PERIODIC AND LIMIT-PERIODIC DISCRETE SCHRÖDINGER OPERATORS

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Abstract. The theory of discrete periodic and limit-periodic Schrödinger operators is developed. In particular, the Floquet–Bloch decomposition is discussed. Furthermore, it is shown that an arbitrarily small potential can add a gap for even periods. In dimension two, it is shown that for coprime periods small potential terms don’t add gaps thus proving a Bethe–Sommerfeld type statement. Furthermore limit-periodic potentials whose spectrum is an interval are constructed.

1. Introduction

My aim in this paper is two fold. The first three sections discuss the basic theory of discrete periodic Schrödinger operators on $\mathbb{Z}^d$. The reason for writing this is that there are good reference on $\mathbb{Z}$ [20], [22] and for continuum operators [19], [21], but as far as I know no reference on $\mathbb{Z}^d$ for $d \geq 2$. Then in the second part, I present new results on Schrödinger operators on $\mathbb{Z}^2$:

(i) Theorem 6.1 shows that if the periods are coprime, then small enough perturbations do not add gaps in the spectrum.

(ii) Theorem 5.3 valid in any dimension shows that there exist arbitrarily small perturbations of even periods adding one gap in the spectrum.

(iii) Finally Theorem 7.1 exhibits a large class of limit-periodic potentials whose spectrum is an interval.

The main ingredient in the proof of (i) and (iii) is Theorem 5.1 which asserts that any energy can be an eigenvalue of multiplicity at least two for at most finitely many operators in the Floquet–Bloch decomposition.

Both (i) and (iii) are phenomena appearing in dimension two. In dimension one, one generally has gaps, see Avila [1], Avron–Simon [2], Damanik–Gan [5], [6], Krüger–Gan [7]. It is an interesting task to prove statements analog to (i) and (iii) in dimensions three and higher.

I consider (i) an analog of the Bethe–Sommerfeld conjecture for continuous Schrödinger operators. This conjecture states that for $d \geq 2$ and any periodic function $V : \mathbb{R}^d \to \mathbb{R}$ the spectrum of the operator

$$-\sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} + V$$

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only contains finitely many gaps. This conjecture has been solved completely by Parnovski in [16], see also [3, 17] for more recent work. For some earlier work see the books [9] and [21]. In [10], [11], [12], [13], Karpeshina and Lee have derived analogous statements to (iii) in the continuum setting. In fact using their KAM-type methods, Karpeshina and Lee are able to prove absolutely continuous spectrum.

A difference between the continuum case and the discrete case considered here, is that the discrete statement depends on the underlying period lattice as (ii) shows. I have included Questions 4.5, 6.2, and 6.4 in order to highlight some problems that would allow us to gain further understanding of higher dimensional operators. These questions do not address how to construct operators with pure-point spectrum, since this has already been solved by Pöschel in [15].

2. Periodic discrete Schrödinger operators

In this section, I discuss the spectral theory of discrete periodic Schrödinger operators. Since, I am unaware of a source of this, the discussion is somewhat detailed. Discussions in the continuous case can be found in Reed–Simon [19], Skriganov [21].

2.1. Periodic functions. We recall that given periods \( p = \{p_j\}_{j=1}^{d} \in (\mathbb{Z}_+)^d \), a function \( f : \mathbb{Z}^d \rightarrow \mathbb{C} \) is called \( p \)-periodic if

\[
(2.1) \quad f(n + p_j b_j) = f(n)
\]

for all \( 1 \leq j \leq d \) and \( n \in \mathbb{Z}^d \), where \( b_j \) denotes the standard basis of \( \mathbb{Z}^d \). Define

\[
(2.2) \quad \mathbb{B} = \times_{j=1}^{d} \left\{ 0, \frac{1}{p_j}, \ldots, \frac{p_j-1}{p_j} \right\} \subseteq \mathbb{T}^d.
\]

We denote \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \) and \( e(x) = e^{2\pi i x} \) as usual. Define the Fourier coefficients \( \hat{f} : \mathbb{B} \rightarrow \mathbb{C} \) of a \( p \)-periodic function \( f \) by

\[
(2.3) \quad \hat{f}(k) = \frac{1}{P} \sum_{n \in \mathbb{B}} f(n) e(-k \cdot n), \quad k \cdot n = \sum_{j=1}^{d} k_j n_j
\]

with \( P = \prod_{j=1}^{d} p_j \). One easily checks that

\[
(2.4) \quad f(n) = \sum_{k \in \mathbb{B}} \hat{f}(k) e(k \cdot n).
\]

Recall that for a function \( u \in \ell^1(\mathbb{Z}^d) \), its Fourier transform \( Fu = \hat{u} : \mathbb{T}^d \rightarrow \mathbb{C} \) is given by \( Fu(x) = \hat{u}(x) = \sum_{n \in \mathbb{Z}^d} e(-n \cdot x) u(n) \). Then \( F \) is extended to \( \ell^2(\mathbb{Z}^d) \rightarrow \ell^2(\mathbb{Z}^d) \) by continuity. Plancherel’s identity shows that this map is unitary. We have that

**Lemma 2.1.** Let \( V \) be a \( p \)-periodic function and \( u \in \ell^2(\mathbb{Z}^d) \). Then

\[
(2.5) \quad \hat{V} u(x) = \sum_{k \in \mathbb{B}} \hat{V}(k) \hat{u}(x-k).
\]

**Proof.** This is a computation. \( \square \)
The computations of this section seem to depend on the period. However, the choices where made in a such a way, that they are compatible. For example if one views \( f : \mathbb{Z}^d \to \mathbb{C} \) as a \( 2p \)-periodic function instead of a \( p \)-periodic one, then the Fourier coefficients stay the same.

2.2. Momentum representation of the periodic Schrödinger operator.

The discrete Laplacian \( \Delta : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d) \) is defined by \( \Delta u(n) = \sum_{j=1}^d (u(n - b_j) + u(n + b_j)) \). We recall that its Fourier transform is given by

\[
F \Delta F^{-1} \hat{u}(\mathbf{x}) = \left( \sum_{j=1}^d 2 \cos(2\pi x_j) \right) \hat{u}(\mathbf{x}).
\]

In particular, if \( V(n) = \sum_{k \in B} \hat{V}(k) e(k \cdot n) \) is a \( p \)-periodic potential, then the Schrödinger equation \((\Delta + V)u = Eu\) takes the form

\[
(2.6) \quad \left( \sum_{j=1}^d 2 \cos(2\pi x_j) \right) \hat{u}(\mathbf{x}) + \sum_{k \in B} \hat{V}(k) \hat{u}(\mathbf{x} - k) = E \hat{u}(\mathbf{x})
\]

in Fourier variables. It is easy to see that the equations involving \( \{\hat{u}(\mathbf{x} + k)\}_{k \in B} \) for

\[
(2.7) \quad \mathbf{x} \in \mathbb{V} = \times_{j=1}^d [0, \frac{1}{p_j}), \quad B = \times_{j=1}^d \left\{ 0, \frac{1}{p_j}, \ldots, \frac{p_j - 1}{p_j} \right\}
\]

are all independent of each other. Define the space \( L^2(\mathbb{V} \times B) \) as the set of all maps \( f : \mathbb{V} \times B \to \mathbb{C} \) with norm

\[
(2.8) \quad \|f\|_{L^2(\mathbb{V} \times B)}^2 = \sum_{k \in B} \int_{\mathbb{V}} |f(x, k)|^2 d\mathbf{x}.
\]

