Distributed Evaluation and Convergence of Self-Appraisals in Social Networks

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Abstract—We consider in this paper a networked system of opinion dynamics in continuous time, where the agents are able to evaluate their self-appraisal in a distributed way. In the model we formulate, the underlying network topology is described by a strongly connected digraph. For each ordered pair of agents \((i, j)\), we assign a function of self-appraisal to agent \(i\), which measures the level of importance of agent \(i\) to agent \(j\). Thus by communicating only with her neighbors, each agent is able to calculate the difference between her level of importance to others and others’ level of importance to her. The dynamical system of self-appraisals is then designed to drive these differences to zero. We show that for almost all initial conditions, the solution of this dynamical system asymptotically converges to an equilibrium which is exponentially stable.

I. INTRODUCTION

According to Wikipedia, a social network is defined as a social structure made up of actors, such as agents and organizations, and the dyadic ties between these actors. This is a formal definition from sociology. Roughly speaking, a social network consists of a group of people and the relationships between agents. The concept of social networks is familiar to most people because of the emergence of online social networking services such as Facebook, Twitter, and Google+. Many social behaviors spread through social networks of interacting agents. Examples are opinion dynamics and demonstrations [4]. In this paper, we will focus on an important issue in opinion dynamics.

Over the past decades, there has been considerable attention in understanding how an agent’s opinion evolves over time. In social science, various models have been proposed to illustrate opinion dynamics. Notable among them are the three classical models: the DeGroot model [1], the Friedkin-Johnsen model [5], and the Krause model [6]. In the DeGroot model, each agent has a fixed set of neighbors and takes the convex combination of her own opinion and the opinions of her neighbors. The Friedkin-Johnsen model is a variation of the DeGroot model in which each agent adheres to her initial opinion to a certain degree. The Krause model defines the neighbor sets in a different way; each agent takes those agents whose opinions differ from her by no more than a certain confidence level as her neighbors. It is worth emphasizing that the Krause model is a nonlinear system, while the first two models lead to linear systems.

Recently, with the rapid expansion of large-scale online networks, much attention has been drawn in control society to model and analyze processes of opinion formation, aiming to extend the classical models for taking into account more variations [7]–[14]. In the work of [7] and [8], the effects of stubborn agents who never update their opinions are investigated in a randomized pair-wise updating process. In [9], the opinion formation process is reformulated as a local interaction game and the concept of the stubbornness of an agent regarding her initial opinion is introduced. The work of [10] proposes a variation of the Krause model, which involves a continuum of agents. In [11], another variation of the Krause model is proposed in which the neighbors of an agent are defined to be those agents whose influence range contains this agent. The work of [12] takes into account exogenous factors, such as the influence of media, and each agent is assumed to update her opinion via the opinions of the population inside the agent’s confidence range and the information from an exogenous input in that range. In the literature, both discrete-time [1], [5] and continuous-time [15], [16] approaches have been adopted on how each agent updates her opinion.

A particularly interesting recent work is the DeGroot-Friedkin model proposed by Jia et al. [13], [14]. The DeGroot-Friedkin model borrows the concept of reflected appraisal from sociology [17], [18] and studies the evolution of self-confidence, i.e., how confident an agent is for her opinions on a sequence of issues. Briefly speaking, reflected appraisal describes the phenomenon that agents’ self-appraisals on some dimension (e.g., self-confidence, self-esteem) are influenced by the appraisals of other agents on them. The DeGroot-Friedkin model with stubborn agents is addressed in [19]. Following the work of [13], [14], a modified DeGroot-Friedkin model is proposed in [20] in which each agent updates its level of self-confidence in a more frequent manner. Specifically, all the agents in the network update their own levels of self-confidence right after each time of discussion, instead of waiting for the opinion process to reach a consensus on any particular issue which could take infinite time. The complete analysis of the modified DeGroot-Friedkin model is carried out in some special cases, yet a complete understanding of the system behavior for the most general case remains open.

Motivated by the discrete-time dynamical system in the modified DeGroot-Friedkin model, we introduce in this paper a continuous-time self-appraisal model whereby the agents in a social network are able to evaluate their self-appraisals.
in a distributed way. For each ordered pair of agents \(i\) and \(j\), we assign a function of self-appraisal to agent \(i\), which measures the level of importance of agent \(i\) to agent \(j\). With local interaction with her neighbors only, each agent is able to calculate the difference between her level of importance to others and others’ level of importance to her. The proposed dynamical system of self-appraisals aims to drive these differences to zero. We show that for almost all initial conditions, the solution of the dynamical system asymptotically converges to an equilibrium which is exponentially stable under some quite general assumptions.

