HOMOTOPY TYPE OF FROBENIUS COMPLEXES II

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Abstract. A finitely generated additive submonoid Λ of \( \mathbb{N}^d \) has the partial order defined by \( \lambda \leq \lambda + \mu \) for \( \lambda, \mu \in \Lambda \). The Frobenius complex is the order complex of an open interval of \( \Lambda \). In this paper, we express the homotopy type of the Frobenius complex of \( \Lambda[\rho/r] \) in terms of those of \( \Lambda \), where \( \Lambda[\rho/r] \) is the additive monoid which is added the \( r \)-th part of \( \rho \) to \( \Lambda \). As applications, we determine the homotopy type of the Frobenius complex of some submonoids of \( \mathbb{N} \), for example, the submonoid generated by a finite geometric sequence. We also state an application to the multi-graded Poincaré series.

1. Introduction

We consider a finitely generated additive monoid \( \Lambda \) which is cancellative and has no non-zero invertible element, for example, a finitely generated additive submonoid of \( \mathbb{N}^d \). Then the partial order \( \leq \) on \( \Lambda \) is defined by \( \lambda \leq \lambda + \mu \) for \( \lambda, \mu \in \Lambda \). The Frobenius complex \( \mathcal{F}(\lambda; \Lambda) \) is the order complex \( \| (0, \lambda) \| \) of the open interval of \( \Lambda \).

Laudal and Sletsjøe [LS85] proved that the graded component \( \text{Tor}_{i,\Lambda}^{K[\Lambda]}(K, K) \) of the torsion group over the monoid algebra \( K[\Lambda] \) is isomorphic to the reduced homology group \( \tilde{H}_{i-2}(\mathcal{F}(\lambda; \Lambda); K) \) of the Frobenius complex with coefficients in \( K \), and showed, as an application, that the Poincaré series of a saturated rational submonoid \( \Lambda \) of \( \mathbb{N}^2 \) is given by a rational function.

The multi-graded Poincaré series \( P_{\Lambda}(t, z) \) of \( \Lambda \) is defined by

\[
P_{\Lambda}(t, z) = \sum_{i \in \mathbb{N}} \sum_{\lambda \in \Lambda} \dim_K \text{Tor}_{i,\Lambda}^{K[\Lambda]}(K, K) \cdot t^i z^\lambda,
\]

and it is an open question whether \( P_{\Lambda}(t, z) \) is given by a rational function or not for an additive monoid \( \Lambda \) (see [PRS98]).

Clark and Ehrenborg [CE12] determined the homotopy type of the Frobenius complexes of some additive submonoids of \( \mathbb{N} \). For example, it is shown that the Frobenius complex of the submonoid of \( \mathbb{N} \) generated by two elements is either contractible or homotopy equivalent to a sphere, and the multi-graded Poincaré series is determined and proved to be rational. The proof is based on discrete Morse theory.

In this paper, we establish a method to calculate the homotopy type of the Frobenius complexes. We construct the additive monoid \( \Lambda[\rho/r] \) which is added the \( r \)-th part of \( \rho \) to \( \Lambda \), and show that for \( \lambda \in \Lambda \) and \( k \in \mathbb{N}^{<r} \) the Frobenius complex

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of \( \Lambda[\rho/r] \) satisfies
\[
\mathcal{F}(\lambda + k\rho/r; \Lambda[\rho/r]) \simeq \begin{cases} 
\bigvee_{\ell \rho \leq \lambda} \Sigma^{2\ell + k} \mathcal{F}(\lambda - \ell\rho; \Lambda) & \text{if } k \leq 1, \\
\text{pt} & \text{if } k \geq 2,
\end{cases}
\]
when \( \rho \) is a reducible element of \( \Lambda \) and \( r \geq 2 \). This is the main theorem of this paper. The proof is based on the basic technique of homotopy theory for posets and CW complexes. Applying the main theorem, we derive a formula for the multi-graded Poincaré series of \( \Lambda[\rho/r] \), namely
\[
P_{\Lambda[\rho/r]}(t, z) = \frac{P_{\Lambda}(t, z) \cdot (1 + tz^{\rho/r})}{1 - t^2z^\rho}.
\]

As applications, we show that the Frobenius complex \( \mathcal{F}(\lambda; \Lambda) \) of \( \Lambda \) is homotopy equivalent to a wedge of spheres (the dimensions may not coincide), and determine the multi-graded Poincaré series of \( \Lambda \), which is proved to be rational, for some submonoids \( \Lambda \) of \( \mathbb{N} \), for example, the submonoid of \( \mathbb{N} \) generated by the geometric sequence \( p^n, p^{n-1}q, \ldots, pq^{n-1}, q^n \). This gives an answer to a question raised by Clark and Ehrenborg (CE12, Question 6.4).

This paper is organized in the following way. Section 2 presents some preliminaries. In Section 3 the main theorem is stated and proved. Section 4 is devoted to applications to the multi-graded Poincaré series. In Section 5 we calculate some examples using the results of previous sections.

2. Preliminaries

2.1. Topology. In this paper, \( S^n \) denotes the \( n \)-sphere for \( n \geq 0 \), namely
\[
S^n = \{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \cdots + x_n^2 = 1 \},
\]
\( S^{-1} \) the empty space, and \( \text{pt} \) the one-point space. We introduce a formal symbol \( S^{-2} \). Fix a field \( K \). We define the reduced Betti number \( \tilde{\beta}_i(X) \) for a non-empty topological space \( X \) and \( i \in \mathbb{Z} \) by
\[
\tilde{\beta}_i(X) = \begin{cases} 
\dim_K \tilde{H}_i(X; K) & \text{if } i \geq 0, \\
0 & \text{if } i < 0.
\end{cases}
\]
For the empty space \( S^{-1} \), we define
\[
\tilde{\beta}_i(S^{-1}) = \delta_{i,-1} \quad (i \in \mathbb{Z}),
\]
where \( \delta_{i,j} \) denotes the Kronecker’s delta. We also define
\[
\tilde{\beta}_i(S^{-2}) = \delta_{i,-2} \quad (i \in \mathbb{Z}).
\]

By \( \Sigma X \) we denote the (unreduced) suspension of a topological space \( X \). Note that \( \Sigma S^{-1} \approx S^0 \). We also define \( \Sigma S^{-2} = S^{-1} \). Then
\[
\tilde{\beta}_i(\Sigma X) = \tilde{\beta}_{i-1}(X) \quad (i \in \mathbb{Z})
\]
holds when \( X \) is either a topological space or \( S^{-2} \).

We use the two following lemmas in the proof of the main theorem.

**Lemma 2.1.** Let \( X_1 \) and \( X_2 \) be subcomplexes of a CW complex \( X \). If both \( X_1 \) and \( X_2 \) are contractible, then the union \( X_1 \cup X_2 \) is homotopy equivalent to the suspension \( \Sigma(X_1 \cap X_2) \) of the intersection.
Lemma 2.2. Let $X_1$ and $X_2$ be subcomplexes of a CW complex $X$. If $X_2$ is contractible and if the inclusion $X_1 \cap X_2 \hookrightarrow X_1$ is homotopic to the constant map to a point $x_1$ of $X_1$, then the union $X_1 \cup X_2$ is homotopy equivalent to the wedge $X_1 \vee \Sigma(X_1 \cap X_2)$ of $X_1$ and the suspension of the intersection, where we let $X_1$ be pointed at $x_1$ and $\Sigma(X_1 \cap X_2)$ at one end point.

