THE LIFESPAN OF SMALL SOLUTIONS TO CUBIC DERIVATIVE NONLINEAR SCHRÖDINGER EQUATIONS IN ONE SPACE DIMENSION

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Abstract. Consider the initial value problem for cubic derivative nonlinear Schrödinger equations in one space dimension. We provide a detailed lower bound estimate for the lifespan of the solution, which can be computed explicitly from the initial data and the nonlinear term. This is an extension and a refinement of the previous work by one of the authors [H. Sunagawa: Osaka J. Math. 43 (2006), 771–789], in which the gauge-invariant nonlinearity was treated.

1. Introduction and the main result. This paper is concerned with the lifespan of solutions to cubic derivative nonlinear Schrödinger equations in one space dimension with small initial data:

\[
\begin{cases}
  i\partial_t u + \frac{1}{2} \partial_x^2 u = N(u, \partial_x u), & t > 0, \ x \in \mathbb{R}, \\
  u(0, x) = \varepsilon \varphi(x), & x \in \mathbb{R},
\end{cases}
\]

where \(i = \sqrt{-1}\), \(u = u(t, x)\) is a \(\mathbb{C}\)-valued unknown function, \(\varepsilon > 0\) is a small parameter which is responsible for the size of the initial data, and \(\varphi\) is a prescribed \(\mathbb{C}\)-valued function which belongs to \(H^3 \cap H^{2,1}(\mathbb{R})\). Here and later on as well, \(H^s\) denotes the standard \(L^2\)-based Sobolev space of order \(s\), and the weighted Sobolev space \(H^{s,\sigma}\) is defined by \(\{\phi \in L^2 \mid \langle \cdot \rangle^\sigma \phi \in H^s\}\), equipped with the norm \(\|\phi\|_{H^{s,\sigma}} = \|\langle \cdot \rangle^\sigma \phi\|_{H^s}\), where \(\langle x \rangle = \sqrt{1 + x^2}\). Throughout this paper, the nonlinear term \(N(u, \partial_x u)\) is always assumed to be a cubic homogeneous polynomial in \((u, \partial_x u, \partial_x^2 u)\) with complex coefficients. We will often write \(u_x\) for \(\partial_x u\).

From the perturbative point of view, cubic nonlinear Schrödinger equations in one space dimension are of special interest because the best possible decay in \(L^2_x\) of general cubic nonlinear terms is \(O(t^{-1})\), so the cubic nonlinearity must be regarded as a long-range perturbation. In general, standard perturbative approach is valid only for \(t \lesssim \exp(o(\varepsilon^{-2}))\), and our problem is to make clear how the nonlinearity affects the behavior of the solutions for \(t \gtrsim \exp(O(\varepsilon^{-2}))\). Let us recall some known
results briefly. The most well-studied case is the gauge-invariant case, that is the case where \( N \) satisfies
\[
N(e^{i\theta} z, e^{i\theta} \zeta) = e^{i\theta} N(z, \zeta), \quad (z, \zeta) \in \mathbb{C} \times \mathbb{C}, \quad \theta \in \mathbb{R}.
\] (1.2)

There are a lot of works devoted to large-time behavior of the solution to (1.1) under (1.2) (see e.g., [40], [25], [35], [7], [8], [18], [36], [30], [17] and the references cited therein). On the other hand, if (1.2) is violated, the situation becomes delicate due to the appearance of oscillation structure. It is pointed out in [10] (see also [9], [39], [33]) that contribution of non-gauge-invariant terms may be regarded as a short-range perturbation if at least one derivative of \( u \) is included, whereas, as studied in [11], [12], [13], [14], [34], [15] etc., it turns out that contribution of non-gauge-invariant cubic terms without derivative is quite difficult to handle. In what follows, let us assume that \( N \) satisfies
\[
N(e^{i\theta}, 0) = e^{i\theta} N(1, 0), \quad \theta \in \mathbb{R},
\] (1.3)

to exclude the worst terms \( u^3, \pi^2 u \) and \( \pi^3 \) (see the appendix for explicit representation of \( N \) satisfying (1.3)). We also define \( \nu : \mathbb{R} \to \mathbb{C} \) by
\[
\nu(\xi) := \frac{1}{2\pi i} \oint_{|z|=1} N(z, i\xi z) \frac{dz}{z^2}, \quad \xi \in \mathbb{R}.
\]

Roughly speaking, this contour integral extracts the contribution of the gauge-invariant part in \( N \). Remark that \( \nu(\xi) \) coincides with \( N(1, i\xi) \) in the gauge-invariant case (see also (A.7) below). Typical previous results on global existence and large-time asymptotic behavior of solutions to (1.1) under (1.3) can be summarized in terms of \( \nu(\xi) \) as follows:

(i) If \( \text{Im} \nu(\xi) \leq 0 \) for all \( \xi \in \mathbb{R} \), then the solution exists globally in \( C([0, \infty); H^3 \cap H^{2,1}(\mathbb{R})) \) for sufficiently small \( \varepsilon \). Moreover the solution satisfies
\[
\|u(t)\|_{L^\infty} \leq \frac{C\varepsilon}{\sqrt{1 + t}}, \quad t \geq 0,
\]
where the constant \( C \) is independent of \( \varepsilon \) ([10], [17], etc.).

(ii) If \( \text{Im} \nu(\xi) = 0 \) for all \( \xi \in \mathbb{R} \), then the solution has a logarithmic oscillating factor in the asymptotic profile, i.e., it holds that
\[
u(t, x) = \frac{1}{\sqrt{t}} \alpha(x/t) \exp \left( i \frac{x^2}{2t} - i\alpha(x/t)^2 \text{Re} \nu(x/t) \log t \right) + o(t^{-1/2})
\]
as \( t \to +\infty \) uniformly in \( x \in \mathbb{R} \), where \( \alpha(\xi) \) is a suitable \( \mathbb{C} \)-valued function of \( \xi \in \mathbb{R} \) satisfying \( |\alpha(\xi)| \lesssim \varepsilon \). In particular, the solution is asymptotically free if and only if \( \nu(\xi) \) vanishes identically on \( \mathbb{R} \) ([40], [25], [7], [8], [18], [10], etc.).

(iii) If \( \sup_{\xi \in \mathbb{R}} \text{Im} \nu(\xi) < 0 \), then the solution gains an additional logarithmic time-decay:
\[
\|u(t)\|_{L^\infty} \leq \frac{C'\varepsilon}{\sqrt{(1 + t)(1 + \varepsilon^2 \log(2 + t))}}, \quad t \geq 0,
\]
where the constant \( C' \) is independent of \( \varepsilon \) ([36], [17], [28], etc.).

Now, let us turn our attentions to the remaining case: \( \text{Im} \nu(\xi_0) > 0 \) for some \( \xi_0 \in \mathbb{R} \). To the authors’ knowledge, there is no global existence result in that case, and many interesting problems are left unsolved especially when we focus on the issue of small data blow-up. In the previous paper [38], lower bounds for the lifespan...
$T_c$ of the solution to (1.1) are considered in detail under the assumption (1.2). It is proved in [38] that

$$
\liminf_{\varepsilon \to +0} \varepsilon^2 \log T_c \geq \frac{1}{2 \sup_{|\xi| \in \mathbb{R}} \left( |\hat{\varphi}(\xi)|^2 \, \text{Im} \, N(1, i\xi) \right)} =: \tau_0,
$$

where $\hat{\varphi}$ denotes the Fourier transform of $\varphi$, i.e.,

$$
\hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iy\xi} \varphi(y) \, dy, \quad \xi \in \mathbb{R},
$$

by constructing an approximate solution $u_a$ which blows up at the time $t = \exp(\tau_0/\varepsilon^2)$ and getting an a priori estimate not for the solution $u$ itself but for the difference $u - u_a$. What is important in (1.4) is that this is quite analogous to the famous results due to John [23] and Hörmander [20] which concern quasilinear wave equations in three space dimensions (see [21], [32], [4] for analogous results on the Klein-Gordon case, and also [31], [24], [5], [30], [37], [41], [22], etc. for related works of them). Remember that the detailed lifespan estimates obtained in [23] and [20] are fairly sharp and have close connection with the so-called null condition introduced by Klainerman [27] and Christodoulou [3]. However, the approach exploited in [38] has the following two drawbacks:

- it heavily relies on the gauge-invariance (1.2),
- it requires higher regularity and faster decay as $|x| \to \infty$ for $\varphi$ than those for $u(t, \cdot)$.

