Codimension-2 brane–bulk matching: examples from six and ten dimensions

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Abstract. Experience with Randall–Sundrum models teaches the importance of following how branes back-react onto the bulk geometry, since this can dramatically affect the system’s low-energy properties. Yet the practical use of this observation for model building is so far mostly restricted to branes having only one transverse dimension (codimension-1) in the bulk space, since this is where tools for following back-reaction are well developed. This is likely to be a serious limitation since experience also tells us that one dimension is rarely representative of what happens in higher dimensions. Here we summarize recent progress in developing the matching conditions that describe how codimension-2 branes couple to bulk metric, gauge and scalar fields. These matching conditions are then applied to three situations: D7-branes in F-theory compactifications of ten-dimensional (10D) Type IIB string vacua; 3-branes coupled to bulk axions in unwarped and non-supersymmetric six-dimensional (6D) systems; and 3-branes coupled to chiral, gauged 6D supergravity. For each it is shown how the resulting brane–bulk dynamics are reproduced by the scalar potential for the low-energy moduli in the dimensionally reduced, on-brane effective theory. For 6D supergravity, we show that the only 4D-maximally symmetric bulk geometries supported by positive-tension branes are flat.

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1. Introduction

Space-filling branes, situated around extra dimensions, provide a remarkable framework for approaching phenomenological problems. Besides being well motivated—for instance, arising very naturally within string theory—branes lead to novel kinds of low-energy physics that can cut to the core of many of the naturalness issues that currently plague particle physics and cosmology.

The realization that not all particles need to ‘see’ the same number of dimensions (because brane-bound particles are trapped to move only along the branes) is the first type of brane-related insight to have had a major impact on physics, leading to the recognition that the scale of gravity can be much smaller than the Planck scale [1]. A second major revelation came with the realization that the back-reaction of branes on their environment can strongly influence their low-energy properties, such as by providing deep gravitational potential wells within the extra dimensions that redshift the energy of those branes that live within them [2].

Although branes can, in principle, have a great variety of dimensions, almost all of the detailed exploration of brane–bulk back-reaction is specialized to the case of codimension-1 branes, i.e. those branes that span just one dimension less than the dimension of the full spacetime. This is partially because tools for describing how branes back-react to their surroundings are only well developed for codimension-1 surfaces, since in this case the problem can be expressed in terms of the Israel junction conditions [3]. This restriction to codimension-1 objects is potentially very limiting because the special nature of kinematics in one dimension makes it unlikely that back-reaction for codimension-1 branes is representative of back-reaction for branes with higher codimension.

The main obstacle to understanding how properties of higher-codimension branes are related to the bulk geometries that they source is the fact that these bulk geometries typically diverge at the position of their sources. (The most familiar example of this for a codimension-3 object is the divergence of the Coulomb potential of a nucleus evaluated at the nuclear position.) It is one of the special features of codimension-1 objects that the bulk fields that they source
typically do not diverge at their positions. They instead cause discontinuities of derivatives across their surfaces, whose properties are captured by the Israel junction conditions. The next-simplest case consists of codimension-2 objects, whose back-reaction is complicated enough to allow the possibility of bulk fields diverging at the positions of the sources. Although bulk fields can diverge for codimension-2 sources, they need not do so in time-independent situations. (For instance, they can instead give rise to conical singularities, such as for cosmic strings in four-dimensional (4D) spacetime [4]. When bulk fields do not diverge, the relation between bulk and brane properties is more easy to formulate and so is better studied [5].) The potential for divergent bulk configurations makes codimension-2 branes more representative of systems with more generic codimension than are codimension-1 branes. But dynamics in two dimensions is still simple enough to allow explicit closed-form solutions to be known for the bulk configurations sourced by codimension-2 branes, allowing a detailed study of their properties.

Tools for describing how bulk fields respond to the properties of source branes were recently developed in the general case, including where the bulk fields diverge [6–8], opening up the properties of codimension-2 branes for phenomenological exploration. These tools—summarized (and slightly generalized) in section 2 below for a fairly general class of scalar–tensor–Maxwell theories in $n$ extra dimensions—boil down to a set of matching conditions that relate the near-brane limit of the radial derivatives of the bulk fields to the action for the brane in question.

In section 3, we apply these tools to three kinds of examples: compact geometries sourced by D7 branes in F-theory compactifications of 10D Type IIB supergravity; 3-branes coupled to a bulk axion within unwarped, non-supersymmetric 6D scalar/Maxwell/Einstein theory; and 3-branes coupled to 6D chiral gauged supergravity. We draw the following lessons from these comparisons:

- F-theory compactifications [10] of 10D Type IIB supergravity sourced by D7-branes serve as a reality check, since string theory tells us the detailed form of both the brane and the bulk actions [9], and explicit solutions are known for the transverse spacetimes that are sourced by these branes [21]. We verify the codimension-2 brane/bulk matching conditions by checking that the asymptotic forms for the solutions are related to the known brane actions in the prescribed way.
- In 6D axion-Maxwell–Einstein theory, flux-compactified solutions are known for the bulk that interpolates between two 3-branes, and these are simple enough to allow the explicit calculation of how branes contribute to the low-energy axion potential [11]. From the perspective of six dimensions, the resulting axion stabilization arises through the requirement that both branes be consistent in their demands on the bulk. We show that the stabilized value agrees precisely with the result of minimizing the low-energy axion potential as seen by an observer who has integrated out the extra dimensions below the Kaluza–Klein (KK) scale. We also show how this potential gives the same value for the curvature of the maximally symmetric on-brane geometry, as is calculated from the higher-dimensional field equations.
- Stable flux compactifications are also known for 6D chiral gauged supergravity [12], having up to two singularities that represent the positions of two source branes [13]. These solutions are known in explicit closed form for the most general solutions having a flat

\[\text{For a review with references, see [9].}\]
on-brane geometry and axial symmetry in the bulk; and in a slightly more implicit form for solutions with de Sitter or anti-de Sitter on-brane geometry. In this case, we use the matching conditions to show that the only bulk configurations that can be supported by positive-tension branes have flat induced on-brane geometries, with (possibly warped) bulk geometries with non-singular limits as the source branes are approached. We also show how geometries that diverge at the brane positions can arise from specific kinds of negative-tension branes, while no maximally symmetric solutions exist at all for many kinds of brane sources (presumably corresponding to time-dependent runaway bulk geometries, such as those considered in [14]).

Section 4 briefly summarizes some of the implications of these results.

2. The bulk–brane system

We start by describing the bulk–brane framework within which we work. This starts with a statement of the scalar-metric–Maxwell system whose equations we use, followed by a statement of how the near-brane boundary conditions of the bulk fields are related to the action of the branes that are their source. Finally, we describe the contribution of each brane to the low-energy scalar potential that is valid over distances much longer than the size of the extra dimensions, and identify a constraint that allows a simple description of this contribution given the properties of the brane tension.

2.1. The bulk

The starting point is the statement of the equations of motion that govern the bulk.

2.1.1. General formulation. We assume the following action for the $n$-dimensional bulk physics, describing a general scalar-tensor theory coupled to a Maxwell field,$^5$

$$ S = \int_{\mathcal{M}} d^n x \, \mathcal{L}_b + \int_{\partial \mathcal{M}} d^{n-1} x \, \mathcal{L}_{\text{GH}}, \quad (2.1) $$

where

$$ \mathcal{L}_b = -\sqrt{-g} \left\{ \frac{1}{2\kappa^2} g^{MN} \left[ R_{MN} + G_{AB}(\phi) \partial_M \phi^A \partial_N \phi^B \right] + \frac{1}{4} f(\phi) F_{MN} F^{MN} + V(\phi) \right\}, \quad (2.2) $$

and the Gibbons–Hawking Lagrangian [17] is

$$ \mathcal{L}_{\text{GH}} = \frac{1}{\kappa^2} \sqrt{-\hat{\gamma}} \, K, \quad (2.3) $$

and is required in the presence of boundaries in order to make the Einstein action well posed. Here $F = dA$ is the field strength of the Maxwell field, $R$ is the Ricci scalar for the 6D spacetime metric, $g_{MN}$, and $G_{AB}$ is the metric of the target space within which the scalar fields, $\phi^A$, $A = 1, \ldots, N$, take values. $\hat{\gamma}_{ij} = g_{MN} \partial_i x^M \partial_j x^N$ is the induced metric, and $K$ is the trace, $\hat{\gamma}^{ij} K_{ij}$, of the extrinsic curvature, of the boundary surface, $\partial \mathcal{M}$.

$^5$ Our metric is mostly plus, with Weinberg’s curvature conventions [15], which differ from those of MTW [16] only by an overall sign in the definition of the Riemann tensor.
This bulk action is chosen to be general enough to include the bosonic part of the supersymmetric theories of interest. Its field equations are

\[
\frac{1}{2\kappa^2} \left( R_{MN} + G_{AB} \partial_M \phi^A \partial_N \phi^B \right) + \frac{f}{2} F_M^\rho F_N^\rho + \frac{1}{n-2} \left[ V - \frac{f}{4} F^\rho_\rho F^\rho_\rho \right] g_{MN} = 0, \quad (2.4)
\]

\[
G_{AB} \Box \phi^B - \kappa^2 \left[ \frac{\partial V}{\partial \phi^A} + \frac{1}{4} \frac{\partial f}{\partial \phi^A} F_{MN} F^{MN} \right] = 0, \quad (2.5)
\]

and

\[
\nabla_M \left( f F^M^\nu \right) = 0, \quad (2.6)
\]

where

\[
\Box \phi^A := g^{MN} \left[ \nabla_M \partial_N \phi^A + \Gamma^A_{BC}(\phi) \partial_M \phi^B \partial_N \phi^C \right], \quad (2.7)
\]

with \( \Gamma^A_{BC}(\phi) \) being the Christoffel connection built from the metric \( G_{AB} \).

### 2.1.2. Metric ansätze

Our interest is in configurations whose geometries are maximally symmetric in the brane directions, for which it is convenient to specialize to the metric

\[
ds^2 = g_{MN} dx^M dx^N = e^{2W} \hat{g}_{\mu\nu} dx^\mu dx^\nu + g_{mn} dx^m dx^n = e^{2W} \hat{g}_{\mu\nu} dx^\mu dx^\nu + e^{2C} dz d\bar{z},
\]

where \( \hat{g}_{\mu\nu}(x) \) denotes a maximally symmetric \((n-2)\)-dimensional metric. The coordinates are \( x^\mu = \{x^m, x^\tau\} \), with \( x^\mu, \mu = 0, \ldots, n-3 \), labelling the brane directions, and \( m = n-2, n-1 \) (or \( z = x^{n-2} + ix^{n-1} \)) being coordinates for the two dimensions transverse to the branes.

For some applications, particularly very near a brane, it is useful to further specialize to the most general ansatz consistent with cylindrical symmetry in the two transverse dimensions, \( \{x^m, m = n-2, n-1\} \). This leads to the following metric:

\[
ds^2 = d\rho^2 + e^{2B} d\theta^2 + e^{2W} \hat{g}_{\mu\nu} dx^\mu dx^\nu = e^{2C} \left( dr^2 + r^2 d\theta^2 \right) + e^{2W} \hat{g}_{\mu\nu} dx^\mu dx^\nu, \quad (2.9)
\]

where \( \theta \) labels the direction of cylindrical symmetry, and the functions \( B = B(\rho) \) and \( W = W(\rho) \) depend on the proper distance, \( \rho \), only—or \( C = C(r) \) is a function only of \( r \). The functions \( W \) and \( C \) are generally singular at the positions of any source branes. For instance, if \( e^C = (\ell/r)^a \) for \( r^2 = |z|^2 \), then the proper distance becomes \( \rho = [\ell/(1-a)](\ell/r)^{a-1} \) and \( e^B = \ell(\ell/r)^{a-1} = (1-a)\rho \), showing that the metric in this case has a conical singularity at \( r = \rho = 0 \), with defect angle \( \delta = 2\pi a \).

