_g \) can be extended to a vector field near \( p \) by allowing \( \eta_{2,j,k,...} \) to be zero near \( p \). Similarly for boundary points of \( f_{sys} \).

Making all choices in a mapping class group equivariant way, and using a partition of unity to ensure the vector field approaches zero smoothly outside of a neighbourhood of \( \mathcal{P}_g \), a vector field \( X_g \) satisfying the conditions of the lemma is obtained. □

Recall that \( T_\alpha g \) is the \( \alpha \)-thick part of \( \mathcal{T}_g \).

**Tubular versus regular neighbourhoods.** Due to the fact that \( \mathcal{P}_g \) is not an embedded submanifold, and there is a triangulation of \( T_\alpha^g \) compatible with the stratification for which \( \mathcal{P}_g \) is a subcomplex, convention suggests one works with regular neighbourhoods of \( \mathcal{P}_g \) as opposed to tubular neighbourhoods. However, the construction of the deformation retraction made use of flows generated by smooth vector fields. Moreover, in the next section, the argument uses pre-images of points under the deformation retraction. This concept is not well defined for regular neighbourhoods for which the notion of deformation retraction is replaced by the more combinatorial notion that the inclusion is a homotopy equivalence, where the homotopy equivalence is usually obtained as a sequence of simplicial collapses. For these reasons, tubular neighbourhoods of \( \mathcal{P}_g \) will now be defined. As a semi-analytic subset of \( \mathcal{T}_g \), \( \mathcal{P}_g \) has properties in common with embedded submanifolds, that guarantee existence of an analogue of tubular neighbourhoods. The technicalities as discussed in detail in [16].

Let \( \mathfrak{T}(\mathcal{P}_g) \) be a neighbourhood of \( \mathcal{P}_g \subset \mathcal{T}_g \) for which there is a deformation retraction \( \phi \) of \( \mathfrak{T}(\mathcal{P}_g) \) onto \( \mathcal{P}_g \). Then \( \mathfrak{T}(\mathcal{P}_g) \) is a **tubular neighbourhood** of \( \mathcal{P}_g \) if the following conditions are satisfied:

1. When \( \sigma \) is the interior of a locally top dimensional simplex of \( \mathcal{P}_g \), \( \mathfrak{T}(\mathcal{P}_g) \) contains \( \sigma \times B^{\text{codim} \sigma} \), where \( B^{\text{codim} \sigma} \) is a ball of dimension equal to the codimension of \( \sigma \) in \( \mathcal{T}_g \). Also, \( \sigma \times B^{\text{codim} \sigma} \mapsto \mathcal{P}_g = (\sigma,0) \simeq \sigma \).
2. For a point \( x \in \sigma \), \( (x, B^{\text{codim} \sigma}) \) is the pre-image of the point \( x \) under the deformation retraction \( \phi \).
3. When \( \sigma_f \) is the interior of a face of a locally top dimensional simplex, \( \mathfrak{T}(\mathcal{P}_g) \) contains the preimage \( \phi^{-1}(\sigma_f) \) of \( \sigma_f \) under \( \phi \).
4. The inclusion maps of a face \( \sigma_f \) into larger dimensional simplices of \( \mathcal{P}_g \) induce maps between boundary points of \( \phi^{-1}(\sigma_f) \) to boundary points of the sets from (1). This is done in such a way that \( \phi \) is defined on the disjoint union \( \mathfrak{T}(\mathcal{P}_g) \) of the closure of the pieces from (1) and (3) modulo the identifications given by the maps.

The next proposition can be used in place of Claim 7.

