Lattice QCD as a theory of interacting surfaces

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Abstract.

Pure gauge lattice QCD at arbitrary $D$ is considered. Exact integration over link variables in an arbitrary $D$-volume leads naturally to an appearance of a set of surfaces filling the volume and gives an exact expression for functional of their boundaries. The interaction between each two surfaces is proportional to their common area and is realized by a non-local matrix differential operator acting on their boundaries. The surface self-interaction is given by the QCD$_2$ functional of boundary. Partition functions and observables (Wilson loop averages) are written as an averages over all configurations of an integer-valued field living on a surfaces.
1 Introduction.

Pure gauge QCD (quantum gluodynamics) in $D$ dimensions has been reformulated by K.Wilson in terms of collective (lattice) variables [1]:

$$S = \frac{N}{\lambda_o} \sum_f \text{tr} \left( U_f + U_f^\dagger \right)$$

where sum goes over all faces $f$ of $D$-dimensional lattice, $U_f = \prod_{i \in f} U_i$ with $U_i$ being unitary matrix ($U(N)$ or $SU(N)$) attached to $i$-th link of a lattice, $\lambda_o$ is the bar (lattice) coupling constant. In continuum limit, at any $D \leq 4$, $\lambda_o$ goes to zero. At $D = 2 \lambda_o \sim \epsilon^2$, at $D = 3 \lambda_o \sim \epsilon$ and at $D = 4 \lambda_o \sim -\frac{1}{\log \epsilon}$ where $\epsilon$ is the linear size of a lattice.

The partition function is defined as an integral over all link variables,

$$Z = \int \prod_i dU_i \, e^{S},$$

while the simplest observable, the Wilson loop average, is

$$W(C) = \frac{1}{NZ} \int \prod_i dU_i \, e^{S} \text{tr} \left( \prod_{j \in C} U_j \right) ,$$

where $C$ is a closed contour.

In order to integrate exactly over unitary matrices one makes the group Fourier transformation of the face variables, first manifestly used in $D = 2$ by A.A.Migdal [2]:

$$e^{\frac{N}{\lambda_o} \text{tr} \left( U_f + U_f^\dagger \right)} = \sum_r d_r \Lambda_r \left( \frac{N}{\lambda_o} \right) \chi_r (U_f) ,$$

$$d_r \Lambda_r \left( \frac{N}{\lambda_o} \right) = \int dU \chi_r (U) \, e^{\frac{N}{\lambda_o} \text{tr} \left( U + U^\dagger \right)} ,$$

where $r$ is irreducible representation of the gauge group, $\chi_r (U)$ is its character and $d_r = \chi_r (I)$ is its dimension,

$$d_r = \prod_{1 \leq i < j \leq N} \left( 1 + \frac{n_i - n_j}{j - i} \right) .$$

Here and below we use the standard parametrization of $r$ by its highest weight components, $n_1 \geq ... \geq n_N$, associated with a lengths of lines in the Young table.

Direct calculation of (5) gives

$$\Lambda_r \left( \frac{N}{\lambda_o} \right) = \frac{1}{d_r} \det_{ij} I_{n_i - i + j} \left( \frac{2N}{\lambda_o} \right) ,$$

where $I_n (x)$ is the modified Bessel function.
In $D = 2$, an exact solution of the model can be obtained using only orthogonality of characters,

$$
\int dU \chi_{r_1}(AU) \chi_{r_2}(U^\dagger B) = \delta_{r_1,r_2} \frac{\chi_{r_1}(AB)}{d_{r_1}} .
$$

(8)

The result of integration inside a disk of area $A$ (therefore, its lattice area is $A/\epsilon^2$) gives functional of boundary $\Gamma$:

$$
Z_{\text{latt}}(\Gamma) = \sum_r d_r A^{r/\epsilon^2} \chi_r(\Gamma) .
$$

(9)

In the continuum limit, $\epsilon \to 0$ and $\lambda_o = \lambda \epsilon^2$, we need an asymptotic expansion of $\Lambda_r$, which can be found from (8) by the saddle point method to give:

$$
\Lambda_r \sim 1 - \frac{\lambda_o C_2(r)}{2N} + O(\lambda_o^2)
$$

(10)

up to representation-independent factor. Here, $C_2(r)$ is the eigenvalue of the second Casimir operator,

$$
C_2(r) = \sum_{i=1}^N n_i(n_i + N + 1 - 2i) .
$$

(11)

