The Tensor Track VII: From Quantum Gravity to Artificial Intelligence

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Abstract
Assuming some familiarity with quantum field theory and with the tensor track approach that one of us presented in the previous series “Tensor Track I-VI”, we provide, as usual, the developments in quantum gravity of the last two years. Next we present in some detail two algorithms inspired by Random Tensor Theory which has been developed in the quantum gravity context. One is devoted to the detection and recovery of a signal in a random tensor (that can be associated to the noise) with new theoretical guarantees for more general cases such as tensors with different dimensions. The other, SMPI, is more ambitious but maybe less rigorous; it is devoted to significantly and fundamentally improve the performance of algorithms for Tensor principal component analysis but without complete theoretical guarantees yet. Then we sketch all sorts of application relevant to information theory and artificial intelligence and provide their corresponding bibliography.

Keywords: Quantum Gravity, Tensors Models, Data Analysis, Artificial Intelligence.
1 Introduction

Random tensors originated in theoretical physics. Matrix models had some success in quantizing the strong interaction, as t’Hooft made the fundamental observation [1] that their $1/N$ expansion is a topological expansion.

In 1990’s the dominating theory was string theory. Random matrix models were seen at this time as a successful theory for quantizing gravity but only in two dimensions. The inventors of random tensor models wanted to replicate the success of matrix models for dimensions $D = 3, 4, ...$ [2–4].

In 2010 a new kind of $1/N$ expansion was discovered for these random tensor models ([5], for a review, consult [6]). It relies heavily on the Gurau degree of a $D + 1$-colored graph $G$, with is a concept partly topological and partly combinatorial. The Gurau degree $\omega$ is a positive number $\omega(G) \in \mathbb{N}$. To define it, we need a new notion, that of the jackets. For the moment, it suffices to say that a jacket $J_\sigma$ is associated to a colored graph and a cyclic permutation $\sigma$ on the colors. There are three inequivalent jackets for $D = 3$ [5], twelve inequivalent jackets for $D = 4$ and so on [7]. To each jacket is associated a combinatorial map, hence a matrix model, thus a degree of the ordinary $1/N$ graphs of the corresponding matrix model. The Gurau degree is proportional to the sum of the ordinary (t’Hooft) degree of the jackets:

$$\omega(G) = \frac{1}{(D-1)!} \sum_\sigma g(J_\sigma)$$

Initially, the tensor track is an attempt to quantize gravity in dimension $D > 2$ by combining random tensor models to discrete geometry and the renormalisation group [8]. The tensor track lies at the crossroad of several closely related approaches to quantize gravity, most notably causal dynamical triangulations [9], quantum field theory on non-noncommutative spaces [10–12], and group field theory [13].

Random tensors share with random matrices the fact that they are a zero-dimensional world, and, as such, they are background-independent; it makes no references whatsoever of any particular space-time. Moreover, random tensors models, based on the field theory of Feynman, are amenable to renormalisation group techniques [14]. Simple models even share with non-Abelian gauge theories the property of asymptotic freedom [15, 16].

Random tensors are expected to play a growing role in many areas of mathematics, physics, and computer science. To our knowledge, communities using random tensors have mostly grown apart, developing their own tools and results. Nowadays there is an increasing circle of mathematicians and physicists working on random tensors, and these people have been lately inclined towards more applications linked to data analysis and artificial intelligence. In this way, the line which separated them from computer scientists becomes a little bit blurred.

This review is organized as follows. Section II is devoted to a brief summary of the tensor track. In section III we review in more depth two tensors-inspired
algorithms. In Section IV we made the rather bold step of applying this stuff to artificial intelligence. Section V presents our future perspectives.

2 Quantum gravity

Let us review briefly the previous chapters of the tensor track [8, 17–19].

