Strong contraction, the mirabolic group and the Kirillov conjecture

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Abstract. We lift any (infinitesimal) unitary irreducible representation of $GL_n(\mathbb{R})$ to a family of representations that strongly contracts to a certain type of (infinitesimal) unitary irreducible representations of $\mathbb{R}^n \rtimes M_n$, with $M_n$ being the mirabolic subgroup of $GL_n(\mathbb{R})$. For the case of $n = 2$ we obtain the full unitary dual of $\mathbb{R}^2 \rtimes M_2$ as a strong contraction. We demonstrate the role of the Kirillov conjecture and Kirillov model for these contractions.

Dedicated to I.E. Segal (1918-1998) in commemoration of the centenary of his birth.

1. Introduction

Contraction of Lie algebras first appeared in the work of Segal [1] and Inönü and Wigner [2]. Over the years, many applications to mathematical physics where found e.g., see [3, 4, 5, 6, 7, 8] and references therein. Several years ago, the authors, together with their collaborators, introduced the notion of strong contraction for representations of Lie algebras [9]. This involves a new setup for contraction of representations which utilize representations that are realized on certain spaces of functions. Many contractions, and most notably Inönü-Wigner contractions, are being performed with respect to a subgroup (or sub-Lie-algebra). Here we shall focus on the contraction of $gl_n(\mathbb{R})$ with respect to the Lie algebra $m_n$ consisting of all $n \times n$ real matrices having their last row equal to zero. This is the Lie algebra of the mirabolic group $M_n$, consisting of all invertible $n \times n$ matrices having their last row given by $(0, 0, ..., 0, 1)$. The Lie algebra obtained via this contraction is the semidirect product $\mathbb{R}^n \rtimes m_n$. It is the Lie algebra of $\mathbb{R}^n \rtimes M_n$. The mirabolic group $M_n$ has the following remarkable property: any unitary irreducible representation of $GL_n(\mathbb{R})$ restricted to $M_n$ remains irreducible. This is known as the Kirillov conjecture, proven in the p-adic case in [10] and for real groups in [11]. An important feature of the irreducible representations of $GL_n(\mathbb{R})$ is the existence of their Kirillov model which is a realization on a certain space of functions on $GL_{n-1}(\mathbb{R})$ [12]. The Kirillov model arises in various contexts in automorphic representation theory and representation theory of finite groups of Lie type, e.g., see [13, 14]. Recently, an explicit formulas for the space of $K$-finite vector in the Kirillov model of $GL_2(\mathbb{R})$ was found [15]. This was used in [9] to obtain the skew-Hermitian irreducible representations of the Poincaré Lie algebra $\mathfrak{so}(1, 1)$ as strong contractions of representations of $gl_2(\mathbb{R})$.

The first purpose of this paper is to demonstrate how the Kirillov conjecture and the Kirillov model are useful for strong contractions in the context of contraction of $gl_n(\mathbb{R})$ with respect to
In particular we shall use the Kirillov conjecture to prove the following result.

**Theorem 1.** Let \( \pi : \text{GL}_n(\mathbb{R}) \to \mathcal{U}(\mathcal{H}) \) be a unitary irreducible representation, realized on a Hilbert space of functions \( \mathcal{H} \). Let \( \pi_0 : \mathbb{R}^n \times M_n \to \mathcal{U}(\mathcal{H}) \) be the unitary irreducible representation given by the restriction of \( \pi \) to \( M_n \), extended trivially to \( \mathbb{R}^n \times M_n \). Let \( d\pi \) and \( d\pi_0 \) be the representations of Lie algebras associated with \( \pi \) and \( \pi_0 \) (respectively) on the space of smooth vectors of \( \mathcal{H} \). Then the constant family of representations \( \{d\pi_\epsilon = d\pi\}_{\epsilon \in \mathbb{R}} \) strongly contract to \( d\pi_0 \).

The second purpose is to obtain the full unitary dual of \( \mathbb{R}^2 \times M_2 \) as a strong contraction. More precisely we prove the following result.

**Theorem 2.** For any unitary irreducible representation \( \pi_0 \) of \( \mathbb{R}^2 \times M_2 \) the following hold.