Thus, the continuum limit of (9) is defined by the substitution

$$
\Lambda^{A/\epsilon^2} \to \exp \left( -\frac{\lambda A}{2N} C_2(r) \right)
$$

and takes the form

$$
Z(\Gamma) = \sum_r d_r \exp \left( -\frac{A}{2N} C_2(r) \right) \chi_r(\Gamma) .
$$

(12)

The same can be done in the case of non-trivial topology. The results are:

(i) functional of boundaries $\Gamma_i$, $i = 1, ..., n$ of a sphere with $n$ holes and of area $A$ (continuum coupling constant $\lambda$ is absorbed into the area):

$$
Z(\Gamma_1, ..., \Gamma_n) = \sum_r d_r^{2-n} \exp \left( -\frac{A}{2N} C_2(r) \right) \prod_{i=1}^n \chi_r(\Gamma_i) ,
$$

(13)

(ii) partition function for a closed surface of a genus $g$ and of area $A$:

$$
Z_g(A) = \sum_r d_r^{2(1-g)} \exp \left( -\frac{A}{2N} C_2(r) \right) ,
$$

(14)

(iii) Wilson loop average:

$$
W_g(C) = \sum_{r_1, ..., r_m} \Phi_{r_1, ..., r_m} \prod_{k=1}^m d_{r_k}^{2(1-g_k)} \exp \left( -\frac{A_k}{2N} C_2(r_k) \right) ,
$$

(15)

1By “boundary”, here and below, we equally imply the geometrical boundary and the product of unitary matrices attached to it.

2See also [4].
where $m$ is the number of windows, $A_k$ is the area of a window, $g_k$ is the “genus per window” and coefficient $\Phi_{r_1...r_m}$ is the $U(N)$ ($SU(N)$) group factor dependent on the contour topology (see Ref.[3] for details).

In $D > 2$, the orthogonality condition (8) is not enough to perform an integration over link variables since there are more than two plaquettes match at each link. Formally, we still could integrate using known formulas for tensor product of irreducible representations. This results in a sums over internal spaces of representations. The Clebsch-Gordan coefficients entering in these expansions are not known in any compact and general form. Besides, after integration we should perform a summation of a resulting expressions weighted with these Clebsch-Gordan coefficients, which makes the problem to be extremely difficult. However, the problem becomes less hopeless if one guesses that the three steps, namely, expansion into representations, integration over link variables and then back re-summation, could be performed in one step if some adequate variables are found, which should be a combination of one-link integral and of a sum over representations.

In the next two sections we present such a variables and describe the procedure of integration in an arbitrary (lattice) $D$-volume.

2 One-link integral.

Though our final results will not depend on choice of a lattice, we start for concreteness from the quadrilated regular lattice.

There are $2D - 2$ plaquettes interacting through each link on a quadrilated $D$-dimensional lattice. The one-link integral has the form

$$\int dU \prod_{k=1}^{2D-2} e^{\frac{\lambda_0}{N} \text{tr} W_k U + \text{h.c.}} ,$$

(16)

where by $W_k$ we denote a product of three other unitary matrices in a $k$-th plaquette (see left side of Fig.1 for 3D example). In the heat kernel framework, (16) is equivalent to

$$\sum_{r_1...r_{2D-2}} \prod_{k=1}^{2D-2} d_{r_k} e^{-\frac{\lambda_0}{2N} C_2(r_k)} \int dU \prod_{k=1}^{2D-2} \chi_{r_k}(W_k U) .$$

(17)

Expressions (16) and (17) are identical up to the $O(\lambda_0^2)$ order. The difference appears in order $O(\lambda_0^3)$.

Making shift $U \rightarrow W_j^\dagger U$ (the Haar measure $dU$ is invariant with respect to such transformations) we see that the integral (16), (17) depends on $2D - 1$ boundaries $W_k W_j^\dagger$, $j \neq k$. The shift can be made in $2D - 2$ possible directions corresponding to $2D - 2$ possible choices of $j$. The picture corresponding to one of a four possible directions of gluing in $D = 3$ is presented on Fig.4.
Thus, after one-link integration we obtain $2D - 1$ two-plaquette surfaces with some interaction between them. We are going now to calculate this interaction and to continue the procedure consequently for all links.

