Composite fermion basis for $M$-component Bose gases

V Skogvoll and O Liabøtrø

Department of Physics, University of Oslo, PO Box 1048 Blindern, 0316 Oslo, Norway

E-mail: vidarsko@gmail.com

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Abstract

The composite fermion (CF) formalism produces wave functions that are not always linearly independent. This is especially so in the low angular momentum regime in the lowest Landau level, where a subclass of CF states, known as simple states, gives a good description of the low energy spectrum. For the two-component Bose gas, explicit bases avoiding the large number of redundant states have been found. We generalize one of these bases to the $M$-component Bose gas and prove its validity. We also show that the numbers of linearly independent simple states for different values of angular momentum are given by coefficients of $q$-multinomials.

Keywords: composite fermions, Bose–Einstein condensate, basis sets, combinatorics, $q$-multinomials

(Some figures may appear in colour only in the online journal)

1. Introduction

Rapidly rotating atomic gases and the associated quantum phenomena have been studied quite extensively, for review see e.g. [1–3]. Over the years, different groups have been able to engineer multi-component Bose–Einstein condensates. Examples include using hyperfine states of $^{87}\text{Rb}$ [4, 5] to create two-component mixtures, the use of optical traps to create spin-1 Bose–Einstein condensates with $^{87}\text{Rb}$ [6] and $^{23}\text{Na}$ [7–9] with relatively small spin interaction contributions, and the use of Feshbach resonances to make two-component Bose condensates in $^{87}\text{Rb} – ^{39}\text{Kr}$ [10], $^{87}\text{Rb} – ^{41}\text{K}$ [11], and $^{87}\text{Rb} – ^{85}\text{Rb}$ [12]. The focus of this paper is multi-species Bose gases in the lowest Landau level (LLL) at low angular momentum. This regime was realized for single species systems in a recent experiment [13]. Hope is that such experiments could also be run for multi-component Bose gases and a good theoretical understanding of such systems is therefore of great interest.

The history of constructing trial wave functions, from the Laughlin wave function [14] to the conceptualization of composite fermion (CF) formalism [15, 16] and trial wave functions...
addressing non-Abelian quantum Hall states [17–19], has shown success and taught us much about the quantum structures of such systems. The adaptation and application of the CF formalism has also shown promise when applied to weakly interacting rotating multi-component Bose gases [20]. In particular the space of trial wave functions spanned by ‘simple states’ (see section 3) has been shown to give significant overlap with the low energy part of the spectrum for contact interaction.

Simple states result from a projection to the LLL, which generally leads to linear dependencies between the resulting wave functions. The ratio of the number of \textit{a priori} non-zero simple states to linearly independent ones can quickly get very large, especially in the low angular momentum regime, e.g. $10^7/8$ for a mixture of $3 + 2 + 1$ bosons at $L = 4$. Recent developments [21] have been able to shed light on these linear dependencies for two-component gases, even giving explicit bases [22].

In this article we present an explicit basis for the space of simple states for the $M$-component Bose gas and the proof of its validity. This basis generalizes the two-component basis of [22]. Additionally we show that the numbers of basis states are given by coefficients of $q$-multinomials.

2. $M$-Component rotating Bose gases

In this paper, $\alpha, \beta$ will label the different species of particles while $i, j$ will label particles of the same species. There are $M$ boson species in the system and $N_\alpha$ particles of species $\alpha$. There are $A = \sum_{\alpha=1}^M N_\alpha$ particles in total. Whenever we iterate over all particles, we will use indices $\mu, \nu$. The particles of the first species have indices $\mu = 1, \ldots, N_1$, the second $\mu = N_1 + 1, \ldots, N_1 + N_2$ and so on. In general we can relate the species $\alpha$ and the particle number $i$ within that species to its index $\mu$ by

\[
(\alpha, i) \leftrightarrow \mu = \left(\sum_{\beta=1}^{\alpha-1} N_\beta\right) + i.
\]  