In the Hermitian matrix ensemble (GUE) perturbed by a quadratic interaction, the $1/N$ expansion is well known. The free partition function is $\int dM e^{-\frac{1}{2N} \text{Tr} M^2}$, where

$$dM = \prod_k dM_{kk} \prod_{i<j} d\text{Re} M_{ij} d\text{Im} M_{ij}, \tag{2}$$

and the expectation values of $U(N)$ invariants

$$< \text{Tr} M^{p_1} \text{Tr} M^{p_2} \cdots \text{Tr} M^{p_k} > \tag{3}$$

is entirely determined by the propagator

$$C_{ij,kl} = \frac{1}{N} \delta_{il} \delta_{jk} \tag{4}$$

and by Wick’s rule.

Any scalar function of a tensorial quantum field theory is a big functional integral on a Gaussian measure and an interactive part. In the tensorial case this interactive part is a sum of invariants of the tensor. For example the partition function is a scalar function of $N$, defined by

$$Z(N) = \int d\mu(T) e^{-\sum_{I_{\text{inv}}} S_{I_{\text{inv}}}(T)} \tag{5}$$

so as the free energy. The partition function and the corresponding free energy are related by a normalized logarithm

$$F(N) = \frac{1}{ND} \log Z(N) \tag{6}$$

The invariants themselves can be classified in terms of graphs. Of course these graphs depend upon the group symmetries of the tensor. For matrix models the expectation values of the invariants can be classified by ribbon diagrams. In the case of tensor models Figure 1 depicts a partial list of connected invariants for $\otimes_{i=1}^3 U(N)$.

To generalize the $1/N$ expansion to the tensor case, the first step is to choose an invariant, for example a quartic invariant, and to normalize the
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![Fig. 1: Examples of $U(N)$ invariants](image)

(a) Melon graph  
(b) Pillow graph  
(c) $K_{3,3}$ graph

$S_{Inv_0}(T)$ for that invariant:

$$S_{Inv_0}(T) \rightarrow \frac{\lambda}{N^\alpha} S_{Inv_0}(T)$$  \hspace{1cm} (7)

Now the partition function and the corresponding free energy depend on two variables. In a quantum field theory the usual form of perturbation theory is to expand in power series in $\lambda$, and the perturbation is indexed by Feynman amplitudes associated to Feynman graphs. For instance the perturbation of the free energy is of the following form

$$F(\lambda, N) = \sum_G \frac{(-\lambda)^{v(G)}}{\text{sym}(G)} A(G, N)$$  \hspace{1cm} (8)

Once this is done, the hard step is to find $\alpha$ such that $1/N$ expansion exists, i.e. such that the perturbation of the free energy is of the following form

$$F(\lambda, N) = \sum_G \frac{(-\lambda)^{v(G)}}{\text{sym}(G)} A(G, N) = \sum_{\omega \in \mathbb{N}} N^{-\omega} F_{\omega}(\lambda)$$  \hspace{1cm} (9)

In the case of $\bigotimes_{i=1}^D U(N)$ and a quartic $D$-melonic invariant, the $1/N$ expansion is governed by the Gurau degree and $F_0(\lambda)$ is formed by all the $D+1$ melonic graphs (the 0 color being associated to the Feynman propagators [20]). The family of $D+1$ melonic graphs [21] which lead the $1/N$ expansion of random tensors models can perhaps be called too trivial from a topologist point of view; it corresponds to some triangulations of the sphere $S_D$. But, as the Gurau degree is not only purely topological, the interplay between combinatorics and topology in sub-leading terms can be amazingly complex, even for a geometrically-oriented person!

Now that we have been able to identify the leading terms in the $1/N$ expansion, the second step is to be able to resum them, i.e. to explicitly compute $F_0(\lambda)$. This step has been performed for the first time by a paper by Gurau and Ryan [22]. From a probabilistic and statistical mechanical point of view, it corresponds to Aldous phase of branched polymers.

Once we have been able to compute explicitly $F_0(\lambda)$, many possibilities are open to us: modifying the symmetry of the main tensor, include the
renormalization group by modifying in a specific way our propagators, a non-perturbative treatment of some simple models...

