Loop Grassmannian cohomology, the principal nilpotent and Kostant theorem

VICTOR GINZBURG

Abstract

Given a complex projective algebraic variety, write $H^\bullet(X, \mathbb{C})$ for its cohomology with complex coefficients and $IH^\bullet(X, \mathbb{C})$ for its Intersection cohomology. We first show that under some fairly general conditions the canonical map $H^\bullet(X, \mathbb{C}) \rightarrow IH^\bullet(X, \mathbb{C})$ is injective.

Now let $\text{Gr} := G((z))/G[[z]]$ be the loop Grassmannian for a complex semisimple group $G$, and let $X$ be the closure of a $G[[z]]$-orbit in $\text{Gr}$. We prove, using the general result above, a conjecture of D. Peterson describing the cohomology algebra $H^\bullet(X, \mathbb{C})$ in terms of the centralizer of the principal nilpotent in the Langlands dual of $\text{Lie}(G)$.

In the last section we give a new "topological" proof of Kostant’s theorem about the polynomial algebra of a semisimple Lie algebra, based on purity of the equivariant intersection cohomology groups of $G[[z]]$-orbits on $\text{Gr}$.

1 Main results.

The purpose of this note is to prove a result relating the cohomology of some Schubert varieties in the affine Grassmannian to the centralizer of a principal nilpotent in the Langlands dual semisimple Lie algebra. This result was communicated to me, as a conjecture, by Dale Peterson in Summer 1997.

Our proof of Peterson’s conjecture is based on a general geometric result about intersection cohomology of algebraic varieties with a $\mathbb{C}^*$-action, which we explain now.

Let $X$ be a smooth complex projective variety with an algebraic $\mathbb{C}^*$-action. Assume $\mathbb{C}^*$ acts on $X$ with isolated fixed points, and write $W$ for the fixed point set. For each fixed point $w \in W$ let $C_w$ be the closure of the cell, $IC(X_w, \mathbb{C})$ the corresponding intersection cohomology complex. We use the "naive" normalization in which the restriction of $IC(X_w, \mathbb{C})$ to $C_w$ is the constant sheaf concentrated in degree 0 (not in degree $-\dim C_w$ as in [BBD]). Thus non-zero cohomology sheaves, $\mathcal{H}^{i}IC(X_w, \mathbb{C})$, may occur only in degrees $0 \leq i \leq \dim C_w$; in particular, we have $\mathcal{H}^{0}IC(X_w, \mathbb{C}) = \mathcal{H}^{0}j_*\mathcal{C}_{C_w}$, is the direct image of the constant sheaf on $C_w$. Using standard truncation functors, $\tau_{\leq j}$, $j \in \mathbb{Z}$, on the derived category, see e.g., [BBD], we may...
Proposition 1.4

Rewrite this isomorphism in the form $\tau_{\leq 0} IC(X_w, \mathbb{C}) = H^0 j_* C_{C_w}$. Therefore, one obtains by adjunction canonical morphisms

$$C_{X_w} \to H^0 j_* C_{C_w} = \tau_{\leq 0} IC(X_w, \mathbb{C}) \to IC(X_w, \mathbb{C}).$$

(1.1)

The composition of these morphisms induces a natural map on hyper-cohomology

$$\varphi : H^\bullet(X_w, \mathbb{C}) = H^\bullet(C_{X_w}) \to H^\bullet(IC(X_w, \mathbb{C})) = H^\bullet(X_w, \mathbb{C}).$$

(1.2)

Our general geometric result is

**Theorem 1.3** Assume that the decomposition $X = \sqcup_{w \in W} C_w$ is a stratification of $X$. Then, for any $w \in W$, the map $\varphi : H^\bullet(X_w, \mathbb{C}) \to IH^\bullet(X_w, \mathbb{C})$ is injective, or equivalently, the dual map to homology $IH^\bullet(X_w, \mathbb{C}) \to H_\bullet(X_w, \mathbb{C})$ is surjective.

We now recall some basic notation concerning loop groups. Let $\mathbb{C}((z))$ be the field of formal Laurent power series, and $\mathbb{C}[[z]] \subset \mathbb{C}((z))$ its ring of integers, that is the ring of formal power series regular at $z = 0$. Fix a complex connected semisimple group $G$ with trivial center, i.e., of adjoint type, and write $G((z))$, resp. $G[[z]]$, for the set of its $\mathbb{C}((z))$-rational, resp. $\mathbb{C}[[z]]$-rational, points. The coset space $Gr := G((z))/G[[z]]$ is called the loop Grassmannian. The space $Gr$ has a natural structure of a direct limit of a sequence of projective varieties of increasing dimension, see e.g. [G2, §1.2] or [Lu]. Furthermore, all orbits of the left $G[[z]]$-action on $Gr$ are finite dimensional. Choosing a maximal torus and a Borel subgroup $T \subset B \subset G$ gives a natural labelling of the $G[[z]]$-orbits in $Gr$ by antidominant coweights $\lambda \in \text{Hom}(\mathbb{C}^*, T)$. We write $O_\lambda$ for the $G[[z]]$-orbit corresponding to an anti-dominant coweight $\lambda$. The closure, $\overline{O}_\lambda \subset Gr$ is known to be a finite dimensional projective variety, singular in general.

