Feasible Set in a Discrete Epidemic Model

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Abstract. This paper aims at the study of the delimitation of the sets of feasible trajectories (discrete trajectories having a biological sense) generated by a family of two-dimensional map $T$ related to a discrete epidemic model. The inverse of $T$ has vanishing denominator giving rise to non-classical singularities: a non definition line, a focal point and a prefocal line. In particular, we give an answer to one of the open problems proposed by Kocic and Ladas.

1. Introduction
Many classical epidemic models have been proposed and studied [1-4]. The deterministic epidemic model

$$x_{n+1} = (1 - \sum_{j=1}^{k-1} x_{n-j})(1 - e^{-Ax_n}), \quad n = 0, 1, \cdots$$

(1.1)

is the special case of an epidemic model which is derived in [5, 6], where $x_n$ is the parasitoid population size in generation $n$, $A$ is the instantaneous search rate i.e. average encounters per host per unit of search time. Two research projects are proposed in [7]. One has been partly solved in [8]. The other is to choose the initial conditions of Eq. (1.1) to keep the solutions positive for all $n=0,1,\cdots$. The main purpose of this paper is to give an answer to this and other questions concerning Eq.(1.1).

For the sake of simplicity we shall firstly focus on the simplest case, obtained for $k=2$ in (1.1).

$$x_{n+1} = (1-x_n - x_{n-1})(1 - e^{-Ax_n}), \quad n = 0, 1, \cdots$$

(1.2)

However, many of the techniques can be generated to recurrence (1.1) with higher values, i.e. recurrence order greater than 2. Some possible generations are given in the paper. We shall make use of some results on iterated plane maps with a denominator which can be vanish, given in [9-11] where new kinds of singularities, called focal points and prefocal curves, have been defined. As shown in the paper, some interesting results of the recurrence (1.2) can be evidenced in the light of these concepts.

The plan of the work is as follows. In Sec.2 where some properties and the terminology introduced in [9-11] are briefly described and extensively used. In Sec.3 we present the conditions of existence and stability for fixed points. Sec.4 concerns some definitions of feasible sets, feasible domains and basins, and the delimitation of them is presented in detail. Some extensions are given for higher order recurrence (1.1). We end this paper with some conclusions and discussions in Sec.5.
2. Properties of the equivalent two-dimensional map

In order to use the concepts of the singularities, let us rewrite the second-order recurrence in (1.2) as a two-dimensional system of the first order, i.e. an iterated map of the plane. As usual, this is obtained by letting \( (x_{n+1}, x_n) = (x, y) \) so that (1.2) can be written as \( M(x_{n+1}, x_n) \rightarrow (x', y') \), where \( M(x, y) \rightarrow (x', y') \) is the two-dimensional map given by

\[
M : \begin{cases} 
  x' = y \\
  y' = (1-x-y)(1-e^{-Ay})
\end{cases}
\]  
\tag{2.1}

The Jacobian matrix of map (2.1) is

\[
J = \begin{pmatrix} 0 & 1 \\
  e^{-Ay} - 1 & A(1-x-y)e^{-Ay}
\end{pmatrix}
\]  
\tag{2.2}

Apparently, its Jacobian determinant \( \det(J) = 1 - e^{-Ay} \) vanishes on the line given by \( y=0 \), so that the map (2.1) is invertible except on the \( x \) axis,

\[
\text{the inverse of } M \text{ is } M^{-1} : \begin{cases} 
  x = 1 - x' - \frac{y'}{1-e^{-Ay}} \\
  y = x'
\end{cases}
\]  
\tag{2.3}

