Weight function for the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_3)$

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Abstract

We give a precise expression for the universal weight function of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_3)$. The calculations use the technique of projecting products of Drinfeld currents on the intersections of Borel subalgebras.

1 Introduction

The ideology of a nested Bethe ansatz [1] prescribes two steps for describing the transfer-matrix eigenvectors in finite-dimensional representations of a quantum affine algebra. First, specific rational functions with values in the representation should be constructed; second, a system of Bethe equations for these functions should be solved. These rational vector-valued functions are called off-shell (nested) Bethe vectors. They can serve as a generating system of vectors of a finite-dimensional representation of a quantum affine algebra. We use the equivalent name ‘weight function’, which came from applications in difference Knizhnik-Zamolodchikov equations [9], [10].

A general construction of a weight function for a quantum affine algebra $U_q(\mathfrak{g})$ was recently suggested [4]. This construction uses the existence of two different types of Borel subalgebras in a quantum affine algebra. One type is related to the realization of $U_q(\mathfrak{g})$ as a quantized Kac-Moody algebra, and the other comes from the current realization of $U_q(\mathfrak{g})$ proposed by Drinfeld [2]. The weight function is defined as the projection of a product of Drinfeld currents on the intersection of Borel subalgebras of $U_q(\mathfrak{g})$ of different types (see Sec. 3.1).

Our goal in this paper is to develop a technique for calculation the weight function starting from the definition in [4]. According to this definition, for calculation the weight function, the product of Drinfeld currents must be arranged in a normally ordered form. Then only those terms are kept that belong to the intersection of Borel subalgebras of different types. The normal-ordering procedure requires investigating the current adjoint action and the composed root currents, introduced in [3]. The final result is a precise universal expression for the weight function of $U_q(\widehat{\mathfrak{sl}}_3)$, which can then be specialized to any finite-dimensional representation of $U_q(\widehat{\mathfrak{sl}}_3)$.

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This paper is organized as follows. In Section 2 we introduce the main objects of the investigation. Section 3 is devoted to formulating the main results. They contain a precise expression for the weight function of $U_q(\hat{sl}_3)$ (Theorems 1 and 2). As a particular case, we give an expression for the weight function of $U_q(\hat{sl}_2)$ in an integral form (Theorem 2). The kernel of the integral is a well-known partition function, which coincides with the partition function of the six-vertex model on a finite square lattice with fixed domain-wall boundary conditions. Later, we need a combinatorial identity for this kernel, which we prove by observing the self-adjointness of the projection operators (Proposition 3.5).

Sections 4 and 5 are devoted to proving the main statements, which includes studying the analytic properties of composed currents and related products of currents and of their projections (see Propositions 4.1 and 5.1). As a particular case, we give an expression for the analytic continuation of the products of currents and of their projections (see Proposition 5.1). In the appendices, we give the necessary properties of the opposite projection operator, commutation relations between currents and their projections, and another proof of the main result.

## 2 Basic notation

### 2.1 $U_q(\hat{sl}_3)$ in Chevalley generators

The quantum affine algebra $U_q(\hat{sl}_3)$ is generated by Chevalley\(^3\) generators $e_{\pm \alpha_i}$ and $k_{\alpha_i}^{\pm 1}$, where $i = 0, 1, 2$ and $\prod_{i=0}^2 k_{\alpha_i} = 1$, subject to the relations

\[
k_{\alpha_i}e_{\pm \alpha_j}k_{\alpha_i}^{-1} = q_i^{\pm a_{ij}}e_{\pm \alpha_j}, \quad [e_{\alpha_i}, e_{-\alpha_j}] = \delta_{ij}\frac{k_{\alpha_i} - k_{\alpha_i}^{-1}}{q_i - q_i^{-1}},
\]

\[
e_{\pm \alpha_i}^2 + [2]q_i e_{\pm \alpha_i}e_{\pm \alpha_j}e_{\pm \alpha_i} + e_{\pm \alpha_j}e_{\pm \alpha_i}^2 = 0, \quad i \neq j, \quad (\alpha_i, \alpha_j) = -1,
\]

where $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$ is the Gauss $q$-number and $a_{ij} = (\alpha_i, \alpha_j)$ is symmetrized Cartan matrix of the affine algebra $sl_3$,

\[
a_{ij} = (\alpha_i, \alpha_j) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.
\]

One of the possible Hopf structures (which we call the standard Hopf structure) is given by the formulas

\[
\Delta(e_{\alpha_i}) = e_{\alpha_i} \otimes 1 + k_{\alpha_i} \otimes e_{\alpha_i}, \quad \Delta(e_{-\alpha_i}) = 1 \otimes e_{-\alpha_i} + e_{-\alpha_i} \otimes k_{\alpha_i}^{-1},
\]

\[
\Delta(k_{\alpha_i}) = k_{\alpha_i} \otimes k_{\alpha_i}, \quad \varepsilon(e_{\pm \alpha_i}) = 0, \quad \varepsilon(k_{\alpha_i}^{\pm 1}) = 1,
\]

\[
a(e_{\alpha_i}) = -k_{\alpha_i}^{-1}e_{\alpha_i}, \quad a(e_{-\alpha_i}) = -e_{-\alpha_i}k_{\alpha_i}, \quad a(k_{\alpha_i}^{\pm 1}) = k_{\alpha_i}^{\pm 1},
\]

where $\Delta$, $\varepsilon$ and $a$ are the respective comultiplication, counit, and antipode maps.

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\(^3\)In what follows we do not use a grading operator and set the central charge equal to zero. Such an algebra is usually denoted by $U_q'(\hat{sl}_3)$.
2.2 Current realization of the algebra $U_q(\mathfrak{sl}_3)$

As does any quantum affine algebra, $U_q(\mathfrak{sl}_3)$ admits a current realization \cite{[2]}. In this description (we again assume that the central charge is zero), $U_q(\mathfrak{sl}_3)$ is generated by the elements $e_i[n]$ and $f_i[n]$, where $i = \alpha, \beta$ and $n \in \mathbb{Z}$, and $k_i^{\pm 1}$ and $h_i[n]$, where $i = \alpha, \beta$ and $n \in \mathbb{Z} \setminus \{0\}$. They are gathered in the generating functions

$$
e_i(z) = \sum_{n \in \mathbb{Z}} e_i[n] z^{-n}, \quad f_i(z) = \sum_{n \in \mathbb{Z}} f_i[n] z^{-n},$$

$$
\psi_i^\pm(z) = \sum_{n > 0} \psi_i^\pm[n] z^{\mp n} = k_i^{\pm 1} \exp \left( \pm (q - q^{-1}) \sum_{n > 0} h_i[\pm n] z^{\mp n} \right),
$$

which satisfy the relations

$$
(z - q^{(i,j)}w)e_i(z)e_j(w) = e_j(w)e_i(z) (q^{(i,j)} z - w),
$$

$$
(z - q^{-(i,j)}w)f_i(z)f_j(w) = f_j(w)f_i(z) (q^{-(i,j)} z - w),
$$

$$
\psi_i^\pm(z) e_j(w) \left( \psi_i^\pm(z) \right)^{-1} = \frac{(q^{(i,j)} z - w)}{(z - q^{(i,j)}w)} e_j(w),
$$

$$
\psi_i^\pm(z) f_j(w) \left( \psi_i^\pm(z) \right)^{-1} = \frac{(q^{-(i,j)} z - w)}{(z - q^{-(i,j)}w)} f_j(w),
$$

$$
\psi_i^\pm(z) \psi_j^\pm(w) = \psi_j^\pm(w) \psi_i^\pm(z), \quad \mu, \nu = \pm,
$$

$$
[e_i(z), f_j(w)] = \frac{\delta_{ij} \delta(z/w)}{q - q^{-1}} \left( \psi_i^+(z) - \psi_i^-(w) \right),
$$

where $i, j = \alpha, \beta$, $\delta(z) = \sum_{k \in \mathbb{Z}} z^k$, $(\alpha, \alpha) = (\beta, \beta) = 2$, $(\alpha, \beta) = -1$, and

$$
\text{Sym}_{z_1, z_2} (e_i(z_1)e_i(z_2)e_j(w) - (q + q^{-1}) e_i(z_1) e_j(w) e_i(z_2) + e_j(w) e_i(z_1) e_i(z_2)) = 0, \quad (2.10)
$$

$$
\text{Sym}_{z_1, z_2} (f_i(z_1)f_i(z_2)f_j(w) - (q + q^{-1}) f_i(z_1) f_j(w) f_i(z_2) + f_j(w) f_i(z_1) f_i(z_2)) = 0, \quad (2.11)
$$

where $i, j = \alpha, \beta$, $i \neq j$. The assignment

$$
k_{\alpha_1} \mapsto k_{\alpha}, \quad k_{\alpha_2} \mapsto k_{\beta}, \quad k_{\alpha_0} \mapsto k_{\alpha}^{-1} k_{\beta}^{-1},
$$

$$
e_{\alpha_1} \mapsto e_{\alpha}[0], \quad e_{\alpha_2} \mapsto e_{\beta}[0], \quad e_{-\alpha_1} \mapsto f_{\alpha}[0], \quad e_{-\alpha_2} \mapsto f_{\beta}[0],
$$

$$
e_{\alpha_0} \mapsto f_{\beta}[1] f_{\alpha}[0] - q f_{\alpha}[0] f_{\beta}[1], \quad e_{-\alpha_0} \mapsto e_{\alpha}[0] e_{\beta}[-1] - q^{-1} e_{\beta}[-1] e_{\alpha}[0]
$$

establishes the isomorphism of the two realizations.

The algebra $U_q(\mathfrak{sl}_3)$ admits a natural completion $\overline{U_q(\mathfrak{sl}_3)} = U_q(D(\mathfrak{sl}_3))$, which can be described as the minimal extension of $U_q(\mathfrak{sl}_3)$ and which acts in all representations of $U_q(\mathfrak{sl}_3)$ that are highest-weight representations of $U_q(\mathfrak{b}_+)$ (see Sec. 2.2 in \cite{[7]} for the details).

In a highest-weight representation of $U_q(\mathfrak{sl}_3)$, any matrix coefficients of an arbitrary product of the currents $a_1(z_1), \ldots, a_n(z_n)$ are formal power series in the space

$$
\mathbb{C}[z_1, z_1^{-1}, \ldots, z_m, z_m^{-1}]
$$

\[
\begin{bmatrix}
\frac{z_2}{z_1}, \frac{z_3}{z_2}, \ldots, \frac{z_m}{z_{m-1}}
\end{bmatrix}
\]
and converge to a rational function in the domain $|z_1| \gg |z_2| \gg \cdots \gg |z_n|$ (see [3], [4]). This observation and commutation relations (2.6), which dictate the rule for the analytic continuation from the above domain, allow considering products of currents as meromorphic functions with values in $\mathcal{U}_q(\widehat{\mathfrak{sl}_3})$. In what follows, we freely use this analytic language and replace formal integrals with contour integrals in this formalism. An integral without the contour specified always means a formal integral.

Another Hopf structure in $U_q(\widehat{\mathfrak{sl}_3})$ is naturally related to the current realization. In terms of currents, it is given by

\[\Delta^{(D)} e_i(z) = e_i(z) \otimes 1 + \psi_i^-(z) \otimes e_i(z),\]
\[\Delta^{(D)} f_i(z) = 1 \otimes f_i(z) + f_i(z) \otimes \psi_i^+(z),\]
\[\Delta^{(D)} \psi_i^\pm(z) = \psi_i^\pm(z) \otimes \psi_i^\mp(z),\]
\[a(e_i(z)) = - (\psi_i^-)^{-1} e_i(z), \quad a(f_i(z)) = - f_i(z) (\psi_i^+)^{-1},\]
\[a(\psi_i^-) = (\psi_i^\pm)\psi_i^\mp(z), \quad \varepsilon(e_i(z)) = \varepsilon(f_i(z)) = 0, \quad \varepsilon(\psi_i^\mp(z)) = 1.\]

The comultiplications $\Delta$ in Sec. 2.1 and $\Delta^{(D)}$ are related by the twist, which can be described explicitly (see [2]).

### 2.3 Borel subalgebras of $U_q(\widehat{\mathfrak{sl}_3})$

We let $U_q(\mathfrak{b}_\pm)$ denote the subalgebra of $U_q(\widehat{\mathfrak{sl}_3})$ generated by the elements $e_{\alpha_i}$ and $k_{\alpha_i}^{\pm 1}$, $i = 0, 1, 2$. We also let $U_q(\mathfrak{b}_-)$ denote the subalgebra of $U_q(\widehat{\mathfrak{sl}_3})$ generated by the elements $e_{-\alpha_i}$ and $k_{\alpha_i}^{\pm 1}$, $i = 0, 1, 2$.

The algebras $U_q(\mathfrak{b}_\pm)$ are Hopf subalgebras of $U_q(\widehat{\mathfrak{sl}_3})$ with respect to the standard comultiplication $\Delta$ and serve as $q$-deformations of the enveloping algebras of opposite Borel subalgebras of the Lie algebra $\widehat{\mathfrak{sl}_3}$. We call them the standard Borel subalgebras. They contain subalgebras $U_q(\mathfrak{n}_\pm)$ generated by the elements $e_{\pm \alpha_i}$, $i = 0, 1, 2$.

