A Possible Lattice Chiral Gauge Theory

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Abstract

We analyze the dynamics of an $SU_L(2)\otimes U_R(1)$ chiral gauge theory on a lattice with a large multifermion coupling $1 \ll g_2 < \infty$. It is shown that no spontaneous symmetry breaking occurs; the “spectator” fermion $\psi_R^a(x)$ is a free mode; doublers are decoupled as massive Dirac fermions consistently with the chiral gauge symmetry. Whether right-handed three-fermion states disappear and chiral fermions emerge in the low-energy limit are discussed. Provided right-handed three-fermion states disappear, we discuss the chiral gauge coupling, Ward identities, the gauge anomaly and anomalous $U_L(1)$ global current within the gauge-invariant prescription of renormalization of the gauge perturbation theory.

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1. Introduction

Since the “no-go” theorem [1] of Nielsen and Ninomiya was demonstrated in 1981, the problem of chiral fermion “doubling” and “vector-like” phenomenon on a lattice still exists if one insists on preserving chiral gauge symmetries. Actually, the essential spirit of the “no-go” theorem of Nielsen and Ninomiya is that, under certain prerequisites, the paradox concerning chiral gauge symmetries, vector-like doubling and anomalies are unavoidably entwined. In fact, if taking the attitude of retaining the chiral gauge symmetries at short distances, we regard the “no-go” theorem is not a real barrier, rather, its prerequisites hint what could be the possible choices of Nature for the Standard Model at short distances [2, 3, 4, 5] to get resolution of this paradox. That was the reason and motivation that we persist on preserving exact chiral gauge symmetries by multifermion couplings, which violate bilinearity, one of the prerequisites of the “no-go” theorem.

Eichten and Preskill (EP) [2] were pioneers on the idea of multifermion couplings ten years ago. The crucial points of their idea can be briefly described as follows. Multifermion couplings are introduced such that, in the phase space of strong-couplings, Weyl states composing three elementary Weyl fermions (three-fermion states) are bound. Then, these three-fermion states pair up with elementary Weyl fermions to be Dirac fermions. Such Dirac fermions can be massive without violating chiral gauge symmetries due to the appropriate quantum numbers and chirality carried by these three-fermion states. The binding thresholds of such three-fermion states depend on elementary Weyl modes residing in different regions of the Brillouin zone. If one assumes that the spontaneous symmetry breaking of the Nambu-Jona Lasinio (NJL) type [6] does not occur and such binding thresholds separate the weak-coupling symmetric phase from the strong-coupling symmetric phase, there are two possibilities to realize the continuum limit of chiral fermions in phase space. One is of crossing over the binding threshold of the three-fermion state of chiral fermions; another is of a wedge between two thresholds, where the three-fermion state of chiral fermions has not been formed, provided all doublers sitting in various edges of the Brillouin zone have been bound to be massive Dirac fermions and decouple.

To visualize this idea, EP proposed a model [2] of multifermion couplings with $SU(5)$ and $SO(10)$ chiral symmetries and suggested the possible regions in phase space to define the continuum limit of chiral fermions. However, the same model of multifermion couplings with $SO(10)$ chiral symmetry was studied in ref. [7], where it was pointed out that such models of multifermion couplings fail to give chiral fermions in the continuum limit. The reasons are that an NJL spontaneous symmetry breaking phase separating the strong-coupling symmetric phase from the weak-coupling symmetric phase, the right-handed Weyl states do not completely disassociate from the left-handed chiral fermions and the phase structure of such a model of multifermion couplings is similar to that of the
Smit-Swift (Wilson-Yukawa) model \[8\], which has been very carefully studied and shown to fail \[9\].

We should not be surprised that a particular class of multifermion coupling or corresponding Yukawa coupling models does not work. This does not mean that EP’s idea is definitely wrong in all possible classes of multifermion coupling or corresponding Yukawa coupling models, unless there is another generalized “no-go” theorem on interacting theories. Actually, Nielsen and Ninomiya gave an interesting comment on EP’s idea based on their intuition of anomalies \[11\]. In this paper, we present a possible lattice chiral gauge theory with an extremely large multifermion coupling in section 2. In successive sections, we attempt to study the problems concerning the lattice chiral gauge theory listed as fellow:

1. the “spectator” fermion $\psi_R$ is a free mode and decoupled;
2. no NJL spontaneous chiral symmetry breaking occurs;
3. all doublers $p = \tilde{p} + \pi_A$ are strongly bound to be massive Dirac fermions and decoupled consistently with the $SU_L(2) \otimes U_R(1)$ chiral symmetry;
4. the chiral fermions ($p = \tilde{p}$) of $\psi^i_L(x)$ and $\psi_R(x)$ have not yet been bound to the three-fermion state, an undoubled chiral mode of $\psi^i_L(x)$ exists in the low-energy spectrum;
5. the chiral gauge coupling and Ward identities;
6. the vacuum functional and chiral gauge anomaly;
7. the anomalous global current (fermion number non-conservation).

We discuss the chance that EP’s idea can be realized in this model, and the possibilities that the model can go to failure in the end of each section.

2. Formulation and the large multifermion coupling $1 \ll g_2 < \infty$

Let us consider the following fermion action of the $SU_L(2) \otimes U_R(1)$ chiral symmetries on a lattice with one external multifermion coupling $g_2 \gg 1$.

$$
S = \frac{1}{2a} \sum_x \left( \bar{\psi}^i_L(x) \gamma_\mu D^\mu \psi^i_L(x) + \bar{\psi}_R(x) \gamma_\mu \partial^\mu \psi_R(x) \right) \\
+ g_2 \bar{\psi}^i_L(x) \cdot \Delta \psi_R(x) \bar{\psi}_R(x) \cdot \psi^i_L(x),
$$

where “$a$” is the lattice spacing; $\psi^i_L (i = 1, 2)$ is an $SU_L(2)$ gauged doublet, $\psi_R$ is an $SU_L(2)$ singlet and both are two-component Weyl fermions. The $\psi_R$ is treated

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1 Ref.\[10\] discussed the constraints on the existence of chiral fermions in interacting lattice theories.

2 All momenta are scaled to be dimensionless and the physical momentum $\tilde{p} \simeq 0$ and $\pi_A$ runs over fifteen lattice momenta $\pi_A \neq 0$. 

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2
as a “spectator” fermion. The $\Delta$ that is a high order differential operator on the lattice obeys

$$\Delta(x)\psi_R(x) = \sum_\mu (\psi_R(x + \mu) + \psi_R(x - \mu) - 2\psi_R(x)). \tag{2}$$

The multifermion coupling $g_2$ is a dimension-10 operator relevant only for doublers $p = \bar{p} + \pi_A$, but irrelevant for chiral fermions $p = \bar{p}$ of $\psi_L^i$ and $\psi_R$. The action (1) preserves the global chiral symmetry $SU_L(2) \otimes U_R(1)$ and the $\psi_L^i(x)$ can be gauged to have the exact local $SU_L(2)$ chiral gauge symmetry. In addition, the action (1) possesses a $\psi_R$-shift-symmetry (2) ($\epsilon =$ const.),

$$\psi_R(x) \rightarrow \psi_R(x) + \epsilon. \tag{3}$$

The global symmetry $U_L(1)$ relating to the conservation of the fermion number of $\psi_L^i(x)$ is explicit in eq. (1). However, this global symmetry will be discussed and shown to be anomalous in section 8.

Our goal is to seek a possible segment $1 \ll g_2 \infty$, where an undoubled $SU_L(2)$-chiral-gauged fermion content is exhibited in the continuum limit consistently with $SU_L(2) \otimes U_R(1)$ chiral symmetry and the theory has the correct features of the chiral gauge coupling, Ward identities, gauge anomaly and anomalous global current, which are listed in introduction section. We are bound to demonstrate these properties of the theory in this segment.

To prove the first and second points concerning free “spectator” fermion $\psi_R(x)$ and no NJL spontaneous symmetry breaking, we need the Ward identity stemming from the $\psi_R$-shift-symmetry (3). Considering the generating functional $W(\eta, J)$ of the theory,

$$W(\eta, J) = -\ln Z(\eta, J), \quad \int_\phi = \int [d\psi_L^i d\psi_R dA_\mu] \tag{4}$$

$$Z(\eta, J) = \int_\phi \exp \left( -S + \int_x \left( \psi_L^i \eta_L^i + \bar{\eta}_L^i \psi_L^i + \bar{\psi}_R \eta_R + \bar{\eta}_R \psi_R + A_\mu J^\mu \right) \right),$$

we define the generating functional of one-particle irreducible vertices (the effective action $\Gamma(\psi_L^i, \bar{\psi}_R^i, A_\mu)$ as the Legendre transform of $W(\eta, J)$

$$\Gamma(\psi_L^i, \bar{\psi}_R^i, A_\mu) = W(\eta, J) - \int_x \left( \bar{\psi}_L^i \eta_L^i + \bar{\eta}_L^i \psi_L^i + \bar{\psi}_R \eta_R + \bar{\eta}_R \psi_R + A_\mu J^\mu \right), \tag{5}$$

and with the relations (for short $X = L$ and the $SU_L(2)$ index “$i$” is dropped for $X = R$)

$$A'_\mu(x) = \langle A_\mu(x) \rangle = \frac{\delta W}{\delta J_\mu(x)}, \quad J_\mu(x) = -\frac{\delta \Gamma}{\delta A'_\mu(x)};$$

$$\psi_X^i(x) = \langle \psi_X^i(x) \rangle = -\frac{\delta W}{\delta \bar{\eta}_X^i(x)}, \quad \bar{\eta}_X^i(x) = \frac{\delta \Gamma}{\delta \psi_X^i(x)};$$

$$\bar{\psi}_X^i(x) = \langle \bar{\psi}_X^i(x) \rangle = \frac{\delta W}{\delta \bar{\eta}_X^i(x)}, \quad \eta_X^i(x) = -\frac{\delta \Gamma}{\delta \bar{\psi}_X^i(x)}; \tag{6}$$
in which the fermionic derivatives are left-derivatives. In eq. (6), the \langle \cdots \rangle indicates an expectation value with respect to the partition functional \( Z(\eta, J) \).