For the map (2.3), the non definition set \( \delta \) coinciding with the locus of points in which at least one (here the first component of Eq.(2.3)) denominator vanishes is given by \( \delta = \{ (x', y') \in \mathbb{R}^2 | x' = 0 \} \). In general, the two-dimensional recurrence sequence obtained by the iteration of \( M^{-1} \) is well defined provided that the initial condition belongs to the good set \( G \) given by \( G = \mathbb{R}^2 \setminus \Lambda \), where \( \Lambda \) (a set of zero Lebesgue measure in \( \mathbb{R}^2 \)) is the union of the images of any rank of the line \( \delta \), that is \( \Lambda = \bigcup_{n=0}^{\infty} M^{n}(\delta) \). In other words, sequences generated by the recurrence (2.3) can be obtained by the iteration of the two-dimensional map \( M^{-1} : G \rightarrow G \), which is not defined in the whole plane. Following the terminology introduced in [9-11], the line will be called set of non definition, and its point \( Q=(0,0) \) is the line of equation \( y=0 \). This means that a one-to-one correspondence exists between the slopes \( m \) of arcs \( \eta \) through the focal point \( Q \) and the points \( (x(m),0) \) where their preimages \( M^{-1}(\eta) \) cross the prefocal set \( \delta \). In this case such correspondence is very simple, given by \( m \rightarrow x(m) = 1 - m/A \). This implies that the preimage by \( M \) of an arc \( \eta \) crossing through \( Q \), is an arc \( M^{-1}(\eta) \) which crosses the prefocal line at the point \( (1-m/A, 0) \), and conversely, the image \( M(\eta) \) of an arbitrary arc \( \eta \) which crosses the line \( y=0 \) at a point \( (x, 0) \) is an arc which crosses through \( Q \) with a slope \( m=A(1-x) \) in \( Q \) (see the qualitative picture in Fig.1(a)). Instead, if an arc \( \gamma \) crosses the set of non definition \( \delta \) at a non focal point \( (0, \xi) \), \( \xi \neq 0 \), then its image by \( M^{-1} \) is an unbounded arc, doubly asymptotic to the line of equation \( y=0 \). We shall see that the properties of focal points and prefocal sets of \( M^{-1} \) will help us to understand the peculiar properties of the recurrence (2.1).
Figure 1 (a) shows the noticeable qualitative change in the shape of the image and preimages, due to a contact between an arc and the singular set $\delta_s$ and $\delta_0$. (b) One dimensional bifurcation diagram of the system (1.2).

Existence and stability of fixed points

Proposition 1

1. Assume $A>1$, the original point $(0, 0)$ is saddle fixed point.
2. Assume $A>1$, map (2.1) has a unique positive fixed point $x^* \in (0, 1/3)$.
3. The positive fixed point $(\bar{x}, \bar{x})$ of map (2.1) is stable for $A < \frac{1 - \bar{x}}{(1 - 2\bar{x})(1 - 3\bar{x})}$.

Proof: (1) The multipliers (or eigenvalues) of the fixed point $(0, 0)$ are $A>1$ and 0, therefore, the original point $(0, 0)$ is saddle fixed point.

(2) The positive fixed point $(\bar{x}, \bar{x})$ of map (2.1) satisfies the following equation:

$$(1 - 2\bar{x})(1 - e^{-\lambda \bar{x}}) = \bar{x} \quad (3.1)$$

From Eq.(3.1) we have $e^{-\lambda \bar{x}} = (3\bar{x} - 1)/(2\bar{x} - 1)$, this implies that the necessary condition of existence for a positive fixed point is $0 < (3\bar{x} - 1)/(2\bar{x} - 1) < 1$. Therefore, the necessary condition of existence for a positive fixed point is $0 < \bar{x} < 1/3$.

Let $g(x) = (1-2x)(1-e^{-\lambda x})$, then $g'(x) = -2 + (A - 2Ax + 2)e^{-\lambda x}$ and $g''(x) = -A(4 + A - 2Ax)e^{-\lambda x} < 0$ for $0 < x < 1/3$, hence $g(x)$ is a convex function. Noticing that $g(0) = 0$ and $g'(0) = -A > 1$, by the properties of $g(x)$ we obtain that map (2.1) has a unique positive fixed point.

(3) From Eq.(2.2) (Jacobian matrix of map(2.1)), the characteristic equation of the Jacobian matrix $J$ at positive fixed point $(\bar{x}, \bar{x})$ can be given by

$$\lambda^2 - A(1 - 2\bar{x})e^{-\lambda x} \lambda - (e^{-\lambda \bar{x}} - 1) = 0 \quad (3.2)$$

A simple analysis shows that the condition of stability for positive fixed point is

$$[A(1 - 2\bar{x}) + 1]e^{-\lambda \bar{x}} < 2 \quad (3.3)$$

Substituting $e^{-\lambda \bar{x}} = (3\bar{x} - 1)/(2\bar{x} - 1)$ into Eq.(3.3) we obtain that the positive fixed point is stable for
Simulation shows that the positive fixed point begins at $A\approx 1.1625$ then loses its stability giving birth to a 3-cycle through fold bifurcation for $A > F(\bar{x}) \approx 27.75$. This can be seen in the fig.1(b).