**Proposition 13.** For small enough \( \epsilon \), every vector field \( X_\epsilon \) constructed in Section 3 generates a flow that maps \( \mathcal{T}_g \) into a tubular neighbourhood of \( \mathcal{P}_g \).
Proof. First of all, it will be shown how to construct vector fields satisfying Claim 7. Fix some \( \epsilon' \) in the definition of \( \mathcal{P}_{g,\epsilon'} \), and \( \epsilon \) in the construction of \( X_\epsilon \) both sufficiently small such that

- The zeros of \( X_\epsilon \) are contained in \( N(g, \delta) \) from Lemma 12
- \( X_\epsilon \) is zero on \( \mathcal{P}_{g,\epsilon'} \)

Then the vector field \( V_\epsilon := X_g + X_\epsilon \) has the property that \( f_{sys} \) is everywhere increasing in the direction of \( V_\epsilon \), and on \( \mathcal{P}_{g,\epsilon'} \), \( g_{sys} \) is decreasing in the direction of \( V_\epsilon \). Adding the vector fields \( X_g \) and \( X_\epsilon \) does not create zeros, because the lengths of the systoles are always strictly increasing in the direction of the sum whenever one or both of the vector fields is nonzero.

Claim 7 then follows for a vector field \( V_\epsilon \). This will be shown first for a \( K \) given by the \( \alpha \)-thick part of \( T_g \). This suffices to prove the claim, because any \( K \) is contained within the \( \alpha \)-thick part \( T_g^\alpha \) of \( T_g \) for sufficiently small \( \alpha \). Recall that for \( t > t' \), \( \phi_t(T_g^\alpha) \subset \phi_{t'}(T_g^\alpha) \). Define a continuous function \( g_{sys}^{sup}(t) \) equal to the supremum of \( g_{sys} \) over \( \phi_t(T_g) \). Then \( \phi_{t}(T_g^{\alpha}) \) is contained in \( \mathcal{P}_{g,\epsilon_{sys}^{sup}(t')} \) for \( t > t' \).

As \( t \) increases, \( g_{sys}^{sup}(t) \) is nonincreasing because a supremum does not increase when the calculation is restricted to a subset. If \( g_{sys}^{sup}(t) \) can be shown to decrease below \( \epsilon' \), then since \( \mathcal{P}_{g,\epsilon'} \) was constructed to be invariant under the flow, this shows that any point of \( T_g^\alpha \) eventually flows into and becomes trapped within \( \mathcal{P}_{g,\epsilon'} \).

Since the closure of \( T_g^\alpha \setminus \mathcal{P}_{g,\epsilon'} \) is compact modulo the action of \( \Gamma_g \), there is a uniform lower bound on the rate at which \( f_{sys} \circ \gamma \) is increasing along the intersection of any flowline with \( \overline{T_g^\alpha \setminus \mathcal{P}_{g,\epsilon'}} \). Once \( f_{sys} \) becomes sufficiently large along \( \gamma(t) \), by construction \( g_{sys} \) is decreasing along \( \gamma(t) \) and eventually enters \( \mathcal{P}_{g,\epsilon'} \).

That the time required for any \( K \) to flow into \( \mathcal{P}_{g,\epsilon'} \) is uniformly bounded independently of \( K \) follows from Lemma 6 and compactness of the thick part of \( T_g \) modulo the action of the mapping class group.

Now suppose \( V_{\epsilon'}^{bad} \) is any vector field constructed according to Section 3 with parameter \( \epsilon \). Define \( X(s) : [0, 1] \to (1 - s)V_{\epsilon'}^{bad} + sV_\epsilon \). For each value of \( s \in [0, 1] \), \( X(s) \) determines a \( \phi_{t,s} \). As the flow lines depend smoothly on the vector field, letting \( s \) vary from 0 to 1 gives a smooth mapping class group-equivariant isotopy between the flows generated by \( V_{\epsilon'}^{bad} \) and \( V_\epsilon \).

It remains to show that the interior of \( \phi_{T,1}(T_g^\alpha) \) is a tubular neighbourhood of \( \mathcal{P}_g \). This is done by using the flow generated by \( X(s') \) for \( s' \in (0, 1) \) chosen sufficiently large to ensure that \( \phi_{T,1}(T_g^\alpha) \) is invariant under this flow. Note that \( \mathcal{P}_{g,\epsilon'} \) is a tubular neighbourhood for sufficiently small \( \epsilon' \) and the flowlines of the vector field \( X(s') \) intersect \( \partial \mathcal{P}_{g,\epsilon'} \) transversely. This implies that distinct points on \( \partial T_g^\alpha \) trace out disjoint paths in \( T_g^\alpha \setminus \mathcal{P}_g \) as the second parameter in the definition of deformation retraction varies from 0 to 1. Consequently, the pre-image of a point in the interior of a simplex of \( \mathcal{P}_g \) is a closed ball, and the remaining properties of a tubular neighbourhood can be readily verified. \( \square \)
Corollary 14. For sufficiently small $\alpha$, $T_\alpha^g$ is a tubular neighbourhood of $P_g$.