The Hamiltonian of an $M$-component Bose gas in a harmonic oscillator trap of strength $\omega$ under rotation with angular velocity $\Omega$ and a species-dependent contact interaction is given by

\[
\hat{H} = \sum_{\mu=1}^A \left(\frac{\hat{p}_\mu^2}{2m} + \frac{1}{2}m\omega^2\hat{r}_\mu^2 - \Omega\hat{l}_\mu\right) + \sum_{\mu=1}^A \sum_{\nu=\mu+1}^A 2\pi g_{\mu\nu} \delta(\hat{r}_\mu - \hat{r}_\nu),
\]  

where $g_{\mu\nu} = g_{\mu'\nu'}$ if $\mu$ is of the same species as $\mu'$ and $\nu$ is of the same species as $\nu'$. In the weak interaction limit, this reduces to the Lowest Landau level problem [1] in the effective magnetic field $2m\omega$ and the Hamiltonian is given by

\[
\hat{H} = \hat{H}_0 + \hat{W} = \sum_{\mu=1}^A (\omega - \Omega)\hat{l}_\mu + \sum_{\mu<\nu=1}^A 2\pi g_{\mu\nu} \delta(\hat{r}_\mu - \hat{r}_\nu).
\]  

In the ideal limit $(\omega - \Omega) \to 0$, the Landau levels flatten and the eigenstates are solely determined by the interaction $\hat{W}$.

Single-particle eigenstates in the lowest Landau level with angular momentum $l$ are given by

\[
\psi_{0,l}(z_\mu) = N_l c^l_\mu \exp\left(-|z_\mu|^2/4\right),
\]
where \( z_\mu = x_\mu + iy_\mu \) is the complex position variable of particle \( \mu \) in units of the magnetic length \( \sqrt{\hbar/(2m\omega)} \). The simultaneous many-body eigenstates of \( \hat{H}_0 \) and the total angular momentum \( \hat{L} = \sum_\mu \hat{l}_\mu \), with eigenvalues \( L\hbar(\omega - \Omega) \) and \( L \) respectively, are products of a ubiquitous Gaussian factor, \( \exp \left(-\sum_{\mu=1}^A |z_\mu|^2/4 \right) \), and a homogeneous polynomial \( P \) of degree \( L \), symmetric in variables of the same species.

### 3. Simple states

A CF trial wave function for the bosonic \( M \)-component system is easily generalizable from the two-component case [16] and is on the form

\[
\Psi_{\text{CF}} = \mathcal{P}_{\text{LLL}} \left[ \left( \prod_{\alpha=1}^M \Phi_\alpha \right) \mathcal{J}^q \right] \exp \left(-\sum_{\mu=1}^A |z_\mu|^2/4 \right),
\]

where \( \mathcal{P}_{\text{LLL}} \) denotes the projection to the LLL and \( \Phi_\alpha \) is a Slater determinant for the particles of species \( \alpha \) and consists of orbitals from degenerate Landau-like levels called \( \Lambda \)-levels [22]. \( \mathcal{J} \) is the Jastrow factor given by

\[
\mathcal{J} = \prod_{\mu < \nu} (z_\mu - z_\nu) = \sum_{\tau \in \Delta} (-1)^{|\tau|} \prod_{\mu=1}^A z^{\mu-1}_{\tau(\mu)},
\]

and \( q \) is an odd number to ensure the correct overall symmetry of the CF wave function. In this paper, we limit our attention to \( q = 1 \). Simple states are CF trial wave functions where only the lowest angular momentum state of each \( \Lambda \)-level is available to each species. This minimizes the CF cyclotron energy for a given \( L \) [21]. The Slater determinants then only consist of powers of \( \bar{z} \). These complex conjugated variables then translate to powers of \( \partial_z \) after projection to the LLL by the projection of Girvin and Jach [23] (called method I in [16]). The polynomial part of a simple state can consequently be represented by an array \( n \in \mathbb{N}^A \) where \( n_\alpha \) are occupation numbers which correspond to occupied \( \Lambda \)-levels for the different particles. We name these polynomials ‘simple polynomials’ and they take the form

\[
P(n) = \sum_{\sigma \in S_{\oplus n_\alpha}} (-1)^{|\sigma|} \prod_{\mu=1}^A \partial_{n_\sigma}^{n_\alpha(\mu)} \mathcal{J}
\]

where

\[
S_{\oplus n_\alpha} = \bigoplus_{\alpha=1}^M S_{n_\alpha},
\]

and \( S_{n_\alpha} \) is the symmetric group of \( N_\alpha \) elements. The degree of \( P(n) \) and the angular momentum of the corresponding simple state is

\[
L = \frac{A(A-1)}{2} - \sum_{\mu=1}^A n_\mu.
\]

By the anti-symmetries of equation (7), a simple polynomial \( P(n) \) is anti-symmetric under the interchange of elements \( n_\mu \leftrightarrow n_\nu \) when \( \mu \) and \( \nu \) belong to the same species. This allows us to represent a non-zero simple polynomial pictorially (up to a sign) in a grid \( \{1,2,\ldots,M\} \times \{0,1,2,\ldots,A-1\} \) where \( \sigma \) at position \((\alpha,n)\) corresponds to some \( n_{\alpha,\sigma} = n \),
Figure 1. Pictorial representation of $P(n)$ for $(N_1, N_2, N_3) = (2, 2, 3)$ given by $n = (0, 2; 1, 2; 0, 2, 3)$. The semicolons in $n$ indicate which occupation numbers belong to which species.

since repeated values of $n$ within a species is 0 due to anti-symmetry. Figure 1 shows an example of such a pictorial representation.

