An Introduction to the $q$-Laguerre–Hahn Orthogonal $q$-Polynomials

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Abstract. Orthogonal $q$-polynomials associated with $q$-Laguerre–Hahn form will be studied as a generalization of the $q$-semiclassical forms via a suitable $q$-difference equation. The concept of class and a criterion to determinate it will be given. The $q$-Riccati equation satisfied by the corresponding formal Stieltjes series is obtained. Also, the structure relation is established. Some illustrative examples are highlighted.

Key words: orthogonal $q$-polynomials; $q$-Laguerre–Hahn form; $q$-difference operator; $q$-difference equation; $q$-Riccati equation

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1 Introduction and preliminary results

The concept of the usual Laguerre–Hahn polynomials were extensively studied by several authors [1, 2, 4, 6, 8, 9, 10, 15, 18]. They constitute a very remarkable family of orthogonal polynomials taking consideration of most of the monic orthogonal polynomials sequences (MOPS) found in literature. In particular, semiclassical orthogonal polynomials are Laguerre–Hahn MOPS [15, 20]. The Laguerre–Hahn set of form (linear functional) is invariant under the standard perturbations of forms [2, 9, 18, 20]. It is well known that a usual Laguerre–Hahn polynomial satisfies a fourth order differential equation with polynomials coefficients but the converse remains not proved until now [20]. Discrete Laguerre–Hahn polynomials were studied in [13]. These families are already extensions of discrete semiclassical polynomials [19]. In literature, analysis and characterization of the $q$-Laguerre–Hahn orthogonal $q$-polynomials have not been yet presented in a unified way. However, several authors have studied the fourth order $q$-difference equation related to some examples of $q$-Laguerre–Hahn orthogonal $q$-polynomials such as the co-recursive and the $r$th associated of $q$-classical polynomials [11, 12]. More generally, the fourth order difference equation of Laguerre–Hahn orthogonal on special non-uniform lattices polynomials was established in [4]. For other relevant works in the domain of orthogonal $q$-polynomials and $q$-difference equation theory see [3, 21] and [5].

So the aim of this contribution is to establish a basic theory of $q$-Laguerre–Hahn orthogonal $q$-polynomials. We give some characterization theorems for this case such as the structure relation and the $q$-Riccati equation. We extend the concept of the class of the usual Laguerre–Hahn forms to the $q$-Laguerre–Hahn case. Moreover, we show that some standard transformation and perturbation carried out on the $q$-Laguerre–Hahn forms lead to new $q$-Laguerre–Hahn forms; the class of the resulting forms is analyzed and some examples are treated.

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The product defined as before is commutative [20]. Particu-
larly, the inverse exists is defined by

\[ (Tu, f) = (u, Tf), \quad u \in \mathcal{P}', \quad f \in \mathcal{P}. \]

For instance, for any form \( u \), any polynomial \( g \) and any \((a, c) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C} \), we let \( H_q u, gu, h_a u, Du, (x - c)^{-1}u \) and \( \delta_c \), be the forms defined as usually [20] and [16] for the results related to the operator \( H_q \)

\[
\begin{align*}
    &\langle H_q u, f \rangle := -\langle u, H_q f \rangle, \\
    &\langle gu, f \rangle := \langle u, g f \rangle, \\
    &\langle h_a u, f \rangle := \langle u, h_a f \rangle, \\
    &\langle Du, f \rangle := -\langle u, f' \rangle, \\
    &\langle (x - c)^{-1} u, f \rangle := \langle u, \theta c f \rangle, \\
    &\langle \delta_c, f \rangle := f(c),
\end{align*}
\]

where for all \( f \in \mathcal{P} \) and \( q \in \tilde{\mathbb{C}} := \{ z \in \mathbb{C}, \ z \neq 0, \ z^n \neq 1, \ n \geq 1 \} \) [16]

\[
(H_q f)(x) = \frac{f(qx) - f(x)}{(q - 1)x}, \quad (h_a f)(x) = f(ax), \quad (\theta c f)(x) = \frac{f(x) - f(c)}{x - c}.
\]

In particular, this yields to

\[
(H_q u)_n = -[n]_q (u)_{n-1}, \quad n \geq 0,
\]

where \((u)_{-1} = 0 \) and \([n]_q := \frac{q^n - 1}{q - 1}, \ n \geq 0 \) [15]. It is obvious that when \( q \to 1 \), we meet again the derivative \( D \).

For \( f \in \mathcal{P} \) and \( u \in \mathcal{P}' \), the product \( uf \) is the polynomial [20]

\[
(uf)(x) := \langle u, \frac{x f(x) - \zeta f(\zeta)}{x - \zeta} \rangle = \sum_{i=0}^{n} \left( \sum_{j=i}^{n} (u)_{j-i} f_j \right) x^i,
\]

where \( f(x) = \sum_{i=0}^{n} f_i x^i \). This allows us to define the Cauchy’s product of two forms:

\[
\langle uv, f \rangle := \langle u, vf \rangle, \quad f \in \mathcal{P}.
\]

The product defined as before is commutative [20]. Particularly, the inverse \( u^{-1} \) of \( u \) if there exists is defined by \( uu^{-1} = \delta_0 \).

The Stieltjes formal series of \( u \in \mathcal{P}' \) is defined by

\[
S(u)(z) := -\sum_{n \geq 0} \frac{(u)_n}{z^{n+1}}.
\]

A form \( u \) is said to be regular whenever there is a sequence of monic polynomials \( \{P_n\}_{n \geq 0} \), \( \deg P_n = n, \ n \geq 0 \) such that \( \langle u, P_n P_m \rangle = r_n \delta_{n,m} \) with \( r_n \neq 0 \) for any \( n, m \geq 0 \). In this case, \( \{P_n\}_{n \geq 0} \) is called a monic orthogonal polynomials sequence MOPS and it is characterized by the following three-term recurrence relation (Favard’s theorem)

\[
P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\
P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1} P_n(x), \quad n \geq 0,
\]

(1.1)

where \( \beta_n = \frac{(u,x^2 f_n^2)}{r_n} \in \mathbb{C}, \ \gamma_{n+1} = \frac{r_{n+1}}{r_n} \in \mathbb{C} \setminus \{0\}, \ n \geq 0. \)
The shifted MOPS \( \{ \tilde{P}_n := a^{-n}(h_a P_n) \}_{n \geq 0} \) is then orthogonal with respect to \( \tilde{u} = h_{a-1} u \) and satisfies (1.1) with [20]

\[
\tilde{\beta}_n = \frac{\beta_n}{a}, \quad \tilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, \quad n \geq 0.
\]

Moreover, the form \( u \) is said to be normalized if \( (u)_0 = 1 \). In this paper, we suppose that any form will be normalized.

The form \( u \) is said to be positive definite if and only if \( \beta_n \in \mathbb{R} \) and \( \gamma_{n+1} > 0 \) for all \( n \geq 0 \). When \( u \) is regular, \( \{ P_n \}_{n \geq 0} \) is a symmetrical MOPS if and only if \( \beta_n = 0, n \geq 0 \) or equivalently \( (u)_{2n+1} = 0, n \geq 0 \).

Given a regular form \( u \) and the corresponding MOPS \( \{ P_n \}_{n \geq 0} \), we define the associated sequence of the first kind \( \{ P_n^{(1)} \}_{n \geq 0} \) by [20] equations (2.8) and (2.9)

\[
P_n^{(1)}(x) = \langle u, \frac{P_{n+1}(x) - P_{n+1}(\xi)}{x - \xi} \rangle = (u_0 P_{n+1})(x), \quad n \geq 0.
\]

The following well known results (see [16, 17, 20]) will be needed in the sequel.

**Lemma 1.** Let \( u \in \mathcal{P}' \). \( u \) is regular if and only if \( \Delta_n(u) \neq 0, n \geq 0 \) where

\[
\Delta_n(u) := \det \left( (u)_{\mu+\nu} \right)_{\mu, \nu = 0}^{n}, \quad n \geq 0
\]

are the Hankel determinants.

**Lemma 2.** For \( f, g \in \mathcal{P}, u, v \in \mathcal{P}' \), \( (a, b, c) \in \mathbb{C} \setminus \{ 0 \} \times \mathbb{C}^2 \), and \( n \geq 1 \), we have

\[
(x - c)((x - c)^{-1} u) = u, \quad (x - c)^{-1}((x - c)u) = u - (u)_{0}\delta_c,
\]

\[
(u_0 f)(x) = a_n x^{n-1}(u)_0 + \text{lower order terms}, \quad f(x) = \sum_{k=0}^{n} a_k x^k,
\]

\[
u_0(f g) = (u_0 f) + (f u)\theta_0 g,
\]

\[
u_0(f P_{k+1}) = f P_k^{(1)}, \quad k + 1 \geq \deg f,
\]

\[
\theta_b - \theta_c = (b - c) \theta_b \circ \theta_c, \quad \theta_b \circ \theta_c = \theta_c \circ \theta_b,
\]

\[
h_a(gu) = (h_{a-1})(h_{a}u), \quad h_a(uv) = (h_{a}u)(h_{a}v), \quad h_a(x^{-1}u) = ax^{-1}h_{a}u,
\]

\[
h_{q^{-1}} \circ H_q = H_{q^{-1}}, \quad H_q \circ h_{q^{-1}} = q^{-1}H_{q-1}, \quad \text{in } \mathcal{P},
\]

\[
h_{q^{-1}} \circ H_q = q^{-1}H_{q^{-1}}, \quad H_q \circ h_{q^{-1}} = H_{q^{-1}}, \quad \text{in } \mathcal{P}',
\]

\[
H_q(fg)(x) = \left( h_{q}f \right)((h_{q}g)(x) + g(x)(h_{q}f)(x),
\]

\[
H_q(gu) = (h_{q^{-1}}g)H_q u + q^{-1}(H_{q^{-1}}g)u,
\]

\[
H_{q^{-1}}(u_0 f)(x) = q(H_{q}u)\theta_0(h_{q^{-1}}f)(x) + (u_0 h_{q^{-1}}f)(x),
\]

\[
S(fu)(z) = f(z)S(u)(z) + (u_0 f)(z),
\]

\[
S(uv)(z) = -z S(u)(z)S(v)(z),
\]

\[
S(x^{-n}u)(z) = z^{-n}S(u)(z), \quad S(u^{-1})(z) = z^{-2}(S(u)(z))^{-1},
\]

\[
S(h_{q}u)(z) = q^{-1}(H_{q^{-1}}(S(u)))(z), \quad (h_{q^{-1}}S(u))(z) = q S(h_{q}u)(z).
\]

**Definition 1.** A form \( u \) is called \( q \)-Laguerre–Hahn when it is regular and satisfies the \( q \)-difference equation

\[
H_q(\Phi u) + \Psi u + B(x^{-1}u(h_{q}u)) = 0,
\]

where \( \Phi, \Psi, B \) are polynomials, with \( \Phi \) monic. The corresponding orthogonal sequence \( \{ P_n \}_{n \geq 0} \) is called \( q \)-Laguerre–Hahn MOPS.
Remark 1. When \( B = 0 \) and the form \( u \) is regular then \( u \) is \( q \)-semiclassical [17]. When \( u \) is regular and not \( q \)-semiclassical then \( u \) is called a strict \( q \)-Laguerre–Hahn form.

