Indecomposable manipulations with simple modules in category \( \mathcal{O} \)

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We study the problem of indecomposability of translations of simple modules in the principal block of BGG category \( \mathcal{O} \) for \( \mathfrak{sl}_n \), as conjectured in [KiM1]. We describe some general techniques and prove a few general results which may be applied to study various special cases of this problem. We apply our results to verify indecomposability for \( n \leq 6 \). We also study the problem of indecomposability of shufflings and twistings of simple modules and obtain some partial results.

1. Introduction

Consider the simple complex Lie algebra \( \mathfrak{sl}_n \), for \( n \geq 2 \), with a fixed standard triangular decomposition \( \mathfrak{sl}_n = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \) and let \( \mathcal{O} \) be the associated
BGG category $\mathcal{O}$ as in [BGG2] [Hu]. The Weyl group $W$ of $\mathfrak{sl}_n$ indexes naturally both simple modules $L(w)$, where $w \in W$, in the principal block $\mathcal{O}_0$ of $\mathcal{O}$ and indecomposable projective functors $\theta_w$, where $w \in W$, in $\mathcal{O}_0$, see [BG]. The following conjecture is formulated in [KiM1] Conjecture 2.

**Conjecture 1.** For all $x, y \in W$, the module $\theta_x L(y)$ is either indecomposable or zero.

The present paper arose as a result of various, so far unsuccessful, attempts to either prove or disprove this conjecture. If true, the statement of Conjecture 1 would be a type A phenomenon, as it is well-known that it fails already in type $B_2$, see e.g. [KiM1 Subsection 5.1].

As mentioned in [KiM1], the question posed by Conjecture 1 has a history related to classification of projective functors on parabolic category $\mathcal{O}$ in type A. That problem was solved in [KiM1] using results from higher representation theory, namely, classification of simple transitive 2-representations of the 2-category of Soergel bimodules over the coinvariant algebra of $W$, see [MM5]. The latter motivates one of the approaches which we try in the present paper. The other approaches are more “classical” and directed towards understanding the endomorphism algebra of $\theta_x L(y)$, by comparing it, using homological methods, with the endomorphism algebra of some other indecomposable modules.

Here is a short summary of our main results related to Conjecture 1:

- We show that, to prove Conjecture 1 it is enough to consider the case when both $x$ and $y$ are involutions.

- We show that, to prove Conjecture 1 it is enough to consider the case when every simple reflection of $W$ appears in a reduced decomposition of $x$.

- We verify Conjecture 1 for $n = 2, 3, 4, 5, 6$.

The second item here is a biproduct of a general observation that category $\mathcal{O}$ has an internal “recursion” in the sense of an equivalence of certain abelian subquotients and blocks of category $\mathcal{O}$ for smaller Lie algebras, see Theorems 32 and 37.

The paper is organized as follows: In Section 2 we have collected all preliminaries and generalities on category $\mathcal{O}$. In Section 3 we describe some special cases of Conjecture 1 and also recall the results and observations.
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from [KiM1]. In Section 4 we describe two homological approaches to Conjecture 1. One of them, presented in Subsection 4.1, compares the endomorphism algebra \( \theta_x L(y) \) to that of a (twisted) indecomposable projective module in \( \mathcal{O}_0 \). The other one, presented in Subsection 4.2, explores the possibility of an inductive proof with respect to the Bruhat order. Section 5 outlines a higher representation theoretic approach which is based on understanding the action on \( \mathcal{O}_0 \) of the 2-full 2-subcategory of the 2-category of projective functors generated by \( \theta_x \). None of the above approaches seem to work in full generality for the moment. However, a common feature of the above approaches is that they all work in the special case of \( x \) being the longest element of a parabolic subalgebra (this case is “easy” and was already described in [KiM1]). All the approaches mentioned about motivate a number of question, of homological or higher representation theoretical nature, respectively, which we emphasize and study (or describe the answer to) in special cases.

Section 6 contains two general observations which, in particular, turn out to be useful for the study of Conjecture 1. We observe a “recursive” structure of \( \mathcal{O}_0 \) in the sense that, for any parabolic subgroup of \( W \), the category \( \mathcal{O}_0 \) turns out to have a filtration by Serre subcategories such that the corresponding subquotients are equivalent to the category \( \mathcal{O}_0 \) considered with respect to the Lie algebra associated with the parabolic subgroup. We show two ways to construct such a filtration (using left, respectively, right cosets of \( W \) with respect to the parabolic subgroup). One of these ways is well-coordinated with the action of twisting functors while the other one is well-coordinated with the action of projective functors. The latter in particular implies that, for fixed \( x \) and \( y \), Conjecture 1 has the same answer for all \( n \) for which the elements \( x \) and \( y \) make sense (i.e. may be defined using the obvious chain of inclusions for symmetric groups). Another consequence is that if, for fixed \( x \) and \( n \), Conjecture 1 is true for all \( y \), then it is true, for the same \( x \), for all \( n \) and for all \( y \) for which it can be formulated.

In Section 7 we prove Conjecture 1 in the cases \( n = 2, 3, 4, 5, 6 \). The cases \( n = 2, 3, 4 \) are easy and the case \( n = 5 \) can be dealt with using the results from Section 6. The case \( n = 6 \) is substantially more difficult and requires a number of case-by-case studies. After application of all general methods which we know (in particular, of those from Section 6), for \( n = 6 \), we are left with five cases which we have to go through on a case-by-case basis and using various tricks and explicit computations with Kazhdan-Lusztig polynomials (for this we use explicit tables from [Go]). So far we did not manage to extend the arguments we used in these specific cases to any more general situation.
However, the versatility of difficulties that show up for \( n = 6 \) suggests that the general case of Conjecture 1 might be really difficult.

Finally, in Section 8 we study the problem of indecomposability of shufflings of simple modules in category \( \mathcal{O} \). We show that it is equivalent to the problem of indecomposability of twistings of simple modules in category \( \mathcal{O} \) and also establish this indecomposability in a number of cases. We conjecture that any shuffling (or twisting) of a simple module in category \( \mathcal{O} \) in any Lie type is either indecomposable or zero.

2. Category \( \mathcal{O} \)

2.1. The Lie algebra and weights

We work over the ground field \( \mathbb{C} \) of complex numbers. Some of our results will be also valid outside type A. Therefore we introduce all main objects in wider generality and then, eventually, restrict to type A when it is necessary.

Let \( \mathfrak{g} \) be a reductive Lie algebra with a fixed triangular decomposition

\[
\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.
\] (1)

Here \( \mathfrak{h} \) is a fixed Cartan subalgebra and \( \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+ \) is the corresponding Borel subalgebra. We denote by \( R = R_+ \cup R_- \) the corresponding root system of \( \mathfrak{g} \) in \( \mathfrak{h}^* \) decomposed into positive and negative roots and by \( W \) the associated Weyl group. The half of the sum of all positive roots is denoted by \( \rho \in \mathfrak{h}^* \) and the \( W \)-invariant form on \( \mathfrak{h}^* \) is denoted \( \langle \cdot, \cdot \rangle \).

Let \( \Lambda \subset \mathfrak{h}^* \) denote the set of integral weights, that is weights which appear in finite dimensional \( \mathfrak{g} \)-modules. The natural partial order on \( \Lambda \) is defined by setting \( \lambda \geq \mu \) if and only if \( \lambda - \mu \) is a (possibly empty) sum of positive roots. The dominant weights form the subset

\[
\Lambda^+ = \{ \lambda \mid \langle \lambda + \rho, \alpha \rangle \geq 0, \text{ for all } \alpha \in R_+ \}.
\]

The subset \( \Lambda^{++} \) of regular dominant weights in \( \Lambda^+ \) consists of those weights for which the above inequality is strict.

For a given parabolic subalgebra \( \mathfrak{p} \) satisfying \( \mathfrak{b} \subset \mathfrak{p} \subset \mathfrak{g} \), we have the corresponding parabolic decomposition

\[
\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}^+,
\]
where $I$ is the *Levi subalgebra* of $p$ and $u^+$ is the *nilpotent radical* of $p$. As $\langle \cdot, \cdot \rangle$ is non-degenerate, there exists an element $H_I \in h$, such that $\lambda(H_I) = (\lambda, \rho(l))$, with $\rho(l)$ being the half of the sum of all positive roots of $I$. It then follows that the action of $\text{ad} H_I$ on $g$ yields a $\mathbb{Z}$-grading where

$$g_0 = I, \quad u^- = \bigoplus_{i < 0} g_i \quad \text{and} \quad u^+ = \bigoplus_{i > 0} g_i.$$  

For a Lie algebra $a$, we denote by $U(a)$ the universal enveloping algebra of $a$. We set $U := U(g)$.

### 2.2. The Weyl group and parabolic subgroups

The Weyl group $W$ is partially ordered with respect to the *Bruhat order* $\preceq$, see [Hu, Section 0.4]. We choose the convention where the identity element $e$ of $W$ is the minimum element and the *longest element* $w_0$ in $W$ is the maximum element. The *length function* on $W$ is denoted by $\ell$ and the set of *simple reflections* in $W$ is denoted by $S$.

We consider the *dot action* of $W$ on $h^*$ given by $w \cdot \lambda = w(\lambda + \rho) - \rho$, for $w \in W$ and $\lambda \in h^*$. The stabilizer of $\lambda \in h^*$ with respect to this action is denoted by $W_\lambda \subset W$. Then we let $X_\lambda$ denote the set of longest representatives in $W$ of the cosets in $W/W_\lambda$.

Given a parabolic subalgebra $p$ of $g$ as in the previous subsection, the Weyl group of $I$ with respect to $h$ is denoted by $W_p$ and has longest element $w_0^p$. The set of shortest coset representatives in $W/W_p$ is denoted by $X_p$ and the corresponding set for $W_p \backslash W$ is denoted by $pX$.

We now define another partial order on $\Lambda$. This one only has relations inside Weyl group orbits. For any $\lambda \in \Lambda^+$, we set

$$w_1 \cdot \lambda \preceq w_2 \cdot \lambda \iff w_1 \geq w_2, \quad \text{for all } w_1, w_2 \in X_\lambda.$$  

The partial order $\preceq$ is also generated by the relations $r \cdot \mu \prec \mu$, where $\mu \in \Lambda$ and $r \in W$ is a (not necessarily simple) reflection such that $r \cdot \mu \leq \mu$.

A subset $K \subset \Lambda$ is called *saturated* if it is an ideal for the partial order $\preceq$. Concretely, for any $\lambda \in K$ and $\mu \preceq \lambda$, the requirement is that $\mu \in K$.

For $w \in W$, the *support* $\text{supp}(w)$ of $w$ is the set of simple reflections which appear in (any) reduced decomposition of $w$. 
2.3. BGG category \( \mathcal{O} \)

Consider the *BGG category* \( \mathcal{O} \) associated to the triangular decomposition \([1]\), see \([\text{BGG2, Hu, Ja}]\). Simple objects in \( \mathcal{O} \) are, up to isomorphism, simple highest weight modules \( L(\mu) \), where \( \mu \in \mathfrak{h}^* \). The module \( L(\mu) \) is the simple top of the Verma module \( \Delta(\mu) \) and has highest weight \( \mu \). The projective cover of \( L(\mu) \) in \( \mathcal{O} \) is denoted \( P(\mu) \). The injective envelope of \( L(\mu) \) in \( \mathcal{O} \) is denoted \( I(\mu) \).

We will only consider the *integral part* \( \mathcal{O}_\Lambda \) of \( \mathcal{O} \) which consists of all modules with weights in \( \Lambda \). The category \( \mathcal{O}_\Lambda \) decomposes as a direct sum of its *indecomposable* blocks as follows:

\[
\mathcal{O}_\Lambda = \bigoplus_{\lambda \in \Lambda^+} \mathcal{O}_\lambda,
\]

where \( \mathcal{O}_\lambda \), for \( \lambda \in \Lambda^+ \), is the Serre subcategory of \( \mathcal{O} \) generated by all simples of the form \( L(x \cdot \lambda) \), where \( x \in X_\lambda \). For \( \lambda = 0 \), the corresponding block \( \mathcal{O}_0 \) is called the *principal block*. Note that the orbit \( W \cdot 0 \) is regular and hence isomorphism classes of simples in \( \mathcal{O}_0 \) are in bijection with elements in \( W \). For \( w \in W \), we will often denote \( L(w \cdot 0) \) simply by \( L(w) \) and similarly for all other structural modules.

By \([\text{Hu} \text{, Theorem 5.1}]\), for all \( \mu, \nu \in \Lambda \), we have

\[
(2) \quad [\Delta(\mu) : L(\nu)] \neq 0 \iff \nu \leq \mu \iff \Delta(\nu) \subset \Delta(\mu).
\]

For any \( K \subset \Lambda \), we consider the Serre subcategory \( \mathcal{O}^K \) of \( \mathcal{O}_\Lambda \) generated by \( \{ L(\mu) \mid \mu \in K \} \). The projective cover of \( L(\mu) \) in \( \mathcal{O}^K \) will be denoted by \( P^K(\mu) \) and is, by construction, the largest quotient of \( P(\mu) \) which belongs to \( \mathcal{O}^K \). If \( K \) is saturated, then \( \Delta(\mu) \in \mathcal{O}^K \) if and only if \( \mu \in K \).

2.4. Projective functors

A *projective functor* on \( \mathcal{O} \) is a direct summand of a functor of the form \(- \otimes V\), where \( V \) is a finite dimensional \( \mathfrak{g} \)-module. By \([\text{BG} \text{, Theorem 3.3}]\), isomorphism classes of indecomposable projective functors on \( \mathcal{O}_0 \) are in bijection with elements in \( W \). For \( w \in W \), we denote by \( \theta_w \) the unique (up to isomorphism) indecomposable projective functor on \( \mathcal{O}_0 \) such that

\[
(3) \quad \theta_w \Delta(\varepsilon) \cong P(w).
\]
For any \( w \in W \), the pair \((\theta_w, \theta_{w^{-1}})\) is an adjoint pair of functors, see [BG, Section 3]. By [Ja, 4.12], for any \( w \in W \) and \( s \in S \) such that \( \ell(ws) > \ell(w) \), we have an isomorphism and a short exact sequence as follows:

\[
\theta_s \Delta(w) \cong \theta_s \Delta(ws), \quad \text{and} \quad 0 \to \Delta(w) \to \theta_s \Delta(w) \to \Delta(ws) \to 0.
\]

Let \( \leq_L, \leq_R \) and \( \leq_J \) denote the Kazhdan-Lusztig left, right and two-sided pre-orders on \( W \), respectively. Let \( \sim_L, \sim_R \) and \( \sim_J \) denote the corresponding equivalence classes (also known as cells), see [KL] for details. For \( x, y \in W \), we have

\[
\theta_x L(y) \neq 0 \quad \text{if and only if} \quad x^{-1} \leq_L y \quad \text{if and only if} \quad x \leq_R y^{-1},
\]

see [KiM1, equation (1)].

### 2.5. Twisting functors

For every simple reflection \( s \) in \( W \), we have the corresponding right exact twisting functor \( T_s \) on \( O_\Lambda \), see e.g. [Ar, AS, KhM]. By [AS, Lemma 2.1(5)], we have

\[
T_s \theta \cong \theta T_s,
\]

for any projective functor \( \theta \). For any \( \mu \in \Lambda \) with \( s \cdot \mu \leq \mu \), we have

\[
T_s \Delta(\mu) \cong \Delta(s \cdot \mu),
\]

see e.g. [CM, Lemma 5.7]. In general, for any \( \mu \in \Lambda \), we have

\[
[T_s \Delta(\mu)] = [\Delta(s \cdot \mu)]
\]

in the Grothendieck group of \( O \), see [AS, Lemma 2.1(3)].