Proof. This argument is the same as the proof that $\phi_{T,1}(T_\alpha^g)$ is a tubular neighbourhood of $P_g$ given above. \qed

6. A deformation retraction of the Thurston spine

This section shows that there is a $\Gamma_g$-equivariant deformation retraction of the Thurston spine onto a CW-complex of dimension equal to the virtual cohomological dimension of $\Gamma_g$. The exposition begins with a brief survey of properties of fixed point sets of the action of subgroups of $\Gamma_g$ on $T_g$.

By Fricke’s theorem, the mapping class group acts properly discontinuously on $T_g$. The stabiliser of any point of $T_g$ is therefore finite. As $\Gamma_g$ acts by isometry with respect to a number of metrics, such as the Teichmüller metric and the Weil-Petersson metric, each connected component of a fixed point set of a finite subgroup of $\Gamma_g$ is a closed, totally geodesic submanifold with respect to these metrics.

As the action of $\Gamma_g$ preserves $f_{sys}$, every 0-dimensional fixed point set of any finite subgroup of $\Gamma_g$ is a critical point of $f_{sys}$. If a 0-dimensional fixed point set were not a critical point, since the action of $\Gamma_g$ maps the unit cone of increase to itself, Brouwer’s fixed point theorem implies the existence of a vector that is fixed by the action of the subgroup. A Weil-Petersson geodesic through $p$ with this tangent vector is therefore also contained in the fixed point set, contradicting the assumption that it is 0-dimensional. This applies to any $\Gamma_g$-equivariant Morse function, not just $f_{sys}$. Away from critical points of $f_{sys}$, the fixed point sets of finite subgroups intersect the level sets of $f_{sys}$ transversely, according to the piecewise-smooth definition of transversality given in [1]. This, too, holds for any $\Gamma_g$-equivariant Morse function, not just $f_{sys}$.

Theorem 15 (Theorem 1 of the Introduction). There is a mapping class group-equivariant deformation retraction of the Thurston spine of a closed orientable surface of genus $g$ onto a CW-complex of dimension equal to $4g - 5$.

Proof. In [24], analogues of the stable and unstable manifolds of a topological Morse function were defined. In contrast to the case for Morse functions, the stable and unstable manifolds of critical points of $f_{sys}$, although defined, are not uniquely defined. Nevertheless, since $P_g$ contains all the critical points, and is fixed by an $f_{sys}$-increasing flow, it also $\Gamma_g$-equivariantly deformation retracts on a union of some notion of the unstable manifolds of the critical points.

Theorem 9 will first be used to perform a $\Gamma_g$-equivariant deformation retraction of $P_g$ onto a CW-complex $P_g^X$, using the flow of a vector field $X$ on $P_g$. This deformation retraction is the largest $\Gamma_g$-equivariant deformation retraction possible that does not cancel critical points and is constructed in such a way as to preserve the level set structure of $f_{sys}$. By Theorem 9, the restriction of $f_{sys}$ to $P_g$ only has stationary points at critical and boundary points of $f_{sys}$. Around these points, the local behaviour of $f_{sys}$ is well understood as explained in
Section 5 of [17]. What remains after the first deformation retraction will be interpreted as a union of unstable manifolds of critical points. The second deformation retraction then uses the topology of the boundary of the thick part of \( \mathcal{T}_g \), \( \mathcal{T}_g^M \), to construct a deformation retraction onto a CW-complex of the required dimension. The \( \Gamma_g \)-equivariance of the second deformation retraction is possible as a result of the observation that the fixed point sets of the action of \( \Gamma_g \) behave like unstable manifolds of critical points.

