Measurement of the Wigner distribution function of non-separable laser beams employing a toroidal mirror

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Abstract
We present a new experimental approach to determine the Wigner distribution function of non-separable laser beams. A rotatable toroidal mirror is employed to focus an IR laser beam at a wavelength of 1064 nm. For different rotation angles of the mirror, intensity profiles of the reflected beam are captured. The resulting data suffices to fully reconstruct the four-dimensional Wigner distribution from which the global degree of coherence is derived. A comparison to theoretical expectations reveals validity of the applied formalism. Since the setup dispenses with any transmissive optics it can be applied to a broad spectral range, especially to extreme UV and soft x-ray radiation.

Keywords: coherence, Wigner distribution function, free-electron laser, soft x-ray, extreme ultraviolet

1. Introduction
Coherent electromagnetic radiation in the extreme UV (EUV) and soft x-ray range is the basis of remarkable scientific investigations. For instance, with coherent diffractive imaging (CDI) it is possible to reconstruct biological cells, proteins or viruses from their diffraction image [1–3]. Conventional CDI demands a high degree of spatial and temporal coherence of the employed beam. In fact, even the newest generation synchrotron or free-electron laser (FEL) sources...
produce rather partially coherent radiation [4–6]. Spence et al [7] formulated a weaker requirement on the spatial coherence properties of applied beams, i.e. the lateral coherence length should be larger than twice the size of the illuminated object. This demand can further be relaxed under an a priori knowledge of the mutual coherence function of the incident beam employing advanced algorithms which take partial coherence into account [8, 9].

Many different experimental methods serve to characterize the propagation of FEL beams, e.g. the analysis of intensity profiles or wavefronts [10–13]. A standard approach to gain information on the spatial coherence properties is Young’s double pinhole experiment [14] which is commonly conducted at FELs [4–6]. However, measuring the entire mutual coherence function $\Gamma(\vec{x}, \vec{s})$ is an elaborate task since each single point $(\vec{x}, \vec{s})$ in the four-dimensional (4D) configuration space requires one experiment. For an adequate rasterization, this exceeds hundreds of thousands of shots.

In order to entirely recover $\Gamma(\vec{x}, \vec{s})$ with reasonable effort we follow the formalism of the Wigner distribution function (WDF) $h(\vec{x}, \vec{u})$ which is defined as the two-dimensional Fourier-transform of the mutual coherence function. That method is well established for visible and UV wavelengths [15–19] and has been applied to synchrotron [20] and FEL sources [21, 22] too.

The reconstruction of $h(\vec{x}, \vec{u})$ for a separable beam (i.e. the Wigner distribution decomposes into a product of horizontal and vertical function) requires information of the three-dimensional (3D) radiation field. In general, this is realized by a caustic scan behind a focusing optic [21]. If the beam is non-separable, the same 3D data set can be employed to derive the 4D Wigner distribution [22]. However, due to a lack of data, gaps remain in the phase space possibly leading to reconstruction errors. This issue can be addressed by an interpolation procedure, but obviously, in order to provide physical access to the entire 4D phase space of the beam, one further degree of freedom needs to be introduced in the setup. This can involve an astigmatic focusing element being rotatable about the optical axis. To our best knowledge, in the field of FEL and synchrotron sources, a measurement of the entire 4D phase space has not been realized yet.

In the present study we employ a rotatable toroidal mirror to recover the Wigner distribution of non-separable beams. The measurement is demonstrated for several modes of an IR laser generated by an adjustable resonator. For each mode we derive the WDF $h(\vec{x}, \vec{u})$ and the global degree of coherence $K$. Validity of the obtained data is proved by comparison with the theory.