Lemma 3. Let \( u \) be a regular form. If \( u \) is a strict \( q \)-Laguerre–Hahn form satisfying (1.17) and there exist two polynomials \( \Delta \) and \( \Omega \) such that

\[
\Delta u + \Omega(x^{-1}u(h_qu)) = 0
\]

then \( \Delta = \Omega = 0 \).

Proof. The operation \( \Delta \times (1.17) - B \times (1.18) \) gives

\[
\Omega H_q(\Phi u) + (\Omega \Psi - \Delta B) u = 0.
\]

According to (1.9) and (1.11), the above equation becomes

\[
H_q((h_q \Omega)\Phi u) + (\Omega \Psi - (H_q \Omega) \Phi - \Delta B)u = 0.
\]

Then \( \Delta = \Omega = 0 \) because the form \( u \) is regular and not \( q \)-semiclassical.

Lemma 4. Consider the sequence \( \{\hat{P}_n\}_{n \geq 0} \) obtained by shifting \( P_n \), i.e. \( \hat{P}_n(x) = a^{-n}P_n(ax) \), \( n \geq 0, a \neq 0 \). When \( u \) satisfies (1.17), then \( \hat{u} = h_{a-1}u \) fulfills the \( q \)-difference equation

\[
H_q(\hat{\Phi} \hat{u}) + \hat{\Psi} \hat{u} + \hat{B}(x^{-1}\hat{u}(h_q \hat{u})) = 0,
\]

where \( \hat{\Phi}(x) = a^{-\deg \Phi \Phi(ax), \hat{\Psi}(x) = a^{1-\deg \Phi \Psi(ax), \hat{B}(x) = a^{-\deg B B(ax).}} \)

Proof. With \( u = h_a \hat{u} \), we have \( \Psi u = \Psi(h_a \hat{u}) = h_a((h_a \Psi)\hat{u}) \) from (1.7). Further,

\[
H_q(\Phi u) = H_q(\Phi(h_a \hat{u})) = H_q(h_a((h_a \Phi)\hat{u})) = a^{-1}h_a(H_q((h_a \Phi)\hat{u}))
\]

from (1.7) and (1.9).

Moreover, by virtue of (1.7) an other time we get

\[
B(x^{-1}u(h_qu)) = B(x^{-1}(h_a \hat{u})(h_q \hat{u})) = B(x^{-1}h_a(\hat{u}h_q \hat{u})) = a^{-1}h_a((h_a B)(x^{-1}u(h_q \hat{u}))).
\]

Equation (1.17) becomes

\[
h_a(H_q(\Phi(ax) \hat{u}) + a\Psi(ax) \hat{u} + B(ax)(x^{-1}u(h_q \hat{u}))) = 0.
\]

Hence the desired result.

2 Class of a \( q \)-Laguerre–Hahn form

It is obvious that a \( q \)-Laguerre–Hahn form satisfies an infinite number of \( q \)-difference equations type (1.17). Indeed, multiplying (1.17) by a polynomial \( \chi \) and taking into account (1.7), (1.11) we obtain

\[
H_q((h_q \chi) \Phi u) + \{ \chi \Psi - \Phi(h_q \chi) \} u + (\chi B)(x^{-1}u(h_qu)) = 0.
\]

Put \( t = \deg \Phi, p = \deg \Psi, r = \deg B \) with \( d = \max(t, r) \) and \( s = \max(p - 1, d - 2) \). Thus, there exists \( u \rightarrow h(u) \subset \mathbb{N} \cup \{-1\} \) from the set of \( q \)-Laguerre–Hahn forms into the subsets of \( \mathbb{N} \cup \{-1\} \).

Definition 2. The minimum element of \( h(u) \) will be called the class of \( u \). When \( u \) is of class \( s \), the sequence \( \{P_n\}_{n \geq 0} \) orthogonal with respect to \( u \) is said to be of class \( s \).
Proposition 1. The number \( s \) is an integer positive or zero. In other words, if \( p \equiv 0 \), then \( d \geq 2 \) or if \( 0 \leq d \leq 1 \), then necessarily \( p \geq 1 \).

Proof. Let us show that in case \( s = -1 \), the form \( u \) is not regular, which is a contradiction. Indeed, when \( s = -1 \), we have

\[
\Phi(x) = c_1 x + c_0, \quad \Psi(x) = a_0, \quad B(x) = b_1 x + b_0
\]

with \( c_1 = 1 \) or \( c_1 = 0 \) and \( c_0 = 1 \), and where \( a_0 \neq 0 \).

The condition \( \langle H_q(\Phi u) + \Psi u + B(x^{-1} u(h_q u)), x^n \rangle = 0, 0 \leq n \leq 4 \) gives successively

\[
a_0 + b_1 = 0,
\]

\[
(qb_1 - c_1)(u)_1 + b_0 - c_0 = 0, \tag{2.2}
\]

\[
(q^2 b_1 - (1 + q)c_1)((u)_2 - (u)_1^2) = 0, \tag{2.3}
\]

\[
(q^3 b_1 - (1 + q + q^2)c_1)(u)_3 + \{ (1 + q^2)b_0 + q(1 + q)b_1(u)_1 - (1 + q + q^2)c_0 \}(u)_2
\]

\[
+ qb_0(u)_1^2 = 0, \tag{2.4}
\]

\[
(q^4 b_1 - (1 + q)(1 + q^2)c_1)(u)_4 + \{ (1 + q^3)b_0 + q(1 + q^2)b_1(u)_1 - (1 + q)(1 + q^2)c_0 \}(u)_3
\]

\[
+ q^2 b_1(u)_2^2 + q(1 + q)b_0(u)_1(u)_2 = 0. \tag{2.5}
\]

Suppose \( q^2 b_1 - (1 + q)c_1 \neq 0 \). From (2.3)

\[
\Delta_1 = \begin{vmatrix}
1 & (u)_1 \\
(u)_1 & (u)_2
\end{vmatrix}
\]

Contradiction.

Suppose \( q^2 b_1 = (1 + q)c_1 = 0 \) implies \( b_1 = 0 = c_1 \) implies (2.2) \( b_0 = c_0 = 1 \). Thus (2.4)

\[
(u)_2 - (u)_1^2 = 0, \quad \text{hence} \quad \Delta_1 = 0. \quad \text{Contradiction.}
\]

Suppose \( q^2 b_1 = (1 + q)c_1 \neq 0 \) with \( c_1 = 1 \). From (2.2) and (2.4), (2.5), we have

\[
(u)_1 = q(c_0 - b_0),
\]

\[
(u)_3 = q(c_0 - 2b_0)(u)_2 + q^3 b_0(c_0 - b_0)^2, \tag{2.6}
\]

\[
(u)_4 = (u)_2^2 + q^2 b_0^2(u)_2 - q^4 b_0^2(c_0 - b_0)^2.
\]

On the other hand, let us consider the Hankel determinant

\[
\Delta_2 = \begin{vmatrix}
1 & (u)_1 & (u)_2 \\
(u)_1 & (u)_2 & (u)_3 \\
(u)_2 & (u)_3 & (u)_4
\end{vmatrix}
\]

With (2.0), we get \( \Delta_2 = 0 \). Contradiction.

\[\blacksquare\]

Proposition 2. Let \( u \) be a strict \( q \)-Laguerre–Hahn form satisfying

\[
H_q(\Phi_1 u) + \Psi_1 u + B_1(x^{-1} u(h_q u)) = 0, \tag{2.7}
\]

and

\[
H_q(\Phi_2 u) + \Psi_2 u + B_2(x^{-1} u(h_q u)) = 0, \tag{2.8}
\]

where \( \Phi_1, \Psi_1, B_1, \Phi_2, \Psi_2, B_2 \) are polynomials, \( \Phi_1, \Phi_2 \) monic and \( \deg \Phi_i = t_i \), \( \deg \Psi_i = p_i \), \( \deg B_i = r_i \), \( d_i = \max(t_i, r_i) \), \( s_i = \max(p_i - 1, d_i - 2) \) for \( i \in \{1, 2\} \). Let \( \Phi = \gcd(\Phi_1, \Phi_2) \). Then, there exist two polynomials \( \Psi \) and \( B \) such that

\[
H_q(\Phi u) + \Psi u + B(x^{-1} u(h_q u)) = 0, \tag{2.9}
\]
with

\[ s = \max(p - 1, d - 2) = s_1 - t_1 + t = s_2 - t_2 + t, \]  

(2.10)

where \( t = \deg \Phi, \ p = \deg \Psi, \ r = \deg B \) and \( d = \max(t, r) \).

