Desingularization of singular hyperkähler varieties I.

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Let $M$ be a singular hyperkähler variety, obtained as a moduli space of stable holomorphic bundles on a compact hyperkähler manifold [alg-geom/9307008]. Consider $M$ as a complex variety in one of the complex structures induced by the hyperkähler structure. We show that normalization of $M$ is smooth, hyperkähler and does not depend on the choice of induced complex structure.

0 Introduction

The structure of this paper is following.

- In the first section, we give a compendium of definitions and results from hyperkähler geometry, all known from literature.

- Section 2 deals with the real analytic varieties underlying complex varieties. We define almost complex structures on a real analytic variety. This notion is used in order to define hypercomplex varieties. We show that a hyperkähler manifold is always hypercomplex.

- In Section 3, we give a definition of a singular hyperkähler variety, following [V-bun] and [V3]. We cite basic properties and list the examples of such manifolds.

- In Section 4, we define locally homogeneous singularities. A space with locally homogeneous singularities (SLHS) is an analytic space $X$ such that for all $x \in X$, the $x$-completion of a local ring $\mathcal{O}_x X$ is isomorphic to an $x$-completion of associated graded ring $(\mathcal{O}_x X)_{gr}$. We show that a complex variety is SLHS if and only if the underlying real analytic variety is SLHS. This allows us to define invariantly the notion of a hyperkähler SLHS. The natural examples of hyperkähler SLHS include the moduli spaces of stable holomorphic bundles, considered in [V-bun]. We conjecture that every hyperkähler variety is a space with locally homogeneous singularities.

- In Section 5, we study the tangent cone of a singular hyperkähler manifold $M$ in the point $x \in M$. We show that its reduction, which is a closed
subvariety of $T_x M$, is a union of linear subspaces $L_i \subset T_x M$. These subspaces are invariant under the natural quaternion action in $T_x M$. This implies that a normalization of $(M, I)$ is smooth. Here, as usually, $(M, I)$ denotes $M$ considered as a complex variety, with $I$ a complex structure induced by the singular hyperkähler structure on $M$.

- In Section 6, we formulate and prove the desingularization theorem for hyperkähler varieties with locally homogeneous singularities. For each such variety $M$ we construct a finite surjective morphism $\tilde{M} \to M$ of hyperkähler varieties, such that $\tilde{M}$ is smooth and $n$ is an isomorphism outside of singularities of $M$. The $\tilde{M}$ is obtained as a normalization of $M$; thus, our construction is canonical and functorial.

1 Hyperkähler manifolds

1.1 Definitions

This subsection contains a compression of the basic definitions from hyperkähler geometry, found, for instance, in [Bes] or in [Beau].

**Definition 1.1:** ([Bes]) A hyperkähler manifold is a Riemannian manifold $M$ endowed with three complex structures $I$, $J$ and $K$, such that the following holds.

(i) the metric on $M$ is Kähler with respect to these complex structures and

(ii) $I$, $J$ and $K$, considered as endomorphisms of a real tangent bundle, satisfy the relation $I \circ J = -J \circ I = K$.

The notion of a hyperkähler manifold was introduced by E. Calabi ([K]).

Clearly, hyperkähler manifold has the natural action of quaternion algebra $\mathbb{H}$ in its real tangent bundle $TM$. Therefore its complex dimension is even. For each quaternion $L \in \mathbb{H}$, $L^2 = -1$, the corresponding automorphism of $TM$ is an almost complex structure. It is easy to check that this almost complex structure is integrable ([Bes]).

**Definition 1.2:** Let $M$ be a hyperkähler manifold, $L$ a quaternion satisfying $L^2 = -1$. The corresponding complex structure on $M$ is called an induced complex structure. The $M$ considered as a complex manifold is denoted by $(M, L)$.

Let $M$ be a hyperkähler manifold. We identify the group $SU(2)$ with the group of unitary quaternions. This gives a canonical action of $SU(2)$ on the tangent bundle, and all its tensor powers. In particular, we obtain a natural action of $SU(2)$ on the bundle of differential forms.
Lemma 1.3: The action of $SU(2)$ on differential forms commutes with the Laplacian.

Proof: This is Proposition 1.1 of [V-bun]. □

Thus, for compact $M$, we may speak of the natural action of $SU(2)$ in cohomology.

1.2 Trianalytic subvarieties in compact hyperkähler manifolds.

In this subsection, we give a definition and a few basic properties of trianalytic subvarieties of hyperkähler manifolds. We follow [V2].

Let $M$ be a compact hyperkähler manifold, $\dim R M = 2m$.

