THE ∗-VARIATION OF BANACH-MAZUR GAME AND FORCING AXIOMS

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Abstract. We introduce a property of posets which strengthens the \((\omega_1 + 1)\)-strategic closedness. This property is defined using a variation of Banach-Mazur game on posets, where the first player chooses a countable set of conditions instead of a single condition at each turn. We prove PFA is preserved under any forcing over a poset with this property. As an application we reproduce a proof of Magidor’s theorem about the consistency of PFA with some weak variations of the square principles. We also argue how different this property is from the \((\omega_1 + 1)\)-operational closedness, which we introduced in our previous work, by observing which portions of MA\(^{+}(\omega_1\text{-closed})\) are preserved or destroyed under forcing over posets with either property.

1. Introduction

As a part of studies on consequences of various forcing axioms, some studies have been devoted to understanding what kind of forcing preserves those axioms. As one of the earliest comprehensive results in this area, Larson \[12\] proved that MM is preserved under forcing over any poset such that every pairwise compatible subset of size at most \(\omega_1\) has a common extension. In fact his proof also works for PFA instead of MM, and for any \(\omega_2\)-directed closed poset (that is, a poset such that every directed subset of size at most \(\omega_1\) has a common extension). As for PFA, König and the author \[10\] extended Larson’ theorem by showing that it is preserved under forcing over any \(\omega_2\)-closed poset, although later in \[11\] they showed that it is not the case for MM.

Can we still find any reasonable broader class of posets preserving PFA? There are some limitations observed from known results. Caicedo and Velickovic \[2\] proved the following theorem.

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Theorem 1.1 (Caicedo and Velickovic [2]). Suppose that BPFA holds both in the universe and in an inner model $M$, and that $M$ computes $\omega_2$ correctly. Then $M$ contains all subsets of $\omega_1$.

Note that by Theorem 1.1 it is observed that any forcing preserving both PFA and $\omega_2$ adds no new subsets of $\omega_1$.

One natural generalization of the $\omega_2$-closedness still adding no new subsets of $\omega_1$ is the $(\omega_1 + 1)$-strategic closedness, defined in terms of the existence of a winning strategy for the second player in the corresponding (generalized) Banach-Mazur game of length $(\omega_1 + 1)$.

The $(\omega_1 + 1)$-strategic closedness is, however, not enough to preserve PFA. In fact, the natural poset adding a $\Box_{\omega_1}$-sequence is $(\omega_1 + 1)$-strategically closed, whereas $\Box_{\omega_1}$ fails under PFA as proved by Todorcevic [17].

Considering these facts, one possible approach to obtain a generalization of the $\omega_2$-closedness which remain to preserve PFA is to strengthen the notion of strategic closedness in some appropriate way.

In our previous paper [21], we proved that PFA is preserved under forcing with any operationally closed poset, that is, a poset such that the second player wins the corresponding Banach-Mazur game even when at each turn she is only allowed to use the Boolean infimum of preceding moves and the ordinal number of the turn to decide her move, not allowed to use full information about the preceding moves.

In this paper we introduce another strengthening of the $(\omega_1 + 1)$-strategic closedness preserving PFA in the following way. We introduce a variation of Banach-Mazur game where the first player chooses a countable set of conditions at each turn, instead of a single condition. At each moment, the Boolean infimum of all conditions he has chosen by the time plays the same role as his move in the usual Banach-Mazur game. We say a poset is $^*$-tactically closed if the second player wins even when at each turn she is only allowed to use the set of conditions the opponent has chosen by the time to decide her move. We also introduce the notion of $^*$-operational closedness which generalizes both operational closedness and $^*$-tactical closedness, and prove that PFA is preserved even under forcing with any poset with this property.

As an application of this result we give a proof of the following well-known theorem originally proved by Magidor.

Theorem 1.2 (Magidor [13]). PFA is consistent with the statement that $\Box_{\kappa, \omega_2}$ holds for all cardinals $\kappa$ such that $\kappa \geq \omega_2$.

Since now we have two seemingly different ways to strengthen the strategic closedness obtaining the preservation of PFA, it is natural to ask if they are really different. As an answer to this question, we show
that under $\text{MA}^+ (\omega_1$-closed), another well-known forcing axiom, neither operational nor $\ast$-tactical closedness implies the other, by showing that forcing over any poset with either property preserves some consequence of $\text{MA}^+ (\omega_1$-closed), which can be destroyed by forcing over some poset with the other property.

This paper is organized as follows. In the rest of this section we introduce and prove some lemmata about forcing which will be used in later sections. In §2 we quickly review the notion of generalized Banach-Mazur games, and then introduce the $\ast$-variation of the games and the notion of $\ast$-tactical closedness of posets. In §3 we prove the preservation of PFA under any $\ast$-tactically closed forcing, and as its application we give a proof of Theorem 1.2. In §4 we introduce a combinatorial principle named as SCP$^-$ (where SCP stands for the \textit{setwise climbability property}), and prove that it is equivalent to $\text{MA}_{\omega_2}$ for $\ast$-tactically closed posets. In §5 we show that the Chang’s Conjecture (CC) holds in any generic extension by any $\ast$-tactically closed forcing, whenever $\text{MA}^+ (\omega_1$-closed) is assumed in the ground model. Since it is known that there exists an operationally closed poset which forces the failure of CC, this result shows that the operational closedness does not imply the $\ast$-tactical closedness. In §6 we show that SCP$^-$ fails in any generic extension by any operationally closed forcing, whenever $\text{MA}^+ (\omega_1$-closed) is assumed in the ground model. Since the natural poset forcing SCP$^-$ is $\ast$-tactically closed, this shows that the $\ast$-tactical closedness does not imply the operational closedness, either.

Our notations are mostly standard. We adopt the same convention as [21] for posets: Each of our poset $\mathbb{P}$ is reflexive, transitive and separative (not necessarily antisymmetric), with one greatest element $1_\mathbb{P}$ being specified. We will use the following notations in later sections: For sets $M, N$ and an ordinal $\delta < \omega_2$, we denote $M \prec_\delta N$ if $M \prec N$ (as $\in$-structures) and $M \cap \delta = N \cap \delta$ hold. We also denote $M \prec_\delta^s N$ if $M \prec_\delta N$ and $M \cap \omega_2 \subseteq N \cap \omega_2$ hold.

We end this section with introducing a couple of lemmata which will be used in §5 and §6.

\textbf{Definition 1.3.} Let $\mathbb{P}$ be a poset, $\theta$ a sufficiently large regular cardinal and $N$ a countable elementary submodel of $H_\theta$.

1. We say $q \in \mathbb{P}$ is $(N, \mathbb{P})$-\textit{strongly generic} if for every $D \in N$ which is a dense subset of $\mathbb{P}$ there exists an $r \in D \cap N$ such that $r \geq_\mathbb{P} q$.
2. For an $(N, \mathbb{P})$-strongly generic $q$, we denote

$$q \upharpoonright N := \bigwedge \{ r \in N \cap \mathbb{P} \mid r \geq_\mathbb{P} q \},$$
where the meet in the right-hand side is computed in the boolean completion \( B(\mathbb{P}) \) of \( \mathbb{P} \).

(3) We say a \( \leq \mathbb{P} \)-descending sequence \( \langle p_n \mid n < \omega \rangle \subseteq N \) is \((N, \mathbb{P})\)-generic if for every dense subset \( D \in N \) of \( \mathbb{P} \) there exists an \( n < \omega \) such that \( p_n \in D \).

**Lemma 1.4.** Let \( \mathbb{P}, \theta \) and \( N \) be as in Definition 1.3. Suppose \( \langle p_n \mid n < \omega \rangle \) is an \((N, \mathbb{P})\)-generic sequence and \( q \) is a common extension of the \( p_n \)'s. then \( q \) is \((N, \mathbb{P})\)-strongly generic and it holds that

\[
q \upharpoonright N = \bigwedge \{p_n \mid n < \omega\}.
\]

**Proof.** By definitions it is easy to see that \( q \) is \((N, \mathbb{P})\)-strongly generic and \( q \upharpoonright N \leq \mathbb{P} p_n \) for each \( n < \omega \). For each \( r \in N \cap \mathbb{P} \) such that \( r \geq \mathbb{P} q \),

\[
E_r = \{p \in \mathbb{P} \mid p \leq \mathbb{P} r \lor p \perp \mathbb{P} r\}
\]

is a dense subset of \( \mathbb{P} \) and \( E_r \in N \) holds, and thus there exists an \( n < \omega \) such that \( p_n \in E_r \). Since \( r \) and \( p_n \) has \( q \) as a common extension, \( p_n \leq \mathbb{P} r \) must be the case. This gives the required equality. \( \square \)

Recall that for posets \( \mathbb{P} \) and \( \mathbb{R} \), a mapping \( \pi : \mathbb{R} \to \mathbb{P} \) is said to be a *projection* if it satisfies the following conditions:

(a) \( \pi \) is order-preserving.
(b) \( \pi(1_\mathbb{R}) = 1_\mathbb{P} \).
(c) \( \forall r \in \mathbb{R} \forall p \in \mathbb{P}[\pi(r) \geq \mathbb{P} p \Rightarrow \exists r' \leq \mathbb{R} \ r[p \geq \mathbb{P} \pi(r')]] \).

For basic properties of projections see [3]. Note that, for a projection \( \pi : \mathbb{R} \to \mathbb{P} \), whenever \( G \) is an \( \mathbb{R} \)-generic filter over \( V \), \( \pi''G \) generates a \( \mathbb{P} \)-generic filter \( \pi_*(G) \) over \( V \), and \( V[G] \) contains \( V[\pi_*(G)] \). Knowing this, in later sections we often abusively use each \( \mathbb{P} \)-name \( \tau \) to denote the \( \mathbb{R} \)-name representing \( \tau_{\pi_*(G)} \) in \( V[G] \) whenever \( G \) is \( \mathbb{R} \)-generic over \( V \).

**Lemma 1.5.** Let \( \mathbb{P} \) and \( \mathbb{R} \) be posets and \( \pi : \mathbb{R} \to \mathbb{P} \) a projection. Suppose \( \theta \) is a large enough regular cardinal, \( N \) is a countable elementary submodel of \( \mathcal{A} = (H_\theta, \in) \) with \( \mathbb{P}, \mathbb{R}, \pi \in N \), and \( q \in \mathbb{P} \) is an \((N, \mathbb{P})\)-strongly generic condition. Then for any dense subset \( D \in N \) of \( \mathbb{R} \) and any \( p \in N \cap \mathbb{R} \) satisfying \( \pi(p) \geq \mathbb{P} q \) there exists \( p' \leq \mathbb{R} p \) such that \( p' \in N \cap D \) and that \( \pi(p') \geq \mathbb{P} q \).

**Proof.** Note that \( D_p = \{r \in D \mid r \leq \mathbb{R} p\} \) is dense below \( p \) and \( D_p \in N \) holds. Since \( \pi \) is a projection and is in \( N \), \( \pi''D_p \) is dense below \( \pi(p) \) and \( \pi''D_p \in N \) holds. Since \( q \) is \((N, \mathbb{P})\)-strongly generic, there exists an \( r \in \pi''D_p \cap N \) such that \( r \geq \mathbb{R} q \). But then again since \( \pi \in N \) we have that there exists \( p' \in D_p \cap N \) such that \( \pi(p') = r \). This \( p' \) satisfies all requirements. \( \square \)
2. A variation of generalized Banach-Mazur game

Let us first quickly review some basics of Banach-Mazur games on posets, introduced by Jech \[9\] and generalized by Foreman \[5\]. Our notations mostly follows \[21\]. For a poset \(P\) and an ordinal \(\alpha\), the two-player game \(G_\alpha(P)\) is played as follows: Player I and II take turns to choose \(P\)-conditions one-by-one, so that each move is stronger than all preceding moves. Their turns take place in a well-ordered timeline. Player I goes first in the beginning of the game, whereas at other limit turns Player II goes first. Therefore a play of this game can be displayed as follows:

\[
\begin{align*}
\text{I : } & a_0 \quad a_1 \quad a_2 \quad \cdots \quad a_{\omega+1} \quad \cdots \\
\text{II : } & b_0 \quad b_1 \quad b_2 \quad \cdots \quad b_\omega \quad b_{\omega+1} \quad \cdots
\end{align*}
\]

Throughout this paper, we use the above numbering of turns, that is, we consider that Player I skips his limit turns. Player II wins this game if she was able to take turns \(\alpha\) times, without getting unable to make legitimate moves on the way. \(\alpha\) is called as the length of the game \(G_\alpha(P)\). In this paper we are mainly interested in games of length \((\omega_1 + 1)\).

In each turn of Player II during a play of \(G_\alpha(P)\), we call the sequence \(s\) of preceding moves of Player I (in the chronological order) as the current status of the turn. We call \(\bigwedge s\) (computed in \(B(P)\)) as the current position of the turn. This terminology makes sense because \(\bigwedge s\) gives the exact upper bound of possible moves of Player II of the turn. We also call the order type of preceding moves of Player II as the ordinal number of the turn.

A strategy (of Player II for \(G_\alpha(P)\)) is a function which, in each turn of Player II, takes the current status of the turn as an argument, and gives a \(P\)-condition as a suggestion for Player II’s move of the turn. A strategy \(\sigma\) is a winning strategy if Player II wins any play of \(G_\alpha(P)\) as far as she plays as \(\sigma\) suggests.

A strategy is called an operation (resp. tactic) if its suggestion depends only on the current position and the ordinal number (resp. only on the current position) of each turn.

We say \(P\) is \(\alpha\)-strategically (resp. \(\alpha\)-operationally, \(\alpha\)-tactically) closed if there exists a winning strategy (resp. operation, tactic).

Now we introduce a variation of \(G_{\omega_1+1}(P)\) to define new game closedness properties of posets.

