Tensor network formulation of two dimensional gravity

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ABSTRACT: We show how to formulate a lattice gauge theory whose naive continuum limit corresponds to two dimensional (Euclidean) quantum gravity including a positive cosmological constant. More precisely the resultant continuum theory corresponds to gravity in a first order formalism in which the local frame and spin connection are treated as independent fields. Recasting this lattice theory as a tensor network allows us to study the theory at strong coupling without encountering a sign problem. In two dimensions this tensor network is exactly soluble and we show that the system has a series of critical points associated with first order phase transitions. The construction generalizes in principle to four dimensions and other signs of the cosmological constant.
1. Introduction

We will motivate our work by starting with a discussion of how gravity in four dimensions can be written as a gauge theory. As we will show this argument holds equally well in two dimensions which also exhibits further simplifications and in discrete form can be rewritten as a soluble tensor network theory.

2. Review of the Palatini-Cartan formulation of Einstein gravity

It is well known that continuum Einstein gravity can be rewritten in a first order formalism using the local frame field $e_{\mu}(x)$ and a spin connection $\omega_{\mu}(x)$ [1]. The metric is given by

$$g_{\mu\nu} = e^{a}_{\mu} e^{b}_{\nu} \eta_{ab}$$  \hspace{1cm} (2.1)

and is clearly invariant under local Lorentz rotations of the frame field $e^{a}_{\mu} \rightarrow \Lambda^{a}_{b} e^{b}_{\mu}$. Notice that the world indices - here $\mu$ - are unaffected by this local transformation which acts only on the tangent space - the roman indices. In order to write down derivatives one must introduce a gauge field which is precisely the spin connection $\omega_{\mu}(x)$ transforming as

$$\omega^{ab}_{\mu} \rightarrow D^{ac}_{\mu} \phi^{cb} = \partial_{\mu} \phi^{ab} + \left[\omega_{\mu}, \phi\right]^{ab}$$  \hspace{1cm} (2.2)
where \( \omega_{\mu} = \sum_{a<b} \omega_{\mu}^{ab}(x)T^{ab} \) is summed over the generators of the Lorentz group. A natural locally Lorentz invariant action can then be written down in terms of \( e_{\mu} \) and the usual Yang-Mills curvature \( R_{\mu\nu} = [D_{\mu}, D_{\nu}] \) as

\[
S = \frac{1}{\ell_p^2} \int d^4x \epsilon^{\mu\nu\rho\lambda} \epsilon_{abcd} \left( e_{\mu}^{a} e_{\nu}^{b} R_{\lambda\rho}^{cd} - \frac{1}{\ell_p^2} e_{\mu}^{a} e_{\nu}^{b} e_{\lambda}^{c} e_{\rho}^{d} \right) \quad (2.3)
\]

Notice that since \( g_{\mu\nu} \) is not a fundamental field in this approach the only tensor available to contract world indices is the invariant tensor \( \epsilon^{\mu\nu\rho\lambda} \) which automatically guarantees that the theory is independent of coordinate transformations. Rather remarkably this action reduces to the usual Einstein-Hilbert action provided

\[
\det(e_{\mu}^{a}) \neq 0 \quad (2.4)
\]

\[
T_{\mu\nu}^{a} = D_{[\mu} e_{\nu]}^{a} = 0 \quad (2.5)
\]

where the first line guarantees that we can invert the frame field considered as a \( 4 \times 4 \) matrix and the second is the usual vanishing torsion condition required to achieve a theory that depends only on the metric by suppling an additional condition that expresses the spin connection in terms of the frame field. Notice that the first term in the eqn 2.3 reduces to the usual Ricci scalar of the metric theory once one employs the relation \( e^{b}_{a} e_{\lambda}^{a} = \delta^{b}_{a} \) relating the frame to its (matrix) inverse:

\[
\epsilon^{\mu\nu\rho\lambda} \epsilon_{abcd} e_{\mu}^{a} e_{\nu}^{b} R_{\lambda\rho}^{cd} \quad = \quad (\epsilon^{\mu\nu\rho\lambda} \epsilon_{abcd} e_{\mu}^{a} e_{\nu}^{b} e_{\lambda}^{c} e_{\rho}^{d}) (e^{\lambda}_{c} e^{\rho}_{d} R_{\lambda\rho}^{cd}) = 24 \sqrt{-g} R. \quad (2.6)
\]

Thus in the metric language the action becomes

\[
S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - 2\Lambda) \quad (2.7)
\]

Here, \( G = \frac{\ell_p^2}{M_{Pl}^2} \) is the gravitational constant and \( \Lambda = \frac{1}{2\ell_p^2} \) is the cosmological constant.