2.2. Posets. A partially ordered set (poset, in short) is a set $P$ together with a partial order $\leq_P$ on $P$. By $<_P$ we denote the strict order associated to $\leq_P$, namely

$$x <_P y \iff x \leq_P y \text{ and } x \neq y \quad (x, y \in P).$$

By $\|P\|$ we denotes the order complex of $P$, that is, $\|P\|$ is the abstract simplicial complex whose vertices are elements of $P$ and whose simplices are finite chains of $P$. We also denote the geometric realization of the order complex $\|P\|$ by the same symbol $\|P\|$.

Let $P$ and $Q$ be posets. A poset map from $P$ to $Q$ is a set map $f : P \to Q$ such that $x \leq_P y$ implies $f(x) \leq_Q f(y)$ for any $x, y \in P$. By $\|f\|$ we denotes the continuous map from $\|P\|$ to $\|Q\|$ induced by $f$. For poset maps $f, g : P \to Q$ we write $f \leq g$ if $f(x) \leq_Q g(x)$ holds for each $x \in P$. Note that if $f \leq g$, then the induced map $\|f\|$ is homotopic to $\|g\|$.

Let $P$ be a poset and $a, b \in P$. We write

$$P_{\geq a} = \{ x \in P \mid a \leq_P x \}$$
$$P_{> a} = \{ x \in P \mid a <_P x \}$$
$$[a, b)_P = \{ x \in P \mid a \leq_P x <_P b \}, \quad \text{and}$$
$$(a, b)_P = \{ x \in P \mid a <_P x <_P b \}.$$

Similarly, $P^{\leq b}, P^{<_b}, [a, b)_P$ and $(a, b)_P$ are defined.

The following lemmas are useful to calculate homotopy types of posets.

Lemma 2.3. If a poset $P$ has either a minimum element or a maximum element, then the order complex $\|P\|$ is contractible.

Proof. We show only the case where $P$ has a minimum element $m$. Let $c : P \to P$ be the constant map to $m$. Then $c$ is a poset map and satisfies $c \leq \text{id}_P$. Thus the identity map of $\|P\|$ is homotopic to the constant map $\|c\|$. Hence $\|P\|$ is contractible. \hfill \square

Lemma 2.4. Let $P$ and $Q$ be posets, $f : P \to Q$ and $g : Q \to P$ poset maps, and $a \in P$. Assume that $gf \leq \text{id}_P$ and $fg \leq \text{id}_Q$. Then we have

$$\|P^{<_a}\| \simeq \begin{cases} \|Q^{<_f(a)}\| & \text{if } gf(a) = a, \\ \text{pt} & \text{if } gf(a) <_P a. \end{cases}$$

Proof. We first show the case $gf(a) <_P a$. We have

$$f(P^{<_a}) \subset f(P^{\leq a}) \subset Q^{<_f(a)} \quad \text{and}$$
$$g(Q^{<_f(a)}) \subset P^{\leq gf(a)} \subset P^{<_a}.$$

Thus $f$ and $g$ induce poset maps between $P^{<_a}$ and $Q^{<_f(a)}$. Moreover, the induced maps between the order complexes are homotopy inverses of each other. Hence

$$\|P^{<_a}\| \simeq \|Q^{<_f(a)}\| \simeq \text{pt}.$$
We next show the case $gf(a) = a$. We have
\[
 f(P^{\leq a}) \subset Q^{\leq f(a)} \quad \text{and} \quad g(Q^{\leq f(a)}) \subset P^{\leq gf(a)} = P^{\leq a}.
\]
Let $x \in P^{\leq a}$ and $f(x) = f(a)$. Then we have $a = gf(a) = gf(x) \leq_P x$, and so $x = a$. Conversely, let $y \in Q^{\leq f(a)}$ and $g(y) = a$. Then we have $f(a) = fg(y) \leq_Q y$, and so $y = f(a)$. Thus $f$ and $g$ induce poset maps between $P^{<a}$ and $Q^{<f(a)}$. Moreover, the induced maps between the order complexes are homotopy inverses of each other. Hence
\[
\|P^{<a}\| \simeq \|Q^{<f(a)}\|.
\]
\[\square\]

2.3. Frobenius complexes. In this paper, $\mathbb{N}$ denotes the additive monoid of non-negative integers. An affine monoid is an additive monoid $\Lambda$ which satisfies three following conditions:

(1) $\Lambda$ is cancellative, that is, $\lambda + \nu = \mu + \nu$ implies $\lambda = \mu$ for any $\lambda, \mu, \nu \in \Lambda$.

(2) $\Lambda$ has no non-zero invertible element, that is, $\lambda + \mu = 0$ implies $\lambda = \mu = 0$ for any $\lambda, \mu \in \Lambda$.

(3) $\Lambda$ is finitely generated, that is, there exist finite elements $\alpha_1, \ldots, \alpha_d$ of $\Lambda$ such that for any element $\lambda$ of $\Lambda$ can be written as $\lambda = m_1\alpha_1 + \cdots + m_d\alpha_d$ for some $m_1, \ldots, m_d \in \mathbb{N}$.

For example, a finitely generated submonoid of $\mathbb{N}^d$ is an affine monoid. By $\Lambda_+$ we denote the set of all non-zero elements of $\Lambda$.

An affine monoid $\Lambda$ has the partial order $\leq_\Lambda$ defined by
\[
\lambda \leq_\Lambda \mu \iff \text{there exists } \nu \in \Lambda \text{ such that } \lambda + \nu = \mu \quad (\lambda, \mu \in \Lambda).
\]
Such an element $\nu$ is unique if exists, since $\Lambda$ is cancellative. Thus we let $\mu - \lambda = \nu$ if $\lambda + \nu = \mu$. By $<_\Lambda$ we denote the strict order associated to $\leq_\Lambda$, namely
\[
\lambda <_\Lambda \mu \iff \lambda \leq_\Lambda \mu \text{ and } \lambda \neq \mu
\]
\[\iff \text{there exists } \nu \in \Lambda_+ \text{ such that } \lambda + \nu = \mu \quad (\lambda, \mu \in \Lambda).
\]
For an affine monoid $\Lambda$ and $\lambda \in \Lambda$, the Frobenius complex $\mathcal{F}(\lambda; \Lambda)$ is defined by
\[
\mathcal{F}(\lambda; \Lambda) = \begin{cases} 
\| (0, \lambda) \| & \text{if } \lambda \in \Lambda_+, \\
S^{-2} & \text{if } \lambda = 0,
\end{cases}
\]
where $S^{-2}$ is the formal symbol introduced in Subsection 2.1.

2.4. Multi-graded Poincaré series. Let $\Lambda$ be an affine monoid, and fix a field $K$. The multi-graded Poincaré series $P_\Lambda(t, z)$ of $\Lambda$ is defined by
\[
P_\Lambda(t, z) = P^{|\Lambda|}_K(t, z) = \sum_{i \in \mathbb{N}} \sum_{\lambda \in \Lambda} \beta_i(\lambda; \Lambda) t^i z^\lambda,
\]
where we let
\[
\beta_i(\lambda; \Lambda) = \dim_K \text{Tor}_{i, \lambda}^K(K, K) \quad (i \in \mathbb{N}, \lambda \in \Lambda).
\]
Laudal and Sletsjøe [LS85] proved the formula
\[
(2.1) \quad \beta_i(\lambda; \Lambda) = \bar{\beta}_{i-2}(\mathcal{F}(\lambda; \Lambda)) \quad (i \in \mathbb{N}, \lambda \in \Lambda).
\]
See [PRS98] for more details.
2.5. Locally finiteness of affine monoids.

