EXTREME VALUES PROBLEM OF UNCERTAIN HEAT EQUATION

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Abstract. Uncertain heat equation is a class of uncertain partial differential equations involving Liu processes. This paper first gives the uncertainty distributions and the inverse uncertainty distributions of extreme values of solutions for uncertain heat equations. Numerical methods are designed to gain the inverse uncertainty distributions of extreme values of solutions.

1. Introduction. To handle with the human degrees of belief, uncertainty theory is a new mathematical system that is based on four axioms (normality axiom, duality axiom, subadditivity axiom and product axiom). It was established by Liu [3] in 2007 and perfected by Liu [5] in 2009. As an important concept in uncertainty theory, uncertain process describes the dynamical behavior of uncertain phenomena, it is a series of uncertain variables indexed by totally ordered set. A fundamental uncertain process is a Liu process [5] to model white noise in uncertain system. Based on Liu process, uncertain calculus was built by Liu [5] to deal with the integration and differentiation of functions for uncertain processes.

On the strength of uncertain calculus, uncertain differential equation was first presented by Liu [4]. The existence and uniqueness theorem of an uncertain differential equation was gave by Chen and Liu [1]. In 2013, Yao and Chen [15] did a particularly important work to reduce an uncertain differential equation to a family of ordinary differential equations. This results made it possible to analyze uncertain differential equation, and lots of significative works have been obtained such as the inverse uncertainty distribution of solution; expected value of solution; extreme value of solution; first hitting time of solution and time integral of solution. At present, uncertain differential equation theory has been widely applied in many fields by many researchers, such as uncertain finance (Liu [5, 7], Chen and Gao [2], Liu et al. [9]), uncertain control (Zhu [16]), and uncertain differential game (Yang and Gao [10, 11]).

Uncertain partial differential equation, as an extension of uncertain differential equation, was first proposed by Yang and Yao [12]. They derived uncertain heat equation whose heat source is often affected by the uncertain interference. In addition, they also obtained the solution and inverse uncertainty distribution of solution...
for a special class of uncertain heat equations. Later then, under linear growth condition and Lipschitz condition, Yang and Ni [13] proved an existence and uniqueness theorem of solutions for general uncertain heat equations. Yang [14] defined a concept of $\alpha$-path for uncertain heat equation, and showed that the solution of an uncertain heat equation can be represented by a spectrum of solutions for ordinary heat equations. Besides, he also got the inverse uncertainty distribution of solution and expected value of solution for an uncertain heat equation via $\alpha$-path.

This paper aims at giving the extreme values of solution for an uncertain heat equation. The remainder of this paper is structured as follows. Section 2 introduces some works of uncertain heat equation. Section 3 obtains the uncertainty distributions and the inverse uncertainty distributions of extreme values of solution for an uncertain heat equation. And some numerical methods are designed to get the inverse uncertainty distributions of extreme values of solution. Section 4 gives several examples to expound the efficiency of the proposed numerical methods. At last, Section 5 addresses a brief summary.

2. Uncertain heat equation. As a classical partial differential equation, heat equation describes the variation of temperature in a given region over time. However, heat source often suffers the interference of noise in practice. In order to describe the noise, two processes are used, one is a Wiener process that is based on probability theory, another is a Liu process that is based on uncertainty theory. If we consider noise as Wiener process, then heat equation turns into stochastic heat equation. Nevertheless, Yang and Yao [12] pointed that it is unreasonable to model the heat conduction process via stochastic heat equation. Therefore, Yang and Yao [12] proposed an one-dimensional uncertain heat equation whose the noise of heat source is described by Liu process as follows,

$$\begin{align*}
\frac{\partial U_{t,x}}{\partial t} - a^2 \frac{\partial^2 U_{t,x}}{\partial x^2} &= f(t, x) + \sigma(t, x) \dot{C}_t \\
U_{0,x} &= \varphi(x), \quad t > 0, x \in \mathbb{R}
\end{align*}$$

