Positively Curved Surfaces in the Three-sphere

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Abstract

In this talk I will discuss an example of the use of fully nonlinear parabolic flows to prove geometric results. I will emphasise the fact that there is a wide variety of geometric parabolic equations to choose from, and to get the best results it can be very important to choose the best flow. I will illustrate this in the setting of surfaces in a three-dimensional sphere.

There are quite a few relevant results for surfaces in the sphere satisfying various kinds of curvature equations, including totally umbillic surfaces, minimal surfaces and constant mean curvature surfaces, and intrinsically flat surfaces. Parabolic flows can strengthen such results by allowing classes of surfaces satisfying curvature inequalities rather than equalities: This was first done by Huisken, who used mean curvature flow to deform certain classes of surfaces to totally umbillic surfaces. This motivates the question “What is the optimal result of this kind?” — that is, what is the weakest pointwise curvature condition which defines a class of surfaces which retracts to the space of great spheres?

The answer to this question can be guessed in view of the examples. To prove it requires a surprising choice of evolution equation, forced by the requirement that the pointwise curvature condition be preserved.

I will conclude by mentioning some other geometric situations in which strong results can be proved by choosing the best possible evolution equation.

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1. Introduction

My aim in this talk is to demonstrate the use of fully nonlinear parabolic evolution equations as tools for proving results in differential geometry. I will emphasise the fact that there is a wide variety of flows which are geometrically defined and

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potentially applicable to geometric problems, and that there is great benefit to be had by choosing the flow carefully. I will focus on a particular application, relating to surfaces in the 3-sphere, but the method has much wider applicability.

There are some well-known examples of geometric evolution equations of the kind I want to consider: Eells and Sampson \[8\] used a heat flow to prove existence of harmonic maps into non-positively curved targets; Hamilton considered the flow of Riemannian metrics in the direction of their Ricci tensor, and proved that it deforms metrics of positive Ricci curvature on three-manifolds \[12\] and metrics of positive curvature operator on four-manifolds \[13\] to constant curvature metrics. The Ricci flow also gives results in higher dimensions, proved by Huisken \[14\], Nishikawa \[24\] and Margerin \[19\]–\[21\], if the curvature tensor is suitably pinched.

The mean curvature flow of submanifolds of Euclidean space is also well-known as the gradient descent flow of the area functional, and because it arises in models of interfaces such as in annealing metals. The examples I will concentrate on are closest to the last example, as they are evolution equations describing submanifolds moving with curvature-dependent velocity. There are many parabolic flows of this kind, particularly for the codimension one (hypersurface) case: William Firey \[11\] introduced the motion by Gauss curvature as a model for pebbles wearing away as they tumble, and other flows which have been considered include motion by powers of Gauss curvature \[28\], \[6\], the square root of the scalar curvature \[7\], the harmonic mean of the principal curvatures \[2\]–\[3\], and the reciprocal of the mean curvature \[17\]. More generally, one can take the velocity to be a function of the principal curvatures which is monotone increasing in each argument.

This gives a huge variety of flows to choose from, so it makes sense to choose the flow carefully to suit the problem. I will illustrate a strategy for choosing the flow by asking that some desired curvature inequality be preserved under the flow.

I will begin, in the next two sections, by discussing some old results concerning surfaces in the three-sphere. This motivates the results of the later sections.

2. Constant mean curvature surfaces

There is a well-known result of Simons \[27\] which says that a minimal hypersurface in a $S^{n+1}$ with the squared norm of the second fundamental form $|A|^2$ less than $n$ is in fact totally geodesic (hence a great $n$-sphere). This result comes from an application of Simons’ identity which relates the second derivatives of mean curvature to the Laplacian of the second fundamental form:

$$\nabla_i \nabla_j H = \Delta h_{ij} + |A|^2 h_{ij} - H h_i^p h_{pj} + H g_{ij} - nh_{ij}.$$  

From this we can deduce if the hypersurface is minimal (so $H = 0$)

$$0 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^2(|A|^2 - n).$$

If $|A|^2 < n$ at a maximum, then the maximum principle implies $|A|^2$ is identically zero, and the result follows. Also, if the maximum of $|A|^2$ is equal to $n$, then $M$ must be a product $S^k(a) \times S^{n-k}(b)$ in $R^{k+1} \times R^{n+1-k}$, with radii $a$ and $b$ determined by the fact that $M$ lies in $S^{n+1} \subset R^{n+2}$ and is minimal.
Simons’ argument was taken up by other authors ([25], [5], [1]) in the slightly more general setting of constant mean curvature hypersurfaces. The results are similar: If the hypersurface has constant mean curvature $H$, and $|A|$ is bounded by a constant depending on $n$ and $H$, then the hypersurface is totally umbilic, hence a geodesic sphere in $S^{n+1}$; if the inequality is not strict then the only extra possibilities are products of spheres. The argument is similar to that above, but complicated by the non-vanishing of the mean curvature.

