A BRIEF SURVEY ON INTRINSICALLY KNOTTED AND LINKED GRAPHS

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1. Introduction

In the early 1980’s, Sachs [36, 37] showed that if G is one of the seven graphs in Figure 1 known as the Petersen family graphs, then every spatial embedding of G, i.e., embedding of G in S^3 or R^3, contains a nontrivial link — specifically, two cycles that have odd linking number. Henceforth, spatial embedding will be shortened to embedding; and we will not distinguish between an embedding and its image. A graph is **intrinsically linked** (IL) if every embedding of it contains a nontrivial link. For example, Figure 1 shows a specific embedding of the first graph in the Petersen family, K_6, the complete graph on six vertices, with a nontrivial 2-component link highlighted. At about the same time, Conway and Gordon [4] also showed that K_6 is IL. They further showed that K_7 is **intrinsically knotted** (IK), i.e. every spatial embedding of it contains a nontrivial knot.

![Figure 1. Left: The Petersen family graphs [42]. Right: An embedding of K_6, with a nontrivial link highlighted.](image)

A graph H is a **minor** of another graph G if H can be obtained from a subgraph of G by contracting zero or more edges. It’s not difficult to see that if G has a linkless (resp. knotless) embedding, i.e., G is not IL (IK), then every minor of G has a linkless (knotless) embedding [6, 30]. So we say the property of having a linkless (knotless) embedding is **minor closed** (also called hereditary).

A graph G with a given property is said to be **minor minimal** with respect to that property if no minor of G, other than G itself, has that property. Kuratowski [26] and Wagner [41] showed that a graph is nonplanar, i.e. cannot be embedded in R^2, iff it contains K_5 or K_{3,3} (the complete bipartite graph on 3 + 3 vertices).

*Date:* June 15, 2020.

2000 Mathematics Subject Classification. Primary 05C10, Secondary 57M15, 57M25.

*Key words and phrases.* spatial graphs, intrinsically knotted, intrinsically linked.
as a minor. Equivalently, these two graphs are the only minor minimal nonplanar graphs. It’s easy to check that each of the seven graphs in the Petersen family is minor minimal IL (MMIL), i.e., if any edge is deleted or contracted, the resulting graph has a linkless embedding. Sachs conjectured that these are the only MMIL graphs, which was later proved by Robertson, Seymour, and Thomas:

**Theorem 1.** [35] A graph is IL iff it contains a Petersen family graph as a minor.

This gives us an algorithm for deciding whether any given graph \( G \) is IL: check whether one of the Petersen family graphs is a minor of \( G \).

In contrast, finding all minor minimal IK (MMIK) graphs has turned out more difficult. Robertson and Seymour’s Graph Minor Theorem [34] says that in any infinite set of (finite) graphs, at least one is a minor of another. It follows that for any property whatsoever (minor closed or not), there are only finitely many graphs that are minor minimal with respect to that property. In particular, there are only finitely many MMIK graphs. If we knew the finite set of all MMIK graphs, we would be able to decide whether or not any given graph is IK. So far there are at least 264 known MMIK graphs [9], and, for all we know, this could be just the tip of the iceberg — we don’t even have an upper bound on the number of MMIK graphs.

For \( n \geq 3 \), a graph is **intrinsically \( n \)-linked** (I\( n \)L) if every spatial embedding of it contains a nonsplit \( n \)-link (a link with \( n \) components). It was shown in [12] that \( K_{10} \) is I3L; and it was shown in [5] that removing from \( K_{10} \) four edges that share one vertex, or two nonadjacent edges, yields I3L graphs; but it’s not known if they are MMI3L. Examples of MMI\( n \)L graphs were given for every \( n \geq 3 \) in [8].

Other “measures of complexity” have also been studied. For example: given any pair of positive integers \( \lambda \) and \( n \geq 2 \), every embedding of a sufficiently large complete graph contains a 2-link with linking number at least \( \lambda \) in magnitude [7, 38]; contains a nonsplit \( n \)-link with all linking numbers even [13]; contains a knot \( K \) such that the magnitude of the second coefficient of its Conway polynomial, i.e. \( |a_2(K)| \), is larger than \( \lambda \) [7, 35]; and contains a nonsplit \( n \)-link \( L \) such that for every component \( C \) of \( L \), \( |a_2(C)| > \lambda \) and for any two components \( C \) and \( C' \) of \( L \), \( |\text{lk}(C, C')| > \lambda \) [10]. No minor minimal graphs with respect to any of these properties are known.

