MULTIPLICITIES IN THE MIXED TRACE COCHARACTER SEQUENCE OF TWO $3 \times 3$ MATRICES

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Abstract. We find explicitly the multiplicities in the (mixed) trace cocharacter sequence of two $3 \times 3$ matrices over a field of characteristic 0 and show that asymptotically they behave as polynomials of seventh degree. As a consequence we obtain also the multiplicities of certain irreducible characters in the cocharacter sequence of the polynomial identities of $3 \times 3$ matrices.

INTRODUCTION

All considerations in this paper are over an arbitrary field $F$ of characteristic 0. Let $n \geq 2$ be a fixed integer. Consider the $d$ generic $n \times n$ matrices $X_1, \ldots, X_d$, $d \geq 2$. The main results of our paper concern the case $n = 3$ and $d = 2$. There are several important algebras related to $X_1, \ldots, X_d$. Among them are the algebra $R_d$, generated by $X_1, \ldots, X_d$, the pure (or commutative) trace algebra $C_d$ generated by the traces of all products $\text{tr}(X_{i_1} \cdots X_{i_k})$, and the mixed (or noncommutative) trace algebra $T_d$ generated by $R_d$ and $C_d$ regarding the elements of $C_d$ as scalar matrices. We denote by $C, R$, and $T$ the corresponding algebras related to a countable set $\{X_1, X_2, \ldots\}$ of generic matrices.

The algebra $R$ is one of the most important objects in the theory of algebras with polynomial identities. It is isomorphic to the factor algebra $F\langle x_1, x_2, \ldots \rangle/I(M_n(F))$ of the free associative algebra $F\langle x_1, x_2, \ldots \rangle$ modulo the ideal $I(M_n(F))$ of the polynomial identities of the $n \times n$ matrix algebra $M_n(F)$. The algebra $C_d$ has a natural interpretation in classical invariant theory, as the algebra of invariants of the general linear group $GL_n(F)$ acting by simultaneous conjugation on $d$ matrices of size $n$. The algebra $T_d$ is known as the algebra of matrix concominants and also consists of the invariant functions under a suitable action of $GL_n(F)$. See e.g. the books [12], [11], or [6] as a background on $R_d$, $C_d$, and $T_d$ and their application to invariant theory, structure theory of PI-algebras, and theory of finite dimensional division algebras.

The algebras $R_d, C_d, T_d$ as well as $R, C, T$ are graded by multidegree. The symmetric group $S_k$ acts naturally on the multilinear elements of degree $k$ of $R, T, C$. The corresponding $S_k$-characters $\chi_k(R) = \chi_k(M_n(F)), \chi_k(C), \chi_k(T)$ are called, respectively, the cocharacter of the polynomial identities, the pure trace cocharacter,
and the mixed trace cocharacter of $n \times n$ matrices. They decompose as

$$\chi_k(M_n(F)) = \sum_{\lambda \vdash n} m_\lambda(M_n(F)) \chi_\lambda,$$

$$\chi_k(C) = \sum_{\lambda \vdash n} m_\lambda(C) \chi_\lambda,$$

$$\chi_k(T) = \sum_{\lambda \vdash n} m_\lambda(T) \chi_\lambda,$$

where $\lambda$ is a partition of $k$ and $\chi_\lambda$ is the related irreducible $S_k$-character. If some of the multiplicities $m_\lambda(M_n(F)), m_\lambda(C), m_\lambda(T)$ is nonzero, then $\lambda = (\lambda_1, \ldots, \lambda_{n^2})$ is a partition in not more than $n^2$ parts.

The algebras $R_d, C_d, T_d$ are also $GL_d(F)$-modules with $GL_d(F)$-action induced by the canonical $GL_d(F)$-action on the vector space with basis $\{X_1, \ldots, X_d\}$ and decompose as

$$R_d = \sum_{k \geq 0} \sum_{\lambda \vdash k} m_\lambda(M_n(F)) W_\lambda, \quad C_d = \sum_{k \geq 0} \sum_{\lambda \vdash k} m_\lambda(C) W_\lambda, \quad T_d = \sum_{k \geq 0} \sum_{\lambda \vdash k} m_\lambda(T) W_\lambda,$$

where $W_\lambda$ is the irreducible $GL_d(F)$-module related to $\lambda = (\lambda_1, \ldots, \lambda_d)$, and the multiplicities of $W_\lambda$ are the same as the multiplicities of $\chi_\lambda$ in the corresponding $S_k$-cocharacter.

