The Erdős-Pósa Property for Directed Graphs

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Abstract. A classical result by Erdős and Pósa states that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $k$, every graph $G$ contains $k$ pairwise vertex disjoint cycles or a set $T$ of at most $f(k)$ vertices such that $G - T$ is acyclic. The generalisation of this result to directed graphs is known as Younger’s conjecture and was proved by Reed, Robertson, Seymour and Thomas in 1996.

This so-called Erdős-Pósa property can naturally be generalised to arbitrary graphs and digraphs. Robertson and Seymour proved that a graph $H$ has the Erdős-Pósa property if, and only if, $H$ is planar.

In this paper we study the corresponding problem for digraphs. We obtain a complete characterisation of the class of strongly connected digraphs which have the Erdős-Pósa property (both for topological and butterfly minors). We also generalise this result to classes of digraphs which are not strongly connected. In particular, we study the class of vertex-cyclic digraphs (digraphs without trivial strong components). For this natural class of digraphs we obtain a nearly complete characterisation of the digraphs within this class with the Erdős-Pósa property. In particular we give positive and algorithmic examples of digraphs with the Erdős-Pósa property by using directed tree decompositions in a novel way.

1 Introduction

A classical result by Erdős and Pósa states that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $k$, every graph $G$ contains $k$ pairwise vertex disjoint cycles or a set $T$ of at most $f(k)$ vertices such that $G - T$ is acyclic.

In [8], Robertson and Seymour considered a natural generalisation of this result to arbitrary graphs: a graph $H$ has the Erdős-Pósa property if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that every graph $G$ either has $k$ disjoint copies of $H$ as a minor or contains a set $T$ of at most $f(k)$ vertices such that $H$ is not a minor of $G - T$. They showed that a graph $H$ has the Erdős-Pósa property in this sense if, and only if, it is planar. In fact,
their proof implies that $H$ has the Erdős-Pósa-property for minors if, and only if, there is an integer $h = f(|H|)$ such that a $(h \times h)$-grid contains $H$ as a minor.

It follows from the grid theorem (see [3]), which says that if the tree width of a given graph $G$ is at least $f(t)$, then $G$ has a $(t \times t)$-grid as a minor, that there is no $(h \times h)$-grid minor, then the tree width of $G$ is at most $f(h)$. Finally, it is proved in [8] that the Erdős-Pósa-property holds within any graph of bounded tree-width. These facts imply the above characterization.

In [9], Younger conjectured the natural generalisation of Erdős and Pósa’s original result to directed graphs and directed cycles. This conjecture has received considerable attention by the research community, and it has been open for nearly a quarter of a century. Following several partial results, Younger’s conjecture was eventually confirmed by Reed et al. in [7].

In this paper we consider the generalisation of Younger’s conjecture to arbitrary digraphs $H$. Whereas for undirected graphs, there is an agreed notion of minor, for directed graphs there are several competing proposals. Here we study the Erdős-Pósa property for two common notions of directed minors, topological minors and butterfly minors. See Section 2 for details.

**Definition 1.1.** A digraph $H$ has the Erdős-Pósa property for topological minors if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $k \geq 0$, every digraph $G$ either contains $k$ disjoint subgraphs each containing $H$ as a topological minor or there is a set $S \subseteq V(G)$ of at most $f(k)$ vertices such that $G - S$ does not contain $H$ as a topological minor. The definition for butterfly minors is analogous.

For both concepts of minors we give a complete characterisation of the strongly connected digraphs which have the Erdős-Pósa property. It turns out that our characterisation is essentially the analogy of the above mentioned Robertson-Seymour theorem for undirected graphs. We prove that a digraph $H$ has the Erdős-Pósa-property for topological minors (butterfly minors), if, and only if, there is an integer $h = f(|H|)$ such that a cylindrical wall (grid) of order $h$ contains $H$ as a topological minor (butterfly minor).

Note that if $H$ is a cycle, then this is exactly Younger’s conjecture. Hence, the first main result of this paper is the following (see Section 2 and 3 for details).

**Theorem 4.1.** Let $H$ be a strongly connected digraph. $H$ has the Erdős-Pósa property for butterfly (topological) minors if, and only if, there is a cylindrical grid (wall) $G_{c}$, for some constant $c = c(H)$, such that $H$ is a butterfly (topological) minor of $G_{c}$.

Furthermore, for every fixed strongly connected digraph $H$ satisfying these conditions and every $k$ there is a polynomial time algorithm which, given a digraph $G$ as input, either computes $k$ disjoint (butterfly or topological) models of $H$ in $G$ or a set $S$ of $\leq h(k)$ vertices such that $G - S$ does not contain a model of $H$.

This result is particularly interesting as there is no similar classification known for undirected graphs in terms of topological minors.

If $H$ is not strongly connected, then our techniques described above fails. In fact they fail already in the bounded directed tree width case. Nevertheless, we are able to provide a far reaching characterisation of the Erdős-Pósa property for a much larger class of digraphs. In particular, we study the much more general class of vertex-cyclic
digraphs, i.e. digraphs without trivial strong components (components consisting of a single vertex only). For this natural class of digraphs we obtain the following result (see Section 5 for details).

**Theorem 5.2** Let $H$ be a weakly connected vertex-cyclic digraph. If $H$ has the Erdős-Pósa property for butterfly (topological) minors, then it is ultra-homogeneous with respect to butterfly (topological) embeddings, its maximum degree is at most 3 and every strong component of $H$ is a butterfly (topological) minor of some cylindrical grid (wall) $G_k$.

We also obtain a positive result as an example of a digraph satisfying the properties in the previous theorem. This theorem is probably the most challenging result of the paper using directed tree decompositions algorithmically in a novel way which may be of independent interest.

**Theorem 5.6** Let $H$ be a digraph consisting of two disjoint cycles joined by a single edge. There is a function $h : \mathbb{N} \rightarrow \mathbb{N}$ such that for every integer $k$ and every graph $G$ either there are $k$ distinct topological models of $H$ in $G$ or there is a set $S \subseteq V(G)$ such that $|S| \leq h(|H| + k)$ and $H \not\preceq G - S$.

Furthermore, for every $H$ and $k$ there is a polynomial-time algorithm which either finds $k$ distinct topological models of $H$ in $G$ or finds a set $S \subseteq G$ of vertices of size at most $h(|H| + k)$ which hits every topological model of $H$ in $G$.

