Trapped bosons, thermodynamic limit and condensation: a study in the framework of resolvent algebras

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The virtues of resolvent algebras, compared to other approaches for the treatment of canonical quantum systems, are exemplified by infinite systems of non-relativistic bosons. Within this framework, equilibrium states of trapped and untrapped bosons are defined on a fixed C*-algebra for all physically meaningful values of the temperature and chemical potential. Moreover, the algebra provides the tools for their analysis without having to rely on ad hoc prescriptions for the test of pertinent features, such as the appearance of Bose-Einstein condensates. The method is illustrated in case of non-interacting systems in any number of spatial dimensions and sheds new light on the appearance of condensates. Yet the framework also covers interactions and thus provides a universal basis for the analysis of bosonic systems.

Dedicated to Jakob Yngvason on the occasion of his 75th birthday

1. INTRODUCTION

The algebraic approach to the discussion of infinite bosonic systems appears to be hampered by the fact that bosons can accumulate unlimitedly in finite regions. It might therefore seem hopeless to find observables which still give meaningful results in such singular states. Indeed, algebras of polynomials of the underlying Bose fields do not work. Proceeding to the common Weyl algebra generated by their exponentials, the unitary Weyl operators, does not solve this problem either. These operators can often not be interpreted as meaningful observables in such singular states in view of their unrestrained fluctuations and discontinuous behavior under symmetry transformations. In applications one encounters this problem by observing, for example, that the limits of certain sequences of equilibrium states on the Weyl algebra no longer satisfy the distinctive KMS condition.

A surprisingly simple solution of these well known problems was presented in [9]. Instead of dealing with exponentials of the Bose fields, one considers their resolvents. The resulting C*-algebra is, such as the Weyl algebra, defined by a few relations, encoding the algebraic properties of the field. But in contrast to the Weyl algebra, this resolvent algebra is not simple, i.e. it has a non-trivial ideal structure [4]. As a matter of fact, these ideals are a necessary ingredient for the solution of the preceding problems, since they comprise those observables which become singular, hence physically meaningless in singular states. These observables are annihilated in the corresponding representations, i.e. they are members of some ideal of the resolvent algebra and do not cause any problems in these states. On the other hand, the resolvent algebra has faithful irreducible representations induced by states of physical interest [9]. It therefore covers in a meaningful manner all possible states of bosonic systems.

It ought to be mentioned that resolvent algebras had already appeared in disguise much earlier in an investigation by Kastler of potentially interesting algebras for the description of canonical quantum systems [17]. This relationship was recently uncovered by Georgescu and Iftimovici [14], who also added further structural results to this framework. In spite of the work of Kastler, the utility of the resolvent algebra for the discussion of infinite bosonic systems remained unnoticed for decades, however. Only more recently it has found applications to problems of physical interest, cf. for example [6, 8, 10, 11, 16].
Within this framework, we complement here these results by an analysis of trapped and untrapped equilibrium states of non-interacting bosons for given temperature and chemical potential. Trapped states are described by Gibbs-von Neumann density operators on Fock space and are called Gibbs-von Neumann states, for short. We study in detail their thermodynamic and infinite particle number limits, the latter being obtained by proceeding to the maximal admissible value of the chemical potential, and the appearance of Bose-Einstein condensates. These topics have been widely discussed in the literature from various points of view, cf. \[2, 18, 21\] and references quoted there. It is the primary purpose of our article to illustrate the virtues of the resolvent algebras, compared to these other approaches, and to shed new light on some well known results. For the convenience of the reader, we begin by briefly recalling from \[2\] some relevant definitions and pertinent facts.

As already mentioned, the resolvent algebra can abstractly be defined by relations. Since it is faithfully represented on Fock space \[2\], we can deal here with this concrete representation and proceed from it later to disjoint representations of physical interest. The bosonic Fock space is denoted by \(\mathcal{F}\); it is the infinite direct sum \(\bigoplus_{n=0}^{\infty} \mathcal{F}_n\) of \(n\)-particle spaces which are generated by \(n\)-fold symmetric tensor products of single particle states in \(\mathcal{F}_1 \simeq L^2(\mathbb{R}^s)\) in \(s\) spatial dimensions. The zero particle state in \(\mathcal{F}\) (vacuum) is denoted by \(\Omega\).

Let \(\mathcal{D}(\mathbb{R}^s) \subset L^2(\mathbb{R}^s)\) be some space of complex-valued test functions on configuration space. As a matter of fact, one may proceed from any dense subspace of \(L^2(\mathbb{R}^s)\), our results do not depend on its particular choice. We will comment on this point further below. The Bose field underlying Fock space is denoted by \(\phi\). It is defined as a real linear map from \(\mathcal{D}(\mathbb{R}^s)\) to selfadjoint operators, acting on a common core in \(\mathcal{F}\), viz. the domain of the particle number operator \(N\). The corresponding annihilation and creation operators are given by \(a(f) = 2^{-1/2}(\phi(f) + i\phi(if))\) and \(a(f)^* = 2^{-1/2}(\phi(f) - i\phi(if)), \ f \in \mathcal{D}(\mathbb{R}^s)\). In order to avoid the subtleties associated with unbounded operators, it is appropriate to proceed to bounded functions of the field. Continuous functions of the field, vanishing at infinity, are most convenient and prominent examples are the resolvents

\[
R(\lambda, f) = (i\lambda 1 - \phi(f))^{-1}, \quad f \in \mathcal{D}(\mathbb{R}^s), \ Re \lambda \neq 0.
\]

Moreover, these resolvents comprise all algebraic properties of the field. For example, the canonical commutation relations are encoded in the relation

\[
[R(\lambda, f), R(\mu, g)] = i \text{Im} \langle f, g \rangle R(\lambda, f)R(\mu, g)^2 R(\lambda, f),
\]

where \(\langle f, g \rangle\) denotes the (sesquilinear) scalar product of \(f, g\) in \(L^2(\mathbb{R}^s)\), fixing the symplectic form \(\text{Im} \langle f, g \rangle\). The resolvent algebra \(\mathfrak{R}\) is defined as the norm-closed subalgebra of the algebra \(\mathcal{B}(\mathcal{F})\) of bounded operators on \(\mathcal{F}\), which is generated by the resolvents; it is thus a unital \(C^*\)-algebra \[2\].

In the discussion of infinite bosonic systems, occupation numbers of particle states play a prominent role. The corresponding observables can be identified within \(\mathfrak{R}\) with the help of the gauge group generated by the particle number operator \(N\). It operates by automorphisms \(\gamma\) on \(\mathfrak{R}\) and acts on the basic resolvents according to

\[
\gamma(u)(R(\lambda, f)) = e^{iuN} R(\lambda, f) e^{-iuN} = R(\lambda, e^{iu} f), \quad 0 \leq u \leq 2\pi.
\]

Even though this action is discontinuous in \(u\) with regard to the norm topology, harmonic analysis of the gauge group is possible in the \(C^*\)-algebra \(\mathfrak{R}\) in the following sense. One finds \[2\] that for any \(R \in \mathfrak{R}\) the integrals

\[
R_m = (1/2\pi) \int_0^{2\pi} du e^{imu} \gamma(u)(R), \quad m \in \mathbb{Z},
\]

being defined on \(\mathcal{F}\) in the strong operator topology, are elements of the resolvent algebra \(\mathfrak{R}\). In particular, the operators corresponding to \(m = 0\) constitute a norm closed subalgebra \(\mathfrak{A} \subset \mathfrak{R}\), which contains the gauge invariant observables. The operators corresponding to arbitrary \(m \in \mathbb{Z}\) transform as tensors under the action of the gauge group and generate a norm closed subalgebra \(\mathfrak{A} \subset \mathfrak{R}\) on which this group acts pointwise norm continuously. Thus,
harmonic analysis and synthesis of the gauge group is possible on $\mathfrak{A}$. It is therefore convenient to focus on these subalgebras in applications.

Let us mention as an aside that the Weyl algebra, although being stable under the action of the gauge group, does not contain a single gauge invariant observable, apart from multiples of the identity. The integrals of Weyl operators, analogous to relation (1.4), are not contained in the Weyl algebra and therefore cannot be used in the analysis of non-Fock states on it.

The primary motivation for the introduction of resolvent algebras in $\mathfrak{R}$ was the observation that they admit the automorphic action of a multitude of dynamics of physical interest, in contrast to the Weyl algebra. For bosonic systems with a finite number of degrees of freedom, described by finite dimensional test function spaces, the preceding framework is fully satisfactory. Yet for infinite systems it is in general too restrictive. This can already be seen in case of non-interacting systems: let $H$ be the second quantization of any given selfadjoint operator $h$ on the single particle space $\mathcal{F}_1$. Thus $H$ commutes with the particle number operator $N$ and one may therefore ask whether the algebra of observables $\mathfrak{A}$ is stable under the adjoint action of $e^{itH}$, $t \in \mathbb{R}$. It turns out that for any given $n \in \mathbb{N}$ one has

$$\text{Ad } e^{itH}(A) | \bigoplus_{m=0}^n \mathcal{F}_m \in \mathfrak{A} | \bigoplus_{m=0}^n \mathcal{F}_m, \quad t \in \mathbb{R}, \; A \in \mathfrak{A},$$

i.e. the restrictions of the observable algebra to states with limited particle numbers are stable under all of these dynamics. Yet the hoped for inclusion $\text{Ad } e^{itH}(\mathfrak{A}) \subset \mathfrak{A}$ holds only if the chosen test function space is stable under the action of the single particle dynamics, i.e. $e^{ith}\mathcal{D}(\mathbb{R}^*) \subset \mathcal{D}(\mathbb{R}^*)$, $t \in \mathbb{R}$. As an aside, relation (1.5) still holds in case of dynamics involving sufficiently regular pair interactions, but the desired inclusion never applies.

These problems can be solved by slightly extending the observable algebra. Since this algebra works well for finite systems, it seems natural to try to find an extension without altering the observables on the subspaces of Fock space with limited particle number. Adopting this point of view, the maximal admissible extension is the C*-algebra $\overline{\mathfrak{A}} \supset \mathfrak{A}$ which is generated by all bounded, gauge invariant operators on $\mathcal{F}$ whose action on any given subspace $\bigoplus_{m=0}^n \mathcal{F}_m$ coincides with the action of some operator in $\mathfrak{A}$, possibly depending on $n \in \mathbb{N}$. Alternatively, $\overline{\mathfrak{A}}$ can be presented as projective limit of the observable algebras on the subspaces of $\mathcal{F}$ with limited particle number. We will refer to $\overline{\mathfrak{A}}$ as canonical extension of $\mathfrak{A}$. It is only a small subalgebra of the algebra of all bounded, gauge invariant operators on $\mathcal{F}$; for example, it does not contain any operator with continuous spectrum. Nevertheless, this algebra is stable under the action of non-interacting dynamics as well as interacting dynamics with continuous two-body potentials vanishing at infinity. In a similar manner, the field algebra $\mathfrak{F}$, generated by all tensors under the action of the gauge group, can be extended to a C*-algebra $\mathfrak{F}_c$ which is stable under these dynamics. As a matter of fact, this extension is obtained by adding to $\mathfrak{F}_c$ a single isometry. For a thorough discussion of these facts, cf. [6, 7].

The algebras $\overline{\mathfrak{A}}$, respectively $\mathfrak{F}_c$, provide in a sense maximal arenas for the discussion of the dynamics of infinite bosonic systems. Yet in applications it is frequently more convenient to proceed to suitable subalgebras. For given dynamics, a minimal choice for the observables would be the C*-algebra, which is generated on $\mathcal{F}$ by the operators $\{\text{Ad } e^{itH}(A) : A \in \mathfrak{A}, \; t \in \mathbb{R}\}$. If one needs to have control on the continuity properties of the dynamics, pointwise in the norm topology (characterizing C*-dynamical systems), one is led to consider the C*-algebra $\mathfrak{R}_c$, generated by

$$\left\{ \int dt \; f(t) \; \text{Ad } e^{itH}(A) : A \in \mathfrak{A}, \; f \in L^1(\mathbb{R}) \right\},$$

where the integrals are defined in the strong operator topology. It was shown in [6] that the algebra $\mathfrak{R}_c$ is also contained in $\overline{\mathfrak{A}}$. In a similar manner one can proceed for given dynamics to convenient subalgebras of $\mathfrak{F}_c$, cf. [7].