Making the parameter \( \epsilon \) in eq. (3) to be spacetime dependent, and varying the generating function (4) according to the transformation rule (3) for arbitrary \( \epsilon(x) \neq 0 \), we arrive at the Ward identity corresponding to the \( \psi_R \)-shift-symmetry of the action (1):

\[
\frac{1}{2a} \gamma_{\mu} \partial^{\mu} \psi'_R(x) + g_2 \Delta \left( \bar{\psi}_L^i(x) \cdot \Delta \psi_R(x) \psi_L^j(x) \right) - \frac{\delta \Gamma}{\delta \psi'_R(x)} = 0. \tag{7}
\]

Based on this Ward identity (7), one can get all one-particle irreducible vertices containing at least one external “spectator” fermion \( \psi_R \).

Taking functional derivatives of eq. (7) with respect to appropriate “prime” fields (6) and then putting external sources \( J = \eta = 0 \), one can derive:

\[
\int_x e^{-ipx} \frac{\delta^{(2)} \Gamma}{\partial \psi'_R(x) \partial \psi'_R(0)} = \frac{i}{a} \gamma_{\mu} \sin(p^{\mu}), \tag{8}
\]

\[
\int_x e^{-ipx} \frac{\delta^{(2)} \Gamma}{\partial \bar{\psi}^i_L(x) \partial \psi'_R(0)} = \frac{1}{2} \sum_i (p) = 2g_2 w(p) \langle \bar{\psi}_L^i(0) \cdot \Delta \psi_R(0) \rangle_o, \tag{9}
\]

where the \( \langle \cdots \rangle_o \) indicates an expectation value with respect to the partition functional \( Z(J, \eta) \) with vanishing external sources \( J = \eta = 0 \). The \( w(p) \) is the Fourier transformation of the differential operator \( \frac{1}{2} \Delta(x) \),

\[
w(p) = \sum_{\mu} (1 - \cos p_{\mu}) \rightarrow O(p^2) \quad p \rightarrow 0, \tag{10}
\]

which is the Wilson factor [13]. The expectation value \( \langle \bar{\psi}_L^i(0) \cdot \Delta \psi_R(0) \rangle_o \) can be zero (symmetric phase) and non-zero (broken phase) depending the value of the multifermion coupling \( g_2 \) [14]. Other 1PI interacting vertices containing more than two external “spectator” fermion \( \psi_R \) identically vanish. Eq. (8) indicates an absence of the wave function renormalization of \( \psi_R(x) \). Thus, we can conclude that the “spectator” fermion \( \psi_R(x) \) is free mode for vanishing eq. (9) in the continuum limit \( p \rightarrow 0 \). This conclusion is independent of the finite value of the multifermion coupling \( g_2 \).

On the other hand, in the continuum limit \( p \rightarrow 0 \), one can have similar conclusion without using the Ward identity of the \( \psi_R \)-shift-symmetry to exactly obtain 1PI functions (8,9). Due to the structure of the multifermion interaction (4), every external “spectator” fermion \( \psi_R \) with momentum \( p \) associates with a Wilson factor \( w(p) \), the 1PI interacting vertices with an external “spectator” fermion \( \psi_R \) goes to zero \( (O(p^2)) \) as \( p \rightarrow 0 \). From this point of view, one can see that eqs. (8,9) must be right in \( p \rightarrow 0 \).

As another important consequence of the Ward identity of the shift-symmetry, eq(4) shows, for any finite values of the multifermion coupling \( g_2 \), the normal
modes \((p \to 0)\) of \(\psi_L(x)\) and \(\psi_R\) do not undergo NJL spontaneous symmetry breaking in the continuum limit. In equation (11), one has
\[
p_\mu = 0, \quad \Sigma^i(p) = 0.
\]
In ref. [14], we arrived at the same conclusion by explicit large-\(N_c\) calculations.

It is due to these properties, we can have a possibility of finding a segment
\[
1 \ll g_2 < \infty
\]
without NJL spontaneous symmetry breaking for normal modes and in meantime, doublers decouple as massive Dirac fermions via the EP mechanism that will be discussed in section 4. This gives us, as we will see in section 5, a loophole to have chiral fermions in the continuum limit.

3. No NJL spontaneous symmetry breaking

In previous section, we have shown the vanishing of the spontaneous symmetry breaking for the normal mode sector (11) for any values of \(g_2\). In this section, as \(g_2 \to \infty\), we show the total absence of the spontaneous symmetry breaking that means not only eq. (11) but also
\[
\Sigma(p) = 0 \quad p \neq 0,
\]
which indicates doublers do not undergo the spontaneous symmetry breaking as well. Note that in ref. [14], it was shown by large-\(N_c\) calculations that doublers acquire the NJL spontaneous symmetry breaking masses for certain finite value of multifermion coupling \(g_2\), which is inbetween the strong-coupling symmetric phase and the weak-coupling symmetric phase.

Adopting the technique of strong-coupling expansion in powers of \(1/g_2\), we make a rescaling of the fermion fields,
\[
\psi^i_L(x) \to (g_2)^{1/4}\psi^i_L(x); \quad \psi_R(x) \to (g_2)^{1/4}\psi_R(x),
\]
and rewrite the action (11) and partition function (14) in terms of new fermion fields
\[
S_f(x) = \frac{1}{2ag_2^2} \sum_\mu \left( \bar{\psi}^i_L(x)\gamma_\mu \psi^i_L(x + \mu) + \bar{\psi}_R(x)\gamma_\mu \psi_R(x + \mu) \right)
\]
\[
S_2(x) = \bar{\psi}^i_L(x) \cdot [\Delta \psi_R(x)] \cdot \bar{\psi}^i_L(x),
\]
where the gauge field is perturbatively eliminated and
\[
\sum_\mu \bar{\psi}^i_L(x)\gamma_\mu \psi^i_L(x + \mu) = \sum_\mu \left( \bar{\psi}^i_L(x)\gamma_\mu \psi^i_L(x + \mu) - \bar{\psi}^i_L(x)\gamma_\mu \psi^i_L(x - \mu) \right).
\]
For the coupling $g_2 \to \infty$, the kinetic terms $S_f(x)$ can be dropped and we consider the strong-coupling limit. With $S_2(x)$ given in eq. (16), the integral of $e^{-S_2(x)}$ is given

$$Z = \prod_{\alpha i \sigma \xi} \left\{ \int [d\bar{\psi}_R(x) d\psi_R(x)] [d\bar{\psi}_L(x) d\psi_L(x)] \exp (-S_2(x)) \right\}^{4N} \left( \det \Delta^2(x) \right)^4,$$

(18)

where the determinant is taken only over the lattice-space-time and "$N$" is the number of lattice sites. For the non-zero eigenvalues of the operator $\Delta^2(x)$, eq. (18) shows an existence of the sensible strong-coupling limit. However, as for the zero eigenvalue of the operator $\Delta^2(x)$, this strong-coupling limit should not be analytic and the strong-coupling expansion in powers of $\frac{1}{g_2}$ breaks down.

To show eq.(13), i.e. no NJL symmetry breaking occurs in this segment (12), we need to calculate the two-point functions:

$$S_{j RL}(x) \equiv \langle \psi_R(0), \bar{\psi}_j(x) \rangle,$$

(19)

$$S_{j RR}(x) \equiv \langle \psi_R(0), [\bar{\psi}_j(x) \cdot \psi_R(x)] \bar{\psi}_R(x) \rangle.$$

(20)

Using the strong-coupling expansion in $O(\frac{1}{g_2})$ and at non-trivial leading order, we get recursion relations [14]:

$$S_{j RL}(x) = \frac{1}{g_2 \Delta^2(x)} \left( \frac{1}{2a} \right)^3 \sum_{\mu} S_{j RR}(x + \mu) \gamma_\mu,$$

(21)

$$S_{j RR}(x) = \frac{1}{g_2 \Delta^2(x)} \left( \frac{1}{2a} \right)^3 \sum_{\mu} S_{j RL}(x + \mu) \gamma_\mu.$$

(22)

These recursion relations are not valid where the operator $\Delta^2(x)$ has zero eigenvalue. For $p \neq 0$ and $\Delta^2(p) \neq 0$, the Fourier transformation of these recursion relations leads to

$$S_{j RL}(p) = \frac{1}{4g_2w^2(p)} \left( \frac{i}{4a^3} \right) \sum_{\mu} \sin p_\mu S_{j RR}(p) \gamma_\mu,$$

(23)

$$S_{j RR}(p) = \frac{i}{4g_2w^2(p)a} \sum_{\mu} \sin p_\mu S_{j RL}(p) \gamma_\mu,$$

(24)

where $S_{j RR}(p)$ and $S_{j RL}(p)$ are the Fourier transformation of eqs.(13,20). The solution to these recursion relations is

$$\left( (8ag_2w^2(p))^2 + \frac{1}{a^2} \sum_{\mu} \sin^2 p_\mu \right) S_{j RL}(p) = 0.$$

(25)

Clearly, for $p \neq 0$, we must have

$$\Sigma_j(p) \sim S_{j RL}(p) = 0, \quad S_{j RR}(p) = 0.$$

(26)
This demonstration can be straightforwardly generalized to show the vanishing of all n-point functions that are not $SU_L(2) \otimes U_R(1)$ chiral symmetric. Together with eq. (11), we conclude the segment $1 \ll g_2 < \infty$ is entirely symmetric and no NJL spontaneous symmetry breaking takes place.

