A COUNTDOWN PROCESS, WITH APPLICATION TO THE RANK OF MATRICES OVER $\mathbb{F}_q(n)$

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Abstract. Motivated by the work of Fulman and Goldstein, comparing the distribution of the corank of random matrices in $\mathbb{F}_q[n]$ with the limit distribution as $n \to \infty$, we define a countdown process, driven by independent geometric random variables related to random integer partitions. Analysis of this process leads to sharper bounds on the total variation distance.

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1. Introduction

Fulman and Goldstein [6] used Stein’s method to get lower and upper bounds on the total variation distance, between the rank distribution for random $n$ by $n + m$ matrices over the finite field $\mathbb{F}_q$, and its limit for $n \to \infty$. For $m \geq 0$, with notation that suppresses the dependence on $m$ from the random rank, [6] proved that the distance satisfies the upper and lower bounds

$$\frac{1}{8q^{n+n+1}} \leq d_{TV}(Q_{q,n}, Q_q) \leq \frac{3}{q^{n+n+1}},$$

so that the upper bound is 24 times the lower bound. We provide a sharper upper bound with a very simple proof in Theorem 3.5. With more computation, Theorem 6.2 provides matching upper and lower bounds, for $m \geq 0$, which differ by a factor of 2. Theorem 6.2 also gives an explicit asymptotic
formula, for all $m \in \mathbb{Z}$. It is necessary to deal with $m < 0$ as a separate case from $m \geq 0$; although matrix transpose provides a handle on the rank distribution, there is a subtle effect on total variation distance, so that for $m < 0$ the lower bound is of order $1/q^{n+1}$ rather than $1/q^{m+n+1}$. Note that [6] also used Stein’s method to handle five other classes of matrices: symmetric, symmetric with zero diagonal, skew symmetric, skew centrosymmetric, and Hermitian matrices, but our method only handles the simplest case.

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2. A Markov chain from linear algebra

Write $F_q$ for the finite field with $q$ elements. The well-known formula for the number of nonsingular $n$ by $n$ matrices over $F_q$, that
\[
|GL(n, q)| = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-2})(q^n - q^{n-1})
\]
has a well-known, and somewhat prettier probabilistic interpretation, by comparing with the number of all $n$ by $n$ matrices over $F_q$,
\[
P(\text{nonsingular}) = \frac{|GL(n, q)|}{|F_q^n|} = \frac{|GL(n, q)|}{q^{n^2}} = g_n(q^{-1})
\]
where
\[
g_n(x) := (1 - x^n)(1 - x^{n-1}) \cdots (1 - x^2)(1 - x).
\]
This function $g_n$ may be viewed as a perturbation of a simpler object, the Euler function
\[
g(x) := \prod_{i \geq 1}(1 - x^i), \quad \text{for } x \text{ with } |x| < 1,
\]
sometimes called the reciprocal of the partition function, and famous for its role in the Euler pentagonal number theorem [2].

Implicit in (2) is a story for $n$ by $n + m$ matrices, allowing $m \in \mathbb{Z}$ to be negative but requiring both $n \geq 0$ and $n + m \geq 0$, thinking of $m$ as time. In this story, one thinks about an entire process, evolving in time, and a natural question arises: what is the time to hit zero, that is, how many length $n$ columns are needed to span a space of dimension $n$? The process story is given in detail in the following paragraph.

For fixed $n$, consider independent random vectors $v_1, v_2, \ldots$, distributed uniformly over the $q^n$ values in $F_q^n$. With $A_k$ taken to be the space spanned by the first $k$ of these vectors, so that $A_0$ is the singleton set containing only the all zero vector, consider the corank of $A_k$, for $k = 0, 1, 2, \ldots$. (We say corank, thinking of the $n$ by $k$ matrix with columns $v_1, \ldots, v_k$; the term codimension might be more correct, but no confusion arises from using the simpler word.) As $k$ increases, this corank decreases from $n$ down to zero. Given that the corank of $A_k$ is $i$, and otherwise independent of $v_1, v_2, \ldots, v_k$, the chance that $v_{k+1} \in A_k$ so that corank($A_{k+1}$) = corank($A_k$) = $i$ rather than corank($A_{k+1}$) = $i - 1$, is exactly $q^{n-i}/q^n = q^{-i}$, regardless of the
value of \( n \). Trivially, this conditional independence leads to a Markov chain, which is a pure death process, with independent, geometrically distributed holding times. We celebrate these observations as a formal statement, for future reference.

**Proposition 2.1.** For any \( n \geq 1 \), in the preceding story over \( \mathbb{F}_q \), write \( Y_k := n - \) the dimension of \( A_k \). Then, with \( x := 1/q, Y_0, Y_1, Y_2, \ldots \) is a Markov chain on \( \mathbb{Z}_+ \), with transition probabilities

\[
p(i, j) = \begin{cases} x^i & \text{if } j = i \\ 1 - x^i & \text{if } j = i - 1 \end{cases}
\]

starting at \( n \).

**Proof.** The proof is given by the previous paragraph. \( \square \)

For the sake of comparing the distribution for \( n \) with its limit distribution as \( n \to \infty \), it is convenient and natural to shift the time, replacing \( k \) by \( t = k - n \), so that the growing spaces \( A_0, A_1, A_2, \ldots \) have coranks decreasing, from \( n \) down to zero, with the time-shift taken so that, in the case corresponding to a nonsingular matrix, corank zero is hit at time \( t = 0 \).

### 2.1. Counting down from infinity.

The (deterministic) countdown process, with all zero delays, is

\[
x = (x_t)_{t \in \mathbb{Z}} := \phi(0) \quad \text{with } x_{-t} = t, x_t = 0 \text{ for } t = 0, 1, 2, \ldots.
\]

The space of allowable delays is

\[
\Omega := \{ z = (z_1, z_2, \ldots) : z_1 + z_2 + \ldots < \infty \} \subset (\mathbb{Z}_+)^N,
\]

with least element \( 0 := (0, 0, \ldots) \in \Omega \). For general \( z \in \Omega \), the value \( x = \phi(z) \) of the deterministic countdown process is that perturbation of the path given by (6) such that

\[
z_i \text{ is the delay at height } i, \ i = 1, 2, \ldots, \ \text{and } 0 = \lim_{t \to \infty} x_{-t} - t.
\]

We use the indicator notation \( 1(Q) = 1 \) if statement \( Q \) is true, \( 1(Q) = 0 \) if statement \( Q \) is false. We also write \( \mathbb{N} = \{1, 2, \ldots\} \) and \( \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \).

