Stability and convergence analysis of high-order numerical schemes with DtN-type absorbing boundary conditions for nonlocal wave equations

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The stability and convergence analysis of high-order numerical approximations for the one- and two-dimensional nonlocal wave equations on unbounded spatial domains are considered. We first use the quadrature-based finite difference schemes to discretize the spatially nonlocal operator, and apply the explicit difference scheme to approximate the temporal derivative to achieve a fully discrete infinity system. After that, we construct the Dirichlet-to-Neumann (DtN)-type absorbing boundary conditions (ABCs) to reduce the infinite discrete system into a finite discrete system. To do so, we first adopt the idea in [Du, Zhang and Zheng, Commun. Comput. Phys., 24(4):1049–1072, 2018 and Du, Han, Zhang and Zheng, SIAM J. Sci. Comp., 40(3):A1430–A1445, 2018] to derive the Dirichlet-to-Dirichlet (DtD)-type mappings for one- and two-dimensional cases, respectively. We then use the discrete nonlocal Green’s first identity to achieve the discrete DtN-type mappings from the DtD-type mappings. The resulting DtN-type mappings make it possible to perform the stability and convergence analysis of the reduced problem. Numerical experiments are provided to demonstrate the accuracy and effectiveness of the proposed approach.

Keywords: nonlocal wave equation, artificial boundary method, absorbing boundary conditions, stability and convergence analysis, DtN-type map

1. Introduction

Recently, nonlocal models have received much attention in various research areas, such as the peridynamic theory of continuum mechanics (see Silling, 2000), image processing (see, e.g., Buades et al., 2005; Gilboa & Osher, 2008; Lou et al., 2010), biology (see, e.g., Painter et al., 2015) and diffusion processes (see, e.g., D’Elia et al., 2017; Ignat & Rossi, 2007). While most existing nonlocal models are formulated on bounded domains with volume constraints (see Emmrich & Weckner, 2007; Du et al., 2013; Tian & Du, 2013, 2014; Zhou & Du, 2010), the models on infinite domains are more reasonable when describing wave propagation in an exceedingly large sample (see, e.g., Weckner & Abeyaratne, 2005; Weckner & Emmrich, 2005). In this work, we consider the computation of the $d$-dimensional

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nonlocal wave equation with \(d = 1, 2\), given as

\[
\frac{\partial^2}{\partial t^2} u(x, t) + L_\delta u(x, t) = f(x, t), \quad x \in \mathbb{R}^d, \\
u(x, 0) = \varphi(x), \quad x \in \mathbb{R}^d, \\
\frac{\partial}{\partial t} u(x, 0) = \psi(x), \quad x \in \mathbb{R}^d,
\]

where the body force \(f(x, t)\), the initial values \(\varphi(x)\) and \(\psi(x)\) are the given compactly supported functions, and the nonlocal operator \(L_\delta\) is defined as

\[
L_\delta u(x) = \int_{B_\delta(x)} (u(x) - u(y)) \gamma(x - y)dy.
\]

In the definition above, \(B_\delta(x)\) is an interval \((d = 1)\) or a square \((d = 2)\) centered at \(x\) with side length \(2\delta\), and the radial kernel function \(\gamma(\alpha)\) satisfies

- nonnegativity: \(\gamma(\alpha) \geq 0, \alpha \in \mathbb{R}^d\),
- finite horizon: \(\gamma(\alpha) = 0, \alpha \in \mathbb{R}^d \setminus B_\delta(0)\).

Here horizon parameter \(\delta\) is used to measure the range of nonlocal interaction. Moreover, if the second-order moment of the kernel satisfies

\[
\frac{1}{2} \int_{\alpha \in B_\delta(0)} ||\alpha||^2 \gamma(\alpha) d\alpha = d,
\]

then the nonlocal operator \(L_\delta\) converges to the classical Laplace operator \(-\Delta\) when \(\delta \to 0\) (see Du et al., 2018a, 2019a; Du & Zhou, 2011; Du, 2019). Consequently, as \(\delta \to 0\), the solution of nonlocal model (1.1) converges to that of the following local model

\[
\frac{\partial^2}{\partial t^2} u(x, t) - \Delta u(x, t) = f(x, t), \quad x \in \mathbb{R}^d, t > 0, \\
u(x, 0) = \varphi(x), \quad x \in \mathbb{R}^d, \\
\frac{\partial}{\partial t} u(x, 0) = \psi(x), \quad x \in \mathbb{R}^d.
\]

Several tools have been developed to solve problems defined on the unbounded domains, such as the artificial boundary method (ABM) (see Han & Wu, 2013), perfectly matched layer method (see Berenger, 1994), infinite element or boundary element method (see Ying & Han, 1980; Yu, 1993) and so on. Among the above successful approaches, we here use the ABM to deal with the problem (1.1). The key ingredient of ABM is to design appropriate absorbing/artificial boundary conditions (ABCs), also called transparent or nonreflecting boundary conditions in literatures, on the artificial boundaries satisfied by the solution of the original problem, which reduce the original unbounded problem to an initial-boundary-value problem on bounded computational domains of interest. The ideal ABCs can efficiently absorb/annihilate waves on artificial boundaries, and do not produce the reflected or nonphysical waves to disrupt the waves in the computational domain.

The ABM has been well studied to solve local problems on unbounded spatial domains (see Grote & Keller, 1995; Hagstrom, 1999; Lubich & Schädle, 2002; Teng, 2003; Givoli, 2004, 1991). For local problems, the well-posedness requires values of the solution along only the boundary of considered domain \(\Omega\). For nonlocal problems, it requires values of the solution over a layer with a thickness of \(\delta\) outside of \(\Omega\) due to the nonlocal interactions. This brings essential difficulties to the design of ABCs

\[
L_\delta u(x) = \int_{B_\delta(x)} (u(x) - u(y)) \gamma(x - y)dy.
\]
for nonlocal problems, compared with local problems (see Zhang, 2021). Recently, much effort and great progress have been made for nonlocal problems (see Zheng et al., 2017; Zhang et al., 2017; Du et al., 2018a,b; Yan et al., 2020; Zheng et al., 2020; Shojaei et al., 2020; Ji et al., 2021b,a; Wang et al., 2022). For 1D nonlocal diffusion equations, Zhang et al. (2017) derive the continuous Dirichlet-to-Neumann (DtN)-type ABCs (global in time) and high-order Padé approximate ABCs (local in time). Zheng et al. (2017) construct the Dirichlet-to-Dirichlet (DtD)-type ABCs using the Laplace transform in the spatial direction; Furthermore, Zheng et al. (2020) propose the discrete DtN-type ABCs for the stability and convergence analysis and develop a fast convolution algorithm to efficiently implement the ABCs. In addition, Shojaei et al. (2020) construct the approximated Dirichlet-type ABCs derived from exponential basis functions for both 1D, 2D and 3D cases. For nonlocal Schrödinger equations, Yan et al. (2020) construct the exact ABCs using the $z$-transform for the spatially discretized 1D system. Ji et al. (2021b) develop an exact boundary conditions by accurately computing the Green’s functions of the semi-discrete nonlocal Schrödinger equations. As for nonlocal wave equations given by (1.1), Du et al. (2018a,b) construct the DtD-type ABCs using the spatial Laplace transform for 1D case and the idea of integral equation method for 2D case, respectively, but the stability and convergence analysis of the proposed schemes remains open.

The aim of this work is to construct numerical schemes with the rigorous stability and convergence analysis for nonlocal wave equation (1.1) on unbounded domains. A typical procedure of solving the problem on an unbounded domain by the ABM is first to derive suitable ABCs to restrict problem on a bounded domain, then approximate the reduced initial-boundary-value problem. The resulting ABCs for the continuum model usually involve convolution operations and other complicated forms, therefore, are hard to be approximated without loss of accuracy of the whole numerical scheme, not to mention their stability and convergence analysis. An alternative procedure is first of all to fully discretize the original problem on the unbounded domain, and then directly construct the exact discrete ABCs for the fully discrete infinite system.

In this work, we adopt the second strategy to discretize the continuum models into infinite discrete systems over the whole space. The explicit finite difference (FD) scheme is used to approximate the temporal derivative. And the spatially discrete schemes here we used are the quadrature-based difference schemes, which can be arbitrarily high-order. After that, we apply the DtD-type ABCs developed in Du et al. (2018a,b) to reduce the infinite system to a finite system on the bounded domain. The resulting discrete DtD-type ABCs are exact and are tractable for practical implementations, but it is hard to obtain their stability and convergence analysis. To this end, we further construct the discrete DtN-type ABCs based on the discrete nonlocal Green’s first identity. The DtN-type ABCs is useful to establish the stability and convergence of the reduced finite system. Using the energy method, we prove that under the CFL condition of the nonlocal case, the proposed numerical scheme has an optimal convergence order of $O(\tau^2 + h^q)$, where $\tau$ and $h$ denote the time step size and spatial mesh size.

The paper is organized as follows. In section 2, a fully discrete scheme is presented to approximate the nonlocal wave equation (1.1) to obtain the infinite discrete system. In section 3, the DtN-type ABCs are constructed based on the DtD-type ABCs, which reduce the infinite discrete system to a finite discrete systems. In section 4, the stability and convergence of the proposed numerical schemes are analyzed, and numerical experiments are provided to demonstrate our theoretical analysis in section 5. The conclusion is drawn in section 6.
2. Fully discrete wave system

In this section, we discretize the nonlocal operator (1.2) using the high-order quadrature-based FD scheme and approximate the temporal derivative using the explicit FD scheme to achieve a fully discrete wave system over the whole space.