Introduce the map \( W = \tilde{W} F : \ell^2(\mathbb{Z}^d) \to L^2(\mathbb{V} \times B) \) where

\[
(2.9) \quad (\tilde{W} \hat{u})(\mathbf{x}, k) = \frac{\hat{u}(\mathbf{x} + k)}{\left( \prod_{j=1}^d p_j \right)^{1/2}}.
\]

Lemma 2.2. The map \( W : \ell^2(\mathbb{Z}^d) \to L^2(\mathbb{V} \times B) \) is unitary. Furthermore, if the support of \( u \)

\[
(2.10) \quad \text{supp}(u) = \{ n \in \mathbb{Z}^d : \ u(n) \neq 0 \}
\]

is finite, then

\[
(2.11) \quad |Wu(\mathbf{x}, k)| \leq \left( \frac{\#(\text{supp}(u))}{\prod_{j=1}^d p_j} \right)^{1/2} \|u\|.
\]

Proof. Since \( F \) is unitary, it suffices to check that \( \tilde{W} \) is unitary, but this follows directly. For the second claim, observe that

\[
Wu(\mathbf{x}, k) = \frac{1}{p_1^{1/2}} \sum_{n \in \text{supp}(u)} e(-n \cdot (\mathbf{x} + k)) u(n)
\]

which implies claim bu Cauchy–Schwarz. \( \square \)
Also define for \( \theta \in V \) the operator \( \hat{H}_{\theta} : \ell^2(B) \to \ell^2(B) \)

\[
\hat{H}_{\theta} v(\ell) = \left( \sum_{j=1}^{d} 2 \cos \left( 2\pi (\ell_j + \theta_j) \right) \right) v(\ell) + \sum_{k \in B} \hat{V}(k) v(\ell - k).
\]

We define the operator \( \hat{V}^* : \ell^2(B) \to \ell^2(B) \) by

\[
\hat{V}^* \psi(\ell) = \sum_{k \in B} \hat{V}(k) v(\ell - k).
\]

This way \( \hat{H}_{\theta} = \hat{H}_{\theta}^0 + \hat{V}^* \). We define

\[
\hat{H}_\theta : L^2(V \times B) \to L^2(V \times B)
\]

\[
(\hat{H}_\theta u)(\ell, k) = (\hat{H}_{\theta}^0 u(\ell, .))(k).
\]

The following proposition provides the first form of Floquet–Bloch decomposition of the periodic operator \( H = \Delta + V \). We will encounter a second one in Subsection 2.4.

**Proposition 2.3.** We have that

\[
W H W^{-1} = \hat{H}_\theta.
\]

**Proof.** This follows from the preceding computations.

We note that this gives a decomposition of \( H \) as a direct integral in the sense of Section XIII.16. of [19].

We furthermore wish to point out at this point the following periodicity of \( \hat{H}_{\theta} \).

First observe that our definition of \( \hat{H}_{\theta} \) makes sense for any \( \theta \in \mathbb{R}^d \) and even \( \theta \in \mathbb{C}^d \). We have that

**Lemma 2.4.** Let \( 1 \leq j \leq d \). The operators

\[
\hat{H}_{\theta, j}, \quad \hat{H}_{\theta, j + b_j},
\]

are unitarily equivalent.

**Proof.** The unitary equivalence is given by \( U \) defined by \( U \psi(n) = \psi(n + \frac{1}{p_j} b_j) \).

2.3. **Analytic parametrization of eigenvalues and absolutely continuous spectrum.** To simplify the notation, we will now fix \( \theta^+ = \{ \theta_j \}_{j=0}^{d-2} \) and define

\[
A(t) = \hat{H}_{\theta(t, \theta^+)}.
\]

Clearly \( A(t) \) is an analytic family of operators defined for every \( t \in \mathbb{C} \).

**Proposition 2.5.** The eigenvalues \( \lambda_j(t) \) of \( A(t) \) can be chosen to be analytic functions of \( t \). Furthermore, each of these \( \lambda_j(t) \) is a non-constant function of \( t \).

**Proof.** The first claim follows from \( t \mapsto A(t) \) being an analytic map and for example Theorem II.6.1. in [8].

For the second claim observe that as \( \text{Im}(t) \to \infty \), we have that \( \| A(t)^{-1} \| \to 0 \), since \( A(t) \) is dominated by the diagonal containing values of size \( \gtrsim e^t \). Now

\[
\| A(t)^{-1} \| = \max_j \frac{1}{|\lambda_j(t)|}
\]

implies that \( |\lambda_j(t)| \to \infty \) as \( \text{Im}(t) \to \infty \). This implies that these are non-constant.
The next theorem shows that the spectrum is absolutely continuous. It will be used in the Section 4 to study the integrated density of states and the spectral measure. Furthermore, it provides essential information on the nature of the spectrum.

**Theorem 2.6.** The spectrum of $H$ is purely absolutely continuous.

**Proof.** See [19] Theorem XIII.100. □

2.4. **Space basis.** In this section, we derive a second type of Floquet–Bloch decomposition. Let $\Phi \in \mathcal{V}$ and define the space $\ell^2_{\Phi}(\mathbb{Z}^d)$ as the set of all functions $u \colon \mathbb{Z}^d \to \mathbb{C}$ such that

$$u(n + p_j b_j) = e(p_j \theta_j)u(n), \quad j = 1, \ldots, n, \quad n \in \mathbb{Z}^d,$$

with the norm

$$\|u\|^2 = \sum_{1 \leq n_i \leq p_1} \cdots \sum_{1 \leq n_d \leq p_d} |u(n)|^2.$$

Define

$$U_{p, \Phi} : \ell^2(\mathbb{B}) \to \ell^2_{\Phi}(\mathbb{Z}^d),$$

$$U_{p, \Phi} \varphi(u) = \sum_{k \in \mathbb{B}} \varphi(k)e\left(\sum_{j=1}^d (\theta_j + k_j) \cdot n_j\right).$$

One can check that this operator is unitary. Denote by $H_{p, \Phi}$ the restriction of $H$ to $\ell^2_{\Phi}(\mathbb{Z}^d)$.

**Lemma 2.7.** We have

$$\tilde{H}_{p, \Phi} = U_{p, \Phi}^* H_{p, \Phi} U_{p, \Phi}$$

In particular $\tilde{H}_{p, \Phi}$ and $H_{p, \Phi}$ have the same eigenvalues.

**Proof.** This is a computation. □

Define $\mathbb{P} = \times_{j=1}^d \{1, \ldots, p_j\}$, $\Delta^j_{\ell^2_{\Phi}} : \ell^2(\mathbb{P}) \to \ell^2(\mathbb{P})$ by

$$\Delta^j_{\ell^2_{\Phi}} u(n) = \begin{cases} u(n - b_j) + u(n + b_j), & 2 \leq n_j \leq p_j - 1 \\ u(n - b_j) + e(\theta_j)u(n + (p_j - 1) \cdot b_j), & n_j = 1 \\ u(n - b_j) + e(-\theta_j)u(n - (p_j - 1) \cdot b_j), & n_j = p_j, \end{cases}$$

and $\Delta_{\ell^2_{\Phi}} = \sum_{j=1}^d \Delta^j_{\ell^2_{\Phi}}$. We note that $\ell^2_{\Phi}(\mathbb{Z}^d)$ is isomorphic to $\ell^2(\mathbb{P})$ and that $H \psi = E \psi$ if and only if

$$(\Delta_{\ell^2_{\Phi}} + V) \tilde{\psi} = E \tilde{\psi}$$

where $\tilde{\psi}$ denotes the restriction of $\psi$ to $\mathbb{P}$.

