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Asratian, A., Casselgren, C. J., Petrosyan, P. A., (2019), Cyclic deficiency of graphs, Discrete Applied Mathematics, 266, 171-185. https://doi.org/10.1016/j.dam.2018.12.003

Original publication available at:
https://doi.org/10.1016/j.dam.2018.12.003

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Cyclic Deficiency of Graphs

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A proper edge coloring of a graph $G$ with colors $1, 2, \ldots, t$ is called a cyclic interval $t$-coloring if for each vertex $v$ of $G$ the edges incident to $v$ are colored by consecutive colors, under the condition that color 1 is considered as consecutive to color $t$. In this paper we introduce and investigate a new notion, the cyclic deficiency of a graph $G$, defined as the minimum number of pendant edges whose attachment to $G$ yields a graph admitting a cyclic interval coloring; this number can be considered as a measure of closeness of $G$ of being cyclically interval colorable. We determine or bound the cyclic deficiency of several families of graphs. In particular, we present examples of graphs of bounded maximum degree with arbitrarily large cyclic deficiency, and graphs whose cyclic deficiency approaches the number of vertices. Finally, we conjecture that the cyclic deficiency of any graph does not exceed the number of vertices, and we present several results supporting this conjecture.

Keywords: edge coloring, interval edge coloring, cyclic interval edge coloring, deficiency, cyclic deficiency

1. Introduction

We use [37] for terminology and notation not defined here. All graphs in this paper are finite, undirected, and contain no multiple edges or loops; a multigraph may have both multiple edges and loops. $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph $G$, respectively. A proper $t$-edge coloring of a graph $G$ is a mapping $\alpha : E(G) \rightarrow \{1, \ldots, t\}$ such that $\alpha(e) \neq \alpha(e')$ for every pair of adjacent edges $e$ and $e'$ in $G$.

A proper $t$-edge coloring of a graph $G$ is called an interval $t$-coloring if the colors of the edges incident to every vertex $v$ of $G$ form an interval of integers. This notion was
introduced by Asratian and Kamalian [3] (available in English as [4]), motivated by the problem of constructing timetables without “gaps” for teachers and classes. Generally, it is an NP-complete problem to determine whether a bipartite graph has an interval coloring [35]. However some classes of graphs have been proved to admit interval colorings; it is known, for example, that trees, regular and complete bipartite graphs [3,20,23], doubly convex bipartite graphs [5,24], grids [16], and outerplanar bipartite graphs [17] have interval colorings. Additionally, all \((2,b)\)-biregular graphs [20,21,25] and \((3,6)\)-biregular graphs [11] admit interval colorings, where an \((a,b)\)-biregular graph is a bipartite graph where the vertices in one part all have degree \(a\) and the vertices in the other part all have degree \(b\).

In [18,19] Giaro et al. introduced and investigated the deficiency \(\text{def}(G)\) of a graph \(G\) defined as the minimum number of pendant edges whose attachment to \(G\) makes the resulting graph interval colorable. In [18] it was proved that there are bipartite graphs whose deficiency approaches the number of vertices, and in [19] it was proved that if \(G\) is an \(r\)-regular graph with an odd number of vertices, then \(\text{def}(G) \geq \frac{r^2}{2}\); the last result was recently generalized in [8]. Furthermore in [19] the deficiency of complete graphs, wheels and broken wheels was determined. Schwartz [34] obtained tight bounds on the deficiency of some families of regular graphs. Recently, Petrosyan and Khachatryan [32] proved that for near-complete graphs \(\text{def}(K_{2n+1} - e) = n - 1\) (where \(e\) is an edge of \(K_{2n+1}\)), thereby confirming a conjecture of Borowiecka-Olszewska et al. [8]. Further results on deficiency appear in [1,8–10,14,27,32].

Another type of proper \(t\)-edge colorings, a cyclic interval \(t\)-coloring, was introduced by de Werra and Solot [36]. A proper \(t\)-edge coloring \(\alpha : E(G) \rightarrow \{1, \ldots, t\}\) of a graph \(G\) is called a cyclic interval \(t\)-coloring if the colors of the edges incident to every vertex \(v\) of \(G\) either form an interval of integers or the set \(\{1, \ldots, t\} \setminus \{\alpha(e) : e \text{ is incident to } v\}\) is an interval of integers. This notion was motivated by scheduling problems arising in flexible manufacturing systems, in particular the so-called cylindrical open shop scheduling problem. Clearly, any interval \(t\)-coloring of a graph \(G\) is also a cyclic interval \(t\)-coloring. Therefore all above mentioned classes of graphs which admit interval edge colorings, also admit cyclic interval colorings.

Generally, the problem of determining whether a given bipartite graph admits a cyclic interval coloring is \(NP\)-complete [29]. Nevertheless, all graphs with maximum degree at most 3 [30], complete multipartite graphs [2], outerplanar bipartite graphs [36], bipartite graphs with maximum degree 4 and Eulerian bipartite graphs with maximum degree not exceeding 8 [2], and some families of biregular bipartite graphs [2,11,12] admit cyclic interval colorings.

In this paper we introduce and investigate a new notion, the cyclic deficiency of a graph \(G\), denoted by \(\text{def}_c(G)\), defined as the minimum number of pendant edges that has to be attached to \(G\) in order to obtain a graph admitting a cyclic interval coloring. This number can be considered as a measure of closeness of a graph \(G\) of being cyclically interval colorable. Clearly, \(\text{def}_c(G) = 0\) if and only if \(G\) admits a cyclic interval coloring. Hence, there are infinite families of graphs with cyclic deficiency 0 but arbitrarily large deficiency, e.g. regular graphs with an odd number of vertices.

We present examples of graphs of bounded maximum degree with arbitrarily large cyclic deficiency, and graphs whose cyclic deficiency approaches the number of vertices. We
determine or bound the cyclic deficiency of several families of graphs and describe three different methods for constructing graphs with large cyclic deficiency. Our constructions generalize earlier constructions by Hertz, Erdős, Malafiejki and Sevastjanov (see e.g. [18,19,22]). Since the inequality \( \text{def}_c(G) \leq \text{def}(G) \) holds for any graph \( G \), our constructions yield new classes of graphs with large deficiency. Finally, we conjecture that the cyclic deficiency of any graph does not exceed the number of vertices, and we present several results supporting this conjecture. In particular we prove that the conjecture is true for graphs with maximum degree at most 5, bipartite graphs with maximum degree at most 8, and graphs where the difference between maximum and minimum degrees does not exceed 2.

2. Definitions and auxiliary results

A graph \( G \) is cyclically interval colorable if it has a cyclic interval \( t \)-coloring for some positive integer \( t \). The set of all cyclically interval colorable graphs is denoted by \( \mathcal{N}_c \). For a graph \( G \in \mathcal{N}_c \), the maximum number of colors in a cyclic interval coloring of \( G \) is denoted by \( W_c(G) \).

The degree of a vertex \( v \) of a graph \( G \) is denoted by \( d_G(v) \). \( \Delta(G) \) and \( \delta(G) \) denote the maximum and minimum degrees of \( G \), respectively. A graph \( G \) is even if the degree of every vertex of \( G \) is even. The diameter of a graph \( G \) we denote by \( \text{diam}(G) \).

We shall need a classic result from factor theory. A 2-factor of a multigraph \( G \) (where loops are allowed) is a 2-regular spanning subgraph of \( G \).

**Theorem 2.1 (Petersen’s Theorem).** Let \( G \) be a 2r-regular multigraph (where loops are allowed). Then \( G \) has a decomposition into edge-disjoint 2-factors.

The chromatic index \( \chi'(G) \) of a graph \( G \) is the minimum number \( t \) for which there exists a proper \( t \)-edge coloring of \( G \).

**Theorem 2.2 (Vizing’s Theorem).** For any graph \( G \), \( \chi'(G) = \Delta(G) \) or \( \chi'(G) = \Delta(G) + 1 \).

A graph \( G \) is said to be Class 1 if \( \chi'(G) = \Delta(G) \), and Class 2 if \( \chi'(G) = \Delta(G) + 1 \).

