Crossing Numbers and Stress of Random Graphs

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Abstract. Consider a random geometric graph over a random point process in $\mathbb{R}^d$. Two points are connected by an edge if and only if their distance is bounded by a prescribed distance parameter. We show that projecting the graph onto a two dimensional plane is expected to yield a constant-factor crossing number (and rectilinear crossing number) approximation. We also show that the crossing number is positively correlated to the stress of the graph’s projection.

1 Introduction

An undirected abstract graph $G_0$ consists of vertices and edges connecting vertex pairs. An injection of $G_0$ into $\mathbb{R}^d$ is an injective map from the vertices of $G_0$ to $\mathbb{R}^d$, and edges onto curves between their corresponding end points but not containing any other vertex point. For $d \geq 3$, we may assume that distinct edges do not share any point (other than a common end point). For $d = 2$, we call the injection a drawing, and it may be necessary to have points where curves cross. A drawing is good if no pair of edges crosses more than once, nor meets tangentially, and no three edges share the same crossing point. Given a drawing $D$, we define its crossing number $\text{cr}(D)$ as the number points where edges cross. The crossing number $\text{cr}(G_0)$ of the graph itself is the smallest $\text{cr}(D)$ over all its good drawings $D$. We may restrict our attention to the rectilinear crossing number $\text{cr}(G_0)$, where edge curves are straight lines; note that $\text{cr}(G_0) \geq \text{cr}(G_0)$.

The crossing number and its variants have been studied for several decades, see, e.g., [29], but still many questions are widely open. We know the crossing numbers only for very few graph classes; already for $\text{cr}(K_n)$, i.e., on complete graphs with $n$ vertices, we only have conjectures, and for $\text{cr}(K_n)$ not even them. Since deciding $\text{cr}(G_0)$ is NP-complete [14] (and $\text{cr}$ even $\exists \mathbb{R}$-complete [4]), several attempts for approximation algorithms have been undertaken. The problem does not allow a PTAS unless $P=\text{NP}$ [6]. For general graphs, we currently do not know whether there is an $\alpha$-approximation for any constant $\alpha$. However, we can achieve constant ratios for dense graphs [13] and for bounded pathwidth graphs [3]. Other strong algorithms deal with graphs of maximum bounded degree and achieve either slightly sublinear ratios [12], or constant ratios for further restrictions such as embeddability on low-genus surfaces [15–17] or a bounded number of graph elements to remove to obtain planarity [7,9–11].
We will make use of the crossing lemma, originally due to [2, 24]: There are constants\(^4\) \(d \geq 4, c \geq \frac{1}{16}\) such that any abstract graph \(G_0\) on \(n\) vertices and \(m \geq dn\) edges has \(\text{cr}(G_0) \geq cm^3/n^2\). In particular for (dense) graphs with \(m = \Theta(n^2)\), this yields the asymptotically tight maximum of \(\Theta(m^2)\) crossings.

**Random Geometric Graphs (RGGs).** We always consider a geometric graph \(G\) as input, i.e., an abstract graph \(G_0\) together with a straight-line injection into \(\mathbb{R}^d\), for some \(d \geq 2\); we identify the vertices with their points. For a 2-dimensional plane \(L\), the postfix operator \(|_L\) denotes the projection onto \(L\).

Given a set of points \(V\) in \(\mathbb{R}^d\), the unit-ball graph (unit-disk graph if \(d = 2\)) is the geometric graph using \(V\) as vertices that has an edge between two points iff balls of radius 1 centered at these points touch or overlap. Thus, points are adjacent iff their distance is \(\leq 2\). In general, we may use arbitrary threshold distances \(\delta > 0\). We are interested in random geometric graphs (RGGs), i.e., when using a Poisson point process to obtain \(V\) for the above graph class.

**Stress.** When drawing (in particular large) graphs with straight lines in practice, stress is a well-known and successful concept, see, e.g., [5, 19, 20]: let \(G\) be a geometric graph, \(d_0, d_1\) two distance functions on vertex pairs—(at least) the latter of which depends on an injection—and \(w\) weights. We have:

\[
\text{stress}(G) := \sum_{v_1, v_2 \in V(G), v_1 \neq v_2} w(v_1, v_2) \cdot (d_0(v_1, v_2) - d_1(v_1, v_2))^2.
\]

In a typical scenario, \(G\) is injected into \(\mathbb{R}^2\), \(d_0\) encodes the graph-theoretic distances (number of edges on the shortest path) or some given similarity matrix, and \(d_1\) is the Euclidean distance in \(\mathbb{R}^2\). Intuitively, in a drawing of 0 (or low) stress, the vertices' geometric distances \(d_1\) are (nearly) identical to their “desired” distance according to \(d_0\). A typical weight function \(w(v_1, v_2) := d_0(v_1, v_2)^{-2}\) softens the effect of “bad” geometric injections for vertices that are far away from each other anyhow. It has been observed empirically that low-stress drawings tend to be visually pleasing and to have a low number of crossings, see, e.g., [8, 21]. While it may seem worthwhile to approximate the crossing number by minimizing a drawing’s stress, there is no sound mathematical basis for this approach.

There are different ways to find (close to) minimal-stress drawings in 2D [5]. One way is multidimensional scaling, cf. [19], where we start with an injection of an abstract graph \(G_0\) into some high-dimensional space \(\mathbb{R}^d\) and asking for a projection of it onto \(\mathbb{R}^2\) with minimal stress. It should be understood that Euclidean distances in a unit-ball graph in \(\mathbb{R}^d\) by construction closely correspond to the graph-theoretic distances. In fact, for such graphs it seems reasonable to use the distances in \(\mathbb{R}^d\) as the given metrics \(d_0\), and seek an injection into \(\mathbb{R}^2\)—whose resulting distances form \(d_1\)—by means of projection.

\(^3\) Incidentally, the lemma allows an intriguingly elegant proof using stochastics [1].

\(^4\) The currently best constants \(d = 7, c = \frac{1}{20}\) are due to [18].
Contribution. We consider RGGs for large $t$ and investigate the mean, variance, and corresponding law of large numbers both for their rectilinear crossing number and their minimal stress when projecting them onto the plane. We also prove, for the first time, a positive correlation between these two measures.

While our technical proofs make heavy use of stochastic machinery (several details of which have to be deferred to the appendix), the consequences are very algorithmic: We give a surprisingly simple algorithm that yields an expected constant approximation ratio for random geometric graphs even in the pure abstract setting. In fact, we can state the algorithm already now; the remainder of this paper deals with the proof of its properties and correctness:

Given a random geometric graph $G$ in $\mathbb{R}^d$ (see below for details), we pick a random 2-dimensional plane $L$ in $\mathbb{R}^d$ to obtain a straight-line drawing $G|_L$ that yields a crossing number approximation both for $\overline{cr}(G)$ and for $cr(G_0)$.

Throughout this paper, we prefer to work within the setting of a Poisson point process because of the strong mathematical tools from the Malliavin calculus that are available in this case. It is straightforward to de-Poissonize our results: this yields asymptotically the same results—even with the same constants—for $n$ uniform random points instead of a Poisson point process; we omit the details.

2 Notations and Tools from Stochastic Geometry

Let $W \subset \mathbb{R}^d$ be a convex set of volume $\text{vol}_d(W) = 1$. Choose a Poisson distributed random variable $n$ with parameter $t$, i.e., $E(n) = t$. Next choose $n$ points $V = \{v_1, \ldots, v_n\}$ independently in $W$ according to the uniform distribution. Those points form a Poisson point process $V$ in $W$ of intensity $t$. A Poisson point process has several nice properties, e.g., for disjoint subsets $A, B \subset W$, the sets $V \cap A$ and $V \cap B$ are independent (thus also their size is independent).