Let $\tilde{\mathfrak{g}}$ be the complex semisimple Lie algebra dual to $\text{Lie } G$ in the sense of Langlands. That is, $\tilde{\mathfrak{g}}$ has a Cartan subalgebra $\mathfrak{t}$ identified with $(\text{Lie } T)^*$, the dual of $Lie T$, and the root system of $(\tilde{\mathfrak{g}}, \mathfrak{t})$ is dual to that of $(G, T)$. Thus, the coweight lattice $\text{Hom}(\mathbb{C}^*, T)$ becomes identified canonically with the weight lattice $\mathfrak{t}^* \subset \tilde{\mathfrak{g}}^*$. Fix a principal $\mathfrak{sl}_2$-triple $(h, e, f) \subset \tilde{\mathfrak{g}}$, such that $h \in \mathfrak{t}$ and such that $e \in \tilde{\mathfrak{g}}$ is a principal nilpotent contained in the span of positive root vectors. Then the centralizer of $h$ in $\tilde{\mathfrak{g}}$ equals $\mathfrak{t}$. Further, the centralizer algebra $\tilde{g}^c$ is an abelian Lie subalgebra in $\tilde{\mathfrak{g}}$ whose dimension equals dim $\mathfrak{t}$. The space $\tilde{g}^c$ is stable under the adjoint $h$-action on $\tilde{\mathfrak{g}}$, and the weight decomposition with respect to $ad h$ puts a grading on $\tilde{g}^c$. Because $\tilde{g}^c$ is abelian we identify the enveloping algebra $U(\tilde{g}^c)$ with the symmetric algebra $S(\tilde{g}^c)$, and view it as a graded algebra with the grading induced from that on $\tilde{g}^c$. The following result has been proved in [G2, Proposition 1.7.2] (and independently proved by Peterson in the simply-laced case).

**Proposition 1.4** There is a natural graded algebra isomorphism $\varphi : H^\bullet(Gr, \mathbb{C}) \to U(\tilde{g}^c)$.

Given an anti-dominant weight $\lambda \in \mathfrak{t}_c^*$, let $V_\lambda$ denote the irreducible representation of $\tilde{g}$ with lowest weight $\lambda$. Choose $v_\lambda$, a lowest weight vector, and write $\text{Ann}_{U(\tilde{g}^c)}(v_\lambda)$ for the annihilator of $v_\lambda$ in $U(\tilde{g}^c)$. On the other hand let $i : O_\lambda \hookrightarrow Gr$ be the imbedding of the projective variety $O_\lambda$ labelled by $\lambda$.

With this understood, the result conjectured by Peterson reeds...
Theorem 1.5 The restriction map $i^* : H^\bullet(\Gr, \C) \to H^\bullet(\overline{\Theta}, \C)$ is surjective and induces, via Proposition \[4\], a graded algebra isomorphism

$$H^\bullet(\overline{\Theta}, \C) \simeq U(\mathfrak{g}^e)/\Ann_{U(\mathfrak{g}^e)}(v_\lambda).$$

This generalizes \[G2, Proposition 1.9\] as well as the result \[G2, Proposition 1.8.1\], saying that if $\lambda$ is minuscule then $U(\mathfrak{g}^e) \cdot v_\lambda = V_\lambda$. Our geometric proof of Theorem 1.5 also implies the following

Corollary 1.6 For any vector $v \in V_\lambda$ one has: $\Ann_{U(\mathfrak{g}^e)}(v) \supseteq \Ann_{U(\mathfrak{g}^e)}(v_\lambda).$ \hfill $\square$

2 Proof of Theorem 1.3.

The strategy of the proof follows the pattern of \[G1, \S3].

First of all, we enumerate the strata $\{C_w\}_{w \in W}$ in a convenient way. To that end, write $\xi$ for the vector field on $X$ generating the $S^1$-action on $X$ arising from the $\C^*$-action by restriction to the unit circle. Choose an $S^1$-equivariant Kähler form $\omega$ on $X$, and let $i_\xi \omega$ be the 1-form obtained by contraction. This form is exact since $H^1(X, \C) = 0$, hence, there is a function $f \in C^\infty(X)$ such that $i_\xi \omega = df$. The function $f$ is known to be a Morse function whose critical points are precisely the fixed points of the $\C^*$-action. Moreover, the Bialynicki-Birula decomposition coincides with the cell-decomposition associated to $f$ by Morse theory, see e.g. \[CG, ch.2\]. We enumerate all the fixed points $\{w_1, \ldots, w_N\} = W$ in such a way that $f(w_1) \leq f(w_2) \leq \ldots \leq f(w_N)$, and put $C_n := C_{w_n}$, $n = 1, \ldots, N$.

The sets $X_k := \sqcup_{n \leq k} C_n$ form an increasing filtration of $X$ by closed algebraic subvarieties.

Note that $X_{w_k}$ is an irreducible component of $X_k$, so that $IC(X_{w_k}, \C)$ is a direct summand of $IC(X_k, \C)$. Therefore, we may (and will) replace $X_{w_k}$ by $X_k$ in some arguments below.

In addition to the Bialynicki-Birula decomposition $X = \sqcup_{w \in W} C_w$ considered so far, which is often referred to as the plus-decomposition, one also has the dual minus-decomposition $X = \sqcup_{w \in W} C_w^-$, where

$$C_w^- = \{x \in X \mid \lim_{t \to \infty} t \cdot x = w\}$$

is the repulsing set at $w \in W$. Let $\overline{C_w}$ denote the closure of $C_w^-$, and write $c_n \in H^\bullet(X, \C)$ for the Poincaré dual of the fundamental class of $\overline{C_w}$. Recall (see e.g. \[G1, p.488\]) that the closure $\overline{C_n}$ does not intersect $X_{n-1}$ and meets $X_n$ transversally in a single point, $w_n$. Therefore, $\langle c_n, [C_n] \rangle = 1$, and the classes $\{c_n\}_{n=1,\ldots,N}$ form a basis of $H^\bullet(X, \C)$.

