A semi-classical study of the Jaynes–Cummings model

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Abstract. We consider the Jaynes–Cummings model of a single quantum spin $s$ coupled to a harmonic oscillator in a parameter regime where the underlying classical dynamics exhibits an unstable equilibrium point. This state of the model is relevant to the physics of cold atom systems, in non-equilibrium situations obtained by fast sweeping through a Feshbach resonance. We show that in this integrable system with two degrees of freedom, for any initial condition close to the unstable point, the classical dynamics is controlled by a singularity of the focus–focus type. In particular, it displays the expected monodromy, which forbids the existence of global action-angle coordinates. Explicit calculations of the joint spectrum of conserved quantities reveal the monodromy at the quantum level, as a dislocation in the lattice of eigenvalues. We perform a detailed semi-classical analysis of the associated eigenstates. Whereas most of the levels are well described by the usual Bohr–Sommerfeld quantization rules, properly adapted to polar coordinates, we show how these rules are modified in the vicinity of the critical level. The spectral decomposition of the classically unstable state is computed, and is found to be dominated by the critical WKB states. This provides a useful tool with which to analyze the quantum dynamics starting from this particular state, which exhibits an aperiodic sequence of solitonic pulses with a rather well defined characteristic frequency.

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1. Introduction

The Jaynes–Cummings model was originally introduced to describe the near resonant interaction between a two-level atom and a quantized mode of the electromagnetic field [1]. When the field is treated classically, the populations of the two levels exhibit periodic Rabi oscillations whose frequency is proportional to the field intensity. The full quantum treatment shows that the possible oscillation frequencies are quantized, and are determined by the total photon number stored in the mode. For an initial coherent state of the field, residual quantum fluctuations in the photon number lead to a gradual blurring and a subsequent collapse of the Rabi oscillations after a finite time. On even longer timescales, the model predicts a revival of the oscillations, followed later by a second collapse, and so on. These collapses and revivals have been analyzed in detail by Narozhny et al [3], building on the exact solution of the Heisenberg equations of motion given by Ackerhalt and Rzazewski [2]. Such complex time evolution has been observed experimentally with the Rydberg micromaser [4]. More recently, direct experimental evidence that the possible oscillation frequencies are quantized has been achieved with Rydberg atoms interacting with the small coherent fields stored in a cavity with a large quality factor [5]. Another interesting feature of the Jaynes–Cummings model is that it provides a way to prepare
the field in a linear superposition of coherent states [6]. For the non-resonant case, closely related ideas were used to measure experimentally the decoherence of Schrödinger cat states of the field in a cavity [7].

In the present paper, we shall consider the generalization of the Jaynes–Cummings model where the two-level atom is replaced by a single spin $s$. One motivation for this is from the phenomenon of superradiance, where a population of identical two-level atoms interacts coherently with the quantized electromagnetic field. As shown by Dicke [8], this phenomenon can be viewed as the result of a cooperative behavior, where individual atomic dipoles build up to make a macroscopic effective spin. In the large $s$ limit, and for most initial conditions, a semi-classical approach is quite reliable. However, the corresponding classical Hamiltonian system with two degrees of freedom is known to exhibit an unstable equilibrium point, for a large region in its parameter space. As shown by Bonifacio and Preparata [9], the subsequent evolution of the system, starting from such a state, is dominated by quantum fluctuations as it would be for a quantum pendulum initially prepared with the highest possible potential energy. These authors have found that, at short times, the evolution of the system is almost periodic, with solitonic pulses of photons separating quieter time intervals where most of the energy is stored in the macroscopic spin. Because the stationary states in the quantum system have eigenenergies that are not strictly equidistant, this quasiperiodic behavior gives way, at longer times, to a rather complicated pattern, that is reminiscent of the collapses and revivals in the $S = 1/2$ Jaynes–Cummings model.

An additional motivation for studying the large $s$ Jaynes–Cummings model comes from recent developments in cold atom physics. It has been shown that the sign and the strength of the two-body interaction can be tuned at will in the vicinity of a Feshbach resonance, and this has enabled various groups to explore the whole crossover from the Bose–Einstein condensation of tightly bound molecules [10] to the BCS condensate of weakly bound atomic pairs [11, 12]. In the case of a fast sweeping of the external magnetic field through a Feshbach resonance, some coherent macroscopic oscillations in the molecular condensate population have been predicted theoretically [13], from a description of the low energy dynamics in terms of a collection of $N$ spins $1/2$ coupled to a single harmonic oscillator. This model has been shown to be integrable [14] by Yuzbashyan et al who emphasized its connection with the original integrable Gaudin model [15]. It turns out that in the crossover region, the free Fermi sea is unstable towards the formation of a pair condensate, and that this instability is manifested by the appearance of two pairs of conjugated complex frequencies in a linear analysis. This shows that for any value of $N$ (which is also half of the total number of atoms), these coherent oscillations are well captured by an effective model with two degrees of freedom, one spin, and one oscillator. Therefore, there is a close connection between the quantum dynamics in the neighborhood of the classical unstable point of the Jaynes–Cummings model, and the evolution of a cold Fermi gas after an attractive interaction has been switched on suddenly.

The problem of quantizing a classical system in the vicinity of an unstable equilibrium point has been a subject of recent interest, especially in the mathematical community. The Bohr–Sommerfeld quantization principle suggests that the density of states exhibits, in the $\hbar$ going to zero limit, some singularity at the energy of the classical equilibrium point, in close analogy to the Van Hove singularities for the energy spectrum of a quantum particle moving in a periodic potential. This intuitive expectation has been confirmed by

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rigorous [16]–[18] and numerical [19] studies. In particular, for a critical level of a system with one degree of freedom, there are typically $\ln h$ eigenvalues in an energy interval of width proportional to $h$ around the critical value [17,18]. A phase space analysis of the corresponding wavefunctions shows that they are concentrated along the classical unstable orbits which leave the critical point in the remote past and return to it in the remote future [16]. To find the eigenstates requires an extension of the usual Bohr–Sommerfeld rules, because matching the components of the wavefunction which propagate towards the critical point or away from it is a special fully quantum problem, which can be solved by reduction to a normal form [16,20].

Finally, we will show that the spin $s$ Jaynes–Cummings model is an example of an integrable system for which it is impossible to define global action-angle coordinates. By the Arnold–Liouville theorem, classical integrability implies that phase space is foliated by $n$-dimensional invariant tori. Angle coordinates are introduced by constructing $n$ independent periodic Hamiltonian flows on these tori. Hence each invariant torus is equipped with a lattice of symplectic translations which act as the identity on this torus, the lattice of periods of these flows, which is equivalent to the data of $n$ independent cycles on the torus. To get the angles we still must choose an origin on these cycles. All this can be done in a continuous way for close enough nearby tori showing the existence of local action-angle variables. Globally, a first obstruction can come from the impossibility of choosing an origin on each torus in a consistent way. In the case of the Jaynes–Cummings model, this obstruction is absent (see [21], page 702). The second obstruction to the existence of global action-angle variables comes from the impossibility of choosing a basis of cycles on the tori in a uniform way. More precisely, along each curve in the manifold of regular invariant tori, the lattice associated with each torus can be followed by continuity. This adiabatic process, when carried along a closed loop, induces an automorphism on the lattice attached to initial (and final) torus, which is called the monodromy [21]. Several simple dynamical systems, including the spherical pendulum [21] or the champagne bottle potential [22], have been shown to exhibit such phenomenon. After quantization, classical monodromy induces topological defects such as dislocations in the lattice of common eigenvalues of the mutually commuting conserved operators [23]. Interesting applications have been found, especially in molecular physics [24]. In the Jaynes–Cummings model, the monodromy is directly associated with the unstable critical point, because it belongs to a singular invariant manifold, namely a pinched torus. This implies that the set of regular tori is not simply connected, which allows for a non-trivial monodromy, of the so-called focus–focus type [25]–[27]. The quantization of a generic system with such a singularity has been studied in detail by Vũ Ngọc [28]. Here, we present an explicit semi-classical analysis of the common energy and angular momentum eigenstates in the vicinity of their critical values, which illustrates all the concepts just mentioned.

In section 2 we introduce the classical Jaynes–Cummings model and describe its stationary points, stable and unstable. We then explain and compute the classical monodromy when we loop around the unstable point. We also introduce a reduced system that will be important in subsequent considerations. In section 3 we define the quantum model and explain that the appearance of a default in the joint spectrum of the two commuting quantities is directly related to the monodromy phenomenon in the classical theory. In section 4 we perform the above reduction directly on the quantum system and study its Bohr–Sommerfeld quantization. The reduction procedure favors some natural
coordinates which however introduce some subtleties in the semi-classical quantization: there is a subprincipal symbol and moreover the integral of this subprincipal symbol on certain trajectories may have unexpected jumps. We explain this phenomenon. The result is that the Bohr–Sommerfeld quantization works very well everywhere, except for energies around the critical one. In section 5 we therefore turn to the semi-classical analysis of the system around the unstable stationary point. We derive the singular Bohr–Sommerfeld rules. Finally in section 6 we apply the previous results to the calculation of the time evolution of the molecule formation rate, starting from the unstable state. For this we need the decomposition of the unstable state on the eigenstates basis. We find that only a few states contribute and there is a drastic reduction of the dimensionality of the relevant Hilbert space. We remark that to compute these coefficients, it is enough to solve the time-dependent Schrödinger equation for small time. We perform this analysis and we show that we can extend it by gluing it to the time-dependent WKB wavefunction. This gives new insights on the old result of Bonifacio and Preparata.

2. The classical one-spin system

2.1. Stationary points and their stability

Hence, we consider the following Hamiltonian:

\[ H = 2\epsilon s^z + \omega \bar{b}b + g(bs^- + bs^+) \]  

Here \( s^z = s^3, s^\pm = s^1 \pm is^2 \) are spin variables, and \( b, \bar{b} \) describe a harmonic oscillator. The Poisson brackets read

\[ \{s^a, s^b\} = -\epsilon_{abc} s^c, \quad \{b, \bar{b}\} = i. \]  

Changing the sign of \( g \) amounts to making the change \((b, \bar{b}) \rightarrow (-b, -\bar{b})\) which is a symplectic transformation. Rescaling the time we can assume \( g = 1 \).

The Poisson bracket of the spin variables is degenerate. To obtain a symplectic manifold, we fix the value of the Casimir function

\[ \vec{s} \cdot \vec{s} = (s^z)^2 + s^+ s^- = s_{cl}^2 \]  

so that the corresponding phase space becomes the product of a sphere with a plane and has total dimension 4. Let us write

\[ H = H_0 + \omega H_1 \]

with

\[ H_1 = \bar{b}b + s^z, \quad H_0 = 2\kappa s^z + \bar{b}s^- + bs^+, \quad \kappa = \epsilon - \omega/2. \]  

Clearly we have

\[ \{H_0, H_1\} = 0 \]

so the system is integrable\(^1\). Hence we can solve simultaneously the evolution equations

\[ \partial_0 f = \{H_0, f\}, \quad \partial_1 f = \{H_1, f\}. \]

\(^1\) This remains true for the \(N\)-spin system. In the integrability community this model is known to be a limiting case of the Gaudin model [14].

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In physical terms, the conservation of $H_1$ is given by $\delta s$ according to whether the spin is up or down. Then configuration. We assume that $\epsilon<0$ so that among these two configurations, the one with minimal energy is the spin up. It looks like being a minimum; however it becomes unstable for some values of the parameters. To see this, we perform the analysis of the small fluctuations around this critical points (or stationary points, i.e. all time derivatives equal zero) of both time evolutions $t_0$ and $t_1$ and hence are very special. Any Hamiltonian, a function of $H_0$ and $H_1$, will have these two points among its critical points. However, it may have more.

For instance when $\omega = H_0 + \omega H_1$ an additional family of stationary points exist when $\omega^2 > s_{cl}^2$. They are given by $s^\pm = \sqrt{s_{cl}^2 - \epsilon^2 \omega^2 e^{\pm i \varphi}}$, $s^z = -\epsilon \omega$, $b = -\frac{1}{\omega} \sqrt{s_{cl}^2 - \epsilon^2 \omega^2 e^{-i \varphi}} \forall \varphi$ and their energy is given by $E' = -\frac{1}{\omega} (s_{cl}^2 + \omega^2 \epsilon^2)$.

The energies of the configurations equation (9) are $E = \pm 2 \epsilon s_{cl}$ and so $E - E' = \frac{1}{\omega} (s_{cl} \pm \epsilon \omega)^2 > 0$ and we see that $E'$ represents the degenerate ground states of $H$, breaking rotational invariance around the $z$ axis.

We are mostly interested in the configurations of equation (9). To fix ideas we assume $\epsilon<0$ so that among these two configurations, the one with minimal energy is the spin up. It looks like being a minimum; however it becomes unstable for some values of the parameters. To see this, we perform the analysis of the small fluctuations around this configuration. We assume that $b, \bar{b}, s^\pm$ are first order and $s^z = s_{cl} e + \delta s^z$ where $e = \pm 1$ according to whether the spin is up or down. Then $\delta s^z$ is determined by saying that the spin has length $s_{cl}$:

$\delta s^z = -\frac{\epsilon}{2s_{cl}} s^- s^+$. 

Once $f(t_0, t_1)$ is known, the solution of the equation of motion $\partial_t f = \{H, f\} = (\partial_t + \omega \partial_{t_1}) f$ is given by $f(t, \omega t)$. The equations of motion read

\begin{align*}
\partial_t b &= -ib \\
\partial_t s^z &= 0 \\
\partial_t s^+ &= i(bs^- - bs^+) \\
\partial_t s^- &= -is^-
\end{align*}

The $t_1$ evolution is simply a simultaneous rotation around the $z$ axis of the spin and a rotation of the same angle of the harmonic oscillator part:

$b(t_0, t_1) = e^{-it_1 \bar{b}(t_0)}$, $s^\pm(t_0, t_1) = e^{\pm i t_1} s^\pm(t_0)$, $s^z(t_0, t_1) = s^z(t_0)$.

In physical terms, the conservation of $H_1$ corresponds to the conservation of the total number of particles in the system, when this model (with $N$ quantum spins $1/2$) is used to describe the coherent dynamics between a Fermi gas and a condensate of molecules [13].

We wrote the full equations of motion to emphasize the fact that the points are all the critical points (or stationary points, i.e. all time derivatives equal zero) of both time evolutions $t_0$ and $t_1$ and hence are very special. Any Hamiltonian, a function of $H_0$ and $H_1$, will have these two points among its critical points. However, it may have more.

For instance when $H = H_0 + \omega H_1$ an additional family of stationary points exist when $\omega^2 < s_{cl}^2$. They are given by $s^\pm = \sqrt{s_{cl}^2 - \epsilon^2 \omega^2 e^{\pm i \varphi}}$, $s^z = -\epsilon \omega$, $b = -\frac{1}{\omega} \sqrt{s_{cl}^2 - \epsilon^2 \omega^2 e^{-i \varphi}} \forall \varphi$ and their energy is given by $E' = -\frac{1}{\omega} (s_{cl}^2 + \omega^2 \epsilon^2)$.

The energies of the configurations equation (9) are $E = \pm 2 \epsilon s_{cl}$ and so $E - E' = \frac{1}{\omega} (s_{cl} \pm \epsilon \omega)^2 > 0$ and we see that $E'$ represents the degenerate ground states of $H$, breaking rotational invariance around the $z$ axis.
This is of second order and is compatible with equation (5). The linearized equations of motion (with respect to \( H \)) are

\[
\dot{b} = -i\omega b - is^- \tag{11}
\]

\[
\dot{s}^- = -2i\epsilon s^- + 2is_cl^\epsilon b \tag{12}
\]

and their complex conjugate. We look for eigenmodes in the form

\[
b(t) = b(0)e^{-2iEt}, \quad s^- = s^-(0)e^{-2iEt}.
\]

We get from equation (12)

\[
s^-(0) = -s_cl\frac{e}{E - \epsilon}b(0).
\]

Inserting into equation (11), we obtain the self-consistency equation for \( E \):

\[
E = \frac{\omega}{2} - \frac{s_cl}{2} \frac{e}{E - \epsilon}. \tag{13}
\]

The discriminant of this second-degree equation for \( E \) is \( \kappa^2 - 2s_cl\epsilon \). If \( \epsilon = -1 \) it is positive and we have a local energy maximum. However if \( \epsilon = +1 \), the roots become complex when \( \kappa^2 \leq 2s_cl \). In that case one of the modes exponentially increases with time, i.e. the point is unstable. Since we are in a situation where \( \kappa \leq 0 \), the transition occurs when \( \kappa = -\sqrt{2s_cl} \).

