Empirical likelihood inference for threshold autoregressive conditional heteroscedasticity model

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Abstract

This paper considers the parameter estimation problem of a first-order threshold autoregressive conditional heteroscedasticity model by using the empirical likelihood method. We obtain the empirical likelihood ratio statistic based on the estimating equation of the least squares estimation and construct the confidence region for the model parameters. Simulation studies indicate that the empirical likelihood method outperforms the normal approximation-based method in terms of coverage probability.

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Keywords: Empirical likelihood; Threshold autoregressive model; Conditional heteroscedasticity; Confidence region; Least squares method

1 Introduction

Consider the following first-order threshold autoregressive conditional heteroscedasticity model:

\[ X_t = \theta_1 X^+_t + \theta_2 X^-_t + \epsilon_t, \]  

where \( X^+_t = \max(X_t, 0), \) \( X^-_t = \min(X_t, 0), \) \( \epsilon_t = \sqrt{h_t} \epsilon_t, \) \( h_t = \alpha_0 + \alpha_1 (\epsilon^+_{t-1})^2 + \alpha_2 (\epsilon^-_{t-1})^2, \) \( \epsilon_t \) is a sequence of independent and identically distributed random variables satisfying \( E\epsilon_t = 0 \) and \( \text{Var}(\epsilon_t) = 1. \) \( \theta_1, \theta_2, \alpha_0, \alpha_1, \) and \( \alpha_2 \) are the model parameters with \( \alpha_0 > 0, \) \( 0 \leq \alpha_j < 1, \) \( j = 1, 2. \)

When \( \theta_1 = \theta_2, \) model (1) becomes the usual autoregressive model whose innovation is a conditional heteroscedasticity process. Threshold autoregressive model is a nonlinear time series model. Because the threshold autoregressive model can explain nonlinear features such as asymmetry and limit cycles, it is widely used in time series modeling (see Tong [1]). Petruccelli and Woolford [2] first defined model (1) and investigated its properties and the parameter estimation problems, but they assumed that the error sequence is a sequence of independent and identically distributed random variables. Brockwell et al. [3] and Hwang and Basawa [4] further generalized the model coefficients to be random.
variables. Hwang and Woo [5] first considered the parameter estimation problems when the error sequence is a conditional heteroscedasticity process and proposed to use the conditional least squares method to estimate the model parameters. In this paper, we use the empirical likelihood method to estimate the model parameters.

Similar to the parametric likelihood, Owen [6–8] introduced empirical likelihood method. It is a nonparametric likelihood method which establishes a likelihood function through placing positive probability on every one of the observed data values, but often makes no assumptions on the data-generating mechanism. Empirical likelihood has many advantages compared with the normal approximation method. For example, the limiting distribution of empirical likelihood ratio statistic is a chi-squared distribution. Therefore, we need not estimate the asymptotic variance when we construct the confidence region. Moreover, the confidence region is completely decided by the data themselves because we make no assumptions on the probability distribution of the data. These attract the attention of statisticians to make inference for all kinds of statistical models using the empirical likelihood method, such as linear regressive model [9–13], generalized linear models [14–17], and partially linear models [18–21]. In recent years, empirical likelihood method is also applied to make statistical inference about time series models, such as autoregressive model [22–24], random coefficient autoregressive model [25–28], and integer-valued autoregressive model [29–32].

In this paper, we obtain the limiting distribution of empirical log-likelihood ratio statistic and construct the confidence region for the parameters in model (1) by using the empirical likelihood method. Some simulation studies indicate that the empirical likelihood method has a higher coverage probability compared with the normal approximation-based method.

This paper is organized as follows. In Sect. 2, we present the main methods and results. Some simulation results and real data analysis are given in Sect. 3. Section 4 is concerned with the proofs of the main results. Moreover, the symbols “$d$” and “$p$” denote convergence in distribution and convergence in probability, respectively. $O_p(1)$ means a term which is bounded in probability. $o_p(1)$ means a term which converges to zero in probability. “Almost surely” and “independent identical distributed” are denoted by “a.s.” and “i.i.d.”, respectively.

