Determining Static Stresses of Deformed Solitons

V. D. Tsukanov

Institute of Theoretical Physics, National Science Center
"Kharkov Institute of Physics and Technology"
61108, Kharkov, Ukraine

An equation for the quasi-static soliton ansatz depending on an arbitrary set of collective variables is covariantly derived on the basis of the variational approach to the method of collective variables. The field configuration and the static stresses of the deformed $\Phi^4$-kink that are produced by the excitation of the internal soliton mode are exactly determined. The kink interaction potential at large distances is considered for the example of the nonlinear Klein-Gordon system. A general approach to the problem of exactly determining the intersoliton potential for the entire set of physically admissible two-soliton configurations is discussed.

1 Introduction

The dynamic questions in solving some soliton problems are relegated to the background, at least, at the initial stage, and the problem of determining the static stresses in the system becomes the key point in the investigation. This problem directly reduces to finding quasi-static soliton configurations described using a set of non-Goldstone collective coordinates. In this context, the problem of determining the intersoliton potential in various models can be mentioned. The non-Goldstone collective coordinates in these problems are the separation parameter and also the variables describing the relative orientation of the soliton pair in the internal space in multidimensional cases. Furthermore, the non-Goldstone coordinates, which in essence are nonlinear amplitudes, describe the dynamics of internal soliton modes. The problem of determining the quasi-static configurations in these problems has been repeatedly discussed, and the ways to solve it were related to various model constructions. In this connection, we mention the investigations of the skyrmion interaction begun as early as 1962 [1, 2, 3, 4, 5]. Moreover, the soliton interaction was also studied for some other models [6, 7, 8], where some important properties of intersoliton potentials were established. At the same time, it is clear that an approach based on model semiphenomenological notions does not ensure a comprehensive investigation of the problem, for which a consistent theoretical scheme is required.

Touching upon this problem, we indicate some difficulties related to introducing non-Goldstone collective coordinates. For example, the notion of the distance between extended objects undergoing mutual deformation acquires its unambiguous
visual meaning only if these objects are at sufficiently large distances from each other. Therefore, a consistent theory must not only describe the process of deformation of solitons as they approach each other but also answer the question of interpreting the ambiguity in determining the intersoliton distance at the stage where the solitons lose their identity as a result of this deformation. The difficulties in constructing a theoretical scheme suitable for describing intersoliton interaction that are due to the above-mentioned factor have been extensively mentioned in the literature and were most distinctly stated in [9]. Some simple considerations were later presented in [10] in favor of the universal variational approach to the method of collective variables, which permits stating various problems of soliton dynamics in the framework of the Hamiltonian procedure. In [10], the general equation for quasi-static configurations determined by an arbitrary set of collective coordinates was also derived. It is essential that the resulting equation is invariant under reparameterizing the collective variables, which ensures that the physical quantities are independent of the way in which the intersoliton distance is defined.

In Sec. 2, we present a new covariant method for deriving an equation for the quasi-static ansatz using basic notions related to Riemannian manifolds. The following two sections are devoted to elaborating solution methods for this equation in some specific cases. In Sec. 3, the problem of determining the configurations and static stresses of the $\Phi^4$-kink whose deformation is produced by the excitation of the internal soliton mode is considered. This problem can be solved exactly. Similar problems were considered in many papers. The corresponding references can be found in [11, 12]. It is unlikely that there is a model with a nontrivial potential that admits an exact solution of the problem of determining the intersoliton interaction. A solution of this problem that allows calculating an asymptotic expression for the intersoliton interaction can most likely be found for an arbitrary model. An example of a calculation of this kind is given in Sec. 4 where the equation for the quasi-static ansatz is used to determine the kink interaction for the nonlinear Klein-Gordon equation in the domain of large intersoliton distances. We stress that, apart from independent interest, knowing the intersoliton potential in the asymptotic domain permits introducing a physically acceptable gauge in the numerical analysis of the equation throughout the range of intersoliton distances. In Sec. 5, we discuss the investigation results and indicate the main elements of the suggested theoretical scheme.