As previously discovered [20] for the two-component Bose gas and also seen for the $M$-component Bose gas, the space of simple states overlaps significantly with the low-energy part of the delta-potential interaction spectrum. Figure 2 shows an example of this when the interaction is species-independent. Note that a completely species-independent interaction is not often physically realizable, but results are similar for small deviations from this ideal case. Perhaps most striking is the large overlap of the ground state with the space of simple states when the dimension of this space is but a small fraction of the dimension of the relevant sector of the Hilbert space. Seeing that the space of simple states has this characteristic, an explicit basis for it is of great interest.

We define the elementary differentiation polynomial of degree $R$ by

$$d_R = \frac{N_R}{A!} \prod_{\tau \in S_A} \partial_{\tau(\nu)},$$

(10)

where $N_R = \frac{1}{A!(A-R)!}$, $d_R$ is the sum over all unique products of $R$ out of $A$ first order partial derivatives. The action of $d_R$ on a simple polynomial is given by

$$d_R P(n) = \frac{N_R}{A!} \prod_{\tau \in S_A} \sum_{\sigma \in S_{A-R}} (-1)^{|\sigma|} \prod_{\mu=1}^A \partial_{\tau(\nu)}^{\sigma(\nu)} \mathcal{J}$$

$$= \frac{N_R}{A!} \prod_{\tau \in S_A} \sum_{\sigma \in S_{A-R}} (-1)^{|\sigma|} \prod_{\nu=1}^R \partial_{\tau(\nu)} \prod_{\mu=1}^A \partial_{\tau(\mu)}^{\sigma(\nu)} \mathcal{J}$$

$$= \frac{N_R}{A!} \prod_{\tau \in S_A} \sum_{\sigma \in S_{A-R}} (-1)^{|\sigma|} \prod_{\nu=1}^R \partial_{\tau(\nu)} \prod_{\mu=R+1}^A \partial_{\tau(\mu)}^{\sigma(\nu)} \mathcal{J}$$

$$= \frac{N_R}{A!} \prod_{\tau \in S_A} P(n + \sum_{\nu=1}^R \sigma(\nu)),$$

(11)
where $e_\nu \in \mathbb{N}^A$ is a unit vector in the $\nu$'th direction. Lemma A.1 states that simple polynomials obey

$$\hat{d}_\mu P(n) = 0.$$  \hspace{1cm} (12)

This is an equivalent formulation of generalized translation invariance, as was used in [21, 22].

### 4. A basis for the simple states

We now present our main result.  

**Theorem 1.** The set $B_L = \{ P(n) \mid$ equations $(13)-(15) \}$, where

$$n_{\alpha,i} < \sum_\beta N_\beta,$$  \hspace{1cm} (13)

$$n_{\alpha,i} < n_{\alpha,j} \iff i < j$$  \hspace{1cm} (14)

$$\sum_{\mu=1}^A n_\mu = \frac{A(A-1)}{2} - L$$  \hspace{1cm} (15)

is a basis for the space of simple polynomials of degree $L$. Consequently the corresponding set of simple states $\{ P(n) \exp \left( - \sum_{\mu=1}^A |\tau_{\mu}|^2 / 4 \right) \}$ is a basis for the space of simple states with angular momentum $L$.

Pictorially, a simple polynomial in $B_L$ corresponds to a figure where no $\nu$'s occupy spaces above the step function which increases by $N_\alpha$ for each step $\alpha - 1 \to \alpha$, see figure 3. An example of two polynomials from the same space where one is not in $B_L$ is given in figure 4.

**Proof.** To prove that $B_L$ is a basis we need to show that (i) it spans the whole space of simple polynomials and that (ii) it is a linearly independent set. We refer to appendix A for the lemmata used in this section.

(i) Spanning of the space of simple polynomials.

Equation (14) picks a unique permutation within each species which fixes the otherwise ambiguous sign of the pictorial representation. Equation (15) ensures that the polynomials have the correct degree. The non-trivial result is that equation (13) gives a basis under these conditions.

