Covariant $q$-differential Calculus
and its Deformations at $q^N = 1$

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Abstract. We construct the generalized version of covariant $\mathbb{Z}_3$-graded differential calculus introduced by one of us (R.K.) in [2], [3], and then extended to the case of arbitrary $\mathbb{Z}_N$ grading in ([1], [5], [6]). Here our main purpose is to establish the recurrence formulae for the $N$-th power of covariant $q$-differential $D_q = d_q + A$ and to analyze more closely the particular case of $q$ being an $N$th primitive root of unity. The generalized notions of connection and curvature are introduced and several examples of realization are displayed for $N = 3$ and 4. Finally we briefly discuss the idea of infinitesimal deformations of the parameter $q$ in the complex plane.

1 Introduction

The idea of investigating the generalizations of exterior differential calculus by postulating $d^N = 0$ instead of the usual $d^2 = 0$, leading to generalized Grassmann algebras with $n$-th order constitutive relations is not really a new one, and has been mentioned here and there quite a long time ago, but only recently it has been given the attention it really deserves. In a recent series of articles ([1], [2], [3], [4], [5], [6], [7]), the case $N = 3$ has been investigated in more detail (cf. also [8], [9]), then a general theory of $q$-differentials with $q = N$ and $d^N = 0$ has been worked out, introducing also the $q$-analog of the homological algebras.

Besides the universal construction in terms of tensor products of algebras and linear spaces, some simple concrete realizations have been found, namely a finite (matrix) version in which the operator $d$ is identified with a $\mathbb{Z}_N$ commutator with a grade 1 element of the $\mathbb{Z}_N$-graded algebra, and a differential calculus on the infinite space of functions generated by a finite number of variables $x^k$ has been also defined ([2], [3]). Finally, the notion of the covariant differential has been discussed, and generalized for the $\mathbb{Z}_3$-graded case. In particular, it enabled us to introduce the generalized notion of the curvature form, which in the $\mathbb{Z}_3$-graded case was equal to the 3-form $D^2 A$, the second covariant differential of the connection 1-form $A$.

In the present article we shall investigate the general $q$-deformed differential calculus, with special attention being paid to the case when $q$ is equal to an $N$-th primitive root of unity, $q^N = 1$. We shall show how in this case the $N$-th covariant differential acting on an appropriate module $\mathcal{H}$ reduces itself to an automorphism.
of \( \mathcal{H} \), i.e. \( D^N \Phi = \Omega \Phi \), with \( \Omega = D^{N-1} A \). We shall also derive a simple recurrence formula for \( D^N \Phi \) for an arbitrary value of the parameter \( q \).

Next, we discuss briefly two realizations of this \( q \)-differential calculus with \( q \) an \( N \)-th primitive root of unity: the \( N \times N \) complex matrix representation, and a generalized Grassmannian spanned by all the differential forms of the type \( dx^k, d^2 x^k, \ldots, d^{N-1} x^k \). The explicit expressions are found for \( \Omega = D^{N-1} A \) in the matrix case, and for a few low values of \( N \) in the generalized Grassmannian case. In the case of matrix realization, the general form of a matrix representing the “pure gauge” configuration is given.

Then we consider the \textit{infinitesimal deformations} of the complex parameter \( q \) itself. It seems worthwhile to know what happens when the consecutive infinitesimal deformations form a polygon in \( \mathbb{C}^1 \), so that after \( N \) steps we come back to the initial value of \( q \). In the case when \( q \) tends to the \( N \)-th primitive root of unity, certain combinations of products \( N \) deformed differentials yield 0-degree operators with interesting properties.

Finally, we look at the infinitesimal deformations of the covariant \( q \)-differential \( D_q \), in which not only the first term \( d_q \) is transformed into \( d_q + \epsilon \), but parallelly also the connection one-form \( A \) is replaced by \( A + \epsilon \Lambda \). Here again, we compute the \( N \)-th order products of the deformed covariant differentials.

## 2 Universal \( q \)-differential and its properties

Let \( \mathcal{A} \) be an associative algebra with unit element, generated by two elements denoted by \( U \) and \( \eta \), respectively. We attribute the \textit{grade} 1 to the element \( \eta \), and \textit{grade} 0 to the element \( U \): \( \deg(U) = 0 \), \( \deg(\eta) = 1 \).

\( q \) being a complex number different from 0, we shall impose the following commutation relation between the generators \( U \) and \( \eta \):

\[
U \eta = q \eta U
\]

(1)

A general element belonging to \( \mathcal{A} \) can now be represented by a finite sum of various powers of the generators \( U \) and \( \eta \) in ordered products:

\[
B \in \mathcal{A}, \quad B = \sum_{m,n} b_{mn} U^m \eta^n = \sum_n \beta_n(U) \eta^n
\]

(2)

An element of \( \mathcal{A} \) has a well-defined degree \( n \) if it is a monomial of \( n \)-th order in the generator \( \eta \). The algebra \( \mathcal{A} \) acquires a natural \( \mathbb{Z} \)-grading and can be represented as an infinite sum of subspaces with well-defined grades:

\[
\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3 \oplus \ldots \oplus \mathcal{A}_k \oplus \mathcal{A}_{k+1} \oplus \ldots
\]