A formal version of the informal specification (8), naming the domain and codomain, is that

\[
\phi : \Omega \to (\mathbb{Z}_+)^\mathbb{Z}
\]

satisfies

\[
x = \phi(z) \text{ satisfies } \forall i \geq 1, 1 + z_i = \sum_{t \in \mathbb{Z}} 1(x_t = i), \ \forall t, x_t - x_{t-1} \in \{0, 1\},
\]

and \( 0 = \lim_{t \to \infty} (x_{-t} - t) = \lim_{t \to \infty} x_t \).

Clearly, the map \( \phi \) is a bijection between \( \Omega \), and the image, \( \phi(\Omega) \).

**Observation 2.2.** Suppose that \( x = (x_t)_{t \in \mathbb{Z}} = \phi(z) \), where \( z \in \Omega \). Then the hitting time to zero, for the trajectory \( x \), is

\[
h_0(x) := \min \{ t : x_t = 0 \} = z_1 + z_2 + \ldots.
\]
The process stays on the line $x = -t$ for all $t \leq -4$. The largest $i$ for which $z_i$ is nonzero is $i = 4$ with $z_4 = 2$, so there are two delays at height four, and therefore three points $(t, x_t)$ for which $x_t = 4$. Similarly, $z_2 = 3$ causes the process to be delayed 3 times at height 2, and $z_1 = 1$ causes $x$ to spend one extra unit of time at height 1, before dropping down permanently to the $t$-axis.

Consider the two circled points in Figure 1 at $(3, 2)$ and $(4, 1)$. Here, a “death” has occurred at time 3, and $x_t$ has decreased as $t$ increased. Both of these points are on the line $x = -t + 5$, whereas the process started on the line $x = -t$. The process has moved from the line $x = -t$ for all sufficiently large $x$ to the line $x = -t + 5$ for $x = 2, 1$ because there were 5 delays at heights 2 and above, corresponding to the fact that $z_2 + z_3 + z_4 + \cdots = 5$. In general, we have the following observation, which will be important later.

**Observation 2.3.** Suppose that $x = (x_t)_{t \in \mathbb{Z}} = \phi(z)$, where $z = (z_1, z_2, \ldots)$. Then

$$x_t = k \text{ and } x_{t+1} = k - 1 \text{ if and only if } t + k = z_k + z_{k+1} + z_{k+2} + \ldots.$$
COUNTDOWN FOR CORANK, FROM \( \infty \) TO ZERO

\[ t + x = 1, \text{ at height } x = i_0 \text{ and times } t = i_0, i_0 + 1, \text{ does not show up in the frame of the picture.} \]

2.2. Geometrically distributed delays, or \textit{driving noise}. Fix \( x \in (0, 1) \). Let \( Z \) be a process of independent geometrically distributed random variables, with

\[ Z = (Z_1, Z_2, \ldots), \quad P(Z_i \geq k) = x^k, \quad k = 0, 1, 2, \ldots. \]

(This process is natural to the study of random integer partitions; see Remark 4.2 for some details.) For the process with all coordinates indexed by \( i > n \) zeroed out, we write

\[ Z^{(n)} = (Z_1, Z_2, \ldots, Z_n, 0, 0, \ldots). \]

Taking the sum of all coordinates, in each of the two processes specified by (10) and (11), we have

\[ S := Z_1 + Z_2 + \cdots, \quad S_n := Z_1 + \cdots + Z_n, \]

with \( S_0 := 0 \). Applying the countdown function \( \phi \), defined by (8) – (9), to each of the two processes specified by (10) and (11), we have

\[ X := \phi(Z), \quad X^{(n)} := \phi(Z^{(n)}). \]

We will be interested in comparing \( X \) with \( X^{(n)} \), and a first step is to compare \( S \) with \( S_n \), so we also define

\[ R_n := S - S_n = Z_{n+1} + Z_{n+2} + \cdots, \]

for \( n = 0, 1, 2, \ldots \). Note that \( S = R_0 \).

\textbf{Proposition 2.4.} For any \( x \in (0, 1) \), for any \( n \geq 1 \), \( X^{(n)}_1, X^{(n)}_{-n+1}, X^{(n)}_{-n+2}, \ldots \) is a Markov chain on \( \mathbb{Z}_+ \), with transition probabilities given by (5), starting at \( n \).

\textbf{Proof.} Obvious; it corresponds to the “memoryless” property of geometric distributions. \qed
Proposition 2.5. For $x = 1/q$ where $q$ is a prime power, for $n \geq 1$, the segment $X^{(n)}_m, X^{(n)}_{m+1}, X^{(n)}_{m+2}, \ldots$ of $X^{(n)}$ is a realization of $Y_0, Y_1, Y_2, \ldots$, the Markov chain for linear algebra over $\mathbb{F}_q$ as in Proposition 2.1. In particular, for integers $m, n$ with $n \geq 1$ and $n + m \geq 0$,

$$X^{(n)}_m = d Y_{n+m}$$

(15) $= d n - \text{the rank of a random } n \times n + m \text{ matrix } M \text{ over } \mathbb{F}_q$.

Proof. Obvious again, apart from the trickiness of the time shift by $n$ connecting the two processes. □

Proposition 2.6. For $x = 1/q$ where $q$ is a prime power, for $m \in \mathbb{Z}$, $X_m$, from the process $X$ defined by (10) and (13), is distributed as the limit, upon $n \to \infty$, of $n$ minus the rank of a random $n \times n + m$ matrix $M$ over $\mathbb{F}_q$.

Proof. Obvious from Proposition 2.5 and the coupling, with $\mathbb{P}(X_m \neq X^{(n)}_m) \leq \mathbb{P}(X \neq X^{(n)}) = \mathbb{P}(Z \neq Z^{(n)}) = \mathbb{P}(Z_{n+1} + Z_{n+2} + \cdots \neq 0)$. □

3. Easy bounds on total variation distance

The total variation distance between random elements, say $X, Y$ in a space $S$ is defined, in general, by

$$d_{TV}(X, Y) = \sup_{B \subset S} |\mathbb{P}(X \in B) - \mathbb{P}(Y \in B)|,$$

(16) where the supremum is taken over measurable subsets of $S$. In case $S$ is discrete, this is equivalent to

$$d_{TV}(X, Y) = \sum_k \max(0, \mathbb{P}(X = k) - \mathbb{P}(Y = k)).$$

(17) Another characterization of total variation distance is that the total variation distance between $X$ and $Y$ is equal to the infimum, over all couplings, of $\mathbb{P}(X \neq Y)$; it is understood that the marginal distributions of $X$ and $Y$ have been specified, and coupling means to choose any joint distribution for $(X,Y)$ having the given marginals.