2 Preliminaries

In this section we briefly recall necessary definitions and fix our notation. We denote the set of non-negative integers by $\mathbb{N}$. For $n \in \mathbb{N}$ we write $[n]$ for the set of integers $\{1, \ldots, n\}$.

We refer the reader to [12] for basic concepts of graph and digraph theory. All graphs and digraphs in this paper are finite without loops. We denote the vertex set of $G$ by $V(G)$ and its edge set by $E(G)$. If $G$ is a digraph and $S \subseteq V(G)$ or $S \subseteq E(G)$, then $G[S]$ denotes the subgraph of $G$ induced by $S$. For $S \subseteq V(G)$ we write $G - S$ for the subgraph of $G$ induced by $V(G) - S$. For vertices $v \in V(G)$, edges $e \in E(G)$ or sets $F \subseteq E(G)$ we define $G - v, G - e, G - F$ analogously.

The *in-degree* $d^+_G(v)$ of a vertex $v$ in a digraph $G$ is the number of edges with head $v$ in $G$. The *out-degree* $d^-_G(v)$ is the number of edges with tail $v$. By degree $d_G(v)$ of $v$ we mean the sum $d^+_G(v) + d^-_G(v)$. We usually drop the index if $G$ is clear from the context.

A strong component (or component) in a digraph $G$ is a maximal strongly connected subgraph. The block graph of a digraph $G$ is the digraph obtained from $G$ by contracting each strong component into a single vertex. We call a digraph weakly connected if its underlying undirected graph is connected.
3 Directed Minors, Directed Grids and Directed Tree-Width

Directed Minors. In this section we introduce two different kinds of minors, butterfly minors (see \[4\]) and topological minors, and establish some basic properties needed below.

Definition 3.1 (butterfly minor). Let \( G \) be a digraph. An edge \( e = (u, v) \in E(G) \) is butterfly-contractible if \( e \) is the only outgoing edge of \( u \) or the only incoming edge of \( v \). In this case the graph \( G' \) obtained from \( G \) by butterfly-contracting \( e \) is the graph with vertex set \( (V(G) - \{u, v\}) \cup \{x_{u,v}\} \), where \( x_{u,v} \) is a fresh vertex. The edges of \( G' \) are the same as the edges of \( G \) except for the edges incident with \( u \) or \( v \). Instead, the new vertex \( x_{u,v} \) has the same neighbours as \( u \) and \( v \), eliminating parallel edges. A digraph \( H \) is a butterfly-minor of \( G \), denoted \( H \preceq_b G \), if it can be obtained from a subgraph of \( G \) by butterfly contraction.

We aim at an alternative characterisation of butterfly minors. Let \( H, G \) be digraphs such that \( H \preceq_b G \). Let \( G' \) be a subgraph of \( G \) such that \( H \) can be obtained from \( G \) by butterfly contraction and let \( F \subseteq E(G') \) be the set of edges contracted in \( G' \) to form \( H \). Then we can partition \( F \) into disjoint sets \( F_1, \ldots, F_h \) such that the edges in each \( F_i \) are contracted to form a single vertex. Hence, \( h = |V(H)| \) and no two edges \( e_1, e_2 \) from different sets \( F_i \neq F_j \) share an endpoint. The edges of \( G' \) not in any \( F_i \) are in one-to-one correspondence to the edges of \( H \). Hence, we can also define butterfly minors by a map \( \mu \) which assigns to every edge \( e \in E(H) \) an edge \( e \in E(G) \) and to every \( v \in V(H) \) a subgraph \( \mu(v) \subseteq G \) which is \( G'[F_i] \) for some \( i \) as above. We call this a butterfly model or image of \( H \) in \( G \). As shown in the following lemma, we can always choose an image such that \( \mu(v) \) are particularly simple.

Lemma 3.2. Let \( H, G \) be digraphs such that \( H \preceq_b G \). Then there is a function \( \mu \) which maps \( E(H) \) to \( E(G) \) and vertices \( v \in V(H) \) to subgraphs \( \mu(v) \subseteq G \) such that

- \( \mu(u) \cap \mu(v) = \emptyset \) for any \( u \neq v \in E(H) \),
- for all \( e = (u, v) \in E(H) \) the edge \( \mu(e) \) has its tail in \( \mu(u) \) and its head in \( \mu(v) \),
- for all \( v \in V(H) \), \( \mu(v) \) is the union of an in-branching \( T_i \) and out-branching \( T_o \) which only have their roots in common and such that for every \( e \in E(H) \), if \( v \) is the head of \( e \) then the head of \( \mu(e) \) is a vertex in \( T_i \) and if \( v \) is the tail of \( e \) then the tail of \( \mu(e) \) is in \( T_o \).

We call a map \( \mu \) as above a tree-like model of \( H \) in \( G \). We define \( \mu(H) := \bigcup_{f \in E(H) \cup V(H)} \mu(f) \).

Proof. Suppose the claim was false. Then there are digraphs \( H, G' \) such that \( H \preceq_b G' \) but \( H \) has no tree-like model in \( G' \). We call such a pair a counter example. Choose such a pair and fix \( H \). Within all \( G \) such that \( (H, G) \) is a counter example let \( G \) be a digraph of minimal order and, subject to this, with a minimal number of edges.

For any model \( \mu \) of \( H \) in \( G \) let us call the complexity of \( \mu \) the number of edges that are contracted. Let \( \mu \) be an image of \( H \) in \( G \) of minimal complexity. We prove by induction on the complexity that \( \mu \) is tree-like. Clearly, if the complexity is 0, i.e. no edges need to be contracted, then \( \mu \) is tree-like. So suppose the complexity is at least 1. Let \( G' := \mu(H) \subseteq G \) be the minimal subgraph of \( G \) containing all of \( \mu \). By the choice of \( G \), we have \( G' = G \). Choose an edge \( e = (u, v) \in E(\mu(v)) \) for some \( v \in V(H) \) that
is butterfly-contracible in \( G \) and let \( G^* \) be the digraph obtained from \( G \) by contracting \( e \).

Let \( x \) be the new vertex generated by contracting \( e \). Then \( H \preceq_G G^* \) and, as \( G^* \) has lower order than \( G \), there is a tree-like model \( \mu^* \) of \( H \) in \( G^* \). If \( x \) is not in \( \bigcup_{v \in V(H)} \mu(v) \), then \( \mu^* \) is a model of \( H \) in a proper subgraph of \( G \), contradicting the choice of \( G \). So there is a \( z \in V(H) \) such that \( x \in \mu^*(z) \). Let \( F^* = E(\mu^*(v)) \).

