A stochastic control approach to Sine-Gordon EQFT

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Dedicated to the memory of Mikhail Barashkov

Abstract

We study the Sine-Gordon model for $\beta^2 < 4\pi$ in infinite volume. We give a variational characterization of it’s Laplace transform, and deduce from this large deviations. Along the way we obtain estimates which are strong enough to obtain a proof of the Osterwalder-Schrader axioms including exponential decay of correlations as a byproduct. Our method is based on the Boue-Dupuis formula with an emphasis on the stochastic control structure of the problem.

1 Introduction

In this article we investigate the Sine Gordon measure on the plane for $\beta^2 < 4\pi$, that is

$$\nu_{\text{SG}} = \frac{1}{Z} \exp \left( -\lambda \int_{\mathbb{R}^2} \cos(\beta \varphi) d\mu \right) d\mu$$

where $\mu$ is the Gaussian Free Field on $\mathbb{R}^2$, that is the Gaussian measure on $\mathcal{S}'(\mathbb{R}^2)$ (tempered distributions) with covariance

$$\int_{\mathcal{S}'(\mathbb{R}^2)} \langle f, \varphi \rangle_{L^2(\mathbb{R}^2)} \langle g, \varphi \rangle_{L^2(\mathbb{R}^2)} d\mu = \langle f, (m^2 - \Delta)^{-1} g \rangle_{L^2(\mathbb{R}^2)}.$$ 

Measures of this form are of interest in mathematical physics since they allow the construction of relativistic Quantum Field Theories (QFTs) via the Osterwalder-Schrader Reconstruction Theorem, provided one is able to prove that they satisfy the Osterwalder-Schrader Axioms.

Expression (1) has no rigorous meaning since $\mu$ is known to be supported on genuine distributions so $\cos(\beta \varphi)$ is ill defined. Nevertheless if $\varphi$ is sampled from $\mu$ one can give rigorous meaning to the Wick ordered cosine $\llbracket \cos(\beta \varphi) \rrbracket$ as a random distribution. It is important to note that the process of Wick ordering “multiplies the cosine by an infinite constant” and thus the resulting potential becomes nonconvex even if $\lambda \ll m$. In finite volume this is sufficient to define a density with respect to free field, for the sine gordon measure provided $\beta^2 < 4\pi$. For $4\pi \leqslant \beta^2 < 8\pi$ the construction becomes more complicated and the partition function $Z$ acquires further divergencies. A further complication arises since samples of $\mu$ are not expected to have any decay at infinity so the expression $\int_{\mathbb{R}^2} \llbracket \cos(\beta \varphi) \rrbracket d\mu$ loses its meaning even though $\llbracket \cos(\beta \varphi) \rrbracket$ is a well defined random distribution.

Despite the difficulties the Sine-Gordon measures has been studied extensively in the mathematical literature. In [27] the measure was constructed in the full space for $\beta < 4\pi$ and $\lambda \ll m$ and the authors were able verify the
Osterwalder-Schrader axioms, exponential decay of the correlation functions as well as nontriviality of the scattering matrix of the corresponding Quantum Field Theory. In the full range $0 < \beta^2 < 8\pi$ the measure was constructed in \cite{22, 38, 23}. A markov property for the Sine-Gordon model was shown in \cite{4}. The markov property allows one to employ an alternative route to construct the corresponding QFT via Nelson’s reconstruction \cite{41, 42}. In \cite{17} Brydges and Kennedy gave an elegant construction of the Sine-Gordon measure using the Polchinski equation \cite{45}. Their method was later explored in \cite{10} to prove a Logarathmic Sobolev Inequality for $\beta^2 < 6\pi$ and in \cite{12, 32} to construct a coupling between the Sine-Gordon measure and the Free Field, and subsequently study the maximum of the Sine-Gordon field for $\beta^2 < 6\pi$. In \cite{43} the authors showed that the Sine gordon measure is invariant under a flow a wave quation with Sinus nonlinearity (in finite volume). For $m = 0$ the Sine-Gordon model also exhibits the Coleman Correspondence, a relation with the fermionic Thirring model \cite{27, 13, 14}.

To the Sine-Gordon measure one can associate Langevin dynamics, that is the stochastic PDE

$$(\partial_t + (m^2 - \Delta))u - \lambda \beta [\sin(\beta \varphi)] = \xi(t, x) \quad \xi \text{ Space time white noise.}$$

These have been studied in \cite{3, 30}. These Langevin dynamics have been employed to study the measure in the context of the $\Phi^4_3$ model using techniques from stochastic PDE’s, see \cite{5, 29}. This approach is known as Stochastic Quantization (SQ). An elliptic version of the SQ equations have also recently been developed \cite{6}.

In \cite{7} a variational approach to study EQFTs was proposed, and the $\Phi^4_3$ model in finite volume was studied. Here we aim to pursue this point of view in infinite volume. The variational approach is an interpretation of the Polchinski equation in terms of stochastic control, and thus most closely resembles \cite{18}. More precisely it studies the stochastic control problem whose associated HJB equation is the Polchinski equation, see also Appendix C. On the other hand, as was seen in \cite{7}, the variational method also allows to deploy techniques developed in the study of stochastic quantization. Furthermore it allows to derive a variational representation of the measure, without making reference to a limiting process, even if the measure is not absolutely continuous with respect to the GFF. Let us mention that the variational approach has also been used to study phase transition for the $\Phi^4_3$ model \cite{20}.

In \cite{8} M.Gubinelli and the author studied the $\Phi^4_2$ and Hoergh-Krohn (with exponential interaction) models in infinite volume using the variational method. Through studying the Euler-Lagrange equation of the corresponding variational problem it is possible to derive a variational formula for the Hoergh-Krohn model in infinite volume. This required convexity of the renormalized interaction and thus we were not able this result for the $\Phi^4_2$ model although we obtained partial results in this direction. In the present paper we will study this problem for the Sine-Gordon model in the case $\beta^2 < 4\pi$. Although in this case the renormalized interaction is not convex we will be able to circumvent this using the boundedness of the sinus.

The starting point of our analysis will be the Boue-Dupuis representation \cite{16} of the Laplace transform of the approximate Sine-Gordon measure. See also the papers of Üstünel \cite{47} and Zhang \cite{48} for extensions to the infinite dimensional setting, where extensions and further results on the variational formula are obtained. The Boue-Dupuis formula has been used to derive Large Deviation Principles (LDPs) in different contexts \cite{19}. A byproduct of our approach will be to derive a large deviations principle for the Sine-Gordon model in the semiclassical limit. A similar result has been obtained for the Liouville measure in \cite{39, 37} and in \cite{8} for $\Phi^4_2$.

1.1 Results

We will now give an overview of our results and a rough outline of the proofs. We want to study the laplace transform, for some sufficiently nice functional $f$ (see below for details)

$$- \log \int \exp \left( -f(\varphi) - \lambda \int_{\mathbb{R}^2} \rho[\cos(\beta \varphi)] dx \right) d\mu = W^\rho(f).$$

Here we have introduced an "infrared cutoff" to deal with the divergence due to infinite volume. It is chosen according to the following definition:
**Definition 1** We denote by

\[ C = \{ \rho \in C_c^\infty(\mathbb{R}^2) : 0 \leq \rho \leq 1, |\nabla \rho| \leq 1 \} \]

the space of spatial cutoffs. We will say that family of spatial cutoffs \( \rho^N \in C \) converges to 1 or \( \rho^N \to 1 \) if for any \( K > 0 \) there exists \( N_0 \in \mathbb{N} \) such that \( \rho^N(x) = 1 \) for any \( x \in B(0, K) \) and \( N \geq N_0 \). Usually we will drop the index \( N \) and simply write \( \rho \to 1 \) in this case.

We will first study the measures

\[ d\nu^\rho_{SG} = \frac{1}{Z^\rho} \exp \left( -\lambda \int_{\mathbb{R}^2} \rho[\cos(\beta \varphi)] dx \right) d\mu \quad Z^\rho = \exp(W^\rho(0)) \]

The BD formula gives

\[ W^\rho(f) = \inf_u F^{f,\rho}(u) = \inf_{u \in H_a} \mathbb{E} \left[ f(W_\infty + I_\infty(u)) + \lambda \int \rho[\cos(\beta(W_\infty + I_\infty(u))] + \frac{1}{2} \int_0^\infty \|u_s\|^2_{L^2} ds \right] \]

Here

- \( W_t \) is a Gaussian process, such that \( \text{Law}(W_\infty) = \mu \) and \( W_t \) is smooth almost surely for \( t < \infty \), see Section 2.2 for details.
- \( I : L^2(\mathbb{R}_+ \times \mathbb{R}^2) \to L^\infty(\mathbb{R}_+, H^1(\mathbb{R}^2)) \) is a linear map given by

\[ I_t(u) = \int_0^t J_s u_s ds \quad J_t = \left( \frac{1}{t^2} e^{-\frac{(m^2 - \Delta)}{t}} \right)^{1/2} \]

also see Section 2.2 for details.
- \( H_a \) is the space of processes adapted to \( W_t \) such that

\[ \mathbb{E} \left[ \int_0^\infty \|u_s\|^2_{L^2} ds \right] < \infty \]

The minimization problem on r.h.s of (1.1) is a stochastic control problem. There is two ways to study the minimizer:

- One can ignore the control structure and study the Euler Lagrange equations for the minimizer as was done in [8], which bears strong resemblance with the study of Stochastic Quantization.
- Alternatively one can analyse the value function of the control problem and use the verification theorem from stochastic control theory (Proposition 4 below) to infer properties of the minimizer. We will use both points of view here. First we will use the control point of view to derive an \( L^\infty \) bound on the optimizer:

**Theorem 1** The minimizer \( \bar{u}^\rho \) in \( H_a \) of the functional

\[ F^\rho(u) = \mathbb{E} \left[ \lambda \int \rho[\cos(\beta(W_\infty + I_\infty(u)))] dx + \frac{1}{2} \int_0^\infty \|u_t\|^2_{L^2} dt \right] \]

satisfies

\[ \|\bar{u}^\rho_t\|_{L^\infty} \leq C(t)^{\beta^2/8\pi - 1} . \]
Once this is established we will want to study the dependence of the optimizer on a local perturbation \( f \). We will show that, provided \( f \) is sufficiently nice, with \( w(x) = \exp(\gamma |x|) \) and \( u^{f,\rho} \) being the minimizer for \( u^{f,\rho} \) is

\[
E \left[ \int_0^\infty \int w|u^{f,\rho} - \bar{u}^\rho|^2 \, dx \, dt \right] < \infty \tag{2}
\]

Where we will establish (2) studying the Euler-Lagrange equations for \( F^{f,\rho} \). For this convexity of \( F^\rho \) is crucial.

Observe that the second term \( \frac{1}{2} \int_0^\infty \|u_t\|_{L^2}^2 \, dt \) is strictly convex, but the first term seemingly breaks convexity since the cosine is multiplied with an infinite constant. However applying Itô’s formula one can calculate

\[
\lambda \int \rho \|\cos(\beta(W_\infty + I_\infty(u)))\| \, dx
= \lambda \beta \int \rho [\sin(\beta(W_t + I_t(u)))] J_t u \, dx + \text{martingale}
= \lambda \beta \int J_t (\rho \alpha(t)(\sin(\beta(W_t + I_t(u)))) \, u \, dx + \text{martingale}
\]

Below we will show that \( \alpha(t) \leq (t)^{3/8} \) and \( \|J_t u\|_{L^p} \leq t^{-1} \|u\|_{L^p} \) so one can hope for the functional to be convex provided that \( \lambda \) is small enough and \( u \) stays in a bounded region, which is guaranteed by Theorem 1. Expanding this heuristic we will prove (2), which in turn will allow us to remove the cutoff \( \rho \) in the variational formula and prove the following theorem:

**Theorem 2** Define \( \mathbb{D}^f \) to be the space

\[
\mathbb{D}^f = \left\{ v \in H_a : E \left[ \int_0^\infty \int \|wv\|_{L^2(\mathbb{R}^2)}^2 \, dx \, dt \right] \leq C \right\}.
\]

Then there exists a \( \lambda_0 \) such that for any \(-\lambda_0 < \lambda < \lambda_0\)

\[
\lim_{\rho \to 1} (W^{\rho}(f) - W^{\rho}(0)) = \inf_{v \in \mathbb{D}^f} \bar{F}^f(v)
\]

where

\[
\bar{F}^f(v) = E[f(W_\infty + I_\infty(\bar{u}) + I_\infty(v)) + \lambda \int \left[ \cos(\beta(W_\infty + I_\infty(\bar{u}) + I_\infty(v))) \right] \, dx + \lambda \int_0^\infty \bar{u}_t v_t \, dx \, dt + \frac{1}{2} \int_0^\infty \|v_t\|_{L^2}^2 \, dt
\]

and \( \bar{u} \) is the limit of \( \bar{u}^\rho \) from Theorem 7 which will be shown to exist in Proposition 7.

It turns out that the minimizer of eq (1.1) for \( f = 0 \) provides a coupling of the Sine-Gordon measure with the Gaussian Free Field, so from Theorem 1 we will be able to deduce the following:

**Theorem 3**
1. There exists random variables \( I^\rho \in L^\infty(\mathbb{P} \times \mathbb{R}^2) \) and

\[
\sup_{\rho \in C} \| I^\rho \|_{L^\infty(\mathbb{P} \times \mathbb{R}^2)} < \infty
\]

such that the measures \( \nu^\rho_{SG} \) satisfy

\[
\nu^\rho_{SG} = \text{Law}(W_\infty + I^\rho).
\]

2. If \( \lambda \) is small enough we have that there exists an \( I \in L^\infty(\mathbb{P} \times \mathbb{R}^2) \) such that for any \( \gamma > 0 \)

\[
\| I^\rho - I \|_{L^\infty(\mathbb{P},L^2,-\gamma)} \to 0,
\]

and the Law of \((W,I)\) is euclidean invariant.

**Proof** Theorem 3 follows from Lemma 1, Theorem 1 and Proposition 7, note that Euclidean invariance follows easily from the uniquenss of \( I \).

An analogue of Theorem 3 has been established in finite volume in [12] even in the larger range \( \beta^2 < 6\pi \). We expect their techniques to extend to the infinite volume case at least for weak coupling (when \( \lambda \) is small enough).

Both Theorem 7 and Theorem 3 imply that if \( \lambda \) is small enough \( \nu^\rho_{SG} \) converges to a unique measure \( \nu_{SG} \) as \( \rho \to 1 \).

Theorem 4 \( \nu_{SG} \) satisfies the Osterwalder-Schrader axioms. Furthermore the correlations decay exponentially and \( \nu_{SG} \) is non-Gaussian.

**Proof** Euclidean invariance and follows directly from Theorem 3, observe that analyticity also follows since Theorem 3 implies that \( \nu_{SG} \) has gaussian tails. The remaining axioms are shown in Section 6.

Finally we will discuss large deviations for \( \nu_{SG} \) in a semi-classical limit similarly to [39, 37, 8]. For this we have to introduce Planck’s constant into the measure. We want to look at the measure formally given by

\[
d\nu_{SG,h} = \frac{1}{Z_h} e^{-\frac{1}{\hbar} \int \lambda \cos(\beta \varphi(x)) + \frac{1}{2} m \varphi^2(x) + \frac{1}{2} \| \nabla \varphi(x) \|^2} dx d\varphi.
\]

This can be rewritten as

\[
\nu_{SG,h}(d\varphi) = \frac{1}{Z_h} e^{-\frac{1}{\hbar} \int \cos(\hbar^{1/2} \beta \varphi) \mu(d\varphi)}.
\]

where \( Z_h \) is normalization constant and we are interested in the limit \( h \to 0 \). These measure can be made sense of in the same way as \( \nu_{SG} \). We will prove

Theorem 5 \( \nu_{SG,h} \) satisfies a large deviations principle with rate function

\[
I(\varphi) = \lambda \int (\cos(\beta \varphi(x)) - 1) dx + \frac{1}{2} m^2 \int \varphi^2(x) dx + \frac{1}{2} \int |\nabla \varphi(x)|^2 dx.
\]

or equivalently

\[
\lim_{h \to 0} -h \log \int e^{-\frac{1}{h} I(\varphi)} d\nu_{SG,h} = \inf_{\varphi \in H^{-1}(\langle x \rangle - \nu)} \{ f(\varphi) + I(\varphi) \}.
\]
Let us remark that for singular SPDE’s large deviations have been studied in [31] and the more precise study of Laplace asymptotics has been carried out in [26, 35]. It would be interesting to investigate Laplace asymptotics in our context.

Structure of the article. In Section 2 we collect some useful estimates, introduce some useful processes and collect some facts on stochastic control. Details and proofs will be given in the corresponding appendices. In Section 3 we will prove Theorem 1. In Section 4 we will prove Theorem 2. In Section 5 we will show Theorem 5. Finally in Section 6 we will establish Theorem 4. The appendices contain supplementary material on stochastic control and technical estimates.

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2 Preliminaries

In this section we collect some useful notions, estimates and facts which will be used in the rest of the paper.

2.1 Weighted spaces

First recall the definition of Littlewood–Paley blocks. Let \( \chi, \varrho \) be smooth radial functions \( \mathbb{R}^2 \rightarrow \mathbb{R} \) such that

- \( \text{supp} \chi \subseteq B(0, R) \), \( \text{supp} \varrho \subseteq B(0, 2R) \setminus B(0, R) \);
- \( 0 \leq \chi, \varrho \leq 1 \), \( \chi(\xi) + \sum_{j \geq 0} \varrho(2^{-j} \xi) = 1 \) for any \( \xi \in \mathbb{R}^d \);
- \( \text{supp} \varrho(2^{-j} \cdot) \cap \text{supp} \varrho(2^{-i} \cdot) = \emptyset \) if \( |i - j| > 1 \).

Introduce the notations \( \varrho_{-1} = \chi \), \( \varrho_j = \varrho(2^{-j} \cdot) \) for \( j \geq 0 \). For any \( f \in \mathcal{S}'(\mathbb{R}^2) \) we define the operators \( \Delta_j f := \varrho_j(D)f \), \( j \geq -1 \).

**Definition 2** Let \( s \in \mathbb{R}, p, q \in [1, \infty] \). For a Schwartz distribution \( f \in \mathcal{S}'(\mathbb{R}^2) \) define the norm

\[
\|f\|_{B^s_{p,q}} = \|(2^{js}\|\Delta_j f\|_{L^p})_{j \geq -1}\|_{L^q}
\]

where \( \|\|_p \) denotes the normalized \( L^p(\Lambda) \) norm. The space \( B^s_{p,q} \) is the set of functions \( f \in \mathcal{S}'(\mathbb{R}^2) \) such that \( \|f\|_{B^s_{p,q}} < \infty \) moreover \( H^s = B^s_{2,2} \) are the usual Sobolev spaces, and we denote by \( \mathcal{C}^s \) the closure of smooth functions in the \( B^s_{\infty, \infty} \) norm.

**Definition 3** A polynomial weight \( \rho \) is a function \( \rho : \mathbb{R}^2 \rightarrow [0, \infty) \) of the form \( \rho(x) = c(x)^{-\sigma} \) for \( \sigma, c \geq 0 \). For a polynomial weight \( \rho \) let

\[
\|f\|_{L^p(\rho)} = \left( \int |f(x)|^p \rho(x) \, dx \right)^{1/p}
\]

and by \( L^p(\rho) \) the space of functions for which this norm is finite.
Definition 4 For a polynomial weight \( \rho \) let
\[
\| f \|_{L^p(\rho)} = \left( \int |f(x)|^p \rho(x) \, dx \right)^{1/p}
\]
and by \( L^p(\rho) \) the space of functions for which this norm is finite. Similarly we define the weighted Besov spaces \( B_{p,q}^s(\rho) \) as the set of elements of \( \mathcal{D}'(\mathbb{R}^d) \) for which the norm
\[
\| f \|_{B_{p,q}^s(\rho)} = \| (2^j s \| \Delta_j f \|_{L^p(\rho)})_{j \geq -1} \|_{\ell^q}.
\]
We also introduce some spaces with exponential weights:

Definition 5 For a set \( z \in \mathbb{R}^2 \), \( r \in \mathbb{R} \) we define the weighted \( L^p \) spaces
\[
\| f \|_{L^p,z} = \left( \int \exp(rp|x|)f^p(x) \, dx \right)^{1/p}
\]
And
\[
\| f \|_{W^{1,p},z} = \| f \|_{L^p,z} + \left( \int \exp(rp|x-z|)(\nabla f(x))^p \, dx \right)^{1/p}
\]
We will also set \( H_{z}^{1,r} = W_{z}^{1,2,r} \). Furthermore we will set
\[
\| f \|_{L^p,z} = \| f \|_{L^p,0} \quad \text{and} \quad \| f \|_{W^{1,p},z} = \| f \|_{W^{1,p},0}.
\]

Notation 1 Throughout the chapter we will frequently compute Gradients and Hessian of functionals on \( f : L^2(\mathbb{R}^2) \rightarrow \mathbb{R} \). We will always interpret \( \nabla f(\varphi) \), to be an element \( L^2(\mathbb{R}^2) \) by the Riesz representation theorem. Similarly we will always interpret \( \operatorname{Hess} f(\varphi) \) to be an operator on \( L^2(\mathbb{R}) \).\]

Definition 6 For a Frechet differentiable functional \( G : L^2((x)^{-n}) \rightarrow \mathbb{R} \) and \( x \in \mathbb{R}^2 \) we define the quantities
\[
|G|_{1,\infty} = \sup_{\varphi \in \mathcal{D}'((x)^{-n})} \| \nabla G(\varphi) \|_{L^\infty}
\]
\[
|G|_{1,2,r} = \sup_{\varphi \in \mathcal{D}'((x)^{-n})} \| \nabla G(\varphi) \|_{L^2,z}^r.
\]

Proposition 1 We have the following estimates
1. \( \| f \|_{L^2((x)^{-n})} \leq C \langle d(0,y) \rangle^{-n/2} \| f \|_{L^2,0}\gamma} \)
2. Let \( s \in \{0,1\} \) \( r > 0 \) and \( f \in W_p^{s,r} \) is supported on \( B(0,N)^e \), \( N \geq 1 \). Then
\[
\| f \|_{W_p^{s,-r}} \leq \exp(-\kappa N) \| f \|_{W_p^{s,r}}.
\]
For a proof see Appendix B.
2.2 Heat Kernel decomposition and martingale Cutoff

We consider the decomposition (with $L = (m^2 - \Delta)$)

$$L^{-1} = \int_0^\infty J_t^2 dt$$

where

$$J_t = \left(\frac{1}{t^2} e^{-L/t}\right)^{1/2}.$$

We denote by

$$C_t = \int_0^t J_s^2 ds = L^{-1} e^{-L/t},$$

and by $K_t(x, y)$ the kernel of $C_t$. From the definitions one can see that

$$K_t(x, x) = \int_0^t e^{-m^2/s} \left(\frac{1}{4\pi s} \right) \frac{1}{4\pi s} e^{-4s|x-y|^2} ds$$

so

$$K_t(x, x) = \int_0^t e^{-m^2/s} \left(\frac{1}{4\pi s} \right) \frac{1}{4\pi s} e^{-4s|x-y|^2} ds$$

where $\sup_{t \in \mathbb{R}_+} C(t) < \infty$. Let $0 \leq s < t$ and $u \in L^2([s, t], L^2(\mathbb{R}^2))$. For later use we introduce the notation

$$I_{s,t}(u) = \int_s^t J_t^2 dt.$$

and we set $I_T(u) = I_{0,T}(u)$.

