Can wormholes have negative temperatures?

Soon-Tae Hong* and Sung-Won Kim†

Department of Science Education, Ewha Womans University, Seoul 120-750 Korea

(Dated: March 24, 2022)

We study (3+1) Morris-Thorne wormhole to investigate its higher dimensional embedding structures and thermodynamic properties. It is shown that the wormhole is embedded in (5+2) global embedding Minkowski space. This embedding enables us to construct the wormhole entropy and wormhole temperature by exploiting Unruh effects. We also propose a possibility of negative temperature originated from exotic matter distribution of the wormhole.

PACS numbers: 02.40.-k; 04.20.-q; 04.50.+h; 05.70.-a

1. Introduction. Since the cosmic microwave background was discovered, there have been many ideas and proposals to figure out how the universe has evolved. The standard big bang scenario has led to the inflationary cosmology [1] and nowadays to the M-theory cosmology with bouncing universes [2]. There have been also considerable discussions on the theoretical existence of wormhole geometry, since Morris and Thorne (MT) proposed a possibility of traversable wormhole, through which observers can pass travelling between two universes as a short cut [3]. According to the Einstein field equations, the MT wormhole needs the exotic matter, which violates the weak energy condition.

On the other hand, it has been discovered the novel aspects that the thermodynamics of higher dimensional black holes can often be interpreted in terms of lower dimensional black hole solutions [4]. In fact, a slightly modified solution of (2+1) dimensional Banados-Teitelboim-Zanelli black hole [5, 6] yields a solution to the string theory, so-called the black string [7]. Since the thermal Hawking effects [8] on a curved manifold were studied as Unruh effects [9] in a higher flat dimensional space-time, following the global embedding Minkowski space (GEMS) approach [10] several authors recently have shown that this approach could yield a unified derivation of temperature for various curved manifolds in (2+1) dimensions [5, 6, 11, 12] and in (3+1) dimensions [11, 13, 14]. Moreover, the MT wormhole has been described in terms of its embedding profile surface geometry [3].

In this paper we will analyze the geometries of the MT wormhole manifolds [3] to construct their higher dimensional flat embeddings, which will be shown to be related with the embedding profile surface geometry of the wormhole. In these GEMS embeddings, we will investigate the Hawking temperature and entropy via the Unruh effects to propose a possibility of “negative temperature” associated with the “exotic matter.” Recently, the exotic matter was introduced in the Friedmann-Robertson-Walker model where the bouncing universe could be initiated by the negative energy density at the big crunch [15]. Moreover, a possibility of negative temperature has been proposed in the de Sitter geometry even though it was forbidden due to thermodynamic instability [16].

2. GEMS geometries of wormholes. In order to study the GEMS structure for wormholes, we start with the static MT wormhole four-metric of the form [3, 17]

\[
\text{ds}^2 = e^{2\Phi(r)}dt^2 - \left(1 - \frac{b(r)}{r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)
\] (1)

where the arbitrary smooth functions \(\Phi(r)\) and \(b(r)\) are the lapse and wormhole shape functions. Note that, in order for the wormhole to be maintained, the wormhole function \(b(r)\) should be positive and satisfy the flaring-out condition, which will be explicitly given later in the GEMS structure. The lapse function \(\Phi(r)\) is finite everywhere.

After some algebra, we obtain the (5+2) GEMS structure \(ds^2 = \eta_{MN}dx^Mdx^N\) with the flat Minkowski metric

\[
\eta_{MN} = \text{diag} (+1, -1, -1, -1, -1, 1),
\] (2)

where the coordinate transformations are given, with two additional space-like and one time-like dimensions, as follows

\[
x^0 = k_S^{-1}e^{\Phi(r)} \sinh \kappa t, \\
x^1 = k_S^{-1}e^{\Phi(r)} \cosh \kappa t, \\
x^2 = r \sin \theta \cos \phi,
\]

*Electronic address: soonhong@ewha.ac.kr
†Electronic address: sungwon@ewha.ac.kr
\[ x^3 = r \sin \theta \sin \phi, \]
\[ x^4 = r \cos \theta, \]
\[ x^5 = \int \frac{dr}{\left(1 - \frac{b(r)}{r}\right)^{1/2}}, \]
\[ x^6 = \int dr \left[1 + k_S^{-2}(\Phi'(r))^2 e^{2\Phi(r)}\right]^{1/2}, \]

(3)

where the coordinate \(x^5 \in (\infty, +\infty)\) corresponds to the proper radial distance measured by static observers [3] and \(k_S\) is the surface gravity, which will be discussed later.