Theorem 3.1. For any $n \geq 1$ and $x \in (0, 1)$, the total variation distance between the processes $X$ and $X^{(n)}$ defined by (10) and (13) is

$$d_{TV}(X, X^{(n)}) = \mathbb{P}(R_n > 0)$$

(18) $$= 1 - \prod_{i=n+1}^{\infty} (1 - x^i),$$

and

$$\frac{x^{n+1}}{1 - x} - \frac{x^{2n+3}}{(1 - x)^2} \leq d_{TV}(X, X^{(n)}) \leq \frac{x^{n+1}}{1 - x}.$$

(19)
Proof. Let \( A = \{ x \in (\mathbb{Z}_+)^\mathbb{Z} : x_t \neq -t \text{ for some } t < -n \} \) be the set of paths in which a delay happened at some height above \( n \). Then \( \mathbb{P}(X(n) \in A) = 0 \), but \( \mathbb{P}(X \in A) = \mathbb{P}(Z_{n+1} + Z_{n+2} + \cdots > 0) = \mathbb{P}(R_n > 0) \), showing that

\[
d_{TV}(X, X(n)) \geq \mathbb{P}(X \in A) - \mathbb{P}(X(n) \in A) = \mathbb{P}(R_n > 0).
\]

Conversely, the total variation distance between \( X \) and \( X(n) \) is equal to the infimum, over all couplings, of \( \mathbb{P}(X \neq X(n)) \). In the coupling given by (13), \( X \) and \( X(n) \) are unequal if and only if \( Z_{n+1} + Z_{n+2} + \cdots > 0 \), proving the corresponding upper bound, and hence the equality in (18).

The lower and upper bounds in (19) follow from the exact expression on the right side of (18) and from the “Bonferroni” inequalities

\[
\sum_{1 \leq i \leq m} p_i - \sum_{1 \leq i < j \leq m} p_i p_j \leq 1 - \prod_{i=1}^{m} (1 - p_i) \leq \sum_{1 \leq i \leq m} p_i
\]

valid for all \( m \in \mathbb{N} \), provided \( 0 \leq p_i \leq 1 \) for \( 1 \leq i \leq m \). To use these to prove (19), set \( p_i = x^{n+i} \), let \( m \to \infty \), then use the facts that \( \sum_{i>n} x^i = x^{n+1}/(1 - x) \) and \( \sum_{j<i>n} x^i x^j = x^{2n+3}/(1 - x)^2 \).

The classical (first two) Bonferroni inequalities are

\[
\sum_{1 \leq i \leq m} \mathbb{P}(B_i) - \sum_{1 \leq i < j \leq m} \mathbb{P}(B_i \cap B_j) \leq \mathbb{P}\left( \bigcup_{i=1}^{m} B_i \right) \leq \sum_{1 \leq i \leq m} \mathbb{P}(B_i),
\]

and with \( B_i := \{ Z_{n+i} \geq 1 \} \) we have \( p_i = \mathbb{P}(B_i) \), and for \( i < j \), \( p_i p_j = \mathbb{P}(B_i \cap B_j) \), using the independence of \( Z_1, Z_2, \ldots \), we have exactly (20).

\[ \square \]

Corollary 3.2. For any \( n \geq 1 \), \( t \in \mathbb{Z} \), and \( x \in (0,1) \),

\[
d_{TV}(X_t, X_t^{(n)}) \leq u(n) := 1 - \prod_{i=n+1}^{\infty} (1 - x^i) \leq \frac{x^{n+1}}{1 - x}
\]

and

\[
d_{TV}(S, S_n) \leq u(n) := 1 - \prod_{i=n+1}^{\infty} (1 - x^i) \leq \frac{x^{n+1}}{1 - x}.
\]

Proof. These upper bounds are an immediate corollary of Theorem 3.1, since with the deterministic function \( e_t \), “extract coordinate \( t \)”, we have \( X_t = e_t(X) \) and \( X_t^{(n)} = e_t(X^{(n)}) \), and with deterministic functional \( h_0 \), the hitting time to zero from Observation 2.2, we have \( S = h_0(X) \) and \( S_n = h_0(X^{(n)}) \).

\[ \square \]

Note the the upper bound \( u(n) \) in (21) does not vary with \( t \in \mathbb{Z} \). In the next proposition, we give a sort of matching lower bound; this lower bound varies with \( t \). The upper bound in (21) is quite poor for \( t > 0 \), as will eventually be seen from Theorem 6.2. The next proposition gives our “easy” lower bound for cases with \( t \leq 0 \).
Proposition 3.3. For $x \in (0, 1)$, for integers $n, t$ with $n \geq 1, t \leq 0$,

$$d_{TV}(X_t, X_t^{(n)}) \geq \mathbb{P}(X_t^{(n)} + t = 0) - \mathbb{P}(X_t + t = 0)$$

$$= \left( \prod_{-t < i \leq n} (1 - x^i) \right) \left( 1 - \prod_{i=n+1}^{\infty} (1 - x^i) \right)$$

$$\geq \ell(t, n) := \left( \prod_{-t < i \leq n} (1 - x^i) \right) \left( \frac{x^{n+1}}{1 - x} - \frac{x^{2n+3}}{(1 - x)^2} \right).$$

Proof. For $t \leq 0$, that the event $\{X_t + t = 0\}$ equals the event that $(Z_{-t+1} = Z_{-t+2} = \cdots = 0)$, and $\{X_t^{(n)} + t = 0\}$ equals the event that $(Z_{-t+1} = Z_{-t+2} = \cdots = Z_n = 0)$. The inequality is the same that we used in getting (19) from (18).

