A Higher Dimensional Stationary Rotating Black Hole Must be Axisymmetric

Stefan Hollands1, Akihiro Ishibashi2, Robert M. Wald2

1 Institut für Theoretische Physik, Universität Göttingen, D-37077 Göttingen, Germany. E-mail: hollands@theorie.physik.uni-goe.de
2 Enrico Fermi Institute and Department of Physics, The University of Chicago, Chicago, IL 60637, USA. E-mail: akihiro@midway.uchicago.edu; rmwa@midway.uchicago.edu; akihiro@oscar.uchicago.edu

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Abstract: A key result in the proof of black hole uniqueness in 4-dimensions is that a stationary black hole that is “rotating”—i.e., is such that the stationary Killing field is not everywhere normal to the horizon—must be axisymmetric. The proof of this result in 4-dimensions relies on the fact that the orbits of the stationary Killing field on the horizon have the property that they must return to the same null geodesic generator of the horizon after a certain period, $P$. This latter property follows, in turn, from the fact that the cross-sections of the horizon are two-dimensional spheres. However, in space-times of dimension greater than 4, it is no longer true that the orbits of the stationary Killing field on the horizon must return to the same null geodesic generator. In this paper, we prove that, nevertheless, a higher dimensional stationary black hole that is rotating must be axisymmetric. No assumptions are made concerning the topology of the horizon cross-sections other than that they are compact. However, we assume that the horizon is non-degenerate and, as in the 4-dimensional proof, that the spacetime is analytic.

1. Introduction

Consider an $n$-dimensional stationary spacetime containing a black hole. Since the event horizon of the black hole must be mapped into itself by the action of any isometry, the asymptotically timelike Killing field $t^a$ must be tangent to the horizon. Therefore, we have two cases to consider: (i) $t^a$ is normal to the horizon, i.e., tangent to the null geodesic generators of the horizon; (ii) $t^a$ is not normal to the horizon. In 4-dimensions it is known that in case (i), for suitably regular non-extremal vacuum or Einstein-Maxwell black holes, the black hole must be static [42, 5]. Furthermore, in 4-dimensions it is known that in case (ii), under fairly general assumptions about the nature of the matter content but assuming analyticity of the spacetime and non-extremality of the black hole, there must exist an additional Killing field that is normal to the horizon. It can then be
shown that the black hole must be axisymmetric\(^1\) as well as stationary \([18, 19]\). This latter result is often referred to as a “rigidity theorem,” since it implies that the horizon generators of a “rotating” black hole (i.e., a black hole for which \(t^a\) is not normal to the horizon) must rotate rigidly with respect to infinity. A proof of the rigidity theorem in 4-dimensions which partially eliminates the analyticity assumption was given by Friedrich, Racz, and Wald \([9, 32]\), based upon an argument of Isenberg and Moncrief \([27, 20]\) concerning the properties of spacetimes with a compact null surface with closed generators. The above results for both cases (i) and (ii) are critical steps in the proofs of black hole uniqueness in 4-dimensions, since they allow one to apply Israel’s theorems \([23, 24]\) in case (i) and the Carter-Robinson-Mazur-Bunting theorems \([2, 36, 25, 1]\) in case (ii).

Many attempts to unify the forces and equations of nature involve the consideration of spacetimes with \(n > 4\) dimensions. Therefore, it is of considerable interest to consider a generalization of the rigidity theorem to higher dimensions, especially in view of the fact that there seems to be a larger variety of black hole solutions (see e.g., \([7, 12, 15]\)), the classification of which has not been achieved yet.\(^2\) The purpose of this paper is to present a proof of the rigidity theorem in higher dimensions for non-extremal black holes.

The dimensionality of the spacetime enters the proof of the rigidity theorem in 4-dimensions in the following key way: The expansion and shear of the null geodesic generators of the horizon of a stationary black hole can be shown to vanish (see below). The induced (degenerate) metric on the \((n - 1)\)-dimensional horizon gives rise to a Riemannian metric, \(\gamma_{ab}\), on an arbitrary \((n - 2)\)-dimensional cross-section, \(\Sigma\), of the horizon. On account of the vanishing shear and expansion, all cross-sections of the horizon are isometric, and the projection of the stationary Killing field \(t^a\) onto \(\Sigma\) gives rise to a Killing field, \(s^a\), of \(\gamma_{ab}\) on \(\Sigma\). In case (ii), \(s^a\) does not vanish identically. Now, when \(n = 4\), it is known that \(\Sigma\) must have the topology of a 2-sphere, \(S^2\). Since the Euler characteristic of \(S^2\) is nonzero, it follows that \(s^a\) must vanish at some point \(p \in \Sigma\).

However, since \(\Sigma\) is 2-dimensional, it then follows that the isometries generated by \(s^a\) simply rotate the tangent space at \(p\). It then follows that all of the orbits of \(s^a\) are periodic with a fixed period \(P\), from which it follows that, after period \(P\), the orbits of \(t^a\) on the horizon must return to the same generator. Consequently, if we identify points in spacetime that differ by the action of the stationary isometry of parameter \(P\), the horizon becomes a compact null surface with closed null geodesic generators. The theorem of Isenberg and Moncrief \([27, 20]\) then provides the desired additional Killing field normal to this null surface.

In \(n > 4\) dimensions, the Euler characteristic of \(\Sigma\) may vanish, and, even if it is non-vanishing, if \(n > 5\) there is no reason that the isometries generated by \(s^a\) need have closed orbits even when \(s^a\) vanishes at some point \(p \in \Sigma\). Thus, for example, even in the 5-dimensional Myers-Perry black hole solution \([30]\) with cross section topology \(\Sigma = S^3\), one can choose the rotational parameters of the solution so that the orbits of the stationary Killing field \(t^a\) do not map horizon generators into themselves.

One possible approach to generalizing the rigidity theorem to higher dimensions would be to choose an arbitrary \(P > 0\) and identify points in the spacetime that differ

\(^1\) In this paper, by “axisymmetric” we mean that spacetime possesses one-parameter group of isometries isomorphic to \(U(1)\) whose orbits are spacelike. We do not require that the Killing field vanishes on an “axis.”

\(^2\) There have recently appeared several works on general properties of a class of stationary, axisymmetric vacuum solutions, including an \(n\)-dimensional generalization of the Weyl solutions for the static case (see e.g., \([3, 6, 16, 17]\), and see also \([26, 43]\) and references therein for some techniques of generating such solutions in 5-dimensions).