Let us start from the heat kernel representation (17) and consider for simplicity the case of only three plaquettes,

$$
\sum_{r_1, r_2, r_3} d_{r_1} d_{r_2} d_{r_3} e^{-\frac{\lambda_0}{2N} \left[ C_2(r_1) + C_2(r_2) + C_2(r_3) \right]} \int dU \chi_{r_1}(W_1 U) \chi_{r_2}(W_2 U) \chi_{r_3}(U) ,
$$

where $W_1$ and $W_2$ are the boundaries of the two-plaquette surfaces.

We are going now to calculate exactly the quantity

$$
\sum_{r} d_r e^{-\frac{\lambda_0}{2N} C_2(r)} \int dU \chi_{r_1}(W_1 U) \chi_{r_2}(W_2 U) \chi_r(U) .
$$

A direct strategy would be to integrate first over $U$ and then to take the sum over $r$ and over its internal sub-space. As we have mentioned in Introduction this way is extremely difficult. Instead of that, we first replace the sum over $r$ by the original Wilson term $\exp \frac{\lambda_0}{N} \text{tr} (U + U^\dagger)$ and then derive the heat kernel exponent as a first order non-zero term in $\lambda_0$-expansion of an integral over $U$. This will give an adequate variables for (19).

To derive $\lambda_0$-expansion of the integral

$$
\frac{1}{f(\lambda_0)} \int dU \ e^{N \text{tr} (U + U^\dagger)} \chi_{r_1}(W_1 U) \chi_{r_2}(W_2 U) , \quad f(\lambda_0) = \int dU \ e^{N \text{tr} (U + U^\dagger)} ,
$$

we diagonalize $U$, $U = \Omega u \Omega^\dagger$ where $u = \text{diag} \{ e^{i\phi_1}, ..., e^{i\phi_N} \}$, and integrate over diagonal $u$ near the saddle point $u = I$. Then, (20) takes the form:

$$
\chi_{r_1}(W_1) \chi_{r_2}(W_2) \left( 1 - \frac{\lambda_0}{2N} \left[ C_2(r_1) + C_2(r_2) - 2C_1(r_1)C_1(r_2) \right] \right) + \lambda_0 \frac{N + 1}{N} \sum_{k=1}^N \int d\Omega \frac{\partial}{\partial \phi_k} \bigg|_{u=I} \chi_{r_1}(W_1 \Omega u \Omega^\dagger) \frac{\partial}{\partial \phi_k} \bigg|_{u=I} \chi_{r_2}(W_2 \Omega u \Omega^\dagger) + O(\lambda_0^2) , 
$$

Figure 1: Gluing in three-dimensions.
where the relation
\[
\sum_{k=1}^{N} \frac{\partial}{\partial \phi_k} \bigg|_{u=I} \chi_r(W \Omega u \Omega^\dagger) = i C_1(r) \chi_r(W),
\]
\[C_1(r) = \sum_{k=1}^{N} n_k \] (22)

has been used \((C_1(r) \) is the first Casimir operator eigenvalue). Indeed,
\[
\sum_{k=1}^{N} \frac{\partial}{\partial \phi_k} \bigg|_{u=I} \chi_r(W \Omega u \Omega^\dagger) = \sum_{\alpha} \tau_{\alpha \alpha}^r (\Omega^\dagger W \Omega) \sum_{k=1}^{N} \frac{\partial}{\partial \phi_k} \bigg|_{u=I} \tau_{\alpha \alpha}^r (u),
\]
(23)

where \(\tau_{\alpha \beta}^r\) is the matrix element of an \(r\)-th irreducible representation. There are only diagonal matrix elements in (23) since \(u\) is diagonal. The matrix label \(\alpha\) is parametrized by the following Gelfand-Zetlin patterns \([5]\):
\[
\alpha = \begin{pmatrix}
n_1^N & n_2^N & n_3^N & \ldots & n_N^N \\
n_1^{N-1} & n_2^{N-1} & \ldots & n_{N-1}^{N-1} \\
\vdots & \vdots & \ddots & \vdots \\
n_1^2 & n_2^2 & & \\
n_1^1 & & & \\
\end{pmatrix},
\]
(24)

where
\[
n_j^i \geq n_j^{i-1} \geq n_{j+1}^i,
\]
(25)

