Effects of cavity birefringence on remote entanglement generation

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The generation of entanglement between distant atoms via single photons is the basis for networked quantum computing, a promising route to large-scale trapped ion and atom processors. Locating the emitter within an optical cavity provides an efficient matter-light interface, but mirror-induced birefringence within the cavity introduces time-dependence to the produced photons’ polarisation. We show that such polarisation oscillation effects can lead to severe loss of fidelity in the context of interference-based remote entanglement schemes. While total elimination of birefringence-inducing mirror ellipticity is preferable, we propose two remedies for systems where this cannot be achieved: one which restores high entanglement fidelities in the presence of polarisation oscillation via local qubit corrections, and one which suppresses polarisation oscillations for experiments with linearly polarised atomic emission projection by employing highly elliptical mirrors. We conclude that even modest cavity birefringence can be detrimental to remote entanglement performance and should be carefully considered when designing such experiments.

PACS numbers:
Keywords:

I. INTRODUCTION

The interference of single photons formed the basis of many applications in quantum optics, from Young’s double slit experiment to ascertain the wave-particle duality 1, to Michelson-type interferometers as a cosmic probe 2, to Bell’s inequality experiments via two-photon correlations 3. More recently, the interference of photons has proved to be an indispensable tool in quantum communications and quantum information processing, and notably in the pursuit of scalable quantum computing via quantum networked architectures of small processors (nodes) interacting via quantum channels 4, 5. Entanglement across the nodes is generated via deterministic direct entanglement transfer 6, 7 (where a mediating photon’s entanglement with its emitter is transferred to an entanglement between the emitter and the recipient) or via measurement based entanglement 8–12. With the former, successful entanglement is detected a posteriori (–accompanied with its collapse). The latter, albeit an inherently probabilist protocol, is favoured for scaling as successful entanglement is heralded upon detection of readout photons 5. It has led to successful elementary implementations with ions 13, neutral atoms 14, defect centres in diamonds 15 and micromechanical oscillators 16. Usually, the polarisation of the photons in such processes is well defined and controlled. The rate of entanglement generation in these experiments has been primarily limited by the collection efficiency of photons from the emitters. The use of optical cavities has further improved our abilities to control the temporal shape of the photon 17 and has significantly improved collection efficiencies through the Purcell effect 18, both of which play a key role in enhancing entanglement fidelities and rates 19, 20. To this end, recent experiments have ventured toward the integration of miniature optical cavities 21–27 with more emerging. Such cavities with small radii of curvatures are likely to exhibit birefringence effects owing to limited precision in manufacturing and coating capabilities for the required small radii of curvatures 28–31. Recently, it was demonstrated that the polarisation of photons from birefringent emitter-cavity systems undergoes polarisation oscillation 32 - the produced photon’s polarisation changes on the timescale of its length. The effects have been harnessed as a means of enhancing extraction rates 33.

In this paper, we point out that polarisation oscillation in measurement based entanglement schemes, where the qubit is often encoded in the photons’ polarisation degree of freedom, can cause irreversible loss in the entanglement fidelity and therefore must be carefully considered in such implementations.

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arXiv:2008.11712v1  [quant-ph]  26 Aug 2020
Figure 1: Measurement based entanglement setup for polarisation encoded photonic qubits. (a) Two nodes composed of an emitter and a cavity emit into the input modes (a) and (b) of a non-polarising beam splitter (NPBS). At each node, the emitter is in superposition of the $|\uparrow\rangle$ and $|\downarrow\rangle$ states, entangled with orthogonal photon polarisations (see (c)). The cavities at nodes 1 and 2 are birefringent-free but aren’t necessarily identical, hence emit temporally distinguishable photons. For simplicity we only depict the temporal profile of one of the polarisations ($\alpha_H$ and $\beta_H$) of the superposed states. The output at modes (c) and (d) are directed to polarising beam splitters for detecting the orthogonal components at the photodetector (PD). A quarter wave plate (QWP) allows for the mapping of the cavity emission onto a desired polarisation basis. In setup (b), node 2 is birefringent, hence its emission’s polarisation changes along its pulse length with $\beta'_H$ ($\beta'_V$) showing the $H$ ($V$) polarised component (where the superscript denotes the expected polarisation in the absence of birefringence). (c) Energy level scheme of the emitter-cavity system at different node types. A bichromatic pump (black arrows) with detunings $\delta_1$ and $\delta_2$ transfers the emitter population from the ground state $|g\rangle$ to the qubit state $|\downarrow\rangle$ ($|\uparrow\rangle$) with the emission of a $\sigma^-$ ($\sigma^+$) photon mediated by a cavity, with frequency $\omega_{cav}$, coupled to the $|e\rangle \leftrightarrow |\downarrow\rangle$ and $|e\rangle \leftrightarrow |\uparrow\rangle$ transitions. The qubit states are shown to have a Zeeman splitting of $\Delta$. In the birefringent node, the cavity is shown to exhibit linear birefringence with energy $\omega_H$ ($\omega_V$) for the horizontal (vertical) polarisation eigenmode, and with splitting $\Omega_B$ (with units setting $\hbar = 1$).