From now on we fix some $w = w_k \in W$ and let $i_w : X_w \hookrightarrow X$ denote the inclusion. It follows that the classes $i_w^* c_n$ such that $w_n \in X_w$ form a basis of $H^\bullet(X_w, \C)$. Abusing the notation we will often write $c_n$ instead of $i_w^* c_n$.

Proving the theorem amounts to showing that, for all $n$ such that $w_n \in X_w$, the classes in $\zeta(c_n) \in H^\bullet(X_w, \C)$ are linearly independent. Assume to the contrary, that there exists a non-trivial linear relation:

$$\sum_{\{n \mid w_n \in X_w\}} \lambda_n \cdot \zeta(c_n) = 0. \quad (2.1)$$
Let $n$ be the minimal index such that $\lambda_n \neq 0$. We keep this choice of $n$ from now on. Put $d = 2\dim_c X_n = 2(\dim_c X - \dim_c C_n)$, so that the cohomology class $c_n \in H^\bullet(X)$ has degree $d$.

We have natural diagrams of inclusions

$$i_n : X_n \hookrightarrow X \quad , \quad X_{n-1} \hookrightarrow X_n \hookrightarrow C_n . \quad (2.2)$$

Writing $\mathcal{L} := IC(X_w)$ for the intersection complex on $X_w$ we get canonical morphisms

$$\mathcal{L} \to (i_n)_* i_n^* \mathcal{L} \to (i_n u)_* (i_n u)^* \mathcal{L} .$$

These sheaf morphisms induce natural maps on hyper-cohomology:

$$IH^\bullet(X_w) = H^\bullet \mathcal{L} \to H^\bullet(i_n^* \mathcal{L}) \to H^\bullet(u^* i_n^* \mathcal{L}).$$

Observe that the cohomology class $c_n \in H^d(X)$ acts by multiplication on each of the hyper-cohomology groups above. We consider the following diagram, see [G1, (3.8a)]:

$$
\begin{array}{cccccc}
H^\bullet(X_w) & \xrightarrow{\sim} & H^\bullet \mathcal{L} & \xrightarrow{i_n^*} & H^\bullet(i_n^* \mathcal{L}) & \xrightarrow{u^*} & H^\bullet(u^* i_n^* \mathcal{L}) \\
\downarrow_{c_n u} & & \downarrow_{c_n u} & & \downarrow_{c_n u} & & \downarrow_{c_n u} \\
H^{\bullet+d}(X_w) & \xrightarrow{\sim} & H^{\bullet+d} \mathcal{L} & \xrightarrow{i_n^*} & H^{\bullet+d}(i_n^* \mathcal{L}) & \xrightarrow{u^*} & H^{\bullet+d}(u^* i_n^* \mathcal{L})
\end{array} \quad (2.3)
$$

The group $H^{\bullet+d}(-)$ at the bottom right corner of the diagram stands for the cohomology with compact support, the rightmost vertical map is essentially the standard Thom isomorphism $H^d(\mathbb{R}^d) \overset{\sim}{\to} H^d_c(\mathbb{R}^d)$, and the maps $u^*$ and $u_!$ are induced by the inclusion $u$ in (2.2). The first two squares in (2.3) clearly commute. Further, for any constructible complex $\mathcal{L}$ on $X_n$ which is constant along the stratification $X_n = \bigcup_{j \leq n} X_j$, the action on $H^\bullet(\mathcal{L})$ of the Poincaré dual of the fundamental class of the submanifold $\varepsilon : C_n \hookrightarrow X$ is given by the composition of the following natural maps (see, e.g., proof of the ‘hard Lefschetz theorem’ in [BBD]):

$$H^\bullet(\mathcal{L}) \xrightarrow{\varepsilon^!} H^\bullet(\varepsilon^* \mathcal{L}) \simeq H^{\bullet+d}(\varepsilon_! \mathcal{L}) \xrightarrow{\varepsilon^!} H^{\bullet+d}(\mathcal{L}).$$

In the case $\mathcal{L} = i_n^* \mathcal{L}$ the composition above amounts to going, in diagram (2.3), along the arrow $u^*$ followed by the rightmost vertical arrow, and finally along the arrow $u_!$. This shows that the right square in (2.3) commutes. Thus, (2.3) is a commutative diagram.

We will make use of the following result, due to Soergel [S, Lemma 19]:

**Lemma 2.4** The map $u_!$ in diagram (2.3) is injective. □

**Remark.** This result was proved in [S] by showing that the hyper-cohomology long exact sequence associated to the distinguished triangle $u_! u^! \mathcal{L} \to \mathcal{L} \to v_* v^* \mathcal{L}$ splits, provided $\mathcal{L}$ is pointwise pure. The pointwise purity of $\mathcal{L}$ (as well as the above Lemma) was verified in [G1, Lemma 3.5] under the additional technical condition [G1, (1.2)]. This additional condition is however not necessary and can be avoided as follows. One first argues that, since $C_n^{-}$ is an algebraic subvariety transverse to all the strata $C_j$, the restriction $\varepsilon^* \mathcal{L}$ is pure. The result then follows by a standard argument as, e.g., in the proof of [G1, Lemma 3.5]. The reason we assumed condition [G1, (1.2)] was that in [G1], in addition to the
injectivity of the map $w$, we also used surjectivity of the map $u^*$ in diagram (2.3). That surjectivity plays no role in the present paper. ◊

