Convergence of Numerical Solution of The Tamed Milstein Method for NSDDEs

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Abstract

In this paper, we apply the tamed technique to the Milstein numerical scheme to investigate Neutral Stochastic Delay Differential Equations(NSDDEs) with highly nonlinear coefficients. Under the local Lipschitz condition and Khasminskii condition, the tamed Milstein numerical solution converges strongly to the exact solution.

Key words: neutral stochastic differential delay equations; tamed Milstein scheme; local Lipschitz; strong convergence.

1 Introduction

As an important class of differential dynamical systems, Neutral Stochastic Delay Differential Equations(NSDDEs) play an important role in many fields, such as biology, finance and automatic control \cite{13}. In general, although some linear stochastic differential equations(SDEs) can be solved explicitly, solutions to SDEs with highly nonlinear coefficients are difficult to obtain. Therefore, numerical schemes are of great importance in the study of nonlinear SDEs. In recent years, many literatures related to numerical schemes have emerged, see \cite{8,14}.

Most convergence results on the numerical schemes for SDEs require the coefficients satisfying the local Lipschitz condition and linear growth condition. When the drift term or diffusion term does not meet the linear growth condition, the Euler-Maruyama(EM) method will divergence in finite time, see \cite{4}. Later, Hutzenthaler \cite{5} proposed a tamed EM method with strong convergence of order $1/2$. Then, Sabanis \cite{10} extended the strong convergence results of the tamed Euler scheme to SDEs whose drift term satisfies local conditions and superlinear growth conditions. Wang and Gan \cite{12} further extended a tamed Milstein scheme to SDEs with commutative noise.

NSDDEs is one of the special class of ordinary SDEs, which can be used to describe important phenomena in life. When their coefficients do not meet the linear growth, the classical EM scheme does not converge. Therefore, scholars have given many modified numerical schemes for NSDDEs with highly nonlinear coefficients, and studied their convergence. Ji \cite{6} analyzed the convergence rate of tamed EM for NSDDEs with linear

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growth of diffusion term. Tan and Yuan [11] studied the strong convergence of the tamed theta scheme for NSDDEs with one-side Lipschitz drift coefficients. Yan et al. [13] investigated the strong convergence of the split-step theta method for the NSDDEs, where the corresponding coefficients may be highly nonlinear with respect to the delay variables. Lan and Wang [7] investigated the strong convergence of the split-step theta method for the NSDDEs, where the corresponding coefficients may be highly nonlinear with respect to the delay variables. For a class of highly nonlinear SDDEs with nonlinear growth conditions, Zhang et al. [15] proposed the truncated Milstein method and gave its convergence rate. Deng et al. [2] developed two types of explicit tamed EM schemes for NSDDEs, in which both drift and diffusion coefficients can grow superlinearly, and investigate the strong convergence, mean-square stability.

Inspired by [2, 6, 12], we develop tamed Milstein scheme for NSDDEs with highly nonlinear coefficients. The rest of the paper is organized as follows. In the section 2, we present some preliminaries and assumptions on the NSDDEs. The tamed Milstein scheme is proposed in section 3 and the strong convergence of the tamed Milstein is given in section 4.

2 Preliminaries

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathcal{F}_0$ contains all $P$-null sets). Let $B(t)$ be an $m$-dimensional Brownian motion. $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^n$. The inner product of $x, y$ in $\mathbb{R}^n$ is denoted by $(x, y)$ or $x^T y$. If $A$ is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$. Let $\tau > 0$ be a constant and denote $C([-\tau, 0] ; \mathbb{R}^n)$ the space of all continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\xi\| = \sup_{-\tau \leq \theta \leq 0} |\xi(\theta)|$.

In this paper, we study the numerical approximation of the neutral stochastic differential delay equations (NSDDEs)

$$d[x(t) - D(x(t - \tau))] = b(x(t), x(t - \tau)) dt + \sigma(x(t), x(t - \tau)) dB(t), \ t \geq 0,$$

with the initial data

$$\{x(\theta) : -\tau \leq \theta \leq 0\} = \xi \in L^p_{\mathcal{F}_0} \left([-\tau, 0]; \mathbb{R}^n\right),$$

that is $\xi$ is an $\mathcal{F}_0$-measurable $C([-\tau, 0]; \mathbb{R}^n)$-valued random variable and $\mathbb{E}\|\xi\|^p < \infty$. Here

$$D : \mathbb{R}^n \rightarrow \mathbb{R}^n, \ b : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \ \sigma : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}.$$ 

We assume that the coefficients $D$, $b$ and $\sigma$ are Borel-measurable and satisfy the following conditions:

(A1) For any $s, t \in [-\tau, 0]$ and $q > 0$, there exists a positive constant $\theta > 0$ such that

$$\mathbb{E} |\xi(t) - \xi(s)|^q \leq \theta |t - s|^q.$$

(A2) $D(0) = 0$ and there exists a constant $\kappa \in (0, 1)$ such that

$$|D(x) - D(y)| \leq \kappa |x - y| \text{ for all } x, y \in \mathbb{R}^n.$$ 

(A3) For every constant $R > 0$, there exists a positive constant $K_R$, such that

$$|b(x, y) - b(\bar{x}, \bar{y})| \vee |\sigma(x, y) - \sigma(\bar{x}, \bar{y})| \leq K_R (|x - \bar{x}| + |y - \bar{y}|),$$

\text{ for all } x, y \in \mathbb{R}^n.$$
for all $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq R$.

(A4) There are constants $R > 0$ and $\bar{K}_R > 0$ such that

$$|\sigma_i(x, y)\sigma(x, y) - \sigma_i(\bar{x}, \bar{y})\sigma(\bar{x}, \bar{y})| \leq \bar{K}_R(|x - \bar{x}| + |y - \bar{y}|), \quad i = 1, 2.$$ 

for all $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq R$ where

$$\sigma_i(x_1, x_2) = \frac{\partial \sigma(x_1, x_2)}{\partial x_i}.$$

(A5) There exist constants $p > 2$ and $K_1 > 0$ such that

$$(x - D(y))^T b(x, y) + \frac{p-1}{2} |\sigma(x, y)|^2 \leq K_1 (1 + |x|^2 + |y|^2),$$

for $x, y \in \mathbb{R}^n$.

Lemma 2.1 Let assumption (A3) and (A4) hold, the equation (2.1) have a unique global solution $x(t), t \in [0, T]$, and there exists a positive constant $C$ such that for any $p \geq 2$

$$\mathbb{E}\left( \sup_{0 \leq t \leq T} |x(t)|^p \right) \leq C,$$

where the positive constant $C := C(p, T, K_1, ||\xi||, \kappa)$.

