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Electronic and photonic counting statistics as probes of non-equilibrium quantum dynamics

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Abstract

We analyse the full counting statistics (FCS) of photons flowing in and out of a microwave cavity coupled to a voltage-biased Josephson junction. Tunnelling of Cooper pairs generates a coherent flow of photons into the cavity whilst at the same time photons can also leak out incoherently. We use a very general unitary transformation method to demonstrate that there is a simple connection between the FCS of the charges and the photons in the long time limit, revealing that all the cumulants of the coherent and the incoherent processes match in that limit. We also explore some of the interesting features in the counting statistics of the charges and photons which arise from the strongly nonlinear dynamics of the system. These include very narrow distributions associated with the emergence of coherent transport and regimes where counting of either an odd or an even number of photons leaving the cavity can result in strongly non-classical cat states within the cavity.

1. Introduction

The radiation produced by the flow of charges through mesoscopic conductors [1–4] can be very different to that produced by a classical current [5–7], allowing them to be used to produce sources for non-classical light from the microwave up to the low terahertz regime [8–14]. A particular strength of these systems is the high degree of tunability that can be achieved through circuit design [13, 15] and they can thus be seen as providing a toolbox to explore charge-light interactions in the far from equilibrium regime. Particular progress has been achieved for cavity-coupled conductors such as Josephson junction (JJ) devices and semiconductor quantum dots, where the coupling was predominantly to a single cavity mode. Recent experiments were able to generate large non-equilibrium photon populations leading to strongly coupled charge and photon dynamics [16–19].

For systems consisting of a mesoscopic conductor coupled to a cavity mode, illustrated schematically in figure 1(a), charge flow generates a coherent photon flux entering the mode that is balanced by a leakage of radiation out into the wider electromagnetic environment and the mode itself can be thought of as a conductor, albeit a photonic one rather than an electrical one. From this perspective it is natural to ask how the counting statistics of the photons flowing in and out of the mode are related to the statistics of the charge current [20–22] and how they respond to the nonlinear quantum dynamics of the system. Indeed, measurements of photon statistics may reveal details of charge-charge correlations such as transitions between incoherent and coherent charge flow [21].

Another interesting question relates to the conditional evolution of the system. What aspects of the dynamics would change if it were possible to measure the photons leaking out of the cavity in real time? Although the full counting statistics (FCS) of photon emission, and even aspects of the conditioned dynamics were considered some time ago in the context of canonical quantum optical systems, such as the degenerate parametric amplifier [23], the prospect of actually being able to detect and count microwave photons leaking out of cavity in the near future [24–26], together with the wide range of nonlinear dynamics that such systems can access makes it worth returning to this question [27].
In this article we address these issues focussing on a specific cavity-coupled conductor, a JJ biased at sub-gap voltages, where the connection between the flow of an electrical current and the generation of photons is particularly simple and the coupled quantum dynamics especially rich \[2, 8, 9, 21, 28–35\]. A dc current flows at resonances where the energy available to tunnelling Cooper pairs matches that required to generate one or more cavity photons so that all of the energy from the voltage source is converted into photons \[2, 16\], unlike in other conductors \[17, 20\]. Elsewhere we have explored how the current noise is linked to fluctuations in the flux of photons leaving the cavity \[21\]. Here we go much further, using a very general unitary transformation method to show that the FCS of photons and charges are related in a strikingly simple way. From this it follows directly that whilst the counting statistics are the same in the limit of long times, their behaviour is very different for short times where coherences play an important role. For the JJ-cavity system specifically, we find that the counting statistics provide a rich source of information about the complex nonlinear dynamics of the system. Considering higher cumulants beyond the noise and the tails of the distribution function, corresponding to large deviations from the mean, allows insights into rare, atypical behaviour and the new dynamical domains thereby explored.

We also investigate the interesting states of the cavity that emerge when the dynamics is conditioned on measurements of the photons leaving the cavity.