**Definition 2.1.** For a poset \(P\), \(G^*(P)\) denotes the following game: Players take turns in the same way as in \(G_{\omega_1+1}(P)\), but Player I chooses
a subset of $\mathbb{P}$ of size at most countable at each turn, instead of a single condition. Therefore a play of $G^*(\mathbb{P})$ can be displayed as follows:

\[
\begin{array}{ccccccc}
I : & A_0 & A_1 & A_2 & \cdots & A_{\omega+1} & \cdots \\
II : & b_0 & b_1 & b_2 & \cdots & b_\omega & b_{\omega+1} & \cdots,
\end{array}
\]

where $A_\gamma \in \mathbb{P}^{\leq \aleph_0} \setminus \{\emptyset\}$ for each $\gamma \in \omega_1 \setminus \text{Lim}$, and $b_\gamma \in \mathbb{P}$ for each $\gamma < \omega_1$. They must obey the following rules (Player I is responsible for (a)–(c) and Player II is for (d):

(a) $\langle A_\gamma \mid \gamma \in \omega_1 \setminus \text{Lim} \rangle$ is $\subseteq$-increasing.
(b) For each $\gamma \in \omega_1 \setminus \text{Lim}$, $A_\gamma$ has a common extension in $\mathbb{P}$.
(c) For each $\gamma < \omega_1$, it holds that $\bigwedge A_{\gamma+1} \leq \mathbb{B}(\mathbb{P}) b_\gamma$.
(d) For each $\gamma < \omega_1$, $b_\gamma$ is a common extension of $A_\gamma$ (for limit $\gamma$ we define $A_\gamma = \bigcup\{A_\xi \mid \xi \in \gamma \setminus \text{Lim}\}$).

Again, Player II wins this game if she was able to make her $\omega_1$-th move.

Note that, in the above play of $G^*(\mathbb{P})$, if we replace each $A_\gamma$ with its boolean infimum, we will obtain a play of $G^{\omega_1+1}(\mathbb{B}(\mathbb{P}))$. In fact, as long as players play with perfect recall, $G^*(\mathbb{P})$ is essentially an equivalent game to $G^{\omega_1+1}(\mathbb{B}(\mathbb{P}))$, and even to $G^{\omega_1+1}(\mathbb{P})$. The point of introduction of $G^*(\mathbb{P})$ lies in cases when players have limited memory on preceding moves.

**Definition 2.2.** Let $\mathbb{P}$ be a poset. We let $\mathbb{P}^{\leq \aleph_0}$ denote the set $\{A \in \mathbb{P}^{\leq \aleph_0} \setminus \{\emptyset\} \mid A$ has a common extension in $\mathbb{P}\}$.

(1) A function from $\omega_1 \times \mathbb{P}^{\leq \aleph_0} \to \mathbb{P}$ is called a $*$-operation for $G^*(\mathbb{P})$.

In a play of $G^*(\mathbb{P})$, we say Player II *plays according to* a $*$-operation $\tau$ if, for each $\delta < \omega_1$ she chooses $\tau(\delta, A_\delta)$ as her $\delta$-th move as far as it is defined, where $A_\delta$ denotes the union of subsets of $\mathbb{P}$ chosen by Player I by the time. $\tau$ is said to be a winning $*$-operation for $G^*(\mathbb{P})$ if Player II wins any play of $G^*(\mathbb{P})$ as far as she plays according to $\tau$. $\mathbb{P}$ is $*$-operationally closed if there exists a winning $*$-operation for $G^*(\mathbb{P})$.

(2) A $*$-operation $\tau$ for $G^*(\mathbb{P})$ is called a $*$-tactic if the values of $\tau$ do not depend on its first argument. We often consider a $*$-tactic simply as a function defined on $\mathbb{P}^{\leq \aleph_0}$. $\mathbb{P}$ is $*$-tactically closed if there exists a winning $*$-tactic for $G^*(\mathbb{P})$.

A nontrivial example of a $*$-tactically closed forcing is found among the class of natural posets forcing the following combinatorial principles introduced by Schimmerling [16] which are weaker variations of Jensen’s square principles.
Definition 2.3. For an uncountable cardinal $\kappa$ and a cardinal $\lambda$ with $1 \leq \lambda \leq \kappa$, $\square_{\kappa,\lambda}$ denotes the following statement: There exists a sequence $\vec{C} = \langle C_\alpha \mid \alpha \in \kappa^+ \cap \text{Lim} \rangle$ such that for every limit $\alpha < \kappa^+$
(i) $1 \leq |C_\alpha| \leq \lambda$.
(ii) $C_\alpha$ consists of club subsets of $\alpha$ of order type $\leq \kappa$.
(iii) For every $C \in C_\alpha$ and $\beta \in \text{l.p.}(C)$, $C \cap \beta \in C_\beta$ holds.

Note that $\square_{\kappa,1}$ is the original Jensen’s square principle $\square_{\kappa}$.

Definition 2.4. Let $\kappa$ and $\lambda$ be as in Definition 2.3. $P_{\square_{\kappa,\lambda}}$ denotes the following poset. A condition $p \in P_{\square_{\kappa,\lambda}}$ is either the empty sequence $1_{P_{\square_{\kappa,\lambda}}} = \langle \rangle$, or of the form
(1) $p = \langle C_\alpha^p \mid \alpha \in (\alpha^p + 1) \cap \text{Lim} \rangle$ (where $\alpha^p \in \kappa^+ \cap \text{Lim}$)
which satisfies (i)–(iii) of Definition 2.3 for every limit $\alpha \leq \alpha^p$. $P_{\square_{\kappa,\lambda}}$ is ordered by initial segment.

Note that, since $P_{\square_{\kappa,\lambda}}$ is $(\kappa + 1)$-strategically closed, it preserves cardinalities below $\kappa^+$ and thus forces $\square_{\kappa,\lambda}$.

Theorem 2.5. Let $\kappa$ be an uncountable cardinal, and $\lambda$ a cardinal satisfying $2^{\aleph_0} \leq \lambda \leq \kappa$. Then $P_{\square_{\kappa,\lambda}}$ is $*$-tactically closed.

Proof. We define a $*$-tactic $\tau : [P_{\square_{\kappa,\lambda}}]^{<\aleph_0} \to P$ as follows. Let $A \in [P_{\square_{\kappa,\lambda}}]^{<\aleph_0}$ be arbitrary.
(1) If $A$ has the strongest condition $p$, if $p$ is of the form as in (1) of Definition 2.4, then we let $\tau(A) := p \upharpoonright \langle C_\alpha^p \mid \alpha \in \alpha^p + \omega \rangle$ (where $C_\alpha^p := \{ \{\alpha^p + n \mid n < \omega\} \}$).
If $p = 1_{P_{\square_{\kappa,\lambda}}}$ we let $\tau(A) := \langle C_\omega^p \rangle$ (where $C_\omega^p = \{\omega\}$).

In both cases it is easy to check that $\tau(A)$ forms a $P_{\square_{\kappa,\lambda}}$-condition which extends $p$.
(2) If $A$ has no strongest condition, then $q := \bigcup A$ is of the form
$q = \langle C_\alpha^q \mid \alpha \in \beta \cap \text{Lim} \rangle$ for some limit $\beta$.
Let $a = \{ \gamma \in \beta \cap \text{Lim} \mid q \upharpoonright (\gamma + 1) \in A \}$. Then $a$ is a countable subset of $\beta \cap \text{Lim}$ which is unbounded in $\beta$. In this case we set $\tau(A) := q \upharpoonright \langle C_\beta^q \rangle$, where
$C_\beta^q := \{ C \subseteq_{\text{club}} \beta \cap \text{cl}(a) \mid \forall \gamma \in \beta \cap \text{l.p.}(C)[C \cap \gamma \in C_\gamma] \}$.

Note that $|C_\beta^q| \leq 2^{\aleph_0}$ holds since $\text{cl}(a)$ is countable. On the other hand, $C_\beta^q$ is nonempty since $\beta \cap \text{cl}(a)$ has a subset of order type $\omega$ which is unbounded in $\beta$. Now it is easy to check that $\tau(A)$ forms a $P_{\square_{\kappa,\lambda}}$-condition which extends all conditions in $A$. 

Let us show that $\tau$ is a winning $\ast$-tactic. Consider any play of the game $G^*(\mathbb{P}_{\diamond,\lambda})$ where Player II plays according to $\tau$. Note that since $\mathbb{P}_{\diamond,\lambda}$ is $\omega_1$-closed, the play never ends throughout the first $\omega_1$ turns of both players. Let us display the play as follows:

I : \hspace{0.5cm} A_0 \hspace{0.5cm} A_1 \hspace{0.5cm} A_2 \hspace{0.5cm} \cdots \hspace{0.5cm} A_{\omega+1} \hspace{0.5cm} \cdots \\
II : \hspace{1.0cm} q_0 \hspace{0.5cm} q_1 \hspace{0.5cm} q_2 \hspace{0.5cm} \cdots \hspace{0.5cm} q_\omega \hspace{0.5cm} q_{\omega+1} \hspace{0.5cm} \cdots \\

Let us also set $A_\gamma := \bigcup\{A_\xi \mid \xi \in \gamma \setminus \text{Lim}\}$ for each limit $\gamma < \omega_1$. Note that then $q_\gamma = \tau(A_\gamma)$ holds for every $\gamma < \omega_1$. It is enough to show that $q_\gamma$'s have a common extension in $\mathbb{P}_{\diamond,\lambda}$. Since $\langle q_\gamma \mid \gamma < \omega_1 \rangle$ is a descending sequence in $\mathbb{P}_{\diamond,\lambda}$, $q := \bigcup\{q_\gamma \mid \gamma < \omega_1\}$ is of the form

$$ q = \langle C^q_\alpha \mid \alpha \in \beta^q \cap \text{Lim} \rangle \quad \text{for some limit } \beta^q < \kappa^+, $$

and for each $\gamma < \omega_1$, $q_\gamma = q \upharpoonright (\beta^q_\gamma + 1)$ for some limit $\beta^q_\gamma < \beta^q$. By the definition of $\tau$, it is easy to observe that $\langle \beta^q_\gamma \mid \gamma < \omega_1 \rangle$ is continuous, strongly increasing, and thus we have $\beta^q = \sup\{\beta^q_\gamma \mid \gamma < \omega_1\}$ and $\text{cf}\beta^q = \omega_1$.

So it is enough to show that there exists a club subset $C$ of $\beta$ such that $\text{o.t.}(C) = \omega_1$ and $C \cap \alpha \in C^q_\alpha$ holds for each $\alpha \in \text{l.p.}(C)$, because if it is the case, then

$$ q \cap \langle C^q_\gamma \rangle \quad (\text{where } C^q_\beta := \{C\}) $$

forms a $\mathbb{P}_{\diamond,\lambda}$-condition which extends all $q_\gamma$'s. For each $\gamma < \omega_1$, pick a subset $a_\gamma$ of $(\beta^q_{\omega(\gamma+1)} \cap \text{Lim})$ such that $a_\gamma$ is unbounded in $\beta^q_{\omega(\gamma+1)}$, o.t.$(a_\gamma) = \omega$ and that $q \upharpoonright (\xi + 1) \in A_{\omega(\gamma+1)}$ for every $\xi \in a_\gamma$ (this is possible because $A_{\omega(\gamma+1)}$ has no strongest condition). Then let

$$ C := \bigcup\{a_\gamma \mid \gamma < \omega_1\} \cup \{\beta^q_\gamma \mid \gamma \in \omega_1 \cap \text{Lim}\}. $$

It is easy to see that $C$ is a club subset of $\beta^q$ of order type $\omega_1$, and that any limit point of $C$ other than $\beta^q$ is of the form $\beta^q_\gamma$ for some limit $\gamma < \omega_1$. It is also easy to see that, for each limit $\gamma < \omega_1$,

$$ C \cap \beta^q_\gamma = \bigcup\{a_\delta \mid \delta < \gamma\} \cup \{\beta^q_\delta \mid \delta \in \gamma \cap \text{Lim}\} $$

is unbounded in $\beta^q_\gamma$ and is included by $\text{cl}(\{\xi \in \beta^q_\gamma \cap \text{Lim} \mid q \upharpoonright (\xi + 1) \in A_\gamma\})$. Now we have $C \cap \beta^q_\gamma \in C^q_\gamma$ for every limit $\gamma < \omega_1$, by induction on $\gamma$. \hfill $\Box$ (Theorem 2.5)

The following iteration lemma will be used in the next section.

**Lemma 2.6.** Suppose $\langle \mathbb{P}_\alpha, \dot{Q}_\beta \mid \alpha \leq \gamma, \beta < \beta \rangle$ is an iterated forcing construction such that:

(i) $\mathbb{P}_0$ is a trivial poset.
(ii) For each $\alpha < \gamma$ it holds that $\Vdash_{P_{\alpha}} "\hat{Q}_{\alpha} is *-tactically closed."$

(iii) For each limit ordinal $\xi \leq \gamma$, $P_\xi$ is either the direct limit or the inverse limit of $\langle P_\zeta \mid \zeta < \xi \rangle$, and the latter is the case whenever $\text{cf} \xi \leq \omega_1$ holds.

Then $P_\gamma$ is *-tactically closed.

Proof. Straightforward, considering the fact that every *-tactically closed poset $P$ has a winning *-tactic $\tau$ which is 'defensive', that is, $\tau(\{1_P\}) = 1_P$ holds. \hfill \Box

3. Preservation of PFA

In this section we prove the following theorem.

Theorem 3.1. PFA is preserved under any *-operationally closed forcing.

Note that Theorem 3.1 generalizes [21, Theorem 10] which claims that PFA is preserved under any operationally closed forcing, and since the basic structure of our proof is the same as the one given there, we will expose our proof somewhat briefly, rather focusing on differences from the older proof.

Proof. Suppose PFA holds in $V$. Let $P$ be any *-operationally closed poset, and $\sigma$ a winning *-operation for $G^*(P)$. We will show that PFA remains true in $V^P$. Let $\hat{Q}$ be any $P$-name for a proper poset, and $\langle \hat{D}_\xi \mid \xi < \omega_1 \rangle$ $P$-names for a dense subset of $\hat{Q}$. It is enough to show that there exists a $P$-name $\hat{F}$ such that

$\Vdash_P "\hat{F}$ is a filter on $\hat{Q}$ and $\hat{F} \cap \hat{D}_\xi \neq \emptyset$ for every $\xi < \omega_1."$

For any $P$-generic filter $G$ over $V$ and any $\hat{Q}_G$-generic filter $H$ over $V[G]$, we define a poset $R = R_{G,H}$ within $V[G][H]$ as follows: A condition of $R$ is either the empty sequence $1_R = \langle \rangle$, or of the form $P = \langle A_\xi^P \mid \xi \leq \alpha^P \rangle$ ($\alpha^P < \omega_1$) such that

(a) $P$ is a $\subseteq$-continuous increasing sequence of elements of $[P]_{\leq \omega_0} \cap V$.

