Hamiltonian analysis of the Higgs mechanism for gravity

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Abstract
In this paper, we perform the canonical description of the Higgs mechanism for gravity and provide the Hamiltonian definition of massive gravities.

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1. Introduction

Construction of massive gravity is plagued with theoretical problems and issues that have not been resolved from the pioneering work of Fierz and Pauli [1]. One such well-known problem of massive gravity is that there is no smooth massless limit in the perturbation theory of massive gravity in the sense that the massless limit in massive gravity exists but does not agree with the results derived from general relativity (GR) that describes massless gravitons. This pathological behavior of massive gravity is known as van Dam–Veltman–Zakharov discontinuity [6, 7].

Even if the construction of massive gravities is an interesting theoretical challenge, there is another more stronger reason for their formulation. Since Einstein’s theory of gravity is a well-tested theory, it seems to be natural to add to the action

\footnote{For extensive reviews of various aspects of IR modification of gravities, see [2–5].}
of GR a term which will, in the linearized approximation, give a mass to gravitons without modifying the kinetic terms coming from GR.

A very interesting formulation of the massive theory of gravity is based on the Brout–Englert–Higgs mechanism for gravity. In more details, we consider a diffeomorphism invariant action with a usual Einstein–Hilbert term together with the function of the metric coupled to $D$ scalar fields for $D$-dimensional gravity [8]. Then gravitons acquire a mass due to a mechanism which may be thought of as the Brout–Englert–Higgs mechanism for gravity when the vacuum expectation value of each scalar field breaks one coordinate reparameterization invariance.

Our goal is to develop the Hamiltonian description of the systems studied in [8–14]. With the help of the $(D - 1) + 1$ split formalism for the spacetime metric, we find the corresponding Hamiltonian and show that it is a linear combination of the Hamiltonian and diffeomorphism constraints. The consistency of the theory demands that these constraints are preserved during the time evolution of the system. In order to check whether they are preserved or not, we calculate the Poisson brackets of these constraints. However, it turns out that it is a non-trivial task to calculate the Poisson bracket of the Hamiltonian constraint for the scalar field whose dynamics is governed by general action. Despite of this fact we show that the Poisson brackets of the scalar field Hamiltonian constraints are proportional to the diffeomorphism constraints. In other words, we show that the Poisson bracket of the scalar field Hamiltonian constraints takes exactly the same form as the Poisson bracket of GR Hamiltonian constraints. Collecting all these results, we obtain that the Poisson algebra of constraints is closed and hence these constraints are consistent with the time evolution of the system. This result is crucial for the possibility of fixing the gauge in the framework of the Hamiltonian formalism. We fix the gauge by introducing $D$ gauge-fixed functions that correspond to the gauge fixing conditions introduced in [8–14]. Then the system of these gauge fixing functions and original constraints form the set of the second-class constraints that can be explicitly solved. Further, since the original Hamiltonian is given as the linear combination of constraints, we find that in the process of the gauge fixing it strongly vanishes. On the other hand, the gauge fixing implies that the reduced phase space is spanned by the spatial components of the metric $h_{ij}$ and their conjugate momenta $p^i$. Then we argue that the gauge fixing condition $t = \phi^0$ naturally introduces the Hamiltonian on the reduced phase space equal to $-p_0$ where $p_0$ is the momentum conjugate to $\phi^0$. Note that $p_0$ is the function of the reduced phase space variables $h_{ij}$, $p^i$ as a result of solving the Hamiltonian constraint. In other words, we claim that the definition of the massive gravity is given by the reduced phase space variables $h_{ij}$, $p^i$ together with the gauge-fixed Hamiltonian $H_{\text{fix}} = -p_0(h_{ij}, p^i)$, where all non-dynamical modes are absent.

Experiences from many areas of theoretical physics teach us that in some cases the Lagrangian formulation of a given theory is much more efficient than the Hamiltonian ones. For that reason, we feel that it is useful to determine the Lagrangian formulation of the gauge-fixed theory as well. It turns out that in order to find this Lagrangian, it is convenient to introduce new non-dynamical modes so that the Legendre transformation from the Hamiltonian to Lagrangian formulation can be easily performed. Interestingly, this Lagrangian can be written in such a form that resembles the standard Einstein–Hilbert action together with the potential term that explicitly breaks the diffeomorphism invariance of given theory.

The structure of this paper is as follows. In section 2, we perform the Hamiltonian analysis of the system introduced in [8–14] and we perform the gauge fixing of this system that leads to the new Hamiltonian formulation of the massive gravity. Then in section 3, we give two examples of scalar potentials that allow us to find the explicit form of the gauge-fixed

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2 This interesting proposal of formulation of the massive gravity was also studied in [9–14].
Hamiltonian. In section 4, we outline our results and suggest possible extension of this work. The explicit calculation of the Poisson bracket of the constrains is given in the appendix.

2. Hamiltonian analysis of GR with massless scalars

In this section, we develop the Hamiltonian formalism for the following action:

$$S = \frac{1}{16\pi G} \int d^Dx \sqrt{-g} [R - L(H^{AB})],$$

where $G$ is D-dimensional Newton’s constant and the induced internal metric is defined as

$$H^{AB} = g^{\mu\nu} \nabla_\mu \phi^A \nabla_\nu \phi^B,$$

where $\phi^A$ are real D scalar fields with $A = 0, \ldots, D - 1$ and where the indices $A, B, \ldots$ are raised and lowered using the metric $\eta_{AB} = \text{diag}(-1, 1, \ldots, 1)$. Finally, $L$ is a generic function of $H^{AB}$. The detailed analysis of the properties of the function $L$ was performed in [10, 11, 13, 14]. It is important to stress that the form of the potential $L$ is not arbitrary. In fact, it was argued in [10, 11, 13, 14] that this potential has to lead to the equations of motion that possess the following vacuum solution:

$$\phi^A = x^\mu \delta^A_\mu, \quad g_{\mu\nu} = \eta_{\mu\nu}.$$  

(3)

This requirement leads to the constraint on the potential $L$,

$$\frac{\delta L(H_{\mu})}{\delta H^{AB}} = \frac{1}{2} \eta_{AB} L(H_{\nu}),$$

where $H^A_B = \eta^{A_B}$.

