Explicit determinantal formulas for solutions to the generalized Sylvester quaternion matrix equation and its special cases.

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Abstract

Within the framework of the theory of quaternion column-row determinants and using determinantal representations of the Moore-Penrose inverse previously obtained by the author, we get explicit determinantal representation formulas of solutions (analogs of Cramer’s rule) to the quaternion two-sided generalized Sylvester matrix equation \( A_1X_1B_1 + A_2X_2B_2 = C \) and its all special cases when its first term or both terms are one-sided. Finally, we derive determinantal representations of two like-Lyapunov equations.

Keywords Matrix equation; Sylvester matrix equation; Lyapunov matrix equation; Cramer Rule; quaternion matrix; noncommutative determinant

Mathematics subject classifications 15A24, 15A09, 15A15, 15B33.

1 Introduction

Let \( \mathbb{H}^{m \times n} \) and \( \mathbb{H}^{m \times n}_r \) stand for the set of all \( m \times n \) matrices and matrices with rank \( r \), respectively, over the quaternion skew field \( \mathbb{H} = \{ a_0 + a_1i + a_2j + a_3k \mid i^2 = j^2 = k^2 = -1, a_0, a_1, a_2, a_3 \in \mathbb{R} \} \), where \( \mathbb{R} \) is the real number field.

In this paper, we investigate the two-sided coupled generalized Sylvester matrix equation over \( \mathbb{H} \),

\[
A_1X_1B_1 + A_2X_2B_2 = C. \tag{1.1}
\]

Since Sylvester-type matrix equations have wide applications in several fields (see, e.g. \[3\]–\[5\], \[11\]–\[13\], \[52\]), these equations are thoroughly studied and there are many important results about them (see, e.g. \[6\]–\[8\], \[14\]–\[16\], \[25\]–\[27\], \[46\]–\[48\], \[53\]). Mansour \[28\] studied the solvability condition of (1.1) in the operator algebra. Liping \[26\] has begun investigations of a similar equation over the quaternion skew field. Baksalary and Kala \[2\] derived the general solution to (1.1) expressed in terms of generalized inverses that has been extended to the quaternion skew field in \[14\]–\[15\]. Quaternion matrix equations similar to Eq. (1.1) have been recently investigated by many authors (see, e.g. \[16\]–\[18\], \[31\]–\[33\], \[36\]–\[38\], \[51\]).

The main goal of this paper is to derive determinantal representations of the general solution to Eq. (1.1) and its all special cases over the quaternion skew field using previously obtained determinantal representations of the Moore-Penrose inverse. Evidently, determinantal representation of a solution of a matrix equation (which is expressed in terms of generalized inverses) gives a direct method of its finding analogous to classical Cramer’s rule (when a solution is
expressed by an usual inverse) that has important theoretical and practical significance [12].

For $A \in \mathbb{H}^{m \times n}$, the symbol $A^*$ stands for conjugate transpose (Hermitian adjoint) of $A$. A matrix $A \in \mathbb{H}^{n \times n}$ is Hermitian if $A^* = A$. The definition of the Moore-Penrose inverse matrix has been extended to quaternion matrices as follows.

**Definition 1.1.** The Moore-Penrose inverse of $A \in \mathbb{H}^{m \times n}$, denoted by $A^\dagger$, is the unique matrix $A^\dagger \in \mathbb{H}^{n \times m}$ satisfying the following four equations

1. $AA^\dagger A = A$, 2. $A^\dagger AA^\dagger = A^\dagger$, 3. $(AA^\dagger)^* = AA^\dagger$, 4. $(A^\dagger A)^* = A^\dagger A$.

The problem for determinantal representation of quaternion generalized inverses as well as solutions and generalized inverse solutions of quaternion matrix equations only now can be solved due to the theory of column-row determinants introduced in [13,14]. Within the framework of the theory of column-row determinants, determinantal representations of various generalized quaternion inverses and generalized inverse solutions to quaternion matrix equations have been derived by the author (see, e.g. [15–24]) and by other researchers (see, e.g. [37–40]).

The paper is organized as follows. In Subsection 2, we start with introduction of row-column determinants, and determinantal representations of the Moore-Penrose inverse and of solutions to the quaternion matrix equation $AXB = C$ and its special cases previously obtained within the framework of the theory of row-column determinants. The explicit determinantal representation solution to Eq. (1.1) is derived in Section 3. In Subsection 4, we give Cramer’s rules to special cases of (1.1) when only one term of the equation is two-sided. In Subsection 5 and 6, we get Cramer’s rules to special cases of (1.1) when both terms of the equation are one-sided, and to some two similar Lyapunov equations, respectively. A numerical example to illustrate the main results is considered in Section 7. Finally, in Section 8, the conclusions are drawn.

# 2 Preliminaries. Elements of the theory of row-column determinants.

For $A \in \mathbb{H}^{n \times n}$, we define $n$ row determinants and $n$ column determinants. Suppose $S_n$ is the symmetric group on the set $I_n = \{1, \ldots, n\}$.

**Definition 2.1.** The $i$th row determinant of $A = (a_{ij}) \in \mathbb{H}^{n \times n}$ is defined for all $i = 1, \ldots, n$ by putting

$$
\text{rdet}_i A = \sum_{\sigma \in S_n} (-1)^{n-r} (a_{i_1 i_2} a_{i_1 i_1+1} \ldots a_{i_{k_1} i_{l_1}} \ldots (a_{i_{k_r} i_{s_1}+1} \ldots a_{i_{k_r}+s i_{k_r}}),
$$

where $i_{k_1} < i_{k_2} < \cdots < i_{k_r}$ and $i_{k_t} < i_{k_t+s}$ for all $t = 2, \ldots, r$ and $s = 1, \ldots, l_t$. 


Definition 2.2. The $j$th column determinant of $A = (a_{ij}) \in \mathbb{H}^{n \times n}$ is defined for all $j = 1, \ldots, n$ by putting
\[
\text{cdet}_j A = \sum_{\tau \in \mathcal{S}_n} (-1)^{n-r} (a_{j_{k_r} j_{k_r+1}} \cdots a_{j_{k_{r+1}} j_{k_r}}) \cdots (a_{j_{k_{t+1}} j_{k_t}} \cdots a_{j_{k_1+1} j_{k_1}} a_{j_{k_t} j_{k_1}}),
\]
where $j_{k_1} < j_{k_2} < \cdots < j_{k_r}$ and $j_{k_t} < j_{k_{t+1}}$ for $t = 2, \ldots, r$ and $s = 1, \ldots, l_i$.

Theorem 2.3. Let $A = (a_{ij})$ be a submatrix of $\mathbb{H}^{m \times n}$ whose rows are indexed by $\alpha$ and whose columns are indexed by $\beta$. Similarly, let $A^\alpha_{\beta}$ be a principal submatrix of $A$ whose rows and columns are indexed by $\alpha$. If $A$ is Hermitian, then $|A|_{\alpha_{\beta}}$ denotes the corresponding principal minor of det $A$. For $1 \leq k \leq n$, the collection of strictly increasing sequences of $k$ integers chosen from $\{1, \ldots, n\}$ is denoted by $L_{k,n} := \{\alpha : \alpha = (\alpha_1, \ldots, \alpha_k), 1 \leq \alpha_1 < \cdots < \alpha_k \leq n\}$. For fixed $i \in \alpha$ and $j \in \beta$, let $I_{r,m}\{i\} := \{\alpha : \alpha \in L_{r,m}, i \in \alpha\}$, $J_{r,n}\{j\} := \{\beta : \beta \in L_{r,n}, j \in \beta\}$.

Let $a_{ij}$ be the $j$th column and $a_{ij}$, be the $i$th row of $A$. Suppose $A_{\alpha, \beta}$ (b) denotes the matrix obtained from $A$ by replacing its $j$th column with the column b, and $A_{\alpha, \beta}$, denotes the matrix obtained from $A$ by replacing its $i$th row with the row b. Denote the $j$th column and the $i$th row of $A^*$ by $a^*_{ij}$ and $a^*_{ji}$, respectively.

Theorem 2.3. [13] If $A \in \mathbb{H}^{m \times n}$, then the Moore-Penrose inverse $A^\dagger = (a^\dagger_{ij}) \in \mathbb{H}^{m \times n}$ have the following determinantal representations
\[
a^\dagger_{ij} = \sum_{\beta \in J_{r,n}\{i\}} \frac{\text{rdet}_i ((A^*A)_i (a^*_{ij}))^\beta}{\sum_{\beta \in J_{r,n}} |A^*A|_\beta^\beta}, \quad (2.1)
\]
or
\[
a^\dagger_{ij} = \sum_{\alpha \in I_{r,m}\{j\}} \frac{\text{rdet}_j ((AA^*)_j (a^*_{ij}))^\alpha}{\sum_{\alpha \in I_{r,m}} |AA^*|_\alpha^\alpha}. \quad (2.2)
\]

