An Algorithm for Ennola’s Second Theorem and Counting Smooth Numbers in Practice

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Abstract

Let \( \Psi(x, y) \) count the number of positive integers \( n \leq x \) such that every prime divisor of \( n \) is at most \( y \). There are a number of applications where values of \( \Psi(x, y) \) are needed, such as in optimizing integer factoring and discrete logarithm algorithms \([9, 19]\) and generating factored smooth numbers uniformly at random \([3]\). Note that such numbers are useful in at least one post-quantum cryptography protocol \([8, 21]\).

Given inputs \( x \) and \( y \), what is the best way to estimate \( \Psi(x, y) \)? We address this problem in three ways: with a new algorithm to estimate \( \Psi(x, y) \), with a performance improvement to an established algorithm, and with empirically based advice on how to choose an algorithm to estimate \( \Psi \) for the given inputs.

Our new algorithm to estimate \( \Psi(x, y) \) is based on Ennola’s second theorem \([10]\), which applies when \( y < (\log x)^{3/4-\epsilon} \) for \( \epsilon > 0 \). It takes \( O(y^2/\log y) \) arithmetic operations of precomputation and \( O(y \log y) \) operations per evaluation of \( \Psi \).

We show how to speed up Algorithm HT \([16]\), which is based on the saddle-point method of Hildebrand and Tenenbaum \([14]\), by a factor proportional to \( \log \log x \), by applying Newton’s method in a new way.

And finally we give our empirical advice based on five algorithms to compute estimates for \( \Psi(x, y) \). The challenge here is that the bound-
aries of the ranges of applicability, as given in theorems, often include unknown constants or small values of \( \epsilon > 0 \), for example, that cannot be programmed directly.

1 Introduction

Let \( \Psi(x, y) \) count the number of integers \( n \leq x \) such that the largest prime divisor of \( n \) is \( \leq y \). There are a variety of algorithms to estimate the value of \( \Psi(x, y) \) in the literature \[4, 5, 6, 7, 10, 16, 17, 20, 22, 24, 25, 27\]. All these methods have one drawback or another. Some have very slow runtimes, some are inaccurate in practice, and some have very limited ranges of applicability.

To make this even more difficult, in most cases, the theorems that provide a region of the \( x/y \) plain where the algorithm’s accuracy has a guarantee is not specified explicitly. This makes the boundaries of such regions impossible to program.

In this paper, we try to determine the best way to estimate \( \Psi(x, y) \) for specific values of \( x \) and \( y \) in practice. To do this, we implemented many algorithms for \( \Psi(x, y) \) and explored their boundaries of applicability empirically. We present these results in §4.

In the process of our study, we noticed that the second theorem in Ennola’s paper \[10\] had not been tried, to our knowledge. The first theorem in that paper is well known and is quoted, for example, in \[26, \S 5.2\]. We found that the range of applicability of this second theorem, in practice, far, far exceeds its proven guarantee. So we begin with an exposition of our algorithm based on Ennola’s second theorem, and we analyze its running time below in §2. We believe this algorithm is completely new.

We also show, in §3 how to trim a factor proportional to \( \log \log x \) from the running time of Algorithm HT \[16\].

We conclude in §5 with some comments.

2 An Algorithm Based on Ennola’s Second Theorem

We begin this section by reviewing Ennola’s second theorem, then we present the algorithm, we give an analysis of the running time and the space used, and we conclude with some practical notes and data on the algorithm’s accuracy in practice.
2.1 Ennola’s Theorem

Let $2 \leq y < x$. Ennola’s theorem applies when $y$ is very small, namely when $y \leq (\log x)^{3/4-\epsilon}$ for $\epsilon > 0$.

We need a few definitions.

First, define the sequence $c_n$ with $c_0 = 1$, $c_1 = 1/2$, $c_{2k+1} = 0$ for $k \geq 1$, and

$$c_{2k} = (-1)^k \frac{2\zeta(2k)}{(2\pi)^{2k}}$$

for $k \geq 1$; here $\zeta$ is the Riemann zeta function.