Given two points \( x, x' \) of \( \mathcal{T}_g \), \( x' \) will be said to be above \( x \) if \( f_{\text{sys}}(x') > f_{\text{sys}}(x) \). Whenever \( x \) is on the boundary of a simplex \( \sigma \) of \( \mathcal{P}_g \), \( \sigma \) will be said to be above \( x \) if the cone of increase of \( f_{\text{sys}} \) at \( x \) has nonempty intersection with the tangent cone at \( x \).

Assuming a triangulation of \( \mathcal{P}_g \) compatible with the stratification, a vector field \( X \) on \( \mathcal{P}_g \) will now be defined, smooth on the interior of each simplex, satisfying the following constraints:

1. \( f_{\text{sys}} \) is everywhere increasing in the direction of \( X \), except at critical points or boundary points of \( f_{\text{sys}} \), where \( X = 0 \).
2. \( X \) is \( \Gamma_g \)-equivariant.
3. At any point \( x \in \mathcal{P}_g \), \( X(x) \) is in the tangent cone to \( \mathcal{P}_g \) at \( x \).
4. If there is more than one simplex above \( x \), whenever \( X(x) \) is in the tangent cone to the simplices \( \{\sigma_1, \ldots, \sigma_n\} \), it is in the intersection of the tangent cones at \( x \) to the simplices \( \{\sigma_1, \ldots, \sigma_n\} \), and hence in the tangent cone to a simplex on the boundary of each of \( \{\sigma_1, \ldots, \sigma_n\} \).
5. On a neighbourhood of a critical point or boundary point \( p \) of \( f_{\text{sys}} \), \( |X(x)| \geq \sinh(d(x, p)) \), where \( d(x, p) \) is the Weil-Petersson distance in \( \mathcal{T}_g \) between \( x \) and \( p \), and \( |X(x)| \) is the Weil-Petersson length of the vector.

Conditions (1), (3) and (4) are possible as a result of Theorem 9. Condition (2) is possible because all quantities and constraints are defined in terms of \( \Gamma_g \)-equivariant quantities. Condition (5) can be achieved by rescaling with the help of a \( \Gamma_g \)-equivariant partition of unity.

The flow generated by \( X \) gives a \( \Gamma_g \)-equivariant deformation retraction of \( \mathcal{P}_g \) onto a CW-complex \( \mathcal{P}_g^X \), which will be thought of as a piecewise-smooth analogue of a Morse-Smale complex of \( f_{\text{sys}} \). Condition (5) above was included to ensure that the retraction occurs in finite time, giving a deformation retraction rather than merely a retraction. Flows defined on simplicial complexes are a standard tool in optimisation and combinatorics, but differ from flows defined on manifolds, as the existence and uniqueness theorem for ODEs only holds on the interiors of simplices where the vector field is smooth.

The complex \( \mathcal{P}_g^X \) is analogous to a Morse-Smale complex, but due to the fact that it is obtained from a flow on a simplicial complex, rather than on a manifold where the existence and uniqueness theorem for ODEs hold everywhere, the unstable manifolds for distinct critical points might not be disjoint.
An unstable manifold $\mathcal{M}(p)$ of a critical point $p$ of $f_{\text{sys}}$ of index $j$ is a codimension $j$ ball with boundary on the unstable manifolds of larger index critical points. On $\mathcal{M}(p)$, the level sets are codimension 1 spheres centered on the critical point. Flowlines originate at the critical point, and end on a critical point on the boundary, merge with a flowline contained in an unstable manifold on the boundary of $\mathcal{M}(p)$, or merge with a flowline of a critical point $p_1 \neq p$ for which $\mathcal{M}(p) \cap \mathcal{M}(p_2)$ is nonempty. It follows from Condition (4) above that, while unstable manifolds can flow together as $f_{\text{sys}}$ increases, as illustrated schematically in Figure 3 (a), they cannot flow apart, as shown in Figure 3 (b). More precisely, once a flowline is contained in the intersection of two distinct unstable manifolds, it stays in the intersection, until it reaches the boundary in the form of an unstable manifold of a higher index critical point.

