Causality and Self-consistency in Classical Electrodynamics

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We present a pedagogical review of old inconsistencies of Classical Electrodynamics and of some new ideas that solve them. Problems with the electron equation of motion and with the non-integrable singularity of its self-field energy tensor are well known. They are consequences, we show, of neglecting terms that are null off the charge world-line but that give a non null contribution on its world-line. The electron self-field energy tensor is integrable without the use of any kind of renormalization; there is no causality violation and no conflict with energy conservation in the electron equation of motion, when its meaning is properly considered.

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I. INTRODUCTION

Classical Electrodynamics of a point electron is based on the Lienard-Wiechert solution; its many old and unsolved problems [1–3] make of it a non-consistent theory. One can mention the field singularity or the self-energy problem; the non-integrable singularities of its energy tensor; the causality-violating behavior of solutions of the Lorentz-Dirac equation [4–7]; etc. Here, we will discuss these problems. We will show that their solution is connected to a more strict implementation of causality (extended causality) which is explained in section II. In section III we review and discuss the singularities and non-integrability of the electron self-field energy tensor. Some helpful mathematical results are presented in section IV. They are useful in the working out of some limiting processes. In section V the electron equation of motion, which does not have the Schott term, is derived and its physical meaning is discussed. Section VI is included like an appendix of section V for showing an alternative way to the electron equation of motion that illuminates its physical meaning.

The retarded Lienard-Wiechert solution

\[ A(x) = \frac{V}{\rho} \mid_{\tau_{ret}}, \text{ for } \rho > 0, \]  

is the retarded solution to the wave equation

\[ \Box A(x) = 4\pi J(x) \]  

and to

\[ \partial.A = \frac{\partial A^\mu}{\partial x^\mu} = 0, \]  

where J, given by

\[ J(x) = \int drV \delta^4[x - z(\tau)], \]  

is the current for a point electron that describes a given trajectory \( z(\tau) \), parameterized by its proper-time \( \tau \); \( V = \frac{d\tau}{d\tau} \). The electron charge and the speed of light are taken as 1.

\[ \rho := -V_{\alpha}R^\alpha = -V\eta.R = -V.R, \]  

where \( \eta \) is the Minkowski metric tensor with signature +2, and \( R := x - z(\tau) \). \( \rho \) is the invariant distance (in the charge rest frame) between \( z(\tau_{ret}) \), the position of the charge at the retarded time, and \( x \), its self-field event (See figure 1). The constraints

\[ R^2 = 0, \]  

and
must be satisfied. The constraint $R^2 = 0$ requires that $x$ and $z(\tau)$ belong to a same light-cone; it has two solutions, \( \tau_{\text{ret}} \) and \( \tau_{\text{adv}} \), which are the points where \( J \) intercepts the past and the future light-cone of $x$ (see figure 1), and they correspond, respectively to the advanced and the retarded solutions. The retarded solution describes a signal emitted at $z(\tau_{\text{ret}})$ and that is being observed at $x$, with $x^0 > z^0(\tau_{\text{ret}})$, while the advanced solution also observed at $x$, \( z(\tau_{\text{adv}}) \), with $x^0 < z^0(\tau_{\text{adv}})$. $R^0 > 0$ is a restriction to the retarded solution $R$ as it excludes the causality violating advanced solution, and justifies the restriction $[\tau_{\text{ret}}]$ in $[\tau]$. But this is not the only available interpretation; we will show below another one that does not have problems with causality violation and, remarkably allows the description of particle creation and annihilation still in a classical physics context.

II. CAUSALITY AND SPACETIME GEOMETRY

When working with variations or derivatives of $A$ the constraint (9) must be considered in the neighbourhoods of $x$ and of $z$: $x + dx$ and $z(\tau_{\text{ret}} + d\tau)$ must also belong to a same light-cone. A differentiation of (9) ($R.dR = 0 \rightarrow R.(dx - Vd\tau) = 0 \rightarrow R.dx + \rho d\tau = 0$) generates the constraint

\[
d\tau + K.dx = 0,
\]

where $K$, defined for $\rho > 0$, by

\[
K := \frac{R}{\rho},
\]

is a null 4-vector, $K^2 = 0$, and represents a light-cone generator, a tangent to the light-cone. The constraint (8) defines a family of hyperplanes tangent to and enveloped by the light-cone defined by $R^2 = 0$. Together, these two constraints require that $x$ and $z(\tau_{\text{ret}})$ belong to a same straight line, the $x$-lightcone generator tangent to $K^\mu$, or equivalently, orthogonal to $K_\mu$. See figures 1 and 2.

There is a geometric and physical interpretation of the two constraints (8) and (9). $R^2 = 0$ assures that $A(x)$ is a signal that propagates with the speed of light, on a light-cone; in field theory it corresponds to the implementation of the so called *local causality*: only points inside or on a same light-cone can be causally connected. It defines for a physical object, at a point, *its physical spacetime*, that is the regions of the space-time manifold that it can have access to.

But together, (8) and (9) produce a much more restrictive constraint: a massless physical object cannot leave, by itself, its light-cone generator(labelled by $K$). Or, in other words, the part of a wavefront of $A(x)$ that moves along a light-cone generator must remain in this same generator. This is in direct contradiction to the Huyghens Principle that assumes that the signal at a point of a wavefront is made of contributions from all points of previous wavefronts; each point of a wavefront acts as a secondary source emitting signal to all space directions. The Huyghens Principle is appropriate for a description of light as a continuous wavy manifestation, but not for a discrete one.

