Some Curvature Properties on a Special Paracontact Kenmotsu Manifold with Respect to Semi-Symmetric Connection

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Abstract The object of the present paper is to study some properties of curvature tensor $\tilde{R}$ of a semi-symmetric non-metric connection $\tilde{\nabla}$ in a type of special para contact Kenmotsu (briefly SP-Kenmotsu) manifold. We have deduced the expressions for curvature tensor $\tilde{R}$ and the Ricci tensor $\tilde{S}$ of $M_n$ with respect to semi-symmetric non-metric connection $\tilde{\nabla}$. It is proved that in an SP-Kenmotsu manifold if the curvature tensor of the semi-symmetric non-metric connection vanishes then the manifold is projectively flat.

Keywords: curvature tensor, ricci tensor, projective curvature tensor, non-metric connection, sp-kemotsu manifold

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1. Introduction

Friedmann and Schouten [1,2] introduced the idea of semi-symmetric linear connection on a differentiable manifold. Hayden [3] introduced semi-symmetric metric connection on a Riemannian manifold and it was further developed by Yano [4]. Semi-symmetric connections play an important role in the study of Riemannian manifolds. There are various physical problems involving the semi-symmetric metric connection. For example, if a man is moving on the surface of the earth always facing one definite point, say Jerusalem or Mekka or the North pole, then this displacement is semi-symmetric and metric [1]. In 1975, Prvanović [5] introduced the concept of semi-symmetric non-metric connection with the name pseudo-metric, which was further studied by Andonie [6,7]. The study of semi-symmetric non-metric connection is much older than the nomenclature ‘non-metric’ was introduced. In 1992, Agashe and Chafle [8] introduced a semi-symmetric connection $\tilde{\nabla}$ satisfying $\tilde{\nabla}X g \neq 0$ on a Riemannian manifold, and called such a connection as semi-symmetric non-metric connection. Later, the curvature properties of the connection in an SP-Sasakian manifold were studied by Bhagwat Prasad [9], and many others.

On the other hand, in 1976, Sato [10] defined the notions of an almost paracontact Riemannian manifold. After that, T. Adati and K. Matsumoto [11] defined and studied para-Sasakian and SP-Sasakian manifolds which are regarded as a special kind of an almost contact Riemannian manifolds. Before Sato, in 1972, Kenmotsu [12] defined a class of almost contact Riemannian manifolds satisfying some special conditions. In 1995, Sinha and Sai Prasad [13] have defined a class of almost paracontact metric manifolds namely para Kenmotsu (briefly P-Kenmotsu) and special para Kenmotsu (briefly SP-Kenmotsu) manifolds.

In 1970, Pokhariyal and Mishra [14] have introduced new tensor fields, called W and E-tensor fields in a Riemannian manifold and studied their properties. In the present paper, we consider the W-curvature tensor of a semi-symmetric non-metric connection and obtained a relation connecting the curvature tensors of $M_n$ with respect to semi-symmetric non-metric connection and the Riemannian connection. It is proved that in an SP-Kenmotsu manifold if the curvature tensor of the semi-symmetric non-metric connection vanishes then the manifold is projectively flat.
for all vector fields X and Y on M_n. Then the manifold M_n corresponds to a straight line in the Euclidean space, then M_n is said to be locally projectively flat. For n \geq 3, M_n is locally projectively flat if and only if the projective curvature tensor W vanishes. For n = 2, the projective curvature tensor identically vanishes.

2. Curvature Tensor

The manifold M_n is considered to be an SP-Kenmotsu manifold. Let us denote the curvature tensor of the semi-symmetric non-metric connection \( \tilde{\nabla} \) by \( \tilde{R} \) and the curvature tensor of \( \nabla \) by R. By straightforward calculation, we get

\[
\tilde{R}(X,Y,Z) = R(X,Y,Z) + g(X,Y)Z - g(Y,Z)X.
\]

As a consequence of equations (1.11) and (1.14), equation (2.1) reduces to

\[
\tilde{R}(X,Y,Z) = R(X,Y,Z) + g(X,Y)Z - g(Y,Z)X
\]

which is the relation between the curvature tensors of M_n with respect to the semi-symmetric non-metric connection \( \tilde{\nabla} \) and the Riemannian connection \( \nabla \).