Also the continuous-time self-appraisal model has its own mathematics value as it can be viewed as a prototype of a general class of dynamical systems as

\[
\dot{x}_i = -f(x_i) + \sum_j c_{ji} f(x_j) \tag{1}
\]

with \(f\) a smooth function, and \(c_{ji}\) positive constant. In the case \(f(x) = x\), system (1), with appropriate values of \(c_{ij}\), can be realized as a consensus process, or as a forward equation for a Markov process. Yet, less is known when the function \(f\) is nonlinear. In this paper, we will address the system behavior of this type of differential equation in the unit simplex with the function \(f\) being a particular quadratic function.

The rest of this paper is organized as follows. In section II, we will describe in detail the self-appraisal model as well as the motivations behind it. Also we will state the main theorem of the paper at the end of the section. In section III, we will focus on some basic properties of the dynamical systems. In particular, we will show that the unit-simplex (i.e., the convex hull spanned by the standard basis of \(\mathbb{R}^n\)) is a positive invariant set, i.e., if we start with any initial condition in the simplex, then the solution of the dynamical system will remain in the simplex for all time. In the same section, we will also characterize the set of equilibria of the dynamical system in the simplex. We show that there will be \((n+1)\) equilibria in total, with \(n\) equilibria being the \(n\) vertices and one in the interior of the simplex. We then show in section IV that all vertices are repellers, i.e., we are able to construct a closed neighborhood of each vertex of the simplex such that any solution of the dynamical system can stay in that closed neighborhood only in a finite amount of time. On the other hand, we show in section V that the interior equilibrium is exponentially stable. The issue of the global convergence of the dynamical system will also be addressed and established in the last section.

II. THE CONTINUOUS-TIME SELF-APPRaisal Model AND THE MAIN Theorem

Consider a group of \(n\) interacting agents among whom there is a process of opinion formation taking place. The opinion of each agent may initially differ from each other, but by communicating with her neighbors and taking account into their opinions, all the agents may reach an agreement, or say a consensus state. Of course, it matters in the social network what consensus state the opinion process will converge to. This certainly depends on the update rule of each agent. For example, it matters how much an agent \(i\) insists on her own opinion, and how much she will take the opinions from others. In this paper, we assume that the self-consistency of an agent is determined by her self-appraisal, and we will focus on how the self-appraisal evolves over time.

Before stepping further, we will first introduce some basic definitions, as well as some notations of this paper. By convention, the neighbor relation in this paper is characterized by a directed graph (or simply digraph) \(G = (V,E)\), with \(V = \{1, \ldots, n\}\) the set of vertices and \(E\) the set of edges. If \(i \rightarrow j\) is an edge of \(G\), then we assume that agent \(j\) knows the opinion held by agent \(i\). We assume, in this paper, that the digraph \(G\) is strongly connected, i.e., for any ordered pair \((i,j)\) of vertices, there is a path from \(i\) to \(j\) in \(G\). For each fixed \(i\), an agent \(k\) is said to be an incoming neighbor of \(i\) if \(k \rightarrow i \in E\), and similarly an agent \(j\) is said to be an outgoing neighbor of \(i\) if \(i \rightarrow j \in E\). We denote by \(V^+_i\) (resp. \(V^-_i\)) the set of incoming (resp. outgoing) neighbors of agent \(i\). A key assumption we will take for the graph \(G\) is that each agent \(i\) has at least two incoming neighbors, in other words the number of agent she could take opinions from is at least two.

To introduce the self-appraisal model, we first consider the following continuous-time opinion dynamical system as

\[
\dot{z}_i(t) = (1 - x_i(t)) \cdot ( -z_i(t) + \sum_{j \in V^-_i} c_{ij} z_j(t)) \tag{2}
\]

where \(z_i(t)\) is a real number (or a vector) representing the opinion of agent \(i\) on certain ongoing issue at time \(t\). The number \(x_i(t) \in [0,1]\) represents the self-appraisal of agent \(i\) in the social network. Each \(c_{ij}\) with \(i \rightarrow j \in E\) is strictly positive, and we assume in this paper that

\[
\sum_{j=1}^{n} c_{ij} = 1, \quad \forall i \in V \tag{3}
\]

With these properties of \(c_{ij}\), we realize that system (2) is a continuous-time linear consensus process [21], scaled by the nonnegative factor \((1 - x_i(t))\) which models the insistence of agent \(i\) on her own opinion. To see this, we note that the larger the value of \(x_i(t)\), then the smaller the magnitude of \(\dot{z}_i(t)\). The value of \((1 - x_i(t))\) can be thought as a measure of the total amount of opinions agent \(i\) is willing to take from others. The number \(c_{ij}(1-x_i(t))\) is then the amount of opinion agent \(i\) takes from agent \(j\). As a reminder, we have assumed that each vertex \(i\) of \(G\) has at least two outgoing neighbors. We will now make another key assumption that each \(c_{ij}\), with \(i \rightarrow j \in E\), is less than \(1/2\). This is the case when there is no dominant neighbor for each agent \(i\). In other words, when agent \(i\) takes opinions from her outgoing neighbors, there is no such agent whose opinion is dominant over all the other neighbors.