If \( q \) is a primitive \( N \)-th root of unity, \( q^N = 1 \), and if the following supplementary conditions are imposed on the generators \( U \) and \( \eta \):

\[
U^N = 1, \quad \eta^N = 1,
\]

(3)

\[
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\]
then our algebra becomes finite and can be represented as algebra of complex $N \times N$ matrices. Its canonical representation will be introduced later in the Section 4. In that case the matrix $U$ is called the grading matrix, the matrix $\eta$ has an inverse which is equal to $\eta^T$, and Eq. (1) can be written as:

$$U^{-1} \eta U = q \eta,$$

consequently,

$$U^{-1} \eta^m U = q^m \eta^m$$  \hspace{1cm} (4)

The $q$-differential of an element $B$ of grade $b$ in $\mathcal{A}$ can be defined now as follows (for any value of the parameter $q$):

$$d_q B = \eta B - q^b B \eta$$  \hspace{1cm} (5)

The grades add up under the associative multiplication in our algebra; that is why the $q$-differential satisfies the $q$-deformed Leibniz rule. For a product of two matrices $A B$, with deg($A$) = $a$ and deg($B$) = $b$ one has

$$d_q (A B) = (d_q A) B + q^a A (d_q B)$$  \hspace{1cm} (6)

The proof is straightforward by applying the definition:

$$d_q (A B) = \eta (AB) - q^{a+b} (AB) \eta = \eta AB - q^a A \eta B + q^a \eta A B - q^a q^b A B \eta = [\eta A - q^a A \eta] B + q^a A [\eta B - q^b B \eta] = (d_q A) B + q^a A (d_q B)$$

Now we evaluate the consecutive powers of the operator $d_q$:

$$d_q^2 B = d_q (\eta B - q^b B \eta) = \eta (\eta B - q^b B \eta) - q^{b+1} (\eta B - q^b B \eta) \eta = \eta^2 B - q^b (1 + q) \eta B \eta + q^{2b+1} B \eta^2;$$

$$d_q^3 B = \eta^3 B + q^b (1 + q + q^2) \eta^2 B \eta + q^{2b+1} (1 + q + q^2) \eta B \eta^2 - q^{3b+3} B \eta^3$$

It is enough to check the action of consecutive powers of the operator $d_q$ on the two generators $U$ and $\eta$ in order to be able to extend them on an arbitrary element $B$ ($b = \text{deg}(B)$) of the entire algebra $\mathcal{A}$. We find out easily that for the element of degree one, $\eta$, we have

$$d_q \eta = \eta^2 - q \eta^2,$$

$$d_q^2 \eta = (1 - q) \eta^3 - (1 - q) q^2 \eta^3 = (1 - q) (1 - q^2) \eta^3,$$

$$d_q^N \eta = (1 - q) (1 - q^2) \cdots (1 - q^N) \eta^{N+1}$$  \hspace{1cm} (7)
If the condition \( q^N = 1 \) is imposed, the last expression vanishes, implying \( d^N \eta = 0 \). When the operator \( d_q \) acts of the 0-degree generator \( U \), we find

\[
d_q U = \eta U - U \eta,
\]

\[
d_q^2 U = \eta^2 U - \eta U \eta - q \eta U \eta + q U \eta^2
\]

Using the \( q \)-deformed commutation relation we can write

\[
-\eta U \eta = -q \eta^2 U \quad \text{and} \quad -q \eta U \eta = -U \eta^2
\]

so that one gets

\[
d_q^2 U = (1 - q) (\eta^2 U - U \eta^2).
\]

Similarly, one has for \( d^N U \):

\[
d_q^N U = (1 - q)(1 - q^2) \ldots (1 - q^{N-1}) (\eta^N U - U \eta^N)
\]

which vanishes because we have assumed \( \eta^N = 1 \), so that it does commute with any element of the algebra.

It is easy to see that independently of the grade of \( B \), one has \( d^N B = 0 \) when \( q \) is the \( N \)-th primitive root of unity and when \( \eta^N = 1 \). This is obvious in the two particular cases shown above, with \( q = -1 \) and \( q = j = e^{2\pi i/3} \). The general formula can be easily proved by recurrence using the properties of the generalized binomial symbols

\[
\left[\begin{array}{c} N \\ k \end{array}\right]_q = \frac{[N]_q!}{[k]_q!([N-k])_q!},
\]

with \([k]_q = (1 + q + q^2 + \ldots + q^k)\) and \([k]_q! = [1]_q [2]_q \ldots [k]_q\). Classical recurrence relations remain valid, their proof by induction obvious:

\[
[n+1]_q [n]_q! = [n+1]_q! \quad \text{and} \quad \left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{[n]_q!}{[n-k]_q! [k]_q!}
\]

\[q^{k+1} \left[\begin{array}{c} n \\ k+1 \end{array}\right]_q + \left[\begin{array}{c} n \\ k \end{array}\right]_q = \left[\begin{array}{c} n+1 \\ k+1 \end{array}\right]_q \]

