TWO EXAMPLES OF SURFACES WITH NORMAL CROSSING SINGULARITIES

JÁNOS KOLLÁR

Let $S$ be a surface over $\mathbb{C}$ with only normal crossing singularities, abbreviated as $\text{nc}$. That is, each point of $S$ is analytically isomorphic to one of 3 local models: smooth point $(x = 0) \subset \mathbb{C}^3$, double nc $(xy = 0) \subset \mathbb{C}^3$ or triple nc $(xyz = 0) \subset \mathbb{C}^3$.

The normalization $n : \bar{S} \to S$ is smooth and the preimage of the singular locus $D \subset S$ is a nc curve $\bar{D} \subset \bar{S}$. The dualizing sheaf (or canonical line bundle) $\omega_S$ is locally free and $n^*\omega_S \cong \omega_{\bar{S}}(\bar{D})$.

The aim of this note is to give examples of nc surfaces whose canonical line bundle exhibits unexpected behavior.

**Proposition 1.** There is an irreducible, projective, nc surface $T_1$ of general type given in (5) whose canonical ring
$$\sum_{m \geq 0} H^0(T_1, \omega_{T_1}^m)$$
is not finitely generated.

**Proposition 2.** There is an irreducible, nc surface $T_2$ given in (6) such that $\omega_{T_2}$ is not ample yet its pull back to the normalization $n^*\omega_{T_2}$ is ample.

The latter answers in the negative a problem left unresolved in [?], III.2.6.2 and posed explicitly in [?], 1.12.

I found both of these examples while studying the minimal model program for semi-log-canonical surfaces. The key observation is that the minimal model program leads to singularities that satisfy the numerical conditions of log canonicity, yet no reflexive power of their dualizing sheaf is locally free. The pluricanonical forms behave unexpectedly near such singularities, and this lies at the heart of both of the examples.

Semi-log-canonical surfaces naturally appear as semi-stable limits of smooth surfaces of general type. The surface $T_1$ does not arise this way, but, as far as I know, there could be examples of nc surfaces which are smoothable yet whose canonical ring is not finitely generated. Indeed, if $g : X \to (c \in C)$ is such a family with nc fiber $X_c$, then the relative minimal model program produces $g^m : X^m \to C$ such that $K_{X^m/C} = g^m$-nef, hence the canonical ring of the central fiber $X^m_c$ is finitely generated. Even if $X \dasharrow X^m$ does not contract any divisor, and hence $X_c$ is birational to $X^m_c$, the canonical ring of $X_c$ can be different from the canonical ring of $X^m_c$. The reason is that flips in $X \dasharrow X^m$ may correspond to blow ups $X_c \leftarrow X^m_c$. Even for normal log canonical surfaces $(S, \Delta)$, the canonical ring is a birational invariant only if we declare that all new curves appear in the boundary $\Delta$ with coefficient 1. As we go from $X_c$ to $X^m_c$, the coefficients of the new curves are dictated by the 3-fold $X$ and are typically less than 1. Thus all we can assert is that the canonical ring of $X^m_c$ is a subring of the canonical ring of $X_c$. 

1
3 (Gluing along curves). Let $S$ be a surface, $C \subset S$ a curve and $g : C \to C'$ a finite morphism which is locally analytically a closed immersion. (Note that this condition holds in the nc case.)

For each $c' \in C'$ glue the local branches of $S$ as dictated by $g : C \to C'$ and finally glue this to $S \setminus C$. The resulting surface is denoted by $S/(g)$. Let $n : S \to S/(g)$ be the corresponding morphism. If $S$ is normal then $n$ is the normalization of $S/(g)$.

There is no problem doing this as a complex analytic space, but the (quasi)projectivity of $S/(g)$ can be quite tricky. We use the following simple criterion:

Claim. Assume that $S$ is projective and there is an ample divisor $H$ such that $H$ intersects $C$ transversally and $H \cap C = g^{-1}(g(H \cap C))$. Then $n(H)$ is a Cartier divisor on $S$ and $n(H)$ is ample by [?, Exc III.5.7].

The existence of $S$ is a very special case of a general gluing result [?].

Let $X$ be a scheme, $Y \subset X$ a closed subscheme and $f : Y \to Y'$ a finite morphism. Then there is a unique $F : X \to X'$ such that $F|_Y$ factors through $f$ and $F$ is maximal with this property. In general, $X'$ is only an algebraic space.

4 (Computing sections of $\omega^m_S$). Let $S$ be a reduced surface and $Z \subset S$ a finite set of points such that $S \setminus Z$ has only smooth and double nc points. As usual, $\omega^m_S$ denotes the double dual of $\omega_S^m$.

