Improvements on lower bounds for the blow-up time under local nonlinear Neumann conditions

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Abstract

This paper studies the heat equation \( u_t = \Delta u \) in a bounded domain \( \Omega \subset \mathbb{R}^n (n \geq 2) \) with positive initial data and a local nonlinear Neumann boundary condition: the normal derivative \( \partial u / \partial n = u^q \) on partial boundary \( \Gamma_1 \subseteq \partial \Omega \) for some \( q > 1 \), while \( \partial u / \partial n = 0 \) on the other part. We investigate the lower bound of the blow-up time \( T^* \) of \( u \) in several aspects. First, \( T^* \) is proved to be at least of order \((q - 1)^{-1}\) as \( q \to 1^+ \). Since the existing upper bound is of order \((q - 1)^{-1}\), this result is sharp. Secondly, if \( \Omega \) is convex and \(|\Gamma_1|\) denotes the surface area of \( \Gamma_1 \), then \( T^* \) is shown to be at least of order \(|\Gamma_1|^{-\frac{n-1}{n-1}}\) for \( n \geq 3 \) and \(|\Gamma_1|^{-1}/\ln(|\Gamma_1|^{-1})\) for \( n = 2 \) as \(|\Gamma_1| \to 0\), while the previous result is \(|\Gamma_1|^{-\alpha}\) for any \( \alpha < \frac{n-1}{n-1} \). Finally, we generalize the results for convex domains to the domains with only local convexity near \( \Gamma_1 \).

1 Introduction

1.1 Problem and notations

In this paper, unless otherwise stated, \( \Omega \) represents a bounded open subset in \( \mathbb{R}^n (n \geq 2) \) with \( C^2 \) boundary \( \partial \Omega \). \( \Gamma_1 \) and \( \Gamma_2 \) denote two disjoint relatively open subsets of \( \partial \Omega \). \( \partial \Gamma_1 = \partial \Gamma_2 = \bar{\Gamma} \) is a common \( C^1 \) boundary of \( \Gamma_1 \) and \( \Gamma_2 \). Moreover, \( \Gamma_1 \neq \emptyset \) and \( \partial \Omega = \Gamma_1 \cup \bar{\Gamma} \cup \Gamma_2 \). We study the following problem:

\[
\begin{cases}
  u_t(x,t) = \Delta u(x,t) & \text{in } \Omega \times (0,T], \\
  \frac{\partial u(x,t)}{\partial n(x,z)} = u^q(x,t) & \text{on } \Gamma_1 \times (0,T], \\
  \frac{\partial u(x,t)}{\partial n(x,z)} = 0 & \text{on } \Gamma_2 \times (0,T], \\
  u(x,0) = u_0(x) & \text{in } \Omega,
\end{cases}
\]

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where
\[ q > 1, \quad u_0 \in C^1(\Omega), \quad u_0(x) \geq 0, \quad u_0(x) \neq 0. \] (1.2)

The normal derivative in (1.1) is understood in the following way: for any \((x, t) \in \partial \Omega \times (0, T)\),
\[
\frac{\partial u(x, t)}{\partial n(x)} \triangleq \lim_{h \to 0^+} (Du)(x_h, t) \cdot \vec{n}(x),
\] (1.3)

where \(Du\) denotes the spatial derivative of \(u\), \(\vec{n}(x)\) denotes the exterior unit normal vector at \(x\) and \(x_h \triangleq x - h \vec{n}(x)\) for \(x \in \partial \Omega\). Since \(\partial \Omega\) is \(C^2\), \(x_h\) belongs to \(\Omega\) when \(h\) is positive and sufficiently small.

Throughout this paper, we write
\[ M_0 = \max_{x \in \Omega} u_0(x) \] (1.4)

and denote \(M(t)\) to be the supremum of the solution \(u\) to (1.1) on \(\Omega \times [0, t]\):
\[ M(t) = \sup_{(x, \tau) \in \Omega \times [0, t]} u(x, \tau). \] (1.5)

\(|\Gamma_1|\) represents the surface area of \(\Gamma_1\), that is
\[ |\Gamma_1| = \int_{\Gamma_1} dS(x), \]

where \(dS(x)\) means the surface integral with respect to the variable \(x\). \(\Phi\) refers to the fundamental solution to the heat equation:
\[ \Phi(x, t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right), \quad \forall (x, t) \in \mathbb{R}^n \times (0, \infty). \] (1.6)

In addition, \(C = C(a, b \ldots)\) and \(C_i = C_i(a, b \ldots)\) represent positive constants which only depend on the parameters \(a, b \ldots\). One should note that \(C\) and \(C_i\) may stand for different constants from line to line. However, \(C^* = C^*(a, b \ldots)\) and \(C_i^* = C_i^*(a, b \ldots)\) will represent the constants which are fixed.

When \(\Gamma_1 = \partial \Omega\), the problem (1.1) and more general parabolic equations with Neumann boundary conditions have been studied quite a lot. In addition, the Cauchy problems and the Dirichlet boundary value problems related to the nonlinear blow-up phenomenon of the parabolic type were also investigated. We refer the readers to the surveys \([5, 14]\) and the books \([6, 9, 23]\). The topics include the local and global existence and uniqueness of the solutions \([1-4, 11, 17, 25, 27]\); nonexistence of global solutions and the upper bound estimates for the blow-up time \([10, 13, 15, 17, 18, 24, 25, 27]\); lower bound estimates for the blow-up time \([16, 19, 22, 27, 28]\); blow-up sets, blow-up rate and the asymptotic behaviour of the solutions near the blow-up time \([7, 8, 10, 11, 17, 18, 24, 26]\).

For the research on the bounds of the blow-up time, the upper bound is usually related to the nonexistence of the global solutions and various methods have been developed. Meanwhile, the lower bound was not studied as much in the past but was paid more attention in recent years. However, the lower bound can be argued to be more useful in practice, since it provides an estimate of the safe time. As an instance, for the problem (1.1), which was proposed in [27] to describe the re-entry process to the atmosphere of the Columbia Space Shuttle, the lower bound of the blow-up time would provide a safe time of the landing of the shuttle. In contrast to the upper bound case, not many methods have been explored to deal with the lower bound. In addition, when \(\Gamma_1\) is a proper subset of \(\partial \Omega\), to the authors’ knowledge, only two papers \([27, 28]\) investigated the relation between the lower bound
of the blow-up time and the surface area $|\Gamma_1|$. The purpose of this work is to further improve the lower bound estimate of the blow-up time in terms of $|\Gamma_1|$, especially when $|\Gamma_1| \to 0$. Before presenting the main results of this paper, let us review what has been known.

The recent paper [27] studied systematically. According to it (see Theorem 1.3 in [27]), $u$ has a unique classical solution on its time interval $(0, T^*)$ and is expected to be small. As discussed in [28], by sending $q \to 1^+$ or $M_0 \to 0^+$, the order of the lower bound in (1.9) is of order $(q-1)^{-1}$ or $M_0^{-(q-1)}$, both of which are optimal.

In [28], by assuming $\Omega$ is convex, it obtained a lower bound of polynomial order $|\Gamma_1|^{-\alpha}$ for any $\alpha < \frac{1}{n-1}$. More precisely, for any $\alpha \in [0, \frac{1}{n-1})$, there exists $C = C(n, \Omega, \alpha)$ such that

$$T^* \geq \frac{C}{(q-1)M_0^{q-1}|\Gamma_1|^{\alpha}} \left( \min \left\{ 1, \frac{1}{qM_0^{q-1}|\Gamma_1|^{\alpha}} \right\} \right)^{\frac{1+\alpha}{1-(n-1)\alpha}}. \quad (1.9)$$

In addition to the relation between $T^*$ and $|\Gamma_1|$, (1.9) also provides sharp dependence of $T^*$ on $q$ and $M_0$. As discussed in [28], by sending $q \to 1^+$ or $M_0 \to 0^+$, the order of the lower bound in (1.9) is $(q-1)^{-1}$ or $M_0^{-(q-1)}$, both of which are optimal.

Based on the idea in [28], this paper will provide a unified method to enhance the lower bound of $T^*$ in several aspects (especially the asymptotic behaviour of $T^*$ as $|\Gamma_1| \to 0^+$) according to the geometric assumptions on $\Omega$.

### 1.2 Main results

Noticing that the lower bound in (1.8) is negative unless $|\Gamma_1|$ or $M_0$ is sufficiently small or $q$ is sufficiently close to 1, so it is desirable to derive a lower bound which is always positive. The first result below fulfills this expectation. Moreover, it obtains better asymptotic behavior of the lower bound when $|\Gamma_1| \to 0^+$ or $q \to 1^+$.

**Theorem 1.1.** Assume (1.2). Let $T^*$ be the maximal existence time for (1.1). Then there exists a
constant $C = C(n, \Omega)$ such that

$$T^* \geq \frac{C}{q-1} \ln \left( 1 + (2M_0)^{-4(q-1)} |\Gamma_1|^{-\frac{2}{q-1}} \right),$$

(1.10)

where $M_0$ is given by (1.4).

Let us compare (1.10) with (1.8) in more detail on the asymptotic behavior.

• As $|\Gamma_1| \to 0^+$, the order of (1.10) is $\ln |\Gamma_1|^{-1}$ while the order of (1.8) is only $\left( \ln |\Gamma_1|^{-1} \right)^{\frac{1}{q-1}}$.

• As $q \to 1^+$, the order of (1.10) is $(q-1)^{-1}$, which is optimal since the order of the upper bound (1.7) is also $(q-1)^{-1}$. However, the order of (1.8) is only $(\ln \frac{1}{q-1})^{\frac{1}{q-1}}$.

When the domain $\Omega$ is convex, for any $\alpha < \frac{1}{n-1}$, [28] derives the lower bound (1.9) which is of order $|\Gamma_1|^{-\alpha}$ as $|\Gamma_1| \to 0^+$. The next result in this paper improves the order to be $|\Gamma_1|^{-1/(n-1)}$ for $n \geq 3$ and $(|\Gamma_1| \ln \frac{1}{|\Gamma_1|})^{-1}$ for $n = 2$ as $|\Gamma_1| \to 0^+$.

**Theorem 1.2.** Assume (1.2). Let $T^*$ be the maximal existence time for (1.1) and $M_0$ be defined as in (1.4). Assume $\Omega$ is convex. Then there exist constants $Y_0 = Y_0(n, \Omega)$ and $C = C(n, \Omega)$ such that the following statements hold.

