Uniqueness of the Fock representation of the Gowdy $S^1 \times S^2$ and $S^3$ models

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Abstract
After a suitable gauge fixing, the local gravitational degrees of freedom of the Gowdy $S^1 \times S^2$ and $S^3$ cosmologies are encoded in an axisymmetric field on the sphere $S^2$. Recently, it has been shown that a standard field parametrization of these reduced models admits no Fock quantization with a unitary dynamics. This lack of unitarity is surpassed by a convenient redefinition of the field and the choice of an adequate complex structure. The result is a Fock quantization where both the dynamics and the $SO(3)$-symmetries of the field equations are unitarily implemented. The present work proves that this Fock representation is in fact unique inasmuch as, up to equivalence, there exists no other possible choice of $SO(3)$-invariant complex structure leading to a unitary implementation of the time evolution.

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1. Introduction

In a series of papers [1–5], a Fock quantization of the linearly polarized Gowdy $T^3$ cosmologies [6] has been put forward [2, 3] and shown to be unique under natural conditions [4, 5]. With respect to previous proposals [7, 8], a crucial step was the redefinition of the scalar field that effectively parametrizes the local degrees of freedom of the model [2, 3]. This new parametrization allowed the construction of a Fock quantization with unitary dynamics, in contrast to the situation found in [1, 5, 8, 9] when the seemingly more natural parametrization adopted in [7] is used.

In addition to the unitary dynamics, the quantization introduced in [2, 3] provides a unitary representation of the group of symmetries of the (reduced) model, which in the $T^3$
case is actually a gauge group. This was achieved by means of a complex structure that is invariant under the action of these symmetries\textsuperscript{4}. Moreover, it was shown that the conditions of unitary implementation of the dynamics and invariance of the complex structure completely fix the quantization, i.e. any two Fock representations satisfying these conditions are unitarily equivalent [4].

More recently, part of the results obtained originally in the context of the $T^3$ model was extended to the linearly polarized Gowdy $S^1 \times S^2$ and $S^3$ models [12, 13]. As shown in [12], the local degrees of freedom of these models are effectively described by an axisymmetric scalar field on $S^2$ (more precisely in a spacetime $(0, \pi) \times S^2$), obeying the same field equation in both cases. Starting from this formulation, the issue of unitary evolution was then discussed, restricting the considerations to Fock representations of the scalar field determined by $SO(3)$-invariant complex structures [13]. Firstly, it was found that, like in the $T^3$ case, the seemingly natural field parametrization of these models does not admit a quantization with unitary dynamics. Secondly, it was seen that a field redefinition of the type considered in the $T^3$ case again allows for a unitary implementation of the dynamics.

The aim of the present work is to show that the uniqueness theorem presented in [4], directly applicable to the Gowdy $T^3$ case as well as to more general circumstances, is again valid in the Gowdy $S^1 \times S^2$ and $S^3$ cases (when the new field parametrization is adopted). Specifically, we will show that, among the set of complex structures considered in [13], those that allow a unitary implementation of the scalar field dynamics define a unique unitary equivalence class of representations. Let us stress that restricting attention to complex structures (or states) that remain invariant under symmetry groups is a standard practice in quantum field theory, as a natural way to ensure the unitary implementation of those groups. This applies both to cases of gauge groups, or simply of symmetries leading to conservation laws.

The paper is organized as follows. In section 2, we briefly review the quantization of the $S^1 \times S^2$ and $S^3$ models along the lines of [13]. In section 3, we show the uniqueness of the quantization. This is the main section of the paper. The proof of this uniqueness result is an adaptation of that presented in [4]. To avoid unnecessary repetitions, only the essential technical arguments are explained, obviating a discussion of the framework that can be found in [1–5]. We present our conclusions in section 4, together with a brief discussion of other relevant points.

2. The quantization of the $S^1 \times S^2$ and $S^3$ models

In this section, we briefly review the quantization of the Gowdy $S^1 \times S^2$ and $S^3$ models discussed in [12, 13].

In the classical theory, once the reduction, gauge fixing and deparametrization of the models have been performed, the effective configuration variable for both the Gowdy $S^1 \times S^2$ and $S^3$ linearly polarized cosmologies is an axisymmetric field on the sphere $S^2$, which after a mode decomposition in terms of spherical harmonics can be written as

$$
\phi(t, s) = \sum_{\ell=0}^{\infty} [a_\ell(t)Y_{\ell 0}(s) + a^{*}_\ell(t)Y^*_{\ell 0}(s)].
$$

(1)

Here, $s \in S^2$, $t \in (0, \pi)$ is the (internal) time, $Y_{\ell 0}$ is the ($\ell$, $m = 0$) spherical harmonic and the symbol $*$ denotes the complex conjugation.