We observe first that $1 \in H^0(X_w)$ and we have $\mathcal{X}(c_n) = c_n \cup \mathcal{X}(1)$. Using diagram (2.3) we find

$$i_n^* \mathcal{X}(c_n) = c_n \cup i_n^* \mathcal{X}(1) = w(c_n \cup u^* i_n^* \mathcal{X}(1)).$$

(2.5)

Now, it is immediate from (1.1) that the class $u^* i_n^* \mathcal{X}(1)$ is a generator of the 0-th hyper-cohomology group of the complex $u^* i_n^* \mathcal{C}$, hence non-zero. Therefore, the class $c_n \cup u^* i_n^* \mathcal{X}(1)$ is again non-zero, by the Thom isomorphism. Hence, the RHS of (2.5) is non-zero, due to Lemma 2.4. We conclude that

$$i_n^* \mathcal{X}(c_n) \neq 0 \quad (2.6)$$

To complete the proof of Theorem 1.3 we apply the map $i_n^*$ to the linear relation (2.1). Bearing in mind our choice of $n$ we obtain

$$0 = i_n^* \left( \sum_{\{m \mid w_m \in X_w\}} \lambda_m \cdot \mathcal{X}(c_m) \right) = \sum_{m \geq n} \lambda_m \cdot i_n^* \mathcal{X}(c_m) = \lambda_n \cdot i_n^* \mathcal{X}(c_n),$$

where the last equality is due to the fact that, for any $m > n$, the fundamental class of $\overline{C_m}$ does not intersect $X_n$, whence $i_n^* c_m = 0$. Thus, the equation above yields $\lambda_n \cdot i_n^* \mathcal{X}(c_n) = 0$, and in view of (2.4) we deduce $\lambda_n = 0$. The contradiction completes the proof of the Theorem. ◊

**Question.** We do not know whether Theorem 1.3 is a formal consequence of the main theorem of [G1], in view of the similarity between the proofs of the two theorems. ◊

It often happens in applications that the $\mathbb{C}^*$-action on $X$ can be extended to an algebraic action of a complex torus $T \supset \mathbb{C}^*$. Then each stratum of the Bialynicki-Birula decomposition $X = \sqcup_{w \in W} C_w$ is $T$-stable since the actions of $T$ and $\mathbb{C}^*$ commute. Therefore, for any $w \in W$, we may consider $T$-equivariant cohomology groups $H_T^*(X_w, \mathbb{C})$ and $T$-equivariant intersection cohomology groups $IH_T^*(X_w, \mathbb{C})$, cf. [G2, 8.3].

**Corollary 2.7** If the decomposition $X = \sqcup_{w \in W} C_w$ is a stratification of $X$ then, for any $w \in W$, the natural map $H_T^*(X_w, \mathbb{C}) \to IH_T^*(X_w, \mathbb{C})$ is injective.

**Proof.** It is known that both $H_T^*(X_w, \mathbb{C})$ and $IH_T^*(X_w, \mathbb{C})$ are finitely generated modules over $H_T^*(pt) \simeq \mathbb{C}[\text{Lie } T]$. Hence to prove injectivity it suffices to show that, for any maximal ideal $\mathfrak{m} \subset \mathbb{C}[\text{Lie } T]$, the localized map $H_T^*(X_w, \mathbb{C})_{(\mathfrak{m})} \to IH_T^*(X_w, \mathbb{C})_{(\mathfrak{m})}$ is injective. Any maximal ideal in $\mathbb{C}[\text{Lie } T]$ consists of the polynomials vanishing at a given point $t \in \text{Lie } T$. Therefore, we must show that, for any $t \in \text{Lie } T$, the localized map $H_T^*(X_w, \mathbb{C})_{t} \to IH_T^*(X_w, \mathbb{C})_{t}$ is injective. But the latter map may be replaced, due to the Localization theorem in equivariant cohomology, cf. [G2, Thm.8.6], by a similar map, $H^*(X_w^t, \mathbb{C}) \to IH^*(X_w^t, \mathbb{C})$, between the corresponding non-equivariant cohomology groups of the $t$-fixed point set, $X_w^t$. The result now follows from Theorem 1.3 applied to the $\mathbb{C}^*$-manifold $X^t$. ◊
3 The loop Grassmannian.

We would like to apply Theorem 1.3 to \( X = \text{Gr} \), the loop Grassmannian.

Recall that we have fixed \( T \subset B \subset G \), a maximal torus and a Borel subgroup in \( G \). Define an Iwahori subgroup \( I \subset G[[z]] \) to be formed by all loops \( f \in G[[z]] \) such that \( f(0) \in B \). It is known, see e.g., \([11]\), that \( I \)-orbits form a cell-decomposition of \( \text{Gr} \) that refines the stratification by \( G[[z]] \)-orbits, \( \text{Gr} = \bigcup \lambda \Omega_{\lambda} \). In particular, for any \( \lambda \), the variety \( \Omega_{\lambda} \) is the closure of a single \( I \)-orbit. It is known further that the decomposition of \( \text{Gr} \) into \( I \)-orbits coincides with the Bialynicki-Birula decomposition, \( \text{Gr} = \bigcup \lambda C_{\lambda} \), with respect to an appropriate one-parameter subgroup \( \mathbb{C}^* \subset T \). Thus, we are in the setup of Theorem 1.3 except that the variety \( \text{Gr} \) is neither finite-dimensional, nor smooth.