(1)

where $a^2$ is a constant thermal diffusivity $(a > 0)$, $\dot{C}_t = dC_t/dt$ denotes the time white noise, $C_t$ is a Liu process, $f(t, x)$ is a heat source, $\sigma(t, x)$ is a diffusion term of heat source, and $\varphi(x)$ is a given initial temperature at time $t = 0$. They proved that the solution of uncertain heat equation (1) is

$$
U_{t,x} = \int_{-\infty}^{+\infty} K(t, x-y)\varphi(y)dy + \int_{0}^{t} \int_{-\infty}^{+\infty} K(t-s, x-y)f(s, y)dyds 
+ \int_{0}^{t} \int_{-\infty}^{+\infty} K(t-s, x-y)\sigma(s, y)dydC_s
\tag{2}
$$

where

$$K(t, x) = \frac{1}{2a\sqrt{\pi t}} \exp \left( -\frac{x^2}{4a^2t} \right).
$$

For a general uncertain heat equation,

$$\begin{align*}
\frac{\partial U_{t,x}}{\partial t} - a^2 \frac{\partial^2 U_{t,x}}{\partial x^2} &= f(t, x, U_{t,x}) + \sigma(t, x, U_{t,x}) \dot{C}_t \\
U_{0,x} &= \varphi(x), \quad t > 0, x \in \mathbb{R}
\end{align*}$$

(3)

where $\varphi(x)$ is a bounded real-valued function. Yang and Ni [13] proved an existence and uniqueness theorem of solution for uncertain heat equation (3) under linear
growth condition
\[|f(t, x, u)| + |\sigma(t, x, u)| \leq L(1 + |u|), \quad \forall x \in \mathbb{R}, t \geq 0\]
and Lipschitz condition
\[|f(t, x, u) - f(t, x, v)| + |\sigma(t, x, u) - \sigma(t, x, v)| \leq L|u - v|, \quad \forall x \in \mathbb{R}, t \geq 0\]
for some constant \(L\). As a corollary, the solution (2) is unique for uncertain heat equation (1) if \(f(t, x), \sigma(t, x)\) and \(\varphi(x)\) are bounded functions.

Yang [14] defined a concept of \(\alpha\)-path, and found a connection between an uncertain heat equation and a family ordinary heat equations.

**Definition 2.1.** (Yang [14]) Let \(\alpha\) be a number with \(0 < \alpha < 1\). An uncertain heat equation
\[
\frac{\partial U_{t,x}}{\partial t} - a^2 \frac{\partial^2 U_{t,x}}{\partial x^2} = f(t, x, U_{t,x}) + \sigma(t, x, U_{t,x})\hat{C}_t
\]
is said to have an \(\alpha\)-path \(U_{t,x}^\alpha\) if it solves the corresponding partial differential equation
\[
\frac{\partial U_{t,x}^\alpha}{\partial t} - a^2 \frac{\partial^2 U_{t,x}^\alpha}{\partial x^2} = f(t, x, U_{t,x}^\alpha) + |\sigma(t, x, U_{t,x}^\alpha)| \Phi^{-1}(\alpha)
\]
where \(\Phi^{-1}(\alpha)\) is the inverse standard normal uncertainty distribution, i.e.,
\[
\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.
\]

**Theorem 2.2.** (Yang [14]) Let \(U_{t,x}\) and \(U_{t,x}^\alpha\) be the solution and \(\alpha\)-path of the uncertain heat equation
\[
\frac{\partial U_{t,x}}{\partial t} - a^2 \frac{\partial^2 U_{t,x}}{\partial x^2} = f(t, x, U_{t,x}) + \sigma(t, x, U_{t,x})\hat{C}_t,
\]
respectively. Then
\[
\mathcal{M}\{U_{t,x} \leq U_{t,x}^\alpha, \forall t, x\} = \alpha,
\]
\[
\mathcal{M}\{U_{t,x} > U_{t,x}^\alpha, \forall t, x\} = 1 - \alpha.
\]

As a corollary, the solution \(U_{t,x}\) has an inverse uncertainty distribution \(\Phi_t^{-1}(\alpha) = U_{t,x}^\alpha\).

3. **Extreme value of solution.** This section investigates the extreme values of the solution for an uncertain heat equation and obtains their inverse uncertainty distributions. Besides, some numerical methods are designed to get the inverse uncertainty distributions.