II. METHOD

A. Non-birefringent system

We consider the entangler depicted in Fig. 1. An emitter-cavity system at node (a) generates the balanced emitter-photon entangled state

$$\left|\psi^{(a)}\right\rangle = \frac{1}{\sqrt{2}} \left|\uparrow,H^{(a)}\right\rangle + \left|\downarrow,V^{(a)}\right\rangle$$

(1)

$$= \frac{1}{\sqrt{2}} \left( a^\dagger_H \left|\uparrow\right\rangle + a^\dagger_V \left|\downarrow\right\rangle \right) |0\rangle$$

(2)

where $a^\dagger_x$ is the $x$-polarised photon creation operator at mode (a) of the non-polarising beam splitter, with $x = \{H\text{(horizontal)}, V\text{(vertical)}\}$. The photon’s temporal waveform, which has so far not been considered, plays a key role in role in interference based processes, and consequently on entanglement fidelities as we shall later see. We incorporate
the photons waveform to the emitter-photon state by introducing the photon-wavepacket creation operator [34],

$$a_x^\dagger \rightarrow A_x^\alpha := \int \alpha_x^\alpha(t)a_x^\dagger(t)\, dt,$$

which holds under normalization conditions \( \int |\alpha_x(t)|^2\, dt = 1 \) and where \( \alpha_x(t) \) is the temporal wavepacket amplitude of the x-polarised emission as depicted in Fig[4]a). With the beam splitter transformation

$$\begin{pmatrix} a_x(t) \\ b_x(t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_x(t) \\ d_x(t) \end{pmatrix},$$

(4)

where \(b(t), c(t)\) and \(d(t)\) are annihilation operators for modes (b), (c) and (d) respectively, the wave packet creation operators transform as

$$A_x^\alpha = \frac{1}{\sqrt{2}} (C_x^\alpha - D_x^\alpha).$$

(5)

for the wavepacket from node (a), where \(C_x^\alpha := \int \alpha_x^\alpha(t)c_x^\dagger(t)\, dt\) and similarly, \(D_x^\alpha := \int \alpha_x^\alpha(t)d_x^\dagger(t)\, dt\). We define the wavepacket creation operator for node (b), \(B_x^\alpha\), which transforms as \(B_x^\alpha = \frac{1}{\sqrt{2}} (C_x^\alpha + D_x^\alpha)\). The output state thus becomes

$$|\phi_{out}\rangle = \frac{1}{4} \left[ (C_H^\alpha - D_H^\alpha)(C_V^\beta + D_V^\beta)|\uparrow\downarrow\rangle + (C_V^\alpha - D_V^\alpha)(C_H^\beta + D_H^\beta)|\downarrow\uparrow\rangle + C_H^\alpha C_V^\beta |\uparrow\downarrow\rangle + C_V^\alpha C_H^\beta |\downarrow\uparrow\rangle + C_H^\alpha D_V^\beta |\uparrow\downarrow\rangle - D_H^\alpha C_V^\beta |\downarrow\uparrow\rangle - D_H^\alpha D_V^\beta |\uparrow\downarrow\rangle - D_V^\alpha D_H^\beta |\downarrow\uparrow\rangle \right] |0\rangle$$

(6)

where the last 4 lines correspond to relevant pair of terms for detection of a \(H\) and \(V\) photons leading to successful heralds. The projected state upon the detection event of a \(H\) and \(V\) photons at node (c) at respective times \(t_H\) and \(t_V\) is

$$|\phi_{H,V}(t_H,t_V)\rangle = c_H(t_H)c_V(t_V)|\phi_{out}\rangle$$

$$= \frac{1}{4} c_H(t_H)c_V(t_V) \left[ (C_H^\alpha C_V^\beta |\uparrow\downarrow\rangle + C_V^\alpha C_H^\beta |\downarrow\uparrow\rangle) |0\rangle \right. \right.$$

$$= \frac{1}{4} \left[ \int \alpha_H^\dagger(s_H)\delta(t_H - s_H)\beta_V^\dagger(s_V)\delta(t_V - s_V)\, ds_H\, ds_V |\uparrow\downarrow\rangle \right. \left. + \int \alpha_V^\dagger(s_V)\delta(t_V - s_V)\beta_H^\dagger(s_H)\delta(t_H - s_H)\, ds_H\, ds_V |\downarrow\uparrow\rangle \right] |0\rangle$$