4. Three-fermion states and vector-like spectrum

We turn to the third point that concerns about decoupling of doublers. On this extreme strong coupling symmetric segment (12), the $\psi^i_L$ and $\psi_R$ in (1) are bound up to form three-fermion states $[2, 14]$:

$$\Psi^i_R = \frac{1}{2a}(\bar{\psi}_R \cdot \psi^i_L) \psi_R; \quad \Psi^n_L = \frac{1}{2a}(\bar{\psi}^n_L \cdot \psi_R) \psi^i_L.$$  (27)

These bound states are Weyl fermions and respectively pair up with the $\bar{\psi}_R$ and $\bar{\psi}^i_L$ to be massive, neutral $\Psi_n$ and charged $\Psi^i_c$ Dirac fermions,

$$\Psi^i_c = (\psi^i_L, \psi^i_R), \quad \Psi_n = (\Psi^n_L, \psi_R).$$  (28)

These three-fermion states (27) carry the appropriate quantum numbers of the chiral gauge group that accommodates the $\psi^i_L$ and $\psi_R$. The $\Psi^i_R$ is $SU_L(2)$-covariant and $U_R(1)$ invariant. The $\Psi^n_L$ is $SU_L(2)$-invariant and $U_R(1)$-covariant. Thus, the spectrum of the massive composite Dirac fermions $\Psi^i_c$ and $\Psi_n$ (28) is vector-like, consistently with the $SU_L(2) \otimes U_R(1)$ chiral symmetry.

In order to study 1PI vertex functions containing the external legs of three-fermion states (27), we define: (i) the right-handed composite “primed” field as

$$\Psi^n_R \equiv \langle \Psi^i_R \rangle = \frac{1}{2a} \frac{\delta^{(3)}W(\eta)}{\delta \eta_R(x) \delta \bar{\eta}_L(x) \delta \bar{\eta}_R(x)};$$  (29)

and (ii) the left-handed composite “primed” field as

$$\Psi^n_L \equiv \langle \Psi^n_L \rangle = \frac{1}{2a} \frac{\delta^{(3)}W(\eta)}{\delta \eta_L(x) \delta \eta_R(x) \delta \bar{\eta}_L(x)}.$$  (30)

Thus, 1PI vertex functions containing the external legs of three-fermion states (27),

$$\frac{\delta^{(2)} \Gamma}{\delta \Psi^n_R(x) \psi^i_L(y)}; \quad \frac{\delta^{(2)} \Gamma}{\delta \psi^i_R(y) \Psi^n_L(x)}; \cdots,$$  (31)

are the truncation of the Green functions

$$\langle \Psi^i_R(x) \bar{\psi}^i_L(0) \rangle = \frac{1}{2a} \frac{\delta^{(4)}W(\eta)}{\delta \eta_R(x) \delta \bar{\eta}_L(x) \delta \bar{\eta}_R(0) \delta \eta_L^i(0)},$$

$$\langle \Psi^n_L(x) \bar{\psi}_R(0) \rangle = \frac{1}{2a} \frac{\delta^{(4)}W(\eta)}{\delta \eta_L(x) \delta \eta_R(x) \delta \bar{\eta}_L(x) \delta \eta_R(0) \delta \eta_L^i(0)}, \cdots.$$  (32)
Taking functional derivative of the Ward identity eq.(7) with respect to $\Psi^\mu_L(x)$ and then putting external sources $J = \eta = 0$, we can derive

$$
\int_x e^{-ipx} \frac{\delta^{(2)} \Gamma}{\delta \Psi^\mu_L(x) \delta \psi_R(0)} = aM(p),
$$

(33)

where

$$
M(p) = 8ag_2w^2(p).
$$

(34)

In the basis of the 1PI vertex functions eqs.(8,33), we can determine the inverse propagator of the neutral composite Dirac fermion $\Psi_n(x)$ to be,

$$
S_n^{-1}(p) = \sum_\mu \gamma_\mu f_\mu(p)P_L + \frac{i}{a} \sum_\mu \gamma_\mu \sin p_\mu P_R + M(p),
$$

(35)

where $f_\mu(p)$ remains unknown.

To obtain the inverse propagator of the charged Dirac fermion $\Psi^i_L(x)$ and $f_\mu(p)$ in eq.(33), we have to use the strong-coupling expansion in powers of $\frac{1}{g_2}$ to compute the following two-point functions with insertions of appropriate composite operators,

$$
\begin{align*}
S^{ij}_{LL}(x) &= \langle \psi^i_L(0)\bar{\psi}^j_L(x) \rangle, \\
S^{ij}_{RL}(x) &= \langle \psi^i_L(0)\bar{\psi}^j_R(x) \rangle, \\
S^{ij}_{LR}(x) &= \langle \psi^i_R(0)\bar{\psi}^j_L(x) \rangle, \\
S^{ij}_{RR}(x) &= \langle \psi^i_R(0)\bar{\psi}^j_R(x) \rangle.
\end{align*}
$$

(36)

In the lowest nontrivial order, we obtain following recursion relations [14],

$$
\begin{align*}
S^{ij}_{LL}(x) &= \frac{1}{g_2 \Delta^2(x)} \left( \frac{1}{2a} \right)^2 \sum_\mu S^{ij}_{RL}(x + \mu) \gamma_\mu, \\
S^{ij}_{RL}(x) &= \frac{1}{2a} \left( \frac{\delta(x)\delta_{ij}}{g_2 \Delta^2(x)} + \frac{1}{g_2 \Delta^2(x)} \right) \sum_\mu S^{ij}_{LL}(x + \mu) \gamma_\mu, \\
S^{ij}_{LR}(x) &= \frac{1}{2a} \sum_\mu \gamma_\mu S^{ij}_{RL}(x + \mu).
\end{align*}
$$

(37, 38, 39)

It is suggested by eq.(38) that the states coupling to operators $\psi^i_L(x)$ and $\Psi^i_R(x)$ are mixed in the lowest nontrivial order of the expansion in $\frac{1}{g_2}$, producing a massive four-component Dirac fermion. To find masses, we make the Fourier transformation of these recursion equations for $p \neq 0$ and $\Delta^2(p) = 4w^2(p) \neq 0$,

$$
\begin{align*}
S^{ij}_{LL}(p) &= \frac{1}{4g_2w^2(p)} \sum_\mu \sin p_\mu S^{ij}_{RL}(p) \gamma_\mu, \\
S^{ij}_{RL}(p) &= \frac{1}{2a} \left( \frac{\delta_{ij}}{8g_2w^2(p)} + \frac{i}{4g_2w^2(p)a} \sum_\mu \sin p_\mu S^{ij}_{LL}(p) \gamma_\mu \right), \\
S^{ij}_{LR}(p) &= \frac{1}{2a} \left[ \frac{1}{4g_2w^2(p)} \left( \frac{i}{a} \right) \sum_\mu \sin p_\mu \gamma_\mu S^{ij}_{RL}(p) \right], \\
S^{ij}_{RR}(p) &= \frac{1}{2a} \sum_\mu \gamma_\mu S^{ij}_{RL}(p).
\end{align*}
$$

(40, 41, 42, 43)
and obtain

\begin{align*}
S_{ij}^{LL}(p) &= P_L \frac{\delta_{ij} \frac{i}{a} \sum_\mu \sin p^\mu \gamma_\mu}{\frac{i}{a} \sum_\mu \sin^2 p_\mu + M^2(p)} P_R, \\
S_{ij}^{RL}(p) &= P_L \frac{\delta_{ij} M(p) \sum_\mu \sin^2 p_\mu + M^2(p)}{\frac{i}{a} \sum_\mu \sin^2 p_\mu + M^2(p)} P_L, \\
S_{ij}^{LR}(p) &= P_R \frac{\delta_{ij} \frac{i}{a} \sum_\mu \sin p^\mu \gamma_\mu}{\frac{i}{a} \sum_\mu \sin^2 p_\mu + M^2(p)} P_L, \\
S_{ij}^{RR}(p) &= P_R \frac{\delta_{ij} \frac{i}{a} \sum_\mu \sin p^\mu \gamma_\mu \sin 2 p_\mu + M^2(p)}{\frac{i}{a} \sum_\mu \sin^2 p_\mu + M^2(p)} P_R.
\end{align*}

Finally we see that the field \( \psi^L_i(x) \) couples to \( \Psi^R_i(x) \) to form Dirac massive states with masses \( M(p) \), given by the locations of the poles of eqs. (44-47). From these equations, we obtain the propagator of a massive Dirac fermion

\begin{align*}
S_{ij}^{c}(p) &= S_{ij}^{LL}(p) + S_{ij}^{RL}(p) + S_{ij}^{LR}(p) + S_{ij}^{RR}(p) \\
&= \delta_{ij} \frac{\frac{i}{a} \sum_\mu \sin p^\mu \gamma_\mu + M(p)}{\frac{i}{a} \sum_\mu \sin^2 p_\mu + M^2(p)} P_R, \quad p \neq 0, \\
S_{ij}^{-1}(p) &= \delta_{ij} \left( \frac{i}{a} \sum_\mu \gamma_\mu \sin p^\mu P_L + \frac{i}{a} \sum_\mu \gamma_\mu \sin p^\mu P_R + M(p) \right) .
\end{align*}

Similar result can be obtained for the neutral composite Dirac fermion \( \tilde{\Psi}^R_i(x) \), with

\begin{equation}
f_\mu(p) = \frac{i}{a} \sum_\mu \gamma_\mu \sin p^\mu ,
\end{equation}

and the propagator of the neutral Dirac fermion is,

\begin{align*}
S_n(p) &= \frac{\frac{i}{a} \sum_\mu \sin p^\mu \gamma_\mu + M(p)}{\frac{i}{a} \sum_\mu \sin^2 p_\mu + M^2(p)} P_R, \quad p \neq 0, \\
S_n^{-1}(p) &= \left( \frac{i}{a} \sum_\mu \gamma_\mu \sin p^\mu P_L + \frac{i}{a} \sum_\mu \gamma_\mu \sin p^\mu P_R + M(p) \right) .
\end{align*}

Eqs. (48) and (51) show that all \( SU_L(2) \) charged and neutral doublers \( p = \tilde{p} + \pi_A \) are decoupled as massive Dirac fermions consistently with the \( SU_L(2) \otimes U_R(1) \) chiral symmetries.