Remark 2.4. Note that for an arbitrary full-rank matrix, $A \in \mathbb{H}^{m \times n}$, a column-vector $d_{\alpha, j}$, and a row-vector $d_{i, k}$, with appropriate sizes, we put, respectively, if
If $r = n$, then
\[
\text{cdet}_i((\mathbf{A}^*\mathbf{A})_i(\mathbf{d}_j)) = \sum_{\beta \in J_{n,n}(i)} \text{cdet}_i((\mathbf{A}^*\mathbf{A})_i(\mathbf{d}_j))_\beta,
\]
\[
\text{det}(\mathbf{A}^*\mathbf{A}) = \sum_{\beta \in J_{n,n}} |\mathbf{A}^*\mathbf{A}|^\beta_
\]
if $r = m$, then
\[
\text{rdet}_j((\mathbf{A}\mathbf{A}^*)_j(\mathbf{d}_i)) = \sum_{\alpha \in I_{m,m}(j)} \text{rdet}_j((\mathbf{A}\mathbf{A}^*)_j(\mathbf{d}_i))_\alpha,
\]
\[
\text{det}(\mathbf{A}\mathbf{A}^*) = \sum_{\alpha \in I_{m,m}} |\mathbf{A}\mathbf{A}^*|^\alpha_m.
\]

**Corollary 2.5.** If $\mathbf{A} \in \mathbb{H}_{m \times n}^r$, then the projection matrix $\mathbf{A}^!\mathbf{A} =: \mathbf{P}_\mathbf{A} = (p_{ij})_{n \times n}$ have the determinantal representation
\[
p_{ij} = \frac{\sum_{\beta \in J_{r,n}(i)} \text{cdet}_i((\mathbf{A}^*\mathbf{A})_i(\hat{\mathbf{a}}_j))_\beta}{\sum_{\beta \in J_{r,n}} |\mathbf{A}^*\mathbf{A}|^\beta_i},
\]
where $\hat{\mathbf{a}}_j$ denotes the $j$th column of $\mathbf{A}^*\mathbf{A} \in \mathbb{H}_{n \times n}^r$.

**Corollary 2.6.** If $\mathbf{A} \in \mathbb{H}_{m \times n}^r$, then the projection matrix $\mathbf{A}\mathbf{A}^! =: \mathbf{Q}_\mathbf{A} = (q_{ij})_{m \times m}$ have the determinantal representation
\[
q_{ij} = \frac{\sum_{\alpha \in I_{m,m}(j)} \text{rdet}_j((\mathbf{A}\mathbf{A}^*)_j(\hat{\mathbf{a}}_i))_\alpha}{\sum_{\alpha \in I_{m,m}} |\mathbf{A}\mathbf{A}^*|^\alpha_m},
\]
where $\hat{\mathbf{a}}_i$ denotes the $i$th row of $\mathbf{A}\mathbf{A}^* \in \mathbb{H}_{m \times m}^r$.

The following important orthogonal projectors $\mathbf{L}_\mathbf{A} := \mathbf{I} - \mathbf{A}^!\mathbf{A}$ and $\mathbf{R}_\mathbf{A} := \mathbf{I} - \mathbf{A}\mathbf{A}^!$ induced from $\mathbf{A}$ will be used below.

**Theorem 2.7.** [44] Let $\mathbf{A} \in \mathbb{H}_{m \times n}^r$, $\mathbf{B} \in \mathbb{H}_{r \times s}^r$, $\mathbf{C} \in \mathbb{H}_{m \times s}$ be known and $\mathbf{X} \in \mathbb{H}_{n \times r}$ be unknown. Then the matrix equation
\[
\mathbf{AXB} = \mathbf{C}
\]
is consistent if and only if $\mathbf{A}\mathbf{A}^!\mathbf{B}\mathbf{B} = \mathbf{C}$. In this case, its general solution can be expressed as
\[
\mathbf{X} = \mathbf{A}^!\mathbf{B} + \mathbf{L}_\mathbf{A}\mathbf{V} + \mathbf{W}\mathbf{B},
\]
where $\mathbf{V}, \mathbf{W}$ are arbitrary matrices over $\mathbb{H}$ with appropriate dimensions.
Theorem 2.8. \[10\] Let \(A \in \mathbb{H}_k^{m \times n}, \ B \in \mathbb{H}_r^{p \times s}\). Then the partial solution \(X^0 = A^1CB^1 = (x_{ij}^0) \in \mathbb{H}^{n \times r}\) have determinantal representations,

\[
x_{ij}^0 = \sum_{\beta \in J_{1,n}\{i\}} \frac{\text{cdet}_i \left((A^*A)_{j} (\mathbf{d}^B)\right)_{\beta}}{\sum_{\beta \in J_{1,n}} |A^*A|_{\beta} \sum_{\alpha \in I_{r,m}} |BB^*|_{\alpha}^m},
\]
or

\[
x_{ij}^0 = \sum_{\alpha \in I_{r,m}\{j\}} \frac{\text{rdet}_j \left((BB^*)_{i} (\mathbf{d}^A)\right)_{\alpha}}{\sum_{\beta \in J_{1,n}} |A^*A|_{\beta} \sum_{\alpha \in I_{r,m}} |BB^*|_{\alpha}^m}.
\]

where

\[
d_{ij}^B = \left[\sum_{\alpha \in I_{r,m}\{j\}} \text{rdet}_j \left((BB^*)_{i} (\mathbf{c}_k)\right)_{\alpha}\right] \in \mathbb{H}^{n \times 1}, \ k = 1, \ldots, n,
\]

\[
d_{ij}^A = \left[\sum_{\beta \in J_{1,n}\{i\}} \text{cdet}_i \left((A^*A)_{i} (\mathbf{c}_l)\right)_{\beta}\right] \in \mathbb{H}^{1 \times r}, \ l = 1, \ldots, r,
\]

are the column vector and the row vector, respectively. \(\mathbf{c}_i\) and \(\mathbf{c}_j\) are the \(i\)th row and the \(j\)th column of \(C = A^*CB^*\).

Corollary 2.9. Let \(A \in \mathbb{H}_k^{m \times n}, \ C \in \mathbb{H}_r^{p \times s}\) be known and \(X \in \mathbb{H}^{n \times r}\) be unknown. Then the matrix equation \(AX = C\) is consistent if and only if \(AA^1C = C\). In this case, its general solution can be expressed as \(X = A^1C + L_AV\), where \(V\) is an arbitrary matrix over \(\mathbb{H}\) with appropriate dimensions. The partial solution \(X^0 = A^1C\) has the following determinantal representation,

\[
x_{ij}^0 = \sum_{\beta \in J_{1,n}\{i\}} \frac{\text{cdet}_i \left((A^*A)_{i} (\mathbf{c}_j)\right)_{\beta}}{\sum_{\beta \in J_{1,n}} |A^*A|_{\beta}^m}.
\]

where \(\mathbf{c}_j\) is the \(j\)th column of \(C = A^*C\).

Corollary 2.10. Let \(B \in \mathbb{H}_r^{p \times s}, \ C \in \mathbb{H}_r^{p \times s}\) be given, and \(X \in \mathbb{H}^{n \times r}\) be unknown. Then the equation \(XB = C\) is solvable if and only if \(C = CB^1B\) and its general solution is \(X = CB^1 + WR_B\), where \(W\) is a any matrix with conformable dimension. Moreover, its partial solution \(X = CB^1\) has the determinantal representation,

\[
x_{ij} = \sum_{\alpha \in I_{r,m}\{j\}} \frac{\text{rdet}_j \left((BB^*)_{i} (\mathbf{c}_l)\right)_{\alpha}}{\sum_{\alpha \in I_{r,m}} |BB^*|_{\alpha}^m}.
\]

where \(\mathbf{c}_l\) is the \(i\)th row of \(C = CB^*\).
3 Determinantal representations of a partial solution to the generalized Sylvester equation (1.1).

Lemma 3.1. Let $A_1 \in \mathbb{H}^{m \times n}$, $B_1 \in \mathbb{H}^{r \times s}$, $A_2 \in \mathbb{H}^{m \times p}$, $B_2 \in \mathbb{H}^{q \times s}$, $C \in \mathbb{H}^{m \times s}$. Put $M = R_A A_2$, $N = B_2 B_B$, $S = A_2 L_M$. Then the following results are equivalent.

(i) Eq. (1.1) has a solution $(X_1, X_2)$, where $X_1 \in \mathbb{H}^{n \times r}$, $X_2 \in \mathbb{H}^{p \times q}$.

(ii) 

\[ R_M R_A C = 0, \quad R_A C L_B = 0, \quad C L_B L_N = 0, \quad R_A C L_B = 0. \]  

(3.1)

(iii) 

\[ Q_M R_A C P_B = R_A C, \quad Q_A C L_B P_N = C L_B. \]

(iv) \[ \text{rank} \begin{bmatrix} A_1 & A_2 \end{bmatrix} = \text{rank} \begin{bmatrix} A_1 & A_2 \end{bmatrix}, \quad \text{rank} \begin{bmatrix} B_1^* & B_2^* \end{bmatrix} = \text{rank} \begin{bmatrix} B_1^* & B_2^* \end{bmatrix}, \]  

\[ \text{rank} \begin{bmatrix} A_1 & C \\ 0 & B_2 \end{bmatrix} = \text{rank} \begin{bmatrix} A_1 & 0 \\ 0 & B_2 \end{bmatrix}, \quad \text{rank} \begin{bmatrix} A_2 & C \\ 0 & B_1 \end{bmatrix} = \text{rank} \begin{bmatrix} A_2 & 0 \\ 0 & B_1 \end{bmatrix}. \]

In that case, the general solution of (1.1) can be expressed as the following,

\[ X_1 = A_1^* C B_1^1 - A_1^* A_2 M^1 \bar{R}_A C B_1^1 - A_1^* S A_2^1 C L_B L_N^1 B_2^1 B_1^1 - \\
A_1^* S V R N B_2^1 + L_A U + Z R_B, \]  

(3.2)

\[ X_2 = M^1 R_A C B_2^1 + L_M S^1 S A_2^1 C L_B L_N^1 + L_M (V - S^1 S V N N^1) + W R_B, \]  

(3.3)

where $U$, $V$, $Z$, and $W$ are arbitrary matrices of suitable shapes over $\mathbb{H}$.

