CONTACT KÄHLER MANIFOLDS: SYMMETRIES AND
DEFORMATIONS

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ABSTRACT. We study complex compact Kähler manifolds $X$ carrying a
contact structure. If $X$ is almost homogeneous and $b_2(X) \geq 2$, then $X$ is a
projectivised tangent bundle (this was known in the projective case even
without assumption on the existence of vector fields). We further show
that a global projective deformation of the projectivised tangent bundle
over a projective space is again of this type unless it is the projectivisation
of a special unstable bundle over a projective space. Examples for these
bundles are given in any dimension.

1. INTRODUCTION

A contact structure on a complex manifold $X$ is in some sense the opposite
of a foliation: there is a vector bundle sequence

$$0 \rightarrow F \rightarrow T_X \rightarrow L \rightarrow 0,$$

where $T_X$ is the tangent bundle and $L$ a line bundle, with the additional
property that the induced map

$$\bigwedge^2 F \rightarrow L, \; v \wedge w \mapsto \frac{[v, w]}{F}$$

is everywhere non-degenerate.

Suppose now that $X$ is compact and Kähler or projective. If $b_2(X) = 1$,
then at least conjecturally the structure is well-understood: $X$ should arise
as minimal orbit in the projectivised Lie algebra of contact automorphisms.

Beauville [Be98] proved this conjecture under the additional assumption
that the group of contact automorphisms is reductive and that the contact
line bundle $L$ has “enough” sections.

If $b_2(X) \geq 2$ and $X$ is projective, then, due to [KPSW00] and [De02], $X$
is a projectivized tangent bundle $\mathbb{P}(T_Y)$ (in the sense of Grothendieck, taking
hyperplanes) over a projective manifold $Y$ (and conversely every such pro-
jectivised tangent bundle carries a contact structure). If $X$ is only Kähler,
the analogous conclusion is unknown. By [De02], the canonical bundle $K_X$
is still not pseudo-effective in the Kähler setting, but—unlike in the projec-
tive case—it is not known whether this implies uniruledness of $X$.

If however $X$ has enough symmetries, then we are able to deal with this
situation:

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Theorem 1.1. Let $X$ be an almost homogeneous compact Kähler manifold carrying a contact structure. If $b_2(X) \geq 2$, then there is a compact Kähler manifold $Y$ such that $X \cong \mathbb{P}(T_Y)$.

Here a manifold is said to be almost homogeneous, if the group of holomorphic automorphisms acts with an open orbit. Equivalently, the holomorphic vector fields generate the tangent bundle $T_X$ at some (hence at the general) point.

In this setting it might be interesting to try to classify all compact almost homogeneous Kähler manifolds $X$ of the form $X = \mathbb{P}(T_Y)$. Section 4 studies this question in dimension 3.

In the second part of the paper we treat the deformation problem for projective contact manifolds. We consider a family 

$$\pi: \mathcal{X} \to \Delta$$

of projective manifolds over the 1-dimensional disc $\Delta \subset \mathbb{C}$. Suppose that all $X_t = \pi^{-1}(t)$ are contact for $t \neq 0$. Is then $X_0$ also a contact manifold?

Suppose first that $b_2(X_t) = 1$. Then—as discussed above—$X_t$ should be homogeneous for $t \neq 0$. Assuming homogeneity, the situation is well-understood by the work of Hwang and Mok. In fact, then $X_0$ is again homogeneous with one surprising 7-dimensional exception, discovered by Pasquier-Perrin [PP10] and elaborated further by Hwang [Hw10]. Therefore one has rigidity and the contact structure survives unless the Pasquier-Perrin case happens, where the contact structure does not survive. We refer to [Hw10] and the references given at the beginning of section 5. Therefore—up to the homogeneity conjecture—the situation is well-understood.

If $b_2(X_t) \geq 2$, the situation gets even more difficult; so we will assume that $X_t$ is homogeneous for $t \neq 0$. We give a short argument in sect. 2, showing that then $X_t$ is either $\mathbb{P}(T_{\mathbb{P}^n})$ or a product of a torus and $\mathbb{P}^n$. Then we investigate the global projective rigidity of $\mathbb{P}(T_{\mathbb{P}^n})$:

Theorem 1.2. Let $\pi: \mathcal{X} \to \Delta$ be a projective family of compact manifolds. If $X_t \cong \mathbb{P}(T_{\mathbb{P}^n})$ for $t \neq 0$, then either $X_0 \cong \mathbb{P}(T_{\mathbb{P}^n})$ or $X_0 \cong \mathbb{P}(V)$ with some unstable vector bundle $V$ on $\mathbb{P}^n$.

The assumption that $X_0$ is projective is indispensable for our proof. In general, $X_0$ is only Moishezon, and in particular methods from Mori theory fail. In case $X_0$ is even assumed to be Fano, the theorem was proved by Wiśniewski [Wi91a], in this case $X_0 \cong \mathbb{P}(T_{\mathbb{P}^n})$. The case $X_0 \cong \mathbb{P}(V)$ with an unstable bundle really occurs; we provide examples in all dimensions in section 6. In this case $X_0$ is no longer a contact manifold.

In general, without homogeneity assumption, $X_t$ is the projectivisation of the tangent bundle of some projective variety $Y_t$; here we have only some
If however $X_t$ is again homogeneous ($t \neq 0$) and not the projectivization of the tangent bundle of a projective space, then $X_t$ is a product of a torus $A_t$ and a projective space, and we obtain a rather clear picture, described in Section 7.

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2. HOMOGENEOUS KÄHLER CONTACT MANIFOLDS

We first study homogeneous manifolds which are projectivized tangent bundles.

**Proposition 2.1.** Let $Y$ be compact Kähler. Then $X = \mathbb{P}(T_Y)$ is homogeneous if and only if $Y$ is a torus or $Y = \mathbb{P}_n$.

*Proof.* One direction being clear, assume that $X$ is homogeneous; thus $Y$ is homogeneous, too. The theorem of Borel-Remmert [BR62] says that $Y \cong A \times G/P$ where $G/P$ is a rational homogeneous manifold ($G$ a semi-simple complex Lie group and $P$ a parabolic subgroup) and $A$ a torus, one factor possibly of dimension 0. Let $d = \dim A \geq 0$.

We first assume that $d > 0$. Then $T_Y = \mathcal{O}^d_Y \oplus T_{G/P}$ leading to an inclusion

$$Z := \mathbb{P}(\mathcal{O}^d_Y) \subset X$$

with normal bundle

$$N_{Z/X} = \mathcal{O}_Z(1) \otimes \pi^* q^* (\Omega^1_{G/P}) = p^* (\mathcal{O}(1)) \otimes \pi^* q^* (\Omega^1_{G/P}).$$

Here $\pi: X \to Y$, $p: Z = \mathbb{P}_{d-1} \times Y \to \mathbb{P}_{d-1}$ and $q: Y \to G/P$ are the projections. Now, $X$ being homogeneous, $N_{Z/X}$ is spanned. This is only possible when $\dim G/P = 0$ so that $Y = A$.

If $d = 0$, then $X$ is rational homogeneous, hence Fano. This is to say that $T_Y$ is ample, hence $Y = \mathbb{P}_n$ (we do not need Mori’s theorem here because $Y$ is already homogeneous). \qed

Proposition 2.1 is now applied to obtain

**Proposition 2.2.** Let $X$ be a homogeneous compact Kähler manifold with contact structure and $\dim X = 2n - 1$. Then either $X$ is a Fano manifold (and therefore $X \cong \mathbb{P}(T_{\mathbb{P}_n})$) by Prop. 2.1, unless $b_2(X) = 1$ or

$$X \cong A \times \mathbb{P}_{n-1} = \mathbb{P}(T_A),$$

where $A$ denotes a complex torus of dimension $n$ and $T_A$ its holomorphic tangent bundle.
Proof. Again by the theorem of Borel-Remmert, \( X \cong A \times G/P \) where \( G/P \) is rational-homogeneous and \( A \) a torus, one factor possibly of dimension 0. If \( A \) does not appear, then \( X \) is Fano with \( b_2(X) \geq 2 \) and therefore by \([KPSW00]\) of the form \( X = \mathbb{P}(T_Y) \). Then we conclude by Prop. 2.1

So we may assume \( \dim A > 0 \). Since a torus does not admit a contact structure, it follows that the factor \( G/P \) is nontrivial, i.e. \( \dim G/P \geq 1 \).

We consider the projection \( \pi: X \to A \) of \( \mathbb{P}(T_Y) \) over \( A \). Every fiber is \( G/P \) and in particular a Fano manifold. We may therefore use the arguments of \([KPSW00]\), Proposition 2.11, to conclude that every fiber is \( \mathbb{P}^{n-1} \). Note that the arguments used in \([KPSW00]\), Proposition 2.11 do not use the assumption that \( X \) is projective. This completes the proof.

\[ \square \]

3. THE ALMOST HOMOGENEOUS CASE

The aim of this section is to generalize the previous section to almost homogeneous contact manifolds.