(7)

where we have used \([c_x(t_x),c_x^\dagger(s_x)] = \delta(t_x - s_x)\). The fidelity of the conditional atomic state for the above detection event, \(|\psi^+\rangle = (0)c_H(t_H)c_V(t_V)|\psi_{out}\rangle\), with the Bell state \(|\Psi^+_{\text{Bell}}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\) is given by

$$F = \frac{1}{\text{Tr}(|\psi^+\rangle\langle\psi^+|)} |\langle\Psi^+_{\text{Bell}}|\psi^+\rangle|^2$$

$$= \frac{1}{2} \left( 1 + \frac{\alpha_H^\dagger(t_H)\beta_H(t_H)\beta_V^\dagger(t_V)\alpha_V(t_V) + \text{c.c.}}{|\alpha_H^\dagger(t_H)\alpha_V(t_V)|^2 + |\beta_H(t_H)\beta_V(t_V)|^2} \right).$$

(8)

Integrating over all detection events, we find the fidelity to be

$$F = \frac{1}{2} \left( 1 + \text{Re} \left\{ \int dt_H \alpha_H^\dagger(t_H)\beta_H(t_H) \int dt_V \beta_V^\dagger(t_V)\alpha_V(t_V) \right\} \right).$$

(9)

When the emission wavepackets from the emitter-cavity nodes are identical, the fidelity with the Bell state becomes unity.
B. Introducing birefringence

We now consider the case where the second emitter-cavity node suffers from birefringence. It has been shown in [2] that the polarisation of a photon emitted from a birefringent cavity will generally exhibit a time dependence. The different path length for each eigenmode of the cavity leads to a relative path difference, which accumulates during the roundtrips in the cavity, and results in an effective rotation of the polarisation states. If the path difference accumulates faster than the cavity decay rate, a time-dependent polarisation will be observed as depicted in Fig. 1(b). The birefringence transforms the waveform $\beta_H(t)$ to a superposition of an $H$ component with waveform $\beta^H_{t}(t)$, and a $V$ component, $\beta^V_{t}(t)$. The photon-wavepacket creation operator from the non-birefringent system can thus be substituted as the following for the birefringent node:

\begin{align}
B^\delta_H &= B^\delta_H + B^\delta_{V}, \\
B^\delta_V &= B^\delta_{V} + B^\delta_{V},
\end{align}

where $B^\delta_x$ are wavepacket creation operators, with $B^\delta_H = h^\delta_H \int \beta^H_{H*}(t)\alpha^H_{H}(t)dt$ and $B^\delta_V = v^\delta_H \int \beta^V_{V*}(t)\alpha^V_{V}(t)dt$ where the coefficients $h^\delta_H$ and $v^\delta_H$, are introduced for normalisation while retaining the $\int |\beta^H_{H}(t)|^2 dt = 1$ conditions required for the photon-wavepacket creation operator formalism. This transformation makes no assumptions about the relative decay rate for each cavity eigenmode nor about potential back action of the emission onto the emitter-cavity dynamics.

The resulting state at node (b) thus becomes

$$
|\psi^{(b)}\rangle = \frac{1}{\sqrt{2}} \left( (B^\delta_H + B^\delta_V) |\uparrow\rangle + (B^\delta_H - B^\delta_V) |\downarrow\rangle \right).
$$

$B^\delta_H$ and $B^\delta_V$ transform similarly to $B^\delta$ under the beam splitter. If we consider again the detection event of a $H$- and a $V$-polarised photons at node (c), the relevant terms in the expansion of $|\phi_{out}\rangle = |\psi^{(1)}\rangle \otimes |\psi^{(2)}\rangle$ are

$$
C^\delta_H C^\delta_V |\uparrow\rangle, C^\delta_H C^\delta_V |\downarrow\rangle, C^\delta_V C^\delta_H |\uparrow\rangle, C^\delta_V C^\delta_H |\downarrow\rangle.
$$