The purpose of this paper is to improve these two points. To state the main result, let us define $\tilde{\tau}_0 \in (0, +\infty]$ by

$$
\frac{1}{\tilde{\tau}_0} = 2 \sup_{|\xi| \in \mathbb{R}} \left( |\hat{\varphi}(\xi)|^2 \, \text{Im} \, \nu(\xi) \right),
$$

where we associate $1/\tilde{\tau}_0 = 0$ with $\tilde{\tau}_0 = +\infty$. Remark that the right-hand side of (1.5) is always non-negative if $\varphi \in H^{2,1}(\mathbb{R})$, because $\text{Im} \, \nu(\xi) = O(|\xi|^3)$ and $|\hat{\varphi}(\xi)|^2 = O(|\xi|^{-2})$ as $|\xi| \to \infty$. In particular, we can easily check that $\tilde{\tau}_0 = +\infty$ if $\text{Im} \, \nu(\xi) \leq 0$ for all $\xi \in \mathbb{R}$. We note also that $\tilde{\tau}_0$ coincides with $\tau_0$ if (1.2) is satisfied.

The main result of this paper is as follows:

**Theorem 1.1.** Assume that $\varphi \in H^3 \cap H^{2,1}(\mathbb{R})$. Suppose that the nonlinear term $N$ satisfies (1.3). Let $T_c$ be the supremum of $T > 0$ such that (1.1) admits a unique solution in $C([0, T); H^3 \cap H^{2,1}(\mathbb{R}))$. Then we have

$$
\liminf_{\varepsilon \to +0} \varepsilon^2 \log T_c \geq \tilde{\tau}_0,
$$

where $\tilde{\tau}_0 \in (0, +\infty]$ is given by (1.5).

**Remark 1.1.** The above theorem concerns only the forward Cauchy problem (i.e., for $t > 0$). For the backward Cauchy problem, we can obtain a similar lower bound $\tilde{\tau}_0'$ which can be written explicitly in terms of $N$ and $\varphi$. Indeed, if $u(t, x)$ is a solution to (1.1) for $t < 0$, then $u(-t, x)$ is also a solution to the Schrödinger equation with another cubic derivative nonlinearity for $t > 0$. However, it should be noted that $\tilde{\tau}_0'$ does not coincide with $\tilde{\tau}_0$ in general. For example, when $N = -i|u|^2 u$ and $\varphi \neq 0$, we can check that $\tilde{\tau}_0 = +\infty$ and $\tilde{\tau}_0' < +\infty$, whence the small data global existence is expected only for the positive time direction. On the other hand, if $\text{Im} \, \nu(\xi) = 0$ for all $\xi \in \mathbb{R}$, then we have $\tilde{\tau}_0 = \tilde{\tau}_0' = +\infty$. In fact, the solution exists globally in both time directions in that case.
We close this section with the contents of this paper: Section 2 is devoted to a lemma on some ordinary differential equation. In Section 3, we recall basic properties of the operators $\mathcal{J}$ and $\mathcal{Z}$, as well as the smoothing property of the linear Schrödinger equations. After that, we will get an a priori estimate in Section 4, and the main theorem will be proved in Section 5. The proof of technical lemmas will be given in the appendix.

2. A lemma on ODE. In this section we introduce a lemma on some ordinary differential equation, keeping in mind an application to (4.11) below.

Let $\kappa, \theta_0 : \mathbb{R} \to \mathbb{C}$ be continuous functions satisfying
\[
\sup_{\xi \in \mathbb{R}} |\kappa(\xi)| < \infty, \quad \sup_{\xi \in \mathbb{R}} |\theta_0(\xi)| < \infty, \quad \sup_{\xi \in \mathbb{R}} (|\theta_0(\xi)|^2 \text{Im} \kappa(\xi)) \geq 0.
\]

We set $C_1 = \sup_{\xi \in \mathbb{R}} |\kappa(\xi)|$ and define $\tau_1 \in (0, +\infty]$ by
\[
\tau_1 = \frac{1}{2 \sup_{\xi \in \mathbb{R}} (|\theta_0(\xi)|^2 \text{Im} \kappa(\xi))},
\]
where $1/0$ is understood as $+\infty$. Let $\beta_0(t, \xi)$ be a solution to
\[
\left\{
\begin{array}{l}
  i\partial_t \beta_0(t, \xi) = \frac{\kappa(\xi)}{t} |\beta_0(t, \xi)|^2 \beta_0(t, \xi), \\
  \beta_0(1, \xi) = \varepsilon \theta_0(\xi),
\end{array}
\right.
\]  

(2.1)

where $\varepsilon > 0$ is a parameter. Then it is easy to see that
\[
|\beta_0(t, \xi)|^2 = \frac{\varepsilon^2 |\theta_0(\xi)|^2}{1 - 2\varepsilon^2 |\theta_0(\xi)|^2 \text{Im} \kappa(\xi) \log t}
\]
as long as the denominator is strictly positive. In view of this expression, we can see that
\[
\sup_{(t, \xi) \in [1, \varepsilon^{1/2}] \times \mathbb{R}} |\beta_0(t, \xi)| \leq C_2 \varepsilon
\]  

(2.2)

for $\sigma \in (0, \tau_1)$, where
\[
C_2 = \frac{1}{\sqrt{1 - \sigma/\tau_1}} \sup_{\xi \in \mathbb{R}} |\theta_0(\xi)| (\in (0, \infty)),
\]

while
\[
\sup_{\xi \in \mathbb{R}} |\beta_0(t, \xi)| \to +\infty \quad \text{as} \quad t \to \exp(\tau_1/\varepsilon^2) - 0
\]
if $\tau_1 < \infty$.

Next we consider a perturbation of (2.1). For this purpose, let $T > 1$ and let $\theta_1 : \mathbb{R} \to \mathbb{C}$, $\rho : [1, T) \times \mathbb{R} \to \mathbb{C}$ be continuous functions satisfying
\[
\sup_{\xi \in \mathbb{R}} |\theta_1(\xi)| \leq C_3 \varepsilon^{1+\delta}, \quad \sup_{(t, \xi) \in [1, T) \times \mathbb{R}} t^{1+\mu} |\rho(t, \xi)| \leq C_4 \varepsilon^{1+\delta}
\]
with some $C_3, C_4, \delta, \mu > 0$. Let $\beta : [1, T) \times \mathbb{R} \to \mathbb{C}$ be a smooth function satisfying
\[
\left\{
\begin{array}{l}
  i\partial_t \beta(t, \xi) = \frac{\kappa(\xi)}{t} |\beta(t, \xi)|^2 \beta(t, \xi) + \rho(t, \xi), \quad (t, \xi) \in (1, T) \times \mathbb{R}, \\
  \beta(1, \xi) = \varepsilon \theta_0(\xi) + \theta_1(\xi).
\end{array}
\right.
\]

The following lemma asserts that an estimate similar to (2.2) remains valid if (2.1) is perturbed by $\rho$ and $\theta_1$:
Lemma 2.1. Let \( \sigma \in (0, \tau_1) \). We set \( T_* = \min \{ T, e^{\sigma/\varepsilon^2} \} \). For \( \varepsilon \in (0, M^{-1/\delta}) \), we have
\[
\sup_{(t, \xi) \in [1, T_*) \times \mathbb{R}} |\beta(t, \xi)| \leq C_2 \varepsilon + M \varepsilon^{1+\delta},
\]
where
\[
M = \left( 2C_3 + \frac{C_4}{\mu} \right) e^{C_1(1+3C_2+3C_2^2)\sigma}.
\]
Proof. We put \( w(t, \xi) = \beta(t, \xi) - \beta_0(t, \xi) \) and
\[
T_* = \sup \left\{ \tilde{T} \in [1, T_*] \mid \sup_{(t, \xi) \in [1, \tilde{T}] \times \mathbb{R}} |w(t, \xi)| \leq M \varepsilon^{1+\delta} \right\}.
\]
Note that \( T_* > 1 \), because of the estimate
\[
|w(1, \xi)| = |\theta_1(\xi)| \leq C_3 \varepsilon^{1+\delta} \leq \frac{M}{2} \varepsilon^{1+\delta}
\]
and the continuity of \( w \). Since \( w \) satisfies
\[
i \Delta_t w = \frac{\kappa(\xi)}{t} \left( |w + \beta_0|^2 (w + \beta_0) - |\beta_0|^2 \beta_0 \right) + \rho,
\]
we see that
\[
\partial_t |w|^2 = 2 \text{Im} \left( \nabla \cdot \tilde{w} \right)
\leq \frac{2}{t} \left( \tilde{C} \left( M^2 \varepsilon^{2+2\delta} + 3C_2 M \varepsilon^{2+\delta} + 3C_2^2 \varepsilon^2 \right) |w|^2 + |w||\rho| \right)
\leq \frac{2}{t} \tilde{C} \varepsilon^2 |w|^2 + \frac{C_4 \varepsilon^{1+\delta}}{t^{1+\mu}} |w|
\]
for \( t \in [1, T_*] \), where \( \tilde{C} = C_1 (1 + 3C_2 + 3C_2^2) \). By the Gronwall-type argument, we obtain
\[
|w(t, \xi)| \leq \left( |\theta_1(\xi)| + \int_1^\infty \frac{C_4 \varepsilon^{1+\delta}}{2s^{1+\mu} + \tilde{C} \varepsilon^2} ds \right) e^{\tilde{C} \varepsilon^2 \log t}
\leq \left( C_3 \varepsilon^{1+\delta} + \frac{C_4 \varepsilon^{1+\delta}}{2(\mu + \tilde{C} \varepsilon^2)} \right) e^{\tilde{C} \sigma}
\leq \frac{M}{2} \varepsilon^{1+\delta}
\]
for \( (t, \xi) \in [1, T_*) \times \mathbb{R} \). This contradicts the definition of \( T_* \) if \( T_* < T_* \). Therefore we conclude \( T_* = T_* \). In other words, we have
\[
\sup_{(t, \xi) \in [1, T_*) \times \mathbb{R}} |w(t, \xi)| \leq M \varepsilon^{1+\delta},
\]
whence
\[
|\beta(t, \xi)| \leq |\beta_0(t, \xi)| + |w(t, \xi)| \leq C_2 \varepsilon + M \varepsilon^{1+\delta}
\]
for \( (t, \xi) \in [1, T_*) \times \mathbb{R} \). This completes the proof. \( \square \)