These three-fermion states are constituted by a soft fermionic pair \( \bar{\psi}^L_i(x) \cdot \psi^R_i(x) \) together with chiral fermions \( \psi^L_i(x) \) or \( \psi^R_i(x) \). The structure of the multifermion interaction (11) shows, for each external “spectator” fermion \( \psi^R_i(x) \), there is a Wilson factor \( w(p) \) that is not vanishing for \( p \sim \pi_A \). It can be shown that, in order to

\footnotetext[3] {The momentum of this fermionic pair is very small \( q = p' - p \ll 1 \), where \( p' \) and \( p \) are the momenta of constituent fermions.}
have an enough strong interacting strength to bind up such soft fermionic pair and then three-fermion states with momentum \( p \simeq \pi_A \), these three constituent fermions must have roughly equal momenta

\[
(p, -p, p)
\]

\(|p| \sim \pi_A \) modulo \( 2\pi \). With such bound states mixing with elementary states, it is possible to just fill the Dirac sea for single fermion modes using the bound constituents, one could for low energy physics purposes get rid of doublers \([11]\). It is also pointed out in ref.\([11]\), the configuration \((53)\) is hard to be localized because of the Heisenberg uncertainty principal. We really should consider a superposition of different momenta. This makes it very hard to fill some constituent momentum states up precisely with bound constituents. We leave this discussion open for future work to make sure the decoupling of doublers.

5. Chiral fermions in the low-energy region

In this section, we discuss the fourth point that the normal modes \((p = \bar{p} \sim 0)\) of the \( \psi^L(x) \) and \( \psi^R(x) \) are massless and chiral in the low-energy limit. This is most difficult point to show for the time being, since the strong coupling expansion in powers of \( \frac{1}{g^2} \) breaks down for \( p \to 0 \). In the basis of the continuity \([1, 10]\) of the spectrum \((48, 51)\) in the momentum space due to the locality of the theory \((1)\), one may argue that the vector-like spectrum \((48, 51)\), which is obtained in \( p \neq 0 \), can be continuously extrapolated on to \( p \to 0 \), and we fail to have chiral fermions in the low-energy region.

However, we would like to look at this point based on the point of view that is the essential idea presented in the original paper of Eichten and Preskill \([2]\) to have chiral fermions in continuum limit. The question of whether the spectra of normal modes \((p \to 0)\) of \( \psi^L \) and \( \psi^R \) are chiral is crucially related to the question of whether the normal modes \((p \sim 0)\) of the three-fermion states \((28)\) have been composed in the segment \( 1 \ll g^2 < \infty \). The effective multifermion coupling for these normal modes becomes small and the binding energy of these three-fermion states becomes small as \( p \to 0 \). The continuity of the spectrum \((48, 51)\) in the momentum space breaks down when the spectrum meets a threshold, if there exists a such threshold in \( p \to 0 \), where the binding energy of these three-fermion states goes to zero. In the following discussion, we adopt the 1+1 dimension case to illustrate this threshold phenomenon.

We take the charged Dirac fermion \((48)\) on its mass shell and consider that the time direction is continuous and one space is discrete. We obtain the dispersion relation corresponding to this Dirac fermion \((48)\) for \( p \neq 0 \),

\[
E(p) = \pm \sqrt{\sin^2 p + (8a^2gw^2(p))^2},
\]
where \( E(p) \) is the dimensionless energy of the state “\( p \)”. In eq.(54), the “+” sign corresponds to the dispersion relation of the right-handed three-fermion state \( \Psi_R^i(x) \). Due to the locality of the theory, the spectrum of this bound state \( \Psi_R^i(x) \) is continuous in the momentum space \([1, 10]\). The vector-like spectrum (54) that we obtained by the strong coupling expansion at \( p \sim O(1) \), can be analytically continued to low momentum states \( p \to 0 \), unless this bound state \( \Psi_R^i \) hits the energy threshold of disappearing into its constituents, if there exists such a threshold. Note that eq.(54) obtained from the strong coupling expansion at \( p = 0 \) is not analytic (18).

For a given total momentum \( p \) in the low-energy region, we consider a system that is not a bound state and contains the same constituents as the bound state \( \Psi_R^i(x) \) does, i.e., three free chiral fermions: right-handed fermions \( \bar{\psi}_R \) and \( \psi_R \) with momenta \( p_1 \) and \( p_2 \); a left-handed fermion \( \psi_L^i \) with momentum \( p_3 \), where

\[
p = p_1 + p_2 + p_3 > 0, \quad |p_i| \ll \frac{\pi}{2}, \quad i = 1, 2, 3.
\] (55)

As we have shown that the NJL spontaneous symmetry breaking does not occur (11) for the states \( |p_i| \to 0 \) in the segment \( (1 \ll g_2 < \infty) \), the total energy of such a system is given by

\[
E_t = E_1(p_1) + E_2(p_2) + E_3(p_3)
\]

\[
E_1(p_1) = \sqrt{\sin^2 p_1}, \quad p_1 > 0
\]

\[
E_2(p_2) = \sqrt{\sin^2 p_2}, \quad p_2 > 0
\]

\[
E_3(p_3) = -\sqrt{\sin^2 p_3}, \quad p_3 < 0
\] (56)

where all negative-energy states have been filled. There is no any definite relationship between the total energy \( E_t \) and the total momentum \( p \), since this system is not a bound state. The total energy \( E_t \) level of such a system is continuous because of relative degrees of freedom \( (p_1, p_2, p_3) \) within the system.

The lowest energy \( \min E_t \) (the threshold) of such a system and corresponding configuration can be determined by minimizing the following total energy with a legandre multiple \( \lambda \) (the constraint (55)),

\[
E_t = E_1(p_1) + E_2(p_2) + E_3(p_3) + \lambda p.
\] (57)

One obtains the threshold of this system and the corresponding configuration in the momentum space

\[
\min E_t(p) = 3|\sin p|, \quad p_3 = -p, \quad p_1 = p_2 = p.
\] (58)

Note that this configuration is the same as that (53) of the three-fermion state that we discussed in the end of previous section.
On the other hand, the three-fermion state $\Psi_R \sim (\bar{\psi}_R \cdot \psi^*_L) \psi_R$, as a composite particle, has the definite relationship between its energy and momentum that is given by the dispersion relation (54) with the “+” sign. Given the same momentum “$p$” as eq.(55), this composite fermion states is stable, only if only there is an energy gap $\delta(p)$ (binding energy) between the threshold (58) and the energy (54) of the three-fermion state, i.e.,

$$\delta(p) = \min E_t(p) - E(p) > 0.$$  

(59)

The three-fermion state disappears into its constituents, when the energy gap $\delta$ goes to zero,

$$\delta(p) = \min E_t(p) - E(p) = 0.$$  

(60)

The same discussions can be applied to the neutral three-fermion state $\Psi^n_L$ (51). This discussion is very much like the case of hydrogen, a bound state composed by an electron and a proton, where the energy gap between the first energy-level ($n=1$) and the continuous spectrum is 13.6 eV. Hydrogen turns into a free electron and a free proton as the energy-gap disappears ($n \gg 1$).

Substituting eqs.(54) and (58) into eq.(60), we obtain in the continuum limit $p \to 0$, the energy-gap

$$\delta(p) \to 0, \quad p \to 0$$  

(61)

where the three-fermion-state spectrum dissolves into free chiral fermion spectra. Obviously, this plausible speculation needs receiving either a rigorously analytical proof or a numerical evidence, which are subject to future work. Nevertheless, We assume there exist a threshold in momentum space. The low-energy fermion states “$p$” below this threshold $\epsilon$

$$|p| < \epsilon \ll \frac{\pi}{2},$$  

(62)

are massless and chiral. This threshold $\epsilon$ certainly depends on the multifermion coupling $g_2$. It is right now not clear to us how to determine this threshold.

To end these discussions, we would like to point out the fact that the normal modes do not undergo the NJL spontaneous symmetry breaking (11) for any finite value of the multifermion coupling $g_2$ is extremely crucial. This means, respect to normal modes, there is no a broken phase separating the strong symmetric phase from the weak symmetric phase. In the other words, there is no a mass-gap in eq.(56). Otherwise, the system (56) would be massive, the energy gap (59) would never be zero and we end up with vector-like spectrum in the low-energy $p \to 0$ region. This is the main reason for the failure of EP’s approach, as pointed out in ref.[7]. Thus, we might have a chance to realize EP’s idea that (i) the chiral continuum limit can be defined on a phase transition from one symmetric phase to another symmetric phase; (ii) there is a region in the phase space $g_2$ where doublers are gauge-invariantly decoupled and normal modes are chiral (non NJL-generated masses). The later is a possible case (the segment $1 \ll g_2 < \infty$) that we have discovered.
If what we expect can be convincingly confirmed, in the continuum limit, undoubled low-energy chiral fermions $\psi^i_L(x)$ and $\psi_R(x)$ exist consistently with the $SU_L(2) \otimes U_R(1)$ symmetry ($\tilde{p}$ is the dimensionful momentum),

$$ S^1_L(\tilde{p})^{ij} = i\gamma_\mu \tilde{p}^\mu \delta_{ij} P_L; \quad S^1_R(\tilde{p}) = i\gamma_\mu \tilde{p}^\mu P_R, \quad (63) $$

where $\tilde{Z}_2$ is the finite wave-function renormalization constant of the elementary interpolating field $\psi^i_L(x)$. The spectrum of the theory in this segment is the following. It consists of 15 copies of $SU(2)$-QCD charged Dirac doublers eq.(48) and 15 copies of $SU(2)$ neutral Dirac doublers eq.(51). They are very massive and decoupled. Beside, the low energy spectrum contain the massless normal modes eqs.(63) for $p = \tilde{p}$. This is very analogous to the mirror fermion model, except doubler masses are not generated by the spontaneous symmetry breaking.