**Proposition 2** Let $-m < \gamma < m$. We have the following estimates

1. $\|J_t f\|_{L^\infty} \leq t^{-1} \|f\|_{L^\infty}$
2. $\|I_{s,t}(u)\|_{L^2(\mathbb{R}^2)} \leq C\langle s \rangle^{-1/2} \|u\|_{L^2(\mathbb{R}_+, L^2(\gamma))}$
3. $\|I_{s,t}(u)\|_{W^{1,\infty}} \leq C\langle t \rangle^{1/2} \|u\|_{L^\infty([s, t] \times \mathbb{R}^2)}$

For a proof see Appendix B.

We now define regularized GFF as

$$W_{s,t} = \int_s^t Q_t dX_t$$

where $X_s$ is a cylindrical Brownian motion on $L^2$. We set $W_t = W_{0,t}$. We can calculate:

$$\mathbb{E}[W_t(x)W_t(y)] = K_t(x, y)$$

and it is clear that $W_t$ is a martingale.
2.3 Stochastic Control

We are interested in studying the quantities

\[ V_{t,T}(\varphi) = -\log \mathbb{E}[\exp(-V_T(\varphi + W_{t,T}))] \]

where \( Z_{t,T} = \exp(-V_{t,T}) \), for \( \varphi \in L^2(\mathbb{R}^2) \).

For the rest of this article we will denote by \( C_n(\mathbb{L}^2(\mathbb{R}^2)) \) functions \( \mathbb{L}^2(\rho) \rightarrow \mathbb{R} \) which are \( n \) times continuously Fréchet differentiable with bounded derivatives. The following proposition is known as the Boue-Dupuis [16] or Borell [15] formula and is a stochastic control representation of the Polchinski equation.

**Proposition 3** Assume that \( V_T \in C^2(\mathbb{L}^2(\langle x \rangle^{-n})) \). Then

\[ V_{t,T}(\varphi) = -\log \mathbb{E}[e^{-V_T(\varphi + W_{t,T})}] = \inf_{u \in H_a} \mathbb{E}[V_T(W_{t,T} + I_{t,T}(u)) + \frac{1}{2} \int_t^T \|u_s\|^2_{\mathbb{L}^2} ds] \quad (4) \]

**Proposition 4 (Verification)** Assume that \( V_{t,T} \) is defined as in (4). If \( V_T \) is in \( C^2(\mathbb{L}^2(\langle x \rangle^{-n})) \) then so is \( V_{t,T} \) and the equation

\[ dY_{s,t} = -J^2_t \nabla V_{t,T}(Y_{s,t}) dt + J_t dX_t, \quad (5) \]

possess a unique solution in \( \mathbb{L}^2(\mathbb{P}C([0,T], \mathbb{L}^2(\mathbb{R}^2))) \). The infimum on the r.h.s is attained by

\[ u_s = -J_s \nabla V_{s,T}(Y_{t,s}) = -J_s \nabla V_{s,T}(W_{t,s} + I_{t,s}(u)). \]

2.4 Wick ordered cosine

Since the Gaussian Free Field in 2 dimensions is supported on distributions \( \cos(\beta W_\infty) \) is ill defined. However we can correct this by Wick ordering, that is considering instead \( \alpha(T) \cos(\beta W_T) \), with a \( \alpha(T) \rightarrow \infty \) and hoping to obtain a nontrivial limit. That this is indeed possible is the content of the following proposition from [34].

**Proposition 5** Assume that \( \beta^2 < 4\pi \). Then there exists a (differentiable) function \( \alpha(T) \) such that

- \( \alpha(T) \leq C(T)^{\beta^2/8\pi} \)
- \( \left[ \cos(\beta W_T) \right] := \alpha_T \cos(\beta W_T) \) is a martingale in \( T \).
- For an \( p \in [1, \infty) \), as \( T \rightarrow \infty \left[ \cos(\beta W_T) \right] \) converges in \( L^p(\mathbb{P}, B_{p,p}^{-\beta^2/4\pi-2\delta}(\langle x \rangle^{-n})) \) to a limit \( \left[ \cos(\beta W_\infty) \right] \).

For a proof see Appendix A. We can introduce the approximate measures given by

\[ \int f(\varphi) \nu_{SG}^{T,p}(d\varphi) = \frac{1}{Z_T} \mathbb{E}[f(W_T) \exp(-\lambda \int \rho \|\cos(\beta W_T)\| dx)] \]

where \( Z_T \) is a normalization constant. We shall see that \( \nu_{SG}^{T,p} \rightarrow \nu_{SG}^p \) weakly on \( H^{-1}(\langle x \rangle^{-n}) \).
2.5 An Envelope theorem

Lemma 1 Let \( T \in [0, \infty] \). Let \( f, g : H^{-1}(x^{-n}) \to \mathbb{R} \) be continuous bounded functions. Let \( u^g \) be a minimizer for \( F_{T,\rho}^T(u) = \mathbb{E} \left[ g(W_T + I_T(u)) + \int \rho \| \cos(\beta(W_T + I_T(u))) \| dx + \frac{1}{2} \int_0^\infty \| u_t \|^2_{L^2} dt \right] \).

Then \( \int f(\varphi)\nu_{SG}^{T,\rho,g}(d\varphi) = \mathbb{E}\left[ f(W_T + I_T(u^g)) \right] \),

where \( d\nu_{SG}^{T,\rho,g} = \frac{1}{Z_g} \exp(-g(\varphi)) d\nu_{SG}^{T,\rho,g} \) and \( Z_g = \int \exp(-g(\varphi)) d\nu_{SG}^{T,\rho,g} \).

Proof This is a version of the envelope theorem \([40, 2]\). We have

\[ -\log \int \exp(f(\varphi)) d\nu_{SG}^{T,\rho,g} = \left. \frac{d}{dv} \right|_{v=0} - \log \int \exp(-vf(\varphi)) d\nu_{SG}^{T,\rho,g} \]

and the r.h.s is well known to be differentiable in \( v \) since it’s the cumulant generating function. Now

\[ -\log \int \exp(f(\varphi)) d\nu_{SG}^{T,\rho,g} = \inf_{u \in H_a} \mathbb{E} \left[ f(W_T + I_T(u)) + g(W_T + I_T(u)) + \int \rho \| \cos(\beta(W_T + I_T(u))) \| dx + \frac{1}{2} \int_0^T \| u_t \|^2_{L^2} dt \right] \]

For \( T < \infty \) this equality follows from Corollary \([2]\) in Appendix \([4]\). For \( T = \infty \) we have from Theorem 1.3 in \([34]\) that

\[ \mathbb{E} \left[ \exp \left( p \left\| \cos(\beta W_\infty) \right\| dx \right) \right] < \infty \]

so \( \left\| \cos(\beta W_\infty) \right\| dx \) is a tame functional in the language of Üstünel \([47]\), which implies the variational formula (see \([47]\)). Now note that

\[ \lim_{v \to 0} \inf_{u \in H_a} F^{v,f}(u) - \inf_{u \in H_a} F^0(u) = \lim_{v \to 0} \frac{F^{v,f}(u^g) - F^0(u^g)}{v} = \frac{F(W_\infty + I_\infty(u^{T,\rho,g}))}{v} \]

which implies the statement. \( \Box \)

Note that Lemma \([1]\) and Proposition \([6]\) below imply that \( \nu_{SG}^{T,\rho} \to \nu_{SG}^g \) weakly (e.g on \( H^{-1}(x^{-n}) \)).
3 Proof of Theorem 1

Lemma 2 Assume that $|V|_{1,\infty} + |V|_{2,2} < \infty$ then

$$ V_{t,T}(\varphi) = \inf_{u \in \mathbb{R}^n} \mathbb{E} \left[ V_T(W_t + I_{t,T}(u) + \varphi) + \frac{1}{2} \int_t^T \|u_s\|_{L^2}^2 dt \right] $$

(6)

satisfies

$$ \nabla V_T(\varphi) = \mathbb{E}[\nabla V_T(W_t + I_{t,T}(u^\varphi) + \varphi)] $$

where $u^\varphi$ is the optimizer on the r.h.s of (6), in particular

$$ |V_{t,T}|_{1,\infty} \leq |V_T|_{1,\infty}. $$

Proof We have

$$ \langle \nabla V_{t,T}(\varphi), \psi \rangle_{L^2(\mathbb{R}^2)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (V_{t,T}(\varphi + \varepsilon \psi) - V_{t,T}(\varphi)) $$

which implies

$$ \langle \nabla V_{t,T}(\varphi), \psi \rangle_{L^2(\mathbb{R}^2)} \leq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \mathbb{E} \left[ V_T(W_t + I_{t,T}(u^\varphi) + \varphi + \varepsilon \psi) + \frac{1}{2} \int_t^T \|u_s^\varphi\|_{L^2}^2 dt \right] - \mathbb{E} \left[ V_T(W_t + I_{t,T}(u^\varphi) + \varphi + \varepsilon \psi) + \frac{1}{2} \int_t^T \|u_s^\varphi\|_{L^2}^2 dt \right] \right) $$

on the other hand we have by differentiability,

$$ \langle \nabla V_{t,T}(\varphi), \psi \rangle_{L^2(\mathbb{R}^2)} \geq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \mathbb{E} \left[ V_T(W_t + I_{t,T}(u^\varphi) + \varphi + \varepsilon \psi) + \frac{1}{2} \int_t^T \|u_s^\varphi\|_{L^2}^2 dt \right] - \mathbb{E} \left[ V_T(W_t + I_{t,T}(u^\varphi) + \varphi - \varepsilon \psi) + \frac{1}{2} \int_t^T \|u_s^\varphi\|_{L^2}^2 dt \right] \right) $$

and we get the statement.

Corollary 1 Assume that $t \geq T/2$ and $|V|_{1,\infty} \leq C T^{1/2 - \delta}$. Then with $u^\varphi$ minimizing

$$ F(u) = \mathbb{E} \left[ V_T(W_t + I_{t,T}(u) + \varphi) + \frac{1}{2} \int_t^T \|u_t\|_{L^2}^2 dt \right]. $$

Then $u^\varphi$ satisfies

$$ \|u_s^\varphi\|_{L^\infty} \leq C \lambda^{-1/2}. $$
Proof By Proposition 4 $u_\varphi$ satisfies
\[ \| u_\varphi \|_{L^\infty} = \| J_\varphi \nabla V_{s,T}(W_s + I_s(u_\varphi)) \|_{L^\infty} \leq s^{-1} \lambda T^{1/2 - \delta} \leq \lambda(s)^{-1/2 - \delta}, \]
since $s \geq T/2$. □

We now show that $u_\varphi$ also satisfies an Euler-Lagrange equation.

Lemma 3 Denote by $u_\varphi$ the minizer of the r.h.s of (6). It exists by Proposition 4 since $V_T \in C^2(L^2(\mathbb{R}^2), \mathbb{R})$. We show that $u_\varphi$ satisfies, for any $h \in H^a$
\[ E \left[ \int \nabla V_T(W_t,T + I_t,T(u_\varphi) + \varphi) I_t,T(h)dx + \int_t^T \int u_\varphi h dx dt \right]. \tag{7} \]

Proof Indeed since $u_\varphi$ is a minimizer we have for any $h \in H^a$
\[ E \left[ V_T(W_t,T + I_t,T(u_\varphi \pm \varepsilon h) + \varepsilon h_s)^2 \right] - E \left[ V_T(W_t,T + I_t,T(u_\varphi) + \varepsilon h_s)^2 \right] \geq 0 \]
and taking $\varepsilon \to 0$ we get the claim since
\[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (V_T(W_t,T + I_t,T(u_\varphi \pm \varepsilon h) + \varphi) - V_T(\varphi + I_t,T(u) + \varphi)) = \pm \int \nabla V_T(W_t,T + I_t,T(u_\varphi) + \varphi) I_t,T(h)dx \]
and
\[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \int_t^T \| u_\varphi \pm \varepsilon h_s \|_{L^2}^2 dt - \int_t^T \| u_\varphi \|_{L^2}^2 dt \right) = \pm 2 \int_t^T \int u_\varphi h_s dx ds \]
so
\[ \pm E \left[ \int \nabla V_T(W_t,T + I_t,T(u_\varphi) + \varphi) I_t,T(u_\varphi)dx + \int_t^T \int u_\varphi h_s dx ds \right] \geq 0 \]
which implies the claim. □

Lemma 4 Assume that
\[ \nabla V_T(\varphi) = \lambda \alpha T \beta \rho \sin(\beta \varphi) + R_T(\varphi) \]
with $\sup_{\varphi \in L^2(\mathbb{R}^2)} \| R_T(\varphi) \|_{L^\infty} \leq C$ and $\beta^2 < 4\pi$. Then for $t \geq T/2$
\[ \nabla V_t(T(\varphi) = \lambda \alpha t \beta \rho \sin(\beta \varphi) + R_t(T(\varphi) \]
where
1. The following inequality holds
\[ \sup_{\varphi \in L^2((x)^n)} \| R_{t,T}(\varphi) \|_{L^\infty} \leq C \lambda^2(t)^{-\delta} + \sup_{\varphi \in L^2((x)^n)} \| R_T(\varphi) \|_{L^\infty}. \]

2. There exists a constant dependent on \( \rho \) such that
\[ \sup_{\varphi \in L^2((x)^n)} \| R_{t,T}(\varphi) \|_{L^2} \leq C \rho \lambda^2(t)^{-\delta} + \sup_{\varphi \in L^2((x)^n)} \| R_T(\varphi) \|_{L^2}. \]

3. There exists a constant \( C \) (independent of \( T, \lambda, \beta \)) such that for \( t \geq T/2 \land C \)
\[ \sup_{\varphi, \psi \in L^2((x)^n)} \frac{\| R_{t,T}(\varphi) - R_{t,T}(\psi) \|_{L^2}}{\| \varphi - \psi \|_{L^2}} \leq C \left( \sup_{\varphi, \psi \in L^2((x)^n)} \frac{\| R_T(\varphi) - R_T(\psi) \|_{L^2}}{\| \varphi - \psi \|_{L^2}} \right). \]

Proof of 1. Step 1
We first establish the estimate for \( t \geq T/2 \). By Lemma\[2\]
\[
\nabla V_{t,T}(\varphi) = \mathbb{E}[\lambda \alpha T \beta \rho \sin(\beta(W_{t,T} + I_{t,T}(u^\varphi) + \varphi))] + \mathbb{E}[R_T(W_{t,T} + I_{t,T}(u^\varphi) + \varphi)]
\]
\[ = \mathbb{E}[\lambda \alpha T \beta \rho \sin(\beta(W_{t,T} + \varphi))] + \mathbb{E} \left[ \lambda \int_0^1 \alpha T \beta \sin(\beta(W_{t,T} + sI_{t,T}(u^\varphi) + \varphi))I_{t,T}(u^\varphi)ds \right]
\]
\[ + \mathbb{E}[R_T(W_{t,T} + I_{t,T}(u^\varphi) + \varphi)] \]

Now note that by the martingale property of the renormalized sine
\[ \mathbb{E}[\alpha T \sin(\beta W_{t,T} + \beta \varphi)] = \alpha T \beta \sin(\beta \varphi) \]
so it remains to estimate the second term. Since
\[
\mathbb{E} \left[ \lambda \int_0^1 \alpha T \beta \rho \sin(\beta(W_{t,T} + sI_{t,T}(u^\varphi) + \varphi))I_{t,T}(u^\varphi)ds \right]_{L^\infty}
\]
\[ \leq \lambda \mathbb{E} \sup_{s \in [0,1]} \alpha T \beta \rho \sin(\beta(W_{t,T} + sI_{t,T}(u^\varphi) + \varphi))I_{t,T}(u^\varphi)_{L^\infty}
\]
\[ \leq C \lambda^2(t)^{-1/2} \alpha T
\]
\[ \leq C \lambda^2(t)^{-\delta}, \]

where in the second to last line we used Corollary[1]. This gives the first inequality for \( t \geq T/2 \).

Proof of 1. Step 2
By Proposition[4] below we have that
\[ V_{t,T}(\varphi) = \inf_{u \in H_u} \mathbb{E} \left[ V_{T/2,T}(W_{t,T/2} + I_{t,T/2}(u) + \varphi) + \frac{1}{2} \int_t^{T/2} \| u_t \|_{L^2}^2 dt \right] \]
so applying Step 1 we get for \( t \geq T/4 \)
\[ \nabla V_{t,T}(\varphi) = \lambda \beta \alpha T \sin(\beta \varphi) + R_{t,T}(\varphi) \]

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which is the desired statement.

Observe that

\[
\sup_{\varphi \in L^2((x)^\alpha)} \| R_{t,T}(\varphi) \|_{L^\infty} \leq C\lambda^2 \left( (T/2)^{-\delta} + (T/4)^{-\delta} \right).
\]

Proceeding like this inductively we obtain for \( T/2^{i-1} \geq t \geq T/2^i \)

\[
\sup_{\varphi \in L^2((x)^\alpha)} \| R_{t,T}(\varphi) \|_{L^\infty} \leq C\lambda^2 \sum_{k=1}^{i} (T/2^k)^{-\delta}
\leq 2C\lambda^2 \sum_{k=1}^{i} (2t/2^k)^{-\delta}
\leq C\lambda^2 (t)^{-\delta} \sum_{k=1}^{i} 2^{i-k}\delta
\]

\[
\tilde{k} = (i-k) = C\lambda^2 (t)^{-\delta} \sum_{k=1}^{i} 2^{-\tilde{k}\delta}
\leq C\lambda^2 (t)^{-\delta}
\]

which is the desired statement.

**Proof of 2.** The proof of the second assertion is analogous since in that case \( \rho \sin(\cdot) \) is in \( L^2(\mathbb{R}^2) \).

**Proof of 3. Step 1** We first proof the statement for \( t \geq T/2 \wedge C \). For this we have to first estimate the difference

\[
I_{t,T}(u^{\rho,\rho}) - I_{t,T}(u^{\psi,\rho}).
\]

Observe that

\[
F_{t,T}(u) - F_{t,T}(\varphi)
= E \left[ V_T(W_{t,T} + I_{t,T}(u) + \psi) + \frac{1}{2} \int_0^\infty \| u_t \|_{L^2}^2 dt \right]
\]

\[
= E \left[ V_T(W_{t,T} + I_{t,T}(u) + \varphi) \right.
+ \int_0^1 \int \nabla V_T(W_{t,T} + I_{t,T}(u) + (1-\theta)\varphi + \theta\psi)(\varphi - \psi) dx d\theta + \frac{1}{2} \int_0^\infty \| u_t \|_{L^2}^2 dt \left. \right]
\]

\[
= F_{t,T}(\varphi) + E \left[ \int_0^1 \int \nabla V_T(W_{t,T} + I_{t,T}(u) + (1-\theta)\varphi + \theta\psi)(\varphi - \psi) dx \right]
\]

\[
=: F_{t,T}(\varphi) + K(u, \varphi, \psi),
\]

and the last line is to be taken as a definition for \( K \). By our assumption we can estimate

\[
|K(u, \varphi, \psi) - K(v, \varphi, \psi)|
\leq E \left[ \int_0^1 \int (\nabla V_T(W_{t,T} + I_{t,T}(u) + (1-\theta)\varphi + \theta\psi) - \nabla V_T(W_{t,T} + I_{t,T}(v) + (1-\theta)\varphi + \theta\psi))(\varphi - \psi) dx \right]
\leq C(T)^{\beta^2/4\pi} E[\| I_{t,T}(u-v) \|_{L^2(\mathbb{R}^2)}] \| \varphi - \psi \|_{L^2(\mathbb{R}^2)}
\]

\[
\leq Ct^{-1/2} (T)^{\beta^2/4\pi} E[\| u-v \|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}] \| \varphi - \psi \|_{L^2(\mathbb{R}^2)}
\]

\[
\leq Ct^{-\delta} E[\| u-v \|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}] \| \varphi - \psi \|_{L^2(\mathbb{R}^2)}
\]

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where in the second to last line we used Lemma 21 and in the last line the assumption $T \leq 2t$. Furthermore $u^{\psi, \rho}$ is a minimizer implies

$$F^{\rho, \psi}(u^{\psi, \rho}) - F^{\rho, \psi}(u^{\varphi, \rho}) \leq 0.$$  

On the other hand using the semiconvexity of $V_T$ (which follows from the assumption) and the Euler Lagrange equation for $u^{\varphi, \rho}$ (Equation (7)) we get

$$F^{\rho, \varphi}(u) - F^{\rho, \varphi}(u^{\varphi, \rho}) = \mathbb{E} \left[ V_T(W_{t,T} + I_{t,T}(u + \varphi) - V_T(W_{t,T} + I_{t,T}(u^{\varphi, \rho} + \varphi)) + \frac{1}{2} \int_0^\infty \|u_t\|_{L^2}^2 dt - \frac{1}{2} \int_0^\infty \|u_t^{\varphi, \rho}\|_{L^2}^2 dt \right]$$

$$\geq \mathbb{E} \left[ \int_0^1 \nabla V_T(W_{t,T} + I_{t,T}(u^{\varphi, \rho}) + \varphi) I_{t,T}(u - u^{\varphi, \rho}) dx + \int_0^\infty \int_0^\infty u_t^{\varphi, \rho}(u_t - u_t^{\varphi, \rho}) dx dt \\
- T^{1/2} \|I_{t,T}(u - u^{\varphi, \rho})\|_{L^2}^2 + \frac{1}{2} \int_0^\infty \|u_t - u_t^{\varphi, \rho}\|_{L^2}^2 dt \right]$$

$$= -CT^{1/2} \mathbb{E} \left[ \|I_{t,T}(u - u^{\varphi, \rho})\|_{L^2}^2 \right] + \frac{1}{2} \mathbb{E} \left[ \int_0^\infty \|u_t - u_t^{\varphi, \rho}\|_{L^2}^2 dt \right]$$

$$\geq \frac{1}{4} \int_0^\infty \|u_t - u_t^{\varphi, \rho}\|_{L^2}^2 dt.$$  

In the last line we used Lemma 21 and the assumption that $t \geq T/2 \wedge C$. Combining everything we get

$$0 \geq F^{\psi, \rho}(u^{\psi, \rho}) - F^{\rho, \psi}(u^{\varphi, \rho})$$

$$= F^{\rho, \psi}(u^{\psi, \rho}) - F^{\rho, \psi}(u^{\varphi, \rho})$$

$$+ K(u^{\psi, \rho}, \varphi, \psi) - K(u^{\varphi, \rho}, \varphi, \psi)$$

$$\geq \frac{1}{4} \mathbb{E} \left[ \int_0^\infty \|u_t^{\psi, \rho} - u_t^{\varphi, \rho}\|_{L^2}^2 dt \right] - \|K(u^{\psi, \rho}, \varphi, \psi) - K(u^{\varphi, \rho}, \varphi, \psi)\|_1$$

$$\geq \frac{1}{4} \mathbb{E} \left[ \|u^{\psi, \rho} - u^{\varphi, \rho}\|_{L^2(\mathbb{R} \times \mathbb{R}^2)}^2 \right] - Ct^{-\delta} \mathbb{E} \left[ \|u^{\psi, \rho} - u^{\varphi, \rho}\|_{L^2(\mathbb{R} \times \mathbb{R}^2)}^2 \right]^{1/2} \|\varphi - \psi\|_{L^2(\mathbb{R}^2)}$$

which implies

$$\|u^{\psi, \rho} - u^{\varphi, \rho}\|_{L^2(\mathbb{R} \times \mathbb{R}^2)} \leq Ct^{-\delta} \|\varphi - \psi\|_{L^2(\mathbb{R}^2)}.$$  

Note that this implies $\|I_{t,T}(u^{\psi, \rho} - u^{\varphi, \rho})\|_{L^2(\mathbb{R}^2)} \leq Ct^{-1/2 - \delta} \|\varphi - \psi\|_{L^2(\mathbb{R}^2)}.$
Proof of 3. Step 2 Recall that we have to estimate

\[ \left\| \lambda E \left[ \int_0^1 \alpha_T \beta \sin(\beta(W_{t,T} + s I_{t,T}(u^{\psi,\rho})) I_{t,T}(u^{\psi,\rho}) ds \right] \right. \]

\[ -\lambda \left\| \int_0^1 \alpha_T \beta \sin(\beta(W_{t,T} + s I_{t,T}(u^{\psi,\rho})) I_{t,T}(u^{\psi,\rho}) ds \right\|_{L^2} \]

\[ \leq \lambda \left\| \int_0^1 \alpha_T \beta \sin(\beta(W_{t,T} + s I_{t,T}(u^{\psi,\rho})) I_{t,T}(u^{\psi,\rho}) ds \right\|_{L^2} \]

\[ + \lambda \left\| \int_0^1 \alpha_T \beta \sin(\beta(W_{t,T} + s I_{t,T}(u^{\psi,\rho})) I_{t,T}(u^{\psi,\rho}) ds \right\|_{L^2} \]

\[ \leq C \lambda \left( T^{1/2} \left\| I_{t,T}(u^{\psi,\rho} - u^{\psi,\rho}) \right\|_{L^2} + T^{1/2} \left\| I_{t,T}(u^{\psi,\rho}) \right\|_{L^\infty} \right) \left( \left\| I_{t,T}(u^{\psi,\rho} - u^{\psi,\rho}) \right\|_{L^2} + \left\| \psi - \psi \right\|_{L^2} \right) \]

Where in the last line we have used Corollary [1].