3. Preliminaries related to the Schrödinger operator. This section is devoted to preliminaries related to the operator \( L = i \partial_t + \frac{1}{2} \partial_x^2 \). In what follows, we denote several positive constants by \( C \), which may vary from one line to another.
3.1. The operators $\mathcal{J}$ and $\mathcal{Z}$. We introduce $\mathcal{J} = x + it\partial_x$ and $\mathcal{Z} = x\partial_x + 2t\partial_t$, which have good compatibility with $\mathcal{L}$. The following relations will be used repeatedly in the subsequent sections:

$$\left[\partial_x, \mathcal{J}\right] = 1, \quad \left[\mathcal{L}, \mathcal{J}\right] = 0, \quad \left[\partial_x, \mathcal{Z}\right] = \partial_x, \quad \left[\mathcal{L}, \mathcal{Z}\right] = 2\mathcal{L},$$

where $\left[\cdot, \cdot\right]$ stands for the commutator of two linear operators. Another important relation is

$$\mathcal{J}\partial_x = \mathcal{Z} + 2it\mathcal{L}, \quad (3.1)$$

which will play the key role in our analysis. Next we set

$$(U(t)\phi)(x) = e^{i\frac{t}{2}\partial_x^2} \phi(x) = e^{-i\pi/4}\sqrt{2\pi t} \int_{\mathbb{R}} e^{-iy^2/2t} \phi(y) dy$$

for $t > 0$. We will occasionally abbreviate $U(t)\phi$ to $U\phi$ if it causes no confusion. Also we introduce

$$(G\phi)(\xi) = e^{-i\pi/4}\hat{\phi}(\xi) = e^{-i\pi/4}\sqrt{2\pi} \int_{\mathbb{R}} e^{-iy\xi} \phi(y) dy.$$ 

The following lemma is well-known (see the series of papers by Hayashi and Naumkin [7]–[15] for the proof):

Lemma 3.1. We have

$$\|v\|_{L^\infty} \leq t^{-1/2} \left\|GU^{-1}v\right\|_{L^\infty} + Ct^{-3/4}(\|v\|_{L^2} + \|\mathcal{J}v\|_{L^2})$$

for $t > 0$.

3.2. A smoothing property. In this subsection, we recall a smoothing property of the linear Schrödinger equations, which will be used effectively in Step 3 of §4.1. As is well-known, the standard energy method causes a derivative loss when the nonlinear term involves the derivatives of the unknown function. Smoothing effect is a useful tool to overcome this difficulty. Among various kinds of smoothing properties, we will follow the approach of [16], whose original idea is due to Doi [6] (see also [2] and the references cited therein for the history and more information of this subject). Let $H$ be the Hilbert transform, that is,

$$Hv(x) := \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{v(y)}{x-y} dy.$$ 

With a non-negative weight function $\Phi(x)$, let us define the operator $S_\Phi$ by

$$S_\Phi v(x) := \left\{ \cosh \left( \int_{-\infty}^{x} \Phi(y) dy \right) \right\} v(x) - i \left\{ \sinh \left( \int_{-\infty}^{x} \Phi(y) dy \right) \right\} Hv(x).$$

Note that $S_\Phi$ is $L^2$-automorphism and both $\|S_\Phi\|_{L^2 \to L^2}$ and $\|S_\Phi^{-1}\|_{L^2 \to L^2}$ are dominated by $C \exp(\|\Phi\|_{L^1})$. Roughly speaking, the operator $S_\Phi$ is chosen so that

$$[\mathcal{L}, S_\Phi] \simeq -i\Phi S_\Phi [\partial_x] + \text{‘harmless terms’},$$

and the first term in the right-hand side enables us to gain the half-derivative $|\partial_x|^{1/2}$. More precisely, we have the following two lemmas:
Lemma 3.2 ([16]). Let \( v = v(t,x) \) and \( \psi = \psi(t,x) \) be \( \mathbb{C} \)-valued smooth functions. We set \( \Phi(t,x) = \psi(\psi^2 + \psi_x^2) \) with \( \eta \geq 1 \). Then there exists the constant \( C \), which is independent of \( \eta \), such that
\[
\frac{d}{dt} \| S_\Phi v(t) \|_{L^2}^2 + \left\| \sqrt{\Phi(t)} S_\Phi \partial_x^{1/2} v(t) \right\|_{L^2}^2 \\
\leq C \eta \| \psi(t) \|_{W^{2,\infty}}^2 + \eta^3 \| \psi(t) \|_{W^{1,\infty}}^2 + \eta \| L \psi(t) \|_{H^1} \| v(t) \|_{L^2}^2 \\
+ 2 \left| \langle S_\Phi v(t), S_\Phi L v(t) \rangle \right|,
\]
where \( W^{k,\infty} \) denotes the \( L^\infty \)-based Sobolev space of order \( k \).

Lemma 3.3 ([16]). Let \( v = v(x) \) and \( \psi = \psi(x) \) be \( \mathbb{C} \)-valued smooth functions. Suppose that \( q_1 \) and \( q_2 \) are quadratic homogeneous polynomials in \( (\psi, \bar{\psi}, \psi_x, \bar{\psi}_x) \). We set \( \Phi(x) = \eta (|\psi|^2 + |\psi_x|^2) \) with \( \eta \geq 1 \). Then there exists the constant \( C \), which is independent of \( \eta \), such that
\[
\left| \langle S_\Phi v, S_\Phi q_1 (\psi, \psi_x) \partial_x v \rangle_{L^2} \right| + \left| \langle S_\Phi v, S_\Phi q_2 (\psi, \psi_x) \overline{\partial_x v} \rangle_{L^2} \right| \\
\leq C \eta^{-1} e^{C\eta \| \psi \|_{L^2}^4} \left\| \sqrt{\Phi(t)} \partial_x^{1/2} v \right\|_{L^2}^2 \\
+ C e^{C \eta \| \psi \|_{H^1}^2} (1 + \eta^2 \| \psi \|_{H^1}^4 + \eta^2 \| \psi \|_{W^{1,\infty}}^4) \| \psi \|_{W^{2,\infty}}^2 \| v \|_{L^2}^2.
\]

For the proof, see Section 2 in [16] (see also the appendix of [28]).

4. A priori estimate. Throughout this section, we fix \( \sigma \in (0, \tilde{\tau}_0) \) and \( T \in (0, e^{\sigma/\varepsilon^2}) \), where \( \tilde{\tau}_0 \) is defined by (1.5). Let \( u \in C([0,T]; H^3 \cap H^{2,1}) \) be a solution to (1.1) for \( t \in [0, T) \), and we set \( \alpha(t, \xi) = \mathcal{G} [\mathcal{U}(t)^{-1} u(t, \cdot)](\xi) \), where \( \mathcal{G} \) and \( \mathcal{U} \) are given in Section 3. We also put
\[
E(T) = \sup_{t \in [0,T]} \left( 1 + t \right)^{-\gamma} (\| u(t) \|_{H^3} + \| J u(t) \|_{H^2}) + \sup_{\xi \in \mathbb{R}} (\xi^2 | \alpha(t, \xi) |)
\]
with \( \gamma \in (0, 1/12) \). The goal of this section is to prove the following:

Lemma 4.1. Assume that \( N \) satisfies (1.3). Let \( \sigma, T \) and \( \gamma \) be as above. Then there exist positive constants \( \varepsilon_0 \) and \( K \), not depending on \( T \), such that
\[
E(T) \leq \varepsilon^{2/3}
\]
implies
\[
E(T) \leq K \varepsilon,
\]
provided that \( \varepsilon \in (0, \varepsilon_0) \).

We divide the proof of this lemma into two subsections. We remark that many parts of the proof below are similar to that of Section 3 in [10], although we need modifications to fit for our purpose.

4.1. \( L^2 \)-estimates. In this part, we consider the bound for \( \| u(t) \|_{H^3} + \| J u(t) \|_{H^2} \). By virtue of the inequality
\[
\| u(t) \|_{H^3} + \| J u(t) \|_{H^2} \\
\leq C \left( \| u(t) \|_{L^2} + \| \partial_x^2 u(t) \|_{L^2} + \| J u(t) \|_{L^2} + \| \partial_x J \partial_x u(t) \|_{L^2} \right),
\]
(4.2)
it suffices to show that each term in the right-hand side can be dominated by $C\varepsilon(1 + t)\gamma$. We are going to estimate these four terms by separate ways.