**Proof.** With \( \Phi = \gcd(\Phi_1, \Phi_2) \), there exist two co-prime polynomials \( \tilde{\Phi}_1, \tilde{\Phi}_2 \) such that

\[ \Phi_1 = \Phi \tilde{\Phi}_1, \quad \Phi_2 = \Phi \tilde{\Phi}_2. \]  

(2.11)

Taking into account equations (1.11) become for \( i \in \{1, 2\} \)

\[ (h_{q^{-1}} \tilde{\Phi}_1) H_q(\Phi u) + \{\Psi_i + q^{-1} H_{q^{-1}} \tilde{\Phi}_i\} u + B_i(x^{-1} u h_q u) = 0. \]  

(2.12)

The operation \( (h_{q^{-1}} \tilde{\Phi}_1) \times (2.12_{-1}) - (h_{q^{-1}} \tilde{\Phi}_1) \times (2.12_{-2}) \) gives

\[ \{(h_{q^{-1}} \tilde{\Phi}_1) (\Psi_1 + q^{-1} \Phi (H_{q^{-1}} \tilde{\Phi}_1)) - (h_{q^{-1}} \tilde{\Phi}_1) (\Psi_2 + q^{-1} \Phi (H_{q^{-1}} \tilde{\Phi}_2))\} u \]
\[ + \{(h_{q^{-1}} \tilde{\Phi}_2) B_1 - (h_{q^{-1}} \tilde{\Phi}_1) B_2\} (x^{-1} u h_q u) = 0. \]

From the fact that \( u \) is a strict \( q \)-Laguerre–Hahn form and by virtue of Lemma 3 we get

\[ (h_{q^{-1}} \tilde{\Phi}_1) (\Psi_2 + q^{-1} \Phi (H_{q^{-1}} \tilde{\Phi}_2)) = (h_{q^{-1}} \tilde{\Phi}_2) (\Psi_1 + q^{-1} \Phi (H_{q^{-1}} \tilde{\Phi}_1)), \]
\[ (h_{q^{-1}} \tilde{\Phi}_1) B_2 = (h_{q^{-1}} \tilde{\Phi}_2) B_1. \]

Thus, there exist two polynomials \( \Psi \) and \( B \) such that

\[ \Psi_1 + q^{-1} \Phi (H_{q^{-1}} \tilde{\Phi}_1) = (h_{q^{-1}} \tilde{\Phi}_1) \Psi, \quad \Psi_2 + q^{-1} \Phi (H_{q^{-1}} \tilde{\Phi}_2) = (h_{q^{-1}} \tilde{\Phi}_2) \Psi, \]
\[ B_1 = (h_{q^{-1}} \tilde{\Phi}_1) B, \quad B_2 = (h_{q^{-1}} \tilde{\Phi}_2) B. \]  

(2.13)

Then, formulas (2.7), (2.8) become

\[ (h_{q^{-1}} \tilde{\Phi}_i) \{H_q(\Phi u) + \Psi u + B(x^{-1} u h_q u)\} = 0, \quad i \in \{1, 2\}. \]  

(2.14)

But the polynomials \( h_{q^{-1}} \tilde{\Phi}_1 \) and \( h_{q^{-1}} \tilde{\Phi}_2 \) are also co-prime. Using the Bezzout identity, there exist two polynomials \( A_1 \) and \( A_2 \) such that

\[ A_1(h_{q^{-1}} \tilde{\Phi}_1) + A_2(h_{q^{-1}} \tilde{\Phi}_2) = 1. \]

Consequently, the operation \( A_1 \times (2.14_{-1}) + A_2 \times (2.14_{-2}) \) leads to (2.9). With (2.11) and (2.13) it is easy to prove (2.10). \( \blacksquare \)

**Proposition 3.** For any \( q \)-Laguerre–Hahn form \( u \), the triplet \((\Phi, \Psi, B)\) \((\Phi \) monic\) which realizes the minimum of \( h(u) \) is unique.

**Proof.** If \( s_1 = s_2 \) in (2.9), (2.10) and \( s_1 = s_2 = s = \min h(u) \), then \( t_1 = t = t_2 \). Consequently, \( \Phi_1 = \Phi = \Phi_2, \ B_1 = B = B_2 \) and \( \Psi_1 = \Psi = \Psi_2 \). \( \blacksquare \)

Then, it’s necessary to give a criterion which allows us to simplify the class. For this, let us recall the following lemma:
Lemma 5. Consider u a regular form, Φ, Ψ and B three polynomials, Φ monic. For any zero \( c \) of Φ, denoting
\[
\Phi(x) = (x - c)\Phi_c(x),
\]
\[
q\Phi(x) + \Phi_c(x) = (x - cq)\Phi_{cq}(x) + r_{cq},
\]
\[
qB(x) = (x - cq)B_{cq}(x) + b_{cq}.
\]

The following statements are equivalent:
\[
H_q(\Phi u) + \Psi u + B(x^{-1}uh_q u) = 0,
\]
\[
H_q(\Phi_c u) + \Psi_{cq} u + B_{cq}(x^{-1}uh_q u) + r_{cq}(x - cq)^{-1}u + b_{cq}(x - cq)^{-1}(x^{-1}uh_q u)
\]
\[
\quad - \{ \langle u, \Psi_{cq} \rangle + \langle x^{-1}uh_q u, B_{cq} \rangle \} \delta_c = 0.
\]

Proof. The proof is obtained straightforwardly by using the relations in (1.2) and in (2.16). ■

Proposition 4. A regular form \( u \) -Laguerre–Hahn satisfying (1.17) is of class \( s \) if and only if
\[
\prod_{c \in \mathbb{Z}_q} \left\{ |q(h_q\Psi)(c) + (H_q\Phi)(c)| + |q(h_q B)(c)| 
\right\}
\[
+ \left\{ \langle u, q(\theta_{cq}\Psi) + (\theta_{cq} \circ \phi_{\Phi}) + q(h_q u(\theta_0 \circ \theta_{cq} B)) \rangle \right\} > 0,
\]
where \( \mathbb{Z}_q \) is the set of roots of \( \Phi \).

Proof. Let \( c \) be a root of \( \Phi \): \( \Phi(x) = (x - c)\Phi_c(x) \). On account of (2.15) we have
\[
r_{cq} = q\Psi(cq) + \Phi_c(cq) = q(h_q\Psi)(c) + (H_q\Phi)(c),
\]
\[
b_{cq} = qB(cq) = q(h_q B)(c),
\]
\[
\Psi_{cq}(x) = q(\theta_{cq}\Psi)(x) + (\theta_{cq} \circ \phi_{\Phi})(x) = q(\theta_{cq}\Psi)(x) + (\theta_{cq} \circ \phi_{\Phi})(x),
\]
\[
B_{cq}(x) = q(\theta_{cq} B)(x).
\]

Therefore,
\[
\langle u, \Psi_{cq} \rangle + \langle x^{-1}uh_q u, B_{cq} \rangle = \langle u, q(\theta_{cq}\Psi) + (\theta_{cq} \circ \phi_{\Phi}) \rangle + \langle uh_q u, q\theta_0 \circ \theta_{cq} B \rangle
\]
\[
\quad = \langle u, q(\theta_{cq}\Psi) + (\theta_{cq} \circ \phi_{\Phi}) \rangle + \langle u, q(h_q u(\theta_0 \circ \theta_{cq} B)) \rangle
\]
\[
\quad = \langle u, q(\theta_{cq}\Psi) + (\theta_{cq} \circ \phi_{\Phi}) + q(h_q u(\theta_0 \circ \theta_{cq} B)) \rangle.
\]

The condition (2.17) is necessary. Let us suppose that \( c \) fulfils the conditions
\[
r_{cq} = 0, \quad b_{cq} = 0, \quad \langle u, q(\theta_{cq}\Psi) + (\theta_{cq} \circ \phi_{\Phi}) + q(h_q u(\theta_0 \circ \theta_{cq} B)) \rangle = 0.
\]

Then on account of Lemma 5 (2.16) becomes
\[
H_q(\Phi_c u) + \Psi_{cq} u + B_{cq}(x^{-1}uh_q u) = 0
\]
with \( s_c = \max(\max(\deg \Phi_c, \deg B_{cq}) - 2, \deg \Psi_c - 1) < s \), what contradicts with \( s := \min h(u) \).

The condition (2.17) is sufficient. Let us suppose \( u \) to be of class \( \tilde{s} < s \). There exist three polynomials \( \tilde{\Phi} \) (monic) \( \deg \tilde{\Phi} = \tilde{t} \), \( \tilde{\Psi} \), \( \deg \tilde{\Phi} = \tilde{p} \), \( B \), \( \deg B = \tilde{r} \) such that
\[
H_q(\tilde{\Phi} u) + \tilde{\Psi} u + \tilde{B}(x^{-1}uh_q u) = 0
\]
with \( \tilde{s} = \max(\tilde{d} - 2, \tilde{p} - 1) \) where \( \tilde{d} := \max(\tilde{t}, \tilde{r}) \). By Proposition 2 it exists a polynomial \( \chi \) such that
\[
\Phi = \chi \tilde{\Phi}, \quad \Psi = (h_q^{-1}\chi)\tilde{\Psi} - q^{-1}(H_q^{-1}\chi)\tilde{\Phi}, \quad B = (h_q^{-1}\chi)\tilde{B}.
\]
Since $\tilde{s} < s$ hence $\deg \chi \geq 1$. Let $c$ be a zero of $\chi : \chi(x) = (x - c)\chi_c(x)$. On account of (1.10) we have
\[q\Psi(x) + \Phi_c(x) = (x - cq)\{(h_{q^{-1}}\chi_c)(x)\tilde{\Psi}(x) - q^{-1}(H_{q^{-1}}\chi_c)(x)\tilde{\Phi}(x)\}.\]
Thus $r_{cq} = 0$ and $b_{cq} = 0$. Moreover, with (1.8) we have
\[
\langle u, q(\theta_{cq}\Psi) + (\theta_{cq} \circ \theta_c\Phi) + q(h_qu(\theta_0 \circ \theta_{cq}B)) \rangle
= \langle u, (h_{q^{-1}}\chi_c)\tilde{\Psi} - q^{-1}(H_{q^{-1}}\chi_c)\tilde{\Phi} + (h_qu)\theta_0((h_{q^{-1}}\chi_c)\tilde{B}) \rangle
= \langle \tilde{\Psi}u, h_{q^{-1}}\chi_c \rangle + \langle H_q(\tilde{\Phi}u), h_{q^{-1}}\chi_c \rangle + (\tilde{B}(x^{-1}uh_qu), h_{q^{-1}}\chi_c) = 0.
\]
This is contradictory with (2.17). Consequently, $\tilde{s} = s$, $\tilde{\Phi} = \Phi$, $\tilde{\Psi} = \Psi$ and $\tilde{B} = B$.