Some simplifications of (3.2) and (3.3) can be derived due to the quaternionic analogues of the following propositions.

Lemma 3.2. If $A \in \mathbb{H}^{m \times n}$ is Hermitian and idempotent, then the following equation holds for any matrix $B \in \mathbb{H}^{m \times n}$,

\[ (A B) = (B A)^*, \]  

(3.4)

\[ (A B)^* A = (A B)^*. \]  

(3.5)

Since $R_A$, $L_B$, and $L_M$ are projectors, then by (3.4) and (3.5), we have, respectively,

\[ M^1 R_A = (R_A A_2)^1 R_A = (R_A A_2)^1 = M^1, \]  

\[ L_B N^1 = L_B (B_2 B_B)^1 = (B_2 B_B)^1 = N^1, \]  

\[ L_M S^1 = L_M (A_2 L_M)^1 = (A_2 L_M)^1 = S^1. \]  

(3.6)
Using (3.6), we obtain the following expressions of (3.2) and (3.3),

\[
X_1 = A^\dagger CB^\dagger - A^\dagger A_2 M^\dagger CB^\dagger - A^\dagger SA_2^\dagger CN^\dagger B_2 B_1^\dagger - A^\dagger SVR_N B_2 B_1^\dagger + L_{A_1}, U + ZR_{B_1},
\]

\[
X_2 = M^\dagger CB^\dagger + P_S A_2^\dagger CN^\dagger + L_M(V - P_S VQ_N) + WR_{B_2}.
\]

By putting \(U, V, Z,\) and \(W\) as zero-matrices of suitable shapes, we obtain the following partial solution to (1.1),

\[
X_1 = A^\dagger CB^\dagger - A^\dagger A_2 M^\dagger CB^\dagger - A^\dagger SA_2^\dagger CN^\dagger B_2 B_1^\dagger,
\]

\[
X_2 = M^\dagger CB^\dagger + P_S A_2^\dagger CN^\dagger.
\]

Further we give determinantal representations of (3.7)-(3.8). Let \(c\) are the column vector and the row vector, respectively. \(r\) have \(k\)th row and the \(1\)th column of \(A\).

(i) By Theorem 2.8 for the first term of (3.7), \(A_1^\dagger CB_1^\dagger := X_{11} = (x_{ij}^{(11)})\), we have

\[
x_{ij}^{(11)} = \sum_{\beta \in J_{1,n}} \frac{\det_i \left( (A_1^\dagger A_1)_{i,j} \left( d_{B_i}^1 \right) \right)^\alpha}{\det_i \left( (A_1^\dagger A_1)_{i,i} \left( d_{B_i}^1 \right) \right)}
\]

or

\[
x_{ij}^{(11)} = \sum_{\beta \in J_{1,n}} \frac{\det_i \left( (A_1^\dagger A_1)_{i,j} \left( d_{B_i}^1 \right) \right)^\alpha}{\det_i \left( (A_1^\dagger A_1)_{i,i} \left( d_{B_i}^1 \right) \right)}
\]

where

\[
d_{B_i}^1 = \left[ \sum_{a \in J_{1,n}} \det_i \left( (B_1^\dagger B_1^\dagger)_{j,j} \left( c^{(1)} \right) \right) \right] \in \mathbb{H}^{n \times 1}, \quad k = 1, \ldots, n,
\]

\[
d_{A_i}^1 = \left[ \sum_{\beta \in J_{1,n}} \det_i \left( (A_1^\dagger A_1)_{i,j} \left( c^{(1)} \right) \right) \right] \in \mathbb{H}^{1 \times r}, \quad l = 1, \ldots, r,
\]

are the column vector and the row vector, respectively. \(c_k^{(1)}\) and \(c_l^{(1)}\) are the \(k\)th row and the \(l\)th column of \(C_1 := A_1^\dagger CB_1^\dagger\).

(ii) Using the determinantal representation (2.1) for \(A_1^\dagger\) and by Theorem 2.8 we obtain the following representation of the second term of (3.7), \(A_1^\dagger A_2 M^\dagger CB_1^\dagger := X_{12} = (x_{ij}^{(12)})\),

\[
x_{ij}^{(12)} = \sum_{\beta \in J_{1,n}} \frac{\det_i \left( (A_1^\dagger A_1)_{i,j} \left( a^{(2)} \right) \right) \varphi_{ij}}{\det_i \left( (A_1^\dagger A_1)_{i,i} \left( a^{(2)} \right) \right)}
\]

\[
= \sum_{\beta \in J_{1,n}} \frac{\det_i \left( (A_1^\dagger A_1)_{i,j} \left( a^{(2)} \right) \right) \varphi_{ij}}{\det_i \left( (A_1^\dagger A_1)_{i,i} \left( a^{(2)} \right) \right)}
\]
where

\[ \varphi_{ij} = \sum_{\beta \in \mathcal{J}_{3,r,p}(t)} \text{cdet}_t \left( (M^*M)^{\alpha}_t \left( \psi_{j}^{B_1} \right)^{\beta}_\beta \right) \]

or

\[ \varphi_{ij} = \sum_{\alpha \in \mathcal{I}_{3,r}(j)} \text{rdet}_j \left( (B_1B_1^*)^{\alpha}_j \left( \psi_{i}^{M} \right)^{\alpha}_\alpha \right) \]

and

\[ \psi_{j}^{B_1} = \left[ \sum_{\alpha \in \mathcal{I}_{3,r}(j)} \text{rdet}_j \left( (B_1B_1^*)^{\alpha}_j \left( c_k^{(2)} \right)^{\alpha}_\alpha \right) \right] \in \mathbb{H}^{p \times 1}, \ k = 1, \ldots, p, \]

\[ \psi_{i}^{M} = \left[ \sum_{\beta \in \mathcal{J}_{3,p}(t)} \text{cdet}_t \left( (M^*M)^{\beta}_t \left( c_i^{(2)} \right)^{\beta}_\beta \right) \right] \in \mathbb{H}^{1 \times r}, \ l = 1, \ldots, r, \]

are the column vector and the row vector, respectively. \( \tilde{a}_{ij}^{(2)} \) is the \( i \)th column of \( \tilde{A}_2 := A_3^\dagger A_2 \), \( c_k^{(2)} \) and \( c_i^{(2)} \) are the \( k \)th row and the \( t \)th column of \( C_2 := M^*CB_1^\dagger \), respectively.

(iii) For the third term of (3.7), \( A_3^\dagger S_{\alpha} A_{\alpha}^\dagger C N_{\alpha} B_{2} B_1^\dagger := X_{13} = (x_{ij}^{(13)}) \), we use the determinantal representations (2.1) to \( A_3^\dagger \) and (2.2) to \( B_1 \), respectively. Then, due to Theorem 2.8 for \( A_3^\dagger C N_{\alpha} \), we have

\[ x_{ij}^{(13)} = \sum_{\alpha=1}^{\alpha_q} \sum_{\beta=1}^{\beta_q} \text{cdet}_t \left( (A_3^\dagger A_1)^{\beta}_t \left( \tilde{k}_{1}^{(2)} \right)^{\beta}_\beta \right) \eta_f \sum_{\alpha \in \mathcal{I}_{3,r}(j)} \text{rdet}_j \left( (B_1B_1^*)^{\alpha}_j \left( \tilde{k}_{1}^{(2)} \right)^{\alpha}_\alpha \right) \]

\[ \sum_{\beta \in \mathcal{J}_{3,p}(t)} |A_1^\dagger A_1|^{\beta}_\beta \sum_{\beta \in \mathcal{J}_{3,p}} |A_2^\dagger A_2|^{\beta}_\beta \sum_{\alpha \in \mathcal{I}_{3,r}} |N_{\alpha} |^{\alpha}_\alpha \sum_{\alpha \in \mathcal{I}_{3,r}} |B_1B_1^*|^{\alpha}_\alpha \]

\[ (3.12) \]

where

\[ \eta_f = \sum_{\beta \in \mathcal{J}_{3,r,p}(t)} \text{cdet}_t \left( (A_3^\dagger A_2)^{\beta}_t \left( \zeta_{1}^{(2)} \right)^{\beta}_\beta \right), \]

\[ (3.13) \]

or

\[ \eta_f = \sum_{\alpha \in \mathcal{I}_{3,r}(j)} \text{rdet}_j \left( (N_{\alpha}^* f)^{\alpha}_j \left( \zeta_{1}^{(2)} \right)^{\alpha}_\alpha \right), \]

\[ (3.14) \]

and

\[ \zeta_{1}^{N} = \left[ \sum_{\alpha \in \mathcal{I}_{3,r}(j)} \text{rdet}_j \left( (N_{\alpha}^* f)^{\alpha}_j \left( c_k^{(3)} \right)^{\alpha}_\alpha \right) \right] \in \mathbb{H}^{p \times 1}, \ k = 1, \ldots, p, \]

\[ \zeta_{1}^{A_2} = \left[ \sum_{\beta \in \mathcal{J}_{3,r,p}(t)} \text{cdet}_t \left( (A_3^\dagger A_2)^{\beta}_t \left( c_i^{(3)} \right)^{\beta}_\beta \right) \right] \in \mathbb{H}^{1 \times q}, \ l = 1, \ldots, q, \]
are the column vector and the row vector, respectively. \( \bar{s}_i \) is the \( i \)th column of \( \bar{s} := A^1_S \bar{b} \), \( \bar{b}^{(2)}_j \) is the \( j \)th row of \( \bar{B}_2 := B_2 B_1^\dagger \), \( c_k^{(3)} \) and \( c_i^{(3)} \) are the \( k \)th row and the \( i \)th column of \( C_3 := A_3^2 C \bar{N}^\ast \).