The projected atomic state for this detection event at respective times $t_H$ and $t_V$ for the $H$ and $V$ polarised photons is

$$
|\Psi^+\rangle = \langle 0\rangle c_H(t_H) c_V(t_V) |\phi_{out}\rangle
= \frac{1}{4} c_H(t_H) c_V(t_V) \left[ C^\delta_H |\uparrow\rangle \left( C^\delta_V |\uparrow\rangle + C^\delta_V |\downarrow\rangle \right) + C^\delta_V |\downarrow\rangle \left( C^\delta_H |\uparrow\rangle + C^\delta_H |\downarrow\rangle \right) \right].
$$

We can clearly see from this that the entangled state is not in the desired Bell basis $\{|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle\}$, but rather in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ with $|\downarrow\rangle = (C^\delta_H |\uparrow\rangle + C^\delta_V |\downarrow\rangle)$ and $|\downarrow\rangle = (C^\delta_H |\uparrow\rangle + C^\delta_H |\downarrow\rangle)$. We will see that this is more than a trivial rotation in bases and that it necessitates a careful evaluation owing to complications introduced by the arbitrary detection times, $t_H$ and $t_V$. Simplifying the terms in Eq. (11), we find the conditional atomic state for the detection of $H$ and $V$ photons at node (c) at respective times $t_H$ and $t_V$ to be:

$$
|\Psi^+\rangle = \langle 0\rangle c_H(t_H) c_V(t_V) |\phi_{out}\rangle
= \frac{1}{4} \left[ \alpha^*_{H}(t_H) |\uparrow\rangle \left( v^\delta_H \beta^H_{H*}(t_V) |\downarrow\rangle + v^\delta_V \beta^V_{V*}(t_V) |\downarrow\rangle \right) + \alpha^*_{V}(t_V) |\downarrow\rangle \left( h^\delta_H \beta^H_{H*}(t_H) |\uparrow\rangle + h^\delta_V \beta^V_{H*}(t_H) |\downarrow\rangle \right) \right]
$$

The fidelity with the Bell state $|\Psi^+_{Bell}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ where $\{|\uparrow\rangle, |\downarrow\rangle\}$ is a measurement basis of node 2 that can be suitably chosen to maximise the fidelity, is given by

$$
F = \frac{1}{\text{Tr}\{\rho\}} \left| \langle \Psi^+_{Bell} | \Psi^+\rangle \right|^2
$$

where $\rho = |\Psi^+\rangle \langle \Psi^+|$. For the traditional choice of the basis $\{|\uparrow\rangle, |\downarrow\rangle, |\uparrow\rangle, |\downarrow\rangle\}$, the fidelity becomes

$$
F = \frac{1}{2} \left[ \frac{\text{Re}\left\{ \alpha^*_{H}(t_H) \beta^H_{H}(t_H) \alpha^*_{V}(t_V) \beta^V_{V}(t_V) \beta^H_{H*}(t_V) v^\delta_{V} \beta^V_{V*}(t_V) h^\delta_{V} \beta^V_{H*}(t_H) \right\}}{8 \text{Tr}\{\rho\}} - \frac{|\alpha_{H}(t_H) v^\delta_{H} \beta^H_{H*}(t_V)|^2 + |\alpha_{V}(t_V) h^\delta_{V} \beta^V_{H*}(t_H)|^2}{16 \text{Tr}\{\rho\}} \right]
$$
which closely resembles the form for the non-birefringent case with an extra reduction term. In addition to the condition of identical $H$ and $V$ waveforms from the nodes, the maximum fidelity in the presence of birefringence requires $|h_\beta H| = |v_\beta V| = 1$, as expected. This traditional choice of $\{|\uparrow\rangle, |\downarrow\rangle\}$ doesn’t maximise the fidelity in general. For instance, one could select $|\uparrow\rangle = (v_\beta H \beta_{VH}^*(t_V)|\uparrow\rangle + v_\beta V \beta_{HV}^*(t_V)|\downarrow\rangle$; in this scenario, the fidelity will be higher than for the traditional choice of $|\uparrow\rangle$, provided the detection times are close, $t_H \simeq t_V$ (in addition to the cavity decay rates for both eigenmodes being similar). To show this, we simplify Eq. (12) to the normalised state

\[ |\phi_{H,V}\rangle = c_1 |\uparrow\rangle |\downarrow\rangle + c_2 |\downarrow\rangle \left( \sqrt{1-p(\delta, \Delta)} |\uparrow\rangle + \sqrt{p(\delta, \Delta)} |\downarrow\rangle \right) \]