The Hilbert (or Poincaré) series of $R_d$ is

$$H(R_d) = H(R_d, t_1, \ldots, t_d) = \sum \dim R_d^{(k)} k_1^{1} \cdots k_d^{k_d},$$

where $R_d^{(k)}$ is the homogeneous component of multidegree $k = (k_1, \ldots, k_d)$. Similarly one defines the Hilbert series of $C_d$ and $T_d$. These series are symmetric functions and decompose as infinite linear combinations of Schur functions $S_\lambda(t_1, \ldots, t_d)$. Since the Hilbert series play the role of characters of $GL_d(F)$ and the Schur functions are the characters of the corresponding irreducible $GL_d(F)$-modules, the multiplicities $m_\lambda(M_n(F)), m_\lambda(C), m_\lambda(T)$ can be obtained from the Hilbert series of $R_d, C_d, T_d$ for $d = n^2$. For small $d$, the Hilbert series give the multiplicities for $\lambda = (\lambda_1, \ldots, \lambda_d)$ only. See the book by Macdonald \[13\] as a standard reading on theory of symmetric functions and the book by one of the authors \[5\] for the applications of representation theory of $S_k$ and $GL_d(F)$ to PI-algebras.

The algebras $C$ and $T$ are described in terms of invariant theory and are easier to study than the algebra $R$. Since the three algebras are very close to each other, the standard way to investigate the polynomial identities of $M_n(F)$ is via $C$ and $T$. The multiplicities of the cocharacters of $M_n(F), C, T$ are explicitly found for $n = 2$ only, by Formanek \[9\] for $M_2(F), C, T$, and, with different methods, by Drensky \[4\] for $M_3(F)$, see also Procesi \[14\]. In particular,

$$m_\lambda(T) = (\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_3 - \lambda_4 + 1),$$

if $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $m_\lambda(T) = 0$ if $\lambda_5 \neq 0$.

It follows from classical invariant theory that the Hilbert series of $C_d$ and $T_d$ can be expressed as multiple integrals but for $n \geq 3$ their direct evaluation is quite difficult and was given by Teranishi \[15, 16\] for $C_2$ and $n = 3$ and $n = 4$ only. Van den Bergh \[18\] found a graph theoretical approach for the calculation of $H(C_d)$ and $H(T_d)$. Berele and Stembridge \[2\] calculated the Hilbert series of $C_d$ and $T_d$ for $n = 3, d \leq 3$ and of $T_2$ for $n = 4$, correcting also some typographical errors in
the expression of $H(C_2)$ for $n = 4$ in [10]. Using the Hilbert series of $C_2$, $n = 3$, Berele [1] found an asymptotic expression of $m_{(\lambda_1, \lambda_2)}(C)$. The explicit form of the generating function of $m_{(\lambda_1, \lambda_2)}(C)$ was found by Drensky and Genov [7] correcting also a technical error in [1]. The proof of [7] is quite technical. Later Drensky and Genov [8] suggested a method to find the coefficients of the Schur functions in the expansion of a class of rational symmetric functions in two variables. This significantly simplified their proof of the formula for the generating function of $m_{(\lambda_1, \lambda_2)}(C)$. A recent general result of Carbonara, Carini and Remmel [3] states that, for any $n$, there exists a positive integer $u_n$, such that for a fixed partition $\lambda = (\lambda_2, \ldots, \lambda_n)$, there exist $u_n$ polynomials $P_{n,k}^\lambda(\lambda_1)$, $k = 0, 1, \ldots, u_n-1$, of degree $n-1$ and with the same leading term, with the property that $m_\lambda(C) = P_{n,k}^\lambda(\lambda_1)$, for $\lambda_1$ sufficiently large, where $\lambda = (\lambda_1, \lambda) = (\lambda_1, \ldots, \lambda_n)$, $k \equiv \lambda_1 \mod u_n$. Since the cocharacter sequences of $C$ and $T$ are related by $\chi_k(T) = \chi_{k+1}(C) \downarrow S_k$, and the multiplicities $m_\lambda(T)$ can be expressed in terms of the multiplicities of $C$ by the branching theorem, this easily implies a similar statement for $m_\lambda(T)$. For $n = 3$ the expression of $u_n$ given in [3] is $u_3 = 6$.

