We construct an asymptotically free gauge theory with an $SU(2) \times SU(2) \times U(1)$ vectorlike flavor symmetry. We show that this gauge theory has properties which provide insight into aspects of low-energy QCD.

1 Introduction

Consider an asymptotically free gauge theory with global symmetry group $G$. At short distances this gauge theory allows a perturbative definition. One fundamental question at long distances relates to how the symmetry group $G$ is realized in the vacuum. Virtually no progress has been made in answering this question from first principles in QCD, the gauge theory responsible for all hadronic and nuclear phenomena. One might hope that ultimately lattice methods will provide insight. Unfortunately, there is at present no lattice formulation of gauge theory which preserves $G$ if $G$ is chiral, and therefore at present the lattice cannot reveal how QCD flavor symmetries are realized in the vacuum.

For purposes of this paper, QCD has two massless flavors and global symmetry group $G = SU(2) \times SU(2) \times U(1)$. This global symmetry group can be decomposed into $H = SU(2)_V \times U(1)$ which corresponds to isospin and baryon number and is vectorlike since it does not imply massless quarks, and $SU(2)_A$ which is chiral and therefore exact only in the limit of vanishing quark masses. No-go theorems formulated for QCD-like gauge theories imply that vectorlike symmetries are not spontaneously broken. Therefore, on theoretical grounds we expect that in the low-energy theory $H$ will be unbroken, and $SU(2)_A$ can be unbroken, completely broken or partially broken by the vacuum. Observation reveals that $SU(2)_A$ is broken completely in the vacuum.

In this paper we will motivate study of other gauge theories with QCD flavor symmetries by considering general properties of the nonlinear realization of $G$. We will show that the manifold possible $G$ transformation properties of the hadronic mass-squared matrix suggest the existence of other gauge theories with precisely the same symmetry breaking pattern as QCD. We will construct
an asymptotically free gauge theory with a $G = SU(2) \times SU(2) \times U(1)$ flavor symmetry. This gauge theory is different from QCD both in its matter content and in that the entire global symmetry group is vectorlike. Therefore the masses of the quarks in this theory are unconstrained by symmetries. We will show that this gauge theory has interesting properties which provide insight into several mysterious aspects of low-energy QCD.

2 The Two Faces of the Nonlinear Realization

2.1 Chiral Perturbation Theory

Consider an underlying theory with flavor symmetry group $G = SU(2) \times SU(2) \times U(1)$ which is spontaneously broken to $H = SU(2)_V \times U(1)$. The assumption that $SU(2)_A$ is broken by the vacuum leads to a wealth of information about the low-energy theory, which is true for all underlying theories with the same pattern of symmetry breaking. There are 3 Goldstone bosons, identified with pions. Using the theory of nonlinear realizations it is straightforward to construct the most general low-energy lagrangian of pions living on the coset space $G/H = SU(2) \times SU(2)/SU(2)$. The basic physics underlying this construction is simple. Since it costs no energy to move from one point on the vacuum manifold to another, pions couple to themselves and other hadrons only through derivative interactions. Hence the assumed pattern of symmetry breaking implies that scattering amplitudes involving pions are expansions in powers of pion energy. The parameters that appear in this expansion are unconstrained by the pattern of symmetry breaking, but can be fit to one experiment in order to predict another. This method is known as chiral perturbation theory.

2.2 Asymptotic Matching

Chiral perturbation theory is not the only information encoded in the nonlinear realization. The scattering amplitudes involving pions are polynomials in energy. Therefore, in order to get reasonable asymptotic behavior, the axial couplings of the hadrons must be related in special ways. This leads to an apparent paradox. Presumably couplings can be related only by symmetry. However, $G$ is the only symmetry in the problem and $G$ is spontaneously broken! What then relates the axial couplings?