This paper is organized as follows. In section 2 we introduce our theoretical model of the JJ-cavity system, after which we detail the unitary transformation method for connecting charge and photon FCS in section 3. Then in sections 4 and 5 we explore the behaviour of the counting statistics at resonances of the system where one or two photons are generated at a time, respectively. The conditioned dynamics of the system are discussed in section 6 and we conclude in section 7. Details about aspects of the calculations carried out are provided in appendix.

2. Josephson-cavity system

The model system we study consists of a JJ in series with an LC oscillator to which a (sub-gap) voltage bias, \(V\), is applied \[2, 16\]; a possible realization is shown in figure 1 (b). The oscillator is one of the modes of a high-\(Q\) superconducting microwave cavity which is assumed to be weakly coupled to a transmission line through which photons leak out of the system \[30\]. The Hamiltonian of the JJ-oscillator system takes the time-dependent form \[30, 31\]

\[
H = \hbar \omega_0 a^\dagger a - E_J \cos [\omega_J t + \Delta_0 (a + a^\dagger)],
\]

where \(a\) is the lowering operator for the oscillator which has frequency \(\omega_0\) = \(1/\sqrt{LC}\), \(E_J\) is the Josephson energy of the junction and \(\omega_J = 2eV/\hbar\) the Josephson frequency set by the bias voltage. The quantity \(\Delta_0 = (2e^2/h)^{1/4}(L/C)^{1/4}\) characterizes the strength of the quantum fluctuations in the oscillator; it measures the strength of the zero point fluctuations in the flux of the oscillator (in units of the flux quantum)\(^3\). While early experiments operated in the low impedance regime \(\Delta_0 \ll 1\) \[2, 16\], recent progress in circuit design allows for \(\Delta_0 \sim \mathcal{O}(1)\) \[13, 15\].

\(^3\)This treatment can also be generalized to include the effects of low frequency voltage fluctuations \[31\].
We will focus on situations where the Josephson frequency, \( \omega_J = 2eV / h \), is close to an integer, \( p \), times the oscillator frequency. In such cases, one can perform a rotating wave approximation and obtain the effective Hamiltonian \([30, 31]\) (in a frame rotating at frequency \( \omega_J / p \)):

\[
H_{\text{RWA}}^{(p)} = \hbar \delta^{(p)}a^\dagger a - \frac{\hbar}{2} [(-i)^p \tilde{E}_J / (a^\dagger a)] = \frac{\hbar}{2} (2\Delta_0 \sqrt{a^\dagger a}) (a^\dagger a)^{p/2},
\]

where the renormalized Josephson energy is defined as \( \tilde{E}_J = E_J e^{-\Delta_0 / 2} \), the detuning is given by \( \delta^{(p)} = \omega_0 - \omega_J / p \) and colons imply normal ordering. Physically, the RWA Hamiltonian describes the behaviour close to resonances where the transport of a single Cooper pair across the junction leads to the generation of \( p \) photons.

Including weak coupling between the oscillator and its surroundings \([30, 31]\) (i.e. the transmission line), assumed to be at zero temperature for simplicity\(^4\), we can write down a master equation for the oscillator \( \dot{\rho} = \mathcal{L}[\rho] \), with the Liouvillian

\[
\mathcal{L}[\rho] = -\frac{i}{\hbar} [H_{\text{RWA}}^{(p)} \rho] + \gamma \left( 2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a \right),
\]

where \( \gamma \) is the energy relaxation rate and the dissipative terms account for the irreversible loss of photons. However, if the photons leaving the cavity are measured by a detector over a time interval \([0, t] \) then we can define a probability for observing \( N \) photons through \( P(N, t) = \text{Tr}[\rho_N(t)] \) with \( \rho_N(t) \) the density operator conditioned on this measurement, related to the unconditioned density operator by \( \rho(t) = \sum_N \rho_N(t) \). The set of conditioned density operators then evolve according to \([20, 36-40]\)