2 Equation for the quasi-static ansatz

We consider the question of finding the quasi-static ansatz for a soliton system described by an arbitrary set of collective variables. Let

$$L = \sum g_{ik} \dot{q}^i \dot{q}^k - H(q, 0)$$

be a nondegenerate Lagrangian quadratic with respect to velocities that admits soliton solutions, let $q^i$ be the field coordinates, and let $i \equiv \{i, x\}$ be the set of discrete
indices and spatial coordinates. It is convenient to interpret the basic configuration space as a Riemannian manifold with a metric \( g_{ik} \) generated by the kinetic term. In this case, if \( Q \) are collective variables, then the quasi-static ansatz \( q^i_c(Q) \) describing the deformed soliton state can be interpreted as a submanifold parameterized by the variables \( Q \). We assume that this ansatz is determined by the stationarity condition for the Hamilton function defined on \( q^i_c(Q) \) for the system in question relative to arbitrary infinitesimal variations \( \delta q^i_c(Q) \) orthogonal to the manifold surface, i.e.,

\[
\frac{\partial H(q_c,0)}{\partial q^i_c} \delta q^i_c = 0, \tag{1}
\]

where the variations \( \delta q^i_c(Q) \) satisfy the conditions

\[
g_{ik}(q_c) \frac{\partial q^i_c(Q)}{\partial Q} \delta q^k_c(Q) = 0.
\]

To cancel the variations in Eq. (1), we introduce the operator \( \mathcal{P} \) of projection onto the subspace tangent to the submanifold \( q^i_c(Q) \) at the point \( Q \)

\[
\mathcal{P}^i_k = g_{kl}(q_c) \frac{\partial q^l_c}{\partial Q} g^{QQ} \frac{\partial q^i_c}{\partial Q}, \quad \mathcal{P}^2 = \mathcal{P}, \tag{2}
\]

where

\[
g_{QQ} = \frac{\partial q^i_c}{\partial Q} g_{ik}(q_c) \frac{\partial q^k_c}{\partial Q}
\]

is the metric on the submanifold \( q^i_c(Q) \) and \( g^{QQ} \) is the matrix inverse to \( g_{QQ} \). Because the variations \( \delta q^i_c(Q) \) in the orthogonal directions to \( q^i_c(Q) \) are arbitrary, Eq. (1) can then be brought to the form

\[
(1 - \mathcal{P}) \frac{\partial H(q_c,0)}{\partial q^i_c} = 0.
\]

Substituting expression (2) for the operator \( \mathcal{P} \) in this formula, we obtain

\[
\frac{\partial H(q_c,0)}{\partial q^i_c} = g_{ik}(q_c) \frac{\partial q^k_c}{\partial Q} g^{QQ} \frac{\partial E(Q)}{\partial Q}, \tag{3}
\]

where \( E(Q) \equiv H(q_c(Q),0) \) is the energy of the quasi-static configuration. By the definition of the tensor \( g^{QQ} \), the right-hand side of Eq. (3), which takes the existence of static stresses in the collective subsystem into account, is invariant under an arbitrary nondegenerate change of variables \( Q \to Q'(Q) \). This indicates that the variational procedure under consideration is covariant. In this connection, we note that in relation to the problem of determining two-soliton configurations, for example, Eq. (3) exactly describes these configurations in the domain of distances where the deformed solitons lose their identity. The choice of the gauge only determines the specific form of the parameterization describing the configurations. For
convenience, it can be required that the separation parameter coincide with the natural intersoliton distance if this distance notably exceeds the soliton size. We note that if the collective coordinates are Goldstone variables, i.e., if the coordinates $Q$ coincide with the degeneration parameters of vacuum solutions, then the energy $E$ does not depend on $Q$, and Eq. (3) becomes the equation $\partial H(q_c,0)/\partial q_c^i = 0$ for static solutions.

3 Static stresses of the deformed $\Phi^4$-kink.

We use Eq. (3) to determine the configuration of the deformed kink in the $\Phi^4$ model. The Lagrangian of the model and the corresponding static kink solution $u_c(x - X)$ defined to within the spatial coordinate $X$ have the forms

$$L = (1/2) \int dx \left( \Phi^2(x) - \Phi'^2(x) - (\Phi^2(x) - 1)^2 \right), \quad u_c(x) = \tanh x. \quad (4)$$

The eigenfunctions of the operator $\mathcal{L}$ describing linear fluctuations on the background of the kink solution include the mode $\sigma(x)$ belonging to the discrete-spectrum states,

$$\mathcal{L}\sigma(x) = 3\sigma(x), \quad \sigma(x) = \sqrt{3/2} \frac{\tanh x}{\cosh x}, \quad \mathcal{L} = -\frac{\partial^2}{\partial x^2} + 4 - 6 \cosh^{-2} x. \quad (5)$$