The next result gives a sufficient condition for a graph to be Class 1 (see, for example, [15]).

**Theorem 2.3** If \( G \) is a graph where no two vertices of maximum degree are adjacent, then \( G \) is Class 1.

We denote by \( \mathbb{N} \) the the set of all positive integers. If \( t \in \mathbb{N} \) and \( A \) is a subset of the set \( \{1, \ldots, t\} \), then \( A \) is called a cyclic interval modulo \( t \) if either \( A \) or \( \{1, \ldots, t\} \setminus A \) is an interval of integers. The deficiency modulo \( t \) of a given set \( A \) of integers is the minimum size of a set \( B \) of positive integers such that \( A \cup B \) is a cyclic interval modulo \( t \). A set of positive integers \( A \) is called near-cyclic modulo \( t \) (or just near-cyclic) if there is an integer \( k \) such that \( A \cup \{k\} \) is a cyclic interval modulo \( t \).

If \( \alpha \) is a proper edge coloring of \( G \) and \( v \in V(G) \), then \( S_G(v, \alpha) \) (or \( S(v, \alpha) \)) denotes the set of colors appearing on edges incident to \( v \).
**Definition 1** For a given graph $G$ and a proper $t$-edge coloring $\alpha$ of $G$, the *cyclic deficiency of $\alpha$ at a vertex* $v \in V(G)$, denoted $\text{def}_c(v, \alpha)$ is the deficiency modulo $t$ of the set $S(v, \alpha)$. The cyclic deficiency $\text{def}_c(G, \alpha)$ of the edge coloring $\alpha$ is the sum of cyclic deficiencies of all vertices in $G$. The cyclic deficiency $\text{def}_c(G)$ of $G$ can then be defined as the minimum of $\text{def}_c(G, \alpha)$ taken over all proper edge colorings $\alpha$ of $G$.

![Figure 1. A cyclically interval non-colorable graph with 17 vertices.](image)

Clearly, $\text{def}_c(G) = 0$ if and only if $G$ has a cyclic interval coloring. In particular, this implies that the problem of computing the cyclic deficiency of a given graph is $NP$-complete.

In [28] it was shown that among all connected graphs with at most 6 vertices there are only seven Class 1 graphs without interval colorings. On the other hand, in [6] it was shown that among all connected graphs with at most 6 vertices there are only eight Class 2 graphs. For all these graphs we constructed cyclic interval colorings, so all connected graphs with at most 6 vertices are cyclically interval colorable. However, there exists a connected graph with 17 vertices that has no cyclic interval coloring [33]. (See Figure 1)

Finally, we need the notion of a projective plane.

**Definition 2** A *finite projective plane* $\pi(n)$ of order $n$ ($n \geq 2$) has $n^2 + n + 1$ points and $n^2 + n + 1$ lines, and satisfies the following properties:

**P1** any two points determine a line;

**P2** any two lines determine a point;
P3 every point is incident to \( n + 1 \) lines;
P4 every line is incident to \( n + 1 \) points.

3. Comparison of deficiency and cyclic deficiency

Clearly, \( \text{def}_c(G) \leq \text{def}(G) \) for every graph \( G \), since any interval coloring of \( G \) is also a cyclic interval coloring of \( G \). In particular, \( \text{def}_c(G) = \text{def}(G) = 0 \) for every graph \( G \) which admits an interval coloring.

In this section we will show that the difference between the deficiency and cyclic deficiency can be arbitrarily large, even for graphs with large deficiency.

Figure 2. The graph \( S_{7,7,7} \).

We begin by considering generalizations of two families of bipartite graphs with large deficiency introduced by Giaro et al. [18]. For any \( a, b, c \in \mathbb{N} \), define the graph \( S_{a,b,c} \) as follows:

\[
V(S_{a,b,c}) = \{u_0, u_1, u_2, u_3, v_1, v_2, v_3\} \cup \{x_1, \ldots, x_a, y_1, \ldots, y_b, z_1, \ldots, z_c\}
\]

\[
E(S_{a,b,c}) = \{u_1v_1, v_1u_2, u_2v_2, v_2u_3, v_3u_1\} \cup \{u_0x_i, u_1x_i : 1 \leq i \leq a\}
\]

\[
\cup \{u_0y_j, u_2y_j : 1 \leq j \leq b\} \cup \{u_0z_k, u_3z_k : 1 \leq k \leq c\}.
\]
Figure 2 shows the graph $S_{7,7,7}$.

Next we define a family of graphs $M_{a,b,c}$ $(a, b, c \in \mathbb{N})$. We set

$$V(M_{a,b,c}) = \{u_0, u_1, u_2, u_3\} \cup \{x_1, \ldots, x_a, y_1, \ldots, y_b, z_1, \ldots, z_c\}$$

and

$$E(M_{a,b,c}) = \{u_0x_i, u_1x_i, u_2x_i : 1 \leq i \leq a\} \cup \{u_0y_j, u_2y_j, u_3y_j : 1 \leq j \leq b\}$$

$$\cup \{u_0z_k, u_3z_k, u_1z_k : 1 \leq k \leq c\}.$$

Figure 3 shows the graph $M_{5,5,5}$.

Clearly, $S_{a,b,c}$ and $M_{a,b,c}$ are connected bipartite graphs. Giaro et al. [18] showed that the graphs $S_k = S_{k,k,k}$ and $M_k = M_{k,k,k}$ satisfy $\text{def}(S_k) \geq k - 6$ and $\text{def}(M_k) \geq k - 4$ for each $k \geq 6$; that is, the deficiencies of $S_k$ and $M_k$ grow with the number of vertices. Note that quite recently, a further generalization of the graphs $\{S_{a,b,c}\}$ was considered in [7] by Borowiecka-Olszewska et al. Moreover, they proved that in fact $\text{def}(S_k) = \max\{0, k - 6\}$, and, additionally, it follows from the results in [7] that $\text{def}(S_{a,b,c}) = 0$ if $\min\{a, b, c\} \leq 6$.

Here we shall prove that all graphs in the families $\{S_{a,b,c}\}$ and $\{M_{a,b,c}\}$ have cyclic deficiency 0. In the latter case, this result was first obtained in [30] for the case when $a = b = c$.

Figure 3. The graph $M_{5,5,5}$.

**Theorem 3.1** For any $a, b, c \in \mathbb{N}$, $\text{def}_c(S_{a,b,c}) = \text{def}_c(M_{a,b,c}) = 0$. 
Cyclic Deficiency of Graphs

**Proof.** By the remark above, we may assume that $\min\{a, b, c\} \geq 6$. For the proof, we construct edge colorings of the graphs $S_{a,b,c}$ and $M_{a,b,c}$ with cyclic deficiency zero. We first construct an edge coloring $\alpha$ of the graph $S_{a,b,c}$. We define this coloring as follows:

1. For $1 \leq i \leq a$, let $\alpha(u_0x_i) = i$ and $\alpha(u_1x_i) = i + 1$;
2. For $1 \leq j \leq b$, let $\alpha(u_0y_j) = a + j$ and $\alpha(u_2y_j) = a + 1 + j$;
3. For $1 \leq k \leq c$, let $\alpha(u_0z_k) = a + b + k$;
4. For $1 \leq k \leq c - 1$, let $\alpha(u_3z_k) = a + b + 1 + k$, and $\alpha(u_3z_c) = 1$;
5. $\alpha(u_1v_1) = a + 2, \alpha(v_1u_2) = a + 1, \alpha(u_2v_2) = a + b + 2, \alpha(v_2u_3) = a + b + 1, \alpha(u_3v_3) = 2, \alpha(v_3u_1) = 1$.

It is not difficult to see that $\alpha$ is a cyclic interval $(a + b + c)$-coloring of $S_{a,b,c}$.

