The Koszul map $K: \mathbf{U}(\mathfrak{gl}(n)) \to \mathbf{Sym}(\mathfrak{gl}(n)) \cong \mathbb{C}[M_{n,n}]$

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c-contents

1 Introduction

2 Determinantal Young bitableau and permanental Young *-bitableau in the polynomial algebra $\mathbb{C}[M_{n,n}]$

3 The superalgebraic approach to the enveloping algebra $\mathbf{U}(\mathfrak{gl}(n))$

4 The bitableaux correspondence maps $B$ and $B^*$ and the Koszul map $K$

5 Expansion formulae for column Capelli bitableaux and column Capelli *-bitableaux

6 The BCK Theorem

7 Laplace expansions

8 Expansion formulae for column Capelli bitableaux and column Capelli *-bitableaux

9 The BCK Theorem

10 Laplace expansions

11 Expansion formulae for column Capelli bitableaux and column Capelli *-bitableaux

12 The BCK Theorem

13 Expansion formulae for column Capelli bitableaux and column Capelli *-bitableaux

14 The BCK Theorem

15 Expansion formulae for column Capelli bitableaux and column Capelli *-bitableaux

16 The BCK Theorem

17 Expansion formulae for column Capelli bitableaux and column Capelli *-bitableaux

1 Introduction

The Bitableau correspondence isomorphism/Koszul map Theorem (BCK Theorem, for short, Theorem 6.5 below) describes a relevant pair of mutually inverse vector space isomorphisms, the Koszul map (15, see also [2])

$$K: \mathbf{U}(\mathfrak{gl}(n)) \to \mathbb{C}[M_{n,n}] \cong \mathbf{Sym}(\mathfrak{gl}(n)),$$

and the bitableaux correspondence isomorphism

$$B: \mathbb{C}[M_{n,n}] \cong \mathbf{Sym}(\mathfrak{gl}(n)) \to \mathbf{U}(\mathfrak{gl}(n)),$$

that deeply link the enveloping algebra $\mathbf{U}(\mathfrak{gl}(n))$ of the general linear Lie algebra $\mathfrak{gl}(n)$ and the $\mathbb{C}$-algebra $\mathbb{C}[M_{n,n}]$ of polynomials in the entries of a “generic” square matrix of order $n$. The BCK Theorem can be regarded as a sharpened version of the PBW Theorem for the enveloping algebra $\mathbf{U}(\mathfrak{gl}(n))$.

The main objects in $\mathbf{U}(\mathfrak{gl}(n))$ - Capelli bitableaux, Capelli *-bitableaux and right Young-Capelli bitableaux - are introduced in section 3 by means of
the method of virtual variables, which is in turn a superalgebraic extension of Capelli’s method of variabili ausiliarie [10].

The isomorphism $B$ maps any (determinantal) bitableau $(S|T)$ in $\mathbb{C}[M_{n,n}]$ to the Capelli bitableau $[S|T]$ in $\mathfrak{U}(gl(n))$ (Theorem 4.1 below) and any (permanental) $\ast$-bitableau $(S|T)^\ast$ in $\mathbb{C}[M_{n,n}]$ to the Capelli $\ast$-bitableau $[S|T]^\ast$ in $\mathfrak{U}(gl(n))$ (Theorem 4.2 below).

Since the standard bitableaux are a basis of $\mathbb{C}[M_{n,n}]$ ([13], [12], [11], [14]), then the standard Capelli bitableaux are a basis of $\mathfrak{U}(gl(n))$ [3]. Since the costandard $\ast$-bitableaux are a basis of $\mathbb{C}[M_{n,n}]$, then the costandard Capelli $\ast$-bitableaux are a basis of $\mathfrak{U}(gl(n))$ [3].

In the polynomial algebra $\mathbb{C}[M_{n,n}]$, column bitableaux and column $\ast$-bitableaux are, up to a sign, the same monomials. Their images in $\mathfrak{U}(gl(n))$ - under the isomorphism $B$ - are the column Capelli bitableaux and the column Capelli $\ast$-bitableaux, respectively.

Although column Capelli bitableaux and column Capelli $\ast$-bitableaux are far from being “monomials” in the enveloping algebra $\mathfrak{U}(gl(n))$, their images under the Koszul isomorphism $K$ are indeed (commutative) monomials in the polynomial algebra $\mathbb{C}[M_{n,n}]$. Therefore, column Capelli bitableaux and column Capelli $\ast$-bitableaux play the same crucial role in $\mathfrak{U}(gl(n))$ that monomials play in $\mathbb{C}[M_{n,n}]$.

Capelli bitableaux and right Young-Capelli bitableaux expand - up to a global sign - into column Capelli bitableaux just in the same way as (determinantal) bitableaux and right symmetrized bitableaux (see, e.g. [1], [3], [4]) expand into the corresponding monomials in $\mathbb{C}[M_{n,n}]$. Capelli $\ast$-bitableaux expand - up to a global sign - into column Capelli bitableaux just in the same way as (permanental) $\ast$-bitableaux expand into the corresponding monomials in $\mathbb{C}[M_{n,n}]$.

The expressions of column Capelli bitableaux and column Capelli $\ast$-bitableaux in $\mathfrak{U}(gl(n))$ can be simply computed (Proposition 5.3 below). Some results in this paper appeared in the present author’s notes [5], [6] (in the more general setting of superalgebras), in a rather cumbersome notation and with sketchy proofs due to limitations of space. The emphasis is here on the role of column Capelli bitableaux and $\ast$-bitableaux, and this leads to a simpler presentation.

The starting point of the present approach is twofold:

- The linear operator $B : \mathbb{C}[M_{n,n}] \to \mathfrak{U}(gl(n))$ that maps any (determinantal) bitableau $(S|T)$ in $\mathbb{C}[M_{n,n}]$ to the Capelli bitableau $[S|T]$ in $\mathfrak{U}(gl(n))$ (Theorem 4.1 below).

- The linear operator $B^\ast : \mathbb{C}[M_{n,n}] \to \mathfrak{U}(gl(n))$ that maps any (permanental) bitableau $(S|T)^\ast$ in $\mathbb{C}[M_{n,n}]$ to the Capelli bitableau $[S|T]^\ast$ in $\mathfrak{U}(gl(n))$ (Theorem 4.2 below).

The Koszul map $K : \mathfrak{U}(gl(n)) \to \mathbb{C}[M_{n,n}]$ is defined in section 3. By using the identities of subsection 5.3 we prove that $K$ maps any column Capelli bitableau $[S|T]$ in $\mathfrak{U}(gl(n))$ to the corresponding column bitableau $(S|T)$ in...
\[ \mathbb{C}[M_{n,n}] \] and maps any column Capelli *-bitableau \([S|T]^*\) in \(U(gl(n))\) to the corresponding column *-bitableau \((S|T)^*\) in \(\mathbb{C}[M_{n,n}]\) (Propositions 6.1 and 6.3 below).

This implies that both \(B\) and \(B^*\) are inverses of \(K\) and, therefore, \(B^* = B\) and both \(B\) and \(K\) are linear isomorphims (Theorem 6.5 below).

Bitableaux and *-bitableaux in \(\mathbb{C}[M_{n,n}]\) can be expanded into monomials (column bitableaux and column *-bitableaux), via the Laplace rules. The BCK Theorem implies that Capelli bitableaux and Capelli *-bitableaux expand in the same way into column Capelli bitableaux and column Capelli *-bitableaux, respectively. By combining these expansions with Proposition 5.3, the explicit forms for Capelli bitableaux and Capelli *-bitableaux as elements of \(U(gl(n))\) can be easily computed.