The level set of the unstable point, i.e. the set of points in the four-dimensional phase space of the spin-oscillator system, which have the same values of \( H_0 \) and \( H_1 \) as the critical point \( H_0 = 2\kappa s_cl, H_1 = s_cl \), has the topology of a pinched two-dimensional torus; see figure 1. This type of stationary point is known in the mathematical literature as a focus–focus singularity [25]–[27]. The above perturbation analysis shows that in the immediate vicinity of the critical point, the pinched torus has the shape of two cones that meet precisely at the critical point. One of these cones is associated with the unstable small perturbations, namely those which are exponentially amplified, whereas the other cone corresponds to perturbations which are exponentially attenuated. Of course, these two cones are connected, so any initial condition located on the unstable cone gives rise to a trajectory which eventually reaches the stable cone in a finite time. Note that this longitudinal motion from one cone to the other is superimposed on a spiraling motion around the closed cycle of the pinched torus which is generated by the action of \( H_1 \).

An important object is the image of the phase space in \( R^2 \) under the map \((H_1, H_0)\). It is shown in figure 2. It is a convex domain in \( R^2 \). The upper and lower green boundaries are obtained as follows. We set \( x = \bar{b}b \) and

\[
H_1 = x + s^\pm \equiv m - s_cl, \quad \text{Max}(0, m - 2s_cl) \leq x \leq m \tag{14}
\]

where the parameter \( m \) is introduced to ease the comparison with the quantum case. Then we set

\[
b = \sqrt{x}e^{i\theta}, \quad \bar{b} = \sqrt{x}e^{-i\theta}, \quad s^\pm = \sqrt{s_cl^2 - (s^-)^2}e^{\pm i\varphi} = \sqrt{m - x)(2s_cl - m + x)}e^{\pm i\varphi}
\]

so that

\[
H_0 = 2\kappa(m - s_cl - x) + 2\sqrt{x(m - x)(2s_cl - m + x)} \cos(\theta - \varphi).
\]

It follows that

\[
2\kappa(m - s_cl - x) - 2\sqrt{x(m - x)(2s_cl - m + x)} \leq H_0 \leq 2\kappa(m - s_cl - x) + 2\sqrt{x(m - x)(2s_cl - m + x)}.
\]
Figure 1. The image in phase space of the level set of the critical unstable point $H_0 = 2\kappa s_{cl}, H_1 = s_{cl}$ is a pinched two-dimensional torus.

Figure 2. The image in $\mathbb{R}^2$ of the phase space by the map $(H_1 \equiv m - s_{cl}, H_0)$. The green dots are the critical points. The stable one is the point on the vertical axis ($m = 0, -2\kappa s_{cl}$) and the unstable one is located at ($m = 2s_{cl}, 2\kappa s_{cl}$). The upper and lower green boundaries are obtained as explained in the text.

To find the green curves in figure 2 we have to minimize the left-hand side and maximize the right-hand side in the above inequalities when $x$ varies within the bounds given in equation (14).

In the unstable case, because an initial condition close to the critical point leads to a large trajectory along the pinched torus, it is interesting to find the complete solutions...
of the equations of motion. They are easily obtained as follows. Remark that
\[
(\dot{s}^z)^2 = -(\dot{b}s^- - bs^+)^2 = -(\dot{b}s^- + bs^+)^2 + 4\dot{b}bs^+s^- = -\left(H_0 - 2\kappa s^z\right)^2 + 4(H_1 - s^z)(s_{cl}^2 - (s^z)^2)
\]
or
\[
(\dot{s}^z)^2 = 4(s^z)^3 - 4(H_1 + \kappa^2)(s^z)^2 + 4(\kappa H_0 - s_{cl}^2)s^z + (4s_{cl}^2 H_1 - H_0^2).
\]
Hence, \(s^z\) satisfies an equation of the Weierstrass type and is solved by elliptic functions. Indeed, setting
\[
s^z = \frac{1}{3}(H_1 + \kappa^2) - X, \quad \dot{s}^z = iY
\]
the equation becomes
\[
Y^2 = 4X^3 - g_2 X - g_3
\]
with
\[
g_2 = \frac{4}{3}(H_1^2 - 3H_0\kappa + 2H_1\kappa^2 + \kappa^4 + 3s_{cl}^2)
\]
\[
g_3 = \frac{1}{27}(-27H_0^2 - 8H_1^3 + 36H_0H_1\kappa - 24H_1^2\kappa^2 + 36H_0\kappa^3 - 24H_1\kappa^4 - 8\kappa^6 + 72H_1s_{cl}^2 - 36\kappa^2 s_{cl}^4).
\]
Therefore, the general solution of the one-spin system reads
\[
s^z = \frac{1}{3}(H_1 + \kappa^2) - \wp(it + \alpha)
\]
and
\[
\bar{b}b = H_1 - s^z = \frac{1}{3}(2H_1 - \kappa^2) + \wp(it + \alpha)
\]
where \(\alpha\) is an integration constant and \(\wp(\theta)\) is the Weierstrass function associated with the curve equation (15). Initial conditions can be chosen such that when \(t = 0\), we start from a point intersecting the real axis \(Y = 0\). This happens when \(\alpha\) is half a period:
\[
\alpha = \omega_1 \quad \text{or} \quad \omega_2 \quad \text{or} \quad \omega_3 = -\omega_1 - \omega_2.
\]
This general solution however is not very useful because the physics that we are interested in lies on Liouville tori specified by very particular values of the conserved quantities \(H_0, H_1\). For the configuration equation (9) with spin up, the values of the Hamiltonians are
\[
H_0 = 2\kappa s_{cl}, \quad H_1 = s_{cl}.
\]
In that case we have
\[
g_2 = \frac{4}{3}\Omega^4, \quad g_3 = \frac{8}{27}\Omega^6
\]
and
\[
4X^3 - g_2 X - g_3 = 4 \left(X + \frac{1}{3}\Omega^4\right)^2 \left(X - \frac{1}{3}\Omega^2\right)
\]
where we have defined
\[
\Omega^2 = 2s_{cl} - \kappa^2 > 0 \text{ in the unstable case.}
\]

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Hence, we are precisely in the case where the elliptic curve becomes degenerate. Then the solution of the equations of motion is expressed in terms of trigonometric functions:

\[ X(t) = \frac{2}{3} \Omega^2 - \Omega^2 \tanh^2(\Omega(t - t_0)) \]

and

\[ x(t) \equiv \bar{b}b(t) = \frac{\Omega^2}{\cosh^2 \Omega(t - t_0)}. \tag{19} \]

This solution represents a single solitonic pulse centered at time \( t_0 \). When the initial condition is close but not identical to the unstable configuration with spin up and \( b = \bar{b} = 0 \), this unique soliton is replaced by a periodic sequence of pulses that can be described in terms of the Weierstrass function as shown above. Note that in the context of cold fermionic atoms, \( \bar{b}b \) represents the number of molecular bound states that have been formed in the system.

### 2.2. The normal form and monodromy

The dynamics in the vicinity of the unstable equilibrium can also be visualized through an appropriate choice of canonical variables in which the quadratic parts in the expansions of \( H_0 \) and \( H_1 \) take a simple form. Let us introduce the angle \( \nu \in [0, \pi/2] \) such that \( |\kappa| = \sqrt{2s_{cl}} \cos \nu \) and \( \Omega = \sqrt{2s_{cl}} \sin \nu \). As we have shown, instability occurs when \( \kappa^2 < 2s_{cl} \), which guarantees that \( \nu \) is real. Let us then define two complex coordinates \( A_s \) and \( A_u \) by

\[
\begin{pmatrix} A_s \\ A_u \end{pmatrix} = \frac{1}{\sqrt{2\sin \nu}} \begin{pmatrix} e^{-i\nu/2} & e^{i\nu/2} \\ e^{i\nu/2} & e^{-i\nu/2} \end{pmatrix} \begin{pmatrix} b \\ s^- \end{pmatrix}. \tag{20} \]

The classical Poisson brackets for these variables are

\[
\{ A_s, \bar{A}_s \} = \{ A_u, \bar{A}_u \} = 0 \tag{21} \\
\{ A_s, \bar{A}_u \} = \{ \bar{A}_s, A_u \} = 1. \tag{22} 
\]

Here we have approximated the exact relation \( \{ s^+, s^- \} = 2i s^z \) by \( \{ s^+, s^- \} = 2i s_{cl} \), which is appropriate for capturing the linearized flow near the unstable fixed point. After subtracting their values at the critical point which do not influence the dynamics, the corresponding quadratic Hamiltonian \( H_0 \) and global rotation generator \( H_1 \) read

\[
H_0 = 2\Omega \text{Re}(\bar{A}_s A_u) + \kappa H_1 \tag{23} \\
H_1 = 2\text{Im}(\bar{A}_s A_u). \tag{24} 
\]

With the above Poisson brackets, the linearized equations of motion take the form

\[
\dot{A}_s = (-\Omega - i\kappa)A_s \tag{25} \\
\dot{A}_u = (\Omega - i\kappa)A_u. \tag{26} 
\]

With these new variables, the pinched torus appears, in the neighborhood of the unstable point, as the union of two planes intersecting transversely. These are defined by \( A_u = 0 \).
which corresponds to the stable branch, and by $A_s = 0$ which gives the unstable branch. The global rotations generated by $H_1$ multiply both $A_s$ and $A_u$ by the same phase factor $e^{-i\chi}$. We may then visualize these two planes as two cones whose common tip is the critical point, as depicted in figure 1. The stable cone is obtained from the half-line where $A_s$ is real and positive and $A_u = 0$, after the action of all possible global rotations. A similar description holds for the unstable cone. As shown by equation (19), any trajectory starting form the unstable branch eventually reaches the stable one. This important property will play a crucial role below in the computation of the monodromy.

For latter applications, especially in the quantum case, it is useful to write down $A_s$ and $A_u$ explicitly in terms of their real and imaginary parts:

$$A_s = (P_1 - iP_2)/\sqrt{2}$$
$$A_u = (X_1 - iX_2)/\sqrt{2}.$$  

This reproduces the above Poisson brackets, provided we set $\{X_i, X_j\} = 0$, $\{P_i, P_j\} = 0$, $\{P_i, X_j\} = \delta_{ij}$ for $i, j \in \{1, 2\}$. The quadratic Hamiltonian now reads

$$H_0 = \Omega(X_1 P_1 + X_2 P_2) + \kappa H_1$$
$$H_1 = X_1 P_2 - X_2 P_1.$$  

which is the standard normal form for the focus–focus singularity [25]–[27].

We are now in a position to define and compute the monodromy attached to a closed path around the unstable point in the $(H_0, H_1)$ plane as in figure 2. Let us consider a one-parameter family of regular invariant tori which are close to the pinched torus. This family can be described via a curve in the $(\bar{A}_s, \bar{A}_u)$ plane as in figure 2. Let us consider now the path $\chi \in [0, 2\pi) \to \bar{A}_s \bar{A}_u = \eta^2 e^{i\chi}$ where $\eta > 0$ is assumed to be small. To construct the monodromy, we need to define, for each value of $\chi$, two vector fields on the corresponding torus, which are generated by $\chi$-dependent linear combinations of $H_0$ and $H_1$ and whose flows are $2\pi$ periodic. Furthermore we require that the periodic orbits of these two flows generate a cycle basis $(e_1, e_2)$ in the two-dimensional homology of the torus. This basis evolves continuously as $\chi$ increases from 0 to $2\pi$. The monodromy expresses the fact that $(e_1, e_2)$ can turn into a different basis after one closed loop in the $(H_0, H_1)$ plane. The discussion to follow has been to a large extent inspired by Cushman and Duistermat [27].

One of these two flows can be chosen as the $2\pi$ global rotation generated by $H_1$. The corresponding orbits are circles which provide the first basic cycle $e_1$. For the other one, we first emphasize that the flow generated by $H_0$ is not periodic in general. So, to get the other basic cycle $e_2$, we have to consider the flow generated by an appropriate linear combination of $H_0$ and $H_1$. To construct it, it is convenient to consider an initial condition $(A_s(0), A_u(0))$ such that $\bar{A}_s(0) A_u(0) = \eta^2 e^{i\chi}$ and $|A_s(0)| \ll |A_u(0)|$, that is close to the unstable manifold of the pinched torus. Let us pick a small enough $\zeta > 0$ so that the linearized equations of motion are still accurate when $|A_u(0)|^2 + |A_u(0)|^2 < \zeta^2$, and let us also assume that $\eta \ll \zeta$. Using the global rotation invariance of the dynamics, we may choose $A_u(0) = \zeta$ and $A_s(0) = (\eta^2/\zeta) e^{-i\chi}$. If we let the system evolve starting from this initial condition, $|A_s|$ will first decrease and $|A_u|$ will increase, so the trajectory
becomes closer to the unstable torus, along its unstable manifold. After a finite time \( t_1 \), the trajectory reappears in the neighborhood of the unstable fixed point, but now, near the stable manifold. This behavior can be seen either as a consequence of equation (19) for the trajectories on the pinched torus and extended to nearby trajectories by a continuity argument, or can be checked directly from the explicit solution of the classical dynamics equations (16) and (17) in terms of the Weierstrass function. We may thus choose \( t_1 \) such that \( |A_s(t_1)| = \zeta \). A very important property of \( A_s(t_1) \) is that it has a well defined limit when \( \eta \) goes to zero, because in this limit, one recovers the trajectory on the pinched torus such that \( A_s(0) = 0 \) and \( A_u(0) = \zeta \). As a result, when \( \eta \) is small enough, the argument of \( A_s(t_1) \) in polar coordinates weakly depends on \( \chi \) and the winding number of \( A_s(t_1) \) when \( \chi \) goes from 0 to \( 2\pi \) vanishes. We then let the system evolve until time \( t_2 \) using the linearized flow. The time \( t_2 \) is chosen in order to recover \( |A_u(t_2)| = \zeta \). Using equation (26) and the fact that \( \bar{A}_u A_u \) is conserved, this gives \( t_2 - t_1 = (2/\Omega)\ln(\zeta/\eta) \). Furthermore, from \( \bar{A}_u(t_1)A_u(t_1) = \eta^2 e^{i\chi} \) and equation (26), we deduce

\[
A_u(t_2) = e^{i\chi} \exp \left( -2i\frac{\kappa}{\Omega} \ln \frac{\zeta}{\eta} \right) A_u(t_1). \tag{31}
\]

In general, we have no reason to expect that \( A_u(t_2) \) should be equal to \( A_u(0) \) so, to get a periodic flow on the torus, the evolution generated by \( H_0 \) during the time \( t_2 \) has to be followed by a global rotation of angle \( \beta(\chi) \) given by

\[
e^{i\beta(\chi)} = e^{i\chi} \exp \left( -2i\frac{\kappa}{\Omega} \ln \frac{\zeta}{\eta} \right) A_s(t_1)/\zeta. \tag{32}
\]

The sign of \( \beta \) reflects the fact that the \( H_1 \) flow applied during the time \( \beta \) multiplies \( A_s \) and \( A_u \) by \( e^{-i\beta} \). Note that because the flows associated with \( H_0 \) and \( H_1 \) commute, the composition of the \( H_0 \) flow during time \( t_2 \) and of the \( H_1 \) flow along an angle \( \beta \) can also be viewed as the flow generated by the linear combination \( H_0 + (\beta/t_2)H_1 \) during time \( t_2 \). We have established the periodicity of this flow for the orbit starting from \( A_u(0) = \zeta \) and \( A_u(0) = (\eta^2/\zeta)e^{-i\chi} \). But any other orbit on the invariant torus characterized by \( \bar{A}_u A_u = \eta^2 e^{i\chi} \) can be deduced from this one by a global rotation, which commutes with the flow of \( H_0 + (\beta/t_2)H_1 \). This establishes the periodicity of this latter flow, and therefore allows us to construct the other basic cycle \( e_2(\chi) \), which depends smoothly on \( \chi \).