By [KhM, Theorem 2], twisting functors satisfy braid relations, which allows us to unambiguously define

\[
T_w := T_{s_1} T_{s_2} \cdots T_{s_k},
\]

where \( w = s_1 \cdots s_k \) is a reduced expression. By [AS, Theorem 2.2 and Corollary 4.2], \( T_w \) induces an isomorphism

\[
T_w : \text{Hom}_O(M, N) \to \text{Hom}_O(T_w M, T_w N)
\]
for any two modules $M, N$ with $\Delta$-flag. More generally, the left derived functor of $T_w$ is an autoequivalence of $D^b(O)$.

The functor $T_s$, where $s \in S$, is defined in [Ar] via tensoring with a certain semi-infinite $U$-$U$-bimodule. This means that $T_w$ is defined as a functor on the category of all $U$-modules. Furthermore, the definition is applicable to any Lie algebra with a fixed $\mathfrak{sl}_2$-subalgebra. From the definition in [Ar] Sections 2.1 and 2.3, we have that

$$T_s \circ \text{Ind}^g_k \cong \text{Ind}^g_k \circ T_s,$$

for any subalgebra $\mathfrak{k} \subset g$ which contains the $\mathfrak{sl}_2$-subalgebra corresponding to the simple reflection $s$.

Let $G_s$, where $s \in S$, denote Joseph’s completion functor which is right adjoint to $T_s$, see [KhM]. For a reduced expression $w = s_1s_2 \cdots s_k$, we define

$$G_w := G_{s_k} \cdots G_{s_2} G_{s_1},$$

and have that $(T_w, G_w)$ is an adjoint pair.

### 2.6. Graded versions

The principal origin of the grading on $O$ comes from the center of $O$ which is isomorphic to the coinvariant algebra

$$C := S(h)/\langle S(h)W \rangle$$

naturally graded by setting the degree of $h$ to be 2.

We denote by $O^\mathbb{Z}$ the $\mathbb{Z}$-graded version of $O$ where in each block of $O$ we fix the corresponding Koszul grading, see [BGS]. We denote by (1) the functor which decreases the grading by 1.

All structural modules (simples, projectives, Vermas etc.) admit graded lifts. We fix the standard graded lift, which we will denote by the same symbol as the corresponding ungraded module, as follows:

- for simple modules, their standard graded lifts are concentrated in degree zero;
- for Verma modules, their standard graded lifts are such that the natural projection onto the simple top is homogeneous of degree zero;
- for projectives modules, their standard graded lifts are such that the natural projection onto the simple top is homogeneous of degree zero.
Both projective and twisting functors admit graded lifts. For $\theta_w$, where $w \in W$, we fix its standard graded lift, which we will denote by the same symbol, such that (3) holds in $O^Z$. For $T_s$, where $s \in S$, we fix its standard graded lift, which we will denote by the same symbol, such that (7) holds in $O^Z$. By (9), this uniquely defines standard graded lifts of all $T_w$, where $w \in W$. We refer the reader to [MO, Ma2, KiM1] for details.

### 2.7. Shuffling functors

The shuffling functor $C_s$ corresponding to a simple reflection $s$, see [Ca, MS2], is the endofunctor of $O_0$ defined as the cokernel of the adjunction morphism from the identity functor to the projective functor $\theta_s$. The graded lift of $C_s$ is defined by the exact sequence

$$\text{Id}(-1) \to \theta_s \to C_s \to 0.$$

For any $w \in W$ with reduced expression $w = s_1 s_2 \cdots s_m$, we can define the functor

$$C_w = C_{s_m} C_{s_{m-1}} \cdots C_{s_1},$$

where the resulting functor does not depend on the choice of a reduced expression, see [MS1] Lemma 5.10 or [KhM, Theorem 2] and [MOS, Section 6.5]. The functor $C_w$ is right exact and the corresponding left derived functor $\mathcal{L}C_w$ is an auto-equivalence of $D^b(O_0)$, see [MS1, Theorem 5.7]. The cohomology functors of $C_w$ vanish on modules with Verma flag, see [MS1, Proposition 5.3].

### 3. Some special cases and general reductions

#### 3.1. Koszul-Ringel duality reduction

For $M \in \mathcal{O}$, denote by $n(M)$ the number of indecomposable summands of $M$. Then Conjecture 1 can be reformulated as the conjecture that, in type A, the function

$$f : W \times W \to \{0, 1, 2, \ldots \}$$

defined via $f(x, y) := n(\theta_x L(y))$,

has only values 0 and 1.

By [So2], the category $\mathcal{O}_0$ is Koszul self-dual. By [So3], the category $\mathcal{O}_0$ is Ringel self-dual. By [Ma2, Theorem 16], the composition of Koszul and Ringel dualities on $D^b(\mathcal{O}_0)$ maps $\theta_x L(y)$ to $\theta_y^{-1} w_0 L(w_0 x^{-1})$. Both of
these dualities are equivalences of certain derived categories which are Krull-Schmidt. Therefore, for all \( x, y \in W \), we have

\[
(12) \quad f(x, y) = f(y^{-1}w_0, w_0x^{-1}).
\]

### 3.2. Invariance under Kazhdan-Lusztig cells

**Proposition 2.** Assume that we are in type A. Let \( x, x', y, y' \in W \) be such that \( x \sim_R x' \) and \( y \sim_L y' \). Then \( f(x, y) = f(x', y') \).

**Proof.** That \( f(x', y) = f(x', y') \), is proved in the proof of [KiM1, Proposition 11]. Applying (12) to this statement (for all \( x', y \) and \( y' \)), we get \( f(x,y) = f(x',y) \) and the claim follows. \( \square \)

Let \( I(W) \) denote the set of all involutions in \( W \). Let \( I'(W) \) denote the set of all elements in \( W \) of the form \( w w_0 \), where \( w \in I(W) \).

**Corollary 3.** Assume that \( g \) is of type A. The following claims are equivalent.

(a) Conjecture [1] is true.

(b) The assertion of Conjecture [1] is true for all \( x, y \in I(W) \).

(c) The assertion of Conjecture [1] is true for all \( x \in I(W) \) and \( y \in I'(W) \).

**Proof.** In type A, any left and any right cell contains a unique involution (the Duflo involution). If \( L \) is a left (right) cell, then \( Lw_0 \) is a left (right) cell as well, see [BB, Page 179]. Therefore any left and any right cell contains a unique element in \( I'(W) \). Hence equivalence of (a)–(c) follows directly from Proposition 2. \( \square \)

### 3.3. Special cases

In [KiM1, Section 5], the following special cases (here \( x, y \in W \)) are given:

- In type A, from \( x \sim_J y \) it follows that \( f(x,y) \in \{0,1\} \).
- In any type, from \( x = w_0^p \), for some \( p \), it follows that \( f(x,y) \in \{0,1\} \).
- In any type, from \( y = w_0^p w_0, \) for some \( p \), it follows that we have \( f(x,y) \in \{0,1\} \) (combine the previous case with (12)).
- In any type, from \( y \in I(W) \) and \( y \sim_R w_0^p w_0 \), for some \( p \), it follows that \( f(x,y) \in \{0,1\} \).
3.4. Regular vs singular blocks

One can also ask the question of whether $\theta L(\lambda)$ is indecomposable or zero for any indecomposable projective functor $\theta$ and any simple highest weight module $L(\lambda)$. Conjecture \cite{1} formally refers only to regular $\lambda$. The general case, however, follows easily from the regular one.

**Lemma 4.** Assume that we are in type A. If Conjecture \cite{1} is true, then $\theta L(\lambda)$ is indecomposable or zero for any indecomposable projective functor $\theta$ and any simple highest weight module $L(\lambda)$.

**Proof.** If $\lambda$ is singular, we may write $L(\lambda)$ as the translation to the wall of a regular simple highest weight module, say $L(\lambda) = \theta^{\text{out}} L(\mu)$. If we assume $\theta L(\lambda) \cong M \oplus N$ with non-zero $M$ and $N$, then, translating $M$ and $N$ out of the wall, we get a non-trivial decomposition $\theta^{\text{out}} M \oplus \theta^{\text{out}} N$. This means that $\theta^{\text{out}} \theta^{\text{out}} L(\mu)$ decomposes. As the projective functor $\theta^{\text{out}} \theta^{\text{out}}$ is indecomposable (as it sends the dominant Verma module to an indecomposable projective module, which is the translation out of the wall of an indecomposable projective module in our singular block), the claim follows. \qed

4. Homological approach

To simplify notation, in this section we write Hom and Ext instead of $\text{Hom}_\mathcal{O}$ and $\text{Ext}_\mathcal{O}$, respectively.

4.1. Via twisted projective modules

Recall that a module is indecomposable if and only if its endomorphism algebra is local. Therefore, to prove that some module is indecomposable it is enough to show that the endomorphism algebra of this module is positively graded with one-dimensional degree zero component. This ensures that the endomorphism algebra is local and can be checked on the level of graded vector spaces.

For $w \in W$, we will denote by $K(w)$ the kernel of the natural projection $\Delta(w) \twoheadrightarrow L(w)$. Note that $K(w)$ is naturally graded and lives in strictly positive degrees.

**Proposition 5.** Assume that $\mathfrak{g}$ is of any type and let $x, y \in W$. Assume that the graded vector space $\text{Ext}^1(\theta_x \Delta(y), \theta_x K(y))$ is either zero or lives in strictly positive degrees. Then $f(x, y) \in \{0, 1\}$. 


Proof. Applying the bifunctor Hom(\(-, -\)) to the short exact sequence
\[
\theta_x K(y) \hookrightarrow \theta_x \Delta(y) \twoheadrightarrow \theta_x L(y),
\]
we get the following diagram in which the bottom row is exact:
\[
\begin{array}{ccc}
\text{Hom}(\theta_x L(y), \theta_x L(y)) & \rightarrow & \text{Hom}(\theta_x \Delta(y), \theta_x \Delta(y)) \\
\downarrow & & \downarrow \\
\text{Hom}(\theta_x \Delta(y), \theta_x L(y)) & \rightarrow & \text{Ext}^1(\theta_x \Delta(y), \theta_x K(y))
\end{array}
\]
Using Subsection 2.5, we have:
\[
\text{End}(\theta_x \Delta(y)) \cong \text{End}(\theta_x T_y \Delta(e)) \\
\cong \text{End}(T_y \theta_x \Delta(e)) \\
\cong \text{End}(T_y P(x)) \\
\cong \text{End}(P(x)).
\]
As \(P(x)\) is an indecomposable projective module over a Koszul algebra, \(\text{End}(P(x))\) is positively graded and local. In particular, it lives in non-negative degrees and the degree zero component is one-dimensional.

If \(\text{Ext}^1(\theta_x \Delta(y), \theta_x K(y)) = 0\) or if \(\text{Ext}^1(\theta_x \Delta(y), \theta_x K(y))\) lives in strictly positive degrees, we obtain that \(\text{End}(\theta_x L(y))\) is finite dimensional, non-negatively graded and the degree zero component is one-dimensional. Therefore \(\text{End}(\theta_x L(y))\) is local and the claim follows. \(\square\)

The observations below should be compared with the results of Subsection 3.3.

Lemma 6. Assume that \(\mathfrak{g}\) is of any type and let \(x, y \in W\) be such that \(x = w_0^p\), for some \(p\). Then we have
\[
\text{Ext}^1(\theta_x \Delta(y), \theta_x K(y)) = 0 = \text{Ext}^1(\theta_x \Delta(y), K(y)).
\]
Proof. The functor \(\theta_x = \theta_{w_0^p}\) is the translation functor through the intersection of all walls which correspond to simple reflections of \(I\). This functor can be factorized \(\theta_x = \theta^{out} \theta^{on}\), where \(\theta^{on}\) is the translation to this intersection and \(\theta^{out}\) is the translation out of this intersection. The two latter functors are biadjoint and the composition \(\theta^{on} \theta^{out}\) is isomorphic to a direct sum of \(|W^p|\) copies of the identity functor for the singular block on the intersection.
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of these walls. Therefore, using biadjunction and additivity, it is enough to prove that $\text{Ext}^1(\theta_x \Delta(y), K(y)) = 0$. Again, using biadjunction, we have

$$\text{Ext}^1(\theta_x \Delta(y), K(y)) = \text{Ext}^1(\theta^{\text{on}} \Delta(y), \theta^{\text{on}} K(y)).$$

Now, $\theta^{\text{on}} \Delta(y)$ is a Verma module in the singular block. If $\theta_x L(y) = 0$, then $\theta_x \Delta(y) = \theta_x K(y)$ and we get the assertion of the lemma due to the fact that Verma modules have no self-extensions in $\mathcal{O}$. If $\theta_x L(y) \neq 0$, we note that each composition subquotient $L(z)$ of $K(y)$ satisfies $z \succ y$ and this is preserved by translation to the singular block. Therefore the assertion of the lemma follows from the fact that the singular block is a highest weight category and hence $\theta^{\text{on}} \Delta(y)$ is relatively projective with respect to $\theta^{\text{on}} K(y)$ because of the ordering property above. □

**Lemma 7.** Assume that $\mathfrak{g}$ is of any type and let $x, y \in W$ be such that $y = w_0^p w_0$, for some $p$. Then $\text{Ext}^1(\theta_x \Delta(y), K(y)) = 0$.

**Proof.** Under our assumptions on $y$, we may consider the BGG resolution of $K(y)$, that is a complex

$$X_\bullet : 0 \to X_{-k} \to \cdots \to X_{-1} \to X_0 \to 0$$

obtained, using the parabolic induction and then truncation, from the classical BGG-resolution, see [BGG1]. Here

$$X_0 = \bigoplus_{w \in W : \ell(w) = 1} \Delta(\ell y), \quad X_{-1} = \bigoplus_{w \in W : \ell(w) = 2} \Delta(\ell y), \quad \ldots$$

Note that each $\Delta(y')$ appearing in this resolution satisfies $y' \succ y$ and can be written as $T_y \Delta(y'')$, where $y'' \preceq w_0^p w_0$. Therefore $X_\bullet$ has the form $T_y X_\bullet$ for some complex $X_\bullet$ of Verma modules concentrated in non-positive positions. Now, using Subsection 2.5, $\text{Ext}^1(\theta_x \Delta(y), K(y))$ can be rewritten as

$$\text{Hom}_{\mathcal{D}^b(\mathcal{O})}(P(x), Y_\bullet[1])$$

and the latter is zero as the shifted complex $Y_\bullet[1]$ has only zero in position zero. □

Lemma 7 implies, by adjunction, that $\text{Ext}^1(\theta_x \Delta(y), \theta_{x'} K(y)) = 0$, for all $x' \in W$ and all $x, y$ as in the formulation of the lemma.

The above observations naturally raise the following questions:
Question 8. For which \( x, y \in W \) does the space \( \text{Ext}^1(\theta_x \Delta(y), \theta_x K(y)) \) live in strictly positive degrees?

Question 9. For which \( x, y \in W \) do we have \( \text{Ext}^1(\theta_x \Delta(y), \theta_x K(y)) = 0 \)?