The purpose of the present paper is to calculate, for $n = 3$, the generating function of $m_{(\lambda_1, \lambda_2)}(T)$ and to find explicit expressions for $m_{(\lambda_1, \lambda_2)}(T)$. As in the case of $m_{(\lambda_1, \lambda_2)}(C)$, see [1], the multiplicity $m_{(\lambda_1, \lambda_2)}(T)$ behaves asymptotically as a polynomial of degree 7 in $\lambda_1$ and $\lambda_2$. Our approach is to apply the method of Drensky and Genov [8] to the explicit form of the Hilbert series of $T_2$ found by Berele and Stembridge [2]. As a consequence of some results of Formanek [9, 10] we obtain the exact values of the multiplicities $m_\lambda(M_3(F))$ for $\lambda = (\lambda_1, \ldots, \lambda_9)$ when $\lambda_3 = \ldots = \lambda_9 \geq 2$. The calculations have been performed independently by the first and the third authors in Palermo and by the second author in Sofia, as an additional warranty of their correctness. In both cases we have used Maple.

1. Preliminaries

Let $F[[x, y]]$ be the algebra of formal power series in two variables and let

$$\text{Sym}[[x, y]] = F[[x, y]]^{S_2}$$

be the subalgebra of symmetric functions. The set of Schur functions

$$\{S_\lambda(x, y) \mid \lambda = (\lambda_1, \lambda_2) \in \mathbb{Z} \times \mathbb{Z}, \lambda_1 \geq \lambda_2 \geq 0\}$$

forms a basis of $\text{Sym}[[x, y]]$ as a topological vector space with its usual formal power series topology. Hence for any symmetric function $f(x, y)$ in two variables,

$$f(x, y) = \sum_{i,j \geq 0} a(i, j)x^iy^j = \sum_{\lambda} m(\lambda)S_\lambda(x, y),$$

where $a(i, j) = a(j, i) \in F$ for all $i, j \geq 0$ and $m(\lambda) \in F$ is the multiplicity of $S_\lambda(x, y)$ in the decomposition of $f(x, y)$. The Schur function $S_\lambda(x, y)$ has the following simple expression

$$S_\lambda(x, y) = (xy)^{\lambda_2}(x^p + x^{p-1}y + \cdots + xy^{p-1} + y^p) = \frac{(xy)^{\lambda_2}(x^{p+1} - y^{p+1})}{x - y}, \quad (1)$$

where we have denoted $p = \lambda_1 - \lambda_2$. This easily gives that the coefficients $a(i, j)$ and the multiplicities $m(\lambda)$ of $f(x, y)$ are related by

$$m(\lambda_1, \lambda_2) = a(\lambda_1, \lambda_2) - a(\lambda_1 + 1, \lambda_2 - 1),$$
where $a(\lambda_1 + 1, \lambda_2 - 1) = 0$ if $\lambda_2 = 0$.

Drensky and Genov [7] introduced the multiplicity series of $f(x, y)$, namely

$$M(f)(t, u) = \sum_{\lambda_1 \geq \lambda_2 \geq 0} m(\lambda_1, \lambda_2) t^{\lambda_1} u^{\lambda_2} \in F[[t, u]].$$

Introducing a new variable $v = tu$, the series $M(f)(t, u)$ accepts the more convenient form

$$M'(f)(t, v) = M(f)(t, u) = \sum_{\lambda_1 \geq \lambda_2 \geq 0} m(\lambda_1, \lambda_2) t^{\lambda_1 - \lambda_2} v^{\lambda_2} \in F[[t, v]]$$

and the mapping $M' : \text{Sym}[[x, y]] \to F[[t, v]]$ is a continuous linear bijection.