(i) There is a realization of \( \pi_0 \) on a Hilbert space of functions \( \mathcal{H} \).
(ii) There is a dense \((\mathbb{R}^2 \times m_2)\)-invariant subspace \( \mathcal{H}^\infty \) of \( \mathcal{H} \).
(iii) There is a family of representations \( \{d\pi_\epsilon \colon \mathfrak{g}_2 \to \text{End}(\mathcal{H}^\infty)\}_{\epsilon \in \mathbb{R} \neq 0} \), or a sequence of representations \( \{d\pi_{\epsilon n} : \mathfrak{gl}_2(\mathbb{R}) \to \text{End}(\mathcal{H}^\infty)\}_{\epsilon \in \mathbb{N}} \) that strongly contracts to \( d\pi_0 : \mathbb{R}^2 \times m_2 \to \text{End}(\mathcal{H}^\infty) \).

### 2. Contractions

In this section we introduce notations and review some generalities on Inönü-Wigner contraction [2]. We spell out an Inönü-Wigner contraction [2] of \( \mathfrak{gl}_n(\mathbb{R}) \) with respect to the Lie algebra of the mirabolic group \( M_n \). Then we recall the notion of strong contraction of representations of Lie algebras.

#### 2.1. Contraction of Lie algebras

Given a real Lie algebra \( \mathfrak{g} = (V, [\cdot, \cdot]) \) (with \( V \) the underlying vector space, \( [\cdot, \cdot] \) the Lie brackets of \( \mathfrak{g} \)) and a decomposition \( V = \mathfrak{t} \oplus s \) with \( \mathfrak{t} \) being a subalgebra and \( s \) a vector space complement to \( \mathfrak{t} \), there is a corresponding Inönü-Wigner contraction; For every nonzero \( \epsilon \in \mathbb{R} \) we have a linear invertible map \( t_\epsilon : V \to V \) given by \( t_\epsilon(X_\mathfrak{t} + X_\mathfrak{s}) = X_\mathfrak{t} + \epsilon X_\mathfrak{s} \), for \( X_\mathfrak{t} \in \mathfrak{t} \) and \( X_\mathfrak{s} \in s \). For \( X, Y \in V \), the formula \( [X, Y]_\epsilon := t_\epsilon^{-1}[t_\epsilon(X), t_\epsilon(Y)] \) defines Lie brackets on \( V \). We denote the corresponding Lie algebra by \( \mathfrak{g}_\epsilon = (V, [\cdot, \cdot]_\epsilon) \). Moreover, for every \( X, Y \in V \),

\[
[X, Y]_0 := \lim_{\epsilon \to 0} t_\epsilon^{-1}[t_\epsilon(X), t_\epsilon(Y)]
\]

converges and defines Lie brackets on \( V \). The obtained Lie algebra \( \mathfrak{g}_0 = (V, [\cdot, \cdot]_0) \) is called the contraction of \( \mathfrak{g} = \mathfrak{g}_1 \). The Lie algebra \( \mathfrak{g}_0 \) is a semidirect product of \( \mathfrak{t} \) and the abelian ideal \( s \). This contraction is denoted by \( \mathfrak{g} \xrightarrow{t_\epsilon} \mathfrak{g}_0 \).

#### 2.2. The case of \( m_n \subset \mathfrak{gl}_n(\mathbb{R}) \)

Consider the Lie algebra of \( n \times n \) real matrices, \( \mathfrak{gl}_n(\mathbb{R}) \). We denote its underlying vector space by \( V_n \), and we shall use the standard basis \( \{e_{ij}\}_{1 \leq i, j \leq n} \) of \( V_n \) to define a subalgebra and a vector space complement. As the subalgebra \( \mathfrak{t} \) we take \( m_n = \text{span}_\mathbb{R}\{e_{ij}\}_{1 \leq i \leq n-1, 1 \leq j \leq n} \) and choose the complement \( s_n := \text{span}_\mathbb{R}\{e_{ij}\}_{1 \leq j \leq n} \). For \( \epsilon \neq 0 \), the corresponding contraction maps \( t_\epsilon : V_n \to V_n \) are given by

\[
t_\epsilon(e_{ij}) = \begin{cases} e_{ij}, & i \neq n \\ \epsilon e_{ij}, & i = n. \end{cases}
\]
For every \( \epsilon \in \mathbb{R} \), the Lie brackets \([\cdot, \cdot]_\epsilon\) on \( V_n \) are explicitly given by
\[
\left[ e_{ij}, e_{kl} \right]_\epsilon = \begin{cases} 
\epsilon \delta_{jk} e_{il} - \delta_{il} e_{kj} & , i \neq n, j \neq n \\
\epsilon \delta_{kl} e_{il} - \delta_{il} e_{kj} & , i = n, j \neq n \\
\epsilon \delta_{il} e_{kj} - \delta_{jk} e_{il} & , i \neq n, j = n \\
e(\delta_{jk} e_{il} - \delta_{il} e_{kj}) & , i = k = n.
\end{cases}
\]

The group \( \mathbb{R}^n \times M_n \) with product given by
\[
(v, A)(u, B) = (v + (A^{-1})^T u, AB)
\]
for \((v, A), (u, B) \in \mathbb{R}^n \times M_n\), is a Lie group containing \( M_n \) as a subgroup and having \((\text{gl}_n(\mathbb{R}))_0\) as its Lie algebra.