The bulk scalars are similarly just functions of \( \rho \), \( \phi^A = \phi^A(\rho) \), and a gauge can be chosen so that the only nonzero component for the Maxwell field is \( A_\mu = A_\theta(\rho) \delta^\mu_\theta \), and so

\[
F_{\rho\theta} = -F_{\theta\rho} = A_\rho', \quad (2.10)
\]

where the prime denotes differentiation with respect to \( \rho \).

The Einstein equations subject to this ansatz reduce to

\[
\frac{1}{n-2} e^{-2W} \hat{R} + W'' + (n-2)(W')^2 + W'B' - \frac{1}{n-2} \kappa^2 e^{-2B} f(A_\rho')^2 + \frac{2\kappa^2 V}{n-2} = 0(\mu\nu), \quad (2.11)
\]
\[
B'' + (B')^2 + (n - 2) W' B' + \frac{n - 3}{n - 2} \kappa^2 e^{-2B} f(A'_\theta)^2 + \frac{2\kappa^2 V}{n - 2} = 0 \quad (\theta \theta),
\]
(2.12)

\[
(n - 2) \left[ W'' + (W')^2 \right] + B'' + (B')^2 + G_{AB} \phi'' \phi'' + \frac{n - 3}{n - 2} \kappa^2 e^{-2B} f(A'_\theta)^2 + \frac{2\kappa^2 V}{n - 2} = 0 \quad (\rho \rho),
\]
(2.13)

while the dilaton and Maxwell equations become
\[
e^{-B - 4W} \left( e^{2W} G_{AB} \phi'' \right)'' + G_{AB} \Gamma^c_{\theta \phi} \phi'' \phi'' - \kappa^2 \left[ \frac{\partial V}{\partial \phi^A} + \frac{1}{4} \frac{\partial f}{\partial \phi^A} e^{-2B} f(A'_\theta)^2 \right] = 0,
\]
(2.14)
and
\[
\left( e^{-B + 4W} f A'_\theta \right)' = 0.
\]
(2.15)

### 2.2. Boundary conditions for codimension-2 branes

#### 2.2.1. General formulation.

Suppose an \((n - 2)\)-dimensional, space-filling, codimension-2 brane is located at a position, \(x^m = x^m_0\), within the two extra dimensions, with brane action
\[
S_b = - \int_{x_b} d^{n-2}x \sqrt{-g} \left[ L_b (\phi^A, A_\theta, g_{\theta \theta}) + \cdots \right],
\]
(2.16)

where \(L_b\) denotes the brane Lagrangian, which is potentially a function of the bulk scalars, \(\phi^A\), and the tangential components of the bulk Maxwell field and metric, \(A_\mu\) and \(g_{\mu \nu}\), but not their derivatives. (Ellipsis denotes the possible subdominant, higher-derivative effective interactions that can also be present.) We imagine the geometry surrounding the brane to be given by the axisymmetric \(\textit{ansatz}\) of equation (2.9), with the brane located at \(\rho = 0\), so \(\theta\) denotes the angular direction about its position. Because our interest is in maximally symmetric solutions along the brane directions, we do not entertain a dependence of \(T_b\) on any components of \(A_\mu\) and \(g_{\mu \nu}\) apart from \(A_\theta\) and \(g_{\theta \theta}\).

The induced metric on the brane is \(\gamma_{\mu \nu} = g_{MN} \partial_\mu x^M \partial_\nu x^N = e^{2W} g_{\mu \nu}\). Because of the warp factor appearing in this metric, for later purposes it is convenient to define the ‘warped’ tension, \(T_b\), by \(T_b = e^{-2W} L_b\), so that the brane action becomes
\[
S_b = - \int_{x_b} d^{n-2}x \sqrt{-g} \left[ T_b (\phi, A_\theta, g_{\theta \theta}, W) + \cdots \right].
\]
(2.17)

The back-reaction of such a brane onto the bulk geometry dictates the asymptotic near-brane behaviour of the bulk fields nearby, through codimension-2 matching conditions that generalize [6–8] the more familiar ones that are encountered for codimension-1 branes. For the bulk scalars, these state
\[
\lim_{\rho \rightarrow 0} \oint_{x_b} d\theta \left[ \frac{1}{\kappa^2} \sqrt{-g} G_{AB} \partial_\rho \phi^A \right] = - \frac{\delta S_b}{\delta \phi^A},
\]
(2.18)

where the integration is about a small circle of proper radius \(\rho\) encircling the brane position, \(x_b\), which is taken to be situated at \(\rho = 0\). Similarly, the Maxwell matching condition is
\[
\lim_{\rho \rightarrow 0} \oint_{x_b} d\theta \left[ \sqrt{-g} f F^{\rho M} \right] = - \frac{\delta S_b}{\delta A_M}.
\]
(2.19)

\(^6\) A familiar example of this from electrostatics is the \(1/\rho\) dependence of the Coulomb potential that occurs in the immediate vicinity of a point charge situated at \(\rho = 0\).
Finally, the metric matching condition is

$$\lim_{\rho \to 0} \int_{x_\rho} \, d\theta \left[ \frac{1}{2\kappa^2} \sqrt{-g} \left( K_{ij} - K g^{ij} \right) - (\text{flat}) \right] = -\frac{\delta S_\rho}{\delta g_{ij}},$$

(2.20)

where $K_{ij}$ is the extrinsic curvature of the fixed-$\rho$ surface, for which the local coordinates are those appropriate for surfaces of constant $\rho$: $\{x^i, i = 0, 1, \ldots, n - 2\}$. Here ‘flat’ denotes the same result evaluated near the origin of a space for which the brane location $\rho = 0$ is non-singular.

### 2.2.2. Axially symmetric ansatz

Specialized to the ansatz of equation (2.9), the scalar-field matching condition becomes

$$\left[ \frac{2\pi}{\kappa^2} e^{B+(n-2)W} \left( -\sqrt{-g} \mathcal{G}_{\alpha\beta} \phi^\nu \right) \right]_{x_\rho} = \frac{\partial}{\partial \phi^\nu} \left[ \frac{\partial}{\partial \phi^\nu} \sqrt{-g} T_\nu \right].$$

(2.21)

With the same ansatz, the corresponding result for the Maxwell field reduces to

$$\left[ 2\pi \sqrt{-g} e^{-B+(n-2)W} f A_\rho \right]_{x_\rho} = \frac{\partial}{\partial A_\rho} \left[ \sqrt{-g} T_\rho \right] := \sqrt{-g} J_\rho(\phi),$$

(2.22)

where the last equality defines the quantity $J_\rho$.

Finally, for fixed-$\rho$ surfaces in this ansatz, $K_{ij} = \frac{1}{2} \delta_{ij} g_{ij}$, and the comparison ‘flat’ metric is $d\tau^2 = d\rho^2 + \rho^2 d\theta^2 + e^{2W} \delta_{\mu\nu} \, dx^\mu dx^\nu$, with $W_{\text{flat}} \to 0$ as $\rho \to 0$. Since $K_{\theta\theta} = B' e^{2B}$ and $K_{\mu\nu} = W' e^{2W} \delta_{\mu\nu}$, we have $K = g^{ij} K_{ij} = B' + (n-2)W'$, and so the $(\mu\nu)$ components of the metric matching conditions give

$$\left[ -\frac{2\pi}{\kappa^2} \sqrt{-g} e^{(n-2)W} \left( e^B \left( (n-3)W' + B' \right) - 1 \right) \right]_{x_\rho} = \sqrt{-g} T_\rho(\phi),$$

(2.23)

while the $(\theta\theta)$ components are

$$\left[ \frac{2\pi}{\kappa^2} \sqrt{-g} e^{B+(n-2)W} \left( (n-2)W' \right) \right]_{x_\rho} = -2 \frac{\partial}{\partial g_{\theta\theta}} \left[ \frac{\partial}{\partial g_{\theta\theta}} \sqrt{-g} T_{\rho} \right]$$

$$:= (n-2) \sqrt{-g} U_{\rho}(\phi),$$

(2.24)

where the last equality defines $U_{\rho}$. Just as $T_\rho$ physically represents the brane tension, $J_\rho$ can be interpreted as describing microscopic axial currents within the brane, or equivalently any microscopic magnetic flux these currents enclose within the brane. Once the dimensions transverse to the brane are dimensionally reduced, $U_{\rho}$ turns out [6, 7] to be related to the brane contribution to the scalar potential within the low-energy 4D effective theory defined below the KK scale (as is seen in more detail later).

### 2.3. The brane constraint

These matching conditions, when combined with the bulk equations of motion, imply an important constraint relating the quantities $T_\rho$, $J_\rho$ and $U_\rho$ [6, 7, 18]. This constraint comes from eliminating second derivatives, $\delta^2_{\rho}$, of the fields from the field equations and so can be regarded as the ‘Hamiltonian’ constraint on the initial data when integrating the field equations in the $\rho$
direction. When written in the form given above, the relevant combination of Einstein equations is \((n - 2)(\mu \nu) + (\partial \theta) - (\rho \rho)\), which implies
\[
(n - 3)(n - 2) \left( W' \right)^2 + 2(n - 2) W' B' - G_{ab} \phi^a \phi^b - \kappa^2 e^{-2B} f (A'_b)^2 + e^{-2W} \hat{R} + 2\kappa^2 V = 0.
\]
(2.25)

To turn this into a constraint on brane properties, multiply it through by \(e^{2B+2(n-2)W}\) and take the limit \(x \to x_b\), using the above matching conditions to eliminate the derivatives \(\phi^a'\), \(B'\), \(W'\) and \(A'_b\) in favour of the brane functions \(T_b\), \(J_b\) and \(U_b\). The required matching conditions are

\[
\left[ e^B \phi^a \right]_{x_b} = e^{-(n-2)W} G_{ab} \frac{\partial T_b}{\partial \phi^b} \quad \text{with} \quad T_b := \frac{\kappa^2 T_b}{2\pi},
\]
\[
\left[ \kappa A'_b \right]_{x_b} = e^{-(n-2)W} T_b \quad \text{with} \quad J_b := \frac{\kappa e^B J_b}{2\pi},
\]
\[
\left[ e^B W' \right]_{x_b} = e^{-(n-2)W} U_b \quad \text{with} \quad U_b := \frac{\kappa^2 U_b}{2\pi},
\]
and

\[
\left[ e^B B' - 1 \right]_{x_b} = -e^{-(n-2)W} \left[ T_b + (n - 3) U_b \right].
\]

(2.26)

where each one of \(U_b\), \(T_b\) and \(J_b\) is dimensionless (keeping in mind \(e^B\) has dimensions of length). Using equations (2.26) in equation (2.25), we find the desired constraint:

\[
(n - 3)(n - 2) \left( U_b \right)^2 + 2(n - 2) U_b \left[ e^{(n-2)W} - T_b - (n - 3) U_b \right] - G_{ab} \frac{\partial T_b}{\partial \phi^a} \frac{\partial T_b}{\partial \phi^b} - \frac{(J_b)^2}{f} + e^{2B+2(n-2)W} \left[ e^{-2W} \hat{R} + 2\kappa^2 V \right]_{x_b} = 0.
\]
(2.27)

This crucially simplifies once we use the fact that, near the brane, \(e^B \to 0\) as \(\rho \to 0\). (This states that the circumstances of small circles about the brane must vanish as the radii of the circles vanish. If not true, the object at \(\rho = 0\) would not be interpreted as a codimension-2 brane.) The key observation [6, 7] is that the quantities \(\kappa e^{2B} J_b\), \(e^{2B+2W} \hat{R}\) and \(\kappa^2 e^{2B} V\) also tend to vanish in this limit (as would be true, for instance, if \(e^{-2W} \hat{R}\), \(V\) and \(J_b\) were bounded at the brane positions), implying that the constraint becomes

\[
(n - 2) U_b \left[ 2 e^{(n-2)W} - 2 T_b - (n - 3) U_b \right] - (T_b)^2 \simeq 0,
\]
(2.28)

where \((T_b)^2 = G_{ab} \partial_a T_b \partial_b T_b\).