\textsuperscript{4} Let us recall that a quantization of the Fock type is determined by a complex structure on the space of classical solutions, and that symplectic transformations which leave the complex structure invariant are implemented by unitary transformations which leave the vacuum invariant (up to a phase), see e.g. [10, 11].
The field $\phi$ obeys the equation

$$\dot{\phi} + \cot t \phi - \Delta_{S^2} \phi = 0,$$  \hspace{1cm} (2)

where $\Delta_{S^2}$ denotes the Laplace–Beltrami operator on $S^2$ and the dot stands for the time derivative. The field equation (2) is invariant under the group $SO(3)$, acting as rotations on $S^2$.

Given equation (1), the dynamics of the system can be described in terms of the infinite set of modes $\{y_\ell\}$, which from (2) satisfy the equations of motion:

$$\dot{y}_\ell + \cot t y_\ell + \ell (\ell + 1) y_\ell = 0.$$  \hspace{1cm} (3)

Independent solutions of these equations are, for each mode, the functions $P_\ell(\cos t)$ and $Q_\ell(\cos t)$, where $P_\ell$ and $Q_\ell$ denote the first and second class Legendre functions [14].

Endowing the space of solutions with a complex structure $\bar{J}$ (compatible with the symplectic form) one can construct a Fock representation of the field $\phi$. In order to preserve the $SO(3)$-symmetry in the quantum description, one restricts the attention to the set of complex structures which descend from $SO(3)$-invariant ones under the restriction of axisymmetry (since the field $\phi$ must also be axisymmetric owing to the Killing symmetries of the models). The result is a family $\{\bar{J}\}$ of $SO(3)$-invariant Fock representations. However, time evolution fails to be implemented as a unitary transformation in each member of $\{\bar{J}\}$ [13].

In order to arrive at a unitary theory, a time-dependent transformation of the basic field is performed, namely

$$\xi := \sqrt{\sin t} \phi,$$  \hspace{1cm} (4)

which is analogous to the transformation proposed in [2, 3]. The field $\xi$ can be expanded in terms of the new modes $z_\ell(t) := \sqrt{\sin t} y_\ell(t)$. Since relation (4) is simply a time-dependent scaling, the $SO(3)$-transformations again define dynamical symmetries of the field $\xi$.

Turning now to the Fock quantizations of the field $\xi$, these are determined by the possible complex structures on the space of classical solutions $\{z_\ell, \ell = 0, 1, 2, \ldots\}$ to the mode equations

$$\ddot{z}_\ell + \left[\frac{1}{t} (1 + \csc^2 t) + \ell (\ell + 1)\right] z_\ell = 0.$$  \hspace{1cm} (5)

Owing to the commented $SO(3)$-symmetries, we will restrict our attention to the class of complex structures which descend from $SO(3)$-invariant ones. As shown in [13], this class is parametrized by sequences of real pairs $\{(\rho_\ell, v_\ell)\}$, where $\rho_\ell > 0 \forall \ell$. To be precise, let us consider the complex combinations of classical solutions

$$z_\ell^\rho(t) = \rho_\ell P_\ell(\cos t) + \left(v_\ell + \frac{i}{\rho_\ell}\right) Q_\ell(\cos t) \sqrt{\frac{\sin t}{2}}.$$  \hspace{1cm} (6)

Then, the complex structure $J$ defined by the pairs $(\rho_\ell, v_\ell)$ is such that

$$J(z_\ell^\rho) = iz_\ell^{\rho*}, \quad J(z_\ell^{\rho*}) = -iz_\ell^{\rho}.$$  \hspace{1cm} (7)

The unitarity of the dynamics of the field $\xi$ depends on the quantum representation, and therefore on the complex structure $J$ which determines it. For each $J$ and each pair $t_0, t_1 \in (0, \pi)$, the symplectic transformation defined by classical evolution from time $t_0$ to time $t_1$ is determined by the Bogoliubov coefficients [13]$^5$:

$$\alpha_\ell^J(t_0, t_1) = z_\ell^J(t_1)[z_\ell^{\rho*}(t_0) - \frac{1}{2} \cot t_0 z_\ell^{\rho*}(t_0)] - z_\ell^J(t_0)[\dot{z}_\ell^J(t_1) - \frac{1}{2} \cot t_1 z_\ell^{\rho*}(t_1)],$$

$$\beta_\ell^J(t_0, t_1) = z_\ell^J(t_1)[\dot{z}_\ell^J(t_0) - \frac{1}{2} \cot t_0 \dot{z}_\ell^{\rho*}(t_0)] - z_\ell^J(t_0)[\dot{z}_\ell^J(t_1) - \frac{1}{2} \cot t_1 \dot{z}_\ell^{\rho*}(t_1)].$$  \hspace{1cm} (8)

$^5$ Reference [13] adopts a non-standard notation for the Bogoliubov coefficients, which we also follow here to avoid confusion. The standard coefficients are $-i\alpha_\ell^J$ and $-i\beta_\ell^J$. 

