Study on Application of New Kinetic Energy Theorem of Rigid Body Fixed-Axis Rotation

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Abstract: The classical kinetic energy theorem of rigid body fixed-axis rotation cannot correctly reflect non-conservative force doing dissipative work, non-conservative reacting force doing equivalent dissipative work at the same time, instead convert the system's mechanical energy into the non-mechanical energy of the bearer. Now these defects are overcome by the new kinetic energy theorem of rigid body fixed-axis rotation. The meaning of new kinetic energy theorem of rigid body fixed-axis rotation is interpreted, the procedures to solve practical problems on new kinetic energy theorem of rigid body fixed-axis rotation and computing methods of the terms in the new kinetic energy theorem of rigid body fixed-axis rotation are studied, as well as the physical significance of the results is described.

1. Introduction

On the basis of teaching and practices on general physical theories, the author found classical kinetic energy theorem failed to describe the defect that both non-conservative frictional force and non-conservative friction reacting force do dissipative work, put forth an assumption (2002), demonstrated it with mathematical physic methods (2003 ~ 2004), created three new theorems and a new theory (Jan. 2005) through derivation in the particle-function domain, demonstrated with experiments [1], and finally resolved difficulties encountered by classical kinetic energy theorem and classical work-and-power principle in Newtonian mechanics over the course of its practices and applications from its birth 300 years ago.

New theorems and new principle included in the new particle work-and-power theory are [2]: new kinetic energy theorem on particles; new kinetic energy theorem on particle system; new work-and-power theorem on particle system; new kinetic energy theory on rigid-body fixed-axis rotation. These three new theorems and a new principle replaced corresponding classical kinetic energy theorem and the classical work-and-power principle; all of them apply to describe and predict not only kinetic characteristics of particles (particle system) only under action of conservative mechanic force but also kinetic characteristics of particles (particle system) only under action of non-conservative mechanic force but also kinetic characteristics of particles (particle system) only under action of non-conservative mechanic force or rotary characteristics of rigid-body fixed-axis rotation only under action of non-conservative mechanic torque, as well as kinetic characteristics of particle (particle system) under joint action of conservative mechanic force (torque) and non-conservative mechanic force (torque) or rigid-body fixed-axis rotation [3]; all of them can reflect the law of mechanic motion on material[4]: Both conservative and non-conservative forces can do non-dissipative work to transfer mechanical energy between interacting bodies; non-conservative force can also do dissipative work to convert mechanical energy into non-mechanical energy of a bearer; non-conservative reacting force does dissipative work while non-conservative force does dissipative work. All these objective laws are
reflected in the new work-energy theories [5]. Classical work-energy theories do not correctly describe the objective laws of non-conservative reacting force doing equivalent dissipative work while non-conservative force doing dissipative work. The fact promotes the new development of work-energy equation [6]. New kinetic energy theorem of rigid body fixed-axis rotation [7] is part of the new work-energy theories, objectively reflecting the transferring and transforming processes of mechanical work and mechanical energy and their corresponding relationship. Only by learning and understanding, the new theorem can be correctly used to effectively explain objective phenomena, guide practices, and predict the future.

2. Study on Application of New Kinetic Energy Theorem of Rigid Body Fixed-axis Rotation

2.1 New kinetic energy theorem of rigid body fixed-axis rotation

New kinetic energy theorem of rigid body fixed-axis rotation can be expressed by: that the sum of $A_{cm}$ (infinitesimal), the work done by all conservative external torques, $A_{cmi}$ (infinitesimal), the implicated work done by all non-conservative torques, $A_{cmd}$ (infinitesimal), the dissipative work done by all non-conservative reacting torques during the rigid body fixed-axis rotation, equals the kinetic energy increment of rigid body fixed-axis rotation (infinitesimal).