Our goal is to develop the Hamiltonian formalism for the system defined by action (1). As usual in the study of the Hamiltonian formalism for gravity, we introduce the $(D - 1) + 1$ formalism. Explicitly, let us consider the $D$-dimensional manifold $\mathcal{M}$ with the coordinates $x^\mu, \mu = 0, \ldots, D - 1$, where $x^\mu = (t, \mathbf{x})$, $\mathbf{x} = (x^1, \ldots, x^{D-1})$. We presume that this spacetime is endowed with the metric $g_{\mu\nu}(x^\rho)$ with the signature $(-, +, \ldots, +)$. Suppose that $\mathcal{M}$ can be foliated by a family of space-like surfaces $\Sigma_t$ defined by $t = x^0$. Let $h_{ij}, i, j = 1, \ldots, D$, denote the metric on $\Sigma_t$ with the inverse $h^{ij}$ so that $h_{ij} h^{kj} = \delta^k_i$. We further introduce the operator $\nabla_i$ that is covariant derivative defined with the metric $h_{ij}$. We define the lapse function $N = 1/\sqrt{-g^{00}}$ and the shift function $N^i = -g^{0i}/g^{00}$. In terms of these variables, we write the components of the metric $g_{\mu\nu}$ as

$$g_{00} = -N^2 + N_i h^{ij} N_j, \quad g_{0i} = N_i, \quad g_{ij} = h_{ij},$$

$$g^{00} = -\frac{1}{N^2}, \quad g^{0i} = \frac{N^i}{N^2}, \quad g^{ij} = h^{ij} - \frac{N^i N^j}{N^2}.$$  

(5)

Then it is easy to see that

$$\sqrt{-g} = N \sqrt{\det h}.$$  

(6)

In the $(D - 1) + 1$ formalism, $H^{AB}$ takes the form

$$H^{AB} = -\nabla_\nu \phi^A \nabla_\nu \phi^B + h^{ij} \phi^A \partial_i \phi^B,$$

(7)

where

$$\nabla_\nu \phi^A = \frac{1}{N} (\partial_\nu \phi^A - N^i \partial_i \phi^A).$$  

(8)

Then from (1) we easily find the momenta conjugate to $\phi^A$,

$$p_A = \frac{\sqrt{\det h}}{8\pi G} \frac{\delta L}{\delta H^{AB}} \nabla_\nu \phi^B.$$  

(9)
Using this result, we find the following matrix equation:

$$K_{AB} = D_{AC}(H^{CD} - H^{CD})D_{DB}(H^{AB}),$$

(10)

where we introduced the following matrices:

$$K_{AB} = \left(\frac{16\pi G}{2\sqrt{\det h}}\right)^2 p_A p_B, \quad V^{AB} = h^{ij}\partial_i\phi^A\partial_j\phi^B, \quad D_{AB}(H^{AB}) = \frac{\delta L}{\delta H_{AB}}.$$  

(11)

For further purposes, we also introduce the matrix ˜$D^{AB}$ inverse to $D_{AB}$ so that

$$D_{AB} \tilde{D}^{BC} = \delta^C_A.$$  

(12)

that will be useful below. Collecting all these results, we find the Hamiltonian of the scalar fields in the form

$$H^\phi = \int d^d x (\partial_t \phi^A p_A - L) = \int d^d x (N(x)\mathcal{H}_T^\phi(x) + N^i(x)\mathcal{H}_i^\phi(x)), $$

(13)

where

$$\mathcal{H}_T^\phi = \frac{1}{16\pi G} (2\sqrt{\det h} K_{AB} \tilde{D}^{BA} + V(J)), \quad \mathcal{H}_i^\phi = p_A \partial_i \phi^A.$$  

(14)

and where $d = D - 1$.

In the same way we should proceed with the Hamiltonian analysis of the GR action. Since the procedure is well known, we immediately write the final result

$$H^{GR} = \int d^d x (N(x)\mathcal{H}_{T}^{GR}(x) + N^i(x)\mathcal{H}_i^{GR}(x)), $$

(15)

where

$$\mathcal{H}_T^{GR} = \frac{16\pi G}{\sqrt{\det h}} \pi^{ij} G^{ijkl} \pi_{kl} - \frac{\sqrt{\det h}}{16\pi G} R^{(D-1)}, \quad \mathcal{H}_i = -2h_{ik} \nabla_l \pi^{lk}. $$

(16)

Let us explain the notation used here. $\pi^{ij}$ are momenta conjugate to $h_{ij}$. The generalized metric $G^{ijkl}$ is defined as

$$G_{ijkl} = \frac{1}{2} (h_{ik} h_{jl} + h_{il} h_{jk}) - h_{ij} h_{kl}. $$

(17)

The inverse metric $G^{ijkl}$ is equal to

$$G^{ijkl} = \frac{1}{2} (h^{ik} h^{jl} + h^{il} h^{jk}) - h^{ij} h^{kl}. $$

(18)

