BLASCHKE PRODUCTS AND DOMAINS OF ELLIPTICITY

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ABSTRACT. Elliptic Möbius transformations of the unit disk are those for which there is a fixed point in $D$. It is not hard to classify which Möbius transformations are elliptic in terms of the parameters. The set of parameters can be identified with the solid torus $S^1 \times D$, and the set of elliptic parameters is called the domain of ellipticity. In this paper, we study the domain of ellipticity for finite Blaschke products of the form $B(z) = e^{i\theta} \left( \frac{z-w}{1-w\bar{z}} \right)^n$ for $n \geq 2$. We will also study the set corresponding to the Mandelbrot set for this family, and show how it is related to the domain of ellipticity.

1. Introduction

1.1. Möbius transformations. Every Möbius transformation of the unit disk $D$ can be written in the form

$$A(z) = e^{i\theta} \left( \frac{z-w}{1-w\bar{z}} \right)$$

for some $\theta \in [0, 2\pi)$ and $w \in D$. This can be written alternatively as

$$A(z) = \frac{Ce^{i\theta/2}z - Ce^{i\theta/2}w}{Ce^{-i\theta/2} - Ce^{-i\theta/2}w},$$

where $C = (1 - |w|^2)^{-1/2}$. The significance of writing $A$ in this form is that the matrix representing $A$, given by

$$\begin{pmatrix}
Ce^{i\theta/2} & -Ce^{i\theta/2}w \\
-Ce^{-i\theta/2}w & Ce^{-i\theta/2}
\end{pmatrix},$$

has determinant 1 and trace-squared equal to

$$\tau(A) = \left( \frac{e^{i\theta/2}}{\sqrt{1 - |w|^2}} + \frac{e^{-i\theta/2}}{\sqrt{1 - |w|^2}} \right)^2 = \frac{2(1 + \cos \theta)}{1 - |w|^2}. \tag{1.1}$$

We recall that Möbius transformations of $D$ can be classified as follows:

(i) $A$ is called hyperbolic if $A$ has two fixed points on $\partial D$ and none in $D$,

(ii) $A$ is called parabolic if $A$ has one fixed point on $\partial D$ and none in $D$,

(iii) $A$ is called elliptic if $A$ has no fixed points on $\partial D$ and one in $D$.

Note that by the Schwarz-Pick Lemma (see for example [2]), if $A$ is not the identity, $A$ can have a maximum of one fixed point in $D$, and so these three cases provide a complete classification of Möbius transformations of $D$. This classification can also be expressed in terms of $\tau$:

(i) $A$ is hyperbolic if and only if $\tau(A) > 4$,

(ii) $A$ is parabolic if and only if $\tau(A) = 4$,

(iii) $A$ is elliptic if and only if $0 \leq \tau(A) < 4$. 

We see from (1.1) that \( \tau(A) \) is real when \( A \) is represented in the normalized form as given above. Hence if we fix \( \theta \in [0, 2\pi) \), the set of \( w \)-values for which \( A \) is elliptic is given by the disk

\[ \{ w \in \mathbb{D} : |w| < \sin(\theta/2) \} . \]

Note this set is empty when \( \theta = 0 \). Since the set of parameters for Möbius transformations of \( \mathbb{D} \) can be parameterized by the solid torus \( S^1 \times \mathbb{D} \), the domain of ellipticity is given by the open set

\[ E := \{ (e^{i\theta}, w) \in S^1 \times \mathbb{D} : |w| < \sin(\theta/2) \} . \]

The boundary of \( E \) gives the set of parabolic parameters by the classification in terms of \( \tau \).

1.2. Blaschke products. A finite Blaschke product is a function \( B : \mathbb{D} \to \mathbb{D} \) of the form

\[ B(z) = e^{i\theta} \prod_{i=1}^{n} \left( \frac{z - w_i}{1 - \bar{w}_i z} \right) , \]

for some \( \theta \in [0, 2\pi) \) and \( w_i \in \mathbb{D} \) for \( i = 1, \ldots, n \). We call a Blaschke product non-trivial if \( n \geq 2 \).

Every finite degree self-mapping of \( \mathbb{D} \) is a finite Blaschke product \([2] \text{ p.19}\), and so they can be viewed as analogues for polynomials in the disk. Again by the Schwarz-Pick Lemma, \( B \) can have at most one fixed point in \( \mathbb{D} \). If \( z_0 \) is a fixed point of \( B \), then it is straightforward to show that \( 1/z_0 \) is also a fixed point of \( B \). Hence all but possibly two (with the convention that infinity is a fixed point if some \( w_i = 0 \)) of the fixed points of \( B \) must lie on \( \partial \mathbb{D} \).