Let $V^k_\frac{1}{1}$, $k \geq 1$, be the set of all ordered $k$-tuples over $V$ with pairwise distinct elements. We will consider $V$ as the vertex set of a geometric graph $G$ for the distances parameter $(\delta_t)_{t>0}$ with edges $E = \{\{u, v\} \mid u, v \in V, u \neq v, \|u - v\| \leq \delta_t\}$, i.e., we have an edge between two distinct points if and only if their distance is at most $\delta_t$. Such random geometric graphs (RGG) have been extensively investigated, see, e.g., [26, 28], but nothing is known about the stress or crossing number of its underlying abstract graph $G_0$.

A U-statistic $U(k, f) := \sum_{v \in V^k_\frac{1}{1}} f(v)$ is the sum over $f(v)$ for all $k$-tuples $v$. Here, $f$ is a measurable non-negative real-valued function, and $f(v)$ only depends on $v$ and is independent of the rest of $V$. The number of edges in $G$ is a U-statistic as $m = \frac{1}{2} \sum_{v, u \in V, v \neq u} 1(\|v - u\| \leq \delta_t)$. Likewise, the stress of a geometric graph as well as the crossing number of a straight-line drawing is a U-statistic, using 2- and 4-tuples of $V$, respectively. The well-known multivariate Slivnyak-Mecke formula tells us how to compute the expectation $E_V$ over all realizations of the Poisson process $V$; for U-statistics we have, see [30, Cor. 3.2.3]:

$$E_V \sum_{(v_1, \ldots, v_k) \in V^k_\frac{1}{1}} f(v_1, \ldots, v_k) = t^k \int_{W^k} f(v_1, \ldots, v_k) dv_1 \cdots dv_k.$$ (2)
We already know $E_V n = E_V |V| = t$. Solving the above formula for the expected number of edges, we obtain
\[
E_V m = E_V |E| = \frac{\kappa_d}{2} t^2 \delta_t^d + O(t^2 \delta_t^{d+1} \surf(W)),
\]
where $\kappa_d = \text{vol}(B_d)$ is the volume of the unit ball $B_d$ in $\mathbb{R}^d$, and $\surf(W)$ the surface area of $W$. For $n$ and $m$, central limit theorems and concentration inequalities are well known as $t \to \infty$, see, e.g., [26, 28].

The expected degree $E_V \deg(v)$ of a typical vertex $v$ is approximately of order $\kappa_d t \delta_t^d$ (this can be made precise using Palm distributions). This naturally leads to three different asymptotic regimes as introduced in Penrose’s book [26]:

- in the sparse regime we have $\lim_{t \to \infty} t \delta_t^d = 0$, thus $E_V \deg(v)$ tends to zero;
- in the thermodynamic regime we have $\lim_{t \to \infty} t \delta_t^d = c > 0$, thus $E_V \deg(v)$ is asymptotically constant;
- in the dense regime we have $\lim_{t \to \infty} t \delta_t^d = \infty$, thus $E_V \deg(v) \to \infty$.

Observe that in standard graph theoretic terms, the thermodynamic regime leads to sparse graphs, i.e., via (3) we obtain $E_V m = \Theta(t) = \Theta(E_V n)$. Similarly, the dense regime—together with $\delta_t \to c$—leads to dense graphs, i.e., $E_V m = \Theta(t^2) = \Theta((E_V n)^2)$. Recall that to employ the crossing lemma, we want $m \geq 4n$. Also, the lemma already shows that any good (straight-line) drawing of a dense graph $G_0$ already gives a constant-factor approximation for $\cr(G_0)$ (and $\overline{\cr}(G_0)$). In the following we thus assume a constant $0 < c \leq t \delta_t^d$ and $\delta_t \to 0$, i.e., $m = o(n^2)$.

The Slivnyak-Mecke formula is a classical tool to compute expectations and will thus be used extensively throughout this paper. Yet, suitable tools to compute variances came up only recently. They emerged in connection with the development of the Malliavin calculus for Poisson point processes [22, 25]. An important operator for functions $g(V)$ of Poisson point processes is the difference (also called add-one-cost) operator,
\[
D_v g(V) := g(V \cup \{v\}) - g(V),
\]
which considers the change in the function value when adding a single further point $v$. We know that there is a Poincaré inequality for Poisson functionals [22, 31], yielding the upper bound in (4) below. On the other hand, the isometry property of the Wiener-Itô chaos expansion [23] of an (square integrable) $L^2$-function $g(V)$ leads to the lower bound in (4):
\[
t \int_W (E_V D_v g(V))^2 \, dv \leq \Var_V g(V) \leq t \int_W E_V (D_v g(V))^2 \, dv. \tag{4}
\]

Often, in particular in the cases we are interested in in this paper, the bounds are sharp in the order of $t$ and often even sharp in the occurring constant. This is due to the fact that the Wiener-Itô chaos expansion, the Poincaré inequality, and the lower bound are particularly well-behaved for Poisson U-statistics [27].
3 Rectilinear Crossing Number of an RGG

Let \( \mathcal{L} \) be the set of all two-dimensional linear planes and \( L \in \mathcal{L} \) be a random plane chosen according to a (uniform) Haar probability measure on \( \mathcal{L} \). The drawing \( G_L := G|_L \) is the projection of \( G \) onto \( L \). Let \([u, v]\) denote the segment between vertex points \( u, v \in V \) if their distance is at most \( \delta \) and \( \emptyset \) otherwise. The rectilinear crossing number of \( G_L \) is a U-statistic of order 4:

\[
\mathfrak{c}(G_L) = \frac{1}{8} \sum_{(v_1, v_2, v_3, v_4) \in V^4} \mathbf{1}([v_1, v_2]|_L \cap [v_3, v_4]|_L \neq \emptyset).
\]

Keep in mind that even for the best possible projection we only obtain

\[
\min_{L \in \mathcal{L}} \mathfrak{c}(G|_L) \geq \mathfrak{c}(G_0).
\]

To analyze \( \mathbb{E}_V \min_{L \in \mathcal{L}} \mathfrak{c}(G|_L) \) is more complicated than \( \mathbb{E}_L, V \mathfrak{c}(G|_L) \); fortunately, we will not require it.

3.1 The Expectation of the Rectilinear Crossing Numbers

For the expectation with respect to the underlying Poisson point process the Slivnyak-Mecke formula (2) gives

\[
\mathbb{E}_V \mathfrak{c}(G_L) = \frac{1}{8} t^4 \int \int \mathbf{1}([v_1, v_2]|_L \cap [v_3, v_4]|_L \neq \emptyset) dv_3 dv_2 dv_4 dv_1.
\]

Let \( c_d \) be the constant given by the expectation of the event that two independent edges cross. In Appendix A, we prove in Proposition 15 that \( c_d \leq 2 \pi \kappa^2_d \), that \( \frac{I_W(v_1)}{\delta^{d+2}_t} \) is bounded by \( c_d \) times the volume of the maximal \((d-2)\)-dimensional section of \( W \), and that

\[
\lim_{\delta_t \to 0} \frac{I_W(v_1)}{\delta^{d+2}_t} = c_d \text{vol}_{d-2}((v_1 + L^\perp) \cap W),
\]

where \( L^\perp \) is the \( d - 2 \) dimensional hyperplane perpendicular to \( L \). Using the dominated convergence theorem of Lebesgue and Fubini’s theorem we obtain

\[
\lim_{t \to \infty} \frac{\mathbb{E}_V \mathfrak{c}(G_L)}{t^4 \delta^{2d+2}_t} = \frac{1}{8} c_d \int_W \text{vol}_{d-2}((v_1 + L^\perp) \cap W) dv_1
\]

\[
= \frac{1}{8} c_d \int_{W \cap (v_1^L + L^\perp) \cap W} \text{vol}_{d-2}((v_1^L + L^\perp) \cap W) dv_1 dv_1^L
\]

\[
= \frac{1}{8} c_d \int_{W \cap L} \text{vol}_{d-2}((v_1^L + L^\perp) \cap W) dv_1^L.
\]

\[= I^{(2)}(W, L)\]
Theorem 1. Let $G_L$ be the projection of an RGG onto a two-dimensional plane $L$. Then, as $t \to \infty$ and $\delta t \to 0$,

$$E_V\, \mathcal{C}(G_L) = \frac{1}{8} c_d t^{4} \delta^{2d+2} \int (W, L) + o(\delta^{2d+2} t^{4}).$$

