Nonlinear random gravity. I. Stochastic gravitational waves and spontaneous conformal fluctuations due to the quantum vacuum

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We investigate the problem of metric fluctuations in the presence of the vacuum fluctuations of matter fields and critically assess the usual assertion that vacuum energy implies a Planckian cosmological constant. To this end, a new stochastic classical approach to the quantum fluctuations of spacetime is developed. The work extends conceptually Boyer’s random electrodynamics to a theory of random gravity but has a considerably richer structure for inheriting non-linearity from general relativity. Attention is drawn to subtleties in choosing boundary conditions for metric fluctuations in relation to their dynamical consequences. We point out that those compatible with the observed Lorentz invariance must allow for spontaneous conformal fluctuations, in addition to stochastic gravitational waves due to zero point gravitons. This is implemented through an effective metric defined in terms of the random spacetime metric modulo a fluctuating conformal factor. It satisfies an effective Einstein equation coupled to an effective stress-energy tensor incorporating gravitational self energy of metric fluctuations as well as matter fields. The effective Einstein equation is expanded perturbatively up to second order non-linearity. In the process of regularizing divergent integrals, a UV-cutoff is introduced whose specific value, however, does not enter into the resulting description of random gravity. By assuming certain physically reasonable statistical properties of the conformal fluctuations, it is shown that the averaged effective metric satisfies the empty space Einstein equation with an effective cosmological constant. This effective cosmological constant vanishes when only the massless matter fields are included. More generally, a finite effective cosmological constant compatible with the observational constraints can be obtained as long as the bare masses of the massive matter fields are nearly zero, or the conformal invariance of matter is restored at some high energy scale.

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I. INTRODUCTION

Zero point energy and vacuum fluctuations have fascinated physicists1,2,3,4,5 and the wider scientific audience3,4,5 ever since their prediction by quantum field theory. Although the precise quantum nature of vacuum is not fully understood, vacuum fluctuations of the electromagnetic field have long received support from experiments. They include the Casimir effect10,11,12, Lamb shift of atomic energy levels13,14 and spontaneous emission from atoms15,16,17. The fundamental fluctuation-dissipation theorem18 provides a further basis for the vacuum fluctuations manifested as superconductive current noise in the Josephson junctions19,20,21.

Even so, elements of doubt still surround the physical reality of the quantum vacuum. Some argue that e.g. the Casimir effect could be explained without the vacuum energy22. Laboratory measurements of vacuum fluctuations can at best detect differences in the vacuum energy by modifying their boundary conditions22. It is generally believed that if vacuum energy is real, then it must gravitate in accordance with general relativity, and that only through the resulting gravity the net vacuum energy could be determined. Recently, it is shown that the part of vacuum energy responsible for the Casimir effect does indeed gravitate24. However, the gravitational consequence of the total vacuum energy remains controversial25,26,27,28.

By Lorentz invariance, after any appropriate regularization, the vacuum energy can only couple to the Einstein equation via an effective cosmological constant. There is a common perception that an exceedingly large cosmological constant must necessarily be built up from all ground states up to the Planck energy density scale29. While a sizable cosmological constant could have driven the cosmic inflation in the early universe, observations indicate its present value to be just under the critical density value30,31.

Despite the early suggested link between the vacuum energy of elementary particles and the cosmological constant by Zel’dovich32, a detailed mechanism is lacking. This motivates the “dark energy” models such as the quintessence fields and their extensions29,31. Leaving aside the original mysterious cancelation of the huge vac-
uum energy, these models indeed offer arguably the most popular current approach to the “cosmological constant problem”. Nevertheless, efforts to account for the observed cosmic acceleration using cosmological perturbation back-reaction without resorting to dark energy have continued. The need for a better understanding of nonlinear metric perturbations and quantum gravity effects is clearly highlighted by the recent progress in this direction.

Spacetime metric fluctuations are an integral part of the quantum vacuum. However, a fundamental difficulty in studying these effects is the absence of a consistent quantum theory of gravity with an appropriate classical limit. Recently, progress has been made using a stochastic semiclassical gravity approach. However, these results are limited to linear metric fluctuations and are explicitly evaluated only for the vacuum states of conformally invariant fields. Moffat’s stochastic gravity is nonlinear and capable of including general matter sources. It assumes a fluctuating gravitational coupling strength with certain statistical properties.

In this paper, we address the issue of metric fluctuations and their implications on cosmic acceleration without invoking additional hypothetical fields. We shall develop a new stochastic classical gravity approach without modifying the Einstein equation. Rather, we seek appropriate fluctuating solutions to these equations in the spirit of Boyer. However, we do not advocate the stochastic classical approach to replace the ultimate need for quantum gravity. The new approach should nonetheless allow us to model low energy metric fluctuations due to the quantum vacuum and assess their effect on classical spacetime at a larger scale. A follow-up paper will explore the possibility of decoherence effects due to nonlinear metric fluctuations.

Classical stochastic behaviour is evident in many observed quantum vacuum phenomena. The effects involved may be explained using classical electromagnetic fields satisfying Maxwell’s equations, as pointed out by Welton and others. The fields are however subject to fluctuating boundary conditions, formulated in Boyer’s random electrodynamics, to mimic vacuum fluctuations. York championed a novel gravitational analogue in which black hole entropy and radiance are derived from quasinormal mode metric fluctuations. These fluctuations are prescribed by a classical Vaidya geometry satisfying the Einstein equation with amplitudes set to the quantum zero point level. An improved quantum treatment of the problem using path integral is recently provided.

The stochastic classical approximation to quantum fluctuations is physically justified for low energy vacuum effects. However, new subtleties arise when interactions and nonlinearities are included. One hopes to derive these effects from a full quantum theory. But, when it comes to gravity, there isn’t one ready at hand! Given the classical nature of the low energy quantum fluctuations of matter, it seems inconsistent if they do not couple to gravity classically. A naive stochastic classical treatment of general relativity in vacuum could be applied perturbatively as follows. For linearized gravity, the Boyer type boundary condition results in a sea of zero point gravitational waves. The effective stress-energy tensor of these gravitational waves then enters into the higher order Einstein equation as a source. This effective energy density is positive and formally infinite. Even after a cutoff at e.g. the Planck scale, there appear to be problems still. The resulting effective stress-energy tensor resembles that of a massless spin-2 radiation fluid. It cannot be interpreted as a cosmological constant contribution, thereby breaking the Lorentz invariance of the resulting vacuum state. Its large energy density would lead either to a rapid expansion of the universe as discussed by Weinberg or to a drastic collapse of spacetime as noticed by Pauli. It seems that adding the vacuum energy of matter could serve only to worsen the situation.

What has not been taken onboard in the argument above is that nonlinear metric perturbations may themselves fluctuate. All experiments done so far on vacuum fluctuations have clearly demonstrated the crucial dependence of their physical consequences on the choice of boundary conditions. A physically acceptable boundary condition for the spacetime metric should allow for its higher order fluctuations in the presence of nonlinear gravitational interactions. However, cosmological observations yield constraints on classical, averaged quantities. Hence the boundary conditions on the nonlinear metric perturbations can practically be imposed only after an appropriate averaging procedure. The statistical properties of the fluctuating part of the metric are determined by requiring that the resulting averaged classical equation matches the Einstein equation in empty space with an effective cosmological constant.

This paper is devoted to developing and presenting this general framework by showing that such classical boundary conditions are attainable if a stochastic conformal modulation of the metric appears at the microscopic level. It is known from the canonical analysis of general relativity that an oscillatory conformal modulation of a metric induces an effective stress-energy tensor with a negative kinetic term. This suggests that conformational fluctuations may yield a regularization mechanism for the overall amount of vacuum energy. We will show that, provided the conformal fluctuations satisfy certain statistical conditions, the vacuum structure of spacetime can be made compatible with Lorentz invariance at a classical scale together with an effective cosmological constant.

In what follows, all calculations will be done locally with respect to a physical inertial laboratory frame. When viewed at this laboratory scale, (apparently) empty spacetime appears to be flat and described by a Minkowski metric. We shall further assume that there exists a cutoff scale limiting the laboratory observer access to the microscopic structure of spacetime and its
effects upon physical probes. In particular, this cut-off scale also fixes the energy scale which is accessible by the laboratory observer. Throughout this paper we use the signature convention \((-+,+,+,+\) and, unless otherwise stated, we work in dimensionless units with \(c = G = \hbar = 1\). Moreover, when a tensor, e.g. \(A_{ab}\), is a functional of other tensors, e.g. \(B_{cd}, C_{e}, \ldots\), we will sometimes use the notation \(A_{ab}[B, C, \ldots]\), so that the expression \([B, C, \ldots]\) indicates functional dependence upon the tensors \(B_{cd}, C_{e}, \ldots\). When outside the brackets, the symbol \(B := g^{ab}B_{ab}\) will denote the trace. With these conventions, the Einstein equation with a cosmological constant \(\Lambda\) reads \(G_{ab}[g] + \Lambda g_{ab} = 8\pi T_{ab}\).

II. MICROSCOPIC RANDOM CLASSICAL GRAVITY

A. Random scale and stochastic classical approach

We consider the microscopic structure of spacetime at a scale \(l := \lambda L_{P}\), where \(L_{P} \approx 10^{-35}\) m is the Planck scale and where the dimensionless parameter \(\lambda \gtrsim 1\). It sets the benchmark between the full quantum gravity domain and a semiclassical domain, in which spacetime properties still inherit traces of the underlying quantum gravity physics, though being expected to be treatable by semiclassical means.

To study spacetime at such a small scale we extend Boyer’s framework of random electrodynamics \([40, 47]\) and describe spacetime vacuum fluctuations by means of stochastic classical fields. This kind of approach is necessary, as a definite quantum gravity theory is still lacking. Boyer’s work shows that having a stochastic classical field results by choosing appropriate fluctuating boundary conditions for the classical field equations. The classical-random field can simulate quantum vacuum fluctuations in the sense that a variety of physical phenomena, e.g. Casimir force between two plates, can be accounted for within the classical-random scenario.

Even in empty spacetime the vacuum energy of matter fields is still a source of gravity. Therefore we consider the Einstein equation for the spacetime metric \(\gamma_{ab}\)

\[
G_{ab}[\gamma] = 8\pi T_{ab}[\psi, \gamma],
\]

where \(T_{ab}\) is a model stress-energy tensor describing the overall vacuum energy contributions coming from all matter fields, collectively denoted by \(\psi\). The metric \(\gamma_{ab}\) and the matter fields \(\psi\) are considered to be randomly fluctuating at the scale \(l\). Accordingly we will refer to \(l\) as to the random scale and interpret it as the typical scale above which quantum vacuum properties can be approximately described by means of classical stochastic fields. Below \(l\) and closer to the Planck scale a full theory of quantum spacetime is required.

