A dual rigidity of the sphere and the hyperbolic plane
Magdalena Caballero and Rafael M. Rubio
Departamento de Matemáticas, Campus de Rabanales,
Universidad de Córdoba, 14071 Córdoba, Spain,
E-mails: magdalena.caballero@uco.es, rmrubio@uco.es

Abstract
There are several well-known characterizations of the sphere as a regular surface in
the Euclidean space. By means of a purely synthetic technique, we get a rigidity result for
the sphere without any curvature conditions, nor completeness or compactness. As well
as a dual result for the hyperbolic plane, the spacelike sphere in the Minkowski space.

2010 MSC: 53A05, 53A35, 53C24.
Keywords: Euclidean and Lorentzian Geometries, Sphere and Hyperbolic Plane.

1 Introduction
In 1897 Hadamard proved that any compact connected regular surface with positive Gaussian
curvature in the three-dimensional Euclidean space \( \mathbb{E}^3 \) is a topological sphere, \([3]\). His result
motivated the search for conditions to conclude that such a surface is necessarily a round
sphere (an Euclidean sphere). Two answers were given by Liebmann. The first one, in 1899
\([6]\), proved the rigidity of the sphere, conjectured by F. Minding in 1939,

\[
\text{If } S \text{ is a compact and connected regular surface in } \mathbb{E}^3 \text{ with constant Gaussian curvature } K, \text{ then } M \text{ is a sphere of radius } 1/\sqrt{K}.
\]

The second one involves the mean curvature, \([7]\).

\[
\text{Any compact and connected regular surface in } \mathbb{E}^3 \text{ with positive Gaussian curvature and constant mean curvature is a sphere.}
\]

Shortly after, Hilbert gave a simpler proof of the first result, \([4]\). His ideas were used by Chern in \([2]\) to get a more general characterization of the sphere, concerning Weingarten surfaces. He obtained the previous two results by Liebmann as corollaries.

Another result on the sphere involving the Gaussian curvature, which is a direct consequence of a result by Hopf \([5]\), asserts that it is the only complete and simply connected regular surface in \( \mathbb{E}^3 \) with positive constant Gaussian curvature.

The last characterization of the sphere we will mention is the Alexandrov theorem, which assures that a compact and connected regular surface of constant mean curvature in \( \mathbb{E}^3 \) is a sphere, \([1]\).
Now consider the Minkowski space $\mathbb{L}^3$. A regular surface in this space is called spacelike if its induced metric is Riemannian. In this setting, the hyperbolic plane $\mathbb{H}^2$ can be realized as one connected component of the hyperboloid of two sheets, and so it can be viewed as the spacelike sphere in $\mathbb{L}^3$. Analogously to the (Euclidean) sphere, the hyperbolic plane can be characterized as the only spacelike regular surface in $\mathbb{L}^3$ which is complete, simply connected and with negative constant Gaussian curvature, [5].

In this work, we are interested in surfaces foliated by circles. R. López proved that a surface in $\mathbb{E}^3$ with constant Gaussian curvature and foliated by pieces of circles is included in a sphere, or the planes containing the circles of the foliation are parallel, [8]. In [9], the same author obtained the dual result in $\mathbb{L}^3$. It states that a spacelike surface in $\mathbb{L}^3$ with constant Gaussian curvature and foliated by pieces of circles must be a portion of a hyperbolic plane, unless the planes of the foliation are parallel.

This paper is devoted to prove natural dual characterizations of the sphere in $\mathbb{E}^3$ and the hyperbolic plane in $\mathbb{L}^3$. In our results neither the Gaussian curvature nor the mean curvature appear. Neither completeness nor compactness hypotheses are required. We only need a hypothesis on the intersection of the surfaces by planes.

More specifically, we say that a regular surface $S$ in the Euclidean space $\mathbb{E}^3$ satisfying the $\mathcal{P}$ property if for each affine plane $\Pi$ intersecting $S$, the set $\Pi \cap S$ is a circle (including the degenerate case with radius zero).

Analogously, we say that a spacelike regular surface $S$ in the Minkowski space $\mathbb{L}^3$ satisfies the $\mathcal{P}^*$ property if for each spacelike affine plane $\Pi$ intersecting $S$, the set $\Pi \cap S$ is a circle (including the degenerate case with radius zero). Notice that a circle in a spacelike affine plane $\Pi$ of $\mathbb{L}^3$ is the locus of the points in $\Pi$ at a constant distance from a fixed point in $\Pi$, where the distance considered is the one associated to the induced metric.