Proof: By $(a + b)^p \leq (1 + \varepsilon)^{p-1}(a^p + \varepsilon^{1-p}b^p)$ where $a, b, \varepsilon > 0, p \geq 2$

$$|x(t)|^p = |x(t) - D(x(t - \tau)) + D(x(t - \tau))|^p \leq (1 + \varepsilon)^{p-1}\left(|x(t) - D(x(t - \tau))|^p + \varepsilon^{1-p}|D(x(t - \tau))|^p\right).$$

Let $\varepsilon = \frac{\kappa}{1 - \kappa}$, then

$$|x(t)|^p \leq (1 - \kappa)^{1-p}|x(t) - D(x(t - \tau))|^p + \kappa^{1-p}D(x(t - \tau))^p \leq (1 - \kappa)^{1-p}|x(t) - D(x(t - \tau))|^p + \kappa x(t - \tau)^p \leq (1 - \kappa)^{1-p}|x(t) - D(x(t - \tau))|^p + \kappa(||\xi||^p + x(t))^p \leq (1 - \kappa)^{-p}|x(t) - D(x(t - \tau))|^p + \frac{\kappa}{1 - \kappa}||\xi||^p. \quad (2.4)$$

An application of Itô formula yields

$$|x(t) - D(x(t - \tau))|^p \leq |\xi(0) - D(\xi(-\tau))|^p + \int_0^t p|x(s) - D(x(s - \tau))|^p - \frac{1}{2} |\sigma(x(s), x(s - \tau))|^2$$

$$+ (x(s) - D(x(s - \tau)))^T b(x(s), x(s - \tau)) ds + \int_0^t p|x(s) - D(x(s - \tau))|^{p-2} (x(s) - D(x(s - \tau)))^T \sigma(x(s), x(s - \tau)) dB(s) = |\xi(0) - D(\xi(-\tau))|^p + I_1(t) + I_2(t) + I_3(t). \quad (2.5)$$
Under (A2) and (A5), we have the following estimate
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} I_1(t) \right) + \mathbb{E} \left( \sup_{0 \leq t \leq T} I_2(t) \right) \\
= \mathbb{E} \left( \sup_{0 \leq t \leq T} \int_0^t p |x(s) - D(x(s - \tau))|^{p-2} \left[ (x(s) - D(x(s - \tau)))^T b(x(s), x(s - \tau)) + \frac{p-1}{2} |\sigma(x(s), x(s - \tau)|^2 \right] ds \right) \\
\leq C \mathbb{E} \int_0^T (x(s))^{p-2} + |D(x(s - \tau))|^{p-2} \left( 1 + |x(s)|^2 + |x(s - \tau)|^2 \right) ds \\
\leq C \mathbb{E} \int_0^T \left( 1 + |x(s)|^p + |x(s - \tau)|^p \right) ds \\
\leq C + C \int_0^T \mathbb{E} \left( \sup_{0 \leq u \leq s} |x(u)|^p \right) ds. \tag{2.6}
\]

For \( I_3(t) \), by the Burkholder-Davis-Gundy (BDG) inequality and H"older inequality which yields
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} I_3(t) \right) \\
= \mathbb{E} \left( \sup_{0 \leq t \leq T} \int_0^t p |x(s) - D(x(s - \tau))|^{p-2} (x(s) - D(x(s - \tau)))^T \sigma(x(s), x(s - \tau)) dB(s) \right) \\
\leq C_p \mathbb{E} \left( \int_0^T \left( |x(s) - D(x(s - \tau))|^{2p-2} |\sigma(x(s), x(s - \tau)|^2 \right) ds \right) \frac{1}{2} \\
\leq C_p \mathbb{E} \left( \sup_{0 \leq t \leq T} |x(t) - D(x(t - \tau))|^{p-1} \left( \int_0^T |\sigma(x(s), x(s - \tau)|^2 ds \right) \right)^{1/2}. \tag{2.7}
\]

Then by inequality \( a^{p-1}b \leq \frac{p-1}{p} a^p + \frac{1}{p} b^p \) and (A3), we could obtain
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} I_3(t) \right) \\
\leq C_p \mathbb{E} \left( \sup_{0 \leq t \leq T} |x(t) - D(x(t - \tau))|^p \right) + C_p \mathbb{E} \left( \int_0^T |\sigma(x(s), x(s - \tau)|^2 ds \right)^{p/2} \\
\leq C_p \mathbb{E} \left( \sup_{0 \leq t \leq T} |x(t) - D(x(t - \tau))|^p \right) + C_p \mathbb{E} \left( \int_0^T (1 + |x(s)|^2 + |x(s - \tau)|^2 ds \right)^{p/2} \\
\leq C_p + C_p \mathbb{E} \left( \sup_{0 \leq t \leq T} |x(t) - D(x(t - \tau))|^p \right) + C_p \int_0^T \mathbb{E} \left( \sup_{0 \leq u \leq s} |x(u)|^p \right) ds. \tag{2.7}
\]

Now, substituting \(2.6) and \(2.7\) into \(2.5\), it derives that
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |x(t) - D(x(t - \tau))|^p \right) \leq C + C \int_0^T \mathbb{E} \left( \sup_{0 \leq u \leq s} |x(u)|^p \right) ds. \tag{2.8}
\]

Then, substituting into \(2.4\), it derives that
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |x(t)|^p \right) \leq C + C \int_0^T \mathbb{E} \left( \sup_{0 \leq u \leq s} |x(u)|^p \right) ds.
\]

An application of Gronwall inequality yields
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |x(t)|^p \right) \leq C.
\]
3 Tamed Milstein Scheme

Fix $0 < \tau < T$, without loss of gengrality, we assume that $T$ and $\tau$ are rational numbers, and the step size $\Delta \in (0, 1)$ be fraction of $\tau$ and $T$, so that there exist two positive integers $M$ and $m$ such that $\Delta = T/M = \tau/m, t_k = k \Delta : k = -m, \ldots, 0, 1, 2, \ldots M-1$.

Applying the classic Milstein scheme [9] to (2.1) with initial data (2.2), we have the discrete-time Milstein scheme

$$\begin{align*}
Y_{t_k} = & \xi (t_k), \quad k = -m, -m + 1, \ldots, 0 \\
Y_{t_{k+1}} = & D(Y_{t_{k+1-m}}) + Y_{t_k} - D(Y_{t_{k-m}}) + b_h(Y_{t_k}, Y_{t_{k-m}}) \Delta + \sigma(Y_{t_k}, Y_{t_{k-m}}) \Delta B_{t_k} \\
& + \sigma_1(Y_{t_k}, Y_{t_{k-m}}) \sigma(Y_{t_k}, Y_{t_{k-m}}) l_1 \\
& + \sigma_2(Y_{t_k}, Y_{t_{k-m}}) \sigma(Y_{t_{k-1}}, Y_{t_{k-2}}) l_2, \quad k = 0, 1, \ldots, M-1,
\end{align*}$$

Under assumption (A4) and (A3)

Here

$$l_1 = \int_{t_k}^{t_{k+1}} \int_{t_k}^s dB(t)dB(s) = \frac{(\Delta B_{t_k})^2 - \Delta}{2}, \quad l_2 = \int_{t_k}^{t_{k+1}} \int_{t_k}^s dB(t - \tau)dB(s).$$

$$\Delta B_{t_k} = B_{t_{k+1}} - B_{t_k}, \quad \sigma_1(x, y) = \frac{\partial \sigma(x, y)}{\partial x}, \quad \sigma_2(x, y) = \frac{\partial \sigma(x, y)}{\partial y}.$$

Defining the tamed drift term

$$b_h(x, y) = \frac{b(x, y)}{1 + \Delta^\alpha |b(x, y)|},$$

for all $x, y \in \mathbb{R}^n$ and $\alpha \in \left(0, \frac{1}{2}\right]$.

