Supersymmetric Biorthogonal Quantum Systems

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August 11, 2018

Abstract

We discuss supersymmetric biorthogonal systems, with emphasis given to the periodic solutions that occur at spectral singularities of PT symmetric models. For these periodic solutions, the dual functions are associated polynomials that obey inhomogeneous equations. We construct in detail some explicit examples for the supersymmetric pairs of potentials \( V_{\pm}(z) = -U(z)^2 \pm z \frac{d}{dz} U(z) \) where \( U(z) \equiv \sum_{k>0} \nu_k z^k \).

In particular, we consider the cases generated by \( U(z) = z \) and \( z/(1-z) \). We also briefly consider the effects of magnetic vector potentials on the partition functions of these systems.

1 Introduction

There has been some recent theoretical interest in non-hermitian Schrödinger equations, in the guise of “PT symmetric theories” [1]. Several previous authors have considered supersymmetry in this context [2]. Here, we consider a few elementary soluble examples and explore them in some detail. We believe exact solvability permits the underlying structure to be appreciated more completely. Some recent papers [3, 7] have also touched on the relevance of biorthogonal systems [4] for PT symmetric models. As in [7], we wish again to stress the importance of such systems, and their generality, only here in the context of supersymmetric examples. We are not aware of any previous systematic discussion of supersymmetric biorthogonal systems along the lines of that given here.

We consider models whose Hamiltonians are of the form

\[
H_{\pm} = \left( z \frac{d}{dz} + \nu \mp U \right) \left( z \frac{d}{dz} + \nu \pm U \right) = \left( z \frac{d}{dz} + \nu \right)^2 - U(z)^2 \pm z \frac{d}{dz} U(z) \tag{1}
\]

\[
U(z) \equiv \sum_{k>0} \nu_k z^k \tag{2}
\]

where the exponents \( k \) in the “superpotential” \( U \) are of the same sign, and where \( \nu \) and \( \nu_k \) have arbitrary values. We look for eigenfunctions and associated functions that have particular analytic behavior near \( z = 0 \). The connection to PT symmetric theories is achieved by writing \( z = me^{ix} \), as explained in [7].
2 General Theory for $\nu = 0$

First we consider $\nu = 0$. For bounded functions of $x \in (\infty, +\infty)$, with $z = me^{ix}$, the spectrum of any Hamiltonian of the form \[ H = \left( z \frac{d}{dz} \right)^2 + \sum_{k>0} \mu_k z^k \] is known to be the real, positive half-line, for any choice of $\mu$s such that $\sum_{k>0} |\mu_k| < \infty$, and not just for those $\mu$s which are real or $x$-translationally equivalent to real values. So $\mathbb{P}$T symmetry is not required for real energy eigenvalues in these models [6]. Here, we will restrict our attention to $2\pi$-periodic functions of $x$, with $z = me^{ix}$, and their duals, to obtain a discrete subset of real energy eigenvalues, namely just $\{E_n = n^2 \mid n = 0, 1, \cdots \}$.

In this situation the general theory for supersymmetric pairs of non-hermitian Hamiltonians goes as follows. For a given $U(z)$ we may construct pairs of finite polynomials in $z^{-1}$, 

$$\chi^\pm_n(z) = \frac{1}{z^n} \sum_{j=0}^n c^\pm_{n,j} z^j$$

that satisfy the inhomogeneous equations,

$$\left( z \frac{d}{dz} \pm U(z) \right) \chi^\pm_n(z) + n \chi^\mp_n(z) = \Lambda^\pm_n(z) = \sum_{k>0} \lambda^\pm_n(k) z^k$$

where $U(z)$ and $\Lambda^\pm_n(z)$ are analytic about the origin, at which point they all vanish. A priori the $\Lambda^\pm_n(z)$ need not be given, for a given $U(z)$ they may be determined along with $\chi^\pm_n(z)$. Clearly, $\chi^\pm_n(z) \sim_{n \to 0} c^\pm_{n,0}/z^n$ and, by convention, we normalize so that the coefficient of the most negative power of $z$ is just $c^\pm_{n,0} = 1$. This choice also fixes the normalization of $\Lambda^\pm_n$. For a given $U(z)$ the four functions $\chi^\pm_n(z)$ and $\Lambda^\pm_n(z)$ are now completely specified.

Alternatively, we construct pairs of functions $\psi^\pm_n(z)$ analytic about the origin, hence given by series in non-negative integer powers of $z$, that satisfy the equations

$$\left( z \frac{d}{dz} \pm U(z) \right) \psi^\pm_n(z) = n \psi^\mp_n(z)$$

with $\psi^\pm_n(z) \sim_{n \to 0} z^n$. Usually these are infinite series, but again, for a given $U(z)$, both functions $\psi^\pm_n(z)$ are now completely specified.

It follows that the $\psi^\pm_n(z)$ are eigenfunctions of $H_\pm$ with eigenvalues $n^2$,

$$H_\pm \psi^\pm_n(z) = n^2 \psi^\pm_n(z)$$

$$H_\pm = \left( z \frac{d}{dz} \mp U(z) \right) \left( z \frac{d}{dz} \pm U(z) \right) = \left( z \frac{d}{dz} \right)^2 - U(z)^2 \pm z \frac{d}{dz} U(z)$$

while the associated polynomials obey the inhomogeneous equations

$$(H_\pm - k^2) \chi^\pm_k(z) = \left( z \frac{d}{dz} \mp U(z) \right) \Lambda^\pm_n(z) - k \Lambda^\pm_n(z)$$

Note the RHS of this last equation involves only positive powers, $z^n$ for $n \geq 1$. Moreover, it also follows that these functions are biorthonormal systems such that

$$\frac{1}{2\pi i} \int \frac{dz}{z} \chi^\pm_k(z) \psi^\pm_n(z) = \delta_{kn}, \quad \sum_{n=0}^{\infty} \chi^\pm_k(w) \psi^\pm_n(z) = \frac{1}{1 - \frac{w}{z}}$$

These relations encode orthonormality and completeness on analytic functions. For more or less obvious reasons, we will call the full set of functions $\{\chi^\pm_j, \psi^\pm_k, \chi^\pm_n\}$ an analytic, supersymmetric biorthogonal system corresponding to the given function $U(z)$. 


The dual polynomials and their inhomogeneities. We start the construction with the dual polynomial and explicitly compute the first few. This serves to illustrate how the various functions are uniquely determined, given that \( \chi^\pm_n(z) \) is not usually fruitful. – Morse and Feshbach \[4\] p 931.