• **Case 1:** $n \geq 3$. Denote

$$Y = M_0^{q-1} |\Gamma_1|^{\frac{1}{q-1}}.$$

If $Y \leq Y_0/q$, then

$$T^* \geq \frac{C}{(q-1)Y}.$$  

(1.11)

• **Case 2:** $n = 2$. Denote

$$Y = M_0^{q-1} |\Gamma_1| \ln \left( \frac{1}{|\Gamma_1|} + 1 \right).$$

If $Y \leq Y_0/q$, then

$$T^* \geq \frac{C}{(q-1)Y}.$$  

(1.12)

In some practical situations, the convexity of domain $\Omega$ is not expected. However, the local convexity near $\Gamma_1$ is usually reasonable. Taking the model in [27] as an example again, since $\Gamma_1$ is on the left wing of the shuttle, the region near $\Gamma_1$ is indeed convex although the whole shuttle is not. Thus it is desirable to generalize Theorem 1.2 to the domains with only local convexity near $\Gamma_1$. The third result realizes this goal. Before the statement of the third result, let us explain the meaning of the local convexity near $\Gamma_1$.

**Definition 1.3 (Local convexity near partial boundary).** Let $\Omega$ be a bounded open subset in $\mathbb{R}^n$ and $\Gamma \subseteq \partial \Omega$. We say $\Omega$ is locally convex near $\Gamma$ if there exists $d > 0$ such that $Conv([\Gamma]_d) \subseteq \overline{\Omega}$, where

$$[\Gamma]_d \triangleq \{ x \in \partial \Omega : \text{dist}(x, \Gamma) < d \}$$

denotes the boundary part whose distance to $\Gamma$ is within $d$ and $Conv([\Gamma]_d)$ means the convex hull of $[\Gamma]_d$.

Based on this definition, the local convexity near $\Gamma_1$ in this paper means $Conv([\Gamma_1]_d) \subseteq \overline{\Omega}$ for some $d > 0$. 

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Theorem 1.4. Assume (1.2). Let $T^*$ be the maximal existence time for (1.1) and $M_0$ be defined as in (1.4). Assume $\text{Conv}([\Gamma_1]_d) \subseteq \overline{\Omega}$ for some $d > 0$. Then there exist constants $Y_0 = Y_0(n, \Omega, d)$ and $C = C(n, \Omega, d)$ such that the following statements hold.

• Case 1: $n \geq 3$. Denote $Y = M_0^{q-1}|\Gamma_1|^{\frac{1}{n-1}}$. If $Y \leq Y_0/q$, then
  $$T^* \geq \frac{C}{(q-1)Y|\ln Y|}. \quad (1.14)$$

• Case 2: $n = 2$. Denote $Y = M_0^{q-1}|\Gamma_1||\Gamma_1|^{\ln(\frac{1}{|\Gamma_1|} + 1)}$. If $Y \leq Y_0/q$, then
  $$T^* \geq \frac{C}{(q-1)Y|\ln Y|}. \quad (1.15)$$

To compare Theorem 1.4 with Theorem 1.2, the estimates in Theorem 1.4 are almost identical to those in Theorem 1.2 except an extra term $|\ln Y|$ in the denominator. If we look at the proofs, this extra term is due to the lack of the global convexity of $\Omega$. The outlines of the proofs for Theorem 1.2 and Theorem 1.4 are very similar, but the computations in the latter one will be much more complicated due to the lack of the global convexity again.

1.3 Outline of the approach

Although this paper deals with domains with three different geometrical assumptions, the methods share many similarities and follow the same outline. Let $M(t)$ be the same as in (1.5). The basic idea is to chop the range of $M(t)$ into small pieces $[M_{k-1}, M_k]$ ($k \geq 1$) and derive a lower bound $t_{k*}$ for $t_k$, the time that $M(t)$ increases from $M_{k-1}$ to $M_k$. Suppose such lower bound $t_{k*}$ can be found for $L$ steps ($L$ may be finite or infinite), then $\sum_{k=1}^{L} t_{k*}$ becomes a lower bound for $T^*$. The analysis will be based on the representation formula (3.1).

The common part of the proofs for Theorems 1.1, 1.2 and 1.4 is the second paragraph in the proof of Theorem 1.1 in Section 3. After the equation (3.10), the proofs will be slightly different due to the geometric properties of the domains. For convenience, we write down the equation (3.10) as below.

$$M_k \leq \left[1 + 4(I_1 + I_2)\right]M_{k-1} + 4I_3M_k^q,$$

where $I_1$, $I_2$ and $I_3$ are defined as in (3.7). For the estimates on $I_1 + I_2$ and $I_3$, we will argue in different ways under the following three cases.

1. For a general domain $\Omega$, Lemma 3.1 implies $I_1 + I_2 \leq C\sqrt{t_k}$ for some constant $C = C(n, \Omega)$ and we will use (2.7) to bound $I_3$.

2. For any convex domain $\Omega$, the identities (2.1) and (2.2) yield $I_1 + I_2 = 0$ and we will apply Lemma 2.7 and Lemma 2.10 to bound $I_3$.

3. For any domain $\Omega$ that is locally convex near $\Gamma_1$, that is $\text{Conv}([\Gamma_1]_d) \subseteq \overline{\Omega}$ for some $d > 0$, the identity (2.1) and Corollary 2.2 lead to

$$I_1 + I_2 \leq C_t k \exp \left(- \frac{d^2}{8t_k}\right)$$
for some constant $C = C(n, \Omega, d)$. On the other hand, we will exploit Lemma 2.7 and Lemma 2.10 again to bound $I_3$.

Several remarks will be made in sequel.

- First, since small $t_k$ is of interest, the bound for $I_1 + I_2$ in Case (3) is exponential decay as $t_k \to 0$. Due to this fast decay, the result in Case (3) is very close to that in Case (2). In addition, either the result in Case (2) or Case (3) is far better than that in Case (1) where the estimate on $I_1 + I_2$ only decays like $\sqrt{t_k}$.

- Secondly, (2.7) implies that

$$I_3 \leq \frac{C}{1 - (n-1)\alpha} |\Gamma_1|^{\frac{1-(n-1)\alpha}{\alpha}} t_k^{\frac{1-(n-1)\alpha}{\alpha}}$$

for some constant $C = C(n, \Omega)$ and for any $\alpha \in [0, \frac{1}{n-1})$. Lemma 2.7 and Lemma 2.10 push the power of $|\Gamma_1|$ a little bit further. More precisely, Lemma 2.7 implies

$$I_3 \leq C|\Gamma_1|^{\frac{1}{2n-2r}}$$

when $n \geq 3$ and Lemma 2.10 yields

$$I_3 \leq C|\Gamma_1| \ln \left( \frac{1}{|\Gamma_1|} + 1 \right)$$

when $n = 2$.

- Thirdly, for general domains, our method will not gain better lower bound for $T^*$ (regarding the order of $|\Gamma_1|^{-1}$) by increasing the power of $|\Gamma_1|$ in the estimate of $I_3$, so we just choose $\alpha = \frac{1}{2(n-1)}$ in (2.7) instead of exploiting Lemma 2.7 and Lemma 2.10.

- Finally, for convex domains or the domains with local convexity near $\Gamma_1$, the power of $|\Gamma_1|$ in the bound of $I_3$ makes a difference in the final lower bound estimate of $T^*$ (regarding the order of $|\Gamma_1|^{-1}$), so we apply Lemma 2.7 and Lemma 2.10 instead of (2.7).

### 1.4 Organization

The organization of this paper is as follows. Section 2 presents some preliminary results which will be used later. Section 3 verifies Theorem 1.1 for general domain $\Omega$. Section 4 provides the proof for Theorem 1.2 when the domain $\Omega$ is convex. Section 5 justifies Theorem 1.4 for the domain $\Omega$ that is locally convex near $\Gamma_1$.

### 2 Auxiliary lemmas

#### 2.1 One identity and its related results

In [28], it mentioned an elementary identity (see Lemma 2.2 in [28]) about the heat kernel, namely for any $x \in \partial \Omega$ and $t > 0$,

$$\int_{\Omega} \Phi(x - y, t) dy - \int_0^t \int_{\partial \Omega} \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} dS(y) d\tau = \frac{1}{2}, \quad \forall x \in \partial \Omega, t > 0, \quad (2.1)$$
where 
\[
\frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} \triangleq D_y \Phi(x - y, t - \tau) \cdot \vec{n}(y)
\]
is the normal derivative. In addition, if \(\Omega\) is convex, then
\[
\int_{\Omega} \Phi(x - y, t) \, dy + \int_0^t \int_{\partial \Omega} \left| \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} \right| \, dS(y) \, d\tau = \frac{1}{2}, \quad \forall x \in \partial \Omega, \ t > 0. \quad (2.2)
\]

This subsection will derive an intermediate result, Corollary 2.2, when the convexity is only assumed near \(\Gamma_1\) rather than in the whole domain. Before presenting Corollary 2.2, we first show an auxiliary lemma.

