Entropy of quantum fields for nonextreme black holes in the extreme limit

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Nonextreme black hole in a cavity within the framework of the canonical or grand canonical ensemble can approach the extreme limit with a finite temperature measured on a boundary located at a finite proper distance from the horizon. In spite of this finite temperature, it is shown that the one-loop contribution $S_q$ of quantum fields to the thermodynamic entropy due to equilibrium Hawking radiation vanishes in the limit under consideration. The same is true for the finite temperature version of the Bertotti-Robinson spacetime into which a classical Reissner-Nordström black hole turns in the extreme limit. The result $S_q = 0$ is attributed to the nature of a horizon for the Bertotti-Robinson spacetime.

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The present paper is devoted to the description of thermodynamic properties of black holes near the extreme limit in the framework of the Euclidean path integral approach. There is good reason to believe that such properties depend strongly on the topological sector to which the black hole belongs. Therefore, truly extreme black holes and nonextreme ones near the extreme state may have little in common. Thermodynamics in the first case was considered in [1] where it was argued that the zero-loop entropy $S$ for extreme black holes equals zero, the horizon being situated infinitely far away from any point outside it. The second situation was discussed in [2], [3] for the black hole in a finite size cavity in the framework of the grand canonical ensemble. Then the entropy $S$ is equal to its standard Bekenstein-Hawking value $A/4$ ($A$ is the surface area of the horizon). The aim of the
present paper is to extend the previous result indicated above and examine the possible role of quantum corrections $S_q$ to the entropy for the limiting case considered in [2]. For this case the limit is taken in such a way that the temperature measured on the boundary is finite and the proper distance between the horizon and any other point inside a cavity stays finite. I show that properties of the black hole in such an extreme limit are rather unexpected: although the temperature $T \neq 0$, $S_q = 0$. In other words, the entropy of a dressed black hole comes entirely from the spacetime geometry ($S = A/4$) as it does for a bare hole.

The problem considered below has, apart from a general interest to properties of near-extreme black holes, one more motivation connected with the status of the third law of thermodynamics (Nernst theorem) in black hole thermodynamics. (This issue has been recently touched upon in [4]). This theorem asserts that the entropy should go to zero (or some universal constant) when temperature approaches zero. The results of either the previous paper [2] or the present one show that one should be very careful in the formulation of the third law in the context of black hole thermodynamics due to the crucial difference between the Hawking temperature $T_H$ and temperature $T$ which determines properties of the gravitational ensemble and is imposed on the boundary of the cavity inside which the hole is situated [3]. In the case under discussion the zero-loop entropy $S \neq 0$, $S_q = 0$; $T \neq 0$, $T_H = 0$.

Throughout the paper we mean by $S_q$ the one-loop contribution of a hot gas of quantum radiation. This quantity is finite, being the linear functional of an renormalized stress-energy tensor $T^\nu_\mu$ in the Hartle-Hawking state [3], [4]. We do not calculate here quantum corrections to the entropy of the black hole itself and do not consider here in detail the mechanism of renormalization: it is assumed right from the very beginning that such renormalization is already performed, so we deal with the finite $S_q$ and $T^\nu_\mu$ in the Hartle-Hawking state. Thus, the total entropy $S = A/4G + S_q$ where $G$ is the renormalized gravitational constant. In particular, it means that information about the structure of divergent corrections to the statistical-mechanical entropy $S$ of the black hole itself near the extreme state [9], [10], [11] is essentially insufficient for determining $S_q$. 

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The proof of the statement that $S_q = 0$ in the limit under discussion is based on recovering $S_q$ from $T_{\mu}^{\nu}$. Under some physically reasonable restrictions on the behavior of the metric near the horizon we derive the formula for $S_q$ in terms of $T_{\mu}^{\nu}$ for a spherically-symmetric black hole spacetime which probably can be useful for further applications as well. After this formula is obtained for an arbitrary relation between $r_+$ and $r_B$ the limiting transition to $r_+ = r_B$ corresponding to the state under consideration is made.