**Proposition 2.5.** An affine monoid \( \Lambda \) is locally finite, that is, for any \( \lambda \in \Lambda \) the subset \( \Lambda^{\leq \lambda} \) is finite.

**Proof.** Take a finite generating system \( \{\alpha_1, \ldots, \alpha_d\} \) of \( \Lambda \). We can assume that each \( \alpha_i \) is non-zero. Let \( \pi : \mathbb{N}^d \to \Lambda \) be the map defined by

\[
\pi(n_1, \ldots, n_d) = \sum_{i=1}^d n_i \alpha_i \quad ((n_1, \ldots, n_d) \in \mathbb{N}^d).
\]

Then \( \pi \) is surjective monoid homomorphism, and satisfies that \( \pi(\vec{n}) = 0 \) implies \( \vec{n} = \vec{0} \) for any \( \vec{n} \in \mathbb{N}^d \) since \( \Lambda \) has no non-zero invertible element.

Let \( \lambda \in \Lambda \). We now show that

\[
\Lambda^{\leq \lambda} \subset \bigcup_{\ell \in \pi^{-1}(\lambda)} \pi((\mathbb{N}^d)^{\leq \ell}).
\]

Let \( \mu \in \Lambda^{\leq \lambda} \). Take \( \vec{m} \in \pi^{-1}(\mu) \) and \( \vec{n} \in \pi^{-1}(\lambda - \mu) \). Then we have

\[
\pi(\vec{m} + \vec{n}) = \pi(\vec{m}) + \pi(\vec{n}) = \mu + (\lambda - \mu) = \lambda.
\]

Hence

\[
\mu = \pi(\vec{m}) \in \pi((\mathbb{N}^d)^{\leq \ell}) \subset \bigcup_{\ell \in \pi^{-1}(\lambda)} \pi((\mathbb{N}^d)^{\leq \ell}).
\]

Since each \( (\mathbb{N}^d)^{\leq \ell} \) is finite, it suffices to show that \( \pi^{-1}(\lambda) \) is finite. We now show that \( \pi^{-1}(\lambda) \) is an anti-chain of \( \mathbb{N}^d \), that is, \( \vec{m} \leq_{\mathbb{N}^d} \vec{n} \) implies \( \vec{m} = \vec{n} \) for any \( \vec{m}, \vec{n} \in \pi^{-1}(\lambda) \). Let \( \vec{m}, \vec{n} \in \pi^{-1}(\lambda) \) and assume that \( \vec{m} \leq_{\mathbb{N}^d} \vec{n} \). Then we have

\[
\lambda = \pi(\vec{n}) = \pi(\vec{m}) + \pi(\vec{n} - \vec{m}) = \lambda + \pi(\vec{n} - \vec{m}).
\]

Since \( \Lambda \) is cancellative, we obtain \( \pi(\vec{n} - \vec{m}) = 0 \), which implies \( \vec{m} = \vec{n} \). By the lemma below, the anti-chain \( \pi^{-1}(\lambda) \) of \( \mathbb{N}^d \) is finite. \( \square \)

**Lemma 2.6.** Any anti-chain of \( \mathbb{N}^d \) is finite.

**Proof.** The proof is by induction on \( d \). If \( d = 1 \), any anti-chain of \( \mathbb{N} \) is either the empty set or a singleton since the order of \( \mathbb{N} \) is total.

Let \( d \geq 2 \) and \( A \) an anti-chain of \( \mathbb{N}^d \). Assume that \( A \) is not empty. Take \( \vec{a} = (a_1, \ldots, a_d) \in A \). Let \( A_i = \{(n_1, \ldots, n_d) \in A \mid n_i < a_i\} \). We now show that

\[
A = A_1 \cup \cdots \cup A_d \cup \{\vec{a}\}.
\]

Let \( \vec{n} = (n_1, \ldots, n_d) \in A \setminus (\bigcup_{i=1}^d A_i) \). Then we have \( a_i \leq n_i \) for each \( i \), which implies \( \vec{a} \leq_{\mathbb{N}^d} \vec{n} \). Hence \( \vec{n} = \vec{a} \) since \( A \) is an anti-chain.

Let \( A_{i,k} = \{(n_1, \ldots, n_d) \in A \mid n_i = k\} \). Then \( A_{i,k} \) is finite since \( A_{i,k} \) is an anti-chain of \( \{ (n_1, \ldots, n_d) \in \mathbb{N}^d \mid n_i = k \} \cong \mathbb{N}^{d-1} \). Thus each \( A_i = A_{i,0} \cup \cdots \cup A_{i,a_i-1} \) is finite. Hence so is \( A \). \( \square \)

**Corollary 2.7.** For an affine monoid \( \Lambda \) and \( \lambda \in \Lambda_+ \) the Frobenius complex \( F(\lambda; \Lambda) \) is a finite simplicial complex.

**Proof.** The open interval \( (0, \lambda)_\Lambda \) is finite since \( (0, \lambda)_\Lambda \subset \Lambda^{\leq \lambda} \). Thus the order complex \( \| (0, \lambda)_\Lambda \| \) is a finite simplicial complex. \( \square \)

**Proposition 2.8.** Let \( \Lambda \) be an affine monoid and \( \rho \in \Lambda_+ \). Then for any \( \lambda \in \Lambda \) the set of all \( \ell \in \mathbb{N} \) satisfying \( \ell \rho \leq_\Lambda \lambda \) is finite.
Proof. Let \( A \) be the set of all \( \ell \in \mathbb{N} \) satisfying \( \ell \rho \leq \lambda \), and \( f : \mathbb{N} \rightarrow \Lambda \) the map defined by \( f(\ell) = \ell \rho \). Since \( \Lambda \) is an affine monoid and \( \rho \) is non-zero, the map \( f \) is injective. Moreover, the image of \( A \) by \( f \) is contained in the finite subset \( \Lambda^{\leq \lambda} \). Hence \( A \) is finite. \( \square \)

3. The main theorem

Let \( \Lambda \) be an affine monoid, \( \rho \in \Lambda \), and \( r \in \mathbb{N} \). Assume that \( \rho \) is reducible, that is, there exist \( \sigma, \tau \in \Lambda^+ \) such that \( \rho = \sigma + \tau \), and that \( r \geq 2 \).

Let us construct the additive monoid \( \Lambda[\rho/r] \) which is added the \( r \)-th part of \( \rho \) to \( \Lambda \). Consider the direct sum \( \Lambda \oplus \mathbb{N} \) of \( \Lambda \) and the free additive monoid \( \mathbb{N} \) generated by a formal element \( \alpha \), and the equivalence relation \( \sim \) on \( \Lambda \oplus \mathbb{N} \) generated by

\[
(\lambda + \rho) + k\alpha \sim \lambda + (r + k)\alpha \quad (\lambda \in \Lambda, \ k \in \mathbb{N}).
\]

Define \( \Lambda[\rho/r] \) to be the quotient \( (\Lambda \oplus \mathbb{N}/\mathbb{N})/\sim \). Since the equivalence relation \( \sim \) is closed by addition of \( \Lambda \oplus \mathbb{N} \), that is, \( x \sim x' \) and \( y \sim y' \) implies \( x + y \sim x' + y' \) for any \( x, x', y, y' \in \Lambda \oplus \mathbb{N} \), the quotient \( \Lambda[\rho/r] \) has the canonical additive monoid structure. We denote the equivalence class of \( \lambda + k\alpha \) modulo \( \sim \) simply by \( \lambda + k\rho/r \).