Finally we can estimate

\[ \left\| E\left[ R_T (W_{t,T} + I_{t,T}(u^{\psi})) - R_T (W_{t,T} + I_{t,T}(u^{\psi})) \right] \right\|_{L^2} \]

\[ \leq \left\| R_T \right\|_{L^2 \to L^2} \left( \left\| I_{t,T}(u^{\psi}) - I_{t,T}(u^{\psi}) \right\|_{L^2} + \left\| \psi - \psi \right\|_{L^2} \right) \]

where we have used the notation

\[ \left\| R_T \right\|_{L^2 \to L^2} = \sup_{\varphi, \psi \in L^2(\mathbb{R})} \frac{\left\| R_T(\varphi) - R_T(\psi) \right\|_{L^2}}{\left\| \varphi - \psi \right\|_{L^2}}. \]

Proof of 3. Step 3 Now proceeding as in the proof of Assertion 1, Step 2, we obtain for \( t \geq T/2^i \wedge C \)

\[ \left\| I_{t,T} \right\|_{L^2 \to L^2} \leq \prod_{k=1}^i \left( 1 + \langle T/2^k \rangle^{-\delta} \right) \]

Now observe that

\[ \log \prod_{k=1}^i \left( 1 + \langle T/2^k \rangle^{-\delta} \right) \leq \sum_{k=1}^i \langle T/2^k \rangle^{-\delta} \leq C \sum_{k=1}^i \langle T/2^k \rangle^{-\delta} \leq C \]

This gives the statement for \( t \geq C \). Now finally we can conclude by Proposition [13]. \( \square \)

Proof of Theorem [1] Let \( u^{T,\rho} \) be the minimizer for the functional

\[ F^T(u) = \mathbb{E} \left[ \int \rho \left[ \cos(\beta(W_T + I_T(u))) \right] dx + \frac{1}{2} \int_0^T \left\| u_t \right\|_2^2 dt \right]. \]

By Proposition [14] we have that

\[ \left\| u^{T,\rho} \right\|_{L^\infty} = \left\| Q_T \nabla V_{t,T}(W_T + I_T(u)) \right\|_{L^\infty} \]

where

\[ V_{t,T}(\varphi) = \inf_{u \in \mathbb{H}_a} \mathbb{E} \left[ \int \rho \left[ \cos(\beta(W_{t,T} + I_{t,T}(u) + \varphi)) \right] dx + \frac{1}{2} \int_t^T \left\| u_t \right\|_2^2 dt \right]. \]
Now by Lemma 1
\[ \|Q_t \nabla V_{t,T}(W_t + I_t(u))\|_{L^\infty} \leq t^{-1} \sup_{\varphi \in L^2((x)^n)} \|\nabla V_{t,T}(\varphi)\|_{L^\infty} \leq Ct^{-1}(\alpha(t) + t^{-3}) \leq Ct^{-1/2 - \delta}, \]
which implies the statement. \[\square\]

**Proposition 6** Let \( u^{T,\rho} \) be the minimizer for the functional
\[ F^T(u) = \mathbb{E} \left[ \int \rho[\cos(\beta(W_T + I_T(u)))]dx + \frac{1}{2} \int_0^\infty \|u_t\|_{L^2}dt \right], \]
then as \( T \to \infty \) \( u^{T,\rho} \) converges to \( u^{\infty,\rho} \) in \( \mathbb{H}_a \) to a minimizer of \( F^\infty(u) \).

**Proof** We first show that \( u^{T,\rho} \) is a Cauchy-sequence. By Proposition 3 \( u^{T,\rho} \) satisfies the equation
\[ u_t^{T,\rho} = J_t V_{t,T}(W_t + I_t(u^{T,\rho})). \]
so
\[ (u_t^{T_1,\rho} - u_t^{T_2,\rho}) = J_t(V_{t,T_1}(W_t + I_t(u^{T_1,\rho})) - V_{t,T_2}(W_t + I_t(u^{T_2,\rho}))). \]
Now note that by Lemma 1
\[ \nabla V_{t,T_1}(\varphi) = \lambda_\alpha \nabla \sin(\beta \varphi) + R_{T_1},(\varphi) \]
with \( \|R_{T_1},(\varphi)\|_{L^2} \leq C_\rho T_1^{-\delta}. \) This implies by Proposition 14 that with \( T_1 \leq T_2 \)
\[ \sup_{\varphi \in L^2((x)^n)} \|\nabla V_{t,T_1}(\varphi) - \nabla V_{t,T_2}(\varphi)\|_{L^2} \leq C|t|^{-\delta}(T_1)^{-\delta}, \]
and
\[ \|Q_t(V_{t,T_1}(W_t + I_t(u^{T_1,\rho})) - V_{t,T_2}(W_t + I_t(u^{T_2,\rho})))\|_{L^2} \leq Ct^{-1}\|V_{t,T_1}(W_t + I_t(u^{T_1,\rho})) - V_{t,T_2}(W_t + I_t(u^{T_2,\rho})))\|_{L^2} \]
\[ \leq Ct^{-1}\|V_{t,T_1}(W_t + I_t(u^{T_1,\rho})) - V_{t,T_2}(W_t + I_t(u^{T_2,\rho})))\|_{L^2} \]
\[ + Ct^{-1}\sup_{\varphi \in L^2((x)^n)} \|V_{t,T_1}(\varphi) - V_{t,T_2}(\varphi)\|_{L^2} \]
\[ \leq Ct^{-1}t^{3/8}\|I_t(u^{T_1,\rho} - u^{T_2,\rho})\|_{L^2} + Ct^{-1}\sup_{\varphi \in L^2((x)^n)} \|V_{t,T_1}(\varphi) - V_{t,T_2}(\varphi)\|_{L^2} \]
\[ \leq Ct^{-1}t^{1-\delta/8}\|u^{T_1,\rho} - u^{T_2,\rho}\|_{L^2} + Ct^{-1}(T_1)^{-\delta}. \]
Gronwall’s lemma now gives
\[ \|u^{T_1,\rho} - u^{T_2,\rho}\|_{L^2} \leq Ct^{-1}(T_1)^{-\delta} \]
which implies that \( u^{T,\rho} \) is a Cauchy sequence in \( L^2(\mathbb{R}_+ \times \mathbb{R}^2) \) (since \( t^{-1} \) is in \( L^2 \)). To prove that the limit is a minimizer of \( F^{\infty,\rho} \) it is sufficient to check that
\[ \lim_{T \to \infty} F^{T,\rho}(u^T) = F^{\infty,\rho}(u) \]

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for any sequence $u^T \to u$ in $\mathbb{H}_a$. Indeed this implies

$$F^\infty,\rho(u) \geq \inf_{u \in \mathbb{H}_a} F^\infty,\rho(u) = \inf_{u \in \mathbb{H}_a} \lim_{T \to \infty} F^T,\rho(u) \geq \lim_{T \to \infty} F^T,\rho(u^T,\rho) = F^\infty,\rho(u^\infty,\rho),$$

which gives the claim. To show this observe that

$$|F^{T,\rho}(u^T) - F^{\infty,\rho}(u)|$$

$$\leq \mathbb{E} \left[ \int \rho(\|\cos(\beta(W_\infty + I_\infty(u)))\| - [\cos(\beta(W_T + I_T(u)))]^2 dx \right] + \mathbb{E}[\|u - u^T\|_{L^2(\mathbb{R}_+ \times \mathbb{R}^2)}^2]$$

and recall that for $T \in [0, \infty]$

$$\langle \cos(\beta(W_T + I_T(u))) \rangle = \langle \cos(\beta(W_T)) \rangle \sin(\beta I_T(u)) + [\sin(\beta(W_T))] \cos(\beta I_T(u)).$$

We have that if $u^T \to u$ in $\mathbb{H}_a$ then $\|I_T(u^T) - I_\infty(u)\| \leq \|I_T(u - u^T)\| + \|I_T_\infty(u)\| \to 0 \mathbb{P} - a.s.$

Furthermore $\|I_T(u^T) - I_\infty(u)\| \leq \|u^T\|_{L^2(\mathbb{R}_+ \times \mathbb{R}^2)} + \|u\|_{L^2(\mathbb{R}_+ \times \mathbb{R}^2)}$ and thereby uniformly in $L^2(\mathbb{P})$. Finally recalling that $\|\cos(\beta(W_T))\| \to \|\cos(\beta(W_\infty))\|$ in $LP(\mathbb{P}, H^1_{loc}(\mathbb{R}^2))$ we can conclude that

$$\int \rho(\|\cos(\beta(W_T + I_T(u)))\| - [\cos(\beta(W_\infty + I_\infty(u)))]^2 dx \right]$$

is $\mathbb{P} - a.s$ and is uniformly integrable, which implies the claim.

\[\square\]

### 4 Variational characterization

**Theorem 6** Let $\bar{u}^{\rho,f}$ be the minimizer of

$$F^{\rho,f}(u) = \mathbb{E} \left[ f(W_\infty + I_\infty(u)) + \int \rho(x) [\cos(\beta(W_\infty + I_\infty(u)))] dx + \frac{1}{2} \int \|u_t\|_{L^2}^2 dt \right] \quad (8)$$

Then there exists $\gamma, \delta > 0$ such that for any $\lambda(\beta + \beta^2) < \delta$ and $w(x) = \exp(\gamma|x|)$, there exists $C > 0$ such that

$$\mathbb{E} \left[ \int_0^\infty \|w(\bar{u}^{\rho,f} - \bar{u})\|_{L^2}^2 dt \right] \leq C |f|_{1, \gamma}$$

Before proceeding to the proof we establish and EL-equation for the minimers of (8), similarly to (8).

**Lemma 5** There exists a minimer $\bar{u}^{\rho,f}$ of (8) in $\mathbb{H}_a$ and it satisfies the equation

$$\mathbb{E} \left[ \beta^2 \int_0^\infty \int J_t(\rho[\cos(\beta(W_t + I_t(\bar{u}^{\rho,f})))]) I_t(h_t) \left( \bar{u}^{\rho,f}_t \right) dx dt \right. $$

$$+ \beta \int_0^\infty \left[ J_t(\rho[\sin(\beta(W_t + I_t(\bar{u}^{\rho,f})))]) h_t dx \right. $$

$$= \mathbb{E} \left[ \int_0^\infty \bar{u}^{\rho,f}_t h_t dx dt \right] - \mathbb{E} \left[ \int \nabla f(W_\infty + I_\infty(\bar{u}^{\rho,f})) I_\infty(h) dx \right].$$

for any $h \in \mathbb{H}_a$. 18
Indeed by Ito’s formula
\[ \int_0^T \rho \| \cos(\beta(W_t + I_T(u))) \| dx = \int_0^T \rho \| \cos(\beta(W_t + I_t(u))) \| J_t u_t dx + \text{martingale}. \]

Indeed by Ito’s formula
\[ \int_0^T \rho \| \cos(\beta(W_t + I_T(u))) \| dx - \int_0^T \rho dx = \int_0^T \rho \| \sin(\beta(W_t + I_t(u))) \| dW_t + \int_0^T J_t(\rho \| \sin(\beta(W_t + I_t(u))) \|) u_t dt. \]

A priori the first term on the r.h.s might only be a local martingale but we can see that by Ito isometry it’s quadratic variation is
\[ \int (J_t(\rho \| \sin(\beta(W_t + I_t(u))) \|))^2 dx \leq t^{-2} t^{2.1/4\pi} \| \rho \|_{L^2}^2 \leq t^{-1.2} \| \rho \|_{L^2}^2 \]

wich is integrable in time. Thus we can conclude that the first part is indeed an martingale. From this we deduce that
\[ \mathbb{E} \left[ f(W_\infty + I_\infty(u)) + \lambda \int_0^T \rho \| \cos(\beta(W_\infty + I_\infty(u))) \| dx + \frac{1}{2} \int \| u_t \|_{L^2}^2 dt \right] = \mathbb{E} \left[ f(W_\infty + I_\infty(u)) + \lambda \int_0^\infty J_t(\rho \| \sin(\beta(W_t + I_t(u))) \|)(u_t) dx + \frac{1}{2} \int \| u_t \|_{L^2}^2 dt + \int \rho dx \right]. \]

Now assume that \( u \in \mathbb{H}_a \) minimizes the functional. Then for any \( h \in \mathbb{H}_a \) we have
\[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (F(u + \varepsilon h) - F(u)) \geq 0 \quad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (F(u) - F(u - \varepsilon h)) \leq 0. \]

It is clear that
\[ \frac{1}{2} \left( \int \| u_t + h_t \|_{L^2}^2 dt + \int \| u_t \|_{L^2}^2 dt \right) = \int \int u_t h_t dx dt. \]

Now consider the limit
\[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \mathbb{E} \left[ \int J_t(\rho \| \sin(\beta(W_t + I_t(u + \varepsilon h))) \|)(u_t + \varepsilon h_t) dx dt \right] \right) \]

Recall that since \( \| \sin(\beta(W_t + I_t(u + \varepsilon h))) \| = \alpha(t) \sin(\beta(W_t + I_t(u + \varepsilon h))) \) and \( \alpha(t) \leq Ct^{3/8\pi} \)
\[
\frac{1}{\varepsilon} \int (J_t \rho [\sin(\beta(W_t + I_t(u + \varepsilon h)))]) (u_t + \varepsilon h_t) \right) - J_t([\sin(\beta(W_t + I_t(u)))])(u_t) dx \right) \]
\[
\leq \frac{1}{\varepsilon} \left( \int J_t(\rho [\sin(\beta(W_t + I_t(u + \varepsilon h)))])(u_t) - J_t(\rho [\sin(\beta(W_t + I_t(u)))])(u_t) dx \right) \]
\[
+ \left( \int J_t(\rho [\sin(\beta(W_t + I_t(u + \varepsilon h)))])(h_t) dx \right) \]
\[
\leq C t^{3/8\pi - 1} \| I_t(h) \|_{L^2(\mathbb{R}^2)} \| u_t \|_{L^2(\mathbb{R}^2)} + C t^{3/8\pi} \left( \int \rho dx \right)^{1/2} \| h_t \|_{L^2(\mathbb{R}^2)}.
\]
Note that the last term does not depend on $\varepsilon$ and is integrable in time and probability, since $u,h \in H_u$. By dominated convergence we thus have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( E \left[ \int_0^\infty \int J_t \rho \{ \sin(\beta(W_t + I_t(u + \varepsilon h))) \} (u_t + \varepsilon h_t) dx dt \right] \right)$$

$$- \int_0^\infty \int J_t \rho \{ \sin(\beta(W_t + I_t(u))) \} (u_t) dx dt \right)$$

$$= E \int_0^\infty \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int J_t \rho \{ \sin(\beta(W_t + I_t(u + \varepsilon h))) \} (u_t + \varepsilon h_t) dx$$

$$- \int J_t \rho \{ \sin(\beta(W_t + I_t(u))) \} (u_t) dx \right) dt$$

and so it remains to show

$$\lim_{\varepsilon \to 0} \left( \frac{1}{\varepsilon} \int J_t \rho \{ \sin(\beta(W_t + I_t(u + \varepsilon h))) \} (u_t + \varepsilon h_t) dx - \int J_t \rho \{ \sin(\beta(W_t + I_t(u))) \} (u_t) dx \right)$$

$$= \left( \int \frac{1}{\varepsilon} (J_t \rho \{ \sin(\beta(W_t + I_t(u + \varepsilon h))) \} (u_t + \varepsilon h) - J_t \rho \{ \sin(\beta(W_t + I_t(u))) \} (u_t)) dx \right)$$

from which the statement follows by chain rule. Now

$$\frac{1}{\varepsilon} |J_t \rho \{ \sin(\beta(W_t + I_t(u + \varepsilon h))) \} (u_t + \varepsilon h_t) - J_t \rho \{ \sin(\beta(W_t + I_t(u))) \} (u_t)|$$

$$\leq |J_t \rho \{ \sin(\beta(W_t + I_t(u + \varepsilon h))) \} (h_t)|$$

$$+ \frac{1}{\varepsilon} |J_t \rho \{ \sin(\beta(W_t + I_t(u + \varepsilon h))) \} - J_t \rho \{ \sin(\beta(W_t + I_t(u))) \} (u_t)|$$

$$\leq \mu^{\beta/8} \| J_t \rho \{ \sin(\beta(W_t + I_t(u + \varepsilon h))) \} (u_t) \| u_t$$

which is integrable in $\mathbb{R}^2$, so we can conclude by dominated convergence. \[ \square \]

**Lemma 6** Let $\bar{u}^{\rho,g}$ satisfy the equation

$$E \left[ \lambda^2 \int_0^\infty \int J_t \rho \{ \cos(\beta(W_t + I_t(\bar{u}^{\rho,g}))) \} I_t(h_t) (\bar{u}^{\rho,g}) dx dt \right]$$

$$+ \lambda \beta \left( \int_0^\infty \int J_t \rho \{ \sin(\beta(W_t + I_t(\bar{u}^{\rho,g}))) \} I_t(h_t) dx dt \right)$$

$$= E \left[ \int_0^\infty \int \bar{u}^{\rho,g} h_t dx dt \right] - E[g(W,\bar{u}^{\rho,g},h)].$$ 

and assume that for $0 < \gamma < 2m$ and $z \in \mathbb{R}^2$, $w(x) = \exp(\gamma |x - z|)$, we have

$$E[g(W,\bar{u}^{\rho,g},h)] \leq C_g E\left[ w^{-1/2} \left\| \bar{u}^{\rho,g} \right\|^2_{L^2(\mathbb{R}_+ \times \mathbb{R}^2)} \right]^{1/2}.$$ 

Then there exists a $\kappa > 0$ such that for $\lambda(\beta^2 + \beta) \leq \kappa$

$$E\left[ \left\| w^{1/2}(\bar{u}^{\rho,g} - \bar{u}^{\rho,0}) \right\|^2_{L^2(\mathbb{R}_+ \times \mathbb{R}^2)} \right] \leq 2C_g.$$

**Proof** Taking differences of the EL equations we get

$$E \left[ \int_0^\infty \int (\bar{u}^{\rho,g} - \bar{u}^{\rho,0}) h_t dx dt \right] = I + II + III$$

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with

\[
I = E \left[ \lambda \beta^2 \int_0^\infty \int J_t \left( \rho \left( \left\| \cos(\beta(W_t + I_t(\tilde{u}^\rho))) \right\| - \left\| \cos(\beta(W_t + I_t(\tilde{u}^\rho))) \right\| I_t(h) \right) \right) (\tilde{u}^\rho_t) dx dt \right]
\]

\[
= E \left[ \lambda \beta^2 \int_0^\infty \int J_t \left( \rho \left( \left\| \cos(\beta(W_t + I_t(\tilde{u}^\rho))) \right\| - \left\| \cos(\beta(W_t + I_t(\tilde{u}^\rho))) \right\| I_t(h) \right) \right) (\tilde{u}^\rho_t)
+ J_t (\rho \left( \left\| \cos(\beta(W_t + I_t(\tilde{u}^\rho))) \right\| I_t(h) \right) (\tilde{u}^\rho_t - u^\rho_t)) dx dt \right]
\]

\[
= I_a + I_b
\]

\[
II = \lambda \beta E \left[ \int_0^\infty \int J_t \left( \left\| \sin(\beta(W_t + I_t(\tilde{u}^\rho))) \right\| - \left\| \sin(\beta(W_t + I_t(\tilde{u}^\rho))) \right\| h_t dx dt \right]
\]

\[
III = E [g(W, u^\rho, h)]
\]