2.1 Discretization of the nonlocal operator

Here we extend the second-order quadrature-based FD scheme approximating spatially nonlocal operators (see Tian & Du, 2013; Du et al., 2019a, 2018a) to arbitrarily high-order scheme. First we state some norm notations used in the whole paper. The notations \( |\cdot|_1 \) and \( \|\cdot\| \) stand for the \( \ell_1 \) norm and standard Euclidean norm (i.e., \( \ell^2 \)-norm) in the \( d \)-dimensional vector space, respectively. And \( |\cdot|_\infty \) represents the maximum norm, concretely, for a vector \( \mathbf{x} \in \mathbb{R}^d \) or a matrix \( A \in \mathbb{R}^{m \times n} \),

\[
|\mathbf{x}|_\infty = \max_{i=1, \ldots, d} |x_i|, \quad |A|_\infty = \max_{i=1, \ldots, m; j=1, \ldots, n} |A_{i,j}|.
\]

Let \( \{\mathbf{x}_k = k h\}_{k \in \mathbb{Z}^d} \) be the set of nodes (grid points) of the uniform rectangular grid \( \mathcal{T}_h \) over the whole space with mesh size \( h \), where \( k \) denotes a multiindex. The nonlocal operator (1.2) acting on \( u(x_k) \) can be written as

\[
\mathcal{L}_\delta u(x_k) = \int_{B_\delta(x_k)} \frac{u(x_k) - u(y)}{w(x_k - y)} w(y) \gamma(x_k - y) dy,
\]

(2.1)

where the weight function \( w(z) = \|z\|^2 / |z|_1 \) is introduced to ensure the approximate scheme is asymptotically compatible (AC) (see Du et al., 2019a), a concept proposed by Tian & Du (2014), which means that the solution of the scheme converges to that of the corresponding local continuum models when both horizon \( \delta \) and mesh size \( h \) tend to zero, regardless of how \( \delta \) and \( h \) may or may not be dependent (see Tian & Du (2020), Tian & Du (2013), Tian & Du (2014) for further information). The property of AC is vital in multiscale modelling and computation. In this work, we focus on the case of fixed \( \delta \), so whether the numerical scheme is AC is not our main concern.

We use the idea of composite integration rule to compute the integral (2.1). First we divide the integral domain \( B_\delta(x_k) \) into \( (2L/p)^d \) equal small domains \( T^k_l \), where \( L = [\delta/h] \). For an example, the 1D domain \( T^k_1 \) is given as

\[
T^k_1 = [x_k - Lh + (i - 1) ph, x_k - Lh + iph], \quad i = 1, 2, \ldots, 2L/p.
\]

For simplicity, we always choose \( h \) that can make \( \delta \) an integral multiple of \( h \) and \( L \) an integral multiple of \( p \). Then on each small domain \( T^k_1 \), we use the \( p \)-th degree Lagrange interpolation to approximate the integrand part \( \frac{u(x_k) - u(y)}{w(x_k - y)} \). The rest part \( w(x_k - y) \gamma(x_k - y) \) can be regarded as the integral weight.

Let \( u_k \) be the approximation of \( u(x_k) \). \( \Phi_{k,p}(x) \) be the \( p \)-th degree (1D) or bi\( p \)th degree (2D) Lagrange polynomial at point \( x_k \) on each divided small domain \( T^k_1 \), then one obtains the discretization for (2.1) as

\[
\mathcal{L}_{\delta,h} u_k = \sum_{m \in \mathbb{Z}^d, m \neq k} \frac{u_k - u_m}{w(x_k - x_m)} \int_{B_\delta(x_k)} \Phi_{m,p}(y) w(y) \gamma(x_k - y) dy
\]

\[
= \sum_{m \in \mathbb{Z}^d, m \neq k} \frac{u_k - u_m}{w(x_k - x_m)} \int_{B_\delta(0)} \Phi_{m-k,p}(s) w(s) \gamma(s) ds
\]

(2.2)
where

\[ a_m = \begin{cases} \frac{1}{w(z)} \int_{B_{\delta}(0)} \Phi_{\kappa,p}(s) w(s) \gamma(s) ds, & m \neq 0, \\ 0, & m = 0. \end{cases} \tag{2.3} \]

It is obvious that \( a_m = a_{-m} \).

According to the finite horizon assumption (1.4) of kernel, the coefficient \( a_m \) satisfies

\[ a_m = 0, \quad |m| \geq L. \]

For further study, the following equivalent form of the \( L_{\delta,h}u \) is needed

\[ L_{\delta,h}u_k = \sum_{|m| \leq L} c_m u_{k+m}, \quad k \in \mathbb{Z}^d \tag{2.4} \]

with

\[ c_m = \begin{cases} -a_m, & m \neq 0, \\ \sum_{m \in \mathbb{Z}^d, m \neq 0} a_m, & m = 0. \end{cases} \tag{2.5} \]

where the property \( a_m = a_{-m} \) is used.

On the truncation error of quadrature-based FD approximation (2.2), we have the following lemma.

**Lemma 2.1** If \( u \in C_b^{p+3}(\mathbb{R}^d) \) and \( w(s) \gamma(s) \) is integrable in \( B_{\delta}(0) \), then it holds that

\[ |L_{\delta,h}u - L_{\delta}u|_\infty \leq Ch^q, \tag{2.6} \]

where \( C \) is a constant independent of \( h \). And the order \( q \) is given as

\[ q = \begin{cases} p + 1, & p \text{ is odd}, \\ p + 2, & p \text{ is even} \end{cases} \quad \text{and} \quad u \in C_b^{p+4}(\mathbb{R}^d). \tag{2.7} \]

For brevity, the proof of this lemma is given in Appendix.

**Remark 2.1** We discuss the symbol of the coefficient \( a \) given in (2.3). For the case of \( p = 1 \), all coefficients \( a_m \) are non-negative since the basis function \( \Phi_{\kappa,m}(s) \) is non-negative. When \( p \geq 2 \), the situation is complicated. The sign of \( a_m \) depends on the kernel, and it cannot be guaranteed to be always non-negative. Through the direct calculation, one has that the coefficients are non-negative when \( p \leq 6 \) for the constant kernel. And for the common used kernels \( \gamma_1(s) = C|s|^{-\frac{1}{2}} \) and \( \gamma_2(s) = C|s|^{-\frac{3}{2}} \), the corresponding coefficients are non-negative when \( p \leq 7 \) and \( p \leq 3 \), respectively.

**Remark 2.2** The weight function \( w(z) \) is introduced to guarantee the numerical approximation with the linear interpolation (i.e., \( p = 1 \)) is asymptotically compatible in Du et al. (2019a). Here we keep the weight \( w(z) \) in discretization since the introduction of \( w(z) \) also can relax the requirement for kernel function (see Du et al., 2019a).

### 2.2 Fully discrete wave system

Let \( \mathcal{F} = \{ t_n | t_n = n\tau; \quad 0 \leq n \leq N \} \) be a uniform partition of \([0,T]\) with the time step size \( \tau = T/N \), and \( u_k^{(n)} \) be the approximation of \( u(x_k,t_n) \). Define the second-order approximation for time derivative by

\[ \mathcal{D}_\tau u^{(n)} = \frac{1}{\tau^2} (u^{(n+1)} - 2u^{(n)} + u^{(n-1)}). \tag{2.8} \]
Using the explicit finite difference method to discretize the temporal derivative for problem (1.1), we have the fully discrete system on the whole space as

\[ \mathcal{D}_t u_k^{(n)} + \mathcal{L}_{\delta,h} u_k^{(n)} = f_k^{(n)}, \quad k \in \mathbb{Z}^d, n \geq 1, \quad (2.9) \]

\[ u_k^{(0)} = \varphi_k, \quad k \in \mathbb{Z}^d, \quad (2.10) \]

\[ u_k^{(1)} = \varphi_k + \tau \psi_k + \frac{\tau^2}{2} ( -\mathcal{L}_{\delta,h} \varphi_k + f_k^{(0)} ), \quad k \in \mathbb{Z}^d. \quad (2.11) \]

### 3. Design of absorbing boundary conditions

We now consider the construction of DtN-type ABCs for the fully discrete system (2.9)-(2.11) based on the DtD-type mappings proposed by Du et al. (2018a,b). We first streamline the useful notations and tools. Let \( \Omega = \{ x \in \mathbb{R}^d : |x|_\infty < \beta \} \) be the computational domain of interest, where \( \beta \) is a positive real number. Set \( M = \lfloor \beta/h \rfloor \). To clearly address the index in various grid domains, we define

\[ \mathbb{K} = \{ k \in \mathbb{Z}^d : |k|_\infty < M \}, \quad \mathbb{K}^c = \{ k \in \mathbb{Z}^d : |k|_\infty \geq M \}, \]

\[ \mathbb{K}^- = \{ k \in \mathbb{Z}^d : |k|_\infty \geq M - L \}, \quad \mathbb{K}^- = \{ k \in \mathbb{Z}^d : |k|_\infty < M \}, \quad (3.1) \]

\[ \mathbb{K}^+ = \{ k \in \mathbb{Z}^d : |k|_\infty < M + L \}, \quad \mathbb{K}^+ = \{ k \in \mathbb{Z}^d : |k|_\infty < M + L \}. \]

We also introduce the \( z \)-transform and its inverse transform for a bounded infinite sequence \( \{u^{(n)}\}_{n=0}^{+\infty} \) as

\[ \hat{u}(z) = \sum_{n=0}^{+\infty} z^{-n} u^{(n)}, \quad |z| > 1, \quad (3.2) \]

\[ u^{(n)} = \frac{1}{2\pi i} \int_{C_{\rho}} \hat{u}(z) z^{n-1} dz, \quad n \geq 0, \quad \rho > 1, \quad (3.3) \]

where \( z \) is a continuous complex variable, \( C_{\rho} \) is a counterclockwise circle with a radius of \( \rho \).