In the following sections we discuss the above, and a few other topics, in greater detail.

## 2. IL graphs

The proof that \( K_6 \) is IL is short and beautiful: There are 20 triangles (3-cycles) in \( K_6 \). For each triangle, there is exactly one triangle disjoint from it. Thus there are exactly 10 pairs of disjoint triangles, i.e. 2-links, in \( K_6 \). Any pair of disjoint edges is contained in exactly two such links. So, given any embedding of \( K_6 \), any crossing change between any two disjoint edges affects the linking number of exactly two links, and the magnitude of each of their linking numbers changes by 1. Thus, the sum of all linking numbers does not change parity under any crossing change. Now, in the embedding of \( K_6 \) shown in Figure 1, the sum of all ten linking numbers is odd. Therefore the same is true in every embedding of \( K_6 \), since any embedding can be obtained from any other embedding by isotopy and crossing changes. Hence every embedding contains at least one link with odd linking number.
Sachs observed that a similar argument can be used to show all seven graphs in the Petersen family are IL. He also observed that each of these graphs can be obtained from any other by one or more \( \nabla Y \) and \( Y \nabla \) moves, as defined in Figure 2. Furthermore, this family is closed under \( \nabla Y \) and \( Y \nabla \) moves. It follows, by Theorem 1, that \( \nabla Y \) and \( Y \nabla \) moves preserve the property of being MMIL.

\[ \begin{align*}
\nabla Y & \quad \text{move} \\
Y \nabla & \quad \text{move}
\end{align*} \]

**Figure 2.** \( \nabla Y \) and \( Y \nabla \) moves.

In fact, Sachs observed that a \( \nabla Y \) move on any IL graph yields an IL graph; or, equivalently, that a \( Y \nabla \) move on a “linklessly embeddable” graph yields a linklessly embeddable graph. The proof of this is elementary and straightforward, which we outline here. Suppose \( G' \) is obtained from a linklessly embeddable graph \( G \) by a \( Y \nabla \) move. Take a linkless embedding \( \Gamma \) of \( G \), and replace the \( Y \) involved in the \( Y \nabla \) move by a triangle \( \nabla \) whose edges are “close and parallel” to the edges of the \( Y \). This gives an embedding \( \Gamma' \) of \( G' \). It’s easy to show that any link in \( \Gamma' \) that doesn’t have \( \nabla \) as a component is isotopic to a link in \( \Gamma \), and hence is trivial. And any link in \( \Gamma' \) that does have \( \nabla \) as a component is also trivial since \( \nabla \) bounds a disk whose interior is disjoint from \( \Gamma \).

It is also true that a \( Y \nabla \) move on any IL graph yields an IL graph, but the only known proofs of it rely on Theorem 1 or the following result of [35]: If \( G \) has a linkless embedding, then it has a paneled embedding, i.e., an embedding \( \Gamma \) such that every cycle in \( \Gamma \) bounds a disk whose interior is disjoint from \( \Gamma \).

Let’s say a graph is \( \mathbb{Z}_2 \)-IL if every embedding of it contains a 2-link with linking number nonzero mod 2. Thus, each of the Petersen family graphs is \( \mathbb{Z}_2 \)-IL. This, together with Theorem 1, implies that \( G \) is IL iff it is \( \mathbb{Z}_2 \)-IL.

It is possible to determine if a graph \( G \) is \( \mathbb{Z}_2 \)-IL by simply solving a system of linear equations, without even using Theorem 1. We give an outline here. First, pick an arbitrary embedding \( \Gamma \) of \( G \), and compute the linking numbers mod 2 for all 2-links (pairs of disjoint cycles) in \( \Gamma \). An arbitrary embedding \( \Gamma' \) of \( G \) can be obtained from \( \Gamma \) by adding some number of full twists between each pair of disjoint edges, plus isotopy (adding twists is equivalent to letting edges “pass through” each other). Say there are \( d \) pairs of disjoint edges in \( G \). Let \( x_1, \cdots, x_d \) be variables representing the number of full twists to be added to the \( d \) disjoint pairs of edges to obtain \( \Gamma' \) from \( \Gamma \). Then the linking number of any 2-link in \( \Gamma' \) can be written in terms of \( x_1, \cdots, x_d \) and the linking number of that 2-link in \( \Gamma \). Setting each of these expressions equal to zero gives us a system of linear equations in \( d \) variables. This system of equations has a solution in \( \mathbb{Z}_2 \) iff \( G \) is not \( \mathbb{Z}_2 \)-IL. Note that the number of cycles in a graph can grow exponentially with the graph’s size, so this algorithm is exponential in time and space. In [25, 40], polynomial time algorithms are given for finding linkless embeddings of graphs.
3. IK Graphs