3.1. Almost homogeneous projectivized tangent bundles. We begin with the following general observation.

Lemma 3.1. Let \( Y \) be a compact complex manifold and let \( X = \mathbb{P}(T_Y) \) be its projectivised tangent bundle. If \( X \) is almost homogeneous, then \( Y \) is almost homogeneous.

We already mentioned that if \( X \) is homogeneous, so is \( Y \).

Proof. Let \( \pi: X \to Y \) be the bundle projection and consider the relative tangent sequence

\[ 0 \to T_{X/Y} \to T_X \to \pi^*T_Y \to 0. \]

Since at a general point of \( X \) the tangent bundle \( T_X \) is spanned by global sections, so is \( \pi^*T_Y \). So if \( y \in Y \) is general, if \( x \in \pi^{-1}(y) \) is general and \( v \in (\pi^*T_Y)_x \), then there exists

\[ s \in H^0(X, \pi^*(T_Y)) \]

such that \( s(x) = v \). Since \( s = \pi^*(t) \) with \( t \in H^0(Y, T_Y) \), we obtain \( t(y) = v \in T_{Y,y} \). Thus \( Y \) is almost homogeneous.

\[ \square \]

Remark 3.2. Note that, conversely, the projectivized tangent bundle \( X = \mathbb{P}(T_Y) \) of an almost homogeneous manifold \( Y \) is in general not almost homogeneous. This is illustrated by the following examples.

Example 3.3. We start in a quite general setting with a projective manifold \( Y \) of dimension \( n \). We assume that \( Y \) is almost homogeneous with \( h^0(Y, T_Y) = n \). Furthermore we assume

\[ h^0(Y, \Omega^1_Y \otimes T_Y) = h^0(Y, \text{End}(T_Y)) = 1, \]  

(1)
an assumption which is e.g. satisfied if $T_Y$ is stable for some polarization. We let $X = \mathbb{P}(T_Y)$ be the projectivized tangent bundle with projection $\pi : X = \mathbb{P}(T_Y) \to Y$ and hyperplane bundle $\mathcal{O}_X(1)$. Pushing forward the relative Euler sequence to $Y$ yields

$$0 \to \mathcal{O}_Y \to \Omega^1_Y \otimes \pi_* (\mathcal{O}_X(1)) \to \pi_* T_{X/Y} \to 0.$$ 

Since $\pi_* (\mathcal{O}_X(1)) = T_Y$, we obtain

$$0 \to \mathcal{O}_Y \to \Omega^1_Y \otimes T_Y \to \pi_* T_{X/Y} \to 0.$$ 

This sequence splits via the trace map $\Omega^1_Y \otimes T_Y \simeq \text{End}(T_Y) \to \mathcal{O}_Y$, so we obtain the exact sequence

$$0 \to H^0(Y, \mathcal{O}_Y) \to H^0(Y, \Omega^1_Y \otimes T_Y) \to H^0(Y, \pi_* T_{X/Y}) \to 0.$$ 

Using assumption (1) we find

$$H^0(X, T_{X/Y}) = H^0(Y, \pi_* T_{X/Y}) = 0.$$ 

Now the relative tangent sequence with respect to $\pi : X \to Y$ yields an exact sequence

$$0 \to H^0(X, T_{X/Y}) \to H^0(X, T_X) \to H^0(X, \pi^* (T_Y)) \simeq H^0(Y, T_Y)$$

and therefore

$$h^0(T_X) \leq h^0(T_Y).$$

Hence $h^0(T_X) \leq n$, and $X$ cannot be almost homogeneous.

Notice that an inequality $h^0(T_X) \leq 2n - 2$ suffices to conclude that $X$ is not almost homogeneous. Therefore we could weaken the assumptions $h^0(T_Y) = n$ and $h^0(\text{End}(T_Y)) = 1$ to

$$h^0(T_Y) + h^0(\text{End}(T_Y)) \leq 2n - 2.$$ 

We give two specific examples.

First, let $Y$ be a del Pezzo surface of degree six, i.e., a three-point blow-up of $\mathbb{P}_2$. Its automorphisms group is $(\mathbb{C}^*)^2 \rtimes S_3$. In particular, $Y$ is almost homogeneous and $h^0(T_Y) = 2$. Since $h^0(\text{End}(T_{\mathbb{P}_2})) = 1$ and $Y$ is a blow up of $\mathbb{P}_2$, each endomorphism of $T_Y$ induces an endomorphism of $T_{\mathbb{P}_2}$ and it follows that

$$h^0(T_Y \otimes \Omega^1_Y) = h^0(\text{End}(T_Y)) = 1. \quad (2)$$

Hence the assumptions of our previous considerations are fulfilled and $X = \mathbb{P}(T_Y)$ is not almost homogeneous.

Here is an example with $b_2(Y) = 1$. We let $Y$ be the Mukai-Umemura Fano threefold of type $V_{22}$, [MUK83]. Here $h^0(T_Y) = 3$ and $Y$ is almost homogeneous with $\text{Aut}^0(Y) = \text{SL}_2(\mathbb{C})$. Since $T_Y$ is known to be stable (see e.g. [PW95]), again all assumptions are satisfied and $X = \mathbb{P}(T_Y)$ is not almost homogeneous.
3.2. The Albanese map for almost homogeneous manifolds. A well-known theorem of Barth-Oeljeklaus determines the structure of the Albanese map of an almost homogeneous Kähler manifold.

**Theorem 3.4 ([BO74]):** Let $X$ be an almost homogeneous compact Kähler manifold. Then the Albanese map $\alpha : X \to A$ is a fiber bundle. The fibers are connected, simply-connected and projective.

**Remark 3.5.** The fibers $X_\alpha$ of $\alpha$ are almost homogeneous.

Proof. Let $x, y \in X_\alpha$ be two general points. Then there exists $f \in \text{Aut}(X)$ with $f(x) = y.$ Since the automorphism $f$ is fiber preserving, we obtain an automorphism of $X_\alpha$ mapping $x$ to $y.$ □

3.3. The case $q(X) = 0.$ If the irregularity of $X$ is $q(X) = 0,$ the Albanese map is trivial, and it follows that $X$ itself is simply-connected and projective.

**Lemma 3.6.** Let $X$ be an almost homogeneous compact Kähler manifold with contact structure. If $q(X) = 0$ and $b_2(X) \geq 2,$ then $X \cong \mathbb{P}(T_Y)$ is a projectivised tangent bundle.

Proof. $X$ being projective, the results of [KPSW00] apply. Combining them with [Deo2] (cf. Corollary 4) yields the desired result. □

**Remark 3.7.** The case where $q(X) = 0$ and $b_2(X) = 1$ remains to be studied. Here $X$ is an almost homogeneous Fano manifold. It would be interesting to find out whether the results of [Be98] apply. I.e., one has to check whether $\text{Aut}(X)$ is reductive and whether the map associated with the contact line bundle $L$ is generically finite.

In order to study the second property, consider the long exact sequence

$$0 \to H^0(X, F) \to H^0(X, T_X) \to H^0(X, L) \to \ldots$$

If $H^0(X, F) \neq 0$ then $X$ has more than one contact structure [Le95], Prop.2.2, hence Corollary 4.5 of [Ke01] implies that $X \cong \mathbb{P}_{2n+1}$ or $X \cong \mathbb{P}(T_Y).$

If $H^0(X, F) = 0$ then $L$ has “many sections” and the map associated with $L$ is expected to be generically finite.

3.4. The case $q(X) \geq 1.$ If the irregularity of $X$ is positive, then the Albanese map $\alpha : X \to A$ is a fiber bundle. We denote its fiber by $X_\alpha.$

**Lemma 3.8.** Let $X$ be an almost homogeneous compact Kähler manifold with contact structure and $q(X) \geq 1.$ If the fiber $X_\alpha$ of the Albanese map fulfills $b_2(X_\alpha) = 1,$ then $X \cong \mathbb{P}(T_A) = \mathbb{P}_n \times A,$ where $A$ is the Albanese torus of $X.$
Proof. Since $b_2(X_a) = 1$, then $X_a$ (being uniruled) is a Fano manifold. We may therefore apply Proposition 2.11 of [KPSW00] (which works perfectly in our situation) to conclude that $\alpha: X \to A$ is a $\mathbb{P}_n$-bundle. The proof of Theorem 2.12 in [KPSW00] can now be adapted to conclude that $X \cong \mathbb{P}(T_A)$. To be more specific, we already know in our situation that $X = \mathbb{P}(E)$ with $E = \alpha^*(L)$. The only thing to be verified is the isomorphism $E \cong T_A$. But this is seen as in the last part of the proof of Theorem 2.12 in [KPSW00], since section 2.1 of [KPSW00] works on any manifold.