and find fidelity

\[ F = \frac{1}{2} \left( 1 + 2 \text{Re} \left\{ c_1^* c_2 \sqrt{1-p} \right\} - |c_2|^2 p \right) \]

where $p$ is a function of $\delta = t_V - t_H$ and $\Delta$, a function of the decay rates and the relative coupling strength to the cavity eigenmodes, governing the overlap between the states $|\uparrow\rangle$ and $|\downarrow\rangle$. For identical cavity mode decay rates and effective coupling strengths, the fidelity is generally maximised when $t_H = t_V$, a condition where $p$ is minimised. This is a crude assessment which assumes a driving pulse which produces photons with no ‘ringing’ on their waveforms.

The approach of choosing an appropriate basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ naturally assumes post-selection of detection time-stamps to filter desired fidelities. An alternative, and complementary, approach to restore the loss in fidelity from the birefringence effects is to make local corrections to the stationary qubits, conditioned on the detection time-stamps. We elaborate on this technique later.

First, we consider if the effects of birefringence, namely the distortion of the photonic waveforms, can be restored before the interference based measurement. The mentioned methods require strict time-stamping of photon detection events. In an ideal scenario where there is no birefringence, the heralding of fully entangled states is independent of the relative detection times. This is a tremendous advantage which removes the necessity of extra, and often noise-injecting, local operations. Naturally, one wonders if one can introduce a serial device to restore time-independence of the photon’s polarisation. Phase retarders are routinely employed in laboratories to manipulate the polarisation of an incoming beam. However, typically, the input beam has a fixed polarisation (compared to the timescale required to adjust the phase retarder’s settings). In the considered case, the length of photons are desired to be short to maximise repetition rate (note, fidelity can be traded for rate and vice versa) and are typically on the order of 500 ns. The polarisation of the produced photons can oscillate on shorter timescales as demonstrated in 32. To the best of the authors’ knowledge, there are no commercially available linear devices capable of performing a time-dependent rotation of such photons’ polarisation to produce a time independent polarisation 41. For the sake of completeness, we point out that such a device could also be another emitter-cavity system with complementary properties. However, if such process could be performed efficiently, the need for a heralding entangler would be defeated. It thus becomes necessary to seek other options in order to recover the desired dynamics.

III. RESULTS

A. Raw fidelity

We numerically study the fidelity of the entangled states produced from the system depicted in Fig. 1(b) with the corresponding Bell states in the case of degenerate and non-degenerate qubit states. The emitter is modelled as a 4 level atom (Fig. 1(c)). A v-stirap (vacuum-stimulated Raman adiabatic passage) scheme is employed to transfer the population from the ground-state, $|g\rangle$, to the stationary qubit states, $|\uparrow\rangle$ and $|\downarrow\rangle$ via the intermediate state $|e\rangle$. We consider only adiabatically produced photons - that is, spontaneous emission does not repopulate the ground state 13. The system at one node exhibits no birefringence whilst the system at the second node exhibits birefringence (Fig. 1(b)) characterised by $\Omega_B$, the frequency splitting of the eigenmodes. Fig. 2 summarises the results. Here, we set $g = \kappa = \gamma/0.6$, where $g$ is the emitter-cavity coupling strength (set to be the same for each of the v-stirap branches), $\kappa$ is the cavity decay rate (assumed to be identical for both modes), and $\gamma$ is the spontaneous emission rate of the $|e\rangle$ state. The qubit states are degenerate and a transition to either qubit state from the $|e\rangle$ state is equally probable. The relation $g \sim \kappa$ is shown to be near ideal for optimal extraction efficiency 20, and the choice $\gamma = 0.6g$ is loosely based on currently achievable desired coupling regimes 26.