Essentially the same argument that shows the $\nabla Y$ move preserves ILness also shows the $Y \nabla$ move preserves IKness. However, the $Y \nabla$ move does not necessarily preserve IKness [11]. For example, there are twenty graphs that can be obtained from $K_7$ by zero or more $\nabla Y$ and $Y \nabla$ moves. Six of these graphs cannot be obtained from $K_7$ by $\nabla Y$ moves only — they require $Y \nabla$ moves also. And it turns out all these six graphs have knotless embeddings [11, 19, 20].

Given two disjoint graphs $G_1$ and $G_2$, let $G_1 \ast G_2$, the cone of $G_1$ with $G_2$, be the graph obtained by adding all edges from vertices of $G_1$ to vertices of $G_2$, i.e.,

$$G_1 \ast G_2 = G_1 \cup G_2 \cup \{v_1v_2 \mid v_i \in V(G_i)\}.$$

For about twenty years, the only known IK graphs were $K_7$ and its descendants, i.e., graphs obtained from $K_7$ by $\nabla Y$ moves only. It was suspected that $K_{3,3,1,1}$ (the complete 4-partite graph on $3 + 3 + 1 + 1$ vertices) is also IK. Recall that $K_5$ and $K_{3,3}$ are minor minimal nonplanar. Coning with one vertex on each of these graphs gives $K_6$ and $K_{3,3,1,1}$, both of which are in the Petersen family and hence MMIL. Coning again gives $K_7$ and $K_{3,3,1,1}$; and $K_7$ was shown to be MMIK; so it was natural to ask if $K_{3,3,1,1}$ is too. Foisy [15] proved that $K_{3,3,1,1}$ indeed is IK. His technique, partially outlined below, led to finding many more MMIK graphs later on [16, 17, 19].

Figure 3 shows a multi-graph (i.e. double edges and loops are allowed) commonly called $D_4$, with four of its cycles labeled $C_1, \ldots, C_4$. Let’s say an embedding of $D_4$ is double linked mod 2 if $\text{lk}(C_1, C_3)$ and $\text{lk}(C_2, C_4)$ are both nonzero mod 2. To show $K_{3,3,1,1}$ is IK, Foisy proved the following key lemma. A more general version of the lemma was proved, independently, by Taniyama and Yasuhara [39].

$$\text{Figure 3. The } D_4 \text{ graph.}$$

**Lemma 2 (D₄ Lemma).** [15, 39] Every embedding of $D_4$ that is double linked mod 2 contains a knot $K$ with $a_2(K) \neq 0 \mod 2$.

Foisy proved that $K_{3,3,1,1}$ is IK by showing that every embedding of it contains as a minor a $D_4$ that is double linked mod 2.

Let’s say a graph $G$ is $ID_4$ mod 2 if every embedding of $G$ contains as a minor a $D_4$ that is double linked mod 2; and $G$ is $Ia_2$ mod 2 if every embedding of $G$ contains a knot $K$ such that $a_2(K) \neq 0 \mod 2$. Thus, the $D_4$ Lemma says if $G$ is $ID_4$ mod 2 then it’s $Ia_2$ mod 2. We also know that if $G$ is $Ia_2$ mod 2 then it’s IK. Let’s abbreviate these two implications as: $ID_4 \mod 2 \implies Ia_2 \mod 2 \implies IK$. It is natural to ask if the converse of each of these implications is also true.
**Question 1.** (a) \( \text{IK} \iff \text{I}a_2 \mod 2 \) (b) \( \text{IA}_2 \iff \text{ID}_4 \mod 2 \) (c) \( \text{IK} \iff \text{ID}_4 \mod 2 \)

The question “\( \text{IL} \iff \text{I}n\text{L} \mod 2 \)” is also open.