So $X \cong \mathbb{P}(T_A)$ and $X \cong \mathbb{P}_n \times A$. □

It remains to study the case where the fiber $X_a$ fulfills $b_2(X_a) \geq 2$. In this case we consider a relative Mori contraction (over $A$; the projection is a projective morphism, [Na87], (4.12))

$$\varphi: X \to Y.$$ 

**Lemma 3.9.** We have $\dim X > \dim Y$.

**Proof.** The lemma follows from the fact that the restriction map $\varphi_a = \varphi|_{X_a}$ is not birational. This can be shown by the same arguments as in Lemma 2.10 of [KPSW00] using the length of the contraction and the restriction of the contact line bundle to the fiber $X_a$. Again the projectivity of $X$ is not needed in Lemma 2.10. □

As above, we may now apply Proposition 2.11 of [KPSW00] and conclude that the general fiber of $\varphi$ is $\mathbb{P}_n$. It remains to check that $\varphi$ is a $\mathbb{P}_n$-bundle and $X \cong \mathbb{P}(T_Y)$. This is done again as in Theorem 2.12 of [KPSW00] with Fujita’s result generalized to the Kähler setting by Lemma 3.10. Also the compactness assumption in [Fu85] is not necessary, this will be important later.

**Lemma 3.10.** Let $X$ be a complex manifold, $f: X \to S$ a proper surjective map to a normal complex space $S$. Let $L$ be a relatively ample line bundle on $X$ such that $(F, L_F) \cong (\mathbb{P}_r, \mathcal{O}(1))$ for a general fiber $F$ of $f$. If $f$ is equidimensional, then $f$ is a $\mathbb{P}_r$-bundle.

In total, we obtain

**Theorem 3.11.** Let $X$ be a compact almost homogeneous Kähler contact manifold, $b_2(X) \geq 2$. Then $X = \mathbb{P}(T_Y)$ with a compact Kähler manifold $Y$.

The arguments above actually also show the following.

**Theorem 3.12.** Let $X$ be a compact Kähler contact manifold. Let $\varphi: X \to Y$ be a surjective map with connected fibers such that $-K_X$ is $\varphi$-ample and such that $\rho(X/Y) = 1$ (we do not require the normal variety $Y$ to be Kähler). Then $Y$ is smooth and $X = \mathbb{P}(T_Y)$. 

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**Contact Kähler Manifolds: Symmetries and Deformations**

7

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One might wonder whether this is still true when $X$ is Moishezon or bimeromorphic to a Kähler manifold. Although there is no apparent reason why the theorem should not hold in this context, at least the methods of proof completely fail. More generally, also the assumption that $X$ is almost homogeneous should be unnecessary. If $X$ is still Kähler, a Mori theory in the non-algebraic case seems unavoidable. Already the question whether $X$ is uniruled is hard.

3.5. **Conclusion and open questions.** (1) In all but one case we find that a compact almost homogeneous Kähler contact manifold $X$ has the structure of a projectivised tangent bundle. The remaining case where $q(X) = 0$ and $b_2(X) = 1$ is discussed in Remark [3.7].

(2) Can one classify all $Y$ (necessarily almost homogeneous) such that $\mathbb{P}(T_Y)$ is almost homogeneous? The case where $\dim Y = 2$ will be treated in the next section. One might also expect that if $Y = G/P$, then $X$ should be almost homogeneous. In case $Y$ is a Grassmannian or a quadric, this has been checked by Goldstein [Go83]. Of course, if $Y = \mathbb{P}_n$, then $X$ is even homogeneous.

4. **Almost homogeneous contact threefolds**

In this section we specialize to almost homogeneous contact manifolds in dimension 3.

**Theorem 4.1.** Let $X$ be a smooth compact Kähler threefold which is of the form $X = \mathbb{P}(T_Y)$ for some compact (Kähler) surface $Y$.

(1) If $X$ is almost homogeneous, then $Y$ is a minimal surface or a blow-up of $\mathbb{P}_2$ or $Y = \mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$ for some $n \geq 0$, $n \neq 1$.

(2) If $Y$ is minimal, then $X$ is almost homogeneous if and only if $Y$ is one of the following surfaces.

- $Y = \mathbb{P}_2$
- $Y = \mathbb{F}_n$ for some $n \geq 0$, $n \neq 1$
- $Y$ is a torus
- $Y = \mathbb{P}(\mathcal{E})$ with $\mathcal{E}$ a vector bundle of rank 2 over an elliptic curve which is either a direct sum of two topologically trivial line bundles or the non-split extension of two trivial line bundles.

**Proof.** Suppose $X$ is almost homogeneous. Then $Y$ is almost homogeneous, too (Lemma 3.1). By Potters’ classification [Po68], $Y$ is one of the following.

(1) $Y = \mathbb{P}_2$

(2) $Y = \mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$ for some $n \geq 0$, $n \neq 1$

(3) $Y$ is a torus
(4) \( Y = \mathbb{P}(\mathcal{E}) \) with \( \mathcal{E} \) a vector bundle of rank 2 over an elliptic curve which is either a direct sum of two topologically trivial line bundles or the non-split extension of two trivial line bundles

(5) \( Y \) is a certain blow-up of \( \mathbb{P}_2 \) or of \( \mathbb{P}_n \).

This already shows the first claim of the theorem, and it suffices to assume \( Y \) to be a minimal surface of the list and to check whether \( X = \mathbb{P}(T_Y) \) is almost homogeneous. In cases (1) and (3) this is clear; \( X \) is even homogeneous.

To proceed further, consider the tangent bundle sequence

\[
0 \to T_{X/Y} \to T_X \to \pi^*(T_Y) \to 0.
\]

Notice

\[
h^0(T_{X/Y}) = h^0(-K_{X/Y}) = h^0(S^2T_Y \otimes K_Y).
\]

Applying \( \pi_* \) and observing that the connecting morphism

\[
T_Y \to R^1\pi_*(T_{X/Y})
\]

(induced by the Kodaira-Spencer maps) vanishes since \( \pi \) is locally trivial, it follows that

\[
H^0(X, T_X) \to H^0(X, \pi^*(T_Y)) = H^0(Y, T_Y)
\]

is surjective. If therefore

\[
H^0(X, T_{X/Y}) \simeq H^0(Y, S^2T_Y \otimes K_Y) \neq 0,
\]

the tangent bundle \( T_X \) is obviously spanned and therefore \( X \) is almost homogeneous.

In case (4), \((\ast)\) is now easily verified: Let \( p: \mathbb{P}(\mathcal{E}) \to C \) be the \( \mathbb{P}_1 \)-fibration over the elliptic curve \( C \). The tangent bundle sequence reads

\[
0 \to -K_Y \to T_Y \to \mathcal{O}_Y \to 0.
\]

Since \( T_Y \) is generically spanned, the map \( H^0(\mathcal{O}_Y) \to H^1(-K_Y) \) must vanish, so that the sequence splits:

\[
T_Y \simeq -K_Y \oplus \mathcal{O}_Y.
\]

Thus \( S^2T_Y \otimes K_Y \simeq -K_Y \oplus \mathcal{O}_Y \otimes K_Y \) and \((\ast)\) follows.

Now if \( Y = \mathbb{P}_n \) as in (2), let \( p: Y \to \mathbb{P}_1 \) be the natural projection. The relative tangent sequence then reads

\[
0 \to T_{Y/\mathbb{P}_1} \to T_Y \to p^*\mathcal{O}_{\mathbb{P}_1}(2) \to 0.
\]

Taking the second symmetric power and tensorizing with \( K_Y \) yields

\[
0 \to T_Y \otimes T_{Y/\mathbb{P}_1} \otimes K_Y \to S^2T_Y \otimes K_Y \to p^*\mathcal{O}_{\mathbb{P}_1}(4) \otimes K_Y \to 0,
\]

so, by \((\ast\ast)\), we obtain an inclusion

\[
H^0(T_{Y/\mathbb{P}_1}^{\otimes 2} \otimes K_Y) \subset H^0(S^2T_Y \otimes K_Y).
\]
Now by the relative Euler sequence, \( T_{Y/F_1} \simeq O_Y(2) \otimes p^*O_{P_1}(n) \), and thus
\[
H^0(T_{Y/F_1}^\otimes 2 \otimes K_Y) \simeq H^0(O_Y(2) \otimes p^*O_{P_1}(n-2)).
\]
Now since
\[
p_*(O_Y(2) \otimes p^*O_{P_1}(n-2)) \simeq O_{P_1}(n-2) \oplus O_{P_1}(-2) \oplus O_{P_1}(-n-2),
\]
we have shown \( \Box \) to be true for \( n \geq 2 \). If \( n = 0 \), i.e., \( Y \simeq P_1 \times P_1 \), the sequence \( \Box \Box \) splits and an easy calculation shows that \( \Box \) is satisfied also in this case.

\[ \square \]

**Remark 4.2.** The case that \( Y \) is a non-minimal rational surface in Theorem 4.1 could be further studied, but this is a rather tedious task.