Let me look closer at the situation for surfaces in the three-sphere: The intrinsic curvature of the surface is given by $1 + \kappa_1 \kappa_2 = 1 + \frac{1}{2} H^2 - \frac{1}{2} |A|^2$. If $M$ is minimal, then $H = 0$, so $|A|^2 < 2$ is equivalent to positivity of the intrinsic curvature. This is also true for constant mean curvature surfaces: In two dimensions, the curvature condition from [25] and [5] is equivalent to positivity of the intrinsic curvature.

3. Flat tori

The condition of positive intrinsic curvature seems natural in view of the results on constant mean curvature surfaces. For surfaces in space, positive curvature is a rather restrictive condition — a compact surface satisfying this condition is the boundary of a convex region. In the 3-sphere it seems somewhat less restrictive, as we can see by considering the ‘boundary’ case of flat surfaces, where there are the beautiful results of Weiner [32] and Enomoto [9] which classify flat tori in the 3-sphere by their Gauss maps. It was known for some time that there are many examples of these (see [26]), since the inverse image of any smooth curve in $S^2$ under the Hopf projection is a flat torus in $S^3$. These examples are all invariant under the action of $U(1)$ on $C^2 \simeq \mathbb{R}^4$, but Weiner and Enomoto showed that there are many examples which are not symmetric.

The Gauss map of a surface in $S^3$ can be thought of in several ways: One can consider the tangent plane of the surface as a subspace of $R^4$, which gives a map from the surface to the Grassmannian $G_{2,4}$ of 2-planes in $R^4$. The latter is a metric product $S^2 \times S^2$, and the projections onto each factor are called the self-dual and anti-self-dual Gauss maps. Alternatively, since $S^3$ is a group, one can map the unit normal of the surface by either left or right translations to the Lie algebra — this again gives two maps to $S^2$, and of course these are the same as before: The self-dual Gauss map is the same as the left-translation Gauss map, and the anti-self-dual Gauss map is the same as the right-translation Gauss map.

Enomoto [9] observed that if $M^2$ is intrinsically flat in $S^3$, then both Gauss maps are degenerate (their images are just curves in $S^2$). Weiner gave the complete classification result: The image curves $\gamma_1$ and $\gamma_2$ necessarily have zero total curvature, and if $I_1$ and $I_2$ are subintervals of $\gamma_1$ and $\gamma_2$ respectively, then $|\int_{I_1} \kappa \, ds| + |\int_{I_2} \kappa \, ds| < \pi$. Conversely, if $\gamma_1$ and $\gamma_2$ are any curves satisfying these conditions, then there is a flat torus with these curves as the images of the two Gauss maps, and the torus is unique up to motion by unit speed in the normal direction.

This gives a very large family of flat tori in the 3-sphere, and from these we see that surfaces with positive intrinsic curvature in $S^3$ can look quite complicated: The
Surface can look metrically like a long thin cylinder with caps on the ends, placed in $S^3$ by ‘winding around’ a flat torus many times before closing off the ends.

4. Curvature flow

Curvature flow can give powerful generalisations of results like those from [27], [25] and [5]: Huisken [16] extended techniques developed earlier for convex hypersurfaces in Euclidean space [14] to prove the following result:

**Theorem:** Let $M^i_0 = x_0(M)$ be a hypersurface in $S^{n+1}$ which satisfies

$$|A|^2 < \frac{1}{n-1} H^2 + 2$$

if $n > 2$, and

$$|A|^2 < \frac{3}{4} H^2 + \frac{4}{3}$$

if $n = 2$. Then there exists a smooth family of hypersurfaces $\{M_t = x_t(M)\}_{0 \leq t < T}$ which satisfy the same curvature condition and move by mean curvature flow with initial data $M_0$. Either $T < \infty$ and $M_t$ is asymptotic to a family of geodesic spheres shrinking to their common centre, or $T = \infty$ and $M_t$ approaches a great sphere.

This includes the result that there are no minimal surfaces with $|A|^2 < n$ except great spheres. It also implies the stronger statement that every hypersurface satisfying $|A|^2 < \frac{1}{n-1} H^2 + 2$ can be deformed, keeping this condition, to a great sphere (except in the case $n = 2$). The condition $|A|^2 < \frac{1}{n-1} H^2 + 2$ is the same as that arrived at by Okumura [25] for constant mean curvature surfaces (Cheng and Nakagawa [5] improved this for higher dimensions, but in two dimensions it is sharp). The proof of the above result is significantly more difficult than that for the constant mean curvature case.