\[
\dot{\rho}_N = -\frac{i}{\hbar} [H_{\text{RWA}}^{(p)} \rho_N] - \frac{\gamma}{2} (a^\dagger a \rho_N + \rho_N a^\dagger a) + \gamma a \rho_{N-1} a^\dagger,
\]

different members of the set linked by quantum jumps corresponding to the detection of a photon. Equivalently, we can define \( \rho_x = \sum_N e^{-\gamma N/2} \rho_N \) which evolves according to the single equation

\[
\dot{\rho}_x = \mathcal{L}_x \rho_x = -\frac{i}{\hbar} [H_{\text{RWA}}^{(p)} \rho_x] - \frac{\gamma}{2} (2 a^\dagger a \rho_x - a^\dagger a \rho_x - \rho_x a^\dagger a).
\]

In contrast to the photons leaking out of the cavity, the flow of Cooper pairs across the JJ is coherent, described by the current operator \([21]\)

\[
I(t) = \frac{2eE_J}{h} \sin[\omega_J t + \Delta_0 (a + a^\dagger)].
\]

Close to one of the \( p \)-photon resonances (and within the framework of the rotating wave approximation) we can identify the operator corresponding to just the dc current by moving to the rotating frame in which the density operator reaches a steady-state in the limit of long times. The dc current is given by the part of the transformed current operator which does not oscillate in time \([21]\),

\[
I_{CP}^{(p)} = \frac{i \rho_{-1} \tilde{E}_J}{\hbar} \left[ J_p (2 \Delta_0 \sqrt{a^\dagger a}) (a^\dagger a)^{p/2} + (-1)^{p-1} (a^\dagger a)^p \right].
\]

In this system the photons that leak out of the cavity are all generated through the work done by the voltage source, and hence (within the framework of the RWA and on-resonance) we must have the simple balance condition \( \hbar (\omega_J / p) \langle I_{CP} \rangle = V \langle I_{CP} \rangle \), where the average photon flux is simply given by \( \langle I_{ph} \rangle = \gamma \langle a^\dagger a \rangle \). It is readily shown that this condition on the averages is indeed met \([21, 41]\) and one can also go beyond averages to look at how fluctuations in the Cooper pair and photon current match up \([21]\), but in the following we make a much more general connection between the dynamics of charges and photons at the level of the FCS.

### 3. Counting statistics of photons and charges

The JJ-cavity system couples together the transport of Cooper pairs with the generation of photons. The coherent dynamics of Cooper pairs (described by the current operator \( I \)) travelling through the JJ is clearly linked to a coherent flow of photons in (and out) of the cavity, but there is also an incoherent flow of photons leaking out of the cavity due to dissipation (see figure 1). The cumulants of the incoherent and coherent photon flows (and hence also that of Cooper pairs) can be formally connected in the limit of large times by mapping from the counting statistics of one to the other via a unitary transformation method.

\(^4\) Including finite temperature effects and absorption of photons from the transmission line, it can be straightforwardly checked that the central results of this paper are not affected for \( k_B T \ll \hbar \omega_0 \) as realized in typical experimental situations. More specifically, the unitary-transformation method still applies and the relation between the FCS of charge and photons still holds exactly.
We start by considering the generating function
\[ F_{ph}(\chi, t) = \ln \left[ \sum_{N=0}^{\infty} P(N, t) e^{-i\chi N} \right], \tag{8} \]
defined in terms of the photon counting distribution \( P(N, t) \). The moment generating function is then given by
\[ e^{-F_{ph}(\chi, t)} = \text{Tr}\{\exp[\mathcal{L}_{\chi} t] \rho(0)\}, \tag{9} \]
where \( \rho(0) \) is the density matrix at the start of the counting, for which the steady-state of equation (3) is the most natural choice. The mean photon current and its moments follow from the cumulants
\[ \langle n(t) \rangle \rightarrow t\lambda(\chi) \text{ where } \lambda \text{ is the eigenvalue of the counting Liouvillian } \mathcal{L}_{\chi} \text{ with the least negative real part}. \]