We define a set \( F \subseteq E(G) \) as follows. Every edge in \( F^* \) is either an edge in \( G \) or has \( x \) as one endpoint. If \( e = (w, x) \in F^* \) then \((w, u) \in E(G) \) or \((w, v) \in E(G) \) (or both). If \((w, u) \in E(G) \) exists, we add it to \( F \), otherwise we add \((w, v) \). Similarly, if \((x, w) \in F^* \), for some \( w \in V(G^*) \), then at least one of \((u, w) \) or \((v, w) \) is in \( E(G) \). If \((v, w) \in E(G) \) exists, we add it to \( F \) and otherwise we add \((u, w) \). For all \( v' \neq z \in V(H) \) we set \( \mu(v') := \mu^*(v') \) and we set \( \mu(z) := G[F] \).

Finally, for all edges \( e \in V(H) \), if \( \mu^*(e) \) does not contain \( x \) as an endpoint we set \( \mu(e) := \mu^*(e) \). If \( \mu^*(e) := e' \) with \( e' = (w, x) \) then \((w, u) \) or \((w, v) \) exist in \( E(G) \). If \((w, u) \in E(G) \), we set \( \mu(e) := (w, u) \) and otherwise we set \( \mu(e) := (w, v) \). If \( e = (x, w) \) we proceed analogously, setting \( \mu(e) := (v, w) \) if it exists and otherwise \( \mu(e) := (u, w) \).

We claim that \( \mu \) is a tree-like model of \( H \) in \( G \). Suppose not. We know that \( \mu^* \) is a tree-like model. Hence, for every \( v' \neq z \), \( \mu(v') \) is tree-like. So only \( \mu(z) \) may violate the tree-condition. Furthermore, the edges in \( \mu^*(z) \) induce a tree-like model, i.e. \( \mu^*(v) \) consists of the union of an in-branching \( T_i \) and an out-branching \( T_o \) as in the statement of the lemma. One of the vertices in \( \mu^*(z) \) is the fresh vertex \( x \). Suppose first that \( x \in V(T_i) \setminus V(T_o) \). If all incoming edges of \( x \) in \( \mu^* \) have been replaced by edges with head \( u \) and the unique out-going edge by an edge with tail \( v \), then \( \mu(v) \) is tree-like. So at least one incoming edge of \( x \) has been replaced by an edge \( e_i = (w, v) \) or the unique out-going edge \( e_o \) of \( x \) has been replaced by \((u, w) \), for some \( w \in V(G) \). If only the out-going edge has been replaced by \((u, w) \), then \( v \) has no incoming and only one out-going edge to \( u \), so we can simply delete \( v \) from \( \mu(z) \) and obtain a model. But this would violate the choice of \( G \). Hence, at least one edge \((w, x) \) has been replaced by \((w, v) \). However, if the out-going edge of \( x \) has been replaced by an edge \((v, w) \), then we still have a tree-like model. Hence, the only case where \( \mu \) is not tree-like is if the out-going edge of \( x \) in \( \mu^* \) has been replaced by an edge \((u, w) \) and at least one in-coming edge of \( x \) has been replaced by \((w, v) \). However, in this case the edge \((u, v) \) would not have been butterfly-contracible in \( G \) as it would neither be the only out-going edge of \( u \) nor the only incoming edge of \( v \), contradicting the choice of the edge \((u, v) \).

The other cases, i.e. if \( x \in V(T_o) \setminus V(T_i) \) or \( x \) is the root of \( T_i \) and of \( T_o \) are similar. This concludes the proof.

Hence by Lemma [3.2] we can from now on assume that butterfly-models are always tree-like as in the previous lemma. We will now define the other kind of minors considered in this paper.

**Definition 3.3 (topological minor).** Let \( H, G \) be digraphs. \( H \) is a topological minor of \( G \), denoted \( H \preceq_t G \), if there is a mapping \( \mu \) which maps every vertex \( v \in V(H) \) to a vertex \( \mu(v) \in V(G) \) and assigns to every edge \( e \in E(H) \) a directed path \( \mu(e) \subseteq G \) such that

1. \( \mu(v) \neq \mu(w) \) for all \( v \neq w \in V(H) \) and
2. if \( e = (v, w) \in E(H) \) then \( \mu(e) \) is a path linking \( \mu(v) \) to \( \mu(w) \) and \( \mu(e) \cap (\bigcup_{v \in V(H)} \mu(v) \cup \bigcup_{e' \neq e \in E(H)} \mu(e')) = \{ \mu(v), \mu(w) \} \).

We call \( \mu \) a topological model of \( H \) in \( G \) and define \( \mu(H) := \bigcup_{f \in E(H), \nu \in V(H)} \mu(f) \).

That is, \( H \) is a topological minor of \( G \) if \( H \) is a subdivision of a subgraph of \( G \). We also need the following result.

**Lemma 3.4.** Let \( H \) be a digraph of maximum degree at most 3. If \( H \preceq_{tb} G \), for some digraph \( G \), then \( H \preceq_t G \).

**Proof.** Let \( H \preceq_{tb} G \). Hence, there is a tree-like model \( \mu \) of \( H \) in \( G \). Clearly, for \( v \in V(H) \), we can choose the in-branching \( T_i \) and the out-branching \( T_o \), comprising \( \mu(v) \) so that there are at most 3 leaves. For, if a leaf of \( T_o \) is not the tail of an edge \( \mu(e) \), for some \( e \in E(H) \), then we can delete it from the model, unless it is the only vertex of \( T_o \). Similarly, we can delete leaves of \( T_i \), which are not the head of any \( \mu(e), e \in E(H) \). But this implies that \( T_i \cup T_o \) has only at most 3 leaves and therefore contains only one vertex \( v' \) of degree \( \geq 2 \). We can therefore map \( v \) to \( v' \) and edges of \( H \) to corresponding paths to obtain \( H \) as a topological minor of \( G \).

**Directed Tree-Width.** We briefly recall the definition of directed tree width from [4]. By an arborescence we mean a directed graph \( R \) such that \( R \) has a vertex \( r_0 \), called the root of \( R \), with the property that for every vertex \( r \in V(R) \) there is a unique directed path from \( r_0 \) to \( r \). Thus every arborescence arises from a tree by selecting a root and directing all edges away from the root. If \( r, r' \in V(R) \) we write \( r' > r \) if \( r' \neq r \) and there exists a directed path in \( R \) with initial vertex \( r \) and terminal vertex \( r' \). If \( e \in E(R) \) we write \( r' > e \) if either \( r' = r \) or \( r' > r \), where \( r \) is the head of \( e \).