(b) $\sigma(\xi, A_\xi^P) \geq_{R(P)} \bigwedge A_{\xi+1}^P$ for every $\xi < \alpha^P$.

(c) $\sigma(\alpha^P, A_{\alpha^P}) \in G$.

For this $P$ we write $\alpha^P$ and $A_{\alpha^P}$ as $l(P)$ and $A^P$ respectively.

$R$ is ordered by initial segment. Let $\hat{R}$ denote the canonical $(P \ast \hat{Q})$-name representing $R_{G,H}$.

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1 We conjecture that the similar statement as Lemma 2.6 for *-operationally closed posets is also valid, but at present we do not have a proof, because it is not clear to us whether this ‘defensiveness’ assumption is valid for *-operations.
Claim $S := P \ast \dot{Q} \ast \dot{R}$ is proper.

Proof of Claim. Let $\theta$ be a sufficiently large regular cardinal, and $N$ an arbitrary countable elementary submodel of $\langle H_{\theta}, \in \rangle$ containing all relevant objects. Set $\delta := N \cap \omega_1$. Let $\langle p, \dot{q}, \dot{P} \rangle \in S \cap N$ be arbitrary. It is enough to show that there exists an $(N, P)$-generic $p' \leq_P p$ such that whenever $G$ is a $P$-generic filter over $V$ with $p' \in G$, there exists an $(N[G], \dot{Q}_G)$-generic $q' \leq_{\dot{Q}_G} \dot{q}_G$ satisfying the following: Whenever $H$ is a $\dot{Q}_G$-generic filter over $V[G]$ with $q' \in H$, there exists an $(N[G][H], \dot{R}_{G*H})$-generic $P' \leq_{\dot{R}_{G*H}} \dot{P}_{G*H}$.

First pick an $(N, P)$-generic sequence $\langle p_n \mid n < \omega \rangle$ with $p_0 = p$. Set $A := \{s \in N \cap P \mid \exists n < \omega [s \geq_P p_n]\}$. Since $P$ is $(\omega + 1)$-strategically closed, an easy density argument derives that $p_n$'s have a common extension in $P$, and therefore $A \in [\mathbb{P}]^{\omega_1}$. Set $p' := \sigma(\delta, A)$.

Suppose $G$ is any $P$-generic filter over $V$ with $p' \in G$. Then by the $(N, P)$-genericity of $\langle p_n \mid n < \omega \rangle$ it is easy to see that

$$A = N \cap G.$$  

Since $\dot{q}_G \in N[G]$ and $\dot{Q}_G$ is proper in $V[G]$, we can pick an $(N[G], \dot{Q}_G)$-generic $q' \leq_{\dot{Q}_G} \dot{q}_G$. Now suppose $H$ is any $\dot{Q}_G$-generic filter over $V[G]$ with $q' \in H$. Since $\dot{P}_{G*H} \in N[G][H]$, it is possible to pick an $(N[G][H], \dot{R}_{G*H})$-generic sequence $\langle P_n \mid n < \omega \rangle$ with $P_0 = \dot{P}_{G*H}$. Set $\bar{P} := \bigcup \{P_n \mid n < \omega\}$. Again by an easy density argument we have that $\bar{P}$ is of the form $\langle A_\gamma \mid \gamma < \delta \rangle$.

Subclaim $\bigcup \{A_\gamma \mid \gamma < \delta\} = A$.

Proof of Subclaim. Since $P_n \in N[G][H]$ for each $n < \omega$ and $\delta \subseteq N[G][H]$, we have $A_\gamma \subseteq N[G][H]$ for each $\gamma < \delta$. Since $\rho' \in G$ is $(N, P)$-generic and $q' \in H$ is $(N[G], \dot{Q}_G)$-generic, we have $N[G][H] \cap V = N$. Note that $A_\gamma \subseteq V$ and is countable for each $\gamma < \delta$, and therefore $A_\gamma \subseteq N$ holds for each $\gamma < \delta$. Moreover $A_\gamma \subseteq G$ also holds for each $\gamma < \delta$ by the definition of $\dot{R}$. Thus by (2) we have $\bigcup \{A_\gamma \mid \gamma < \delta\} \subseteq A$. For the inclusion of the other direction, by (2) and the $(N[G][H], \dot{R}_{G*H})$-genericity of the sequence $\langle P_n \mid n < \omega \rangle$, it is enough to show the following density lemma:

Lemma 3.2. In $V[G][H]$, $D_a = \{R \in \dot{R}_{G*H} \mid a \in A^R\}$ is dense in $\dot{R}_{G*H}$ for each $a \in G$.

Proof of Lemma 3.2. Suppose $R \in \dot{R}_{G*H}$ and $a \in G$. By the definition of $\dot{R}_{G*H}$ it holds that $\sigma(l(R), A^R) \in G$. Therefore $a$ and $\sigma(l(R), A^R)$
are compatible, and since $\sigma(l(R) + 1, A^R \cup \{a, b\}) \leq_p b$ holds for each common extension $b$ of $a$ and $\sigma(l(R), A^R)$ we have
\[
\{\sigma(l(R) + 1, A^P \cup \{a, b\}) \mid b \leq_p a \land b \leq_p \sigma(l(R), A^R)\}
\]
is dense below the boolean meet of $a$ and $\sigma(l(R), A^R)$. But this set is defined in $V$, and thus by the genericity of $G$ there exists some common extension $b$ of $a$ and $\sigma(l(R), A^R)$ such that $\sigma(l(R) + 1, A^R \cup \{a, b\}) \in G$. Therefore we have $R' := R^\ast(A^R \cup \{a, b\}) \in \mathbb{R}_{G*H}$ and thus $R' \in D_a$.
\[\Box\text{(Lemma 3.2)}\]
\[\Box\text{(Subclaim)}\]

Since $\sigma(\delta, A) = p'$ belongs to $G$, by letting $A_\delta := A$ and $P' := \langle A_\gamma \mid \gamma \leq \delta \rangle$ we have that $P'$ is an $\mathbb{R}_{G*H}$-condition which extends all $P_n$’s, and thus is an $(\mathcal{N}[G][H], \mathbb{R}_{G*H})$-generic condition. This finishes the proof of our claim.
\[\Box\text{(Claim)}\]

Note that, by the same density argument as above, for any $\mathcal{S}$-generic filter $G*H*I$ over $V, \bigcup I$ is an $\omega_1$-sequence of elements of $[\mathbb{P}]^\leq_{\aleph_0}$ and thus is an $(\mathcal{N}[G][H], \mathbb{R}_{G*H})$-generic condition. For each $\gamma < \omega_1$ let $\check{A}_\gamma$ be the canonical $\mathcal{S}$-name representing the $\gamma$-th entry of this sequence.

In $V$, apply PFA to $\mathcal{S}$ with a sufficiently rich family of $\aleph_1$-many dense subsets of $\mathcal{S}$ we can find a directed subset $\{s_\gamma = \langle p_\gamma, \check{q}_\gamma, \check{P}_\gamma \rangle \mid \gamma < \omega_1 \}$ of $\mathcal{S}$, $\{A_\gamma \mid \gamma < \omega_1 \} \subseteq [\mathbb{P}]^\leq_{\aleph_0}$ and $\{\xi_\gamma \mid \gamma < \omega_1 \} \subseteq \omega_1$ such that for each $\gamma < \omega_1$ it holds that
\[(3)\quad s_\gamma \Vdash \check{A}_\gamma = \check{A}_\gamma \land l(\check{P}_\gamma) = \check{\xi}_\gamma \land \sigma(\check{\xi}_\gamma, A^\check{P}_\gamma) = \check{p}_\gamma,\]
and
\[(4)\quad p_\gamma \Vdash \check{q}_\gamma \in \check{D}_\gamma.\]

By the directedness and genericity of the set $\{s_\gamma \mid \gamma < \omega_1 \}$ together with (2) we have that

\[
\begin{array}{cccccc}
I : & A_0 & A_1 & A_2 & \cdots & A_{\omega+1} & \cdots \\
II : & r_0 & r_1 & r_2 & \cdots & r_\omega & \cdots
\end{array}
\]
(where $r_\gamma$ denotes $\sigma(\gamma, A_\gamma)$ for each $\gamma < \omega_1$) forms a play of $G^\ast(\mathbb{P})$ where Player II plays according to $\sigma$\footnote{The only nontrivial part of this statement is the $\subseteq$-continuity of $\{A_\gamma \mid \gamma < \omega_1 \}$. This can be worked out as follows. Work in $V$. At first for each element of $[\mathbb{P}]^\leq_{\aleph_0}$ fix its enumeration of order type $\omega$. For each limit $\gamma < \omega_1$ and $n < \omega$ let $D_{\gamma,n}$ be the dense subset of $\mathcal{S}$ consisting of the conditions which decide the least ordinal $\xi < \gamma$ such that the $n$-th element of $A_\gamma$ belongs to $A_\xi$. Then we may put these dense subsets in the family to which PFA is applied.}. We also have that, for each $\gamma < \omega_1$ there exists some $\xi_\gamma < \omega_1$ such that $p_\gamma = r_\xi$. Therefore
$p, s, \gamma$'s have a common extension $p$ in $\mathbb{P}$. By (4) and the directness of $s, \gamma$'s we have:

$$p \models \{ \dot{q}_\gamma \mid \gamma < \omega_1 \} \text{ is directed} \land \forall \gamma < \omega_1[\dot{q}_\gamma \in \dot{D}_\gamma].$$

Note that we can pick such $p$ below any given condition of $\mathbb{P}$. This suffices for our conclusion. □ (Theorem 3.1)

As an immediate corollary of Theorem 3.1 together with Theorem 2.5, we can reproduce a proof of the following well-known theorem first proved by Magidor [13].

**Theorem 3.3** (Magidor). The statement that $\square_\kappa,\omega_2$ holds for every cardinal $\kappa \geq \omega_2$ is relatively consistent with ZFC + PFA.

**Proof.** We may assume that in our ground model it holds that ZFC + PFA + "$2^\kappa = \kappa^+$ for every cardinal $\kappa \geq \omega_2$", since the last statement is relatively consistent to ZFC + PFA (see [21, Proof of Theorem 17]). Let

$$\langle \mathbb{P}_\alpha, \dot{Q}_\alpha \mid \alpha \in \text{Ord} \rangle$$

be the proper class iterated forcing construction with the Easton support such that $\mathbb{P}_0$ is trivial and that $\models_{\mathbb{P}_\alpha} \dot{Q}_\alpha = \mathbb{P}_{\omega_2, \omega_2}$ for every ordinal $\alpha$, and let $\mathbb{P}_\infty$ be its direct limit. By standard arguments we have that forcing with $\mathbb{P}_\infty$ preserves ZFC and cofinalities (for iterations with the Easton support see [1]; for treatment of proper class forcing consult [7]), and therefore forces $\square_\kappa,\omega_2$ for every cardinal $\kappa \geq \omega_2$. Now by Theorem 2.5 and Lemma 2.6 $\mathbb{P}_\infty$ is $*$-tactically closed, and thus by Theorem 3.1 PFA holds in this extension. □

4. **The Setwise Climbability Properties**

It has been observed that Jensen’s square principles and some of their variations can be characterized as a Martin-type axiom for a suitable class of posets. For example, Velleman [20], and Ishiu and the author [8] observed that $\square_{\omega_1}$ is equivalent to MA$_{\omega_2}$ for the class of $(\omega_1 + 1)$-strategically closed posets. For another example, in [21] the author introduced the following fragment of $\square_{\omega_1}$ and observed that it is equivalent to MA$_{\omega_2}$ for the class of operationally closed posets.

**Definition 4.1.** CP$_{\omega_1}$ (the climbability property) is the following statement: There exists a function $f : \omega_2 \rightarrow \omega_1$ such that for each $\beta \in S_1^2$, there exists a club subset $C$ of $\beta$ with o.t.$C = \omega_1$ such that $f(\alpha) = \text{o.t.}(C \cap \alpha)$ holds for every $\alpha \in C$.

3As for a written proof of this theorem, it is announced in a recent paper by Cummings and Magidor [4] which argues the weak square principles derived from the Martin’s Maximum, that it will be dealt with in a further publication by Magidor.
In this section we introduce two more combinatorial principles named as the *setwise climability properties*, and show that they are equivalent to MA$_{\omega_2}$ for the class of *-tactically closed posets and that of *-operationally closed posets respectively.

**Definition 4.2.** (1) SCP is the following statement: There exists a sequence $\langle z_\alpha \mid \alpha \in S_0^2 \rangle$ and a function $f : \omega_2 \to \omega_1$ satisfying:

(a) For each $\alpha \in S_0^2$, $z_\alpha$ is a countable cofinal subset of $\alpha$.

(b) For each $\beta \in S_1^2$, there exists a club subset $C$ of $\beta \cap S_0^2$ with o.t.$C = \omega_1$ satisfying:

(i) $\langle z_\alpha \mid \alpha \in C \rangle$ is increasing and continuous with respect to inclusion.

(ii) For each $\alpha \in C$, $f(\alpha) = o.t.(C \cap \alpha)$ holds.

(2) SCP$^-$ is the statement obtained by removing all references to the function $f$ in the above statement of SCP.

Now we introduce natural posets respectively for SCP and SCP$^-$.  

**Definition 4.3.** We define posets $P_{SCP}$ and $P_{SCP}^-$ as follows:

(1) A condition $p$ of $P_{SCP}$ is of the form

$p = \langle \langle z_\alpha^p \mid \alpha \in S_0^2 \land \alpha \leq \beta^p \rangle, f^p \rangle$

satisfying

(a) $\beta^p$ is an ordinal in $S_0^2$.

(b) $f^p : \beta^p + 1 \to \omega_1$.

(c) For each $\alpha \in S_0^2$ with $\alpha \leq \beta^p$, $z_\alpha^p$ is a countable cofinal subset of $\alpha$.

(d) For each $\beta \in \beta^p \cap S_1^2$, there exists a club subset $C$ of $\beta^p \cap S_0^2$ with o.t.$C = \omega_1$ satisfying:

(i) $\langle z_\alpha^p \mid \alpha \in C \rangle$ is increasing and continuous with respect to inclusion.