Fig. 2(a) show the photons produced from the birefringent system for the $\Omega_B$ setting given at the top of the columns, and when the qubits states are degenerate ($\Delta = 0$). If there were no birefringence, the production of a $H(V)$ photon would only be associated with the atomic $|\uparrow\rangle(|\downarrow\rangle)$ state. However, polarisation oscillations create a
Figure 2: Entanglement dynamics for degenerate qubit states when the birefringence at the birefringent node is set to 1/10, 1/3, 2/3 and 1 in units of $\kappa$. Populations in the emitter-photon state space (a), raw fidelities for detections of $H$ and $V$ photons at respective times $t_H$ and $t_V$ (b), fidelities post-correction for the same detection events (c), corresponding detection probability densities (d), magnitude of local qubit rotation required for fidelity correction at the birefringent node, with arrows indicating the azimuthal rotation direction (e) and fidelity-success rate trade-offs (f), all for the degree of birefringence quoted at the head of the corresponding column. The blue contours in (b) show the detection event that need to be retained for an average raw fidelity of 99.9% (no post-selection would be required for $\Omega_B = 1/10\kappa$). Likewise, the red contours in (c) show the post-selectable region for an average fidelity of 99.9% (no post-selection is required for $\Omega_B \leq \kappa/3$). The respective success probabilities for these contours are marked by the dashed lines in (f).
Figure 3: Entanglement dynamics for non-degenerate qubit states when the birefringence at the birefringent node is set to 1/10, 1/3, 2/3 and 1 in units of $\kappa$. Populations in the emitter-photon state space (a), raw fidelities for detections of $H$ and $V$ photons at respective times $t_H$ and $t_V$ (b), fidelities post-correction for the same detection events (c), corresponding detection probability densities (d), magnitude of local qubit rotation required for fidelity correction at the birefringent node, with arrows indicating the azimuthal rotation direction (e) and fidelity-rate trade-offs (f), all for the degree of birefringence quoted at the head of the corresponding column. The blue contours in (b) show the detection event that need to be retained for an average raw fidelity of 99.9% (no post-selection would be required for $\Omega_B = 0.1\kappa$). Likewise, the red contours in (c) show the post-selectable region for an average fidelity of 99.9% (no post-selection is required for $\Omega_B \leq \kappa/10$). The respective success probabilities for these contours are marked by the dashed lines in (f).
non-zero population in the $|\uparrow V\rangle$ and $|\downarrow H\rangle$ states. We look at the fidelity landscape for all detection times (Fig. 2(b)). We note that the fidelity is highest when detection events occur at times corresponding to the onset of the photonic pulse. This is intuitive as these timescales are shorter than those required for polarisation oscillations to take effect. However, these sets of events happen with very low probability (see Fig. 2(d)), which further restricts the fidelity versus entanglement generation rate trade-off. We also note a symmetry about the diagonal $t_H = t_V$; this is the case when the cavity decay rate has no polarisation dependency and the effective couplings to the stationary qubit states are identical; the fidelity generally falls away from this diagonal.

We study the fidelity versus entanglement rate trade-off attainable by post selection of detection events. The blue contours in Fig. 2(b) show the set of detection events yielding an average fidelity equal to or greater than 99%. Note that this average fidelity is obtained by weighing the raw fidelities (Fig. 2(b)) with the corresponding probability densities for each event (Fig. 2(c)). In a similar way, we produce Fig. 2(f) which show the probability of success associated with a minimum raw fidelity (blue solid lines). The blue dashed vertical lines are analogous to the blue contours in Fig. 2(b), marking the 99% raw average fidelity boundary. It is immediately noticeable that birefringence causes a great detriment to achievable minimum fidelity for a given success rate. When the birefringence is $\Omega_B = 0.1\kappa$, post selection is not required to retain a high average fidelity. On the other hand, for larger birefringence, acute post selection becomes necessary reducing the probability of successful high-fidelity heralding close to nil.

The theoretical maximum probability of heralding from $H$ and $V$ photon detections at different nodes is 25% when there is no birefringence. However, when placing restrictions on the maximum emission length (with the aim of enhancing emission rates), stronger driving pumps are required which in turn reduces photon extraction efficiencies due to spontaneous emission scattering. This results in an overall lower success rate. In the absence of birefringence, all detection events can, in principle, give unit fidelity if one is not concerned with the length of the emitted photons. On the other hand, when birefringence enters the picture (with all other parameters, notably the driving powers and detunings, kept the same), the maximum attainable success rate can drop due to reduced photon emission probabilities.

Fig. 3 is a reproduction of the dynamics in Fig. 2 where the degeneracy of the qubit states is lifted by $\Delta = 5\kappa$ and the transition $|e\rangle \leftrightarrow |f\rangle$ state is $5/4$ times more probable than the $|e\rangle \leftrightarrow |\downarrow\rangle$ transition. The Rabi frequencies of the drives have been adjusted to produce photons of the similar length as in Fig. 2(a), and with their detunings appropriately adjusted for resonant Raman transitions. The resulting photonic profiles Fig. 3(a) are significantly distorted due to a beating effect emanating from the bichromatic drive. This leads to a worsened fidelity-rate relationship in contrast to when $\Delta \approx 0$. The probability density landscapes (Fig. 3(d)) become patterned, and so do the high fidelity regions in Fig. 3(b) adding a layer of complexity for post-selection of high-fidelity detection events.