Several examples/applications are provided throughout the paper. In particular, we show, in few lines, that the row Capelli bitableau \([12 \ldots n|12 \ldots n]\) equals the famous Capelli determinant [9] in \(U(gl(n))\) (see Proposition 7.4).

2 Determinantal Young bitableau and permanental Young *-bitableau in the polynomial algebra \(\mathbb{C}[M_{n,n}]\)

Let

\[ \mathbb{C}[M_{n,n}] = \mathbb{C}[[i|j]]_{i,j=1, \ldots, n} \]

be the polynomial algebra in the (commutative) “generic” entries \((i|j)\) of the matrix:

\[
M_{n,n} = [(i|j)]_{i,j=1, \ldots, n} = \begin{pmatrix} (1|1) & \ldots & (1|n) \\ (2|1) & \ldots & (2|n) \\ \vdots & \ddots & \vdots \\ (n|1) & \ldots & (n|n) \end{pmatrix}.
\]

Given the standard basis \(\{e_{ij}; i, j = 1, 2, \ldots, n\}\) of the general linear Lie algebra \(gl(n)\), the map \(e_{ij} \rightarrow (i|j)\) induces an isomorphism \(\text{Sym}(gl(n)) \cong \mathbb{C}[M_{n,n}]\).

Let \(\omega = i_1 i_2 \cdots i_p, \varpi = j_1 j_2 \cdots j_p\) be words on the alphabet \(\{1, 2, \ldots, n\}\).

Following [14] and [3], the biproduct of the two words \(\omega\) and \(\varpi\)

\[
(\omega|\varpi) = (i_1 i_2 \cdots i_p|j_1 j_2 \cdots j_p) \quad (1)
\]

is the signed minor:

\[
(\omega|\varpi) = (-1)^{\binom{p}{2}} \det\left( (i_r|j_s) \right)_{r,s=1,2, \ldots, p} \in \mathbb{C}[M_{n,n}].
\]

Let \(S = (\omega_1, \omega_2, \ldots, \omega_p)\) and \(T = (\varpi_1, \varpi_2, \ldots, \varpi_p)\) be Young tableaux on \(\{1, 2, \ldots, n\}\) of the same shape \(\lambda\).
Following again [14] and [3], the (determinantal) Young bitableau

\[(S|T) = \begin{pmatrix}
  \omega_1 & \varpi_1 \\
  \omega_2 & \varpi_2 \\
  \vdots & \vdots \\
  \omega_p & \varpi_p \\
\end{pmatrix}\]  

(2)

is the signed product of the biproducts of the pairs of corresponding rows:

\[(S|T) = \pm (\omega_1|\varpi_1)(\omega_2|\varpi_2)\cdots(\omega_p|\varpi_p),\]  

(3)

where

\[\pm = (-1)^{\ell(\omega_2)\ell(\varpi_1)+\ell(\omega_3)\ell(\varpi_1)+\cdots+\ell(\omega_p)\ell(\varpi_1)+\ell(\varpi_2)+\cdots+\ell(\varpi_{p-1})},\]  

(4)

and the symbol \(\ell(w)\) denotes the length of the word \(w\).

The *-biproduct of the two words \(\omega\) and \(\varpi\)

\[(\omega|\varpi) = (i_1i_2\cdots i_p | j_1j_2\cdots j_p)\]  

(5)

is the is the permanent:

\[(\omega|\varpi)^* = \text{per} \begin{pmatrix} (i_r | j_s) \end{pmatrix}_{r,s=1,2,\ldots,p} \in \mathbb{C}[M_{n,n}].\]

Let \(S = (\omega_1, \omega_2, \ldots, \omega_p)\) and \(T = (\varpi_1, \varpi_2, \ldots, \varpi_p)\) be Young tableaux on \(\{1, 2, \ldots, n\}\) of the same shape \(\lambda\).

Following again [14] and [3], the (permanental) Young *-bitableau

\[(S|T)^* = \begin{pmatrix}
  \omega_1 & \varpi_1^* \\
  \omega_2 & \varpi_2^* \\
  \vdots & \vdots \\
  \omega_p & \varpi_p^* \\
\end{pmatrix}\]  

(6)

is the product of the *-biproducts of the pairs of corresponding rows:

\[(S|T)^* = (\omega_1|\varpi_1)^*(\omega_2|\varpi_2)^*\cdots(\omega_p|\varpi_p)^*.\]  

(7)

A column Young tableau of depth \(h\) is a tableau of shape \((1^h)\). Then for a column Young bitableau, we have:

\[\begin{pmatrix} i_1 & j_1 \\
  i_2 & j_2 \\
  \vdots & \vdots \\
  i_h & j_h \end{pmatrix} = (-1)^{\binom{h}{2}}(i_1,j_1)(i_2,j_2)\cdots(i_h,j_h)\]

and for a column Young *-bitableau, we have:

\[\begin{pmatrix} i_1 & j_1 \\
  i_2 & j_2 \\
  \vdots & \vdots \\
  i_h & j_h \end{pmatrix}^* = (i_1,j_1)(i_2,j_2)\cdots(i_h,j_h).\]  

4
3 The superalgebraic approach to the enveloping algebra $U(gl(n))$

We follow [3], [4], [5], [6].

Let $X = \{a_1, \ldots, a_{m_0}\} \cup \{\beta_1, \ldots, \beta_{m_1}\} \cup L$, $L = \{1, 2, \ldots, n\}$ ( $m_0, m_1$ “sufficiently large” ) denote the union of the alphabets of virtual positive, virtual negative and proper negative symbols, respectively. Let $U(gl(m_0|m_1+n))$ denote the enveloping algebra of the general linear Lie superalgebra $gl(m_0|m_1+n)$, with basis $\{e_{a,b}; a,b \in X\}$, $|e_{a,b}| = |a| + |b| \in \mathbb{Z}_2$. The general linear Lie algebra $gl(n)$ (with basis $\{e_{a,b}; a,b \in L\}$, $|e_{a,b}| = |a| + |b| = 0 \in \mathbb{Z}_2$) is regarded as the subalgebra $gl(0|n)$ of $gl(m_0|m_1+n)$.

We recall ([5], [6]) that a product

$$e_{a_1b_1} \cdots e_{a_kb_k} \in U(gl(m_0|m_1+n))$$

is an irregular expression whenever there exists a right subword

$$e_{a_2b_2} \cdots e_{a_nb_n} e_{a_1b_1},$$

$i \leq m$ and a virtual symbol $\gamma \in A_0 \cup A_1$ such that

$$\# \{j; b_j = \gamma, \gamma \leq i\} > \# \{j; a_j = \gamma, j < i\}.$$  \hspace{1cm} (8)

The meaning of an irregular expression - in terms of the action of $U(gl(m_0|m_1+n))$ - is that there exists a virtual symbol $\gamma$ and a right subsequence in which the symbol $\gamma$ is annihilated more times than it was already created.

Let $\text{Irr}$ be the left ideal of $U(gl(m_0|m_1+n))$ generated by the set of irregular expressions ([5], [6], see also [1]).

**Proposition 3.1.** ([5], [2]) The sum $U(gl(0|n)) + \text{Irr}$ is a direct sum of vector subspaces of $U(gl(m_0|m_1+n))$.

