The intermediate level-sets of the four-dimensional membrane model

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May 10, 2022

Abstract

In this paper, we consider the discrete membrane model in four dimensions. We confirm the existence of the scaling limit of the intermediate (i.e., a multiple of the expected maximum) level-sets of the model, and show that it is equal in law to the sub-critical Gaussian multiplicative chaos (GMC) measure of the continuum membrane model.

1 Introduction and main results

The membrane model, also known as the discrete bilaplacian field, is a Gaussian lattice model whose Hamiltonian is given through the bilaplacian operator. It gets its name from counterpart continuum models in mechanics that describe the behaviour of thin structures (e.g. membranes) that react elastically to external forces. In [20] Sakagawa investigates the entropic repulsion phenomenon for a class of finite-range Gaussian fields, initiating the study of the membrane model within the probability community. The study of the extremal theory of the membrane model is largely inspired by that of the planar Gaussian free field (GFF) – e.g., maximum and entropic repulsion in [5, 9], intermediate level-sets in [3], extremal process in [2, 4]; see [1] for a more thorough review. Kurt [15, 16] studies the maximum and the entropic repulsion of the membrane model in its critical dimension (4D). Schweiger [21] analyzes the asymptotics of the Green function corresponding to the membrane model on the cubic lattice. An object closely related to extremal is the set of high points, a.k.a. level sets. In [7] Cipriani studies the high points of the membrane model in four dimension. There is also an on-going work [12] by the first author and collaborators investigating the extremal process in four dimension, aiming at establishing similar characterizations as those for the planar GFF. In supercritical dimensions, [19] studies the percolation of the level-sets, inspired by [6] for similar results on supercritical GFF. The membrane model also has a scaling limit, which is introduced and studied in [8] by Cipriani, Dan and Hazra.

We now describe our model and results in detail. The discrete membrane model (with Dirichlet boundary conditions) on a finite subset \( V \subset \mathbb{Z}^d \) is a centered Gaussian field \( \{ h_x \} \), which vanishes outside \( V \), whose Hamiltonian is given by \( \frac{1}{2} \sum_{x \in V} |\Delta h_x|^2 \), where \( \Delta \) denotes the discrete Laplacian operator. We will give its precise definition in the next section.

In 2009, Kurt [15] shows that the extreme value of the 4D membrane model in square lattices \( D_N := (0, N)^4 \cap \mathbb{Z}^4 \) grows as

\[
\max_{x \in D_N} h_x \sim 2\sqrt{2\gamma \log N}, \quad N \to \infty,
\]

where “\( \sim \)” means that the ratio tends to one and \( \gamma := 8/\pi^2 \) satisfies that the asymptotic of the Green function with respect to the 4D membrane model (see Definition 2.2) satisfies \( G_{D_N}(x, x) \sim \gamma \log N + O(1) \) as \( N \to \infty \) for \( x \) “deep” inside \( D_N \). [15] also discussed the intermediate level-sets

\[
\{ x \in D_N : h_x \geq 2\lambda \sqrt{2\gamma \log N} \}, \quad \lambda \in (0, 1),
\]

∗Beijing International Center for Mathematical Research, Peking University. Research supported by the National Key R&D Program of China (No. 2021YFA1002700 and No. 2020YFA0712900) and NSFC (No. 12071012).
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and shows the size of the intermediate level-sets is \( N^{4(1-\lambda^2)+o(1)} \), where \( o(1) \to 0 \) in probability as \( N \to \infty \).

In this paper, we are going to show that the point measure induced by the intermediate level-sets, namely

\[
\sum_{x \in D_N} \delta_{x/N} \otimes \delta_{h^N_{x/a_N}}
\]

with \( a_N \) a scale sequence such that for some \( \lambda \in (0, 1) \),

\[
\lim_{N \to \infty} \frac{a_N}{\log N} = 2\lambda \sqrt{2\gamma}, \tag{1.1}
\]

has a scaling limit which is non-trivial and can be characterized in an explicit way. More precisely, setting scaling sequence

\[
K_N := \frac{N^4}{\log N} e^{-\frac{a^2_N}{4\log N}}, \tag{1.2}
\]

we have:

**Theorem 1.1.** Set \( D = (0, 1)^4 \) and \( D_N = (0, N)^4 \cap \mathbb{Z}^4 \). For each \( \lambda \in (0, 1) \) and each sample \( h^{D_N} \) of the membrane model in \( D_N \), define the rescaled point measure

\[
\eta^D_N := \frac{1}{K_N} \sum_{x \in D_N} \delta_{x/N} \otimes \delta_{h^N_{x/a_N}}, \tag{1.3}
\]

with \( a_N, K_N \) as in (1.1) and (1.2). Then, there is a random Borel measure \( Z^D_\lambda \) on \( D \) with \( \mathbb{E}[Z^D_\lambda(\bar{D})] \in \mathbb{R}_+ \) such that

\[
\eta^D_N \xrightarrow{law} Z^D_\lambda(\partial x) \otimes e^{-\pi\lambda h} dh \tag{1.4}
\]

with respect to the topology of vague convergence of measures on \( \bar{D} \times \mathbb{R} \).

As a direct consequence of Theorem 1.1, we can give sharp asymptotics for the size of the intermediate level-sets:

**Corollary 1.2.** For \( a_N, K_N \) as in (1.1) and (1.2),

\[
\frac{1}{K_N} \# \{ x \in D_N : h^{D_N}(x) \geq a_N \} \xrightarrow{law} (\pi^2)^{-1} Z^D_\lambda(\bar{D}).
\]

In fact, the measure \( Z^D_\lambda \) can be characterized by the scaling limit of membrane model. In [8], Cipriani introduces the continuum membrane model which serves as a counterpart of the membrane model in the continuum. The following theorem shows that \( Z^D_\lambda \) has the law of the Gaussian multiplicative chaos corresponding to the continuum membrane model on \( D \):

**Theorem 1.3.** For \( D = (0, 1)^4 \), the random measure \( Z^D_\lambda \) has the following law:

\[
Z^D_\lambda(dx) \xrightarrow{law} \sqrt{\pi} e^{\frac{\alpha^2}{4} s_D(x)} \mu^{D, \pi\lambda}_\infty(dx)
\]

where the \( \mu^{D, \pi\lambda}_\infty \) is the subcritical GMC measure on \( D \) defined in (2.11) and the function \( s_D(x) \) is defined in (2.4).

The proof of Theorem 1.1 is largely inspired by [3], which discusses scaling limits of intermediate level-sets of DGFF: we first prove that \( \eta^D_N \) is tight with respect to the vague topology via a truncated second moment argument, then show that the sub-sequential limits of \( \eta^D_N \) can be factorized as the right-hand side of (1.4), and finally we prove the uniqueness of the sub-sequential limits by showing that they must satisfy some axiomatic characterization (see Proposition 5.1 and 5.3). Although many intermediate propositions and lemmas are have parallel in [3], in order to make calculations explicit and keep this paper self-contained we still include their proofs here. Theorem 1.3 is an application of Shamov’s approach to Gaussian Multiplicative Chaos (see [22] for more details).
In principle, the results in Theorem 1.1-1.3 may be also extended to general nice domains in \( \mathbb{R}^4 \). However, the analysis of the asymptotics of the Green function corresponding to the discrete bilaplacian operator is much more difficult than the case of GFF, for which the random walk representation plays a crucial role. Although a representation by intersections of two independent random walks exists for the discrete bilaplacian operator, it does not satisfy the boundary condition of our model. Deriving these asymptotics falls outside the scope of this work, therefore we do not pursue this generalization any further.

Another possible direction for generalization is to extend the results above to the Gaussian polylaplacian models (in their critical dimension), which are defined by replacing the discrete bilaplacian in (2.1) by \( \nabla \Delta \frac{e^{n}}{n} \) or \( \Delta^2 \) depending on whether the parameter \( n \) (which also indicates the respective critical dimension) is odd or even. The key points in the proof are the Gibbs-Markov property and the asymptotics of the Green function. The Gibbs-Markov property holds for all such models. The \( n \)-dimensional Gaussian polylaplacian model with even parameter \( n \) should be log-correlated by applying the methods in [13, 17, 18]. Then the similar scaling limit results of the intermediate level-sets should follow when one derives sufficiently sharp asymptotics of the Green function. When \( n \) is odd, since the operator is no longer local, the scheme above does not work any more. It is an interesting question whether one can still obtain similar results in this case.

We now briefly describe how this paper is organized. In Section 2, we will give some brief introduction on the discrete and continuum membrane model, and clarify the notation in this paper. In Section 3, we will prove the existence and factorization property of the sub-sequential limits. In Section 5, we will estimate the first moment and the second moment of the size of truncated level-sets. In Section 4, we will introduce our notation, and then give some basic results on the membrane model.

Acknowledgements: The authors thank Hao Ge and Jiaxi Zhao for helpful suggestions during various stages of this work.

## 2 Notation and Preliminaries

In this section, we will introduce our notation, and then give some basic results on the membrane model.

### 2.1 Notation

In this paper, \( \nabla, \Delta \) denotes the discrete gradient and Laplacian operator, which is defined by:

\[
\nabla f(x) := (f(x + e_1) - f(x), \ldots, f(x + e_d) - f(x)),
\]

for \( e_i \) the \( i \)-th unit coordinate vector in \( \mathbb{R}^d \), and

\[
\Delta f(x) = \frac{1}{2d} \sum_{y, |y-x|} (f(y) - f(x)).
\]

And \( \tilde{\nabla}, \tilde{\Delta} \) denotes the continuum gradient and Laplacian operator. \( G, \tilde{G} \) denotes the discrete and continuum Green function, which will be defined in the next subsection. The space \( \mathcal{H}_0^2 \) is the completion of the test function space with the inner product \( \langle \cdot, \cdot \rangle_\Delta \).

The constants in this paper, such as \( c, c', \tilde{c}, \ldots \), are universal, but may change from place to place. And \( c(r, s, \ldots) \) denotes a constant depending only on \( r, s, \ldots \), whose precise value may change from place to place. Moreover, these constants are assumed to be positive without further specification.

Let \( \mathcal{D} \) be the collection of all open dyadic cubes and their finite unions. The dyadic cubes takes the form

\[(i2^{-n}, (i+1)2^{-n}) \times (j2^{-n}, (j+1)2^{-n}) \times (k2^{-n}, (k+1)2^{-n}) \times (l2^{-n}, (l+1)2^{-n}), \quad i, j, k, l, n \in \mathbb{Z}.
\]

For \( D \in \mathcal{D} \), let \( D_N \) denote its lattice approximation, which is a subset of \( \mathbb{Z}^4 \), such that \( D_N / N \subset D \) and for all \( x \in D' \), the \( l^\infty \) distance between \( N x \) and \( D_N \) is larger than 1. One can assume that every domain
2.2 Preliminaries

We start with the definition of the (discrete) membrane model:

**Definition 2.1.** The discrete membrane model (with Dirichlet boundary condition) on a finite subset \( V \) of \( \mathbb{Z}^d \) is a centered Gaussian field denoting by \( h^V \), vanishes outside \( V \), and has the probability density function

\[
P(dh^V) \propto \exp \left( -\frac{1}{2} \sum_{x \in \mathbb{Z}^d} |\Delta h_x|^2 \right) \prod_{x \in V} dh_x \prod_{x \notin V} \delta_0(dh_x),
\]

(2.1)

where \( \delta_0 \) is the Dirac point mass at 0.

Next, we will define the Green function of the membrane model.

**Definition 2.2.** The Green function \( G^V(x,y) \) is the correlation function of the membrane model, that is, \( G^V(x,y) := \mathbb{E}[h^V_x h^V_y] \).