Next we define an edge coloring $\beta$ of the graph $M_{a,b,c}$ as follows:

1'. For $2 \leq i \leq a$, let $\beta(u_0x_i) = i - 1$, and $\beta(u_0x_1) = a + b + c + 1$;
2'. For $1 \leq i \leq a$, let $\beta(u_1x_i) = i$ and $\beta(u_2x_i) = i + 1$;
3'. For $1 \leq j \leq b$, let $\beta(u_0y_j) = a - 1 + j$;
4'. For $1 \leq j \leq b$, let $\beta(u_2y_j) = a + 1 + j$ and $\beta(u_3y_j) = a + j$;
5'. For $1 \leq k \leq c$, let $\beta(u_0z_k) = a + b - 1 + k$;
6'. For $1 \leq k \leq c$, let $\beta(u_3z_k) = a + b + k$ and $\beta(u_1z_k) = a + b + 1 + k$.

It is easy to verify that $\beta$ is a cyclic interval $(a + b + c + 1)$-coloring of $M_{a,b,c}$. Hence, $\def_c(S_{a,b,c}) = \def_c(M_{a,b,c}) = 0$. □

The main result of this section is the following.

**Theorem 3.2** For any positive integers $m, n$ ($m \leq n$), there exists a connected graph $G$ of bounded maximum degree such that $\def_c(G) = m$ and $\def(G) = n$.

**Proof.** We shall construct a connected graph $G_{m,n}$ satisfying the conditions of the theorem. Let $H$ be the graph shown in Figure 4 and $F$ be the graph shown in Figure 5. In [18,32], it was proved that $\def(K_5) = 2$ and $\def(K_9-e) = 3$. This implies that $\def(H) \geq 1$ and $\def(F) \geq 1$. Let us consider the graph $G_{m,n}$ shown in Figure 6; this graph contains $m$ copies of $H$ and $n - m$ copies of $F$. Clearly, $G_{m,n}$ is a connected graph and $\Delta(G_{m,n}) = 12$ for any $m, n \in \mathbb{N}$. Since the graph $G_{m,n}$ contains $m$ copies of $H$ and $n - m$ copies of $F$, $\def(G_{m,n}) \geq n$. On the other hand, the coloring in Figure 7 yields that $\def(G_{m,n}) \leq n$. (The vertices with non-zero deficiency appear at the top of $K_5$ and $K_9 - e$ in Figure 7.) Thus $\def(G_{m,n}) = n$ for any $m, n \in \mathbb{N}$.

Let us now show that $\def_c(G_{m,n}) = m$ for any $m, n \in \mathbb{N}$. Since the edge coloring of $F$ in Figure 8 is a cyclic interval coloring, it follows that this graph is cyclically interval
colorable. Using this fact, it is easily seen from the edge colorings in Figure 8 that 
\( \text{def}_c(G_{m,n}) \leq m \). (The vertices with non-zero cyclic deficiency appear at the top of \( K_5 \) in Figure 8.)

To show that \( \text{def}_c(G_{m,n}) \geq m \), let \( J \) be the graph in Figure 1. In [33], it was proved that \( J \notin \mathcal{K}_c \), thus \( \text{def}_c(J) \geq 1 \). Let \( \alpha \) be a proper \( t \)-edge coloring of \( G_{m,n} \) with a minimum cyclic deficiency, that is \( \text{def}_c(G_{m,n}, \alpha) = \text{def}_c(G_{m,n}) \). Suppose, for a contradiction, that \( \text{def}_c(G_{m,n}) < m \). Since \( G_{m,n} \) contains \( m \) copies of \( J \), this implies that there exists a copy \( J' \) of the graph \( J \) in \( G_{m,n} \) such that for every \( v \in V(J') \), the set \( S_{J'}(v, \alpha) \) is a cyclic interval modulo \( t \). This implies that \( J \) has a cyclic interval coloring, a contradiction. \( \square \)

The above theorem has some immediate consequences.

**Corollary 3.3** For any \( n \in \mathbb{N} \), there exists a connected graph \( G \) of bounded maximum degree such that \( \text{def}_c(G) \geq n \).

**Corollary 3.4** For any \( n \in \mathbb{N} \), there exists a connected graph \( G \) of bounded maximum degree such that \( \text{def}(G) - \text{def}_c(G) \geq n \).
Figure 7. def($G_{m,n}$) $\leq n$. 
Figure 8. $\text{def}_c(G_{m,n}) \leq m$. 
4. Graphs with large cyclic deficiency

In this section we shall describe three methods for constructing classes of graphs with large cyclic deficiency. Our methods generalize previous constructions of graphs without interval colorings by Hertz, Giaro et al. and Erdős (see e.g. [18,22,19]) and give new large cyclic deficiency. Our methods generalize previous constructions of graphs without is, def, Cyclic Deficiency of Graphs

4.1. Constructions using subdivisions

Let $G$ be a graph with $V(G) = \{v_1, \ldots, v_n\}$. Define the graphs $S(G)$ and $\hat{G}$ as follows:

$$V(S(G)) = \{v_1, \ldots, v_n\} \cup \{w_{ij} : v_iv_j \in E(G)\},$$

$$E(S(G)) = \{v_iw_{ij}, v_jw_{ij} : v_iv_j \in E(G)\},$$

$$V(\hat{G}) = V(S(G)) \cup \{u\}, \ u \notin V(S(G)), \ E(\hat{G}) = E(S(G)) \cup \{uw_{ij} : v_iv_j \in E(G)\}.$$ 

In other words, $S(G)$ is the graph obtained by subdividing every edge of $G$, and $\hat{G}$ is the graph obtained from $S(G)$ by connecting every inserted vertex to a new vertex $u$. Clearly, $S(G)$ and $\hat{G}$ are bipartite graphs.

We will use the following two results.

Theorem 4.1 [13] If $G$ is a connected graph with at least two vertices, then $\text{diam}(S(G)) \leq 2\text{diam}(G) + 2$.

Theorem 4.2 [33] If $G$ is a connected bipartite graph and $G \in \mathcal{R}_c$, then $W_c(G) \leq 1 + 2\text{diam}(G) (\Delta(G) - 1)$.

The following is the main result of this subsection.

Theorem 4.3 If $G$ is a connected graph with $\Delta(G) \geq 3$, then

$$\text{def}_c(\hat{G}) \geq \frac{(\text{def}_c(G) - 1)}{4(\text{diam}(G) + 2)} - \Delta(G) + 1.$$ 

Proof. Let $\alpha$ be a proper $t$-edge coloring of $\hat{G}$ with the minimum cyclic deficiency, that is, $\text{def}_c(\hat{G}, \alpha) = \text{def}_c(\hat{G})$. Clearly, $t \geq \Delta(\hat{G}) = |E(G)|$.

Define an auxiliary graph $\hat{G}'$ as follows: for each vertex $v \in V(\hat{G})$ with $\text{def}_c(v, \alpha) > 0$, we attach $\text{def}_c(v, \alpha)$ pendant edges at vertex $v$. Clearly, $|V(\hat{G}')| = |V(\hat{G})| + \text{def}_c(\hat{G})$. Next, define the graph $\hat{G}^+$ from $\hat{G}'$ as follows: we remove the vertex $u$ from $\hat{G}'$ and all isolated vertices from $\hat{G}' - u$ (if such vertices exist) and add a new vertex $v_{ij}$ for each inserted vertex $w_{ij}$; then we add new edges $u_{ij}w_{ij}$ ($v_iv_j \in E(G)$). The graph $\hat{G}^+$ is connected and bipartite, and has edge set $E(\hat{G}' - u) \cup \{u_{ij}w_{ij} : v_iv_j \in E(G)\}$. Moreover, by Theorem 4.1, we obtain that $\text{diam}(\hat{G}^+) \leq 2\text{diam}(G) + 4$.