We determine all the coefficients in the expansion for \( \chi^\pm_n(z) \), and \( \Lambda^\pm_n(z) \) to eliminate all \( z^{-k} \) terms on the LHS of (5), for \( k = 0, 1, \ldots, n \), and then we sweep all the remaining positive powers of \( z \) into the \( \Lambda^\pm_n(z) \). We find

\[
\begin{align*}
\chi_0^\pm(z) &= 1, \\
\chi_1^\pm(z) &= z \pm v_1, \\
\chi_2^\pm(z) &= \frac{1}{z^2} \pm \frac{1}{5} v_1 \frac{1}{z} \pm \frac{1}{3} v_2 - \frac{1}{30} v_1^2, \\
\chi_3^\pm(z) &= \left( \frac{1}{z^3} \pm \frac{1}{5} v_1 \frac{1}{z^2} \pm \frac{1}{10} v_2 \frac{1}{z} - \frac{1}{5} v_1^2 \right) \frac{1}{z} \\
&\quad \pm \frac{1}{3} v_3 - \frac{1}{20} v_2 v_1 + \frac{1}{3} v_1^2 \right)
\end{align*}
\]

etc. When the sum \( \sum_{k>0} \) that defines \( U \) is finite, the process is clearly finite mathematics all the way, for any \( n \). Note that \( \chi_n^\pm(z) \) and \( \Lambda_n^\pm(z) \) are obtained from \( \chi_n^\pm(z) \) and \( \Lambda_n^\pm(z) \), or vice versa, just by flipping the signs of all the \( v \). Also note that all negative powers of \( z \) can be expressed as finite sums of \( \chi_n^\pm(z) | n \geq 0 \) or \( \Lambda_n^\pm(z) | n \geq 0 \).

More systematically, we solve (5) as follows. We impose the condition that the RHS involve only positive powers of \( z \) so as to obtain an inhomogeneity that will be orthogonal to the span of \( \{z^n \mid n \geq 0\} \). Under contour integration \[4\] (11).

This leads to \( n \) equations that fix the coefficients \( c_{n,k} \) for \( k = 1, \ldots, n \) in terms of \( c_{n,0} \), the latter being an overall choice of normalization. Thus

\[
\sum_{j=0}^{n} c_{n,j}^\pm (j - n) z^j = \sum_{j=0}^{n} v_k z^j + n \sum_{j=0}^{n} c_{n,j}^\pm z^j + \sum_{j=0}^{n} \lambda_n^\pm (k) z^{k+n}
\]

so

\[
\lambda_n^\pm (k) = \pm \sum_{j=0}^{n} v_k c_{n,k-j}^\pm - nc_{n,j}^\pm = 0 \quad \text{for} \quad j = 0, \ldots, n
\]

and

\[
\chi_n^\pm(z) = \pm \sum_{j=0}^{n} \psi_{k+n-j} c_{n,j}^\pm
\]

We determine all the coefficients in the \( \chi_n^\pm(z) \) from the pair of equations

\[
\left( \begin{array}{cc}
    n-j & -n \\
    -n & n-j
\end{array} \right) \left( \begin{array}{c}
    c_{n,j}^+ \\
    c_{n,j}^-
\end{array} \right) = \sum_{k=1}^{j} v_k \left( \begin{array}{c}
    c_{n,j-k}^+ \\
    c_{n,j-k}^-
\end{array} \right) \quad \text{for} \quad j = 0, \ldots, n
\]

where \( c_{n,0}^+ = 1 \). Now \( \det \left( \begin{array}{cc}
    n-j & -n \\
    -n & n-j
\end{array} \right) = j(2n-j) > 0 \) for \( 0 < j \leq n \), and

\[
\left( \begin{array}{cc}
    n-j & -n \\
    -n & n-j
\end{array} \right)^{-1} = \frac{1}{j(2n-j)} \left( \begin{array}{cc}
    j-n & n \\
    -n & j-n
\end{array} \right).
\]

So we have the recursion relations

\[
c_{n,j}^\pm = \frac{\pm 1}{j(2n-j)} \sum_{k=1}^{j} v_k \left( (j-n) c_{n,j-k}^\pm + nc_{n,j-k}^\pm \right)
\]

Each \( c_{n,j}^\pm \) depends on only the first \( j \) coefficients in the expansion for \( U(z) \), i.e. just on \( v_{k \leq j} \). From the \( c_{n,j}^\pm \) we then determine the \( \lambda_n^\pm (k) \) using (13). Note that \( \lambda_n^\pm (k) \neq 0 \) for \( k > n \) is possible here, depending on the

\[\text{So these systems are exceptions to the statement: “A direct and frontal attack on the problem of determining [the dual polynomials] is not usually fruitful.” – Morse and Feshbach 4 p 931.}\]
values of the \( \nu \). The \( \lambda^\pm_n (k) \) will depend on all the \( \nu \), in general, with \( k \) taking on all values up to and including the highest power of \( z \) appearing in \( U \).

For example, for \( n = 2 \):

\[
\begin{align*}
c_{2,0}^+ &= 1, \quad c_{2,1}^+ = \frac{1}{3} v_1, \quad c_{2,2}^+ = \frac{1}{2} v_2 - \frac{1}{6} v_1^2, \\
\pm \lambda_2^+ (k) &= v_{k+2} + v_{k+1} c_{2,1}^+ + v_k c_{2,2}^+ \\
&= v_{k+2} \pm \frac{1}{3} v_1 v_{k+1} + \left( \pm \frac{1}{2} v_2 - \frac{1}{6} v_1^2 \right) v_k
\end{align*}
\]

(15)

And for \( n = 3 \):

\[
\begin{align*}
c_{3,0}^+ &= 1, \quad c_{3,1}^+ = \frac{1}{5} v_1, \quad c_{3,2}^+ = \frac{1}{3} v_2 - \frac{1}{10} v_1^2, \quad c_{3,3}^+ = -\frac{1}{3} v_3 - \frac{3}{20} v_2 v_1 + \frac{1}{30} v_1^3, \\
\pm \lambda_3^+ (k) &= v_{k+3} + v_{k+2} c_{3,1}^+ + v_{k+1} c_{3,2}^+ + v_k c_{3,3}^+ \\
&= v_{k+3} \pm \frac{1}{5} v_1 v_{k+2} + \left( \pm \frac{1}{4} v_2 - \frac{1}{10} v_1^2 \right) v_{k+1} + \left( \pm \frac{1}{3} v_3 - \frac{3}{20} v_2 v_1 + \frac{1}{30} v_1^3 \right) v_k
\end{align*}
\]

(16)

Hence the table given earlier.

**Energy eigenfunctions**  The energy eigenfunctions, when written as series,

\[
\psi^\pm_n (z) = z^n \sum_{j=0}^{\infty} a^\pm_{n,j} z^j
\]

(19)

can be determined just as the dual polynomials were by direct solution of (6), or else the eigenfunctions can be determined by imposing the bi-orthonormalizations in (10). As discussed in [7], these orthogonality conditions amount to a set of triangular equations which can always be solved, sequentially, for the \( a^\pm_{n,j} \) in terms of the \( c^\pm \)s. Namely

\[
a^\pm_{n,0} = \frac{1}{c^\pm_{n,0}}, \quad \sum_{j=0}^{k-n} c^\pm_{k,k-n-j} a^\pm_{n,j} = 0 \quad \text{for} \quad k > n
\]

(20)

The series for \( \psi^\pm_n \) is a development in the minors that invert these triangular equations. By considering all \( k > n \) in succession, we obtain all \( a^\pm_{n,j} \) in terms of \( c^\pm_{k,l} \), or vice versa.

