Hecke algebras as subalgebras of Clifford geometric algebras of multivectors

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Abstract
Clifford geometric algebras of multivectors are introduced which exhibit a bilinear form which is not necessarily symmetric. Looking at a subset of bi-vectors in $\mathcal{C}(\mathbb{K}^{2n}, B)$, we proof that these elements generate the Hecke algebra $H_{\mathbb{K}}(n + 1, q)$ if the bilinear form $B$ is chosen appropriately. This shows, that $q$-quantization can be generated by Clifford multivector objects which describe usually composite entities. This contrasts current approaches which give deformed versions of Clifford algebras by deforming the one-vector variables. Our example shows, that it is not evident from a mathematical point of view, that $q$-deformation is in any sense more elementary than the undeformed structure.

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1 Introduction
Recent developments in theoretical physics employ the so called noncommutative geometry [1] or in a more special case $q$-deformed geometry [2, 3, 4]. The underlying structure is either the $\mathcal{C}^*$-theory which incorporates also topological and convergence aspects or else Hopf algebras, which model the algebraic aspects of a theory [5, 6]. It is convenient to speak about $q$-symmetry since the spaces on which $q$-symmetry acts tend to be braided. It is thus convenient to study braided monoidal categories, e.g [7, 8] and many others. The main idea is to introduce a braided tensor product algebra structure

\[(a \otimes b)(c \otimes d) = a\Psi(b \otimes c)d,\]

where $\Psi$ is a braiding. If $\Psi$ occurs to be trivial or minus the flip operator $\Psi(a \otimes b) = -b \otimes a$, one deals with the ordinary tensor product (bosons) or a $\mathbb{Z}_2$-graded version of it (fermions). A general braiding leads thus to a general or braid statistics. The central relations obeyed by braid groups are the Artin braid relations [8]

\[b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, \quad b_i b_k = b_k b_i, \quad |i - k| \geq 2.\]
The first of them is actually equivalent to the so called quantum Yang–Baxter equation, which is the special case of the Yang–Baxter equation \([10]\) (in standard notation)

\[ R_{12}(u)R_{13}(u + v)R_{23}(v) = R_{23}(v)R_{13}(u + v)R_{12}(v) \]  

(3)

with the spectral parameters set to \(v = u + v = u\).

There was a great progress in the theory of (quantum) statistical mechanics, which originated in the development of the inverse scattering method \([11]\) and the star triangle relation \([12]\), both methods having their roots in braided symmetries, see e.g. \([13]\). There are lots of models now solvable with this methods, Ising \([14]\) and \(N\)-state Potts models \([15, 16]\), Vertex \([17]\) and IRF models \([18]\) may be prominent examples. A further example might be given by the (fractional) quantum Hall effect \([19]\). Furthermore, the unexpected connection between link invariants and type III subfactors of von Neumann algebras unveiled by V. Jones pushed low dimensional topology far ahead, see \([20, 21]\). There is even a connection of the Jones polynomial to quantum field theory \([22]\).

A further branch of applications arises from the common believe, that \(q\)-symmetry being more general than the usual bosonic or fermionic ones and thus is more fundamental, see e.g. \([2, 23, 24]\). The natural thing to do is thus to provide \(q\)-deformed versions of physical relevant groups e.g. the Poincaré group \([25]\). There is a strong believe, that the fundamental constant \(\hbar\) is involved in this construction and that space-time should behave “\(q\)-symmetric” at small scales.

The above mentioned situations when \(q\)-symmetry leads to explicit results shear the feature of being effective or composite models. There is no recent evidence that \(q\)-symmetry has to be used in fundamental interactions. Moreover, it might be expected that a \(q\)-deformed Poincaré group has an underlying structure which generates this symmetry.

From a mathematical point, there is no harm in \(q\)-deforming all structures which can be done so. But a physical application requires an interpretation which seems currently not obvious, but relays on rather abstract developments as quantum plains and \(q\)-deformed or noncommutative geometry.

We are thus in the perplexing situation, that because of its generality \(q\)-deformation can be applied to nearly every mathematical structure which is currently used in physics. But we don’t know in which cases it might be reasonable. To be able to decide this question, it is a valuable advantage to have an embedding of the mathematical structure which lays at the heart of \(q\)-deformation, the Hecke algebra, in a larger framework. From this outstanding point of view it might be possible to decide if \(q\)-symmetry has to be applied to e.g. gravitation or not.