By observation, one has

$$|b_h(x, y)| \leq \frac{1}{\Delta^\alpha} \wedge |b(x, y)|.$$  

(3.3)

**Remark 3.1** When $b$ is replaced by $b_h$, assumptions A3 and A4 are still true.

Under assumption (A3)

$$|b_h(x, y) - b_h(\bar{x}, \bar{y})|$$

$$= \left| \frac{b(x, y)}{1 + \Delta^\alpha |b(x, y)|} - \frac{b(\bar{x}, \bar{y})}{1 + \Delta^\alpha |b(\bar{x}, \bar{y})|} \right|$$

$$= \left| \frac{b(x, y) - b(\bar{x}, \bar{y})}{(1 + \Delta^\alpha |b(x, y)|)(1 + \Delta^\alpha |b(\bar{x}, \bar{y})|)} + \frac{\Delta^\alpha |b(x, y)| |b(\bar{x}, \bar{y})| - |b(x, y)| |b(\bar{x}, \bar{y})|}{(1 + \Delta^\alpha |b(x, y)|)(1 + \Delta^\alpha |b(\bar{x}, \bar{y})|)} \right|$$

$$\leq \frac{1}{(1 + \Delta^\alpha |b(x, y)|)(1 + \Delta^\alpha |b(\bar{x}, \bar{y})|)} K_R \left(|x - \bar{x}| + |y - \bar{y}|\right)$$

$$\leq K_R \left(|x - \bar{x}| + |y - \bar{y}|\right).$$  

(3.4)

Under assumption (A4)

$$(x - D(y))^T b_h(x, y)$$

$$= \frac{1}{1 + \Delta^\alpha |b(x, y)|} (x - D(y))^T b(x, y)$$

$$\leq \frac{K_1}{1 + \Delta^\alpha |b(x, y)|} (1 + |x|^2 + |y|^2)$$

$$\leq K_1 (1 + |x|^2 + |y|^2).$$  

(3.5)
Define tamed Milstein scheme continuous-time step process as

\[
\bar{y}(t) = \sum_{k=0}^{\infty} Y_k I_{[t_k, t_{k+1})}(t), \quad t \geq 0.
\]  

(3.6)

where \( I_{[t_k, t_{k+1})}(t) \) is the indicator function on \([t_k, t_{k+1})\). Then, we could define the corresponding continuous-time tamed Milstein scheme

\[
\begin{cases}
y(t) = \xi(t), t \in [-\tau, 0]; \\
y(t) = D(y(t - \tau)) + \xi(0) - D(\xi(-\tau)) \\
\quad + \int_0^t b_h(y(s), y(s - \tau))ds + \int_0^t \sigma(y(s), y(s - \tau))dB(s) \\
\quad + \int_0^t \sigma_1(y(s), y(s - \tau))\sigma(y(s), y(s - \tau))\Delta \bar{B}(s)dB(s) \\
\quad + \int_0^t \sigma_2(y(s), y(s - \tau))\sigma(y(s - \tau), y(s - 2\tau))\Delta \bar{B}(s - \tau)dB(s), t \in (0, T].
\end{cases}
\]  

(3.7)

Here \( \Delta \bar{B}(t) = \sum_{k=0}^{\infty} (B_{t_{k+1}} - B_{t_k}) I_{[t_k, t_{k+1})}(t), t \geq 0. \)

Then \( \bar{y}(t) = Y_k = y(t_k) \), for any \( t \in [t_k, t_{k+1}) \) with \( k \geq 0 \), we could conclude that

\[
\mathbb{E}|\bar{y}(t)| \leq \mathbb{E}\left( \sup_{0 \leq t \leq T} |y(t)| \right). 
\]  

(3.8)

(B1) There exists a positive constant \( N_R \) depending on \( R \), such that for any \( x, y \in \mathbb{R}^d \),

\[
\sup_{|x|\sqrt{|y|} \leq R} |b(x, y) - b_h(x, y)| \leq N_R \Delta. 
\]  

(3.9)

4 Convergence of the tamed Milstein

In this section, we show that the tamed Milstein scheme converges to the exact solution under certain conditions, i.e. we have the following main result:

Theorem 4.1 Let (A1) (A2) remark 3.1 and (B1) hold, then for any \( p \geq 2 \)

\[
\lim_{\Delta \rightarrow 0} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |x(t) - y(t)|^p \right] = 0. 
\]  

(4.1)

4.1 Moment properties of tamed Milstein method

To prove our main results, in this subsection we investigate the boundedness of moments tamed Milstein approximation in the following section.

Lemma 4.1 Under (B1), for all \( t \in [0, T] \), we have

\[
\mathbb{E} \sup_{0 < t \leq T} |y(t)|^p \leq C, 
\]  

(4.2)

where the positive constant \( C := C(p, T, K_R, K_1, ||\xi||, \kappa) \).
Proof: Using the similar way as in (2.31),

\[ |y(t)|^p \leq (1 - \kappa)^-p|y(t) - D(\bar{y}(t - \tau))|^p + \frac{\kappa}{1 - \kappa}|D(\bar{y}(t - \tau))|^p. \]  

(4.3)

By applying the Itô formula to (3.7), it derives that

\[ |y(t) - D(\bar{y}(t - \tau))|^p \leq |\xi(0) - D(\xi(-\tau))|^p \]

\[ + \int_0^t p|y(s) - D(\bar{y}(s - \tau))|^p \left[ (\bar{y}(s) - D(\bar{y}(s - \tau)))^T b_h(\bar{y}(s), \bar{y}(s - \tau)) \right. \]

\[ + (p - 1)|\sigma(\bar{y}(s), \bar{y}(s - \tau))|^2 + (p - 1)|\sigma_1(\bar{y}(s), \bar{y}(s - \tau))|\sigma(\bar{y}(s), \bar{y}(s - \tau))\Delta B(s) \]

\[ + \sigma_2(\bar{y}(s), \bar{y}(s - \tau))\sigma(\bar{y}(s), \bar{y}(s - \tau))\Delta B(s) \]

\[ + (y(s) - \bar{y}(s))^T b_h(\bar{y}(s), \bar{y}(s - \tau)) \right] ds \]

\[ + \int_0^t p|y(s) - D(\bar{y}(s - \tau))|^p \left[ (\bar{y}(s) - D(\bar{y}(s - \tau)))^T \sigma(\bar{y}(s), \bar{y}(s - \tau)) \right. \]

\[ + \sigma_1(\bar{y}(s), \bar{y}(s - \tau))\sigma(\bar{y}(s), \bar{y}(s - \tau))\Delta B(s) \]

\[ + \sigma_2(\bar{y}(s), \bar{y}(s - \tau))\sigma(\bar{y}(s), \bar{y}(s - \tau))\Delta B(s) \]

\[ + (y(s) - \bar{y}(s))^T b_h(\bar{y}(s), \bar{y}(s - \tau)) \right] dB(s) \]

\[ = |\xi(0) - D(\xi(-\tau))|^p + \sum_{i=1}^T H_i(t). \]  

(4.4)