6. Gauge coupling vertices and Ward identities

Whether this chiral continuum theory in the scaling region $1 \ll g_2 < \infty$ could be altered, as the $SU_L(2)$ chiral gauge coupling $g$ is perturbatively turned on and the action (1) is $SU_L(2)$-chirally gauged. One should expect a slight change of scaling segment. We should be able to re-tune the multifermion couplings $g_2$ to compensate these perturbative changes, due the fact that the $SU_L(2)$-chiral gauge interaction does not spoil the $\psi_R$-shift-symmetry. As a consequence, the Ward identity associating with the $\psi_R$-shift-symmetry remains valid when the chiral gauge interaction reacts.

Based on the Ward identity of the $\psi_R$-shift-symmetry, we take functional derivatives with respect of the gauge field $A'_\mu$, and we arrive at the following Ward identities,

$$ \frac{\delta^{(2)} \Gamma}{\delta A'_\mu \delta \psi'_R} = \frac{\delta^{(3)} \Gamma}{\delta A'_\mu \delta \psi'_R \delta \psi'_R} = \frac{\delta^{(3)} \Gamma}{\delta A'_L \delta \Psi'_L \delta \psi'_R} = \ldots = 0. \quad (64) $$

As a result of these Ward identities and identical vanishing of 1PI functions containing external gauge fields, “spectator” fermions $\psi_R(x)$ and neutral composite field $\Psi'_L(x)$, we find that absolute non interacting between the gauge field and the “spectator” fermion $\psi_R$ and the neutral three-fermion states $\Psi'_L(x)$. Thus, we disregard those neutral modes.

In order to find the interacting vertex between the gauge boson and the charged Dirac fermion $\Psi'_c(x)$, we need to consider the following three-point functions,

$$ \langle \Psi'_c(x_1) \bar{\Psi}'_c(x) A'^c_\nu(y) \rangle = \langle \psi'_L(x_1) \bar{\psi}'_L(x) A'^c_\nu(y) \rangle + \langle \psi'_L(x_1) \bar{\Psi}'_R(x) A'^c_\nu(y) \rangle + \langle \Psi'_R(x_1) \bar{\psi}'_L(x) A'^c_\nu(y) \rangle + \langle \Psi'_R(x_1) \bar{\Psi}'_R(x) A'^c_\nu(y) \rangle, \quad (65) $$
where we omit henceforth the SU_L(2) indices i and j. Assuming the vertex functions to be \( \Lambda^a_\mu(p, p') \) and \( q = p' + p \), we can write the three-point functions in the momentum space:

\[
\int_{x_1 x y} e^{i(p'x - px_1 - qy)} \langle \bar{\psi}_L(x_1) \bar{\Psi}_L(x) A^a_\mu(y) \rangle = G_\nu_\mu (q) S_{LL}(p) \Lambda^{b}_\mu LL(p, p') S_{LL}(p'); (66)
\]

\[
\int_{x_1 x y} e^{i(p'x - px_1 - qy)} \langle \bar{\psi}_L(x_1) \bar{\Psi}_R(x) A^a_\mu(y) \rangle = C_\nu_\mu (q) S_{LR}(p) \Lambda^{b}_\mu LR(p, p') S_{RR}(p'); (67)
\]

\[
\int_{x_1 x y} e^{i(p'x - px_1 - qy)} \langle \Psi_R(x_1) \bar{\Psi}_R(x) A^a_\mu(y) \rangle = C_\nu_\mu (q) S_{RR}(p) \Lambda^{b}_\mu RR(p, p') S_{RR}(p'); (68)
\]

\[
\int_{x_1 x y} e^{i(p'x - px_1 - qy)} \langle \bar{\Psi}_c(x_1) \bar{\Psi}_c(x) A^a_\mu(y) \rangle = C_\nu_\mu (q) S_c(p) \Lambda^{b}_\mu c(p, p') S_c(p'); (69)
\]

where \( G_\nu_\mu (q) \) is the propagator of the gauge boson; the \( S_{LL}(p), S_{RR}(p) \) and \( S_c(p) \) are the propagators of chiral fermions \( \psi_L(x), \Psi_R(x) \) and Dirac fermion \( \Psi_c(x) \) given in eqs.(14-18).

Using the small gauge coupling expansion, one can directly calculate the three-point function

\[
\langle \psi_L(x_1) \bar{\psi}_L(x) A^a_\mu(y) \rangle = \frac{i g}{2} \left( \frac{\tau^a}{2} \right) \sum \langle \psi_L(x_1) \bar{\psi}_L(x) \rangle \gamma_\mu
\]

\[
\left[ \langle \psi_L(z + \rho) \bar{\psi}_L(x) \rangle A^b_\mu (z + \frac{\rho}{2}) A^a_\mu (y) + \langle \psi_L(z - \rho) \bar{\psi}_L(x) \rangle A^b_\mu (z - \frac{\rho}{2}) A^a_\mu (y) \right]; (70)
\]

and obtains

\[
\Lambda^{(1)}_{\mu LL}(p, p') = i g \left( \frac{\tau^a}{2} \right) \cos \left( \frac{p + p'}{2} \right) \gamma_\mu P_L,
\]

\[
\Lambda^{(2)}_{\mu LL}(p, p') = -i g^2 \left( \frac{\tau^a p^b}{4} \right) \sin \left( \frac{p + p'}{2} \right) \gamma_\mu \delta_{\mu \nu} P_L,
\]

\[
\ldots
\]

By the strong coupling expansion in powers of \( \frac{1}{g^2} \), we try to compute the other three-point functions in eqs.(65) in terms of \( \langle \psi_L(x_1) \bar{\psi}_L(x) A^a_\mu(y) \rangle \). Analogously to recursion relations (37-39), we obtain the following recursion relations at the nontrivial order,

\[
\langle \psi_L(x_1) \bar{\psi}_L(x) A^a_\mu(y) \rangle = \frac{1}{g^2 \Delta^2(x)} \left( \frac{1}{2a} \right)^2 \sum \langle \psi_L(x_1) \bar{\Psi}_R(x + \rho) A^a_\mu(y) \rangle \gamma_\rho (72)
\]

\[
\langle \psi_L(x_1) \bar{\psi}_L(x) A^a_\mu(y) \rangle = \frac{1}{g^2 \Delta^2(x_1)} \left( \frac{1}{2a} \right)^2 \sum \gamma_\rho \langle \Psi_R(x_1 + \rho) \bar{\Psi}_L(x) A^a_\mu(y) \rangle (73)
\]

\[
\langle \Psi_R(x_1) \bar{\Psi}_R(x) A^a_\mu(y) \rangle = \frac{1}{g^2 \Delta^2(x)} \left( \frac{1}{2a} \right)^2 \sum \gamma_\rho \langle \psi_L(x_1) \bar{\Psi}_R(x + \rho) A^a_\mu(y) \rangle. (74)
\]
We make the Fourier transform in both sides of above recursion relations, obtain 

\[ S_{LL}(p)\Lambda_{\mu LL}(p,p')S_{LL}(p') = \frac{i}{a M(p')} S_{LL}(p)\Lambda_{\mu LL}(p,p') S_{RR}(p') \sum_{\rho} \sin p'_\rho \gamma^\rho (75) \]

\[ S_{LL}(p)\Lambda_{\mu LL}(p,p')S_{LL}(p') = \frac{i}{a M(p)} \sum_{\rho} \sin p_\rho \gamma^\rho S_{RR}(p)\Lambda_{\mu RL}(p,p') S_{LL}(p') (76) \]

\[ S_{RR}(p)\Lambda_{\mu RR}(p,p')S_{RR}(p') = \frac{i}{a M(p')} \sum_{\rho} \sin p'_\rho \gamma^\rho S_{LL}(p)\Lambda_{\mu LR}(p,p') S_{RR}(p') (77) \]

In these equations, the propagator of gauge boson \( G_{\nu\rho}(q) \) is eliminated from the both sides of equations.

Using these recursion relations (75-77), \( S_{LL}(p) \) and \( S_{RR}(p) \) in eqs. (44,47), we can compute the vertex functions \( \Lambda_{\mu RL}(p,p'), \Lambda_{\mu LR}(p,p') \) and \( \Lambda_{\mu RR}(p,p') \) in terms of the vertex function \( \Lambda_{\mu LL}(p,p') \) (71) that is obtained from perturbative calculations in powers of the small gauge coupling.

\[ M(p')\Lambda_{\mu LL}(p,p') = \Lambda_{\mu LR}(p,p') \left( \frac{i}{a} \right) \sum_{\rho} \sin p'_\rho \gamma^\rho, \quad (78) \]

\[ M(p)\Lambda_{\mu LL}(p,p') = \left( \frac{i}{a} \right) \sum_{\rho} \sin p_\rho \gamma^\rho \Lambda_{\mu RL}(p,p'), \quad (79) \]

\[ M(p')\Lambda_{\mu RR}(p,p') = \left( \frac{i}{a} \right) \sum_{\rho} \sin p'_\rho \gamma^\rho \Lambda_{\mu LR}(p,p'). \quad (80) \]

Taking \( \Lambda_{\mu LL}(p,p') \) to be eq. (71) at the leading order, we obtain

\[ \Lambda_{\mu RR}^{(1)}(p,p') = ig \left( \frac{\tau^a}{2} \right) \cos \left( \frac{p + p'}{2} \right) \gamma_\mu P_R, \quad (81) \]

\[ \Lambda_{\mu LR}^{(1)}(p,p') \left( \frac{i}{a} \right) \sin p'_\mu = \frac{1}{2} M(p') \left( \frac{i}{a} \right) \sin \left( \frac{p + p'}{2} \right) A_{\mu LR}(p,p'), \quad (82) \]

\[ \left( \frac{i}{a} \right) \sin p_\mu \Lambda_{\mu RL}^{(1)}(p,p') = \frac{1}{2} M(p) \left( \frac{i}{a} \right) \sin \left( \frac{p + p'}{2} \right) A_{\mu RL}(p,p'). \quad (83) \]

Thus, the coupling (83) between the gauge field and Dirac fermion (48) is given by

\[ \Lambda_{\mu}^{(1)} = \Lambda_{\mu LL}^{(1)} + \Lambda_{\mu LR}^{(1)} + \Lambda_{\mu RL}^{(1)} + \Lambda_{\mu RR}^{(1)} \quad (84) \]

These calculations can be straightforwardly generalized to higher orders of the perturbative expansion in powers of the gauge coupling. One can check that these results precisely obey the following Ward identity of the exact \( SU_2(2) \) chiral gauge symmetry \( p', p \neq 0 \)

\[ \left( \frac{i}{a} \right) (\sin p_\mu - \sin p'_\mu) \Lambda_{\mu}^{(1)}(p,p') = S^{-1}_c(p) - S^{-1}_c(p'). \quad (85) \]
where the gauge coupling $g$ and generator $\frac{a}{2}$ are eliminated from the vertex $\Lambda_{\mu c}$. These results are what we expected, since we are in the symmetric phase ($1 \ll g^2 < \infty$) where the exact $SU_L(2)$ chiral gauge symmetry is realized by the vector-like spectrum excluding the low-energy states $p' \neq 0$ and $p \neq 0$.