The virtual algebra $\text{Virt}(m_0+m_1,n)$ is the subalgebra

$$\text{Virt}(m_0+m_1,n) = U(gl(0|n)) \oplus \text{Irr} \subset U(gl(m_0|m_1+n)).$$

**Proposition 3.2.** The left ideal $\text{Irr}$ of $U(gl(m_0|m_1+n))$ is a two sided ideal of $\text{Virt}(m_0+m_1,n)$.

The Capelli devirtualization epimorphism is the projection

$$p : \text{Virt}(m_0+m_1,n) = U(gl(0|n)) \oplus \text{Irr} \rightarrow U(gl(0|n)) = U(gl(n))$$

with $\text{Ker}(p) = \text{Irr}$.

In a formal way, balanced monomials are elements of the algebra $U(gl(m_0|m_1+n))$ of the form:

- $e_{\gamma_1} \cdots e_{\gamma_k} \cdot e_{\gamma_{p_1}j_1} \cdots e_{\gamma_{p_k}j_k}$
- $e_{\alpha_1} \cdots e_{\alpha_k} \cdot e_{\gamma_{q_1}j_1} \cdots e_{\gamma_{q_k}j_k}$
Proposition 3.3. (3, 11, 1, 2) Every balanced monomial belongs to \( \text{Virt}(m_0 + m_1, n) \). Hence its image under the Capelli epimorphism \( p \) belongs to \( U(gl(n)) \).

Let \( S \) and \( T \) be the Young tableaux

\[
S = \begin{pmatrix}
i_{p_1} & \cdots & i_{p_{\lambda_1}} \\
i_{q_1} & \cdots & i_{q_{\lambda_2}} \\
\cdots \\
i_{r_1} & \cdots & i_{r_{\lambda_m}}
\end{pmatrix}, \quad T = \begin{pmatrix}
j_{s_1} & \cdots & j_{s_{\lambda_1}} \\
j_{t_1} & \cdots & j_{t_{\lambda_2}} \\
\cdots \\
j_{v_1} & \cdots & j_{v_{\lambda_m}}
\end{pmatrix}.
\]

(9)

To the pair \((S, T)\), we associate the bitableau monomial:

\[
e_{S,T} = e_{i_{p_1},j_{s_1}} \cdots e_{i_{p_{\lambda_1}},j_{s_{\lambda_1}}} e_{i_{q_1},j_{t_1}} \cdots e_{i_{q_{\lambda_2}},j_{t_{\lambda_2}}} \cdots e_{i_{r_1},j_{v_1}} \cdots e_{i_{r_{\lambda_m}},j_{v_{\lambda_m}}}.
\]

in \( U(gl(m_0|m_1 + n)) \).

Let \( \beta_1, \ldots, \beta_{\lambda_1} \in A_1, \alpha_1, \ldots, \alpha_p \in A_0 \) be sets of negative and positive virtual symbols, respectively. Set

\[
D_\lambda = \begin{pmatrix}
\beta_1 & \cdots & \beta_{\lambda_1} \\
\beta_1 & \cdots & \beta_{\lambda_2} \\
\cdots \\
\beta_1 & \cdots & \beta_{\lambda_p}
\end{pmatrix}, \quad C_\lambda = \begin{pmatrix}
\alpha_1 & \cdots & \alpha_1 \\
\alpha_2 & \cdots & \alpha_2 \\
\cdots \\
\alpha_p & \cdots & \alpha_p
\end{pmatrix}.
\]

The tableaux \( D_\lambda \) and \( C_\lambda \) are called the virtual Deryugts and Coderuylts tableaux of shape \( \lambda \), respectively.

Given a pair of Young tableaux \( S, T \) of the same shape \( \lambda \) on the proper alphabet \( L \), consider the elements

\[
e_{S,C_\lambda}, e_{C_\lambda,T} \in U(gl(m_0|m_1 + n)),
\]

\[
e_{S,D_\lambda}, e_{D_\lambda,T} \in U(gl(m_0|m_1 + n)),
\]

(11)

(12)

Since elements (10), (11) and (12) are balanced monomials in \( U(gl(m_0|m_1 + n)) \), they belong to the subalgebra \( \text{Virt}(m_0 + m_1, n) \).

We set

\[
p(e_{S,C_\lambda}, e_{C_\lambda,T}) = [S|T] \in U(gl(n)),
\]

\[
\text{and so on},
\]

where \( i_1, \ldots, i_k, j_1, \ldots, j_k \in L \), i.e., the \( i_1, \ldots, i_k, j_1, \ldots, j_k \) are \( k \) proper (negative) symbols, and the \( \gamma_{p_1}, \ldots, \gamma_{p_k}, \theta_{q_1}, \ldots, \theta_{q_k} \) are virtual symbols. In plain words, a balanced monomial is product of two or more factors where the rightmost one annihilates the \( k \) proper symbols \( j_1, \ldots, j_k \) and creates some virtual symbols; the leftmost one annihilates all the virtual symbols and creates the \( k \) proper symbols \( i_1, \ldots, i_k \): between these two factors, there might be further factors that annihilate and create virtual symbols only.
and call the element \([S|T]\) a Capelli bitableau \([5], [6]\).

We set

\[
p(S, \overline{D}_\lambda e_{\overline{D}_\lambda}T) = \begin{bmatrix} S \\ T \end{bmatrix} \in U(gl(n)),
\]

and call the element \([S|T]\) a \(\ast\)-bitableau \([5], [6]\).

We set

\[
p(e_{S, C_\lambda} e_{C_\lambda D_\lambda} e_{D_\lambda} T) = \begin{bmatrix} S \\ T \end{bmatrix} \in U(gl(n)),
\]

and call the element \([S|T]\) a right Young-Capelli bitableau \([4]\).

The bitableaux correspondence maps \(\mathcal{B}\) and \(\mathcal{B}^\ast\)
and the Koszul map \(\mathcal{K}\)

**Theorem 4.1.** The bitableaux correspondence map

\[
\mathcal{B}: (S|T) \mapsto [S|T]
\]

uniquely extends to a linear map

\[
\mathcal{B}: \mathbb{C}[M_{n,n}] \cong \text{Sym}(gl(n)) \rightarrow U(gl(n)).
\]

**Theorem 4.2.** The \(\ast\)-bitableaux correspondence map

\[
\mathcal{B}^\ast: (S|T)^\ast \mapsto [S|T]^\ast
\]

uniquely extends to a linear map

\[
\mathcal{B}^\ast: \mathbb{C}[M_{n,n}] \cong \text{Sym}(gl(n)) \rightarrow U(gl(n)).
\]

The linear isomorphisms \(\mathcal{B}\) and \(\mathcal{B}^\ast\) were introduced in [6], Theorem 1 and
Theorem 3. Eqs. (13), (14) indeed define linear operators since bitableaux in \(\mathbb{C}[M_{n,n}]\)
and Capelli bitableaux in \(U(gl(n))\) are ruled by the same straightening laws as well as \(\ast\)-bitableaux and Capelli \(\ast\)-bitableaux (see [4], Proposition 7).