The Denjoy-Wolff Theorem \([3] \text{ p.58}\) states that if \( f : \mathbb{D} \to \mathbb{D} \) is holomorphic then there is some \( z_0 \in \overline{\mathbb{D}} \) such that \( f^n(z) \to z_0 \) for every \( z \in \mathbb{D} \). We call such a point a Denjoy-Wolff point of \( f \). There is a classification of finite Blaschke products in analogy with that for Möbius transformations:

(i) \( B \) is called hyperbolic if the Denjoy-Wolff point \( z_0 \) of \( B \) lies on \( \partial \mathbb{D} \) and \( |B'(z_0)| < 1 \),

(ii) \( B \) is called parabolic if the Denjoy-Wolff point \( z_0 \) of \( B \) lies on \( \partial \mathbb{D} \) and \( |B'(z_0)| = 1 \),

(iii) \( B \) is called elliptic if the Denjoy-Wolff point \( z_0 \) of \( B \) lies in \( \mathbb{D} \). In this case, we must have \( |B'(z_0)| < 1 \).

1.3. Dynamics of Blaschke products. The Fatou set of a rational map \( f : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \) is the set

\[ F(f) = \{ z \in \overline{\mathbb{C}} : \text{ the family } (f^n)_{n=1}^{\infty} \text{ is normal in some neighbourhood of } z \} . \]

This is the domain of stable behaviour of the iterates of \( f \). The set of chaotic behaviour is the Julia set \( J(f) \) and is given by \( \overline{\mathbb{C}} \setminus F(f) \). The Fatou set is always open and hence the Julia set is always closed.

For Blaschke products, the Julia set is always contained in \( \partial \mathbb{D} \) and is either the whole of \( \partial \mathbb{D} \) or a Cantor subset of \( \partial \mathbb{D} \). These two cases can be characterized as follows; see \([3] \text{ p.58}\) and \([1, 4]\) as well as \([7]\) for a discussion of this characterization.

**Theorem 1.1.** Let \( B \) be a non-trivial finite Blaschke product. Then:

(i) if \( B \) is elliptic, \( J(B) = \partial \mathbb{D} \),

(ii) if \( B \) is hyperbolic, \( J(B) \) is a Cantor subset of \( \mathbb{D} \),

(iii) if \( B \) is parabolic and \( z_0 \in \partial \mathbb{D} \) is the Denjoy-Wolff point of \( B \), \( J(B) = \partial \mathbb{D} \) if \( B''(z_0) = 0 \) and \( J(B) \) is a Cantor subset of \( \partial \mathbb{D} \) if \( B''(z_0) \neq 0 \).
We remark that the parabolic case is the most subtle. Denoting by $d$ the hyperbolic distance in $\mathbb{D}$, a finite Blaschke product is said to be of zero hyperbolic step if there exists $z \in \mathbb{D}$ such that $\lim_{n \to \infty} d(B^n(z), B^{n+1}(z)) = 0$. If this holds for some $z \in \mathbb{D}$, then it holds for every point in $\mathbb{D}$. If $B$ is not of zero hyperbolic step, then since $d(B^n(z), B^{n+1}(z))$ is non-increasing by the Schwarz-Pick Lemma, we have $\lim_{n \to \infty} d(B^n(z), B^{n+1}(z)) > 0$. In this case, $B$ is said to be of positive hyperbolic step. See [1], as well as [7], for more details.

**Theorem 1.2 ([1] [4]).** A parabolic finite Blaschke product $B$ is of zero hyperbolic step if and only if $J(B) = \partial \mathbb{D}$ if and only if $B^n(z_0) = 0$, where $z_0$ is the Denjoy-Wolff point of $B$.