Now, we consider each term of (3.8).

(i) Due to Theorem 2.8 for the first term \( M^\dagger CB_2^\dagger =: X_{21} = (x_{gf}^{(21)}) \) of (3.8), we have

\[
x_{gf}^{(21)} = \frac{\sum_{\beta \in J_{r_5,p} \{g\}} \text{cdet}_g \left( (M \ast M)_g \left( d_{gf}^{B_2} \right) \right)_\beta}{\sum_{\beta \in J_{r_5,p}} |M \ast M|_{\beta}^\beta \sum_{\alpha \in I_{r_4,q}} |B_2 B_2^\ast|_\alpha^\alpha}, \tag{3.15}
\]

or

\[
x_{gf}^{(21)} = \frac{\sum_{\alpha \in I_{r_4,q} \{f\}} \text{rdet}_f \left( (B_2 B_2^\ast)_f, (d_{gf}^{M})_\alpha \right)^\alpha}{\sum_{\beta \in J_{r_5,p}} |M \ast M|_{\beta}^\beta \sum_{\alpha \in I_{r_4,q}} |B_2 B_2^\ast|_\alpha^\alpha}, \tag{3.16}
\]

where

\[
d_{gf}^{B_2} = \left[ \sum_{\alpha \in I_{r_4,q} \{f\}} \text{rdet}_f \left( (B_2 B_2^\ast)_f, (c_k^{(4)})_\alpha \right)^\alpha \right] \in \mathbb{H}^{p \times 1}, \quad k = 1, \ldots, p,
\]

\[
d_{gf}^{M} = \left[ \sum_{\beta \in J_{r_5,p} \{g\}} \text{cdet}_g \left( (M \ast M)_g \left( c_i^{(4)} \right) \right)_\beta \right] \in \mathbb{H}^{1 \times q}, \quad l = 1, \ldots, q,
\]

are the column vector and the row vector, respectively. \( c_k^{(4)} \) and \( c_i^{(4)} \) are the \( k \)th row and the \( i \)th column of \( C_4 := M \ast C B_2^\dagger \).

(ii) Finally, for the second term \( P_S A_3^2 C N^\dagger =: X_{22} = (x_{gf}^{(22)}) \) of (3.8) using (2.9) for a determinantal representation of \( P_S \), and due to Theorem 2.8 for \( A_3^2 C N^\dagger \), we obtain

\[
x_{gf}^{(22)} = \frac{\sum_{\beta \in J_{r_7,p} \{g\}} \sum_{\alpha \in I_{r_3,q}} \text{cdet}_g \left( (S \ast S)_g \left( \bar{s}_l \right) \right)_\beta \eta_{gf}}{\sum_{\beta \in J_{r_7,p}} |S \ast S|_{\beta}^\beta \sum_{\alpha \in I_{r_3,q}} |A_3^2 A_2|_\alpha^\alpha \sum_{\beta \in J_{r_5,p}} |N N^\ast|_\alpha^\alpha}, \tag{3.17}
\]

where \( \eta_{gf} \) is (3.13) or (3.14).

So, we prove the following theorem.

**Theorem 3.3.** Let \( A_1 \in \mathbb{H}^{m \times n}_{r_1}, B_1 \in \mathbb{H}^{r \times s}_{r_2}, A_2 \in \mathbb{H}^{m \times p}_{r_3}, B_2 \in \mathbb{H}^{q \times s}_{r_4} \), rank \( M = r_5 \), rank \( N = r_6 \), rank \( S = r_7 \). Then the pair solution (7.7)-(3.8), \( X_1 = (x_{ij}^{(1)}) \in \mathbb{H}^{n \times r}, X_2 = (x_{gf}^{(2)}) \in \mathbb{H}^{p \times q} \) to Eq. (1.1) by the components

\[
x_{ij}^{(1)} = x_{ij}^{(11)} - x_{ij}^{(12)} - x_{ij}^{(13)}, \quad x_{gf}^{(2)} = x_{gf}^{(21)} + x_{gf}^{(22)},
\]

has the determinantal representation, where the term \( x_{ij}^{(11)} \) is (3.9) or (3.10), \( x_{ij}^{(12)} \) is (3.17), \( x_{ij}^{(13)} \) is (3.16), \( x_{gf}^{(21)} \) is (3.12) or (3.14), \( x_{gf}^{(22)} \) is (3.17).
4 Cramer’s Rules for special cases of (1.1) with only one two-sided term.

In this section, we consider all special cases of (1.1) when only second term is two-sided.

1. Let in Eq. (1.1) the matrix $B_1$ be vanish, i.e. $B_1 = I_s$. Then, we have the equation

$$A_1X_1 + A_2X_2B_2 = C,$$

where $A_1 \in \mathbb{H}^{m \times n}$, $A_2 \in \mathbb{H}^{m \times p}$, $B_2 \in \mathbb{H}^{q \times s}$, and $X_2 \in \mathbb{H}^{p \times q}$ are to be determined. Since $L_{B_1} = R_{B_1} = 0$, $N = B_2L_{B_1} = 0$, and $L_N = R_N = I$ and taking into account the simplifications by (3.4) and (3.5), then we derive the following analog of Lemma 3.1.

**Lemma 4.1.** Let $M = R_{A_1}A_2$, $S = A_2L_M$. Then the following results are equivalent.

(i) Eq. (4.1) is solvable.

(ii) $RM_{RA_1}C = 0, RA_1CL_{B_2} = 0$.

(iii) $QM_{RA_1}CP_{B_2} = R_{A_1}C$.

(iv) $\text{rank} [A_1 \ A_2 \ C] = \text{rank} [A_1 \ A_2]$, $\text{rank} \begin{bmatrix} A_1 & C \\ 0 & B_2 \end{bmatrix} = \text{rank} \begin{bmatrix} A_1 & 0 \\ 0 & B_2 \end{bmatrix}$.

In that case, the general solution of (4.1) can be expressed as follows,

$$X_1 = A_1^\dagger C - A_1^\dagger A_2M^\dagger C - A_1^\dagger S\ V\ B_2 + L_{A_1}U,$$

$$X_2 = M^\dagger CB_2^\dagger + L_MV + WR_{B_2},$$

where $U$, $V$, and $W$ are arbitrary matrices of suitable shapes over $\mathbb{H}$.

By putting $U$, $V$, and $W$ as zero-matrices of suitable shapes, we obtain the following partial solution of (4.1),

$$X_1 = A_1^\dagger C - A_1^\dagger A_2M^\dagger C,$$

$$X_2 = M^\dagger CB_2^\dagger.$$  

Further we give determinantal representations of (4.2)–(4.3).

**Theorem 4.2.** Let $A_1 \in \mathbb{H}^{m \times n}$, $A_2 \in \mathbb{H}^{m \times p}$, $B_2 \in \mathbb{H}^{q \times s}$, and $\text{rank} M = \min\{\text{rank}\ A_2, \text{rank} R_{A_1}\} = r_4$. Then the solution $X_1 = (x_{ij}^{(1)}) \in \mathbb{H}^{n \times s}$ from
has the determinantal representation

\[
x_{ij}^{(1)} = \frac{\sum_{\beta \in J_{r,n}(i)} \text{cdet} \left( (A_1^* A_1)_j \left( c_{(1)}^j \right) \right)^\beta}{\sum_{\beta \in J_{r,n}} |A_1^* A_1|^\beta_{\beta}} - \sum_{t=1}^p \sum_{\beta \in J_{r,n}(i)} \text{cdet} \left( (A_1^* A_1)_j \left( \tilde{a}_{(2)}^j \right) \right)^\beta \sum_{\beta \in J_{r,p}(t)} \text{cdet} \left( (M^* M)_j \left( c_{(2)}^j \right) \right)^\beta
\]

where \( \tilde{a}_{(2)}^j \) is the \( t \)-th column of \( \tilde{A}_2 := A_1^* A_2 \), \( c_{(1)}^j \) and \( c_{(2)}^j \) are the \( j \)-th columns of \( C_1 := A_1^* C \) and \( C_2 := M^* C \), respectively. The solution \( X_2 = \left(x_{gf}^{(2)}\right) \in \mathbb{H}^{p \times q} \) from (4.3) has the determinantal representations

\[
x_{gf}^{(2)} = \frac{\sum_{\beta \in J_{r,p}(g)} \text{cdet}_g \left( (M^* M)_g \left( d_{B^2 f}^g \right) \right)^\beta}{\sum_{\beta \in J_{r,p}} |M^* M|^\beta_{\beta} \sum_{\alpha \in I_{r,q}} |B_2 B_2^\alpha|^\alpha_{\alpha}}, \quad (4.5)
\]

or

\[
x_{gf}^{(2)} = \frac{\sum_{\alpha \in I_{r,q}(f)} \text{rdet}_f \left( (B_2 B_2^\alpha)_f \left( d_{g}^\alpha \right) \right)^\alpha}{\sum_{\beta \in J_{r,p}} |M^* M|^\beta_{\beta} \sum_{\alpha \in I_{r,q}} |B_2 B_2^\alpha|^\alpha_{\alpha}}, \quad (4.6)
\]

where

\[
d_{B^2 f} = \left[ \sum_{\alpha \in I_{r,q}(f)} \text{rdet}_f \left( (B_2 B_2^\alpha)_f \left( c_{(3)}^k \right) \right)^\alpha \right]_{k=1, \ldots, p} \in \mathbb{H}^{p \times 1},
\]

\[
d_{g}^\alpha = \left[ \sum_{\beta \in J_{r,p}(g)} \text{cdet}_g \left( (M^* M)_g \left( c_{(3)}^l \right) \right)^\beta \right]_{l=1, \ldots, q} \in \mathbb{H}^{1 \times q},
\]

are the column vector and the row vector, respectively. \( c_{(3)}^k \) and \( c_{(3)}^l \) are the \( k \)-th row and the \( l \)-th column of \( C_3 := M^* CB_2^\alpha \).