We extend the proper $t$-edge coloring $\alpha$ of $\hat{G}$ to a proper $t$-edge coloring $\beta$ of $\hat{G}^+$ as follows: for each vertex $v \in V(\hat{G}' - u)$ with $\text{def}_c(v, \alpha) > 0$, we color the attached edges incident to $v$ using $\text{def}_c(v, \alpha)$ distinct colors to obtain a cyclic interval modulo $t$, and for each inserted vertex $w_{ij}$, we color the edge $u_{ij}w_{ij}$ with color $\alpha(uw_{ij})$. By the definition of $\beta$ and the construction of $\hat{G}^+$, we obtain that $\beta$ is a cyclic interval $t$-coloring. By Theorem 4.2, we have
\[ |E(G)| \leq t \leq 1 + 2\text{diam}(\hat{G}^+) \left( \Delta(\hat{G}^+) - 1 \right) \leq 1 + 4(\text{diam}(G) + 2) \left( \Delta(G) + \text{def}_c(\hat{G}) - 1 \right). \]

Note that the third inequality follows from the fact that \( \Delta(G) \geq 3. \) From the preceding equation we now deduce that
\[
\text{def}_c(\hat{G}) \geq \frac{(|E(G)| - 1)}{4(\text{diam}(G) + 2)} - \Delta(G) + 1. \quad \square
\]

Using Theorem 4.3, we can generate infinite families of graphs with large cyclic deficiency. Let us consider some examples.

For the complete graph \( K_n \) we have \( \text{diam}(K_n) = 1 \) and \( \Delta(K_n) = n - 1; \) so, by Theorem 4.3, if \( n \geq 4, \) then \( \text{def}_c(\hat{K}_n) \geq \frac{(n^2 - n - 2)}{24} - n + 2. \)

Next, for the complete bipartite graph \( K_{m,n}, \) it holds that \( \text{diam}(K_{m,n}) \leq 2 \) and \( \Delta(K_{m,n}) = \max\{m, n\}; \) thus by Theorem 4.3, if \( \max\{m, n\} \geq 3, \) then \( \text{def}_c(\hat{K}_{m,n}) \geq \frac{(mn - 1)}{16} - \max\{m, n\} + 1. \)

Finally, for the hypercube \( Q_n, \) \( |E(Q_n)| = n2^{n-1} \) and \( \text{diam}(Q_n) = \Delta(Q_n) = n. \) Using Theorem 4.3 we deduce that \( \text{def}_c(\hat{Q}_n) \geq \frac{(n2^{n-1} - 1)}{4(n+2)} - n + 1 \) \( (n \geq 3). \)

4.2. Constructions using trees

Our next construction uses techniques first described in [31] and generalizes the family of so-called Hertz graphs first described in [18].

Let \( T \) be a tree and let \( P \) be the set of all paths in \( T. \) We set \( F(T) = \{v : v \in V(T) \land d_T(v) = 1\}, \) and define \( M(T) \) as follows:

\[
M(T) = \max_{P \in P} |E(P)| + |\{uw : uw \in E(T), u \in V(P), w \notin V(P)\}|.
\]

Now let us define the graph \( \tilde{T} \) as follows:
\[
V(\tilde{T}) = V(T) \cup \{u\}, u \notin V(T), E(\tilde{T}) = E(T) \cup \{uv : v \in F(T)\}.
\]

Clearly, \( \tilde{T} \) is a connected graph with \( \Delta(\tilde{T}) = |F(T)|. \) Moreover, if \( T \) is a tree in which the distance between any two pendant vertices is even, then \( \tilde{T} \) is a connected bipartite graph.

In [26], Kamalian proved the following result.

**Theorem 4.4** If \( T \) is a tree, then \( T \in \mathfrak{R}_c \) and \( W_c(T) = M(T). \)

Using this theorem we prove the following.

**Theorem 4.5** If \( T \) is a tree, then \( \text{def}_c(\tilde{T}) \geq |F(T)| - M(T) - 2. \)

**Proof.** Let \( F(T) = \{v_1, \ldots, v_p\} \) and \( \alpha \) be a proper \( t \)-edge coloring of \( \tilde{T} \) with the minimum cyclic deficiency, that is \( \text{def}_c(\tilde{T}, \alpha) = \text{def}_c(\tilde{T}). \) Clearly, \( t \geq |F(T)|. \)

Define an auxiliary graph \( \tilde{T'} \) as follows: for each vertex \( v \in V(\tilde{T}) \) with \( \text{def}_c(v, \alpha) > 0, \) we attach \( \text{def}_c(v, \alpha) \) pendant edges at vertex \( v. \) Clearly, \( |V(\tilde{T'})| = |V(\tilde{T})| + \text{def}_c(\tilde{T}). \) Next, for
a graph $\tilde{T}'$ and $F(T) = \{v_1, \ldots, v_p\}$, define the graph $T^+$ as follows: we remove the vertex $u$ from $\tilde{T}'$ and all isolated vertices from $\tilde{T}' - u$ (if such vertices exist) and add new vertices $u_1, \ldots, u_p$, then we add new edges $u_1v_1, \ldots, u_pv_p$. Clearly, $T^+$ is a tree with edge set $E(\tilde{T}'-u) \cup \{u_1v_1, \ldots, u_pv_p\}$. Moreover, it is easy to see that $M(T^+) \leq M(T) + 2 + \text{def}_c(\tilde{T})$.

We now extend a proper $t$-edge coloring $\alpha$ of $\tilde{T}$ to a proper $t$-edge coloring $\beta$ of $T^+$ as follows: for each vertex $v \in V(\tilde{T}' - u)$ with $\text{def}_c(v, \alpha) > 0$, we color the attached edges incident to $v$ using $\text{def}_c(v, \alpha)$ distinct colors to obtain a cyclic interval modulo $t$, and for each $1 \leq i \leq p$, we color the edge $u_iv_i$ with color $\alpha_{uv_i}$. By the definition of $\beta$ and the construction of $T^+$, we obtain that $T^+$ has a cyclic interval $t$-coloring. Since $W_c(T^+) = M(T^+)$ (by Theorem 4.4), we have

$$|F(T)| \leq t \leq W_c(T^+) = M(T^+) \leq M(T) + 2 + \text{def}_c(\tilde{T}).$$

Hence

$$\text{def}_c(\tilde{T}) \geq |F(T)| - M(T) - 2.$$

\[ \square \]

We note the following corollaries.

**Corollary 4.6** If $T$ is a tree, then $\text{def}(\tilde{T}) \geq |F(T)| - M(T) - 2$.

**Corollary 4.7** If $T$ is a tree in which the distance between any two pendant vertices is even, then the graph $\tilde{T}$ is bipartite, and $\text{def}_c(\tilde{T}) \geq |F(T)| - M(T) - 2$.

Our constructions by trees generalize the so-called Hertz’s graphs $H_{p,q}$ first described in [18]. Hertz’s graphs are known to have a high deficiency so let us specifically consider the cyclic deficiency of such graphs.

In [18] the Hertz’s graph $H_{p,q}$ ($p, q \geq 2$) was defined as follows:

$$V(H_{p,q}) = \{a, b_1, b_2, \ldots, b_p, d\} \cup \{c_j^{(i)} : 1 \leq i \leq p, 1 \leq j \leq q\}$$

and

$$E(H_{p,q}) = E_1 \cup E_2 \cup E_3,$$

where

$$E_1 = \{ab_i : 1 \leq i \leq p\}, \quad E_2 = \{b_ic_j^{(i)} : 1 \leq i \leq p, 1 \leq j \leq q\},$$

$$E_3 = \{c_j^{(i)}d : 1 \leq i \leq p, 1 \leq j \leq q\}.$$

The graph $H_{p,q}$ is bipartite with maximum degree $\Delta(H_{p,q}) = pq$ and $|V(H_{p,q})| = pq + p + 2$. For Hertz’s graphs, Giaro, Kubale and Malafiejski proved the following theorem.