1.4. Statement of results. The set of degree $n$ Blaschke products for $n \geq 2$ can be parameterized by $S^1 \times \mathbb{D}^n$, that is, $(e^{i\theta}, w_1, \ldots, w_n)$ represents the Blaschke product given by (1.3). We call the set of ellipticity

$$E_n = \left\{ (e^{i\theta}, w_1, \ldots, w_n) \in S^1 \times \mathbb{D}^n : B(z) = e^{i\theta} \prod_{i=1}^{n} \left( \frac{z - w_i}{1 - \overline{w}_i z} \right) \text{ is elliptic} \right\}.$$ 

For $n = 1$, $E_1$ is given by (1.2). In this paper, we will study the subset of $E_n$ given by

$$\{(e^{i\theta}, w_1, \ldots, w_n) \in E_n : w_1 = w_2 = \ldots = w_n\}.$$ 

In other words, this subset consists of elliptic Blaschke products that can be written in the form

$$B(z) = e^{i\theta} \left( \frac{z - w}{1 - \overline{w} z} \right)^n$$

for some $\theta \in [0, 2\pi)$ and $w \in \mathbb{D}$. We denote by $D_n$ the subset of $S^1 \times \mathbb{D}$ corresponding to these Blaschke products. We will show that the set of elliptic parameters is a domain and describe some of its geometric properties in $S^1 \times \mathbb{D}$.

**Theorem 1.3.** Let $\theta \in [0, 2\pi)$ be fixed. Then the subset $\{w \in \mathbb{D} : (e^{i\theta}, w) \in D_n\}$ is an open set in $\mathbb{D}$ which is starlike about 0. The set $D_n$ is open in $S^1 \times \mathbb{D}$ and contains the torus $S^1 \times \{z \in \mathbb{D} : |z| < \frac{n-1}{n+1}\}$.

We next discuss how the connectedness locus for $E_n$ is related to $D_n$. We denote by $\mathcal{M}_n$ the set

$$\mathcal{M}_n = \left\{ (e^{i\theta}, w) \in S^1 \times \mathbb{D} : J \left( e^{i\theta} \left( \frac{z - w}{1 - \overline{w} z} \right)^n \right) = \partial \mathbb{D} \right\}.$$ 

Since every elliptic parameter gives rise to a connected Julia set, the question of which parabolic elements give a connected Julia set remains. It follows from the proof of Theorem 1.3 that the parabolic elements are parameterized by the relative boundary of $D_n$ in $S^1 \times \mathbb{D}$.

**Theorem 1.4.** The set $\mathcal{M}_n$ is the union of $D_n$ together with $n - 1$ disjoint curves on the relative boundary of $D_n$ in $S^1 \times \mathbb{D}$, given by $\partial D_n \cap \{ (e^{i\theta}, w) : |w| = \frac{n-1}{n+1} \}$.

We leave open the question of describing the topological structure of $E_n$ and how its boundary splits into parabolic parameters with zero and positive hyperbolic step. We remark that the set $\mathcal{M}_n$ plays the same role for the family of Blaschke products considered in this paper as the Mandelbrot set does for the family of quadratic polynomials of the form $z^2 + c$, namely the subset of parameter space for which the Julia set is connected. Now $z^2 + c_1$ and $z^2 + c_2$ are not conjugate unless $c_1 = c_2$. Analogous to this fact, two distinct elements of $\mathcal{M}_n$
restricted to $\partial \mathbb{D}$ are not absolutely continuously conjugate by a result of Shub and Sullivan [9] (see also [3]).

An application of these results will be given in [6]. There, it will be shown that in the neighbourhood of a fixed point of a quasiregular mapping in the plane with constant complex dilatation and of any local index, the behaviour of the iterates can be determined by a conjugate of a Blaschke product. Hence knowing when such a Blaschke product has Julia set equal to $\partial \mathbb{D}$ or a Cantor subset of $\partial \mathbb{D}$ has consequences for the dynamics of the quasiregular mapping.

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2. Proof of Theorem 1.3

2.1. Fixing notation and outline of the proof. Let $\theta \in [0, 2\pi)$ and $n \in \mathbb{N}$ with $n \geq 2$ be fixed. We write $w \in \mathbb{D}$ in polar form $w = se^{i\psi}$ and will analyze what happens when $\psi$ is fixed and $s \in [0, 1)$ is varied. We denote by $A$ the Möbius transformation

$$A(z) = \left( \frac{z - w}{1 - \overline{w}z} \right)$$

and by $B$ the Blaschke product

$$B(z) = e^{i\theta}A(z)^n.$$ If $z_0$ is a fixed point of a Blaschke product, then $1/\overline{z_0}$ is also a fixed point. It follows that a Blaschke product of degree $n$ either has $n - 1$ repelling fixed points on $\partial \mathbb{D}$ and a pair of attracting fixed points in $\mathbb{D}$ and $\mathbb{C} \setminus \mathbb{D}$, or $n$ repelling fixed points on $\partial \mathbb{D}$ and one attracting fixed point on $\partial \mathbb{D}$, or all fixed points on $\partial \mathbb{D}$ with one of them neutral. Therefore, we will be interested in studying the set of points on $\partial \mathbb{D}$ where $|B'(z)| \leq 1$ and when this set contains a fixed point. This will then determine whether the corresponding Blaschke product is elliptic, hyperbolic or parabolic.