What is important about this last form of the constraint is that the on-brane curvature drops out in this limit, meaning that equation (2.28) cannot be read as being solved for \(\hat{R}\). Instead, this constraint expresses a consistency condition for the brane action and junction conditions, imposed by the bulk equations of motion. In practice, it provides a very simple method for computing the quantity \(U_b(\phi)\) once expressions for \(T_b(\phi)\) are given, since solving equation (2.28) implies

\[
U_b = \frac{1}{n - 3} \left[ (e^{(n-2)W} - T_b) \pm \sqrt{(e^{(n-2)W} - T_b)^2 - \left( \frac{n - 3}{n - 2} \right) (T_b)^2} \right].
\]
(2.29)
Here the root is chosen for which \( U_b \to 0 \) when \((T_b^0)^2 \to 0\), and so is \( \pm \) according to whether the sign of \((e^{(a-2)W} - T_b)\) is \( \mp \). This means that \( U_b \) has the same sign as does \((e^{(a-2)W} - T_b)\).

Note also that requiring the square root is never complex requires

\[
\frac{n-3}{n-2} (T_b^0)^2 \leq (e^{(a-2)W} - T_b)^2. \tag{2.30}
\]

This last condition can be nontrivial, even though control over the semiclassical approximation requires \(|T_b| \ll 1 \) and \((T_b^0)^2 \ll 1\). This is because it can happen that \( e^W \to 0 \) at the brane, in which case equation (2.30) becomes a constraint on the size of \((T_b^0)^2/T_b^2\).

For \((T_b^0)^2 \ll (e^{(a-2)W} - T_b)^2\), equation (2.29) becomes

\[
U_b \simeq \frac{(T_b^0)^2}{2(n-2)(e^{(a-2)W} - T_b)^2} + \frac{(n-3)(T_b^0)^4}{8(n-2)^2(e^{(a-2)W} - T_b)^3} + \cdots. \tag{2.31}
\]

2.4. The classical low-energy on-brane effective action

Over distances much longer than the size of the two compact dimensions transverse to the brane, the classical bulk dynamics are governed by the motion of the massless KK states. The dynamics are effectively \( d \)-dimensional, with \( d = n-2 \). To understand the dynamics from this \( d \)-dimensional perspective, it is useful to integrate out the extra dimensions to obtain the low-energy lower-dimensional effective theory. At the classical level, this amounts to eliminating all of the massive KK states as functions of their massless counterparts, using the bulk classical equations of motion.

In the present instance, the massless KK states consist of the on-brane metric and Maxwell fields, \( \hat{g}_{\mu\nu} \) and \( A_\mu \), as well as any \( d \)-dimensional scalars, \( \varphi^a \), descending from \( \phi^A \), and/or from moduli in the metric components, \( g_{mn} \), in the extra dimensions. To obtain the low-energy potential, \( V_{\text{eff}}(\varphi) \), for the various \( d \)-dimensional scalars, \( \varphi^a \), we eliminate the massive KK modes in the action, as functions of \( \hat{g}_{\mu\nu} \) and \( \varphi^a \). The transverse metric, \( g_{mn} \), is eliminated by using the trace reversed \((mn)\) Einstein equations, which single out the kinetic terms for \( g_{mn} \):

\[
\frac{1}{2\kappa^2} \left( \mathcal{R}_{mn} + \mathcal{G}_{\alpha\beta} \partial_\alpha \phi^A \partial_\beta \phi^A \right) + \frac{1}{2} F_{m}^{\mu} F_{n\mu} + \frac{1}{n-2} \left[ V - \frac{f}{4} F_{\rho\varphi} F^{\rho\varphi} \right] g_{mn} = 0. \tag{2.32}
\]

These comprise two independent equations, which we take to be the sum and difference of the \((\rho\rho)\) and \((\theta\theta)\) components. The difference gives

\[
(n-2) \left( W'' + (W')^2 - W' B' \right) + \mathcal{G}_{AB} \phi^A \phi^B = 0, \tag{2.33}
\]

while the sum is equivalent to contracting equation (2.32) with \( g^{mn} \), to give

\[
\frac{1}{2\kappa^2} \left( \mathcal{R}_{(2)} + \mathcal{G}_{AB} \partial_\alpha \phi^A \partial_\beta \phi^B \right) = - \frac{n-3}{2(n-2)} f F_{mn} F^{mn} - \frac{2}{n-2} V, \tag{2.34}
\]

where we write the higher-dimensional curvature scalar as

\[
\mathcal{R} = g^{MN} \mathcal{R}^p_{MPN} = \mathcal{R}_{(n-2)} + \mathcal{R}_{(2)}
\]

where \( \mathcal{R}_{(2)} = g^{mn} \mathcal{R}^p_{mn} = R_{(2)} + (n-2)(\Box W + \nabla W \cdot \nabla W) \)

\[
= R_{(2)} + (n-2) \left[ W'' + (W')^2 + B' W' \right], \tag{2.35}
\]

and \( \mathcal{R}_{(n-2)} = g^{\mu\nu} \mathcal{R}_{\mu\nu} = e^{-2W} \hat{g}^{\mu\nu} \hat{R}_{\mu\nu} + (n-2)[\Box W + (n-4) \nabla W \cdot \nabla W] \)

\[
eq e^{-2W} \hat{g}^{\mu\nu} \hat{R}_{\mu\nu} + (n-2) \left[ W'' + (n-4)(W')^2 + B' W' \right].
\]
Here \( R_{(2)} = g^{mn} R_{mpn} \) and \( \hat{g}^{\mu\nu} \hat{R}_{\mu\nu} \), respectively, denote the curvature scalars built from the 2D metric, \( g_{mn} \), and the 4D metric, \( \hat{g}_{\mu\nu} \).

Using equation (2.34) to eliminate \( R_{(2)} \) from the bulk action then yields the bulk contribution to the lower-dimensional Lagrangian density. Using \( \sqrt{-g} = \sqrt{-\hat{g}} \sqrt{g_2} e^{(n-2)W} \), we find

\[
\mathcal{L}_{\text{eff}}(\varphi) = - \int d^2 x \sqrt{g_2} e^{(n-2)W} \left[ \frac{1}{2\kappa^2} R_{(n-2)} + \frac{4-n}{4(n-2)} f F_{mn} F^{mn} + \frac{n-4}{n-2} V \right]
\]

\[
= - \int d^2 x \sqrt{g_2} e^{(n-2)W} \left\{ \frac{1}{2\kappa^2} \left[ e^{-2W} \hat{g}^{\mu\nu} \hat{R}_{\mu\nu} + (n-2) \left( W'' + (n-4)(W')^2 + B'W' \right) \right]
\right. \\
+ \frac{4-n}{4(n-2)} f F_{mn} F^{mn} + \frac{n-4}{n-2} V \right\}
\]

\[
= - \int d^2 x \sqrt{g_2} e^{(n-2)W} \left\{ \frac{1}{2\kappa^2} \left[ e^{-2W} \hat{g}^{\mu\nu} \hat{R}_{\mu\nu} + (n-2) (n-5)(W')^2 + 2W'B' \right]
\right. \\
- G_{\hat{A}B} \phi^{\hat{A}} \phi^{B'} \left. \right\} + \frac{4-n}{4(n-2)} f F_{mn} F^{mn} + \frac{n-4}{n-2} V \right\}
\]

\[
= - \int d^{n-2} x \sqrt{-\hat{g}} \left[ \frac{1}{2\kappa^2} \hat{g}^{\mu\nu} \hat{R}_{\mu\nu} + V_B \right].
\]

(2.36)

where the second to last equality uses the second independent bulk field equation, equation (2.33), the last equality defines the bulk potential, \( V_B \), and the lower-dimensional Newton’s constant, \( \kappa_5^2 = 8\pi G_N \), is given by

\[
\frac{1}{\kappa_5^2(\varphi)} := \frac{1}{\kappa^2} \int d^2 x \sqrt{g_2} e^{(n-4)W}.
\]

(2.37)

In general, this depends on the low-energy scalar fields, a dependence that can be removed by performing a Weyl rescaling to reach the lower-dimension Einstein frame.

To obtain the complete low-energy scalar potential, \( V_{\text{eff}} \), the bulk contribution, \( V_B \), must be combined with two other contributions, both associated with the source branes. The first of these comes from the boundary terms of the bulk action [6, 7], such as the Gibbons–Hawking term for the metric, evaluated at a small surface, \( \Sigma_b \), situated a short proper distance, \( \rho = \epsilon \), from the position of each of the source branes:

\[
S_{\text{GH}} = \sum_{b=0}^{\infty} \lim_{\epsilon \to 0} \oint_{\Sigma_b} d\theta \ d^{n-2} x \frac{1}{\kappa^2} \sqrt{-\hat{g}} K
\]

\[
= \frac{2\pi}{\kappa^2} \sum_{b=0}^{\infty} (-1)^b \int_{\rho=\rho_b} d^{n-2} x \sqrt{-\hat{g}} e^{B+(n-2)W} \left[ B' + (n-2)W' \right]
\]

(2.36)

Although, in principle, the extra dimensional part of the trace reversed (\( \mu \nu \)) Einstein equation, \( E_{\mu\nu}(x, y) = 0 \) could also be used to eliminate massive KK modes, this cannot be used to eliminate \( R_{(n-2)} \) from \( V_B \) because the integration in equation (2.36) projects onto the zero-mode component of \( E_{\mu\nu} = 0 \).
appropriately warped. The complete low-energy scalar potential is therefore

\[
- \sum_{b=0}^{1} \int_{\rho=\rho_b} d^{n-2}x \sqrt{-g} \left( \left[ -T_b - (n-3)U_b \right] + (n-2)U_b \right)
\]

\[
- \sum_{b=0}^{1} \int_{\rho=\rho_b} d^{n-2}x \sqrt{-g} \left( U_b - T_b \right).
\]

(2.38)

Here we use the axisymmetric ansatz, as is appropriate very near the source branes. The relative sign, \((-)^b\), and the overall sign in the second line arise because primes denote \(d/d\rho\), while the derivatives appearing in the Gibbons–Hawking action and matching conditions are outward directed, and this is in the \(d\rho\) direction for one brane and \(-d\rho\) for the other. The second to last line uses the matching conditions described earlier to exchange \(W'\) and \(B'\) for terms involving the brane action, using the fact that the contribution of \(e^{\beta K}\) flat cancels between the two branes.