A very interesting approach to \(q\)-symmetry by spinors and thereby also with the help of Clifford algebras can be found in \([6]\). This approach, however, takes \(q\)-symmetry as an elementary property. In the same spirit, the Clifford algebra of a Hecke braid was constructed in \([26]\).

We will provide a theorem, which shows that Hecke algebras, can be obtained as subalgebras of certain Clifford algebras. This subalgebras are generated by bivector elements and thus by objects which are composed. Furthermore, since the interpretation of Clifford algebraic expressions is well known, we come to the end that \(q\)-symmetry is tightly interwoven with composite structures, as was suggested already in \([27]\). This relation is seen from the fact that \(q\)-symmetry is obtained in this approach as a multivector symmetry. It is this relation, that makes us so suspicious against a \(q\)-deformation of space-time as long as one does not have a microscopic description of these entities. Hopefully our approach will open a possibility to clarify this situation.
2 Clifford geometric algebra of multivectors

There are many possibilities to introduce Clifford algebras, each of them emphasize a different point of view. In our case, it is of utmost importance to have the Clifford algebra build over a graded linear space. This grading is obtained from the space underlying a Grassmann algebra. The Clifford algebra is then related to the endomorphism algebra of this Grassmann algebra. This construction, the Chevalley deformation [28], was originally invented to be able to treat Clifford algebras over fields of \( \text{char} = 2 \), see appendix of [29] by Lounesto and [36]. However, we use this construction in an entirely different context. With help of the construction of M. Riesz [29], one is able to reconstruct the multivector structure and thereby a correspondence between the linear spaces underlying the Clifford algebra and the Grassmann algebra in use. This reconstruction depends on an automorphism \( J \), which is arbitrary, see [31]. In fact this is just the reversed direction of our construction given below following Chevalley.

Let \( T(V) \) be the tensor algebra build over the \( K \)-linear space \( V \). The field \( K \) will be either \( \mathbb{R} \) or \( \mathbb{C} \). With \( V^0 \cong K \) we have

\[
T(V) = K \oplus V \oplus V \otimes K \oplus V \otimes \ldots
\]

The tensor algebra is associative and unital. In \( T(V) \) one has bilateral or two-sided ideals, which can be used to construct new algebras by factorization. As an example we define the Grassmann algebra in this way.

**Definition 1** The Grassmann algebra \( \bigwedge(V) \) is the factor algebra of the tensor algebra w.r.t. the bilateral ideal

\[
I_{Gr} = \{ y \mid y = a \otimes x \otimes x \otimes b, \ a, b \in T(V), x \in V \}
\]

\[
\bigwedge(V) = \pi(T(V)) = \frac{T(V)}{I_{Gr}} = K \oplus V \oplus V \wedge V \oplus \ldots.
\]

The canonical projection \( \pi : T(V) \mapsto \bigwedge(V) \) maps the tensor product \( \otimes \) onto the exterior or wedge product denoted by \( \wedge \).

One may note, that the factorization preserves the grading naturally inherited by the tensor algebra, since the ideal \( I_{Gr} \) is homogeneous. Defining homogeneous parts of \( T(V) \) and \( \bigwedge(V) \) by \( T^k(V) = V \otimes \ldots \otimes V \) and \( \bigwedge^k(V) = V \wedge \ldots \wedge V \), \( k \)-factors, we obtain \( \pi(T^k(V)) = \bigwedge^k(V) \).

Proceeding to Clifford algebras requires a further structure, the quadratic form.

**Definition 2** The map \( Q : V \mapsto K \), satisfying \( (\alpha \in K, x, y \in V) \)

\[
i) \quad Q(\alpha x) = \alpha^2 Q(x)
\]

\[
ii) \quad B_p(x, y) = \frac{1}{2}(Q(x + y) - Q(x) - Q(y)),
\]

where \( B_p(x, y) \) is a symmetric bilinear form is called a quadratic form.

It is tempting to introduce an ideal \( I_C \)

\[
I_C = \{ y \mid y = a \otimes (x \otimes x - Q(x) I) \otimes b, \ a, b \in T(V), x \in V \}
\]

(7)

to obtain the Clifford algebra by a factorization procedure. However, since we are interested in arbitrary bilinear forms underlying a Clifford algebra, we will take another approach, which is much more reasonable for such a structure. Furthermore,
the Clifford algebra does not have an intrinsic multivector structure, but is only $\mathbb{Z}_2$ graded, since the ideal $I_{c_2}$ is inhomogeneous.