Since we are considering here vacuum metric fluctuations in an otherwise empty universe, these can in practice be assigned as random perturbations about some background metric \(g^{B}_{ab}\). In the following calculations we work locally and with respect to a physical inertial laboratory frame, whose typical scale \(L_{lab}\) is much larger than the random scale, yet small enough for the background metric to appear Minkowski in the appropriate coordinate system. With this choice we have \(g^{B}_{ab} = \eta_{ab}\) and

\[
\gamma_{ab} = \eta_{ab} + \gamma^{(1)}_{ab} + \gamma^{(2)}_{ab} + \ldots,
\]

where \(\gamma^{(n)}_{ab}\) indicates a small fluctuating term. Here and henceforth we follow the standard notation where a superscript \((n)\) denotes an \(n\)-th order perturbation. In the subsequent expansion of field equations, matter fields \(\psi\) will be treated as first order quantities.

The classical equation \((1)\) will be analyzed explicitly in sections III and IV for the first and second order metric perturbations. Boyer’s type fluctuating boundary conditions will be imposed on the linearized Einstein equation, so that the resulting randomly fluctuating tensor \(\gamma^{(1)}_{ab}\) can be interpreted as describing graviton fluctuations at the random scale \(l\). Its induced stress energy tensor is quadratic in the fluctuations and, together with the part of \(T_{ab}\), which is quadratic in the matter fields \(\psi\), it will enter the second order equation for the higher order metric perturbations. These are not expected to describe a physical field having its own amount of vacuum fluctuations and, accordingly, they will not be linked to Boyer’s type random boundary conditions. They will nonetheless be fluctuating as a result of their coupling to matter and graviton fluctuations.

B. Macroscopic Lorentz invariance of vacuum

Once the vacuum properties of spacetime at the random scale \(l\) are described in terms of a stochastic classical metric, all derived physical quantities, e.g. the Einstein tensor, are fluctuating also. As already noted by Boyer \([40]\) vacuum must be Lorentz invariant when viewed at some appropriate macroscopic scale. As a result the only way it can possibly contribute to the Einstein equation is through an effective cosmological constant term. Lorentz invariance also implies that vacuum statistical properties must be the same regardless of space position and direction, i.e. homogeneity and isotropy hold in a statistical sense.

In order to include these properties into our formalism we want to recover classical and smooth quantities from the fluctuating fields. This can be obtained as a result of an averaging process. To this end we follow the spacetime averaging procedure described in \([50, 54]\), so that fluctuating tensors average to tensors. This process involves a spacetime averaging over regions whose typical dimensions are large in comparison to the fluctuations typical wavelengths but smaller than the scale over which the background geometry changes significantly. Accordingly we introduce an averaging classical scale \(L\) such
that \( l \ll L \ll L_{\text{lab}} \). For completeness we also define a upper breakdown scale \( L_{\text{max}} \geq L_{\text{lab}} \), as that scale at which the background deviations from a flat geometry start to be significant.

While keeping in mind that we are here only considering ‘apparently’ empty spacetime, a final comment about the involved physical scales is in order. The following hierarchy holds, with \( L_P \ll l \ll L \ll L_{\text{lab}} \ll L_{\text{max}} \). The precise characterization of the classical scale \( L \) is that of the smallest scale at which classical, Lorentz invariant spacetime starts to emerge as a result of the averaging process. Though much larger than \( l \), the classical scale \( L \) is still expected to be very small in comparison to the laboratory scale (table I). The limits of the presently suggested theory are thus clearly set: (i) below the random scale \( l \) the random fields approximation breaks down and one should consider the more general case of small fluctuations upon a curved geometry; (ii) spacetime starts to be smooth and classical when viewed at the classical scale \( L \); moreover, from \( L \) and up to the upper breakdown scale \( L_{\text{max}} \), the averaged fluctuations look like small, classical metric perturbation upon a flat background; (iii) beyond the scale \( L_{\text{max}} \), the approximation of flat Minkowski background breakdowns down.

| Scale | \( L_P \) | \( l \) | \( L \) | \( L_{\text{max}} \) |
|-------|-----------|------|------|-------|
| Order of magn. (m) | \( \approx 10^{-35} \) | \( \gtrsim 10^{-35} \) | \( \gtrsim 10^{-35} \) | \( \gtrsim 1 \) |
| Physical domain | Quantum gravity | Random gravity | Classical gravity | Classical gravity |
| Background | None | \( \eta_{\text{lab}} \) | \( \eta_{\text{lab}} \) | \( \eta_{\text{lab}} \) |

TABLE I: A guide to relevant physical scales. The current formulation of the random gravity approach is supposed to yield a valid description of spacetime from the random scale \( l \), up to the breakdown scale \( L_{\text{max}} \). Below \( l \) a full quantum gravity theory would be needed. Above \( L_{\text{max}} \) the flat background approximation breaks down and one should consider the more general case of small fluctuations upon a curved geometry \( g_{\text{lab}}^B \).

C. Conformal fluctuations and the effective Einstein equation

The linearized Einstein equation with fluctuating boundary conditions will be connected to the vacuum fluctuations of graviton and will pose no problems. The second order equation will contain a source term, quadratic in the fluctuating fields \( \psi \) and \( \gamma_{\text{lab}}^{(1)} \), whose physical effect upon the properties of empty spacetime must be carefully assessed. To this end, and without loss of generality, we introduce an effective metric \( g_{ab} \), conformally related to the metric \( \gamma_{ab} \) via

\[
\gamma_{ab} = e^{2\alpha} g_{ab}. \tag{3}
\]

Hereafter we refer to \( \alpha \) as to the conformal field.

The equation satisfied by the effective metric \( g_{ab} \) is found in a standard way from \( G_{\text{lab}}[\gamma] = 8\pi T_{ab}[\psi, \gamma] \) by re-expressing the Ricci tensor \( R_{ab}[\gamma] \) and Ricci scalar \( R[\gamma] \) in terms of \( g_{ab} \), its compatible covariant derivative \( \nabla_{\alpha} \) and the conformal field \( \alpha \) as

\[
R_{ab}[\gamma] = R_{ab}[g] - 2\nabla_a \alpha_b - g_{ab} \Box \alpha + 2\alpha_a \alpha_b - 2g_{ab} \alpha_c \alpha_c.
\]

and

\[
R[\gamma] = e^{-2\alpha} \left\{ R[g] - 6\Box \alpha - 6\alpha_c \alpha_c \right\}.
\]

Here \( \alpha, \alpha_a := \nabla_a \alpha = \partial_a \alpha, \Box := \nabla^c \nabla_c \) while \( R[g] \) and \( R[\gamma] \) are the Ricci tensor and Ricci scalar for the effective metric \( g_{ab} \). The Einstein tensor \( G_{ab}[\gamma] = R_{ab}[\gamma] - \frac{1}{2} R[\gamma] \gamma_{ab} \) follows as

\[
G_{ab}[\gamma] = G_{ab}[g] - 2\nabla_a \alpha_b + 2g_{ab} \Box \alpha + 2\alpha_a \alpha_b + g_{ab} \alpha_c \alpha_c, \tag{4}
\]

and we have the following effective Einstein equation for the effective metric \( g_{ab} \)

\[
G_{ab}[g] = 8\pi (T_{ab} + \Sigma_{ab}). \tag{5}
\]

The effective stress energy tensor for the conformal field is

\[
\Sigma_{ab} := \Sigma_{ab}^1 + \Sigma_{ab}^2, \tag{6}
\]

where we have conveniently split \( \Sigma_{ab} \) into the two parts:

\[
\Sigma_{ab}^1 := \frac{1}{4\pi} \left( \nabla_a \alpha_b - g_{ab} \Box \alpha \right), \tag{7}
\]

and

\[
\Sigma_{ab}^2 := -\frac{1}{4\pi} \left( \alpha_a \alpha_b + \frac{1}{2} g_{ab} \alpha_c \alpha_c \right). \tag{8}
\]

The effective stress energy tensor defined above depends on \( \alpha \) only through its derivatives. Any physical situation with \( \alpha = \text{const.} \) would yield \( \Sigma_{ab} = 0 \). In this case the two metrics \( \gamma_{ab} \) and \( g_{ab} \) are equivalent and simply describe the same spacetime with a different choice of physical units. However, if \( \alpha \) varies throughout spacetime, the induced non vanishing stress energy tensor will affect the metric tensor \( g_{ab} \). In particular, conformal spacetime modulations induced by a randomly fluctuating \( \alpha \) would contribute, together with \( \psi \) and \( \gamma_{\text{lab}}^{(1)} \), to the vacuum structure of spacetime.

Since the metric \( \gamma_{ab} \) satisfies its own Einstein equation \( G_{ab}[\gamma] = 8\pi T_{ab}[\psi, \gamma] \), there is a fundamental arbitrariness connected to the introduction of the conformal split. This arbitrariness has two main implications: (i) a complete freedom in the choice of a possible dynamical equation for \( \alpha \), whose Boyer’s type boundary conditions would lead to a randomly fluctuating conformal field; (ii) once an equation is chosen the statistical amplitude of the conformal fluctuations would still be completely arbitrary.
The arbitrariness in the introduction of $\alpha$ will be lifted by imposing that the averaged second order Einstein equation for the metric $g_{ab}$ preserves vacuum Lorentz invariance at the classical scale and reproduces the appropriate empty space classical Einstein equation with an effective cosmological constant. We will show indeed that these requirements suggest a simple wave equation holding for $\alpha$ and fix precisely the statistical amount of the conformal fluctuations amplitude.

D. Characterization of the spacetime fluctuations

Within our framework, spacetime at the random scale presents two types of fluctuations: (i) conformal, scale fluctuations induced by the randomly fluctuating conformal field $\alpha$; (ii) effective metric fluctuations as described by $g_{ab}$. We now proceed to set up the relevant stochastic properties of $\alpha$ and $g_{ab}$.

(i) Conformal fluctuations. The field $\alpha$ describes small fluctuations in the local scale of spacetime. We assign it as a first order, stationary and stochastic field satisfying

$$\langle \alpha \rangle = 0, \quad \langle \alpha_{ab} \rangle = 0, \quad (9)$$

where $\langle \cdot \rangle$ indicates averaging. The particular choice $\langle \alpha \rangle = 0$ corresponds to the fact that the expectation value of the quantum field operator $\hat{\alpha}$ (object of the underlying quantum theory) in the appropriate vacuum state $|0\rangle$ should vanish.

In section III we will interpret the first order metric fluctuations in $\gamma_{ab}^{(1)}$ as describing zero point graviton fluctuations. Accordingly, we introduce the convenient notation

$$\beta_{ab} := \gamma_{ab}^{(1)}. \quad (10)$$

It is our choice to prescribe the conformal fluctuations in $\alpha$ to be in addition to and strongly uncorrelated with the graviton fluctuations. This choice serves to simplify the technical derivations presented in this paper. A more detailed analysis shows that our main results are nevertheless general enough without assuming the statistical independence of the conformal and graviton fluctuations. Further comments upon this particular choice are given near the end of section III. Accordingly, we impose the statistical property:

$$\langle \alpha, \cdots \beta_{ab}, \cdots \rangle = 0, \quad (11)$$

where “,$\cdots$” is a shorthand to indicate the derivatives. In particular this implies that

$$\langle \alpha \beta_{ab} \rangle = 0. \quad (12)$$

When $\beta_{ab}$ is split into its independent traceless $\beta_{ab}^\ast$ and trace $\beta := \eta^{ab} \beta_{ab}$ parts as

$$\beta_{ab} = \beta_{ab}^\ast + \frac{1}{4} \eta_{ab} \beta \quad (13)$$

the above relation also implies

$$\langle \alpha \beta_{ab}^\ast \rangle = \langle \alpha \beta \rangle = 0. \quad (14)$$

Note that indeces are here raised by $\eta_{ab}$ as we are working with first order quantities.