This limiting transition is the characteristic feature of the configuration under discussion. The points with $r > r_+$ of the original spacetime (say, of the Reissner-Nordström (RN) black hole) lie at infinite proper distance from $r_+$, as usual. However, since the relevant manifold is obtained in the limiting case under discussion by suitable expanding the vicinity of the horizon into the region with a finite Euclidean four-volume, all points of this manifold pick up the value $r = r_+$ for which the metric coefficient $g_{rr} \to \infty$. As a result, the proper distance $l = \int_{r_+}^{r} dr \sqrt{g_{rr}}$ is finite for the configuration at hand (see [2], [3] for details).

We suggest also another approach to the problem. It is shown in [3] that the limiting geometry of a RN black hole near this state coincides with the Bertotti-Robinson (BR) spacetime with a non-zero surface gravity. Starting from the BR spacetime right from the very beginning, we show that $S_q = 0$ in agreement with the result obtained by the limiting transition from the RN metric. Bearing in mind physical relevance of BR spacetimes as useful tool for discussing many aspects of the Hawking effect [14], this result can be of interest on its own.

Let us consider the Euclidean RN metric

$$ds^2 = f d\tau^2 + f^{-1} dr^2 + r^2 d\omega^2$$

$$d\omega^2 = d\phi^2 \sin^2 \theta + d\theta^2, \quad f = (1 - r_+/r)(1 - e^2/r_+r).$$

Here the Euclidean time $0 \leq \tau \leq \beta_0 = \beta_H$ where the inverse Hawking temperature

$$\beta_H = 4\pi r_+(1 - e^2/r_+^2)^{-1}$$

At finite temperature the one-loop action $I_q$ of quantum fields has the standard form [15].
\[ I_q = - \int d^4x \sqrt{g} T^0_0 - S_q \]  

which for the metric reduces to

\[ I_q = \beta_H M_q - S_q, \]  

\[ M_q = -4\pi \int_{r_+}^{r_B} dr r^2 T^0_0. \]  

It is assumed that the black hole is enclosed in the cavity of a radius \( r_B \). Properties of quantum fields in thermal equilibrium between a black hole and its Hawking radiation are described by the renormalized stress-energy tensor in the Hartle-Hawking state. In this state all diagonal components of \( T^\mu_\nu \) are regular at the horizon (off-diagonal components for our metric are equal to zero).

Now we will recover \( S_q \) from \( T^\mu_\nu \). Let us consider the variation of the metric between two equilibrium states with different \( r_+ \) but the same \( e \). One cannot use the standard formula

\[ \delta I_q = \frac{1}{2} \int d^4x \sqrt{g} T^\mu_\nu \delta g^\mu\nu \]  

directly, since when \( r_+ \) varies the limits of integration either in \( \tau \) or in \( r \) change themselves along with the metric. Therefore, it is convenient to introduce new variables which take their values between fixed limits. Let \( \tau = \tilde{\tau} \beta_H / 2\pi, \ y = (r - r_+)/(r_B - r_+). \) Then \( 0 \leq \tilde{\tau} \leq 2\pi, \ 0 \leq y \leq 1. \) Now we may use the above equation immediately. In so doing,

\[ \delta g^{yy} = (\frac{\delta f}{f} + \frac{2\delta r_+}{r_B - r_+}) g^{yy}, \ \delta g^{00} = -(\frac{\delta f}{f} + \frac{2\delta \beta_H}{\beta_H}) g^{00}, \ \delta g^{aa} = \frac{2(y - 1)}{r} g^{aa} \delta r_+ \]  

where \( a = \theta, \phi. \)

As \( f = f(r, r_+) = f(r_+, y, r_+) \) we have (the dependence on \( e \) is omitted for shortness) to take into account the full dependence of \( f \) on \( r_+ \) in calculating \( \delta f \). Then, substituting \( \delta g^\mu\nu \) into \( \delta I_q \) we obtain

\[ \frac{\delta I_q}{4\pi} = -\delta \beta_H \int_{r_+}^{r_B} dr r^2 T^0_0 + \delta r_+ \beta_H / 2 \int_{r_+}^{r_B} dr r^2 h(T^r_r - T^0_0) + \gamma \beta_H \delta r_+ \]  