There is a standard way, see e.g., \([10]\), to go around this difficulty. Specifically, the space \( \text{Gr} \) may be imbedded into a slightly larger infinite dimensional variety \( \tilde{\text{Gr}} \), which is a union of \( G[z^{-1}] \)-orbits of finite codimension. Thus, \( \tilde{\text{Gr}} \) has the structure of a direct limit of infinite-dimensional smooth open subsets, hence may be regarded as a smooth variety (see \([10]\) or \([22]\), §6.1-6.4) for more details about such smooth infinite dimensional varieties). Although the variety \( \text{Gr} \) is by no means compact, there is an explicit minus-decomposition \( \tilde{\text{Gr}} = \bigcup \lambda C_{\lambda} \), cf., \([22]\) 6.4, that enjoys all the properties of the minus decomposition for a \( \mathbb{C}^* \)-action on a smooth projective variety, that were exploited in the proof of Theorem 1.3 above. Therefore the proof of the theorem goes through. We conclude that theorem 1.3 holds for \( \text{Gr} \) in the sense that, for any \( \lambda \), the canonical map \( H^\bullet(\Omega_{\lambda}, \mathbb{C}) \rightarrow IH^\bullet(\Omega_{\lambda}, \mathbb{C}) \) is injective.

We now recall the main result of \([22]\). Let \( P(\text{Gr}) \) be the category of semisimple \( G[[z]] \)-equivariant perverse sheaves on \( \text{Gr} \) with compact support. Also, write \( \text{Rep}(G^*) \) for the category of finite dimensional representations of \( G^* \), the Langlands dual of \( G \). Then we have (see \([22]\), Theorem 1.4.1 and Theorem 1.7.6):

**Theorem 3.1** (i) There is an equivalence of the categories \( P(\text{Gr}) \) and \( \text{Rep}(G^*) \) which sends \( IC(\Omega_{\lambda}, \mathbb{C}) \) to \( V_{\lambda} \).

(ii) For any \( \mathcal{L} \in P(\text{Gr}) \), the hyper-cohomology \( H^\bullet(\mathcal{L}) \) gets identified, under the equivalence, with the underlying vector space of the corresponding representation of \( G^* \).

(iii) Furthermore, for any \( u \in H^\bullet(\text{Gr}, \mathbb{C}) \), the natural action of \( u \) on the hyper-cohomology \( H^\bullet(\mathcal{L}) \) corresponds, via (ii) and the isomorphism \( \varphi : H^\bullet(\text{Gr}, \mathbb{C}) \rightarrow U(\hat{g}^e) \) of Proposition 1.4, to the natural action of \( \varphi(u) \in U(\hat{g}^e) \) in the corresponding \( G^* \)-module. □

**Proof of Theorem 1.5.** Fix \( \lambda \), and let \( IH^\bullet(\Omega_{\lambda}, \mathbb{C}) \rightarrow V_{\lambda} \) be the identification of Theorem 3.1 (ii). According to \([22]\) this map sends the unit, \( 1 \in IH^0(\Omega_{\lambda}, \mathbb{C}) \), to a lowest weight vector \( v_{\lambda} \in V_{\lambda} \). Therefore, Theorem 3.1 implies that the map \( \varphi \) of 3.1 (iii) induces a graded algebra isomorphism:

\[
\varphi : H^\bullet(\text{Gr}, \mathbb{C})/\text{Ann}_{H^\bullet(\text{Gr}, \mathbb{C})}(\pi(1)) \simeq U(\hat{g}^e)/\text{Ann}_{U(\hat{g}^e)}(v_{\lambda}) .
\]

(3.2)

Let \( i : \Omega_{\lambda} \hookrightarrow \text{Gr} \) denote the imbedding. Note that the \( H^\bullet(\text{Gr}, \mathbb{C}) \)-action on \( IH^\bullet(\Omega_{\lambda}, \mathbb{C}) \) factors through the restriction map \( i^* : H^\bullet(\text{Gr}, \mathbb{C}) \rightarrow H^\bullet(\Omega_{\lambda}, \mathbb{C}) \). The restriction map is
surjective since the dual map on homology $i_* : H_*(\mathfrak{g}, \mathbb{C}) \to H_*(\mathfrak{g}, \mathbb{C})$ is injective (because both spaces have compatible cell-decompositions by $I$-orbits of even real dimension). Combining the surjectivity observation above with (3.2) we obtain the following chain of algebra isomorphisms:

$$H^\bullet(\mathfrak{g}, \mathbb{C}) / \text{Ann} H^\bullet(\mathfrak{g}, \mathbb{C})(\kappa(1)) \simeq U(\mathfrak{g}) / \text{Ann} U(\mathfrak{g})(\nu_{\lambda}).$$

But $\text{Ann} H^\bullet(\mathfrak{g}, \mathbb{C})(\kappa(1)) = 0$ because of Theorem 1.3 applied to the variety $\mathfrak{g}$. The isomorphism of Theorem 1.5 follows. □

4 "Topological" proof of Kostant’s theorem.