The second contribution to the 4D scalar potential comes from the contribution of the brane action itself (equation (2.16)). Combining these with \(V_{AB}\) above gives the full 4D scalar potential in the classical limit, as in [7]

\[
- \int d^{n-2}x \sqrt{-g} V_{\text{eff}} = - \int d^{n-2}x \sqrt{-g} V_B + \sum_{b=0}^{1} \left[ S_b + \lim_{\epsilon \to 0} S_{\text{GH}} \right]
\]

\[
= - \int d^{n-2}x \sqrt{-g} V_B - \int_{\rho=\rho_b} d^{n-2}x \sqrt{-g} \left[ T_b + \left( U_b - T_b \right) \right],
\]

(2.39)

where the notation \(W_b\) is a reminder that \(W\) is evaluated at the brane. This shows that (within the classical approximation) the effect of the Gibbons–Hawking terms is to ensure that the net contribution of each brane to the low-energy scalar potential is given by the quantity \(U_b\), appropriately warped. The complete low-energy scalar potential is therefore

\[
V_{\text{eff}} = V_S + \sum_{b} U_b
\]

\[
= \sum_{b} U_b + \int d^2x \sqrt{g} \ e^{(n-2)W} \left\{ \frac{1}{2\kappa^2} \left[ (n-2) \left\{ (n-5)(W')^2 + 2W'B' - \mathcal{G}_{AB} \partial^\mu \phi^A \partial^\nu \phi^B \right\} \right] + \frac{4-n}{4(n-2)} f \ F_{mn} F^{mn} + \frac{n-4}{n-2} \ \frac{V}{F} \right\}.
\]

2.4.1. Stationary points. For some purposes it is sufficient to obtain the value of the potential, \(V_{\text{eff}}(\phi_0)\), evaluated at its stationary point, where \(V_{\text{eff}}(\phi_0) = 0\). This can be obtained from the higher-dimensional action by eliminating fields using all of the equations of motion, and not just those of the massive KK modes. In this case, we may directly use the equation of motion,

\[
\frac{1}{2\kappa^2} (R + \mathcal{G}_{AB} \partial^\mu \phi^A \partial^\nu \phi^B) = - \frac{(n-4)}{4(n-2)} f \ F_{MN} F^{MN} - \frac{nV}{n-2},
\]

(2.40)

rather than equation (2.34) for \(R_{(2)}\). Using this to eliminate \(R\) from the bulk action yields

\[
S_{\text{ext}} = - \int d^nx \sqrt{-g} \left[ \frac{1}{2\kappa^2} (R + \mathcal{G}_{AB} \partial^\mu \phi^A \partial^\nu \phi^B) + \frac{1}{4} f \ F_{MN} F^{MN} + V \right],
\]

\[
= - \frac{2}{n-2} \int d^nx \sqrt{-g} \left[ \frac{1}{4} f \ F^{mn} F_{mn} - V \right].
\]

(2.41)
When comparing with the low-energy theory, we must also evaluate the low-energy action at its stationary point. That is, we evaluate the action

$$S_{\text{eff}} = - \int d^{n-2}x \sqrt{-g} \left[ \frac{1}{2\kappa_n^2} \hat{R}_{(n-2)} + V_{\text{eff}} \right]$$

as the solution to the low-energy field equations,

$$\frac{1}{2\kappa_n^2} \hat{R}_{(n-2)} = - \frac{(n-2)}{n-4} V_{\text{eff}},$$

leading to

$$S_{\text{ext}} = \frac{2}{n-4} \int d^{n-2}x \sqrt{-g} V_{\text{eff}}(\psi_0).$$

Using the previous results for $V_{\text{ext}}$ and the brane contribution then gives

$$\frac{2}{n-4} V_{\text{eff}}(\psi_0) = - \sum_b e^{(n-2)\omega_b} U_b - \frac{2}{n-2} \int d^2x \sqrt{g_2} e^{(n-2)W} \left[ \frac{1}{4} f F_{mn} F^{mn} - V \right].$$

In many cases of interest, the bulk contribution to this expression can itself also be written as a sum of contributions localized at the position of each brane. This is true, in particular, whenever the bulk action, $S_B = \int d^n x L_B$, enjoys a classical scaling symmetry, under which $L_B[\lambda^n \psi_i] \equiv \lambda L_B[\psi_i]$, for arbitrary real, constant $\lambda$. (This type of scale symmetry generically holds for higher-dimensional supergravity theories in particular.) When this is true, the Lagrange density satisfies the identity

$$L_B \equiv \sum_i p_i \left[ \psi_i \frac{\partial L_B}{\partial \psi_i} + \partial_\mu \psi_i \frac{\partial L_B}{\partial (\partial_\mu \psi_i)} \right] = \sum_i \left\{ \partial_\mu \left[ p_i \frac{\partial L_B}{\partial (\partial_\mu \psi_i)} \right] + p_i \psi_i \left[ \frac{\partial L_B}{\partial \psi_i} - \partial_\mu \left( \frac{\partial L_B}{\partial (\partial_\mu \psi_i)} \right) \right] \right\},$$

which shows [20] that the action becomes a total derivative whenever it is evaluated at an arbitrary classical solution. Whenever this is true, the entire low-energy potential can be interpreted as the sum over brane contributions, as was done for the Gibbons–Hawking term above.

3. Examples

It is instructive to test the above construction by applying it to situations for which explicit solutions are known for the higher-dimensional theory. We do so in this section using F-theory compactifications of 10D Type IIB supergravity to eight dimensions in the presence of space-filling D7 branes, and using compactifications to four dimensions of supersymmetric and non-supersymmetric 6D theories.

3.1. D7 branes in F-theory

We start with F-theory [10] compactifications of Type IIB supergravity to eight dimensions, which serves as an example where explicit forms for the bulk and brane actions are known, as are closed-form expressions for the bulk sourced by various space-filling brane configurations [21].

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This provides a check on the validity of the matching conditions and on the low-energy on-brane scalar potential.

The bulk fields to be followed in this case are the metric, $g_{MN}$, and the axio-dilaton,
\[ \tau = C_0 + i e^{-\phi}, \tag{3.1} \]
where $C_0$ is the Ramond–Ramond scalar and $\phi$ is the 10D dilaton, for which the string coupling is $g_s = e^\phi$. The bulk action for these fields in the 10D Einstein frame is
\[ S_B = -\frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} g^{MN} \left[ R_{MN} + \frac{\partial_M \bar{\tau} \partial_N \tau}{2(\text{Im } \tau)^2} \right], \tag{3.2} \]
which is invariant under PSL(2,$\mathbb{R}$) transformations
\[ \tau \rightarrow a\tau + b \]
where $a$ through $d$ satisfy $ad - bc = 1$. Quantum effects are expected to break this to PSL(2,$\mathbb{Z}$), for which the parameters are restricted to be integers. Since $e^\phi \geq 0$, the field $\tau$ lives in the upper-half $\tau$ plane, but because of the symmetry it suffices to consider $\tau$ to live within the fundamental domain, $\mathcal{F}$, defined by modding out the upper half-plane by a PSL(2,$\mathbb{Z}$).

3.1.1. Bulk solutions. The scalar field equation for this action is
\[ \bar{\partial} \tau + 2 \frac{\partial \tau}{\bar{\tau} - \tau} = 0, \tag{3.4} \]
which is satisfied by any holomorphic function, $\tau = \tau(z)$, for which $\bar{\partial} \tau = 0$.

Explicit solutions to the field equations of this model are known [21], for which two of the dimensions are compactified. Using complex coordinates, $z = x^8 + ix^9$, for the compact dimensions, the solutions are given by
\[ j(\tau(z)) = P(z) \quad \text{and} \quad ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + e^{2C(z, \bar{z})} d\bar{z} dz, \tag{3.5} \]
where the properties of the functions $j(\tau)$, $P(z)$ and $C(z, \bar{z})$ are now described.

The function $j(\tau)$ is the standard bijection from the fundamental domain, $\mathcal{F}$, to the complex sphere, given in terms of Jacobi $\vartheta$-functions by
\[ j(\tau) = \frac{1728 \left[ E_4(\tau) \right]^3}{\left[ E_4(\tau) \right]^3 - \left[ E_6(\tau) \right]^2}, \tag{3.6} \]
where $E_k(\tau)$ are the Eisenstein modular forms [22]. For large $\text{Im } \tau$, $j(\tau)$ diverges exponentially quickly, and the factor of 1728 is chosen so that it asymptotes to $j(\tau) \simeq e^{-2\pi i \tau} + \ldots$.

$P(z)$ is a holomorphic function, whose singularities occur at the locations of the source branes, $z = z_i$ for $i = 1, \ldots, N$. Since the singularities of the metric turn out to be conical when $P(z)$ has isolated poles as $z \rightarrow z_i$, it is convenient to choose $P(z)$ to be a ratio of polynomials. The simplest case could be taken as $P = 1/z$, describing a source at $z = 0$, but it turns out that the metric obtained from the Einstein equations is not compact in this case. The metric is compact when $P(z)$ has 24 poles, such as for the choice
\[ P(z) = \frac{4(24f)^3}{27g^2 + 4f^3}, \tag{3.7} \]

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with \( f(z) \) being a polynomial of degree 8 and \( g(z) \) a polynomial of degree 12. This gives a compactification of Type IIB supergravity on \( CP^1 \), corresponding to an F-theory reduction on K3 [10].

Finally, the metric function \( C(z, \bar{z}) \) is chosen by solving the Einstein equation. Using \( \mathcal{R}_{\bar{z}z} = 2 \frac{\partial \bar{\partial}}{\partial \bar{\partial}} C \) and \( \bar{\partial} \tau = 0 \), this equation of motion is

\[
2 \frac{\partial \bar{\partial}}{\partial \bar{\partial}} C = \frac{\partial \tau \bar{\partial} \tau}{(\tau - \bar{\tau})^2} = \partial \bar{\partial} \ln \left( \text{Im} \tau \right). \tag{3.8}
\]

The required solution is

\[
e^{2C(z, \bar{z})} = (\text{Im} \tau) \left| \eta^2(\tau) \prod_{i=1}^{N} (z - z_i)^{-1/12} \right|^2, \tag{3.9}
\]

where \( \eta(\tau) = q^{1/24} \prod_k (1 - q^k) \), for \( q = e^{2\pi i \tau} \), denotes the Dedekind \( \eta \)-function, and the product runs over the singularities of \( P(z) \). The first factor of this expression is chosen to satisfy equation (3.8), and the holomorphic factors are chosen to ensure invariance under PSL(2,\( \mathbb{Z} \)), and by the requirement that the result does not vanish anywhere.