**Lemma 2.1.** Let \(\Omega\) and \(\Gamma_1\) be the same as in (1.1). Then for any \(d > 0\), there exists \(C = C(n, \Omega, d)\) such that for any \(x \in \overline{\Gamma}_1\) and \(t > 0\),
\[
\int_0^t \int_{\partial \Omega \setminus \Gamma_1} \left| \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} \right| \, dS(y) \, d\tau \leq C t \exp \left( -\frac{d^2}{8t} \right). \quad (2.3)
\]

**Proof.** In this proof, \(C\) denotes a constant which depends only on \(n, \Omega\) and \(d\). By a change of variable in \(\tau\) and the definition of \(\Phi\),
\[
\int_0^t \int_{\partial \Omega \setminus \Gamma_1} \left| \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} \right| \, dS(y) \, d\tau = \int_0^t \int_{\partial \Omega \setminus \Gamma_1} \left| \frac{\partial \Phi(x - y, \tau)}{\partial n(y)} \right| \, dS(y) \, d\tau \leq C \int_0^t \int_{\partial \Omega \setminus \Gamma_1} \frac{|(x - y) \cdot \vec{n}(y)|}{\tau^{\frac{n}{2} + 1}} \exp \left( -\frac{|x - y|^2}{4\tau} \right) \, dS(y) \, d\tau. \quad (2.4)
\]
Since \(\partial \Omega\) is assumed to be \(C^2\), then \(|(x - y) \cdot \vec{n}(y)| \leq C|x - y|^2\). In addition, \(|x - y| \geq d\) for any \(x \in \overline{\Gamma}_1\) and \(y \in \partial \Omega \setminus \Gamma_1\). As a result,
\[
\frac{|(x - y) \cdot \vec{n}(y)|}{\tau^{\frac{n}{2} + 1}} \exp \left( -\frac{|x - y|^2}{4\tau} \right) \leq C|x - y|^{-n} \left( \frac{|x - y|^2}{\tau} \right)^{1 + \frac{n}{2}} \exp \left( -\frac{|x - y|^2}{4\tau} \right) \leq C|x - y|^{-n} \exp \left( -\frac{|x - y|^2}{8\tau} \right) \leq C d^{-n} \exp \left( -\frac{d^2}{8\tau} \right). \quad (2.5)
\]
Plugging (2.5) into (2.4),
\[
\int_0^t \int_{\partial \Omega \setminus \Gamma_1} \left| \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} \right| \, dS(y) \, d\tau \leq C d^{-n} \int_0^t \int_{\partial \Omega \setminus \Gamma_1} \exp \left( -\frac{d^2}{8\tau} \right) \, dS(y) \, d\tau \leq C d^{-n} |\partial \Omega| \int_0^t \exp \left( -\frac{d^2}{8\tau} \right) \, d\tau \leq C d^{-n} |\partial \Omega| t \exp \left( -\frac{d^2}{8\tau} \right).
\]
By exploiting Lemma 2.1, the following (2.6) is a variant of the identity (2.2), and it will play the same role in the proof of Theorem 1.4 as (2.2) will do in the proof of Theorem 1.2.

**Corollary 2.2.** Let $\Omega$ and $\Gamma_1$ be the same as in (1.3). Assume there exists $d > 0$ such that $\text{Conv}(\Gamma_1^d) \subseteq \overline{\Omega}$. Then there exists $C = C(n, \Omega, d)$ such that for any $x \in \Gamma_1$ and $t > 0$,

$$
\int_\Omega \Phi(x - y, t) \, dy + \int_0^t \int_{\partial \Omega} \left| \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} \right| \, dS(y) \, d\tau \leq \frac{1}{2} + C \, t \exp \left( - \frac{d^2}{8t} \right).  \tag{2.6}
$$

**Proof.** Since $x \in \Gamma_1$ and $\text{Conv}(\Gamma_1^d) \subseteq \overline{\Omega}$, we have

$$
\frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} = \frac{C(x - y) \cdot \tilde{r}(y)}{(t - \tau)^{n/2 + 1}} \exp \left( - \frac{|x - y|^2}{4(t - \tau)} \right) \leq 0, \quad \forall y \in [\Gamma_1]^d.
$$

As a result,

$$
\begin{align*}
\int_0^t \int_{\partial \Omega} \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} \, dS(y) \, d\tau &+ \int_0^t \int_{\partial \Omega} \left| \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} \right| \, dS(y) \, d\tau \\
&= \int_0^t \int_{\partial \Omega \setminus \Gamma_1^d} \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} \, dS(y) \, d\tau + \int_0^t \int_{\partial \Omega \setminus \Gamma_1^d} \left| \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} \right| \, dS(y) \, d\tau \\
&\leq C \, t \exp \left( - \frac{d^2}{8t} \right),
\end{align*}
$$

where the last inequality is due to Lemma 2.1. Therefore

$$
\begin{align*}
\int_\Omega \Phi(x - y, t) \, dy &+ \int_0^t \int_{\partial \Omega} \left| \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} \right| \, dS(y) \, d\tau \\
&\leq \int_\Omega \Phi(x - y, t) \, dy - \int_0^t \int_{\partial \Omega} \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} \, dS(y) \, d\tau + C \, t \exp \left( - \frac{d^2}{8t} \right) \\
&= \frac{1}{2} + C \, t \exp \left( - \frac{d^2}{8t} \right),
\end{align*}
$$

where the last equality is because of (2.1). \hfill \Box

### 2.2 Estimate for the boundary-time integral of the heat kernel

The estimate for the boundary-time integral of the heat kernel is a basic tool in the derivation of the lower bound in (1.9). More precisely (see Lemma 2.3 in [28]), there exists $C = C(n, \Omega)$ such that for any $\Gamma \subseteq \partial \Omega$, $\alpha \in \left[ 0, \frac{1}{n-1} \right)$, $x \in \partial \Omega$ and $t > 0$,

$$
\int_\Gamma \Phi(x - y, t - \tau) \, dS(y) \, d\tau \leq \frac{C}{1 - (n-1)\alpha} \, |\Gamma|^\alpha \, t^{\frac{1-(n-1)\alpha}{n-1}}. \tag{2.7}
$$

According to the method in [28], the power $\alpha$ in (2.7) determines the power on $|\Gamma_1|^{-1}$ of the lower bound for $T^*$ in (1.9). However, the range of the power $\alpha$ in (2.7) missed $\frac{1}{n-1}$ since the coefficient will blow up as $\alpha \nearrow \frac{1}{n-1}$. So it is natural to ask whether $\alpha$ can be taken as $\frac{1}{n-1}$ by other methods. In this subsection, the above expectation will be justified for $n \geq 3$ in Lemma 2.7 and for $n = 2$ (with an extra log term and bounded time $t$) in Lemma 2.10.

We first introduce a simple fact which can be regarded as a rearrangement result.
Lemma 2.3. Let \( n \geq 1 \) and \( f : (0, \infty) \to [0, \infty) \) be a decreasing function. Then for any bounded subset \( U \) of \( \mathbb{R}^n \) and for any \( x \in \mathbb{R}^n \),

\[
\int_U f(|x - y|) \, dy \leq \int_{B_R(0)} f(|z|) \, dz
\]

where \( R \) satisfies \(|B_R(0)| = |U|\) (namely the volume of \( B_R(0) \) equals the volume of \( U \)).

Proof. Define \( U_1 = U - \{x\} \).

Then by a change of variable \( z = y - x \),

\[
\int_U f(|x - y|) \, dy = \int_{U_1} f(|z|) \, dz
\]

\[
= \int_{U_1 \cap B_R(0)} f(|z|) \, dz + \int_{U_1 \setminus B_R(0)} f(|z|) \, dz
\]

\[\triangleq J_1 + J_2,\]

(2.9)

Since \( f \) is decreasing,

\[J_2 \leq f(R)|U_1 \setminus B_R(0)|.\]

Due to the definition of \( R \), \(|B_R(0)| = |U| = |U_1|\). So we have \(|B_R(0) \setminus U_1| = |U_1 \setminus B_R(0)|\). As a result,

\[J_2 \leq f(R)|B_R(0) \setminus U_1| \leq \int_{B_R(0) \setminus U_1} f(|z|) \, dz,\]

(2.10)

where the last inequality is again due to the decay of \( f \). Combining (2.9) and (2.10), we finish the proof.

Definition 2.4. Let \( \Omega \) be a bounded, open subset of \( \mathbb{R}^n \) with \( C^1 \) boundary. Let \( \Gamma \) be a subset of \( \partial \Omega \). We say \( \Gamma \) is given by a graph if (upon relabelling and reorienting the coordinates axes) there exists a bounded subset \( U \subseteq \mathbb{R}^{n-1} \) and a \( C^1 \) function \( \phi : \mathbb{R}^{n-1} \to \mathbb{R} \) such that

\[\Gamma = \{(\tilde{y}, \phi(\tilde{y})) : \tilde{y} \in U\}.\]

In the following, for any \( x \in \mathbb{R}^n \), we will decompose it to be \( x = (\tilde{x}, x_n) \), where \( \tilde{x} \) denotes the first \( n - 1 \) components of \( x \).

Lemma 2.5. Let \( \Omega \) be a bounded, open subset of \( \mathbb{R}^n (n \geq 3) \) with \( C^1 \) boundary. Let \( \Gamma \) be a subset of \( \partial \Omega \) that is given by a graph as in Definition 2.4. Then there exists a constant \( C = C(n, ||\nabla \phi||_{L^\infty(U)}) \), where \( \phi \) and \( U \) are the same as in Definition 2.4, such that for any \( x \in \mathbb{R}^n \),

\[
\int_{\Gamma} \frac{1}{|x - y|^{n-2}} \, dS(y) \leq C|\Gamma|^{1/(n-1)}.
\]

Proof. By Definition 2.4, without loss of generality, we can assume there exists a \( C^1 \) function \( \phi : \mathbb{R}^{n-1} \to \mathbb{R} \) and a bounded subset \( U \) of \( \mathbb{R}^{n-1} \) such that

\[\Gamma = \{(\tilde{y}, \phi(\tilde{y})) : \tilde{y} \in U\}.\]

(2.11)
Thus,

\[
\int_{\Gamma} \frac{1}{|x - y|^{n-2}} dS(y) = \int_{U} \frac{\sqrt{1 + |\nabla \phi(y)|^2}}{|(x, x_n) - (y, \phi(y))|^{n-2}} d\tilde{y} \\
\leq \int_{U} \frac{\sqrt{1 + |\nabla \phi(y)|^2}}{|x - \tilde{y}|^{n-2}} d\tilde{y} \\
\leq C \int_{U} \frac{1}{|x - \tilde{y}|^{n-2}} d\tilde{y}.
\]

Define

\[
f(r) = \frac{1}{r^{n-2}}, \quad \forall \ r > 0.
\]

Then it follows from Lemma 2.3 that

\[
\int_{U} \frac{1}{|x - y|^{n-2}} d\tilde{y} \leq \int_{B_R(0)} f(|\tilde{z}|) d\tilde{z} = CR = C|U|^{1/(n-1)}.
\]

Again by the parametrization (2.11), it is readily seen that $|U| \leq |\Gamma|$. Hence,

\[
\int_{\Gamma} \frac{1}{|x - y|^{n-2}} dS(y) \leq C|U|^{1/(n-1)} \leq C|\Gamma|^{1/(n-1)}.
\]

**Corollary 2.6.** Let $\Omega$ be a bounded open subset of $\mathbb{R}^n (n \geq 3)$ with $C^1$ boundary. Let $\Gamma$ be any subset of $\partial \Omega$. Then there exists a constant $C = C(n, \Omega)$ such that for any $x \in \mathbb{R}^n$,

\[
\int_{\Gamma} \frac{1}{|x - y|^{n-2}} dS(y) \leq C|\Gamma|^{1/(n-1)}.
\]

**Proof.** Since $\partial \Omega$ is $C^1$, for any point $x_0 \in \partial \Omega$, the boundary part of $\Omega$ near $x_0$ is given by a graph as in Definition 2.4. Therefore we can split $\partial \Omega$ into finite pieces:

\[
\partial \Omega = \bigcup_{i=1}^{K} A_i, \tag{2.12}
\]

where each $A_i (1 \leq i \leq K)$ is given by the graph of some $C^1$ function $\phi_i$ on some bounded set $U_i \subseteq \mathbb{R}^{n-1}$. The number of total pieces $K$ and $||\nabla \phi_i||_{L^\infty(U_i)}$ only depend on $\Omega$.