**Theorem 4.8** [18] For any positive integers $p \geq 4$, $q \geq 3$,

$$\text{def}(H_{p,q}) = pq - p - 2q - 2.$$
Note that this result was recently generalized by Borowiecka-Olszewska et al. [9]. Using Theorems 4.5 and 4.8 we show that the following result holds.

**Theorem 4.9** For any positive integers \( p \geq 4, q \geq 3 \), we have

\[
def_c(H_{p,q}) = \text{def}(H_{p,q}) = pq - p - 2q - 2.
\]

**Proof.** Let us consider the tree \( T = H_{p,q} - d \). Since \( M(T) = p + 2q \), \( |F(T)| = pq \) and taking into account that the graph \( H_{p,q} \) is isomorphic to \( \tilde{T} \), by Theorem 4.5, we obtain that \( \text{def}_c(H_{p,q}) \geq pq - p - 2q - 2 \). On the other hand, by Theorem 4.8, we have \( \text{def}_c(H_{p,q}) \leq \text{def}(H_{p,q}) = pq - p - 2q - 2 \). Hence, \( \text{def}_c(H_{p,q}) = \text{def}(H_{p,q}) = pq - p - 2q - 2 \) for any \( p \geq 4, q \geq 3 \). \( \square \)

![Figure 9. The tree used for constructing the smallest example of a bipartite graph with no cyclic interval coloring.](image)

Finally, let us remark that the above technique can be used for constructing the smallest, in terms of maximum degree, currently known example of a bipartite graph with no cyclic interval coloring (cf. [2]). To this end, consider the tree \( T \) shown in Figure 9. Since \( M(T) = 11 \) and \( |F(T)| = 14 \), the bipartite graph \( \tilde{T} \) with \( |V(\tilde{T})| = 21 \) and \( \Delta(\tilde{T}) = 14 \) has no cyclic interval coloring.

### 4.3. Constructions using finite projective planes

In the last part of this section we use finite projective planes (see Definition 2) for constructing bipartite graphs with large cyclic deficiency. This family of graphs was first described in [31].

Let \( \pi(n) \) be a finite projective plane of order \( n \geq 2 \), \( \{1, 2, \ldots, n^2 + n + 1\} \) be the set of points and \( L = \{l_1, l_2, \ldots, l_{n^2+n+1}\} \) the set of lines of \( \pi(n) \). Let \( A_i = \{k \in l_i : 1 \leq k \leq n^2 + n + 1\} \) for every \( 1 \leq i \leq n^2 + n + 1 \); then \( |A_i| = n + 1 \) for every \( i \), and \( A_i \neq A_j \) if \( i \neq j \). For a sequence of \( n^2 + n + 1 \) integers \( r_1, r_2, \ldots, r_{n^2+n+1} \in \mathbb{N} \), we define the graph \( \text{Erd}(r_1, \ldots, r_{n^2+n+1}) \) as follows:
Cyclic Deficiency of Graphs

\[ V(\text{Erd}(r_1, \ldots, r_{n^2+n+1})) = \{u\} \cup \{1, \ldots, n^2 + n + 1\} \]
\[ \cup \left\{v^{(l_i)}_1, \ldots, v^{(l_i)}_{r_i} : 1 \leq i \leq n^2 + n + 1\right\}, \]
\[ E(\text{Erd}(r_1, \ldots, r_{n^2+n+1})) = \bigcup_{i=1}^{n^2+n+1} \left( \left\{uv^{(l_i)}_1, \ldots, uv^{(l_i)}_{r_i}\right\} \cup \left\{v^{(l_i)}_1 k, \ldots, v^{(l_i)}_{r_i} k : k \in A_i\right\} \right). \]

Clearly, \(\text{Erd}(r_1, r_2, \ldots, r_{n^2+n+1})\) is a connected bipartite graph where the number of vertices is \(n^2 + n + 2 + \sum r_i\) and the maximum degree is \(\sum r_i\).

Note that the above graph with parameters \(n = 3\) and \(r_1 = r_2 = \cdots = r_{13} = 1\) (see Figure 10) was described in 1991 by Erdős [22].

![Figure 10. Erdős’ graph.](image)

**Theorem 4.10** For any sequence \(r_1, r_2, \ldots, r_{n^2+n+1} \in \mathbb{N}\) with \(r_1 \geq r_2 \geq \cdots \geq r_{n^2+n+1}\), we have
\[
\text{def}_c(\text{Erd}(r_1, r_2, \ldots, r_{n^2+n+1})) \geq \frac{1}{10} \left( \sum_{i=n+2}^{n^2+n+1} r_i - 9 \sum_{i=1}^{n+1} r_i - 1 \right).
\]

**Proof.** Let \(\alpha\) be a proper \(t\)-edge coloring of \(\text{Erd}(r_1, \ldots, r_{n^2+n+1})\) with the minimum cyclic deficiency, that is,
\[
\text{def}_c(\text{Erd}(r_1, r_2, \ldots, r_{n^2+n+1}), \alpha) = \text{def}_c(\text{Erd}(r_1, r_2, \ldots, r_{n^2+n+1})).
\]
Clearly, \( t \geq \sum_{i=1}^{n^2+n+1} r_i \), because the maximum degree of \( \text{Erd}(r_1, \ldots, r_{n^2+n+1}) \) is \( \sum_{i=1}^{n^2+n+1} r_i \).

Define an auxiliary graph \( \text{Erd}(\overline{\tau}) \) as follows: for each vertex \( v \in V(\text{Erd}(r_1, \ldots, r_{n^2+n+1})) \) with \( \text{def}_c(v, \alpha) > 0 \), we attach \( \text{def}_c(v, \alpha) \) pendant edges at vertex \( v \). Clearly, \( |V(\text{Erd}(\overline{\tau}))| = |V(\text{Erd}(r_1, \ldots, r_{n^2+n+1}))| + \text{def}_c(\text{Erd}(r_1, \ldots, r_{n^2+n+1})) \). Next, from \( \text{Erd}(\overline{\tau}) \) we define the graph \( \text{Erd}^+(\overline{\tau}) \) as follows: we remove the vertex \( u \) from \( \text{Erd}(\overline{\tau}) \) and all isolated vertices from \( \text{Erd}(\overline{\tau}) - u \) (if such vertices exist) and add new vertices \( w_1^{(l_1)}, \ldots, w_i^{(l_i)} \) for \( 1 \leq i \leq n^2+n+1 \); then we add new edges

\[
\begin{align*}
v_1^{(l_1)} w_1^{(l_1)}, & \ldots, v_i^{(l_i)} w_i^{(l_i)}, \quad 1 \leq i \leq n^2+n+1.
\end{align*}
\]

The graph \( \text{Erd}^+(\overline{\tau}) \) is connected bipartite graph and has diameter at most 5.

We now extend the proper \( t \)-edge coloring \( \alpha \) of \( \text{Erd}(r_1, \ldots, r_{n^2+n+1}) \) to a proper \( t \)-edge coloring \( \beta \) of \( \text{Erd}^+(\overline{\tau}) \) as follows: for each vertex \( v \in V(\text{Erd}(\overline{\tau}) - u) \) with \( \text{def}_c(v, \alpha) > 0 \), we color the attached edges incident to \( v \) using \( \text{def}_c(v, \alpha) \) distinct colors to obtain a cyclic interval modulo \( t \) at \( v \), and we color the edges \( v_1^{(l_1)} w_1^{(l_1)}, \ldots, v_i^{(l_i)} w_i^{(l_i)} \) with colors \( \alpha(v_1^{(l_1)} w_1^{(l_1)}), \ldots, \alpha(v_i^{(l_i)} w_i^{(l_i)}) \) (\( 1 \leq i \leq n^2+n+1 \)), respectively. By the definition of \( \beta \) and the construction of \( \text{Erd}^+(\overline{\tau}) \), we obtain a cyclic interval \( t \)-coloring of \( \text{Erd}^+(\overline{\tau}) \). Now, by Theorem 4.2, we have

\[
\sum_{i=1}^{n^2+n+1} r_i \leq t \leq 1 + 2 \text{diam}(\text{Erd}^+(\overline{\tau})) \left( \Delta(\text{Erd}^+(\overline{\tau})) - 1 \right)
\]

\[
\leq 1 + 10 \left( \Delta(\text{Erd}(r_1, \ldots, r_{n^2+n+1}) - u) + \text{def}_c(\text{Erd}(r_1, \ldots, r_{n^2+n+1})) \right)
\]

\[
= 1 + 10 \sum_{i=1}^{n+1} r_i + 10 \text{def}_c(\text{Erd}(r_1, \ldots, r_{n^2+n+1})).
\]