In contradistinction, the constraints (8) and (9), together, imply that a point on a wavefront propagates, on its light-cone generator, independently of all the other wavefront points. Each point of a wavefront, therefore, can be treated as an entity by itself. It is so justified the naming of a CLASSICAL PHOTON to each point of an electromagnetic wavefront. This corresponds to an EXTENDED CAUSALITY concept and it is readily extensible to massive objects too (8). It is appropriate for descriptions of particle-like fields with discrete interactions, that is, localized and propagating like a particle. Usually field theories are based on a local-causality implementation, but it possible to build a theory basing on this extended causality. This is being discussed elsewhere (8).

Armed with this extend-causality concept we can present another physical interpretation of the two Lienard-Wiechert solutions. At the event $x$ there are two classical photons. One, that was emitted by the electron current $J$, at $z(\tau_{\text{ret}})$ with $x^0 > z^0(\tau_{\text{ret}})$, and is moving in the $K$ generator of the $x$-light-cone, $K^\mu := (K^0, \vec{K})$; \( J \) is its source. The other one, moving on a $\vec{K}$-generator, $\vec{K}^\mu := (K^0, -\vec{K})$, will be absorbed by $J$ at $z(\tau_{\text{adv}})$, with $x^0 < z^0(\tau_{\text{adv}})$. $J$ is its sink. See figure 2. They are both retarded solutions and correspond, respectively, to the creation and destruction of a “classical photon”. Exactly this: creation and destruction of particles in classical physics! This interpretation is only possible with these concepts of extended causality and of classical photon.

III. ENERGY TENSOR AND INTEGRABILITY

When taking derivatives of $A(x)$ we must consider the restriction (8), or equivalently, $K_\mu = -\frac{\partial \tau_{\text{ret}}}{\partial x^\nu}$. This can turn, for the untrained, a trivial calculation into a mess. The best and more fruitful approach, in our opinion, is to take $x$ and $\tau_{\text{ret}}$ as 5 independent parameters, and introduce a new derivative operator $\nabla$, replacing the usual one:
\[
\frac{\partial}{\partial x^\mu} \Rightarrow \nabla_\mu := \frac{\partial}{\partial x^\mu} + \frac{\partial \tau}{\partial x^\mu} \frac{\partial}{\partial \tau} = \frac{\partial}{\partial x^\mu} - K_\mu \frac{\partial}{\partial \tau},
\]
(10)
or \( \nabla_\mu := \partial_\mu - K_\mu \partial_\tau \), in a shorter notation. The geometric meaning of \( \nabla \) is quite clear; it is the derivative allowed by the restriction (8), that is, displacements on the hyperplane \( d\tau + K.d\mathbf{x} = 0 \) only. The constraints (8) and (9) together restricts \( \nabla \) to displacements along the \( K \) light-cone generator only. Therefore, \( \partial_\mu A(x) \), with the restriction \( \big|_{\tau_{ret}} \) implicit is equivalent to \( \nabla_\mu A(x) \) without any restriction.

\[
\partial_\mu A(x) \big|_{\tau_{ret}} = \nabla_\mu A(x)
\]
(11)

This corresponds to a geometrization of the extended causality concept. Therefore we can write

\[
\nabla_\mu A^\nu = \nabla_\mu \frac{V^\nu}{\rho} = -\frac{K_\mu \dot{a}^\nu}{\rho} - \frac{V^\nu}{\rho^2} \nabla_\mu \rho = -K_\mu \frac{\dot{a}^\nu}{\rho} - \frac{V^\nu (K_\mu E - V_\mu)}{\rho^2},
\]
(12)

with

\[
E = 1 + \dot{a}.R = 1 + \rho \dot{a}_K,
\]
(13)
as \( \nabla_\mu V^\nu = -K_\mu \dot{a}^\nu \) and

\[
\nabla_\mu \rho = K_\mu E - V_\mu,
\]
(14)

where \( \dot{a}_K := \dot{a}K \). For notation simplicity we use \( [A, B] \) standing for \( [A_\mu, B_\nu] := A_\mu B_\nu - B_\mu A_\nu \) and \( (A,B) \) for \( (A_\mu, B_\nu) := A_\mu B_\nu + A_\nu B_\mu \), and we are omitting, from now on, the always implicit restriction \( \big|_{\tau_{ret}} \).

We observe that the Lorentz gauge condition is automatically satisfied

\[
\nabla.A = \frac{\rho \dot{a}_K + V \nabla \rho}{\rho^2} = 0,
\]
(15)
as \( V.K = -1, V^2 = -1 \), and \( V.\nabla \rho = 1 - E = -\rho \dot{a}_K \).