Finally, $R^{(D-1)}$ and $\nabla_l$ respectively are the Ricci curvature and covariant derivative calculated using the metric $h_{ij}$. Finally, due to the fact that action (1) does not contain the time derivative of $N, N^i$, we find that the corresponding conjugate momenta $\pi_N, \pi_i$ are the primary constraints of the theory

$$\pi_i \approx 0, \quad \pi_N \approx 0.$$  

(19)

The condition of preservation of these constraints implies the existence of the secondary ones

$$\mathcal{H}_T = \mathcal{H}_T^{GR} + \mathcal{H}_T^\phi \approx 0, \quad \mathcal{H}_i = \mathcal{H}_i^{GR} + \mathcal{H}_i^\phi \approx 0 $$

(20)

or their smeared form

$$H_T(M) = \int d^d x M(x) \mathcal{H}_T(x), \quad H_i(M^i) = \int d^d x M^i(x) \mathcal{H}_i(x). $$

(21)

where $M(x)$ and $M^i(x)$ are arbitrary functions.
As the next step we have to show that these constraints are preserved during the time evolution of the systems. Since these calculations are very long and technical, we have included them into the appendix where an interested reader could find more details. The upshot of this analysis is that these constraints are really preserved during the time evolutions and no additional constraints are generated. Further, these constraints are the first-class constraints and obey the algebra of constraints (A.1) presented in the appendix.

Since we showed that this system possesses \(D\) first-class constraints \(H_T(x) \approx 0\), \(H_i(x) \approx 0\), we can proceed to their gauge fixing. This procedure is an analog of the Higgs mechanism used in the construction of the massive gravity [8, 10–13]. In these models, the vacuum expectation values of the scalar fields coincide with \(D\) spacetime coordinates. The result of this fixing is the complete breaking of the spacetime diffeomorphism and emergence of the mass term for the graviton in the action when we study the small fluctuations of gravity above the flat spacetime.

The standard way of how to fix the gauge freedom in the Hamiltonian framework is to introduce \(D\) gauge-fixed functions that can be interpreted as additional constraints imposed on the system and that have non-zero Poisson brackets with the original first-class constraints. As a result, the extended system of original constraints together with gauge-fixed functions forms the collection of the second-class constraints with no gauge freedom left. (For the review of this formalism, see [19–21].)

By analogy with fixing the gauge given in [8, 10–13], we introduce the following \(D\) gauge fixing functions:

\[
G^A(x) = \phi^A(x) - x^A. \tag{22}
\]

Clearly,

\[
[G^A(x), H_i(y)] = \delta^A_i \delta(x - y) \tag{23}
\]

and

\[
[G^A(x), H_T(y)] = \frac{(16 \pi G)^2}{2 \det h} \frac{\delta H^\phi_T(x)}{\delta K_{AN}(x)} \delta K_{0N}(x) p_P(x) \delta(x - y). \tag{24}
\]

Due to the fact that the Poisson brackets (23) and (24) are non-zero on the constraint surface, we see that the collection of constraints \((G^A, H_T, H_i)\) is the system of the second-class constraints. Alternatively, the requirement of the time preservation of the constraints \(G^A(x) \approx 0\) during the time evolution of the system implies the following consistency equation:

\[
\frac{dG^i(x)}{dt} = [G^i(x), H] = N^i(x) + N(x) \frac{(16 \pi G)^2}{2 \det h} \frac{\delta H^\phi_T(x)}{\delta K_{AN}(x)} p_P(x) = 0, \tag{25}
\]

where we used (23) and (24). We see that these equations determine \(N\) and \(N^i\) as the functions of the canonical variables \(h_{ij}\), \(p^{ij}\) with no gauge freedom left.

The fact that \(G^A, H_T, H_i\) are the second-class constrains implies that they vanish strongly and can be explicitly solved for \(p_A\). As a result, the reduced phase space is spanned by \(h_{ij}\) and \(p^{ij}\). Further, since the Hamiltonian of the original system was given as a linear combination of the constraints \(H_T, H_i\), we now see that it vanishes strongly.
On the other hand, let us write the original action (1) in the form
\[ S = \int dt d^4x (\mathcal{L}_A - H) = \int dt \left( \int d^4x (\partial_t h_{ij} \pi^{ij} + p_0(p^{ij}, g_{ij})) \right), \] (26)
where we used the fact that \( H = 0 \) and imposed the gauge fixing functions (22). We see from (26) that it is natural to interpret \(-p_0\) as the Hamiltonian density of the reduced theory
\[ \mathcal{H}_{\text{fix}} = -p_0(h_{ij}(x), \mathcal{H}^{\text{GR}}(x), \mathcal{H}^{\text{GR}}_i(x)) \equiv \frac{-16\pi G}{2\sqrt{\det h}} p_0, \] (27)
where we also used the fact that from the constraints \( H_i = 0 \), we can express \( p_i \) as
\[ p_i(x) = -\mathcal{H}^{\text{GR}}_i(x). \] (28)
Finally, note that \( p_0 \) can be derived from the Hamiltonian constraint \( \mathcal{H}_T = \mathcal{H}^{\text{GR}} + \mathcal{H}^{\phi T}_T(x) = 0 \) at least in principle.

In summary, we found the Hamiltonian formulation of massive gravity where the physical degrees of freedom are \( h_{ij}, p^{ij} \) and where the Hamiltonian is given in (27). Note also that the explicit form of this Hamiltonian depends on the form of the function \( L(H^{AB}) \). We give simple examples of two solvable potentials in the next section. Generally, however it is very difficult to find the explicit form of the gauge-fixed Hamiltonian due to the complicated structure of the function \( L(H^{AB}) \).