The general formula for \( d_q^N B \) can be also established by induction and reads as follows:

\[
d_q^N B = \sum_{k=0}^{N} (-1)^k q^{k+b+\frac{k(k-1)}{2}} \left[\begin{array}{c} N \\ k \end{array}\right]_q \eta^{N-k} B \eta^k
\]
Now we can easily check that \(d_q^N B = 0\). Indeed, the \(q\)-binomial coefficients \(\binom{N}{k}\) vanish for \(0 < k < N\) because of the common factor \([N]_q = (1 + q + q^2 + \ldots + q^{N-1}) = 0\), the only remaining term being equal to

\[
d_q^N B = \eta^N B + (-1)^N q^{N/2} B \eta^N
\]

Now, because \(\eta^N = 1\) is the unit element of the algebra \(A\), commuting with any \(B \in A\), and because of \(q^{N/2} = 1\), we can write

\[
d_q^N B = \left(1 + (-1)^N q^{N/2}\right) B
\]

This expression vanishes identically whether \(N\) is even or odd number. If \(N\) is odd, then \((N - 1)/2 = c\) is an integer, so that \(q^{cN} = 1\), while \((-1)^N = -1\). If \(N\) is even, then \(q^N = -1\), \((-1)^N = 1\), but \((q^N)^{(N-1)/2} = (-1)^{(N-1)} = -1\), so in both cases we have a factor \(1 + (-1)\) in front of \(B\), which makes this expression vanish, thus completing the proof that \(d_q^N B = 0\).

### 3 Covariant \(q\)-differential and its successive powers

Defining the covariant \(q\)-differential acting on an element of the module \(\Phi \in H\) as above, i.e.

\[
D \Phi = d \Phi + A \Phi,
\]

\(A\) denoting the connection 1-form, which is an element of degree 1 belonging to the algebra \(A\), and using the \(q\)-Leibniz rule, we get the following explicit expressions for the consecutive powers of \(D\) acting on \(\Phi\), i.e. the formulae for \(D^N \Phi\):

\[
D \Phi = d \Phi + A \Phi;
\]

\[
D^2 \Phi = d^2 \Phi + (1 + q) A d \Phi + (D A) \Phi;
\]

In the limit of \(q = -1\) we have \(d^2 \Phi = 0\), so that only the last term of the above expression survives, yielding the well-known formula for curvature in the \(\mathbb{Z}_2\)-graded case, \(D^2 \Phi = (D A) \Phi\). Now,

\[
D^3 \Phi = d^3 \Phi + (1 + q + q^2) A d^2 \Phi + (1 + q + q^2) (D A) d \Phi + (D^2 A) \Phi;
\]

Here again, when \(q = j = e^{2\pi i/3}\) is the primitive 3-rd root of unity, we have \(d^3 \Phi = 0\) by definition, and the coefficient \((1 + q + q^2)\) vanishes, leaving only \(D^3 \Phi = (D^2 A) \Phi\). Similarly, \(D^4 \Phi = (D^3 A) \Phi\), and so on.

The general formula can be quite easily established with the help of notations.
that have become standard by now. Using the definitions of the $q$-entire numbers and the $q$-factorials introduced in the previous section (9, 10 and 11), we get

$$D^N \Phi = d^N \Phi + \sum_{k=1}^{N-1} \left[ N \atop k \right]_q (D^{k-1} A) d^{N-k} \Phi + (D^{N-1}) \Phi \quad (16)$$

The proof uses simple recurrence. Acting again with the operator $D$ and separating the two first and the two last terms we get:

$$D^{N+1} \Phi = d^{N+1} \Phi + A d^N \Phi + \sum_{k=1}^{N-1} \left[ N \atop k \right]_q (D^{k+1} A) d^{N-k-1} \Phi +$$

$$+ \sum_{k=1}^{N-1} q^k \left[ N \atop k \right]_q (D^{k-1} A) d^{N-k+1} \Phi + +q^N (D^{N-1} A) d\Phi + (D^N) \Phi \quad (17)$$

Leaving the first and last terms unchanged, and including the term $q^N (D^N A) \Phi$ in the second sum, and then shifting the summation index from $(k - 1)$ to $k$, we can re-write the above formula as

$$D^{N+1} \Phi =$$

$$d^{N+1} \Phi + \sum_{k=1}^{N+1} \left( q^{k+1} \left[ N \atop k+1 \right]_q + \left[ N \atop k \right]_q \right) (D^{k} A) d^{N-k} \Phi + (D^N) \Phi$$

and we get the same formula again if we use the fact that the recurrence formula (11) holds for any value of $q$. In particular, when $q$ is a primitive $N$-th root of unity, the formula reduces to

$$D^N \Phi = (D^{N-1} A) \Phi = \Omega \Phi \quad (18)$$

Let $S$ be an automorphism of the algebra $A$ in which the 1-form $A$ takes its values. If the module $\mathcal{H}$ is a free one, it induces automatically an automorphism of $\mathcal{H}$. It is easy to prove the following generalization of the “pure gauge” connexions and the fact that the corresponding curvature form must vanish:

If $A = S^{-1} dS$ and $d^N = 0$, then $\Omega = D^{N-1} A = 0$ and vice-versa.