### 2.3. Strong contractions

Below we recall the definition of strong contraction in the special case of a common underlying inner product space of functions, for all representations that take part in the contraction procedure. This will be enough for the purposes of this paper. For the general definition see [8].

Keeping the previous notation, suppose that \( t_\epsilon : V \to V \) is a family of linear invertible maps that realizes a contraction from \( g = g_1 = (V, \langle \cdot, \cdot \rangle_1) \) to \( g_0 = (V, \langle \cdot, \cdot \rangle_0) \). Let \( X \) be a topological space, and \( \mu \) a positive Borel measure on \( X \). Let \( W \) be a subspace of \( L^2(X, d\mu) \).

Let \( \pi_0 : g_0 \to \text{End}(W) \) be a representation of \( g_0 \) and for every \( \epsilon \neq 0 \), \( \pi_\epsilon : g \to \text{End}(W) \) a representation of \( g \). The representation \( \pi_0 \) is a strong contraction of the family \( \{\pi_\epsilon\}_{\epsilon \neq 0} \) if the following hold.

(i) For every \( f \in W \), every \( Y \in V \) and every \( x \in X \),
\[
\lim_{\epsilon \to 0} (\pi_\epsilon(t_\epsilon Y)f)(x) = (\pi_0 Y f)(x).
\]

(ii) For every \( f \in W \) and every \( Y \in V \),
\[
\lim_{\epsilon \to 0} \| (\pi_\epsilon(t_\epsilon Y)f) - (\pi_0 Y f) \| = 0.
\]

Similarly, a sequence of representations \( \pi_n : g \to \text{End}(W) \) strongly contract to \( \pi_0 : g_0 \to \text{End}(W) \) if there is a sequence of real numbers \( \epsilon_n \) converging to zero such that

(i) For every \( f \in W \), every \( Y \in V \) and every \( x \in X \),
\[
\lim_{n \to \infty} (\pi_n(t_{\epsilon_n} Y)f)(x) = (\pi_0 Y f)(x).
\]

(ii) For every \( f \in W \) and every \( Y \in V \),
\[
\lim_{n \to \infty} \| (\pi_n(t_{\epsilon_n} Y)f) - (\pi_0 Y f) \| = 0.
\]

### 3. Strong contraction of constant families of unitary irreducible representations of \( \text{GL}_n(\mathbb{R}) \)

This section deals with strong contractions of constant families of unitary irreducible representations of \( \text{GL}_n(\mathbb{R}) \). We shall start our discussion proving a simple lemma which is applicable in more general context.
In this section, using the Mackey machine \cite{17,18,19}, we describe the unitary dual of $G_0$. The unitary dual of $G_0$ is typically reducible. Lemma 1 implies the following result for the above mentioned contraction with respect to the mirabolic subgroup.

**Theorem 1.** Let $\pi : G_0 \to \mathcal{U}(\mathcal{H})$ be a unitary irreducible representation, realized on a Hilbert space of functions $\mathcal{H}$. Then $\pi_0$ is unitary irreducible and the constant family of representations \( \{d\pi_\epsilon : g_0 \to \text{End}(\mathcal{H}^\infty)\}_{\epsilon \in \mathbb{R} \neq 0} \) with $d\pi_\epsilon = d\pi$, strongly contracts to $d\pi_0 : g_0 \to \text{End}(\mathcal{H}^\infty)$.

The proof is elementary and is included just for completeness.