**Proof.** By using Corollary 2.9 to the both terms of (4.2) and Theorem 2.8 to (4.3), we evidently obtain the determinantal representations (4.4) and (4.5)- (4.6), respectively.

2. Let now in Eq. (1.1) the matrix \( A_1 \) be vanish, i.e. \( A_1 = I_m \). Then we have the equation

\[
X_1 B_1 + A_2 X_2 B_2 = C,
\]

(4.7)
Lemma 4.3. Let $B_1 \in \mathbb{H}^{r \times s}$, $A_2 \in \mathbb{H}^{m \times p}$, $B_2 \in \mathbb{H}^{q \times s}$, $C \in \mathbb{H}^{m \times r}$ be given, $X_1 \in \mathbb{H}^{m \times r}$ and $X_2 \in \mathbb{H}^{p \times q}$ are to be determined. Since $L_{A_1} = 0$, $R_{A_1} = 0$, $M = A_2 L_{A_1} = 0$, $L_M = I$, and $S = A_2 L_M = A_2$ and taking into account the simplifications by (3.4) and (3.5), then we derive the following lemma similar to Lemma 3.1.

Theorem 4.4. Further we give determinantal representations of (4.8)-(4.9).

In that case, the general solution of (4.7) can be expressed as follows

\[
X_1 = CB_1^\dagger - Q_{A_2} CN^\dagger B_2^\dagger - A_2 VR_N B_2^\dagger + ZR_{B_1},
\]

\[
X_2 = A_2^\dagger CN^\dagger + V - P_{A_2} VQ_N + WR_{B_2},
\]

where $V$, $Z$ and $W$ are arbitrary matrices of suitable shapes over $\mathbb{H}$.

By putting $V$, $Z$, and $W$ as zero-matrices of suitable shapes, we obtain the following partial solution of (4.7),

\[
X_1 = CB_1^\dagger - Q_{A_2} CN^\dagger B_2^\dagger,
\]

\[
X_2 = A_2^\dagger CN^\dagger.
\]

Further we give determinantal representations of (4.8)-(4.9).

Theorem 4.4. Let $B_1 \in \mathbb{H}^{r \times s}$, $A_2 \in \mathbb{H}^{m \times p}$, $B_2 \in \mathbb{H}^{q \times s}$, and $\text{rank} N = r_4$. Then the solution (4.8) has the determinantal representation

\[
x_{ij}^{(1)} = \sum_{\alpha \in I_{r_1 \times r}} \frac{\text{rdet}_j \left( (B_1 B_1^\dagger)_{\alpha} \left( c_{i.}^{(1)} \right) \right)_{\alpha}}{\sum_{\alpha \in I_{1 \times r}} |B_1 B_1^\dagger|_{\alpha}} - \sum_{l=1}^{m} \sum_{t=1}^{n} x_{il}^{(11)} x_{lj}^{(13)} - \sum_{l=1}^{m} \sum_{t=1}^{n} x_{lt}^{(12)} x_{lj}^{(13)},
\]

\[
\sum_{\alpha \in I_{r_2 \times m}} |A_2 A_2^\dagger|_{\alpha} \sum_{\alpha \in I_{r_4 \times q}} |NN^*|_{\alpha} \sum_{\alpha \in I_{r_1 \times r}} |B_1 B_1^\dagger|_{\alpha},
\]

where

\[
x_{il}^{(11)} = \sum_{\alpha \in I_{r_2 \times m}} \text{rdet}_j \left( (A_2 A_2^\dagger)_{\alpha} \left( a_{i.}^{(2)} \right) \right)_{\alpha},
\]

\[
x_{lt}^{(12)} = \sum_{\alpha \in I_{r_4 \times q}} \text{rdet}_t \left( (NN^*)_{\alpha} \left( c_{i.}^{(2)} \right) \right)_{\alpha},
\]

\[
x_{lj}^{(13)} = \sum_{\alpha \in I_{r_1 \times r}} \text{rdet}_j \left( (B_1 B_1^\dagger)_{\alpha} \left( b_{i.}^{(2)} \right) \right)_{\alpha},
\]
Lemma 4.5. The following results are equivalent.

(i) Eq. (4.13) is solvable.
(ii) \( CL_{B_2} = 0 \).

(iii) \( \text{rank} \begin{bmatrix} A_2 & C \end{bmatrix} = \text{rank}[A_2], \text{rank} \begin{bmatrix} B_2^* & C^* \end{bmatrix} = \text{rank}[B_2] \).

In that case, the general solution of (1.1) can be expressed as the following,

\[ X_1 = C - A_2 VB_2, \]
\[ X_2 = V + WR_{B_2}, \]

where \( V \) and \( W \) are arbitrary matrices of suitable shapes over \( \mathbb{H} \).

Since determinantal representations of (4.14)-(4.15) are evident, we omit them.

5 Cramer’s rules for special cases of (1.1) with both one-sided terms.

In this section, we consider all special cases of Eq.(1.1) when its both terms are one-sided.

1. Let the matrices \( B_1 \) and \( A_2 \) be vanish in Eq.(1.1), i.e. \( B_1 = I_s \) and \( A_2 = I_m \). Then we have the equation

\[ A_1 X_1 + X_2 B_2 = C, \]  

where \( A_1 \in \mathbb{H}^{m \times n}, B_2 \in \mathbb{H}^{q \times s}, C \in \mathbb{H}^{m \times s} \) be given, \( X_1 \in \mathbb{H}^{n \times s} \) and \( X_2 \in \mathbb{H}^{m \times q} \) are to be determined. This equation is the classical Sylvester equation.

So, \( L_{B_1} = R_{B_1} = 0, N = B_2 L_{B_1} = 0, L_{A_2} = R_{A_2} = 0, P_{A_2} = I, L_N = R_N = I, M = R_{A_1}, \) and \( S = L_M \). Since \( R_{A_1} \) is the orthogonal projector onto the kernel of \( A_1 \), the we have

\[ A_1^\dagger M^\dagger = A_1^\dagger R_{A_1}^\dagger = A_1^\dagger (I - A_1 A_1^\dagger)^\dagger = 0, \]
\[ A_1^\dagger L_M = A_1^\dagger (I - R_{A_1}^\dagger R_{A_1}) = A_1^\dagger, \]
\[ M^\dagger R_{A_1} = R_{A_1}^\dagger R_{A_1} = R_{A_1}. \]

Due to (5.2) and taking into account of simplifications by (3.4) and (3.5), we have the following analog of Lemma 3.1.

Lemma 5.1. The following results are equivalent.

(i) Eq. (5.1) is solvable.

(ii) \( R_{A_1} CL_{B_2} = 0 \).

(iii) \( \text{rank} \begin{bmatrix} A_1 & C \\ 0 & B_2 \end{bmatrix} = \text{rank} \begin{bmatrix} A_1 & 0 \\ 0 & B_2 \end{bmatrix} \).

In that case, the general solution of (5.1) can be expressed as follows

\[ X_1 = A_1^\dagger C - A_1^\dagger VB_2 + L_{A_1} U, \]
\[ X_2 = R_{A_1} CB_2^\dagger + A_1 A_1^\dagger V + WR_{B_2}, \]

where \( U, V, \) and \( W \) are arbitrary matrices of suitable shapes over \( \mathbb{H} \).
The denoting \( V_1 := A^1_1 V \) in (5.3)-(5.4) gives the expression of the general solution of (5.1) that has been first derived in [1].

By putting \( U, V \), and \( W \) as zero-matrices of suitable shapes, the following partial solution of (5.1) can be obtained,

\[
X_1 = A^1_1 C, \quad (5.5)
\]

\[
X_2 = CB^\dagger_2 - QA_1 CB^\dagger_2. \quad (5.6)
\]

**Theorem 5.2.** Let \( A_1 \in \mathbb{H}^n \), \( B_2 \in \mathbb{H}^q \). Then the solution (5.6) has the determinantal representation

\[
x^\alpha_{ij} = \frac{\sum_{\beta \in J_{n,q}} c_{ij}^{(1)} (A_1^\dagger A_1)^{\beta}_{ij} c^{(1)}_{ij}}{\sum_{\beta \in J_{n,q}} |A_1^\dagger A_1|_{\beta}^2}, \quad (5.7)
\]

where \( c_{ij}^{(1)} \) is the \( j \)th columns of \( C_1 := A_1^\dagger C \). The determinantal representation of (5.6) is

\[
x^\alpha_{gj} = \frac{\sum_{\alpha \in I_{n,q}} \text{r} det_f \left( (B_2 B_2^\dagger) f \left( c^{(2)}_{gj} \right) \right)_{\alpha} A_1^\dagger A_1}{\sum_{\alpha \in I_{n,q}} |B_2 B_2^\dagger|_{\alpha}^2} - \sum_{\alpha \in I_{n,q}} |B_2 B_2^\dagger|_{\alpha}^2
\]

\[
\sum_{l=1}^m \sum_{\alpha \in I_{n,q}} \text{r} det_l \left( (A_1^\dagger A_1)^l \left( a^{(1)}_{gj} \right) \right)_{\alpha} \sum_{\alpha \in I_{n,q}} \text{r} det_f \left( (B_2 B_2^\dagger) f \left( c^{(2)}_{gj} \right) \right)_{\alpha}
\]

\[
\sum_{\alpha \in I_{n,q}} |A_1^\dagger A_1|_{\alpha}^2 \sum_{\alpha \in I_{n,q}} |B_2 B_2^\dagger|_{\alpha}^2,
\]

(5.8)

where \( c^{(2)}_{gj} \) and \( a^{(1)}_{gj} \) are the \( g \)th rows of \( C_2 := CB_2^\dagger \) and \( A_1^\dagger A_1 \), respectively.

