The arithmeticity of a Kodaira fibration is determined by its universal cover

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Abstract. Let $S \to C$ be a Kodaira fibration. Here we show that whether or not the algebraic surface $S$ is defined over a number field depends only on the biholomorphic class of its universal cover.

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1. Introduction and statement of results

Let $X \subset \mathbb{P}^n$ be a complex projective variety and $k$ a subfield of the field of the complex numbers $\mathbb{C}$. We shall say that $X$ is defined over $k$ or that $k$ is a field of definition for $X$ if there exists a collection of homogenous polynomials $f_0, \ldots, f_m$ with coefficients in $k$ so that the variety they define is isomorphic to $X$. We will say that $X$ is arithmetic if it is defined over $\mathbb{Q}$ or equivalently over a number field.

While it is classically known that there are only three simply connected Riemann surfaces, there is a huge amount of possibilities for the holomorphic universal cover of a complex surface $S$. It would be interesting to understand the extent to which the arithmeticity of a projective surface can be read off from its holomorphic universal cover. In this short note we study this question for a very important class of complex surfaces known in the literature as Kodaira fibrations.

A Kodaira fibration consists of a non-singular compact complex surface $S$, a compact Riemann surface $C$ and a surjective holomorphic map $S \to C$ everywhere of maximal rank such that the fibers are connected and not mutually isomorphic Riemann surfaces. The genera $g$ of the fibre and $b$ of $C$ are called the genus of the fibration and of the base respectively. It is known that such a surface $S$ must be an algebraic surface of general type and that necessarily $g \geq 3$ and $b \geq 2$. We notice

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that an important theorem by Arakelov [1] implies that, up to isomorphism, there are only finitely many Kodaira fibrations over a given algebraic curve $C$.

In 1967, Kodaira [13] used fibrations of this kind to show that the signature of a differentiable fiber bundle need not be multiplicative. Soon after Kas [12] studied the deformation space of the surfaces constructed by Kodaira, and two years later Atiyah [2] and Hirzebruch [10] studied further properties concerning the signature of Kodaira fibrations in a volume dedicated to Kodaira himself.

Explicit constructions of Kodaira surfaces have been made by González-Diez and Harvey [8], Bryan and Donagi [4], Zaal [16] and Catanese and Rollenske [6].

We now state the main results of the paper.

**Theorem 1.1.** Let $k$ be an algebraically closed subfield of the complex numbers and $S_1 \to C_1$ and $S_2 \to C_2$ two Kodaira fibrations so that their respective holomorphic universal covers are biholomorphically equivalent. Then $S_1$ is defined over $k$ if and only if $S_2$ is defined over $k$. In particular, $S_1$ is arithmetic if and only if $S_2$ is arithmetic.

To prove this theorem we will have to show first the following result which is interesting in its own right.

**Theorem 1.2.** Let $k$ be an algebraically closed subfield of the complex numbers and $S \to C$ a Kodaira fibration. Then $S$ is defined over $k$ if and only if $C$ is defined over $k$. In particular, $S$ is an arithmetic surface if and only if $C$ is an arithmetic curve.

Theorem 1.1 states that the arithmeticity of a Kodaira fibration can be recognized in its holomorphic universal cover. We anticipate that the holomorphic universal cover of $S$ is a contractible bounded domain $\mathcal{B} \subset \mathbb{C}^2$ (see Section 2). Clearly, Theorem 1.1 implies that the biholomorphism class of $\mathcal{B}$ varies together with the variation of $S$ in moduli space. We note that in general Kodaira surfaces are not rigid ([12], [6]).

## 2. Uniformization of Kodaira Surfaces

It is well-known that the universal cover of a Riemann surface is biholomorphically equivalent to the projective line $\mathbb{P}^1$, the complex plane $\mathbb{C}$ or the upper half-plane $\mathbb{H}$. Understanding universal covers of complex manifolds of higher dimension seems to be a very complicated task. However, thanks to the work of Bers [3] and Griffiths [9] on uniformization of algebraic varieties, it is possible to describe the universal cover of a Kodaira fibration $f : S \to C$ in a very explicit way.