With the consensus process in mind, we then propose the
dynamics of the self-appraisal $x_i$ as follows.

$$\dot{x}_i = -(1-x_i)x_i + \sum_{j \in V_i^+} c_{ij}(1-x_j)x_j$$ (4)

We assume that the sum of the self-appraisals is scaled to be one, i.e., the vector $\vec{x} = (x_1, \cdots, x_n)$ is located in the unit simplex denoted by $\Delta^{n-1}$. We will show in the next section that if an initial condition $\vec{x}(0)$ other than a vertex of $\Delta^{n-1}$, the solution $\vec{x}(t)$ of system (4) converges to the equilibrium $\vec{x}^\ast$. In other words, $x_i^\ast$ will be the steady state of the self-appraisal of agent $i$ for all initial conditions other than vertices.

III. SIMPLEX AS A POSITIVE INVARIANT SET AND THE ASSOCIATED EQUILIBRIA

In this section, we will focus on establishing some basic properties of system (4). The first result we will establish in this section is stated below.

**Theorem 1:** The unit simplex $\Delta^{n-1}$ is a positive invariant set of system (4).

We first note that the system equation can be recast into the following matrix form as

$$\dot{x} = C^\top (I - X)x$$ (6)

where $X := \text{diag}(x_1, \cdots, x_n)$ is a diagonal matrix, and $C$ is an infinitesimal stochastic matrix (i.e., a matrix with nonnegative off-diagonal entries and zero-row-sum) defined as follows. Let $C_{ij}$ be the $ij$-th entry of $C$, then

$$C_{ij} = \begin{cases} c_{ij} & \text{if } i \to j \in E \\ 0 & \text{otherwise} \end{cases}$$ (7)

We note that since the digraph $\mathbb{G}$ is strongly connected, and each $c_{ij}$ is positive for $i \to j \in E$. Thus, the matrix $C$ is irreducible, i.e., it is not similar via permutation to a block upper-triangular matrix. A relevant property of an irreducible, infinitesimal stochastic matrix is that it only has a simple zero eigenvalue and all the other eigenvalues have negative real parts.

Let $\vec{e} \in \mathbb{R}^n$ be a vector of all ones, and let $\phi(x) := \langle \vec{e}, \vec{x} \rangle$ with $\langle \cdot, \cdot \rangle$ the standard inner-product of two vectors in $\mathbb{R}^n$. We then have

$$\frac{d}{dt} \phi(\vec{x}(t)) = \langle \vec{e}, C^\top (I - X)\vec{x} \rangle = 0$$ (8)

This holds because $C$ has zero-row-sum, and hence $C\vec{e} = 0$. So if we start with an initial condition $\vec{x}(0) \in \Delta^{n-1}$, then $\phi(\vec{x}(t)) = 1$ for all $t \geq 0$. So to prove Theorem 2 it suffices to show that each entry $x_i(t)$ of $\vec{x}(t)$ will be nonnegative along the evolution.

We will now introduce a key notion for proving Theorem 2. Let $\vec{x} \in \Delta^{n-1}$ but other than a vertex, and we assume that $x_i = 0$ for some $i \in V$. Let

$$\gamma := j_k \to \cdots \to j_1 \to i$$ (9)

be a path in $\mathbb{G}$ satisfying the condition that

$$\begin{cases} x_{j_k} > 0 \\ x_{j_l} = 0 \quad \forall l < k \end{cases}$$ (10)

The length of a path is defined to be the number of edges contained in the path. So for example, in the case above, the length of $\gamma$ is $k$. Suppose $k$ is the least number for a path $\gamma$ to satisfy condition (10), then we say $\gamma$ is a critical path.
related to vertex \( i \) at the point \( \bar{x} \). It is clear that all critical paths have the same length.

**Lemma 3:** Suppose \( \bar{x}(t) \in \Delta^{n-1} \) but not a vertex, and \( x_i(t) = 0 \) for some \( i \in V \). Let \( k \) be the length of a critical path related to \( i \) at \( \bar{x}(t) \). Then

\[
\frac{d^k}{dt^k} x_i(t) > 0
\]

This, in particular, implies that \( x_i(t + \epsilon) > 0 \) for sufficiently small \( \epsilon \).

**Proof:** The proof of the lemma is carried out by induction on the length of the critical path.

**Base case.** We assume that \( k = 1 \). Then there must exist a \( j' \in V_i^+ \) such that \( x_{j'}(t) > 0 \). So then

\[
\frac{d}{dt} x_i = -(1 - x_i)x_i + \sum_{j \in V_i^+} c_{ji} (1 - x_j) x_j \\
\geq c_{ji} (1 - x_j) x_{j'} > 0
\]

The last inequality holds because \( \bar{x}(t) \) is not a vertex, and hence \( 0 < x_{j'}(t) < 1 \). This establishes the base case.