Let $n : \tilde{S} \to S$ denote the normalization and $\tilde{Z} := n^{-1}(Z)$. Then $\tilde{S} \setminus \tilde{Z}$ is the normalization of $S \setminus Z$. Let $D \subset S$ be the singular locus and $\tilde{D} := n^{-1}(D)$ its preimage in $\tilde{S}$. $D \setminus \tilde{Z}$ is a smooth curve and there is an involution $\sigma : D \setminus \tilde{Z} \to D \setminus \tilde{Z}$ such that $\sigma(D \setminus \tilde{Z}) = (D \setminus \tilde{Z})/\sigma$.

We say that $S$ is obtained from $\tilde{S}$ by the gluing $\sigma$. Note that $\sigma$ determines $\tilde{D} \to D$ only on a dense open set. If we assume in addition that $S$ satisfies Serre’s condition $S_2$, then $\tilde{S}, \tilde{D}$ and $\sigma$ determine $S$ uniquely.

From this description it is easy to compute the pluricanonical sections:

$$H^0(S, \omega^m_S) = \{ s \in H^0(\tilde{S} \setminus \tilde{Z}, \omega^m_{\tilde{S}\setminus\tilde{Z}}(m\tilde{D})) : s|_{\tilde{D}\setminus\tilde{Z}} \text{ is } ((-1)^m\sigma)\text{-invariant} \}.$$ (See (71) about the sign $(-1)^m$.)

Example 5. Let $A$ be an elliptic curve with 4 distinct points $p_1, p_2, q_1, q_2 \in A$ such that $p_1 + p_2 \sim q_1 + q_2$.

Let $f : S \to P$ be a genus 2 (irrational) pencil such that $\omega_S$ is ample and there are 2 fibers $F_p \cong A/(p_1 \text{ identified with } p_2)$ and $F_q \cong A/(q_1 \text{ identified with } q_2)$.

Let $S_1 \to S$ be obtained by blowing up $q_1, q_2 \in F_p$ and $p_1, p_2 \in F_q$. The corresponding exceptional curves are $E_{q_1}, E_{q_2}, E_{p_1}, E_{p_2}$.

Set $B := A/(p_1 \text{ identified with } p_2, q_1 \text{ identified with } q_2)$ and let $C'$ be the union of the 2-nodal curve $B$ plus transversal copies $\mathbb{P}^1_{p_i}, \mathbb{P}^1_{q_i}$ through the nodes.

Let $C := F_p + F_q + E_{p_1} + E_{p_2} + E_{q_1} + E_{q_2}$ and define $g : C \to C'$ to be the identity of $A$ on $F_p$ and on $F_q$ and isomorphisms $E_{p_i}, E_{q_i}, \mathbb{P}^1_{p_i}, \mathbb{P}^1_{q_i}$.

Set $T_1 := S_1/(g)$. $T_1$ has 2 triple points at the 2 nodes of $B$.

Example 6. Let $C := (z^2 = x^6 + 2y^6) \subset \mathbb{P}(1, 1, 3)$ and $E := (z^2 = xy(x^2 + y^2)) \subset \mathbb{P}(1, 1, 2)$ with (hyper)elliptic involutions $\tau_C, \tau_E$. Let $p \in E$ denote $(0:1:0)$ and $q \in E$ denote $(1:0:0)$; both are fixed by $\tau_E$.

Set $S := C \times E/(\tau_C, \tau_E)$. Consider the curves $D_p := C \times p/(\tau_C, \tau_E) \cong \mathbb{P}^1_{xy}$ and $D_q := C \times q/(\tau_C, \tau_E) \cong \mathbb{P}^1_{xy}$.
Let \( \sigma : D_p \to D_q \) be the isomorphism which sends \((x,y)\) to \((y,x)\) and let \( T := S/\langle \sigma \rangle \) be the surface obtained by gluing \( D_p \) to \( D_q \) using \( \sigma \). A key property is that \( \sigma \) maps nodes to smooth points.

The surface \( S \) has 12 ordinary double points, let \( Z \subset T \) be their images. The surface we are looking for is \( T_2 := T \setminus Z \).

We start the proofs by the key local computations. It is then easy to read off the required global properties.

7 (Local computation 1). Let \( C_1 := (xy = 0) \subset \mathbb{C}^2 \) := \( S_1 \). Let \( C_{21} := (u_1 = 0) \subset \mathbb{C}^2 \) and \( C_{22} := (v_2 = 0) \subset \mathbb{C}^2 \) := \( S_{22} \). Set \( S_2 := S_{21} \coprod S_{22} \) and \( C_2 := C_{21} \coprod C_{22} \).