Definition 1.4: Let $N \subset M$ be a closed subset of $M$. Then $N$ is called trianalytic if $N$ is a complex analytic subset of $(M, L)$ for any induced complex structure $L$.

Let $I$ be an induced complex structure on $M$, and $N \subset (M, I)$ be a closed analytic subvariety of $(M, I)$, $\dim C N = n$. Denote by $[N] \in H_{2n}(M)$ the homology class represented by $N$. Let $\langle N \rangle \in H^{2m-2n}(M)$ denote the Poincare dual cohomology class. Recall that the hyperkähler structure induces the action of the group $SU(2)$ on the space $H^{2m-2n}(M)$.

Theorem 1.5: Assume that $\langle N \rangle \in H^{2m-2n}(M)$ is invariant with respect to the action of $SU(2)$ on $H^{2m-2n}(M)$. Then $N$ is trianalytic.

Proof: This is Theorem 4.1 of [V2]. □

Remark 1.6: Trianalytic subvarieties have an action of quaternion algebra in the tangent bundle. In particular, the real dimension of such subvarieties is divisible by 4.

1.3 Totally geodesic submanifolds.

Proposition 1.7: Let $X \xleftarrow{\phi} M$ be an embedding of Riemannian manifolds (not necessarily compact) compatible with the Riemannian structure. Then the following conditions are equivalent.

(i) Every geodesic line in $X$ is geodesic in $M$.

(ii) Consider the Levi-Civita connection $\nabla$ on $TM$, and restriction of $\nabla$ to $TM|_X$. Consider the orthogonal decomposition

$$TM|_X = TX \oplus TX^\perp. \quad (1.1)$$

Then, this decomposition is preserved by the connection $\nabla$. 

Proof: Well known; see, for instance, [Bes].

**Proposition 1.8:** Let \( X \subset M \) be a trianalytic submanifold of a hyperkähler manifold \( M \), where \( M \) is not necessarily compact. Then \( X \) is totally geodesic.

**Proof:** This is [V3], Corollary 5.4.

## 2 Real analytic varieties

Let \( X \) be a complex analytic variety. The “real analytic variety underlying \( X \)” (denoted by \( X_\mathbb{R} \)) is the following object. By definition, \( X_\mathbb{R} \) is a ringed space with the same topology as \( X \), but with a different structure sheaf, denoted by \( O_{X_\mathbb{R}} \). Let \( C(X, \mathbb{R}) \) be a sheaf of continuous \( \mathbb{R} \)-valued functions on \( X \). Then \( O_{X_\mathbb{R}} \) is a subsheaf of \( C(X, \mathbb{R}) \), defined as follows. Let \( A \subset C(X, \mathbb{R}) \) be an arbitrary subsheaf of \( C(X, \mathbb{R}) \). By \( \text{Ser}(A) \subset C(X, \mathbb{R}) \), we denote the sheaf of all functions which can be locally represented by the absolutely convergent series \( \sum P_i(a_1, ..., a_n) \), where \( a_1, ..., a_n \) are sections of \( A \) and \( P_i \) are polynomials with coefficients in \( \mathbb{R} \). By definition, \( O_{X_\mathbb{R}} = \text{Ser}(\text{Re}O_X) \), where \( \text{Re}O_X \) is a sheaf of real parts of holomorphic functions.

Another interesting sheaf associated with \( X_\mathbb{R} \) is a sheaf \( O_{X_\mathbb{R}} \otimes \mathbb{C} \subset \mathbb{C}(X, \mathbb{C}) \) of complex-valued real analytic functions.

Consider the sheaf \( O_X \) of holomorphic functions on \( X \) as a subsheaf of the sheaf \( C(X, \mathbb{C}) \) of continuous \( \mathbb{C} \)-valued functions on \( X \). The sheaf \( C(X, \mathbb{C}) \) has a natural automorphism \( f \mapsto \overline{f} \), where \( \overline{f} \) is complex conjugation. By definition, the section \( f \) of \( C(X, \mathbb{C}) \) is called **antiholomorphic** if \( \overline{f} \) is holomorphic. Let \( O_X \) be the sheaf of holomorphic functions, and \( \overline{O}_X \) be the sheaf of antiholomorphic functions on \( X \). Let \( O_X \otimes \mathbb{C} \overline{O}_X \rightarrow C(X, \mathbb{C}) \) be the natural multiplication map. Clearly, the image of \( i \) belongs to the subsheaf \( O_{X_\mathbb{R}} \otimes \mathbb{C} \subset C(X, \mathbb{C}) \).

