A NEW CONDITION TO IDENTIFY ISOTROPIC DIELECTRIC–MAGNETIC MATERIALS DISPLAYING NEGATIVE PHASE VELOCITY

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ABSTRACT: The derivation of a new condition for characterizing isotropic dielectric–magnetic materials exhibiting negative phase velocity, and the equivalence of that condition with previously derived conditions, are presented.

Keywords: negative phase velocity, power flow

1. INTRODUCTION

Non–dissipative mediums with both simultaneously negative permittivity and permeability were first investigated by Veselago [1] in 1968. These mediums support electromagnetic wave propagation in which the phase velocity is antiparallel to the direction of energy flow, and other unusual electromagnetic effects such as the reversal of the Doppler effect and Cerenkov radiation. After the publication of Veselago’s work, more than three decades went by for the actual realization of artificial materials that are effectively isotropic, homogeneous, and possess negative real permittivity and permeability in some frequency range [2, 3].

When dissipation is included in the analysis, a general condition for the
constitutive parameters of an isotropic dielectric–magnetic medium to have phase velocity directed oppositely to the power flow, was reported about two years ago [4]: Most importantly, according to that condition, the real parts of both the permittivity and the permeability need not be both negative.

In this communication, we derive a new condition for characterizing isotropic materials with negative phase velocity. Although this new condition looks very different from its predecessor [4], we also show here the equivalence between both conditions.

2. THE NEW CONDITION

Let us consider a linear isotropic dielectric–magnetic medium characterized by complex–valued relative permittivity and relative permeability scalars $\epsilon = \epsilon_r + i\epsilon_i$ and $\mu = \mu_r + i\mu_i$. An $\exp(-i\omega t)$ time–dependence is implicit, with $\omega$ as the angular frequency.

The wave equation gives the square of the complex–valued refractive index $n = n_r + in_i$ as

$$n^2 = \epsilon\mu \Rightarrow n_r^2 - n_i^2 + 2in_r n_i = \mu_r \epsilon_r - \mu_i \epsilon_i + i(\mu_i \epsilon_r + \mu_r \epsilon_i). \quad (1)$$

The sign of $n_r$ gives the phase velocity direction, whereas the sign of the real part of $n/\mu$, i.e.,

$$\text{Re}\left(\frac{n}{\mu}\right) = n_r \mu_r + n_i \mu_i, \quad (2)$$

gives the direction of power flow [4]. Therefore, for this medium to have negative phase velocity and positive power flow, the following conditions should hold simultaneously:

$$n_r < 0, \quad (3)$$

$$n_r \mu_r + n_i \mu_i > 0, \quad (4)$$

Equation (1) yields the biquadratic equation

$$n_r^4 - (\mu_r \epsilon_r - \mu_i \epsilon_i) n_r^2 - \frac{1}{4}(\mu_i \epsilon_r + \mu_r \epsilon_i) = 0. \quad (5)$$

This equation has only two real–valued solutions for $n_r$, viz.,

$$n_r = \pm \left(\frac{\sqrt{\mu |\mu| + \mu_r \epsilon_r - \mu_i \epsilon_i}}{2}\right)^{1/2}. \quad (6)$$
Noting that the relation
\[ \mu_i \epsilon_i - \mu_r \epsilon_r < \sqrt{(\mu_i \epsilon_i - \mu_r \epsilon_r)^2 + (\mu_i \epsilon_r + \mu_r \epsilon_i)^2} \]  
holds for all values of the constitutive parameters \( \epsilon_r, \epsilon_i \) and \( \mu_r, \mu_i \), we see that
\[ 0 < |\epsilon| |\mu| + \mu_r \epsilon_r - \mu_i \epsilon_i ; \]  
hence, the right side of (6) is always positive.

As the negative square root must be chosen in (6) in order to satisfy the condition (3), therefore
\[ n_r = -\frac{1}{\sqrt{2}} \left( |\epsilon| |\mu| + \mu_r \epsilon_r - \mu_i \epsilon_i \right)^{1/2} , \]  
\[ n_i = -\frac{1}{\sqrt{2}} \frac{\mu_i \epsilon_r + \mu_r \epsilon_i}{\left( |\epsilon| |\mu| + \mu_r \epsilon_r - \mu_i \epsilon_i \right)^{1/2}} . \]  

On using these expressions and (2) in the condition (4), a condition for power flow and phase velocity in opposite directions is finally derived as follows:
\[ \mu_r \left( |\epsilon| |\mu| + \mu_r \epsilon_r - \mu_i \epsilon_i \right)^{1/2} + \mu_i \frac{\mu_i \epsilon_r + \mu_r \epsilon_i}{\left( |\epsilon| |\mu| + \mu_r \epsilon_r - \mu_i \epsilon_i \right)^{1/2}} < 0 . \]  

This condition can be rewritten in the very simple form
\[ \epsilon_r |\mu| + \mu_r |\epsilon| < 0 , \]  
which is the chief contribution of this communication.

3. EQUIVALENCE WITH PREVIOUSLY DERIVED CONDITION

The general condition derived for the phase velocity to be oppositely directed to the power flow about two years ago [4] is as follows:
\[ \left( |\epsilon| - \epsilon_r \right) \left( |\mu| - \mu_r \right) > \epsilon_i \mu_i . \]  
Although it looks very different, this condition, which can be rewritten as
\[ \epsilon_r |\mu| + \mu_r |\epsilon| < |\epsilon||\mu| + \mu_r \epsilon_r - \mu_i \epsilon_i , \]
is completely equivalent to the new condition (12).

Clearly, if (12) is satisfied, then, taking into account the validity of (8), (14) also is. To show that the reverse is also true, we start from (14) and assume that (12) does not hold. As the left side of the inequality (14) is non–negative, squaring both sides does not change the sense of the inequality and we get

\[
\left( \varepsilon_r \mu_l + \mu_r \varepsilon_i \right)^2 < \left( |\varepsilon| \mu | + \mu_r \varepsilon_r - \mu_i \varepsilon_i \right)^2.
\]

(15)

Simplification of this inequality leads to

\[
\varepsilon_i \mu_i \left( |\varepsilon| \mu | + \mu_r \varepsilon_r - \mu_i \varepsilon_i \right) < 0.
\]

(16)

But causality dictates that \( \varepsilon_i \geq 0 \) and \( \mu_i \geq 0 \); hence, we must conclude that

\[
|\varepsilon| \mu | + \mu_r \varepsilon_r - \mu_i \varepsilon_i < 0,
\]

(17)
in contradiction with Eq. (8). Therefore, we must accept the validity of the condition (12). This completes the demonstration of the equivalence between conditions (12) and (13).

4. CONCLUDING REMARKS

We note in passing that both conditions (12) and (13) are also equivalent to the condition

\[
\varepsilon_r \mu_i + \mu_r \varepsilon_i < 0,
\]

(18)

reported very recently \[5\]. This condition is due to R. Ruppin.

To conclude, we have here derived a simple new condition for the constitutive parameters of a linear isotropic dielectric–magnetic medium to have phase velocity opposite to the direction of power flow, and we have demonstrated its equivalence with previously derived conditions.

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