The presented method is especially suitable for stable beam sources, e.g. seeded FELs, since a large number of separate intensity profiles contributes to the reconstruction of $h(\vec{x}, \vec{u})$. However, in case of fluctuations, the approach describes the beam in terms of an average. Due to the absence of any transmissive optics, the presented setup can also be applied to radiation in the EUV and soft x-ray range. Thus, for the first time, a comprehensive characterization of synchrotron and FEL beams is feasible, possibly serving to further improve beam quality. As described above, knowledge of the entire mutual coherence function can also open the possibility to conduct CDI experiments under comparably poor coherence conditions. Hence, reconstructions of large samples with dimensions exceeding the coherence length become attainable. Moreover, requirements on spatial and spectral filtering of synchrotron or FEL beams can be relaxed, resulting in a more efficient use of the available photons.
2. Theory

The Wigner distribution $h(\vec{x}, \vec{u})$ of a quasi-monochromatic paraxial beam is defined in terms of a two-dimensional Fourier transform of the mutual coherence function $\Gamma(\vec{x}, \vec{s})$ [23]

$$h(\vec{x}, \vec{u}) = \left(\frac{k}{2\pi}\right)^2 \int \Gamma(\vec{x}, \vec{s}) e^{ik\vec{u}\cdot\vec{s}} d^2s,$$

(1)

where $\vec{x} = (x, y)$ and $\vec{s} = (s_x, s_y)$ are two-dimensional spatial and $\vec{u} = (u, v)$ angular coordinates in a plane perpendicular to the direction of beam propagation and $k$ is the mean wave number.

The propagation of the Wigner distribution through static and lossless systems, signified by a $4 \times 4$ optical ray propagation matrix

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

(2)

is discussed elsewhere [19] resulting in the projection slice theorem of tomography [24]

$$\tilde{h}_{\text{ref}} (A_{\{p\}}^T \vec{w}, B_{\{p\}}^T \vec{w}) = \tilde{I}_{\{p\}}(\vec{w}),$$

(3)

with the $2 \times 2$ sub matrices $A$ and $B$, the reciprocal coordinate $\vec{w}$ and a set of parameters $\{p\}$ which is discussed later. The Fourier transformed intensity distribution $\tilde{I}_{\{p\}}(\vec{w})$ represents a two-dimensional plane in the 4D reciprocal phase space. In the aligned astigmatic case, i.e. $A$ and $B$ being diagonal matrices, the tilt angles of such a plane relative to the axes $w_x$ and $w_y$ are

$$\phi_t = \tan^{-1}(B_{11}/A_{11}) \quad \text{and} \quad \phi_q = \tan^{-1}(B_{22}/A_{22}).$$

(4)

The optical system being employed to generate projections of the phase space defines the parameter set $\{p\}$. Here, a toroidal mirror is used as focusing element which is rotatable about its symmetry axis. Beam profiles are measured at different positions $z$ of the resulting caustic by a movable detector. Thus, two variable parameters contribute to $\{p\}$, i.e. the rotation angle $\phi$ of the mirror and the detector position $z$. The corresponding ray transformation matrix $S_{\{\phi, z\}}$ is composed of a propagation from the mean waist position in the beam system $z_0$ to the toroidal mirror at $z = 0$ and a subsequent propagation to the detector plane $z$:

$$S_{\{\phi, z\}} = S_{\text{prop}}(z) \cdot S_{\text{mirror}}(R_1, R_2, \alpha, \phi) \cdot S_{\text{prop}}(-z_0),$$

(5)

with

$$S_{\text{mirror}}(R_1, R_2, \alpha, \phi) = S_{\text{tilt}}(\alpha) \cdot S_{\text{rot}}(\phi) \cdot S_{\text{toroid}}(R_1, R_2) \cdot S_{\text{rot}}^{-1}(\phi) \cdot S_{\text{tilt}}^{-1}(\alpha),$$

(6)

describing the tilted and rotated mirror. The involved matrices can be found in the appendix. Applying the introduced system matrix $S_{\{\phi, z\}}$ to the projection slice theorem given in equation (3), the Wigner distribution is reconstructed as described in more detail in section 4.

Finally, the global degree of coherence $K$ is deduced following

$$K = \frac{\lambda^2}{P^2} \int h(\vec{x}, \vec{u})^2 d^2x d^2u,$$

(7)

with the wavelength $\lambda$ and the total power of the beam $P = \int h(\vec{x}, \vec{u}) d^2x d^2u$. 