Let \( \pi : \Lambda \oplus \mathbb{N} \rightarrow \Lambda \times \mathbb{N}^{<r} \) be the map defined by

\[
\pi(\lambda + (r + k)\alpha) = (\lambda + \ell \rho, k) \quad (\lambda \in \Lambda, \ \ell \in \mathbb{N}, \ k \in \mathbb{N}^{<r}).
\]

Then \( x \sim y \) is equivalent to \( \pi(x) = \pi(y) \) for any \( x, y \in \Lambda \oplus \mathbb{N} \). We can check that there is a bijection \( \Lambda \times \mathbb{N}^{<r} \cong \Lambda[\rho/r] \) which sends \( (\lambda, k) \) to \( \lambda + k\rho/r \), and that \( \Lambda[\rho/r] \) is an affine monoid. Note that for \( \lambda, \lambda' \in \Lambda \) and \( k, k' \in \mathbb{N}^{<r} \) we have

\[
\lambda + k\rho/r \leq \Lambda[\rho/r] \lambda' + k'\rho/r \iff \begin{cases} \lambda \leq \Lambda \lambda' & \text{if } k \leq k', \\ \lambda + \rho \leq \Lambda \lambda' & \text{if } k > k'. \end{cases}
\]

Define the function \( \ell_{\rho} : \Lambda \rightarrow \mathbb{N} \) by

\[
\ell_{\rho}(\lambda) = \max\{ \ell \in \mathbb{N} \mid \ell \rho \leq \lambda \} \quad (\lambda \in \Lambda).
\]

By Proposition 2.8 the function \( \ell_{\rho} \) is well-defined since \( \rho \in \Lambda^+ \). Note that \( \ell_{\rho} \) satisfies that

\[
\ell_{\rho} \leq \Lambda \lambda \iff \ell \leq \ell_{\rho}(\lambda) \quad (\ell \in \mathbb{N}, \ \lambda \in \Lambda).
\]

Consequently,

\[
\ell_{\rho}(\lambda + \rho) = \ell_{\rho}(\lambda) + 1 \quad (\lambda \in \Lambda)
\]

holds since \( \ell \rho \leq \Lambda \lambda \) is equivalent to \( (\ell + 1)\rho \leq \Lambda \lambda + \rho \).

The following is the main theorem of this paper.

**Theorem 3.1.** For \( \lambda \in \Lambda \) and \( k \in \mathbb{N}^{<r} \) the Frobenius complex of \( \Lambda[\rho/r] \) satisfies

\[
\mathcal{F}(\lambda + k\rho/r; \Lambda[\rho/r]) \simeq \begin{cases} \ell_{\rho}(\lambda) \Sigma^{2\ell_{\rho}(\lambda)} \mathcal{F}(\lambda - \ell \rho; \Lambda) & \text{if } k \leq 1, \\ \mathcal{F}(\lambda + k\rho/2; \Lambda[\rho/2]) & \text{if } k \geq 2. \end{cases}
\]

We will prove Theorem 3.1 in several steps.

**Proposition 3.2.** For \( \lambda \in \Lambda \) and \( k \in \mathbb{N}^{<r} \) the Frobenius complex of \( \Lambda[\rho/r] \) satisfies

\[
\mathcal{F}(\lambda + k\rho/r; \Lambda[\rho/r]) \simeq \begin{cases} \mathcal{F}(\lambda + k\rho/2; \Lambda[\rho/2]) & \text{if } k \leq 1, \\ \mathcal{F}(\lambda + k\rho/2; \Lambda[\rho/2]) & \text{if } k \geq 2. \end{cases}
\]
Proof. Let $f : \Lambda[\rho/r] \to \Lambda[\rho/2]$ and $g : \Lambda[\rho/2] \to \Lambda[\rho/r]$ be the maps defined by

\[
f(\lambda + k \rho/r) = \lambda + \min\{k, 1\} \rho/2 \quad (\lambda \in \Lambda, \ k \in \mathbb{N}^{<r}) \quad \text{and}
\]
\[
g(\lambda + k \rho/2) = \lambda + k \rho/r \quad (\lambda \in \Lambda, \ k \in \mathbb{N}^{\leq 2}).
\]

Then $f$ and $g$ are poset maps, and satisfy $gf \leq \text{id}_{\Lambda[\rho/r]}$ and $fg = \text{id}_{\Lambda[\rho/2]}$. Moreover, $f$ and $g$ induce poset maps between $\Lambda[\rho/r]_+$ and $\Lambda[\rho/2]_+$. Let $\lambda \in \Lambda$ and $k \in \mathbb{N}^{<r}$. Note that

\[
gf(\lambda + k \rho/r) = \lambda + k \rho/r \iff k \leq 1.
\]

We now apply Lemma 2.4 to $f : \Lambda[\rho/r]_+ \to \Lambda[\rho/2]_+$ and $g : \Lambda[\rho/2]_+ \to \Lambda[\rho/r]_+$. If $k \leq 1$, then we have

\[
\mathcal{F}(\lambda + k \rho/r; \Lambda[\rho/r]) = \|\Lambda[\rho/r]_{\lambda + k \rho/r}^{<}\| \\
\simeq \|\Lambda[\rho/2]_{\lambda + k \rho/r}^{<}\| \\
= \mathcal{F}(\lambda + k \rho/2; \Lambda[\rho/2]).
\]

If $k \geq 2$, then we have

\[
\mathcal{F}(\lambda + k \rho/r; \Lambda[\rho/r]) = \|\Lambda[\rho/r]_{\lambda + k \rho/r}^{<}\| \simeq \text{pt}. \quad \square
\]

By the previous proposition, we need only consider the case $r = 2$. We define some subsets and maps to observe $\Lambda[\rho/2]$. Let

\[
U_1 = \Lambda[\rho/2] \setminus \{0, \rho/2\},
\]
\[
U_2 = \Lambda[\rho/2]_{\geq \rho/2}, \quad \text{and}
\]
\[
U_{12} = U_1 \cap U_2.
\]

Then $U_1, U_2$ and $U_{12}$ are upper subsets of $\Lambda[\rho/2]$ and satisfy $U_1 \cup U_2 = \Lambda[\rho/2]_+$ and $U_{12} = \Lambda[\rho/2]_{> \rho/2}$. Let $h : \Lambda[\rho/2] \to \Lambda$ and $i : \Lambda \to \Lambda[\rho/2]$ be the maps defined by

\[
h(\lambda + k \rho/2) = \lambda \quad (\lambda \in \Lambda, \ k \in \mathbb{N}^{<2}) \quad \text{and}
\]
\[
i(\lambda) = \lambda \quad (\lambda \in \Lambda).
\]

Then $h$ and $i$ are poset maps, and satisfy $ih \leq \text{id}_{\Lambda[\rho/2]}$ and $hi = \text{id}_{\Lambda}$. Moreover, $h$ and $i$ induces poset maps between $U_1$ and $\Lambda_+$. 

