Exact solutions for isometric embeddings of pseudo-Riemannian manifolds

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Abstract. Embeddings into higher dimensions are of direct importance in the study of higher dimensional theories of our Universe, in high energy physics and in classical general relativity. Theorems have been established that guarantee the existence of local and global codimension-1 embeddings between pseudo-Riemannian manifolds, particularly for Einstein embedding spaces. A technique has been provided to determine solutions to such embeddings. However, general solutions have not yet been found and most known explicit solutions are for embedded spaces with relatively simple Ricci curvature. Motivated by this, we have considered isometric embeddings of 4-dimensional pseudo-Riemannian spacetimes into 5-dimensional Einstein manifolds. We have applied the technique to treat specific 4-dimensional cases of interest in astrophysics and cosmology (including the global monopole exterior and Vaidya-de Sitter-class solutions), and provided novel physical insights into, for example, Einstein-Gauss-Bonnet gravity. Since difficulties arise in solving the 5-dimensional equations for given 4-dimensional spaces, we have also investigated embedded spaces, which admit bulks with a particular metric form. These analyses help to provide insight to the general embedding problem.

1. Introduction

Inspired by string theory, D-brane theory and Horava-Witten theory, there has been much interest in extra dimensional effective theories, evidenced by phenomenological models such as the braneworld scenarios [1], which attempt to tackle problems such as the mass-hierarchy problem, the origins of the primordial spectrum, the inflationary field and dark energy. Einstein-Gauss-Bonnet (EGB) gravity [2, 3] takes a complementary approach, building a natural five-dimensional (5D) theory by considering additional terms in the action that only contribute for more than four dimensions. Induced matter theory [4] attempts to describe energy-momentum on the brane in terms of purely geometric effects in higher dimensions. This higher dimensional physics is complemented by the recent proof of several existence theorems for both local [5, 6, 7] and global embeddings [8].

The Dahia-Romero theorem [5] proves that a 4D analytic pseudo-Riemannian manifold $M$ with metric $ds^2 = g_{ik} dx^i dx^k$ has a local analytic isometric embedding into a 5D Einstein manifold $N$ with metric (in Gaussian normal coordinates, without loss of generality):

$$d\tilde{s}^2 = \tilde{g}_{\alpha\beta}(x^1, y) dx^\alpha dx^\beta = \tilde{g}_{ik}(x^1, y) dx^i dx^k + \epsilon dy^2, \quad \epsilon^2 = 1, \quad \tilde{g}_{ik}(x^1, 0) = g_{ik},$$

(1)

along the hypersurface $y = 0$. Here, $y$ denotes the fifth coordinate, $\tilde{R}_{\alpha\beta} = \frac{2\Lambda}{3} \tilde{g}_{\alpha\beta}$ and $\Lambda$ is the 5D cosmological constant. The metric in Eq. (1) is a solution to the Codazzi, Gauss and...
Ricci curvature propagation equations (equivalent to the field equations) are given respectively by:
\[
\tilde{R}_{ik} = \tilde{g}^{jk}(\nabla_j \tilde{\Omega}_{ik} - \nabla_i \tilde{\Omega}_{jk}) = 0, \tag{2}
\]
\[
\tilde{G}^m_n = -\frac{1}{2} \tilde{g}^{ik} \tilde{g}^{jm}[\tilde{R}_{ikjm} + \epsilon(\tilde{\Omega}_{ik} \tilde{\Omega}_{jm} - \tilde{\Omega}_{jk} \tilde{\Omega}_{im})] = \Lambda, \quad \text{and} \tag{3}
\]
\[
\tilde{R}_{ik} = \tilde{R}_{ik} + \frac{\epsilon}{2} \frac{\partial^2 \tilde{g}_{ik}}{\partial y^2} + \frac{\epsilon \tilde{g}^{lm}}{4} \left( \frac{\partial \tilde{g}_{ik} \partial \tilde{g}_{jm}}{\partial y} - \frac{1}{2} \frac{\partial \tilde{g}_{jm} \partial \tilde{g}_{jk}}{\partial y} \right) = -\frac{2\Lambda}{3} \tilde{g}_{ik}, \tag{4}
\]
with the condition \( \tilde{g}_{ik}(x^2,0) = g_{ik} \). The term \( \tilde{\Omega}_{ik} = -\frac{1}{2} \frac{\partial \tilde{g}_{ik}}{\partial y} \) is the extrinsic curvature of the surface \( y = 0 \) with \( \tilde{\Omega}_{ik}(x^2,0) = \tilde{\Omega}_{ik} \). By appealing to the Cauchy-Kowalewskaja theorem, it can be shown that a unique analytic solution \( \tilde{g}_{ik} \) exists, and that Eqs. (2) and (3) need only be solved at \( y = 0 \) [5]. The propagation Eq. (4) is used to specify the rest of the bulk. The embedded and embedding spaces are well-behaved with regards to stability and causality [9].