For unit-disk graphs, i.e., $d = 2$, the choice of $L$ is unique and the projection superfluous. There the expected crossing number is asymptotically $c_2 \frac{m^3}{n^2}$ and thus of order $\Theta(m^3/n^2)$ which is asymptotically optimal as witnessed by the crossing lemma. In general, the expectation is of order

$$t^4 \delta^{2d+2} = \Theta \left( \frac{m^3}{n^2} \left( \frac{m}{n^2} \right)^{\frac{2-d}{d}} \right).$$

The extra factor $m/n^2$ can be understood as the probability that two vertices are connected via an edge, thus measures the “density” of the graph.

3.2 The Variance of the Rectilinear Crossing Numbers

By the variance inequalities (4) for functionals of Poisson point processes we are interested in the moments of the difference operator of the crossing numbers:

$$E_V D_v \, \mathcal{C}(G_L) = \frac{1}{8} E_V \sum_{(v_2, \ldots, v_4) \in V_3^4} \mathbb{1}\{[v, v_2]_L \cap [v_3, v_4]_L \neq \emptyset\} = \frac{1}{8} t^3 I_W(v) \quad (6)$$

$$E_V (D_v \, \mathcal{C}(G_L))^2 = E_V \left( \frac{1}{8} \sum_{(v_2, \ldots, v_4) \in V_3^4} \mathbb{1}\{[v, v_2]_L \cap [v_3, v_4]_L \neq \emptyset\}\right)^2 \quad (7)$$

Plugging (7) into the Poincaré inequality (4) gives

$$\text{Var}_V \, \mathcal{C}(G_L) \leq \frac{1}{64} t \int_W E_V \left( \sum_{(v_2, \ldots, v_4) \in V_3^4} \mathbb{1}\{[v, v_2]_L \cap [v_3, v_4]_L \neq \emptyset\}\right)^2 dv.$$

Using calculations from integral geometry (see Appendix B), there is a constant $0 < c_d' \leq 2 \pi \kappa_d c_d$ (given by the expectation of the event that two pairs of independent edges cross) such that

$$\text{Var}_V \, \mathcal{C}(G_L) \leq \frac{1}{64} \left( c_d'' + c_d'\frac{t \delta^{2d+4}}{t \delta t^d} \right) t^7 \delta^{4d+4} \int_W \text{vol}_{d-2}\{(v + L)^\perp \cap W\}^2 (1 + o(1)) dv$$

$$+ O(\max\{t^6 \delta^{4d+2} \delta^{3d+2}, t^5 \delta^{3d+2}, t^4 \delta^{2d+2}\}).$$

We use that $t \delta_t^d \geq c > 0$, assume $d \geq 3$, and use Fubini’s theorem again.

$$\lim_{t \to \infty} \frac{\text{Var}_V \, \mathcal{C}(G_L)}{t^7 \delta^{4d+4}} \leq \frac{1}{64} \left( c_d'' + c_d' \lim_{t \to \infty} \frac{1}{t \delta_t^d} \right) \int_{W \cap L} \text{vol}_{d-2}\{(v + L)^\perp \cap W\}^3 dv.$$

$$= I^3(W, L)$$
On the other hand, (6) and the lower bound in (4) gives in our case
\[
\text{Var}_V \overline{\mathrm{cr}}(G_L) \geq t \int W (\text{Var}_V D_v \overline{\mathrm{cr}}(G_L))^2 dv \\
\geq \frac{1}{64} t^7 \int W (v)^2 dv = \frac{1}{64} c_d^2 t^7 \delta_t^{4d+4} I(3)(W, L)(1 + o(1)).
\]
Thus our bounds have the correct order and, in the dense regime where \( t \delta_t \to \infty \), are even sharp. Using \( 0 < c_d' \leq 2\pi \kappa_d c_d \) we obtain:

**Theorem 2.** Let \( G_L \) be the projection of an RGG in \( \mathbb{R}^d \), \( d \geq 3 \), onto a two-dimensional plane \( L \). Then, as \( t \to \infty \) and \( \delta_t \to 0 \),
\[
\frac{1}{64} c_d^2 I(3)(W, L) \leq \lim_{t \to \infty} \frac{\text{Var}_V \overline{\mathrm{cr}}(G_L)}{t^7 \delta_t^{4d+4}} \leq \frac{1}{64} (c_d^2 + 2\pi \kappa_d c_d \lim_{t \to \infty} \frac{1}{t \delta_t^2}) I(3)(W, L).
\]

Theorem 1 and Theorem 2 show for the standard deviation
\[
\sigma(\overline{\mathrm{cr}}(G_L)) = \sqrt{\text{Var}_V \overline{\mathrm{cr}}(G_L)} = \Theta(t^d \delta_t^{2d+2} t^{-\frac{1}{2}}) = \Theta(\text{E}_V \overline{\mathrm{cr}}(G_L)(\text{E}_V n)^{-\frac{1}{2}}),
\]
which is smaller than the expectation by a factor \( (\text{E}_V n)^{-\frac{1}{2}} = t^{-\frac{d}{2}} \). Or, equivalently, the coefficient of variation \( \frac{\sigma(\overline{\mathrm{cr}}(G_L))}{\text{E}(\overline{\mathrm{cr}}(G_L))} \) is of order \( t^{-\frac{d}{2}} \). As \( t \to \infty \), our bounds on the expectation and variance together with Chebyshev’s inequality lead to
\[
P\left( \left| \frac{\overline{\mathrm{cr}}(G_L)}{t^d \delta_t^{2d+2}} - \frac{\text{E}_V \overline{\mathrm{cr}}(G_L)}{t^d \delta_t^{2d+2}} \right| \geq \varepsilon \right) \leq \frac{\text{Var}_V \overline{\mathrm{cr}}(G_L)}{t^d \delta_t^{2d+2} \varepsilon^2} \to 0.
\]

**Corollary 3 (Law of Large Numbers).** For given \( L \), the normalized random crossing number converges in probability (with respect to the Poisson point process \( V \)) as \( t \to \infty \),
\[
\overline{\mathrm{cr}}(G_L) \xrightarrow{t^d \delta_t^{2d+2}} \frac{1}{8} c_d I(2)(W, L).
\]

Until now we fixed a plane \( L \) and computed the variance with respect to the random points \( V \). Theorem 1 and Theorem 2 allow to compute the expectation and variance with respect to \( V \) and a randomly chosen plane \( L \). For the expectation we obtain from Theorem 1 and by Fubini’s theorem
\[
\text{E}_{L, V} \overline{\mathrm{cr}}(G_L) = \frac{1}{8} c_d t^d \delta_t^{2d+2} \int_L I(2)(W, L) dL + o(t^d \delta_t^{2d+2}), \tag{8}
\]
as \( t \to \infty \) and \( \delta_t \to 0 \), where \( dL \) denotes integration with respect to the Haar measure on \( L \). For simplicity we assume in the following that \( \lim_{t \to \infty} t \delta_t^{-1} = 0 \). We use the variance decomposition \( \text{Var}_{L, V} X = \text{E}_{L} \text{Var}_{V} X + \text{Var}_{L} \text{E}_{V} X \). By
\[
\text{E}_L \text{Var}_V \overline{\mathrm{cr}}(G_L) = \frac{1}{64} c_d^2 t^7 \delta_t^{4d+4} \int_L I(3)(W, L) dL + o(t^7 \delta_t^{4d+4}), \quad \text{and}
\]