The formulas are as follows:

Differential formula:

$$\int dA_{cm} + \int dA_{cmi} + \int dA_{cmd} = \int \left( \frac{1}{2} J \omega^2 \right)$$  \hspace{1cm} (1)

Integral formula:

$$A_{cm} + A_{cmi} + A_{cmd} = \frac{1}{2} J \omega^2 - \frac{1}{2} J \omega_0^2$$  \hspace{1cm} (2)

When using the new kinetic energy theorem of rigid body fixed-axis rotation to solve practical problems, it is necessary to know the stress conditions of the rigid bodies under discussion and the rotations of the applying rigid bodies, and correctly use the new definition to mechanical work in the new kinetic energy theorem of rigid body fixed-axis rotation, so as to calculate terms in the formulas of the new kinetic energy theorem of rigid body fixed-axis rotation. The updating significances of the new kinetic energy theorem of rigid body fixed-axis rotation are as follows.

2.2 Study on application of new kinetic energy theorem of rigid body fixed-axis rotation

The new kinetic energy theorem of rigid fixed-axis rotation can describe the transfer and transformation of the mechanical energy of the rotating rigid bodies, as well as the mechanical energy transformation of the applying rigid bodies.

An example. A phonograph turntable rotates around an axis through the center of the vertical disk at uniform angular velocity $\omega_0 = \omega_0^k$ [8]. A phonorecord on the turntable will rotate with the turntable under friction torque. Given that phonorecord radius is $R$, mass is $m$, friction coefficient between the record and the turntable is $\mu$. Compute: (a) torque on the record, (b) time for the record to reach angular velocity $\omega_0 = \omega_0^k$, (c) work done by friction torque during this period, and the final state function of the record.

Solution. Shown in ‘Figure 1’. A cylindrical coordinate system is established on the phonorecord as the research object.

The research objects and related applying bodies are numbered: the phonorecord is No.0, turntable No.1, rotating axis No. 2, the earth No.3.

Force analysis on the record: The forces on the record during its angular velocity increasing from zero to $\omega_0 = \omega_0^k$ include: the turntable’s sliding friction force $f_u$, support force $N_u$, the axis’ radial binding force $f_v$, and the earth's gravity $g = mgk$. 

Figure 1. Rotation of a phonorecord

The reacting forces of the above forces include the record’s sliding friction force on the turntable $f_n$, positive pressure $k$, the record’s radial counter binding force on the rotating axis $T$, the record’s gravity on the earth $g$.

The meaning of the right subscripts of the rigid body forces (torques), $F_{ij}$: $i$ is serial number of a bearer, $j$ is serial number of an applying body ($i, j=0, 1, 2, 3$; and $i \neq j$).

2.2.1 Torques on the phonorecord: An infinitesimal area $dS = drdl$ with a length of $dl$ and a width of $dr$ is taken on the record.

First, non-conservative torque on the record is calculated. The infinitesimal sliding friction force of the turntable on the infinitesimal area is $\tau = \mu \frac{mg}{\pi R^2} dr dl$. The infinitesimal sliding friction torque of the infinitesimal sliding friction force to fixed axis Oz is

$$dM'_{(r_0, 0, 1, 1)} = (r \times dF_n) \cdot k = [(-r_0) \times (\mu \frac{mg}{\pi R^2} dr dl \tau)] \cdot k = \mu \frac{mg}{\pi R^2} r dr dl$$  (3)

The infinitesimal support force of the turntable on the infinitesimal area is $dN_n = \frac{mg}{\pi R^2} dr dl (-k)$, and the infinitesimal non-conservative torque of the infinitesimal support force to fixed axis Oz is

$$dM'_{(r_0, 0, 2, 2)} = (r \times dN_n) \cdot k = 0$$  (4)

The infinitesimal radial binding force of the turntable on the infinitesimal area is $dT_n = dT_{n, r} = \omega r \frac{m}{\pi R^2} dr dl \cdot n$, and the infinitesimal non-conservative torque of the infinitesimal radial binding force to fixed axis Oz is

$$dM'_{(r_0, 0, 2, 2)} = (r \times dT_{n, r}) \cdot k = 0$$  (5)

On the infinitesimal area $dS$ of the annular record with radius $r$, width $dr$ and length $2\pi r$, every infinitesimal non-conservative torque of the turntable to fixed axis Oz successively according to (3), (4), (5), is

$$dM_{0, 1, 1} = \int dM'_{(r_0, 0, 1, 1)} = \int \left[ (\mu \frac{mg}{\pi R^2} r dr dl \tau) \right] dr$$  (6)

$$dM_{0, 2, 2} = \int dM'_{(r_0, 0, 2, 2)} = 0$$  (7)

$$dM_{0, 0, 3} = \int dM'_{(r_0, 0, 0, 3)} = 0$$  (8)
Every non-conservative torque of the turntable to fixed axis Oz on the turntable successively according to (6), (7), (8) is

\[ M_{\text{non-cons}} = \int dM_{\text{non-cons}}(r) = \int \frac{2\mu mg}{R^2} r^2 dr = \frac{2}{3} R \mu mg \]  
\[ (9) \]