The proof is by straightforward calculation; indeed, if $A = S^{-1} dS$, then $D\Phi = (d + S^{-1} dS)\Phi = d\Phi + S^{-1} dS \Phi$ (we remind that both $S^{-1}$ and $S$ are of degree 0 in the sense of differential forms);

$D^2 \Phi = (d + S^{-1} dS)^2 \Phi = d^2 \Phi + S^{-1} d^2 S \Phi + (1 + q) S^{-1} dS d\Phi$;

$D^3 \Phi = d^3 \Phi + S^{-1} d^3 S \Phi + (1 + q + q^2) S^{-1} d^2 S d\Phi + (1 + q + q^2) S^{-1} dS d^2 \Phi$;

and the general formula is

$$D^N \Phi = d^N \Phi + S^{-1} d^N S \Phi + \sum_{i=1}^{N-1} \left[ N \atop k \right]_q S^{-1} d^k S d^{N-k} \Phi \quad (19)$$
from which we see that \( D^N \Phi = 0 \), because for \( q \) which is a primitive \( N \)-th root of unity one has \( d^N = 0 \), and all the symbols \( \left[ \frac{N}{k} \right]_q \) vanish. This means that \( \Omega = D^{N-1} A = 0 \), because we have already checked that \( D^N \Phi = (D^{N-1} A) \phi = (\Omega) \Phi \).

The above formulae generalize the notions of connection and curvature in a universal way, independent of the realization. In order to make these formulae useful, one must express the curvature in a more explicit manner, which shall depend on the realization chosen. In the next section we show several examples of such realizations, along with the explicit calculations of the generalized curvature forms.

4 Matrix and functional realizations of \( q \)-differential calculus

As was stated above, when \( q \) is a primitive \( n \)-th root of unity, the \( q \)-differential algebra can be faithfully represented by \( n \times n \) complex matrices. Most of the results obtained in the previous sections are independent of realization we choose; nevertheless, some of them can be given explicitly and lead to the formulae specific for each particular realization. Thus, in the matrix realization, the “pure gauge” connection 1-form \( A \) and the matrix \( \eta \) inducing the exterior \( \mathbb{Z}_N \)-graded differential can be chosen as follows:

\[
A = \begin{pmatrix}
0 & \alpha & 0 & \ldots & 0 \\
0 & 0 & \beta & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & \phi \\
\omega & 0 & 0 & \ldots & 0 \\
\end{pmatrix}, \quad \eta = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & 1 \\
1 & 0 & 0 & \ldots & 0 \\
\end{pmatrix}, \quad (20)
\]

whereas the grading matrix \( U \) (which is the 0-degree generator of our algebra) is chosen then as

\[
U = \text{diag}(q, q^2, q^3, \ldots q^N)
\]

The matrix \( \Omega = D^{N-1} A \) is of degree \( N \), i.e. it is diagonal; moreover, it is easy to check that it is proportional to the unit \( N \times N \) matrix with the coefficient \( (\alpha + 1)(\beta + 1)(\gamma + 1)\ldots(\omega + 1) \). Now, taking an arbitrary 0-degree matrix \( S = \text{diag}(a, b, c, \ldots, y, z) \), its inverse being \( S^{-1} = \text{diag}(a^{-1}, b^{-1}, c^{-1}, \ldots, y^{-1}, z^{-1}) \), and identifying \( S^{-1} dS \) with \( A \), we get

\[
\alpha = \left( \frac{b}{a} - 1 \right), \quad \beta = \left( \frac{c}{b} - 1 \right), \ldots, \omega = \left( \frac{z}{a} - 1 \right)
\]

therefore \( (\alpha + 1)(\beta + 1)\ldots(\omega + 1) = \left( \frac{b}{a} \right) \left( \frac{c}{b} \right) \left( \frac{d}{c} \right) \ldots \left( \frac{y}{z} \right) \left( \frac{z}{a} \right) = 1 \)

and it becomes clear that there is a one-to-one correspondence between the “pure gauge” \( S^{-1} dS \) and null-curvature connections.

This realization of the \( d^N = 0 \) differential calculus seems to be quite trivial. Hopefully, a more sophisticated version is at hand. Indeed, is it easy to see that for
a given $N$, one can introduce $N$ degree 1 linearly independent complex matrices
$\eta_k$, $k = 1, 2, \ldots, N$ that can be chosen as follows:

$$\eta_k = \begin{pmatrix}
0 & q^k & 0 & \cdots & 0 \\
0 & 0 & q^{2k} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & q^{(N-1)k} \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix}$$

(21)

Direct calculus shows that

$$\eta_k^N = (q^k q^{2k} \cdots q^{(N-1)k}) 1_{N \times N} = q^{\frac{N(N-1)}{2}} 1_{N \times N}$$

which amounts to

$$\eta_k^N = 1_{N \times N} \text{ if } N \text{ is odd or if } k \text{ is even,}$$

$$\eta_k^N = -1_{N \times N} \text{ if } N \text{ is even and } k \text{ is odd.}$$