**Theorem 3.1.** Let \(U_{t,x}\) and \(U_{t,x}^\alpha\) be the solution and \(\alpha\)-path of the uncertain heat equation
\[
\frac{\partial U_{t,x}}{\partial t} - a^2 \frac{\partial^2 U_{t,x}}{\partial x^2} = f(t, x, U_{t,x}) + \sigma(t, x, U_{t,x})\hat{C}_t,
\]
respectively. Then for any time \(t > 0\) and any point \(x \in \mathbb{R}\), and strictly increasing function \(J(\cdot)\), the supremum
\[
\sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x})
\]
has an inverse uncertainty distribution
\[
\Phi_t^{-1}(\alpha) = \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}^\alpha);
\]
and the infimum
\[ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) \]
has an inverse uncertainty distribution
\[ \Phi_t^{-1}(\alpha) = \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}^\alpha). \]

**Proof.** Since \( J(\cdot) \) is a strictly increasing function, it is always true that
\[ \left\{ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) \leq \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^\alpha) \right\} \supset \{ U_{s,x} \leq U_{s,x}^\alpha, \forall s, x \}. \]
By monotonicity theorem and Theorem 2.2, we get
\[ \mathcal{M} \left\{ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) \leq \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^\alpha) \right\} \geq \mathcal{M} \{ U_{s,x} \leq U_{s,x}^\alpha, \forall s, x \} = \alpha. \quad (4) \]

Similarly, we also obtain
\[ \mathcal{M} \left\{ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) > \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^\alpha) \right\} \geq \mathcal{M} \{ U_{s,x} > U_{s,x}^\alpha, \forall s, x \} = 1 - \alpha. \quad (5) \]

In addition, since
\[ \left\{ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) \leq \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^\alpha) \right\} \]
and
\[ \left\{ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) > \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^\alpha) \right\} \]
are opposite events, the duality axiom makes
\[ \mathcal{M} \left\{ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) \leq \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^\alpha) \right\} + \mathcal{M} \left\{ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) > \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^\alpha) \right\} = 1. \quad (6) \]

It follows from equations (4)-(6) that
\[ \mathcal{M} \left\{ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) \leq \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^\alpha) \right\} = \alpha. \]
Hence the supremum
\[ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) \]
has an inverse uncertainty distribution
\[ \Phi_t^{-1}(\alpha) = \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}^\alpha). \]

**Next,** it is easy to verify that
\[ \left\{ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) \leq \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^\alpha) \right\} \supset \{ U_{s,x} \leq U_{s,x}^\alpha, \forall s, x \}. \]
By monotonicity theorem and Theorem 2.2, we get

\[ M \left\{ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) \leq \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}^\alpha) \right\} \geq M\{ U_{s,x} \leq U_{s,x}^\alpha, \forall s, x \} = \alpha. \quad (7) \]

Similarly, we also obtain

\[ M \left\{ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) > \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}^\alpha) \right\} \geq M\{ U_{s,x} > U_{s,x}^\alpha, \forall s, x \} = 1 - \alpha. \quad (8) \]

In addition, since

\[ \left\{ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) \right\} \text{ and } \left\{ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}^\alpha) \right\} \]

are opposite events, the duality axiom makes

\[ M \left\{ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) \leq \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}^\alpha) \right\} \]

\[ + M \left\{ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) > \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}^\alpha) \right\} = 1. \quad (9) \]

It follows from equations (7)-(9) that

\[ M \left\{ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) \leq \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}^\alpha) \right\} = \alpha. \]

Hence the infimum

\[ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) \]

has an inverse uncertainty distribution

\[ \Phi_t^{-1}(\alpha) = \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}^\alpha). \]

Thus the theorem is proved.  \( \square \)

Assume \( J(\cdot) \) is a strictly increasing function, and \( U_{t,x} \) is a solution of uncertain heat equation (3). Then the following procedure provides the inverse uncertainty distributions of the supremum and infimum.

**Step 0.** Domain \( Q = \{(t, x)|0 \leq t \leq T, 0 \leq x \leq l\}. \)

**Step 1.** Determine time step \( \tau = T/n \) and space step \( h = l/m \), where \( n \) and \( m \) are two positive integers. Write \( t_i = i\tau \) (\( i = 0, 1, \ldots, n \)), \( x_j = jh \) (\( j = 0, 1, \ldots, m \)), \( U_{i,j}^\alpha = U_{i,x}^\alpha \) and

\[ F_{i,j}^\alpha = f(t_i, x_j, U_{i,j}^\alpha) + |\sigma(t_i, x_j, U_{i,j}^\alpha)\Phi_t^{-1}(\alpha)|. \]

**Step 2.** Fix \( \alpha \) on \( (0, 1) \). Set \( i = 0 \) and \( H = \max\{J(\varphi(x_j)), j = 0, 1, \ldots, m\} \) or \( H = \min\{J(\varphi(x_j)), j = 0, 1, \ldots, m\} \).