Hence

\[
\text{def}_c(\text{Erd}(r_1, \ldots, r_{n^2+n+1})) \geq \frac{1}{10} \left( \frac{n^2+n+1}{n+2} - 9 \sum_{i=1}^{n+1} r_i - 1 \right). \quad \square
\]

Using Theorem 4.10 we can generate infinite families of graphs with large cyclic deficiency. For example, if \( r_1 = r_2 = \cdots = r_{n^2+n+1} = k \), where \( k \) is some constant, then

\[
\text{def}_c(\text{Erd}(r_1, \ldots, r_{n^2+n+1})) \geq \frac{1}{10} (n^2k - 9k(n+1) - 1).
\]

5. Upper bounds on \( \text{def}_c(G) \)

In this section we present some upper bounds on the cyclic deficiency of graphs. Theorem 2.2 implies that if every vertex of a graph \( G \) has degree 1 or \( \Delta(G) \), then any proper edge coloring of \( G \) with \( \chi'(G) \) colors is a cyclic interval coloring of \( G \). This implies that
the graph obtained from \( G \) by attaching \( \Delta(G) - d_G(v) \) pendant edges at each vertex \( v \) with \( 1 < d_G(v) < \Delta(G) \) is cyclically interval colorable. Therefore, for any graph \( G \)

\[
\text{def}_c(G) \leq \sum_{v \in V(G), \ 2 \leq d_G(v) \leq \Delta(G) - 1} (\Delta(G) - d_G(v)).
\]

In \([2]\) we proved that all bipartite graphs with maximum degree 4 admit cyclic interval colorings. For bipartite graphs with maximum degree \( \Delta(G) \geq 5 \) the above upper bound on the cyclic deficiency can be slightly improved.

**Proposition 5.1** If \( G \) is a bipartite graph with maximum degree \( \Delta(G) \geq 5 \), then

\[
\text{def}_c(G) \leq \left\{ \begin{array}{ll}
\sum_{v \in V(G), \ 3 \leq d_G(v) \leq \Delta(G) - 3} (\Delta(G) - 2 - d_G(v)), & \text{if } \Delta(G) \text{ is even}, \\
\sum_{v \in V(G), \ 3 \leq d_G(v) \leq \Delta(G) - 2} (\Delta(G) - 1 - d_G(v)), & \text{if } \Delta(G) \text{ is odd}.
\end{array} \right.
\]

**Proof.** In the proof we follow the idea from the proof of Theorem 1 in \([2]\).

First we consider the case when \( \Delta(G) \) is even. In this case we construct a new multigraph \( G^* \) as follows: first we take two isomorphic copies \( G_1 \) and \( G_2 \) of the graph \( G \) and join by an edge every vertex with an odd vertex degree in \( G \) with its copy in \( G_2 \); then for each vertex \( u \in V(G_1) \cup V(G_2) \) of degree \( 2k \), we add \( \Delta(G) - k \) loops at \( u \) \( (1 \leq k < \Delta(G)/2) \).

Clearly, \( G^* \) is a \( \Delta(G) \)-regular multigraph. By Petersen’s theorem, \( G^* \) can be represented as a union of edge-disjoint 2-factors \( F_1, \ldots, F_{\Delta(G)/2} \). By removing all loops from 2-factors \( F_1, \ldots, F_{\Delta(G)/2} \) of \( G^* \), we obtain that the resulting graph \( G' \) is a union of edge-disjoint even subgraphs \( F'_1, \ldots, F'_{\Delta(G)/2} \). Since \( G' \) is bipartite, for each \( i \) \((1 \leq i \leq \Delta(G)/2)\), \( F'_i \) is a collection of even cycles in \( G' \), and we can color the edges of \( F'_i \) alternately with colors 2\( i \) – 1 and 2\( i \). The resulting coloring \( \alpha \) is a proper edge coloring of \( G' \) with colors 1, ..., \( \Delta(G) \).

Since for each vertex \( u \in V(G') \) with \( d_{G'}(u) = 2k \) \((1 \leq k \leq \Delta(G)/2)\), there are \( k \) even subgraphs \( F'_{i_1}, F'_{i_2}, \ldots, F'_{i_k} \) such that \( d_{F'_{i_1}}(u) = d_{F'_{i_2}}(u) = \cdots = d_{F'_{i_k}}(u) = 2 \), we obtain that \( S_{G'}(u, \alpha) = \{2i_1 - 1, 2i_1, 2i_2 - 1, 2i_2, \ldots, 2i_k - 1, 2i_k\} \).

Consider the restriction of \( \alpha \) to the graph \( G_1 \), and let \( \varphi \) be the corresponding edge coloring of \( G \). For any vertex \( v \) of \( G \) with \( d_G(v) \in \{2, \Delta(G) - 2, \Delta(G) - 1\} \), \( S_G(v, \varphi) \) is a cyclic interval modulo \( \Delta(G) \). Moreover, by the construction of \( \varphi \), if a vertex \( v \) has even degree, then the cyclic deficiency of \( S_G(v, \varphi) \) is at most \( \Delta(G) - 2 - d_G(v) \), because color \( 2r - 1 \) is in \( S_G(v, \varphi) \) if and only if color 2\( r \) is in \( S_G(v, \varphi) \); if \( v \) has odd degree, then there is exactly one integer \( r \) with the property that precisely one of the colors 2\( r \) – 1 and 2\( r \) is in \( S_G(v, \varphi) \), and a similar argument as in the even case applies.

We thus deduce that

\[
\text{def}_c(G, \varphi) \leq \sum_{v \in V(G), \ 3 \leq d_G(v) \leq \Delta(G) - 3} (\Delta(G) - 2 - d_G(v)).
\]

Next we consider the case when \( \Delta(G) \) is odd. In this case we construct a new graph \( G^{**} \) as follows: we take two isomorphic copies of the graph \( G \) and join by an edge every
vertex of maximum degree with its copy. Clearly, $G^{**}$ is a bipartite graph with $\Delta(G^{**}) = \Delta(G) + 1$. Moreover, since $G^{**}$ has even maximum degree, the desired result now follows by constructing an edge coloring $\varphi$ of $G^{**}$ as above, and taking the restriction of this coloring to one of the copies of $G$. □

In the following, for a graph $G$, we denote by $V_k(G)$ (or just $V_k$) the set of vertices of degree $k$ in $G$.

**Corollary 5.2** If $G$ is a bipartite graph with $\Delta(G) \leq 6$, then $\text{def}_c(G) \leq |V_3|$.

**Corollary 5.3** If $G$ is a bipartite graph where, for some $r \geq 2$, all vertex degrees are in the set $\{2r-3, 2r-2, 2r-1, 2r\}$, then $\text{def}_c(G) \leq |V_{2r-3}|$.

In the preceding section we proved that there are families of bipartite graphs $G_n$ such that $\lim_{n \to \infty} \frac{|\text{def}_c(G_n)|}{|V(G)|} = 1$. On the other hand, we do not know graphs $G$ with $\text{def}_c(G) > |V(G)|$. Hence, the following conjecture seems natural.