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with $\alpha^J_\ell(t_0,t_1)$ and $\beta^J_\ell(t_0,t_1)$ being the linear and antilinear part of the transformation, respectively.

It follows from well-known general results \cite{11, 15} that the evolution from $t_0$ to $t_1$ is unitarily implementable in the Fock representation defined by the complex structure $J$ iff the sequence $\{\beta^J_\ell(t_0,t_1)\}$ is square summable (SQS); i.e., the dynamics is unitarily implementable iff $\sum_{\ell=0}^{\infty} |\beta^J_\ell(t_0,t_1)|^2 < \infty$ for all $t_0, t_1 \in (0, \pi)$.

Employing the asymptotic expansion of the Legendre functions for large values of $\ell$,\(^6\) given e.g. in \cite{16}, one can see that this condition of square summability is satisfied for a large subcase of $SO(3)$-invariant complex structures, which includes in particular the complex structure determined by $\rho_\ell = \sqrt{\pi/2}$ and $\nu_\ell = 0 \forall \ell$.

### 3. Uniqueness of the quantization

Let $J_F$ denote the complex structure defined by the particular values $\rho_\ell = \sqrt{\pi/2}$ and $\nu_\ell = 0 \forall \ell$. We will call $\{z^J_\ell, z^{J*}_\ell\}$ the set of complex classical solutions associated with $J_F$. On the other hand, let us introduce the following parameters $A_\ell$ and $B_\ell$:

\[
A_\ell := \frac{1}{\sqrt{2\pi}} \left[ \rho_\ell - i \frac{\pi}{2} \left( \nu_\ell + \frac{i}{\rho_\ell} \right) \right], \\
B_\ell := \frac{1}{\sqrt{2\pi}} \left[ \rho_\ell + i \frac{\pi}{2} \left( \nu_\ell + \frac{i}{\rho_\ell} \right) \right].
\]

These parameters provide the transformation from $\{z^J_\ell, z^{J*}_\ell\}$ to the set of solutions $\{z^J_\ell, z^{J*}_\ell\}$ which corresponds to the complex structure $J$ determined by $(\rho_\ell, \nu_\ell)$, namely

\[
z^J_\ell = A_\ell z^{J*}_\ell + B_\ell z^{J*}_\ell.
\]

Note also that

\[
|A_\ell|^2 - |B_\ell|^2 = 1 \ \forall \ell.
\]

It then follows that $|A_\ell| \geq 1 \ \forall \ell$, and that the sequence $\{B_\ell/A_\ell\}$ is bounded.

More importantly, as a consequence of transformation (10), one concludes that the complex structures $J$ (with parameters $(A_\ell, B_\ell)$) and $J_F$ determine unitarily equivalent Fock representations iff the sequence $\{B_\ell\}$ is SQS (see e.g. \cite{4} for details).

We will now prove that if a complex structure $J$ is such that the sequence $\{\beta^J_\ell(t_0,t_1)\}$ is SQS $\forall t_0, t_1$, then the sequence $\{B_\ell\}$ is necessarily SQS, so that the representations determined by $J$ and $J_F$ are equivalent. In other words, we will prove that the Fock representation selected by $J_F$ is the unique (up to unitary equivalence) $SO(3)$-invariant Fock representation where the dynamics is implemented as a unitary transformation.

In order to simplify the notation, the sequences $\{\beta^J_\ell(t_0,t_1)\}$ and $\{\alpha^J_\ell(t_0,t_1)\}$ will be respectively denoted from now on by $\{\beta_\ell(t_0,t_1)\}$ and $\{\alpha_\ell(t_0,t_1)\}$. It is not difficult to see that the coefficients $\beta^J_\ell(t_0,t_1)$ and $\alpha^J_\ell(t_0,t_1)$ are related by

\[
\beta^J_\ell(t_0,t_1) = A_\ell^2 \beta_\ell(t_0,t_1) + B_\ell^2 \beta^*_\ell(t_0,t_1) + 2 A_\ell B_\ell \text{Re}[\alpha_\ell(t_0,t_1)].
\]