From the perspective of matrix models, to go further in the two parameters approach require a particular technique, namely double scaling. The first step of applying this technique to tensors has been done [23, 24]. The initial papers have been followed by mixed results, some results suggest the universality of branched polymers, others pointing to the fact that some simple and natural restrictions change that universality class. But, from the perspective of quantum gravity, the main goal is to resum the sub-leading terms in order to find a more interesting phase of geometry pondered by Einstein-Hilbert action. In this vein, we would like to highlight one contribution, that of Lionni and Marckert [25]. In this paper they use new combinatorial bijections to uncover a random phase in higher dimensions.

Let us come to the Sachdev-Ye-Kitaev (SYK) model [26, 27]. Discovered by Kitaev, it is a quartic model of N Majorana fermions coupled by a disordered tensor. It is a model of condensed matter, hence it depends on time though a Hamiltonian. The disordered tensor is centered Gaussian iid

\[ < J_{abcd} > = 0, \quad < J^2_{abcd} > = \frac{\lambda^2}{N^3}, \]  

and the Hamiltonian is simply \( H = J_{abcd} \psi_a \psi_b \psi_c \psi_d \). This model posses three important properties: it is solvable at large \( N \), there is a conformal symmetry at strong coupling, hence it can be a fixed point of the renormalization group, and, above all from quantum gravity, it is maximally chaotic in the sense of [28]. Hence the SYK model, although very simple, offers a path to the main theoretical concepts of quantum gravity, such as Bekenstein-Hawking entropy and holography.

SYK became a very active field, from the early papers to nowadays. At large \( N \) the Schwinger-Dyson equation for the 2-point function is closed. The conformal symmetry can be broken and the corresponding subject goes under the name of near-AdS2/near-CFT1 correspondence. This entails a relationship with Jackiw-Teitelboim two-dimensional quantum gravity.

Witten has found a genuine field theory model (with no disorder), in which the tensors plays a much more fundamental role [29]. In a nutshell, he discovered that his model has the same melonic limit as the tensors models pioneered by Gurau. Klebanov and Tarnopolsky [30], when combined with an earlier work of Carrozza and Tanasa [31], allows on a big simplification of the group symmetry of the main tensor, from \( U(N)^D(D-1)/2 \) to \( O(N)^D \) (see Figure 2).

The action of the KTCT model is

\[ S = \int dt \frac{i}{2} \psi_{i_1,i_2,i_3} \partial_t \psi_{i_1,i_2,i_3} + \frac{\lambda}{4N^{3/2}} \psi_{i_1,i_2,i_3} \psi_{i_4,i_5,i_6} \psi_{i_1,i_2,i_6} \psi_{i_1,i_5,i_6}. \]  

Unlike the initial SYK model, these tensor models fit in the framework of local quantum field theory with \( D = 1 \). Hence there is a possibility to extend them in \( D > 1 \)! We would like to stressed one recent contributions
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Fig. 2: Examples of $O(N)$ graphs and their associated invariants

in this domain. For $p = 3$ and $p = 5$ there exist a melonic large $N$ limit for $p$-irreducible tensors in the sense of Young tableaux [32–34]. Carrozza and Harribey overcome huge difficulties to solve the case $p = 5$ [34].

3 Algorithms

Given the richness of the tools developed in the subject of random tensors, it is very appealing to attempt to take advantage of them for various other situations outside of quantum gravity.

In this section, we consider an important problem, Tensor PCA, where these tools were successfully used to introduce a novel framework that achieved new theoretical and practical improvements. Tensor PCA [35] consists in detecting and retrieving a spike $v_0 \otimes^k$ from noise-corrupted multi-linear measurements in the form of a tensor $T$

$$T = \beta v_0 \otimes^k + Z,$$

with $Z \in (\mathbb{R}^n)^{\otimes k}$ a pure Gaussian noise tensor of order $k$ and dimension $n$ with identically independent distributed (iid) standard Gaussian entries: $Z_{i_1,i_2,\ldots,i_k} \sim \mathcal{N}(0,1)$ and $\beta$ is the signal-to-noise ratio.

