Leader Election and Shape Formation with Self-Organizing Programmable Matter

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Abstract. Many proposals have already been made for realizing programmable matter, ranging from shape-changing molecules, DNA tiles, and synthetic cells to reconfigurable modular robotics. We are particularly interested in programmable matter consisting of simple computational elements, called \textit{particles}, that can establish and release bonds and can actively move in a self-organized way, and in the feasibility of solving basic problems relevant for programmable matter with them. As a model for such self-organizing particle systems, we will use a general form of the amoebot model first proposed in [22]. Based on that model, we present efficient local-control algorithms for leader election and path formation requiring only particles with constant size memory, and we also discuss the limitations of solving these problems within the amoebot model.

1 Introduction

There have been many conceptions of programmable matter, and thus many avenues of research using that name, each pursuing solutions for specific application scenarios with their own, special capabilities and constraints. However, most of the research in this area has been done by scientists from the natural sciences and the robotics area and not so much from theoretical computer science, so not much is known yet about which assumptions or primitives allow the development of distributed algorithms that can solve relevant problems for programmable matter in a self-organizing way. Notable exceptions are DNA tile assembly and population protocols, but certain classes of problems that involve active movement of the particles, such as re-arranging the shape of programmable matter, are out of scope for these. To study shape formation, we focus instead on a general form of the amoebot model first proposed in [22].

1.1 General Amoebot model

In the general amoebot model, programmable matter consists of a uniform set of simple computational units called particles that can move and bond to other
particles and use these bonds to exchange information. The particles have very limited memory, act asynchronously, and they achieve locomotion by expanding and contracting, which resembles the behavior of amoeba [3].

As a base for our amoebot model, we assume that we have a set of particles that aim at maintaining a connected structure at all times. This is needed to prevent the particles from drifting apart in an uncontrolled manner like in fluids and because in our case particles communicate only via bonds. The shape and positions of the bonds of the particles mandate that they can only assume discrete positions in the particle structure. This justifies the use of a possibly infinite, undirected graph \( G = (V,E) \), where \( V \) represents all possible positions of a particle (relative to the other particles in their structure) and \( E \) represents all possible transitions between positions.

Each particle occupies either a single node or a pair of adjacent nodes in \( G \), and every node can be occupied by at most one particle. Two particles occupying adjacent nodes are connected, and we refer to such particles as neighbors. Particles are anonymous but the bonds of each particle have unique labels, which implies that a particle can uniquely identify each of its outgoing edges. Each particle has a local memory in which it can store some bounded amount of information, and any pair of connected particles has a bounded shared memory that can be read and written by both of them and that can be accessed using the edge label associated with that connection.

Particles move through expansions and contractions: If a particle occupies one node (i.e., it is contracted), it can expand to an unoccupied adjacent node to occupy two nodes. If a particle occupies two nodes (i.e., it is expanded), it can contract to one of these nodes to occupy only a single node. Performing movements via expansions and contractions has various advantages. For example, it would easily allow a particle to abort a movement if its movement is in conflict with other movements. A particle always knows whether it is contracted or expanded — in the latter, it also knows along which edge it expands — and this information will be available to neighboring particles. In a handover, two scenarios are possible: a) a contracted particle \( p \) can "push" a neighboring expanded particle \( q \) and expand into the neighboring node previously occupied by \( q \), forcing \( q \) to contract, or b) an expanded particle \( p \) can "pull" a neighboring contracted particle \( q \) to a cell occupied by it thereby expanding that particle to that cell, which allows \( p \) to contract to its other cell. The ability to use a handover allows the system to stay connected while particles move (e.g., for particles moving in a worm-like fashion). Note that while expansions and contractions may represent the way particles physically move in space (resembling loosely the movement of amoeba), they can also be interpreted as a particle "looking ahead" and establishing new connections (by expanding) before it fully moves to a new position and severs the old connections it had (by contracting). In Figure 1 we illustrate a set of particles (some contracted, some expanded) using the infinite regular triangular grid graph \( G_{eqt} \) as the underlying graph \( G \).

We assume the standard asynchronous computation model, i.e., only one particle can be active at a time. Whenever a particle is active, it can perform an
Fig. 1. The left part shows an example of a particle structure in $G_{eqt}$. A contracted particle is depicted as a black dot, and an expanded particle is depicted as two black dots connected by an edge. The right part shows a particle structure with 3 borders. The outer border is depicted in red and the two inner borders are depicted in green and blue, respectively.

arbitrary bounded amount of computation (involving its local memory as well as the shared memories with its neighboring particles) followed by no or a single movement. A round is over once every particle has been activated at least once.

1.2 Our Contributions

In this paper we focus on the problem of solving leader election and shape formation for particles with constant memory. For the shape formation, we just focus on forming a path of particles, that is, we are searching for a local-control algorithm so that for any initial set $A \subseteq V$ of positions occupied by particles where $G|_A$ (i.e., the subgraph of $G$ induced by $A$) is connected, the particles will eventually rearrange themselves into a set of occupied positions $A'$ such that $G|_{A'}$ forms a path.

Suppose that there is a distributed protocol for the path formation problem in the general amoebot model. Since it is possible to decide when $G|_{A'}$ forms a path, it is also possible to design a distributed protocol in which the nodes can decide when a leader has been elected: once the path has been formed, its two endpoints contend for leadership using tokens with random bits sent back and forth until one of them wins. On the other hand, one can deduce from [29] that in the general amoebot model there is no distributed protocol that can decide when a leader has been elected. Hence, there cannot be a distributed protocol for path formation in the general amoebot model. So additional assumptions on the particle system are needed to solve these problems.