Despite these existing results, it has been difficult to obtain general solutions to the embedding equations as well as explicit solutions for chosen embedded spaces, particularly those with non-trivial energy-momentum and/or Ricci curvature. This motivates the investigation of embeddings for 4D non-vacuum spacetimes relevant in astrophysics and cosmology. We have focused on 5D Einstein spaces, because of their role in high-energy physics and their geometric simplicity.

Above and in the following, we have consistently adopted the following notational conventions: Roman lower case indices label the coordinates of the embedded (lower dimensional) space, Roman upper case indices label its spatial coordinates, and Greek indices label the coordinates of the embedding (higher dimensional bulk) space. A tilde denotes quantities pertaining to the bulk, an overbar denotes quantities obtained from the \( n \)-dimensional component of the bulk metric, and \( d\Psi^2 = d\theta^2 + \sin^2 \theta d\phi^2 \).

2. Results
We have applied the above methodology to investigate embeddings of 4D spherically symmetric spaces into 5D Einstein bulks [10, 11, 12]. Here, we have presented a summary of our findings. First, we examined an embedding of the 4D exterior global monopole [10], which is a pointlike topological defect formed during symmetry breaking phenomena in the early universe [13], and occurs as a limit in EGB Kaluza-Klein black holes [3]. The gravitational field exterior to a 4D global monopole is a non-vacuum space with simple metric:
\[
ds^2 = -dt^2 + K^{-1}dr^2 + r^2 d\Psi^2, \quad K = 1 - 8\pi G\eta^2, \quad \eta \sim 10^{16} \text{GeV}, \quad \text{and} \quad r > r_c,
\]
but non-trivial Ricci curvature \( R = \frac{2(K-1)}{r^2} \), and energy-momentum \( T_{00} = \frac{-\eta^2}{r^2}, \quad T_{11} = \frac{\eta^2}{Kr^2} \). The embedding Eqs. (2), (3) and (4) with \( \tilde{\Lambda} = 0 \) can be solved to obtain a Ricci flat bulk:
\[
ds^2 = -dt^2 + K^{-1}dr^2 + (r - \alpha y)^2 d\Psi^2 + \epsilon dy^2, \quad \alpha^2 = \epsilon(1 - K), \tag{5}
\]
that embeds the exterior global monopole at \( y = 0 \). Considering 5D Ricci flat bulks of the form:
\[
\bar{g}_{ik} = \text{diag} \left[-e^{-B(y,r)}, \quad e^{B(y,r)}, \quad (r - \sigma y)^2, \quad (r - \sigma y)^2 \sin^2 \theta] \right., \quad \sigma \in \mathbb{R}, \tag{6}
\]
we have found only the 4D solutions \( ds^2 = -(1 - \epsilon \sigma^2)dt^2 + (1 - \epsilon \sigma^2)^{-1}dr^2 + r^2 d\Psi^2 \). Thus, the global monopole metric is the canonical example for embeddings of type given in Eq. (6). Applying the transformation \( R = r - \alpha y, \quad Y = \alpha(\epsilon K)^{-1/2}r + (\epsilon K)^{1/2}y \) to Eq. (5) yields 5D Minkowski space with no deficits, where the surface \( Y = \alpha(\epsilon K)^{-1/2}R \) corresponds to the monopole metric. This transformation involves a rotation of \( (r, y) \) to \( (R, Y) \) by different small
angles for the $r$ and $y$ axes, which are related to the energy scale $\eta$. By the global existence theorems [8], we have deduced that the monopole exterior can be embedded into a 5D globally flat space. With the imposition of a deficit angle, various 5D cosmic string solutions may also be constructed. Since 5D Minkowski space is stable, this potentially alleviates the stability problem for the 4D monopole exterior metric, provided we regard gravity as being five-dimensional.