(the top level numbers are coincide with the highest weight components: \(n_k^N = n_k, \ k = 1, \ldots, N\)). Then
\[
\tau_{\alpha \alpha}^r (u) = \prod_{j=1}^{N} e^{i \phi_j \delta c_j},
\]
(26)

where \(c_j\) is a sum of numbers of \(j\)-th level,
\[
c_j = \sum_{k=1}^{j} n_k^j, \quad c_0 = 0
\]
(27)

and \(\delta c_j = c_j - c_{j-1}\). Besides,
\[
\sum_{k=1}^{N} \delta c_k = c_N \equiv C_1(r).
\]
(28)

Then, the relation (22) becomes obvious.

To make further calculation we use the formula (Weyl second formula for characters):
\[
\chi_n(W) = \det_{ij} \sigma_{n_{i-i+j}}(W),
\]
(29)

where numbers \(n_i\)'s are the highest weight components of \(r\) and \(\sigma_n(W)\) is the character of representation \(\{n, 0, \ldots, 0\}\) \([5]\). This character can be written as
\[
\sigma_n(W) = w_{n,1}(\text{tr } W)^n + \sum_{q=2}^{n} w_{n,q}(\text{tr } W)^{n-q} \text{tr } W^q
\]
(30)

\[3\]In formula (29), \(\sigma_n\) with \(n < 0\) might occur. The convention for that case is \(\sigma_n = 0\).
(an exact form of a coefficients $w_{n,i}$ is not needed for the following consideration). Since

$$- i \frac{\partial}{\partial \phi_k} \bigg|_{\Phi = I} \sigma_n (W \Omega u \Omega^\dagger) = (\Omega^\dagger W \Omega)_{kk} \sigma_{n-1} (W) + \sum_{q=2}^{n} q w_q (\Omega^\dagger W^q \Omega)_{kk} (\text{tr } W)^{n-q}$$

(31)

and

$$\sum_{k=1}^{N} d\Omega (\Omega^\dagger W_1 \Omega)_{kk} (\Omega^\dagger W_2 \Omega)_{kk} = \frac{1}{N + 1} \left( \text{tr } W_1 \text{ tr } W_2 + \text{tr } W_1 W_2 \right),$$

(32)

we find:

$$\begin{align*}
(N + 1) \sum_{k=1}^{N} \int d\Omega & \left. \frac{\partial}{\partial \phi_k} \right|_{\Phi = I} \sigma_n (W_1 \Omega u \Omega^\dagger) \left. \frac{\partial}{\partial \phi_k} \right|_{\Phi = I} \sigma_m (W_2 \Omega u \Omega^\dagger) \\
& = \left( \text{tr } \partial_{W_1} \text{ tr } \partial_{W_2} + \text{tr } \partial_{W_1} \partial_{W_2} \right) \sigma_n (W_1) \sigma_m (W_2),
\end{align*}$$

(33)

where matrix elements of $\partial_{W}$ are defined by

$$\begin{align*}
(\partial_{W})_{jk} &= i \sum_{n=1}^{N} W_{jn} \frac{\partial}{\partial W_{kn}}, \\
\frac{\partial}{\partial W_{kn}} W_{jm} &= \delta_{jk} \delta_{mn},
\end{align*}$$

(34)

i.e., matrix elements $W_{jk}$ of fundamental representation has to be considered as an independent variables (in other words, an action of derivative (34) is defined on the $\text{GL}(N)$ group). An important property of derivative $\partial_{W}$ is its invariance under right group transformations $^4$

$$\partial_{W} = \partial_{W V}, \quad V \in \text{GL}(N).$$

(35)

It is not difficult now to generalize (33) to the case of arbitrary representations:

$$\begin{align*}
(N + 1) \sum_{k=1}^{N} \int d\Omega & \left. \frac{\partial}{\partial \phi_k} \right|_{\Phi = I} \chi_{r_1} (W_1 \Omega u \Omega^\dagger) \left. \frac{\partial}{\partial \phi_k} \right|_{\Phi = I} \chi_{r_2} (W_2 \Omega u \Omega^\dagger) \\
& = \left( \text{tr } \partial_{W_1} \text{ tr } \partial_{W_2} + \text{tr } \partial_{W_1} \partial_{W_2} \right) \chi_{r_1} (W_1) \chi_{r_2} (W_2).
\end{align*}$$

