Propagation of Optical Dark Solitons in Metamaterials

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Abstract. This paper studies the interaction between the formation of a single dark soliton and a pair of fundamental-order dark solitons. Based on the nonlinear Schrodinger’s equation combined with the lossless Drude model, the impacts of the nonlinearity, group velocity dispersion, third-order dispersion, and the self-steepening on the propagation of a dark soliton under the normal dispersion are studied using the distributed Fourier method. By selecting the ratio of group velocity dispersion length and the nonlinear length, the conditions for the stable transmission of dark solitons in metamaterials are obtained. On this basis, this paper also studies the interaction of a pair of fundamental-order dark soliton under the different conditions of the same amplitude with the same phase, the same amplitude with different phases, and different amplitudes with the same phase. In this paper, it is shown that the fundamental-order dark soliton pairs exhibit different characteristics of the interaction under the different conditions.

Keywords: Metamaterial; Nonlinear Schrödinger equation; Dark solitons; Interaction.

1. Introduction
Metamaterials are different from conventional materials. They are artificially manufactured with some characteristics that conventional materials do not have. There are many previous researches focusing on the nonlinearity of the metamaterials, such as self-steepening \cite{6}, saturation nonlinearity and so on. The interaction of electromagnetic wave with a metamaterial not only modulates the metamaterial but modulates the electromagnetic wave. Such modulation depends on the physical and chemical structures of the metamaterial. Optical solitons are a type of special case of the interaction of light with nonlinear dispersive media. Both bright and dark solitons can be produced when the impacts of the third order nonlinearity and the group velocity dispersion are balanced. For bright solitons, they should be with the positive nonlinearity in anomalous dispersive region or the negative nonlinearity in normal dispersive region. Hailan Liu et. al. found that, for dark solitons, these conditions are opposite \cite{2}. The work by W. Zhao and E. Bourkoff shows that compared with bright soliton, dark soliton is more stable against loss and background noise \cite{3}. This paper focuses on the studies of the propagation of single dark soliton and dark soliton pairs in metamaterials. The previous works select parameters arbitrarily in the study of the formation of a single dark soliton in metamaterials \cite{2,3}. The arbitrariness in the parameter selection makes them loss the practicality. In our work the determination of all parameters is based on Drude model which dominates the electromagnetic behaviours of metamaterials. On the other hand, researches on the propagation of a pair of solitons are of special significance as a pair of solitons is a compression of a sequence of signal pulses. The interaction within a pair of solitons which affects the quality of signal transmission should be prohibited. The previous work on the interaction between dark solitons shows that two dark solitons in a pair repel each other from the beginning and this repulsion is quite weak.
compared to bright solitons, and it is also not periodic \cite{5}. This paper also studies the complicated cases of a pair of solitons with different amplitudes and different phases that the previous researches did not deal with. In a word, the contribution of this paper is two-fold. Firstly, the paper studies the formation of a single dark soliton in real Drude model. Secondly, the paper studies the interactions of dark soliton pairs under the conditions of different amplitudes with the same phase and the same amplitude with different phases. In Section 2, we will introduce the basics about the Schrödinger equation, which is the theoretical model of this paper. In Section 3, we will introduce the formation conditions of a single dark soliton under various situations. In Section 4, we will introduce the interaction between fundamental-order dark soliton pairs. Lastly in Section 5, we will summarize our work.

2. Theoretical Model

For a single light pulse, its transmission in metamaterials can be described by the nonlinear Schrödinger equation:

\[ \frac{\partial A}{\partial z} = -\frac{i}{2} \beta_2 \frac{\partial^2 A}{\partial \tau^2} + \frac{1}{6} \beta_3 \frac{\partial^3 A}{\partial \tau^3} + i \Gamma_1 (1 + i \xi (\frac{\partial}{\partial \tau}) |A|^2) \]  