The Cartan formalism offer several advantages over the conventional metric approach; it is explicitly independent of any background metric, it employs the familiar formalism of gauge theories and it naturally includes fermions via the spin connection. It’s main disadvantage is that it contains two independent fields - the frame and spin connection. However we will see in the next section that it is possible to enlarge the gauge symmetry in such a way that both fields play an equivalent role and where the equations of motion automatically ensure the torsion free condition.

3. Cartan gravity as spontaneously broken (anti)de Sitter gravity

While it is well known that gravity can be recast as a gauge theory of the Lorentz group as reviewed in the previous section it is less well known that it can be recovered from a theory in which the Lorentz symmetry is extended to a full de Sitter symmetry with the frame \( e_{\mu}^{a} \) supplying the additional gauge fields [2]. The key physical requirement is that the vacuum of
the theory must be correspond to a phase in which the de Sitter symmetry is spontaneously broken down to Lorentz symmetry.

For simplicity we will restrict the discussion from this point on to the Euclidean theory in which the $SO(4)$ Lorentz symmetry is embedded in the larger $SO(5)$ (Euclidean) de Sitter symmetry. The action that is required is

$$S_M = \kappa \int d^4 x \epsilon^{\mu\nu\lambda\rho} \epsilon_{ABCDE} \phi^E F_{\mu\nu}^{AB} F_{\lambda\rho}^{CD} \quad A, B, \ldots = 1 \ldots 5 \quad (3.1)$$

The curvature $F$ is the usual $SO(5)$ Yang-Mills term while the scalar field is a new degree of freedom which transforms in the fundamental representation of $SO(5)$. We impose the condition that $\phi^A \phi^A = 1$ corresponding to a phase in which the $SO(5)$ is spontaneously broken to $SO(4)$. If we make the gauge choice $\phi^A = \delta^5A$ we can decompose this action under the unbroken $SO(4)$ subgroup by identifying the fields of the broken generators with the frame $e_\mu$ as in

$$A_\mu = \omega_\mu^{ab} T^{ab} + \frac{1}{\ell} e^a_\mu T^5 a, b = 1 \ldots 4 \quad (3.2)$$

In a similar way the $SO(5)$ curvature decomposes under the unbroken subgroup as

$$F_{\mu\nu} = (R_{\mu\nu}^{ab} - \frac{1}{\ell^2} e^a_{[\mu} e^b_{\nu]} T^{ab} + D_{[\mu} e^a_{\nu]} T^5 a, b = 1 \ldots 4 \quad (3.3)$$

where $R$ is the $SO(4)$ curvature as before.

There are several advantages to this construction. Firstly, the classical equations of motion of the $SO(5)$ theory ensure that $F = 0$ and also $D\phi = 0$. The former ensures that

$$R^{ab}_{\mu\nu} - \frac{1}{\ell^2} e^a_{[\mu} e^b_{\nu]} = 0 \quad (3.4)$$

$$D_{[\mu} e^a_{\nu]} = 0 \quad (3.5)$$

Thus the torsion free condition and Einstein’s equation including a cosmological constant emerge automatically in the classical limit. The enlarged gauge symmetry also helps to constrain counter terms in the theory and requires that the correct measure for path integration be invariant under the de Sitter symmetry.\footnote{It should be noted that essentially the same construction works in odd dimensions and generates Witten’s representation of 3d gravity as a Chern-Simons gauge theory. In that case however there is no need for an additional scalar to break the symmetry [3].}

1 It should also be clear that remaining dimensionless constant in the theory $\kappa = \left(\frac{\ell}{\ell_p}\right)^2 \sim \frac{1}{\ell^4}\lambda$.  