3. Experiment

The experimental setup is schematically shown in figure 1. The toroidal mirror at $z = 0$ separates the beam system $z < 0$ from the camera system $z > 0$. Variables in the camera system are denoted by a ', as for instance the waist positions $z_{0,x}'$ and $z_{0,y}'$. A diode pumped Nd:YVO4 laser provides the test beam operating at its fundamental wavelength $\lambda = 1064$ nm in continuous wave mode. The hemispherical resonator is adjustable in order to generate different Hermite–Gaussian modes and their superpositions, also denoted as TEM modes. Within the scope of this investigation this involves TEM$_{00}$, TEM$_{10}$, TEM$_{02}$, TEM$_{03}$ and an uncorrelated superposition of TEM$_{10}$ and TEM$_{01}$. Astigmatic focusing of the beam is realized by a toroidal mirror under an incidence angle of $\alpha = 11^\circ$ (aluminum alloy with the radii $R_1 = 300$ mm and $R_2 = 200$ mm, roughness $\leq 12$ nm and shape accuracy $\leq 1$ µm). A motorized rotation stage allows rotation within the range $-\pi \ldots \pi$ about its symmetry axis controlled by a PC. A subsequent CCD camera (12 bit dynamic range, $1280 \times 1024$ pixel resolution on a $2/3''$ chip) mounted on a motorized linear stage is movable within distances of $70$ mm...200 mm to the toroidal mirror. One measurement takes 30 min and involves 420 beam profiles at 42 different $z$-positions and ten equidistant angles. In order to approach isotropic mapping in reciprocal phase space, i.e. equidistant spacing $\Delta \varphi_{x,y}$ between consecutive planes defined by equation (4), two sets of $z$-positions are employed, distributed as

$$z_{x,i} = \frac{z_R \tan \varphi_i - z_0}{1 - 2/R_1 (z_R \tan \varphi_i - z_0)},$$

$$z_{y,i} = \frac{z_R \tan \varphi_i - z_0}{1 - 2/R_2 (z_R \tan \varphi_i - z_0)},$$

with the mean Rayleigh length $z_R$ in the beam system. In an approximation the incidence angle on the mirror has been assumed to be $\alpha = 0$ for this calculation. Practically, this results in tight sampling of the beam waists behind the toroid becoming coarser further out.

4. Evaluation

Before we retrieve the Wigner distribution, it is necessary to derive waist position $z_0$ and Rayleigh length $z_R$ in the beam system. Thus, in a preprocessing these parameters are obtained...
for the camera system (denoted by ′) and then back-propagated to the beam system by matrix methods. Finally, the Wigner distribution $h(\vec{x}, \vec{u})$ is reconstructed at mean waist position $z_0$ in the beam system.

### 4.1. Preprocessing

Data evaluation starts with background correction, centering of the obtained beam profiles and evaluation of the second order beam moments. For the aligned case, i.e. $\phi = 0$ the beam parameters $z_{0,x}, z_{R,y}$ and $d_{0,x}$ for $x$- and $y$-direction are derived by a least squares fit [10]. Then, a $2 \times 2$ beam matrix at horizontal waist position in the camera system is generated by

$$B_{\text{cam}} = \begin{pmatrix} \langle x^2 \rangle' & \langle xu \rangle' \\ \langle xu \rangle' & \langle u^2 \rangle' \end{pmatrix} = \begin{pmatrix} (d_{0,x}'/4)^2 & 0 \\ 0 & (d_{0,x}'/4z_{R,x}')^2 \end{pmatrix}$$