Next, define $d_n$ to be the coefficient of $s^n$ in the following power series:

$$\prod_{p \leq y} \left( \sum_{k=0}^{\infty} c_k (\log p)^k s^k \right).$$

Finally, define

$$R(t) := \sum_{n=0}^{\pi(y)} \frac{d_n t^n}{(\pi(y) - n)!}.$$  \hfill (2)

**Theorem 1** (Ennola \cite{10}) Let $2 \leq y \leq (\log x)^{0}$ with $0 < \theta < 3/4$. Then

$$\Psi(x,y) = \left( \prod_{p \leq y} \frac{\log x}{\log p} \right) R(1/\log x) \left( 1 + O((\log x)^{-1/8+\epsilon}) \right)$$

Ennola also gave an upper bound for the tail of the sum defining $R(t)$, which will enable us to compute with only the first few terms. Let $m$ be an integer with $2 \leq m \leq \pi(y)$. In practice, we ended up using $m = \pi(y)$ almost always.

Define

$$R_m(t) := \sum_{n=0}^{m} \frac{d_n t^n}{(\pi(y) - n)!}. \hfill (3)$$

Then $R(t) = R_{\pi(y)}(t)$. Ennola \cite{10} (27), p.9 proved

**Lemma 1.1** Let $m \geq n_0$ where $n_0 = \max\{e \log y, ey^2/(\log x \log y)\}$. If $0 < t < 1$, then

$$R(t) - R_m(t) = \sum_{n=m+1}^{\pi(y)} \frac{d_n t^n}{(\pi(y) - n)!}.$$
\[ \leq \sum_{n=m+1}^{\pi(y)} \frac{|d_n| t^n}{(\pi(y) - n)!} \]
\[ < \frac{1}{2^m \pi(y)!} \]

Note that since \( d_0 = 1 \), the first term of the sum is \( 1/\pi(y)! \), and in fact Ennola showed that \( R(1/\log x) \gg 1/\pi(y)! \) [10] (23),(28)]. Given this, it is not too surprising that Ennola was able to show that if \( m \geq n_0 \) then we have

\[ R(1/\log x) = R_m(1/\log x)(1 + O(2^{-m})). \]

Below, we will choose \( m \geq \log_2 \log x \), allowing us to compute \( R_m \) in place of \( R \) with no ill effect on the overall relative error.

2.2 The Algorithm

2.2.1 Precomputation

Computing the \( d_n \) coefficients is the bottleneck of the algorithm, but we can precompute them since they are the same for all \((x, y)\) input pairs with the same \( y \) value. And in fact, if we know beforehand the maximum \( y \) we will need to accept as input, we can find the \( d_n \) coefficients for all \( y' \leq y \) along the way for no extra cost, aside from the space needed to store the coefficients.

Normally we use \( m = \pi(y) \) for precomputation, unless we have a reasonably tight range of values for \( x \) that we can bound beforehand. We address this situation further below and give a tighter runtime analysis for precomputation for when this is the case.

1. Compute the list of primes up to \( y \) using a sieve.

2. Compute \( \zeta(2k) \) for integers \( k, 0 < 2k \leq m \). We do this by noting that \( \zeta(2) = \pi^2/6 \) and using the recurrence

\[ \zeta(2k) = \frac{1}{k+1/2} \sum_{j=1}^{k-1} \zeta(2j) \zeta(2k - 2j). \]

3. Compute the \( c_n \) for \( 0 \leq n \leq m \) using \([1]\).

4. Next, we compute the \( d_n \) for \( 0 \leq n \leq m \). Recall that \( d_k \) is defined as the coefficient of \( s^k \) in \([2]\).
Define the polynomial $f_{p,m}(x) := \sum_{k=0}^{m} c_k (\log p)^k x^k$ for $p$ a prime. Compute $F$ as follows:

\[
F := f_{2,m}(x);
\]
for each prime $p$, $2 < p \leq y$:
\[
F := F \cdot f_{p,m}(x) \text{ mod } x^{m+1};
\]

The coefficients $d_n$ for $n \leq m$ are then the coefficients of $F$.