2 Methods and main results

For model (1), Hwang and Woo [5] obtained the least square estimation of the model parameters and its limiting properties. Now we use the empirical likelihood method to estimate the model parameters. Before giving the main results, we assume that the following conditions are true:

(A1) Probability density function $f(\cdot)$ of $e_t$ has its support on $(-\infty, +\infty)$. $\theta_{\max} + \sqrt{\alpha_{\max}} < 1$, where $\theta_{\max} = \max\{|\theta_1|, |\theta_2|\}$, $\alpha_{\max} = \max\{\alpha_1, \alpha_2\}$.

(A2) $E(X_t^2) < \infty$.

According to Theorem 1 in Hwang and Woo [5], if (A1) holds, then $\{X_t, t \geq 1\}$ is geometrically ergodic, and the sequence $\{X_t\}$ has a unique stationary distribution.

Hwang and Woo [5] used the least square method to estimate the model parameters. Let $\theta = (\theta_1, \theta_2)^T$. Based on the observation data $\{X_0, X_1, \ldots, X_n\}$, the least square estimation $\hat{\theta}$ of $\theta$ can be obtained by minimizing $Q(\theta) = \sum_{t=1}^{n}(X_t - E(X_t | \mathcal{F}_{t-1}))^2 = \sum_{t=1}^{n}(X_t - \theta_1 X_{t-1} - \theta_2 X_{t-2} - \theta_3)$. 

This paper is organized as follows. In Sect. 2, we present the main methods and results. Some simulation results and real data analysis are given in Sect. 3. Section 4 is concerned with the proofs of the main results. Moreover, the symbols “$d$” and “$p$” denote convergence in distribution and convergence in probability, respectively. $O_p(1)$ means a term which is bounded in probability. $o_p(1)$ means a term which converges to zero in probability. “Almost surely” and “independent identical distributed” are denoted by “a.s.” and “i.i.d.”, respectively.
\(\theta_2 X_{t-1}^{-1}\) with respect to \(\theta\). Solving

\[
\frac{\partial Q(\theta)}{\partial \theta} = -2 \sum_{t=1}^{n} (X_t - \theta_1 X_{t-1}^* - \theta_2 X_{t-1}^{-1}) (X_{t-1}^* X_{t-1}^{-1}) = 0
\]

(2)

for \(\theta\), we know that

\[
\theta^* = \left( \frac{\sum_{t=1}^{n} X_t X_{t-1}^* / \sum_{t=1}^{n} (X_{t-1}^*)^2}{\sum_{t=1}^{n} X_t X_{t-1}^{-1} / \sum_{t=1}^{n} (X_{t-1}^{-1})^2} \right).
\]

Let \(X_t = (X_{t-1}^*, X_{t-1}^{-1})^T\). Then the estimating equation (2) can be written as

\[
\sum_{t=1}^{n} (X_t - X_t^* \theta) X_t = 0.
\]

(3)

Further, let \(H_t(\theta) = (X_t - X_t^* \theta) X_t\). By (3), we can obtain the following empirical likelihood ratio statistic:

\[
L(\theta) = \max \left\{ \prod_{t=1}^{n} np_t : \sum_{t=1}^{n} p_t H_t(\theta) = 0, p_t \geq 0, \sum_{t=1}^{n} p_t = 1 \right\}.
\]

(4)

By using the Lagrange multiplier method, it is easy to know that

\[
p_t = \frac{1}{n} \frac{1}{1 + b^*(\theta) H_t(\theta)},
\]

where the Lagrange multiplier \(b(\theta)\) satisfies

\[
\frac{1}{n} \sum_{t=1}^{n} \frac{H_t(\theta)}{1 + b^*(\theta) H_t(\theta)} = 0.
\]

(6)

Therefore, we have

\[
-2 \log(L(\theta)) = 2 \sum_{t=1}^{n} \log \left( 1 + b^*(\theta) H_t(\theta) \right).
\]

(7)

The following theorem indicates that the limiting distribution of \(-2 \log(L(\theta))\) is a chi-squared distribution.

**Theorem 2.1** If (A1) and (A2) hold, then when \(n \to \infty\),

\[
-2 \log(L(\theta)) \xrightarrow{d} \chi^2(2),
\]

(8)

where \(\chi^2(2)\) is the chi-squared distribution with two degrees of freedom.