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Theorem 4.

\[
\text{Var}_L E_V \overline{\text{cr}}(G_L) = E_L (E_V \overline{\text{cr}}(G_L))^2 - (E_{L,V} \overline{\text{cr}}(G_L))^2 = \\
\frac{1}{64} c_d^2 t^8 \delta_t^{d+4} \left[ \int \mathcal{L} I(2)(W, L)^2 dL - \left( \int \mathcal{L} I(2)(W, L) dL \right)^2 \right] + o(t^8 \delta_t^{d+4})
\]

we obtain

\[
\text{Var}_{L,V} \overline{\text{cr}}(G_L) = \frac{1}{64} c_d^2 t^8 \delta_t^{d+4} \left[ \int \mathcal{L} I(2)(W, L)^2 dL - \left( \int \mathcal{L} I(2)(W, L) dL \right)^2 \right] + o(t^8 \delta_t^{d+4}).
\]

Hölder’s inequality implies that the term in brackets is positive as long as \( I(2)(W, L) \) is not a constant function.

3.3 The Rotation Invariant Case

If \( W \) is the ball \( B \) of unit volume and thus \( V \) is rotation invariant, then \( I(2)(B, L) = I(2)(B) \) is a constant function independent of \( L \), and the leading term in (9) is vanishing. From (8) we see that in this case the expectation is independent of \( L \).

\[
E_V \overline{\text{cr}}(G_L) = E_L E_V \overline{\text{cr}}(G_L) = t^4 d_t^{2d+2} I(2)(B) + o(t^4 d_t^{2d+2})
\]

For the variance this implies \( \text{Var}_L E_V \overline{\text{cr}}(G_L) = 0 \), and hence

\[
\text{Var}_{L,V} \overline{\text{cr}}(G_L) = E_L \text{Var}_V \overline{\text{cr}}(G_L) = \frac{1}{64} c_d^2 t^7 \delta_t^{d+4} I(3)(B) + o(t^7 \delta_t^{d+4}).
\]

In this case the variance \( \text{Var}_{L,V} \) is of the order \( t^{-1} \)—and thus surprisingly significantly—smaller than in the general case.

Theorem 4. Let \( G_L \) be the projection of an RGG in the ball \( B \subset \mathbb{R}^d \), \( d \geq 3 \), onto a two-dimensional uniformly chosen random plane \( L \). Then

\[
E_{L,V} \overline{\text{cr}}(G_L) = \frac{1}{8} c_d t^4 d_t^{2d+2} I(2)(B) + o(t^4 d_t^{2d+2}) \quad \text{and}
\]

\[
\text{Var}_{L,V} \overline{\text{cr}}(G_L) = \frac{1}{64} c_d^2 t^7 \delta_t^{d+4} I(3)(B) + o(t^7 \delta_t^{d+4}),
\]

as \( t \to \infty \), \( \delta_t \to 0 \) and \( t\delta_t^d \to \infty \).

Again, Chebychev’s inequality immediately yields a law of large numbers which states that with high probability the crossing number of \( G_L \) in a random direction is very close to \( \frac{1}{8} c_d t^4 d_t^{2d+2} I(2)(B) \).

Corollary 5 (Law of Large Numbers). Let \( G_L \) be the projection of an RGG in \( B \subset \mathbb{R}^d \), \( d \geq 3 \), onto a random two-dimensional plane \( L \). Then the normalized random crossing number converges in probability (with respect to the Poisson point process \( V \) and to \( L \)), as \( t \to \infty \),

\[
\frac{\overline{\text{cr}}(G_L)}{t^4 d_t^{2d+2}} \to \frac{1}{8} c_d I(2)(B).
\]
As known by the crossing lemma, the optimal crossing number is of order \( \frac{m^3}{n} \). In our setting this means that we are looking for the optimal direction of projection which leads to a crossing number of order \( t^4 \delta_t^{3d} \), much smaller than the expectation \( \mathbb{E}_V \overline{\text{cr}}(G_L) \). Chebychev’s inequality shows that if \( W = B \) it is difficult to find this optimal direction and to reach this order of magnitude; using \( \delta_t \to 0 \) in the last step we have:

\[
P_{L,V}(\overline{\text{cr}}(G_L) \leq c t^4 \delta_t^{3d}) \leq \mathbb{P}_{L,V}(\overline{\text{cr}}(G_L) - \mathbb{E}_{L,V}\overline{\text{cr}}(G_L) \geq \mathbb{E}_{L,V}\overline{\text{cr}}(G_L) - c t^4 \delta_t^{3d}) \leq \frac{\text{Var}_{L,V}(\overline{\text{cr}}(G_L))}{(\mathbb{E}_{L,V}\overline{\text{cr}}(G_L) - c t^4 \delta_t^{3d})^2} = O(t^{-1}).
\]

Hence a computational naïve approach of minimizing the crossing numbers by just projecting onto a sample of random planes seems to be expensive. This suggests to combine the search for an optimal choice of the direction of projection with other quantities of the RGG. It is a long standing assumption in graph drawing that there is a connection between the crossing number and the stress of a graph. Therefore the next section is devoted to investigations concerning the stress of RGGs.

4 The Stress of an RGG

According to (1) we define the stress of \( G_L \) as

\[
\text{stress}(G, G_L) := \frac{1}{2} \sum_{(v_1, v_2) \in V^2} w(v_1, v_2)(d_0(v_1, v_2) - d_L(v_1, v_2))^2,
\]

where \( w(v_1, v_2) \) a positive weight-function and \( d_0 \) resp. \( d_L \) are the distances between \( v_1 \) and \( v_2 \), resp \( v_1|L \) and \( v_2|L \). As \( \overline{\text{cr}}(G) \), stress is a U-statistic, but now of order two. Using the Slivnyak-Mecke formula, it is immediate that

\[
\mathbb{E}_V \text{stress}(G, G_L) = \frac{1}{2} \int_{W^2} w(v_1, v_2)(d_0(v_1, v_2) - d_L(v_1, v_2))^2 dv_1 dv_2.
\]

For the variance, the Poincaré inequality (4) implies

\[
\text{Var}_V \text{stress}(G, G_L) \leq t \int_W \mathbb{E}_V(D_v(\text{stress}(G, G_L)))^2 dv
\]

\[
= \frac{1}{4} t \int_W \mathbb{E}_V \left( \sum_{v_1 \in V} w(v, v_1)(d_0(v, v_1) - d_L(v, v_1))^2 \right)^2 dv
\]

\[
= \frac{1}{4} t \mathbb{E}_V \left( \sum_{v_1 \in V} w(v, v_1)(d_0(v, v_1) - d_L(v, v_1))^2 \right)^2.
\]
\[
\begin{align*}
= \frac{1}{4} t^3 \int \prod_{i=1}^{2} \left( w(v, v_i)(d_0(v, v_i) - d_L(v, v_i))^2 \right) dv_1 dv_2 dv \\
\mathcal{S}^{(2)}(W, L) \\
+ \frac{1}{4} t^2 \int \frac{w(v, v_1)^2(d_0(v, v_1) - d_L(v, v_1))^4}{W^2} dv_1 dv.
\end{align*}
\]

Hence the standard deviation of the stress is smaller than the expectation by a factor \( t^{-\frac{1}{2}} \) and thus the stress is concentrated around its mean. Again the computation of the lower bound for the variance in (4) is asymptotically sharp.