Therefore, we consider a geometric version of the amoebot model: We assume that $G$ is equal to $G_{eqt}$ (see Figure 1) and that for each particle the bonds are labeled in a consecutive way in clockwise direction so that every particle has the same sense of clockwise direction, but the particles may not have a common sense of orientation in a sense that they have different offsets of the labelings. We show that in this case the leader election problem can be solved, i.e., at the end we have exactly one leader and the leader knows that it is the only leader left. Moreover, the runtime for our leader election algorithm is worst-case
optimal in a sense that it needs at most $O(L)$ rounds with high probability, where $L$ is the maximum length of a border between the particle structure and an empty region (inside or outside of it) in $G_{eqt}$. Based on the leader election algorithm, we present a distributed algorithm for the path formation problem. Both algorithms assume that the system is in a well-initialized state. It would certainly be desirable to have algorithms that can tolerate any initial state, but at the end of the paper we show that there are certain limitations on solving leader election and path formation in a self-stabilizing fashion.

1.3 Related Work

Many approaches related to programmable matter have recently been proposed. One can distinguish between active and passive systems. In passive systems the particles either do not have any intelligence at all (but just move and bond based on their structural properties or due to chemical interactions with the environment), or they have limited computational capabilities but cannot control their movements. Examples of research on passive systems are DNA computing \cite{19,21,39}, tile self-assembly systems in general (e.g., see the surveys in \cite{23,35,40}), population protocols \cite{4}, and slime molds \cite{10,33}. We will not describe these models in detail as they are only of little relevance for our approach. On the other hand in active systems, computational particles can control the way they act and move in order to solve a specific task. Robotic swarms, and modular robotic systems are some examples of active programmable matter systems.

In the area of swarm robotics it is usually assumed that there is a collection of autonomous robots that have limited sensing, often including vision, and communication ranges, and that can freely move in a given area. They follow a variety of goals, as for example graph exploration (e.g., \cite{24}), gathering problems (e.g., \cite{2,17}), shape formation problems (e.g., \cite{25,37}), and to understand the global effects of local behavior in natural swarms like social insects, birds, or fish (see e.g., \cite{12,8}). Surveys of recent results in swarm robotics can be found in \cite{31,34}; other samples of representative work can be found in e.g., \cite{7,19,20,18,28,32,5,36}. While the analytical techniques developed in the area of swarm robotics and natural swarms are of some relevance for this work, the individual units in those systems have more powerful communication and processing capabilities than in the systems we consider.

The field of modular self-reconfigurable robotic systems focuses on intra-robotic aspects such as the design, fabrication, motion planning, and control of autonomous kinematic machines with variable morphology (see e.g., \cite{26,42}). Metamorphic robots form a subclass of self-reconfigurable robots that share some of the characteristics of our geometric model \cite{16}. The hardware development in the field of self-reconfigurable robotics has been complemented by a number of algorithmic advances (e.g., \cite{11,35,37}), but so far mechanisms that automatically scale from a few to hundreds or thousands of individual units are still under investigation, and no rigorous theoretical foundation is available yet.
The \textit{nubot} model \cite{woods2003nubots,woods2004self,woods2005self}, by Woods et al. aims at providing the theoretical framework that would allow for a more rigorous algorithmic study of biomolecular-inspired systems, more specifically of self-assembly systems with active molecular components. While bio-molecular inspired systems share many similarities with our self-organizing particle systems, there are many differences that do not allow us to translate the algorithms and other results under the nubot model to our systems — e.g., there is always an arbitrarily large supply of "extra" particles that can be added to the system as needed, and the system allows for an additional (non-local) notion of rigid-body movement.

2 Leader Election in the Geometric Amoebot Model

In this section we show how leader election can be achieved when $G$ provides geometric information. In particular, we assume that $G$ is equal to the equilateral triangular graph $G_{\text{eqt}}$ (see Figure 1) and that the edges in $G$ are labeled in a consecutive way in clockwise direction around a node so that every particle has the same sense of clockwise rotation. However, we do not assume that the labeling is uniform throughout the graph, so the particles do not necessarily share a common sense of direction in the grid. Our approach organizes the particle system into a set of cycles and executes an algorithm on each cycle independently.

2.1 Organization into Cycles

Let $A \subseteq V$ be any initial distribution of contracted particles such that $G_{\text{eqt}}|_A$ is connected. Consider the graph $G_{\text{eqt}}|_{V\setminus A}$ induced by the unoccupied nodes in $G_{\text{eqt}}$. We call a connected component of $G_{\text{eqt}}|_{V\setminus A}$ an empty region. Let $N(R)$ be the neighborhood of an empty region $R$ in $G_{\text{eqt}}$. Then all nodes in $N(R)$ are occupied and we call the graph $G_{\text{eqt}}|_{N(R)}$ a border. Since $G_{\text{eqt}}|_A$ is a connected finite graph, exactly one empty region has infinite size while the remaining empty regions have finite size. We define the border corresponding to the infinite empty region to be the unique outer border and refer to a border that corresponds to a finite empty region as an inner border, see Figure 1.

With these definitions in place we can organize particles along the borders into cycles as follows. Let $p$ be a particle occupying a node $v$ and let $N(v)$ be the neighborhood of $v$. For each connected component of unoccupied nodes in $N(v)$, $p$ simulates a distinct agent. Hence, $p$ will only simulate any agents if $v$ is part of a border, and $p$ simulates at most 3 agents due to the structure of $G_{\text{eqt}}$. For each face in the planar graph $G_{\text{eqt}}|_{A}$ corresponding to an empty region, neighboring particles connect their agents associated with that face in a cycle as shown in Figure 2. The common sense of clockwise rotation can be used to give each cycle a unique orientation, as indicated by the direction of the edges in Figure 2.