Because the winding number of \( A_s(t_1) \) vanishes provided \( \eta \) is small enough, we have the crucial relation

\[
\beta(2\pi) = \beta(0) + 2\pi. \tag{33}
\]

In other words, when we follow by continuity the periodic flow generated by \( H_0 + (\beta/t_2)H_1 \) during time \( t_2 \), the final flow is deduced from the initial one by a global rotation of \( 2\pi \). To summarize, as \( \chi \) increases smoothly from 0 to \( 2\pi \), \( e_1 \) is left unchanged, whereas \( e_2 \) evolves into \( e_2 + e_1 \). In this basis, the monodromy matrix of the focus–focus singularity is then

\[
M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{34}
\]
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Figure 3. The sphere represents the reduced phase space obtained by fixing the conserved quantity $H_1$ to its value $s_{cl}$ at the critical point. Here, this point corresponds to the north pole on the sphere. The red curve is the image of the pinched torus after the reduction procedure. It is obtained by setting the other integral of motion $H_0$ to its critical value $H_0 = 2\kappa s_{cl}$. For this, we have used the expression for $H_0$ given in equation (37). The singular nature of the pinched torus is reflected by the cusp of the red curve at the north pole.

2.3. The reduced system

As we have discussed before, the common level set of $H_0$ and $H_1$ which contains the unstable critical point is a pinched torus. Since we are mostly interested in the time evolution of the oscillator energy $\bar{b}b$, and because this quantity is invariant under the Hamiltonian action of $H_1$, it is natural to reduce the dynamics to the orbits of $H_1$. For an initial condition on the pinched torus, this amounts to discarding the spiraling motion around the torus, and concentrating on the longitudinal motion from the stable cone to the unstable one. In other words, this reduction procedure reduces the pinched torus into a curve which has a cusp at the critical point. This is illustrated by the thick curve in figure 3.

Another description of this trajectory is given by the thick curve in figure 4 that will be discussed below. In practice, the first thing to do to implement this reduction is to fix the value of $H_1$ to its critical value

$$H_1 = \bar{b}b + s^2 = s_{cl}. \quad (35)$$

This defines a submanifold $\mathcal{H}_{s_{cl}}$ in phase space. Since $H_0$ and $H_1$ Poisson commute we may consider the reduced system obtained by performing a Hamiltonian reduction by the one-parameter group generated by $H_1$. The reduced phase space is the quotient of $\mathcal{H}_{s_{cl}}$ by the flow generated by $H_1$ (the trivial $t_1$ phases in equation (8)). It is of dimension 2.
Figure 4. The phase portrait corresponding to the Hamiltonian $H_0$, equation (37), in the ($\theta, x$) variables, which are natural coordinates for the reduced system. The thick red line corresponds to the critical separatrix $H_0 = 2\kappa s_{cl}$, which is the image of the pinched critical torus after symplectic reduction. Note that the unstable critical point is mapped onto the vertical axis at $x = 0$, so the critical separatrix is indeed a loop. The blue dots correspond to the minimal and maximal values of $H_0$ when $H_1 = s_{cl}$.

Explicitly, we set
\[ x = \bar{b} > 0 \implies b = \sqrt{x}e^{i\theta}, \quad \bar{b} = \sqrt{x}e^{-i\theta}. \]
From equation (35) and from the condition $(s^z)^2 + s^z s^\pm = s_{cl}^2$ we deduce
\[ s^z = s_{cl} - x, \quad \text{and} \quad s^\pm = \sqrt{x(2s_{cl} - x)}e^{\mp i\varphi}. \tag{36} \]

The reduced Hamiltonian $H_0$ reads
\[ H_0 = 2\kappa s_{cl} + 2x\sqrt{2s_{cl} - x} \cos(\theta - \varphi) - 2\kappa x. \tag{37} \]
Notice that $x$ and $\theta - \varphi$ are invariant by the $H_1$ flow, and they can be taken as coordinates on the reduced phase space. These coordinates are canonically conjugate as we easily see by writing the symplectic form
\[ \omega = -i\delta b \wedge \delta \bar{b} + \frac{1}{s^z} \delta s^+ \wedge \delta s^- = \delta x \wedge \delta(\theta - \varphi). \]

In the following, when talking about this reduced system, we will simply set $\varphi = 0$. These coordinates are very convenient but one has to be aware that they are singular at $x = 0$ and $2s_{cl}$. The whole segment $x = 0, 0 \leq \theta < 2\pi$ should be identified with one point and similarly for the segment $x = 2s_{cl}, 0 \leq \theta < 2\pi$.

The equations of motion of the reduced system read
\[ \dot{x} = 2x\sqrt{2s_{cl} - x} \sin \theta \]
\[ \dot{\theta} = -2\kappa + 2 \left( \frac{\sqrt{2s_{cl} - x}}{2\sqrt{2s_{cl} - x}} - \frac{x}{2\sqrt{2s_{cl} - x}} \right) \cos \theta. \]
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From this we see that the critical points are given by
\[ x = 0, \quad \cos \theta = \frac{\kappa}{\sqrt{2s_{cl}}}, \quad \text{or} \quad z^2 - \frac{2\kappa}{\sqrt{2s_{cl}}}z + 1 = 0, \quad z = e^{i\theta}. \]

In our case \( \kappa \) is negative, so the solutions of the quadratic equations will be defined as
\[ z_\pm = \frac{\kappa \pm \Omega}{\sqrt{2s_{cl}}} = -e^{\pm i\nu}, \quad \Omega^2 = 2s_{cl} - \kappa^2, \quad 0 \leq \nu \leq \pi/2. \]

These points exist precisely in the unstable regime \( \kappa^2 \leq 2s_{cl} \). They correspond to only one point in phase space whose energy is given by
\[ H_0 = 2\kappa s_{cl} \equiv E_c \] (38)
and hence they correspond to the unstable point. The image of the pinched critical torus after the reduction procedure is shown in figure 4 as the thick line connecting \( z_+ \) and \( z_- \). Note that this line intersects the \( x = 0 \) segment with a finite angle. This angle reflects precisely the pinching of the torus at the unstable critical point in the original four-dimensional phase space.

Another critical point is obtained by setting \( \theta = \pi \). Then
\[ x = 2s_{cl} - X^2, \quad \text{with} \quad X = \frac{1}{4}(-\kappa + \sqrt{\kappa^2 + 6s_{cl}}). \] (39)
Since \( 0 \leq x \leq 2s_{cl} \), we should also have \( 0 \leq X^2 \leq 2s_{cl} \), which is the case when \( \kappa \geq -\sqrt{2s_{cl}} \), that is to say in the unstable region. The energy is given by
\[ H_0 = 2\kappa s_{cl} - 2(X + \kappa)(2s_{cl} - X^2). \] (40)
We see from that equation that, in the unstable region, \( H_0 - 2\kappa s_{cl} \) is always negative and the configuration equation (39) represents the ground state.

Finally, the energy maximum of this reduced system is obtained for \( \theta = 0 \) and \( x = 2s_{cl} - Y^2 \), with \( Y = (\kappa + \sqrt{\kappa^2 + 6})/3 \). In the phase portrait shown in figure 4, one sees a curve which contains the point at \( x = 2s_{cl} \) and which looks like a separatrix. However, we notice that, unlike the situation around \( x = 0 \), it does not correspond to any stationary point of the unreduced Hamiltonian. In fact it is simple to show that the curve is actually tangent to the vertical line \( x = 2s_{cl} \). On the sphere, this is the trajectory passing through the south pole.

3. The quantum one-spin system

3.1. The energy spectrum

We now consider the quantum system. We set as in the classical case
\[ H = H_0 + \omega H_1 \]
with
\[ H_0 = (2\epsilon - \omega)s^z + b^\dagger s^- + bs^+ = 2\kappa s^z + b^\dagger s^- + bs^+, \quad H_1 = b^\dagger b + s^z \]
where as before
\[ \kappa = \epsilon - \omega/2. \] (41)

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We impose the commutation relations
\[ [b, b^\dagger] = \hbar, \quad [s^+, s^-] = 2\hbar s^z, \quad [s^z, s^\pm] = \pm \hbar s^\pm. \]

We assume that the spin acts on a spin \( s \) representation:
\[ s^z |m\rangle = \hbar m |m\rangle, \quad s^\pm |m\rangle = \hbar \sqrt{s(s+1) - m(m+1)} |m\pm1\rangle, \]
where \( 2s \) is integer. Of course
\[ (s^z)^2 + \frac{1}{2}(s^+ s^- + s^- s^+) = \hbar^2 s(s+1). \]

We still have
\[ [H_0, H_1] = 0 \]
and the quantum system is integrable. The Hamiltonian \( H \) can be diagonalized using the Bethe ansatz, but we will not follow this path here. For related studies along these lines for the Neumann model see \[29\]. Note that recently, the non-equilibrium dynamics of a similar quantum integrable model (the Richardson model) has been studied using a combination of analytical and numerical tools \[30\]. Let
\[ e_n = (2b^\dagger)^n|0\rangle \otimes (s^+)^{M-n}|s\rangle, \quad \text{Sup}(0, M-2s) \leq n \leq M. \quad (42) \]

For all these states one has
\[ H_1 e_n = \hbar (M-s) e_n. \]

Since \( H_0 \) commutes with \( H_1 \) we can restrict \( H_0 \) to the subspace spanned by the \( e_n \). Hence, we write
\[ \Psi = \sum_{n=\text{Sup}(0, M-2s)}^{M} p_n \frac{e_n}{||e_n||}, \]
where the norm \( ||e_n|| \) of the vector \( e_n \) is given by
\[ ||e_n||^2 = \hbar^{2M-n-2n} \frac{(2s)! (M-n)! n!}{(2s-M+n)!}. \]

Using
\[ H_0 e_n = 2\hbar \kappa (M-s-n) e_n + \frac{\hbar^2}{2} (M-n)(2s+1-M+n) e_{n+1} + 2\hbar n e_{n-1}, \]
the Schrödinger equation
\[ i\hbar \frac{\partial \Psi}{\partial t} = H_0 \Psi \]
becomes
\[ i\hbar \frac{\partial p_n}{\partial t} = \hbar^{3/2} \sqrt{(n+1)(2s+1-M+n)(M-n)} p_{n+1} + \hbar^{3/2} \sqrt{n(2s-M+n)(M+1-n)} p_{n-1} + 2\hbar \kappa (M-n-s) p_n. \]
Figure 5. The lattice of the joint spectrum of $H_0$ and $H_1$ (or equivalently the integer $M$) in the unstable regime. The green point represents the classical unstable point ($H_1 = s_{cl}$, or $M = 2s$, and $H_0 = 2\kappa s_{cl}$). The parameters are $\hbar^{-1} = s = 10$, and $\kappa = -0.5\sqrt{2s_{cl}}$. This figure demonstrates the non-trivial quantum monodromy in this system. For this purpose, we follow by continuity an elementary cell of this lattice along two closed paths. When the path encloses the image of the critical point in the $(H_0, H_1)$ plane, we see that an initial square cell does not preserve its shape at the end of the cycle, but evolves into a parallelogram with a tilt. In other words, the lattice of joint eigenvalues exhibits a dislocation located at the critical value of $H_0$ and $H_1$.

and the eigenvector equation $H_0 \Psi = E \Psi$ reads

$$
\hbar^{3/2} \sqrt{(n+1)(2s+1-M+n)(M-n)p_{n+1} + \hbar^{3/2} \sqrt{n(2s-M+n)(M+1-n)p_{n-1}} + 2\hbar n(M-n-s)p_n = E p_n.
$$

(43)

In this basis, the Hamiltonian is represented by a symmetric Jacobi matrix and it is easy to diagonalize it numerically. Varying $M = 0, 1, \ldots$ we construct the lattice of the joint spectrum of $H_0, H_1$. We see in figure 5 that the lattice has a defect located near the unstable classical point. This defect induces a quantum monodromy when we transport a cell of the lattice around it. The rule for transporting a cell is as follows. We start with a cell and we choose the next cell in one of the four possible positions (east, west, north, south) in such a way that two edges of the new cell prolongate two sides of the original cell, and the two cells have a common edge. We apply these rules on a path which closes. If the path encloses the unstable point, the last cell is different from the initial cell.

The precise form of this lattice defect can be related to the classical monodromy matrix $M$, as shown by Vũ Ngọc [23]. Choosing locally a cycle basis $e_i$ on the Arnold–Liouville tori is equivalent to specifying local action-angle coordinates $(J_i, \phi_i)$ such that $e_i$ is obtained from the periodic orbit generated by $J_i$. After one closed circuit in the base space of this fibration by tori, the basic cycles are changed into $e'_i = \sum_j M_{ij} e_j$ and
therefore, the new local action variables $J'_i$ are deduced from the initial ones:

$$J'_i = \sum_{j=1}^{n} M_{ji} J_j.$$  \hfill (44)

Heuristically, the Bohr–Sommerfeld quantization principle may be viewed as the requirement that the quantum wavefunctions should be $2\pi$ periodic in each of the $n$ phase variables $\phi_i$, which implies that $J_i$ should be an integer multiple of $\hbar$. When $\hbar$ is small, the discrete set of common eigenvalues of the conserved operators has locally the appearance of a regular lattice, whose basis vectors $v_1, \ldots, v_n$ are approximately given by $dJ_i(v_j) = \hbar \delta_{ij}$. Because of the monodromy, this lattice undergoes a smooth deformation as one moves around a critical value of the mutually commuting conserved quantities. After one complete turn, the basic lattice vectors are changed into $v'_1, \ldots, v'_n$, which are also approximately given by $dJ'_i(v'_j) = \hbar \delta_{ij}$. Taking into account the definition of the classical monodromy matrix in equation (44), this leads to the semi-classical monodromy on the joint spectrum

$$v'_i = \sum_{j=1}^{n} (M^{-1})_{ij} v_j.$$  \hfill (45)

The discussion above relates to the action-angle variables $(J_1, J_2)$. In the $(H_0, H_1)$ variables the basis $v, v'$ should be transformed by the Jacobian matrix of the mapping $(J_1, J_2) \rightarrow (H_0(J_1, J_2), H_1(J_1, J_2))$.

In figure 5, we see that after a clockwise turn around the singular value in the $(H_1, H_0)$ plane (which corresponds to a positive winding in the $(H_0, H_1)$ plane), the lattice vectors $v_M$ and $v_H$ are transformed into $v_M' - v_H'$ and $v_H'$ respectively. This is in perfect agreement with the classical monodromy matrix computed in section 2.2.

When $M = 2s$, i.e. when $H_1$ takes its critical value corresponding to the unstable point,

$$H_1 e_n = \hbar s e_n,$$

the Schrödinger equation simplifies to

$$i\hbar \frac{\partial p_n}{\partial t} = \hbar^{3/2} (n + 1) \sqrt{(2s - n)p_{n+1} + \hbar^{3/2} n \sqrt{(2s + 1 - n)p_{n-1} + 2\hbar \kappa (s - n)p_n}}.$$  \hfill (46)

Setting

$$x = \hbar n, \quad s_{cl} = \hbar s$$

this equation becomes

$$i\hbar \frac{\partial p(x)}{\partial t} = (x + \hbar)\sqrt{(2s_{cl} - x)p(x + \hbar) + x\sqrt{(2s_{cl} + \hbar - x)p(x - \hbar) + 2\kappa (s_{cl} - x)p(x)}}.$$  \hfill (47)

Introducing the shift operator

$$e^{i\theta} p(x) = p(x + \hbar)$$

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Figure 6. The eigenvalues of the reduced $H_0$ in the stable regime on the left and in the unstable case on the right. The origin of the energy axis is the classical critical energy $E_c = 2\hbar s \kappa$. In the unstable case the ground state energy lies below this line and close to the classical value given by equation (40). ($\hbar^{-1} = s = 30$, $\kappa = -2\sqrt{2s_{cl}}$ (stable case), $\kappa = -0.5\sqrt{2s_{cl}}$ (unstable case).)

this can be rewritten as

$$i\hbar \frac{\partial p(x)}{\partial t} = \left[ \sqrt{(2s_{cl} - x)e^{i\theta} x + xe^{-i\theta}(2s_{cl} - x) + 2\kappa(s_{cl} - x)} \right] p(x)$$

(48)

and we recognize the quantum version of the classical reduced Hamiltonian equation (37), but with a very specific ordering of the operators.