Let \( G \) be a digraph and let \( Z \subseteq V(G) \). We say that a set \( S \subseteq (V(G) - Z) \) is \( Z \)-normal if there is no directed walk in \( G - Z \) with the first and the last vertex in \( S \) that uses a vertex of \( G - (Z \cup S) \). It follows that every \( Z \)-normal set is the union of the vertex sets of strongly connected components of \( G - Z \). It is straightforward to check that a set \( S \) is \( Z \)-normal if, and only if, the vertex sets of the strongly connected components of \( G - Z \) can be numbered \( S_1, S_2, \ldots, S_d \) in such a way that

1. if \( 1 \leq i < j \leq d \), then no edge of \( G \) has head in \( S_i \) and tail in \( S_j \), and
2. either \( S = \emptyset \), or \( S = S_i \cup S_{i+1} \cup \cdots \cup S_j \) for some integers \( i, j \) with \( 1 \leq i \leq j \leq d \).

**Definition 3.5.** A directed tree decomposition of a digraph \( G \) is a triple \((T, \beta, \gamma)\), where \( T \) is an arborescence, \( \beta : V(T) \to 2^{V(G)} \) and \( \gamma : E(T) \to 2^{V(G)} \) are functions such that

1. \( \{ \beta(t) : t \in V(T) \} \) is a partition of \( V(G) \) and
2. if \( e \in E(T) \), then \( \bigcup \{ \beta(t) : t \in V(T), t > e \} \) is \( \gamma(e) \)-normal.

For any \( t \in V(T) \) we define \( \Gamma(t) := \beta(t) \cup \bigcup \{ \gamma(e) : e \sim t \} \), where \( e \sim t \) if \( e \) is incident with \( t \).

The width of \((T, \beta, \gamma)\) is the least integer \( w \) such that \( |\Gamma(t)| \leq w + 1 \) for all \( t \in V(T) \). The directed tree width of \( G \) is the least integer \( w \) such that \( G \) has a directed tree decomposition of width \( w \).
The sets $\beta(t)$ are called the bags and the sets $\gamma(e)$ are called the guards of the directed tree decomposition. If $t \in V(T)$ we write $T_t$ for the subtree of $T$ rooted at $t$ (i.e. the subtree containing all vertices $s$ such that the unique path from the root of $T$ to $s$ contains $t$) and we define $\beta(T_t) := \bigcup_{s \in V(T_t)} \beta(s)$. It is easy to see that the directed tree width of a subdigraph of $G$ is at most the directed tree width of $G$.

We close the section on directed tree-width by the following lemma, which we need below.

**Lemma 3.6.** Let $T := (T, \beta, \gamma)$ be a directed tree decomposition of a digraph $G$ and let $H$ be a strongly connected subgraph of $G$. Let $S \subseteq T$ be the subgraph of $T$ induced by $\beta^{-1}(H) := \{ t \in V(T) : \beta(t) \cap V(H) \neq \emptyset \}$ and let $U \subseteq T$ be the (inclusion) minimal subtree of $T$ containing all of $S$. Then $\Gamma(t) \cap V(H) \neq \emptyset$ for every $t \in V(U)$.

**Proof.** Let $S$ and $U$ be as defined in the statement of the lemma. Towards a contradiction suppose that there is some $u \in V(U)$ such that $\Gamma(u) \cap V(H) = \emptyset$. Clearly, $u \notin V(S)$. By construction of $U$ this implies that there are vertices $s, t \in U$ and $v, v' \in V(H)$ with $v \in \beta(s), v' \in \beta(t)$ and $s, t$ are in different components of $U \setminus u$. Let $P_1, P_2$ be two paths in $H$ with $P_1$ linking $v$ to $v'$ and $P_2$ linking $v'$ to $v$.

As $T$ is a tree at least one of $s, t$ must be in the subtree of $T$ rooted at a child of $u$. Let $c$ be this child and assume w.l.o.g. that $s$ is in the subtree of $T$ rooted at $c$. But then $P_1 \cdot P_2$ is a directed walk starting and ending in $\beta(T_c)$ which contains a vertex, namely $v'$, not in $\beta(T_c)$. Hence, by the definition of directed tree-decompositions, $P_1 \cdot P_2 \cap \Gamma(u) \neq \emptyset$, contradicting the assumption that $\Gamma(u) \cap V(H) = \emptyset$. \qed

The following theorem follows from [4], see e.g. [6] for details. A linkage in a digraph $G$ is a set $L$ of pairwise internally vertex disjoint directed paths. The order $|L|$ is the number of paths in $L$. Let $\sigma := \{(s_1, t_1), \ldots, (s_k, t_k)\}$ be a set of $k$ pairs of vertices in $G$. A $\sigma$-linkage is a linkage $L := \{P_1, \ldots, P_k\}$ of order $k$ such that $P_i$ links $s_i$ to $t_i$.

**Theorem 3.7.** Let $G$ be a digraph and $T := (T, \beta, \gamma)$ be a directed tree-decomposition of $G$ of width $w$. Let $k \geq 1$ and $\sigma$ be a set of $k$ pairs of vertices in $G$. Then it can be decided in time $O(|V(G)||E(G)|^{O(k+w)})$ whether $G$ contains a $\sigma$-linkage.

From this, we obtain the following algorithmic result that will be needed later.

**Theorem 3.8.** Let $H$ be a fixed digraph. There is an algorithm running in time $|G|^{O(|E(H)|)}$ which, given a digraph $G$ of directed tree-width at most $w$ as input, computes a butterfly model (topological model) of $H$ in $G$ or determines that $H \not\subseteq_G G$.

**Proof.** The proof for both minor models is nearly identical. We therefore only consider the more complicated cases of butterfly minors. Let $H$ be given and let $G$ be a digraph. If $H \not\subseteq_G G$ then, by Lemma 3.2 there is a tree-like model $H$ in $G$. Hence, every edge $e \in E(H)$ is mapped to an edge $\mu(e) \in E(G)$ and every vertex $v \in V(H)$ is mapped to the union $\mu(v)$ of an in- and out-branching $T_i \cup T_o$. Clearly, the branchings can be chosen so that they have at most $d_H(v)$ leaves and therefore they contain at most $d_H(v)$ vertices of degree more than 2. In total, therefore there are at most $2|E(H)|$ vertices of degree more than 2 in $\bigcup_{v \in V(H)} \mu(v)$. Hence, any tree-like model of $H$ in $G$ consists of the $2|E(H)|$ endpoints of the edges $\mu(e), e \in E(H)$, of the at most $2|E(H)|$ vertices of
The main result of this section is the following theorem.