**Step 1.** Estimate for $\|u(t)\|_{L^2}$. First we remark that (4.1) yields

$$
\|u(t)\|_{W^{2,\infty}} \leq \frac{C\varepsilon^{2/3}}{(1 + t)^{1/2}}
$$

for $t \in [0, T)$. Indeed, the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ yields

$$
\|u(t)\|_{W^{2,\infty}} \leq C\|u(t)\|_{H^3} \leq C\varepsilon^{2/3}
$$

for $t \leq 1$, while it follows from Lemma 3.1 that

$$
\|u(t)\|_{W^{2,\infty}} \leq \frac{C}{t^{1/2}} \sup_{\xi \in \mathbb{R}} (\xi^2 |\alpha(t, \xi)|) + \frac{C}{t^{3/4}} (\|u(t)\|_{H^2} + \|Ju(t)\|_{H^2}) \leq \frac{C\varepsilon^{2/3}}{t^{1/2}}
$$

for $t \in [1, T)$. Now, by the standard energy method, we have

$$
d\|u(t)\|_{L^2} \leq \|N(u, u_x)\|_{L^2} \leq C\|u(t)\|_{W^{1,\infty}}^2 \|u(t)\|_{H^1} \leq \frac{C\varepsilon^2}{(1 + t)^{1-\gamma}},
$$

whence

$$
\|u(t)\|_{L^2} \leq \|u(0)\|_{L^2} + \int_0^t \frac{C\varepsilon^2}{(1 + \tau)^{1-\gamma}} d\tau 
\leq C\varepsilon + C\varepsilon^2 (1 + t)^\gamma 
\leq C\varepsilon (1 + t)^\gamma
$$

(4.3)

for $t \in [0, T)$.

**Step 2.** Estimate for $\|Ju(t)\|_{L^2}$. If $t \leq 1$, there is no difficulty because we do not have to pay attentions to possible growth in $t$. Indeed, since

$$
\|Ju_N(u(t), u_x(t))\|_{L^2} \leq C(1 + t)\|u\|_{W^{1,\infty}}^2 (\|Ju\|_{H^1} + \|u\|_{H^2}) \leq C\varepsilon^2,
$$

we have

$$
\|Ju(t)\|_{L^2} \leq C\|u(0)\|_{H^{0.1}} + \int_0^1 \|Ju_N(u(\tau), u_x(\tau))\|_{L^2} d\tau \leq C\varepsilon + C\varepsilon^2
$$

for $t \leq 1$. To consider the case of $t \geq 1$, let us first recall a remarkable lemma due to Hayashi–Naumkin [10]:

**Lemma 4.2.** Assume that $N$ satisfies (1.3). Then the following decomposition holds:

$$
Ju_N(u, u_x) = L(tP) + Q,
$$

where $P$ is a cubic homogeneous polynomial in $(u, \overline{u}, u_x, \overline{u}_x)$, and $Q$ satisfies

$$
\|Q\|_{L^2} \leq C\|u\|_{W^{2,\infty}}^2 (\|u\|_{H^1} + \|Ju\|_{H^2} + \|Zu\|_{H^1}) + \frac{C}{t} \|u\|_{W^{2,\infty}} \|Ju\|_{H^2}^2 + \|u\|_{H^1}^2
$$

for $t \geq 1$.

For the convenience of the readers, we shall give a sketch of the proof in the appendix. Now we are going to apply this lemma. Let $t \in [1, T)$. Since the above decomposition allows us to rewrite the original equation as

$$
L(Ju - tP) = Q,
$$
the standard energy method gives us
\[ \|J(t) - tP\|_{L^2} \leq C(\varepsilon + \varepsilon^2) + \int_1^t \|Q(\tau)\|_{L^2} d\tau. \]

By the relation (3.1), we have
\[ \|Z\|_{H^1} \leq \|J\|_{H^1} + 2t\|N(u, u_x)\|_{H^1} \leq C\varepsilon^{2/3}(1 + t)^{\gamma} + C\varepsilon^2 t(1 + t)^{-1+\gamma} \leq C\varepsilon^{2/3}(1 + t)^\gamma, \]
which leads to
\[ \|Q(t)\|_{L^2} \leq \frac{C\varepsilon^2}{(1 + t)^{1-\gamma}}. \]

Also we have
\[ \|P\|_{L^2} \leq C\|u\|_{W^{1,\infty}}^2 \|u\|_{H^1} \leq \frac{C\varepsilon^2}{(1 + t)^{1-\gamma}}. \]

Summing up, we have
\[ \|J(t)\|_{L^2} \leq t\|P\|_{L^2} + \|J(t) - tP\|_{L^2} \leq C\varepsilon(1 + t)^\gamma \]
for \( t \in [1, T] \).

**Step 3.** Estimate for \( \|\partial_x^2 u(t)\|_{L^2} \). We apply Lemma 3.2 with \( v = \partial_x^2 u \), \( \psi = u \) and \( \eta = \varepsilon^{-2/3} \). Then we obtain
\[ \frac{d}{dt}\|S\|_{L^2}^2 \|S\partial_x^2 u(t)\|_{L^2}^2 \leq CB(t)\|\partial_x^2 u(t)\|_{L^2}^2 + 2\|\langle S\partial_x^2 u, S\partial_x^2 N(u, u_x)\rangle\|_{L^2}^2, \]
where
\[ B(t) = e^{C\varepsilon^{-2/3}}\|u\|_{H^1}^2 \left( \varepsilon^{-2/3} \|u(t)\|_{W^{2,\infty}}^2 + \varepsilon^{-2} \|u(t)\|_{W^{1,\infty}}^6 \right) \]
\[ + \varepsilon^{-2/3} \|u(t)\|_{H^1} \|N(u, u_x)\|_{H^1}. \]

Since (4.1) yields
\[ \|u(t)\|_{H^1} \leq C\|\alpha(t)\|_{H^{0,1}} \leq C \left( \int_{\mathbb{R}} \frac{d\xi}{(\xi^2)^{1/2}} \right)^{1/2} \sup_{\xi \in \mathbb{R}} (|\xi|^2 |\alpha(t, \xi)|) \leq C\varepsilon^{2/3}, \]
we see that \( B(t) \) can be dominated by \( C\varepsilon^{2/3}(1 + t)^{-1} \). Also we observe that the usual Leibniz rule leads to
\[ \frac{\partial_x^2 N(u, u_x)}{u_x} = q_1(u, u_x)\partial_x^2 u + q_2(u, u_x)\partial_x(q_1^2 u) + \rho_1, \]
where \( q_1, q_2 \) are defined by
\[ q_1(z, \xi) = \frac{\partial N}{\partial u}(z, \xi), \quad q_2(z, \xi) = \frac{\partial N}{\partial u}(z, \xi), \]
and \( \rho_1 \) satisfies
\[ \|\rho_1\|_{L^2} \leq C\|u\|_{W^{2,\infty}}^2 \|u\|_{H^1}. \]

By Lemma 3.3, we have
\[ \|\langle S\partial_x^2 u, S\partial_x^2 N(u, u_x)\rangle\|_{L^2} \leq \|\langle S\partial_x^2 u, S\partial_x q_1(u, u_x)\partial_x(q_1^2 u)\rangle\|_{L^2} \]
\[ + \|\langle S\partial_x^2 u, S\partial_x q_2(u, u_x)\partial_x(q_1^2 u)\rangle\|_{L^2} \]
\[ \leq \|\langle S\partial_x^2 u, S\partial_x^2 N(u, u_x)\rangle\|_{L^2}. \]
which yields
\[ + C\|S_F\partial_x^3 u\|_{L^2}\|S_F\rho_1\|_{L^2} \leq C\varepsilon^{2/3}e^{C\varepsilon^{-2/3}\|u\|_{H^1}^2} \left( \sqrt{\|S_F\|_{L^2}} \right)^2 \||\partial_x^3 u\|_{L^2}^2 + Ce^{C\varepsilon^{-2/3}\|u\|_{H^1}^2} \left( 1 + \varepsilon^{-4/3}\|u\|_{H^1}^4 + \varepsilon^{-4/3}\|u\|_{W^{1,\infty}}^4 \right) \||u\|_{W^{2,\infty}}^2 \|\partial_x^3 u\|_{L^2}^2 \\
+ Ce^{C\varepsilon^{-2/3}\|u\|_{H^1}^2} \|u\|_{W^{2,\infty}}^2 \|u\|_{H^3}^2 \leq C_0 \varepsilon^{2/3} \left\| \sqrt{\|S_F\|_{L^2}} \|\partial_x^3 u\|_{L^2}^2 \right\|_{L^2}^2 + C\varepsilon^{8/3} \left( 1 + t \right)^{1-2\gamma} \]
with some positive constant $C_0$ not depending on $\varepsilon$. Piecing the above estimates all together, we obtain
\[ \frac{d}{dt} \|S_F\partial_x^3 u(t)\|_{L^2}^2 \leq (2C_0 \varepsilon^{2/3} - 1) \left\| \sqrt{\|S_F\|_{L^2}} \|\partial_x^3 u\|_{L^2}^2 \right\|_{L^2}^2 + Ce^{C\varepsilon^{-2/3}\|u(t)\|_{H^1}^2} \|S_F\partial_x^3 u(t)\|_{L^2}^2 \leq C\varepsilon(1 + t)^\gamma \] (4.6)
provided that $\varepsilon \leq (2C_0)^{-3/2}$. Integrating with respect to $t$, we have
\[ \|S_F\partial_x^3 u(t)\|_{L^2}^2 \leq Ce^{C\varepsilon^{2/3}\varepsilon^{2} + C\varepsilon^2(1 + t)^{2\gamma}} \leq C\varepsilon^2(1 + t)^{2\gamma}, \]
whence
\[ \|\partial_x^3 u(t)\|_{L^2} \leq e^{C\varepsilon^{2/3}\|u(t)\|_{H^1}^2} \|S_F\partial_x^3 u(t)\|_{L^2} \leq C\varepsilon(1 + t)^\gamma \] (4.7)
for $t \in [0, T)$.