\[
\text{Var}_{V} \text{stress}(G, G_L) \geq \frac{1}{4} t^2 \int \left( \mathbb{E}_v \sum_{v_1 \in V} w(v, v_1)(d_0(v, v_1) - d_L(v, v_1))^2 \right) dv = \frac{1}{4} t^3 \mathcal{S}^{(2)}(W, L).
\]

**Theorem 6.** Let \( G_L \) be the projection of an RGG in \( \mathbb{R}^d, d \geq 3 \), onto a two-dimensional plane \( L \). Then

\[
\mathbb{E}_V \text{stress}(G, G_L) = \frac{1}{2} t^2 \mathcal{S}^{(1)}(W, L) \quad \text{and}
\]

\[
\text{Var}_{V} \text{stress}(G, G_L) = \frac{1}{4} t^3 \mathcal{S}^{(2)}(W, L) + O(t^2).
\]

The discussions from Section 3.2 and Section 3.3 lead to analogous results for the stress of the RGG. Using Chebychev’s inequality we could derive a law of large numbers. Taking expectations with respect to a uniform plane \( L \) we obtain:

\[
\mathbb{E}_{L,V} \text{stress}(G, G_L) = \frac{1}{2} t^2 \int_{L} \mathcal{S}^{(1)}(W, L) dL,
\]

\[
\text{Var}_{L,V} \text{stress}(G, G_L) = \frac{1}{4} t^4 \left[ \int_{L} \mathcal{S}^{(1)}(W, L)^2 dL - \left( \int_{L} \mathcal{S}^{(1)}(W, L) dL \right)^2 \right] + O(t^3).
\]

Again, the term in brackets is only vanishing if \( W = B \). In this case

\[
\text{Var}_{L,V} \text{stress}(G, G_L) = \mathbb{E}_L \text{Var}_V \text{stress}(G, G_L) = \frac{1}{4} t^3 \mathcal{S}^{(2)}(B) + O(t^2).
\]

### 5 Correlation between Crossing Number and Stress

It seems to be widely conjectured that the crossing number and the stress should be positively correlated. Yet it also seems that a rigorous proof is still missing.
It is the aim of this section to provide the first proof of this conjecture, in the case where the graph is a random geometric graph.

Clearly, by the definition of $\text{cr}$ and stress we have

$$D_v \text{cr}(G_L) \geq 0 \quad \text{and} \quad D_v \text{stress}(G, G_L) \geq 0,$$

for all $v$ and all realizations of $V$. Such a functional $F$ satisfying $D_v(F) \geq 0$ is called increasing. The Harris-FKG inequality for Poisson point processes [22] links this fact to the correlation of $\text{cr}(G_L)$ and stress$(G, G_L)$.

**Theorem 7.** Because stress and $\text{cr}$ are increasing we have

$$E \text{cr}(G_L) \text{stress}(G, G_L) \geq E \text{cr}(G_L) E \text{stress}(G, G_L),$$

and thus the correlation is positive.

We immediately obtain that the covariance is positive and is of order at most

$$\text{Cov}_V (\text{cr}(G_L), \text{stress}(G, G_L)) \leq \sqrt{\text{Var}_V \text{cr}(G_L) \text{Var}_V \text{stress}(G, G_L)} \leq \frac{1}{16} \left(1 + \frac{2\pi \kappa d}{c_d} \lim_{t \to \infty} \frac{1}{t^{d+2}} \right)^{1/2} I^{(3)}(W, L)^{1/2} S^{(2)}(W, L)^{1/2} + o(t^{5\delta_2 + 2}).$$

In Appendix C we use Mehler’s formula to prove a lower bound:

$$\text{Cov}_V (\text{cr}(G_L), \text{stress}(G, G_L)) \geq \frac{t^5}{16} \int \frac{I_W(v, v_1) (d_0(v, v_1) - d_L(v, v_1))^2 dv_1 dv}{W^2}.$$

We combine this bound with (5), divide by the standard deviations from Theorem 2 and Theorem 6 and obtain the asymptotics for the correlation coefficient:

**Theorem 8.** Let $G_L$ be the projection of an RGG in $\mathbb{R}^d$, $d \geq 3$, onto a two-dimensional plane $L$. Then

$$\lim_{t \to \infty} \text{Corr}_V (\text{cr}(G_L), \text{stress}(G, G_L)) = \frac{1}{W^2} \int_{W^2} \text{vol}_{d-2}((v + L^\perp) \cap W) w(v, v_1) (d_0(v, v_1) - d_L(v, v_1))^2 dv_1 dv \geq \frac{1}{W^2} \left(1 + \frac{2\pi \kappa d}{c_d} \lim_{t \to \infty} \frac{1}{t^{d+2}} \right)^{1/2} I^{(3)}(W, L)^{1/2} S^{(2)}(W, L)^{1/2}.$$

It can be shown that this bound is even tight and asymptotically gives the correct correlation coefficient.

### 5.1 The Rotation Invariant Case

In principle the bounds for the covariance in the Poisson point process $V$ given above can be used to compute covariance bounds in $L$ and $V$ when $L$ is not fixed but random. For this we could use the covariance decomposition

$$\text{Cov}_{L,V}(X, Y) = E_L \text{Cov}_V(X, Y) + \text{Cov}_L(E_V X, E_V Y).$$

Here we concentrate again on the case when $W = B$ is the ball of unit volume and thus $V$ is rotation invariant. Then $\text{Cov}_L(E_V \text{cr}(G_L), E_V \text{stress}(G_L)) = 0$, and as an immediate consequence of Theorem 8 we obtain
**Corollary 9.** Let $G_L$ be the projection of an RGG in $B \subset \mathbb{R}^d$, $d \geq 3$, onto a two-dimensional random plane $L$. Then the correlation between the crossing number and the stress of the RGG is positive with

$$\lim_{t \to \infty} \text{Corr}_{L,V}(\text{cr}(G_L), \text{stress}(G, G_L)) \geq \frac{\int_{B^2} \text{vol}_{d-2}(v + L^\perp) \cap B)w(v, v_1)(d_0(v, v_1) - d_L(v, v_1))^2dv_1dv}{(1 + \frac{2\pi\kappa_d}{\epsilon_d}\lim_{t \to \infty} \frac{1}{\epsilon_d})^{\frac{1}{2}}I(3)(B)^{\frac{1}{2}}S(2)(B)^{\frac{1}{2}}}.\$$

In particular, the correlation does not vanish as $t \to \infty$. This gives the first proof we are aware of, that there is a strict positive correlation between the crossing number and the stress of a graph. Hence, at least for RGGs, the method to optimize the stress to obtain good crossing numbers can be supported by rigorous mathematics.

**6 Consequences and Conclusion**

Apart from providing precise asymptotics for the crossing numbers of drawings of random geometric graphs, the main findings are the positive covariance and the non-vanishing correlation between the stress and the crossing number of the drawing of a random geometric graph. Of interest would be whether $\text{Cov}_L(\text{cr}(G_L), \text{stress}(G, G_L)) > 0$ for arbitrary graphs $G$. Yet there are simple examples of graphs $G$ where this is wrong. Yet we could ask in a slightly weaker form whether at least $\mathbb{E}_V \text{Cov}_L(\text{cr}(G_L), \text{stress}(G_L)) > 0$, but we have not been able to prove that.

We may coarsely summarize the gist of all the above findings algorithmically in the context of crossing number approximation, ignoring precise numeric terms that can be found above. We yield the first (expected) crossing number approximations for a rich class of randomized graphs:

**Corollary 10.** Let $G$ be a random geometric graph in $\mathbb{R}^2$ (unit-disk graph) as defined above. With high probability, the number of crossings in its natural straight-line drawing is at most a constant factor away from $\text{cr}(G_0)$ and $\overline{\text{cr}}(G_0)$.