where \( h = f^{-1} \frac{\partial f}{\partial r_+}, \gamma = \int_{r_+}^{r_B} dr r^2 \{ T^y_y (1 - y)[ f'(T^y_y - T^0_0)/2f - (T^\theta_\theta + T^\phi_\phi)/r] \}, \) prime denotes derivative with respect to \( r. \)
Now take into account that $T_y^y = T_r^r$ and make use the conservation law according to which

$$r^{-2}(T_r^r r^2)' = (T_0^y - T_r^y) f'/2f + r^{-1}(T_0^\theta + T_0^\phi)$$  \hspace{1cm} (8)

Then $\gamma = \int_{r_+}^{r_B} dr [T_r^r r^2 (y - 1)]' = T_r^r (r_+) r_+^2$ as $y = 1$ at $r = r_B$ and $y = 0$ at $r = r_+$. Thus, eventually we have

$$\delta I_q/4\pi = -\delta \beta H \int_{r_+}^{r_B} dr r^2 T_0^0 + \delta r + \beta H T_r^r (r_+) r_+^2 + \delta r_+ \beta H /2 \int_{r_+}^{r_B} dr r^2 h(T_r^r - T_0^0)$$ \hspace{1cm} (9)

For the Schwarzschild metric ($e = 0$) one can use the scale properties of it and after integration in 9 reproduce the result of [6] and [7]. It gives $S_q$ in terms of one-dimensional integrals from $T_\nu^\mu$. Although for the RN metric the expression for $S_q$ is more complicated, it is rather convenient for analysis.

Comparing [4] and [3] one can get

$$\frac{\partial}{\partial r_+} S_q = -4\pi \beta H g(r_+, r_B), \ g = g_1 + g_2, \ g_1 = \int_{r_+}^{r_B} dr r^2 \frac{\partial}{\partial r_+} T_0^0, \ g_2 = 1/2 \int_{r_+}^{r_B} dr r^2 h(T_r^r - T_0^0)$$ \hspace{1cm} (10)

Integrating [10] we can write down

$$S_q = 4\pi \int_{r_+}^{r_B} d\tilde{r}_+ \beta_H (\tilde{r}_+) g(\tilde{r}_+, r_B)$$ \hspace{1cm} (11)

The constant of integration is chosen to make sure $S_q \to 0$ when $r_+ \to r_B$ for a generic nonextreme hole: when there is no room for radiation its entropy must be zero.

The expression [11] for the entropy is a linear functional of $T_\mu^\nu$. In the Hartle-Hawking state $T_\mu^\nu = (T_\mu^\nu)_{th} + (T_\mu^\nu)_B$ where $(T_\mu^\nu)_B$ is the stress-energy tensor in the Boulware state, $(T_\mu^\nu)_{th} = \alpha T^4 diag(-1, 1/3, 1/3, 1/3)$ is that of thermal radiation [12]. Therefore, the entropy under discussion $S_q = S_{th} + S_B$ where $S_{th}$ is the entropy of thermal radiation (statistical-mechanical entropy of the gas of massless quanta) and $S_B$ is the renormalization constant. Both $S_{th}$ and $S_B$ are divergent but their sum is finite due to the finiteness of the $T_\mu^\nu$ in the Hartle-Hawking state. Such splitting explains the renormalization procedure for the thermal atmosphere of
a black hole developed in [13] and shows that the entropy under discussion harmonizes with the statistical-mechanical entropy after such a renormalization (cf. discussion in [8] where, however, the relevance of splitting properties of $T_{\mu}^{\nu}$ in the Hartle-Hawking state were not pointed out).

The limit we are operating with is just $r_+ = r_B$. However, the state the black hole approaches is now extreme ($m = e = r_+$). For such a state the Euclidean four-volume is finite [2], [3] and whether $S_q = 0$ or $S_q \neq 0$ is not obvious in advance. Indeed, we deal with the competition of two factors: the integration region shrinks ($r_+ \to r_B$) but $\beta_H(r_+) \to \infty$ (cf. the calculation of the proper distance between the horizon and boundary in [2]). It is the evaluation of the expression $11$ that we now turn to.