In the present article we restrict our attention to systems of non-interacting bosons. Since we are interested in trapped systems and their thermodynamic limit, we need to consider different dynamics. An algebra which covers
them all is obtained by choosing as test function space underlying the resolvent algebra $\mathcal{R}$ the space of all square integrable functions $L^2(\mathbb{R}^s)$. We will deal in the following with this algebra and the corresponding observable and field subalgebras $\mathfrak{A}$, respectively $\mathfrak{g}$. It is note-worthy that the latter two algebras are contained in the canonically extended algebras obtained for any initial choice of a test function space $\mathcal{D}(\mathbb{R}^s)$. We also note that in the present case of non-interacting systems most computations can conveniently be performed within the resolvent algebra $\mathcal{R}$, because it is stable under the dynamics. But in case of interaction it is advantageous to proceed from the outset to its observable, respectively field subalgebras, where one has better control of the action of the dynamics.

Within this framework we consider Gibbs-von Neumann states, i.e. trapped thermal equilibrium states of bosons for given temperature and chemical potential, which are faithful and normal with regard to the Fock representation. By proceeding to the limits of zero trapping potential (thermodynamic limit), respectively infinite particle number (maximal chemical potential), we obtain states which still satisfy the KMS-condition for the limit dynamics. The resulting representations are not normal with regard to the Fock representation, however. Moreover, in the infinite particle number limit they are also not faithful, since observables, which are sensitive to particles in the respective ground states become singular, i.e. the corresponding operators in the resolvent algebra disappear in the limit. This is an indication for the appearance of condensates.

The study of condensates requires a more detailed analysis, however. To this end we adopt a local point of view: let $O \subset \mathbb{R}^s$ be any bounded region and let $N(O)$ be the corresponding number operator on Fock space $\mathcal{F}$, counting the number of particles in $O$. Thus for any given normal state $\omega$, described by a density matrix on Fock space, the quantity $\omega(N(O))$ is the expected number of particles in $O$. For normal thermal states, being of interest here, this quantity is finite. Now given any sequence $\omega_\nu$ of normal states, $\nu \in \mathbb{N}$, these expectation values may (a) either stay finite or (b) approach infinity in the limit. In case (a), the limit states have a finite particle density in $O$; in order to increase it one needs to add to them further particles. In case (b), the appearance of an infinity of particles in $O$ can have different reasons. We attribute it here to the formation of a Bose-Einstein condensate whenever there is some wave function $g$ such that the number of particles in $O$ having wave functions in the orthogonal complement of $g$ stays finite in the limit. So these particles have a critical density. The divergence of $N(O)$ can then be attributed entirely to particles with wave function $g$, creating condensates in the approximating states which grow unboundedly.

The computations needed in the analysis of equilibrium states in order to decide which of these cases is at hand can conveniently be performed with aid of the resolvent algebra. It turns out that for trapped thermal systems, which are confined by some regular potential, case (b) always occurs in the limit of infinite particle numbers. Wheras the density of particles in excited states, forming a thermal cloud, remains bounded in the approximating states, these states exhibit growing densities of Bose-Einstein condensates of particles with wave function $g$ carrying minimal energy. This feature occurs in any number of dimensions and does not imply the spontaneous breakdown of gauge invariance.

In case of the thermodynamic limit states, containing an infinity of particles from the outset, the local properties of the states depend on the number of dimensions: in $s = 1, 2$ dimensions, the states obtained for maximal chemical potential also belong to case (b) without including proper condensates. Yet, as we shall see, they have properties which resemble quasi-condensates. If $s > 2$ one arrives in the limit at case (a), i.e. the particle density in bounded regions stays finite. In order to increase this density, one can modify the limit states without destroying their equilibrium property. This is accomplished by shifting the quantum field by a classical field which is created by the collective effects of particles with a suitable (improper) wave function $g$. It can be described by the adjoint action of Weyl operators, depending on $g$. In this manner, the local density of particles with this wave function can be arbitrarily increased without affecting the density of the thermal cloud, escorting them. Hence the resulting states describe condensates as well. These states are no longer gauge invariant, however, due to the action of the Weyl operators. So, in summary, the resolvent algebra provides an adequate framework for the study of trapped and untrapped thermal states and of the differing manifestations of condensation.
Our article is organized as follows. In the subsequent section we consider quasifree states on the resolvent algebra which are fixed by specifying their one and two-point functions. For appropriate sequences of such states, we will study their convergence to limit states which are disjoint from the Fock representation. The insights gained in this section will be used in Sect. 3 in the analysis of trapped equilibrium states and of their thermodynamic and infinite particle number limits. The particle density is recovered from the observable algebra in Sect. 4 and the formation and analysis of condensates is discussed in Sec. 5. The article concludes with a summary and an outlook on interacting systems.

2. QUASIFREE STATES

In this section we consider certain specific quasifree states on the resolvent algebra which are normal with regard to the Fock representation. We will proceed from them to limit states, inducing inequivalent representations by the GNS construction [15]. In this analysis we make use of the fact that the Fock representation of the resolvent algebra extends to a regular representation of the Weyl algebra and vice versa. Regular representations of resolvent algebras are defined by the property that all resolvents have trivial kernels. They are in one-to-one correspondence to regular representations of the Weyl algebra, in contrast to the singular representations [9, Cor. 4.4].

The unitary Weyl operators in the Fock representation are denoted by $W(f)$, $f \in L^2(\mathbb{R}^s)$. They satisfy the common Weyl relations
\begin{equation}
W(f)W(g) = e^{-(i/2)\text{Im}(f,g)} W(f + g), \quad f, g \in L^2(\mathbb{R}^s).
\end{equation}

The quasifree states $\omega$ of interest here are conveniently defined on the Weyl operators by the equality
\begin{equation}
\omega(W(f)) = e^{i l_\omega(f)} e^{-(1/2)(f,f)_\omega}, \quad f \in L^2(\mathbb{R}^s).
\end{equation}

Here $(f, g)_\omega = \overline{(g, f)}_\omega$ is a bilinear form on $L^2(\mathbb{R}^s)$, regarded as a real vector space. Its real part is a scalar product, its imaginary part is given by $(1/2) \text{Im} (f, g)$, where $(1/4) |\text{Im} (f, g)|^2 \leq (f, f)_\omega (g, g)_\omega$ for $f, g \in L^2(\mathbb{R}^s)$. To simplify terminology, we refer to the forms $(\cdot, \cdot)_\omega$ as scalar products. We assume in the following that they are gauge invariant (stable under multiplication of its entries with equal phase factors). The symbol $l_\omega(\cdot)$ denotes a real linear functional on $L^2(\mathbb{R}^s)$, which is fixed by the one-point function of the field. The two-point function of the field is related to these entities by the formula
\begin{equation}
\omega(\phi(f)\phi(g)) = (f, g)_\omega + l_\omega(f)l_\omega(g), \quad f, g \in L^2(\mathbb{R}^s).
\end{equation}

With the preceding conventions, a quasifree state is gauge invariant if and only if the linear functional vanishes.

We are interested here in certain specific sequences of such quasifree states and their limits. As is well known, any sequence of states has limit points. But the resulting expectation values of the Weyl operators have in general the form only on subspaces $D \subset L^2(\mathbb{R}^s)$ and vanish on their complements. Moreover, the underlying linear functionals may attain non-linear limits. Since the Weyl algebra is simple, such states lead to singular GNS representations, often defying a meaningful physical interpretation. In particular, it may happen that generators of the spacetime symmetries are not defined.

As we shall see, the situation is better for quasifree states on the resolvent algebra. The expectation values of the resolvents in these state are obtained from equation (2.2) by a Laplace transformation. For $\text{Re} \lambda_1 > 0, \ldots, \text{Re} \lambda_n > 0$
one has, cf. [Eq. 17],
\[ i^n \omega(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n)) \]
\[ = \int_{\mathbb{R}^n_+} du_1 \cdots du_n e^{-\sum_j u_j \lambda_j} \omega(W(-u_1 f_1) \cdots W(-u_n f_n)) \]
\[ = \int_{\mathbb{R}^n_+} du_1 \cdots du_n e^{-\sum_j u_j (\lambda_j + i\omega(f_j))} e^{-(i/2) \sum_{k<l} \text{Im}(u_k f_k, u_l f_l)} e^{-(1/2) \sum_{k,l} \langle u_k f_k, u_l f_l \rangle}. \]

Making use of the equality \( R(-\lambda, f) = -R(\lambda, f) \) one obtains analogous relations for any choice of the signs of \( \text{Re} \lambda_1, \ldots, \text{Re} \lambda_n \).

Let us turn now to the analysis of sequences of quasifree states \( \omega_{\nu} \) on the resolvent algebra, which are fixed by specific sequences of scalar products and linear functionals. Having the applications in mind, we assume that the underlying sequences of scalar products \( \langle f, f \rangle_{\nu}, f \in L^2(\mathbb{R}^s) \), have (possibly infinite) limits as \( \nu \in \mathbb{N} \) tends to infinity. With this input the following proposition obtains.

**Proposition 2.1.** Let \( \omega_{\nu}, \nu \in \mathbb{N} \), be a sequence of quasifree states on the resolvent algebra \( \mathfrak{K} \) with properties specified above.

(i) If the states are gauge invariant, the sequence converges, pointwise on \( \mathfrak{K} \), to some state \( \omega_\infty \). There is some complex subspace \( D \subset L^2(\mathbb{R}^s) \) with scalar product \( (\cdot, \cdot)_\infty \), such that for \( f_1, \ldots, f_n \in D \) and \( \text{Re} \lambda_1 > 0, \ldots, \text{Re} \lambda_n > 0, n \in \mathbb{N} \),
\[ i^n \omega_\infty(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n)) \]
\[ = \int_{\mathbb{R}^n_+} du_1 \cdots du_n e^{-\sum_j u_j \lambda_j} e^{-(i/2) \sum_{k<l} \text{Im}(u_k f_k, u_l f_l)} e^{-(1/2) \sum_{k,l} \langle u_k f_k, u_l f_l \rangle}. \]
If \( \{f_1, \ldots, f_n\} \cap (L^2(\mathbb{R}^s) \setminus D) \neq \emptyset \) one has
\[ \omega_\infty(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n)) = 0. \]

(ii) If the states are not gauge invariant, there exists a state \( \omega_\infty \) on \( \mathfrak{K} \), which is a weak-* limit point of the sequence.
It has the following properties: as in (i), there is some complex subspace \( D \subset L^2(\mathbb{R}^s) \) with a scalar product \( (\cdot, \cdot)_\infty \) and a real subspace \( D_\infty \subset \mathcal{D} \) with a real linear functional \( l_\infty \) such that for \( f_1, \ldots, f_n \in D_\infty \) and \( \text{Re} \lambda_1 > 0, \ldots, \text{Re} \lambda_n > 0, n \in \mathbb{N} \),
\[ i^n \omega_\infty(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n)) \]
\[ = \int_{\mathbb{R}^n_+} du_1 \cdots du_n e^{-\sum_j u_j (\lambda_j + i l_\infty(f_j))} e^{-(i/2) \sum_{k<l} \text{Im}(u_k f_k, u_l f_l)} e^{-(1/2) \sum_{k,l} \langle u_k f_k, u_l f_l \rangle}. \]
If \( \{f_1, \ldots, f_n\} \cap (L^2(\mathbb{R}^s) \setminus D_\infty) \neq \emptyset \) one has
\[ \omega_\infty(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n)) = 0. \]

(iii) Given any finite dimensional complex subspace \( K \subset L^2(\mathbb{R}^s) \), let \( \mathfrak{K}(K) \subset \mathfrak{K} \) be the subalgebra which is generated by resolvents \( R(\lambda, f) \) with \( f \in K \), \( \text{Re} \lambda \neq 0 \). There exists a subset \( \mathbb{I} \subset \mathbb{N} \) such that the sequence \( \omega_{\nu} | \mathfrak{K}(K) \), \( \nu \in \mathbb{I} \), converges pointwise to \( \omega_\infty \).