**Claim 2.1:** The sheaf homomorphism \( i : O_X \otimes \mathbb{C} \overline{O}_X \rightarrow O_{X_\mathbb{R}} \otimes \mathbb{C} \subset C(X, \mathbb{C}) \) is injective. For each point \( x \in X \), \( i \) induces an isomorphism on \( x \)-completions of \( O_X \otimes \mathbb{C} \overline{O}_X \) and \( O_{X_\mathbb{R}} \otimes \mathbb{C} \).

**Proof:** Well known (see, for instance, [GMT]).

Let \( \Omega^1(O_{X_\mathbb{R}}) \), \( \Omega^1(O_X \otimes \mathbb{C} \overline{O}_X) \), \( \Omega^1(O_{X_\mathbb{R}} \otimes \mathbb{C}) \) be the sheaves of Kähler differentials associated with the corresponding ring sheaves. There are natural sheaf maps

\[
\Omega^1(O_{X_\mathbb{R}}) \otimes \mathbb{C} \rightarrow \Omega^1(O_{X_\mathbb{R}} \otimes \mathbb{C}) \tag{2.1}
\]

and

\[
\Omega^1(O_{X_\mathbb{R}} \otimes \mathbb{C}) \rightarrow \Omega^1(O_X \otimes \mathbb{C} \overline{O}_X), \tag{2.2}
\]

corresponding to the monomorphisms.
Claim 2.2: The map (2.1) is an isomorphism. Tensoring both sides of (2.2) by $O_X \otimes \mathbb{C}$ produces an isomorphism

$$\Omega^1(O_X \otimes \mathbb{C}) \otimes_{O_X \otimes \mathbb{C}} (O_X \otimes \mathbb{C}) = \Omega^1(O_X \otimes \mathbb{C}).$$

Proof: Clear. ■

According to the general results about differentials (see, for example, [1], Chapter II, Ex. 8.3), the sheaf $\Omega^1(O_X \otimes \mathbb{C})$ admits a canonical decomposition:

$$\Omega^1(O_X \otimes \mathbb{C}) \cong \Omega^1(O_X \otimes \mathbb{C}) \oplus O_X \otimes \mathbb{C} \Omega^1(O_X \otimes \mathbb{C}).$$

Let $I$ be an endomorphism of $\Omega^1(O_X \otimes \mathbb{C})$ which acts as a multiplication by $\sqrt{-1}$ on

$$\Omega^1(O_X \otimes \mathbb{C}) \cong \Omega^1(O_X \otimes \mathbb{C})$$

and as a multiplication by $-\sqrt{-1}$ on

$$O_X \otimes \mathbb{C} \Omega^1(O_X \otimes \mathbb{C}).$$

Let $I$ be the corresponding $O_X \otimes \mathbb{C}$-linear endomorphism of

$$\Omega^1(O_X \otimes \mathbb{C}) \cong \Omega^1(O_X \otimes \mathbb{C}).$$

As easy check ensures that $I$ is real, that is, comes from the $O_X$-linear endomorphism of $\Omega^1(O_X \otimes \mathbb{C})$. Denote this $O_X$-linear endomorphism by

$$I : \Omega^1(O_X) \longrightarrow \Omega^1(O_X),$$

$I^2 = -1$. The endomorphism $I$ is called a complex structure operator. In the case when $X$ is smooth, $I$ coincides with the usual complex structure operator on the cotangent space.

Definition 2.3: Let $X$, $Y$ be complex analytic varieties, and

$$f : X \longrightarrow Y$$

be a morphism of underlying real analytic varieties. Let $f^* \Omega^1_Y \longrightarrow \Omega^1_X$ be the natural map of sheaves of differentials associated with $f$. Let
be the complex structure operators, and

\[ f^* I_Y : f^* \Omega^1_{Y_R} \longrightarrow f^* \Omega^1_{Y_R} \]

be \( \mathcal{O}_{X_R} \)-linear automorphism of \( f^* \Omega^1_{Y_R} \) defined as a pullback of \( I_Y \). We say that \( f \) **commutes with the complex structure** if

\[ P \circ f^* I_Y = I_X \circ P. \tag{2.3} \]

**Theorem 2.4:** Let \( X, Y \) be complex analytic varieties, and \( f_R : X_R \longrightarrow Y_R \) be a morphism of underlying real analytic varieties, which commutes with the complex structure. Then there exist a morphism \( f : X \longrightarrow Y \) of complex analytic varieties, such that \( f_R \) is its underlying morphism.