For $\underline{d}, \underline{p} \in (\mathbb{Z}_+)^d$, we define

$$(\underline{d} \cdot \underline{p})_j = d_j \cdot p_j.$$
Lemma 2.8. Let $d, p \in (\mathbb{Z}_+)^d$, $\theta \in \mathcal{V}_{d^*p}$. Then

$$
(2.25) \quad \ell^2_{d^*p\theta}(\mathbb{Z}^d) = \bigoplus_{\phi \in \Phi} \ell^2_{p\phi}(\mathbb{Z}^d)
$$

where

$$
(2.26) \quad \Phi = \left\{ (\theta_j + \ell_j p_j d_j)_{j=1}^d, \quad \ell \in \times_{j=1}^d \{0, \ldots, d_j-1\} \right\}.
$$

Proof. If $u \in \ell^2_{d^*p\theta}(\mathbb{Z}^d)$, then

$$
u(n + d_j p_j b_j) = e(p_j d_j \varphi_j) u(n)$$

which is equal to $e(p_j d_j \theta_j) u(n)$ if and only if $\varphi \in \Phi$. Hence, we have that

$$
\ell^2_{d^*p\theta}(\mathbb{Z}^d) \supseteq \bigoplus_{\phi \in \Phi} \ell^2_{p\phi}(\mathbb{Z}^d).
$$

A counting argument shows that the dimensions of the spaces agree. Hence, equality holds. \[\square\]

This lemma implies that the eigenvalues of $H_{d^*p\theta}$ are the union over the eigenvalues of $H_{p\phi}$ with $\phi \in \Phi$.

3. Bands and Gaps

The goal of this section is to introduce the language of bands and gaps and to prove basic results about them.

Lemma 3.1. The operator $\hat{H}_{p\theta}$ has $P = \prod_{j=1}^P p_j$ eigenvalues. Orders these

$$
(3.1) \quad E_1(\theta) \leq E_2(\theta) \leq \cdots \leq E_P(\theta)
$$

in increasing order. Then

$$
(3.2) \quad E_j(\theta) = \min_{\dim(\mathcal{V}) = j-1} \max_{\|\psi\| = 1} \langle \psi, \hat{H}_{p\theta} \psi \rangle
$$

in particular the $E_j : \mathcal{V} \to \mathbb{R}$ are continuous functions.

Proof. The number of eigenvalues is $P$, since $\ell^2(\mathbb{R})$ is $P$-dimensional. \[\square\]

We will now relate the properties of the functions $E_j(\theta)$ to the spectrum of $H$.

Define

$$
(3.3) \quad E_j^- = \min_{\theta \in \mathcal{V}} E_j(\theta), \quad E_j^+ = \max_{\theta \in \mathcal{V}} E_j(\theta).
$$

Definition 3.2. The intervals $[E_j^-, E_j^+]$ are called bands. If $E_j^+ < E_{j+1}^-$, then $(E_j^+, E_{j+1}^-)$ is called a gap.

It is clear that the bands are subset of the spectrum $\sigma(H)$ of $H$ and that gaps of the resolvent. We furthermore note
Theorem 3.3. The spectrum of the $p$-periodic Schrödinger operator $H = \Delta + V$ is given by
\begin{equation}
\sigma(\Delta + V) = \bigcup_{j=1}^{P} [E^{-j}, E^{+j}] .
\end{equation}

In particular, it contains at most $P - 1 = \prod_{j=1}^{d} p_j - 1$ many gaps.

Proof. See Theorem XIII.85 in [19] and use that the functions $E_j$ are continuous. □

One should point out that all these gaps can occur as the following is a somewhat degenerate example shows. Let $m_\ell$ be an enumeration of $P = \times_{j=1}^{d} \{1, \ldots, p_j\}$ and define a potential $V$ by
\begin{equation}
V(\underline{n}) = (4d + 1) \ell
\end{equation}
whenever $n_j = m_j^\ell \pmod{p_j}$ for $j = 1, \ldots, d$. It is then relatively easy to check that
\begin{equation}
\sigma(\Delta + V) \cap ((4d + 1)\ell + 2d, (2d + 1)\ell + 2d + 1) = \emptyset
\end{equation}
for $\ell = 1, \ldots, P - 1$ and
\begin{equation}
\sigma(\Delta + V) \cap ((4d + 1)\ell - 2d, (4d + 1)\ell + 2d) \neq \emptyset
\end{equation}
for $\ell = 1, \ldots, P$. Hence $\sigma(\Delta + V)$ contains at least $P - 1$ many gaps, and by Theorem 3.3 exactly $P - 1$.

3.1. Non-constancy. We will now discuss further properties of the eigenvalue parametrization from Lemma 3.1. For $X$ a topological space, we will call a point $x \in X$ a point of increase of a function $f : X \to \mathbb{R}$ if for any open set $x \in U$, we have
\begin{equation}
\inf_{y \in U} f(y) < f(x) < \sup_{y \in U} f(y).
\end{equation}
Or in words, we can find $y, \tilde{y}$ arbitrarily close to $x$ such that $f(y) < f(x) < f(\tilde{y})$.

The main result is
\begin{thm}
Let $[E^{-j}, E^{+j}]$ be a band. Then for any $E \in (E^{-j}, E^{+j})$, there exist infinitely many points $\underline{\theta}$ such that $E_j(\underline{\theta}) = E$ and $\underline{\theta}$ is a point of increase.
\end{thm}

A refinement of the following argument, noting that $E_j^{-1}(E)$ is the boundary of an open set, shows that the set of $\underline{\theta}$ described in this theorem has Hausdorff dimension at least $d - 1$. For the proof, we will need the following lemma

Lemma 3.5. The eigenvalues $E_j(\underline{\theta})$ defined in Lemma 3.1 are not constant on an open set.

Proof. Assume that $E_j$ was constant on an open set. This would in imply that $\lambda(\ell)$ as in Proposition 2.5 was constant for some $\ell$. A contradiction. □

Proof of Theorem 3.4. The two sets $A = E_j^{-1}((-\infty, E))$ and $B = E_j^{-1}(E, \infty)$ are disjoint and open in $V$. Hence,
\begin{equation}
C = [0, 1]^d \setminus (A \cup B) = E_j^{-1}(\{E\})
\end{equation}
must be infinite. By the previous lemma \( C \) does not contain an open set. This implies that also
\[
\tilde{C} = C \cap \mathcal{A} \cap \mathcal{B}
\]
is infinite, and by definition contains points of increase of \( E_j \). Hence, we are done.

We furthermore note the obvious lemma

**Lemma 3.6.** Let \( \theta \) be a point of increase of \( E_j \), then \( E_j(\theta) \) is in the interior of a band.

### 3.2. Stability of the spectrum being an interval.

We start with

**Definition 3.7.** Let \( H \) be a \( p \)-periodic Schrödinger operator and \( \delta \in \mathbb{R} \). We say that the bands of \( H \) are \( \delta \)-overlapping if
\[
E_{j+1}^+ - E_{j}^+ \geq \delta
\]
for \( j = 1, \ldots, P - 1 \). The bands of \( H \) are called overlapping if they are \( \delta \)-overlapping for some \( \delta > 0 \).