**Remark:** Resolvents assigned in (i) to test functions in \( L^2(\mathbb{R}^s) \setminus D \), respectively in (ii) to test functions in \( L^2(\mathbb{R}^s) \setminus D_\infty \), generate ideals in \( \mathfrak{K} \). Note that these sets are not fixed from the outset and depend on the limit state.
Proof. Let \( \mathcal{D} \subset L^2(\mathbb{R}^n) \) be the subset of all functions for which the sequences \( \langle f, l \rangle_{\nu}, f \in \mathcal{D} \), have a finite limit for \( \nu \) tending to infinity. It follows from the triangle inequality and the polarization identity that \( \mathcal{D} \) is a complex linear subspace of \( L^2(\mathbb{R}^n) \) and that there is some scalar product \( \langle \cdot, \cdot \rangle_\infty \) on \( \mathcal{D} \) such that
\[
\lim_{l \to \infty} \langle f, g \rangle_{\nu} = \langle f, g \rangle_\infty, \quad f, g \in \mathcal{D}.
\]
Now the function appearing in the exponents of the last line of equation \([2.4]\)
\[
u
\]
\[
\sum_{k,l} u_k u_l \langle f_k, f_l \rangle_{\nu} + (1/2) \sum_k u_k^2 \langle f_k, f_k \rangle_{\nu},
\]
has a non-negative real part. If \( f_1, \ldots, f_n \in \mathcal{D} \), it converges in the limit of large \( \nu \) uniformly on compact subsets of \( \mathbb{R}^n \) to a similar expression, where the sequence of scalar products is replaced by their limits. On the other hand, if \( f_k \in L^2(\mathbb{R}^n) \setminus \mathcal{D} \) for some \( k \), the real part of this function tends to \( +\infty \) for almost all \( u_1, \ldots, u_n \). Now the modulus of the integrand in equation \([2.4]\) is bounded by the integrable function \( e^{-\sum u_k \Re \lambda_k} \). It therefore follows from the dominated convergence theorem that the expectation values \( \omega_{\nu}(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n)) \) in gauge invariant states converge in the limit of large \( \nu \) to the expressions given in part (i) of the statement. Since the finite sums of products of resolvents are norm dense in the resolvent algebra, it is then also clear that the sequence of gauge invariant states converges pointwise on \( \mathcal{R} \), completing the proof of part (i) of the statement.

Turning to part (ii), the complex subspace \( \mathcal{D} \subset L^2(\mathbb{R}^n) \) and the limit scalar product \( \langle \cdot, \cdot \rangle_\infty \) are defined as in the preceding step. It also follows from that argument that the expectation values \( \omega_{\nu}(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n)) \) converge to 0 if \( f_j \in L^2(\mathbb{R}^n) \setminus \mathcal{D} \) for some \( j \). Moreover, if \( f_1, \ldots, f_n \in \mathcal{D} \), one can replace in equation \([2.4]\) the scalar products \( \langle \cdot, \cdot \rangle_{\nu} \) by their limit values \( \langle \cdot, \cdot \rangle_\infty \), making use of the dominated convergence theorem. The resulting error vanishes in the limit of large \( \nu \). So we can focus on the sequence of real linear functionals \( l_\nu, \nu \in \mathbb{N} \), restricted to \( \mathcal{D} \).

In order to exhibit the properties of this sequence, we proceed to the two-point compactification of \( \mathbb{R} \), given by the function \( x \mapsto c(x) = x/\sqrt{x^2 + 1} \), where \( \pm \infty \) are mapped to \( \pm 1 \). We then consider the sequence of functionals on \( \mathcal{D} \) given by \( f \mapsto c(l_\nu(f)) \in [-1, 1], \nu \in \mathbb{N} \). According to Tychonoff’s theorem the cartesian product \( \prod_{\nu \in \mathbb{N}} [-1, 1] \), equipped with the product topology, is compact. Hence the sequences \( \nu \mapsto c(l_\nu(f)) \) have some limit (accumulation) point in this product, which is denoted by \( c(l_\infty(f)) \), \( f \in \mathcal{D} \). Thus, given any finite family of functions \( f_j \in \mathcal{D} \), there exists some subset \( \mathbb{I} \subset \mathbb{N} \) such that \( \lim_{\nu \in \mathbb{I}} c(l_\nu(f_j)) = c(l_\infty(f_j)), j = 1, \ldots, n \).

Now let \( \mathcal{D}_\infty \subset \mathcal{D} \) be the set of all \( f \in \mathcal{D} \) for which \( |c(l_\infty(f))| < 1 \). Applying to the corresponding approximating sequences the inverse function \( c^{-1} \), it follows that for any finite number of elements \( f_j \in \mathcal{D}_\infty \) one has \( \lim_{\nu \in \mathbb{I}} l_\nu(f_j) = l_\infty(f_j), j = 1, \ldots, n \), for the appropriate choice of index set \( \mathbb{I} \subset \mathbb{N} \). Since all functionals \( l_\nu \) are real linear on \( \mathcal{D} \), this implies that \( \mathcal{D}_\infty \) is a real subspace of \( \mathcal{D} \) and that the limit \( l_\infty \) is a real linear functional on this space. Thus the sequences \( l_\nu(f_j), \nu \in \mathbb{I} \), converge to \( l_\infty(f) \) for all \( f \) in the real linear span of the functions \( f_1, \ldots, f_n \).

Next, let \( f_1, \ldots, f_n \in \mathcal{D} \) be a finite family of functions among which there is some \( f_k \in \mathcal{D} \setminus \mathcal{D}_\infty \), i.e. \( c(l_\infty(f_k)) \in \{ \pm 1 \} \). As before, there is some subset \( \mathbb{I} \subset \mathbb{N} \) such that \( \lim_{\nu \in \mathbb{I}} c(l_\nu(f_k)) = c(l_\infty(f_k)), j = 1, \ldots, n \). Applying again the inverse function \( c^{-1} \), it follows that the sequence \( l_\nu(f_k), \nu \in \mathbb{I} \), diverges. As a matter of fact, since \( f_1, \ldots, f_n \) span a finite dimensional space, the sequence of functions
\[
u
\]
\[
s_j \mapsto \sum_{j=1}^n u_j l_\nu(f_j), \quad \nu \in \mathbb{I},
\]

tends to \( \pm \infty \) for almost all \( u_1, \ldots, u_n \in \mathbb{R}^n \).

With this information, we can turn now to the analysis of the expectation values of products of resolvents with functions \( f_j \in \mathcal{D}, j = 1, \ldots, n \), in the states \( \omega_\nu, \nu \in \mathbb{N} \). Recalling that we may replace the scalar products in equation
by their limit values, we only need to consider the Fourier transforms of
\[ u_1, \ldots, u_n \mapsto e^{-\sum_j u_j \lambda_j - (i/2) \sum_{k<l} u_k u_l \text{Im}(f_k f_l) - (1/2) \sum_{k,l} u_k u_l \langle f_k f_l \rangle} \]
(2.12)
at the points \( l_\nu(f_1), \ldots, l_\nu(f_n) \), \( \nu \in \mathbb{N} \). Since the functions are elements of \( L^1(\mathbb{R}_+^n) \), their Fourier transforms are continuous and vanish at infinity. Thus the limit state \( \omega_\infty \) can be defined on any sum of products of resolvents as follows: there exists for the functions underlying the resolvents a subset \( I \subset \mathbb{N} \) such that the sequences \( c(l_\nu(f_j)) \), \( \nu \in \mathbb{I} \), converge to \( c(l_\infty(f_j)) \), \( j = 1, \ldots, n \). In view of the preceding results it is then clear that the sub-sequence of states \( \omega_\nu \), \( \nu \in \mathbb{I} \), converges on the products of resolvents to the expression given in equation (2.7) whenever \( f_1, \ldots, f_n \in \mathcal{D}_\infty \).

If \( f_k \in \mathcal{L}^2(\mathbb{R}^s) \setminus \mathcal{D}_\infty \) for some \( k \), the corresponding sequence converges to 0. Thus \( \omega_\infty \) is a limit point of the sequence \( \omega_\nu, \nu \in \mathbb{N} \), with properties stated in (ii).

As to part (iii), the statement follows from (i) for gauge invariant sequences of states. In the non-gauge invariant case we can proceed as in the proof of part (ii). Let \( \mathcal{D}_{K\infty} \equiv K \cap \mathcal{D}_\infty \). Since this space is finite dimensional, there exists a subset \( \mathbb{I} \subset \mathbb{N} \) such that \( \lim_{\nu \in \mathbb{I}} l_\nu(g) = l_\infty(g) \) for \( g \in \mathcal{D}_{K\infty} \); if \( g \in K \setminus \mathcal{D}_{K\infty} \), the sequence \( l_\nu(g) \), \( \nu \in \mathbb{I} \), diverges. Now let \( g_j \in K, j = 1, \ldots, n \), be an arbitrary finite number of functions and consider an \( n \)-fold product of resolvents involving these functions. It follows from the preceding arguments that the expectation value of this product in the sequence of states \( \omega_\nu \), \( \nu \in \mathbb{N} \), vanishes in the limit if \( g_k \in K \setminus \mathcal{D}_{K\infty} \) for some \( k \). On the other hand, the subsequence of states \( \omega_\nu \), \( \nu \in \mathbb{I} \), converges according to the preceding arguments to \( \omega_\infty \) on all products of resolvents with \( g_j \in \mathcal{D}_{K\infty} \), \( j = 1, \ldots, n \). This completes the proof of the statement.

The limit states \( \omega_\infty \), established in this proposition, are defined on the full resolvent algebra \( \mathcal{R} \) and thus on its observable and field subalgebras \( \mathfrak{A} \), respectively \( \mathfrak{G} \). As we have seen, these states induce in general non-faithful GNS representations, where certain specific resolvents are trivially represented. Yet, as can be inferred from the relations given in the proposition, the non-trivially represented subalgebras \( \mathcal{R}(\mathcal{D}) \) in case (i) and \( \mathcal{R}(\mathcal{D}_\infty) \) in case (ii) are regular, i.e. the underlying resolvents with functions \( f \) in the respective subspaces of \( L^2(\mathbb{R}^s) \) have trivial kernels. This is a consequence of the fact that the expectation values of \( i\lambda R(\lambda, f) \) converge to 1 for \( \lambda \) tending to infinity, cf. [9, Prop. 4.5]. It is this feature which allows for the continuous unitary implementation of symmetry transformations of \( \mathcal{R} \) in the limit representations in cases, where it fails for the Weyl algebra.

3. TRAPPED AND UNTRAPPED EQUILIBRIUM STATES

In this section we analyze the properties of Gibbs-von Neumann states on the resolvent algebra \( \mathcal{R} \) and of their thermodynamic and infinite particle number limits. To this end we proceed for given length \( L > 0 \), from the single particle Hamiltonians \( h_L \equiv P^2 + V_L(Q) \), defined on a suitable domain in \( L^2(\mathbb{R}^s) \). Here \( P \) denotes the momentum operator, \( Q \) the position operator and \( x \mapsto V_L(x) = L^{-2} V(x/L) \) is a (scaled) potential, where \( V \) is continuous and non-negative, tending to infinity for large \( |x| \). To simplify the discussion, we assume that \( L \mapsto V_L(x) \) is monotonically decreasing for fixed \( x \in \mathbb{R}^s \).