Using the above theorem, we can construct the empirical likelihood ratio confidence region for the parameter \(\theta\). For \(0 < \delta < 1\), the 100\((1 - \delta)\)\% asymptotic confidence region for the parameter \(\theta\) is

\[
C(\delta) = \{ \theta : L(\theta) \leq \chi^2_\delta(2) \},
\]

(9)
where \( \chi_\delta^2(2) \) is the upper \( \delta \) quantile of chi-squared distribution with two degrees of freedom.

### 3 Simulation studies

In this section, we carry out some simulation studies to compare the performances of our empirical likelihood (EL) method with the least square (LS) method proposed by Hwang and Woo [5] through random simulation. Consider the simulation results of model (1) in the following error sequence:

- **Sequence I:** \( \{e_t\} \) is a sequence of independent and identically distributed (i.i.d.) standard normal distribution \( N(0,1) \) random variables.
- **Sequence II:** \( \{e_t\} \) is an independent noise sequence with \( \epsilon \)-contamination distribution, and the distribution function of \( \{e_t\} \) is

  \[
  F_{e_t}(x) = \epsilon \Phi\left(\frac{x}{\sigma_1}\right) + (1 - \epsilon)\Phi\left(\frac{x}{\sigma_2}\right),
  \]

  where \( \sigma_i > 0 \) (\( i = 1, 2 \)), \( \epsilon \) is a fixed constant satisfying \( 0 < \epsilon < 1 \) and \( \Phi(x) \) is the distribution function of the standard normal random variable.

- **Sequence III:** \( \{e_t\} \) is a sequence of independent and identically distributed (i.i.d.) mixing random variable sequence, and the distribution function of \( \{e_t\} \) is

  \[
  F_{e_t}(x) = \epsilon \Phi\left(\frac{x}{\sigma}\right) + (1 - \epsilon)T_k(x),
  \]

  where \( \sigma > 0 \), \( 0 < \epsilon < 1 \), \( T_k(x) \) is the distribution function of \( T \) distribution with \( k \) degrees of freedom, \( \Phi(x) \) is the distribution function of standard normal random variable.

We calculate the coverage probabilities of the empirical likelihood and the least square methods for different model parameters. The nominal confidence level \( 1 - \delta \) is chosen to be 0.90. All simulation studies are based on 1000 repetitions, and the sample sizes considered in these simulations are \( n = 100, 300, \) and 500. The simulation results for sequence I are presented in Table 1. For sequence II, we simulate \( (\epsilon, \sigma_1, \sigma_2) = (0.9, 1, 3) \) and \( (\epsilon, \sigma_1, \sigma_2) = (0.75, 1, \sqrt{7}) \), and the simulation results are presented in Table 2 and Table 3, respectively. For sequence III, we simulate \( (\epsilon, \sigma, k) = (0.2, 1, 6) \) and \( (\epsilon, \sigma, k) = (0.5, 1, 3) \), and the simulation results are presented in Table 4 and Table 5, respectively. The first figures in parentheses are the simulation results obtained by the empirical likelihood method, and the second figures are the simulation results obtained by the least square method.

From the simulation results in Tables 1–5, it can be seen that, for different error distribution, the confidence region constructed by the empirical likelihood method has a higher coverage probabilities for different parameters, sample sizes, pollution levels, and pollution distributions. Moreover, the confidence region constructed by the empirical likelihood method is closer to the confidence level 0.90. This shows that the empirical likelihood method is more robust than the least square method.

### 4 Real data analysis

In this section, we use our method to fit student teacher ratio (number of teachers = 1) data in Chinese universities, which are provided by the website of China National Bureau of Statistics (http://data.stats.gov.cn/easyquery.htm?cn=C01&zb=A060E01&sj=2019). Student teacher ratio is an important index to measure the level of universities. There are
Let the yearly counts of student-teacher ratio in China over the period from 1949 to 2018. To correlation function (PACF) for the series with 70 available observations, which are denoted by $X_t$. The observations represent the yearly counts of student-teacher ratio in China over the period from 1949 to 2018. Let $Y_t = X_t - X_{t-1}$. The plot of sample path, autocorrelation function (ACF), and partial autocorrelation function (PACF) for the series $\{Y_t\}$ are given in Figs. 1, 2, and 3, respectively.
The corresponding plots of sample autocorrelation function (ACF) and partial autocorrelation function (PACF) indicate an $AR(1)$-like autocorrelation structure.