The eigenvector equation $H_0 \Psi = E \Psi$ reads

$$h^{3/2}(n + 1)\sqrt{(2s - n)p_{n+1} + h^{3/2}n\sqrt{(2s + 1 - n)p_{n-1}} + (2\hbar\kappa(s - n) - E)p_n = 0.}$$

(49)

The eigenvalues are shown in figure 6 in the stable case and in the unstable case.

4. Bohr–Sommerfeld quantization

In this section we examine the standard semi-classical analysis of the one-spin system. There are two complications as compared to the usual formulae. One comes from the fact that the phase space of the spin degrees of freedom is a 2-sphere and hence has a non-trivial topology. The second one is related to the fact that the Schrödinger equation is a difference equation and its symbol has a subprincipal part. Let us recall the Schrödinger equation:

$$(x + \hbar)\sqrt{2s_{cl} - xp(x + \hbar)} + x\sqrt{2s_{cl} + \hbar - xp(x - \hbar)} - 2\kappa xp(x) = \hbar\epsilon p(x)$$

(50)

where

$$\hbar\epsilon = E - 2\hbar\kappa s, \quad x = \hbar n.$$

The first thing to do is to compute the Weyl symbol $h(x, \theta)$ of the Hamiltonian operator. In standard notation, it is defined by

$$(Hp)(x) = \frac{1}{2\pi\hbar} \int e^{-i(x+y)/(2\hbar)} \left( \frac{x + y}{2} \right) p(y) \, d\theta dy.$$
It is straightforward to check that, for the Hamiltonian of equation (50),

\[ h(x, \theta) = 2 \left( x + \frac{\hbar}{2} \right) \sqrt{2s_{cl} + \frac{\hbar}{2} - x \cos \theta + 2\kappa (s_{cl} - x)} \].

Expanding in \( \hbar \) we find the principal and subprincipal symbols

\[ h(x, \theta) = h^0(x, \theta) + \hbar h^1(x, \theta) + \ldots \]

where

\[ h^0(x, \theta) = 2x \sqrt{2s_{cl} - x \cos \theta + 2\kappa (s_{cl} - x)} \]

\[ h^1(x, \theta) = \left( \sqrt{2s_{cl} - x} + \frac{x}{2\sqrt{2s_{cl} - x}} \right) \cos \theta \].

We recognize, of course, that \( h^0(x, \theta) \) is the classical Hamiltonian of the system equation (37). In particular the symplectic form reads

\[ \omega = dx \wedge d\theta \].

Note that the definition of the Weyl symbol that we have used here is motivated by the simpler case where the classical phase space can be viewed as the cotangent bundle of a smooth manifold. In our case, the \( \theta \) coordinate is \( 2\pi \) periodic, and we are not strictly speaking dealing with the cotangent bundle over the line parameterized by the \( x \) coordinate. To justify this procedure, we may first embed our system in a larger phase space, where both \( \theta \) and \( x \) run from \( -\infty \) to \( \infty \). Because the symbol \( h(x, \theta) \) is \( 2\pi \) periodic in \( \theta \), the unitary operator \( T \) associated with the \( 2\pi \) translation of \( \theta \) commutes with the quantum Hamiltonian \( H \) associated with the symbol \( h(x, \theta) \). It is then possible to perform the semi-classical analysis in the enlarged Hilbert space and to project afterward the states thus obtained onto the subspace of \( 2\pi \) periodic wavefunctions, which are eigenvectors of \( T \) with the eigenvalue 1. Imposing the periodicity in \( \theta \) forces \( x \) to be an integer multiple of \( \hbar \). From equation (50), we see that the physical subspace, spanned by state vectors \( |x = n\hbar\rangle \) with \( 0 \leq n \leq 2s \), is stable under the action of \( H \).

An alternative approach would be to use a quantization scheme, such as Berezin–Toeplitz quantization [31], which allows one to work directly with a compact classical phase space such as the sphere. When \( H_1 \) takes its critical value corresponding to the unstable point, it is easy to show that the eigenvector equation (49) can be cast into the Schrödinger equation form for a pure spin \( s \) Hamiltonian given by

\[ H_{eff} = 2\kappa s^z + s^+ \sqrt{s_{cl} - s_z} + \sqrt{s_{cl} - s_z} s^- \].

(51)

Starting from the known Toeplitz symbols of the basic spin operators, one may derive Bohr–Sommerfeld quantization rules directly from this Hamiltonian \( H_{eff} \) [32], but we shall not explore this further in this paper.

Returning to our main discussion we now choose a 1-form \( \gamma \) such that

\[ \gamma(X_{h^0}) = -h^1(x, \theta) \]

(52)

where \( X_{h^0} \) is the Hamiltonian vector field associated with the Hamiltonian \( h^0(x, \theta) \), which is tangent to the variety \( h^0(x, \theta) = E \). Using \( x \) as the coordinate on this manifold, we find

\[ X_{h^0} = 2x \sqrt{2s_{cl} - x} \sin \theta \partial_x \].

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Hence, we may choose
\[ \gamma = -\frac{1}{2} \left( \frac{1}{x} + \frac{1}{2(2s_{cl} - x)} \right) \cot \theta \, dx. \]
Under these circumstances, the Bohr–Sommerfeld quantization conditions involve the form \( \gamma \) and read (see e.g. [23])
\[ \Phi_{Reg}(\epsilon) = \frac{1}{2\pi \hbar} \int_{C(E)} \alpha + \frac{1}{2\pi} \int_{C(E)} \gamma + \mu_{C(E)} \frac{1}{4} = n \] (53)
where \( C(E) \) is the classical trajectory of energy \( E \), \( \alpha \) is the canonical 1-form \( \omega = d\alpha \), \( \mu_{C(E)} \) is the Maslov index of this trajectory and \( n \) is an integer. Note that the integral of \( \gamma \) over \( C(E) \) is completely specified by the constraint equation (52).

We now come to the fact that the 2-sphere has a non-trivial topology. A consequence is that the canonical 1-form \( \alpha \) does not exist globally. In the coordinates \( x, \theta \) we may choose
\[ \alpha = x \, d\theta \]
but this form is singular in the vicinity of the south pole where \( x = 2s_{cl} \). Let us consider a closed path on the sphere parameterized by the segment at constant \( x, \theta \) running from 0 to \( 2\pi \). The integral of \( \alpha \) around this path is \( 2\pi x \). When \( x \) goes to \( 2s_{cl} \), the integral goes to \( 4\pi s_{cl} \) which is the total area of the sphere. But the limit path for \( x = 2s_{cl} \) corresponds on the sphere to a trivial loop, fixed at the south pole. The consistency of Bohr–Sommerfeld quantization then requires \( 4\pi s_{cl} \) to be an integer multiple of \( 2\pi \hbar \). Hence we recover the quantization condition of the spin: \( 2s \) should be an integer.

Before checking the validity of equation (53) in our system, it is quite instructive to plot the integral of \( \gamma \) along the curve \( C(E) \) as a function of \( E \). This is shown in figure 7. Here, we see two singularities at the values of \( E \) where \( C(E) \) goes through either the north or the south pole. At the south pole \( \int_{C(E)} \gamma \) merely jumps by \( \pi \), whereas at the north pole, a jump of \( \pi \) is superimposed onto a logarithmic divergence. We can easily find an asymptotic formula for this integral for energies close to the critical value \( E_c = 2s_{cl}\hbar \):
\[ \int \gamma = \frac{\kappa}{\Omega} \log \frac{\sqrt{2s_{cl}|\Delta E|}}{16\Omega^4} + 2(\pi - \nu) - \pi\Theta(\Delta E) + O(\Delta E, \Delta E \log |\Delta E|) \] (54)
where \( \Delta E = E - E_c \), and \( \Theta(\Delta E) = 1 \) if \( \Delta E > 0 \) and 0 if \( \Delta E < 0 \).

We wish now to show that these jumps are merely an effect of working with polar coordinates \( (x, \theta) \). It is instructive to consider first a quantum system with one degree of freedom, whose classical phase space is the \( (q, p) \) plane. From any smooth function \( h(p, q) \), we define the quantum operator \( \mathcal{O}_{W,h} \) by
\[ (\mathcal{O}_{W,h}(\Psi))(q) = \frac{1}{2\pi \hbar} \int e^{(i/\hbar)(q-q')p} \hbar \left( \frac{p + q'}{2} \right) \Psi(q') \, dp \, dq'. \]
Polar coordinates are introduced at the classical level by
\[ (p, q) = \sqrt{2x}(\cos \theta, \sin \theta). \]
This definition implies that \( dp \wedge dq = dx \wedge d\theta \). At the quantum level, we start with the usual \( (\hat{p}, \hat{q}) \) operators. In the Hilbert space \( L^2(q) \), we introduce the eigenvector basis \( \{|n\} \) of \( \hat{p}^2 + \hat{q}^2 \), such that \( (\hat{p}^2 + \hat{q}^2)|n\rangle = \hbar(2n + 1)|n\rangle \). We may then set \( \hat{x} = \sum_{n \geq 0} \hbar n |n\rangle \langle n| \).
To define $\hat{\theta}$, such that $[\hat{\theta}, \hat{x}] = i\hbar$, it is natural to enlarge the Hilbert space, allowing $n$ to take also negative integer values as well. Then we set $\exp(i\hat{\theta})|n\rangle = |n + 1\rangle$. From $a|n\rangle = \sqrt{n}|n - 1\rangle$ and $a^+|n\rangle = \sqrt{n + 1}|n + 1\rangle$, we see that, in the physical subspace $n \geq 0$, we have $a = \hbar^{-1/2}\exp(-i\hat{\theta})\hat{x}^{1/2}$ and $a^+ = \hbar^{-1/2}\hat{x}^{1/2}\exp(i\hat{\theta})$. The Weyl symbols, in the $(x, \theta)$ variables, of these operators are $\hbar^{-1/2}(x + \hbar/2)\hat{x}^{1/2}$ and $\hbar^{-1/2}(a + a^+)\hat{x}^{1/2}\exp(\mp i\hat{\theta})$. Using $\hat{p} = (\hbar/2)^{1/2}(a + a^+)$ and $\hat{q} = (\hbar/2)^{1/2}i(a - a^+)$, we deduce that the Weyl symbols of $\hat{p}$ and $\hat{q}$ in the $(x, \theta)$ variables are respectively $\sqrt{2x + \hbar}\cos \theta$ and $\sqrt{2x + \hbar}\sin \theta$. So the Weyl symbol is not invariant under the nonlinear change of coordinates from $(p, q)$ to $(x, \theta)$. For linear functions of $p$ and $q$ the symbol in $(x, \theta)$, denoted by $h_{\text{pol}}(x, \theta)$, is obtained from $h(p, q)$ by substituting $p$ and $q$ by their classical expressions as functions of $(x, \theta)$ and then by shifting $x$ to $x + \hbar/2$. Such a simple rule does not hold for more complicated functions $h(p, q)$. Nevertheless, one can show that the substitution of $x \to x + \hbar/2$ gives the symbol $h_{\text{pol}}(x, \theta)$ up to first order in $\hbar$. Practically, this means that

$$h_{\text{pol}}(x, \theta) = h(\sqrt{2x + \hbar}\cos \theta, \sqrt{2x + \hbar}\sin \theta) \mod O(\hbar^2)$$

or, more explicitly,

$$h_{\text{pol}}(x, \theta) = h(p, q) + \frac{\hbar}{4x} \left( p \frac{\partial h}{\partial p}(p, q) + q \frac{\partial h}{\partial q}(p, q) \right) + O(\hbar^2)$$

where $(p, q) = \sqrt{2x}(\cos \theta, \sin \theta)$. This has the following important consequence: even if $h(p, q)$ has no subprincipal part (i.e. it does not depend on $\hbar$), $h_{\text{pol}}(x, \theta)$ does acquire a
Figure 8. In this figure we see that \(-\frac{1}{2} \oint d\theta\) equals zero if the integral is taken over the small ellipse, while it is equal to \(-\pi\) if it is taken over the big ellipse enclosing the origin. This explains why the integral of the subprincipal symbol in polar coordinates jumps by \(\pm \pi\) when the orbit crosses the origin of the polar coordinates.

The subprincipal part \(h_{\text{pol}}^1(x, \theta)\). Now, along any classical trajectory associated with \(h(p, q)\), we have

\[
p \frac{\partial h}{\partial p}(p, q) + q \frac{\partial h}{\partial q}(p, q) = p\dot{q} - q\dot{p} = 2x\dot{\theta},
\]

so \(h_{\text{pol}}^1(x, \theta) = \dot{\theta}/2\). This striking result implies that, when we work in polar coordinates, the \(\gamma\) form associated with \(h_{\text{pol}}^1(x, \theta)\) may be chosen as \(-d\theta/2\). This immediately explains why \(\int_{C(E)} \gamma\) jumps by \(-\pi\) when the \(h(p, q) = E\) orbit crosses the origin of the polar coordinates, because then \(\int_{C(E)} d\theta\) jumps from 0 to \(2\pi\); see figure 8. Note that, in general, we expect \(h(p, q)\) to have also a subprincipal part \(h^1(p, q)\). But this gives an additional term to \(h_{\text{pol}}^1(x, \theta)\) which is deduced from \(h^1(p, q)\) simply by the classical change of variables from \((p, q)\) to \((x, \theta)\). This term has no reason to display any singularity when the classical trajectory goes through the origin.

Remark that around the south pole, if we set

\[
 x = 2s_{\text{cl}} - y, \quad p = \sqrt{2y} \cos \theta, \quad q = \sqrt{2y} \sin \theta
\]

then we can expand

\[
h_0 = -2s_{\text{cl}}(\kappa + s_{\text{cl}}) + \kappa \left( p + \frac{\sqrt{2s_{\text{cl}}}}{\kappa} \right)^2 + q^2
\]

and we see that our system is equivalent to a shifted harmonic oscillator.

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Note that on the sphere, polar coordinates have two singularities, at the north and at the south poles, which correspond to \( x = 0 \) and \( 2s_{cl} \) respectively. To generalize the above analysis to the sphere, we may thus use two charts \((p, q)\) and \((p', q')\) such that

\[
(p, q) = \sqrt{2x} (\cos \theta, \sin \theta) \quad (p', q') = \sqrt{2x'} (\cos \theta', \sin \theta')
\]

\[ x + x' = 2s_{cl} \quad \theta + \theta' = 0. \]

We therefore get \( dp \wedge dq = dx \wedge d\theta = dx' \wedge d\theta' = dp' \wedge dq' \). It is interesting to note that the Weyl symbols \( h_{pol}(x, \theta) \) and \( h_{pol}'(x', \theta') \) are obtained from each other by the classical transformation from \((x, \theta)\) to \((x', \theta')\). On the other hand, \( h(p, q) \) and \( h'(p', q') \) corresponding to the same quantum Hamiltonian in the physical subspace \( 0 \leq x = 2s_{cl} - x' \leq 2s_{cl} \) do not coincide, because shifting \( x \) to \( x + \hbar / 2 \) amounts to shifting \( x' \) to \( x' - \hbar / 2 \) instead of \( x' + \hbar / 2 \). This implies that it is impossible to construct a quantum operator in the 2s + 1-dimensional Hilbert space of a spin s for which both subprincipal parts of \( h(p, q) \) and \( h'(p', q') \) would vanish.

To formulate the Bohr–Sommerfeld rule, it is convenient to use the \((p, q)\) coordinates when we consider a set of classical orbits which remain at a finite distance from the south pole. In these coordinates, the subleading term \( \int_{C(E)} \gamma \) and the Maslov index are both continuous when \( C(E) \) crosses the north pole. Going back to \((x, \theta)\) coordinates, we see that the jump in \( \int_{C(E)} \gamma \) has to be compensated by a jump in the Maslov index. So the Bohr–Sommerfeld formula (53) can be expressed in \((x, \theta)\) coordinates, provided these jumps in the Maslov index are taken into account.