In order to have control on the spectral properties of \( h_L \), we assume that the potentials comply with the bounds required in [10, Thm. XIII.81]. Simple examples are \( x \mapsto L^{-2-\eta} |x|^\eta \) for \( \eta > 1 \). Thus \( h_L \) is non-negative and has discrete spectrum. It is also apparent that each \( h_L \) is mapped to \( (1/L^2) h_1 \) by the adjoint action of a unitary scale transformation depending on \( L \). The ground state energy of \( h_L \) is denoted by \( \epsilon_{L,1} \equiv L^{-2} \epsilon_1 \), where \( \epsilon_1 \) is the positive, non-degenerate ground state energy of \( h_1 \) [10, Thm. XIII.47]. Applying the estimates on the asymptotic number of eigenstates of \( h_1 \), given in the above reference, all single particle partition functions
\[ \beta \mapsto \text{Tr} \mathcal{F}_1 e^{-\beta h_L} = \text{Tr} \mathcal{F}_1 e^{-\beta/(L^2)} h_1, \quad \beta > 0, \]
(3.1)
turn out to be finite.
If $H_L$ is the second quantization of $h_L$, often denoted by $d\Gamma(h_L)$, it follows from standard arguments that the operators $e^{-\beta (H_L - \mu N)}$ are of trace class on $\mathcal{F}$ for arbitrary inverse temperature $\beta > 0$ and chemical potential $\mu < \epsilon_{L,1}$. Thus, putting $Z = \text{Tr}_F e^{-\beta (H_L - \mu N)}$, the corresponding states on the resolvent algebra,

$$\omega_{\beta,\mu,L}(R) = Z^{-1} \text{Tr}_F e^{-\beta (H_L - \mu N)} R, \quad R \in \mathcal{R},$$

are trapped, gauge invariant equilibrium states (Gibbs-von Neumann states). They satisfy the KMS condition on $\mathcal{R}$ with regard to the adjoint action of the unitary group $t \mapsto e^{it (H_L - \mu N)}$ for the given $\beta$ and $\mu$. Note that these actions coincide on the gauge invariant subalgebra $\mathfrak{A} \subset \mathcal{R}$ for all values of the chemical potential $\mu$.

It is well known that these equilibrium states determine quasifree states on the Weyl algebra and, since they are regular, also on the resolvent algebra. As a matter of fact, any KMS state on the Weyl algebra for given non-interacting time evolution is a quasifree state under quite general conditions [20]. In the case at hand the equilibrium states are fixed by the scalar products, $f,g \in L^2(\mathbb{R}^s)$,

$$(f,g)_{\beta,\mu,L} = (1/2)\left(\langle f,e^{\beta(h_L-\mu)}(e^{\beta(h_L-\mu)}-1)^{-1}g\rangle + \langle g,(e^{\beta(h_L-\mu)}-1)^{-1}f\rangle\right).$$

The thermodynamic limit is obtained by letting $L$ tend to infinity, where the limit of the single particle Hamiltonians is denoted by $h_\infty = P^2$ and the ground state energy is replaced by their limit $\epsilon_{\infty,1} = 0$. The subsequent lemma enters in our proof that the limits of certain basic sequences of equilibrium states exist on the resolvent algebra and satisfy the KMS condition for the corresponding limit dynamics.

**Lemma 3.1.** Let $(\cdot, \cdot)_{\beta,\mu,L}$ be the scalar products defined above.

(i) Let $\beta > 0$, $\mu < 0$, and $t \in \mathbb{R}$. Then

$$\lim_{L \to \infty} \langle f,e^{it(h_L-\mu)}g \rangle_{\beta,\mu,L} = \langle f,e^{it(h_\infty-\mu)}g \rangle_{\beta,\mu,\infty}, \quad f,g \in L^2(\mathbb{R}^s).$$

(ii) Let $L \in \mathbb{R}_+ \cup +\infty$. There exists a complex subspace $\mathcal{D}_L \subset L^2(\mathbb{R}^s)$, which is stable under the action of $e^{it h_L}$, $t \in \mathbb{R}$, such that the limits

$$\lim_{\mu \nearrow \epsilon_{L,1}} \langle f,e^{it(h_L-\mu)}g \rangle_{\beta,\mu,L} = \langle f,e^{it(h_L-\epsilon_{L,1})}g \rangle_{\beta,\epsilon_{L,1},L}, \quad f \in \mathcal{D}_L,$$

exist and are continuous in $t$. On the other hand,

$$\lim_{\mu \nearrow \epsilon_{L,1}} \langle f,f \rangle_{\beta,\mu,L} = +\infty, \quad f \in L^2(\mathbb{R}^s) \setminus \mathcal{D}_L.$$  

If $L < \infty$, $\mathcal{D}_L = (1-E_{L,1})L^2(\mathbb{R}^s)$, where $E_{L,1}$ is the projection onto the ground state of $h_L$. If $L = +\infty$, $\mathcal{D}_L$ is the domain of $|P|^{-1}$.

**Proof.** (i) The monotonicity of the potentials implies that $V_{L_1} \geq V_{L_2} \geq 0$, hence $(h_{L_1} - \mu) \geq (h_{L_2} - \mu) > (h_\infty - \mu)$ if $L_2 \geq L_1$. Proceeding to the inverses, the sequence of resolvents $(h_L - \mu)^{-1}$ is monotonically increasing with increasing $L$ and converges to $(h_\infty - \mu)^{-1}$ in the strong operator topology. Now the functions $\epsilon \mapsto e^{\beta(\epsilon - \mu)}(e^{\beta(\epsilon - \mu)} - 1)^{-1}e^{it(\epsilon - \mu)}$ and $\epsilon \mapsto e^{it(\epsilon - \mu)}(e^{\beta(\epsilon - \mu)} - 1)^{-1}$ are bounded and continuous on the closure of $\mathbb{R}_+$ since $\mu < 0$. It therefore follows from functional calculus that the corresponding sequences of operators, where $\epsilon$ is replaced by $h_L$, converges for large $L$ in the strong operator topology on $L^2(\mathbb{R}^s)$ to the operator, where $\epsilon$ is replaced by $h_\infty$. This completes the proof of part (i).

(ii) Let $L$ be finite. Since the ground state of $h_L$ is simple and isolated from the rest of the spectrum, there is some $\delta > 0$ such that for $\mu$ in a neighborhood of $\epsilon_{L,1}$ one has $(h_L - \mu)(1-E_{L,1}) \geq \delta (1-E_{L,1})$. Hence the operator functions $\mu \mapsto (e^{\beta(h_L-\mu)} - 1)^{-1}(1-E_{L,1})$ as well as $\mu \mapsto e^{\beta(h_L-\mu)}(e^{\beta(h_L-\mu)} - 1)^{-1}(1-E_{L,1})$ are norm continuous at
μ = ϵ_{L,1}. So the scalar products in equation (3.5) converge for \( f, g \in (1 - E_1)L^2(\mathbb{R}^\ast) \). On the other hand, given any \( f \in L^2(\mathbb{R}^\ast) \) such that \( E_{L,1}f \neq 0 \) one has

\[
(f, (e^{\beta(h_{L,1} - \mu)} - 1)^{-1}f) \geq (e^{\beta(\epsilon_{L,1} - \mu)} - 1)^{-1}(f, E_{L,1}f),
\]

(3.7)

which tends to \(+\infty\) in the limit \( \mu \nearrow \epsilon_{L,1} \). Thus \( \mathcal{D}_L = (1 - E_{L,1})L^2(\mathbb{R}^\ast) \), being stable under the action of \( e^{ith_L}, t \in \mathbb{R} \), is the maximal complex subspace on which relation (3.5) holds.

Next, let \( L = +\infty \). The operator function \( \mu \mapsto (e^{\beta(h_{\infty} - \mu)} - 1)^{-1}h_{\infty} \) converges in the limit \( \mu \nearrow 0 \) in the strong operator topology to the bounded operator \( (e^{\beta h_{\infty}} - 1)^{-1}h_{\infty} \). So for \( f, g \in R \) in the domain of \( |P|^{-1} = h_{\infty}^{-1/2} \) the scalar products in (3.5) also converge in this limit. On the other hand, if \( f \) does not lie in this domain, it follows from the lower bound of the function

\[
\epsilon \mapsto (e^{\beta \epsilon} - 1)^{-1} \geq (\beta \epsilon)^{-1}e^{-\beta \epsilon}, \quad \epsilon > 0,
\]

(3.8)

that the expectation values \( (f, (e^{\beta(h_{\infty} - \mu)} - 1)^{-1}f) \) diverge in the limit \( \mu \nearrow 0 \). Thus \( \mathcal{D}_L \) is equal to the domain of \( |P|^{-1} \), completing the proof.

With the help of this lemma and results obtained in the preceding section we arrive at the following statement on pertinent limits of equilibrium states.

**Proposition 3.2.** Let \( \omega_{\beta,\mu,L} \) be the equilibrium states on the resolvent algebra \( \mathfrak{R} \), defined in equation (3.2) for \( \beta > 0, \mu < \epsilon_{L,1} \) and \( L > 0 \).

(i) Let \( \beta > 0, \mu < 0 \). The thermodynamic limit of the states exists, pointwise on \( \mathfrak{R} \),

\[
\lim_{L \to \infty} \omega_{\beta,\mu,L}(R) = \omega_{\beta,\mu,\infty}(R), \quad R \in \mathfrak{R}.
\]

(3.9)

The limit states \( \omega_{\beta,\mu,\infty} \) satisfy the KMS condition at inverse temperature \( \beta \) and chemical potential \( \mu \) for the dynamics induced by the adjoint action of the unitary group \( t \mapsto e^{i(H_{\infty} - \mu N)} \) on \( \mathfrak{R} \), where \( H_{\infty} = d\Gamma(h_{\infty}) \).

(ii) Let \( L \in \mathbb{R}^+ \cup +\infty \) be fixed and let \( \beta > 0, \mu < \epsilon_{L,1} \). The infinite particle number limit of the states exists, pointwise on \( \mathfrak{R} \),

\[
\lim_{\mu \nearrow \epsilon_{L,1}} \omega_{\beta,\mu,L}(R) = \omega_{\beta,\epsilon_{L,1},L}(R), \quad R \in \mathfrak{R}.
\]

(3.10)

The limit state \( \omega_{\beta,\epsilon_{L,1},L} \) satisfies the KMS condition at inverse temperature \( \beta \) for the dynamics induced by the adjoint action of the unitary group \( t \to e^{i(H_{L,1} - \epsilon_{L,1}N)} \) on \( \mathfrak{R} \). The subalgebra \( \mathfrak{R}(\mathcal{D}_L) \subset \mathfrak{R} \), which is generated by resolvents assigned to functions in the complex subspace \( \mathcal{D}_L \subset L^2(\mathbb{R}^\ast) \), defined in the preceding lemma, is stable under the dynamics. Its GNS representation, induced by the limit state, is regular. All resolvents assigned to elements of \( L^2(\mathbb{R}^\ast) \setminus \mathcal{D}_L \) are trivially represented there and thus generate an ideal of \( \mathfrak{R} \).

(iii) The states \( \omega_{\beta,\mu,\infty}, \mu \leq 0 \), are mixing, viz. for any \( R_1, R_2 \in \mathfrak{R} \) one has

\[
\lim_{t \to \infty} \omega_{\beta,\mu,\infty}(R_1, Ad e^{it(H_{\infty} - \mu N)}(R_2)) = \omega_{\beta,\mu,\infty}(R_1) \omega_{\beta,\mu,\infty}(R_2).
\]

(3.11)

So these states describe pure phases.