![Figure 3](image)

**Figure 3.** Part (a) shows two unstable manifolds flowing together, Part (b) shows two unstable manifolds flowing apart, contradicting Condition (4). The arrows indicate the direction of increase of $f_{\text{sys}}$.

The intersection of the interiors of two distinct unstable manifolds $\mathcal{M}(p_1)$ and $\mathcal{M}(p_2)$ will be called a flap. Flaps do not contain critical points. Up to homotopy equivalence, the existence of flaps does not change the topology of $\mathcal{P}^X_g$. There is a homotopy equivalence from $\mathcal{P}^X_g$ to a complex without flaps, constructed by collapsing the intersection of the flowlines with the flaps. The theorem will first be proven under the assumption that there are no flaps, and then generalised.

The next step is to understand the way fixed point sets of $\Gamma_g$ intersect the level sets within unstable manifolds of critical points, to show that they do not obstruct the construction of a $\Gamma_g$-equivariant deformation retraction. A fundamental domain of the action of $\Gamma_g$ on $\mathcal{M}(p)$ will be constructed.

Denote by $G(p)$ the subgroup of $\Gamma_g$ that stabilises $\mathcal{M}(p)$ as a set, not necessarily pointwise. The group $G(p)$ is finite as a consequence of Fricke’s theorem. The intersection of the fixed point sets of finite groups $G_1 \subset G(p)$ and $G_2 \subset G(p)$ is the fixed point set of the subgroup of $\Gamma_g$ generated by $G_1 \cup G_2$.

In the absence of flaps, $\mathcal{M}(p)$ is an open cell. The unstable manifold $\mathcal{M}(p)$ and any unstable manifold on the boundary of $\mathcal{M}(p)$ each contain exactly one critical point. All fixed
point sets must be transverse to the level sets both of $\mathcal{M}(p)$ and of the unstable manifolds on the boundary of $\mathcal{M}(p)$ or intersect the unstable manifold in a critical point. As the flow used to construct $P_g^X$ is $\Gamma_g$-equivariant, $G(p)$ maps flowlines to flowlines. Moreover, $G(p)$ maps critical points to critical points of the same index, unstable manifolds to unstable manifolds, level sets to level sets, and fixes $p$. The fixed point set of any subgroup of $G(p)$ therefore contain $p$. If a flowline $\gamma(t)$ is tangent to a fixed point set at time $t_0$, it is tangent at all times $t > t_0$ in its domain, otherwise the vector field $X$ would not be $\Gamma_g$-equivariant, or not single valued. This same argument shows that the assumption of no flaps ensures that the only fixed point sets of $\Gamma_g$ that intersect the interior of $\mathcal{M}(p)$ are fixed point sets of subgroups of $G(p)$.

A fundamental domain of the action of $G(p)$ on $\mathcal{M}(p)$ can therefore be chosen to be invariant under the flow on $\mathcal{M}(p)$. This ensures that the interior of the fundamental domain is $\Gamma_g$-equivariantly contractible by collapsing flowlines.

Let $p$ be a critical point of $f_{\text{sys}}$ of smallest index, i.e. $\mathcal{M}(p)$ has maximal dimension, and let $q$ be an interior point of $\mathcal{M}(p)$. By Corollary 14 the pre-image of $q$ under the deformation retraction is a ball $\mathcal{B}(q)$. This ball has empty boundary and intersects $P_g$ in the single point $q$. By construction, the dimension of $\mathcal{B}(q)$ is equal to the codimension of $P_g^X$ in $\mathcal{T}_g$.