Aiming to remove counting from the dissipative part of equation (5), we search for a suitable transformation of the (super)operators \( a_i^{(+)\dagger}/a_i^{(-)\dagger} \) acting from the left/right on an element of Liouville space. Clearly, a transformation of the form
\[ a \rightarrow \mathcal{U}(a) \mathcal{U}^{-1} = e^{i\chi/2} a \cdot a \xrightarrow{\mathcal{U}} e^{-i\chi/2} \cdot a; \quad a^{\dagger} \xrightarrow{\mathcal{U}} e^{i\chi/2} a^{\dagger}; \quad \cdot \xrightarrow{\mathcal{U}} e^{i\chi/2} \cdot a; \tag{11} \]
accomplishes the desired goal: it eliminates the counting factor from the dissipative part, while new counting factors appear in the coherent terms (see equations (13) and (14) below).

For long counting times \( F_{ph}(\chi, t \rightarrow \infty) \rightarrow t\lambda(\chi) \), where \( \lambda \) is the eigenvalue of the counting Liouvillian \( \mathcal{L}_{\chi} \) with the least negative real part.

Since the matrix \( \mathcal{U} \) is unitary, the eigenvalues of the Liouvillian remain unchanged by the transformation \( \mathcal{U} \). Such a transformation yields
\[ \mathcal{L}^{ph}_{\chi}(\rho) = \mathcal{U} \mathcal{L}_{\chi} \mathcal{U}^{-1}[\rho] = -\frac{i}{\hbar} (H_{\chi} \rho - \rho H_{-\chi}) + \frac{\eta}{2} (2\Delta a^{\dagger} - a^{\dagger} a - a a^{\dagger}), \tag{13} \]
where
\[ H_{\chi} = \hbar \delta^{(p)} a^{\dagger} a - \frac{(-1)^p \hbar}{2} (e^{-i\chi/2} a^{(+)\dagger})^p \cdot \left[ e^{i\chi/2} (a^{(-)\dagger})^p + (-1)^p e^{i\chi/2} (a^{(-)\dagger})^p \right] \frac{\mathcal{F}_p(2\Delta a^{\dagger} a^{(-)\dagger})^{p/2}}{(a^{(-)\dagger})^{p/2}}; \tag{14} \]
with the counting field \( \chi \) now appearing in the Hamiltonian terms. The transformed Liouvillian, \( \mathcal{L}^{ph}_{\chi} \), can be interpreted as describing a counting process in its own right. The counting field that appears is \( \eta = \eta \chi \) and it counts the transfer of packets of \( \rho \) photons \([40, 42-44, 46]\). We can also define the corresponding generating function
\[ F_{\chi}(\eta, t) \rightarrow \ln \left( \text{Tr}\{\exp[\mathcal{L}^{ph}_{\chi} t] \rho(0)\} \right), \tag{15} \]
from which a distribution, \( \tilde{P}(M, t) \), describing the coherent transfer of \( M \) packets of \( \rho \) photons can be constructed:
\[ \tilde{P}(M, t) = \frac{1}{4\pi} \int_{0}^{4\pi} \text{d} \eta e^{F_{\chi}(\eta, t)} e^{i\chi \eta}. \tag{16} \]

Note that the integration runs over \( 4\pi \) as \( F_{\chi}(\eta, t) \) is a function of \( \eta/2 \) and since the packages of \( \rho \) photons can be absorbed as well as emitted, \( M \) can take on both positive and negative integer values. We will explore the properties of \( \tilde{P}(M, t) \) and its connection to the incoherent counting distribution, \( P(N, t) \), through specific examples in the following sections.