**Remark:** The limit states on \( \mathfrak{R} \) in this proposition are not normal relative to the Fock representation. But the adjoint action of the given unitary time translations on \( \mathcal{F} \) leaves \( \mathfrak{R} \) invariant, thereby defining automorphisms of this algebra. The action of these automorphisms on \( \mathfrak{R} \) is implemented in the limit representations by the adjoint action of modified unitary operators, generated by Liouvillians which depend on the thermal parameters.
Proof. (i) The existence of the limit follows from the first parts of Lemma 3.1 and Proposition 2.1, note that the scalar products in the lemma converge on all of \( L^2(\mathbb{R}^s) \). The proof that the limit states satisfy the KMS condition is obtained by standard arguments. For subsequent reference, we sketch it here for arbitrary unitary groups \( t \mapsto e^{it h} \) on \( L^2(\mathbb{R}^s) \) with positive generator \( h \), leaving some complex subspace \( D \subset L^2(\mathbb{R}^s) \) invariant. Given \( \beta > 0 \), the corresponding scalar products are

\[
\langle f, g \rangle_{\beta} = (1/2) \left( \langle f, e^{\beta h} (e^{\beta h} - 1)^{-1} g \rangle + \langle g, (e^{\beta h} - 1)^{-1} f \rangle \right), \quad f, g \in D.
\]

Thus the functions \( t \mapsto \langle f, e^{it h} g \rangle_{\beta} \) can continuously be extended to the strip \( \{ z \in \mathbb{C} : 0 \leq \text{Im} \, z \leq \beta \} \), are analytic in its interior, and bounded. Their boundary values at the upper rim of the strip are \( t \mapsto \langle f, e^{i(\beta-t) h} g \rangle_{\beta} = \langle e^{it h} g, f \rangle_{\beta} \).

So these scalar products, representing the two-point function in the corresponding quasifree state, satisfy the KMS condition. Making use of this information in equation (2.4), cf. also equation (2.10), it follows once again from the dominated convergence theorem that the correlation functions of products of resolvents in \( \mathcal{R}(D) \) are continuous in time and satisfy the KMS condition at inverse temperature \( \beta \). Since the sums of products of resolvents are stable under the action of the dynamics and dense in \( \mathcal{R} \), the KMS condition holds on this algebra. In the case at hand \( D = L^2(\mathbb{R}^s) \), completing the proof of the first part.

(ii) The convergence of the functionals now follows from the second part of Lemma 3.1 and the first part of Proposition 2.1. Moreover, resolvents containing functions \( f \in L^2(\mathbb{R}^s) \setminus D_L \) are trivially represented in the GNS representations induced by the limit state. So they generate an ideal in \( \mathcal{R} \). The subalgebra \( \mathcal{R}(D_L) \subset \mathcal{R} \) is regularly represented, cf. the remarks at the end of Sect. 2. Since \( D_L \) is stable under the limit dynamics, products of resolvents containing a factor with a function \( f \in L^2(\mathbb{R}^s) \setminus D_L \) are annihilated in the state at all times. The KMS condition is then trivially satisfied. For products containing only resolvents with functions in \( D_L \), the KMS property follows from the preceding argument.

(iii) Given \( \mu \leq 0 \), let \( D_{\mu, \infty} \subset L^2(\mathbb{R}^s) \) be the subspace fixed by the state \( \omega_{\beta, \mu, \infty} \), cf. the preceding steps. The statement holds trivially if in expectation values of products of resolvents in this state one of the underlying functions is not contained in \( D_{\mu, \infty} \). If \( f, g \in D_{\mu, \infty} \) it follows from the familiar spectral properties of \( h_{\infty} = P^2 \) that \( \lim_{t \to \infty} \langle f, e^{i(h_{\infty} - \mu) t} g \rangle_{\beta, \mu} = 0 \). Making use of this information in relation (2.4) one sees that products of resolvents at different times in the state, involving only functions in \( D_{\mu, \infty} \), become uncorrelated in the limit of large time differences. This completes the proof of the statement.

In the remainder of this section we construct states out of the preceding ones by adding to the quantum field some classical background field. Let \( f \mapsto l(f) \) be a real linear functional on \( L^2(\mathbb{R}^s) \). Plugging this functional into equation (2.4), where the scalar product is given by equation (3.3), one obtains quasifree states \( \omega_{\beta, \mu, L; l} \) on the resolvent algebra. They are normal with regard to the Fock representation if \( l(f) = \text{Im} \langle c, f \rangle \) for some \( c \in L^2(\mathbb{R}^s) \) and are obtained by composing the KMS states \( \omega_{\beta, \mu, L} \) with the adjoint action \( \text{Ad} \, W(e) = W(e)^* \cdot W(e) \) of the Weyl operator corresponding to \( c \). The resulting states are neither gauge invariant, nor stationary, and consequently do not satisfy the KMS condition.

Being interested in the phenomenon of condensation, we restrict our attention to certain specific functionals, which are determined by eigenvectors of \( h_L \). Let \( e_{L,k} \) be such an eigenvector corresponding to the eigenvalue \( \epsilon_{L,k} = L^{-2} \epsilon_k \), where \( \epsilon_k \) is the \( k \)-th eigenvalue of \( h_1 \), and let \( l_{L,k}(f) = \text{Im} \langle e_{L,k}, f \rangle \), \( k \in \mathbb{N} \). The time dependence of the resulting expectation values in the states \( \omega_{\beta, \mu, L; l_{L,k}} \) is conveniently determined for Weyl operators \( W(f) \), cf. equation (2.2),

\[
t \mapsto \omega_{\beta, \mu, L; l_{L,k}} \left( \text{Ad} \, e^{it(H_L - \mu L)}(W(f)) \right) = e^{iL, k(t)(f)} e^{-(1/2)(f, f)_{\beta, \mu, L}}.
\]

Here

\[
l_{L,k}(t)(f) = \text{Im} \langle e_{L,k}, e^{i(H_L - \mu)L} f \rangle = \text{Im} e^{it(\epsilon_{L,k} - \mu)} \langle e_{L,k}, f \rangle.
\]
So the states change periodically in time with period $\tau = 2\pi/(\epsilon_{L,k} - \mu)$. Taking a mean of these states over the time interval $\tau$, one obtains the averaged states

$$\overline{\omega}_{\beta,\mu,L;l_{L,k}} = \frac{1}{\tau} \int_0^\tau dt \omega_{\beta,\mu,L;l_{L,k}} \circ \text{Ad} e^{iH_L (\beta - \mu) N}.$$  \hfill (3.15)

In expectation values of Weyl operators, the phase factor in equation (3.13) is then replaced by $\omega \tau$. So the states change periodically in time with period $\tau$. To see this, let $l_{L,k}$ be a (generalized) sequence of functionals which approximates $l_{\infty,k}$ in the limit of large $L$ and diverges for some $f \in L^2(\mathbb{R}^s)$. Then the resulting averaged state vanishes on products of resolvents containing a resolvent with a function $\zeta f$ for some $\zeta \in T$. This follows from the decay properties of the Bessel function for large arguments. Thus the limit functional $l_{\infty,k}$ contributes to the thermodynamic limit of the averaged state only for elements in the maximal complex subspace of its domain, denoted by $\mathcal{D}_\infty = \bigcap_{\zeta \in T} \zeta \mathcal{D}_\infty$.

We turn now to the determination of the thermodynamic as well as infinite particle number limits of the states $\overline{\omega}_{\beta,\mu,L;l_{L,k}}$. The underlying scalar products $\langle \cdot, \cdot \rangle_{\beta,\mu,L}$ were already analyzed in Lemma 3.1. So we must only have a closer look at the functionals $f \mapsto l_{L,k}(f)$. As discussed in Sec. 2, the functionals $l_{L,k}(f)$ approach their limits $l_{\infty,k}(f)$ in general only in the weak topology, viz. for any given finite number of functions $f$ one has to proceed to specific subsequence of the scaling factors $L_i$, $i \in I$. In order to retain control of the continuity properties of the limit states with regard to the time translations, they need to be adjusted in the approximating correlation functions according to the respective scale $L$. This is also natural from the point of view of physics where one studies the dynamical properties of systems for differing external constraints, such as the length scale $L$. Applying this rule to the functionals, it follows from the equality

$$l_{L,k}(e^{itH_L} f) = \cos(t\epsilon_k/L^2) l_{L,k}(f) + \sin(t\epsilon_k/L^2) l_{L,k}(if), \quad f \in \mathcal{D}_\infty,$$

that $\lim_{t \in T} l_{L,k}(e^{itH_L} f) = l_{\infty,k}(f), t \in \mathbb{R}$. So the limit of these particular functionals, contributing to the correlation functions in the thermodynamic limit, turns out to be invariant under the action of the limit dynamics.

**Proposition 3.3.** Let $\overline{\omega}_{\beta,\mu,L;l_{L,k}}$ be the averaged states on the resolvent algebra $\mathcal{A}$, which are determined by relations $3.13$ and $3.15$ for $\beta > 0$, $\mu < \epsilon_{L,1}$, $L > 0$, and functionals $l_{L,k}$ specified above.

(i) Let $\beta > 0$, $\mu < 0$. There exists a state $\overline{\omega}_{\beta,\mu,\infty;l_{\infty,k}}$ on $\mathcal{A}$ which is a weak* limit point of the given states for large $L$. It has the following properties: there is a complex subspace $\mathcal{D}_\infty \subset L^2(\mathbb{R}^s)$ such that the algebra $\mathcal{A}(\mathcal{D}_\infty)$ is regularly represented in the GNS representation induced by the limit state; resolvents assigned to functions in $L^2(\mathbb{R}^s) \setminus \mathcal{D}_\infty$ are trivially represented there. The state is gauge invariant and stationary with regard to the limit dynamics $t \mapsto \text{Ad} e^{it(H_\infty - \mu N)}$, and the correlation functions depend continuously on time. If the underlying limit functional $l_{\infty,k}$ is different from 0, the limit state does not satisfy the KMS condition.

(ii) Let $L \in \mathbb{R}_+ \cup +\infty$ be fixed and let $\beta > 0$, $\mu < \epsilon_{L,1}$. The infinite particle number limit of the states exists, pointwise on $\mathcal{A}$,

$$\lim_{\mu \searrow \epsilon_{L,1}} \overline{\omega}_{\beta,\mu,L;l_{L,k}}(R) = \overline{\omega}_{\beta,\epsilon_{L,1},L;l_{L,k}}(R), \quad R \in \mathcal{A}.$$

(3.18)
The algebra \( \mathcal{R}(D_L \cap T_\infty) \) is regularly represented in the GNS representation induced by the limit state, where \( D_L \) was defined in Lemma 5.1. Resolvents assigned to functions in \( L^2(\mathbb{R}^s) \setminus (D_L \cap T_\infty) \) are trivially represented there.

At level \( k = 1 \), all limit states satisfy the KMS condition at inverse temperature \( \beta \) for the dynamics given by \( t \mapsto \text{Ad} e^{it(H_L - \epsilon_{L,1} N)} \); if \( L < \infty \) they are unique and do not depend on the normalization of the wave functions \( e_{L,1} \) entering in \( l_{L,1} \). For level \( k > 1 \) and finite \( L \) the limit states do not satisfy the KMS condition. In the thermodynamic limit \( L = +\infty \), there exist several limit states satisfying the KMS condition for any given level \( k \); they depend on the normalization of the wave functions \( e_{L,k} \) in the approximating functionals \( l_{L,k} \), cf. the remark below.

(iii) Let \( \beta > 0 \), \( \mu = 0 \), and \( l_{\infty,k} \neq 0 \) for some \( k \in \mathbb{N} \). The corresponding KMS state \( \overline{\omega}_{\beta,0,\infty;l_{\infty,k}} \) is not mixing, i.e., describes a mixture of phases. It can (centrally) be decomposed into a mean over the gauge group,

\[
\overline{\omega}_{\beta,0,\infty;l_{\infty,k}} = (1/2\pi) \int_{0}^{2\pi} du \omega_{\beta,0,\infty;l_{\infty,k}} \circ \gamma_u, \tag{3.19}
\]

where \( \omega_{\beta,0,\infty;l_{\infty,k}} \) is a KMS state, describing a pure phase. This state is not gauge invariant.