2.2 Algorithm

The leader election algorithm operates independently on each cycle. For simplicity and ease of presentation we assume for now that agents have a global view.
of the cycle they are part of and that agents act synchronously. At any given
time, some subset of agents on a cycle will consider themselves candidates, i.e.
potential leaders of the system. Initially, every agent considers itself a candidate.
Between any two candidates on a cycle there is a (possibly empty) sequence of
non-candidate agents. We call such a sequence a segment and specifically refer to
the segment coming after (before) a candidate $c$ in the direction of the cycle as
the front segment (back segment) of $c$. We denote the lengths of those segments
as $|fs(c)|$ and $|bs(c)|$. We use front candidate ($fc(c)$) and back candidate ($bc(c)$)
to denote the candidates at the end of these segments. We drop the $c$ in paran-
theses if it is clear from the context. We define the distance $d(c_1,c_2)$ between
candidates $c_1$ and $c_2$ as the number of agents between $c_1$ and $c_2$ when going from
$c_1$ to $c_2$ in the direction of the cycle. We say a candidate $c_1$ covers a candidate
$c_2$ (or $c_2$ is covered by $c_1$) if $|fs(c_1)| > d(c_2,c_1)$. The leader election progresses in
phases. In each phase, each candidate executes Algorithm 1. A phase consists of
three synchronized subphases, i.e., agents can only progress to the next subphase
once all agents have finished the current subphase.

Consider the execution of Algorithm 1 by a candidate $c$. If the algorithm
returns "not leader" then $c$ revokes its candidacy and becomes part of a segment.
If the algorithm returns "leader", $c$ will become the leader of the particle system.
The transferral of candidacy in subphase 2 means that $c$ withdraws its own
candidacy but at the same time promotes the agent at position $pos$ (i.e., the
front candidate of $c$ in subphase 1) to be a candidate.

2.3 Correctness

In order to show the correctness of our algorithm, we show that it satisfies the
following conditions, that relate to the entire particle system (not just a single
cycle):

1. Safety: There always exists at least one candidate.
Algorithm 1 Leader Election in the Geometric Amoebot Model

Subphase 1:
\( \text{pos} \leftarrow \text{position of } fc \)
if covered by any candidate or \( |fs| < |bs| \) then
\( \text{return not leader} \)

Subphase 2:
if coin flip results in heads then
\( \text{transfer candidacy to agent at pos} \)

Subphase 3:
if only candidate on border then
if outside border then
\( \text{return leader} \)
else
\( \text{return not leader} \)

2. Liveness: In each phase at least one candidate withdraws leadership with at least constant probability as long as there is more than one candidate left.

Lemma 1. Algorithm 1 satisfies the safety condition.

Proof. We will show by induction that on the cycle associated with the outer border there will always be at least one candidate. Initially, this holds trivially. So assume that it holds before a phase. Let \( c \) be the candidate with the longest front segment. Then there is no candidate covering \( c \) and also \( |fs| < |bs| \) cannot be true. Hence, \( c \) will not withdraw candidacy in subphase 1. In subphase 2, the candidacy of \( c \) might be transferred but will not vanish. Let \( c' \) be the agent that received the candidacy if it was transferred and \( c' = c \) otherwise. In subphase 3, \( c' \) will not withdraw candidacy because it lies on the outer border. Hence, there is still a candidate after the phase. \( \square \)

Lemma 2. Algorithm 1 satisfies the liveness condition.

Proof. Assume that there are two or more candidates in the system. First we consider the case that there is a cycle with two or more candidates. If there are segments of different lengths on that cycle, we have \( |fs| < |bs| \) for at least one candidate which will therefore withdraw its candidacy in subphase 1. If all segments are of equal length, we have that in subphase 2 with probability at least \( \frac{1}{4} \) there is a candidate \( c \) that transfers candidacy while \( fc(c) \) does not. Hence, the number of candidates is reduced with probability at least \( \frac{1}{4} \). Now consider the case that all cycles have at most one candidate. Then there is a cycle corresponding to an inner border that has exactly one candidate. That candidate will withdraw candidacy in subphase 3 and thereby reduce the number of candidates in the system. \( \square \)

The following Theorem is a direct consequence of Lemmas 1 and 2.
Theorem 1. Algorithm \( \mathcal{A} \) eventually elects a single leader for any connected particle system on \( G_{eqt} \).

2.4 Runtime Analysis

For a cycle of agents let \( L \) be the length of the cycle and let \( l_i \) be the longest segment length before phase \( i \) of the execution of Algorithm \( \mathcal{A} \). We define \( l_i = L \) if there is no candidate on the cycle. It is easy to see that if \( l_i \geq L/2 \) then in phase \( i+1 \) either the leader is elected (outer border) or all candidates on the cycle vanish (inner border). For the case \( l_i < L/2 \), Lemma 3 provides the key insight of our analysis.

Lemma 3. For any phase \( i \) such that \( l_i < L/2 \) it holds \( l_{i+1} \geq l_i \) in any case and \( l_{i+1} \geq 2l_i \) with probability at least \( 1/4 \).

Proof. Consider a candidate \( c \) such that \( |fs(c)| = l_i \). Subphase 1 can only increase segment lengths and \( c \) will not withdraw leadership. So after subphase 1 we have \( |fs(c)| \geq l_i \). Also, we have \( |bs(c)| \geq l_i \) because \( c \) covers any candidate \( c' \) such that \( d(c', c) < l_i \). For subphase 2 we have to distinguish two cases based on the outcome of the coin flip of \( c \). If \( c \) does not transfer candidacy, we still have \( |bs(c)| \geq l_i \) so that \( |fs(bc(c))| \geq l_i \). For the case that \( c \) does transfer candidacy, we have to further distinguish two cases based on the outcome of the coin flip of \( fc(c) \). Let \( c' \) be the agent that received the candidacy from \( c \). If \( fc(c) \) also transfers its candidacy, after subphase 2 we have \( |fs(c')| \geq l_i \). This is because \( fc(c) \) transfers its candidacy by distance of at most \( l_i \) while \( c \) transfers its candidacy by a distance of exactly \( l_i \) by assumption. If \( fc(c) \) does not transfer candidacy, after subphase 2 we have \( |fs(c')| \geq 2l_i \) because the front segment of \( c' \) now spans both the back segment and the front segment of \( c \) before subphase 2. The probability that \( c \) transfers candidacy while \( fc(c) \) does not is \( 1/4 \). \( \square \)

Let \( L_{\text{max}} \) be the length of the longest cycle in the particle system. Based on Lemma 3, it is easy to see that under complete synchronisation of subphases and with the agents having a global view of the cycle, our algorithm requires on expectation \( O(\log(L_{\text{max}})) \) phases to elect a leader. For now assume that our algorithm can be realized as a local-control protocol such that phase \( i \) requires \( O(l_i) \) rounds. Theorem 2 gives a bound on the number of rounds required by the algorithm based on this assumption. In the following section we establish that there is indeed such a realization and argue that the synchronisation of subphases is not required.