### 5. Deformations I: The rational case

We consider a family \( \pi: \mathcal{X} \to \Delta \) of compact manifolds over the unit disc \( \Delta \subset \mathbb{C} \). As usual, we let \( X_t = \pi^{-1}(t) \). We shall assume \( X_t \) to be a projective manifold for all \( t \), so we are only interested in projective families here. If now \( X_t \) is a contact manifold for \( t \neq 0 \), when is \( X_0 \) still a contact manifold? If \( b_2(X_t) = 1 \), there is a counterexample due to [PP10], see also [Hw10]. Here the \( X_t \) are 7-dimensional rational-homogeneous contact manifolds and \( X_0 \) is a non-homogeneous non-contact manifold. If one believes that any Fano contact manifold with \( b_2 = 1 \) is rational-homogeneous, then due to the results of Hwang and Mok, this is the only example where a limit of contact manifolds with \( b_2 = 1 \) is not contact.

If \( b_2(X_t) \geq 2 \), it is no longer true that the limit \( X_0 \) is always a contact manifold, as can be seen from the following example: We let \( \mathcal{Y} \to \Delta \) be a family of compact manifolds such that \( Y_t \simeq P_1 \times P_1 \) for \( t \neq 0 \) and \( Y_0 \simeq P_2 \). Then there exist line bundles \( L_1 \) and \( L_2 \) on \( \mathcal{Y} \) such that \( L_1|Y_t \simeq O_{P_1 \times P_1}(2,0) \) and \( L_2|Y_t \simeq O_{P_1 \times P_1}(0,2) \) for every \( t \neq 0 \). If we let \( \mathcal{X} := P(L_1 \oplus L_2) \), then \( X_t \simeq P(T_{Y_t}) \) for \( t \neq 0 \), but \( X_0 \not\simeq P(T_{Y_0}) \).

However \( P(T_{P_1 \times P_1}) \) is not homogeneous; in fact by Proposition 2.1, \( P(T_{P_n}) \) is the only homogeneous rational contact manifold with \( b_2 \geq 2 \). In this prominent case we prove global projective rigidity, i.e., \( X_0 = P(T_{P_n}) \), unless \( X_0 \) is the projectivization of some unstable bundle, so that both contact structures survive in the limit. In the “unstable case”, the contact structure does not survive. The special case where \( X_0 \) is Fano is due to Wiśniewski [Wi91a]; here global rigidity always holds.

There is a slightly different point of view, asking whether projective limits of rational-homogeneous manifolds are again rational-homogeneous. As before, if \( b_2(X_t) = 1 \), this is true by the results of Hwang and Mok with the 7-dimensional exception. In case \( b_2(X_t) \geq 2 \), this is false in general (e.g. for \( P_1 \times P_1 \)), but the picture under which circumstances global rigidity is still true is completely open.
Theorem 5.1. Let $\pi: \mathcal{X} \to \Delta$ be a family of compact manifolds. Assume $X_t \simeq \mathbb{P}(T_{\mathbb{P}^n})$ for $t \neq 0$. If $X_0$ is projective, then either $X_0 \simeq \mathbb{P}(T_{\mathbb{P}^n})$ or $X_0 \simeq \mathbb{P}(V)$ with some unstable vector bundle $V$ on $\mathbb{P}^n$.

Proof. Since $K_X$ is not $\pi$-nef, there exists a relative Mori contraction (see [Na87], (4.12), we may shrink $\Delta$)
\[ \Phi: \mathcal{X} \to \mathcal{Y} \]
over $\Delta$. Put $\Delta^* = \Delta \setminus \{0\}$ and $\mathcal{X}^* = \mathcal{X} \setminus X_0; \mathcal{Y}^* = \mathcal{Y} \setminus Y_0$. Now $\phi_t = \Phi|X_t$ is a Mori contraction for any $t$ (cp. [KM92], (12.3.4), but this is pretty clear in our simple situation), unless possibly $\phi_t$ is biholomorphic for $t \neq 0$.

Now since $\mathcal{X}, \Delta$ and $\pi$ are smooth, the latter case cannot occur by [Wi91b], (1.3), so $\phi_t$ is the contraction of an extremal ray for any $t \in \Delta$. Let $\tau: \mathcal{Y} \to \Delta$ be the induced projection and set $Y_t = \tau^{-1}(t)$, so that $Y_t \simeq \mathbb{P}_n$ for $t \neq 0$. Since $\mathcal{Y}$ is normal, the normal variety $Y_0$ must also have dimension $n$.

From the exponential sequence, Hodge decomposition and the topological triviality of the family $\mathcal{X}$, it follows that
\[ \text{Pic}(\mathcal{X}) \simeq H^2(\mathcal{X}, \mathbb{Z}) \simeq \mathbb{Z}^2 \]
and that
\[ \text{Pic}(X_0) \simeq H^2(X_0, \mathbb{Z}) \simeq \mathbb{Z}^2. \]
Furthermore, the restriction $\text{Pic}(\mathcal{X}) \to \text{Pic}(X_0)$ is bijective. As an immediate consequence, we can write
\[ -K_X = n\mathcal{H} \]
with a line bundle $\mathcal{H}$ on $\mathcal{X}$. Let $\mathcal{H}_t = \mathcal{H}|X_t$ so that $\mathcal{H}_t \simeq \mathcal{O}_{\mathbb{P}(T_{\mathbb{P}^n})}(1)$ for $t \neq 0$.

Claim 5.2. $Y_0 \simeq \mathbb{P}_n$.

In fact, by our previous considerations, there is a unique line bundle $\mathcal{L}$ on $\mathcal{X}$ such that
\[ \mathcal{L}|X_t = \phi_t^*(\mathcal{O}_{\mathbb{P}_n}(1)) \]
for $t \neq 0$. Moreover $\mathcal{L}|X_0 = \phi_0^*(\mathcal{L}')$ with some ample line bundle $\mathcal{L}'$ on $Y_0$. Therefore by semi-continuity,
\[ h^0(\mathcal{L}') = h^0(\mathcal{L}|X_0) \geq n + 1 \]
and
\[ c_1(\mathcal{L}')^n = 1. \]
Hence by results of Fujita [Fu90], (I.1.1), see also [BS95], (III.3.1), we have $(Y_0, \mathcal{L}') \simeq (\mathbb{P}_n, \mathcal{O}(1))$.

In particular we obtain

Sub-Corollary 5.3. $\mathcal{Y}$ is smooth and $\mathcal{Y} \simeq \mathbb{P}_n \times \Delta$. 


Next we notice that the general fiber of $\varphi_0$ must be $\mathbb{P}^{n-1}$, since it is a smooth degeneration of fibers of $\varphi_t$ (by the classical theorem of Hirzebruch–Kodaira).

One main difficulty is that $\varphi_0$ might not be equidimensional. If we know equidimensionality, we may apply [Fu85, 2.12] to conclude that $X_0 = \mathbb{P}(E_0)$ with a locally free sheaf $E_0$ on $Y_0$.

We introduce the torsion free sheaf

$$F = \Phi^*(\mathcal{H}) \otimes \mathcal{O}_Y(-1).$$

Since

$$\text{codim } \Phi^{-1}(\text{Sing}(F)) \geq 2,$$

the sheaf $F$ is actually reflexive and of course locally free outside $Y_0$. In the following Sublemma we will prove that $F$ is actually locally free.

**Sub-Lemma 5.4.** $F$ is locally free and therefore $X = \mathbb{P}(F)$.

**Proof.** As explained above, it is sufficient to show that

$$\varphi_0 : X_0 \to \mathbb{P}_n$$

is equidimensional. So let $F_0$ be an irreducible component of a fiber of $\varphi_0$. Then $F_0$ gives rise to a class

$$[F_0] \in H^{2k}(X_0, \mathbb{Q}),$$

where we denote by $k$ the codimension of $F_0$ in $X$. Obviously $k \leq n$, and we must exclude the case that $k < n$.

So we assume in the following that $k < n$. Then, since $X_0$ is homeomorphic to $\mathbb{P}(T_{\mathbb{P}_n})$, the Leray–Hirsch theorem gives

$$\dim H^{2k}(X_0, \mathbb{Q}) = k + 1.$$  

Now if we denote by $H$ the class of a hyperplane in $\mathbb{P}_n$, and by $L$ the class of an ample divisor on $X_0$, then the classes

$$L^k, L^{k-1}.(\varphi_0^*H), \ldots, L.(\varphi_0^*H)^{k-1}, (\varphi_0^*H)^k$$

form a basis of $H^{2k}(X_0, \mathbb{Q})$, which can be seen as follows: By the dimension formula given above, it is sufficient to show linear independence, so assume that we are given $\lambda_0, \ldots, \lambda_k \in \mathbb{Q}$ such that

$$\sum_{\ell=0}^{k} \lambda_\ell L^{k-\ell}.(\varphi_0^*H)^\ell = 0.$$  

Now let $\ell_0 \in \{0, \ldots, k\}$. By induction, we assume that $\lambda_\ell = 0$ for all $\ell < \ell_0$.