To save the $d_k$ coefficients for $y' < y$, one simply pulls their values off $F$ in the last step immediately after all primes $p \leq y'$ have been processed.

### 2.2.2 The Algorithm

With the $d_n$ coefficients precomputed, the algorithm is as follows.

1. Set $m := \lfloor \min\{\pi(y), \max\{\log_2 \log x, e \log y, ey^2/(\log x \log y)\}\} \rfloor$.
2. Compute $R_m(1/\log x)$ using (3).
3. Compute $T := \prod_{p \leq y} \frac{\log x}{\log p}$ and output the estimate $T \cdot R_m(1/\log x)$.

### 2.3 Complexity Analysis

We assume basic arithmetic operations on integers and floating-point numbers, such as addition and multiplication, take constant time. We also assume that basic special functions, like log and exp take constant time. In practice, we used the standard long double data type in C/C++.

To maintain a relative error of $1+O(1/\log x)$, we need at least $O(\log \log x)$ bits of precision in all our calculations. We will measure space in the number of machine words, under the assumption that each word holds one floating point number of the necessary precision.

#### 2.3.1 Precomputation

1. The primes up to $y$ can be found using $O(y/\log \log y)$ arithmetic operations; see [1]. Storing these primes takes $O(y/\log y)$ words of space.
2. This takes $\sum_{k=1}^{m/2} (k - 1) = O(m^2)$ time. $O(m)$ words are required to store the $\zeta$ function values.
3. This is $O(m)$ operations if you are careful about how the powers of $2\pi$ are computed. Again, $O(m)$ words of space are needed for the $c_n$ values.
4. Computing each $f_{p,m}(x)$ takes $O(m)$ operations, if the terms are computed from low degree to high.

$O(m \log m)$ operations are needed for the convolutions to compute a single polynomial product, using FFT techniques.

The total time, then, is $O(\pi(y)m \log m)$ for this step. It uses $O(m)$ words of space.

In practice we used a simple $O(m^2)$ algorithm for polynomial multiplication.

The total time for precomputation is $O(\pi(y)m \log m)$ or, when $m = \pi(y)$, $O(\pi(y)^2 \log y) = O(y^2/\log y)$ for all precomputation up to $y$.

The total space is $O(\pi(y)^2)$ to store all the $d_n$ coefficients for all $y' \leq y$.

### 2.3.2 The Algorithm

1. This is constant time and space.

2. Computing $R_m(1/\log x)$ takes $O(m + \pi(y))$ operations. Note that powers of $1/\log x$ and the factorial denominators should be computed in opposite directions first.

   Again, $O(m)$ words of space suffice for this step.

3. Computing $T$ takes $O(\pi(y))$ operations and constant space.

So after precomputation, the time is $O(\pi(y))$ operations to compute an estimate for $\Psi(x, y)$, independent of $x$ (or $m$).

If we know $x$, or have a bound on its range relative to $y$, then computing $m$ may save a bit of precomputation time in some cases, as the following table shows.

| Ranges for $x, y$ | $m$ | Time (ops) |
|-------------------|-----|------------|
| $\pi(y) \leq \log_2 \log x$ | $m = \pi(y)$ | $O((\log \log x)^2 \log \log \log x)$ |
| $\log_2 \log x < \pi(y) \leq \sqrt{\log x}$ | $m = \log_2 \log x$ | $O(\pi(y) \log \log x \log \log \log x)$ or $O(\sqrt{\log x} \log \log x \log \log \log x)$ |
| $\sqrt{\log x} < \pi(y) \leq \pi(\log x)$ | $m = ey^2/(\log x \log y)$ | $O(y^2/\log x \log y)$ |
| $\log x \ll y$ | $m = \pi(y)$ | $O(y^2/\log y)$ |

All cases are bounded by $O(y^2/\log y)$. Note that the last row of the table is outside the guaranteed range given in Ennola’s second theorem.
2.4 Practical Notes

2.4.1 Computing $T \cdot R_m(1/\log x)$

In practice, computing steps 2 and 3 of the main algorithm, especially 2, can lead to overflow or underflow if using fixed precision floating point numbers, such as the `long double` datatype in C++. In addition to this, if one is careful, it is possible to evaluate $R$ in time linear in $\pi(y)$, as we will now show.