As shown in lemma A.2 by induction, a consequence of lemma A.1 is that for all $R \leq n_\mu$

$$P(n) = (-1)^R \hat{d}_R^{\mu} P(n - Re_\mu),$$  \hspace{1cm} (16)

where

$$\hat{d}_R^{\mu} = \hat{d}_R^{\nu} \sum_{\tau \in S_A} \prod_{\nu=1}^R \partial_{\tau(\nu)} (1 - \delta_{\mu\tau(\nu)}).$$  \hspace{1cm} (17)
The action of $\hat{d}_\mu^\mu$ on a simple polynomial gives the sum of all simple polynomials where $R$ occupation numbers excluding the $\mu$'th are increased by 1. So lemma A.2 means that we can write a simple polynomial $P(n)$ as the sum of all simple polynomials where one, the $\mu$'th, occupation number of $n$ has been reduced by $R$ units and $R$ other (not $\mu$) have been increased by one.

We now introduce a partial ordering $\prec$ of the simple polynomials as follows:

$$P(n') \prec P(n) \quad \text{if for an } \alpha \quad \sum_i n'_{\alpha i} > \sum_i n_{\alpha i} \quad \text{and} \quad \sum_i n'_{\beta i} = \sum_i n_{\beta i} \quad \forall \beta > \alpha.$$  \hfill (18)

This means that a polynomial is ‘smaller’ wrt. $\prec$ than another if the sum of its occupation numbers for the last species is bigger than the other. And if those are equal, the one with the largest occupation number sum for the next to last species is ‘smaller’ and so on. For a given total sum of occupation numbers in $n$, there is a limit as to how ‘small’ a state can get since $n_{\alpha i} \geq 0$. Now consider a simple polynomial $P(n^*)$ of degree $L$ which is not in $B_L$ because it violates equation (13). This implies that it contains some exponent

$$n^*_\mu = n^*_{\alpha i} \geq \sum_{\beta \leq \alpha} N_{\beta i}.$$  \hfill (19)

Figure 2. The exact diagonalization (dots) of the homogeneous zero-range interaction ($g_{\mu \nu} = g$) within the Hilbert Space of translationally invariant states for $(N_1, N_2, N_3) = (3, 2, 2)$ together with the diagonalization (rings) within (a) the subspace of simple states and (b) a random subspace of equal dimension. The random subspaces are spanned by state vectors chosen uniformly on the unit sphere and are included for comparison. Numbers denote the squared absolute values of the ground state projection onto the subspaces.
We rewrite $P(n^*)$ according to lemma A.2

$$P(n^*) = (-1)^{n^*} d_{n^*_\mu} P(n^* - n^*_\mu e_\mu).$$

(20)

The right hand side of this equation is a sum of polynomials where the $\mu$th occupation number is $n^*_\mu$ less than in $P(n^*)$ and $n^*_\mu$ other occupation numbers are 1 more than in $P(n^*)$. The number of occupation numbers for species $\beta \leq \alpha$ which can be increased in this way is $\sum_{\beta \leq \alpha} N_\beta - 1$. We see from equation (19) that this is less than $n^*_\mu$ and by the pigeon hole principle at least one occupation number must have increased for a species $\beta > \alpha$. This means that we have rewritten $P(n^*)$ in terms of polynomials that are ‘smaller’ wrt. $\prec$. For any simple polynomial not in $B_L$ because it violates equation (13) we can perform this iteration and since there is a lower limit to how ‘small’ a state can be, at some point we must have rewritten $P(n^*)$ in terms of polynomials that obey equation (13).

Figure 3. The step function that limits the configuration space of the basis polynomials when represented pictorially.

Figure 4. Example of two simple polynomials for the system $(N_1, N_2, N_3) = (2, 2, 3)$, both of degree $L = 9$. The left polynomial is not in $B_L$ whereas the right one is.
(iii) Linear independence of the polynomials. To prove the linear independence of the basis states we must return to the full form of a simple state given by \( \mathbf{n} \).

\[
P(\mathbf{n}) = \sum_{\sigma \in \mathcal{S}_{\mathcal{G}_{n_0}}} \sum_{\tau \in \mathcal{S}_L} (-1)^{\tau + |\tau|} \prod_{\mu=1}^{A} \partial_{\sigma(\mu)}^\mu \prod_{\nu=1}^{A} z^{\nu-1}_{\tau(\nu)}
\]

\[
= \sum_{\tau \in \mathcal{S}_L} (-1)^{|\tau|} \sum_{\sigma \in \mathcal{S}_{\mathcal{G}_{n_0}}} \prod_{\mu=1}^{A} \partial_{\sigma(\mu)}^\mu z^{\tau(\mu)-1}_{\sigma(\mu)}
\]

\[
= \sum_{\tau \in \mathcal{S}_L} t_{\mathbf{n},\tau}.
\]