3.1.2. Brane sources. The presence of branes in these solutions is signalled by singularities where \( P(z) \approx c_i / (z - z_i) \), for which \( q = e^{2\pi i \tau} \approx (z - z_i) / c_i \), and so the above solution implies

\[
\tau(z) \approx \frac{1}{2\pi i} \ln(z - z_i) + \cdots \quad \text{and} \quad e^{2C(z, \bar{z})} \approx k \text{Im} \tau, \tag{3.10}
\]

for constant \( k \). As \( z \to \infty \), on the other hand, \( P(z) \) remains bounded and so \( \tau \) approaches some finite value. In this case, the metric function becomes

\[
e^{2C(z, \bar{z})} \propto (z \bar{z})^{-N/12}, \tag{3.11}
\]

and so, if we change coordinates to \( z = 1/w \), we have \( e^{2C} \, dz \, d\bar{z} \approx |w|^{(N-24)/6} dw \, d\bar{w} \), which is non-singular because \( N = 24 \). But each individual brane contributed to this an amount \( e^{2C} \approx |w|^{1/6} dw \, d\bar{w} \propto r^{1/6} (dr^2 + r^2 d\theta^2) \), which we saw below equation (2.9) corresponds to a deficit angle of \( \delta = \pi/6 \).

3.1.3. Matching conditions. We are now in a situation to use these solutions to test the matching conditions found in earlier sections. We can do so even though the geometry involved is not axisymmetric, because it becomes effectively axisymmetric in the near-brane limit.

To this end we assume a brane action of the form

\[
S_b = - \int d^6 x \sqrt{-g} \ T_b(\tau, \bar{\tau}), \tag{3.12}
\]

where for a D7-brane in the Einstein frame we expect

\[
T_b = T_s e^\phi = \frac{T_s}{\text{Im} \tau} = \frac{2i T_s}{\tau - \bar{\tau}}, \tag{3.13}
\]

for constant \( T_s \).

Keeping in mind that \( W = 0 \) for the bulk solutions given above, the matching condition for the bulk scalar, equation (2.21), becomes

\[
\frac{2\pi}{\kappa^2} \left[ \frac{e^\phi}{4 (\text{Im} \tau)^2} \partial_\tau \tau \right]_{\text{sb}} = \frac{2\pi}{\kappa^2} \left[ \frac{r}{4 (\text{Im} \tau)^2} \partial_r \tau \right]_{\text{sb}} = \frac{\partial}{\partial \bar{\tau}} T_b = \frac{T_s}{2i (\text{Im} \tau)^2}. \tag{3.14}
\]
This uses the change of variables \( d\rho = e^C \, dr \) and \( e^B = r \, e^C \) to convert from proper distance to conformally flat coordinates near the brane. Using the near-brane limit \( \tau \simeq \ln r/2\pi i \) to evaluate \([r \, \partial/\partial r]_\mathrm{nb} \simeq 1/(2\pi i)\), we find that the matching condition becomes \( T_s = 1/(2\kappa^2)\).

Note that, since \( e^\phi \) is the string coupling constant, this semiclassical reasoning presupposes that \( \text{Im} \, \tau = e^{-\phi} \) is large near the brane, so that \( \kappa^2 T_b = \kappa^2 T_s/\text{Im} \, \tau = 1/(2\text{Im} \, \tau) \ll 1 \). This is automatically satisfied as \( r \to 0 \) because \( \text{Im} \, \tau \simeq -(\ln r)/2\pi \).

The metric matching conditions can be understood in a similar way. First, matching the on-brane components of the metric gives, from equation (2.23),

\[
-\frac{2\pi}{\kappa^2} \left[ e^B \frac{\partial}{\partial r} B - 1 \right] = -\frac{2\pi}{\kappa^2} \left[ r \, \partial_r B - 1 \right] = -\frac{2\pi}{\kappa^2} \left[ r \, \partial, C \right] = T_b(\tau, \bar{\tau}) = \frac{T_s}{\text{Im} \, \tau},
\]

which again uses \( e^\phi = r \, \partial_r \) as well as \( B = C + \text{Im} \, r \). Using equation (3.9) gives \( e^{2C} \simeq \text{Im} \, \tau \) near the brane, and so \( r \, \partial_r C \simeq \frac{1}{2} \left[ r \, \partial_r \right. \text{Im} \, \tau \left. \right]/\text{Im} \, \tau \) to get \([r \, \partial, C]_\mathrm{nb} = -1/(4\pi \, \text{Im} \, \tau)\). Once again the dependence on \( \text{Im} \, \tau \) is consistent on both sides and so the matching condition boils down to the statement \( 2\kappa^2 T_s = 1\), as above.

A further check comes from using the values for \( \kappa^2 \) and \( T_s \) for a D7-brane predicted in string theory [9]. Using \( T_s = 2\pi/\ell_s^8 \) and \( \kappa^2 = \ell_s^8/4\pi \), where \( \ell_s = 2\pi \sqrt{\alpha'} \) is the string length, we have

\[
2\kappa^2 T_s = 2 \left( \frac{\ell_s^8}{4\pi} \right) \left( \frac{2\pi}{\ell_s^8} \right) = 1,
\]

as required.

Finally, the absence of warping in the bulk solution, \( W = 0 \), implies that the remaining metric matching condition, equation (2.24), degenerates to \( U_b = 0 \). To compute \( U_b \) in the present instance, we use the constraint, equation (2.29), specialized to \( n = 10 \) dimensions

\[
U_b = \frac{1}{7} \left[ (1 - T_b) - \sqrt{(1 - T_b)^2 - \frac{7}{8} (T_b')^2} \right],
\]

where \( T_b = \kappa^2 T_b/2\pi = \kappa^2 T_s/(2\pi \, \text{Im} \, \tau) \), and use

\[
(T_b')^2 = 2 \left( \text{Im} \, \tau \right)^2 \frac{\partial T_b}{\partial \tau} \frac{\partial T_b}{\partial \bar{\tau}} = \frac{1}{2 \left( \text{Im} \, \tau \right)^2} \left( \kappa^2 T_s \right)^2 = \frac{1}{8\pi^2 \left( \text{Im} \, \tau \right)^2}.
\]

Clearly \((T_b')^2 = 0\) because \( \text{Im} \, \tau \to \infty \) as one approaches the brane, and this in turn ensures \( U_b = 0 \), as desired.

As a final check we compute the effective scalar potential, \( V_{\text{eff}} \), for the KK scalar zero mode in the 8D theory on the brane, after dimensional reduction. Because \( U_b = 0 \), this simply amounts to evaluating the action, equation (3.2), at the classical solution to the extra dimensional Einstein equations, which state

\[
\mathcal{R}_{mn} + \frac{1}{4 \left( \text{Im} \, \tau \right)^2} \left[ \partial_m \tau \, \partial_n \bar{\tau} + \partial_n \tau \, \partial_m \bar{\tau} \right] = 0.
\]

We see that \( V_{\text{eff}} = 0 \) in the effective theory, which is consistent with the maximally symmetric on-brane geometry being flat.
3.2. Brane–axion couplings in six dimensions (6D)

We next apply the above matching conditions to the example of two branes coupled to a bulk Goldstone mode (axion), \( \phi \), in six dimensions. Since 6D examples with flat on-brane geometries have already been discussed in some detail in [7], we concentrate here on solutions to the higher-dimensional equations for which the on-brane geometry is known to be curved. Our purpose is to provide a nontrivial example for which the shape of the full low-energy potential, \( V_{\text{eff}}(\phi) \), and its value at its stationary point, \( V_{\text{eff}}(\phi_0) \), can be explicitly computed directly from the higher-dimensional theory. Because this allows a check on how \( V_{\text{eff}} \) varies from its minimum, it allows us to verify that the extremal point is actually a local minimum of the low-energy potential.

The simplest such system starts with gravity coupled to a single bulk scalar and Maxwell field, with the bulk Lagrangian density given by

\[
L_B = -\sqrt{-g} \left\{ \frac{1}{2\kappa^2} g^{MN} \left[ R_{MN} + \partial_M \phi \partial_N \phi \right] + \frac{1}{4} F_{MN} F^{MN} + \Lambda \right\},
\]

(3.20)

where \( \Lambda \) is a bulk cosmological constant whose value can be chosen to obtain any desired curvature on the brane. Note that the choices \( f(\phi) = 1 \) and \( V(\phi) = \Lambda \) ensure the action has a shift symmetry, \( \phi \rightarrow \phi + \xi \), that guarantees the existence of a scalar KK zero mode having a constant profile across the bulk. This is the only such classically massless scalar KK mode, because the presence of the bulk cosmological term, \( \Lambda \), breaks the rigid scaling symmetry that the Einstein action normally has. This breaking ensures that the presence of \( \Lambda \) removes the ‘breathing’ mode corresponding to rigid expansions of the extra dimensional geometry, which would have otherwise been a low-energy scalar zero mode.

3.2.1. Bulk solutions. The field equations in this case admit explicit solutions for which the 4D on-brane geometry is maximally symmetric and the extra dimensions are axially symmetric [5, 11]. Using the ansatz of equation (2.9), a simple solution is

\[
ds^2 = \hat{g}_{\mu \nu} \, dx^\mu \, dx^\nu + d\rho^2 + \alpha^2 L^2 \sin^2\left(\frac{\rho}{L}\right) \, d\theta^2,
\]

(3.21)

\[F_{\rho \theta} = \alpha B_0 L \sin\left(\frac{\rho}{L}\right),\]

(3.22)

with \( \phi = \phi_0 \) being constant. The bulk field equations imply the following relation amongst the constants \( B_0, L \) and \( \Lambda \):

\[R_{(2)} = -\frac{2}{L^2} = -\kappa^2 \left(\frac{3B_0^2}{2} + \Lambda\right),\]

(3.23)

and the curvature of the on-brane metric is given by

\[\hat{R} = 2\kappa^2 \left(\frac{B_0^2}{2} - \Lambda\right).\]

(3.24)

When \( \alpha = 1 \), the extra dimensional metric describes a sphere of radius \( L \). When \( \alpha \neq 1 \), the geometry would still look like a sphere if we redefine \( \theta \rightarrow \alpha \theta \), although \( \theta \) is then not periodic with period \( 2\pi \). This indicates that there are conical singularities at both \( \rho = 0 \) and \( \rho = \pi L \), with defect angle given by \( \delta = 2\pi (1 - \alpha) \).
3.2.2. Brane properties. We now look for a pair of brane sources located at these two singularities that can support this geometry. We again take codimension-2 brane actions of the form

$$S_b = -\int d^4x \sqrt{-\gamma} \ T_b(\phi).$$  \hspace{0.5cm} (3.25)

Because the bulk solution has constant scalar, \(\phi = \phi_0\), its derivative, \(\partial_\phi \phi\), vanishes at both branes. This is only consistent with the scalar matching condition if \(T_b'(\phi)\) also vanishes for both branes when evaluated at the same place: \(\phi = \phi_0\). The vanishing of \(T_b'(\phi)\) at \(\phi = \phi_0\) also ensures that \(U_b(\phi)\) vanishes there, and this is consistent with the \((\theta \theta)\) matching condition, equation (2.24), because \(W = 0\) throughout the bulk in the classical solution ensures \(\partial_\phi W = 0\) at the brane positions.

Finally, the \((\mu \nu)\) matching condition, equation (2.23), reads

$$-\frac{2\pi}{\kappa^2} \left[ e^B B' - 1 \right]_{x_b} = T_b(\phi_0).$$  \hspace{0.5cm} (3.26)

Using \(e^B = \alpha L \sin(\rho/L)\) gives \(e^B B' \to \alpha\) as \(\rho \to 0\), and so this matching condition gives the usual expression for the defect angle in terms of the brane tension,

$$\delta = 2\pi (1 - \alpha) = \kappa^2 T_b(\phi_0),$$  \hspace{0.5cm} (3.27)

and so \(T_b = \kappa^2 T_b/2\pi = 1 - \alpha\).