**Remark 2.** When $q \rightarrow 1$ we recover again the criterion which allows us to simplify a usual Laguerre–Hahn form [6].

**Remark 3.** When $B = 0$ and $s = 0$, the form $u$ is usually called $q$-classical [16]. When $B = 0$ and $s = 1$, the symmetrical $q$-semiclassical orthogonal $q$-polynomials of class one are exhaustively described in [14].

**Proposition 5.** Let $u$ be a symmetrical $q$-Laguerre–Hahn form of class $s$ satisfying (1.17). The following statements hold

(i) If $s$ is odd, then the polynomials $\Phi$ and $B$ are odd and $\Psi$ is even.

(ii) If $s$ is even, then the polynomials $\Phi$ and $B$ are even and $\Psi$ is odd.

**Proof.** Writing
\[
\Phi(x) = \Phi^e(x^2) + x\Phi^o(x^2), \quad \Psi(x) = \Psi^e(x^2) + x\Psi^o(x^2), \quad B(x) = B^e(x^2) + xB^o(x^2),
\]
then (1.17) becomes
\[
H_q(\Phi^e(x^2)u) + x\Phi^o(x^2)u + B^e(x^2)(x^{-1}uh_qu)
+ H_q(x\Phi^o(x^2)u) + \Psi^e(x^2)u + xB^o(x^2)(x^{-1}uh_qu) = 0.
\]
Denoting
\[
w^e = H_q(\Phi^e(x^2)u) + x\Psi^o(x^2)u + B^e(x^2)(x^{-1}uh_qu),
w^o = H_q(x\Phi^o(x^2)u) + \Psi^e(x^2)u + xB^o(x^2)(x^{-1}uh_qu).
\]
Then,
\[
w^o + w^e = 0. \tag{2.19}
\]
From (2.19) we get
\[\langle w^o \rangle_n = -(w^e)_n, \quad n \geq 0. \tag{2.20}\]
From definitions in (2.18) and (2.20) we can write for $n \geq 0$
\[
\langle w^e \rangle_n = \langle u, x^{2n+1}\Psi^o(x^2) - [2n]q^{2n-1}\Phi^e(x^2) \rangle + \langle uh_qu, x^{2n-1}B^e(x^2) \rangle.
\]
\[(w^o)_{2n+1} = \langle u, x^{2n+1} \Phi^o(x^2) \rangle = [2n+1]q^{2n+1}\Phi^o(x^2) + \langle uh^e_q u, x^{2n+1} B^o(x^2) \rangle. \quad (2.21)\]

Now, with the fact that \(u\) is a symmetrical form then \(uh^e_q u\) is also a symmetrical form. Indeed,

\[(uh^e_q u)_{2n+1} = \sum_{k=0}^{2n+1} (h^e_q u)_{k}(2n+1-k) = \sum_{k=0}^{2n+1} q^k(u)_{k}(2n+1-k) = 0, \quad n \geq 0.\]

Thus (2.21) gives

\[(w^o)_{2n+1} = 0 = (w^e)_{2n}, \quad n \geq 0. \quad (2.22)\]

On account of (2.19) and (2.22) we deduce \(w^o = w^e = 0\). Consequently \(u\) satisfies two \(q\)-difference equations

\[H_q(\Phi^e(x^2)u) + x^2 \Phi^o(x^2)u + B^e(x^2)(x^{-1}uh^e_q u) = 0, \quad (2.23)\]

and

\[H_q(x^2 \Phi^o(x^2)u) + \Phi^e(x^2)u + x^2 B^e(x^2)(x^{-1}uh^e_q u) = 0. \quad (2.24)\]

(i) If \(s = 2k+1, \) with \(s = \max(d-2, p-1)\) we get \(d \leq 2k+3, \) \(p \leq 2k+2\) then \(\deg(x^2 \Phi^o(x^2)) \leq 2k+1, \) \(\deg(\Phi^o(x^2)) \leq 2k+2 \) and \(\deg(B^o(x^2)) \leq 2k+2.\) So, in accordance with (2.23), we obtain the contradiction \(s = 2k+1 \leq 2.\) Necessary \(\Phi^e = B^e = \Psi^o = 0.\)

(ii) If \(s = 2k, \) with \(s = \max(d-2, p-1)\) we get \(d \leq 2k+2, \) \(p \leq 2k+1\) then \(\deg(\Psi^e(x^2)) \leq 2k, \) \(\deg(x \Phi^o(x^2)) \leq 2k+1 \) and \(\deg(x^2 B^o(x^2)) \leq 2k+1.\) So, in accordance with (2.24), we obtain the contradiction \(s = 2k \leq 2k-1.\) Necessary \(\Phi^o = B^o = \Psi^o = 0.\) Hence the desired result.  

3 Different characterizations of \(q\)-Laguerre–Hahn forms

One of the most important characterizations of the \(q\)-Laguerre–Hahn forms is given in terms of a non homogeneous second order \(q\)-difference equation so called \(q\)-Riccati equation fulfilled by its formal Stieltjes series. See also [6] [8] [10] [15] for the usual case and [13] for the discrete one.

**Proposition 6.** Let \(u\) be a regular form. The following statement are equivalents:

(a) \(u\) belongs to the \(q\)-Laguerre–Hahn class, satisfying (1.17).

(b) The Stieltjes formal series \(S(u)\) satisfies the \(q\)-Riccati equation

\[(h_{q^{-1}}(\Phi)(z)H_{q^{-1}}(S(u))(z) = B(z)S(u)(z)(h_{q^{-1}}S(u))(z) + C(z)S(u)(z) + D(z), \quad (3.1)\]

where \(\Phi\) and \(B\) are polynomials defined in (1.17) and

\[C(z) = -(h_{q^{-1}}(\Phi)(z) - q(\Psi(z)),\]

\[D(z) = \{-h_{q^{-1}}(u \theta^2_0 \Phi)(z) + q(u \theta^2_0 \Psi)(z) + q(uh^e_q u)(\theta^2_0 B)(z)\}. \quad (3.2)\]

**Proof.** (a) \(\Rightarrow\) (b). Suppose that (a) is satisfied, then there exist three polynomials \(\Phi\) (monic), \(\Psi\) and \(B\) such that \(H_q(\Phi u) + \Psi u + B(x^{-1}uh^e_q u) = 0.\) From (1.11) the above \(q\)-difference equation becomes

\[(h_{q^{-1}}(\Phi)(H_q u) + \{\Psi + q^{-1}(H_{q^{-1}}(\Phi))\} u + B(x^{-1}uh^e_q u) = 0.\]
From definition of $S(u)$ and the linearity of $S$ we obtain
\[
S((h_{q^{-1}}\Phi)(H_q u))(z) + S(\Psi u)(z) + q^{-1}S((H_{q^{-1}}\Phi)u)(z) + S(B(x^{-1}uh_q u))(z) = 0. \tag{3.3}
\]

Moreover,
\[
S(\Psi u)(z) & = \Psi(z)S(u)(z) + (u\theta_0\Psi)(z), \\
q^{-1}S((H_{q^{-1}}\Phi)u)(z) & = q^{-1}(H_{q^{-1}}\Phi)(z)S(u)(z) + q^{-1}(u\theta_0(H_{q^{-1}}\Phi))(z), \\
S((h_{q^{-1}}\Phi)(H_q u))(z) & = (h_{q^{-1}}\Phi)(z)S(H_q u)(z) + (H_q u)\theta_0(h_{q^{-1}}\Phi)(z), \\
S(B(x^{-1}uh_q u))(z) & = B(z)S(x^{-1}uh_q u)(z) + ((x^{-1}uh_q u)\theta_0 B)(z),
\]

and
\[
(u\theta_0(H_{q^{-1}}\Phi))(z) + q((H_q u)\theta_0(h_{q^{-1}}\Phi))(z) & = H_{q^{-1}}(u\theta_0\Phi)(z).
\]

(3.3) becomes
\[
(h_{q^{-1}}\Phi)(z)H_{q^{-1}}(S(u))(z) = B(z)S(u)(z)(h_{q^{-1}}S(u))(z) - (H_{q^{-1}}\Phi + q\Psi)(z)S(u)(z) - \left\{ H_{q^{-1}}(u\theta_0\Phi) + qu\theta_0\Psi + q(uh_q u)\theta_0^2 B \right\}(z).
\]

The previous relation gives \textcolor{red}{(3.1)} with \textcolor{red}{(3.2)}.  