It turns out that every known MMIK graph \( G \) is \( \text{IA}_2 \mod 2 \) and \( \text{ID}_4 \mod 2 \). But this is not necessarily evidence that the answer to either part of Question 1 is yes, because most of the known MMIK graphs were found by looking for graphs that are \( \text{ID}_4 \mod 2 \).

Determining if a graph is \( \text{ID}_4 \mod 2 \) can be done by solving systems of linear equations \([39]\). If it is true that \( \text{IK} \iff \text{ID}_4 \mod 2 \), then the algorithm of \([39]\) can be used to decide whether an arbitrary graph is IK.

There may be (a lot) more MMIK graphs than have been found so far; and trying to find some of them might not be too hard. For example, one can start with a non-MMIK graph \( G \) obtained by a \( Y \forall \) move from a known MMIK graph, and keep adding new edges to \( G \) or expanding vertices of \( G \) into edges (the reverse of contracting edges) until one obtains an IK graph. But finding more and more MMIK graphs doesn’t seem to have advanced our understanding of IK graphs very much. In trying to understand IK graphs better, another approach has been to try to classify all IK graphs with a given number of edges. For example, it has been shown that there are no IK graphs with 20 or fewer edges, and the only IK graphs with exactly 21 edges are \( K_7 \) and its descendants \([1, 23, 24, 28]\); IK graphs with 22 edges have also been partially classified \([21, 22]\). However, this approach doesn’t seem to have led to significant insights or advances in the theory either.

4. Miscellaneous facts and open problems

In \([31]\) it was shown that if \( G \) is IK and \( e \) is an edge of a 3-cycle in \( G \), then \( G\setminus e \) is IL. The following related questions might be useful in trying to answer Question 1.

**Question 2.** Suppose \( G \) is IK. (a) Is \( G \setminus e \), or \( G/e \), IL for every edge, or for some edge, \( e \) of \( G \)? (b) Does \( G \) have at least two distinct nonsplit links?

Sachs observed that a graph \( G \) is non-planar iff the graph \( G \ast v \), the cone of \( G \) with one vertex \( v \), is IL. This can be seen as follows. If \( G \) is nonplanar, then it contains \( K_5 \) or \( K_{3,3} \) as a minor. So \( G \ast v \) contains \( K_5 \ast v = K_6 \) or \( K_{3,3} \ast v = K_{3,3,1} \) as a minor, and hence \( G \ast v \) is IL. Conversely, if \( G \) is planar, it is easy to construct a linkless embedding of \( G \ast v \): embed \( G \) in the plane, put \( v \) above the plane, and connect \( v \) with straight edges to all vertices of \( G \).

A graph \( G \) is said to be \( n \)-**apex** if there exist vertices \( v_1, \ldots, v_n \) in \( G \) such that \( G - \{v_1, \ldots, v_n\} \), i.e., the graph obtained by removing \( \{v_1, \ldots, v_n\} \) and all edges incident to them, is planar. A 1-apex graph is called **apex**. Thus, by above, apex graphs are not IL. It can similarly be shown that 2-apex graphs are not IK. In fact, \( G \) is planar iff the graph \( G \ast v \ast w \) (i.e., \( G \ast K_2 \)) is not IK \([31, 32]\). The reason is similar to the one given above: If \( G \) is nonplanar, then \( G \ast v \ast w \) contains \( K_5 \ast v \ast w = K_7 \) or \( K_{3,3} \ast v \ast w = K_{3,3,1,1} \) as a minor; and since both of these graphs are IK, \( G \ast v \ast w \) is IK too. If \( G \) is planar, we can construct a knotless embedding of \( G \ast v \ast w \) as follows: embed \( G \) in the plane, put \( v \) above the plane, put \( w \) below the plane, and connect \( v \) and \( w \) to all vertices of \( G \) and to each other with straight edges. The

\( ^1 \)That \( K_7 \) and its descendants are \( \text{ID}_4 \) is not in the literature but is believed to be true if the computer program of \([39]\) is correct.
list of all minor minimal non-\(n\)-apex graphs is not known for any \(n\), even \(n = 1\). A detailed survey of results on apex and 2-apex graphs can be found in [9].