Due to the fact that any $\alpha$-thick subset of $\mathcal{T}_g$ is invariant under Thurston’s flow, $\mathcal{B}(q)$ intersects the thick part of $\mathcal{T}_g$, $\mathcal{T}^{\alpha}_g$, in a connected set. The boundary of $\mathcal{T}^{\alpha}_g$ is piecewise smooth, and $\mathcal{B}(q)$ intersects each of the top dimensional pieces in such a way that the inward facing normal to each piece makes an angle with the tangent space to $\mathcal{B}(q)$ that is bounded away from $\pi/2$. In this sense, $\mathcal{B}(q)$ intersects $\partial \mathcal{T}^{\alpha}_g$ transversely; alternatively, one could use the definition of piecewise linear transversality from [1]. The intersection of $\mathcal{B}(q)$ with $\partial \mathcal{T}^{\alpha}_g$ is therefore a sphere $S^{\alpha}(q)$ of dimension 1 less than the codimension of $\mathcal{M}(p)$ in $\mathcal{T}_g$.

The dimension of $\mathcal{M}(p)$ cannot be less than $4g - 5$, as this is the virtual cohomological dimension of $\Gamma_g$, and gives a lower bound on the dimension of any spine. Assume the dimension of $\mathcal{M}(p)$ is greater than $4g - 5$. In this case, $S^{\alpha}(q)$ has dimension less than $2g - 2$. Since $\partial \mathcal{T}^{\alpha}_g$ is $\Gamma_g$-equivariantly homotopy equivalent to a wedge of spheres $\vee^\infty S^{2g-2}$, $S^{\alpha}(q)$ is contractible in $\partial \mathcal{T}^{\alpha}_g$. Moreover $\mathcal{B}(q) \cap \mathcal{T}^{\alpha}_g$ can be homotoped relative to its boundary $S^{\alpha}(q)$ into $\partial \mathcal{T}^{\alpha}_g$.

The homotopy of $\mathcal{B}(q) \cap \mathcal{T}^{\alpha}_g$ into $\partial \mathcal{T}^{\alpha}_g$ moves the point $q$ off $P_g^X$. This implies that $P_g^X$ has nonempty boundary, and without loss of generality it is possible to assume that $\partial P_g^X \cap \partial \mathcal{M}(p)$ is nonempty.

The boundary of $\mathcal{M}(p)$ consists of unstable manifolds of critical points of larger index than $p$. There is consequently an unstable manifold $\mathcal{M}(p^b)$ in $\partial \mathcal{M}(p) \cap \partial P_g^X$, where $p^b$ has index 1 more than $p$, and $\mathcal{M}(p^b)$ is not on the boundary of any unstable manifold other than $\mathcal{M}(p)$. To make the deformation retraction $\Gamma_g$-equivariant, all deformation retractions
coming from $\Gamma_g$-orbits of $\mathcal{M}(p^b)$ will be performed simultaneously.

When there is only one element of the orbit of $\mathcal{M}(p^b)$ on $\partial\mathcal{M}(p)$, the image of the deformation retraction is the CW-complex obtained by deleting the orbits of the interiors of $\mathcal{M}(p)$ and $\mathcal{M}(p^b)$ from $\mathcal{P}_g^X$.

When more than one element of the orbit of $\mathcal{M}(p^b)$ is on the boundary of $\mathcal{M}(p)$, there is a nontrivial finite subgroup $G(p)$ of $\Gamma_g$ that fixes the $\mathcal{M}(p)$. A fundamental domain $\mathcal{D}(p)$ of the action of $G(p)$ on $\mathcal{M}(p)$ is constructed as explained above. A fundamental domain of $\mathcal{M}(p^b)$ under the action of the subgroup of $G(p)$ that stabilises $\mathcal{M}(p^b)$, will be denoted by $\mathcal{D}(p^b)$. Here is assumed that $\mathcal{D}(p^b)$ is on the boundary of $\mathcal{D}(p)$.

Claim - A $\Gamma_g$-equivariant deformation retraction of $\mathcal{P}_g$ is obtained by deleting the $\Gamma_g$-orbits of the interiors of each of $\mathcal{D}(p^b)$ and $\mathcal{D}(p)$.

To prove the claim, there are two issues to address. First of all, the possibility that the stabiliser subgroup of $\mathcal{M}(p^b)$ is not contained in $G(p)$ was deliberately ignored. If this were the case, the point $q$ of the ball $\mathcal{B}(q)$ could not have been homotoped off $\mathcal{P}_g$ through $\mathcal{M}(p^b)$, because $\mathcal{M}(p^b)$ would then be on the boundary of both $\mathcal{M}(p)$ and the image of $\mathcal{M}(p)$ under the stabiliser subgroup of $\mathcal{M}(p^b)$.