Theorem 2. Algorithm \( \mathcal{A} \) requires \( O(L_{\text{max}}) \) rounds on expectation.

Proof. Let the random variable \( X_i \) describe the number of rounds during the execution of Algorithm \( \mathcal{A} \) such that \( l_i \in [2^{i-1}, 2^i) \). Then, under the assumption that phase \( i \) requires \( O(l_i) \) rounds, the total runtime of our algorithm is

\[
T = \sum_{i=1}^{\lceil \log(L_{\text{max}}) \rceil} X_i \cdot O(2^i).
\]
Since $E(X_i) \leq 4$ due to Lemma 3, the expected runtime is

$$E(T) \leq \sum_{i=1}^{\lceil\log(L_{\text{max}})\rceil} E(X_i) \cdot O(2^i) = O(L_{\text{max}}).$$

\[\square\]

2.5 Asynchronous Local-Control Protocol

In this section we present details on how the algorithm can be realized as an asynchronous local-control protocol. The protocol heavily relies on token passing. We assume that tokens are messages of constant size. The tokens of each subphase of our algorithm are independent of each other, so an agent has to handle distinct tokens for each phase at the same time. If not otherwise specified, we assume that tokens of a single subphase move through the agents in a well-ordered manner (i.e., a token does not surpass another token in front of it but waits until the agent is free to hold it). For the first subphase we will specify the token passing scheme in detail, for the other phases we give a more general overview. Furthermore, a candidate that revokes candidacy in any subphase of the algorithm will annul the work it has done so far.

Candidate Elimination In subphase 1 of Algorithm 1 candidates use differences in length of their front and back segments to eliminate other candidates. To implement this behavior in a local-control protocol, a candidate will virtually push its front segment into its back segment and eliminate all candidates that are covered. If a candidate detects that its front segment is too short to cover any candidate it revokes candidacy.

Consider a candidate $c$ at the beginning of subphase 1. It sends a starting token along its front segment (i.e., in direction of the cycle). Each agent that receives the token forwards it and sends a cover token in opposing direction of the cycle. Candidate $fc(c)$ consumes the starting token and sends a special end of segment (EOS) cover token. As long as the cover tokens stay on the front segment of $c$ we consider them to be passive, i.e., they are only forwarded by agents in a pipelined fashion. Once a token passes $c$ on its path it becomes active.

An active cover token matches with the first agent on its path that has not been matched by another active token of $c$. Once an agent is matched it deletes all other tokens it holds (like cover tokens from another candidate), except for passive tokens. If a candidate is matched it revokes candidacy immediately. Additionally, if it is in subphase 1 it consumes its own passive cover tokens instead of forwarding them to its back segment (see Figure 3 for examples). If the the candidate is in subphase 2 or 3, no additional work has to be done (except for the aforementioned annulment). Once the EOS token has matched, the last matched agent sends a finish token back to $c$. As soon as $c$ receives this token, subphase 1 is complete. If the tokens of $c$ have not matched with another candidate (which can be detected by the finish token) $c$
revokes candidacy and annuls all tokens sent out in subphase 1 (i.e., it unmatches all agents), otherwise the agents stay matched for subphase 2. Note, that in case the EOS token has matched with another candidate (i.e., both segments have equal length), both candidates stay candidates.

**Fig. 3.** Two examples for the token-agent-matching process. Agents are nodes on the lowest line, which is part of a cycle directed to the right. Colored lines depict unmatched cover tokens occupying consecutive agents (agents below token lines are already matched). Tokens on the lower lane are still passive, tokens in the upper lane are active. **Case (a):** The red and blue candidate have already matched part of their back segments. Passive tokens of blue are not canceled out by active tokens of red. **Case (b):** The red tokens have already matched up to the blue candidate and beyond. Blue revoked leadership, the passive tokens of blue (dotted lines) cannot become active and are simply consumed by the blue candidate.

**Candidate Transferral** In case a candidate has adjacent segments of equal length or has eliminated other candidates in subphase 1, it transfers candidacy to the agent at position pos (i.e., its $f_c$ before subphase 1) with probability $\frac{1}{2}$. In order to do so in a local-control protocol, the candidate uses the matched agents in its back segment from subphase 1. By using a token passing scheme similar to the first subphase, the matched agents from the first subphase create cover tokens that match with the front segment of the candidate. Candidacy is transferred onward on the newly matched agents (which unmatches the agents implicitly) until position pos.

**Solitude Verification** A candidate that wants to determine whether it is the only candidate left, tests if its front segment ends in another candidate or in itself. To do so, it enforces its own orientation on all agents in the segment. Thereby, every agent in the segment is able to determine the direction of its outgoing edge along the cycle. These edge directions can be seen as vectors in the two dimensional plane and in case the segment is the whole cycle, these vectors cancel out component wise (see Figure 4). By a simple token passing scheme agents will try match their edge directions component wise with an edge direction in the opposing direction from another agent. Finally, the candidate
inspects the segment and if all agents are matched it is the only candidate on the cycle.

![Diagram](image)

**Fig. 4.** An example of solitude verification: the candidate (red node) has enforced its orientation (red arrows) on all nodes. All agents determined the offset to their predecessors (black arrows and numbers at nodes). Since the candidate is the last one, the vectors cancel out (component wise).