(ii) For each $\alpha \in C$, $f(\alpha) = o.t.(C \cap \alpha)$.

For $p, q \in P_{SCP}$, we let $p \leq_{P_{SCP}} q$ if $z_q = z_p \upharpoonright (\beta^q + 1)$ and $f^q = q^p \upharpoonright (\beta^q + 1)$.

(2) A condition $p$ of $P_{SCP}^-$ is of the form

$p = \langle z_\alpha^p \mid \alpha \in S_0^2 \land \alpha \leq \beta^p \rangle$

satisfying (a), (c) and (d)(i) in (1) above.

Both $P_{SCP}$ and $P_{SCP}^-$ are ordered by initial segment.

**Lemma 4.4.** (1) $P_{SCP}$ is *-operationally closed.

(2) $P_{SCP}^-$ is *-tactically closed.

*Proof.* We will only show (1). (2) is easier. We will define a *-operation $\tau : \omega_1 \times [P_{SCP}]_{\leq \omega}^+ \to P_{SCP}$. Let $\delta < \omega$ and $A \in [P_{SCP}]_{\leq \omega}^+$. If $\delta$ is
0 or a successor ordinal, set \( \tau(\delta, A) \) so that it properly extends the boolean infimum of \( A \). This assures that, whenever Player II plays according to \( \tau \), for each limit \( \eta < \omega_1 \), the union of Player I’s moves made before the \( \eta \)-th turn of Player II has no strongest condition. So for the case \( \delta \) is a nonzero limit ordinal, we may define \( \tau(\delta, A) \) only for \( A \) with no strongest condition. For such \( A \), there exist \( \gamma \in S^2_0 \), \( \vec{z} = \langle z_\alpha : \alpha \in S^2_0, \alpha < \gamma \rangle \) and a function \( f : \gamma \to \omega_1 \) such that the following holds:

\[
A = \{ \langle \vec{z} \upharpoonright (\beta^p + 1), f \upharpoonright (\beta^p + 1) \rangle \mid p \in A \}.
\]

Then we set

\[
\tau(\delta, A) := \langle \vec{z}^* \langle z_\gamma \rangle, f \cup \{ \langle \gamma, \vec{d} \rangle \} \rangle,
\]

where \( z_\gamma = \{ \beta^p \mid p \in A \} \) and \( \vec{d} \) is such that \( \delta = \omega(1 + \vec{d}) \).

We will show that \( \tau \) is a winning \(*\)-operation. Consider any play of \( G^* (\P_{\text{SCP}}) \) where Player II plays according to \( \tau \). It is easy to see that \( \P_{\text{SCP}} \) is \( \omega_1 \)-closed, and thus it is enough to show that Player II can make her \( \omega_1 \)-th move. For \( \delta < \omega_1 \), let \( A_\delta \) denote the union of Player I’s moves made before the \( \delta \)-th move of Player II. Then there exist an ordinal \( \gamma \in S^2_1 \), \( \vec{z} = \langle z_\alpha : \alpha \in S^2_0, \alpha < \gamma \rangle \) and a function \( f : \gamma \to \omega_1 \) such that, for each \( \delta < \omega_1 \) the following holds:

\[
A_\delta = \{ \langle \vec{z} \upharpoonright (\beta^p + 1), f \upharpoonright (\beta^p + 1) \rangle \mid p \in A_\delta \}.
\]

Let \( \gamma_\xi := \text{sup}\{ \beta^p \mid p \in A_{\omega(1+\xi)} \} \} \) for \( \xi < \omega_1 \). Then by the definition of \( \tau \), \( C = \{ \gamma_\xi \mid \xi < \omega_1 \} \) is a club subset of \( \gamma \). Moreover, for each \( \xi < \omega_1 \) we have \( z_{\gamma_\xi} = \{ \beta^p \mid p \in A_{\omega(1+\xi)} \} \), and thus \( \langle z_{\gamma_\xi} \mid \xi < \omega_1 \rangle \) is increasing and continuous with respect to inclusion. Furthermore, for each \( \xi < \omega_1 \) we have \( f(\gamma_\xi) = \xi = o.t.(C \cap \gamma_\xi) \). These facts assure that \( \langle \vec{z}, f \rangle \) can be extended to a condition of \( \P_{\text{SCP}} \), which is a common extension of all moves of Player I. This shows that Player II wins the game. \( \square \)

In particular, both \( \P_{\text{SCP}} \) and \( \P_{\text{SCP}^-} \) are \( (\omega_1 + 1) \)-strategically closed, and therefore preserve cardinalities below \( \omega_2 \). Moreover, a simple induction argument using this closedness property gives the following density lemma:

**Lemma 4.5.** For each \( \beta < \omega_2 \), \( D_\beta = \{ p \in \P_{\text{SCP}} \mid \beta^p > \beta \} \) and \( D^-_\beta = \{ p \in \P_{\text{SCP}^-} \mid \beta^p > \beta \} \) are dense respectively in \( \P_{\text{SCP}} \) and \( \P_{\text{SCP}^-} \). \( \square \)

These facts assures that \( \P_{\text{SCP}} \) forces SCP and that \( \P_{\text{SCP}^-} \) forces SCP\(^-\).

**Theorem 4.6.** (1) The following are equivalent:

(a) SCP.

(b) Every \(*\)-operationally closed poset is \( \omega_2 \)-strategically closed.
(c) $P_{SCP}$ is $\omega_2$-strategically closed.
(d) $MA_{\omega_2}$ ($\ast$-operationally closed).
(e) $MA_{\omega_2}(P_{SCP})$.

(2) The statement obtained by replacing each occurrence of ‘SCP’ and ‘$\ast$-operationally closed’ in (1) by ‘SCP$^{-1}$’ and ‘$\ast$-tactically closed’ respectively is also valid.

Proof. Again we only show (1), since (2) is easier. Since $MA_{\omega_2}$ is valid for every $\omega_2$-strategically closed poset, (b) implies (d) and (c) implies (e) respectively. (b) implies (c) and (d) implies (e) by Lemma 4.4(1). (e) implies (a) by Lemma 4.5, since a filter which intersects every $D_\beta$ generates a witness for SCP. Now assume (a) and show (b) holds.

Suppose $\langle z_\alpha | \alpha \in S^2_0 \rangle$ and $f$ witness SCP. Let $P$ be any $\ast$-operationally closed poset, and $\tau$ a winning $\ast$-operation for $G^\ast(P)$. We may assume that $\tau(\delta, A)$ is a common extension of $A$ for every $\delta < \omega_1$ and $A \in [P]^\omega_{\leq \omega}$. We will describe how Player II wins $G_{\omega_2}(P)$. We use symbols $a_\gamma$, and $b_\gamma$ to denote the $\gamma$-th move of Player I and Player II respectively. We will use the following lemma (for proof see Ishiu and Yoshinobu [8, Lemma 2.3 and the proof of Theorem 3.3]).

Lemma 4.7 (Ishiu and Yoshinobu). There exists a tree $T = \langle S, <_T \rangle$ (where $S = \omega_2 \setminus \text{Lim}$) of height $\omega$ such that

1. For every $\beta, \gamma \in S$, $\beta <_T \gamma$ implies $\beta < \gamma$.
2. For every $\alpha \in S^2_0$, there exists a cofinal branch $b$ of $T$ with $\text{sup } b = \alpha$.

Let $T$ be a tree as above. In a play of $G_{\omega_2}(P)$, for each $\gamma < \omega_2$, Player II may choose her $\gamma$-th move in the following way:

$$b_\gamma = \begin{cases} \tau(\text{ht}_T(\gamma), \{a_\xi \mid \xi \leq_T \gamma \}) & \text{if } \gamma \in S, \\ \tau(f(\gamma), \{a_\xi \mid \xi \in z_\gamma \}) & \text{if } \gamma \in S^2_0, \\ \text{any common extension of } \{a_\xi \mid \xi < \gamma \} & \text{if } \gamma \in S^1_0. \end{cases}$$

Let us show that this is a winning strategy. Consider any play of $G_{\omega_2}(P)$ where Player II plays according to this strategy. It is enough to show that, for each $\gamma < \omega_2$,

(i) The $\gamma$-th turn of Player II exists, that is, the current position of the turn is positive.
(ii) The $\gamma$-th move of Player II is legitimate, that is, her move extends the current position of the turn.

We will show (1) and (2) by induction on $\gamma$. So by induction hypothesis we may suppose that all moves preceding the $\gamma$-th move of Player II have been legitimately made.
Case 1 $\gamma \in S$.
In this case (i) is immediate, since the current position is $a_\gamma$. (ii) follows from the following inequality:
\[
\tau(\text{ht}_T(\gamma), \{a_\xi \mid \xi \leq T \gamma\}) \leq B(P) \bigwedge \{a_\xi \mid \xi \leq T \gamma\} = a_\gamma.
\]

Case 2 $\gamma \in S_0^2$.
By the definition of $T$, there is a cofinal branch $b = \langle \xi_n \mid n < \omega \rangle$ of $T$ such that $\sup b = \gamma$. For each $n < \omega$ it holds that $\xi_n \in S$ and $\text{ht}_T(\xi_n) = n$, and thus we have $b_{\xi_n} = \tau(n, \{a_{\xi_m} \mid m \leq n\})$. Therefore, setting $A_n = \{a_{\xi_m} \mid m \leq n\}$ for each $n < \omega$,
\[
\begin{array}{ccccccc}
I &: A_0 & A_1 & A_2 & \cdots \\
II &: b_{\xi_0} & b_{\xi_1} & b_{\xi_2} & \cdots
\end{array}
\]
forms a part of a play of $G^*(P)$ where Player II plays according to $\tau$. Therefore $\bigwedge \{b_{\xi_n} \mid n < \omega\}$ is positive. Since $\{b_{\xi_n} \mid n < \omega\}$ is cofinal in the moves made before the $\gamma$-th turn of Player II, $\bigwedge \{b_{\xi_n} \mid n < \omega\}$ is equal to the current position of the turn. This shows (ii). For (i), note that we have
\[
\tau(f(\gamma), \{a_\xi \mid \xi \in z_\gamma\}) \leq B(P) \bigwedge \{a_\xi \mid \xi \in z_\gamma\}.
\]
Since $z_\gamma$ is cofinal in $\gamma$, the right hand side of the above inequality is equal to the current position of the $\gamma$-th turn of Player II.

Case 3 $\gamma \in S_1^2$.
By the definitions of $\langle z_\alpha \mid \alpha \in S_0^2 \rangle$ and $f$ there exists a club subset $C$ of $\gamma \cap S_0^2$ such that $\langle z_\alpha \mid \alpha \in C \rangle$ is increasing and continuous with respect to the inclusion, and that $f(\alpha) = \text{o.t.}(C \cap \alpha)$ holds for each $\alpha \in C$. Therefore, letting $\langle \alpha_\xi \mid \xi < \omega_1 \rangle$ be the increasing enumeration of $C$, it holds that $b_{\alpha_\xi} = \tau(\xi, \{a_\eta \mid \eta \in z_{\alpha_\xi}\})$. Therefore, setting $A_\xi = \{a_\eta \mid \eta \in z_{\alpha_\xi}\}$ for each $\xi < \omega_1$,
\[
\begin{array}{ccccccc}
I &: A_0 & A_1 & A_2 & \cdots & A_{\omega+1} & \cdots \\
II &: b_{\alpha_0} & b_{\alpha_1} & b_{\alpha_2} & \cdots & b_{\alpha_\omega} & b_{\alpha_{\omega+1}} & \cdots
\end{array}
\]
forms a play of $G^*(P)$ where Player II plays according to $\tau$. This implies that $\bigwedge \{b_{\alpha_\xi} \mid \xi < \omega_1\}$ is positive. Since $\{b_{\alpha_\xi} \mid \xi < \omega_1\}$ is cofinal in the moves made before the $\gamma$-th turn of Player II, $\bigwedge \{b_{\alpha_\xi} \mid \xi < \omega_1\}$ is equal to the current position of the turn. This shows (i). In this case (ii) is clear.

5. Operations versus $\ast$-tactics (1): preservation under $\ast$-tactically closed forcing

In this section we show the following theorem.
Theorem 5.1. Assume $\text{MA}^+ (\omega_1\text{-closed})$. Then for every $\ast$-tactically closed poset $\mathbb{P}$, it holds that

$$\models_\mathbb{P} \neg \text{CP}.$$  

Note that since CP can be forced by an $(\omega_1 + 1)$-operationally closed forcing, this theorem shows that $(\omega_1 + 1)$-operationally closed does not imply $\ast$-tactical closedness. Since $\text{SCP}^{-}$ can be forced by a $\ast$-tactically closed forcing, this theorem also shows that $\text{SCP}^{-}$ does not imply CP.

Since CP negates the Chang’s Conjecture (CC) (see [21] for proof), to prove Theorem 5.1 it is enough to show the following:

Lemma 5.2. Assume $\text{MA}^+ (\omega_1\text{-closed})$. Then for every $(\omega_1 + 1)$-$\ast$-tactically closed poset $\mathbb{P}$, it holds that

$$\models_\mathbb{P} \text{CC}.$$  

We will give a proof of Lemma 5.2 below. Our proof is based on and generalizes that of Miyamoto [14, Theorem 1.3], which obtains a model of CC with some weak fragment of $\square \omega_1$, assuming (an axiom equivalent to) $\text{MA}^+ (\omega_1\text{-closed})$ in the ground model. Note also that Miyamoto’s argument was extracted from Sakai [15], which obtains a model of CC with $\square \omega_1$, starting from the ground model with a measurable cardinal.

We will use the following equivalent form of $\text{MA}^+ (\omega_1\text{-closed})$, introduced by Miyamoto [14].