The subalgebra $U_q(\mathfrak{n}_+)$ is a left coideal of $U_q(\mathfrak{b}_+)$ with respect to the standard comultiplication, and the subalgebra $U_q(\mathfrak{n}_-)$ is a right coideal of $U_q(\mathfrak{b}_-)$ with respect to the standard comultiplication, i.e.,

\[\Delta(U_q(\mathfrak{n}_+)) \subset U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{n}_+), \quad \Delta(U_q(\mathfrak{n}_-)) \subset U_q(\mathfrak{n}_-) \otimes U_q(\mathfrak{b}_-).\]

Borel subalgebras of another type are related to the current realization of $U_q(\widehat{\mathfrak{sl}_3})$. We let $U_F$ denote the subalgebra of $U_q(\widehat{\mathfrak{sl}_3})$ generated by the elements $k_{\alpha_i}^{\pm 1}$ and $f_i[n]$, where $i = \alpha, \beta$, and $n \in \mathbb{Z}$, and $h_i[n]$, where $i = \alpha, \beta$, and $n > 0$. Its completion $\overline{U}_F$ is a Hopf subalgebra of $U_q(\widehat{\mathfrak{sl}_3})$ with respect to the comultiplication $\Delta^{(D)}$. We call $U_F$ the current Borel subalgebra. It contains the subalgebra $U_f$ generated by the elements $f_i[n]$, where $i = \alpha, \beta$, and $n \in \mathbb{Z}$. The completed algebra $\overline{U}_f$ is a right coideal of $\overline{U}_F$ with respect to the comultiplication $\Delta^{(D)}$ and serves as a $q$-deformed enveloping algebra of the algebra of currents valued in $\mathfrak{n}_-$.

The opposite current Borel subalgebra $U_E$ is generated by the elements $k_{\alpha_i}^{\pm 1}$ and $e_i[n]$, where $i = \alpha, \beta$, $n \in \mathbb{Z}$, and by the elements $h_i[n]$, $i = \alpha, \beta$, $n < 0$. 
2.4 Projections  $P^\pm$ on intersections of Borel subalgebras

We let $U_F^+$ and $U_F^-$ denote the subalgebras of the current Borel algebra $U_F$,

$$U_F^- = U_F \cap U_q(n_-), \quad U_F^+ = U_F \cap U_q(b_+) .$$

(2.14)

For any $x \in U_q(\mathfrak{sl}_3)$, we let $\text{ad}_x : U_q(\mathfrak{sl}_3) \to U_q(\mathfrak{sl}_3)$ be the operator of the adjoint action of $x$ in $U_q(\mathfrak{sl}_3)$. It is defined by the relation

$$\text{ad}_x(y) = \sum_j a(x'_j) \cdot y \cdot x''_j, \quad \text{if} \quad \Delta(x) = \sum_j x'_j \otimes x''_j .$$

For $i = \alpha, \beta$, let $S_i$ be the operator $\text{ad}_{f_i[0]}$ such that

$$S_i(y) = y f_i[0] - f_i[0] k_i y k_i^{-1} .$$

(2.15)

We call $S_i$ the screening operators.

**Proposition 2.1**

(i) The algebra $U_f^-$ is generated by the elements $f_i[n]$, where $i = \alpha, \beta$ and $n \leq 0$; the algebra $U_F^+$ is generated by the elements $k_i^{\pm 1}$, $f_i[n]$, and $h_i[n]$, where $i = \alpha, \beta$ and $n > 0$, and by the element

$$f_{\alpha+\beta}[1] = f_{\beta}[1] f_{\alpha}[0] - q f_{\alpha}[0] f_{\beta}[1] = - (f_{\alpha}[1] f_{\beta}[0] - q f_{\beta}[0] f_{\alpha}[1]) .$$

(2.16)

(ii) The subalgebras $U_f^-$ and $U_F^+$ are invariant under the action of the screening operators $S_i$, $i = \alpha, \beta$.

(iii) The subalgebra $U_F^+$ is a left coideal of $U_F$ with respect to the comultiplication $\Delta^{(D)}$; the subalgebra $U_f^-$ is a right coideal of $U_f$ with respect to the comultiplication $\Delta^{(D)}$.

(iv) The multiplication in $U_F$ establishes an isomorphism of the vector spaces $U_F$ and $U_f^- \otimes U_F^+$.

**Proof.** Statement (ii) can be verified as follows:

$$S_{\alpha} (f_{\beta}[1] f_{\alpha}[0] - q f_{\alpha}[0] f_{\beta}[1]) =$$

$$= (f_{\beta}[1] f_{\alpha}[0] - q f_{\alpha}[0] f_{\beta}[1]) f_{\alpha}[0] - q^{-1} f_{\alpha}[0] (f_{\beta}[1] f_{\alpha}[0] - q f_{\alpha}[0] f_{\beta}[1]) =$$

$$= f_{\beta}[1] f_{\alpha}[0] f_{\alpha}[0] - (q + q^{-1}) f_{\alpha}[0] f_{\beta}[1] f_{\alpha}[0] + f_{\alpha}[0] f_{\alpha}[0] f_{\beta}[1] = 0 .$$

To prove (iii), we use formula (2.14) written in terms of modes as

$$\Delta^{(D)} (f_i[n]) = 1 \otimes f_i[n] + \sum_{k \geq 0} f_i[n - k] \otimes \psi_i^+[k] .$$

We must show that $\Delta^{(D)} (f_{\beta}[1] f_{\alpha}[0] - q f_{\alpha}[0] f_{\beta}[1]) \in U_F \otimes U_F^+$ . The formula above shows that it suffices to verify that $\psi_{\beta}^+[k] f_{\alpha}[0] - q f_{\alpha}[0] \psi_{\beta}^+[k] \in U_F^+$ . But this holds because of the relation

$$\psi_{\beta}^+(z) f_{\alpha}[0] - q f_{\alpha}[0] \psi_{\beta}^+(z) = (q^2 - 1) \sum_{n=1}^{\infty} (qz)^{-n} \psi_{\beta}^+(z) f_{\alpha}[n] .$$
We define the operators $P = P^+: U_F \to U_F^+$ and $P^-: U_F \to U_F^-$ by the relations

\[ P(f_1 f_2) = P^+(f_1 f_2) = \varepsilon(f_1) f_2, \quad P^-(f_1 f_2) = f_1 \varepsilon(f_2) \]  

(2.17)

for any $f_1 \in U_F^-$ and $f_2 \in U_F^+$. Proposition 4.3 implies that the algebras $U_F^-$ and $U_F^+$ satisfy conditions (i) and (ii) in Sec. 4.1 in [8] (also see Section 6 in [5]) with respect to the comultiplication $(\Delta^{(D)})^{\text{op}}$. By [8], the operators $P^\pm$ are therefore well-defined projection operators, $(P^\pm)^2 = P^\pm$, which admit extensions to the completed algebra $\overline{U}_F$ such that for any $f \in \overline{U}_F$, the canonical decomposition

\[ f = \sum_i P^-(f_i'') \cdot P^+(f_i'), \quad \text{if} \quad \Delta^{(D)}(f) = \sum_i f_i' \otimes f_i'' \]  

(2.18)

holds. We call expressions of the form $f = \sum_i \tilde{f}_i \tilde{f}_i'$, where $\tilde{f}_i \in U^-_F$ and $\tilde{f}_i' \in U^+_F$, the normally ordered expansion. The normally ordered expansion is compatible with the action of the algebra $U_q(\hat{\mathfrak{sl}}_3)$ in highest-weight representations. Expression (2.18) gives an ordered expansion of an arbitrary element $f \in \overline{U}_F$.

### 2.5 Composite current and strings

We define the generating function of the elements in $\overline{U}_q(\hat{\mathfrak{sl}}_3)$

\[ f_{\alpha+\beta}(z) = \oint f_\alpha(z) f_\beta(w) \frac{dw}{w} - \oint \frac{q^{-1} - z/w}{1 - q^{-1} z/w} f_\beta(w) f_\alpha(z) \frac{dw}{w}, \]  

(2.19)

where the formal integral $\oint g(w) \frac{dw}{w}$ of a Laurent series $g(w) = \sum_{k \in \mathbb{Z}} g_k w^{-k}$ means taking its coefficient $g_0$.

We can also write the formal integral in the analytic language [6],

\[ f_{\alpha+\beta}(z) = - \text{res}_{w=zq^{-1}} f_\alpha(z) f_\beta(w) \frac{dw}{w}, \]  

(2.20)

such that the relation

\[ f_\alpha(z) f_\beta(w) = \frac{1 - qz/w}{q - z/w} f_\beta(w) f_\alpha(z) + \delta(zq^{-1}/w) f_{\alpha+\beta}(z) \]  

(2.21)

holds in the algebra $\overline{U}_q(\hat{\mathfrak{sl}}_3)$. For any $a, b \in \mathbb{Z}_{\geq 0}$, the products

\[ f_\alpha(u_1) \cdots f_\alpha(u_a) f_{\alpha+\beta}(u_{a+1}) \cdots f_{\alpha+\beta}(u_{a+b}), \quad f_{\alpha+\beta}(u_1) \cdots f_{\alpha+\beta}(u_a) f_\beta(u_{a+1}) \cdots f_\beta(u_{a+b}) \]  

(2.22)

are called strings. The products

\[ f_{\alpha+\beta}(u_{a+b}) \cdots f_{\alpha+\beta}(u_{a+1}) f_\alpha(u_a) \cdots f_\alpha(u_1), \quad f_\beta(u_{a+b}) \cdots f_\beta(u_{a+1}) f_{\alpha+\beta}(u_a) \cdots f_{\alpha+\beta}(u_1) \]  

(2.23)

are called opposite strings to strings (2.22). The strings have nice analytic properties, which are crucial for their use in this paper. These properties are listed in Proposition 4.3.
3 Main results

3.1 Universal weight function

Let \( V \) be a representation of \( U_q(\hat{sl}_3) \) and \( v \) be a vector in \( V \). We call \( v \) a highest-weight vector with respect to the current Borel subalgebra \( U_E \), if

\[
e_i(z)v = 0, \quad \psi_i^\pm(z)v = \lambda_i(z)v, \quad i = \alpha, \beta,
\]

where \( \lambda_i(z) \) is a meromorphic function decomposed into a series in \( z^{-1} \) for \( \psi_i^+(z) \) and into a series in \( z \) for \( \psi_i^-(z) \). The representation \( V \) is called a representation with the highest-weight vector \( v \in V \) with respect to \( U_E \) if it is generated by \( v \) over \( U_q(\hat{sl}_3) \).

Let \( \Pi \) denote the two-element set \( \{\alpha, \beta\} \) of positive simple roots of the Lie algebra \( sl_3 \). An ordered set \( I = \{a_1, \ldots, a_{|I|}\} \), together with a map \( \iota : I \to \Pi \), is called an ordered \( \Pi \)-multiset.

We suppose that for any ordered \( \Pi \)-multiset \( I \), \( |I| = n \), a formal series \( W(t_{i_1}, \ldots, t_{i_n}) \in U\{t_{i_1}, \ldots, t_{i_n}\}, i_k \in I, \) is chosen, where

\[
U\{t_{i_1}, \ldots, t_{i_n}\} = U_q(\hat{sl}_3)[t_{i_1}, t_{i_1}^{-1}, \ldots, t_{i_n}, t_{i_n}^{-1}] \left[ \frac{t_{i_2}}{t_{i_1}}, \frac{t_{i_3}}{t_{i_2}}, \ldots, \frac{t_{i_{n-1}}}{t_{i_n}}, 1/t_{i_n} \right],
\]

i.e., \( W(t_{i_1}, \ldots, t_{i_n}) \) is a formal power series in the variables \( t_{i_2}/t_{i_1}, t_{i_3}/t_{i_2}, \ldots, t_{i_{n-1}}/t_{i_n}, 1/t_{i_n} \) with coefficients in polynomials \( U_q(\hat{sl}_3)[t_{i_1}, t_{i_1}^{-1}, \ldots, t_{i_n}, t_{i_n}^{-1}] \) such that

1) for any representation \( V \) that is highest-weight with respect to \( U_E \) with the highest weight vector \( v \), the function

\[
w_V(t_{i_1}, \ldots, t_{i_n}) = W(t_{i_1}, \ldots, t_{i_n})v
\]

converges in the domain \( |t_{i_1}| \gg \cdots \gg |t_{i_n}| \) to a meromorphic \( V \)-valued function,

2) if \( I = \emptyset \), then \( W = 1 \) and \( w_V = v \), and

3) if \( V = V_1 \otimes V_2 \) is a tensor product of highest-weight representations with the highest-weight vectors \( v_1 \) and \( v_2 \) and the highest-weight series \( \{\lambda_i^{(1)}(z)\} \) and \( \{\lambda_i^{(2)}(z)\}, i = \alpha, \beta \), then for any ordered \( \Pi \)-multiset \( I \), we have

\[
w_V \left( \{t_a | a \in I\} \right) = \sum_{I_1 \sqcup I_2} w_{V_1} \left( \{t_a | a \in I_1\} \right) \otimes w_{V_2} \left( \{t_a | a \in I_2\} \right) \times \prod_{a \in I_1} \lambda_i^{(2)}(t_a) \times \prod_{a < b, a \in I_1, b \in I_2} \frac{q^{-(\iota(a), \iota(b))}t_a - t_b}{t_a - q^{-(\iota(a), \iota(b))}t_b}.
\]

A collection \( W(t_{i_1}, \ldots, t_{i_n}) \) is called a universal weight function. A collection \( w(t_{i_1}, \ldots, t_{i_n}) \) is called a weight function.

If solutions of the corresponding system of Bethe equations are chosen as parameters in the weight function, then a set of Bethe vectors is obtained. The weight function with free parameters was systematically used to investigate solutions of \( q \)-difference Knizhnick-Zamolodchikov equations [9, 10].
Let $I = \{i_1, \ldots, i_n\}$ be an ordered $\Pi$-multiset. We set
\begin{equation}
W(t_{i_1}, \ldots, t_{i_n}) = P\left(f_{i(i_1)}(t_{i_1}) \cdots f_{i(i_n)}(t_{i_n})\right).
\end{equation}

The main result in paper [1] can be formulated as follows in the particular case of $U_q(\widehat{sl}_n)$.

**Theorem.** [1] The collection $W(t_{i_1}, \ldots, t_{i_n})$ defined in (3.4) is a universal weight function.

We again note that all the expressions for universal weight functions $W(t_{i_1}, \ldots, t_{i_n})$ are to be understood as formal series in the variables $t_{i_2}/t_{i_1}, t_{i_3}/t_{i_2}, \ldots, t_{i_n}/t_{i_{n-1}}, 1/t_{i_n}$. If we deal with a weight function $w(t_{i_1}, \ldots, t_{i_n})$ that is a vector-valued rational function, then there is no difference in the choice of the domain where this function is expanded (see Sec. 5.1 for more details).