For a submanifold on an equatorial slice (\(\theta = \pi/2\)) at a fixed moment of time, the MT wormhole metric (1) is reduced into

\[ ds^2 = -\left(1 - \frac{b(r)}{r}\right)^{-1} dr^2 - r^2 d\phi^2, \]

(4)

and its (3+1) GEMS structure is given by

\[ ds^2 = -(dx^2)^2 - (dx^3)^2 - (dx^5)^2 + (dx^6)^2 \]

with \((x^2, x^3)\) given by (5) and \(z\) defined as

\[ z = \int \frac{dr}{\left(\frac{r}{b(r)} - 1\right)^{1/2}}, \]

(7)

which describes the embedding profile surface of the MT wormhole geometry [3].

3. Thermodynamics in wormhole GEMS. Consider the thermodynamic properties of the static MT wormhole described by the GEMS coordinate transformations (3). Introducing the Killing vector \(\xi = \partial_t\) we evaluate the surface gravity \(k_S\) at radius \(r\)

\[ k_S = \Phi'(r) e^{\Phi(r)} \left(1 - \frac{b(r)}{r}\right)^{1/2}. \]

(8)

In the GEMS structure, the Hawking temperature is attainable through the relation \(T_H = a_7/2\pi\) with the (5+2) acceleration \(a_7\). For the Unruh detectors moving according to constant \(r, \theta, \phi\) as in the Schwarzschild black hole GEMS [11, 14], \(a_7\) is described by the Rindler-like motion in the embedded flat space as follows

\[ a_7^{-2} = (x^1)^2 - (x^0)^2 \]

(9)

in the \((x^0, x^1)\) plane to yield

\[ a_7^{-2} = k_S^{-2} e^{2\Phi(r)}. \]

(10)

Substituting the surface gravity \(k_S\) in (8) into (10), we obtain the Hawking temperature

\[ T_H = \frac{a_7}{2\pi} = \frac{1}{2\pi} \Phi'(r) \left(1 - \frac{b(r)}{r}\right)^{1/2}, \]

(11)
which is consistent with the fact that the $a_7$ is also attainable from the relation

$$a_7 = \frac{k_S}{g_{00}^{1/2}}. \quad (12)$$

Moreover, the desired wormhole temperature is given by

$$T_0 = \frac{k_S}{2\pi} = \frac{1}{2\pi} \Phi'(r)e^{\Phi(r)} \left(1 - \frac{b(r)}{r}\right)^{1/2}. \quad (13)$$

Here one notes that, even though the event horizon does not exist in the wormhole, in constructing the above Hawking and wormhole temperatures we have taken the limit that $r$ approaches $r = b(r)$, corresponding to the event horizons of the black holes, to yield the suprema of these wormhole temperatures. This continuous limiting procedure makes sense in ensuring the existence of the Hawking and wormhole temperatures since the temperatures of the geometrical objects are purely defined in terms of their geometrical surface gravities [18] and the suprema of the wormhole temperatures are well-defined.

In the orthonormal basis with the metric $g_{\mu\nu} = \text{diag} (+1, -1, -1, -1)$, the wormhole stress-energy tensor is given by [3],

$$T_{\mu\nu} = \text{diag}(\rho(r), -\tau(r), p(r), p(r)), \quad (14)$$

where $\rho$ is the total density of mass-energy, $\tau$ is the tension per unit area measured in the radial direction, and $p$ is the pressure measured in lateral direction. Note that, in an ordinary perfect fluid, $-\tau = p$. In the orthonormal basis, one can evaluate the non-vanishing Ricci components and the Einstein curvature:

$$R_{\bar{0}\bar{0}} = (\Phi'' + (\Phi')^2) \left(1 - \frac{b}{r}\right) + \Phi' \left(-\frac{3b}{2r^2} - \frac{b'}{2r} + \frac{2}{r}\right),$$

$$R_{11} = -(\Phi'' + (\Phi')^2) \left(1 - \frac{b}{r}\right) + \left(\Phi' + \frac{2}{r}\right) \left(-\frac{b}{2r^2} + \frac{b'}{2r}\right),$$

$$R_{22} = R_{33} = \frac{1}{r^2} \left(\Phi''(-r + b) + \frac{b}{2r} + \frac{b'}{2}\right),$$

$$R = 2(\Phi'' + (\Phi')^2) \left(1 - \frac{b}{r}\right) + \Phi' \left(-\frac{3b}{r^2} - \frac{b'}{r} + \frac{4}{r}\right) - \frac{2b'}{r^2}. \quad (15)$$