As far as $g_1$ is concerned, it is essential that the geometry of the spacetime is regular everywhere. More precisely, it takes the form of the Bertotti-Robinson metric [3] (see below). Correspondingly, $T_{\mu}^{\nu}$ approach their BR values. Therefore, the integrand in $g_1$ remains finite and $\lim_{r_+ \to r_B} g_1 = 0$. Consider the behavior of $g_2$. For the RN metric the quantity $h = -r^{-1}(1 - r_+/r)^{-1}(1 - e^2/r_+r)^{-1}(1 - e^2/r_+^2) \propto (1 - r_+/r)^{-1}$ when $r \to r_+$. The regularity condition at the horizon demands $T_0^0 = T_r^r$ at $r = r_+$. Therefore, $g_2 \to 0$ along with $g_1$. Thus, $\lim_{r_+ \to r_B} g(r_+, r_B) = 0$.

To show that $S_q = 0$ it is sufficient to prove that $\lim_{r_+ \to r_B} \int_{r_+}^{r_B} d\tilde{r}_+ \beta_H(\tilde{r}_+) \equiv C < \infty$. Direct calculation with $2$ taken into account shows that

$$C = 2\pi r_B^2 \lim \ln [(r_B - e)/(r_+ - e)]$$

(12)

Up to this point we only used the limiting relation $r_+ = m = e = r_B$ between parameters but did not consider in which way they approach this limit. Now we specify it by demand that our limit correspond to the finite local temperature on the boundary. Then, as was shown in [2], $e = pxr_+, x = r_+/r_B$ where the parameter $p = 1 - \varepsilon, x = 1 - \varepsilon \alpha, \varepsilon \to 0$ but $\alpha$ remains the finite nonzero quantity. Either $p$ or $\alpha$ can be expressed in terms of boundary data but their concrete values are irrelevant for our purposes. The only conclusion to be drawn from these properties is that either the numerator or the denominator in $12$ inside the logarithm
have the same order in $\varepsilon$, so their ratio is a finite number. Thus, $C < \infty$ and $S_q = 0$. It is worth noting that although $M_q \to 0$ according to 4, the product $M_q \beta H \propto (r_B - r_+) / (r_+ - e)$ remains finite for the same reason, so $I_q \neq 0$.

As follows from the method of derivation, the result $S_q = 0$ as the black hole approaches the extreme state ($T_0 \to 0$) in the non-extreme topological sector remains valid for the more general type of metrics:

$$ds^2 = f d\tau^2 + V^{-1} dr^2 + r^2 d\omega^2$$

What is needed is only the character of asymptotic behavior: $f \propto A(r - r_+)(r - r_-)$, $V \propto B(r - r_+)(r - r_-)$ near the horizon ($A$ and $B$ are constants, $r_-$ is the radius of an inner horizon), the limit under discussion corresponding to $r_+ = r_-$. As a matter of fact, the above proof relied on general properties of spacetime and regularity of $T_\mu^\nu$ near the horizon but did not use explicitly the concrete form of the limiting geometry. Its character was not obvious in advance since the coordinate $r$ becomes singular and $f \to 0$. It was shown in [3] that the metric turns in this limit into

$$ds^2 = r_+^2 (d\tau^2 \sinh^2 x + dx^2 + d\omega^2)$$

which is the version of the BR spacetime [16], [17] with the non-zero surface gravity. Here $0 \leq \tau \leq 2\pi$. Bearing in mind the physical significance of the issue of quantum corrections to the black hole geometry near the extreme state, below we present another proof of the property $S_q = 0$ starting from 14 right from the very beginning. As it relies on 14 directly independently of where it originates from, such a proof can be of interest for studying thermal properties of the BR spacetime itself.

The quantum stress-energy tensor of massless scalar fields regular at the horizon reads [18], [19]

$$T_\mu^\nu = r_+^{-1} A \delta_\mu^\nu$$

where $A$ is a pure number whose value is irrelevant for our purposes. Let us consider the finite portion of 14 with the boundary at $x = x_B$. Then the action is the function of two
independent variables $x_B$ and $r_+$. Now we can recover the action from the stress-energy tensor in the manner similar to that described above. As the metric 14 differs from 1, I sketch below the main points of calculations.