**Proposition 3.3.** For $\lambda \in \Lambda$ the Frobenius complex of $\Lambda[\rho/2]$ satisfies

\[
\mathcal{F}(\lambda + \rho/2; \Lambda[\rho/2]) \simeq \Sigma \mathcal{F}(\lambda; \Lambda[\rho/2]).
\]

**Proof.** If $\lambda = 0$, then we have

\[
\mathcal{F}(\rho/2; \Lambda[\rho/2]) = \|(0, \rho/2)_{\Lambda[\rho/2]}\| = S^{-1} = \Sigma S^{-2} = \Sigma \mathcal{F}(0; \Lambda[\rho/2]).
\]

Assume that $\lambda \in \Lambda_+$. Then we have

\[
\mathcal{F}(\lambda + \rho/2; \Lambda[\rho/2]) = \|\Lambda[\rho/2]_{\lambda + \rho/2}^{<}\| = \|U_1^{\lambda + \rho/2}\| \cup \|U_2^{\lambda + \rho/2}\|.
\]

Applying Lemma 2.4 to $h : U_1 \to \Lambda_+$ and $i : \Lambda_+ \to U_1$, we obtain

\[
\|U_1^{\lambda + \rho/2}\| \simeq \text{pt}
\]

since $ih(\lambda + \rho/2) = \lambda_{\Lambda[\rho/2]} \lambda + \rho/2$. On the other hand, we have

\[
\|U_2^{\lambda + \rho/2}\| = \|[\rho/2, \lambda + \rho/2]_{\Lambda[\rho/2]}\| \simeq \text{pt}.
\]
We also have
\[ \|U_1^<\lambda+\rho/2\| \cap \|U_2^<\lambda+\rho/2\| = \|U_1^<\lambda+\rho/2\| \]
\[ = \| (\rho/2, \lambda + \rho/2)_{\Lambda[\rho/2]} \| \]
\[ \approx \|(0, \lambda)_{\Lambda[\rho/2]}\| \]
\[ = \mathcal{F}(\lambda; \Lambda[\rho/2]), \]
where the homeomorphism \[ \|(0, \lambda)_{\Lambda[\rho/2]}\| \approx \|(\rho/2, \lambda + \rho/2)_{\Lambda[\rho/2]}\| \]
is given by the poset isomorphism
\[ (0, \lambda)_{\Lambda[\rho/2]} \cong (\rho/2, \lambda + \rho/2)_{\Lambda[\rho/2]} \]
which sends \( \gamma \) to \( \gamma + \rho/2 \). Applying Lemma 2.1, we obtain
\[ \mathcal{F}(\lambda + \rho/2; \Lambda[\rho/2]) = \|U_1^<\lambda+\rho/2\| \cup \|U_2^<\lambda+\rho/2\| \]
\[ \cong \Sigma(\|U_1^<\lambda+\rho/2\| \cap \|U_2^<\lambda+\rho/2\|) \]
\[ \approx \Sigma \mathcal{F}(\lambda; \Lambda[\rho/2]). \]
\[ \square \]

The following proposition completes the proof of Theorem 3.1.

**Proposition 3.4.** For \( \lambda \in \Lambda \) the Frobenius complex of \( \Lambda[\rho/2] \) satisfies

\[ \mathcal{F}(\lambda; \Lambda[\rho/2]) \simeq \ell_{\rho}(\lambda) \]

\[ \sum_{\ell=0}^{\ell_{\rho}(\lambda)} \mathcal{F}(\lambda - \ell\rho; \Lambda). \]

**Proof.** The proof is by induction on \( \ell_{\rho}(\lambda) \). Assume that \( \ell_{\rho}(\lambda) = 0 \), that is, \( \rho \not\leq \lambda \). If \( \lambda = 0 \), the both sides of (3.1) are \( S^{-2} \). Let \( \lambda \in \Lambda_+ \). Then \( i : \Lambda \to \Lambda[\rho/2] \) induces a poset isomorphism \( (0, \lambda)_\Lambda \cong (0, \lambda)_{\Lambda[\rho/2]} \). Hence
\[ \mathcal{F}(\lambda; \Lambda[\rho/2]) = \|(0, \lambda)_{\Lambda[\rho/2]}\| \approx \|(0, \lambda)\| = \mathcal{F}(\lambda; \Lambda). \]

Assume that \( \ell_{\rho}(\lambda) \geq 1 \), that is, \( \rho \leq \lambda \). We have
\[ \mathcal{F}(\lambda; \Lambda[\rho/2]) = \|\Lambda[\rho/2]\| \cdot \|\Lambda_+^{\leq \lambda}\| = \|U_1^{<\lambda}\| \cup \|U_2^{<\lambda}\|. \]
Applying Lemma 2.4 to \( h : U_1 \to \Lambda_+ \) and \( i : \Lambda_+ \to U_1 \), we obtain
\[ \|U_1^{<\lambda}\| \simeq \|\Lambda_+^{<h(\lambda)}\| = \mathcal{F}(\lambda; \Lambda) \]
since \( ih(\lambda) = \lambda \). On the other hand, we have
\[ \|U_2^{<\lambda}\| = \|\rho/2, \lambda\|_{\Lambda[\rho/2]} \approx pt. \]
We also have
\[ \|U_1^{<\lambda}\| \cap \|U_2^{<\lambda}\| = \|U_1^{<\lambda}\| \]
\[ = \|\rho/2, \lambda\|_{\Lambda[\rho/2]} \]
\[ \approx \|(0, \lambda - \rho/2)_{\Lambda[\rho/2]}\| \]
\[ = \mathcal{F}((\lambda - \rho) + \rho/2; \Lambda[\rho/2]) \]
\[ \approx \Sigma \mathcal{F}(\lambda - \rho; \Lambda[\rho/2]). \]
Since $\ell_\rho(\lambda - \rho) = \ell_\rho(\lambda) - 1$, we can apply the inductive hypothesis to $\lambda - \rho$, and conclude that

$$
\Sigma F(\lambda - \rho; \Lambda[\rho/2]) \cong \Sigma \bigvee_{\ell=0}^{\ell_\rho(\lambda - \rho)} \Sigma^{2\ell} F((\lambda - \rho) - \ell\rho; \Lambda)
$$

$$
\cong \bigvee_{\ell=1}^{\ell_\rho(\lambda)} \Sigma^{2\ell - 1} F(\lambda - \ell\rho; \Lambda).
$$

Since $\rho$ is reducible, we can take $\sigma \in \Lambda$ such that $0 <_\Lambda \sigma <_\Lambda \rho$. We now show that the inclusion $\|U_{12}^{<\lambda}\| \cap \|U_{2}^{<\lambda}\| = \|U_{12}^{<\lambda}\| \hookrightarrow \|U_{1}^{<\lambda}\|$ is homotopic to the constant map to $\sigma$. Note that

$$
U_{12}^{<\lambda} = (\rho/2, \lambda)_{\Lambda[\rho/2]}
$$

$$
= \{ \mu \mid \mu \in [\rho, \lambda)_{\Lambda} \} \sqcup \{ \mu + \rho/2 \mid \mu \in (0, \lambda - \rho]_{\Lambda} \},
$$

and

$$
U_{1}^{<\lambda} = \{ \mu \mid \mu \in (0, \lambda)_{\Lambda} \} \sqcup \{ \mu + \rho/2 \mid \mu \in (0, \lambda - \rho]_{\Lambda} \}.
$$

Let $g_1, g_2, g_3, g_4 : U_{12}^{<\lambda} \to U_{1}^{<\lambda}$ be the maps defined by

$$
g_1(\mu) = \mu
$$

$$
g_2(\mu) = \mu
$$

$$
g_3(\mu) = \mu
$$

$$
g_4(\mu) = \sigma \quad \text{ (} \mu \in [\rho, \lambda)_{\Lambda} \text{) and}
$$

$$
g_1(\mu + \rho/2) = \mu + \rho/2
$$

$$
g_2(\mu + \rho/2) = \mu
$$

$$
g_3(\mu + \rho/2) = \mu + \sigma
$$

$$
g_4(\mu + \rho/2) = \sigma \quad \text{ (} \mu \in (0, \lambda - \rho]_{\Lambda} \text{)}.
$$