**Step 3.** Calculate \( U_{i+1,j}^\alpha \) (\( j = 0, 1, \ldots, m \)) by the following recursion formula

\[
\begin{aligned}
U_{i+1,0}^\alpha &= rU_{i,0}^\alpha + (1-r)U_{i,1}^\alpha + \tau F_{i,0}^\alpha \\
U_{i+1,j}^\alpha &= rU_{i,j+1}^\alpha + (1-2r)U_{i,j}^\alpha + rU_{i,j-1}^\alpha + \tau F_{i,j}^\alpha, \quad j = 1, 2, \ldots, m-1 \\
U_{i+1,m}^\alpha &= (1-r)U_{i,m}^\alpha + rU_{i,m-1}^\alpha + \tau F_{i,m}^\alpha
\end{aligned}
\]
where \( r = a^2 \tau / h^2 \), \( U^\alpha_{0,j} = \varphi(x_j) \).

**Step 4.** Set

\[
H \leftarrow \max \{ H, J(\varphi(U^\alpha_{i+1,j})), j = 0, 1, \ldots, m \}
\]
or

\[
H \leftarrow \min \{ H, J(\varphi(U^\alpha_{i+1,j})), j = 0, 1, \ldots, m \}
\]

and \( i \leftarrow i + 1 \).

**Step 5.** Repeat Steps 3 and 4 for \( n \) times.

**Step 6.** The inverse uncertainty distribution of

\[
\sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) \quad \text{or} \quad \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x})
\]
is determined by

\[
\Phi^{-1}_t(\alpha) = H.
\]

**Theorem 3.2.** Let \( U_{t,x} \) be the solution of the uncertain heat equation

\[
\frac{\partial U_{t,x}}{\partial t} - a^2 \frac{\partial^2 U_{t,x}}{\partial x^2} = f(t, x, U_{t,x}) + \sigma(t, x, U_{t,x}) \dot{C}_t.
\]

Suppose \( U_{t,x} \) has an uncertainty distribution \( \Phi_{t,x}(y) \) at each time \( t \) and each point \( x \). Then for a strictly increasing function \( J(\cdot) \), the supremum

\[
\sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x})
\]
has an uncertainty distribution

\[
\Phi_t(y) = \inf_{0 \leq s \leq t, x \in \mathbb{R}} \Phi_{t,x}(J^{-1}(y));
\]

and the infimum

\[
\inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x})
\]
has an uncertainty distribution

\[
\Phi_t(y) = \sup_{0 \leq s \leq t, x \in \mathbb{R}} \Phi_{t,x}(J^{-1}(y)).
\]

**Proof.** Since \( U^\alpha_{t,x} = \Phi_{t,x}^{-1}(\alpha) \), we have

\[
\mathbb{M} \left\{ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) \leq \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(\Phi^{-1}_t(\alpha)) \right\} = \alpha
\]
by Theorem 3.1. Write

\[
y = \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(\Phi^{-1}_t(\alpha)),
\]
that is,

\[
\alpha = \inf_{0 \leq s \leq t, x \in \mathbb{R}} \Phi_{t,x}(J^{-1}(y)).
\]

Then

\[
\mathbb{M} \left\{ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) \leq y \right\} = \inf_{0 \leq s \leq t, x \in \mathbb{R}} \Phi_{t,x}(J^{-1}(y)).
\]

Hence the supremum

\[
\sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x})
\]
has an uncertainty distribution
\[ \Phi_t(y) = \inf_{0 \leq s \leq t, x \in \mathbb{R}} \phi_{t,x}(J^{-1}(y)). \]