### 3.2 Reduction to projections of strings

Let $S_n$ be the group of permutations on $n$ elements. For any set $\vec{t} = \{t_1, \ldots, t_n\}$ of variables $t_1, \ldots, t_n$ and any $\sigma \in S_n$, we let $\sigma \vec{t}$ denote the set $\{t_{\sigma(1)}, \ldots, t_{\sigma(n)}\}$. We keep the symbol $\check{\omega}$ for the longest element of the group $S_n$. In this notation, the set $\check{\omega} \vec{t}$ means the set $\vec{t}$ with the reversed order: $\check{\omega} \vec{t} = \{t_n, \ldots, t_1\}$.

The group $S_n$ acts naturally in the space of vector-valued meromorphic functions of $n$ variables $\vec{t} = \{t_1, \ldots, t_n\}$ by the rule $F(\vec{t}) \mapsto \sigma F(\vec{t})$, where
\begin{equation}
\sigma F(\vec{t}) = \sigma F(t_1, \ldots, t_n) = F(t_{\sigma(1)}, \ldots, t_{\sigma(n)}) = F(\check{\omega} \vec{t}).
\end{equation}

We now suppose that $F(\vec{t})$ is a series in the domain $|t_1| \gg \cdots \gg |t_n|$ with values in the vector space $V$, i.e., $F(\vec{t})$ belongs to the space
\begin{equation}
V[t_{1}, t_{-1}, \ldots, t_{-1}]^n \left[ \frac{t_2}{t_1}, \frac{t_3}{t_2}, \ldots, \frac{t_n}{t_{n-1}}, \frac{1}{t_n} \right].
\end{equation}

We suppose that this series converges in the domain $|t_1| \gg \cdots \gg |t_n|$ to an analytic function and that for any $\sigma \in S_n$, this analytic function admits an analytic continuation to the domain $|t_{\sigma(1)}| \gg \cdots \gg |t_{\sigma(n)}|$. We then set $\sigma F(\vec{t})$ equal to the formal series representing the analytic continuation of the function $F(t_{\sigma(1)}, \ldots, t_{\sigma(n)})$ to the domain $|t_1| \gg \cdots \gg |t_n|$. Therefore, $\sigma F(\vec{t})$ is again a series in (3.5).

With this convention, the symmetrization $\text{Sym}_n^\sigma F(t_1, \ldots, t_n)$ of a function $F(t_1, \ldots, t_n)$, as well as of a series $F(t_1, \ldots, t_n)$ in a domain $|t_1| \gg \cdots \gg |t_n|$, is the sum $\text{Sym}_n^\sigma F(t_1, \ldots, t_n) = \sum_{\sigma \in S_n} \sigma F(t_1, \ldots, t_n)$. The $q$-symmetrization of a function $F(\vec{t})$ of $n$ variables or of a series $F(\vec{t})$ in a domain $|t_1| \gg \cdots \gg |t_n|$ is defined as
\begin{equation}
\text{Sym}_n^\sigma F(\vec{t}) = \sum_{\sigma \in S_n} \prod_{i < i'} \frac{q^{-1} - q t_{\sigma(i)}/t_{\sigma(i')}}{q^{-1} - q t_{\sigma(i)}/t_{\sigma(i')}} \sigma F(\vec{t}).
\end{equation}

Symmetrization of a series that is convergent in a different asymptotic zone is defined in analogously.

Universal weight function [3.4] allows analytic continuations to different asymptotic zones because the operator $P$ extends to a projection operator in the completed algebra $\overline{U}_F$, where the analytic continuation of the products of currents is well defined.
Let $I = \{i_1, ..., i_n\}$ be an ordered $\Pi$-multiset. For any permutation $\sigma \in S_n$, we let $^{\sigma}I$ denote an ordered $\Pi$-multiset $\sigma I = \{i_{\sigma(1)}, ..., i_{\sigma(n)}\}$ that differs from $I$ by permutations of the elements but has the same map $\iota: I \to \Pi$. Let $W(t_{i_{\sigma(1)}} \cdots t_{i_{\sigma(n)}})$ be a universal weight function corresponding to the ordered set $^{\sigma}I$ and $\tilde{W}(t_{i_1} \cdots t_{i_n})$ be the analytic continuation of the weight function $W(t_{i_1} \cdots t_{i_n})$ to the domain $|t_{i_{\sigma(1)}}| \gg \cdots \gg |t_{i_{\sigma(n)}}|$.

**Proposition 3.1** Universal weight function (3.8) satisfies the relations

$$W(t_{i_{\sigma(1)}} \cdots t_{i_{\sigma(n)}}) = \prod_{i < j, \sigma^{-1}(i) > \sigma^{-1}(j)} q^{(\iota(i), \iota(j))} \frac{t_{ij}}{1 - q^{\iota(i), \iota(j)} \frac{t_{ij}}{t_{ik}}} \tilde{W}(t_1 \cdots t_n). \quad (3.7)$$

This proposition is a direct consequence of Proposition 5.1. It follows from Proposition 3.1 that the universal weight function for $U_q(\mathfrak{sl}_3)$ is completely defined by the expression

$$W(t_1, ..., t_a, s_1, ..., s_b) = P(f_\alpha(t_1) \cdots f_\alpha(t_a) f_\beta(s_1) \cdots f_\beta(s_b)). \quad (3.8)$$

In this paper, we suggest an explicit expression for function (3.8) in terms of the current generators of $U_q(\mathfrak{sl}_3)$.

For the sets of variables $\vec{t} = \{t_1, ..., t_k\}$ and $\vec{s} = \{s_1, ..., s_k\}$, we define the series

$$Y(\vec{t}; \vec{s}) = \prod_{i=1}^{k} \frac{1}{1 - s_i/t_i} \prod_{j=1}^{i-1} \frac{q - q^{-1}s_j/t_i}{1 - s_j/t_i} = \prod_{i=1}^{k} \frac{1}{1 - s_i/t_i} \prod_{j=i+1}^{k} \frac{q - q^{-1}s_i/t_j}{1 - s_i/t_j}. \quad (3.9)$$

$$Z(\vec{t}; \vec{s}) = Y(\vec{t}; \vec{s}) \prod_{i=1}^{k} \frac{s_i}{t_i}$$

**Theorem 1** Universal weight function (3.8) can be written as

$$W(t_1, ..., t_a, s_1, ..., s_b) = P( f_\alpha(t_1) \cdots f_\alpha(t_a) f_\beta(s_1) \cdots f_\beta(s_b) ) = \sum_{k=0}^{\min\{a,b\}} \frac{1}{k!(a-k)!(b-k)!} \text{Sym}_a \text{Sym}_b \left( P(f_\alpha(t_1) \cdots f_\alpha(t_{a-k}) f_{\alpha+\beta}(t_{a-k+1}) \cdots f_{\alpha+\beta}(t_a) \right) \times P(f_\beta(s_{k+1}) \cdots f_\beta(s_b)) Y(q^{-1}t_{a-k+1}, \ldots, q^{-1}t_a; s_1, \ldots, s_k). \quad (3.10)$$

This theorem reduces calculation the weight function to calculating the projections of strings.

### 3.3 Projections of strings

We first describe projections of single currents. For any current $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n}$, let $a^{\pm}(z)$ denote the currents $(a(z) = a^{+}(z) - a^{-}(z))$

$$a^{+}(z) = \int \frac{a(w) \, dw}{1 - \frac{w}{z}} = \sum_{n > 0} a_n z^{-n} ;$$

$$a^{-}(z) = -\int \frac{a(w) \, dw}{1 - \frac{z}{w}} = -\sum_{n < 0} a_n z^{-n}. \quad (3.11)$$
Proposition 3.2 \( \text{Projections of the currents } f_\alpha(z), \ f_\beta(z), \ \text{and } f_{\alpha+\beta}(z) \) can be written as

\[
P(f_\alpha(t)) = f_\alpha^+(t), \quad P(f_\beta(t)) = f_\beta^+(t), \quad P(f_{\alpha+\beta}(t)) = S_\beta(f_\alpha^+(t)) = f_\alpha^+(t) f_\beta[0] - q f_\beta[0] f_\alpha^+(t).
\] (3.12)

There are also analogous formulas for the opposite projection:

\[
P^-(f_\alpha(t)) = -f_\alpha^-(t), \quad P^-(f_\beta(t)) = -f_\beta^-(t), \quad P^-(f_{\alpha+\beta}(t)) = q^{-1} S_\alpha(f_\beta^-(q^{-1} t)) = q^{-1} f_\beta^-(q^{-1} t) f_\alpha[0] - f_\alpha[0] f_\beta^-(q^{-1} t).
\] (3.13)

We define a set of rational functions of the variable \( s \) depending on parameters \( s_1, \ldots, s_b \):

\[
\varphi_{s_j}(s; s_1, \ldots, s_b) = \prod_{i=1, i \neq j}^{b} \frac{s - s_i}{s_j - s_i} \prod_{i=1}^{b} \frac{q^{-1} s_j - q s_i}{q^{-1} s - q s_i}.
\] (3.14)

As functions of \( s \), they have simple poles at the points \( s = q^2 s_i, \ i = 1, \ldots, b \), tend to zero as \( s \to \infty \), and have properties \( \varphi_{s_j}(s_i; s_1, \ldots, s_b) = \delta_{ij} \). Set (3.14) is uniquely defined by these properties.

We define the combination of currents

\[
f_\gamma(t; t_1, \ldots, t_b) = f_\gamma(t) - \sum_{m=1}^{b} \varphi_{t_m}(t; t_1, \ldots, t_b) f_\gamma(t_m),
\] (3.15)

where \( \gamma \) coincides either with the simple root \( \alpha \), the simple root \( \beta \), or the composite root \( \alpha + \beta \).

Theorem 2 \( \text{The projection of the string } (2.22) \) has the factored form

\[
P(f_\alpha(t_1) \cdots f_\alpha(t_{a-k}) f_{\alpha+\beta}(t_{a-k+1}) \cdots f_{\alpha+\beta}(t_a)) =
\prod_{1 \leq i < j \leq a - k < \alpha} q t_i - q^{-1} t_j
\prod_{1 \leq i < j \leq a} q^{-1} t_i - q t_j
\times P(f_{\alpha+\beta}(t_a)) P(f_{\alpha+\beta}(t_{a-1}; t_a)) \cdots P(f_{\alpha+\beta}(t_{a-k+1}; t_{a-k+2}, \ldots, t_a))
\times P(f_{\alpha}(t_{a-k}; t_{a-k+1}, \ldots, t_a)) \cdots P(f_\alpha(t_1; t_2, \ldots, t_a)).
\] (3.16)

3.4 Examples

We give several explicit examples illustrating Theorems 1 and 2. The second, third and forth examples are given with the corollary to Theorem 3 taken into account:

\[
P(f_\alpha(t_1) f_{\alpha+\beta}(t_2)) = \frac{q^{-1} t_1 - q t_2}{t_1 - t_2} P(f_{\alpha+\beta}(t_2)) \left( f_{\alpha}^+(t_1) - \frac{(q - q^{-1}) t_1}{q t_2 - q^{-1} t_2} f_{\alpha}^+(t_2) \right),
\]

\[
P(f_\alpha(t_1) f_{\alpha}(t_2)) = f_{\alpha}^+(t_1) \left( f_{\alpha}^+(t_2) - \frac{(q - q^{-1}) t_1}{q t_1 - q^{-1} t_2} f_{\alpha}^+(t_1) \right),
\]

\[
P(f_{\alpha+\beta}(t_1) f_{\alpha+\beta}(t_2)) = P(f_{\alpha+\beta}(t_1)) \left( P(f_{\alpha+\beta}(t_2)) - \frac{(q - q^{-1}) t_1}{q t_1 - q^{-1} t_2} P(f_{\alpha+\beta}(t_1)) \right),
\]
\[ P(f_α(t_1)f_α(t_2)f_α(t_3)) = f_α^+(t_1) \left( f_α^+(t_2) - \frac{(q-q^{-1})t_1}{mt_1 - q^{-1}t_2} f_α^+(t_1) \right) \times \]
\[ \times \left( f_α^+(t_3) - \frac{t_1 - t_3}{t_2 - t_3} (q - q^{-1})t_3 f_α^+(t_3) - \frac{t_2 - t_3}{t_2 - t_1} (q - q^{-1})t_1 f_α^+(t_1) \right) \]

and

\[ P(f_α(t_1)f_α(t_2)f_β(s_1)f_β(s_2)) = P(f_α(t_1)f_α(t_2)) P(f_β(s_1)f_β(s_2)) + \]
\[ + \frac{1}{2} \text{Sym}_{t_1,t_2} \left( \text{Sym}_{t_1,t_2} \left( \frac{t_2}{t_2 - qs_1} f_β^+(s_2) \right) \right) \]
\[ + \frac{1}{2} \text{Sym}_{t_1,t_2} \left( \text{Sym}_{t_1,t_2} \left( \frac{t_1}{t_1 - qs_1} \text{Sym}_{s_1,s_2} \left( \frac{t_2}{t_2 - q_1 s_1} f_β^+(s_2) \right) \right) \right) \]  

(3.17)

We note that in Theorem 2 and in the examples considered above, the normal ordering of the roots is changed from \(\alpha, \alpha + \beta, \beta\) to \(\alpha + \beta, \alpha, \beta\). The correct normal ordering \(\alpha, \alpha + \beta, \beta\) can be restored by using the commutation relations given in Proposition 5.8. The result of this calculations is that the second line in the formula (3.17) can be replaced with the expression

\[ \text{Sym}_{s_1,s_2} \left( \text{Sym}_{t_1,t_2} \left( f_α^+(t_1) P(f_α(t_2); t_1) \right) \frac{q^{-1}t_1 - qt_2}{t_1 - t_2} \frac{qt_1 - s_1}{t_1 - qs_1} \frac{t_2}{t_2 - qs_1} f_β^+(s_2) \right) \].