Despite of this fact we now show that it is possible to find the Lagrangian density for given gauge-fixed theory. To do this, we introduce four modes \( A, B, C_i, D^i \) and corresponding conjugate momenta \( (p_A, p_B, p^i, p_i) \) with non-zero Poisson brackets
\[ \{A(x), p_A(y)\} = \delta(x - y), \quad \{B(x), p_B(y)\} = \delta(x - y), \]
\[ \{C_i(x), p^i(y)\} = \delta_i^j \delta(x - y), \quad \{D^i(x), p_j(y)\} = \delta_i^j \delta(x - y). \] (29)
With the help of these additional modes, we rewrite the Hamiltonian for gauge-fixed theory as
\[ \mathcal{H}_{\text{fix}} = -p_0(h_{ij}, A, C_i) + B(\mathcal{H}^{\text{GR}} - A) + D^i(\mathcal{H}^{\text{GR}} - C_i) + v_A p_A + v_B p_B + v^i p_i + v_i p_i, \] (30)
where the Lagrange multipliers \( v_A, v_B, v^i, v_i \) ensure that \( p_A, p_B, p_i \) and \( p^i \) are primary constraints of the theory
\[ p_A \approx 0, \quad p_B \approx 0, \quad p_i \approx 0, \quad p^i \approx 0. \] (31)
Then the fact that these constraints have to be preserved during the time evolution of the system implies the secondary constraints
\[ \partial_t p_A = \{p_A, \mathcal{H}_{\text{fix}}\} = \frac{\delta p_0}{\delta A} - B \equiv \Phi_A \approx 0, \]
\[ \partial_t p_B = \{p_A, \mathcal{H}_{\text{fix}}\} = -A + \mathcal{H}^{\text{GR}} \equiv \Phi_B \approx 0, \]
\[ \partial_t p_i = \{p_i, \mathcal{H}_{\text{fix}}\} = -C_i + \mathcal{H}^{\text{GR}}_i \equiv \Phi_i \approx 0, \] (32)
\[ \partial_t p^i = \{p^i, \mathcal{H}_{\text{fix}}\} = \frac{\delta p_0}{\delta C_i} - D^i \equiv \Phi^i \approx 0. \]
It can be shown that the collections of the constraints \( (p_A, p_B, p_i, p^i, \Phi_A, \Phi_B, \Phi^i, \Phi_i) \) are the second-class constraints. The solution of these constraints reduces (30) into the original form of the gauge-fixed Hamiltonian (27).

The main advantage of the extended form of the Hamiltonian density (30) is that it allows us to find the corresponding Lagrangian in a relatively straightforward way. In fact, from (30)
we easily obtain the time derivatives of the canonical variables $h_{ij}$, $A$, $B$, $D^i$, $C_i$,

$$
\partial_t h_{ij} = [h_{ij}, H_{\text{fix}}] = B \frac{32\pi G}{\sqrt{\det h}} G_{ijkl} \pi^{kl} + 2 \nabla_i D_j,
$$

$$
\partial_t A = [A, H_{\text{fix}}] = v_A,
\partial_t B = [B, H_{\text{fix}}] = v_B,
\partial_t D^i = [D^i, H_{\text{fix}}] = v^i,
\partial_t C_i = [C_i, H_{\text{fix}}] = v_i.
$$

(33)

It turns out that it is useful to introduce the following object:

$$
\hat{K}_{ij} = \frac{1}{2B} (\partial_t h_{ij} - \nabla_i D_j - \nabla_j D_i)
$$

(34)

that due to the first equation in (33) is related to $\pi^{ij}$ as

$$
\hat{K}_{ij} = \frac{16\pi G}{\sqrt{\det h}} G_{ijkl} \pi^{kl}.
$$

Then it is easy to find the corresponding Lagrangian

$$
L_{\text{fix}} = \int d^d x \left( \frac{\sqrt{\det h}}{16\pi G} \hat{B}(\hat{K}_{ij}) \hat{K}_{ij} + R^{(D-1)} + \frac{\sqrt{\det h}}{8\pi G} \tilde{p}_0(h_{ij}, A, C_i) + B A + D^i C_i \right),
$$

(36)

where $\hat{g}$ is a $D$-dimensional metric with components

$$
\hat{g}_{00} = -B^2 + D_i h^{ij} D_j, \quad \hat{g}_{0i} = D_i, \quad \hat{g}_{ij} = h_{ij},
$$

(37)

and where $\hat{R}$ is $D+1$ Ricci scalar built from this metric. We see that the last form of Lagrangian (36) can be interpreted as the sum of the GR action with additional potential terms that breaks the full diffeomorphism invariance of the theory. It is important to stress that this potential term depends on the auxiliary fields $A$ and $C_i$. In principle, these terms could be integrated out; however, we expect that the resulting Lagrangian would be very complicated.

3. Examples of the potentials $L(H^{AB})$

In this section, we give two solvable examples of the scalar function $L(H^{AB})$ that allow us to find the explicit form of the Lagrangian for massive gravity.