**Proof:** By Corollary 9.4, \([V3]\), the map \( f \), defined on the sets of points of \( X \) and \( Y \), is meromorphic; to prove Theorem 2.4 we need to show it is holomorphic. Let \( \Gamma \subset X \times Y \) be the graph of \( f \). Since \( f \) is meromorphic, \( \Gamma \) is a complex subvariety of \( X \times Y \). It will suffice to show that the natural projections \( \pi_1 : \Gamma \longrightarrow X, \pi_2 : \Gamma \longrightarrow Y \) are isomorphisms. By \([V3]\), Lemma 9.12, the morphisms \( \pi_i \) are flat. Since \( f_R \) induces isomorphism of Zariski tangent spaces, same is true of \( \pi_i \). Thus, \( \pi_i \) are unramified. Therefore, the maps \( \pi_i \) are etale. Since they are one-to-one on points, \( \pi_i \) etale implies \( \pi_i \) is an isomorphism.

**Definition 2.5:** Let \( M \) be a real analytic variety, and

\[ I : \Omega^1(\mathcal{O}_M) \longrightarrow \Omega^1(\mathcal{O}_M) \]

be an endomorphism satisfying \( I^2 = -1 \). Then \( I \) is called **an almost complex structure on** \( M \). If there exist a complex analytic structure \( \mathcal{C} \) on \( M \) such that \( I \) appears as the complex structure operator associated with \( \mathcal{C} \), we say that \( I \) is **integrable**. Theorem 2.4 implies that this complex structure is unique if it exists.

**Definition 2.6:** (Hypercomplex variety) Let \( M \) be a real analytic variety equipped with almost complex structures \( I, J \) and \( K \), such that \( I \circ J = -J \circ I = K \). Assume that for all \( a, b, c \in \mathbb{R} \), such that \( a^2 + b^2 + c^2 = 1 \), the almost complex structure \( aI + bJ + cK \) is integrable. Then \( M \) is called a **hypercomplex variety**.
Claim 2.7: Let $M$ be a hyperkähler manifold. Then $M$ is hypercomplex.

Proof: Let $I$, $J$ be induced complex structures. We need to identify $(M,I)_\mathbb{R}$ and $(M,J)_\mathbb{R}$ in a natural way. These varieties are canonically identified as $C^\infty$-manifolds; we need only to show that this identification is real analytic. This is \cite{V3}, Proposition 6.5. $lacksquare$

The following proposition will be used further on in this paper.

Proposition 2.8: Let $M$ be a complex variety, $x \in X$ a point, and $Z_xM \subset T_xM$ be the reduction of the Zariski tangent cone to $M$ in $x$, considered as a closed subvariety of the Zariski tangent space $T_xM$. Let $Z_xM_\mathbb{R} \subset T_xM_\mathbb{R}$ be the Zariski tangent cone for the underlying real analytic variety $M_\mathbb{R}$. Then $(Z_xM)_\mathbb{R} \subset (T_xM)_\mathbb{R} = T_xM_\mathbb{R}$ coincides with $Z_xM_\mathbb{R}$.

Proof: For each $v \in T_xM$, the point $v$ belongs to $Z_xM$ if and only if there exists a real analytic path $\gamma : [0,1] \rightarrow M$, $\gamma(0) = x$ satisfying $\frac{d\gamma}{dt} = v$. The same holds true for $Z_xM_\mathbb{R}$. Thus, $v \in Z_xM$ if and only if $v \in Z_xM_\mathbb{R}$. $lacksquare$

3 Singular hyperkähler varieties.

In this section, we follow \cite{V3}, Section 10. For more examples, motivations and reference, the reader is advised to check \cite{V3}.

Definition 3.1: (\cite{V-bun}, Definition 6.5) Let $M$ be a hypercomplex variety $M$. The following data define a structure of hyperkähler variety on $M$.

(i) For every $x \in M$, we have an $\mathbb{R}$-linear symmetric positively defined bilinear form $s_x : T_xM \times T_xM \rightarrow \mathbb{R}$ on the corresponding real Zariski tangent space.

(ii) For each triple of induced complex structures $I$, $J$, $K$, such that $I \circ J = K$, we have a holomorphic differential 2-form $\Omega \in \Omega^2(M,I)$.

(iii) Fix a triple of induced complex structure $I$, $J$, $K$, such that $I \circ J = K$. Consider the corresponding differential 2-form $\Omega$ of (ii). Let $J : T_xM \rightarrow T_xM$ be an endomorphism of the real Zariski tangent spaces defined by $J$, and $Re \Omega |_{(a,J(b))}$ the real part of $\Omega$, considered as a bilinear form on $T_xM$. Let $r_x$ be a bilinear form $r_x : T_xM \times T_xM \rightarrow \mathbb{R}$ defined by $r_x(a,b) = -Re \Omega |_{(a,J(b))}$. Then $r_x$ is equal to the form $s_x$ of (i). In particular, $r_x$ is independent from the choice of $I$, $J$, $K$.