**Inductive step.** We will assume that the lemma holds for any \( k \leq m \) with \( m \geq 1 \), and we prove for the case \( k = m + 1 \). Let \( W_i^+ \subseteq V_i^+ \) be defined as follows: if \( j \in W_i^+ \), then \( x_i(t) = 0 \) and there is a critical path of length \( m \) related to \( j \) at \( \bar{x}(t) \). The set \( W_i^+ \) is nonempty because the length of a critical path related to \( i \) at \( \bar{x}(t) \) is \((m+1)\). It is also clear that there does not exist a \( j \in V_i^+ \), together with a critical path related to \( j \) at \( \bar{x}(t) \) whose length is less than \( m \). This holds because otherwise, there will be a critical path related to \( i \) at \( \bar{x}(t) \) whose length is less than \((m+1)\). By induction, we have

\[
\frac{d^l}{dt^l} x_j(t) = 0, \quad \forall j \in V_i^+ \text{ and } \forall l < m
\]

This, in particular, implies that

\[
\frac{d^l}{dt^l} x_i(t) = 0, \quad \forall l \leq m
\]

On the other hand, we have

\[
\frac{d^m}{dt^m} x_j(t) = \begin{cases} > 0 & \text{if } j \in W_i^+ \\ = 0 & \text{otherwise} \end{cases}
\]

So then, by direct computation we will have

\[
\frac{d^{m+1}}{dt^{m+1}} x_i(t) = \sum_{j \in W_i^+} (1 - x_j(t)) \frac{d^m}{dt^m} x_j(t) > 0
\]

This then completes the proof.

**Theorem 4:** There are \((n+1)\) equilibria of system \( \mathbf{4} \). The \( n \) vertices of \( \Delta^{n-1} \) are all equilibria, and the rest equilibrium \( \bar{x}^\ast \) is in the interior of \( \Delta^{n-1} \).

**Proof:** It is clear that if \( \bar{x} = \bar{C} \), then \((I - X)\bar{x} = 0\), thus the vector field \( f \) vanishes at \( \bar{C} \). In the rest of the proof, we will show that there is only one equilibrium in the interior of \( \Delta^{n-1} \). Since \( C \) is an irreducible, infinitesimal stochastic matrix, there is a simple zero eigenvalue. Moreover, if we let \( \bar{v} \) be the left-eigenvector of \( C \) for the zero eigenvalue, and we scale \( \bar{v} \) so that \( \langle \bar{v}, \bar{v} \rangle = 1 \), then all entries of \( \bar{v} \) strictly positive. On the other hand, if \( \bar{x}^\ast \) is an equilibrium of system \( \mathbf{4} \), then we must have

\[
(I - X^\ast)\bar{x}^\ast = \alpha \bar{v}
\]

for some positive number \( \alpha \). To be explicit, we let \( x_i^\ast \) (resp. \( v_i \)) be the \( i \)-th entry of \( \bar{x}^\ast \) (resp. \( \bar{v} \)), so then

\[
(1 - x_i^\ast) x_i^\ast = \alpha v_i
\]

for all \( i \in V \). We now note a fact (which we will prove in the next section) that if \( \bar{x}^\ast \) is an equilibrium of system \( \mathbf{4} \), then \( x_i^\ast \leq 1/2 \). This, in particular, implies that for a fixed \( \alpha \), we can solve for each \( x_i^\ast \) explicitly as

\[
x_i^\ast = \frac{1}{2} (1 - \sqrt{1 - 4\alpha v_i})
\]

It now suffices to show that there is a unique \( \alpha \) with the resulting vector \( \bar{x}^\ast \) contained in \( \Delta^{n-1} \).

To establish this fact, we introduce a function \( \psi(\alpha) \) given by

\[
\psi(\alpha) := \sum_{i \in V} \frac{1}{2} (1 - \sqrt{1 - 4\alpha v_i})
\]

It is clear that \( \psi(0) = 0 \), and \( \psi \) is strictly monotonically increasing in \( \alpha \) as long as \((1 - 4\alpha v_i)\) remains nonnegative for all \( i \in V \). For convenience, we assume that \( v_1 \geq v_i \) for all \( i > 1 \), and let \( \alpha_1 := 1/(4v_1) \). It then suffices to show that \( \psi(\alpha_1) > 1 \) because if this holds, then there will be a unique \( \alpha \in (0, \alpha_1) \) with \( \psi(\alpha) = 1 \), and hence, the equilibrium \( \bar{x}^\ast \) solved by expression \((19)\) is also unique in the interior of \( \Delta^{n-1} \). First we re-write \( \psi(\alpha_1) \) as

\[
\psi(\alpha_1) = \frac{n}{2} - \frac{1}{2} \sum_{i > 1} \sqrt{1 - v_i/v_1}
\]