Now setting \( h = w(\tilde{u}^{\rho,g} - \tilde{u}^\rho) \) we can estimate

\[
\frac{1}{\lambda \beta^2} |I_a|
\]

\[
= \left| E \left[ \int_0^\infty \int J_t \left( \rho \left( \left\| \cos(\beta(W_t + I_t(\tilde{u}^\rho))) \right\| - \left\| \cos(\beta(W_t + I_t(\tilde{u}^\rho))) \right\| I_t(h) \right) \right) (\tilde{u}^\rho_t) dx dt \right]\right|
\]

\[
= \left| E \left[ \int_0^\infty \int J_t \left( w^{1/2} \rho \left( \left\| \cos(\beta(W_t + I_t(\tilde{u}^\rho))) \right\| - w^{1/2} \rho \left\| \cos(\beta(W_t + I_t(\tilde{u}^\rho))) \right\| I_t(h) \right) \right) (\tilde{u}^\rho_t) dx dt \right]\right|
\]

\[
\leq E \left[ \left( \left\| \left( J_t w^{1/2} \rho \left( \left\| \cos(\beta(W_t + I_t(\tilde{u}^\rho))) \right\| - J_t w^{1/2} \rho \left\| \cos(\beta(W_t + I_t(\tilde{u}^\rho))) \right\| I_t(h) \right) \right) (\tilde{u}^\rho_t)^2 \right\|_{L^1(\mathbb{R}^+, L^2(\mathbb{R}))} \right]^{1/2}
\times E [\|w^{1/2} I_t(w(\tilde{u}^{\rho,g} - \tilde{u}^\rho))\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))}]^{1/2}
\]

\[
\leq E \left[ \left( \left\| \left( J_t w^{1/2} \rho \left( \left\| \cos(\beta(W_t + I_t(\tilde{u}^\rho))) \right\| - J_t w^{1/2} \rho \left\| \cos(\beta(W_t + I_t(\tilde{u}^\rho))) \right\| I_t(h) \right) \right) (\tilde{u}^\rho_t)^2 \right\|_{L^1(\mathbb{R}^+, L^2(\mathbb{R}))} \right]^{1/2}
\times E [\|w^{1/2} I_t(w(\tilde{u}^{\rho,g} - \tilde{u}^\rho))\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))}]^{1/2}
\]

\[
\leq E \left[ \left( \left\| \left( J_t w^{1/2} \rho \left( \left\| \cos(\beta(W_t + I_t(\tilde{u}^\rho))) \right\| - J_t w^{1/2} \rho \left\| \cos(\beta(W_t + I_t(\tilde{u}^\rho))) \right\| I_t(h) \right) \right) (\tilde{u}^\rho_t)^2 \right\|_{L^1(\mathbb{R}^+, L^2(\mathbb{R}))} \right]^{1/2}
\times E [\|w^{1/2} I_t(w(\tilde{u}^{\rho,g} - \tilde{u}^\rho))\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))}]^{1/2}
\]

\[
\leq C \left[ \|u^\rho\|_{L^2(\mathbb{R}^+, L^\infty(\mathbb{R}))} E \left[ \left\| \left( J_t w^{1/2} \rho \left( \left\| \cos(\beta(W_t + I_t(\tilde{u}^\rho))) \right\| - J_t w^{1/2} \rho \left\| \cos(\beta(W_t + I_t(\tilde{u}^\rho))) \right\| I_t(h) \right) \right) (\tilde{u}^\rho_t)^2 \right\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))} \right]^{1/2}
\times E [\|w^{1/2} I_t(w(\tilde{u}^{\rho,g} - \tilde{u}^\rho))\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))}]^{1/2}
\]

\[
\leq C \left[ \|u^\rho\|_{L^2(\mathbb{R}^+, L^\infty(\mathbb{R}))} E \left[ \left\| \left( J_t w^{1/2} \rho \left( \left\| \cos(\beta(W_t + I_t(\tilde{u}^\rho))) \right\| - J_t w^{1/2} \rho \left\| \cos(\beta(W_t + I_t(\tilde{u}^\rho))) \right\| I_t(h) \right) \right) (\tilde{u}^\rho_t)^2 \right\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))} \right]^{1/2}
\times E [\|w^{1/2} I_t(w(\tilde{u}^{\rho,g} - \tilde{u}^\rho))\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))}]^{1/2}
\]

since we have

\[
E \left[ \left\| w^{1/2} I_t(w(\tilde{u}^{\rho,g} - \tilde{u}^\rho)) \right\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))}^2 \right] \leq E \left[ \left\| w^{1/2} w(\tilde{u}^{\rho,g} - \tilde{u}^\rho) \right\|_{L^2(\mathbb{R}^+, \mathbb{R}^2)}^2 \right] \leq E \left[ \left\| w^{1/2} (\tilde{u}^{\rho,g} - \tilde{u}^\rho) \right\|_{L^2(\mathbb{R}^+, \mathbb{R}^2)}^2 \right] \]
and
\begin{align*}
\frac{1}{\sqrt{2\pi}}|I_0| &= \mathbb{E}\left[\int_0^\infty J_t(\rho\cos(\beta(W_t + I_t(\tilde{u}^{\rho,g})))\parallel I_t(w(\tilde{u}^{\rho,g} - \tilde{u}^\rho)))dxdtdf\right] \\
&= \mathbb{E}\left[\int_0^\infty w^{-1/2} J_t(\rho\cos(\beta(W_t + I_t(\tilde{u}^{\rho,g})))\parallel I_t(w(\tilde{u}^{\rho,g} - \tilde{u}^\rho)))dxdtdf\right] \\
&\leq \mathbb{E}[\parallel J_t(\rho\cos(\beta(W_t + I_t(\tilde{u}^{\rho,g})))\parallel w^{-1/2} I_t(w(\tilde{u}^{\rho,g} - \tilde{u}^\rho)))\parallel^2_{L^2([0,\infty])}]^{1/2} \\
&\leq \mathbb{E}[\parallel w^{1/2}(\tilde{u}^{\rho,g} - \tilde{u}^\rho))\parallel^2_{L^2([0,\infty])}^{1/2} \mathbb{E}[\parallel w^{-1/2} I_t(w(\tilde{u}^{\rho,g} - \tilde{u}^\rho)))\parallel^2_{L^2([0,\infty])}]^{1/2} \\
&\leq \mathbb{E}[\parallel w^{1/2}(\tilde{u}^{\rho,g} - \tilde{u}^\rho))\parallel^2_{L^2([0,\infty])}^{1/2} \mathbb{E}[\parallel w^{-1/2} I_t(w(\tilde{u}^{\rho,g} - \tilde{u}^\rho)))\parallel^2_{L^2([0,\infty])}]^{1/2} \\
&\leq \mathbb{E}[\parallel w^{1/2}(\tilde{u}^{\rho,g} - \tilde{u}^\rho))\parallel^2_{L^2([0,\infty])}]^{1/2},
\end{align*}

where we have used Lemma 21. So in total we get that
\[ |I| \leq C\lambda\beta^2 \mathbb{E}[\parallel w^{1/2}(u^{\rho,f} - u^\rho)\parallel^2_{L^2([0,\infty])}] \]

similarly
\begin{align*}
|II| &= \lambda\beta \mathbb{E}\left[\int_0^\infty J_t(\sin(\beta(W_t + I_t(\tilde{u}^{\rho,f}))) - \sin(\beta(W_t + I_t(\tilde{u}^\rho)))\parallel w(\tilde{u}^{\rho,f} - \tilde{u}^\rho))dxdtdf\right] \\
&\leq \lambda\beta \mathbb{E}\left[\int_0^\infty \parallel w^{1/2}(\tilde{u}^{\rho,f} - \tilde{u}^\rho))\parallel^2_{L^2([0,\infty])} \mathbb{E}[\parallel w^{-1/2} I_t(w(\tilde{u}^{\rho,g} - \tilde{u}^\rho)))\parallel^2_{L^2([0,\infty])}]^{1/2} \\
&\leq \lambda\beta \mathbb{E}[\parallel w^{1/2}(\tilde{u}^{\rho,g} - \tilde{u}^\rho))\parallel^2_{L^2([0,\infty])}^{1/2} \mathbb{E}[\parallel w^{-1/2} I_t(w(\tilde{u}^{\rho,g} - \tilde{u}^\rho)))\parallel^2_{L^2([0,\infty])}]^{1/2} \\
&\leq \lambda\beta \mathbb{E}[\parallel w^{1/2}(\tilde{u}^{\rho,g} - \tilde{u}^\rho))\parallel^2_{L^2([0,\infty])}]^{1/2}.
\end{align*}

By assumption
\[ |III| \leq C_g\mathbb{E}[\parallel w^{1/2}(\tilde{u}^{\rho,f} - \tilde{u}^\rho))\parallel^2_{L^2([0,\infty])}]^{1/2} = \mathbb{E}[\parallel w^{1/2}(\tilde{u}^{\rho,f} - \tilde{u}^\rho))\parallel^2_{L^2([0,\infty])}]^{1/2} \]

All together we obtain
\[ \mathbb{E}[\parallel w^{1/2}(\tilde{u}^{\rho,f} - \tilde{u}^\rho))\parallel^2_{L^2([0,\infty])}] = |I + II + III| \]
\[ \leq C\lambda(\beta + \beta^2) \mathbb{E}[\parallel w^{1/2}(\tilde{u}^{\rho,f} - \tilde{u}^\rho))\parallel^2_{L^2([0,\infty])}] + C_g\mathbb{E}[\parallel w^{1/2}(\tilde{u}^{\rho,f} - \tilde{u}^\rho))\parallel^2_{L^2([0,\infty])}]^{1/2} \]

Provided that \( C\lambda(\beta + \beta^2) < 1/2 \) this implies
\[ \mathbb{E}[\parallel w^{1/2}(\tilde{u}^{\rho,f} - \tilde{u}^\rho))\parallel^2_{L^2([0,\infty])}]^{1/2} \leq 2C_g \]

which is the claim.

\[ \square \]

**Proof of Theorem 6** By Lemma 5 and Lemma 5 it is sufficient to verify that with
\[ \mathbb{E}\left[\int \nabla f(W_\infty + I_\infty(u^{\rho,f}))I_\infty(h)\right] \leq |f|_{1,\gamma}E[\parallel h\parallel^2_{L^2([0,\infty])}]^{1/2}. \]

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However this holds since with $w(x) = \exp(2\gamma|x|)$

$$E \left[ \int \nabla f(W_\infty + I_\infty(u^{\rho,f}))I_\infty(h) \right]$$

\[ \leq E \left[ \int w \nabla f(W_\infty + I_\infty(u^{\rho,f}))w^{-1}I_\infty(h) \right] \]

\[ \leq |f|_{1,\gamma} E[\|w^{-1}I_\infty(h)\|_{L^2}] \]

\[ \leq |f|_{1,\gamma} E[\|I_\infty(h)\|^2_{L^2(\mathbb{R}^+;L^2,-\gamma)}]^{1/2} \]

\[ \square \]

**Proposition 7** There exists an $\bar{u} \in L^2(\mathbb{R}^+, L^\infty(\mathbb{R}^2))$ such that for any $0 < \gamma < m$

$$\lim_{\rho \to 1} E \left[ \int_0^\infty \|\bar{u}^{\rho,0} - \bar{u}_{s}\|^2_{L^2(\mathbb{R}^2)} ds \right] = 0.$$

**Proof** We first show that if $\text{supp}(\rho) \subseteq B(y, 1) \subset \mathbb{R}^2$ then for $w(x) = \exp(-2\gamma|x-y|)$ we have if $\|u\|_{L^\infty(\mathbb{R}^2)} \leq C$ then:

$$E \left[ \int_0^\infty \int J_\rho[\sin(W_t + I_t(u))]h_t dx dt + \int_0^\infty \int J_\rho[\sin(W_t + I_t(u))]I_t(h)u_t dx dt \right]$$

\[ \leq C E[\|w^{-1/2}h\|^2_{L^2(\mathbb{R}^+)\times\mathbb{R}^2}]^{1/2}. \]

Indeed

$$= E \left[ \int_0^\infty \int J_\rho[\sin(W_t + I_t(u))]h_t dx dt + \int_0^\infty \int J_\rho[\sin(W_t + I_t(u))]I_t(h)u_t dx dt \right]$$

\[ \leq E[\|w^{-1/2}J_\rho[\sin(W_t + I_t(u))]\|_{L^2(\mathbb{R}^+;L^\infty(\mathbb{R}^2))}\|w^{-1/2}h\|^2_{L^2(\mathbb{R}^+)\times\mathbb{R}^2}] \]

\[ \leq E[\|w^{-1/2}h\|^2_{L^2(\mathbb{R}^+)\times\mathbb{R}^2}]^{1/2}. \]

Now suppose that $|x| \geq N$, $(\rho_1 - \rho_2) \subset B(x, 1)$ and applying Lemma 6 we get with $w(y) = \exp(\gamma|x-y|)$

$$E[\|w^{-1/2}(\bar{u}^{\rho_1,0} - \bar{u}^{\rho_2,0})\|^2_{L^2(\mathbb{R}^+;L^2(\mathbb{R}^2))}]^{1/2} \leq C$$

which implies by Lemma 15 from Appendix B that

$$E[\|w^{\rho_1,0} - w^{\rho_2,0}\|^2_{L^2(\mathbb{R}^+;L^2,-\gamma)}]^{1/2} \leq C \exp(-\gamma|x|).$$

(9)

Now suppose that that $\rho_1, \rho_2 = 1$ on $B(0, N)$. We can decompose $\rho_2 - \rho_1 = \sum_{i \in \mathbb{Z}^2} \chi_i(\rho_2 - \rho_1)$ as $\sum_{i \in \mathbb{Z}^2} \rho_i$ where $\chi_i$ is a partition of unity with $\text{supp} \chi_i \subset B(i, 2)$. Applying estimate (6) iteratively we get

$$E[\|w^{\rho_2,0} - w^{\rho_1,0}\|^2_{L^2(\mathbb{R}^+;L^2,-\gamma)}]^{1/2} \leq C \sum_{i \in \mathbb{Z}^2, |i| \geq N} E[\|w^{\rho_i,0} + \rho_i - \rho_1,0\|^2_{L^2(\mathbb{R}^+;L^2,-\gamma)}]^{1/2}$$

\[ \leq C \sum_{i \in \mathbb{Z}^2, |i| \geq N} \exp(-\gamma|i|) \]

\[ \leq C \exp(-\gamma N). \]
This shows that \( w^\rho \) is a Couchy-sequence in \( L^2(\mathbb{P}, L^2(\mathbb{R}^+, L^2;\gamma)) \) which implies that it converges in this space. Since \( w^\rho \in \mathbb{H}_a \) the limit is also adapted to \( W_t \).

\[ \square \]

**Theorem 7** Define \( \mathbb{D}^f \) to be the space

\[
\mathbb{D}^f = \left\{ v \in \mathbb{H}_a : \mathbb{E} \left[ \int_0^\infty \int \| v \|_{L^2(\mathbb{R}^2)}^2 \, dx \, dt \right] \leq C \right\} .
\]

Then for \( C \) large enough

\[
\lim_{\rho \to 1} (W^\rho(f) - W^\rho(0)) = \inf_{v \in \mathbb{D}} \tilde{F}^f(v)
\]

where

\[
\tilde{F}^f(v) = \mathbb{E} [f(W_\infty + I_\infty(u) + I_\infty(v)) + \lambda \int \| \cos(\beta(W_\infty + I_\infty(u) + I_\infty(v))) \| - \| \cos(\beta(W_\infty + I_\infty(u))) \| \, dx + \int_0^\infty \bar{u}_t v_t \, dx \, dt + \frac{1}{2} \int_0^\infty \| v_t \|_{L^2(\mathbb{R}^2)}^2 \, dt]
\]

and \( \bar{u} \) has been introduced in Proposition 4.

**Proof** We have

\[
(W^\rho(f) - W^\rho(0)) = \inf_{u \in \mathbb{H}_a} (F^{I,\rho}(u) - F^0,\rho(u^\rho)) = \inf_{v \in \mathbb{H}_a} \tilde{F}^{I,\rho}(v)
\]

where

\[
\tilde{F}^{I,\rho}(v) = F^{I,\rho}(w^\rho + v) - F^0,\rho(w^\rho).
\]

We can restrict the functional on the the space \( \mathbb{D}^f \) without changing the infimum by Theorem 6. We now claim that \( F^f(v) - \tilde{F}^{I,\rho}(v) \) goes to 0 uniformly on \( \mathbb{D}^f \). Indeed we can estimate

\[
\begin{align*}
\tilde{F}^f(v) - \tilde{F}^{I,\rho}(v) & = \mathbb{E} [f(W_\infty + I_\infty(v)) - f(W_\infty + I_\infty(u) + I_\infty(\bar{u}^\rho))] \\
& + \lambda \int \rho \| \cos(\beta W_\infty) \| (\cos(\beta(I_\infty(v) + I_\infty(\bar{u}^\rho))) - \cos(\beta I_\infty(\bar{u}^\rho))) \\
& + \lambda \int \rho \| \sin(\beta W_\infty) \| ((\sin(\beta(I_\infty(v) + I_\infty(\bar{u}^\rho))) - \sin(\beta I_\infty(\bar{u}^\rho))) \\
& + \lambda (1 - \rho) \| \cos(\beta W_\infty) \| (\cos(\beta(I_\infty(v) + I_\infty(\bar{u}^\rho))) - \cos(\beta I_\infty(\bar{u}^\rho))) \\
& + \lambda (1 - \rho) \| \sin(\beta W_\infty) \| ((\sin(\beta(I_\infty(v) + I_\infty(\bar{u}^\rho))) - \sin(\beta I_\infty(\bar{u}^\rho))) \\
& + \int_0^\infty v_t (\bar{u}_t - \bar{u}_t^\rho) \, dx \, dt \\
& + \int_0^\infty \| v_t \|_{L^2(\mathbb{R}^2)}^2 \, dt
\end{align*}
\]
By Interpolation with $L^\infty$, for $q$ close enough to 1:

$$\|((\cos(\beta(I_\infty(v) + I_\infty(\bar{u}))) - \cos(\beta I_\infty(\bar{u})))) - \cos(\beta(I_\infty(v) + I_\infty(\bar{u}))) - \cos(\beta I_\infty(\bar{u})))\|_{B_{1,q}^{-\frac{1}{2}}(\mathbb{x})^p}$$

$$\leq 4\beta \int_0^1 \|(\sin(\theta \beta I_\infty(v)) + \beta I_\infty(\bar{u}))) - \sin(\theta \beta I_\infty(v)) + \beta I_\infty(\bar{u}))I_\infty(v))\|_{W^{1,1},\rho}^{1-\delta} \, d\theta$$

$$\leq C\|I_\infty(\bar{u}) - I_\infty(\bar{u})\|_{H^{1,\gamma}}^{1-\delta_1} \|I_\infty(v)\|_{H^{1,\gamma}}^{1-\delta_1} + C\|I_\infty(\bar{u}) - I_\infty(\bar{u})\|_{L^2,\gamma}^{1-\delta_1} \|I_\infty(v)\|_{H^{1,\gamma}}^{2(1-\delta_1)}$$

Where we have used the embedding $W^{1,1,\gamma} \hookrightarrow B_{1,1}(\mathbb{x})^{2k}$ and subsequent interpolation with $L^\infty$. We have also applied applied Lemma [7] using that $\|I(\bar{u})\|_{W^{1,\infty}} \leq C$ from Theorem [1] and Lemma [20]. It is clear that

$$\|I_\infty(\bar{u}) - I_\infty(\bar{u})\|_{L^2,\gamma} \leq \|I_\infty(\bar{u})\|_{L^\infty}^{1-\delta} \|I_\infty(\bar{u})\|_{L^\infty}^{1-\delta} \|I_\infty(v)\|_{L^\infty}^{1-\delta}.$$

We can then use this estimate to obtain for $p$ large enough such that $1/p + 1/q = 1$

$$\lambda \mathbb{E} \left[ \int \rho \|\cos(\beta W_\infty)\|((\cos(\beta(I_\infty(v) + I_\infty(\bar{u}))) - \cos(\beta I_\infty(\bar{u})))) - \cos(\beta(I_\infty(v) + I_\infty(\bar{u}))) - \cos(\beta I_\infty(\bar{u})))\right]$$

$$\leq C \mathbb{E}[\|\cos(\beta W_\infty)\|^p_{B_{p,q}^{-\frac{1}{2}}(\mathbb{x})^{-\frac{1}{2}}}]^{1/p}$$

$$\times \mathbb{E}[\|\|I_\infty(\bar{u}) - I_\infty(\bar{u})\|^q_{H^{1,\gamma}} \|I_\infty(v)\|^{q(1-\delta_1)}_{H^{1,\gamma}} + \|I_\infty(\bar{u}) - I_\infty(\bar{u})\|^q_{H^{1,\gamma}} \|I_\infty(v)\|^{q(1-\delta_1)}_{H^{1,\gamma}})]^{1/q}$$

$$\leq \mathbb{E}[\|\|\cos(\beta W_\infty)\|^p_{B_{p,q}^{-\frac{1}{2}}(\mathbb{x})^{-\frac{1}{2}}}]^{1/p}$$

$$\times \mathbb{E}[\|\|I_\infty(\bar{u}) - I_\infty(\bar{u})\|_{H^{1,\gamma}}^{2(1-\delta_1)}]^{1/2q} \mathbb{E}[\|I_\infty(v)\|^{2q(1-\delta_1)}_{H^{1,\gamma}}]^{1/2q}$$

$$+ \mathbb{E}[\|\|I_\infty(\bar{u}) - I_\infty(\bar{u})\|^q_{H^{1,\gamma}}]^{1/q} \mathbb{E}[\|I_\infty(v)\|^{q(1-\delta_1)}_{H^{1,\gamma}}]^{1/q},$$

provided that we choose $q < 1/(1 - \delta_1)$ and $\delta_2 = 2(1 - q(1 - \delta_1))/(1 - \delta_1)$. Now for $v \in \mathbb{D}$ the last line is bounded by

$$C(\mathbb{E}[\|I_\infty(\bar{u}) - I_\infty(\bar{u})\|^{2q(1-\delta_1)}_{H^{1,\gamma}}]^{1/2q} + \mathbb{E}[\|I_\infty(\bar{u}) - I_\infty(\bar{u})\|^q_{H^{1,\gamma}}]^{1/q})$$

which goes to 0. We can proceed analogously for the sinus term. To estimate

$$\beta \int_0^1 \int (1 - \rho) \cos(\beta W_\infty)((\sin(\beta I_\infty(v)) + \beta I_\infty(\bar{u})))I_\infty(v)) \, d\theta$$

it is not hard to see that that $\|(1 - \rho)f\|_{W^{1,1}(\mathbb{x})^k} \leq N^{-k/2} \|f\|_{W^{1,1}(\mathbb{x})^{k/2}}$, so interpolating between $W^{1,1,\gamma/2}$ and $L^\infty$ we have

$$\leq \mathbb{E}[\|\cos(\beta W_\infty)\|^p_{B_{p,q}^{-\frac{1}{2}}(\mathbb{x})^{-\frac{1}{2}}}]^{1/p}$$

$$\times \mathbb{E}[\|\|I_\infty(\bar{u}) - I_\infty(\bar{u})\|_{H^{1,\gamma}}]^{(1-\delta)q} \mathbb{E}[\|I_\infty(v)\|^{(1-\delta)q}_{W^{1,1,\gamma/2}}]^{1/q}$$
Now
\[
\mathbb{E}[(\sin(\beta t I_\infty(v) + I_\infty(\bar{u})))I_\infty(v)]^{1-\delta}
\]
can be estimated analogously to the above computations. Clearly
\[
\int_0^\infty \int v(\bar{u} - u^p) dxdt \leq \mathbb{E}[\|v\|_{L^2(R^2, L^{2,\gamma})}^{1/2}]\mathbb{E}[\|\bar{u} - u^p\|^2_{L^2(R^2, L^{2,\gamma})}]^{1/2}
\]
Finally by definition of \(f\)
\[
\mathbb{E}[\|f(W_\infty + I_\infty(v) + I_\infty(\bar{u})) - f(W_\infty + I_\infty(v) + I_\infty(\bar{u}^p))\| \leq \mathbb{E}[\|\bar{u} - u^p\|^2_{L^2(R^2, L^{2,\gamma})}]^{1/2}
\]
which allows us to conclude.