The Green function solves the following difference equation:

\[
\begin{align*}
\Delta^2 G^V(x,y) &= \delta_x(y), & y & \in V, \\
G^V(x,y) &= 0, & y & \in \partial_2 V,
\end{align*}
\]

where \( \partial_2 V \) denotes the points in \( V^c \) whose graph distance to \( V \) is less or equal than 2.

The estimation of the Green function is important throughout the proof. We now paraphrase the asymptotics from [21, Observation 1.5]. Before we state it, we need to define the continuum Green function \( \tilde{G} \):

**Definition 2.3.** The continuum Green function of Bilaplacian operator on a domain \( D \) with Dirichlet boundary condition on \( \partial D \) is a function \( \tilde{G}^D(x,y) = \tilde{G}^D_y(x) \) that solves the following equations:

\[
\begin{align*}
\tilde{\Delta}^2 \tilde{G}_y(x) &= \delta_y(x), & x & \in D, \\
\tilde{G}_y(x) &= 0, & x & \in \partial D, \\
\frac{\partial \tilde{G}_y}{\partial n}(x) &= 0, & x & \in \partial D.
\end{align*}
\]

Recall \( D = (0,1)^4 \) and \( D_N = (0,N)^4 \cap \mathbb{Z}^4 \). Letting \( d(x) \) be the \( l^\infty \) distance from \( x \) to \( \partial D \), denoting \( \Gamma(x,y) := \gamma \log |x-y| \) as the fundamental solution, setting

\[
a^D(x,y) := \tilde{G}^D(x,y) - \Gamma(x,y)(x,y \in D, x \neq y),
\]

(2.3)

and thanks to [21, Lemma 3.5], \( a^D(x,y) \) can be continuously extended to \( D \times D \), and we define

\[
s_D(x) := a^D(x,x).
\]

(2.4)

We then have:

**Theorem 2.4.** There exists \( \theta > 0 \), such that:

\[
G^{D_N}([xN], [yN]) = \gamma \log N + s_D(x) + o(1)
\]

(2.5)

with \( o(1) \to 0 \) as \( N \to \infty \) for all \( x \in D \) with \( d(x) \geq \frac{(\log N)^{\theta}}{N} \). And

\[
G^{D_N}([xN], [yN]) = \tilde{G}^D(x,y) + o(1).
\]

(2.6)
with \( o(1) \to 0 \) as \( N \to \infty \) for all \( x, y \in \{(x, y) : x, y \in D, x \neq y \} \) with \( \min\{d(x), d(y)\} \geq \frac{(\log N)^g}{N} \). Moreover

\[
|G^D_N(x, y) - \gamma \log \left( 2 + N \frac{\max(d(x/N), d(y/N))}{1 + |x - y|} \right) | = O(1). \tag{2.7}
\]

Since the above asymptotics are scale-invariant, and the Green function of the union of disjoint sets is as simple as the Green function on each set. Therefore the above asymptotics (2.5)-(2.7) are valid on \( \mathcal{D} \).

Similar to the Gaussian free field, the membrane model also enjoys the Gibbs-Markov property. We formalize it in the following theorem and its proof will be given in the Appendix.

**Theorem 2.5.** For \( U \subset V \subset \mathbb{Z}^d \), let \( h^V \) be the membrane model in \( V \). Then we may write:

\[
h^V \overset{\text{law}}{=} h^U + \varphi^{V,U}, \tag{2.8}
\]

where \( h^U \) is the membrane model in \( U \), and \( \varphi^{V,U} \) has discrete biharmonic sample paths in \( U \), and \( h^U, \varphi^{V,U} \) are independent. Moreover, we have:

\[
\varphi^{V,U} \overset{\text{law}}{=} \mathbb{E}[h^V|\sigma(h^V_z : z \in V\setminus U)]. \tag{2.9}
\]

Calculating the variance on each side of (2.8), together with (2.9), we have \( G^V(x, x) - G^U(x, x) = \text{var}[\varphi^{V,U}(x)], \) which gives

\[
G^V(x, x) \geq G^U(x, x), \quad x \in U \subset V. \tag{2.10}
\]

The continuum membrane model, introduced in [8] can be seen as the scaling limit of the discrete membrane model. We now give two equivalent definitions of the continuum membrane model.

**Definition 2.6.** A continuum membrane model on a bounded, open \( D \subset \mathbb{R}^d \) is a functional \( h : D \to \mathbb{R} \) to a random variable such that

(1) \( h \) is a.s. linear, i.e.,

\[
h(af + bg) = ah(f) + bh(g) \quad \text{a.s.}
\]

for all bounded measurable \( f, g \) and all \( a, b \in \mathbb{R} \), and

(2) for all bounded measurable \( f \), \( h(f) \overset{\text{law}}{=} \mathcal{N}(0, \sigma^2(f)) \), where

\[
\sigma^2(f) = \int_{D \times D} \tilde{G}^D(x, y)f(x)f(y)dx dy.
\]

Equivalently, \( h \) is a centered Gaussian process on the space of bounded, measurable function \( f : D \to \mathbb{R} \) such that the covariance structure is given by:

\[
\text{cov}(h(f), h(g)) = \int_{D \times D} \tilde{G}^D(x, y)f(x)g(y)dx dy.
\]

Alternatively, the continuum membrane model can be defined as the canonical Gaussian process on Hilbert space \( H^2_D(D) \) with the inner product \( \langle f, g \rangle_D = \int_D \Delta f(x) \Delta g(x)dx \). If we let \( \{f_k\} \) be an orthonormal basis of the Hilbert space, then with \( Z_k \) i.i.d. standard normals, we will denote the partial sums by

\[
\phi_n(x) := \sum_{j=1}^n Z_k f_k(x).
\]

For any smooth function \( f \in H^2_D(D) \), let \( h_n(f) := \int_D f(x)\phi_n(x)dx \), then the \( L^2 \)-limit of \( h_n \) in \( H^2_D(D) \) satisfies Definition 2.6.

Finally, we will define the Gaussian multiplicative chaos associated with the continuum membrane model. For each \( \beta \in \mathbb{R}_+ \), define the random measure

\[
\mu_n^{D,\beta}(dx) := 1_D(x)e^{\beta \phi_n(x)} - \frac{1}{2} \Delta^2 \mathbb{E}[\phi_n(x)^2] \ dx.
\]
Thanks to Kahane [14] (see also Shamov’s approach [22]), there exists a.a. finite random Borel measure \( \mu_{\infty}^{D,\beta} \), supported on \( D \), and for each measurable \( A \subset D \),

\[
\lim_{n \to \infty} \mu_n^{D,\beta}(A) = \mu_{\infty}^{D,\beta}(A), \quad \text{a.s.} \tag{2.11}
\]

Then the measure \( \mu_{\infty}^{D,\beta} \) is defined as the GMC associated with the continuum membrane model, and such measure is unique regardless of the choice of the orthonormal basis.

### 3 Moments of the size of level-sets

We will start our proof by calculating the first and the second moments of the size of level-sets. Lemma 3.1 and 3.2 calculate the first moments of the size of level-sets, and when we turn to the second moment estimations, a truncation is needed for all \( \lambda \in (0,1) \). Lemma 3.3 discusses the difference between the truncated and untruncated level-sets, and Proposition 3.4 calculates the truncated second moments.

For each \( b \in \mathbb{R} \), define

\[
\Gamma_N^D(b) := \{ x \in D_N : h^{D_N}(x) \geq a_N + b \}.
\]

First of all, we will give a general size bound of \( \Gamma_N^D(b) \):

**Lemma 3.1.** For all \( D \in \mathcal{D} \) there is a \( c = c(D) \) such that for all \( N \geq 1 \), all \( b \in [-\log N, \log N] \), all sequences \( a_N \) such that \( 0 \leq a_N \leq 4\sqrt{2\gamma \log N} \) and all \( A \subset D_N \), we have

\[
\mathbb{E}|\Gamma_N^D(b) \cap A| \leq cK_N \frac{|A|}{N^\gamma} e^{-\frac{a_N b}{\frac{1}{2} \log N}}. \tag{3.1}
\]

**Proof.** We will prove (3.1) by summing over \( x \in D_N \). In fact, it suffices to prove that uniformly for \( x \in D_N \) and \( |b| \leq \log N \):

\[
\mathbb{P}(h^{D_N}(x) \geq a_N + b) \leq \frac{c(D)}{\sqrt{\log N}} e^{-\frac{a_N^2}{\frac{1}{2} \gamma \log N}} e^{-\frac{a_N b}{\frac{1}{2} \log N}}.
\]

Without loss of generality, we can assume that \( D \) contains only one dyadic square.

By Theorem 2.5, for lattice domains \( U \subset V \subset \mathbb{Z}^d \), we can write \( h^V \overset{\text{law}}{=} \varphi^{V,U} + h^U \), and the two parts in the right-hand side are independent. Then we have

\[
\mathbb{P}(h^U(x) \geq a) \leq \mathbb{P}(h^U(x) \geq a, \varphi^{V,U}(x) \geq 0) + \mathbb{P}(h^U(x) \geq a, \varphi^{V,U}(x) \leq 0) = 2\mathbb{P}(h^U(x) \geq a, \varphi^{V,U}(x) \geq 0) \leq 2\mathbb{P}(h^V(x) \geq a).
\]

We then enlarge \( D_N \) to a larger lattice \( \tilde{D}_N \), whose diameter is 2 times larger than that of \( D_N \), and ensures that \( \tilde{D}_N \) and \( D_N \) have the same center. Then the variance of \( h^{D_N}(x)(x \in D_N) \) is uniformly within a constant of \( \gamma \log N \) thanks to (2.7). So for some \( c' \),

\[
\mathbb{P}(h^{D_N}(x) \geq a_N + b) \leq \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\gamma \log N - c'}} \int_b^\infty e^{-\frac{(s_N + b)^2}{2(\gamma \log N + c')}} ds \leq \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\gamma \log N - c'}} \int_b^\infty e^{-\frac{a_N^2 + 2a_N b}{2(\gamma \log N + c')}} ds \leq C \frac{a_N}{\sqrt{\log N}} e^{-\frac{a_N b}{\frac{1}{2} \log N}} e^{-\frac{a_N^2}{\frac{1}{2} \gamma \log N}}.
\]

This completes the proof of the lemma. \( \square \)

Thanks to Lemma 3.1, more accurate asymptotics of the above expectation can be calculated for open sets \( A \) whose closure lie inside \( D \).
Lemma 3.2. For all \( b \in \mathbb{R} \) and all open sets \( A \) whose closure lie inside \( D \),
\[
\mathbb{E}\{x \in \Gamma_N^D(b) : x/N \in A\} = \frac{e^{-\pi \lambda b}}{4\lambda \sqrt{\pi}} \left( \int_A e^{\frac{n^2}{2} s_D(x)} dx + o(1) \right) K_N,
\]
with \( o(1) \to 0 \) as \( N \to \infty \) uniformly on compact sets in \( D \).