Let $\pi : \mathbb{H} \to C$ be the universal covering map of $C$ and $\Gamma$ the covering group so that $C \cong \mathbb{H}/\Gamma$. By considering the pull-back $h : \pi^*S \to \mathbb{H}$ of $f$ by $\pi$, we obtain a new fibration in which, for each $t \in \mathbb{H}$, the fiber $h^{-1}(t)$ agrees
with the Riemann surface $f^{-1}(\pi(t))$. Teichmüller theory enables us to choose uniformizations $h^{-1}(t) = D_t/K_t$ possessing the following properties:

(a) $K_t$ is a Kleinian group acting on a bounded domain $D_t$ of $\mathbb{C}$ which is biholomorphically equivalent to a disk.

(b) The union of all these disks $B := \bigcup_{t \in \mathbb{H}} D_t$ is a contractible bounded domain of $\mathbb{C}^2$ which is biholomorphic to the universal cover of $S$, that is, $S \cong B/G$, where $G < \text{Aut}(B)$ is the covering group.

(c) The group $G$ is endowed with a surjective homomorphism of groups $\Theta : G \rightarrow \Gamma$ which induces an exact sequence of groups

$$1 \longrightarrow K \longrightarrow G \longrightarrow \Gamma \longrightarrow 1$$

where, for each $t \in \mathbb{H}$, the subgroup $K$ preserves $D_t$ and acts on it as the Kleinian group $K_t$.

We note that $B$ carries itself a fibration structure $B \rightarrow \mathbb{H}$ whose fiber over $t \in \mathbb{H}$ is $D_t$ (i.e. $B$ is a Bergman domain in Bers’ terminology).

In [14] and [15] Shabat studied the automorphism groups of universal covers of families of Riemann surfaces and proved a deep result which in the case of Kodaira fibrations amounts to the following theorem.

**Theorem (Shabat).** Let $f : S \rightarrow C$ be a Kodaira fibration and $B$ the holomorphic universal cover of $S$. Then:

(a) the covering group $G$ of $S$ has finite index in $\text{Aut}(B)$.

(b) $\text{Aut}(B)$ is a discrete group.

3. Proof of Theorems 1.1 and 1.2

We denote by $\text{Gal}(\mathbb{C})$ the group of field automorphisms of $\mathbb{C}$. The natural action of $\text{Gal}(\mathbb{C})$ on the ring of polynomials $\mathbb{C}[x_0, \ldots, x_n]$ induces a well-defined action $\sigma(X) \rightarrow X^\sigma$ on the set of isomorphism classes of algebraic varieties. Throughout this section $k$ will denote an algebraically closed subfield of $\mathbb{C}$ and $\text{Gal}(\mathbb{C}/k)$ the subgroup of $\text{Gal}(\mathbb{C})$ consisting of all automorphisms $\sigma$ fixing the elements of $k$.

3.1. **Proof of Theorem 1.2.** Let $f : S \rightarrow C$ be a Kodaira fibration. Let us assume that the curve $C$ is defined over $k$. Then $C^\sigma = C$ for all $\sigma \in \text{Gal}(\mathbb{C}/k)$, and so, by Arakelov’s finiteness Theorem, there are only finitely many pairwise non-isomorphic Kodaira fibrations $f^\sigma : S^\sigma \rightarrow C^\sigma$ with $\sigma \in \text{Gal}(\mathbb{C}/k)$. This implies that $S$ is defined over $k$ [7, Crit. 2.1].

In order to prove the converse, we begin by recalling that a complex manifold $X$ is named hyperbolic if every holomorphic map $\mathbb{C} \rightarrow X$ is a constant map. We claim
that Kodaira fibrations are hyperbolic. In fact, let \( f : S \rightarrow C \) be a Kodaira fibration and \( \varphi : \mathbb{C} \rightarrow S \) a non-constant holomorphic map. As \( C \) has genus greater than one, the map \( f \circ \varphi : \mathbb{C} \rightarrow C \) must be constant and therefore the image of \( \varphi \) has to be contained in a fiber \( f^{-1}(x) \) for some \( x \in C \). Since the fibers are also hyperbolic, \( \varphi \) must be constant too.

Let us now assume that \( S \) is defined over \( k \), so that \( S^\sigma = S \) for any \( \sigma \in \text{Gal}(\mathbb{C}/k) \). Now as \( S \) is a Kähler hyperbolic manifold, the canonical divisor \( K_S \) is ample [5] and this implies that only finitely many curves \( R \) of genus greater than one can arise as the image of a surjective morphism \( S \rightarrow R \) [11]. In particular the family \( \{C^\sigma : \sigma \in \text{Gal}(\mathbb{C}/k)\} \) itself contains only finitely many isomorphism classes of curves. It then follows that \( C \) is defined over \( k \) [7, Crit. 2.1], as required.