(ii) Effective metric fluctuations. Since empty spacetime with vacuum fluctuations appears approximately flat at the laboratory scale $L_{lab}$, we assume a microscopic random effective metric of the form

$$g_{ab} = \eta_{ab} + g_{ab}^{(1)} + g_{ab}^{(2)} \cdots, \quad (15)$$

where $\eta_{ab}$ is the Minkowski metric and $g_{ab}^{(n)}$ a stochastic tensor describing $n$th order fluctuations. These are small in the sense that, in some appropriate coordinate system, they satisfy $|g_{ab}^{(n)}| \ll 1$. We shall also use the shorthand

$$g_{ab} := g_{ab}^{(1)}. \quad (16)$$

For each order $n$ of the expansion, we have the classical (non-fluctuating) part of $g_{ab}^{(n)}$ given by

$$g_{ab}^{(n)C} := \langle g_{ab}^{(n)} \rangle \quad (17)$$

and define the fluctuating part of $g_{ab}^{(n)}$ by

$$g_{ab}^{(n)F} := g_{ab}^{(n)} - \langle g_{ab}^{(n)} \rangle \quad (18)$$

so that $g_{ab}^{(n)} = g_{ab}^{(n)C} + g_{ab}^{(n)F}$. It follows that $g_{ab}^{(n)F}$ has a zero mean:

$$\langle g_{ab}^{(n)F} \rangle = 0. \quad (19)$$

The average of the random effective metric at the scale $L$, yields the classical effective metric

$$\langle g_{ab} \rangle := g_{ab}^{C} = \eta_{ab} + g_{ab}^{C} + g_{ab}^{(2)C} + \mathcal{O}(3), \quad (20)$$

where the terms $g_{ab}^{(n)C}$ could account for some large scale deviation from flat spacetime due to vacuum fluctuations. In terms of the metric $\gamma_{ab}$ we have

$$\gamma_{ab} = e^{2\alpha} g_{ab}$$

$$= (1 + 2\alpha + 2\alpha^2 + \cdots) \times (\eta_{ab} + g_{ab} + g_{ab}^{(2)} + \cdots).$$

Keeping terms up to second order we have

$$\gamma_{ab} = \eta_{ab} + \beta_{ab} + \gamma_{ab}^{(2)} + \mathcal{O}(3), \quad (21)$$

with linear fluctuations,

$$\beta_{ab} = g_{ab} + 2\alpha \eta_{ab} \quad (22)$$

and with nonlinear metric fluctuations given by

$$\gamma_{ab}^{(2)} = g_{ab}^{(2)} + 2\alpha^2 \eta_{ab} + 2\alpha q_{ab}. \quad (23)$$
After averaging, the corresponding classical metric
\( \gamma_{ab}^C := \langle \eta_{ab} \rangle \) is, in the second order,
\[
\gamma_{ab}^C = \eta_{ab} + g_{ab}^C + g_{ab}^{(2)C} + 2 \langle \alpha^2 \rangle \eta_{ab} + 2 \langle \alpha q_{ab} \rangle + \mathcal{O}(3).
\]
(24)
The mean of the cross term involving \( \alpha \) and \( q_{ab} \) gives
\[
\langle \alpha q_{ab} \rangle = \langle \alpha \beta_{ab} \rangle - 2 \eta_{ab} \langle \alpha^2 \rangle = -2 \eta_{ab} \langle \alpha^2 \rangle,
\]
where we have used (12). Then
\[
\gamma_{ab}^C = (1 - 2 \langle \alpha^2 \rangle) \eta_{ab} + g_{ab}^C + g_{ab}^{(2)C} + \mathcal{O}(3).
\]
(26)
Comparing with (20) we see that, apart for the re-scaling factor \( 1 - 2 \langle \alpha^2 \rangle \) and up to second order, the classical properties of the spacetime metric can be described by the classical perturbations \( g_{ab}^C \) and \( g_{ab}^{(2)C} \) of the effective metric \( g_{ab} \). The scaling factor should be positive in order for the metric not to become singular. Within the current second order approximation, this implies the condition \( \langle \alpha^2 \rangle < 1/2 \). This bound is however likely to be an artifact due to the expansion truncation, since the full conformal factor \( \exp(2\alpha) \) is always positive.

It is worth further investigating the correlation properties between the conformal fluctuations \( \alpha \) and the effective metric linear perturbation \( q_{ab} \). By collecting all the traceless and trace parts in (22) we have
\[
\beta_{ab}^* + \frac{1}{4} \eta_{ab} \beta = q_{ab}^* + \frac{1}{4} \eta_{ab} (q + 8 \alpha),
\]
(27)where the traces are obtained as \( \beta = \eta^{ab} \beta_{ab} \) and \( q = \eta^{ab} q_{ab} \). The above equation implies the two algebraic constraints
\[
\begin{aligned}
q_{ab} &= \beta_{ab}^* \\
q &= \beta - 8 \alpha.
\end{aligned}
\]
(28)The first constraint involving the traceless parts implies
\[
\langle \alpha q_{ab}^* \rangle = 0,
\]
(29)as we know from (14) that \( \alpha \) and \( \beta_{ab}^* \) are uncorrelated. From this property and equation (28) we get
\[
\langle \alpha q \rangle = -8 \langle \alpha^2 \rangle.
\]
(30)The last two equations are important in that they show that the conformal field is really just correlated to the trace part of the linear metric perturbation \( q_{ab} \).

We conclude this section by remarking that, if the metric perturbation \( \beta_{ab} \) was put into a \( \mathbb{T} \mathbb{T} \) gauge, then \( \beta = 0 \) identically and, as a consequence, \( q = -8 \alpha \). In this case, apart for a multiplicative factor, \( \alpha \) would coincide with the trace of \( q_{ab} \). It follows that, once \( \beta = 0, \) \( q_{ab} \) cannot be put into a \( \mathbb{T} \mathbb{T} \) gauge as well, or it would be \( \alpha = 0 \), leading to the trivial case \( q_{ab} \equiv \gamma_{ab} \). Considering that gravitons would be described in the \( \mathbb{T} \mathbb{T} \) gauge by \( \beta_{ab}^* \) and also in view of the identity \( q_{ab}^* \equiv \beta_{ab}^* \), we can interpret the traceless part of the metric fluctuation \( q_{ab} \) as describing graviton in the spacetime \( (\mathcal{M}, g_{ab}) \). We also remark that the assignment of \( \alpha \) as an external fluctuating field effectively attaches physical meaning to the trace fluctuations of \( q_{ab} \).

### E. Matter field stress-energy tensors

Vacuum carries contributions from all physical fields of nature. As a result the random gravity framework is not complete without inclusion of the matter fields. These are considered to be in their ground state. We remark that the vacuum state for the various matter fields is well defined on a flat Minkowski background. However, since we work at the random scale \( l \), we represent matter fields vacuum fluctuations by means of first order and stochastic quantities. At this stage we adopt a minimum approach by neglecting the non-gravitational interactions between various matter fields components. The stress energy tensor \( T_{ab} \) in (15) describes matter fields and, ideally, carries contributions from all sectors of the Standard Model. Then \( T_{ab} = \sum_j T_{ab}^j \) where the index \( j \) runs over all matter fields.

The detailed microscopic expression of the generic component \( T_{ab}^j \) will depend quadratically upon the corresponding fluctuating matter field, as well as on the random metric \( \gamma_{ab} = \eta_{ab} + \sum_n \gamma_{ab}^{(n)} \). As a result the stress energy tensor at the random scale \( l \) is a also a stochastic quantity. The dependence on the \( \gamma_{ab}^{(n)} \) would account for the back-reaction of gravity fluctuations upon matter fields. However, as long as we work up to second order, only the flat classic background \( \eta_{ab} \) will appear but not the metric fluctuations \( \gamma_{ab}^{(n)} \). In this sense it is like having fluctuating fields on a Minkowski background and the effect of gravity upon the stress energy tensor would only appear as a third order effect. Within this framework, in an appropriate coordinate system, matter fields stress energy tensor component are quadratic fluctuating quantities defined on a flat background.

The corresponding energy density contribution can be defined at the classical scale \( L \) in a statistical sense through an averaging procedure. Provided the high frequency components are cutoff at the random scale \( l = \lambda L_p \), the average will be well defined and finite. Then the quantity \( \langle T_{ab} \rangle \) gives rise to a macroscopic stress-energy tensor at the classical scale \( L \). Because of homogeneity and isotropy holding at the classical scale, its most general form is
\[
\langle T_{ab} \rangle = \begin{pmatrix} \rho & 0 & 0 \\ 0 & p_1 & 0 \\ 0 & 0 & p \end{pmatrix}
\]
(31)where \( \rho := \langle T_{00} \rangle \) is the energy density and \( p := \frac{1}{3} \langle T_{ij} \rangle \), \( i = 1, 2, 3 \) is the pressure. The dominant energy condi-
tion, i.e. $\rho \geq 0$ and $\rho \geq p$, is normally thought to be valid for all known reasonable forms of matter [13], at least as long as the adiabatic speed of sound $dp/d\rho$ is less than the speed of light. This is true for massless fields since, in this case, $\rho = 3p$. In appendix A we show that the stronger condition $\rho > 3p$ is satisfied by massive fields in their vacuum state, at least in the ideal case in which interactions can be neglected and the field masses are much smaller than the Planck mass.

F. From random gravity to classical gravity

The physical picture is such that, at the random scale $l = \lambda L_{P}$, spacetime and matter fields are randomly fluctuating. As a result we have the random Einstein equation for the effective metric $g_{ab}$

$$G_{ab}[g] = 8\pi T_{ab}^{\text{eff}}[\psi, \alpha, g],$$

with $\psi$ denoting collectively all matter fields and where the effective stress energy tensor induced by the conformal fluctuations is

$$T_{ab}^{\text{eff}}[\psi, \alpha, g] = \Sigma_{1}^{a} [\alpha, g] + \Sigma_{2}^{a} [\alpha, g] + T_{ab}[\psi, \alpha, g],$$

the linear and quadratic parts in $\alpha$ being given in equations (7) and (8).

At the classical scale $L$ and above, the averaged microscopic equation will yield a corresponding classical equation in terms of smooth, non-random quantities $\langle G_{ab}[g] \rangle = 8\pi \langle T_{ab}^{\text{eff}}[\psi, \alpha, g] \rangle$. We will now proceed to apply an order by order expansion scheme to equation (32) and study under which conditions it takes the only form which is compatible with Lorentz invariance of vacuum, i.e. that of the empty space classical Einstein equation with an effective cosmological constant term.