Given \(i, j = 1, 2, \ldots, n\), let

\[
\rho_{ij}: \mathbb{C}[M_{n,n}] \rightarrow \mathbb{C}[M_{n,n}]
\]

be the linear operator

\[
\rho_{ij}(p) = D_{ij}(p) + (ij) \cdot p, \quad \text{for every } p \in \mathbb{C}[M_{n,n}],
\]

where \(D_{ij}\) denotes the polarization operator defined by the following conditions:

- \(D_{ij}\) is a derivation,
Proposition 4.3. We have:

\[
[p_{ij}, p_{hk}] = p_{ij}p_{hk} - p_{hk}p_{ij} = \delta_{jh}p_{ik} - \delta_{ik}p_{hj}.
\]

By the universal property of \( U(gl(n)) \), Proposition 4.3 implies

Proposition 4.4. The map

\[ e_{ij} \mapsto \rho_{ij}, \quad e_{ij} \in gl(n) \]

defines an associative algebra morphism

\[ \tau : U(gl(n)) \to \text{End}_C[C[M_{n,n}]]. \]

Let \( \varepsilon_1 \) be the linear map evaluation at 1

\[ \varepsilon_1 : \text{End}_C[C[M_{n,n}]] \to C[M_{n,n}], \]

\[ \varepsilon_1(\rho) = \rho(1) \in C[M_{n,n}], \quad \text{for every } \rho \in \text{End}_C[C[M_{n,n}]]. \]

The Koszul map is the (linear) composition map

\[ \mathcal{K} : U(gl(n)) \to C[M_{n,n}] \cong \text{Sym}(gl(n)), \]

\[ \mathcal{K} = \varepsilon_1 \circ \tau. \]

Proposition 4.5. We have:

1. \( \mathcal{K}(e_{i_1,j_1}e_{i_2,j_2} \cdots e_{i_h,j_h}) = \rho_{i_1,j_1}\rho_{i_2,j_2} \cdots \rho_{i_h,j_h}(1), \quad e_{i_p,j_p} \in gl(n), \quad p = 12, \ldots, h. \)

2. \( \mathcal{K}(e_{ij}P) = \rho_{ij}(\mathcal{K}(P)), \quad \text{for every } P \in U(gl(n)), \quad e_{ij} \in gl(n). \)

5 Expansion formulae for column Capelli bitableaux and column Capelli *-bitableaux

Consider the column Capelli bitableau

\[
\begin{array}{|c|c|}
\hline
i_1 & j_1 \\
\hline
i_2 & j_2 \\
\hline
\vdots & \vdots \\
\hline
i_h & j_h \\
\hline
\end{array}
\]

\[ = p\left(e_{i_1,\alpha_1} \cdots e_{i_h,\alpha_h} e_{\alpha_1,j_1} \cdots e_{\alpha_h,j_h}\right) \in U(gl(n)), \]
(where \( \alpha_1, \ldots, \alpha_h \) are arbitrary distinct positive virtual symbols) and the column Capelli *-bitableau

\[
\begin{bmatrix}
  i_1 & j_1 \\
  i_2 & j_2 \\
  \vdots & \vdots \\
  i_h & j_h \\
\end{bmatrix}^* = p(e_{i_1, \beta_1} \cdots e_{i_h, \beta_h} e_{\beta_1, j_1} \cdots e_{\beta_h, j_h}) \in U(gl(n))
\]

(where \( \beta_1, \ldots, \beta_h \) are arbitrary distinct negative virtual symbols).

Remember that the proper symbols \( i_1, i_2, \ldots, i_h, \ j_1, j_2, \ldots, j_h \in L = \{1, 2, \ldots, n\} \) are assumed to be negative.

**Remark 5.1.** The families of column Capelli bitableaux and the column Capelli *-bitableaux are systems of linear generators of \( U(gl(n)) \).

**Remark 5.2.** Both column Capelli bitableaux and column Capelli *-bitableaux are row-commutative, in symbols:

1. \[
\begin{bmatrix}
  i_1 & j_1 \\
  i_2 & j_2 \\
  \vdots & \vdots \\
  i_h & j_h \\
\end{bmatrix} = \begin{bmatrix}
  i_{\sigma(1)} & j_{\sigma(1)} \\
  i_{\sigma(2)} & j_{\sigma(2)} \\
  \vdots & \vdots \\
  i_{\sigma(h)} & j_{\sigma(h)} \\
\end{bmatrix}, \quad \sigma \in vS_h,
\]

2. \[
\begin{bmatrix}
  i_1 & j_1 \\
  i_2 & j_2 \\
  \vdots & \vdots \\
  i_h & j_h \\
\end{bmatrix}^* = \begin{bmatrix}
  i_{\sigma(1)} & j_{\sigma(1)} \\
  i_{\sigma(2)} & j_{\sigma(2)} \\
  \vdots & \vdots \\
  i_{\sigma(h)} & j_{\sigma(h)} \\
\end{bmatrix}^*, \quad \sigma \in vS_h,
\]

We recall two basic expansion formulae, that describe the effect of picking out (on the left hand side) the first row of column Capelli bitableaux and column Capelli *-bitableaux. These formulae play a crucial role in the theory of the Koszul map \( K \), and provide a simple way to compute the actual forms of column Capelli bitableaux and column Capelli *-bitableaux as elements of \( U(gl(n)) \) are row-commutative, in symbols:

**Proposition 5.3.** We have:
1. 
\[
\begin{bmatrix}
i_1 & j_1 \\
i_2 & j_2 \\
\vdots & \vdots \\
i_h & j_h
\end{bmatrix}
= (-1)^{h-1} e_{i_1,j_1} \begin{bmatrix}
i_2 & j_2 \\
\vdots & \vdots \\
i_h & j_h
\end{bmatrix}
+ (-1)^{h-2} \sum_{k=2}^{h} \delta_{i_k,j_1} \begin{bmatrix}
i_1 & j_k \\
\vdots & \vdots \\
i_h & j_h
\end{bmatrix}
\in U(gl(n)).
\]

2. 
\[
\begin{bmatrix}
i_1 & j_1 \\
i_2 & j_2 \\
\vdots & \vdots \\
i_h & j_h
\end{bmatrix}^*
= e_{i_1,j_1} \begin{bmatrix}
i_2 & j_2 \\
\vdots & \vdots \\
i_h & j_h
\end{bmatrix}^* - \sum_{k=2}^{h} \delta_{i_k,j_1} \begin{bmatrix}
i_1 & j_k \\
\vdots & \vdots \\
i_h & j_h
\end{bmatrix}^*
\in U(gl(n)).
\]

For a proof, see e.g. [8].

Example 5.4.
\[
\begin{bmatrix}
1 & 2 \\
2 & 3 \\
3 & 4 \\
2 & 3
\end{bmatrix}
= -e_{12} \begin{bmatrix}
2 & 3 \\
3 & 4 \\
2 & 3 \\
1 & 3
\end{bmatrix}
+ \begin{bmatrix}
1 & 3 \\
3 & 4 \\
2 & 3 \\
1 & 3
\end{bmatrix}
= -e_{12} \begin{bmatrix}
2 & 3 \\
3 & 4 \\
2 & 3 \\
1 & 3
\end{bmatrix}
+ 2 \begin{bmatrix}
1 & 3 \\
3 & 4 \\
2 & 3 \\
1 & 3
\end{bmatrix}
= -e_{12} e_{23} \begin{bmatrix}
3 & 4 \\
2 & 3 \\
2 & 3 \\
1 & 3
\end{bmatrix}
+ 2 (e_{13} \begin{bmatrix}
3 & 4 \\
2 & 3 \\
1 & 4 \\
2 & 3
\end{bmatrix}
= e_{12} e_{23} e_{34} e_{23} - e_{12} e_{24} e_{23} - 2e_{13} e_{34} e_{23} + 2e_{14} e_{23} \in U(gl(4)).
\]

\[\square\]
6 The BCK Theorem

Proposition 6.1.