In what follows, based on the observation data $\{Y_t\}$, we give the figure of the empirical likelihood ratio confidence region when the confidence level is 0.95 (see Fig. 4). After a
simple calculation, we know that the least square estimation $\theta^* = (0.1616, 0.6191)$, and it is denoted by $\star$ in Fig. 4. From Fig. 4, we can see that the least square estimation $\theta^*$ is in the empirical likelihood ratio confidence region. Moreover, the empirical likelihood ratio confidence region is relatively small although the confidence level is 0.95.

5 Proofs

In order to establish Theorem 2.1, we first prove the following lemmas.
Lemma 5.1  If (A1) and (A2) hold, then

\[
\frac{1}{\sqrt{n}} \sum_{t=1}^{n} H_t(\theta) \xrightarrow{d} N(0, D) \quad \text{as } n \to \infty,
\]

(10)

where

\[
D = \begin{pmatrix}
E(\epsilon_1^2(X_0)^2) & 0 \\
0 & E(\epsilon_1^2(X_0)^2)
\end{pmatrix}.
\]
Proof. Note that

\[
\sqrt{n}(\theta^* - \theta) = \sqrt{n}\left(\left(\sum_{t=1}^{n} X_t X_t^\tau\right)^{-1} \sum_{t=1}^{n} X_t X_t - \theta\right)
\]

\[
= \sqrt{n}\left(\left(\sum_{t=1}^{n} X_t X_t^\tau\right)^{-1} \sum_{t=1}^{n} X_t X_t - \left(\sum_{t=1}^{n} X_t X_t^\tau\right)^{-1} \sum_{t=1}^{n} X_t X_t^\tau \theta\right)
\]

\[
= \sqrt{n}\left(\sum_{t=1}^{n} X_t X_t^\tau\right)^{-1} \left(\sum_{t=1}^{n} X_t X_t - \sum_{t=1}^{n} X_t X_t^\tau \theta\right)
\]

\[
= \sqrt{n}\left(\sum_{t=1}^{n} X_t X_t^\tau\right)^{-1} \left(\sum_{t=1}^{n} X_t (X_t - X_t^\tau \theta)\right)
\]

\[
= \left(\frac{1}{n} \sum_{t=1}^{n} X_t X_t^\tau\right)^{-1} \frac{1}{\sqrt{n}} \left(\sum_{t=1}^{n} X_t (X_t - X_t^\tau \theta)\right).
\]

Therefore we have

\[
\frac{1}{\sqrt{n}} \sum_{t=1}^{n} H_t(\theta) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} (X_t - X_t^\tau \theta) X_t
\]

\[
= \frac{1}{n} \sum_{t=1}^{n} X_t X_t^\tau \left(\frac{1}{n} \sum_{t=1}^{n} X_t X_t^\tau\right)^{-1} \frac{1}{\sqrt{n}} \left(\sum_{t=1}^{n} X_t (X_t - X_t^\tau \theta)\right)
\]

\[
= \frac{1}{n} \sum_{t=1}^{n} X_t X_t^\tau \sqrt{n}(\theta^* - \theta) .
\]

(11)

Figure 4 The figure of the empirical likelihood ratio confidence region
By the ergodic theorem, we have
\[
\frac{1}{n} \sum_{t=1}^{n} X_t X_t' \xrightarrow{a.s.} E(X_t X_t')
\]
\[
= E\left(\begin{pmatrix} X_{t-1}'^2 & 0 \\ 0 & (X_{t-1})^2 \end{pmatrix}\right)
\]
\[
= \left(\begin{array}{rr}
E(X_{t-1})^2 & 0 \\
0 & E(X_{t-1})^2
\end{array}\right)
\]
\[\triangleq W.
\]

According to the result of Lemma 1 in Hwang and Woo [5], we have
\[
\sqrt{n}(\theta^* - \theta) \xrightarrow{d} N(0, W^{-1}DW^{-1}) \quad \text{as} \quad n \to \infty. \tag{12}
\]

Combining with (11), we know that Lemma 5.1 holds. \hfill \square

**Lemma 5.2** If (A1) and (A2) hold, then
\[
\frac{1}{n} \sum_{t=1}^{n} H_t(\theta) H_t^*(\theta) \xrightarrow{p} D \quad \text{as} \quad n \to \infty. \tag{13}
\]