Consider the relation between exact formulas, asymptotics, lower bounds, and upper bounds. We use the notation $a_n \sim b_n$ to mean that $a_n$ is asymptotically equal to $b_n$, formally defined by $\lim_{n \to \infty} a_n / b_n = 1$. It is obvious from (19) that exact expression for the distance between processes may be described asymptotically, with

$$d_{TV}(X, X^{(n)}) \sim \frac{x^{n+1}}{1 - x}.$$  

It is more difficult to give asymptotics for the distance between the marginals, $d_{TV}(X_t, X_t^{(n)})$. Corollary 3.2 and Proposition 3.3 provide upper and lower bounds, $u(n)$ and $\ell(t, n)$, with

$$r(t) := \lim_{n \to \infty} \frac{\ell(t, n)}{u(n)} = \prod_{-t < i} (1 - x^i)$$

which, with the notation from (4) and the display above (4) is $r(t) = g(x)/g_{-t}(x)$. At $t = 0$ the product $g_{-t}$ has no factors; it is identically 1, and we have $r(t) = g(x)$. As $t \to -\infty$, $g_{-t}$ acquires more and more of the factors of $g$, and $r(t) \to 1$, so in a sense, the upper and lower bounds combined come close to giving the asymptotic total variation distance.

We will completely handle the task of giving asymptotics for $d_{TV}(X_t, X_t^{(n)})$, with Theorem 6.2.

Proposition 3.4. For $x = 1/q$ where $q$ is a prime power, for any integers $t, n$ with $n \geq 0, n + t \geq 0$

$$X_t^{(n)} \overset{d}{=} X_{-t}^{(n+t)} - t.$$
Proof. Consider a random \( n \times (n + t) \) matrix \( M \) over \( \mathbb{F}_q \). We exploit the fundamental result that row rank equals column rank.

\[
X_t^{(n)} = d \quad \text{column rank of } M = n - \text{row rank of } M = (n + t - \text{column rank of } M^T) - t = d \quad X_t^{(n+1)} - t.
\]

\[\square\]

**Theorem 3.5.** For any prime power \( q \), for any \( m \geq 0 \), for every \( n \geq 1 \), the total variation distance, between the \( Q_{q,n} := (n \text{ minus the rank of a random } n \times n + m \text{ matrix over } \mathbb{F}_q) \), and \( Q_q := \text{ the distributional limit upon } n \rightarrow \infty \) of \( Q_{q,n} \), satisfies

\[
d_{TV}(Q_{q,n}, Q_q) \leq \frac{q}{q-1} \frac{1}{q^{n+m+1}}.
\]

Proof. Combine Propositions 2.5 and 2.6, together with Proposition 3.4 and the bound (21) from Corollary 3.2 applied at \( t = m \). Note that we are using \( x = 1/q \), so that \( 1/(1-x) = q/(q-1) \). \[\square\]

**Theorem 3.6.** Exactly as in Theorem 3.5, except that now we take \( m < 0 \). For every \( n \) with \( n + m \geq 0 \),

\[
d_{TV}(Q_{q,n}, Q_q) \leq \frac{q}{q-1} \frac{1}{q^{n+1}}.
\]

Proof. Just as the proof of Theorem 3.5, except that we do not invoke Proposition 3.4. \[\square\]

## 4. A Technical Lemma

For the computations in the next section, we will require the following fact.

**Lemma 4.1.** For all \( n \geq m \geq 0 \), and \( |x| < 1, |y| < 1 \),

\[
\prod_{i=m}^{n} \frac{1}{1 - yx^i} = \sum_{k \geq 0} y^k x^m k \cdot \prod_{k=n-m+1}^{n+k} \frac{1-x^i}{\prod_{i=1}^{k} 1-x^i}.
\]

Proof. It suffices to prove this in the case \( m = 0 \), namely to show that

\[\text{Lemma 4.1:}\]

\[
\prod_{i=0}^{n} \frac{1}{1 - yx^i} = \sum_{k \geq 0} y^k \cdot \prod_{i=k+1}^{n+k} \frac{1-x^i}{\prod_{i=1}^{k} 1-x^i},
\]

since the general result follows from replacing \( n \) with \( n-m \) and \( y \) with \( yx^m \) in (24).

Letting \( F_n(x, y) \) denote the right hand side of (24), elementary power series manipulations obtain that \( (1-yx^n)F_n(x, y) = F_{n-1}(x, y) \), which implies

\[
F_n(x, y) = \frac{1}{1-yx^n} F_{n-1}(x, y).
\]
Iterating the latter relation \( n \) times yields that
\[
F_n(x, y) = \left( \prod_{i=1}^{n} \frac{1}{1 - yx^i} \right) F_0(x, y),
\]
which combined with the base case \( F_0(x, y) = \sum_{k \geq 0} y^k = \frac{1}{1-y} \) proves (24). \( \square \)

**Remark 4.2.** A more conceptual proof of this result can be given, relating the result to random integer partitions, where a partition of \( r \) is given weight \( x^r \); see [1, 5, 4, 3]. (In contrast with the linear algebra applications involving \( x = 1/q \), bounded away from 1, taking \( x = \exp(-\pi/\sqrt{6r}) \) leads to excellent approximations for a random partition of a large integer \( r \); see [8].) In more detail, \( Z_i \) is interpreted as the number of parts of size \( i \), so that \( r = \sum iZ_i \) is the size of the partition, and \( k = \sum Z_i \) is the number of parts. We are considering partitions where all part sizes lie in the range \( m \) to \( n \), and such a partition \( \lambda \) of size \( r \), with exactly \( k \) parts, is in bijective correspondence with a partition \( X' \) of \( r-km \) with at most \( k \) parts, each of size at most \( n-m \), by removing \( m \) from each part of \( \lambda \).

5. **Distributional Results**

We compute the distributions of \( X^{(n)}_r \), and \( X_t \), for all for \( x \in (0, 1) \), \( n \geq 0 \), and \( t \in \mathbb{Z} \). The explicit formulas lead to an interesting symmetry, stated in Corollary 5.6, which in case \( x = 1/q \), where \( q \) is a prime power, was already proved, in Proposition 3.4.