Remark 3.2:

(a) It is clear how to define a morphism of hyperkähler varieties.
(b) For $M$ non-singular, Definition 3.1 is equivalent to the usual one (Definition 1.1). If $M$ is non-singular, the form $s_x$ becomes the usual Riemann form, and $\Omega$ becomes the standard holomorphically symplectic form.

(c) It is easy to check the following. Let $X$ be a hypercomplex subvariety of a hyperkähler variety $M$. Then, restricting the forms $s_x$ and $\Omega$ to $X$, we obtain a hyperkähler structure on $X$. In particular, trianalytic subvarieties of hyperkähler manifolds are always hyperkähler, in the sense of Definition 3.1.

Caution: Not everything which is seemingly hyperkähler satisfies the conditions of Definition 3.1. Take a quotient $M/G$ as a hyperkähler manifold by an action of finite group $G$, acting in accordance with hyperkähler structure. Then $M/G$ is, generally speaking, not hyperkähler (see [V3], Section 10 for details).

The following theorem, proven in [V-bun] (Theorem 6.3), gives a convenient way to construct examples of hyperkähler varieties.

**Theorem 3.3:** Let $M$ be a compact hyperkähler manifold, $I$ an induced complex structure and $B$ a stable holomorphic bundle over $(M, I)$. Let $\text{Def}(B)$ be a reduction of the deformation space of stable holomorphic structures on $B$. Assume that $c_1(B), c_2(B)$ are $SU(2)$-invariant, with respect to the standard action of $SU(2)$ on $H^*(M)$. Then $\text{Def}(B)$ has a natural structure of a hyperkähler variety.

4 Spaces with locally homogeneous singularities

**Definition 4.1:** (local rings with LHS) Let $A$ be a local ring. Denote by $m$ its maximal ideal. Let $A_{gr}$ be the corresponding associated graded ring. Let $\hat{A}, \hat{A}_{gr}$ be the $m$-adic completion of $A, A_{gr}$. Let $(\hat{A})_{gr}, (\hat{A}_{gr})_{gr}$ be the associated graded rings, which are naturally isomorphic to $A_{gr}$. We say that $A$ has **locally homogeneous singularities** (LHS) if there exists an isomorphism $\rho: \hat{A} \rightarrow \hat{A}_{gr}$ which induces the standard isomorphism $i: (A)_{gr} \rightarrow (A_{gr})_{gr}$ on associated graded rings.

**Definition 4.2:** (SLHS) Let $X$ be a complex or real analytic space. Then $X$ is called be a space with locally homogeneous singularities (SLHS) if for each $x \in M$, the local ring $O_x, M$ has locally homogeneous singularities.

By **system of coordinates** on a complex space $X$, defined in a neighbourhood $U$ of $x \in X$, we understand a closed embedding $U \hookrightarrow B$ where $B$ is an

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1 The deformation space might have nilpotents in the structure sheaf. We take its reduction to avoid this.
open subset of $\mathbb{C}^n$. Clearly, a system of coordinates can be considered as a set of functions $f_1, ..., f_n$ on $U$. Then $U \subset B$ is defined by a system of equations on $f_1, ..., f_n$.

**Remark 4.3:** Let $X$ be a complex space. Assume that for each $x \in X$, there exist a system of coordinates $f_1, ..., f_n$ in a neighbourhood $U$ of $x$, such that $U \subset B$ is defined by a system of homogeneous polynomial equations. Then $X$ is a space with locally homogeneous singularities. This explains the term.

**Claim 4.4:** Let $X$ be a complex analytic space with locally homogeneous singularities, and $X_r$ its reduction (same space, with structure sheaf factorized by nilradical). Then $X_r$ is also a space with locally homogeneous singularities.

**Proof:** Clear.

**Lemma 4.5:** Let $A_1, A_2$ be local rings over $\mathbb{C}$, with $A_i/m_i = \mathbb{C}$, where $m_i$ is the maximal ideal of $A_i$. Then $A_1 \otimes \mathbb{C} A_2$ is LHS if and only if $A_1$ and $A_2$ are LHS.