where $A$ is the complex amplitude envelope of the pulsed electric field, $\beta_2$ is the second-order dispersion, $\beta_3$ is the third-order dispersion. The detailed derivation of these two dispersion parameters can be found in the previous work \cite{1}. $\Gamma_1 = \varepsilon_0 \mu_0 \chi^{(3)} \omega^2 \mu_r(\omega) / 2 \beta_0$ is the third-order nonlinear coefficient, $\varepsilon_0$ is the dielectric constant in vacuum, $\mu_0$ is the permeability constant in vacuum, $\mu_r(\omega)$ is the permeability coefficient, $\omega$ is the pulse frequency, $\chi^{(3)}$ is the third-order electrical susceptibility, $\beta_0$ is the propagation constant corresponding to the center frequency $\omega_0$, $S = 1 / \omega_0 [1 + (\partial |\omega \mu_r(\omega)| / \partial \omega)_{\omega=\omega_0} / \mu_r(\omega) - \omega_0 \beta_3 / \beta_0]$ is the total self-steepening coefficient and $\beta_1$ is the group velocity.

In order to facilitate the calculation of the interaction between the self-steepening and the fundamental-order dark soliton pair, we normalize equation (1):

\[ \partial_t U = -\frac{\nabla n(\beta_2)}{2} \partial_t^2 U + \frac{b_3}{6} \partial_t^3 U + i N^2 \times [|U|^2 U + i S_1 (\partial_t |U|^2 U)] \]  

Assuming that $\xi = Z / L_{d2}$ is the normalized distance, $\tau = T / T_0$ is the normalized time, $U = A / A_0$ is the normalized pulse electric field complex amplitude envelope, where $A_0$ is the initial pulse amplitude and $T_0$ pulse width, $L_{d2} = T_0^2 / |\beta_2|$ is the second-order dispersion length, and the definition is $N^2 = [L_{d2} / |L_{NL}|]$. $N$ is the soliton order, $L_{NL} = (\Gamma_1 A_0^2)^{-1}$ is nonlinear length, $S_1 = S / T_0$, $b_3 = L_{d2} / L_{d3}$, $L_{d3} = T_0^3 / |\beta_3|$ is the third-order dispersion length.

3. The Formation of a Single Dark Soliton

3.1. The Effect of Pulse Amplitude on Pulse Transmission

The nonlinear length $L_{NL}$ is related to the pulse amplitude $A_0$. In the normal dispersion area, the positive second-order dispersion and positive nonlinearity are selected as parameters, as shown in table 1. Ignoring the effects of third-order dispersion and self-steepening, we change the pulse amplitude so that the ratio of the second-order dispersion length to the nonlinear length satisfies $|L_{d2}| / |L_{NL}| > 1$, $|L_{d2}| / |L_{NL}| < 1$ and $|L_{d2}| / |L_{NL}| > 1$. In figure 1a, $|L_{d2}| / |L_{NL}| < 1$, which means the nonlinear length is greater than the second-order dispersion length, and the second-order dispersion plays a major role. The pulse has broadened and raised during the transmission process. In figure 1b, $|L_{d2}| / |L_{NL}| > 1$, which means the second-order dispersion length is greater than the nonlinear length, and the nonlinearity plays a major role. The pulse becomes narrower during transmission, and is no longer convex but concave. When the pulse amplitude is adjusted to ensure that the dispersion length is equal to the nonlinear length, that is, $|L_{d2}| / |L_{NL}| = 1$, not only the bright soliton with stable transmission \cite{1}, but also the dark soliton with stable transmission can be obtained.
Table 1. Data with different ratios of group velocity dispersion and nonlinearity.

| $\bar{\omega}$ | $\bar{\omega}_p$ | $\beta_2$ (ps$^2$/km) | $\Gamma_1$ ($W \cdot km$)$^{-1}$ | $A_0$ | $|L_{d2}|/|L_{NL}|$ | Figure |
|-------|--------|-----------------|----------------------------|-------|----------------|------|
| 0.70698 | 0.80000 | 3.84088 | 16.20512 | 0.0156 | 0.0256691 | Figure 1a |
| 0.70698 | 0.80000 | 3.84088 | 16.20512 | 0.0974 | 2.0379386 | Figure 1b |
| 0.70698 | 0.80000 | 3.84088 | 16.20512 | 0.1390 | 1 | Figure 1c |

Figure 1. Pulse propagation when $\bar{\omega} = 0.70698$ with (a) $|L_{d2}|/|L_{NL}| < 1$, (b) $|L_{d2}|/|L_{NL}| > 1$ and (c) $|L_{d2}|/|L_{NL}| = 1$.