Secondly, within $\mathcal{D}(p)$, the deformation retraction is achieved by first collapsing the flowlines in $\mathcal{D}(p)$ with endpoints on $\mathcal{D}(p^b)$, retracting along level sets, and then collapsing any remaining pieces of flowlines, as shown in Figure 4. This concludes the proof of the claim.

Now consider the case with flaps. There could be a finite subgroup or subgroups of $\Gamma_g$, not contained in $G(p)$, whose fixed point set intersects the interior of $\mathcal{M}(p)$. These are fixed point sets of groups that permute the unstable manifolds that meet along a flap, as illustrated in Figure 5. As before, if a flowline $\gamma(t)$ is tangent to one such fixed point set at time $t_0$, it is tangent at all times $t > t_0$ in its domain. In this case, the point $\gamma(t_0)$ is
where two unstable manifolds meet to make a flap, not a critical point. As above, the fact that \( \Gamma_g \) maps flowlines to flowlines makes it possible to construct a fundamental domain of the action of \( \Gamma_g \) on \( \mathcal{M}(p) \) with the property that the interior of the fundamental domain is \( \Gamma_g \)-equivariantly contractible by collapsing flowlines.

![Figure 5](image)

**Figure 5.** Suppose \( \mathcal{M}(p_1) \) and \( \mathcal{M}(p_2) \) intersect along a flow, as shown. It is possible that there is a finite subgroup \( G(p_1, p_2) \) of \( \Gamma_g \) that interchanges the critical points \( p_1 \) and \( p_2 \). In this case, the fixed point set of \( G(p_1, p_2) \) shown in red can intersect both \( \mathcal{M}(p_1) \) and \( \mathcal{M}(p_2) \), even though \( G(p_1, p_2) \) is not contained in \( G(p_1) \) or \( G(p_2) \).

When constructing the deformation retraction of the \( \Gamma_g \)-orbit of \( \mathcal{M}(p) \), first construct the fundamental domain \( D(p) \) as above. If \( D(p^b) \) is disjoint from the boundary of any flaps, the deformation retraction is obtained by deleting the orbit of the interior of \( D(p^b) \) and of the interior of \( D(p) \backslash \)flaps. This deformation retraction is achieved by iterating the same steps as illustrated in Figure 4, using the fact that flaps do not contain critical points. If \( D(p^b) \) is on the boundary of one or more flaps, for any \( q \in D(p^b) \) first collapse any smooth subinterval of a flowline with endpoint \( \gamma(t_0) \) on \( D(p^b) \) and contained in the intersection of all flaps with boundary point \( \gamma(t_0) \). Then retract along level sets, and repeat.

Since the intersection of flaps with fundamental domains are not necessarily connected, and flaps can intersect along smaller flaps, this deformation retraction does not necessarily remove the entire interior of the intersection of a flap with a fundamental domain. However, it does preserve the property that the dimensions of any flaps are no larger than the dimensions of the remnants of any unstable manifolds in which they are contained. If \( D(p^b) \) intersects both the boundary of one or more flaps and the boundary of \( D(p) \backslash \)flaps, the above deformation retractions are performed both on the flaps and the complement of the flaps in the orbit of \( D(p) \). Whenever a deformation retraction creates flowlines whose initial point is not on a critical point, a deformation retraction of the flap is then performed by iterating the steps as above.