**Step 4.** Estimate for $\|\partial_x \mathcal{J} \partial_x u(t)\|_{L^2}$. By using the commutation relation $[\mathcal{L}, \partial_x Z] = 2\partial_x \mathcal{L}$ and the Leibniz rule for $Z$, we have
\[ \mathcal{L}\partial_x Z u = q_1(u, u_x)\partial_x(\partial_x Z u) + q_2(u, u_x)\partial_x(\partial_x Z u) + \rho_2, \]
where $q_1, q_2$ are given by (4.5), and $\rho_2$ satisfies
\[ \|\rho_2\|_{L^2} \leq C\|\|u\|_{W^{2,\infty}} + \|Z u\|_{H^1}\|. \]

Since the relation (3.1) leads to
\[ \|Z u\|_{H^1} \leq \|\mathcal{J} \partial_x u\|_{H^1} + 2t \|N(u, u_x)\|_{H^1} \leq C\varepsilon^{2/3}(1 + t)^\gamma, \]
we see that
\[ \|\rho_2\|_{L^2} \leq \frac{C\varepsilon^2}{(1 + t)^{1-\gamma}}. \]

Thus, as in the derivation of (4.6), we have
\[ \frac{d}{dt} \|S_F\partial_x Z u(t)\|_{L^2}^2 \leq \frac{C\varepsilon^2}{(1 + t)^{1-2\gamma}}, \]
which yields
\[ \|\partial_x Z u(t)\|_{L^2} \leq C\varepsilon(1 + t)^\gamma. \]

Finally, by using the relation (3.1) again, we obtain
\[ \|\partial_x \mathcal{J} \partial_x u(t)\|_{L^2} \leq \|\partial_x Z u\|_{L^2} + 2t \|\partial_x N(u, u_x)\|_{L^2} \leq C\varepsilon(1 + t)^\gamma + 2t \frac{C\varepsilon^2}{(1 + t)^{1-\gamma}} \leq C\varepsilon(1 + t)^\gamma \] (4.8)
for \( t \in [0, T) \).

**Final step.** Substituting (4.3), (4.4), (4.7), (4.8) into (4.2), we arrive at the desired estimate
\[
\|u(t)\|_{H^3} + \|Ju(t)\|_{H^2} \leq C\varepsilon(1 + t)^{\gamma}
\]
for \( t \in [0, T) \).

4.2. **Estimates for** \( \alpha \). In this part, we will show \( \langle \xi \rangle^2 |\alpha(t, \xi)| \leq C\varepsilon \) for \( (t, \xi) \in [0, T) \times \mathbb{R} \) under the assumption (4.1). If \( t \leq 1 \), the Sobolev embedding yields this estimate immediately. Hence we may assume \( T > 1 \) and \( t \in [1, T) \) in what follows.

Now let us introduce a useful lemma, which is due to Hayashi–Naumkin [10] though the expression is slightly different. We write \( \alpha_\omega(t, \xi) = \alpha(t, \xi/\omega) \) for \( \omega \in \mathbb{R}\setminus\{0\} \).

**Lemma 4.3.** Assume that \( N \) satisfies (1.3). Then, for \( l \in \{0, 1, 2\} \), the following decomposition holds:
\[
\mathcal{G}U^{-1}\partial_x^l N(u, u_x) = \left(\frac{\langle \xi \rangle}{t}\right)^3 |\alpha|^2 \alpha + \frac{\xi e^{it\xi^2}}{t} \mu_{1,l}(\xi) \alpha_3^3 \\
+ \frac{\xi e^{it\xi^2}}{t} \mu_{2,l}(\xi) (\alpha_{-3})^3 + \frac{\xi e^{it\xi^2}}{t} \mu_{3,l}(\xi) (\alpha_{-1})^2 (\alpha_{-1}) + R_l, \quad (4.9)
\]
where \( \mu_{1,l}(\xi), \mu_{2,l}(\xi), \mu_{3,l}(\xi) \) are polynomials in \( \xi \) of order at most \( 2+l \), and \( R_l(t, \xi) \) satisfies
\[
\sum_{l=0}^{2} \|R_l(t)\|_{L^\infty} \leq \frac{C}{t^{5/4}} (\|u(t)\|_{H^3} + \|Ju(t)\|_{H^2})^3
\]
for \( t \geq 1 \).

The proof of this lemma will be given in the appendix. It follows from this lemma that
\[
\langle \xi \rangle^2 i\partial_t \alpha = \mathcal{G}U^{-1}(1 - \partial_x^2)\mathcal{L}u \\
= \mathcal{G}U^{-1}N(u, u_x) - \mathcal{G}U^{-1}\partial_x^2 N(u, u_x) \\
= \left(\frac{\langle \xi \rangle}{t}\right)^3 |\alpha|^2 \alpha + V + R_0 - R_2, \quad (4.10)
\]
where
\[
V(t, \xi) = \frac{\xi e^{it\xi^2}}{t} p_1(\xi) \alpha_3^3 + \frac{\xi e^{it\xi^2}}{t} p_2(\xi) (\alpha_{-3})^3 + \frac{\xi e^{it\xi^2}}{t} p_3(\xi) (\alpha_{-1})^2 (\alpha_{-1})
\]
with \( p_k(\xi) = \mu_{k,0}(\xi) - \mu_{k,2}(\xi) \) \( (k = 1, 2, 3) \). We deduce from (4.1) and (4.10) that
\[
|\partial_t \alpha(t, \xi)| \leq \frac{C\varepsilon^2}{\langle \xi \rangle^2 t^2}, \quad (t, \xi) \in [1, T) \times \mathbb{R}.
\]

Also, by using the identity
\[
\frac{\xi e^{it\xi^2}}{t} A(t, \xi) = i\partial_t \left( -it\xi e^{it\xi^2} A(t, \xi) - t e^{it\xi^2} \partial_t \left( \frac{\xi A(t, \xi)}{t(1 + i\omega t\xi^2)} \right) \right),
\]
we see that \( V \) can be rewritten as \( i\partial_t V_1 + V_2 \) with suitable \( V_1, V_2 \) satisfying
\[
|V_1(t, \xi)| \leq \frac{C\varepsilon^2}{t^{1/2}}, \quad |V_2(t, \xi)| \leq \frac{C\varepsilon^2}{t^{3/2}}.
\]
Note that
\[ \sup_{\xi \in \mathbb{R}} \left| \frac{\xi}{1 + i\omega t\xi^2} \right| \leq \frac{C}{t^{1/2}} \]
if \( \omega \in \mathbb{R} \setminus \{0\} \). Now we set \( \beta(t, \xi) = \langle \xi \rangle^2 \alpha(t, \xi) - V_1(t, \xi) \) and \( \kappa(t, \xi) = \langle \xi \rangle^{-4} \nu(t, \xi) \). Then we have
\[
i \partial_t \beta(t, \xi) = \frac{\kappa(t, \xi)}{t} |\beta(t, \xi)|^2 \beta(t, \xi) + \rho(t, \xi), \tag{4.11} \]
where
\[ \rho(t, \xi) = \frac{\kappa(t, \xi)}{t} \left( |\langle \xi \rangle^2 \alpha|^2 \langle \xi \rangle^2 \alpha - |\beta|^2 \beta \right) + V_2(t, \xi) + R_0(t, \xi) - R_2(t, \xi). \]
Remark that \( \rho \) can be regarded as a remainder because we have
\[ |\rho(t, \xi)| \leq \frac{C}{t} \cdot \left( C\varepsilon^{2/3} \right)^2 \cdot \frac{C\varepsilon^2}{t^{1/2}} + C\varepsilon^2 \cdot \frac{C\varepsilon^2}{t^{3/2}} + \frac{C}{t^{5/4}} \cdot \left( C\varepsilon^{2/3} \hat{\gamma} \right)^3 \leq \frac{C\varepsilon}{t^{1+\mu}} \]
with \( \mu = 1/4 - 3\gamma > 0 \). Moreover we have
\[
\sup_{\xi \in \mathbb{R}} |\beta(1, \xi) - \varepsilon \langle \xi \rangle^2 \hat{\varphi}(\xi)| \leq C \left\| \left( 1 - \partial_x^2 \right) \left( H(1)^{-1}u(1, \cdot) - \varepsilon \varphi \right) \right\|_{H^0,1} + \sup_{\xi \in \mathbb{R}} |V_1(1, \xi)|
\leq C \int_0^1 \left\| N(u(t), u_x(t)) \right\|_{H^2,1} dt + C\varepsilon^2
\leq C \left( \sup_{t \in [0,1]} \| u(t) \|_{H^3 \cap H^2,1} \right)^3 + C\varepsilon^2
\leq C\varepsilon^2.
\]
Therefore we can apply Lemma 2.1 with \( \theta_0(\xi) = \langle \xi \rangle^2 \hat{\varphi}(\xi) \) and \( \tau_1 = \tau_0 \) to obtain
\[ |\beta(t, \xi)| \leq C\varepsilon \]
whence
\[ \langle \xi \rangle^2 |\alpha(t, \xi)| \leq |\beta(t, \xi)| + |V_1(t, \xi)| \leq C\varepsilon \]
for \((t, \xi) \in [1, T) \times \mathbb{R}\), as desired. \(\square\)