**Proof ("if" part):** Let $\rho_i : \hat{A}_i \rightarrow (\hat{A}_i)_{gr}$ be the maps given by LHS condition. Consider the map

$$\rho_1 \otimes \rho_2 : \hat{A}_1 \otimes \mathbb{C} \hat{A}_2 \rightarrow (\hat{A}_1)_{gr} \otimes \mathbb{C} (\hat{A}_2)_{gr}. \quad (4.1)$$

Denote the functor of adic completions of local rings by $B \rightarrow \hat{B}$. Clearly, $\hat{A}_1 \otimes \mathbb{C} \hat{A}_2 = \hat{A}_1 \otimes \mathbb{C} \hat{A}_2$, and $(\hat{A}_1)_{gr} \otimes \mathbb{C} (\hat{A}_2)_{gr} = (\hat{A}_1)_{gr} \otimes \mathbb{C} (\hat{A}_2)_{gr}$. Plugging these isomorphisms into the completion of both sides of (4.1), we obtain that a completion of $\rho_1 \otimes \rho_2$ provides an LHS map for $A_1 \otimes \mathbb{C} A_2$.

"only if" part: Let

$$\rho : A_1 \otimes \mathbb{C} A_2 \rightarrow ((A_1) \otimes \mathbb{C} (A_2))_{gr}$$

be the LHS map for $A_1 \otimes \mathbb{C} A_2$. There are natural maps

$$u : \hat{A}_1 \rightarrow A_1 \otimes \mathbb{C} A_2$$

and

$$v : ((A_1) \otimes \mathbb{C} (A_2))_{gr} \rightarrow (\hat{A}_1)_{gr}.$$ 

The $u$ comes from the natural embedding $a \rightarrow a \otimes 1 \in A_1 \otimes \mathbb{C} A_2$ and $v$ from the natural surjection $a \otimes b \rightarrow a \otimes \pi(b) \in A_1 \otimes \mathbb{C}$, where $\pi : A_2 \rightarrow \mathbb{C}$ is the standard quotient map. It is clear that $u \circ v$ induces identity on the associated graded ring of $A_1$. [Lemma 4.3] is proven.

**Proposition 4.6:** Let $M$ be a complex variety, $M_\mathbb{R}$ the underlying real analytic variety. Then $M_\mathbb{R}$ is a space with locally homogeneous singularities (SLHS) if and only if $M$ is SLHS.
Proof: By Claim 2.1, $\hat{\mathcal{O}}(\mathcal{M}_R) \otimes \mathbb{C} = \hat{\mathcal{O}}_x \mathcal{M} \otimes \hat{\mathcal{O}}_x \mathcal{M}$. Thus, Proposition 4.6 is implied immediately by Lemma 4.5.

**Corollary 4.7:** Let $\mathcal{M}$ be a hyperkähler variety, $I_1$, $I_2$ induced complex structures. Then $(\mathcal{M}, I_1)$ is a space with locally homogeneous singularities if and only if $(\mathcal{M}, I_2)$ is SLHS.

**Proof:** The real analytic variety underlying $(\mathcal{M}, I_1)$ coincides with that underlying $(\mathcal{M}, I_2)$. Applying Proposition 4.6, we immediately obtain Corollary 4.7.

**Definition 4.8:** Let $\mathcal{M}$ be a hyperkähler variety. Then $\mathcal{M}$ is called a space with locally homogeneous singularities (SLHS) if the underlying real analytic variety is SLHS or, equivalently, for some induced complex structure $I$ the $(\mathcal{M}, I)$ is SLHS.

**Theorem 4.9:** Let $\mathcal{M}$ be a compact hyperkähler manifold, $I$ an induced complex structure and $\mathcal{B}$ a stable holomorphic bundle over $(\mathcal{M}, I)$. Assume that $c_1(\mathcal{B})$, $c_2(\mathcal{B})$ are $SU(2)$-invariant, with respect to the standard action of $SU(2)$ on $H^*(\mathcal{M})$. Let $\text{Def}(\mathcal{B})$ be a reduction of a deformation space of stable holomorphic structures on $\mathcal{B}$, which is a hyperkähler variety by Theorem 3.3. Then $\text{Def}(\mathcal{B})$ is a space with locally homogeneous singularities (SLHS).

**Proof:** Let $x$ be a point of $\text{Def}(\mathcal{B})$, corresponding to a stable holomorphic bundle $\mathcal{B}$. In [V-bun], Section 7, the neighbourhood $U$ of $x$ in $\text{Def}(\mathcal{B})$ was described explicitly as follows. We constructed a locally closed holomorphic embedding $U \hookrightarrow H^1(\text{End}(\mathcal{B}))$. We proved that $v \in H^1(\text{End}(\mathcal{B}))$ belongs to the image of $\varphi$ if and only if $v^2 = 0$. Here $v^2 \in H^2(\text{End}(\mathcal{B}))$ is the square of $v$, taken with respect to the product

$$H^1(\text{End}(\mathcal{B})) \times H^1(\text{End}(\mathcal{B})) \rightarrow H^2(\text{End}(\mathcal{B}))$$

associated with the algebraic structure on $\text{End}(\mathcal{B})$. Clearly, the relation $v^2 = 0$ is homogeneous. This relation defines a locally closed SLHS subspace $Y$ of $H^1(\text{End}(\mathcal{B}))$, such that $\varphi(U)$ is its reduction. Applying Claim 4.4, we obtain that $\varphi(U)$ is also a space with locally homogeneous singularities.