The relation between symmetric functions and their multiplicity series is given, see (11) in [7], by

$$f(x, y) = \frac{xM'(f)(x, xy) - yM'(f)(y, xy)}{x - y}. \quad (2)$$

The symmetric functions depending on the product $xy$ only behave like constants under the mapping $M'$. If $a(z) \in F[[z]]$ is a formal power series in one variable $z$, then

$$M'(a(xy)f(x, y)) = a(v)M'(f(x, y)), \quad (3)$$

for any symmetric function $f(x, y)$, see (10) in [7].

We consider two linear operators $Y$ and $Y_a$ of $F[[t, v]]$. If $h(t, v) \in F[[t, v]]$, there exists a unique $f(x, y) \in \text{Sym}[[x, y]]$ such that $h(t, v) = M'(f)$. As in [7], we define

$$Y(h)(t, v) = M'\left(\frac{f(x, y)}{(1 - x)(1 - y)}\right).$$

Similarly, if $a(z) \in F[[z]]$, then

$$Y_a(h)(t, v) = M'\left(\frac{f(x, y)}{(1 - a(xy)x)(1 - a(xy)y)}\right).$$

Clearly, $Y = Y_1$. The next lemma gives relations between $h(t, v), Y(h)$, and $Y_a(h)$.

**Lemma 1.** If $h(t, v) \in F[[t, v]]$ and $a(z) \in F[[z]]$, then

$$Y(h)(t, v) = \frac{th(t, v) - vh(v, v)}{(1 - t)(t - v)}, \quad (4)$$

$$Y_a(h)(t, v) = \frac{th(t, v) - a(v)vh(a(v), v)}{(1 - a(v)t)(t - a(v)v)}. \quad (5)$$

**Proof.** The first equation is established in Proposition 2 of [7] and the second in Lemma 1 of [8]. Since [8] is an announcement only, we give the proof for completeness of the exposition. Since the mapping $M'$ is linear and continuous, it is sufficient to verify (5) for the image $h(t, v) = M'(S_\lambda(x, y))$ of the Schur functions only. By (1) and (3) we may assume that $\lambda_2 = 0$, i.e. $\lambda = (p, 0)$, and we obtain that $h(t, v) = t^p$. Direct verification shows that the right hand side of (5) has the form

$$h_a(t, v) = \frac{t^p - (a(v)v)^p}{(1 - a(v)t)(t - a(v)v)}.$$ 

Further calculations show that

$$xh_a(x, xy) = \frac{x^p - (xy)^p a(xy)^p}{(1 - a(xy)x)(1 - a(xy)y)},$$
\begin{align*}
\frac{xh_a(x,xy) - yh_a(y,xy)}{x-y} &= \frac{x^{p+1} - y^{p+1}}{(x-y)(1-a(xy)x)(1-a(xy)y)} \\
 &= \frac{S_{(p,0)}(x,y)}{(1-a(xy)x)(1-a(xy)y)},
\end{align*}

i.e. \( S_{(p,0)}(x,y) \) and \( h_a(t,v) \) satisfy the relation (2). Since \( M' \) is a bijection, this completes the proof. \qed

Finally, we need the following equality which is a translation in our language of a partial case of a result of Thrall [17], see also (8) in [7]:

\begin{equation}
M' \left( \frac{1}{(1-x^2)(1-xy)(1-y^2)} \right) = \frac{1}{(1-t^2)(1-v^2)}.
\end{equation}

2. MAIN RESULTS

Till the end of the paper we consider 3 \times 3 matrices only, i.e. we fix \( n = 3 \). Most of our considerations are for \( d = 2 \) and we replace the variables \( t_1, t_2 \) with \( x, y \). The Hilbert series of \( T_2 \) found by Berele and Stembridge [2] is

\begin{equation}
H(T_2, x, y) = \frac{1}{(1-x^2)(1-y^2)(1-x^2)(1-y^2)(1-x^2y)(1-xy^2)}.
\end{equation}

The first of our main results gives the multiplicity series of this Hilbert series.