In the first case, we follow [8] and consider the function $L(H^{AB})$ in the form

$$
L = \Lambda + H^{AB} \eta^{BA}.
$$

(38)

Then it is easy to see that

$$
\frac{\delta L}{\delta H_{AB}} = \eta_{BA}
$$

(39)

and

$$
J^{AB} \eta_{BA} = V^{AB} \eta_{BA} - K^{AB} \eta_{BA}.
$$

(40)

Using these results, it is a straightforward exercise to find the scalar field Hamiltonian density

$$
\mathcal{H}_T^\phi = \frac{\sqrt{\det h}}{16\pi G} (K_{AB} \eta^{BA} + V^{AB} \eta_{BA} + \Lambda).
$$

(41)

We would like to stress that a similar model with $L = \Lambda + H_{AB} \delta^{AB}$ has been analyzed previously in [22] with results that overlap ours.
Then following the general procedure outlined in the previous section, we find

$$\mathcal{H}_{\text{fix}} = \frac{\sqrt{\det h}}{8\pi G} \sqrt{\frac{16\pi G}{\sqrt{\det h}}} \mathcal{H}_{\text{GR}}^{T} + \left( \frac{8\pi G}{\sqrt{\det h}} \right)^{2} \mathcal{H}_{i}^{\text{GR}} \delta^{ij} \mathcal{H}_{j}^{\text{GR}} + h^{ij} \delta_{ji} + \Lambda. \quad (42)$$

Finally, we find the Lagrangian density of the gauge-fixed theory in the form

$$\mathcal{L}_{\text{fix}} = \frac{\sqrt{\det h}}{16\pi G} B(\mathring{K}_{ij} G^{ijkl} \mathring{K}_{kl} + R^{(D-1)} + \Lambda) + D_{i} C^{i} + AB$$

$$- \frac{\sqrt{\det h}}{8\pi G} \left( A + \left( \frac{8\pi G}{\sqrt{\det h}} \right)^{2} C_{i} \delta^{ij} C_{j} + \frac{1}{\Omega} h^{ij} \delta_{ji}. \quad (43)$$

We would like to stress that we can integrate out the auxiliary fields $A, B, C^{i}, D_{i}$ from Lagrangian (43) and then to derive the Lagrangian density for the dynamical modes $h_{ij}$ only. However, the resulting Lagrangian would be very complicated and hence we prefer to work with the extended Lagrangian (43).

As the second example of exactly solvable theory, we consider the Lagrangian function $L(H^{AB})$ in the form

$$L(H^{AB}) = \frac{1}{\Omega} + 4 K^{AB} \eta^{BA} \quad (44)$$

where $\Lambda$ and $\Omega$ are constants. From (44), we easily find

$$D_{AB} = \frac{1}{2} \sqrt{\Omega + H^{AB} \eta^{BA}} \quad (45)$$

and hence

$$J^{AB} \eta_{BA} = \frac{V^{AB} \eta_{BA} - 4 \Omega K_{AB} \eta^{BA}}{1 + 4 K_{AB} \eta^{BA}}. \quad (46)$$

After some calculation we obtain the Hamiltonian density for the scalar field in the form

$$\mathcal{H}_{\text{fix}} = \frac{\sqrt{\det h}}{16\pi G} \sqrt{V^{AB} \eta_{BA} + 4 \Omega K_{AB} \eta^{BA}} - \frac{\sqrt{\det h}}{16\pi G} \Lambda. \quad (47)$$

Following the analysis presented in section 2, we find the gauge-fixed Hamiltonian density in the form

$$\mathcal{H}_{\text{fix}} = \frac{\sqrt{\det h}}{16\pi G} \sqrt{1 + \left( \frac{16\pi G}{\sqrt{\det h}} \right)^{2} \mathcal{H}_{i}^{\text{GR}} \delta^{ij} \mathcal{H}_{j}^{\text{GR}} - \left( \frac{16\pi G}{\sqrt{\det h}} \right)^{2} \mathcal{H}_{T}^{\text{GR}} - \Lambda} \frac{1}{\Omega + h^{ij} \delta_{ji}} \quad (48)$$

and the corresponding Lagrangian

$$L_{\text{fix}} = \int a^{0} x \left( \frac{1}{16\pi G} \sqrt{\det \bar{g}(\mathring{R} + \Lambda) + BA + D^{i} C_{i}} \right.$$\n
$$- \frac{\sqrt{\det h}}{16\pi G} \sqrt{1 + \left( \frac{16\pi G}{\sqrt{\det h}} \right)^{2} C_{i} \delta^{ij} C_{j} - \left( \frac{16\pi G}{\sqrt{\det h}} \right)^{2} \frac{1}{\Omega + h^{ij} \delta_{ji}}} \right). \quad (49)$$

In this section, we give two explicit examples of Hamiltonians and Lagrangians for massive gravity. Clearly it would be desirable to understand the properties of these models further.
4. Conclusion

This paper was devoted to the study of the Higgs mechanism for gravity from the point of view of the Hamiltonian formalism. We performed the fixing of the spacetime diffeomorphism and we argued that the resulting Hamiltonian corresponds to the Hamiltonian of the massive gravity. The main advantage of our approach is that this theory is defined on the reduced phase space spanned by the physical degrees of freedom $h_{ij}, p_{ij}$ only. On the other hand, the price we pay for this property is that it is difficult to find the form of the gauge-fixed Hamiltonian for the general potential $L(H^{AB})$. In fact, we are not able to find the explicit form of the gauge-fixed Hamiltonian for the specific form of the scalar actions introduced in [9–14]. On the other hand, introducing additional auxiliary fields we can determine Lagrangian for massive gravity that has the form of the ordinary GR action with specific potential terms that break diffeomorphism invariance. Then we can ask the question how this Lagrangian is related to the original Lagrangian where we fix the gauge as in [9–14]. One can hope that these actions could be related by some field redefinitions. However, finding this redefinition seems to be very complicated due to the presence of the auxiliary fields $A, C_i$ in Lagrangian (36) whose explicit integration would lead to very obscure form of the Lagrangian.