**Remark:** If \( L = \infty \), the space \( T_\infty \), determined by the limit functional \( l_{\infty,k} \), must have a non-trivial intersection with the domain of \( |P|^{-1} \) in order to arrive at non-trivial states in the limit of infinite particle number. Let us briefly comment on this point in case of potentials \( x \mapsto V(x) \) which are real analytic. There the corresponding (real) functions \( x \mapsto e_k(x) \) of \( h_1 \) are also real analytic, bounded, and the complement of their nodal sets consists of at most \( k \) connected components, cf. [11, Ch. 3.4]. If \( k = 1 \), the function \( x \mapsto e_1(x) \) can be chosen to be positive, so for test functions \( f \) one obtains in the limit \( l_{\infty,1}(f) = e_1(0) \int dx \text{Im} f(x) \). In \( s > 2 \) dimensions, any test function lies in the domain of \( |P|^{-1} \) and the limit functional is non-trivial. But in \( s = 1, 2 \) dimensions, this domain condition implies that \( f \) must be the (partial) derivative of another test function, \( f = -\partial g \), so the integral vanishes. In that case one can renormalize the approximating functionals \( l_{L,1} \) by an additional factor \( L \), giving in the limit \( l_{\infty,1}(f) = D e_1(0) \cdot \int dx \text{Im} g(x) \). If this limit is still zero, one can continue this procedure with higher powers of \( L \) and smaller subsets of test functions. It yields a non-trivial result after a finite number of steps since \( e_1 \) is real analytic. Thus also in \( s = 1, 2 \) dimensions the ground state wave functions gives rise to functionals in the thermodynamic limit with non-trivial domains \( T_\infty \). In a similar manner the existence of functionals involving the excited states \( e_k, k > 1 \), can be established, including also those cases, where the nodal sets pass through the origin.

**Proof.** (i) According to the arguments given in the proof of the second part of Proposition 2.1 and Lemma 3.1, the thermodynamic limits of the time dependent correlation functions in the given states coincide with those in the states \( \overline{\omega}_{\beta,\mu,\infty;l_{L,k}} \), where the underlying scalar products \( \langle \cdot, \cdot \rangle_{\beta,\mu,L} \) are replaced by their limit \( \langle \cdot, \cdot \rangle_{\beta,\mu,\infty} \). The \( L \)-dependence of the correlation functions in the latter states arises from the term

\[
J_0(|\langle e_{L,k}, \sum_j u_j e^{it_j(h_{L,\mu} - \mu)} f_j \rangle|) = J_0(|\sum_j u_j e^{it_j(h_{L,\mu} - \mu)} \langle e_{L,k}, f_j \rangle|), \tag{3.20}
\]

where \( f_j \in L^2(\mathbb{R}^s), t_j \in \mathbb{R} \) for \( j = 1, \ldots, n \). It replaces in equation (2.4) the phase factor depending on the functionals \( l_{L,k} \). Since \( \langle e_{L,k}, f \rangle = (l_{L,k}(f) + i l_{L,k}(f)) \) it follows, as explained, that the term \( 3.20 \) vanishes in the thermodynamic limit for almost all \( u_1, \ldots, u_n \) if \( f_k \in L^2(\mathbb{R}^s) \setminus T_\infty \) for some \( k \). If all functions are contained in \( T_\infty \), the limit is given by

\[
J_0(|\sum_j u_j e^{-it_j \mu} \langle e_{\infty,k}, f_j \rangle|), \]  

where \( f \mapsto \langle e_{\infty,k}, f_j \rangle \) is the complex linear functional on \( T_\infty \), determined by \( l_{\infty,k} \).

The representation of the algebra \( \mathcal{R}(D_L) \) induced by the limit state is regular, as can be seen by inspection of the expectation values of the underlying resolvents in equation (2.4), cf. the remark at the end of Sect. 2. It is also clear that all resolvents assigned to functions \( f \in L^2(\mathbb{R}^s) \setminus T_\infty \) are trivially represented there.

The limit state is gauge invariant, being approximated by such states. Putting \( t_j = t, j = 1, \ldots, n \), in the Bessel factor and the scalar products appearing in the correlation functions of the limit state, derived from (2.4), it is
apparent that this state is stationary under the action of the limit dynamics. Moreover, the continuity properties of the time translations on $L^2(\mathbb{R}^s)$ and of the Bessel factor imply that the correlation functions are continuous with regard to separate time translations of the underlying operators. But if $l_{\infty,k} \neq 0$ and $\mu < 0$, the time dependence of the Bessel factor impedes the KMS property of the limit state.

(ii) Adopting the arguments given in the proof of Proposition 3.1, the convergence of the states follows from the corresponding properties of the scalar products, established in Lemma 3.1, and the continuity properties of the Bessel factor with regard to $\mu$. The properties of the resulting algebras are then established as in the preceding step.

If $k = 1$, the Bessel factor is time independent for $\mu = \epsilon_{L,1}$ and any finite or infinite length $L$. The KMS property of the resulting states therefore follows from the corresponding properties of the scalar products, established in the proof of Proposition 3.1. Now for finite $L$ the domain $\mathcal{D}_L$ is equal to $(1 - E_{L,1}) L^2(\mathbb{R}^s)$, which lies in the kernel of $l_{L,1}$, hence the KMS state is not modified by this functional. If $k > 1$, the phase factors in the Bessel factor oscillate with frequency $(\epsilon_k - \epsilon_1)/L^2$, destroying the KMS property. In the thermodynamic limit $L = +\infty$ and $\mu = 0$, this factor is independent of time for any $k \in \mathbb{N}$. So the KMS property of the states follows again from the KMS property of the scalar products. Examples of non-trivial limit functionals which modify the states were presented in the preceding remark.

(iii) Proceeding as in the proof of the third part of Proposition 3.2, there appears in relation 2.4 the Bessel factor $J_0(\sum_j u_j \langle e_{\infty,k}, f_j \rangle)$ in the expectation values of products of resolvents at different times in the averaged state. It is also time independent. But, in contrast to the exponential function, it does not split into a product of factors containing only functions entering in corresponding factors of the product of resolvents. This implies after a moments reflection that the state is not mixing. The decomposition of the state into pure phases is accomplished by decomposing the Bessel factor into its exponential contributions, involving the gauge transformed test functions,

$$J_0(\sum_j u_j \langle e_{\infty,k}, f_j \rangle) = (1/2\pi) \int_0^{2\pi} du \sum_j e^{i \epsilon_{\infty,k}(e^{iu}f_j)}, \quad f_j \in \mathcal{D}_L \cap \mathcal{D}_\infty. \quad (3.21)$$

This mean can be interchanged with the integration in equation 2.4 for the averaged state. Since the scalar products are invariant under simultaneous gauge transformations of its entries one arrives at equality 3.10. By arguments given in the proof of the third part of Proposition 3.2 one sees that the state $\omega_{s,0,\infty,l_{\infty,k}}$, appearing on the right hand side of this equality, is mixing. Since $l_{\infty,k} \neq 0$ it is not gauge invariant, however.

This proposition shows that equilibrium states of physical interest, which are limits of Gibbs-von Neumann states, can be defined on the resolvent algebra without running into any mathematical problems. The results shed light on some basic points. First, the infinite particle number limit of trapped equilibrium states exists for any value of the temperature and dimension. The resulting equilibrium states include an infinity of particles in the ground state which, as we shall see in more detail, describe a condensate. The preceding results imply that this condensate cannot be changed by adding to it more of these particles with the help of Weyl operators, the limit states are unique. Second, the thermodynamic limit of trapped equilibrium states exists for any value of the temperature, chemical potential and dimension of space. We will see that for vanishing chemical potential these states describe quasi condensates of zero energy modes (improper states) in one and two dimensions. In higher dimensions the states exhibit a maximal (critical) local density, however. Yet the preceding results imply that Weyl operators creating such modes act non-trivially on the states. So one can increase their density unlimitedly, leading to the formation of condensates.

In common (textbook) discussions of the phenomenon of Bose-Einstein condensation one proceeds from the (grand) canonical ensemble with given sharp (mean) particle number and sharp boundaries (boxes). Anticipating that the states of interest are homogeneous, the particle density is defined as the quotient of the number of particles and the volume of the boxes. In order to exhibit condensates, one then compares the density of particles in the ground
4. PARTICLE DENSITIES

Heuristically, the observable determining the particle density is represented by the function \( x \mapsto a^*(x)a(x) \), which can be given a rigorous meaning in the sense of operator valued distributions in the Fock representation. Its integral is the particle number operator \( N \). Since we are dealing here also with disjoint representations, there arises the question of how one can decide whether this density is still meaningful there. It is another virtue of the resolvent algebra that this physically important question can be answered within its framework.

As we have seen, the (limit) states \( \omega \) of interest typically determine some complex subspace \( D \subset L^2(\mathbb{R}^n) \) for which the corresponding resolvent algebra \( \mathcal{R}(D) \) is regularly represented in the induced GNS representation; resolvents attached to the complement of \( D \) are trivially represented there. Now let \( K \subset D \) be any finite dimensional complex subspace. Then, a fortiori, the restriction of the representation to the subalgebra \( \mathcal{R}(K) \) is regular. This implies by the Stone-von Neumann theorem that this restriction is quasi equivalent to the Fock representation of \( \mathcal{R}(K) \), cf. [3, Thm. 4.5] and its subsequent remark. In the Fock representation of \( \mathcal{R}(K) \), the particle number operator \( N_K \) is densely defined. Choosing some orthonormal basis \( e_1, \ldots, e_n \in K \), it can be presented in the familiar form

\[
N_K = \sum_{j=1}^n a^*(e_j)a(e_j).
\]

In fact, it can be approximated on its domain by observables in \( \mathcal{R}(K) \subset \mathcal{R}(K) \) in the strong operator topology. We briefly sketch the argument: given \( f \in K \) and \( \varepsilon > 0 \), the operators

\[
a^*(f)a(f)(1 + \varepsilon a^*(f)a(f))^{-1} = (1/\varepsilon) \left( 1 - (1 + \varepsilon a^*(f)a(f))^{-1} \right)
\]

are elements of \( \mathcal{A}(K) \). This follows from [3, Lem. 3.1] according to which the operators \( (1 + \varepsilon a^*(f)a(f))^{-1} \) are contained in \( \mathcal{R}(K) \); in fact, they are members of some compact ideal of \( \mathcal{R}(K) \) and since they are gauge invariant, they are contained in \( \mathcal{A}(K) \). It is also clear that \( a^*(f)a(f) \leq \|f\|^2 N_K \). Hence the operators (4.1) converge in the strong operator topology on the domain of \( N_K \) to the unbounded operator \( a^*(f)a(f) \) in the limit \( \varepsilon \searrow 0 \).

We make use of this fact in order to determine the local particle properties of states \( \omega \) on the corresponding regular subalgebras \( \mathcal{R}(D) \subset \mathcal{R} \). Let \( O \subset \mathbb{R}^n \) be any open bounded region and let \( D(O) \subset L^2(O) \cap D \) be the subspace of functions having support in \( O \); depending on the given state, it may happen that \( D(O) = L^2(O) \). Since the operator sequence (4.1) is monotonically increasing for decreasing \( \varepsilon \), the following definition is meaningful.

**Definition:** Let \( \omega \) be a state on the resolvent algebra which is regular on \( \mathcal{R}(D) \) and annihilates the ideal in \( \mathcal{R} \) which is generated by resolvents with functions in \( L^2(\mathbb{R}^n) \setminus D \). Let \( O \subset \mathbb{R}^n \) be an open bounded region and let \( \mathcal{R}(D(O)) \subset \mathcal{R}(D) \) be the corresponding regular subalgebra. Then

(i) \( \omega \) admits a (partial) particle interpretation in \( O \) if for any \( f \in D(O) \)

\[
\omega(a^*(f)a(f)) \doteq \lim_{\varepsilon \searrow 0} \omega(a^*(f)a(f)(1 + \varepsilon a^*(f)a(f))^{-1}) < \infty.
\]
The number of particles in \( O \) with a wave function \( f \in L^2(O) \backslash \mathcal{D}(O) \) is not defined, \textit{i.e.} the corresponding approximations \( \{ \tilde{x}_n \} \) tend to infinity.