Definition 5.3. $\text{FA}^*(\omega_1\text{-closed})$ denotes the following statement: For any $\omega_1\text{-closed}$ poset $\mathbb{P}$, any family of dense subsets $\langle D_i \mid i < \omega_1 \rangle$, any regular cardinal $\theta \geq (2^{\text{rc}(\mathbb{P})})^+$, any structure $\mathfrak{A} = \langle H_\theta, \in, \langle \mathbb{P} \rangle, \ldots \rangle$, any countable $\mathbb{N} \prec \mathcal{N}$ and any $(\mathbb{N}, \mathbb{P})$-generic condition $p \in \mathbb{P}$, there exists a directed subset $F$ of $\mathbb{P}$ of size at most $\aleph_1$ satisfying the following:

(i) $F \cap D_i \neq \emptyset$ for every $i < \omega_1$.
(ii) $q \leq_\mathbb{P} p$ holds for some $q \in F$.
(iii) $\mathbb{N} \prec_{\omega_1} N(F) = \{ g(F) \mid g : \mathbb{P} \rightarrow H_\theta \land g \in \mathbb{N} \}.$

Lemma 5.4 (Miyamoto and Usuba). $\text{FA}^*(\omega_1\text{-closed})$ is equivalent to $\text{MA}^+ (\omega_1\text{-closed})$.

See Usuba [18] for a proof of Lemma 5.4.

Lemma 5.5. Assume $\text{FA}^*(\omega_1\text{-closed})$. Let $\mathbb{P}$, $\langle D_i \mid i < \omega_1 \rangle$, $\theta$, $\mathfrak{A}$, $\mathbb{N}$ and $p$ be as in Definition 5.3. If $\mathbb{P}$ collapses $\omega_2$, then $F$ in the conclusion of Definition 5.3 can be taken so that $\mathbb{N} \prec_{\omega_1} N(F)$ holds.

Proof. Pick a $\mathbb{P}$-name $\dot{f} \in \mathbb{N}$ for a map from $\omega_1$ onto $\omega_2^\mathbb{V}$, and choose $p' \leq_\mathbb{P} p$ so that there exists $\alpha < \omega_2$ with $\alpha \geq \sup(\mathbb{N} \cap \omega_2)$ and $i < \omega_1$. 

such that \( p' \models \tilde{f}(\check{i}) = \check{\alpha}. \) Now apply \( FA^*(\omega_1\text{-closed}) \) for \( p' \) instead of \( p \) to obtain \( F \). Then \( F \) also satisfies (\[\square\]) for \( p \). Note that

\[
\beta := \sup\{\eta < \omega_2 \mid \exists q \in F \exists i < \omega_1[q \models \tilde{f}(\check{i}) = \check{\eta}]\}
\]

satisfies \( \sup(N \cap \omega_2) \leq \alpha \leq \beta < \omega_2 \) and \( \beta \in N(F) \), and therefore we have \( N \prec_{\omega_1}^s N(F) \). \( \square \)

We will also use the following auxiliary poset which is induced from a given \( * \)-tactically closed poset.

**Definition 5.6.** Let \( \mathbb{P} \) be a \( * \)-tactically closed poset, and fix a winning \( * \)-tactic \( \sigma \) for \( G^*(\mathbb{P}) \). We define a poset \( \mathbb{R} = \mathbb{R}(\mathbb{P}, \sigma) \) as follows: A condition \( P \) of \( \mathbb{R} \) is either the empty sequence \( \langle \rangle \), or of the form \( P = \langle A^P_\xi \mid \xi \leq \alpha^P \rangle \) (\( \alpha^P < \omega_1 \)) satisfying the following:

(a) \( P \) is a \( \subseteq \)-continuous increasing sequence of elements of \( [\mathbb{P}]^{\leq \omega} \).

(b) \( \sigma(A^P_\xi) \supseteq \mathbb{R}(\mathbb{P}) \land A^P_{\xi+1} \) for every \( \xi < \alpha^P \).

\( \mathbb{R} \) is ordered by initial segment. Note that \( P = \langle A^P_\xi \mid \xi \leq \alpha^P \rangle \) is an \( \mathbb{R} \)-condition if and only if

\[
\begin{align*}
\text{I} : \quad & A^P_0 \quad A^P_1 \quad \cdots \\
\text{II} : \quad & \sigma(A^P_0) \quad \sigma(A^P_1) \quad \cdots \quad \sigma(A^P_{\alpha^P})
\end{align*}
\]

forms a part of a play of \( G^*(\mathbb{P}) \) where Player II plays according to \( \sigma \).

Therefore it is clear that \( \mathbb{R} \) is \( \omega_1 \)-closed. In fact, whenever \( \langle P_n \mid n < \omega \rangle \) is a strictly descending sequence of \( \mathbb{R} \)-conditions, \( \bigcup\{P_n \mid n < \omega\} \) is of the form \( \langle A_\xi \mid \xi < \gamma \rangle \) for some limit \( \gamma < \omega_1 \), and letting \( A_\gamma = \bigcup\{A_\xi \mid \xi < \gamma\} \) we have \( A_\gamma \in [\mathbb{P}]^{\leq \omega} \) because \( \sigma \) is a winning \( * \)-tactic, and \( P_\omega = \langle A_\xi \mid \xi \leq \gamma \rangle \) is the greatest common extension of \( \{P_n \mid n < \omega\} \).

For \( P = \langle A^P_\xi \mid \xi \leq \alpha^P \rangle \in \mathbb{R} \) we denote \( A^P_{\alpha^P} \) simply as \( A^P \). We define \( \pi : \mathbb{R} \rightarrow \mathbb{P} \) by \( \pi(1_{\mathbb{R}}) = 1_\mathbb{P} \) and \( \pi(P) = \sigma(A^P) \) for other \( P \in \mathbb{R} \). It is easy to check that \( \pi : \mathbb{R} \rightarrow \mathbb{P} \) is a projection. We also denote \( \alpha^P \) as \( \text{lh}(P) \).

**Lemma 5.7.** Let \( \mathbb{P}, \sigma \) and \( \mathbb{R} \) be as above. Then the following subsets of \( \mathbb{R} \) are dense and open in \( \mathbb{R} \).

(1) \( D^\text{lh}_\alpha = \{P \in \mathbb{R} \mid \text{lh}(P) \geq \alpha\} \) for \( \alpha < \omega_1 \).

(2) \( E_a = \{P \in \mathbb{R} \mid a \in A^P \vee a \perp_\mathbb{P} \pi(P)\} \) for \( a \in \mathbb{P} \).

**Proof.** Straightforward. \( \square \)

**Lemma 5.8.** Let \( \mathbb{P}, \sigma \) and \( \mathbb{R} \) be as in Definition 5.6. If \( \mathbb{P} \) is non-atomic, then \( \mathbb{R} \) collapses \( \omega_2 \).

**Proof.** Note first that \( \mathbb{R} \) is a tree of height \( \omega_1 \). Since \( \mathbb{P} \) is non-atomic and \( (\omega_1 + 1\text{-strategically closed}) \), each condition of \( \mathbb{P} \) has at least \( 2^{\aleph_1} \geq \aleph_2 \) pairwise incompatible extensions. This implies that each node of \( \mathbb{R} \) has
at least $\aleph_2$ distinct immediate successors. Using this fact one can let each node of $\mathbb{R}$ code a function from a countable ordinal to $\omega_2$, so that each $\mathbb{R}$-generic filter codes a function from $\omega_1$ to $\omega_2^\gamma$, which is surjective by genericity.

Proof of Lemma 5.2. Since MA$^+$ ($\omega_1$-closed) implies CC, as shown in Foreman, Magidor and Shelah [6], we may assume that $\mathbb{P}$ is non-atomic.

Fix a sufficiently large regular cardinal $\theta$. Let $\dot{F} \in H_\theta$ be any $\mathbb{P}$-name for a function $[\omega_2]^\omega \to \omega_2$. It is enough to show that, for any $p \in \mathbb{P}$, there exist $q \leq \mathbb{P} p$ and $N \prec (H_\theta, \in, \{\mathbb{P}\}, \{\dot{F}\})$ such that $N \cap \omega_1 < \omega_1$, $|N \cap \omega_2| = \aleph_1$ and that $q$ is $(\mathbb{N}, \mathbb{P})$-generic. Fix a winning $*$-tactic $\sigma$ for $G^*(\mathbb{P})$ and let $\mathfrak{A} = \langle H_\theta, \in, \{\mathbb{P}\}, \{\dot{F}\}, \{\sigma\} \rangle$.

By recursion we will simultaneously define $p_\gamma \in \mathbb{P}$ for $\gamma < \omega_1 \setminus \text{Lim}$, a countable elementary substructure $N_\gamma$ of $\mathfrak{A}$, $A_\gamma \subseteq [\mathbb{P}]^\omega_{\text{Lim}}$ and $P_\gamma \in \mathbb{R}$ respectively for $\gamma < \omega_1$ so that the following requirements are satisfied:

For each $\xi < \omega_1$,

(i) $\sigma(A_\xi) \geq \mathbb{P} p_\xi$ if $\xi = \zeta + 1$.
(ii) $N_\xi \prec^* \omega_{\text{Lim}}N_\xi$ if $\xi = \zeta + 1$.
(iii) $N_\xi = \bigcup \{N_\zeta | \zeta < \xi\}$ if $\xi \in \text{Lim}$.
(iv) $p_\xi \in A_\xi$ if $\xi \notin \text{Lim}$.
(v) $A_\xi \subseteq A_\xi$ if $\xi = \zeta + 1$.
(vi) $A_\xi = \bigcup \{A_\zeta | \zeta < \xi\}$ if $\xi \in \text{Lim}$.
(vii) $P_\xi$ is $(N_\zeta, \mathbb{R})$-strongly generic.
(viii) $A_\xi = A_\xi^\mathbb{P}$.
(ix) $A_\xi \subseteq N_\xi$.

Let $\gamma < \omega_1$ and suppose $\langle p_\xi | \xi \in \gamma \setminus \text{Lim} \rangle$, $\langle N_\xi | \xi < \gamma \rangle$, $\langle A_\xi | \xi < \gamma \rangle$ and $\langle P_\xi | \xi < \gamma \rangle$ are already defined and satisfy (i)–(ix) for every $\xi < \gamma$.

We will define $p_\gamma$ (if $\gamma \notin \text{Lim}$), $N_\gamma$, $A_\gamma$ and $P_\gamma$ so that (i)–(ix) for $\xi = \gamma$ hold.

Case 1 $\gamma = 0$.

First let $p_0 = p$, and pick a countable $N_0 \prec \mathfrak{A}$ such that $p_0 \in N_0$. Let $\delta = N_0 \cap \omega_1$. Now pick an $(N_0, \mathbb{R})$-generic sequence $\langle p_{0,n} | n < \omega \rangle$ such that $P_{0,0} = \langle p_0 \rangle$. Now let $P_0$ be the greatest common extension of $\langle P_{0,n} | n < \omega \rangle$ and let $A_0 := A_0^\mathbb{P}$. Then it is easy to see (iv), (vii), (viii) and (ix) for $\xi = 0$.

Case 2 $\gamma = \zeta + 1$.

Since $P_\zeta$ is $(N_\zeta, \mathbb{R})$-generic by (vii) for $\xi = \zeta$, we can apply Lemma 5.5 to $\mathbb{R}$, dense subsets $D_\alpha^\mathbb{R}$ ($\alpha < \omega_1$), $N_\zeta$ and $P_\zeta$ to get a directed subset $F$ of $\mathbb{R}$ satisfying:

1. $F \cap D_\alpha^\mathbb{R} \neq \emptyset$ for every $\alpha < \omega_1$.
2. $Q \in F$ for some $Q \leq_\mathbb{R} P_\zeta$. 

(3) $N_\gamma \prec_{\omega_1} N_\gamma(F)$.

Now let $N_\gamma := N_\gamma(F)$. By (3) we have (i) for $\xi = \gamma$. By (i) and the directedness of $F$, $\bigcup F$ is of the form $\langle A_i^F \mid i < \omega_1 \rangle$. Note that

$$
\begin{align*}
\text{I} : & \quad A_0^F \quad A_1^F \quad \cdots \quad A_{\omega+1}^F \quad \cdots \\
\text{II} : & \quad \sigma(A_0^F) \quad \sigma(A_1^F) \quad \cdots \quad \sigma(A_{\omega+1}^F) \quad \cdots 
\end{align*}
$$

forms a play of $G^*(\mathbb{P})$ where Player II plays according to $\sigma$. Since $\sigma$ is a winning $*$-tactic, $\bigcup \{A_i^F \mid i < \omega_1 \}$ has a common extension in $\mathbb{P}$. Since $F \in N_\gamma$, we can pick such a common extension $p_\gamma$ within $N_\gamma$. By (2) $P_\gamma$ is an initial segment of $\bigcup F$. By (i) and (iii) for $\xi \leq \zeta$ we have $N_\zeta \cap \omega_1 = \delta$, and thus by Lemma 5.7(1) and (vii) for $\xi = \zeta$ we have $A_\zeta = A^{P_\zeta} = A^F_\delta$. Therefore it holds that $\sigma(A_\zeta) = \sigma(A^F_\delta) \geq \sigma(\mathbb{P}) \wedge A^F_{\delta+1} \geq \mathbb{B}(\mathbb{P}) p_\gamma$. This gives (ii) for $\xi = \gamma$.

Now construct $A_\gamma$ and $P_\gamma$ in the same way as $A_0$ and $P_0$: Pick an $(N_\gamma, \mathbb{R})$-generic sequence $\langle P_{\gamma,n} \mid n < \omega \rangle$ such that $P_{\gamma,0} = \{p_\gamma\}$. Then let $P_\gamma$ be the greatest common extension of $\{P_{\gamma,n} \mid n < \omega \}$ and let $A_\gamma := A^P_\gamma$. Then we have (iv), (vii), (viii) and (ix) for $\xi = \gamma$.

Let $q \in A_\gamma$ be arbitrary. Note that $E_q$ as in Lemma 5.7(2) is dense open in $\mathbb{R}$ and belongs to $N_\zeta \subseteq N_\gamma$. Thus we have $P_\gamma \in E_q$ by (vii) for $\xi = \gamma$. So either $q \in A_\gamma$ or $q \perp \pi(P_\gamma)$ holds. By (i) for $\xi = \gamma$, $p_\gamma$ is a common extension of $A_\zeta$ and thus $q \geq_{\mathbb{P}} p_\gamma$ holds. On the other hand, since $p_\gamma \in A_\gamma$ by (iv) for $\xi = \gamma$ and $\pi(P_\gamma) = \sigma(A_\gamma)$ is a common extension of $A_\gamma$, we have $p_\gamma \geq_{\mathbb{P}} \pi(P_\gamma)$. Thus $q \geq_{\mathbb{P}} \pi(P_\gamma)$ holds and in particular they are compatible. Therefore $q \in A_\gamma$ holds. This shows $A_\zeta \subseteq A_\gamma$, that is (v) for $\xi = \gamma$.