The results presented in Fig. 2(f) and Fig. 3(f) help deduce the tolerable degree of birefringence for desired fidelities (optionally conditioned on a minimum success rate), and provide specifications for the machining of cavities.

We have seen that the uniform distribution of the fidelity against success probability for the non-birefringent pair of nodes is significantly impaired when one node suffers from birefringence. In the following we assess if and how the fidelities can be recovered.

### B. Restoring fidelity

Heralded entanglement schemes are employed because efficient deterministic global gate operations are presently not available for remote systems. As such, whilst it would be theoretically possible to restore the loss in fidelities via global operations, it would defeat the practical need of employing heralded entanglers. However, local operations are well within the remit of experiments [36,38]. We therefore find, for each detection time pairs $t_H$ and $t_V$, the local qubit rotation that would maximise the fidelity of the resulting state with the Bell state. The resulting fidelities, required corrections and success probability-fidelity trade-offs are presented in Fig. 2(c), Fig. 2(e) and Fig. 2(f) (red solid lines), respectively, when the qubits are degenerate. The corresponding results for the non-degenerate case are presented in Fig. 3(c), Fig. 3(e) and Fig. 3(f) (red solid lines). Fig. 2(e) and Fig. 3(e) show the magnitude of the qubit rotation required to optimise fidelity for each detection event. The overlaid black arrows show the azimuthal direction of the required qubit rotation[49]. Fig. 2(e) and Fig. 3(e) show the fidelity landscapes for each detection time pairs following the qubit corrections. A sizeable difference is noticed in the region with average fidelity of 99% or greater (red contours) when compared to the raw fidelities (Fig. 2(b) and Fig. 3(b)). The fidelity is near-fully restored in the absence of energy splitting between the qubit states (Fig. 2(f)). When the degeneracy between the qubit states is lifted however, restoration to fidelities greater than or equal to 99% is limited to a small fraction of the detection events (see red dashed lines in Fig. 3(f)).

When the degeneracy between the qubits is lifted, the photonic temporal profiles are heavily distorted from the preferred Gaussian-like profile [19]. Furthermore, the population profile in the $|\uparrow\rangle |H\rangle$ and $|\downarrow\rangle |V\rangle$ states are no longer identical. These result in a practically non-trivial complexity in the qubit corrections landscape required to optimise fidelity (Fig. 3(e)).
In addition to $\Omega_B$, a large birefringence, $\delta_B$, is deliberately induced in the cavity’s $H-V$ basis. **Top:** Here the atom is emitting circularly polarised photons. Polarisation oscillations would result as the $\sigma^\pm$ emission couple to both cavity eigenmodes. **Bottom:** The emission is mapped to $H$ and $V$ polarised photons by an appropriate choice of quantisation axis as in [39]. The drive detunings and/or the Zeeman splitting are adjusted for resonant emission into the $H$ and $V$ modes. Note that the states $|\downarrow\rangle$ and $|\uparrow\rangle$ here do not necessarily correspond to the same Zeeman sublevel states as those in the $\sigma^\pm$ emission scheme. (b-c) Fidelity vs deliberate birefringence, $\delta_B$, superposed on an undesired birefringence of $\Omega_B = 1.0\kappa$ with (red) and without (blue) local qubit corrections to restore fidelity. The green traces show the associated success probability. In (b), $\Delta = 0$ whilst in (c), $\Delta = 5\kappa$.

**C. Deliberate birefringence**

We have seen that birefringence can lead to severe loss in fidelity in measurement based entanglement, some of which can be recovered via local operations. Extra local qubit operations are often themselves a source of noise; experimentalists may therefore naturally prefer to avoid them by opting for non-birefringent cavities. However, due to machining tolerances, it is impossible to eliminate birefringence fully. In the case of non-negligible birefringence, we propose an alternative to the local qubit corrections presented above. This entails matching-up the atomic emissions’ basis as well as frequencies to those of the cavity’s birefringence (Fig 4). We show that by selecting appropriate Zeeman splitting and quantisation axis together with a large birefringence, one can reduce polarisation oscillations and therefore be able to obtain high fidelities without the need for local qubit corrections. We extend this approach to demonstrate that even in the presence of off-axis undesired birefringence (emerging from machining tolerances, for example), adding large birefringence can be used to suppress polarisation oscillation from the undesired birefringence.