From the Einstein field equations $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu}$ and (15), one can obtain

$$b' = 8\pi Gr^2\rho,$$

$$\Phi' = \frac{b - 8\pi Gr^3\tau}{2r(r - b)},$$

$$\tau' = (\rho - \tau)\Phi' - \frac{2}{r}(p + \tau), \quad (16)$$

which yield, together with (11) and (13), the Hawking temperature, wormhole temperature and wormhole shape function,

$$T_H = \frac{1}{4\pi r^2} \frac{b(r) - 8\pi Gr^3\tau(r)}{(1 - \frac{b(r)}{r})^{1/2}},$$

$$T_0 = \frac{1}{4\pi r^2} \frac{b(r) - 8\pi Gr^3\tau(r)e^{\Phi(r)}}{(1 - \frac{b(r)}{r})^{1/2}},$$

$$b(r) = \int_0^r dr' 8\pi Gr'^2\rho(r'), \quad (17)$$

where we have extended the lower bound of $b(r)$ to zero even though there exists no exotic matter up to the throat $r = r_0$. Note that in the Hawking and wormhole temperatures in (17), one cannot exclude the possibility of the negative temperature in the case of $b(r) < 8\pi Gr^3\tau(r)$. Recently, the negative temperature has been proposed in
the de Sitter geometry [16]. However, due to the thermodynamic instability of the de Sitter space, the negative temperature was prohibited. In the wormhole case, there could exist the “negative temperature” originated from the “exotic matter,” different from the de Sitter case which does not allow the exotic matter.

Next, for the MT wormhole geometry to be connectible to asymptotically flat space-time, the embedding profile surface \( z = z(r) \) in (7) flares outward from the throat at \( r = b \) to yield the flaring-out condition at or near the throat

\[
\frac{d^2 r}{dz^2} = \frac{b - b' r}{2b^2} > 0, \tag{18}
\]

from which one obtains the constraint at or near the throat \([3]\),

\[
\frac{\tau - \rho}{|\rho|} > 0. \tag{19}
\]

For the limit of \( x^5 \to \pm \infty \), one has two regions (or universes), and at \( x^5 = 0 \) the wormhole shape function \( b(r) \) has a minimum value at \( r = r_0 \). In order to investigate the thermodynamics in the wormhole geometry, one may consider a specific case that the exotic matter resides only on the neck of the wormhole,

\[
\rho(r) = \rho_0 \delta(r - r_0), \quad \tau(r) = \tau_0 \delta(r - r_0), \tag{20}
\]

and the two universes contact at \( x^5 = 0 \). Note that the wormhole shape function is then given by

\[
b(r) = 8\pi Gr_0^2 \rho_0 > 0, \tag{21}
\]

to yield the Hawking temperature and wormhole temperature,

\[
T_H = \frac{b_0}{4\pi r^2} \left( \frac{1 - b_0}{r_0} \right)^{1/2} - \frac{2Gr_0 \tau_0}{\left( 1 - \frac{b_0}{r_0} \right)^{1/2}} \delta(r - r_0),
\]

\[
T_0 = \frac{b_0}{4\pi r^2} - 2Gr_0 \tau_0 \delta(r - r_0), \tag{22}
\]

showing that the wormhole temperature is positive (negative) outside (inside) the exotic matter distribution, since at \( r \neq r_0 \) one has only the positive first term while at \( r = r_0 \) the negative second term associated with the delta function and the suppressed first term. Note that, even though the metric (1) has smooth geometry, the temperature (11) can be discontinuous since the factor \( \Phi'(r) \) in (16) is associated with the exotic matter to yield discontinuity as in the temperature (22).

For the further investigation of the wormhole temperature outside the exotic matter distribution, we assume for brevity that \( \rho, \tau \) and \( \rho \) vanish at all radii \( r > r_0 \) for some surface radius \( r = r_0 \) with constant distribution for some finite region. In this distribution \( \Phi' \) is also negative which means “negative temperature” by the third relation of (16). One can thus have outside the cut-off at \( r = r_0 \) the geometry of the standard Schwarzschild form [3]

\[
b(r) = b(r_0) = \text{const} = B > 0,
\]

\[
\Phi(r) = \frac{1}{2} \ln \left( \frac{1 - \frac{B}{r}}{r} \right). \tag{23}
\]

Exploiting (11) and (23), we arrive at the Hawking and wormhole temperatures in the region \( r > r_0 \)

\[
T_H = \frac{B}{4\pi r^2} \left( \frac{1 - \frac{B}{r}}{r} \right)^{1/2},
\]

\[
T_0 = \frac{B}{4\pi r^2}, \tag{24}
\]

which are positive definite since \( \tau \) vanishes in this region. Note that \( T_H \) and \( T_0 \) vanish asymptotically consistent with the cosmological phenomenology.