**Conjecture 4.10:** Let $\mathcal{M}$ be a hyperkähler variety. Then $\mathcal{M}$ is a space with locally homogeneous singularities.

There is a rather convoluted argument which might prove Conjecture 4.10. This argument will be a subject of forthcoming paper [V-ne].

**5 Tangent cone of a hyperkähler variety**

Let $\mathcal{M}$ be a hyperkähler variety, $I$ an induced complex structure and $Z_x(\mathcal{M}, I)$ be a reduction of a Zariski tangent cone to $(\mathcal{M}, I)$ in $x \in \mathcal{M}$. Consider $Z_x(\mathcal{M}, I)$...
as a closed subvariety in the Zariski tangent space $T_xM$. The space $T_xM$ has a natural metric and quaternionic structure. This makes $T_xM$ into a hyperkähler manifold, isomorphic to $\mathbb{H}^n$.

**Theorem 5.1:** Under these assumptions, the following assertions hold:

(i) The subvariety $Z_x(M, I) \subset T_xM$ is independent from the choice of induced complex structure $I$.

(ii) Moreover, $Z_x(M, I)$ is a trianalytic subvariety of $T_xM$.

**Proof:** [Theorem 5.1] (i) is implied by Proposition 2.8. By Theorem 5.1 (i), the Zariski tangent cone $Z_x(M, I)$ is a hypercomplex subvariety of $TM$. According to [Remark 3.2] (c), this implies that $Z_x(M)$ is hyperkähler. ■

Further on, we denote the Zariski tangent cone to a hyperkähler variety by $Z_xM$. The Zariski tangent cone is equipped with a natural hyperkähler structure.

The following theorem shows that the Zariski tangent cone $Z_xM \subset T_xM$ is a union of planes $L_i \subset T_xM$.

**Theorem 5.2:** Let $M$ be a hyperkähler variety, $I$ an induced complex structure and $x \in M$ a point. Consider the reduction of the Zariski tangent cone (denoted by $Z_xM$) as a subvariety of the quaternionic space $T_xM$. Let $Z_x(M, I) = \bigcup L_i$ be the irreducible decomposition of the complex variety $Z_x(M, I)$. Then

(i) The decomposition $Z_x(M, I) = \bigcup L_i$ is independent from the choice of induced complex structure $I$.

(ii) For every $i$, the variety $L_i$ is a linear subspace of $T_xM$, invariant under quaternion action.

**Proof:** Let $L_i$ be an irreducible component of $Z_x(M, I)$, $Z_x^{ns}(M, I)$ be the non-singular part of $Z_x(M, I)$, and $L_i^{ns} := Z_x^{ns}(M, I) \cap L_i$. Then $L_i$ is a closure of $L_i^{ns}$ in $T_xM$. Clearly from [Theorem 5.1] $L_i^{ns}(M)$ is a hyperkähler submanifold in $T_xM$. By [Proposition 1.8] $L_i^{ns}$ is totally geodesic. A totally geodesic submanifold of a flat manifold is again flat. Therefore, $L_i^{ns}$ is an open subset of a linear subspace $\tilde{L}_i \subset T_xM$. Since $L_i^{ns}$ is a hyperkähler submanifold, $\tilde{L}_i$ is invariant with respect to quaternions. The closure $\bar{L}_i$ of $L_i^{ns}$ is a complex analytic subvariety of $T_x(M, I)$. Therefore, $\bar{L}_i = L_i$. This proves [Theorem 5.2] (ii). From the above argument, it is clear that $Z_x^{ns}(M, I) = \bigsqcup L_i^{ns}$ (disconnected sum). Taking connected components of $Z_x^{ns}M$ for each induced complex structure, we obtain the same decomposition $Z_x(M, I) = \cup L_i$, with $L_i$ being closure of connected components. This proves [Theorem 5.2] (ii). ■
Corollary 5.3: Let $M$ be a hyperkähler variety, and $I$ an induced complex structure. Assume that $M$ is a space with locally homogeneous singularities. Then the normalization of $(M, I)$ is smooth.