**Case 3** $\gamma \in \text{Lim}$.

Let $N_\gamma := \bigcup \{N_\zeta \mid \zeta < \gamma \}$ and $A_\gamma := \bigcup \{A_\zeta \mid \zeta < \gamma \}$. Then by induction hypothesis we immediately have (iii), (vi) and (ix) for $\xi = \gamma$. So it is enough to find a $P_\gamma$ satisfying (vii) and (viii) for $\xi = \gamma$. Let $\langle D_n \mid n < \omega \rangle$ be an enumeration of the dense subsets of $\mathbb{R}$ in $N_\gamma$. Fix a strictly increasing sequence $\langle \gamma_n \mid n < \omega \rangle$ of ordinals converging to $\gamma$ so that $D_n \in N_{\gamma_n}$ holds for every $n < \omega$.

By induction we construct a descending sequence $\langle P_{\gamma,n} \mid n < \omega \rangle$ in $\mathbb{R}$ satisfying the following requirements for each $n < \omega$:

- $P_{\gamma,n} \in N_{\gamma_n}$.
- $P_{\gamma,n+1} \in D_n$.
- $\pi(P_{\gamma,n}) \geq_{\mathbb{P}} \pi(P_{\gamma,n+1})$.

First let $P_{\gamma,0} := 1_{\mathbb{R}}$. It is clear that $P_{\gamma,0} \in N_{\gamma_0}$ and $\pi(P_{\gamma,0}) = 1_{\mathbb{P}} \geq_{\mathbb{P}} \pi(P_{\gamma,0})$. Suppose $P_{\gamma,n} \in N_{\gamma_n}$ was defined and satisfies $\pi(P_{\gamma,n}) \geq_{\mathbb{P}} \pi(P_{\gamma,n})$. Note that since (vii) for $\xi = \gamma_n$ holds and $\pi$ is a projection, $\pi(P_{\gamma,n})$ is $(N_{\gamma_n}, \mathbb{P})$-strongly generic. Thus since $D_n \in N_{\gamma_n}$, by
Lemma 1.5 we can pick $P_{\gamma,n+1} \leq \mathbb{R} P_{\gamma,n}$ such that $P_{\gamma,n+1} \in D_n \cap N_\gamma$ and that $\pi(P_{\gamma,n+1}) \geq_\mathbb{P} \pi(P_{\gamma,n})$. But by (iii) and (iii) for $\xi < \gamma$ we have $N_\gamma \subseteq N_{\gamma,n+1}$, and by (i), (iv), (v), (vi) and (viii) for $\xi < \gamma$ we have

$$\pi(P_{\gamma,n}) = \sigma(A_{\gamma,n}) \geq_\mathbb{P} p_{\gamma,n+1} \geq_\mathbb{P} \bigwedge A_{\gamma,n}$$

Therefore we have $P_{\gamma,n+1} \in N_{\gamma,n+1}$ and $\pi(P_{\gamma,n+1}) \geq_\mathbb{P} \pi(P_{\gamma,n})$, which finishes the construction.

Note that $\langle P_{\gamma,n} \mid n < \omega \rangle$ is an $(N_\gamma, \mathbb{R})$-generic sequence. Now let $P_\gamma$ be the greatest common extension of $\{P_{\gamma,n} \mid n < \omega \}$. Then $P_\gamma$ is $(N_\gamma, \mathbb{R})$-strongly generic. So it is enough to show the following.

**Claim** $A^{P_\gamma} = A_\gamma$.

**Proof of Claim.** Pick any $q \in A^{P_\gamma}$. Then $q \in A^{P_{\gamma,n}}$ for some $n < \omega$. Since $P_{\gamma,n} \in N_\gamma$ and $A^{P_{\gamma,n}}$ is countable, $q \in N_\gamma$. Since $P_{\gamma,n}$ is $(N_\gamma, \mathbb{R})$-strongly generic, we have $P_{\gamma,n} \in E_q$. Therefore either $q \in A_\gamma$ or $q \perp_\mathbb{P} \pi(P_{\gamma,n})$ holds. But since $q \in A^{P_{\gamma,n}}$ by our construction of $P_{\gamma,n}$ we have

$$q \geq_\mathbb{P} \bigwedge A^{P_{\gamma,n}} \geq_\mathbb{P} \pi(P_{\gamma,n}) \geq_\mathbb{P} \pi(P_{\gamma,n}),$$

and therefore $q \in A_{\gamma,n} \subseteq A_\gamma$ must be the case. This shows that $A^{P_\gamma} \subseteq A_\gamma$.

Now pick any $r \in A_\gamma$. Since $A_\gamma \subseteq N_\gamma$, $E_r = D_m$ for some $m < \omega$. By (vii) and (viii) for $\xi < \gamma$ we can pick $n > m$ so that $r \in A_{\gamma,n}$ holds. Since $E_r$ is open, $P_{\gamma,n+1} \geq_\mathbb{R} P_{\gamma,n} \in E_r$ holds and thus either $r \in A^{P_{\gamma,n}}$ or $r \perp_\mathbb{P} \pi(P_{\gamma,n})$ holds. On the other hand, by our construction of $P_{\gamma,n}$ and (i), (iv), (v) and (viii) for $\xi < \gamma$ we have

$$\pi(P_{\gamma,n}) \geq_\mathbb{P} \pi(P_{\gamma,n}) = \sigma(A_{\gamma,n}) \geq_\mathbb{P} p_{\gamma,n+1} \in A_{\gamma,n+1}.$$

Therefore both $r$ and $\pi(P_{\gamma,n})$ are in $A_{\gamma,n+1}$ (by (v) for $\xi < \gamma$) and thus are compatible. So $r \in A^{P_{\gamma,n}} \subseteq A^{P_\gamma}$ must be the case. This shows that $A_\gamma \subseteq A^{P_\gamma}$.

We have just finished our construction of $p_\gamma$‘s, $N_\gamma$‘s, $A_\gamma$‘s and $P_\gamma$’s. Let $N := \bigcup\{N_\gamma \mid \gamma < \omega_1\}$. By (iii) and (iii) we have that $N \cap \omega_1 = \delta < \omega_1$ and that $|N \cap \omega_2| = \aleph_1$. By (i), (iv), (v) and (vi) we have that

$$I: \quad A_0 \quad A_1 \quad \cdots \quad A_{\omega+1} \quad \cdots$$

$$II: \quad \sigma(A_0) \quad \sigma(A_1) \quad \cdots \quad \sigma(A_\omega) \quad \cdots$$

forms a play of $G^*(\mathbb{P})$ where Player II plays according to $\sigma$. Therefore there exists a common extension $q \in \mathbb{P}$ of $A := \bigcup\{A_\gamma \mid \gamma < \omega_1\}$. By $p = p_0 \in A_0 \subseteq A$ we have $q \leq_\mathbb{P} p$. For each $\gamma < \omega_1$, by
we have $P_\gamma$ is strongly $(N_\gamma, \mathbb{R})$-generic, and since $\pi$ is a projection it follows that $\pi(P_\gamma) = \sigma(A_\gamma)$ is strongly $(N_\gamma, \mathbb{P})$-generic. But $\sigma(A_\gamma) \geq_{\mathbb{B}(\mathbb{P})} A_{\gamma+1} \geq_{\mathbb{B}(\mathbb{P})} q$ holds and thus $q$ is also strongly $(N_\gamma, \mathbb{P})$-generic. Therefore $q$ is $(N, \mathbb{P})$-generic. This completes our proof of Lemma 5.2. □

(Lemma 5.2)

6. Operations versus $*$-tactics (2): preservation under operationally closed forcing

In this section we show the following theorem, which can be considered as a counterpart of Theorem 5.1.

Theorem 6.1. Assume MA$^+$($\omega_1$-closed). Then for any ($\omega_1+1$)-operationally closed poset $\mathbb{P}$, it holds that

$$\models_\mathbb{P} \neg \text{SCP}^-.$$  

Since SCP$^-$ can be forced by a $*$-tactically closed poset, Theorem 6.1 shows that $*$-tactically closedness does not imply ($\omega_1+1$)-operational closedness. Since CP can be forced by an ($\omega_1+1$)-operationally closed poset, it also shows that CP does not imply SCP$^-$. 

Note that, since a poset is ($\omega_1+1$)-operationally closed poset if and only if its Boolean completion is (see [21, Lemma 8]), to prove Theorem 6.1 we may assume that $\mathbb{P} = \mathbb{B} \setminus \{0_\mathbb{B}\}$ for a complete Boolean algebra $\mathbb{B}$. So in the rest of this section we fix such $\mathbb{B}$ and $\mathbb{P}$, and we also fix a winning operation $\tau : (\omega_1+1) \times \mathbb{P} \to \mathbb{P}$.

We define an auxiliary poset which plays a similar role as $R$ in §5.

Definition 6.2. We define a poset $\mathbb{S} = \mathbb{S}(\mathbb{P}, \tau)$ as follows: A condition $s$ in $\mathbb{S}$ is either the empty sequence $1_\mathbb{S} = \langle \rangle$, or of the form $\langle a_\gamma^s \mid \gamma \leq \alpha^s \rangle$ ($\zeta^s < \omega_1$) satisfying

(a) $a_{\gamma+1}^s \leq_{\mathbb{P}} \tau(\gamma, a_\gamma^s)$ for each $\gamma < \alpha^s$, and
(b) $a_\gamma^s = \bigwedge \{a_\xi^s \mid \xi < \gamma\}$ for each nonzero limit $\gamma \leq \alpha^s$.

$\mathbb{S}$ is ordered by initial segment. Note that $s = \langle a_\gamma^s \mid \gamma \leq \alpha^s \rangle$ is an $\mathbb{S}$-condition if and only if

$$\begin{align*}
\text{I:} & \quad a_0^s \quad a_1^s \quad \cdots \quad a_{\omega_1}^s \quad \cdots \\
\text{II:} & \quad b_0^s \quad b_1^s \quad \cdots \quad b_{\omega}^s \quad b_{\omega+1}^s \quad \cdots \quad b_{\alpha^s}^s
\end{align*}$$

(where $b_\gamma^s := \tau(\gamma, a_\gamma^s)$) forms a part of a play of $G_{\omega_1+1}(\mathbb{P})$ where Player II plays according to $\tau$. Therefore it is clear that $\mathbb{S}$ is $\omega_1$-closed. For $s = \langle a_\gamma^s \mid \gamma \leq \alpha^s \rangle \in \mathbb{S}$ we denote $\alpha^s$ and $\tau(\alpha^s, a_{\alpha^s}^s)$ as $lh(s)$ and $lm(s)$ respectively. We also set $lm(1_\mathbb{S}) = 1_\mathbb{P}$. Then it is easy to see that $lm : \mathbb{S} \to \mathbb{P}$ is a projection.
Lemma 6.3. For each $\zeta < \omega_1$, $D^\text{lh}_\zeta := \{ s \in S \mid \text{lh}(s) \geq \zeta \}$ is dense in $S$.

Proof. Straightforward. \hfill \Box

Lemma 6.4. $S$ collapses $\omega_2$.

Proof. Similar to the proof of Lemma 5.8. \hfill \Box

The following is the key lemma to prove Theorem 6.1, though it is provable in ZFC and does not require $\text{MA}^+$(\omega_1\text{-closed}).

Lemma 6.5. Suppose $\dot{Z}$ is a $\mathbb{P}$-name such that

\begin{equation}
\vdash_{\mathbb{P}} \text{“}\dot{Z} \text{ is a function on } S^2_0 \text{ such that for every } \alpha \in S^2_0 \dot{Z}(\alpha) \text{ is a countable unbounded subset of } \alpha.”
\end{equation}

Then it holds that

\begin{equation}
\vdash_{S} \text{“}\{x \in [(\omega_2V)]^\omega \mid \dot{Z}(\sup x) \notin x\} \text{ is stationary in } [(\omega_2V)]^\omega.”
\end{equation}

Let us assume Lemma 6.5 for a while and prove Theorem 6.1 first.

Proof of Theorem 6.1 (up to Lemma 6.5). Assume $\text{MA}^+(\omega_1\text{-closed})$ holds in $V$. Suppose for a contradiction that there exists a $\mathbb{P}$-name $\dot{Z}$ satisfying (5) of Lemma 6.5 and $p \in \mathbb{P}$ such that:

\begin{equation}
p \vdash_{\mathbb{P}} \text{“For every } \beta \in S^2_1 \text{ there exists a club subset } C \text{ of } \beta \cap S^2_0 \text{ such that o.t.C = } \omega_1 \text{ and that } \langle \dot{Z}(\alpha) \mid \alpha \in C \rangle \text{ is } \subseteq \text{-continuous increasing.”}
\end{equation}

By Lemma 6.5 we have (6). Since $S$ is $\omega_1\text{-closed}$ and collapses $\omega_2$, one can choose an $S$-name $\dot{N}$ such that

\begin{equation}
\vdash_{S} \text{“}\dot{N} \text{ is a function on } \omega_1 \text{ such that } \langle \dot{N}(\zeta) \mid \zeta < \omega_1 \rangle \text{ is a } \subseteq \text{-continuous increasing sequence of countable sets satisfying} \end{equation}

\begin{equation}
\bigcup \{ \dot{N}(\zeta) \mid \zeta < \omega_1 \} = (\omega_2V) \text{ and that for each } \zeta < \omega_1 \end{equation}

\begin{equation}
\text{it holds that } \sup \dot{N}(\zeta) \notin \check{\dot{N}}(\zeta), \text{ sup } \dot{N}(\zeta) < \sup \hat{N}(\zeta + 1) \text{ and } \dot{Z}(\sup \dot{N}(\zeta)) \subseteq \hat{N}(\zeta + 1).”
\end{equation}

Therefore, in $V^S$, $\{ \dot{N}(\zeta) \mid \zeta < \omega_1 \}$ forms a club subset of $[(\omega_2V)]^{\leq \omega}$ and thus by (6) we have

\begin{equation}
\vdash_{S} \text{“}\check{T} := \{ \zeta < \omega_1 \mid \dot{Z}(\sup \dot{N}(\zeta)) \notin \hat{N}(\zeta)\} \text{ is stationary.”}
\end{equation}

Note that, for each $\zeta < \omega_1$,

\begin{equation}
D^0_\zeta := \{ s \in S \mid s \text{ decides the values of } \check{N}(\zeta) \text{ and } \dot{Z}(\sup \check{N}(\zeta))\}
\end{equation}

is a dense subset of $S$. 