When $p', p \to 0$ and reach the threshold $\epsilon$ (62) that we discussed in section 5, the right-handed three-fermion state $\Psi^i_R(x)$ is supposed to disappear. The 1PI vertex functions $\Lambda_{\mu RR}$, $\Lambda_{\mu RL}$, and $\Lambda_{\mu LR}$ relevant to $\Psi^i_R(x)$ vanish on this threshold. The coupling vertex (84) between the gauge boson and fermion turns out to be chiral in consistent with the $SU_L(2)$ chiral gauge symmetry, $(p', p \to 0)$

\[
\left( \frac{i}{a} \right) (\sin p_\mu - \sin p'_\mu) \Lambda^{(1)}_{\mu LL}(p, p') = S^{-1}_L(p) - S^{-1}_L(p'),
\]

where the propagator of chiral fermion $S^{-1}_L(p)$ is given by eq.(63) and this Ward identity is realized by the chiral spectrum. Here we stress again that the disappearance of the three-fermion (right-handed) state is essential point to obtain continuum chiral gauge coupling in the low-energy limit. However, we have to confess that similar to the threshold (62), eq.(86) is a plausible speculation for the time being, since we need more evidence and computations to show whether or not it is true.

The Ward identities (85) and (86) play an extremely important role to guarantee that the gauge perturbation theory in the scaling region ($1 \ll g^2 < \infty$) is gauge symmetric. To all orders of gauge coupling perturbation theory, gauge boson masses vanish and the gauge boson propagator is gauge-invariantly transverse. The gauge perturbation theory can be described in the normal renormalization prescription as that of the QCD and QED theory. In fact, due to the manifest $SU_L(2)$ chiral gauge symmetry and corresponding Ward identities that are respected by the spectrum (vector-spectrum for $p \neq 0$ and chiral-spectrum for $p = 0$) in this possible scaling regime, we should then apply the Rome approach [16] (which is based on the conventional wisdom of quantum field theories) to perturbation theory in the small gauge coupling. It is expected that the Rome approach would work in the same way but all gauge-variant counterterms are prohibited.

7. The vacuum functional and gauge anomaly

Provided the scenario of the gauge coupling and spectrum given in above section, one should expect that the gauge field should not only chirally couple to the massless chiral fermion of the $\psi^i_L$ in the low-energy regime, but also vectorially couple to the massive doublers of Dirac fermion $\Psi^i_c$ in the high-energy regime. In this section, we discuss the gauge anomaly and the renormalization of gauge perturbation theory.
We consider the following $n$-point 1PI functional:

$$
\Gamma^{(n)}_{\{\mu\}} = \frac{\delta^{(n)} \Gamma(A')}{\delta A'_{\mu_1}(x_1) \cdots \delta A'_{\mu_j}(x_j) \cdots \delta A'_{\mu_n}(x_n)},
$$

(87)

where $j = 1 \cdots n$, $(n \geq 2)$ and $\Gamma(A')$ is the vacuum functional. The perturbative computation of the 1PI vertex functions $\Gamma^{(n)}_{\{\mu\}}$ can be straightforwardly performed by adopting the method presented in ref. [17] for lattice QCD. Dividing the integration of internal momenta into 16 hypercubes where 16 modes live, we have 16 contributions to the truncated vertex functions. The region where is the chiral fermion modes of continuum limit is defined as

$$
\Omega = [0, \epsilon]^4, \quad p < \epsilon \ll \frac{\pi}{2}, \quad p \to 0
$$

(88)

where the $\epsilon$ is the energy-threshold given by (62), on which $\Psi_R(x)$ disappears, in section 5.

As a first example, we deal with the vacuum polarization

$$
\Pi_{\mu\nu}(p) = \sum_{i=1}^{16} \Pi_{\mu\nu}^i(p), \quad \Pi_{\mu\nu}^d(p) = \sum_{i=2}^{16} \Pi_{\mu\nu}^i(p).
$$

(89)

For the contributions $\Pi_{\mu\nu}^d(p)$ from the 15 doublers $(i = 2, \ldots, 16)$, we make a Taylor expansion in terms of external physical momenta $p = \bar{p}$ and the following equation is mutatis mutandis valid [17],

$$
\Pi_{\mu\nu}^d(p) = \Pi_{\mu\nu}^0(0) + \Pi_{\mu\nu}^{d(2)}(p)(\delta_{\mu\nu}p^2 - p_\mu p_\nu)
$$

$$
\quad + \sum_{i=2}^{16} \left(1 - p_\rho |_\phi \partial_\rho - \frac{1}{2} p_\rho p_\sigma |_\phi \partial_\rho \partial_\sigma\right) \Pi_{\mu\nu}^{con}(p, m_i),
$$

(90)

where $|_\phi f(p) = f(0)$ and $m_i$ are doubler masses. The first and second terms are specific for the lattice regularization. Since the 15 doublers are gauged as an $SU(2)$ QCD-like gauge theory with propagator (48) and interacting vertex (84), the Ward identities (85) associated with this vectorial gauge symmetry result in the vanishing of the first divergent term $\Pi_{\mu\nu}^0(0)$ and the gauge invariance of the second finite contact term in eq.(90). We recall that in Roma approach, this was achieved by enforcing Ward identities and gauge-variant counterterms. The third term in eq.(90) corresponds to the relativistic contribution of the 15 doublers. The $\Pi_{\mu\nu}^{con}(p, m_i)$ is logarithmicly divergent and evaluated in some continuum regularization. For doubler masses $m_i$ of $O(a^{-1})$, the third term in eq.(90) is just finite and gauge-invariant contributions.

We turn to the contribution $\Pi_{\mu\nu}^n(p)$ of the massless chiral mode that is in the first hypercube $\Omega = [-\epsilon, \epsilon]^4$ (88). We can use some regularization to calculate this contribution,

$$
\Pi_{\mu\nu}^n(p) = \Pi_{\mu\nu}^{n(2)}(p)(\delta_{\mu\nu}p^2 - p_\mu p_\nu).
$$

(91)
The spectrum eq. (63) and gauge-coupling vertex eq. (71) with respect to the chiral mode is $SU_L(2)$ chiral-gauge symmetric. The Ward identity associated with this chiral gauge symmetry render eq. (91) to be gauge invariant. The $\epsilon$-dependence ($\ell n \epsilon$) in eq. (91) has to be exactly cancelled out from those contributions (90) from doublers, because the continuity of 1PI vertex functions in momentum space.

In summary, the total vacuum polarization $\Pi_{\mu\nu}(p)$ contains two parts: (i) the vacuum polarization of the chiral mode $\psi_i^L$ in some continuum regularization; (ii) gauge invariant finite terms stemming from doublers’ contributions. The second part is the same as the perturbative lattice QCD, and can be subtracted away in normal renormalization prescription.

The second example is the 1PI vertex functions $\Gamma^{(n)}_{\{\mu\}}(\{p\})(n \geq 4)$,

$$
\Gamma^{(n)}_{\{\mu\}}(\{p\}) = \sum_{i=1}^{16} \Gamma^{(n)\mu}_i(\{p\}, m_i) \quad n \geq 4,
$$

where internal momentum integral is analogously divided into the contributions from sixteen sub-regions of the Brillouin zone where sixteen modes live. Based on gauge invariance and power counting, one concludes that up to some gauge invariant finite terms, the $\Gamma^{(n)}_{\{\mu\}}(\{p\})(n \geq 4)$ contain the 15 continuum expressions for 15 massive ($m_i$) Dirac doublers and one for the massless Weyl mode. The 15 doubler contributions vanish for $m_i \sim O(a^{-1})$. The $n$-point 1PI vertex functions (92) end up with their continuum counterpart for the Weyl fermion and some gauge invariant finite terms. These finite gauge invariant terms come from doublers’ contributions are similar to those in the lattice QCD, and can be subtracted away in the normal renormalization prescription.