(36)

Finally, since

$$\text{tr } \partial_{W} \chi_r (W) = i C_1 (r) \chi_r (W)$$

(compare with (23)), equation (20) takes the form $^5$

$$\int dU \ e^{\sum_{n=1}^{N} \text{tr } (U + U^\dagger)} \chi_{r_1} (W_1 U) \chi_{r_2} (W_2 U)$$

$$= \left( 1 - \frac{2}{N} \left[ C_2 (r_1) + C_2 (r_2) - 2 \text{tr } \partial_{W_1} \partial_{W_2} \right] \right) \chi_{r_1} (W_1) \chi_{r_2} (W_2) + O(\lambda_0^2).$$

(38)

$^4$We could define $\partial_{W}$ as

$$\begin{align*}
(\partial_{W})_{jk} &= i \sum_{n=1}^{N} W_{nj} \frac{\partial}{\partial W_{nk}}.
\end{align*}$$

Then, $\partial_{W}$ is invariant under the left shift $W \rightarrow V W$.

$^5$In what follows we put $f(\lambda_0) = 1$ since this factor is representation-independent and, therefore, precisely cancels in any physical quantity (normalized to partition function).
Therefore,
\[
\sum_r d_r e^{-\frac{\lambda_0}{2N} C_2(r)} \int dU \chi_{r_1}(W_1 U) \chi_{r_2}(W_2 U) \chi_r(U)
\]
\[= e^{-\frac{\lambda_0}{2N} (C_2(r_1) + C_2(r_2) - 2 \text{tr} \partial_{W_1} \partial_{W_2})} \chi_{r_1}(W_1) \chi_{r_2}(W_2) .
\]

(39)

As explained above, the result (39) is exact, i.e., valid for arbitrary \(\lambda_0\). It can be also viewed as follows. Equation (19) can be written as
\[
\sum_r d_r e^{\frac{\lambda_0}{2N} \Delta_{W-I}} \int dU \chi_{r_1}(W_1 U) \chi_{r_2}(W_2 U) \chi_r(W U)
\]
\[= \sum_{k=1}^{\infty} \frac{\lambda_0^k}{k! (2N)^k} \Delta_{W-I}^k \int dU \chi_{r_1}(W_1 U) \chi_{r_2}(W_2 U) \sum_r d_r \chi_r(W U) ,
\]
where \(\Delta_W\) is an invariant Laplace-Beltrami operator which is the differential operator with respect to parameters of \(W\). A character of an arbitrary irreducible representation is the eigenfunction of \(\Delta\),
\[
\Delta_W \chi_r(W) = -C_2(r) \chi_r(W) .
\]

(41)

Using the completeness condition for a characters,
\[
\sum_r d_r \chi_r(W) = \delta(W, I) ,
\]
we write (40) as
\[
\sum_{k=1}^{\infty} \frac{\lambda_0^k}{k! (2N)^k} \Delta_{W-I}^k \chi_{r_1}(W_1 W) \chi_{r_2}(W_2 W) .
\]

(43)

Actually, we have checked above that
\[
\Delta_W \chi_{r_1}(W_1 W) \chi_{r_2}(W_2 W) = -(C_2(r_1) + C_2(r_2) - 2 \text{tr} \partial_{W_1} \partial_{W_2}) \chi_{r_1}(W_1 W) \chi_{r_2}(W_2 W) .
\]

(44)

Therefore,
\[
\Delta_{W}^k \chi_{r_1}(W_1 W) \chi_{r_2}(W_2 W) = (-)^k (C_2(r_1) + C_2(r_2) - 2 \text{tr} \partial_{W_1} \partial_{W_2})^k \chi_{r_1}(W_1 W) \chi_{r_2}(W_2 W) ,
\]

(45)

which confirms (39).

A generalization to the case of arbitrary number \(P\) of plaquettes joining at one link is straightforward:
\[
\sum_r d_r e^{-\frac{\lambda_0}{2N} C_2(r)} \int dU \chi_{r}(U) \prod_{p=1}^{P} \chi_{r_p}(W_p U)
\]
\[= \exp -\frac{\lambda_0}{2N} \left( \sum_{p=1}^{P} C_2(r_p) - 2 \sum_{<pq>} \text{tr} \partial_{W_p} \partial_{W_q} \right) \prod_{p=1}^{P} \chi_{r_p}(W_p) .
\]

(46)
Operator $\exp \frac{1}{N} \text{tr} \partial_{W_i} \partial_{W_j}$ acts on a character in a simple way. In Appendix we give several examples corresponding to several low-dimensional representations. It is worth mentioning, that although the formula (29) is written for non-negative signatures ($n_N \geq 0$) only, the results (39), (46) are valid for an arbitrary representation.