III. FIRST ORDER ANALYSIS OF THE RANDOM EINSTEIN EQUATION

A. Linear solution and graviton vacuum fluctuations

We now proceed to linearize the microscopic random Einstein equation (32) and study the structure of its effective stress energy tensor, including vacuum conformal fluctuations. The terms $\Sigma_{2}^{a}$ and $T_{ab}$ contain fluctuations starting from second order so that only the term $\Sigma_{1}^{a}$ contributes. Using equation (7) we have the linearized random Einstein equation

$$G_{ab}^{(1)}[q] = 2 \partial_{\alpha} \partial_{\beta} \alpha - 2 \eta_{ab} \partial^{\gamma} \partial_{\gamma} \alpha,$$

where $G_{ab}^{(1)}$ is the usual linear operator resulting from the linearized Einstein tensor [51] and $q_{ab}$ represents the first order metric fluctuations. Here the indices are shifted using $\eta_{ab}$ and $\eta^{ab}$.

This equation can be simplified. Indeed, from $q_{ab} = \beta_{ab} - 2\alpha \eta_{ab}$, it is readily verified using the explicit expression for $G_{ab}^{(1)}$ given in appendix B that

$$G_{ab}^{(1)}[q] = G_{ab}^{(1)}[\beta - 2\alpha \eta] = G_{ab}^{(1)}[\beta] + 2 \partial_{a} \partial_{b} \alpha - 2 \eta_{ab} \partial^{c} \partial_{c} \alpha. \quad (35)$$

Comparing with (34) we see that the first order approximation to the microscopic Einstein equation takes the form

$$G_{ab}^{(1)}[\beta] = 0,$$

with $\beta_{ab} = q_{ab} + 2\alpha \eta_{ab}$ and

$$\langle \beta_{ab} \rangle = \langle q_{ab} \rangle = \eta_{ab}^{C}. \quad (37)$$

In the Lorentz gauge, equation (36) is the ordinary wave equation and it is usually considered to describe weak gravitational waves (GWs). Within our current random framework, it can be assigned fluctuating boundary conditions. As a result, the fluctuating tensor $\beta_{ab}$ is thought to represent the vacuum fluctuations of graviton. Note that, as explained in section II C, the conformal field $\alpha$ can in principle be assigned arbitrarily. Once this is done, $q_{ab}$ is fixed from $q_{ab} = \beta_{ab} - 2\alpha \eta_{ab}$ in such a way that (34) is automatically satisfied. We remark how the conformal fluctuations act in equation (34) as a fluctuating forcing source term, implying that the first order effective metric fluctuations $q_{ab}$ correlate to $\alpha$. This observation rules out the possibility $\langle q_{ab} \rangle = 0$ and motivates our choice $\langle \alpha \beta_{ab} \rangle = 0$ in section II D as, in fact, a physically reasonable scenario.

The corresponding equation holding at the classical scale $L$ is found by taking the mean in (34) or (36). Since $G_{ab}^{(1)}$ is a linear operator it commutes with the average operation $\langle \cdots \rangle$ and $\langle G_{ab}^{(1)}[\beta] \rangle = \langle G_{ab}^{(1)}[\beta] \rangle$, implying

$$G_{ab}^{(1)}[\eta^{C}] = 0,$$

where we have used (37). The first order, classical, metric correction $q_{ab}^{C}$ would describe classical gravitational waves propagating on a Minkowski background. Since we want to characterize spacetime vacuum properties only, we set these GWs to zero by choosing

$$q_{ab}^{C} = 0. \quad (39)$$

This represents the first order solution to our problem. We have here the important result that, in the first order, vacuum fluctuations do not give any visible effect on the structure of spacetime.

B. Drift of classical spacetime due to non linear vacuum effects

After taking into account the first order solution, the metric structure follows from equations (21)-(23) as

$$\gamma_{ab} = \eta_{ab} + \beta_{ab}^{F} + \gamma_{ab}^{(2)} + O(3), \quad (40)$$
with the zero mean, linear fluctuations,
\[ \beta_{ab}^F = g_{ab}^F + 2\alpha \eta_{ab} \]  
(41)
describing graviton vacuum fluctuations, and with second order fluctuations
\[ \gamma_{ab}^{(2)} = g_{ab}^{(2)} + 2\alpha^2 \eta_{ab} + 2\alpha \eta_{ab} g_{ab}^F. \]  
(42)
After averaging, the full classical metric follows as
\[ \langle \gamma_{ab} \rangle = (1 - 2\langle \alpha^2 \rangle) \eta_{ab} + g_{ab}^{(2)C} + \mathcal{O}(3), \]  
(43)
where we have used (25). This result shows that a deviation from a flat Minkowski background can arise as a second order drift effect due to nonlinear vacuum fluctuations. The classical equation satisfied by \( g_{ab}^{(2)C} \) is found by a second order analysis of the averaged random Einstein equation.

IV. SECOND ORDER ANALYSIS OF THE RANDOM EINSTEIN EQUATION

A. Second order equation and gravitons effective stress energy tensor in vacuum

Expanding the fluctuating Einstein tensor in the random equation (42) up to second order yields the result
\[ G_{ab} = G_{ab}^{(1)}[g] + G_{ab}^{(1)}[g^{(2)}] + G_{ab}^{(2)}[q] + \mathcal{O}(3), \]  
(44)
where \( G_{ab}^{(1)} \) is the linear operator introduced above while \( G_{ab}^{(2)} \) is quadratic in its argument and its explicit structure is reported in appendix B.

Because observations of the spacetime metric involve classical, macroscopic scales, we want to analyze the structure of the averaged second order equation. Taking the mean in (44), the first term vanishes by virtue of the first order solution. Then, using the fact that \( q_{ab} = g_{ab}^F \) we get
\[ \langle G_{ab} \rangle = G_{ab}^{(1)}[g^{(2)C}] + \langle G_{ab}^{(2)}[q^F] \rangle + \mathcal{O}(3). \]  
(45)
Using \( q_{ab}^F = \beta_{ab}^F - 2\alpha \eta_{ab} \) we obtain
\[ \langle G_{ab} \rangle = G_{ab}^{(1)}[g^{(2)C}] + \left( G_{ab}^{(2)}[\beta^F] + \langle G_{ab}^{(2)}[2\alpha \eta] \rangle \right). \]  
(46)
A fourth term deriving from \( \langle G_{ab}^{(2)}[q^F] \rangle \) and containing products of the derivatives of \( \beta_{ab}^F \) and \( \alpha \) averages to zero because of the fact that these two quantities are strongly uncorrelated as expressed in (11).

The second term in (46) gives rise to the effective stress-energy tensor
\[ \langle T_{ab}^{GW}[\beta^F] \rangle := -\frac{1}{8\pi} \left( G_{ab}^{(2)}[\beta^F] \right), \]  
(47)
which is quadratic in the first order fluctuations \( \beta_{ab}^F \) and acts, together with matter and conformal field fluctuations effective tensors, as an extra source for the classical, second order metric drift term \( g_{ab}^{(2)C} \).

The structure of this effective stress-energy tensor has been studied extensively in \cite{50}, where it is shown to be positive definite, traceless, and well defined in describing the energy content of linear gravitational fluctuations, as long as the wavelengths of the fluctuations involved are shorter than the typical scale over which the geometry of the background metric varies significantly. In our case this means that the graviton fluctuations wavelength contained in \( \beta_{ab}^F \) should be shorter than the breakdown scale \( L_{\text{max}} \). This would technically imply an infrared cutoff. However, since in the current version of the theory we are working with a flat background we will ignore this. Within our present framework, \( \langle T_{ab}^{GW} \rangle \) will be connected to the energy density due to graviton vacuum fluctuations. Given the properties of \( \langle T_{ab}^{GW} \rangle \) and the fact that it effectively describes gravitons as massless spin-2 particles, we can avoid considering its explicit averaged structure by simply including gravitons within the collection of matter fields, as described collectively by \( \psi \).

With this in mind, the averaged second order equation takes the form
\[ G_{ab}^{(1)}[g^{(2)C}] = 8\pi \left( T_{ab}^{\text{CF}(2)}[\alpha, \beta^F] + T_{ab}^{(2)}[\psi] \right) \]  
(48)
where \( \psi \) now includes gravitons as represented by \( \beta_{ab}^F \) and the superscripts \( (2) \) indicate second order quantities. The first term on the r.h.s. is the second order, averaged effective stress-energy tensor due to the conformal fluctuations, given by
\[ \langle T_{ab}^{\text{CF}(2)}[\alpha, \beta^F] \rangle := \left( \Sigma_{ab}^{1(2)}[\alpha, \beta^F] + \Sigma_{ab}^{2(2)}[\alpha] \right) \]  
(49)
- \frac{1}{2\pi} G_{ab}^{(2)}[\alpha \eta].

B. Effective stress energy tensor analysis

1. Conformal field

We analyze the three contributions to \( \langle T_{ab}^{\text{CF}(2)} \rangle \) given in (49). More details on the derivations of the following equations are given in appendix B. From (7), the first term can be seen to contain second order terms involving products of \( \alpha \) and \( g_{ab}^F \). Using equation (11) to replace \( g_{ab}^F \) and using the fact that \( \alpha \) and \( \beta_{ab}^F \) are uncorrelated, this term turns out to be independent of \( \beta_{ab}^F \) and takes the form:
\[ \langle \Sigma_{ab}^{1(2)}[\alpha, \beta^F] \rangle = \frac{1}{4\pi} \left( 2\alpha_{,a} \alpha_{,b} + \eta_{ab} \alpha_{,c} \alpha_{,c} \right). \]  
(50)
The second term in (49) follows from (8) replacing \( g_{ab} \) with \( \eta_{ab} \):
\[ \langle \Sigma_{ab}^{2(2)}[\alpha] \rangle = -\frac{1}{4\pi} \left( \alpha_{,a} \alpha_{,b} + \frac{1}{2} \eta_{ab} \alpha_{,c} \alpha_{,c} \right). \]  
(51)
Using the explicit form of the non linear operator \( G_{ab}^{(2)} \), the third term can be calculated to be:
\[
-\frac{1}{2\pi} \langle G_{ab}^{(2)}[\alpha]\rangle = -\frac{3}{4\pi} \langle \alpha, \alpha_b \rangle.
\] (52)

Collecting the above results, equation (49) becomes
\[
\langle T_{ab}^{\text{CF}(2)}[\alpha] \rangle = -\frac{1}{2\pi} \langle \alpha, \alpha_b \rangle - \frac{1}{4} \eta_{ab} \alpha_c \alpha_c.
\] (53)

Considering the stochastic properties of the conformal fluctuations in different spacetime directions to be uncorrelated and since \( \langle \alpha_c \rangle = 0 \) we have \( \langle \alpha, \alpha_b \rangle = \langle \alpha, \alpha_b \rangle = \delta_{ab} \langle \alpha^2 \rangle \), where \( \delta_{ab} \) is the four dimensional Kronecker symbol. Isotropy at the classical scale implies that, for all three space directions,
\[
\langle \alpha_i^2 \rangle = \langle |\nabla \alpha_i|^2 \rangle / 3, \quad i = 1, 2, 3,
\] (54)
where \( \langle |\nabla \alpha_i|^2 \rangle = \langle \alpha_i^2 \rangle + \langle \alpha_i \rangle + \langle \alpha_i^2 \rangle \). Then it is readily verified that the average of the tensor in (53) has the structure
\[
\langle T_{ab}^{\text{CF}(2)}[\alpha] \rangle = \begin{pmatrix}
\rho_\alpha & 0 & 0 & 0 \\
0 & \frac{\rho_\alpha}{3} & 0 & 0 \\
0 & 0 & \frac{\rho_\alpha}{3} & 0 \\
0 & 0 & 0 & \rho_\alpha \\
\end{pmatrix},
\] (55)

where
\[
\rho_\alpha := -\frac{3}{8\pi} \left( \langle \alpha_i^2 \rangle + \frac{\langle |\nabla \alpha_i|^2 \rangle}{3} \right) \leq 0
\] (56)

Quite remarkably it is traceless:
\[
\langle T_{ab}^{\text{CF}(2)}[\alpha] \rangle = 0.
\] (57)

The classical, average quantity \( \rho_\alpha \) corresponds to an effective, negative energy density connected to the vacuum conformal fluctuations.