It is well known that a Riemannian manifold is of constant curvature if and only if it is projectively flat or conformally flat [15] and in general, the necessary and sufficient condition for a manifold with a symmetric linear connection to be projectively flat is that the projective curvature tensor with respect to it vanishes identically on a manifold [16].

As an example, if M_n is a Riemannian manifold with vanishing curvature tensor with respect to semi-symmetric non-metric connection, then M_n is projectively flat [8]. Analogous to this, we prove the following for an SP-Kenmotsu manifold which is Riemannian.

**Theorem 2.1:** If in an SP-Kenmotsu manifold M_n the curvature tensor of a semi-symmetric non-metric connection \( \tilde{\nabla} \) vanishes, then the manifold is projectively flat.

**Proof:** Since \( \tilde{R} = 0 \), then equation (2.2) gives

\[
R(X,Y,Z) = g(Y,Z)X - g(X,Z)Y.
\]

On contracting the above equation, we get

\[
\text{Ric}(Y,Z) = (n-1)g(Y,Z).
\]

Then, by (2.3) and (2.4), we get

\[
R(X,Y,Z) - \frac{1}{n-1}\left[Ric(Y,Z)X + Ric(X,Z)Y\right] = 0
\]

or \( W = 0 \) from (1.15), proves that the manifold is projectively flat.

**Theorem 2.2:** If in an SP-Kenmotsu manifold M_n the Ric tensor of a semi-symmetric non-metric connection \( \tilde{\nabla} \) vanishes, then the curvature tensor of \( \tilde{\nabla} \) is equal to the projective curvature tensor of the manifold M_n.

**Proof:** From equation (2.2), we have

\[
\tilde{R}(X,Y,Z,U) = R(X,Y,Z,U) + g(X,Z)g(Y,U) - g(Y,Z)g(X,U).
\]
On contracting the above equation, we get
\[ \mathring{\text{Ric}}(Y, Z) = \text{Ric}(Y, Z) - (n-1)g(Y, Z). \quad (2.7) \]
Since \( \mathring{\text{Ric}} = 0 \), we have
\[ g(Y, Z) = \frac{1}{n-1}[\text{Ric}(Y, Z)]. \quad (2.8) \]
From equations (2.2) and (2.8), we have \( \mathring{\mathring{R}} = W \).

**Theorem 2.3**: In an SP-Kenmotsu manifold the projective curvature tensor of a semi-symmetric non-metric connection \( \mathring{\nabla} \) is equal to the projective curvature tensor of the manifold.

**Proof**: From equations (2.2) and (2.7), we get
\[ \mathring{\mathring{R}}(X, Y, Z) = R(X, Y, Z) + \frac{1}{n-1}\left[\mathring{\text{Ric}}(Y, Z)\right]X - \frac{1}{n-1}\left[\mathring{\text{Ric}}(X, Z) - \text{Ric}(X, Z)\right]Y. \quad (2.9) \]
The terms of the equation (2.9) can be rearranged as
\[ \mathring{\mathring{R}}(X, Y, Z) - \frac{1}{n-1}\left[\mathring{\text{Ric}}(Y, Z)X - \text{Ric}(X, Z)Y\right] = R(X, Y, Z) - \frac{1}{n-1}\left[\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y\right], \quad (2.10) \]
which is \( 'W = W \), where \( 'W \) is the Weyl projective curvature tensor with respect to the semi-symmetric non-metric connection.

**Theorem 2.4**: In an SP-Kenmotsu manifold with semi-symmetric non-metric connection \( \mathring{\nabla} \) we have
a) \( \mathring{\mathring{R}}(X, Y, Z) + \mathring{\mathring{R}}(Y, Z, X) + \mathring{\mathring{R}}(Z, X, Y) = 0 \)
b) \( \mathring{\mathring{R}}(X, Y, Z, U) + \mathring{\mathring{R}}(X, Y, U, Z) = 0 \)

**Proof**: Using the Bianchi’s first identity with respect to the Riemannian connection equation (2.2) gives (a). From equation (2.6) we get (b).

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**Competing Interest**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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