### 3.5 Universal weight function for \(U_q(\hat{\mathfrak{sl}}_2)\)

The currents \(e_α(z), f_α(z), \text{ and } \psi^±_α(z)\), as well as Chevalley generators \(e_{±α_i}\) and \(k_{±α_i}\), \(i = 0, 1\), generate a Hopf subalgebra \(U_q(\hat{\mathfrak{sl}}_2)\) in \(U_q(\hat{\mathfrak{sl}}_3)\). For this algebra, the weight function and projection operators can be defined independently. We observe that the corresponding projection of the product \(f_α(t_1) \cdots f_α(t_n)\) coincides with its projection inside the algebra \(U_q(\hat{\mathfrak{sl}}_3)\). As a corollary of Theorem 2, we obtain the description of the weight function for \(U_q(\hat{\mathfrak{sl}}_2)\). It also admits a simple integral presentation.

**Theorem 3**

(i) **Universal weight function** (3.3) can be written as

\[ W(t_1, \ldots, t_a) = P(f_α(t_1) \cdots f_α(t_a)) = \]
\[ = \prod_{1 \leq i < j \leq a} \frac{q^{t_i - t_j}}{q^{t_i} - q^{-1}t_j} f_α^+(t_a) f_α^+(t_{a-1}; t_a) \cdots f_α^+(t_1; t_2, \ldots, t_a). \]  

(3.18)

(ii) **Weight function** (3.18) admits the integral representation

\[ P(f_α(t_1) \cdots f_α(t_a)) = \prod_{i < j} \frac{t_i - t_j}{q^{t_i} - q^{-1}t_j} \int \int \cdots \int Z(\vec{t}; \vec{w}) f_α(w_a) dw_a \cdots f_α(w_1) dw_1. \]  

(3.19)

The currents \(f_α^+(t_i; t_\ell+1, \ldots, t_a)\) are defined by formula (3.15), and the kernel \(Z(\vec{t}; \vec{w})\) of the integral transformation is defined in (3.9).
Proof. Statement (i) is a particular case of Theorem 2. Statement (ii) is obtained from (i) by substituting expressions (3.11) and the elementary identity

\[ \frac{1}{t - w} - \sum_{m=1}^{b} \varphi_{i_m}(t; t_1, \ldots, t_b) \frac{1}{t_m - w} = \frac{1}{t - w} \prod_{i=1}^{b} \left( \frac{t - t_i}{w - t_i} q^{-1} w - qt_i \right). \]

We note that because the factor before the integral in (3.19) has the same $q$-symmetry properties as the product of currents $f_0(t_1)f_0(t_2) \cdots f_0(t_a)$, the integral is itself symmetric under permutations of the parameters $t_1, \ldots, t_a$. This means that we can use the kernel

\[ Z(t_1, \ldots, t_a; w_1, \ldots, w_a) \]

in (3.19) instead of the kernel $Z(t_1, \ldots, t_a; w_1, \ldots, w_a)$ for any permutation $\sigma \in S_a$.

**Corollary 3.3** The projection of the product of currents can be written in the “direct” order

\[ P \left( f_0(t_1) \cdots f_0(t_a) \right) = \prod_{1 \leq i < j \leq a} \frac{t_i - t_j}{q t_i - q^{-1} t_j} \int \cdots \int Z(\tilde{w}; \tilde{w}) f_0(w_1) \frac{dw_1}{w_1} \cdots f_0(w_a) \frac{dw_a}{w_a} \] (3.20)

\[ = f^+_0(t_1) f^+_0(t_2) \cdots f^+_0(t_a-1, t_1, \ldots, t_{a-2}) f^+_0(t_{a}; t_1, \ldots, t_a). \]

To prove this corollary, it is sufficient to rename the parameters in integral (3.19) $t_i \rightarrow t_{a+1-i}$, $i = 1, \ldots, a$ and calculate the integral, or to rename the variables in (3.18) and analytically continue the result to the original domain.

### 3.6 Combinatorial identity for the kernels $Y(\tilde{t}; \tilde{s})$ and $Z(\tilde{t}; \tilde{s})$

The opposite current Borel subalgebra $U_E$ (see Sec. 2.3) also admits a decomposition into a product of its intersections with Borel subalgebras,

\[ U^+_e = U_E \cap U_q(b_+) = U_E \cap U_q(n_+), \quad U^-_E = U_E \cap U_q(b_-), \]

such that the relations

\[ \tilde{P}^+(e_1 e_2) = e_1 \varepsilon(e_2), \quad \tilde{P}(e_1 e_2) = \tilde{P}^-(e_1 e_2) = \varepsilon(e_1) e_2, \] (3.21)

where $e_1 \in U^+_e$ and $e_2 \in U^-_E$, define projection operators $\tilde{P}^\pm$ that map $U_E$ to their subalgebras $U^-_E$ and $U^+_E$ and have properties analogous to the properties of the projection operator $P^\pm$.

In particular, the projection $\tilde{P} \left( e_0(s_1) \cdots e_0(s_b) \right)$ admits an integral representation with the factored kernel $Z(\tilde{w}; \tilde{t})$ (see (3.9)):

\[ \tilde{P} \left( e_0(s_1) \cdots e_0(s_a) \right) = \prod_{i < j} \frac{s_i - s_j}{q^{-1} s_i - q s_j} \int \cdots \int Z(\tilde{w}; \tilde{t}) e_0(w_1) \frac{dw_1}{w_1} \cdots e_0(w_a) \frac{dw_a}{w_a}. \] (3.22)

We let the symbol $\tilde{\text{Sym}}_n g(s_1, \ldots, s_n)$ denote $q^{-1}$-symmetrization of a function $g(s_1, \ldots, s_n)$:

\[ \tilde{\text{Sym}}_n g(s_1, \ldots, s_n) = \sum_{\nu \in S_n} \prod_{i < j} \frac{q s_i - q^{-1} s_j}{q^{-1} s_i - q s_j} g(s_{\nu(1)}, \ldots, s_{\nu(n)}) = \prod_{i < j} \frac{q s_i - q^{-1} s_j}{q^{-1} s_i - q s_j} \tilde{\text{Sym}}_n g(s_n, \ldots, s_1). \] (3.23)
The current Borel subalgebras $U_F$ and $U_E$ are Hopf dual with respect to the Hopf structure $\Delta^{(D)}$. Let $\langle , \rangle : U_E \otimes U_F \to \mathbb{C}$ be the corresponding Hopf pairing. It has the properties $\langle ab, x \rangle = \langle b \otimes a, \Delta^{(D)}(x) \rangle$ and $\langle a, xy \rangle = \langle \Delta^{(D)}(a), x \otimes y \rangle$; for the algebra $U_q(\mathfrak{sl}_2)$, it is given by

$$\langle e_\alpha(s_1) \cdots e_\alpha(s_n), f_\alpha(t_1) \cdots f_\alpha(t_n) \rangle = (q^{-1} - q)^{-n} \text{Sym}_n \left( \prod_{k=1}^n \delta \left( \frac{s_k}{t_k} \right) \right). \tag{3.24}$$

We note that for obvious reasons, the right-hand side of equality (3.24) can be rewritten as a $q^{-1}$-symmetrization in the variables $s_1, \ldots, s_n$ in the domain $|s_1| \ll |s_2| \ll \ldots \ll |s_n|$.

**Proposition 3.4** The operators $P^\pm$ and $\tilde{P}^\pm$ are adjoint with respect to the Hopf pairing $\langle , \rangle$ for any $f \in U_F$ and $e \in U_E$ we have

$$\langle e, P^+(f) \rangle = \langle \tilde{P}^-(e), f \rangle, \quad \langle e, P^-(f) \rangle = \langle \tilde{P}^+(e), f \rangle.$$  

**Proof.** We let $\tilde{R} \in U_E \otimes U_F$ denote the tensor of Hopf pairing (3.24). In the notation in [8], $\tilde{R}$ coincides with $(\mathcal{R}^{(2)})^{-1}$. It was established in Sec. 4.2 in that paper that the two pairs $(U_F^-, U_F^+)$ and $(U_E^+, U_E^-)$ of subalgebras of current Borel algebras form a biorthogonal decomposition of a quantum affine algebra (see Sec. 4.1 in [8], for the definition). This implies that the tensor $R \in U_E \otimes U_F$ of the Hopf pairing admits a decomposition $R = R_1 R_2$, where

$$R_1 = (1 \otimes P^-)\tilde{R} = (\tilde{P}^+ \otimes 1)\tilde{R}, \quad R_2 = (1 \otimes P^+)\tilde{R} = (\tilde{P}^- \otimes 1)\tilde{R},$$

such that $e \in U_E$, $f \in U_F$ and the equalities

$$\langle e, P^+(f) \rangle = \langle \tilde{P}^-(e), f \rangle = \langle R_2, f \otimes e \rangle, \quad \langle e, P^-(f) \rangle = \langle \tilde{P}^+(e), f \rangle = \langle R_1, f \otimes e \rangle \tag{3.25}$$

hold. \qed

**Proposition 3.5**

(i) For any sets of variables $\bar{t} = \{t_1, \ldots, t_n\}$ and $\bar{s} = \{s_1, \ldots, s_n\}$, we have the equalities

$$\langle e_\alpha(s_1) \cdots e_\alpha(s_n), P(f_\alpha(t_1) \cdots f_\alpha(t_n)) \rangle = (q^{-1} - q)^{-n} \prod_{i<j} \frac{t_i - t_j}{q t_i - q^{-1} t_j} \text{Sym}_n Z(\bar{t}, \bar{s}), \tag{3.26}$$

$$\langle \tilde{P}(e_\alpha(s_1) \cdots e_\alpha(s_n)), f_\alpha(t_1) \cdots f_\alpha(t_n) \rangle = (q^{-1} - q)^{-n} \prod_{i<j} \frac{s_i - s_j}{q^{-1} s_i - q s_j} \text{Sym}_n Z(\bar{t}, \bar{s}). \tag{3.27}$$

(ii) For any two permutations $\sigma, \sigma' \in S_n$, we have the following identity in the ring of functions, symmetric with respect to both sets of variables $\bar{t}$ and $\bar{s}$:

$$\prod_{i<j} \frac{q t_i - q^{-1} t_j}{t_i - t_j} \text{Sym}_n Z(\bar{t}, \sigma \bar{s}) = \prod_{i<j} \frac{q s_i - q^{-1} s_j}{s_i - s_j} \text{Sym}_n Z(\sigma' \bar{t}, s), \tag{3.28}$$

$$\prod_{i<j} \frac{q t_i - q^{-1} t_j}{t_i - t_j} \text{Sym}_n Y(\bar{t}, \sigma \bar{s}) = \prod_{i<j} \frac{q s_i - q^{-1} s_j}{s_i - s_j} \text{Sym}_n Y(\sigma' \bar{t}, s). \tag{3.29}$$
Proof. Statement (i) can be obtained by substituting integral presentations \([3.19]\) and \([3.22]\) in the corresponding Hopf pairings. From (i), after the \(q^{-1}\) - symmetrization is replaced with \(q\)-symmetrization according to \([3.23]\), we obtain (ii) for \(\sigma = \sigma' = 1\) (we recall that the functions \(Z(\bar{t}; s)\) and \(Y(\bar{t}; \bar{s})\) differ by a simple factor, symmetric with respect to both set of variables; see \([3.9]\)). Next, both sides of equality \([3.26]\) are \(q\)-symmetric with respect to the variables \(t\). Because the product \(\prod_{i<j} \frac{t_i - t_j}{q_{i,j} - q_{j,i}}\) is also \(q\)-symmetric, the remaining factor \(\hat{\text{Sym}}_n^\sigma Z(\bar{t}; \bar{s})\) is symmetric with respect to the variables \(t\), and for any \(\sigma \in S_n\), we hence have

\[
\hat{\text{Sym}}_n^\sigma Z(\bar{t}; \bar{s}) = \hat{\text{Sym}}_n^\sigma Z(\bar{t}; \bar{s}) \quad \hat{\text{Sym}}_n^\sigma Z(\bar{t}; \bar{s}) = \hat{\text{Sym}}_n^\sigma Z(\bar{t}; \bar{s}),
\]

(3.30)

which implies the statement (ii). \(\square\)

Identity \([3.31]\) has several proofs via direct calculations. Our proof is based on interpreting both sides of this identity as specific matrix elements of a \(U_q(\frak{sl}_2)\) weight function. In what follows, we use identity \([3.29]\) in the form

\[
\prod_{i<j} \frac{qt_i - q^{-1}s_j}{t_i - t_j} \text{Sym}^\sigma_\alpha Y(t_1, \ldots, t_n; s_n, \ldots, s_1) = \prod_{i<j} \frac{qs_i - q^{-1}s_j}{s_i - s_j} \text{Sym}^\sigma_\alpha Y(t_1, \ldots, t_n; s_1, \ldots, s_n). \quad (3.31)
\]

We let \(Z(\bar{t}, \bar{s})\) denote the left- or the right-hand side of identity \([3.31]\) divided by the product \(\prod_{i=1}^n t_i\). As follows from the above considerations, this function \(Z(\bar{t}, \bar{s})\) is symmetric with respect to both sets of variables \(\bar{t}\) and \(\bar{s}\) and has a “physical” meaning. It coincides with the partition function of the complete inhomogeneous six-vertex model on a square lattice with domain-wall boundary conditions. As pointed out to us by N. Slavnov, this function has the determinant representation

\[
Z(\bar{t}, \bar{s}) = \frac{\prod_{i,j=1}^n (qt_i - q^{-1}s_j)}{\prod_{i<j} (t_i - t_j)(s_j - s_i)} \det \left| \frac{1}{(t_i - s_j)(qt_i - q^{-1}s_j)} \right|_{i,j=1,\ldots,n}. \quad (3.32)
\]