Then $g_1, g_2, g_3$ and $g_4$ are poset maps and satisfy

$$
g_1 \geq g_2 \leq g_3 \geq g_4.
$$

Hence

$$
\text{(inclusion)} = \|g_1\| \cong \|g_2\| \cong \|g_3\| \cong \|g_4\| = \text{(constant)}.
$$

Applying Lemma 2.2, we obtain

$$
F(\lambda; \Lambda[\rho/2]) = \|U_{12}^{<\lambda}\| \cup \|U_{2}^{<\lambda}\|
$$

$$
\cong \|U_{1}^{<\lambda}\| \vee \Sigma (\|U_{1}^{<\lambda}\| \cap \|U_{2}^{<\lambda}\|)
$$

$$
\cong F(\lambda; \Lambda) \vee \Sigma \bigvee_{\ell=1}^{\ell_\rho(\lambda)} \Sigma^{2\ell - 1} F(\lambda - \ell\rho; \Lambda)
$$

$$
\cong \bigvee_{\ell=0}^{\ell_\rho(\lambda)} \Sigma^{2\ell} F(\lambda - \ell\rho; \Lambda).
$$

4. Applications

Let $\Lambda$ be an affine monoid, $\rho \in \Lambda$ and $r \in \mathbb{N}$. Assume that $\rho$ is reducible and that $r \geq 2$. 
Theorem 4.1. The multi-graded Poincaré series of $\Lambda[\rho/r]$ satisfies

$$P_{\Lambda[\rho/r]}(t, z) = \frac{P_{\Lambda}(t, z) \cdot (1 + tz^{\rho/r})}{1 - t^2z^\rho}.$$ 

In particular, if $r = 2$, then

$$P_{\Lambda[\rho/2]}(t, z) = \frac{P_{\Lambda}(t, z)}{1 - tz^{\rho/2}}.$$ 

Proof. Combining the equation (2.1) and Theorem 3.1, we obtain

$$\beta_i(\lambda + k\rho/r; \Lambda[\rho/r]) = \begin{cases} 
\ell_r(\lambda) \sum_{\ell=0}^{r} \beta_{i - 2\ell - k}(\lambda - \ell\rho; \Lambda) & \text{if } k \leq 1, \\
0 & \text{if } k \geq 2
\end{cases}$$

for $\lambda \in \Lambda$ and $k \in \mathbb{N}^{<r}$, where we let $\beta_i(\lambda; \Lambda) = 0$ for $i < 0$. Thus we have

$$P_{\Lambda[\rho/r]}(t, z) = \sum_{i\in\mathbb{N}} \sum_{\lambda\in\Lambda} \sum_{k=0}^{r-1} \beta_i(\lambda + k\rho/r; \Lambda[\rho/r]) t^i z^{\lambda + k\rho/r}$$

$$= \sum_{i\in\mathbb{N}} \sum_{\lambda\in\Lambda} \sum_{k=0}^{1} \beta_i(\lambda) \sum_{\ell=0}^{\infty} \sum_{\lambda' \in \Lambda} \beta_{i-2\ell-k}(\lambda - \ell\rho; \Lambda) t^\ell z^{\lambda + k\rho/r}$$

$$= \sum_{\ell=0}^{\infty} \sum_{i\in\mathbb{N}} \sum_{\lambda\in\Lambda} \beta_{i-2\ell-k}(\lambda - \ell\rho; \Lambda) t^\ell z^{\lambda + k\rho/r}$$

$$= \sum_{\ell=0}^{\infty} \sum_{j \in \mathbb{N}} \sum_{\mu \in \Lambda} \beta_j(\mu; \Lambda) t^\ell z^\mu \cdot \sum_{k=0}^{\infty} (t^2z^\rho)^k$$

$$= \frac{P_{\Lambda}(t, z) \cdot (1 + tz^{\rho/r})}{1 - t^2z^\rho}.$$ 

\[\square\]

Corollary 4.2. If the multi-graded Poincaré series $P_{\Lambda}(t, z)$ of $\Lambda$ is given by a rational function, then so is $P_{\Lambda[\rho/r]}(t, z)$.

We say that $X$ has the homotopy type of a wedge of spheres if $X$ is $S^{-2}$ or a topological space which is either

- empty,
- contractible,
- homotopy equivalent to a sphere $S^n$ for some $n \geq 0$, or
- homotopy equivalent to a wedge $\bigvee_{i=1}^m S^{n_i}$ of spheres for some $n_1, \ldots, n_m \geq 0$.

Note that for such an $X$ the reduced Betti numbers are independent of the choice of a field $K$, and the homotopy type is determined only by the reduced Betti numbers.

Corollary 4.3. Assume that there is a generating system $\{\alpha_1, \ldots, \alpha_d\}$ of $\Lambda$ which satisfies $\alpha_i \leq \Lambda \rho$ for each $i$. If $\mathcal{F}(\Lambda; \Lambda)$ has the homotopy type of a wedge of spheres for each $\lambda \in \Lambda$, then so has $\mathcal{F}(\gamma; \Lambda[\rho/r])$ for each $\gamma \in \Lambda[\rho/r]$. 

Proof. We first show that $F(\lambda; \Lambda)$ is 0-connected for each $\lambda \in \Lambda_{>\rho}$. We can assume that each $\alpha_i$ is non-zero. For any element $\mu = \sum_i m_i \alpha_i$ of $\Lambda_+$, since $(m_1, \ldots, m_d)$ is non-zero, $\alpha_i \leq \mu$ holds for some $i$. Hence

$$F(\lambda; \Lambda) = \|(0, \lambda)\| = \sum_{i=1}^d \|(\alpha_i, \lambda)\|.$$ 

Each $\|(\alpha_i, \lambda)\|$ is contractible and contains $\rho$ since $\alpha_i \leq \Lambda \rho \lesssim \lambda$. Thus $F(\lambda; \Lambda)$ is 0-connected.

Consider $X = F(\lambda + k\rho/r; \Lambda[\rho/r])$ for $\lambda \in \Lambda$ and $k \in \mathbb{N}_{<r}$. If $k \geq 2$, then $X$ is contractible. Let $k \leq 1$. If $\ell_\rho(\lambda) = 0$, then we have

$$X \simeq \Sigma^k F(\lambda; \Lambda).$$

Since $F(\lambda; \Lambda)$ has the homotopy type of a wedge of spheres, so has $X$. Let $\ell_\rho(\lambda) \geq 1$, that is, $\rho \lesssim \Lambda \lambda$. Then we have

$$X \simeq \bigvee_{\ell = 0}^{\ell_\rho(\lambda)} \Sigma^{2\ell + k} F(\lambda - \ell \rho; \Lambda).$$

If either $\lambda \in \Lambda_{>\rho}$ or $k = 1$ holds, then each $\Sigma^{2\ell + k} F(\lambda - \ell \rho; \Lambda)$ is a 0-connected topological space which has the homotopy type of a wedge of spheres. Thus so has $X$. If both $\lambda = \rho$ and $k = 0$ hold, then we have

$$X \simeq F(\rho; \Lambda) \vee \Sigma^2 F(0; \Lambda) = F(\rho; \Lambda) \vee S^0.$$ 

Since $F(\rho; \Lambda)$ is a non-empty topological space which has the homotopy type of a wedge of spheres, so has $X$. \hfill \Box

5. Examples

By $\langle a_1, \ldots, a_d \rangle$ we denote the submonoid of $\mathbb{N}$ generated by $a_1, \ldots, a_d \in \mathbb{N}$. We first show some convenient propositions.