**Lemma 5.1.** For all \( x \in (0, 1) \), for all \( k \in \mathbb{Z}^+ \), and \( n \geq m > 0 \),
\[
\mathbb{P}(S_n - S_{m-1} = k) = x^{mk} \cdot \frac{\prod_{i=m-n-m+1}^{n-m+k+1}(1-x^i)\prod_{i=1}^{m-1}(1-x^i)}{\prod_{i=1}^{n}(1-x^i)}.
\]

**Proof.** Let \( G(s) := \mathbb{E} [s^{(S_n-S_{m-1})}] \) be the probability generating function for \( S_n - S_{m-1} \), so \( G(s) = \sum_{k=0}^{\infty} \mathbb{P}(S_n - S_{m-1} = k)s^k \). Since \( S_n - S_{m-1} \) is a sum \( Z_m + Z_{m+1} + \cdots + Z_n \) of independent geometric random variables, each with probability generating function \( \mathbb{E} [s^{Z_i}] = \frac{1-x^i}{1-sx^i} \), it follows that
\[
\sum_{k=0}^{\infty} \mathbb{P}(S_n - S_{m-1} = k)s^k = G(s) = \prod_{i=m-n-m+1}^{n} \frac{1-x^i}{1-sx^i} = \frac{g_n(x)}{g_{m-1}(x)} \prod_{i=m}^{n} \frac{1}{1-sx^i},
\]
where \( g_n \) is given by (3). Using Lemma 4.1, we can rewrite the product on the right hand side of the previous equation as
\[
\frac{g_n(x)}{g_{m-1}(x)} \prod_{i=m}^{n} \frac{1}{1-sx^i} = \frac{g_n(x)}{g_{m-1}(x)} \sum_{k \geq 0} s^{k} x^{mk} \frac{\prod_{i=n-m-k+1}^{n-m+k+1} 1-x^i}{\prod_{i=1}^{k} 1-x^i}.
\]
Finally, combining the last two equations, then equating the coefficients of \( s^{k} \), proves the lemma. \( \square \)

**Corollary 5.2.** The distribution of \( S_n \) is logconcave.
Proof. Using Theorem 5.1, for $k \geq 0$,

$$
P(S_n = k) = x^k (1 - x^n) \prod_{i=k+1}^{n+k-1} (1 - x^i).
$$

Cancellation of some common factors leads to

$$
(25) \quad \frac{P(S_n = k + 1)}{P(S_n = k)} = x \frac{1 - x^{n+k}}{1 - x^{k+1}}.
$$

Using (25), one can verify that

$$
P(S_n = k + 1)^2 \geq P(S_n = k) P(S_n = k + 2)
$$

holds for all $k \in \mathbb{Z}$, which is precisely the condition that the distribution is log concave.

\[\Box\]

Remark 5.3. In the special case $x = 1/q$ where $q$ is a prime power, the product formula (26) for distribution of $X_t^{(n)}$, given below in Theorem 5.5, governs the number of rectangular matrices of a given rank over $\mathbb{F}_q$, and this case can be traced back to 1893 [7]; it also appears as [9, p. 157, problem 192b].

The product formula has an easy combinatorial proof. Recall that the $q$-binomial coefficients, defined by

$$
\binom{n}{k}_q = \prod_{i=0}^{k-1} \frac{1 - q^{n-i}}{1 - q^{i+1}},
$$

give the number of $k$-dimensional subspaces of $\mathbb{F}_q^n$. Consider an $n \times (n + t)$ matrix with rank $n - k$ as a linear map from $\mathbb{F}_q^{n+t}$ to $\mathbb{F}_q^n$. There are $\binom{n+t}{t+k}_q$ choices for the $(t + k)$-dimensional kernel $K$ of this map, $\binom{n}{n-k}_q$ choices for the $(n-k)$-dimensional image, $I$, and there are $\prod_{i=0}^{n-k-1} (q^{n-k} - q^i)$ ways to specify the nonsingular linear transformation from a fixed complement of $K$ to $I$. This proof is due to Dennis Stanton, private communication.

For symmetric, skew-symmetric, and Hermitian matrices, there are analogous product formulas for the number of matrices of a given rank, due to Carlitz and Hodges, see [10, page 661].

It would seem natural that there should be a transfer principle, so that knowing the result for $x = 1/q$ implies the result for all $x \in (0, 1)$, but we don’t know of such a principle. Such a transfer principle would also apply to reflection symmetry, allowing Proposition 3.4 to imply Corollary 5.6. We believe that our proof of Theorem 5.5, exploiting the Markov property of the countdown process, has both simplicity and novelty.

Open Problem 5.4. Is there a transfer principle, allowing results for the Markov chain defined by (5) with parameter $x \in (0, 1)$ to be deduced, with no extra computation, from the combinatorial and linear algebraic results corresponding to the cases $x = 1/q$ where $q$ must be a prime power?
Theorem 5.5. For all \( x \in (0, 1) \), for all \( n \geq k \geq 0 \), and for all \( t \geq -k \),

\[
(26) \quad \mathbb{P}(X_t^{(n)} = k) = x^{k(t+k)} \cdot \frac{\prod_{i=n-k+1}^{n+t}(1-x^i) \prod_{i=k+1}^{n+t}(1-x^i)}{\prod_{i=1}^{t+k}(1-x^i)},
\]

and

\[
\mathbb{P}(X_t = k) = x^{k(t+k)} \cdot \frac{\prod_{i=k+1}^{\infty}(1-x^i)}{\prod_{i=1}^{t+k}(1-x^i)}.
\]

Proof. Recall from Proposition 2.4 that \( \{X_t^{(n)}\}_{t=n}^{\infty} \) is a time-homogenous Markov process, with transition probabilities

\[
(27) \quad \mathbb{P}(X_{t+1}^{(n)} = k | X_t^{(n)} = k) = x^k, \quad \mathbb{P}(X_{t+1}^{(n)} = k-1 | X_t^{(n)} = k) = 1 - x^k.
\]

In case \( k \geq 1 \), let \( D_{t,k} = \{X_t^{(n)} = k, X_{t+1}^{(n)} = k-1\} \) be the event that there is a “death” at time \( t \) and height \( k \), and \( V_{t,k} = \{X_t^{(n)} = k, X_{t+1}^{(n)} = k\} \) be the event of a survival. Provided \( k \geq 1 \), (27) implies that the ratio of the probabilities of survival to death is given by \( \mathbb{P}(V_{t,k})/\mathbb{P}(D_{t,k}) = x^k/(1-x^k) \).

Combined with the fact that \( \{X_t^{(n)} = k\} \) is the disjoint union of \( D_{t,k} \) and \( V_{t,k} \), we get that

\[
(28) \quad \mathbb{P}(X_t^{(n)} = k) = \frac{1}{1-x^k} \mathbb{P}(D_{t,k}).
\]

The reason that (28) is useful comes from observation 2.3, which implies that \( D_{k,t} \) occurs if and only if \( t+k = Z_k + Z_{k+1} + Z_{k+2} + \cdots + Z_n \), so

\[
(29) \quad \mathbb{P}(D_{t,k}) = \mathbb{P}(S_n - S_{k-1} = t+k).
\]

Combining (28), (29) and Lemma 5.1 proves (26) for all \( k \geq 1 \).