For vectors \( u = \{u_k\}_{k \in \mathcal{F}} \) and \( v = \{v_k\}_{k \in \mathcal{F}} \) (\( \mathcal{F} \) indicates any subset of \( \mathbb{Z}^d \), such as \( \mathbb{K}, \mathbb{K}^c \)), the \( \ell^2 \)-inner product and norm are respectively given as

\[ (u,v)_{\mathcal{F}} = \sum_{k \in \mathcal{F}} u_k v_k \quad \text{and} \quad \|u\|_{\mathcal{F}} = \sqrt{(u,u)_{\mathcal{F}}}. \quad (3.4) \]

And we denote \( L^2 \)-inner product by \( \langle \cdot, \cdot \rangle \), i.e.,

\[ (f,g) = \int_{\Omega} f(x)g(x)dx, \quad \forall f,g \in L^2(\Omega). \]

The discrete \( L^2 \)-inner product and norm in \( \mathcal{F} \subset \mathbb{Z}^d \) are defined as

\[ (u,v)_{h,\mathcal{F}} = h^d \sum_{k \in \mathcal{F}} u_k v_k, \quad \|u\|_{h,\mathcal{F}} = \sqrt{(u,u)_{h,\mathcal{F}}}. \]

And we define a discrete bilinear form \( \langle \cdot, \cdot \rangle_{h,\mathcal{F}} \) by

\[ \langle u,v \rangle_{h,\mathcal{F}} = \frac{h^d}{2} \sum_{k \in \mathcal{F}} \sum_{m \in \mathcal{F}} a_{k-m}(u_k-u_m)(v_k-v_m). \quad (3.5) \]
Then $|u|^2 := \sqrt{(u,v)_{h,\mathbb{F}}}$ is a discrete seminorm. For brevity, for any vector confined on the index set $\mathbb{K}$, we omit the subscripts $\mathbb{K}$ in the notation below, such as

$$(u,v)_{h,\mathbb{K}} := (u,v)_h, \quad \|u\|_{h,\mathbb{K}} := \|u\|_h.$$ 

### 3.1 DtD-type absorbing boundary conditions

To construct the DtN-type ABCs, we briefly review the design of DtD-type ABCs. As the initial data $\varphi$, $\psi$ and the source function $f$ are compactly supported, we assume that

$$f(x,t) = \varphi(x) = \psi(x) = 0, \quad |x|_\infty > \beta.$$ 

(3.6)

The problem (2.9)-(2.11) is equivalent to the following two subproblems. The first subproblem is defined on the index set $\mathbb{K}$ as

$$\mathcal{D}_\tau u_k^{(n)} + \mathcal{L}_{\delta,h} u_k^{(n)} = f_k^{(n)}, \quad k \in \mathbb{K}, n \geq 1,$$

$$u_k^{(0)} = \varphi_k, \quad k \in \mathbb{K},$$

$$u_k^{(1)} = \varphi_k + \tau \psi_k + \frac{\tau^2}{2} (-\mathcal{L}_{\delta,h} \varphi_k + f_k^{(0)}), \quad k \in \mathbb{K}.$$ 

(3.7)

The second subproblem is defined on the index set $\mathbb{K}^c$ as

$$\mathcal{D}_\tau u_k^{(n)} + \mathcal{L}_{\delta,h} u_k^{(n)} = 0, \quad k \in \mathbb{K}^c, n \geq 1,$$

$$u_k^{(0)} = 0, \quad u_k^{(1)} = 0, \quad k \in \mathbb{K}^c.$$ 

(3.8)

Problems (3.7) and (3.8) can not be solved independently, since they are related through the boundary. Following the idea presented in Du et al. (2018a,b), the value of $\{u_k\}_{k \in \mathbb{K}^c}$ can be expressed by $\{u_k\}_{k \in \mathbb{K}^c}$ through considering the exterior problem (3.8). One may apply the $z$-transform to (3.8) to have

$$s\hat{u}_k + \sum_{|m|_\infty \leq L} c_m \hat{u}_{k+m} = 0, \quad k \in \mathbb{K}^c,$$

$$\lim_{|k|_\infty \to +\infty} \hat{u}_k = 0,$$

(3.9)

where $s = (z^{-1} - 2 + z)/\tau^2$.

To investigate the well-posedness of problem (3.9), we introduce the sequence space equipped with the $\ell^2$-norm

$$\ell^2 = \{\hat{u} = \{u_k\}_{k \in \mathbb{Z}^d} : |\hat{u}|^2 = \sum_{k \in \mathbb{Z}^d} |u_k|^2 < +\infty\},$$

and define the linear operator $\mathcal{L}_{\delta,h}$ on $\ell^2$ as

$$\mathcal{L}_{\delta,h} \hat{u} = \left\{ \sum_{|m|_\infty \leq L} c_m \hat{u}_{k+m} \right\}_{k \in \mathbb{Z}^d}, \quad \forall \hat{u} = \{u_k\}_{k \in \mathbb{Z}^d} \in \ell^2.$$ 

One can verify that $\mathcal{L}_{\delta,h}$ is nonnegative and symmetric. Therefore, the spectrum set $\sigma(\mathcal{L}_{\delta,h})$ is located in the positive-half real axis.
For all \( s \notin \sigma(-L_{\delta,k}) \) and prescribed the boundary data \( \hat{u}_k \) with all \( k \in K_\gamma^+ \), the exterior problem (3.9) admits a unique solution. Accordingly, we expect that the values of \( u_k \) on \( K_\gamma^+ \) can be expressed by the values on \( K_\gamma^- \), i.e., there exists a matrix function with entries

\[
\mathcal{X}_{k,m} = \mathcal{X}_{k,m}(z), \quad k \in K_\gamma^+, \ m \in K_\gamma^-,
\]

such that

\[
\hat{u}_k = \sum_{m \in K_\gamma^-} \mathcal{X}_{k,m} \hat{u}_m, \quad k \in K_\gamma^+.
\]  

(3.10)

Applying the inverse \( z \)-transform to (3.10), one has the following DtD-type ABCs:

\[
u_k^{(n)} = \sum_{m \in K_\gamma^-} (\mathcal{X}_{k,m} * u_m)^{(n)} = \sum_{m \in K_\gamma^-} \left( \sum_{j=0}^{n} \mathcal{X}_{k,m}^{(n-j)} u_m^{(j)} \right), \quad k \in K_\gamma^+, \]

(3.11)

where

\[
\mathcal{X}_{k,m}^{(j)} = \frac{1}{2\pi i} \int_{C_\rho} \mathcal{X}_{k,m}(z) z^{-j} dz, \quad \rho > 1, \ j \geq 0.
\]  

(3.12)

In simulations, we utilize the trapezoidal rule to approximate the contour integral in (3.12), i.e.,

\[
\mathcal{X}_{k,m}(3.10) \approx \mathcal{X}_{k,m}^{(j)} = \frac{\rho^j}{P} \sum_{\rho=1}^{P} \mathcal{X}_{k,m}(\rho e^{2\pi i p/P}) e^{2\pi i j p/P}, \quad j \geq 0,
\]

(3.13)

where \( P \) is a positive integer. And for any \( \varepsilon > 0 \), one can take \( P \) large enough such that

\[
|\mathcal{X}^{(j)} - \mathcal{X}^{(j)}_{3.10}|_\infty \leq \varepsilon, \quad \forall j.
\]  

(3.14)

So far, we achieve a discrete initial-boundary-value problem with the DtD-type ABCs

\[
\mathcal{D}_t u_k^{(n)} + \mathcal{L}_{\delta,k} u_k^{(n)} = f_k^{(n)}, \quad k \in \mathbb{K}, \ n \geq 1,
\]

\[
\sum_{m \in \mathbb{K}^-} \sum_{j=0}^{n} \mathcal{X}_{k,m}^{(n-j)} u_m^{(j)}, \quad k \in \mathbb{K}^+, \ n \geq 1,
\]

(3.15)

\[
\begin{aligned}
\nu_k^{(0)} &= \Phi_k, \quad k \in \mathbb{K}, \\
\nu_k^{(1)} &= \Phi_k + \tau \psi_k + \frac{\tau^2}{2} \left( -\mathcal{L}_{\delta,k} \Phi_k + f_k^{(0)} \right), \quad k \in \mathbb{K}.
\end{aligned}
\]

In the following, we use the recently developed methods in Du et al. (2018a,b) to address how to achieve the formula of \( \mathcal{X}_{k,m} \) for the 1D and 2D cases, respectively.