(b) $\Rightarrow$ (a). Let $u \in P'$ regular with its formal Stieltjes series $S(u)$ satisfying \textcolor{red}{(3.1)}. Likewise as in the previous implication, formula \textcolor{red}{(3.1)} leads to
\[
S\{H_q(\Phi u) - q^{-1}(C + H_{q^{-1}}\Phi)u + B(x^{-1}uh_q u)\} = q^{-1}D - q^{-1}u\theta_0 C + ((uh_q u)\theta_0^2 B) + ((H_q u)\theta_0(h_{q^{-1}}\Phi)),
\]
which implies
\[
S\{H_q(\Phi u) - q^{-1}(C + H_{q^{-1}}\Phi)u + B(x^{-1}uh_q u)\} = 0, \\
D(z) = (u\theta_0 C)(z) - q((uh_q u)(\theta_0^2 B))(z) - q((H_q u)\theta_0(h_{q^{-1}}\Phi))(z).
\]

According to \textcolor{red}{(3.2)} and \textcolor{red}{(1.12)} we deduce that
\[
H_q(\Phi u) + \Psi u + B(x^{-1}uh_q u) = 0,
\]
with
\[
\Psi = -q^{-1}(C + H_{q^{-1}}\Phi). \tag{3.4}
\]

We are going to give the criterion which allows us to simplify the class of $q$-Laguerre–Hahn form in terms of the coefficients corresponding to the previous characterization.
Proposition 7. A regular form $u$ $q$-Laguerre–Hahn satisfying (3.1) is of class $s$ if and only if

$$\prod_{c \in Z_\Phi} \{ |B(cq)| + |C(cq)| + |D(cq)| \} > 0,$$

(3.5)

where $Z_\Phi$ is the set of roots of $\Phi$ with

$$s = \max (\deg B - 2, \deg C - 1, \deg D).$$

(3.6)

Proof. By comparing (2.17) and (3.5), it is enough to prove the following equalities

$$|C(cq)| = |q(h_q \Psi)(c) + (H_q \Phi)(c)|,$$

$$|D(cq)| = |\langle u, q(\theta_{cq} \Psi) + (\theta_{cq} \circ \theta_c \Phi) + q(h_q u(\theta_0 \circ \theta_{cq} B)) \rangle|.$$ 

Indeed, on account of (3.2), the definition of the polynomial $uf$, the definition of the product form $uv$ and (1.8) we have

$$C(cq) = -(H_{q^{-1}} \Phi)(cq) - q\Psi(cq) = -(H_q \Phi)(c) - q(h_q \Psi)(c),$$

and

$$D(cq) = -\{H_{q^{-1}}(u\theta_0 \Phi)(cq) + q(u\theta_0 \Psi)(cq) + q(uh_q u)(\theta_0^{-2} B)(cq)\}$$

$$= -\{H_q(u\theta_0 \Phi)(c) + \langle u, q(\theta_{cq} \Psi) + q(h_q u(\theta_0 \circ \theta_{cq} B)) \rangle\}.$$ 

Moreover,

$$H_q(u\theta_0 \Phi)(c) \text{ by (1.9) } \frac{(u\theta_0 \Phi)(cq) - (u\theta_0 \Phi)(c)}{(q - 1)c} \Rightarrow \langle u, \frac{\theta_{cq} \Phi - \theta_c \Phi}{cq - c} \rangle = \langle u, \theta_{cq} \circ \theta_c \Phi \rangle. $$

Thus (2.17) is equivalent to (3.5). To prove (3.6), according to the definition of the class we may write

$$s = \max (\deg B - 2, \deg \Phi - 2, \deg \Psi - 1).$$

(3.7)

- If $\deg \Psi \neq \max (\deg B - 1, \deg \Phi - 1)$, on account of (3.2) and (3.7) we get the following implications

$$\deg B \leq \deg \Phi \Rightarrow \begin{cases} 
\deg C = s + 1, \\
\deg D \leq s,
\end{cases} \Rightarrow \max (\deg B - 2, \deg C - 1, \deg D) = s,$$

$$\deg B > \deg \Phi \Rightarrow \begin{cases} 
\deg C \leq s + 1, \\
\deg D = s,
\end{cases} \Rightarrow \max (\deg B - 2, \deg C - 1, \deg D) = s.$$

- If $\deg \Psi = \max (\deg B - 1, \deg \Phi - 1)$ and $\deg B > \deg \Phi$ then $s + 1 = \deg \Psi = \deg B - 1 > \deg \Phi - 1$. Consequently, $\max (\deg B - 2, \deg C - 1, \deg D) = s$.

- If $\deg \Psi = \max (\deg B - 1, \deg \Phi - 1)$ and $\deg B = \deg \Phi$ then $\deg \Psi = \deg B - 1 = \deg \Phi - 1$ which implies $\deg B - 2 = s$, $\deg C - 1 \leq s$, $\deg D \leq s$. Therefore $\max (\deg B - 2, \deg C - 1, \deg D) = s$.

- If $\deg \Psi = \max (\deg B - 1, \deg \Phi - 1)$ and $\deg B < \deg \Phi$ then $\deg \Psi = \deg \Phi - 1$ and $s = \deg \Psi - 1$. Writing $\Phi(x) = x^{p+1} + \text{lower order terms}$, $\Psi(x) = a_p x^p + \cdots + a_0$, by virtue of (3.2) and (1.3), it is worth noting that $C(z) = -(p+1)z^{p-1} + \text{lower order terms}$ and $D(z) = -(p+1)z^{p-1} + \text{lower order terms}$ with $a_p \neq [p]^q$ assuming either $\deg C = s$ or $\deg D = s$. Thus, $\max (\deg B - 2, \deg C - 1, \deg D) = s$.

Hence the desired result (3.6).
An other important characterization of the $q$-Laguerre–Hahn forms is the structure relation. See also [11,15] for the usual case and [13] for the discrete one.

**Proposition 8.** Let $u$ be a regular form and $\{P_n\}_{n \geq 0}$ be its MOPS. The following statements are equivalent:

(i) $u$ is a $q$-Laguerre–Hahn form satisfying (1.17).

(ii) There exist an integer $s \geq 0$, two polynomials $\Phi$ (monic), $B$ with $t = \deg \Phi \leq s + 2$, $r = \deg B \leq s + 2$ and a sequence of complex numbers $\{\lambda_{n,\nu}\}_{n,\nu \geq 0}$ such that

$$
\Phi(x)(H_qP_{n+1})(x) - h_q(BP^{(1)}_n)(x) = \sum_{\nu = n-s}^{n+d} \lambda_{n,\nu}P_\nu(x), \quad n > s, \quad \lambda_{n,n-s} \neq 0, \quad (3.8)
$$

where $d = \max(t,r)$ and $\{P_n^{(1)}\}_{n \geq 0}$ be the associated sequence of the first kind for the sequence $\{P_n\}_{n \geq 0}$.

**Proof.** (i) $\Rightarrow$ (ii). Beginning with the expression $\Phi(x)(H_qP_{n+1})(x) - h_q(BP^{(1)}_n)(x)$ which is a polynomial of degree at most $n + d$. Then, there exists a sequence of complex numbers $\{\lambda_{n,\nu}\}_{n,\nu \geq 0}$ such that

$$
\Phi(x)(H_qP_{n+1})(x) - (h_qB)(x)(h_qP^{(1)}_n)(x) = \sum_{\nu = 0}^{n+d} \lambda_{n,\nu}P_\nu(x), \quad n \geq 0. \quad (3.9)
$$

Multiplying both sides of (3.9) by $P_m$, $0 \leq m \leq n + d$ and applying $u$ we get

$$
\langle u, \Phi P_m(H_qP_{n+1}) \rangle - \langle h_qu, B(h_q^{-1}P_m)(u\theta_0P_{n+1}) \rangle = \lambda_{n,m}\langle u, P^2_m \rangle, \quad n \geq 0, \quad 0 \leq m \leq n + d. \quad (3.10)
$$

On the other hand, applying $H_q(\Phi u) + \Psi u + B(x^{-1}uh_qu) = 0$ to $P_{n+1}(h_q^{-1}P_m)$, on account of the definitions, (1.10) and (1.5) we obtain

$$
0 = \langle h_q(\Phi u) + \Psi u + B(x^{-1}uh_qu), P_{n+1}(h_q^{-1}P_m) \rangle
= \langle u, \Psi P_{n+1}(h_q^{-1}P_m) - \Phi H_q(P_{n+1}(h_q^{-1}P_m)) \rangle + \langle h_qu, u\theta_0(BP_{n+1}(h_q^{-1}P_m)) \rangle
= \langle u, \{ \Psi(h_q^{-1}P_m) - q^{-1}\Phi(H_q^{-1}P_m) \} P_{n+1} - \Phi P_m(H_qP_{n+1}) \rangle
+ \langle h_qu, u\theta_0(BP_{n+1}(h_q^{-1}P_m)) \rangle.
$$

Thus, for $n \geq 0$, $0 \leq m \leq n + d$

$$
\langle u, \Phi P_m(H_qP_{n+1}) \rangle = \langle u, \{ \Psi(h_q^{-1}P_m) - q^{-1}\Phi(H_q^{-1}P_m) \} P_{n+1} \rangle
+ \langle h_qu, u\theta_0(BP_{n+1}(h_q^{-1}P_m)) \rangle. \quad (3.11)
$$

Using (3.10), (3.11) to eliminate $\langle u, \Phi P_m(H_qP_{n+1}) \rangle$ we get for $n \geq 0$, $0 \leq m \leq n + d$

$$
\langle u, \{ \Psi(h_q^{-1}P_m) - q^{-1}\Phi(H_q^{-1}P_m) \} P_{n+1} \rangle
+ \langle h_qu, u\theta_0(BP_{n+1}(h_q^{-1}P_m)) \rangle
= \lambda_{n,m}\langle u, P^2_m \rangle. \quad (3.12)
$$

Moreover, by virtue of (1.5) we have $B(u\theta_0P_{n+1}) = u\theta_0(BP_{n+1})$, $n > s$. Therefore, taking into account (1.4) and definitions, (3.12) yields for $n > s$, $0 \leq m \leq n + d$

$$
\langle u, \{ \Psi(h_q^{-1}P_m) - q^{-1}\Phi(H_q^{-1}P_m) + B((h_qu)\theta_0(h_q^{-1}P_m)) \} P_{n+1} \rangle = \lambda_{n,m}\langle u, P^2_m \rangle.
$$
with
\[
\deg \left\{ \Psi(h_{q^{-1}}P_m) - q^{-1}\Phi(H_{q^{-1}}P_m) + B((h_q u)\theta_0(h_{q^{-1}}P_m)) \right\} \leq m + s + 1.
\]

Consequently, the orthogonality of \( \{P_n\}_{n \geq 0} \) with respect to \( u \) gives
\[
\lambda_{n,m} = 0, \quad 0 \leq m \leq n - s - 1, \quad n \geq s + 1, \quad \lambda_{n,n-s} \neq 0.
\]

Hence the desired result (3.8).