For any $1 \leq i \leq K$, $\Gamma \cap A_i$ is also a boundary part given by a graph. Therefore by Lemma 2.5 there exists a constant $C = C(n, \Omega)$ such that for any $1 \leq i \leq K$,

\[
\int_{\Gamma \cap A_i} \frac{1}{|x - y|^{n-2}} dS(y) \leq C|\Gamma \cap A_i|^{1/(n-1)}.
\]
Hence,
\[
\int_{\Gamma} \frac{1}{|x - y|^{n-2}} dS(y) \leq \sum_{i=1}^{K} \int_{\Gamma \cap A_i} \frac{1}{|x - y|^{n-2}} dS(y) \leq C \sum_{i=1}^{K} |\Gamma \cap A_i|^{1/(n-1)} \leq CK|\Gamma|^{1/(n-1)} = C|\Gamma|^{1/(n-1)}.
\]

Lemma 2.5 and Corollary 2.6 will be applied to show the desired Lemma 2.7 which pushes the power \(\alpha\) in (2.7) to \(\frac{1}{n-1}\) when \(n \geq 3\).

**Lemma 2.7.** Let \(\Omega\) be a bounded open subset of \(\mathbb{R}^n (n \geq 3)\) with \(C^1\) boundary. Let \(\Gamma\) be any subset of \(\partial \Omega\). Then there exists \(C = C(n, \Omega)\) such that for any \(x \in \mathbb{R}^n\) and \(t \geq 0\),
\[
\int_0^t \int_{\Gamma} \Phi(x - y, t - \tau) dS(y) d\tau \leq C|\Gamma|^{1/(n-1)}.
\]

**Proof.** In this proof, unless otherwise stated, \(C\) represents constants which only depend on \(n\) and \(\Omega\). First, by the explicit formula (1.6) of \(\Phi\) and a change of variable in \(\tau\), we have
\[
\int_0^t \int_{\Gamma} \Phi(x - y, t - \tau) dS(y) d\tau = C \int_{\Gamma} \int_0^t \tau^{-n/2} e^{-|x - y|^2/(4\tau)} d\tau dS(y).
\]
Then by the change of variable \(s = |x - y|^2/(4\tau)\) for \(\tau\),
\[
\int_{\Gamma} \int_0^t \tau^{-n/2} e^{-|x - y|^2/(4\tau)} d\tau dS(y) \leq C \int_{\Gamma} \frac{1}{|x - y|^{n-2}} \int_{|x - y|^2/(4\tau)}^{\infty} s^{\frac{n}{2}-2} e^{-s} ds dS(y).
\]
Since \(n \geq 3\), \(s^{\frac{n}{2}-2} e^{-s}\) is integrable on \((0, \infty)\). As a result,
\[
\int_{\Gamma} \frac{1}{|x - y|^{n-2}} \int_{|x - y|^2/(4\tau)}^{\infty} s^{\frac{n}{2}-2} e^{-s} ds dS(y) \leq \int_{\Gamma} \frac{1}{|x - y|^{n-2}} \int_0^{\infty} s^{\frac{n}{2}-2} e^{-s} ds dS(y) = C \int_{\Gamma} \frac{1}{|x - y|^{n-2}} dS(y).
\]
Now applying Corollary 2.6
\[
\int_{\Gamma} \frac{1}{|x - y|^{n-2}} dS(y) \leq C|\Gamma|^{1/(n-1)}.
\]

The following Lemma 2.8, Corollary 2.9, and Lemma 2.10 are parallel results as Lemma 2.5, Corollary 2.6, and Lemma 2.7, but they deal with dimension \(n = 2\) rather than \(n \geq 3\).
Lemma 2.8. Let $\Omega$ be a bounded, open subset of $\mathbb{R}^2$ with $C^1$ boundary. Let $\Gamma$ be any subset of $\partial \Omega$ that is given by a graph as in Definition 2.4. Then there exists a constant $C = C(\Omega, ||\nabla \phi||_{L^\infty(U)})$, where $\phi$ and $U$ are the same as those in Definition 2.4 such that for any $x \in \overline{\Omega}$,

$$\int_{\Gamma} \ln \left( \frac{d_\Omega}{|x-y|} \right) dS(y) \leq C|\Gamma| \ln \left( \frac{1}{|\Gamma|} + 1 \right),$$

where $d_\Omega$ denotes the diameter of $\Omega$.

Proof. By Definition 2.4 without loss of generality, we can assume there exists a $C^1$ function $\phi : \mathbb{R} \to \mathbb{R}$ and a bounded set $U \subseteq \mathbb{R}$ such that

$$\Gamma = \{(\tilde{y}, \phi(\tilde{y})) : \tilde{y} \in U\}. \tag{2.15}$$

In addition, we define

$$f(r) = \begin{cases} \ln \left( \frac{d_\Omega}{r} \right), & 0 < r \leq d_\Omega, \\ 0, & r > d_\Omega. \end{cases} \tag{2.16}$$

Since $x = (\hat{x}, x_n) \in \overline{\Omega}$, then for any $(\tilde{y}, \phi(\tilde{y})) \in \Gamma$,

$$|\hat{x} - \tilde{y}| \leq |(\hat{x}, x_n) - (\tilde{y}, \phi(\tilde{y}))| \leq d_\Omega.$$

As a result,

$$\int_{\Gamma} \ln \left( \frac{d_\Omega}{|x-y|} \right) dS(y) = \int_U \ln \left( \frac{d_\Omega}{|\tilde{x}, x_n| - (\tilde{y}, \phi(\tilde{y}))} \right) \sqrt{1 + |\nabla \phi(\tilde{y})|^2} \, d\tilde{y} \leq C \int_U \ln \left( \frac{d_\Omega}{|\hat{x} - \tilde{y}|} \right) d\tilde{y} = C \int_U f(|\hat{x} - \tilde{y}|) \, d\tilde{y}. \tag{2.17}$$

Now it follows from Lemma 2.3 that

$$\int_U f(|\hat{x} - \tilde{y}|) \, d\tilde{y} \leq \int_{B_R(0)} f(|\tilde{z}|) \, d\tilde{z} = 2 \int_0^R f(r) \, dr, \tag{2.18}$$

where $|B_R(0)| = |U|$, namely $2R = |U|$. For any $\tilde{y}_1, \tilde{y}_2 \in U$, we have

$$|\tilde{y}_1 - \tilde{y}_2| \leq |(\tilde{y}_1, \phi(\tilde{y}_1)) - (\tilde{y}_2, \phi(\tilde{y}_2))| \leq d_\Omega,$$

which implies $\text{diam}(U) \leq d_\Omega$. Moreover, since $U \subseteq \mathbb{R}$, then $|U| \leq \text{diam}(U)$. Thus, $R = |U|/2 \leq d_\Omega/2$. So it follows from (2.16) that

$$\int_0^R f(r) \, dr = \int_0^R \ln \left( \frac{d_\Omega}{r} \right) \, dr = R \left[ \ln \left( \frac{d_\Omega}{R} \right) + 1 \right]. \tag{2.19}$$
Again by the parametrization (2.15), it is readily seen that $|U| \leq |\Gamma|$. Therefore,

$$R \leq \min \left\{ \frac{|\Gamma|}{2}, \frac{d_\Omega}{2} \right\}.$$ 

Define

$$g(r) = r \left[ \ln \left( \frac{d_\Omega}{r} \right) + 1 \right], \quad \forall r > 0.$$ 

Then $g$ is increasing when $r \in (0, d_\Omega]$ and (2.19) implies $\int_0^R f(r) \, dr = g(R)$. Next, we will estimate $g(R)$ in the following two situations.

- $|\Gamma| \leq d_\Omega$.

$$g(R) \leq g(|\Gamma|) = |\Gamma| \left[ \ln \left( \frac{d_\Omega}{|\Gamma|} \right) + 1 \right] \leq C|\Gamma| \ln \left( \frac{1}{|\Gamma|} + 1 \right)$$

for some constant $C$ only depending on $\Omega$.

- $|\Gamma| > d_\Omega$.

$$g(R) \leq g(d_\Omega) = d_\Omega.$$

Define

$$h(r) = r \ln \left( \frac{1}{r} + 1 \right), \quad \forall r > 0.$$ (2.21)

Then

$$h''(r) = -\frac{1}{r(1+r)^2} < 0, \quad \forall r > 0.$$ 

This implies $h'(r) > 0$ for any $r > 0$, since $\lim_{r \to \infty} h'(r) = 0$. Hence, $h$ is an increasing function and

$$|\Gamma| \ln \left( \frac{1}{|\Gamma|} + 1 \right) = h(|\Gamma|) \geq h(d_\Omega) = d_\Omega \ln \left( \frac{1}{d_\Omega} + 1 \right).$$

Thus,

$$g(R) \leq C|\Gamma| \ln \left( \frac{1}{|\Gamma|} + 1 \right),$$

where $C = 1/\ln \left( \frac{1}{d_\Omega} + 1 \right)$ is a constant depending only on $\Omega$.

Combining (2.17), (2.18), (2.20) and (2.22), the conclusion follows.