Question 10. For which \( x, y \in W \) do we have \( \text{Ext}^1(\theta_x \Delta(y), K(y)) = 0 \)?

None of these questions seems to be easy in full generality, for instance, due to the following example.

Example 11. For \( g = \mathfrak{sl}_3 \), we have \( W = \{ e, s, t, st, ts, w_0 \} \). Using the adjunction \((T_s, G_s)\), we obtain that \( \text{Ext}^1(\theta_{ts} \Delta(s), K(s)) \) is isomorphic to \( \text{Hom}(P(ts), \mathcal{R}^1 G_s K(s)) \), where \( \mathcal{R}^1 G_s \) is the functor of taking the maximal \( s \)-finite quotient. In the case of \( K(s) \), this quotient is isomorphic to \( L(ts) \). Tracking the grading, one sees that this \( L(ts) \) does live in degree zero. Hence \( \text{Ext}^1(\theta_{ts} \Delta(s), K(s)) \) is one-dimensional and is concentrated in degree zero. At the same time, as \( \theta_{ts} L(ts) = \theta_s \theta_t L(ts) = 0 \), it follows that \( \text{Ext}^1(\theta_{ts} \Delta(s), \theta_{ts} K(s)) = 0 \).

We finish this subsection with the following observation.

Proposition 12. Assume that \( g \) is of any type and let \( x, y \in W \). Then the graded vector space \( \text{Ext}^1(\theta_x \Delta(y), K(y)) \), if non-zero, lives in non-negative degrees.

Proof. By adjunction, we can rewrite \( \text{Ext}^1(\theta_x \Delta(y), K(y)) \) as

\[
\text{Hom}_{D^b(O)}(P(x), \mathcal{R}G_y K(y)[1])
\]

and hence reduce the statement to that of module \( \mathcal{R}^1 G_y K(y) \) living in non-negative degrees. Note that \( K(y) \) lives in strictly positive degrees, by construction.

By [AS, KhM, MS2], the functor \( \mathcal{R}G_s \) has derived length two with the zero component \( G_s \) and the first homology component \( Z_s(1) \), where \( Z_s \) is the corresponding Zuckerman functor of taking the maximal \( s \)-finite quotient. Applying \( G_s \) to a module concentrated in degree 0, either gives 0 or a module concentrated in degrees 0 and 1 (this is the dual of [AS, Section 6]). Therefore \( G_s \) cannot lower the minimum non-zero degree of a graded module. In turn, \( Z_s(1) \) may lower this minimum degree by one.

Now, writing \( \mathcal{R}G_y \) as a composition of various \( \mathcal{R}G_s \), where \( s \) is a simple reflection, we can use the previous paragraph and the fact that \( K(y) \) lives
in strictly positive degrees to deduce that $R^1G_yK(y)$ lives in non-negative degrees. □

4.2. By induction with respect to the Bruhat order

In this subsection we work in type A.

A (much more subtle) variation of the approach described in the previous subsection is to prove Conjecture 1 by the downward induction on $y$ with respect to the Bruhat order. The basis of the induction, that is the case $y = w_0$, is covered by Subsection 3.3. So, the trick is to prove the induction step. Note also that, by Ringel self-duality of $O_0$, we also know that each algebra $\text{End}(\theta_xL(w_0))$ is non-negatively graded with one dimensional zero component.

Let $y \in W$ and $s \in S$ be such that $sy \preceq y$. Then $\text{Ext}^1(L(sy), L(y))$ is one-dimensional. Let $M_{y,s}$ be an indecomposable module which fits into a short exact sequence

$$0 \rightarrow L(y) \rightarrow M_{y,s} \rightarrow L(sy) \rightarrow 0.$$  

(13)

Due to Proposition 2 without loss of generality we may assume that $y$ and $sy$ belong to different left cells and hence also to different two-sided cells.

**Theorem 13.** Assume that we are in type A. Assume that $y$ and $s$ are as above and $x \in W$. Then, under the vanishing condition

$$\text{Ext}^1(\theta_xM_{y,s}, \theta_xL(y)) = 0,$$

the fact that $\text{End}(\theta_xL(y))$ is non-negatively graded with one dimensional zero component implies that $\text{End}(\theta_xL(sy))$ is non-negatively graded with one dimensional zero component.

**Proof.** Applying the bifunctor $\text{Hom}(\_, \_)$ to the image of the short exact sequence (13) under $\theta_x$, we obtain the following commutative diagram:

\[
\begin{array}{cccc}
\text{Hom}(\theta_xL(sy), \theta_xL(y)) & \rightarrow & \text{Hom}(\theta_xL(sy), \theta_xM_{y,s}) & \rightarrow & \text{Hom}(\theta_xL(sy), \theta_xL(sy)) \\
\downarrow & & \downarrow & & \downarrow \\
\text{Hom}(\theta_xM_{y,s}, \theta_xL(y)) & \rightarrow & \text{Hom}(\theta_xM_{y,s}, \theta_xM_{y,s}) & \rightarrow & \text{Hom}(\theta_xM_{y,s}, \theta_xL(sy)) \\
\downarrow & & \downarrow & & \downarrow \\
\text{Hom}(\theta_xL(y), \theta_xL(y)) & \rightarrow & \text{Hom}(\theta_xL(y), \theta_xM_{y,s}) & \rightarrow & \text{Hom}(\theta_xL(y), \theta_xL(sy)) \\
\end{array}
\]
By our assumptions, \( sy < J y \). Recall that all top components and all socular components of \( \theta_x L(y) \) belong to the right cell of \( L(y) \) and all composition subquotients of \( \theta_x L(y) \) are less than or equal to \( y \) with respect to the right order. Further, each simple in the socle of \( \theta_x I(y) \) is greater than or equal to \( y \) with respect to the two-sided order. It follows that \( \theta_x L(sy) \) cannot contribute to the socle of \( \theta_x M_{y,s} \) and that none of the socular or top components of \( \theta_x L(y) \) appears in \( \theta_x L(sy) \). Taking all this into account, we get

\[
\text{(14)} \quad \text{Hom}(\theta_x L(sy), \theta_x L(y)) = \text{Hom}(\theta_x L(sy), \theta_x M_{y,s}) = \text{Hom}(\theta_x L(y), \theta_x L(sy)) = 0.
\]

Therefore

\[
\text{End}(\theta_x L(sy)) \cong \text{Hom}(\theta_x M_{y,s}, \theta_x L(sy))
\]

and, under the assumption \( \text{Ext}^1(\theta_x M_{y,s}, \theta_x L(y)) = 0 \), we get a surjection

\[
\text{(15)} \quad \text{End}(\theta_x M_{y,s}) \rightarrow \text{End}(\theta_x L(sy)).
\]

Furthermore, \( \text{(14)} \) also yields

\[
\text{End}(\theta_x L(y)) \cong \text{Hom}(\theta_x L(y), \theta_x M_{y,s})
\]

and

\[
\text{End}(\theta_x M_{y,s}) \hookrightarrow \text{Hom}(\theta_x L(y), \theta_x M_{y,s}).
\]

Therefore \( \text{Hom}(\theta_x L(y), \theta_x M_{y,s}) \) is non-negatively graded with one dimensional zero component by the inductive assumption, which implies that \( \text{End}(\theta_x M_{y,s}) \) is non-negatively graded with one dimensional zero component and, via \( \text{Hom}(\theta_x L(y), \theta_x M_{y,s}) \), that the endomorphism algebra of \( \theta_x L(sy) \) is non-negatively graded with one dimensional zero component. \( \square \)

As an immediate consequence from Theorem 13 we have:

\textbf{Corollary 14.} Assume that \( \text{Ext}^1(\theta_x M_{y,s}, \theta_x L(y)) = 0 \), for all \( x, y \) and \( s \) as in Theorem 13. Then Conjecture 1 is true. \( \square \)

This motivates the following questions:

\textbf{Question 15.} For which \( x, y \in W \) and \( s \in S \) does \( \text{Ext}^1(\theta_x M_{y,s}, \theta_x L(y)) \) vanish?

\textbf{Question 16.} For which \( x, y \in W \) and \( s \in S \) does \( \text{Ext}^1(\theta_x M_{y,s}, L(y)) \) vanish?
These questions, in turn, motivate the following related, but not equivalent questions:

**Question 17.** For which \(x, y \in W\) do we have \(\text{Ext}^1(\theta_x L(y), \theta_x L(y)) = 0\)?

**Question 18.** For which \(x, y \in W\) do we have \(\text{Ext}^1(\theta_x L(y), L(y)) = 0\)?

Again, all the above questions can be answered in the special case \(x = w_0^p\), for some \(p\).

**Lemma 19.** Assume that we are in type A. If \(x = w_0^p\), for some \(p\), then

\[
\text{Ext}^1(\theta_x L(y), L(y)) = 0, \quad \text{for all } y \in W.
\]

In particular, we have the equality \(\text{Ext}^1(\theta_x L(y), \theta_x L(y)) = 0\).

**Proof.** Similarly to the proof of Lemma 6, it is enough to prove the first vanishing property \(\text{Ext}^1(\theta_x L(y), L(y)) = 0\). By adjunction, this is equivalent to

\[
\text{Ext}^1(\theta^m L(y), \theta^m L(y)) = 0.
\]

If \(\theta^m L(y) = 0\), we are done. In case we have \(\theta^m L(y) \neq 0\), the module \(\theta^m L(y)\) is a simple module in the singular block. As category \(\mathcal{O}\) has finite projective dimension, simple modules cannot have non-zero first extensions. The claim follows.

**Lemma 20.** Assume that we are in type A. If \(x = w_0^p\), for some \(p\), then

\[
\text{Ext}^1(\theta_x M_{y,s}, L(y)) = 0, \quad \text{for all } y \in W \text{ and } s \in S \text{ as above.}
\]

In particular, \(\text{Ext}^1(\theta_x M_{y,s}, \theta_x L(y)) = 0\).

**Proof.** As above, it is enough to prove \(\text{Ext}^1(\theta_x M_{y,s}, L(y)) = 0\) which is equivalent to \(\text{Ext}^1(\theta^m M_{y,s}, \theta^m L(y)) = 0\). If \(\theta^m L(y) = 0\), the claim is clear. If \(\theta^m L(sy) = 0\), the claim reduces to Lemma 19 due to exactness of \(\theta^m\). If \(\theta^m M_{y,s}\) has length two, then it is an indecomposable module and the claim follows from the fact that \(\text{Ext}^1(\theta^m L(sy), \theta^m L(y))\) is one-dimensional.
5. Higher representation theoretic approach

5.1. Basics on 2-categories and 2-representations

In this subsection we recall some basics on finitary 2-categories and their 2-representations following [MM1, MM3, MM5, MM6]. For generalities on 2-categories, we refer the reader to [McL].

A 2-category is a category enriched over the category of small categories. A finitary 2-category is a 2-category \( \mathcal{C} \) for which all relevant structural information is finite. In particular, such \( \mathcal{C} \) has finitely many objects, its morphisms categories are additive, \( \mathbb{C} \)-linear Krull-Schmidt categories with finitely many isomorphism classes of indecomposable objects and finite dimensional morphism spaces. All identity 1-morphisms in \( \mathcal{C} \) are indecomposable.

Functors between 2-categories preserving all relevant 2-structure are called 2-functors. A 2-representation of a 2-category is a 2-functor to a suitable 2-category. Quite often, a 2-representation of a 2-category \( \mathcal{C} \) can be viewed as a functorial action of \( \mathcal{C} \) on some category. All 2-representations of \( \mathcal{C} \) form a 2-category where 1-morphisms are 2-natural transformations and 2-morphisms are modifications. For finitary 2-categories, the most natural 2-representations are functorial actions, by additive functors, on additive \( \mathbb{C} \)-linear Krull-Schmidt categories.

A finitary 2-category \( \mathcal{C} \) is called fiat if it has a weak involution and the corresponding adjunction morphisms.

For two indecomposable 1-morphisms \( F \) and \( G \) of a finitary 2-category \( \mathcal{C} \), one writes \( F \geq_L G \) provided that \( F \) is isomorphic to a direct summand of \( H \circ G \), for some 1-morphism \( H \). The defines the left preorder \( \geq_L \) and the corresponding equivalence classes are called left cells. The right preorder \( \geq_R \) and the corresponding right cells, and also the two-sided preorder \( \geq_J \) and the corresponding two-sided cells are defined similarly using multiplication from the right, or from both sides, respectively.

5.2. Ingredients

We denote by \( \mathcal{P} \) the fiat 2-category of projective functors acting on \( \mathcal{O}_0 \), see [MM1, Subsection 7.1] for details. Up to isomorphism, indecomposable 1-morphisms in \( \mathcal{P} \) are exactly \( \theta_w \), \( w \in W \). Following the conventions of [MM5], due to the right nature of the action of \( \mathcal{P} \) on \( \mathcal{O}_0 \), left cells of \( \mathcal{P} \) are indexed by right Kazhdan-Lusztig cells in \( W \), right cells of \( \mathcal{P} \) are indexed by left Kazhdan-Lusztig cells in \( W \) and two-sided cells of \( \mathcal{P} \) are indexed
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by two-sided Kazhdan-Lusztig cells in $W$. The notion of Duflo involution for $\mathcal{P}$, see [MM1, Section 4], corresponds to the notion of Duflo involution in $W$.

If $g$ is of type $A$, all two-sided cells in $\mathcal{P}$ are strongly regular in the sense that the intersection of any left and any right cell inside the same two-sided cell consists of exactly one indecomposable 1-morphism.

5.3. 2-subcategories of $\mathcal{P}$ generated by involutions

For $x \in I(W)$, we denote by $\mathcal{P}_x$ the 2-full 2-subcategory of $\mathcal{P}$ whose 1-morphisms are all endofunctors of $O_0$ which belong to the additive closure of all endofunctors of the form

$$\theta^k_x := \begin{cases} \theta_{k-1} \circ \theta_x & k \geq 1; \\ \theta_e & k = 0. \end{cases}$$

Lemma 21. Let $A$ be a finite dimensional, basic, non-simple, associative, unital algebra over an algebraically closed field $k$ and $\{e_1, e_2, \ldots, e_m\}$ a complete set of pairwise orthogonal primitive idempotents in $A$. Then, for any $i, j, k \in \{1, 2, \ldots, m\}$, we have

$$(Ae_i \otimes_k e_j A) \otimes_A (Ae_j \otimes_k e_k A) \cong (Ae_i \otimes_k e_k A)^{\otimes \dim(e_i, e_j, e_k)} \not= 0.$$ 

Proof. This is a direct computation which follows directly from the definitions. $\square$

We recall that, by [MM3, Theorem 13], given a fiat 2-category $\mathcal{C}$ and a strongly regular two-sided cell $\mathcal{J}$ in $\mathcal{C}$, there is a unique maximal 2-ideal $\mathcal{I}$ in $\mathcal{C}$ such that $\mathcal{J}$ becomes a unique maximal two-sided cell in the $\mathcal{J}$-simple quotient $\mathcal{C}/\mathcal{I}$. Furthermore, there is a finite dimensional algebra $A$ such that 1-morphisms in $\mathcal{J}$ in the quotient $\mathcal{C}/\mathcal{I}$ can be viewed as indecomposable projective functors for a finite dimensional associative algebra. With the notation as in Lemma 21 we have that left cells inside $\mathcal{J}$ have the form $\{Ae_i \otimes_k e_j A : i = 1, \ldots, m\}$, where $j$ is arbitrary, right cells inside $\mathcal{J}$ have the form $\{Ae_i \otimes_k e_j A : j = 1, \ldots, m\}$, where $i$ is arbitrary, and each left or right cell contains a unique Duflo involution, namely a 1-morphism of the form $Ae_i \otimes_k e_i A$. Finally, the adjoint of $Ae_i \otimes_k e_j A$ is given by $Ae_j \otimes_k e_i A$, for all $i$ and $j$. We refer to [MM5, Section 5] for details.
Proposition 22. The 2-category $\mathcal{P}_x$ is fiat. In type $A$, we additionally have that every two-sided cells of $\mathcal{P}_x$ is strongly regular.