\( t_{\mathbf{n},\tau} \) is a symmetric polynomial within each species given by

\[
t_{\mathbf{n},\tau} = (-1)^{|\tau|} K_{\mathbf{n},\tau} \sum_{\sigma \in \mathcal{S}_{\mathcal{G}_{n_0}}} \prod_{\mu=1}^{A} z^{\tau(\mu)-1-n_\mu}_{\sigma(\mu)},
\]

where \( K_{\mathbf{n},\tau} \) is a constant obtained from differentiation, zero if \( n_\mu > \tau(\mu) - 1 \) for any \( \mu \). We define an ordering of non-zero terms by

\[
t_{\mathbf{n},\tau} < t_{\mathbf{n}',\tau'}
\]

if for the last particle species the \( k \)th least exponent of \( t_{\mathbf{n},\tau} \) is greater than the \( k \)th least exponent of \( t_{\mathbf{n}',\tau'} \) and their \( k - 1 \) least exponents are pairwise equal. If this is the case for all exponents for particles in the exponents in the last species, the terms are sorted according to the \( M - 1 \) particle species and so on. Consider \( t_{\mathbf{n},\mathcal{I}} \) where \( \mathcal{I} \) is the identity permutation

\[
t_{\mathbf{n},\mathcal{I}} = K_{\mathbf{n},\mathcal{I}} \sum_{\sigma \in \mathcal{S}_{\mathcal{G}_{n_0}}} \prod_{\mu=1}^{A} z^{\mu-1-n_\mu}_{\sigma(\mu)}.
\]

This term is non-zero for all simple polynomials that obey equation (13) while zero for simple polynomials that do not because for the latter at least one \( n_\mu \) is greater than \( \mu - 1 \). For any other term \( t_{\mathbf{n},\tau} \) in \( P(\mathbf{n}) \)

\[
t_{\mathbf{n},\mathcal{I}} < t_{\mathbf{n},\tau}.
\]

Since every simple polynomial in \( \mathcal{B}_L \) has a unique \( t_{\mathbf{n},\mathcal{I}} \), we can impose a complete ordering on the polynomials in \( \mathcal{B}_L \) by

\[
P(\mathbf{n}) < P(\mathbf{n}') \iff t_{\mathbf{n},\mathcal{I}} < t_{\mathbf{n}',\mathcal{I}}.
\]

The smallest simple polynomial \( P(\mathbf{n}_0) \) with respect to the ordering then has the property that \( t_{\mathbf{n}_0,\mathcal{I}} \) does not occur in \( P(\mathbf{n}) \) for any \( P(\mathbf{n}) > P(\mathbf{n}_0) \). In general \( t_{\mathbf{n},\mathcal{I}} \) is not a term in \( P(\mathbf{n}') \) for any \( P(\mathbf{n}') > P(\mathbf{n}) \). We numerate the simple states in \( \mathcal{B}_L \) from 0 to \( T \) according to this relation. Consider then a linear dependence relation with all \( T \) simple states from the basis \( \mathcal{B}_L \)

\[
c_0 P(\mathbf{n}_0) + c_1 P(\mathbf{n}_1) + ... + c_T P(\mathbf{n}_T) = 0.
\]

Since \( t_{\mathbf{n}_0,\mathcal{I}} \) is in \( P(\mathbf{n}_0) \), but not in \( P(\mathbf{n}_i) \) for \( i > 0 \), \( c_0 \) must be zero. By induction all coefficients must be 0 and the states are linearly independent by definition. \( \square \)

5. On the number of states

We now address the problem of counting the simple polynomials in \( \mathcal{B}_L \). To do so we utilize a one-to-one correspondence between simple polynomials and a certain class of multipartitions...
which we have called simple multipartitions (SMPs). We refer to appendix B for the mathematical details of how to arrive at the result of this section.

We use the $q$-analog of the factorial to present a closed form expression for the number of simple polynomials of degree $L$, which we denote $k_L(N_1, ..., N_M)$.