For convenience, we again choose the normalizations \( c^\pm_{n,0} = 1 \). The results of the recursion relations (20) are the pair of correlated series

\[
\begin{align*}
\chi_n^\pm (z) &= \frac{1}{z^n} \left( 1 + c_{n,1}^\pm z + c_{n,2}^\pm z^2 + \cdots + c_{n,n}^\pm z^n \right) \\
\psi_n^\pm (z) &= z^n \begin{vmatrix}
1 & 1 & z^2 & 1 & \cdots \\
0 & 1 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{vmatrix}
\begin{vmatrix}
c_{n+1,1}^\pm & c_{n+1,2}^\pm & \cdots & c_{n+1,n}^\pm \\
1 & c_{n+1,2}^\pm & \cdots & c_{n+1,n}^\pm \\
0 & c_{n+1,2}^\pm & \cdots & c_{n+1,n}^\pm \\
1 & \cdots & \cdots & \cdots \\
\end{vmatrix}
\end{align*}
\]

(21)

\[
\begin{align*}
\chi_n^\pm (z) &= \frac{1}{z^n} \left( 1 + c_{n,1}^\pm z + c_{n,2}^\pm z^2 + \cdots + c_{n,n}^\pm z^n \right) \\
\psi_n^\pm (z) &= z^n \begin{vmatrix}
1 & 1 & z^2 & 1 & \cdots \\
0 & 1 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{vmatrix}
\begin{vmatrix}
c_{n+1,1}^\pm & c_{n+1,2}^\pm & \cdots & c_{n+1,n}^\pm \\
1 & c_{n+1,2}^\pm & \cdots & c_{n+1,n}^\pm \\
0 & c_{n+1,2}^\pm & \cdots & c_{n+1,n}^\pm \\
1 & \cdots & \cdots & \cdots \\
\end{vmatrix}
\end{align*}
\]

(22)
Alternatively the \( c^\pm \)'s may be expressed in terms of the \( a^\pm \)'s.

\[
\psi^\pm_n(z) = z^n \left( 1 + a^\pm_{n,1} z + a^\pm_{n,2} z^2 + a^\pm_{n,3} z^3 + a^\pm_{n,4} z^4 + \cdots \right) \quad (23)
\]

\[
\chi^\pm_n(z) = \frac{1}{z^n} \left( 1 - a^\pm_{n-1,1} z + a^\pm_{n-2,1} z^2 + a^\pm_{n-3,2} z^3 + a^\pm_{n-4,3} z^4 + \cdots \right)
\]

\[
= \frac{1}{z^n} \left( 1 - a^\pm_{n-1,1} z + a^\pm_{n-2,1} z^2 + a^\pm_{n-3,2} z^3 + a^\pm_{n-4,3} z^4 + \cdots \right)
\]

\[

\begin{bmatrix}
 1 & z & z^2 & z^3 & \cdots & z^{n-1} & z^n \\
 a^\pm_{n-1,1} & 1 & 0 & 0 & \cdots & 0 & 0  \\
a^\pm_{n-2,2} & a^\pm_{n-2,1} & 1 & 0 & \cdots & 0 & 0  \\
a^\pm_{n-3,3} & a^\pm_{n-3,2} & a^\pm_{n-3,1} & 1 & \cdots & 0 & 0  \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots  \\
a^\pm_{0,n} & a^\pm_{0,n-1} & a^\pm_{0,n-2} & a^\pm_{0,n-3} & \cdots & a^\pm_{0,1} & 1  \\
\end{bmatrix}
\]

\[

\begin{bmatrix}
 1 & 0 & \cdots & 0 \\
a^\pm_{n-1,1} & 1 & \cdots & 0  \\
a^\pm_{n-2,2} & a^\pm_{n-2,1} & 1 & \cdots  \\
a^\pm_{n-3,3} & a^\pm_{n-3,2} & a^\pm_{n-3,1} & 1 & \ddots  \\
 \vdots & \vdots & \vdots & \ddots & \ddots  \\
a^\pm_{0,n} & a^\pm_{0,n-1} & a^\pm_{0,n-2} & a^\pm_{0,n-3} & \cdots & a^\pm_{0,1} & 1  \\
\end{bmatrix}
\]

\[

\begin{bmatrix}
 1 & z & z^2 & z^3 & \cdots & z^{n-1} & z^n \\
 a^\pm_{n-1,1} & 1 & 0 & 0 & \cdots & 0 & 0  \\
a^\pm_{n-2,2} & a^\pm_{n-2,1} & 1 & 0 & \cdots & 0 & 0  \\
a^\pm_{n-3,3} & a^\pm_{n-3,2} & a^\pm_{n-3,1} & 1 & \cdots & 0 & 0  \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots  \\
a^\pm_{0,n} & a^\pm_{0,n-1} & a^\pm_{0,n-2} & a^\pm_{0,n-3} & \cdots & a^\pm_{0,1} & 1  \\
\end{bmatrix}
\]

**Orthogonality and Completeness**  With either pair of these correlated series, the orthogonality relations in (10) are easily checked. They amount to the obvious statements that

\[
\frac{1}{2\pi i} \int \frac{dz}{z} \chi^\pm_n(z) \psi^\pm_n(z) = 1, \quad \frac{1}{2\pi i} \int \frac{dz}{z} \chi^\pm_n(z) \psi^\pm_{n+k}(z) = 0 \quad \text{for} \quad k > 0 \quad (25)
\]

as well as more involved cancellations to show \( \frac{1}{2\pi i} \int \frac{dz}{z} \chi^\pm_n(z) \psi^\pm_n(z) = 0 \) for \( k > 0 \). In particular, there is a complete cancellation of all the contributions to this latter contour integral upon expansion of the \((k+1) \times (k+1)\) determinant that appears in the first line of (24). Exploiting the multi-linearity of the determinant, and performing the integrations entry by entry,
which vanishes since the 1st and the \((k+1)\)th rows are identical. Similarly

\[
\frac{1}{2\pi i} \oint \frac{dz}{z} \chi_{n+k}^\pm (z) \psi_n^\pm (z) = \begin{bmatrix}
\psi_{n+k}^\pm & \psi_{n+k-1}^\pm & \psi_{n+k-2}^\pm & \cdots & 1 \\
\psi_{n+k+1}^\pm & 0 & 0 & \cdots & 0 \\
\psi_{n+k+2}^\pm & \psi_{n+k+1}^\pm & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\psi_{n+k}^\pm & \psi_{n+k-1}^\pm & \psi_{n+k-2}^\pm & \cdots & 1 
\end{bmatrix} = 0
\]  

(27)

for \(k > 0\).

The correlations between coefficients in \((21)\) and \((22)\), or in \((23)\) and \((24)\), also imply the completeness relation in \((11)\) by guaranteeing that all terms of the form \(z^k w^{-l}\) for \(k \neq l\) cancel out in the sum

\[
\sum_{n=0}^{\infty} \chi_n^\pm (w) \psi_n^\pm (z) = 0
\]

which follow from \((20)\). Terms of the form \((z/w)^k\) as required to give \(\frac{1}{\pi i}\) are provided just by the leading terms in \((21)\) and \((22)\).

The coefficients of \(z^{n+k}\) in \(\psi_n^\pm (z)\) are again finite polynomials in the \(\psi_n\). While convergence of this series, as written, is certainly not obvious for arbitrary \(\psi_n\), it is clear that convergence can be determined on a case-by-case basis from the explicit form of the coefficients. Moreover, when the \(\psi_n\) of \(\psi_n\) are complete on the span of \(\psi_n\), just as all negative powers of \(\psi_n\) can be expressed as series of \(\psi_n\), and the completeness of \(\chi_n^\pm (z)\), where all positive powers of \(z\) can be expressed as finite sums of \(\chi_n^\pm (z)\).