**Inner Outer Border Test** The last candidate of a cycle can decide whether its cycle corresponds to an inner or the outer border as follows. A cycle corresponding to an inner border has counter-clockwise rotation while a cycle corresponding to the outer border has clockwise rotation, see Figure 2. The candidate sends a token along the cycle that sums the angles of the turns the cycle takes, see Figure 5. When the token returns to the candidate its value represents the external angle of the polygon corresponding to the cycle while respecting the rotation of the cycle. So it is $-360^\circ$ for an inner border and $360^\circ$ for the outer border. The token can represent the angle as an integer $k$ such that the angle is $k \cdot 60^\circ$. Furthermore, to distinguish the two possible final values of $k$ it is sufficient to store the $k$ modulo 5, so that the token only requires 3 bits of memory.

**Fig. 5.** The angle between the directions a token enters and exits an agent.

angle of the polygon corresponding to the cycle while respecting the rotation of the cycle. So it is $-360^\circ$ for an inner border and $360^\circ$ for the outer border. The token can represent the angle as an integer $k$ such that the angle is $k \cdot 60^\circ$. Furthermore, to distinguish the two possible final values of $k$ it is sufficient to store the $k$ modulo 5, so that the token only requires 3 bits of memory.

**Safety, Liveness & Runtime** Here, we give an intuition on how the proofs of Section 2.3 and 2.4 have to be adapted for the local-control protocol. The main obstacle is asynchronous behavior of agents, since subphases are not coordinated anymore.

For safety consider a candidate $c$ with the longest front segment. If $c$ is in the first or third subphase it cannot revoke candidacy. However, in subphase 2 it is possible that $c$ gets covered by another candidate $c'$ while transferring
candidacy. Since we cannot make any assumptions about front segment length of \( c' \), also \( c' \) can be covered by another candidate in the same round. We can inductively continue this argument. However, there has to exist a candidate \( c^* \) that does not revoke candidacy in the same round, since \( c \) itself is in phase 2. Therefore, it cannot cover any candidate and we cannot close the cycle to get a cyclic relation of covering each other.

Liveness can be shown analogous to the proof in the synchronous scenario since the candidate with the longest front segment either covers candidates in subphase 1 or (in case all segments have equal length) revokes candidacy with probability \( \frac{1}{4} \) in subphase 2.

Note that the different local-control protocols for the subphases all require at most \( O(l_i) \) rounds for each candidate, since the number of tokens used for each subphase and the distance they travel is dependent on the length of the front segment. In order to show the desired runtime one has to argue that the longest segment still doubles after a constant number of rounds in expectation. In case all segments have equal length, we can directly apply the analysis of the synchronous case, since candidates will never be covered by another candidate. In case segment lengths are not equal, it is possible that a candidate with the longest segment gets covered by another candidate while transferring candidacy. However, it is either covered by its front candidate or by a candidate that has already covered its front candidate. Both cases cannot prevent the process of doubling \( l_i \).

3 Path Formation in the Geometric Amoebot Model

Next we consider the path formation problem under the geometric amoebot model as introduced at the beginning of Section 2. We aim at solving the path formation problem by arranging the particles in a straight line in \( G_{eqt} \). We assume that initially we have an arbitrary connected structure of contracted particles with a unique leader. The leader is used as the starting point for the line of particles and specifies the direction in which this line will grow. As the line grows, every particle touched by the line that is already in a valid line position becomes part of the line. Any other particle connected to the line becomes the root of a tree of particles. Every root aims at traveling around the line in a clockwise manner until it joins the line. As a root particle moves, the other particles in its tree follow in a worm-like fashion (i.e., via a series of handover operations).

Before we give a detailed description of the algorithm, we provide some preliminaries. We define the set of \textit{states} that a particle can be in as \textit{inactive}, \textit{follower}, \textit{root}, and \textit{retired}. Initially, all particles are \textit{inactive}, except the leader particle, which is always in a \textit{retired} state. In addition to its state, each particle \( p \) may maintain a constant number of \textit{flags} in its shared memory. For an expanded particle, we denote the node the particle last expanded into as the \textit{head} of the.

\footnote{For a simulation video of the Line Formation Algorithm please see [http://sops.cs.upb.de](http://sops.cs.upb.de).}
particle and call the other occupied node its tail. In our algorithm, we assume that every time a particle contracts, it contracts out of its tail.

The spanning forest algorithm, given in Algorithm 2, is a basic building block we use for shape formation problems. This algorithm aims at organizing the particles as a spanning forest, where the particles that represent the roots of the trees determine the direction of movement, whom the remaining particles follow. Each particle $p$ continuously runs Algorithm 2 until $p$ becomes retired. If particle $p$ is a follower, it stores a flag $p.parent$ in its shared memory corresponding to the edge adjacent to its parent $p'$ in the spanning forest (any particle $q$ can then easily check if $p$ is a child of $q$). If $p$ is the leader particle, then it sets the flag $p.linedir$ in the shared memories corresponding to two of its edges in opposite directions (i.e., an edge $i$ and the edge that appears three positions after $i$ in clockwise order), denoting that it would like to extend the line through the directions given by these edges.

### 3.1 Correctness of the Spanning Forrest

Before we show correctness and runtime of the line formation algorithm, we prove some general properties concerning the correctness of the spanning forest itself. We say followers and roots particles are active. As specified in Algorithm 2, only followers can set the flag $p.parent$. The first three lemmas demonstrate some properties that hold during the execution of the spanning forest procedure and will be used in Section 3.2 to analyze the complete algorithm, when we incorporate the check for retirement of particles according to the line formation problem, and the propagation of the line direction.

The configuration of the system of particles at time $t$ consists, for every particle $p$, of the current state of $p$, including whether the particle is expanded or contracted, of any flags in $p$’s shared memory, and of the node(s) $p$ occupies in $G$ (given by the relative position of $p$ according to the other particles) at time $t$, as well as the labeling of the bonds of $p$. Following the asynchronous model, the system of particles progresses by performing atomic actions, each of which affects the configuration of one or two particles.