(22)

(A similar, though slightly more complicated formula is valid for a totally symmetrized product of any $N$ generators, which is always proportional to the unit $N \times N$ matrix, but which in certain cases may vanish). Now we can define $N$ independent $q$-derivations denoted $d_k$, $k = 1, 2, \ldots, N$:

$$d_k B = [\eta_k, B]_q = \eta_k B - q^{\deg(B)} B \eta_k$$

(23)

Obviously, the $N$-th power of each of these $q$-differentials vanishes:

$$d_k^N B = 0 \text{ for any } B \in A;$$

but we have also the following generalization of this fact:

$$\sum_{\pi_\alpha(k_1 k_2 \ldots k_N)} d_{\pi_\alpha(k_1)} d_{\pi_\alpha(k_2)} \cdots d_{\pi_\alpha(k_N)} B = 0, \quad k = 1, 2, \ldots, N.$$  

(24)

where $\sum_{\pi_\alpha(k_1 k_2 \ldots k_N)}$ means the sum over all the $N!$ permutations of $N$ indices. In the case when some of the indices are repeated, one still has:

$$\sum_{symmetrized} d_{(k_1}^{m_1} d_{k_2}^{m_2} \cdots d_{k_p)}^{m_p} B = 0 \quad \text{if} \quad m_1 + m_2 + \cdots + m_p = N$$

(25)

At this point we can introduce a generalization of the covariant differential: it is enough to introduce $N$ independent degree 1 matrices (“one-forms”) $A_k$, $k = 1, 2, \ldots, N$ and to define

$$D_k \Phi = d_k \Phi + A_k \Phi$$
As in the case of a single \( q \)-differential \( dq \), we can show that the sum over all permutations of \( N \) consecutive applications of the covariant differentials \( D_k \) leads to an automorphism of the free module \( \mathcal{H} \):

\[
\sum_{\text{all permutations}} D_1 D_2 \ldots D_N B = \Omega_{1 \ldots N} B
\]

Furthermore it can be proved by direct calculus that the totally symmetric quantity \( \Omega_{1 \ldots N} \) is proportional to a unit \( N \times N \) matrix, and it is equal to

\[
\Omega_{1 \ldots (N-1)} = \sum_{\text{all permutations}} (D_1 D_2 \ldots D_{N-1}) A_N
\]

and similarly in the case of the repeated indices.

As an illustration, let us show how this scheme works in the simplest case when \( N = 2 \). Then we set

\[
\eta_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_k = \begin{pmatrix} 0 & \alpha_k \\ \beta_k & 0 \end{pmatrix}, \quad k = 1, 2.
\]

Direct calculus shows that all the components of the curvature 2-form are proportional to the \( 2 \times 2 \) unit matrix:

\[
\begin{align*}
\Omega_{12} &= \Omega_{21} = (\alpha_1 + \alpha_2 + \beta_1 - \beta_2 + \alpha_1 \beta_2 + \alpha_2 \beta_1) \mathbf{1}; \\
\Omega_{11} &= (\alpha_1 - \beta_2) \mathbf{1}, \\
\Omega_{22} &= (\alpha_2 + \beta_2) \mathbf{1}
\end{align*}
\]

If \( U \) is a diagonal (degree 0) matrix, the gauge principle applies now to the components of the connection \( A_k \) as in the classical gauge theory, i.e. the curvature \( N \)-form transforms homogeneously when the connection 1-form undergoes a gauge transformation:

\[
A_k \rightarrow \tilde{A}_k = U^{-1} A U + U^{-1} d_k U, \quad \Omega \rightarrow \tilde{\Omega} = U^{-1} \Omega U.
\]

In the matrix realization the curvature \( \Omega \) is always proportional to the unit matrix and commutes with all the elements of the algebra, and is \emph{invariant} under gauge transformations: \( \tilde{\Omega} = \Omega \).

An interesting problem arises if we consider the notion of connection forms with given symmetry properties. The transformation matrices \( U \) belong to the group \( GL(N, \mathbb{R}) \). The elements of this group can also act directly on the set of \( N \) matrices \( A_k \) inducing a linear transformation:

\[
A_k \rightarrow \tilde{A}_k = M_{kj}^i A_j
\]

A connection 1-form is said to be \emph{symmetric} with respect to a given subgroup of the \( GL(N, \mathbb{R}) \) group if for any given element of this subgroup identified as linear transformation \( M^j_k \) a corresponding representation \( U(M) \) exists, satisfying

\[
M^j_k A_j = U^{-1} A_k U + U^{-1} d_k U
\]
Such connections have been considered in the classical case by N. Manton ([10]) and used in the analysis of Higgs mechanism in electroweak theory ([11]).

Now, in the usual $\mathbb{Z}_2$-graded case, the most important realization of the Grassmann algebra was the algebra of exterior differential forms defined on a differential manifold. Here a similar realization can be conceived in a $\mathbb{Z}_N$-graded case starting with more explicit calculus on the example of $N = 3$, which has been introduced already in ([3]).