To check equation (53) we evaluate \( \Phi_{Reg}(\epsilon_n) \) for the exact values \( \epsilon_n \) and we define the defect \( \delta_n \) by

\[
\Phi_{Reg}(\epsilon_n) = n + \delta_n.
\]

We define

\[
I_{-1} = \frac{1}{2\pi\hbar} \int_{C(E)} \alpha, \quad I_0 = \frac{1}{2\pi} \int_{C(E)} \gamma
\]

\[
\delta_n = \begin{cases} 
I_{-1} + I_0 - (n + \frac{1}{2}), & \text{if } \epsilon_n \leq 2\hbar\kappa s \\
I_{-1} + I_0 + \frac{1}{2} - (n + \frac{1}{2}), & \text{if } 2\hbar\kappa s \leq \epsilon_n \leq -2\hbar\kappa s \\
I_{-1} + I_0 + 1 - (n + \frac{1}{2}), & \text{if } \epsilon_n \geq -2\hbar\kappa s.
\end{cases}
\]

The excellent accuracy of this Bohr–Sommerfeld rule is clearly visible in figure 9, which shows the defect \( \delta_n \) as a function of \( n \). We see that it remains small everywhere, and we emphasize that it does not exhibit any singularity when \( \epsilon_n \) crosses the value \(-2\kappa s_{cl}\), corresponding to the south pole. On the other hand, something special happens around \( \epsilon_n = 2\kappa s_{cl} \) (the energy of the north pole), which cannot be attributed to the use of polar coordinates, but which reflects the crossing through the singular orbit associated with the pinched torus. A detailed description of the energy spectrum in the vicinity of the classical unstable point is the goal of section 5.

5. The generalized Bohr–Sommerfeld rule

This region of the spectrum requires a different quantization rule [16]–[20], [28,34] because the classical motion near the turning point at \( x_{\min} \) is strongly affected by the presence
Figure 9. To test the quality of the Bohr–Sommerfeld quantization, we plot the default $\delta_n$ as a function of $n$ which labels the energy eigenstates. We see that the accuracy of the usual Bohr–Sommerfeld rule is very good away from the critical energy. In particular there is nothing special at the south pole.

of the unstable point at $x = 0$. Therefore, we first need to analyze the small $x$ behavior of the energy eigenstates, when $E$ is close to $2\kappa s_{\text{cl}}$. In the intermediate regime, between $x_{\text{min}}(E)$ and $x_{\text{max}}(E)$, a standard WKB analysis is quite accurate. Near $x_{\text{max}}(E)$, we have a regular turning point, which can be described, as usual, by an Airy function. Gluing the wavefunction of the eigenstates between these three different regimes will give us the generalized Bohr–Sommerfeld quantization rules.

5.1. Small $x$ analysis

Here, we shall recast the normal form around the singular point, studied in section 2.2, within the framework of the reduced system. For this, we assume $n \ll 2s$ or $x = \hbar n \ll 2s_{\text{cl}}$. In that approximation the Schrödinger equation (49) becomes

$$E p_n = \hbar \sqrt{2sh(n+1)p_{n+1} + \hbar \sqrt{2sh}p_{n-1} + 2\hbar \kappa(s-n)p_n}.$$

(55)

This equation is linear in $n$ and can be solved by using the Laplace–Fourier transform. Letting

$$\Psi_{\text{Small}}(\theta) = \sum_{n=0}^{\infty} p_n e^{i n \theta}$$

(56)

the equation reads

$$H_0 \Psi_{\text{Small}} = E \Psi_{\text{Small}}, \quad E = E_{\text{c}} + \hbar \epsilon$$

with

$$H_0 = -i\hbar (\sqrt{2s_{\text{cl}}(e^{i \theta} + e^{-i \theta})} - 2\kappa) \partial_\theta + \hbar (2\kappa s + \sqrt{2s_{\text{cl}}e^{i \theta}}).$$

(57)
Let us set
\[ z = e^{i\theta}, \quad \frac{d}{d\theta} = iz \frac{d}{dz}. \]

The equation becomes
\[
\left[ (\sqrt{2}s_{\text{cl}}(z + z^{-1}) - 2\kappa)z \frac{d}{dz} + \sqrt{2}s_{\text{cl}}z \right] \Psi_{\text{Small}} = \epsilon \Psi_{\text{Small}}.
\]

We are looking for a solution which is analytic in a neighborhood of the origin in the \( z \)-plane. This can be written as
\[
\Psi_{\text{Small}}(z, \epsilon) = \left( 1 - \frac{z}{z_+} \right)^{\Delta_+} \left( 1 - \frac{z}{z_-} \right)^{\Delta_-}
\]
where \( z \) are the solutions of the second-order equation already introduced before:
\[
z^2 - 2\frac{\kappa}{\sqrt{2}s_{\text{cl}}} z + 1 = 0, \quad z_{\pm} = \frac{\kappa \pm i\Omega}{\sqrt{2}s_{\text{cl}}} = -e^{\pm i\nu}
\]
and
\[
\Delta_{\pm} = -\frac{1}{2} \pm \frac{\epsilon - \kappa}{2\Omega} \equiv -\frac{1}{2} \mp if(E).
\]

Note that, when \( \hbar \) is small, the leading term in \( f(E) \) is \( (E - E_c)/2\hbar\Omega \). It is quite instructive to express this wavefunction directly in the \( \theta \) variable. Defining
\[
\chi(\theta) = (\cos \theta + \cos \nu)^{-1/2} \exp i \left[ f(E) \log \left( \frac{\cos(\theta - \nu)/2}{\cos(\theta + \nu)/2} \right) - \frac{\theta}{2} \right]
\]
we have
\[
\Psi_{\text{Small}}(\theta) = \begin{cases} 
\chi(\theta), & -\pi + \nu \leq \theta \leq \pi - \nu \\
i \exp(-\pi f(E)) \chi(\theta), & \pi - \nu \leq \theta \leq \pi \\
-i \exp(-\pi f(E)) \chi(\theta), & -\pi \leq \theta \leq -\pi + \nu.
\end{cases}
\]

Notice that this has precisely the expected form in the semi-classical limit \( \hbar \to 0 \). Indeed, when \( E > E_c \), the wavefunction is mostly confined to the interval \(-\pi + \nu \leq \theta \leq \pi - \nu\), and it is exponentially small in the classically forbidden regions \( \pi - \nu \leq \theta \leq \pi \) and \(-\pi \leq \theta \leq -\pi + \nu\). Such an exponentially small prefactor is reminiscent of the behavior of a tunneling amplitude. This is consistent with our treatment of \( \theta \) as an unbounded variable, subjected to a \( 2\pi \) periodic Hamiltonian. Enforcing integer values of \( n \) requires the wavefunction \( \Psi_{\text{Small}}(\theta) \) to be \( 2\pi \) periodic in \( \theta \). If \( \Psi_{\text{Small}}(\theta) \) were identically zero in the classically forbidden intervals, its Fourier transform would develop a tail for negative values of \( n \). So these evanescent parts of the wavefunction are required in order to ensure that the wavefunction belongs to the physical Hilbert space \( n \geq 0 \). Note that when \( E < E_c \), the classically forbidden region becomes the intervals \(-\pi \leq \theta \leq -\pi + \nu\) and \( \pi - \nu \leq \theta \leq \pi \).

In the classically allowed regions, the semi-classical wavefunction is expected to take the textbook form
\[
\Psi_{\text{Small}}(\theta) = a(\theta) \exp i \left( \frac{S_0(\theta)}{\hbar} + S_1(\theta) \right) \tag{58}
\]
where the three functions $a$, $S_0$ and $S_1$ take real values. These functions satisfy the standard equations [33], involving the principal and subprincipal symbols $h^0(x,\theta) = E + 2\sqrt{2c_3}x(\cos \theta + \cos \nu)$ and $h^1(x,\theta) = \sqrt{2c_3}\cos \theta$:

$$h^0(S_0'(\theta),\theta) = E$$  \hspace{1cm} (59)

$$\frac{d}{d\theta} \left( a^2(\theta) \partial_x h^0(S_0'(\theta),\theta) \right) = 0$$  \hspace{1cm} (60)

$$\partial_x h^0(S_0'(\theta),\theta) S_1'(\theta) = -h^1(S_0'(\theta),\theta).$$  \hspace{1cm} (61)

From the above expressions for $\Psi^{\text{Small}}(\theta)$, we can check that it has exactly this semi-classical form. Most likely, this occurs because $\Psi^{\text{Small}}(\theta)$ is an eigenstate of a quantum Hamiltonian which is derived by symplectic reduction from the quadratic Hamiltonian of the normal form discussed in section 2.2. Quadratic Hamiltonians are particularly well-behaved with respect to the $h \to 0$ limit in the sense that the Bohr–Sommerfeld formula gives the exact energy spectrum, and also that the full quantum propagator obeys equations similar to (59) and (60).

Let us now discuss the $x = nh$ representation which is quite useful from the physical standpoint.

$$p^{\text{Small}}(x,\epsilon) = \int_{C_0} \frac{dz}{2i\pi} z^{-(x/h)-1} \Psi(z) = \int_{C_0} \frac{dz}{2i\pi} z^{-(x/h)-1} \left( 1 - \frac{z}{z_+} \right)^{\Delta_+} \left( 1 - \frac{z}{z_-} \right)^{\Delta_-}$$

where $C_0$ is a small contour around the origin. When $E$ is sufficiently far (in a sense to be made precise below) from its critical value $E_c$, we may use the saddle point approximation to evaluate $\Psi^{\text{Small}}(x,\epsilon)$. This yields again the expected semi-classical form, namely

$$p^{\text{Small}}(x) = b(x)e^{\pm i\pi/4} \exp -i \left( \frac{W_0(x)}{h} + W_1(x) \right)$$  \hspace{1cm} (62)

where $b(x)$, $W_0(x)$ and $S_1(x)$ satisfy

$$h^0(x, W_0'(x)) = E$$  \hspace{1cm} (63)

$$\frac{d}{dx} \left( b^2(x) \partial_x h^0(x, W_0'(x)) \right) = 0$$  \hspace{1cm} (64)

$$\partial_x h^0(x, W_0'(x)) W_1'(x) = -h^1(x, W_0'(x)).$$  \hspace{1cm} (65)

Note that $W_0(x)$ is the Legendre transform of $S_0(\theta)$, that is $S_0'(\theta) = x$, $W_0'(x) = \theta$ and $S_0(\theta) + W_0(x) = x\theta$. It is interesting to write down explicitly this semi-classical wavefunction when $x \gg 1$. It reads

$$p^{\text{Small}}_{\text{sc}}(x) = B_+(E) e^{-i(\pi - \nu)x/h} + B_-(E) e^{i(\pi - \nu)x/h}$$

where $B_-(E) = \bar{B}_+(E)$ and

$$\frac{B_+(E)}{B_-(E)} = e^{i(\nu - \pi/2)} \exp i \left( 2f(E) \log \frac{4h\Omega}{|E - E_c|} + \frac{E - E_c}{h\Omega} \right).$$  \hspace{1cm} (66)

We note that the phase factor diverges when $E$ reaches $E_c$. This behavior is induced by the contribution of the subprincipal symbol to the phase of the wavefunction. The prefactor

\[ \text{doi:10.1088/1742-5468/2009/07/P07011} \]
remains fixed. In this case, the two exponents $\Delta$ and $\Delta'$ when we integrate over $p$, the dominant contribution to $S_0''(\theta(x))$ is the exponential factor $e^{i\Delta(\theta(x))}$. This exponential factor goes like $\exp(i((\nu - \pi)/2))$ in the above expression does not jump when $E$ crosses the critical value $E_c$. This is consistent with the analysis of section 4, because we do find a $\pi$ jump in the Maslov index that is compensated by a $\pi$ jump in the contribution of the subprincipal symbol.

Let us now make precise the validity domain of the stationary phase approximation. When we integrate over $\theta$ to compute the Fourier transform $p_{\text{small}}(x)$, we get an oscillating integral whose phase factor can be approximated by $\exp(iS_0''(\theta(x))(\theta - \theta(x))^2/(2\hbar))$ where $\theta(x)$ is one of the two saddle points defined implicitly by $S_0''(\theta(x)) = x$. This oscillating Gaussian has a typical width $\langle \Delta \theta^2 \rangle = \hbar/|S_0''(\theta(x))|$. We find that

$$S_0''(\theta(x)) = \frac{E - E_c}{2\sqrt{2s_\text{cl}}} \frac{\sin \theta}{(\cos \theta + \cos \nu)^2}.$$  

When $E$ goes to $E_c$ at fixed $x$, the phase factor becomes $e^{i(\nu - \pi)/2}$, because $\cos \theta + \cos \nu = E/(2\sqrt{2s_\text{cl}}x)$. So $|S_0''(\theta(x))| \approx 2\Omega x^2/|E - E_c|$. On the other hand, we have seen that the amplitude of $\Psi(\theta)$ diverges when $\theta = \pm(\pi - \nu)$. The distance $\delta\theta$ between $\theta(x)$ and the closest singularity goes like $|E - E_c|/2\Omega x$. The stationary phase approximation holds as long as $\theta(x)$ is far enough from the singularities, that is if $\langle \Delta \theta^2 \rangle \ll (\delta\theta)^2$, or equivalently,

$$|E - E_c| \gg 2\hbar \Omega. \quad (67)$$

When this condition is not fulfilled, the stationary phase approximation breaks down. The dominant contribution to $p_{\text{small}}(x)$ comes from the vicinity of $\theta = \pm(\pi - \nu)$. We have therefore to consider a different asymptotic regime, where $\hbar$ goes to 0 while $\epsilon = (E - E_c)/\hbar$ remains fixed. In this case, the two exponents $\Delta_+$, $\Delta_-$ are fixed, and the only large parameter in the Fourier integral giving $p_{\text{small}}(x)$ is $x/\hbar$. One then obtains

$$p_{\text{small}}(x) = A_+(E) \frac{e^{-i(\nu - \pi/2)x/\hbar}}{(x/\hbar)^{(1/2)-if(E)}} + A_-(E) \frac{e^{i(\nu - \pi/2)x/\hbar}}{(x/\hbar)^{(1/2)+if(E)}} \equiv p_+^{\text{small}}(x) + p_-^{\text{small}}(x) \quad (68)$$

with $A_-(E) = \tilde{A}_+(E)$ and

$$\frac{A_+(E)}{A_-(E)} = e^{i(\nu - \pi/2)} \frac{\Gamma((1/2) - if(E))}{\Gamma((1/2) + if(E))} \exp i(2f(E)\log(2\sin \nu)). \quad (69)$$

This analysis first shows that, in spite of the breakdown of the stationary phase approximation, $p_{\text{small}}(x)$ is still given at large $x$ by a sum of two semi-classical wavefunctions associated with the same principal and subprincipal symbols $\hbar^0$ and $\hbar^1$. This implies that there will be a perfect matching between $p_{\text{small}}(x)$ and the WKB wavefunctions built at finite $x$ from the full classical Hamiltonian. This will be discussed in more detail below. The only modification to the conventional WKB analysis lies in the phase factor $A_+/A_-$, between the ongoing and outgoing amplitudes, which differs markedly from the semi-classical $B_+/B_-$. In particular, we see that the full quantum treatment at fixed $\epsilon$ provides a regularization of the divergence coming from the subprincipal symbol. Indeed, the singular factor $\log(4\hbar \Omega/|E - E_c|)$ is replaced by the constant $\log(2\sin \nu)$. Such a phenomenon has been demonstrated before for various models [20, 28].