In describing nonstationary states, it is usually assumed that the $\Phi^4$-kink possesses an internal degree of freedom whose small oscillations can be identified with the mode $\sigma(x)$. Therefore, apart from the translational coordinate $X$, the set of collective coordinates describing the kink state must also include the variable $\tau$ related to the excitation of the internal mode of the $\Phi^4$-kink. For this set of collective coordinates, we can expect that the directing vectors $\Phi'_c \equiv \partial \Phi_c/\partial x$ and $\Phi_{c\tau} \equiv \partial \Phi_c/\partial \tau$ constructed on the solutions of Eq. (3) are orthogonal in the tangent space,

$$\int dx \Phi'_c(x)\Phi_{c\tau}(x) \equiv < \Phi'_c \Phi_{c\tau} > = 0. \quad (6)$$

Also, taking into account that the energy is independent of the cyclic variable, we can represent Eq. (3) in the case under consideration as

$$-\Phi''_c(x) + 2\Phi'_c(x)(\Phi^2_c(x) - 1) = < \Phi'^2_c \Phi_{c\tau} >^{-1} \frac{\partial E}{\partial \tau}. \quad (7)$$

The right-hand side of Eq. (7) is invariant under the reparameterization $\tau \rightarrow \tau'(\tau)$. If the collective variable is fixed using the gauge

$$- < \Phi'^2_{c\tau} >^{-1} \partial E/\partial \tau = 1, \quad (8)$$

then Eq. (7) becomes the nonlinear diffusion equation

$$\frac{\partial \Phi_c(x)}{\partial \tau} - \Phi''_c(x) + 2\Phi'_c(x)(\Phi^2_c(x) - 1) = 0. \quad (9)$$
Equation (9) was investigated in [13], where its exact solutions were found. In particular, the solutions of the type of a deformed kink that tend to $+1$ as $x \to +\infty$ can be written in the form

$$\Phi_c(x, \tau) = \frac{e^{2x} - 1}{e^{2x} + 1 \pm e^{x-3\tau}}. \quad (10)$$

Integrating, we can easily verify that these solutions satisfy orthogonality condition (3) and gauge condition (8). We now note that the physical domain of small deformations, where solutions (10) only slightly differ from static kink solution (9), corresponds to $\tau \to \infty$. In other words, the collective variable $\tau$ defined by gauge condition (8) is inconvenient from the physical standpoint. It plays only an intermediate role as it permits representing Eq. (7) in the form of a nonlinear diffusion equation. It is therefore advisable to combine the two solutions in (10) to form one solution,

$$\Phi_c(x, q) = \frac{e^{2x} - 1}{e^{2x} + 1 - \sqrt{6} q e^x} \approx u_c(x) + q\sigma(x) + O(q^2), \quad (11)$$

where the new collective variable $q$, which takes both positive and negative values, has the meaning of the amplitude of the internal mode under small deformations [10]. The configuration energy $E(q) \equiv H(\Phi_c, 0)$ determining the static stresses of the deformed kink has the form

$$E(q) = \frac{4}{3} + \frac{q^2}{1-q^2} - \frac{q^3}{(1-q^2)^{3/2}} \arccos q \approx \frac{4}{3} + \frac{3}{2} q^2 + O(q^3), \quad \sqrt{3}=\sqrt{3/2} q.$$ 

Here, the first term is the kink mass, and the quadratic term with respect to $q$ describes the potential of a harmonic oscillator with the internal mode frequency $\omega^2 = 3$. The range of the collective variable where the energy remains real is confined to the interval $-\sqrt{2/3} < q < +\sqrt{2/3}$. Solution (11) has no singularities with respect to the spatial coordinate in this range of $q$. We also stress that the choice of the physical collective variable tending to the amplitude of the internal mode in the limit of small deformations is ambiguous. Namely, it is determined for convenience and is connected with explicit form (10) of the solution for the deformed kink in the situation in question.