Long ago Weinberg\(^3\), and independently Wess and Zumino\(^4\), answered this question by considering pion-hadron scattering in helicity conserving (collinear) Lorentz frames. The fundamental observation is that the nonlinear realization also constrains the form of scattering amplitudes involving any number of pi-
ons and hadrons expanded in inverse powers of energy. Consider the elastic scattering process \( \pi \alpha \rightarrow \pi \beta \) where \( \alpha \) and \( \beta \) are any single-hadron states. It is possible to extract the coefficient of the scattering amplitude that scales as the zeroth power of energy. The crossing-odd and -even amplitudes are, respectively,

\[
M_{\beta b, \alpha a}^{(-)}(\omega) \propto \{ i \epsilon_{abc} T_c - [X^\lambda_a, X^\lambda_b] \}_{\beta a} \omega^0 + O(1/\omega) + O(\omega) \tag{1}
\]

\[
M_{\beta b, \alpha a}^{(+)}(\omega) \propto [X^\lambda_a, [M^2, X^\lambda_b]]_{\beta a} \omega^0 + O(1/\omega) + O(\omega) \tag{2}
\]

where \( T_a \) is the isospin matrix, and \( X^\lambda_a \) is an axial vector coupling matrix, related to the matrix element of the process \( \alpha(p, \lambda) \rightarrow \beta(p', \lambda') + \pi(q, a) \) in any frame in which the momenta are collinear, and \( \lambda \) is helicity —which is conserved in the collinear frame. The coefficients of positive powers of energy always contain counterterms which can be assigned any value and so are uninteresting. It is the coefficients of the zeroth power of \( \omega \) which are interesting, since they involve only the axial vector coupling matrix and the hadronic mass-squared matrix in algebraic form.

The values taken by these coefficients determine how the low-energy theory matches to the underlying theory. Since \( SU(2) \times SU(2) \) is a symmetry of the underlying theory we have the matching condition \( M^{(-)} = 0 \). Together with Eq. (1), the defining relations, \([T_a, T_b] = i \epsilon_{abc} T_c \) and \([T_a, X^\lambda_b] = i \epsilon_{abc} X^\lambda_c \) close the chiral algebra and so for each helicity, \( \lambda \), hadrons fall into representations of \( SU(2) \times SU(2) \), in spite of the fact that the group is spontaneously broken. This statement of chiral symmetry is simply a generalization of the Adler-Weisberger sum rule. The matching condition \( M^{(-)} = 0 \) is equivalent to the statement that the crossing-odd amplitude satisfies an unsubtracted dispersion relation.

The matching condition for the crossing-even amplitude is especially interesting since it determines the asymptotic behavior of the total cross-section. It is clear from Eq. (2) that the matching condition is determined by the transformation property with respect to \( SU(2) \times SU(2) \) of the hadronic mass-squared matrix. In general one can write

\[
\hat{M}^2 = \sum_{R} \hat{M}_R^2 \tag{3}
\]

where \( R \) is a representation of \( SU(2) \times SU(2) \). For instance, if \( M^{(+)} = 0 \) then \( \hat{M}^2 \) transforms as the singlet representation, \( R = (1, 1) \). As we will show, the correct symmetry property of the mass-squared matrix is difficult to infer from the QCD lagrangian because symmetry breaking is inherently nonperturbative.
3 Asymptotic Matching in QCD

Can we learn anything about the transformation properties of $\hat{M}^2$ directly from the QCD lagrangian? In two-flavor massless QCD the degrees of freedom relevant to $G$ are the Weyl fermions $Q_L$ and $Q_R$ which transform as $(2, 1)$ and $(1, 2)$ with respect to $SU(2) \times SU(2)$. Consider a generic mass term $MQ\bar{Q}$ which might represent a current quark mass or a constituent mass induced by a $<QQ>$ condensate (equivalently we could consider a sigma model). We assign $M$ spurion transformation property $(2, 2)$ with respect to $SU(2) \times SU(2)$. In order to make contact with the results of the previous section formulated in helicity conserving Lorentz frames we can express this operator in light-front coordinates which gives $M^2Q^\dagger Q$. Here $M^2$ transforms as $(1, 1)$. This then implies that $M^2$ transforms like a singlet giving the matching condition $\mathcal{M}^{(+)} = 0$, and total hadronic cross sections should fall off rapidly asymptotically. This is of course in violent disagreement with experiment. However, there is no contradiction. The QCD lagrangian does not reveal the correct transformation properties of the mass-squared matrix because anything we infer from the QCD lagrangian and its degrees of freedom is tied to perturbation theory, and the correct transformation properties are nonperturbative. Presumably if there was a means of defining QCD at long distance that respected $G$, the correct matching condition would be manifest.