4 Analytic properties of strings

4.1 Properties of the current \(f_{\alpha+\beta}(z)\)

The current \(f_{\alpha+\beta}(z)\) is defined in Sec. 2.5 by relation \([2.20]\). We first note that because of relations \([3.6]\), it admits the analytic representation

\[
f_{\alpha+\beta}(z) = \text{res}_{w=z} f_\alpha(w)f_\beta(q^{-1}z) \frac{dw}{w} = (q - q^{-1})f_\beta(q^{-1}z)f_\alpha(z) \quad (4.1)
\]

in addition to \([2.20]\). We can also obtain the current \(f_{\alpha+\beta}(z)\) as a result of the adjoint action related to the comultiplication \(\Delta^{(D)}\). We define the left and right adjoint actions with respect to the coalgebra structure \(\Delta^{(D)}\):

\[
\text{ad}^{(D)}_x(y) = \sum_j a^{(D)}(x'_j) \cdot y \cdot x''_j, \quad \text{ad}^{(D)}_x(y) = \sum_j x'_j \cdot y \cdot (a^{(D)})^{-1}(x''_j) \quad (4.2)
\]

if \(\Delta^{(D)}(x) = \sum_j x'_j \otimes x''_j\). We call them the current adjoint actions. We have

\[
\text{ad}^{(D)}_{f_\alpha(z)}(y) = yf_\alpha(z) - f_\alpha(z)(\psi^+_\alpha(z))^{-1}y\psi^+_\alpha(z),
\]

\[
\text{ad}^{(D)}_{f_\beta(z)}(y) = f_\beta(z)y - \psi^+_\beta(z)y\psi^+_\beta(z)^{-1}f_\beta(z). \quad (4.3)
\]
Proposition 4.1 We have the equalities
\[ \text{ad}^{(D)}_{f_\alpha(w)}(f_\alpha(z)) = \delta(z^{-1}w)f_{\alpha+\beta}(z), \quad \text{ad}^{(\hat{D})}_{f_\alpha(w)}(f_\beta(z)) = \delta(qz/w)f_{\alpha+\beta}(qz), \quad (4.4) \]
and hence
\[ f_{\alpha+\beta}(z) = \oint \frac{dw}{w} \text{ad}^{(D)}_{f_\beta(w)}(f_\alpha(z)) = \oint \frac{dw}{w} \text{ad}^{(\hat{D})}_{f_\alpha(w)}(f_\beta(q^{-1}z)). \quad (4.5) \]

Proposition 4.2 [4] The following relations hold in \( U_q(\hat{sl}_3) \):
\[ f_\alpha(z)f_{\alpha+\beta}(w) = \frac{q^{-1}z - qw}{z - w} f_{\alpha+\beta}(w)f_\alpha(z), \]
\[ f_{\alpha+\beta}(qz)f_\beta(w) = \frac{q^{-1}z - qw}{z - w} f_\beta(w)f_{\alpha+\beta}(qz), \]
\[ \frac{qz - q^{-1}w}{z - w} f_{\alpha+\beta}(z)f_{\alpha+\beta}(w) = \frac{zq^{-1} - qw}{z - w} f_{\alpha+\beta}(w)f_{\alpha+\beta}(z). \quad (4.6) \]

We note that in the analytic language, both sides of all relations (4.6) are analytic functions in \((C^*)^2\). This means, for instance, that the product \( f_\alpha(z)f_{\alpha+\beta}(w) \) has no zeroes and poles, while the product \( f_{\alpha+\beta}(w)f_{\alpha}(z) \) has a simple zero at \( z = w \) and a simple pole at \( z = q^2w \).

Proof. The proof combines relations (4.6) and the Serre relations in the analytic form [3]. Namely, in the algebra \( U_q(\hat{sl}_3) \),

(i) the products \( f_\alpha(z)f_\alpha(w) \) and \( f_\beta(z)f_\beta(w) \) have a simple zero at \( z = w \), and

(ii) the products \((z_1 - qz_2)(z_2 - qz_3)(z_1 - q^{-2}z_3)\) vanish on the lines \( z_2 = qz_1 = q^{-1}z_3 \) and \( z_2 = q^{-2}z_1 = qz_3 \).

The properties of products of currents given in Proposition 4.2 admit a straightforward generalization to strings.

Proposition 4.3

(i) Strings (2.2b) only have simple poles on the hyperplanes \( u_i = q^{-2}u_j \) and simple zeros on the hyperplanes \( u_i = u_j \), where either \( 1 \leq i < j \leq a \) or \( a + 1 \leq i < j \leq a + b \).

(ii) Opposite strings (2.2b) only have simple poles on the hyperplanes \( u_i = q^2u_j \) and simple zeros on the hyperplanes \( u_i = u_j \) for all pairs \( i < j \).

(iii) The strings and opposite strings are related. In particular,
\[ f_\alpha(t_{k+1}) \cdots f_\alpha(t_0)f_{\alpha+\beta}(t_1) \cdots f_{\alpha+\beta}(t_k) = \prod_{i,j: 1 \leq i \leq k < j \leq a} \frac{q^{-1}t_i - qt_j}{t_i - t_j} f_{\alpha+\beta}(t_1) \cdots f_{\alpha+\beta}(t_k)f_{\alpha}(t_{k+1}) \cdots f_{\alpha}(t_n). \quad (4.7) \]

4.2 Screening operators and projections of \( f_{\alpha+\beta}(z) \)

Let \( \tilde{S}_i \) denote the screening operators \( \tilde{S}_i = \text{ad}_{f_i[0]} \) with respect to the adjoint action \( \text{ad} \) in \( U_q(\hat{sl}_3) : \text{ad}_x(y) = \sum_i x'_i \cdot y \cdot a(x'_i) \), where \( \Delta(x) = \sum_i x'_i \otimes x''_i \), such that
\[ \tilde{S}_i(y) = f_i[0]y - k_i^{-1}y k_i f_i[0]. \quad (4.8) \]

We have the following relations between screening and projection operators.
Proposition 4.4

(i) For any \( y \in U_F \) and \( i = \alpha, \beta \), we have equalities

\[
P^+ \left( \int \frac{dw}{w} \, \text{ad}^{(D)}_{f_i(w)}(y) \right) = S_i \left( P^+(y) \right), \quad P^- \left( \int \frac{dw}{w} \, \tilde{\text{ad}}^{(D)}_{f_i(w)}(y) \right) = \tilde{S}_i \left( P^-(y) \right).
\]

(ii) The screenings operators \( S_i \) and \( \tilde{S}_i \) are related as

\[
\tilde{S}_i(y) = -q^{-2}k_i^{-1}S_i(y)k_i.
\]

(iii) The screening operators \( S_i \) and \( \tilde{S}_i \) commute with projections \( P^\pm \): for any \( y \in U_F \),

\[
P^\pm S_i(y) = S_i P^\pm(y), \quad P^\pm \tilde{S}_i(y) = \tilde{S}_i P^\pm(y).
\]

Proof. Statements (i) and (ii) are obvious. We prove the equality \( PS_{f_i[0]}(y) = S_{f_i[0]} P(y) \), where \( y = y_1y_2 \) and \( y_1 \in U_F^- \) and \( y_2 \in U_F^+ \). The adjoint action has the property \( \text{ad}_\varepsilon(y_1 \cdot y_2) = \sum \text{ad}_{x_i}(y_1) \cdot \text{ad}_{x_i'}(y_2) \), if \( \Delta(x) = \sum x_i' \otimes x_i'' \). This implies the relation

\[
S_i(y) = S_i(y_1)k_iy_2k_i^{-1} + y_1S_i(y_2),
\]

\[
PS_i(y) = \varepsilon(S_i(y_1))k_iy_2k_i^{-1} + \varepsilon(y_1)S_i(y_2) = S_i P(y)
\]

because the screening operator \( S_i \) preserves the subalgebras \( U_F^+ \) and \( U_F^- \) and \( \varepsilon(S_i(y)) = 0 \) for any \( y \in U_f \) except \( y = 1 \).

The properties of the screenings operators and of the adjoint actions allow calculating the projections of the current \( f_{\alpha+\beta}(z) \) and establishing the corresponding normally ordered decomposition \( \ref{eq:2.1.1} \) for it.

Proposition 4.5

(i) The projections of the current \( f_{\alpha+\beta}(z) \) are

\[
P^+ \left( f_{\alpha+\beta}(z) \right) = S_\beta \left( f_\alpha^+(z) \right), \quad P^- \left( f_{\alpha+\beta}(z) \right) = -\tilde{S}_\alpha \left( f_\beta^-(q^{-1}z) \right).
\]

(ii) We have the normally ordered expansion

\[
f_{\alpha+\beta}(z) - P \left( f_{\alpha+\beta}(z) \right) = (q^{-1} - q) \left( f_\beta^-(q^{-1}z) f_\alpha(z) \right)^+ - f_{\alpha+\beta}(z).
\]

Proof. Statement (i) follows from Proposition \( \ref{prop:4.4} \) and relation \( \ref{eq:4.5} \). Next, formal integral \( \ref{eq:2.1.9} \) can be written as

\[
f_{\alpha+\beta}(w) = f_\alpha(w)f_\beta[0] - q^{-1}f_\beta[0]f_\alpha(w) - (q^{-1} - q) \sum_{k > 0} f_\beta[-k]f_\alpha(w) \left( q^{-1}w \right)^k =
\]

\[
f_\alpha(w)f_\beta[0] - qf_\beta[0]f_\alpha(w) - (q^{-1} - q) \sum_{k > 0} f_\beta[-k]f_\alpha(w) \left( q^{-1}w \right)^k =
\]

\[
S_\beta \left( f_\alpha(w) \right) + (q^{-1} - q)f_\beta^-(q^{-1}w)f_\alpha(w).
\]

Applying the operation \( \oint \frac{dw}{z(w)} \) see \( \ref{eq:3.1.1} \) to both sides of \( \ref{eq:4.14} \) proves Statement (ii). \( \square \)
4.3 Proof of Theorem 4.3

We first calculate the projection of the opposite string \( P(f_{a+\beta}(t_1) \cdots f_{a+\beta}(t_k) f_a(t_{k+1}) \cdots f_a(t_a)) \).

**Proposition 4.6**

(i) The projection of the string \( f_{a+\beta}(t_1) \cdots f_{a+\beta}(t_k) f_a(t_{k+1}) \cdots f_a(t_a) \) with \( a > k \) can be written as

\[
P(f_{a+\beta}(t_1) \cdots f_{a+\beta}(t_k) f_a(t_{k+1}) \cdots f_a(t_a)) = P(f_{a+\beta}(t_1) \cdots f_{a+\beta}(t_k) f_a(t_{k+1}) \cdots f_a(t_a-1)) f^+_a(t_a) + \sum_{j=1}^{a-1} X_j(t_1, \ldots, t_a-1) \frac{a-1}{t_a-q^2 t_j}.
\]

(ii) The projection of the string \( f_{a+\beta}(t_1) \cdots f_{a+\beta}(t_k) \) can be represented as

\[
P(f_{a+\beta}(t_1) \cdots f_{a+\beta}(t_k)) = P(f_{a+\beta}(t_1) \cdots f_{a+\beta}(t_{k-1})) P(f_a(t_k)) + \sum_{j=1}^{k-1} X_j(t_1, \ldots, t_{k-1}) \frac{k-1}{t_k-q^2 t_j}.
\]

**Proof.** Relation (4.16) is proved based on inductively using the following lemma, which shows that as the current \( f^\alpha_z(t_a) \) is moved to the left of the string, only simple poles at the points \( t_a = q^2 t_j \) appear and the corresponding operator-valued coefficient \( X_j \) at \( (t_a-q^2 t_j)^{-1} \) is therefore independent of \( t_a \).

**Lemma 4.7** The following relations hold:

\[
f_a(z) f^\alpha_z(w) = \frac{q^{-1}z - qw}{qz - q^{-1}w} f^\alpha_z(w) f_a(z) + \frac{z(q - q^{-1})}{qz - q^{-1}w} (1 + q^2) f^+_a(q^2 z) f_a(z),
\]

\[
f_{a+\beta}(z) f^\alpha_z(w) = \frac{z - w}{qz - q^{-1}w} f^\alpha_z(w) f_{a+\beta}(z) + \frac{z(q - q^{-1})}{qz - q^{-1}w} f_{a+\beta}(z) f^\alpha_z(z).
\]

**Proof.** These relations follow from applying the integral transformation \(- \oint \frac{du}{a+\beta} \frac{1}{1-w/u}\) to the relations

\[
f_a(z) f_a(u) = \frac{q^{-1}z - qu}{qz - q^{-1}u} f_a(u) f_a(z) + (q^2 - q^2) \frac{\delta(q^2 z/u)}{u} f_a(q^2 z) f_a(z),
\]

\[
f_a(u) f_{a+\beta}(z) = \frac{q^{-1}u - qz}{u - z} f_{a+\beta}(z) f_a(u).
\]

To prove Statement (ii), we use relation (4.14) and the normally ordered expansion

\[
f_{a+\beta}(z_k) = P(f_{a+\beta}(z_k)) + (q^{-1} - q) (f^\beta_{-}(q^{-1} z_k) f_a(z_k))^+ - f^\alpha_{a+\beta}(z_k)
\]

substituted in the left-hand side of relation (4.17). We then inductively move the currents \( f^\beta_{-}(q^{-1} z_k) \) and \( f^\alpha_{a+\beta}(z_k) \) to the left, using relations (4.14) and observing that \( P(f^\beta_{-}(z_k) \cdot F) = P(f^\alpha_{a+\beta}(z_k) \cdot F) = 0 \) for any element \( F \in U_F \).