**One-dimensional case.** For the 1D case, \( \mathbb{K} = (-M,M) \cap \mathbb{Z} \), then we divide \( \mathbb{K}^- \) into two index subsets \( \mathbb{K}^- = \{ k \in \mathbb{Z} : k \geq M \} \) and \( \mathbb{K}^- = \{ k \in \mathbb{Z} : k \leq -M \} \). Similarly, let \( \mathbb{K}_\gamma^{+} = \{ M, \ldots, M+L-1 \} \), \( \mathbb{K}_\gamma^+ = \{ -M-L+1, \ldots, -M \} \), then \( \mathbb{K}_\gamma^+ = \mathbb{K}_\gamma^+ \cup \mathbb{K}_\gamma^+ \).
We first consider the right exterior problem, i.e., the discrete problem (3.9) restricted to $K_{c^r}$. Let us introduce a family of vectors as

$$
\vec{U}_{M,q} = [\hat{u}_{M+(q-1)L}, \ldots, \hat{u}_{M+qL-1}]^T, \; q = 0, 1, \ldots
$$

Then, the discrete problem (3.9) restricted on $K_{c^r}$ can be rewritten as

$$
\begin{align*}
& s\vec{U}_{M,q} + A\vec{U}_{M,q-1} + B\vec{U}_{M,q} + A^T\vec{U}_{M,q+1} = 0, \quad q \geq 1, \\
& \lim_{q \to +\infty} \vec{U}_{M,q} = 0,
\end{align*}
$$

(3.16)

where the coefficient matrices $A$ and $B$ are given as

$$
A = \begin{pmatrix}
c_L & \cdots & c_2 & c_1 \\
c_L & \cdots & c_2 & \cdots \\
\vdots & \ddots & \vdots & \ddots \\
c_L & \cdots & \cdots & \cdots
\end{pmatrix},
B = \begin{pmatrix}
c_0 & c_1 & \cdots & c_{L-1} \\
c_1 & c_0 & c_1 & \cdots \\
\vdots & \cdots & \cdots & \cdots \\
c_{L-1} & \cdots & c_1 & c_0
\end{pmatrix}.
$$

(3.18)

Set $A_0 = -(s+B)^{-1}A$ and $B_0 = -(s+B)^{-1}A^T$. Eq. (3.16) can be further written as

$$
\vec{U}_{M,q} = A_0\vec{U}_{M,q-1} + B_0\vec{U}_{M,q+1}, \quad q \geq 1.
$$

(3.19)

Prescribed $\vec{U}_{M,0}$, from (3.19) with boundary condition (3.17), one can express $\vec{U}_{M,1}$ by $\vec{U}_{M,0}$ as

$$
\vec{U}_{M,1} = \vec{X}_r(s)\vec{U}_{M,0}.
$$

(3.20)

Specifically for $L = 1$, both $A_0$ and $B_0$ degenerate to scalars. So the mapping $\vec{X}_r(s)$ can be computed analytically as

$$
\vec{X}_r(s) = \frac{c_0 + s - \sqrt{2c_0s + s^2}}{c_0}.
$$

(3.21)

However, for the case of $L \geq 2$, it is nontrivial to find the exact expression of $\vec{X}_r(s)$. We use the iterative technique proposed in Du et al. (2018a) to numerically calculate $\vec{X}_r(s)$ to have

$$
\vec{X}_r(s) = A_0 + B_0[A_1 + B_1[\cdots + B_{m-1}[A_m + B_m[\cdots]]]],
$$

(3.22)

where $A_0$ and $B_0$ are given in (3.18), and $A_m, B_m \; (m \geq 1)$ are computed iteratively by

$$
(A_m \; B_m) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & -B_{m-1} & 0 \\ -A_{m-1} & I & -B_{m-1} \\ 0 & -A_{m-1} & I \end{pmatrix}^{-1} \begin{pmatrix} A_{m-1} & 0 \\ 0 & 0 \\ 0 & B_{m-1} \end{pmatrix}.
$$

(3.23)

By analogy with the design of DtD-type mapping on the right, one can derive a DtD-type mapping on the left as

$$
\vec{U}_{M,1} = \vec{X}_l(s)\vec{U}_{M,0},
$$

(3.24)

where

$$
\vec{U}_{M,q} = [\hat{u}_{M-(q-1)L}, \ldots, \hat{u}_{M-qL+1}]^T, \quad q = 0, 1.
$$
**Remark 3.1** By truncating the series terms in the formula (3.22), we obtain the approximation of the operator \( \hat{\mathcal{K}} \). The truncation criterion is to introduce a tolerance error, which is set as \( \varepsilon := 10^{-14} \), such that the \( L^2 \)-norms of \( A_m \) and \( B_m \) in (3.23) are less than the given tolerance \( \varepsilon \). This provides an efficient way of evaluating \( \hat{\mathcal{K}} \) in (3.22) for the problem considered in this paper. It turns out that the maximum number of the iteration to obtain the converged \( \hat{\mathcal{K}} \) for the given \( \varepsilon \) is less than 20 in all simulations.

**Two-dimensional case.** We utilize the methodology of the nonlocal potential theory to design the DtD-type ABCs for two-dimensional case (see Du et al., 2018b). Let \( G_k = G_k(z) \) be the fundamental solution of the equation (3.9) with \( s(z) \notin \sigma(-\mathcal{L}_{\delta,h}) \), this is, \( G_k \) satisfies the governing equation

\[
sg_k + \sum_{m,|m| \leq L} c_m G_{k+m} = \delta_{k,0}, \quad k \in \mathbb{Z}^2,
\]

(3.25)

and the boundary condition

\[
\lim_{|k| \to +\infty} G_k = 0,
\]

(3.26)

where \( \delta_{k,0} \) stands for the Kronecker symbol. The two-dimensional discrete Fourier transform (DFT) of \( \{G_k\}_{k \in \mathbb{Z}^2} \) is defined as

\[
(\mathcal{F}G)_\xi = \sum_{k \in \mathbb{Z}^2} G_k e^{-i \xi \cdot k}, \quad \xi \in \mathbb{R}^2.
\]

(3.27)

By performing the DFT to (3.25), one has

\[
(\mathcal{F}G)_\xi = \left( s + \sum_{m,|m| \leq L} e^{i m \cdot \xi} c_m \right)^{-1}, \quad \xi \in \mathbb{R}^2.
\]

(3.27)

Then using the inverse Fourier transform on \( \{(\mathcal{F}G)_\xi\}_{\xi \in \mathbb{R}^2} \) yields

\[
G_k = \frac{1}{4\pi^2} \int_{[0,2\pi]^2} \left( s + \sum_{m,|m| \leq L} e^{i m \cdot \xi} c_m \right)^{-1} e^{i k \cdot \xi} d\xi, \quad k \in \mathbb{Z}^2.
\]

Following the idea of the potential theory, we assume the solution of (3.9) can be expressed as

\[
\hat{u}_k = \sum_{m \in \mathbb{K}_\gamma} G_{k+m} q_m, \quad |k|_{\infty} > M - L,
\]

(3.28)

where \( q_m \) is the potential to be determined. Confining (3.28) to the boundary layer \( \mathbb{K}_\gamma \) produces

\[
\hat{u}_k = \sum_{m \in \mathbb{K}_\gamma} G_{k+m} q_m, \quad k \in \mathbb{K}_\gamma.
\]

(3.29)

Denote \( (G_{k,m}^{-1}) \) by the inverse matrix of the matrix \( G_{k+m} \) with \( k,m \in \mathbb{K}_\gamma \). Thus, the potential \( q_m \) can be expressed by the fundamental solution \( G_k \) and the value of \( \hat{u}_k \) on \( k \in \mathbb{K}_\gamma \) in the form of

\[
q_m = \sum_{l \in \mathbb{K}_\gamma} G_{m,l}^{-1} \hat{u}_l, \quad m \in \mathbb{K}_\gamma.
\]
Substituting (3.29) into (3.28) and restricting to the boundary \( \mathcal{K}_+ \) yields

\[
\hat{u}_k = \sum_{m \in \mathbb{K}_+} \mathcal{G}_{k,m} u_m, \quad k \in \mathbb{K}_+^+,
\]

with

\[
\mathcal{G}_{k,m} = \sum_{l \in \mathbb{K}_+} G_{k,l} G_{l,m}^{-1}, \quad k \in \mathbb{K}_+^+, \quad m \in \mathbb{K}_+^-.
\]

We point out that our procedure of adopting the potential theory to construct ABCs for the two-dimensional discrete system is similar to the difference potential method (DPM) proposed by Ryaben’kii et al. (see, Ryaben’kii & Tsynkov, 2006; Tsynkov, 1996, and references therein). The DPM is also based on the potential theory, and concretely, which needs to formulate an appropriate auxiliary problem (to simplify the numerical implementation), and then construct the boundary equation with projection. The DPM has been successfully applied to design ABCs at irregular artificial boundaries for local problems.