**Lemma 7** Assume that \(\|f^1\|_{W^{1,\infty}} + \|f^2\|_{L^{1,\infty}} \leq C\). Then
\[
\|((\cos(f^1 + g) - \cos(f^2 + g))g)\|_{W^{1,\gamma}} \leq C(\|f^1 - f^2\|_{H^{1,-\gamma}} \|g\|_{H^{1,2\gamma}} + \|f^1 - f^2\|_{L^{2,-\gamma}} \|g\|_{H^{1,2\gamma}})
\]

**Proof** Set \(w(x) = \exp(\gamma x)\). Then with \(1/p + 1/q + 1/2 = 1\) and \(q\) close enough to 2 we have
\[
\|\nabla((\cos(f^1 + g) - \cos(f^2 + g))g)\|_{L^{1,\gamma}} \leq \int_{R^2} w(x)(\cos(f^1 + g) - \cos(f^2 + g))g\nabla g dx + \int_{R^2} w(x)(\cos(f^1 + g) - \cos(f^2 + g))g^2 \nabla g dx
\]
\[
\leq \mathbb{E}[\|\nabla g\|_{L^{2,\gamma}} \|f^1 - f^2\|_{L^{2,-\gamma}} \|g\|_{L^q} \|\nabla g\|_{L^{2,\gamma}} + \|f^1 - f^2\|_{L^{2,-\gamma}} \|g\|_{L^q} \|\nabla g\|_{L^{2,\gamma}} + \|f^1 - f^2\|_{L^{2,\gamma}} \|f^1 - f^2\|_{L^{2,-\gamma}} \|g\|_{L^{2,\gamma}} + \|f^1 - f^2\|_{L^{2,-\gamma}} \|g\|_{L^{2,\gamma}}]
\]
Now using the Sobolev embedding
\[
\|g\|_{L^q} \leq \|g\|_{H^1} \leq \|g\|_{H^{1,2\gamma}}
\]
we have
\[
\|\nabla g\|_{L^{2,\gamma}} \|\nabla f^1 - \nabla f^2\|_{L^{2,-\gamma}} + \|f^1 - f^2\|_{L^{p,-\gamma}} \|g\|_{L^q} \|\nabla g\|_{L^{2,\gamma}} + \|\nabla f^1\|_{L^{\infty}} \|f^1 - f^2\|_{L^{2,-\gamma}} \|g\|_{L^q} \|\nabla g\|_{L^{2,\gamma}} + \|\nabla f^1\|_{L^{\infty}} \|f^1 - f^2\|_{L^{2,-\gamma}} \|g\|_{L^q} \|\nabla g\|_{L^{2,\gamma}}
\]
\[
\leq C(\|f^1 - f^2\|_{H^{1,-\gamma}} \|g\|_{H^{1,2\gamma}} + \|f^1 - f^2\|_{L^{2,-\gamma}} \|g\|_{H^{1,2\gamma}})
\]
where in the last line we have applied the assumption \(\|f^1\|_{W^{1,\infty}} + \|f^2\|_{W^{1,\infty}} \leq C\).

Now using that
\[
\|((\cos(f^1 + g) - \cos(f^2 + g))g\|_{L^{1,\gamma}} \leq \|f^1 - f^2\|_{L^{2,-\gamma}} \|g\|_{L^{2,\gamma}}
\]
we can conclude. □

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5 Large deviations

In this section we want to discuss a Laplace principle for the Sine-Gordon measure in the “semiclassical limit” as described in the introduction. We introduce the family \( \nu_{S_G, h}^{T, \rho} \) of measures given by

\[
\int_{\mathcal{C}^r (\mathbb{R}^2)} g(\phi) \nu_{S_G, h}^{T, \rho} (d\phi) = \frac{E \left[ g(h^{1/2} W_T e^{-\frac{4}{h} V_h^{T, \rho}(h^{1/2} W_T)} \right]}{Z_h^{T, \rho}},
\]

where similarly as above

\[
V_h^{T, \rho}(\varphi) := \lambda h^{h/2} \int_{\mathbb{R}^2} \cos(\beta \varphi(x)) d x Z_h^{T, \rho} := E[e^{-V_h^{T, \rho}(W_0, T)}]
\]

for any bounded measurable \( g : H^{-1}((\mathcal{C}^r (\mathbb{R}^2) \to \mathbb{R}. \) Here \( \alpha_h(T) = e^{\frac{d^2}{2} h K_T(0)} \) and \( \alpha^h(T) \cos(h^{1/2} \beta W_T) \) enjoys the same properties as \( \alpha(T) \cos(\beta W_T) \). It will also be convenient to introduce the unnormalized measures \( \tilde{\nu}_{S_G, h}^{T, \rho} = Z_h^{T, \rho} \nu_{S_G, h}^{T, \rho} \).

Note that this corresponds (modulo a normalization constant) to the measure heuristically defined by

\[
e^{-\frac{1}{h} \int_{\mathbb{R}^2} \lambda h^{h/2} \cos(\beta \varphi(x)) + \frac{1}{2} m^2 \varphi(x)^2 + \frac{1}{2} |\nabla \varphi(x)|^2 d x d \varphi}.
\]

Our goal is now to show that \( \nu \) given as the weak limit of \( \nu_{S_G, h}^{T, \rho} \) as \( T \to \infty, \rho \to 1 \) satisfies a Laplace principle as \( h \to 0 \). We recall the definition of the Laplace principle.

**Definition 7** A sequence of Borel measures \( \nu_{\varepsilon} \) on a metric space \( S \) satisfies the Laplace principle with rate function \( I \) if for any continuous bounded function \( f : S \to \mathbb{R} \)

\[
- \lim_{\varepsilon \to 0} \varepsilon \log \int e^{-\frac{1}{\varepsilon} f(x)} \nu_{\varepsilon} (dx) = \inf_{x \in S} \{ f(x) + I(x) \}.
\]

**Definition 8** For a metric space \( S \) and let \( I : S \to \mathbb{R} \) be a rate function. A set \( D \subseteq C(S) \) is called rate function determining if any exponentially tight sequence \( \nu_{\varepsilon} \) of measures on \( S \) such that

\[
- \lim_{\varepsilon \to 0} \varepsilon \log \int e^{-\frac{1}{\varepsilon} f} d \nu_{\varepsilon} = \inf_{x \in S} \{ f(x) + I(x) \},
\]

for all \( f \in D \) satisfies a large deviations principle with rate function \( I \).

**Lemma 8** Assume that \( D \subseteq C(S) \) is bounded below, i.e \( f \geq -C \) for any \( f \in D \) with \( C \) independent of \( f \). Furthermore assume that \( D \) isolates points i.e for each compact set \( K \subseteq S, x \in S \) and \( \varepsilon > 0 \) there exists \( f \in D \) such that

- \( |f(x)| < \varepsilon \)
- \( \inf_{y \in K} f(y) \geq 0 \)
- \( \inf_{y \in K \cap B(x, \varepsilon)} f(y) \geq \varepsilon^{-1} \)

Then \( D \) is rate function determining.
For a proof see [25] Proposition 3.20.

**Lemma 9** Let $S = H^{-1}((x)^{-n})$ for any $\gamma > 0$ then
\[ D = C^2(L^2(\mathbb{R}^2), \mathbb{R}^+ \cap C(H^{-1}((x)^{-n})) \cap \{ |f|_{1,2,m} \leq \infty \} \cap \{ f \geq 0 \} \]
is rate function determining.

**Proof** We want to verify the assumptions of Lemma [8]. By translating it is enough to verify the assumptions for $x = 0 \in H^{-1}((x)^{-n})$. Furthermore we can assume that $K \subseteq B(0, N)$ for some $N > 0$. Now choose $\chi \in C_c^\infty (\mathbb{R}, \mathbb{R}_+)$ such that $\chi(0) = 0$ and $\chi(y) \geq \varepsilon^{-1}$ if $\lambda \geq |y|^2 > \varepsilon$. $f(\phi) = \chi(||\phi||_{H^{-1}, -m})$ satisfies the requirement of Lemma [8]. Clearly $f \in C^2(L^2(\mathbb{R}^2), \mathbb{R}_+) \cap C(H^{-1}((x)^{-n}))$, furthermore
\[
\nabla f(\phi) = 2\chi'(||\phi||_{H^{-1}, -m})(w(1 - \Delta)^{-1}w\phi)
\]
where $w(y) = \exp(-my)$. This implies that $|f|_{1,2,m} \leq \infty$ since
\[
\|w(1 - \Delta)^{-1}w\phi\|_{L^2,m} \leq \|(1 - \Delta)^{-1}w\phi\|_{L^2} \leq \|\phi\|_{H^{-1}, -m}.
\]

From the Boue-Dupuis formula we obtain

**Proposition 8**
\[
\int h \log \int e^{-\frac{1}{2}f} dw_{SG,H}^T(d\phi) = \inf_{u \in \mathbb{R}_+} \mathbb{E} \left[ f(h^{1/2}W_\infty + I_\infty(u)) + \lambda \int \rho(x) ||\cos(h^{1/2}\beta W_\infty + \beta I_\infty(u))||dx + \frac{1}{2} \int_0^\infty \|u_t\|_{L^2(\mathbb{R}^2)}^2 dt \right].
\]

Repeating the proof of Proposition [2] word for word we get:

**Proposition 9** Let $\bar{u}^{h,\rho}$ be the minimizer of
\[
F^{h,\rho}(u) = \mathbb{E} \left[ \lambda \int \rho(x) ||\cos(h^{1/2}\beta W_\infty + \beta I_\infty(u))||dx + \frac{1}{2} \int_0^\infty \|u_t\|_{L^2(\mathbb{R}^2)}^2 dt \right] .
\]
Then $\bar{u}^{h,\rho}$ converges in $L^2(\mathbb{P}, L^2(\mathbb{R}_+, L^2;\gamma))$, we denote the limit by $\bar{u}^h$, more precisely we have
\[
\lim_{\rho \to 1} \sup_{h \geq 1} \mathbb{E} \left[ \int_0^\infty \| \bar{u}_t^h - \bar{u}_t^{h,\rho} \|_{L^2(\mathbb{R}^2)}^2 dt \right] = 0.
\]

**Notation 2** Denote
\[
G_{h}^F(v) = \mathbb{E} \left[ f(hW_\infty + I_\infty(\bar{u}^h) + I_\infty(v)) + \lambda \int [\cos(\beta W_\infty + \beta I_\infty(\bar{u}^h) + \beta I_\infty(v))] - ||\cos(\beta W_\infty + \beta I_\infty(\bar{u}^h))|| \right]
\int \int \bar{u}_t^h v_t dtdx + \frac{1}{2} \int_0^\infty \|v_t\|_{L^2(\mathbb{R}^2)}^2 dt \right] .
\]

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In complete analogy with Theorem we can deduce the following proposition.

Proposition 10

\[
\begin{align*}
    h \log \int e^{-\frac{1}{h}I} d\nu_{SG, h}^T (d\phi) & =: \inf_{e \in \mathcal{D}_T} G^f_h (v)
\end{align*}
\]

Proposition 11 With the notation of Proposition we have:

\[
\lim_{h \to 0} \mathbb{E} \left[ \int_0^\infty \| u_t \|_{L^2}^2 dt \right] = 0.
\]

Proof Keeping in mind it is sufficient to show that

\[
\lim_{h \to 0} \mathbb{E} \left[ \int_0^\infty \| \bar{u}^h_{t \rho} \|_{L^2}^2 dt \right] = 0.
\]

By a simple modification of Lemma we can assume that \( \bar{u}^h_{\rho} \) satisfies, for any \( h \in \mathbb{H}_a \),

\[
\mathbb{E} \left[ \lambda \beta \int \rho(x) [\sin(h^{1/2} \beta W_\infty + \beta I_\infty(\bar{u}^h_{\rho}))] I_\infty(h) dx + \int_0^\infty \int \bar{u}^h_{t \rho} h_t dx \right] = 0.
\]

By choosing \( h = \bar{u}^h_{\rho} \) we get

\[
\mathbb{E} \left[ \| \bar{u}^h_{\rho} \|_{L^2}^2 \right] = -\mathbb{E} \left[ \lambda \beta \int \rho(x) [\sin(h^{1/2} \beta W_\infty + \beta I_\infty(\bar{u}^h_{\rho}))] I_\infty(\bar{u}^h_{\rho}) dx \right].
\]

Expanding we get

\[
[\sin(h^{1/2} \beta W_\infty + \beta I_\infty(\bar{u}^h_{\rho}))] = [\sin(h^{1/2} \beta W_\infty)] \cos(\beta I_\infty(\bar{u}^h_{\rho})) + [\cos(h^{1/2} \beta W_\infty)] \sin(\beta I_\infty(\bar{u}^h_{\rho})).
\]

Now

\[
\| \rho [\sin(h^{1/2} \beta W_\infty)] \cos(\beta I_\infty(\bar{u}^h_{\rho})) \|_{W^{-\beta/4 \pi - \delta, 2}} \leq \| \rho [\cos(h^{1/2} \beta W_\infty)] \|_{W^{-\beta/4 \pi - \delta, 2}} \| \cos(\beta I_\infty(\bar{u}^h_{\rho})) \|_{L^\infty} 
\]

where we have used that \( \| \bar{u}^h_{\rho} \|_{L^\infty} \leq C(t)^{-1/2 - \delta} \) and Lemma Furthermore

\[
\| \rho [\cos(h^{1/2} \beta W_\infty)] - 1 \|_{W^{-\beta/4 \pi - \delta, 2}} \| \cos(\beta I_\infty(\bar{u}^h_{\rho})) \|_{L^\infty} 
\]

These two estimates imply

\[
\int \rho \left( [\sin(h^{1/2} \beta W_\infty)] \cos(\beta I_\infty(\bar{u}^h_{\rho})) + [\cos(h^{1/2} \beta W_\infty)] - 1 \right) \sin(\beta I_\infty(\bar{u}^h_{\rho})) I_\infty(\bar{u}^h_{\rho}) dx 
\]

\[
\leq C \left( \| \rho [\sin(h^{1/2} \beta W_\infty)] \|_{W^{-\beta/4 \pi - \delta, 2}} + \| \rho [\cos(h^{1/2} \beta W_\infty)] - 1 \|_{W^{-\beta/4 \pi - \delta, 2}} \right) \| \bar{u}^h_{\rho} \|_{L^2 (\mathbb{R} \times \mathbb{R}^2)}
\]

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Finally
\[ \lambda \beta \left| \int \rho \sin(\beta I_\infty(\bar{u}^h))I_\infty(\bar{u}^h) \right| \leq \lambda \beta \|I_\infty(\bar{u}^h)\|_{L^2(R^2)}^2 \leq C \lambda \beta \|\bar{u}^h\|_{L^2(R^+_x \times R^2)}^2 \]

Now putting everything together we obtain for \( C \lambda \beta \leq 1/2 \)
\[
\mathbb{E}[\|\bar{u}^h\|^2_{L^2(R^+_x \times R^2)}]^{1/2} \\
\leq C \mathbb{E} \left[ \|\rho [\sin(h^{1/2}B_{W})] \|_{W^{-\beta,2}_{-4,-2}}^2 + \|\rho (\cos(h^{1/2}B_{W})) - 1 \|_{W^{-\beta,2}_{-4,-2}}^2 \right]^{1/2}
\]
and the r.h.s goes to 0 as \( h \to 0 \). Remark [1] below.

**Proposition 12** Assume that \( |f|_{1,2,m} < \infty \) and \( f : H^{-1}(\langle x \rangle^{-n}) \to \mathbb{R} \) be Lipschitz continuous.

\[
\lim_{h \to 0} \sup_{v \in \mathcal{D}} |G_h^f(v) - G_0^f(v)| = 0
\]

where
\[
G_h^f(v) = \mathbb{E} \left[ f(I_0,\infty(v)) + \lambda \int (\cos(\beta I_0,\infty(v)) - 1) + \frac{1}{2} \int_0^\infty \|v_t\|_{L^2}^2 dt \right].
\]

**Proof** By Lipschitz continuity of \( f \)
\[
|h^{1/2}W_{\infty} + I_\infty(v) + I_\infty(\bar{u}^h) - f(I_\infty(v))| \\
\leq h^{1/2} \mathbb{E}[\|W_{\infty}\|_{H^{-1}(\langle x \rangle^{-n})}] + \mathbb{E}[\|I_\infty(\bar{u}^h)\|_{H^{-1}(\langle x \rangle^{-n})}] \\
\to 0.
\]

Furthermore
\[
\left| \int [\sin(h^{1/2}B_{W})] (\sin(\beta I_\infty(v) + \beta I_\infty(\bar{u}^h)) - \sin(\beta I_\infty(\bar{u}^h))) \right| \\
\leq \|\sin(h^{1/2}B_{W})\|_{B_{p,p}^{1,q}(\langle x \rangle^{-n})} \\
\times \|\sin(\beta I_\infty(v) + \beta I_\infty(\bar{u}^h)) - \sin(\beta I_\infty(\bar{u}^h))\|_{B_{p,q}^{1,q}(\langle x \rangle^{-n})}
\]

Now for \( q \) close enough to 1 we have for any \( \gamma > 0 \)
\[
\| (\sin(\beta I_\infty(v) + \beta I_\infty(\bar{u}^h)) - \sin(\beta I_\infty(\bar{u}^h))) \|_{B_{p,q}^{1,q}(\langle x \rangle^{-n})} \\
\leq \| (\cos(\beta I_\infty(v) + \beta I_\infty(\bar{u}^h)) - \cos(\beta I_\infty(\bar{u}^h)) \nabla I_\infty(\bar{u}^h)) \|_{L^{1,\gamma}}^{1-\delta} \\
+ \beta \| (\cos(\beta I_\infty(v) + \beta I_\infty(\bar{u}^h)) \nabla I_\infty(v)) \|_{L^{1,\gamma}}^{1-\delta} \\
+ \beta \| (\sin(\beta I_\infty(v) + \beta I_\infty(\bar{u}^h)) - \sin(\beta I_\infty(\bar{u}^h))) \|_{L^{1,\gamma}}^{1-\delta} \\
\leq C \| I_\infty(v) \|_{L^{2,\gamma}} \| \nabla I_\infty(\bar{u}^h) \|_{L^{2,-\gamma}} + \| \nabla I_\infty(v) \|_{L^{2,\gamma}} + \| I_\infty(\bar{u}^h) \|_{L^{2,\gamma}}^{1-\delta}
\]

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By Remark 1. For the second term by fundamental theorem of calculus we can write

\[ E \left[ \int \left[ \sin(h^{1/2} \beta W_\infty) \right] (\sin(\beta I_\infty(v) + \beta I_\infty(u^b)) - \sin(\beta I_\infty(u^b))) \right] \]

\[ \leq C E \left[ \left\| \sin(h^{1/2} W_\infty) \right\|_{B_{p+4}^{1+\delta}}^{1/\delta} \right] \times E \left[ \left\| I_\infty(v) \right\|_{L^2, \gamma} || \nabla I_\infty(u^b) ||_{L^2, \gamma} + || \nabla I_\infty(v) ||_{L^2, \gamma} + || I_\infty(v) ||_{L^2, \gamma} \right] \]

and since by Remark 1

\[ E \left[ \left\| \sin(h^{1/2} \beta W_\infty) \right\|_{B_{p+4}^{1+\delta}(x) - n}^{1/\delta} \right] \rightarrow 0, \]

as \( h \rightarrow 0 \), we have uniform convergence of this term to 0. We now rewrite

\[ \left| \int \left[ \cos(h^{1/2} \beta W_\infty) \right] (\cos(\beta I_\infty(v) + \beta I_\infty(u^b)) - \cos(\beta I_\infty(u^b))) \right| \]

\[ - \int (\cos(\beta I_\infty(v)) - 1) \]

\[ \leq \left| \int \left[ \cos(h^{1/2} \beta W_\infty) \right] - 1 \right| \left( \cos(\beta I_\infty(v) + \beta I_\infty(u^b)) - \cos(\beta I_\infty(u^b)) \right) \]

\[ + \int (\cos(\beta I_\infty(v) + \beta I_\infty(u^b)) - \cos(\beta I_\infty(u^b))) \]

\[ - \int (\cos(\beta I_\infty(v)) - 1) \]

The first term can be estimated in the same way as the sinus term, provided we replace \( \left\| \sin(h^{1/2} \beta W_\infty) \right\| \) with \( \left\| \cos(h^{1/2} \beta W_\infty) \right\| - 1 \) which also satisfies

\[ E \left[ \left\| \cos(h^{1/2} \beta W_\infty) \right\| - 1 \right]^{1/\delta}_{B_{p+4}^{1+\delta}(x) - n} \rightarrow 0. \]

by Remark 1. For the second term by fundamental theorem of calculus we can write

\[ (\cos(\beta I_\infty(v) + I_\infty(u^b)) - \cos(I_\infty(u^b))) - (\cos(I_\infty(v)) - 1) \]

\[ = - \beta \int_0^1 (\cos(\theta \beta I_\infty(v) + \beta I_\infty(u^b)) - \cos(\theta \beta I_\infty(u^b))) I_\infty(v) d\theta \]

\[ = - \beta \int_0^1 \int_0^1 (\cos(\theta \beta I_\infty(v) + \xi \beta I_\infty(u^b)) I_\infty(u^b) I_\infty(v) d\theta d\xi \]

and so

\[ E \left[ \int_0^1 \int_0^1 (\cos(\theta \beta I_\infty(v) + \xi \beta I_\infty(u^b)) I_\infty(u^b) I_\infty(v)) d\theta d\xi \right] \]

\[ \leq E \left[ || I_\infty(u^b) ||_{L^2, \gamma} || I_\infty(v) ||_{L^2, \gamma} \right] \]

\[ \leq E \left[ || I_\infty(u^b) ||_{L^2, \gamma}^{1/2} E || I_\infty(v) ||_{L^2, \gamma}^{1/2} \right] \]

which implies also that term converges to 0. Finally

\[ E \left[ \int_0^\infty v_t u^b_t dt \right] \leq E \left[ || v ||_{L^2(\mathbb{R}^+, L^2, \gamma)} || u^b ||_{L^2(\mathbb{R}^+, L^2, \gamma)} \right] \]

\[ \leq E \left[ || v ||_{L^2(\mathbb{R}^+, L^2, \gamma)}^{1/2} E || u^b ||_{L^2(\mathbb{R}^+, L^2, \gamma)}^{1/2} \right] \]
and we can conclude.

We now relate $G^f$ to the rate function.

**Lemma 10**

$$\inf_{u \in D^f} G^f_0(u) = \inf_{\psi \in H^1(\mathbb{R}^2)} \{f(\psi) + I(\psi)\}$$

**Proof** By Lemma [11] below it is enough to show that

$$\inf_{u \in D^f} G^f_0(u) = \inf_{\|\psi\|_{H^{1,\gamma}}} \{f(\psi) + I(\psi)\}$$

for some $\gamma > 0$.

**Step 1.** First we prove

$$\inf_{u \in H^a} F(u) \leq \inf_{\|\psi\|_{H^{1,\gamma}}} \{f(\psi) + I(\psi)\}.$$

Restricting the infimum to processes of the form

$$u_s = J_s(m^2 - \Delta)\psi$$

with $\psi \in H^2(\mathbb{R}^2) \cap H^{1,2\gamma}$, we see that

$$I_{0,\infty}(u) = \int_0^{\infty} J_s u_s ds = \int_0^{\infty} J_s^2(m^2 - \Delta)\psi ds = \psi.$$

We also compute

$$\int_0^{\infty} \int_{\mathbb{R}^2} u_s^2 ds = \int_0^{\infty} \langle J_s^2(m^2 - \Delta)\psi, (m^2 - \Delta)\psi \rangle_{L^2(\mathbb{R}^2)} = \langle \psi, (m^2 - \Delta)\psi \rangle_{L^2(\mathbb{R}^2)}$$

and with $w(x) = \exp(\gamma x)$

$$\int_0^{\infty} \int_{\mathbb{R}^2} w u_s^2 ds = \int_0^{\infty} \langle w J_s^2(m^2 - \Delta)\psi, (m^2 - \Delta)\psi \rangle_{L^2(\mathbb{R}^2)} = \langle w\psi, (m^2 - \Delta)\psi \rangle_{L^2(\mathbb{R}^2)}$$

from which we can deduce that $\|wu\|_{L^2(\mathbb{R}^2)} \leq C\|\psi\|_{H^{1,\gamma}}$ and $u$ is in $D^f$. So

$$\inf_{u \in D^f} F(u) \leq \inf_{\|\psi\|_{H^{1,\gamma}}} \{f(\psi) + I(\psi)\} \leq \inf_{\|\psi\|_{H^{1,\gamma}}} \{f(\psi) + I(\psi)\}$$

where the last equality follows from the density of the $H^2$ in $H^{1,2\gamma}$ and continuity of the functional in $H^1$. 