The $q$-analog of a positive integer is $[n]_q = \sum_{i=0}^{n-1} q^i$, and the $q$-factorial $[n]_q!$ is defined recursively by $[n]_q! = [n]_q[n-1]_q!$ with $[0]_q! = 1$. The numbers $k_L(N_1, ..., N_M)$ then appear as the coefficients of the $q$-multinomial

$$\sum_{L=0}^{L_{\text{MAX}}} k_L(N_1, ..., N_M) q^L = \frac{[N_1 + ... + N_M]_q!}{[N_1]_q! \times ... \times [N_M]_q!}$$

where

$$L_{\text{MAX}} = \sum_{\alpha=1}^{M} \sum_{\beta=\alpha+1}^{M} N_\alpha N_\beta$$

is the largest possible degree of a simple polynomial. We see that equation (28) reflects the fact that the number of states in $B_L$ is independent of the order $N_1, ..., N_M$, even though the actual states of $B_L$ depend on said order. It is instructive to see an example of how to use equation (28). Consider the system $(N_1, N_2, N_3) = (3, 2, 1)$. The corresponding $q$-multinomial is given by

$$\frac{[3 + 2 + 1]_q!}{[3]_q[2]_q[1]_q} = \frac{[6]_q[5]_q[4]_q[3]_q!}{[3]_q[2]_q[1]_q[1]_q} = \frac{[6]_q[5]_q[4]_q[3]_q!}{[2]_q[1]_q[1]_q} = \frac{(1 + q + q^2 + q^3 + q^4 + q^5)(1 + q + q^2 + q^3) (1 + q + q^2 + q^3)}{(1 + q)(1)(1)}$$

$$= 1 + 2q + 4q^2 + 6q^3 + 8q^4 + 9q^5 + 9q^6 + 8q^7 + 6q^8 + 4q^9 + 2q^{10} + q^{11}.$$  

Which means, for instance, that the number of simple polynomials of degree $L = 4$ is 8.

### 6. Summary

We have found a basis for the space of $M$-component simple states generalizing one of the bases found for the two-component simple states [22] along with a proof of its validity. Redundant states due to linear dependencies is a common issue with the CF formalism and our result gives a contribution to the understanding of these dependencies. The more general problem of understanding the linear dependencies between all compact states is still open.

The numbers of basis states for different values of angular momentum turn out to be given by coefficients of $q$-multinomials. Linking these numbers to a certain class of multipartitions has lead to an interesting visual interpretation of the fact that the $q$-multinomials are independent of the order $N_1, ..., N_M$ (see appendix B and figure B1).

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Appendix A. Lemmata

Lemma A.1 (Symmetric translation invariance).

\[ \tilde{\partial}_\nu P(n) = 0 \quad (A.1) \]

For \( R \geq 1 \).

**Proof.** \( \tilde{\partial}_\nu \) consists only of differential operators and commutes with the differential operators in \( P(n) \). We therefore only need to show that \( \tilde{\partial}_\nu P = 0 \).

\[
\tilde{\partial}_\nu P = N R \sum_{\tau \in S_R} \prod_{\mu=1}^R \partial_{\tau(\mu)} \sum_{\sigma \in S_k} (-1)^{|\sigma|} \prod_{\nu=1}^A z^\nu_{\sigma(\nu)} = N R \sum_{\tau \in S_R} \sum_{\sigma \in S_k} (-1)^{|\sigma|} \prod_{\mu=1}^R \partial_{\tau(\mu)} \prod_{\nu=1}^A z^\nu_{\sigma(\nu)}, \quad (A.2)
\]

Consider a term \( t_1 \) in this sum given by \( \tau = \tau_1 \) and \( \sigma = \sigma_1 \) and consider the lowest value \( \nu' \) for which \( \sigma_1(\nu') = \tau_1(\mu_1) \) for some \( \mu_1 \in \{1, 2, ..., R\} \). If \( \nu' = 1 \), then \( t_1 \) is zero due to differentiation of the constant \( z^0_{\tau_1(1)} \). Else, if \( \nu' > 1 \), we may consider the factor \( z^{\nu'-2}_{\sigma_1(\nu'-1)} \) and write

\[
t_1 = (-1)^{|\sigma_1|} z^{\nu'-2}_{\sigma_1(\nu'-1)} \partial_{\tau_1(\mu_1)} z^{\nu'-1}_{\sigma_1(\nu')} \times \ldots \quad (A.3)
\]