**Lemma 4.** For a follower $p$, the node indicated by $p.parent$ is occupied by an active particle.

**Proof.** Consider a follower $p$ in any configuration during the execution of Algorithm 2. Note that $p$ can only become a follower from an inactive state, and once it leaves the follower state it will not switch to that state again. Consider the first configuration $c_1$ in which $p$ is a follower. In the configuration $c_0$ immediately before $c_1$, $p$ must be inactive and it becomes a follower because of an active particle $p'$ occupying the position indicated by $p.parent$ in $c_0$. The particle $p'$ is still adjacent to the edge flagged by $p.parent$ in $c_1$. Now assume that $p.parent$ points to an active particle $p'$ in a configuration $c_i$, and that $p$ is still a follower in the next configuration $c_i+1$ that results from executing an action $a$. If $a$ affects $p$ and $p'$, the action must be a handover in which $p$ updates its flag $p.parent$ such that
Algorithm 2 Line Formation Algorithm

SpanningForest \((p)\):
Particle \(p\) acts as follows, depending on its current state:

**inactive:** If \(p\) is connected to a retired particle, then \(p\) becomes a root particle. Otherwise, if an adjacent particle \(p'\) is a root or a follower, \(p\) sets the flag \(p.parent\) on the shared memory corresponding to the edge to \(p'\) and becomes a follower. If none of the above applies, it remains inactive.

**follower:** If \(p\) is contracted and connected to a retired particle, then \(p\) becomes a root particle. Otherwise, it considers the following three cases: (i) if \(p\) is contracted and \(p\)'s parent \(p'\) (given by the flag \(p.parent\)) is expanded, then \(p\) expands in a handover with \(p'\), adjusting \(p.parent\) to still point to \(p'\) after the handover; (ii) if \(p\) is expanded and has a contracted child particle \(p'\), then \(p\) executes a handover with \(p'\); (iii) if \(p\) is expanded, has no children, and \(p\) has no inactive neighbor, then \(p\) contracts.

**root:** Particle \(p\) may become retired following CheckRetire \((p)\). Otherwise, it considers the following three cases: (i) if \(p\) is contracted, it tries to expand in the direction given by LineDir \((p)\); (ii) If \(p\) is expanded and has a child \(p'\), then \(p\) executes a handover contraction with \(p'\); (iii) if \(p\) is expanded and has no children, and no inactive neighbor, then \(p\) contracts.

**retired:** \(p\) performs no further action.

CheckRetire \((p)\):
if \(p\) is a contracted root then
    if \(p\) has an adjacent edge \(i\) to \(p'\) with a flag \(p'.linedir\), where \(p'\) is retired then
        Let \(i'\) be the edge opposite to \(i\) in clockwise order
        \(p\) sets the flag \(p.linedir\) in the shared memory of edges \(i\) and \(i'\)
        \(p\) becomes retired.

LineDir \((p)\):
Let \(i\) be the label of an edge connected to a retired particle.
while edge \(i\) points to a retired particle do
    \(i \leftarrow\) label of next edge in clockwise direction
return \(i\)

\(p.parent\) may be moved to the edge that now connects \(p\) to \(p'\) in \(c_{i+1}\). If \(a\) affects \(p\) but not \(p'\), it must be a contraction in which \(p.parent\) does not change and still points to \(p'\). If \(a\) affects \(p'\) but not \(p\), there are multiple possibilities. The particle \(p'\) might switch from follower to root state, or from root to retired state, or it might expand, none of which violates the lemma. Furthermore, \(p'\) might contract. If \(p.parent\) points to the head of \(p'\), \(p'\) is still adjacent to the edge flagged by \(p.parent\) in \(c_{i+1}\). Otherwise, \(p\) is a child adjacent to the tail of \(p'\) in \(c_i\) and therefore the contraction must be part of a handover. As \(p\) is not involved in the action, the handover must be between \(p'\) and a third active particle \(p''\). It is easy to see that after such a handover \(p.parent\) points to either \(p'\) or \(p''\). Finally, if \(a\) affects neither \(p\) nor \(p'\), \(p.parent\) will still point to \(p'\) in \(c_{i+1}\).

Based on Lemma 3, we define a directed graph \(A(c)\) for a configuration \(c\) as follows. \(A(c)\) contains the same nodes as the nodes occupied in \(G_{eqt}\) by the set
of particles in $c$. For every expanded particle $p$ in $c$, $A(c)$ contains a directed edge from the tail to the head of $p$, and for every follower $p'$ in $c$, $A(c)$ contains a directed edge from the head of $p'$ to $p'.\text{parent}$.

**Lemma 5.** The graph $A(c)$ is a forest, and if there is at least one active particle, every connected component of inactive particles contains a particle that is connected to an active particle.

**Proof.** In an initial configuration $c_0$, all particles are inactive and therefore the lemma holds trivially. Now assume that the lemma holds for a configuration $c_i$. We will show that it also holds for the next configuration $c_{i+1}$ that results from executing an action $a$. If $a$ affects an inactive particle $p$, this particle either becomes a follower or a root. In the former case $p$ joins an existing tree, and in the latter case $p$ forms a new tree in $A(c_{i+1})$. In either case, $A(c_{i+1})$ is a forest and the connected component of inactive particles that $p$ belongs to in $c_i$ is either non-existent or connected to $p$ in $c_{i+1}$. If $a$ affects only a single particle $p$ that is in state follower, this particle can contract or become a root. In the former case, $p$ has no child $p'$ such that $p'.\text{parent}$ is the tail of $p$ and also $p$ has no inactive neighbors. Therefore, the contraction of $p$ does not disconnect any follower or inactive particle and, accordingly, does not violate the lemma. In the latter case, $p$ becomes a root of a tree which also does not violates the lemma. If $a$ involves only a single particle $p$ that is in state root, $p$ can expand or contract or become retired. An expansion and becoming retired trivially cannot violate the lemma and the argument for the contraction is the same as for the contraction of a follower above. Finally, if $a$ involves two active particles in $c_i$, these particles perform a handover. While such a handover can change the parent relation among the nodes, it cannot violate the lemma. \hfill $\square$

The following lemma shows that the spanning forest always makes progress, by showing that as long as the roots keep moving, the remaining particles will eventually follow.