To figure out further the negative temperature inside the exotic matter distribution, for the wormhole mass-energy density \( \rho \) and tension \( \tau \) one may take an ansatz of the forms [19]

\[
\rho(r) = \rho_0 r^\beta, \quad \tau(r) = \tau_0 r^\beta, \tag{25}
\]
which, together with the inequality (19), yields

\[ b(r) < \frac{8\pi G r^3}{\beta + 3} \tau(r), \]  

(26)

so that the Hawking and wormhole temperatures (17) are negative definite except the case \(-3 < \beta < -2\). However, the powers in the interval \(-3 < \beta < -2\) cannot take place in the tension \(\tau(r)\) of the MT wormhole matter distribution satisfying the asymptotic flaring-out condition which holds in the interval \(\beta < -3\). The Hawking and wormhole temperatures within the exotic matter thus become negative at least when one assumes the energy density and the tension of the form (25).

In the wormhole geometry (20), one may have a manifold of the form \(S^2 \times h\) with \(h \in (-\delta, +\delta)\) (\(\delta \to 0\)) and the wormhole has two-sphere boundaries of radius \(b_0\). Moreover, the entropy seen by an accelerated observer in the Minkowski space is attainable from the transverse area to the observer [20]. In an embedded higher dimensional flat manifold, there exist embedding constraints to yield the finite transverse area or entropy. In the static wormhole GEMS of interest, one can thus formulate the entropy lower bound:

\[ S = 2 \int dx^2 dx^3 dx^4 dx^5 dx^6 \delta [((x^2)^2 + (x^3)^2 + (x^4)^2 - r^2] \delta (x^5) \delta (x^6 - f(r)) \]  

(27)

where \(f(r)\) can be read off from (3). Here the integration over \(x^6\) subject to the constraint \(\delta [x^6 - f(r)]\) is unity and the constraint \(\delta (x^5)\) leads to \(r = b_0\) so that the \(x^5\) integral yields unity and the remaining integrals over \(x^i\) \((i = 2, 3, 4)\) with constraint \(\delta [(x^2)^2 + (x^3)^2 + (x^4)^2 - r^2]\) produce the area \(4\pi b_0^2\). The factor 2 originates from the fact that one has two boundaries at \(h = \pm \delta\). We thus arrive at the entropy lower bound

\[ S = 8\pi b_0^2, \]  

(28)

which is consistent with the holographic description that all the microscopic quantum information is deposited on the upper and lower two-sphere boundaries of radius \(b_0\), and with the fact that the entropy is extensive quantity. Moreover, for the other geometries of the standard Schwarzschild form (23) and of the ansatz (25), one can have additional area contributions from the nonvanishing surfaces for the wormhole shape functions at \(r \neq r_0\), to yield the increased entropies. The entropies for any cases are then constrained by the general “lower bound” value (28).

4. Conclusion. We have studied the (3+1) Morris-Thorne wormhole to obtain the (5+2) higher dimensional flat embedding structure. We have thus shown on these flat embedding geometries that the wormhole temperature has negative (positive) values inside (outside) the exotic matter distribution accumulated mostly around the wormhole shape radius, and the wormhole entropy lower bound is twice the throat area of the wormhole. Note that for the definition of the exotic matter one has the constraint condition (19) between \(\rho\) and \(\tau\), and in this paper we have assumed several forms of ansatz for these variables to evaluate the wormhole temperatures inside the exotic matter distributions. It will be interesting in further investigation to study the wormhole temperature without any ad hoc ansatz.

The authors would like to acknowledge financial support in part from the Korea Science and Engineering Foundation Grant R01-2000-00015.
[12] S.T. Hong, Y.W. Kim and Y.J. Park, Phys. Rev. D62, 024024 (2000); S.T. Hong, W.T. Kim, Y.W. Kim and Y.J. Park, Phys. Rev. D62, 064021 (2000).
[13] S.W. Hawking and H.S. Reall, Phys. Rev. D61, 024014 (1999); L. Andrianopoli, M. Derix, G.W. Gibbons, C. Herdeiro, A. Santambrogio and A.V. Proeyen, Class. Quant. Grav. 17, 1875 (2000).
[14] Y.W. Kim, Y.J. Park and K.S. Soh, Phys. Rev. D62, 104020 (2000).
[15] J.C. Hwang and H. Noh, Phys. Rev. D65, 124010 (2002).
[16] V. Balasubramanian, J. de Boer and D. Minic, Class. Quant. Grav. 19, 5655 (2002).
[17] S.-W. Kim and H. Lee, Phys. Rev. D63, 064014 (2001).
[18] R.M. Wald, Living Rev.Rel. 4, 6 (2001).
[19] S.-W. Kim, Phys. Rev. D53, 6889 (1996); S.-W. Kim and S.P. Kim, Phys. Rev. D58, 087703 (1998).
[20] R. Laflamme, Phys. Lett. B196, 449 (1987).