We postulate that by definition the differential $df$ of a function $f$ coincides with the usual one:

$$df = \frac{\partial f}{\partial \xi^k} d\xi^k = (\partial_k f) d\xi^k$$

(28)

When computing formally higher-order differentials, we shall suppose that our exterior differential operator $d$ obeys the generalized graded Leibniz rule:

$$d (\omega \phi) = d \omega \phi + q^{\deg(\omega)} \omega d \phi$$

(29)

where we suppose that $q$ is an $N$-th order root of unity, instead of $-1$ in the $\mathbb{Z}_2$-graded case, and that the grades add up modulo $N$ under the associative multiplication of exterior forms; the functions are of grade 0, and the operator $d$ raises the grade of any form by 1, which means that the linear operator $d$ applied to $\xi^k$ produces a 1-form whose $\mathbb{Z}_N$-grade is 1 by definition; when applied two times, by iteration, it will produce a new entity, which we shall call a 1-form of grade 2, denoted by $d^2 \xi^k$. Finally, we require that $d^N = 0$.

Let $F$ denote the algebra of functions $C^\infty (\xi^k)$, over which the $\mathbb{Z}_N$-graded algebra generated by the forms $d\xi^i$, $d^2 \xi^k$, $d^3 \xi^k$, etc., behaves as a left module. In other words, we shall be able to multiply the forms $d\xi^i$, $d^2 \xi^k$, $d^3 \xi^k$, etc. by the functions on the left only; right multiplication will just not be considered here. That is why we shall write by definition, e.g.

$$d(\xi^i \xi^k) := \xi^i d\xi^k + \xi^k d\xi^i$$

(30)

This amounts to suppose that the coordinates (functions) commute with the 1-forms, but do not necessarily commute with the forms of higher order. With thus established $\mathbb{Z}_3$-graded Leibniz rule, the postulate $d^3 = 0$ suggests in an almost unique way the ternary and binary commutation rules for the differentials $d\xi^i$ and $d^2 \xi^k$. Consider the differentials of a function of the coordinates $\xi^k$:

$$df := (\partial_i f) d\xi^i$$
$$d^2 f := (\partial_k \partial_i f) d\xi^k d\xi^i + (\partial_i f) d^2 \xi^i$$
$$d^3 f := (\partial_m \partial_n \partial_i f) d\xi^m d\xi^k d\xi^i + (\partial_k \partial_i f) d^2 \xi^k d\xi^i + j (\partial_i f) d\xi^i d^2 \xi^k + (\partial_k \partial_i f) d\xi^k d^2 \xi^i$$

(we remind that the last part of the differential, $(\partial_i f) d^3 \xi^i$, vanishes by virtue of the postulate $d^4 \xi^i = 0$). Supposing that partial derivatives commute, exchanging
the summation indices $i$ et $k$ in the last expression and replacing $1 + j$ by $-j^2$, we arrive at the following two conditions that lead to the vanishing of $d^3 f$:

$$d\xi^m d\xi^k d\xi^i + d\xi^k d\xi^i d\xi^m + d\xi^i d\xi^m d\xi^k = 0; \quad d^2\xi^k d\xi^i - j^2 d\xi^i d^2\xi^k = 0. \quad (31)$$

which lead in turn to the following choice of relations:

$$d\xi^i d\xi^k d\xi^m = j d\xi^k d\xi^m d\xi^i, \quad \text{and} \quad d\xi^i d^2\xi^k = j d^2\xi^k d\xi^i. \quad (32)$$

Strictly speaking, the above formulæ hold only for the symmetric part of the above expression; we choose to impose stronger relations in order to make the resulting space of forms finite-dimensional.

Extending these rules to all the expressions with a well-defined grade, and applying the associativity of the $\mathbb{Z}_3$-exterior product, we see that all products of the type $d\xi^i d\xi^k d\xi^m d\xi^n$ and $d\xi^i d\xi^k d^2\xi^m$ must vanish, and along with them, also the monomials of higher order containing these as factors.

Still, this is not sufficient in order to satisfy the rule $d^3 = 0$ on all the forms spanned by the generators $d\xi^1$ and $d^2\xi^k$. It can be proved without much pain that the expressions containing $d^2\xi^i d^2\xi^k$ must vanish, too; so we set forward the additional rule declaring that any expression containing four or more operators $d$ must identically vanish. With this set of rules we can check that $d^3 = 0$ on all the forms, whatever their grade or degree.

As in the case of the matrix algebra realization, it is very easy to introduce a $\mathbb{Z}_N$-graded generalization. Replacing $j$ by $N$-th primitive root of unity, $q$, and introducing higher ”multi-differentials”, $d^3 \xi^k$, $d^4 \xi^k$, ..., up to $d^{N-1} \xi^k$, we compute next differentials as follows: we impose on the operator $d$ the $q$-graded Leibniz rule and we require that $d^N = 0$, we can impose the following minimal set of generalized commutation rules on the products of forms of order $N$:

$$d\xi^{k_1} d\xi^{k_2} \ldots d\xi^{k_N} = q d\xi^{k_2} \ldots d\xi^{k_N} d\xi^{k_1} = q^2 d\xi^{k_3} \ldots d\xi^{k_N} d\xi^{k_1} d\xi^{k_2} \ldots \text{etc.}, \quad (33)$$

As a corollary, one can conjecture that for $N \geq 3$ any product of more than $N$ such 1-forms must vanish, which has been proved for the general $\mathbb{Z}_N$-graded Grassmann algebras.