**Corollary 11.** Let $G$ be a random geometric graph in $\mathbb{R}^d$ (unit-ball graph) as defined above. We obtain a straight-line drawing $D$ by projecting it onto a randomly chosen 2D plane. With high probability, the number of crossings in $D$ is at most a factor $\alpha$ away from $\text{cr}(G_0)$ and $\overline{\text{cr}}(G_0)$. Thereby, $\alpha$ is only dependent on the graph’s density.

**Corollary 12.** Let $G$ be a random geometric graph and use its natural distances in $\mathbb{R}^d$ as input for stress minimization. The stress is positively correlated to the crossing number. Loosely speaking, a drawing of $G$ with close to minimal stress is expected to yield a close to minimal number of crossings.
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A Towards the Expectation of Crossing Number

From now on—for all calculations in Appendix A, B, and C—\([u, v]\) denotes the segment between vertex points \(u, v \in V\). If their distance has to be at most \(\delta_t\) this condition is explicitly given by an indicator function or the domain of the integral.

The following integrals show up in several calculations and play an essential role in determining the asymptotic behavior of the moments of \(\tau(G_L)\).

**Lemma 13.** Let \(K\) be a convex body and \(L\) a 2-dimensional plane in \(\mathbb{R}^d\). Define

\[
J_{L}^{(1)}(K) := \int_{(\delta_t B_d) \times K \times (\delta_t B_d)} \mathbb{1}([0, x]_L \cap (y + [0, z])_L \neq \emptyset) \, dy \, dx.
\]

Then

\[
J_{L}^{(1)}(K) = c_d \delta_t^{2d+2} \text{vol}_{d-2}(L^\perp \cap K) \,(1 + o(1))
\]

as \(\delta_t \to 0\), with

\[
c_d = \int_{B_d \times (2B_d) \times B_d} \mathbb{1}([0, x]_L \cap (y^L + [0, z]_L) \neq \emptyset) \, dz \, dy \, dx \leq 2\pi \kappa_2^3.
\]

Further we have

\[
J_{L}^{(1)}(K) \leq c_d \delta_t^{2d+2} M_{d-2}(K)
\]

where \(M_{d-2}(K)\) is the volume of the maximal \((d-2)\)-dimensional section of \(K\).

**Proof.** We write \(y = (y^L, y^L^\perp)\) with \(y^L \in L, y^L^\perp \in L^\perp\). Clearly, for \([y, z]\) to meet \([0, x]\) we need that \(y\) is at least contained in a cylinder of radius \(2\delta_t\) above the origin, \(y^L \in 2\delta_t B_2 \subset L\). Using Fubini’s theorem we obtain

\[
J_{L}^{(1)}(K) = \int_{(\delta_t B_d) \times K \times (\delta_t B_d)} \mathbb{1}([0, x]_L \cap (y + [0, z])_L \neq \emptyset) \, dy \, dx
\]

\[
= \int_{(\delta_t B_d) \times K \times (\delta_t B_d)} \mathbb{1}([0, x]_L \cap (y^L + [0, z]_L) \neq \emptyset) \int_{L^\perp} \mathbb{1}((y^L, y^L^\perp) \in K) \, dy^L \, dz \, dy^L \, dx
\]

\[
= \int_{(\delta_t B_d) \times K \times (\delta_t B_d)} \mathbb{1}([0, x]_L \cap (y^L + [0, z]_L) \neq \emptyset) \int_{L^\perp} \mathbb{1}((y^L + L^\perp) \cap K) \, dy^L \, dx
\]

\[
\leq \delta_t^{2d+2} \int_{B_d \times (2B_d) \times B_d} \mathbb{1}([0, x]_L \cap (y^L + [0, z]_L) \neq \emptyset) \int_{L^\perp} \mathbb{1}((y^L + L^\perp) \cap K) \, dy^L \, dx
\]

\[
\leq c_d \delta_t^{2d+2} \max_{u \in \delta_t B_d} \text{vol}_{d-2}((u + L^\perp) \cap K).
\]
with
\[ c_d = \int_{B_d \times (2B_d) \times B_d} 1((0, x|_L] \cap (y^L + [0, z|_L]) \neq \emptyset) dy^L \, dx. \]

Analogously,
\[ J^{(1)}_L(K) \geq c_d \delta_t^{2d+2} 1(2\delta_t B_2 \subset K|_L) \min_{u \in \delta_t B_d} \text{vol}_{d-2}((u + L^\perp) \cap K). \]

We give a simple estimate for the constant $c_d$. It is immediate that if $y^L + [0, z|_L]$ meets $[0, x|_L]$ then $-y^L + [0, z|_L]$ cannot meet $[0, x|_L]$. Hence the indicator function equals one on at most half of $2B_2$. This gives
\[ c_d \leq \frac{4\pi}{2} \kappa_d^2 \]

This finishes the proof of the lemma.

In the next section we need to determine a closely related integral.

**Lemma 14.** Let $K$ be a convex body and $L$ a 2-dimensional plane in $\mathbb{R}^d$. Define
\[ J^{(2)}_L(K) := \int_{(\delta_t B_d) \times K^2 \times (\delta_t B_d)^2} 1([0, x]|_L] \cap (y_1 + [0, z_1]|_L] \neq \emptyset) \, dz_1 \, dz_2 \, dy_1 \, dy_2 \, dx. \]

Then
\[ J^{(2)}_L(K) = c'_d \delta_t^{3d+4} \text{vol}_{d-2}(L^\perp \cap K)^2 (1 + o(1)) \]
as $\delta_t \to 0$, with
\[ c'_d = \int_{B_d \times (2B_d)^2 \times B_d^2} 1((0, x|_L] \cap (y^L_1 + [0, z_1|_L]) \neq \emptyset) \, dz_1 \, dz_2 \, dy^L_1 \, dy^L_2 \, dx \leq 2\pi \kappa_d c_d. \]

Further we have
\[ J^{(2)}_L(K) \leq c'_d \delta_t^{3d+4} \text{M}_{d-2}(K)^2. \]
Proving Lemma 13 we estimate
\[ J_L^{(2)}(K) = \int_{B_d \times (2B_2)^2 \times B_3^2} \mathbf{1}([0, x] \cap (y_1^L + [0, z_1]L) \neq \emptyset) \, dy_1 \, dz_1 \, dy_2 \, dx \]
\[ = \int_{(B_3 d_b)^2 \times (B_3 d_B)^2} \mathbf{1}([0, x] \cap (y_1^L + [0, z_1]L) \neq \emptyset) \, dy_1 \, dz_1 \, dy_2 \, dx \]
\[ \leq \delta^{3d+4} \int_{B_d \times (2B_2)^2 \times B_3^2} \mathbf{1}([0, x] \cap (y_1^L + [0, z_1]L) \neq \emptyset) \, dy_1 \, dz_1 \, dy_2 \, dx \]
\[ \leq c_d \delta^{3d+4} \left( \max_{u \in \delta B_d} \text{vol}_{d-2}((u + L^\perp) \cap K) \right)^2 \]

with \( c_d \) as in (11).