\[
K\left(\begin{bmatrix}
  i_1 & j_1 \\
  i_2 & j_2 \\
  \vdots & \vdots \\
  i_h & j_h
\end{bmatrix}\right) = \left(\begin{bmatrix}
  i_1 & j_1 \\
  i_2 & j_2 \\
  \vdots & \vdots \\
  i_h & j_h
\end{bmatrix}\right)
\]

\[= (-1)^{(i_2)}(i_1:j_1)(i_2:j_2)\ldots(i_h:j_h) \in \mathbb{C}[M_{n,n}] \cong \text{Sym}(gl(n))\]

Proof.

\[
K\left(\begin{bmatrix}
  i_1 & j_1 \\
  i_2 & j_2 \\
  \vdots & \vdots \\
  i_h & j_h
\end{bmatrix}\right) =
\]

\[= (-1)^{h-1}K(e_{i_1j_1}) \left(\begin{bmatrix}
  i_2 & j_2 \\
  \vdots & \vdots \\
  i_h & j_h
\end{bmatrix}\right) + (-1)^{h-2}K(\sum_{k=2}^h \delta_{i_kj_1}) \left(\begin{bmatrix}
  i_2 & j_2 \\
  \vdots & \vdots \\
  i_h & j_h
\end{bmatrix}\right)
\]

\[= (-1)^{h-1}p_{i_1j_1}(K(\left(\begin{bmatrix}
  i_2 & j_2 \\
  \vdots & \vdots \\
  i_h & j_h
\end{bmatrix}\right)) + (-1)^{h-2}K(\sum_{k=2}^h \delta_{i_kj_1}) \left(\begin{bmatrix}
  i_2 & j_2 \\
  \vdots & \vdots \\
  i_h & j_h
\end{bmatrix}\right)
\]

\[= (-1)^{h-1}D_{i_1j_1} \left(\begin{bmatrix}
  i_2 & j_2 \\
  \vdots & \vdots \\
  i_h & j_h
\end{bmatrix}\right) + (-1)^{h-1}(i_1:j_1) \left(\begin{bmatrix}
  i_2 & j_2 \\
  \vdots & \vdots \\
  i_h & j_h
\end{bmatrix}\right)
\]

\[+ (-1)^{h-2} \sum_{k=2}^h \delta_{i_kj_1} \left(\begin{bmatrix}
  i_2 & j_2 \\
  \vdots & \vdots \\
  i_h & j_h
\end{bmatrix}\right)
\]

\[= (-1)^{h-1}(i_1:j_1) \left(\begin{bmatrix}
  i_2 & j_2 \\
  \vdots & \vdots \\
  i_h & j_h
\end{bmatrix}\right) = \left(\begin{bmatrix}
  i_1 & j_1 \\
  i_2 & j_2 \\
  \vdots & \vdots \\
  i_h & j_h
\end{bmatrix}\right).
\]
Example 6.2. Consider the column Capelli bitableau
\[
\begin{bmatrix}
1 & 2 \\
2 & 1 \\
3 & 1
\end{bmatrix}
= e_{12} \begin{bmatrix}
2 & 1 \\
3 & 1
\end{bmatrix}
- \begin{bmatrix}
1 & 1 \\
3 & 1
\end{bmatrix}
= -e_{12}e_{21}e_{31} + e_{11}e_{31} \in U(gl(n)).
\]

We have
\[
\mathcal{K}\left( \begin{bmatrix}
1 & 2 \\
2 & 1 \\
3 & 1
\end{bmatrix} \right) = \mathcal{K}\left( -e_{12}e_{21}e_{31} + e_{11}e_{31} \right)
= \begin{pmatrix}
1 & 2 \\
2 & 1 \\
3 & 1
\end{pmatrix}
= -(1|2)(2|1)(3|1) \in \mathbb{C}[M_{n,n}] \cong \text{Sym}(gl(n)).
\]

\[\square\]

Proposition 6.3.
\[
\mathcal{K}\left( \begin{bmatrix}
i_1 & j_1 \\
i_2 & j_2 \\
\vdots & \vdots \\
i_h & j_h
\end{bmatrix} \right)^* = \begin{pmatrix}
i_1 & j_1 \\
i_2 & j_2 \\
\vdots & \vdots \\
i_h & j_h
\end{pmatrix}^* 
= (i_1|j_1)(i_2|j_2) \ldots (i_h|j_h) \in \mathbb{C}[M_{n,n}] \cong \text{Sym}(gl(n)).
\]

Proof.
\[
\mathcal{K}\left( \begin{bmatrix}
i_1 & j_1 \\
i_2 & j_2 \\
\vdots & \vdots \\
i_h & j_h
\end{bmatrix} \right)^*
= \mathcal{K}(e_{i_1,j_1} \begin{bmatrix}
i_2 & j_2 \\
i_h & j_h
\end{bmatrix}^*)
- \mathcal{K}\left( \sum_{k=2}^{h} \delta_{i_k,j_1} \begin{bmatrix}
i_1 & j_k \\
i_h & j_h
\end{bmatrix} \right)
\]
Corollary 6.4.

\[\begin{align*}
&\rho_{i_1j_1}(K(\begin{bmatrix} i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{bmatrix}^* ) - K(\sum_{k=2}^h \delta_{i_kj_1} \begin{bmatrix} i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{bmatrix})) \\
&= D_{i_1j_1} \left( \begin{bmatrix} i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{bmatrix} \right)^* + (i_1|j_1) \left( \begin{bmatrix} i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{bmatrix} \right)^* \\
&\quad - \sum_{k=2}^h \delta_{i_kj_1} \begin{bmatrix} i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{bmatrix} \\
&= (i_1|j_1) \left( \begin{bmatrix} i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{bmatrix} \right)^* = \left( \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_h \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \\ \vdots \\ j_h \end{bmatrix} \right)^* .
\end{align*}\]

Since

\[\begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_h \\ j_1 \\ j_2 \\ \vdots \\ j_h \end{bmatrix}^* = (i_1|j_1)(i_2|j_2) \cdots (i_h|j_h) = (-1)^{\binom{h}{2}} \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_h \\ j_1 \\ j_2 \\ \vdots \\ j_h \end{bmatrix},\]

we have

Corollary 6.4.

\[\begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_h \\ j_1 \\ j_2 \\ \vdots \\ j_h \end{bmatrix}^* = (-1)^{\binom{h}{2}} \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_h \\ j_1 \\ j_2 \\ \vdots \\ j_h \end{bmatrix} \in U(gl(n)).\]

\[\square\]
Since, Theorem 4.1 specializes to
\[ B(\begin{array}{c|c}
i_1 & j_1 \\
i_2 & j_2 \\
\vdots & \vdots \\
i_h & j_h \end{array}) = \begin{array}{c|c}
i_1 & j_1 \\
i_2 & j_2 \\
\vdots & \vdots \\
i_h & j_h \end{array} \]
and, Theorem 4.2 specializes to
\[ B^*(\begin{array}{c|c}
i_1 & j_1 \\
i_2 & j_2 \\
\vdots & \vdots \\
i_h & j_h \end{array})^* = \begin{array}{c|c}
i_1 & j_1 \\
i_2 & j_2 \\
\vdots & \vdots \\
i_h & j_h \end{array}^* \]
from Remark 5.1 we have

**Theorem 6.5. (BCK Theorem)**

1. \( B = \kappa^{-1} \),
2. \( B^* = \kappa^{-1} \),
3. \( B^* = B \),
4. \( B, \kappa \) are linear isomorphisms.