2. Matter fields

The second order, averaged matter stress-energy tensor is given in equation (41) in terms of the matter vacuum energy density \( \rho \) and pressure \( p \). As already discussed, it now also includes gravitons as massless particles. Performing a trace decomposition it has the structure
\[
\langle T_{ab}^{(2)}[\psi] \rangle = \begin{pmatrix}
\rho_\psi' & 0 & 0 & 0 \\
0 & \frac{\rho_\psi'}{3} & 0 & 0 \\
0 & 0 & \frac{\rho_\psi'}{3} & 0 \\
0 & 0 & 0 & \rho_\psi' \\
\end{pmatrix} + \frac{1}{4} \eta_{ab} \langle T^{(2)} \rangle
\] (58)

where
\[
\rho_\psi' := \frac{3}{4} (\rho + p),
\] (59)
is the energy density of the traceless part and
\[
\langle T^{(2)} \rangle = 3p - \rho.
\] (60)
is the trace part, with \( T^{(2)} = \eta^{ab} T_{ab}^{(2)} \).

To proceed, it is conceptually useful to separate the fields \( \psi \) into massless fields (including gravitons) and massive fields, collectively denoted by \( \Psi \) and \( \Upsilon \) respectively. Massless fields contained in \( \Psi \) must have a traceless stress-energy tensor. This implies the usual equation of state
\[
\rho_\psi = 3p_\psi,
\] (61)
where \( \rho_\psi \) and \( p_\psi \) denote the contribution to the vacuum energy density and pressure coming form all massless fields, including graviton. In a similar way \( \rho_\Upsilon \) and \( p_\Upsilon \) will indicate the corresponding contributions from the massive fields. With this notation equations (54) and (60) can be re-expressed as
\[
\rho_\psi := \rho_\psi + \frac{3}{4} (\rho_\Upsilon + p_\Upsilon) \geq 0,
\] (62)
\[
\langle T^{(2)} \rangle = 3p_\Upsilon - \rho_\Upsilon,
\] (63)
showing explicitly that only the massive fields contribute to the trace part of matter vacuum energy density. Within our level of approximation, trace anomalies due to either matter interactions or metric fluctuations are not taken into account.

Finally, the averaged equation holding at the classical scale \( L \) and fixing the classical second order metric perturbation \( g_{ab}^{(2)C} \) is
\[
G_{ab}^{(1)}[g^{(2)C}] = 8\pi \left( \langle T_{ab}^{\text{CF}(2)}[\alpha] + T_{ab}^{(2)}[\Psi, \Upsilon] \rangle \right),
\] (64)
where \( \langle T_{ab}^{\text{CF}(2)}[\alpha] \rangle \) is given in (53) - (57) and \( \langle T_{ab}^{(2)}[\Psi, \Upsilon] \rangle \) in (58), (62) and (63).

C. Vacuum energy balance equation

The classical, second order equation (54) with no conformal fluctuations \( \alpha = \text{const.} \Rightarrow \Sigma_{ab} = 0 \) would lead to \( G_{ab}^{(1)}[g^{(2)C}] = 8\pi \left( \langle T_{ab}^{(2)}[\Psi, \Upsilon] \rangle \right) \). Given the structure of the vacuum, averaged matter stress energy tensor in (53) we immediately notice a problem connected to its traceless part. Indeed, even with the adopted ultraviolet cutoff at the random scale \( l \), the energy density \( \rho_\psi \) will attain a huge value and, more interestingly, will break the Lorentz invariance of vacuum at the classical scale. Even in the idealized situation where the matter fields were to be neglected, the traceless effective stress energy tensor associated with graviton, included in \( \Psi \), would still cause the same problem.
The above conclusion can be avoided once the effects of the nonlinear metric fluctuations are carefully taken into account. When these are prescribed to include some amount of conformal fluctuations, Lorentz invariance can be restored if the following vacuum energy balance equation holds

\[ \rho_a + \rho_\psi^* = 0. \] (65)

Then the only residual vacuum effect at the classical scale would come in the form of an effective cosmological constant \( \Lambda_M \) as defined by

\[ \Lambda_M := -2\pi G \left( T^{(2)}(\Upsilon) \right) = \frac{8\pi G}{c^4} \rho_M, \] (66)

where the matter related quantity \( \rho_M \) has the dimensions of an energy density and is defined as

\[ \rho_M := \frac{1}{4}(\rho_T - 3p_T). \] (67)

For clarity we have explicitly displayed in (66) the factors that convert a cosmological constant into the corresponding energy density.

We will study the implications of equations (65) and (66) in the following sections. They embody the choice of the classical boundary conditions for the classical metric perturbation \( g_{ab}^{(2)C} \) in the sense that, once they can be shown to hold, then the classical second order Einstein equation (64) in empty space reads

\[ C^{(1)}_{ab}[g^{(2)C}] = -\Lambda_M \eta_{ab}. \] (68)

This formally corresponds to the linear approximation of classical Einstein equation with a cosmological constant \( \Lambda \) and an approximatively flat metric. Indeed, if \( g_{ab} \approx \eta_{ab} + h_{ab}, \) with \( |h_{ab}| \ll 1, \) then the linear terms of \( G_{ab} = -\Lambda g_{ab} \) would yield \( C^{(1)}[h] = -\Lambda \eta_{ab}. \) This has precisely the same form as in (68), provided that we identify \( h_{ab} = g_{ab}^{(2)C} \) and \( \Lambda = \Lambda_M. \)

D. Effective cosmological constant

The current estimated amount of observed vacuum energy in the present epoch universe is [31]

\[ \rho_{\text{vacuum}}^{\text{obs}} \approx 5 \times 10^{-11} \text{ J m}^{-3} \approx 10^{-125} \rho_P, \] (69)

where \( \rho_P \approx 10^{19} \text{ GeV/L}^3 \) is the Planck energy density. This value is deduced from the observed large scale properties of the universe, including the approximatively flat geometry, as suggested by CMB anisotropies [57], and the present, small, cosmic acceleration indicated by type Ia supernovae observations [58]. Defining the Planck cosmological constant as

\[ \Lambda_P := \frac{8\pi G}{c^4} \rho_P, \] (70)

then (69) implies the following observational constraint on the present epoch cosmological constant

\[ 0 \leq \Lambda_M^{\text{obs}} \lesssim 10^{-125} \Lambda_P. \] (71)

The upper bound is extremely small, especially in comparison to the Quantum Field Theory (QFT) theoretical expectations usually reported in the literature. There it is commonly argued that, applying QFT within the Standard Model and up to the Planck scale, the expected cosmological constant would result to be of the order of \( \Lambda_P, \) hence the famous discrepancy of about 120 orders of magnitude with the observed value [31].

Our random gravity framework also predicts that an effective cosmological constant should indeed emerge due to vacuum energy effects. However, at odds with the usual claims, only the massive fields are expected to give a contribution. A necessary requirement for the observational constraint not to be violated is that \( \Lambda_M \) be positive. This implies that massive matter fields in the vacuum state should satisfy the following condition

\[ \rho_M = \frac{\Lambda_M c^4}{8\pi G} = \frac{1}{4}(\rho_T - 3p_T) \geq 0. \] (72)

In appendix A it is shown that, in a first approximation that treats matter fields in their vacuum state as a collection of non-interacting harmonic oscillators, \( \rho_M \) is positive and proportional to a squared mass parameter \( M^2, \) defined in equation (A22) as a weighted average of the squared masses of all particles comprised in \( \Upsilon. \) Specifically, we obtain in (A24) and (A25) that

\[ \rho_M = \frac{N_T \rho_P}{8\lambda^2} \left( \frac{M}{M_P} \right)^2 \] (73)

\[ \Lambda_M = \frac{N_T \Lambda_P}{8\lambda^2} \left( \frac{M}{M_P} \right)^2. \] (74)

The above clearly shows that \( \rho_M \propto \frac{\rho_P}{M^2} \left( \frac{M}{M_P} \right)^2, \) where \( M_P \approx 10^{19} \text{ GeV} \) is the Planck mass. This value can still be large and formally diverges as \( \lambda \) decreases. If a Planck cutoff scale \( \lambda \sim 1 \) were to be applied applied as per the “usual practice”, we would end up with an effective cosmological constant \( \Lambda_M \) much smaller than the Planck value \( \Lambda_P, \) roughly by a factor of \( (M/M_P)^2. \) Estimating this using the maximum mass scale of the Standard Model, namely \( M = M_H \approx 100 \text{ GeV}, \) then the reduction factor is \( (M_P/M_H)^2 \approx 10^{-34} \) and the resulting actual value for \( \Lambda_M \) still would exceed the observational value \( \Lambda_{\text{obs}} \) by about 90 orders of magnitude. However our present result, indicating that the cosmological constant seems to be related to the matter fields masses, opens up new a prospect for considerations that will be discussed in the final section of this paper.
V. ANALYSIS OF THE BALANCE EQUATION

A. Lorentz invariance of vacuum and conformal fluctuations

Turning now to the issue of Lorentz invariance, we can use \(55\) and the balance equation \(65\) to get

\[ 3 \langle \alpha^2 \rangle + \langle |\nabla \alpha|^2 \rangle = 8\pi \rho_\psi. \quad (75) \]

In principle, this equation can always be satisfied and comes in the form of a statistical condition for the conformal fluctuations \(\alpha\). It requires that their “intensity” be precisely adjusted so as to balance the traceless part of the vacuum energy contribution due to all the other fluctuating fields. At the present stage this requirement is imposed “from the outside”, on the basis that Lorentz invariance of vacuum should hold at the classical scale. However we would expect that a high energy, underlying quantum theory of gravity that incorporates conformal fluctuations as an essential ingredient, could lead to a more detailed and fundamental “internal” explanation for the balancing mechanism to hold. Some preliminary but promising indications along these lines are given in \([53, 59]\), where the canonical formulation of gravity is enriched to include conformal symmetry in both the geometrodynamics and connection variables approaches.