Then intersecting (4) with $L^{n-k+1+\ell_0}.(\varphi_0^*H)^{n-\ell_0}$ yields

$$\lambda_{\ell_0} L^{n-1}.(\varphi_0^*H)^n = 0,$$

thus $\lambda_{\ell_0} = 0$ since $L^{n-1}.(\varphi_0^*H)^n > 0$. 

So (3) is indeed a basis of $H^2(X_0, \mathbb{Q})$ and we can write

$$[F_0] = \sum_{\ell=0}^{k} \alpha_{\ell} L^{k-\ell} (\phi_0^* H)^{\ell}$$

for some $\alpha_0, \ldots, \alpha_k \in \mathbb{Q}$. We now let $\ell_0 \in \{0, \ldots, k\}$ and assume that $\alpha_{\ell} = 0$ for $\ell < \ell_0$. We observe that $[F_0].(\phi_0^* H)^{n-\ell_0} = 0$ since $F_0$ is contained in a fiber of $\phi_0$ and $\ell_0 \leq k < n$. Hence, intersecting (5) with $L^{n-k-1+\ell_0}(\phi_0^* H)^{n-\ell_0}$ yields

$$0 = \alpha_{\ell_0} L^{n-\ell} (\phi_0^* H)^{n},$$

so we deduce $\alpha_{\ell_0} = 0$ as before. Therefore by induction, we have $[F_0] = 0$, which is impossible, $X_0$ being projective.

Now we set $V = F|X_0$. If the bundle $V$ is semi-stable, then $V \simeq T_{P_{n}}$ and the theorem is settled.

Suppose in Theorem 5.1 that $X_0 \simeq \mathbb{P}(V)$ with an unstable bundle $V$ (we will show in section 6 that this can indeed occur). Then $X_0$ does not carry a contact structure. In fact, otherwise $X_0 \simeq \mathbb{P}(T_S)$ with some projective variety $S$, [KPSW00]. Hence $X_0$ has two extremal contractions, and therefore $X_0$ is Fano. Hence $T_S$ is ample and thus $S \simeq \mathbb{P}_n$ (or apply Wiśniewski’s theorem). Therefore we may state the following

**Corollary 5.5.** Let $\pi : \mathcal{X} \rightarrow \Delta$ be a family of compact manifolds. Assume $X_t \simeq \mathbb{P}(T_{P_{n}})$ for $t \neq 0$. If $X_0$ is a projective contact manifold, then $X_0 \simeq \mathbb{P}(T_{P_{n}})$.

In the situation of Theorem 5.1 we had two contact structures on $X_t$. This phenomenon is quite unique because of the following result [KPSW00], Prop. 2.13.

**Theorem 5.6.** Let $X$ be a projective contact manifold of dimension $2n - 1$ admitting two extremal rays in the cone of curves $\overline{NE}(X)$. Then $X \simeq \mathbb{P}(T_{P_{n}})$.

Here is an extension of Theorem 5.6 to the non-algebraic case.

**Theorem 5.7.** Let $X$ be a compact contact Kähler manifold admitting two contractions $\phi_1 : X \rightarrow Y_1$ to normal compact Kähler spaces $Y_i$. This is to say that $-K_X$ is $\phi_1$-ample and that $\rho(X/Y_1) = 1$. Then $X$ is projective and therefore $X \simeq \mathbb{P}(T_{P_{n}})$.

**Proof.** We already know by Theorem 3.13 that $X \simeq \mathbb{P}(T_{Y_1})$. Let $F \simeq \mathbb{P}_{n-1}$ be a fiber of $\phi_2$. Then the restriction $\phi_1|F$ is finite. We claim that $Y_1$ must be projective. In fact, consider the rational quotient, say $f : Y_1 \dashrightarrow Z$, which is an almost holomorphic map to a compact Kähler manifold $Z$. By construction, the map $f$ contracts the images $\phi_1(F)$, hence $\dim Z \leq 1$. But then $Z$ is projective and therefore $Y_1$ is projective, too (e.g. by arguing that $y_1$ cannot carry a holomorphic 2-form).

By symmetry, $Y_2$ is projective, too. Since the morphisms $\phi_i$ induce a finite map $X \rightarrow Y_1 \times Y_2$ (onto the image of $X$), the variety $X$ is also projective. □
Any projective contact manifold $X$ with $b_2(X) \geq 2$ is of the form $X = \mathbb{P}(T_Y)$. Therefore it is natural ask for generalizations of Theorem [5.1] substituting the projective space by other projective varieties.

**Proposition 5.8.** Let $\pi: \mathcal{X} \to \Delta$ be a projective family of compact manifolds $X_t$ of dimension $2n - 1$. Assume that $X_t \simeq \mathbb{P}(T_{Y_t})$ for $t \neq 0$ with (necessarily projective) manifolds $Y_t \neq \mathbb{P}_n$. Assume that $H^q(X_t, \mathcal{O}_{X_t}) = 0$ for $q = 1, 2$ for some (hence all) $t$. Then the following statements hold.

1. There exists a relative contraction $\Phi: \mathcal{X} \to \mathcal{Y}$ over $\Delta$ such that $\Phi|X_t$ is the given $\mathbb{P}_{n-1}$-bundle structure for $t \neq 0$.
2. If $\phi_0 := \Phi|X_0$ is equidimensional, then $X_0 \simeq \mathbb{P}(E_0)$ with a rank-$n$ bundle $E$ over the projective manifold $Y_0$. Assume that $Y_t = \tau^{-1}(t)$ for $t \neq 0$. In other words, $\mathcal{X} \simeq \mathbb{P}(E)$ such that $E = T_{Y/\Delta}$ over $\Delta \setminus \{0\}$.

**Proof.** Since $Y_t \neq \mathbb{P}_n$ by assumption, every $X_t$, $t \neq 0$, has a unique Mori contraction, the projection $\psi_t: X_t \to Y_t$, by Theorem [5.6]. As in the proof of Theorem [5.1], we obtain a relative Mori contraction

$\Phi: \mathcal{X} \to \mathcal{Y}$

over $\Delta$, and necessarily $\Phi|X_t = \phi_t$ for all $t \neq 0$ (we use again [Wi91b], (1.3)). This already shows Claim (1).

If $\phi_0$ is equidimensional, we apply—as in the proof of Theorem [5.1]—[BS95], (III.3.2.1), to conclude that there exists a locally free sheaf $E_0$ of rank $n$ on $Y_0$ such that $X_0 \simeq \mathbb{P}(E_0)$, proving (2).

**Theorem 5.9.** Let $\pi: \mathcal{X} \to \Delta$ be a projective family of compact manifolds $X_t$ of dimension $2n - 1$. Assume that $X_t \simeq \mathbb{P}(T_{Y_t})$ for $t \neq 0$ with (necessarily projective) manifolds $Y_t(\neq \mathbb{P}_n)$. Assume that $H^q(X_t, \mathcal{O}_{X_t}) = 0$ for $q = 1, 2$ for some (hence all) $t$. Assume moreover that

1. $\dim X_0 \leq 5$,
2. $b_2(Y_t) = 1$ for some $t \neq 0$ and all $1 \leq j < \frac{n}{2}$.

Then there exists a relative contraction $\Phi: \mathcal{X} \to \mathcal{Y}$ over $\Delta$ such that $\Phi|X_t$ is the given $\mathbb{P}_{n-1}$-bundle structure for $t \neq 0$. Moreover there is a locally free sheaf $E$ on $\mathcal{Y}$ such that $\mathcal{X} \simeq \mathbb{P}(E)$ and $E|Y_t \simeq T_{Y-t}$ for all $t \neq 0$.