**Lemma 6.** An expanded particle eventually contracts.

**Proof.** Consider an expanded particle $p$ in a configuration $c$. Note that $p$ must be active. If there is an enabled action that includes the contraction of $p$, that action will remain enabled until $p$ eventually contracts when $p$ is validated in the current round. So assume that there is no enabled action that includes the contraction of $p$. According to Lemma 5 and the transition rule from inactive to active particles, at some point in time all particles in the system will be active. If the contraction of $p$ becomes part of an enabled action before this happens, $p$ will eventually contract. So assume that all particles are active but still $p$ cannot contract. If $p$ has no children, the isolated contraction of $p$ is an enabled action which contradicts our assumption. Therefore, $p$ must have children.

Furthermore, $p$ must read at least one child $p'$ having its $p'.\text{parent}$ flag pointing towards $p$ over its tail and all children having their parent flags pointing towards $p'$s tail must be expanded as otherwise $p$ could again contract as part of
a handover. If \( p' \) would contract, a handover between \( p' \) and \( p \) would become an enabled action. We can apply the complete argument presented in this proof so far to \( p' \) and so on backwards along a branch in a tree in \( A(c) \) until we reach a particle that can contract. We will reach such a particle by Lemma 5. Therefore, we found a sequence of expanded particles that starts with \( p' \) and ends with a particle that eventually contracts. The contraction of that last particle will allow the particle before it in the sequence to contract and so on. Finally, the contraction of \( p \) will become part of an enabled action and therefore \( p \) will eventually contract.

\[ \Box \]

### 3.2 Correctness and Complexity of Line Formation

Now, we can show that the algorithmic primitives as developed in the Section 3 solve the line formation problem.

**Theorem 3.** Algorithm 2 correctly solves the line formation problem.

**Proof.** We need to show that the algorithm terminates and that when it does, the formed shape is a straight line. As a result of Lemma 5, every particle \( p \) eventually becomes active. According to the algorithm proposed for line formation problem, if \( p \) is adjacent to the leader particle, it becomes a root and moves in a clockwise manner around the current line structure until it eventually reaches one of the valid positions that can extend the line and becomes retired. By contradiction, assume \( p \) never becomes retired. Since the number of particles is bounded (and therefore the size of the current line structure is bounded), there must be an infinite number of configurations \( c_i \) where \( p \) had a root particle blocking its desired clockwise movement around the line structure. Let \( p' \) be the last root \( p \) sees as its clockwise neighbor over the line structure (since once a particle becomes a root, it will stay connected to the line structure and always attempt to move in a clockwise manner, \( p' \) is well-defined). Applying the same argument inductively to \( p' \), we will get an infinite sequence of roots on the line structure that never touch a valid spot pointed by one of \( p.linedir \) flags of an already retired particle, a contradiction, since the current line structure (and the number of retired particles) is bounded. Therefore, every root eventually changes into a retired state.

From Algorithm 2, every follower in the neighborhood of a retired particle becomes a root. For every root \( q \) with at least one follower child, let \( c \) be the first configuration when \( q \) becomes retired. If \( q \) still has any child in \( c \) then all of its children \( p \) become roots. Applying this argument recursively we will reach to a configuration such that there exists no root \( q \) having a follower child which proves that eventually every follower becomes a root.

Putting it all together, eventually all particles become retired and the algorithm terminates.

Note that it also follows from the argument above that the set of retired particles at the end of the algorithm forms a connected structure (since the particles start from a connected configuration and never get disconnected through
the process). The connected structure must form a line, since a root particle may only become retired if it is contracted and it now occupies a previously unoccupied spot following the direction of the line. □

Finally, we evaluate the performance of our algorithm in terms of rounds until the line is formed and also in terms of the number of movements (expansions and contractions) of the particles, i.e., the total work performed by the algorithm.

**Theorem 4.** Algorithm 2 solves the line formation problem in worst-case optimal $O(n)$ number of rounds and $O(n^2)$ work.

The theorem follows from Lemmas 7, 8 and 9.

**Lemma 7.** The worst-case work required by Algorithm 2 to solve the line formation problem is $\Omega(n^2)$.

**Proof.** Consider the configuration depicted in Figure 6. The particle labeled $i$ requires at least $2i$ movements before it becomes adjacent to the leader particle and becomes retired. Therefore, proposed algorithm requires at least $\sum_{i=0}^{n-c-1} 2i = \Omega(n^2)$. □

![Fig. 6. Worst-case configuration concerning work. The leader is shown in black and the other system particles are shown in grey. There are $c$ particles adjacent to the leader, where $c$ is a constant, and $n-c-1$ are located on a straight line (numbered 0, 1, 2, ...). We use two arrows representing the edges that the leader selects for line formation.

In the following, we will show a matching upper bound:

**Lemma 8.** Algorithm 2 terminates in $O(n^2)$ work.

**Proof.** To prove the upper bound, we simply show that every particle executes $O(n)$ movements. The theorem then follows. Consider a particle $p$. While $p$ is in inactive or a retired state, it does not move. Let $c$ be the first configuration when $p$ becomes a follower. Consider the directed path in $A(c)$ from the head of $p$ to its root $p'$. There always is a such a path since every follower belongs to a tree in $A(c)$ by Lemma 5.

Let $P = (a_0, a_1, \ldots, a_m)$ be that path in $A(c)$ where $a_0$ is the head of $p$ and $a_m$ is a child of $p'$. According to Algorithm 2, $p$ attempts to follow $P$ by...
sequentially expanding into the nodes $a_0, a_1, \ldots, a_m$. The length of this path is bounded by $2n$ and, therefore, the number of movements $p$ executes while being a follower is $O(n)$. Once $p$ becomes a root, it only performs expansions and contractions around the currently constructed line structure, until it reaches one of the valid positions on the line. Since the total number of retired particles is at most $n$, this leads to an additional $O(n)$ movements by $p$. Therefore, the number of movements a particle $p$ executes is $O(n)$, which concludes the theorem. □

Now we evaluate our algorithm in terms of the number of the rounds it takes until it terminates in a line configuration (note that Theorem 8 also follows trivially from Theorem 9).