Let $V^*$ be the space of linear forms on $V$, i.e. $V^* \simeq \text{lin}[V, K]$. Elements $\omega \in V^*$ act on elements $x \in V$, but there is no natural identification between $V$ and $V^*$. However, we can find a set of $x_i$ which span $V$ and dual elements $\omega_k$ acting on the $x_i$ in a canonical way

$$\omega_k(x_i) = \delta_{ki}. \quad (8)$$

This allows to introduce a map $\ast : V \mapsto V^*$, $x^*_i = \omega_i$ which may be called Euclidean dual isomorphism \cite{30}. The two spaces $(V^*, V)$ connected by this duality constitute a pairing $\langle . \mid . : \rangle : V^* \times V \mapsto K$. $V^*$ are isomorphic to $V$ in finite dimensions, so it is natural to build a Grassmann algebra $\Lambda(V^*)$ over it. This is the algebra of Grassmann multiforms.

It is further a natural thing to extend the pairing of the grade-one space and its dual to the whole algebras $\Lambda(V)$ and $\Lambda(V^*)$, as can be seen by its frequent occurrence in literature \cite{31, 32, 33, 34, 35, 36} and others. This can be done by the

**Definition 3** Let $\tau, \eta \in \Lambda(V^*)$, $\omega \in V^*$, $u, v \in \Lambda(V)$ and $x \in V$, then we can define a canonical action of $\Lambda(V^*)$ on $\Lambda(V)$ requiring

i) $\omega(x) = \langle \omega \mid x \rangle$

ii) $\omega(u \wedge v) = w(u) \wedge v + \hat{u} \wedge \omega(v)$

iii) $(\tau \wedge \eta)(u) = \tau(\eta(u)) \quad (9)$

where $\hat{u}$ is the main involution $\hat{V} = -V$ extended to $\Lambda(V)$.

In fact we have given by definition 3 an isomorphism between the Grassmann algebra $\Lambda(V^*)$ and the dual Grassmann algebra $\Lambda(V)^*$. This can be made much clearer in writing

$$y \cdot x = \omega_y(x) = \langle \omega_y \mid x \rangle = B(y, x), \quad (10)$$

where we have used the canonical identification of $V$ and $V^*$ via the pairing. One should be very careful in the distinction of $\Lambda(V^*)$ and $[\Lambda(V)]^*$, since they are isomorphic but not equivalent. Furthermore, we emphasize that in writing $y \cdot$ we make explicitly use of a special dual isomorphism encoded in the contraction

$$y \cdot : V \mapsto V^*$$

$$y \mapsto y \cdot = \omega_y. \quad (11)$$

Since there is no natural, that is mathematically motivated or even better functorial relation between $V$ and $V^*$, we are called to seek for physically motivated reasons to select a pairing. This freedom will enable us in section 3 to give a proof of our main theorem.

**Theorem 4** Let $(V, Q)$ be a pair of a $K$-linear space $V$ and $Q$ a quadratic form $Q$ as defined in 2. There exists an injection $\gamma$ called Clifford map from $V$ into the associative unital algebra $\cal{C}(V, Q)$ which satisfies

$$\gamma x \gamma x = Q(x) \mathbb{I}. \quad (12)$$

**Definition 5** The (smallest) algebra $\cal{C}(V, Q)$ generated by $\mathbb{I}$ and $\gamma x_i$, $\{x_i\}$ span $V$ is called (the) Clifford algebra of $Q$ over $V$. \hfill \blacksquare
By polarization of the relation (12) we get the usual commutation relations; \( x, y \in V \)

\[
\gamma_x \gamma_y + \gamma_y \gamma_x = 2B_p(x, y) \mathbb{I},
\]

where \( B_p(x, y) \) is the symmetric polar form of \( Q \) as defined in (6).

**Remarks:** i) We could have obtained this result directly by factorization of the tensor algebra with the ideal (7). ii) There exists Clifford algebras with are universal, in this case it is convenient to speak from the Clifford algebra over \( (V, Q) \). iii) If \( V \cong \mathbb{K}^n \cong \mathbb{C}^n \) or \( \mathbb{R}^n \), we denote \( \text{CL}(V, Q) \) also by \( \text{CL}(\mathbb{C}^n) \cong \mathcal{O}_n \) or \( \mathcal{O}(\mathbb{R}_{p,q}) \) where the pair \( p, q \) enumerate the number of positive and negative eigenvalues of \( Q \). We can as well give the dimension \( n \) and signature \( s = p - q \) to classify all quadratic forms over \( \mathbb{R} \). In the case of the complex field, on remains with the dimension as can be seen e.g. from the Weyl unitary trick, letting \( x_i \rightarrow ix_i \) which flips the sign. We do not use sesquilinear forms here, which could be included nevertheless.