### 3 Functional of boundaries and set $\Sigma$

Using property (35) we proceed further link by link and integrate out all link variables inside an arbitrary $D$-volume. In this way we obtain some set $\Sigma$ of (interacting) surfaces $S_i$. Suppose, all of them are disks (it is clear, at any $D$ it can be done, at least in a small enough volume). Let $\Gamma_i$ is the boundary of $i$-th disk and $A_i = A(S_i)$ is its (lattice) area. To each $S_i$ corresponds the sum over irreducible representations $r_i$. Corresponding functional of boundaries $Z_\Sigma(\{\Gamma\})$ has the form of a sum over all configurations $\{r\}$:

\[
Z_\Sigma(\{\Gamma\}) = \sum_{\{r\}} e^{-S_\Sigma(\{r\},\{\partial_{\Gamma}\})} \prod_{i \in \Sigma} d_{r_i} \chi_{r_i}(\Gamma_i),
\]

\[
S_\Sigma(\{r\}, \{r\}) = \lambda_o \frac{2}{N} \sum_{i \in \Sigma} A_i C_2(r_i) - \lambda_o \frac{1}{N} \sum_{<ij>} A_{ij} \text{tr} \partial_{\Gamma_i} \partial_{\Gamma_j},
\]

where $A_{ij} = A(S_i \cap S_j)$. Formula (47) generalizes the expression (12) for the functional of boundary in $D = 2$. The latter corresponds to (47) with $\Sigma$ containing only one surface.

There is an infinite number of equivalent sets $\Sigma$. Actually, each set is defined by the local gauge fixing, i.e., all such a sets are gauge equivalent. It is clear, observables are independent on the choice which should be dictated just by convenience of calculations. In $D = 3$, for example, it is possible to chose $\Sigma$ containing surfaces of only disk topology. The example is drawn (in projection) in Fig.2(a). In this case, all disks are compressed and has the form of a closed from one side cylinders of one plaquette width, which fill densely a 3D volume. Choosing another direction of gluing at (at least) one link (see Fig.1) we obtain another set $\Sigma'$ which differ from $\Sigma$ not only by smooth deformation of a surfaces but also by an appearance of a compact surfaces (see Fig.2(b)).

To see what happens when the compact surfaces appear in the set we, first, consider the situation when the surfaces with a boundaries are not only disks (1-holed spheres) but also spheres with an arbitrary number of holes. It is not difficult to see that in such a case we have to replace an expression $d_{r_i} \chi_{r_i}(\Gamma_i)$ which appears in (17) and corresponds to $i$-th disk by $d_{r_i}^{2-n_i} \prod_{k=1}^{n_i} \chi_{r_i}(\Gamma_{i,k})$ where $\Gamma_{i,k}$ is the boundary of $k$-th hole ($k = 1, ..., n_i$) on $i$-th surface.

Thus, the most general expression for functional of boundaries is:

\[
Z_\Sigma(\{\Gamma\}) = \sum_{\{r\}} e^{-S_\Sigma(\{r\},\{\partial_{\Gamma}\})} \prod_{i \in \Sigma} d_{r_i}^{2-n_i} \prod_{k=1}^{n_i} \chi_{r_i}(\Gamma_{i,k}).
\]

8
Figure 2: Two examples of gauge equivalent sets of surfaces (projection from $D=3$). This equally can be considered as a set of curves for the Principal Chiral Field model.

A compactification of a surfaces will become clear in the next section, where we will consider a partition functions.

4 Evolution operator, partition functions and loop averages.

The expression (49) can be equally written as

$$Z_{\Sigma}(\{\Gamma\}) = \hat{R} \prod_{i \in \Sigma} Z(\Gamma_i),$$

$$\hat{R} = \exp \frac{\lambda_o}{N} \sum_{<ij>} A_{ij} \text{tr} \partial_{\Gamma_i} \partial_{\Gamma_j},$$

where the 2D boundary functional $Z(\Gamma_i)$ is given by (12) in the case of a disk or by (13) in a general case (then, $\Gamma_i$ means the set of boundaries $\{\Gamma_{i,1}, ..., \Gamma_{i,n_i}\}$ corresponding to $i$-th surface). Thus, the differential operator $\hat{R}$ describes an evolution of $\text{QCD}_2$ in $\text{QCD}_D$.