**Conjecture 5.4** For any graph $G$, $\text{def}_c(G) \leq |V(G)|$.

Note that the above results imply that this conjecture holds for bipartite graphs with maximum degree at most 6. Moreover, in [2] we proved that every bipartite graph $G$ where all vertex degrees are in the set $\{1, 2, 4, 6, 7, 8\}$ has a cyclic interval coloring. This result implies that for every bipartite graph $G$ with $\Delta(G) \leq 8$, we have $\text{def}_c(G) \leq |V_3| + |V_5|$. To see this, let $G$ be such a graph and take two copies $G_1$ and $G_2$ of $G$ and join by an edge every vertex of degree 3 or 5 in $G_1$ with its copy in $G_2$. By the result in [2], the obtained graph has a cyclic interval coloring and by taking the restriction of this coloring to e.g. $G_1$, we deduce that $\text{def}_c(G) \leq |V_3| + |V_5|$. Thus Conjecture 5.4 holds for any bipartite graph with maximum degree at most 8.

Next, we consider biregular bipartite graphs; such graphs have been conjectured to always admit cyclic interval colorings [11]. Corollary 5.3 implies that $(2r-3, 2r)$-biregular and $(2r-3, 2r-1)$-biregular graphs satisfy Conjecture 5.4. We shall use the following proposition for establishing that Conjecture 5.4 holds for $(2r - 4, 2r)$-biregular graphs. Since any bipartite graph with maximum degree 8 satisfies Conjecture 5.4, we assume that $r \geq 5$.

**Proposition 5.5** If $G$ is a bipartite graph where, for some $r \geq 5$, all vertex degrees are in the set $\{2r-4, 2r-3, 2r-2, 2r-1, 2r\}$ and

$$\frac{r-5}{r-1}|V_{2r-4}| \leq |V_{2r-2}| + |V_{2r-1}| + |V_{2r}|,$$

then $\text{def}_c(G) \leq |V(G)|$.

**Proof.** Let $G'$ be a multigraph obtained by adding two loops at a vertex of degree $2r-4$ in $G$ and one loop at any vertex of degree $2r-3$ or $2r-2$ in $G$. Let $H$ be the multigraph obtained from two copies $G_1$ and $G_2$ of $G'$ where any vertex of odd degree in $G_1$ is joined by an edge to its corresponding vertex in $G_2$. Clearly, $H$ is $2r$-regular.
Proceeding as in the proof of Proposition 5.1, let $\alpha$ be the cyclic interval $2r$-coloring of $H$ obtained by decomposing $H$ into 2-factors and properly edge coloring each factor of $H$ by $2$ consecutive colors.

Denote by $\varphi$ the edge coloring of $G$ corresponding to the restriction of $\alpha$ to $G_1$. Then for any vertex $v$ of $G$, $S_G(v, \varphi)$ is a cyclic interval modulo $2r$ if $v$ has degree $2r - 2$, $2r - 1$ or $2r$ in $G$. Moreover, if $v$ has degree $2r - 3$, then $S_G(v, \varphi)$ has cyclic deficiency at most 1, and if $v$ has degree $2r - 4$, then $S_G(v, \varphi)$ has cyclic deficiency at most 2.

Consider a vertex $v$ of degree $2r - 4$ in $G$. If for some $i \in \{1, \ldots, 2r\}$, the colors in $\{i, i + 1, i + 2, i + 3\}$ (where numbers are taken modulo $2r$) do not appear in $S_G(v, \varphi)$, then $S_G(v, \varphi)$ is a cyclic interval modulo $2r$. Thus, by permuting colors and using an averaging argument, we may assume that at least $\frac{r}{(2)} |V_{2r - 4}|$ vertices $v$ in $V_{2r - 4}$ satisfy that $S_G(v, \varphi)$ is a cyclic interval modulo $2r$. Indeed, if we choose a permutation of the numbers $1, \ldots, r$ uniformly at random, where each number $i$ corresponds to a pair of colors $2i - 1, 2i$, and apply this permutation to the colors used by $\varphi$, then the probability that a given vertex of degree $2r - 4$ gets a cyclic interval of colors on its incident edges is $\frac{r}{(2)}$, because all color sets are equally likely. Let $X$ be a random variable counting the number of vertices of degree $2r - 4$ that do get cyclic intervals of colors on their incident edges; then, by linearity of expectation, we have that $E(X) = \frac{r}{(2)} |V_{2r - 4}|$. Hence, the probability that there is some permutation such that $X \geq \frac{r}{(2)} |V_{2r - 4}|$ is greater than zero; thus, we may assume that at least $\frac{r}{(2)} |V_{2r - 4}|$ vertices $v$ in $V_{2r - 4}$ satisfy that $S_G(v, \varphi)$ is a cyclic interval modulo $2r$.

Hence, if

$$|V_{2r - 4}| \left(1 - \frac{2}{r - 1}\right) \leq \frac{2}{r - 1} |V_{2r - 4}| + |V_{2r - 2}| + |V_{2r - 1}| + |V_{2r}|,$$

then the number of vertices with cyclic deficiency zero is at least the number of vertices with cyclic deficiency two, which implies that $\text{def}_c(G) \leq |V(G)|$. Now, (1) holds by assumption, so the required result follows. $\square$

**Corollary 5.6** If $G$ is a $(2r - 4, 2r)$-biregular graph, then $\text{def}_c(G) \leq |V(G)|$.

**Proof.** The remark before Proposition 5.5 implies that it suffices to consider the case when $r \geq 5$. Let $X$ and $Y$ be the parts of $G$, where vertices in $X$ have degree $2r - 4$. Since $G$ is $(2r - 4, 2r)$-biregular, we have $(2r - 4)|X| = 2r|Y|$, which implies that

$$|Y| = \frac{r - 2}{r} |X| \geq \frac{r - 5}{r - 1} |X|,$$

for every $r \geq 5$, and so the result follows from Proposition 5.5. $\square$

In the following we shall present some further results supporting Conjecture 5.4. We begin by showing that any graph with maximum degree at most 5 satisfies this conjecture. We shall use the following observation, the proof is just straightforward case analysis and is left to the reader.
Observation 5.7 If $\varphi$ is a proper 6-edge coloring of a graph $G$ with maximum degree 5, then for every vertex $v$ of $G$, $S_G(v, \varphi)$ is near-cyclic modulo 6 unless

$$S_G(v, \varphi) \in \{\{1, 4\}, \{2, 5\}, \{3, 6\}, \{1, 3, 5\}, \{2, 4, 6\}\}.$$ 

Theorem 5.8 If $G$ is a graph with maximum degree at most 5, then $\text{def}_5(G) \leq |V(G)|$.

Proof. If $\Delta(G) \leq 4$, consider a proper 5-edge coloring $\alpha$ of $G$; such a coloring exists by Theorem 2.2. For a vertex $v$ of degree 1 or 4, clearly $S_G(v, \alpha)$ is a cyclic interval modulo 5. Moreover, straightforward case analysis shows that for any vertex $v$ of degree 2 or 3 in $G$, $S_G(v, \alpha)$ is near-cyclic. Hence, $\text{def}_5(G) \leq |V(G)|$ for any graph $G$ of maximum degree at most 4.

Let us now consider a graph $G$ with maximum degree 5. Let $M$ be a maximum matching of $G[V_5]$. Then, by Theorem 2.3, $G - M$ is 5-edge colorable; let $\varphi$ be a proper 5-edge coloring of $G - M$. We define some subsets of $V(G)$:

- $A_{i,j,k}(\varphi)$ is the set of all vertices $v$ for which $S_G(v, \varphi) = \{i, j, k\}$, where $1 \leq i < j < k \leq 5$;
- $A_{i,j}(\varphi)$ is the set of all vertices $v$ for which $S_G(v, \varphi) = \{i, j\}$, where $1 \leq i < j \leq 5$.

If $|A_{1,3,5}(\varphi)| > |A_{2,3,4}(\varphi)|$, then by permuting the colors of $\varphi$ we can construct a coloring $\varphi'$ such that $|A_{1,3,5}(\varphi')| < |A_{2,3,4}(\varphi')|$. Thus we may assume that $|A_{1,3,5}(\varphi)| \leq |A_{2,3,4}(\varphi)|$.