3.1 Motivation

The motivation for Tensor PCA is two-fold:

- Practical applications: algorithms addressing Tensor PCA could easily be generalized to Tensor decomposition, which have applications in Tensor faces [36], Hyperspectral imagery [37], DNN compression [38, 39], multimodal data fusion [40], wireless communication [41], computer vision [42], natural language processing [43], etc. More details could be found in Section 4.
Theoretical interest: Tensor PCA is often used as a prototypical inference problem for the theoretical study of the computational hardness of optimization in high-dimensional non-convex landscapes, in particular using the well spread gradient descent algorithm and its variants ([44–47]). Tensor PCA is also considered as an interesting study case of statistical-algorithmic gaps that appears in various other problems (see references in [44] and [48]). Indeed, while information theory shows that is theoretically possible to recover the signal for $\beta \sim O(1)$, all existent algorithms have been shown or conjectured to have an algorithmic threshold of at least $\beta \sim O(n^{(k-2)/4})$.

3.2 Random Tensor Theory for Tensor PCA

3.2.1 The framework

Matrix data analysis and principal component analysis (PCA) is mostly stated in the “quantum-mechanics” language of eigenvalues rather than in the “quantum-field theoretic” language of invariants and (Feynman) graphs. For tensors the quantum-field language is the natural one. An important task in tensor data analysis is therefore to translate the results of matrix data analysis and PCA into the quantum-field theoretic language of invariants and graphs.

An original connection have been made in [49] between tensorial data analysis and the random tensor theory developed last decade in the context of quantum gravity by the theoretical physics community.

This connection is based on the introduction of matrices that are built out of a graph and an edge as illustrated in Figure 3. Indeed, given a graph invariant, we call "cutting an edge" the fact of not performing a sum over the index associated to this edge, which gives us a matrix.

The eigenvector associated to the largest eigenvalue of this matrix can then be proven to be correlated to the signal vector $\mathbf{v}$ for a range of SNR $\beta$. The algorithm 1 is based on this simple fact.

3.2.2 The algorithms

It appears that the two state of the art (SOTA) methods, which are the Tensor Unfolding method and the Homotopy-inspired method, are equivalent to the algorithms associated to the graphs of degree 2. This striking fact incites us to investigate the algorithm associated to the tetrahedral graph which is a graph of degree 4 as illustrated in Figure 4.
Algorithm 1 Recovery algorithm associated to the graph $\mathcal{G}$ and edge $e$

**Input:** The tensor $T = \beta v^{\otimes k} + Z$  
**Goal:** Estimate $v_0$.  
Calculate the matrix $M_{\mathcal{G},e}(T)$  
Compute its top eigenvector by matrix power iteration (repeat $v_i \leftarrow M_{ij}v_j$).  
**Output:** Obtaining an estimated vector $v$

- **Invariant of degree 2**  
  - Tensor Unfolding  
  - Homotopy method

- **Invariant of degree 4**  
  - Tetrahedral

**Fig. 4:** Methods associated to invariant graphs

**Novel theoretical threshold:** Let’s first consider the more general case where the tensor $T$ has axes of different dimensions $n_i$ ($T \in \bigotimes_{i=1}^{k} \mathbb{R}^{n_i}$). We can assume without any loss of generality that $n_1 \geq n_2 \geq \cdots \geq n_k$.

$$T = \beta v_1 \otimes \cdots \otimes v_k + Z \quad \text{where} \quad v_i \in \mathbb{R}^{n_i}, \quad n_i \in \mathbb{N}. \quad (13)$$

Our framework allows us to derive a new algorithmic threshold for this more general case: the threshold for $v_1$ is given by max $\left( (\prod_{i=1}^{k} n_i)^{1/4}, n_1^{1/2} \right)$ while the thresholds for $v_j, j \geq 2$ are equal to $\left( \prod_{i=1}^{k} n_i \right)^{1/4}$. It is, to the best of our knowledge, the first generalization of the threshold $\beta = n^{k/4}$ derived in [35] when $n_i = n \forall i \in [k]$.