Proof. Since the closure of \( A \) lies in \( D \), all the points in \( A \) lies “deep” inside \( D \). Thanks to (2.5) we have
\[
G^D_N([xN], [xN]) = \gamma \log N + s_D(x) + o(1),
\]
with \( o(1) \to 0 \) as \( N \to \infty \) uniformly in \( \bar{A} \). Then using the following Gaussian tail bound approximation from [11, Theorem 1.2.6]:
\[
\left( \frac{1}{x} - \frac{1}{x^3} \right) \frac{e^{x^2}}{\sqrt{2\pi}} \leq \mathbb{P}(X \geq \sigma x) \leq \frac{1}{x} \frac{e^{x^2}}{\sqrt{2\pi}}, \tag{3.2}
\]
with \( X \sim N(0, \sigma^2) \). We have:
\[
\mathbb{P}(h^D_N(x) \geq a_N + b) = (1 + o(1)) \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\gamma \log N + s_D(x) + o(1)}}{a_N + b} e^{-\frac{s_D(x)^2}{2(\gamma \log N + s_D(x) + o(1))}}
\]
\[
= (1 + o(1)) \frac{1}{4\lambda \sqrt{\pi} \log N} e^{-\frac{s_D(x)^2}{2\lambda^2 \log N}} e^{-\frac{\pi \lambda b}{4\lambda^2} e^{\frac{n^2}{2} s_D(x)}}.
\]
By the continuity of \( s_D(x) \), we can replace the Riemann sum by integrating on \( A \), thus completes the proof. \( \square \)

Then, we need to deal with a truncated version of the measures \( \eta^D_N \) for the proof for all \( \lambda \in (0, 1) \). Let \( \{D_N\} \) be the lattice approximation of a domain \( D \in \mathcal{D} \). Denote \( \Lambda_r(x) := \{z \in \mathbb{Z}^1 : \|z - x\| \leq r\} \) and, for \( N \geq 1 \) and \( x \in D_N \), write
\[
n(x) := \max\{n \geq 0 : \Lambda_{c^{n+1}}(x) \subset D_N\}.
\]
Thanks to Theorem 2.4, we have \( \log N - c(\epsilon) \leq n(x) \leq \log N + c'(\epsilon) \) for all \( x \in D_N \) with \( d(x, D^c_N) > \epsilon N \). We now define
\[
\Delta^k(x) := \begin{cases} 
\emptyset, & k = 0, \\
\Lambda_{c^k}(x), & 1 \leq k \leq n(x) - 1, \\
D_N, & k = n(x).
\end{cases}
\]
For \( U \subseteq V \), recall \( \varphi^{V,U} \) as in (2.9). We then set
\[
S_k(x) := \varphi^{D_N, \Delta^k(x)}(x), \quad k = 0, 1, \ldots, n(x).
\]
By definition, we have \( S_0(x) = h^D_N \) and \( S_n(x) = 0 \).

Next, for some sequence \( a_N \) satisfying (1.1) and \( M > 0 \), let
\[
T_{N,M}(x) := \bigcap_{k=k_N}^{n(x)} \left\{ \left| S_k(x) - a_N \frac{n(x) - k}{n(x)} \right| \leq M(n(x) - k)^{3/4} \right\}
\]
where
\[
k_N := \left\lfloor \frac{1}{8} \log K_N \right\rfloor = \frac{1}{2}(1 - \lambda^2 + o(1)) \log N.
\]
Then we can define the truncated point measure
\[
\eta^D_{N,M} := \frac{1}{K_N} \sum_{x \in D_N} 1_{T_{N,M}(x)} \delta_x/N \otimes \delta_k^D(x) - a_N.
\]
By the definition of \( \eta^D_{N,M} \), we immediately get \( \langle \eta^D_{N,M}, f \rangle \leq \langle \eta^D_N, f \rangle \) for any measurable \( f \geq 0 \).
First of all, we need to deal with the difference between $\hat{\eta}_N^{D,M}$ and $\eta_N^D$ when $M$ is sufficiently large. For this, we define the truncated level-sets
\[
\hat{\Gamma}_N^{D,M}(b) := \{x \in D_N : h^{D_N}(x) \geq a_N + b, T_N(x)\}.
\]

**Proposition 3.3.** For each $\lambda \in (0, 1)$ and each $b_0 > 0$ there exist $c, \bar{c}$ such that for any $D \in \mathcal{D}$, any $b \in [-b_0, b_0]$, any $M \geq 1$ and large $N$,
\[
\mathbb{E}[\Gamma_N^D(b) \setminus \hat{\Gamma}_N^{D,M}(b)] \leq ce^{-\bar{c}M^2} (\text{diam}D)^{4+4\lambda^2} K_N.
\]

We postpone its proof till the end of this section.

A direct consequence of Proposition 3.3 is that for any bounded, measurable
\[
\lim_{M \to \infty} \limsup_{N \to \infty} |\langle \hat{\eta}_N^{D,M}, f \rangle - \langle \eta_N^D, f \rangle| = 0.
\]

By Lemma 3.1 and the domination of $\hat{\eta}_N^{D,M}$ by $\eta_N^D$, the family of truncated measures $\{\hat{\eta}_N^{D,M} : N \geq 1\}$ is tight with respect to the vague topology. Then a sub-sequential weak limit $\hat{\eta}^{D,M}$ exists and we can study its properties, which will be discussed in the next section.

We then need to calculate the truncated second moments. For $b, b' \in \mathbb{R}$ with $b < b'$, let
\[
\hat{\Gamma}_N^{D,M}(b, b') := \hat{\Gamma}_N^{D,M}(b) \setminus \hat{\Gamma}_N^{D,M}(b').
\]

Then we have:

**Proposition 3.4.** Let $\lambda \in (0, 1)$. For all $\varepsilon, M > 0$ and all $b < b'$, there is $c = c(\varepsilon, M, b, b')$ such that for all $D \in \mathcal{D}$ and all large $N$,
\[
\mathbb{E}[\hat{\Gamma}_N^{D,M}(b, b') \cap D_N^c] \leq c(\text{diam}D)^{8+8\lambda^2} K_N^2.
\]

In order to prove Proposition 3.3 and 3.4, we need the following lemmas:

**Lemma 3.5.** For all $\varepsilon, r > 0$, there is $c = c(\varepsilon, r)$ such that for all $D \in \mathcal{D}$ with $\text{diam}D \leq r$ and all large $N$, we have:

1. For all $x \in D_N$ and all $k \leq m \leq n(x)$,
\[
\text{var}[S_k(x) - S_m(x)] = (m - k)\gamma + o(1)
\]

where $o(1) \to 0$ when $N \to \infty$.

2. For all $x \in D_N$ and all $k$ such that $k \leq n(x)$,
\[
\text{var}[S_k(x)] - (n(x) - k)\gamma \in [0, c].
\]

3. For all $l \geq 1$ there is $c' = c'(\varepsilon, l)$ such that for all $x \in D_N$, all $k$, where $k \leq n(x)$, all $m$ with $k - l \leq m \leq k$ and all $y \in D_N$ satisfies $\Delta^{m+1}(y) \subseteq \Delta^k(x) \setminus \{x\}$, we have
\[
\mathbb{E}[S_k(x)S_m(y)] \leq (n(x) - k)\gamma + c', \quad \text{var}[S_m(y) - S_k(x)] \in [\gamma/2, c].
\]

**Proof.** By (2.5) and (2.8),
\[
\text{var}[S_k(x) - S_m(x)] = \text{var} \left[ \varphi^{\Lambda^m} \Lambda^{k}(0, 0) \right] = G^{\Lambda^m}(0, 0) - G^{\Lambda^k}(0, 0) = (m - k)\gamma + o(1),
\]

which gives (3.4). For (3.5), if $k = n(x)$ the claim is automatically true, thus we only need to consider the case when $k < n(x)$. Since $n(x) \geq \log N - c$ for $c = c(\varepsilon) > 0$, we may find $\tilde{c} = \tilde{c}(\varepsilon, r) > 0$ such that $D_N \subseteq \Lambda^m(x)$. Thanks to (2.10), we have
\[
\text{var}[S_k(x)] \leq G^{\Lambda^m(x) + \tilde{c} \varepsilon}(0, 0) - G^{\Lambda^k}(0, 0) \leq (n(x) + \tilde{c} - k)\gamma + o(1).
\]
By definition \( D_N \supseteq \Lambda_{\varepsilon^{n(x)+1}}(x) \), we thus have
\[
\var[ S_k(x) ] \geq G^{\Lambda_{\varepsilon^{n(x)+1}}}(0,0) - G^{\Lambda_{\varepsilon}}(0,0) \geq (n(x) + 1 - k)\gamma + o(1).
\]
This completes the proof of (3.5).

For (3.6), notice that \( \mathbb{E}[S_k(x)S_m(y)] = \mathbb{E}[\varphi^{D_N,\Delta^k_x}(x)\varphi^{D_N,\Delta^m_y}(y)] \) since Theorem 2.5 permits us to write \( \varphi^{V,W} \equiv \varphi^{U,V} + \varphi^{V,W} \) for \( W \subset V \subset U \). Then, by Theorem 2.4 and 2.5, this expectation equals
\[
\mathbb{E}[h^{D_N}(x)h^{D_N}(y)] - \mathbb{E}[h^{\Delta^k_x}(x)h^{\Delta^k_y}(y)] \leq (n(x) - k)\gamma + c'
\]
where \( c' \) does not depend on the choice of \( l \).

Finally, we have
\[
\var[ S_m(y) - S_k(x) ] = \var[ S_m(y) ] + \var[ S_k(x) ] - 2\mathbb{E}[S_m(y)S_k(x)],
\]
together with (3.5) and the first equation of (3.6) we get the upper bound of the second equation of (3.6). For the lower bound the second equation of (3.6), we have
\[
\var[ S_m(y) - S_k(x) ] = \var[ \varphi^{D_N,\Delta^k}(y) - \varphi^{D_N,\Delta^k}(x) + \varphi^{\Delta^k_x}(x) ] \geq \var[ \varphi^{\Delta^k_x}(x) ] \geq \var[ \varphi^{\Delta^{m+1}_x}(x) ] \geq \gamma + o(1).
\]
This proves (3.6).

**Lemma 3.6.** If \( x, y \in D_N \) satisfy \( \Delta^k_x \subseteq \Delta^m_y \), then for all \( k' < k \) and \( m' > m \), \( S_{k'}(x) - S_k(x) \) and \( S_m(y) - S_{m'}(y) \) are independent. Particularly, for any \( x \in D_N \), the process \( \{ S_k(x) \}_{k=0}^{n(x)} \) has independent increments.

We omit its proof since it is a trivial consequence of Theorem 2.5.

Now we are ready for:

**Proof of Proposition 3.3.** For any \( b < b' \) and \( \varepsilon > 0 \), we have:
\[
\mathbb{E}[\Gamma^{D_N}_N(b)\Gamma^{D,M}_N(b)] \leq \mathbb{E}[\Gamma^{D_N}_N(b)\mathbb{1}_{D_N^x}] + \mathbb{E}[\Gamma^{D_N}_N(b')] + \sum_{x \in D_N^x} \sum_{k=k_N}^{n(x)} \mathbb{P}(h^{D_N}(x) - a_N \in [b, b') \cap T^k_{N,M}(x)),
\]
where \( T^k_{N,M}(x) := \{ |S_k(x) - a_Nn(x) - \varepsilon| > M(n(x) - k)^{\varepsilon/4} \} \). The first two terms on the right-hand side of (3.7) can be bounded by Lemma 3.1 by taking \( b' \) sufficiently large and \( \varepsilon \) small enough. Now we turn to the estimation of the third term on the right-hand side of (3.7). For given \( x \in D_N^x \) and \( s \in [b, b'] \), we have
\[
(S_k(x)|h^{D_N}(x) = a_N + s) \equiv N(\var[ S_k(x) ] \var[ S_0(x) ](a_N + s), \var[ S_k(x) ] \var[ S_0(x) ] - \var[ S_k(x) ] \var[ S_0(x) ] - \var[ S_k(x) ] \var[ S_0(x) ] - \var[ S_k(x) ] \var[ S_0(x) ] - \var[ S_k(x) ] \var[ S_0(x) ])
\]
By Lemma 3.5, we have:
\[
\left| \frac{\var[ S_k(x) ]}{\var[ S_0(x) ]} - \frac{n(x) - k}{n(x)} \right| \leq \frac{c}{n(x)}
\]
and
\[
\frac{\var[ S_k(x) ] \var[ S_0(x) ] - \var[ S_k(x) ] \var[ S_0(x) ]}{\var[ S_0(x) ]} \leq c'(n(x) - k).
\]
By (3.2), we then have
\[
\mathbb{P}(T^k_{N,M}(x)|h^{D_N}(x) = a_N + s) \leq ce^{-\varepsilon M^2(n(x)-k)^{\varepsilon/2}}.
\]
By the uniform control above, we can bound the third term of (3.7) by
\[ c \sum_{x \in D_N^+} \sum_{k=1}^{\infty} e^{-c_d \|x\|^2} \mathbb{P}(h_{D_n}(x) - a_N \in [b, b']) \leq c c_d \mathbb{E}[\Gamma_{D_n}^D(b)]. \]

Since \( \tilde{G}^D(r, x) = \tilde{G}^D(x, y) \), we get
\[ s_{r, D}(x) = s_D(x) + \gamma \log r \quad (3.8) \]
by (2.3) and (2.4). By Lemma 3.2 and (3.8), we complete the proof of the lemma.