**Proof** Notice that
\[
\frac{1}{n} \sum_{t=1}^{n} H_t(\theta) H_t^*(\theta) = \frac{1}{n} \sum_{t=1}^{n} (X_t - X_t^* \theta)^2 X_t X_t'
\]
\[
= \frac{1}{n} \sum_{t=1}^{n} \left(\begin{array}{rr}
(X_t - X_t^* \theta)^2 & 0 \\
0 & (X_{t-1})^2
\end{array}\right)
\]
\[
= \frac{1}{n} \sum_{t=1}^{n} \left(\begin{array}{rr}
(X_t - X_t^* \theta)^2 & 0 \\
0 & (X_t - X_t^* \theta)^2
\end{array}\right)
\]
\[
= \frac{1}{n} \sum_{t=1}^{n} \left(\begin{array}{rr}
\varepsilon_t^2 (X_{t-1})^2 & 0 \\
0 & \varepsilon_t^2 (X_{t-1})^2
\end{array}\right)
\]
Therefore, according to the ergodic theorem, Lemma 5.2 is established. \hfill \square

**Lemma 5.3** If (A1) and (A2) hold, then
\[
\max_{1 \leq t \leq n} \|H_t(\theta)\| = o_p(n^2) \quad \text{as} \quad n \to \infty. \tag{14}
\]

**Proof** Based on assumption (A2), we know that \(E(H_t(\theta) H_t^*(\theta)) < \infty\), which implies that
\[
\sum_{n=1}^{\infty} P(H_t(\theta) H_t^*(\theta) > n) < \infty.
\]
Further by assumption (A1), we know that the model is stationary, from which we can conclude that

\[ \sum_{n=1}^{\infty} P(H_n(\theta)H_n^*(\theta) > n) < \infty. \]

By using the Borel–Cantelli lemma, we know that \( \|H_1(\theta)\| > n^{\frac{1}{2}} \) holds for finite \( n \), which implies that \( \max_{1 \leq t \leq n} \|H_t(\theta)\| > n^{\frac{1}{2}} \) holds for finite \( n \). Similarly, we can obtain that for \( \forall \epsilon > 0 \), \( \max_{1 \leq t \leq n} \|H_t(\theta)\| > \epsilon n^{\frac{1}{2}} \) holds for finite \( n \). Thus, Lemma 5.3 is established.

**Lemma 5.4** If (A1) and (A2) hold, then

\[ \|b\| = O_p(n^{-\frac{1}{4}}). \]  \hspace{1cm} (15)

**Proof** Let \( b = \|b\| \bar{\xi} \). By (6), we have

\[
0 = \frac{1}{n} \sum_{t=1}^{n} \bar{\xi}^T H_t(\theta) \frac{1 + b^*(\theta)H_t(\theta)}{1 + b^*(\theta)H_t(\theta)}
\]

\[
= \bar{\xi} \frac{1}{n} \sum_{t=1}^{n} H_t(\theta) \left( 1 - \frac{b^*(\theta)H_t(\theta)}{1 + b^*(\theta)H_t(\theta)} \right)
\]

\[
= \bar{\xi} \frac{1}{n} \sum_{t=1}^{n} H_t(\theta) - \frac{\bar{\xi}^T}{n} \sum_{t=1}^{n} H_t(\theta)H_t^*(\theta)b(\theta)
\]

\[
= \bar{\xi} \frac{1}{n} \sum_{t=1}^{n} H_t(\theta) - \frac{\bar{\xi}^T}{n} \sum_{t=1}^{n} H_t(\theta)H_t^*(\theta) \frac{\|b(\theta)\|}{1 + b^*(\theta)H_t(\theta)}
\]

\[
= \bar{\xi} \frac{1}{n} \sum_{t=1}^{n} H_t(\theta) - \|b(\theta)\| \bar{\xi} \frac{1}{n} \sum_{t=1}^{n} \frac{H_t(\theta)H_t^*(\theta)}{1 + b^*(\theta)H_t(\theta)} \bar{\xi}.
\]

Hence we have

\[
\bar{\xi} \frac{1}{n} \sum_{t=1}^{n} H_t(\theta) = \|b(\theta)\| \bar{\xi} \bar{D}_n \bar{\xi},
\]  \hspace{1cm} (16)

where

\[
\bar{D}_n = \frac{1}{n} \sum_{t=1}^{n} \frac{H_t(\theta)H_t^*(\theta)}{1 + b^*(\theta)H_t(\theta)}.
\]