Proof. As $x$ is an involution, $\theta_x$ is self-adjoint, and hence all $\theta^k_x$ are self-adjoint as well. This implies that $\mathcal{P}_x$ is stable under taking adjoint functors and hence, being a 2-full subcategory of a fiat 2-category, is fiat.

By [KiM2, Corollary 19], every two-sided cell of a fiat 2-category is regular. That the intersection of a left and right cell inside the same two-sided cell of a fiat 2-category is non-empty, is proved in [MM1, Proposition 28]. That, in type $A$, such an intersection for $\mathcal{P}_x$ cannot contain more than one element follows from the corresponding property of $\mathcal{P}$. This shows that any two-sided cell of $\mathcal{P}_x$ is strongly regular. The claim follows. $\square$

The example below shows how the case of $x = w_0^p$, for some $p$, distinguishes itself with respect to $\mathcal{P}_x$ among other involutions.

Example 23. If $x = w_0^p$, for some $p$, then $\theta_x \circ \theta_x \cong \theta^{\oplus |W_x|}_x$ and hence $\theta_x$ is the only indecomposable 1-morphism in $\mathcal{P}_x$, up to isomorphism, which is not isomorphic to the identity.

Example 23 should be compared with the following statement:

Lemma 24. Assume that $\mathfrak{g}$ is of any type. Let $x \in I(W)$ and $s_1, s_2, \ldots, s_k$ be the list of simple reflections which appear in a reduced decomposition of $x$. Let $W'$ be the parabolic subgroup of $W$ generated by these reflections and $w'_0$ be the longest element in $W'$. Then the 2-category $\mathcal{P}_x$ contains $\theta_{w'_0}$.

Proof. Without loss of generality we may assume $S = \{s_1, s_2, \ldots, s_k\}$, which means that $W = W'$ and hence $w_0 = w'_0$. It is enough to show that, for any $w \in W$, the Verma flag of $\theta_x \Delta(w)$ contains all $\Delta(ws)$, where $s \in S$ and $ws > w$. Indeed, then, by induction, we will have that $\theta^{(w_0)}_x \Delta(e)$ contains $\Delta(w_0)$ and hence $\theta_{w_0}$ is a direct summand of $\theta^{(w_0)}_x$.

That $\theta_x \Delta(e)$ contains all $\Delta(s)$, where $s \in S$, follows directly from our assumptions as they guarantee that $x > s$, for all $s \in S$. For an arbitrary $w$ the necessary claim is now obtained by applying $T_w$. $\square$

The above raises the following natural question:

Question 25. Which $\theta_z$, where $z \in W$, appear in $\mathcal{P}_x$, for a fixed $x \in I(W)$?
From [MM1, Section 5] we know that, for any $x \in I(W)$, we have

$$\theta_x \circ \theta_x = \theta_x^{\oplus k} \oplus \theta,$$

where $k = \dim \text{End}(\theta_x L(x)) > 0$ and each indecomposable direct summand of $\theta$ is strictly $J$-bigger that $\theta_x$. Further, if $\theta_z$ appears in $P_x$, then, clearly supp$(z) \subset$ supp$(x)$. As any element different from $w'_0$ and contained in the same $J$-cell as $w'_0$ must have strictly bigger support, it follows that all $\theta_z$ appearing in $P_x$ and different from $\theta_x$ and $\theta_{w'_0}$ have the property $x < J z < J w'_0$.

5.4. The case $x = w'_0$

This subsection provides higher representation theoretic proof for various, mostly already known, results. Thus the value of this subsection is in the corresponding methods and arguments, rather than the results. We hope that some of these ideas and methods could be extended to attack more general cases of Conjecture 1.

**Proposition 26.** Let $A$ be a finite dimensional, connected, associative and unital $k$-algebra and $F$ an indecomposable and self-adjoint endofunctor of $A$-mod satisfying $F \circ F \cong F \oplus k$, for some $k \in \{1, 2, \ldots\}$. Let, further, $L$ be a simple $A$-module such that $FL \neq 0$. Then we have the following:

(a) All indecomposable summands of $FL$ are isomorphic.

(b) For every simple $A$-module $L'$ appearing in the top or socle of $FL$, we have $\text{add}(FL) = \text{add}(FL')$.

(c) Every simple $A$-module $L'$ appearing in the top or socle of $FL$ is isomorphic to $L$.

**Proof.** Let $FL = M_1 \oplus M_2 \oplus \ldots \oplus M_m$, where all $M_i$ are indecomposable. If $F \cong \text{Id}_A$-mod, then the claim is clear, so we assume that $F \not\cong \text{Id}_A$-mod.

Let $\mathcal{C}$ be the 2-full 2-subcategory of the 2-category of right exact endofunctors of $A$-mod whose 1-morphisms are all endofunctors in $\text{add} \left( \text{Id}_A \text{-mod} \oplus F \right)$. Then $\mathcal{C}$ is fiat and has two two-sided cells, one consisting of $\text{Id}_A$-mod and the other one consisting of $F$. By [MM5, Theorem 18], the only simple transitive 2-representations of $\mathcal{C}$ are cell 2-representations. As every $M_i$ is in the image of $F$, this $M_i$ corresponds to the cell 2-representation of the cell containing $F$. Now, from [ChM, Theorem 25] or [MM6, Theorem 4], it follows that $FM_i \in \text{add}(M_i)$, for every $i$. 
Pick some $M_i$ and let $L'$ be a simple in the top of $M_i$. As $F$ is self-adjoint, by adjunction, $F L' \neq 0$, moreover $L$ appears in the socle of $F L'$ (and hence also in the top by the self-duality of $F L'$). Applying $F$ to $M_i \rightarrow L'$, we get

$$M_i^\oplus k \cong FM_i \rightarrow F L' \rightarrow L. \tag{16}$$

Applying $F$ to (16) again, we get

$$M_i^\oplus k \rightarrow F L \cong M_1 \oplus M_2 \oplus \ldots M_m.$$

In particular, each $M_j$ is a homomorphic image of $M_i^\oplus k$. Now we will need:

**Lemma 27.** Let $B$ be a finite dimensional algebra and $X$ an indecomposable $B$-module. Then, for any positive integer $n$, any surjection $X^\oplus n \rightarrow X$ splits.

*Proof.* Let $D$ denote the local endomorphism algebra of $X$ and $m$ the unique maximal ideal in $D$. Then any $\alpha : X^\oplus n \rightarrow X$ is given by a $1 \times n$ matrix with coefficients in $D$. If one of those coefficients is not in $m$, then this coefficient is an isomorphism and hence $\alpha$ is a surjection and is, obviously, split. In particular, if $m = 0$, then the claim is clear. So, in what follows we assume $m \neq 0$.

Assume now that $\alpha$ is a surjection and that all coefficients in the corresponding matrix are in $m$. As $m$ is nilpotent, it contains a non-zero element $f$ such that $f m = 0$. As $\alpha$ is a surjection, we have $f \circ \alpha \neq 0$. On the other hand, $fg = 0$ for any coefficient $g$ in the matrix of $\alpha$ as $g \in m$. This is a contradiction which implies our statement. \hfill $\square$

**Lemma 28.** Let $B$ be a finite dimensional algebra and $X$ and $Y$ indecomposable $B$-modules such that $X$ is a quotient of some $Y^\oplus m$ and $Y$ is a quotient of some $X^\oplus n$. Then $X \cong Y$.

*Proof.* By our assumptions, we have $Y^\oplus mn \rightarrow X^\oplus n \rightarrow Y$. By Lemma 27, the epimorphism $Y^\oplus mn \rightarrow Y$ splits. Composing this splitting with the map $Y^\oplus mn \rightarrow X^\oplus n$, we get a splitting for $X^\oplus n \rightarrow Y$. As $X$ is indecomposable, it follows that $X \cong Y$. \hfill $\square$

As each $M_j$ is a homomorphism image of $M_i^\oplus k$, from Lemma 28 we obtain that all $M_i$ are isomorphic. This proves claim (a).

To prove claim (b), let $L'$ be a simple module such that $F L \rightarrow L'$ or, equivalently $L' \hookrightarrow F L$. By claim (a), we have $F L \cong M^\oplus m$ and $F L' \cong N^\oplus m'$,
for some indecomposable $M$ and $N$ and some positive integers $m$ and $m'$. Applying $F$ to $FL \to L'$, we get $F \oplus kL \simeq F \circ FL \to FL'$, in particular, $M \oplus p \to N$, for some $p$.

By adjunction, $FL' \to L$. Therefore the previous paragraph gives that $N \oplus q \to M$, for some $q$. Therefore $M \simeq N$ by Lemma 28 implying claim (b).

The 2-category $C$ is fiat with strongly regular two-sided cells, it has two two-sided cells and hence two equivalence classes of cell 2-representations. One of these is annihilated by $F$, while, in the other one, we have $F \oplus k \to M$, where $P$ is the unique, up to isomorphism, indecomposable object.

Let $I$ be an indexing set of the isomorphism classes of indecomposable $A$-modules. Let $Q := (m_{i,j})_{i,j \in I}$ be the matrix describing the multiplicity of $P(i)$ in $F \oplus k \to M$ for $i, j \in I$. Then, from the previous paragraph and [ChM, Theorem 25], it follows that one can order elements in $I$ such that $Q$ has the form

\[
\begin{pmatrix}
0 & 0 \\
* & kE
\end{pmatrix},
\]

where $E$ is the identity matrix. By adjunction, the transpose of this matrix gives the matrix counting the decomposition multiplicities $[FL(i) : L(j)]$. Consequently, $FP(i) \cong P(i) \oplus k$ provided that $FL(i) \neq 0$. This means that $P(i) \oplus k \to FL(i)$ and proves claim (c).

Proposition 26 applies, in particular, to the situation when $A$-mod $\cong O_0$ and $F \cong \theta_{w^p_0}$, for some $p$. The proof of Proposition 26 raises the following natural questions.

**Question 29.** For which $x \in I(W)$ and $y \in W$, does the module $L(y)$ appear in the top of any summand in $\theta_xL(y)$?

**Question 30.** For which $x \in I(W)$ and $y \in W$, none of the summands in $\theta_xL(y)$ is isomorphic to a summand of $\theta_{x'}L(y)$, for some $x' \in I(W)$ such that $x < x'$ and $\theta_{x'}$ is a 1-morphism in $P_x$?

**Corollary 31.** Assume that $g$ is of any type. Assume that $x = w^p_0$, for some $p$. Let $y \in W$ be such that $\theta_xL(y) \neq 0$. Then the module $\theta_xL(y)$ is indecomposable.

**Proof.** Let us first consider the case $y = x$. As $x = w^p_0$, the 1-morphism $\theta_x$ is the Duflo involution in its left (and right) cell of $P$. Therefore $\theta_xL(x)$ is indecomposable and has simple top $L(x)$, see [MMI Subsection 4.5]. By self-duality, it also has simple socle $L(x)$. From the Kazhdan-Lusztig combinatorics we get that, as a graded module, $\theta_xL(x)$ is concentrated between
degrees $\pm \ell(w_p^0)$ with simple top and socle in the respective extremal degrees. As $O_0$ is Koszul, it follows that $\theta_x L(x)$ has Loewy length $2\ell(w_p^0) + 1$. The additive closure of $\theta_x L(x)$ carries, by [MM1, Subsection 4.5], the natural structure of a 2-representation of $P_x$ which is equivalent to the cell 2-representation of $P_x$ corresponding to the cell $\{\theta_x\}$.

Now consider the case of general $y$. From the Kazhdan-Lusztig combinatorics we get that, as a graded module, $\theta_x L(y)$ is concentrated between degrees $\pm \ell(w_p^0)$ and both extremal degrees are one-dimensional. As $O_0$ is Koszul, it follows that the Loewy length of $\theta_x L(y)$ is at most $2\ell(w_p^0) + 1$.

Assume that $\theta_x L(y)$ decomposes. Then, by Proposition 26(a), the module $\theta_x L(y)$ is a direct sum of several copies of the same indecomposable module $N$. We claim that the Loewy length of $N$ is strictly smaller than $2\ell(w_p^0) + 1$. By the previous paragraph, the Loewy length of $N$ is at most $2\ell(w_p^0) + 1$. If the Loewy length of $N$ were $2\ell(w_p^0) + 1$, then every submodule of codimension one in $N \oplus N$ would also have Loewy length $2\ell(w_p^0) + 1$. At the same time, the submodule

$$\bigoplus_{i=-\ell(w_p^0)+1}^{\ell(w_p^0)} (\theta_x L(y))_i$$

has Loewy length at most $2\ell(w_p^0)$, a contradiction.

The above means that both $N$ and $\theta_x L(y)$ have Loewy length at most $2\ell(w_p^0)$. At the same time, the additive closure of $N$ carries the structure of a 2-representation of $P_x$ whose unique simple transitive quotient is the cell 2-representation of $P_x$ corresponding to the cell $\{\theta_x\}$. Therefore some quotient of $N$ must have Loewy length $2\ell(w_p^0) + 1$ by the first paragraph of this proof to provide an equivalence with the cell 2-representation. This is a contradiction which completes the proof of this corollary. □

6. Equivalences

6.1. Serre subquotient categories

For an arbitrary abelian category $C$, a non-empty full subcategory $B$ of $C$ is called a Serre subcategory provided that, for every short exact sequence

$$0 \rightarrow Y_1 \rightarrow X \rightarrow Y_2 \rightarrow 0$$

in $C$, we have $X \in B$ if and only if $Y_1, Y_2 \in B$. 
For a Serre subcategory $B \subset C$, we have the *Serre quotient category* $C/B$, which is defined as follows:

- the objects of $C/B$ are those of $C$;
- for any $X,Y \in C$, we have

$$\text{Hom}_{C/B}(X,Y) := \lim_{\longrightarrow} \text{Hom}_C(X',Y/Y'),$$

where $X'$, resp. $Y'$, runs over all sub-objects in $C$ (ordered by inclusion) of $X$, resp. $Y$, such that $X/X' \in B$, resp. $Y' \in B$.

We have the corresponding exact functor $\pi : C \to C/B$, which is the identity on objects and maps a morphism $f : X \to Y$ to the corresponding element in the direct limit.