The result seems very satisfying, except when $n = 2$ where the method does not seem to work for Okumura’s condition $|A|^2 < H^2 + 2$. The latter is exactly the condition of positive intrinsic curvature. This raises several questions: Does mean curvature flow in fact preserve this condition? If not, is there any flow which does?

5. The optimal result

5.1. Choosing the evolution equation

Now we can illustrate the method: The previous questions can be answered in a rather systematic way. The idea is to write down the conditions required for an arbitrary flow by a function $F$ of curvature to preserve positive intrinsic curvature.

We can write down an evolution equation for an arbitrary function $G$ of the principal curvatures $\kappa_1$ and $\kappa_2$, and see what conditions are required for the flow to preserve the condition $G \geq 0$. For convenience we can write $G$ in the form

$$G(\kappa_1, \kappa_2) = (\kappa_1 - \kappa_2)^2 - \varphi(\kappa_1 + \kappa_2)^2$$

(5.1)
so that in the case we are interested in, \( \varphi(x) = \sqrt{4 + x^2} \). We can also write

\[
F = f(\kappa_1 + \kappa_2, G).
\]

(5.2)

Then the evolution equation for \( G \) is as follows:

\[
\frac{\partial G}{\partial t} = \hat{F}^{ij} \nabla_i \nabla_j G + Q(h, \nabla h) + Z(h),
\]

(5.3)

where \( \hat{F} \) is the matrix of derivatives of \( F \) with respect to the components of the second fundamental form, which is positive definite as long as \( F \) is an increasing function of each of the principal curvatures. The second term is a quadratic function of the components of the derivative of the second fundamental form, with coefficients depending on curvature \( h \), explicitly given by

\[
Q = \left( \hat{G}^{ij} \bar{F}^{kl, mn} - \bar{F}^{ij} \hat{G}^{kl, mn} \right) \nabla_i h_{kl} \nabla_j h_{mn},
\]

where \( \bar{F} \) is the second derivative of \( F \) with respect to the components of \( h \). The last term \( Z \) depends on the curvature alone, and has the form

\[
Z = \hat{G}^{ij} \left( F(h^2_{ij} + g_{ij}) + \hat{F}^{kl} (h_{ij} h^2_{kl} - h_{kl} h^2_{ij} + g_{ij} h_{kl} - g_{kl} h_{ij}) \right)
\]

\[
= F \left( \hat{G}^1(1 + \kappa_1^2) + \hat{G}^2(1 + \kappa_2^2) \right) + (1 + \kappa_1 \kappa_2)(\kappa_2 - \kappa_1)(\hat{G}^1 \hat{F}^2 - \hat{F}^1 \hat{G}^2).
\]

To show that \( G \geq 0 \) is preserved (with \( G = 1 + \kappa_1 \kappa_2 \)), we consider the situation at a point where \( G \) first attains a zero minimum. Then the first term on the right-hand side of (5.3) is non-negative; we consider each of the other terms. The last term is simplest: Substituting the forms of \( F \) and \( G \) from (5.1) and (5.2), we find

\[
Z = G \left( f H + \frac{\partial f}{\partial H} \varphi^2 \right),
\]

so \( Z \) vanishes at a zero of \( G \), no matter what speed \( F \) we use. This is another indication of the fact that the condition of positive intrinsic curvature is optimal. The gradient terms are the most complicated, but we can simplify them significantly by observing two things: First, \( \nabla h \) is a totally symmetric 3-tensor, by the Codazzi equation. Second, at a minimum of \( G \), the gradients of \( G \) vanish. It follows that there are only two independent components of \( \nabla h \), and one finds that these never mix in the expression for \( Q \), so that

\[
Q = \alpha (\nabla_1 h_{22})^2 + \beta (\nabla_2 h_{11})^2.
\]

Since we have no further information about \( \nabla h \) (that is, no reason to expect that the magnitudes of these remaining components should vanish) we must impose the condition that \( \alpha \) and \( \beta \) are non-negative. This gives two conditions, which we can interpret as conditions on the first and second derivatives of \( F \). A fact which is perhaps not obvious is that these conditions only involve the restriction of \( F \) to the
boundary of the set \( \{ G = 0 \} \) in the curvature plane, so we can consider \( F \) as defined by (5.2) with \( G = 0 \). Then the conditions can be written explicitly as follows:

\[
\frac{\varphi''}{1 - \varphi'} - \frac{1 - \varphi'}{\varphi} \leq \frac{f''}{f'} \leq \frac{\varphi''}{1 + \varphi'} + \frac{1 + \varphi'}{\varphi}.
\]