**Proof.** We can assume that $\mathcal{H}$ is a subspace of $L^2(X)$ for some measurable space $X$ with a positive Borel measure $\mu$. For every $Y \in \mathfrak{t}$, every function $f \in \mathcal{H}$ and every $x \in X$, we have

\[
\lim_{\epsilon \to 0} (d\pi_\epsilon(t_x Y)f)(x) = (d\pi(Y)f)(x) = (d\pi_0(Y)f)(x),
\]

\[
\lim_{\epsilon \to 0} \| (d\pi_\epsilon(t_x Y)f) - (d\pi_0(Y)f) \| = 0.
\]

For every $Y \in \mathfrak{s}$, every function $f \in \mathcal{H}$ and every $x \in X$, we have

\[
\lim_{\epsilon \to 0} (d\pi_\epsilon(t_x Y)f)(x) = \lim_{\epsilon \to 0} (d\pi(Y)f)(x) = 0 = (d\pi_0(Y)f)(x),
\]

\[
\lim_{\epsilon \to 0} \| (d\pi_\epsilon(t_x Y)f) - (d\pi_0(Y)f) \| = \lim_{\epsilon \to 0} \| (\epsilon d\pi(Y)f) - 0 \| = \lim_{\epsilon \to 0} \| \epsilon (d\pi(Y)f) \| = 0.
\]

We keep the above notation and let $G = \text{GL}_n(\mathbb{R})$, $K = M_n$ and $S = \mathbb{R}^n$. Then the Kirillov conjecture implies that $\pi|_{M_n}$ (and hence also $\pi_0$) is unitary irreducible for any unitary irreducible $\pi$. In general, that is for other groups, $\pi_0$ is typically reducible. Lemma 1 implies the following result for the above mentioned contraction with respect to the mirabolic subgroup.

**Lemma 1.** Let $\pi : G_0 \to \mathcal{U}(\mathcal{H})$ be a unitary irreducible representation, realized on a Hilbert space of functions $\mathcal{H}$. Then $\pi_0$ is unitary irreducible and the constant family of representations \( \{d\pi_\epsilon : g_0 \to \text{End}(\mathcal{H}^\infty)\}_{\epsilon \in \mathbb{R} \neq 0} \) with $d\pi_\epsilon = d\pi$, strongly contracts to $d\pi_0 : g_0 \to \text{End}(\mathcal{H}^\infty)$.

**4. The unitary dual of $\mathbb{R}^2 \times M_2$**

In this section, using the Mackey machine \cite{17,18,19}, we describe the unitary dual of $\mathbb{R}^2 \times M_2$. We shall explicitly write the associated representation of $g_0 = \mathbb{R}^2 \times m_2$ on a corresponding dense subspace of the space of smooth vectors. These realizations are used in section 5.

The character group $\mathbb{R}^2$ consists of all functions of the form

\[
X_u : \mathbb{R}^2 \to \mathbb{C}
\]

\[
X_u(v) = e^{i(u \cdot v)},
\]

with $u, v \in \mathbb{R}^2$. The mirabolic group $M_2$ acts on $\mathbb{R}^2$ by $A \cdot X_u = X_{Au}$. Unitary irreducible representations of $\mathbb{R}^2 \times M_2$ are parameterized by an orbit of some $X_u$ in $\mathbb{R}^2$ and a unitary irreducible representation of the stabilizer of $X_u$. Below we give an exhaustive list of these representations up to equivalence.
(i) **The orbit of the trivial character** $\chi_{(0,0)}$. In this case we obtain unitary irreducible representations that are trivial on $\mathbb{R}^2$. Such a representation is given by a unitary irreducible representation of $M_2$. Explicitly, we have exactly the following cases.

(a) For $\lambda \in \mathbb{R}$ and $\sigma \in \{0, 1\}$, the representation $\eta_{(0,0)}^{\lambda,\sigma} : M_2 \to \text{GL}(\mathbb{C}) \cong \mathbb{C}^*$ given by

$$
\eta_{(0,0)}^{\lambda,\sigma} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \text{sgn}(a)^\sigma |a|^{i\lambda}.
$$

The associated representation $d\eta_{(0,0)}^{\lambda,\sigma}$ of $g_0$ on $\mathbb{C}$ is given by

$$
e_{11} \mapsto i\lambda, \ e_{12} \mapsto 0, \ e_{21} \mapsto 0, \ e_{22} \mapsto 0.
$$

(b) The representation $\eta_{(0,0)} : M_2 \to \mathcal{U}(L^2(\mathbb{R}^*, \frac{dx}{|x|}))$ given by

$$
\left( \eta_{(0,0)} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} f \right) (x) = e^{ibx} f(ax).
$$

The associated representation $d\eta_{(0,0)}$ of $g_0$ on $C_c^\infty(\mathbb{R}^*, \frac{dx}{|x|})$, the inner product space of smooth compactly supported functions in $L^2(\mathbb{R}^*, \frac{dx}{|x|})$, is given by

$$
e_{11} \mapsto x\partial_x, \ e_{12} \mapsto ix, \ e_{21} \mapsto 0, \ e_{22} \mapsto 0.
$$