Similarly, we have
\[ M \left\{ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) \leq \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(\Phi_{t,x}^{-1}(\alpha)) \right\} = \alpha \]
by Theorem 3.1. Write
\[ y = \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(\Phi_{t,x}^{-1}(\alpha)), \]
that is,
\[ \alpha = \sup_{0 \leq s \leq t, x \in \mathbb{R}} \Phi_{t,x}(J^{-1}(y)). \]
Then
\[ M \left\{ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) \leq y \right\} = \sup_{0 \leq s \leq t, x \in \mathbb{R}} \Phi_{t,x}(J^{-1}(y)). \]
Hence the infimum
\[ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) \]
has an uncertainty distribution
\[ \Phi_t(y) = \sup_{0 \leq s \leq t, x \in \mathbb{R}} \Phi_{t,x}(J^{-1}(y)). \]
Thus the theorem is proved. \( \square \)

**Theorem 3.3.** Let \( U_{t,x} \) and \( U_{t,x}^{\alpha} \) be the solution and \( \alpha \)-path of the uncertain heat equation
\[ \frac{\partial U_{t,x}}{\partial t} - a^2 \frac{\partial^2 U_{t,x}}{\partial x^2} = f(t, x, U_{t,x}) + \sigma(t, x, U_{t,x}) \dot{C}_t, \]
respectively. Then for any time \( t > 0 \) and any point \( x \in \mathbb{R} \), and strictly decreasing function \( J(\cdot) \), the supremum
\[ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) \]
has an inverse uncertainty distribution
\[ \Phi^{-1}_t(\alpha) = \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}^{1-\alpha}); \]
and the infimum
\[ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) \]
has an inverse uncertainty distribution
\[ \Phi^{-1}_t(\alpha) = \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}^{1-\alpha}). \]

**Proof.** Since \( J(\cdot) \) is a strictly decreasing function, it is always true that
\[ \left\{ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) \leq \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^{1-\alpha}) \right\} \supset \left\{ U_{s,x} \geq U_{s,x}^{1-\alpha}, \forall s, x \right\}. \]
By monotonicity theorem and Theorem 2.2, we get
\[ M \left\{ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) \leq \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^{1-\alpha}) \right\} \geq M\{U_{s,x} \geq U_{s,x}^{1-\alpha}, \forall s, x \} = \alpha. \]
Similarly, we also obtain
\[
M \left\{ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) > \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^{1-\alpha}) \right\} \geq M \{ U_{s,x} < U_{s,x}^{1-\alpha}, \forall s, x \} = 1 - \alpha. \tag{11}
\]

In addition, since
\[
\left\{ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) \leq \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^{1-\alpha}) \right\}
\]
and
\[
\left\{ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) > \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^{1-\alpha}) \right\}
\]
are opposite events, the duality axiom makes
\[
M \left\{ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) \leq \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^{1-\alpha}) \right\} + M \left\{ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) > \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^{1-\alpha}) \right\} = 1. \tag{12}
\]

It follows from equations (10)-(12) that
\[
M \left\{ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) \leq \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^{1-\alpha}) \right\} = \alpha.
\]

Hence the supremum
\[
\sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x})
\]
has an inverse uncertainty distribution
\[
\Phi_{t}^{-1}(\alpha) = \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}^{1-\alpha}).
\]

Next, it is easy to verify that
\[
\left\{ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) \leq \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^{1-\alpha}) \right\} \supset \{ U_{s,x} \geq U_{s,x}^{1-\alpha}, \forall s, x \}.
\]

By monotonicity theorem and Theorem 2.2, we get
\[
M \left\{ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) \leq \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^{1-\alpha}) \right\} \geq M \{ U_{s,x} \geq U_{s,x}^{1-\alpha}, \forall s, x \} = \alpha. \tag{13}
\]

Similarly, we also obtain
\[
M \left\{ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) > \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^{1-\alpha}) \right\} \geq M \{ U_{s,x} < U_{s,x}^{1-\alpha}, \forall s, x \} = 1 - \alpha. \tag{14}
\]

In addition, since
\[
\left\{ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) \leq \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^{1-\alpha}) \right\}
\]
and
\[
\left\{ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) > \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^{1-\alpha}) \right\}
\]
are opposite events, the duality axiom makes
\[
M \left\{ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) \leq \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^{1-\alpha}) \right\} + M \left\{ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) > \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^{1-\alpha}) \right\} = 1.
\]
(15)
It follows from equations (13)-(15) that
\[
M \left\{ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}) \leq \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{t,x}^{1-\alpha}) \right\} = \alpha.
\]
Hence the infimum
\[
\inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x})
\]
has an inverse uncertainty distribution
\[
\Phi^{-1}(\alpha) = \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}^{1-\alpha}).
\]
Thus the theorem is proved. □

Assume \( J(\cdot) \) is a strictly decreasing function, and \( U_{t,x} \) is a solution of uncertain heat equation (3). Then the following procedure provides the inverse uncertainty distributions of the supremum and infimum.