where we have written out only the permutation sign and the variables and differentiation operators of type \( \sigma_1(\nu') \) and \( \sigma_1(\nu' - 1) \). We can now pick another term, \( t_2 \), given by permutations \( \sigma_2 \) and \( \tau_2 \). The permutation \( \sigma_2 \) is the same as \( \sigma_1 \) except that \( \sigma_2(\nu') = \sigma_1(\nu' - 1) \) and \( \sigma_2(\nu' - 1) = \sigma_1(\nu') \). We find the \( \mu_2 > R \) such that \( \tau_1(\mu_2) = \sigma(\nu' - 1) \) and define \( \tau_2 \) to be equal to \( \tau_1 \) except that \( \tau_2(\mu_1) = \tau_1(\mu_2) \) and \( \tau_2(\mu_2) = \tau_1(\mu_1) \). This gives

\[
t_2 = (-1)^{|\sigma_1|} z^{\nu'-2}_{\sigma_1(\nu'-1)} \partial_{\tau_1(\mu_1)} z^{\nu'-1}_{\sigma_2(\nu')} \times \ldots \quad (A.4)
\]

where the hidden part is the same as in equation (A.3). We also have that

\[
z^{\nu'-2}_{\sigma_1(\nu'-1)} \partial_{\tau_1(\mu_1)} z^{\nu'-1}_{\sigma_1(\nu')} = z^{\nu'-2}_{\sigma_2(\nu'-1)} \partial_{\tau_1(\mu_1)} z^{\nu'-1}_{\sigma_2(\nu')} = (\nu' - 1) z^{\nu'-2}_{\sigma_1(\nu'-1)} z^{\nu'-2}_{\sigma_1(\nu')} \quad (A.5)
\]

The permutations \( \sigma_1 \) and \( \sigma_2 \) have different parities, and we can therefore conclude that \( t_1 + t_2 = 0 \) and that all terms of \( \tilde{\partial}_\nu P \) are cancelled in this way.

Lemma A.2 (Simple state identity).

\[ P(n) = (-1)^{R} \tilde{\partial}_\nu R P(n - Re_\mu) \quad (A.6) \]
where
\[
\hat{d}_R^{-\mu} = N_R \sum_{\tau \in S_n} \prod_{\nu=1}^{R} \partial_{\tau(\nu)} (1 - \delta_{\mu\tau(\nu)}).
\]  
(A.7)

In words, the lemma states that we can write a simple polynomial \( P(n) \) as the sum of all simple polynomials where one, the \( \mu \)'th, occupation number of \( n \) has been reduced by \( R \) units and \( R \) other (not \( \mu \)) have been increased by one.

**Proof.** The proof is based on induction. We define the operator
\[
\hat{d}_R^\mu = N_R \sum_{\tau \in S_n} \prod_{\nu=1}^{R} \partial_{\tau(\nu)} \delta_{\mu\tau(\nu)}
\]  
(A.8)

which when acting on \( P(n) \) gives the sum of all simple polynomials where \( R \) occupation numbers, always including \( \mu \), have been increased by 1. By definition
\[
\hat{d}_R = \hat{d}_R^{-\mu} + \hat{d}_R^\mu.
\]  
(A.9)

Consider the polynomial \( P(n - 1e_\mu) \). Lemma A.1 proves the base case, since
\[
0 = \hat{d}_1 P(n - 1e_\mu)
= \hat{d}_1^\mu P(n - 1e_\mu) + \hat{d}_1^{-\mu} P(n - 1e_\mu)
= P(n) + \hat{d}_1^{-\mu} P(n - 1e_\mu)
\]  
(A.10)

\[
P(n) = (-1)^1 \hat{d}_1^{-\mu} P(n - 1e_\mu).
\]  
(A.11)

Now, we assume the Lemma holds for \( R \), i.e.
\[
P(n) = (-1)^R \hat{d}_R^{-\mu} P(n - Re_\mu).
\]  
(A.12)

Consider the polynomial \( P(n - (R + 1)e_\mu) \). Lemma A.1 gives
\[
0 = \hat{d}_{R+1} P(n - (R + 1)e_\mu)
= \hat{d}_{R+1}^\mu P(n - (R + 1)e_\mu) + \hat{d}_{R+1}^{-\mu} P(n - (R + 1)e_\mu)
= \hat{d}_R^{-\mu} P(n - Re_\mu) + \hat{d}_{R+1}^{-\mu} P(n - (R + 1)e_\mu)
\]  
(A.13)

since the sum of simple polynomials where \( R + 1 \) occupation in \( n \), always including \( \mu \), have been increased by 1 is the same as the sum of simple polynomials where \( R \) occupation numbers in \( (n + e_\mu) \), not including \( \mu \), have been increased by 1. By the induction hypothesis, this is
\[
(-1)^R P(n) + \hat{d}_{R+1}^{-\mu} P(n - (R + 1)e_\mu) = 0
\]  
(A.14)

\[
P(n) = (-1)^{R+1} \hat{d}_{R+1}^{-\mu} P(n - (R + 1)e_\mu).
\]  
(A.15)