We then make a key observation that \( v_1 \leq 1/3 \). This follows from the fact that

\[
v_1 = \sum_{j \in V_1^+} c_{ji} v_j \leq \frac{1}{2} \sum_{j > 1} v_j = \frac{1}{2} (1 - v_1)
\]

The first equality holds because \( C^\top \bar{v} = 0 \), the middle inequality holds because \( c_{ji} \leq 1/2 \) by our earlier assumption, and the
last equality holds because $\sum_{i=1}^n v_i = 1$. Next, we consider the following optimization problem as

$$\begin{align*}
\max & \sum_{i=2}^{n-1} \sqrt{1-w_i} \\
\text{s.t.} & \sum_{k=2}^k w_i \geq 2 \quad \text{and} \quad 0 \leq w_i \leq 1
\end{align*}$$

(23)

with $w_i := v_i/v_1$. Since the function $\sum_{i=2}^{n-1} \sqrt{1-w_i}$ is concave in $w_i$, the maximum is achieved if and only if

$$w_i = \frac{2}{n-1}, \quad \forall i > 1$$

(24)

In other words, we need to choose $v_1 = 1/3$ and $v_i = 2/(3(n-1))$ for this to happen. So then

$$\max_{i \geq 2} \sum_{i=2}^{n-1} \sqrt{1-w_i} = (n-1)\sqrt{1-\frac{2}{n-1}} < (n-2)$$

(25)

The last inequality holds because

$$1-\frac{2}{n-1} < 1-\frac{2}{n-1} + \frac{1}{(n-1)^2} = \left(\frac{n-2}{n-1}\right)^2$$

(26)

which implies that

$$\psi(\alpha_i) > \frac{n}{2} - \frac{1}{2}(n-2) = 1$$

(27)

This then completes the proof. ■

We here note that the proof of the uniqueness of the interior equilibrium can be carried out in a slightly more general case where the condition $c_{ij} \leq 1/2$ is not required anymore. Also we note that the set of equilibria of system (4) coincides with the set of equilibria of the opinion system studied in [13], [14], though with a completely different dynamical system.

IV. VERTICES AS REPPELLERS

In this section, we will mainly focus on the system behavior around a vertex $\vec{c}_i$ of $\Delta^{n-1}$. Let $\varepsilon$ be a real number in the open unit interval $(0, 1)$, and let $P_i(\varepsilon)$ be a closed neighborhood of $\vec{c}_i$ in $\Delta^{n-1}$ defined as

$$P_i(\varepsilon) = \{ \vec{x} \in \Delta^{n-1} | \vec{x}_i \geq \varepsilon \}$$

(28)

It is clear that for any $\varepsilon \in (0, 1)$, the set $P_i(\varepsilon)$ is nonempty.

In the rest of the paper, we let

$$f(\vec{x}) := C^T(I-X)\vec{x}$$

(29)

be the vector field of system (4), and we let $f_i(\vec{x})$ be the $i$-th component of $f$ at $\vec{x}$. The main result we will establish in this section is stated below.

**Theorem 5:** Suppose for a fixed vertex $i$, there is a real number $\varepsilon \in (0, 1)$ such that $c_{ij} \leq \varepsilon$ for all $j \in V_i^+$. Let $P_i(\varepsilon)$ be the closed neighborhood of $\vec{c}_i$ defined by expression (28), then $f_i(\vec{x}) < 0$ for any $\vec{x} \in P_i(\varepsilon)$. Thus, the set $\Delta^{n-1} - P_i(\varepsilon)$, as the complement of $P_i(\varepsilon)$ in $\Delta^{n-1}$, satisfies the next two conditions

a). the open set $\Delta^{n-1} - P_i(\varepsilon)$ is positive invariant;

b). for any initial condition $\vec{x}(0)$ other than $\vec{c}_i$, the solution of system (4) will enter into $\Delta^{n-1} - P_i(\varepsilon)$ in finite time.

**Proof:** For simplicity, we will assume $i = 1$ in the rest of this proof. It is clear that

$$f_1(\vec{x}) = -(1-x_1)x_1 + \sum_{j \in V_1^+} c_{j1}(1-x_j)x_j$$

$$\leq -(1-x_1)x_1 + \varepsilon \sum_{j \in V_1^+} (1-x_j)x_j$$

(30)

We then consider the following optimization problem as

$$\max_{j \in V_1^+} (1-x_j)x_j$$

s.t. $\sum_{j \in V_1^+} x_j \leq 1 - x_1$ and $x_j \geq 0$

(31)