Now we turn to the truncated second moment estimation.

**Proof of Proposition 3.4.** Writing
\[ \mathbb{E}[|\hat{\Gamma}_{D_n}^D(b, b') \cap D_n^+|^2] = \sum_{x, y \in D_n^N} \mathbb{P}(x, y \in \hat{\Gamma}_{D_n}^D(b, b')) = \sum_{x, y \in D_n^N} \mathbb{P}(x, y \in \hat{\Gamma}_{D_n}^D(b, b')) + \sum_{x, y \in D_n^N} \mathbb{P}(x, y \in \hat{\Gamma}_{D_n}^D(b, b')), \quad (3.9) \]
we will separately estimate the two terms on the right-hand side of (3.9). The first term on the right-hand side of (3.9) can be bounded by \( K_{D_n}^{1/2} \mathbb{E}[\hat{\Gamma}_{D_n}^D((b, b')] = O(K_{D_n}^{1/2}) \) thanks to Proposition 3.3.

In order to estimate the second term on the right-hand side of (3.9), for any \( x, y \) such that \( \|x - y\| > K_{D_n}^{1/8} \), we need to estimate the probability that \( x, y \in \hat{\Gamma}_{D_n}^D(b, b') \). Denote \( k := k(x, y) = (\log^+ \|x - y\|) + 1 \) \& (n(x), and let \( l = l(x, y) \geq 1 \) be the smallest integer such that
\[ \Delta^{k-l}(x) \cap \Delta^{k-l-1}(y) = \emptyset, \quad \Delta^{k-l+1}(x) \cup \Delta^{k-l+1}(y) \subseteq \Delta^k(x). \]
Since \( n(x) \leq \log N + c(D) \) and \( n(y) \geq \log N - c'(D) \), we have \( l \leq \tilde{c}(D, \varepsilon) \).

Writing
\[ h_{D_n}(x) = S_k(x) + (S_{k-l}(x) - S_k(x)) + (S_0(x) - S_{k-l}(x)), \quad (3.10) \]
and
\[ h_{D_n}(y) = S_k(y) + (S_{k-l}(y) - S_k(y)) + (S_0(y) - S_{k-l}(y)), \quad (3.11) \]
Lemma 3.6 implies that the three terms on the right-hand side of (3.10) are independent, and the third term is independent of the first two terms on the right-hand side of (3.11), and is also independent of the first two terms in (3.10). Given \( t \in [-M(n(x) - k)^{3/4}, M(n(x) - k)^{3/4}] , s_1, s_2 \in [b, b') \) \& \( u_1, u_2 \in [-n(x)^{3/4}, n(x)^{3/4}] \), conditioned on \( S_k(x) - a_N \binom{n(x) - k}{n(x)} = t, S_{k-l}(x) - S_k(x) = u_1, S_{k-l}(y) - S_k(x) = u_2 \), we have
\[ \mathbb{P}^*(h_{D_n}(x) - a_N \in ds_1, h_{D_n}(y) - a_N \in ds_2) = \mathbb{P}^*(S_0(x) - S_{k-l}(x) - a_N \frac{h_{D_n}(x) - t}{\gamma n(x)} + u_1 \in ds_1, S_{k-l}(y) - a_N \frac{h_{D_n}(y) - t}{\gamma n(x)} + u_2 \in ds_2) \leq \frac{c}{k} \exp \left( -\frac{k a_n^2}{\gamma n(x)} (t - u_1 + s_1)^2 + \frac{k a_n}{\gamma n(x)} (t - u_2 + s_2)^2 \right) \frac{\gamma n(x)}{2k} ds_1 ds_2, \quad (3.12) \]
where \( \mathbb{P}^* \) denotes the conditional probability. The third line of (3.12) follows from applying Lemma 3.5 to replace the variance of \( S_0(x) - S_{k-l}(x) \) and \( S_0(y) - S_{k-l}(y) \) by \( k \gamma \), which only contributes a harmless multiplicative factor. Next, we will integrate out the conditional probability in (3.12) with respect to \( S_{k-l}(y) - S_k(x) \) and \( S_{k-l}(x) - S_k(x) \) given \( S_k(x) \). By Lemma 3.5, there exists \( c > 0 \) and \( c'' > c' > 0 \) such that \( E[(S_{k-l}(y) - S_k(x)) S_k(x)] \leq c \) and \( \text{var}(S_{k-l}(y) - S_k(x)) \in [c', c''] \). For \( t \) s.t. \( |t| \leq M(n(x) - k)^{3/4} \),
we have
\[ \left| \mathbb{E} \left[ S_{k-l}(y) - S_k(x) \big| S_k(x) = \frac{n(x) - k}{n(x)} = t \right] \right| \leq \frac{ct}{n(x) - k + 1} \leq cM, \]
and
\[ \text{var}[S_{k-l}(y) - S_k(x) | S_k(x)] \leq \text{var}[S_{k-l}(y) - S_k(x)] \leq c''. \]
For \( S_{k-l}(x) - S_k(x) \), the corresponding conditional expectation and variance have similar bounds. Since \( a_N/n(x) \) is bounded, we can get
\[ \mathbb{E} \left[ \frac{n(x) - k}{n(x)} \left( S_{k-l}(y) - S_k(x) \right) + \frac{n(x) - k}{n(x)} (S_{k-l}(y) - S_k(x)) \right| S_k(x) = \frac{n(x) - k}{n(x)} = t \] \[ \leq \hat{c} \]
by Cauchy-Schwarz inequality. Then we have
\[ \mathbb{P} \left( h^{D_N}(x) - a_N \in [b, b'), h^{D_N}(y) - a_N \in [b, b') \big| S_k(x) - \frac{n(x) - k}{n(x)} = t \right) \]
\[ \leq \mathbb{P} \left( |S_{k-l}(x) - S_k(x)| \bigvee |S_{k-l}(y) - S_k(x)| > n(x)^{3/4} \bigg| S_k(x) - \frac{n(x) - k}{n(x)} = t \right) + \]
\[ \frac{c}{k} \int_{[b, b') \times [b, b')} e^{-\frac{k^2}{n(x)} + a_N/\sqrt{n(x)}} (2t - s_1 - s_2) \, ds_1 \, ds_2 \]
(3.13)
with \( c' = c'(b, b', M) \). Note that if \( k = n(x) \) then \( S_k(x) = t = 0 \) and the condition vanishes. Therefore in this right-hand side of (3.13) also bounds the left-hand side without the conditioning. When \( k < n(x) \), we integrate the left-hand side of (3.13) with respect to \( S_k(x) - \frac{n(x) - k}{n(x)} \) to get
\[ \mathbb{P}(x, y \in \hat{\Gamma}_{N}^{D,M}(b, b')) \]
\[ \leq \mathbb{P} \left( h^{D_N}(x) - a_N \in [b, b'), h^{D_N}(y) - a_N \in [b, b'), |S_k(x) - \frac{n(x) - k}{n(x)}| \leq M(n(x) - k)^{3/4} \right) \]
\[ \leq \frac{c}{k} \int_{|t| \leq M(n(x) - k)^{3/4}} e^{-\frac{k^2}{n(x)} + a_N/\sqrt{n(x)} + \frac{M(n(x)-k)^3}{4}} \, dt \]
(3.14)
Finally, for the estimation of the second term on the right-hand side of (3.9), we will write:
\[ \sum_{x, y \in D^*_N \setminus K^{1/s}_N} \mathbb{P}(x, y \in \hat{\Gamma}_{N}^{D,M}(b, b')) = \sum_{k=n}^{N} \sum_{x, y \in D^*_N \setminus K^{1/s}_N \setminus K^{1/s}_N} \mathbb{P}(x, y \in \hat{\Gamma}_{N}^{D,M}(b, b')) \]
where we set \( n = \max_{x \in D^*_N \setminus n(x)} \). Note that, for a given \( k \), the sum contains \( O(N^4e^{4k}) \) pairs of \( (x, y) \). Since a change in \( n \) by an additive constant only contributes a harmless multiplicative factor, by (3.14) we have
\[ \sum_{x, y \in D^*_N \setminus K^{1/s}_N \setminus K^{1/s}_N} \mathbb{P}(x, y \in \hat{\Gamma}_{N}^{D,M}(b, b')) \leq c' \frac{K_N}{N^4} \sum_{k=n}^{n} \frac{\sqrt{n}}{k \sqrt{n} - k + 1} N^4 e^{-k^2/2n^2 \gamma n} + \frac{M(n-k)^3}{4}. \]
(3.15)
Since \( \frac{k^2}{2n^2} \rightarrow 4\lambda^2 < 4 \) as \( N \rightarrow \infty \), the exponent on the right-hand side of (3.15) grows linearly with \( k \). Then we can dominate the sum by the \( k = n \) term. As \( n = \log N + O(1) \), the right-hand side is \( O(K^2_N) \) by some simple calculation. In order to complete the proof, we only need to rescale the domain and show the diameter dependence since all bounds above are regardless of the diameter of the domain.
We now rescale the domain with a factor \( r < 1 \). Assuming \( D' := r^{-1}D \), note that the lattice approxima-
tion and the scale sequence changes as \( D'_N := D_{[N/r] - j} \), \( a'_N := a_{[N/r] - j} \), \( k'_N := k_{[N/r] - j} \) for some \( j \). We then get
\[
\hat{\Gamma}_{[N/r] - j}^{D,M}(b) = \hat{\Gamma}_N^{D',M}(b).
\]
Note that
\[
K'_N := \frac{N^4}{\sqrt{\log N}} e^{-\frac{s^2}{4\sqrt{\log N}}} = (\frac{4^4}{4^4} + o(1))K_{[N/r] - j},
\]
as all integers can be represented by \([N/r] - j\) with some \( N \) and \( j \), the claim for \( D' \) follows from the claim of \( D \), which completes the proof. \( \square \)

### 4 Existence and factorization of sub-sequential limits

In this part, we will prove that sub-sequential weak limit of the measures \( \{\eta^D_N\} \) exists, and can be factorized as in (4.8). Recall that \( \hat{\eta}^{D,M}_N \) is the truncated point measure defined in (3.3). Since \( D \times (\mathbb{R} \cup \{\infty\}) \) is separable, we have:

**Proposition 4.1.** For \( \lambda \in (0, 1) \) and large \( M \), the family of measures \( \{\hat{\eta}^{D,M}_N : N \geq 1\} \) is tight with respect to the topology of vague convergence on the space of Radon measures on \( D \times (\mathbb{R} \cup \{\infty\}) \). Moreover, for any sub-sequential weak limit \( \hat{\eta}^{D,M} \) of these measures and any \( b \in \mathbb{R} \),
\[
\mathbb{P}(\hat{\eta}^{D,M}(D \times [b, \infty))) < \infty = 1, \tag{4.1}
\]
and, for any non-empty open set \( A \subset D \),
\[
\mathbb{P}(\hat{\eta}^{D,M}(A \times [b, \infty))) > 0 > 0. \tag{4.2}
\]
Moreover, we have \( \hat{\eta}^{D,M}(A \times \mathbb{R}) = 0 \) a.s. for any \( A \) with \( \mu(A) = 0 \), with \( \mu \) the Lebesgue measure.