**Corollary 2.9.** Let $\Omega$ be a bounded, open subset of $\mathbb{R}^2$ with $C^1$ boundary. Let $\Gamma$ be any subset of $\partial \Omega$. Then there exists a constant $C = C(\Omega)$ such that for any $x \in \overline{\Omega}$,

$$\int_{\Gamma} \ln \left( \frac{d_\Omega}{|x-y|} \right) \, dS(y) \leq C|\Gamma| \ln \left( \frac{1}{|\Gamma|} + 1 \right),$$

where $d_\Omega$ denotes the diameter of $\Omega$.

**Proof.** Similar to the proof of Corollary 2.6, we first decompose $\partial \Omega$ as that in (2.12). Then

$$\int_{\Gamma} \ln \left( \frac{d_\Omega}{|x-y|} \right) \, dS(y) \leq \sum_{i=1}^K \int_{\Gamma \cap A_i} \ln \left( \frac{d_\Omega}{|x-y|} \right) \, dS(y).$$
Since each $\Gamma \cap A_i$ is given by a graph, we can apply Lemma 2.8 to conclude there exists a constant $C = C(\Omega)$ such that for each $1 \leq i \leq K$,
\[
\int_{\Gamma \cap A_i} \ln \left( \frac{d\Omega}{|x-y|} \right) dS(y) \leq C|\Gamma \cap A_i| \ln \left( \frac{1}{|\Gamma \cap A_i|} + 1 \right).
\]
Recalling the function $h$ defined in (2.21) is an increasing function, so
\[
|\Gamma \cap A_i| \ln \left( \frac{1}{|\Gamma \cap A_i|} + 1 \right) \leq |\Gamma| \ln \left( \frac{1}{|\Gamma|} + 1 \right).
\]
As a result,
\[
\int_{\Gamma} \ln \left( \frac{d\Omega}{|x-y|} \right) dS(y) \leq C|\Gamma| \ln \left( \frac{1}{|\Gamma|} + 1 \right).
\]

Next, Lemma 2.8 and Corollary 2.9 will be applied to show our desired Lemma 2.10 which is an improvement of (2.7) when $n = 2$.

**Lemma 2.10.** Let $\Omega$ be a bounded, open subset of $\mathbb{R}^2$ with $C^1$ boundary. Let $\Gamma$ be any subset of $\partial \Omega$. Then there exists $C = C(\Omega)$ such that for any $x \in \Omega$ and $t \in [0,1]$,
\[
\int_{t}^{1} \int_{\Gamma} \Phi(x-y, t-\tau) dS(y) d\tau \leq C|\Gamma| \ln \left( \frac{1}{|\Gamma|} + 1 \right). \tag{2.23}
\]

**Proof.** We proceed similarly as that in the proof of Lemma 2.7 until (2.14). Next, the situation is different since $s^{n/2-2}e^{-s}$ is not integrable near $s = 0$ when $n = 2$. For convenience, we rewrite (2.14) when $n = 2$ as following:
\[
\int_{\Gamma} \int_{0}^{t} \int_{\Gamma} \Phi(x-y, t-\tau) dS(y) d\tau ds \leq C \int_{\Gamma} \int_{|x-y|^{2}/4t}^{\infty} s^{-1}e^{-s} ds dS(y). \tag{2.24}
\]
Since $t \leq 1$ and $x \in \Omega$, $|x-y|^{2}/4t \geq |x-y|^{2}/4$. Thus,
\[
\int_{|x-y|^{2}/4t}^{\infty} s^{-1}e^{-s} ds \leq \int_{|x-y|^{2}/4}^{\infty} s^{-1}e^{-s} ds = \int_{|x-y|^{2}/4}^{d_{\Omega}} s^{-1}e^{-s} ds + \int_{d_{\Omega}}^{\infty} s^{-1}e^{-s} ds \leq \int_{|x-y|^{2}/4}^{d_{\Omega}} s^{-1} ds + \frac{1}{d_{\Omega}^{2}} \int_{d_{\Omega}}^{\infty} e^{-s} ds = 2 \ln \left( \frac{d\Omega}{|x-y|} \right) + C.
\]
As a result,
\[
\int_{\Gamma} \int_{|x-y|^{2}/4t}^{\infty} s^{-1}e^{-s} ds dS(y) \leq 2 \int_{\Gamma} \ln \left( \frac{d\Omega}{|x-y|} \right) dS(y) + C|\Gamma|. \tag{2.25}
\]
Now applying Corollary 2.9
\[
\int_{\Gamma} \ln \left( \frac{d\Omega}{|x-y|} \right) dS(y) \leq C|\Gamma| \ln \left( \frac{1}{|\Gamma|} + 1 \right).
\]
Finally noticing that
\[ |\Gamma| \leq \frac{1}{\ln \left(\frac{1}{|\partial\Omega|} + 1\right)} |\Gamma| \ln \left(\frac{1}{|\Gamma|} + 1\right) = C|\Gamma| \ln \left(\frac{1}{|\Gamma|} + 1\right), \]
the lemma is proved. \(\square\)

3 Proof of Theorem 1.1

The starting point of the proofs in this paper is the representation formula of the solution \(u\) (see Lemma A.1 in [28]): for any \(T \in [0, T^*]\) and \((x, t) \in \partial\Omega \times [0, T^* - T)\),

\[
u(x, T + t) = 2 \int_{\Omega} \Phi(x - y, t) u(y, T) \, dy - 2 \int_0^t \oint_{\partial\Omega} \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} u(y, T + \tau) \, dS(y) \, d\tau + 2 \int_0^t \int_{\Gamma_1} \Phi(x - y, t - \tau) \, u^q(y, T + \tau) \, dS(y) \, d\tau. \tag{3.1} \]

To estimate the integral of \(\frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)}\) on \(\partial\Omega \times [0, t]\), we apply the lemma below.

**Lemma 3.1.** There exists \(C = C(n, \Omega)\) such that for any \(x \in \partial\Omega\) and \(t > 0\),

\[
\int_0^t \oint_{\partial\Omega} \left| \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} \right| \, dS(y) \, d\tau \leq C \sqrt{t}. \tag{3.2} \]

**Proof.** By the definition of \(\Phi\),

\[
\left| \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} \right| = C |(x - y) \cdot n(y)| \exp \left( -\frac{|x - y|^2}{4(t - \tau)} \right). \]

Since \(\partial\Omega\) is assumed to be \(C^2\), there exists a constant \(C\) such that \(|(x - y) \cdot n(y)| \leq C|x - y|^2\) for any \(x, y \in \partial\Omega\). As a result,

\[
\left| \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} \right| \leq C |x - y|^2 \exp \left( -\frac{|x - y|^2}{4(t - \tau)} \right). \]

Noticing the term

\[
\frac{|x - y|^2}{t - \tau} \exp \left( -\frac{|x - y|^2}{8(t - \tau)} \right)
\]

is bounded by some constant, so

\[
\left| \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} \right| \leq \frac{C}{(t - \tau)^{n/2}} \exp \left( -\frac{|x - y|^2}{8(t - \tau)} \right) \leq \frac{C}{(t - \tau)^{n/2}} \exp \left( -\frac{|x - y|^2}{8(t - \tau)} \right),
\]

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Thus,
\[
\int_0^t \int_{\partial \Omega} \left| \frac{\partial \Phi(x-y, t-\tau)}{\partial n(y)} \right| dS(y) d\tau
\leq C \int_0^t \int_{\partial \Omega} \frac{1}{(t-\tau)^n} \exp \left( -\frac{|x-y|^2}{8(t-\tau)} \right) dS(y) d\tau
= C \int_0^t \int_{\partial \Omega} \frac{1}{\tau^{n/2}} \exp \left( -\frac{|x-y|^2}{8\tau} \right) dS(y) d\tau.
\]

By the change of variable \( \sigma = 2\tau \),
\[
\int_0^t \int_{\partial \Omega} \frac{1}{\tau^{n/2}} \exp \left( -\frac{|x-y|^2}{8\tau} \right) dS(y) d\tau
= C \int_0^{2t} \int_{\partial \Omega} \frac{1}{\sigma^{n/2}} \exp \left( -\frac{|x-y|^2}{4\sigma} \right) dS(y) d\sigma
= C \int_0^{2t} \int_{\partial \Omega} \Phi(x-y, \sigma) dS(y) d\sigma
= C \int_0^{2t} \int_{\partial \Omega} \Phi(x-y, 2t-\sigma) dS(y) d\sigma.
\]

Finally, invoking (2.7) with \( \Gamma = \partial \Omega \) and \( \alpha = 0 \), the proof is finished.

\[\Box\]

**Proof of Theorem 1.1** In this proof, \( C \) will denote constants which only depend on \( n \) and \( \Omega \), the values of \( C \) may be different in different places. But \( C^* \) and \( C^*_i (i \geq 1) \) will represent fixed constants which only depend on \( n \) and \( \Omega \). \( M(t) \) represents the same function as in (1.5).

For any strictly increasing sequence \( \{M_k\}_{k \geq 0} \) whose initial term is the same as the \( M_0 \) defined in (1.4), we denote \( T_k \) to be the first time that \( M(t) \) reaches \( M_k \). Obviously, \( T_0 = 0 \). For any \( k \geq 1 \), define
\[ t_k = T_k - T_{k-1} \quad (3.3) \]
to be the time spent in the \( k \)th step. By the maximum principle and the Hopf lemma, there exists \( x^k \in \overline{\Gamma}_1 \) such that
\[ u(x^k, T_k) = M_k. \quad (3.4) \]

Applying the representation formula (3.1) with \( T = T_{k-1} \) and \( (x, t) = (x^k, t_k) \), then
\[
u(x^k, T_k) = 2 \int_{\partial \Omega} \Phi(x^k - y, t_k) u(y, T_{k-1}) dy
- 2 \int_0^{t_k} \int_{\partial \Omega} \frac{\partial \Phi(x^k - y, t_k - \tau)}{\partial n(y)} u(y, T_{k-1} + \tau) dS(y) d\tau
+ 2 \int_0^{t_k} \int_{\Gamma_1} \Phi(x^k - y, t_k - \tau) u^q(y, T_{k-1} + \tau) dS(y) d\tau. \quad (3.5)\]
Combining (3.4) and (3.5),

\[ M_k \leq 2M_{k-1} \int_{\Omega} \Phi(x^k - y, t_k) \, dy \\
+ 2M_k \int_0^{t_k} \int_{\partial \Omega} \left| \frac{\partial \Phi(x^k - y, t_k - \tau)}{\partial n(y)} \right| \, dS(y) \, d\tau \\
+ 2M_k^q \int_0^{t_k} \int_{\Gamma_1} \Phi(x^k - y, t_k - \tau) \, dS(y) \, d\tau. \]