**Tensor PCA experiments:** In Figure 5 right, we compare the recovery performance. For every algorithm we use two variants: the simple algorithm outputting $v$ and an algorithm where we apply 100 power iterations on $v$: $v_i \leftarrow T_{ijk}v_jv_k$, distinguishable by a prefix ”p-”. We compare our method (tetrahedral) to other algorithmic methods: the melonic (tensor unfolding) and the homotopy. They give the state of art respectively for the symmetric and asymmetric tensor. We see that our new algorithm is able to achieve better performance. However, it should be noted that the tetrahedral method is more costly in computational power given that it is based on a graph of degree 4. In conclusion, this novel framework is not only able to cover the two SOTA methods as special case of algorithms, but it is also able to offer novel algorithms with larger computational cost but better performance. Providing a large variety of algorithms with different performance/computational cost ratio is a very interesting feature as it is adaptable to the resources and precision objective of the situation.
Fig. 5: In the right: (a) Comparison of different methods for symmetric recovery. n=150. In the left: (b) Recovery of a spike with different dimensions

We see in the Figure 5 left, that the threshold $(n_1 n_2 n_3)^{1/4}$ for the three vectors matches perfectly with the experiences when $n_3 < n_1 n_2$. We also see that when $n_3 > n_1 n_2$ the recovery of $v_3$ (in green and with the diamond and square markers) have a different asymptotic behavior than $v_1$ and $v_2$ (it becomes $n_3^{1/2}$ since $n_3^{1/2} \geq (n_1 n_2 n_3)^{1/4}$), corresponding to our theoretical predictions.

3.3 Selective Multiple Power Iteration for high dimensional non convex landscapes.

The previous framework is able to provide algorithms with theoretical guarantees that are easily obtained than to Random Tensor Theory and that can address more general situations like a tensor with different dimensions.

In contrast, the second algorithm SMPI [50] (Selective Multiple Power Iteration) does not have theoretical guarantees yet, but it provides a surprising fundamental improvement over all existent algorithms. It consists in applying, in parallel, the power iteration method with $m_{iter}$ iterations to $m_{init}$ different random initialization. Then, SMPI uses the maximum likelihood estimator to select the output vector in this subset by choosing the vector that maximizes $T(v, v, v)$. It is described in the Algorithm 2.

Interestingly, Power Iteration could also be studied through tensor invariants, as it was already partially investigated in [51]. The associated graph is illustrated in Figure 6.

3.3.1 Comparison with the state of the art.

We plot in Figure 7 the correlation between the output of each algorithm and the signal vector; for SMPI (blue), TensorLy (TenLy, red) [52] which a recent practical Python package widely used and the State-of-the-art represented here by the Unfolding (Unf, green) [35] and Homotopy-based (Hom, orange) [53] methods for four values of the dimension of each axe of the tensor ($n = 100, 200, 400$).
Algorithm 2 Selective Multiple Power Iteration

1: **Input:** The tensor $T = Z + \beta v_0^{\otimes k}$, $m_{\text{init}} > 10n$, $m_{\text{iter}} > 10n$, $\Lambda$
2: **Goal:** Estimate $v_0$.
3: for $i = 0$ to $m_{\text{init}}$ do
4:   Generate a random vector $v_{i,0}$
5:   for $j = 0$ to $m_{\text{iter}}$ do
6:     $v_{i,j+1} = \frac{\langle T(:, v_{i,j}, v_{i,j}) \rangle}{\|T(:, v_{i,j}, v_{i,j})\|}$
7:     if $j > \Lambda$ and $|\langle v_{i,j-\Lambda}, v_{i,j} \rangle| \geq 1 - \varepsilon$ then
8:       $v_{i,m_{\text{iter}}} = v_{i,j}$
9:       break
10:   end if
11: end for
12: end for
13: Select the vector $v = \arg \max_{1 \leq i \leq m_{\text{init}}} T(v_{i,m_{\text{iter}}}, v_{i,m_{\text{iter}}}, v_{i,m_{\text{iter}}})$
14: **Output:** the estimated vector $v$

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**Fig. 6:** The graph associated to the power iteration method with 3 iterations for an initialization $v$. The cross represents the vector $v$ and the black dot the tensor $T$

We observe that SMPI already outperforms existent algorithms for $n = 100$ and the only one whose performance does not decrease as $n$ grows. This is remarkable as it is conjectured that the algorithmic threshold (the $\beta$ above which the algorithm is successful) scales as $n^{1/4}$ even if the information theory theory is at $O(1)$ (this fact is denoted the statistical-computational gap).