**Proof.** By definition, \( \hat{\eta}^{D,M}_N(D \times [b, \infty)) = |\hat{\Gamma}^{D,M}_N(b)| \leq |\Gamma_N^D(b)|. \) Then by Lemma 3.1, the family of measures \( \{\hat{\eta}^{D,M}_N(D \times [b, \infty)) : N \geq 1\} \) is tight for each \( b \in \mathbb{R} \). Thus for any function \( f : D \times (\mathbb{R} \cup \{\infty\}) \to \mathbb{R} \) which is continuous with compact support, \( \{(\eta^{D,M}_N, f) : N \geq 1\} \) is tight, which completes the proof of the first statement.

Now we let \( \hat{\eta}^{D,M} \) be a sub-sequential weak limit of the measures \( \{\hat{\eta}^{D,M}_N : N \geq 1\} \). By Fatou’s lemma, Lemma 3.1 and the domination of \( \hat{\eta}^{D,M}_N \) by \( \eta^D_N \), we have \( \mathbb{E}\hat{\eta}^{D,M}(D \times [b, \infty)) < \infty \) for each \( b \in \mathbb{R} \). Lemma 3.1 also gives \( \hat{\eta}^{D,M}(A \times \mathbb{R}) = 0 \) a.s. for any \( A \) with \( \mu(A) = 0 \).

For (4.2), denote \( X_N := \hat{\eta}^{D,M}_N(A \times [b, \infty)) \). Note that \( \{X_N : N \geq 1\} \) is uniformly integrable since \( \sup_{N \geq 1} \mathbb{E}X_N^\lambda < \infty \) thanks to Proposition 3.4. Lemma 3.2 implies that \( \inf_{N \geq 1} \mathbb{E}X_N^\lambda > 0 \). Obviously \( \exp(\frac{\lambda s_D(x)}{2}) > 0 \), then any distributional limit of \( X_N \) has positive expectation, which completes the proof. \( \square \)

The claim (4.2) indicates that every sub-sequential weak limit \( \hat{\eta}^{D,M} \) of measures \( \{\hat{\eta}^{D,M}_N : N \geq 1\} \) has positive total mass with positive probability when \( M \) is large enough. Claims (4.1) and (4.2) remain valid with \( \hat{\eta}^{D,M}_N \) replaced by \( \eta^D \) thanks to the domination of \( \hat{\eta}^{D,M}_N \) by \( \eta^D \).

Then, for any \( b \in \mathbb{R} \) and function \( f : D \times (\mathbb{R} \cup \{\infty\}) \to \mathbb{R} \), define
\[
f_b(x, h) := f(x, h + b)e^{-\pi h h}.
\]

The following proposition will lead to the factorization property:

**Proposition 4.2.** Let \( \lambda \in (0, 1) \), and \( \eta^D \) be any sub-sequential limit of \( \{\eta^D_N : N \geq 1\} \). Then for any \( b \in \mathbb{R} \) and any \( f : D \times (\mathbb{R} \cup \{\infty\}) \to \mathbb{R} \) of the form \( f(x, h) = 1_A(x)1_{[0, \infty)}(h) \) with \( A \subset D \) open,
\[
\langle \eta^D, f_b \rangle = \langle \eta^D, f \rangle, \text{ a.s.}
\]

In order to prove the proposition, we need the following lemma:
Lemma 4.3. Let $\lambda \in (0,1)$. For each open $A \subset D$ and each $b \in \mathbb{R}$, denoting $A_N := \{x \in \mathbb{Z}^4 : x/N \in A\}$, we have

$$\lim_{N \to \infty} \frac{1}{K_N} E\left| \tilde{\Gamma}^{D,M}_N(0) \cap A_N \right| - e^{\pi \lambda b} \left| \tilde{\Gamma}^{D,M}_N(b) \cap A_N \right| = 0.$$ 

Proof. By Cauchy-Schwarz inequality, it suffices to prove

$$\lim_{N \to \infty} \frac{1}{K_N} E \left( \left| \tilde{\Gamma}^{D,M}_N(0) \cap A_N \right| - e^{\pi \lambda b} \left| \tilde{\Gamma}^{D,M}_N(b) \cap A_N \right| \right)^2 = 0. \quad (4.3)$$

The expectation in (4.3) can be written as

$$\sum_{x,y \in A_N} \mathbb{E} \left[ (1_{\{h^D_N(x) \geq a_N\}} - e^{\pi \lambda b} 1_{\{h^D_N(x) \geq a_N + b\}}) \right] \left( 1_{\{h^D_N(y) \geq a_N\}} - e^{\pi \lambda b} 1_{\{h^D_N(y) \geq a_N + b\}} \right) 1_{T_{N,M}(x)} 1_{T_{N,M}(y)}. \quad (4.4)$$

We first consider the terms corresponding to $x, y$ such that

$$m := \|x - y\|$$

satisfies

$$m \geq \frac{3}{2} k_N, \quad \|x - y\| \in [e^m + 2e^{k_N}, e^{m+1} - 2e^{k_N}]. \quad (4.6)$$

Denoting $\mathcal{F} := \sigma(h^D_N) : z \in D_N \backslash (\Delta^{k_N}(x) \cup \Delta^{k_N}(y))$, then the term in (4.4) corresponding to such $x, y$ can be written as

$$\mathbb{E} \left[ (\mathbb{P}(h^D_N(x) \geq a_N | \mathcal{F}) - e^{\pi \lambda b} \mathbb{P}(h^D_N(x) \geq a_N + b | \mathcal{F})) 1_{T_{N,M}(x)} \right] \mathbb{E} \left[ (\mathbb{P}(h^D_N(y) \geq a_N | \mathcal{F}) - e^{\pi \lambda b} \mathbb{P}(h^D_N(y) \geq a_N + b | \mathcal{F})) 1_{T_{N,M}(y)} \right]. \quad (4.7)$$

Thanks to Theorem 2.5, we can write $h^D_N(x) = (S_0(x) - S_{k_N}(x)) + S_{k_N}(x)$, where the two terms on the right-hand side are independent of each other, and $(S_0(x) - S_{k_N}(x))$ has the law of the membrane model in $\Lambda_{k_N}(x)$. Using a similar decomposition for $h^D_N(y)$, thanks to the fact that $S_{k_N}(x), S_{k_N}(y)$ is $\mathcal{F}$-measurable, then (4.7) is bounded by

$$\mathbb{E} \left[ 1_{T_{N,M}(x)} F \left( S_{k_N}(x) - a_N \frac{\log N - k_N}{\log N} \right) 1_{T_{N,M}(y)} F \left( S_{k_N}(y) - a_N \frac{\log N - k_N}{\log N} \right) \right],$$

where

$$F(u) = \mathbb{P} \left( \tilde{h}(0) \geq a_N \frac{\log N - k_N}{\log N} - u \right) - e^{\pi \lambda b} \mathbb{P} \left( \tilde{h}(0) \geq a_N \frac{\log N - k_N}{\log N} - u + b \right)$$

with $\tilde{h}$ denoting the membrane model on $\Lambda_{k_N}$. Since $a_N k_N / \log N \sim 2\sqrt{2} \gamma k_N$ and $\mathbb{E}[\tilde{h}^2(0)] = \gamma k_N + O(1)$, assuming that $|u| < k_N^{7/8}$, thanks to (3.2), we then have

$$\mathbb{P} \left( \tilde{h}(0) \geq a_N \frac{\log N - k_N}{\log N} - u \right) \sim a_N \frac{\log N - k_N}{\log N} - u \quad \text{and} \quad \mathbb{P} \left( \tilde{h}(0) \geq a_N \frac{\log N - k_N}{\log N} - u + b \right) \sim e^{\pi \lambda b}$$

as $N \to \infty$. Since on $T_{N,M}(x)$ we have $|S_{k_N}(x) - a_N \log N - k_N| < k_N^{7/8}$ when $N$ is sufficiently large, we have $F(u) = o(1) \mathbb{P} \left( \tilde{h}(0) \geq a_N \frac{\log N - k_N}{\log N} - u \right)$ with $o(1) \to 0$ as $N \to \infty$ uniformly in $|u| < k_N^{7/8}$. Similarly, on $T_{N,M}(y)$ we have $|S_{k_N}(y) - a_N \log N - k_N| < k_N^{7/8}$ when $N$ is sufficiently large. Then we can show that the expectation is bounded by

$$o(1) \mathbb{P}(h^D_N(x) \geq a_N, h^D_N(y) \geq a_N, T_{N,M}(x), T_{N,M}(y)).$$

As in Proposition 3.4 the sum of the corresponding terms is thus at most $o(K_N^2)$ in this case.

We now consider the rest of the terms. Recall (4.5), if $x, y$ s.t. $m < \frac{3}{2} k_N$, then we can bound the corresponding term by $4e^{2\pi \lambda (b/2)} \mathbb{P}(h^D_N(x) \geq a_N + b \land 0)$. Thanks to Lemma 3.1, the sum of these terms
is at most \( o(K_N^2) \). If \( x, y \) satisfy \( m \geq \frac{3}{4}K_N \) but not the second condition in (4.6), then we can bound the corresponding term by \( 4e^{2\pi \lambda b(0)}P(h^{D_N}(x) \geq a_N - b \wedge 0, \eta^{D_N}(y) \geq a_N - b \wedge 0) \). Note that, for a fixed \( m \), the total number of pairs of this type is at most of order \( N^4e^{m+k_N} = o(1)N^4e^{2m} \), thanks to Proposition 3.4, the sum in this case is thus at most \( o(K_N^2) \) as well, thus completing the proof. \( \square \)

We can now prove Proposition 4.2.

**Proof of Proposition 4.2.** Let \( f(x, h) = 1_A(x)1_{[0, \infty)}(h) \). Lemma 4.3 is equivalent to

\[
\lim_{N \to \infty} \mathbb{E} \left| \langle \hat{\eta}_N^{D,M}, f \rangle - \langle \hat{\eta}_N^{D,M}, f_b \rangle \right| = 0, \quad b \in \mathbb{R}.
\]

Taking the distributional limit of \( \langle \hat{\eta}_N^{D,M}, f - f_b \rangle \), by Fatou’s lemma, we have \( \langle \hat{\eta}_N^{D,M}, f - f_b \rangle = 0 \) a.s. Then letting \( M \to \infty \), we complete the proof of the proposition. \( \square \)

The proposition now leads to the following factorization property:

**Proposition 4.4.** Let \( \eta^D \) be any sub-sequential limit of \( \eta_N^D \) along a deterministic subsequence \( \{N_k\} \).

Then, with probability one,

\[
\eta^D(dx dh) = Z_{\lambda}^D(dx) \otimes e^{-\pi \lambda b} dh \quad (4.8)
\]

with \( Z_{\lambda}^D \) a finite random Borel measure on \( \tilde{D} \).