3.2 DtN-type absorbing boundary conditions

In order to construct the DtN-type ABC based on the DtD-type mapping (3.11), we now introduce the formula of nonlocal Neumann boundary. By analogy with the classical Green’s first identity

\[
(-\Delta u, v)_{\Omega} = (\nabla u, \nabla v)_{\Omega} - (\partial_n u, v)_{\partial \Omega},
\]

the nonlocal Green’s first identity is given as:

\[
(\mathcal{L}_\delta u, v) = \int_{x \in \Omega} \int_{y \in \mathbb{R}^d} (u(x) - u(y)) v(x) \gamma(|x - y|) dy dx
\]

\[
= \frac{1}{2} \int_{x \in \Omega} \int_{y \in \Omega} (u(x) - u(y)) (v(x) - v(y)) \gamma(|x - y|) dy dx
\]

\[
+ \int_{x \in \Omega} \int_{y \in \partial \Omega} (u(x) - u(y)) v(x) \gamma(|x - y|) dy dx.
\]

From (3.31), we have the nonlocal Neumann boundary (see the details in Du et al. (2012, 2019b))

\[
\mathcal{N} u(x) = -\int_{y \in \Omega_\delta} (u(x) - u(y)) \gamma(|x - y|) dy, \quad x \in \Omega_\delta^-,
\]

where \( \Omega_\delta^- = \{ x \in \Omega : \text{dist}(x, \partial \Omega) \leq \delta \} \) and the finite horizon property (1.4) is used to truncate the interaction domain. To obtain the formula of discrete Neumann boundary, we perform the discrete nonlocal Green’s first identity as

\[
(\mathcal{L}_\delta, h u, v)_h = h^d \sum_{k \in \mathbb{K}} \mathcal{L}_\delta, h u_k \cdot v_k
\]

\[
= h^d \sum_{k \in \mathbb{K}} \sum_{m \in \mathbb{K}} a_{k,m} (u_k - u_m) v_k + h^d \sum_{k \in \mathbb{K}} \sum_{m \in \mathbb{K}} a_{k,m} (u_k - u_m) v_k
\]

\[
= \frac{1}{2} h^d \sum_{k \in \mathbb{K}} \sum_{m \in \mathbb{K}} a_{k,m} (u_k - u_m) (v_k - v_m) + h^d \sum_{k \in \mathbb{K}} \sum_{m \in \mathbb{K}} a_{k,m} (u_k - u_m) v_k
\]

\[
= (u, v)_h - (\mathcal{N} u, v)_{\mathbb{K}_+}.
\]
In the above, the symbol $\langle \cdot, \cdot \rangle_K$ and $\langle \cdot, \cdot \rangle_{K^c}$ are defined in (3.5) and (3.4), respectively. And the discrete nonlocal Neumann boundary, denoted by $\mathcal{N}_K u$, is formulated as
\begin{equation}
\mathcal{N}_K u_k = -h^d \sum_{m \in K^c} a_{k-m}(u_k - u_m), \quad k \in K^-.
\end{equation}

Thus, we can reformulate the DtD-type mapping (3.11) into the following DtN-type mapping
\begin{equation}
\mathcal{N}_K u_k^{(n)} = -h^d \sum_{m \in K^c} a_{k-m} \left( u_k^{(n)} - \sum_{l \in K^-} \mathcal{N}_K u_l^{(n)} \right),
\end{equation}
\begin{equation}
\quad : = \gamma_k^{(n)}(u_k^{(n)}), \quad k \in K^-.
\end{equation}

Finally, we obtain a numerical scheme with the DtN-type ABCs for the nonlocal problem (1.1)
\begin{equation}
\mathcal{D}_\tau u_k^{(n)} + \mathcal{L}_{\delta,h} u_k^{(n)} = f_k^{(n)}, \quad k \in K, n \geq 1,
\end{equation}
\begin{equation}
\mathcal{N}_K u_k^{(n)} = \gamma_k^{(n)}(u_k^{(n)}), \quad k \in K^c, n \geq 1,
\end{equation}
\begin{equation}
\begin{aligned}
u_k^{(0)} & = \phi_k, \\
\quad u_k^{(1)} & = \phi_k + \tau \psi_k + \frac{\tau^2}{2} \left( -\mathcal{L}_{\delta,h} \phi_k + f_k^{(0)} \right), \quad k \in K.
\end{aligned}
\end{equation}

**Remark 3.2.** We can also derive the Neumann boundary (3.34) through considering the discrete nonlocal Green’s first identity on the exterior domain $K^c$,
\begin{equation}
\langle \mathcal{L}_{\delta,h} u, v \rangle_h_{K^c} = \frac{h^d}{2} \sum_{k \in K} \sum_{m \in K^c} a_{k-m}(u_k - u_m)(v_k - v_m) + h^d \sum_{k \in K^c} \sum_{m \in K^-} a_{k-m}(u_k - u_m)v_k
\end{equation}
\begin{equation}
\quad = \langle u, v \rangle_{h_{K^c}} + \frac{h^d}{2} \sum_{k \in K^c} \sum_{m \in K^-} a_{k-m}(u_k - u_m)(v_k - v_m) + \langle \mathcal{N}_K u, v \rangle_{K^-}.
\end{equation}

This formula serves to bridge the interior and exterior problems, which will be used in stability analysis of the numerical scheme (3.36) in next section.

**4. Stability and convergence analysis**

We now consider the stability of the following discrete system
\begin{equation}
\mathcal{D}_\tau \phi_k^{(n)} + \mathcal{L}_{\delta,h} \phi_k^{(n)} = g_k^{(n)}, \quad k \in K, n \geq 1,
\end{equation}
\begin{equation}
\mathcal{N}_K \phi_k^{(n)} = \gamma_k^{(n)}(\phi_k^{(n)}) + g_b^{(n)}, \quad k \in K^c, n \geq 1,
\end{equation}
\begin{equation}
\begin{aligned}
\phi_k^{(0)} & = \mu_k^{(0)}, \\
\quad \phi_k^{(1)} & = \mu_k^{(1)}, \quad k \in K,
\end{aligned}
\end{equation}
where $\mu = \{\mu_k\}_{k \in K}$ are the initial values, $g = \{g_k\}_{k \in K}$ and $g_b = \{g_b,k\}_{k \in K^c}$ are the interior and boundary perturbation terms, respectively.

Define the discrete energy norm
\begin{equation}
\| \phi^{(n)} \|_E^2 = \| D^F \phi^{(n-1)} \|_h^2 + \frac{1}{4} \| \phi^{(n)} + \phi^{(n-1)} \|_h^2, \quad n \geq 1,
\end{equation}
where the forward difference operator $D^F$ is given as $D^F u^{(n)} = \frac{1}{\tau} (u^{(n+1)} - u^{(n)})$. 
4.1 Stability analysis

THEOREM 4.1 Take \( S = 2((2L + 1)^d - 1) |a|_\infty \), where \( a \) is the coefficient of the discrete nonlocal operator defined in (2.3). When \( a \geq 0 \), there exist positive constants \( C \) and \( \tau_0 \) such that for \( \tau \leq \min\{ \tau_0, 2/\sqrt{S} \} \), the solution of (4.1)-(4.3) satisfies the following stability estimate for \( l \geq 2 \):

\[
\| \phi^{(l)} \|_E^2 \leq \| D_\tau^F \mu^{(0)} \|_{\hat{h}}^2 + C \tau \sum_{n=1}^{l-1} \left( \| g^{(n)} \|_{\hat{h}}^2 + h^{-d} \| g_b^{(n)} \|_{\hat{h}}^2 \right). \tag{4.5}
\]

Proof. Taking the \( L^2 \)-inner product between (4.1) and \( (\phi^{(n+1)} - \phi^{(n-1)}) \) on \( \mathbb{K} \) yields

\[
(D_\tau \phi^{(n)}, \phi^{(n+1)} - \phi^{(n-1)})_h + (L_{\delta,h} \phi^{(n)}, \phi^{(n+1)} - \phi^{(n-1)})_h = (g^{(n)}, \phi^{(n+1)} - \phi^{(n-1)})_h. \tag{4.6}
\]

The first term in the above equation can be written as

\[
(D_\tau \phi^{(n)}, \phi^{(n+1)} - \phi^{(n-1)})_h = (D_\tau^F \phi^{(n)} - D_\tau^F \phi^{(n-1)}, D_\tau^F \phi^{(n)} + D_\tau^F \phi^{(n-1)})_h = \| D_\tau^F \phi^{(n)} \|_{\hat{h}}^2 - \| D_\tau^F \phi^{(n-1)} \|_{\hat{h}}^2.
\]

Applying the discrete nonlocal Green’s first identity (3.33) to the second term of (4.6), one has

\[
(L_{\delta,h} \phi^{(n)}, \phi^{(n+1)} - \phi^{(n-1)})_h = (\phi^{(n)}, \phi^{(n+1)} - \phi^{(n-1)})_h - (\phi^{(n)} \phi^{(n+1)} - \phi^{(n)} \phi^{(n-1)})_K. \\
= \frac{1}{4} \left( |\phi^{(n+1)}|^2 - |\phi^{(n)}|^2 + |\phi^{(n-1)}|^2 - |\phi^{(n)}}{1} |^2 - |\phi^{(n)} - \phi^{(n-1)}|^2 \right)_K - \left( g^{(n)}, \phi^{(n+1)} - \phi^{(n-1)} \right)_K.
\]

where the fact is used in the last equality that

\[
a(b - c) = \frac{1}{4} ((a + b)^2 - (a + c)^2 - (a - b)^2 + (a - c)^2).
\]