The gluing is defined by \( \sigma : C_1 \setminus (0,0) \to C_2 \) sending \((0,y) \mapsto (0,y) \in C_{21}\) and \((x,0) \mapsto (x,0) \in C_{22} \).

Note that \( T := (S_1 \coprod S_2) / \sigma \) is not a nc surface. Rather, it has a triple point with local finite generation fails since the \( \mathcal{O} \). A local model is given by \((t_1 = t_2 = 0) \cup (t_2 = t_3 = 0) \cup (t_3 = t_4 = 0) \subset \mathbb{C}^4 \).

The isomorphism is given by \((x,y) \mapsto (0,x,y,0), (u_1, v_1) \mapsto (v_1, u_1, 0, 0)\) and \((u_2, v_2) \mapsto (0,0, u_2, v_2) \).

A local generator of \( \omega_{S_{21}}(C_{21}) \) is \( u_1^{-1} dv_1 \wedge dv_1 \), and the restriction \( \omega_{S_{21}}(C_{21})|_{C_{21}} = \omega_{C_{21}} \) is given by the Poincaré residue map

\[
\frac{df}{f} \wedge dg|_{(f=0)} = dg|_{(f=0)}.
\]

Thus \( \omega_{S_{21}}(mC_{21})|_{C_{21}} = (dv_1)^m \cdot \mathcal{O}_{C_{21}} \). The situation on \( C_{22} \) is similar.

On the other hand, a local generator of \( \omega_{S_1}(C_1) \) is \((xy)^{-1} dx \wedge dy \). Its restriction to \( C_1 \) gives a local generator \( \eta \) of \( \omega_{C_1} \). Note that

\[
\eta|_{(y=0)} = -\frac{dx}{x} \quad \text{and} \quad \eta|_{(x=0)} = \frac{dy}{y}.
\]

Thus

\[
\omega_{S_1}^m(mC_1)|_{C_1} = \eta^m \cdot \mathcal{O}_{C_1}.
\]

The interesting feature appears when we compute that

\[
\sigma^*(dv_1)^m = y^m \cdot \eta|_{(x=0)} \quad \text{and} \quad \sigma^*(du_2)^m = (-x)^m \cdot \eta|_{(y=0)}.
\]

Thus a local section of \( \omega_{S_1}^m(mC_1) \) satisfies the gluing condition (4.1) iff it is contained in

\[
(xy, x^m, y^m) \cdot \left( dx \wedge dy \right)^m.
\]

Local finite generation fails since the \( \mathcal{O}_{S_1} \)-algebra

\[
\sum_{m \geq 0} (xy, x^m, y^m) \cdot W^m \subset \mathbb{C}[x, y, W] \quad \text{is not finitely generated},
\]

where \( W \) is a formal variable (or weight) taking care of the grading. Indeed, for every \( m \), the element \( xy \cdot W^m \) needs to be added as a new generator.

8 (Proof of (4.1)). As we noted,

\[
n^* \omega_T = \omega_{S_1}(F_p + F_q + E_{p_1} + E_{p_2} + E_{q_1} + E_{q_2}),
\]

this line bundle has negative degree along the 4 curves \( E_{q_1}, E_{q_2}, E_{p_1}, E_{p_2} \).
That is, the surface \((S_1, F_p + F_q + E_{p_1} + E_{p_2} + E_{q_1} + E_{q_2})\) is not log-minimal. Its log-minimal model is \((S, F_p + F_q)\). Let \(T\) be the surface obtained from \(S\) by gluing \(F_p\) to \(F_q\) by the identity of \(A\). (Note that both \(F_p\) and \(F_q\) are birational to \(A\).) \(T\) is singular along the 2-nodal curve \(B\). At the nodes \(P, Q\) of \(B\) we get a singularity as in \((7)\).

Thus, instead of thinking of \(\sum_{m \geq 0} H^0(T_1, \omega_{T_1}^m)\) as a subalgebra of
\[
\sum_{m \geq 0} H^0(S_1, \omega_{S_1}^m(m(F_p + F_q + E_{p_1} + E_{p_2} + E_{q_1} + E_{q_2})))
\]
we work with
\[
\sum_{m \geq 0} H^0(T, \omega_T^m) = \sum_{m \geq 0} H^0(T_1, \omega_{T_1}^m),
\]
and use the representation
\[
H^0(T_1, \omega_{T_1}^m) = \{ s \in H^0(S, \omega_S^m(m(F_1 + F_2)) : s|_{F_p + F_q} \text{ is } (-1)^m-s\text{-invariant} \}.
\]

Near the two triple points \(P, Q \in T\), we are in the situation described in \((7)\). In particular, we know that the \(\mathcal{O}_T\)-algebra \(\sum_{m \geq 0} \omega_T^m\) is not finitely generated, not even locally near \(P\) or \(Q\).