Since the set of (determinantal) standard bitableaux
\[ \left\{ (S|T); \text{sh}(S) = \text{sh}(T) = \lambda, \lambda_1 \leq n, S, T \text{ standard} \right\} \]
and the set of (permanental) costandard *-bitableaux
\[ \left\{ (U|V)^*; \text{sh}(U) = \text{sh}(V) = \mu, \tilde{\mu}_1 \leq n, U, V \text{ costandard} \right\} \]
are (linear) bases of \( \mathbb{C}[M_{n,n}] \cong \text{Sym}(gl(n)) \), then

**Corollary 6.6.** The set of (determinantal) standard Capelli bitableaux
\[ \left\{ [S|T]; \text{sh}(S) = \text{sh}(T) = \lambda, \lambda_1 \leq n, S, T \text{ standard} \right\} \]
and the set of (permanental) costandard Capelli *-bitableaux
\[ \left\{ [U|V]^*; \text{sh}(U) = \text{sh}(V) = \mu, \tilde{\mu}_1 \leq n, U, V \text{ costandard} \right\} \]
are (linear) bases of \( U(gl(n)) \).
The bitableaux correspondence isomorphism $B$ and the Koszul isomorphism $K$ well-behave with respect to right symmetrized bitableaux
\[(S | T) \in \mathbb{C}[M_{n,n}]
\]
and right Young-Capelli bitableaux
\[[S | T] \in U(gl(n)).
\]
In plain words, any right Young-Capelli bitableaux $[S | T]$ is the image - with respect to the linear operator $B$ - of the right symmetrized bitableaux $(S | T)$.

**Theorem 6.7.** We have:
\[B: (S | T) \mapsto [S | T],
\]
\[K: [S | T] \mapsto (S | T).
\]

Since the set
\[\{(S | T); \text{sh}(S) = \text{sh}(T) = \lambda + h, \lambda_1 \leq n, S, T \text{ standard}\}\]
is the Gordan-Capelli basis of $\mathbb{C}[M_{n,n}]$ (see, e.g. [1], [3], [4]), then

**Corollary 6.8.** The set of right Young-Capelli bitableaux
\[\{[S | T]; \text{sh}(S) = \text{sh}(T) = \lambda, \lambda_1 \leq n, S, T \text{ standard}\}\]
is a (linear) basis of $U(gl(n))$.

## 7 Laplace expansions

### 7.1 Laplace expansions in $\mathbb{C}[M_{n,n}]$

Recall that
\[(i_1i_2\cdots i_h|j_1j_2\cdots j_h) = (-1)^{\binom{h}{2}} \det([i_s,j_t])_{s,t=1,2,\ldots,h} \in \mathbb{C}[M_{n,n}],
\]
and, therefore, the biproduct $(i_1i_2\cdots i_h|j_1j_2\cdots j_h) \in \mathbb{C}[M_{n,n}]$ expands into column bitableaux as follows:

\[(i_1i_2\cdots i_h|j_1j_2\cdots j_h) = \sum_{\sigma \in S_h} (-1)^{|\sigma|} \begin{vmatrix} i_{\sigma(1)} & j_1 \\ i_{\sigma(2)} & j_2 \\ \vdots & \vdots \\ i_{\sigma(h)} & j_h \end{vmatrix}
\]

\[= \sum_{\sigma \in S_h} (-1)^{|\sigma|} \begin{vmatrix} i_1 & j_{\sigma(1)} \\ i_2 & j_{\sigma(2)} \\ \vdots & \vdots \\ i_h & j_{\sigma(h)} \end{vmatrix}.
\]
Notice that, in the passage from monomials to column bitableaux, the sign $(-1)^{(2)}$ disappears.

Recall that

$$ (i_1i_2 \cdots i_h|j_1j_2 \cdots j_h)^* = \text{per}[(i_s|j_t)]_{s,t=1,2,\ldots,h} \in \mathbb{C}[M_{n,n}], $$

and, therefore, the *-biprodut $(i_1i_2 \cdots i_h|j_1j_2 \cdots j_h)^* \in \mathbb{C}[M_{n,n}]$ expands into column *-bitableaux as follows:

$$ (i_1i_2 \cdots i_h|j_1j_2 \cdots j_h)^* = \sum_{\sigma \in S_h} \begin{pmatrix} i_{\sigma(1)} & j_1 \\ i_{\sigma(2)} & j_2 \\ \vdots & \vdots \\ i_{\sigma(h)} & j_h \end{pmatrix}^* \quad = \quad \sum_{\sigma \in S_h} \begin{pmatrix} i_1 & j_{\sigma(1)} \\ i_2 & j_{\sigma(2)} \\ \vdots & \vdots \\ i_h & j_{\sigma(h)} \end{pmatrix}. $$

The preceding arguments extend to bitableaux and to *-bitableaux of any shape $\lambda$, $\lambda_1 \leq n$. Given the Young tableaux of of eq. (15)

$$ S = \begin{pmatrix} i_{p_1} & \cdots & \cdots & i_{p_{\lambda_1}} \\ i_{q_1} & \cdots & \cdots & i_{q_{\lambda_2}} \\ \vdots & \vdots & \cdots & \vdots \\ i_{r_1} & \cdots & i_{r_{\lambda_m}} \end{pmatrix}, \quad T = \begin{pmatrix} j_{s_1} & \cdots & \cdots & j_{s_{\lambda_1}} \\ j_{t_1} & \cdots & \cdots & j_{t_{\lambda_2}} \\ \vdots & \vdots & \cdots & \vdots \\ j_{v_1} & \cdots & j_{v_{\lambda_m}} \end{pmatrix}, \quad (15) $$

, a simple sign computation shows that

$$ (S|T) = \sum_{\sigma_1,\ldots,\sigma_m} (-1)^{\sum_{k=1}^m |\sigma_k|} \begin{pmatrix} i_{p_{\sigma_1(1)}} & j_{s_1} \\ \vdots & \vdots \\ i_{p_{\sigma_1(\lambda_1)}} & j_{s_{\lambda_1}} \\ \vdots & \vdots \\ i_{r_{\sigma_m(1)}} & j_{v_1} \\ \vdots & \vdots \\ i_{r_{\sigma_m(\lambda_m)}} & j_{v_{\lambda_m}} \end{pmatrix} $$

$$ = \sum_{\sigma_1,\ldots,\sigma_m} (-1)^{\sum_{k=1}^m |\sigma_k|} \begin{pmatrix} i_{p_1} & j_{s_{\sigma_1(1)}} \\ \vdots & \vdots \\ i_{p_{\lambda_1}} & j_{s_{\sigma_1(\lambda_1)}} \\ \vdots & \vdots \\ i_{r_1} & j_{v_{\sigma_m(1)}} \\ \vdots & \vdots \\ i_{r_{\lambda_m}} & j_{v_{\sigma_m(\lambda_m)}} \end{pmatrix} $$

where the multiple sums range over all permutations $\sigma_1 \in S_{\lambda_1}, \ldots, \sigma_m \in S_{\lambda_m}$.