As explained above, one cannot use directly eq.5 since the integration region itself changes under variation of $x_B$. Let us introduce the variable $y$ according to $x = y x_B$, $0 \leq y \leq 1$. Then eq. 5 is applicable. Let $x_B$ be varied while keeping $r_+$ fixed. Then after simple transformations one gets

$$
\left( \frac{\partial I}{\partial x_B} \right)_{r_+} = -8 \pi^2 \sinh x_B
$$

(16)

Integrating this equation we obtain

$$
I = -8 \pi^2 (\cosh x_B - \xi)
$$

(17)

Here the quantity $\xi$ is in fact constant not depending on $r_+$ as it follows from dimension grounds. One should put $\xi = 1$ that the action obey the boundary condition $I(x_B = 0) = 0$: when there is no room for radiation its action must be zero. Then the action 17 can be rewritten in the form

$$
I_q = -r_+^{-4} \int d^4x \sqrt{g} A
$$

(18)

where $g$ is the determinant of the metric 14, $0 \leq x \leq x_B$, $0 \leq \tau \leq 2\pi$. One can easily check that the action 18 reproduces correctly conformal anomaly of the tensor 15. For this purpose it is necessary to take into account that this action depends on $r_+$ either via the metric or via the factor $r_+^{-4}$ as a parameter. Then the variation of the metric leads to the conformal anomaly term which is compensated by the variation of the $r_+^{-4}$ factor to give

$$
\frac{\partial I}{\partial r_+} = 0
$$

in agreement with eq.17.

Comparing eq.18 with the general form 3 and taking into account 15, we conclude that $S_q = 0$ in agreement with the above calculations for RN black holes. It is worth noting that although back reaction of quantum fields will certainly change the geometry 14 it will be the effect of the next order in the Planck constant for $S_q$, so at least in the one-loop approximation (quantum fields propagating on a classical background) $S_q = 0$. 

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The above result $S_q = 0$ admits qualitative physical explanation. Let us pass from (14) to its Lorentzian version by substitution $\tau = it$. Then introducing new variables according to

$$t_1 = r_+ e^t \coth x, \quad \rho = r_+ e^t (\sinh x)^{-1}$$

we obtain the metric in the form

$$ds^2 = \rho^{-2} r_+^2 (-dt_1^2 + d\rho^2 + \rho^2 d\omega^2)$$

The metric (14) is equivalent only to the part $t_1 > \rho$ of the metric (20) that resembles the relation between the Rindler space (analog of 14) and the Minkowski one (analog of 20). The event horizon of an accelerated observer in the flat spacetime represents the pure kinematic effect in the sense that by suitable coordinate transformation the metric turns into explicitly flat form and does not have an event horizon for an inertial observer at infinity in contrast to black hole spacetimes. For this reason, in spite of effects of interaction between an accelerated detector and Rindler quanta [21] the renormalized tensor $T^\nu_\mu = 0$ in the Hartle-Hawking state which corresponds in this case to the Minkowski vacuum of an inertial observer [20], so $S_q = 0$ for the thermodynamic entropy in the Rindler space. (To avoid confusion, I recall that I discuss only finite renormalized quantities which contribute to thermodynamics and do not consider in detail the mechanism of compensation between divergent zero-point vacuum oscillations in curved background and thermal excitations.)

It goes without saying that relationship between 14 and 20 is more complicated. In particular, the BR spacetime has three non-commuting Killing vectors, the metric 20 is geodesically incomplete itself [14], etc. However, one can think that qualitatively the reason for $S_q = 0$ is the same: the horizon in the BR spacetime 14 is not a true black hole horizon and does not reveal itself in $S_q$ which, being coordinate independent quantity, can be calculated in the reference frame 20 where the horizon at $\rho = \infty$ is ”too weak” ($T_H = 0$) to gain entropy of quantum fields.