By assumption (A5), (3.5) and Young’s inequality \( a^{p - 2}b \leq \frac{p - 2}{p}a^p + \frac{2}{p}b^{p - 2} \), it derives that

\[ \mathbb{E}\left( \sup_{0 \leq t \leq T} H_1(t) \right) + \mathbb{E}\left( \sup_{0 \leq t \leq T} H_2(t) \right) \]

\[ = \mathbb{E}\left( \sup_{0 \leq t \leq T} \int_0^t p|y(s) - D(\bar{y}(s - \tau))|^p \left[ (\bar{y}(s) - D(\bar{y}(s - \tau)))^T b_h(\bar{y}(s), \bar{y}(s - \tau)) \right. \right. \]

\[ + (p - 1)|\sigma(\bar{y}(s), \bar{y}(s - \tau))|^2 \right. \]

\[ \left. \leq \mathbb{E}\int_0^T p|y(s) - D(\bar{y}(s - \tau))|^p \left[ (\bar{y}(s) - D(\bar{y}(s - \tau)))^T b_h(\bar{y}(s), \bar{y}(s - \tau)) \right. \right. \]

\[ + (p - 1)|\sigma(\bar{y}(s), \bar{y}(s - \tau))|^2 \right. \]

\[ \leq C\mathbb{E}\int_0^T |y(s) - D(\bar{y}(s - \tau))|^p \left( 1 + |\bar{y}(s)|^2 + |\bar{y}(s - \tau)|^2 \right) ds \]

\[ \leq C\mathbb{E}\int_0^T \left[ |y(s)|^{p - 2} + |D(\bar{y}(s - \tau))|^{p - 2} \right] \left( 1 + |\bar{y}(s)|^2 + |\bar{y}(s - \tau)|^2 \right) ds \]

\[ \leq C\mathbb{E}\int_0^T \left[ 1 + |y(s)|^p + |\bar{y}(s)|^p + |\bar{y}(s - \tau)|^p \right] ds \]

\[ \leq C + C \int_0^T \mathbb{E} \sup_{0 \leq u \leq s} |y(u)|^p ds. \]  

(4.5)
By Young’s inequality and assumption \( (A4) \), it derives that

\[
\mathbb{E}( \sup_{0 \leq t \leq T} H_3(t) ) \\
= \mathbb{E} \left( \sup_{0 \leq t \leq T} \int_0^t p(p-1)|y(s) - D(\bar{y}(s - \tau))|^p \sigma_1(\bar{y}(s), \bar{y}(s - \tau)) \right. \\
\left. + \sigma_2(\bar{y}(s), \bar{y}(s - \tau)) \sigma(\bar{y}(s), \bar{y}(s - \tau)) \Delta \bar{B}(s) \right) \\
\leq C \mathbb{E} \int_0^T |y(s) - D(\bar{y}(s - \tau))|^p ds + C \mathbb{E} \int_0^T |\sigma_1(\bar{y}(s), \bar{y}(s - \tau)) \sigma(\bar{y}(s), \bar{y}(s - \tau)) \Delta \bar{B}(s)|^p ds \\
\leq C \mathbb{E} \int_0^T |y(s) - D(\bar{y}(s - \tau))|^p ds + C \mathbb{E} \int_0^T |\sigma_2(\bar{y}(s), \bar{y}(s - \tau)) \sigma(\bar{y}(s), \bar{y}(s - \tau)) \Delta \bar{B}(s)|^p ds \\
\leq C \mathbb{E} \int_0^T |y(s) - D(\bar{y}(s - \tau))|^p ds + C \mathbb{E} \int_0^T |\sigma_2(\bar{y}(s), \bar{y}(s - \tau)) \sigma(\bar{y}(s), \bar{y}(s - \tau)) \Delta \bar{B}(s)|^p ds \\
\leq C \mathbb{E} \int_0^T |y(s) - D(\bar{y}(s - \tau))|^p ds + C \mathbb{E} \int_0^T |\sigma_2(\bar{y}(s), \bar{y}(s - \tau)) \sigma(\bar{y}(s), \bar{y}(s - \tau)) \Delta \bar{B}(s)|^p ds \\
\leq C_p + C_p \mathbb{E} \left( \sup_{0 \leq t \leq T} |y(t) - D(\bar{y}(t - \tau))| \right) + C_p \int_0^T \mathbb{E}\left( \sup_{0 \leq u \leq s} |y(u)|^p \right) ds. \quad (4.6)
\]

By Young’s inequality and \( (3.33) \) then

\[
\mathbb{E}( \sup_{0 \leq t \leq T} H_4(t) ) \\
= p \mathbb{E} \left( \sup_{0 \leq t \leq T} \int_0^t |y(s) - D(\bar{y}(s - \tau))|^p \bar{y}(s) ds \right) \\
\leq p \mathbb{E} \int_0^T |y(s) - D(\bar{y}(s - \tau))|^p \bar{y}(s) ds \\
\leq p \mathbb{E} \left( \sup_{0 \leq t \leq T} |y(t) - D(\bar{y}(t - \tau))| \right)^p \mathbb{E}\left( \int_0^T |y(s) - \bar{y}(s)| b_h(\bar{y}(s), \bar{y}(s - \tau)) ds \right) \\
\leq (p - 2) \mathbb{E} \left( \sup_{0 \leq t \leq T} |y(t) - D(\bar{y}(t - \tau))|^p \right) + 2 \mathbb{E} \int_0^T |y(s) - \bar{y}(s)|^{p/2} b_h(\bar{y}(s), \bar{y}(s - \tau)) ds \\
\leq (p - 2) \mathbb{E} \left( \sup_{0 \leq t \leq T} |y(t) - D(\bar{y}(t - \tau))|^p \right) + 2 \mathbb{E} \int_0^T |y(s) - \bar{y}(s)|^{p/2} \Delta^\alpha \bar{B}(s) ds \\
\leq (p - 2) \mathbb{E} \left( \sup_{0 \leq t \leq T} |y(t) - D(\bar{y}(t - \tau))|^p \right) + 2 \mathbb{E} \left( \sup_{0 \leq u \leq s} |y(u)|^{p/2} \Delta^\alpha \bar{B}(s) \right) ds \\
\leq C_p \mathbb{E} \left( \sup_{0 \leq t \leq T} |y(t) - D(\bar{y}(t - \tau))|^p \right) + C_p \mathbb{E} \left( \sup_{0 \leq u \leq s} |y(u)|^{p} \right) ds. \quad (4.7)
\]
By the BDG inequality and Young’s inequality, it derives that
\[
\mathbb{E}( \sup_{0 \leq t \leq T} H_5(t) ) = \mathbb{E}( \sup_{0 \leq t \leq T} p \int_0^t |y(s) - D(\bar{g}(s - \tau))|^p - 2(y(s) - D(\bar{g}(s - \tau)), \sigma(\bar{g}(s), \bar{y}(s - \tau))dB(s)) ) \\
\leq C_p \mathbb{E}( \int_0^T (|y(s) - D(\bar{g}(s - \tau))|^{2p-2} |\sigma(\bar{g}(s), \bar{y}(s - \tau))|^2)ds )^{1/2} \\
\leq C_p \mathbb{E}( \sup_{0 \leq t \leq T} |y(t) - D(\bar{g}(t - \tau))|^{p-1} (\int_0^T |\sigma(\bar{g}(s), \bar{y}(s - \tau))|^2 ds )^{1/2} ) \\
\leq C_p + C_p \mathbb{E}( \sup_{0 \leq t \leq T} |y(t) - D(\bar{g}(t - \tau))|^p ) + C_p \int_0^T \mathbb{E}( \sup_{0 \leq u \leq s} |y(u)|^p ) ds. \tag{4.8}
\]