B. Characterizing \(\alpha\) using the wave equation

The conformal metric fluctuations provide a mechanism to restore Lorentz invariance of vacuum at the classical scale provided they have suitable statistical properties as expressed by the averages \(\langle \alpha^2 \rangle\) and \(\langle |\nabla \alpha|^2 \rangle\). We recall at this point that since the conformal field \(\alpha\) appears in our present formalism as an external field that we can prescribe at will, we are free to assign in such a way that it satisfies an extra dynamical constraint without affecting the Einstein equation. To be guided in our choice we notice that the second order effective stress-energy tensor in \(65\) is traceless and depends solely on the first derivatives of \(\alpha\). If it had to have a trace term, the simplest and most natural combination containing first derivatives of \(\alpha\) that would serve the purpose is

\[ \alpha^c \alpha_c \equiv -\alpha_\alpha^2 + |\nabla \alpha|^2. \quad (76) \]

Therefore a simple and reasonable statistical condition to impose on the conformal fluctuations appears quite naturally to be

\[ \langle \alpha^c \alpha_c \rangle \equiv -\langle \alpha_\alpha^2 \rangle + \langle |\nabla \alpha|^2 \rangle = 0. \quad (77) \]

Then a natural choice for the extra dynamical constraint is suggested by the fact that (i) \(\alpha\) is a scalar field and (ii) it is expected to satisfy the statistical relation \(77\). It is

\[ \partial^c \partial_c \alpha = 0, \quad (78) \]

i.e. the simple wave equation on Minkowski spacetime for a massless scalar field. Indeed it is readily verified that, provided the statistical properties of \(\alpha\) are stationary, equation \(78\) guarantees the condition \(\langle \alpha^2 \rangle - \langle |\nabla \alpha|^2 \rangle = 0\) to hold. Our choice is further motivated by the fact that the usual stress-energy tensor for a massless scalar field \(\phi\) on a flat spacetime has the form

\[ T_{\alpha \beta}^{\text{KG}} := \left( \phi_\alpha \phi_\beta - \frac{1}{4} \eta_{\alpha \beta} \phi^2 \right) + \frac{1}{4} \eta_{\alpha \beta} (\phi^2). \quad (79) \]

Provided \(\langle \phi^2 \rangle = 0\) and modulo the irrelevant rescaling factor of \(1/2\pi\), its average matches precisely the expression in \(55\), but with the negative sign.

Once the wave equation for \(\alpha\) is introduced, it can be solved with Boyer’s type (random) boundary condition and the overall (classical) statistical amplitudes set in such a way that the balance equation is satisfied. It is precisely in this sense that the conformal random fluctuations are independent of graviton’s and those of other matter fields. However, their overall amount is fixed at the classical level by the requirement of Lorentz invariance alone. Using \(77\) the balance equation takes the very simple form

\[ \langle |\nabla \alpha|^2 \rangle = 2\pi \rho_\psi. \quad (80) \]

An important physical justification of the wave equation \(79\) for \(\alpha\) is that its solution admits a Lorentz invariant spectral density suitable for describing vacuum fluctuations. Further, the form of this spectral density allows the effective stress-energy tensor of the conformal field to regularize the vacuum energy of matter as detailed below.

The formal analogy between the stress-energy tensors \(79\) and \(65\) and equation \(80\) above suggest that we introduce a rescaled conformal field as

\[ \varphi := \frac{\alpha}{\sqrt{2\pi}}. \quad (81) \]

With this notation the balance equation reads

\[ \langle |\nabla \varphi|^2 \rangle = \rho_\psi^* . \quad (82) \]

C. Isotropic power spectrum of the conformal fluctuations

The wave equation implies the usual dispersion relation \(\omega^2 = k^2\) relating frequency \(\omega\) and wave number magnitude \(k := |k|\). The conformal field can accordingly be expressed in Fourier components whose corresponding power spectral density \(S_\varphi(k)\) can be defined in a statistical sense \(41\). Since vacuum is isotropic at the classical scale we have accordingly \(S_\varphi = S_\varphi(k)\). It is then straightforward to show that

\[ \langle \varphi^2 \rangle = \frac{1}{(2\pi)^3} \int d^3 k S_\varphi(k), \quad (83) \]
\[ \langle |\nabla \varphi|^2 \rangle = \frac{1}{(2\pi)^3} \int d^3k k^2 S_\varphi(k), \]  
\[ \langle \varphi^2 \rangle = \frac{1}{(2\pi)^3} \int d^3k \omega^2(k) S_\varphi(k), \]

which, given the dispersion relation, are of course compatible with (77).

Since \( \int d^3k \equiv 4\pi \int dk k^2 \) these imply an integration over the wavelengths of the conformal fluctuations. In the present context we shall adopt a UV, i.e. high frequency, cutoff. We take the corresponding minimum wavelength to be equal to the random scale \( l = \lambda L_P \). This is because our random formalism based on the concept of classical fluctuating fields breaks down for scales shorter that \( l \).

We thus have the wave number cutoff

\[ k_\lambda := \frac{2\pi}{\lambda L_P}. \]  

From the physical viewpoint the dimensionless cutoff parameter \( \lambda \) should be in the range of \( \lambda \gtrsim 10^9 \), below which the quantum spacetime effects are expected to be important. However, as far as Lorentz invariance of vacuum is concerned, it can also be thought as a regularization parameter whose precise value is not critical. This is so because the same UV cutoff is to be applied to the calculation of the matter related energy density and pressure. As a result, although the resulting expressions for \( \langle |\nabla \varphi|^2 \rangle \) and \( \rho^*_\psi \) formally diverge as \( \lambda \to 0 \), their ratio and the related balancing mechanism turn out to be cutoff independent.

The statistical quantities defined above must be invariant under Lorentz transformations. To this end we take

\[ S_\varphi(k) := \frac{S_0 h G}{c^2 \omega(k)}, \]

where, being \( \varphi \) dimensionless, the combination of \( h, G \) and \( c \) gives the right dimensions for a power spectrum (i.e. \( L^3 \)), and \( S_0 \) is a dimensionless constant that controls the overall ‘normalization’. Our choice is motivated by the fact that this yields the Lorenz invariant measure \( d^3k/\omega(k) \) [60] and it implies an energy spectrum \( \rho(\omega) \propto \omega^3 \), which is shown by Boyer (see ref. below) to be the only Lorentz invariant vacuum energy spectrum for a massless field. Of course Lorentz invariance is preserved provided the cutoff \( k_\lambda \) is given by the same number for all inertial observers, as discussed in details by Boyer [61].

D. Explicit solution of the balance equation

Thanks to equations (83)-85, all the averages involving the conformal fluctuations simply depend upon the spectral parameter \( S_0 \) and, of course, the cutoff parameter \( \lambda \). Although it functions here as a regularization parameter, we recall that a specific value for \( \lambda \) may mark an effective transition scale from the random to the purely quantum domain. In principle, this effective cutoff value could be estimated, for example, through high precision measurement of the decoherence suffered by massive quantum particles propagating in vacuum [42, 43, 68, 69]. This possibility will be further investigated in the follow up paper [44].

Restoring full physical units by inserting back the constants \( c, \hbar \) and \( G \), equation (83) is

\[ \rho_P L_P^2 \langle |\nabla \varphi|^2 \rangle = \rho^*_\psi \]

where \( L_P \) is the Planck length. The dispersion relation with restored physical constants is \( \omega(k) = ck \) and the integrals (83)-(84) yield

\[ \langle \varphi^2 \rangle = \frac{S_0}{\lambda^2}, \]  

\[ \langle |\nabla \varphi|^2 \rangle = \frac{2\pi^2 S_0}{L_P^2 \lambda^4}. \]

Substituting (90) into (88) we have immediately

\[ S_0 = \frac{\lambda^4 \rho^*_\psi}{2\pi^2 \rho_P}. \]  

As discussed above, the cutoff parameter \( \lambda \) dependence is only apparent. The matter fields “traceless” energy density \( \rho^*_\psi \) is calculated in appendix A as

\[ \rho^*_\psi = \frac{\pi^2 N \rho_P}{\lambda^4}, \]  

where \( N \) is the total number of independent matter fields modes. Substituting into (91), this leads to the cutoff independent result

\[ S_0 = \frac{N}{2}, \]

representing precisely the power spectrum normalization that is needed to balance matter fields, traceless vacuum energy density as described by \( \rho^*_\psi \) and restore Lorentz invariance.

Plugging this value back into equation (83), the average conformal fluctuations squared amplitude follows as

\[ \langle \alpha^2 \rangle = \frac{N}{2\lambda^2}. \]

VI. INTERPRETATION AND DISCUSSION

If vacuum energy has any physical reality then it should act as a source of gravity according to the Einstein equation. Quantum field theory predicts a large amount of vacuum energy that may lead to a Planck size cosmological constant. However current observations of a
small cosmic acceleration suggest that the overall amount of vacuum energy in the universe must be many order of magnitudes smaller than predicted by the standard approach.

We have provided a general random gravity framework in which spacetime vacuum fluctuations are described by means of a stochastic metric satisfying the ordinary Einstein field equation. As it was shown by Boyer for the electromagnetic field, this is equivalent to specifying appropriate fluctuating boundary conditions. The resulting fluctuating classical fields mimic quantum fluctuations, in the sense that a variety of related phenomena can be accounted for. In this work we assume this to be valid in the case of spacetime fluctuations, up to some energy scale which is set by the random scale close to the Planck scale. These are described by fluctuating metric perturbations defined on a flat background metric. The resulting random gravity theory is built so as to provide a description of the apparent empty spacetime from the random scale up to some large scale \( L_\text{max} \), where the deviations of the background from a flat Minkowski start to be significant. To this end, a spacetime averaging procedure is employed to describe the passage from the fluctuating metric to a classical metric that describes spacetime beyond some classical scale \( L \gg l \).

It turns out that, in order for the resulting theory to be compatible with the Lorentz invariance of vacuum at the classical scale, the fluctuating metric must execute conformal fluctuations spontaneously. This yields a metric of the form \( \gamma_{ab} = \exp(2\alpha)g_{ab} \). This equation must properly be interpreted as a simple statement about the two metric tensors \( \gamma_{ab} \) and \( g_{ab} \) being \emph{conformally related}. The issue of which of them actually “contains” the conformal fluctuations is an interesting one. In the present framework this question is partially answered by prescribing the fluctuations in \( \alpha \) to be independent from the fluctuations in the first order metric perturbation \( \beta_{ab} = \gamma_{ab}^{(1)} \), supposed to mimic graviton vacuum fluctuations. Once this is done, \( \alpha \) appears to be related to the trace of \( g_{ab} = g_{ab}^{(1)} \), i.e. the first order perturbation of the effective metric \( g_{ab} \). In principle we can cast \( \beta_{ab} \) into a \( TT \) gauge, so that \( \beta = 0 \) and \( \alpha \equiv -q/8 \), where \( q \) denotes the traces. In this sense we can claim that conformal fluctuations are actually “contained”, to \emph{first order}, within the effective metric \( g_{ab} \) but \emph{not} within the metric \( \gamma_{ab} \). However, conformal fluctuations propagate through nonlinearly coupling in such a way that \( \alpha \) should affect the higher order metric perturbations of both metric tensors \( g_{ab} \) and \( \gamma_{ab} \).