Second, the conservative torque on the record is calculated.

\[ \frac{dm}{dR} = \frac{m}{\pi R^2} drdl \]  

with length \( dl \) and width \( dr \) on the record, the infinitesimal conservative torque of the infinitesimal gravity \( d\theta_c = \frac{mg}{\pi R} drdlk \) to fixed axis Oz is

\[ dM_{\text{cons}} = (r \times d\theta_c) \cdot k = 0 \]  
\[ (12) \]

From the above formula, the gravity torque of the earth to fixed axis Oz on the record is

\[ M_{\text{cons}} = \int dM_{\text{cons}} = \int \left( \frac{mg}{\pi R} drdl \right) = 0 \]  
\[ (13) \]

Resultant external torque to fixed axis Oz on the record according to (9), (10), (11), (13) is

\[ M = \sum M_{\text{non-cons}} + M_{\text{cons}} = M_{\text{non-cons12}} + M_{\text{non-cons12}} + M_{\text{non-cons12}} + M_{\text{non-cons12}} = \frac{2}{3} R \mu mg \]  
\[ (14) \]

2.2.2 Time for the record to reach angular velocity \( \omega_i = \omega_0 k \) : Substitute (14) of the above resultant external torque to fixed axis Oz on the record into the rotation theorem yields

\[ \begin{align*}
\frac{2}{3} R \mu mg &= J \beta \\
J &= \frac{mR^2}{2} \\
\beta &= \frac{4\mu g}{3R}
\end{align*} \]  
\[ (15) \]

That is, the record rotates at uniform variable angular velocity. From

\[ \begin{align*}
\omega &= \omega_0 + \beta \Delta t \\
\omega_0 &= 0, \omega = \omega_0
\end{align*} \]  
\[ (17) \]

The time for the record to reach angular velocity \( \omega_i = \omega_0 k \) is

\[ \Delta t = \frac{\omega_i}{\beta} = \frac{3\omega_i R}{4\mu g} \]  
\[ (18) \]

In the time of \( \Delta t \), the absolute angular displacement of the record rotation according to

\[ \omega_i - \omega_0 = \frac{2\beta \Delta \theta}{8\mu g} \]  
\[ (19) \]
2.2.3 In the time of $\Delta t$, work done by non-conservative torque is calculated by means of the new definition of torque work: First, absolute angular velocity, relative angular velocity and convected angular velocity of the record are derived.

There is only friction torque during the record rotating at an angular acceleration, so, only relative motion and convected motion of the record to the turntable are required.

At a moment $t \in [0, \frac{3\omega R}{4\mu} \Delta t]$ when the record rotates at uniform angular acceleration, according to (16) and (17), absolute angular velocity of the record is

$$\omega_t(t) = \frac{4\mu g}{3R} t$$  \hspace{1cm} (20)

Relative angular velocity of the record to the turntable, according to $\omega_{01} = \omega_0 - \omega_1$, is

$$\omega_k = \frac{4\mu g}{3R} t - \omega_{k0} \leq 0$$ \hspace{1cm} (21)

Convected angular velocity of the record (i.e., absolute angular velocity of the applying turntable) is

$$\omega_t = \omega_t k$$ \hspace{1cm} (22)

Second, non-dissipative work and dissipative work of non-conservative torque, and dissipative work of non-conservative reactive torque are calculated.