In particular, if the bands of \( H \) are overlapping, then the spectrum of \( H \) is an interval. We allow for negative values of \( \delta \) so statements like the next theorem become possible without restrictions on \( \|V\|_{\infty} \).

**Theorem 3.8.** Let the bands of \( H \) be \( \delta \)-overlapping. Then the bands of \( H + V \) are \( \delta - 2\|V\|_{\infty}\)-overlapping.

For \( p \)-periodic \( V : \mathbb{Z}^d \to \mathbb{R} \), we denote by \( E_j(\theta, V) \) the eigenvalues of \( H_{\theta, \theta} + V \) as defined in Lemma 3.1. We note the following simple lemma.

**Lemma 3.9.** We have
\[
E_j(\theta, 0) - \|V\|_{\infty} \leq E_j(\theta, V) \leq E_j(\theta, 0) + \|V\|_{\infty}.
\]
In particular, if
\[
\|V\|_{\infty} \leq \frac{1}{2} \min_{1 \leq j \leq P - 1} (E_{j+1}^+ - E_j^-)
\]
then the spectrum of \( H + V \) is an interval.

**Proof.** The first claim follows from the min-max principle [3.2]. The second one from the first, since it implies \( E_j^+ - E_{j+1}^- \geq 0 \) which exactly says that \( \sigma(H + V) \) is an interval.

**Proof of Theorem 3.8.** This follows from the previous lemma.

The following lemma will be used in our construction of limit-periodic potentials.

**Lemma 3.10.** Let \( H \) be a \( p \)-periodic Schrödinger operator and \( \mathcal{V} \) a compact set of \( p \)-periodic potentials. Assume that for every \( V \in \mathcal{V} \), the bands of \( H + V \) are overlapping. Then there exists \( \delta > 0 \) such that for all such \( V \), the bands of \( H + V \) are \( \delta \)-overlapping.

**Proof.** It is easy to see that the map \( V \mapsto E_j(\theta, V) \) is continuous. Hence, also the map
\[
g : V \mapsto \inf_j (E_{j+1}^- - E_j^+).
\]
is continuous. Since \( \mathcal{V} \) is compact and \( g(V) > 0 \), it follows that \( \inf_{V \in \mathcal{V}} g(V) > 0 \) which is the claim.
3.3. An upper bound on the length of bands. We have that

**Theorem 3.11.** Let \( H \) be \( p \)-periodic, then the length of bands is bounded by

\[
E_j^+ - E_j^- \leq 4\pi \sum_{j=1}^{d} \frac{1}{p_j}.
\]

**Proof.** Let \( E_j^\pm_j = E_j(\theta_\pm) \). Since \( \theta_\pm \in \mathcal{V} \), we have \(|(\theta_- - \theta_+)_j| \leq \frac{1}{p_j}\). Define the family of operators

\[
A(t) = \hat{H}_{p,\theta_+ + t(\theta_+ - \theta_-)}, \quad t \in [0, 1]
\]

which is clearly analytic. Denote by \( \lambda_\ell(t) \) an analytic parametrization of the eigenvalues of \( A(t) \). We have that \( |\lambda_\ell'(t)| \leq \|A'(t)\| \) for some \( \ell = \ell(t) \) except at finitely many points, we obtain that

\[
|E_j^+ - E_j^-| \leq \sup_{t \in [0, 1]} \|A'(t)\|.
\]

One can easily compute that \( \|\partial_{\theta_j} \hat{H}_{p,\theta}\| \leq 4\pi \). Hence, we obtain that

\[
A'(t) = \sum_{j=1}^{d} (\theta_+ - \theta_-)_j \cdot \partial_{\theta_j} \hat{H}_{p,\theta}
\]

satisfies \( \|A'(t)\| \leq 4\pi \sum_{j=1}^{d} \frac{1}{p_j} \), which is the claim. \( \square \)

4. The integrated density of states and spectral measures

The goal of this section is to investigate two quantities related to the spectrum of \( H \): the integrated density of states and the spectral measures.

4.1. The integrated density of states. Let \( \ell \geq 1 \) and denote by \( \Lambda_\ell \), the rectangle

\[
\Lambda_\ell = \times_{j=1}^{d} \{1, \ldots, \ell \cdot p_j\}
\]

and by \( \#\Lambda_\ell \) the number of elements in \( \Lambda_\ell \). We denote by \( H^{\Lambda_\ell} \) the restriction of \( H \) to \( \ell^2(\Lambda_\ell) \) and by

\[
\text{tr} \left( P(-\infty, E) \left( H^{\Lambda_\ell} \right) \right)
\]

the number of eigenvalues of \( H^{\Lambda_\ell} \) less than \( E \).

**Theorem 4.1.** The limit

\[
k(E) = \lim_{\ell \to \infty} \frac{1}{\#\Lambda_\ell} \text{tr} \left( P(-\infty, E) \left( H^{\Lambda_\ell} \right) \right)
\]

exists. Furthermore with \( E_j(\theta) \) as in Lemma 3.1 we have

\[
k(E) = \frac{1}{P} \int_{\mathcal{V}} \#(j : E_j(\theta) \leq E) d\theta.
\]

By Lemma 2.4 we have that

\[
\text{tr} \left( P(-\infty, E) \left( H^{\Lambda_\ell} \right) \right) = \text{tr} \left( P(-\infty, E) \left( H^{\Lambda_\ell + n \cdot \mathbf{e}_j} \right) \right)
\]

for any \( n \in \mathbb{Z}^d \). Hence, the limit in Theorem 4.1 is somewhat more general than we claim here. We first prove
Lemma 4.2. Let \( \vartheta \in \mathcal{V} \). Then
\[
(4.6) \quad \lim_{\ell \to \infty} \frac{1}{\# \Lambda_{\ell}} \left( \text{tr} \left( P_{(-\infty, E)} \left( H_{\Lambda_{\ell}} \right) \right) - \text{tr} \left( P_{(-\infty, E)} \left( H_{\ell \cdot \vartheta} \right) \right) \right) = 0.
\]

Proof. \( H_{\Lambda_{\ell}} \) and \( H_{\ell \cdot \vartheta} \) differ by a rank
\[
r_{\ell} = 2^{\ell d - 1} \prod_{j=1}^{d} p_j
\]
perturbation. Hence,
\[
\frac{1}{\# \Lambda_{\ell}} \left( \text{tr} \left( P_{(-\infty, E)} \left( H_{\Lambda_{\ell}} \right) \right) - \text{tr} \left( P_{(-\infty, E)} \left( H_{\ell \cdot \vartheta} \right) \right) \right) = O \left( \frac{1}{\ell} \right)
\]
and the claim follows. \( \square \)

By Lemma 2.8, we have that
\[
(4.7) \quad \text{tr} \left( P_{(-\infty, E)} \left( H_{\ell \cdot \vartheta} \right) \right) = \sum_{\ell \vartheta \equiv \vartheta \pmod{1}} \text{tr} \left( P_{(-\infty, E)} \left( H_{\vartheta} \right) \right).
\]

Proof of Theorem 4.1. By the previous two results, it suffices to show that
\[
\lim_{\ell \to \infty} \frac{1}{\ell d} \sum_{\ell \vartheta \equiv \vartheta \pmod{1}} \text{tr} \left( P_{(-\infty, E)} \left( H_{\vartheta} \right) \right) = \int_{\mathcal{V}} \# \{ j : E_j(\vartheta) \leq E \} \, d\vartheta.
\]

For this, observe that
\[
\text{tr} \left( P_{(-\infty, E)} \left( H_{\vartheta} \right) \right) = \# \{ j : E_j(\vartheta) < E \}.
\]
Furthermore, by Theorem XIII.83.(e) in [19] and Theorem 2.6 the measure of \( \vartheta \in \mathcal{V} \) such that \( E_j(\vartheta) = E \) for some \( j \), is 0. Hence, the result follows by a convergence theorem for integrals. \( \square \)

The results of Craig–Simon [4] imply that the integrated density of states is log Hölder continuous. Furthermore, it follows from Theorem 2.6 that it is absolutely continuous. It would be interesting to obtain further regularity results, see the discussion in the next subsection.