The next important step in our investigation would be to analyze the spectrum of fluctuations around the flat spacetime background. It would also be very interesting to study the classical solutions corresponding to these forms of massive gravities. We hope to return to the analysis of these problems in the future.

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Appendix. Algebra of Hamiltonian constraints

In this appendix, we explicitly prove the stability of constraints (20), or equivalently, we show that these constraints are preserved during the time evolution of the system. In fact, the careful analysis of these constraints was performed in 1970s in the geometrodynamics program [15–18]. It was argued there that for the consistency of theory, the Poisson brackets of constraints should have the form

$$\{H_S(N^i), H_S(M^j)\} = H_S(N^i \partial_j M^j - M^j \partial_i N^j),$$

$$\{H_S(N^i), H_T(M)\} = H_T(N^i \partial M),$$

$$\{H_T(N), H_T(M)\} = H_S((M \partial_j N - N \partial_j M) h^{ij}).$$

The aim of this appendix is to show that the Poisson brackets of constraints obey algebra (A.1). In fact, it is well known that the GR constraints

$$H^{GR}_T(N) = \int d^D x N(x) \mathcal{H}^{GR}_T(x), \quad H^{GR}_S(N^i) = \int d^D x N^i(x) \mathcal{H}^{GR}_S(x)$$

obey the Poisson brackets’ relations (A.1). Further in the case of the scalar field diffeomorphism constraint, we easily find

$$\{H^S_S(N^i), H^S_S(M^j)\} = H^S_S(N^i \partial_j M^j - M^j \partial_i N^j).$$
Then using the fact that the mixed Poisson brackets \( \{ H_\phi (N^i), H_\text{GR}^S (M) \} \) are trivially zero, we find that the Poisson brackets of diffeomorphism constraints obey the first equation in (A.1).

As the next step, we calculate the Poisson bracket between the diffeomorphism constraint \( H_5 (N^i) \) and \( H_\phi^T (M) \). Using the Poisson brackets
\[
\{ H_5 (N^i), \phi^A \} = -N^i \partial_i \phi^A,
\{ H_5 (N^i), p_A \} = -\partial_i (N^i p_A),
\{ H_5 (N^i), h_{ij} \} = -N^k \partial_k h_{ij} - \partial_i N^k h_{kj} - h_{ik} \partial_j N^k,
\{ H_5 (N^i), \sqrt{\det h} \} = -N^k \partial_k \sqrt{\det h} - \partial_k N^k \sqrt{\det h},
\]
we easily find
\[
\{ H_5 (N^i), K_{AB} \} = -N^k \partial_k K_{AB},
\{ H_5 (N^i), V^{AB} \} = -N^k \partial_k V^{AB}.
\]
(A.4)

Collecting all these results, we obtain
\[
\{ H_5 (N^i), \mathcal{H}_T^\phi \} = -N^k \partial_k \mathcal{H}_T^\phi - \partial_k N^k \mathcal{H}_T^\phi.
\]
(A.6)

Finally, using the fact that \( \{ H_5^{GR} (N^i), H_\text{GR}^T (M) \} = H_\text{GR}^T (N^i \partial_i M) \) and equation (A.6), we find that
\[
\{ H_5 (N^i), H_T (M) \} = H_T (N^i \partial_i M).
\]
(A.7)

As the last step, we calculate the Poisson brackets of the Hamiltonian constraints. While it is well known that the Poisson bracket of the GR Hamiltonian constraints takes precisely the form given in (A.1), it is far from obvious that for the system defined by action (1) it holds as well. In the case of the simple scalar action \( \sim g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi \), the result is well known, see for example [15–18]. Further, it was shown very elegantly in [15] that the scalar field action in the form \( L (-g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi) \) leads to the Hamiltonian formulation where the constraints obey the Hamiltonian brackets (A.1). On the other hand, the scalar field Hamiltonian constraint (14) is more general than the form of the Hamiltonian constraints studied in [15] so that we perform the explicit calculation of the Poisson brackets of the smeared form of the Hamiltonian constraints (14) below.

Before we proceed to this calculation, we discuss the calculation of the mixed Poisson brackets between the GR Hamiltonian constraint and scalar field Hamiltonian constraint. The crucial point is that the constraint \( H_\phi^T \) depends on \( g_{ij} \) only. Then it is easy to see that
\[
\{ H_\text{GR}^T (N), H_\phi^T (M) \} + \{ H_\phi^T (N), H_\text{GR}^T (M) \}
= \int d^Dx d^Dy N(x) M(y) \{ \{ \mathcal{H}_T^\phi (x), \mathcal{H}_T^\phi (y) \} \} = 0
\]
using the fact that \( \{ \mathcal{H}_T^\phi (x), \mathcal{H}_T^\phi (y) \} \sim \delta (x - y). \)

\[
\text{Poisson brackets of Hamiltonian constraints for the scalar field}
\]