Proof: The normalization of $Z_x M$ is smooth by Theorem 5.2. The normalization is compatible with the adic completions ([M], Chapter 9, Proposition 24.E). Therefore, the integral closure of the completion of $O_{Z_x M}$ is a regular ring. Now, from the definition of locally homogeneous intersections, it follows that the integral closure of $O_x M'$ is also a regular ring, where $O_x M'$ is an adic completion of the local ring of holomorphic functions on $(M, I)$ in a neighbourhood of $x$. Applying [M], Chapter 9, Proposition 24.E again, we obtain that the integral closure of $O_x M$ is regular. This proves Corollary 5.3.

6 Desingularization of hyperkähler varieties

Theorem 6.1: Let $M$ be a hyperkähler or a hypercomplex variety. Assume that $M$ is a space with locally homogeneous singularities, and $I$ an induced complex structure. Let $\widetilde{(M, I)} \longrightarrow (M, I)$ be the normalization of $(M, I)$. Then $\widetilde{(M, I)}$ is smooth and has a natural hyperkähler structure $\mathcal{H}$, such that the associated map $n : \widetilde{(M, I)} \longrightarrow (M, I)$ agrees with $\mathcal{H}$. Moreover, the hyperkähler manifold $\widetilde{M} := (M, I)$ is independent from the choice of induced complex structure $I$.

Proof: The variety $\widetilde{(M, I)}$ is smooth by Corollary 5.3. Let $x \in M$, and $U \subset M$ be a neighbourhood of $x$. Let $\mathcal{R}_x(U)$ be the set of irreducible components of $U$ which contain $x$. There is a natural map $\tau : \mathcal{R}_x(U) \longrightarrow \text{Irr}(\text{Spec} O_x M')$, where $\text{Irr}(\text{Spec} O_x M')$ is a set of irreducible components of $\text{Spec} O_x M'$, where $O_x M'$ is a completion of $O_x M$ in $x$. Since $O_x M$ is Henselian ([R], VII.4), there exist a neighbourhood $U' \subset U$ such that $\tau : \mathcal{R}_x(U) \longrightarrow \text{Irr}(\text{Spec} O_x M')$ is a bijection. Fix such an $U$. Since $M$ is a space locally with locally homogeneous singularities, the irreducible decomposition of $U$ coincides with the irreducible decomposition of the tangent cone $Z_x M$.

Let $\coprod U_i \xrightarrow{u} U$ be the morphism mapping a disjoint union of irreducible components of $U$ to $U$. By Theorem 5.2, the $x$-completion of $O_{U_i}$ is regular. Shrinking $U_i$ if necessary, we may assume that $U_i$ is smooth. Then, the morphism $u$ coincides with the normalization of $U$.

For each variety $X$, we denote by $X^{\text{ns}} \subset X$ the set of non-singular points of $X$. Clearly, $u(U_i) \cap U^{\text{ns}}$ is a connected component of $U^{\text{ns}}$. Therefore, $u(U_i)$ is trianalytic in $U$. By Remark 3.2(c), $U_i$ has a natural hyperkähler structure, which agrees with the map $u$. This gives a hyperkähler structure on the normalization $\widetilde{U} := \coprod U_i$. Gluing these hyperkähler structures, we obtain a hyperkähler structure $\mathcal{H}$ on the smooth manifold $(\widetilde{M}, \widetilde{I})$. Consider the normalization map $n : \widetilde{(M, I)} \longrightarrow (M, I)$, and let $\widetilde{M}^n := n^{-1}(M^{\text{ns}})$. Then, $n \bigg|_{\widetilde{M}^n} \widetilde{M}^n \longrightarrow M^{\text{ns}}$ is a...
finite covering which is compatible with the hyperkähler structure. Thus, $\mathcal{H}|_{\tilde{M}_n}$ can be obtained as a pullback from $M$. Clearly, a hyperkähler structure on a manifold is uniquely defined by its restriction to an open dense subset. We obtain that $\mathcal{H}$ is independent from the choice of $I$. \[ \blacksquare \]

**Remark 6.2:** The desingularization argument works well for hypercomplex varieties. The word “hyperkähler” in this article can be in most cases replaced by “hypercomplex”, because we never use the metric structure.

**Acknowledgements:** It is a pleasure to acknowledge the help of P. Deligne, who pointed out an error in the original argument. Deligne also suggested the term “locally homogeneous singularities”. I am grateful to A. Beilinson, D. Kaledin, D. Kazhdan, T. Pantev and S.-T. Yau for enlightening discussions.

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