2.2. The subset of $\partial \mathbb{D}$ where $|B'(z)| < 1$. The derivative of $B$ is

$$B'(z) = e^{i\theta}nA(z)^{n-1}A'(z).$$

Since for $|z| = 1$ we have $|A(z)| = 1$, it follows that if $z = e^{i\phi}$,

$$|B'(z)| = \frac{n(1 - |w|^2)}{|1 - \overline{w}z|^2} = \frac{n(1 - s^2)}{|1 - se^{i(\phi - \psi)}|^2} = \frac{n(1 - s^2)}{1 + s^2 - 2s \cos(\phi - \psi)}.$$ 

Definition 2.1. Denote by $K = K(s)$ the set of points on $\partial \mathbb{D}$ where $|B'(z)| \leq 1$. More precisely,

$$K = \left\{ e^{i\phi} \subset \partial \mathbb{D} : \frac{n(1 - s^2)}{1 + s^2 - 2s \cos(\phi - \psi)} \leq 1 \right\}.$$ 

Lemma 2.2. The set $K$ is empty for $s < \frac{n-1}{n+1}$, the single point $e^{i(\psi + \pi)}$ for $s = \frac{n-1}{n+1}$ and is an arc in $\partial \mathbb{D}$ centred at $e^{i(\psi + \pi)}$ for $s > \frac{n-1}{n+1}$.

Proof. By (2.2), if $s$ is fixed, the smallest value that $|B'(z)|$ takes on $\partial \mathbb{D}$ is when $z = e^{i(\psi + \pi)}$ and is given by $\frac{n(1-s)}{1+s}$. This is decreasing in $s$ and is equal to 1 when $s = \frac{n-1}{n+1}$. Hence the lemma follows from this observation and from the symmetry of the expression in (2.2). \[\square\]
**Definition 2.3.** If $s > \frac{n-1}{n+1}$, so that $K$ is a non-empty arc, we denote the endpoints of $K$ by $e^{i\phi_1}$ and $e^{i\phi_2}$ with $\phi_1 < \phi_2$ under the convention that $\arg(z) \in [\psi, \psi + 2\pi)$.

In particular, $B'(e^{i\phi_1}) = B'(e^{i\phi_2}) = 1$. Given an arc $I$ in $\partial\mathbb{D}$, we denote by $|I|$ the arc-length of $I$.

![Diagram](image_url)

**Figure 1.** Diagram showing the action of $B$ on $K$.

**Lemma 2.4.** With the notation above, if $s \geq \frac{n-1}{n+1}$, $|K| = 2\pi - 2\cos^{-1}(t)$, where

$$t = t(s) = \frac{1 - n + (1 + n)s^2}{2s},$$

and $|B(K)| = n(2\pi - 2\cos^{-1}(u))$, where

$$u = u(s) = \frac{1 - n - (1 + n)s^2}{2ns}.$$

See Figure 1 for $K$ and $B(K)$ when $s > \frac{n-1}{n+1}$.

**Proof.** By Lemma 2.2 since $K$ is an arc and the endpoints of $K$ are those $e^{i\phi}$ for which

$$\cos(\phi - \psi) = \frac{1 - n + (1 + n)s^2}{2s},$$

the first part follows. For the second part, since $B(z) = e^{i\theta}A(z)^n$, $B(K)$ will be an arc. To find $|B(K)|$, we just need to find $|A(K)|$ and then multiply by $n$. Note that since $|B'(z)| \leq 1$ for $z \in K$, $|B(K)| \leq |K| < 2\pi$. Now,