(ii) \( \omega \) is locally normal on \( \mathfrak{H}(\mathcal{D}(O)) \) if it has a particle interpretation in \( O \) and, for some orthonormal basis \( e_j \in \mathcal{D}(O), j \in \mathbb{N} \), one has

\[
\omega(N_{\omega}(O)) = \lim_{n \to \infty} \sum_{j=1}^{n} \omega(a^*(e_j)a(e_j)) < \infty. \tag{4.3}
\]

\textbf{Remark:} Here \( N_{\omega}(O) \) is the \textit{regular} number operator counting particles in the region \( O \) with wave functions in \( \mathcal{D}(O) \). This operator is defined in the GNS representation of \( \mathfrak{H}(\mathcal{D}(O)) \) induced by \( \omega \). In fact, this representation is quasi equivalent to the Fock representation of this algebra \[12\].

The regular number operators \( N_{\omega}(O) \) in this definition are physically significant order parameters, allowing to establish the appearance, respectively absence, of condensates in equilibrium states \( \omega \) within the region \( O \). In trapped systems, described by Gibbs-von Neumann states \( \omega \), they are defined for functions in the largest possible space \( \mathcal{D}(O) = L^2(O) \), hence \( N_{\omega}(O) = N(O) \). But also in the thermodynamic limit the resulting states are frequently locally normal in this strong sense. Keeping the temperature fixed and proceeding to the maximal possible value of the chemical potential \( \mu \), one can then check whether the corresponding expectation values of \( N_{\omega_j}(O) \) stay finite, or tend to infinity. In the former case the limit system has a critical density in \( O \) at the given temperature. In the latter case, there are no limitations on the density, signaling condensation as we shall see.

In computations of the expectation values of local particle number operators it is sometimes convenient to proceed to their local densities, instead of controlling infinite sums of occupation numbers. We therefore recall here briefly this standard device. Let \( d_{\varepsilon}, \varepsilon > 0 \), be a sequence of test functions on \( \mathbb{R}^s \) which converges to the Dirac measure at the origin if \( \varepsilon \) tends to 0 and let \( h \) be a positive test function with compact support. As is well known, one can proceed in Fock space to the limit

\[
\lim_{\varepsilon \searrow 0} \int d^s x \ h(x) \ AdU(x)(a^{\dagger}(d_{\varepsilon})a(d_{\varepsilon})) \doteq \int d^s x \ h(x) a^{\dagger}(x)a(x), \tag{4.4}
\]

where one has convergence in the sense of sesquilinear forms between vectors with finite particle number and regular wave functions. The resulting form can be extended to a selfadjoint operator and if \( O_1 \subset \text{supp} \ h \subset O_2 \) one obtains for it the bounds

\[
\inf_{x \in O_1} h(x) N(O_1) \leq \int d^s x \ h(x) a^{\dagger}(x)a(x) \leq \sup_{x \in O_2} h(x) N(O_2). \tag{4.5}
\]

Similar formulas can be established for the regular number operators in non-Fock states. Whereas the assignment \( x \mapsto \omega(a^{\dagger}(x)a(x)) \) is in general only defined in the sense of distributions, the resulting expectation values can often be presented by continuous functions. The desired information about condensation can then directly be extracted from the respective two-point functions, as we will see in the subsequent section.

5. PROBING CONDENSATION

We analyze now the trapped equilibrium states which we have constructed, making use of the local order parameters put forward in the preceding section. In order to highlight the virtues of our approach, we begin by considering the simple example of harmonic trapping potentials. Let \( h_1 = (P^2 + Q^2) \) be the unscaled Hamiltonian on \( \mathbb{R}^s \). Its normalized ground state is denoted by \( e_1 \in L^2(\mathbb{R}^s) \), carrying the energy \( e_1 = s \). The corresponding equilibrium states \( \omega_{\beta,\mu,1} \) on the resolvent algebra for given \( \beta > 0 \) and \( \mu < \mu_1 = s \) are fixed by equation \[17\] with \( L = 1 \). We want to determine in the limit of maximal chemical potential \( \mu \) the expected number of particles with given wave function in
D = (1 − E₁)L²(ℝ¹), where E₁ is the projection onto the ground state of h₁. For f, g ∈ L²(ℝ¹) we obtain in the limit μ ↠ ϵ₁, recalling that the limit state is regular on R(D),

\[ \omega_{β,ϵ₁,1}(a^*(1 − E₁)f) = \langle g, (e^{β(h₁−ϵ₁)} − 1)^{-1}(1 − E₁)f \rangle. \tag{5.1} \]

It follows from this equality that we can extend the limit state on R(D) to a quasifree state on the full resolvent algebra R, denoted by \( \omega_{β,ϵ₁,1} \), which does not contain a single particle with wave function \( ϵ₁ \). This fact allows us to define its particle density, as outlined in the preceding section: we have

\[ \langle g, (e^{β(h₁−ϵ₁)} − 1)^{-1}(1 − E₁)f \rangle = \sum_{n=1}^{∞} \langle g, e^{-nβ(h₁−ϵ₁)}(1 − E₁)f \rangle. \]

The kernels of the operators \( e^{-nβ(h₁−ϵ₁)}(1 − E₁) \) on \( L²(ℝ¹) \) are given by

\[
x, y ↦ \langle x | e^{-nβ(h₁−ϵ₁)}(1 − E₁)|y \rangle = e^{nβ₁}(2π sinh(2nβ))^{-s/2} e^{-(1/2)coth(2βn)(x²+y²) − 2xy sinh(2βn))} - π^{-s/2} e^{−(x²+y²)/2}, \tag{5.2} \]

where in the second line the Mehler formula has been used; the contribution in the last line is due to the projection. The sum of these kernels is absolutely convergent and the result is continuous in \( x, y \). In particular, the regular particle density in the (extended) limit state is given by

\[
x ↦ \omega_{β,ϵ₁,1}(a^*(x)a(x)) = π^{-s/2} \sum_{n=1}^{∞} \left((1 − e^{-4nβ})^{-s/2} e^{(1 − coth(2nβ)+1/sinh(2nβ)) x² − 1} \right) e^{-x²}. \tag{5.3} \]

It implies \( \omega_{β,ϵ₁,1}(N_μ(O)) \leq ∫ _O dx \omega_{β,ϵ₁,1}(a^*(x)a(x)) \), where the inequality is due to the completion of the basis in \( D(O) \) to a basis in \( L²(O) \). One obtains equality in the limit \( O ↠ ℝ¹ \). So for all temperatures, the regular local observables in the trapped equilibrium states indicate a maximal (critical) local density of the thermal cloud. On the other hand, the number of particles in the approximating states \( ω_{β,µ,1} \) with a wave function \( f \) having some overlap with the ground state wave function, \( E₁f \neq 0 \), diverges in the limit. So we conclude that the particles in the ground state form condensates in the approximating states whenever the total expected number of particles in the trap exceeds the critical number \( N_μ \) of particles in the thermal cloud. By comparing the density of all particles with the critical density \( [5.3] \) of the cloud, one can determine the amount of condensate which is formed locally.

As was already mentioned, one can increase particle densities by composing states with Weyl automorphisms. In order to arrive at stationary states, one must choose eigenfunctions of \( h₁ \) in the underlying Weyl operators. According to Proposition [5.3] choosing the ground state wave function \( ϵ₁ \) and taking a time average does not alter the limit states. For (arbitrarily normalized) excited states \( ϵ_k \) one obtains, bearing in mind that the original state is gauge invariant,

\[
ω_{β,µ,1} ∘ Ad W(ϵ_k) \left(a^*(f)a(f)\right) = ω_{β,µ,1}(a^*(f)a(f)) + |⟨ϵ_k, f⟩|^2. \tag{5.4} \]

In this manner, the density of particles with wave function \( ϵ_k \) can be made arbitrarily large in any given region \( O \). Yet, as we have seen in Proposition [5.8], choosing the ground state wave function \( ϵ₁ \) and taking a time average does not alter the limit states. For (arbitrarily normalized) excited states \( ϵ_k \) one obtains, bearing in mind that the original state is gauge invariant,

\[
ω_{β,µ,1} ∘ Ad W(ϵ_k) \left(a^*(f)a(f)\right) = ω_{β,µ,1}(a^*(f)a(f)) + |⟨ϵ_k, f⟩|^2. \tag{5.4} \]

In this manner, the density of particles with wave function \( ϵ_k \) can be made arbitrarily large in any given region \( O \). Yet, as we have seen in Proposition [5.8], the resulting states as well as their time averages are not in equilibrium since the correlations between operators at different times exhibit oscillations which violate the KMS condition. Due to the lack of interaction, these oscillations are not suppressed by time averages, the added particles do not equilibrate in the thermal background and feel the external forces forever. It is an interesting question whether this feature disappears in trapped interacting theories.

For systems of particles trapped by non-harmonic forces, there are no such simple formulas for the density of particles with regular wave functions. Yet one can rely there on the method of counting the number of particles locally. This
 amounts to computing for bounded regions \( O \subset \mathbb{R}^s \) and corresponding projections \( E(O) \) onto \( L^2(O) \subset L^2(\mathbb{R}^s) \) the expectation values of the corresponding regular number operators \( N_\omega(O) \),

\[
\omega_{\beta, \epsilon_1}(N_\omega(O)) \leq \text{Tr} \ E(O) \left( e^{\beta(h_1-\epsilon_1)} - 1 \right)^{-1} (1 - E_1) \ E(O).
\]  

Here \( \epsilon_1 \) and \( E_1 \) are the eigenvalue and projection, respectively, fixed by the ground state \( e_1 \) of \( h_1 \). Making use of the estimate \( e \mapsto (e^\epsilon - 1)^{-1} \leq e^{-k\epsilon}/(1 - k\epsilon) \) for \( \epsilon > 0 \) and \( 0 < k < 1 \), one obtains the upper bound, putting \( k = 1/2 \),

\[
\omega_{\beta, \epsilon_1}(N_\omega(O)) \leq 2(1/\beta(e_2 - \epsilon_1))^{-1} e^{(\beta/2)e_1} \text{Tr} \ e^{-(\beta/2)h_1}.
\]  

The operators \( e^{-(\beta/2)h_1} \) are of trace class, so it follows again that, no matter how the trapping potential is chosen, all particles with a wave function in \( D = (1 - E_1)L^2(O) \) have a critical density for any value of the temperature. As a matter of fact, since the upper bound does not depend on the size of \( O \), the total number of these particles in the limit state is finite. On the other hand, the number of particles in \( O \) having a wave function which overlaps with \( e_1 \) diverges if the chemical potential approaches its maximal value. In more detail, for any \( f \in L^2(O) \) with \( E_1 f \neq 0 \) one has, cf. Proposition 3.2

\[
\lim_{\mu \nearrow \epsilon_1} \omega_{\beta, \mu}(1 + a^*(f)a(f))^{-1} = 0.
\]  

Thus the density of particles in the ground state increases unlimitedly in the approximating states, in contrast to the density of all other particles in the thermal cloud escorting them. So the particles in the ground state form Bose-Einstein condensates which can be detected by observations in any given region \( O \), as outlined in the introduction.

In order to suppress the effects of the trapping potentials and to determine the inherent properties of the condensates, it is convenient to proceed to the thermodynamic limit. Let us briefly discuss this familiar topic from the present point of view. As was shown in Proposition 3.2 one obtains in the thermodynamic limit always the same gauge invariant limit state \( \omega_{\beta, \mu, \infty} \) for any given trapping potential and \( \beta > 0 \), \( \mu < 0 \). The underlying scalar product is fixed by

\[
\langle f, f \rangle_{\beta, \mu, \infty} = (1/2) \int dp \frac{e^{\beta(p^2 - \mu)} + 1}{e^{\beta(p^2 - \mu)} - 1} |\tilde{f}(p)|^2, \quad f \in L^2(\mathbb{R}^s),
\]  

where the tilde \( \sim \) denotes Fourier transformation.

To probe for condensates in the states \( \omega_{\beta, \mu, \infty} \), we proceed again to the limit of maximal chemical potential, \( \mu \nearrow 0 \). In this limit the regular observables involve functions \( f \) in the domain of \( |P|^{-1} \). Plugging these functions into the preceding equation, the integrals remain finite in the limit. Hence in terms of the regular observables, the limit states admit a particle interpretation in all regions \( O \). The question of whether the corresponding particle numbers are summable, \( \mu \nearrow 0 \), involves the number of dimensions, however.