The definition of the limit in (4.3) is not as general as possible. One can show that if \( \Lambda_t \) is a Folner sequence for \( \mathbb{Z}^d \), then
\[
(4.8) \quad k(E) = \lim_{t \to \infty} \frac{1}{\# \Lambda_t} \text{tr} \left( P_{(-\infty, E)} \left( H_{\Lambda_t} \right) \right).
\]

The function \( k \) defined in Theorem 4.1 is clearly increasing. Hence, there exists a measure \( \nu \) such that \( k(E) = \nu(\infty, E) \). This measure is called density of states.

Lemma 4.3. For \( \text{Im}(z) > 0 \), we have
\[
(4.9) \quad \int \frac{d\nu(t)}{t - z} = \frac{1}{P} \sum_{j=1}^{P} \int_{\mathcal{V}} \frac{d\vartheta}{E_j(\vartheta) - z}
\]
where \( P = \prod_{j=1}^{d} p_j \).
Proof. We have that
\[ \int \frac{d\nu(t)}{t - z} = - \int \frac{k(t)dt}{t - z} = \int k(E) \sum_{j=1}^{P} \int_{\mathbb{R}} \chi(-\infty, E_j(p)) (E) d\theta. \]

The claim then follows by Fubini and a quick computation. \( \square \)

4.2. Spectral measures. By Theorem 2.6 all the spectral measures are absolutely continuous. The goal of this section will be to give more quantitative information.

Given \( u \in \ell^2(\mathbb{Z}^d) \), we denote by \( \mu^u \) the measure satisfying
\[ \langle u, (H - z)^{-1} u \rangle = \int \frac{d\mu^u(t)}{t - z} \]
for \( \text{Im}(z) > 0 \). The main result of this section is

**Theorem 4.4.** Let \( H \) be \( p \)-periodic and \( u \in \ell^2(\mathbb{Z}^d) \) with finite support. Then the spectral measure \( \mu^u \) is absolutely continuous with respect to the density of states \( \nu \).

Furthermore there exists \( C = C(p, u) > 0 \), such that
\[ \left\| \frac{d\mu^u}{d\nu} \right\|_{L^\infty(\mathbb{R})} \leq C. \]

Here \( \frac{d\mu^u}{d\nu} \) denotes the Radon-Nikodym derivative of \( \mu^u \) with respect to \( \nu \). At this point, I would like to ask

**Question 4.5.** Let \( B > 0 \). Do there exist \( q > 1 \) and \( C > 0 \) such that we have for all \( p \)-periodic \( V \) with \( \|V\|_\infty \leq B \) that
\[ \left\| \frac{d\nu}{dE} \right\|_{L^q(\mathbb{R})} \leq C? \]

This is true in dimension one, see [6]. A positive answer to this question would allow us to obtain an uniform \( L^q \) bound on all spectral measures. This result would then in turn allow us to carry over the construction of limit-periodic potentials with absolutely continuous spectrum of Avron–Simon [2] (see also [6]) to the multi-dimensional case.

We now begin with the proof of Theorem 4.4. Denote \( u_k(k) = (Wu)(k) \) and by \( E_j(p), \psi_j(p) \) the eigenvalues and eigenfunctions of \( \hat{H}_{p,p} \). We clearly have that
\[ \int \frac{d\mu^u(t)}{t - z} = \sum_{j=1}^{P} \int_{\mathbb{R}} |\langle \psi_j(p), u_k \rangle|^2 \frac{d\theta}{E_j(p) - z}. \]

On the other hand, we have seen in the previous part of this section that
\[ \int \frac{d\nu(t)}{t - z} = \frac{1}{\prod_{j=1}^{d} \sum_{j=1}^{P} \int_{\mathbb{R}} \frac{d\theta}{E_j(p) - z}}. \]

We are now ready for

**Proof of Theorem 4.4.** The densities of a measure are given by
\[ \frac{d\mu^u(E)}{dE} = \lim_{\varepsilon \to 0} \frac{1}{\pi} \text{Im} \left( \langle u, (H - (E + i\varepsilon))^{-1} u \rangle \right). \]
Hence, we conclude from the previous two formulas that
\[
\frac{d\mu^u}{dE} \leq \frac{1}{\pi} \sup_{1 \leq j \leq P} |\langle \psi_j(\theta), u_\theta \rangle|^2 \cdot \lim_{\varepsilon \to 0} \int_\theta \sum_{j=1}^P \frac{\varepsilon d\theta}{(E_j(\theta) - E)^2 + \varepsilon^2}
\]
\[
\leq C(u, p) \frac{d\nu}{dE},
\]
where we used \(|\langle \psi_j(\theta), u_\theta \rangle|\) is uniformly bounded, by Lemma 2.2 and \(\psi_j(\theta)\) being normalized. \(\square\)

5. Simplicity of the spectrum for coprime periods

In this section, we restrict ourselves to dimension two. For this reason, we will denote the periods of the potential \(V\) by \((p, q)\) and the angles in the Floquet–Bloch decomposition by \(\theta \in [0, \frac{1}{p}), \varphi \in [0, \frac{1}{q})\). We furthermore recall that \(p, q \geq 2\) are called coprime, if they have no common divisor.

The following theorem will be proven in this section. It is a technical result that will allow the constructions in the following sections.

**Theorem 5.1.** Let \(p, q \geq 2\) be co-prime, \(E \in \mathbb{R}\), and \(V : \mathbb{Z}^2 \to \mathbb{R}\) be \((p, q)\)-periodic. Then the set
\[
\{(\theta, \varphi) : E is an eigenvalue of multiplicity \geq 2 of \hat{H}_{(p, q), (\theta, \varphi)}\}
\]
is finite.

The analog statement in dimensions \(d \geq 2\), is that the set of \(\theta\) such that \(E\) is an eigenvalue of multiplicity at least two, has dimension less than \(d - 2\). I expect that proving this result and then using it would be somewhat more involved, than what is done here.

The proof of this theorem has essentially two parts. First an algebraic reduction is performed allowing us to prove the claim by proving a statement when \(\text{Im}(\theta)\) or \(\text{Im}(\varphi)\) is large. In this regime the operator is essentially diagonal, and the analysis of this takes up the second step.