The Maxwell field \( F_{\mu \nu} := \nabla_\mu A_\nu - \nabla_\nu A_\mu \), is found to be

\[
F = \frac{1}{\rho^2} [K, W],
\]
(16)

with

\[
W^\mu = \rho \dot{a}^\mu + EV^\mu.
\]
(17)
The electron self-field energy-momentum tensor, \( 4\pi \Theta = F.F - \frac{\eta}{4} F^2 \), is

\[
4\pi \rho^4 \Theta_{\mu \nu} = [K^\mu, W^\alpha][K_\alpha, W_\nu] - \frac{\eta \mu \nu}{4} [K^\alpha, W^\beta][K_\beta, W_\alpha],
\]
(18)
or in an expanded expression

\[
-4\pi \rho^4 \Theta = (K, W) + KKW^2 + WWK^2 + \frac{\eta}{2}(1 - K^2W^2),
\]
(19)
as \( K.W = -1 \). We will use rather compact expressions like (18) instead of (19) also because they make easier the calculation of some limits that we will have to do later. With \( W^2 = \rho^2 \dot{a}^2 - E^2 = \rho^2 \dot{a}^2 - (1 + \rho \dot{a}_K)^2 \), \( \Theta \) may be written, according to its powers of \( \rho \), as \( \Theta = \Theta_2 + \Theta_3 + \Theta_4 \), with

\[
4\pi \rho^2 \Theta_2 = [K, a + V a_K],[K, a + V a_K] - \frac{\eta}{4}[K, a + V a_K]^2,
\]
(20)
or,

\[
-4\pi \rho^2 \Theta_2 = KK(a^2 - a_K^2) + K^2(a + V a_K)(a + V a_K) - \frac{\eta}{2}K^2(a^2 - a_K^2).
\]
(21)
or

\[ 4\pi \rho^3 \Theta_3 = -(K + VK^2, a + V\alpha_K) + (2KK - \eta K^2)\alpha_K. \]

\[ 4\pi \rho^3 \Theta_4 = [K, V]_n [K, V] - \frac{\eta}{2}[K, V]^2. \]  \hspace{1cm} (22)

or

\[ 4\pi \rho^3 \Theta_4 = KK - (K, V) - K^2VV - \frac{\eta}{2}(1 + K^2). \]

If we neglect the \(K^2\)-terms in (23-22) we have:

\[ 4\pi \rho^3 \Theta_2|_{K^2=0} = -KK\left(\alpha^2 - \alpha_{K^2}\right), \] \hspace{1cm} (23)

\[ 4\pi \rho^3 \Theta_3|_{K^2=0} = 2KK\alpha_K - \left(K, \alpha + V\alpha_K\right), \] \hspace{1cm} (24)

\[ 4\pi \rho^3 \Theta_4|_{K^2=0} = KK - (K, V) - \frac{\eta}{2}, \] \hspace{1cm} (25)

which are the usual expressions that one finds, for example in [1–3, 5–7]. Observe that

\[ K, \Theta_2|_{K^2=0} = 0, \] \hspace{1cm} (26)

which is important in the identification of \(\Theta_2\) with the radiated part of \(\Theta\), and that

\[ K, \Theta_3|_{K^2=0} = 0. \] \hspace{1cm} (27)

The presence of non-integrable singularities in the electron self-field energy tensor is a major problem. \(\Theta_2|_{K^2=0}\), although singular at \(\rho = 0\), is nonetheless integrable. By that it is meant that it produces a finite flux through a spacelike hypersurface \(\sigma\) of normal \(n\), that is, \(\int d^3\sigma \Theta_2n\) exists, while \(\Theta_3|_{K^2=0}\) and \(\Theta_4|_{K^2=0}\) are not integrable; they generate, respectively, the problematic Schott term in the LDE and a divergent term, the electron bound 4-momentum, which includes the so called electron self-energy. Previous attempts, based on distribution theory, for taming these singularities have relied on modifications of the Maxwell theory with addition of extra terms to \(\Theta\) on the electron world-line (see for example the reviews [3, 4]). They redefine \(\Theta_3\) and \(\Theta_4\) at the electron world-line in order to make them integrable without changing them at \(\rho > 0\), so to preserve the standard results of Classical Electrodynamics. But this is always an ad hoc introduction of something strange to the theory. Another unsatisfactory aspect of this procedure is that it regularizes the above integral but leaves an unexplained and unphysical discontinuity in the flux of 4-momentum, \(\int dx^4 \Theta^\mu\nu\nabla_\mu \rho \delta(\rho - \epsilon_1)\), through a cylindrical hypersurface \(\rho = \text{const}\) enclosing the charge world-line. It is particularly interesting that, as we will show now, instead of adding anything we should actually not drop out the null \(K^2\)-terms. Their contribution (not null, in an appropriate limit) cancel the infinities. The same problem happens in the derivations of the electron equation of motion from these incomplete expressions of \(\Theta\). The Schott term in the Lorentz-Dirac equation is a consequence; it does not appear in the equations when the full expression of \(\Theta\) is correctly used.

By force of the constraints [1] and [8], as \(x\) and \(z(\tau_{rel})\) must remain on a same straight-line, the lightcone-generator \(K\), the limit \(\rho \to 0\) necessarily implies also on \(x^\mu \to x(\tau_{rel})^\mu\) or \(R^\mu \to 0\).

At \(z(\tau_{rel})\), \(K = \frac{\dot{\rho}}{\rho}\) produces a \((\frac{\partial}{\partial \tau})\)-type of indeterminacy, which can be evaluated at neighboring points \(\tau = \tau_{rel} \pm d\tau\) by the L’Hospital rule and \(\frac{d}{d\tau}\) (see figure 3). This application of the L’Hospital rule corresponds then to finding two simultaneous limits: \(\rho \to 0\) and \(\tau \to \tau_{rel}\).