**Proof.** Using Corollary 2.9 to (5.5) and Corollaries 2.10 and 2.6 to (5.6), we evidently obtain the determinantal representations (5.7) and (5.8), respectively. \( \square \)

2. Let in Eq. (1.1) the matrices \( A_1 \) and \( B_2 \) be vanish, i.e. \( A_1 = I_m \) and \( B_2 = I_q \). Then we have the equation

\[
X_1B_1 + A_2X_2 = C, \quad (5.9)
\]

where \( B_1 \in \mathbb{H}^{r \times s}, A_2 \in \mathbb{H}^{m \times p}, C \in \mathbb{H}^{r \times s} \), \( X_1 \in \mathbb{H}^{m \times r} \) and \( X_2 \in \mathbb{H}^{p \times s} \) are to be determined. So, \( L_{A_1} = R_{A_1} = 0, M = R_{A_1}, A_2 = 0, L_{B_2} = R_{B_2} = 0, P_{B_2} = I, L_M = R_M = I, N = L_{B_1}, \) and \( S = A_2 \). Since \( L_{B_1} \) is the orthogonal projector onto the kernel of \( B_1 \), then we have

\[
N^\dagger B_1^\dagger = L_{B_1}^\dagger B_1^\dagger = (I - B_1^\dagger B_1)^\dagger B_1^\dagger = 0, R_{A_1}^\dagger B_1^\dagger = (I - L_{B_1} L_{B_1}^\dagger) B_1^\dagger = B_1^\dagger,
\]

\[
L_{B_1} N^\dagger = L_{B_1} L_{B_1}^\dagger = L_{B_1}. \quad (5.10)
\]
Due to (5.10) and taking into account simplifications by (3.4) and (3.5), the analog of Lemma 3.1 follows.

**Lemma 5.3.** The following results are equivalent.

(i) Eq. (5.9) is solvable.

(ii) \( R_{A_2}C_{LB_1} = 0 \).

(iii) \( \text{rank} \begin{bmatrix} A_2 & C \\ 0 & B_1 \end{bmatrix} = \text{rank} \begin{bmatrix} A_2 & 0 \\ 0 & B_1 \end{bmatrix} \).

In that case, the general pair solution of (5.9) is

\[
X_1 = CB_1^\dagger - A_2VB_1^\dagger + ZR_{B_1}, \quad X_2 = A_2^\dagger CL_{B_1} + L A_2 VL_{B_1},
\]

where \( V \) and \( Z \) are arbitrary matrices over \( \mathbb{H} \) of suitable shapes.

By putting \( V \) and \( Z \) as zero-matrices of suitable shapes, we have the following partial pair solution of (5.9),

\[
X_1 = CB_1^\dagger, \quad X_2 = A_2^\dagger C - A_2^\dagger CP_{B_1}, \quad (5.11)
\]

\[
X_2 = A_2^\dagger C - A_2^\dagger CP_{B_1}, \quad (5.12)
\]

**Theorem 5.4.** Let \( B_1 \in \mathbb{H}^{r \times s}, A_2 \in \mathbb{H}^{m \times p} \). Then (5.11) has the determinantal representation

\[
X_{ij}^{(1)} = \sum_{\alpha \in I_{r_1}, r \setminus \{j\}} \text{rdet}_j \left( (B_1B_1^\dagger)_{\alpha} \left( c_{i}^{(1)} \right) \right)_{\alpha},
\]

where \( c_{i}^{(1)} \) is the \( j \)th row of \( C_1 := CB_1^\dagger \). The solution (5.12) has the determinantal representation

\[
X_{ij}^{(2)} = \sum_{\beta \in I_{r_2}, p \setminus \{g\}} \text{cdet}_g \left( (A_2^\dagger A_2)_{\beta} \left( e_{i}^{(2)} \right) \right)_{\beta} \]

\[
\sum_{l=1}^{\hat{s}} \sum_{\beta \in I_{r_1}, \{1\}} \text{cdet}_l \left( (B_1^\dagger B_1)_{l} \left( b_{i}^{(1)} \right) \right)_{\beta}
\]

\[
\sum_{\beta \in I_{r_2}, p \setminus \{g\}} \left| A_2^\dagger A_2 \right|_{\beta} \sum_{\beta \in I_{r_1}, \{1\}} \left| B_1^\dagger B_1 \right|_{\beta},
\]

where \( e_{i}^{(2)} \) and \( b_{i}^{(1)} \) are the \( f \)th columns of \( C_2 := A_2^\dagger C \) and \( B_1^\dagger B_1 \), respectively.

**Proof.** Using Corollary 2.10 to (5.11) and Corollaries 2.9 and 2.5 to (5.12), we evidently obtain the determinantal representations (5.13) and (5.14), respectively.

\[\square\]
3. Let the matrices \( B_1 \) and \( B_2 \) be vanish in Eq.(1.1), i.e. \( B_1 = B_2 = I_r \). Then we have the equation
\[
A_1 X_1 + A_2 X_2 = C, \tag{5.15}
\]
where \( A_1 \in \mathbb{H}^{m \times n} \), \( A_2 \in \mathbb{H}^{m \times p} \), \( C \in \mathbb{H}^{m \times r} \) be given, \( X_1 \in \mathbb{H}^{n \times r} \) and \( X_2 \in \mathbb{H}^{p \times r} \) are to be determined. So, \( L_{B_1} = R_{B_1} = 0 \), \( L_{B_2} = R_{B_2} = 0 \), \( P_{B_2} = I \), and \( N = 0 \).

Due to (3.4) and (3.5), the following analog of Lemma 3.1 can be obtained.

**Lemma 5.5.** Let \( M = R A_1 A_2, \ S = A_2 L_M \). The following results are equivalent.

(i) Eq. (5.15) is solvable.

(ii) \( R M R A_1 C = 0 \).

(iii) \( \text{rank} \left[ A_1 A_2 C \right] = \text{rank} \left[ A_1 A_2 \right] \).

In that case, the general solution of (5.15) can be expressed as follows
\[
X_1 = A_1^\dagger C - A_2^\dagger A_1 M^\dagger C - A_2^\dagger S V + L_{A_1} U, \tag{5.16}
\]
\[
X_2 = M^\dagger C + L_{M} V. \tag{5.17}
\]

where \( U \) and \( V \) are arbitrary matrices over \( \mathbb{H} \) of suitable shapes.

By putting \( U \) and \( V \) as zero-matrices of suitable shapes, we have the following partial pair solution of (5.15),
\[
X_1 = A_1^\dagger C - A_2^\dagger A_1 M^\dagger C, \tag{5.16}
\]
\[
X_2 = M^\dagger C. \tag{5.17}
\]

The next theorem can be proved similarly to Theorem 4.2.

**Theorem 5.6.** Let \( A_1 \in \mathbb{H}^{m \times n} \), \( A_2 \in \mathbb{H}^{m \times p} \), and \( \text{rank} M = r_4 \). Then the determinantal representation of (5.16) is the same as (4.4), and the solution (5.17) has the determinantal representation
\[
x_{(2)}^{(2)} f = \sum_{\beta \in J_{r_4, p}} \frac{\text{cdet}_g \left( \left( M^* M \right) g \left( c_f^{(2)} \right) \right)}{\left( M^* M \right) g \left( c_f^{(2)} \right) \beta} \sum_{\beta \in J_{r_4, p}} |M^* M\beta|^{-1},
\]

where \( c_f^{(2)} \) is the \( f \)th column of \( C_2 := M^* C \).

4. Let, now, the matrices \( A_1 \) and \( A_2 \) be vanish in Eq.(1.1), i.e. \( A_1 = A_2 = I_m \). Then we have the equation
\[
X_1 B_1 + X_2 B_2 = C, \tag{5.18}
\]
where \( B_1 \in \mathbb{H}^{r \times s} \), \( B_2 \in \mathbb{H}^{p \times s} \), \( C \in \mathbb{H}^{m \times s} \) be given, \( X_1 \in \mathbb{H}^{m \times r} \) and \( X_2 \in \mathbb{H}^{m \times q} \) are to be determined. Since \( L_{A_1} = R_{A_1} = 0 \), \( M = A_2 L_{A_1} = 0 \), \( L_M = I \), \( L_{A_2} = R_{A_2} = 0 \), \( P_{A_2} = Q_{A_2} = I \), and \( S = A_2 L_M = I \) and taking into account simplifications by (3.4) and (3.5), then we derive the analog of Lemma 5.1.
Lemma 5.7. Let $N = B_2L_{B_1}$. Then the following results are equivalent.