In this section, we determine the Poisson bracket between \( H_\phi^T (N) \) and \( H_\phi^T (M) \). As a warm example, we begin with the single scalar field action
\[
S = \int d^Dx \sqrt{-g} L(-I/2),
\]
where \( I = g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi \). In the \((D - 1) + 1\) formalism, this action takes the form
\[
S = \int d^Dx \sqrt{\det h} L \left( \frac{1}{2} (\nabla^2 \phi - h^{ij} \partial_i \phi \partial_j \phi) \right).
\]
(A.10)
Then it is easy to see that the momentum conjugate to $\phi$ is equal to

$$p = \sqrt{\det h} L'(J/2) \nabla_{\phi} \phi$$

(A.11)

and consequently

$$K = \left[ \frac{p}{\sqrt{\det h}} \right]^2 = L'^2(J/2)(I + V),$$

(A.12)

where $V = h^{ij} \partial_i \phi \partial_j \phi$ and where $L'(x) = \frac{dL(x)}{dx}$. Let us presume that equation (A.12) can be solved for $I$ and we denote this solution as $I = J(K, V)$. Then (A.11) implies

$$\nabla_{\phi} \phi = \frac{1}{\sqrt{h} L'(J/2)^\pi}$$

(A.13)

and hence the Hamiltonian takes the form

$$H = \int d^d x (N(x) \mathcal{H}_T(x) + N'(x) \mathcal{H}_r(x)),$$

(A.14)

where

$$\mathcal{H}_T = \frac{p^2}{\sqrt{\det h} L'(J/2)} - \sqrt{\det h} L(J/2).$$

(A.15)

Our goal is to calculate the Poisson bracket $\{H^\phi_T(N), H^\phi_r(M)\}$. To do this we start with the calculation of the following Poisson bracket:

$$\{\phi, H_T\} = \int d^d x \{\phi, N(x) \mathcal{H}_T(x)\} = N \frac{2p}{\sqrt{\det h} L'^2(J/2)} \nabla_{\phi} \phi - N \frac{p^3}{(\sqrt{\det h})^3 L'^2(J/2)} \frac{\delta J}{\delta K} - N \frac{p}{\sqrt{\det h} L'(J/2)} \frac{\delta J}{\delta K},$$

(A.16)

where we used

$$\{\phi(x), J(y)\} = 2 \frac{p}{\sqrt{\det h} \delta K} \delta(x - y).$$

(A.17)

To proceed further we note that $J$ obeys the equation

$$K = L'^2(J/2)(J + V).$$

(A.18)

When we differentiate this equation with respect to $K$ and use the fact that $J = J(K, V)$, we find

$$KL''(J/2) \frac{\delta J}{\delta K} = L'(J/2) \left( 1 - L'^2(J/2) \frac{\delta J}{\delta K} \right).$$

(A.19)

Inserting this result into (A.16) and performing the appropriate manipulation, we find that the Poisson bracket (A.16) takes the form

$$\{\phi, H_T(N)\} = N \frac{\pi}{\sqrt{\det h} L'(J/2)} = N \nabla_{\phi} \phi$$

(A.20)

that agrees with the result derived in [15].

As the next step, we determine the Poisson bracket between $p$ and $H_T(N)$.

$$\{p, H_T(N)\} = \int d^d x N(x) \left( -\frac{p^2}{2\sqrt{\det h} L'(J/2)^\frac{1}{2}} \frac{\delta J}{\delta K} \frac{dJ}{dV}(x)\{p, V(x)\} \right)$$

$$- \frac{1}{2} \sqrt{\det h} L'(J/2)^\frac{1}{2} \frac{dJ}{dV}(x)\{p, V(x)\} \right).$$

(A.21)
With the help of relation (A.18), we find
\[ KL''(J/2) \frac{dJ}{dV} = -L^3(J/2) \left( 1 + \frac{dJ}{dV} \right), \quad \frac{dJ}{dV} = -L \frac{dJ}{dK}. \]  
(A.22)
Inserting these expressions into (A.21), we find
\[ \{ p, H_T(N) \} = \frac{1}{2} \int d^d x N \sqrt{\det h} L'(x) [p, V(x)] = \frac{1}{2} [N \sqrt{\det h} h^{ij} L'(x) \partial_j \phi], \]  
(A.23)
where we also used
\[ \{ p(x), V(y) \} = -2 h^{ij}(y) \partial_j \phi(y) \delta(x - y). \]  
(A.24)
Now we are ready to determine the Poisson bracket \( \{ H_T(N), H_T(M) \} \). To do this, we consider the following expression:
\[ \{ \{ \phi, H_T(N) \}, H_T(M) \} - \{ \{ \phi, H_T(M) \}, H_T(N) \}. \]  
(A.25)
In the first step, we use (A.20) and find
\[ \left\{ \frac{1}{L'}, H_T(N) \right\} = -2N \frac{L''(J/2)}{L^3(J/2) \frac{dK}{dJ}} \left( \frac{p}{\sqrt{\det h} L'(x/2)} + \frac{p}{\det h} \partial_i [\sqrt{\det h} h^{ij} \partial_j \phi] \right). \]  
(A.26)
Then it is easy to see that \( \frac{1}{2} \int d^d x N \sqrt{\det h} \{ L', H_T(N) \} - N \left\{ \frac{1}{L'}, H_T(N) \right\} \) = 0. Finally, with the help of (A.23), we find that (A.25) is equal to
\[ \{ \{ \phi, H_T(N) \}, H_T(M) \} - \{ \{ \phi, H_T(M) \}, H_T(N) \} = (N \partial_i M - M \partial_i N) h^{ij} \partial_j \phi. \]  
(A.28)
With the help of the Jacobi identity, we can rewrite this expression into the form
\[ \{ \phi, \{ H_T(N), H_T(M) \} \} = (N \partial_i M - M \partial_i N) h^{ij} \partial_j \phi \approx \{ \phi, H_T(N \partial_i M - M \partial_i N) h^{ij} \partial_j \phi + f(\phi) \} \]  
(A.29)
for the arbitrary function \( f(\phi) \). On the other hand, performing the same step with \( \pi \) we find that \( f \) does not depend on \( \phi \) as well and hence could be taken to vanish at least at classical level. In summary, we proved that the Poisson bracket of the Hamiltonian constraints of the scalar field with general action (A.9) takes the desired form
\[ \{ H_T(N), H_T(M) \} = H_T((N \partial_i M - M \partial_i N) h^{ij}). \]  
(A.30)
Now we are ready to proceed to the calculation of the Poisson bracket of the Hamiltonian constraint when the action for the scalar fields is given in (1). We again start with the calculation of the Poisson bracket between \( \phi^X \) and \( H_T^\phi \),
\[ \{ \phi^X, H_T^\phi \} = N \sqrt{\det h} \left( 2[\phi^X, K_{AB}] \tilde{D}^{BA} \right. \]
\[ \left. - 2K_{AB} \tilde{D}^{BC} \frac{\delta O_{CD}}{\delta K_{MN}} \tilde{D}^{DA} \{ \phi^X, K_{MN} \} + D_{AB} \frac{\delta f^{BA}}{\delta K_{MN}} \{ \phi^X, K_{MN} \} \right). \]  
(A.31)
To proceed we note that equation (12) implies the following relation:
\[ K_{AB} = D_{AC} (V^{CD} - J^{CD}) D_{DB}. \]  
(A.32)
where $D_{AB}$ depends on $J$. Then we calculate the Poisson bracket between $\phi^X$ and the relation given above, multiply the result with $\tilde{D}^{BA}$ and finally take the traces over capital indices. As a result, we find