Define

$$f_n := \frac{(\log x)^{\pi(y)-n}}{(\pi(y) - n)!}.$$ 

Then we have $f_n = f_{n+1} \cdot (\log x)/(\pi(y) - n)$ when $0 \leq n < \pi(y)$, and $f_{\pi(y)} = 1$. This gives us

$$T \cdot R_m(1/\log x) = \prod_{p \leq y} \frac{\log x}{\log p} \sum_{n=0}^{m} d_n (\log x)^{\pi(y)-n} (\pi(y) - n)!$$

$$= \prod_{p \leq y} \frac{1}{\log p} \sum_{n=0}^{m} d_n (\log x)^{\pi(y)-n} (\pi(y) - n)!$$

$$= \prod_{p \leq y} \frac{1}{\log p} \sum_{n=0}^{m} d_n \cdot f_n.$$ 

The following pseudocode fragment will compute this:

```plaintext
fn := 1;
if m = \pi(y)
   then sum := d_{\pi(y)} \cdot fn
   else sum := 0;
endif;
for n = \pi(y) - 1 downto 0 do:
   fn := fn \cdot (\log x)/(\pi(y) - n);
   if n \leq m then sum := sum + d_n \cdot fn; endif;
endfor;
P := 1;
for each prime p \leq y do:
P := P \cdot 1/(\log p);
endfor;
output sum \cdot P;
```
Accuracy

We conclude this section with empirical results on the accuracy of this new algorithm. In the table below we give the ratio of the value given by our new algorithm over the exact value of $\Psi(x,y)$ for various values of $x, y$.

| $y$   | $x = 2^{15}$ | $x = 2^{20}$ | $x = 2^{25}$ | $x = 2^{30}$ | $x = 2^{33}$ |
|-------|--------------|--------------|--------------|--------------|--------------|
| $2^{5}$ | 0.999972     | 1.00009      | 1.00001      | 0.999978     | 0.999998     |
| $2^{10}$ | 1.0083       | 0.995777     | 1.00115      | 0.999969     | 1.00052      |
| $2^{15}$ | --           | 1.00219      | 1.00472      | 0.994501     | 1.00183      |

Note that when $y = 32 = 2^5$, for Ennola’s second theorem to apply, we would require that $y < (\log x)^{3/4}$; this would imply $x > \exp 32^{4/3}$, a 44-digit number. So this method seems to apply to a much wider $x, y$ range than is currently proven. And, although its preprocessing makes it very slow for larger $y$, it seems to be as accurate, if not more accurate, than Algorithm HT [16].

3 Improving Algorithm HT

At a high level, Algorithm HT [16] uses Newton’s method to find the zero, $\alpha$, of a continuous function. Our idea to improve the algorithm is to first find an approximation to $\alpha$, called $\alpha_f$, using the version of Algorithm HT that assumes the Riemann Hypothesis to bound the error when estimating the distribution of primes, allowing for faster summing of functions of primes [22]. Then, starting from $\alpha_f$, Newton’s method is applied in the context of the original Algorithm HT, allowing for much faster convergence, often requiring only one iteration in practice, yet providing the same level of accuracy as Algorithm HT.

We begin this section with a review of Algorithms HT and HT-fast, then present our new twist, Algorithm HT$\alpha$, and wrap up the section with some implementation results.