Summing index \( n \) from 1 to \( l - 1 \) in (4.6) and combining with initial conditions (4.3), one obtains

\[
|| D_\tau^F \phi^{(l-1)} ||_{\hat{h}}^2 - || D_\tau^F \mu^{(0)} ||_{\hat{h}}^2 + \frac{1}{4} \left( |\phi^{(l)} + \phi^{(l-1)}|^2 - |\phi^{(l)} - \phi^{(l-1)}|^2 \right)_h - \sum_{n=1}^{l-1} \left( g^{(n)}, \phi^{(n+1)} - \phi^{(n-1)} \right)_K \\
= \sum_{n=1}^{l-1} \left( g^{(n)}, \phi^{(n+1)} - \phi^{(n-1)} \right)_h + \sum_{n=1}^{l-1} \left( g_b^{(n)}, \phi^{(n+1)} - \phi^{(n-1)} \right)_K. \tag{4.7}
\]

To estimate the second term associated with the boundary on the left side of the above equation, we consider the following exterior problem

\[
D_\tau^F \tilde{\phi}^{(n)} + L_{\delta,h} \tilde{\phi}^{(n)} = 0, \quad \tilde{\phi}^{(n)} |_{K^c} = \phi^{(n)} |_{K^c}, \tag{4.8}
\]

\[
\tilde{\phi}^{(n)} |_{K^c} = \phi^{(n)} |_{K^c}, \tag{4.9}
\]

\[
\tilde{\phi}^{(0)} |_{K^c} = \phi^{(1)} |_{K^c} = 0, \quad \tilde{\phi}^{(0)} |_{K^c} = \phi^{(1)} |_{K^c}. \tag{4.10}
\]
Taking the $L^2$-inner product between (4.8) and $\phi^{(n+1)} - \phi^{(n-1)}$ on the domain $K^c$, one has

$$(\mathcal{D}_k \phi^{(n)}), \phi^{(n+1)} - \phi^{(n-1)} + (\mathcal{D}_k \phi^{(n)}), \phi^{(n+1)} - \phi^{(n-1)} = 0.$$  

Summing index $n$ from 1 to $l - 1$ and combining with initial conditions and (3.37), one has

$$\sum_{n=1}^{l-1}(\phi^{(n)}), \phi^{(n+1)} - \phi^{(n-1)} = 0.$$  

Substituting the left-hand side of (4.11) into (4.7), one obtains

$$\sum_{k \in K^c} \sum_{m \in \mathbb{R}} \alpha_{k-m} \left( (\phi_k^{(l)} - \phi^m_{m+1})^2 + (\phi_k^{(l)} - \phi^m_{m+1})^2 \right) = - \sum_{n=1}^{l-1}(\phi^{(n)}), \phi^{(n+1)} - \phi^{(n-1)}.$$  

Substituting the left-hand side of (4.11) into (4.7), one obtains

$$\sum_{n=1}^{l-1}(g^{(n)}), \phi^{(n+1)} - \phi^{(n-1)} + \sum_{n=1}^{l-1}(g^{(n)}), \phi^{(n+1)} - \phi^{(n-1)} = 0.$$  

Note that

$$|\phi^{(l)} - \phi^{(l-1)}|^2 \leq h^d \sum_{k \in K^c} \sum_{m \in \mathbb{R}} \alpha_{k-m}((\phi_k^{(l)} - \phi^m_{m-1})^2 + (\phi_k^{(l)} - \phi^m_{m+1})^2)$$

$$\leq h^d \sum_{k \in K^c} \sum_{m \in \mathbb{R}} \alpha_{k-m}((\phi_k^{(l)} - \phi^m_{m-1})^2 + (\phi_k^{(l)} - \phi^m_{m+1})^2)$$

$$\leq S h^d \sum_{k \in K^c} (\phi_k^{(l)} - \phi_k^{(l-1)})^2$$

$$= S \tau^2 \|\mathcal{D}_k \phi^{(l-1)}\|_{h}^2.$$  

where $S = 2((2L+1)d - 1)|a|_{\infty}$ and the property of $a \geq 0$ is used. Similarly, one also has

$$|\phi^{(l)} - \phi^{(l-1)}|^2 \leq S \tau^2 \|\mathcal{D}_k \phi^{(l-1)}\|_{h}^2.$$  

and

$$h^d \sum_{k \in K^c} \sum_{m \in \mathbb{R}} \alpha_{k-m}((\phi_k^{(l)} - \phi^m_{m-1})^2 + (\phi_k^{(l)} - \phi^m_{m+1})^2)$$

$$\leq S \tau^2 (\|\mathcal{D}_k \phi^{(l-1)}\|_{h}^2 + \|\mathcal{D}_k \phi^{(l-1)}\|_{h}^2).$$  

(4.15)
Plugging (4.13), (4.14) and (4.15) into (4.12), one yields
\[
(1 - S\tau^2/4)\|D_t^F \phi^{(l-1)}\|_h^2 + \frac{1}{4}\|\phi^{(l-1)}\|_h^2 + (1 - S\tau^2/4)\|D_t^F \phi^{(l-1)}\|_{\mathbb{H}^\infty}
+ \frac{1}{4}\|\phi^{(l)}\|_h^2 + \frac{j^d}{4} \sum_{k \in \mathbb{K}_i^l} \sum_{m \in \mathbb{K}_j^l} a_{k-m} \left( (\tilde{\phi}_k^{(l)} - \tilde{\phi}_m^{(l)}) + (\tilde{\phi}_k^{(l-1)} - \tilde{\phi}_m^{(l-1)}) \right)^2
\leq \|D_t^F \phi^{(l)}\|_h^2 + \sum_{n=1}^{l-1} \left( g^{(n)}, \phi^{(n+1)} - \phi^{(n-1)} \right)_h + \sum_{n=1}^{l-1} \left( g^{(n)}_b, \phi^{(n+1)} - \phi^{(n-1)} \right)_{\mathbb{K}_i^l}.
\]

If \((1 - S\tau^2/4) > 0\), then there exists a positive constant \(C\), s.t.,
\[
\|\phi^{(l)}\|_E^2 \leq C \left( \|D_t^F \phi^{(l)}\|_h^2 + \sum_{n=1}^{l-1} \left( g^{(n)}, \phi^{(n+1)} - \phi^{(n-1)} \right)_h \right)
+ C\tau \sum_{n=1}^{l-1} \left( g^{(n)}_b, D_t^F \phi^{(n+1)} + D_t^F \phi^{(n-1)} \right)_{\mathbb{K}_i^l}
\leq C \|D_t^F \phi^{(l)}\|_h^2 + C\tau \sum_{n=1}^{l-1} \|g^{(n)}\|_h^2 + C\tau h^{-d} \sum_{n=1}^{l-1} \|g^{(n)}_b\|_h^2
+ C\tau \sum_{n=0}^{l-1} \|D_t^F \phi^{(n)}\|_h^2.
\]

Applying the discrete Gronwall’s inequality (see Quarteroni & Valli, 1994) for positive constant \(\tau_0\) such that \(\tau \leq \tau_0\), one obtains (4.5).

\[
\Box
\]

4.2 Convergence analysis

We now analyze the error of numerical scheme (3.36) based on the above stability analysis. Let \(u^{(n)}_s = \{u(x_k, t_n)\}_{k \in \mathbb{K}}\) be the vector whose entries are the nodal values of exact solutions of problem (1.1) at the time \(t_n\), and \(u^{(n)} = \{u^{(n)}_k\}_{k \in \mathbb{K}}\) whose entries are the nodal values of solutions of the numerical scheme (3.36). Denote the error by \(\tilde{u}^{(n)} = u^{(n)} - u^{(n)}_s\). To perform the error estimate, we further introduce \(\bar{u}^{(n)} = \{\bar{u}^{(n)}_k\}_{k \in \mathbb{K}}\) whose entries are the nodal values of numerical solutions of scheme (3.36) with replacing the approximate \(\bar{\mathcal{H}}\) by the exact \(\mathcal{H}\). Then the error \(\tilde{u}^{(n)}\) can be divided into two parts, i.e.,
\[
\tilde{u}^{(n)} = (u_s^{(n)} - \bar{u}^{(n)}) + (\bar{u}^{(n)} - u^{(n)}) := \tilde{u}_{1}^{(n)} + \tilde{u}_{2}^{(n)}.
\]

We now consider these two errors separately. Note that the solution \(\bar{u}^{(n)}\) is the same as the solution of fully discrete system (2.9) confined on the computational domain since the discrete ABCs are exact.
According to the stability analysis in Theorem 4.1, we have
\[ \|\phi^{1,(n)}\|_E \leq C_1(\tau^2 + h^q), \quad 2 \leq n \leq N, \]  
(4.16)
where \( C_1 \) is a positive constant, \( q \) is given in Lemma 2.1.