The expression for functional of boundaries (49) can serve as the building block in construction of partition functions and observables on arbitrary $D$-manifolds including those of non-trivial topology.

It is clear, the partition function can be obtained from (49) by putting all boundaries $\Gamma_i$ be equal to $I$, i.e.,

$$Z_{\Sigma} = \sum_{\{r\}} e^{-S_\Sigma(\{\Gamma\}, \{\partial_{\Gamma} \})} \prod_{i \in \Sigma} d_{r_i}^{2-n_i} \prod_{k=1}^{n_i} \chi_{r_i}(\Gamma_{i,k}),$$

where $S_\Sigma(\{\Gamma\}, \{\partial_{\Gamma} \})$ is defined in (48). This is a general expression for the partition function of $\text{U}(N)$ (and $\text{SU}(N)$) lattice quantum gauge theory in $D$-dimensions.

To write the Wilson loop average (3) in the same terms let us consider an arbitrary
surface $S_C$ such that $C = \partial S_C$. Then, the result is

$$W(C) = \frac{1}{NZ} \sum_{\{r\}} e^{-S_{\Sigma}(\{\{\partial_{r_t}=1\}\})} \prod_j \frac{\lambda_o}{N} A(S_C \cap S_j) \text{tr} \partial C = 1 \text{ tr} \prod_{i \in \Sigma} d_{r_i}^{2-n_i} \prod_{k=1}^{n_i} \chi_{r_i}(\Gamma_{i,k}) .$$

(53)

Differentiation with respect to $C$ is easy to perform and we obtain an equivalent expression:

$$W(C) = \frac{1}{NZ} \sum_{\{r\}} e^{-S_{\Sigma}(\{\{\partial_{r_t}=1\}\})} \text{tr} \left( \prod_j \frac{\lambda_o}{N} A(S_C \cap S_j) \partial C = 1 \right) \prod_{i \in \Sigma} d_{r_i}^{2-n_i} \prod_{k=1}^{n_i} \chi_{r_i}(\Gamma_{i,k}) .$$

(54)

Thus, the loop average takes the form of an average of differential operator

$$\frac{1}{N} \text{tr} \left( \prod_j \frac{\lambda_o}{N} A(S_C \cap S_j) \partial C \right) .$$

(55)

It is not difficult to check that the result is independent on a choice of $S_C$.

## 5 Discussion.

We represented pure gauge lattice QCD$_D$ as a statistical model of integer-valued scalar fields living on a set of interacting surfaces. Apparently, modulo some possible (and hopefully fruitful) re-writing, the expressions derived here for the functional of boundaries (49), for the partition function (52) and for loop averages (53) cannot be simplified further, unless continuum limit is taken. This can be seen already in the U(1) case (QED), where partition function takes the simple form:

$$Z_{\Sigma} = \sum_{\{n\}} e^{-S_{\Sigma}(\{n\})} ,$$

$$S_{\Sigma}(\{n\}) = \frac{\lambda_o}{2} \sum_{i \in \Sigma} A_i n_i^2 - \lambda_o \sum_{<ij>} A_{ij} n_i n_j .$$

(56)

By “simplification” we mean an essential reduction of configuration space $\{r\}$. Such a simplification is expected in continuum limit. We hope that the derived integrated version of the model is a better starting point for analytical study the continuum limit than the original, non-integrated version. The problem might perhaps be solvable by methods of elementary combinatorics, where the only subtlety is the calculation of the surface entropy factor which manifests itself in a change of the set $\Sigma$ under refinement of the lattice.

The method of integration over unitary matrices and the formula (39) (or, in general, (46)) can be applied without any changes to the Principal Chiral Field (PCF)

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6I wish to thank M.Karliner for discussion of this point.
model (for recent developments in the model and for references see [3]). The set $\Sigma$ in this case becomes a set of curves (see Fig.2) and the partition function is again of the form (52), after we replace areas of a surfaces by lengths of curves. Apparently, due to its respective simplicity, the PCF model might become the first case where the continuum limit can be taken.