One can look at the result $S_q$ from the more general viewpoint. Let we have a static spacetime $M_1$. Perform transformation from the original reference frame $(x^i, t)$ in which the
metric does not depend on time to the new one \((\tilde{x}^i, \tilde{t})\), both coordinate systems being non-degenerate. Let the metric in the new frame also do not depend on time, the coordinates \((\tilde{x}^i, \tilde{t})\) covering only the part \(M_2\) of \(M_1\). Let the spacetime \(M_1\) has the horizon for observers following orbits of the timelike Killing vector such that the associated Hawking temperature \(T_H = 0\) or does not have a horizon at all whereas there is a horizon in \(M_2\) with the corresponding Hawking temperature \(\tilde{T}_H \neq 0\). The relationship between two sections of the BR spacetime corresponds to the first case while the relationship between the Minkowski and Rindler spacetimes corresponds to the second one. It is plausible that the property \(S_q = 0\) reflects the kinematic nature of a horizon in \(M_2\) in contrast to the black hole case and follows directly from the general assumptions sketched above. It is of interest to elucidate whether this statement can be proven rigorously without the reference to an explicit form of the metric.

*Note added*. In the recent paper [22] R.B.Mann and S.N.Solodukhin considered quantum corrections to the entropy of a black hole itself and the mechanism of renormalization in the extreme limit in question and found that such corrections due to massless scalar fields have the universal behavior. In fact, the result \(S_q = 0\) for the entropy of Hawking radiation derived in our paper can also be considered as a manifestation of universality inherent to the extreme limit.

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[1] S. W. Hawking, G. T. Horowitz, and S.F. Ross, Phys.Rev. **D 51**, 4302 (1995).

[2] O. B. Zaslavskii, Phys.Rev.Lett. **76**, 2211 (1996).

[3] O. B. Zaslavskii, Phys. Rev. **D 56**, 2188 (1997).

[4] R. M. Wald, Phys. Rev. **D 56**, 6467 (1997).

[5] J. W. York, Jr., Phys. Rev. **D 33**, 2092 (1986).

[6] O. B. Zaslavskii, Phys. Lett. **181 A**, 105 (1993).
[7] O. B. Zaslavskii, Class. Quant. Grav. **13**, L23 (1996).

[8] V. P. Frolov, Phys. Rev. Lett. **74**, 3319 (1995).

[9] S. N. Solodukhin, Phys. Rev. **D 51**, 618 (1995).

[10] J. G. Demers, R. Lafrance, and R. C. Myers, Phys. Rev. **D 52**, 2245 (1995).

[11] G. Cognola, L. Vanzo, and S. Zerbini, Phys. Rev. **D 52**, 4548 (1995).

[12] S. M. Christensen and S. A. Fulling, Phys. Rev. **D15** (1977) 2088; P. Candelas, Phys. Rev. D 15 (1980) 2185; V. P. Frolov and K. S. Thorne, Phys. Rev. D **39** (1989) 2125.

[13] W. H. Zurek and K. S. Thorne, Phys. Rev. Lett. **54** (1985) 2171.

[14] A. S. Lapedes, Phys. Rev. **D 17**, 2556 (1978).

[15] J. D. Dowker and G. Kennedy, J. Phys. A: Math. Gen. **11**, 895 (1978).

[16] B. Bertotti, Phys. Rev. **116**, 1331 (1959).

[17] I. Robinson, Bull. Acad. Pol. Sci. **7**, 351 (1959).

[18] P. R. Anderson, W. A. Hiscock, and D. J. Loranz, Phys. Rev. Lett. **74**, 4365 (1995).

[19] L. A. Kofman and V. Sahni, Phys. Lett. **127 B**, 197 (1983).

[20] V. L. Ginzburg and V. P. Frolov, In: Quantum theory and gravitation (Proc. P. N. Lebedev Phys. Inst. Vol. 197)/Ed. M. A. Markov. - M.: Nauka, 1989.

[21] N. D. Birrel and P. C. W. Davies. *Quantum Fields in Curved Space* (Cambridge University press, Cambridge, England, 1982).

[22] R. B. Mann and S. N. Solodukhin, "Universality of Quantum Entropy for Extreme Black Holes", preprint [hep-th/9709064](http://arxiv.org/abs/hep-th/9709064).