Let \( D_n = \frac{1}{n} \sum_{t=1}^{n} H_t(\theta)H_t^*(\theta) \). From (5), we can see that \( 1 + b^*(\theta)H_t(\theta) > 0 \). Thus we have

\[
\|b(\theta)\| \bar{\xi} D_n \bar{\xi} \leq \|b(\theta)\| \bar{\xi} \frac{1}{n} \sum_{t=1}^{n} \frac{H_t(\theta)H_t^*(\theta)}{1 + b^*(\theta)H_t(\theta)} \bar{\xi} \left( 1 + \max_{1 \leq t \leq n} b^*(\theta)H_t(\theta) \right)
\]

\[
\leq \|b(\theta)\| \bar{\xi} \bar{D}_n \bar{\xi} \left( 1 + \max_{1 \leq t \leq n} b^*(\theta)H_t(\theta) \right)
\]

\[
\leq \|b(\theta)\| \bar{\xi} \bar{D}_n \bar{\xi} \left( 1 + \|b(\theta)\| \max_{1 \leq t \leq n} H_t(\theta) \right)
\]
\[ \sum_{t=1}^{n} H_t(\theta) \left( 1 + \| b(\theta) \| \max_{1 \leq t \leq n} \| H_t(\theta) \| \right), \]  
which implies that
\[ \left\| b(\theta) \right\| \left( \zeta^T D_n \zeta - \max_{1 \leq t \leq n} \| H_t(\theta) \| \right) \leq \zeta^T \frac{1}{n} \sum_{t=1}^{n} H_t(\theta). \]  
According to Lemma 5.1, we have
\[ \frac{1}{\sqrt{n}} \sum_{t=1}^{n} H_t(\theta) = O_p(1), \]  
which implies that
\[ \zeta^T \frac{1}{n} \sum_{t=1}^{n} H_t(\theta) = O_p(n^{-\frac{1}{2}}). \]  
Further, by Lemma 5.3, we obtain
\[ \max_{1 \leq t \leq n} \| H_t(\theta) \| \zeta^T \frac{1}{n} \sum_{t=1}^{n} H_t(\theta) = \frac{1}{\sqrt{n}} \zeta^T \max_{1 \leq t \leq n} \| H_t(\theta) \| \frac{1}{\sqrt{n}} \sum_{t=1}^{n} H_t(\theta) \]
\[ = \frac{1}{\sqrt{n}} \sigma_p(\frac{1}{n}) O_p(1) \]
\[ = o_p(1). \]  
Note that \( D \) is a positive definite matrix. Thus we have
\[ \zeta^T D_n \zeta \xrightarrow{p} \zeta^T D \zeta > 0 \]  
and
\[ \sigma_{\text{min}} + o_p(1) \leq \zeta^T D_n \zeta \leq \sigma_{\text{max}} + o_p(1), \]  
where \( \sigma_{\text{max}} \) and \( \sigma_{\text{min}} \) are the smallest and the largest eigenvalue of \( D \), respectively. Combining with (18)–(23), we can obtain that
\[ \left\| b(\theta) \right\| \left( \zeta^T D_n \zeta + o_p(1) \right) = O_p(n^{-\frac{1}{2}}). \]  
Combined with (22), we know that \( \left\| b(\theta) \right\| = O_p(n^{-\frac{1}{2}}). \) Lemma 5.4 is established. \( \square \)

**Lemma 5.5** If \((A1)\) and \((A2)\) hold, then
\[ b(\theta) = \left( \sum_{t=1}^{n} H_t(\theta) H_t^2(\theta) \right)^{-1} \sum_{t=1}^{n} H_t(\theta) + B_n, \]  

\[ \sum_{t=1}^{n} H_t(\theta) \left( 1 + \| b(\theta) \| \max_{1 \leq t \leq n} \| H_t(\theta) \| \right), \]
where

\[ B_n = \left( \sum_{t=1}^{n} H_t(\theta)H_t^\tau(\theta) \right)^{-1} \sum_{t=1}^{n} \frac{H_t(\theta)(b^*(\theta)H_t(\theta))^2}{1 + b^*(\theta)H_t(\theta)}, \]  

(26)

and

\[ \|B_n\| = o_p\left(n^{-\frac{1}{2}}\right). \]  