Using the similar approach as in (4.8), we can estimate \( H_6(t) \) as
\[
\mathbb{E}( \sup_{0 \leq t \leq T} H_6(t) ) = \mathbb{E}( \sup_{0 \leq t \leq T} p \int_0^t |y(s) - D(\bar{g}(s - \tau))|^{p-1} \sigma(\bar{g}(s), \bar{g}(s - \tau)) \Delta \bar{B}(s) ) \\
\leq C_p \mathbb{E}( \sup_{0 \leq t \leq T} |y(t) - D(\bar{g}(t - \tau))|^{p-1} (\int_0^T |\sigma(\bar{g}(s), \bar{g}(s - \tau))|^2 ds )^{1/2} ) \\
\leq C_p + C_p \mathbb{E}( \sup_{0 \leq t \leq T} |y(t) - D(\bar{g}(t - \tau))|^p ) + C_p \int_0^T \mathbb{E}( \sup_{0 \leq u \leq s} |y(u)|^p ) ds. \tag{4.9}
\]

Using the similar approach as in (4.9), we can estimate \( H_7(t) \) as
\[
\mathbb{E}( \sup_{0 \leq t \leq T} H_7(t) ) \leq C_p + C_p \mathbb{E}( \sup_{0 \leq t \leq T} |y(t) - D(\bar{g}(t - \tau))|^p ) + C_p \int_0^T \mathbb{E}( \sup_{0 \leq u \leq s} |y(u)|^p ) ds. \tag{4.10}
\]

Now, substituting (4.5), (4.6), (4.7), (4.8), (4.9) and (4.10) into (4.4), it derives that
\[
\mathbb{E}( \sup_{0 \leq t \leq T} |y(t) - D(\bar{g}(t - \tau))|^p ) \leq C + C \int_0^T \mathbb{E}( \sup_{0 \leq u \leq s} |y(u)|^p ) ds. \tag{4.11}
\]

Then, substituting into (4.3)
\[
\mathbb{E}( \sup_{0 \leq t \leq T} |y(t)|^p ) \leq C + C \int_0^T \mathbb{E}( \sup_{0 \leq u \leq s} |y(u)|^p ) ds.
\]

An application of Gronwall inequality yields
\[
\sup_{0 \leq t \leq T} \mathbb{E} |y(t)|^p \leq C.
\]

**Lemma 4.2** Under (3.5), then \( \forall t > 0, p \geq 2, \)
\[
\mathbb{E} \left[ \sup_{t_k \leq t \leq t_{k+1}} |y(t) - \bar{y}(t)|^p \right] \leq C, \tag{4.12}
\]

where the positive constant \( C := C(p, T, K_R, ||\xi||) \).
Proof: By $|\sum_{i=1}^{n} x_i|^p \leq n^{p-1} \sum_{i=1}^{n} |x_i|^p$ it derives that

$$|y(t) - \bar{y}(t)|^p = |y(t) - y(t_k)|^p \leq 5^{p-1} \left[ |D(y(t-\tau)) - D(y(t_k-\tau))|^p + \int_{t_k}^{t} b_h(\bar{y}(s), \bar{y}(s-\tau))ds \right]^p + \left[ \int_{t_k}^{t} \sigma(\bar{y}(s), \bar{y}(s-\tau))dB(s) \right]^p,$$

By the BDG inequality, Hölder inequality and (A3), it derives that

$$\mathbb{E} \left[ \sup_{t_k \leq t \leq t_{k+1}} (J_1(t)) \right] \leq \mathbb{E} \left[ \sup_{t_k \leq t \leq t_{k+1}} \left| \int_{t_k}^{t} b_h(\bar{y}(s), \bar{y}(s-\tau))ds \right|^p \right] \leq \Delta^{p-1} \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \left| b_h(\bar{y}(s), \bar{y}(s-\tau)) \right|^p ds \right] \leq K_{R} \Delta^{p-1} \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \left| \bar{y}(s) \right|^p + \left| \bar{y}(s-\tau) \right|^p ds \right] \leq K_{R} \Delta^{p-1} C_{p} \left[ \tau ||\xi||^p + \mathbb{E} \int_{t_k}^{t} \sup_{0 \leq u \leq s} |y(u)|^p ds \right] \leq C + C \mathbb{E} \int_{t_k}^{t} \sup_{0 \leq u \leq s} |y(u)|^p ds. \quad (4.14)$$

By the BDG inequality, Hölder inequality and (A3), it derives that

$$\mathbb{E} \left[ \sup_{t_k \leq t \leq t_{k+1}} (J_2(t)) \right] = \mathbb{E} \left[ \sup_{t_k \leq t \leq t_{k+1}} \left| \int_{t_k}^{t} \sigma(\bar{y}(s), \bar{y}(s-\tau))dB(s) \right|^p \right] \leq C_{p} \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \left| \sigma(\bar{y}(s), \bar{y}(s-\tau)) \right|^2 ds \right]^{\frac{p}{2}} \leq C_{p} \Delta^{\frac{p-2}{2}} \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \left| \sigma(\bar{y}(s), \bar{y}(s-\tau)) \right|^p ds \right] \leq K_{R} C_{p} \Delta^{\frac{p-2}{2}} \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} (1 + \left| \bar{y}(s) \right|^p + \left| \bar{y}(s-\tau) \right|^p) ds \right] \leq K_{R} \Delta^{\frac{p-2}{2}} C_{p} \left[ \tau ||\xi||^p + \mathbb{E} \int_{t_k}^{t} \sup_{0 \leq u \leq s} |y(u)|^p ds \right] \leq C + C \mathbb{E} \int_{t_k}^{t} \sup_{0 \leq u \leq s} |y(u)|^p ds. \quad (4.15)$$
Using the similar approach as in (4.15), it estimates \( J_3(t) \) and \( J_4(t) \) as

\[
\mathbb{E}\left[ \sup_{t_k \leq t \leq t_{k+1}} (J_3(t)) \right] + \mathbb{E}\left[ \sup_{t_k \leq t \leq t_{k+1}} (J_4(t)) \right] \\
= \mathbb{E}\left[ \sup_{t_k \leq t \leq t_{k+1}} \left| \int_{t_k}^{t} \sigma_1(\bar{y}(s), \bar{y}(s - \tau)) \sigma(\bar{y}(s), \bar{y}(s - \tau)) \Delta \bar{B}(s) dB(s) \right|^p \right] \\
+ \mathbb{E}\left[ \sup_{t_k \leq t \leq t_{k+1}} \left| \int_{t_k}^{t} \sigma_2(\bar{y}(s), \bar{y}(s - \tau)) \sigma(\bar{y}(s - \tau), \bar{y}(s - 2\tau)) \Delta \bar{B}(s) dB(s - \tau) \right|^p \right] \\
\leq C_p \Delta^{p \over 2} \int_{t_k}^{t} |\sigma_1(\bar{y}(s), \bar{y}(s - \tau)) \sigma(\bar{y}(s), \bar{y}(s - \tau)) \Delta \bar{B}(s)|^p ds \\
+ C_p \Delta^{p \over 2} \int_{t_k}^{t} |\sigma_2(\bar{y}(s), \bar{y}(s - \tau)) \sigma(\bar{y}(s - \tau), \bar{y}(s - 2\tau)) \Delta \bar{B}(s - \tau)|^p ds \\
\leq K_R C_p \Delta^{p \over 2} \left[ \tau||\xi||^p + \int_{t_k}^{t} \mathbb{E} \sup_{0 \leq u \leq s} |y(u)|^p ds + (\Delta \bar{B}(s))^p ds \right] \\
\leq C + C \int_{t_k}^{t} \mathbb{E} \sup_{0 \leq u \leq s} |y(u)|^p ds.
\] (4.16)