Replacing the term \( \int_{\Omega} \Phi(x^k - y, t_k) \, dy \) by the identity (2.1), then

\[ M_k \leq 2M_{k-1} \left[ \frac{1}{2} + \int_0^{t_k} \int_{\partial \Omega} \left| \frac{\partial \Phi(x^k - y, t_k - \tau)}{\partial n(y)} \right| \, dS(y) \, d\tau \right] \\
+ 2M_k \int_0^{t_k} \int_{\partial \Omega} \left| \frac{\partial \Phi(x^k - y, t_k - \tau)}{\partial n(y)} \right| \, dS(y) \, d\tau \\
+ 2M_k^q \int_0^{t_k} \int_{\Gamma_1} \Phi(x^k - y, t_k - \tau) \, dS(y) \, d\tau. \]

Moving the term on the second line of the right hand side to the left, we obtain

\[ (1 - 2I_1)M_k \leq (1 + 2I_2)M_{k-1} + 2I_3M_k^q, \quad (3.6) \]

where

\[ I_1 = \int_0^{t_k} \int_{\partial \Omega} \left| \frac{\partial \Phi(x^k - y, t_k - \tau)}{\partial n(y)} \right| \, dS(y) \, d\tau, \]
\[ I_2 = \int_0^{t_k} \int_{\partial \Omega} \left| \frac{\partial \Phi(x^k - y, t_k - \tau)}{\partial n(y)} \right| \, dS(y) \, d\tau, \]
\[ I_3 = \int_0^{t_k} \int_{\Gamma_1} \Phi(x^k - y, t_k - \tau) \, dS(y) \, d\tau. \]

It is readily seen that \(|I_2| \leq I_1\). In addition, by Lemma \ref{lemma3.1}, there exists a constant \( C^* \) such that

\[ I_1 \leq C^* \sqrt{t_k}. \quad (3.8) \]

If \( t_k \) satisfies

\[ t_k \leq \frac{1}{16(C^*)^2}, \quad (3.9) \]

then \(|I_2| \leq I_1 \leq \frac{1}{4}\). As a result, \( 1 - 2I_1 \geq \frac{1}{2} \) and

\[ \frac{1 + 2I_2}{1 - 2I_1} = 1 + \frac{2(I_1 + I_2)}{1 - 2I_1} \leq 1 + 4(I_1 + I_2). \]

Hence by dividing \( 1 - 2I_1 \) from both sides of (3.6), we obtain

\[ M_k \leq \left[ 1 + 4(I_1 + I_2) \right] M_{k-1} + 4I_3M_k^q. \quad (3.10) \]

In the following, by choosing a suitable sequence \( \{M_k\}_{k \geq 0} \) and obtaining a lower bound \( t_{k_*} \) for
each \( t_k \), the sum of all \( t_k \) becomes a lower bound for \( T^* \). First, due to the estimate (3.8) again,

\[
I_1 + I_2 \leq 2I_1 \leq 2C^* \sqrt{t_k}.
\] (3.11)

Next in order to estimate \( I_3 \), we apply (2.7) for \( \Gamma = \Gamma_1 \) and \( \alpha = \frac{1}{2(n-1)} \), then there exists some constant \( C \) such that

\[
I_3 \leq C|\Gamma_1|^\alpha t_k^{1/4}.
\] (3.12)

Plugging (3.11) and (3.12) into (3.10) yields

\[
M_k \leq (1 + C\sqrt{t_k})M_{k-1} + C|\Gamma_1|^\alpha t_k^{1/4} M_k^q.
\] (3.13)

Define

\[
M_k = 2^k M_0.
\] (3.14)

Then

\[
2^k M_0 \leq (1 + C\sqrt{t_k})2^{k-1} M_0 + C|\Gamma_1|^\alpha t_k^{1/4} 2^k M_0^q.
\]

Subtracting \( 2^{k-1} M_0 \) from both sides, we obtain

\[
2^{k-1} M_0 \leq C\sqrt{t_k} 2^{k-1} M_0 + C|\Gamma_1|^\alpha t_k^{1/4} 2^{qk} M_0^q.
\]

Dividing by \( 2^{k-1} M_0 \),

\[
1 \leq C\sqrt{t_k} + C|\Gamma_1|^\alpha t_k^{1/4} 2^{(q-1)k} M_0^{q-1}.
\]

Thus,

\[
\sqrt{t_k} + |\Gamma_1|^\alpha M_0^{q-1} 2^{(q-1)k} t_k^{1/4} - \frac{1}{C} \geq 0.
\]

Regarding the left hand side of the above inequality to be a quadratic function in \( t_k^{1/4} \), then \( t_k^{1/4} \) has to be greater than its positive root, that is

\[
t_k^{1/4} \geq \frac{1}{2} \left( -|\Gamma_1|^{\alpha} M_0^{q-1} 2^{(q-1)k} + \sqrt{||\Gamma_1|^{2\alpha} M_0^{2(q-1)} 2^{2(q-1)k} + \frac{4}{C}} \right).
\]

Consequently,

\[
t_k^{1/4} \geq \frac{2}{C \left( ||\Gamma_1|^{2\alpha} M_0^{2(q-1)} 2^{2(q-1)k} + \frac{4}{C} \right)} \geq \frac{1}{C \sqrt{||\Gamma_1|^{2\alpha} M_0^{2(q-1)} 2^{2(q-1)k} + \frac{4}{C}}}.
\]

Hence, there exists \( C_1^* \) such that

\[
t_k \geq \frac{1}{C_1^* \left( ||\Gamma_1|^{4\alpha} M_0^{4(q-1)} 2^{4(q-1)k} + 1 \right)}.
\] (3.15)

As a summary of the above paragraph, by choosing \( M_k = 2^k M_0 \), then (3.9) implies (3.15). Therefore,

\[
t_k \geq \min \left\{ \frac{1}{16(C^*)^2}, \frac{1}{C_1^* \left( ||\Gamma_1|^{4\alpha} M_0^{4(q-1)} 2^{4(q-1)k} + 1 \right)} \right\}.
\]
Denoting
\[ C^*_2 = \min \left\{ \frac{1}{16(C^*)^2}, \frac{1}{C^*_1} \right\}, \]
then
\[ t_k \geq \frac{C^*_2}{|\Gamma_1|^{4\alpha} M_0^{4(q-1)}/2^{4(q-1)k} + 1}. \]  (3.16)
Hence,
\[ T^* = \sum_{k=1}^{\infty} t_k \geq C^*_2 \sum_{k=1}^{\infty} \frac{1}{|\Gamma_1|^{4\alpha} M_0^{4(q-1)}/2^{4(q-1)k} + 1} \]
\[ \geq C^*_2 \int_1^{\infty} \frac{1}{|\Gamma_1|^{4\alpha} M_0^{4(q-1)}/2^{4(q-1)x} + 1} \, dx \]
\[ = \frac{C^*_2}{4(q-1) \ln(2)} \ln \left(1 + \frac{1}{|\Gamma_1|^{4\alpha} M_0^{4(q-1)}/2^{4(q-1)}}\right). \]

Recalling \( \alpha = \frac{1}{2n-1} \), (1.10) follows.

4 Proof of Theorem 1.2

Define
\[ E_q = (q - 1)^{q-1}/q^q, \quad \forall q > 1. \]  (4.1)
By elementary calculus,
\[ \frac{1}{3q} < E_q < \min \left\{ \frac{1}{q}, \frac{1}{(q-1)e} \right\} < 1. \]  (4.2)
The lemma below is a simple generalization of Lemma 3.2 in [28].

Lemma 4.1. For any \( q > 1 \) and \( m > 0 \), write \( E_q \) as in (4.1) and define \( g : (m, \infty) \to \mathbb{R} \) by
\[ g(\lambda) = \frac{\lambda - m}{\lambda^q}, \quad \forall \lambda > m. \]  (4.3)
Then the following two claims hold.

1. For any \( y \in (0, m^{1-q} E_q) \), there exists unique \( \lambda \in (m, \frac{q}{q-1} m) \) such that \( g(\lambda) = y \).

2. For any \( y > m^{1-q} E_q \), there does not exist \( \lambda > m \) such that \( g(\lambda) = y \).

Proof. Since \( g \) is strictly increasing on the interval \( (m, \frac{q}{q-1} m) \) and strictly decreasing on the interval \( [\frac{q}{q-1} m, \infty) \), it reaches the maximum at \( \lambda = \frac{q}{q-1} m \). Noticing that
\[ g \left( \frac{q}{q-1} m \right) = m^{1-q} E_q, \]
then the claims (1) and (2) follow directly.

Now we can carry out the main proof in this section.

Proof of Theorem 1.2. We will demonstrate detailed proof for the case \( n \geq 3 \), the proof for the case \( n = 2 \) is similar and will be briefly mentioned at the end. In the proof below, \( C \) will denote the
constants which only depend on \( n \) and \( \Omega \), the values of \( C \) may be different in different places. But \( C^* \) will represent a fixed constant which only depends on \( n \) and \( \Omega \). Let \( M(t) \) be defined as in (1.5).

**Step 1.** The first part is exactly the same as the second paragraph in the proof of Theorem 1.1 namely we adopt the same notations and the same estimates from (3.3) through (3.10). In particular, we make the assumption (3.9).

**Step 2.** In this step, we will find a constant \( t_* > 0 \) and a finite strictly increasing sequence \( \{ M_k \}_{0 \leq k \leq L} \) such that \( t_k \geq t_* \) for \( 1 \leq k \leq L \). Then in Step 3, a lower bound for \( L t_* \) will be derived.

Due to the convexity of \( \Omega \), the normal derivative \( \frac{\partial \Phi(x_k - y, t_k - \tau)}{\partial n(y)} \) in (3.7) is always nonpositive. As a result,

\[
I_1 + I_2 = \int_0^{t_k} \int_{\partial \Omega} \frac{\partial \Phi(x_k - y, t_k - \tau)}{\partial n(y)} + \left| \frac{\partial \Phi(x_k - y, t_k - \tau)}{\partial n(y)} \right| dS(y) d\tau = 0. \tag{4.4}
\]

To estimate \( I_3 \), we apply Lemma 2.7 to conclude

\[
I_3 \leq C |\Gamma_1|^{1/(n-1)} \tag{4.5}
\]

for some constant \( C = C(n, \Omega) \). Hence plugging (4.4) and (4.5) into (3.10), we get

\[
M_k \leq M_{k-1} + C^* |\Gamma_1|^{1/(n-1)} M_k^{q-1} \tag{4.6}
\]

for some constant \( C^* = C^*(n, \Omega) \). As a summary, the argument so far claims that if (3.9) holds, then \( M_k \) will satisfy (4.6).