**Proof.** By the previous proposition it suffices to show that $\phi_0 = \Phi|X_0$ is equidimensional.

1. First suppose that $\dim X_0 \leq 5$. Then $1 \leq \dim Y_0 \leq 3$. The case $\dim Y_0 = 1$ is trivial. If $\dim Y_0 = 2$, then all fibers must have codimension 2, because $\phi_0$ does not contract a divisor (the relative Picard number being 1). If $\dim Y_0 = 3$, then by [AW97], (5.1), we cannot have a 3-dimensional fiber. Since again there is no 4-dimensional fiber, $\phi_0$ must be equidimensional also in this case.
(2) If \( b_{2j}(Y_t) = 1 \) for some \( t \) and all \( 1 \leq j \leq \frac{n}{2} \), then \( b_{2k}(X_t) = k + 1 \) for \( k < n \) and we may simply argue as in Sublemma 5.4 to conclude that \( \phi_0 \) is equidimensional (the smoothness of \( Y_0 \) is not essential in the reasoning of Sublemma 5.4).

\[ \square \]

6. DEGENERATIONS OF \( TP_n \)

In view of Theorem 5.1, we can ask the question which bundles can occur as degenerations of \( TP_n \), i.e., for which rank-\( n \) bundles \( V \) on \( P_n \) there exists a rank-\( n \) bundle \( V \) on \( P_n \times \Delta \) such that

\[
\forall_t := \mathcal{V}|\mathbb{P}_n \times \{t\} \cong \begin{cases} T_{P_n}, & \text{for } t \neq 0, \\ V, & \text{for } t = 0. \end{cases}
\]

In the case that \( n \geq 3 \) is odd, it was already observed by Hwang in [Hw06] that one can easily construct a nontrivial degeneration of \( TP_n \) as follows: We consider the null correlation bundle on \( P_n \), which is a rank-\( (n-1) \) bundle \( N \) on \( P_n \) given by a short exact sequence

\[
0 \to N \to TP_n(-1) \to O_{P_n}(1) \to 0.
\]

(cf. [OSS80], (I.4.2)). The existence of this sequence now implies that \( TP_n \) can be degenerated to \( N(1) \oplus O_{P_n}(2) \).

When \( n \) is even, matters become more complicated, but we can still obtain nontrivial degenerations:

**Proposition 6.1.** Let \( n \geq 2 \). Then there exists a rank-\( n \) bundle \( \mathcal{V} \) on \( P_n \times \Delta \) such that \( \forall_t \cong TP_n \) for \( t \neq 0 \) and \( h^0(\mathcal{V}_0(-2)) = 1 \), so in particular \( \mathcal{V}_0 \neq TP_n \).

**Proof.** We construct an inclusion of vector bundles

\[
A: \Omega^1_{P_n \times \Delta/\Delta}(2) \oplus O_{P_n \times \Delta} \to O_{P_n \times \Delta}(1)^{\oplus(n+1)} \oplus \Omega^1_{P_n \times \Delta/\Delta}(2)
\]

via a family \( A = (A_t)_{t \in \Delta} \) of matrices

\[
A_t = \begin{pmatrix} \alpha_t & \beta_t \\ \gamma_t & \delta_t \end{pmatrix}
\]

of sheaf homomorphisms

\[
\alpha_t: \Omega^1_{P_n}(2) \to O_{P_n}(1)^{\oplus(n+1)}, \quad \beta_t: O_{P_n} \to O_{P_n}(1)^{\oplus(n+1)},
\]

\[
\gamma_t: \Omega^1_{P_n}(2) \to \Omega^1_{P_n}(2), \quad \delta_t: O_{P_n} \to \Omega^1_{P_n}(2),
\]

which we define as follows: We take \( \alpha_t \) and \( \beta_t \) to be the natural inclusions coming from the Euler sequence and its dual, where we choose the coordinates on \( P_n \) such that

\[
\beta_t(O_{P_n}) \not\subseteq \alpha_t(\Omega^1_{P_n}(2)).
\]

This implies that the map

\[
\alpha_t \oplus \beta_t: \Omega^1_{P_n}(2) \oplus O_{P_n} \to O_{P_n}(1)^{\oplus(n+1)}
\]

is generically surjective. Since $\Omega^1_{\mathbb{P}_n}(2) \simeq \Lambda^{n-1}(T_{\mathbb{P}_n}(-1))$ is globally generated, a general section in $H^0(\Omega^1_{\mathbb{P}_n}(2))$ has only finitely many zeroes. Since $\Omega^1_{\mathbb{P}_n}(2)$ is homogeneous, we can thus choose the map $\delta_t$ in such a way that its zeroes are disjoint from the locus where $\alpha_t \oplus \beta_t$ is not surjective. Finally we let $\gamma_t = t \cdot \text{id}$.

Now in order to show that $A$ is an inclusion of vector bundles, we need to show that for any point $(p, t) \in \mathbb{P}_n \times \Delta$, the matrix

$$A_t(p) = \begin{pmatrix} \alpha_t(p) \\ \beta_t(p) \\ \gamma_t(p) \\ \delta_t(p) \end{pmatrix} \in C^{(2n+1) \times (n+1)}$$

has rank $n+1$. For semicontinuity reasons, shrinking $\Delta$ if necessary, we can assume $t = 0$, then the rank condition follows easily from the choice of $\alpha_0, \beta_0, \gamma_0, \delta_0$.

We now let

$$\mathcal{V} := \text{coker } A.$$

It remains to investigate the properties of the bundles $\mathcal{V}_t := \mathcal{V}|_{\mathbb{P}_n \times \{t\}}$. For each $t \in \Delta$, we have an exact sequence of vector bundles

$$0 \to \Omega^1_{\mathbb{P}_n}(2) \oplus \mathcal{O}_{\mathbb{P}_n} \to \mathcal{O}_{\mathbb{P}_n}(1)^{\oplus (n+1)} \oplus \Omega^1_{\mathbb{P}_n}(2) \to \mathcal{V}_t \to 0. \tag{6}$$

We want to calculate $H^q(\mathcal{V}_t(-1 - k))$ for $k = 0, \ldots, n$. From the Bott formula we obtain for $(k, q) \in \{0, \ldots, n\}^2$:

$$h^q(\Omega^1_{\mathbb{P}_n}(1 - k)) = \begin{cases} 1, & \text{for } (k, q) = (1, 1), \\
0, & \text{otherwise.} \end{cases}$$

Now if we tensorize (6) with $\mathcal{O}_{\mathbb{P}_n}(-1 - k)$, take the long exact cohomology sequence and observe that $H^q(\delta_0) = \text{id}$ for every $q$, we get for $(k, q) \in \{0, \ldots, n\}^2$:

$$h^q(\mathcal{V}_0(-1 - k)) = \begin{cases} n + 1, & \text{for } (k, q) = (0, 0), \\
1, & \text{for } (k, q) \in \{(1, 0), (1, 1), (n, n - 1)\}, \\
0, & \text{otherwise.} \end{cases}$$

Similarly, if we observe that $H^q(\delta_t) = \text{id}$ for $t \neq 0$, we obtain for $t \neq 0$, $(k, q) \in \{0, \ldots, n\}^2$:

$$h^q(\mathcal{V}_0(-1 - k)) = \begin{cases} n + 1, & \text{for } (k, q) = (0, 0), \\
1, & \text{for } (k, q) \in \{(n, n - 1)\}, \\
0, & \text{otherwise.} \end{cases}$$

The proposition now follows from Lemma 6.2. \(\square\)
Lemma 6.2. Let $V$ be a vector bundle on $\mathbb{P}_n$ such that for any $(k, q) \in \{0, \ldots, n\}^2$, we have

$$h^q(V(-1-k)) = \begin{cases} n+1, & \text{for } (k, q) = (0, 0), \\ 1, & \text{for } (k, q) = (n, n-1), \\ 0, & \text{otherwise}. \end{cases}$$

Then $V \simeq T_{\mathbb{P}_n}$.

Proof. We consider the Beilinson spectral sequence for the bundle $V(-1)$, which has $E_1$-term

$$E_1^{pq} = H^q(V(-1+p)) \otimes \Omega_{\mathbb{P}_n}^{-p}$$

(cf. [OSS80], (II.3.1.3)).

By assumption, $E_1^{pq} = 0$ for $(p, q) \notin \{(0, 0), (-n, n-1)\}$ and

$$E_1^{0,0} = \mathcal{O}_{\mathbb{P}_n}^{\oplus(n+1)}, \quad E_1^{-n,n-1} = \mathcal{O}_{\mathbb{P}_n}(-1).$$

In particular, the only nonzero differential occurs at the $E_2$-term, namely

$$d_n^{-n,n-1}: E_n^{-n,n-1} \to E_n^{0,0}.$$ 

Since $E_2^{pq} = 0$ for $p+q \neq 0$ and $E_{\infty}^{-p,p}$ are the quotients of a filtration of $V(-1)$, the differential $d_n^{-n,n-1}$ induces a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}_n}(-1) \to \mathcal{O}_{\mathbb{P}_n}^{\oplus(n+1)} \to V(-1) \to 0. \quad (7)$$

Now since $V$ is locally free, the map $d_n^{-n,n-1}$ cannot have zeroes, so $(7)$ must be an Euler sequence, whence $V(-1) \simeq T_{\mathbb{P}_n}(-1)$. \hfill \square

7. Deformations II: Positive Irregularity

A homogeneous compact contact Kähler manifold $X$ of dimension $2n + 1$ with $b_2(X) \geq 2$ is either $\mathbb{P}(T_{\mathbb{P}_n})$ or a product $A \times \mathbb{P}_n$ with a torus $A$ of dimension $n + 1$. Here we study in general the Kähler deformations of $A \times \mathbb{P}_n$, where $A$ is an $m$-dimensional torus.

