A BIJECTION BETWEEN TWO DIFFERENT CLASSES OF PARTITIONS ENUMERATED BY $p_\nu(n)$

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Abstract
In this paper, we give a purely bijective proof that two different partition classes that are both combinatorial interpretations of the partition function $p_\nu(n)$, a partition function related to the third order mock theta function $\nu(q)$, are equinumerous. In doing so, we give a partial solution to a combinatorial problem proposed in a paper by Andrews.

1. Introduction and Notation
Consider the third order mock theta function $\nu(q)$, which was first defined by Watson [4] and may be defined as follows:

$$\nu(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q; q^2)_{n+1}},$$

where the q-Pochhammer symbol $(a; q)_n$ is defined as usual

$$(a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k).$$

The partition function $p_\nu(n)$ may be defined as the partition function for which $\nu(-q)$ is the generating function, and a number of combinatorial interpretations have been given for this partition function. Among these is the number of self-conjugate odd Ferrers graphs of $2n+1$ and the number of self-conjugate partitions of $4n+1$ into odd parts [2, 3]. Odd Ferrers graphs, introduced by Andrews in [1], may be defined as Ferrers graphs in which a 2 is placed in every box, except the surrounding border, where 1s are placed in each box. For example, the following
odd Ferrers graph represents the partition 7 + 7 + 3 + 1:

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 1 1 1 1 1 1 1
 1 2 2 2
 1 2
 1
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Let $O_{2n+1}$ be the set of self-conjugate odd Ferrers graphs for $2n+1$, let $S_{4n+1}$ be the set of self-conjugate partitions of $4n+1$ into odd parts, let $O = \cup_{n>0} O_{2n+1}$ and let $S = \cup_{n>0} S_{4n+1}$. The following theorem has previously been proven through non-bijective means [2].

**Theorem 1.** For all $n$, the number of partitions in the class $O_{2n+1}$ is equal to the number of partitions in the class $S_{4n+1}$.

We will give a purely bijective proof of this theorem by describing a bijection $\phi$ such that $\phi(\lambda) = \mu$, where $\lambda$ and $\mu$ are both partitions, $\lambda \in O_{2n+1}$, and $\mu \in S_{4n+1}$, and use the case where $\lambda = 3 + 5 + 3$, representable as the following odd Ferrers graph

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 1 1 1
 1 2 2
 1 2
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as an example (note that in this example case, $\lambda \in O_{11}$, and that $\mu \in S_{21}$). In doing so, we give a partial solution to the combinatorial challenge proposed by Andrews [2] asking for bijections between the various classes of partitions enumerated by $p_\nu(n)$.

2. A Bijection Between $O_{2n+1}$ and $S_{4n+1}$

Consider the fact that the Ferrers diagrams of self-conjugate partitions may be thought of as being made up of “hooks” of other self-conjugate partitions in which every part other than the greatest part is equal to 1. For example, the Ferrers digram of the self-conjugate partition $4 + 4 + 2 + 2$

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can be thought of as consisting of the following “hooks”: 

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Let \( h_i \) denote the \( i \)th “hook” in a self-conjugate partition \( \pi \), where \( i > 0 \). Note that, where \(|\pi|\) may denote the sum of the parts of \( \pi \), where \(|h_i|\) may denote the sum of the parts in each hook in the Ferrers diagram of \( \pi \), and where \( n \) may denote the number of hooks in \( \pi \), that \( \sum_{i=1}^{n} |h_i| = |\pi| \). Additionally, for \( \lambda \in \mathcal{O} \), let \( t = \sum_{i=2}^{n} |h_i| \), or the sum of the 2s in the odd Ferrers diagram. We will distinguish between the hooks in \( \lambda \) and the hooks in \( \mu \) by using \( h_i \) to denote the \( i \)th hook in the former and \( \eta_i \) to denote the \( i \)th hook in the latter. The map \( \phi(\lambda) = \mu \) may be described as follows.

Step 1: Create \( \eta_1 \) by creating a hook with the largest part equal to \(|h_1|\). Note that \(|\eta_1| = 2|h_1| - 1\). For the example case for \( \lambda \) given above, \( \eta_1 \) would be the following.

Step 2: For each \( h_i \) where \( i > 1 \), create \( \eta_{2i-2} \) such that \(|\eta_{2i-2}| = |h_i| + 1\), and \( \eta_{2i-1} \) such that \(|\eta_{2i-1}| = |h_i| - 1\). For example, in the example case of \( \lambda \) given above, \(|h_2| = 6\), so we create \( \eta_2 \) and \( \eta_3 \) such that \(|\eta_2| = 7\) and \(|\eta_3| = 5\), and since the number of hooks in \( \lambda \) is equal to 2, the creation of these hooks completes the bijection resulting in the following partition, that being \( 5 + 5 + 5 + 3 + 3 \).

The map described evidently always results in a self-conjugate partition. The map described also always results in a partition of \( 4n+1 \), because in creating \( \eta_1 \) we create a partition of size \( 2h_1 - 1 \), and in adding every \( \eta_i \) such that \( i > 1 \), we add 2\( t \) to this partition, thus making a partition of size \( 2(h_1 + t) - 1 \). We know that \( h_1 + t = |\lambda| = 2n + 1 \), so substituting \( 2n + 1 \) for \( h_1 + t \) in the previous expression reveals that the sum of the parts in the newly created partition is always equal to \( 4n + 1 \). Additionally, we know that the newly created partition is always a partition into odd parts because it always creates a partition in which the greatest part of \( \eta_1 \) is odd, the number of hooks is odd, and in which the greatest part of each hook alternates in parity, where the greatest part of \( \eta_{2i-2} \) is always one greater than the
greatest part of $\eta_{2n-1}$. The inverse map is obvious, so $\phi$ is a bijection, and thus $|\mathcal{O}_{2n+1}| = |\mathcal{S}_{4n+1}|$ for all $n$.

3. Further Remarks

Recall the natural bijection that exists between the class of self-conjugate partitions of $n$ and the class of partitions of $n$ into distinct odd parts that maps a self-conjugate partition onto a partition into distinct odd parts by making the sum of the parts in each of the hooks in the self conjugate partition into a part in the newly created partition. Where $D_{2n+1}$ may denote the set of partitions of $2n + 1$ into distinct parts in which there is one odd part which is greater than half the greatest even part and every other part is even and is of the form $4k + 2$ where $k \in \mathbb{N}$, and where $DO_{4n+1}$ may denote the set of partitions of $4n + 1$ into an odd number of distinct odd parts such that, when ordered from largest to smallest, the parts alternate between being of the form $4k + 1$ and being of the form $4k + 3$ where again $k \in \mathbb{N}$, an analogous bijection exists between $O_{2n+1}$ and $D_{2n+1}$ and between $S_{4n+1}$ and $DO_{4n+1}$. Thus, the bijection given above induces one between $D_{2n+1}$ and $DO_{4n+1}$.

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References

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