(27)

Proof. By (6), we have

\[
0 = \frac{1}{n} \sum_{t=1}^{n} \frac{H_t(\theta)}{1 + b^*(\theta)H_t(\theta)} = \frac{1}{n} \sum_{t=1}^{n} \frac{H_t(\theta)}{1 + b^*(\theta)H_t(\theta) + \left(\frac{b^*(\theta)H_t(\theta)}{1 + b^*(\theta)H_t(\theta)}\right)^2} \\
= \frac{1}{n} \sum_{t=1}^{n} H_t(\theta) - \frac{1}{n} \sum_{t=1}^{n} H_t(\theta)H_t^\tau(\theta)b(\theta) + \frac{1}{n} \sum_{t=1}^{n} \frac{H_t(\theta)(b^*(\theta)H_t(\theta))^2}{1 + b^*(\theta)H_t(\theta)}.
\]

Thus, (25) can be established.

In what follows, we consider (27). Note that

\[
\|B_n\| = \left\| \left( \sum_{t=1}^{n} H_t(\theta)H_t^\tau(\theta) \right)^{-1} \sum_{t=1}^{n} \frac{H_t(\theta)(b^*(\theta)H_t(\theta))^2}{1 + b^*(\theta)H_t(\theta)} \right\|
\]

\[
= \left\| \left( \frac{1}{n} \sum_{t=1}^{n} H_t(\theta)H_t^\tau(\theta) \right)^{-1} \frac{1}{n} \sum_{t=1}^{n} \frac{H_t(\theta)(b^*(\theta)H_t(\theta))^2}{1 + b^*(\theta)H_t(\theta)} \right\|
\]

\[
\leq \left\| \left( \frac{1}{n} \sum_{t=1}^{n} H_t(\theta)H_t^\tau(\theta) \right)^{-1} \left\| \frac{1}{n} \sum_{t=1}^{n} \frac{H_t(\theta)(b^*(\theta)H_t(\theta))^2}{1 + b^*(\theta)H_t(\theta)} \right\| \right\|
\]

\[
\leq \left\| \left( \frac{1}{n} \sum_{t=1}^{n} H_t(\theta)H_t^\tau(\theta) \right)^{-1} \|b^*(\theta)\|^2 \frac{1}{n} \sum_{t=1}^{n} \frac{\|H_t(\theta)\|^3}{1 + b^*(\theta)H_t(\theta)} \right\|
\]

\[
\leq O_p(1)O_p\left(n^{-\frac{1}{2}}\right)O_p\left(n^{-\frac{1}{2}}\right)O_p(1)
\]

\[
= o_p\left(n^{-\frac{1}{2}}\right).
\]

(28)

So (27) holds. \[\square\]

Lemma 5.6 If (A1) and (A2) hold, then

\[
-2\log(L(\theta)) = \left( \sum_{t=1}^{n} H_t(\theta) \right)^\tau \left( \sum_{t=1}^{n} H_t(\theta)H_t^\tau(\theta) \right)^{-1} \sum_{t=1}^{n} H_t(\theta)
\]

\[- B_n^\tau \sum_{t=1}^{n} H_t(\theta)H_t^\tau(\theta)B_n + 2 \sum_{t=1}^{n} \eta_t, \]

(29)
where

\[ B_n^t \sum_{t=1}^{n} H_t(\theta)H_t^\tau(\theta)B_n = o_p(1), \quad (30) \]

\[ \sum_{t=1}^{n} \eta_t = o_p(1). \quad (31) \]

**Proof** We expand

\[-2 \log(L(\theta)) = 2 \sum_{t=1}^{n} \log(1 + b^\tau(\theta)H_t(\theta))\]

\[ = 2 \sum_{t=1}^{n} b^\tau(\theta)H_t(\theta) - \sum_{t=1}^{n} \{b^\tau(\theta)H_t(\theta)\}^2 + 2 \sum_{t=1}^{n} \eta_t \]

\[ = 2 \left( \left( \sum_{t=1}^{n} H_t(\theta)H_t^\tau(\theta) \right)^{-1} \sum_{t=1}^{n} H_t(\theta) + B_n \right) \sum_{t=1}^{n} H_t(\theta) \]

\[ - \left( \left( \sum_{t=1}^{n} H_t(\theta)H_t^\tau(\theta) \right)^{-1} \sum_{t=1}^{n} H_t(\theta) + B_n \right) \left( \sum_{t=1}^{n} H_t(\theta)H_t^\tau(\theta) \right) \]

\[ \times \left( \left( \sum_{t=1}^{n} H_t(\theta)H_t^\tau(\theta) \right)^{-1} \sum_{t=1}^{n} H_t(\theta) + B_n \right) \]

\[ + 2 \sum_{t=1}^{n} \eta_t. \quad (32) \]

After a simple algebraic operation, we know that (29) holds.