Now apply $\text{MA}^+ (\omega_1\text{-closed})$ to get a filter $F$ on $\mathbb{S}$ such that $\langle p \rangle \in F$, $F$ intersects $D^\text{lh}_\zeta$ and $D^0_\zeta$ for all $\zeta < \omega_1$, and that
\[ T = \{ \zeta < \omega_1 \mid \exists s \in F \cup \mathbb{S} \text{ "} \zeta \in \check{T} \text{"} \} \]
is stationary in $\omega_1$. Note that, since $F$ intersects $D^\text{lh}_\zeta$ for all $\zeta < \omega_1$, $\bigcup F$ is of the form $\langle a_\gamma | \gamma < \omega_1 \rangle$ with $a_0 = p$, and
\[
\begin{align*}
I : & \quad a_0 \ a_1 \ \cdots \ a_{\omega+1} \ \cdots \\
II : & \quad b_0 \ b_1 \ \cdots \ b_\omega \ b_{\omega+1} \ \cdots
\end{align*}
\]
(where $b_\gamma := \tau(\gamma, a_\gamma)$) forms a play of $G_{\omega+1}(\mathbb{P})$ where Player II plays according to $\tau$. Therefore this sequence has a common extension $q \in \mathbb{P}$.

Note also that each element of $F$ is an initial segment of $\bigcup F$, and thus $F$ is $\omega_1$-directed. For each $\zeta < \omega_1$ let $N_\zeta$ and $z_\zeta$ be such that
\[
\text{(9)} \quad r_\zeta \vDash \mathbb{S} \text{ "} \dot{N}(\check{\zeta}) = \check{N}_\zeta \text{ and } \dot{Z}(\sup \dot{N}(\check{\zeta})) = \check{z}_\zeta \text{"}
\]
holds for some $r_\zeta \in F$. Such $N_\zeta$ and $z_\zeta$ uniquely exist by the directedness of $F$ and the fact that $F$ intersects $D^0_\zeta$. Moreover, by the definition of $T$, for each $\zeta \in T$ we may assume
\[
\text{(10)} \quad r_\zeta \vDash \mathbb{S} \text{ "} \dot{Z}(\sup \dot{N}(\check{\zeta})) \notin \dot{N}(\check{\zeta}) \text{."}
\]
Let $\alpha_\zeta = \sup N_\zeta$ for $\zeta < \omega_1$. By (8), (9), (10) and the $\omega_1$-directedness of $F$ we have the following:

(a) $\langle N_\zeta \mid \zeta < \omega_1 \rangle$ forms a $\subseteq$-increasing continuous sequence of countable subsets of $\omega_2$ and $\langle \alpha_\zeta \mid \zeta < \omega_1 \rangle$ is a continuous strictly increasing sequence.

(b) For each $\zeta < \omega_1$ it holds that $\sup z_\zeta = \alpha_\zeta$ and $z_\zeta \subseteq N_{\zeta+1}$, and therefore $\bigcup \{ z_\zeta \mid \zeta < \zeta \} \subseteq N_\zeta$ if $\zeta$ is limit.

(c) For each $\zeta \in T$, $z_\zeta \notin N_\zeta$ holds.

Now for each $\zeta < \omega_1$, by (9) and the fact that $\sup z_\zeta = \alpha_\zeta$, it holds that
\[
\text{(11)} \quad r_\zeta \vDash \mathbb{S} \text{ "} \dot{Z}(\check{\alpha_\zeta}) = \check{z}_\zeta \text{."}
\]
Since $\dot{Z}$ is a $\mathbb{P}$-name and $\text{lm}$ is a projection, by absoluteness we have
\[
\text{lm}(r_\zeta) \vDash \mathbb{P} \text{ "} \dot{Z}(\check{\alpha_\zeta}) = \check{z}_\zeta \text{."}
\]
Since $q$ extends $\text{lm}(r_\zeta)$ for all $\zeta < \omega_1$ we have
\[
\text{(12)} \quad q \vDash \mathbb{P} \text{ "} \forall \zeta < \omega_1 [\dot{Z}(\check{\alpha_\zeta}) = \check{z}_\zeta] \text{."}
\]
Since $q$ also extends $p$, by (7) we have
\[
q \vDash \mathbb{P} \text{ "} \text{There exists a club subset } C \text{ of } \omega_1 \text{ such that } \langle \dot{Z}(\check{\alpha_\zeta}) \mid \zeta \in C \rangle \text{ is a } \subseteq \text{-continuous increasing sequence."
}
By (12), (13) and the fact that $P$ is $\omega_2$-Baire, there exist $q' \leq_P q$ and a club subset $C_0$ (in $V$) of $\omega_1$ such that

$q' \Vdash \text{"}(z_\zeta \mid \zeta \in C_0)\text{" is a } \subseteq\text{-continuous increasing sequence."

By absoluteness, $\langle z_\zeta \mid \zeta \in C_0 \rangle$ is a $\subseteq\text{-continuous increasing sequence in } V$. Now since $T$ is stationary in $\omega_1$, there is an ordinal $\zeta_0 \in T \cap 1.p.(C_0)$. Thus on the one hand, by (c) we have that $z_{\zeta_0} \not\in N_{\zeta_0}$, and on the other hand, by (b) we have

$$z_{\zeta_0} = \bigcup \{z_\xi \mid \xi \in \zeta_0 \cap C_0\} \subseteq \bigcup \{z_\xi \mid \xi < \zeta_0\} \subseteq N_{\zeta_0}.$$  

This is a contradiction and finishes the proof of Theorem 6.1.

Now let us go back to the proof of Lemma 6.5. To this end, we introduce another type of two player game, which is a variation of the one introduced by Velickovic [19, p.272].

**Definition 6.6.** Let $\theta$ be a sufficiently large regular cardinal, and $\mathfrak{A}$ a model of the form $\langle H_\theta, \in, P, \tau, \ldots \rangle$. Let $C$ denote the set of countable elementary submodels of $\mathfrak{A}$. For $p \in P$, $\mathcal{G}(\mathfrak{A}, p)$ denotes the following two-player game. Players choose their moves as follows:

I : $\langle N_0, p_0 \rangle \langle N_1, p_1 \rangle \cdots$

II : $\langle \eta_0, q_0 \rangle \langle \eta_1, q_1 \rangle \cdots,$

where $N_i \in C$, $\eta_i \in [\omega_1, \omega_2)$ and $p_i, q_i \in P$ for each $i < \omega$. Players must follow the following rules:

(a) $p_i$ is $(N_i, P)$-strongly generic for each $i < \omega$.
(b) $p_0 \Vdash N_0 \leq_P p$, and $p_{i+1} \Vdash N_{i+1} \leq_P q_i$ for each $i < \omega$.
(c) $N_{i+1} \supseteq_{\eta_i} N_i$ for each $i < \omega$.
(d) $q_i \leq_P p_i$ for each $i < \omega$.

Note that (a)–(c) are required to Player I, whereas (d) is to Player II. Player I wins if and only if he successfully finished his $\omega$ turns without getting unable to make a legal move on the way.

**Lemma 6.7.** Player I has a winning strategy for $\mathcal{G}(\mathfrak{A}, p)$.

**Proof.** Note that $\mathcal{G}(\mathfrak{A}, p)$ is open to Player II, and thus is determined. So it is enough to show that Player II does not have a winning strategy. Let $v$ be any strategy for Player II.

By a simultaneous induction on $\omega_1 \cdot n + \gamma$ (for $n < \omega$ and $\gamma < \omega_1$), we will choose $N^n_\gamma \in C$, $r^n_\gamma, p^n_\gamma, q^n_\gamma \in P$, $\eta^n_\gamma \in \omega_2$ and a partial play $R^n_\gamma$ of $\mathcal{G}(\mathfrak{A}, p)$ so that the following requirements are satisfied: For each $m < \omega$ and $\xi < \omega_1$,

(i) $N^m_\xi \supseteq \langle N^m_\gamma, r^m_\gamma, p^m_\gamma, q^m_\gamma, \eta^m_\gamma, R^m_\gamma \rangle$ if $\xi = \zeta + 1.$
(ii) \( N^m_\xi \supseteq \langle N^m_\zeta, r^m_\zeta, p^m_\zeta, q^m_\zeta, \eta^m_\zeta, R^m_\zeta | \zeta < \omega_1 \rangle \) if \( m = l + 1 \).

(iii) \( N^m_\xi = \bigcup \{ N^m_\zeta | \zeta < \xi \} \) if \( \xi \) is limit.

(iv) \( r^m_\zeta \leq p^m_\zeta \) if \( \zeta = \zeta + 1 \).

(v) \( r^m_\zeta = \bigwedge \{ r^m_\zeta | \zeta < \xi \} \) if \( \xi \) is limit.

(vi) \( p^m_\zeta = \tau(\xi, r^m_\zeta) \).

(vii) \( p^m_\zeta \) is \( (N^m_\xi, \mathbb{P}) \)-strongly generic and \( p^m_\zeta \upharpoonright N^m_\xi = r^m_\zeta \).

(viii) \( q^m_\zeta \leq p^m_\zeta \).

(ix) \( R^m_\xi \) is a part of a play of \( \mathcal{G}(\mathfrak{A}, p) \) where Player II plays according to \( v \), ending with Player I’s move \( \langle N^m_\zeta, p^m_\zeta \rangle \) followed by Player II’s move \( \langle \eta^m_\zeta, q^m_\zeta \rangle \).

(x) \( R^m_\xi \) extends \( R^m_\delta \) if \( m = l + 1 \), where \( \delta^m_\zeta \) denotes \( N^m_\zeta \cap \omega_1 \).

Let \( n < \omega \) and \( \gamma < \omega_1 \), and suppose that

\[
\langle N^m_\zeta, r^m_\zeta, p^m_\zeta, q^m_\zeta, \eta^m_\zeta, R^m_\zeta | \omega_1 \cdot m + \xi < \omega_1 \cdot n + \gamma \rangle
\]

is already defined and satisfy (i)–(ix) for every pair of \( m \) and \( \xi \) such that \( \omega_1 \cdot m + \xi < \omega_1 \cdot n + \gamma \). We will define \( N^m_\xi, r^m_\xi, p^m_\xi, q^m_\xi, \eta^m_\xi \) and \( R^m_\xi \) so that (i)–(ix) for \( m = n \) and \( \xi = \gamma \) hold. Throughout the cases below, for example, “(i) for \( m = n \)” is simply expressed as “(i)”. Other combinations of \( m \) and \( \xi \), which appear as induction hypotheses, are explicitly specified.

**Case 1** \( n = \gamma = 0 \).

Pick \( N^0_0 \in \mathcal{C} \) so that \( p \in N^0_0 \). Pick an \((N^0_0, \mathcal{P})\)-generic sequence \( s^0_0 \) beginning with \( p \). Then set \( r^0_0 = \bigwedge s^0_0 \) and \( p^0_0 = \tau(0, r^0_0) \). Then by Lemma 1.4, we have (vi) and (viii), and the latter assures that \( \langle N^0_0, p^0_0 \rangle \) forms a legal 0-th move of Player I in \( \mathcal{G}(\mathfrak{A}, p) \). Set \( \langle \eta^0_0, q^0_0 \rangle = v(\langle N^0_0, p^0_0 \rangle) \) and \( R^0_0 = \langle \langle N^0_0, p^0_0 \rangle, \langle \eta^0_0, q^0_0 \rangle \rangle \). Then (viii) and (ix) are satisfied.

**Case 2** \( n = 0 \) and \( \gamma = \zeta + 1 \).

Pick \( N^0_0 \in \mathcal{C} \) so that \( \langle N^0_\zeta, r^0_\zeta, p^0_\zeta, q^0_\zeta, \eta^0_\zeta, R^0_\zeta \rangle \in N^0_0 \). Then (i) is satisfied. Pick an \((N^0_\zeta, \mathcal{P})\)-generic sequence \( s^0_\zeta \) beginning with \( q^0_\zeta \). Then set \( r^0_\zeta = \bigwedge s^0_\zeta \) and \( p^0_\zeta = \tau(\gamma, r^0_\zeta) \). Then \( \langle N^0_\zeta, p^0_\zeta \rangle = v(\langle N^0_\zeta, p^0_\zeta \rangle) \) and \( R^0_\zeta = \langle \langle N^0_\zeta, p^0_\zeta \rangle, \langle \eta^0_\zeta, q^0_\zeta \rangle \rangle \). Then (ix) is clear, and (vi)–(ix) are obtained in the same way as in Case 1.

**Case 3** \( n = 0 \) and \( \gamma \) is limit.

Let \( N^0_0 = \bigcup \{ N^0_\zeta | \zeta < \gamma \} \) and \( r^0_\zeta = \bigwedge \{ r^0_\zeta | \zeta < \gamma \} \). Then define \( p^0_\zeta, q^0_\zeta, \eta^0_\zeta \) and \( R^0_\zeta \) in the same way as in Case 2. By (i) and (iii) for \( m = 0 \) and \( \xi < \gamma \), \( \langle N^0_\zeta | \zeta < \gamma \rangle \) forms a \( \subseteq \)-increasing continuous chain of elements of \( \mathcal{C} \), and thus we have \( N^0_0 \in \mathcal{C} \) and (iii). By (iv), (v), (vi)
and (viii) for $m = 0$ and $\xi < \gamma$ we have that

$$
\begin{align*}
\text{I:} & \quad r_0^0 \quad r_1^0 \quad \cdots \quad \quad r_{\omega+1}^0 \\
\text{II:} & \quad p_0^0 \quad p_1^0 \quad \cdots \quad p_{\omega}^0 \quad p_{\omega+1}^0 \\
\end{align*}
$$

forms a part of a play of $G_{\omega_1+1}(\mathbb{P})$ where Player II plays according to $\tau$, and thus we have $r_0^0 \in \mathbb{P}$ and (vii). (vi) is clear by the construction. Pick a strictly increasing sequence $\langle \gamma_j \mid j < \omega \rangle$ of ordinals converging to $\gamma$. Then by the above observation we have $r_0^0 = \bigwedge \{ p_0^{\gamma_j} \mid j < \omega \}$. Now by (ii) and (vii) for $m = 0$ and $\xi < \gamma$ and (iii) for $m = 0$ and $\xi = \gamma$, we have that $\langle p_0^{\gamma_j} \mid j < \omega \rangle$ is an $(N_\gamma^m, \mathbb{P})$-generic sequence, and thus by Lemma 1.4 we have (vii). (viii) and (ix) are obtained in the same way as in the former cases.