3. The feasible set and feasible domain

One of research projects proposed in [7] is to choose the initial conditions of Eq.(1.1) so that the solutions remain positive for all $n=0,1\ldots$. This is related to the following concepts:

**Definition 1** We call $S$ the feasible set, the set of points whose trajectory starting from $S$ totally belongs to the first quadrant $\Re^2_+ = \{(x,y) \in \Re^2 \mid x \geq 0, y \geq 0\}$. The full trajectory is called a feasible trajectory.

**Definition 2** We call $D$ the feasible domain, if a full trajectory starting from $D$ totally belongs to the first quadrant and converges to any one of the different bounded attractors belonging to $\Re^2_+$. The full trajectory is called a converging feasible trajectory.

Since only converging feasible trajectory can represent reasonable time evolutions of the epidemic model, the first important problem to solve is the delimitation of the set of initial conditions that generate a converging feasible trajectory for given parameters. In this section, we shall give an answer to this question.

**Definition 3** We denote by $B(A)$ the basin of attraction of an attractor $A$ (or the basin), defined as the set of all the points whose trajectories converge to the bounded attractor $A$.

From definitions 1-3, we have $D = B \cap S$, if there is only one attractor in $\Re^2_+$. So in such case, to determine the feasible domain, we only need to know the feasible set and the attracting basin. The basin can be obtained by many classical methods, so an exact determination of the feasible set $S$ for the model (1.2) is the main goal of this section. From definition 1, we know that $M^k(S) \subseteq \Re^2_+, k=0,1,\ldots$. This implies $S \subseteq M^k(\Re^2_+), k=0,1,2,\ldots$. Therefore, we expect that the feasible set boundary can be determined by all the preimages of the two axes which are the boundaries of $\Re^2_+$ (see [12-16]).

From the Eq.(2.3) (the inverse of the model (2.1)), the rank-1 preimage of $Y-Q$, does not exist or is infinite and $M^{-1}(Q) = \{y=0\}$, hence, we obtain

$$\partial S \subseteq \bigcup_{k=0}^{\infty} M^{-k}(X) \bigcup Y$$ (4.1)

This means that the feasible set boundary is mainly related to the axis $X=\{y=0\}$. Denoting by $\omega$ the $x$-axis, we can analytically obtain the preimages of $\omega$ from the equation of (2.1). Here we only give the rank-1 and rank-2 preimages

$$\omega^{-1}_+: x+y=1, \quad \omega^{-1}_-: e^{-\beta y} = \frac{x+y-1}{x}$$ (4.2)

since the higher rank preimages are complex.

Because the $\omega$ crosses the non definition line $\delta$ at the focal point $Q(0,0)$ with the slope $m=0$, we know, from the analysis of Sec.2, its preimage $\omega^{-1}_+$ intersects the prefocal line $\delta_0 : y=0$ at point $(x(m),0)=(1,0)$. However, the $\omega^{-1}_-$ also crosses the non definition line at non focal point $(0,1)$, so its preimage $\omega^{-1}_-$ takes the form of $M^{-1}(\delta)$ in fig1(a) and has two branches asymptotic to the line $y=0$ (see fig.2(a,c)). From Eq.(4.2), We have $\omega^{-1}_- : x=(y-1) e^{\beta y}/(1-e^{\beta y})$. It is easy to find that the curve $\omega^{-1}_-$ has other two asymptotical lines $x=0, x+y=1$. 

4
To gain a good interpretation of the feasible set structure, we first give the following proposition:

**Proposition 2** Consider a map $T$, If a set $\Omega$ entirely belongs to the non-feasible set $C(S)$, then all its preimages belong to the set $C(S)$ forever.

**Proof:** Suppose the assertion is not true, then there exists $k$ such that $T^k(\Omega) \cap S \neq \emptyset$, hence, there are points in $S$ giving birth to the points of $\Omega \subseteq C(S)$ after $k$ iterations of $T$, which contradicts the definition of the set $S$. This completes the proof.

From proposition 2, we know that once a preimage of $\omega$ totally belongs to $C(S)$, all its preimages of higher order can not belong to the feasible set boundaries.