4 Kink interaction.

An important role in studying the properties of soliton states is played by the solutions of Eq. (9) that are defined by the asymptotics $\Phi_c(x) \to -1$ as $x \to \pm\infty$. These solutions describe the kink-antikink configurations and permit establishing the properties of the intersoliton interaction. In this case, the collective variable relates to the separation parameter $r$, $2r$ is the intersoliton distance. The excitations of internal modes are not taken into account. Unfortunately, it seems that exact solutions of Eq. (9) with given boundary conditions have not yet been found.
Approximate solutions in the domain of large distances can be obtained in the general form if the corresponding equation

$$-\Phi''(x) + U'(\Phi_c(x)) = \frac{\partial \Phi_c(x)}{\partial r} < \Phi^2_{cr} >^{-1} \frac{\partial E}{\partial r},$$

for the nonlinear Klein-Gordon model is considered instead of (7). Here, $U(\Phi)$ is the potential of the model. If the origin is placed at the center of the bisoliton pair and its symmetry $\Phi_c(x, r) = \pm \Phi_c(-x, r)$ is taken into account, then the entire configuration can be described by considering the domain $x \leq 0$. The even functions describe the kink-antikink states and the odd ones describe the kink-kink states (in the case of a periodic potential $U(\Phi)$). If the intersoliton distance is large, then to the left of the point $x = 0$, the ansatz $\Phi_c(x, r)$ only slightly differs from the one-kink solution $u_c(x + r)$ of the static Klein-Gordon equation

$$-u''_c(x) + U'(u_c(x)) = 0.$$

Therefore, representing the ansatz $\Phi_c(x, r)$ in the form

$$\Phi_c(x, r) = u_c(x + r) + \eta(x + r, r), \quad x \leq 0,$$

where $\eta(x + r, r)$ tends to zero as $r \to \infty$, and taking into consideration that in the domain of large values of $r$,

$$< \Phi^2_{cr} > \approx 2 \int_{-\infty}^{\infty} dx u^2_c(x) \equiv 2M,$$

where $M$ is the kink mass, we derive an equation for $\eta(x)$ from (12),

$$\mathcal{L} \eta(x) = \frac{1}{2M} \frac{\partial E}{\partial r} u'_c(x), \quad x \leq r, \quad \mathcal{L} \equiv -\frac{\partial^2}{\partial x^2} + U''(u_c(x)).$$

Here, $\partial E/\partial r$ is a small parameter in the domain of large values of $r$. Multiplying Eq. (15) by the zero mode $u'_c(x)$ of the operator $\mathcal{L}$ and integrating by parts from $-\infty$ to $x$, we can reduce the order of the equation. As a result, we obtain the linear first-order equation for $\eta(x)$

$$-u'_c(x) \eta'(x) + u''_c(x) \eta(x) = \frac{M(x) \partial E}{2M} \frac{\partial r}{\partial r}, \quad M(x) = \int_{-\infty}^{x} dx' u^2_c(x').$$

The solution of Eq. (16) can be written as

$$\eta(x) = u'_c(x) \left( \frac{\eta(r)}{u'_c(r)} + \frac{1}{2M} \frac{\partial E}{\partial r} \int_{x}^{r} dx' \frac{M(x')}{u^2_c(x')} \right).$$
The function $\eta(r)$ playing the role of the integration constant in (17) is found from
the boundary conditions. For the kink-antikink and kink-kink pairs, these conditions
have the respective forms $\Phi'(0, r) = 0$ and $\Phi_c(0, r) = 0$. Using these conditions and
Eq. (16), we obtain the following expressions for the boundary values $\eta(r)$ and $\eta'(r)$
in these two cases:

\begin{align*}
\eta'(r) &= -u'_c(r), \\
\eta(r) &= \frac{1}{u''_c(r)} \left( \frac{M(r) \partial E}{2M} - u'^2_c(r) \right) \tag{18}
\end{align*}

for the kink-antikink pair and

\begin{align*}
\eta(r) &= -u_c(r), \\
\eta'(r) &= -\frac{1}{u'_c(r)} \left( \frac{M(r) \partial E}{2M} + u''_c(r)u_c(r) \right) \tag{19}
\end{align*}