Since the function $\sum_{j \in V_1^+} (1-x_j)x_j$ is concave in $x_j$ for $j \in V_1^+$, the maximum is achieved when

$$x_j = \frac{1-x_1}{d_1}$$

(32)

for each $j \in V_1^+$. We here use $d_1$ to denote the cardinality of $V_1^+$. So then,

$$\sum_{j \in V_1^+} (1-x_j)x_j \leq (1-x_1)(1-\frac{1-x_1}{d_1})$$

(33)

By combining expression (30) and expression (33), we then have

$$f_1(\vec{x}) \leq -(1-x_1)(-x_1 + \varepsilon - \varepsilon \frac{1-x_1}{d_1})$$

$$\leq -(1-\frac{\varepsilon}{d_1})(1-x_1)(x_1 - \varepsilon \frac{d_1-1}{d_1-\varepsilon})$$

(34)

It is clear that $(1-\varepsilon/d_1) > 0$ because $\varepsilon < 1$ while $d_1 \geq 1$. On the other hand, we know that if $x_1 \geq \varepsilon$, then

$$x_1 - \varepsilon \frac{d_1-1}{d_1-\varepsilon} \geq \varepsilon(1 - \frac{d_1-1}{d_1-\varepsilon}) \geq \frac{\varepsilon(1-\varepsilon)}{d_1-\varepsilon} > 0$$

(35)

This then shows that

$$f_1(\vec{x}) < 0, \quad \text{if } x_1 \in [\varepsilon, 1)$$

(36)

By the condition, it is clear that $\Delta^{n-1} - P_i(\varepsilon)$ is a positive invariant set. To establish condition b), we fix an initial condition $\vec{x}(0) \in P_i(\varepsilon)$ other than $\vec{c}_i$, and we let

$$\nu := \inf \{ |f_1(\vec{x})| \| \vec{x} \| \in P_i(\varepsilon) - P_i(x_1(0)) \}$$

(37)

The set $P_i(\varepsilon) - P_i(x_1(0))$ is compact over which $f_1(\vec{x})$ is strictly positive, so $\nu$ exists and it is positive. So if we start with $\vec{x}(0)$, then the solution $\vec{x}(t)$ will enter into $\Delta^{n-1} - P_i(\varepsilon)$ in no more than $(x_1(0) - \varepsilon)/\nu + 1$ units of time. ■

**Remark 1:** If all the assumptions in the statement of Theorem 5 hold, then the vertex $\vec{c}_i$ is unstable. In fact, there will be no solution $\vec{x}(t)$ of system (4) converging to $\vec{c}_i$. Also, we note that the value of $\vec{x}_i(t)$ will be less than $\varepsilon$.

**Remark 2:** By expression (25), we may sharpen the statement of Theorem 5 by expanding the set $P_i(\varepsilon)$ to $P_i(\varepsilon - \delta_1)$ for some positive number $\delta_1$. For example, if we let

$$\delta_1 := \frac{1}{2} \varepsilon(1-\varepsilon)$$

(38)

then $0 < \delta_1 < \varepsilon$, and we have

$$f_1(\vec{x}) \leq -\frac{1}{2}(1-\frac{\varepsilon}{d_1})(1-x_1)\delta_1 < 0$$

(39)
for any \( x_i \geq \varepsilon - \delta_i \). Then \( P_i(\varepsilon - \delta_i) \) contains \( P_i(\varepsilon) \) as a proper subset, and the set \( \Delta^{n-1} - P_i(\varepsilon - \delta_i) \) satisfies the two properties in the statement of Theorem 6. The fact that we can replace \( P_i(\varepsilon) \) with \( P_i(\varepsilon - \delta) \) may not be so appealing in terms of evaluating \( x_i \), especially when \( d_1 \gg 1 \). But it will be useful for proving the convergence of system (4) in the next section.

We have assumed in this paper that \( c_{ij} \leq 1/2 \) for all \( i \rightarrow j \in E \), so by Remark 1 we have

\[
x_i^* < 1/2
\]

for all \( i \in V \). Also by Theorem 5 and Remark 2 there is a positive number

\[
\delta := \min_{i \in V} \delta_i
\]

such that each \( P_i(1/2 - \delta) \) satisfies the two properties stated in Theorem 5. We will fix the number \( \delta \) in the rest of this paper, and for convenience we let

\[
\alpha := 1/2 - \delta
\]

Now consider the following set as

\[
Q := \{ \bar{x} \in \Delta^{n-1} | x_i < \alpha, \forall i \in V \}
\]

It is clear that we can write \( Q \) as

\[
Q = \bigcap_{i \in V} (\Delta^{n-1} - P_i(\alpha))
\]

Thus, a corollary of Theorem 6 about \( Q \) is that

**Corollary 6:** The set \( Q \), defined by expression (43), is positive invariant. For any initial condition \( \bar{x}(0) \) other than a vertex \( \bar{e}_i \), the solution \( \bar{x}(t) \) will enter into \( Q \) in finite time.