4. Two dimensions

We devote the remainder of this paper to an exploration of this approach in the case of two dimensions. There are several studies of gauge theoretic formulations of gravity in two dimensions [4, 5, 6, 7] and a great deal is known about the metric theory through Liouville
theory and matrix models [8, 9]. In two dimensions, the Einstein-Hilbert action is a topological invariant and the the Einstein tensor is identically zero. To obtain a non-trivial analog of the Einstein equation, Jackiw and Teitelboim proposed a solution, $R - 2\Lambda = 0$, where $R$ is the Ricci scalar and $\Lambda$ is the cosmological constant. The proposed Lagrangian of the Jackiw-Teitelboim (dilaton) gravity [10] is,

$$\mathcal{L} = \sqrt{-g} \phi (R - \Lambda)$$  \hspace{1cm} (4.1)

Note the introduction of the scalar field which acts as a Lagrange multiplier needed to enforce the equation of motion. It is analogous to the scalar that appeared in the previous gauge theoretic approach to four dimensional gravity.

Returning to this latter construction it is easy to see that the analogous Lagrangian to eqn. 3.1 in two dimensions is,

$$S = \int d^2 x \epsilon_{\mu\nu} \epsilon_{abc} \epsilon^c F_{ab}^{\mu\nu}, \ a, b = 0, 1, 2 \hspace{1cm} (4.2)$$

where $F$ takes its values in $SO(3)$. Exploiting the homomorphism $SO(3) \sim SU(2)$ this can be rewritten as

$$S = \int d^2 x \epsilon_{\mu\nu} \text{Tr} (\phi F)$$  \hspace{1cm} (4.3)

where $\phi$ is now in the adjoint of $SU(2)$. Picking a unitary gauge allows us to simplify the action further to

$$S = \kappa \int d^2 x \epsilon_{\mu\nu} \text{Tr} (\sigma_3 F)$$  \hspace{1cm} (4.4)

where clearly exhibits the remaining exact $SO(2) \sim U(1)$ Lorentz symmetry. Again the classical equations correspond to vanishing torsion and $R = \frac{1}{\ell^2}$. Thus we have shown that a natural candidate for two dimensional gravity takes the form of an $SU(2)$ gauge theory. It is then natural to discretize it on a lattice and look for a non-trivial continuum limit. We turn to this in the next section.

5. Lattice model and tensor network representation

While the model could be discretized on any lattice it is simplest to pick a simple square lattice, place group elements of $SU(2)$ on the links as in lattice QCD and write an action of the form

$$S = -\kappa \sum_x \text{Tr} (i\sigma_3 \left[ U_P - U_P^\dagger \right])$$  \hspace{1cm} (5.1)

where $U_P$ is the usual Wilson plaquette operator.

At this point one should be worried that by picking a particular lattice we have lost the original coordinate invariance of the continuum theory. Of course this is necessarily true in any discrete model and one should now worry about all possible $SU(2)$ invariant counter terms that could be induced via quantum corrections that break the diffeomorphism
invariance. Perhaps the most important of these is the usual Wilson term and so we have added this term with an independent coupling $\beta$ to our action. The idea is that by tuning in the $(\kappa, \beta)$ plane we might hope to restore coordinate invariance in the continuum limit. The presence of the $\beta$ term has another consequence; it allows us to control the lattice spacing in the model since one expects that as $\beta \to \infty$ all the gauge links will be driven close to the unit matrix (up to gauge transformations) and our lattice expressions go over to their continuum counterparts. The final action we hence study takes the form

$$S = -\mu \sum_x \text{ReTr} \left( MU_P \right)$$

(5.2)

where both gravity and Wilson terms have been combined into a single operator depending on the (constant) $SU(2)$ matrix $M = e^{i\sigma_3 \theta}$ which is a function of the couplings $(\kappa, \beta)$.

$$\cos \theta = \frac{\beta}{\mu}$$

(5.3)

$$\mu = \sqrt{\beta^2 + \kappa^2}$$

(5.4)

To proceed further we employ the character expansion:

$$e^{\text{Tr} \left( MU_P \right)} = \sum_j \frac{2(2j + 1)I_{2j+1}(\mu)}{\mu} \chi^j \left( MU_P \right)$$

(5.5)

with $I_n$ the modified Bessel function and where the sum runs over all irreducible representations of $SU(2)$ labeled by $j$. Expanding the character $\chi^j$ on products of Wigner D-matrices yields an expression for the partition function:

$$Z = \int \prod_i DU_i \prod_p 2(2j + 1)I_{2j+1}(\mu) D_{ab}^j(M)D_{bc}^j(U_1)D_{cd}^j(U_2)D_{de}^j(U_3)D_{ea}^j(U_4)$$