(i) Eq. (5.18) is solvable.

(ii) $CL_{B_2}L_N = 0$.

(iii) $\text{rank } [B_1 B_2 C] = \text{rank } [B_1^* B_2^*]$.

In that case, the general solution of (5.18) can be expressed as follows

$$X_1 = CB_1^* - CN^*B_1B_2^* - VR_NB_2B_1^* + ZR_{B_1},$$

$$X_2 = CN^* + VR_N + WR_{B_2},$$

where $V$, $Z$ and $W$ are arbitrary matrices over $\mathbb{H}$ of suitable shapes.

By putting $V$, $Z$, and $W$ as zero-matrices of suitable shapes, we obtain the following partial solution to (5.18),

$$X_1 = CB_1^* - CN^*B_2B_1^*, \quad (5.19)$$

$$X_2 = CN^*, \quad (5.20)$$

Theorem 5.8. Let $B_1 \in \mathbb{H}^{r_1 \times s}$, $B_2 \in \mathbb{H}^{q_2 \times s}$, and $\text{rank } N = r_3$. Then the pair solution (5.19)-(5.20) has the determinantal representation,

$$x_{_{1j}}^{(1)} = \sum_{\alpha \in I_{r_1 \times r_2 \times s_1 \times s_2 \times s_3}} \frac{\text{rdet}_j \left( (B_1^*B_1^*)_{\alpha} \left( c_{_{1j}}^{(1)} \right) \right)^{\alpha}}{\sum_{\alpha \in I_{r_1 \times r_2 \times s_1 \times s_2 \times s_3}} |B_1^*B_1^*|^{\alpha}},$$

$$x_{_{2f}}^{(2)} = \sum_{\alpha \in I_{r_1 \times r_2 \times s_1 \times s_2 \times s_3}} \frac{\text{rdet}_f \left( (NN^*)_{\alpha} \left( c_{_{2f}}^{(2)} \right) \right)^{\alpha}}{\sum_{\alpha \in I_{r_1 \times r_2 \times s_1 \times s_2 \times s_3}} |NN^*|^{\alpha}},$$

where $c_{_{1j}}^{(1)}$ and $c_{_{2f}}^{(2)}$ are the $i$th and $l$th rows of $C_1 := CB_1^*$ and $C_2 := CN^*$, respectively, and $\tilde{b}_{_{2l}}^{(2)}$ is the $t$th row of $B_2 := B_2^*$.

Proof. Using Corollary 2.10 to (5.20) and the both terms of (5.19), we evidently obtain the determinantal representations (5.21) and (5.22). \qed
6 Cramer’s rules for like-Lyapunov equations.

The well-known Lyapunov equation is $AX + XA^* = B$. In this section, we consider some like Lyapunov equations.

1. Consider the following matrix equation,

$$AX + X^*B = C,$$  \hspace{1cm} (6.1)

where $A \in \mathbb{H}^{m \times n}$, $B \in \mathbb{H}^{n \times m}$, and $C \in \mathbb{H}^{m \times m}$. Due to $[30]$, the following lemma can be expanded from the complex field to $\mathbb{H}$.

**Lemma 6.1.** If Eq. (6.1) has solution, and

$$A^*C \left( I - \frac{1}{2}P_B \right) = \left( I - \frac{1}{2}Q_A \right) CB^*,$$

then

$$X_0 = A^*C \left( I - \frac{1}{2}Q_B \right)$$  \hspace{1cm} (6.2)

is a solution to (6.1) and $-RACL_B = 0$.

**Proof.** The proof is similar to ($[30]$, Theorem 1). $\square$

**Theorem 6.2.** Let $A \in \mathbb{H}^{m \times n}$ and $B \in \mathbb{H}^{n \times m}$. Then the solution $X_0 = (x_{ij})$ to (6.1) has the following determinantal representation,

$$x_{ij} = \frac{\sum_{\beta \in J_{1,n}(i)} \text{cdet}_i \left( (A^*A)_{ij} \left( c_{j}^{(1)} \right) \right)_{\beta} \sum_{\alpha \in I_{2,n}(j)} \text{rdet}_j((BB^*)_{,j} \left( d_{i}^{(2)} \right) \alpha) - 2 \sum_{\beta \in J_{1,n}(i)} |A^*A|_{\beta} \sum_{\alpha \in I_{2,n}} |BB^*|_{\alpha}}{\sum_{\beta \in J_{1,n}(i)} |A^*A|_{\beta}}$$  \hspace{1cm} (6.3)

or

$$x_{ij} = \frac{\sum_{\beta \in J_{1,n}(i)} \text{cdet}_i \left( (A^*A)_{ij} \left( c_{j}^{(1)} \right) \right)_{\beta} \sum_{\alpha \in I_{2,n}(j)} \text{rdet}_j((BB^*)_{,j} \left( d_{i}^{(2)} \right) \alpha) - 2 \sum_{\beta \in J_{1,n}(i)} |A^*A|_{\beta} \sum_{\alpha \in I_{2,n}} |BB^*|_{\alpha}}{\sum_{\beta \in J_{1,n}(i)} |A^*A|_{\beta}}$$  \hspace{1cm} (6.4)

where $c_{j}^{(1)}$ is the column vector of $C_1 := A^*C$, and

$$d_{i}^{A} = \left[ \sum_{\beta \in J_{1,n}(i)} \text{cdet}_i \left( (A^*A)_{,i} \left( c_{k}^{(2)} \right) \right)_{\beta} \right]_{\beta} \in \mathbb{H}^{1 \times n}, \ \ k = 1, \ldots, n,$$

$$d_{i}^{B} = \left[ \sum_{\alpha \in I_{2,n}(j)} \text{rdet}_j((BB^*)_{,j} \left( c_{l}^{(2)} \right) \alpha) \right]_{\alpha} \in \mathbb{H}^{n \times 1}, \ \ l = 1, \ldots, n,$$

are the row vector and the column vector, respectively. $c_{k}^{(2)}$ and $c_{l}^{(2)}$ are the $k$th column and the $l$th row of $C_2 := A^*CBB^*$.  


Proof. Using Corollary 2.9 to the first term of (6.2) and Theorem 2.8 to the second term, we get (6.3)-(6.4).

2. Finally, consider the following matrix equation,

$$AX + X^*A^* = B,$$

(6.5)

where $A \in \mathbb{H}^{m \times n}$ and $B \in \mathbb{H}^{m \times m}$. Hodges [10] found the explicit solution to (6.5) and expressed it in terms of the Moore-Penrose inverse over a finite field. Djordjević [7] extended these results to the infinite dimensional settings. Due to [7, 10], the following lemma can be expanded to $\mathbb{H}$.

**Lemma 6.3.** The following statements are equivalent.

(i) There exists a solution $X \in \mathbb{H}^{n \times m}$ to Eq. (6.5).

(ii) $B^* = B$ or $R_ABR_A = 0$.

In that case, the general solution to (6.5) can be expressed as the following,

$$X = A^\dagger B \left( I - \frac{1}{2} Q_A \right) + L_A Y + P_A Z A^*,$$

(6.6)

where $Z \in \mathbb{H}^{m \times m}$ satisfies $A(Z + Z^*)A^* = 0$ and $Y \in \mathbb{H}^{m \times m}$ is arbitrary.

**Proof.** The proof is similar to (7, Theorem 2.2).

Note that, in [7], the equation $A^*X + X^*A = B$ has been considered instead of (6.5) with $A$ and $B$ as operators of Hilbert spaces. The result obtained in [7] would be equal to (6.6) by substituting $A$ with $A^*$ and taking into account

$$P_A^* = (A^*)^\dagger = A^* = (AA^\dagger)^* = Q_A,$$

so $Q_A = P_A$, $L_A = I - P_A = I - Q_A = R_A$, and $R_A = L_A$.

By putting $Z = Y = 0$, we have the following partial solution to (6.5),

$$X = A^\dagger B \left( I - \frac{1}{2} Q_A \right).$$

(6.7)

The following theorem on determinantal representations of (6.7) can be proven similar to Theorem 6.2.

**Theorem 6.4.** Let $A = (a_{ij}) \in \mathbb{H}_{r_1}^{m \times n}$. Then (6.7) has the determinantal representations

$$x_{ij} = \frac{\sum_{\beta \in J_{r_1,n}(i)} \text{cdet}_i\left( (A^*A)_j \left( (1)_{b_{ij}} \right) \right)_{\beta}}{\sum_{\beta \in J_{r_1,n}} |A^*A|_{\beta}^2} - \frac{\sum_{\alpha \in I_{r_1,m}(j)} \text{rdet}_j((AA^*)_j \left( (1)_{d_{\alpha}} \right))_{\alpha}}{2 \sum_{\beta \in J_{r_1,n}} |A^*A|_{\beta}^2 \sum_{\alpha \in I_{r_1,m}} |AA^*|_{\alpha}^2},$$

(6.8)
where \( b^{(1)}_j \) is the column vector of \( B_1 := A^* B \), and

\[
\begin{align*}
d_i &= \left[ \sum_{\beta \in J_{r_1,n}(i)} \cdet_{i} \left( (A^* A)_i \left( b^{(2)}_{(k)} \right) \right)^{\beta} \right] \in \mathbb{H}^{1 \times n}, \quad k = 1, \ldots, n, \\
d_j &= \left[ \sum_{\alpha \in I_{r_1,m}(j)} \rdet_{j} \left( (A A^*)_j \left( b^{(2)}_{(l)} \right) \right) \right] \in \mathbb{H}^{n \times 1}, \quad l = 1, \ldots, n,
\end{align*}
\]

are the row vector and the column vector, respectively. \( b^{(2)}_k \) and \( b^{(2)}_l \) are the \( k \)th column and the \( l \)th row of \( B_2 := A^* B A A^* \).