**Lemma 9.** Algorithm 2 is optimal in terms of worst-case number of rounds, i.e., it terminates in $O(n)$ rounds.

**Proof.** We give a proof sketch for Theorem 9. First we need to prove that the number of the rounds needed for all the particles in a spanning forest containing a single tree that consists of a single path of length $k$ to touch a particle on the line is $\Theta(k)$. Second, we show that given any arbitrary tree $T$ of size $k'$ we can always construct another tree $T'$ that has exactly "one less branch" than $T$ by removing a "V" in $T$ (consisting of the paths $P = p_1, \ldots, p_x$ and $Q = q_1, \ldots, q_y$, $x, y > 1$, in $T$, such that $p_1$ and $q_1$ are leaves of $T$ and $p_x = q_y$, and where all the other $p_i$'s, $1 < i < x$ and $q_j$'s, $1 < j < y$, have exactly one child) and replacing it by the path $p_1, \ldots, p_{x-1}, q_1, \ldots, q_{y-1}$ to be attached directly to node $q_y$ (i.e., $q_y$ will be the parent of $q_{y-1}$). Since we have $x + y - 1$ nodes in $P \cup Q$, we must have that the number of rounds in $T$ when a node in $P \cup Q$ has an enabled action that will contribute to bringing all the nodes in $P \cup Q$ past the depth occupied by $p_x$ in $T$ is $\Omega(x + y)$. We use the results that the number of rounds required to bring all the particles in a path of length $k$ is $\Theta(k)$ past the depth of the path’s root to claim that the number of rounds when a node in $P \cup Q$ executes an enabled action that will contribute to bringing all the nodes in $P \cup Q$ past the depth occupied by $p_x$ in $T'$ (which is the same depth as that of $p_x$ in $T$) is $\Theta(x + y)$ and hence that the total number of required rounds by our algorithm for $T'$ is no less than that for $T$. We apply this argument recursively until we are left with a spanning forest formed by trees consisting of simple paths. Moreover, if at any point during the algorithm any two roots $r_1$ and $r_2$ (of path-trees) become adjacent, we keep the root that is furthest in CW direction around the current line structure — say root $r_1$ w.l.o.g. — and linearize the rest of the path following $r_1$ using the same "de-branching" process above. Hence we can now assume, from a worst-case number of rounds perspective, that at any point in time during the execution of the algorithm, we have trees consisting of single paths in the forest such that no two roots are adjacent. Then using the result for single path-trees, we can conclude that the total number of rounds required by our algorithm until all particles touch the line structure will be, in the worst-case, asymptotically equal to the length of the longest path-tree that we formed in our "de-branching" process, i.e., $O(n)$. 


In order to conclude the proof, we must present a bound on the number of rounds necessary until the last particle that touches the line reaches its final position on the line. Since any set of adjacent particles touching the line forms a path, basically the same argument used for the number of rounds required until all particles touch the line will be enough to show that in an additional $O(n)$ rounds after the last particle touched the line, all the particles will have reached their final line position.

It remains to show a sketch on how we can prove that it takes $\Theta(k)$ rounds until all the $k$ particles in a path-tree touch the line. First, for a path with $k$ particles whose root is connected to a retired particle, it will take at most $O(k)$ rounds for all the particles to become active. After all particles become active, we can show by induction that in another $O(k)$ rounds, we will reach a configuration $c$ where the particles appear in alternating expanded and contracted state in the path. Then we can show that in every three rounds, the leaf particle $p$ of the path gets one position closer to touching a particle on the line.

To see that the worst-case guarantees on the number of rounds is optimal, we again have to consider a forest consisting of a single path (as depicted in Figure 6).

4 **Self-stabilizing Leader Election and Shape Formation**

Consider the variant of the geometric amoebot model in which faults can occur that arbitrarily corrupt the local memory of a particle. For an algorithm to solve the self-stabilizing leader election problem it has to satisfy the following requirements: First, once no more faults occur, the particle system converges towards a valid goal configuration, i.e., eventually a unique leader will establish. Second, once a valid goal configuration is reached the system has to remain in a valid goal configuration as long as no faults occur. Third, even in presence of faults the system has to remain connected. An analog definition can be made for self-stabilizing shape formation.

Our leader election algorithm can be extended to a self-stabilizing leader election algorithm with $O(\log^* n)$ memory using the results of [6,30]. However, it is not possible to design a self-stabilizing algorithm for the line formation. The reason for this is that even a much simpler problem called movement problem cannot be solved in a self-stabilizing manner. It is easy to see that if the movement problem cannot be solved in a self-stabilizing manner, then also the line formation problem cannot be solved in a self-stabilizing manner.

In the movement problem we are given an initial distribution $A$ of particles that can be in a contracted as well as expanded state, and the goal is to change the set of nodes occupied by the particles without causing disconnectivity. For the ring of expanded particles it holds that for any protocol $P$ there is an initial state so that $P$ does not solve the movement problem. To show this we consider two cases: suppose that there is any state $s$ for some particle in the ring that would cause that particle to contract. In this case set two particles on opposite sides of the ring to that state, and the ring will break due to their contractions.
Otherwise, \( P \) would not move any particle of the ring, so also in this case it would not solve the movement problem.

5 Conclusion

All of the algorithms presented for the geometric amoebot model can be trivially extended when \( G \) is any other regular grid graph, namely the square regular grid graph or the hexagonal regular grid graph.

As future work, we would like to identify the minimum set of key geometric properties that \( G \) must have in order for the proposed algorithms to work. Also, if in the geometric amoebot model, the particles had a common sense of direction, we would like to investigate whether leader election could be solved deterministically using a slight modification of our algorithm: For each border the last remaining candidate is deterministically chosen to be the "East-most" particle of the set of the "South-most" particles. This algorithm would be similar to the one proposed in [27] for tile self-assembly systems.

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