(9)

and correspondingly for the vertical direction. In order to back-propagate these beam matrices from camera to beam system we employ the $2 \times 2$ propagation matrices

$$S_x = S_{\text{prop}}(z_{0,x}') \cdot S_{\text{toroid},x} = \begin{pmatrix} 1 & z_{0,x}' \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -2/R_1 \cos(\alpha) & 1 \end{pmatrix},$$

$$S_y = S_{\text{prop}}(z_{0,y}') \cdot S_{\text{toroid},y} = \begin{pmatrix} 1 & z_{0,y}' \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -2 \cos(\alpha)/R_2 & 1 \end{pmatrix}$$

(10)

and the propagation relation

$$B_{\text{beam}} = S^{-1} \cdot B_{\text{cam}} \cdot (S^{-1})^T.$$  

(11)

From the resulting beam matrix the beam parameters in $x$-direction are calculated by

$$z_{0,x} = -\frac{\langle xu \rangle}{\langle u^2 \rangle},$$

$$z_{R,x} = \sqrt{\frac{\langle x^2 \rangle}{\langle u^2 \rangle} - \frac{\langle xu \rangle^2}{\langle u^2 \rangle^2}}.$$

(12)

With a corresponding relation for the vertical parameters $z_{0,y}$ and $z_{R,y}$, the mean values $z_0$ and $z_R$ are computed.

### 4.2. Reconstruction of the Wigner distribution

Reconstruction of the Wigner distribution is done on a 4D regular grid containing $129^4$ cells. Following equation (3) the Fourier space of $h$ is filled with data from intensity profiles measured for different parameter sets $\{\phi, z\}$. Here, matrix $B$ is divided by the mean Rayleigh length $z_R$ which reasonably scales the angular coordinates $u$ and $v$, leading to a uniform distribution of $h$ in the resulting phase space. If more than one value contributes to a cell, the arithmetic average is applied. Finally, a 4D Fourier transform of $\tilde{h}$ results the WDF $h(\vec{x}, \vec{u})$. 

5
In this section it should be proved that the presented setup is capable of capturing the WDF of complex and non-separable beams. The aim is to generate specific Hermite–Gaussian beams, also denoted as TEM modes, which can be described theoretically. This way, a comparison can be drawn between experimental results and theoretical descriptions.

The WDF of a Hermite–Gaussian beam is given by

\[
\langle h_{nm}(\vec{x}, \vec{u}) \rangle = h_n(x, u) \cdot h_m(y, v)
\]

with

\[
\theta_\lambda = \frac{4\lambda}{(\pi d_{0,x})}
\]

sets the divergence in relation to the waist diameter and \(L_n\) denotes the Laguerre polynomial of degree \(n\). Single TEM modes result in a global degree of coherence \(K = 1\) and in a high order, they exhibit complex intensity profiles as can be seen in figure 2. However, they are always separable. In contrast, the uncorrelated superposition of a TEM\(_{10}\) and TEM\(_{01}\) beam

\[
h_{sp}(\vec{x}, \vec{u}) = h_{10}(\vec{x}, \vec{u}) + h_{01}(\vec{x}, \vec{u})
\]

is non-separable. Here, the global degree of coherence is \(K = 0.5\). In this case, the intensity profile shows a ring structure as depicted in figure 2(d).

WDF of Hermite–Gaussian beams are depicted in figure 3 as derived from theory and experiment up to order \(n = 3\). Qualitatively, it is revealed that the measured data sets suffice to successfully reconstruct the WDF of the chosen modes. Sub-figures 3(a) and (b) show the vertical and horizontal projection resulting for a TEM\(_{10}\) beam, corresponding to a Hermite–Gaussian mode of the order \(n = 1\) and \(m = 0\). Obviously, the 4D distribution separates properly into horizontal and vertical mode. Sub-figures 3(c) and (d) are produced from a TEM\(_{02}\) and TEM\(_{03}\) beam.