In the first order approximation \( \alpha \) does not affect graviton fluctuations and it just acts as a linear source of random noise for \( g_{ab} \). This happens in such a way that, at the classical scale, the classical tensors \( \langle \beta_{ab} \rangle \) and \( \langle g_{ab} \rangle \) coincide and satisfy the usual linearized equation for GWs with no extra sources. In the second order the conformal fluctuations described by \( \alpha \) induce a traceless, negative definite, effective stress energy tensor which acts as a source for the effective metric \( g_{ab} \). Under some circumstances, this can counterbalance the predominant part of vacuum energy due to matter fields and coming from the traceless part of their associated vacuum stress energy tensor. Without such a counterbalancing mechanism, the associated stress-energy tensor attains a large, close to Planckian magnitude, thereby breaking Lorentz invariance, even with an UV cutoff set at the random scale \( l = \lambda L_p \).

We showed that Lorentz invariance can be restored by the conformal fluctuations by means of a simple balancing mechanism, in such a way that the large, traceless, matter fields contribution is canceled out by the negative, conformal contribution. This regularization process fixes the averaged conformal fluctuation squared amplitude unambiguously once these are chosen to have a Lorentz invariant power spectrum and to satisfy certain specific statistical properties. These are shown to be compatible with a simple massless wave equation ruling the conformal fluctuations dynamics. This extra dynamical constraint is consistent with the Einstein equation since the conformal field is originally prescribed as an external field and, in principle, can be assigned arbitrarily. The main choice related to its statistical independence with graviton fluctuations at the random level, together with the Lorentz invariance requirement at the classical level, effectively attach physical meaning to the conformal field fluctuations and fix their statistical properties.

The dimensionless cutoff parameter \( \lambda \) introduced in the process is likely to mark the transition from the random domain to the purely quantum domain at the Planck scale. In this sense it sets the small scale breakdown limit to our current theoretical approach. However it is very interesting to note that, in a similar way as to what happens with the Casimir effect, the spectral density of the conformal fluctuations that results from the balance mechanism is cutoff independent. This is a strong result, “almost” entirely based upon general relativity alone. The “almost” simply referring to the fact that, to implement the regularization process, it has been necessary to modify the classical boundary conditions and enter into the random fields domain.

The final outcome of the regularization process is (i) a Lorentz invariant “empty” spacetime at all classical scales and (ii) an effective cosmological constant \( \Lambda_M \), emerging as a second order nonlinear effect due to the trace part of the matter fields vacuum stress-energy tensor. The effective cosmological constant thus appears as the only remnant effect, built up from the quantum vacuum, which could in principle have an effect at a large scale. At odds with what commonly stated, the massless fields do not seem to play any role in contributing to the effective cosmological constant, at least within the present second order nonlinear approach. Our final equation for \( \Lambda_M \) shows that it is proportional to a squared mass parameter \( M^2 \), obtained as a weighted average over all massive fields.

Even when a usual cutoff at the Planck scale (i.e. \( \lambda \sim 1 \)) is applied, our formula yields a leftover, effec-
tive cosmological constant which is roughly 30 orders of magnitude lower than the usual Planckian value, though still much larger than the almost vanishing value \( \Lambda^{\text{obs}} \approx (8\pi G/c^4)10^{-125}\rho_P \) suggested by cosmological observations. Our current result could however be reinterpreted under a slightly different and interesting perspective: by identifying \( \Lambda_M = \Lambda^{\text{obs}} \), we get an equation for the matter field averaged effective mass \( M \), namely

\[
M_{\lambda} \approx \sqrt{\frac{\Lambda^{\text{obs}}}{(\Lambda_P/\lambda)^4}} \equiv \sqrt{\frac{\rho_{\text{vacuum}}^{\text{obs}}}{(\rho_P/\lambda)^4}} \quad (95)
\]

Once the numbers are plugged in this yields the result

\[
M \approx \lambda M_P \sqrt{10^{-125}} \approx \lambda \times 10^{-44} \text{ GeV}.
\]

This result shows that, for any realistic value of the cutoff parameter, including the Planckian value \( \lambda \approx 1 \), the average matter fields effective mass matching the currently observed cosmological constant value is basically vanishing. This result is interesting since it links to some scenarios discussed in the literature and suggesting that, at the most fundamental level, matter fields bare masses may in fact be zero [62, 63]. The observed particle masses would appear as a result of interactions. One example would be a Higgs, complex scalar field based interaction [64, 65, 66], but other possibilities are being considered [63, 67].

The particular approach used in this paper, where matter interactions are not included, could be relevant in this perspective if we interpret the effective mass \( M \) as the average bare mass of the matter fields in their vacuum state. If this is zero as suggested by some authors and, we add, if the actual mass generating mechanisms are not effective in the vacuum state, then a possible solution to the riddle of the cosmological constant problem could be “almost” at hand. We say “almost” since any sound statement would first require a solid and deep understanding of the nature of vacuum and, especially, of the meaning of mass and how it comes to be. At this stage we point out however that, with the conformal fluctuations as a possible mechanism to yield Lorentz invariance and neutralize most part of the matter fields vacuum energy density, the possible connection between cosmological constant and effective (bare) matter fields mass is very interesting.

A widely accepted present view is this: While physics may well restore conformal invariance at a unified high energy scale where all fundamental particles become massless, these particles nonetheless gain an effective mass by settling into a local minimum of the Higgs potential (or any other mass generating mechanism) at low energies. This implies that the effective cosmological constant \( \Lambda_M \) is built from nonzero mass up to the Higgs (or equivalent) energy scale, say \( M_H \sim 100 \text{GeV} \), corresponding to an effective cutoff at \( \lambda \approx M_P/M_H \) in [74]. This equation, together with \( M \sim M_H \) then yields the estimated effective cosmological constant to be

\[
\Lambda_M \sim \Lambda_P \left( \frac{M_H}{M_P} \right)^4 \sim 10^{-68} \Lambda_P \sim 10^{57} \Lambda^{\text{obs}} \quad (96)
\]

which still exceeds the observational value by about 57 orders of magnitude. However, this estimate assumes a simplistic zero value of the local minimum of the Higgs potential. It has been suggested that, it is the local maximum, where the Higgs field vanishes, that should be zero [29]. While other “calibrations” may exist, it is interesting to note that if this is true, then the equilibrium potential energy of the Higgs field would amount to an additional effective cosmological constant, that is (roughly within the same order of magnitude) equal and opposite to \( \Lambda_M \) given in [66]. Whether or not this possible cancelation would result in a net effective cosmological constant even closer to the observational value is an intriguing issue for future investigation, where further interaction effects ought to be considered as well.

We conclude with the following comment. We have shown that general relativity permits spontaneous fluctuations of metric, and the conformal fluctuations in particular. These bear dynamical consequences on the evolution of the averaged metric. Once the spontaneously fluctuating conformal field is set to regulate the net amount of vacuum energy, it appears to have somehow a life on its own. In this sense the conformal field is as real as the vacuum energy. One then wonders why we never have detected it. In this respect we notice that even standard gravitational waves are so weak that, to date, still fail to be detected. Hopefully projects such as LIGO and LISA will allow a direct unambiguous detection of GWs. In the case of vacuum conformal fluctuations, we are talking about tiny modulations of space time that happen at the random scale, i.e. \( \gtrsim 10^{-35} \text{ m} \). These modulations average out at the classical scale. This makes them ideally very difficult to detect directly, even though there is a possibility that matter wave interferometry could provide experimental evidence in measuring the related decoherence on massive, non-relativistic quantum particles [68, 69]. At this stage, it is clear however that, once the role of the conformal fluctuations in regularizing the vacuum energy of the universe has been clarified, an indirect evidence of their reality could be, after all, the fact that observations seem to hint at a vacuum which is compatible with Lorentz invariance at the classical scale and a positive, small, cosmological constant.

The inclusion of spontaneous conformal spacetime fluctuations as a fundamental ingredient of the quantum vacuum seems to yield promising results. More efforts will be needed to clarify their deeper, quantum, meaning and cast further light upon the significance and the mutual relation between, quantum vacuum, cosmological constant, as well as the generation mechanism and the ultimate nature of mass.
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APPENDIX A: ENERGY DENSITY AND PRESSURE OF MATTER FIELDS USING THE FREE FIELD APPROXIMATION

Here we estimate the matter fields related quantities \( \rho_c \) and \( \rho_M \) in full physical units. From their definitions in (62) and (72), we need evaluate the three quantities \( \rho_P \), \( \rho_T \) and \( \rho_T \). These can in principle be obtained with quantum field theory. However this is not a trivial task. Within the Standard Model, vacuum energy density is estimated to be given by, at least, three main contributions \[ \phi \text{ : vacuum zero-point energy plus virtual particles fluctuations, } \]

\[ (ii) \text{ QCD gluon and quark condensates} \]

\[ (iii) \text{ Higgs field.} \]

Through the following calculation we want to get a first approximation estimate, by neglecting fields interactions and by describing the free-field configuration as a collection of decoupled harmonic oscillators of frequency

\[ \omega_k = \sqrt{c^2 k^2 + m^2 c^4}, \]  

(A1)

where \( k \) is the norm of the spatial wave vector \( k \) and \( m \) is some effective mass of the field quanta under examination. This is quite accurate for the EM field but it is likely to be quite a crude approximation for e.g. the QCD sector. By doing so we are neglecting higher order contributions to the vacuum energy density as well as the nonlinear and strong coupling effects of QCD and a detailed treatment of the Higgs fields. Nonetheless we expect to obtain a meaningful lower bound estimate to the vacuum energy.

We consider an arbitrary matter field of effective mass \( m \) within the Standard Model. The field is comprised in \( \Psi \) or \( \Upsilon \) depending on whether it is \( m = 0 \) or \( m \neq 0 \). We calculate a lower bound to the associated vacuum energy. The field is comprised in the vacuum energy.