3.2.3. The 4D perspective. We now show how the above picture is reproduced in the low-energy 4D effective theory below the KK scale. Although we cannot ask in the low-energy theory about the profiles of bulk fields within the extra dimensions, we can use it to understand the curvature, \(\hat{R}\), of the 4D on-brane geometry and the value, \(\phi_0\), to which the low-energy scalar field is fixed.

To this end, we can explore the scalar potential, \(V_{\text{eff}}\), for the KK zero mode of the scalar, \(\phi\), as it is moved away from \(\phi_0\). To do so requires more information about the shape of \(T_b(\phi)\), so we choose for simplicity

$$T_b(\phi) = M_b^4 + \frac{\mu_b^4}{2} (\phi - \phi_0)^2,$$  \hspace{0.5cm} (3.28)

although any choice for \(T_b(\phi)\) would do, so long as both tensions share a common zero for \(\partial T_b/\partial \phi\).

With this choice we have

$$T_b = \frac{\kappa^2 M_b^4}{2\pi} + \frac{\kappa^2 \mu_b^4}{4\pi} (\phi - \phi_0)^2, \hspace{0.5cm} T_b' = \frac{\kappa^2 \mu_b^4}{2\pi} (\phi - \phi_0),$$  \hspace{0.5cm} (3.29)

and so to lowest nontrivial order in \(\kappa^2\)

$$U_b = \frac{1}{3} \left[ (1 - T_b) - \sqrt{(1 - T_b)^2 - \frac{3}{4} (T_b')^2} \right]$$

$$\approx \frac{(T_b')^2}{8(1 - T_b)} + \frac{3(T_b')^4}{128(1 - T_b)^3} + \cdots.$$  \hspace{0.5cm} (3.30)
Specialized to the above tension, this becomes
\[ U_b \simeq \frac{\kappa^2 \mu_b^8}{16\pi} (\phi - \phi_0)^2 + \ldots. \] (3.31)

Note [26] that because \( U_b \) is quadratic in \( T_b' \), both it and its derivative, \( U_b' \), naturally vanish at zeros of \( T_b' \). Furthermore, the coefficient of \( (\phi - \phi_0)^2 \) in \( U_b \) is suppressed relative to the same term in \( T_b \) by an additional power of the small dimensionless factor \( \kappa^2 \mu_b^8 / 8\pi \ll 1 \). The full expression for the effective potential (2.40) in this case reduces to

\[ V_{\text{eff}} = \sum_b U_b + V_s(\phi_0) + \frac{1}{2} V_{\text{eff}}''(\phi_0)(\phi - \phi_0)^2 + \ldots \]

\[ = \sum_b U_b + \int d^2x \sqrt{g_2} \ e^{4W} \left\{ -\frac{1}{8} F_{mn} F^{mn} + \frac{1}{2} \Lambda \right\} + \frac{1}{2} V_{\text{eff}}''(\phi_0)(\phi - \phi_0)^2 + \ldots \]

\[ = \sum_b U_b + \frac{\pi}{2} \left( \Lambda - \frac{B_0^2}{2} \int \rho e^{B} \right) + \frac{1}{2} V_{\text{eff}}''(\phi_0)(\phi - \phi_0)^2 + \ldots \]

\[ = \left( \Lambda - \frac{B_0^2}{2} \right) 2\pi \alpha L^2 + \frac{1}{2} \left[ V_{\text{eff}}''(\phi_0) + \sum_b \kappa^2 \mu_b^8 / 8\pi \right] (\phi - \phi_0)^2 + \ldots \]

assuming that both \( W' \) and \( \phi' \) vanish when \( \phi = \phi_0 \). More explicit progress requires the calculation of \( V_{\text{eff}}''(\phi_0) \), although this can be expected to be non-negative if the bulk solution is stable. This shows that \( V_{\text{eff}}(\phi) \) is minimized at \( \phi = \phi_0 \), and this is how the 4D theory understands the value at which \( \phi \) is stabilized.

The value of the potential at this minimum has a direct physical interpretation, since it sets the value of the 4D curvature through the 4D Einstein equations. These read, as usual,
\[ \hat{R}_{\mu\nu} - \frac{1}{2} \hat{g}_{\mu\nu} - \kappa_5^2 V_{\text{eff}} \hat{g}_{\mu\nu} = 0, \] (3.32)

where the 4D Newton coupling is
\[ \frac{1}{\kappa_5^2} = \frac{2\pi}{\kappa^2} \int_0^{\pi L} d\rho e^B = \frac{4\pi \alpha L^2}{\kappa^2}, \] (3.33)

and so
\[ \hat{R} = -4\kappa_5^2 V_{\text{eff}}(\phi_0) = 2\kappa^2 \left( \Lambda - \frac{B_0^2}{2} \right), \] (3.34)

in agreement with the higher-dimensional result, equation (3.24). Note that this agreement requires, in particular, that the brane tensions \( T_b(\phi_0) = M_b^4 \) drop out of the low-energy potential.

Finally, note that evaluating the potential, equation (3.32), at its minimum by evaluating the action at the classical solution gives a result that agrees with the general expression (2.45), which in the present instance is evaluated to be

\[ V_{\text{eff}}(\phi_0) = -\sum_b e^{4W_b} U_b - \frac{1}{2} \int d^2x \sqrt{g_2} \ e^{4W} \left[ \frac{1}{4} f F_{mn} F^{mn} - V \right] \]

\[ = \frac{1}{2} \left( 4\pi \alpha L^2 \right) \left( \Lambda - \frac{B_0^2}{2} \right). \] (3.35)
3.3. Warped and unwarped supersymmetric examples

A large class of examples of explicit flux compactifications with nontrivial warping and scalar profiles in the extra dimensions is provided by solutions \([13, 14, 19, 20, 23–25]\) to chiral 6D supergravity \([12]\). Our goal with this example is to identify the properties of the branes that are required to source the known solutions. In general, the existence of solutions hinges on the consistency of these brane properties with the form of the intervening bulk, but these solutions are not known in closed form in the case where the on-brane dimensions are curved. In this situation, it is much easier to investigate the existence of solutions using the equivalent formulation in terms of minima of the low-energy scalar potential, since it is much easier to determine when such solutions exist.

The solutions of interest take as their starting point the following bosonic part of the supersymmetric action

\[
\mathcal{L}_B = -\sqrt{-g} \left\{ \frac{1}{2\kappa^2} g^{MN} \left[ R_{MN} + \partial_M \phi \partial_N \phi \right] + \frac{1}{4} e^{-\phi} F_{MN} F^{MN} + \frac{2g^2}{\kappa^4} e^\phi \right\},
\]

where the constant \(g\) denotes the 6D gauge coupling for the Maxwell field. Because this Lagrangian enjoys the property \(\mathcal{L}_B \rightarrow \lambda^2 \mathcal{L}_B\) when \(e^\phi \rightarrow \lambda^{-1} e^\phi\) and \(g_{MN} \rightarrow \lambda g_{MN}\), the arguments of section 2.4 imply that it becomes a total derivative once evaluated at an arbitrary classical solution \([20]\):

\[
\mathcal{L}_B(g_{MN}^c, A_M^c, \phi^c) = \frac{1}{2\kappa^2} \sqrt{-g^c} \Box \phi^c.
\]

3.3.1. Bulk solutions. For this system, it is useful to choose a slightly different metric ansatz \([23]\),

\[
ds^2 = W^2 \hat{g}_{\mu\nu} dx^\mu dx^\nu + a^2 \left( W^8 d\eta^2 + d\theta^2 \right),
\]

where \(a = a(\eta), W = W(\eta)\) and \(\hat{g}_{\mu\nu}\) is a maximally symmetric 4D de Sitter metric, with \(\hat{R} = -12 H^2\). With these choices the proper circumference of a circle along which \(\theta\) varies from zero to \(2\pi\) at fixed \(\eta = 2\pi a(\eta)\), and \(d\rho = a W^4 d\eta\). The dilaton is similarly taken to depend only on \(\eta, \phi = \phi(\eta)\), and the Maxwell field is given by \(A_\theta = A_\theta(\eta)\), so that

\[
F_{\eta\theta} = Q a^2 e^\phi.
\]

In this case, the content of Maxwell’s equations is that \(Q\) must be a constant, while the dilaton and the trace-reversed Einstein equations become

\[
\phi'' = \frac{2g^2}{\kappa^2} a^2 W^8 e^\phi - \frac{\kappa^2 Q^2}{2} a^2 e^\phi,
\]

and

\[
(\mu \nu) : \frac{W''}{W} - \left( \frac{W'}{W} \right)^2 + \frac{1}{2} \phi'' = \left( \frac{W'}{W} + \frac{1}{2} \phi' \right)' = 3 H^2 a^2 W^6,
\]

\[
(\theta \theta) : \frac{a''}{a} - \left( \frac{a'}{a} \right)^2 + \frac{1}{2} \phi'' = \left( \frac{a'}{a} + \frac{1}{2} \phi' \right)' = -\kappa^2 Q^2 a^2 e^\phi.
\]
In all of these equations, primes denote \(d/d\eta\). The ‘Hamiltonian constraint’—i.e. the \((\eta\eta)\) Einstein equation—in these variables is, similarly,

\[
\frac{1}{2} (\phi')^2 - \frac{4a'\mathcal{W}}{a\mathcal{W}} - \frac{6(\mathcal{W}')^2}{\mathcal{W}^2} = \frac{2g^2}{\kappa^2} a^2 \mathcal{W}^8 \phi^6 - 6H^2 a^2 \mathcal{W}^6 - \frac{\kappa^2}{2} Q^2 a^2 \phi^6. \tag{3.43}
\]

The scale invariance of the full 6D field equations under \(e^\phi \to e^\phi/\lambda\) and \(g_{MN} \to \lambda g_{MN}\) can be seen from the invariance of the above equations under

\[
\{\phi, a, \mathcal{W}, H\} \to \{\phi + \phi_0, a e^{-\phi_0/2}, \mathcal{W}, H e^{\phi_0/2}\}, \tag{3.44}
\]

with \(\phi_0\) being an arbitrary real constant. In the case of \(H = 0\), this symmetry implies the existence of a one-parameter family of classical solutions, and a corresponding flat direction (labelled \(\phi_0\)) that represents a classically massless KK zero mode coming from a combination of the metric and \(\phi\) fields.