\( (ii) \Rightarrow (i) \). Let \( v \) be the form defined by
\[
v := H_q(\Phi u) + B\left(x^{-1}u h_q u\right) + \left(\sum_{i=0}^{s+1} a_i x^i\right) u
\]
with \( a_i \in \mathbb{C}, 0 \leq i \leq s + 1 \). From definitions and the hypothesis of \( (ii) \) we may write successively
\[
\langle v, P_{n+1} \rangle = \langle H_q(\Phi u) + B\left(x^{-1}u h_q u\right), P_{n+1}\rangle + \langle u, P_{n+1} \sum_{i=0}^{s+1} a_i x^i \rangle
\]
\[
= -\langle u, \Phi(H_q P_{n+1}) - (h_q u)\theta_0(BP_{n+1}) \rangle + \langle u, P_{n+1} \sum_{i=0}^{s+1} a_i x^i \rangle
\]
\[
= -\langle u, \sum_{\nu=0}^{n+d} \lambda_{n,\nu} P_{\nu} \rangle + \langle u, P_{n+1} \sum_{i=0}^{s+1} a_i x^i \rangle
\]
\[
= -\sum_{\nu=0}^{n+d} \lambda_{n,\nu} \langle u, P_{\nu} \rangle + \sum_{i=0}^{s+1} a_i \langle u, x^i P_{n+1} \rangle, \quad n > s.
\]

From assumption of orthogonality of \( \{P_n\}_{n \geq 0} \) with respect to \( u \) we get
\[
\langle v, P_n \rangle = 0, \quad n \geq s + 2.
\]

In order to get \( \langle v, P_n \rangle = 0 \), for any \( n \geq 0 \), we shall choose \( a_i \) with \( i = 0, 1, \ldots, s + 1 \), such that \( \langle v, P_i \rangle = 0 \), for \( i = 0, 1, \ldots, s + 1 \). These coefficients \( a_i \) are determined in a unique way. Thus, we have deduced the existence of polynomial \( \Psi(x) = \sum_{i=0}^{s+1} a_i x^i \) such that \( \langle v, P_n \rangle = 0 \), for any \( n \geq 0 \). This leads to \( H_q(\Phi u) + \Psi u + B\left(x^{-1}u h_q u\right) = 0 \) and the point \( (i) \) is then proved.

\section{Applications}

\subsection{The co-recursive of a q-Laguerre–Hahn form}

Let \( \mu \) be a complex number, \( u \) a regular form and \( \{P_n\}_{n \geq 0} \) be its corresponding MOPS satisfying (1.1). We define the co-recursive \( \{P_n^{[\mu]}\}_{n \geq 0} \) of \( \{P_n\}_{n \geq 0} \) as the family of monic polynomials satisfying the following three-term recurrence relation [20, Definition 4.2]
\[
P_0^{[\mu]}(x) = 1, \quad P_1^{[\mu]}(x) = x - \beta_0 - \mu,
\]
\[
P_n^{[\mu]}(x) = (x - \beta_{n+1})P_{n+1}^{[\mu]}(x) - \gamma_n P_n^{[\mu]}(x), \quad n \geq 0.
\]

Denoting by \( u^{[\mu]} \) its corresponding regular form. It is well known that [20, equation (4.14)]
\[
u^{[\mu]} = u(\delta - \mu x^{-1} u)^{-1}.
\]
Proposition 9. If \( u \) is a \( q \)-Laguerre–Hahn form of class \( s \), then \( u^{[\mu]} \) is a \( q \)-Laguerre–Hahn form of the same class \( s \).

Proof. The relation linking \( S(u) \) and \( S(u^{[\mu]}) \) is \([20]\) equation (4.15) \( S(u^{[\mu]}) = \frac{S(u)}{1 + \mu S(u)} \) or equivalently

\[
S(u) = \frac{S(u^{[\mu]})}{1 - \mu S(u^{[\mu]})}. \tag{4.1}
\]

From definitions and by virtue of (4.1) we have

\[
h_{q^{-1}}S(u) = \frac{h_{q^{-1}}S(u^{[\mu]})}{1 - \mu h_{q^{-1}}S(u^{[\mu]})}
\]

and

\[
(H_{q^{-1}}S(u))(z) = \frac{h_{q^{-1}}S(u^{[\mu]}))(z)}{1 - \mu h_{q^{-1}}S(u^{[\mu]}))(z)} = \frac{S(u^{[\mu]}))(z)}{(1 - \mu h_{q^{-1}}S(u^{[\mu]}))(z)}.
\]

Replacing the above results in (3.1) the \( q \)-Riccati equation becomes

\[
(h_{q^{-1}}\Phi) = \frac{H_{q^{-1}}S(u^{[\mu]})}{(1 - \mu h_{q^{-1}}S(u^{[\mu]})}(1 - \mu S(u^{[\mu]}))
\]

\[
= B \frac{S(u^{[\mu]})}{1 - \mu S(u^{[\mu]})} \frac{h_{q^{-1}}S(u^{[\mu]})}{1 - \mu h_{q^{-1}}S(u^{[\mu]})} + C \frac{S(u^{[\mu]})}{1 - \mu S(u^{[\mu]})} + D.
\]

Equivalently

\[
(h_{q^{-1}}\Phi)H_{q^{-1}}S(u^{[\mu]}) = BS(u^{[\mu]})h_{q^{-1}}S(u^{[\mu]}) + CS(u^{[\mu]})(1 - \mu h_{q^{-1}}S(u^{[\mu]}) + D(1 - \mu h_{q^{-1}}S(u^{[\mu]}) (1 - \mu S(u^{[\mu]})).
\]

Therefore the \( q \)-Riccati equation satisfied by \( S(u^{[\mu]}) \)

\[
(h_{q^{-1}}\Phi^{[\mu]})H_{q^{-1}}S(u^{[\mu]}) = B^{[\mu]}S(u^{[\mu]})h_{q^{-1}}S(u^{[\mu]}) + C^{[\mu]}S(u^{[\mu]}) + D^{[\mu]}, \tag{4.2}
\]

where

\[
K\Phi^{[\mu]}(x) = \Phi(x) + \mu(1 - q)x(h_{q}D)(x), \quad KB^{[\mu]}(x) = B(x) - \mu C(x) + \mu^{2}D(x),
\]

\[
KC^{[\mu]}(x) = C(x) - 2\mu D(x), \quad KD^{[\mu]}(x) = D(x), \tag{4.3}
\]

the non zero constant \( K \) is chosen such that the polynomial \( \Phi^{[\mu]} \) is monic. \( u^{[\mu]} \) is then a \( q \)-Laguerre–Hahn form.

On account of (3.2), (3.4) and (4.3) we get

\[
K\Psi^{[\mu]} = \Psi + \mu(q^{-1}D + h_{q}D). \tag{4.4}
\]

As a consequence, the regular form \( u^{[\mu]} \) fulfills the following \( q \)-difference equation

\[
H_{q}(\Phi^{[\mu]}u^{[\mu]}) + \Psi^{[\mu]}u^{[\mu]} + B^{[\mu]}(x^{-1}u^{[\mu]}h_{q}u^{[\mu]}) = 0. \tag{4.5}
\]

We suppose that the \( q \)-Riccati equation (3.1) of \( u \) is irreducible of class \( s \). With respect to the class, we use the result (3.5) of Proposition 7 and get for every zero \( c \) of \( \Phi^{[\mu]} \):
• If $D(cq) \neq 0$, then $D^{[\nu]}(cq) = K^{-1}D(cq) \neq 0$ and equation (4.2) is not reducible.

• We suppose that $D(cq) = 0$. From the fact that $\Phi^{[\nu]}(c) = 0$, the first relation in (4.3) leads to $\Phi(c) = 0$ and the third equality in (4.3) gives $C^{[\nu]}(cq) = K^{-1}C(cq)$.

If $C(cq) \neq 0$, then the equation (4.2) is still not reducible. If $C(cq) = 0 = D(cq)$, then $B^{[\nu]}(cq) = K^{-1}B(cq) \neq 0$ since $u$ is of class $s$. We conclude that

$$|B^{[\nu]}(cq)| + |C^{[\nu]}(cq)| + |D^{[\nu]}(cq)| > 0.$$ 

Consequently, the class $s^{[\nu]}$ of $u^{[\nu]}$ is given by $s^{[\nu]} = \max(\deg B^{[\nu]} - 2, \deg C^{[\nu]} - 1, \deg D^{[\nu]})$. Accordingly to the last equality in (4.3) and (3.6) we get $s^{[\nu]} = \max(\deg B^{[\nu]} - 2, \deg C^{[\nu]} - 1, \deg D)$. A discussion on the degree leads to $s^{[\nu]} = s$.