Based on the above observation, if we choose

\[
\delta_1 = 2C^* |\Gamma_1|^{1/(n-1)} \tag{4.7}
\]

and define \( M_k \) to be the solution (if it exists) to

\[
\frac{M_k - M_{k-1}}{M_k^{q-1}} = \delta_1, \tag{4.8}
\]

then (3.9) can not hold since otherwise (4.6) will be violated. Consequently \( t_k > t_* \), where

\[
t_* = \frac{1}{16(C^*)^2}. \tag{4.9}
\]

Due to Lemma 4.1, the existence of a solution \( M_k \) to (4.8) is equivalent to the inequality \( M_k^{q-1} \delta_1 \leq E_q \). In addition, as long as such a solution exists, \( M_k \) can be chosen to satisfy

\[
M_{k-1} < M_k \leq \frac{q}{q-1} M_{k-1}. \tag{4.10}
\]

Thus, the strategy of constructing \( \{ M_k \} \) is summarized as below. First, define \( M_0 \) and \( \delta_1 \) as in (1.4) and (4.7). Next suppose \( M_{k-1} \) has been constructed for some \( k \geq 1 \), then whether defining \( M_k \) depends on how large \( M_{k-1} \) is.

- If \( M_{k-1}^{q-1} \delta_1 \leq E_q \), then we define \( M_k \in (M_{k-1}, \frac{q}{q-1} M_{k-1}] \) to be the solution to (4.8).
- If \( M_{k-1}^{q-1} \delta_1 > E_q \), then there does not exist \( M_k > M_{k-1} \) which solves (4.8). So we do not define
Let \( M_k \) and stop the construction.

According to this construction, if \( \{M_k\}_{1 \leq k \leq L_0} \) have been defined, then \( T_k - T_{k-1} \geq t_* \) for any \( 1 \leq k \leq L_0 \). Therefore, \( T_k \geq kt_* \) for any \( 1 \leq k \leq L_0 \). Since \( T^* \) is finite, \( L_0 \leq T^*/t_* < \infty \), which means the cardinality of \( \{M_k\} \) has to be finite (actually this fact can also be justified by analysing the construction directly, see Lemma 4.2). So we can assume the constructed sequence is \( \{M_k\}_{0 \leq k \leq L} \) for some finite \( L \).

**Step 3.** By Lemma 4.2

\[
L > \frac{1}{10(q-1)} \left( \frac{1}{M_0^{q-1}\delta_1} - 3q \right).
\]

To obtain an effective lower bound, \( \frac{1}{M_0^{q-1}\delta_1} \) should be greater than \( 3q \). If requiring

\[
\frac{1}{M_0^{q-1}\delta_1} \geq 6q,
\]

then

\[
L > \frac{1}{20(q-1)M_0^{q-1}\delta_1} = \frac{1}{40C^*(q-1)M_0^{q-1}|\Gamma_1|^{1/(n-1)}}.
\]

Denote \( Y = M_0^{q-1}|\Gamma_1|^{1/(n-1)} \). Then

\[
T^* \geq Lt_* > \frac{C}{(q-1)Y}
\]

for some constant \( C \). Finally, noticing that (4.10) is equivalent to

\[
Y \leq \frac{1}{12C^*q},
\]

the proof for the case \( n \geq 3 \) is finished by setting \( Y_0 = 1/(12C^*) \).

When \( n = 2 \), the process is almost identical as the above except two differences. First, Lemma 2.10 will be applied instead of Lemma 2.7, so the term \( |\Gamma_1|^{1/(n-1)} \) in the above proof needs to be replaced by \( |\Gamma_1| \ln \left( \frac{\log \sqrt{n}}{\log \log n} + 1 \right) \). Secondly, due to the restriction \( t \leq 1 \) in Lemma 2.10, \( t_k \) should satisfy both \( t_k \leq 1 \) and (3.9) in order to justify (4.6). Consequently the choice of \( t_* \) will be

\[
t_* = \min \left\{ \frac{1}{16(C^*)^2}, 1 \right\}
\]

instead of (4.9). Fortunately, this additional requirement will not bring major changes to the proof. Actually, without loss of generality, we can choose \( C^* \) to be larger than \( 1/4 \), which makes \( \frac{1}{16(C^*)^2} \leq 1 \). As a result, (4.11) coincides with (4.9). Then the rest of the proof is the same. \( \Box \)

The following lemma has been applied in the proof of Theorem 1.2 and will be used again in the proof of Theorem 1.3 so we state it separately for convenience. It is a generalization of Lemma 3.3 in [28], but its statement and proof are much simpler.

**Lemma 4.2.** Given \( q > 1 \), \( M_0 > 0 \) and \( \delta_1 > 0 \), denote \( E_q \) as (4.1) and construct a (finite) sequence \( \{M_k\}_{k \geq 0} \) inductively as follows. Suppose \( M_{k-1} \) has been constructed for some \( k \geq 1 \), then based on Lemma 4.1, whether defining \( M_k \) depends on how large \( M_{k-1} \) is.

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\( \text{If } M_{k-1}^{q-1} \delta_1 \leq E_q, \text{ then we define } M_k \in \left( M_{k-1}, \frac{q}{q-1} M_{k-1} \right) \text{ to be the solution to} \)

\[
\frac{M_k - M_{k-1}}{M_k^q} = \delta_1. \tag{4.12}
\]

\( \text{If } M_{k-1}^{q-1} \delta_1 > E_q, \text{ then there does not exist } M_k > M_{k-1} \text{ which solves } (4.12). \text{ So we do not define } M_k \text{ and stop the construction.} \)

We claim this construction stops in finite steps and if the last term is denoted as \( M_L \), then

\[
L > \frac{1}{10(q-1)} \left( \frac{1}{M_0^{q-1} \delta_1} - 3q \right). \tag{4.13}
\]

**Proof.** First, we will show the construction has to stop in finite steps. In fact, it follows from (4.12) that the sequence \( \{M_k\} \) is strictly increasing and

\[
M_k = M_{k-1} + M_k^q \delta_1 \geq (1 + M_0^{q-1} \delta_1) M_{k-1}.
\]

As a result,

\[
M_k \geq (1 + M_0^{q-1} \delta_1)^k M_0.
\]

Thus \( M_k^{q-1} \) will exceed \( E_q/\delta_1 \) when \( k \) is sufficiently large, which forces the construction to stop.

Next suppose the constructed sequence is \( \{M_k\}_{0 \leq k \leq L} \). The lower bound (4.13) for \( L \) will be justified below.

**Case 1.** \( M_0^{q-1} \delta_1 > E_q \). In this case, it follows from (4.12) that

\[
\frac{1}{M_0^{q-1} \delta_1} < \frac{1}{E_q} < 3q.
\]

Thus (4.13) holds automatically since the right hand side of (4.13) is negative.

**Case 2.** \( M_0^{q-1} \delta_1 \leq E_q \). In this case, it is evident from the construction that \( L \geq 1 \). Moreover,

\[
M_{L-1}^{q-1} \delta_1 \leq E_q \quad \text{and} \quad M_L^{q-1} \delta_1 > E_q.
\]

According to the recursive relation (4.12),

\[
M_{k-1} = M_k (1 - M_k^{q-1} \delta_1).
\]

Raising both sides to the power \( q-1 \) and multiplying by \( \delta_1 \),

\[
M_{k-1}^{q-1} \delta_1 = M_k^{q-1} (1 - M_k^{q-1} \delta_1)^{q-1} \delta_1.
\]

Let \( x_k = M_k^{q-1} \delta_1 \). Then

\[
x_{k-1} = x_k (1 - x_k)^{q-1}, \quad \forall 1 \leq k \leq L. \tag{4.14}
\]

Moreover,

\[
x_0 = M_0^{q-1} \delta_1, \quad x_{L-1} \leq E_q \quad \text{and} \quad x_L > E_q.
\]
Noticing that $M_L \leq \frac{q}{q-1} M_{L-1}$, so

$$x_L = \left( \frac{M_L}{M_{L-1}} \right)^{q-1} x_{L-1} \leq \left( \frac{q}{q-1} \right)^{q-1} E_q = \frac{1}{q}.$$

Since the right hand side of (4.14) is nonlinear in $x_k$, it seems impossible to express $x_k$ as an explicit formula in terms of $x_{k-1}$. This motivates us to consider the “reversed” relation of (4.14), namely a new sequence $\{y_k\}_{0 \leq k \leq L}$ defined in the following way:

$$y_k \triangleq y_{k-1}(1 - y_{k-1})^{q-1}, \quad \forall 1 \leq k \leq L. \quad (4.15)$$

To analyse the sequence $\{y_k\}$, we define $h : (0, 1) \to \mathbb{R}$ by

$$h(t) = t(1-t)^{q-1}$$

so that $y_k = h(y_{k-1})$ for $1 \leq k \leq L$. It is easy to see that $h$ is strictly increasing on $(0, 1/q]$ and strictly decreasing on $[1/q, 1)$. Noticing $0 < y_0 < x_L \leq 1/q$, so

$$y_1 = h(y_0) < h(x_L) = x_{L-1}.$$

Keep doing this, we get $y_k < x_{L-k}$ for any $0 \leq k \leq L$. In particular, $y_L < x_0 = M_0^{q-1} \delta_1$.