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Step 2. We now prove the converse inequality

\[ \inf_{u \in D^\circ} F_0^\rho(u) \geq \inf_{\psi \in H^1(\mathbb{R}^2)} \{ f(\psi) + I(\psi) \}. \]

Recall that from Lemma 22 \[ \|u\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} \geq \|(m^2 - \Delta)^{1/2} I^\infty_0(u)\|_{L^2} \]
so

\[ \inf_{u \in D^\circ} F(u) \geq \inf_{u \in D^\circ} E \left[ f(I^\infty_0(u)) + \frac{1}{2} \int_{\mathbb{R}^2} ((m^2 - \Delta) I^\infty_0(u) I^\infty_0(u)) \right] \]

\[ \geq \inf_{\psi \in H^1(\mathbb{R}^2)} \{ f(\psi) + I(\psi) \}. \]

which proves the statement.

\[ \blacksquare \]

Lemma 11 Assume that \( 2\gamma^2 + \lambda < m^2 \). Then for \( \rho \in C^\infty(\mathbb{R}^2) \) and \( \rho, |\nabla \rho| \leq 1 \) (note that this includes the \( \rho = 1 \) case.)

\[ \inf_{\psi \in H^1(\mathbb{R}^2)} f(\psi) + I^\rho(\psi) = \inf_{\|\psi\|_{H^{1,1}} \in C[f_{1,2,\gamma}]} f(\psi) + I^\rho(\psi) \]

Proof By a standard argument we obtain that any minimizer of \( f(\psi) + I(\psi) \) satisfies the Euler Lagrange equation

\[ \nabla f(\varphi) + \lambda \rho \sin(\beta \varphi) + m^2 \varphi - \Delta \varphi = 0. \] (12)

Now multiplying (12) with \( w\varphi \) where \( w(x) = \exp(2\gamma |x|) \) and integrating we obtain

\[ 0 = \int w \nabla f(\varphi) \varphi + \lambda \int w \rho \sin(\beta \varphi) \varphi + m^2 \int w \varphi^2 - \int w \varphi \Delta \varphi \]

\[ = \int \rho \nabla f(\varphi) \varphi + \lambda \int w \rho \sin(\beta \varphi) \varphi + m^2 \int w \varphi^2 + \int w |\nabla \varphi|^2 + \int \varphi \nabla w \cdot \nabla \varphi \]

now observe that \( \nabla w = 2\gamma \frac{x}{|x|} \exp(2\gamma |x|) \) so \( |\nabla w| \leq 2\gamma w \)

\[ \int |\varphi \nabla w \cdot \nabla \varphi| \leq 2\gamma^2 \int w \varphi^2 + \frac{1}{2} \int w |\nabla \varphi|^2 \]

note also that

\[ \lambda \int |\rho w \sin(\beta \varphi) \varphi| \leq \lambda \int w \varphi^2. \]

Since \( m^2 \int w \varphi^2 + \int w |\nabla \varphi|^2 > 0 \) we have

\[ 0 = \int w \nabla f(\varphi) \varphi + \lambda \int w \rho \sin(\beta \varphi) \varphi + m^2 \int w \varphi^2 + \int w |\nabla \varphi|^2 + \int \varphi \nabla w \cdot \nabla \varphi \]

\[ \geq (m^2 - 2\gamma^2 + \lambda - \delta) \int w \varphi^2 + \frac{1}{2} \int w |\nabla \varphi|^2 - C |f|^2_{1,2,m} \]

which implies

\[ \int w \varphi^2 + \frac{1}{2} \int w |\nabla \varphi|^2 \leq C \gamma |f|^2_{1,2,m}. \]

\[ \blacksquare \]
6 Osterwalder Schrader Axioms

In this section we complete the proof of Theorem 4.

6.1 Reflection Positivity

To prove Reflection Positivity we prove that the measure $\nu_{SG}$ is a limit of reflection positive measures, since reflection positivity is clearly preserved by weak limits. We denote by $\nu_{SG} := \lim_{\rho \to \infty} \nu_{SG}$. Since $\nu_{SG} \to \nu_{SG}$ as $\rho \to 1$ it is enough to construct a sequence $\nu_{SG} \to \nu_{SG}$ such that $\nu_{SG}$ is reflection positive. We can take $\rho$ being invariant under the time reflection $\Theta f(x_1, x_2) = \frac{1}{2} f(-x_1, x_2)$. To construct $\nu_{SG}$ we cannot smooth in the “physical time” direction since this would destroy reflection positivity. Instead define $\theta = \delta_0 \otimes \eta$, $\theta \in \mathcal{F}'(\mathbb{R}^2)$ where $\eta \in C_c^\infty(\mathbb{R})$. Also set $\theta^\varepsilon = \varepsilon^{-2} \theta(\cdot / \varepsilon) = \delta_0 \otimes \eta_\varepsilon$ where $\eta_\varepsilon = \varepsilon^{-1} \eta(\cdot / \varepsilon)$. Finally we set $W^\varepsilon_T = \theta^\varepsilon * W_T, T \in [0, \infty]$. We define

$$\nu_{SG}^\varepsilon = e^{-\lambda f} \rho \alpha^\varepsilon \cos(\beta W^\varepsilon) d\mathbb{P}.$$

We will now proceed in three steps: In Step 1 we show that for the correct choice of $\alpha^\varepsilon$

$$\alpha^\varepsilon \cos(\beta W^\varepsilon) \to [\cos(\beta W_\infty)].$$

In Step 2 we show that for any $p > 1$

$$\sup_{\varepsilon} \mathbb{E} \left[ e^{-\lambda p f} \rho \alpha^\varepsilon \cos(\beta W^\varepsilon) \right] < \infty.$$

Steps 1 and 2 together imply that $\nu_{SG}^\varepsilon \to \nu_{SG}$. In Step 3 we prove that $\nu_{SG}^\varepsilon$ is indeed reflection positive.

**Step 1.** Observe that

$$\mathbb{E}[\theta^\varepsilon * W_{T_1}(x) \theta^\varepsilon * W_{T_2}(y)] = (\theta^\varepsilon \otimes \theta^\varepsilon * K_{T_1} \wedge T_2)(x, y).$$

Now observe that for $T \in [0, \infty]$, $K_T(x, y) = K_T(x - y)$ with $K_T(x) \leq -\frac{1}{4\pi} \log(T \wedge |x|) + g(x)$ with $g$ a bounded function. Furthermore

$$(\theta^\varepsilon \otimes \theta^\varepsilon * K_T)(x, y) = (\theta^\varepsilon * \theta^\varepsilon * K_T)(x - y).$$

Then it not hard to see that

$$K^\varepsilon(x) = \theta^\varepsilon * \theta^\varepsilon * K = \frac{1}{4\pi} \log \left( \frac{1}{|x| \wedge \varepsilon} \right) + g^\varepsilon(x).$$

with $\sup_{\varepsilon} \|g^\varepsilon\|_{L^\infty} < \infty$. From this we can deduce that $\theta^\varepsilon * W_\infty(x)$ is in $L^2_{loc}(\mathbb{R}^2)$ almost surely since for any bounded $U \subseteq \mathbb{R}^2$

$$\mathbb{E} \left[ \int_U ((\theta^\varepsilon * W_\infty)(x))^2 dx \right] = |U| K^\varepsilon(0).$$

We claim that for any $f \in C_c^\infty(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2} f^\varepsilon := \int f e^{2i\varepsilon x} \hat{K}^\varepsilon(0) \hat{e}^{i\beta W^\varepsilon} \to \int f e^{i\beta W_\infty}$$

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where the convergence is in $L^2(\mathbb{P})$. To prove this we calculate

$$
\mathbb{E} \left[ \left| \int_{\mathbb{R}^2} e^{\frac{1}{2} \beta^2 K^e(x)} e^{i \beta W^e(x)} f(x) - e^{\frac{1}{2} \beta^2 K^T(x)} e^{i \beta W^T(x)} f(x) \, dx \right|^2 \right] 
$$

$$
= \mathbb{E} \left[ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\beta^2 K^e(x)} e^{i \beta (W^e(x) - W^e(y))} - e^{\beta^2 (K^T(x) + K^T(y))} e^{i \beta (W^T(x) - W^T(y))} 
\beta^2 (K^T(x) + K^T(y)) e^{i \beta (W^T(x) - W^T(y))} f(x) f(y) \, dx \, dy \right] 
$$

$$
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\beta^2 K^e(x-y)} + e^{\beta^2 \bar{K}^T(x-y)} - 2 e^{\beta^2 \mathbb{E}[W^T(x)W^e(y)]} f(x) f(y) \, dx \, dy.
$$

W.l.o.g we can take $f \geq 0$. Now since $K^e(x-y) \leq -\frac{1}{4\pi} \log |x-y| + C$, $K_T(x-y) \leq -\frac{1}{4\pi} \log |x-y| + C$. We have by dominated convergence and Fatou’s lemma

$$
\lim_{\varepsilon \to 0} \lim_{T \to \infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\beta^2 K^e(x-y)} + e^{\beta^2 K_T(x-y)} - 2 e^{\beta^2 \mathbb{E}[W^T(x)W^e(y)]} f(x) f(y) \, dx \, dy = 0
$$

which proves the claim. This clearly implies

$$
\int \rho e^{\frac{1}{2} \beta^2 K^e} \cos(\beta W^e) \to \int \rho \left[ \cos(\beta W^e) \right]
$$

in $L^2(\mathbb{P})$. In particular we can select a subsequence (not relabeled) such that this implies that $\mathbb{P} \rightarrow a.s$

$$
\rho \left[ e^{-\lambda} f \rho e^{\frac{1}{2} \beta^2 K^e} \cos(\beta W^e) \right] \to e^{-\lambda} f \rho \left[ \cos(\beta W^e) \right].
$$

**Step 2.** Step 1 will imply that $\nu_{SG}^{\varepsilon,0} \to \nu_{SG}^{\varepsilon}$ as soon as we have established that

$$
\sup_{\varepsilon} \mathbb{E} \left[ e^{-\lambda} f \rho e^{\beta^2 K^e} \cos(\beta W^e) \right] < \infty.
$$

From Corollary we know

$$
- \log \mathbb{E} \left[ e^{-\lambda} f \rho e^{\beta^2 K^e} \cos(\beta W^e) \right] = \inf_{u \in \mathcal{U}_0} \mathbb{E} \left[ \lambda \rho \int_{\mathbb{R}^2} e^{\beta^2 K^e} \cos(\beta (W^e + I^e(u))) + \frac{1}{2} \int_0^\infty \| u_t \|^2 \, dt \right]
$$

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with \( I^\varepsilon(u) = \theta^\varepsilon * I_\infty(u) \). Expanding the cosine we get
\[
\left| \int \rho e^{\frac{\beta^2}{2}K^\varepsilon(0)} \cos(\beta(W^\varepsilon_\infty + I^\varepsilon(u))) \right|^2 = \left| \int \rho e^{\frac{\beta^2}{2}K^\varepsilon(0)} \cos(\beta W^\varepsilon_\infty) \cos(\beta I^\varepsilon(u)) \right|^2 + \left| \int \rho e^{\frac{\beta^2}{2}K^\varepsilon(0)} \sin(\beta W^\varepsilon_\infty) \sin(\beta I^\varepsilon(u)) \right|^2
\]
\[
\leq \mathbb{E} \left( \left| \int \rho e^{\frac{\beta^2}{2}K^\varepsilon(0)} \cos(\beta W^\varepsilon_\infty) \right|^2 \right) H^{-1} + \mathbb{E} \left( \left| \int \rho e^{\frac{\beta^2}{2}K^\varepsilon(0)} \sin(\beta W^\varepsilon_\infty) \right|^2 \right) H^{-1}
\]
\[
\leq C \left( \mathbb{E} \left( \left| \int \rho e^{\frac{\beta^2}{2}K^\varepsilon(0)} \cos(\beta W^\varepsilon_\infty) \right|^2 \right) H^{-1} + \mathbb{E} \left( \left| \int \rho e^{\frac{\beta^2}{2}K^\varepsilon(0)} \sin(\beta W^\varepsilon_\infty) \right|^2 \right) H^{-1} \right)
\]
where in the last line we have used Lemma 23. This implies by Young’s inequality
\[
\inf_{u \in \mathcal{U}_M} \mathbb{E} \left( \lambda \int \rho e^{\beta^2 K^\varepsilon(0)} \cos(\beta(W^\varepsilon_\infty + I^\varepsilon(u))) + \frac{1}{2} \int_0^\infty \|u_I\|_2^2 dt \right)
\geq -C \left( \mathbb{E} \left( \left| \int \rho e^{\frac{\beta^2}{2}K^\varepsilon(0)} \cos(\beta W^\varepsilon_\infty) \right|^2 \right) H^{-1} + \mathbb{E} \left( \left| \int \rho e^{\frac{\beta^2}{2}K^\varepsilon(0)} \sin(\beta W^\varepsilon_\infty) \right|^2 \right) H^{-1} \right) + \frac{1}{4} \mathbb{E} \left( \int_0^\infty \|u_I\|_2^2 dt \right).
\]
Now note that from a simple calculation we get
\[
\mathbb{E} \left( \left| e^{\frac{\beta^2}{2}K^\varepsilon(0)} \cos(\beta W^\varepsilon_\infty(x)) e^{\frac{\beta^2}{2}K^\varepsilon(0)} \cos(\beta W^\varepsilon_\infty(y)) \right| \right) \leq C \frac{1}{|x-y|^{\beta^2/2\pi}},
\]
from which we can conclude by Lemma ?? that \( \sup_x \mathbb{E} \left( \left| \int \rho e^{\frac{\beta^2}{2}K^\varepsilon(0)} \cos(\beta W^\varepsilon_\infty) \right|^2 \right) H^{-1} < \infty \), so we can deduce that
\[
\sup_x \mathbb{E} \left( e^{-\lambda \rho \int \rho e^{\beta^2 K^\varepsilon(0)} \cos(\beta W^\varepsilon_\infty)} \right) < \infty.
\]
Step 3. We now show that \( \nu^\varepsilon_{SG} \) are reflection positive. We can write
\[
\nu^\varepsilon_{SG} = e^{-\lambda S^\varepsilon(\phi)} \mu^\varepsilon_{SG}, \quad \text{with} \quad S^\varepsilon(\phi) = e^{\frac{1}{2} \beta^2 K^\varepsilon(0)} \int \rho \cos(\beta \phi)
\]
where \( \mu^\varepsilon_{SG} = \text{Law}(W^\varepsilon_\infty) \) is the gaussian measure with covariance operator
\[
C^\varepsilon(f) = \theta^\varepsilon * (m^2 - \Delta)^{-1} * \theta^\varepsilon f.
\]
We claim that \( \mu^\varepsilon_{SG} \) is reflection positive. Since it is Gaussian by Theorem 6.2.2 in [23] it is enough to show that
\[
\langle f, \Pi^\varepsilon \Theta C^\varepsilon \Pi^\varepsilon f \rangle_{L^2} \geq 0.
\]
Where $\Pi_+$ is the projection on $L^2(\mathbb{R}_+ \times \mathbb{R})$. Since the convolution with $\theta^c$ commutes with $\Pi_+$ we have

\[
\langle f, \Pi_+ \Theta \psi \Pi_+ f \rangle = \langle \Pi_+ (\theta^c * f), \Theta (m^2 - \Delta)^{-1} \Pi_+ (\theta^c * f) \rangle \geq 0,
\]

where in the last line we have used reflection positivity of $(m^2 - \Delta)^{-1}$. Now finally we prove that $\nu^{SG}_\rho$ is indeed reflection positive. Write

\[
S_\rho^{\psi, +}(\phi) = e^{\frac{2i}{\rho} K^+(0)} \int_{\mathbb{R}_+ \times \mathbb{R}} \rho \cos(\beta \phi).
\]

Observe that provided $\rho$ is symmetric

\[
S_\rho^{\psi}(\phi) = S_\rho^{\psi, +}(\phi) + S_\rho^{\psi, -}(\Theta \phi).
\]

Then

\[
\int F(\phi) \Theta F(\phi) d\nu^{SG}_\rho = \int F(\phi) e^{-\lambda S_\rho^{\psi, +}(\phi)} \Theta F(\phi) e^{-\lambda S_\rho^{\psi, +}(\phi)} d\mu^{SG}_\rho \geq 0
\]

by reflection positivity of $\mu^{SG}_\rho$.

### 6.2 Exponential clustering

In this section we want to study expectations under the Sine Gordon measure of the form

\[
\int_{\mathcal{M}(\mathbb{R}^2)} \prod_{i=1}^k \langle \psi_i, \phi \rangle_{L^2(\mathbb{R}^2)} \nu^{SG}_\rho(d\phi).
\]

Our goal is to show that there exist constants $C = C(\{\psi_i\}_{i=1}^k)$ and an $m_p > 0$ independent of $\psi$, such that for any $a \in \mathbb{R}^2$ and supp $\psi_i \subset B(0, 1)$

\[
\left| \int_{\mathcal{M}(\mathbb{R}^2)} \prod_{i=1}^l \langle \psi_i, \phi \rangle_{L^2(\mathbb{R}^2)} \prod_{i=l+1}^k \langle \psi_i(\cdot + a), \phi \rangle_{\nu^{SG}_\rho(d\phi)} 
- \int_{\mathcal{M}(\mathbb{R}^2)} \prod_{i=1}^l \langle \psi_i, \phi \rangle_{L^2(\mathbb{R}^2)} \nu^{SG}_\rho(d\phi) \int_{\mathcal{M}(\mathbb{R}^2)} \prod_{i=l+1}^k \langle \psi_i, \phi \rangle_{L^2(\mathbb{R}^2)} \nu^{SG}_\rho(d\phi) \right| 
\leq C \exp(-m_p|a|).
\]

In this subsection all constants will be allowed to depend on $\psi_i$. The idea of proof is similar to the analogous statement in [9]. First note that a simple computation gives, for $f, g : H^{-1}(\mathbb{R}^2) \rightarrow \mathbb{R}$ continuous,bounded

\[
\frac{d}{dt} \frac{d}{ds} \left( -\log \int_{\mathcal{M}(\mathbb{R}^2)} e^{-tf-sg} d\nu^{SG}_\rho \right) = \int_{\mathcal{M}(\mathbb{R}^2)} f g d\nu^{SG}_\rho - \int_{\mathcal{M}(\mathbb{R}^2)} f d\nu^{SG}_\rho \int_{\mathcal{M}(\mathbb{R}^2)} g d\nu^{SG}_\rho.
\]

**Lemma 12** Assume that $0 < \gamma < m$ and $f, g : H^{-1}(\mathbb{R}^2) \rightarrow \mathbb{R}^2$ are Frechet-differentiable such that $|f|_{1,2,\gamma}^\rho + |g|_{1,2,\gamma}^\rho$

\[
\frac{d}{dt} \frac{d}{ds} \left( -\log \int_{\mathcal{M}(\mathbb{R}^2)} e^{-tf-sg} d\nu^{SG}_\rho \right) \leq C |f|_{1,2,\gamma}^\rho |g|_{1,2,\gamma}^\rho \exp(-\gamma|x-z|).
\]

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Proof By weak convergence it is enough to prove the statement for $\nu_{SG}^\rho$ with $C, \gamma$ uniform in $\rho$. By Lemma 1 we have

$$\frac{d}{ds} \frac{d}{dt} \left( - \log \int_{\mathcal{S}(R^2)} e^{-tf - sg} d\nu_{SG}^\rho \right) = \lim_{s \to 0} \frac{1}{s} (E[f(W_s + I_s(u^{s,g} \cdot \rho))] - E[f(W_s + I_s(u^{0,\rho}))]).$$

Now from Theorem 6 we get

$$\|I_s(u^{s,g,\rho}) - I_s(u^{0,\rho})\|_{L^{2,\gamma}(B)} \leq s|g|_{1,2}^B,$$

so we have by Lemma 15

$$|E[f(W_s + I_s(u^{s,g,\rho}))] - E[f(W_s + I_s(u^{0,\rho}))]| \leq C |f|_{1,2, y} \|I_s(u^{s,g,\rho}) - I_s(u^{0,\rho})\|_{L^{2,-\gamma}}$$

which implies the statement.