(5.6)

with $U_1, U_2, \ldots$ denoting the links around a given plaquette. For a two dimensional torus we can then integrate out the individual gauge links $U_\mu(x)$ using the result $\int DUD^{*k}D^{*k} = \frac{1}{2j+1} \delta_{jk} \delta_{ac} \delta_{bd}$. Clearly the result of this integration ensures that only a single representation survives and the resultant expression can be organized as a product over all sites

$$Z = \sum_j f_j^N$$

(5.9)
where

\[ f_j = \frac{1}{2j+1} U_{2j} \left( \frac{\beta}{\mu} \right) \frac{2I_{2j+1}(\mu)}{\mu} \]  

(5.10)

where the character can be expressed in terms of Chebyshev polynomials of the second kind \( U_{2j} \). With the partition function written in this form, an obvious tensor network formulation can be built. Consider the tensor located at lattice plaquette \( x \),

\[ T^{(x)}_{ijkl} = \begin{cases} f_r & \text{if } i = j = k = l = 2r \\ 0 & \text{otherwise.} \end{cases} \]  

(5.11)

where each index is associated with an adjacent plaquette, of which there are four in two-dimensions. This tensor is very diagonal, with the only nonzero entries being those where all four indices are identical. By contracting this tensor with itself one reconstructs the partition function,

\[ Z = \prod_{n=1}^{2N} \sum_{i_n=0}^{\infty} T_{i_1i_2i_3i_4} \cdots T_{i_{4i_{2N-2}i_{2N-1}i_{2N}}} = \text{Tr} \left[ \prod_{x=1}^{N} T^{(x)} \right] \]  

(5.12)

with the trace being interpreted as a tensor trace. Since the tensor is diagonal, the \( N \) tensors simply reproduce the \( N \)th power of the \( f_r \)s, and the \( 2N \) sums for each link, simply reduce to one sum over representations. A different although necessarily equivalent formulation can be done by expanding the Boltzmann weight with Eq. (5.2), and separately expanding the Boltzmann weight with the Wilson action. The same steps from above can be followed and the gauge fields can be integrated over to give a more complicated tensor in terms of Clebsch-Gordan coefficients.

Critical points of the system correspond to zeros of \( Z \) in the plane of complex coupling. In general phase transitions occur when these so-called Fisher zeros pinch the axes in the thermodynamic limit. In the next section we examine this in more detail.

6. Fisher zeros

Since \( \beta \) corresponds to an irrelevant operator in the language of the renormalization group we will focus our analysis on the plane of complex \( \kappa \). In fact simple dimensional analysis indicates that \( \beta \) contains a factor of the lattice spacing squared \( a^2 \). Keeping the physical area of the geometry fixed as the number of sites \( N \) is increased requires \( a^2 = c/N \). Hence we keep \( c \) fixed and scale \( \beta = c/N \). In practice we fix \( c = 1 \) in our work.

In practice we truncate the expansion in representation \( j \) at some \( j_{\max} \). Setting \( j_{\max} = 1 \) we show in figure 1 lines where the real and imaginary parts of \( Z \) vanish when \( N = 16 \). Where these curves cross corresponds to zeros of \( Z \). We observe that rings of zeroes develop centered at discrete intervals along the imaginary \( \kappa \) axis. If we focus on the leading ring we can see that the density of zeroes along the ring increases with \( N \) - see figure 2 which shows results for \( N = 36 \). Indeed, we observe that the number of zeroes is precisely \( N \). Notice that while there are no zeroes on the imaginary axis the set of zeroes approach the
**Figure 1:** Zeros of the partition function in the complex $\kappa$ plane with $j_{\text{max}} = 1$, $\beta = \frac{1}{N}$. for $N = 16$

**Figure 2:** Zeros of the partition function in the complex $\kappa$ plane with $j_{\text{max}} = 1$, $\beta = \frac{1}{N}$ for $N = 36$

imaginary axis as $N$ increases. This is precisely the behavior required of a Fisher zero in the thermodynamic limit. The only twist over the usual scenario is that the convergence to zero on the imaginary axis implies that the system only develops a phase transition when the coupling
Figure 3:

$\kappa$ is pure imaginary. On reflection this actually should not be surprising; the gravity term resembles a topological term since it employs an epsilon tensor to contract spacetime indices. On Wick rotation to Euclidean space such a term naturally acquires a factor of the square root of minus one. Notice also that such an action would be impossible to simulate using Monte Carlo methods because of a dramatic sign problem highlighting the advantages of the tensor network approach we adopt here. Actually the rate at which the zeroes approach the imaginary axis yields the correlation length exponent $\nu$ associated with the phase transition that arises in the thermodynamic limit [11],

$$\kappa_{\text{zero}}(N) = \kappa_c(\infty) + AN^{-\frac{1}{2\nu}}$$  \hspace{1cm} (6.1)

Figure 3 shows a plot of $\text{Im} \kappa$ for the zero closest to the axis as a function of $L = \sqrt{N}$ together with a fit to the form given in eqn. 6.1. The fitted exponent is $\nu = 0.2495$ and $\kappa_c(\infty) = 3.51833$.

Let us now try to understand this structure using analytical arguments. In the limit $N \to \infty$ the partition function formally truncates to just the leading term $j_{\text{max}} = 0$. The free energy is then

$$f = \frac{1}{N} \ln Z = \ln \frac{I_1(\kappa)}{\kappa}$$  \hspace{1cm} (6.2)

This clearly possesses no zeroes on the real $\kappa$ line. However, taking $\kappa \to i\kappa$ takes $I_1(\kappa) \to J_1(\kappa)$ and it appears that the free energy possesses a series of logarithmic singularities along the imaginary axis corresponding to the zeroes of the first Bessel function. Furthermore

$$\frac{\partial f}{\partial \kappa} = \det e = \frac{1}{J_1(\kappa)}(J_0(\kappa) - J_2(\kappa)) - \frac{1}{\kappa}$$  \hspace{1cm} (6.3)
where we have neglected the Riemann term as it is a topological invariant and does not scale with $N$. Clearly at points where $J_1 = 0$ the mean area measured in units of the lattice spacing diverges. This is one crucial requirement of a sensible continuum limit.

However while the theory truncates to the leading term $J_1(\kappa)$ for generic values of $\kappa$ in the large $N$ limit this procedure fails precisely in regions close to the zeroes of $J_1$. If the second term $J_3$ is kept in the analysis it is easy to see that

$$Z = \left(\frac{2}{\kappa} J_1(\kappa)\right)^N, \quad \text{for} \quad \left|\frac{3J_1}{J_3}\right| > 1 \quad (6.4)$$

$$Z = \left(\frac{2}{3\kappa} J_3(\kappa)\right)^N, \quad \text{for} \quad \left|\frac{3J_1}{J_3}\right| < 1. \quad (6.5)$$

Since the zeroes of $J_3$ never coincide with those of $J_1$ there will always be windows in $\kappa$ around each zero of $J_1$ where the free energy changes from behaving like $J_1$ to $J_3$ and an exact zero is avoided. The upper and lower limits of this window can be found by solving the equation $\left|\frac{J_1}{3J_3}\right| = 1$. Since the Bessel functions are analytic this interval becomes a curve in the complex plane corresponding to the rings observed in the Fisher zero analysis. The occurrence of $N$ zeroes then corresponds to the solutions of $(3J_1(z-z_0)/J_3(z-z_0))^N = 1$ with $z_0$ a zero of $J_1$. Close to $z_0$ this can be approximated by a linear function of $z-z_0$ which is then proportional to an $N$th root of unity. It is interesting to note that, from this expression we obtain a critical coupling constant $\kappa_c = 3.51832$ where the phase transition occurs. This matches exactly from the finite size scaling analysis described in the first part of this section.

It should be clear that while the free energy is continuous at this boundary in the large $N$ limit its derivative will not be - the jump in the slope being $\frac{\partial}{\partial \kappa} \ln(J_3(\kappa)/J_1(\kappa))$. Thus one expects a series of finite jumps in the value of $\langle \det e \rangle$ as a function of (imaginary) coupling $\kappa$. The existence of such first order phase transitions hence preclude the existence of a continuum limit in this lattice theory. This conclusion remains even for larger $j_{\max}$. Inclusion of the higher order Bessels does not change the contour plots of the zeros of the partition function significantly near the first zero of $J_1$. The largest terms in the expansion near the first zero of $J_1$ arise from $J_3$ and all others are exponentially suppressed as $N \to \infty$. It is possible that some higher Bessels $J_m$ and $J_n$ with $m, n > 3$ will dominate near some zero of $J_1$ further from the origin so that the window of convergence will be controlled by $(n J_m(\kappa))/(m J_n(\kappa))$. However the essential conclusion of a discontinuous first derivative of the free energy will continue to hold.