**Proposition 5.1.** Let $\Lambda$ be a finitely generated submonoid $\langle a_1, \ldots, a_d \rangle$ of $\mathbb{N}$, $\rho$ a reducible element of $\Lambda$, and $r \in \mathbb{N}_{\geq 2}$. Assume that $\Lambda \subset \langle r \rangle$ and let $\rho = rb$. If $r$ and $b$ are relatively prime, then there exists a monoid isomorphism $\Lambda[\rho/r] \cong \Lambda + \langle b \rangle$ which sends $\lambda \in \Lambda$ to $\lambda$ and $\rho/r$ to $b$.

**Proof.** Let $\tilde{F} : \Lambda \oplus \mathbb{N} \alpha \to \mathbb{N}$ be the monoid homomorphism which sends $\lambda \in \Lambda$ to $\lambda$ and $\alpha$ to $b$. Then $\tilde{F}$ induces a surjective monoid homomorphism

$$F : \Lambda[\rho/r] \to \Lambda + \langle b \rangle$$

which sends $\lambda \in \Lambda$ to $\lambda$ and $\rho/r$ to $b$. It suffices to show that $F$ is injective, that is, $\tilde{F}(x) = \tilde{F}(y)$ implies $x \sim y$ for any $x, y \in \Lambda \oplus \mathbb{N} \alpha$. Let $\lambda, \lambda' \in \Lambda$ and $k, k' \in \mathbb{N}$, and assume that $\lambda + kb = \lambda' + k'b$. We can assume that $k \leq k'$. Then we have

$$\lambda - \lambda' = (k' - k)b.$$ 

Since $\lambda - \lambda'$ is a multiple of $r$, there exists $\ell \in \mathbb{N}$ such that $k' - k = \ell r$. Hence $\lambda - \lambda' = \ell rb = \ell \rho$. Thus we have

$$\lambda + k\alpha = (\lambda' + \ell \rho) + k\alpha \sim \lambda' + (\ell r + k)\alpha = \lambda' + k'\alpha.$$ \hfill \Box

**Proposition 5.2.** Let $\Lambda$ be a finitely generated submonoid of $\mathbb{N}$ and $p$ a positive integer. Then the Frobenius complex $F(p\lambda; p\Lambda)$ of $p\Lambda$ is homeomorphic to $F(\lambda; \Lambda)$ for each $\lambda \in \Lambda$, and the multi-graded Poincaré series of $p\Lambda$ satisfies

$$P_{p\Lambda}(t, z) = P_{\Lambda}(t, z^p).$$
Proof. By the monoid isomorphism \( \Lambda \cong p\Lambda \) which sends \( \lambda \) to \( p\lambda \), \( \mathcal{F}(\lambda; \Lambda) \) is homeomorphic to \( \mathcal{F}(p\lambda; p\Lambda) \) for each \( \lambda \in \Lambda \). By the equation \( (2.1) \), we have

\[
\beta_i(\lambda; \Lambda) = \tilde{\beta}_{i-2}(\mathcal{F}(\lambda; \Lambda)) = \tilde{\beta}_{i-2}(\mathcal{F}(p\lambda; p\Lambda)) = \beta_i(p\lambda; p\Lambda)
\]

for \( i \in \mathbb{N} \) and \( \lambda \in \Lambda \). Hence

\[
P_{p\Lambda}(t, z) = \sum_{i \in \mathbb{N}} \sum_{\lambda \in \Lambda} \beta_i(p\lambda, p\Lambda) t^i z^{p\lambda} = \sum_{i \in \mathbb{N}} \sum_{\lambda \in \Lambda} \beta_i(\lambda, \Lambda) t^i (z^p)^\lambda = P_{\Lambda}(t, z^p).
\]

Let us calculate the simplest case.

Example 5.3. The Frobenius complex \( \mathcal{F}(n; \mathbb{N}) \) of \( \mathbb{N} \) has the homotopy type of a wedge of spheres for each \( n \in \mathbb{N} \), and the multi-graded Poincaré series of \( \mathbb{N} \) satisfies

\[
P_N(t, z) = 1 + t z.
\]

Proof. We have

\[
\mathcal{F}(0, N) = S^{-2}, \quad \mathcal{F}(1, N) = \|\langle 0, 1 \rangle_N\| = S^{-1}, \quad \text{and} \quad \mathcal{F}(n, N) = \|\langle 0, n \rangle_N\| = \|\langle 1, n \rangle_N\| \simeq pt \quad (n \geq 2).
\]

Thus \( \mathcal{F}(n; \mathbb{N}) \) has the homotopy type of a wedge of spheres for each \( n \in \mathbb{N} \), and the multi-graded Poincaré series satisfies

\[
P_N(t, z) = \sum_{i \in \mathbb{N}} \sum_{n \in \mathbb{N}} \beta_i(n; \mathbb{N}) t^i z^n = \sum_{i \in \mathbb{N}} \sum_{n \in \mathbb{N}} \tilde{\beta}_{i-2}(\mathcal{F}(n; \mathbb{N})) t^i z^n = 1 + t z.
\]

We now calculate some examples using the result of the previous sections.

Example 5.4. Let \( a \) and \( b \) be integers with \( 2 \leq a < b \), and assume that \( b \notin \langle a \rangle \). Then the Frobenius complex \( \mathcal{F}(\lambda; \langle a, b \rangle) \) of \( \langle a, b \rangle \) has the homotopy type of a wedge of spheres for each \( \lambda \in \langle a, b \rangle \), and the multi-graded Poincaré series of \( \langle a, b \rangle \) satisfies

\[
P_{\langle a, b \rangle}(t, z) = (1 + t z^a) \cdot (1 + t z^b) \cdot \frac{1}{1 - t^2 z^m},
\]

where \( m \) is the least common multiple of \( a \) and \( b \).

Proof. We first show the case where \( a \) and \( b \) are relatively prime. Let \( \rho = ab \in \langle a \rangle \). By Proposition [5.1] there exists a monoid isomorphism \( \langle a \rangle [\rho/a] \cong \langle a, b \rangle \) which sends \( a \) to \( a \) and \( \rho/a \) to \( b \). Note that \( \langle a \rangle = a\mathbb{N} \cong \mathbb{N} \). Then we can apply Theorem [4.4] and Corollary [4.3] to \( \langle a \rangle [\rho/a] \cong \langle a, b \rangle \). Thus \( \mathcal{F}(\lambda; \langle a, b \rangle) \) has the homotopy type of
a wedge of spheres for each \( \lambda \in \langle a, b \rangle \), and the multi-graded Poincaré series satisfies

\[
P_{\langle a, b \rangle}(t, z) = \frac{P_{\langle a \rangle}(t, z) \cdot (1 + tz^a)}{1 - t^2z^b} = \frac{P_{\langle b \rangle}(t, z^b) \cdot (1 + tz^b)}{1 - t^2z^{ab}} = \frac{(1 + tz^a) \cdot (1 + tz^b)}{1 - t^2z^{ab}}.
\]

We now turn to general cases. Let \( d \) be the greatest common divisor of \( a \) and \( b \), and let \( a = a'd \) and \( b = b'd \). Then \( a' \) and \( b' \) are relatively prime, and satisfy \( 2 \leq a' < b' \) by the assumptions \( a < b \) and \( b \notin \langle a \rangle \). Note that \( \langle a, b \rangle = d(a', b') \). Thus \( F(\lambda; \langle a, b \rangle) \) has the homotopy type of a wedge of spheres for each \( \lambda \in \langle a, b \rangle \), and the multi-graded Poincaré series satisfies

\[
P_{\langle a, b \rangle}(t, z) = P_{\langle a', b' \rangle}(t, z^d) = \frac{(1 + t(z^d)^{a'}) \cdot (1 + t(z^d)^{b'})}{1 - t^2(z^d)^{a'b'}} = \frac{(1 + tz^a) \cdot (1 + tz^b)}{1 - t^2z^{ab}}.
\]

\[\square\]