We model each independent field component as a scalar field \( \phi \) with mass \( m \) and associated Klein-Gordon stress-energy tensor

\[ T_{ab}^{KG}[\phi] = \phi_a \phi_b - \frac{1}{2} \eta_{ab} \left[ \phi^c \phi^c + \frac{m^2 c^2}{\hbar^2} \phi^2 \right]. \]  

(A2)

From this we see that \( \phi^2 \) has the dimension of force. Regarding \( \phi \) as in its zero-point fluctuating state, we shall assume it to be stationary over spacetime and having an isotropic spectral density \( S_\phi(k) \), so that the mean squared field is given by

\[ \langle \phi^2 \rangle = \frac{1}{(2\pi)^3} \int d^3 k S_\phi(k). \]  

(A3)

Furthermore, the mean squared time derivative \( \phi_t = c\phi,0 \) satisfies

\[ \langle \phi_t^2 \rangle = \frac{1}{(2\pi)^3} \int d^3 k \omega_k^2 S_\phi(k) \]  

(A4)

and the mean squared gradient \( \langle \nabla \phi \rangle = \phi, i \) for \( i = 1, 2, 3 \) satisfies

\[ \langle \nabla \phi \rangle = \frac{1}{(2\pi)^3} \int d^3 k k^2 S_\phi(k). \]  

(A5)

It follows from (A1), (A3), (A4), (A5) that

\[ \langle \phi^2 \phi_a \rangle = \frac{1}{c^2} \langle \phi_t^2 \rangle + \langle \nabla \phi \rangle^2 = -\frac{m^2 c^2}{\hbar^2} \langle \phi^2 \rangle. \]  

(A6)

Using (A4), (A5) and (A6) we see that the mean stress-energy tensor given by (A2) becomes

\[ \langle T_{ab}^{KG} \rangle = \phi_a \phi_b, \]  

(A7)

which yields the effective energy density

\[ \rho = \langle T_{00} \rangle = \frac{1}{c^2} \langle \phi_t^2 \rangle = \frac{1}{(2\pi)^3} \int d^3 k \omega_k^2 S_\phi(k) \]  

(A8)

and the effective pressure

\[ p = \frac{1}{3} \langle T_{ii} \rangle = \frac{1}{3} \langle \nabla \phi \rangle^2 = \frac{1}{3(2\pi)^3} \int d^3 k k^2 S_\phi(k). \]  

(A9)

A useful combination follows from (A1), (A8) and (A9) as

\[ \rho - 3p = \frac{1}{(2\pi)^3} \int d^3 k \frac{m^2 c^2}{\hbar^2} S_\phi(k). \]  

(A10)

The spectral density \( S_\phi(k) \) itself can be determined through the well-known zero point energy density expression

\[ \rho = \frac{1}{(2\pi)^3} \int d^3 k \frac{\hbar \omega_k}{2}, \]  

(A11)

that adds up contributions from all wave modes of \( \phi \). Comparing (A8) and (A11) we see that the spectral density \( S_\phi(k) \) takes the Lorentz invariant form

\[ S_\phi(k) = \frac{\hbar c^2}{2\omega_k}. \]  

(A12)

Substituting (A1) and (A12) into (A9) and (A10) and integrating using \( \int d^3 k = 4\pi \int dk k^2 \) up to the cutoff value \( \lambda = \frac{2\pi c}{L} \), we obtain

\[ \rho = \frac{\hbar}{4\pi^2} \int_0^{k_\lambda} dk k^2 \sqrt{c^2 k^2 + m^2 c^4}, \]  

(A13)
and
\[
p = \frac{\hbar c^2}{12\pi^2} \int_0^{k_\Lambda} dk k^4 \sqrt{c^2 k^2 + \frac{m^2 c^4}{\hbar^2}}. \tag{A14}
\]
Introducing the dimensionless variable \( y := \frac{4k}{m c} \), we can rewrite (A13) and (A14) as
\[
\rho = \frac{\rho_p^4}{4\pi^2} \left( \frac{m_P}{\lambda} \right)^4 \int_0^{2\pi} dy y^2 \sqrt{1 + y^2}, \tag{A15}
\]
and
\[
p = \frac{\rho_p^4}{12\pi^2} \left( \frac{m_P}{\lambda} \right)^4 \int_0^{2\pi} dy y^2 \sqrt{1 + y^2}, \tag{A16}
\]
in terms of the Planck energy density \( \rho_P \) and the effective mass of the field in units of \( M_P/\lambda \), i.e. \( m_\lambda := \frac{m}{M_P/\lambda} \).

For \( m_\lambda \ll 1 \) we can approximate (A15), (A16) by
\[
\rho = \frac{\pi^2 \rho_p}{\lambda^4} + \frac{\rho_p^4}{4\lambda^4} \left( \frac{m}{M_P} \right)^2, \tag{A17}
\]
and
\[
p = \frac{\pi^2 \rho_p}{3\lambda^4} - \frac{\rho_p^4}{12\lambda^4} \left( \frac{m}{M_P} \right)^2, \tag{A18}
\]
up to adding \( \mathcal{O}(m_\lambda^4) \) terms. This approximation is physically well justified. The heaviest particles in the Standard Model are the quark top, with \( m_t \approx 173 \) GeV, the weak bosons \( W^\pm \), with \( m_{W^\pm} \approx 80 \) GeV and the Z boson with \( m_Z \approx 91 \) GeV. All the other particles have \( m \lesssim 1 \) GeV. Being the Planck mass of the order of \( 10^{18} \) GeV, it will be \( m_\lambda \ll 1 \) as long as the cutoff parameter satisfies \( \lambda \lesssim 10^{15} \). This is a safe upper bound, as the stochastic classical conformal fluctuations are expected to have a cutoff which should not exceed \( \lambda \approx 10^2 - 10^3 \). [51]

Within this approximation, by subtracting (A17) and (A18) we get
\[
\rho = \frac{3}{2} p = \frac{\rho_p^4}{2\lambda^4} \left( \frac{m}{M_P} \right)^2. \tag{A19}
\]
This relation can also be obtained directly from the right hand side of (A10) by following through the steps leading from (A13) to (A18).

The expressions (A17) and (A18) are the sums of two terms, one of which depends upon the effective mass of the field. Collecting the contribution from all massless fields we have the corresponding energy density \( c^2 \rho_P \) and pressure \( p_P \) given by
\[
\rho_P = 3 p_P = \frac{N_P \pi^2 \rho_p}{\lambda^4}, \tag{A20}
\]
where \( N_P \) is the total number of independent components for all massless fields.

In the massive case, the total energy density \( \rho_T \) follows from all contributing fields with individual effective mass \( m_i \) and number of independent components \( N_{T_i} \). Using (A17) we have
\[
\rho_T = \frac{N_T \pi^2 \rho_p}{\lambda^4} + \frac{N_T^2 \rho_P}{4\lambda^2} \left( \frac{M}{M_P} \right)^2, \tag{A21}
\]
where \( N_T := \sum_i N_{T_i} \) is the total number of independent components of the massive fields and
\[
M^2 := \frac{\sum_i N_{T_i} m_i^2}{N_T} \tag{A22}
\]
is the weighted average squared effective mass. The total pressure of the massive matter fields then follows from (A18) to be
\[
p_T = \frac{N_T \pi^2 \rho_p}{3\lambda^4} - \frac{N_T^2 \rho_P}{12\lambda^4} \left( \frac{M}{M_P} \right)^2. \tag{A23}
\]
We therefore have
\[
\rho_M := \frac{1}{4} (\rho_T - 3p_T) = \frac{N_T \rho_P}{8\lambda^2} \left( \frac{M}{M_P} \right)^2, \tag{A24}
\]
corresponding to the effective cosmological constant
\[
\Lambda_M = \frac{N_T \Lambda_P}{8\lambda^2} \left( \frac{M}{M_P} \right)^2. \tag{A25}
\]
Finally the leading contribution of \( \rho_\psi \) can be obtained by substituting (A20), (A21) and (A23) into (22) to be
\[
\rho_\psi = \frac{\pi^2 N P \rho_P}{\lambda^4}, \tag{A26}
\]
up to adding \( \mathcal{O}(M_\lambda^4) \) terms, where
\[
M_\lambda := \frac{M}{M_P/\lambda}, \tag{A27}
\]
and
\[
N := N_\psi + N_T \tag{A28}
\]
is the total number of independent components for massless and massive fields. The ratio between the traceless part and effective cosmological constant energy densities for the matter fields scales as
\[
\frac{\rho_M}{\rho_\psi} \propto M_\lambda^2 \ll 1, \tag{A29}
\]
where the inequality holds within the Standard Model, with \( M \approx 10^2 \) GeV, and for any realistic value of the cutoff parameter.

**APPENDIX B: SOME TECHNICAL DERIVATIONS**

The linearized Einstein tensor has the explicit form
\[
G^{(i)}_{ab} [h] := \frac{1}{2} \partial^c \partial_b h_{ac} + \frac{1}{2} \partial^c \partial_a h_{bc} - \frac{1}{2} \partial^c \partial_b h_{ac} - \frac{1}{2} \partial^c \partial_a h_{bc} - \frac{1}{2} \partial_c \partial_b h - \frac{1}{2} \partial_c \partial_a h_{ab}. \tag{B1}
\]
where \( h := \eta^{ab}h_{ab} \) denotes the trace. It is straightforward to verify that, for \( h_{ab} \equiv 2\alpha\eta_{ab} \) this yields

\[
G^{(1)}_{ab}[2\alpha\eta_{ab}] = 2\eta_{ab}\partial^c\partial_d\alpha - 2\partial_a\partial_b\alpha,
\]

(B2)
as used in \([54]\).

Turning now to the calculation of \(< T^{CF}_{ab}(2) > \) in \([49]\), the full explicit expression of \(- < G^{(2)}_{ab} > /8\pi \) can be simplified to yield the following nonlinear differential operator \([11]\)

\[
\partial_a\tilde{h}_{cd}\partial_b\tilde{h}^{cd} - \frac{1}{2}\partial_a\tilde{h}\partial_b\tilde{h} - \partial_a\tilde{h}_{bc}\partial_d\tilde{h}^{cd} - \partial_b\tilde{h}_{ac}\partial_d\tilde{h}^{cd},
\]

(B3)

where

\[
\tilde{h}_{ab} := h_{ab} - \frac{1}{2}\eta_{ab}h
\]

(B4)
is the trace reversed metric perturbation. Given this expression it can easily be verified that, for \( h_{ab} \equiv 2\eta_{ab}\alpha \),

\[
- \frac{1}{8\pi} < G^{(2)}_{ab}[2\eta\alpha] > = - \frac{1}{2\pi} < G^{(2)}_{ab}[\eta\alpha] > = - \frac{3}{4\pi} \langle \alpha, \alpha \rangle.
\]

(B5)

To conclude, the term \( \Sigma^{(1)}_{ab} \) follows from taking the second order terms in equation \([41]\). Expressing the covariant derivatives using the linearized connection

\[
\Gamma^{c(1)}_{ab} := \frac{1}{2}\eta^{cd}(\partial_aq_{bd} + \partial_bq_{ad} - \partial_dq_{ab}),
\]

(B6)
where \( q_{ab} := g^{(1)}_{ab} \) is the first order effective metric perturbation, gives

\[
8\pi\Sigma^{(2)}_{ab} := (2\nabla_a\alpha, b - 2g_{ab}\Box\alpha)^{(2)}
\]

\[
= - \eta^{cd}(\partial_aq_{bd} + \partial_bq_{ad} - \partial_dq_{ab})\alpha_c + \eta_{ab}\eta^{de}(\partial_dq_{ef} + \partial_eq_{df} - \partial_fq_{de})\alpha_c
\]

\[
- 2\partial_bq^{cd}\partial_d\alpha + 2\eta_{ab}\eta^{de}\partial_d\alpha_c.
\]

(B7)

Using \( q_{ab} = \beta_{ab} - 2\alpha\eta_{ab} \) to eliminate \( q_{ab} \), equation \([B7]\) gives a series of terms involving products of \( \beta_{ab} \) and \( \alpha \), whose corresponding averages vanish thanks to \([11]\). What is left is a sequence of terms, quadratic in \( \alpha \), which are obtained from \([B7]\) upon substitution of \( q_{ab} \) with \(-2\alpha\eta_{ab} \). Their average can be evaluated to yield the final result

\[
8\pi < \Sigma^{(2)}_{ab} > = 4(\alpha, a\alpha, b + 2\eta_{ab}\alpha^c\alpha_c).
\]

(B8)

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