5. **Proof of the main theorem.** Now we are in a position to prove Theorem 1.1. First we state a standard local existence result without proof. Let \( t_0 \geq 0 \) be fixed, and consider the initial value problem
\[
\begin{aligned}
L u &= N(u, u_x), & t > t_0, & x \in \mathbb{R}, \\
u(t_0, x) &= \psi(x), & x \in \mathbb{R}.
\end{aligned} \tag{5.1}
\]

**Lemma 5.1.** Let \( N \) be a cubic homogeneous polynomial in \((u, u_x, u_{xx})\). Let \( \psi \in H^3 \cap H^{2,1}(\mathbb{R}) \). Then there exists \( T_0 = T_0(\| \psi \|_{H^3}) > 0 \), independent of \( t_0 \), such that (5.1) has a unique solution \( u \in C([t_0, t_0 + T_0); H^3 \cap H^{2,1}(\mathbb{R})) \).

See [26], [19], [1], [25], [16], etc., for more details on local existence theorems.

**Proof of Theorem 1.1.** Let \( T_0 \) be the lifespan defined in the statement of Theorem 1.1. We remark that Lemma 5.1 with \( t_0 = 0 \) and \( \psi = \varepsilon \varphi \) implies \( T_0 > 0 \). Next we set
\[ T^* = \sup \{ T \in [0, T_0^*) \mid E(T) \leq \varepsilon^{2/3} \}. \]
Note that $T^* > 0$ if $\varepsilon$ is suitably small, because of the estimate $E(0) \leq C\varepsilon \leq (1/2)\varepsilon^{2/3}$ and the continuity of $[0, T_\varepsilon) \ni T \mapsto E(T)$. Now, we take $\sigma \in (0, \tilde{\tau}_0)$ and assume $T^* \leq e^{\sigma^2/\varepsilon^2}$. Then Lemma 4.1 with $T = T^*$ yields

$$E(T^*) \leq K\varepsilon \leq \frac{1}{2}\varepsilon^{2/3}$$

if $\varepsilon \leq \min\{\varepsilon_0, (2K)^{-3}\}$. By the continuity of $[0, T_\varepsilon) \ni T \mapsto E(T)$, we can choose $\Delta > 0$ such that

$$E(T^* + \Delta) \leq \varepsilon^{2/3}.$$ 

This contradicts the definition of $T^*$. Therefore we must have $T^* \geq e^{\sigma^2/\varepsilon^2}$ if $\varepsilon$ is suitably small. Consequently, we have

$$\lim_{\varepsilon \to +0} \varepsilon^2 \log T_\varepsilon \geq \sigma.$$ 

Since $\sigma \in (0, \tilde{\tau}_0)$ is arbitrary, we arrive at the desired conclusion. \hspace{1cm} \square

**Appendix A. Proof of Lemmas 4.2 and 4.3.** In this appendix, we will prove Lemmas 4.2 and 4.3 along the idea of [10].

**A.1. Proof of Lemma 4.2.** First we observe that the nonlinear term $N$ satisfying (1.3) can be written as $N = F + G$, where

$$F = a_1 u^2 u_x + a_2 u u_x^2 + a_3 u^3 + b_1 u^2 u_x + b_2 u u_x^2 + b_3 u_x^3$$

$$+ c_1 u^2 u_x + c_2 |u|^2 \overline{u} \overline{u_x} + c_3 u u_x^2 + c_4 |u|^2 \overline{u} + c_5 |u|^2 u_x^2$$

(A.1)

and

$$G = \lambda_1 |u|^2 u_x + \lambda_2 |u|^2 u_x^2 + \lambda_3 u^2 \overline{u} \overline{u} + \lambda_4 |u|^2 u_x + \lambda_5 \overline{u} \overline{u_x} + \lambda_6 |u|^2 u_x$$

(A.2)

with $a_j, b_j, c_j, \lambda_j \in \mathbb{C}$. Note that $G$ is gauge-invariant, while $F$ is not. By using the identities

$$\phi \partial_x \psi = (\partial_x \phi)\psi + \frac{1}{it}(\phi J \psi - (J \phi)\psi)$$

(A.3)

and

$$\phi \partial_x \overline{\psi} = -(\partial_x \phi)\overline{\psi} + \frac{1}{it}((J \phi)\overline{\psi} - \phi J \overline{\psi}),$$

(A.4)

we see that $F$ can be rewritten as $\partial_x F_1 + \frac{1}{it} F_2$, where

$$F_1 = \frac{a_1}{3} u^3 + \frac{a_2}{3} u^2 u_x + \frac{a_3}{3} u_x^3 + b_1 \overline{u} u_x + b_2 \overline{u_x} u_x + b_3 u_x u_{xx}$$

$$+ (c_2 - c_1)|u|^2 \overline{u} + c_3 |u|^2 \overline{u_x} + c_4 \overline{u} u_x^2 + c_5 |u|^2 u_x$$

and

$$F_2 = \frac{a_2}{3} u (u J u_x - u_x J u) - \frac{2a_3}{3} u_x (u J u_x - u_x J u)$$

$$- \frac{b_2}{3} u (u J u_x - u_x J u) + \frac{2b_3}{3} u_x (u J u_x - u_x J u)$$

$$+ (c_2 - 2c_1) u (u J u_x - u_x J u) - c_3 \overline{u} J u - u J \overline{u_x}$$

$$- c_4 \overline{u_x} J u - u_x J \overline{u}.$$ 

We deduce from the relation (3.1) that

$$J N(u, u_x) = (2 + 2i \mathcal{L}) F_1 + \frac{1}{it} J F_2 + J G = \mathcal{L}(t P) + Q,$$
where \( P = 2iF_1 \) and \( Q = (Z + 2)F_1 + \frac{1}{4\pi} \mathcal{J}F_2 + \mathcal{J}G \). By the Leibniz rule for \( Z \), we have
\[
\|(Z + 2)F_1\|_{L^2} \leq C\|u\|_{W^{1,\infty}}^3 (\|Zu\|_{H^1} + \|u\|_{H^1}).
\]
On the other hand, since \( G \) is gauge-invariant, we can use the identity
\[
\mathcal{J}(f_1 f_2 \overline{f_3}) = (\mathcal{J} f_1) f_2 \overline{f_3} + f_1 (\mathcal{J} f_2 \overline{f_3}) - f_1 f_2 \overline{\mathcal{J} f_3}
\]
to obtain
\[
\|\mathcal{J}G\|_{L^2} \leq C\|u\|_{W^{1,\infty}}^3 (\|\mathcal{J}u\|_{H^1} + \|u\|_{L^2}).
\]
In order to get the \( L^2 \)-bound for \( \frac{1}{4\pi} \mathcal{J}F_2 \), we apply the identities
\[
\mathcal{J}(f_1 f_2 \overline{f_3}) = it\{\partial_x (f_1 f_2 \overline{f_3}) - (\partial_x f_1) f_2 \overline{f_3} + (\mathcal{J} f_1) f_2 \overline{f_3} - f_1 f_2 \overline{\mathcal{J} f_3},
\]
\[
\mathcal{J}(f_1 f_2 \overline{f_3}) = it\{\partial_x (f_1 f_2 \overline{f_3}) + (\partial_x f_1) f_2 \overline{f_3} + (\mathcal{J} f_1) f_2 \overline{f_3} - f_1 f_2 \overline{\mathcal{J} f_3}
\]
to each term of \( F_2 \) multiplied by \( \mathcal{J} \), and use the inequality
\[
\|\mathcal{J} \partial_x \phi\|_{L^\infty} \leq C (\|\mathcal{J} \phi\|_{H^2} + \|\phi\|_{H^1}).
\]
Then we have
\[
\|\mathcal{J} F_2\|_{L^2} \leq Ct\|u\|_{W^{2,\infty}}^3 (\|\mathcal{J}u\|_{H^2} + \|u\|_{H^1}) + C\|u\|_{W^{2,\infty}} (\|\mathcal{J}u\|_{H^2} + \|u\|_{H^1})^2.
\]
Piecing them together, we arrive at the desired decomposition. \( \square \)