From Eq.(2.1), we have $M^{-1}(\Omega) = \{(x,y)|y \geq 0, x+y \leq 1\}$, $M^{-1}(\{(x,y)|y<0, x+y<1\}) \cup \{(x,y)|y>0, x+y>1\} = \{(x,y)|y>0\}$. This implies that only the points in the triangle, bounded by $\omega$, $\omega_1$, and $Y$ axis, denoted by $D = \{(x,y)|x \geq 0, y \geq 0, x+y \leq 1\}$, can generate the feasible trajectories. That is, the set $C(D)$ is the non-feasible set. Now we shall prove the curve $\omega_2$ tally belongs to the set $C(D)$. The second equation of Eq.(4.2) implies that $\omega_2 \subseteq \{(x,y)|x+y=1\}$, i.e. $\omega_2 \subseteq D_1 \cup D_2$, where $D_1 = \{(x,y)|x>0, x+y>1\}$, $D_2 = \{(x,y)|x<0, x+y<1\}$. Therefore, $\omega_2 \subseteq C(D)$ for $A>1$. From proposition 2, the higher preimages of $\omega_2$ belong to $C(D)$ forever thus can not become the boundaries of feasible set. Therefore, we gain the following proposition:

**Proposition 3** The feasible sets of system (1.2) are the triangle $D$ bounded by $x$ and $y$ axis and the line $\omega_1 = M(X): x+y=1$, that is, $\partial D \subseteq \left( \bigcup_{n=0}^{\infty} M^{-n}(X) \right) \bigcup Y = \partial \omega_1 \bigcup Y$
The basins of attraction $B(A)$ are illustrated by cyan regions (see Fig.2(b)(d)). The boundaries of basin $B(A)$ from $B(\infty)$ can be determined by the insets (i.e. the stable manifolds) of the saddle point $O(0,0)$ which is on the basin boundaries. Because the original point is not only the saddle fixed point but also the focal point, associated prefocal line $\{y=0\}$, i.e. $M(\{y=0\})=O(0,0)$. This means that x axis is just the local stable manifolds of the saddle fixed point. Therefore, the basin boundaries, as seen in fig.2(a)(d), can be determined by the preimages of x axis. The basin of positive fixed point $(0.1974, 0.1974)$ at $A=2$ is unbounded and non connected region (see Fig.2(b)).

Because there is only one attractor in for the epidemic model (1.2), the feasible domain is the feasible set, that is, $D=B\cap S=S$. In fact, for many biological models, there is only one attractor in $\mathbb{R}^2$ (denoted by $A'$), which implies that we usually have $S \subseteq B(A')$. Therefore, the feasible domain is usually the feasible set, namely $D=S$. But we may have cases in which two different attractors exist in $\mathbb{R}^2$, and in such cases $D$ is different from $B \cap S$ (because $S$ also includes stable sets of saddles belonging to $\mathbb{R}^2$ which are not in $D$).

We have considered the simplest case of the family of recurrence given in (1.1), obtained for $k=2$. However, from the proofs of the propositions, we know that similar results continue to hold, with obvious changes, even for any $k>2$. The recurrence (1.1) of order $k$ is equivalent to a system of $k$ equation of first order. Identifying $(x_0, \ldots, x_{k-1})=(y_0, \ldots, y_k)$, we get a $k$ dimensional map

$$T_k: \mathbb{R}^k \rightarrow \mathbb{R}^k, (y_0, \ldots, y_k) \rightarrow T_k(y_0, \ldots, y_k)$$

We have the following proposition:

**Proposition 4** The feasible sets of system (1.1) are super tetrahedron

$$S = \left\{(y_0, \ldots, y_{k-1}, y_k) \mid y_0 \geq 0, \ldots, y_{k-1} \geq 0, y_k \geq 0, \sum_{j=1}^{j=k} y_j \leq 1, \sum_{j=1}^{j=k} y_j = 1 \right\},$$

bounded by the super planes $y_i=0, \ldots, y_{k-1}=0, y_k=0$ and $\sum_{j=1}^{j=k} y_j = 1$. Feasible domains are the feasible sets, i.e. $D=S$.

4. Summary and discussion

In this paper we investigate the global properties of a discrete epidemic model by the interaction between the computer experiment and the mathematical analysis. We give an answer to one of the open problems, proposed by Kocic and Ladas, from another perspective of the global dynamical behavior by a study of the domains of feasible trajectories and their bifurcations.

For the biologist these results may be interesting, since the delimitation of the feasible domains and basins permits them to understand which initial conditions are suitable for the biological model, what kind of exogenous shock can be recovered by the endogenous dynamics of the biological system, and which one will cause severe crashes and extinctions of the biological system, namely an irreversible departure from the attractor.

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