We will now use Chevalley deformations to construct the Clifford algebra of multivectors. The main idea is, that one can decompose the Clifford map

\[
\gamma_x = x \mathcal{J} + x \wedge.
\]

There is thus a natural action of \( \gamma_x \) on \( \bigwedge(V) \).

**Theorem 6 (Chevalley)** Let \( \bigwedge(V) \) be the Grassmann algebra over \( V \) and \( \gamma : V \mapsto \text{End}(\bigwedge(V)) \), be defined as in (14), then \( \gamma \) is a Clifford map.

We have shown that \( \mathcal{O} \) is a subalgebra of the endomorphism algebra of \( \bigwedge(V) \),

\[
\mathcal{O} \subseteq \bigwedge(V).
\]

It is possible to interpret \( x \mathcal{J} \) and \( x \wedge \) as annihilating and creation operators (on the Grassmann algebra) \( \bigwedge(V) \).

With help of the relations (1) we can then lift this Clifford map to multivector actions. **No symmetry requirement has to be made on the contraction.** This leads to the

**Definition 7 (Clifford algebra of multivectors)** Let \( B : V \times V \mapsto \mathbb{K} \) be an arbitrary bilinear form. The Clifford algebra \( \mathcal{O}(V, B) \) obtained from lifting the Clifford map

\[
\gamma_x = x \mathcal{J} + x \wedge = \langle x \mid \cdot + \cdot x \wedge = B(x, \cdot) + x \wedge
\]

\((16)\)
to \( \text{End}(\bigwedge(V)) \) using the relations (14) is called Clifford algebra of multivectors.

Note, that \( B(x, \cdot) = \omega_x \) is a map from \( V \mapsto V^* \) and incorporates a dual isomorphism. It is clear from the construction that \( \mathcal{O}(V, B) \) has a multivector structure or say a \( \mathbb{Z}_n \)-grading inherited from the Grassmann algebra \( \bigwedge(V) \).

\( B \) admits a decomposition into symmetric and antisymmetric parts \( B = G + F \). The symmetric part \( G = B_p \) corresponds to a quadratic form \( Q \), see definition (2).

**Theorem 8** The Clifford algebra \( \mathcal{O}(V, Q) \cong \mathcal{O}(V, G) \) is isomorphic as Clifford algebra to \( \mathcal{O}(V, B) \), if \( B \) admits a decomposition \( B = G + F \), \( G^T = G \), \( F^T = -F \).

A proof can be found for low dimensions in [26] and in general in [8]. However, this result was implicitly known to physicists, see [26, 39]. In fact, this is the old Wick rule of QFT. We will insist on the \( \mathbb{Z}_n \)-grading and therefore carefully distinguish Clifford algebras of multivectors with a common quadratic form \( Q \) but different contractions \( B \). Only this generalization makes it possible to find Hecke algebras as subalgebras in Clifford algebras.
We give some further notations. Let \{j_i\} be a set of elements spanning \( V \cong \langle j_1, \ldots, j_n \rangle \) and \{\partial_k\} be a set of dual elements. Building the Grassmann algebras \( \wedge(V) \) and \( \wedge(V^*) \) and defining the action of the forms via (3), one obtains the relations

\[
\begin{align*}
    i) & \quad j_i \wedge j_i = 0 = \partial_i \wedge \partial_i \\
    ii) & \quad \partial_i j_k + j_k \partial_i = B_{ik} + B_{ki} = 2G_{ik}.
\end{align*}
\]

The space \( V = V \oplus V^* \) is thus spanned by (note the reversed order of indices for the \( \partial \) elements)

\[
\{e_1, \ldots, e_{2n}\} = \{j_1, \ldots, j_n, \partial_n, \ldots, \partial_1\}. \tag{18}
\]