### 3.3.2 Difference between SMPI and other power-iteration based algorithms

Power iteration has been studied in various works with different settings. However, [50] experiments suggest that the success of SMPI is related to five essential non-trivial features. It turns out that, to the best of our knowledge, no other work studied Power Iteration with the five features simultaneously present, as summarized in the following Table 1. Thus, further theoretical investigations are required to unveil the theoretical insights behind this
Fig. 7: Comparison of the results of SMPI with TensorLy (TenLy) and the State-of-the-art represented here by the Unfolding (Unf) and Homotopy-based (Hom) methods for four values of the dimension of each axe of the tensor \((n = 100, 200, 400)\). The results consist of the correlation between the output of each algorithm and the signal vector.

experimental success, which could lead to new important insights on the statistical-computational gap.

| Algorithm                | Symmetry | Discreet step size | Poly. nb of initialisat. | Poly. nb of iterations | No stop condition |
|--------------------------|----------|--------------------|--------------------------|------------------------|-------------------|
| Wang et al., 2017 [54]   | Yes      | Yes                | No                       | No                     | No                |
| Huang et al., 2020 [55]  | No       | Yes                | Yes                      | Yes                    | Yes               |
| Ben Arous et al., 2020 [44] | Yes     | No                 | Yes                      | Yes                    | Yes               |
| Dudeja et al., 2022 [56] | Yes      | Yes                | Yes                      | No                     | Yes               |
| SMPI, 2021 [50]          | Yes      | Yes                | Yes                      | Yes                    | Yes               |

4 Artificial Intelligence

The importance of random matrices in Artificial Intelligence (AI) is now well-established. In view of this, random tensors and more generally, tensor tools are expected to play a growing role in many AI subjects. In this section, we give some examples of AI applications where tensors seem promising.
4.1 Application to image processing

Image processing is an important topic in computer science due to its uncountable applications. Since 2012, the performances in this domain have drastically increased with the revolution of deep learning [57]. It is worth to note that deep learning have first demonstrated is strength on image processing (recognition of handwritten digits via the MNIST database [58], large-scale image classification in the ImageNet database [59], image synthesis using GAN architectures [60],....). However, it is also in this context that appeared one of the main weaknesses of deep learning, namely its sensitivity to small changes at the input level [61–63]. This problem is so important that it opens a completely new area of research known as adversarial examples [64]. Indeed, these intriguing properties are clearly not suited for critical applications (involving human lives). This limitation of deep learning is amplified by its need of large amount of data, its enormous energy cost [65, 66] and its black box behavior [67], which all together limit the domain of applications for the deep learning approach.

As a scientific challenge, it is important to find and develop alternative or complementary methods to deep learning models that are sufficiently powerful in order to deal robustly with image processing tasks, which is for example needed to address critical applications (e.g. autonomous vehicle). With this in mind, we argue that it would be interesting to develop and spread tensor methods for image processing in order to gain more robustness and theoretical guarantee in practical applications. Of course, the subject of tensor methods is not new in image processing [42]. Indeed, its usefulness has already been shown for example for facial image representation [36, 68–70], for human motion analysis [71], for classification of images of Handwritten digit [72], for the general task of image denoising [73] and for background-foreground separation in videos [74–76].