### 3.2. Proof of Theorem 1.1

Let \( f_2 : S_2 \rightarrow C_2 \) be a Kodaira fibration and \( S_1 \) an arbitrary non-singular complex surface. Let us denote by \( \mathcal{B}_i \) the uniformizing group of \( S_i \) and suppose that there exists an isomorphism \( \alpha : \mathcal{B}_1 \rightarrow \mathcal{B}_2 \) between them. Let \( \mathcal{G}_i \) be the uniformizing group of \( S_i \) so that \( \mathcal{B}_i/\mathcal{G}_i \cong S_i \). By Shabat’s Theorem \( \mathcal{G}_2 \) has finite index in \( \text{Aut}(\mathcal{B}_2) \). We claim that \( \mathcal{G}_1 \) has finite index in \( \text{Aut}(\mathcal{B}_1) \) too. In fact, as \( \mathcal{B}_1/\text{Aut}(\mathcal{B}_1) \cong \mathcal{B}_2/\text{Aut}(\mathcal{B}_2) \) and as \( \text{Aut}(\mathcal{B}_2) \) is a discrete group, the projection map \( S_1 = \mathcal{B}_1/\mathcal{G}_1 \rightarrow \mathcal{B}_1/\text{Aut}(\mathcal{B}_1) \) is a holomorphic map between (normal) compact complex surfaces; from here the claim follows.

By replacing \( \mathcal{G}_1 \) by \( \alpha \mathcal{G}_1 \alpha^{-1} \) we can assume that \( \mathcal{B}_1 = \mathcal{B}_2 \), so we denote \( \mathcal{B}_i \) simply by \( \mathcal{B} \). As both \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) have finite index in \( \text{Aut}(\mathcal{B}) \), so must do their intersection \( \mathcal{G}_{12} = \mathcal{G}_1 \cap \mathcal{G}_2 \). The complex surface \( S_{12} := \mathcal{B}/\mathcal{G}_{12} \) is endowed with two finite degree covers \( \pi'_i : S_{12} \rightarrow S_i \) with \( i = 1, 2 \); in particular, \( S_{12} \) is also a projective surface. Moreover, if we denote by \( \Theta_{12} \) the restriction of the epimorphism \( \Theta : \mathcal{G}_2 \rightarrow \Gamma_2 \) introduced in the previous section to \( \mathcal{G}_{12} \), then we obtain an exact sequence of groups

\[
1 \longrightarrow K_{12} \longrightarrow \mathcal{G}_{12} \overset{\Theta_{12}}{\longrightarrow} \Gamma_{12} \longrightarrow 1
\]

where \( \Gamma_{12} = \Theta_{12}(\mathcal{G}_{12}) \) and \( K_{12} = \ker(\Theta_{12}) = \mathcal{K} \cap \mathcal{G}_{12} \). As in Section 2, this sequence defines a Kodaira fibration \( f_{12} : S_{12} \rightarrow C_{12} := \mathcal{H}/\Gamma_{12} \) whose fiber over \([\mathcal{H}]\Gamma_1 \) is isomorphic to the Riemann surface \( D_1/K_{12}^{12} \) where \( K_{12}^{12} \) is the Kleinian group that realizes the action of \( K_{12} \) on \( D_1 \). We have the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{B} & & \mathcal{B} \\
S_{12} & \xrightarrow{\pi'_2} & S_2 \\
\downarrow f_{12} & & \downarrow f \\
C_{12} & \xrightarrow{p} & C_2
\end{array}
\]

where \( p \) is the projection induced by the finite index inclusion \( \Gamma_{12} < \Gamma_2 \).
Let us now assume that $S_2$ is defined over $k$. Then Theorem 1.2 ensures that $C_2$ is also defined over $k$. Furthermore, being an unbranched cover of $C_2$, the curve $C_{12}$ must also be defined over $k$ [7, Th. 4.1]. Again, by Theorem 1.2 we conclude that $S_{12}$ is defined over $k$. Now, as $S_1$ is a surface of general type arising as the image (by $\pi_1'$) of a surface defined over $k$, it must be defined over $k$ as well [7, Prop. 3.2]. This proves Theorem 1.1.

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