**Theorem 2.** The multiplicity series of the Hilbert series \( H(T_2, x, y) \) of the mixed trace algebra \( T_2 \) of two generic \( 3 \times 3 \) matrices is

\begin{equation}
M'(H(T_2, x, y))(t,v) = \frac{h_3(v)t^3 + h_2(v)t^2 + h_1(v)t + h_0(v)}{(1-v)^t(1+v)^t(1+v^2)(1+t)(1-vt)},
\end{equation}

where the polynomials \( h_i(v) \in F[v] \), \( i = 0, 1, 2, 3 \), are

\begin{align*}
h_3(v) &= v^2(v^4 - v^3 + 3v^2 - v + 1), \\
h_2(v) &= v(2v^4 - 4v^3 + v^2 - v - 1), \\
h_1(v) &= v(-v^4 - v^3 + v^2 - 4v + 2), \\
h_0(v) &= v^4 - v^3 + 3v^2 - v + 1.
\end{align*}

It has also the expression

\begin{equation}
M'(H(T_2, x, y))(t,v) = \frac{a_3(v)}{(1-t)^3} + \frac{a_2(v)}{(1-t)^2} + \frac{a_1(v)}{1-t} + \frac{b(v)}{1+v} + \frac{c(v)}{1-vt},
\end{equation}

where

\begin{align*}
a_3(v) &= \frac{1}{2(1-v)(1+v)^2(1+v^2)^2}, \\
a_2(v) &= \frac{(3v^2 - 2v + 1)}{2^2(1-v)(1+v)^3(1+v^2)}, \\
a_1(v) &= \frac{(v^4 - 6v^3 + 14v^2 - 6v + 1)}{2^2(1-v)(1+v)^2}, \\
b(v) &= \frac{1}{2^4(1-v)(1+v)^2(1+v^2)^2}, \\
c(v) &= \frac{-v^4}{(1-v)(1+v)^2(1+v^2)^2}.
\end{align*}

**Proof.** Direct calculations, which we have performed using Maple, show that the expressions from the right hand sides of (8) and (9) are equal. So, the proof will be completed, if we show that the replacement of any of these expressions in (2) gives the Hilbert series in (7). This has been done also using Maple.

We think that it is interesting to know how we have calculated (8) and (9). We rewrite \( H(T_2, x, y) \) in the form

\begin{align*}
\frac{1}{1-xy} \left( \frac{1}{1-(x)(1-y)} \right)^2 \frac{1}{(1-(xy)x)(1-(xy)y)} \left( \frac{1}{(1-x^2)(1-xy)(1-y^2)} \right).
\end{align*}
It follows from (3), (4), and (5) that
\[ M'(H(T_2, x, y))(t, v) = \frac{1}{1 - v} Y^2 Y_x M' \left( \frac{1}{(1 - x^2)(1 - xy)(1 - y^2)} \right). \]
Taking into account (6), we obtain
\[ M'(H(T_2, x, y))(t, v) = \frac{1}{1 - v} Y^2 Y_x \left( \frac{1}{(1 - t^2)(1 - v^2)} \right). \]
Now we calculate consecutively
\[ w_0(t, v) = \frac{1}{(1 - t^2)(1 - v^2)}, \]
\[ w_1 = Y_x(w_0) = \frac{1 + v^2 t}{(1 - v^2)^2(1 + v^2)(1 - t^2)(1 - vt)}, \]
\[ w_2 = Y(w_1) = \frac{(-v^2(v^2 - v + 1)t^2 - v(v^2 - 1)t + (v^2 - v + 1))}{(1 - v)^4(1 + v^2)(1 - t^2)(1 - t)(1 - vt)}, \]
\[ w_3 = Y(w_2) = \frac{h_3(v)t^3 + h_2(v)t^2 + h_1(v)t + h_0(v)}{(1 - v)^6(1 + v^2)(1 - t^2)(1 - t)(1 - vt)}, \]
\[ M'(H(T_2, x, y))(t, v) = w_3 = \frac{w_3}{1 - v}, \]
as in (8). The expression (9) is obtained considering (8) as a rational function in \( t \) with coefficients from \( F(v) \) and presenting it as a sum of elementary fractions. □