We now use Statement (ii) in Proposition 4.3. It states that the string \( f_{a+\beta}(t_1) \cdots f_{a+\beta}(t_k) f_a(t_{k+1}) \cdots f_a(t_a) \) has simple zeroes at hyperplanes \( t_a = t_i, i = 1, \ldots, a - 1 \). Substituting these
conditions in (4.16) and (4.17) gives a systems of $a - 1$ linear equations over the field of rational functions $\mathbb{C}(t_1, \ldots, t_{a-1})$ for the operators $X_j(t_1, \ldots, t_{a-1})$:

$$\sum_{j=1}^{a-1} \frac{X_j(t_1, \ldots, t_{a-1})}{t_i - q^2 t_j} = X \cdot f^+_\alpha(t_i), \quad i = 1, \ldots, a - 1,$$

(4.21)

where $X = P(f_{\alpha+\beta}(t_1) \cdots f_{\alpha+\beta}(t_k) f_{\alpha}(t_{k+1}) \cdots f_{\alpha}(t_{a-1}))$. The determinant of the matrix $B_{i,j} = (t_i - q^2 t_j)^{-1}$ of this system is nonzero in $\mathbb{C}(t_1, \ldots, t_{a-1})$,

$$\det(B) = (-q^2)^{a-1} \prod_{i \neq j} (t_i - t_j)^2 \prod_{i,j} (t_i - q^2 t_j),$$

and the system hence has a unique solution over $\mathbb{C}(t_1, \ldots, t_{a-1})$. This implies that the operators $X_j$ are linear combinations over $\mathbb{C}(t_1, \ldots, t_{a-1})$ of the operators $X \cdot f^+_\alpha(t_j), \; j = 1, \ldots, a - 1$, and the projection of the string can therefore be represented as

$$P(f_{\alpha+\beta}(t_1) \cdots f_{\alpha+\beta}(t_k) f_{\alpha}(t_{k+1}) \cdots f_{\alpha}(t_{a-1})) = X \cdot f^+_\alpha(t_a) - \sum_{j=1}^{a-1} \varphi_{t_j}(t_a; t_1, \ldots, t_{a-1})X \cdot f^+_\alpha(t_j),$$

(4.22)

where $\varphi_{t_j}(t_a; t_1, \ldots, t_{a-1}) = A_j(t_a; t_1, \ldots, t_{a-1})/\prod_{m=1}^{a-1} (t_a - q^2 t_m)$ are rational functions whose denominators are polynomials in $t_a$ of degree less then $a - 1$. System (4.21) is satisfied if the rational functions $\varphi_{t_j}(t_a; t_1, \ldots, t_{a-1})$ have the property

$$\varphi_{t_j}(t_i; t_1, \ldots, t_{a-1}) = \delta_{i,j}, \quad i, j = 1, \ldots, a - 1.$$

This interpolation problem has a unique solution given by formula (3.14).

Relation (4.22) now appears as a recursive relation between projections of strings of different lengths. The corresponding relation for the strings $f_{\alpha+\beta}(t_1) \cdots f_{\alpha+\beta}(t_k)$ looks the same,

$$P(f_{\alpha+\beta}(t_1) \cdots f_{\alpha+\beta}(t_a)) = X' \cdot P(f_{\alpha+\beta}(t_a)) + \sum_{j=1}^{a-1} \varphi_{t_j}(t_a; t_1, \ldots, t_{a-1})X' \cdot P(f_{\alpha+\beta}(t_j)),$$

where $X' = P(f_{\alpha+\beta}(t_1) \cdots f_{\alpha+\beta}(t_{a-1}))$ and the rational functions $\varphi_{t_j}(t_a; t_1, \ldots, t_{a-1})$ are given by relation (3.14). Successively applying the recursive relations and relation (4.17) gives the statement of Theorem 2.

5 Current adjoint action and symmetrization

5.1 Projections and analytic continuation

We recall that in the completed algebra $\overline{U_q(\mathfrak{sl}_3)}$, any product of currents $f_{i(1)}(t_1) \cdots f_{i(n)}(t_n)$ can be considered an analytic function in a domain $|t_1| \gg |t_2| \gg \cdots \gg |t_n|$, admitting an analytic continuation to a meromorphic function in $(\mathbb{C}^*)^n$. Because of commutation relations (2.6), the analytic continuation of this product to the domain $|t_{\sigma(1)}| \gg |t_{\sigma(2)}| \gg \cdots \gg |t_{\sigma(n)}|$ for any $\sigma \in S_n$ is given by the series

$$\prod_{l > k} \frac{q^{(\sigma(k)), \, \sigma(l))} - \frac{t_{\sigma(l)}}{t_{\sigma(k)}}}{1 - q^{(\sigma(k)), \, \sigma(l))} \frac{t_{\sigma(l)}}{t_{\sigma(k)}} f_{i(\sigma(1))}(t_{\sigma(1)}) \cdots f_{i(\sigma(n))}(t_{\sigma(n)}).$$

(5.1)
Let $\sigma P\left(f_{i(1)}(t_1) \cdots f_{i(n)}(t_n)\right)$ denote the analytic continuation of the projection of the current product $P\left(f_{i(\sigma(1))}(t_{\sigma(1)}) \cdots f_{i(\sigma(n))}(t_{\sigma(n)})\right)$ from the domain $|t_{\sigma(1)}| \gg |t_{\sigma(2)}| \gg \cdots \gg |t_{\sigma(n)}|$ to the domain $|t_1| \gg |t_2| \gg \cdots \gg |t_n|$.

**Proposition 5.1**

(i) The projections $P^\pm$ commute with the analytic continuation.

(ii) We have the identity of formal series in $U\{t_1, ..., t_n\}$ (see (5.3))

$$\sigma P^\pm\left(f_{i(1)}(t_1) \cdots f_{i(n)}(t_n)\right) = \prod_{\sigma^{-1}(k) > \sigma^{-1}(l)} \frac{q^{(l(k)), (l(l))} - t_l/t_k}{1 - q^{(l(k)), (l(l))} t_l/t_k} P^\pm\left(f_{i(1)}(t_1) \cdots f_{i(n)}(t_n)\right). \tag{5.2}$$

**Proof.** Statement (i) is based on the fact that the projection operator preserve the normal ordering in the algebra $U_F$ (with respect to the action in the category of highest-weight representations); in other words, it is continuous in the topology that defines the completion $\overline{U}_F$. Statement (ii) follows from (i) and (5.1).

This proposition provides a powerful tool for the computing weight functions. The crucial point in its application is that in contrast to a product of currents, a projection of a current product admits a simple analytic continuation, equivalent to the analytic continuation of rational functions.

**Example.** We consider $P\left(f_\alpha(t_1)f_\alpha(t_2)\right)$. We have (see Sec. 3.4)

$$P\left(f_\alpha(t_1)f_\alpha(t_2)\right) = f_\alpha^+(t_1)f_\alpha^+(t_2) - \frac{(q - q^{-1})t_1}{qt_1 - q^{-1}t_2} \left(f_\alpha^+(t_1)\right)^2, \tag{5.3}$$

$$P\left(f_\alpha(t_2)f_\alpha(t_1)\right) = f_\alpha^+(t_2)f_\alpha^+(t_1) - \frac{(q - q^{-1})t_2}{qt_2 - q^{-1}t_1} \left(f_\alpha^+(t_2)\right)^2. \tag{5.4}$$

Equality (5.3) is an equality of formal series in the domain $|t_1| \gg |t_2|$, which means that the rational function $t_1/(qt_1 - q^{-1}t_2)$ is expanded in a power series in $t_2/t_1$; equality (5.4) is an equality of formal series in the domain $|t_2| \gg |t_1|$, which means that the rational function $t_2/(qt_2 - q^{-1}t_1)$ is expanded in a power series in $t_1/t_2$. The analytic continuation to the domain $|t_1| \gg |t_2|$ in the right-hand side of (5.4) amounts to the analytic continuation of the rational function $t_2/(qt_2 - q^{-1}t_1)$, which should now be expanded in a power series in $t_2/t_1$. Therefore, the equality

$$(12) \quad P\left(f_\alpha(t_1)f_\alpha(t_2)\right) = \frac{q^2 - t_2/t_1}{1 - q^2t_2/t_1} P\left(f_\alpha(t_1)f_\alpha(t_2)\right)$$

implies a relation between formal series in the domain $|t_2| \gg |t_1|,

$$f_\alpha^+(t_2)f_\alpha^+(t_1) - \frac{(q - q^{-1})t_2}{qt_2 - q^{-1}t_1} \left(f_\alpha^+(t_2)\right)^2 = \frac{q^2 - t_2/t_1}{q^2t_2/t_1} f_\alpha^+(t_1)f_\alpha^+(t_2) - \frac{(q - q^{-1})t_1}{q^{-1}t_1 - qt_2} \left(f_\alpha^+(t_1)\right)^2. \tag{5.5}$$

This is one of the basic relations in the Borel subalgebra of $U_q(\mathfrak{sl}_2)$. It also holds in the domain $|t_1| \gg |t_2|$ and can be generalized to a multiple product (see the corollary to Theorem 3).

According to our definition of $q$-symmetrization (see Sec. 3.2), the statements in Proposition 5.1 and the rule of analytic continuation of current product (5.1), projections of current products with the same simple-root index, as well as the current products themselves, are $q$-symmetric:

$$f_\alpha(t_1) \cdots f_\alpha(t_n) = \frac{1}{n!} \text{Sym}^n f_\alpha(t_1) \cdots f_\alpha(t_n),$$

$$P^\pm\left(f_\alpha(t_1) \cdots f_\alpha(t_n)\right) = \frac{1}{n!} \text{Sym}^n P^\pm\left(f_\alpha(t_1) \cdots f_\alpha(t_n)\right). \tag{5.6}$$
We now use these arguments to write a symmetrized version of canonical decomposition (2.18) of a product of currents.

**Proposition 5.2** There is an equality of formal series in the domain \(|t_1| \gg |t_2| \gg \ldots \gg |s_b|\):

\[
f_α(t_1) \cdots f_α(t_a)f_β(s_1) \cdots f_β(s_b) =
\sum_{0 \leq m \leq a, 0 \leq k \leq b} \frac{1}{m!(a-m)!(b-k)!} \sum_{s} \left( \prod_{m+1 \leq \ell \leq a} \prod_{1 \leq \ell \leq k} \frac{qt_\ell - sq_\ell}{t_\ell - q_\ell} \times P^- (f_α(t_1) \cdots f_α(t_m)f_β(s_1) \cdots f_β(s_k)) \cdot P^+ (f_α(t_{m+1}) \cdots f_α(t_a)f_β(s_{k+1}) \cdots f_β(s_b)) \right).
\]

(5.7)

In particular, for currents of the same type, we have

\[
f_β(s_1) \cdots f_β(s_b) = \sum_{0 \leq k \leq b} \frac{1}{k!(b-k)!} \sum_{s} \left( P^- (f_β(s_1) \cdots f_β(s_k)) \cdot P^+ (f_β(s_{k+1}) \cdots f_β(t_b)) \right).
\]

(5.8)

**Proof.** Directly applying expansion (2.18) for the product of currents for formal series in the domain \(|t_1| \gg \cdots \gg |t_n|\) gives the relation

\[
f_\alpha(t_1) \cdots f_\alpha(t_n)(t_n) = \sum_{J \subseteq I} \prod_{\ell \in J, \ell' \in J^c} \frac{t_\ell - q^{(\ell,\ell')}(t')}{q^{(\ell,\ell')}(t_\ell - t_{\ell'})} \times P^- (f_\alpha(t_{j_1}) \cdots f_\alpha(t_{j_k})(t_{j_k})) \cdot P^+ (f_\alpha(t_{j_1'}) \cdots f_\alpha(t_{j_{k'}})(t_{j_{k'}}))
\]

(5.9)

where the set \(I = \{1, \ldots, n\}\), its subset \(J = \{j_1, \ldots, j_k\}\), where \(j_1 < \cdots < j_k\), and \(I \setminus J = \{j_1', \ldots, j_{k'}\}\), where \(j_1' < \cdots < j_{k'}\). Applying the symmetrization procedure based on Proposition 5.1 gives (5.7). \(\square\)

### 5.2 Current adjoint action

Serre relations (2.10) and (2.11) admit different representations. In Sec. 4.11 we represented them as properties of composite currents and strings. Here, we reformulate the Serre relations via the current adjoint action (see (1.2)).

**Lemma 5.3** Serre relations (2.11) for \(i = \beta\) and \(j = \alpha\) can be written as

\[
ad_{f_\beta(s_1)f_\beta(s_2)}^{(D)}(f_\alpha(t)) = 0.
\]

(5.10)

**Proof.** The statement follows from the chain of equalities

\[
ad_{f_\beta(s_1)f_\beta(s_2)}^{(D)}(f_\alpha(t)) = \ad_{f_\beta(s_2)}^{(D)}\left(\ad_{f_\beta(s_1)}^{(D)}(f_\alpha(t))\right) = \ad_{f_\beta(s_2)}^{(D)}(f_{\alpha+\beta}(t)) \delta(t/qs_1) = \left(f_{\alpha+\beta}(t)f_\beta(s_2) - f_\beta(s_2)\psi_\beta^+(s_2)^{-1}f_{\alpha+\beta}(t)\psi_\beta^+(s_2)\right) \delta(t/qs_1) = \left(f_{\alpha+\beta}(t)f_\beta(s_2) - f_\beta(s_2)f_{\alpha+\beta}(t)\frac{t - q^2s_2}{qt - q^2s_2}\right) \delta(t/qs_1) = 0.
\]

This lemma implies the following proposition.
Proposition 5.4 For \( a \geq k \), the identity of formal series

\[
P \left( \text{ad}^{(D)}_{f_{\beta(u_1)} \cdots f_{\beta(u_k)}} (f_{\alpha(t_1)} \cdots f_{\alpha(t_a)}) \right) = \frac{1}{k!(a-k)!} \]

\[
\times \text{Sym}_t \left( \left( P \left( f_{\alpha(t_1)} \cdots f_{\alpha(t_{a-k})} f_{\alpha+\beta(t_{a-k+1})} \cdots f_{\alpha+\beta(t_a)} \right) \right) \times \prod_{i<j}^{k} q^{-1} \frac{t_i - q t_j}{t_i - t_j} \text{Sym}_t \left( \prod_{i=1}^{k} \delta \left( \tilde{t}_i \right) \right) \right),
\]

holds in the domain \(|t_1| \gg \cdots \gg |t_a|\), where \( \tilde{t}_i = t_{a-k+i}, \ i = 1, \ldots, k \).