4.2 Application to Deep Neural Networks speed-up and compression

Higher-order tensor clustering and tensor decompositions such as Tucker, Canonical Polyadic (CP) and Tensor Train (TT) decompositions have been successfully used for Convolutional Neural Networks (CNN) speed-up and compression in [77, 78] and also applied to fully connected neural networks [79] with an impressive achieved compression rate. However, even if these approaches have been addressed individually in multiple papers, as all tensor decompositions have been tried in principle, the combination of tensor decomposition with other approaches such as quantization [80] and distillation [81–84] has not been well studied yet. Related to this combination-based approach, questions such as the following deserved to be explored: Is a “tensorized” neural network more suitable for the quantization than the original model?
Another important research axis aims to take advantage of the recent algorithmic advances in the field of tensor methods [49, 50, 85]. Given the significant improvement of performance they achieved, it pushes us to rethink these new tools for the compression and speed up of Deep Neural Networks (DNN). It will be interesting to study the performance of these decompositions for a pre-trained neural network as well as during the training. For example, in the pre-trained case, there is already interesting aspect that we will investigate with respect to the fine-tuning procedure. Indeed, this approach usually starts with a pretrained CNN and then interleaves the decomposition of convolutional layers with fine-tuning operations [38, 86]. Each decomposition step that leads to a speed-up and compression, unfortunately comes with a drop in accuracy. Thus, the subsequent fine-tuning operation is expected to recover the accuracy drop. One important question is then, how these new tensor algorithms will reduce this drop in accuracy and will it still be necessary to apply the fine-tuning operation?

A third important research axis to investigate is related to the exploitation of the inherent sparsity present in DNN models [87, 88]. To achieve this challenging goal, we need to develop Sparse Tensor Decomposition building on the new tensor methods findings utilizing insights from existent approaches for sparse tensors [89]. Closely related to this point, investigating and developing the Sparse tensor clustering will also be of great interest for the speed-up and compression of DNN.

### 4.3 Application to particle trajectories

In high energy physics, the matrix principal component analysis has been applied to the detection and recovery of particle trajectories emanating from accelerators. A major goal is to fit this MPCA to the continuous flow of data from those accelerators, such as the many accelerators of CERN.

On the other hand Tensor PCA is one essential tool for separating signal from noise. The random tensor theories are making significant and continuous progress in the problems of data analysis. We reason that we have a good occasion to transform the subject and the corresponding algorithms.

### 4.4 Other applications

More generally, it is important to note that there is many other practical applications where tensor decomposition and where SMPI could be an excellent candidate for improving existent performance. Here is a non-exhaustive list of such applications:

Related to image processing, interesting applications of tensor methods have been explored in computer graphics. For example, a framework for image-based rendering dedicated to the realistic mapping of a texture onto a planar
surface is presented in [90]. Another tensor-based application is the face transfer where a mapping is performed to apply a video recorded performances of one individual to facial animations of another [91].

Hyperspectral images are naturally three-dimensional arrays so there have been different types of work exploiting tensor methods for such data. Tensor decomposition have been explored for different tasks such as hyperspectral image denoising [92], hyperspectral image super-resolution [93], hyperspectral image classification [94], for object detection [95] and also for medical hyperspectral image analysis [96].

In telecommunication, CP decomposition is used for tensor-based modulation [41, 97]. It used for massive random access, whereby a large number of transmitters communicate with a single receiver, ant it constitutes a key design challenge for future generations of wireless systems.

Let us come to opportunities in the field of chemical and biology. For instance, in chemometrics [98] three-dimensional arrays may be generated by collecting data tables with a fixed set of objects and variables under different experimental conditions, at different sampling times, etc. One may also refer to the recent review [99] where different types of higher order data in manufacturing processes are presented. Their potential usage is addressed using methods like CP tensor decomposition. The authors also give concrete perspectives on the application of tensorial data analytics to these kind of processes.

We finish this section by briefly mentioning applications where tensorial data analytics has been explored. This is for example the case in geophysics in the context of three-dimensional irregular seismic data reconstruction [100]. Electroencephalogram (EEG) data collected during a cognitive control task may also benefit from tensor methods [101]. Other interesting applications of tensor methods are dedicated to natural language processing [43] and radar data [102]. We can also mention the following works [103] concerning the application of tensor decomposition techniques to many-body tensors which has proven highly beneficial to reduce the computational cost in quantum chemistry and solid-state physics.

5 Conclusion

This review is a modest step in the direction of bring closer the different people working on random tensors and it suggests research in many directions, among which:

- investigate better some tensorial fields models such as $T_D^4$ family and determine their renormalisation group flow,
- understand better the connection between tensors-inspired algorithms and disordered systems,
- figure out the concrete applications relevant to data analysis and to artificial intelligence.
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