An illustration of \( Q \) in the case \( n = 3 \) is made in Figure 1. Yet, we hope that this illustration does not leave you an impression that the set \( Q \) is small when compared with \( \Delta^{n-1} \). Suppose we measure the set \( Q \) by computing the proportion of the volume of \( Q \) over the volume of \( \Delta^{n-1} \), then one can show that

\[
\frac{\operatorname{Vol}(Q)}{\operatorname{Vol}(\Delta^{n-1})} = 1 - n(1 - \alpha)^{n-1}
\]

To evaluate the right hand side of the expression, we first notice that there is an upper bound for \( \delta \) as 1/12 which happens in the case \( \max_{i \in V} d_i = 2 \). In particular, we note that this upper bound does not depend on \( n \). We then have

\[
\frac{\operatorname{Vol}(Q)}{\operatorname{Vol}(\Delta^{n-1})} \geq 1 - n(7/12)^{n-1}
\]

The right hand side of the expression is strictly increasing for \( n \geq 2 \), and moreover

\[
\lim_{n \to \infty} \frac{\operatorname{Vol}(Q)}{\operatorname{Vol}(\Delta^{n-1})} = 1
\]

In fact, if we choose \( n = 10 \), then \( \operatorname{Vol}(Q)/\operatorname{Vol}(\Delta^{n-1}) > 0.9 \).

V. Global Convergence of the Self-Appraisal Model

In this section, we will focus on the global behavior of system (4). In particular, we will establish the convergence of system (4), and then prove that the interior equilibrium \( \bar{x}^* \) is exponentially stable. We recall the set

\[
Q = \{ \bar{x} \in \Delta^{n-1} | x_i < 1/2, \forall i \in V \}
\]

defined by expression (42), and we will now establish the next result about \( Q \).

**Theorem 7:** Let \( \bar{x}(0) \in Q \) be the initial condition of system (4), then the solution \( \bar{x}(t) \) converges to the interior equilibrium \( \bar{x}^* \in Q \).

**Proof:** In this proof, we find it helpful to introduce the following variable as

\[
y_i := \frac{1 - x_i}{1 - x_i^*} x_i
\]

this is well-defined because \( 0 < x_i^* < 1/2 \). The dynamical system for \( y_i \) is then given by

\[
\frac{d}{dt} y_i = (1 - 2x_i)(-y_i + \sum_{j \in V_i^+} c_{ji} y_j)
\]

where \( c_{ji} \) is given by

\[
c_{ji} := c_{ji} \frac{(1 - x_i^*)x_j}{(1 - x_i^*)x_i^*}
\]

We introduce \( \tilde{c}_{ji} \) because it satisfies the condition that

\[
\sum_{j \in V_i^+} \tilde{c}_{ji} = \frac{1}{(1 - x_i^*)x_i^*} \sum_{j \in V_i^+} c_{ji} (1 - x_j^*)x_j^* = 1
\]

and this holds for all \( i \). This equality is important because it enables us to re-write system equation (50) in the following matrix form as

\[
\frac{d}{dt} \bar{y} = A(\bar{x})^\top \bar{y}
\]
where $A(\vec{x})^\top$ is an irreducible, infinitesimal stochastic matrix defined as follows. Let $a_{ij}(\vec{x})$ be the $i$-th entry of $A(\vec{x})$, then

$$a_{ij}(\vec{x}) := \begin{cases} (1 - 2x_i)\delta_{ij} & \text{if } i \to j \in E \\ 0 & \text{otherwise} \end{cases}$$

(54)

for $i \neq j$, and the diagonal entries of $A(\vec{x})$ are then determined by the condition that the column-sum of $A(\vec{x})$ is zero. We will now describe a relevant property about off-diagonal entries of $A(\vec{x})$. First we notice that if $\vec{x} \in Q$, then

$$1 - 2x_i > 1 - 2\alpha \geq 2\delta$$

(55)

with $\alpha$ and $\delta$ defined by expression (41) and expression (42) respectively. This, in particular, implies that there is a fixed positive number $\sigma$ such that if $i \to j \in E$, then

$$a_{ij}(\vec{x}) \geq \sigma, \quad \forall \vec{x} \in Q$$

(56)

In other words, we have found a uniform lower bound $\sigma$ for all nonzero off-diagonal entries of $A(\vec{x})$ for $\vec{x} \in Q$.