Now consider the vertices in $A_{1,4}(\varphi)$ and $A_{2,5}(\varphi)$. If

$$|A_{1,4}(\varphi)| + |A_{2,5}(\varphi)| > |A_{1,2}(\varphi)| + |A_{4,5}(\varphi)|,$$

then by permuting colors 1 and 5 we obtain a coloring $\varphi'$ satisfying

$$|A_{1,4}(\varphi')| + |A_{2,5}(\varphi')| < |A_{1,2}(\varphi')| + |A_{4,5}(\varphi')|,$$

and, moreover, $|A_{1,3,5}(\varphi')| = |A_{1,3,5}(\varphi)|$ and $|A_{2,3,4}(\varphi')| = |A_{2,3,4}(\varphi)|$.

From the preceding paragraphs we thus conclude that there is a proper 5-edge coloring $\alpha$ of $G - M$ such that $|A_{1,3,5}(\alpha)| \leq |A_{2,3,4}(\alpha)|$ and

$$|A_{1,4}(\alpha)| + |A_{2,5}(\alpha)| \leq |A_{1,2}(\alpha)| + |A_{4,5}(\alpha)|.$$

We now define a proper edge coloring $\alpha'$ of $G$ by setting $\alpha'(e) = 6$ if $e \in M$ and $\alpha'(e) = \alpha(e)$ if $e \notin M$. By Observation 5.7, for every vertex $v$ of $G$, $S_G(v, \alpha')$ is near-cyclic modulo 6 unless

$$S_G(v, \alpha') \in \{\{1, 4\}, \{2, 5\}, \{3, 6\}, \{1, 3, 5\}, \{2, 4, 6\}\}.$$ 

Clearly, no vertex $v$ in $G$ satisfies $S_G(v, \alpha') \in \{\{3, 6\}, \{2, 4, 6\}\}$. Thus, a vertex $v$ in $G$ has cyclic deficiency 2 under $\alpha'$ if and only if

$$S_G(v, \alpha') \in \{\{1, 4\}, \{2, 5\}, \{1, 3, 5\}\}.$$ 

Now, since $|A_{1,3,5}(\alpha')| \leq |A_{2,3,4}(\alpha')|$, 

$$|A_{1,4}(\alpha')| + |A_{2,5}(\alpha')| \leq |A_{1,2}(\alpha')| + |A_{4,5}(\alpha')|,$$

and the cyclic deficiencies of the sets $\{2, 3, 4\}, \{1, 2\}$ and $\{4, 5\}$ are all equal to 0, it follows that $\text{def}_5(G) \leq |V(G)|$. $\square$
All regular graphs trivially have cyclic interval colorings, as do also all Class 1 graphs $G$ with $\Delta(G) - \delta(G) \leq 1$. Moreover, any Class 2 graph $G$ with $\Delta(G) - \delta(G) \leq 1$ satisfies Conjecture 5.4. The next proposition shows that a slightly stronger statement is true.

**Theorem 5.9** For any graph $G$ with $\Delta(G) - \delta(G) \leq 2$, $\text{def}_c(G) \leq |V(G)|$.

**Proof.** By the last remark, we may assume that $\Delta(G) - \delta(G) = 2$. Set $k = \Delta(G)$.

If $G$ is Class 1, then it clearly has a proper $k$-edge coloring $\alpha$ such that for any vertex $S_G(v, \alpha)$ is near-cyclic, implying that $\text{def}_c(G) \leq |V(G)|$. Assume, consequently, that $G$ is Class 2.

Let $M$ be a maximum matching of $G[V_k]$. Set $H = G - M$. Note that in $H$ no pair of vertices of degree $k$ are adjacent. Let $M'$ be a minimum matching in $H$ covering all vertices of degree $k$ in $H$; such a matching exists since $H$ is Class 1. Note that the graph $J = H - M'$ has maximum degree at most $k - 1$. Let $M''$ be a maximum matching in $J_{k-1}$, where $J_{k-1}$ is the subgraph of $J$ induced by the vertices of degree $k - 1$ in $J$. Let $\tilde{M} = M \cup M' \cup M''$.

Denote by $G_{\tilde{M}}$ the subgraph of $G$ induced by $\tilde{M}$; that is, the subgraph of $G$ consisting of all vertices which are endpoints of edges in $\tilde{M}$ and with edge set $\tilde{M}$.

**Claim 1** The graph $G_{\tilde{M}}$ is 2-edge colorable.

**Proof.** We first prove that $G_{\tilde{M}}$ has maximum degree 2. Since $M$ is a maximum matching in $G[V_k]$, if a vertex is incident with edges from both $M$ and $M'$, then it has degree $k - 2$ in $G - M \cup M'$, and is therefore not incident with any edge from $M''$. This implies that any vertex of $G$ is incident with at most two edges from $\tilde{M}$.

Now we prove that $G_{\tilde{M}}$ has no odd cycle. Since $H = G - M$ contains no pair of adjacent vertices of degree $k$, and every edge of $M'$ is incident with a vertex of degree $k$ in $H$, one of the ends of an edge in $M'$ has degree at most $k - 2$ in $J = H - M'$. Since $M''$ is a maximum matching in $J_{k-1}$, this means that no edge of $M'$ can be in a cycle of $G_{\tilde{M}}$. Thus, if $G_{\tilde{M}}$ contains a cycle $C$, then $E(C) \subseteq M \cup M''$. However, $M$ and $M''$ are both matchings, so edges in $C$ lie alternately in $M$ and $M''$. Therefore $G_{\tilde{M}}$ contains no odd cycle.

**Claim 2** The graph $G - \tilde{M}$ is $(k - 1)$-edge colorable.

**Proof.** As pointed out above, the graph $J = G - M \cup M'$ has maximum degree $k - 1$, so the graph $G - \tilde{M}$ has maximum degree at most $k - 1$. If $G - \tilde{M}$ has maximum degree $k - 2$, then it is clearly $(k - 1)$-edge colorable. Suppose instead that $G - M'$ has maximum degree $k - 1$. Since $M''$ is a maximum matching in $J_{k-1}$, no pair of vertices of degree $k - 1$ is adjacent in $G - \tilde{M}$. Hence, by Theorem 2.3, $G$ is Class 1.

We continue the proof of Proposition 5.9. Let $\psi$ be a proper $(k - 1)$-edge coloring of $G - \tilde{M}$ using colors $1, \ldots, k - 1$, and let $\varphi$ be a proper 2-edge coloring of $G_{\tilde{M}}$ using colors $k$ and $k + 1$. Denote by $\alpha$ the coloring of $G$ that $\psi$ and $\varphi$ yield. Denote by $U$ the set of vertices of degree $k - 2$ in $G$ that are adjacent to an edge of $M'$. If $v \in U$, then $\text{def}_c(v, \alpha) \leq 2$. If, on the other hand, $v \in V_{k-2} \setminus U$, then $d_{G - \tilde{M}}(v) = k - 2$. Since the
graph $G - \hat{M}$ is $(k - 1)$-edge colorable, we have that $\text{def}_c(v, \alpha) \leq 1$. Moreover, if $v \in V_{k-1}$, then $\text{def}_c(v, \alpha) \leq 1$, and if $v \in V_k$, then $\text{def}_c(v, \alpha) = 0$. Hence

$$\text{def}_c(G) \leq 2|U| + |V_{k-2} \setminus U| + |V_{k-1}|.$$ 

Since $|U| \leq |V_k|$, we deduce that

$$\text{def}_c(G) \leq |V_{k-2}| + |V_{k-1}| + |V_k| = |V(G)|.$$

$\blacksquare$

**Acknowledgement.** We would like to thank Hrant Khachatrian for drawing the main figures in the paper. We would also like to thank the referees for their careful reading and helpful suggestions.

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