**Step 0.** Domain \( Q = \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq l\} \).

**Step 1.** Determine time step \( \tau = T/n \) and space step \( h = l/m \), where \( n \) and \( m \) are two positive integers. Write \( t_i = i \tau \) \((i = 0, 1, \cdots, n)\), \( x_j = jh \) \((j = 0, 1, \cdots, m)\), \( U_{i,j} = U_{t_i,x_j} \) and
\[
F_{i,j}^\alpha = f(t_i, x_j, U_{i,j}^\alpha) + |\sigma(t_i, x_j, U_{i,j}^\alpha)| \Phi^{-1}(\alpha).
\]
**Step 2.** Fix \( \alpha \) on \((0, 1)\). Set \( i = 0 \) and \( H = \max\{J(\varphi(x_j)), j = 0, 1, \cdots, m\} \) or \( H = \min\{J(\varphi(x_j)), j = 0, 1, \cdots, m\} \).

**Step 3.** Calculate \( U_{i+1,j}^{1-\alpha} \) \((j = 0, 1, \cdots, m)\) by the following recursion formula
\[
\begin{align*}
U_{i+1,0}^{1-\alpha} &= rU_{i,1}^{1-\alpha} + (1-r)U_{i,0}^{1-\alpha} + \tau F_{i,0}^{1-\alpha}, \\
U_{i+1,j}^{1-\alpha} &= rU_{i,j+1}^{1-\alpha} + (1-2r)U_{i,j}^{1-\alpha} + rU_{i,j-1}^{1-\alpha} + \tau F_{i,j}^{1-\alpha}, \quad j = 1, 2, \cdots, m-1, \\
U_{i+1,m}^{1-\alpha} &= (1-r)U_{i,m}^{1-\alpha} + rU_{i,m-1}^{1-\alpha} + \tau F_{i,m}^{1-\alpha}
\end{align*}
\]
where \( r = a^2 \tau/h^2, U_{0,j}^{1-\alpha} = \varphi(x_j) \).

**Step 4.** Set
\[
H \leftarrow \max\{H, J(\varphi(U_{i+1,j}^{1-\alpha})), j = 0, 1, \cdots, m\}
\]
or
\[
H \leftarrow \min\{H, J(\varphi(U_{i+1,j}^{1-\alpha})), j = 0, 1, \cdots, m\}
\]
and \( i \leftarrow i + 1 \).

**Step 5.** Repeat Steps 3 and 4 for \( n \) times.

**Step 6.** The inverse uncertainty distribution of
\[
\inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) \quad \text{or} \quad \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x})
\]
is determined by
\[
\Phi^{-1}(\alpha) = H.
\]
Theorem 3.4. Let $U_{t,x}$ be the solution of the uncertain heat equation
\[
\frac{\partial U_{t,x}}{\partial t} - a^2 \frac{\partial^2 U_{t,x}}{\partial x^2} = f(t, x, U_{t,x}) + \sigma(t, x, U_{t,x}) \dot{C}_t.
\]
Suppose $U_{t,x}$ has an uncertainty distribution $\Phi_{t,x}(y)$ at each time $t$ and each point $x$. Then for a strictly decreasing function $J(\cdot)$, the supremum
\[
\sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x})
\]
has an uncertainty distribution
\[
\Phi_t(y) = 1 - \inf_{0 \leq s \leq t, x \in \mathbb{R}} \Phi_{t,x}(J^{-1}(y));
\]
and the infimum
\[
\inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x})
\]
has an uncertainty distribution
\[
\Phi_t(y) = 1 - \sup_{0 \leq s \leq t, x \in \mathbb{R}} \Phi_{t,x}(J^{-1}(y)).
\]