**Lemma 3.** The function \( M'(H(T_2, x, y))(t, v) \) can be presented in the form
\[
\sum_{p, q \geq 0} \left( a_{pq}^+ + (-1)^q a_{pq}^- + (-1)^p b_p^+ + (-1)^{q+p} b_q^- \right) t^p v^q - \frac{1}{64} \sum_{p, r \geq 0} (-1)^{p+r} p_v^{2r+1} t^p v^q - \left\{ \begin{array}{c}
\sum_{p, s \geq 0} \left[ c_s^+ + (-1)^s c_s^- \right] (tv)^p v^s + \\
\sum_{p, w \geq 0} (-1)^w (tv)^p v^{2w},
\end{array} \right. \]
where
\[ a_{pq}^+ = \frac{(84p^2 + 14pq + q^2)q^5}{2^{57} 7!} + \frac{(40p^2 + 12pq + q^2)q^4}{2^{55} 5!} + \frac{(90p^2 + 49pq + 5q^2)q^3}{2^{53} 3^2} + \frac{1800p^2 + 11676pq + 4993q^2}{2^{55} 5!} + \frac{9492p + 11437q}{2^9 3 \cdot 7} + \frac{43}{64}, \]
\[ a_{pq}^- = \frac{(12p^2 + 18pq + 7q^2)q}{2^{103} 3} + \frac{8p^2 + 28pq + 17q^2}{2^9} + \frac{180p + 229q}{2^9 3} + \frac{13}{64}, \]
\[ b_p^+ = \frac{q + 4}{2^8}, \quad b_q^- = \frac{2q^3 + 24q^2 + 85q + 84}{2^{103}}, \]
\[ c_s^+ = \frac{s^7}{2^{57} 7!} + \frac{s^6}{2^{55} 5!}, \quad c_s^- = \frac{19s^5}{2^{56} 6!} + \frac{s^4}{2^{53} 3!} + \frac{391s^3}{2^{56} 6!} + \frac{79s^2}{2^{105}} - \frac{1453s}{2^{57} 5! 7} - \frac{17}{2^{10}}, \]
\[ c_s^- = \frac{s^3 + 9s^2 + 17s - 3}{2^{103}}. \]
Theorem 4. The multiplicities \( m_{(\lambda_1, \lambda_2)}(T) \) of the mixed trace cocharacter of \( 3 \times 3 \) matrices are given by the following formulas, where we have presented \((\lambda_1, \lambda_2)\) in the form \((p + q, q)\):

\[
m_{(\lambda_1, \lambda_2)}(T) = a^+_{pq} + (-1)^qa^-_{pq} + (-1)^pb^+_{pq} + (-1)^{p+q}b^-_{pq} - \frac{(-1)^{p+r} \varepsilon_1}{64} \\
+ \delta \left( c^+_{q-p} + (-1)^{q-p}c^-_{q-p} + \frac{(-1)^{p+w} \varepsilon_2}{64} \right).
\]

**Proof.** We decompose the rational functions \( a_3(v), a_2(v), a_1(v), b(v), c(v) \in F(v) \) as linear combinations of elementary fractions of the form

\[
\frac{1}{(1-v)^k}, \quad \frac{1}{(1+v)^k}, \quad \frac{1}{1+v}, \quad \frac{1}{1+v^2}, \quad \frac{v}{1+v^2}.
\]