As

\[ \dot{\rho} = -(1 + a.R) \] \hspace{1cm} (28)

and

\[ \dot{R} = -V; \] \hspace{1cm} (29)
then
\[
\lim_{\tau \to t_{\text{ret}}} |K|_{R^2 = 0, R.dR = 0} = V. \tag{30}
\]

This double limiting process is of course distinct of the single \((\rho \to 0)\)-limit, which cannot avoid the singularity. For notation simplicity we will keep using just \(\lim_{\rho \to 0}\) but always with the implicit meaning as indicated in (30). For example by
\[
\lim_{\rho \to 0} K^2 = -1. \tag{31}
\]

we mean
\[
\lim_{\rho \to 0} K^2 |_{R^2 = 0, R.dR = 0} = -1. \tag{32}
\]

Classical Electrodynamics alone, with its picture of a continuous emission of radiation, does not give room for a comprehension of these limiting processes. But we know that this classical continuous emission is just an approximate description of an actually discrete quantum process. Figure 4 portrays a classical picture (the electron and the photon trajectories) of such a fundamental quantum process; it helps in the understanding of these two limiting results. In the limit of \(\rho \to 0\) and \(\tau \to t_{\text{ret}}\) there are 3 distinct velocities: \(K\), the photon 4-velocity, and \(V_1\) and \(V_2\), the electron initial and final 4-velocities. This is the reason for the indeterminacy at \(\tau = t_{\text{ret}}\). At \(\tau = t_{\text{ret}} + d\tau\) there is only \(V_2\), and only \(V_1\) at \(\tau = t_{\text{ret}} - d\tau\). In other words, \(t_{\text{ret}}\) is an isolated singular point on the electron world-line; its neighboring \(t_{\text{ret}} \pm d\tau\) are not singular. This is in flagrant contradiction to the Classical-Electrodynamics assumption of a continuous emission process, because in this case, all points in the electron world-line would be singular points, like \(t_{\text{ret}}\). It is remarkable that we can find vestiges of these traits of the quantum nature of the radiation emission process in a classical (Lienard-Wiechert) solution. This is food for thinking on the real physical meaning of the classical and the quantum fields.

IV. SOME USEFUL MATHEMATICAL TOOLS

To find this double limit of something when \(\rho \to 0\) and \(\tau \to t_{\text{ret}}\) will be done so many times in this paper that it is better to do it in a more systematic way. We want to find
\[
\lim_{\rho \to 0} \frac{N(R, \ldots)}{\rho^n}, \tag{33}
\]

where \(N(R, \ldots)\) is a homogeneous function of \(R\), \(N(R, \ldots)|_{R=0} = 0\). Then, we have to apply the L'Hospital rule consecutively until the indeterminacy is resolved. As \(\frac{\partial}{\partial \tau} = -(1 + A.R) 2\), the denominator of (33) at \(R = 0\) will be different of zero only after the \(n^{th}\)-application of the L'Hospital rule, and then, its value will be \((-1)^n n!\).

If \(p\) is the smallest integer such that \(N(R, \ldots)|_{R=0} \neq 0\), where \(N(R)p := \frac{\partial}{\partial \epsilon} N(R, \ldots)\), then
\[
\lim_{\rho \to 0} \frac{N(R, \ldots)}{\rho^n} = \begin{cases} 
\infty, & \text{if } p \leq n \\
(-1)^n \frac{N(0, \ldots)\epsilon}{n!}, & \text{if } p = n \\
0, & \text{if } p > n
\end{cases} \tag{34}
\]

- **Example 1:** \(\left\{ \begin{array}{l} K = \frac{R}{\rho} \\
K^2 = \frac{R.n - R}{\rho^2}
\end{array} \right\} \quad n = 1 \Rightarrow \lim_{\rho \to 0} K = V \\
\lim_{\rho \to 0} K^2 = -1.
\)

- **Example 2:** \(\left\{ \frac{K.A}{\rho} = \frac{|R.A|}{\rho^3} \Rightarrow p = 1 < n = 2 \Rightarrow \lim_{\rho \to 0} \frac{|K.A|}{\rho} \right\} \) diverges

- **Example 3:** \(\left\{ \frac{\partial}{\partial \rho} [K, V] = \frac{|R.V|}{\rho^2} \Rightarrow p = 4 > n = 3 \quad \lim_{\rho \to 0} \frac{\partial}{\partial \rho} [K, V] = 0 \right\}
\)

- **Example 4:** \(\left\{ \frac{[K, V]}{\rho^6} = \frac{|R.R|}{\rho^3} \Rightarrow p = 2 < n = 3 \quad \lim_{\rho \to 0} \frac{|K, V|}{\rho^6} \right\} \) diverges

Finding these limits for more complex functions can be made easier with two helpful expressions,
\[
N_p = \sum_{a=0}^{p} \left( \begin{array}{c} p \\ a \end{array} \right) A_{p-a} B_a \tag{35}
\]
valid when \( N(R) \) has, respectively, the forms \( N_0 = A_0 B_0 \) or \( N_0 = A_0 B_0 C_0 \), where A, B and C represent possibly distinct functions of R, and the subindices indicate the order of \( \partial \). For example: \( A_0 = A; A_1 = \partial_\tau A; A_2 = \partial^2_\tau A \), and so on. So, for using (34-36), we just have to find the \( \tau \)-derivatives of A, B and C that produce the first non-null term at the point limit of \( R \to 0 \).