Proof. Since $U^\alpha_{t,x} = \Phi^{-1}_{t,x}(\alpha)$, we have
\[
\mathcal{M} \left\{ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) \leq \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(\Phi^{-1}_{t,x}(1-\alpha)) \right\} = \alpha
\]
by Theorem 3.3. Write
\[
y = \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(\Phi^{-1}_{t,x}(1-\alpha)),
\]
that is,
\[
\alpha = 1 - \inf_{0 \leq s \leq t, x \in \mathbb{R}} \Phi_{t,x}(J^{-1}(y)).
\]
Then
\[
\mathcal{M} \left\{ \sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) \leq y \right\} = 1 - \inf_{0 \leq s \leq t, x \in \mathbb{R}} \Phi_{t,x}(J^{-1}(y)).
\]
Hence the supremum
\[
\sup_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x})
\]
has an uncertainty distribution
\[
\Phi_t(y) = 1 - \inf_{0 \leq s \leq t, x \in \mathbb{R}} \Phi_{t,x}(J^{-1}(y)).
\]

Similarly, we have
\[
\mathcal{M} \left\{ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) \leq \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(\Phi^{-1}_{t,x}(1-\alpha)) \right\} = \alpha
\]
by Theorem 3.3. Write
\[
y = \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(\Phi^{-1}_{t,x}(1-\alpha)),
\]
that is,
\[
\alpha = 1 - \sup_{0 \leq s \leq t, x \in \mathbb{R}} \Phi_{t,x}(J^{-1}(y)).
\]
Then
\[
\mathcal{M} \left\{ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) \leq y \right\} = 1 - \sup_{0 \leq s \leq t, x \in \mathbb{R}} \Phi_{t,x}(J^{-1}(y)).
\]
Hence the infimum
\[ \inf_{0 \leq s \leq t, x \in \mathbb{R}} J(U_{s,x}) \]
has an uncertainty distribution
\[ \Phi_t(y) = 1 - \sup_{0 \leq s \leq t, x \in \mathbb{R}} \Phi_t(x)(J^{-1}(y)). \]
Thus the theorem is proved. □

4. Numerical experiments. In this section, we give several examples to illustrate the efficiency of the above numerical method.

Example 4.1. Consider the uncertain heat equation,
\[
\begin{cases}
\frac{\partial U_{t,x}}{\partial t} - \frac{\partial^2 U_{t,x}}{\partial x^2} = e^{-t} \sin x + \dot{C}_t \\
U_{0,x} = 0, \quad t > 0, x \in \mathbb{R}
\end{cases}
\]
By equation (2), the solution of uncertain heat equation (16) is
\[
U_{t,x} = \int_0^t \int_{-\infty}^{\infty} K(t-s, x-y)e^{-s} \sin y dy ds + \int_0^t \int_{-\infty}^{\infty} K(t-s, x-y)dy dC_s
\]
= \(te^{-t} \sin x + C_t\).

Further, we can obtain the inverse uncertainty distributions of the supremum
\[ \sup_{0 \leq s \leq 1, 0 \leq x \leq 8} U_{s,x} \]
and the infimum
\[ \inf_{0 \leq s \leq 1, 0 \leq x \leq 8} U_{s,x} \]
under the parameters, \(t = 1, l = 8, n = 200, m = 80\). See Figure 1.

![Figure 1. Inverse Uncertainty Distributions of Extreme Values in Example 4.1](image)

Example 4.2. Consider the uncertain heat equation,
\[
\begin{cases}
\frac{\partial U_{t,x}}{\partial t} - \frac{\partial^2 U_{t,x}}{\partial x^2} = \sin x - e^{-t} \dot{C}_t \\
U_{0,x} = \cos x, \quad t > 0, x \in \mathbb{R}
\end{cases}
\]
By equation (2), the solution of uncertain heat equation (17) is
\[
U_{t,x} = \int_{-\infty}^{+\infty} K(t, x - y) \cos y dy + \int_{0}^{t} \int_{-\infty}^{+\infty} K(t - s, x - y) \sin y dy ds
- \int_{0}^{t} \int_{-\infty}^{+\infty} K(t - s, x - y) e^{-s} dy dC_s
= e^{-t} \cos x - \int_{0}^{t} e^{-s} dC_s.
\]
Further, we can obtain the inverse uncertainty distributions of the supremum
\[
\sup_{0 \leq s \leq 1, 0 \leq x \leq 8} -U_{s,x}
\]
and the infimum
\[
\inf_{0 \leq s \leq 1, 0 \leq x \leq 8} -U_{s,x}
\]
under the parameters, \( t = 1, l = 8, n = 200, m = 80 \). See Figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Figure2.png}
\caption{Inverse Uncertainty Distributions of Extreme Values in Example 4.2}
\end{figure}

5. Conclusion. The main contribution of this paper was first to obtain extreme values of solution for an uncertain heat equation. But beyond that, some numerical methods were designed to get the inverse uncertainty distributions of extreme values of solution. For future work, one may consider the stability of uncertain heat equation, such as, stability in measure, stability in mean, stability in distribution and almost sure stability; one may study the boundary value problem of uncertain heat equation.