Here, $\text{Re}[\cdot]$ denotes the real part. Let us then suppose that $\{\beta^J_\ell(t_0,t_1)\}$ is SQS $\forall t_0, t_1 \in (0, \pi)$, so that the dynamics is unitarily implemented in the Fock representation determined by the $SO(3)$-invariant complex structure $J$. Then, since $|A_\ell| \geq 1$, the sequence $\{\beta^J_\ell(t_0,t_1)/A_\ell^2\}$ is also SQS. We have

\[
\frac{\beta^J_\ell(t_0,t_1)}{A_\ell^2} = \beta_\ell(t_0,t_1) + \frac{B_\ell^2}{A_\ell^2} \beta^*_\ell(t_0,t_1) + 2 \frac{B_\ell}{A_\ell} \text{Re}[\alpha_\ell(t_0,t_1)].
\]

\(^6\) Note that the first subdominant terms in these expansions are of order $O(\ell^{-3/2})$.  

\footnote{Note that the first subdominant terms in these expansions are of order $O(\ell^{-3/2})$.}
Given that \( \{\beta(t_0, t_1)\} \) is SQS and the sequence \( \{B_i^2/A_i^2\} \) is bounded, it follows that \( \{\beta(t_0, t_1) + (B_i^2/A_i^2)\beta^*_i(t_0, t_1)\} \) is SQS. Hence, since the space of SQS sequences is a linear space, one concludes that the sequence \((\{B_i/|A_i|\Re[\alpha(t_0, t_1)]\})\) is SQS \( \forall t_0, t_1 \in (0, \pi) \).

Using the asymptotic expansion of the Legendre functions for large \( \ell \) [16], one can check that the difference between \( \Re[\alpha(t_0, t_1)] \) and \( \sin(\ell + 1/2)(t_1 - t_0) \) is a SQS sequence \( \forall t_0, t_1 \in [\epsilon, \pi - \epsilon] \), where \( \epsilon > 0 \) is arbitrarily small. Thus, from the bounds on \( B_i/A_i \) and linearity, one gets that \((\{B_i/|A_i|\sin(\ell + 1/2)(t_1 - t_0)\})\) is also SQS. Introducing the notation \( T := t_1 - t_0 \), one then concludes that the limit

\[
\lim_{N \to \infty} \sum_{\ell=0}^{N} \frac{|B_i|^2}{|A_i|^2} \sin^2 \left( \left( \frac{\ell + 1}{2} \right) T \right) =: f(T) \tag{14}
\]

exists \( \forall T \in [0, \pi - \epsilon] \), with \( \epsilon := 2\epsilon \) an arbitrarily small positive number.

One can now apply the Luzin theorem [17], which ensures that, for every \( \delta > 0 \), there exists a measurable set \( E_\delta \subset [0, \pi - \epsilon] \) with \( \int_{E_\delta} dT < \delta \) and a function \( \phi_\delta(T) \), continuous on \([0, \pi - \epsilon]\), which coincides with \( f(T) \) on \( E_\delta \). Here, \( E_\delta \) denotes the complement set \([0, \pi - \epsilon]\) \( \setminus E_\delta \). One then gets

\[
\sum_{\ell=0}^{N} \frac{|B_i|^2}{|A_i|^2} \int_{E_\delta} \sin^2 \left( \left( \frac{\ell + 1}{2} \right) T \right) dT \leq \int_{E_\delta} f(T) dT =: I_\delta \quad \forall N, \tag{15}
\]

where \( I_\delta = \int_{E_\delta} \phi_\delta(T) dT \) is some finite number, and the inequality follows from the fact that \( f(T) \) is the limit of an increasing sequence, given by sums of non-negative terms. On the other hand, one finds

\[
\int_{E_\delta} \sin^2 \left( \left( \frac{\ell + 1}{2} \right) T \right) dT = \int_{0}^{\pi} \sin^2 \left( \left( \frac{\ell + 1}{2} \right) T \right) dT - \int_{\pi - \epsilon}^{\pi} \sin^2 \left( \left( \frac{\ell + 1}{2} \right) T \right) dT
\]

\[
- \int_{\pi - \epsilon}^{\pi} \sin^2 \left( \left( \frac{\ell + 1}{2} \right) T \right) dT \geq \frac{\pi}{2} - \epsilon - \delta \quad \forall \ell. \tag{16}
\]

Combining (15) and (16) one obtains

\[
I_\delta \geq \sum_{\ell=0}^{N} \frac{|B_i|^2}{|A_i|^2} \left( \frac{\pi}{2} - \epsilon - \delta \right) \quad \forall N. \tag{17}
\]

Since it is clearly possible to choose \( \delta \) and \( \epsilon \) such that \( \pi - 2\epsilon - 2\delta > 0 \), one concludes that

\[
\sum_{\ell=0}^{N} \frac{|B_i|^2}{|A_i|^2} \leq \frac{2I_\delta}{\pi - 2\delta - 2\epsilon} \quad \forall N, \tag{18}
\]

implying that the infinite sum \( \sum_{\ell=0}^{\infty}(|B_i|^2/|A_i|^2) \) exists.