### A.2. Proof of Lemma 4.3.

Before we proceed to the proof of Lemma 4.3, we introduce some notations. We put
\[
(M(t)\phi)(x) = e^{it^2/2}\phi(x), \quad (D(t)\phi)(x) = t^{-1/2}\phi\left(\frac{x}{t}\right), \quad V(t)\phi = \mathcal{G}M(t)\mathcal{G}^{-1}\phi,
\]
so that \( U(t) \) is decomposed into \( U(t) = M(t)D(t)GM(t) = M(t)D(t)V(t)\mathcal{G} \). Note that
\[
\|(V(t) - 1)\phi\|_{L^\infty} + \|(V(t)^{-1} - 1)\phi\|_{L^\infty} \leq Ct^{-1/4}\|\phi\|_{H^1}, \tag{A.5}
\]
which comes from the inequalities \( |e^\theta - 1| \leq C|\theta|^{1/2} \) and \( \|\phi\|_{L^\infty} \leq C\|\phi\|_{L^2}^{1/2}\|\partial_x \phi\|_{L^2}^{1/2} \).

In what follows, we will occasionally omit \( "(t)" \) from \( M(t), \ D(t), \ V(t) \) if it causes no confusion, and we will write \( D_\omega = D(\omega) \) for \( \omega \in \mathbb{R}\setminus\{0\} \).

#### Lemma A.1. We have
\[
\|G U^{-1}[f_1 f_2 f_3]\|_{L^\infty} + \|G U^{-1}[f_1 f_2 \overline{f_3}]\|_{L^\infty} \leq \frac{C}{4\pi^2}\|f_1\|_{L^2}\|f_2\|_{L^2}\|f_3\|_{L^2} + \|\mathcal{J} f_3\|_{L^2}.
\]

**Proof.** From the relation \( G U^{-1} = V^{-1}D^{-1}M^{-1} \) and the estimate \( \|V^{-1}\phi\|_{L^\infty} \leq Ct^{1/2}\|\phi\|_{L^1} \), it follows that
\[
\|G U^{-1}[f_1 f_2 f_3]\|_{L^\infty} \leq Ct^{1/2}\|D^{-1}M^{-1}[f_1 f_2 f_3]\|_{L^1},
\]
\[
= Ct^{1/2} \cdot t^{-1/2}\|f_1 f_2 f_3\|_{L^1}
\]
\[
\leq C\|f_1\|_{L^2}\|f_2\|_{L^2}\|f_3\|_{L^\infty}
\]
\[
\leq C t^{-1/2}\|f_1\|_{L^2}\|f_2\|_{L^2}\|f_3\|_{L^\infty}^2 + \|\mathcal{J} f_3\|_{L^2}^2.
\]
We have used the inequality \( \|f\|_{L^\infty} \leq Ct^{-1/2}\|f\|_{L^2}^{1/2}\|\mathcal{J} f\|_{L^2}^{1/2} \) in the last line. The estimate for \( \|G U^{-1}(f_1 f_2 \overline{f_3})\|_{L^\infty} \) can be shown in the same way. \( \square \)

Next we set \( (\mathcal{E}_\omega(t)f)(y) = e^{\omega\frac{y^2}{2}}f(y) \) and \( A_\omega(t) = V(t)^{-1}\mathcal{E}_\omega^{-1}(t) - \mathcal{E}_\omega^{-1}(t)D_\omega \).

#### Lemma A.2. For \( \omega \in \mathbb{R}\setminus\{0\} \), we have
\[
\|A_\omega(t)f\|_{L^\infty} \leq Ct^{-1/4}\|f\|_{H^1}.
\]
Hence we deduce from (A.5) that
\[ N \text{ only the case where } N \text{ satisfies (1.3) (or, equivalently, } N = F + G \text{ with (A.1) and (A.2) can be treated in the same way. Note that } \]
\[ \text{if } N \text{ is given by (A.6), whereas } \text{if } N = F + G \text{ with (A.1), (A.2).} \]

First we consider the case of \( l = 0 \). We put \( \alpha^{(s)} = (i\xi)^{s} \alpha \) so that \( \partial_{x}^{s}u = M^{2}V\alpha^{(s)} \). We also set \( (M^{2}(t)f)(y) = e^{i\omega y^{2}/2}f(y) \). Then it follows that
\[ N(u, u_{x}) = \lambda|u_{x}|^{2}u_{x} + au_{x}^{3} + bu_{x}^{2} + c|u_{x}|^{3}u_{x}. \quad (A.6) \]

General cubic terms \( N \) satisfying (1.3) (or, equivalently, \( N = F + G \) with (A.1) and (A.2)) can be treated in the same way. Note that
\[ \nu(\xi) = i\xi^{3} \int_{0}^{2\pi} (\lambda e^{i\theta} - ae^{3i\theta} + be^{-3i\theta} - ce^{-i\theta})e^{-i\theta}d\theta = i\lambda\xi^{3} \]
if \( N \) is given by (A.6), whereas
\[ \nu(\xi) = \lambda_{1} + i(\lambda_{2} - \lambda_{3})\xi + (\lambda_{4} - \lambda_{5})\xi^{2} + i\lambda_{6}\xi^{3} \quad (A.7) \]
if \( N = F + G \) with (A.1), (A.2).

First we consider the case of \( l = 0 \). We put \( \alpha^{(s)} = (i\xi)^{s} \alpha \) so that \( \partial_{x}^{s}u = M^{2}V\alpha^{(s)} \). We also set \( (M^{2}(t)f)(y) = e^{i\omega y^{2}/2}f(y) \). Then it follows that
\[ N(u, u_{x}) = \lambda M^{2}|D\alpha_{(1)}^{(1)}|^{2}D\alpha_{(1)}^{(1)} + aM^{3}|D\alpha_{(1)}^{(1)}|^{3} \]
\[ + bM^{-3}(D\alpha_{(1)}^{(1)})^{3} + cM^{-1}|D\alpha_{(1)}^{(1)}|^{2}D\alpha_{(1)}^{(1)} \]
\[ = \frac{\lambda}{t}M^{2}\left[|\alpha_{(1)}^{(1)}|^{2}\alpha_{(1)}^{(1)}\right] + \frac{a}{t}M^{3}\left[|\alpha_{(1)}^{(1)}|^{3}\right] \]
\[ + \frac{b}{t}M^{-3}D\left[|\alpha_{(1)}^{(1)}|^{3}\right] + \frac{c}{t}M^{-1}D\left[|\alpha_{(1)}^{(1)}|^{2}\alpha_{(1)}^{(1)}\right]. \]

By the relation \( G\mathcal{U}^{-1}M^{2} = \mathcal{V}^{1}E^{1} \), we have
\[ G\mathcal{U}^{-1}N(u, u_{x}) = \frac{\lambda}{t}\mathcal{V}^{-1}\left[|\alpha_{(1)}^{(1)}|^{2}\alpha_{(1)}^{(1)}\right] + \frac{a}{t}\mathcal{V}^{-1}\left[|\alpha_{(1)}^{(1)}|^{3}\right] \]
\[ + \frac{b}{t}\mathcal{V}^{-1}E^{-4}\left[|\alpha_{(1)}^{(1)}|^{3}\right] + \frac{c}{t}\mathcal{V}^{-1}E^{-2}\left[|\alpha_{(1)}^{(1)}|^{2}\alpha_{(1)}^{(1)}\right] \]
\[ = \frac{\lambda}{t}i\xi^{3}|\alpha|^{2}\alpha + a\xi^{2}D_{3}\left[-i\xi^{3}\alpha\right] \]
\[ + \frac{b}{t}\xi^{2}D_{3}\left|\xi^{3}\alpha\right| + \frac{c}{t}E^{2}D_{1}\left[-i\xi^{3}|\alpha|^{2}\alpha\right] + \frac{\Omega_{0}}{t}, \]
where
\[
\Omega_0 = -\lambda \left( |\alpha^{(1)}|^2 \alpha^{(1)} - |\mathcal{V} \alpha^{(1)}|^2 \mathcal{V} \alpha^{(1)} \right) + \lambda (\mathcal{V}^{-1} - 1) \left[ |\mathcal{V} \alpha^{(1)}|^2 \mathcal{V} \alpha^{(1)} \right] \\
- a\mathcal{E} \frac{2}{t} \left( \xi^{(2)} \Omega_3 \left[ |\alpha^{(1)}|^3 - |\mathcal{V} \alpha^{(1)}|^3 \right] + aA_3(t) \left[ |\mathcal{V} \alpha^{(1)}|^3 \right] \right) \\
- b\mathcal{E} \frac{2}{t} \left( \xi^{(2)} \Omega_3 \left[ |\alpha^{(1)}|^3 - |\mathcal{V} \alpha^{(1)}|^3 \right] + bA_{-3}(t) \left[ |\mathcal{V} \alpha^{(1)}|^3 \right] \right) \\
- c\mathcal{E} \frac{2}{t} \left( \xi^{(2)} \Omega_3 \left[ |\alpha^{(1)}|^3 - |\mathcal{V} \alpha^{(1)}|^3 \right] + cA_{-1}(t) \left[ |\mathcal{V} \alpha^{(1)}|^3 \mathcal{V} \alpha^{(1)} \right] \right).
\]