Proof. The proof is a combinatorial exercise involving definition of current adjoint action \((4.12)\), its properties including \((5.10)\), and the relation

\[
\text{ad}^{(D)}_{f_{\beta(s)}} (F_1 \cdot F_2) = F_1 \cdot \text{ad}^{(D)}_{f_{\beta(s)}} (F_2) + \text{ad}^{(D)}_{f_{\beta(s)}} (F_1) \cdot \psi_{\beta}^+(s)^{-1} F_2 \psi_{\beta}^+(s).
\]

We also note the role of Proposition \(5.3\) which allows using the commutation relation between total currents under the projections without paying attention to \(\delta\)-function terms. With this taken into account and after necessary combinatorial rearrangements, we obtain

\[
P \left( \text{ad}^{(D)}_{f_{\beta(u_1)} \cdots f_{\beta(u_k)}} (f_{\alpha(t_1)} \cdots f_{\alpha(t_a)}) \right) = \frac{1}{k!(a-k)!} \text{Sym}_t \left( \prod_{i=1}^{k} \prod_{\ell=1}^{a-k+i} \frac{q t_{a-k+i} - u_\ell}{t_{a-k+i} - qu_\ell} \times \right.
\]

\[
\left. \times P \left( f_{\alpha(t_1)} \cdots f_{\alpha(t_{a-k})} \text{ad}^{(D)}_{f_{\beta(u_1)}} (f_{\alpha(t_{a-k+1})}) \cdots \text{ad}^{(D)}_{f_{\beta(u_k)}} (f_{\alpha(t_a)}) \right) \right).
\]

Because the adjoint action in the last formula produces a product of \(\delta\)-functions \(\prod_{i} \delta(t_{a-k+i}/qu_i)\), \(i = 1, \ldots, k\) (see \((4.3)\)), we can now move the rational function from under the \(q\)-symmetrization with respect to the variables \(u_i\) and replace the symmetrization with the \(q^{-1}\)-symmetrization with respect to the variables \(t_{a-k+i} = \tilde{t}_i\). Proposition \(5.4\) is thus proved. \(\square\)

5.3 Proof of Theorem \([1]\)

Our goal is to reduce an expression \(P \left( f_{\alpha(t_1)} \cdots f_{\alpha(t_a)} f_{\beta(s_1)} \cdots f_{\beta(s_b)} \right)\) to projections of strings. In this expression, we substitute decomposition \((5.8)\) for the current product \(f_{\beta(s_1)} \cdots f_{\beta(s_b)}\):

\[
P \left( f_{\alpha(t_1)} \cdots f_{\alpha(t_a)} f_{\beta(s_1)} \cdots f_{\beta(s_b)} \right) = \sum_{k=0}^{b} \frac{1}{k!(b-k)!} \times \]

\[
\times \text{Sym}_s \left( P^+ \left( f_{\alpha(t_1)} \cdots f_{\alpha(t_a)} \right) P^- \left( f_{\beta(s_1)} \cdots f_{\beta(s_k)} \right) \cdot P^+ \left( f_{\beta(s_{k+1})} \cdots f_{\beta(s_b)} \right) \right). \tag{5.12}
\]

We now use a strengthened coideal property of the subalgebra \(U_f^-\).

Proposition 5.5

(i) For any element \(F \in U_f^-\), we have

\[
\Delta^{(D)} F = 1 \otimes F + F' \otimes F'', \quad \text{such that} \quad F' \in U_f^- \quad \text{and} \quad \varepsilon(F') = 0. \tag{5.13}
\]
(ii) For any product \( f_{i(1)}(t_1) \cdots f_{i(n)}(t_n) \), we have the equality of series in \( U\{t_1, \ldots, t_n\} \) (see (5.13))

\[
P \left( f_{i(1)}(t_1) \cdots f_{i(m)}(t_m) P^- (f_{i(m+1)}(t_{m+1}) \cdots f_{i(n)}(t_n)) \right) = \]

\[
= P \left( \text{ad}^{(D)}_{P^- (f_{i(m+1)}(t_{m+1}) \cdots f_{i(n)}(t_n))} f_{i(1)}(t_1) \cdots f_{i(m)}(t_m) \right).
\]

(5.14)

Proof. It suffices to verify statement (i) for generators of the algebra \( U_f^- \), where it is a direct observation. Statement (ii) is a direct consequence of (i).

We use a particular case of (5.14),

\[
P \left( f_a(t_1) \cdots f_a(t_a) P^- (f_\beta(s_1) \cdots f_\beta(s_k)) \right) = P \left( \text{ad}^{(D)}_{P^- (f_\beta(s_1) \cdots f_\beta(s_k))} f_a(t_1) \cdots f_a(t_a) \right),
\]

and substitute an integral representation of the projection \( P^- (f_\beta(s_1) \cdots f_\beta(s_k)) \) in it:

\[
P^- (f_\beta(s_1) \cdots f_\beta(s_k)) = \prod_{i<j} \frac{s_i - s_j}{qs_i - q^{-1}s_j} \oint \cdots \oint \ Y(\bar{\omega}; \bar{s}) f_\beta(u_1) du_1 \cdots f_\beta(u_k) du_k.
\]

(5.16)

Then the right-hand side of equality (5.14) becomes

\[
\prod_{i<j} \frac{s_i - s_j}{qs_i - q^{-1}s_j} \oint \cdots \oint \prod_{i=1}^k \frac{du_i}{u_i} \ Y(\bar{\omega}; \bar{s}) \ P \left( \text{ad}^{(D)}_{f_\beta(u_1) \cdots f_\beta(u_k)} (f_a(t_1) \cdots f_a(t_a)) \right) =
\]

\[
= \frac{1}{k!(a-k)!} \text{Sym}^k \left( P \left( f_a(t_1) \cdots f_a(t_{a-k}) f_{a+\beta(t_{a-k+1})} \cdots f_{a+\beta(t_a)} \right) \right)
\]

\[
\times \prod_{i<j} \frac{s_i - s_j}{qs_i - q^{-1}s_j} \frac{q^{-1}t_i - q^{-1}t_j}{t_i - t_j} \widetilde{\text{Sym}}^k \left( \oint \cdots \oint \prod_{i=1}^k \frac{du_i}{u_i} \delta \left( \frac{t_i}{qu_i} \right) Y(\bar{\omega}; \bar{s}) \right),
\]

(5.17)

where \( \tilde{t}_i = t_{a-k+i}, \ i = 1, \ldots, k \). After the integration, the last line in (5.17) becomes

\[
\prod_{i<j} \frac{s_i - s_j}{qs_i - q^{-1}s_j} \frac{q^{-1}t_i - q^{-1}t_j}{t_i - t_j} \widetilde{\text{Sym}}^k \left( Y(q^{-1} \cdot (\bar{\omega}; \bar{s}) \right) =
\]

\[
= \prod_{i<j} \frac{s_i - s_j}{qs_i - q^{-1}s_j} \frac{q^{-1}t_i - q^{-1}t_j}{t_i - t_j} \text{Sym}^k \left( Y(q^{-1} \cdot \bar{\omega}; \bar{s}) \right) = \text{Sym}^k \left( Y(q^{-1} \cdot \bar{\omega}; \bar{s}) \right).
\]

(5.18)

In the first equality in (5.18), relation (3.23) between the \( q \)- and \( q^{-1} \)-symmetrizations is used, while the second equality is obtained from combinatorial identity (3.31).

Projection (5.15) is now given by

\[
P \left( \text{ad}^{(D)}_{P^- (f_\beta(s_1) \cdots f_\beta(s_k))} f_a(t_1) \cdots f_a(t_a) \right) = \frac{1}{k!(a-k)!} \]

\[
\times \text{Sym}^k \left( Y(q^{-1} t_{a-k+1}, \ldots, q^{-1} t_a; s_1, \ldots, s_k) \right)
\]

\[
\times P \left( f_a(t_1) \cdots f_a(t_{a-k}) f_{a+\beta}(t_{a-k+1}) \cdots f_{a+\beta}(t_a) \right).
\]

(5.19)

We now return to the proof of Theorem 11. By definition of \( q \)-symmetrization (5.16), the right-hand side of the last formula is \( q \)-symmetric with respect to the variables \( s_1, \ldots, s_k \). Additional \( q \)-symmetrization then cancels the unwanted factorial (see (5.6)), and we obtain the proof of Theorem 11.  \( \square \)
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Appendix A. The projection operator \( P^- \)

In this appendix, we collect the most important formulas for the opposite projection operator \( P^- \).

1. Just as for the projection \( P = P^+ \), an expression \( P^- (f_1(t_1) \cdots f_n(t_n)) \) is a series in the domain \(|t_1| \gg \cdots \gg |t_n|\) admitting an analytic continuation to different asymptotic zones. In the sense of analytic continuation, it has the same properties as the analogous expressions for the projections \( P^+ \).

We recall the formula for the projection \( P^+ \) of the product of currents corresponding to the same root:

\[
P^+ (f_1(t_1) \cdots f_n(t_n)) = f_1^+ (t_1)f_2^+ (t_2; t_1) \cdots f_n^+ (t_n; t_1, \ldots, t_{n-1}).
\]

The structure of this formula was explained in the proof of the Theorem 2. From analogous considerations (see the next appendix), we can find that a similar triangular decomposition holds for the projection \( P^- \) of the product of currents with the same root index,

\[
P^- (f_1(s_1) \cdots f_n(s_n)) = (-1)^{\ell} f_1^- (s_1; s_2, \ldots, s_b) f_2^- (s_2; s_3, \ldots, s_b) \cdots f_n^- (s_{b-1}; s_b) f_n^- (s_b), \tag{A.1}
\]

where the currents \( f_\beta^- (s_k; s_{k+1}, \ldots, s_b) \) are defined as sums

\[
f_\beta^- (s_k; s_{k+1}, \ldots, s_b) = f_\beta^- (s_k) - \sum_{m=k+1}^{b} \varphi_m (s_k; s_{k+1}, \ldots, s_b) f_\beta^- (s_m) \tag{A.2}
\]

and the rational functions, \( \varphi_{s_j} (s; s_1, \ldots, s_b) \)

\[
\varphi_{s_j} (s; s_1, \ldots, s_b) = \prod_{i=1, i \neq j}^{b} \frac{s - s_i}{s_j - s_i} \prod_{i=1}^{b} \frac{qs_j - q^{-1}s_i}{qs - q^{-1}s_i}, \tag{A.3}
\]

as functions of the variable \( s \) have simple poles at the points \( s = q^{-2}s_i, \ i = 1, \ldots, b \), tend to zero as \( s \to \infty \), and have the property \( \varphi_{s_j} (s_i; s_1, \ldots, s_b) = \delta_{ij} \). These conditions define the set of functions \( \{A.3\} \) uniquely.

As can the projection \( P^+ \), projection \( \{A.1\} \) for the product of currents can be written in the reverse order

\[
P^- (f_\beta (s_1) \cdots f_\beta (s_b)) = (-1)^{\ell} \prod_{1 \leq i < j \leq b} \frac{q^{-1}s_i - q s_j}{qs_i - q^{-1}s_j} f_\beta^- (s_b; s_{b-1}, \ldots, s_1) \cdots f_\beta^- (s_2; s_1) f_\beta^- (s_1). \]
Expression (A.1) can also be written as an integral transform of the product of the total currents,

\[
P^-(f_\beta(s_1) \cdots f_\beta(s_b)) = \prod_{i < j} \frac{s_i - s_j}{q s_i - q^{-1} s_j} \int \prod_{k=1}^b \frac{du_k}{u_k} Y(\omega, \overline{\omega}) f_\beta(u_1) \cdots f_\beta(u_b) = \\
= \prod_{i < j} \frac{s_i - s_j}{q s_i - q^{-1} s_j} \int \prod_{k=1}^b \frac{du_k}{u_k} \frac{1}{1 - s_k/u_k} \prod_{i=k+1}^b \frac{q - q^{-1} s_i/u_k}{1 - s_i/u_k} f_\beta(u_1) \cdots f_\beta(u_b),
\]

which was already used in the preceding section in the proof of Theorem 1.