for the kink-kink pair. Solution (17) describes the distortion of the kink under
the effect of the second soliton when the distance between the kinks substantially
exceeds their sizes and depends on the small parameter $\partial E/\partial r$, which determines
the interaction of the solitons. We note that formulas (17-19) include terms of higher
order than the leading asymptotic terms with respect to $r$. However, because the
function $\eta(x)$ contains competing spatial exponentials, it is inadvisable to select
these terms in considering the field variable $\eta(x)$. Formulas (17-19) should therefore
be used to correctly determine the leading approximation term for the integrated
characteristic, the interkink potential $E(r)$. Because $(1/2)u'^2_c = U(u_c)$, substituting
(13) in the formula for the energy $E(r) = H(\Phi_c, 0)$, expanding with respect to $\eta(x)$,
and integrating by parts with regard to Eq. (16) result in

\begin{align*}
E(r) - 2M - 2 \int^\infty_r dxu'^2_c(x) + \frac{1}{2M} \partial E \int^r_{-\infty} dx \eta(x)u'_c(x) \\
&\quad + \eta(r)(2u'_c(r) + \eta'(r)). 
\end{align*} \tag{20}

Substituting expression (17) for $\eta(x)$ in (20) and using boundary values (18) and
(14) for the kink-antikink and kink-kink pairs, we see that the linear terms with
respect to $\partial E/\partial r$ cancel. In this case, relation (20) becomes

\begin{align*}
E(r) - 2M = A(r) + B(r) \left( \frac{1}{2M} \partial E \right)^2 \tag{21},
\end{align*}

where the functions $A(r)$ and $B(r)$ are given by the formulas

\begin{align*}
A(r) &= \frac{u'^3_c(r)}{u''_c(r)} - 2 \int^\infty_r dxu'^2_c(x), \\
B(r) &= \frac{M^2(r)}{u'_c(r)u''_c(r)} + \int^r_{-\infty} dx \frac{M^2(x)}{u'^2_c(x)}
\end{align*}

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for the kink-antikink pair and

\[
A(r) = u_c^2(r) \frac{u''_c(r)}{u'_c(r)} + 2 \int_r^\infty dx u_c(x) u''_c(x),
\]

\[
B(r) = \int_{-\infty}^r dx \frac{M^2(x)}{u_c^2(x)}
\]

for the kink-kink pair. The asymptotic expression for the kink solution has the form

\[
u_c(x) \big|_{x \to \pm \infty} \to u_\pm \mp a \exp(\mp mx),
\]

where \( m = \sqrt{U''(u_\pm)} \) is the pion mass. In a case of an antisymmetric kink-kink configuration the vacuum constant \( u_+ \) is equal to zero. Taking into consideration that \( M(\infty) = M \) and \( M(-\infty) = 0 \) and calculating the leading approximation terms with respect to \( e^{-mr} \) for the coefficients \( A(r) \) and \( B(r) \), we see that the functions \( A(r) \) vanish and \( B(r) \) coincide to within the sign. Therefore the leading approximation for Eq. (21) has the form

\[
E(r) - 2M = \mp \frac{1}{8m^2a^2} e^{2mr} \left( \frac{\partial E}{\partial r} \right)^2.
\]

whence we find the interaction energy of the bisoliton pair at large distances,

\[
E(R) = 2M \mp 2ma^2e^{-2mr}.
\]

This formula describes attraction (repulsion) for the kink-antikink (kink-kink) configuration. The expression for the energy in (22) in the case of the \( \Phi^4 \) and sine-Gordon systems was found in [6] using an approximate equation of type (12) whose right-hand side was imitated with a phenomenological delta-shaped source [15].

To determine the soliton configuration and interaction throughout the range of distances, the gauge should be fixed, and Eq. (12) should be solved numerically. We note that the representation of the solution in form (13) partly fixes the gauge in the domain of large intersoliton distances. Formulas (14) and (22) obtained using expansion (13) can be used to introduce the natural gauge

\[
< \Phi^2_c >^{-1} \frac{\partial E}{\partial r} = \pm \frac{2m^2a^2}{M} e^{-2mr},
\]

ensuring the coincidence of the doubled parameter \( r \) with the true intersoliton distance in the asymptotic domain \( r \to \infty \). After gauge (23) is fixed, Eq. (12) takes the form of a nonlinear diffusion equation,