We consider the case where there’s an off-axis/undesired birefringence of $\Omega = 1.0\kappa$ as in the last columns of Figs. 2 and 3. We impose an additional linear birefringence, $\delta_B$, up to $\delta_B = 10\kappa$, achievable by engineering deliberate ellipticity onto the cavity mirrors as in [40] and implemented in [27]. As $\delta_B$ increases, the eigenmodes in the $H-V$ basis becomes more and more spectrally resolved, and the degree of polarisation oscillation between these eigenmodes falls. In the limit of large birefringence, this becomes equivalent to coupling the atomic emissions to 2 independent cavities. We thus find that (Fig. 4b-c) high ion-ion entanglement fidelity does not require local qubit corrections for large $\delta_B$. In the case of degenerate qubit states (Fig. 4b), we note a marginal drop in success probability, likely due a beating effect in the photonic outputs due to the bichromatic drive. A reduced degree of adiabaticity leads to reduced emission probability. In the case of non-generate qubits, the drive frequencies need not change greatly for different $\delta_B$; the dynamics in success probability is thus not dominated by frequency beating effects, but likely due to changes in effective cooperativity.
IV. CONCLUSION

Birefringent cavities are known to impose time-dependence on the polarisation of the emitted photons, but the resulting effect on the fidelity of interference-based entanglement-swapping experiments had not previously been investigated. In this paper, we have provided a detailed study of the impact on the performance of heralded, two-photon schemes used to generate entanglement remotely between emitters at distant nodes. We have shown that, in the presence of birefringence, the detection of orthogonal photons no longer heralds fully entangled states, resulting in significantly lower entanglement fidelity. As such, we conclude that the best approach for high entanglement fidelity and rate is the use of non-birefringent cavities. Where cavity birefringence cannot be made negligible a remedy is the use of local qubit operations to attempt to restore orthogonality in the collapse state subspace of the emitters. However, applying the appropriate correction requires sampling from a continuum of unitary correction pulses areas and angles on the basis of the photon arrival times. This approach can near-fully recover fidelities when the qubit states are degenerate, and in this regime the pulse parameters vary slowly with photon time. However, when a large splitting separates the qubit energies and the system is driven with a bichromatic Raman field, recovery of fidelities greater than 99% is possible only for a fraction of the detection space with the optimal pulse parameters varying rapidly with the arrival times, making reliable correction onerous. Considering this situation, we presented an alternative approach to suppress polarisation oscillations. By inducing a large, deliberate birefringence co-linear with the emission bases, the coupling between orthogonal emissions due to stray birefringence can be reduced and hence high entanglement fidelities retained. In summary, uncontrolled birefringence can lead to significant loss in fidelity and success rates in measurement-based entanglement and these effects should be carefully considered and mitigated when designing such experiments.

Acknowledgements

The authors are grateful for helpful discussions with A. Kuhn and T. Barrett. This work was funded by the United Kingdom Engineering and Physical Sciences Research Council “Networked Quantum Information Technology” and “Quantum Computing and Simulation” Hubs.

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[41] UPC068 Ultra-Fast Pockels Cell, Leysop LTD.

[42] For simplicity, we consider the naturally desired balanced entangled state (which maximises entanglement generation rates) without loss of generality on the effects of birefringence. The formulation can simply be extended to an unbalanced entangled state.

[43] When the cavity decay rates are not polarisation dependent, we have $\beta_H^H(t) = \beta_V^V(t)$, $\beta_H^V(t) = \beta_V^H(t)$, $h^{\beta,H} = v^{\beta,V}$ and $h^{\beta,V} = v^{\beta,H}$, assuming all other parameters for the production of $H$ and $V$ photons are identical.

[44] The fastest identified optics were Pockels cells, typically employed in photon routing and Q-switching, with $\sim$-nanosecond optical rise times \cite{UPC068} and modulation bandwidths generally limited to hundreds of kHz.

[45] The coherently produced photons are usually the dominant contributors to the temporally mixed photonic output. In cavity-assisted Raman transition schemes the effective spontaneous emission rate falls with the square of the pump detuning whilst the effective coupling falls linearly with the detuning. Hence, temporal mode mixing could be sufficiently eliminated.

[46] The altitude contributions aren’t generally negligible but the azimuthal contribution were deemed sufficient to gauge a sense of the variability in the required correction for different $\Omega_B$ settings.