(ii) **The orbit of the character** $\chi_{(1,0)}$. For $\lambda \in \mathbb{R}$, the representation $\eta_{(1,0)}^{\lambda} : M_2 \to \mathcal{U}(L^2(\mathbb{R}^*, \frac{dx}{|x|}))$ given by

$$
\left( \eta_{(1,0)}^{\lambda} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} f \right) (x) = e^{iv_1 x - i\lambda bx} f(xa).
$$

The associated representation $d\eta_{(1,0)}^{\lambda}$ of $g_0$ on $C_c^\infty(\mathbb{R}^*, \frac{dx}{|x|})$, is given by

$$
e_{11} \mapsto x\partial_x, \ e_{12} \mapsto i\lambda x, \ e_{21} \mapsto \frac{i}{x}, \ e_{22} \mapsto 0.
$$

(iii) **The orbit of the character** $\chi_{(0,\beta)}$ with $\beta \neq 0$. For $\lambda \in \mathbb{R}$ and $\sigma \in \{0, 1\}$, the representation $\eta_{(0,\beta)}^{\lambda,\sigma} : M_2 \to \mathcal{U}(L^2(\mathbb{R}, dx))$ given by

$$
\left( \eta_{(0,\beta)}^{\lambda,\sigma} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} f \right) (x) = e^{ib\beta(xv_1 + v_2)} \text{sgn}(a)^\sigma |a|^{i\lambda - 1/2} f \left( \frac{b + x}{a} \right).
$$

The associated representation $d\eta_{(0,\beta)}^{\lambda,\sigma}$ of $g_0$ on $C_c^\infty(\mathbb{R}, dx)$, the inner product space of smooth compactly supported functions in $L^2(\mathbb{R}, dx)$, is given by

$$
e_{11} \mapsto i\lambda - \frac{1}{2} - x\partial_x, \ e_{12} \mapsto \partial_x, \ e_{21} \mapsto -i\beta x, \ e_{22} \mapsto i\beta.
$$

We summarize the above discussion in a Lemma.

**Lemma 2.** The list below contains exactly one representative from each equivalence class of unitary irreducible representations of $\mathbb{R}^2 \rtimes M_2$:

(i) $\{\eta_{(0,0)}^{\lambda,\sigma} | \lambda \in \mathbb{R}, \sigma \in \{0, 1\}\} \cup \{\eta_{(0,0)}\}$,

(ii) $\{\eta_{(1,0)}^{\lambda} | \lambda \in \mathbb{R}\}$,

(iii) $\{\eta_{(0,\beta)}^{\lambda,\sigma} | \beta \in \mathbb{R}^*, \lambda \in \mathbb{R}, \sigma \in \{0, 1\}\}$. 


5. The unitary dual of $\mathbb{R}^2 \rtimes M_2$ as strong contraction

In this section for any unitary irreducible representation of $\mathbb{R}^2 \rtimes M_2$, in one of the explicit realizations that are given in Section 4, we build a corresponding strong contraction and, by doing so, proving Theorem 2. We shall organize our calculations according to the orbits of $M_2$.

5.1. The orbit of the trivial character $\chi_0$

For every $\mu \in \mathbb{R}$, we let $\pi_\mu$ be the unitary one dimensional representation of $\text{GL}_2(\mathbb{R})$ given by

$$\Lambda \mapsto |\det \Lambda|^\mu.$$ 

The corresponding representation $d\pi_\mu$ of $\text{gl}_2(\mathbb{R})$ is given by

$$e_{11} \mapsto i\mu, \quad e_{12} \mapsto 0, \quad e_{21} \mapsto 0, \quad e_{22} \mapsto i\mu.$$

**Proposition 1.** For every $\lambda \in \mathbb{R}$ and $\sigma \in \{0, 1\}$, the representation $d\pi^\lambda_\sigma: \mathbb{R}^2 \rtimes m_2 \to \text{End}(\mathbb{C})$ is a strong contraction of the (constant) family $\{d\pi_\lambda : \text{gl}_2(\mathbb{R}) \to \text{End}(\mathbb{C})\}_{\sigma \neq 0}$.

We first remark that in this case we can think of the one dimensional vector space $\mathbb{C}$ as the space of square integrable functions on the trivial measurable space consisting of one point only.