3.1 Algorithm HT and Algorithm HT-fast

A theorem from Hildebrand and Tenenbaum [14] gives us

$$\Psi(x,y) \approx HT(x,y,\alpha(x,y))$$

uniformly for $2 \leq y \leq x$, where

$$HT(x,y,s) := \frac{x^s \zeta(s,y)}{s \sqrt{2\pi \phi_2(s,y)}}$$
and \( \alpha(x, y) \) is the unique solution to

\[
\phi_1(\alpha, y) + \log x = 0.
\]

Here,

\[
\zeta(s, y) = \prod_{p \leq y} (1 - p^{-s})^{-1},
\]

\[
\phi_1(s, y) = -\sum_{p \leq y} \frac{\log p}{p^s - 1},
\]

\[
\phi_2(s, y) = \sum_{p \leq y} \frac{p^s (\log p)^2}{(p^s - 1)^2}.
\]

Algorithm HT [10]:

1. Find all primes \( p \leq y \).

2. Starting at \( \alpha_0 := \log(1 + y/(5 \log x))/\log y \), approximate \( \alpha \) by \( \alpha' \), where \( s = \alpha \) is the solution to \( f(s) = 0 \) where

\[
f(s) := \phi_1(s, y) + \log x,
\]

using Newton’s Method. We require that \( |\alpha' - \alpha| < \min\{0.0001, 1/(\overline{u} \log x)\} \)

where \( \overline{u} = \min\{\log x/\log y, y/\log y\} \).

3. Output \( HT(x, y, \alpha') \).

The overall running time is

\[
O\left(y \left[ \frac{\log \log x}{\log y} + \frac{1}{\log \log y} \right] \right).
\]

We have

\[
f(s) = \phi_1(s, y) + \log x \text{ and } f'(s) = \phi_2(s, y)
\]

So, our iteration function \( g \) for Newton’s Method is

\[
g(s) := s - f'(s)/f(s).
\]

- This algorithm has a running time of \( O(\pi(y) \log \log x) \): \( O(\log \log x) \) iterations of Newton’s method to converge, with each iteration requiring a sum over the primes \( \leq y \) to evaluate \( \phi_1 \).

- In practice, 5-6 iterations suffice for Newton’s Method to converge.
Algorithm HT-fast \[22\] estimates the functions \( \zeta, \phi_1, \phi_2 \) using the prime number theorem; the Riemann Hypothesis is used to bound the error. The error is also controlled by evaluating an initial segment over the primes up to \( O(\sqrt{y}) \).

- Set \( z := \min\{y, \max\{1000, 5\sqrt{y}\}\} \). We have \( \phi_1(s, y) \approx -B(s, y, z) \)

\[
B(s, y, z) = \sum_{p \leq z} \frac{\log p}{p^s - 1} + \sum_{k=1}^{[(\log y)/s]} \frac{1}{1 - ks} \left( y^{1-ks} - z^{1-ks} \right).
\]

The functions \( \zeta, \phi_2 \) are similarly approximated. This version of the algorithm is much faster, taking time proportional to \( \sqrt{y} \), but gives estimates that, though still good, are not quite as accurate as Algorithm HT in practice.

### 3.2 Our New Algorithm HT\(\alpha\)

Here are the steps, following the idea outlined at the beginning of this section.

1. Find all primes \( p \leq y \)

2. Starting at \( \alpha_0 := \log(1+y/(5 \log x))/\log y \), compute an approximation \( \alpha_f \) to the solution \( s = \alpha_1 \) of \( f_1(s) = 0 \) where

\[
f_1(s) := -B(s, y, z) + \log x.
\]

We must have \( |\alpha_1 - \alpha_f| < \min\{0.000001, 1/(\pi \log x)\} \).

3. Using \( \alpha_f \) as a starting point, compute the approximation \( \alpha \) to the solution \( s = \alpha_2 \) of \( f_2(s) = 0 \) where

\[
f_2(s) := \phi_1(s, y) + \log x,
\]

as before. We must have \( |\alpha_2 - \alpha| < \min\{0.0001, 1/(\pi \log x)\} \).