In the case of interest, we have \( \varphi = \sqrt{4 + H^2} \), and the first and last quantities are both equal to \(-2H/(4 + H^2)\). The only possibilities for \( F \) are the following:

\[
F = C_1 + C_2 \arctan \left( \frac{H}{2} \right).
\]

This applies only along the curve \( \{ G = 0 \} \), so we are reasonably free to choose \( F \) in the region where \( G > 0 \), as long as it is monotone in both principal curvatures.

### 5.2. The extreme case

The remarkably restricted form of the evolution equation is illuminated somewhat by considering the extreme case of flat surfaces: If the flow preserves positive intrinsic curvature, then it must also preserve zero curvature. As outlined above, the structure of surfaces with zero curvature is very well understood, and in particular the Gauss map \( G : M^2 \to S^2 \times S^2 \) has the remarkable property that the projection onto each factor is one-dimensional. This must be preserved under the flow.

The flow we have ended up with is characterised by the fact that the Gauss map evolves according to the mean curvature flow (now for codimension 2 surfaces in \( S^2 \times S^2 \), which means that each of the two curves coming from the two projections of the Gauss map evolves according to the curve-shortening flow in \( S^2 \). Since each of the curves divides the area of the sphere into two equal parts, the image of the Gauss map never develops singularities (at least in the case where the two curves are homotopic to great circles traversed once), but in fact the flat tori will in general develop singularities — this is analogous to the motion of a curve in the plane with constant normal speed, which develops singularities even though the normal direction stays constant at each point. Incidentally, there has been some very impressive recent progress on mean curvature flow in higher codimension, due to Mu-Tao Wang [29]–[31], who has used it to prove several very interesting results regarding maps between manifolds.

The examples of flat tori can be used to prove that there is no other curvature-driven flow of surfaces which preserves the condition of positive curvature, by giving examples for any other flow of flat tori which do not stay flat.

### 5.3. Regularity

A technical issue which arises is the following: The speed we ended up with is not concave or convex as a function of the second fundamental form. The regularity estimates due to Krylov [18] and Evans [10] for fully nonlinear equations (needed to prove that we get classical solutions of the flow) require concavity, so we cannot
use these. Instead it is possible to adapt the estimates for elliptic equations in two variables (due to Morrey [22] and Nirenberg [23]) to give good \(C^{2,\alpha}\) estimates for solutions of fully nonlinear parabolic equations in two space variables.

### 5.4. Curvature pinching

Now we come to the problem of choosing a good way to extend the speed from the boundary \(\{G = 0\}\) to the interior of the region \(\{G > 0\}\). The idea is to do this in such a way that any compact surface with strictly positive curvature necessarily has very strongly controlled curvature in the future — that is, we want the region \(\{G > 0\}\) to be exhausted by a nested family of regions which stay away from the boundary, and only approach infinity near the ‘umbillic’ line \(\kappa_1 = \kappa_2\). This means that any singularity which occurs will have to be totally umbillic, so occurs only when the surface shrinks to a point while becoming spherical in shape.

This can be done in many ways. One which is relatively simple to describe, but results in solutions which are only \(C^{2,\alpha}\), is as follows: Take

\[
F = \begin{cases} 
\arctan \kappa_1 + \arctan \kappa_2, & \kappa_1 \kappa_2 < 1; \\
\frac{\pi}{4} (\kappa_1 \kappa_2 + 1), & \kappa_1 \kappa_2 > 1.
\end{cases}
\]

This is then a Lipschitz, monotone increasing function of the curvatures, and one can check that the following regions of the curvature plane are preserved:

\[
\Omega_\varepsilon = \left\{ |\kappa_1 - \kappa_2| \leq \frac{1 + \kappa_1 \kappa_2}{\varepsilon} \right\} \cap \{ \kappa_1 \kappa_2 \leq 1 \} \cup \left\{ |\kappa_1 - \kappa_2| \leq \frac{2}{\varepsilon} \right\} \cap \{ \kappa_1 \kappa_2 \geq 1 \}.
\]

This means that the difference between the principal curvatures stays bounded even if the curvature becomes large, which implies very strong control on singularities. This is similar to the estimate used in [4] to prove that worn stones (i.e. convex surfaces moving by their Gauss curvature) become round as they shrink to points.