There is some choice involved in the construction of \( D(p) \). Boundary cells not lying completely along fixed sets are not uniquely determined. Any such boundary cells that remain after the deformation retraction are not stablised by any subgroups of \( \Gamma_g \). As it will not be necessary to worry about subdividing these into fundamental domains in later
steps, the choices are not important. This argument can therefore be iterated with smaller and smaller subcomplexes in place of $P^X_g$, until a subcomplex of the required dimension is obtained.

\begin{thebibliography}{99}

[1] M. Armstrong and E. Zeeman. Transversality for piecewise linear manifolds. *Topology*, 6:433–466, 1967.
[2] A. Barvinok. Lattice points, polyhedra, and complexity. In *Geometric combinatorics*, volume 13 of IAS/Park City Math. Ser., pages 19–62. American Mathematical Society, Providence, RI, 2007.
[3] L. Bers. Nielsen extensions of Riemann surfaces. *Annales Academiae Scientiarum Fennicae, Series A. I. Mathematica*, 2:29–34, 1976.
[4] M. Bestvina. Four questions about mapping class groups. In *Problems on mapping class groups and related topics*, volume 74 of *Proc. Sympos. Pure Math.*, pages 3–9. Amer. Math. Soc., Providence, RI, 2006.
[5] M. Bestvina, K. Bromberg, K. Fujiwara, and J. Souto. Shearing coordinates and convexity of length functions on Teichmüller space. *American Journal of Mathematics*, 135(6):1449–1476, 2013.
[6] M. Bridson and K. Vogtmann. Automorphism groups of free groups, surface groups and free abelian groups. In *Problems on mapping class groups and related topics*, volume 74 of *Proceedings of Symposia in Pure Mathematics*, pages 301–316. American Mathematical Society, Providence, RI, 2006.
[7] M. Brion and M. Vergne. Lattice points in simple polytopes. *Journal of the American Mathematical Society*, 10(2):371–392, 1997.
[8] S. Broughton. The equisymmetric stratification of the moduli space and the Krull dimension of mapping class groups. *Topology and its applications*, 37:101–113, 1990.
[9] K. Brown. *Cohomology of Groups*. Graduate Texts in Mathematics. Springer, New York, 2012.
[10] L. Ji. Well-rounded equivariant deformation retracts of Teichmüller spaces. *Inventiones Mathematicae*, 184:157–176, 1986.
[11] J. Harer. The cohomology of the moduli space of curves. In *Theory of moduli (Montecatini Terme, 1985)*, volume 1377 of *Lecture Notes in Mathematics*, pages 138–221. Springer Berlin, 1988.
[12] W. Harvey. Boundary structure of the modular group. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 245–251. Princeton Univ. Press, Princeton, N.J., 1981.
[13] K. Hofmann. Triangulation of locally semi-algebraic spaces. ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–University of Michigan.
[14] J. Lee. *Introduction to smooth manifolds*, 2nd Edition, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, 2013.
[15] S. Lojasiewicz. Triangulation of semi-analytic sets. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, 18(4):449–474, 1964.
[16] M. Mirzakhani. Growth of the number of simple closed geodesics on hyperbolic surfaces. *Annals of Mathematics. Second Series*, 168(1):97–125, 2008.
\end{thebibliography}
[24] M. Morse. Topologically non-degenerate functions on a compact $n$-manifold. *Journal d’Analyse Mathématique*, 7:189–208, 1959.

[25] R. Penner. The decorated Teichmüller space of punctured surfaces. *Communications in Mathematical Physics*, 113(2):299–339, 1987.

[26] G. Riera. A formula for the Weil-Petersson product of quadratic differentials. *Journal d’Analyse Mathématique*, 95:105–120, 2005.

[27] P. Schmutz Schaller. Systoles and topological Morse functions for Riemann surfaces. *Journal of Differential Geometry*, 52(3):407–452, 1999.

[28] W. Thurston. A spine for Teichmüller space. Preprint, 1985.

[29] S. Wolpert. Geodesic length functions and the Nielsen problem. *Journal of Differential Geometry*, 25(2):275–296, 1987.

[30] S. Wolpert. Convexity of geodesic-length functions: a reprise. In *Spaces of Kleinian groups*, volume 329 of *London Math. Soc. Lecture Note Ser.*, pages 233–245. Cambridge Univ. Press, Cambridge, 2006.

**Department of Mathematics, Southern University of Science and Technology, Shenzhen, China**

**SUSTech International Center for Mathematics, Southern University of Science and Technology, Shenzhen, China**

*Email address:* ingridmary@sustech.edu.cn