4. Output \( HT(x, y, \alpha) \)

The running times for each of the steps of Algorithm HT\(\alpha\) are as follows:

1. \( O(y/\log \log y) \)

2. \( O(\sqrt{y} \log \log x/\log y) \)
3. $O(y/\log y)$ per iteration

4. $O(y/\log y)$

Observations:

- We can often treat step (1) as a preprocessing step. In practice, we often already have a list of primes available.

- In practice, we observed that step (3) will only run for 1-2 iterations.

- In theory, one iteration suffices for step (3) if $(\log y)^2 \gg \log x$, as the accuracy guarantee for HT-fast matches that of HT to within a factor of $(1 + O(\log y/\log x + 1/\log y))$, which matches $(1 + O(1/\pi))$, the relative error of Algorithm HT, in this case. In general, however, $O(\log\log x)$ iterations may be required if $y$ is extremely small compared to $x$, but in this case Algorithm HT is already fast.

- In no situation should Algorithm HT$_{\alpha}$ be slower than Algorithm HT.

- Algorithm HT-fast relies on the Riemann Hypothesis (RH) for correctness, but HT$_{\alpha}$ only relies on the RH for running time.

Thus, our overall running time is reduced to $O(\pi(y))$, under the assumption we have a list of primes available and $y$ is not extremely small compared to $x$.

For references on prime sieves, see [1, 12, 23].

### 3.3 Implementation Results

Algorithm HT$_{\alpha}$ has comparable error to Algorithm HT and a faster running time.

The following table is a comparison of error, time (in seconds) and Newton’s Method iterations (Its.) per algorithm and by step in the case of HT$_{\alpha}$. Here $x = 2^{30}$:

| $y$ | $HT/\Psi$ | Time | Its. | $HT_{\alpha}/\Psi$ | Time | Its. (2) | Its. (3) |
|-----|-------------|-------|------|---------------------|-------|----------|----------|
| $2^{15}$ | 1.004 | 0.027 | 5 | 1.014 | 0.0006 | 6 | 1 |
| $2^{20}$ | 1.031 | 0.765 | 6 | 1.034 | 0.002 | 6 | 1 |
| $2^{25}$ | 1.018 | 18.599 | 6 | 1.019 | 6.978 | 6 | 1 |

The graphs in Figure[1] give a bit more data visually.
4 Estimating $\Psi(x, y)$

In this section, we give advice on the best way to compute values of $\Psi(x, y)$ for various ranges of $x$ and $y$ in practice.

We considered the following algorithms to estimate $\Psi(x, y)$, roughly ordered by how large $y$ is relative to $x$, starting with methods that work best for small $y$.

- **Buchstab’s Identity**, stated below, directly implies a simple recursive algorithm that gives exact values of $\Psi(x, y)$. However, with a running time roughly proportional to the value of $\Psi(x, y)$, it is only useful for very small inputs.

$$\Psi(x, y) = 1 + \sum_{p \leq y} \Psi(x/p, p)$$

In practice, we supplement with the base case $\Psi(x, 2) = \lfloor \log_2 x \rfloor + 1$ as well.

- **Ennola’s Second Theorem** was discussed in detail in §2. It is provably useful for when $y \leq (\log x)^{3/4-\epsilon}$, but in practice we found it to be perfectly fine so long as its running time is tolerable. Precomputation requires an $O(y^2/\log y)$ running time, which is quite high, but if this information is saved, then the time to compute specific $\Psi(x, y)$ values drops to a more reasonable $O(y \log y)$ time. It is highly accurate in practice.

- **Algorithm HT** (or HT$\alpha$) was discussed above in §3. This method is provably accurate for $2 \leq y \leq x$, but is a bit slow with a running time
roughly linear in $y$. Its running time is similar to Ennola’s method if precomputation is allowed, and much faster if not. Also, less extra space is required for Algorithm HT. Ennola’s method is a bit more accurate, but not provably so.

- **Algorithm HT-fast** is the version of the previous algorithm where sums of primes are estimated using the Riemann Hypothesis to bound the error. It is a bit less accurate than Algorithm HT, but much faster with a running time proportional to $\sqrt{y}$ and the same wide range of applicability.