Remarkably, the non-degenerate energy eigenfunctions \(\chi_n^\pm (z)\) just obtained turn out to be all of the eigenfunctions of \(H^\pm\) which are \(2\pi\)-periodic in \(x\), where \(z = me^{ix}\). Moreover, the fact that \(\psi_n^\pm (z)\) are indeed eigenfunctions, as given in \((17)\), can be deduced in a novel way from \((10)\), the biorthonormality of \(\{\chi_j^\pm (z), \psi_{\nu}^\mp (z)\}\), and the completeness of \(\{\psi_n^\pm (z)\}\) for analytic functions about the origin, as described in \((11)\). In fact, the argument given in \((17)\) can be adapted to the first-order equations. Completeness on analytic functions about \(z = 0\) allows us to write

\[
\left( z \frac{d}{dz} \mp U (z) \right) \psi_n^\pm (z) = \sum_{k=n}^{\infty} b_{n_k}^\pm \chi_k^\mp (z).
\]

(29)

Note the chosen interchange of \(\psi^\pm \leftrightarrow \psi^\mp\) upon LHS \(\leftrightarrow\) RHS. From this expansion and biorthonormality, we have \(b_{n_k}^\pm = \frac{1}{2\pi i} \oint \frac{dz}{z} \chi_k^\mp (z) \left( z \frac{d}{dz} \mp U (z) \right) \psi_n^\pm (z)\). But then, upon integrating by parts and using \((15)\) as well as the orthonormality relations, we also have \(\frac{1}{2\pi i} \oint \frac{dz}{z} \chi_k^\mp (z) \left( z \frac{d}{dz} \mp U (z) \right) \psi_n^\pm (z) = n \delta_{k,n}\). So \(b_{n_k}^\pm = n \delta_{k,n}\), and \((22)\) is obtained. Conversely, given \((10)\) and \((15)\), we may prove the orthogonality relations \(\oint \frac{dz}{z} \chi_k^\mp (z) \psi_n^\pm (z) = 0\) for \(k^2 \neq n^2 \neq 0\) just by inserting \(\left( z \frac{d}{dz} \mp U (z) \right)\) and integrating by parts. That is to say

\[
\oint \frac{dz}{z} \chi_k^\mp (z) \psi_n^\pm (z) = 0
\]

(30)

Thus \((n^2 - k^2) \oint \frac{dz}{z} \chi_k^\mp (z) \psi_n^\pm (z) = 0\).

\footnote{Note that \(z = 0\) is a regular singular point of \((20)\) and/or \((14)\), for any number and any values of the \(\psi_n\) such that \(\sum_{k>0} \nu_k z^k\) converges near the origin. In fact, \((22)\) is exactly the conventional series obtained by expanding about the regular singular point at \(z = 0\), albeit the series was obtained here in an unusual way from the properties of the dual space polynomials.}
3 Examples

As an explicit example, to parallel the discussion in [7], we note that the superpotential

\[ U(z) = \frac{z J_1(z)}{J_0(z)} \]

gives a simple quadratic potential for \( H_+ \)

\[ V_+ = -U^2 + z \frac{d}{dz} U = z^2 \]

but a much more complicated partner potential for \( H_- \)

\[ V_- = -U^2 - z \frac{d}{dz} U = -z^2 \left( 1 + 2 \left( \frac{J_1(z)}{J_0(z)} \right)^2 \right) \]

\[ = -z^2 - \frac{1}{2} z^4 - \frac{1}{8} z^6 - \frac{11}{384} z^8 + O(z^{10}) \]

The complexity of \( H_- \) suggests that we seek a simpler \( U \) to fully illustrate the general theory.

**Complex Morse potentials** Again referring to [7], we consider \( U(z) = \mu z \), hence

\[ H_{\pm} = \left( z \frac{d}{dz} + \nu \right)^2 \pm \mu z - \mu^2 z^2 \]

Note that for this simple example the solution of one Hamiltonian, say \( H_+ \), immediately gives the solution for the other, through the relations \( H_- [z] = H_+ [-z] \), \( \psi_n^-(z) \propto \psi_n^+(z) \). But this is not necessarily the most transparent way to write the solutions for \( H_- \).

When the vector potential is not present, \( \nu = 0 \), it may be best to simply note

\[ \left( z \frac{d}{dz} \mp z \right) \left( \sqrt{\frac{z}{2}} (I_{n-1/2}(z) \mp I_{n+1/2}(z)) \right) = n \left( \sqrt{\frac{z}{2}} (I_{n-1/2}(z) \pm I_{n+1/2}(z)) \right) \]

So then it is obvious that

\[ \psi_n^\pm(z) = Z_n^\pm \sqrt{\frac{z}{2}} (I_{n-1/2}(z) \mp I_{n+1/2}(z)) \]

are eigenfunctions of

\[ H_{\pm} = \left( z \frac{d}{dz} \right)^2 \pm z - z^2 = \left( z \frac{d}{dz} \mp z \right) \left( z \frac{d}{dz} \pm z \right) \]

with eigenvalues \( n^2 \) as given by

\[ H_{\pm} \psi_n^\pm(z) = n^2 \psi_n^\pm(z) \]

and with normalization constants \( Z_n^\pm \). Other ways to write the eigenfunctions for this example are:

\[ \psi_n^\pm(z) = Z_n^\pm \left( z \frac{d}{dz} \mp z + 1 + n \right) \frac{I_{n+1/2}(z)}{\sqrt{z}} \]

\[ \psi_n^\pm(z) = Z_n^\pm \left( \frac{z}{2} \right)^n \frac{\sum k=0 \Gamma(k + \frac{1}{2} + n)}{k! \Gamma(k + \frac{1}{2} + n)} \left( \frac{z}{2} \right)^{2k} \pm \sum k=0 \Gamma(k + \frac{1}{2} + n) \left( \frac{z}{2} \right)^{2k+1} \]

If we choose \( Z_n^+ = Z_n^- \) then the relations between the two sets of eigenfunctions are just

\[ \left( z \frac{d}{dz} \pm z \right) \psi_n^\pm(z) = n \psi_n^\mp(z) \]

---

\[^3\text{Actually, for any constant} \ c \ \text{the superpotential} \ U(z) = z \times \frac{J_1(z) + c Y_1(z)}{J_0(z) + c \nu_0(z)} \ \text{gives a simple quadratic potential for} \ H_+, \ \text{with} \ z^2 = -U^2 + z \frac{d}{dz} U. \ \text{But if} \ c \neq 0, \ U \ \text{involves a logarithm,} \ \ln z, \ \text{and hence is not a periodic function of} \ x \ \text{for} \ z = me^{ix}. \]
Now, what about the dual polynomials $\{\chi_n^\pm | n \geq 0\}$ which are the biorthonormalized duals for $\{\psi_n^\pm | n \geq 0\}$? These are given by

$$
\chi_n^\pm (z) = \frac{1}{Z_n^\pm} \left( \frac{2}{z} \right)^n \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k \Gamma(n-k+\frac{1}{2})}{k!} \left( \frac{z}{2} \right)^{2k} \pm \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^k \Gamma(n-k-\frac{1}{2})}{k!} \left( \frac{z}{2} \right)^{2k+1}
$$

(43)

As illustration of the general theory, for all $k,n \geq 0$ we have the orthonormality and the completeness relation as given in (13). It is straightforward to check these relations by using the explicit series forms for $\chi_n^\pm$ and $\psi_n^\pm$.