**Example 1.** Let $u$ be a $q$-classical form satisfying the $q$-analog of the distributional equation of Pearson type

$$H_q(\phi u) + \psi u = 0, \quad (4.6)$$

where $\phi$ is a monic polynomial of degree at most two and $\psi$ a polynomial of degree one, the co-recurse $u^{[\nu]}$ of $u$ is a $q$-Laguerre–Hahn form of class zero. $u^{[\nu]}$ and the Stieltjes function $S(u^{[\nu]})$ satisfy, respectively, the $q$-difference equation (4.5) and the $q$-Riccati equation (4.2) where on account of (4.3), (4.4)

$$K\Phi^{[\nu]}(x) = \frac{\phi''(0)}{2} x^2 + \left\{ \phi'(0) + \mu(q - 1) \left( \frac{\phi''(0)}{2} + q\psi'(0) \right) \right\} x + \phi(0),$$

$$K\Psi^{[\nu]}(x) = \psi'(0)x + \psi(0) - \mu(q^{-1} + 1) \left( \frac{\phi''(0)}{2} + q\psi'(0) \right),$$

$$KB^{[\nu]}(x) = \mu \left\{ \left( q^{-1} + 1 \right) \frac{\phi''(0)}{2} + q\psi'(0) \right\} x + \phi'(0) + q\psi(0) - \left( \frac{\phi''(0)}{2} + q\psi'(0) \right) \mu, \quad K\gamma_1,$$

$$KC^{[\nu]}(x) = - \left( q\psi'(0) + (q^{-1} + 1)\frac{\phi''(0)}{2} \right) x - \phi'(0) - q\psi(0) + 2\mu \left( \frac{\phi''(0)}{2} + q\psi'(0) \right),$$

$$KD^{[\nu]}(x) = - \frac{\phi''(0)}{2} - q\psi'(0).$$

### 4.2 The associated of a $q$-Laguerre–Hahn form

Let $u$ be a regular form and $\{P_n\}_{n \geq 0}$ its corresponding MOPS satisfying (1.1). The associated sequence of the first kind $\{P_n^{(1)}\}_{n \geq 0}$ of $\{P_n\}_{n \geq 0}$ satisfies the following three-term recurrence relation [20]

$$P_0^{(1)}(x) = 1, \quad P_1^{(1)}(x) = x - \beta_1,$$

$$P_{n+2}^{(1)}(x) = (x - \beta_{n+2})P_{n+1}^{(1)}(x) - \gamma_n + 2P_n^{(1)}(x), \quad n \geq 0.$$ 

Denoting by $u^{(1)}$ its corresponding regular form.

**Proposition 10.** If $u$ is a $q$-Laguerre–Hahn form of class $s$, then $u^{(1)}$ is a $q$-Laguerre–Hahn form of the same class $s$.

**Proof.** We assume that the formal Stieltjes function $S(u)$ of $u$ satisfies (3.1). The relationship between $S(u^{(1)})$ and $S(u)$ is [20] equation (4.7)

$$\gamma_1S(u^{(1)})(z) = -\frac{1}{S(u)(z)} - (z - \alpha_0).$$
Consequently,

\[ S(u)(z) = -\frac{1}{\gamma_1 S(u(1))(z) + (z - \beta_0)}. \]  

(4.7)

From definitions and by virtue of (4.7) we have

\[ h_{q^{-1}}(S(u))(z) = -\frac{1}{\gamma_1 h_{q^{-1}}(S(u(1)))(z) + q^{-1}z - \beta_0} \]

and

\[ H_{q^{-1}}(S(u))(z) = \frac{\gamma_1 H_{q^{-1}}(S(u(1)))(z) + 1}{(\gamma_1 h_{q^{-1}}(S(u(1)))(z) + q^{-1}z - \beta_0)(\gamma_1 S(u(1))(z) + z - \beta_0)}. \]

Substituting in (3.1) the q-Riccati equation becomes

\[ (h_{q^{-1}}\Phi)(z) = \frac{\gamma_1 H_{q^{-1}}(S(u(1)))(z) + 1}{(\gamma_1 h_{q^{-1}}(S(u(1)))(z) + q^{-1}z - \beta_0)(\gamma_1 S(u(1))(z) + z - \beta_0)} B(z) \]

\[ - \frac{C(z)}{(\gamma_1 S(u(1))(z) + z - \beta_0)} + D(z). \]

Equivalently

\[ \gamma_1 \{(h_{q^{-1}}\Phi)(z) + (q^{-1} - 1)z(C(z) - (z - \beta_0)D(z))\} H_{q^{-1}}(S(u(1)))(z) \]

\[ = \gamma_1^2 D(z)S(u(1))(z)h_{q^{-1}}(S(u(1)))(z) + \gamma_1 \{(q^{-1} + 1)z - 2\beta_0)D(z) - C(z)\} S(u(1))(z) \]

\[ + B(z) + (q^{-1}z - \beta_0)(z - \beta_0)D(z) - (q^{-1}z - \beta_0)C(z) - (h_{q^{-1}}\Phi)(z). \]

Therefore the q-Riccati equation satisfied by \( S(u(1)) \)

\[ (h_{q^{-1}}\Phi(1))H_{q^{-1}}S(u(1)) = B(1)S(u(1))h_{q^{-1}}S(u(1)) + C(1)S(u(1)) + D(1), \]

(4.8)

where

\[ K\Phi^{(1)}(x) = \Phi(x) + (q - 1)x((q - 1)x - \beta_0)(h_q D)(x) - (h_q C)(x), \]

\[ KB^{(1)}(x) = \gamma_1 D(x), \quad KC^{(1)}(x) = \gamma_1 \{(q^{-1} + 1)x - 2\beta_0)D(x) - C(x)\}, \]

\[ KD^{(1)}(x) = B(x) + (q^{-1}x - \beta_0)(x - \beta_0)D(x) - (q^{-1}x - \beta_0)C(x) - (h_{q^{-1}}\Phi)(x). \]

(4.9)

\( u^{(1)} \) is then a q-Laguerre–Hahn form.

Moreover, the regular form \( u^{(1)} \) fulfils the q-difference equation

\[ H_q(\Phi^{(1)}u^{(1)}) + \Psi^{(1)}u^{(1)} + B^{(1)}(x^{-1}u^{(1)}h_q u^{(1)}) = 0, \]

(4.10)

with

\[ \Psi^{(1)} = -q^{-1}(C^{(1)} + H_{q^{-1}}\Phi^{(1)}). \]

(4.11)

Likewise, it is straightforward to prove that the class of \( u^{(1)} \) is also \( s \).
Example 2. If \( u \) is a \( q \)-classical form satisfying the \( q \)-analog of the distributional equation of Pearson type \((4.6)\) then the associated \( u^{(1)} \) of \( u \) is a \( q \)-Laguerre–Hahn form of class zero. \( u^{(1)} \) and the formal Stieltjes function \( S(u^{(1)}) \) satisfy, respectively, the \( q \)-difference equation \((4.10)\) and the \( q \)-Riccati equation \((4.8)\) where on account of \((4.9)\) and \((4.11)\)

\[
K\Phi^{(1)}(x) = \frac{q\phi''(0)}{2}x^2 + \left\{ q\phi'(0) + (q-1) \left( q\psi(0) + \beta_0 \left( \frac{\phi''(0)}{2} + q\psi'(0) \right) \right) \right\} x + \phi(0),
\]

\[
K\Psi^{(1)}(x) = -q^{-1} \left\{ (q+1)\frac{\phi''(0)}{2} - \psi'(0) \right\} x + (q+1)\phi'(0)
+ q^2\psi(0) + (q^2 - q + 2) \left( \frac{\phi''(0)}{2} + q\psi'(0) \right) \beta_0,
\]

\[
KB^{(1)}(x) = -\gamma_1 \left( \frac{\phi''(0)}{2} + q\psi'(0) \right),
\]

\[
KC^{(1)}(x) = \gamma_1 \left\{ -\psi'(0)x + \beta_0(\phi''(0) + 2q\psi'(0)) + q\psi(0) + \phi'(0) \right\},
\]

\[
KD^{(1)}(x) = \psi(\beta_0)x - \phi(\beta_0) - q\beta_0\psi(\beta_0).
\]

4.3 The inverse of a \( q \)-Laguerre–Hahn form

Let \( u \) be a regular form and \( \{P_n\}_{n \geq 0} \) its corresponding MOPS satisfying \((1.1)\). Let \( \{P_n^{(1)}\}_{n \geq 0} \) be its associated sequence of the first kind fulfilling \((4.6)\) and orthogonal with respect to the regular form \( u^{(1)} \). The inverse form of \( u \) satisfies \([20, \text{equation (5.27)}]\)

\[
x^2u^{-1} = -\gamma_1 u^{(1)}. \tag{4.12}
\]

The following results can be found in \([2]\)

\[
u^{-1} = \delta - (u^{-1})_1x - \gamma_1x^{-2}u^{(1)}. \tag{4.13}
\]

In general, the form \( u^{-1} \) given by \((4.13)\) is regular if and only if \( \Delta_n \neq 0, n \geq 0 \), with

\[
\Delta_n = \langle u^{(1)}, (P_n^{(1)})^2 \rangle \left\{ \gamma_1 + \sum_{\nu=0}^{n} \frac{\langle \gamma_1 P_{\nu+1}^{(2)}(0) - (u^{-1})_1 P_{\nu+1}^{(1)}(0) \rangle^2}{\langle u^{(1)}, (P_{\nu+1}^{(1)})^2 \rangle} \right\}, \quad n \geq 0,
\]

where \( \{P_n^{(2)}\}_{n \geq 0} \) is the associated sequence of \( \{P_n^{(1)}\}_{n \geq 0} \). In this case, the orthogonal sequence \( \{P_n^{(-)}\}_{n \geq 0} \) relative to \( u^{-1} \) is given by

\[
P_0^{(-)}(x) = 1, \quad P_1^{(-)}(x) = P_1^{(1)}(x) + b_0,
\]

\[
P_n^{(-)}(x) = P_{n+2}^{(1)}(x) + b_{n+1}P_{n+1}^{(1)}(x) + a_nP_n^{(1)}(x), \quad n \geq 0,
\]

where

\[
b_0 = \beta_1 - (u^{-1})_1,
\]

\[
b_{n+1} = \beta_{n+2} - \frac{((u^{-1})_1 P_{n+1}^{(1)}(0) - \gamma_1 P_{n+1}^{(2)}(0))((u^{-1})_1 P_{n+1}^{(1)}(0) - \gamma_1 P_{n+1}^{(2)}(0))}{\Delta_n}, \quad n \geq 0,
\]

\[
a_n = \frac{\Delta_{n+1}}{\Delta_n}, \quad n \geq 0.
\]