**Theorem 4.1.** Let $H$ be a strongly connected digraph. $H$ has the Erdős-Pósa property for butterfly (topological) minors if, and only if, there is a cylindrical grid (wall) $G_c$ for some constant $c = c(H)$, such that $H$ is a butterfly (topological) minor of $G_c$.

Furthermore, for every fixed strongly connected digraph $H$ satisfying these conditions and every $k$ there is a polynomial time algorithm which, given a digraph $G$ as input, either computes $k$ disjoint (butterfly or topological) models of $H$ in $G$ or a set $S$ of $\leq h(k)$ vertices such that $G - S$ does not contain a model of $H$.

**Directed Grids.** A natural dual to directed tree width are cylindrical grids which we define next.

**Definition 3.9 (cylindrical grid and wall).** A cylindrical grid of order $k$, for some $k \geq 1$, is a digraph $G_k$ consisting of $k$ directed cycles $C_1, \ldots, C_k$, pairwise vertex disjoint, together with a set of $2k$ pairwise vertex disjoint paths $P_1, \ldots, P_{2k}$ such that:
- each path $P_i$ has exactly one vertex in common with each cycle $C_j$,
- the paths $P_1, \ldots, P_{2k}$ appear on each $C_i$ in this order,
- for odd $i$ the cycles $C_1, \ldots, C_k$ occur on all $P_i$ in this order and for even $i$ they occur in reverse order $C_k, \ldots, C_1$.

For $1 \leq i \leq k$ and $1 \leq j \leq 2k$ let $x_{i,j}$ be the common vertex of $P_i$ and $C_j$.

A cylindrical wall of order $k$ is the digraph $W_k$ obtained from the cylindrical grid $G_k$ of order $k$ by splitting every vertex $v$ of total degree $4$ as follows: we replace $v$ by two fresh vertices $v_t$, $v_h$ plus an edge $(v_t, v_h)$ so that every edge $(w, v) \in E(G_k)$ is replaced by an edge $(w, v_t)$ and every edge $(v, w) \in E(G_k)$ is replaced by an edge $(v_h, w)$.

We will also need the following result. The second part follows using Lemma 3.4.

**Theorem 3.10 ([8]).** There is a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that every digraph of directed tree width at least $f(k)$ contains a cylindrical grid of order $k$ as a butterfly minor and a cylindrical wall $W_k$ as topological minor.

Finally, we need the following acyclic variant of a cylindrical grid.

**Definition 3.11 (acyclic grid).** An acyclic grid of order $k$ is a pair $(P, Q)$ of sets $P = \{P_1, \ldots, P_k\}$, $Q = \{Q_1, \ldots, Q_k\}$ of pairwise vertex disjoint paths such that:
1. for $1 \leq i \leq k$ and $1 \leq j \leq k$, $P_i \cap Q_j$ is a single vertex $v_{ij}$,
2. for $1 \leq i \leq k$, the vertices $v_{i1}, \ldots, v_{ik}$ are in order in $P_i$, and
3. for $1 \leq j \leq k$, the vertices $v_{1j}, \ldots, v_{kj}$ are in order in $Q_j$.

**4 The Erdős-Pósa Property for Strongly Connected Digraphs**

The main result of this section is the following theorem.

**Theorem 4.1.** Let $H$ be a strongly connected digraph. $H$ has the Erdős-Pósa property for butterfly (topological) minors if, and only if, there is a cylindrical grid (wall) $G_c$ for some constant $c = c(H)$, such that $H$ is a butterfly (topological) minor of $G_c$.

Furthermore, for every fixed strongly connected digraph $H$ satisfying these conditions and every $k$ there is a polynomial time algorithm which, given a digraph $G$ as input, either computes $k$ disjoint (butterfly or topological) models of $H$ in $G$ or a set $S$ of $\leq h(k)$ vertices such that $G - S$ does not contain a model of $H$.
We will split the proof of this theorem into two parts. We first show that strongly connected digraphs have the Erdős-Pósa-property within any class of digraphs of bounded directed tree width. Here, a digraph \( H \) has the Erdős-Pósa-property within a class \( C \) of digraphs if the condition of Definition 3.11 is satisfied for every \( G \in C \).

**Lemma 4.2.** Let \( C \) be a class of digraphs of bounded directed tree width. Then every strongly connected digraph has the Erdős-Pósa-property within \( C \) with respect to butterfly and topological minors.

**Proof.** We prove the case for butterfly minors. Let \( G \) be a strongly connected digraph. If there is a \( \beta \in \beta(T) \) containing \( H \) as butterfly minor, the claim follows. Otherwise, by Lemma 3.6, no model of \( H \) in \( G - \Gamma(t) \) can contain a vertex in \( \beta(T) \setminus \Gamma(t) \) and a vertex of \( G - (\beta(T) \cup \Gamma(t)) \). Hence, all remaining models of \( H \) in \( G - \Gamma(t) \) must be contained in \( G' := G - (\beta(T) \cup \Gamma(t)) \). By induction hypothesis, either \( G' \) contains \( k - 1 \) disjoint models of \( H \) as butterfly minor or a set \( S \) of \( f(k - 1) \) vertices such that \( G' - S \) does not contain \( H \) as a butterfly minor. In the first case we have found \( k \) disjoint copies of \( H \) as butterfly minor in \( G \) and in the second case the set \( S' := S \cup \Gamma(t) \) hits every model of \( H \). As \( |S'| \leq w + 1 + f(k - 1) - k \cdot (w + 1) = f(k) \) the claim follows.

The next theorem follows from the previous lemma and Theorem 3.10.

**Theorem 4.3.** Let \( H \) be a strongly connected digraph. If there is a \( c > 0 \) such that \( H \preceq_{h} G_{c} \) (or \( H \preceq_{h} G_{c} \)), where \( G_{c} \) is the cylindrical grid of order \( c \), then \( H \) has the Erdős-Pósa-property for butterfly minors (resp. topological minors).

Furthermore, for every fixed strongly connected digraph \( H \) satisfying these conditions and every \( k \) there is a polynomial time algorithm which, given a digraph \( G \) as input, either computes \( k \) disjoint models of \( H \) in \( G \) or a set \( S \) of \( h(k) \) vertices such that \( G - S \) does not contain a model of \( H \).