As now $d^2 \neq 0$, $d^3 \neq 0$, ..., $d^{N-1} \neq 0$, we must introduce new independent differentials, $d^2\xi^k$, $d^3\xi^k$, ..., $d^{N-1}\xi^k$. Each kind of these new “one-forms of degree $m$”, with $m = 1, 2 \ldots (N - 1)$ spans a basis of a $D$-dimensional linear space.

We assume that all the products of forms whose total degree is less than $N$ are independent and span new modules over the algebra of functions with appropriate dimensions, e.g. the products of degree 2, $d\xi^k d\xi^m$, span a $D^2$-dimensional linear space; so do the products $d^2\xi^k d^2\xi^m$ and, independently, $d\xi^m d^2\xi^k$ (if $D > 3$), and so on. On the other hand, all other products of degree $N$ must obey the following commutation relations, which are compatible with the cyclic commutation relations for the product of $N$ 1-forms, for example:

$$d^p\xi^k d^{N-p}\xi^l = q^p d^{N-p}\xi^l d^p\xi^k,$$
\[ d^{N-p} \xi^k d\xi^l_1 d\xi^l_2 \ldots d\xi^l_p = q^{N-p} d\xi^l_1 d\xi^l_2 \ldots d\xi^l_p d^{N-p} \xi^k, \text{ etc.} \] (34)

Finally, we shall assume that not only the products of \( N+1 \) and more 1-forms vanish, but along with them, also any other products of all kinds of forms whose total degree is greater than \( N \). This additional assumption is necessary in order to ensure the coordinate-independent character of the condition \( d^N = 0 \), because under a coordinate change all the products of forms of given order mix up and transform into each other, e.g. the terms \( d\xi^j d\xi^k d\xi^l \) with the terms of the type \( d^2 \xi^k d\xi^l \), and similarly for higher order terms.

It is easy to prove that for a given \( N \) it is enough to assume \( d^N \xi^k = 0 \) and the \( N \)-cyclic commutation rule

\[ d\xi^{k_1} d\xi^{k_2} d\xi^{k_3} \ldots d\xi^{k_N} = q d\xi^{k_2} d\xi^{k_3} \ldots d\xi^{k_N} d\xi^{k_1} \]

implemented with its generalization for any product of two exterior forms of the total order adding up to \( N \),

\[ \omega \phi = q^{p(N-p)} \phi \omega = q^{-p^2} \phi \omega \]

whenever \( \deg(\omega) = p \) and \( \deg(\phi) = N - p \), in order to ensure that \( d^N f = 0 \), and in general, \( d^N \omega = 0 \) for any differential form \( \omega \).

We end this section by showing how the gauge-invariant analogs of curvature are expressed via the unique gauge-invariant curvature 2-form \( F_{ik} \) and its covariant derivatives. Indeed, for \( N = 3 \) we have (cf. Ref [3]):

\[ \Omega = D^2 A = \frac{1}{3} (D_i F_{km} + j D_m F_{ki}) d\xi^i d\xi^k d\xi^m + F_{ik} d^2 \xi^i d\xi^k, \] (35)

whereas when \( N = 4 \) we get, with \( \Theta_{iklm} = \frac{1}{4} (D_i D_k F_{lm} + i D_m D_l F_{ki}) \), \( \Phi_{klm} = D_k F_{lm} + i D_m F_{kl}, \) \( \tilde{F}_{ik} = \frac{i}{2} F_{ik} \):

\[ \Omega = D^3 A = \Theta_{jklm} d\xi^j d\xi^k d\xi^l d\xi^m + \Phi_{iklm} d^2 \xi^i d\xi^k d\xi^m + F_{ik} d^3 \xi^i d\xi^k + \tilde{F}_{ik} d^2 \xi^i d^2 \xi^k. \]

5 Deformations of the covariant \( q \)-differential

The \( q \)-differentials defined on the \( Z \)-graded associative unital algebra can be embedded in a bundle over the complex plane \( \mathbb{C}^1 \), with a typical fibre being the space of linear operators defined on the algebra \( A \).

With this view on the \( q \)-differentials, a natural question can be asked now, namely, what would be the effect of one or more \textit{deformations} of the complex parameter \( q \) itself, \( q \to q + \epsilon, \epsilon \in \mathbb{C} \)? In particular, it is interesting to see what will happen if we perform a series of such deformations, returning back to the initial value of \( q \). In other words, if we consider a series of infinitesimal complex deformations \( \epsilon, \epsilon', \epsilon'', \ldots, \epsilon^{(n)} \), such that \( \epsilon + \epsilon' + \epsilon'' + \ldots + \epsilon^{(n)} = 0 \), what will be the result of
consecutive action of $n$ corresponding $(q + \epsilon^{(k)})$-deformed differentials, or of their combinations with various permutations? This question seems quite pertinent in the context of $q$-differential calculus, and nearly as natural as the investigation of consecutive infinitesimal diffeomorphisms of a manifold, induced by two different vector fields, that led Sophus Lie to the definition of the Lie bracket as the commutator of two such automorphisms.