Notice that only the signs of permutations remain.
Similarly,

\[
(S|T)^* = \sum_{\sigma_1, \ldots, \sigma_m} (\sigma_1, \ldots, \sigma_m) \cdot \left( \begin{array}{cc}
i_{p_{\sigma_1(1)}} & j_{s_{\sigma_1(1)}} \\
i_{p_{\sigma_1(\lambda_1)}} & j_{s_{\lambda_1}} \\
\vdots & \vdots \\
i_{r_{\sigma_m(\lambda_m)}} & j_{v_{\lambda_m}}
\end{array} \right)^*
\]

\[
= \sum_{\sigma_1, \ldots, \sigma_m} (\sigma_1, \ldots, \sigma_m) \cdot \left( \begin{array}{ccc}
i_{p_1} & j_{s_{\sigma_1(1)}} \\
i_{p_{\lambda_1}} & j_{s_{\lambda_1}} \\
\vdots & \vdots \\
i_{r_1} & j_{v_{\sigma_m(1)}} \\
i_{r_{\lambda_m}} & j_{v_{\lambda_m}}
\end{array} \right)^*.
\]

### 7.2 Laplace expansions in \( U(gl(n)) \)

Let \( S \) and \( T \) be the Young tableaux

\[
S = \left( \begin{array}{cccccc}
i_{p_1} & \ldots & \ldots & \ldots & \ldots & i_{p_{\lambda_1}} \\
i_{q_1} & \ldots & \ldots & \ldots & \ldots & i_{q_{\lambda_2}} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
i_{r_1} & \ldots & i_{r_{\lambda_m}} & \ldots & \ldots & \ldots
\end{array} \right), \quad T = \left( \begin{array}{cccccc}
j_{s_1} & \ldots & \ldots & \ldots & \ldots & j_{s_{\lambda_1}} \\
j_{t_1} & \ldots & \ldots & \ldots & \ldots & j_{t_{\lambda_2}} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
j_{v_1} & \ldots & j_{v_{\lambda_m}} & \ldots & \ldots & \ldots
\end{array} \right)
\]

of eq. (15).

By Theorem 6.5, the results of subsection 7.1 lead to the following Laplace expansion of Capelli bitableaux into column Capelli bitableaux and of Capelli *-bitableaux into column Capelli *-bitableaux.

**Proposition 7.1.**

\[
[S|T] = \sum_{\sigma_1, \ldots, \sigma_m} (-1)^{\sum_{k=1}^{m} |\sigma_k|} \left( \begin{array}{cc}
i_{p_{\sigma_2(1)}} & j_{s_1} \\
i_{p_{\sigma_2(\lambda_1)}} & j_{s_{\lambda_1}} \\
\vdots & \vdots \\
i_{r_{\sigma_m(\lambda_m)}} & j_{v_{\lambda_m}}
\end{array} \right)
\]

(16)
Example 7.3. One gets the explicit expansions as elements of $\text{Proposition 7.4}$. 

$$
\sum_{\sigma_1, \ldots, \sigma_m} (-1)^{\sum_{k=1}^m |\sigma_k|} \begin{bmatrix}
  i_{p_1} & j_{s_{\sigma_1(1)}} \\
  \vdots & \vdots \\
  i_{p_{\lambda_1}} & j_{s_{\sigma_1(\lambda_1)}} \\
  \vdots & \vdots \\
  i_{r_1} & j_{v_{\sigma_m(1)}} \\
  \vdots & \vdots \\
  i_{r_{\lambda_m}} & j_{v_{\sigma_m(\lambda_m)}}
\end{bmatrix}.
$$

(17)

Proposition 7.2.

$$
[S|T]^* = \sum_{\sigma_1, \ldots, \sigma_m} \begin{bmatrix}
  i_{p_{\sigma_1(1)}} & j_{s_1} \\
  \vdots & \vdots \\
  i_{p_{\sigma_1(\lambda_1)}} & j_{s_{\lambda_1}} \\
  \vdots & \vdots \\
  i_{r_{\sigma_m(1)}} & j_{v_1} \\
  \vdots & \vdots \\
  i_{r_{\sigma_m(\lambda_m)}} & j_{v_{\lambda_m}}
\end{bmatrix}.
$$

(18)

$$
= \sum_{\sigma_1, \ldots, \sigma_m} \begin{bmatrix}
  i_{p_1} & j_{s_{\sigma_1(1)}} \\
  \vdots & \vdots \\
  i_{p_{\lambda_1}} & j_{s_{\sigma_1(\lambda_1)}} \\
  \vdots & \vdots \\
  i_{r_1} & j_{v_{\sigma_m(1)}} \\
  \vdots & \vdots \\
  i_{r_{\lambda_m}} & j_{v_{\sigma_m(\lambda_m)}}
\end{bmatrix}.
$$

(19)

By combining eqs. (18), (22), (24), (19) with the results of Proposition 5.3, one gets the explicit expansions as elements of $\text{U}(gl(n))$.

Example 7.3. We have

$$
[12|12] = \begin{bmatrix}
  1 & 1 \\
  2 & 2 \\
\end{bmatrix} - \begin{bmatrix}
  1 & 1 \\
  2 & 2 \\
\end{bmatrix} = -e_{11}e_{22} + e_{21}e_{12} - e_{22}
$$

$$
= -\text{cDET} \begin{pmatrix}
  e_{11} + 1 & e_{12} \\
  e_{21} & e_{22}
\end{pmatrix} \in \text{U}(gl(n)).
$$

\[ \square \]

Proposition 7.4. Consider the row Capelli bitableau

$$
[n \cdots 21|12 \cdots n] \in \text{U}(gl(n)).
$$

We have:
1.

\[ [n \cdots 21|12 \cdots n] = \text{cdet} \begin{pmatrix} e_{11} + (n - 1) & e_{12} & \cdots & e_{1n} \\ e_{21} & e_{22} + (n - 2) & \cdots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{pmatrix}, \]

the central Capelli column determinant in \( \mathfrak{u}(gl(n)) \).

2.

\[ \mathcal{K} ([n \cdots 21|12 \cdots n]) = \text{det} \begin{pmatrix} (1|1) & \cdots & (1|n) \\ (2|1) & \cdots & (2|n) \\ \vdots & \vdots & \vdots \\ (n|1) & \cdots & (n|n) \end{pmatrix} \in \mathbb{C}[M_{n,n}]. \]

Proof. We have

\[ [n \cdots 21|12 \cdots n] = \sum_{\sigma \in S_n} (-1)^{|\sigma|} \begin{pmatrix} \sigma(n) & 1 \\ \sigma(n - 1) & 2 \\ \vdots & \vdots \\ \sigma(1) & n \end{pmatrix} \]

\[ = \sum_{\sigma \in S_n} (-1)^{|\sigma|} \times \]

\[ ((-1)^{n-1} e_{\sigma(n)1} \begin{pmatrix} \sigma(n - 1) & 2 \\ \sigma(n - 2) & 3 \\ \vdots & \vdots \\ \sigma(1) & n \end{pmatrix} + (-1)^{n-2} \sum_{k=1}^{n-1} \delta_{\sigma(k)1} \begin{pmatrix} \sigma(n - 1) & 2 \\ \vdots & \vdots \\ \sigma(1) & n \end{pmatrix}) \]

\[ = (-1)^{n-1} \sum_{\sigma \in S_n} (-1)^{|\sigma|} (e_{\sigma(n)1} + (n - 1)\delta_{\sigma(n)1}) \begin{pmatrix} \sigma(n - 1) & 2 \\ \sigma(n - 2) & 3 \\ \vdots & \vdots \\ \sigma(1) & n \end{pmatrix}, \]

by the expansion formula for column Capelli bitableaux (Proposition 5.3, item 1).