In case \( k = 0 \), we must prove that

\[
(30) \quad \mathbb{P}(X_t^{(n)} = 0) = \prod_{i=t+1}^{n+t} (1-x^i)
\]

which we prove by induction on \( t \). The base case that \( P(X_0^{(n)} = 0) = \prod_{i=1}^{n+t}(1-x^i) \) holds since \( X_0^{(n)} = 0 \) exactly when \( Z_1 = Z_2 = \cdots = Z_n = 0 \). Assuming that (30) holds for \( t-1 \), note that \( X_t^{(n)} = 0 \) implies that \( X_{t-1}^{(n)} \) is
either 0 or 1. Then
\[
P(X_t^{(n)} = 0) = P(X_{t-1}^{(n)} = 0) + P(D_{t-1,1})
\]
\[
= P(X_{t-1}^{(n)} = 0) + P(S_n - S_0 = t)
\]
\[
= \prod_{i=t}^{n+t-1} (1 - x^i) + x^t \cdot \prod_{i=n}^{n+t-1} (1 - x^i) \prod_{i=1}^{n} (1 - x^i)
\]
\[
= \prod_{i=t}^{n+t-1} (1 - x^i) + x^t \prod_{i=t+1}^{n+t-1} (1 - x^i)
\]
\[
= \left( \prod_{i=t}^{n+t-1} (1 - x^i) \right) \cdot \left( (1 - x^t) + x^t (1 - x^n) \right)
\]
\[
= \prod_{i=t+1}^{n+t} (1 - x^i),
\]
completing the proof by induction.

Finally, \( P(X_t = k) = \lim_{n \to \infty} P(X_t^{(n)} = k) \), since \( X_t^{(n)} \) converges to \( X_t \) almost surely, and therefore in distribution.

\[ \square \]

**Corollary 5.6.** For all \( x \in (0, 1) \), for all \( n \geq 0 \), for all \( t \in \mathbb{Z} \),
\[
X_t^{(n)} \overset{d}{=} X_{-t}^{(n+t)} - t,
\]
and
\[
X_t \overset{d}{=} X_{-t} - t.
\]

**Proof.** It is routine to use Theorem 5.5 to verify that for all \( k \geq 0 \), \( P(X_t^{(n)} = k) = P(X_{-t}^{(n+t)} = t + k) \) for all \( n \geq 0 \) and \( P(X_t = k) = P(X_{-t} = t + k) \). \( \square \)

### 6. Total Variation Distances, for \( 0 < x \leq 1/2 \)

**Theorem 6.1.** Suppose that \( x \in (0, 1) \). Then, in case \( x \leq 1/2 \),
\[
d_{TV}(S, S_n) = \prod_{i=1}^{n}(1 - x^i) \left( 1 - \prod_{i=n+1}^{\infty} (1 - x^i) \right)
\]
\[
\sim \frac{g(x)}{1 - x} \cdot x^{n+1},
\]
where \( g(x) \) is defined in (4).
Proof. From Lemma 5.1, we have that
\[ \mathbb{P}(S_n = k) = x^k (1 - x^n) \prod_{i=k+1}^{n+k-1} (1 - x^i), \]
\[ \mathbb{P}(S = k) = x^k \cdot \prod_{i=k+1}^{\infty} (1 - x^i). \]

The key observation is that \( \mathbb{P}(S_n = k) > \mathbb{P}(S = k) \) when \( k = 0 \), but the reverse inequality holds otherwise. To see this, consider the ratio
\[ \frac{\mathbb{P}(S_n = k)}{\mathbb{P}(S = k)} = \frac{1 - x^n}{\prod_{i=n+1}^{\infty} (1 - x^i)}. \]

When \( k = 0 \), this ratio is \( \frac{1 - x^n}{\prod_{i=n+1}^{\infty} (1 - x^i)} = 1 \), while when \( k \geq 1 \), the ratio is less than one, as shown below:
\[ 1 - x^n \leq 1 - \frac{x^{n+1}}{1 - x} = 1 - \sum_{i=n+1}^{\infty} x^i \leq \prod_{i=n+1}^{\infty} (1 - x^i) \leq \prod_{i=n+k}^{\infty} (1 - x^i). \]

The first inequality above uses the fact that \( \frac{x}{1 - x} \leq 1 \), which follows from \( x \leq \frac{1}{2} \).

We have proven that \( \mathbb{P}(S_n = k) > \mathbb{P}(S = k) \) when \( k = 0 \), but the reverse inequality holds otherwise, which implies that total variation distance is simply given by
\[ d_{TV}(S, S_n) = \mathbb{P}(S_n = 0) - \mathbb{P}(S = 0) = \prod_{i=1}^{n} (1 - x^i) \left( 1 - \prod_{i=n+1}^{\infty} (1 - x^i) \right). \]

The above exact expression for \( d_{TV}(S, S_n) \) implies the asymptotic result \( d_{TV}(S, S_n) \sim x^{n+1} \cdot g(x)/(1 - x) \) by using the bounds
\[ \left( \sum_{i>n} x^i \right) - x^{n+3}/(1 - x)^2 \leq 1 - \prod_{i=n+1}^{\infty} (1 - x^i) \leq \sum_{i>n} x^i, \]
which follow from (20).

Finally, we have exact and asymptotic results for the total variation distance between each coordinate \( X_t^{(n)} \) and \( X_t \) of the two processes, provided that \( x \leq \frac{1}{2} \).

**Theorem 6.2.** Suppose that \( x \in (0, 1/2] \). For all \( n, t \geq 0 \),
\[ d_{TV}(X_t, X_t^{(n)}) = \left( \prod_{i=t+1}^{n+t} 1 - x^i \right) \left( 1 - \prod_{i=n+t+1}^{\infty} \left( 1 - x^i \right) \right) \sim \frac{C_t}{1 - x} \cdot x^{n+t+1}, \]
where \( C_t = \prod_{i=t+1}^{\infty} 1 - x^i \). When \( t < 0 \),

\[
d_{TV}(X_t, X_t^{(n)}) = \left( \prod_{i=|t|+1}^{n} 1 - x^i \right) \left( 1 - \prod_{i=n+1}^{\infty} 1 - x^i \right) \sim \frac{C_{|t|}}{1 - x} \cdot x^{n+1}.
\]

**Proof.** The proof is similar to that of Theorem 6.1.

First, suppose \( t \geq 0 \). For any \( k \in \mathbb{Z}_+ \), consider the ratio between \( P(X_t^{(n)} = k) \) and \( P(X_t = k) \), which we attain using Theorem 5.5:

\[
\frac{P(X_t^{(n)} = k)}{P(X_t = k)} = \frac{\prod_{i=n-k+1}^{n} (1 - x^i)}{\prod_{i=n+t+1}^{\infty} (1 - x^i)}.
\]