So by the induction principle the Lemma holds for all \( R \geq 1. \) \( \square \)
Appendix B. Simple multipartitions (SMPs) and $q$-multinomials

We define an SMP as a multipartition of an integer into $(M - 1)$ partitions in rectangles (i.e. with limits on both the maximal, and the number of non-zero, elements). The dimensions of the rectangles are given by $M$ numbers $N_1, \ldots, N_M$ such that the dimensions of rectangle $\alpha$ are given by

$$\left(\sum_{\beta<\alpha} N_{\beta}\right) \times N_{\alpha+1}. \quad (B.1)$$

We can represent an SMP of an integer $L$ by a vector $p \in \mathbb{N}^A$

$$p = [p_{1,1}, p_{1,2}, \ldots, p_{M,N_M}] \quad (B.2)$$

where $\sum_{\alpha,j} p_{\alpha,j} = L$, $p_{\alpha,j} \leq p_{\alpha,i}$ if $i > j$ and $p_{\alpha,j} \leq \sum_{\beta=1}^{\alpha-1} N_{\beta}$. The possible values of $L$ are within $\{0, \ldots, \sum_{\beta=1}^{M} N_{\beta}\}$. An SMP corresponds to compactly coloring $L$ cells to make $M - 1$ Young diagrams in rectangles of dimensions $N_1 \times N_2$, $(N_1 + N_2) \times N_3$, $\ldots$, $(\sum_{\beta=1}^{M-1} N_{\beta}) \times N_M$ as shown in figure B1.

The one-to-one correspondence between a simple polynomial $P(n) \in B_L$ of degree $L$ and an SMP of $L$ is given by

$$p_{\alpha,j} = \left(\sum_{\beta=1}^{\alpha-1} N_{\beta}\right) + (i - 1) - n_{\alpha,j}. \quad (B.3)$$

Note that since the number of basis states in $B_L$ is independent of the order $N_1, \ldots, N_M$, this one-to-one correspondence imply that the number of possible SMPs of an integer $L$ is independent of the order $N_1, \ldots, N_M$. This is a generalization of the trivial fact that the number
of possible compact colorings of $L$ cells in an $N_1 \times N_2$ Young diagram is the same as in an $N_2 \times N_1$ Young diagram. An example of this with $L = 4$, $(N_1, N_2, N_3) = (1, 2, 3)$ is given in figure B2.

The number of SMPs of $L$ is related to $q$-multinomials (see e.g. [24] for information on these). A $q$-binomial is defined as follows

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!}$$  \hspace{1cm} (B.4)

where $[n]_q$ and $[n]_q!$ have been defined in section 5. The number $k_L(N_1, N_2)$ of compact colorings of $L$ cells in an $N_1 \times N_2$ Young diagram is related to $q$-binomials by (see e.g. [25])

$$\sum_{L=0}^{N_1N_2} k_L(N_1, N_2) q^L = \binom{N_1 + N_2}{N_1}_q = \frac{[N_1 + N_2]_q!}{[N_1]_q! [N_2]_q!}.$$  \hspace{1cm} (B.5)

The product

$$\left(\sum_{L=0}^{N_1N_2} k_L(N_1, N_2) q^L\right)^{N_1 + N_2} \sum_{\ell=0}^{(N_1+N_2)N_3} k_\ell (N_1 + N_2, N_3) q^\ell$$  \hspace{1cm} (B.6)

gives a polynomial in $q$ whose coefficient before $q^L$ is the number of ways to compactly color $L = l + l'$ cells in two Young diagrams of dimensions $N_1 \times N_2$ and $(N_1 + N_2) \times N_3$. We see from the right hand side of equation (B.5) that this product equals

$$\binom{N_1 + N_2}{N_1}_q \binom{N_1 + N_2 + N_3}{N_1 + N_2 + N_3}_q$$

$$= \frac{[N_1 + N_2]_q! [N_1 + N_2 + N_3]_q!}{[N_1]_q! [N_2]_q! [N_3]_q!} = \frac{[N_1 + N_2 + N_3]_q!}{[N_1]_q! [N_2]_q! [N_3]_q!}$$  \hspace{1cm} (B.7)

where we identify the last expression as the $q$-multinomial. We have thus shown equation (28) for the case of three species. The generalization to $M$ species is straightforward.
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