Analogously,
\[ J_L^{(2)}(K) \geq c_d' \delta^{3d+2} \mathbf{1}(2\delta B_2 \subset K) \left( \min_{u \in \delta B_d} \text{vol}_{d-2}((u + L^\perp) \cap K) \right)^2. \]

As in the proof of Lemma 13 we estimate
\[ c_d' = \int_{B_d \times (2B_2)^2 \times B_3^2} \mathbf{1}([0, x] \cap (y_1^L + [0, z_1]L) \neq \emptyset) \, dy_1 \, dz_1 \, dy_2 \, dx \]
\[ \leq 2\pi \kappa_d \int_{B_d \times (2B_2)^2 \times B_3^2} \mathbf{1}([0, x] \cap (y_1^L + [0, z_1]L) \neq \emptyset) \, dy_1 \, dz_1 \, dy_2 \, dx \]
\[ = 2\pi \kappa_d c_d \]

where we used that the integration with respect to \( y_2 \) gives at most half of the volume of \( 2B_2 \), and the integration with respect to \( z_2 \) the volume of \( B_d \). This proves the lemma.

By \( K - \delta \) we denote the inner parallel set \( \{ x \in R^d \mid x + \delta B_d \subset K \} \) of a convex set \( K \).

**Proposition 15.** Let \( v_1 \) be a point in \( W \subset R^d \). Then for
\[ I_W(v_1) = \int_{W^3} \mathbf{1}(\|v_1 - v_2\|, \|v_3 - v_4\| \leq \delta, \|v_1 - v_2\| \leq \delta, \|v_3 - v_4\| \leq \delta) \, dv_1 dv_2 dv_3 dv_4 \]
it holds that

\[
\lim_{\delta_t \to 0} \frac{I_W(v_1)}{\delta_t^{2d+2}} = c_d \text{vol}_{d-2}((v_1 + L^\perp) \cap W)
\]

as \(\delta_t \to 0\), with \(c_d\) given in (10). Further we have

\[
I_W(v_1) \leq c_d \delta_t^{2d+2} M_{d-2}(W)
\]

where \(M_{d-2}(W)\) is the volume of the maximal \((d-2)\)-dimensional section of \(W\).

**Proof.** We substitute \(v_2 = v_1 + x, v_3 = v_1 + y\) and \(v_4 = v_1 + y + z\) and obtain

\[
I_W(v_1) = \int_{(W-v_1)^2} \int_{W-v_1-y} 1([0, x]_L \cap [y, y + z]_L \neq \emptyset, \|x\| \leq \delta_t, \|z\| \leq \delta_t) \, dz \, dy \, dx
\]

\[
= \int_{((W-v_1) \cap \delta_t B_d) \times (W-v_1) \times ((W-v_1-y) \cap \delta_t B_d)} 1([0, x]_L \cap [y, y + z]_L \neq \emptyset) \, dz \, dy \, dx
\]

\[
\leq \int_{(\delta_t B_d) \times (W-v_1) \times (\delta_t B_d)} 1([0, x]_L \cap [y, y + z]_L \neq \emptyset) \, dz \, dy \, dx
\]

\[
= J_L^{(1)}(W - v_1)
\]

and on the other hand

\[
I_W(v_1) \geq \int_{(\delta_t B_d) \times (W-v_1) \times (\delta_t B_d)} 1([0, x]_L \cap [y, y + z]_L \neq \emptyset) \, dz \, dy \, dx
\]

\[
= \int_{(\delta_t B_d) \times (W-v_1) \times (\delta_t B_d)} 1([0, x]_L \cap [y, y + z]_L \neq \emptyset) \, dz \, dy \, dx
\]

\[
= J_L^{(1)}(W - v_1).
\]

Using Lemma 13 this leads to

\[
\lim_{\delta_t \to 0} \frac{J_L^{(1)}(W - v_1)}{\delta_t^{2d+2}} = \lim_{\delta_t \to 0} \frac{J_L^{(1)}(W - v_1)}{\delta_t^{2d+2}} = \lim_{\delta_t \to 0} \frac{I_W(v_1)}{\delta_t^{2d+2}}
\]

\[
= c_d \text{vol}_{d-2}(L^\perp \cap (W - v_1)).
\]

\(\square\)
B Variance of Crossing Number

Recall that \( V \) is a Poisson point process of intensity \( t \). In this section we estimate the variance of the crossing number. In particular we show that only two terms are of interest and the others are of lower order.

\[
\operatorname{Var}_V \varpi(G_L) \leq \frac{1}{64} T \int_W \mathbb{E}_V \left( \sum_{(v_2, \ldots, v_4) \in V_2^3} \mathbb{1}\left( \|v\|_L \cap [v_3, v_4], [v]_L \neq \emptyset, \|v - v_2\| \leq \delta_t, \|v_3 - v_4\| \leq \delta_t \right) \right)^2 dv
\]

\[
= \frac{1}{64} T \int_W \mathbb{E}_V \left( \sum_{(v_2, \ldots, v_4) \in V_2^3} \sum_{(w_2, \ldots, w_4) \in V_2^3} \mathbb{1}\left( \|v\|_L \cap [v_3, v_4], [v]_L \neq \emptyset, \|v - v_2\| \leq \delta_t, \|w_3 - v_4\| \leq \delta_t \right) \mathbb{1}\left( \|v\|_L \cap [v_3, v_4], [v]_L \neq \emptyset, \|v - v_2\| \leq \delta_t, \|w_3 - w_4\| \leq \delta_t \right) \right) dv
\]

Calculating the expectation \( \mathbb{E}_V \) and assuming that all six random points \( v_2, \ldots, v_4, w_3, w_4 \) are different leads to the first term

\[
\frac{1}{64} T \int_W I_W(v)^2 dv. \tag{12}
\]

For all other terms in the expectation \( \mathbb{E}_V \), at least two points of the sets \( \{v_2, \ldots, v_4\}, \{w_2, \ldots, w_4\} \) coincide. As an example we calculate the situation where \( v_2 = w_2 \) and \( v_3, v_4, w_3, w_4 \) are different. The Slivnyak-Mecke formula gives

\[
\mathbb{E}_V \left( \sum_{v_2, \ldots, v_4, w_3, w_4 \in V_2^3} \mathbb{1}\left( \|v, v_2\|_L \cap [v_3, v_4], [v]_L \neq \emptyset, \|v - v_2\| \leq \delta_t, \|v_3 - v_4\| \leq \delta_t \right) \mathbb{1}\left( \|v, v_2\|_L \cap [w_3, w_4], [v]_L \neq \emptyset, \|w_3 - w_4\| \leq \delta_t \right) \right)
\]

\[
= t^5 \int_{W^5} \mathbb{1}\left( \|v, v_2\|_L \cap [v_3, v_4], [v]_L \neq \emptyset, \|v - v_2\| \leq \delta_t, \|v_3 - v_4\| \leq \delta_t \right) \mathbb{1}\left( \|v, v_2\|_L \cap [w_3, w_4], [v]_L \neq \emptyset, \|w_3 - w_4\| \leq \delta_t \right) dv_2dv_3dw_3dw_4.
\]

We substitute \( v_2 = v + x, v_3 = v + y_1 \) and \( v_4 = v + y_1 + z_1, \) and \( w_3 = v + y_2 \) and \( w_4 = v + y_2 + z_2, \) respectively. We obtain

\[
= t^5 \int_{(W-v)^3 \times (W-v-y_2) \times (W-v-y_1)} \mathbb{1}\left( \|v\|_L \cap (y_1 + [0, z_1]), [v]_L \neq \emptyset, \|x\| \leq \delta_t, \|z_1\| \leq \delta_t \right) \mathbb{1}\left( \|v\|_L \cap (y_1 + [0, z_1]), [v]_L \neq \emptyset \right) dz_1dz_2dy_1dy_2dx
\]

\[
\leq t^5 \int_{(\delta, B_d) \times (W-v)^2 \times (\delta, B_d)^2} \mathbb{1}\left( \|v\|_L \cap (y_2 + [0, z_2]), [v]_L \neq \emptyset \right) dz_1dz_2dy_1dy_2dx
\]

\[
= t^5 J_L^{(2)}(W - v).
\]
Analogously to the calculations in Proposition 15, the expectation is also bounded from below by

$$t^5 J_L^{(2)} (W - t_i - v).$$

Together with Lemma 14 this shows that the expectation equals

$$c'_d t^5 \delta^{3d+4} \text{vol}_{d-2}(L^\perp \cap W)^2 (1 + o(1)). \quad (13)$$

More general, if $k$ points coincide, $1 \leq k \leq 3$, the Slivnyak-Mecke formula transforms the sum into a $(6-k)$-fold multiple integration and yields a term $t^{6-k}$ in front of the integrals.