On the other hand, one can verify \( \phi^{2,(n)} \) satisfies Eqs. (4.1)-(4.3) with
\[ g^{(n)} = 0, \]
and
\[ g_{b,k}^{(n)} = h^d \sum_{m \in \mathbb{Z}_+^d} \alpha_{k-m} \sum_{l \in \mathbb{Z}_+^d} (\mathcal{X}_{m,l}^{(n)} - \mathcal{X}_{m,l}^{(n)}) * u_{l}^{(n)}, \quad k \in \mathbb{Z}_+^d. \]  
(4.17)
According to the stability analysis in Theorem 4.1, we have
\[ \|\phi^{(l),2}\|_E \leq \left( C\tau \sum_{n=1}^{l-1} h^{-d}\|g_b^{(n)}\| \right)^{\frac{1}{2}}, \]  
(4.18)
where \( \|g_b^{(n)}\| \) can be further estimated from (4.17) and (3.14) by
\[ \|g_b^{(n)}\| \leq nh^d L^{2d+\frac{1}{2}}|a|_\infty |\mathcal{X} - \mathcal{X}|_\infty |u|_{[0,T_h]} \leq Cnh^d L^{2d+\frac{1}{2}} \varepsilon |a|_\infty. \]  
(4.19)
Since the nonlocal horizon \( \delta \) is fixed, we substitute \( L = \delta/h \) into (4.19) to have
\[ \|g_b^{(n)}\| \leq Cn\delta^{2d+\frac{1}{2}} h^{-d-\frac{1}{2}} \varepsilon |a|_\infty. \]  
(4.20)
The maximum norm of \( a \) depends on the kernels used in the nonlocal operator \( \mathcal{L}_a. \) We here list three popularly and widely used kernel functions as

- constant kernel: \( \gamma(a) = \frac{3}{d} \delta^{-2-d}, \quad |a|_\infty \in [0, \delta]; \)  
  (4.21)
- nonintegrable kernel: \( \gamma(a) = 2|a|^{-1} \delta^{-2}, \quad |a|_\infty \in (0, \delta]; \)  
  (4.22)
- fractional Laplacian kernel: \( \gamma(a) = \frac{2^{2v} \Gamma(v + d/2)}{\pi^{d/2} \Gamma(1 - v)} \|a\|^{-d-2v} (0 < v < 1), \quad |a|_\infty \in (0, \delta]. \)  
  (4.23)
The scheme (2.3) with any \( p \) leads to \( |a|_\infty = \mathcal{O}(h^d) \) for constant kernel (4.21). Similarly, one has \( |a|_\infty = \mathcal{O}(1) \) for kernel (4.22) and \( |a|_\infty = \mathcal{O}(h^{-2v}) \) for kernel (4.23). Without loss of generality, we assume for convenience \( |a|_\infty = \mathcal{O}(h^{-2v}) \), where the index \( r \) is determined by the kernel and the dimension of space.

To ensure that \( \|\phi^{(l),2}\|_E \) has the second-order accuracy in time, one can take \( P \) large enough in (3.13) such that \( \varepsilon = \mathcal{O}(\tau^{2} h^{rac{4d}{3}+1+r}). \) Then the total error \( \phi^{(n)} \) has the following estimate
\[ \|\phi^{(n)}\|_E \leq \|\phi^{1,(n)}\|_E + \|\phi^{2,(n)}\|_E \leq C_1(\tau^2 + h^p) + C_2 \tau^2. \]  
(4.24)
Overall, we obtain the following error estimate of numerical scheme (3.36).
THEOREM 4.2 Assume that the solution of the nonlocal wave equation (1.1) is sufficiently smooth. If \( \tau \leq \min \{ \tau_0, 2/\sqrt{3} \} \) (\( S \) is defined in Theorem 4.1) and taking \( P \) large enough in (3.13) such that \( \varepsilon = O(\tau^3 h^{2+\frac{1}{2}+r}) \), then the following estimate holds

\[
\max_{2 \leq n \leq N} \| \phi^{(n)} \|_E \leq C(\tau^2 + h^q),
\]

where \( q \) depends on the accuracy of the spatially discrete scheme.

REMARK 4.1 For fixed horizon parameter \( \delta \), we now present a fine estimate on the time step size restriction given in Theorems 4.1 and 4.2, i.e., \( \tau \leq \min \{ \tau_0, \frac{2}{\sqrt{3}} \} \). Substituting \( |\alpha|_w = O(h^{-r}) \) and \( L = \delta / h \) into \( S \), one has \( \tau \leq h^d/2^{2+r} \), which implies the time step restriction for different kernels as

\[
\tau \leq \begin{cases} 
    h^d, & \text{kernel (4.21)}, \\
    h^{d/2}, & \text{kernel (4.22)}, \\
    h^{d/2+v}, & \text{kernel (4.23)}.
\end{cases}
\]

5. Numerical Experiments

We now provide two numerical examples to demonstrate the effectiveness of our ABCs and the theoretical analysis. Let \( u_{ref} \) and \( u_h \) be the solutions of problem (1.1) and numerical scheme (3.36), respectively. The \( L^2 \)-error and convergence rate are defined as

\[
L^2\text{-error}(h) = \| u_h - u_{ref} \|_h, \quad \text{(5.1)}
\]

\[
L^2\text{-rate} = \log \left( \frac{L^2\text{-error}(h_1)}{L^2\text{-error}(h_2)} \right) / \log \left( \frac{h_1}{h_2} \right). \quad \text{(5.2)}
\]

EXAMPLE 5.1 We here consider 1D problem (1.1) with \( f(x,t) = 0 \). The initial values are given as

\[
\phi(x) = \exp(-25(x-0.2)^2) + \exp(-25(x+0.2)^2),
\]

\[
\psi(x) = 50x\exp(-25x^2).
\]

We consider all three kernel functions (4.21)-(4.23) listed in section 4. For the convenience of exposition, we denote them by kernel-1, kernel-2 and kernel-3. And we choose \( v = 0.5 \) in kernel-3. In simulations, we set the computational domain \( \Omega = (-2, 2) \), the spatial mesh size \( h = 2^{-7} \), the time step size \( \tau = 2^{-8} \) and the number of quadrature nodes given in (3.13) \( P = 20000 \). And the final time are \( T = 3.5, 10 \) for three kernels, respectively. Figure 1 plots the evolutions of numerical solutions with the linear Lagrange interpolation when \( \delta = 0.25, 0.5 \). One can see that the waves are effectively absorbed when they touch the boundaries, and no reflected wave is generated at boundaries to disrupt the solutions in the computational domain.

To investigate the spatial convergence orders of various approximations such as linear, quadratic and cubic Lagrange interpolations, we set \( \delta = 1/8, T = 2 \) and fix \( \tau = 10^{-5} \), \( P = 80000 \). The \( L^2 \)-errors and convergence rates are shown in Figure 2 by taking \( h = [2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}] \) for linear and quadratic cases, and \( h = [1/24, 1/48, 1/72, 1/96] \) for cubic case. Here the “exact” solutions are computed by pseudo-spectral method over a domain large enough as reference solutions. One can observe that linear interpolation scheme has the second-order convergence rate by comparing it with the second-order slope for all three kernels. And quadratic, cubic Lagrange interpolations have the forth-order convergence rate, expect in a special case where the quadratic interpolation scheme is used to solve the problem (1.1) with
FIG. 1. (Example 5.1:) Evolution of numerical solutions.

the kernel-3. This is caused by the singularity of the kernel-3. We remark that the used time steps in all simulations satisfy the restriction given in (4.26), but this restriction is not sharp, which can be relaxed in the future.

EXAMPLE 5.2 In this example, we consider the two-dimensional problem (1.1) with $f(x,t) = 0$ and the
initial values given as
\[
\varphi(x) = \exp(-25(x - 0.2)^2) + \exp(-25(x + 0.2)^2), \\
\psi(x) = 0.
\]

We choose the constant kernel function (4.21) for \( d = 2 \) and the Gaussian kernel
\[
\gamma(\alpha) = 50\exp(-5\|\alpha\|^2), \quad |\alpha|_{\infty} \leq \delta.
\]

In the simulations, we take the computational domain \( \Omega = (-1, 1)^2 \), \( \delta = 0.5 \), \( h = 2^{-7} \), \( \tau = 10^{-3} \), and \( P = 5000 \). Figure 3 shows the isolines of numerical solutions of scheme (3.36) with the bilinear interpolation at times \( T = 0.1, 0.5, 1 \), respectively. There is no obvious reflection caused by the boundary conditions for both two kernels. To show the error of the numerical solutions, we use the same strategy as that in Example 5.1 to compute the reference solutions. Figure 4 shows the second-order and fourth-order convergence order in \( L^2 \)-error by refining \( h = [1/4, 1/8, 1/12, 1/16, 1/20], \tau = [1/16, 1/24, 1/32, 1/40, 1/48] \) and \( \tau = [1/16, 1/36, 1/64, 1/100, 1/144] \) for linear and quadratic interpolation cases, respectively, and taking the number of quadrature nodes as \( P = [500, 1000, 2000, 4000, 5000] \). The convergence orders are consistent with the theoretical analysis.

6. Conclusion

In this paper we considered the sharp error estimate of arbitrarily high-order schemes in space for multidimensional nonlocal wave equations on unbounded domains. To this end, we first approximated the nonlocal operator with arbitrarily high-order quadrature-based difference schemes, and discretized the time direction with the explicit difference scheme to have a fully discrete infinity system. After that,
we used the methodology in Du et al. (2018a,b) to achieve the DtD-type ABCs for the resulting infinity system, and further presented the formula of nonlocal Neumann data based on the discrete nonlocal Green’s first identity, and finally obtained the DtN-type ABCs. The DtN-type ABCs are available to reduce the infinite system to a finite discrete system, whose solution is equivalent to that of the infinite
system confined on the bounded computational domain. On the other hand, the DtN-type ABCs are also available to present the stability analysis for the reduced finite discrete system. In the practical simulation, the convolution kernel in time arose from the inverse $z$-transform can be approximated with high-order accuracy, i.e., the resulting error can be small enough such that it does not bring the loss of the optimal convergence order. Finally, the efficiency and accuracy of our proposed approach were verified by numerical examples. And we point out that the proposed method above can be extended to solve the classical local wave problems on unbounded domains with arbitrarily high-order schemes in spatial direction.