The above field equations are written so that their right-hand sides tend to zero in the near-brane regions, for which \(a \to 0\). For regions where these right-hand sides are negligible, the equations simplify to

\[
\phi'' \simeq \left(\frac{\mathcal{W}'}{\mathcal{W}}\right) \simeq \left(\frac{a'}{a}\right) \simeq 0, \tag{3.45}
\]

and so, letting \(b = \{0, 1\}\) for the branes at \(\eta = \{-\infty, +\infty\}\), respectively,

\[
\phi \simeq (-)^b q_b \eta, \quad \mathcal{W} \simeq \mathcal{W}_b e^{-b a_b \eta} \quad \text{and} \quad a \simeq a_0 e^{-b a_b \eta}, \tag{3.46}
\]

with different choices for the constants \(a_0, \omega_0\) and \(q_b\) applying for the two limits, \(\eta \to \pm \infty\). For both asymptotic regions these are related by the constraint, equation (3.43), so that

\[
a_0^2 = 4 \omega_0 (2a_b + 3\omega_0). \tag{3.47}
\]

Note that it is only consistent in the near-brane limit to ignore the quantities \(a^2 \mathcal{W}^6\), \(a^2 \phi^6\) and \(a^2 \mathcal{W}^8 \phi^6\) on the right-hand sides of equations (3.41) through (3.43) if

\[
2\alpha_b + 6\omega_0 > 0, \quad 2\alpha_b + q_b > 0 \quad \text{and} \quad 2\alpha_b + 8\omega_0 + q_b > 0. \tag{3.48}
\]

The first of these also guarantees the convergence of the 4D gravitational constant, which is given by (cf equation (2.37))

\[
\frac{1}{\kappa^2} = \frac{2\pi}{\kappa^2} \int_{-\infty}^{\infty} d\eta a^2 \mathcal{W}^6. \tag{3.49}
\]

Furthermore, since our interest is in solutions where \(a \to 0\) at the positions of the brane sources, we demand \(\alpha_b > 0\). This ensures that the circumference of small circles encircling the branes vanishes in the limit that the branes are approached. But if \(\alpha_b > 0\), then \(\omega_0\) must also be non-negative. To see this, suppose that \(\omega_0\) were negative. Then equation (3.47) would imply \(-2\alpha_b - 3\omega_0 > 0\), and so adding this to the first of equations (3.48) would give \(\omega_0 > 0\), in contradiction with the assumption that it is negative. By contrast, the constant \(q_b\) can take either sign.
Solutions to these equations are known to exist for nonzero $H$ [25], although not yet in an explicit closed form. Closed-form solutions are known, however, in the special case where $H$ vanishes, given by [20, 23]

$$e^\theta = \mathcal{W}^{-2} e^{\phi_0 - \lambda_1 \eta}$$

$$\mathcal{W}^4 = \left(\frac{k^2 Q \lambda_2}{2 \lambda_1}\right) \cosh[\lambda_1(\eta - \eta_1)] \cosh[\lambda_2(\eta - \eta_2)]$$

and

$$a^{-4} = \left(\frac{2g\kappa^2 Q^3}{\lambda_1^2 \lambda_2}\right) e^{2(\phi_0 - \lambda_1 \eta)} \cosh^3[\lambda_1(\eta - \eta_1)] \cosh[\lambda_2(\eta - \eta_2)].$$

(3.50)

Here $\eta_i$ and $\lambda_j$ are integration constants, and there is no loss of generality in choosing, say, $\lambda_2 \geq 0$. The equations of motion require the constants to satisfy $\lambda_2^2 = \lambda_1^2 + \lambda_3^2$—and so, in particular, $\lambda_2 \geq |\lambda_1|$ (with equality if, and only if, $\lambda_3 = 0$). $\phi_0$ is an arbitrary constant corresponding to the scale invariance associated with the flat direction.

Because the terms involving $H$ in the equations of motion become negligible in the near-brane limit, the $H = 0$ solutions also provide a more detailed picture of the asymptotic regions at $\eta \rightarrow \pm \infty$. The corresponding metric singularities are generically curvature singularities, except when $\lambda_3 = 0$, in which case they turn out to be conical [24]. The $\lambda_3 = 0$ solutions include the unwarped, constant-dilaton ‘rugby ball’ configurations of [19] as the special case where $\eta_1 = \eta_2$. Note also that the limiting behaviour is as given in equation (3.46), with

$$\alpha_b = \frac{1}{4} \left[3\lambda_1 + \lambda_2 + 2(-)^b \lambda_3\right] \geq 0, \quad \omega_b = \frac{1}{4} (\lambda_2 - \lambda_1) \geq 0,$$

(3.51)

and

$$q_b = (-)^{b+1} \lambda_3 - \frac{1}{2} (\lambda_2 - \lambda_1).$$

(3.52)

Note that the condition $\omega_b \geq 0$ follows from $\lambda_2 \geq |\lambda_1|$, while $\alpha_b \geq 0$ is a consequence of

$$3(\lambda_2 + \lambda_1) - 2\lambda_3 = \sqrt{\lambda_2 + \lambda_1} \left(3\sqrt{\lambda_2 + \lambda_1} - 2\sqrt{\lambda_2 - \lambda_1}\right) \geq 0.$$

(3.53)

A special role is played by the combination

$$\omega_b + \frac{q_b}{2} = (-)^{b+1} \frac{\lambda_3}{2},$$

(3.54)

since this dictates the size of the Hubble constant, $H$. This can be seen by integrating equation (3.41) and using equation (3.49) to obtain [25]

$$3H^2 \int_{-\infty}^{\infty} d\eta \ a^2 \mathcal{W}^6 = \frac{3k^2 H^2}{2\pi \kappa_4^2} = \left[\left(\ln \mathcal{W} + \frac{\phi}{2}\right)\right]_{\eta = \infty}^{\eta = -\infty} = - \sum_b \left(q_b + \omega_b\right).$$

(3.55)

When evaluated for the solutions of equation (3.50), this reduces to the Friedmann equation

$$H^2 = -\frac{2\pi \kappa_4^2}{3k^2} \sum_b \left(q_b + \omega_b\right) = \frac{k_2^2}{3} \left[\frac{2\pi}{k^2} \sum_b (-)^b \frac{\lambda_3}{2}\right] = 0$$

(3.56)

as required. For more general solutions, equations (3.50) hold only approximately in the near-brane region, so the constant $\lambda_3$ could differ for the asymptotic region near each brane.

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3.3.2. Brane properties. As usual, the matching conditions relate the asymptotic bulk solutions to the properties of the source branes. Using $W = e^{W}$, $a = e^{\theta}$ and $aW^d\eta = d\rho$, and taking the brane action to be $S_b = -\int d^4x \sqrt{\gamma} L_b = -\int d^4x \sqrt{\gamma} T_b$, the scalar matching condition, equation (2.21), becomes

$$\frac{2\pi}{\kappa^2} \left[ e^{B+4W} \partial_\theta \phi \right]_{\phi_b} = \frac{\eta}{\partial \phi} \left[ e^{4W} L_b \right] \implies \left[ (-)^b \partial_\theta \phi \right]_{\phi_b} = q_b = \frac{\kappa^2}{2\pi} \left( \frac{\partial T_b}{\partial \phi} \right),$$

(3.58)

where the sign arises because the direction away from the brane is $(-)^b d\eta$ in the two asymptotic regions. The $(\theta \theta)$ metric matching condition, equation (2.24), similarly becomes

$$\frac{2\pi}{\kappa^2} \left[ e^{B+4W} \partial_\rho W \right]_{\phi_b} = U_b(\phi) \implies \left[ (-)^b \left( \frac{\partial_\theta W}{W} \right) \right]_{\phi_b} = \omega_b = \frac{\kappa^2 U_b}{2\pi}.$$  

(3.59)

Finally, the $(\mu \nu)$ components of the metric matching conditions are

$$-\frac{2\pi}{\kappa^2} e^{4W} [e^{B} (3 \partial_\rho W + \partial_\rho B) - 1] = T_b(\phi),$$

(3.60)

and so

$$\left\{ \left[ (-)^b \left( \frac{\partial_\rho W}{W} \right) + \left( \frac{\partial_\theta a}{a} \right) \right] - \frac{W^d}{W^4 \chi_b} \right\} = 3\omega_b + \alpha_b - \frac{W^d(x_b)}{\kappa^2} = -\frac{\kappa^2 T_b}{2\pi}. $$

(3.61)

There are now two qualitatively different cases that are worth considering separately, depending on whether $\omega_b = 0$ or $\omega_b > 0$.

3.3.2.1. Solutions with only conical singularities. If $\omega_b = 0$, then equation (3.47) implies $q_b = 0$ as well, and so both $\phi$ and $W$ asymptote to constants near the brane. Because $\omega_b = 0$ implies that $W \simeq W_b$ is constant in the near-brane regime, the behaviour $a \sim e^{\alpha_b \eta}$ implies that the extra dimensional metric is proportional to

$$e^{2\alpha_b \eta} (W_b^8 d\eta^2 + d\theta^2) = d\rho^2 + \left( \frac{\alpha_b \rho}{W_b^4} \right)^2 d\theta^2,$$

(3.62)

showing that it has only a conical singularity at the brane position, with defect angle $\delta_b = 2\pi (1 - \alpha_b / W_b^4)$.  

When $\omega_b = q_b = 0$, the matching conditions boil down to

$$\frac{\kappa^2 T_b}{2\pi} = \frac{\kappa^2 U_b}{2\pi} = 0 \quad \text{and} \quad \delta_b = \frac{\kappa^2 T_b}{W_b^4} = \kappa^2 L_b.$$  

(3.63)

The last of these relates the tension to the size of the conical defect angle in the usual way, while the first states that the value taken by $\phi$ near each brane must be at a stationary point of the tension on that brane. (Since this is also automatically a zero of $U_b$, the second condition is
redundant.) In order for solutions to exist, the two tensions must be related to one another by the known asymptotic limits of the given bulk solution. That is, if \( \phi_b = \lim \phi(\eta) \) as \( \eta \to -(-1)^{b} \infty \), then \( T_b \) must satisfy \( T_b(\phi_b) = 0 \) at both ends.

Since its right-hand side is non-negative, equation (3.41) shows that it is only possible to have \( \omega_b = q_b = 0 \) at both branes if \( H = 0 \). If \( H = 0 \), the solutions given in equations (3.50) have this property (for both branes) when \( \lambda_3 = 0 \) (and so also \( \lambda_1 = \lambda_2 := \lambda \)). Note that \( V \) and \( e^{\phi} = V^{-2} \) need not be identically constant in this case unless \( \eta_1 = \eta_2 \).

From the point of view of the 4D theory, the result \( H = 0 \) is understood for these solutions in terms of the vanishing of the classical low-energy 4D effective potential,

\[
V_{\text{eff}} = V_{\eta} + \sum_b U_b = 0. \tag{3.64}
\]

This vanishes because equation (3.37) (when \( \phi' = 0 \) near the branes) shows that the bulk contribution to the low-energy potential vanishes, \( V_{\eta} = 0 \), and equation (3.63) implies \( U_b = 0 \) for both branes.

If \( T_b \) should vanish identically, then so must \( U_b \) and \( V_{\text{eff}} \). In this case, the vanishing of \( V_{\text{eff}} \) shows that the flat direction, corresponding to the scaling \( \phi \to \phi + \phi_0 \) and \( g_{MN} \to e^{-\phi_0} g_{MN} \), is not lifted by the classical couplings to the branes. But if \( T_b \) depends nontrivially on \( \phi \), then \( U_b \) becomes nonzero as soon as \( \phi \) differs from its asymptotic value, \( \phi_b \), implying that \( V_{\text{eff}} \) depends nontrivially on \( \phi_0 \). Since \( U_b(\phi_b) \) is given by

\[
U_b = \frac{1}{3} \left[ (V^4 - T_b) - \sqrt{(V^4 - T_b)^2 - \frac{3}{4} (T_b')^2} \right], \tag{3.65}
\]

where \( T_b = T_b(\phi_b + \phi_0) \), it is non-negative (provided \( T_b < V^4 \)). It follows that \( V_{\text{eff}}(\phi_0) \) must be minimized by any configuration for which it vanishes, such as \( \phi_0 = 0 \) (which corresponds to \( \lim \phi = \phi_b \)). This shows how the 4D theory shows that the flat direction, \( \phi_0 \), of the bulk equations becomes fixed at the same value as that chosen by the matching conditions when viewed from the higher-dimensional perspective.