Assume now that $C = A$-$\text{mod}$, for $A$ a finite dimensional, associative and unital algebra over an algebraically closed field $k$. Any Serre subcategory $B$ is then of the form $A/(AeA)$-$\text{mod}$, for some idempotent $e \in A$. The corresponding Serre quotient $C/B$ is then equivalent to $eAe$-$\text{mod}$. In this case the functor $\pi$ is given by multiplication with $e$. In particular, we find that $\pi$ yields isomorphisms

$$\pi : \text{Hom}_C(P,Q) \to \text{Hom}_{C/B}(\pi P, \pi Q), \quad \text{for } P,Q \in \text{add}(Ae).$$

Now consider a right exact functor $F$ on $C = A$-$\text{mod}$, which, up to isomorphism, is of the form $X \otimes_A -$ for some $A$-$A$-bimodule $X$. Assume that $F$ restricts to a functor on the Serre subcategory corresponding to the idempotent $e$ as above, that is $eX(1-e) = 0$. Consequently, there is a right exact endofunctor $\overline{F} \cong eXe \otimes_{eAe} -$ of the Serre subquotient $eAe$-$\text{mod}$ induced by $F$. In other words, the following diagram commutes up to an isomorphism of functors:

$$\begin{array}{ccc}
C & \xrightarrow{F} & C \\
\downarrow{\pi} & & \downarrow{\pi} \\
C/B & \xrightarrow{\overline{F}} & C/B
\end{array}$$

For any right exact functor $F$ on $C = A$-$\text{mod}$, which preserves Serre subcategories $\mathcal{A} \subset \mathcal{B} \subset C$, we will refer to the right exact endofunctor $\overline{F}$ of $\mathcal{B}/\mathcal{A}$ obtained by restriction of $F$ to $\mathcal{B}$ followed by the above procedure simply as the *functor induced from* $F$. 
6.2. Equivalences intertwining twisting functors

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{l}$ the Levi subalgebra of $\mathfrak{p}$.
Let $\lambda \in \Lambda$ be $W^p$-dominant. We define the sets $I_\lambda$ and $J_\lambda$ by

$$I_\lambda = \{ \mu \in \Lambda | \mu \preceq \lambda \} = W^p \cdot \lambda \cap J_\lambda.$$ 

We consider the Serre subquotient $A_\lambda = O^I_\lambda / O^J_\lambda$ (here we use the notation from Subsection 2.3) and use the notation of (17) for induced functors on Serre quotients.

**Theorem 32.** Assume that $\mathfrak{g}$ is of any type. There exists an equivalence of categories

$$\Psi : O_\lambda(\mathfrak{l}) \sim A_\lambda,$$

such that the following diagram commutes, for any $w \in W^p$, up to isomorphism of functors:

$$
\begin{array}{ccc}
O_\lambda(\mathfrak{l}) & \xrightarrow{\text{Ind}_{\mathfrak{p}}^\mathfrak{g}} & O^I_\lambda \\
\downarrow{\Psi} & & \downarrow{\pi} \\
A_\lambda & \xrightarrow{T_w} & O^J_\lambda \\
\downarrow{\pi} & & \downarrow{\Psi} \\
O_\lambda(\mathfrak{l}) & \xrightarrow{\text{Ind}_{\mathfrak{p}}^\mathfrak{g}} & O^J_\lambda
\end{array}
$$

Here $\text{Ind}_{\mathfrak{p}}^\mathfrak{g}$ is the parabolic induction functor which is full and faithful.

We start with the following lemma.

**Lemma 33.** The sets $I_\lambda$ and $J_\lambda$ are saturated.

**Proof.** For $I_\lambda$, this is by construction. The claim for $J_\lambda$ is equivalent to the claim that $W^p \cdot \lambda$ forms an interval for $\preceq$. It is easy to see that for any $\mu_1 \preceq \mu_2$, we have $\mu_1(H_\lambda) \leq \mu_2(H_\lambda)$ with equality holding only if $\mu_1$ and $\mu_2$ are in the same $W^p$-orbit. This implies that any $W^p$-orbit is an interval. $\square$
**Lemma 34.** For any \( y \in W^p \), the twisting functor \( T_y \) on \( O_\lambda \) restricts to an endofunctor of both \( O^{I_\lambda} \) and \( O^{J_\lambda} \).

**Proof.** By induction, we can restrict to the case where \( y = s \) is a simple reflection in \( W^p \). Since the sets \( I_\lambda \) and \( J_\lambda \) are saturated and \( T_s \) is right exact, it suffices to show that \( T_s \Delta(\mu) \) is in \( O^{I_\lambda} \), resp. \( O^{J_\lambda} \), for all \( \mu \in I_\lambda \), resp. \( \mu \in J_\lambda \).

For any \( \mu \in I_\lambda \), equation (2) yields a short exact sequence

\[
0 \to \Delta(\mu) \to \Delta(\lambda) \to N \to 0,
\]

leading, by (7), to the exact sequence

\[
\mathcal{L}_s T_s N \to T_s \Delta(\mu) \to \Delta(s \cdot \lambda).
\]

The left term is in \( O^{I_\lambda} \) since \( \mathcal{L}_s T_s \) is the functor taking the maximal \( s \)-finite submodule of \( N \), see [CM, Proposition 5.10] or the adjoint of [MS2, Theorem 2.2], so \( \mathcal{L}_s T_s N \subset N \in O^{I_\lambda} \). Since the right term is also in \( O^{I_\lambda} \), we find \( T_s \Delta(\mu) \in O^{I_\lambda} \). This concludes the proof for \( I_\lambda \).

By equation (8), the above actually proves that the set \( I_\lambda \) is closed under the left \( W^p \)-action. By definition, this implies that also \( J_\lambda \) is closed under the left \( W^p \)-action. Using equation (8) again thus proves that \( T_s \Delta(\mu) \in O^{J_\lambda} \), for any \( \mu \in J_\lambda \), which concludes the proof. \( \square \)

**Proof of Theorem 32.** We set \( I = I_\lambda \) and \( J = J_\lambda \). We use parabolic induction \( \text{Ind}_p^g \) from \( l \)-modules (interpreted as \( p \)-modules with trivial action of \( u^+ \)) to \( g \)-modules, which yields a functor

\[
\text{Ind}_p^g : O_\lambda(l) \to O^l.
\]

This functor is full and faithful. Indeed, for any \( N_1, N_2 \in O_\lambda(l) \), by adjunction, we have

\[
\text{Hom}_{O^l}(\text{Ind}_p^g N_1, \text{Ind}_p^g N_2) \cong \text{Hom}_{O_\lambda(l)}(N_1, \text{Res}_p^g \text{Ind}_p^g N_2) \cong \text{Hom}_{O_\lambda(l)}(N_1, N_2)
\]

since \( \text{Res}_p^g \text{Ind}_p^g N_2 \cong N_2 \oplus X \) with \( \text{Hom}_{O_\lambda(l)}(N_1, X) = 0 \), where the latter follows by applying \( \text{ad}_{H^l} \).

Similarly we see that \( \text{Ind}_p^g \) is left adjoint to the composition of \( \text{Res}_p^g \) followed by taking the correct \( \text{ad}_{H^l} \)-eigenspace. As the latter functor is exact, it follows that \( \text{Ind}_p^g \) sends projectives to projectives. The easy observation \( [\text{Res}_p^g L(\nu) : L_i(\mu)] = \delta_{\nu \mu} \) implies that all projectives in \( O^l \) belong to the essential image of \( \text{Ind}_p^g \). As \( \text{Ind}_p^g \) is full and faithful, it is an equivalence of
categories. The statement on Verma modules follows from the definition of parabolic induction.

By (11), we have
\[ T_y \circ \text{Ind}^g_p \cong \text{Ind}^g_p \circ T_y, \]
for any \( y \in W_p \). It thus suffices to show that \( T_y \) restricts to \( O_I \) and \( O_J \), which is done in Lemma 34. □

**Corollary 35.** For any \( x \in pX, y \in X_p \) and \( w_1, w_2 \in W_p \), we have
\[ [\Delta(w_1 x) : L(w_2 x)] = [\Delta(w_1) : L(w_2)] = [\Delta(y w_1) : L(y w_2)]. \]

**Proof.** By Theorem 32, for \( \lambda = x \cdot 0 \), we have
\[ [\Delta(w_1 x) : L(w_2 x)] = [\Delta(l(w_1 x \cdot 0)) : L(l(w_2 x \cdot 0))]. \]

By equivalence of regular integral blocks in \( O_l \), see e.g. [Hu, Theorem 7.8], this number is equal to \([\Delta(l(w_1 \cdot 0)) : L(w_2 \cdot 0)]\). Applying Theorem 32 for \( \lambda = 0 \), then yields the first equality in the proposition. The second can be obtained from the first one and the equality
\[ [\Delta(u) : L(v)] = [\Delta(u^{-1}) : L(v^{-1})]. \]

The above is a well-known property of Kazhdan-Lusztig combinatorics. A direct proof is sketched below.

Consider the equivalence \( F : O_0 \to \infty H^1_0 \) of [BG, Theorem 5.9], where the \( \infty H^1_0 \) denotes the category of Harish-Chandra bimodules which admit generalized central character \( \chi_0 \) on the left and central character \( \chi_0 \) on the right, with \( \chi_0 \) being the central character of the trivial module.

By [Ja, Satz 6.34], \( F(L(v)) \) and \( F(L(v^{-1})) \) are linked by the duality on \( \infty H^\infty_0 \), or \( H^1_0 \), which exchanges the left and right action. Furthermore the description of \( \Delta(v) \) as the quotient of \( P(v) \) with respect to the submodule corresponding to all images of \( P(x) \to P(v) \), for \( x \prec v \), implies that \( F(\Delta(v)) \) and \( F(\Delta(v^{-1})) \) are linked by the same duality. □

**Lemma 36.** Consider the equivalence \( \Psi : O_0(l) \to A_0 \) of Theorem 32 for the case \( \lambda = 0 \). For any \( w \in W_p \), the diagram of functors
\[
\begin{array}{ccc}
O_0(l) & \xrightarrow{\Psi} & A_0 & \xleftarrow{\pi} & O_0 \\
\downarrow{\theta_w} & & \downarrow{\theta_w} & & \downarrow{\theta_w} \\
O_0(l) & \xrightarrow{\Psi} & A_0 & \xleftarrow{\pi} & O_0
\end{array}
\]
extending (17), commutes up to isomorphism.

Proof. That $\theta_w$ is well-defined, is a special case of Lemma 38 below. This proves existence of the right-hand side of the diagram.

To study projective functors on $O_0(l)$, it suffices to consider $- \otimes V_0$ for simple finite dimensional $l$-modules $V_0$ on which $H_l$ acts trivially. For any such simple module, we have the corresponding simple $g$-module $V$ with same highest weight.

Recall that we have $\Psi = \pi \circ \text{Ind}^g$. We can then calculate

$$\pi(\text{Ind}^g_{l}(M \otimes V_0)) \cong \pi(\text{Ind}^g_{l}(M \otimes \text{Res}^g_{l} V_0)) \cong \pi(\text{Ind}^g_{l}(M) \otimes V).$$

This shows that $\Psi \circ \theta_w \cong F \circ \Psi$, for some projective functor $F$ on $O_0(g)$. The identification of projective functors then follows immediately from equation (3) and the observation $\Psi(P_l(x)) \cong P(x)$, for all $x \in W^p$. $\square$

6.3. Equivalences intertwining projective functors

Now we fix an element $z \in X^p$. We define the sets $K_z$ and $L_z$ by

$$K_z = \{x \in W \mid z \preceq x\} = zW^p \amalg L_z.$$

Then we again use the notation from Subsection 2.3 to define the Serre subquotient

$$B_z = O_{K_z}/O_{L_z}.$$

Theorem 37. Assume that $g$ is of any type. There exists an equivalence

$$\Phi : O_0(l) \sim B_z,$$

with $\Phi(\Delta_l(y)) \cong \Delta(zy)$, for all $y \in W^p$.

Furthermore, for any $x \in W^p$, the diagram of functors

$$\begin{array}{ccc}
O_0(l) & \xrightarrow{\Phi} & B_z \\
\downarrow \theta_x & & \downarrow \theta_x \\
O_0(l) & \xrightarrow{\Phi} & B_z \\
\end{array}
$$

extending (17), commutes up to isomorphism.

We start with the following lemma.
Lemma 38. For any $x \in W^p$, the functor $\theta_x$ restricts to an endofunctor of both $O^{K_z}$ and $O^{L_z}$.

Proof. It suffices to consider $x = s$, a simple reflection in $W^p$. Since the sets $K_z$ and $L_z$ are saturated, we can just prove that $\theta_x \Delta(y)$ is in $O^{K_z}$, resp. $O^{L_z}$, for any $y \in K_z$, resp. $y \in L_z$. Since $\theta_x \Delta(y)$ is an extension of $\Delta(ys)$ and $\Delta(y)$ by (4), this is equivalent to showing that the sets $K_z$ and $L_z$ are closed under right $W^p$-multiplication. The latter follows from the easy observation that

$$K_z = \coprod_{u \in X^p | z \preceq u} uW^p.$$ 

The claim follows. \hfill \Box

Lemma 39. For any $x \in W^p$, the projective cover of $L(zx)$ in $O^{K_z}$ satisfies

$$P^{K_z}(zx) \cong T_z P(x) \cong \theta_x \Delta(z).$$

Proof. By [Hu, Theorem 3.11], we have a short exact sequence

$$0 \to M \to P(z) \to \Delta(z) \to 0,$$

where $M$ has a $\Delta$-flag with all Verma modules appearing of the form $\Delta(y)$ with $y < z$, so, in particular, $y \not\in K_z$. It thus follows by (7) that

$$P^{K_z}(z) \cong \Delta(z) \cong T_z P(e).$$

Consequently, by equations (3) and (6), we have

$$\theta_x P^{K_z}(z) \cong T_z P(x),$$

for any $x \in W^p$. By adjunction, the module $\theta_x P^{K_z}(z)$ is again projective. We claim that $\theta_x P^{K_z}(z) \cong P^{K_z}(zx)$. To show this we first observe that the top of the module $\theta_x P^{K_z}(z) \cong \theta_x \Delta(z)$ consists of simple modules of the form $L(zx')$ with $x' \in W^p$, by (4). Furthermore, since $\theta_x P^{K_z}(z) \cong T_z P(x)$, for any $x' \in W^p$, we have

$$\dim \text{Hom}_{O^{K_z}}(\theta_x P^{K_z}(z), \Delta(zx')) = \dim \text{Hom}_{O}(T_z P(x), T_z \Delta(x'))$$

$$= \dim \text{Hom}_{O}(P(x), \Delta(x'))$$

$$= [\Delta(zx') : L(zx)].$$

(18)

Here, we used equations (7), (10) and Corollary 35.
If \( P = \bigoplus_{y \in W^p} P^{K^z}(zy)^{\oplus m_y} \) is some projective in \( \mathcal{O}^{K^z} \), then, by projectivity, for any \( x' \in W^p \), we have
\[
\dim \text{Hom}_{\mathcal{O}^{K^z}}(P, \Delta(zx')) = \sum_{y \in W^p} m_y[ \Delta(zx') : L(zy)].
\]
(19) In case \( P = \theta_x P^{K^z}(z) \), comparing (19) with (18), implies that we have \( \theta_x P^{K^z}(z) \cong P^{K^z}(zx) \). The claim follows. \( \square \)

**Proposition 40.** Assume that \( \mathfrak{g} \) is of any type. There exists an equivalence of abelian categories
\[
\Sigma : B_e \to B_z, \quad \text{with} \quad \Sigma(\Delta(y)) \cong \Delta(zy), \quad \text{for all} \quad y \in W^p,
\]
such, for all \( x \in W^p \), we have a diagram
\[
\begin{array}{ccc}
B_e & \xrightarrow{\theta_x} & B_e \\
\downarrow{\Sigma} & & \downarrow{\Sigma} \\
B_z & \xrightarrow{\overline{\theta}_x} & B_z
\end{array}
\]
with \( \overline{\theta}_x \) induced from \( \theta_x \), which commutes up to isomorphism.