The most important contribution to the vacuum functional is the triangle graph $\Gamma_{\mu\nu\alpha}(p, q)$, which is linearly divergent. Again, dividing the integration of internal momenta into 16 hypercubes, one obtains

$$
\Gamma_{\mu\nu\alpha}(p, q) = \sum_{i=1}^{16} \Gamma_{\mu\nu\alpha}^i(p, q)
$$

$$
\Gamma_{\mu\nu\alpha}^i(p, q) = \Gamma_{\mu\nu\alpha}^{(c)}(0) + p_{\rho} \Gamma_{\mu\nu\alpha, \rho}^{(1)}(0) + q_{\rho} \Gamma_{\mu\nu\alpha, \rho}^{(1)}(0)
$$

$$
+ \left( 1 - \left| p_{\rho} \partial_{\rho} - q_{\rho} \partial_{\rho} \right| \right) \Gamma_{\mu\nu\alpha}^{\text{con}}(p, q, m_i),
$$

where $\Gamma_{\mu\nu\alpha}^{\text{con}}(p, q, m_i)$ is evaluated in some continuum regularization. As for the 15 contributions of Dirac doublers ($i = 2 \cdots 15$), the first three terms in eq. (93) are zero for the vector-like Ward identity (85). The non-vanishing contributions are the same as the 15 copies of the $SU(2)$ vectorial gauge theory of massive Dirac fermions. These contributions are gauge-invariant and finite (as $m_i \sim O(a^{-1})$), thus, disassociate from the gauge anomaly.
The non-trivial contribution of the chiral mode in the hypercube \( \Omega = [-\epsilon, \epsilon]^4 \) is given by

\[
\Gamma_{\mu\nu}(p, q) = \int_{\Omega} \frac{d^4 k}{(2\pi)^4} \text{tr} \left[ S(k + \frac{p}{2}) \Gamma_{\mu}(k) S(k - \frac{p}{2}) \Gamma_{\nu}(k - \frac{p + q}{2}) S(k - \frac{p}{2} - q) \Gamma_{\alpha}(k - \frac{q}{2}) \right] + (\nu \leftrightarrow \alpha),
\]

(94)

where the propagator \( S(p) \) and vertex \( \Gamma_{\mu} \) are given by eqs.(63,71). Other contributions containing anomalous vertices (\( \bar{\psi}\psi AA, \bar{\psi}\psi AAA \)) vanish within the hypercube \( \Omega = [-\epsilon, \epsilon]^4 \). As well known, eq. (94) is not gauge invariant. To evaluate eq.(94), one can use some continuum regularizations. As a result, modulo possible local counterterms, we obtain the consistent gauge anomaly for the non-abelian chiral gauge theories as the continuum one:

\[
\delta g \Gamma(A') = -\frac{ig^2}{24\pi^2} \int d^4 x \epsilon^{\alpha\beta\mu\nu} \text{tr} \theta_a(x) \tau_a \left[ A_\alpha(x) \left( \partial_\beta A_\mu + \frac{ig}{2} A_\beta(x) A_\mu(x) \right) \right],
\]

(95)

where the gauge field \( A_\mu = \tau_a A^a_\mu \). The \( SU_L(2) \) chiral gauge theory is anomaly-free for \( \text{tr}(\tau^a, \{\tau^b, \tau^c\}) = 0 \), and the gauge current

\[
J_\mu^a = i\bar{\psi}_L \gamma^\mu \tau_a \psi_L = \frac{\delta \Gamma(A)}{\delta A^a_\mu(x)} \quad \partial_\mu J_\mu^a = 0
\]

(96)

is covariantly conserved and gauge invariant. In the following paragraph, we discuss how we achieve the correct gauge anomaly (95) from the gauge symmetric action (1).

A most subtle property of the naive lattice chiral gauge theory is the appearance of 16 modes. Each mode produces the chiral gauge anomaly with definite axial charge \( Q_5 \), such that the finite (regularized) theory is anomaly-free and the chiral gauge symmetry is perfectly preserved. As has been seen, the 15 doublers decoupled as massive Dirac fermions that are vector-gauge-symmetric (85). Thus, they decouple from the gauge anomaly as well. Only the anomaly associated with the normal (chiral) mode of the \( \psi_i^L \) is left and is the same as the continuum one. The condition is the disappearance of the right-handed three-fermion state \( \Psi^i_R \) in the low-energy limit. It seems surprising that we start from a gauge symmetric action and we end up with the correct gauge anomaly. Normally, one may claim that the anomaly has to come from the explicit breaking of the chiral gauge symmetry in a regularized action (e.g., a Wilson term). This statement is indeed correct if regularized actions are bilinear in fermion fields, since this is nothing but what the “no-go” theorem asserts. However, we run into the dilemma that the gauge anomaly is independent of any explicitly breaking parameters (e.g., the Wilson parameter \( r \) and fermion masses). In fact, the most essential and intrinsic raison d’être of producing the correct gauge anomaly is “decoupling doublers” rather than “explicitly breaking of chiral gauge symmetries ”. If we adopt a bilinear action to decouple doublers, we must explicitly
break chiral gauge symmetry, which is just a superficial artifact in bilinear actions. However, if we give up the bilinearity of regularized actions in fermion fields and find a chiral-gauge-invariant way to decouple doublers, we should not surprise to achieve the correct gauge anomaly \cite{95}. These discussions on the gauge anomaly is not complete yet. It is related again to the possibility of filling three-fermion states \cite{53} into the Dirac sea, further discussions are necessary. Reads are suggested to the papers by Nielsen and Ninomiya \cite{11} and Creutz \cite{18}.

8. The anomalous global current

Non-conservation of fermion numbers is an important feature of the Standard Model. A successful regularization of chiral gauge theories should give this feature in the continuum limit \cite{2,4,19}. In EP’s approach of multifermion couplings, inspired by the origination of the axial anomaly in lattice QCD, it is suggested that the anomalous global current should be originated from the explicit breaking of the global symmetry at the tree level.

In this section, we show a possibility that this global anomaly can be consistently obtained from the explicit symmetric action \cite{11} with the multifermion coupling, if the composite right-handed fermion $\Psi^i_R(x)$ disappears into three chiral constituents in the low-energy limit, and the theory is free from local gauge symmetry breaking and the gauge anomaly.

Our action \cite{11} processes the $U_L(1)$ and $U_R(1)$ global chiral symmetries. At tree level, the theory is invariant under the following transformations:

$$
\psi^i_L \rightarrow e^{i\theta_L} \psi^i_L \quad \psi_R \rightarrow e^{i\theta_R} \psi_R.
$$

These global symmetries lead to the conservation of the singlet chiral fermion currents,

$$
\partial_\mu j^\mu_L(x) = 0, \quad j^\mu_L = i\bar{\psi}^i_L \gamma^\mu \psi^i_L, \quad (98)
$$

$$
\partial_\mu j^\mu_R(x) = 0, \quad j^\mu_R = i\bar{\psi}^i_R \gamma^\mu \psi^i_R + O(a^2), \quad (99)
$$

which are Noether currents. Eqs.\cite{11,18} correspond to the conservation of fermion numbers. However, as we know, eqs.\cite{11,18} should be anomalous.

In order to see whether the conservation of the currents is violated when the chiral gauge field is coupled to chiral fermions, we consider the source currents $\langle j^\mu_L(x) \rangle$ and $\langle j^\mu_R(x) \rangle$ defined as

$$
\langle j^\mu_L(x) \rangle = \frac{\delta \Gamma}{\delta V^L_\mu(x)}; \quad \delta V^L_\mu(x) = -\partial_\mu \theta_L(x); \quad (100)
$$

$$
\langle j^\mu_R(x) \rangle = \frac{\delta \Gamma}{\delta V^R_\mu(x)}; \quad \delta V^R_\mu(x) = -\partial_\mu \theta_R(x). \quad (101)
$$
Under the variations $\delta L$ and $\delta R$ of these $U(1)$-phases $\theta_L(x)$ and $\theta_R(x)$, the effective action $\Gamma$ is transformed (up to $O(\theta_L)$ and $O(\theta_R)$)

$$\delta_L \Gamma = \int d^4x \delta V^L_\mu(x) \langle j^\mu_L(x) \rangle = \int d^4x \theta_L(x) \partial_\mu \langle j^\mu_L(x) \rangle,$$

$$\delta_R \Gamma = \int d^4x \delta V^R_\mu(x) \langle j^\mu_R(x) \rangle = \int d^4x \theta_R(x) \partial_\mu \langle j^\mu_R(x) \rangle,$$

(102)

where

$$\delta_R \Gamma = \Gamma(A_\mu + \delta V^R_\mu(x)) - \Gamma(A_\mu),$$

(103)

$$\delta_L \Gamma = \Gamma(A_\mu + \delta V^L_\mu(x)) - \Gamma(A_\mu).$$

(104)

In our action (1), the $\psi_R$ does not couple to the chiral gauge field and this decoupling strictly holds due to the Ward identity (64). Thus, the $\langle j^\mu_R(x) \rangle$ defined formally in eq.(101) is gauge invariant

$$\delta g \langle j^\mu_R(x) \rangle = 0,$$

(105)

and eq.(103) becomes,

$$\delta_R \Gamma(A) = 0.$$  

(106)

This leads to the conservation of the right-handed current,

$$\partial_\mu \langle j^\mu_R(x) \rangle = 0.$$  

(107)

We find that the Ward identity of the $\psi_R$-shift-symmetry and decoupling between the gauge field and the spectator fermion $\psi_R(x)$ are crucial to the conservation of the right-handed fermion numbers (107).

We have shown in section 7, under a gauge transformation $\delta g$,

$$\delta_g \Gamma_{\mu\alpha}(p, q) \neq 0, \quad \delta_g \Gamma^{(n)}_{\{\mu\}} = 0 \quad (n \neq 3).$$

(108)

The vacuum functional $\Gamma$ is just the same as the continuum counterpart up to some gauge-invariant finite terms. In the anomaly-free $SU_L(2)$ case ($\delta_g \Gamma = 0$), one may conclude that the current $\langle j^\mu_L(x) \rangle$ defined in eq.(100) is gauge invariant.

$$\delta_g \langle j^\mu_L(x) \rangle = \frac{\delta \Gamma}{\delta V^L_\mu(x)} = \frac{\delta \delta g \Gamma}{\delta V^L_\mu(x)} = 0.$$  

(109)

However, this is not true. The order of the differentiations $\delta g$ and $\delta_L$ can not be exchanged. We know that in our action (1), the left-handed variation

$$\delta V^L_\mu(x) = -\partial_\mu \theta_L(x)$$

(110)

can be considered as a commuting $U_L(1)$ factor in the $SU_L(2)$ chiral gauge group, i.e.

$$\tilde{A}_\mu = A_\mu + V^L_\mu, \quad (A_\mu = \frac{\tau^a}{2} A^a_\mu).$$

(111)
Actually, this is a $SU_L(2) \otimes U_L(1)$ chiral gauge group and there is a mixing anomaly $^{20}$.

\[
\delta_L \Gamma = C_1 \frac{ig^2}{32\pi^2} \int d^4x \theta_L \text{tr} \left( F_{\mu\nu} \tilde{F}_{\mu\nu} \right),
\]

\[
\delta_g \Gamma = C_2 \frac{ig}{16\pi^2} \int d^4x \theta_1 \text{tr} \left( \theta_\mu \partial_\mu A_\nu \right),
\]

where $\theta_\mu = \theta_\mu^a \tau^a$ is the $SU_L(2)$ transformation parameter and

\[
F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu.
\]

The reason is that one of the Pauli matrices $\frac{\tau^a}{2}$ in the triangle graph is replaced by the generator (identity) of the $U_L(1)$, i.e., the $U_L(1)$ global current, therefore the vanishing of the $SU_L(2)$ anomaly for $\text{tr}(\tau^a, \{\tau^b, \tau^c\}) = 0$ is no longer true.