While the two generating functions are not identical \(^6\),
\[ e^{F_{\chi}} = \text{Tr}\{\exp[\mathcal{L}_{\chi} \mathcal{U}^{-1} t] \rho(0)\} = \text{Tr}[\mathcal{U}^{-1} \exp[\mathcal{L}_{\chi} \mathcal{U}^{-1} t] \mathcal{U} \rho(0)] = e^{F_{\tilde{\chi}}}, \tag{17} \]
a connection between the two counting processes can be established by considering their behaviour in the limit of long times. In both cases it is dominated by the same eigenvalue \( \lambda \) since the two Liouvillians are related by unitary transformation. In fact the two generating functions only differ by a function of \( \chi \), which is independent of time and hence its contribution becomes irrelevant in the long time limit so that

\(^5\) Different ways of counting for coherent processes are used in the literature \([40, 47]\). Our approach relates to imagining a spin coupled to the current \([47, 48]\).

\(^6\) Note, that \( \mathcal{U} \) can not be cycled under the trace, which is taken in Hilbert space.
It therefore follows that in this limit the cumulants of the coherently transferred bunches (defined as derivatives of \( \mathcal{F}_{\text{ph}} \) with respect to \( \eta \)) obey the relation: \( \kappa_{\text{ph}} (\eta) = \kappa_{\text{ph}} (\eta) / \eta^p \).

Finally, we consider how the photon counting statistics relate to those of the charge (i.e. Cooper pairs). We can write down a current operator for the coherent transfer of \( p \)-photons

\[
I_{\text{ph}} = -\frac{i}{\hbar} \left[ H_{\text{RWA}}^{(p)}, a^d / p \right] = \frac{i e^{-i E_J} \hbar}{2 \Delta_0} \int_0^\infty \frac{dt}{\sqrt{t}} \left[ \left| a^d \right|^p + (-1)^p \left( a^d \right)^p \right].
\]

However, at the \( p \)-photon resonance the transfer of each packet of \( p \) photons is associated with that of a single Cooper pair and indeed, apart from the factor of \( 2e \), the operator given by equation (20) is exactly the same as the one describing the dc Cooper-pair current, equation (7). We thus see immediately that, in the rotating frame and within the RWA, we have the operator relation \( I_{\text{cp}} = 2eI_{\text{ph}} \). Hence in the long time limit, the FCS of Cooper pairs transferred through the JJ will match that of the photons leaving the cavity, up to a scaling factor set by \( p \).

### 4. One-Photon Resonance

We now look in detail at how the counting statistics reflect the nonlinear quantum dynamics of the system, focussing in the first instance on the one photon resonance \( (p = 1) \). Figure 2(a) compares the evolutions of \( P(N, t) \) and \( \tilde{P}(N, t) \) over time (for details of the calculation see appendix). Although these distributions must become identical in the limit of long times, the short time behaviour is radically different. This is hardly surprising as \( \tilde{P}(N, t) \) is a quasiprobability distribution [42, 49], reflecting the coherence of the photon transfer from junction to cavity: not only can photons both enter \( (N > 0) \) and leave the system \( (N < 0) \), the precise number that have done so is ambiguous because of quantum coherences. This essential ambiguity is signalled by the negativity which is present in \( \tilde{P}(N, t) \) for short times. It is dissipation which, over time, sets the precise number that have done so is ambiguous because of quantum coherences. This essential ambiguity is signalled by the negativity which is present in \( \tilde{P}(N, t) \) for short times. It is dissipation which, over time, sets the long time limit, the FCS of Cooper pairs transferred through the JJ will match that of the photons leaving the cavity, up to a scaling factor set by \( p \).

For very low \( E_J \) values, the Hamiltonian (2) can be linearized with respect to the photon number and the system reduces to an oscillator which is damped and driven linearly, leading to a steady-state which is a pure coherent state and hence \( P(N, t) \) is Poissonian in this limit. However, the dynamics of the system becomes strongly nonlinear as \( E_J \) is increased and this is reflected in changes in the shape of the counting distribution, as shown in figure 2(b). Earlier studies at the level of first and second moments of the charge and photonic currents [21, 30] showed that the current noise is suppressed as the system approaches a bifurcation (at \( E_J = E_J^b = h/e \gamma \Delta_0 \) with \( \Delta_0 \Delta_0^2 \approx 1.841 \)), though it rises abruptly at the bifurcation, before slowly dropping away again for even larger \( E_J \). In terms of the
full distribution, this behaviour manifests itself as a marked narrowing below the bifurcation and a broadening beyond the bifurcation. The large deviation function for the photon statistics, figure 2(c), shows clearly that the behaviour remains strongly non-Poissonian over the whole range of $E_f$ studied. This is, in particular, also the case for the limit of very strong driving, where considering only the second cumulant resulting in a Poissonian character. The complete FCS, however, shows that large deviations are much more probable and truly Poissonian emission is only realized in the limit of vanishingly weak driving strength, $E_f \to 0$.