With a little more work we can choose the speed to be a smooth function of the principal curvatures, and then solutions are also smooth.

In the choice above, we also have the nice feature that minimal surfaces do not move. We can with slight modifications arrive at a speed for which constant mean curvature surfaces do not move, for any particular choice of the mean curvature, as long as we are willing to work in the category of oriented surfaces. More generally, we can contrive that for a given monotone increasing function \(\phi\) of the principal curvatures, surfaces satisfying \(\phi = 0\) do not move. Here \(F\) (and \(\phi\)) must be symmetric. We can also choose if desired a speed which is always positive, so that there are no stationary solutions.

### 5.5. The results

The main result for the above speed is the following:

**Theorem 1.** Let \(x_0\) be an immersion of \(S^2\) in \(S^3\), with non-negative intrinsic curvature in the induced metric. Then the flow constructed above deforms \(M_0 = x_0(S^2)\) through a family \(M_t = x_t(S^2)\), with intrinsic curvature strictly positive for
each \( t > 0 \), to either a great sphere (in infinite time) or to a point, with spherical limiting shape (in finite time). If \( M_0 \) is embedded, then so is \( M_t \) for each \( t > 0 \).

This includes in particular Simons’ result on minimal surfaces. If we modify the speed somewhat, then we get the following result, which gives in particular a new result for Weingarten surfaces in the 3-sphere:

**Theorem 2.** Let \( \phi \) be any smooth, strictly monotone function of \( \kappa_1 \) and \( \kappa_2 \) defined on \( \{ \kappa_1 \kappa_2 + 1 \geq 0 \} \). Then there exists a function \( F \) which is smoothly defined on \( \{ \kappa_1 \kappa_2 + 1 \geq 0 \} \), and strictly monotone increasing in each argument, with \( \text{sgn} F = \text{sgn} \phi \) everywhere, such that the following holds: If \( M_0 = x_0(S^2) \) is a smooth compact surface in \( S^3 \) with non-negative intrinsic curvature, then the motion with speed \( F \) deforms \( M_0 \) through a smooth family \( \{ M_t \}_{0 \leq t < T} \), each strictly positively curved, which either converge to a point with spherical limiting shape with \( T < 0 \), or converge to a totally umbilic surface (spherical cap) with \( \phi = 0 \) if \( T = \infty \).

This includes two cases: Either there is some point where \( \phi = 0 \), in which case there is a spherical cap with \( \phi = 0 \) and the above result implies that this is the only surface with \( \phi = 0 \) with positive intrinsic curvature, or \( \phi \) is never zero, in which case all surfaces converge to points. In the latter case a very small geodesic sphere with one choice of orientation will shrink inwards to its centre, while the same sphere with the opposite orientation expands over the equator and eventually contracts to the antipodal point. In this way we have a unique way of associating an oriented surface with the point it eventually contracts to, and we deduce the following:

**Theorem 3.** The space of oriented surfaces with positive intrinsic curvature in \( S^3 \) retracts onto \( S^3 \).

Finally, if we introduce some non-local terms in the speed, we can devise a flow which fixes the enclosed volume, preserves positive intrinsic curvature, and gives convergence to spherical caps, without moving constant mean curvature surfaces.

### 6. Other results by related methods

The methods I outlined above also yield interesting results for a variety of other problems: One which works out similarly, and which has some interesting parallels, is that of surfaces in three-dimensional hyperbolic space. The surfaces of interest are those for which all of the principal curvatures are less than 1 in magnitude. We can find a flow which deforms any such surface in a compact hyperbolic manifold to a minimal surface, while keeping the principal curvatures less than 1 in magnitude. Rather surprisingly, this flow is in a way the hyperbolic analogue of the one we just described for the sphere: Instead of moving with speed equal to the sum of the arctangents of the principal curvatures, we move with speed equal to the sum of the hyperbolic arctangents of the principal curvatures. The resulting flow is very well-behaved, and has the interesting property that the Gauss map of the surface (the map which takes a point of the surface to its tangent plane, thought of as a point in the Grassmannian of spacelike 2-planes in Minkowski space \( R^{3,1} \)), evolves according to mean curvature flow.
The methods also give good results for hypersurfaces in higher-dimensional spheres: Hypersurfaces with positive sectional curvatures can be deformed in such a way as to preserve that condition, and similar results can be deduced. The condition of positive sectional curvature can probably be relaxed: Positive sectional curvature is implied by the condition of Okumura [25] for constant mean curvature hypersurfaces, but not by the sharper condition of Cheng and Nakagawa [5] and Alencar and do Carmo [1].

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