**Proof.** In this case, since the underlying vector space is one dimensional pointwise convergence implies norm convergence. Indeed,

$$\lim_{\epsilon \to 0} d\pi^\lambda_\sigma : (t_\epsilon e_{11}) = i\lambda = d\eta^\lambda_0(0,0)(e_{11}),$$

$$\lim_{\epsilon \to 0} d\pi^\lambda_\sigma : (t_\epsilon e_{12}) = 0 = d\eta^\lambda_0(0,0)(e_{12}),$$

$$\lim_{\epsilon \to 0} d\pi^\lambda_\sigma : (t_\epsilon e_{21}) = 0 = d\eta^\lambda_0(0,0)(e_{21}),$$

$$\lim_{\epsilon \to 0} d\pi^\lambda_\sigma : (t_\epsilon e_{22}) = \lim_{\epsilon \to 0} e_i \lambda = 0 = d\eta^\lambda_0(0,0)(e_{22}).$$

$\square$

**Proposition 2.** There is a unitary irreducible representation $\pi$ of $\text{GL}_2(\mathbb{R})$ on a Hilbert space of functions $\mathcal{H}$ and a dense $\text{gl}_2(\mathbb{R})$-invariant subspace $\mathcal{H}^\infty$ of $\mathcal{H}$, such that the (constant) family of representations $\{d\pi : \text{gl}_2(\mathbb{R}) \to \text{End}(\mathcal{H}^\infty)\}_{\epsilon \neq 0}$ strongly contract to a representation isomorphic to $d\eta_{(0,0)}: \mathbb{R}^2 \rtimes m_2 \to \text{End}(\mathcal{H}^\infty(\mathbb{R}^*, \frac{dx}{|x|}))$.

**Proof.** We can take for $\pi$ any unitary irreducible infinite-dimensional representation of $\text{GL}_2(\mathbb{R})$ realized on a Hilbert space of functions. For example, we can take a unitary irreducible principal series with $\mathcal{H}$ a Hilbert space of functions on $\text{GL}_2(\mathbb{R})$. Theorem 1 guarantees that $\pi_0$ is unitary irreducible and the constant family of representations $\{d\pi_\epsilon : \mathcal{H} \to \text{End}(\mathcal{H}^\infty)\}_{\epsilon \in \mathbb{R} \neq 0}$ with $d\pi_\epsilon = d\pi_0$ strongly contracts to $d\pi_0: \mathbb{R}^2 \rtimes m_2 \to \text{End}(\mathcal{H}^\infty)$. The representation $\pi_0|_{M_2}$ of $M_2$ is unitary irreducible and infinite-dimensional. Up to equivalence there is exactly one such representation. Since $\mathbb{R}^2 \rtimes 1$ acts trivially via $\pi_0$ then $\pi_0$ must be equivalent to $\eta_{(0,0)}$. Hence $d\pi_0$ is equivalent to $d\eta_{(0,0)}$. $\square$
5.2. The orbit of the character $\chi_{\{1,0\}}$.

For every integer $n > 1$ there is a discrete series representation $D^n$ of $GL_2(\mathbb{C})$ in which the scalar matrices act trivially. In the Kirillov model on $L^2(\mathbb{R}, \frac{dx}{|x|})$ the corresponding representation of $gl_2(\mathbb{R})$ on $C^\infty_c(\mathbb{R}^*, \frac{dx}{|x|})$ is given by

$$e_{11} \mapsto x\partial_x, \quad e_{12} \mapsto ix, \quad e_{21} \mapsto -\frac{i}{4x} n^2 - 1 + ix\partial_x, \quad e_{22} \mapsto -x\partial_x.$$ 

For more details, see [15]. For $0 \neq q \in \mathbb{R}$ we twist the above mentioned representation via conjugation by the diagonal matrix diag$(q, 1)$ to obtain the isomorphic representation $dD^{n,q}$ given by

$$e_{11} \mapsto x\partial_x, \quad e_{12} \mapsto iq x, \quad e_{21} \mapsto -\frac{i}{4q} n^2 - 1 + \frac{x}{q} \partial_x, \quad e_{22} \mapsto -x\partial_x.$$ 

**Proposition 3.** For any $0 \neq \lambda \in \mathbb{R}$ the representation $d\eta^{\lambda}_{\{1,0\}}: \mathbb{R}^2 \times m_2 \rightarrow \text{End}(C^\infty_c(\mathbb{R}^*, \frac{dx}{|x|}))$ is a strong contraction of the sequence $\{dD^{n,\lambda}: gl_2(\mathbb{R}) \rightarrow \text{End}(C^\infty_c(\mathbb{R}^*, \frac{dx}{|x|}))\}_{n \in \mathbb{N}}$.