**Proof.** Thanks to Proposition 4.2, we have \( \langle \eta^D, f - f_b \rangle = 0 \) a.s. for each function \( f : \tilde{D} \times \mathbb{R} \to \mathbb{R} \) of the form \( f(x, h) = 1_A(x)1_{[0, \infty)}(h) \) with \( A \subseteq D \) open and each \( b \in \mathbb{R} \). For a Borel set \( A \subseteq \tilde{D} \), let \( Z_{\lambda}^D(A) := \pi \lambda \eta^D(A \times [0, \infty)) \). If \( A \) is open then we have

\[
\eta^D(A \times [b, \infty)) = e^{-\pi \lambda b} \langle \eta^D, f_b \rangle = e^{-\pi \lambda b} \langle \eta^D, f \rangle = (\pi \lambda)^{-1} e^{-\pi \lambda b} Z_{\lambda}^D(A) = \int_{A \times [b, \infty)} Z_{\lambda}^D(dx) e^{-\pi \lambda b} dh
\]

holds almost surely, where the null set may depend on \( A \) or \( b \). In order to complete the proof, we can choose a common null set in the \( \pi \)-system \( \{A \times [b, \infty) : A \subseteq D \text{ open dyadic cube}, b \in \mathbb{Q}\} \) since it is countable. Note that the product Borel \( \sigma \)–algebra on \( \tilde{D} \times \mathbb{R} \) can be generated from the \( \pi \)–system above, and \( \eta^D(\partial A \times \mathbb{R}) = Z_{\lambda}^D(\partial D) = 0 \) thanks to Proposition 4.1, the equality then extends to all the Borel sets on \( \tilde{D} \times \mathbb{R} \). \( \square \)

5 **Uniqueness of the sub-sequential limit**

By the previous sections, any sub-sequential limit \( \eta^D \) of \( \{\eta_N^D : N \geq 1\} \) can be factorized as in Proposition 4.4. In this section, our aim is to show the uniqueness of the measure \( Z_{\lambda}^D \) on the right-hand side of (4.8), and thus also the sub-sequential limit \( \eta^D \), which particularly shows that \( \eta_N^D \) converges to the same distributional limit.

Recall \( \tilde{G}^D(x, y) = \tilde{G}_y^D(x) \) as in (2.2), and \( s_D(x) \) as in (2.4). Then, for two domains \( \tilde{D} \subseteq D \), the function

\[
C^{D, \tilde{D}}(x, y) := \tilde{G}^D(x, y) - \tilde{G}^{\tilde{D}}(x, y)
\]

defines a symmetric, positive semi-definite function from \( \tilde{D}^2 \) to \( \mathbb{R} \). This allow us to define \( \{\Phi^{D, \tilde{D}}(x) : x \in \tilde{D}\} \) to be a centered Gaussian field with covariance \( C^{D, \tilde{D}} \), which has smooth sample paths a.s. The following proposition shows that \( Z_{\lambda}^D \) satisfies some properties (in particular the fifth one, the Gibbs-Markov property in the limit) which, as we are going to show in Proposition 5.3, are sufficient to determine its law uniquely.

**Proposition 5.1.** Let \( \lambda \in (0, 1) \) and let \( \{\eta^D : D \in \mathcal{D}\} \) be the sub-sequential limit of \( \{\eta_N^D : N \geq 1\} \) along a deterministic subsequence \( \{N_k\} \) for \( D \in \mathcal{D} \). Let \( Z_{\lambda}^D \) be the measure factorized from \( \eta^D \), then \( \{Z_{\lambda}^D : D \in \mathcal{D}\} \) satisfies:

1. \( Z_{\lambda}^D \) is supported on \( D \); i.e., \( Z_{\lambda}^D(\mathbb{R}^4 \setminus D) = 0 \) a.s.
(2) For a measurable $A \subset D$ s.t. $\mu(A) = 0$, $Z^D_\lambda(A) = 0$ a.s.

(3) For each measurable $A \subset D$,

$$\mathbb{E}Z^D_\lambda(A) = \frac{\sqrt{\pi}}{4} \int_A e^{\frac{4x^2}{\lambda}(x)} \, dx.$$ 

(4) If $D, \tilde{D} \in \mathcal{D}$ such that $D \cap \tilde{D} = \emptyset$, then

$$Z^D_\lambda(dx) \overset{\text{law}}{=} Z^{\tilde{D}}_\lambda(dx) + Z^\tilde{D}_\lambda(dx).$$

(5) If $D, \tilde{D} \in \mathcal{D}$ such that $\tilde{D} \subset D$ and $\mu(D \setminus \tilde{D}) = 0$, then

$$Z^\tilde{D}_\lambda(dx) \overset{\text{law}}{=} e^{\pi \lambda \Phi^D(x)} Z^\tilde{D}_\lambda(dx).$$

(6) $Z^{a+D}(a + dx) \overset{\text{law}}{=} Z^D_\lambda(dx)$ for each $a \in \mathbb{R}^4$.

Proof. Property (1) is obvious by the definition of the point measure. Property (2) directly follows from Lemma 3.1. Property (3) holds for all $A \subset D$ open since

$$\mathbb{E}Z^D_\lambda(A) = \pi \lambda \mathbb{E}(\eta^D, f) = \pi \lambda \lim_{N \to \infty} \mathbb{E}(\eta^D_N, f) = \frac{\sqrt{\pi}}{4} \int_A e^{\frac{4x^2}{\lambda}(x)} \, dx$$

with $f(x, h) = 1_A(x)1_{[0, \infty)}(h)$ thanks to Lemma 3.2. Property (4) is trivial by decomposing $h^{D_N \cup \tilde{D}_N}$ by the independent sum of $h^{D_N}$ and $h^{\tilde{D}_N}$. Property (6) is trivially true by translation invariance.

Now we turn to property (5). Let $D, \tilde{D} \in \mathcal{D}$ with $\mu(D \setminus \tilde{D}) = 0$. Writing $h^{D_N} \overset{\text{law}}{=} h^{D_N} + \phi^{D_N, \tilde{D}_N}$ thanks to Theorem 2.5, if $f : \tilde{D} \times \mathbb{R} \to \mathbb{R}$ is continuous and compactly-supported in $\tilde{D}$, then

$$\langle \eta^D_N, f \rangle \overset{\text{law}}{=} \langle \tilde{\eta}_N, f \rangle$$

where

$$f_{\phi}(x, h) = f(x, h + \phi^{D_N, \tilde{D}_N}([xN]))$$

with $\phi^{D_N, \tilde{D}_N}$ independent of $h^{\tilde{D}_N}$. Using Theorem 2.4, we have:

$$G^{D_N}([xN], [yN]) - G^{\tilde{D}_N}([xN], [yN]) \rightarrow C^{D, \tilde{D}}(x, y)$$

as $N \to \infty$ locally uniformly in $x, y \in \tilde{D}$. Note that $\phi^{D_N, \tilde{D}_N}$ is a centered Gaussian process with covariance kernel $G^{D_N} - G^{\tilde{D}_N}$ and $\Phi^{D, \tilde{D}}$ is a centered Gaussian process with covariance kernel $C^{D, \tilde{D}}(x, y)$. By the same line of the proof in [2, Theorem B.14], for all $\delta > 0$, one can construct a coupling of $\phi^{D_N, \tilde{D}_N}$ and $\Phi^{D, \tilde{D}}$ s.t.

$$\lim_{N \to \infty} \mathbb{P} \left( \sup_{x \in D, d(x, D') > \delta} \left| \phi^{D_N, \tilde{D}_N}([xN]) - \Phi^{D, \tilde{D}}(x) \right| > \delta \right) = 0.$$ 

Since $f$ is continuous with compact support, letting $f^{\Phi}(x, h) = f(x, h + \Phi^{D, \tilde{D}}(x))$ with $\Phi^{D, \tilde{D}}$ independent of $\eta^D_N$, we thus have

$$\langle \tilde{\eta}_N, f_{\Phi} \rangle \overset{\text{law}}{=} \langle \tilde{\eta}_N, f_{\phi} \rangle + o(1)$$

with $o(1) \to 0$ in probability as $N \to \infty$. Since $x \to \Phi^{D, \tilde{D}}(x)$ is continuous on $\tilde{D}$ almost surely, for any $\eta^D$ (resp. $\tilde{\eta}^D$) that is a sub-sequential limit of $\{\eta^D_N : N \geq 1\}$ (resp. $\{\tilde{\eta}^D_N : N \geq 1\}$), we get

$$\langle \eta^D, f \rangle \overset{\text{law}}{=} \langle \tilde{\eta}^D, f_{\Phi} \rangle.$$ 

Then by Proposition 4.4, we have

$$\langle \tilde{\eta}^D, f_{\Phi} \rangle = \int_{D \times \mathbb{R}} f(x, h + \Phi^{D, \tilde{D}}(x)) Z^D_\lambda(dx) e^{-\pi \lambda h} \, dh = \int_{D \times \mathbb{R}} f(x, h) Z^\tilde{D}_\lambda(dx) e^{-\pi \lambda h - \Phi^{D, \tilde{D}}(x)} \, dh.$$
As the equality holds for any continuous \( f : D \times \mathbb{R} \to \mathbb{R} \) supported on \( \bar{D} \), we have

\[
Z_\lambda^\bar{D} (dx)e^{\pi \lambda \Phi^{\bar{D}}(x)} \overset{\text{law}}{=} Z_\lambda^D (dx),
\]

which is the desired result.

For uniqueness of \( Z_\lambda^D \), we claim that \( Z_{S_n} \) has the same law as the limit of the measures

\[
\begin{align*}
Y_{m}^{S_n}(dx) := & \frac{\sqrt{2}}{4} \sum_{j=1}^{16^m} e^{\pi \lambda \Phi^{S_n}_{\bar{S}_n,m}(x)} e^{2 \gamma^2 (s_{\bar{S}_n,m}(x)-s_{\bar{D}}(x))} 1_{S_{n+m,j}}(x) dx \\
& \text{as } m \to \infty,
\end{align*}
\]

where \( S_n = (0, 2^{-n}]^4 \), and \( \bar{S}_{n,m} := \cup_{j=1}^{16^m} S_{n+m,j} \) with \( S_{n+m,j} (j = 1, \ldots, 16^m) \) the collection of the translations of \( S_{n+m} \) decomposing from \( S_n \). The following lemma gives the (weak) limit of \( Y_{m}^{S_n} \).

**Lemma 5.2.** For each \( \lambda \geq 0 \), there exists \( Y_{\infty}^{S_n} \) an a.s. finite random measure, s.t. for every bounded, measurable \( f : \bar{D} \to \mathbb{R} \),

\[
\lim_{m \to \infty} \langle Y_{m}^{S_n}, f \rangle = \langle Y_{\infty}^{S_n}, f \rangle \quad \text{a.s.}
\]

**Proof.** We can write

\[
\Phi^{S_n}_{\bar{S}_n,m} := \sum_{j=1}^{m} \Phi^{\bar{S}_n,j}_{\bar{S}_n,j}(x),
\]

where \( \bar{S}_{n,0} := S_n \) by the covariance structure of the field, and the terms on the right-hand side are independent. Then we immediately get that \( \langle Y_{m}^{S_n}, f \rangle \) is a martingale adapted to the filtration

\[
\mathcal{F}_m := \sigma(\Phi^{\bar{S}_n,j}_{\bar{S}_n,j}(x) : x \in S'_n, j = 1, 2, \ldots, m) \quad \text{where } S'_n := \bigcap_{m \geq 1} \bar{S}_{n,m},
\]

since

\[
\sum_{j=1}^{16^m} e^{2 \gamma^2 (s_{\bar{S}_n,m}(x)-s_{\bar{D}}(x))} 1_{S_{n+m,j}}(x) = e^{2 \gamma^2 (s_{\bar{D}}(x)-s_{\bar{D}}(x))}, \quad x \in \bar{D},
\]

and, for any \( \bar{D} \subset D \),

\[
E e^{\pi \lambda \Phi^{\bar{D}}(x)} = e^{\frac{1}{2} \pi \lambda^2 C^\bar{D}(x)} = e^{\frac{1}{2} \pi \lambda^2 (s_{\bar{D}}(x)-s_{\bar{D}}(x))}, \quad x \in \bar{D}.
\]

Since \( f \) is bounded, by martingale convergence theorem, we can get

\[
L(f) := \lim_{m \to \infty} \langle Y_{m}^{S_n}, f \rangle
\]

exists almost surely. Then the rest of the proof follows from the proof of [3, Lemma 3.10].