Weinberg made the ansatz $M^2 = \hat{M}^2_{(1, 1)} + \hat{M}^2_{(2, 2)} (\mathcal{R} = (1, 1) \oplus (2, 2))$ which is equivalent to a Regge inspired superconvergent sum rule and consistent with the observed (approximately) constant behavior of total cross-sections; that is, $\mathcal{M}_{ab}^{(+)} \propto \delta_{ab}$. He further showed that the assumption $[\hat{M}^2_{(1, 1)}, \hat{M}^2_{(2, 2)}] = 0$ fixes the reducible representations filled out by mesons in the low-energy theory. This matching condition, $\mathcal{M}_{\alpha\beta}^{(+)} \propto \delta_{\alpha\beta}$, is equivalent to another Regge inspired superconvergent sum rule. These assumptions lead to remarkably accurate predictions in the low-energy theory, and yet are mysterious from the QCD point of view.

4 Gauge Theory with Vectorlike $G$

4.1 Basic Formalism

Given the difficulty in understanding the transformation properties of the hadronic mass-squared matrix from the QCD lagrangian, it is natural to ask whether it is possible to construct other underlying gauge theories with the same flavor symmetries as QCD but with a mass-squared matrix whose transformation properties are nontrivial and manifest. Surprisingly, the answer is yes.
In order to construct an asymptotically free gauge theory with \( \text{vectorlike} \) \( G \) flavor symmetry and all matter transforming nontrivially with respect to the gauge group we require at least four flavors of quarks, isodoublets \( Q \) and \( P \), transforming in the fundamental representation of the gauge group, and one \( G \) four-vector, Lorentz scalar, \( M \), transforming in the adjoint representation of the gauge group. The quantum number assignments are as in Table 1.

We have left the number of colors, \( N \), of the gauge theory arbitrary. The most general renormalizable lagrangian invariant with respect to the flavor symmetries and \( P, C \) and \( T \) is:

\[
\mathcal{L} = \psi \hat{\mathcal{D}} \psi - M_0 \bar{\psi} \sigma_3 \psi + \kappa_1 \bar{\psi}(A + i B \gamma_5 \sigma_3) \psi + \kappa_2 \bar{\psi}(\sigma_3 A + i B \gamma_5) \psi \\
+ \frac{1}{4} \text{tr}(D_\mu \mathcal{M}D^\mu \mathcal{M}^\dagger) - \frac{1}{2} \text{tr}(F_{\mu\nu} F^{\mu\nu}) - V(\mathcal{M} \mathcal{M}^\dagger)
\]

(4)

where \( \psi = (Q \ P)^T \) and \( \mathcal{M} \equiv A + iB \). The trace is over gauge and flavor quantum numbers and the \( \sigma_i \) are Pauli matrices acting in the \( \psi \) space. This lagrangian admits a \( G = SU(2) \times SU(2) \times U(1) \) flavor symmetry and defines the gauge theory for weak couplings. We have chosen \( Q \) and \( P \) to be of like Parity. It is easy to check that the gauge coupling is asymptotically free for all \( N \). The low-energy theory is therefore expected to be in the confined phase.

The gauge theory is by construction anomaly free. Note that if \( \kappa_1 = \kappa_2 = 0 \) the free fermion theory is \( U(4) \) invariant. Therefore, the Yukawa operators break \( U(4) \) explicitly to \( SU(2) \times SU(2) \times U(1) \). There are regions of parameter space where there is enhanced symmetry. When \( \kappa_2 = 0 \) there is a \( Z_2 \) symmetry that cannot be constructed from any \( G \) subgroup: \( \psi \rightarrow \sigma_3 \psi, B \rightarrow -B \).