5.1. Algebraic preparations. Define
\[
\hat{H}_0(u, v)\psi(k, l) = \left(e \left(\frac{k}{p}\right) u + e \left(-\frac{k}{p}\right) \frac{1}{u} + e \left(\frac{l}{q}\right) v + e \left(-\frac{l}{q}\right) \frac{1}{v}\right) \psi(k, l)
\]
acting on \(\ell^2([1, p] \times [1, q])\) and \(\hat{H}(u, v) = \hat{H}_0(u, v) + \hat{V}_*\). We have that \(\hat{H}(\theta, \varphi) = \hat{H}(\varphi, \theta)\) if one identifies \(\ell^2([1, p] \times [1, q])\) with \(\ell^2(\mathbb{B})\) in the obvious way.

**Proposition 5.2.** Let \(E \in \mathbb{R}\). Assume there exist \(u_0, v_0 \in \mathbb{C}\) such that for no \(u, v \in \mathbb{C}\), \(E\) is an eigenvalue of multiplicity at least two of \(\hat{H}(u_0, v)\) or \(\hat{H}(u, v_0)\). Then the set
\[
\{(\theta, \varphi) : E is an eigenvalue of multiplicity at least two of \hat{H}(\theta, \varphi)\}
\]
is finite.

The proof of this result is based on ideas from algebraic geometry in particular Bézout’s theorem and resultants. We recall main properties of resultants for the convenience of the reader. For further details see Chapter 3 of [14]. Given two
polynomials $f(x) = \sum_{j=0}^{d} a_j x^j$ and $g(x) = \sum_{\ell=0}^{D} b_\ell x^\ell$, their resultant $R(f, g)$ is defined as the determinant of the Sylvester matrix

\[
\begin{pmatrix}
a_0 & a_1 & \ldots & a_d & 0 & 0 & \ldots & 0 \\
0 & a_0 & a_1 & \ldots & a_d & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & a_1 & a_2 & \ldots & a_d \\
b_0 & b_1 & \ldots & \ldots & b_\ell & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & b_0 & \ldots & \ldots & \ldots & \ldots & b_D
\end{pmatrix}
\]

We have that $R(f, g) = 0$ if $f$ and $g$ have a common zero.

We now return to the original problem. Define for $E \in \mathbb{R}$ the polynomial

\[
P_E(u, v) = (uv)^p q \det(\widetilde{H}(u, v) - E)
\]

We can write $P_E(u, v) = \sum_{j=0}^{2pq} a_j(u)v^j$ and $\partial_E P_E(u, v) = \sum_{j=0}^{2pq-1} b_j(u)v^j$ for some polynomials $a_j, b_j$ in $u$. Then we can define the resultant of these two polynomials, which will be a function of $u$

\[
f(u) = R(P_E(u, \cdot), \partial_E P_E(u, \cdot)).
\]

Similarly, we can define

\[
g(v) = R(P_E(\cdot, v), \partial_E P_E(\cdot, v)).
\]

Since $f$ and $g$ are polynomials, they are either constant equal to 0 or have finitely many zeros.

**Lemma 5.3.** If $f$ and $g$ are not the constant zero function, then the number of points $(u, v)$ such that $E$ is an eigenvalue of $\widetilde{H}(u, v)$ of multiplicity at least two, is finite.

**Proof.** If $E$ is an eigenvalue of $\widetilde{H}(u, v)$ of multiplicity at least two, then we have that

\[P_E(u, v) = \partial_E P_E(u, v) = 0.
\]

In particular, we have that $f(u) = g(v) = 0$. Hence, the set of $(u, v)$ where $E$ is an eigenvalue of multiplicity at least two is contained in

\[\{(u, v) : f(u) = g(v) = 0\}
\]

which is finite. \qed

**Proof of Proposition 5.2.** Our assumptions imply that $f(u_0) \neq 0$ and $g(v_0) \neq 0$. Hence, we are done. \qed

**Remark 5.4.** The degree of $P_E(u, v)$ is $2p \cdot q$ and of $\partial_E P_E(u, v)$ is $2p \cdot q - 1$. Using this and an inspection of the previous argument shows that the set in (5.3) contains at most $(2pq)^2$ many points.
5.2. Perturbative analysis. Now, we will verify the conditions of Proposition 5.2. We first observe that the claim is symmetric in \( u \) and \( v \), so it suffices to exhibit \( u_0 \). The proof of the existence of \( v_0 \) is then similar.

Next, we will split the operator \((uv)^{p,q}(\tilde{H}(u,v) - E)\) into diagonal and off-diagonal part. For this define
\[
D(u,v)\psi(k,\ell) = d_{k,\ell}(u,v)\psi(k,\ell)
\]
where
\[
d_{k,\ell}(u,v) = u^2ve\left(\frac{k}{p}\right) + ve\left(-\frac{k}{p}\right) + uv^2e\left(\frac{\ell}{q}\right) + ve\left(-\frac{\ell}{q}\right) - Euv,
\]
and
\[
T(u,v)\psi(k,\ell) = uv\sum_{\tilde{k},\tilde{\ell}}\tilde{V}(\tilde{k},\tilde{\ell})\psi(k-\tilde{k},\ell-\tilde{\ell}).
\]

Then, we have that
\[
(uv)^{p,q}(\tilde{H}(u,v) - E) = D(u,v) + T(u,v).
\]
A simple counting argument shows that for every \( u \in C \), \( p(v) = P_E(u,v) = \det(D(u,v) + T(u,v)) \) is a polynomial of degree \( 2p \cdot q \). Hence, we have

**Lemma 5.5.** If there exists \( u \) such that there exist \( 2p \cdot q \) different \( v_j \) such that
\[
0 \in \sigma(D(u,v_j) + T(u,v_j))
\]
is a simple eigenvalue, then the assumptions of Proposition 5.2 hold.

We will show the claim for all sufficiently small \( u \). We begin with the analysis of \( D(u,v) \). The next lemma is a simple computation.

**Lemma 5.6.** Let \( u \) be small, \( k,\ell \) be given. The solutions of
\[
d_{k,\ell}(u,v) = 0
\]
are given by
\[
v_{+}^{k,\ell} = -\frac{1}{u}e\left(-\frac{kq + \ell p}{pq}\right) + O(1), \quad v_{-}^{k,\ell} = ue\left(\frac{kq - \ell p}{pq}\right) + O(u^2).
\]

**Proof.** Let
\[
A = ue\left(\frac{kq - \ell p}{pq}\right) - Ee\left(\frac{\ell}{q}\right) + \frac{1}{u}e\left(-\frac{kq + \ell p}{pq}\right), \quad B = e\left(-\frac{2\ell}{q}\right)
\]
such that \( v_{\pm}^{k,\ell} \) satisfy the quadratic equation \( v^2 + Av + B = 0 \). Since \( u \) is small, we obtain for the roots
\[
v_{\pm}^{k,\ell} = -\frac{1}{2}A(1 \pm \frac{2B}{A^2} + O(B^2/A^4))
\]
which yields the claim after some computations. \( \square \)

The next lemma can be proven by a computation.