Since $\{y_k\}$ is a decreasing positive sequence and $y_0 \leq 1/2$, then $y_k \leq 1/2$ for any $0 \leq k \leq L$. As a result, it follows from (4.15) and the mean value theorem that for any $1 \leq k \leq L$,

$$y_k \geq y_{k-1}[1 - 2(q-1)y_{k-1}]. \quad (4.16)$$

Recalling (4.2) again,

$$y_{k-1} \leq y_0 \leq E_q < \frac{1}{(q-1)e},$$

so

$$1 - 2(q-1)y_{k-1} > 1 - \frac{2}{e} > \frac{1}{5}.$$  

Hence, taking the reciprocal in (4.16) yields

$$\frac{1}{y_k} \leq \frac{1}{y_{k-1}[1 - 2(q-1)y_{k-1}]} = \frac{1}{y_{k-1}} + \frac{2(q-1)}{1 - 2(q-1)y_{k-1}} < \frac{1}{y_{k-1}} + 10(q-1). \quad (4.17)$$

Summing up (4.17) for $k$ from 1 to $L$, then

$$\frac{1}{y_L} < \frac{1}{y_0} + 10(q-1)L. \quad (4.18)$$

Since $y_L < M_0^{q-1} \delta_1$ and

$$y_0 = \min \left\{ \frac{1}{2}, E_q \right\} > \frac{1}{3q},$$

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it follows from (4.18) that
\[
\frac{1}{M_0^q - 1} < 3q + 10(q - 1)L.
\]

Thus,
\[
L > \frac{1}{10(q - 1)} \left( \frac{1}{M_0^q - 1} - 3q \right).
\]

5 Proof of Theorem 1.4

Proof of Theorem 1.4. We will demonstrate detailed proof for the case \( n \geq 3 \), the proof for the case \( n = 2 \) is similar and will be briefly mentioned at the end. In the proof below, \( C \) and \( C_i \) \((i \geq 1)\) will denote the constants which only depend on \( n, \Omega \) and \( d \), the values of \( C \) and \( C_i \) may be different in different places. But \( C^* \) and \( C_i^* \) \((i \geq 1)\) will represent fixed constants which only depend on \( n, \Omega \) and \( d \). Let \( M(t) \) be defined as in (1.5).

Step 1. The first part is exactly the same as the second paragraph in the proof of Theorem 1.1, namely we adopt the same notations and the same estimates from (3.3) through (3.10). In particular, we make the assumption (3.9).

Step 2. In this step, we will find a constant \( t^* > 0 \) and a finite strictly increasing sequence \( \{M_k\}_{0 \leq k \leq L} \) such that \( t_k \geq t^* \) for \( 1 \leq k \leq L \). Then in Step 3, a lower bound for \( Lt^* \) will be derived.

Due to the local convexity near \( \Gamma_1 \), it follows from (2.1) and Corollary 2.2 that
\[
I_1 + I_2 \leq C t_k \exp \left( -\frac{d^2}{8t_k} \right).
\]

Because of the assumption (3.9), the above inequality implies
\[
I_1 + I_2 \leq C \exp \left( -\frac{d^2}{8t_k} \right). \tag{5.1}
\]

On the other hand, we estimate \( I_3 \) in the same way as (4.5). Now plugging (5.1) and (4.5) into (3.10), then
\[
M_k \leq \left[ 1 + C^*_1 \exp \left( -\frac{d^2}{8t_k} \right) \right] M_{k-1} + C^*_2 |\Gamma_1|^{1/(n-1)} M_k^q, \tag{5.2}
\]

for two constants \( C^*_1 \) and \( C^*_2 \). Next if \( t_k \) is so small that
\[
\exp \left( -\frac{d^2}{8t_k} \right) \leq \frac{1}{2C^*_1 \frac{M_k - M_{k-1}}{M_{k-1}}}, \tag{5.3}
\]

which is equivalent to
\[
M_k - \left[ 1 + C^*_1 \exp \left( -\frac{d^2}{8t_k} \right) \right] M_{k-1} \geq \frac{1}{2} (M_k - M_{k-1}).
\]

Then it follows from (5.2) that
\[
\frac{M_k - M_{k-1}}{M_k^q} \leq 2C^*_2 |\Gamma_1|^{1/(n-1)}. \tag{5.4}
\]

As a summary, the argument so far claims if both (3.9) and (5.3) hold, then \( M_k \) will satisfy (5.4).
Based on this observation, if we choose

$$\delta_1 = 4C_2^*|\Gamma_1|^{1/(n-1)} \tag{5.5}$$

and define $M_k$ to be the solution (if it exists) to

$$\frac{M_k - M_{k-1}}{M_k^2} = \delta_1, \tag{5.6}$$

then either (3.9) or (5.3) can not hold since otherwise (5.4) will be violated. The invalidity of (3.9) means

$$t_k > \frac{1}{16(C^*)^2}. \tag{5.7}$$

On the other hand, due to (5.6), the failure of (5.3) implies

$$\exp\left(-\frac{d^2}{8t_k}\right) > \frac{1}{2C^*_1} \frac{M_k - M_{k-1}}{M_{k-1}} = \frac{M_k^q\delta_1}{2C^*_1 M_{k-1}} \geq \frac{M_0^{q-1}\delta_1}{2C^*_1}. \tag{5.8}$$

If

$$M_0^{q-1}\delta_1 \leq C^*_1, \tag{5.9}$$

then the right hand side of (5.8) is smaller than 1. Therefore, (5.8) is equivalent to

$$t_k > \frac{d^2}{8} \left\lfloor \ln \left(\frac{2C^*_1}{M_0^{q-1}\delta_1}\right) \right\rfloor^{-1}. \tag{5.10}$$

In summary, if (5.9) holds, then it follows from (5.7) and (5.10) that $t_k \geq t_*$, where

$$t_* = \min \left\{ \frac{1}{16(C^*)^2}, \frac{d^2}{8} \left\lfloor \ln \left(\frac{2C^*_1}{M_0^{q-1}\delta_1}\right) \right\rfloor^{-1} \right\}. \tag{5.11}$$

Due to Lemma 4.1, the existence of a solution $M_k$ to (5.6) is equivalent to the inequality $M_0^{q-1}\delta_1 \leq E_q$. In addition, as long as such a solution exists, $M_0$ can be chosen to satisfy

$$M_{k-1} < M_k \leq \frac{q}{q-1} M_{k-1}. \tag{5.12}$$

Thus, the strategy of constructing $\{M_k\}$ is summarized as following. First, define $M_0$ and $\delta_1$ as in (1.4) and (5.5). Next suppose $M_{k-1}$ has been constructed for some $k \geq 1$, then whether defining $M_k$ depends on how large $M_{k-1}$ is.

- If $M_{k-1}^{q-1}\delta_1 \leq E_q$, then we define $M_k \in (M_{k-1}, \frac{q}{q-1} M_{k-1}]$ to be the solution to (5.9).
- If $M_{k-1}^{q-1}\delta_1 > E_q$, then there does not exist $M_k > M_{k-1}$ which solves (5.6). So we do not define $M_k$ and stop the construction.

According to this construction, if $\{M_k\}_{1 \leq k \leq L_0}$ have been defined, then $T_k - T_{k-1} \geq t_*$ for any $1 \leq k \leq L_0$. Therefore, $T_k \geq kt_*$ for any $1 \leq k \leq L_0$. Since $T^*$ is finite, $L_0 \leq T^*/t_* < \infty$, which means the cardinality of $\{M_k\}$ has to be finite (actually this fact can also be justified by analysing the construction directly, see Lemma 4.2). So we can assume the constructed sequence is $\{M_k\}_{0 \leq k \leq L}$ for some finite $L$. 

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Step 3. By Lemma 4.2

\[ L > \frac{1}{10(q-1)} \left( \frac{1}{M_0^{-1}\delta} - 3q \right). \]

If

\[ M_0^{-1}\delta \leq \frac{1}{6q}, \tag{5.12} \]

then

\[ L \geq \frac{1}{20(q-1)M_0^{-1}\delta}. \]

Combining the assumptions (5.9) and (5.12), if

\[ M_0^{-1}\delta \leq \min\left\{ C_1^*, \frac{1}{6q} \right\}, \tag{5.13} \]

then

\[ T^* \geq Lt^* \geq \frac{1}{20(q-1)M_0^{-1}\delta} \min\left\{ \frac{1}{16(C^*)^2}, \frac{d^2}{8} \left[ \ln \left( \frac{2C_1^*}{M_0^{-1}\delta} \right) \right]^{-1} \right\}. \tag{5.14} \]

Denote

\[ Y = M_0^{-1}|\Gamma_1|^{1/(n-1)}. \]

Recalling \( \delta_1 = 4C_2^*|\Gamma_1|^{1/(n-1)} \), we can rewrite (5.13) and (5.14) as

\[ Y \leq \min\left\{ \frac{C_1^*}{4C_2^*}, \frac{1}{24C_2^*q} \right\}. \tag{5.13'} \]

and

\[ T^* \geq \frac{1}{80C_2^*(q-1)Y} \min\left\{ \frac{1}{16(C^*)^2}, \frac{d^2}{8} \left[ \ln \left( \frac{C_1^*}{2C_2^*Y} \right) \right]^{-1} \right\}. \tag{5.14'} \]

In order to simplify (5.14'), if

\[ Y \leq \min\left\{ \frac{2C_2^*}{C_1^*}, \frac{C_1^*}{2C_2^*} \exp \left[ -2d^2(C^*)^2 \right] \right\}, \tag{5.15} \]

then

\[ 2d^2(C^*)^2 \leq \ln \left( \frac{C_1^*}{2C_2^*Y} \right) \leq 2 \ln \left( \frac{1}{Y} \right) = 2 \ln |Y|. \]

Taking reciprocal of the above inequality and multiplying by \( d^2/8 \), we obtain

\[ \frac{d^2}{16\ln |Y|} \leq \frac{d^2}{8} \left[ \ln \left( \frac{C_1^*}{2C_2^*Y} \right) \right]^{-1} \leq \frac{1}{16(C^*)^2}. \]

Therefore, (5.14') yields

\[ T^* \geq \frac{1}{80C_2^*(q-1)Y} \frac{d^2}{16\ln |Y|} = \frac{C}{(q-1)Y|\ln Y|}. \]

Combining the assumptions (5.13') and (5.15) together, it suffices to require \( Y \leq Y_0 / q \), where

\[ Y_0 = \min\left\{ \frac{1}{24C_2^*}, \frac{2C_2^*}{C_1^*}, \frac{C_1^*}{4C_2^*} \exp \left[ -2d^2(C^*)^2 \right] \right\}. \]
is a constant which only depends on $n$, $\Omega$ and $d$. Hence, we finish the proof when $n \geq 3$.

When $n = 2$, we can argue in the same way as the last paragraph of the proof for Theorem 1.2 to justify the conclusion.

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