\textbf{A semi-classical study of the Jaynes–Cummings model}

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5.2. WKB analysis

We return to the Schrödinger equation:
\[(x + \hbar)\sqrt{2s_{cl} - xp(x + \hbar)} + x\sqrt{2s_{cl} + \hbar - xp(x - \hbar)} - 2\kappa xp(x) = \hbar\epsilon p(x)\]

where
\[\hbar\epsilon = E - E_c.\]

We try to solve this equation by making the WKB ansatz:
\[p^{WKB}(x) = e^{(-i/\hbar)(W_0(x) + hW_1(x))}b(x).\]

Expanding in \(\hbar\), to order \(\hbar^0\) we find
\[x\sqrt{2s_{cl} - \frac{x}{\cos W'_0} - \kappa x} = 0\]

which is nothing but the Hamilton–Jacobi equation on the critical variety. It is identical to equation (63) where \(h^0(x, \theta)\) is the complete principal symbol, and the energy is taken to be \(E_c\). In this procedure, the energy difference \(\hbar\epsilon\) is viewed as a perturbation, which can be treated by adding the constant term \(-\epsilon\) to the complete subprincipal symbol \(h^1(x, \theta)\).

Alternatively, we may write the Hamilton–Jacobi equation as
\[e^{iW_0} = z(x), \quad z^2(x) - \frac{2\kappa}{\sqrt{2s_{cl} - x}}z(x) + 1 = 0.\]

The solution of the quadratic equation is
\[z_{\pm}(x) = \frac{1}{\sqrt{2s_{cl} - x}}(\kappa \pm i\sqrt{\Omega^2 - x}) = \left(\frac{\kappa \pm i\sqrt{\Omega^2 - x}}{\kappa \mp i\sqrt{\Omega^2 - x}}\right)^{1/2}\]

so \(z_{\pm}(x)\big|_{x=0} = e^{\pm i(\pi - \nu)}\). Hence
\[e^{-(i/\hbar)W_0(x)} = \exp\left(\pm \frac{i}{\hbar} \left(\pi x + \kappa \sqrt{\Omega^2 - x} + \frac{i}{2}(2s_{cl} - x) \log \frac{\kappa + i\sqrt{\Omega^2 - x}}{\kappa - i\sqrt{\Omega^2 - x}}\right)\right)\]

where the two signs refer to the two branches of the classical trajectory. Notice that they also correspond to the two determinations of the square root \(\sqrt{\Omega^2 - x}\).

At the next order in \(\hbar\) the equation for \(b(x)\) is
\[\frac{d}{dx} \left(b^2(x)\sqrt{\Omega^2 - x}\right) = 0.\]

Therefore, we may choose \(b(x) = (x\sqrt{\Omega^2 - x})^{-1/2}\).

The correction \(W_1(x)\) to the phase function satisfies equation (65) with \(h^1(x, \theta)\) replaced by \(h^1(x, \theta) - \epsilon\), that is
\[\partial_\theta h^0(x, W'_0(x))W'_1(x) = -h^1(x, W'_0(x)) + \epsilon\]

\(W'_1(x)\) is the sum of two terms, \(W'_1(x) = -\gamma(x) + \epsilon\delta\beta(x)\), where \(\gamma(x)\) dx is simply the 1-form \(\gamma\) defined by equation (52), evaluated on the critical trajectory, and \(\delta\beta(x) = (\partial_\theta h^0(x, W'_0(x)))^{-1}\). To understand better the meaning of \(\delta\beta(x)\), we start with the action integral \(\int_{C_{(E)}} \beta\), where \(\beta = \theta\) dx is closely related to the canonical 1-form \(\alpha = x\theta\), because \(\alpha + \beta = d(x\theta)\). Note that, by contrast to \(\alpha\), \(\beta\) is defined on the sphere only after
choosing a determination of the longitude $\theta$. Let us now study the variation of this action integral when $E$ moves away from $E_c$ and is changed into $E_c + \hbar \epsilon$. We have

$$\int_{C(E_c + \hbar \epsilon)} \beta - \int_{C(E_c)} \beta = \hbar \epsilon \int_{C(E_c)} \frac{\partial \theta}{\partial E} dx.$$ 

But $\theta(x, E)$ satisfies $\hbar \partial_\theta^0(x, \theta(x, E)) = E$, so

$$\partial_\theta h^0(x, \theta(x, E)) \frac{\partial \theta}{\partial E} (x, E) = 1$$

and finally

$$\int_{C(E_c + \hbar \epsilon)} \beta - \int_{C(E_c)} \beta = \hbar \epsilon \int_{C(E_c)} \delta \beta(x) dx.$$ 

Explicitly, the equation for $W_1(x)$ reads

$$W_1(x) = \frac{\pm 1}{2\sqrt{\Omega^2 - x}} \left( \frac{\kappa}{2(2s_{cl} - x)} + \frac{\kappa - \epsilon}{x} \right), \quad (75)$$

So we have

$$e^{-W_1(x)} = \exp \left( \pm \frac{1}{4} \log \frac{\kappa + i\sqrt{\Omega^2 - x}}{\kappa - i\sqrt{\Omega^2 - x}} \mp i \frac{\epsilon - \kappa}{2\Omega} \log \frac{\Omega + \sqrt{\Omega^2 - x}}{\Omega - \sqrt{\Omega^2 - x}} \right). \quad (76)$$

We can now expand $p^{WKB}(x)$ when $x$ is small. We find

$$p^{WKB}_\pm (x) \simeq A^{WKB}_\pm (\epsilon) \frac{\exp(i(\pi - \nu)(x/\hbar))}{(x/\hbar)^{(1/2) + i f(E)}} \quad (77)$$

where

$$A^{WKB}_\pm (\epsilon) = \frac{1}{\sqrt{\Omega}} \exp \left( \mp i \left( \frac{\kappa \Omega}{\hbar} + \left( 2s + \frac{1}{2} \right) \nu + \frac{\epsilon - \kappa}{2\Omega} \log \left( \frac{4\Omega^2}{\hbar} \right) \right) \right).$$

Notice that the phase factor is the sum of two terms which have simple geometrical interpretations:

$$S_{cl}^\pm = \kappa \Omega + 2s_{cl} \nu = \pi \Omega^2 - \int_0^{\Omega^2} \theta(x) dx = \int_{C_+} \alpha$$

is the classical action computed on the upper half of the classical trajectory, and

$$\frac{\nu}{2} + \frac{\epsilon - \kappa}{2\Omega} \log \left( \frac{4\Omega^2}{\hbar} \right) = \int_0^{\Omega^2} \tilde{\gamma}(x) dx$$

is the regularized integral of the subprincipal symbol $\tilde{\gamma}(x) = \gamma(x) - \epsilon \delta \beta(x)$ on the same trajectory:

$$\int_0^{\Omega^2} \tilde{\gamma}(x) dx = \lim_{x_A \to 0} \left( \int_0^{\Omega^2} \tilde{\gamma}(x) dx + f(E) \log \frac{x_A}{\hbar} \right).$$

Similarly, setting $x = \Omega - |\xi|$ and expanding in $|\xi|/\Omega$ and $|\xi|/\kappa$ we find, to leading order,

$$p^{WKB}_\pm (x) \simeq \frac{1}{|\Omega| |\xi|^{1/4}} \exp \left( \pm \frac{2i}{3\hbar \kappa} |\xi|^{3/2} \pm i \left( \frac{1}{2\kappa} - \frac{\epsilon - \kappa}{\Omega^2} \right) |\xi|^{1/2} \right). \quad (78)$$

The relevant wavefunction is of course a linear combination of $p^{WKB}_\pm (x)$.
5.3. Airy function analysis

Again, we start with the Schrödinger equation (70). We set
\[ p^\text{Airy}(x) = (-1)^n p^\text{Airy}_n(x), \quad x = n\hbar \]
\[ (x + \hbar)\sqrt{2s_{cl} - x}p^\text{Airy}(x + \hbar) + x\sqrt{2s_{cl} + \hbar - x}p^\text{Airy}(x - \hbar) + 2\kappa xp^\text{Airy}(x) = -\hbar c p^\text{Airy}(x) \]

and we expand \( x = \Omega^2 + \xi \), keeping the terms linear in \( \xi \). Remembering that \( \sqrt{\kappa^2} = -\kappa \), we find
\[ \frac{d^2}{d\xi^2} p^\text{Airy} + \frac{1}{\Omega^2} \left( 1 - \frac{\Omega^2}{2\kappa^2} \right) \frac{d}{d\xi} p^\text{Airy} - \frac{1}{\hbar^2\kappa\Omega^2} \left( \hbar \xi - \frac{\hbar \Omega^2}{2\kappa} - \frac{\Omega^2}{\kappa} \xi \right) p^\text{Airy} = 0. \]

The unique solution which decreases exponentially in the classically forbidden region \( \xi > 0 \) is proportional to the Airy function:
\[ p^\text{Airy}(\xi) = e^{-c\xi} \text{Airy}(a^{1/3}(\xi - b)) \]

where
\[ a = \frac{1}{\hbar^2\kappa^2}, \quad b = \hbar \left( \frac{1}{2} - \frac{\kappa(\epsilon - \kappa)}{\Omega^2} \right), \quad c = \frac{1}{2\Omega^2} \left( 1 - \frac{\Omega^2}{2\kappa^2} \right). \]

When \( \xi < 0 \) it behaves like
\[ \frac{\sin((2/3)|X|^{2/3} + \pi/4)}{|X|^{1/4}}, \quad X = a^{1/3}(\xi - b). \]

We expand in \( |\xi|/\kappa \) and \( |\xi|/\Omega \). One has
\[ \frac{2}{3} |X|^{3/2} = -\frac{2}{3\hbar \kappa} |\xi|^{3/2} - \left( \frac{1}{2\kappa} - \frac{\epsilon - \kappa}{\Omega^2} \right) |\xi|^{1/2}. \]

In that approximation \( e^{-c\xi} \approx 1 \), and we get
\[ p^\text{Airy}(\xi) = C \frac{\sin((2/3\hbar \kappa)|\xi|^{3/2} + ((1/2\kappa) - ((\epsilon - \kappa)/\Omega^2))|\xi|^{1/2} - \pi/4)}{\sqrt{\pi}|\xi|^{1/4}}. \]

5.4. Gluing the parts together

The small \( x \) analysis gave
\[ p(x) = p^\text{small}_+(x) + p^\text{small}_-(x) \]
which is valid when \( x \ll 2s_{cl} \). In an intermediate regime, both the small \( x \) and the WKB approximation are valid at the same time, as can be seen by comparing equation (68) and equation (77). So we can glue these two wavefunctions as follows:
\[ p(x) = \frac{A^\text{Small}(\epsilon)}{A^\text{WKB}(\epsilon)} p^\text{WKB}_+(x) + \frac{A^\text{Small}(\epsilon)}{A^\text{WKB}(\epsilon)} p^\text{WKB}_-(x). \]
We can now extend the $p_\pm^{\text{WKB}}(x)$ functions up to the region $x \simeq \Omega^2 - |\xi|$. Recalling the asymptotic form of the WKB wavefunction in that region, equation (78), we find

$$p(x) \simeq \frac{A_+^{\text{Small}}(\epsilon)}{A_+^{\text{WKB}}(\epsilon) \Omega |\xi|^{1/4}} \exp \left( \frac{2i}{3\hbar \kappa} |\xi|^{3/2} + i \left( \frac{1}{2\kappa} - \frac{\epsilon - \kappa}{\Omega^2} \right) |\xi|^{1/2} \right)$$

$$+ \frac{A_-^{\text{Small}}(\epsilon)}{A_-^{\text{WKB}}(\epsilon) \Omega |\xi|^{1/4}} \exp \left( - \frac{2i}{3\hbar \kappa} |\xi|^{3/2} - i \left( \frac{1}{2\kappa} - \frac{\epsilon - \kappa}{\Omega^2} \right) |\xi|^{1/2} \right).$$

This has to be compatible with the Airy asymptotic formula which embodies the boundary conditions, equation (80). Hence we find the condition

$$\frac{A_+^{\text{Small}}(\epsilon)}{A_-^{\text{Small}}(\epsilon)} \frac{A_+^{\text{WKB}}(\epsilon) e^{i(\pi/4)}}{A_-^{\text{WKB}}(\epsilon) e^{-i(\pi/4)}} = -1.$$  \tag{81}

This equation determines the energy parameter $\epsilon$ and is the generalized Bohr–Sommerfeld condition. Given the fact that

$$\frac{A_+^{\text{Small}}(\epsilon)}{A_-^{\text{Small}}(\epsilon)} = e^{i(\nu-(\pi/2))} \frac{\Gamma(1/2 - i((\epsilon - \kappa)/2\Omega))}{\Gamma(1/2 + i((\epsilon - \kappa)/2\Omega))} \exp \left( i \frac{\epsilon - \kappa}{\Omega} \log \left( \frac{2\Omega}{\sqrt{2s_{cl}}} \right) \right)$$

$$\frac{A_+^{\text{WKB}}(\epsilon)}{A_-^{\text{WKB}}(\epsilon)} = \exp \left( -2i \left[ \frac{2\kappa \Omega}{\hbar} + (4s + 1)\nu + \frac{\epsilon - \kappa}{\Omega} \log \left( \frac{4\Omega^2}{\hbar} \right) \right] \right).$$

Equation (81) becomes

$$\frac{\Gamma(1/2 - i((\epsilon - \kappa)/2\Omega))}{\Gamma(1/2 + i((\epsilon - \kappa)/2\Omega))} \exp \left( i \left[ \frac{2\kappa \Omega}{\hbar} + 2(2s + 1)\nu + \frac{\epsilon - \kappa}{\Omega} \log \left( \frac{8\Omega^3}{\hbar \sqrt{2s_{cl}}} \right) \right] \right) = -1.$$  \tag{82}

Taking the logarithm, we find the quantization condition

$$\Phi_{\text{Sing}}(\epsilon_n) = 2\pi \left( n + \frac{1}{2} \right), \quad n \in \mathbb{Z}, \quad E_n = 2\kappa s_{cl} + \hbar \epsilon_n$$  \tag{83}

where

$$\Phi_{\text{Sing}}(\epsilon) = -i \log \frac{\Gamma(1/2 - i((\epsilon - \kappa)/2\Omega))}{\Gamma(1/2 + i((\epsilon - \kappa)/2\Omega))} + \frac{2\kappa \Omega}{\hbar} + 2(2s + 1)\nu + \frac{\epsilon - \kappa}{\Omega} \log \left( \frac{8\Omega^3}{\hbar \sqrt{2s_{cl}}} \right).$$

To test this condition, we can compute $\delta_n = \Phi(\epsilon_n^{\text{exact}}) - 2\pi (n + \frac{1}{2})$ for the exact values of the energies. In figure 10, we plot $\delta_n$ as a function of $n$, using for $\Phi(\epsilon)$ both the usual Bohr–Sommerfeld function:

$$\Phi_{\text{Reg}}(\epsilon) = \frac{1}{\hbar} \int_{C(E)} \alpha + \int_{C(E)} \gamma$$

and the function $\Phi_{\text{Sing}}(\epsilon)$. Near the singularity, the singular Bohr–Sommerfeld condition is much more accurate.

In figure 11 a typical example of the components of the eigenvectors is shown, comparing with the exact result obtained by direct diagonalization of the Jacobi matrix equation (49) and the various results corresponding to the different approximations: small $x$, WKB, and Airy. The agreement is very good.

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Figure 10. To compare the usual and the singular Bohr–Sommerfeld rules, the default $\delta_n$ is plotted as a function of $n$. The red dots are obtained by using the usual Bohr–Sommerfeld rule, while the green dots are obtained by using $\Phi_{\text{Sing}}(\epsilon)$. The dashed vertical lines represent the interval $|E - E_c| < 2\hbar\Omega$ of validity of the singular Bohr–Sommerfeld rule.

Figure 11. The components of an eigenvector close to the critical level. The dots are the exact result. The red curve is the result of the small $x$ analysis. The green curve is the WKB result and the blue curve is the Airy result.