3.3.2.2. Solutions with \( \omega_b > 0 \). On the other hand, if \( \omega_b > 0 \), then \( e^{W} = W \to 0 \) as the brane is approached. In this case, the scalar and \((\mu \nu)\) matching conditions are

\[
q_b = \frac{\kappa^2 T_b}{2\pi} = T_b' \quad \text{and} \quad 3 \omega_b + \alpha_b = -\frac{\kappa^2 T_b}{2\pi} = -T_b. \tag{3.66}
\]

Since \( \alpha_b \) and \( \omega_b \) are both positive, the last of these conditions implies \( T_b < 0 \). The third matching condition in this case is

\[
\omega_b = \frac{\kappa^2 U_b}{2\pi} = U_b = \frac{1}{3} \left[ -T_b - \sqrt{T_b^2 - \frac{3}{4} (T_b')^2} \right], \tag{3.67}
\]

which also requires \( T_b < 0 \) if \( U_b \) and \( \omega_b \) are to be positive.

Because we use coordinates for which the branes are situated at \( \eta \to \pm \infty \), we demand that these matching conditions are satisfied as identities in \( \eta \) in the asymptotic regimes. Use of the asymptotic forms for the bulk solutions in this regime corresponds to expanding the brane tension about the value taken by \( \phi \) at the brane.

This determines the functional form for the brane action, \( T_b(\phi, a, W) = e^{W} L_b(\phi, a) \), required to source the given bulk solution. Because \( e^{\phi} \) and all metric functions behave as
exponentials near the branes—cf equation (3.46)—the brane action must have the form $L_b = -\Lambda_b \, e^{\xi_b} \mathcal{F}(a e^{\phi})$, where $\mathcal{F}(x)$ is an arbitrary function and the powers $\xi_b$ and $\zeta_b$ are chosen to ensure the $\eta$-independence in the near-brane regime of
\[ T_b = -\Lambda_b \, \mathcal{W}^4 \, e^{\xi_b \phi} \, \mathcal{F}(a e^{\phi}) , \tag{3.68} \]
for constant $\Lambda_b$. The parameters $\xi_b$ and $\zeta_b$ therefore satisfy
\[ 4 \, \omega_b + \xi_b \, q_b = \alpha_b + \zeta_b \, q_b = 0 . \tag{3.69} \]
In terms of $\mathcal{F}(x)$, the scalar matching condition becomes
\[ q_b = \frac{k^2}{2\pi} \left( \frac{\partial T_b}{\partial \phi} \right) = -\frac{k^2 \Lambda_b}{2\pi} \, \mathcal{W}^4 \, e^{\xi_b \phi} \left[ \xi_b \, \mathcal{F}(x) + \zeta_b \, x^{4/\eta} \right]_{x=a e^{\phi}} , \tag{3.70} \]
while the metric matching conditions similarly give
\[ 3 \, \omega_b + \alpha_b = -\frac{k^2 T_b}{2\pi} = \frac{k^2 \Lambda_b}{2\pi} \, \mathcal{W}^4 \, e^{\xi_b \phi} \, \mathcal{F}(a e^{\phi}) , \tag{3.71} \]
and so on.

To proceed further requires making choices for the function $\mathcal{F}(x)$. We discuss for simplicity a power law, $\mathcal{F}(x) = x^{\sigma_b}$, to concretely illustrate the brane–bulk interaction.

3.3.2.3. Power-law tension: $\mathcal{F}(x) = x^{\sigma_b}$. Perhaps the simplest choice for the function $\mathcal{F}(x)$ appearing above is a power: $\mathcal{F}(x) = x^{\sigma_b}$, with $\sigma_b$ being a constant. In this case,
\[ T_b = -\Lambda_b \, \mathcal{W}^4 \, a^{\sigma_b} \, e^{\lambda_b \phi} , \tag{3.72} \]
where $\lambda_b = \xi_b + \zeta_b \sigma_b$, and so
\[ 4 \omega_b + \sigma_b \alpha_b + \lambda_b q_b = 0 \tag{3.73} \]
is required to ensure that the $\eta$-dependence cancels in $T_b$ within the near-brane regime. This last equation is to be regarded as being solved for $\sigma_b$.

The scalar matching condition, equation (3.58), then boils down to
\[ q_b = -\lambda_b \, \mathcal{W}^4_b \, a^{\sigma_b} \left( \frac{k^2 \Lambda_b}{2\pi} \right) . \tag{3.74} \]
The $(\mu \nu)$ metric matching condition, equation (3.61), similarly gives
\[ 3 \, \omega_b + \alpha_b = \mathcal{W}^4_b \, a^{\sigma_b} \left( \frac{k^2 \Lambda_b}{2\pi} \right) . \tag{3.75} \]
Combining (3.74) and (3.75) gives the parameter $\lambda_b$ as
\[ \lambda_b = -\frac{q_b}{3 \omega_b + \alpha_b} . \tag{3.76} \]
Clearly $q_b < 0$ implies $\lambda_b > 0$, and vice versa, because $\alpha_b$ and $\omega_b$ are both positive. Note that $\lambda_b > 0$ implies $T_b \to 0$ in the ‘weak-coupling’ limit $e^\phi \to 0$.

Given $\alpha_b$ and $\omega_b$, solving the above conditions gives $q_b = \pm 2 \sqrt{\omega_b (2 \omega_b + 3 \alpha_b)}$ (from equation (3.47)), $\lambda_b$ (from equation (3.76)) and the combination $\mathcal{W}^4_b \, a^{\sigma_b} (k^2 \Lambda_b / 2\pi)$ (from equation (3.75)). The power of $a$ appearing in $T_b$ works out to be
\[ \sigma_b = \frac{4 \omega_b}{3 \omega_b + \alpha_b} > 0. \tag{3.77} \]
One might think that the last matching condition, involving $U_b$, gives an independent equation that can be used to relate $\omega_b$ to $\alpha_b$, but this turns out not to be independent due to the relation between $U_b$ and $T_b$ and the constraint, equation (3.47).
3.3.3. The 4D perspective. In this section, we evaluate the full action at its classical solution to determine the value of $V_{\text{eff}}$ at its minimum. For supergravity, the full bulk action is evaluated to be a total derivative at any classical solution, giving

$$S_{b,\text{ext}} = \frac{1}{2\kappa^2} \int d^6x \sqrt{-g} \Box \phi = \frac{\pi}{\kappa^2} \int d^4x \sqrt{-\hat{g}} \left[ \partial_\eta \phi \right]_\infty^\infty = - \sum_b \frac{T'_b}{2}. \quad (3.78)$$

Adding to this the brane action and the Gibbons–Hawking term, which combine to

$$\sum_b (S_{\text{GH}} + S_b) = - \int d^4x \sqrt{\hat{g}} \, U_b, \quad (3.79)$$

gives the total action evaluated at the classical solution

$$S_{\text{ext}} = - \int d^4x \sqrt{\hat{g}} \sum_b \left( U_b + \frac{T'_b}{2} \right). \quad (3.80)$$

Comparing this with equation (2.44) (for $n = 6$) gives

$$V_{\text{eff}}(\phi_0) = - \sum_b \left( U_b + \frac{T'_b}{2} \right). \quad (3.81)$$

Using this in the 4D Einstein equations gives the 4D curvature

$$\hat{R} = -12H^2 = -4\kappa_N^2 V_{\text{eff}}(\phi_0), \quad (3.82)$$

and so

$$H^2 = \frac{\kappa_N^2}{3} V_{\text{eff}} = - \frac{\kappa_N^2}{3} \sum_b \left( U_b + \frac{T'_b}{2} \right) = - \frac{2\pi \kappa_N^2}{3\kappa^2} \sum_b \left( \omega_b + q_b \right), \quad (3.83)$$

where the last equality uses the matching conditions to rewrite $U_b$ and $T'_b$ in terms of the bulk solution. This agrees with the bulk field equations, equation (3.55), and so shows that the 4D and 6D pictures agree. In order to identify the value of $\phi_0$ itself requires calculating $V_{\text{eff}}$ away from its minimum, which requires a full dimensional reduction of the supergravity action.

4. Conclusions

This paper summarizes the bulk–brane matching conditions for codimension-2 objects (following the presentation given for scalar–tensor theories in [7], with generalizations to include a general coupling to the Maxwell field [6]), and it describes several applications to higher-dimensional brane systems: F-theory compactifications involving space-filling codimension-2 D7-branes situated within ten dimensions; unwarped 3-brane flux compactifications in 6D scalar-Maxwell–Einstein theory; and warped and unwarped 3-brane flux compactifications of 6D chiral gauged supergravity. The latter two cases involve geometries that are maximally symmetric—but possibly curved—in the directions parallel to the branes.

The comparison with the F-theory compactifications provides a sanity check on the junction conditions, since both the brane and the bulk actions are explicitly known for Type IIB string vacua [9], as are explicit solutions for the surrounding bulk geometry [21]. We show that the near-brane asymptotic form of the bulk configurations in this case precisely agrees with what the matching conditions would predict, given the explicit D7-brane action. Furthermore,
this comparison lies within the weak-coupling regime since the bulk solution implies that the
string coupling becomes weak in the near-brane limit.

When applied to 6D systems, the bulk–brane matching conditions can provide a
stabilization mechanism for the bulk scalars (like a bulk axion, or the dilaton) provided that
the brane couplings break the appropriate symmetry that protects the scalar’s mass. When this
is so, the value to which the scalar stabilizes can be understood from the higher-dimensional
point of view as being due to the consistency of the matching conditions at the two branes. Alternatively, it can be regarded as the value that minimizes the effective potential in the low-
energy, on-brane action below the KK scale, although this requires calculation of the potential
away from its minimum.

Although many of the bulk solutions considered in six dimensions (supersymmetric or not)
have de Sitter curvature along the four brane directions [5, 14], we show that for 6D gauged
chiral supergravity, only 4D-flat branes can be sourced by positive-tension branes. To establish
this, we first show that for any 6D theory, a codimension-2 brane tension must be negative
whenever the warp factor tends to zero near the brane. We then prove that the supergravity
field equations imply that the warping vanishes near the brane unless the near-brane geometry
has a conical singularity. Finally, the desired result follows once the field equations are used
to see that any geometry having only conical singularities necessarily is flat in the four brane
directions.

This necessity for negative tension in order to obtain de Sitter and anti-de Sitter branes
echoes the various no-go theorems for finding 4D-de Sitter solutions from extra-dimensional
gravity [27], even though the curvatures of the bulk geometries considered make these theorems
not directly applicable. This suggests that the curvature assumptions made in these theorems
may be somewhat stronger than is necessary.

The relation to 4D de Sitter geometries has potential applications in the search for cosmic
inflation within an extra dimensional context. This is because inflationary configurations often
lay nearby pure de Sitter solutions. In particular, a broad class of time-dependent solutions are
known [14] for the bulk field equations in 6D supergravity, and for some of these the on-
brane 4D geometry is likely to undergo an accelerated expansion. Extension of the arguments
of this paper to these time-dependent situations would be worthwhile, since they could provide
instances of explicit inflationary models for which there is both a higher- and a lower-
dimensional understanding of why the universe accelerates. (By contrast, current inflationary
models typically rely on the low-energy 4D effective theory to conclude that the universe
inflates.) Work along these lines is in progress [28].

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