Next, we consider (30). Note that

\[ \| B_n^t \sum_{t=1}^{n} H_t(\theta)H_t^\tau(\theta)B_n \| \leq \| B_n^t \| \left\| \sum_{t=1}^{n} H_t(\theta)H_t^\tau(\theta) \right\| \| B_n \| \]

\[ = o_p(n^{-\frac{1}{2}})o_p(n^{-\frac{1}{2}})O_p(n) = o_p(1), \quad (33) \]

which implies that (30) holds.

Last, we consider (31). For this, we first prove that there exists a finite real number \( Q > 0 \) such that

\[ P(|\eta_t| \leq Q|b^\tau(\theta)H_t(\theta)|^3, 1 \leq t \leq n) \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty. \quad (34) \]

Consider the third-order Taylor expansion of \( \log(1 + x) \) at \( x = 0 \):

\[ \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \varphi(x), \]

where \( \varphi(x) \rightarrow 0 \) as \( x \rightarrow 0 \). Therefore, there exists \( \iota > 0 \) such that, for any \( |x| < \iota, \frac{\varphi(x)}{x^3} < \frac{1}{6} \).

In addition, note that

\[ \max_{1 \leq t \leq n} \| b^\tau(\theta)H_t(\theta) \| = o_p(1). \]
Therefore, we have
\[
\lim_{n \to \infty} P \left( \max_{1 \leq t \leq n} |b^\tau(\theta)H_t(\theta)|^3 < \kappa^3 \right) = 1.
\]

Let \( A_n = \{ \omega : \max_{1 \leq t \leq n} |b^\tau(\theta)H_t(\theta)|^3 < \kappa^3 \} \). It is easy to prove that, for any \( \omega \in A_n \) and \( 1 \leq t \leq n \),
\[
\frac{\eta_t}{|b^\tau(\theta)H_t(\theta)|^3} = \frac{|b^\tau(\theta)H_t(\theta)|^3 + \sigma(b^\tau(\theta)H_t(\theta))}{|b^\tau(\theta)H_t(\theta)|^3} \leq \frac{1}{3} + \frac{1}{6} = \frac{1}{2}.
\]
Thus we have
\[
P(|\eta_t| \leq Q|b^\tau(\theta)H_t(\theta)|^3, 1 \leq t \leq n) \to 1 \quad \text{as } n \to \infty,
\]
where \( Q = \frac{1}{2} \). This implies that
\[
\left\| \sum_{t=1}^{n} \eta_t \right\| \leq Q \left\| b(\theta) \right\|^3 \left\| \sum_{t=1}^{n} H_t(\theta) \right\|^3 \\
\leq O_p(n^{-\frac{3}{2}})O_p(n^{\frac{3}{2}}) = O_p(1).
\]

\textbf{Proof of Theorem 2.1} By Lemma 5.6, we can conclude that \( -2 \log(L(\theta)) \) and \( (\sum_{t=1}^{n} H_t(\theta) (\sum_{t=1}^{n} H_t(\theta)H_t^\tau(\theta))^{-1} \sum_{t=1}^{n} H_t(\theta)) \) have the same limit distribution. By Lemma 5.1 and Lemma 5.2, we can conclude that
\[
\sum_{t=1}^{n} H_t^\tau(\theta) \left( \sum_{t=1}^{n} H_t(\theta)H_t^\tau(\theta) \right)^{-1} \sum_{t=1}^{n} H_t(\theta) \overset{d}{\to} \chi^2(2).
\]
Hence Theorem 2.1 holds.

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The data used to support the findings of this study are available from the corresponding author upon request.

\textbf{Competing interests}
The authors declare that they have no competing interests.

\textbf{Authors’ contributions}
All authors contributed equally and significantly in this manuscript, and they read and approved the final manuscript.

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