Again choosing both normalization factors to be the same, $Z_n^+ = Z_n^- = Z_n$, the dual polynomials for the complex Morse potential are solutions to the exceptionally simple inhomogeneous pair of equations

$$
\left( \frac{d}{dz} \mp z \right) \chi_n^\pm (z) + n \chi_n^\mp (z) = z \lambda_n^\pm
$$

(44)

where by direct calculation we find

$$
\lambda_n^+ = \frac{(-1)^{\lfloor n/2 \rfloor} \Gamma(\lfloor n/2 \rfloor + \frac{1}{2})}{Z_n} \Gamma(\lfloor n/2 \rfloor + 1), \quad \lambda_n^- = (-1)^{n+1} \lambda_n^+
$$

(45)

That is to say

$$
\lambda_n^\pm = \pm e_n^\pm
$$

(46)

Moreover

$$
(H_\pm - n^2) \chi_n^\pm (z) = \left( \frac{d}{dz} \mp z \right) z \lambda_n^\pm - zn \lambda_n^\mp = z (\lambda_n^+ - n \lambda_n^-) \mp z^2 \lambda_n^\pm
$$

(47)

or with the explicit coefficients

$$
(H_+ - n^2) \chi_n^+ (z) = \frac{(-1)^{\lfloor n/2 \rfloor} \Gamma(\lfloor n/2 \rfloor + \frac{1}{2})}{Z_n} \Gamma(\lfloor n/2 \rfloor + 1) ((1 + (-1)^n) z - z^2)
$$

(48)

$$
(H_+ - n^2) \chi_n^- (z) = \frac{(-1)^{n+1+\lfloor n/2 \rfloor} \Gamma(\lfloor n/2 \rfloor + \frac{1}{2})}{Z_n} \Gamma(\lfloor n/2 \rfloor + 1) ((1 + (-1)^n) z + z^2)
$$

(49)

The coefficients are a bit awkward, particularly the phases, but are dictated by $\chi_n^\pm (z) \bigg|_{z=0} = 2^n \Gamma(n+\frac{1}{2}) \frac{1}{z^{n+\frac{1}{2}}}$.

Singular potentials Now we go on to discuss models with several $\mu$s. In particular, if the sums over $k$ are infinite, the potentials can have fixed singularities for finite values of $z$. We explore the situation for a particular supersymmetric pair of such singular potentials. Namely those generated by the superpotential

$$
U(z) = \frac{z}{1-z}
$$

(50)

Up to the scale of $z$, this is the unique superpotential that reproduces itself to obtain $V_+(z) = U(z)$.

$$
V_+ = z \frac{d}{dz} \left( \frac{z}{1-z} \right)^2 \left( \frac{z}{1-z} \right)^2 = \frac{z}{1-z} = z + z^2 + z^3 + O(z^4)
$$

(51)

$$
V_- = -z \frac{d}{dz} \left( \frac{z}{1-z} \right)^2 \left( \frac{z}{1-z} \right)^2 = -z \frac{1}{(1-z)^2} = -z - 3z^2 - 5z^3 + O(z^4)
$$

What are the exact energy eigenfunctions for these potentials, analytic about $z = 0$? First consider the Hamiltonian with potential $V_+$.

$$
H_+ = z^2 \frac{d^2}{dz^2} + z \frac{d}{dz} + \frac{z}{1-z} = \left( z \frac{d}{dz} - \frac{z}{1-z} \right) \left( z \frac{d}{dz} + \frac{z}{1-z} \right)
$$

(52)

$$
\psi_n^+ (z) = z^n (1-z) \text{hypergeom} \left( \left[ 1 + n + \sqrt{1+n^2}, 1 + n - \sqrt{1+n^2} \right], [1 + 2n], z \right)
$$

(54)

$$
\psi_0^+ (z) = 1 - z
$$
We note the especially simple form for the ground state. Excited states are not such elementary functions.

The dual polynomials in this case are

\[
\chi^+_n(z) = \frac{1}{z^n} \left( 1 + \sum_{k=1}^{n} \frac{z^k}{k!} (2n-k-1)! \frac{(2n-1)!}{(2n-k)!} \frac{\Gamma(k-n+\sqrt{n^2+1}) \Gamma(n+\sqrt{n^2+1})}{\Gamma(1-n+\sqrt{n^2+1}) \Gamma(-k+1+n+\sqrt{n^2+1})} \right)
\]

(55)

\[
\chi^+_n(1) = \frac{\sqrt{1+n^2}}{\Gamma(1+2n) \Gamma(1-n+\sqrt{1+n^2})}
\]

(56)

These are solutions of the inhomogeneous equations

\[
(H - n^2) \chi^+_n(z) = V_+ (z) \chi^+_n(1) = \frac{z}{1-z} \chi^+_n(1)
\]

(57)

where the coefficient of the singular inhomogeneity is just \( \chi^+_n \) evaluated at the singularity. The orthonormality relations between the eigenfunctions and the dual polynomials are again the expected ones, \([10]\). In this case, the contour encloses the origin once in the positive counterclockwise sense, but lies within the unit-radius circle of convergence of the series for \( \psi^+_n(z) \). (Or at least, the contour swerves “to the left” to avoid the singularity at \( z = 1 \).) From the explicit series there also follows the expression for the Cauchy kernel as given in \([10]\).

Now consider the Hamiltonian with the superpartner potential, \( V_- \).

\[
H_- = z^2 \frac{d^2}{dz^2} + z \frac{d}{dz} - z \frac{z^2}{(1-z)^2} = \left( \frac{d}{dz} + \frac{z}{1-z} \right) \left( \frac{d}{dz} - \frac{z}{1-z} \right)
\]

(58)

\[
\frac{z^2}{(1-z)^2} \psi^+_n + z \frac{d}{dz} \psi^+_n - z \frac{z+1}{(1-z)^2} \psi^+_n = n^2 \psi^+_n
\]

(59)

\[
\psi^+_n(z) = z^n (1-z)^2 \text{hypergeom} \left[ 2 + n + \sqrt{1+n^2}, 2 + n - \sqrt{1+n^2} \right], [1+2n], z
\]

(60)

\[
\psi^+_0(z) = \frac{1}{1-z}
\]

Again, we note the especially simple form for the ground state. And again, excited states are not such elementary functions. The dual polynomials are now given by

\[
\chi^+_0(z) = 1 \quad \text{and for} \quad n > 0
\]

\[
\chi^+_n(z) = -\frac{1}{n} \left( z \frac{d}{dz} \chi^+_n(z) + \frac{z}{1-z} \left( \chi^+_n(z) - \chi^+_n(1) \right) \right)
\]

(61)

After simplification of this last expression, we obtain

\[
\chi^+_n(z) = \frac{1}{z^n} \left( 1 - \frac{1}{n} \sum_{k=1}^{n} \frac{z^k}{k!} (k-n+2nk-k^2) \frac{(2n-k-1)!}{(2n-1)!} \frac{\Gamma(k-n+\sqrt{n^2+1}) \Gamma(n+\sqrt{n^2+1})}{\Gamma(1-n+\sqrt{n^2+1}) \Gamma(-k+1+n+\sqrt{n^2+1})} \right)
\]

\[
= \frac{1-1}{z^n} \sum_{j=0}^{n-1} \frac{2(n-j)! (2n-j)!}{j! (2n)!} \frac{\Gamma(1+j-n+\sqrt{n^2+1}) \Gamma(n+\sqrt{n^2+1})}{\Gamma(1-n+\sqrt{n^2+1}) \Gamma(-j+n+\sqrt{n^2+1})}
\]

(62)

Note that \( \chi^-_n(1) = 0 \). Just as \([60]\) has an additional factor of \((1-z)\) compared to \([54]\), so too does \([62]\) compared to \([55]\).