When \( k = 0 \), the numerator is an empty product, which means the above ratio is greater than one. However, for all \( k \geq 1 \), the ratio is less than one, as shown be the following computation:

\[
\prod_{i=n-k+1}^{n} (1 - x^i) \leq 1 - x^n \leq 1 - \frac{x^{n+1}}{1 - x} \leq 1 - \sum_{i=n+t+1}^{\infty} x^i \leq \prod_{i=n+t+1}^{\infty} (1 - x^i).
\]

As in the proof of Theorem 6.1, the total variation distance is simply given by

\[
d_{TV}(X_t, X_t^{(n)}) = P(X_t^{(n)} = 0) - P(X_t = 0) = \prod_{i=|t|+1}^{n+t} (1 - x^i) - \prod_{i=|t|+1}^{\infty} (1 - x^i) = \left( \prod_{i=|t|+1}^{n+t} 1 - x^i \right) \left( 1 - \prod_{i=n+t+1}^{\infty} 1 - x^i \right),
\]

as claimed. The asymptotic result also follows similarly.

In case \( t < 0 \), the result follows from the \( t \geq 0 \) case by using Corollary 5.6, which implies that

\[
(X_t, X_t^{(n)}) = d (X_{-t} + t, X_{-t}^{(n-t)} + t).
\]

so

\[
d_{TV}(X_t, X_t^{(n)}) = d_{TV}(X_{-t} + t, X_{-t}^{(n-t)} + t) = d_{TV}(X_{|t|}, X_{|t|}^{(n-t)}).
\]

We remark that Theorem 6.2 implies the more elementary bounds, for \( t \geq 0 \)

\[
\frac{1}{2(1 - x)} \cdot x^{n+t+1} \leq d_{TV}(X_t, X_t^{(n)}) \leq \frac{1}{1 - x} \cdot x^{n+t+1}.
\]
In the case \( x = 1/q \), this narrows the ratio of upper bound to lower bound in (1) from 24 to 2; see Theorem 3.5 for the notational details of how, with \( t = m \geq 0 \), our \( X_t^{(n)} \) corresponds to \( Q_{q,n} \) and \( X_t \) corresponds to \( Q_q \).

7. Total Variation Distances, allowing \( x > 1/2 \)

For the application to linear algebra, one always has \( x = 1/q \leq 1/2 \), so the results of the previous section are adequate. For the countdown process in general, it is possible to analyze the asymptotic total variation distance, for the hitting time to zero, i.e., \( S \) versus \( S_n \); Theorem 7.9 below contains Theorem 6.1 as a special case. For the analysis of the asymptotic total variation distance for the height at time \( t \), i.e., \( X_t \) versus \( X_t^{(n)} \), there are further obstacles, and we don’t have a generalization for Theorem 6.2. Nevertheless, we believe that the point of view given in the paragraph following Observation 2.3 could be the starting point for such analysis.

Open Problem 7.1. Give asymptotics for \( d_{TV}(X_t, X_t^{(n)}) \), as \( n \to \infty \), for fixed \( t \in \mathbb{Z} \), and allowing \( 0 < x < 1 \).

For our analysis of the asymptotic value of \( d_{TV}(S, S_n) \), we begin with a few general principles, in the form of Lemma 7.2, Lemma 7.4, and Corollary 7.5. After that, we give a bit of concrete calculation in Proposition 7.8, and Theorem 7.9 follows easily.

An integer-valued random variable \( X \) is said to be unimodal if there exists a value \( k_0 \), such that \( \mathbb{P}(X = i) \leq \mathbb{P}(X = i + 1) \) for \( i < k_0 \) and \( \mathbb{P}(X = i) \geq \mathbb{P}(X = i + 1) \) for \( i \geq k_0 \). In this case, we say that the distribution of \( X \) is unimodal, with mode at \( k_0 \). It is a standard fact, easily proved, that if a distribution is log concave, then it must be unimodal.

Lemma 7.2. Let \( X \) be an integer valued random variable whose distribution is unimodal. Then

\[
d_{TV}(X, X + 1) = \sup_{k \in \mathbb{Z}} \mathbb{P}(X = k).
\]

Proof. Starting from (17), we have

\[
d_{TV}(X, X + 1) = \sum_i \max(0, \mathbb{P}(X = i) - \mathbb{P}(X = i - 1)).
\]

When the distribution of \( Y \) is unimodal with mode at \( k_0 \), the above simplifies to

\[
d_{TV}(X, X + 1) = \sum_{i \leq k_0} \mathbb{P}(X = i) - \mathbb{P}(X = i - 1)
\]

and the sum telescopes to give \( d_{TV}(X, X + 1) = \mathbb{P}(X = k_0) \). Obviously, unimodality implies that \( \sup_{k \in \mathbb{Z}} \mathbb{P}(X = k) = \mathbb{P}(X = k_0) \). \( \square \)

Example 7.3. Suppose \( X \) is Poisson with mean \( \lambda \in (0, \infty) \). The distribution of \( X \) is unimodal, with mode at \( k_0 = [\lambda] \), i.e., \( \lambda \) rounded down to an integer. We have \( d_{TV}(X, X + 1) = \mathbb{P}(X = k_0) \). By Stirling’s formula, as \( \lambda \to \infty \), \( d_{TV}(X, X + 1) \sim 1/\sqrt{2 \pi \lambda} \).
Lemma 7.4. Suppose that the distribution of \( Y \) is a mixture of the distributions of \( X \) and \( Z \), with \( P(Y \in B) = (1 - p)P(X \in B) + pP(Z \in B) \) for all measurable \( B \). Then
\[ d_{TV}(X, Y) = p \, d_{TV}(X, Z). \]

Proof. Obvious from (16). \( \square \)

Corollary 7.5. Let \( X \) be an integer valued random variable, whose distribution is unimodal, and let \( U \) be independent of \( X \), with the Bernoulli distribution having parameter \( p \), i.e., \( P(U = 1) = p = 1 - P(U = 0) \). Then
\[ d_{TV}(X, X + U) = p \sup_{k \in \mathbb{Z}} P(X = k). \]

Proof. Lemma 7.4 applies here, with \( Y = X + U \) having distribution a mixture of the distributions of \( X \) and \( Z = X + 1 \). \( \square \)

Example 7.6. Suppose \( X \) is Binomial\((n, p)\) with mean \( p \in (0, 1) \). The distribution of \( X \) is unimodal, with mode at \( k_0 \) equal to \( \lfloor np \rfloor \) or \( \lceil np \rceil \). Suppose \( Y \) is Binomial\((n + 1, p)\). Then \( d_{TV}(X, Y) = p \, P(X = k_0) \). By Stirling’s formula, as \( n \to \infty \), \( d_{TV}(X, Y) \sim p/\sqrt{2 \pi np(1 - p)} \).