From equation (83) we can compute the level spacing between two successive energy levels. It is given by

$$\epsilon_{n+1} - \epsilon_n = \frac{2\pi}{\Phi'_{\text{Sing}}(\epsilon_n)} = \frac{2\pi\Omega}{\log \left(8\Omega^3/\hbar\sqrt{2scl}\right) - \Psi'((\epsilon - \kappa)/2\Omega)}$$

where we defined

$$\Psi'(x) = \frac{i}{2} \frac{d}{dx} \log \frac{\Gamma(1/2 - ix)}{\Gamma(1/2 + ix)}.$$  

This function has a sharp minimum located at $x = 0$ where it takes the value $-(\gamma + 2\log 2)$ where $\gamma$ is Euler’s constant; see figure 12. Hence to leading order, the smallest energy spacing is

$$\Delta\epsilon \simeq \frac{2\pi\Omega}{\log \hbar}.$$
Figure 12. The energy level spacing $\epsilon_{k+1} - \epsilon_k$ as a function the state label $k$. This plot shows clearly the accumulation of energy levels in the vicinity of the critical point.

A detailed analysis of the level spacing in the vicinity of the critical level has been given recently [34], where it has been applied to the description of the long time dynamics of the system.

6. Evolution of the oscillator energy

Once the eigenvectors and eigenvalues are known, we can compute the time evolution of the oscillator energy:

$$\bar{x}(t) = \langle s | e^{i(tH/\hbar)} b^\dagger b e^{-i(tH/\hbar)} | s \rangle.$$

Since the observable $b^\dagger b$ commutes with $H_1$, its matrix elements connecting eigenstates of $H_1$ with different eigenvalues vanish. Because $|s\rangle$ is an eigenvector of $H_1$ with eigenvalue $\hbar s$, we can therefore restrict ourselves to this eigenspace of $H_1$. The time evolution can be computed numerically by first decomposing the initial state on the eigenvector basis:

$$|s\rangle = \sum_n c_n |\Psi(E_n)\rangle \quad (85)$$

so the wavefunction at time $t$ is

$$|\Psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar} |\Psi(E_n)\rangle.$$

In the stable regime $\kappa^2 < 2s_{cl}$, we get quite regular oscillations with a rather small amplitude as shown in figure 13.

By contrast, in the unstable regime $\kappa^2 > 2s_{cl}$, it is energetically favorable to excite the oscillator, and we get a succession of well separated pulses, shown in figure 14. Note that the temporal succession of these pulses displays a rather well defined periodicity, but the fluctuations from one pulse to another are relatively large. The remaining part of this paper is an attempt to understand the main features of this evolution.

In figure 15 we show the coefficients $|c_n|$. Only those eigenstates whose energy is close to the critical classical energy $E_c$ contribute significantly. A good estimate of the

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**Figure 13.** The excitation number of the oscillator as a function of time in the stable case. The green curve is the small time result, equation (97). \((\hbar^{-1} = s = 30, \kappa = -2\sqrt{2s_{cl}}, s_{cl} = 1.)\) Notice that the vertical amplitude is about one hundred times smaller than in the unstable case (compare with figure 14).

**Figure 14.** The excitation number of the oscillator as a function of time in the unstable case. \((\hbar^{-1} = s = 30, \kappa = -0.5\sqrt{2s_{cl}}, s_{cl} = 1.)\) It clearly shows an aperiodic succession of pulses. The green curve is the semi-classical result, equation (101).

The energy width of the initial state \(|s\rangle\) is given by \(\Delta E = (\langle s|H^2|s\rangle - \langle s|H|s\rangle^2)^{1/2}\). From equation (49), we find

\[
\Delta E = \hbar \sqrt{2s_{cl}} = \hbar \Omega / \sin \nu.
\]

(86)

Comparing with the criterion \(|\Delta E| \gg 2\hbar \Omega\) for the validity of the stationary phase approximation, we see that most of the eigenstates which have a significant weight in the spectral decomposition of the initial state actually belong to the singular Bohr–Sommerfeld regime.

An important consequence of this observation is that we can compute \(\bar{x}(t)\) by considering only the few relevant states. The result is shown in figure 16 for a spin \(s = 30\). We retained only six states and superposed the result on the exact curve obtained by keeping the 61 states. We can hardly see any difference between the two curves.
Figure 15. The coefficients of the expansion of the initial state $|s\rangle$ on the eigenvectors $|\Psi(E_n)\rangle$. The red dots are the exact values. We see that only eigenvectors close to the critical level contribute significantly. The curve is given by equation (88), which is deduced from the knowledge of the short time evolution. The agreement between this approximate result and the exact one is remarkable, even in the immediate vicinity of the critical level.

Figure 16. The excitation number of the oscillator as a function of time for $s = 30$. The blue curve is the exact result, and the red curve is obtained by keeping only six energy eigenstates.

We can compute the coefficients $|c_n|$ of the decomposition of the initial state $|s\rangle$. From equation (85), we have

$$\langle s|e^{-i(Ht/\hbar)}|s\rangle = \sum_n e^{-i(E_n t/\hbar)}|c_n|^2 \simeq \int_{-\infty}^{\infty} e^{-i(E t/\hbar)}|c(E)|^2 \rho(E) \, dE \quad (87)$$

where we have approximated the sum over discrete energy levels by an integral. Since we already know that the integral is concentrated around the critical energy $E_c$, the density

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ρ(E) can be computed from the singular Bohr–Sommerfeld phase equation (83):

$$\rho(E) = \frac{dn}{dE} = \frac{1}{2\pi} \frac{d}{dE} \Phi_{\text{Sing}}(E) = \frac{1}{2\pi \hbar} \frac{d}{d\epsilon} \Phi_{\text{Sing}}(\epsilon).$$

On the other hand, since

$$|s\rangle = \frac{e_0}{|e_0|}$$

where $e_0$ is the state equation (42) with the oscillator in its ground state, we get

$$\langle s | e^{-i(Ht/\hbar)} | s \rangle = \frac{1}{2\pi} \int_0^{2\pi} \Psi(\theta, t) \, d\theta = p_0(t)$$

where $\Psi(\theta, t)$ is the solution of the Schrödinger equation with initial condition $\Psi(\theta, t)|_{t=0} = 1$. Inverting the Fourier transform in equation (87), we arrive at

$$|c(E)|^2 \rho(E) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} e^{i(Et/\hbar)} p_0(t) \, dt.$$  

Taking for $p_0(t)$ the small time expression for equation (95), which is valid for times $t < t_0 \simeq |\log \hbar|/2\Omega$, we expect to find an expression valid in the energy interval $|E - E_c| > 2\Omega(|\hbar|/|\log \hbar|)$, i.e. for energies far enough from the critical energy. However because of the factor $1/|\log \hbar|$ this interval covers most of the range of equation (86) where we expect $c(E)$ to be substantially non-zero. Notice also that $\hbar/|\log \hbar|$ is the order of magnitude of the level spacing in the critical region, so the interval where the use of the small time wavefunction is not legitimate contains at most a few levels. Explicitly, computing the Fourier transform of equation (95), we find

$$|c(E)|^2 \rho(E) = \frac{1}{\hbar \sqrt{2s_{\text{cl}}}} \frac{e^{-\nu \alpha(E)/\hbar \Omega}}{1 + e^{-(\pi \alpha(E)/\hbar \Omega)}}$$

where

$$\alpha(E) = E - E_c - \hbar \kappa.$$  

The coefficients $|c(E)|^2$ computed with this formula are shown in figure 15. The agreement with the exact coefficients is excellent.

For longer timescales, we may also infer that the quantum dynamics exhibits the Bohr frequencies that can be deduced from the singular Bohr–Sommerfeld quantization rule (83). As shown before, the typical spacing between these energy levels is given by $\Delta \epsilon \simeq (2\pi \Omega/|\log \hbar|)$. This corresponds to a fundamental period $\Delta t \simeq |\log \hbar|/\Omega$. The fact that the energy levels are not equally spaced, as shown in figure 12, is responsible for the aperiodic behavior on timescales larger than $\Delta t$. Systematic procedures for analyzing the long time behavior, starting from a nearly equidistant spectrum, have been developed [35], but we have not yet attempted to apply them to the present problem. Note that the quantum dynamics in the vicinity of classically unstable equilibria has received a lot of attention in recent years [36,37].

Instead, motivated by the analysis of the previous section, we now present a discussion of the time evolution in the semi-classical regime. As for the stationary levels, we show that we have to treat separately the initial evolution, when $x$ is still small and the system is close to the unstable point, and the motion at later times, for which the usual WBK approach is quite reliable. As we demonstrate, this allows us to predict the appearance of pulses, together with their main period $\Delta t$. This generalizes the early study by Bonifacio and Preparata [9], who focused on the $\kappa = 0$ case.
6.1. Small time analysis

Another expression for the mean oscillator energy is

$$\bar{x}(t) = \hbar \sum_{n=0}^{2s} n |p_n(t)|^2$$

where $p_n(t)$ is the solution of equation (46) with boundary condition

$$p_n(t)|_{t=0} = \delta_{n,0}.$$  

For small time, $p_n(t)$ will be significantly different from zero only for small $n$. So in equation (46) we may assume $n \ll 2s$. The equation becomes

$$i\hbar \frac{\partial p_n}{\partial t} = \hbar \sqrt{2s} \hbar (n+1)p_{n+1} + \hbar \sqrt{2s} \hbar np_{n-1} + 2\hbar \kappa (s-n)p_n. \tag{89}$$

As for the stationary case, this equation is linear in $n$ and can be solved by using the Laplace–Fourier transform. We define

$$\Psi(\theta, t) = \sum_{n=0}^{\infty} p_n(t)e^{i n \theta}, \quad \Psi(\theta, t)|_{t=0} = 1. \tag{90}$$

The time evolution is given by

$$\partial_t \Psi = -\left(\sqrt{2s}\cosh(e^{i \theta} + e^{-i \theta}) - 2\kappa\right) \partial_\theta \Psi - i(2s + \sqrt{2s}\cosh e^{i \theta}) \Psi. \tag{91}$$

This equation can be solved by the method of characteristics. We introduce the function $y(\theta)$ defined by

$$\frac{dy}{d\theta} = \frac{1}{\sqrt{2s}\cosh(e^{i \theta} + e^{-i \theta}) - 2\kappa}.$$

Explicitly

$$y(\theta) = \frac{1}{2\Omega} \log \frac{e^{i \theta} + e^{i \nu}}{e^{i \theta} + e^{-i \nu}}, \quad \text{or else} \quad e^{i \theta} = \frac{1}{\sqrt{2s}\cosh \Omega y} \left(\kappa + i\Omega \frac{\cosh \Omega y}{\sinh \Omega y}\right).$$

The time-dependent Schrödinger equation becomes

$$\partial_t \Psi + \partial_y \Psi = -i(2s + 1)\kappa \Psi + \Omega \frac{\cosh \Omega y}{\sinh \Omega y} \Psi$$

whose solution reads

$$\Psi(y, t) = e^{-i\kappa(2s+1)t} \frac{\sinh \Omega y}{\sinh \Omega(y - t)} \Psi(y - t, 0).$$

Imposing the initial condition equation (90) yields $\Psi(y - t, 0) = 1$ and then

$$\Psi(y, t) = e^{-i\kappa(2s+1)t} \frac{\sinh \Omega y}{\sinh \Omega(y - t)}.$$

Returning to the variable $\theta$, we get

$$\Psi(\theta, t) = e^{-i\kappa(2s+1)t} \frac{e^{i \nu} - e^{-i \nu}}{e^{i \nu - i\Omega t} - e^{-i \nu + i\Omega t}} \frac{1}{1 - (e^{i \nu - i\Omega t} - e^{-i \nu + i\Omega t})/(e^{i \nu - i\Omega t} - e^{-i \nu + i\Omega t})e^{i \theta}}. \tag{92}$$
As for the stationary case, we note that this wavefunction has exactly the form as is dictated by semi-classical analysis, where the full symbol associated with equations (89) and (91) is \( h(x, \theta) = 2\sqrt{2\kappa}(x + h/2)(\cos \theta + \cos \nu) \), after we have subtracted the energy \( E_c \) of the unstable point. The generalization of equation (59) to the time-dependent case is

\[
h^0(\partial_\theta S_0(\theta, t), \theta) + \partial_t S_0(\theta, t) = 0. \tag{93}
\]

But since \( \Psi \) is independent of \( \theta \) at \( t = 0 \), \( \partial_\theta S_0(\theta, t = 0) = 0 \), which means classically that the trajectory begins at \( x = 0 \). But because of the form of \( h^0 \), we get \( h^0(\partial_\theta S_0(\theta, t = 0), \theta) = 0 \) for any \( \theta \), which implies that \( \partial_t S_0(\theta, t = 0) = 0 \). This shows that \( S_0(\theta, t) = 0 \) identically, in agreement with the fact that a classical trajectory starting at \( x = 0 \) stays there for ever! So the time evolution manifested in equation (92) is of purely quantum nature, and it is encoded in the evolution of the amplitude \( a(\theta, t) \) and the subleading phase \( S_1(\theta, t) \). Writing \( \Psi = a(\theta, t) \exp(iS_1(\theta, t)) \), the next order in \( \hbar \) gives

\[
\partial_x h^0(0, \theta)\partial_\theta \Psi + \partial_\theta \Psi + \frac{1}{2} \frac{d}{d\theta} \left( \partial_x h^0(0, \theta) \right) \Psi + i\hbar \frac{1}{2} \partial_t S_0(\theta, t) \Psi = 0. \tag{94}
\]

A simple check shows that this is exactly the same equation as (91) without the term \(-i2\kappa s\Psi \), due to the subtraction of the energy \( E_c \). Expanding in \( e^{i\theta} \) we find

\[
p_n(t) = e^{-i\kappa(2s+1)t} \frac{e^{i\nu} - e^{-i\nu}}{e^{i\nu-\Omega t} - e^{-i\nu+\Omega t}} \left[ \frac{e^{\Omega t} - e^{-\Omega t}}{e^{i\nu-\Omega t} - e^{-i\nu+\Omega t}} \right]^n. \tag{95}
\]

It is instructive to consider the large \( n \) limit. Then \( p_n(t) \) is proportional to \( \exp[-(i/\hbar)(\kappa(2s+1) + \hbar \nu)t + (\pi - \nu)x]] \). This is the dominant phase factor for a WKB state of energy \( E = E_c \) which is concentrated on the outgoing branch of the critical classical trajectory. Note that there also appears a quantum correction to the energy, corresponding to a finite \( \epsilon = \kappa \). On the physical side, this is quite remarkable, because we have just seen that the short time evolution is driven by purely quantum fluctuations, formally described by the subprincipal symbol. Nevertheless, the subsequent evolution is quite close to the classical critical trajectory, because the energy distribution of the initial state is quite narrow, as we have discussed.

Further information is obtained by looking at the probability distribution of the number of emitted quanta:

\[
|p_n(t)|^2 = \frac{\Omega^2}{\Omega^2 + 2s_\kappa \sinh^2 \Omega t} \left[ \frac{2s_\kappa \sinh^2 \Omega t}{\Omega^2 + 2s_\kappa \sinh^2 \Omega t} \right]^n.
\]

This is of the form

\[
|p_n(t)|^2 \simeq e^{-\beta(t)n}, \quad \beta(t) = \log \left( 1 + \frac{\Omega^2}{2s_\kappa \sinh^2 \Omega t} \right).
\]

Hence we have a thermal distribution with a time-dependent effective temperature. Such behavior is due to the strong entanglement between the spin and the oscillator. In fact we somehow expect that quantum effects, such as entanglement, are likely to be magnified in situations where there is an important qualitative difference between the classical and the quantum evolutions.
It is now simple to compute the mean number of molecules produced in this small time regime. We find
\[ \bar{x}(t) = 2\hbar s_{cl} \frac{\sinh^2 \Omega t}{\Omega^2}, \quad \Omega = \sqrt{2s_{cl} - \kappa^2}. \] (96)

The small time approximation is valid as long as the number of molecules is small and will break after a time \( t_0 \) such that
\[ \bar{x}(t_0) \simeq 1. \]
To leading order in \( \hbar \), this timescale is
\[ t_0 \simeq -\frac{1}{2\Omega} \log \hbar. \]
Note that in the stable case, we change \( \Omega \) into \( i\Omega \) and then
\[ \bar{x}(t) = 2\hbar s_{cl} \frac{\sin^2 \Omega t}{\Omega^2}, \quad \Omega = \sqrt{\kappa^2 - 2s_{cl}}. \] (97)

In that case we never leave the small \( n \) regime and this approximation remains valid even for large time as can be seen in figure 13.