\footnote{The symbol \text{cdet} denotes the column determinat of a matrix \( A = [a_{ij}] \) with noncommutative entries: \( \text{cdet}(A) = \sum_{\sigma} (-1)^{|\sigma|} a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n} \).}
Example 7.5. The Capelli bitableau (of shape $\lambda = (2, 2)$):
\[
\begin{bmatrix}
1 & 2 & 2 & 3 \\
2 & 4 & 3 & 4
\end{bmatrix} \in U(gl(4))
\]
(20)
equals (by eq. (16))
\[
\begin{bmatrix}
1 & 2 \\
2 & 3 \\
4 & 4
\end{bmatrix} - \begin{bmatrix}
1 & 3 \\
2 & 2 \\
4 & 4
\end{bmatrix} - \begin{bmatrix}
1 & 2 \\
2 & 3 \\
4 & 4
\end{bmatrix} + \begin{bmatrix}
1 & 3 \\
2 & 2 \\
4 & 4
\end{bmatrix} \in U(gl(4)),
\]
where
\[
\begin{bmatrix}
1 & 2 \\
2 & 3 \\
4 & 4
\end{bmatrix} = e_{12}e_{23}e_{24}e_{44} - 2e_{13}e_{23}e_{44},
\]
(21)
\]
\[ \begin{bmatrix} 1 & 2 & 2 & 4 \\ 2 & 3 & 4 & 3 \end{bmatrix} = -e_{13}e_{22}e_{23}e_{44} + e_{13}e_{23}e_{44}, \quad (22) \]

\[ \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 2 & 4 \\ 4 & 3 \end{bmatrix} = -e_{12}e_{23}e_{24}e_{43} + e_{12}e_{23}e_{24}e_{43} - e_{13}e_{23} + e_{13}e_{23}, \quad (23) \]

\[ \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 2 & 4 \\ 4 & 3 \end{bmatrix} = e_{13}e_{22}e_{24}e_{43} - e_{13}e_{22}e_{23} - e_{13}e_{24}e_{43} - e_{13}e_{23}. \quad (24) \]

This example clarifies the difference between the PBW Theorem and the BCK Theorem.

The PBW Theorem establishes an isomorphism \( \phi \) from the graded object

\[ \text{Gr} \left[ \mathbf{U}(gl(n)) \right] = \bigoplus_{h \in \mathbb{Z}^+} \frac{\mathbf{U}^{(h)}(gl(n))}{\mathbf{U}^{(h-1)}(gl(n))} \]

associated to the filtered algebra \( \mathbf{U}(gl(n)) \) to the symmetric algebra \( \text{Sym}(gl(n)) \cong \mathbb{C}[M_{n,n}] \). By eqs. (21), (22), (24), (19), the image under \( \phi \) of the Capelli bitableau \( (20) - \text{regarded as an element of the quotient space} \mathbf{U}^{(0)}(gl(n)) \mathbf{U}^{(\infty)}(gl(n)) - \) is immediately recognized to be the product determinants

\[ \det \left( \begin{array}{cc} 1 & 2 \\ 2 & 2 \end{array} \right) \times \det \left( \begin{array}{cc} 2 & 4 \\ 4 & 4 \end{array} \right) \in \mathbb{C}[M_{4,4}]. \quad (25) \]

By the BCK Theorem, the product of determinants (25) is indeed the image under the Koszul isomorphism \( \mathcal{K} \) of the Capelli bitableau \( (20) - \text{regarded as an element of} \mathbf{U}(gl(4)) \). Furthermore, the image under \( \mathcal{K} \) of the Capelli *-bitableau

\[ \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 3 & 4 \end{bmatrix}^* \]

equals the product permanents

\[ \text{per} \left( \begin{array}{cc} 1 & 2 \\ 2 & 2 \end{array} \right) \times \text{per} \left( \begin{array}{cc} 2 & 3 \\ 4 & 4 \end{array} \right) \in \mathbb{C}[M_{4,4}]. \]

References

[1] A. Brini, Combinatorics, superalgebras, invariant theory and representation theory, Séminaire Lotharingien de Combinatoire 55 (2007), Article B55g, 117 pp.
[2] A. Brini, Superalgebraic Methods in the Classical Theory of Representations. Capelli’s Identity, the Koszul map and the Center of the Enveloping Algebra $\mathbf{U}(\mathfrak{gl}(n))$, in Topics in Mathematics, Bologna, Quaderni dell’Unione Matematica Italiana n. 15, UMI, 2015, pp. 1 – 27

[3] A. Brini, A. Palareti, A. Teolis, Gordan–Capelli series in superalgebras, Proc. Natl. Acad. Sci. USA 85 (1988), 1330–1333

[4] A. Brini, A. Teolis, Young–Capelli symmetrizers in superalgebras, Proc. Natl. Acad. Sci. USA 86 (1989), 775–778.

[5] A. Brini, A. Teolis, Capelli bitableaux and $\mathbb{Z}$-forms of general linear Lie superalgebras, Proc. Natl. Acad. Sci. USA 87 (1990), 56–60

[6] A. Brini, A. Teolis, Capelli’s theory, Koszul maps, and superalgebras, Proc. Natl. Acad. Sci. USA 90 (1993), 10245–10249

[7] A. Brini, A. Teolis, Central elements in $\mathbf{U}(\mathfrak{gl}(n))$, shifted symmetric functions and the superalgebraic Capelli’s method of virtual variables, preliminary version, Jan. 2018, arXiv: 1608.06780v4, 73 pp.

[8] A. Brini, A. Teolis, Young-Capelli bitableaux, Capelli immanants in $\mathbf{U}(\mathfrak{gl}(n))$ and the Okounkov quantum immanants, July 2018, arXiv: 1807.10045v1, 52 pp.

[9] A. Capelli, Ueber die Zurückführung der Cayley’schen Operation $\Omega$ auf gewöhnliche Polar-Operationen, Math. Ann. 29 (1887), 331-338

[10] A. Capelli, Lezioni sulla teoria delle forme algebriche, Pellerano, Napoli, 1902, available at <https://archive.org/details/lezionisullateo00capegoog>.

[11] C. De Concini, D. Eisenbud, C. Procesi, Young diagrams and determinantal varieties, Invent. Math. 56 (1980), 129–165.

[12] J. Désarménien, J. P. S. Kung, G.-C. Rota, Invariant theory, Young bitableaux and combinatorics, Adv. Math. 27 (1978), 63–92

[13] P. Doubilet, G.-C. Rota, J. A. Stein, On the foundations of combinatorial theory IX. Combinatorial methods in invariant theory, Studies in Appl. Math. 53 (1974), 185–216

[14] F. D. Grosshans, G.-C. Rota and J. A. Stein, Invariant Theory and Superalgebras, AMS, 1987

[15] J.-L. Koszul, Les algèbres de Lie graduées de type $\mathfrak{sl}(n,1)$ et l’opérateur de A. Capelli, C. R. Acad. Sci. Paris Sér. I Math. 292 (1981), no. 2, 139-141

[16] C. Procesi, Lie Groups. An approach through invariants and representations, Universitext, Springer, 2007

[17] H. Weyl, The Classical Groups, 2nd ed., Princeton University Press, 1946