To have a simple notation, we introduce bared indices \( i \in \{1, \ldots, n\} \)

\[
\epsilon_i^\tau = e_{2n+1-i}, \tag{19}
\]

or equivalently

\[
e_i = j_i \quad \epsilon_i = e_{2n+1-i} = \partial_i. \tag{20}
\]

The contraction on \( \mathcal{O}(V, B) \) is then written as

\[
[B(e_i, e_j)] = [B_{ij}] = \begin{bmatrix} B_{ij}^{\partial^\partial} & B_{ij}^{\partial^\tau} \\ B_{ij}^{\tau^\partial} & B_{ij}^{\tau^\tau} \end{bmatrix} \iff \begin{bmatrix} [M_{rs}]^{\tau^\partial} & [M_{rs}]^{\tau^\tau} \\ [N_{vw}]^{\partial^\partial} & [N_{vw}]^{\partial^\tau} \end{bmatrix} \tag{21}
\]

where the overscripts indicate the type of the base element. Indices of blocks run in \{1..n\}. Note, that the matrices \( B^1, B^2 \) and \( N \) are not directly submatrices of \( B \) because of our bared index notation. Introducing a \( n \times n \) matrix \( J = \delta_{i, n+1-k} \) we can identify them as \( B^{\partial^\partial} = B^{\tau^\partial} \), \( B^{\tau^\partial} = J B^2 \), \( B^{\partial^\partial} = J N \).

We could handle the \( 2n \) dimensional complex case as \( \mathcal{O}(\mathbb{R}^{2n+1}, B) \), but we will restrict ourself to the even dimensional case and look at \( \mathcal{O}(\mathbb{C}^{2n}) \cong \mathbb{C} \otimes \mathcal{O}(\mathbb{R}^{2n}) \) as a complexification.

### 3 Hecke algebra as bi-vector subalgebra

**Definition 9** The Hecke algebra \( H_\mathbb{R}(n+1, q) \) has the following presentation

\[
\begin{align*}
    i) & \quad b_i^2 = (1-q)b_i + q \mathbb{I} \quad \text{Hecke condition} \\
    ii) & \quad b_i b_k = b_k b_i \quad |i-k| \geq 2 \quad \text{commutator} \\
    iii) & \quad b_i b_{i+1} b_i = b_{i+1} b_{i} b_{i+1} \quad \text{Artin braid relation} \ [9]
\end{align*}
\]

with generators \( \mathbb{I}, b_i, i \in \{1, \ldots, n\} \), see (6).

Our goal is to find an identification of the \( b_i \) generators as bi-vectors in an appropriate Clifford algebra \( \mathcal{O}(\mathbb{R}^{2n}, B) \) or \( \mathcal{O}(\mathbb{C}^{2n}, B) \). We can formulate our results in the following

**Theorem 10** The Hecke algebra \( H_\mathbb{R}(n+1, q) \) of definition (4) is a subalgebra of the Clifford algebra \( \mathcal{O}(\mathbb{R}^{2n}, B) \) of definition (3) with the following identifications:

\[
\begin{align*}
    i) & \quad b_i := e_i \wedge \epsilon_i = e_i \wedge e_{2n+1-i} \equiv j_i \wedge \partial_i \quad i \in \{1, \ldots, n\} \\
    ii) & \quad B := [B_{ij}] = \begin{bmatrix} [B_{rs}]^{\tau^\tau} & [B_{rs}]^{\tau^\partial} \\ [B_{sv}]^{\partial^\partial} & [B_{sv}]^{\partial^\tau} \end{bmatrix} \tag{24}
\end{align*}
\]
where the submatrices of $B$ satisfy the conditions

\[ B^{ij} \equiv M_{rs} = \frac{1}{2}(M_{rs} - M_{sr}) \]
\[ JB^{\alpha\beta} \equiv N_{\alpha\beta} = \frac{1}{2}(N_{\alpha\beta} - N_{\beta\alpha}) \]
\[ B^{i\beta} \equiv [B_i^1] = [B_{i\pi} + (q - B_{i\pi})\delta_i,\pi] \]
\[ JB^{\alpha j} \equiv [B^2_{\pi}] = [-B_{\pi\pi} + (1 + q)\delta_{\pi\pi} + q\delta_{\pi\pi+1} - \delta_{\pi\pi+1}] \quad (25) \]

Proof: We determine the constraints on the bilinear form $B$ by a direct calculation of the consequences of the relations (23).