The results are

\[
a_3(v) = \frac{1}{8(1-v)^6} + \frac{1}{8(1-v)^5} + \frac{3}{32(1-v)^4} + \frac{1}{16(1-v)^3} + \frac{5}{128(1-v)^2} + \frac{3}{128(1-v)} + \frac{1}{128(1+v)^2} + \frac{1}{128(1+v)},
\]

\[
a_2(v) = \frac{1}{16(1-v)^7} - \frac{1}{32(1-v)^6} + \frac{1}{32(1-v)^5} - \frac{1}{512(1-v)^4} + \frac{1}{32(1-v)^3} - \frac{25}{256(1-v)^2} + \frac{29}{256(1-v)} + \frac{1}{512(1-v)^2} + \frac{1}{512(1-v)},
\]

\[
a_1(v) = \frac{1}{32(1-v)^8} - \frac{1}{32(1-v)^6} - \frac{1}{32(1-v)^5} - \frac{1}{512(1-v)^4} + \frac{1}{32(1-v)^3} - \frac{256(1-v)^2}{1024(1-v)} + \frac{1024(1-v)^2}{1024(1-v)} + \frac{7}{512(1-v)^4} + \frac{7}{256(1+v)^3} + \frac{33}{1024(1+v)^2} - \frac{29}{1024(1+v)},
\]

\[
b(v) = \frac{1}{256(1-v)^2} + \frac{1}{256(1-v)} + \frac{1}{64(1+v)^2} + \frac{32(1+v)^3}{32(1-v)^5} - \frac{1}{1024(1-v)^2} - \frac{256(1+v)^2}{512(1-v)^4} - \frac{64(1+v)^2}{512(1-v)^4} - \frac{9}{256(1-v)^2} + \frac{256(1+v)}{256(1-v)^2} - \frac{1}{64(1+v^2)},
\]

\[
c(v) = -\frac{1}{32(1-v)^8} + \frac{1}{32(1-v)^6} + \frac{1}{32(1-v)^5} + \frac{1}{512(1-v)^4} - \frac{1}{32(1-v)^3} + \frac{256(1-v)^2}{1024(1-v)} - \frac{1}{256(1-v)^2} + \frac{1}{512(1-v)} - \frac{1}{512(1+v)^4} - \frac{1}{512(1+v)^3} + \frac{1}{1024(1+v)^2} + \frac{1}{256(1+v)} + \frac{1}{64(1+v^2)}.
\]

Applying the formula

\[
\frac{1}{(1-z)^k+1} = \sum_{m \geq 0} \binom{k+m}{k} z^m
\]

for \( z = \pm v, -v^2, \pm t, vt \), and expressing the binomial coefficients in (10) as polynomials of degree \( k \) in \( m \), we obtain the presentation of \( M'(H(T_2, x, y))(t,v) \) in the statement of the lemma. \( \square \)

The following theorem gives explicit formulas for the multiplicities \( m_{(\lambda_1, \lambda_2)}(T) \).
Here $\varepsilon_1 = 1$, if $q = 2r + 1$ and $\varepsilon_1 = 0$, if $q = 2r$; $\varepsilon_2 = 1$, if $q = p = 2w$ and $\varepsilon_2 = 0$, if $q - p = 2w + 1$; and $\delta = 0$, if $\lambda_1 > 2\lambda_2$ and $\delta = 1$, if $\lambda_1 \leq 2\lambda_2$. The expressions of $a_{pq}^\pm, b_q^\pm, c_s^\pm$ are given in the above Lemma 3.

Proof. Clearly, the multiplicity $m_{(\lambda_1, \lambda_2)}(T) = m_{(p+q, q)}(T)$ is equal to the coefficient of $t^p v^q$ in the expansion of the formal power series $M'(H(T_2, x, y))(t, v)$. This explains the contribution of $a_{pq}^\pm, b_q^\pm$ and $\varepsilon_1/64$ to $m_{(p+q, q)}(T)$. The coefficient $c_s^\pm$ contributes to $(tv)^s v^t$, which is equal to $t^p v^q$ for $s = q - p$. Hence we need $q \geq p$ which is equivalent to $\lambda_1 \leq 2\lambda_2$. In this case $\delta = 1$. Otherwise $\delta = 0$. The coefficient $\varepsilon_2$ appears by the same reasons as $\varepsilon_1$. □

The expression of $m_{(\lambda_1, \lambda_2)}(T)$ becomes much simpler if we are interested in their asymptotic behavior only. The following corollary follows immediately from Theorem 4, presenting $\lambda$ in the form $(p+q, q)$. It is in the spirit of the description of $m_{(\lambda_1, \lambda_2)}(C)$ given by Berele [1].