Consecutive derivatives of products of functions can become unwieldy. So it is worthy to introduce the concept of “\( \tau \)-order” of a function, meaning the lowest order of the \( \tau \)-derivative of a function that produces a non-null result at \( R = 0 \). Let us represent the “\( \tau \)-order” of \( f(x) \) by \( O[f(x)] \). So, for example, from (28) and (29) we see that

\[
O[R] = 1, \quad O[\rho] = 1, \quad O[a.R] = 2, \tag{37}
\]

For finding the \( N_p \) of (35) and of (36) it is then necessary to consider only the terms with the lowest \( \tau \)-order on each factor. Some combinations of terms have derivatives that cancel parts of each other resulting in a higher \( \tau \)-order term. For example,

\[
\partial_\tau (R^2 + \rho^2) = +2\rho - 2\rho E = -2\rho a.R
\]

\[
\partial^2_\tau (R^2 + \rho^2) = 2E a.R - 2\rho a.R = 2(a.R - \rho a.R) + O(R^1),
\]

\[
\partial^3_\tau (R^2 + \rho^2) = 2(a.R + E a.R - \rho a^2) + O(R^3) = 4a.R - 2\rho a^2 + O(R^3),
\]

\[
\partial^4_\tau (R^2 + \rho^2) = 4a^2 + 2a^2 + O(R^2) = 6a^2 + O(R^2).
\]

So,

\[
O[R^2 + \rho^2] = 4
\]

although

\[
O[R^2] = O[\rho^2] = 2.
\]

Observe that we only have to care with the lowest \( \tau \)-order terms as the other ones, grouped in \( O(R) \), will not survive the limit \( R \to 0 \). Also, we do not care on writing the \( \tau \)-derivatives of factors that will not reduce its \( \tau \)-order. For example in

\[
\partial_\tau (RV + O(R^2)) = -VV + O(R),
\]

the term \( R\dot{a} \) was absorbed in \( O(R) \). In this way we avoid taking unnecessary derivatives.

V. THE ELECTRON EQUATION OF MOTION

The motion of a classical electron [1, 2] is described by the Lorentz-Dirac equation,

\[
m\ddot{a} = F_{ext}V + \frac{2}{3} (\dot{a} - \dot{a}^2 V), \tag{40}
\]
where \( m \) is the electron mass and \( F_{\text{ext}} \) is an external electromagnetic field. The presence of the Schott term, \( \frac{2}{3}e^2 \dot{a} \), is the cause of all of its pathological features, like microscopic non-causality, runaway solutions, preacceleration, and other bizarre effects \([1]\). On the other hand, its presence is apparently necessary for the energy-momentum conservation; without it it would be required a contradictory null radiance for an accelerated charge, as \( \dot{a} . \dot{V} + a^2 = 0 \). This makes of the Lorentz-Dirac equation the greatest paradox of classical field theory as it cannot simultaneously preserve both the causality and the energy conservation \([1, 2]\).

The Lorentz-Dirac equation can be obtained from energy-momentum conservation, that leads to

\[
m \ddot{a} - F_{\text{ext}}^\mu V_\nu = m b_{\text{box}} - \lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to \infty} \int d^4x \nabla_\nu \Theta^{\mu \nu} \theta(\rho - \varepsilon_1) \theta(\varepsilon_2 - \rho) \theta(\tau_2 - \tau) \theta(\tau - \tau_1) =
\]

\[
= - \lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to \infty} \int d^4x \Theta^{\mu \nu} \left( \nabla_\nu \rho \left( \theta(\rho - \varepsilon_1) \delta(\varepsilon_2 - \rho) - \delta(\rho - \varepsilon_1) \theta(\varepsilon_2 - \rho) \right) \theta(\tau_2 - \tau) \theta(\tau - \tau_1) + \right.
\]

\[
+ \left. \theta(\tau_2 - \tau) \delta(\tau - \tau_1) - \delta(\tau_2 - \tau) \theta(\tau - \tau_1) \right) \left( \tau \right),
\]

where \( \tau_2, \tau_1, \varepsilon_2, \) and \( \varepsilon_1 \) are constants with \( \tau_2 > \tau_1 \) and \( \varepsilon_2 > \varepsilon_1 \). \( \theta(\rho - \varepsilon_1) \theta(\varepsilon_2 - \rho) \) defines the spacetime region between two coaxial cylindrical Bhabha tubes surrounding the electron world-line; for each fixed time they are reduced to two spherical surfaces centred at the charge. \( \theta(\tau_2 - \tau) \theta(\tau - \tau_1) \) defines the spacetime region between two light-cones of vertices at \( \tau_2 \) and \( \tau_1 \), respectively. They are necessary for using the Gauss’s theorem in the above intermediary step, as the product of these four Heaviside functions define a closed hypersurface. The terms in the second and third lines of \([10]\) are the flux rates of energy-momentum through the respective hypersurfaces \( \rho = \varepsilon_1, \rho = \varepsilon_2, \tau = \tau_2 \) and \( \tau = \tau_1 \).