**Proof.** Let \( \mathcal{P}_0 \) denote the full additive subcategory of \( \mathcal{O}_0 \) consisting of direct sums of modules isomorphic to elements of \( \{ P(y), y \in W^p \} \). Similarly \( \mathcal{P}_z \), is the full additive subcategory of \( \mathcal{O}_0^{K^z} \) consisting of direct sums of modules isomorphic to elements of \( \{ P^{K^z}(zy), y \in W^p \} \).

By Lemma [39] and equation (6), we have a commuting diagram
\[
\begin{array}{ccc}
\mathcal{P}_0 & \xrightarrow{\theta_x} & \mathcal{P}_0 \\
\downarrow{T_z} & & \downarrow{T_z} \\
\mathcal{P}_z & \xrightarrow{\theta_x} & \mathcal{P}_z
\end{array}
\]
Now \( \mathcal{P}_0 \), resp. \( \mathcal{P}_z \), is equivalent to the category of projective modules in \( B_e \), resp \( B_z \). Under that equivalence, \( \theta_x \) is interchanged with \( \overline{\theta}_x \). We now let \( \Sigma : B_e \to B_z \) denote the equivalence corresponding to the equivalence between the categories of projective modules induced by \( T_z \). By construction this admits the commuting diagram as in the formulation.
Consider the exact sequence
\[(20)\quad P \to P(y) \to \Delta(y) \to 0,\]
in \(\mathcal{O}_0\), where \(P\) is a direct sum of \(P(y')\), with \(y' \in W^p\). By (7), applying \(T_z\) to (20) yields an exact sequence
\[T_zP \to T_zP(y) \to \Delta(zy) \to 0,\]
which is in \(\mathcal{O}_{K_z}\), by Lemma 39. We can interpret (20) as an exact sequence in \(\mathcal{B}_z\), by applying the exact functor \(\pi\), which leads to an exact sequence
\[\Sigma(P) \to \Sigma(P(y)) \to \Sigma(\Delta(y)) \to 0\]
in \(\mathcal{O}_{K_z}/\mathcal{O}_{L_z}\). Comparing both exact sequences using Lemma 39 yields the isomorphism \(\Sigma(\Delta(y)) \cong \Delta(zy)\). \(\square\)

**Proof of Theorem 37.** This is a combination of the equivalences in Lemma 36 and Proposition 40. \(\square\)

**Remark 41.** The equivalences in Theorems 32 and 37 extend to the thick category \(\mathcal{O}\) as in [So4]. After extension, the two equivalences are related by the duality on thick category \(\mathcal{O}\) coming from the duality on Harish-Chandra bimodules used in the proof of Lemma 35.

**Remark 42.** The equivalences in Theorems 32 also intertwines the corresponding shuffling functors.

### 6.4. Application to Conjecture 1

**Corollary 43.** Assume that \(\mathfrak{g}\) is of any type. Let \(\mathfrak{p}\) be a parabolic subalgebra of \(\mathfrak{g}\). Let \(x, y \in W^p\) and \(z \in X^p\). Then \(\theta_x L(y)\) is indecomposable in category \(\mathcal{O}\) for 1 if and only if \(\theta_x L(zy)\) is indecomposable in category \(\mathcal{O}\) for \(\mathfrak{g}\).

**Proof.** We have that \(\theta_x L_l(y)\) is indecomposable if and only if \(\theta_x L(zy)\) is indecomposable in \(\mathcal{B}_z\) by Theorem 37. From equation (4), we have that \(\theta_x \Delta(zy)\) has a Verma flag with highest weights in \(zW^p\), so the top of \(\theta_x \Delta(zy)\) are simples of the form \(L(zw)\), where \(w \in W^p\). The top (and also socle) of \(\theta_x L(zy)\) is a submodule of that for \(\theta_x \Delta(zy)\). Therefore the top (and the socle) of \(\theta_x L(zy)\) consists only of simple modules not in \(\mathcal{O}_{0^+}\). Consequently, the endomorphism algebra of \(\theta_x L(zy)\) in \(\mathcal{O}_0\) is the same as it is in \(\mathcal{B}_z\), so we
find that \( \theta_x L(zy) \) is indecomposable in \( B_z \) if and only if it is indecomposable in \( O_0 \).

As an immediate consequence of Corollary 43, we obtain:

**Corollary 44.** Conjecture 1 is true if and only if it is true under the additional assumption that \( \text{supp}(x) = S \).

7. Conjecture 1 for small values of \( n \)

7.1. The case \( n = 2 \)

For \( n = 2 \), we have \( W = \{e, s\} \) and all elements in \( W \) are of the form \( w_0^p \), for some \( p \). Therefore Conjecture 1 is true in this case due to Subsection 3.3.

7.2. The case \( n = 3 \)

For \( n = 3 \), the Weyl group \( W = \{e, s, t, st, ts, w_0\} \) contains four Kazhdan-Lusztig right cells:

\[
\{e\}, \quad \{s, st\}, \quad \{t, ts\}, \quad \{w_0\}
\]

and \( I(W) = \{e, s, t, w_0\} \). Note that all involutions in \( W \) are of the form \( w_0^p \), for some \( p \). Therefore Conjecture 1 is true in this case due to Subsections 3.2 and 3.3.

7.3. The case \( n = 4 \)

For \( n = 4 \), two-sided cells in \( W \cong S_4 \) are indexed by partitions of 4:

\[
(4), \quad (3,1), \quad (2,2), \quad (2,1,1), \quad (1,1,1,1).
\]

The two-sided cell corresponding to some partition \( \lambda \) will be denoted \( J_\lambda \). We have that \( \theta_x L(y) \neq 0 \) implies \( x \leq_J y \), see [1]. By [GJ Theorem 5.1], the two-sided order on \( W \) coincides with the opposite of the dominance order on partitions.

By Corollary 3 we may assume \( x \in I(W) \) and \( y \in I'(W) \). If \( x \in J_{(4)} \) or \( x \in J_{(3,1)} \), then \( x = w_0^p \), for some \( p \). If \( y \in J_{(1,1,1,1)} \) or \( y \in J_{(2,1,1)} \), then \( y = w_0^p w_0 \), for some \( p \). If \( x, y \in J_{(2,2)} \), then \( x \sim_J y \). Therefore Conjecture 1 is true in this case due to Subsection 3.3.
For $n = 5$, two-sided cells in $W \cong S_5$ are indexed by partitions of 5:

$$(5), \ (4,1), \ (3,2), \ (3,1^2) \ (2^2,1), \ (2,1^3), \ (1^5).$$

The two-sided order coincides with the opposite of the dominance order and is still linear (increasing from left to right). We keep the conventions from the previous subsection. By Corollary 3, we may assume $x \in I(W)$ and $y \in I'(W)$.

We have the following general result.

**Proposition 45.** For any $n \geq 5$, Conjecture 1 is true if we have $x \in J(n)$, $x \in J(n-1,1)$ or $x \in J(n-2,2)$. Similarly, Conjecture 1 is true if $y \in J(1^n)$, $y \in J(2,1^{n-2})$ or $y \in J(2^2,1^{n-4})$.

**Proof.** The second statement follows from the first one using Subsection 3.1, so we only prove the first statement. If $x \in J(n)$ or $x \in J(n-1,1)$, then $x = w^p_0$, for some $p$. So, $\theta_x L(y)$ is either zero or indecomposable, for any $y$, by Subsection 3.3.

The two-sided cell $J(n-2,2)$ has, by the hook formula, $\frac{n(n-3)}{2}$ right Kazhdan-Lusztig cells. If $s$ and $t$ are two commuting simple reflections, then we have $st \in J(n-2,2)$ and it has the form $w^p_0$, for some $p$. Therefore for such $x = st$, the module $\theta_x L(y)$ is either zero or indecomposable, for any $y$, by Subsection 3.3. There are $\frac{(n-2)(n-3)}{2}$ such elements.

Additionally, $J(n-2,2)$ contains $n-3$ involutions of the form $strs$, where $s, r, t$ are simple reflections such that $s$ commutes neither with $r$ nor with $t$. As $n \geq 5$, these involutions have the property that not all simple reflections appear in their reduced expressions. Therefore Corollary 43 reduces these involutions to the case $n = 4$ which is already treated above. As

$$\frac{(n-2)(n-3)}{2} + (n-3) = \frac{n(n-3)}{2},$$

the above covers all Kazhdan-Lusztig right cells in $J(n-2,2)$, completing the proof. \qed

After Proposition 45, it remains to consider the case $x, y \in J(3,1^2)$ which follows from Subsection 3.3. Consequently, Conjecture 1 is true in the case $n = 5$. 
7.5. The case $n = 6$

For $n = 6$, the dominance order on partitions of 6 is as follows:

![Diagram of dominance order](image)

This also gives the two-sided order on two-sided cells. In what follows, we simply write $i_1 \cdots i_k$ for $s_{i_1} \cdots s_{i_k}$, where, as usual, $s_i$ denotes the transposition $(i, i + 1) \in S_6$. We also list elements in their lexicographically minimal reduced expression.

By Corollary 3, we may assume $x \in I(W)$ and $y \in I'(W)$. The cases $x \in J(6)$, $x \in J(5,1)$, $x \in J(4,2)$ and $y \in J(1^6)$, $x \in J(2,1^4)$ follow from Proposition 45.

As $\theta_x L(y)$ is zero unless $x \leq_J y$ (see (5)), the above allows us to restrict the index of the two-sided cells of $x$ and $y$ to the list $(3^2)$, $(4, 1^2)$, $(3, 2, 1)$, $(2^3)$, $(3, 1^3)$. We now give a full list of all those pairs which cannot be dealt with by direct applications of the results in Section 3 and Corollary 43 (here $x \in J(3^2)$ and $y \in J(3,2,1) \cup J(2^3) \cup J(3,1^3)$, or $x \in J(4,1^2)$ and $y \in J(3,2,1) \cup J(3,1^3)$):

- (I) $(123454321, 2321543)$,
- (II) $(123454321, 1324325)$,
- (III) $(123454321, 232432)$,
- (IV) $(2143254, 2321543)$,
- (V) $(2143254, 1324325)$,
- (VI) $(2143254, 232432)$,
- (VII) $(2143254, 121454)$,
- (VIII) $(2143254, 13214543)$,
- (IX) $(321432543, 2321543)$,
- (X) $(321432543, 1324325)$,
- (XI) $(321432543, 232432)$,
We observe that $2321543 \sim L_{1343}$ and that $L(1343)$ is annihilated by $\theta_2, \theta_5$. Therefore $\theta_xL(y) = 0$ in (I) and (IV). Similarly, $1324325 \sim L_{2325}$ and $L(2325)$ is annihilated by $\theta_1, \theta_4$. Therefore $\theta_xL(y) = 0$ in (II) and (V). Further, $L(232432)$ is annihilated by $\theta_1, \theta_5$ and $L(13214543)$ is annihilated by $\theta_2$ and $L(121454)$ is annihilated by $\theta_3$. Therefore $\theta_xL(y) = 0$ in (III), (VIII) and (XII). We also note that (IX) and (X) differ by a symmetry of the root system.

Choosing the shortest elements in the right cell of the first component and in the left cell of the second component and using Proposition 2, the remaining cases (VI), (VII), (IX), (XI) and (XIII) become:

(XIV) $(21435, 232432)$;
(XV) $(21435, 121454)$;
(XVI) $(321435, 1343)$;
(XVII) $(321435, 232432)$;
(XVIII) $(321435, 1214543)$.

Note that both $21435 = 45231$ and $321435 = 345231$ are short-braid hexagon avoiding and hence, by [BW], we have:

\[
\theta_{21435} \cong \theta_1\theta_3\theta_2\theta_5\theta_4, \quad \theta_{321435} \cong \theta_1\theta_5\theta_2\theta_5\theta_4\theta_3.
\]

To proceed, we need the following statement.

**Proposition 46.**

(a) Let $s \in S$ and $y \in W$ be such that $ys \prec y$. Then the module $\theta_sL(y)$ has the following graded picture (the left column gives the degree):

\[
\begin{array}{c|c}
-1 & L(y) \\
0 & \bigoplus_{z \in W} L(z)^{m_z} \\
1 & L(y),
\end{array}
\]

where

- the degree $-1$ contains the simple top of $\theta_sL(y)$,
- the degree $1$ contains the simple socle of $\theta_sL(y)$,
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• the degree 0 contains what is known as the Jantzen middle of \( \theta_s L(y) \).

Furthermore, \( m_z \neq 0 \), for \( z \in W \), implies that \( z \prec z_s \) and \( y_s \preceq z \). Moreover, for each \( z \in W \) such that \( z \prec z_s \), we have

\[
m_z = \dim \text{Ext}^1(L(y), L(z)) = \begin{cases} 
\mu(y,z), & y \prec z; \\
\mu(z,y), & z \prec y,
\end{cases}
\]

where \( \mu \) denotes the Kazhdan-Lusztig \( \mu \)-function, see \([KL]\).

(b) Let \( s, t \in S \) be such that \( sts = tst \) and let \( y \in W \) be such that \( y_s \preceq y \) and \( yt \succeq y \). Then

\[
\theta_l \theta_s L(y) \cong \begin{cases} 
\theta_l L(y_s), & yst \prec y_s; \\
\theta_l L(yt), & \text{otherwise}.
\end{cases}
\]

Proof. Claim (a) is a well-known consequence of the Kazhdan-Lusztig conjecture. We just note that \( \Delta(y) \to L(y) \) and hence \( \theta_s \Delta(y) \to \theta_s L(y) \). As \( \theta_s \Delta(y) \) has \( \Delta(y_s) \) as a submodule and the corresponding quotient is isomorphic to \( \Delta(y) \) (which is, in turn, a submodule of \( \Delta(y_s) \)), it follows that every simple subquotient of \( \theta_s L(y) \) is a simple subquotient of \( \Delta(y_s) \). This justifies the relation \( y_s \preceq z \). The \( \text{Ext}^1 \)-property follows from the observation that, if \( M \in O \) is such that \( M \) has simple top \( L(y) \) and the kernel \( K \) of \( M \to L(y) \) is killed by \( \theta_s \), then, by adjunction, \( \theta_s L(y) \to M \) and hence \( K \) must be a submodule of the Jantzen middle.

To prove claim (b), we note that \( \theta_l L(y) = 0 \) by assumptions and hence, due to claim (a), we just need to show that the Jantzen middle of \( \theta_l L(y) \) contains a unique summand \( L(z) \) such that \( zt \prec z \), moreover, that this \( z \in \{y_s, yt\} \).

From \([Wa, \text{Fact 3.2}]\) it follows that a composition factor \( L(z) \) of \( \theta_s L(y) \), which survives after \( \theta_l \), must satisfy \( |\ell(z) - \ell(y)| = 1 \). If \( z \prec y \), then from claim (a) we have \( y_s \preceq z \) and hence \( z = y_s \). If \( y \prec z \), then \( z = yt \) as \( t \) is in the right descent set of \( z \) but not in the right descent set of \( y \) and \([BB, \text{Proposition 2.2.7}]\) and \([BB, \text{Corollary 2.2.5}]\) give \( y \preceq yt \succeq z \). This establishes \( z \in \{y_s, yt\} \).