We state the following rigidity results:

**Theorem 1.1** Let $S$ be a connected regular surface in the Euclidean space $\mathbb{E}^3$ satisfying the $\mathcal{P}$ property, then $S$ is necessarily an Euclidean sphere.

**Theorem 1.2** Let $S$ be a spacelike connected regular surface in the Minkowski space $\mathbb{L}^3$ satisfying the $\mathcal{P}^*$ property, then $S$ is necessarily a hyperbolic plane.

## 2 The proofs

### Euclidean case.

Let $S$ be a surface in $\mathbb{E}^3$ satisfying the $\mathcal{P}$ property and let $Q \in S$ be an arbitrary point. We consider the tangent plane $T_QS$ and its normal line through $Q$, $\mathcal{L}$. We take the sheaf of affine planes with axis $\mathcal{L}$ and we denote by $\{C_i\}_{i \in I}$ the family of circles obtained when intersecting those planes with $S$.

We consider a plane $\Pi_0$ parallel to $T_QS$ such that $C = \Pi_0 \cap S$ is a non degenerate circle and we denote $P = \Pi_0 \cap \mathcal{L}$. Then $\Pi_0 \cap C_i \neq \emptyset$ for all $i \in I$, and the intersection points of each circle $C_i$ with $\Pi_0$ are the opposite points of a chord of $C_i$ contained in $\Pi_0$ with midpoint $P$. Therefore, the point $P$ must be the center of $C$ and as direct consequence the circles $C_i$ have all the same radius. Thus, the sphere given by $\bigcup_{i \in I} C_i$ is contained in $S$. We finish the proof thanks to the connectedness of $S$.

### Lorentzian case.


Let $S$ be a surface in $\mathbb{L}^3$ satisfying the $P^*$ property and let $\Pi_0$ be a spacelike plane such that $C = \Pi_0 \cap S$ is a non-degenerate circle. We denote its center by $P$ and the normal line through $P$ by $\mathcal{L}$.

Firstly, we prove that $\mathcal{L} \cap S \neq \emptyset$. We proceed by contradiction. Let us assume $\mathcal{L} \cap S = \emptyset$. Therefore, any plane parallel to $\Pi_0$ either does not intersect $S$ or it does it in a non-degenerate circle whose interior contains a point of $\mathcal{L}$. We deduce that any line parallel to $\mathcal{L}$ intersects $S$ at most in one point, otherwise the intersection of $S$ and the plane generated by both lines contains a non-spacelike curve. Thus, we have proved that $S$ is a graph over a domain of $\Pi_0$ not intersecting $\mathcal{L}$ and foliated by circles. Since $S$ is spacelike, it can not be asymptotic to $\mathcal{L}$ or any line parallel to it, and so $\partial S \neq \emptyset$. If $\partial S$ does not contain a point of $\mathcal{L}$, then it contains a circle. In both cases we can find spacelike planes intersecting $S$ in a non closed curve, which is a contradiction.

We notice that $S$ must by closed, otherwise we proceed as before to arrive to a contradiction. We take a point $Q$ at which the distance from $P$ to $\mathcal{L} \cap S$ is attained.

For each $A \in C$ and for each chord perpendicular to the segment $AP$, we call its midpoint $A_m$. We can choose the chord as close to $A$ as necessary so that the plane generated by it and the segment $A_mQ$ is spacelike, we denote it by $\Pi_{A_m}$. We define $\varepsilon_A$ to be the supremum (in the set of all possible chords satisfying the previous property) of the distance from $A$ to $A_m$.

If $\varepsilon = \min_C \varepsilon_A$, we take $0 < \rho < \varepsilon$ and for each $A \in C$ we consider the chord with $d(A_m, A) = \rho$. Hence, all the circles $\Pi_{A_m} \cap S$ have the same radius, and so there is a hyperbolic cap contained in $S$ and containing $C$.

Finally, for each point $A \in S$ there exists a spacelike plane intersecting $S$ in a non-degenerate circle containing $A$. Therefore, there exists a hyperbolic cap contained in $S$ and containing $A$. We finish the proof by using a connectedness argument.

Acknowledgments

The authors are partially supported by the Spanish MICINN Grant with FEDER funds MTM2010-18099.

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