Taking the \( \tau_2 \to \tau_1 \) limit we have

\[
m \ddot{a} - F_{\text{ext}}^\mu V_\nu = \lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to \infty} \int d^4x \Theta^{\mu \nu} \left( \nabla_\nu \rho \left( \theta(\rho - \varepsilon_1) \delta(\varepsilon_2 - \rho) - \delta(\rho - \varepsilon_1) \theta(\varepsilon_2 - \rho) \right) \delta(\tau - \tau_2) \right)
\]

Let us now apply \([31, 32]\) for finding \( \lim_{\rho \to 0} \int_{\tau_2}^{\tau_1} d^3x \Theta \nabla_\rho \delta(\rho - \varepsilon_1) \), which with the explicit use of retarded coordinates (see, for example, p. 20 of \([5]\), \( x = z + \rho K \), can be written as \( \lim_{\rho \to 0} \int_{\tau_2}^{\tau_1} \rho^2 d\rho d^2\Omega \Theta \nabla_\rho \delta(\rho - \varepsilon_1) \). In \([18]\), the definition of \( \Theta \), the second term is the trace of the first one and so we just have to consider this last one because the behaviour of its trace under this limiting process can then easily be inferred. So, as \( K = \frac{\dot{a}}{p} \), and \( \nabla_\rho = (KE - V) \) we have schematically, for the first term of \([18]\) in \( \rho^2 \Theta \nabla_\rho \),

\[
\lim_{\rho \to 0} \frac{N(R, \ldots, \rho^2)}{\rho^4} = \lim_{\rho \to 0} \frac{\rho^2 [K, W] [K, W] (KE - V)}{\rho^4} = \frac{[R, W], [R, W] (RE - V \rho)}{\rho^2}
\]

Then, comparing it with \([33]\) and \([36]\) we have

\[
A_0 = B_0 = [R, W] = [R, \dot{a} \rho + VE] = [R, \dot{a} \rho + V] + \mathcal{O}(R^3)
\]

\[
A_1 = B_1 = [-V, \dot{a} \rho + V] + [R, -\dot{a}E + a] + \mathcal{O}(R^2) = -[V, \dot{a} \rho] + \mathcal{O}(R^2);
\]

\[
A_2 = B_2 = -[a, V] + \mathcal{O}(R);
\]

\[
C_0 = RE - V \rho = R - V \rho + \mathcal{O}(R^3),
\]

\[
C_1 = -V - \dot{a} \rho + VE + \mathcal{O}(R^2) = -\dot{a} \rho + \mathcal{O}(R^2);
\]

\[
C_2 = a + \mathcal{O}(R).
\]

Therefore, for producing a possibly non null \( N_p \), according to \([36]\), \( a, c \) and \( p \) must be given by

\[
c = 2,
\]
\[ p - a = a - c = 2 \implies p = 6 > n = 5. \]

Or in a shorter way

\[ O[[R, W]] = O[RE - V\rho] = 2, \]

and then,

\[ 2O[[R, W]] + O[RE - V\rho] = 6 > n = 5. \]

Then, we conclude from (34) that \( N_p = 0 \),

\[ \lim_{\rho \to 0} \int_{\tau_2} d\rho \Theta \nabla \rho \delta(\rho - \varepsilon_1) = 0. \] (46)

The flux of energy and momentum rate of the electron self-field through the \((\rho = \varepsilon_1)\)-hypersurface in (41) is null at \( \varepsilon_1 = 0 \). This is a new result, a consequence of (30). In the standard approach the contribution from this term produces the problematic Schott term and a diverging expression, the electron bound-momentum which requires mass renormalization [8].

The RHS of (41) is then reduced to

\[ \lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to \infty} \int_{\tau_2} d\rho \Theta \nabla \rho \delta(\varepsilon_2 - \rho) = \lim_{\varepsilon_2 \to \infty} \int_{\tau_2} \rho^2 d\rho d^2\Omega \Theta_2 \nabla \rho \delta(\varepsilon_2 - \rho), \] (47)

as, with (27), only \( \Theta_2 \) from \( \Theta = \Theta_2 + \Theta_3 + \Theta_4 \) survives the passage \( \varepsilon_2 \to \infty \) in the above integral. But from (14), (20) and (26) we have that

\[ \rho^2 \Theta_2 \nabla \rho = -\rho^2 V \Theta_2 = \frac{1}{4\pi} K(\alpha^2 - \alpha_k^2). \] (48)

Then after the angular integration

\[ \frac{1}{4\pi} \int d^2\Omega K(\alpha^2 - \alpha_k^2) = \frac{2}{3} V^\mu \alpha^2, \] (49)

we have

\[ \lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to \infty} \int_{\tau_2} d\rho \Theta \nabla \rho \delta(\varepsilon_2 - \rho) = \frac{2}{3} V^\mu \alpha^2. \] (50)

The last passage is a well known (see, for example, page 111 of [1]) text-book result; \(-\frac{2}{3} V^\mu \alpha^2 \) is the Larmor term for the irradiated energy-momentum rate.