Now, substituting (4.14) (4.15) and (4.16) into (4.13), it derives that

\[
\mathbb{E}|y(t) - \bar{y}(t)|^p \leq C + \int_{t_k}^{t} \mathbb{E} \sup_{0 \leq u \leq s} |y(u)|^p ds.
\]

By (4.2), we could obtain

\[
\mathbb{E}\left[ \sup_{t_k \leq t \leq t_{k+1}} |y(t) - \bar{y}(t)|^p \right] \leq C.
\]

**Lemma 4.3** If assumptions (A2) and (A5) hold, for any real number \( R > ||\xi|| \) define a stopping time \( \tau_R = \inf\{t \geq 0 : |x(t)| \geq R\} \) and let \( \inf \Phi = \infty \) then

\[
P(\tau_R \leq T) \leq \frac{C}{R^2}.
\] (4.17)

where the positive constant \( C := C(T, K_1, ||\xi||, \kappa) \).

**Proof:** By the same way as in (2.4) and Itô formula, it derives that

\[
\mathbb{E}|x(t \wedge \tau_R)|^2 \leq \frac{1}{1 - \kappa} \mathbb{E}|x(t \wedge \tau_R) - D(x(t \wedge \tau_R) - \tau)|^2 + \frac{1}{\kappa} \mathbb{E}|D(x(t \wedge \tau_R - \tau))|^2 \\
\leq \frac{1}{1 - \kappa} \left[ ||\xi(0) - D(\xi(\tau))||^2 + \mathbb{E} \int_{0}^{t \wedge \tau_R} 2|x(s) - D(x(s - \tau))|^T \cdot (b(x(s), x(s - \tau) - \tau)) \right. \\
\left. + |\sigma(x(s), x(s - \tau))|^2 ds \right] + \frac{1}{\kappa} \mathbb{E}|D(x(t \wedge \tau_R - \tau))|^2 \\
= \frac{1}{1 - \kappa} \left[ ||\xi(0) - D(\xi(\tau))||^2 + L_1 + L_2 \right] + L_3.
\] (4.18)

By Assumption (A5), it derives that

\[
L_1 + L_2 = \mathbb{E} \int_{0}^{t \wedge \tau_R} 2|x(s) - D(x(s - \tau))|^T \cdot b(x(s), x(s - \tau)) + |\sigma(x(s), x(s - \tau))|^2 ds \\
\leq \mathbb{E} \int_{0}^{t \wedge \tau_R} 2K_1 (1 + |x(s)|^2 + |x(s - \tau)|) ds \\
\leq 2K_1 T + 2K_1 \tau ||\xi||^2 + 4K_1 \int_{0}^{t} \mathbb{E}|x(s \wedge \tau_R)|^2 ds.
\] (4.19)
By Assumption (A2), it derives that
\[ L_3 = \frac{1}{\kappa} \mathbb{E}|D(x(t \wedge \tau_R - \tau))|^2 \]
\[ \leq \kappa \mathbb{E}|x(t \wedge \tau_R - \tau)|^2 \]
\[ \leq \kappa \left[ ||\xi||^2 + E(x(t \wedge \tau_R))^2 \right]. \quad (4.20) \]

Now, substituting \((4.19)\) and \((4.20)\) into \((4.18)\), it derives that
\[ \mathbb{E}|x(t \wedge \tau_R)|^2 \leq C + C \int_0^t \mathbb{E}|x(s \wedge \tau_R)|^2 ds. \]

The Gronwall inequality yields that
\[ \mathbb{E}|x(t \wedge \tau_R)|^2 \leq C. \]

This implies
\[ R^2 \cdot P(\tau_R \leq T) \leq C. \]

**Lemma 4.4** If assumptions (A1)-(A3) hold, for any real number \(R > ||\xi||\) define a stopping time \(\rho_R = \inf\{t \geq 0 : |x(t)| \geq R\}\) and let \(\inf \Phi = \infty\) then
\[ P(\rho_R \leq T) \leq \frac{C}{R^2}. \quad (4.21) \]

**Proof:** Using the similar approach as in \((2.4)\), we could obtain
\[ \mathbb{E}|y(t \wedge \rho_R)|^2 \leq \frac{1}{1 - \kappa} \mathbb{E}|y(t \wedge \rho_R) - D(\bar{y}(t \wedge \rho_R - \tau))|^2 + \frac{1}{\kappa} \mathbb{E}|D(\bar{y}(t \wedge \rho_R - \tau))|^2 \]
\[ \leq \frac{1}{1 - \kappa} \left[ ||\xi(0) - D(\xi(-\tau))||^2 \right. \]
\[ + 2\mathbb{E} \int_0^{T \wedge \rho_R} \left( (\bar{y}(s) - D(\bar{y}(s - \tau)))^T \cdot b_h(\bar{y}(s), \bar{y}(s - \tau) + |\sigma(\bar{y}(s), \bar{y}(s - \tau))|)^2 \right) ds \]
\[ + 2\mathbb{E} \int_0^{T \wedge \rho_R} |\sigma(\bar{y}(s), \bar{y}(s - \tau)) \cdot \sigma(\bar{y}(s), \bar{y}(s - \tau)) \Delta B(s) \]
\[ + \sigma(\bar{y}(s - \tau), \bar{y}(s - 2\tau)) \Delta \bar{B}(s - \tau)|^2 ds \]
\[ + 2\mathbb{E} \int_0^{T \wedge \rho_R} (y(s) - \bar{y}(s))^T \cdot b_h(\bar{y}(s), \bar{y}(s - \tau) ds \]
\[ + \frac{1}{\kappa} \mathbb{E}|D(\bar{y}(t \wedge \rho_R - \tau))|^2 \]
\[ = \frac{1}{1 - \kappa} \left[ ||\xi(0) - D(\xi(-\tau))||^2 + M_1 + M_2 + M_3 + M_4 \right] + M_5. \quad (4.22) \]