Finally we are able to prove the exponential clustering: Take $\chi^N \in C_c^\infty (\mathbb{R}, \mathbb{R})$ with $\chi^N(x) = 1$ if $|x| \leq N$ and $\chi^N(x) = 0$ if $|x| \geq N + 1$, $\sup_{N \in \mathbb{N}} \|\chi^N\|_{L^\infty} \leq C$. Now define

$$f^N(\phi) = \prod_{i=1}^l \langle \psi_i, \phi \rangle_{L^2(\mathbb{R}^2)} \chi^N(\|\phi\|_{H^{-1,-\gamma}}), \quad g^N(\phi) = \prod_{i=+1}^k \langle \psi_i, \phi \rangle_{L^2(\mathbb{R}^2)} \chi^N(\|\phi\|_{H^{-1,-\gamma}}).$$

Furthermore introduce

$$g^{N,a}(\phi) = \prod_{i=+1}^k \langle \psi_i(\cdot + a), \phi \rangle_{L^2(\mathbb{R}^2)} \chi^N(\|\phi\|_{H^{-1,-\gamma}}).$$

Observe that $f^N, g^N \in C^2(L^2(\mathbb{R}^2))$. Note that with $w(x) = \exp(-\gamma|x-a|)$ by product rule

$$\nabla f^N(\phi)$$

$$= \chi^N(\|\phi\|_{H^{-1,-\gamma}}) \sum_{j=1}^l \prod_{i=0}^l \langle \psi_i, \phi \rangle_{L^2(\mathbb{R}^2)} \psi_j$$

$$+ \chi^N(\|\phi\|_{H^{-1,-\gamma}}) \prod_{i=0}^l \langle \psi_i, \phi \rangle_{L^2(\mathbb{R}^2)} (w(1-\Delta)^{-1} w\phi)$$

so since

$$\|w(1-\Delta)^{-1} w\phi\|_{L^2,B} \leq \|(1-\Delta)^{-1} w\phi\|_{L^2} \leq C \|\phi\|_{H^{-1,-\gamma}}$$

$$|\nabla f^N(\phi)|_{1,2,\gamma} \leq C N^l \left( \prod_{j=1}^l |\psi_j|_{1,2,\gamma} \right)$$

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and now by exponential integrability and translation invariance of \( \nu_{SG} \)

\[
\int_{\mathcal{S}^1(\mathbb{R}^2)} \left| \prod_{i=1}^{l} \left( \psi_i, \phi \right)_{L^2(\mathbb{R}^2)} \right| \prod_{i=l+1}^{k} \left| \psi_i \left( \cdot + a \right), \phi \right| - f^N(\phi)g^{N,a}(\phi) \bigg| \nu_{SG}(d\phi) \\
\leq C \int_{\{\|\phi\|_{H^{1,-\gamma}} \geq N\alpha \|\phi\|_{H^{1,-\gamma}} \geq N\}} \|\phi\|_{H^{1,-\gamma}} \|\phi\|_{H^1_{\alpha}} \nu_{SG}(d\phi) \\
\leq 2\nu_{SG}(\|\phi\|_{H^{1,-\gamma}} \geq N)^{1/2} \int_{\mathcal{S}^1(\mathbb{R}^2)} \|\phi\|_{H^{1,-\gamma}} \nu_{SG}(d\phi) \int \|\phi\|_{H^{1,-\gamma}}^{4k-4l} \nu_{SG}(d\phi) \\
\leq C2\nu_{SG}(\|\phi\|_{H^{1,-\gamma}} \geq N)^{1/2} \int_{\mathcal{S}^1(\mathbb{R}^2)} \|\phi\|_{H^{1,-\gamma}}^{4k-4l} \nu_{SG}(d\phi) \\
\leq Ce^{-N}.
\]

And analogous statements hold for

\[
\int_{\mathcal{S}^1(\mathbb{R}^2)} \left| \prod_{i=1}^{l} \left( \psi_i, \phi \right)_{L^2(\mathbb{R}^2)} - f^N(\phi) \right| \nu_{SG}(d\phi), \int_{\mathcal{S}^1(\mathbb{R}^2)} \bigg| \prod_{i=1}^{l} \left( \psi_i, \phi \right)_{L^2(\mathbb{R}^2)}, \prod_{i=l+1}^{k} \left( \psi_i \left( \cdot + a \right), \phi \right) \bigg| \nu_{SG}(d\phi).
\]

Now by Lemma [12]

\[
\int_{\mathcal{S}^1(\mathbb{R}^2)} f^N(\phi)g^{N,a}(\phi)\nu_{SG}(d\phi) - \int_{\mathcal{S}^1(\mathbb{R}^2)} f^N(\phi)\nu_{SG}(d\phi) = \int_{\mathcal{S}^1(\mathbb{R}^2)} f^N(\phi)g^{N,a}(\phi)\nu_{SG}(d\phi) - \int_{\mathcal{S}^1(\mathbb{R}^2)} f^N(\phi)\nu_{SG}(d\phi) \int_{\mathcal{S}^1(\mathbb{R}^2)} g^{N,a}(\phi)\nu_{SG}(d\phi) \\
\leq \nabla f^N(\phi)_{1,2,\gamma} |\nabla g^{N,a}(\phi)|_{1,2,\gamma} \exp(-\gamma a) \\
= |\nabla f^N(\phi)_{1,2,\gamma} |\nabla g^{N,a}(\phi)|_{1,2,\gamma} \exp(-\gamma a) \\
\leq CN^k \exp(-\gamma a).
\]

Putting things together we have

\[
\int_{\mathcal{S}^1(\mathbb{R}^2)} \prod_{i=1}^{l} \left( \psi_i, \phi \right)_{L^2(\mathbb{R}^2)} \prod_{i=l+1}^{k} \left( \psi_i \left( \cdot + a \right), \phi \right) \nu_{SG}(d\phi) \\
- \int_{\mathcal{S}^1(\mathbb{R}^2)} \prod_{i=1}^{l} \left( \psi_i, \phi \right)_{L^2(\mathbb{R}^2)} \nu_{SG}(d\phi) \int_{\mathcal{S}^1(\mathbb{R}^2)} \prod_{i=l+1}^{k} \left( \psi_i \left( \cdot + a \right), \phi \right) \nu_{SG}(d\phi) \\
\leq C(N^k \exp(-\gamma a) + \exp(-N)) \\
N = \gamma |a| = C((\gamma a)^k \exp(-\gamma |a|) + \exp(-\gamma |a|)) \\
\leq C \exp(-(1-\delta)\gamma |a|).
\]

### 6.3 Non Gaussianity

In this section we prove that \( \nu_{SG} \) is indeed not a Gaussian measure. Assume \( \nu_{SG} \) would be Gaussian, we can regard it as a gaussian measure on the Hilbert space \( H^{-1}(\langle x \rangle^{-n}) \) with \( n \in \mathbb{N} \) sufficiently large. Then there exists a Banach
space \( \mathcal{H} \subseteq \mathcal{F}(\mathbb{R}^2) \) and \( M \in H^{-1}(x, -n) \) such that for any \( \psi \in \mathcal{H} \)
\[
\log \int e^{-(\psi, \phi)} d\nu_{SG} (d\phi) = \|\psi\|^2_\mathcal{H} + (M, \psi)_{H^{-1}(x, -n)}
\]
(This follows easily from Lemma 5.1 in [36]). On the other hand we know that with \( V^p_{\rho,T}(\phi) = \alpha(T) \int \rho(x) \cos(\phi(x)) dx \) by the Cameron-Martin theorem for the Gaussian Free Field
\[
\log \int e^{-(\psi, \phi)} d\nu_{SG} (d\phi) = \lim_{\rho \to 1, T \to \infty} \frac{1}{Z_{\rho,T}} \int e^{-(\psi, \phi)} d\nu_{SG} (d\phi)
\]
\[
= \lim_{\rho \to 1, T \to \infty} \frac{1}{Z_{\rho,T}} \int e^{-(\psi, \phi)} e^{-\lambda V^p_{\rho,T}(\phi)} d\mu_T
\]
\[
= \lim_{\rho \to 1, T \to \infty} \frac{1}{Z_{\rho,T}} \int e^{-(\psi, C_T \phi)} e^{-\lambda V^p_{\rho,T}(C_T \phi)} d\mu
\]
\[
= \lim_{\rho \to 1, T \to \infty} \frac{1}{Z_{\rho,T}} \int e^{(C_T \psi, (m^2 - \Delta)^{-1} C_T \psi)} \frac{1}{Z_{\rho,T}} \int e^{-\lambda V^p_{\rho,T}(\phi + (m^2 - \Delta)^{-1} \psi)} d\mu_T
\]
\[
= \lim_{\rho \to 1, T \to \infty} (\langle C_T \psi, (m^2 - \Delta)^{-1} C_T \psi \rangle + V^p_{0,T}(m^2 - \Delta)^{-1} \psi - V^p_{0,T}(0)).
\]
Recall that since \( \sup_{\phi \in L^2} \| \nabla V^p_{0,T} \|_{L^\infty} \leq C \lambda \) by Lemma [4] we have that for \( \psi \in C^\infty_c \)
\[
\|\psi\|^2_\mathcal{H} - \langle C_T \psi, (m^2 - \Delta)^{-1} C_T \psi \rangle
\]
\[
= \log \int e^{-(\psi, \phi)} d\nu_{SG} (d\phi) - (M, \psi)_{H^{-1}(x, -n)} - \langle C_T \psi, (m^2 - \Delta)^{-1} C_T \psi \rangle
\]
\[
\leq \lim_{\rho \to 1, T \to \infty} \inf \log \int e^{-(\psi, \phi)} d\nu_{SG} (d\phi) - (M, \psi)_{H^{-1}(x, -n)} - \langle C_T \psi, (m^2 - \Delta)^{-1} C_T \psi \rangle
\]
\[
\leq \sup_{T < \infty, \rho \in C^\infty_c (\mathbb{R}^2, [0,1])} |V^p_{0,T}|_{1,\infty} \|(m^2 - \Delta)^{-1} \psi\|_{L^1} - \|M\|_{H^1((x, -n))} \|\psi\|_{H^1((x, -n))}
\]
\[
< \infty.
\]
So in particular \( \mathcal{H} \) contains \( C^\infty_c \) functions. We now show that \( \lim_{\rho \to 1} \lim_{T \to \infty} V^p_{0,T}(\psi) \) is not a quadratic functional which will imply that
\[
\lim_{\rho \to 1} \lim_{T \to \infty} \langle C_T \psi, (m^2 - \Delta)^{-1} C_T \psi \rangle + V^p_{0,T}(\psi) - V^p_{0,T}(0) \neq \|\psi\|^2_\mathcal{H} - (M, \psi)_{H^{-1}(x, -n)}.
\]
giving a contradiction. Observe that
\[
\nabla V^p_{0,T}(\psi) = \lambda \alpha(0) \sin(\psi) + \nabla R_{0,T}(\psi)
\]
with \( \sup_{\psi \in L^2} \| \nabla R_{0,T}(\psi) \|_{L^\infty} \leq C \lambda^2 \), by Lemma [4] Now for a quadratic functional we would have that \( \nabla V(\psi) \) is linear in \( \psi \) so
\[
\lim_{T \to \infty, \rho \to 1} \nabla V^p_{0,T}(\psi + \varphi) + \nabla V^p_{0,T}(\psi - \varphi) - 2 \nabla V^p_{0,T}(\psi) = 0.
\]
(13)
Let us choose \( \psi, \varphi \) such that on \( \varphi, \psi \in C^\infty_c \) and for \( x \in B(0,1) \) \( \psi(x) = \pi/2 \) and \( \varphi(x) = \pi/4 \). Then for any \( x \in B(0,1) \)
\[
\lambda \alpha(0) \sin(\varphi(x) + \psi(x)) + \lambda \alpha(0) \sin(\psi(x) - \varphi(x)) - 2 \lambda \alpha(0) \sin(\psi(x)) = \lambda \left( 2\sqrt{2}/2 - 2 \right) = \lambda \left( \sqrt{2} - 2 \right)
\]
\[
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\]
and since \( \| \nabla R_{0,T}(\psi) \|_{L^\infty} \leq C \lambda^2 \) this implies that for \( \lambda \) sufficiently small and \( x \in B(0,1) \)

\[
\lim_{\rho \to 1} \lim_{T \to \infty} \nabla V_{0,T}^\rho(\psi + \varphi)(x) + \nabla V_{0,T}^\rho(\psi - \varphi)(x) - 2\nabla V_{0,T}^\rho(\psi)(x) > \lambda \left( \sqrt{2} - 2 \right) / 2.
\]

This is clearly a contradiction to (13).

A Wick ordered cosine

We recall the definition of the regularized GFF as

\[
W_t = W_{0,t} = \int_0^t Q_s dX_s
\]

where \( X_s \) is a cylindrical Brownian motion on \( L^2 \). We can calculate:

\[
E[W_t(x)W_t(y)] = K_t(x,y).
\]

Now it is not hard to see from Ito’s formula that the quantity

\[
e^{\frac{i \beta^2}{2} K_t(x,x)} \cos(\beta W_t(x)) =: \alpha(t) \cos(\beta W_t(x))
\]

(14) is a martingale. We will write

\[
\begin{align*}
\cos(\beta W_t)(x) &= \alpha(t) \cos(\beta W_t(x)) \\
\sin(\beta W_t)(x) &= \alpha(t) \sin(\beta W_t(x)) \\
e^{i \beta W_t}(x) &= \alpha(t)e^{i W_t(x)}
\end{align*}
\]

We claim that \( \| \cos(\beta W_t) \| \) bounded in \( L^2(\mathbb{P}, H^{-1+\delta}(x)^{-n}) \) uniformly in \( t, g \). Since it is also a martingale it converges almost surely. We will largely follow [34]. To prove this the following lemma will be helpful:

Lemma 13 Consider the martingale

\[
M_t^{i,x} = K_i * [\cos(\beta W_t)](x).
\]

Then the quadratic variation of \( M_t^{i,x} \), denoted by \( [M_t^{i,x}] \) satisfies for any \( \delta > 0 \),

\[
[\langle M_t^{i,x} \rangle_t] \leq C_\delta 2^{i \beta^2 / 2 \pi + \delta}
\]

where the constant \( C_\delta \) is deterministic and does not depend on \( x \) and \( t \).

Proof We have

\[
K_i * [\cos(\beta W_t)](x) = \int K_i(x-z) \int_0^t [\sin(\beta W_s)](z) dW_s(z) dz
\]
So

\[ |[K_i \ast \cos(\beta W_t)](x)| \leq \tfrac{1}{2\pi} \int |K_i(x - z_1)K_i(x - z_2)| \left| \frac{1}{|z_1 - z_2|^{\beta^2/2\pi}} \right| dz_1 dz_2 \]

\[ \leq C \int |K_i(x - z_1)K_i(x - z_2)| \left| \frac{1}{|z_1 - z_2|^{\beta^2/2\pi}} \right| 1_{|z_1 - z_2| \leq 1} dz_1 dz_2 \]

To estimate term one we write

\[ \int |K_i(x - z_1)| \int |K_i(x - z_2)| \left| \frac{1}{|z_1 - z_2|^{\beta^2/2\pi}} \right| L^p \]

\[ \leq \|K_i\|_{L^p} \|K_i\|_{L^1} \left| \frac{1}{|z_1|^{\beta^2/2\pi}} \right| L^p \]

where we choose \( p = \tfrac{4\pi}{\beta^2 - \delta''} \) for \( \delta'' \) sufficiently small. This implies \( \frac{1}{p'} = 1 - \beta^2 / 4\pi - \delta' \) for some \( \delta' \) which can be made arbitrarily small. Recall that

\[ \|K_i\|_{L^1} \leq C \quad \|K_i\|_{L^\infty} \leq C^{2i} \]

So interpolating with the parameter \( 1 - \beta^2 / 4\pi - \delta' \) we get \( \|K_i\|_{L^p'} \leq \|K_i\|_{L^1} \frac{1}{2^i \beta^2 / 4\pi} \|K_i\|_{L^\infty} \) which implies \( \|K_i\|_{L^p'} \leq 2 \left( \tfrac{\beta^2}{4\pi} + 2^i \right) \). We have chosen \( p \) in such a way that \( \left| \frac{1}{|z|^{\beta^2/2\pi}} \right|_{L^p} < \infty \).

To estimate term II we simply write

\[ \int |K_i(x - z_1)K_i(x - z_2)| \left| \frac{1}{|z_1 - z_2|^{\beta^2/2\pi}} \right| 1_{|z_1 - z_2| \geq 1} dz_1 dz_2 \]

\[ \leq \|K_i\|_{L^2}^2 \]

in total we obtain that

\[ \langle M^{i,x}\rangle \leq C \beta^2 / 2\pi + 2^i \]

\[ \square \]
Lemma 14 For any $p < \infty$ and $\delta > 0$ and $\rho$ such that $\int \rho \, dx < \infty$

$$\sup_{t \geq 0} \mathbb{E} \left[ \left\| \cos(\beta W_t) \right\|^{p}_{B_{p,p}^{-\beta^2/4\pi - \delta}(\rho)} \right] < \infty.$$ 

Proof Using Burkholder’s inequality we obtain

$$\mathbb{E} \left[ \left\| \cos(\beta W_t) \right\|^{p}_{B_{p,p}^{-\beta^2/4\pi - \delta}(\rho)} \right] \leq \mathbb{E} \left( \sum_{i \in \mathbb{N}} 2^{-\beta(i^2/4\pi + \delta)} \left\| K_i * \cos(\beta W_t) \right\|^{p}_{L^p(\rho)} \right)$$

$$= \sum_{i \in \mathbb{N}} 2^{-\beta(i^2/4\pi + \delta)} \int \rho(x) \mathbb{E} \left[ \left\| K_i * \cos(\beta W_t) \right\|^{p}_{L^p(\rho)} \right] dx$$

$$\leq C \sum_{i \in \mathbb{N}} 2^{-\beta(i^2/4\pi + \delta)} \int \rho(x) \mathbb{E} \left( \left| M_{\lambda_i} f \right|^{p/2} \right) dx$$

$$\leq C \sum_{i \in \mathbb{N}} 2^{-\beta(i^2/4\pi + \delta)} 2^{\beta^2 p/4\pi + \delta'}$$

$$\leq \infty$$

if $\delta > \delta'$.

Definition 9 Since $\left\| \cos(\beta W_t) \right\|$ is a martingale and

$$\sup_{t} \mathbb{E} \left[ \left\| \cos(\beta W_t) \right\|^{p}_{B_{p,p}^{-\beta^2/4\pi - 2\delta}(\mathbb{P})} \right] < \infty$$

it converges in $L^p(\mathbb{P}, B_{p,p}^{-\beta^2/4\pi - 2\delta}(\mathbb{P}))$ to a limit. We will denote this limit by $\left\| \cos(\beta W_{\infty}) \right\|$ (and analogously for $\alpha(t) \sin(W_t)$ and $\alpha(t) e^{iW_t}$).

Remark 1 From Lemma 13 we see that $\beta \to 0 \mathbb{E} \left[ \left\| \cos(\beta W_t) \right\| - 1 \right\|^{2}_{L^2((x) - n)} \to 0$. Together with Lemma 14 we can easily deduce from this that

$$\mathbb{E} \left[ \left\| \cos(\beta W_t) \right\|^{2}_{B_{p,p}^{-\beta^2/4\pi - 3\epsilon}(\mathbb{P})} \right] \to 0, \quad \mathbb{E} \left[ \left\| \sin(\beta W_t) \right\|^{2}_{B_{p,p}^{-\beta^2/4\pi - 3\epsilon}(\mathbb{P})} \right] \to 0.$$

B Weighted estimates

Definition 10 For a set $z \in \mathbb{R}^2$, $r \in \mathbb{R}$ we define the weighted $L^p$ spaces

$$\left\| f \right\|_{L^p_{r,r}} = \left( \int \exp(rp|z|) f^p(x) \, dx \right)^{1/p}$$

And

$$\left\| f \right\|_{W^1_{r,r}} = \left\| f \right\|_{L^p_{r,r}} + \left( \int (\exp(rp|z|)) |\nabla f(x)|^p \, dx \right)^{1/p}$$

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We will also set $H^{1, r} = W^{1, 2, r}$. Furthermore we will set
\[
\|f\|_{L^p, r} = \|f\|_{L_0^p, r}, \quad \|f\|_{W^{1, p, r}} = \|f\|_{W_0^{1, p, r}}.
\]

**Lemma 15** Let $r > 0$. Then for $f \in L_0^{2, r_1}$, $g \in L_2^{2, r_2}$
\[
\int fg \, dx \leq \exp(-(r_1 \wedge r_2)|y - z|) \|f\|_{L_0^{2, r_1}(A)} \|g\|_{L_2^{2, r_2}(B)}.
\]

**Proof**
\[
\int fg \, dx \\
\leq \int \exp(r_1|x - y|) \exp(r_2|x - z|) \exp(-r_1 \wedge r_2|y - z|) f(x) g(x) \, dx \\
= \exp(-r_1 \wedge r_2|y - z|) \int \exp(r_1|x - y|) f(x) \exp(r_2|x - z|) g(x) \, dx \\
\leq \exp(-(r_1 \wedge r_2)|y - z|) \|f\|_{L_0^{2, r_1}(A)} \|g\|_{L_2^{2, r_2}(B)}
\]
where we have used that by triangle inequality
\[
r_1|x - y| + r_2|x - z| - r_1 \wedge r_2|y - z| \geq 0.
\]

\[\square\]

**Lemma 16** For any $\gamma > 0$, $n \leq 0$
\[
\|f\|_{L^2((x) - n)} \leq C\langle d(0, y)\rangle^{-n/2} \|f\|_{L_0^2, \gamma}
\]

**Proof**
\[
\int f^2(x)(x)^{-n} \, dx \\
= \int f^2(x)e^{2d(x, A)}e^{-2d|x - y|}(x)^{-n} \, dx \\
\leq \int f^2(x)e^{2d(x, A)}(x - y)^{-n}(x)^{-n} \, dx \\
\leq C\langle d(0, A)\rangle^{-n} \int f^2(x)e^{2|x - y|} \, dx
\]

\[\square\]

**Lemma 17** Let $s \in \{0, 1\}$ $r > 0$ and $f \in W_p^{s, r}$ is supported on $B(0, N)^c$, $N \geq 1$. Then
\[
\|f\|_{W_p^{s, r - \kappa}} \leq N^{-\kappa} \|f\|_{W_p^{s, r}}
\]
Proof

\[
\left( \int f^p \exp((r - \kappa)p|x|)dx \right)^{1/p} = \left( \int_{|x| \geq N} f^p \exp((r - \kappa)p|x|)dx \right)^{1/p} \leq N^{-\kappa} \left( \int f^p \exp(rp|x|)dx \right)^{1/p} = N^{-\kappa} \|f\|_{L^p,r}
\]

This proves the claim with \( s = 0 \). Applying this inequality also to \( \nabla f \) we obtain the full statement. \( \square \)

**Lemma 18**

\[ \|J_t f\|_{L^\infty} \leq t^{-1} \|f\|_{L^\infty} \]

**Proof** This follows directly from Young’s inequality. \( \square \)

**Lemma 19** Assume that \( t/2 \leq s \leq t \), or \( 0 \leq t \leq 1 \) then

\[ \|I_{s,t}(u)\|_{L^\infty} \leq C \|u\|_{L^\infty([s,t] \times \mathbb{R}^2)}. \]

**Proof**

\[
\sup_x \left| \int_s^t \int_{\mathbb{R}^2} e^{-\frac{1}{2}m^2/l} \frac{1}{\sqrt{4\pi l^{1/2}}} e^{-2l|x-y|^2} u_1(y)dy \right| \leq \sup_x \int_s^t \int_{\mathbb{R}^2} e^{-\frac{1}{2}m^2/l} \frac{1}{\sqrt{4\pi l^{1/2}}} e^{-2l|x-y|^2} dy \|u\|_{L^\infty([s,t] \times \mathbb{R}^2)} \leq \int_s^t e^{-\frac{1}{2}m^2/l} l^{-1} dy \|u\|_{L^\infty([s,t] \times \mathbb{R}^2)}
\]

Now in the case \( t/2 \leq s \leq t \)

\[
\int_s^t e^{-\frac{1}{2}m^2/l} l^{-1} dy \|u\|_{L^\infty([s,t] \times \mathbb{R}^2)} \leq \int_{t/2}^t l^{-1} dy \|u\|_{L^\infty([s,t] \times \mathbb{R}^2)} \leq \log 2 \|u\|_{L^\infty([s,t] \times \mathbb{R}^2)}
\]

and in the case \( 0 \leq t \leq 1 \)

\[
\int_s^t e^{-\frac{1}{2}m^2/l} l^{-1} dy \|u\|_{L^\infty([s,t] \times \mathbb{R}^2)} \leq \int_0^1 e^{-\frac{1}{2}m^2/l} dy \|u\|_{L^\infty([s,t] \times \mathbb{R}^2)} \leq C \|u\|_{L^\infty([s,t] \times \mathbb{R}^2)}.
\]

\( \square \)
Lemma 20

\[ \|I_{s,t}(u)\|_{W^{1,\infty}} \leq C\|L(t)^{1/2+\delta} u\|_{L^\infty_t([s,t] \times \mathbb{R}^2)} \]

**Proof** We first treat the case \( s \geq m^2 \). Then

\[
\begin{align*}
&\sup_x \left| \int_s^t \int_{\mathbb{R}^2} \nabla_x e^{-\frac{x^2}{4m^2}} \frac{1}{\sqrt{4\pi}} e^{-2|t-s|^2} u(y) dy \right| dt \\
&= \sup_x \left| \int_s^t \int_{\mathbb{R}^2} e^{-\frac{x^2}{4m^2}} \frac{2(x-y)^t}{\sqrt{\pi}} e^{-2|t-s|^2} u(y) dy \right| dt \\
&\leq \sup_x \left| \int_s^t \int_{\mathbb{R}^2} e^{-\frac{x^2}{4m^2}} \frac{2|x-y|^{1/2-\delta}}{\sqrt{\pi}} e^{-2|t-s|^2} \|L(t)^{1/2+\delta} u\|_{L^\infty_t([s,t] \times \mathbb{R}^2)} \right| dt \\
&\leq \left| \int_{\mathbb{R}^2} \frac{2}{\sqrt{\pi}|x-y|^2-\delta} e^{-m|x-y|^2} dy \right| \|L(t)^{1/2+\delta} u\|_{L^\infty_t([s,t] \times \mathbb{R}^2)} \\
&\leq C\|L(t)^{1/2+\delta} u\|_{L^\infty_t([s,t] \times \mathbb{R}^2)}
\end{align*}
\]

In the case \( s \leq m^2 \) we have \( \exp(-m^2/\ell) \exp(-\ell|x-y|^2) \lesssim \exp(-m|x-y|) \) so we have to estimate

\[
\begin{align*}
&\int_0^1 \int_{\mathbb{R}^2} \exp(-m|x-y|)|x-y|u(y) dy \|u\|_{L^\infty_t([s,t] \times \mathbb{R}^2)} \\
&\lesssim \int_{\mathbb{R}^2} |x-y| \exp(-m|x-y|) dy \|u\|_{L^\infty_t([s,t] \times \mathbb{R}^2)} \\
&\lesssim \|u\|_{L^\infty_t([s,t] \times \mathbb{R}^2)}
\end{align*}
\]

\( \square \)

Lemma 21 Let \( w(x) = \exp(-\gamma|x-z|) \) for \( x, z \in \mathbb{R}^2 \) and \( |\gamma| < m - \kappa \). Then

\[ \|wI_{s,t}(u)\|_{L^2(\mathbb{R}^2)} \leq C(s)^{-1/2}\|wu\|_{L^2(\mathbb{R}^+ \times \mathbb{R}^2)} \]

where the constant depends on \( \kappa \).