Only with positive nonlinearity and positive second-order dispersion in the positive dispersion region can the dark solitons be produced \[2\]. In the anomalous dispersion region, when the positive nonlinearity and the negative second-order dispersion cancel each other out, bright solitons are generated, but dark solitons cannot be generated. The reason is that the positive nonlinearity and negative second-order dispersion are in $|L_{d2}|/|L_{NL}| = 1$ and cannot be offset.

The two graphs in figure 2 explain this point well. Figure 2a only contains negative second-order dispersion in the transmission parameters \[2\]. Figure 2b adds positive nonlinearity on the basis of figure 2a and ensures that the second-order dispersion length and the nonlinear length ratio is 1. The addition of nonlinear should offset the effect of second-order dispersion and make the pulse stable, but as shown in figure 2b, the pulse broadening is more obvious. Therefore, the positive nonlinearity in the negative dispersion region cannot produce dark solitons.

Figure 2. 0 km and 4 km pulse propagations in Negative-Index Region when $\bar{\omega} = 0.70675$. (a) with $\beta_2$ only; (b) with $\beta_2$ and $\Gamma_1$, $|L_{d2}|/|L_{NL}| = 1$.

3.2. Cancellation of Third-order Dispersion

The dark soliton, which can transmit stably after the second-order dispersion and nonlinear cancellation, becomes unstable after adding the third-order dispersion. The pulse in figure 3a has a stable transmission of 100km when there are second-order dispersion, nonlinearity, and $|L_{d2}|/|L_{NL}| = 1$. This is almost the same as the initial pulse. However, after adding the third-order dispersion to the data in figure 3a, the original balance between the second-order dispersion and nonlinearity is broken. There are fluctuations on both sides of the pulse. The overall pulse is elevated and shifts to the right. The left side of the pulse is concave but the right side is convex, as shown in figure 3b.
Table 2. Data for with third-order dispersion and third-order dispersion cancellation.

| $\bar{\omega}$ | $\bar{\omega}_p$ | $\beta_2$ (ps$^2$/km) | $\beta_3$ (ps$^3$/km) | $\Gamma_1$ ($W \cdot km^{-1}$) | $A_0$ | $L_{d2}$ (km) | $L_{d3}$ (km) | Figure |
|----------------|-----------------|---------------------|---------------------|---------------------------|-------|--------------|--------------|--------|
| 0.70688        | 0.8000          | 0.8916              | 2.0714              | 16.2089                   | 0.0469| 28.039       | 60.345       | Figure 3b |
| 0.70720        | 0.8000          | 10.1045             | 2.0879              | 16.1972                   | 0.1580| 5.670        | 60.162       | Figure 3c |

The ratio of the second-order dispersion length to the nonlinear length can be used to determine who plays the main role. On this basis, the third-order dispersion length $L_{d2} = T_0^2/|\beta_3|$ is introduced to determine the magnitude of the third-order dispersion influence. The second-order dispersion length and the third-order dispersion length of figure 3b and figure 3c are shown in table 2. Compared with figure 3b, the figure 3c show that the third-order dispersion length is almost the same. The length of the second-order dispersion in figure 3c is smaller, and the influence of the second-order dispersion is greater. Therefore, the pulse with third-order dispersion in figure 3c is no longer concave and convex obviously on both sides, and the pulse can be transmitted stably for 45km. This is mainly because the second-order dispersion and nonlinearity are not simply cancelled, they all play a role in the transmission of the pulse. Therefore, after adding the third-order dispersion, the broadening of the second-order dispersion cancels the right shift of the third-order dispersion, and the nonlinear concaveness cancels the convexity of the third-order dispersion. However, due to the increase in the influence of the second-order dispersion, when the pulse is transmitted to 45km, the nonlinearity cannot offset the influence of the second-order dispersion. The pulse begins to deform, and severe jitter appears.