i) We try to identify the bi-vector elements $b_i$ from (24) with Hecke generators also denoted by $b_i$ from (23). Since we insist on the multivector structure inherited from the Grassmann multivectors underlying the Clifford multivectors, we have to fulfill in any case the condition

\[ e_i e_i = e_i \wedge e_i = B_{ii} = 0, \quad (B_{ii} = 0). \quad (26) \]

The Hecke relation (23-i) leads with

\[ b_i = j_i \wedge \partial_i = j_i \partial_i - B_{\pi\pi} \]

\[ b_i^2 = (j_i \wedge \partial_i)^2 = (j_i \partial_i - B_{\pi\pi})j_i \wedge \partial_i \]
\[ = j_i[B_{\pi\pi} \partial_i - j_i \partial_i^2] - B_{\pi\pi} j_i \wedge \partial_i \]
\[ = B_{\pi\pi} B_{\pi\pi} - (B_{\pi\pi} - B_{\pi\pi})j_i \wedge \partial_i \]
\[ = B_{\pi\pi} B_{\pi\pi} - (B_{\pi\pi} - B_{\pi\pi})b_i \]
\[ = (1 - q)b_i + q. \quad (27) \]

We get as solutions

\[ B_{ii} = q \quad \text{or} \quad -1 \]
\[ B_{\pi\pi} = 1 \quad \text{or} \quad -q. \quad (29) \]

We will chose $B_{\pi\pi} = q, B_{i\pi} = 1$. The overall minus sign does not matter in our considerations. Including the nilpotency of the Grassmann sources $j$ and $\partial$, we obtained $4n$ constrains on $B$.

ii) The commutator relation (23-ii), which is valid for $|k - i| \geq 2$, can be calculated along the same lines as in (28). This results in

\[ b_i b_k - b_k b_i = (B_{ik} B_{i\pi} - B_{i\pi} B_{ik} - B_{\pi\pi} B_{ik} + B_{\pi\pi} B_{ik}) \mathbb{I} \]
\[ + (B_{ik} B_{i\pi} + B_{i\pi}) j_i \wedge \partial_k - (B_{ik} B_{i\pi} + B_{i\pi}) j_k \wedge \partial_i \]
\[ - (B_{ik} B_{i\pi} + B_{i\pi}) j_i \wedge j_k - (B_{ik} B_{i\pi} + B_{i\pi}) \partial_i \wedge \partial_k \]
\[ \equiv 0. \quad (30) \]

Therefrom we obtain

\[ B_{ik} = -B_{ki} \]
\[ B_{i\pi} = -B_{\pi i} \]
\[ B_{\pi\pi} = -B_{\pi\pi} \quad (31) \]
if \(|i-k| \geq 2\). This leads to \(3n(n-2)/2\) constrains on \(B\).

iii) The third relation is somewhat lengthy and yields

\[
b_i b_{i+1} b_i - b_{i+1} b_i b_{i+1} = (1 + q)(B_{i+1} B_{i+1,i} - B_{i+1} B_{i+1,i})
\]

\[
+ ((B_{i+1} + B_{i+1,i})(B_{i+1,i} + B_{i+1,i})
- (B_{i+1} + B_{i+1,i})(B_{i+1,i} + B_{i+1,i}) - q) j_i \wedge \partial_i
\]

\[
+ ((B_{i+1,i} + B_{i+1,i})(B_{i+1,i} + B_{i+1,i})
- (B_{i+1,i} + B_{i+1,i})(B_{i+1,i} + B_{i+1,i}) + q) j_{i+1} \wedge \partial_{i+1}
\]

\[
- (1 + q)(B_{i+1} + B_{i+1,i}) \partial_i \wedge \partial_{i+1}
\]

\[
+ (1 + q)(B_{i+1,i} + B_{i+1,i}) j_i \wedge j_{i+1}
\]

\[\neq 0. \tag{32}\]

This leads to

\[
B_{i+1,i} = -B_{i+1}
\]

\[
B_{i+1,i} = -B_{i+1,i}
\]

\[
B_{i+1,i+1} = 1 - B_{i+1,i}
\]

\[
B_{i+1,i+1} = q - B_{i+1,i+1} \tag{33}\]

which are \(4(n-1)\) further constrains on \(B\). All in all, we have to impose the constrains given in (26,29,31) and (33) on the bilinear form \(B\) of \(4n^2\) arbitrary \(K\)-valued parameters. We obtain

\[
\#\text{constraints} = \frac{3n^2 + 13n - 8}{2} \tag{34}\]

and remain with

\[
\#\text{degrees of freedom} = \frac{5n^2 - 13n + 8}{2}. \tag{35}\]

The explicit form of \(B\) can be derived from the constrains to be of the form (25). Since we remain with superfluous degrees of freedom, which might be set to zero, we have derived a whole set of Hecke algebra embeddings in \(\mathcal{O}(V,B)\).