$$e^{-i\psi}A(e^{i\phi}) = e^{-i\psi}(e^{i\phi} - se^{i\psi})$$

$$= \frac{e^{i(\phi-\psi)(1 + s^2) - 2s}}{1 + s^2 - 2s \cos(\phi - \psi)}$$

$$= \frac{(1 + s^2) \cos(\phi - \psi) - 2s + i(1 - s^2) \sin(\phi - \psi)}{1 + s^2 - 2s \cos(\phi - \psi)}$$
Hence the endpoints of $K$ are mapped by $e^{-i\psi}A(z)$ onto

$$(1 + s^2)t - s \pm i(1 - s^2)\sqrt{1 - t^2} \over 1 + s^2 - 2st,$$

which has real part

$$(1 + s^2)\left(1 - n + (1 + n)s^2\right) - 2s \over 1 + s^2 - 2s \left(1 - n + (1 + n)s^2\right) = 1 - n - (1 + n)s^2 \over 2ns =: u.$$ 

Since $z \mapsto e^{-i\psi}z$ is just a rotation and does not change arc-length, it follows that $|A(K)| = 2\pi - 2\cos^{-1}(u)$. Since $z \mapsto z^n$ multiplies arc-length by $n$, the second part of the lemma follows.

**Lemma 2.5.** For $s \in \left(\frac{1}{n+1}, 1\right)$, let $p(s) = |K|$ and $q(s) = |B(K)|$. Then $p'(s) > q'(s)$ and so $|K|$ grows faster than $|B(K)|$.

It follows from Lemma 2.4 that as $s \to 1$, $|K| \to 2\pi$ and $|B(K)| \to 0$, but this lemma tells us more.

**Proof.** By Lemma 2.4 we have

$$p'(s) = {2t'(s) \over \sqrt{1 - t^2}}, \quad q'(s) = {2nu'(s) \over \sqrt{1 - u^2}}.$$ 

First, we have

$$t'(s) = \frac{n - 1 + (1 + n)s^2}{2s^2}$$

and

$$1 - t^2 = 1 - \left(\frac{1 - n + (1 + n)s^2}{2s}\right)^2 = \frac{4s^2 - (1 - n)^2 - 2(1 - n^2)s^2 - (1 + n^2)s^4}{4s^2} = \frac{-(1 - n)^2 + (1 + n^2)s^2(1 - s^2)}{4s^2}.$$ 

Therefore after simplifying we have

$$(2.3) \quad p'(s) = \frac{2(n - 1 + (n + 1)s^2)}{s\sqrt{1 - s^2}\sqrt{-(1 - n)^2 + (1 + n^2)s^2}}.$$ 

Next, we have

$$u'(s) = \frac{n - 1 - (1 + n)s^2}{62ns^2}.$$
and

\[ 1 - u^2 = 1 - \left( \frac{1 - n + (1 + n)s^2}{2ns} \right)^2 = \frac{4n^2s^2 - (1 - n)^2 - 2(1 - n^2)s^2 - (1 + n^2)s^4}{4n^2s^2} = \frac{(-(1 - n)^2 + (1 + n)^2s^2)^2}{4n^2s^2}. \]

We therefore have

\[ (2.4) \quad q'(s) = \frac{2n(n - 1 - (n + 1)s^2)}{s\sqrt{1 - s^2}\sqrt{-(1 - n)^2 + (1 + n)^2s^2}}. \]

Comparing (2.3) and (2.4), and denoting \( C \) by a positive function of \( n, s \) we have

\[
p'(s) - q'(s) = C(n - 1 + (n + 1)s^2 - n(n - 1) + n(n + 1)s^2) = C(-(1 - n)^2 + (1 + 1)^2s^2) > 0
\]

for \( s > \frac{n-1}{n+1} \). Since we are restricting to \( s > \frac{n-1}{n+1} \), this proves the lemma.

\[ \square \]

2.3. Classifying Blaschke products. With these lemmas in hand, we can move on to discussing a classification of Blaschke products of the form \( B(z) = e^{i\theta}A(z)^n \).

If \( s = 0 \) then 0 is a superattracting fixed point of \( B \) and \( B \) is elliptic. We want to show that there exists some \( s_0 \) so that for \( 0 \leq s < s_0, B \) is elliptic, for \( s = s_0, B \) is parabolic and for \( s_0 < s < 1, B \) is hyperbolic. We note that there will be cases where \( s_0 = 1 \) and so we always have ellipticity.