**Corollary 5.** For any partition $\lambda = (\lambda_1, \lambda_2)$, the multiplicities $m_{\lambda} = m_{\lambda}(T)$ of the mixed trace cocharacter of $3 \times 3$ matrices satisfy the condition

$$m_{\lambda} = \frac{\lambda_1^2}{7! 2^5} + \frac{(\lambda_1 - \lambda_2)\lambda_2^6}{6! 2^4} + \frac{(\lambda_1 - \lambda_2)^2\lambda_2^3}{5! 2^4} + \mathcal{O}((\lambda_1 + \lambda_2)^6),$$

if $\lambda_1 > 2\lambda_2 \geq 0$ and

$$m_{\lambda} = \frac{\lambda_1^2}{7! 2^5} + \frac{(\lambda_1 - \lambda_2)\lambda_2^6}{6! 2^4} + \frac{(\lambda_1 - \lambda_2)^2\lambda_2^3}{5! 2^4} - \frac{(2\lambda_2 - \lambda_1)^7}{7! 2^5} + \mathcal{O}((\lambda_1 + \lambda_2)^6),$$

if $2\lambda_2 \geq \lambda_1 \geq \lambda_2 > 0$.

If $\lambda_2$ is fixed, then, for $\lambda_1 > 2\lambda_2$,

$$m_{\lambda} = \lambda_1^2 \left( \frac{\lambda_2^2}{5! 2^4} + \frac{\lambda_2^4}{2^5 \cdot 3^3} + \frac{5\lambda_2^3}{2^6} + \frac{13\lambda_2^2}{4! 2^6} + \frac{1633\lambda_2}{5! 2^5} + \frac{15}{2^6} \right) + (-1)^{\lambda_2} \left( \frac{\lambda_2}{2^8} + \frac{1}{2^6} \right) + \mathcal{O}(\lambda_1).$$

Comparing with the asymptotic expression of $m_{(\lambda_1, \lambda_2)}(C)$ in the form of Theorem 15 in [7], we see that

$$m_{(\lambda_1, \lambda_2)}(C) \approx \frac{1}{9} m_{(\lambda_1, \lambda_2)}(T).$$

The formulas for $m_{(\lambda_1, \lambda_2)}(C)$ in [7] agree with the result of [3] that, for a fixed $\lambda_2, \ldots, \lambda_9$, and for $\lambda_1$ sufficiently large, the multiplicity $m_{\lambda_1}(C)$ behaves, depending on $\lambda_1$ mod 6, like a polynomial of second degree in $\lambda_1$, with a leading term which does not depend on $\lambda_1$ mod 6. The second part of Corollary 5 is in the same spirit. The careful study of the form of $M'(H(T_2, x, y))$ from Theorem 2 shows that it is sufficient to consider $\lambda_1$ mod 2 and not mod 6.

Finally, we give an application to the “ordinary” cocharacters of the polynomial identities of $3 \times 3$ matrices.

**Corollary 6.** If $\mu = (\mu_1, \ldots, \mu_9)$ is such that $\mu_3 = \cdots = \mu_9 \geq 2$, then the multiplicity $m_{\mu}(M_3(F))$ is equal to the multiplicity $m_{(\lambda_1, \lambda_2)}(T)$ found in this paper, where $\lambda_1 = \mu_1 - \mu_3$, $\lambda_2 = \mu_2 - \mu_3$.

Proof. By a result of Formanek [9], $m_{\mu}(T) = m_{\lambda}(T)$, for any partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ related by the equalities $\mu_k = \lambda_k + r$ for all $k = 1, \ldots, n^2$ and all $n \geq 2$. Also, for $\lambda_n$ sufficiently large, $m_{\lambda}(M_n(F)) = m_{\lambda}(T)$. 

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Another result of Formanek, in [10], gives that the latter equality holds already for \( \lambda_n^2 \geq 2 \). This immediately completes the proof. \( \square \)

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