Note that in eqs. (112,113), we only consider the triangle diagram ($n = 3$), since

\[
\delta_L \Gamma_{(n)} = 0, \quad \delta_g \Gamma_{(n)} \quad (n \neq 3)
\]

for being gauge-invariant, as we discussed in section 7.

The mixing anomaly (113) has arbitrariness $C_1, C_2$ ($C_1 + C_2 = 1$), which arise because the triangle graphs with one insertion of the $U_L(1)$ global current determine the $\Gamma_{\mu\nu\alpha}(A')$ up to a local counterterm. As the Feynman diagrams determine the effective action $\Gamma(A)$ only up to an arbitrary choice of local counterterms, we are allowed to add local counterterms into the effective action

\[
\Gamma'(A') = \Gamma(A') + \Gamma_{c.t.}(A'),
\]

which is equivalent to the re-definition of the chiral fermion current eq. (100). Due to the fact that the effective action $\Gamma(A')$ we obtained for the $SU_L(2)$ case is free from local gauge-symmetry-breaking terms and the non-local gauge anomaly, the arbitrariness in eq. (113) can be fixed

\[
C_1 = 1, \quad C_2 = 0
\]

by choosing an adequate local counterterm. As a result, the effective action and the left-handed current are gauge invariant,

\[
\delta_L \Gamma'(A) = 0, \quad \delta_g \langle j_L' \rangle = 0,
\]

where $\langle j_L' \rangle$ is the re-definition of the left-handed chiral fermion current eq. (100).

From eq. (102), one obtains

\[
\delta_L \Gamma' = \frac{ig^2}{32\pi^2} \int d^4x \theta_L \text{tr} \left( F_{\mu\nu} \tilde{F}_{\mu\nu} \right); \quad \partial_\mu \langle j_L' \rangle = \frac{ig^2}{32\pi^2} \text{tr} \left( F_{\mu\nu} \tilde{F}_{\mu\nu} \right).
\]
This is just the desired result, which shows the left-handed fermion number is violated by the $SU(2)$ instanton effect.

We stress again that the exact chiral gauge symmetry of a regularized action, gauge invariant 1PI vertex functions (absence of local gauge variant terms and anomaly-free, e.g., the $SU_L(2)$ case and Standard Model) and the right-handed composite fermion $\Psi^R_i(x)$ dissolving into three free chiral fermions plays an extremely crucial role in obtaining the gauge-invariant chiral-fermion-current (118) and its non-conservation (119). The reason for obtaining the correct anomaly (119) coincides with that considered in the approach of (21).

One may ask why this global anomaly comes from an explicit $U_L(1)$ symmetric action (1). This question is raised because of our knowledge of the lattice QCD where the axial current anomaly, 

$$\partial_\mu j^\mu_5 = \frac{ig^2}{16\pi^2} \text{tr} \left( F_{\mu\nu} \tilde{F}^{\mu\nu} \right)$$

(120)

is due to the flavour $SU_L(3) \otimes SU_R(3)$ asymmetric Wilson term. However, a priori, we have no dynamical reason to expect that the non-conservation of fermion number is due to the explicit breaking of the $U_L(1)$ symmetry at the cutoff level. A non-gauge-invariantly regularized action with the $U(1)$-asymmetry, (e.g., Majorana Wilson-Yukawa coupling[22]) does not lead to the correct anomaly eq.(119) [23], unless we force the Ward identities of the chiral gauge symmetry to be obeyed by tuning counterterms [22]. It is still unknown whether chiral gauge symmetric and the $U(1)$ asymmetric model such as $SU(5)$[4] and $SO(10)$[7] give the correct anomaly to the conservation of chiral fermion numbers. It is expected[4,11] that the correct anomaly is unlikely to be produced in these models because explicit-breaking contributions to $\partial_\mu j^\mu_5$ will normally have a quite different $\psi$-field-dependence. In fact, the anomaly (119) disappears as the gauge field turns off, the conservation of the global current must be related to the explicit $U_L(1)$ and $U_R(1)$ symmetries of the action (1). These discussions are not completely clarified and they need further studies. To look at this problem, it is helpful to read the papers by Nielsen and Ninomiya [11] and Creutz [18] about their intuitive understanding of anomalies.

9. Summary

In summary, we present a possible lattice chiral gauge theory with the large multifermion coupling. We discuss how to realize EP’s idea in this model and study relevant problems of regularizing chiral gauge theories, which are listed in the introduction section. We advocate the segment $(1 \ll g^2 < \infty)$ to be the scaling region for defining a continuum chiral gauge theory. In the light of studies presented in this paper, it seems to us that a certain multifermion coupling model
with EP’s idea has chances to work. However, in the multifermion coupling model proposed in this paper, there are many things such as no NJL gauge symmetry breaking, decoupling of doublers, disappearance of the right-handed composite fermion $\Psi_R$, etc. that need to be further clarified and demonstrated. For the controversy of the issue, it is suggested to study multifermion coupling models and solve relevant problems in 1+1 dimensions [24]. It is worthwhile to mention that applying EP’s idea only to mirror fermions residing in one wall of Kaplan’s model [23] is a proposal that may work [25].

In conclusion, we advocate that the further study of EP’s idea and multifermion couplings for regularizing chiral gauge theories be necessary. I thank Profs. G. Preparata, M. Creutz, H.B. Nielsen and E. Eichten for discussions.

References

[1] H.B. Nielsen and M. Ninomiya, Nucl. Phys. B185 (1981) 20, ibid. B193 (1981) 173, Phys. Lett. B105 (1981) 219.

[2] E. Eichten and J. Preskill, Nucl. Phys. B268 (1986) 179.

[3] G. Preparata and S.-S. Xue, Phys. Lett. B264 (1991) 35; ibid B335 (1994) 192; B329 (1994) 87; B325 (1994) 161; B377 (1996) 124; Nucl. Phys. B26 (Proc. Suppl.) (1992) 501; Nucl. Phys. B30 (Proc. Suppl.) (1993) 647.

[4] H.B. Nielsen and S.E. Rugh, Nucl. Phys. 29B (Proc. Suppl.) (1992) 200.

[5] T. Banks, Phys. Lett. B272 (1991) 75; T. Bank and A. Dabholkar, Nucl. Phys. 29B (Proc. Suppl.) (1992) 46.

[6] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122 (1961) 345.

[7] M.F.L. Golterman, D.N. Petcher and E. Rivas, Nucl. Phys. B395 (1993) 597.

[8] J. Smit, Acta Physica Polonica B17 (1986) 531; P.D.V. Swift, Phys. Lett. B145 (1984) 256.

[9] D.N. Petcher, Nucl. Phys.(Proc. Suppl.) B30 (1993) 52, references there in.

[10] Y. Shamir, Phys. Rev. Lett. 71 (1993) 2691; Nucl. Phys.(Proc. Suppl.) B47 (1996) 212.

[11] H.B. Nielsen and M. Ninomiya, Int. J. of Mod. Phys. A6 (1991) 2913; H.B. Nielsen and S.E. Rugh, Nucl. Phys. 29B (Proc. Suppl.) (1992) 200.

[12] M.F.L. Golterman, D.N. Petcher, Phys. Lett. B225 (1989) 159.
[13] K. Wilson, in New phenomena in subnuclear physics (Erice, 1975) ed. A. Zichichi (Plenum, New York, 1977).

[14] S.-S. Xue, Phys. Lett. **B381** (1996) 277 and hep-lat/9605003.

[15] I. Montvay, Nucl. Phys. **B29** (Proc. Suppl.) (1992) 159, references therein.

[16] A. Borrelli, L. Maiani, G.C. Rossi, R. Sisto and M. Testa, Nucl. Phys. **B333** (1990) 335; Phys. Lett. **B221** (1989) 360; L. Maiani, G.C. Rossi, and M. Testa, Phys. Lett. **B261** (1991) 479; *ibid* **B292** (1992) 397; L. Maiani, Nucl. Phys. (Proc.Suppl.) **B29** (1992) 33.

[17] L.H. Karsren and J. Smit, Nucl. Phys. **B183** (1981) 103.

[18] M. Creutz, Nucl. Phys.(Proc. Suppl.) **B42** (1995) 56, Phys. Rev. **D52** (1995) 2951; M. Creutz and I. Horváth, Phys. Rev. **D50** (1994) 2297.

[19] R. Narayanan and H. Neuberger, Nucl. Phys. **B443** (1995) 305, references there in.

[20] J. Preskill, Ann. Phys. **210** (1991) 323.

[21] G. ’t Hooft, Phys. Lett. **B349** (1995) 491; P. Hernández and R. Sundrum, Nucl. Phys. **B455** (1995) 287.

[22] L. Maiani, G.C. Rossi and M. Testa, Phys. Lett. **292** (1992) 397.

[23] S. Aoki, Nucl. Phys. **29B** (Proc. Suppl.) (1992) 171.

[24] Private communications with H.B. Nielsen and E. Eichten.

[25] M. Creutz and S.-S. Xue, in progress.

[26] D. B. Kaplan, Phys. Lett. **B288** (1992) 342.