On the other hand $v_2$ and $w_2$ are $\delta_t$-close to $v$, as well as $v_3, v_4$ and $w_3, w_4$. In each of these cases this yields a term of order $\delta_t^d$. In addition $v_3$ and $w_3$ are contained in a cylinder of base $2\delta_t B_2 \subset L$ yielding terms of order $\delta_t^2$.

Using that everything is symmetric in $v_3, v_4$ and $w_3, w_4$ we list in the following array all cases with the power of $\delta_t$ that occurs.

In the case $k = 1$ we obtain:

| $v_2 = w_2, v_3, v_4, w_3, w_4$ | $v_2 = w_3, v_3, v_4, w_2, w_4$ |
|---------------------------------|---------------------------------|
| $\delta_t^d$                  | $\delta_t^d$                  |
| $\delta_t^d$                  | $\delta_t^d$                  |
| $\delta_t$                    | $\delta_t$                    |
| $\delta_t^2$                  | $\delta_t^2$                  |
| $\delta_t$                    | $\delta_t$                    |
| $\delta_t$                    | $\delta_t$                    |

We have just computed that the first case equals

$$c'_d t^5 \delta^{3d+4} \text{vol}_d (L^\perp \cap W)^2 (1 + o(1))$$

and the remaining cases are of order $t^5 \delta^{4d+2}$. In the case $k = 2$ we obtain

| $v_2 = w_2, v_3 = w_3, v_4, w_4$ | $v_2 = w_3, v_3 = w_2, v_4, w_4$ | $v_2 = w_3, v_3 = w_4, v_4, w_2$ |
|---------------------------------|---------------------------------|---------------------------------|
| $\delta_t^d$                  | $\delta_t^d$                  | $\delta_t^d$                  |
| $\delta_t^d$                  | $\delta_t^d$                  | $\delta_t^d$                  |
| $\delta_t$                    | $\delta_t$                    | $\delta_t$                    |
| $\delta_t^2$                  | $\delta_t^2$                  | $\delta_t^2$                  |
| $\delta_t$                    | $\delta_t$                    | $\delta_t$                    |
| $\delta_t$                    | $\delta_t$                    | $\delta_t$                    |

and thus these cases are of order at most $t^4 \delta^{4d+2}$. In the case $k = 3$ we obtain

| $v_2 = w_2, v_3 = w_3, v_4 = w_4$ | $v_2 = w_3, v_3 = w_2, v_4 = w_4$ |
|---------------------------------|---------------------------------|
| $\delta_t^d$                  | $\delta_t^d$                  |
| $\delta_t^d$                  | $\delta_t^d$                  |
| $\delta_t$                    | $\delta_t$                    |
| $\delta_t^2$                  | $\delta_t^2$                  |
| $\delta_t$                    | $\delta_t$                    |
| $\delta_t$                    | $\delta_t$                    |

and these cases are of order at most $t^3 \delta^{2d+2}$. 
Combining this with (12), Proposition 15, and (13), we have

\[
\text{Var}_V \varpi(G_L) \leq 164 \epsilon^2 d^7 \delta_t^{4d+4} \int_W \text{vol}_{d-2}((v + L^\perp) \cap W)^2 (1 + o(1)) dv \\
+ \frac{1}{64} \epsilon^2 d^6 \delta_t^{3d+4} \int_W \text{vol}_{d-2}((v + L^\perp) \cap W)^2 (1 + o(1)) dv \\
+ O(\max\{t^6 \delta_t^{4d+2}, t^5 \delta_t^{3d+2}, t^4 \delta_t^{2d+2}\}).
\]

This leads to (8).

C Covariance of Crossing Number and Stress

Here we bound the covariance of the crossing number and the stress of a random graph \(G_L\). We apply Mehler’s formula (see e.g. [25, Eq. (84)]) and the Slivnyak-Mecke formula. For \(s \in [0,1]\), let \(V^{(s)}\) be the point process where independently for each chosen vertex one decides with probability \(s\) whether it belongs to \(V^{(s)}\). Thus \(V^{(s)}\) is a Poisson point process with intensity \(st\) and is independent of \(V^{(1-s)} = V - V^{(s)}\). Then the covariance is given by

\[
\text{Cov}_V(\varpi(G_L), \text{stress}(G, G_L)) = t \mathbb{E}_V \int_0^1 \mathbb{E}_V(\varpi(G_L) \mathbb{E}_{V^{(1-s)}}(D_v \text{stress}(G, G_L)|V^{(s)}) ds dv.
\]

The conditional expectation of \(D_v \text{stress}(G, G_L)\) is given by

\[
\mathbb{E}_{V^{(1-s)}}(D_v \text{stress}(G, G_L)|V^{(s)}) = \frac{1}{2} \mathbb{E}_{V^{(1-s)}} \left( \sum_{v_1 \in V^{(s)}} w(v, v_1)(d_0(v, v_1) - d_L(v, v_1))^2 \right) V^{(s)} \\
= \frac{1}{2} \sum_{v_1 \in V^{(s)}} w(v, v_1)(d_0(v, v_1) - d_L(v, v_1))^2 \\
+ \frac{1}{2} t(1 - s) \int_W w(v, v_1)(d_0(v, v_1) - d_L(v, v_1))^2 dv_1.
\]
We multiply by $D_v \mathfrak{m}(G_L)$, ignore the summands where $v_1$ coincides with one of the $v_i$'s and obtain

$$E_V D_v \mathfrak{m}(G_L) E_{V(1-s)} (D_v \text{stress}(G, G_L) | V^{(s)}) \geq$$

$$\geq \frac{1}{16} E_V \left( \sum_{(v_2, \ldots, v_4, v_1) \in V^4} \mathbb{1}_{([v, v_2]|L \cap [v_3, v_4]|L \neq \emptyset)} \mathbb{1}_{(||v - v_2|| \leq \delta_t)} \right)$$

$$\times \left( \mathbb{1}_{(||v_3 - v_4|| \leq \delta_t)} w(v, v_1)(d_0(v, v_1) - d_L(v, v_1))^2 \mathbb{1}_{(v_1 \in V^{(s)})} \right)$$

$$+ \frac{1}{16} E_V \left( \sum_{(v_2, \ldots, v_4) \in V^3} \mathbb{1}_{([v, v_2]|L \cap [v_3, v_4]|L \neq \emptyset)} \mathbb{1}_{(||v - v_2|| \leq \delta_t)} \mathbb{1}_{(||v_3 - v_4|| \leq \delta_t)} t(1-s) \int_{\mathcal{W}} w(v, v_1)(d_0(v, v_1) - d_L(v, v_1))^2 dv_1$$

$$= \frac{1}{16} t^4 I_W(v) \int_{\mathcal{W}} w(v, v_1)(d_0(v, v_1) - d_L(v, v_1))^2 dv_1$$

This gives

$$\text{Cov}_V \left( \mathfrak{m}(G_L), \text{stress}(G, G_L) \right)$$

$$\geq \frac{1}{16} t^5 \int_{\mathcal{W}} I_W(v) \int_{\mathcal{W}} w(v, v_1)(d_0(v, v_1) - d_L(v, v_1))^2 dv_1 dv.$$