Figure 3. 0 km and 4 km pulse propagations in Negative-Index Region when $\bar{\omega}$ = 0.70675. (a) with $\beta_2$ only; (b) with $\beta_2$ and $\Gamma_1$, $|L_{d2}|/|L_{NL}| = 1$. (c) when $\bar{\omega}$ = 0.70720 pulse propagation with third-order dispersion, group velocity dispersion dispersion and nonlinear offset third-order dispersion.

Therefore, in order to obtain the soliton solution for the long-distance stable transmission of pulse, it is necessary to ensure that the second-order dispersion length is equal to the nonlinear length and make the second-order dispersion length much smaller than the third-order dispersion length. In this way, we can maximize the influence of the second-order dispersion effect on pulse transmission, and minimize the effect of the third-order. This allows nonlinear effects and second-order dispersion to eliminate the phenomenon caused by third-order dispersion as much as possible.

3.3. Self-steepening

Dark soliton has better stability against light loss and background noise than bright soliton [3]. In a real Drude model, the self-steepening of the normal dispersion region has almost no effect on the dark soliton. To figure out how the self-steepening affects the transmission, we add it to the scenario where the second-order dispersion and nonlinearity are not canceled. The influence of the third-order dispersion is ignored. As shown in figure 4, the 10km transmission with or without the self-steepening are almost the same. Hence, in a real Drude model, the self-steepening has limited influence on the dark solitons.

Table 3. Third-order dispersion and self-steepening normalized transmission parameters.

| $\bar{\omega}$ | $\bar{\omega}_p$ | $T_0$ | $b_3$ | $S_1$ | $|L_{d2}|/|L_{NL}|$ |
|----------------|-----------------|-------|-------|-------|-------------------|
| 0.7072         | 0.8000          | 5     | 0.0413| -1.63e-04 | 1                 |
Figure 4. Comparison of pulse propagation 10km, with or without self-steepening when group velocity dispersion is not offset with nonlinear. As shown in equation 2, the self-steepening effect is still very insignificant after normalization. Figure 5b is a transmission diagram of 18 dispersion lengths of pulse transmission obtained from the data in table 3. Figure 5a is a transmission diagram of 18 dispersion lengths of pulse transmission obtained after removing the self-steepening. As is shown, the two figures are almost the same, which means, in a real Drude model, the self-steepening has limited influence on the dark solitons.

Figure 5. (a) the normalized propagation of 0 and 18 dispersion lengths when $\omega = 0.7072$ and with third-order dispersion; (b) add self-steepening.

4. Interaction between Fundamental-order Dark Soliton Pairs

4.1. Interaction between Dark Soliton Pairs of the Same Amplitude and Phase

On the basis of obtaining a single dark soliton after normalization, the normalized form of the amplitude of a pair of fundamental-order dark soliton with the same amplitude and phase is:

$$u(\xi = 0, \tau) = \begin{cases} \tanh(\tau + q_0), & -\infty < \tau < 0, \\ -\tanh(\tau - q_0), & 0 \leq \tau < +\infty \end{cases}$$

(3)

where $q_0$ represents the initial distance between two solitons. Figure 6 represents the pulse transmission parameterized by data in Table 4 and the input pulse follows the Equation (3). As shown in Figure 6a and 6b, two dark solitons repel each other when $q_0 = 2$. However, this interaction is obviously suppressed when $q_0 \geq 3$. Figure 6c and 6d are diagrams of pulse transmission with 250 dispersion lengths when $q_0 = 3.5$. As is shown, there is a weak repulsion between the two dark solitons and it is not periodic. What’s more, the pulse after transmitting 250 dispersion almost coincides with the initial pulse.
Figure 6. The fundamental-order dark soliton pair with the same amplitude and phase when \( \bar{\omega} = 0.7070 \). (a) when \( q_0 = 2 \); (c) when \( q_0 = 3.5 \); (b) and (d) is the top view.