Infinitesimal implicated work (i.e., non-dissipative work) done by the sliding friction torque of the turntable on infinitesimal annular area with radius $r$, width $dr$ and length $2\pi r$ taken from the record during time of $dt$ is calculated by infinitesimal non-conservative torque dot product on the rigid body under stress, infinitesimal convected angular displacement (i.e., infinitesimal absolute angular displacement of the corresponding applying body) $d\theta = d\theta tk$; that is, substituting (6) into the following formula yields:

$$dA_{\text{implicated}} = dM_{\text{implicated}} \cdot d\theta = \left(\frac{2\mu mg}{R^2} r dr\right) \left(\omega_t k dr\right)$$ \hspace{1cm} (23)

Total implicated work done by sliding friction torque on the record during $\Delta t = \frac{3\omega R}{4\mu} \Delta t$ of the record rotation at angular acceleration is

$$A_{\text{implicated}} = \left(\frac{2\mu mg}{R^2} r dr\right) \int_0^\Delta t \omega_t k dr = \left(\frac{2}{3} R j R \mu mg \omega_t k \Delta t\right) = \frac{1}{2} m \omega_t^2 R^2$$ \hspace{1cm} (24)

This is the value of the non-dissipative work done by sliding friction torque on the record, transferring the mechanical energy of the turntable to the mechanical energy of the record, i.e., the value of the mechanical energy obtained by the record during the rotation at angular acceleration; non-dissipative works of other non-conservative torques on the record are zero. In addition,

$$\begin{cases}
A_{\text{implicated}} = \sum_j A_{\text{implicated}} = \frac{1}{2} m \omega_t^2 R^2 \\
A_{\text{implicated2}} = A_{\text{implicated}} = 0
\end{cases}$$ \hspace{1cm} (25)

The infinitesimal dissipative work done by the infinitesimal sliding friction torque of the turntable on the infinitesimal annular area during the time of $dr$ is calculated by half of the infinitesimal non-conservative torque on the rigid body bearer dot product infinitesimal relative angular displacement $d\theta_d = d\theta_d k$ of the force point (infinitesimal) to the applying point (infinitesimal), namely

$$dA_{\text{dissipative}} = \frac{1}{2} dM_{\text{dissipative}} \cdot d\theta_d = \frac{1}{2} \left(\frac{2\mu mg}{R^2} r dr\right) \left(\frac{4\mu g}{3R} t - \omega_d\right) dt$$  \hspace{1cm} (26)
From (26), we see, infinitesimal dissipative work done by the infinitesimal sliding counter friction torque on the infinitesimal annular area of the turntable is also

\[ dA_{\text{slide},0} = \frac{1}{2} dM_{\text{slide}} \cdot d\theta = \frac{1}{2} (-dM_{\text{slide}}) \cdot (-d\theta) = dA_{\text{slide},0} \]  

(27)

This formula shows the infinitesimal sliding friction torque and the infinitesimal sliding counter friction torque do equivalent infinitesimal dissipative work synchronously.

During \( \Delta t = \frac{3\omega R}{4\mu} \) of the time of the record rotation at angular acceleration, the total dissipative work done by the sliding friction torque on the record and the total dissipative work done by the sliding counter friction torque on the turntable are respectively

\[ A_{\text{slide}} = \sum_{j=1}^{n} A_{\text{slide}j} = \frac{1}{2} \int \frac{2\mu R^2}{\Delta t} (\frac{4\mu R}{3R} t - \omega t) dt = -\frac{1}{8} m\omega^2 R \]  

(28)

This is the transfer from mechanical energy obtained by the record to non-mechanical energy of the record and the turntable (internal energy and deformation energy, etc.), and dissipative works of other non-conservative forces are zero. That is

\[ \begin{align*}
A_{\text{slide}}^1 = & A_{\text{slide}} = \sum_{j=1}^{n} A_{\text{slide}j} = -\frac{1}{8} m\omega^2 R \\
A_{\text{slide},0} = & A_{\text{slide},0} = 0
\end{align*} \]  

(29)

Thirdly, the non-dissipative work of conservative torque is calculated.