6.2. Periodic soliton pulses

When time increases, the approximation \( n \ll 1 \) is not valid any longer and we must take into account the exact quantum Hamiltonian. In the regime \( \hbar \ll x = nh \ll 2s_{cl} \), we can still perform a WKB approximation. The previous discussion has shown that the semi-classical wavefunction is concentrated on the classical orbit of \( h^0 \) with energy \( E = E_c + \hbar \kappa \), that is \( \epsilon = \kappa \). We set
\[ p(x, t) = e^{-(i/\hbar)(2s_{cl} + h)(t - (i/\hbar)(W_0(x) + \hbar W_1(x)))} b(x, t) \]
where \( W_0(x) \) is given by equation (73), and \( W_1(x) \) by equation (76) in which \( \epsilon \) is replaced by its actual value \( \kappa \). The only source of time dependence, at this level of approximation, arises from the transport equation of the amplitude. Let us define the local velocity \( v(x) \) on the classical trajectory by
\[ v(x) = -\partial_x h^0 (x, W_0'(x)) = \pm 2x\sqrt{\Omega^2 - x}. \]
From this field, we obtain the Hamiltonian flow on the classical trajectory by solving the differential equation \( dx/dt = v(x) \). Let us denote by \( x(x_0, t) \) the solution of this equation which starts from \( x_0 \) at \( t = 0 \). Likewise, we may reverse the flow and define \( x_0(x, t) \). It is convenient to introduce the function \( u(x) \) such that \( du/dx = -1/v(x) \). The solution of the flow is then
\[ u(x_0) = u(x) + t. \] (98)

For our problem, we have
\[ u(x) = -\int_0^x \frac{1}{2x\sqrt{\Omega^2 - x}} \, dx, \quad \text{or else} \quad x = \frac{\Omega^2}{\cosh^2 \Omega u}. \]

The transport equation simply states the conservation of the local density \( b^2(x, t) \):
\[ \partial_t (b^2(x, t)) + \partial_x (b^2(x, t)v(x)) = 0. \]
The evolution of this one-dimensional conserved fluid is characterized by the invariance of $b^2v$ along the flow:

$$b^2(x, t)v(x) = b^2(x_0(x, t), 0)v(x_0(x, t)) \equiv B_0^2(u(x) + t)$$

which simply results from the conservation law and the fact that the velocity field does not depend on time.

Then we have

$$\bar{x}(t) = \int dx \frac{x}{x^2(\Omega^2 - x)^{1/2}} B_0(u(x) + t)^2 = \int du \frac{\Omega^2}{\cosh^2 \Omega u}B_0(u + t)^2. \quad (100)$$

If we assume that $B_0(u)^2$ is peaked around $t_0$, say $B_0(u)^2 = \delta(u - t_0)$, we find

$$\bar{x}(t) = \frac{\Omega^2}{\cosh^2 \Omega(t - t_0)} \quad (101)$$

which is just the classical expression. There is a simple way to evaluate the time $t_0$. In fact there must be a regime where the small time formula and this semi-classical formula should agree, i.e.,

$$\hbar \frac{2s_{\text{cl}}}{\Omega^2} \sinh^2 \Omega t \simeq \frac{\Omega^2}{\cosh^2 \Omega(t - t_0)}, \quad \text{or} \quad \hbar \frac{2s_{\text{cl}}}{4\Omega^2} e^{2\Omega(t - t_0)} \simeq 4\Omega^2 e^{2\Omega(t - t_0)} \quad (102)$$

and this gives the time $t_0$:

$$t_0 = -\frac{1}{2\Omega} \log \frac{2s_{\text{cl}}}{16\Omega^2} \hbar \simeq -\frac{1}{2\Omega} \log \hbar + O(\hbar^0). \quad (103)$$

This reproduces well the position of the first peak. Its height is also well reproduced if instead of the crude approximation $B_0(u)^2 = \delta(u - t_0)$, we glue the wavefunctions given by the small time solution and the WKB solution at a gluing time $t_g$:

$$B_0(u + t_g)^2 \simeq K \frac{\sinh(\Omega u)}{\cosh(\Omega u)^3} \exp \left( -\frac{\Omega^2}{\bar{x}(t_g) \cosh(\Omega u)^2} \right)$$

where the normalization constant $K$ is given by the condition

$$\int_0^\infty B_0(u)^2 du = 1$$

and $\bar{x}(t_g)$ is the value of $\bar{x}$ given by equation (96) taken at time $t = t_g$ (see figure 17). The height of the first peak can be estimated from equation (100). The maximum is obtained when we have the largest overlap between $1/\cosh^2 \Omega u$ and $B_0(u + t)^2$. This happens with a very good approximation, for $t = t_0$ such that the maximum of $B_0(u + t_0)^2$ is at zero:

$$\sinh^2 \Omega(t_0 - t_g) = \frac{1}{4} \left( 1 - \frac{2\Omega}{\bar{x}(t_g)} \right) + \sqrt{\frac{1}{16} \left( 1 - \frac{2\Omega}{\bar{x}(t_g)} \right)^2 + \frac{1}{2}}.$$

The previous equation allows us also to discuss the freedom in the choice of the gluing point $t_g$. Indeed we expect that $t_0$ does not depend on $t_g$ as long as $t_g$ is restricted to a temporal domain where the short time solution and the semi-classical one overlap. If we plot $t_0$ as a function of $t_g$, we see in figure 18 that the curve that we obtain develops a plateau, which means that the value of $t_0$ that we obtain choosing $t_g$ inside this plateau
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Figure 17. Time evolution of the mean oscillator energy $\bar{x}(t)$ for $\kappa = 1/\sqrt{2}$, $s_{cl} = 1$ and $\hbar = 1/300$. The blue line corresponds to the exact solution, the red one to the semi-classical solution and the yellow line to the small time solution.

Figure 18. Time $t_0$ of the first pulse as a function of the gluing time $t_g$ between the short time solution and the WKB solution. The existence of a plateau shows that gluing can be done in an unambiguous way. $\kappa = -1/\sqrt{2}$, $s_{cl} = 1$ and $\hbar = 10^{-4}$. We can recognize a plateau between $t_g = 1$ and 4.

is independent of $t_g$. Moreover, if we plot the value of $t_0$ at the plateau, as a function of $\log(\hbar)$, we recover with a very good accuracy the rough estimation of equation (103); see figure 19.

If we return to the delta-function approximation, it is clear that the situation reproduces itself after a time interval $2t_0$. Hence we can write

$$\bar{x}(t) \simeq \sum_{k=-\infty}^{\infty} \frac{\Omega^2}{\cosh^2 \Omega(t-(2k+1)t_0)} \simeq \Omega^2 \text{cn}^2[\Omega(t-t_0)|k^2]$$

where cn is a Jacobi elliptic function of modulus $k^2$ given approximately by

$$k^2 \simeq 1 - \frac{2\hbar s_{cl}}{\Omega^4}.$$
7. Conclusion

Let us emphasize the main results of this work. First, the Jaynes–Cummings model exhibits, in a large region of its parameter space, an unstable fixed point which corresponds to the focus–focus singularity of an integrable system with two degrees of freedom. As a result, there is a classical monodromy when one considers a loop which encircles the critical value in the \((H_0, H_1)\) plane. At the quantum level, this phenomenon is manifested by a dislocation in the joint spectrum of \(H_0\) and \(H_1\). We then analyzed the eigensubspace of \(H_1\) which corresponds to the critical point. The associated reduced phase space is a sphere. We have shown how to perform a semi-classical analysis using the convenient but singular coordinates \((x, \theta)\) on this sphere. The main result here is that when the classical orbit crosses either the north or the south pole, where the longitude \(\theta\) is not well defined, the action integral associated with the subprincipal symbol jumps by \(\pm \pi\), and this is compensated by a simultaneous jump in the Maslov index. Most of the spectrum is well described by usual Bohr–Sommerfeld quantization rules, with the exception of typically \(|\log \hbar|\) eigenstates in the vicinity of the critical energy, for which special Bohr–Sommerfeld rules have been obtained. Remarkably, the classical unstable equilibrium state, where the spin component \(s^z\) is maximal and the oscillator is in its ground state, has most of its weight on the subspace spanned by these singular semi-classical states. This fact explains rather well the three timescales observed in the evolution of the mean energy of the oscillator. At short time, this energy grows exponentially, reflecting the classical instability of the initial condition. At intermediate times, the energy of the oscillator exhibits a periodic sequence of pulses, which are well described as classical motion along...
the pinched torus containing the unstable point. Finally, the delicate pattern of energy levels, which are not exactly equidistant, governs the aperiodic behavior observed for longer timescales.

This work leaves many unsolved questions. One of them is that of how to develop a detailed analytical description of the long time behavior. This should be \textit{a priori} possible because, as we have seen, the initial state is a linear superposition of only a small number of energy eigenstates, for which the singular WKB analysis developed here provides an accurate modeling. Another interesting direction is the extension to several spins, which is physically relevant to the dynamics of cold atom systems after a fast sweep through a Feshbach resonance. It would be interesting to discuss whether the notion of monodromy can be generalized with more than two degrees of freedom. Finally, it remains to be seen whether the qualitative features of the time evolution starting from the unstable state remain valid for an arbitrary number of spins, as may be conjectured from the structure of the quadratic normal form in the vicinity of the singularity.

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\section*{References}

[1] Jaynes E and Cummings F, \textit{Comparison of quantum and semiclassical radiation theories with application to the beam maser}, 1963 Proc. IEEE \textbf{51} 89
[2] Ackerhalt J and Rzazewski K, \textit{Heisenberg-picture operator perturbation theory}, 1975 Phys. Rev. A \textbf{12} 2549
[3] Narozyński N, Sanchez-Mondragon J and Eberly J, \textit{Coherence versus incoherence: collapse and revival in a simple quantum model}, 1981 Phys. Rev. A \textbf{23} 236
[4] Rempe G, Walther H and Klein N, \textit{Observation of quantum collapse and revival in a one-atom maser}, 1987 Phys. Rev. Lett. \textbf{58} 353
[5] Brune M, Schmidt-Kaler F, Maali A, Dreyer J, Hagley E, Raimond J M and Haroche S, \textit{Quantum Rabi oscillation: a direct test of field quantization in a cavity}, 1991 Phys. Rev. A \textbf{44} 5913
[6] Gea-Banacloche J, \textit{Atom and field-state evolution in the Jaynes–Cummings model for large initial fields}, 1991 Phys. Rev. A \textbf{44} 5913
[7] Brune M, Hagley E, Dreyer J, Maitre X, Maali A, Wunderlich C, Raimond J M and Haroche S, \textit{Observing the progressive decoherence of the ‘meter’ in a quantum measurement}, 1996 Phys. Rev. Lett. \textbf{77} 4887
[8] Dicke R H, \textit{Coherence in spontaneous radiation processes}, 1954 Phys. Rev. \textbf{93} 99
[9] Bonifacio R and Preparata R, \textit{Coherent spontaneous emission}, 1970 Phys. Rev. A \textbf{2} 336
[10] Greiner M, Regal C A and Jin D S, \textit{Emergence of a molecular Bose–Einstein condensate from a Fermi gas}, 2003 Nature \textbf{426} 537
[11] Bourdel T, Khaykovich L, Cubizolles J, Zhang J, Chevy F, Teichmann M, Tarruell L, Kokkelmans S J and Salomon C, \textit{Experimental study of the BEC–BCS crossover region in lithium 6}, 2004 Phys. Rev. Lett. \textbf{93} 050401
[12] Zwierlein M W, Stan C A, Schunck C H, Raupach S M F, Kerman A J and Ketterle W, \textit{Condensation of pairs of fermionic atoms near a Feshbach resonance}, 2004 Phys. Rev. Lett. \textbf{92} 120403
[13] Barankov R A and Levitov L S, \textit{Atom–molecule coexistence and collective dynamics near a Feshbach resonance of cold fermions}, 2004 Phys. Rev. Lett. \textbf{93} 130403

doi:10.1088/1742-5468/2009/07/P07011
A semi-classical study of the Jaynes–Cummings model

[14] Yuzbashyan E, Kuznetsov V and Altshuler B, *Integrable dynamics of coupled Fermi–Bose condensates*, 2005 Phys. Rev. B 72 144524

[15] Gaudin M, 1983 *La Fonction D’onde De Bethe* (Paris: Masson) chapter 13

[16] Colin de Verdière Y and Parisse B, *Equilibre instable en régime semi-classique: I—concentration microlocale*, 1994 Commun. PDE 19 1535

[17] Colin de Verdière Y and Parisse B, *Equilibre instable en régime semi-classique: II—conditions de Bohr–Sommerfeld*, 1994 Ann. Inst. Henri Poincare 61 347

[18] Brummelhuis R, Paul T and Uribe A, *Spectral estimates around a critical level*, 1995 Duke Math. J. 78 477

[19] Child M S, *Quantum states in a champagne bottle*, 1998 J. Phys. A: Math. Gen. 31 657

[20] Colin de Verdière Y and Parisse B, *Singular Bohr–Sommerfeld rules*, 1999 Commun. Math. Phys. 205 459

[21] Bates L, *Monodromy in the champagne bottle*, 1991 J. Appl. Math. Phys. 42 837

[22] Viu Ngoc S, *Quantum monodromy in integrable systems*, 1999 Commun. Math. Phys. 203 465

[23] Sadovskii D A and Zhislinik B I, *Quantum monodromy, its generalizations and molecular manifestations*, 2006 Mol. Phys. 104 2595

[24] Bellon M and Talon M, *Spectrum of the quantum Neumann model*, 2005 Phys. Lett. A 337 360

[25] Zou M, *Monodromy in two degrees of freedom integrable systems*, 1992 J. Geom. Phys. 10 37

[26] Duistermaat J J, *Oscillatory integrals, Lagrange immersions, and unfolding of singularities*, 1974 Commun. Pure Appl. Math. 27 207

[27] Keeling J, *Quantum corrections to the semiclassical collective dynamics in the Tavis–Cummings model*, 2009 arXiv:0901.4245

[28] Micheli A, Jaksh D, Cirac J I and Zoller P, *Many-particle entanglement in two-components Bose–Einstein condensates*, 2003 Phys. Rev. A 67 013607

[29] Boukobza E, Chuchem M, Cohen D and Vardi A, *Phase-diffusion dynamics in weakly coupled Bose–Einstein condensates*, 2008 arXiv:0812.1204

[30] Faribault A, Calabrese P and Caux J S, *Quantum quenches from integrability: the fermionic pairing model*, 2009 J. Stat. Mech. P03018

[31] Berezin F A, *General concept of quantization*, 1975 Commun. Math. Phys. 40 153

[32] Duistermaat J J, *Symbolic calculus for Toeplitz operators with half-form*, 2006 J. Symplec. Geom. 4 171

[33] Duistermaat J J, *Non-Hamiltonian monodromy*, 2001 J. Diff. Eq. 172 42

[34] Charles L, *Semiclassical dynamics of the quartic oscillator*, 1996 Phys. Rev. A 54 2887

[35] Micheli A, Jaksh D, Cirac J I and Zoller P, *Many-particle entanglement in two-components Bose–Einstein condensates*, 2003 Phys. Rev. A 67 013607

[36] Bellon M and Taron M, *The quantum Neumann model: asymptotic analysis*, 2006 Phys. Lett. A 351 283

[37] Bellon M and Taron M, *The quantum Neumann model: refined semiclassical results*, 2006 Phys. Lett. A 356 110

[38] Keeling J, *Quantum corrections to the semiclassical collective dynamics in the Tavis–Cummings model*, 2009 arXiv:0901.4245

[39] Micheli A, Jaksh D, Cirac J I and Zoller P, *Many-particle entanglement in two-components Bose–Einstein condensates*, 2003 Phys. Rev. A 67 013607

[40] Boukobza E, Chuchem M, Cohen D and Vardi A, *Phase-diffusion dynamics in weakly coupled Bose–Einstein condensates*, 2008 arXiv:0812.1204

doi:10.1088/1742-5468/2009/07/P07011