Table 4. Parameters of fundamental-order dark soliton pairs in the same amplitude and phase.

| \( \bar{\omega} \) | \( \bar{\omega}_p \) | \( T_0 \) | \( b_3 \) | \( S_1 \) | \( \frac{[L_{d2}]}{[L_{N1}]} \) |
|------------------|------------------|---------|---------|---------|------------------|
| 0.7070           | 0.8000           | 5       | 0.0942  | -3.242E-05 | 1                |

4.2. Interactions between Dark Soliton Pairs of Different Amplitudes and the Same Phases

The normalized form of the amplitude of a pair of fundamental-order dark solitons with different amplitudes and the same phases is:

\[
u(\xi = 0, \tau) = \begin{cases} 
+ \tanh(\tau + q_0), & -\infty < \tau < 0, \\
-\gamma * \tanh(\gamma * (\tau - q_0)), & 0 \leq \tau < +\infty
\end{cases}
\]  (4)

where \( \gamma \) is relative amplitude. Figure 7a and 7b are the pulse transmission figures when \( \gamma = 1.1, q_0 = 3.5 \). The data is from the table 5. The mutual repulsion between soliton is weak when \( q_0 \geq 3 \). There is an energy flow in figure 7a where one amplitude of the input pulse is high while the other is low. After a certain distance, the amplitude of the two solitons gradually flattened, and the energy flows from the high energy soliton to the low energy soliton. Besides energy flow, there is a time shift in the transmission process of dark solitons of different amplitudes and the same phases. As shown in figure 7b, the dark soliton pair shifts to the left.

Figure 7. (a) The fundamental-order dark soliton pair with different amplitudes and the same phases transmitting 0, 10 and 20 dispersion lengths when \( \bar{\omega} = 0.70786 \). (b) The top view.

Table 5. Parameters of fundamental-order dark soliton pairs in the same amplitude and phase.

| \( \bar{\omega} \) | \( \bar{\omega}_p \) | \( T_0 \) | \( b_3 \) | \( S_1 \) | \( \frac{[L_{d2}]}{[L_{N1}]} \) |
|------------------|------------------|---------|---------|---------|------------------|
| 0.70786          | 0.8000           | 5       | 0.0146  | -3.31E-05 | 1                |

4.3. Interactions between Dark Soliton Pairs of the Same Amplitude and Different Phases

The normalized form of the base-order dark soliton pair of the same amplitude and different phase is:

\[
u(\xi = 0, \tau) = \begin{cases} 
\tanh(\tau + q_0), & -\infty < \tau < 0 \\
\tanh(\tau - q_0) \exp(j\varphi), & 0 \leq \tau < +\infty
\end{cases}
\]  (5)
The two pictures in figure 8 are the transmission diagrams of dark soliton pairs of the same amplitude and different phases generated by the parameters of table 5 when the initial spacing $q_0 = 3.5$. The pulse can keep the waveform of transmission distance short. In the process of transmission, a dark soliton pulse is gradually split, which is bad for the signal transmission. The initial phase difference of the dark soliton pair in figure 8a is $\phi = \pi$, and the split dark soliton will eventually maintain symmetry with the initial dark soliton pair. The initial phase difference of the dark soliton pair in figure 8b is $\phi = \pi/2$, and the split dark soliton pulse is not symmetrical with the initial dark soliton pair after a certain distance.

![Figure 8. The fundamental-order dark soliton pair with the same amplitude and different phases transmitting 1 dispersion length with (a) initial phase difference $\phi = \pi$; (b) initial phase difference $\phi = \pi/2$.](image)

5. Conclusions
This paper studies the conditions for the formation of a single dark soliton in the normal dispersion region and the interaction of fundamental-order dark soliton pairs. Positive nonlinearity can be offset by positive second-order dispersion. Although the self-steepening has almost no effect on the propagation of a dark soliton, the third-order dispersion will make the propagation of the dark soliton unstable. By selecting appropriate parameters of Drude model, the influence of the group velocity dispersion on the pulse can be greater than the third-order dispersion as much as possible, so that the group velocity dispersion and nonlinear effects can eliminate the phenomenon caused by the third-order dispersion. Fundamental-order dark soliton pairs exhibit different interactions under different amplitude and phase conditions. (1) Dark soliton pairs of the same amplitude and phase are mutually exclusive, but the repulsion becomes weak when the soliton distance $q_0 \geq 3$. (2) Energy flows in dark soliton pairs with different amplitudes and same phase from high-energy solitons to low-energy solitons. (3) Dark soliton pairs with the same amplitude and different phases gradually split into new solitons during the propagation process.

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