**Proof.** Let \( g : \mathbb{N} \to \mathbb{N} \) be the function from Theorem 3.10 and let \( g(k, w) = k \cdot w \) be the function as defined in Lemma 4.2. We claim that the function \( h(k) = g(k, f(k \cdot (c + 1))) \) witnesses the Erdős-Pósa-property for \( H \). Towards this aim, let \( G \) be a digraph. If
$dtw(G) \geq f(k \cdot (c+1))$ then $G$ contains $k$ copies of $G_c$ each of which contains $H$ as butterfly minor. Otherwise, $dtw(G) < f(k \cdot (c+1))$ and we can apply Lemma 4.2.

Note that, for every fixed $c$, any tree-like butterfly model of $G_c$ in a graph $G$ has directed tree-width bounded by $O(c)$. Hence, we can compute a model of $H$ in any model of $G_c$ in $G$ by Theorem 3.8.

We now show the converse to the previous result.

**Theorem 4.4.** Every strongly connected digraph $H$ which is not a butterfly minor of some cylindrical grid does not satisfy the Erdős-Pósa property.

To prove the theorem we first define a general construction that will be used later on. Let $G_k = (C_1, \ldots, C_k, P_1, \ldots, P_{2k})$ be a cylindrical grid, where the $C_i$ are the concentric cycles (ordered from the inside out in a fixed embedding of $G_k$ on the plane) and the $P_i$ are the alternating paths, ordered in clockwise order on the cycles $C_j$, so that for odd $i$, the path $P_i$ traverses the cycles in order $C_1, \ldots, C_k$, i.e. from the inside out, whereas for even $i$ the cycles appear on $P_i$ in the reverse order. For $1 \leq i \leq k$ and $1 \leq j \leq 2k$ let $x_{i,j}$ be the common vertex of $P_j$ and $C_i$.

Recall that a cylindrical wall $W_k$ is obtained from $G_k$ by splitting degree 4 vertices. Note that the outer cycle $C_k$ does not have any degree 4 vertices, and therefore the following construction can also be applied to a wall $W_k$.

**Definition 4.5 (The digraphs $G_{n,H}^{k,e}$ and $W_{n,H}^{k,e}$).** Let $H$ be a digraph and let $e \in E(H)$ be an edge. The digraph $G_{n,H}^{k,e}$ is obtained from the disjoint union of $k$ isomorphic copies of $H$, say $H_1, \ldots, H_k$, and the grid $G_k$ as follows. In each copy $H_i$ we delete the edge $e_i = (u_i, v_i)$ corresponding to $e$. Furthermore, in $G_k$ we delete all edges $(x_{i,2i-1}, x_{i,2i})$, for $1 \leq i \leq k$. Finally, for all $1 \leq i \leq k$, we add an edge $(u_i, x_{k,2i})$ and an edge $(x_{2i-1}, v_i)$. We call $G_{n,H}^{k,e}$ the attachment of $H$ to $G_k$ and refer to the graphs $H_i$ with the edge $e_i$ deleted plus the two new edges as the $i$-th copy of $H$ in $G_{n,H}^{k,e}$.

We can apply the same construction using $W_k$ instead of $G_k$. We denote the resulting graph by $W_{n,H}^{k,e}$ and call it the attachment of $H$ to $W_k$.

See Figure 4.1 for a schematic overview of $G_{n,H}^{k,e}$. We are now ready to prove Theorem 4.4.

**Proof of Theorem 4.4.** Let $H$ be a strongly connected digraph such that $H \not\preceq_b G_k$ for all $k \geq 0$. Let $e \in E(H)$. Towards a contradiction, suppose $H$ had the Erdős-Pósa property, witnessed by a function $f : \mathbb{N} \to \mathbb{N}$. Choose a value $k > f(2)$ and let $G := G_{n,H}^{k,e}$.

We first claim that for any set $S \subseteq V(G)$ of at most $f(2)$ vertices, $G - S$ contains $H$ as a butterfly minor. To prove this, let $S$ be such a set. As $|S| < f(2)$, there is an index $1 \leq i \leq k$ such that $S$ does not contain a vertex of $C_i \cup P_{2i-1} \cup P_{2i} \cup H_i$, where $H_i$ is the $i$-th copy of $H$ in $G_{n,H}^{k,e}$. But then, $H \preceq_b C_i \cup P_{2i-1} \cup P_{2i} \cup H_i$.

To complete the proof we show next that $G$ does not contain two disjoint butterfly models of $H$. Let $\mu$ be a tree-like model of $H$ in $G_k$. As $H \not\preceq_b G_k$, by assumption, $\mu$ must contain a vertex $v$ in some copy $H_i$ of $H$ in $G_{n,H}^{k,e}$. But as $H$ is strongly connected and $H_i$ has fewer edges than $H_i$, $\mu$ must also use both edges $(u_i, x_{k,2i})$ and $(x_{2i-1}, v_i)$ and a directed path in $G_k$ linking $x_{k,2i}$ to $x_{2i-1}$. We view $G$ as being embedded in the
plane. Then this path induces a closed curve from \( x_{k,2i} \) to \( x_{k,2i-1} \) in the plane splitting \( G_k \) into two disjoint parts. Furthermore, the part containing the rest of the outer cycle \( C_k \) not on the curve is acyclic. Hence, there cannot be a second model of \( H \) in \( G - \mu(H) \).

Theorem 4.4 and 4.3 together imply the proof of Theorem 4.1 for butterfly minors. To prove it for the topological minors, it is easily seen that the same construction as in the Theorem 4.4 where the grid \( G_k \) is replaced by a wall \( W_k \) proves that if a strongly connected digraph \( H \) is not a topological minor of some fixed directed wall \( W \), then \( H \) does not have the Erdős-Pósa property for topological minors.

We also obtain the following consequence.

**Corollary 4.6.** For strongly connected digraphs, the Erdős-Pósa property (for butterfly and topological minors) is closed under strongly connected subgraphs, i.e. if a strongly connected graph \( H \) does not satisfy the Erdős-Pósa property and \( H \preceq_b G \) then \( G \) does not satisfy Erdős-Pósa property.

5 The EP-Property for Vertex-Cyclic Digraphs

In this section we extend the results of the previous section to the more general class of vertex cyclic digraphs. A digraph is vertex cyclic if it does not contain a trivial strong component, i.e. if every vertex lies on a cycle. Clearly, every strongly connected digraph is vertex cyclic but the converse is not true. For simplicity, in this section we only consider weakly connected digraphs, i.e. where the underlying undirected graph is connected. Many results can be extended to the case of not weakly connected digraphs but we leave this for the full version of the paper.