The non-dissipative work done by the conservative torque on the rigid body bearer is calculated by the classical method of calculating torque work, i.e., the integral of conservative resultant torque dot product and the absolute angular displacement of the rigid body bearer; substituting (12) into the computation formula yields

\[ A_{\text{cons}} = \int dM_{\text{cons}} \cdot d\theta = 0 \]  

(30)

And

\[ A_{\text{cons}} = A_{\text{cons}} = 0 \]  

(31)

2.2.4 Solving unknown quantities by the equations in the new kinetic energy theorem of rigid body fixed-axis rotation: Substituting (25), (29), (31) and \( \omega_0 = 0 \) into (2) of the new kinetic energy theorem of rigid body fixed-axis rotation yields the rotational kinetic energy at the end of the record rotation at angular acceleration which is

\[ 0 + \frac{1}{2} m\omega^2 R^2 - \frac{1}{8} m\omega^2 R^2 - \frac{1}{8} m\omega^2 R^2 = E_1 - 0 \rightarrow \]  

\[ E_1 = \frac{1}{2} I\omega^2 = \frac{1}{4} m\omega^2 R^2 \]  

(32)

(33)

Formulas (32) and (33) show that in this process, the total mechanical energy transferred to the record by the implicated work (non-dissipative work) done by the sliding friction torque on the record is \( \frac{1}{2} m\omega^2 R^2 \), where \( \frac{1}{4} m\omega^2 R^2 \) is transferred to the kinetic energy increment of the record, the others are transferred to non-mechanical energies of the record and the turntable. Their values are respectively \( -\frac{1}{8} m\omega^2 R^2 \). This accords with the process of actual motion energy transfer and conversion of the record.
2.2.5 Discussion:

2.2.5.1 In solving problems by means of the new kinetic energy theorem of rigid body fixed axis rotation, the total non-dissipative work of conservative torque is calculated by the classical calculation method of torque work.

2.2.5.2 The non-dissipative work and dissipative work of non-conservative torque are calculated by the new definition of the corresponding works; the non-dissipative work and dissipative work of each torque are respectively calculated first, then total non-dissipative work and dissipative work are calculated.

2.2.5.3 The work-energy equation about the process of the record rotation at angular acceleration by means of the classical kinetic energy theorem of rigid body fixed axis rotation is

\[ \frac{1}{4} \omega^2 R^2 = E = 0 \]  

(34)

From the above formula, we can see, by means of the classical definition of torque work, only \( \frac{1}{4} \omega R^2 \) of the rotational kinetic energy increment transferred to the record by the sliding friction torque work is calculated, and total mechanical energies transferred to the record and the non-mechanical energies transferred to the record and the turntable cannot be reflected. So the equations in the new kinetic energy theorem of rigid body fixed axis rotation are a more profound and more comprehensive revelation of the nature. The classical kinetic energy theorem equations of rigid body fixed axis rotation do not apply to the conditions where there are non-conservative torques, The interpretation to the physical meaning of the classical kinetic energy theorem equations of rigid body fixed axis rotation only applies to the conditions where there are only conservative torques.

3. Conclusions

Classic work-energy equations cannot correctly reflect the non-conservative force work characteristics or correctly describe the direction and value of mechanical energy transfer; The new kinetic energy theorem of rigid body fixed axis rotation is a more effective theory to explain the natural phenomena and predict the future. Only in the practical applications and theoretical research activities, its due roles can be played.

Reference

[1] Yang,Y .,(2017). Experimental Verification of New Function Theories of Particles (1). https://www.matec-conferences.org/articles/matecconf/abs/2017/42/matecconf_eitce2017_0501/matecconf_eitce2017_05001.html.
[2] Yang.Y .,2015.New Advance in the Particle Work and Energy Theory.In: The 2015 International Conference on Advanced Material Engineering.Guang Zhou.pp.65-83. 
[3] Yang,Y .(2016) Study on Application of New Theorem of Kinetic Energy.J.IJAPM. 6(2):31-44
[4] Yang.Y .,2017.Study on Application of New Work-Energy Principle of Particle System.In: 2017 3rd International Conference on Education and Social Development. Xi ‘An.pp.700-706.
[5] Yang,Y .(2009) Both active force and reactive force making force.J.Manufacturing Automation. 12:126-128
[6] Yang,Y .(2010) Study on New Work-Energy Theories.J. New Curriculum Learning. 33: 184-185
[7] Yang,Y .(2011) New Rotational Kinetic Energy Theorem for Fixed-Axis Rotation of Rigid Object.J. China Science Innovation Herald. 22:81-82
[8] Compiled by seven engineering universities and colleges including Southeast University, adapted by Ma,W.Y . (2006)Physics (Volume One). Higher Education Press,Bei Jing.