4.2 Vectorlike Asymptotic Matching

Assume that \( G = SU(2) \times SU(2) \times U(1) \) is broken spontaneously to \( H = SU(2)_V \times U(1) \) by the condensates \( < \bar{\psi} \psi > \) and \( < \bar{\psi} \sigma_3 \psi > \), where the latter condensate is \( Z_2 \) violating. This is consistent with the observed pattern of symmetry breaking in QCD. Note that due to the presence of the fundamental
scalars the Vafa-Witten theorem does not constrain symmetry breaking in this
gauge theory. Since the pattern of symmetry breaking is, by assumption, the same
as QCD, the low-energy theory is of the same form, as given by the nonlinear
realization. Of course in chiral perturbation theory the values of the low-energy
constants for the two underlying theories can be different. What about asympto-
tic matching? Here we can do better than QCD. Of course we continue to
have the matching condition $\mathcal{M}^{(-)} = 0$. But here we can also determine the
crossing-even matching condition. Allowing constituent quark mass terms in-
duced by the condensates gives $\hat{M} = \hat{M}_{(1,1)} + \hat{M}_{(2,2)}$, where $\hat{M}_{(1,1)} = M_0 \sigma_1$
and $\hat{M}_{(2,2)} = M_1 + M_3 \sigma_3$. Note that $[\hat{M}_{(1,1)}, \hat{M}_{(2,2)}] = 0$ has a nontrivial so-
lution if and only if $M_3 = 0$ which corresponds to the point of unbroken $Z_2$,
assuming $\kappa_2 = 0$ in the lagrangian. One can easily verify that these consid-
erations apply to the mass-squared matrix as well. Therefore, we have the
matching conditions $\mathcal{M}^{(+)\, ab}_{\alpha\beta} \propto \delta_{ab}$, and at the point of enhanced $Z_2$ symmetry,
also $\mathcal{M}^{(+)\, \alpha_0 \beta}_\alpha \propto \delta_{\alpha_0 \beta}$. At low energies this gauge theory with vectorlike flavor sym-
metries gives rise to hadronic scattering amplitudes that exhibit classic Regge
behavior! One might be tempted to believe that, as in QCD, nonperturbative
effects change this picture. This is unlikely. Because $G$ is vectorlike, this gauge
theory admits a lattice definition that respects $G$ at all scales. Therefore, in
contrast with QCD, the lagrangian of the vectorlike theory should exhibit the
symmetry properties of the nonperturbative regime.

5 The Vectorlike Sigma Model

It is easy to construct a sigma model that respects all of the symmetries of
the underlying vectorlike theory. We can simply ignore the gauge degrees
of freedom, identify quarks with baryons (for $N$ odd) and choose $\mathcal{M} = F_\pi +
i 2\pi \tau T_\tau$. The physical baryon states, $\mathcal{N} = (N_+, N_-)^r$, are given by

$$\psi = (\sin \phi + i \sigma_2 \cos \phi) \mathcal{N}$$

where $\cot 2\phi = \kappa_2 F_\pi / M_0$, and have masses: $M_{\pm} = \kappa_1 F_\pi \pm (\cos 2\phi \kappa_2 F_\pi + 
\sin 2\phi M_0)$. The axial vector coupling matrix is given by

$$\hat{g}_A = \begin{pmatrix} -\cos 2\phi & -\sin 2\phi \\ -\sin 2\phi & \cos 2\phi \end{pmatrix}.$$ 

Note that $(\hat{g}_A)^2 = 1$, a statement of the Adler-Weisberger sum rule. Since
$Tr (\hat{g}_A) = 0$, $\pi \to \gamma \gamma$ vanishes for all values of $\phi$, as expected since the
underlying theory is anomaly free.
Note the point of enhanced symmetry. If $\kappa_2 = 0$, the diagonal elements of the axial vector coupling matrix vanish. This is because there is a $Z_2$ symmetry if $\kappa_2 = 0$. We can then assign multiplicatively conserved $Z_2$ charges to each physical state. We assign $N_+^c$ charge $+1$, and $N_-^c$ and $\pi$ charge $-1$. Clearly the diagonal elements of the axialvector coupling matrix vanish because they do not respect the $Z_2$ symmetry. Off-diagonal elements are unity. Here we have $M = \kappa_1 F_\pi$ and $M_3 = \kappa_2 F_\pi \sigma_3$ and so it is clear that the baryon mass-matrix exhibits the claimed symmetry structure, and the point of enhanced $Z_2$ symmetry, $\kappa_2 = 0$, corresponds to the constraint $[M_{(1,1)}, M_{(2,2)}] = 0$. Of course one would find precisely the same $\hat{g}_A$ by simply assuming the equivalent Regge inspired sum rules.

6 Outlook
The nonlinear realization of a broken symmetry encodes symmetry information that is unique to all underlying theories with the same pattern of symmetry breaking (chiral perturbation theory), and symmetry information that is determined by matching to the underlying theory (asymptotic matching). We have shown that while a matching condition which determines the true asymptotic behavior of the total pion-hadron cross-section is mysterious from the QCD point of view, it is easy to construct a new asymptotically free gauge theory with QCD flavor symmetries in which this matching condition is manifest. The fundamental property of this new gauge theory is that all flavor symmetries are vectorlike. This gauge theory therefore admits a lattice formulation that respects QCD-like flavor symmetries and in turn provides a theoretical laboratory which can rigorously address the issue of how these flavor symmetries are realized in the vacuum.

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