Appendix A. Uncertainty theory. This section introduces a few fundamental concepts and theorems in uncertainty theory including uncertain measure, uncertain process and uncertain field.
Definition A.1. (Liu [3]) Let $\mathcal{L}$ be a $\sigma$-algebra on a nonempty set $\Gamma$. A set function $M : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms,
(i) Normality Axiom. $M(\Gamma) = 1$ for the universal set $\Gamma$;
(ii) Duality Axiom. $M(\Lambda) + M(\Lambda^c) = 1$ for any event $\Lambda$;
(iii) Subadditivity Axiom. For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have
\[
M\left(\bigcup_{i=1}^{\infty} \Lambda_i\right) \leq \sum_{i=1}^{\infty} M(\Lambda_i).
\]

Theorem A.2. (Liu [6], Monotonicity Theorem) Uncertain measure $M$ is a monotone increasing set function. That is, for any events $\Lambda_1 \subset \Lambda_2$, we have
\[
M(\Lambda_1) \leq M(\Lambda_2).
\]

Besides, Liu [5] defined the product uncertain measure on the product $\sigma$-algebra $\mathcal{L}$ as follows.
(iv) Product Axiom. Let $(\Gamma_k, L_k, M_k)$ be uncertainty spaces for $k = 1, 2, \cdots$. The product uncertain measure $M$ is an uncertain measure satisfying
\[
M\left(\prod_{k=1}^{\infty} \Lambda_k\right) = \bigwedge_{k=1}^{\infty} M_k(\Lambda_k)
\]
where $\Lambda_k$ are arbitrarily chosen events from $L_k$ for $k = 1, 2, \cdots$, respectively.

An uncertain variable $\xi$ is a measurable function from an uncertainty space $(\Gamma, L, M)$ to the set of real numbers $\mathbb{R}$. The uncertain variables $\xi_1, \xi_2, \cdots, \xi_m$ are said to be independent if
\[
M\left(\bigcap_{i=1}^{m} \{\xi_i \in B_i\}\right) = \bigwedge_{i=1}^{m} M\{\xi_i \in B_i\}
\]
for any Borel sets $B_1, B_2, \cdots, B_m$ of real numbers.

Definition A.3. (Liu [4]) Let $T$ be a totally ordered set (e.g. time) and let $(\Gamma, L, M)$ be an uncertainty space. An uncertain process is a function $X_t(\gamma)$ from $T \times (\Gamma, L, M)$ to the set of real numbers such that $\{X_t \in B\}$ is an event for any Borel set $B$ of real numbers at each time $t$.

Definition A.4. (Liu [5]) An uncertain process $C_t$ is said to be a Liu process if
(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous;
(ii) $C_t$ has stationary and independent increments;
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with an uncertainty distribution
\[
\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3t}}\right)\right)^{-1}, \quad x \in \mathbb{R}.
\]

Definition A.5. (Liu [5]) Let $X_t$ be an uncertain process and let $C_t$ be a Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as
\[
\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|.
\]

Then Liu integral of $X_t$ with respect to $C_t$ is defined as
\[
\int_a^b X_t dC_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})
\]
provided that the limit exists almost surely and is finite. In this case, the uncertain process $X_t$ is said to be Liu integrable.

Let $h(t, c)$ be a continuously differentiable function. Then $Z_t = h(t, C_t)$ has an uncertain differential

$$dZ_t = \frac{\partial h}{\partial t}(t, C_t)dt + \frac{\partial h}{\partial c}(t, C_t)dC_t.$$

**Definition A.6.** (Liu [8]) Let $T$ be a partially ordered set (e.g. time $\times$ space) and let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain field is a function $X_t(\gamma)$ from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{X_t \in B\}$ is an event for any Borel set $B$ of real numbers at each $t$.

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