With (16) and (50) in (12) we could write the electron equation of motion as

\[ m\dot{\alpha}^\mu - F_{\mu\nu}^{ext} V_\nu = -\frac{2}{3} \alpha^2 V^\mu, \] (51)

but it is well known that this could not be a correct equation because it is not self-consistent: its LHS is orthogonal to \( V \),

\[ m\dot{\alpha}.V = 0 \quad \text{and} \quad V.F_{\mu\nu}^{ext}.V = 0, \] (52)

while its RHS is not,

\[ -\frac{2}{3} \alpha^2 V.V = \frac{2}{3} \alpha^2. \] (53)

This seems to be paradoxical until one has a clearer idea of what is happening. We must turn our attention to equation (12), where there is a subtle and very important distinction between its LHS and its RHS. Its LHS is entirely determined by the electron instantaneous position, \( z(\tau) \), while its RHS is determined by the sum of contributions from the electron self-field at all points of a spherical surface. In other words, the LHS is a description of some electron attributes localized at a point (the electron position) while the RHS is a description of the sum of some
electron-self-field attributes over a spherical surface. This distinction is missing in equation (51); it was deleted by the integration process. The LHS of (42) multiplied by $V$ is null, we know, because the force that drives the electron with the 4-velocity $V$ delivers a power $(m\mathbf{\dot{a}}_0 V^0)$ that is equal to the work per unit time realized by this force along the $\vec{V}$ direction ($m\mathbf{\dot{a}}_0 \vec{V}$) (this, we know, is the physical meaning of $m\mathbf{\dot{a}}_0 V = 0$). But this reasoning does not apply to the RHS of (42) multiplied by $V$ because the flux of radiated energy is through a spherical surface not along $V$ (except at $\rho = 0$, because of (30)); in order to make sense, as we are doing a balance of the flux-rate of energy, we have to add this flux-rate from each point of the integration domain. Based on considerations of symmetry one can anticipate that the final result must be null: to each point of a spherical hypersurface $\rho =$ const., $\tau = \tau_2$, that gives a non-null contribution there is another point giving an equal but with opposite sign contribution. The RHS of (41) for the flux of electromagnetic energy-momentum, through the walls of a Bhabha tube around the charge world-line, in the limit of $\rho \to 0$. In particular,

$$\nabla_\nu \Theta^{\mu\nu} = \frac{1}{4\pi} F_{\alpha\nu} F^{\alpha\nu} = \frac{1}{4\pi} F^{\alpha\nu} \Box A^\alpha, \tag{57}$$

and by direct calculation we find that

$$\Box A^\mu = \frac{K^2}{\rho^3} \left(3\rho E\mathbf{\dot{a}}^\mu + \rho^2 \mathbf{\dot{a}}^\mu + (3E^2 + \rho^2 \mathbf{\dot{a}}_0) V^\mu \right). \tag{58}$$

We see then that the integrand of the RHS of (53) is null for $\rho > 0$ as $K^2 = 0$. For simplicity we could then just have used $V$ instead of $X$ in (53), but see the next section for an alternative illuminating calculation. Therefore, we just have to verify that $\rho^2 V^\mu \nabla_\nu \Theta^{\mu\nu}$ is finite, or equivalently that $\rho^3 V^\mu \nabla_\nu \Theta^{\mu\nu} \big|_{\rho=0} = 0$. As

$$V^\mu F_{\nu}^{\mu\nu} = \frac{1}{\rho^3} (EK^\alpha - W^\alpha), \tag{59}$$

then

$$4\pi \rho^5 V^\mu \nabla_\nu \Theta^{\mu\nu} = -K^2 \left(2E\rho^2 \mathbf{\dot{a}}^2 + 3E(1 - E^2) + \rho^2 (\rho \mathbf{\dot{a}} \cdot \mathbf{\dot{a}} - E\mathbf{\dot{a}}_0) \right), \tag{60}$$
and
\[
\lim_{\rho \to 0} \rho^3 V_\mu \nabla_\nu \Theta^{\mu \nu} = \lim_{\rho \to 0} \frac{R^2 \left(2E\rho^2 \dot{a}^2 + 3E(1 - E^2) + \rho^2 (\dot{a}\dot{a} - E\dot{\alpha}_K)\right)}{\rho^4}.
\] (61)

Then,
\[
A_0 = R^2 \implies A_2 = 2 + O(R);
\]
\[
B_0 = 2E\rho^2 \dot{a}^2 + 3E(1 - E^2) + \rho^3 \dot{a}\dot{a} - \rho E\dot{\alpha}_R
\]
In $B_0$ all but $\rho^3 \dot{a}\dot{a}$ are terms of $\tau$-order 2, so
\[
B_0 = 2\rho^2 \dot{a}^2 - 6\dot{\alpha}_R - \rho \dot{\alpha}_R + O(R^3),
\]
\[
B_1 = -5(\rho \dot{a}^2 + \dot{\alpha}_R) + O(R^2),
\]
and
\[
B_2 = O(R).
\]
It is not necessary to go further. Therefore, according to (34) and (35), we have
\[
p-a=2
\]
and
\[
a=3,
\]
and then,
\[
p = 5 > n = 4.
\]
So, both sides of (55) are equally null and there is no contradiction. This is in agreement with the fact that due to (2.4) and to the antisymmetry of $F$,
\[
V_\mu \nabla_\nu \Theta^{\mu \nu} = \frac{1}{4\pi} V_\mu F^\mu_\alpha \nabla_\nu F^{\alpha \nu} = V_\mu F^\mu_\alpha J^\alpha = 0.
\]