\[
-\Phi''_c(x) + U'(\Phi_c(x)) = \pm \frac{2m^2a^2}{M} e^{-2mr} \frac{\partial \Phi_c(x)}{\partial r}.
\]
Equation (24) can be used to find the corrections to potential (22) in the asymptotic domain for the given gauge, and its numerical solution permits reproducing the entire set of physically admissible two-soliton configurations. We stress that solving this equation in the case of the kink-antikink configuration corresponds to describing diffusion in the direction of positive "time" $-r$. This process is stable, and the mutual approach in the kink-antikink pair terminates with its "annihilation" under which $\Phi_c(x)$ goes into the vacuum constant [14]. Actually, the calculation of the evolution of the kink-antikink pair performed in [14] in relation to the investigation of the nonlinear diffusion equation is an example of rigorous determination of the entire set of two-soliton configurations in problems related to studying intersoliton interaction on the basis of Eq. (3).

Constructing the kink-kink configuration reduces to studying diffusion in the direction of negative "time" $r$. In the domain of large intersoliton distances, the presence of nonlinearity in Eq. (24) stabilizes the instability characteristic of this procedure in the case of the linear theory. The stabilization weakens during the mutual approach, and the limiting field configuration is determined by the point where the stability of the process terminates.

5 Conclusion

We have covariantly derived an equation for a quasi-static ansatz on the basis of the variational approach. A characteristic feature of this approach is the invariance under reparameterization with a change of collective variables. The cyclic variables or the variables related to the degeneration of vacuum states are introduced in the standard way. The arbitrariness in the choice of the non-Goldstone variables can be eliminated based on some physical considerations in the domain where these variables can be given an immediate physical interpretation. Otherwise they are chosen for convenience.

As an example, we have considered the static deformations of the $\Phi^4$-kink related to excitation of the internal mode. This problem admits an exact solution. The configurations of the deformed kink, its energy, and the physically admissible range of the amplitude of nonlinear oscillations have been determined. Determining the intersoliton potential for the nonlinear Klein-Gordon equation has also been touched upon. Linearizing the equation for the quasi-static ansatz in the domain of large distances permits establishing the deformation of the kinks and the asymptotic character of their interaction, which allows introducing the natural gauge and representing the equation for the quasi-static ansatz as a nonlinear diffusion equation suitable for numerical determination of the entire set of physically admissible two-soliton configurations.

In relation to determining the intersoliton interaction, we stress that although two-soliton configurations have been comprehensively studied using various qualitative approaches for the majority of actual models, investigations on the basis Eq. (3) permit including the exact effect of soliton deformation at small distances, i.e.,
in the domain most sensitive to the defects characteristic of artificial field constructions. This is particularly important, for instance, in the Skyrme model because the nucleon-nucleon potential in this domain has an insignificant minimum forming the nuclear coupling energy. Although Eq. (3) does not take the spin effects into account, the study of skyrmion-skyrmion interaction on the basis of this equation is a useful stage in the procedure of determining the nucleon-nucleon potential.

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References

[1] T.H.R. Skyrme, *Nucl. Phys. B*, 31, 556 (1962).
[2] A. Jackson, A.D. Jackson and V. Pasquier, *Nucl. Phys. A*, 432, 567 (1985).
[3] E. Sorace and M. Tarlini, *Phys. Rev. D*, 33, 253 (1986).
[4] T.S. Walhout and J. Wambach, *Phys. Rev. Lett.*, 67, 314 (1991).
[5] G. Kalbermann, *Nucl. Phys. A*, 561, 682 (1993).
[6] R. Rajaraman, *Phys. Rev. D*, 15, 2866 (1977).
[7] N.S. Manton, *Nucl. Phys. B*, 126, 525 (1977).
[8] L. Jacobs and C. Rebbi, *Phys. Rev. B*, 19, 4486 (1979).
[9] N.S. Manton, *Phys. Rev. Lett.*, 60, 1916 (1988).
[10] V.D. Tsukanov, *J. Phys. A*, 25, 6099 (1992).
[11] R. Boesch, P. Stancioff and C.R. Willis, *Phys. Rev. B*, 38, 6713 (1988).
[12] R. Boesch and C.R. Willis, *Phys. Rev. B*, 42, 2290 (1990).
[13] A.C. Newell and J.A. Whitehead, *J. Fluid. Mech.*, 38, 239 (1969).
[14] T. Kavahara and M. Tanaka, *Phys. Lett. A*, 97, 311 (1983).
[15] R. Rajaraman. Solitons and Instantons. - Amsterdam: North-Holland, 1982.