Hypersurfaces of constant $y$, which correspond to different global monopoles, and so the embedding space appears to have a 'linelike' structure, although it is not a product metric. This facilitates a comparison with the $r \rightarrow 0$ limit of the EGB black hole [3], in which we have observed how the Gauss-Bonnet term increases the number of permissible (product) vacuum topologies. Furthermore, our (locally flat) metric, with appropriate compactifications, can be viewed as a local embedding into the 5D EGB Kaluza-Klein black hole exterior. This can be made global [10, 8], thereby regaining the model of [3]. The energy-momentum tensor $\bar{T}_{\mu\nu} = - \eta^2 / (r - \alpha y)^2$, $\bar{T}_{11} = \eta^2 / K (r - \alpha y)^2$ for $\bar{g}_{ik}$ reduces to the matter tensor of the global monopole when $y = 0$. This is relevant to induced matter theory [4] as our bulk is empty, but its hypersurfaces of constant $y$ contain matter.

We also have investigated [11, 10] what 4D spherically symmetric spaces in the usual form $ds^2 = -e^{2\nu(t,r)} dt^2 + e^{2\lambda(t,r)} dr^2 + r^2 d\Psi^2$ can embed into 5D Einstein bulks of particular metric forms:

$$ds^2 = A(y) g_{00} dt^2 + B(y) g_{11} dr^2 + C(y) g_{22} d\theta^2 + D(y) g_{33} d\phi^2 + \epsilon dy^2, \quad \text{and}$$

$$ds^2 = A(y, r) ds^2 + \epsilon dy^2. \quad (7)$$

For Eq. (7), the only possible embedded solutions are the general Schwarzschild-de Sitter space, and the Einstein universe given respectively by (with $f, h$ arbitrary functions):

$$ds^2 = -f(t) \left(1 + \frac{k}{r} + \frac{\alpha_1}{3} r^2 \right) dt^2 + \left(1 + \frac{k}{r} + \frac{\alpha_1}{3} r^2 \right)^{-1} dr^2 + r^2 d\Psi^2, \quad \alpha_1, k \in \mathbb{R}, \quad (9)$$

$$ds^2 = -h(t) dt^2 + \left(1 + \frac{\alpha_2}{2} r^2 \right)^{-1} dr^2 + r^2 d\Psi^2, \quad \alpha_2 \in \mathbb{R}, \alpha_2 < 0. \quad (10)$$

A 5D solution in Eq. (7) for embedding metric in Eq. (9) is already known for both $\Lambda = 0$ and $\Lambda \neq 0$ [7], and has $A = B = C = D$. We partially solve the 5D embedding space given in Eq. (7) for the Einstein universe given in Eq. (10) as:

$$A = 1, \quad B = C = D = \left(1 + \sqrt{-\epsilon \alpha_2 / 2} y \right)^2, \quad \Lambda = 0, \quad (11)$$

$$A = \left[ \cosh \left( \sqrt{-2 \epsilon \Lambda / 3} y \right) - a \sqrt{-3 \epsilon / 2} \sinh \left( \sqrt{-2 \epsilon \Lambda / 3} y \right) \right]^2, \quad B = C = D = 1, \quad \Lambda = -3 \alpha_2 / 2, \quad (12)$$

where Eq. (11) is Ricci flat, and Eq. (12) is a particular non-Ricci-flat solution with $a \in \mathbb{R}$ (obtained via Lie symmetry analysis with collaborators [12]). The solution in Eq. (11) can be transformed to Minkowski form so that the embedding space is flat, regaining the result given in Eq. (14). The bulk metric in Eq. (10) cannot have $A = B = C = D$. For ansatz in Eq. (8), we have obtained $\frac{dA}{dr} = 0$, so it reduces to the case of embedding solution in Eq. (9).