Let \( G \) be a digraph and let \( e \in E(G) \). Let \( n \geq 1 \). We define \( G^n_e \) as the digraph obtained from \( G \) by subdividing \( e \) \( n \) times. Given digraphs \( H \) and \( G \), we say that \( H \) is *topologically s-embeddable* in \( G \) if there is an edge \( e \in E(G) \) such that \( H \preceq_t G^n_e \). We say that \( H \) is *butterfly s-embeddable* in \( G \) if there is an edge \( e \in E(G) \) such that \( H \preceq_b G^n_e \).

A digraph \( G \) is ultra-homogeneous with respect to topological (or butterfly) minors, if the block graph of \( G \) is a simple directed path without parallel edges and any two components of \( G \) are pairwise topologically (or butterfly, respectively) s-embeddable.
into each other and furthermore if the length of the block graph is at least 3, then all of the components except the first and the last components, w.r.t. topological order, have the same size and also none of those has smaller size than the first or the last component.

**Definition 5.1.** A digraph $G$ is ultra-homogeneous with respect to topological (or butterfly) minors, if the block graph of $G$ is a simple directed path without parallel edges and any two components of $G$ are pairwise topologically (or butterfly, respectively) $s$-embeddable into each other and furthermore if the length of the block graph is at least 3, then all of the components except the first and the last components, w.r.t. topological order, have the same size and also none of those has smaller size than the first or the last component.

Our main classification result of this section is the following.

**Theorem 5.2.** Let $H$ be a weakly connected vertex-cyclic digraph. If $H$ has the Erdős-Pósa property for butterfly (topological) minors, then it is ultra-homogeneous with respect to butterfly (topological) embeddings, its maximum degree is at most 3 and every strong component of $H$ is a butterfly (topological) minor of some cylindrical grid (wall) $G_k$.

The first result we prove is the following.

**Lemma 5.3.** Let $H$ be a vertex-cyclic digraph. If $H$ contains a vertex of degree at least 4, then $H$ does not have the Erdős-Pósa property for topological minors.

**Proof.** Let $H$ be vertex-cyclic and let $v \in V(H)$ be a vertex of degree at least 4 in $H$. Furthermore, let $e$ be an incident edge of $v$. Towards a contradiction suppose $H$ had the Erdős-Pósa property witnessed by a function $f : \mathbb{N} \to \mathbb{N}$. As in the proof of Theorem 4.4, let $k > f(2)$. Let $W_{H,e}^k$ be the digraph defined in Definition 4.5.

We show first that $W_{H,e}^k$ contains two disjoint topological models of $H$. Let $A_1, \ldots, A_l$ be the strong components of $H$ in topological order, i.e. there is no edge from $A_i$ to $A_j$ whenever $j < i$, and let $A_s$ be the component containing $v$. Let $\mu$ be a topological model of $H$ in $W_{H,e}^k$. Note that in $W_{H,e}^k$ no vertex $w \in V(W_k)$ has degree $\geq 4$. Hence $\mu(v)$ must be in some copy $H'$ of $H$ in $W_{H,e}^k$. More precisely, $\mu(v)$ must be in the strong component of $H'$ corresponding to $A_s$. For, suppose $\mu(v)$ was in a strong component corresponding to some $A_i$ with $i < s$ such that $A_i$ is reachable from $A_s$ in $H$. Then for every $w \in V(H)$ from which $v$ is reachable in $H$, $\mu(w)$ must be in a component $A_j$ in the copy $H'$ such that $A_j$ is reachable from $A_s$. But this is impossible for cardinality reasons. Similarly, we can show that $\mu(v)$ cannot be in any other component except for $A_s$. It follows that every edge and every vertex of $A_i$ must be mapped to either the copy of $A_i$ in $H'$ or to some vertex of the wall or $A_s$ in another copy of $H$. In any case, $\mu(A_i)$ is strongly connected and therefore $\mu(A_i)$ contains both edges connecting $H'$ to the wall and a directed path between them. We can therefore argue as in the proof of Theorem 4.4.

To conclude the argument, we can argue as in the proof of Theorem 4.4 that for every set $S \subseteq V(W_{H,e}^k)$ of order $< k$ the graph $W_{H,e}^k - S$ contains $H$ as topological minor.

□
Note that this result does not necessarily extend to butterfly minors. We now introduce a construction that will frequently be applied below.

**Definition 5.4.** Let \( A_k := ((P_1, \ldots, P_k), (Q_1, \ldots, Q_k)) \) be the acyclic grid of order \( k \) as defined in Definition 5.1. Recall that \( V(P_i) \cap V(Q_j) = \{v_{ij}\} \). Let \( H \) be a digraph and let \( C_1 \) and \( C_2 \) be distinct non-trivial strong components of \( H \) so that there is an edge \( e = (u, v) \) for \( u \in V(C_1) \) and \( v \in V(C_2) \).

1. Let \( e_2 \in E(C_2) \) be an edge incident to \( v \). The left acyclic attachment graph \( A_{e_2}^{n,H,C_1,C_2} \) of \( H \) through \( e \) and \( e_2 \) of order \( n \) is defined as follows. Take a copy of \( A_n = (P, Q) \) and \( n \) disjoint copies \( H_1, \ldots, H_n \) of \( H \). For every \( v \in V(H) \) we write \( v_i \) for its isomorphic copy in \( H_i \) and likewise we write \( e_i \) for the copy of an edge \( e \) in \( H_i \). For all \( 1 \leq i \leq n \), we delete the edges \( e_i' = (u', v_i) \) and \( e_2' = (x, y) \) and instead add the edge \( (u', e_i, v_i) \) and identify the topmost vertex \( v_{1,i} \) of the \( i \)-th column of \( A_k \) with \( x \) and the last vertex \( v_{k,i} \) of this column with \( y \).

2. Now let \( e_1 \in E(C_1) \) be an edge incident to \( u \). The right acyclic attachment graph \( A_{e_1}^{n,H,C_1,C_2} \) of \( H \) through \( e \) and \( e_1 \) of order \( n \) is defined analogously, but now the edge \( e_i \) in the copy \( H_i \) is replaced by an edge \( (v_i, e_i, v_i') \) and the end vertices of the edge \( e_1 = (x, y) \) are identified with \( v_{1,i} \) and \( v_{k,i} \) respectively.