Case 4 $n = l + 1$ and $\gamma = 0$.

First note that by (iv), (vi), (vi) and (viii) for $m = l$ and $\xi < \omega_1$ we have that

$$
\begin{align*}
\text{I:} & \quad r_0^l \quad r_1^l \quad \cdots \quad r_{\omega+1}^l \\
\text{II:} & \quad p_0^l \quad p_1^l \quad \cdots \quad p_{\omega}^l \quad p_{\omega+1}^l \\
\end{align*}
$$

forms a part of a play of $G_{\omega_1+1}(\mathbb{P})$ where Player II plays according to $\tau$. Therefore $\langle r_0^l \mid \xi < \omega_1 \rangle$ has a common extension in $\mathbb{P}$. Now pick $N_0^g \in \mathcal{C}$ so that $\langle N_0^g \cap \gamma_j, p_0^{\gamma_j}, q_j, \eta_j \mid \xi < \omega_1 \rangle \in N_0^g$. This gives (iii). By elementarity there is a common extension $q_\ell \in \mathbb{P}$ of $\langle r_\xi^l \mid \xi < \omega_1 \rangle$ in $N_0^g$. Pick an $(N_0^g, \mathbb{P})$-generic sequence $s_0^n$ beginning with $q_\ell$. Now set $r_0^g = \bigwedge s_0^n$ and $p_0^g = \tau(0, r_0^n)$. Then by Lemma 1.4 we have (vi) and (vii). By (iv) for $m = l$ and $\xi = \delta_0^g$, $R_{\delta_0}^l$ is a part of a play of $\mathcal{G}(\mathfrak{A}, p)$ where Player II plays according to $\nu$, ending with Player I’s move $\langle N_{\delta_0}^g, p_{\delta_0}^g \rangle$ followed by Player II’s move $\langle q_{\delta_0}^g, q_{\delta_0}^g \rangle$. We show that $\langle N_0^g, p_0^g \rangle$ is a legal move of Player I following $R_{\delta_0}^l$. (vii) assures that the rule (c) is satisfied. By the above construction and (iv) for $m = l$ and $\xi = \delta_0^g + 1$ we have that $q_{\delta_0}^g \geq_p r_{\delta_0+1}^g \geq_p q_{\delta_0}^g \geq_p r_{\delta_0}^g = p_0^g \upharpoonright N_0^g$. This assures that the rule (c) is satisfied. For the rule (c), first note that by (i) and (iii) for $m = l$ and $\xi < \omega_1$, $\langle N_0^l \mid \xi < \omega_1 \rangle$ is a $\subseteq$-continuous strictly increasing chain, and thus $\eta^l := \omega_2 \cap \bigcup \{ N_0^l \mid \xi < \omega_1 \}$ is an ordinal below $\omega_2$. Since $\langle N_0^l \mid \xi < \omega_1 \rangle \in N_0^g$, for each $\xi < \delta_0^g$ it holds that $N_0^l \subseteq N_0^g$, and thus we have that $N_0^{l+1} = \bigcup \{ N_0^l \mid \xi < \delta_0^g \} \subseteq N_0^{l+1}$. On the other hand, by elementarity, for each $\alpha \in \eta^l \cap N_0^g$ there exists $\xi \in \omega_1 \cap N_0^g = \delta_0^g$ such that $\alpha \in N_0^l \subseteq N_0^{l+1}$. Moreover, since $\eta^l \cap N_0^g \cap N_0^{l+1} = N_0^{l+1}$, we have $N_0^{l+1} \cap \omega_2 \neq N_0^g \cap \omega_2$. Therefore we have $N_0^{l+1} \prec_\eta^l N_0^g$, in particular $N_0^{l+1} \prec_\eta^l N_0^n$, since $\eta^l \in N_0^{l+1}$ holds by (i) for $m = l$ and $\xi = \delta_0^n + 1$.
and thus $\eta_0^l < \eta^l$ holds. Now let $\langle \eta_0^l, q_0^l \rangle = v(R_{\delta_0}^l \cap \langle \langle N_0^l, p_0^l \rangle \rangle)$ and $R_0^l = R_{\delta_0}^l \cap \langle \langle N_0^l, p_0^l \rangle, \langle \eta_0^l, q_0^l \rangle \rangle$. Now (vi), (ix) and (x) are clear.

Case 5 $n = l + 1$ and $\gamma = \zeta + 1$.

Pick $N_\gamma \in C$ so that $\langle N_\zeta^m, r_\zeta^m, p_\zeta^m, q_\zeta^m, \eta_\zeta^m, R_\zeta^m \rangle \in N_\gamma$. Pick an $(N_\gamma, p)$-generic sequence $s^m_\gamma$ beginning with $q^m_\gamma$. Then set $r^m_\gamma = \bigwedge s^m_\gamma$, $p^m_\gamma = \tau(\gamma, r^m_\gamma)$, $\langle \eta^m_\gamma, q^m_\gamma \rangle = v(R^m_\delta \cap \langle \langle N_\gamma^m, p_\gamma^m \rangle \rangle)$ and $R_\gamma = R^m_\delta \cap \langle \langle N_\gamma^m, p_\gamma^m \rangle, \langle \eta^m_\gamma, q^m_\gamma \rangle \rangle$. (ii) is clear, and this and (ii) for $m = n$ and $\xi = \zeta$ imply (ii). (iv) and (vi)–(x) are obtained in the same way as in the former cases.

Case 6 $n = l + 1$ and $\gamma$ is limit.

Let $N_\gamma = \bigcup \{ N_\zeta^m \mid \zeta < \gamma \}$, $r_\zeta^m = \bigwedge \{ r_\zeta^m \mid \zeta < \gamma \}$ and define $p_\gamma^m$, $q_\gamma^m$, $\eta_\gamma^m$ and $R_\gamma^m$ in the same way as in Case 5. Then (ii), (iii) and (vi)–(x) are obtained in the same way as in the former cases.

Now for each $n < \omega$ let $C_n := \{ \zeta < \omega_1 \mid \delta_\zeta^m = \zeta \}$. For each $n < \omega$, by (ii) and (iii) we have that $\langle \delta_\zeta^m \mid \zeta < \omega_1 \rangle$ is a normal sequence, and thus $C_n$ is a club subset of $\omega_1$. Therefore $C = \bigcap \{ C_n \mid n < \omega \}$ is nonempty. Pick $\gamma \in C$. Then $\delta^m_\gamma = \gamma$ holds for every $n < \omega$, and thus by (x) $\bigcup \{ R_\gamma^m \mid n < \omega \}$ forms a full play in $\mathfrak{G}(\mathfrak{A}, p)$ where Player II plays according to $\nu$. This shows that $\nu$ is not a winning strategy. (Lemma 6.7)

Proof of Lemma 6.4 Suppose $\hat{Z}$ is as in the assumption of the lemma. Let $r$ be any condition in $\mathfrak{S}$, and $\hat{f}$ any $\mathfrak{S}$-name satisfying

$$\|r\|_{\mathfrak{S}} \models \langle \hat{f} : \omega(\omega^Y_1) \rightarrow (\omega^Y_1).$$

Let $\theta$ be a sufficiently large regular cardinal, and set $\mathfrak{A} = \langle H_\theta, \in, \mathbb{P}, \tau, \hat{f}, \{ r \} \rangle$.

To have the conclusion of the lemma, it is enough to show that there exists a countable $N \prec \mathfrak{A}$ and an $(N, \mathfrak{S})$-generic $s$ extending $r$ such that

$$s \|_{\mathfrak{S}} \models \langle \hat{Z}(\tilde{\eta}) \notin \tilde{N} \rangle,$$

where $\eta$ denotes $\sup(N \cap \omega^Y_1)$.

Let $p = \text{Im}(r) \in \mathbb{P}$. By Lemma 6.7 Player I has a winning strategy $\rho$ for the game $\mathfrak{G}(\mathfrak{A}, p)$. We will consider two plays of $\mathfrak{G}(\mathfrak{A}, p)$, named Play $A$ and $B$, played simultaneously in the way described below.

Moves in these plays are denoted as follows:

**Play A**

| I    | $\langle N_0^A, p_0^A \rangle$ | $\langle N_1^A, p_1^A \rangle$ | $\ldots$ |
|------|---------------------------------|---------------------------------|--------|
| II   | $\langle \eta_0^A, q_0^A \rangle$ | $\langle \eta_1^A, q_1^A \rangle$ | $\ldots$ |

**Play B**

| I    | $\langle N_0^B, p_0^B \rangle$ | $\langle N_1^B, p_1^B \rangle$ | $\ldots$ |
|------|---------------------------------|---------------------------------|--------|
| II   | $\langle \eta_0^B, q_0^B \rangle$ | $\langle \eta_1^B, q_1^B \rangle$ | $\ldots$ |
In both plays Player I plays according to $\rho$. Player II chooses her moves as follows:

\[
\begin{cases}
(n^A_n, q^A_n) = (\sup(N^B_n \cap \omega_2), \tau(n, p^B_n)), \\
(n^B_n, q^B_n) = (\sup(N^A_{n+1} \cap \omega_2), p^A_{n+1}).
\end{cases}
\]

By the construction we have:

(i) $N^A_0 = N^B_0$.
(ii) $N^A_0 \prec \sup(N^A_1 \cap \omega_2)$, $N^A_1 \prec \sup(N^A_2 \cap \omega_2)$, $N^A_2 \prec \sup(N^A_3 \cap \omega_2)$, \ldots.
(iii) $N^B_0 \prec \sup(N^B_1 \cap \omega_2)$, $N^B_1 \prec \sup(N^B_2 \cap \omega_2)$, $N^B_2 \prec \sup(N^B_3 \cap \omega_2)$, \ldots.
(iv) $p \geq p^A_0 \upharpoonright N^A_0 = p^B_0 \upharpoonright N^B_0 \geq p^A_0 = p^B_0$.
(v) For every $n < \omega$, $p^A_n$ is $(N^A_n, \mathbb{P})$-strongly generic and $p^B_n$ is $(N^B_n, \mathbb{P})$-strongly generic.
(vi) For every $n < \omega$, it holds that

\[
(p_n^B \geq_p \tau(n, p_n^B)) = (q_n^A \geq_p p_{n+1} \upharpoonright N^A_{n+1} \geq_p p^A_{n+1}) \quad \text{and} \quad (q_n^B \geq_p p_{n+1} \upharpoonright N^B_{n+1} \geq_p p^B_{n+1}).
\]

By (vi), $(p_n^B, q_n^A \mid n < \omega)$ forms a part of a play in $G_{\omega_1+1}(\mathbb{P})$ where Player II plays along $\tau$, and thus we have

\[
q := \bigwedge \{p_n^B \mid n < \omega\} = \bigwedge \{p_n^A \upharpoonright N^A_n \mid n < \omega\} = \bigwedge \{p_n^B \upharpoonright N^A_n \mid n < \omega\} \in \mathbb{P}.
\]

Let $N^A = \bigcup \{N^A_n \mid n < \omega\}$, $N^B = \bigcup \{N^B_n \mid n < \omega\}$. By (ii) and (iii), $N^A, N^B$ are countable elementary submodels of $\mathfrak{A}$. Moreover, by (ii) and (iii) we have

- $N^A \cap \omega_1 = N^B_0 \cap \omega_1 = N^B \cap \omega_1 = N^B \cap \omega_1$ and
- $\sup(N^A \cap \omega_2) = \sup(N^B \cap \omega_2)$.

We will denote these two ordinals as $\delta$ and $\eta$.

Note also that, by (ii), (iii), (vi) and (vi), $(p_n^A \mid n < \omega)$ is an $(N^A, \mathbb{P})$-generic sequence and $(p_n^B \mid n < \omega)$ is an $(N^B, \mathbb{P})$-generic sequence. Therefore by Lemma 1.4 we have

\[
q = q \upharpoonright N^A = q \upharpoonright N^B.
\]

Now let $q' = \tau(\delta, q)$. Since $N^A \cap N^B \cap \omega_2 = N^A_0 \cap \omega_2$ is bounded in $\eta$ by (ii) and (iii), it holds that

$q' \Vdash_{\mathbb{P}} \langle \check{Z}(\check{\eta}) \notin \check{N}^A \lor \check{Z}(\check{\eta}) \notin \check{N}^B, \rangle.$

Therefore, without loss of generality we may assume that there exists $q'' \leq_{\mathbb{P}} q'$ such that

\[
q'' \Vdash_{\mathbb{P}} \langle \check{Z}(\check{\eta}) \notin \check{N}^A, \rangle.
\]
Now since $\text{lm}(r) = p \geq q$ and $r \in N^A$ holds, repeatedly using Lemma 1.5, we obtain an $(N^A, S)$-generic sequence $\langle r_n \mid n < \omega \rangle$ such that $r_0 = r$ and that $\text{lm}(r_n) \geq p$ holds for every $n < \omega$. Let us denote $igcup\{r_n \mid n < \omega\} = \langle a_\gamma, b_\gamma \mid \gamma < \delta \rangle$. By an easy density argument we have $\delta = N^A \cap \omega_1 = \delta$. Since $\text{lm}$ is a projection, $\langle \text{lm}(r_n) \mid n < \omega \rangle$ is an $(N^A, P)$-generic sequence, and thus by Lemma 1.4 and (15) we have

$$\bigwedge\{b_\gamma \mid \gamma < \delta\} = \bigwedge\{\text{lm}(r_n) \mid n < \omega\} = q \restriction N^A = q.$$  

Therefore

$$s = \langle \bigcup\{r_n \mid n < \omega\} \rangle \langle q, q'' \rangle$$

forms an $(N^A, S)$-strongly generic condition satisfying $\text{lm}(s) \leq_p q''$. By (16) and absoluteness we have

$$s \Vdash_S "\tilde{Z}(\tilde{\eta}) \notin N^A."$$

This shows that $s$ is what we need, and completes the proof of the lemma.

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