These results also exhibit biorthonormality and lead to yet another expression for the Cauchy kernel, as in \([10]\), as may be established from the explicit series. The inhomogeneous equation obeyed by \( \chi^+_0 \) is obviously just

\[
H_- \chi^+_0 = V_- \chi^+_0 = -z \frac{z+1}{(1-z)^2}
\]

(63)

On the other hand, the inhomogeneous equations obeyed by \( \chi^+_n \) are a bit more interesting.

\[
(H - n^2) \chi^+_n(z) = V_+ (z) \chi^+_n(1) = \frac{z}{1-z} 2 \frac{d}{dz} \chi^+_n(z)
\]

(64)
This is the same singular inhomogeneity as appears in the superpartner dual polynomial equation, \((67)\), only with a different coefficient. In fact,

\[
2 \frac{d}{dz} \chi_n^- (1) = -n \chi_n^+ (1)
\]

(65)

All this is more transparent upon exploiting the Darboux factorizations of the Hamiltonians, \((52)\) and \((58)\). As expected from the general theory, we have

\[
(H_\pm - n^2) \psi_n^\pm (z) = 0
\]

(66)

\[
(H_+ - n^2) \chi_n^+ (z) = V_0 (z) \chi_n^+ (1) = \frac{z}{1 - z} \chi_n^+ (1)
\]

(67)

\[
(H_- - n^2) \chi_n^- (z) = -V_0 (z) n \chi_n^+ (1) = \frac{z}{1 - z} 2 \chi_n^- (1)
\]

(68)

bearing in mind \((56)\) and \((65)\), as well as

\[
\left( z \frac{d}{dz} \pm \frac{z}{1 - z} \right) \psi_n^\pm (z) = n \psi_n^\mp (z)
\]

(69)

\[
-\left( z \frac{d}{dz} - \frac{z}{1 - z} \right) \chi_n^+ (z) = n \chi_n^- (z) - \frac{z}{1 - z} \chi_n^+ (1)
\]

(70)

\[
-\left( z \frac{d}{dz} + \frac{z}{1 - z} \right) \chi_n^- (z) = n \chi_n^+ (z) + \frac{z}{1 - z} \delta_n 0
\]

(71)

The last of these is strikingly simpler than expected, exhibiting an inhomogeneity only for the case \(n = 0\).

Then there are the complementary, but subsidiary, relations:

\[
\left( -z \frac{d}{dz} \pm \frac{z}{1 - z} \right) \chi_n^\pm (z) = n \chi_n^\mp (z) - \eta_n^\pm (z)
\]

(72)

which allow the functions \(\eta_n^\pm (z)\) to be constructed from \((62)\) and \((55)\). Differentiating again gives

\[
(H_\mp - n^2) \chi_n^\mp (z) = -n \eta_n^\mp (z) - \left( -z \frac{d}{dz} \mp \frac{z}{1 - z} \right) \eta_n^\pm (z)
\]

(73)

4 Magnetic field effects

Now consider \(\nu \neq 0\). For a given \(U (z) \equiv \sum_{k>0} \nu_k z^k\) we define

\[
H_\pm = \left( z \frac{d}{dz} + \nu \right)^2 - U (z)^2 \pm z \frac{d}{dz} U (z) = \left( z \frac{d}{dz} + \nu \mp U \right) \left( z \frac{d}{dz} + \nu \pm U \right)
\]

(74)

\[
\tilde{H}_\pm = \left( z \frac{d}{dz} - \nu \right)^2 - U (z)^2 \pm z \frac{d}{dz} U (z) = \left( z \frac{d}{dz} - \nu \mp U \right) \left( z \frac{d}{dz} - \nu \pm U \right)
\]

(75)

Obviously, \(\tilde{H}_\pm [\nu] = H_\pm [-\nu]\). We seek eigenfunctions and associated functions that have particular analytic behavior near \(z = 0\).

**Energy eigenfunctions** We look for pairs of functions \(\psi_n^\pm (z)\) that satisfy the equations

\[
\left( z \frac{d}{dz} + \nu \pm U (z) \right) \psi_n^\pm (z) = (n + \nu) \psi_n^\mp (z)
\]

(76)

with \(\psi_n^\pm (z) \xrightarrow{z \to 0} z^n\). Usually these are infinite series, but again, for a given \(U (z)\), both functions \(\psi_n^\pm (z)\) are now completely specified. We note that negative integer \(n\) are now admissible, and independent of the corresponding \(|n|\), for generic \(\nu\). It follows that the \(\psi_n^\pm (z)\) are eigenfunctions of \(H_\pm\) with eigenvalues \((n + \nu)^2\),

\[
H_\pm \psi_n^\pm (z) = (n + \nu)^2 \psi_n^\pm (z)
\]

(77)
It also follows for generic $\nu$ that a suitable dual to $\psi_n^\pm(z)$ is provided by $\tilde{\psi}_{-n}^\pm(z)$, where $\tilde{H}_\pm \tilde{\psi}_{-n}^\pm = (-n - \nu)^2 \tilde{\psi}_{-n}^\pm$, so that $\tilde{\psi}_{-n}^\pm$ has the same energy as $\psi_n^\pm$. Nevertheless, as we discuss shortly (cf. remarks about the “right-sector” given below), if we consider subsectors of the spectrum, we may use polynomials as alternative dual functions, just as in the periodic situations discussed above for $\nu \neq 0$.

For example, reconsider the singular potential $U = \frac{1}{z^2}$. For generic $\nu$ the solutions of

$$z^2 \frac{d^2 f}{dz^2} + (1 + 2\nu) z \frac{df}{dz} + \nu^2 f + \frac{z}{1 - z} f = (n + \nu)^2 f$$

which are single-valued about $z = 0$ are

$$\psi_n^+ (z) = z^n (1 - z) \text{hypergeom} \left( \left[ 1 + n + \nu + \sqrt{1 + (n + \nu)^2}, 1 + n + \nu - \sqrt{1 + (n + \nu)^2} \right], \left[ 1 + 2n + 2\nu \right], \frac{z}{(1 - z)^2} \right)$$

where the indices on the hypergeometric function involve $\rho$, the roots of $0 = -1 - n^2 - 2n\nu - 2\nu\rho + \rho^2$. That is to say $\rho = \nu \pm \sqrt{1 + (n + \nu)^2}$. The other solutions of the equation are

$$z^{-n-2\nu} (1 - z) \text{hypergeom} \left( [-\rho + 1 - n, 1 + \rho - n - 2\nu], [1 - 2n - 2\nu], \frac{z}{(1 - z)^2} \right)$$

but these have a branch point at $z = 0$ and are not single-valued unless $2\nu \in \mathbb{Z}$.