Theorem 7.1. Let $\pi: \mathcal{X} \to \Delta$ be a family of compact manifolds over the unit disc $\Delta \subset C$. Assume $X_t \simeq A_t \times \mathbb{P}_n$ for $t \neq 0$, where $A_t$ is a torus of dimension $m$. If $X_0$ is in class $\mathcal{C}$, then the relative Albanese morphism realises $\mathcal{X}$ as a submersion $\alpha: \mathcal{X} \to \mathcal{A}$, where $\pi: \mathcal{A} \to \Delta$ is torus bundle such that $\pi^{-1}(t) \simeq A_t$ for $t \neq 0$. Moreover there is a locally free sheaf $\mathcal{E}$ over $\mathcal{A}$ such that $\mathcal{X} = \mathbb{P}(\mathcal{E})$, $X_t \simeq \mathbb{P}(\mathcal{E}_t)$ for all $t$ and $\mathcal{E}|_{A_t} \simeq \mathcal{O}_{A_t}^{\oplus m+1}$ for $t \neq 0$.

Proof. Let $m = \dim A_t = q(X_t)$ for $t \neq 0$. Hodge decomposition on $X_0$ gives $q(X_0) = m$. Let

$$\alpha: \mathcal{X} \to \mathcal{A}$$

be the relative Albanese map. Then $\mathcal{A} \to \Delta$ is a torus bundle and

$$\alpha_t = \alpha|_{X_t}: X_t \to A_t$$
is the Albanese map for all $t$. Since $\alpha_t$ is surjective for all $t \neq 0$, the map $\alpha$ is surjective, too, and so is $\alpha_0$. We may choose relative vector fields

$$v_1, \ldots, v_m \in H^0(\mathcal{X}, T_{\mathcal{X}/\Delta}),$$

such that for all $t$, the push-forwards $(\alpha_t)_*(v_i|_{X_t})$ form a basis of $H^0(A_t, T_{A_t})$. Since singular fibers are mapped to singular fibers by automorphisms of $X_0$, it follows that the singular locus $S$ of $\alpha_0$, i.e., the set of points $a \in A_0$ such that the fiber over $a$ is singular, is invariant in $A_0$ under the automorphism group. Hence $S = \emptyset$, so that $\alpha_0$ is a submersion like all the other maps $\alpha_t$. Therefore $\alpha$ is a bundle, with fibers $\mathbb{P}^n$ over $\Delta^*$. The global rigidity of the projective space [Si89] applied to local sections of $A$ over $\Delta$, passing through $A_0$, implies that all fibers of $\alpha$ are $\mathbb{P}^n$. The existence of $E$ follows from [El82], (4.3).

□

Remark 7.2. Popovici [Po09] has shown that any global deformation of projective manifolds is in class $C^1$, so that the assumption in Theorem 6.1 that $X_0$ is in class $C^1$ can be removed in case $X_t$ is projective. The Kähler version of Popovici’s theorem is still open.

Example 7.3. We cannot conclude in Theorem 6.1 that $X_0 \cong A_0 \times \mathbb{P}_n$, even if $m = n = 1$. Take e.g. a rank-2 vector bundle $F$ over $\mathbb{P}_1 \times \Delta$ such that $F|_{\mathbb{P}_1 \times \{t\}} = O^2$ for $t \neq 0$ and $F|_{\mathbb{P}_1 \times \{0\}} = O(1) \oplus O(-1)$. Let $\eta: A \to \mathbb{P}_1$ be a two-sheeted covering from an elliptic curve $A$ and set $\mathcal{E} = (\eta \times \text{id})^*(F)$. Then $\mathcal{X} = \mathbb{P}(\mathcal{E})$ is a family of compact surfaces $X_t$ such that $X_t = A \times \mathbb{P}_1$ for $t \neq 0$ but $X_0$ is not a product. Notice also that $X_0$ is not almost homogeneous.

It is a trivial matter to modify this example to obtain a map to a 2-dimensional torus which is a product of elliptic curves. Therefore the limit of a Kähler contact manifold with positive irregularity might not be a contact manifold again.

Corollary 7.4. Assume the situation of Theorem 6.1. Then the following assertions are equivalent.

1. $X_0 \cong A_0 \times \mathbb{P}_n$.
2. $\mathcal{E}_0$ is semi-stable for some Kähler class $\omega$.
3. $X_0$ is homogeneous.
4. $X_0$ is almost homogeneous.

Proof. (1) implies (2). Under the assumption of (1), there is a line bundle $L$ on $A_0$ such that $\mathcal{E}_0 \cong L^\oplus n+1$. Hence $\mathcal{E}$ is semi-stable for actually any choice of $\omega$.

(2) implies (3). From the semi-stability of $\mathcal{E}_0$ and $h^0(\mathcal{E}_0) \geq n + 1$, it follows easily that $\mathcal{E}_0$ is trivial and that $X_0$ is homogeneous as product $A_0 \times \mathbb{P}_n$. In fact, choose $n + 1$ sections of $\mathcal{E}_0$ and consider the induced map $\mu: O_{A_0}^{n+1} \to \mathcal{E}_0$. By the stability of $\mathcal{E}_0$, the map $\mu$ is generically surjective. Hence $\det \mu \neq 0$, hence an isomorphism, so that $\mu$ itself is an isomorphism.
The implication “(3) implies (4)” is obvious. (4) implies (1). Consider the tangent bundle sequence

\[ 0 \rightarrow T_{X_0/A_0} \rightarrow T_{X_0} \rightarrow \alpha_0^*(T_{A_0}) \rightarrow 0. \]

Since \( X_0 \) is almost homogeneous, all vector fields on \( A_0 \) must lift to \( X_0 \). Consequently the connecting map

\[ H^0(X_0, \pi^*(T_{A_0})) \rightarrow H^1(X_0, T_{X_0/A_0}) \]

vanishes, and therefore the tangent bundle sequence splits. Let \( \mathcal{F} = \alpha_0^*(T_{A_0}) \). Then \( \mathcal{F} \subset T_{X_0} \) is clearly a foliation and it has compact leaves (the limits of tori in \( A_t \times \mathbb{P}^n \)). By [Hoe07], 2.4.3, there exists an equi-dimensional holomorphic map \( f : X_0 \rightarrow Z_0 \) to a compact variety \( Z_0 \) such that the set-theoretical fibers \( F \) of \( f \) are leaves of \( \mathcal{F} \). Since the fibers \( F \) have an étale map to \( A_0 \), they must be tori again. It is now immediate that \( Z_0 = \mathbb{P}^n \) and \( X_0 = A_0 \times \mathbb{P}^n \). □

**Corollary 7.5.** Assume in Theorem 6.1 that \( m = 2 \) and \( n = 1 \). Then either

\[ X_0 \cong A_0 \times \mathbb{P}^1, \quad \text{or} \quad X_0 = \mathbb{P}(E_0) \]

and one of the following holds:

1. There is a torus bundle \( p : A_0 \rightarrow B_0 \) to an elliptic curve \( B_0 \) and the rank-2 bundle \( E_0 \) on \( A_0 \) sits in an extension

\[ 0 \rightarrow p^*(L_0) \rightarrow E_0 \rightarrow p^*(L_0^*) \rightarrow 0 \]

with \( \deg L_0 > 0 \).

2. The rank 2-bundle \( E_0 \) sits in an extension

\[ 0 \rightarrow S \rightarrow E_0 \rightarrow I_Z \otimes S^* \rightarrow 0 \]

with an ample line bundle \( S \) and a finite non-empty set \( Z \) of degree \( \deg Z = c_1(S)^2 \).

**Proof.** By Corollary 6.4 we may assume that \( E_0 \) is not semi-stable for some (or any) Kähler class \( \omega \). Let \( S \) be a maximal destabilising subsheaf, which is actually a line bundle, leading to an exact sequence

\[ 0 \rightarrow S \rightarrow E_0 \rightarrow I_Z \otimes S^* \rightarrow 0. \]

Here \( Z \) is a finite set or empty. Taking \( c_2 \) and observing that \( c_2(E_0) = 0 \) gives

\[ c_1(S)^2 = \deg Z. \]

The destabilisation property reads

\[ c_1(S) \cdot \omega > 0. \]

Since \( h^0(E_0) \geq 2 \), we deduce that \( h^0(S) \geq 2 \), in particular, \( S \) is nef, \( S \) being maximal destabilizing.