5. Two-photon resonance

We now turn to the case where each tunnelling Cooper-pair generates two photons ($p = 2$), although the photons still leak out of the cavity one at a time. In this case coherent counting gives rise to a distribution of pairs of photons, whereas incoherent counting leads to one for individual photons, albeit with a clear asymmetry between even and odd photon counting numbers (see figure 3(a)). This asymmetry is easiest to understand in the limit of very low $E_f$, where pairs of photons are excited within the cavity only very rarely [28, 50]. In this regime the time between excitation events is much longer than the typical time for a photon to leak out of the cavity (set by $1/\gamma$) and both photons are likely to have leaked out of the cavity well before the next pair is created. Hence for long times the probability of an even count is much larger than for an odd one (approximately by a factor of $1/n$) with steady-state cavity occupation $\langle n \rangle$ [50]. This asymmetry between even and odd counts persists at larger $E_f$ values though it becomes weaker.

In the limit of very low $E_f$, the on-resonance system Hamiltonian is approximately that of a (sub-threshold) degenerate parametric oscillator (DPO)

\[ H_{\text{DPO}}^{(2)} \approx \frac{E_f}{4} \frac{\Delta_0}{2} (|a|^2 + a^2), \]

for which the FCS of the photons has been derived analytically [28, 50]. In the quadratic approximation the system is unstable above a threshold at $E_f^{(2)} = \hbar \gamma e^{\Delta_0^2/2}/\Delta_0^2$, and the behaviour is controlled by the ratio $E_f/E_f^{(2)}$, with the average photon current diverging as it approaches unity from below. The higher-order terms in the full $p = 2$ Hamiltonian (2) remove the divergences at the threshold and determine the behaviour at larger $E_f$ values. Saturation of the cavity photon number occurs at a level that scales as $\Delta_0^2$ after a second bifurcation at $E_f^{(2)} = \hbar \gamma^2 \frac{e^{\Delta_0^2/2}}{4(\Delta_0^2)^2} \approx 2.4E_f^{(2)}$ with $z_2 = 3.054$ [30].

In the below threshold regime the counting statistics are strongly super-Poissonian [8] (see figure 3(b)). However, the higher order nonlinearities act to suppress the fluctuations below the level predicted by the

\[ \text{Notice that the relative differences between odd and even counting numbers in } P(\mathcal{N}, t) \text{ saturate over time and hence do not feature in the plots of the large deviation function, } \ln[P(\mathcal{N}, t)/\mathcal{N}], \text{ for the limit of large times.} \]
quadratic approximation \([28, 50]\) (i.e. equation \((21)\)), an effect which grows progressively with increasing \(\Delta_0\). Above threshold, \(E_J/E_J^b > 1\), the behaviour of the counting statistics is very different, showing features similar to those seen at the one-photon resonance. In particular, a regime of coherent transport characterized by sub-Poissonian counting statistics emerges once the system is above the threshold, as illustrated in figure 3(b), though it comes to an end as the bifurcation at \(E_J^b\) is approached.

6. Conditioned states

If a certain number of photons, \(N\), is detected leaking out of the cavity over time \(t\), then immediately afterwards the state of the system will be \(\rho_N(t)\) which can be very different from the steady state of the system. In this section we examine how these measurement conditioned states vary with the count number and how they differ between the one- and two-photon resonances.