Given a complex connected semisimple group $G$ with Lie algebra $\mathfrak{g}$ write $\mathbb{C}[\mathfrak{g}]^G \subset \mathbb{C}[\mathfrak{g}]$ for the subring of $adG$-invariant polynomials on $\mathfrak{g}$. In [Ko], B. Kostant established the following fundamental result

Theorem 4.1 There is a graded $G$-stable subspace $H \subset \mathbb{C}[\mathfrak{g}]$ such that the multiplication in $\mathbb{C}[\mathfrak{g}]$ gives rise to a vector space isomorphism

$$\mathbb{C}[\mathfrak{g}]^G \otimes_c H \xrightarrow{\text{mult}} \mathbb{C}[\mathfrak{g}].$$

We are going to show that this theorem may be viewed as a manifestation of “purity” for the equivariant intersection cohomology, $IH^\bullet(\mathfrak{g}, \mathbb{C})$. Recall first, that an element $x \in \mathfrak{g}$ is called regular if $\mathfrak{g}^x$, the centralizer of $x$ in $\mathfrak{g}$, has the minimal possible dimension, $rk \mathfrak{g}$. Given $x \in \mathfrak{g}$ and a finite dimensional rational $G$-module $V$, write $V^{\mathfrak{g}^x}$ for the subspace in $V$ annihilated by the subalgebra $\mathfrak{g}^x$.

Proposition 4.2 For any finite dimensional rational $G$-module $V$ whose weights are contained in the root lattice of $G$, the function: $x \mapsto \dim_c V^{\mathfrak{g}^x}$ is constant on the set of regular elements of $\mathfrak{g}$.

Remark. Note that for $V = \mathfrak{g}$, the adjoint representation, the Proposition amounts to the definition of a regular point. ♦

Of course, Proposition 4.2 follows from Theorem 1.1, by a well-known argument due to Kostant [Ko]. Our main observation is that the results of [G1] and [G2] combined together yield an alternative “topological” proof of Proposition 4.2 independent of Theorem 1.1.

Proof of Proposition 4.2: The natural projection $\pi : \mathfrak{g} = \text{Spec} \mathbb{C}[\mathfrak{g}] \rightarrow \text{Spec} \mathbb{C}[\mathfrak{g}]^G$ sets up a bijection between regular adgoint G-orbits in $\mathfrak{g}$ and (closed) points of the scheme $\text{Spec} \mathbb{C}[\mathfrak{g}]^G \simeq t/W$. Fix a representation $V$, as in the proposition. The function $\delta_V : x \mapsto \dim_c V^{\mathfrak{g}^x}$ is clearly constant on each $G$-orbit, hence, when restricted to regular elements, it may (and will) be viewed as a function on $\text{Spec} \mathbb{C}[\mathfrak{g}]^G$. By semicontinuity, the value of this function at a special (regular) orbit can not be less than its value at the generic orbit. Observe further that there is a $\mathbb{C}^*$-action on $\mathfrak{g}$ by homotheties, preserving the set of regular elements. It induces a natural $\mathbb{C}^*$-action on $\text{Spec} \mathbb{C}[\mathfrak{g}]^G$ with the origin, $m_0$, being the unique attracting fixed point. Thus the point $m_0$ is the "most special" point in
\(Spec \mathbb{C}[g]^G\) in the sense that, for any other point \(m \in Spec \mathbb{C}[g]^G\), we have \(\delta_v(m) \leq \delta_v(m_0)\). Thus, it suffices to show that the value of the function \(\delta_v\) at \(m_0\) equals its generic value. Note that, since the centralizer of a generic element is a Cartan subalgebra, the generic value of \(\delta_v\) is equal to \(dim V(0)\), the zero-weight multiplicity in \(V\). Thus, we must prove that, if \(x\) is a regular nilpotent, then \(dim V^x = dim V(0)\).

To this end, we may replace in all the arguments the Lie algebra \(g\) by \(\check{g}\), its Langlands dual. Thus we let \(V_\lambda\) be a simple finite-dimensional \(G^\vee\)-module with lowest weight \(\lambda\), let \(V_{triv}\) be the trivial \(G^\vee\)-module, and write \(e\) for a regular nilpotent in \(\check{g}\). We have \(dim V_\lambda^e = dim Hom_{\mathfrak{g}}^e(V_{triv}, V_\lambda)\).

The space \(Hom_{\mathfrak{g}}^e(V_{triv}, V_\lambda)\) may be expressed in terms of the geometry of the loop Grassmannian \(Gr\). Specifically, Theorem 1.10.3 of [G2] (whose proof depends on [G1] in an essential way) gives an isomorphism of vector spaces

\[
\text{Ext}^\bullet_{D^b(\mathfrak{g})}(IC_{triv}, IC(\overline{\mathcal{O}}_\lambda)) \cong Hom_{\mathfrak{g}}^e(V_{triv}, V_\lambda),
\]

where \(IC_{triv}\) is the skyscraper sheaf on the one-point orbit \(i_{triv} : \mathcal{O}_{triv} \rightarrow Gr\) corresponding to the trivial representation. The LHS of (4.3) equals, by adjunction, \(\check{\text{IC}}(\overline{\mathcal{O}}_\lambda)^\vee\). Thus, proving the theorem amounts to showing that, for any anti-dominant \(\lambda\) in the root lattice, one has

\[
dim H^\bullet i_{triv}^! IC(\overline{\mathcal{O}}_\lambda) = dim V_\lambda(0)
\]