By \((3.5)\) and \((4.12)\) it derives that
\[ M_1 + M_2 = 2\mathbb{E} \int_0^{T \wedge \rho_R} \left( (\bar{y}(s) - D(\bar{y}(s - \tau)))^T \cdot b_h(\bar{y}(s), \bar{y}(s - \tau) + |\sigma(\bar{y}(s), \bar{y}(s - \tau))|)^2 \right) ds \]
\[ \leq 2K_2 \mathbb{E} \int_0^{T} \left( 1 + \sup_{0 \leq u \leq s} |\bar{y}(u \wedge \rho_R)|^2 + \sup_{0 \leq u \leq s} |\bar{y}(u \wedge \rho_R - \tau)|^2 \right) ds \]
\[ \leq 2TK_2 + 2K_2\tau|\xi|^2 + 4K_2\mathbb{E} \int_0^{T} \sup_{0 \leq u \leq s} |y(u \wedge \rho_R)|^2 du \]
\[ \leq C + C\mathbb{E} \int_0^{T} \sup_{0 \leq u \leq s} |y(u \wedge \rho_R)|^2 du. \quad (4.23) \]
Using the similar approach as in (4.16), we can estimate \( M_3(t) \) as
\[
M_3 = 2\mathbb{E} \int_0^{t \wedge \rho_R} \left| \sigma_1(\bar{y}(s), \bar{y}(s - \tau)) \cdot \sigma(\bar{y}(s), \bar{y}(s - \tau)) \Delta \bar{B}(s) \\
+ \sigma_2(\bar{y}(s), \bar{y}(s - \tau)) \sigma(\bar{y}(s - \tau), \bar{y}(s - 2\tau)) \Delta \bar{B}(s - \tau) \right|^2 ds
\leq C + C \int_0^T \mathbb{E} \sup_{0 \leq u \leq s} |y(u \wedge \rho_R)|^2 ds. \tag{4.24}
\]
By (3.4) and (4.12) it derives that
\[
M_4 = 2\mathbb{E} \int_0^{t \wedge \rho_R} (y(s) - \bar{y}(s))^T \cdot b_h(\bar{y}(s), \bar{y}(s - \tau)) ds
\leq 2K_2C \mathbb{E} \int_0^T \left( 1 + \sup_{0 \leq u \leq s} |\bar{y}(u \wedge \rho_R)|^2 + \sup_{0 \leq u \leq s} |\bar{y}(u \wedge \rho_R - \tau)|^2 \right) ds
\leq 2TK_2 + 2K_2\tau ||\xi||^2 + 4K_2 \mathbb{E} \int_0^T \sup_{0 \leq u \leq s} |y(u \wedge \rho_R)|^2 du
\leq C + C \mathbb{E} \int_0^T \sup_{0 \leq u \leq s} |y(u \wedge \rho_R)|^2 du. \tag{4.25}
\]
By assumption (A3), it derives that
\[
M_5 = \frac{1}{\kappa} \mathbb{E} |D(\bar{y} \wedge \rho_R - \tau)|
\leq \kappa \mathbb{E} |\bar{y}(t \wedge \rho_R - \tau)|^2
\leq \kappa \sup_{0 \leq u \leq t} \mathbb{E} |\bar{y}(u \wedge \rho_R - \tau)|^2
\leq \kappa (||\xi||^2 + \sup_{0 \leq u \leq t} \mathbb{E} |\bar{y}(u \wedge \rho_R)|^2)
\leq C + C \sup_{0 \leq u \leq t} \mathbb{E} |y(u \wedge \rho_R)|^2). \tag{4.26}
\]
Now, substituting (4.23) (4.24) (4.25) and (4.26) into (4.22)
\[
\mathbb{E} |y(t \wedge \rho_R)|^2 \leq C + C \int_0^t \mathbb{E} |y(u \wedge \rho_R)|^2 du.
\]
An application of Gronwall inequality yeilds:
\[
\mathbb{E} |y(t \wedge \rho_R)|^2 \leq C.
\]
This implies
\[
R^2 \cdot P(\rho_R \leq T) \leq C.
\]

4.2 Proof of Therom 4.1

In this section, we give proof of the main theorem of this paper, the strong convergence of the tamed Milstein (3.7) to the solution of (2.1).
Proof: Let $\tau_R$ and $\rho_R$ be the same as before, define: $\theta_R = \tau_R \wedge \rho_R$ and $e(t) = x(t) - y(t)$, by Young’s inequality, it derives that
\[
\mathbb{E}\left[\sup_{0 \leq t \leq T} |e(t)|^p\right] \leq \mathbb{E}\left[\sup_{0 \leq t \leq T} |e(t)|^p I_{\{\tau_R \leq T \text{ or } \rho_R \leq T\}}\right] + \mathbb{E}\left[\sup_{0 \leq t \leq T} |e(t \wedge \theta_R)|^p\right] \\
\leq \frac{p}{q} \mathbb{E}\left[\sup_{0 \leq t \leq T} |x(t) - y(t)|^q\right] + \frac{q - p}{q(p/(q-p)} P(\tau_R \leq T \text{ or } \rho_R \leq T) \\
+ \mathbb{E}\left[\sup_{0 \leq t \leq T} |e(t \wedge \theta_R)|^p\right].
\]

Using the similar approach as (2.4), it derives that
\[
\mathbb{E}\left(\sup_{0 \leq t \leq T} |e(t \wedge \theta_R)|^p\right) \\
\leq \frac{1}{(1-\kappa)^p}\mathbb{E}\left(\sup_{0 \leq t \leq T} |x(t \wedge \theta_R) - y(t \wedge \theta_R) - D(x(t \wedge \theta_R - \tau)) + D(\bar{y}(t \wedge \theta_R - \tau))|^p\right) + C. 
\]

By the Hölder inequality and (A3), for any $t \in [0, T]$, it derives that
\[
|x(t \wedge \theta_R) - y(t \wedge \theta_R) - D(x(t \wedge \theta_R - \tau)) + D(\bar{y}(t \wedge \theta_R - \tau))|^p \\
\leq C_R \int_0^{t \wedge \theta_R} \left[|x(s) - y(s)|^p + |x(s - \tau) - y(s - \tau)|^p\right] ds \\
+ C \int_0^{t \wedge \theta_R} |b(y(s), y(s - \tau)) - b(\bar{y}(s), \bar{y}(s - \tau))|^p ds \\
+ C \int_0^{t \wedge \theta_R} |b(\bar{y}(s), \bar{y}(s - \tau)) - b_\eta(\bar{y}(s), \bar{y}(s - \tau))|^p ds \\
+ C \int_0^{t \wedge \theta_R} (\sigma(x(s), x(s - \tau)) - \sigma(\bar{y}(s), \bar{y}(s - \tau))) dB(s)^p \\
+ C \int_0^{t \wedge \theta_R} (\sigma(\bar{y}(s), \bar{y}(s - \tau))\sigma_1(\bar{y}(s), \bar{y}(s - \tau))) dB(s)^p \\
+ C \int_0^{t \wedge \theta_R} \sigma_2(\bar{y}(s), \bar{y}(s - \tau))\sigma(\bar{y}(s), \bar{y}(s - 2\tau)) dB(s)|^p \\
= \sum_{i=1}^6 Q_i(t). 
\]