The following Proposition give the uniqueness of the law of \( Z_\lambda^D \):}

**Proposition 5.3.** For \( \lambda \in (0, 1) \) and any \( n \in \mathbb{Z} \),

\[
Z_{\lambda}^{S_n}(dx) \overset{\text{law}}{=} Y_{\infty}^{S_n}(dx).
\]

**Proof.** For any bounded, measurable function \( f : S_n \to [0, \infty) \), we are going to show that

\[
E e^{-(Z_{\lambda}^{S_n}, f)} = E e^{-(Y_{\infty}^{S_n}, f)},
\]

which will be done by separately proving the “\( \geq \)” and “\( \leq \)” inequalities.

**Proof of “\( \geq \)”**. Thanks to properties (4) and (5) in Proposition 5.1, we can write

\[
Z_{\lambda}^{S_n}(dx) = \sum_{j=1}^{16^m} e^{\pi \lambda \Phi^{S_n}_{\bar{S}_n,m}(x)} 1_{S_{n+m,j}}(x) Z_{\lambda}^{S_{n+m,j}}(dx),
\]
where $Z^{S_{n \cdot m}^j}, 1 \leq j \leq 16^m$ are independent of others as well as of $\Phi^{S_n, \tilde S_{n \cdot m}}$. Then we have

$$E[(Z^{S_n}_\lambda, f)|\sigma(\Phi^{S_n, \tilde S_{n \cdot m}})] = (Y^{S_n}_m, f).$$

(5.3)

By Jensen’s inequality, we have

$$Ee^{-Z^{S_n}_\lambda, f} \geq Ee^{-Y^{S_n}_m, f}.$$ 

Let $m \to \infty$, by the bounded convergence theorem, we complete the proof of “$\geq$” in (5.2).

**Proof of “$\leq$”**. We now introduce another truncation: For $\delta \in (0, 1/2)$, let $\tilde S^{\delta}_k$ be the translate of $(\delta 2^{-k}, (1- \delta)2^{-k})^4$ with the same center as $S_k$. Let $\tilde S^{\delta}_{n+m, j}$ denote the truncated version of $S_{n+m, j}$, then let $\tilde S^{\delta}_{n, m} := \cup_{j=1}^{16^m} \tilde S^{\delta}_{n+m, j}$ and define $f_{m, \delta}(x) := 1_{\tilde S^{\delta}_{n, m}}(x) f(x)$. The positivity of $f$ then implies $f_{m, \delta} \uparrow f$ as $\delta \downarrow 0$, since $Y^{S_n}_m(S_n \setminus \tilde S^{\delta}_{n, m}) \to 0$, the monotone convergence theorem then shows $E(Y^{S_n}_\infty, f - f_{m, \delta}) \to 0$ as $\delta \downarrow 0, m \to \infty$. In particular, we have

$$E(Y^{S_n}_\infty, f_{m, \delta}) \to E(Y^{S_n}_\infty, f), \quad \delta \downarrow 0, m \to \infty.$$ 

So we can work with $f_{m, \delta}$ instead of $f$. We now define

$$\hat Z^{D, M}_\lambda(A) := \pi \lambda \hat h^{D, M}(A \times [0, \infty))$$

as the “truncated” version of $Z^{D}_\lambda$, and

$$\hat Z^{S, M, m}_\lambda(dx) = \sum_{j=1}^{16^m} e^{\pi \lambda \Phi^{S_{n \cdot m}, M}(x)} 1_{S_{n \cdot m}, j}(x) \hat Z^{S_{n \cdot m}, M}_\lambda(dx).$$

Thanks to Proposition 3.3 we have

$$\hat Z^{D, M}_\lambda(A) \leq Z^{D}_\lambda(A), \quad \hat Z^{D, M}_\lambda(A) \uparrow Z^{D}_\lambda(M \to \infty).$$

For any bounded, measurable $f : \hat D \to \mathbb{R}$ and $\delta > 0$, we then have

$$E e^{-Z^{S_n}_\lambda, f} \leq E e^{-\hat Z^{S_n, M, m}_\lambda, f} \leq E e^{-\hat Z^{S_n, M, m}_\lambda, f_{m, \delta}}.$$ 

Now set

$$\tilde X_j := \int_{\tilde S^{\delta}_{n \cdot m, j}} e^{\pi \lambda \Phi^{S_n, \tilde S_{n \cdot m}, M}} f_{m, \delta}(x) \hat Z^{S_{n \cdot m}, M}_\lambda(dx).$$

By the fact that $\langle \hat Z^{S, M, m}_\lambda, f_{m, \delta} \rangle = \sum_{j=1}^{16^m} \tilde X_j$, thanks to [3, Lemma 3.12], we then get for each $\varepsilon > 0$,

$$E e^{-\hat Z^{S_n, M, m}_\lambda, f_{m, \delta}} \leq E \left[ \exp \left(-e^{-\varepsilon} \sum_{j=1}^{16^m} \tilde X_j 1_{\{X_j \leq \varepsilon\}} |\Phi^{S_n, \tilde S_{n \cdot m}}| \right) \right].$$

(5.4)

Then we need the following lemma:

**Lemma 5.4.** Suppose $\lambda \in (0, 1), \delta \in (0, 1/2)$, for each $\varepsilon > 0$ we have:

$$\sum_{j=1}^{16^m} E[\tilde X_j 1_{\{X_j > \varepsilon\}} |\Phi^{S_n, \tilde S_{n \cdot m}}] \to 0, \quad m \to \infty$$

(5.5)

in probability.

**Proof.** We need to deal with the maximum of the field $\Phi^{S_n, \tilde S_{n \cdot m}}$. It follows the same line of the proof of
Then, let $A_{n,m}$ be the event in (5.6) and applying the uniform bound

$$\text{var}[\Phi S_n, S_n, \gamma(x)] \leq c + \gamma \log(2^m) \tag{5.7}$$

on $A_{n,m}$, we have

$$E[1_{A_{n,m}} \widetilde{X}^j 1_{(X_j > \varepsilon)} | \Phi S_n, \tilde{S}_n, \gamma] \leq \frac{1}{\varepsilon} E[1_{A_{n,m}} \widetilde{X}^j 2 | \Phi S_n, \tilde{S}_n, \gamma] \leq \frac{\|f\|^2}{\varepsilon} e^{4\lambda \log(2^m) + c(\delta) \sqrt{\log(2^m)} e^{8\lambda^2 \log(2^m)}} E[\tilde{Z}^{S_n, M, M}_{\lambda}(S_{n+m, j})^2].$$

Then, Proposition 3.4 and Lemma 3.2 ensure that, for some $c, c' = c(M, n, \delta),

$$E[\tilde{Z}^{S_n, M, M}_{\lambda}(S_{n+m, j})^2] \leq c E[Z^{S_n, M, M}_{\lambda}(S_{n+m, j})]^2 \leq c' \cdot 2^{-m(8+8\lambda^2)}.$$

For any $\zeta > 0$, we thus get

$$\mathbb{P}\left\{ \sum_{j=1}^{16^m} E[\widetilde{X}^j 1_{(X_j > \varepsilon)} | \Phi S_n, \tilde{S}_n, \gamma] \geq \zeta \right\} \leq \mathbb{P}(A_{n,m}) + \frac{c'}{4 \zeta} 2^{-4m(1-\lambda)^2} e^{c(\delta) \sqrt{\log(2^m)}}.$$

Thanks to (5.6), let $m \to \infty$, we finish the proof of Lemma 5.4. \hfill \square

Now come back to the proof of "$\geq". Using (5.4) and (5.5) together, we get

$$\limsup_{m \to \infty} \mathbb{E}e^{-\langle \tilde{Z}^{S_n, M, m}_{\lambda}, f, m, i \rangle} \leq \limsup_{m \to \infty} \mathbb{E} \left[ \exp \left( -e^{-\varepsilon} E[(\tilde{Z}^{S_n, M, M}_{\lambda}, f, m, \delta) | \Phi S_n, \tilde{S}_n, \gamma] \right) \right] \tag{5.8}$$

We now need to prove

$$E[(\tilde{Z}^S_{\lambda}, f, m, \delta) | \Phi S_n, \tilde{S}_n, \gamma] - E[(\tilde{Z}^{S_n, M, m}_{\lambda}, f, m, \delta) | \Phi S_n, \tilde{S}_n, \gamma] \to 0 \tag{5.9}$$

in probability as $m \to \infty$ and then taking $M \to \infty$. Then we will prove (5.9) by showing the left-hand side, after taking expectation, converges to 0.

Invoking (5.7), by the property of conditional expectation, the expectation of the L.H.S. of (5.9) can be bounded by

$$E(\tilde{Z}^S_{\lambda} - \tilde{Z}^{S_n, M}_{\lambda}, f, m, \delta) \leq c \cdot 2^{m(4\lambda^2 + 4)} \|f\| E[\tilde{Z}^{S_n, m}(S_{n+m}) - \tilde{Z}^{S_n, M}_{\lambda}(S_{n+m})].$$

Proposition 3.3 yields

$$E[\tilde{Z}^{S_n, m}(S_{n+m}) - \tilde{Z}^{S_n, M}_{\lambda}(S_{n+m})] \leq c e^{-\varepsilon M^2 2^{-m(4+4\lambda^2)}},$$

which proves (5.9). Recall (5.3), taking $M \to \infty$ and $\varepsilon \downarrow 0$, using (5.9) in (5.8), we finally get

$$\mathbb{E}e^{-\langle \tilde{Z}^S_{\lambda}, f, \rangle} \leq \mathbb{E}e^{-\langle Y^S_{\infty}, f, m, \delta \rangle},$$

Then taking $m \to \infty$ and $\delta \downarrow 0$, we have

$$\mathbb{E}e^{-\langle \tilde{Z}^S_{\lambda}, f, \rangle} \leq \mathbb{E}e^{-\langle Y^S_{\infty}, f, \rangle}.$$

We now finish the proof of "$\leq". Together with the proof of "$\geq"", we complete the proof of Proposition 5.3. \hfill \square
We can finally prove Theorem 1.1:

**Proof of Theorem 1.1.** Proposition 4.4 shows that any sub-sequential limit $\eta^D$ can be factorized as $\eta^D(dx) = Z_D^\lambda(dx) \otimes e^{-\pi \lambda h} dh$, where $Z_D^\lambda$ measures obey properties (1-6) from Proposition 5.1. And Proposition 5.3 finally determined the law of $Z_D^\lambda$ for $D$ being any dyadic cube, particularly for $D = (0, 1)^4$, which completes the proof. \[\square\]

### 6 Characterization of the limit measure

In this part, we will give the proof of Theorem 1.3, which shows that the limit measure $Z_D^\lambda$ has the law of the Gaussian multiplicative chaos on $D$.