For the partner potential, the single-valued solutions of

$$z^2 \frac{d^2 f}{dz^2} + (1 + 2\nu) z \frac{df}{dz} + \nu^2 f - \frac{z}{1 - z} f = (n + \nu)^2 f$$

are

$$\psi_n^- (z) = z^n (1 - z)^2 \text{hypergeom} \left( \left[ 2 + n + \nu + \sqrt{1 + (n + \nu)^2}, 2 + n + \nu - \sqrt{1 + (n + \nu)^2} \right], \left[ 1 + 2n + 2\nu \right], \frac{z}{(1 - z)^2} \right)$$

while the solutions with a cut, for generic $\nu$, are

$$z^{-n-2\nu} (1 - z)^2 \text{hypergeom} \left( [2 + \rho - n - 2\nu, -\rho + 2 - n], [1 - 2n - 2\nu], \frac{z}{(1 - z)^2} \right)$$

where $\rho = \nu \pm \sqrt{1 + (n + \nu)^2}$ as before.

The dual polynomials and their inhomogeneities. On the “right sector” $\{\psi_n^\pm(z) \mid n \geq 0\}$ (this terminology is explained in [1]) we also look for pairs of finite dual polynomials in $z^{-1}$, of the form given in (4). These satisfy the inhomogeneous equations

$$\left( z \frac{d}{dz} - \nu \mp U(z) \right) \chi_n^\pm(z) + (n + \nu) \chi_n^\mp(z) = \Lambda_n^\pm(z) \equiv \sum_{k \geq 0} \lambda_n^\pm(k) z^k \tag{80}$$

where $U(z)$ and $\Lambda_n^\pm(z)$ are all analytic about the origin, where they vanish. A priori the $\chi_n^\pm(z)$ are not given, but are to be determined along with $\chi_n^\pm(z)$. Once more we normalize these polynomials so that $\chi_n^\pm(z) \xrightarrow{z \to 0} 1/z^n$. This choice also fixes the normalization of $\Lambda_n^\pm$. For a given $U(z)$ the four functions $\chi_n^\pm(z)$ and $\Lambda_n^\pm(z)$ are now completely specified.

The associated polynomials also obey the inhomogeneous Hamiltonian equations

$$\tilde{H}_\pm \chi_n^\pm(z) = \left( z \frac{d}{dz} - \nu \mp U(z) \right) \chi_n^\pm(z) = \left( z \frac{d}{dz} - \nu \mp U(z) \right) \left( \chi_n^\pm(z) - (n + \nu) \chi_n^\mp(z) \right) \tag{81}$$

That is to say

$$\left( \tilde{H}_\pm - (n + \nu)^2 \right) \chi_n^\pm(z) = \left( z \frac{d}{dz} - \nu \mp U(z) \right) \Lambda_n^\pm(z) - (n + \nu) \Lambda_n^\mp(z) \tag{83}$$
Note the RHS of this last equation involves only positive powers, $z^n$ for $n \geq 1$.

Moreover, it also follows that for $k, n \geq 0$ and $-1/2 < \nu < 1/2$ these functions are biorthonormal systems as in \((10)\). These relations once again encode orthonormality and completeness on functions analytic around $z = 0$. For example, it is straightforward to show \((n + \nu)^2 \oint dz \chi_k^\pm (z) \psi_n^\pm (z) = (k + \nu)^2 \oint dz \psi_k^\pm (z) \chi_n^\pm (z)\) by inserting the Hamiltonian \((74)\) and integrating by parts to convert to \((75)\). That is

$$0 = \left( (k + \nu)^2 - (n + \nu)^2 \right) \oint dz \chi_k^\pm (z) \psi_n^\pm (z) = (k - n) (k + n + 2 \nu) \oint dz \chi_k^\pm (z) \psi_n^\pm (z) \quad (84)$$

For $k \neq n$ and both $\geq 0$, and $2 \nu \notin \mathbb{Z}, 0$, we conclude $\oint dz \chi_k^\pm (z) \psi_n^\pm (z) = 0$.

Once again we determine the various functions are uniquely determined, given that $\chi_n^\pm (z) \sim_{z \to 0} 1/z^n$. We choose the coefficients in $\chi_n^\pm (z)$ to eliminate all $z^{-k}$ terms in \((80)\), for $k = 0, 1, \ldots, n$, and then we sweep all the remaining positive powers of $z$ into the $\Lambda_n^\pm (z)$,

$$\left( z \frac{d}{dz} - \nu \pm U(z) \right) \chi_n^\pm (z) + (n + \nu) \Lambda_n^\pm (z) = \Lambda_n^\pm (z) \quad (85)$$

$$U(z) = \sum_{k>0} v_k z^k, \quad \chi_n^\pm (z) = \frac{1}{z^n} \sum_{j=0}^n c_{n,j}^\pm z^j, \quad \Lambda_n^\pm (z) = \sum_{k>0} \frac{\lambda_k^\pm (k)}{z^k} \quad (86)$$

$$\sum_{j=0}^n c_{n,j}^\pm (j - n - \nu) z^j + \sum_{k>0} v_k z^k \sum_{j=0}^n c_{n,j}^\pm z^j + (n + \nu) \sum_{j=0}^n c_{n,j}^\pm z^j = \sum_{k>0} \frac{\lambda_k^\pm (k)}{z^k} z^{k+n} \quad (87)$$

So then

$$c_{n,j}^\pm (j - n - \nu) \pm \sum_{k=1}^j v_k c_{n,j-k}^\pm + (n + \nu) c_{n,j}^\pm = 0 \quad \text{for} \quad j = 0, \ldots, n \quad (88)$$

and $\lambda_k^\pm (k) = \pm \sum_{j=0}^n v_{k+n-j} c_{n,j}^\pm \quad (89)$

Therefore the recursion goes like this. First we determine all the coefficients in the $\chi_n^\pm (z)$ from the pair of equations

$$\begin{pmatrix} \nu + n - j & -n - \nu \\ n + \nu & j - n - \nu \end{pmatrix} \begin{pmatrix} c_{n,j}^+ \\ c_{n,j}^- \end{pmatrix} = \sum_{k=1}^j v_k \begin{pmatrix} c_{n,j-k}^+ \\ -c_{n,j-k}^- \end{pmatrix} \quad \text{for} \quad j = 0, \ldots, n \quad (90)$$

where $c_{n,0}^\pm \equiv 1$. Now $\det \begin{pmatrix} \nu + n - j & -n - \nu \\ n + \nu & j - n - \nu \end{pmatrix} = j (2n + 2 \nu - j)$ and $\begin{pmatrix} \nu + n - j & -n - \nu \\ n + \nu & j - n - \nu \end{pmatrix}^{-1} = \frac{1}{j (2n + 2 \nu - j)} \begin{pmatrix} j - n - \nu & n + \nu \\ -n - \nu & \nu + n - j \end{pmatrix}$, so we have the recursion relations

$$c_{n,j}^\pm = \frac{\pm 1}{j (2n + 2 \nu - j)} \sum_{k=1}^j v_k \left( (j - n - \nu) c_{n,j-k}^+ + (n + \nu) c_{n,j-k}^- \right) \quad (91)$$

Each $c_{n,j}^\pm$ depends on only the first $j$ coefficients in the expansion for $U(z)$, i.e. just on $v_{k \leq j}$. Then from the $c_{n,j}^\pm$ we determine the $\lambda_n^\pm (k)$ using \((89)\). Note that $\lambda_n^\pm (k) \neq 0$ for $k > n$ is possible here, depending on the values of the $v$s. For example, for $n = 2$:

$$c_{2,0}^\pm = 1, \quad c_{2,1}^\pm = \frac{\pm 1}{2 \nu + 3} v_1, \quad c_{2,2}^\pm = \frac{\pm 1}{2 (\nu + 1)} v_2 - \frac{1}{2 (2 \nu + 3)} v_1^2 \quad (91)$$

$$\pm \lambda_2^\pm (k) = v_{k+2} + v_{k+1} c_{2,1}^\pm + v_k c_{2,2}^\pm$$

$$= v_{k+2} \pm \frac{1}{2 \nu + 3} v_1 v_{k+1} + \left( \frac{\pm 1}{2 (\nu + 1)} v_2 - \frac{1}{2 (2 \nu + 3)} v_1^2 \right) v_k \quad (92)$$

And for $n = 3$:
This would suggest a phase transition as an indicator for bulk systems governed by these dynamics.