**Proof.** We shall use the sequence $e_n = -\frac{4}{n^2}$. For pointwise convergence observe that for every $f \in C^\infty_c(\mathbb{R}^*)$ and $x \in \mathbb{R},$

$$\lim_{n \rightarrow \infty} \left( dD^{n,\lambda}(t_{e_n}(e_{11})) f \right)(x) = \lim_{n \rightarrow \infty} \left( d\eta^{\lambda}_{\{1,0\}}(e_{11}) f \right)(x),$$

$$\lim_{n \rightarrow \infty} \left( dD^{n,\lambda}(t_{e_n}(e_{12})) f \right)(x) = \left( d\eta^{\lambda}_{\{1,0\}}(e_{12}) f \right)(x),$$

$$\lim_{n \rightarrow \infty} \left( dD^{n,\lambda}(t_{e_n}(e_{21})) f \right)(x) = \lim_{n \rightarrow \infty} -\frac{4\lambda}{n^2\lambda} \left( -\frac{i}{4x} n^2 - 1 + ix\partial_x \right) f(x) = 0 = \left( d\eta^{\lambda}_{\{1,0\}}(e_{22}) f \right)(x).$$

Norm convergence follows from Lebesgue dominated convergence theorem. \qed

**Proposition 4.** The representation $d\eta^0_{\{1,0\}}: \mathbb{R}^2 \times m_2 \rightarrow \text{End}(C^\infty_c(\mathbb{R}^*, \frac{dx}{|x|}))$ is a strong contraction of the sequence $\{dD^{n,\lambda}: gl_2(\mathbb{R}) \rightarrow \text{End}(C^\infty_c(\mathbb{R}^*, \frac{dx}{|x|}))\}_{\lambda \neq 0}.$

Using $e_n = -\frac{4}{n^2}$ the proof is similar to that of Proposition 3.

5.3. The orbit of the character $\chi_{\{0,\beta\}}$ with $0 \neq \beta \in \mathbb{R}$.

In the non-compact picture, the unitary principal series of $SL_2(\mathbb{R})$ are realized on $L^2(\mathbb{R}, dx)$ via

$$\left( \tilde{p}^{\sigma,\nu} \begin{bmatrix} a & b \\ c & d \end{bmatrix} f \right)(x) = \text{sgn}(-bx + d)^\sigma | -bx + d |^{-1 - i\nu} f \left( \frac{ax - c}{-bx + d} \right),$$

here $\sigma \in [0, 1]$ and $\nu \in \mathbb{R}$. See e.g., [16, p. 36]. We twist this representation by precompose $\tilde{p}^{\sigma,\nu}$ with inverse transpose. Explicitly, the new action is $\tilde{p}^{\sigma,\nu}$ given by

$$\left( \tilde{p}^{\sigma,\nu} \begin{bmatrix} a & b \\ c & d \end{bmatrix} f \right)(x) = \text{sgn}(cx + a)^\sigma | cx + a |^{-1 - i\nu} f \left( \frac{dx + b}{cx + a} \right).$$
For every \( \mu \in \mathbb{R} \), we extend \( \tilde{\pi}^{\sigma,v} \) to a unitary irreducible representation \( \tilde{\pi}^{\sigma,v,\mu} \) of \( GL_2(\mathbb{R}) \), by

\[
\left( \tilde{\pi}^{\sigma,v,\mu} \begin{pmatrix} a & b \\ c & d \end{pmatrix} f \right)(x) = |ad - bc|^{1/2 + i\mu} \text{sgn}(cx + a)^{\sigma} |cx + a|^{-1 - i\mu} f \left( \frac{dx + b}{cx + a} \right).
\]

The corresponding representation \( d\tilde{\pi}^{\sigma,v,\mu} \) of \( gl(2,\mathbb{R}) \) on \( C_c^\infty(\mathbb{R}) \) is given by

\[
e_{11} \mapsto i\mu - \frac{1}{2} - i\nu - x\partial_x, \quad e_{12} \mapsto \partial_x, \quad e_{21} \mapsto -(1 + i\nu)x - x^2\partial_x, \quad e_{22} \mapsto \frac{1}{2} + i\mu + x\partial_x.
\]

**Proposition 5.** For \( \lambda \in \mathbb{R} \) and \( \sigma \in \{0, 1\} \), the representation \( d\pi^{\lambda,\sigma}_\epsilon : \mathbb{R}^2 \times M_2 \rightarrow \text{End}(C_c^\infty(\mathbb{R})) \) is a strong contraction of the family \( \{d\tilde{\pi}^{\sigma,v,\mu} : gl(2,\mathbb{R}) \rightarrow \text{End}(C_c^\infty(\mathbb{R}))\}_{\epsilon \neq 0} \).