Finally, since \( \{B_i/A_i\} \) is SQS, the ratio \( B_i/A_i \) necessarily tends to zero. In particular, it then follows from (11) that the sequence \( \{A_i\} \) is bounded. Therefore, the sequence \( \{B_i = A_i(\beta_i/A_i)\} \) is also SQS, as we wanted to prove.

4. Conclusion and further comments

The discussion presented in this work provides a natural extension to the \( S^1 \times S^2 \) and \( S^3 \) topologies of the uniqueness result obtained in [4] for the Fock quantization of the Gowdy \( T^3 \) model. For those other topologies we have proved that, among the set of complex structures that are invariant under the group of \( SO(3) \)-symmetries of the reduced model, there exists a unique unitary equivalence class such that the field evolution is implemented in the quantum
theory as a unitary transformation. We have selected as representative for this unique class the complex structure $J_F$ defined by the particular values $\rho_\ell = \sqrt{\pi/2}$ and $\nu_\ell = 0 \forall \ell$. The associated set of solutions $\{z_\ell^r, z_\ell^{r*}\}$ can be obtained from (6). It is easy to see that, for large $\ell$, these solutions have the asymptotic behavior

$$z_\ell^r = \frac{1}{\sqrt{2\ell + 1}} e^{-\frac{i}{2} (\ell + 1) \pi + i \pi/4} + O(\ell^{-3/2}).$$  \hfill (19)

Disregarding the subdominant correction $O(\ell^{-3/2})$, these are precisely the solutions that one would obtain for equation (5) in the case that the time-dependent potential term (proportional to $\csc^2 t$) could be neglected, a situation that would correspond to a stationary field equation for $\xi$.

The uniqueness proof given here is an extension of our proof for the Gowdy $T^3$ model explained in [4]. Apart from adapting some steps of the demonstration to deal with other topologies, the present proof differs from the previous version in a partial simplification of the arguments, achieved mainly by realizing that the subdominant terms in $\text{Re}[\alpha_\ell(t_0, t_1)]$ for large $\ell$ provide in fact a SQS sequence.

Another issue that we would like to comment on is the freedom in the choice of momentum conjugate to the scalar field $\xi$. The choice made in [13] has the problem of leading to a Hamiltonian that contains a contribution that is linear in the momentum. It is seen in [13] that, in the Fock quantization defined by $J_F$, the natural vacuum of the theory does not belong to the domain of the normal ordered Hamiltonian. Nonetheless, one can introduce a change of momentum of the form $P(N) = P + \cot t Q/2$, where $Q$ and $P$ are the field and its momentum evaluated in the section of constant time under consideration. This change can be understood as a time-dependent canonical transformation. It leads to a new Hamiltonian $H_0$ that is quadratic both in $Q$ and in $P(N)$ and such that its action on the vacuum is well defined. In fact, this time-dependent change of momentum can be alternatively understood as the result of a canonical transformation performed before the deparametrization of the model. The reduced Hamiltonian that one obtains from the Hilbert–Einstein action by means of the canonical transformation and the subsequent deparametrization is precisely the Hamiltonian $H_0$ alluded above.

Finally, the fact that the vacuum of the Fock representation is contained in the domain of the reduced Hamiltonian may be of practical importance for the success of certain quantization approaches. This affects not only the possibility of defining the action of the evolution operator on the vacuum (in the Schrödinger picture) as a formal series in powers of the Hamiltonian, but may also be relevant in quantization schemes that introduce discretizations in which the evolution operator is bound to be substituted by a repeated action of the Hamiltonian, as it may happen to be the case in loop quantum cosmology [18].

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Note added in proof. During the consideration of this work for publication, the authors came to know that a discussion about the uniqueness of the quantization was included in the final version of [13]. That treatment is, however, incomplete and not entirely correct. On one hand, only a small subclass of the set of complex structures that allow unitary dynamics is considered. On the other hand, the first condition in equation (4.22) is in fact not sufficient for a unitary dynamics or for the unitary equivalence of the representations.
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