We interpret the last equation in terms of equivariant cohomology as follows. Let \(T \subset G\) be the maximal torus whose fixed points in \(V\) form the subspace \(V(0)\). The torus \(T\) acts naturally on \(Gr\) preserving all the strata \(\mathcal{O}_\lambda\). It follows that both \(IC(\overline{\mathcal{O}}_\lambda)\) and \(i_{triv}^! IC(\overline{\mathcal{O}}_\lambda)\) are \(T\)-equivariant complexes, see e.g., [G2, 8.3]. We may therefore consider the \(T\)-equivariant hyper-cohomology, \(H^T_\bullet(i_{triv}^! IC(\overline{\mathcal{O}}_\lambda))\), which is a module over \(H^\bullet_T(pt) = \mathbb{C}[t]\). But the complex \(i_{triv}^! IC(\overline{\mathcal{O}}_\lambda)\) is pure, by [G1, Lemma 3.5]. Hence the \(\mathbb{C}[t]\)-module \(H^\bullet_T(i_{triv}^! IC(\overline{\mathcal{O}}_\lambda))\) is free, by [G2, Theorem 8.4.1]. Moreover, the geometric fiber of this free module at the origin \(0 \in t\) is isomorphic to \(H^\bullet_T(i_{triv}^! IC(\overline{\mathcal{O}}_\lambda))\), the non-equivariant cohomology, by [G2, Corollary 8.4.2]. On the other hand, by the geometric construction of a fiber functor on \(P(\mathfrak{g})\) given in [G2, 3.9-3.10], the fixed point decomposition \([G2, 3.6]\) on the equivariant intersection cohomology corresponds, via Theorem 3.1, to the weight decomposition on \(V_\lambda\). Thus, the geometric fiber of \(H^\bullet_T(i_{triv}^! IC(\overline{\mathcal{O}}_\lambda))\) at a general point in \(t\) is precisely the zero-weight subspace, \(V(0)\). Since all fibers of a free module have the same dimension, we conclude that \(dim H^\bullet i_{triv}^! IC(\overline{\mathcal{O}}_\lambda) = dim V_\lambda(0)\), and Proposition 4.2 follows.

\textbf{Proposition 4.2 implies Theorem 4.1:} Given a simple finite dimensional rational \(G\) module \(V\), let \(\mathbb{C}[g]^V = Hom_G(V, \mathbb{C}[g])\) denote the \(V\)-isotypic component of \(\mathbb{C}[g]\). The \(G\)-action on \(\mathbb{C}[g]^V\) being locally finite, one has a \(G\)-stable direct sum decomposition:

\[
\mathbb{C}[g] = \bigoplus_{\text{simple } G\text{-modules } V} V \otimes_{\mathbb{C}} \mathbb{C}[g]^V.
\]

Clearly, for \(V = V_{triv}\), we have \(\mathbb{C}[g]^{V_{triv}} = \mathbb{C}[g]^G\), and for an arbitrary \(V\), \(\mathbb{C}[g]^V\) is a graded \(\mathbb{C}[g]^G\)-module. By the direct sum decomposition above, proving Theorem 4.1 amounts to showing that, for any \(V\), the \(\mathbb{C}[g]^G\)-module \(\mathbb{C}[g]^V\) is free.
Thus we may fix a simple $G$-module $V$ whose weights belong to the root lattice (otherwise $V$ does not occur in $\mathbb{C}[\mathfrak{g}]$), and concentrate our attention on the $\mathbb{C}[\mathfrak{g}]^G$-module $\mathbb{C}[\mathfrak{g}]^V$. The latter is finitely generated, by Hilbert’s classical result on finite generation of $G$-invariants, see e.g., [Ko]. Further, $\mathbb{C}[\mathfrak{g}]^V$ is clearly a graded $\mathbb{C}[\mathfrak{g}]^G$-module. But a finitely generated graded $\mathbb{C}[\mathfrak{g}]^G$-module is free if and only if it is projective. To show $\mathbb{C}[\mathfrak{g}]^V$ is projective we argue as follows.

View $\mathbb{C}[\mathfrak{g}]^V$ as a coherent sheaf on $\text{Spec} \mathbb{C}[\mathfrak{g}]^G$. Let $\mathbb{C}[\mathfrak{g}]^V/m \cdot \mathbb{C}[\mathfrak{g}]^V$ be the geometric fiber of this sheaf at a closed point $m \in \text{Spec} \mathbb{C}[\mathfrak{g}]^G$, regarded as a maximal ideal in $\mathbb{C}[\mathfrak{g}]^G$. It is known that $\mathbb{C}[\mathfrak{g}]^V$ is a projective $\mathbb{C}[\mathfrak{g}]^G$-module if and only if the function $d_v : m \mapsto \dim_c (\mathbb{C}[\mathfrak{g}]^V/m \cdot \mathbb{C}[\mathfrak{g}]^V)$ is constant on the set of closed points of $\text{Spec} \mathbb{C}[\mathfrak{g}]^G$. It suffices to show, due to a semi-continuity argument similar to the one used in the proof of Proposition 4.2, that the value of the function $d_v$ at $m_o$, the origin of $\text{Spec} \mathbb{C}[\mathfrak{g}]^G$, equals its generic value.