$$\tilde{D}^{BA}\{\phi^X, K_{AB}\} = 2\tilde{D}^{BA} \frac{\delta D_{AC}}{\delta K_{MN}}\{\phi^X, K_{MN}\} \tilde{D}^{CD} K_{DB} + \frac{\delta J_{BD}}{\delta K_{MN}} D_{DB}\{\phi^X, K_{MN}\}. \tag{A.33}$$

Using this result in (A.31), we find

$$\{\phi^X, H_T^\phi(N)\} = \frac{N}{2\sqrt{|h|}} \tilde{D}^{BB} \rho_B = N\nabla_n \phi^X. \tag{A.34}$$

In a similar way we proceed with the calculation of the Poisson bracket between $p_X$ and $H_T^\phi$,

$$\{p_X, H_T^\phi(N)\} = 2\partial_I[\sqrt{|h|}^i N D_{XA} \partial_J \phi^A], \tag{A.35}$$

where we used the Poisson bracket between $p_X$ and (A.32) that implies

$$2K_{PB} \tilde{D}^{BA} \frac{\delta D_{AC}}{\delta J_{MN}} \frac{\delta J_{MN}}{\delta V_{PQ}}\{p_X, V_{PQ}\} = \frac{\delta J_{CD}}{\delta V_{PQ}} D_{DC}\{p_X, V_{PQ}\} - \{p_X, V_{CD}\} D_{DC}. \tag{A.36}$$

As in the case of single scalar field, we consider the following expression:

$$\{\{\phi^A, H_T^\phi(N)\} H_M^\phi(M)\} - \{\{\phi^A, H_T^\phi(M)\} H_T^\phi(N)\}, \tag{A.37}$$

where the first double Poisson bracket is equal to

$$\{\{\phi^A, H_T^\phi(N)\} H_M^\phi(M)\} = \frac{N}{\sqrt{|h|}} D^{AB} \partial_I[M \sqrt{|h|}^i D_{BC} \partial_J \phi^C]$$

$$- \frac{N}{\sqrt{|h|}} \tilde{D}^{AC} \left(\frac{\delta D_{CD}}{\delta J_{MN}} \frac{\delta J_{MN}}{\delta V_{PQ}} \{K_{PQ}, H_T^\phi(M)\} \right)$$

$$+ \frac{\delta D_{CD}}{\delta J_{MN}} \frac{\delta J_{MN}}{\delta V_{PQ}} \{V_{PQ}, H_T^\phi(M)\} \right) \tilde{D}^{DB} \rho_B. \tag{A.38}$$

To proceed further we have to determine the Poisson bracket between $D_{AB}$ and $H_T^\phi(N)$. Note that when we compare the variation of (A.32) with respect to $K_{MN}$ with the variation of (A.32) with respect to $V_{MN}$, we find the following relation:

$$\frac{\delta J_{MN}}{\delta V_{PQ}} = -\frac{\delta J_{MN}}{\delta K_{RS}} D_{RP} D_{SQ}. \tag{A.39}$$

Then using this relation in (A.38) and after some manipulation, we find

$$\{\{\phi^A, H_T^\phi(N)\} H_T^\phi(M)\} - \{\{\phi^A, H_T^\phi(M)\} H_T^\phi(N)\} = (N \partial_i M - M \partial_i N) h^{ij} \partial_J \phi^A. \tag{A.40}$$

Performing the same analysis with the conjugate momenta $p_A$, we find the desired result

$$\{H_T^\phi(N), H_T^\phi(M)\} = H_T^\phi((N \partial_i M - M \partial_i N) h^{ij}). \tag{A.41}$$

The upshot of this long analysis is the proof that the Poisson brackets of Hamiltonian constraints $H_T^\phi(N) = H_T^\text{GR}(N) + H_T^\phi(N)$ take exactly the same form as in (A.1). Using this fact we immediately obtain that the Hamiltonian and diffeomorphism constraints are preserved during the time evolution of the system. This is a very important result since only after determining the complete constraint structure of the given theory is it possible to perform the Hamiltonian gauge fixing.
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