**Lemma 5.7.** There exists a constant \( C > 0 \). For \((k,\ell) \neq (\tilde{k},\tilde{\ell})\)
\[
d_{k,\ell}(u,v^{k,\ell}_+) \geq \frac{C}{u}, \quad d_{k,\ell}(u,v^{k,\ell}_-) \geq Cu
\]
and
\[
|v^{k,\ell}_+ - \tilde{v}^{k,\ell}_+| \geq \frac{C}{u}, \quad |v^{k,\ell}_- - \tilde{v}^{k,\ell}_-| \geq \frac{C}{u}, \quad |v^{k,\ell}_- - \tilde{v}^{k,\ell}_+| \geq Cu.
\]

In the following, we will prove...
Proposition 5.8. For $u$ small enough, the $2pq$ zeros of $v \mapsto \det(D(u,v) + T(u,v))$ are given by \( \{ \tilde{v}_+^{k,\ell}, \tilde{v}_-^{k,\ell} \}_{k,\ell} \) with
\[
\tilde{v}_+^{k,\ell} = v_+^{k,\ell} + O(1), \quad \tilde{v}_-^{k,\ell} = v_-^{k,\ell} + O(u^2)
\]
Furthermore $0$ is a simple eigenvalue of $D(u,\tilde{v}_+^{k,\ell}) + T(u,\tilde{v}_+^{k,\ell})$ and $D(u,\tilde{v}_-^{k,\ell}) + T(u,\tilde{v}_-^{k,\ell})$.

As discussed before, this finishes the proof of Theorem 5.1. The proof of this proposition will be given by a perturbative analysis. We will first need

Lemma 5.9. $D(u,v) + T(u,v)$ is normal.

Proof. We have to show that $A(u,v)^*A(u,v) = A(u,v)A(u,v)^*$ with $A(u,v) = D(u,v) + T(u,v)$. Since multiplying an operator by a scalar doesn’t change this condition, it suffices to check that $\tilde{H}(u,v)$ is normal. So we have to show
\[
\tilde{H}(u,v)^*\tilde{H}(u,v) - \tilde{H}(u,v)\tilde{H}(u,v)^* = 0
\]
for all $(u,v) \in \mathbb{C}^2$. For $(u,v) \in \mathbb{R}^2$, $\tilde{H}(u,v)$ is self-adjoint and thus the previous equation holds. By analyticity of $(u,v) \mapsto \tilde{H}(u,v)$ the equation holds for all $(u,v) \in \mathbb{C}^2$ and we are done. \(\square\)

We need the following general fact about normal matrices. It is inspired by Section 9 of [15]. We denote by $\sigma(A)$ the spectrum of $A$.

Proposition 5.10. Let $A$ and $B$ be normal matrices, $\varepsilon > 0$, and $t \in (0, \frac{1}{100})$. Assume that
\begin{enumerate}
  \item[(i)] $0$ is a simple eigenvalue of $A$.
  \item[(ii)] $\sigma(A) \cap \{ z : |z| < \varepsilon \} = \{ 0 \}$.
  \item[(iii)] $\| A - B \| \leq t\varepsilon$.
\end{enumerate}

Then
\[
\{ \lambda \} = \sigma(B) \cap \{ z : |z| < \frac{\varepsilon}{2} \}
\]
with $|\lambda| \leq t\varepsilon$. Denote by $\varphi$ a normalized solution of $B\varphi = \lambda\varphi$. If $\| A\psi \| \leq t\varepsilon$, then there exists a such that
\[
\| \psi - a\varphi \| \leq 16t.
\]

Proof. The first part of the statement follows from eigenvalues being Lipschitz in the perturbation. For the second part, write
\[
\psi = \langle \varphi, \psi \rangle \varphi + \psi^\perp.
\]
Since $\psi^\perp$ is in the orthogonal complement of $\varphi$, we have that
\[
\| (B - \lambda)\psi^\perp \| \geq \frac{\varepsilon}{4} \| \psi^\perp \|.
\]
Hence, we obtain
\[
\| (B - \lambda)\psi \| = \| (B - \lambda)\psi^\perp \| \geq \frac{\varepsilon}{4} \| \psi^\perp \|.
\]
Using that $\| (B - \lambda)\psi \| \leq 4t\varepsilon$, the claim follows. \(\square\)
Define $A_u(v) = D(u,v) + T(u,v)$ for $u$ small enough. It follows from the general theory of normal operators that the eigenvalue $\lambda(v)$ of $A_u(v)$ satisfying

\[(5.20)\quad |\lambda(v^{k,\ell})| = O(u^2)\]

and $v \mapsto \lambda(v)$ is an analytic function whose derivative is given by

\[(5.21)\quad \lambda'(v) = \langle \psi, \partial_v A_u(v) \psi \rangle\]

where $\psi$ is any normalized solution of $(A_u(v) - \lambda(v))\psi = 0$.

**Lemma 5.11.** Let $v = O(u^2)$. We have that $\lambda'(v) = e(-\frac{k}{p}) + O(u)$.

**Proof.** The previous proposition with test function $\psi$ given by

$\psi(n,m) = \begin{cases} 1, & (n,m) = (k,\ell); \\ 0, & \text{otherwise} \end{cases}$

shows this claim. □

**Proof of Proposition 5.8.** By the previous results, we can find $\tau = O(u^2)$ such that

$\lambda(v^{k,\ell} + \tau) = 0$

is a simple eigenvalue of $A_u(v^{k,\ell} + \tau)$.

Repeating the previous the previous considerations for $v^{k,\ell} - \text{finishes the proof.}$ □

6. The spectrum of two dimensional periodic Schr"odinger operators

The next theorem is our discrete analog of the Bethe–Sommerfeld conjecture.

**Theorem 6.1.** Let $p, q \geq 2$ be co-prime. Then there exists $\delta = \delta(p, q) > 0$ such that for any $(p,q)$-periodic $V : \mathbb{Z}^2 \rightarrow \mathbb{R}$ with $\|V\|_\infty \leq \delta$, $\sigma(\Delta + V)$ is an interval.

**Proof.** We have $\sigma(\Delta) = [-2d, 2d]$. For any $E \in (-2d, 2d)$ there exist some $\ell, k \in \mathbb{V}$ such that

$2 \cos(2\pi(k + \theta)) + 2 \cos(2\pi(\ell + \varphi)) = E$

for infinitely many $(\theta, \varphi) \in (0, \frac{1}{p}) \times (0, \frac{1}{q})$. Furthermore, these are all points of increase. By Theorem 6.1 at most finitely many of these do not correspond to simple eigenvalues. Hence, $E_j(\theta, \varphi) = E$ is a simple eigenvalue and increasing. By Lemma 3.6 every $E$ is thus in the interior of a band. This implies that the bands are overlapping. The claim then follows by Theorem 3.8. □

The following example shows that it is necessary that at either $p$ or $q$ is odd for the conclusions of the previous theorem hold. This begs the following question

**Question 6.2.** What is the optimal condition on $p$ and $q$ such that the conclusions of the previous theorem hold?

We now come to the counterexample with even periods. Let $d \geq 1$, $\delta > 0$ and define a 2-periodic potential by

\[(6.1)\quad V_\delta(\mathbb{Z}^2) = \begin{cases} \delta, & \sum_{j=1}^d n_j \mod 2 = 0; \\ -\delta, & \text{otherwise.} \end{cases}\]

Clearly $V_\delta = \delta V_1$ and $\|V\|_\infty \leq \delta$. 