### 7 Examples

In this section, we give an example to illustrate our results.

1. Consider the matrix equation

\[
A_1 X_1 B_1 + A_2 X_2 B_2 = C, \tag{7.1}
\]

with given matrices

\[
A_1 = \begin{bmatrix} i & 1 \\ -1 & i \\ k & -j \end{bmatrix}, \quad A_2 = \begin{bmatrix} i \\ j \\ k \end{bmatrix}, \quad B_1 = \begin{bmatrix} i \\ j \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ -i \\ 2j \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ j \end{bmatrix}.
\]

By this given matrices, the consistency conditions of (3.1) from Lemma 3.1 are fulfilled. So, the system (7.1) is resolvable. Using determinantal representations (2.1)-(2.2) for computing Moore-Penrose inverses, we find that

\[
A_1^\dagger = \frac{1}{6} \begin{bmatrix} i & -1 & -k \\ -1 & 2i & -j \\ i & k & 2 \end{bmatrix}, \quad R_{A_1} = \frac{1}{3} \begin{bmatrix} 2 & i & -j \\ -i & 2 & -k \\ j & k & 2 \end{bmatrix}, \quad B_1^\dagger = \frac{1}{2} \begin{bmatrix} -i & -k \\ -j & i \end{bmatrix}, \quad B_2^\dagger = \frac{1}{2} \begin{bmatrix} -j & i \\ -i & 4j \end{bmatrix}, \quad M = \frac{1}{3} \begin{bmatrix} 2 & 2i & 4j \end{bmatrix}, \quad M^\dagger = \frac{1}{4} \begin{bmatrix} 1 & -i & -j \end{bmatrix}.
\]

So, \( L_{B_1} = \) and \( N = 0 \). By putting free matrices \( U, V, Z, \) and \( W \) as zero-matrices, we first obtain the pair solution by direct matrix multiplications

\[
X_1 = A_1^\dagger C B_1^\dagger - A_1^\dagger A_2 M^\dagger C B_1^\dagger = \frac{1}{8} \begin{bmatrix} 1 & j \\ i & k \end{bmatrix}, \quad X_2 = M^\dagger C B_2^\dagger = \frac{3}{4} \begin{bmatrix} -j & i \end{bmatrix}.
\]
Now, we find the solution to (7.1) by our new proposed approach, namely, by Cramer’s Rule thanks to Theorem 3.3. Since $A_i^* CB_j^* = 0$, then $x_{ij}^{(11)} = 0$ for all $i, j = 1, 2$. Therefore, rank $A_1 = \text{rank} \ A_2 = \text{rank} \ B_1 = \text{rank} \ B_2 = 1$, and

$$M^*M = \begin{bmatrix} 8 & 3 \\ 3 & 1 \end{bmatrix}, \ C_2 = [-4i \ -4k], \ B_1 B_1^* = \begin{bmatrix} 1 & j \\ -j & 1 \end{bmatrix}, \ A_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}. $$

So,

$$ x_{11}^{(1)} = x_{11}^{(12)} = \frac{-(i)(-4i)}{6 \cdot \frac{3}{2}} = \frac{1}{2}. \quad (7.3) $$

Hence, $x_{11}^{(1)}$ obtained by Cramer’s Rule (7.3) and by the matrix method (7.2) are equal. Similarly, we can obtain for $x_{12}^{(1)}, x_{21}^{(1)}, x_{22}^{(1)}, x_{11}^{(2)}, \text{and } x_{12}^{(2)}$.

2. Let us consider the matrix equation

$$AX + X^* A^* = B, \quad (7.4)$$

where

$$A = \begin{bmatrix} 2 & j \\ -k & i \\ i & k \end{bmatrix}, \quad B = \begin{bmatrix} 2 & j & -k \\ -j & 1 & i \\ k & -i & 2 \end{bmatrix}. $$

Since, $B^* = B$, then, by Lemma 6.3, the equation (7.4) is consistent. Since $A^* A = \begin{bmatrix} 6 & 4j \\ -4j & 3 \end{bmatrix}$ and $\det A = 2$, then $\text{rank} A = 2$. By Theorem 2.3 and Lemma 2.4, one can find,

$$A^\dagger = \frac{1}{2} \begin{bmatrix} 2 & -k & i \\ 2j & -2i & -2k \end{bmatrix}, \quad Q_A = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & j \end{bmatrix}. $$

First, we can find the solution to (7.4) by direct calculation. By (6.7),

$$X = A^\dagger B - \frac{1}{2} A^\dagger B Q_A = 0.5 \begin{bmatrix} 4 - i - j & 1 + 2j - k & 2i - j - 2k \\ 2 + 4j + 2k & -2 - 2i + 2j & 2 - 2i - 4k \end{bmatrix} - 0.25 \begin{bmatrix} 4 - i - j & -i + j - 1.5k & 1 + 0.5i - j - k \\ 2 + 4j + 2k & -1 - 3i + k & 2 - i + j - k \end{bmatrix} = 0.25 \begin{bmatrix} 4 - i - j & 2 + i + 3j - 0.5k & -1 - 2.5i + j - 3k \\ 2 + 4j + 2k & -3 - i + 4j - k & 2 - 3i - j - 7k \end{bmatrix}. \quad (7.5) $$

Now, we find the solution to (7.4) by its determinantal representation (6.8).

Since,

$$AA^* = \begin{bmatrix} 5 & 3k & -3i \\ -3k & 2 & 2j \\ 3i & -2j & 2 \end{bmatrix},$$

$$B_1 := A^* B = \begin{bmatrix} 4 + i + j & -1 + 2j + k & -2i + j - 2k \\ 1 - 2j + k & 1 - i + j & 1 + i - 2k \end{bmatrix},$$

$$B_2 = A^* B A A^* = \begin{bmatrix} 29 - i - j & -i + j + 18k & -1 - 18i - k \\ 2 - 19j + 2k & -1 - 12i + k & -i - j - 12k \end{bmatrix}. $$

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and

\[
d_{11}^{(1)} = \sum_{\beta \in J_{2,2}(1)} c_{\det 1} \left( (A^*A)_{1,1} \left( b_{1,1}^{(2)} \right) \right)_{\beta} = \\
\quad c_{\det 1} \begin{bmatrix} 29 - i - j & 4j \\ 2 - 19j + 2k & 3 \end{bmatrix} = 11 - 11i - 11j,
\]

\[
d_{12}^{(1)} = \sum_{\beta \in J_{2,2}(1)} c_{\det 1} \left( (A^*A)_{1,2} \left( b_{1,2}^{(2)} \right) \right)_{\beta} = \\
\quad c_{\det 1} \begin{bmatrix} -i - j + 18k & 4j \\ -1 - 12i + k & 3 \end{bmatrix} = -7i + 7j + 6k,
\]

\[
d_{13}^{(1)} = \sum_{\beta \in J_{2,2}(1)} c_{\det 1} \left( (A^*A)_{1,3} \left( b_{1,3}^{(2)} \right) \right)_{\beta} = \\
\quad c_{\det 1} \begin{bmatrix} -1 - 18i - k & 4j \\ -i - j - 12k & 3 \end{bmatrix} = -7 - 6i - 7k,
\]

and

\[
(AA^*)_{1,1} \left( d_{1,1}^{(1)} \right) = \begin{bmatrix} 11 - 11i - 11j & -7i + 7j + 6k & -7 - 6i - 7k \\
-3k & 2 & 2j \\
3i & -2j & 2 \end{bmatrix},
\]

then

\[
x_{11} = \\
\quad = \sum_{\beta \in J_{2,2}(1)} c_{\det 1} \left( (A^*A)_{1,1} \left( b_{1,1}^{(2)} \right) \right)_{\beta} - \sum_{\alpha \in I_{2,3}(1)} r_{\det 1} \left( (A^*A)_{1,1} \left( d_{1,1}^{(1)} \right) \right)_{\alpha} \\
\quad = \frac{1}{4} c_{\det 1} \begin{bmatrix} 4 + i + j & 4j \\ 1 - 2j + k & 3 \end{bmatrix} - \frac{1}{4} (r_{\det 1} \begin{bmatrix} 11 - 11i - 11j & -7i + 7j + 6k \\ -3k & 2 \end{bmatrix} + \\
+ r_{\det 1} \begin{bmatrix} 11 - 11i - 11j & -7 - 6i - 7k \\ 3i & 2 \end{bmatrix}) = \frac{1}{4} (4 - i - j).
\]

So, \( x_{11} \) obtained by Cramer’s rule and the matrix method \((7.5)\) are equal.

Similarly, we can obtain for all \( x_{ij} \), \( i = 1, 2 \) and \( j = 1, 2, 3 \).

Note that we used Maple with the package CLIFFORD in the calculations.

## 8 Conclusions

Within the framework of the theory of row-column determinants, we have derived explicit formulas for determinantal representations (analogs of Cramer’s
Rule) of solutions to the quaternion two-sided generalized Sylvester matrix equation $A_1X_1B_1 + A_2X_2B_2 = C$ and its all special cases when its first term or both terms are one-sided. Finally, determinantal representations of two like-Lyapunov equations have been obtained. To accomplish that goal, determinantal representations of the Moore-Penrose inverse previously introduced by the author have been used.

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