- The **Dickman $\rho$ estimate** gives

$$
\Psi(x, y) \approx x \cdot \rho(u) + (1 - \gamma) \frac{x}{\log x} \rho(u - 1)
$$

---

Figure 2: Algorithm Recommendations
Figure 3: Algorithm Recommendations (small $x$)

where $u = \log x / \log y$ and $\rho(u)$ is the unique solution of the following equations:

$$
\rho(u) = 1 \quad (0 \leq u \leq 1), \\
\rho'(u) = -\rho(u-1)/u \quad (u \geq 1).
$$

Note that in the literature, one normally sees

$$
\Psi(x, y) \sim x \cdot \rho(u),
$$

but we find adding the second term is worthwhile in improved accuracy. This estimate is valid when $y \geq L(x)$, with $L(x) = (\log x)^{2+\epsilon}$ assuming the Extended Riemann Hypothesis (ERH). Without the ERH, the lower limit on $y$ is much larger, $\exp(\log_2 x)^{5/3+\epsilon}$ for $\epsilon > 0$ [13]. This
method is very fast; with precomputation of the \( \rho \) function, evaluations of \( \Psi(x, y) \) take constant time. \( \rho(u) \) can be computed reasonably quickly using numeric integration; see \[27, 16\]. It is the least accurate of the methods presented here, but its very fast computing time make it desirable, especially for large \( y \), where its accuracy is tolerable in practice.

In Figures 2 and 3 are plots indicating approximately where we recommend one uses each method to compute \( \Psi(x, y) \). They are plotted using a logarithmic scale (base 2) in both \( x \) and \( y \). The key to the graphs is as follows:

- Use Buchstab’s identity — Buchstab takes 1.5 seconds
- Use Ennola’s Second Theorem — \( y = (\log x)^{3/4} \)
- Use HT (Ennola’s is accurate too) — Ennola max (gets slow)
- Use HT (or HT\( \alpha \)) — Switch from HT to HT-Fast
- Use HT-Fast — \( y = L(x) = (\log x)^{2.5} \)
- Use \( x \cdot \rho(u) + (1 - \gamma) \frac{x}{\log x} \rho(u - 1) \) — \( y = x \)

**Methodology and Notes**

- We implemented all the algorithms in C++, we used the standard GNU g++ compiler, and the code was run on a Linux server using standard Intel hardware.

- In \[16\] it was shown that Algorithm HT is extremely accurate in practice, and so we used either Buchstab’s algorithm, or Algorithm HT when Buchstab was too slow, as a baseline for accuracy.

- We collected data on the accuracy and speed of all the algorithms over the range of \( x, y \) values shown in the graphs, except for some algorithms that got too slow.

- After the data was collected, we determined the \( x, y \) ranges where each algorithm was reasonably accurate in practice.

- We varied the value of \( \epsilon \) in the ERH cutoff \( y = L(x) = (\log x)^{2+\epsilon} \) for the Dickman \( \rho \) method to see what worked best in practice.
5 Concluding Remarks

- There are algorithms that give explicit upper and lower bounds on \( \Psi(x, y) \); see \([6, 7, 18, 20]\). Such methods tend to be noticeably slower than the methods used here, which is why we did not consider them. That said, they do have their purposes.

- We currently have no theory as to why the algorithm based on Ennola’s second theorem has, in practice, applicability on a range as wide as that of Algorithm HT. It may be that a closer examination of Ennola’s proof may yield a way to improve the range.

- Is it possible to speed up the algorithm from \( \S2 \) using the prime number theorem to estimate the sums and products over primes, perhaps bounding the error using the ERH as was done in \([20, 22]\)?

- In \([4]\) it was shown how to use LMO summation to improve the running time of Algorithm HT to \( y^{2/3+\epsilon} \). It stands to reason that HT-fast can be done in time \( y^{1/3+\epsilon} \) as well. As far as the authors are aware, this has not yet been implemented.

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