To go from the local infinite generation to global infinite generation we consider the natural map
\[
\rho : \sum_{m \geq 0} H^0(T, \omega_T^m) \to \sum_{m \geq 0} \omega_T^m.
\]
Assume that for all \(m \gg 1\) there are global sections \(t_m \in H^0(T, \omega_T^m)\) such that \(\rho(t_m)\) is not contained in the subalgebra generated by the \(\omega_T^i\) for \(i < m\). Then \(t_m\) is not contained in the subalgebra generated by the \(H^0(T, \omega_T^i)\) for \(i < m\), hence \(\sum_{m \geq 0} H^0(T, \omega_T^m)\) is not finitely generated.

Since \(\omega_S\) is ample and \(F_p, F_q\) are nef, we see that \(\omega_S^m(mF_p + mF_q)(-F_p - F_q)\) is globally generated for \(m \gg 1\). Sections of \(\omega_S^m(mF_p + mF_q)(-F_p - F_q)\) vanish along \(F_p + F_q\), hence they automatically glue and descend to sections of \(\omega_T^m\).

Thus if \(s_m \in H^0(S, \omega_S^m(mF_p + mF_q))\) vanishes along \(F_p + F_q\) with multiplicity 1, then we obtain a corresponding \(t_m \in H^0(T, \omega_T^m)\) which, up to a unit, equals \(xy \cdot W^m\) in \((7)\). Thus \(\sum_{m \geq 0} H^0(T_1, \omega_{T_1}^m)\) is not finitely generated.

Finally, \(T_1\) is projective. To see this first that \(\omega_{T_1}^{-1}\) is relatively ample on \(T_1 \to T\). Thus it is enough to prove that \(T\) is projective.

Note that the pull back of \(\omega_{F_p}\) to \(A\) is \(\mathcal{O}_A(p_1 + p_2)\) and the pull back of \(\omega_{F_q}\) to \(A\) is \(\mathcal{O}_A(q_1 + q_2)\). We assumed that \(p_1 + p_2 \sim q_1 + q_2\), thus there is a divisor \(H \in |\omega_S \otimes f^*(\text{very ample})|\) which intersects \(F_p\) and \(F_q\) transversally in points which are interchanged by \(s\). As noted in \((3)\), \(H\) then descends to an ample divisor on \(T\).

\[\square\]

9 (Local computation 2). Let \(C_1 := (y = 0) \subset \mathbb{C}^2_{x,y} =: S_1\). Let \(C_2 := (v = w = 0) \subset (uv - w^2 = 0) =: S_2 \subset \mathbb{C}^2_{u,v,w}\).

The gluing is defined by \(\tau : C_2 \to C_1\) sending \((u,0,0) \mapsto (u,0) \in C_1\). As in \((7)\), we find that
\[
\omega_{S_1}^m(mC_1)|_{C_1} = (dx)^m \cdot \mathcal{O}_{C_1}.
\]
Let us now glue $S$. Hence, we get that $\omega_S|_{C_2}$ is ample. Finally, is projective since it has a finite map to $P^1$. Since $D := n(D_p) = n(D_q)$ we have 12 points whose local models are as in (9).

Using (11) we see that every global section of $\omega_D^{2m}$ restricted to $D \cap T_2$ is also the restriction of a global section of $\omega_T^{2m}$ to $D \cap T_2$. Since $D \cong P^1$, every global section of $\omega_T^{2m}$ vanishes along the double curve $D$. Thus $\omega_T^2$ is not ample. In fact, $\omega_T^2$ is not even semi- or weakly-positive in any sense.

Finally, $T$ is projective since it has a finite map to $P^1 \times P^1$ given by

$$(x_1:y_1:z_1), (x_2:y_2:s_2) \mapsto ((x_1y_1,x_1y_1^2+y_1^2), (x_2y_2,x_2^2+y_2^2)).$$

Note 11. The explicit computation in (10) is a special case of the following general result:

Let $X$ be a reduced, $S_2$ surface and $F$ a rank 1 sheaf on $X$. Then the $O_X$-algebra $\sum_{m \geq 0} F[m]$ is finitely generated iff $F[m]$ is locally free for some $m > 0$.

It seems that the minimal model of a typical nc surface has such singularities and its canonical ring is not finitely generated.

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Princeton University, Princeton NJ 08544-1000
kollar@math.princeton.edu