Theorem 6.3. We have that
\[
\sigma(\Delta + V_\delta) \cap (-\delta, \delta) = 0.
\]
This theorem shows that given \( p, q \) even. Then there exists a \((p, q)\)-periodic potential \( V \) with \( ||V||_\infty \) arbitrarily small such that the spectrum of \( \Delta + V \) contains a gap. Unfortunately, this example is very specific and only allows to create one gap in the center of spectrum, in order to construct limit-periodic examples with Cantor spectrum, one would need a better mechanism. This brings us to

**Question 6.4. Is there another way to open gaps?**

The key step of the proof is the following lemma

**Lemma 6.5.** For any \( \psi \in \ell^2(\mathbb{Z}^d) \) and \( \delta > 0 \), we have that
\[
\langle \Delta \psi, V_\delta \psi \rangle = 0.
\]

**Proof.** Write \( \Delta = \sum_{j=1}^d \Delta_j \) with \( \Delta_j \psi(n) = \psi(n + b_j) + \psi(n - b_j) \). By linearity, it clearly suffices to show that \( \langle \Delta_j \psi, V_\delta \psi \rangle = 0 \) for each \( j \). Compute
\[
\langle \Delta_j \psi, V_\delta \psi \rangle = \sum_n \psi(n) \psi(n + b_j) (V_\delta(n) + V_\delta(n + b_j)).
\]
Since \( V_\delta(n) + V_\delta(n + b_j) = 0 \), the claim follows. \( \square \)

**Proof of Theorem 6.3.** A computation using the last lemma shows
\[
|| (\Delta + V_\delta) \psi ||^2 = || \Delta \psi ||^2 + \delta^2 || \psi ||^2 \geq \delta^2 || \psi ||^2
\]
for \( \psi \in \ell^2(\mathbb{Z}^d) \). This implies the claim. \( \square \)

### 7. Limit-periodic potentials

We recall that a function \( V : \mathbb{Z}^d \to \mathbb{R} \) is limit-period if it is the limit of periodic functions. The following theorem asserts the existence of limit-periodic potentials whose spectrum is an interval. In fact it shows that the spectrum of all limit-periodic potentials with suitable periods that are sufficiently small in an appropriate sense is an interval.

**Theorem 7.1.** Let \( p, q \geq 2 \) be coprime. There exists a sequence of \( \delta_t > 0 \) such that \( \sum_{t=1}^\infty \delta_t < \infty \) and if \( V_t : \mathbb{Z}^d \to \mathbb{R} \) is a sequence of \((p_t, q_t)\)-periodic potentials satisfying \( ||V_t||_\infty \leq \delta_t \) then the spectrum of
\[
\Delta + \sum_{t=1}^\infty V_t
\]
is an interval.

An important question that this theorem leaves unanswered is the qualitative behavior of the sequence \( \delta_t > 0 \). In the continuous setting Karpeshina and Lee \([10],[11],[12]\) have shown that one can take \( \delta_t = C \exp(-2^m) \) for some constants \( C, \eta > 0 \) to obtain that the spectrum contains a semi-axis.

I expect that using KAM-type techniques as employed by Karpeshina and Lee, one should be able to obtain a similar estimate in our setting. However, such a proof will be much more involved then the one given here. The estimates of Karpeshina and Lee would also allow us to prove that the spectrum is purely absolutely continuous, which is not possible using our methods.
We now begin the proof of Theorem 7.1. We first note that if \( p, q \geq 2 \) are coprime, then also \( p^t, q^t \) are coprime for \( t \geq 1 \).

**Proposition 7.2.** Let \( p, q \geq 2 \) be coprime and \( V : \mathbb{Z}^d \to \mathbb{R} \) be \((p^t, q^t)\)-periodic. Assume the bands of \( \Delta + V \) viewed as a \((p^t, q^t)\)-periodic operator are overlapping, then for \( s \geq t \) also the bands of \( \Delta + V \) viewed as a \((p^s, q^s)\)-periodic operator are overlapping.

This proposition makes no claim over the size of the overlap. In fact as \( s \to \infty \), the size of the overlap goes to 0 as we have seen in Theorem 3.11.

**Proof.** Let \( \sigma(\Delta + V) = [E_0, E_1] \). Let \( E \in (E_0, E_1) \), by Theorem 3.3 there are infinitely many \((\varphi, \theta)\) such that \( E_j(\varphi, \theta) = E \) for some \( j \) and it is increasing. By Theorem 5.1 at most finitely many of them are not simple eigenvalues. Hence, there is \((\varphi, \theta)\) and \( j \) such that \( E_j(\varphi, \theta) = E \) is a simple eigenvalue and \( E_j \) is increasing. By Lemma 3.6 \( E \) is in the interior of a band. The claim follows. \( \square \)

**Lemma 7.3.** Let \( p, q \geq 2 \) be coprime and \( t \geq 1 \). Let \( \mathcal{V} \) be a compact set of \((p^t, q^t)\)-periodic potentials, such that for every \( V \in \mathcal{V} \) the bands of \( \Delta + V \) viewed as a \((p^t, q^t)\)-periodic operator are overlapping.

Then there exists \( \delta > 0 \) such that for every \((p^{t+1}, q^{t+1})\)-periodic \( W \) with \( \|W\|_{\infty} \leq \delta \), the bands of \( \Delta + V + W \) are overlapping.

**Proof.** By the previous proposition, also the bands of \( \Delta + V \) viewed as a \((p^{t+1}, q^{t+1})\)-periodic operator are overlapping for \( V \in \mathcal{V} \). Thus by Lemma 3.10 there exists \( \delta > 0 \) such that for the bands of \( \Delta + V \) viewed as a \((p^{t+1}, q^{t+1})\)-periodic operator are \( \delta \)-overlapping for \( V \in \mathcal{V} \). Take \( \delta = \frac{1}{2} \delta \) and the claim follows by Theorem 3.8. \( \square \)

**Proof of Theorem 7.1.** We have already seen that there exists \( \delta_1 > 0 \) such that the bands of \( \Delta + V_t \) are overlapping for any \((p, q)\)-periodic \( V_t \) with \( \|V_t\|_{\infty} \leq \delta_1 \). We denote the set of all such \( V_t \) by \( \mathcal{V}_1 \), which is clearly compact.

By the previous lemma, there exists \( \delta_2 > 0 \) such that for all \( V_1 \in \mathcal{V}_1 \) and all \((p^2, q^2)\)-periodic \( V_2 \) satisfying \( \|V_2\|_{\infty} \leq \delta_2 \), we have that the bands of \( \Delta + V_1 + V_2 \) viewed as a \((p^2, q^2)\)-periodic operator are overlapping. We denote the set of all such \( V_2 \) by \( \mathcal{V}_2 \). We also see that \( \mathcal{V}_2 \) and \( \mathcal{V}_1 + \mathcal{V}_2 \) are compact.

We can iterate this process to construct sets \( \mathcal{V}_t \) consisting of all \((p^t, q^t)\)-periodic \( V_t \) with \( \|V_t\|_{\infty} \leq \delta_t \) for some sequence \( \delta_t > 0 \). By possibly making \( \delta_t > 0 \) smaller, we can assume that \( \sum_{t=1}^{\infty} \delta_t \) converges. Furthermore, we will have that for every \( T \geq 1 \) and \( V_t \in \mathcal{V}_t \) for \( 1 \leq t \leq T \)

\[
\sigma\left(\Delta + \sum_{t=1}^{T} V_t\right)
\]

is an interval. Since \( \sum_{t=T}^{\infty} V_t \to 0 \) as \( T \to \infty \), it follows that also

\[
\sigma\left(\Delta + \sum_{t=1}^{\infty} V_t\right)
\]

is an interval, which is the claim. \( \square \)
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