We now estimate $Q_1(t)$, by (A1) it derives that
\[
\mathbb{E}\left(\sup_{0 \leq t \leq T} Q_1(t)\right) \\
= \mathbb{E}\left(\sup_{0 \leq t \leq T} C_R \int_0^{t \wedge \theta_R} \left[|x(s) - y(s)|^p + |x(s - \tau) - y(s - \tau)|^p\right] ds\right) \\
\leq \mathbb{E}\left\{\sup_{0 \leq u \leq s} \left[|x(u \wedge \theta_R) - y(u \wedge \theta_R)|^p + |x(u \wedge \theta_R - \tau) - y(u \wedge \theta_R - \tau)|^p\right]\right\} ds \\
\leq \mathbb{E}\left\{\sup_{0 \leq u \leq s} |e(u \wedge \theta_R)|^p ds + C \Delta^p\right\}. 
\]
By (A3) and (4.12), it derives that

\[
\mathbb{E}(\sup_{0 \leq t \leq T} Q_2(t)) \\
= \mathbb{E}(\sup_{0 \leq t \leq T} C \int_0^{t \wedge \theta_R} |b(y(s), y(s - \tau)) - b(\bar{y}(s), \bar{y}(s - \tau))|^p ds) \\
\leq \mathbb{E} \int_0^T |y(s \wedge \theta_R) - \bar{y}(s \wedge \theta_R)|^p ds + C \int_0^T |\xi(s \wedge \theta_R) - \xi(\kappa(s \wedge \theta_R))|^p ds \\
\leq C \Delta^\frac{p}{2}.
\] (4.30)

According to (B1), we could obtain

\[
\mathbb{E}(\sup_{0 \leq t \leq T} Q_3(t)) \\
= \mathbb{E}(\sup_{0 \leq t \leq T} C \int_0^{t \wedge \theta_R} |b(\bar{y}(s), \bar{y}(s - \tau)) - b_h(\bar{y}(s), \bar{y}(s - \tau))|^p ds) \\
\leq C \mathbb{E} \int_0^T |b(\bar{y}(s \wedge \theta_R), \bar{y}(s \wedge \theta_R - \tau)) - b_h(\bar{y}(s \wedge \theta_R), \bar{y}(s \wedge \theta_R - \tau))|^p ds \\
\leq C N_R \Delta^p \Delta^p \leq C.
\] (4.31)

Using similar approach as in (2.7) (4.29) and (4.30) we have

\[
\mathbb{E}(\sup_{0 \leq t \leq T} Q_4(t)) \\
= \mathbb{E}(\sup_{0 \leq t \leq T} C \int_0^{t \wedge \theta_R} \left| \sigma(x(s), x(s - \tau)) - \sigma(\bar{y}(s), \bar{y}(s - \tau)) \right| dB(s)^p \\
\leq C_\nu \Delta^\frac{p-2}{2} \mathbb{E} \int_0^T \left| \sigma(x(s \wedge \theta_R), x(s \wedge \theta_R - \tau)) - \sigma(\bar{y}(s \wedge \theta_R), \bar{y}(s \wedge \theta_R - \tau)) \right|^p ds \\
\leq C_\nu \Delta^\frac{p-2}{2} \mathbb{E} \int_0^T \sup_{0 \leq u \leq s} \left[ |x(u \wedge \theta_R) - y(u \wedge \theta_R)|^p + |x(u \wedge \theta_R - \tau) - y(u \wedge \theta_R - \tau)|^p \right] ds \\
+ C_\nu \Delta^\frac{p-2}{2} \mathbb{E} \int_0^T |y(s \wedge \theta_R) - \bar{y}(s \wedge \theta_R)|^p ds + C \int_0^T |\xi(s \wedge \theta_R) - \xi(\kappa(s \wedge \theta_R))|^p ds \\
\leq \mathbb{E} \int_0^T \sup_{0 \leq u \leq s} |e(u \wedge \theta_R)|^p ds + C \Delta^\frac{p}{2}.
\] (4.32)
By (2.1) and (4.2) and given an \( \epsilon > 0 \), it derives that

\[
E\left( \sup_{0 \leq t \leq T} Q_5(t) \right) \leq CE \int_0^T |\sigma(\bar{y}(s \wedge \theta_R), \bar{y}(s \wedge \theta_R - \tau))\sigma_1(\bar{y}(s \wedge \theta_R), \bar{y}(s \wedge \theta_R - \tau))\Delta \bar{B}(s \wedge \theta_R)|^p ds
\]

\[
\leq CE \int_0^T \left( |\sigma(\bar{y}(s \wedge \theta_R), \bar{y}(s \wedge \theta_R - \tau))\sigma_1(\bar{y}(s \wedge \theta_R), \bar{y}(s \wedge \theta_R - \tau))\Delta \bar{B}(s \wedge \theta_R)|^p
- |\sigma(y(s \wedge \theta_R), y(s \wedge \theta_R - \tau))\sigma_1(y(s \wedge \theta_R), y(s \wedge \theta_R - \tau))\Delta \bar{B}(s \wedge \theta_R)|^p
+ |\sigma(y(s \wedge \theta_R), y(s \wedge \theta_R - \tau))\sigma_1(y(s \wedge \theta_R), y(s \wedge \theta_R - \tau))\Delta \bar{B}(s \wedge \theta_R)|^p \right) ds
\]

\[
\leq C E \int_0^T \left( |y(s \wedge \theta_R) - \bar{y}(s \wedge \theta_R)|^p + |y(s \wedge \theta_R - \tau) - \bar{y}(s \wedge \theta_R - \tau)|^p
+ |y(s \wedge \theta_R)|^p + |y(s \wedge \theta_R - \tau)|^p \right) ds
\]

\[
\leq C \Delta^{\frac{p}{q}} + C. \quad (4.33)
\]

Using the similar approach as in (4.33), we can estimate \( Q_6(t) \) as

\[
E\left( \sup_{0 \leq t \leq T} Q_6(t) \right) \leq C \Delta^{\frac{p}{q}} + C. \quad (4.34)
\]

Substituting (4.29), (4.30), (4.31), (4.32), (4.33) and (4.34) into (4.27)

\[
E\left( \sup_{0 \leq t \leq T} |e(t \wedge \theta_R)|^p \right) \leq C E \int_0^T \sup_{0 \leq u \leq s} |e(u \wedge \theta_R)|^p ds + C. \quad (4.35)
\]

An application of Gronwall inequality yields:

\[
E\left( \sup_{0 \leq t \leq T} |e(t \wedge \theta_R)|^p \right) \leq C. \quad (4.36)
\]

Where the positive constant \( C := C(R, ||\xi||, \kappa, \Delta) \).

By (2.1) (4.2) and given an \( \epsilon > 0 \), there exists \( \eta \) small enough that

\[
\frac{p\eta}{q} E \left[ \sup_{0 \leq t \leq T} |x(t) - y(t)|^q \right] \leq 2p\eta \frac{p\eta}{q} E \left\{ \left[ \sup_{0 \leq t \leq T} |x(t)|^q \right] + \left[ \sup_{0 \leq t \leq T} |y(t)|^q \right] \right\} \leq \frac{\epsilon}{3}.
\]

Choose \( R \) large enough that

\[
\frac{q - p}{qp^{p/(q-p)}} \left[ P(\tau_R \leq T) + P(\rho_R \leq T) \right] \leq \frac{\epsilon}{3}.
\]

And finally by Lemmas 4.1 and 4.2, we choose \( \Delta \) sufficiently small, such that

\[
E\left( \sup_{0 \leq t \leq T} |e(t \wedge \theta_R)|^p \right) \leq \frac{\epsilon}{3}.
\]

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