So then \(\nu\) whereas for \(\nu\) upon defining the theta function

\[
\pm \chi_3^\pm (k) = v_{k+3} + v_{k+2}c_{3,1}^\pm + v_{k+1}c_{3,2}^\pm + v_k c_{3,3}^\pm
\]

\[
= v_{k+3} \pm \frac{1}{2\nu + 5} v_1 v_{k+2} + \left( \frac{\pm 1}{2\nu + 2} v_2 - \frac{1}{2(2\nu + 5)} v_1^2 \right) v_{k+1} + \left( \frac{\pm 1}{(2\nu + 3)} v_3 - \frac{(4\nu + 9)}{6(\nu + 2)(2\nu + 5)} v_2 v_1 \mp \frac{1}{2(2\nu + 3)(2\nu + 5)} v_1^3 \right) v_k
\]

Thus we have the lowest four dual polynomials and their inhomogeneities.

\[
\chi_0^\pm (z) = 1, \quad A_0^\pm (z) = \pm U (z) = \pm \sum_{k>0} v_k z^k
\]

\[
\chi_1^\pm (z) = \frac{1}{z} \pm \frac{1}{\nu + 1} v_1, \quad A_1^\pm (z) = \sum_{k>0} \left( \frac{1}{\nu + 1} v_1 v_k \pm v_{k+1} \right) z^k
\]

\[
\chi_2^\pm (z) = \frac{1}{z^2} \pm \frac{1}{2\nu + 3} v_1 \frac{1}{z} \pm \frac{1}{2(\nu + 1)} v_2 - \frac{1}{2(2\nu + 3)} v_1^2 z^k
\]

\[
A_2^\pm (z) = \sum_{k>0} \left( \frac{1}{2(\nu + 1)} v_2 + \frac{1}{2(2\nu + 3)} v_1^2 v_k + \frac{1}{2\nu + 3} v_1 v_{k+1} \pm v_{k+2} \right) z^k
\]

\[
\chi_3^\pm (z) = \frac{1}{z^3} \pm \frac{1}{2\nu + 5} v_1 \frac{1}{z^2} \pm \left( \frac{1}{2(\nu + 2)} v_2 - \frac{1}{2(2\nu + 5)} v_1^2 \right) \frac{1}{z}
\]

\[
\pm \frac{1}{2\nu + 3} v_3 - \frac{(4\nu + 9)}{6(\nu + 2)(2\nu + 5)} v_2 v_1 \mp \frac{1}{2(2\nu + 3)(2\nu + 5)} v_1^3 z^k
\]

\[
A_3^\pm (z) = \sum_{k>0} \left( \frac{1}{2(\nu + 2)} v_2 + \frac{(4\nu + 9)}{6(\nu + 2)(2\nu + 5)} v_1^2 v_{k+1} + \frac{1}{2\nu + 3} v_1 v_{k+2} \pm v_{k+3} \right) z^k
\]

etc. Note that \(\chi_n^\pm (z)\) and \(A_n^\pm (z)\) are obtained from \(\chi_n^\pm (z)\) and \(A_n^\pm (z)\), or vice versa, just by flipping the signs of all the \(v_s\). These results reduce to those in the previous Table, upon setting \(\nu = 0\).

The partition function is not analytic in \(\nu\) For generic \(\nu\)

\[
\mathcal{Z} [\nu] = \text{trace} \left( e^{-H[\nu]} \right) = \sum_{n=-\infty}^{\infty} e^{-(n+\nu)^2}
\]

\[
e^{-\nu^2} (\vartheta [\nu] + \vartheta [-\nu] - 1)
\]

upon defining the theta function\(\footnote{Or in terms of Jacobi’s functions, \(\vartheta_3 (z, q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2inz}\), so \(\vartheta [\nu] + \vartheta [-\nu] - 1 = \vartheta_3 (z = i\nu, q = 1/e)\).}

\[
\vartheta [\nu] = \sum_{n=0}^{\infty} e^{-n^2 - 2\nu n}
\]

whereas for \(\nu = 0\), the partition function is

\[
\mathcal{Z}_0 = \sum_{n=0}^{\infty} e^{-n^2} = \vartheta [0] = 1.3863 \ldots
\]

So then

\[
\lim_{\nu \to 0} \mathcal{Z} [\nu] = 2\mathcal{Z}_0 - 1 \neq \mathcal{Z}_0
\]

This would suggest a phase transition as an indicator for bulk systems governed by these dynamics.
5 Conclusions

We discussed supersymmetric biorthogonal quantum systems along the lines of [7], paying particular attention to the structure of non-hermitian systems with periodic solutions, for which cases the duals of the energy eigenfunctions are not simply related to the eigenfunctions by either complex conjugation or PT reflection. We worked out the general theory for single particle quantum systems, and we illustrated the general theory with several explicit exact examples.

It remains to investigate many-body or field theoretic extensions of these supersymmetric systems, say by adapting the perturbative methods in [8] on supersymmetric Liouville field theory, or by employing the powerful non-perturbative methods of conformal field theory [9]. This additional study is in progress, and represents one application of the formalism presented in this paper. There is a rich literature on Liouville and super-Liouville theory, models whose importance came to light in the work of Polyakov on string theory [10], but which were subsequently developed much further in the context of conformal field theory and its applications to critical phenomena, as well as to subcritical string theory [11]. In particular, the super-Liouville correlation functions [12] have been shown to exhibit interesting analytic behavior in the exponential coupling constant, similar to the analytic structure of correlators for non-supersymmetric Liouville field theory [13]. The behavior of these correlators is related to properties of various WZNW models [14], and there are particularly intriguing features that correspond to purely imaginary coupling constants – precisely the field theory extensions of the type of models discussed in this paper.

As for other applications of supersymmetric biorthogonal quantum systems, say to non-relativistic situations, an interesting possibility would be to consider driven/dissipative condensates as suggested in [7], but with additional fermions in the condensate [15]. This too is under study.

Acknowledgements We thank C Bender for introducing us to PT symmetric theories and raising our interest in problems involving non-hermitian Hamiltonians. We also thank P G O Freund and A Veitia for useful discussions. One of us (TC) thanks the Aspen Center for Physics for providing the stimulating environment in which parts of this investigation were carried out during June - July 2005, and he also thanks the Institute for Advanced Study, where this work was completed, for its hospitality and support as a visiting Member during January - July 2006. This material is based upon work supported by the National Science Foundation under Grant No’s. 0303550 and 0555603.
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