In summary, the theoretical distributions could well be reproduced in the experiment. However, numerical artifacts cannot entirely be avoided leading to small deviations from a perfect reconstruction. On the other hand, it is possible that the experimental beam contains tiny contributions of deviant modes or amplified spontaneous emission, too.
Figure 3. Wigner distribution functions of different Hermite–Gaussian beams in theory (left) and experiment (right). $h_0$ and $h_1$ represent horizontal and vertical projections from the 4D WDF of a TEM$_{10}$ mode. $h_2$ and $h_3$ result from TEM$_{02}$ and TEM$_{03}$ beams.
The WDF of the non-separable superposition TEM$_{10}$ + TEM$_{01}$ is shown in figure 4. Projections of the 4D WDF are employed for visualization whereas in theory, both projections are identical. Thus, only the horizontal distribution is shown. Again, the theoretical prediction is well resembled by the measurement. Here, the experimental distributions appear slightly sheared in opposing directions indicating a small astigmatic aberration. This could be expected since the laser resonator was tweaked strongly in order to produce this particular beam.

Quantitatively, the derived WDFs are characterized by the global degree of coherence $K$.

Theoretical and experimental values of five different beams are given in table 1. A comparison shows good conformity, only small variations of below 10% can be found. As already mentioned, the investigated beams could contain small amounts of parasitic modes leading to a slight reduction of the coherence. The same effect arises from amplified spontaneous emission, which was weakly present during the experiment. Thus, it can well be that the accuracy of the algorithm is even better than 10%.

The presented results prove the capability of the system to capture the WDF of non-separable and complex beam structures. The use of a reflecting optic allows application to EUV and soft x-ray beams, as produced by FELs and synchrotrons. However, then the employed toroidal mirror has to meet additional requirements, i.e. surface roughness and shape accuracy should amount less than the wavelength of the investigated radiation.

### Table 1. Global degree of coherence $K$ for different Hermite–Gaussian beams in theory and experiment.

| Beam                | TEM$_{00}$ | TEM$_{10}$ | TEM$_{02}$ | TEM$_{03}$ | TEM$_{10}$ + TEM$_{01}$ |
|---------------------|------------|------------|------------|------------|-------------------------|
| Theory              | 1          | 1          | 1          | 1          | 0.5                     |
| Experiment          | 0.95       | 1.06       | 0.98       | 0.90       | 0.46                    |

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### 6. Conclusion

A simple experimental setup to measure the 4D Wigner distribution is proposed. The proper functionality is proved with complex and non-separable beams provided by a near IR laser that...
generates Hermite–Gaussian modes and their superpositions. A comparison is drawn between theoretical expectations and experimental results, revealing good conformity. Application of the system to FEL sources can easily be accomplished, since only reflective optics are employed. In the future, the attainable knowledge of the entire mutual coherence function provides the basis for further developments in the field of coherent imaging. It enables successful CDI experiments with less coherent beams or large samples, exceeding the lateral coherence length. Furthermore, the illumination function can be derived from the WDF [25], and the influence of aberrations can be eliminated. Beyond, CDI reconstruction procedures become more robust by providing a proper initial guess.

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Appendix

Here, for the sake of completeness the 4 × 4 ray transformation matrices contributing to \( S_{[\phi, z]} \) are given in detail:

\[
S_{\text{prop}}(z) = \begin{pmatrix} 1 & 0 & z & 0 \\ 0 & 1 & 0 & z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

\[
S_{\text{tilt}}(\alpha) = \begin{pmatrix} \sqrt{\cos \alpha} & 0 & 0 & 0 \\ 0 & 1/\sqrt{\cos \alpha} & 0 & 0 \\ 0 & 0 & 1/\sqrt{\cos \alpha} & 0 \\ 0 & 0 & 0 & \sqrt{\cos \alpha} \end{pmatrix},
\]

\[
S_{\text{rot}}(\phi) = \begin{pmatrix} \cos \phi & \sin \phi & 0 & 0 \\ -\sin \phi & \cos \phi & 0 & 0 \\ 0 & 0 & \cos \phi & \sin \phi \\ 0 & 0 & -\sin \phi & \cos \phi \end{pmatrix},
\]

\[
S_{\text{toroid}}(R_1, R_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2/R_1 & 0 & 1 & 0 \\ 0 & -2/R_2 & 0 & 1 \end{pmatrix}.
\]

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