7. Conclusions

In this paper we shown how a tensor network formulation can give a useful method to attack gauge theoretic approaches to quantum gravity. In two dimensions we are able to solve the model exactly by mapping it to a dual representation comprising discrete representations of $SU(2)$. In the space of couplings of our current lattice model however we find only first
order phase transitions. We are currently investigating generalizations of the action which may allow for continuous phase transitions. One possibility would be to add gauge invariant kinetic terms for the scalar field. Such terms could arise as a consequence of picking a fixed lattice background. We know from the old work on 2d matrix models that the interesting dynamics in such theories arises from a presence of a propagating scalar - the Liouville field.

Our work has been restricted to two dimensions but there is no problem of principle to generalizing it to four dimensions. For example the discrete analog of eqn. 3.1 is given by [12]

$$ S = \kappa \sum_x \epsilon_{\mu\rho\lambda} \text{Tr} \left( \gamma_5 \left[ U_{\mu\nu}^P - U_{\mu\nu}^{P\dagger} \right] \left[ U_{\rho\lambda}^P - U_{\rho\lambda}^{P\dagger} \right] \right) $$

(7.1)

where the link fields are valued in $\text{Spin}(5)$ and unitary gauge has been used. More subtle is the question of the sign of the cosmological constant. Even in two dimensional Euclidean space an attempt to study anti de Sitter space would necessitate replacing the compact group $SU(2)$ by its non compact cousin $SU(1,1)$. The latter possesses unitary representations labeled by a continuous index in addition to a discrete series of representations which are analogs of those in $SU(2)$. This renders the character expansion and subsequent Haar integration a much more subtle enterprise which the authors hope to pursue in future research.

**Acknowledgments**

This work is supported in part by the U.S. Department of Energy, Office of Science, Office of High Energy Physics, under Award Number DE-SC0019139. The authors are grateful for discussions with Jay Hubisz and Raghav Jha.

**References**

[1] T. Eguchi, P. B. Gilkey and A. J. Hanson, *Gravitation, gauge theories and differential geometry*, *Physics reports* 66 (1980) 213.

[2] S. W. MacDowell and F. Mansouri, *Unified Geometric Theory of Gravity and Supergravity*, *Phys. Rev. Lett.* 38 (1977) 739.

[3] E. Witten, *(2+1)-Dimensional Gravity as an Exactly Soluble System*, *Nucl. Phys.* B311 (1988) 46.

[4] T. Fukuyama and K. Kamimura, *Gauge theory of two-dimensional gravity*, *Physics Letters B* 160 (1985) 259.

[5] K. Isler and C. A. Trugenberger, *Gauge theory of two-dimensional quantum gravity*, *Physical Review Letters* 63 (1989) 834.

[6] A. H. Chamseddine and D. Wyler, *Gauge theory of topological gravity in 1+ 1 dimensions*, *Physics Letters B* 228 (1989) 75.

[7] A. H. Chamseddine, *Topological gravity and supergravity in various dimensions*, *Nuclear Physics B* 346 (1990) 213.
[8] P. H. Ginsparg, *APPLIED CONFORMAL FIELD THEORY*, in *Les Houches Summer School in Theoretical Physics: Fields, Strings, Critical Phenomena Les Houches, France, June 28-August 5, 1988*, pp. 1–168, 1988, hep-th/9108028.

[9] P. H. Ginsparg, *Matrix models of 2-d gravity*, in *Trieste HEP Cosmol.1991:785-826*, pp. 785–826, 1991, hep-th/9112013.

[10] R. Jackiw, *Gauge theories for gravity on a line*, *Theor. Math. Phys.* 92 (1992) 979 [hep-th/9206093].

[11] B. Klaus and C. Roiesnel, *High statistics finite size scaling analysis of U(1) lattice gauge theory with Wilson action*, *Phys. Rev.* D58 (1998) 114509 [hep-lat/9801036].

[12] S. Catterall, D. Ferrante and A. Nicholson, *de Sitter gravity from lattice gauge theory*, *Eur. Phys. J. Plus* 127 (2012) 101 [0912.5525].