**Proof of Theorem 1.3.** In fact, we can rewrite $Y_{S_n}^{S_n}$ defined in (5.1) as

$$Y_{S_n}^{S_n} = \frac{\sqrt{\pi}}{4} \sum_{j=1}^{16} e^{\pi x \Phi_{S_n, \tilde{S}_n, m}(x) - \frac{1}{2} \pi^2 \lambda^2 \mathbb{E}[\Phi_{S_n, \tilde{S}_n, m}(x)^2]} dx.$$ 

Recall that $\Phi_{S_n, \tilde{S}_n, m}$ is the sum of independent fields:

$$\Phi_{S_n, \tilde{S}_n, m} = \sum_{j=1}^{m} \Phi_{S_n, \tilde{S}_n, j} - \frac{1}{2} \pi^2 \lambda^2 \mathbb{E}[\Phi_{S_n, \tilde{S}_n, m}(x)^2].$$

From now on, for brevity of notation, we will write $S$ (resp. $S_j$) instead of $S_n$ (resp. $S_{n,j}$). Now, let $H_k$ denote the subspace of $H^2_{S_n}$, which contains the functions that are biharmonic in $S_k$, with Dirichlet boundary conditions on $\partial S_{k-1}$. We claim that:

$$H^2_{S_n} = \bigoplus_{k=0}^\infty H_k.$$ 

First of all, we need to show that $H_i$ and $H_j$ are orthogonal to each other for all $i \neq j$. Invoking Gauss-Green formula, for $f \in H_i, g \in H_j$ (i < j), we have:

$$\langle f, g \rangle_{\Delta} = \int_S \Delta f g dx = \int_{S_i} g \Delta^2 f dx + \int_{\partial S_j} (\hat{\Delta} f \frac{\partial g}{\partial n} - g \frac{\partial (\hat{\Delta} f)}{\partial n}) d\sigma = 0$$

since $\hat{\Delta}^2 f = 0$ in $S_i$ and $g = \frac{\partial g}{\partial n} = 0$ on $\partial S_i$, thus we prove the orthogonality. Then we need to show that $H^2_{S_n} \setminus \bigoplus_{k=0}^\infty H_k$ is trivial. Assuming $f$ is in the orthocomplement space of $\bigoplus_{k=0}^\infty H_k$, then Gauss-Green formula implies that $f$ is the weak solution of the following boundary value problem:

$$\begin{cases} 
\hat{\Delta}^2 f(x) = 0, & x \in S_\infty, \\
f(x) = 0, & x \in \partial S_\infty, \\
\frac{\partial f}{\partial n}(x) = 0, & x \in \partial S_\infty, 
\end{cases} \quad (6.1)$$

where $S_\infty := \bigcap_{k=0}^\infty S_k$. Thanks to the uniqueness of the solution of (6.1), we conclude that $f = 0$ almost everywhere, and thus complete the proof of the claim above.

Then, let $\{\varphi_{k,j}\}$ denote the orthonormal basis of $H_{k-1}$, a straightforward calculation of covariance shows that:

$$\Phi_{S_{k-1}, S_k} \text{ law } = \sum \varphi_{k,j} Z_{k,j}.$$ 

19
with $Z_{k,j}$ i.i.d. standard normals. Then we get:

$$\Phi_{S,S} \stackrel{law}{=} \sum_{k=0}^{i-1} \varphi_{k,j} Z_{k,j}. \tag{A.1}$$

Denoting $\mathcal{F}_k := \sigma(Z_{i,j} : i + j \leq k)$, we immediately get:

$$E[Y^S_{m}(A) | \mathcal{F}_k] = \int_A \sqrt{\frac{\pi}{4}} e^{-\frac{\Delta^2 \mu S \pi \lambda}{\mu S (k-1)/2}} (dx), \quad m \geq k,$$

where $\mu^{S,\beta}_n$ is the Gaussian multiplicative chaos measure defined through the orthonormal basis $\{\varphi_{k,j}\}_{k,j}$ rearranged through the complete order:

$$(k,j) \preceq (k',j') \Leftrightarrow k < k' \text{ or } k = k', j < j'.$$

Letting $m \to \infty$, since we have proved $Y^S_{m}(A) \to Y^S_{\infty}(A)$ a.s. in Lemma 5.2, we get:

$$E[Y^S_{\infty}(A) | \mathcal{F}_k] = \int_A \sqrt{\frac{\pi}{4}} e^{-\frac{\Delta^2 \mu S \pi \lambda}{\mu S (k-1)/2}} (dx).$$

Then, letting $k \to \infty$ on both sides, by the triviality of $\mathcal{F}_\infty := \sigma(\bigcup_{k \geq 1} \mathcal{F}_k)$, we get:

$$Y^S_{\infty}(A) = \int_A \sqrt{\frac{\pi}{4}} e^{-\frac{\Delta^2 \mu S \pi \lambda}{\mu S (k-1)/2}} (dx).$$

By the arbitrariness of $A$, we complete the proof since $Y^S_{\infty}$ and $Z^S_{\lambda}$ has the same law. \hfill \Box

### A Appendix - the Gibbs-Markov property

In this Appendix, we will give a detailed proof of the Gibbs-Markov property of the membrane model (Theorem 2.5). For a given finite set $V \subset \mathbb{Z}^d$, consider the Hilbert space $H^V := \{ f : \mathbb{Z}^d \to \mathbb{R} : \text{supp}(f) \subset V \}$ endowed with the inner product

$$(f,g)_\Delta := \sum_{x \in \mathbb{Z}^d} \Delta f(x) \cdot \Delta g(x).$$

We then have the following lemma:

**Lemma A.1.** For the setting as above with $V$ finite, let $\{\varphi_n : n = 1, 2, \cdots, |V|\}$ be an orthonormal basis in $H^V$ and let $Z_1, \cdots, Z_{|V|}$ be i.i.d. standard normals. Then

$$\tilde{h}^V_x := \sum_{n=1}^{|V|} \varphi_n(x) Z_n, \quad x \in \mathbb{Z}^d \tag{A.1}$$

has the law of the membrane model in $V$.

**Proof.** Since $\{\tilde{h}^V_x\}_{x \in \mathbb{Z}^d}$ is a centered Gaussian vanishing outside $V$ and

$$E[\tilde{h}^V_x \tilde{h}^V_y] = \sum_{n=1}^{|V|} \varphi_n(x) \varphi_n(y),$$

it suffices to prove that

$$G^V(x,y) = \sum_{n=1}^{|V|} \varphi_n(x) \varphi_n(y).$$
Fixing $y$ on the left-hand side, the right-hand side can be seen as the Fourier expansion of $G^V(x, y)$. Assume that

$$ G^V(x, y) = \sum_{n=1}^{\|V\|} c_n \varphi_n(x), $$

then

$$ c_n = \sum_{x \in \mathbb{Z}^d} \Delta G^V(x, y) \cdot \Delta \varphi_n(x) = \sum_{x \in \mathbb{Z}^d} \Delta^2 G^V(x, y) \cdot \varphi_n(x) = \sum_{x \in \mathbb{Z}^d} \delta_y(x) \varphi_n(x) = \varphi_n(y). $$

Since $y$ is arbitrary, we complete the proof of the lemma.

Then we will prove a decomposition lemma, which leads to the proof of Theorem 2.5.

**Lemma A.2.** Let $U \subset V \subset \mathbb{Z}^d$, and $\mathcal{H}^V, \mathcal{H}^U$ as above. Let $\text{Bih}(U)$ be the space of discrete biharmonic functions in $U$, then we have:

$$ \mathcal{H}^V = \mathcal{H}^U \oplus \text{Bih}(U). $$

**Proof.** First observe that any functions $f \in \mathcal{H}^U$ is orthogonal to any functions in $\text{Bih}(U)$ by the Gauss-Green formula in the discrete case. Then for all $f \in \mathcal{H}^V$, letting $f_0$ be the orthogonal projection of $f$ onto $\mathcal{H}^U$, it suffices to show that $\varphi = f - f_0$ is discrete biharmonic on $U$. Note that $\varphi$ is orthogonal to $\mathcal{H}^U$ by definition, then for any test function $\psi \in \mathcal{H}^U$, by the Gauss-Green formula in the discrete case, we have

$$ \sum_{x \in U} (\Delta^2 \varphi(x)) \psi(x) = \sum_{x \in \mathbb{Z}^d} \Delta \varphi(x) \cdot \Delta \psi(x) = 0. $$

Since $\psi$ is arbitrary, $\varphi$ is discrete biharmonic in $U$. We thus complete the proof.

By the two lemmas above, we can now prove Theorem 2.5.

**Proof of Theorem 2.5.** Let $\{f_n\}$ be an orthonormal basis of $\mathcal{H}^U$, and $\{\phi_n\}$ be an orthonormal basis of $\text{Bih}(U)$. We can write

$$ \tilde{h}^U := \sum X_n f_n, \quad \varphi^{V, U} := \sum Y_n \phi_n, $$

where $X_n, Y_n$ are i.i.d. standard Gaussian variables. Then by Lemma A.1, $\tilde{h}^U$ has the law of the membrane model in $U$. Thanks to Lemma A.2, $\{f_n, \phi_n\}$ is an orthonormal basis of $\mathcal{H}^V$, then again invoking Lemma A.1, $\tilde{h}^U + \varphi^{V, U}$ has the law of the membrane model in $V$, since $X_n$ and $Y_n$ are independent, we get the independence between $\tilde{h}^U$ and $\varphi^{V, U}$. Since $\phi_n \in \text{Bih}(U)$, we prove that $\varphi^{V, U}$ is discrete biharmonic in $U$.

For (2.9), note that $\varphi^{V, U}$ has the law of the biharmonic extension of $h^V(x) (x \in V \setminus U)$ in $U$ since $\phi_n \in \text{Bih}(U)$ and $h^U$ vanishes outside $U$. In order to complete the proof, we only need to show that $\mathbb{E}[h^V | \sigma(h^V_z : z \in V \setminus U)]$ is discrete biharmonic on $U$. Fix $x \in U$, recall Definition 2.1, since

$$ \sum_{y \in \mathbb{Z}^d} |\Delta h_y|^2 = \left(1 + \frac{1}{2d}d\right) h^2_x + h_x \left(\frac{1}{d} \sum_i h_{x\pm e_i} + \frac{1}{d^2} \sum_{i \neq j} h_{x\pm e_i\pm e_j} + \frac{1}{2d^2} \sum_i h_{x\pm 2e_i}\right) + \cdots, $$

where $h_x$ does not occur in $\cdots$ terms. A simple calculation shows that:

$$ \mathbb{E}[h^V_x | \sigma(h^V_z : z \in V \setminus \{x\})] = \frac{d}{2d + 1} \left(\frac{1}{d} \sum_i h_{x\pm e_i} - \frac{1}{d^2} \sum_{i < j} h_{x\pm e_i\pm e_j} - \frac{1}{2d^2} \sum_i h_{x\pm 2e_i}\right). $$

Thanks to the “smaller $\sigma$-algebra always wins” property of conditional expectation, we have

$$ \frac{2d + 1}{2d} \mathbb{E}[h^V | \sigma(h^V_z : z \in V \setminus U)] = \mathbb{E}[\mathbb{E}[h^V | \sigma(h^V_z : z \in V \setminus \{x\})] | \sigma(h^V_z : z \in V \setminus U)] $$

$$ = \mathbb{E}\left[\frac{1}{2d} \sum_i h_{x\pm e_i} - \frac{1}{2d^2} \sum_{i < j} h_{x\pm e_i\pm e_j} - \frac{1}{4d^2} \sum_i h_{x\pm 2e_i} | \sigma(h^V_z : z \in V \setminus U)\right]. $$

By the explicit form of the discrete bilaplacian operator, and the arbitrariness of $x \in U$, we prove that $\mathbb{E}[h^V | \sigma(h^V_z : z \in V \setminus U)]$ is discrete biharmonic on $U$, and thus complete the proof.
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