Set \( z = yt \) and \( z' = y_s \). To complete the proof, it now suffices to show that \( zs \succeq z \) if and only if \( z't \succeq z' \). For the “if” part, we note that \( z't \succeq z' \) means that we have both \( yst \succeq y \) and \( y \preceq y_s \). By \([BB, \text{Corollary 2.2.5}]\) and \([BB, \text{Corollary 2.2.8}]\) we have \( ysts, yt \succeq y, yst \). Once again, we obtain that \( ysts \succeq yt \). The local Hasse diagram for the Bruhat order can, in this
case, be depicted as follows:

\[ \begin{array}{c}
\downarrow \hspace{2cm} \downarrow \\
yts & \text{ys} \\
\downarrow \hspace{2cm} \downarrow \\
yt & \text{yt} \\
\downarrow \hspace{2cm} \downarrow \\
yst & \text{yst} \\
\downarrow \hspace{2cm} \downarrow \\
yst & \text{yst} \\
\end{array} \]

For the “only if” part, if \( zs \geq z \), then we have \( ys \preceq y \preceq yt \preceq yts \). Since

\[ \ell(y) - 1 + 2 \geq \ell(yts) = \ell(ytst) \geq \ell(y) + 2 - 1, \]

we have \( \ell(yts) = \ell(y) + 1 \) which implies \( yst \preceq ys \). \( \square \)

Proposition 46 has an interesting general consequence (compare with [Wa, 3.2]).

**Corollary 47.** Assume that we are in type A. Let \( s, t \) be two simple reflections and \( w \in W \) be such that \( ws \succ w \) and \( wt \prec w \). Then

\[ \sum_{\tilde{w} \in W : \tilde{w}s \prec \tilde{w}, \tilde{w}t \succ \tilde{w}} \mu(w, \tilde{w}) \leq 1. \]  \hfill (22)

**Proof.** If \( t \) and \( s \) commute, then \( ts \) is of the form \( w_0^p \), for some \( p \). If \( t \) and \( s \) do not commute, then \( tst = sts \) and \( \theta_s \theta_t L(w) \) is either indecomposable or zero by Proposition 46(b). On the other hand, by Proposition 16(a), the number of indecomposable direct summands of \( \theta_s \theta_t L(w) \) is given by the left hand side of (22). The claim follows. \( \square \)

We note that the property in Corollary 47 fails outside type A, in fact, already in type \( B_2 \). This is the origin for the failure of Conjecture 1 in type \( B_2 \).

Now we will go through all the remaining cases (XIV)–(XVIII).
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The case (XIV). We claim that Conjecture 1 is true for the pair (XIV). Indeed, by Proposition 46 we have

$$\theta_5 \theta_4 L(232432) \cong \theta_5 L(2324325)$$

and the claim now follows from (21), Corollary 43 and the cases of smaller $n$ as the support of $5231 = 2135$ is not the whole of $S$.

The case (XVI). Next we claim that Conjecture 1 is true for the pair (XVI). Indeed, by Proposition 46 we have

$$\theta_2 \theta_3 L(1343) \cong \theta_2 L(13432) \text{ and } \theta_1 \theta_2 L(13432) \cong \theta_1 L(1343).$$

Taking into account (21) and $321435 = 321453$, we again can reduce indecomposability for our pair $(x, y)$ to indecomposability for a new pair $(x', y')$ in which the support of $x'$ is not the whole of $S$. The claim now follows from Corollary 43 and the cases of smaller $n$.

The case (XVIII). From Proposition 46 we have that

$$\theta_4 \theta_3 L(1214543) \cong \theta_4 L(121454).$$

Now, from (21) it follows that

$$\theta_{21435} L(121454) \cong \theta_{321435} L(1214543).$$

Therefore Conjecture 1 is true for the pair (XVIII) if and only if Conjecture 1 is true for the pair (XV).

The case (XV). We claim that $\theta_{21435} L(121454)$ has simple top $L(1214543)$ and hence is indecomposable. This will establish Conjecture 1 for the pair (XV).

Lemma 48. If $L(z)$ appears in the top of $\theta_{21435} L(121454)$, for some $z \in W$, then $z = 1214543$.

Proof. If $L(z)$ appears in the top of $\theta_{21435} L(121454)$, for some $z \in W$, then $z \sim R 121454$, see the proof of [Ma2, Theorem 6]. The right cell of 121454 consists of the following elements:

$$(23) \{121454, 1214543, 12143543, 12145432, 121435432\}.$$

As $21435 = 24531$, using (21), by adjunction, we also have

$$\text{Hom}(\theta_{21435} L(121454), L(z)) \cong \text{Hom}(\theta_{24} L(121454), \theta_{135} L(z)).$$
which means that \( \theta_{135} L(z) \neq 0 \). It is easy to check that the only \( z \) in the list \([23]\) which has 1, 3 and 5 in its right descent set, is the element \( z = 1214543 \). Therefore the only \( z \) in the list \([23]\) for which \( \theta_{135} L(z) \neq 0 \) is \( z = 1214543 \). The claim follows.

After Lemma 48, the case \([\text{XV}]\) is completed by:

**Lemma 49.** We have \( \dim \text{Hom}(\theta_{21435} L(1214543), L(1214543)) = 1 \).

**Proof.** We use \([21]\) to write \( \theta_{21435} = \theta_{531} \theta_{24} \). Hence, by adjunction, our claim is equivalent to

\[
\dim \text{Hom}(\theta_{24} L(121454), \theta_{135} L(1214543)) = 1.
\]

Note that the fact that this space is non-zero follows, by adjunction, from Lemma 48 and the fact that \( \theta_{21435} L(121454) \neq 0 \).

As \( \theta_{135} \) is of the form \( w_0^p \), for some \( p \), we can factorize \( \theta_{135} = \theta_{135}^\text{out} \theta_{135}^\text{on} \) via translations on and out of the corresponding walls. Therefore, by adjunction, the above is equivalent to

\[
(24) \quad \dim \text{Hom}(\theta_{135}^\text{on} \theta_{24} L(121454), \theta_{135}^\text{on} L(1214543)) = 1.
\]

The module \( \theta_{135}^\text{on} L(1214543) \) is a simple module in a singular block of \( \mathcal{O} \), let us call it \( L(\lambda) \).

As \( \theta_{24} L(121454) \) lives in degrees \( \pm 2, \pm 1, 0 \). Moreover, the degree \( -2 \) contains just its simple top \( L(121454) \) and the degree 2 contains just its simple socle \( L(121454) \). As \( 1214543 \succ 121454 \), we have \( \theta_{135}^\text{on} L(121454) = 0 \). This means that \( M := \theta_{135}^\text{on} \theta_{24} L(121454) \) lives in degrees \( \pm 1, 0 \).

Next we argue that \( M \) can only have \( L(\lambda) \) in the top (up to graded shift). Indeed, any occurrence of some other \( L(\mu) \) would, by adjunction, lead to a contradiction with Lemma 48. This implies that the degree \( -1 \) part of \( M \) just consists of copies of \( L(\lambda) \). In fact, there can only be a single copy as \( \dim \text{Ext}^1(L(121454), L(1214543)) = 1 \) (because 121454 and 1214543 differ by a single reflection) and hence appearance of \( L(1214543) \) with multiplicity higher than one in degree \( -1 \) of \( \theta_{24} L(121454) \) would contradict the fact that \( \theta_{24} L(121454) \) has simple top. Being a translation of a simple module, \( M \) is self-dual, and hence the degree 1 part of \( M \) just consists of a single copy of \( L(\lambda) \) as well. By the even-odd vanishing, \( L(\lambda) \) cannot appear in degree 0.

The module \( L(12134543) \) appears in the degree 0 part of \( \theta_{24} L(121454) \) with non-zero multiplicity. As \( \theta_{135}^\text{on} L(12134543) \neq 0 \), it follows that the degree 0 part of \( M \) is, in fact, non-zero. By the previous paragraph, this degree
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0 part cannot contribute to the top of $M$. By self-duality, this degree 0 part cannot contribute to the socle of $M$ either. The latter, in turn, implies that the unique simple subquotient of the degree 1 part cannot be in the top of the module. This implies (24) and completes the proof.

The arguments in the proof of Lemma 49 motivate the following question:

**Question 50.** Let $n$ be arbitrary and define

$$
\theta_{\text{odd}} := \theta_1 \theta_3 \theta_5 \cdots, \quad \theta_{\text{ev}} := \theta_2 \theta_4 \theta_6 \cdots.
$$

Is it true that, for all $x, y \in W$, we have

$$
\dim \text{Hom}(\theta_{\text{odd}}L(x), \theta_{\text{ev}}L(y)) \leq 1?
$$

We note that $\dim \text{Hom}(\theta_{\text{odd}}L(x), \theta_{\text{ev}}L(y)) > 0$ is only possible for $x, y \in W$ such that $x \sim_R y$. For $x = y = w_0$, we have the following stronger statement.

**Proposition 51.** Let $w \in W$ be such that each simple reflection appears at most once in a reduced expression of $w$. Then $\theta_wL(w_0)$ has simple top. In particular, Question 50 has the positive answer, for $x = w_0$ or $y = w_0$.

We note that the condition “each simple reflection appears at most once in a reduced expression of $w$” does not depend on the choice of a reduced expression as all reduced expressions can be obtained from each other by applying braid relations, see [BB, Theorem 3.3.1].

**Proof.** We start by noting that, for $z \in W$, the condition $w_0 \sim_R z$ implies $z = w_0$. Therefore the second assertion of the proposition follows from the first one by adjunction. To prove the first one, we first note that any $w$ as in the formulation is short-braid hexagon avoiding, cf. [BW] equations (9) and (10)]. Therefore we have the explicit form of the Kazhdan-Lusztig polynomial for such $w$, see [BW, Theorem 1] which implies that $\Delta(e)$ appears only once in the Verma flag of $P(w)$. By [St, Theorem 8.1] and [FKM, Proposition 4], this means that $T(w_0w) \cong \theta_wL(w_0)$ has simple socle and thus also simple top, by self-duality.

**The case (XVII).** In this case we apply the same approach as in the previous case.
Lemma 52. If $L(z)$ appears in the top of $\theta_{321435}L(232432)$, for some $z \in W$, then $z = 23243215$.

Proof. If $L(z)$ appears in the top of $\theta_{321435}L(232432)$, for some $z \in W$, then $z \sim_R 232432$, see the proof of [Ma2, Theorem 6]. The right cell of $232432$ consists of the following elements:

\[(25) \quad \{232432, 2324321, 2324325, 23214321, 23243215, 232432154, 232432543, 232143215 \}.
\]

As $321435 = 342531$, using (21), by adjunction, we also have

$$\dim \text{Hom}(\theta_{321435}L(232432), L(z)) \cong \dim \text{Hom}(\theta_{342}L(232432), \theta_{135}L(z)),$$

which means that $\theta_{135}L(z) \neq 0$. It is easy to check that the only $z$ in the list (25) which has $1, 3$ and $5$ in its right descent set, is the element $z = 23243215$. This means that the only $z$ in the list (25) for which $\theta_{135}L(z) \neq 0$ is $z = 23243215$. The claim follows. \[\square\]

After Lemma 52, the case (XVII) is completed by:

Lemma 53. We have $\dim \text{Hom}(\theta_{321435}L(232432), L(23243215)) = 1$.

Proof. We use (21) to write $\theta_{321435} = \theta_{531}\theta_{324}$. Hence, by adjunction, our claim is equivalent to

$$\dim \text{Hom}(\theta_{324}L(232432), \theta_{135}L(23243215)) = 1.$$ 

Note that the fact that this space is non-zero is obvious. As $135$ is of the form $w_0^p$, for some $p$, we can factorize $\theta_{135} = \theta_{135}^{\text{on}}\theta_{135}^{\text{off}}$ via translations on and out of the corresponding walls. Therefore, by adjunction, the above is equivalent to

\[(26) \quad \dim \text{Hom}(\theta_{135}^{\text{on}}\theta_{324}L(232432), \theta_{135}^{\text{on}}L(23243215)) = 1.
\]

The module $\theta_{135}^{\text{on}}L(23243215)$ is a simple module in a singular block of $O$, let us call it $L(\lambda)$.

As $L(232432)$ is not annihilated by $\theta_2, \theta_3, \theta_4$, we know that the graded module $\theta_{324}L(232432)$ lives in degrees $\pm 3, \pm 2, \pm 1, 0$. Moreover, the degree $-3$ contains just a copy of $L(232432)$ and the degree $3$ contains just a copy of $L(232432)$. As $2324323 \geq 232432$, we have $\theta_{135}^{\text{on}}L(232432) = 0.$
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We want to prove that the graded module $M := \theta_{135}^{\alpha} \theta_{324} L(232432)$ lives in degrees $\pm 1, 0$ with only $L(\lambda)$ in $\deg \pm 1$. We do it by an explicit computation.

From the table of Kazhdan-Lusztig polynomials for $S_6$, see [Go], we have that

$$\{23432, 2321432, 243215432, 213243254\}$$

is the set of all 3-finite permutations $z$ with non-zero $\mu$-functions (whenever the expression $\mu(z, 232432)$ or $\mu(232432, z)$ makes sense). Hence we have that the graded picture of $\theta_{3} L(232432)$ is as follows:

\[
\begin{array}{c|cccc}
-1 & L(232432) \\
0 & L(23432) & L(2321432) & L(243215432) & L(213243254) \\
1 & L(232432) \\
\end{array}
\]

We observe that all permutations corresponding to simple subquotients of the module $\theta_{3} L(232432)$ are 4-free and 5-finite. Therefore, again, by Proposition [46] we have

$$\theta_5 \theta_4 L(232432) \cong \theta_5 L(2324325),$$
$$\theta_5 \theta_4 L(23432) \cong \theta_5 L(234325),$$
$$\theta_5 \theta_4 L(2321432) \cong \theta_5 L(23214325),$$
$$\theta_5 \theta_4 L(243215432) \cong \theta_5 L(2343215432),$$
$$\theta_5 \theta_4 L(213243254) \cong \theta_5 L(21324325).$$

Observing that all simples on the right hand side of the above equations are 2-free and 1-finite and $521 = 215$, we obtain

$$\theta_1 \theta_2 L(2324325) \cong \theta_1 L(23243215),$$
$$\theta_1 \theta_2 L(234325) \cong \theta_1 L(2343215),$$
$$\theta_1 \theta_2 L(23214325) \cong \theta_1 L(23214325),$$
$$\theta_1 \theta_2 L(243215432) \cong \theta_1 L(2343215432),$$
$$\theta_1 \theta_2 L(213243254) \cong \theta_1 L(2132432515).$$

Note that the simple $L(2343215)$ is killed by $\theta_3$ as the expression $23432153$ is reduced. For the case (XVII), since $321435 = 345213$, by tracking the degrees, we have that the graded picture of the module $\theta_{3} L(y)$ is obtained
by applying \( \theta_{135} \) to the following graded picture:

\[
\begin{array}{c|cccc}
-1 & L(23243215) & L(234321543) & L(213243215) \\
0 & L(2321435) & L(23243215) & L(23243215) \\
1 & L(23243215) &
\end{array}
\]

Therefore the graded picture of \( M \) can be obtained by applying \( \theta_{135} \) on the above graded picture, which implies that the graded module \( M \) lives in degree \( \pm 1, 0 \) with \( L(\lambda) \) appearing once in degree 1 and once in degree \(-1\).