Recall the Möbius transformation \( A(z) = \frac{z-se^{i\psi}}{1-se^{-i\psi}} \). The dynamics of \( A \) are easily understood: \( e^{i\psi} \) is a repelling fixed point and \( e^{i(\psi+\pi)} \) is an attracting fixed point. Since \( B(z) = e^{i\theta}A(z)^n \), this means that the arc \( K \), which we recall has centre \( e^{i(\psi+\pi)} \), is contracted by \( A \), expanded for \( z \mapsto z^n \) and then rotated through angle \( \theta \). Hence to find when \( B \) is parabolic, we need to determine when one of the endpoints of \( K \) are fixed by this process.

We first give the case where the endpoints are never fixed.

Lemma 2.6. Suppose that \((n-1)\psi + \theta \in \{2k\pi : k \in \mathbb{Z}\} \) if \( n \) is even or \((n-1)\psi + \theta \in \{(2k+1)\pi : k \in \mathbb{Z}\} \) if \( n \) is odd. Then \( B \) is always elliptic.

Proof. The point \( e^{i(\psi+\pi)} \) is mapped to \( e^{[n(\psi+\pi)+\theta]} \) under \( B \). The conditions of the lemma state that this image is exactly the repelling fixed point of \( A \) and is exactly opposite \( e^{i(\psi+\pi)} \). Now, since \( K \) is symmetric about \( e^{i(\psi+\pi)} \), the arc-length of \( K \) converges to \( 2\pi \) as \( s \to 1 \) and \( K \) is contracted by \( B \), no point of \( K \) can be fixed by \( B \). See Figure 2 Therefore \( B \) is never parabolic or hyperbolic and the conclusion follows.

\[ \square \]

Lemma 2.7. Suppose that \((n-1)\psi + \theta \notin \{2k\pi : k \in \mathbb{Z}\} \) if \( n \) is even or \((n-1)\psi + \theta \notin \{(2k+1)\pi : k \in \mathbb{Z}\} \) if \( n \) is odd. Then there exists \( s_0 \in \left[\frac{n-1}{n+1}, 1\right) \) such that:

- for \( 0 \leq s < s_0, B \) is elliptic,
- for \( s = s_0, B \) is parabolic,
- for \( s_0 < s < 1, B \) is hyperbolic.
Proof. The idea is as follows. Complementary to Lemma 2.6, this time the image of $e^{i(\psi+\pi)}$ is not exactly opposite $e^{i(\psi+\pi)}$. First, if $B$ fixes $e^{i(\psi+\pi)}$, then by Lemma 2.2, for $s = \frac{n-1}{n+1}$, $B$ is parabolic; for $0 \leq s < \frac{n-1}{n+1}$, $B$ is elliptic; and for $s > \frac{n-1}{n+1}$, $B$ is hyperbolic.

Next, $B(e^{i(\psi+\pi)})$ is either in the arc with argument $(\psi+\pi, \psi+2\pi)$ or $(\psi, \psi+\pi)$. Without loss of generality we may assume it is the former, that is, $\phi_0 \in (\psi+\pi, \psi+2\pi)$ where $B(e^{i(\psi+\pi)}) = e^{i\phi_0}$. See Figure 3 for reference to the arguments we give below. For values of $s$ just larger than $\frac{n-1}{n+1}$, $K$ and $B(K)$ are disjoint arcs centred at $e^{i(\psi+\pi)}$ and $e^{i\phi_0}$ respectively. Recall $\phi_1, \phi_2$ are the endpoints of $K$, and that we are assuming that $\phi_2$ and $\phi_0$ are both in the same semicircle with argument in $(\psi+\pi, \psi+2\pi)$. As $s$ increases, $|K|$ increases and we have that both $|K| \to 2\pi$ and $|B(K)| \to 0$ as $s \to 1$. By Lemma 2.5 $|K|$ increases faster than $|B(K)|$ as $s$ increases. Hence there exists one, and only one, $s_0 \in (\frac{n-1}{n+1}, 1)$ such that $B(e^{i\phi_2}) = e^{i\phi_2}$. By construction, when this occurs, $e^{i\phi_2}$ is a fixed point of $B$ for which $B'(e^{i\phi_2}) = 1$ and hence $B$ is parabolic.

For $s < s_0$, if $B$ did have a fixed point in $K$, then it could not be an endpoint by construction. Suppose this fixed point is $e^{i\xi}$, and it must lie between $e^{i(\psi+\pi)}$ and $e^{i\phi_2}$. Then we have that the arc length of the interval in $\partial D$ from $e^{i\xi}$ to $e^{i\phi_2}$ is strictly less than the arc length of the interval from $B(e^{i\xi}) = e^{i\xi}$ to $B(e^{i\phi_2})$. This contradicts the fact that $|B'(z)| \leq 1$ for $z \in K$.