VI. USING THE DIVERGENCE THEOREM

For the sake of a better understanding of $X$ in eq. (55) let us work out its RHS using the divergence theorem. Then we have for the RHS of (55):
\[
\lim_{\varepsilon_2 \to 0} \int_{\mathbb{R}^3} \left\{ \Theta^{\mu \nu} \nabla_\nu X_\mu \theta(\rho - \varepsilon_1)\theta(\varepsilon_2 - \rho) + X_\mu \Theta^{\mu \nu} \nabla_\nu \rho [\delta(\rho - \varepsilon_1)\theta(\varepsilon_2 - \rho) - \theta(\rho - \varepsilon_1)\delta(\varepsilon_2 - \rho)] \right\},
\] (62)

The explicit dependence on $\nabla_\nu X_\mu$ makes clear why we cannot just use $K$ instead of $X$ in (55): although $\lim_{\rho \to 0} K = V$, $\lim_{\rho \to 0} \nabla K \neq \nabla V = -K\dot{\alpha}$.

For working out the first term of (55) we need:
\[
\nabla_\mu K_\nu = \nabla_\mu \left( \frac{R_\nu}{\rho} \right) = \frac{\eta_\mu \nu + K_\nu V_\nu}{\rho} - \frac{K_\nu}{\rho} \nabla_\mu \rho,
\] (63)
\[
W.\nabla \rho = 0
\] (64)
\[
K.W = -1
\] (65)
\[ K \nabla \rho = 1 + K^2 E \quad (66) \]

\[ \Theta^{\mu \nu} \eta_{\mu \nu} = 0. \quad (67) \]

Then, from (19) and \( K^2 = 0 \) we have for the upper limit

\[ \lim_{\varepsilon_2 \to \infty} \int_{\varepsilon_2}^{\infty} d\varepsilon^3 \Theta^{\mu \nu} \nabla_\nu K_\mu = \lim_{\varepsilon_2 \to \infty} \int_{\varepsilon_2}^{\infty} \frac{d\rho}{\rho^3} = - \lim_{\varepsilon_2 \to \infty} \frac{1}{\varepsilon_2^2} = 0. \quad (68) \]

For the lower limit \( \nabla_\nu X_\mu = -K_\nu a_\mu \) and then, from (19),

\[ 4\pi \rho^4 K \Theta a = a_K(K^2 W^2 - 1) = a_K(K^2 + 1) + \rho^2 a_K K^2 (a^2 - a_K^2) - \rho a_K^2 K^2 \quad (69) \]

So,

\[ \lim_{\rho \to 0} 4\pi \int \rho^2 d\rho K \Theta a = - \frac{a_K(K^2 + 1)}{\rho} + a_K K^2 (a^2 - a_K^2) \rho - \rho a_K^2 K^2 \ln \rho = 0, \quad (70) \]

because

\[ \lim_{\rho \to 0} \frac{a_K(K^2 + 1)}{\rho} = \lim_{\rho \to 0} \frac{a_K(R^2 + \rho^2)}{\rho^4} = 0, \quad (71) \]

as

\[ O[a_K] + O[R^2 + \rho^2] = 2 + 4 > 4, \quad (72) \]

and

\[ \lim_{\rho \to 0} a_K K^2 (a^2 - a_K^2) \rho = 0. \quad (73) \]

For evaluating the limit of the last term of (70) we consider that

\[ O[K^2 a_K^2] = 2 = O[\rho^2] \quad (74) \]

to see that

\[ \lim_{\rho \to 0} a_K^2 K^2 \ln \rho \sim \lim_{\rho \to 0} \rho^2 \ln \rho = 0. \quad (75) \]

It is important to use the appropriate values of \( X \) to have consistent results. The use, for example, of \( X = V \) in the upper limit or of \( X = K \) in the lower limit would produce inconsistent results.

For the second term of (62) \( X = V \) and then we have from (19) that

\[ 4\pi \rho^4 V \Theta \nabla \rho = (1 + K^2 E)(W^2 + E) + \frac{\rho a_K}{2}(1 - K^2 W^2). \quad (76) \]

Therefore,

\[ \lim_{\rho \to 0} 4\pi \rho^2 V \Theta \nabla \rho = \lim_{\rho \to 0} \left( \frac{(\rho^2 + R^2 + R^2 a_R)(\rho^2 a^2 - a_R^2 - a_R)}{\rho^3} + \frac{a_R(\rho^2 + R^2 - R^2(\rho^2 a^2 - a_R^2 + 2a_R))}{\rho^4} \right) = 0, \quad (77) \]

because

\[ O[\rho^2 + R^2 + R^2 a_R^2] + O[\rho^2 a^2 - a_R^2] = 4 + 2 > 4 \quad (78) \]

and

\[ O[a_R] + O[\rho^2 + R^2] + a_R^2 = 2 + 4 > 3 \quad (79) \]
Again we only have consistent results if we use the correct values of X in its respective limiting situation.

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LIST OF FIGURE CAPTIONS

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2. Fig. 2.: Creation and annihilation of particles in classical physics.
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4. Fig. 4.: Classical picture of the fundamental quantum process.

FIG. 1. The Lienard-Wiechert solutions.
FIG. 2. Creation and annihilation of particles in classical physics.
FIG. 3. Double limiting process: $\rho \to 0$ along K and $\tau \to \tau_{\text{ret}}$.

FIG. 4. Classical picture of the fundamental quantum process.