By virtue of (A.5) and Lemma A.2, we see that
\[
\|\Omega_0\|_{L^\infty} \leq \frac{C}{t^{1/4}} (\|u\|_{H^2} + \|\mathcal{J} u\|_{H^1})^3.
\]

Therefore we obtain (4.9) with \( l = 0 \) by putting \( \mu_{1,0}(\xi) = -\frac{a}{27\sqrt{3}} \xi^2 \), \( \mu_{2,0}(\xi) = \frac{b}{27\sqrt{3}} \xi^3 \), \( \mu_{3,0}(\xi) = c\xi^3 \).

Next we consider the case of \( l = 1 \). It follow from the identity (A.4) that
\[
\partial_x N(u, u_x) = \lambda |u_{xx}|^2 u_{xx} + 3au_x^2 u_{xx} + 3b\xi u_x^2 u_{xx} + c|u_x|^2 |u_{xx}| + \frac{1}{it} r_1,
\]
where \( r_1 = (\lambda u_x + c\lambda)(|\mathcal{J} u_x|^2 - u_x \mathcal{J} u_x) \). By Lemma A.1, we obtain
\[
\|\mathcal{G}^{-1} r_1\|_{L^\infty} \leq \frac{C}{t^{1/2}} (\|u\|_{H^1} + \|\mathcal{J} u\|_{H^1})^3.
\]

We also set \( h_1 = \lambda |u_{xx}|^2 u_{xx} + 3au_x^2 u_{xx} + 3b\xi u_x^2 u_{xx} + c|u_x|^2 |u_{xx}| \) so that
\[
\mathcal{G}^{-1} \partial_x N(u, u_x) = \mathcal{G}^{-1} h_1 + \frac{1}{it} \mathcal{G}^{-1} r_1.
\]

Then, as in the previous case, we have
\[
\mathcal{G}^{-1} h_1 = -\lambda \left( \xi^4 |\alpha|^3 \alpha + \frac{3a}{t} \mathcal{E} \frac{2}{t} \xi^{(2)} \Omega_3 \left[ \xi^{(2)} \alpha^3 \right] + \frac{3b}{t} \mathcal{E} \frac{2}{t} \xi^{(2)} \Omega_3 \left[ \xi^{(2)} \alpha^3 \right] \right) + \frac{c}{t} \mathcal{E} \frac{2}{t} \xi^{(2)} \Omega_3 \left[ \xi^{(2)} \alpha^3 \right] + \frac{\Omega_1}{t},
\]
where
\[
\Omega_1 = -\lambda \left( |\alpha^{(1)}|^2 \alpha^{(2)} - |\mathcal{V} \alpha^{(1)}|^2 \mathcal{V} \alpha^{(2)} \right) + \lambda (\mathcal{V}^{-1} - 1) \left[ |\mathcal{V} \alpha^{(1)}|^2 \mathcal{V} \alpha^{(2)} \right] \\
- 3a\mathcal{E} \frac{2}{t} \left( \xi^{(2)} \Omega_3 \left[ |\alpha^{(1)}|^3 \alpha^{(2)} - |\mathcal{V} \alpha^{(1)}|^3 \mathcal{V} \alpha^{(2)} \right] + 3aA_3(t) \left[ |\mathcal{V} \alpha^{(1)}|^3 \mathcal{V} \alpha^{(2)} \right] \right) \\
- 3b\mathcal{E} \frac{2}{t} \left( \xi^{(2)} \Omega_3 \left[ |\alpha^{(1)}|^3 \alpha^{(2)} - |\mathcal{V} \alpha^{(1)}|^3 \mathcal{V} \alpha^{(2)} \right] + 3bA_{-3}(t) \left[ |\mathcal{V} \alpha^{(1)}|^3 \mathcal{V} \alpha^{(2)} \right] \right) \\
+ c\mathcal{E} \frac{2}{t} \left( \xi^{(2)} \Omega_3 \left[ |\alpha^{(1)}|^3 \alpha^{(2)} - |\mathcal{V} \alpha^{(1)}|^3 \mathcal{V} \alpha^{(2)} \right] + cA_{-1}(t) \left[ |\mathcal{V} \alpha^{(1)}|^3 \mathcal{V} \alpha^{(2)} \right] \right).
\]

By (A.5) and Lemma A.2, we have
\[
\|\Omega_1\|_{L^\infty} \leq \frac{C}{t^{1/4}} (\|u\|_{H^2} + \|\mathcal{J} u\|_{H^1})^3.
\]

Therefore, by setting \( \mu_{1,1}(\xi) = \frac{a}{27\sqrt{3}} \xi^3 \), \( \mu_{2,1}(\xi) = \frac{b}{27\sqrt{3}} \xi^3 \), \( \mu_{3,1}(\xi) = -i\xi^3 \), we obtain (4.9) with \( l = 1 \).
Finally we consider the case of \( l = 2 \). From the identities (A.3) and (A.4), it follows that
\[
\partial_x^2 N(u, u_x) = \lambda u_x \cdot \overline{u_x} \partial_x(u_{xx}) + c \overline{u_x} \cdot u_x \partial_x(u_{xx}) + (\lambda u_{xx} + c u_{xx}) \partial_x(\|u_x\|^2)
\]
\[
+ 3a u_x (2u^2 + u_x \partial_x(u_{xx})) + 3b u_x (2u^2 + u_x \partial_x(u_{xx})) + \frac{1}{it} \partial_x r_1
\]
\[= h_2 + \frac{1}{it} r_2,
\]
where
\[
h_2 = -\lambda |u_{xx}|^2 u_x + 9a u^2_{xx} u_x + 9b u^2_{xx} u_x - c |u_{xx}|^2 u_x
\]
and
\[
r_2 = \lambda u_x \left( (\mathcal{J} u_{xx}) \overline{u_x} - u_{xx} \mathcal{J} u_x \right) + c \overline{u_x} \left( (\mathcal{J} u_{xx}) u_x - u_{xx} \mathcal{J} u_x \right)
\]
\[
+ (\lambda u_{xx} + c u_{xx}) \left( (\mathcal{J} u_{xx}) u_x - u_{xx} \mathcal{J} u_x \right) + 3a u_x (u_x \mathcal{J} u_{xx} - (\mathcal{J} u_x) u_{xx})
\]
\[
- 3b u_x (u_x \mathcal{J} u_{xx} - (\mathcal{J} u_x) u_{xx}) + \partial_x r_1.
\]

We deduce as before that
\[
\mathcal{G} \mathcal{U}^{-1} h_2 = -\frac{\lambda}{l} \xi^5 |\alpha|^2 \alpha + 9a \frac{\epsilon^2}{l} \mathcal{D}_3 \left[ i \xi^5 \alpha^3 \right]
\]
\[
+ 9b \frac{\epsilon^2}{l} \mathcal{D}_3 \left[ -i \xi^5 \alpha^3 \right] - \frac{c}{l} \epsilon^2 \mathcal{D}_1 \left[ -i \xi^5 |\alpha|^2 \alpha \right] + \frac{\Omega_2}{l}
\]
with
\[
\| \Omega_2 \|_{L^\infty} \leq C \frac{l^{1/4}}{\epsilon^3} \left( \|u\|_{H^2}^2 + \|\mathcal{J} u\|_{H^2} \right)^3.
\]

We also have
\[
\| \mathcal{G} \mathcal{U}^{-1} r_2 \|_{L^\infty} \leq C \frac{l^{1/2}}{\epsilon^2} \left( \|u\|_{H^2}^2 + \|\mathcal{J} u\|_{H^2} \right)^3
\]
by virtue of Lemma A.1. Now we set \( \mu_1 \xi^4 = \frac{a}{2\sqrt{3}} \epsilon^4 \), \( \mu_2 \xi^4 = \frac{b}{2\sqrt{3}} \epsilon^4 \), \( \mu_3 \xi^4 = -c \xi^4 \). Then we arrive at (4.9) with \( l = 2 \). This completes the proof of Lemma 4.3. \( \square \)

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