For $s > s_0$, by Lemma 2.5 the image of the arc $(\psi+\pi, \phi_2)$ strictly contains its image and hence there exists a fixed point with $|B'(z)| < 1$. Hence $B$ is hyperbolic. \qed

It now follows from Lemmas 2.6 and 2.7 that $\{w \in D : (e^{i\theta}, w) \in D_n\}$ is an open subset of $D$ which is starlike about 0. It is clear that the relative boundary of this set in $D$ consists of parabolic elements and so it remains to investigate these parabolic elements.

Proof of Theorem 1.4. By Theorem 1.2 parameters in the relative boundary of $D_n$ give rise to parabolic Blaschke products with $J(B) = \partial D$ only when $B''(z_0) = 0$, where $z_0$ is the
Denjoy-Wolff point of $B$. Differentiating (2.1), we have
\[
B''(z) = ne^{i\theta} A(z)^{n-2} \left( (n-1) |A'(z)|^2 + A(z) A''(z) \right) = ne^{i\theta} A(z)^{n-2} \left( (n-1) \frac{1 - |w|^2}{1 - \overline{w} z} + \left( \frac{z - w}{1 - \overline{w} z} \right) \frac{2 \overline{w} (1 - |w|^2)}{1 - \overline{w} z} \right). 
\]
For $z \in \partial \mathbb{D}$, $A(z) \neq 0$ and so we have to analyze the factor $(n-1)(1 - |w|^2) + 2\overline{w}(z - w)$. This is zero when
\[
z = \frac{2|w|^2 + (1 - n) + (n - 1)|w|^2}{2\overline{w}} = \frac{(1 - n) + (n + 1)|w|^2}{2\overline{w}}.
\]
Since $|z| = 1$, this leads to
\[(n + 1)|w|^2 - 2|w| + (1 - n) = 0,
\]
which has solutions $|w| = 1$ and $|w| = \frac{n-1}{n+1}$. This first of these is inadmissible, but the second is allowable. By Lemmas 2.2 and 2.7, the only situation when a parabolic parameter $(e^{i\theta}, w)$ has $|w| = \frac{n-1}{n+1}$ is when $e^{i(\psi + \pi)}$ is fixed by $B$. Since $e^{i(\psi + \pi)}$ is fixed by $A$, this leads to the condition that $(n - 1)\psi + \theta = 0$ (mod $2\pi$) if $n$ is odd or $(n - 1)\psi + \theta = \pi$ (mod $2\pi$) if $n$ is even.

Hence there are $n - 1$ distinct curves on $\partial D_n$ corresponding to where $\partial D_n$ intersects $\{(e^{i\theta}, w) : |w| = \frac{n-1}{n+1}\}$ for which the corresponding Blaschke product is parabolic with $J(B) = \partial \mathbb{D}$. These curves are given by $\psi = (2k\pi - \theta)/(n - 1)$ for $k = 0, \ldots, n - 2$. For any other parabolic parameter, Theorem 1.2 implies that $J(B)$ is a Cantor subset of $\partial \mathbb{D}$.

Therefore $\mathcal{M}_n$ is the domain of ellipticity $D_n$ together with $n - 1$ distinct curves on $\partial D_n$. \hfill \Box

We end with the observation that when $n = 2$, there is one curve in $\partial D_2 \cap \mathcal{M}_2$, given by $|w| = 1/3$ and $\psi = -\theta$. In particular, any Blaschke product of the form
\[
B(z) = e^{i\theta} \left( \frac{z - e^{-i\theta}/3}{1 - e^{i\theta} z/3} \right)^2
\]
is parabolic with Denjoy-Wolff point $-e^{-i\theta}$ and has $J(B) = \partial \mathbb{D}$.

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Figure 3. Diagram showing the three situations that arise in Lemma 2.7, the elliptic case \((s < s_0)\) with there are no fixed points on \(B\) in \(K\) (top), the parabolic case \((s = s_0)\) where \(B(e^{i\phi_2}) = e^{i\phi_2}\) (middle) and the hyperbolic case \((s > s_0)\) where \(B(K)\) is strictly contained in \(K\) (bottom).