Also, the sequence \( \{P_n^{(-)}\}_{n \geq 0} \) satisfies the three-term recurrence relation

\[
P_0^{(-)}(x) = 1, \quad P_1^{(-)}(x) = x - \beta_0^{(-)},
\]
$P_{n+2}^{(-)}(x) = (x - \beta_{n+1}^{(-)}) P_{n+1}^{(-)}(x) - \gamma_{n+1}^{(-)} P_n^{(-)}(x), \quad n \geq 0,$

with

$$\beta_0^{(-)} = (u^{-1})_1, \quad \beta_{n+1}^{(-)} = \beta_{n+2} + b_n - b_{n+1}, \quad n \geq 0,$$

$$\gamma_1^{(-)} = -\Delta_0, \quad \gamma_{2n}^{(-)} = \gamma_1 \frac{\Delta_1}{\Delta_0}, \quad \gamma_{2n+3}^{(-)} = \frac{\Delta_{n+2}\Delta_n}{\Delta_{n+1}^2} \gamma_{n+2}, \quad n \geq 0.$$

In particular, when $\gamma_1 > 0$ and $u^{(1)}$ is positive definite, then $u^{-1}$ is regular. When $u^{(1)}$ is symmetrical, then $u^{-1}$ is a symmetrical regular form and we have

$$a_{2n} = \frac{\gamma_1 \Lambda_n + 1}{\gamma_1 \Lambda_{n-1} + 1} \gamma_{2n+2}, \quad a_{2n+1} = \gamma_{2n+3}, \quad n \geq 0,$$

$$\gamma_1^{(-)} = -\gamma_1, \quad \gamma_{2n+2}^{(-)} = a_{2n}, \quad \gamma_{2n+3}^{(-)} = \frac{\gamma_{2n+2} \gamma_{2n+3}}{a_{2n}}, \quad n \geq 0,$$

with

$$\Lambda_{-1} = 0, \quad \Lambda_n = \sum_{\nu=0}^n \left( \prod_{k=0}^{\nu} \frac{\gamma_{2k+1}}{\gamma_{2k+2}} \right), \quad n \geq 0, \quad \gamma_0 = 1.$$

**Proposition 11.** If $u$ is a $q$-Laguerre–Hahn form of class $s$, then, when $u^{-1}$ is regular, $u^{-1}$ is a $q$-Laguerre–Hahn form of class at most $s + 2$.

**Proof.** Let $u$ be a $q$-Laguerre–Hahn form of class $s$ satisfying (1.17). It is seen in Proposition 10 that $u^{(1)}$ is also a $q$-Laguerre–Hahn form of class $s$ satisfying the $q$-difference equation (4.10) with polynomials $\Phi^{(1)}, \Psi^{(1)}, B^{(1)}$ respecting (4.9) and (4.11).

Let us suppose $u^{-1}$ is regular that is to say $\Delta_n \neq 0, n \geq 0$. Multiplying (4.10) by $(-\gamma_1)$ and on account of (4.12) and (1.17), the $q$-difference equation (4.10) becomes

$$H_q(x^2\Phi^{(1)}(x)u^{-1}) + x^2\Psi^{(1)}(x)u^{-1} - q^{-2}\gamma_1^{-1}B^{(1)}(x^{-1}(x^2u^{-1})(x^2h_qu^{-1})) = 0.$$

Consequently, the form $u^{-1}$ satisfies the following $q$-difference equation

$$H_q(\Phi^{(-)}u^{-1}) + \Psi^{(-)}u^{-1} + B^{(-)}(x^{-1}u^{-1}h_qu^{-1}) = 0,$$

with

$$K\Phi^{(-)}(x) = x^2\{\Phi^{(1)}(x) + (1 - q)\gamma_1^{-1}x(qx - \beta_0)(h_qB^{(1)}(x))\},$$

$$K\Psi^{(-)}(x) = x\left\{(q^{-1} + 1)((h_q^{-1}\Phi^{(1)})(x) - q^{-1}\Phi^{(1)}(x)) - q^{-3}x(H_q^{-1}\Phi^{(1)}(x)) + \gamma_1^{-1}x((2q^{-1} + q^{-2} - q^{-3})x - (1 + 2q^{-2} - q^{-3})\beta_0)B^{(1)}(x) - (q^{-2} - 1)\gamma_1^{-1}x(qx - \beta_0)(h_qB^{(1)}(x)) - q^{-4}x^2(1 - q)\gamma_1^{-1}(hx - \beta_0)(h_qB^{(1)})(x) - xc^{(1)}(x)\right\},$$

$$KB^{(-)}(x) = -\gamma_1^{-1}q^{-2}x^4B^{(1)}(x).$$

**Example 3.** Let $\mathcal{Y}(b,q^2)$ be the form of Brenke type which is symmetrical $q$-semiclassical of class one such that [14] equation (3.22), $q \leftarrow q^2$

$$H_q(x\mathcal{Y}(b,q^2)) - (b(q-1))^{-1}(q^{-2}x^2 + b - 1)\mathcal{Y}(b,q^2) = 0$$

for $q \in \overline{\mathbb{C}}, b \neq 0, b \neq q, b \neq q^{-2n}, n \geq 0$ and its MOPS $\{P_n\}_{n \geq 0}$ satisfying (1.1) with [7]

$$\beta_n = 0,$$
\( \gamma_{2n+1} = q^{2n+2}(1 - bq^{2n}), \quad \gamma_{2n+2} = bq^{2n+2}(1 - q^{2n+2}), \quad n \geq 0. \) (4.20)

Denoting \( \mathcal{Y}^{(1)}(b, q^2) \) its associated form and \( \mathcal{Y}^{(-1)}(b, q^2) \) its inverse one. Taking into account \( (4.19) \) we have

\[
\Phi(x) = x, \quad \Psi(x) = -(b(q - 1))^{-1}(q^{-2}x^2 + b - 1), \quad B(x) = 0. \quad (4.21)
\]

Also, by virtue of \( (3.2) \) and \( (4.21) \) we get

\[
C(x) = (b(q - 1))^{-1}q^{-1}x^2 + q(q - 1)^{-1}(1 - b^{-1}) - 1, \quad D(x) = (bq(q - 1))^{-1}x. \quad (4.22)
\]

According to Proposition \( 10 \) the form \( \mathcal{Y}^{(1)}(b, q^2) \) is q-Laguerre–Hahn of class one satisfying the q-difference equation \( (4.10) \) and its formal Stieltjes function satisfies the q-Riccati equation \( (4.8) \) where on account of \( (4.20) \), \( (4.22) \) we obtain for \( (4.9), (4.11) \)

\[
K\Phi^{(1)}(x) = b^{-1}x, \\
K\Psi^{(1)}(x) = -q^{-2}(b(q - 1))^{-1}x^2 + q(q - 1)^{-1}(1 - b^{-1}) - (qb)^{-1} - 1, \\
KB^{(1)}(x) = (b^{-1} - 1)q(q - 1)^{-1}x, \\
KC^{(1)}(x) = q^{-2}(b(q - 1))^{-1}x^2 + 1 - q(q - 1)^{-1}(1 - b^{-1}), \\
KD^{(1)}(x) = q^{-2}(b(q - 1))^{-1}x. \quad (4.23)
\]

On the one hand, \( \mathcal{Y}^{(1)}(b, q^2) \) is a symmetrical regular form, then \( \mathcal{Y}^{(-1)}(b, q^2) \) is also a symmetrical regular form and we have for \( (4.14) - (4.16) \) according to \( (4.20) \)

\[
\Lambda_{-1} = 0, \quad \Lambda_0 = \frac{b^{-1} - 1}{1 - q^2}, \quad \Lambda_n = \sum_{\nu=1}^{n+1} b^{-\nu} \frac{(b; q^2)_{\nu}}{(q^2; q^2)_{\nu}}, \quad n \geq 1, \\
\gamma^{(-)}_1 = q^2(b - 1), \quad \gamma^{(-)}_{2n+2} = bq^{2n+2}(1 - q^{2n+2}) \frac{1 + q^2(1 - b)\Lambda_n}{1 + q^2(1 - b)\Lambda_{n-1}}, \quad n \geq 0, \\
\gamma^{(-)}_{2n+3} = q^{2n+4}(1 - bq^{2n+2}) \frac{1 + q^2(1 - b)\Lambda_{n-1}}{1 + q^2(1 - b)\Lambda_n}, \quad n \geq 0,
\]

with \( 7 \)

\[
(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=1}^{n} (1 - aq^{k-1}), \quad n \geq 1.
\]

On the other hand, according to Proposition \( 11 \), \( (4.18) \) and \( (4.23) \), the inverse form \( \mathcal{Y}^{(-1)}(b, q^2) \) is symmetrical q-Laguerre–Hahn satisfying the q-difference equation \( (4.17) \) where

\[
K\Phi^{(-)}(x) = b^{-1}x^3(1 - qx^2), \\
K\Psi^{(-)}(x) = b^{-1}(q - 1)^{-1}x^2(b - q - q^{-3}(q - 1) + (-2q^{-4} + 2q^{-3} + q^{-2} - q^{-1} + q)x^2), \\
KB^{(-)}(x) = -b^{-1}q^{-3}(q - 1)^{-1}x^5.
\]

Thus, according to \( (2.17) \) it is possible to simplify by \( x \) one time uniquely. Consequently, by virtue of \( (2.16) \) the inverse form \( \mathcal{Y}^{(-1)}(b, q^2) \) is q-Laguerre–Hahn of class two fulfilling the q-difference equation

\[
H_q(x^2(x^2 - q^{-1}))\mathcal{Y}^{(-1)}(b, q^2) - q^{-1}x\{1 + q(q - 1)^{-1}(b - q - q^{-3}(q - 1)) \\
+ (q(q - 1)^{-1}(-2q^{-4} + 2q^{-3} + q^{-2} - q^{-1} + q) - q)x^2\}\mathcal{Y}^{(-1)}(b, q^2) \\
+ q^{-3}(q - 1)^{-1}x^4(x^{-1}\mathcal{Y}^{(-1)}(b, q^2)h_q\mathcal{Y}^{(-1)}(b, q^2)) = 0.
\]
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