Figure 4 shows examples of Wigner functions conditioned on counting different numbers of (incoherent) photons together with the Wigner functions of the corresponding steady state for both the one- and two-photon resonance. In the case of the one-photon resonance, conditioning on a count number \(N \neq \langle N \rangle\) leads to an effect which is rather like changing the value of the pump rate, \(E_J\). In particular, as illustrated in figures 4(a)–(c), for the case where the steady state of the system is above the bifurcation conditioning on \(N > \langle N \rangle\) can lead to Wigner functions that are similar to a steady-state Wigner function obtained for a lower pump rate, where the system is below the bifurcation (see \([21]\) for the \(E_J\) dependence of the cavity state). In contrast, conditioning on a count \(N < \langle N \rangle\) leads to Wigner functions that are similar to steady-states for larger pump rates. Related behaviour has been widely studied in the context of biased trajectories, where one aims to make rare dynamics of a given physical system accessible by engineering an alternative system, in which the desired dynamics appear as the typical trajectories. For simple systems a formal mapping \([51]\) yields a constructive way to find such an alternative system. The observed similarity of conditioned Wigner functions to (unconditioned) steady-states at other parameters suggests that a relationship between rare behaviour in a

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**Figure 4.** Wigner functions for steady states and conditioned evolution of the system at the one (a)–(c) and two (d)–(f) photon resonances respectively. (a) is the steady Wigner function for \(E_J/E_J^b = 1.6\) and \(\Delta_0 = 0.5\) at the one-photon resonance. Corresponding Wigner functions for states conditioned on counting \(N = \langle N \rangle / 5\) and \(N = 2\langle N \rangle\) in a time \(\gamma t = 100\) are shown in (b) and (c) respectively. The Wigner function after observation of atypically few/many photons resembles the steady-state for a stronger/weaker driving. (d) is the steady state at the two-photon resonance whilst (e) and (f) are Wigner functions for states conditioned on either odd or even counts after time \(\gamma t = 100\). These observations yield a cavity state showing phase-space interference patterns familiar from even/odd cat states, see also the line cuts in the inset. In this case \(E_J/E_J^b = 1.6\) and \(\Delta_0 = 0.8\).
system with one pumping rate and the typical behaviour at another pumping rate exists in our system, though we have not demonstrated any formal mapping.

For the two-photon case there is an important difference between conditioning on odd and even photon counts. Just above threshold the steady state Wigner function is everywhere positive and consists of two peaks equally spaced from the origin oriented along the squeezing direction of the Hamiltonian (21), see figure 4(d). However, Wigner functions for the conditioned density operators for either even ($\rho_{\text{even}} = \sum \rho_{\text{even}}\rho_N$) or odd ($\rho_{\text{odd}} = \sum \rho_{\text{odd}}\rho_N$) shown in figures 4(e) and (f), resemble Schrödinger cat states displaying a series of interference fringes between the main peaks, together with negative regions. This behaviour is very similar to that predicted by Carmichael for the DPO when pump depletion is included to stabilize the state above threshold which yields an effective two-photon damping [23], whilst non-classical effects resulting from conditioning in a related nonlinear oscillator system were explored elsewhere very recently [27].

The emergence of cat states under conditioning for either odd or even counting numbers is not completely unexpected since the system is driven by a coherent two-photon processes. Oscillators that are driven and damped via mechanisms which only ever involve two-photon processes can evolve naturally into Schrödinger cat states, provided they start in states with either odd or even number-state parity [52–54]. Unfortunately, single photon losses (like the cavity loss mechanism in our system) mix together the different cat states that correspond to odd and even parity, washing out the non-classical features. However, conditioning on an odd or even number of decays acts to stabilise the system within the odd or even subspace, respectively, and hence the main features of the corresponding cat states survive (see figures 4(e) and (f)). Although only very weak negativity appears in the parity conditioned Wigner functions, this can be enhanced significantly by also conditioning on rare events. Note, that the Josephson–cavity platform also allows probing similar effects for higher $p$ resonances and the corresponding higher-order cat states.