Finally, we have investigated [10] the 4D spherically symmetric spaces in retarded time coordinates $ds^2 = \sigma(v,r) dv^2 - 2 \mu(v,r) dv dr + r^2 d\Psi^2$, which can embed into Einstein bulks given in Eq. (1) with:

$$\bar{g}_{ik} = A(y, v) g_{ik}, \quad (13)$$

$$\bar{g}_{ik} = A(y) g_{00} \delta^0_i \delta^0_k + A(y) g_{22} \delta^2_i \delta^2_k + A(y) g_{33} \delta^3_i \delta^3_k + B(y) g_{01} \delta^0_i \delta^1_k + B(y) g_{10} \delta^1_i \delta^0_k. \quad (14)$$
We have determined that $\frac{\partial A}{\partial v} = 0$ for ansatz in Eq. (13), $A = B$ for Eq. (14), and the only 4D solution is the general Vaidya-de Sitter space with constant mass. This is a vacuum space, and so its Einstein bulk is already known [7]. Work on the non-constant mass case is ongoing.

3. Conclusion and future work

We have successfully embedded the exterior global monopole metric, general Schwarzschild-de Sitter space, Einstein universe and general constant mass Vaidya-de Sitter space as the canonical embeddings of types given in Eqs. (6), (7), (8), (13) and (14). These may be promoted to global results in various ways. Full solutions for the exterior global monopole metric and Einstein universe into non-Ricci-flat Einstein spaces have yet to be determined. It would be interesting to consider embedding a 4D global monopole with core mass and/or a cosmological constant term, and to embed global cosmic strings. Further work would include embedding the Reissner-Nordström black hole as another non-vacuum test case, and investigating what spaces can embed into 5D Einstein metrics of different forms. Other points of interest are the comparison of embeddings in general relativity with EGB gravity cases, and the relationship between the conformal geometries of embedding and embedded spaces.

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References

[1] Randall L and Sundrum R 1999 Phys. Rev. Lett. 83 3370, Randall L and Sundrum R 1999 Phys. Rev. Lett. 83 4690, Arkani-Hamed N, Dimopoulos S and Dvali G 1998 Phys. Lett. B 429 263, Arkani-Hamed N, Dimopoulos S and Dvali G 1999 Phys. Rev. D 59 086004, Dvali G, Gabadadze G and Porrati M 2000 Phys. Lett. B 485 208

[2] Dadhich N K 2005 Probing universality of gravity Proc. 11th Reg. Conf. Math. Phys. (Singapore: World Scientific), Dadhich N K 2007 On the Gauss-Bonnet gravity Proc. 12th Reg. Conf. Math. Phys. (Singapore: World Scientific)

[3] Maeda H and Dadhich N K 2006 Phys. Rev. D 74 021501

[4] Wesson P S 1999 Spacetime Matter (Singapore: World Scientific)

[5] Dahia F and Romero C 2002 J. Math. Phys. 43 5804

[6] Dahia F and Romero C 2002 J. Math. Phys. 43 3097

[7] Lidsey J E, Romero C, Tavakol R and Rippl S 1997 Class. Quant. Grav. 14 865, Anderson E and Lidsey J E 2001 Class. Quant. Grav. 18 4831

[8] Katzourakis N I 2005 [math-ph/0407067v4], Moodley J 2008 M.Sc. Thesis, University of KwaZulu-Natal, Moodley J and Amery G 2012 To be submitted to J. Math. Phys.

[9] Dahia F and Romero C 2005 Class. Quant. Grav. 22 5005

[10] Moodley J 2012 Ph.D. Thesis, University of KwaZulu-Natal, Moodley J and Amery G 2012 To be submitted to Commun. Math. Phys., Moodley J and Amery G 2012 In preparation

[11] Moodley J and Amery G 2011 Pramana: J. Phys. 77 533

[12] Okelola M 2011 M.Sc. Thesis, University of KwaZulu-Natal, Okelola M, Govinder K S, Moodley J and Amery G 2012 (In preparation)

[13] Barriola M and Vilenkin A 1989 Phys. Rev. Lett. 63 341

[14] Wesson P S 1994 Ap. J. 436 547