It is well-known that the direct evaluation of the convolution kernel in (3.11) is quite expensive. For the local problems, there are many works on the fast evaluation of ABCs (see, e.g., Zheng, 2007; Jiang & Greengard, 2004; Arnold et al., 2003; Li & Greengard, 2007; Sun et al., 2020). While the operator $\mathcal{K}$ in nonlocal models is more complicated than it in local models, it is difficult to achieve a fast algorithm to the inverse $z$-transform. Recently, Zheng et al. (2020) have developed a fast algorithm by utilizing the discretized contour integrals developed in L´opez-Fernández et al. (2005) for solving the nonlocal heat equation on unbounded domains, but the technique is nontrivial for the wave problem. Thus, further efforts are required to address the fast evaluation of ABCs for nonlocal wave problems.

Additionally, in this work, we have achieved high-order accuracy in space, but only have the second-order accuracy in time. It is natural to ask whether the high-order accuracy in time can be achieved. Fortunately, for the high-order scheme obtained by the modified equation technique (see, e.g., Shubin & Bell, 1987), which is usually adopted to deal with the wave equations, the method of deriving ABCs in this paper seems to be applicable. However, how to analyze the stability of the scheme requires more detailed discussions. In future work, we will extend our method to high-order schemes in time.

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Appendix

The proof of Lemma 2.1. First we consider the case of one-dimension. We review the domain division given in section 2

\[ T_i^k = [x_k + ((i - 1)p - L)h, x_k + (ip - L)h], \quad i = 1, 2, \ldots, 2L/p, \]

then $B_δ(x_k) = \bigcup T_i^k$. The interpolation points in every subdomain $T_i^k$ are given as

\[ s_{i,j} = x_k + ((i - 1)p - L + j)h, \quad j = 0, 1, \ldots, p. \]

For integral

\[ I(f) = \int_{B_δ(0)} f(s)w(s)γ(s)ds, \]
we consider the numerical integration for \( I(f) \)

\[
I_{h,p}(f) = \sum_i \int_{T_i^0} \mathcal{I}_{i,p}(f)(s) w(s) \gamma(s) ds,
\]

(6.1)

where \( \mathcal{I}_{i,p} \) represents the \( p \)th-degree Lagrange interpolation operator on \( T_i^0 \). For simplicity, we denote \( T_i = T_i^0 \). According to the interpolation error of the Lagrange interpolation formula, one has

\[
\mathcal{R}[f] = I(f) - I_{h,p}(f) = \sum_i \int_{T_i} (f(s) - \mathcal{I}_{i,p}(f)(s)) w(s) \gamma(s) ds
\]

(6.2)

where \( \xi_i \in T_i \). Obviously, \( \mathcal{R}[f] = 0 \) for polynomials with degree less than or equal to \( p \). Moreover, when \( p \) is even, numerical integration (6.1) is also accurate for polynomials with degree of \( p + 1 \). Considering \( f(s) = s^{p+1} \), one has

\[
\mathcal{R}[f] = \sum_i \int_{T_i} \prod_{j=0}^{p} (s - s_{i,j}) w(s) \gamma(s) ds.
\]

The above error is zero since the integral domain is symmetric about the origin and the integrand is an odd function.

Based on the symmetry of the kernel, the nonlocal operator (2.1) can be rewritten as

\[
\mathcal{L}_\delta u(x_k) = \frac{1}{2} \int_{B_\delta(0)} \frac{2u(x_k) - u(x_k + s) - u(x_k - s)}{w(s)} w(s) \gamma(s) ds.
\]

Denote

\[
G := G(s;x_k) = \frac{2u(x_k) - u(x_k + s) - u(x_k - s)}{w(s)},
\]

then the numerical scheme (2.2) is

\[
\mathcal{L}_{\delta,h} u(x_k) = \sum_i \int_{T_i} \mathcal{I}_{i,p}[G](s;x_k) w(s) \gamma(s) ds.
\]

(6.3)

When \( p \) is odd, we construct the auxiliary polynomial with degree of \( p \)

\[
H(s;x_k) = -2 \sum_{m=1}^{(p+1)/2} \frac{s^{2m}u^{(2m)}(x_k)}{(2m)!w(s)}.
\]

Let

\[
J(s;x_k) = G(s;x_k) - H(s;x_k).
\]

According to the Taylor’s expansion, one yields

\[
J(s;x_k) = -2 \frac{w(s)}{w(s)} \int_0^s \frac{(s-t)^{p+2}}{(p+2)!} u^{(p+3)}(x_k + t) dt.
\]
Further, we calculate the $p$th-order derivative of $J(s; x_k)$ to have

$$|J^{(p+1)}(s; x_k)| \leq C(p)|u|_\infty |s|.$$  \hfill (6.4)

The truncation error of the approximation (2.2) is given as

$$\begin{align*}
|\mathcal{L}_b u(x_k) - \mathcal{L}_{\delta_b} u(x_k)| &= \left|\frac{1}{2} \sum_t \int_{T_t} (G - \mathcal{A}_{i,p}[G]) w(s) \gamma(s) ds\right| \\
&= \left|\frac{1}{2} \sum_t \int_{T_t} \left((\mathcal{A}_{i,p}[H] - H) - (\mathcal{A}_{i,p}[J] - J)\right) w(s) \gamma(s) ds\right| \\
&\leq \frac{1}{2} \sum_t \int_{T_t} |\mathcal{A}_{i,p}[H] - H| w(s) \gamma(s) ds + \frac{1}{2} \sum_t \int_{T_t} |\mathcal{A}_{i,p}[J] - J| w(s) \gamma(s) ds \\
&=: E_1 + E_2, \hfill (6.5)
\end{align*}$$

where $E_1 = 0$ since $H$ is a polynomial of degree $p$. Next we estimate $E_2$

$$\begin{align*}
E_2 &= \frac{1}{2} \sum_t \int_{T_t} \left|\frac{J^{(p+1)}(\xi; x_k)}{(p+1)!} \prod_{j=0}^{p} (s - s_{i,j})\right| w(s) \gamma(s) ds \\
&\leq C(p)|u|^{(p+3)}|_{\infty} \sum_t \int_{T_t} \left|\xi_i\prod_{j=0}^{p} (s - s_{i,j})\right| w(s) \gamma(s) ds \\
&\leq C(p)\delta|u|^{(p+3)}|_{\infty} h^{p+1} \int_{B_\delta(0)} w(s) \gamma(s) ds. \hfill (6.6)
\end{align*}$$

Then if $u \in C^{p+3}_b(\mathbb{R}^d)$ and $w \gamma$ is integral on domain $B_\delta(0)$, the approximation error of (6.3) is $O(h^{p+1})$ and the estimate constant $C$ is independent of $h$.

When $p$ is even, the numerical error of $O(h^{p+2})$ can be achieved based on the fact that the numerical integration (6.1) has the $(p+1)$th-degree of exactness. We construct the $(p+1)$th-degree polynomial $H_i(s; x_k)$ on $T_t$, which satisfies

$$\begin{align*}
H_i(s_j; x_k) &= G(s_{i,j}; x_k), \quad j = 0, 1, \cdots, p; \\
H'_i(s_{i,s}; x_k) &= G(s_{i,s}; x_k), \quad s_{i,s} = x_k + ((i - 1/2)p - L)h \text{ (midpoint of } T_t). 
\end{align*}$$

According to the error of the Hermite interpolation formula, one has

$$J_i(s; x_k) := G(s; x_k) - H_i(s; x_k) = \frac{G^{(p+2)}(\xi_i; x_k)}{(p+2)!} \prod_{j=0}^{p} (s - s_{i,j}), \quad \xi_i \in T_t.$$

Noting that the value of $H_i(s; x_k)$ only depends on the values of $G$ on the interpolation points, one has

$$\sum_t \int_{T_t} \mathcal{A}_{i,p}[G](s; x_k) w(s) \gamma(s) ds = \sum_t \int_{T_t} \mathcal{A}_{i,p}[H_i](s; x_k) w(s) \gamma(s) ds = \sum_t \int_{T_t} H_i(s; x_k) w(s) \gamma(s) ds.$$
Finally one yields
\[
|\mathcal{L}_\delta u(x_k) - \mathcal{L}_{\delta, h} u(x_k)| = \frac{1}{2} \sum_i \left| \int_{T_i} (G(s; x_k) - J_{i,p}(G)(s; x_k)) w(s) \gamma(s) ds \right|
\]
\[
= \frac{1}{2} \sum_i \int_{T_i} (G(s; x_k) - H_i(s; x_k)) w(s) \gamma(s) ds
\]
\[
\leq \frac{1}{2} \sum_i \int_{T_i} \left| \frac{G^{(p+2)}(\xi_i; x_k)}{(p+2)!} (s-s_{i,s}) \prod_{j=0}^{p} (s-s_{i,j}) \right| w(s) \gamma(s) ds
\]
\[
\leq C(p) \delta h^{p+2} |u^{(p+4)}|_{\infty} \int_{B_\delta(0)} w(s) \gamma(s) ds. \tag{6.7}
\]

This completes the proof. The proof of the two-dimensional case is similar and we omit it here.

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