7. Conclusions

Using the concrete example of a JJ coupled to a cavity, we have shown that a simple relation emerges in the long time limit between the FCS of coherent and incoherent flows of photons into and out of the cavity. Furthermore, since the charge statistics are linked to coherent driving of the cavity, the same approach can be used to link the statistics of the charge flow to that of the photons.

We find that the underlying nonlinear quantum dynamics of the cavity-JJ system leads to the emergence of novel regimes of coherent transport of Cooper pairs and photons, signalled, e.g., by counting distributions which are significantly narrower than for a Poissonian process. The FCS, furthermore, provides more complete information on those aspects of the system’s dynamics, which differ strongly from the average behaviour. We also found that states of the system conditioned on the photon count could have interesting properties, especially at the two-photon resonance where conditioning on even or odd counts can lead to the emergence of strongly non-classical states. We hope that our results can be tested experimentally in the near future, given very recent progress in inferring discrete microwave statistics using continuous measurements [26] as well as the development of counters for individual microwave quanta [24, 25].

Looking beyond the specific JJ–cavity device we focussed on here, the unitary transformation method we introduced for relating different forms of counting statistics is quite general and we expect that it will prove useful in a variety of other contexts. Our approach could be employed to generalize earlier work relating coherent and incoherent electrical current fluctuations [55]; it could also be used to link coherent and incoherent approaches to the photon counting statistics of microwave cavities [49, 56]. Furthermore, the relationship between charge and photon FCS is also interesting in more complex circuit architectures, such as quantum heat engines in which voltage-biased JJs form the key working elements [41], and might provide important information about the relationship between the statistics of heat flow and work done [57]. Our work may also inspire investigations into how the counting statistics of different entities are linked, not just for cavity-conductor set-ups, but also optomechanical devices [58] and other hybrid systems.

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Appendix. Numerical calculation of photon counting distributions

In figures 2 and 3 of the main text we show numerical results for the distributions $P(N, t)$ and $\hat{P}(M, t)$ for the one- and two-photon $(p = 1, 2)$ resonances. These are defined from the corresponding generating functions through equations (8) and (16), respectively, via inversion of the Fourier transformation. The generating functions for counting either coherent photon flow or the photonic leakage from the cavity follow in turn from time evolution with the corresponding Liouvillians (equations (3) and (13) of the main text), which contain the counting field in the coherent or the dissipative part of the Liouvillian.

Numerically, we obtain the quasiprobability distribution $\hat{P}(M, t)$, using the following method: $F_{cph}(p\chi, t)$ for a fixed time $t$ is calculated from time-evolution with $\mathcal{L}^{cph}_\chi$, where a sufficient number of $p\chi \in [0, 4\pi]$ values are sampled from the full period of $F_{cph}(p\chi, t)$ (which is a function of $p\chi/2$). Then the Fourier-integral is numerically evaluated for various values of $M$. The probability distribution $P(N, t)$ is calculated in an analogous manner (with the time-evolution now governed by $\mathcal{L}_\chi$ instead of $\mathcal{L}^{cph}_\chi$). For larger times we used an alternative method based on the $N$-resolved density matrix approach [20, 36], where a set of density matrices $\rho_N(t)$ representing the density matrix of the system after $N$ photons have left the cavity is evolved according to

$$\dot{\rho}_N(t) = \mathcal{L}^{\text{det}}\rho_N(t) + \mathcal{J}[\rho_{N-1}(t)], \quad (A.1)$$

where $\mathcal{J} = \gamma a^\dagger a$ and $\mathcal{L}^{\text{det}} = \mathcal{L} - \mathcal{J}$. The probability distribution then simply follows as $P(N, t) = \text{Tr}\{\rho_N(t)\}$.

The conditioned density matrices visualized in figure 4 follow directly from this $N$-resolved approach. Note that a linkage between FCS and large-deviation function via Legendre transformation can also be exploited.

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