**Example 5.5.** Let \( p, q \) and \( r \) be relatively prime integers with \( 2 \leq p < q < r \). Let \( \Lambda \) be the submonoid \( \langle pq, pr, qr \rangle \) of \( \mathbb{N} \) generated by three elements \( pq, pr \) and \( qr \). Then the Frobenius complex \( F(\lambda; \Lambda) \) of \( \Lambda \) has the homotopy type of a wedge of spheres for each \( \lambda \in \Lambda \), and the multi-graded Poincaré series of \( \Lambda \) satisfies

\[
P_\Lambda(t, z) = \frac{(1 + tz^{pq}) \cdot (1 + tz^{pr}) \cdot (1 + tz^{qr})}{(1 - t^2z^{pq})^2}.
\]

**Proof.** Let \( \rho = p \cdot qr = r \cdot pq = q \cdot pr \in \langle pq, pr \rangle \). By Proposition 5.1 there is a monoid isomorphism \( \langle pq, pr \rangle[\rho/p] \cong \langle pq, pr, qr \rangle = \Lambda \) which sends \( \lambda \in \langle pq, pr \rangle \) to \( \lambda \) and \( \rho/p \) to \( qr \). Then we can apply Theorem 4.1 and Corollary 4.3 to \( \langle pq, pr \rangle \cong \Lambda \). Thus \( F(\lambda; \Lambda) \) has the homotopy type of a wedge of spheres for each \( \lambda \in \Lambda \), and the multi-graded Poincaré series satisfies

\[
P_\Lambda(t, z) = \frac{(1 + tz^{pq}) \cdot (1 + tz^{pr}) \cdot (1 + tz^{qr})}{(1 - t^2z^{pq})^2} \quad \square
\]

**Example 5.6.** Let \( a \) be a positive even number and \( d \) a positive odd number, and assume that \( a + 2d \notin \langle a \rangle \). Let \( \Lambda \) be the submonoid \( \langle a, a + d, a + 2d \rangle \) of \( \mathbb{N} \) generated by the arithmetic sequence \( a, a + d, a + 2d \). Then the Frobenius complex \( F(\lambda; \Lambda) \) of \( \Lambda \) has the homotopy type of a wedge of spheres for each \( \lambda \in \Lambda \), and the multi-graded Poincaré series of \( \Lambda \) satisfies

\[
P_\Lambda(t, z) = \frac{(1 + tz^a) \cdot (1 + tz^{a+2d})}{(1 - t^2z^m) \cdot (1 - tz^{a+d})},
\]

where \( m \) is the least common multiple of \( a \) and \( a + 2d \).

**Proof.** Let \( \rho = 2(a + d) = a + (a + 2d) \in \langle a, a + 2d \rangle \). By Proposition 5.1 there is a monoid isomorphism \( \langle a, a + 2d \rangle[\rho/2] \cong \langle a, a + d, a + 2d \rangle = \Lambda \) which sends \( \lambda \in \langle a, a + 2d \rangle \) to \( \lambda \) and \( \rho/2 \) to \( a + d \). Then we can apply Theorem 4.1 and Corollary 4.3 to \( \langle a, a + 2d \rangle \cong \Lambda \). Thus \( F(\lambda; \Lambda) \) has the homotopy type of a wedge of spheres for each \( \lambda \in \Lambda \), and the multi-graded Poincaré series satisfies

\[
P_\Lambda(t, z) = \frac{P_{\langle a, a + 2d \rangle}(t, z)}{1 - t^2z^{a+d}} = \frac{(1 + tz^a) \cdot (1 + tz^{a+2d})}{(1 - t^2z^m) \cdot (1 - tz^{a+d})}. \quad \square
\]
The following is an answer to a question raised by Clark and Ehrenborg ([CE12], Question 6.4).

**Example 5.7.** Let $p$ and $q$ be relatively prime integers with $2 \leq p < q$. For $n \geq 1$, let $\Lambda_n$ be the submonoid $\langle p^n, p^{n-1}q, \ldots, pq^{n-1}, q^n \rangle$ of $\mathbb{N}$ generated by the geometric sequence $p^n, p^{n-1}q, \ldots, pq^{n-1}, q^n$. Then the Frobenius complex $F(\lambda; \Lambda_n)$ of $\Lambda_n$ has the homotopy type of a wedge of spheres for each $\lambda \in \Lambda_n$, and the multi-graded Poincaré series of $\Lambda_n$ satisfies

$$P_{\Lambda_n}(t, z) = \frac{\prod_{i=0}^{n-1} (1 + tz^{p^{n-i}q^i})}{\prod_{i=1}^{n} (1 - t^2zp^{n-i+1}q^i)}.$$  

In particular, if $p = 2$, then

$$P_{\Lambda_n}(t, z) = \frac{1 + tz^{2^n}}{\prod_{i=1}^{n} (1 - t^2z^{2^{n-i}q^i})}.$$  

**Proof.** The proof is by induction on $n$. If $n = 1$, this follows from Example 5.4. Let $n \geq 2$. Note that $p\Lambda_{n-1} = \langle p^n, p^{n-1}q, \ldots, pq^{n-1} \rangle$ and $\Lambda_n = p\Lambda_{n-1} + \langle q^n \rangle$. Let $\rho = p \cdot q^n = q \cdot pq^{n-1} \in p\Lambda_{n-1}$. By Proposition 5.1, there is a monoid isomorphism $p\Lambda_{n-1}[\rho/p] \cong \Lambda_n$ which sends $\lambda \in p\Lambda_{n-1}$ to $\lambda$ and $\rho/p$ to $q^n$. We now show that $\rho \in p\Lambda_{n-1}$ satisfies the assumption of Corollary 4.3. By the equation

$$q^n - p^n = (q - p) \sum_{i=0}^{n-1} p^{n-i-1}q^i$$

we have

$$\rho = p \cdot q^n = p \cdot p^n + \sum_{i=0}^{n-1} (q - p) \cdot p^{n-i}q^i.$$  

Hence $p^{n-i}q^i \leq p\Lambda_{n-1} \rho$ for each $i \in \{0, \ldots, n-1\}$. Then we can apply Theorem 4.1 and Corollary 4.2 to $p\Lambda_{n-1}[\rho/p] \cong \Lambda_n$. Thus $F(\lambda; \Lambda_n)$ has the homotopy type of a wedge of spheres for each $\lambda \in \Lambda_n$, and the multi-graded Poincaré series satisfies

$$P_{\Lambda_n}(t, z) = \frac{P_{p\Lambda_{n-1}}(t, z) \cdot (1 + tz^{2^n})}{1 - t^2zp^{n+1}q^n}$$

$$= \frac{P_{\Lambda_{n-1}}(t, z^p) \cdot (1 + tz^{2^n})}{1 - t^2zp^{n+1}q^n}$$

$$= \frac{\prod_{i=0}^{n-1} (1 + t(z^p)^{p^{n-i-1}q^i})}{\prod_{i=1}^{n} (1 + t^2(z^p)p^{n-i+1}q^i)} \cdot \frac{1 + tz^{2^n}}{1 - t^2zp^{n+1}q^n}$$

$$= \frac{\prod_{i=0}^{n-1} (1 + t(z^p)^{p^{n-i}q^i})}{\prod_{i=1}^{n} (1 + t^2(z^p)p^{n-i+1}q^i)}.$$  

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