The problem of integration over unitary matrices in QCD is similar to one in matrix models of 2$d$ quantum gravity embedded in $D > 1$ (the simplest case where the problem appears is the closed $D = 1$ target space, i.e., closed chain of hermitian matrices). The present method can be applied, after some modification, to these models as well.

Among possible direct continuations of our analysis let us mention the following:

(i) Expression (49) looks suitable for $1/N$-expansion. The latter has been recently elaborated in a detail for $D = 2$ [7]-[9]. It is tempting to apply this technique to $D > 2$. A straightforward strategy would be to find out a “stringy” interpretation for the evolution operator (51). Then the whole theory could be considered as a set of stringy models (QCD$_2$), interacting through the operator $\hat{R}$. However, a less naive strategy is possible, if one makes the $1/N$-expansion of whole expression (49), together with $\hat{R}$-operator. Then, each term of $1/N$-expansion will take a form of a sum over all surfaces from the set $\Sigma$. This requires some new technique, especially with respect to $\hat{R}$-operator.

(ii) The derived exact expression for functional of boundaries (49), especially written in the form (50), can serve as a starting point for a mean field analysis of the model [8]. An equation for eigenvalues of $\hat{R}$-operator, under mean field assumption, seems solvable, at least on a finite lattices.

(iii) It is not difficult to recognize that the expression (52) for QCD partition function can be interpreted at infinite $N$ as the constrained matrix model. The corresponding technique was worked out for $D = 2$ in [12]-[14]. It would be interesting to solve the large N saddle point equation and to see if there is a region of the coupling constant where unitary constraint can be ignored, which would indicate an existence of large N (apparently, third order) phase transition [15].

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For $1/N$ applied to generalized $D = 2$ Yang-Mills see [10].
For a mean field analysis of the model on a tree see [11].
Appendix.

In this appendix, we will demonstrate on a few examples an action of operator \( \exp \gamma \text{tr} \partial_A \partial_B \) (\( \gamma \) is a parameter) on the function \( \chi_{r_1}(A)\chi_{r_2}(B) \). Thus, we calculate

\[
e^{\gamma \text{tr} \partial_A \partial_B} \chi_{r_1}(A)\chi_{r_2}(B) .
\]

(a) If \( r_1 \) or \( r_2 \) is trivial representation \( \{1, 0, \ldots, 0\} \), then (57) is equal to 1.

(b) The first non-trivial example is \( r_1 = r_2 = \{1, 0, \ldots, 0\} \) (fundamental representation). In this case we have

\[
e^{\gamma \text{tr} \partial_A \partial_B} \text{tr} A \text{tr} B = \sum_{q=0}^{\infty} \frac{\gamma^q}{q!} \text{tr} A \text{tr} B \cosh \gamma - \text{tr} AB \sinh \gamma \quad (58)
\]
since the even-order derivatives result in \( \text{tr} A \text{tr} B \) and odd-order ones result in \( \text{tr} AB \).

(c) \( r_1 = \{1, 0, \ldots, 0\} \), \( r_2 = \{1, 1, 0, \ldots, 0\} \). Then

\[
e^{\gamma \text{tr} \partial_A \partial_B} \text{tr} A \frac{1}{2} (\text{tr}^2 B - \text{tr} B^2) \\
= (\text{tr} B \text{tr} AB - \text{tr} AB^2) \frac{e^{2\gamma} - e^{-\gamma}}{3} + \text{tr} A(\text{tr}^2 B - \text{tr} B^2) \frac{e^{2\gamma} + 2 e^{-\gamma}}{6} .
\]

(d) \( r_1 = \{1, 0, \ldots, 0\} \), \( r_2 = \{2, 0, \ldots, 0\} \). Then

\[
e^{\gamma \text{tr} \partial_A \partial_B} \text{tr} A \frac{1}{2} (\text{tr}^2 B + \text{tr} B^2) \\
= (\text{tr} B \text{tr} AB + \text{tr} AB^2) \frac{e^{-2\gamma} - e^{\gamma}}{3} + \text{tr} A(\text{tr}^2 B + \text{tr} B^2) \frac{e^{-2\gamma} + 2 e^{\gamma}}{6} .
\]

We leave it to reader to substitute these results in the formula (39) to see how it works in these cases.