To this end, consider the natural projection $\pi : \mathfrak{g} = \text{Spec} \mathbb{C}[\mathfrak{g}] \to \text{Spec} \mathbb{C}[\mathfrak{g}]^G$. If $m \in \text{Spec} \mathbb{C}[\mathfrak{g}]^G$ is in general position, then $\pi^{-1}(m)$ is the single adjoint $G$-orbit through a semisimple regular element $h \in \mathfrak{g}$. This orbit is isomorphic, as a $G$-variety, to $G/G^h$, where $G^h$ denotes the centralizer of $h$ in $G$. Therefore, a standard argument involving Frobenius reciprocity, see [Ko] or [CG, §6.7], yields

$$\dim_c (\mathbb{C}[\mathfrak{g}]^V/m \cdot \mathbb{C}[\mathfrak{g}]^V) = \dim_c \mathbb{C}[G/G^h]^V = \dim_c V^{\mathfrak{g}^h}. \quad (4.5)$$

On the other hand, we have set-theoretically: $\pi^{-1}(m_o) = \text{nilpotent variety of } \mathfrak{g}$, see e.g. [CG] ch.3. The nilpotent variety contains a unique open dense $G$-orbit, $N^{\mathfrak{g}^o}$, formed by regular nilpotents. Moreover, the scheme $\pi^{-1}(m_o)$ is reduced at any point of $N^{\mathfrak{g}^o}$, see Lemma 4.6 below and [Ko]. Hence, the scheme imbedding $N^{\mathfrak{g}^o} \hookrightarrow \pi^{-1}(m_o)$ induces an injection: $\mathbb{C}[\mathfrak{g}]^V/m_o \cdot \mathbb{C}[\mathfrak{g}]^V \hookrightarrow \mathbb{C}[N^{\mathfrak{g}^o}]^V$. Thus, we have $\dim (LHS) \leq \dim (RHS)$. Choose a regular nilpotent $e \in N^{\mathfrak{g}^o}$. Then $N^{\mathfrak{g}^o} \simeq G/G^e$, and the Frobenius reciprocity argument mentioned above yields:

$$\dim \mathbb{C}[N^{\mathfrak{g}^o}]^V = \dim \mathbb{C}[G/G^e]^V = \dim V^{\mathfrak{g}^e}.$$ Combining this formula with equation (4.3) and using Proposition 4.2 we obtain

$$\dim (\mathbb{C}[\mathfrak{g}]^V/m_o \cdot \mathbb{C}[\mathfrak{g}]^V) \leq \dim \mathbb{C}[N^{\mathfrak{g}^o}]^V = \dim V^{\mathfrak{g}^e} = \dim V^{\mathfrak{g}^h}.$$ Thus the function $d_v$ is constant, and Theorem 4.1 is proved. $\square$

In the course of our proof of Theorem 4.1 we have used the following result, due to Kostant.

**Lemma 4.6** The zero fiber, $\pi^{-1}(m_o)$, is reduced at any point of $N^{\mathfrak{g}^o}$.

Kostant proved this result by showing that the generators of $\mathbb{C}[\mathfrak{g}]^G$ have linearly independent differentials at any point of $N^{\mathfrak{g}^o}$. The latter has been verified in [Ko2] by a direct computation (see [CG, §6.7] for a slightly different argument). We give an alternative proof of the Lemma, inspired by [BL], which involves no computation and is independent of [Ko2].
**Proof of Lemma:** Set $\tilde{g} = \{ (x, b) \mid x \in b, b = \text{Borel subalgebra in } g \}$. We have the following commutative diagram, see [CG, §3.2]:

\[
\begin{array}{ccc}
\tilde{g} & \xrightarrow{\mu} & g \\
\downarrow{\nu} & & \downarrow{\rho} \\
\rho / W & \xrightarrow{\pi} & t
\end{array}
\]  

(4.7)

In this diagram, the map $\mu$ is proper, $\nu$ is a smooth morphism, and $\pi$ is a finite flat morphism, since $\mathbb{C}[t]$ is free over $\mathbb{C}[t]^W$.

Observe that diagram (4.7) induces a morphism $\psi: \tilde{g} \to g \times_{t/W} t$. Let $g^{\text{reg}} \subset g$ be the Zariski open subset of regular (not necessarily semisimple) elements, and $\tilde{g}^{\text{reg}} := \mu^{-1}(g^{\text{reg}})$.

We claim that the morphism $\psi$ gives an isomorphism:

\[
\psi: \tilde{g}^{\text{reg}} \xrightarrow{\sim} g^{\text{reg}} \times_{t/W} t. 
\]

To prove the claim, note first that since $t$ is finite and flat over $t/W$, the scheme $g^{\text{reg}} \times_{t/W} t$ is finite and flat over $g^{\text{reg}}$. Further, the map $\mu: \tilde{g}^{\text{reg}} \to g^{\text{reg}}$ is proper and has finite fibers, hence this is a finite morphism. Moreover, being a dominant morphism between smooth schemes of the same dimension, this morphism is flat. Thus, both the source and target schemes in (4.8) are finite flat schemes over $g^{\text{reg}}$, a smooth variety. Therefore, both schemes are Cohen-Macaulay (see [BL] or [CG, §2.2]), hence, to show that the map $\psi$ in (4.8) is an isomorphism, it suffices to verify that it is an isomorphism outside a codimension two subvariety. Let $g' \subset g$ be the set of all elements whose semisimple part is either regular, or belongs to at most one root hyperplane in a Cartan subalgebra. Then $\text{codim}(g \setminus g') \geq 2$. On the other hand, proving isomorphism (4.8) for $g'$ amounts, effectively, to an $\mathfrak{s}\mathfrak{l}_2$-computation, which is left to the reader. This proves (4.8).

Now, using (4.8), we may identify the smooth morphism $\nu: \tilde{g}^{\text{reg}} \to t$ with the projection $g^{\text{reg}} \times_{t/W} t \to t$. Applying to this projection the base change with respect to the flat map $t \to t/W$ in diagram (4.7), we deduce that the morphism $\rho: g^{\text{reg}} \to t/W$ is also smooth. Hence its zero-fiber, $N^{\text{reg}}$, is reduced, and the Lemma follows. □

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Department of Mathematics, University of Chicago, Chicago IL 60637, USA; ginzburg@math.uchicago.edu