**Proof.** For pointwise convergence observe that for every \( f \in C_c^\infty(\mathbb{R}^*) \) and \( x \in \mathbb{R} \),

\[
\lim_{\epsilon \to 0} \left( d\tilde{\pi}^{\sigma,v,\epsilon,\lambda+\beta/\epsilon}(t_\epsilon(e_{11}))f \right)(x) = \lim_{\epsilon \to 0} \left( i(\lambda + \frac{\beta}{\epsilon}) - \frac{1}{2} - i\frac{\beta}{\epsilon}x - x\partial_x \right) f(x) = (i\lambda - \frac{1}{2} - x\partial_x) f(x) = \left( d\pi^{\lambda,\sigma}_\epsilon(e_{11})f \right)(x),
\]

\[
\lim_{\epsilon \to 0} \left( d\tilde{\pi}^{\sigma,v,\epsilon,\lambda+\beta/\epsilon}(t_\epsilon(e_{12}))f \right)(x) = \partial_x f(x) = \left( d\pi^{\lambda,\sigma}_\epsilon(e_{12})f \right)(x),
\]

\[
\lim_{\epsilon \to 0} \left( d\tilde{\pi}^{\sigma,v,\epsilon,\lambda+\beta/\epsilon}(t_\epsilon(e_{21}))f \right)(x) = \lim_{\epsilon \to 0} \epsilon \left( - (1 + i\frac{\beta}{\epsilon})x - x^2\partial_x \right) f(x) = -i\beta x f(x) = \left( d\pi^{\lambda,\sigma}_\epsilon(e_{21})f \right)(x),
\]

\[
\lim_{\epsilon \to 0} \left( d\tilde{\pi}^{\sigma,v,\epsilon,\lambda+\beta/\epsilon}(t_\epsilon(e_{22}))f \right)(x) = \lim_{\epsilon \to 0} \epsilon \left( \frac{1}{2} + i(\lambda + \frac{\beta}{\epsilon}) + x\partial_x \right) f(x) = i\beta f(x) = \left( d\pi^{\lambda,\sigma}_\epsilon(e_{22})f \right)(x).
\]

Norm convergence follows from the following lemma which easily proved using Lebesgue dominated convergence theorem.

**Lemma 3.** For every \( \epsilon \in \mathbb{R} \), let \( D_\epsilon \) be a smooth differential operator on \( C_c^\infty(\mathbb{R}) \). Explicitly, \( D_\epsilon = \sum_{i=0}^n d_i(\epsilon) x^i \partial_x^i \) for some smooth functions \( d_i(\epsilon) \in C_c^\infty(\mathbb{R}) \). If for every \( i \in \{1, 2, \ldots, n\} \), and every \( x \in \mathbb{R} \), \( \epsilon \mapsto d_i(\epsilon) x^i \partial_x^i \) is a continuous function of \( \epsilon \) then for every \( f \in C_c^\infty(\mathbb{R}) \),

\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}} |D_\epsilon(f)(x)|^2 \, dx = \int_{\mathbb{R}} |D_0(f)(x)|^2 \, dx.
\]

\[\square\]

Combining Lemma 2 and Propositions 3, we obtain the following result.

**Theorem 2.** For any unitary irreducible representation \( \pi_0 \) of \( \mathbb{R}^2 \rtimes M_2 \) the following hold.

(i) There is a realization of \( \pi_0 \) on a Hilbert space of functions \( \mathcal{H} \).

(ii) There is a dense \( (\mathbb{R}^2 \rtimes M_2) \)-invariant subspace \( \mathcal{H}^{\infty} \) of \( \mathcal{H} \).

(iii) There is a family (or a sequence) of representations \( \{d\pi_\epsilon : gl(2,\mathbb{R}) \rightarrow \text{End}(\mathcal{H}^{\infty})\}_{\epsilon \in \mathbb{R} \neq 0} \) that strongly contract to \( d\pi_0 : \mathbb{R}^2 \rtimes M_2 \rightarrow \text{End}(\mathcal{H}^{\infty}) \).
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