Now fix an initial condition $\vec{x}(0) \in Q$, and let $\vec{x}(t)$ be the solution of system (4). Then the $y$-system (53) can be realized as a time-varying linear system, with $A(\vec{x}(t))$ (or simply $A(t)$) the time-varying infinitesimal stochastic matrix. We then recognize that this is a quite standard time-varying linear consensus model. It is shown in [21] that if $G = (V,E)$ is strongly connected, and if $a_{ij}(t) \geq \sigma$ for some positive $\sigma$ for all $t \geq 0$ and for all $i \to j \in E$, then the transition matrix of the time-varying linear system converges to a rank-one matrix as

$$\Phi(t) = \vec{e} \cdot \vec{v}^T$$

(57)

where $\vec{e}$ is a vector of all ones, and $\vec{v}$ is a vector in the interior of $\Delta^{n-1}$ satisfying the condition that $\langle \vec{v}, \vec{e} \rangle = 1$. By Corollary 6, the set $Q$ is positive invariant, so then the solution $\vec{x}(t)$ is contained in $Q$ for all $t \geq 0$. Thus, the matrix $A(t)$ satisfies the sufficient condition, as stated above, for the convergence of the transition matrix $\Phi(t)$. This then immediately implies that along the evolution of system (53), we have

$$\vec{y}(\infty) = \langle \vec{v}, \vec{y}(0) \rangle \cdot \vec{e}$$

(58)

The convergence of $\vec{y}(t)$ implies the convergence of $\vec{x}(t)$. Since there is only one equilibrium $\vec{x}^*$ inside $Q$, thus we have to conclude that $\vec{x}(\infty) = \vec{x}^*$. ■

By combining Corollary 6 and Theorem 7, we then establish the result of convergence of system (4), i.e., for any initial condition $\vec{x}(0)$ other than a vertex $\vec{e}_i$, the solution of system (4) converges to the unique equilibrium $\vec{x}^* \in Q$. The rest of the section is devoted to establishing the next result.

**Theorem 8:** The interior equilibrium $\vec{x}^*$ of system (4) is exponentially stable.

**Proof:** Let $J(\vec{x}^*)$ be the Jacobian matrix of the vector field $f$ at $\vec{x}^*$, i.e.,

$$J(\vec{x}^*) := \frac{\partial f(\vec{x}^*)}{\partial \vec{x}^*}$$

(59)

Then by computation, we have

$$J(\vec{x}^*) := C^\top (I - 2\vec{x}^*)$$

(60)

Since $x_i^* < 1/2$ for all $i$, the diagonal matrix $(I - 2\vec{x}^*)$ have positive entries. Thus, the matrix $J(\vec{x}^*)^\top$ will be an irreducible, infinitesimal stochastic matrix, and hence it has only one simple zero eigenvalue while all the other eigenvalues of $J(\vec{x}^*)$ have negative real parts. Let $T$ be the linear subspace of $\mathbb{R}^n$ perpendicular to $\vec{e}$, i.e.,

$$T := \{ \vec{v} \in \mathbb{R}^n | \langle \vec{v}, \vec{e} \rangle = 0 \}$$

(61)

Then $T$ can be realized as the tangent space of any point $\vec{x} \in \Delta^{n-1}$. Moreover, this linear subspace $T$ is invariant under $J(\vec{x}^*)$, i.e., for any vector $\vec{v} \in T$, we have

$$J(\vec{x}^*)\vec{v} \in T$$

(62)

This holds because $J(\vec{x}^*)^\top \vec{e} = 0$. So then, all the eigenvalues of $J(\vec{x}^*)$ have negative real parts when restricted to $T$. This then shows that the equilibrium $\vec{x}^*$ is exponentially stable. ■

Theorem 1 is then established by combining Corollary 6, Theorem 7 and Theorem 8.

VI. CONCLUSIONS

In this paper, we have proposed a continuous-time model whereby a number of agents in a social network are able to evaluate their self-appraisals over time in a distributed way. We investigate the global system behavior by introducing several positive invariant sets, and we show that if the initial condition is not one of the vertices of the simplex, then the solution of the system will converge to the unique stable equilibrium $\vec{x}^*$ in the interior of the simplex. This stable equilibrium is interpreted as the steady state of the self-appraisals of the agents, and we relate the value of each $x_i^*$ to the values of $e_{ij}$ as described by Theorem 5. Future work may focus on the case where each $c_{ij}$ is time-variant. For example, we can assume that each $c_{ij}$ also depends on the self-appraisal of $x_j$, then how much opinion agent $i$ takes from agent $j$ will depend on how influential agent $j$ is in the social network. Also we note this model can be developed into many other interesting problems. For example, a question related to the sparse systems is that given the underlying graph $G$, what is the collection of achievable steady states $\vec{x}^*$ is by the choice of $c_{ij}$. Similar questions have been asked and answered in the context of consensus [22]. Another problem related to the optimal control is to assume that there is an agent $i$ who is able to manipulate his own weights $c_{ij}$, and we ask whether there is a choice of these weights so that self-appraisal of agent $i$ is maximized? If further, we assume that there are two such players each of which is trying to maximize her own self-appraisal, then what would be the strategy for each of the player to choose the $c_{ij}$? This is then related to the game theory. By giving these examples, we understand that this self-appraisal model has a rich structure which can be investigated under various assumptions and from various perspectives.
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