Since we have proofed theorem 10 for the quadratic Hecke condition (23-i), we have to give some comments on other choices of the quadratic or higher relations. One finds in literature at least the following types of relations

\[
b_i^2 = (1-q) b_i + q
\]

\[
t_i^2 = (q - q^{-1}) t_i + 1
\]

\[
e_i^2 = \tau e_i
\]

\[
u_i^Q = 1. \tag{36}\]

In general one has a quadratic –or higher order, see \(u_i\)– relation

\[
x_i^2 = g(q) x_i + h(q) \tag{37}\]

where \(g\) and \(h\) are mereomorphic functions of \(q\). The question, if there is a transformation connecting the \(b_i\)'s and in general the \(x_i\)'s addresses the number of equivalence classes of quadratic relations. We can reformulate the above equations into \(Q(x) = E\) where \(E\) is a constant, and have to classify quadratic forms. This can be done over \(\mathbb{R}\) and \(\mathbb{C}\) with help of the Brauer group \(B(K)\), [11, 36]. This group
is trivial for $\mathbb{C}$ since the complex field is algebraically closed and isomorphic to \{-1, +1\} as a multiplicative group in the case of $\mathbb{R}$

$$B(\mathbb{K}) \cong \frac{\mathbb{K}^*}{\mathbb{K}^2}.$$  \hfill (38)

However, this classification does only take (23-i) into account. It is easy to calculate, that a transformation of the type

$$x_i = a(q) + b(q)b_i$$ \hfill (39)

with mereomorphic $a(q), b(q)$ does not alter (23-ii). But (23-iii) is in general not invariant under such a transformation, which is well known in literature. As an example, one arrives at the relations of a Temperly-Lieb algebra \cite{42}, where the third relation is given as, $e_i = (q\mathbb{I} + b_i)/(q + 1), \tau = (2 + q + q^{-1})^{-1}$ compare (3-iii)

$$e_ie_{i+1}e_i - \tau e_i = e_{i+1}e_ie_{i+1} - \tau e_{i+1}.$$ \hfill (40)

Such a relation can easily be obtained in our approach, simply by another choice of the bilinear form $B$, or of course by the above transformation. It remains however to find a connection between traces employed in the phenomenological models and states on our algebra. Such stats were introduced in \cite{43} and provide a very rigid structure on $\mathcal{C}(V, B)$. These states are necessary to be able to calculate invariants and physical outcomes and to be able to show (in)equivalence between different presentations. This intriguing problem will be addressed elsewhere.

### 4 Conclusion

Our main tool were Clifford geometric algebras of multivectors, which provide a generalization of ordinary Clifford algebras of quadratic spaces to such of a pair $(V, B)$. The bilinear form does not having any symmetry i.g. and is thus not bijectively connected to a quadratic form. Every bilinear form $B$ with the same symmetric part $G$ gives rise to the same Clifford algebra. Taking the full $B$ into account allows one to endow the Clifford algebra with a unique $\mathbb{Z}_n$-grading. The Clifford algebra corresponding to $B$ build over the $\mathbb{Z}_n$-graded space $\bigwedge(V)$ is called Clifford algebra of multivectors \cite{8}.

We proved the theorem, that due to an appropriate choice of the bilinear form $B$ in $\mathcal{C}(\mathbb{K}^{2n}, B)$, it is possible to find $n$ bivectors $b_i$ which generate the Hecke algebra $H_{\mathbb{K}}(n + 1, q)$. The proof was straight forward. Since we got a large number of remaining freedoms in the Clifford bilinearform $B$, this parameters might be used to have spectral parameters in the braid relation, which then mutates into the Yang-Baxter equation. This will be considered elsewhere.

Since the Clifford algebra has already an interpretation in physical terms, we have to look at the $b_i$ generators as composite entities. This supports our opinion stated in the introduction and also promoted in \cite{27} that $q$-symmetry might be connected with composite effects. A decision of this conjecture requires further work, especially on the states involved in the calculation of invariants, see \cite{43}.

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