2. Calculation the projection \( P^-(f_\alpha(t_1) \cdots f_\alpha(t_a)f_\beta(s_1) \cdots f_\beta(s_b)) \) also reduces to calculating the projections of strings,

\[
P^-(f_\alpha(t_1) \cdots f_\alpha(t_a)f_\beta(s_1) \cdots f_\beta(s_b)) =
\]

\[
= \sum_{k=0}^{\min(a,b)} \frac{1}{k!(a-k)!(b-k)!} \text{Sym}^a_t \text{Sym}^b_s \left( P^-(f_\alpha(t_1) \cdots f_\alpha(t_{a-k})) \right) \times \text{(A.4)}
\]

\[
P^-(f_\alpha+\beta(qs_1) \cdots f_\alpha+\beta(qsk)f_\beta(s_{k+1}) \cdots f_\beta(s_b)) Z(q^{-1}t_{a-k+1}, ..., q^{-1}t_{a}; s_1, ..., s_k),
\]

where the series \( Z(\overline{t}, \overline{s}) \) is defined in (3.9) and the projection of a string is given by

\[
P^-(f_\alpha+\beta(qs_1) \cdots f_\alpha+\beta(qsk)f_\beta(s_{k+1}) \cdots f_\beta(s_b)) =
\]

\[
= \prod_{1 \leq i < k+1 \leq j \leq b} \frac{qs_i - q^{-1}s_j}{s_i - s_j} \prod_{1 \leq i < j \leq b} \frac{q^{-1}s_i - q^{-1}s_j}{qs_i - q^{-1}s_j} \times P^-(f_\beta(s_{b}; s_{b-1}, ..., s_1)) \cdots P^-(f_\beta(s_{k+1}; s_k, ..., s_1)) \times P^-(f_\alpha+\beta(qs_k; qs_{k-1}, ..., qs_1)) \cdots P^-(f_\alpha+\beta(qs_1)).
\] (A.5)

In this formula, the currents \( f_\alpha(t; t_{i-1}, ..., t_1) \) for the roots \( \gamma = \beta, \alpha + \beta \) are defined by relations (A.2) with coefficient functions (A.3), which are invariant under a simultaneous scaling of all variables. Single current projections are defined by the formulas (3.13) such that \( P^-(f_\alpha(t)) = -f_\alpha^{-}(t) \), \( P^-(f_\alpha+\beta(qs)) = -(f_\alpha[0]f_\beta(s) - q^{-1}f_\beta(s)f_\alpha[0]) \), and \( P^-(f_\beta(s)) = -f_\beta^{-}(s) \). In particular, we have

\[
P^-(f_\alpha(t_1)f_\alpha(t_2)) = \left( f_\alpha^{-}(t_1) - \frac{q - q^{-1}}{qt_1 - q^{-1}t_2} f_\alpha^{-}(t_2) \right) f_\alpha^{-}(t_2) = \\
= \frac{q^{-1}t_1 - qt_2}{qt_1 - q^{-1}t_2} \left( f_\alpha^{-}(t_2) - \frac{q - q^{-1}}{qt_2 - q^{-1}t_1} f_\alpha^{-}(t_1) \right) f_\alpha^{-}(t_1),
\]

\[
P^-(f_\alpha+\beta(qs_1)f_\beta(s_2)) = -\frac{q^{-1}s_1 - q^{-1}s_2}{s_1 - s_2} \left( f_\beta^{-}(s_2) - \frac{q - q^{-1}}{qs_2 - q^{-1}s_1} f_\beta^{-}(s_1) \right) P^-(f_\alpha+\beta(qs_1)).
\]

3. The proof of formula (A.4) is based on one more reformulation of the Serre relations involving the right adjoint action \( \tilde{a}_D^-(y) \) (see [4,2]),

\[
\tilde{a}_D^-(f_\alpha(t_1)f_\alpha(t_2))(f_\beta(s)) = 0,
\] (A.6)

and on an identity analogous to the one proved in Proposition 5.4. Namely, for \( k \leq b \),

\[
P^- \left( \tilde{a}_D^-(f_\alpha(t_1) \cdots f_\alpha(t_k))(f_\beta(s_1) \cdots f_\beta(s_b)) \right) = \frac{1}{k!(b-k)!} \text{Sym}^b \text{Sym}^k \prod_{i=1}^k \prod_{\ell=i+1}^k \frac{q t_\ell - s_i}{q t_\ell - q^{-1}s_i} \times
\]

\[
\prod_{i=1}^k \prod_{\ell=i+1}^k \frac{q t_\ell - s_i}{q t_\ell - q^{-1}s_i} \times P^- \left( \tilde{a}_D^-(f_\alpha(t_1)) \cdots \tilde{a}_D^-(f_\alpha(t_k))(f_\beta(s_k))(f_\beta(s_{k+1}) \cdots f_\beta(s_b)) \right).
\] (A.7)
Appendix B. Commutation relations with projections of currents

Commutation relations (2.21) imply the rules for moving the half-currents \( f^-_\gamma(z) \) to the left and the half-currents \( f^+_\gamma(z) \) to the right through the total currents.

**Proposition 5.6** We have the equalities

\[
 f^+_\alpha(z)f^-_\beta(w) = \frac{qz - w}{z - qw} f^-_\beta(w)f^+_\alpha(z) + \frac{qw(q^{-1} - q)}{z - qw} f^-_\beta(w)f^+_\alpha(qw) + \frac{qw}{z - qw} f^-_{\alpha + \beta}(qw),
\]

(B.1)

\[
 f^-_\alpha(z)f^-_\beta(w) = \frac{qz - w}{z - qw} f^-_\beta(w)f^-_\alpha(z) + \frac{z(q^{-1} - q)}{z - qw} f^-_\beta(q^{-1}z)f^-_\alpha(z) - \frac{z}{z - qw} f^-_{\alpha + \beta}(z).
\]

(B.2)

**Proof.** The proof of relation (B.1), for example, is based on the decomposition of the kernel

\[
\frac{qu - w}{(u - qw)(z - u)} = \frac{qz - w}{z - qw} \frac{1}{z - u} + \frac{qw(q^{-1} - q)}{z - qw} \frac{1}{qw - u}
\]

into the sum of two kernels. \( \square \)

Analogously, from relations (4.6), we have the following proposition.

**Proposition 5.7** The nontrivial commutation relations between the total and half-currents are

\[
 f^+_\alpha(z)f^-_{\alpha + \beta}(w) = \frac{q^{-1}z - qw}{z - w} f^-_{\alpha + \beta}(w)f^+_\alpha(z) + \frac{(q - q^{-1})w}{z - w} f^-_{\alpha + \beta}(w)f^+_\alpha(w),
\]

(B.3)

\[
 f^-_\alpha(z)f^-_{\alpha + \beta}(w) = \frac{q^{-1}z - qw}{z - w} f^-_{\alpha + \beta}(w)f^-_\alpha(z) + \frac{(q - q^{-1})z}{z - w} f^-_{\alpha + \beta}(z)f^-_\alpha(z)
\]

and

\[
 f^+_{\alpha + \beta}(qz)f^-_\beta(w) = \frac{q^{-1}z - qw}{z - w} f^-_\beta(w)f^+_{\alpha + \beta}(qz) + \frac{(q - q^{-1})w}{z - w} f^-_\beta(w)f^+_{\alpha + \beta}(q^{p-i}w),
\]

(B.4)

\[
 f^-_{\alpha + \beta}(qz)f^-_\beta(w) = \frac{q^{-1}z - qw}{z - w} f^-_\beta(w)f^-_{\alpha + \beta}(qz) + \frac{(q - q^{-1})z}{z - w} f^-_\beta(z)f^-_{\alpha + \beta}(qz),
\]

\[
 f^-_{\alpha + \beta}(z)f^-_{\alpha + \beta}(w) = \frac{q^{-1}z - qw}{qz - q^{-1}w} f^-_{\alpha + \beta}(w)f^-_{\alpha + \beta}(z) + \frac{z(q - q^{-1})}{qz - q^{-1}w} (1 + q^2) f^+_{\alpha + \beta}(q^2z)f^-_{\alpha + \beta}(z).
\]

**Remark.** The meaning of the commutation relations in this proposition is that we can move the half-currents \( f^-_\gamma(w) \) to the left and the half-currents \( f^+_\gamma(z) \) to the right through the total currents such that the total currents are unchanged and only shifted half-currents multiplied by rational functions arise. In this paper, we often use these properties of exchange between the total and half-currents.

The following proposition describes the commutation relations between projections of a composite-root current and the projection of simple-root currents.
Proposition 5.8  There are the equalities

\[ P^+ (f_{α+β}(t_1)) f^+_α(t_2) - q f^+_α(t_2) P^+ (f_{α+β}(t_1)) = \]
\[ = \frac{q - q^{-1}}{t_1 - t_2} (f^+_α(t_1) - f^+_α(t_2)) (t_1 P^+ (f_{α+β}(t_1)) - t_2 P^+ (f_{α+β}(t_2))) , \]
\[ f^-_β(s_1) P^- (f_{α+β}(q s_2)) - q^{-1} P^- (f_{α+β}(q s_2)) f^-_β(s_1) = \]
\[ = \frac{q - q^{-1}}{s_1 - s_2} (s_1 P^- (f_{α+β}(q s_1)) - s_2 P^- (f_{α+β}(q s_2))) (f^-_β(s_1) - f^-_β(s_2)) . \]

Proof. The proof consists in applying the screening operator \( S_{f_{i_0}} \) to equality \( [5.11] \) and \( \tilde{S}_{f_{a_0}} \) to the analogous equality where the projection of currents \( f^+_α(t_i) \) is replaced with \( f^-_β(s_i) \). The proof also uses the Serre relations written in terms of the projections of currents as

\[ q f^+_α(t_1) P^+ (f_{α+β}(t_2)) + q f^+_α(t_2) P^+ (f_{α+β}(t_1)) = \]
\[ = P^+ (f_{α+β}(t_1)) f^+_α(t_2) + P^+ (f_{α+β}(t_2)) f^+_α(t_1) , \]
\[ q f^-_β(s_1) P^- (f_{α+β}(s_2)) + q f^-_β(s_2) P^- (f_{α+β}(s_1)) = \]
\[ = P^- (f_{α+β}(s_1)) f^-_β(s_2) + P^- (f_{α+β}(s_2)) f^-_β(s_1) . \]

\[ \square \]

Appendix C. Direct proof of Theorem [11]

The first step in this proof is the same as in [5.12]. We then observe that the first projection \( P^+ \) in [5.12] vanishes for \( k > a \). This explains the upper summation limit in [5.10]. To prove this formula, we calculate the projection

\[ P^+ (f_α(t_1) \cdots f_α(t_a) P^- (f_β(s_1) \cdots f_β(s_k))) \]

or

\[ P^+ (f_α(t_1) \cdots f_α(t_a) f^-_β(s_1; s_2, \ldots, s_k) \cdots f^-_β(s_{k-1}; s_k) f^-_β(s_k)) , \]

moving the half-currents \( f^-_β(s_m; s_{m+1}, \ldots, s_k), m = 1, \ldots, k, \) to the left using commutation relations [1.2]. According to these commutation relations, the composite currents \( f_{α+β}(t_j) \) are created at the positions \( j = 1, \ldots, a \). We next move these composite currents to the right using commutation relations [4.6]. Here, we again use the statements in Proposition 5.11. After these commutations, we obtain the sum over all nonordered subsets \( J = \{j_1, \ldots, j_k\} \in \{1, \ldots, a\} : \)

\[ \prod_{i<j}^k \frac{s_i - s_j}{q s_i - q^{-1} s_j} \sum_J \prod_{m=1}^{j_m-1} \frac{q t_{j_m} - q^{-1} t_{\ell_m}}{t_{j_m} - t_{\ell_m}} \prod_{m=j_m+1}^{a} \frac{q^{-1} t_{j_m} - q t_{\ell_m}}{t_{j_m} - t_{\ell_m}} \times \]
\[ \prod_{m=1}^k \left( \frac{t_{j_m}}{t_{j_m} - q s_m} \prod_{i=m+1}^k \frac{q t_{j_m} - s_i}{t_{j_m} - q s_i} \right) P^+ \left( f_{α+β}(t_{j_1}) \cdots f_{α+β}(t_{j_k}) \right) f_α(t_1) f_α(t_2) \cdots f_α(t_{a-1}) f_α(t_a) . \]

currents depending on \( t_{j_1}, \ldots, t_{j_k} \) omitted.
The next step is to use the commutation relations (4.6) to move the group of the composite currents $f_{\alpha+\beta}(t_{j_1}) \cdots f_{\alpha+\beta}(t_{j_k})$ under projection from left to right,

$$
\prod_{i<j}^{k} \frac{s_i - s_j}{q s_i - q^{-1}s_j} \sum_{J} \prod_{i<j} q^{-1}t_i - qt_j \prod_{i<j} t_i - t_j \prod_{i<j} q^{-1}t_i - qt_j \prod_{m=1}^{k} \left( \frac{t_{jm}}{t_j - q s_m} \prod_{i=m+1}^{k} t_{jm} - q s_i \right) \times P^+ \left( f_{\alpha}(t_1)f_{\alpha}(t_2) \cdots f_{\alpha}(t_{a-1})f_{\alpha}(t_a) f_{\alpha+\beta}(t_{j_1}) \cdots f_{\alpha+\beta}(t_{j_k}) \right).
$$

(C.1)

We now use the fact that the summation in this formula ranges nonordered sets $J$ of size $k$. This means that this summation can be decomposed into two summations: first, over all different but ordered choices \{ $j_1 < j_2 < \cdots < j_k$ \} from the set \{ $1, 2, \ldots, a$ \} and, second, over all permutations among fixed \{ $j_1, j_2, \ldots, j_k$ \}. Because of commutation relations (4.6) between composite currents, this second summation can be written as a $q^{-1}$-symmetrization with respect to a fixed subset \{ $j_1, j_2, \ldots, j_k$ \} of the function

$$
\prod_{m=1}^{k} \left( \frac{1}{t_{jm} - q s_m} \prod_{i=m+1}^{k} t_{jm} - q s_i \right)
$$

because the other ingredients in formula (C.1) are stable under permutation of \{ $j_1, j_2, \ldots, j_k$ \}. Using combinatorial identity (3.31), which can be written as

$$
\prod_{i<j}^{k} \frac{q^{-1}t_i - qt_j}{t_i - t_j} \Sym_t \left( \prod_{m=1}^{k} \left( \frac{1}{t_j - q s_m} \prod_{i=m+1}^{k} t_j - q s_i \right) \right) = \prod_{i<j}^{k} \frac{q s_i - q^{-1}s_j}{s_i - s_j} \Sym_s \left( \prod_{m=1}^{k} \left( \frac{1}{t_m - q s_m} \prod_{i=m+1}^{k} t_m - q s_i \right) \right),
$$

we can now rewrite expression (C.1) as

$$
\sum_{j_1 < \cdots < j_k} \prod_{i<j}^{k} \frac{q^{-1}t_i - qt_j}{qt_i - q^{-1}t_j} \Sym_s \left( \prod_{m=1}^{k} \left( \frac{t_{jm}}{t_j - q s_m} \prod_{i=m+1}^{k} t_{jm} - q s_i \right) \right) \times P^+ \left( f_{\alpha}(t_1)f_{\alpha}(t_2) \cdots f_{\alpha}(t_{a-1})f_{\alpha}(t_a) f_{\alpha+\beta}(t_{j_1}) \cdots f_{\alpha+\beta}(t_{j_k}) \right) .
$$

(C.2)

The summation over all ordered sets $J$ in (C.2) can now be written as a $q$-symmetrization with respect to all the variables $t_1, \ldots, t_a$, and we obtain (5.19). Repeating the argument given at the end of Sec. 5.2, we finish the direct proof of Theorem 1.

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