Example 7.7. Lemma 5.2 states that the distribution of \( S_n \) is logconcave, hence it is unimodal, so Lemma 7.2 and Corollary 7.5 apply. We note that in (25), the limit as \( n \to \infty \) of the ratio is
\[ \frac{P(S = k + 1)}{P(S = k)} = \frac{x}{1 - x^{k+1}}. \]

For \( k = 0, 1, 2, \ldots \), the critical value \( x_k \), where a tie occurs between \( P(S = k) \) and \( P(S = k + 1) \), is the solution \( x_k \) of \( x^{k+1} = 1 - x \). In particular \( x_0 = 1/2 \), and \( x_1 \approx .61803 \) is one less than the golden mean. For \( x \in (x_k, x_{k+1}) \), for all sufficiently large \( n \), the mode of \( S_n \) occurs at \( k \).

For very large \( k \), \( x_k \) is close to 1. A convenient way to analyze the asymptotic relation is to define \( y_k \) by the relation \( x_k = \exp(-1/y_k) \); this leads to \( 1/y_k \doteq 1 - x_k = (x_k)^{k+1} \doteq (x_k)^k = \exp(-k/y_k) \) so that \( y_k \approx k/y_k \) and \( k/y_k \approx \log(y_k) \). Along these lines it can be proved that as \( n \to \infty \), \( k \approx y_k \log y_k \), and even more careful analysis reveals that
\[ k = y_k \log y_k - \frac{1}{2} - \frac{1}{2\pi} y^{-1} + O(k^{-2}). \]

For example, to have \( x_k \approx .999 \) we consider \( y = 1000 \) with \( y \log y \approx 6907.755 \); the exact values nearby are \( x_{6907} = 0.9990004676 \ldots \) and \( x_{6908} = 0.9990005939 \ldots \).

Proposition 7.8. For all \( x \in (0, 1) \), and \( n \geq 1 \), \( P(R_n > 1) \leq x^{2n}/(1 - x)^2 \).

Proof. From Lemma 5.1, we know the distribution of \( R_n = S - S_n \) is given by
\[ P(R_n = k) = x^{(n+1)k} \prod_{i=0}^{\infty} (1 - x^i) / \prod_{i=1}^{k} (1 - x^i). \]
Therefore,
\[
\begin{align*}
\mathbb{P}(U_n \neq R_n) & = 1 - \mathbb{P}(R_n = 0) - \mathbb{P}(R_n = 1) \\
& = 1 - \prod_{i=n+1}^{\infty} (1 - x^i) - \frac{x^{n+1}}{1-x} \prod_{i=n+1}^{\infty} (1 - x^i) \\
& \leq \frac{x^{n+1}}{1-x} - \frac{x^{n+1}}{1-x} \prod_{i=n+1}^{\infty} (1 - x^i) \\
& \leq \left( \frac{x^{n+1}}{1-x} \right)^2.
\end{align*}
\]
\[\square\]

**Theorem 7.9.** For all \(x \in (0, 1)\), as \(n \to \infty\),
\[
d_{TV}(S, S_n) \sim C_{x} \cdot \frac{x^{n+1}}{1-x},
\]
where \(C_{x} = \max_{k \in \mathbb{Z}^+} \mathbb{P}(S = k)\).

**Proof.** Recall from (12) and (14) that \(S = S_n + R_n\), with \(S_n\) independent of \(R_n\). Let \(U_n := \min(R_n, 1)\), and write \(p_n = \mathbb{P}(U_n = 1)\); note that \(S_n\) is independent of \(U_n\). As noted in Example 7.7, the distribution of \(S_n\) is unimodal, so Lemma 7.5 applies, to give
\[
(32) \quad d_{TV}(S_n, S_n + U_n) = p_n \max_k \mathbb{P}(S_n = k).
\]
Note that \(p_n := \mathbb{P}(R_n \geq 1) \geq \mathbb{P}(Z_{n+1} \geq 1) = x^{n+1}, \) and \(\max_k \mathbb{P}(S_n = k) \geq \mathbb{P}(S_n = 0) = g_n(x) \geq g(x) > 0\), with \(g\) specified by (4). Combined, we have \(d_{TV}(S_n, S_n + U_n) \geq x^{n+1}g(x)\).

From our particular coupling, we have
\[
d_{TV}(S, S_n + U_n) = d_{TV}(S_n + R_n, S_n + U_n) \\
\leq \mathbb{P}(S_n + R_n \neq S_n + U_n) \\
\leq \mathbb{P}(R_n > 1).
\]

Now consider the triangle, with vertices at \(A = S\), \(B = S_n\), and \(C = S_n + U_n\). The first paragraph of this proof says the the length of side \(BC\) is at least \(x^{n+1}g(x)\). The second paragraph says that the length of side \(AC\) is at most \(\mathbb{P}(R_n > 1)\), which by Proposition 7.8 is \(O(x^{2n})\), and since \(x \in (0, 1)\), the length of \(AC\) is little oh of the length of \(BC\). Therefore, by the triangle inequality, the length of \(AB\) is asymptotic to the length of \(BC\), so from (32),
\[
d_{TV}(S, S_n) \sim p_n \max_k \mathbb{P}(S_n = k).
\]
Finally, \(p_n = 1 - \mathbb{P}(R_n = 0) = 1 - \prod_{i \geq 0} (1 - x^i) \sim x^{n+1}/(1 - x)\), as in Theorem 3.1, and \(S_n\) converges to \(S\) in distribution, hence, as \(n \to \infty\), \(\max_k \mathbb{P}(S_n = k) \to \max_k \mathbb{P}(S = k) =: C_{x}\). \[\square\]
As noted in Example 7.7, when $x \leq 1/2$, the mode of $S$ occurs at 0, so that

$$d_{TV}(S, S_n) \sim \frac{g(x)}{1-x} \cdot x^{n+1},$$

and we see that Theorem 6.1 is a special case of Theorem 7.9.

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