In \( s = 1, 2 \) dimensions the total number of particles with regular wave functions in \( D(O) \) tends to infinity in the limit, \( \mu \nearrow 0 \), in contrast to the situation in lower dimensions. Moreover, the operators

\[
E(O) \left( e^{\beta P^2} - 1 \right)^{-1} E(O), \quad \beta > 0,
\]  

form a quasi-condensate which manifests itself in any given region \( O \) in \( s = 1, 2 \) dimensions.
being the product of a pair of adjoint Hilbert-Schmidt operators, are of trace class. Thus the expectation values of the (full) particle number operators $N(O)$ in the limit states are finite, so these states are locally normal and possess a critical particle density. It is homogeneous and given by

$$ x \mapsto \omega_{\beta,0,\infty}(a^*(x)a(x)) = (2\pi)^{-s} \int dp (e^{i\beta p^2} - 1)^{-1}. \quad (5.10) $$

In order to increase this density, one has to return to the approximating states and add to them coherent configurations of (suitably renormalized) scaled ground states $e_{L,1}$ of the trapped Hamiltonian $h_L$. The resulting states are normal with regard to the Fock representation. Similarly to relation (5.11), one obtains by this procedure the expectation values

$$ \omega_{\beta,\mu,L} \circ \text{Ad } W(e_{L,1}) (a^*(f)a(f)) = \omega_{\beta,\mu,L}(a^*(f)a(f)) + |\langle e_{L,1}, f \rangle|^2. \quad (5.11) $$

The local properties of the states appearing in the thermodynamic and subsequent infinite particle number limits can be read off from this relation by restricting it to functions $f \in L^2(O)$. One can then replace the functions $x \mapsto e_{L,1}(x) = e_1(x/L)$ by the pointwise products $\chi e_{L,1}$, where $\chi$ is the characteristic function of $O$. Since ground state wave functions are continuous, the functions $\chi e_{L,1}$ converge strongly to $e_1(0)\chi$ in the limit of large $L$. Proceeding first to the thermodynamic limit and then to the limit $\mu \to 0$, it follows from equation (5.11) and the preceding remarks that the resulting limit states satisfy the KMS condition and are locally normal. They describe the pure phases (primary states) in the central decomposition of condensed states, appearing in the thermodynamic limit of Gibbs-von Neumann states in finite volume, cf. [2, Thm. 5.2.32] and [13, Thm. III.3]. Their particle density is given by

$$ x \mapsto \omega_{\beta,0,\infty}(a^*(x)a(x)) + |e_1(0)|^2. \quad (5.12) $$

So the density of the limit states is enlarged by contributions from the (improper) ground state $e_{\infty,1}$. These do not affect the particle content of the original state and create a condensate whose weight is fixed by the normalization of $e_1$.

We conclude this section with a remark which is of relevance in case of interacting systems. As already mentioned, one has to restrict attention there to the field subalgebra $\mathfrak{F} \subset \mathcal{R}$ in order to retain control on the action of the dynamics. Since condensates in $s > 2$ dimensions typically appear as classical (mean) fields shifting the quantum field, there arises the question of whether the field algebra is stable under the adjoint action of Weyl operators, i.e. $\text{Ad } W(e) (\mathfrak{F}) \subset \mathfrak{F}$ for $e \in L^2(\mathbb{R}^s)$. The answer is affirmative, and we briefly sketch the argument for the simple case of tensor fields which are constructed from a single resolvent. So let $\text{Re } \lambda \neq 0$, let $f \in L^2(\mathbb{R}^s)$ and let the norm of $e$ be sufficiently small such that one obtains for the functional $l_e$ the upper bound $\sup_u |l_e(e^{iu}f)|/|\text{Re } \lambda| < 1$. Clearly, $m \in \mathbb{Z}$,

$$ \text{Ad } W(e) \left( \int_0^{2\pi} du e^{imu} R(\lambda, e^{iu}f) \right) = \int_0^{2\pi} du e^{imu} R(\lambda - i l_e(e^{iu}f), e^{iu}f), \quad (5.13) $$

where the integrals are defined in the strong operator topology on $\mathcal{F}$. The resolvent on the right hand side of this equality can be expanded in a Neumann series,

$$ R(\lambda - i l_e(e^{iu}f), e^{iu}f) = \sum_{n=1}^{\infty} (-l_e(e^{iu}f))^{n-1} R(\lambda, e^{iu}f). \quad (5.14) $$

It is absolutely convergent in norm because of the limitations on $e$. Since the functional $l_e$ is real linear, one has $l_e(e^{iu}f) = \cos(u) l_e(f) + \sin(u) l_e(if)$. It follows that each summand in the series (5.14) gives rise in equation (5.13) to a finite sum of tensor operators in $\mathfrak{F}$ whose degree ranges between $m-n+1$ and $m+n-1$. In view of the convergence properties of the series, this shows that the operators (5.14) are contained in $\mathfrak{F}$ in this particular case. In a similar manner one can treat tensor operators built by arbitrary finite sums of products of resolvents. Iterating the adjoint action of the Weyl operators, one finally sees that the constraints on the norm of $e$ can be removed, establishing the statement.
6. CONCLUSIONS

In the present article we have applied the framework of resolvent algebras in a study of thermal properties of trapped and untrapped non-interacting bosons. In particular, we have analyzed the properties of certain specific limit states, such as the thermodynamic limit and the infinite particle number limit, which are of substantial interest for the interpretation of the theory. Compared to other settings, this analysis is greatly simplified by the fact that observables, which become singular in the limit states, automatically disappear. They are members of specific ideals which are annihilated in the states. As a consequence, the limit states retain physically meaningful properties: they satisfy the KMS condition on the full algebra and are regular on the subalgebra generated by the observables which remain non-trivial in the limit.

The limit states lead to representations of the observable algebra which are disjoint from the Fock representation, i.e. they do not admit a particle interpretation on the full algebra. It is therefore natural to focus on local properties of the states which can be determined in bounded regions of space. The resolvent algebra provides the necessary ingredients for such an analysis. In case of the infinite particle number limit it allowed us to determine local subalgebras of observables which retain a meaningful particle interpretation in the limit states and to calculate with their help the respective critical densities of the thermal clouds. On the other hand the algebra contains local observables which indicate the formation of condensates, outrunning these critical densities. They lead to the breakdown of an associated particle picture in the limit states. This local point of view enabled us to treat in a unified manner different manifestations of condensation in trapped and untrapped equilibrium states.

Let us mention as an aside that the concept of localized observables plays an even more prominent role in the algebraic approach to relativistic quantum physics [15]. It is based on the insight that observations are always made in bounded regions and that the physically accessible states are locally normal with respect to each other, in accordance with present results. The ensuing algebraic framework has led to numerous fundamental insights about the properties of the state space of relativistic quantum field theories, its particle interpretation and sector structure, the characterization of thermal states etc. But it also led to new powerful algebraic schemes for the perturbative construction of models. Regarding the present issue of thermal bosons, an intriguing recent article by Brunetti, Fredenhagen and Pinamonti [3] ought to be mentioned here, where states including Bose-Einstein condensates are perturbatively constructed to all orders in an interacting quantum field theory. See also the references quoted there for further related results.

We conclude this article with some remarks pertaining to the problem of condensation in trapped and untrapped interacting systems. To simplify the discussion, we consider two-body interaction potentials which are continuous, vanish rapidly at infinity, and are repulsive. For the trapped systems we restrict ourselves to harmonic trapping potentials at length scale $L$. The resulting Hamiltonians are well defined as selfadjoint operators $H_L$ on Fock space $\mathcal{F}$. The adjoint action of the corresponding unitaries $\text{Ad} e^{itH_L}$, $t \in \mathbb{R}$, maps the algebra of observables $\mathfrak{A}$ into its canonical extension $\overline{\mathfrak{A}}$ and we also consider the regular subalgebra $\mathfrak{A}_c \subset \overline{\mathfrak{A}}$ on which it acts pointwise norm continuously, cf. [6].

The equilibrium states of trapped systems for given $\beta > 0$ and $\mu < \mu_{\text{max}}$ (i.e. the supremum of the admissible chemical potentials, appearing to be infinite by estimates based on the Bogolubov approximation [18, 21]) are described on Fock space by the density operator $e^{-\beta (H_L - \mu N)}$, which is of trace class on $\mathcal{F}$. Putting

$$\omega_{\beta, \mu, L}(A) = Z^{-1} \text{Tr} e^{-\beta (H_L - \mu N)} A, \quad A \in \mathfrak{A},$$

where $Z$ denotes the partition function, one obtains Gibbs-von Neumann states on the observable algebra $\mathfrak{A}$. They extend by continuity to the regular algebra $\overline{\mathfrak{A}}$, and, since they are stationary, the expectation value of any $A \in \mathfrak{A}$ coincides with the expectation value of its time-regularized version $\int dt f(t) \text{Ad} e^{itH_L}(A) \in \overline{\mathfrak{A}}$, where $\int dt f(t) = 1$, cf. equation (1.5).
In order to check whether the bosons form condensates in these trapped states one must proceed to the supremum $\mu_{\text{max}}$ of the chemical potential and determine the corresponding thermal cloud. It is noteworthy that the limit points of the corresponding sequence of states $\mu \mapsto \omega_{\beta,\mu,L}$ on the regular algebra $\mathcal{A}_{c}$ still satisfy the KMS condition at inverse temperature $\beta$ according to standard results for $C^*$-dynamical systems [2, Prop. 5.3.25]. Moreover, by the preceding regularization procedure, the expectation values of the resolvents of the particle number operators $a^*(f) a(f)$, $f \in L^2(\mathbb{R}^s)$, are defined in these limit states. So in order to determine the critical density of the thermal cloud in a given region $O$ it is meaningful to compute the limits

$$\lim_{\mu \mapsto \mu_{\text{max}}} \omega_{\beta,\mu,L}(a^*(f) a(f)), \quad f \in L^2(O).$$

(6.2)

Functions $f$ for which this limit stays finite form again a subspace $\mathcal{D}(O) \subset L^2(O)$ corresponding to regular observables in the limit state. In view of the repulsive nature of the interaction it seems plausible that this subspace is non-trivial. It fixes the particle number operator $N_\omega(O)$ of the thermal cloud, where the limit of $\mu \mapsto \omega_{\beta,\mu,L}(N_\omega(O))$ determines its critical density. The total number of particles in $O$, determined by $N(O)$, is expected to grow indefinitely in the trapped states in this limit. Hence if the critical density of the thermal cloud turns out to be finite, this would indicate the formation of unlimited amounts of condensate for increasing chemical potential. The orthogonal complement of $\mathcal{D}(O)$ in $L^2(O)$ then describes the condensate in $O$. The co-dimension of $\mathcal{D}(O)$ may, however, be larger than one in the presence of interaction due to the more complex structure of ground states.

One may also discuss the issue of condensation in the thermodynamic limit (no trapping potential), cf. [2, Thm. 6.3.31]. There one knows from the outset that $\mu_{\text{max}} = 0$. Moreover, because of the repulsive nature of the interaction, the number of particles in bounded regions ought to be smaller than in the non-interacting theory. We therefore conjecture that in $s > 2$ dimensions the limit states for maximal chemical potential are normal on the local field algebras $\mathfrak{F}(L^2(O))$ with regard to the Fock representation, i.e. they have a critical local density. As was shown in the last part of the preceding section, one can increase this density by the action of Weyl automorphisms. Since the resulting states should be spatially homogeneous, one has to use Weyl operators depending on (improper) eigenstates of the momentum operator with momentum 0. One thereby arrives at states which describe at given time a homogeneous, macroscopically occupied condensate accompanying the thermal background. Due to the interaction between the condensate and the background these initial states are not stationary, however. The remaining intricate problem is then to adjust the thermal background states in a manner such that the composed states are in equilibrium again.

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Data availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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