**Proof** It is enough to prove the inequality for \( s, t \leq 1 \) and \( s, t \geq 1 \), then the general case will follow from \( I_{s,t}(u) = I_{s,1}(u) + I_{1,t}(u) \). In the proof we will use several times that

\[ e^{r|x-z|} e^{-r|x-y|} \lesssim e^{r|y-z|} \]

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For $s, t \geq 1$

\[
\begin{align*}
\int_{\mathbb{R}^2} \left| \int_{s}^{t} \int_{\mathbb{R}^2} e^{2r|x-z|} e^{-\frac{1}{2}m^2/l} \frac{1}{\sqrt{4\pi}} e^{-2l|x-y|^2} u_l(y) dl dy \right|^2 dx \\
\leq \int_{\mathbb{R}^2} \left( \int_{s}^{t} \int_{\mathbb{R}^2} e^{2r|x-z|} \left( \frac{1}{|x-y|^2} e^{-4l|x-y|^2} \right)^{1/2} \left( \int_{s}^{t} u_l^2(y) dl \right)^{1/2} dy \right)^2 dx \\
\leq C \int_{\mathbb{R}^2} \left( \int_{s}^{t} \int_{\mathbb{R}^2} e^{2r|x-z|} \left( \frac{1}{|x-y|^2} e^{-4s|x-y|^2} \right)^{1/2} \left( \int_{s}^{t} u_l^2(y) dl \right)^{1/2} dy \right)^2 dx \\
\leq C \int_{\mathbb{R}^2} \left( \int_{s}^{t} \frac{1}{|x-y|^2} e^{-s|x-y|^2} \left( \int_{s}^{t} e^{2r|x-z|} u_l^2(y) dl \right)^{1/2} dy \right)^2 dx \\
\leq C s^{-1} \|wu\|^2_{L^2(\mathbb{R} \times \mathbb{R}^2)},
\end{align*}
\]

where in the last line we have used Young’s inequality. We now treat the $s, t \leq 1$ case.

\[
\|wI_{s,t}(u)\|^2_{L^2_x} \leq C \int_{s}^{t} \left( \int_{\mathbb{R}^2} e^{2r|x-z|} \left| \int_{\mathbb{R}^2} e^{-\frac{1}{2}m^2/l} \frac{1}{\sqrt{4\pi}} e^{-2l|x-y|^2} u_l(y) dl \right|^2 dx \right) dl
\]

Note that $e^{-\frac{1}{2}m^2/l} \frac{1}{\sqrt{4\pi}} e^{-2l|x-y|^2} \leq C e^{-(m-\kappa)|x-y|}$ so using Jensen’s inequality

\[
\begin{align*}
\|wI_{s,t}(u)\|^2_{L^2_x} &\leq \int_{s}^{t} \left( \int_{\mathbb{R}^2} e^{2r|x-z|} e^{-\frac{1}{2}m^2/l} \frac{1}{\sqrt{4\pi}} e^{-2l|x-y|^2} u_l(y) dl \right)^2 dx \right) dl \\
&\leq C \int_{s}^{t} \left( \int_{\mathbb{R}^2} e^{-(m-\kappa/2)|x-y|} e^{r|x-z|} u_l(y) dl \right)^2 dx \right) dl \\
&\leq C \int_{s}^{t} \left( \int_{\mathbb{R}^2} e^{-(m-\kappa/2-\kappa)|x|} e^{r|x-z|} u_l(y) dl \right)^2 dx \right) dl \\
&\leq C \int_{s}^{t} \left( \int_{\mathbb{R}^2} e^{r|x-z|} u_l(y) dl \right)^2 dx \right) dl \\
&\leq C \|wu\|^2_{L^2_x},
\end{align*}
\]

as long as $m - r - \kappa \geq 0$ and we have used Young’s inequality.

□

In the case where we have no weight we can improve the preceding estimate to have constant 1:

**Lemma 22**

\[
\|(m^2 - \Delta)^{1/2} I_\infty(u)\|_{L^2} \leq \int_{0}^{\infty} \|u_s\|_{L^2} ds
\]

**Proof**
\[
\int_{\mathbb{R}^2} (m^2 - \Delta)^{1/2} I_{0,\infty}(u)^2 \, dx
\]
\[
= \int_{\mathbb{R}^2} (m^2 + |k|^2) \left( \int_0^\infty \frac{1}{t} e^{-t(m^2 + |k|^2)} \, dt \right)^2 \, dk
\]
\[
\leq \int_{\mathbb{R}^2} (m^2 + |k|^2) \left( \int_0^\infty \frac{1}{t^2} e^{-t(m^2 + |k|^2) t} \, dt \right) \int_0^\infty (\mathcal{F} u_s(k))^2 \, dk
\]
\[
= \int_{\mathbb{R}^2} \int_0^\infty (\mathcal{F} u_s(k))^2 \, dk
\]
\[
= \int_0^\infty \|u_s\|_{L^2}^2 \, ds
\]

**Lemma 23** Let \( w_y(x) = \exp(-\gamma|x - z|) \) for \( x, z \in \mathbb{R}^2 \) and \( |\gamma| < m \)

\[
\|w I_{s,t}(u)\|_{L^2(\mathbb{R}^2)} \leq C \|wu\|_{L^2(\mathbb{R}^+ \times \mathbb{R}^2)}
\]

**Proof** Without loss of generality we may set \( z = 0 \). We first discuss the case \( s, t \geq m \)

\[
\int \left| \int_{s}^{t} \int_{\mathbb{R}^2} \exp(|x|) e^{-\frac{1}{2}m^2/|x|} \nabla_x \frac{1}{\sqrt{4\pi}} e^{-2|x-y|^2} u_t(y) \, ddy \right|^2 \, dx 
\]
\[
\leq \int_{\mathbb{R}^2} \int_{s}^{t} \int_{\mathbb{R}^2} e^{-\frac{1}{2}m^2/|x|} \nabla_x \frac{1}{\sqrt{4\pi}} e^{-2|x-y|^2} u_t(y) \exp(|y|) \, ddy \right| \right|^2 \, dx 
\]
\[
+ \int_{\mathbb{R}^2} \left| \int_{s}^{t} \int_{\mathbb{R}^2} e^{-\frac{1}{2}m^2/|x|} (\exp(|x|) - \exp(|y|)) \nabla_x \frac{1}{\sqrt{4\pi}} e^{-2|x-y|^2} u_t(y) \, ddy \right|^2 \, dx 
\]
\[
= I + II
\]

We identify Term I as

\[
\|\nabla_x I_{s,t}(wu)\|_{L^2}
\]

which is bounded by \( \|wu\|_{L^2(\mathbb{R}^+ \times \mathbb{R}^2)} \) from the unweighted estimate. To estimate Term II we have

\[
\int_{\mathbb{R}^2} \int_{s}^{t} \int_{\mathbb{R}^2} e^{-\frac{1}{2}m^2/|x|} (\exp(|x|) - \exp(|y|)) \nabla_x \frac{1}{\sqrt{4\pi}} e^{-2|x-y|^2} u_t(y) \, ddy \right| \right|^2 \, dx 
\]
\[
= \int_{\mathbb{R}^2} \int_{s}^{t} \int_{\mathbb{R}^2} e^{-\frac{1}{2}m^2/|x|} (\exp(|x|) - \exp(|y|)) \frac{2|x-y|}{\sqrt{\pi}} e^{-2|x-y|^2} u_t(y) \, ddy \right| \right|^2 \, dx 
\]
\[
\leq \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \left| (\exp(|x|) - \exp(|y|)) \frac{1}{|x-y|^2} e^{-m|x-y|^2} \|u_t(y)\|_{L^2(\mathbb{R}^+)} \, dy \right| \right)^2 \, dx 
\]
\[
= \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \left| (\exp(|x|) - \exp(|y|))(\exp(-|y|)) \frac{1}{|x-y|^2} e^{m|x-y|^2} \exp(|y|) \|u_t(y)\|_{L^2(\mathbb{R}^+)} \, dy \right| \right)^2 \, dx 
\]

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Now we claim that
\[ \exp(-m/2|x-y|^2) \frac{1}{|x-y|} |(\exp(r|x|) - \exp(r|y|))| \exp(-r|y|) \leq C. \]

Indeed
\[
\exp(-m/2|x-y|^2) \frac{1}{|x-y|} |(\exp(r(|x|-|y|)) - 1)| \\
\leq C \exp(-m/2|x-y|^2) \frac{||x|-|y||}{|x-y|} |(\exp(r(|x|-|y|)))| \\
= C \frac{||x|-|y||}{|x-y|} \exp(r(|x|-|y|)) - m/2|x-y|^2
\]
which is uniformly bounded by reverse triangle inequality. So in total our term is bounded by
\[
\int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \exp(-m/2|x-y|^2) \frac{1}{|x-y|} \exp(r|y|) \|u_t(y)\|_{L^2(\mathbb{R}^2)} \right)^2 dx \\
\leq C \|u\|_{L^2(\mathbb{R}^2)} \|u_t(y)\|_{L^2(\mathbb{R}^2)}
\]
where we were able to Young’s convolution inequality since \( \exp(-m|x|)|x|^{-1} \) is in \( L^1 \). For \( s, t \leq m \) we compute using \( e^{-\frac{1}{2}m^2} / e^{-2t|x-y|^2} \leq e^{-m|x-y|} \). Then for any \( \kappa > 0 \) such that \( m - r > \kappa \) we have
\[
\|\nabla I_{s,t}(u)\|_{L^2, s}^2 \\
= \int_{\mathbb{R}^2} \exp(2r|x|) \left| \int_s^t \int_{\mathbb{R}^2} e^{-\frac{1}{2}m^2} / \sqrt{\pi} e^{-2t|x-y|^2} u_t(y) dy dx \right|^2 dx \\
\leq C \int_{\mathbb{R}^2} \exp(2r|x|) \left| \int_s^t \int_{\mathbb{R}^2} (|x-y| \exp(-m|x-y|) u_t(y))^2 dy dx \right| dx \\
\leq C \kappa \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( \exp(-(m-r-\kappa)|x-y|) \exp(r|y|) \|u(y)\|_2^2 \right) dy dx \\
\leq C \|u\|_{L^2(\mathbb{R}^2)}^2
\]
In the case \( s \leq m, t > m \) we write \( I_{s,t}(u) = I_{s,m}(u) + I_{m,t}(u) \) and we can reduce the problem to the previous two cases.

\[\square\]

\section{Stochastic optimal control}

We consider the decomposition \( (with L = (m^2 - \Delta)) \)
\[
L^{-1} = \int_0^{\infty} J_t^2 dt
\]
where
\[
J_t = \left( \frac{1}{t^2} e^{-L/t} \right)^{1/2}.
\]
We denote by
\[ C_t = \int_0^t J_s^2 \, ds = L^{-1} e^{-L/t}, \]
and by \( K_t(x, y) \) the kernel of \( C_t \). From the definitions one can see that
\[ K_t(x, y) = \int_0^t e^{-m^2/s} \left( \frac{1}{s^2} e^{-4s|x-y|^2} \right) \, ds \]
so
\[ K_t(x, x) = \int_0^t e^{-m^2/s} \left( \frac{1}{4\pi s} \right) \, ds = 1 \text{ for } t \geq 1 \frac{1}{4\pi} \log t + C(t) \]
where \( C(t) < \infty \). Let \( 0 \leq s < t \) and \( u \in L^2([s, t], L^2(\mathbb{R}^2)) \). For later use we introduce the notation
\[ I_{s,t}(u) = \int_s^t J_t u_t \, dt. \]

We are interested in studying the quantities
\[ v_{t,T}(\varphi) = - \log \mathbb{E}[\exp(-V_T(\varphi + W_{t,T}))] \]
where \( W_{t,T} = \int_t^T Q_s \, dX_s \), with \( X \) being a cylindrical Brownian motion on \( L^2(\mathbb{R}^2) \), and \( Z_{t,T} = \exp(-v_{t,T}) \), for \( \varphi \in L^2(\mathbb{R}^2) \).

For the rest of this chapter we will denote by \( C^n(\mathbb{R}^2) \) functions \( L^2(\mathbb{R}^2) \rightarrow \mathbb{R} \) which are \( n \) times continuously Fréchet differentiable with bounded derivatives. Next we can derive a Hamilton-Jacobi-Bellmann equation for \( v_{t,T} \), known in the physics literature as the Polchinski equation.

**Proposition 13** Assume that \( V_T \in C^2(\mathbb{R}^2) \). Then \( v_{t,T} \) satisfies
\[ \frac{\partial}{\partial t} v_{t,T}(\varphi) + \frac{1}{2} \text{Tr}(\hat{C}_t \text{Hess} v_{t,T}(\varphi)) - \frac{1}{2} \| J_t \nabla v_{t,T}(\varphi) \|_{L^2(\mathbb{R}^2)}^2 = 0. \]

Furthermore if \( V_T \in C^2(\mathbb{R}^2) \) then \( v_{t,T} \in C([0, T], C^2(L^2(\mathbb{R}^2))) \cap C^1([0, T], C(L^2(\mathbb{R}^2))). \)

**Proof** Write \( Z_{t,T} = \exp(-v_{t,T}) = \mathbb{E}[\exp(-V_T(\varphi + W_{t,T}))] \). Noting that \( W_{t,T} \) has covariance \( C_T - C_t \) it is not hard to see that
\[ \frac{\partial}{\partial t} Z_{t,T} = \frac{\partial}{\partial t} \mathbb{E}[\exp(-V_T(\varphi + W_{t,T}))] \]
\[ = -\mathbb{E}[(W_{t,T}, (C_T - C_t)^{-1} \hat{C}_t W_{t,T})_{L^2(\mathbb{R}^2)} \exp(-V_T(\varphi + W_{t,T}))]. \]

Now using Gaussian integration by parts (see [11] Exercise 2.1.3)
\[ -\mathbb{E}[(W_{t,T}, (C_T - C_t)^{-1} \hat{C}_t W_{t,T})_{L^2(\mathbb{R}^2)} \exp(-V_T(\varphi + W_{t,T}))] \]
\[ = -\text{Tr}(\hat{C}_t \text{Hess} Z_{t,T}(\varphi)). \]
Applying chain rule we get

\[
\frac{\partial}{\partial t} v_{t,T} = -\frac{\partial}{\partial t} \log Z_{t,T} = -\frac{\partial}{\partial t} Z_{t,T} Z_{t,T}^{-1} \text{Tr}(\dot{C}_t \text{Hess} Z_{t,T}(\varphi)) Z_{t,T} = e^{v_{t,T}} \text{Tr}(\dot{C}_t \text{Hess} v_{t,T}) \]

For the second statement differentiating under the expectation we obtain

\[
Z_{t,T}(\varphi) \in C^2(L^2(\mathbb{R}^2)) ,
\]

so using our first computation we can deduce from this that also

\[Z_{t,T} \in C^1([0,T],C(L^2(\mathbb{R}^2))) \]

Now observing that if \(V_T \in C^2(L^2(\mathbb{R}^2))\) then \(\inf_{t,\varphi} Z_{t,T}(\varphi) > 0\), and using chain rule we can conclude.

\[ \Box \]

**Definition 11** Let \(T > 0\), \(H\) be a Hilbert space and \(V_T : H \to \mathbb{R}\) measurable, bounded below. Let \(X_t\) be a cylindrical process on some Hilbert space \(\Xi\). Let \(\Lambda\) be a Polish space and \(u : [0,T] \to \Lambda\) be a process adapted to \(X_t\). Let \(Y_{s,t}(\varphi, u)\) be a solution to the equation

\[
dY_{s,t}(u, \varphi) = \beta(t, Y_{s,t}(u, \varphi), u_t)dt + \sigma(t, Y_{s,t}(u, \varphi), u_t)dX_t\]

\[
Y_s(u, \varphi) = \varphi.
\]

Where \(\beta : [0,T] \times H \times \Lambda \to H\) and \(\sigma : [0,T] \times H \times \Lambda \to \mathcal{L}(\Xi, H)\) are measurable. Then we say that \(V_{t,T}\) is the value function on the stochastic control problem if

\[
V_{t,T}(\varphi) = \inf_{u \in A([s,T])} \mathbb{E} \left[ V_T(Y_{s,T}(u, \varphi)) + \int_s^T l_t(Y_{t,s}, u_t) dt \right],
\]

with \(l : [0,T] \times H \times \Lambda \to \mathbb{R}\) measurable, bounded below and we denote by \(A([s,t])\) the space of all processes \(u : [s,t] \to \Lambda\) which are adapted to \(X_t\).

**Proposition 14 (Dynamic programming)** \(V_{t,T}\) as defined above satisfies for any \(S < T\)

\[
V_{t,T}(\varphi) = \inf_{u \in A([t,S])} \mathbb{E} \left[ V_S(Y_{t,S}(u, \varphi)) + \int_t^S l_s(Y_{t,s}, u_s) dt \right].
\]

For a proof see [24] Theorem 2.24.

Now assume that \(\sigma(t, Y_t, u_t)\) is self adjoint. We can associate a HJB equation to the control problem from Definition [11]. It is:

\[
\frac{\partial}{\partial t} v(t, \varphi) + \frac{1}{2} \inf_{a \in \Lambda} \left[ \text{Tr}(\sigma^2(t, \varphi, a) \text{Hess} v(t, \varphi)) + \langle \nabla v, \beta(t, \varphi, a) \rangle_H + l(t, \varphi, a) \right] = 0.
\]

\[ v(T, \varphi) = V_T(\varphi) \]

We have the following theorem relating (17) to the solution of the control problem:
**Proposition 15 (Verification)** Assume that \( v \in C([0, T], C^2(H)) \cap C^1([0, T], C(H)) \) and \( v \) solves (17) with \( v(T, \varphi) = V_T(\varphi) \). Furthermore assume that there exists \( u \in A([t, T]) \) and \( Y \) such that \( u, Y \) satisfy (16) and

\[
\begin{align*}
\frac{d}{dt}u_t &\in \arg\min_{a \in \Lambda} \left[ \text{Tr}(\sigma^2(t, Y_t, u_t) \text{Hess} v(t, Y_t)) + \langle \nabla v(t, Y_t), \beta(t, Y_t, a) \rangle_H + l(t, Y_t, a) \right]. 
\end{align*}
\]

Then \( v(t, \varphi) = V_{t,T}(\varphi) \) and the pair \( u, Y \) is optimal.

For a proof see [24] Theorem 2.36. Now consider the case \( H = \Lambda = L^2(\mathbb{R}^2) \) and \( \beta(t, \varphi, a) = J_t a \), \( \sigma(t, \varphi, a) = J_t \), \( l(t, Y_t, a) = \frac{1}{2} \| a \|_{L^2(\mathbb{R}^2)}^2 \).

Then (18) becomes a minimization problem for a quadratic functional and reduces to

\[
\begin{align*}
u_t &= -J_t \nabla v(t, Y_{s,t}).
\end{align*}
\]

This means if we can solve the equation

\[
\frac{d}{dt}Y_{s,t} = -J_t^2 \nabla v(t, Y_{s,t})dt + J_t dX_t,
\]

we can apply the verification theorem. Furthermore in this case (17) takes the form

\[
\begin{align*}
\frac{\partial}{\partial t} v(t, \varphi) + \frac{1}{2} \text{Tr}(\hat{C}_t \text{Hess} v(t, \varphi)) - \frac{1}{2} \| J_t \nabla v(t, \varphi) \|^2_{L^2(\mathbb{R}^2)} = 0,
\end{align*}
\]

since

\[
\begin{align*}
\inf_{a \in \Lambda} \left[ \text{Tr}(\sigma(t, \varphi, a) \text{Hess} v(t, \varphi)) + \langle \nabla v(t, \varphi), \beta(t, \varphi, a) \rangle_H + l(t, \varphi, a) \right]
&= \inf_{a \in \Lambda} \left[ \text{Tr}(J_t^2 \text{Hess} v(t, \varphi)) + \langle \nabla v(t, \varphi), J_t a \rangle_{L^2(\mathbb{R}^2)} + \frac{1}{2} \| a \|_{L^2(\mathbb{R}^2)}^2 \right]
&= \frac{1}{2} \text{Tr}(\hat{C}_t \text{Hess} v(t, \varphi)) - \frac{1}{2} \| J_t \nabla v(t, \varphi) \|^2_{L^2(\mathbb{R}^2)}. \]
\]

**Corollary 2**

\[
- \log \mathbb{E}[e^{-V_{T}(\varphi + W_{s,T})}] = \inf_{u \in H_{\text{ua}}} \mathbb{E} \left[ V_{T}(Y_{s,T}(u, \varphi)) + \frac{1}{2} \int_s^T \| u_t \|_{L^2}^2 dt \right]
\]

where \( H_{\text{ua}} \) is the space of processes adapted to \( X_t \) such that \( \mathbb{E} \left[ \int_0^\infty \| u_t \|_{L^2}^2 dt \right] \) and \( Y_t(u, \varphi) \) satisfies

\[
dY_{s,t}(u, \varphi) = -J_t u_t dt + J_t dW_t \]

\[
Y_{s,s}(u, \varphi) = \varphi.
\]

Note that \( Y_{s,T}(u, \varphi) = \varphi + W_{s,T} + I_{s,T}(u) \). Furthermore the infimum on the r.h.s is attained
Proof As already noted $v_{t,T} = -\log E[e^{-V_T(\varphi + W_{t,T})}]$ satisfies the HJB equation (20) and is in $C([0,T], C^2(L^2(\mathbb{R}^2)))$, so $\nabla v_{t,T}$ is Lipschitz continuous uniformly in $T$ and bounded. By a standard fix-point argument we can then solve (19), and so applying the verification theorem we obtain

$$-\log E[e^{-V_T(\varphi + W_{t,T})}] = \inf_{u \in A(s,T)} E[V_T(Y_{s,T}(u, \varphi)) + \frac{1}{2} \int_s^T \|u_t\|_{L^2}^2 dt].$$

Since $V_T$ is bounded below we can clearly restrict the infimum on the right hand side to $u \in H_n$.  

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