If \( S \) is ample, there is nothing more to prove, hence we may assume that \( S \) is not ample. \( S \) being nef, \( c_1(S)^2 = 0 \) and \( S \) defines a submersion \( p : A_0 \rightarrow \)
$B_0$ to an elliptic curve $B_0$ such that there exists an ample line bundle $\mathcal{L}_0$ with $\mathcal{S} = p^*(\mathcal{L}_0)$. Therefore we obtain an extension

$$0 \to p^*(\mathcal{L}_0) \to \mathcal{E}_0 \to p^*(\mathcal{L}_0^\vee) \to 0,$$

as required.

\[ \square \]

**Remark 7.6.** The second case in Corollary 7.5 really occurs. Take a finite map $f: \mathcal{A} \to \mathbb{P}_2 \times \Delta$ over $\Delta$ and a rank-2 bundle $\mathcal{F}$ on $\mathbb{P}_2 \times \Delta$ such that $\mathcal{F}|\mathbb{P}_2 \times \{t\} \simeq \mathcal{O}^2$ for $t \neq 0$ and such that $\mathcal{F}_0$ is not trivial. For examples see e.g. [Sc83]. Now $\mathcal{E} = f^*(\mathcal{F})$ gives an example we are looking for.

**Corollary 7.7.** Assume in Theorem 6.1 that $m = 2$ and $n = 1$. Let $\Phi: T_{X/\Delta} \to -K_X/2$ be a morphism such that $\Phi|X_t = \phi_t$ is a contact morphism (i.e., defines a contact structure) for $t \neq 0$. Suppose that

$$\phi_0: T_{X_0} \to -K_{X_0}/2$$

does not vanish identically. Then the kernel $\mathcal{F}_0$ of $\phi_0$ is integrable (in contrast to the maximally non-integrable bundle $\mathcal{F}_t$).

**Proof.** We consider a family $(\phi_t)$ of morphisms

$$\phi_t: T_{X_t} \to \mathcal{H}_t$$

such that $\phi_t$ is a contact form for $t \neq 0$ and $-K_{X_t} = 2\mathcal{H}_t$. Consider the (torsion free) kernel $\mathcal{F}_0$ of $\phi_0$. We need to show that the induced map

$$\mu: (\bigwedge^2 \mathcal{F}_0)^{**} = \det \mathcal{F}_0 \to \mathcal{H}_0,$$

vanishes. Since the determinant of the kernel $\mathcal{F}_t$ of $\phi_t$ is isomorphic to $\mathcal{H}_t$, we conclude that

$$\det \mathcal{F}_0 \simeq \mathcal{H}_0 \otimes \mathcal{O}_{X_0}(E) \quad \text{(⋆)}$$

with an effective (possibly vanishing) divisor $E$ on $X_0$. Now the induced map

$$\mu: \det \mathcal{F}_0 \to \mathcal{H}_0$$

must have zeroes, otherwise $X_0$ would be a contact manifold, hence $X_0 \simeq A_0 \times \mathbb{P}_1$. Thus $\mu = 0$ by (⋆), and $\mathcal{F}_0$ is integrable. \[ \square \]
REFERENCES

[AW97] Andreatta, M.; Wiśniewski, J.: A view on contractions of higher-dimensional varieties. Algebraic geometry–Santa Cruz 1995, 153–183, Proc. Sympos. Pure Math., 62, Part 1, Amer. Math. Soc., Providence, RI, 1997.

[BCHM10] Birkar, C.; Cascini, P.; Hacon, C.D.; McKernan, J.: Existence of minimal models for varieties of log general type. J. Amer. Math. Soc. 23 (2010), no. 2, 405–468.

[BO74] Barth, W.; Oeljeklaus, E.: Über die Albanese-Abbildung einer fast-homogenen Kählermannigfaltigkeit. Math. Ann. 211 (1974), 47–62.

[Be98] Beauville, A.: Fano contact manifolds and nilpotent orbits. Comm. Math. Helv. 73 (1998), 566–583.

[BS95] Beltrametti, M.; Sommese, A.J.: The adjunction theory of complex projective varieties. de Gruyter, Berlin, 1995.

[BR62] Borel, A.; Remmert, R.: Über kompakte homogene Kählersche Mannigfaltigkeiten. Math. Ann. 145 (1961/1962), 429–439.

[De02] Demailly, J.P.: On the Frobenius integrability of certain holomorphic p-forms. Complex geometry (Göttingen, 2000), 93–98, Springer, Berlin, 2002.

[El82] Elencwajg, G.: The Brauer groups in complex geometry. Lecture Notes in Math., 917, 222–230, Springer, Berlin-New York, 1982.

[Fu85] Fujita, T.: On polarized manifolds whose adjoint bundles are not semipositive. Adv. Stud. Pure Math. 10 (1985), 167–178.

[Fu90] Fujita, T.: Classification theories of polarized varieties. London Math. Soc. Lecture Notes Ser., 155. Cambridge Univ. Press, 1990.

[Go83] Goldstein, N.: Ampleness in complex homogeneous spaces and a second Lefschetz theorem. Pacific J. Math. 106 (1983), no. 2, 271–291.

[Hoe07] Höring, A.: Uniruled varieties with split tangent bundle. Math. Z. 256 (2007), no. 3, 465–479.

[Hw97] Hwang, J.M.: Rigidity of homogeneous contact manifolds under Fano deformation. J. reine u. angew. Math. 486 (1997), 153–163.

[Hw06] Hwang, J.M.: Rigidity of rational homogeneous spaces. International Congress of Mathematicians. Vol. II, 613–626, Eur. Math. Soc., Zürich, 2006.

[Hw10] Hwang, J.M.: Deformations of the space of lines on the 5-dimensional hyperquadric. Preprint, 2010.

[Ke01] Kebekus, S.: Lines on contact manifolds. J. Reine Angew. Math. 539 (2001), 167–177.

[KM92] Kollár, J.; Mori, S.: Classification of three-dimensional flips. J. Am. Math. Soc. 5 (1992), No. 3, 533–703.

[KMM87] Kawamata, Y.; Matsuda, K.; Matsuki, K.: Introduction to the minimal model problem. Algebraic geometry, Sendai, 1985, 283–360, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.

[KPSW00] Kebekus, S.; Peternell, Th.; Sommese, A.J., Wiśniewski, J.: Projective contact manifolds. Inv. math. 142 (2000), 1–15.

[LS94] LeBrun, C.; Salamon, S.: Strong rigidity of positive quaternion-Kähler manifolds. Inv. math. 118 (1994), 109–132.

[Le95] LeBrun, C.: Fano manifolds, contact structures, and quaternionic geometry. Internat. J. Math. 6 (1995), no. 3, 419–437.

[MU83] Mukai, S.; Umemura, H.: Minimal rational threefolds. Algebraic geometry (Tokyo/Kyoto, 1982), 490–518, Lecture Notes in Math., 1016, Springer, Berlin, 1983.

[Na87] Nakayama, N.: The lower semicontinuity of the plurigenera of complex varieties. Algebraic geometry, Sendai, 1985, 551–590, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.
[OSS80] Okonek, C.; Schneider, M.; Spindler, H.: Vector bundles on complex projective spaces. Progress in Mathematics, 3. Birkhäuser, Boston, Mass., 1980.

[Po68] Potters, J.: On almost homogeneous compact complex surfaces. Inv. math. 8 (1968), 244–268.

[Po09] Popovici, D.: Limits of projective manifolds under holomorphic deformations. arXiv AG/09102032v1

[PP10] Pasquier, B.; Perrin, N.: Local rigidity of quasi-regular varieties. Math. Z. 265 (2010), No. 3, 589–600.

[PW95] Peternell, Th.; Wiśniewski, J.: On stability of tangent bundles of Fano manifolds with $b_2 = 1$. J. Algebraic Geom. 4 (1995), no. 2, 363–384.

[Sc83] Schafft, U.: Nichtsepariertheit instabiler Rang-2-Vektorbündel auf $\mathbb{P}_2$. J. reine angew. Math. 338 (1983), 136–143.

[Si89] Siu, Y.T.: Nondeformability of the complex projective space. J. Reine Angew. Math. 399 (1989), 208–219 (errata: J. Reine Angew. Math. 431 (1992), 65–74).

[Sum97] Summerer, A.: Globale Deformationen der Mannigfaltigkeiten $\mathbb{P}(T_{\mathbb{P}_2})$ und $\mathbb{P}_1 \times \mathbb{P}_2$. Ph.D. thesis, Bayreuth, 1997. Bayreuth. Math. Schr. No. 52 (1997), 173–238.

[Wi91a] Wiśniewski, J.: On Fano manifolds of large index. manuscr. math. 70 (1991), 145–152.

[Wi91b] Wiśniewski, J.: On deformation of nef values. Duke Math. J. 64 (1991), No. 2, 325–332.

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