Let us consider first the simplest case of $N = 2$. According to the scheme exposed above, we shall put $\epsilon' = -\epsilon$. Now, let us compare the consecutive action of the operators $d_{q+\epsilon}$ and $d_{q-\epsilon}$ taken in two different orders. Acting on an arbitrary element $B$ of degree 1, for example, we have by definition:

$$d_{q+\epsilon} B = \eta B - (q + \epsilon) B \eta;$$

and taking into account that now $d_{q+\epsilon} B$ is of degree 2, i.e. $0 \mod 2$, we arrive at the following expression:

$$d_{q-\epsilon} d_{q+\epsilon} B = \epsilon(B\eta^2 - \eta B\eta) = \epsilon(B - \eta B\eta) \quad (q = -1, \eta^2 = 1)$$

Evaluating similar expression with $d_{q+\epsilon} d_{q-\epsilon}$ taken in the reverse order amounts to changing the sign of the parameter $\epsilon$ in the above expression. Therefore adding or subtracting the two expressions we get

$$[d_{q-\epsilon} d_{q+\epsilon} + d_{q+\epsilon} d_{q-\epsilon}] B = 0, \quad (36)$$

$$[d_{q-\epsilon} d_{q+\epsilon} - d_{q+\epsilon} d_{q-\epsilon}] B = 2\epsilon(B - \eta B\eta) \quad (37)$$

When the degree of $B$ is 0, the corresponding formula reads:

$$[d_{q-\epsilon} d_{q+\epsilon} + d_{q+\epsilon} d_{q-\epsilon}] B = 0, \quad (38)$$

$$[d_{q-\epsilon} d_{q+\epsilon} - d_{q+\epsilon} d_{q-\epsilon}] B = 2\epsilon(B + \eta B\eta) \quad (39)$$

It is interesting to note that the operators defined as

$$\mathcal{P}_1(B) = \frac{1}{2}(B - \eta B\eta) \text{ and } \mathcal{P}_2(B) = \frac{1}{2}(B + \eta B\eta)$$

are projectors, i.e. $\mathcal{P}_k^2(B) = \mathcal{P}_k(B)$, $k = 1, 2$. Obviously, $\mathcal{P}_1 \mathcal{P}_2 = \mathcal{P}_2 \mathcal{P}_1 = 0$, so that these operators project onto the subspaces of degree 1 and 0 respectively.

Similar result can be obtained for a series of 3 consecutive deformed $q$-differentials around the value of $q = j = e^{2\pi i/3}$. Let us note

$$d_1 = d_{q+j\epsilon}, \quad d_2 = d_{q+j^2\epsilon} \quad \text{and} \quad d_3 = d_{q+\epsilon}.$$ 

Direct calculus leads to the following:

$$[d_1 d_2 d_3 + d_2 d_3 d_1 + d_3 d_1 d_2] B = 0 \quad (40)$$
whereas the combination that generalizes the *commutator* for the $\mathbb{Z}_3$-graded case produces (up to a factor) projection operators

$$\frac{1}{9} [d_1 d_2 d_3 + jd_2 d_3 d_1 + j^2 d_3 d_1 d_2] B = \frac{1}{3} (B + j\eta B\eta^2 + j^2 \eta^2 B\eta) = P_1(B).$$

if $\deg(B) = 1$, and a similar operator $P_2$ with $j$ and $j^2$ interchanged when $\deg(B) = 2$. Here again, $P_2^2 = P_1$, $P_1^2 = P_2$, $P_1 P_2 = P_2 P_1 = 0$.

Similar projection operators may be obtained by applying the same scheme to the properly symmetrized products of $N$ deformed differentials in the case of a $\mathbb{Z}_N$-graded generalization. More complicated scheme arises when we consider all the $N$ independent “partial” $q$-differentials. A natural and important question to ask is the following: how the covariant $q$-differentials react under these cyclic deformations? What kind of an operator we get applying the $N$-fold products of covariant $q$-differentials deformed as follows:

$$D_q = d_q + A \Rightarrow \tilde{D}_{q+\epsilon} = d_{q+\epsilon} + A + \epsilon \Lambda \quad (41)$$

In the $N = 2$ matrix realization case, the symmetric part of the product yields the usual curvature and no extra terms linear in the deformation parameter $\epsilon$, while the anti-symmetrized product is proportional to $\epsilon$ and is equal to:

$$[D_{q+\epsilon} \tilde{D}_{q-\epsilon} - D_{q-\epsilon} \tilde{D}_{q+\epsilon}] B = D_q B\eta + (D_q \Lambda) B$$

In conclusion, we think that the last idea consisting in a discrete analogue of the holonomy, might be useful for the analysis of new symmetries of the $q$-deformed covariant differential operators and for the classification of their invariant properties. In some sense this is an analog of the geodesic deviation equation in classical differential geometry.

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