The module \( \theta_{135}^n L(2321435) \neq 0 \) appears in the degree 0 part of \( M \) with non-zero multiplicity. By Lemma 52 this degree 0 part cannot contribute to the top of \( M \). By self-duality, this degree 0 part cannot contribute to the socle of \( M \) either. The latter, in turn, implies that the unique simple subquotient of the degree 1 part cannot be in the top of the module. This implies (26) and completes the proof. \( \square \)

Hence Conjecture 1 is true in the case \( n = 6 \).

8. Shuffling and twisting simple modules

8.1. Conjecture and results

Given the close connection between shuffling and translation functors, it is natural to investigate whether shuffling simple modules yields indecomposable modules. This leads us to the following conjecture.

**Conjecture 54.** For \( g \) of any type and for all \( x, y \in W \), the module \( C_x L(y) \) is either indecomposable or zero.

We summarize the evidence in favor of this conjecture in the following proposition. All indecomposability of modules is based on the observation that they have simple top or socle.

**Theorem 55.** Let \( g \) be of any type and consider \( x, y \in W \).

(a) The functor \( C_x \) on \( O_0 \) is indecomposable. More precisely, its endomorphism algebra is isomorphic to the algebra \( \mathcal{C} \) of coinvariants for the Weyl group \( W \).

(b) The complex \( \mathcal{L}C_x L(y) \) is indecomposable in \( D^b(O_0) \).

(c) If \( \theta_x L(y) \) has simple top, then either \( C_x L(y) \) is zero or has simple top.
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(d) The module $C_x L(y)$ has simple top in the following cases:
   (i) $x = w_0^p$, for some parabolic subalgebra $p$, and $y$ is a longest representative in $W/W^p$ (equivalently, $x \leq_R y^{-1}$), in this case the simple top is $L(y)$;
   (ii) $y$ is a Duflo involution and $x \sim_R y$, in this case the simple top is $L(x)$.

(e) If $y = w_0^p w_0$, for some parabolic subalgebra $p$, and $x \leq_R y^{-1}$ (equivalently, $w_0 w_0^p x$ is a shortest representative in $W^p \backslash W$), then $C_x L(y)$ has simple socle $L(yx)$.

(f) If $C_x L(y) \neq 0$, then $x \leq_R y^{-1}$.

These results will be proved below. Despite the partial results in the above proposition, the following question seems unanswered in general.

**Question 56.** When is the module $C_x L(y)$ zero?

We note that the example for $B_2$ where $\theta_x L(y)$ is decomposable, see [KiM1, Section 5.1], does not lead to a counterexample to Conjecture 51.

**Example 57.** If $g = B_2$, then $C_{st} L(ts)$ is indecomposable, despite the fact that $\theta_{st} L(ts)$ is decomposable. The module $C_{st} L(ts)$ is an indecomposable module admitting a short exact sequence

$$0 \to L(e) \oplus L(ts) \to C_{st} L(ts) \to L(t) \oplus L(tst) \to 0.$$ 

8.2. Relation with twisting simple modules

**Lemma 58.** Let $g$ be of any type. For all $x, y \in W$, we have that $C_x L(y)$ is indecomposable if and only if $T_{x^{-1}} L(y^{-1})$ is indecomposable. Moreover,

$$\text{End}_O(C_x L(y)) \cong \text{End}_O(T_{x^{-1}} L(y^{-1})).$$

**Proof.** Let $O_0$ be the full subcategory of $O_0$ of modules which admit a central character. Under the equivalence $O_0 \cong \text{H}^1_0$ of [BG], Soergel’s autoequivalence of $\text{H}^1_0$ from [So1] which exchanges left and right action on a bimodule, maps $C_x L(y)$ to $T_{x^{-1}} L(y^{-1})$, by [Ja Satz 6.34]. The claim follows. □

**Lemma 59.** Let $p$ be a parabolic subalgebra of $g$ and take $x, y \in W^p$ and $z \in X^p$. If $C_x L(y)$ is indecomposable in category $O$ for $l$, then $C_x L(zy)$ is indecomposable in category $O$ for $g$. 

Proof. It follows from Theorem 37 that, for any \( x \in W^p \) and \( \mu \in \mathfrak{h}^* \), we have
\[
\text{End}_{\mathcal{O}(\emptyset)}(C_x L(y)) \cong \text{End}_{\mathcal{B}_s}(C_x L(zy)).
\]

Note that each simple constituent of the top of the module \( C_x L(zy) \) survives, by construction, projection onto \( \mathcal{B}_s \). From the definition of a Serre quotient we thus have that \( \text{End}_{\mathcal{O}(\emptyset)}(C_x L(zy)) \) is a subalgebra of \( \text{End}_{\mathcal{B}_s}(C_x L(zy)) \). Therefore, if \( \text{End}_{\mathcal{B}_s}(C_x L(zy)) \) is local, then so is \( \text{End}_{\mathcal{O}(\emptyset)}(C_x L(zy)) \). The claim follows. \( \square \)

Twisting functors, contrary to shuffling functors, can easily be defined in full generality for singular blocks as well.

**Question 60.** Is the module \( T_x L \) indecomposable or zero, for any simple module \( L \) in \( \mathcal{O} \) and \( x \in W^p \)?

### 8.3. On the (derived) shuffling functor

**Proposition 61.** Let \( \mathfrak{g} \) be of any type. For all \( x \in W \), there exists a linear complex of projective functors on \( \mathcal{O}_0^\mathfrak{g} \), which is isomorphic to \( \mathcal{L}C_x \) as a functor on \( \mathcal{D}^b(\mathcal{O}_0^\mathfrak{g}) \). This complex has length \( \ell(x) \) and its zero term is given by \( \theta_x \).

**Proof.** Consider the endofunctor of \( \mathcal{D}^b(\mathcal{O}_0^\mathfrak{g}) \) given by tensoring with the linear complex
\[
0 \to \text{Id}(\langle -1 \rangle) \to \theta \to 0
\]
and taking the total complex. Using [MS2, Theorem 2], we see that the action of this functor on projective modules coincides with the action of \( \mathcal{L}C_x \). Consequently, these two functors are isomorphic. Now, we can take a reduced expression of \( x \), compose the complexes of the above form corresponding to the factors of this reduced expression, and then form the total complex. This is, by construction, a complex \( P_x^\bullet \) of projective functors of length \( \ell(x) \) which corresponds to the functor \( \mathcal{L}C_x \).

When we evaluate \( P_x^\bullet \) at \( \Delta(e) \), we get a complex of projective modules which is quasi-isomorphic to \( \mathcal{L}C_x \Delta(e) \cong \Delta(x) \). Since \( \mathcal{O}_0 \) is standard Koszul in the sense of [ADL], see [ADL, Corollary 3.8] or [Ma1, Theorem 2.1], \( \Delta(x) \) is quasi-isomorphic to a linear complex of projective modules. Therefore the complex \( P_x^\bullet(\Delta(e)) \) of projective modules is homotopic to a linear complex and thus is a direct sum of a linear part and a number of trivial complexes.
Manipulations with simple modules

of the form

$$0 \rightarrow P \cong P \rightarrow 0.$$  

As all homomorphisms between projective modules in $O_0$ are realizable via natural transformations between projective functors by [BG, Theorem 3.5], it follows that $P^*$ is isomorphic to the direct sum of a linear complex and a number of trivial complexes. The claim follows. $\Box$

**Corollary 62.** The functor $C_x$ on $O_0^Z$ is isomorphic to the cokernel of a natural transformation

$$\theta(-1) \rightarrow \theta_x,$$

for $\theta$ a direct sum of projective functors $\theta_z$, with $z \in W$, without any grading shift.

**Lemma 63.** For any $x \in W$, the algebra $\text{End}(C_x)$ is isomorphic to $\mathbb{C}$.

**Proof.** Consider endomorphism algebras of the functors $\text{Id}$ and $C_x$ as objects in the category of $\mathbb{C}$-linear additive functors on $O_0$. We have a commuting diagram of algebra morphisms

\[
\begin{array}{ccc}
\text{End}(\text{Id}) & \xrightarrow{(\eta_{C_x})} & \text{End}(C_x) \\
\downarrow{\text{Ev}_{P(w_0)}} & & \downarrow{\text{Ev}_{P(w_0)}} \\
\text{End}_{O}(P(w_0)) & \xrightarrow{C_x} & \text{End}_{O}(P(w_0))
\end{array}
\]

where the vertical arrows are evaluation of natural transformations on the module $P(w_0)$ and the upper horizontal arrow maps a natural transformation $\eta : \text{Id} \rightarrow \text{Id}$ to the natural transformation $\eta_{C_x} : \text{Id} \circ C_x \rightarrow \text{Id} \circ C_x$, defined by $(\eta_{C_x})_M = \eta_{C_x} M$, for all $M \in O_0$. We also used the fact that $C_x P(w_0) \cong P(w_0)$ which follows, by induction on the length of $C_x$ from $C_x P(w_0) \cong P(w_0)$, the latter is checked by a direct computation directly from the definitions. Indeed, $\theta_x P(w_0) \cong P(w_0)(1) \oplus P(w_0)(-1)$ and the adjunction morphism $P(w_0)(-1) \rightarrow \theta_x P(w_0)$ is injective as the simple socle of $P(w_0)$ is not killed by $\theta_x$, which yields $C_x P(w_0) \cong P(w_0)$.

It is proved in [So2, Section 2] that $\text{End}(\text{Id}) \cong \mathbb{C}$ and that the left vertical arrow in (27) is an isomorphism. That the lower horizontal arrow is an isomorphism follows from the fact that $L C_x$ is an auto-equivalence of $D^b(O_0)$. It thus suffices to prove that the right vertical arrow is injective.
Consider a short exact sequence

\[ 0 \rightarrow \Delta(e) \rightarrow P(w_0) \rightarrow \text{Coker} \rightarrow 0, \]

where Coker has a Verma flag. Applying all indecomposable projective functors to this sequence and adding all this up gives a short exact sequence

\[ 0 \rightarrow P \rightarrow Q \rightarrow N \rightarrow 0, \]

where \( P \) is a projective generator of \( \mathcal{O}_0 \), the module \( Q \) is a direct sum of copies of \( P(w_0) \) and \( N \) is a module with Verma flag. By Section 2.7, we thus have a short exact sequence

\[ 0 \rightarrow C_x P \rightarrow C_x Q \rightarrow C_x N \rightarrow 0. \]

Evaluating an arbitrary \( \eta \in \text{End}(C_x) \) thus yields a commuting diagram with exact rows

\[
\begin{array}{ccc}
0 & \rightarrow & C_x P \\
\downarrow \eta_P & & \downarrow \eta_Q \\
0 & \rightarrow & C_x Q \\
\end{array}
\]

Since \( P \) is projective generator, \( \eta_P \) is not zero as soon as \( \eta \neq 0 \). It thus follows that \( \eta_Q \) cannot be zero either in this case and the injectivity requested in the previous paragraph is proved. \( \square \)

8.4. Proof of Theorem 55

Claim (a) is proved in Lemma 63. Claim (b) follows from the fact that \( \mathcal{L}C_x \) is an auto-equivalence of the bounded derived category. Claims (c) and (f) follow from the epimorphism \( \theta_x \rightarrow C_x \) in Corollary 62 and the criterion for vanishing of \( \theta_x L(y) \) in (5).

In the following two lemmata we prove the remaining claims (d) and (e).

**Lemma 64.** Let \( g \) be of any type and consider \( x, y \in W \).

(a) If \( x = w_0^p \) for some parabolic subalgebra \( p \), then the module \( C_x L(y) \) has simple top \( L(y) \) if and only if \( x \leq_R y^{-1} \) and is zero otherwise.

(b) If \( y \) is a Duflo involution and \( x \sim_R y \), then the module \( C_x L(y) \) has simple top \( L(x) \).
Proof. Since $\theta_{w_0}L(y)$ either has simple top $L(y)$ or is zero, the corresponding property for $C_{w_0}L(y)$ follows from the paragraph above the lemma. We have

$$C_{w_0}L(y) = \begin{cases} 0 & \text{if } y \neq w_0 \\ \nabla(e) & \text{if } y = w_0. \end{cases}$$

Indeed, the case $y = w_0$ follows from [MS1, Proposition 5.12]. The case $y \neq w_0$ follows by combining the facts that $\theta_{w_0}L(y) = 0$ and, further, that $\theta_{w_0}L(y) \twoheadrightarrow C_{w_0}L(y)$, where the latter follows from Corollary 62. That $C_{w_0}L(y)$ is non-zero if and only if $y$ is a longest representative in $W/W'$ then follows from Theorem 37. The latter condition is equivalent to $w_0^p \leq_R y^{-1}$.

This proves the first claim.

Now assume that $y$ is a Duflo involution. This means that there exists a module $K \subset \Delta(e)$ such that $\theta_x \Delta(e) \cong \theta_x K$, for any $x \sim_R y$, and the top of $K$ is $L(y)$, see e.g. [MM1, Proposition 17]. Consequently, $P(x) \cong \theta_x K$ surjects onto $\theta_x L(y)$, so $\theta_x L(y)$ has simple top $L(x)$. By Corollary 62 we have an exact sequence in $\mathcal{O}_0^\mathbb{Z}$

$$\theta L(y)(-1) \to \theta_x L(y) \to C_x L(y) \to 0.$$ 

The module in the middle term has its simple top in degree $-a(y)$, see [Ma2, Proposition 1(c)]. It is also proved in loc. cit. that the term in the left-hand side is zero in degree $-a(y)$. This means, in particular, that $C_x L(y) \neq 0$ and that it has simple top, which completes the proof. □

For the following lemma, we need to introduce some notation for the parabolic version of category $\mathcal{O}$. Let $\mathcal{O}^p$ denote the full subcategory of modules in $\mathcal{O}$ which are locally $U(l)$-finite. The standard modules in $\mathcal{O}_0^p$ are given by the parabolic Verma modules $\Delta^p(x)$, where $x \in pX$. They can either be defined as the maximal locally $U(l)$-finite quotient modules of the ordinary Verma modules, or as the modules induced from simple finite dimensional $p$-modules. By construction, $\Delta^p(x)$ has simple top $L(x)$. We also write $\nabla^p(x)$ for the dual module of $\Delta^p(x)$.

**Lemma 65.** For a parabolic subalgebra $p$ of $\mathfrak{g}$ and $x \in W$ a shortest representative in $W' \setminus W$, with $W' = w_0W^pW_0$, we have

$$C_x L(w_0^p w_0) \cong C_x L(w_0^p w_0 x).$$

Hence, $C_x L(w_0^p w_0)$ is non-zero and has simple socle $L(w_0^p w_0 x)$. 

Proof. Consider $\lambda \in \Lambda^+$ with $W_{\lambda} = W'$. By [BGS], we have the Koszul duality functor

$$K : D^b(O_0)^Z \to D^b(O_\lambda)^Z,$$

normalized such that $\nabla^p(z)$ is mapped to $\Delta(z^{-1}w_0 \cdot \lambda)$. Under this duality, $\mathcal{L}C_w$ is exchanged with $\mathcal{L}T_w$, see [MOS Section 6.5]. With $w_{0}^\lambda$, the longest element in $W'$, we thus have

$$C_x L(w_0^p w_0) \cong K^{-1}(T_{x^{-1}} \Delta(w_0^\lambda \cdot \lambda)) \cong K^{-1}(\Delta(w_0^\lambda \cdot \lambda)) \cong \nabla^p(w_0^p w_0 x).$$

This completes the proof. □

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