The crossing number \(C(K)\) of a knot \(K\) is the fewest number of crossings among all regular projections of \(K\). It’s easy to see that for every \(n\), the set \(\{K : C(K) \leq n\}\) is finite; so \(A(n) = \max\{|a_2(K)| : C(K) \leq n\}\) is well-defined and finite. As mentioned before, given a fixed \(n\), every embedding of a sufficiently large complete graph contains a knot \(K\) with \(|a_2(K)| > A(n)\); hence, every embedding of a sufficiently large complete graph contains a knot with crossing number larger than \(n\). The bridge number of a knot \(K\) is the minimum number of local maxima with respect to height (\(z\)-coordinate in \(\mathbb{R}^3\)) among all isotopic embeddings of \(K\).

Given any integer \(n \geq 2\), there are infinitely many \(n\)-bridge knots. So the above argument for crossing number doesn’t work for bridge number. This leads to the question: Does there exist, for each \(n\), a graph \(G\) such that every embedding of \(G\) contains a knot with bridge number at least \(n\)?

Suppose \(G^*\) is obtained by a \(\nabla Y\) move from \(G\). It turns out that if \(G\) has any of the following properties, then \(G^*\) has that property too: IL; IK; \(I_02\); \(I_D 1\); InL; nonplanar; non-\(n\)-apex. The proofs for all of these are elementary and short, and most of them are similar to the one we saw for IL. But, curiously, IL is the only property from the above list known to be preserved by \(\nabla Y\) moves.

The complement of a graph \(G\) is a graph \(G^c\) with the same vertices as \(G\) and with exactly those edges not in \(G\). In [2] it was shown that if \(G\) has 9 or more vertices, then \(G\) or \(G^c\) is nonplanar. This result is sharp: there exists a graph \(G\) on 8 vertices such that both \(G\) and \(G^c\) are planar. We can ask a similar question of IL graphs: What is the smallest integer \(v\) such that for every graph \(G\) with \(v\) vertices, \(G\) or \(G^c\) is IL? Here is a partial answer. In [27] it was shown that for all \(n \leq 5\), if \(G\) has \(v\) vertices, \(e\) edges, and \(e > nv - \binom{n+1}{2}\), then \(G\) contains \(K_{n+2}\) as a minor. Now, \(K_{15}\) has 105 edges, so if \(G\) has 15 vertices, then \(G\) or \(G^c\) has at least \(105/2 = 53\) edges. Letting \(n = 4\), we have \(nv - \binom{n+1}{2} = 4(15) - \binom{5}{2} = 50\); since \(53 > 50\), \(G\) or \(G^c\) contains \(K_6\) as a minor, and hence is IL. But this is not sharp. In [33], it is shown that: (i) if \(G\) has 13 vertices, then \(G\) or \(G^c\) is IL, and (ii) there is a graph \(G\) with 10 vertices such that neither \(G\) nor \(G^c\) is IL. The question for graphs with 11 or 12 vertices remains open.

One can similarly show that for every graph \(G\) with 18 or more vertices, \(G\) or \(G^c\) contains a \(K_7\) minor and hence is IK. It is unknown what the minimum number of vertices is that would guarantee that \(G\) or \(G^c\) is IK.

Let’s say a graph is strongly intrinsically linked (SIL) if every embedding of it contains a 2-link with linking number at least 2 in magnitude. Then \(K_{10}\) is SIL, since, by [12], \(K_{10}\) contains a 3-link two of whose linking numbers are nonzero, and by [7], any embedded complete graph that contains such a 3-link contains a “strong” 2-link. What about \(K_9\)? By [12], \(K_9\) does not contain such a 3-link; but it’s not known whether or not \(K_9\) is SIL.

A digraph (directed graph) is a graph each of whose edges is oriented. A consistently oriented cycle in a digraph is a cycle \(x_0, x_1, \ldots, x_n\), where \(x_n = x_0\), such that each edge \(x_i x_{i+1}\) is oriented from \(x_i\) to \(x_{i+1}\). A digraph is said to be InL (resp. IK) if every spatial embedding of it contains a nonsplit \(n\)-link (resp. nontrivial knot) consisting of consistently oriented cycles. In [14], an IK digraph and an I4L digraph are constructed. It is not known whether there exists an InL digraph with \(n \geq 5\). In [18] it was shown that (unlike all the other graph properties
we have discussed) the property of having a linkless embedding is not minor closed for digraphs.

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