It is easily seen that for all \( n > 1 \), \( H \preceq_t A_{e_2}^{n,H,C_1,C_2} \) and hence \( H \preceq_b A_{e_2}^{n,H,C_1,C_2} \), for all choices of \( C_1, C_2, e_1, e_2 \) as in the definition and furthermore, for all \( S \subseteq V(A_{e_2}^{n,H,C_1,C_2}) \) of order \( < n \), \( H \preceq_1 A_{e_2}^{n,H,C_1,C_2} - S \) and hence \( H \preceq_0 A_{e_2}^{n,H,C_1,C_2} - S \). However, for some choices of \( H, C_1, C_2, A_{e_2}^{n,H,C_1,C_2} \) may contain many disjoint models of \( H \), for instance if \( H \) only consists of two cycles \( C_1, C_2 \) connected by an edge.

**Lemma 5.5.** Let \( H \) be a vertex cyclic digraph and let \( C \) be the set of its components. If \( H \) satisfies any of the following conditions, then it does not have the Erdős-Pósa property neither for butterfly nor for topological minors.

1. There are \( C, C_1, C_2 \), all distinct, and edges \( e_1, e_2 \) such that \( e_1 \) links \( C \) to \( C_1 \), for \( l = 1, 2 \), or \( e_1 \) links \( C_1 \) to \( C \), for \( l = 1, 2 \).
2. \( H \) contains two components \( C \) and \( C' \) with two distinct edges linking \( C \) to \( C' \).
3. \( H \) contains two distinct components \( C, C' \) such that \( C \) is not embeddable into \( C' \) (with respect to topological minor).
4. \( H \) contains a strong component \( C \) such that for all \( k \geq 1 \), \( C \not\preceq_h G_k \) (resp. \( C \not\preceq_t W_k \)).
5. \( H \) has three strongly connected components \( C, C', C'' \) such that there is a path from \( C \) to \( C' \) and a path from \( C' \) to \( C'' \) and either \( |C'| < |C''| \) or \( |C'| < |C| \).

**Proof.** We prove the cases for butterfly minors, the cases of topological minors are analogous. Towards a contradiction, suppose that \( H \) has the Erdős-Pósa property witnessed by a function \( f : \mathbb{N} \rightarrow \mathbb{N} \). For each item we construct a counterexample \( A \) such that after deleting \( f(2) \) vertices from \( A \), it still has a model of \( H \).

**Proof of Item 1.** We first consider the case where there is a component \( C \) of \( H \) and two other components \( C_1, C_2 \) with an edge from \( C \) to \( C_1 \) and from \( C \) to \( C_2 \). A terminal component of \( H \) is a strong component without any outgoing edges. Let \( T \) be a terminal component with a minimal number of edges and among these with a minimal number
of vertices. Let \( C \) be a component of \( H \) with edges to two distinct components and such that \( T \) is reachable from \( C \) by a path \( P \) (\( C \) exists as the block graph of \( H \) is not a path, by assumption, and \( G \) is weakly connected) and let \( S \) be the unique component of \( H \) such that \( P \) contains an edge \( e = (s, t) \in E(H) \) with tail \( s \in V(S) \) and head \( t \in V(T) \). Let \( e' = (w, t) \in E(T) \) be an edge with head \( t \), which exists as \( T \) is not a trivial component. Let \( k > f(2) \) and let \( A := A_{x, y}^{k, H, S,T} \) be the left acyclic attachment as defined in Definition 5.4. See Figure 5.1 for an illustration.

Let \( H_1, \ldots, H_k \) be the copies of \( H \) in \( A \). For each vertex \( v \in V(H) \) and edge \( e \in E(H) \) we write \( e^v, e^i \) for the corresponding vertex or edge in \( H_i \). Note that \( e^i, e^i \) do not exist as they were deleted in the construction of \( A \). We denote by \( T^i \) the copy of \( T \) in \( H_i \) with the edge \( e' \) removed and by \( T_i \) the copy of \( T \) in \( H_i \) plus the path \( Q_i \) of the grid by which \( e' \) was replaced.

As noted above, after deleting a set \( D \) of \( f(2) \) vertices from \( A \), \( H \subseteq A - D \). Hence it suffices to show that there are no two distinct butterfly models of \( H \) in \( A \).

Let \( \mu \) be a minimal tree-like butterfly model of \( H \) in \( A \), i.e. a tree-like model such that no proper subgraph of \( \mu(H) \) contains a model of \( H \). As \( \mu \) is tree-like, every \( \mu(v) \) is the union of an in-branching and an out-branching which only share their root \( r_v \). As \( \mu \) is minimal, if \( X \subseteq H \) is strongly connected, then \( \mu(X) \) contains a maximal strongly connected subgraph \( \rho(X) \) which contains every root \( r_v \) for \( v \in V(X) \). It follows that for no component \( X \) of \( H \) we have \( \mu(X) \subseteq T^i \), as \( T \) was the component with the minimal number of edges and \( T^i \) has one edge less. This implies that if for some vertex \( v \in V(H) \), \( \mu(v) \cap T^i \neq \emptyset \), for some \( 1 \leq i \leq k \), then \( T^i \subseteq \mu(X_v) \) where \( X_v \) is the component of \( H \) containing \( v \). Now consider \( \mu(C) \). As \( \mu(C) \) is strongly connected, if \( \mu(C) \) contains a vertex of the acyclic grid \( A_k \) contained in \( A \), then \( \rho(C) = T^i \) for some \( 1 \leq i \leq k \). Let \( C_1, C_2 \) be two components of \( H \) such that \( H \) contains an edge \( e_1 \) from \( C \) to \( C_1 \) and an edge \( e_2 \) from \( C \) to \( C_2 \). But as \( T \) was chosen minimal, \( \rho(C) \cup \rho(C_i) \subseteq T^i \), for \( i \in \{1, 2\} \). Hence, \( \rho(C) \subseteq T^i \) and \( \rho(C_2) \subseteq T^{i_2} \) for some \( j_1 \neq j_2 \) different from \( i \), as otherwise there was no path from \( \mu(C) \) to \( \mu(C_1) \) and \( \mu(C_2) \) in \( A \). But as each of \( \mu(C), \mu(C_1), \mu(C_2) \) contains an entire column of the acyclic grid \( A_k \) in \( A \), this is impossible.

It follows that \( \rho(C) \) must be contained in some \( H_i \setminus \hat{T}_i \). As we cannot have \( \mu(H) \subseteq H_i \setminus \hat{T}_i \), it follows that for some \( j \), \( \hat{T}_j \subseteq \mu(H) \) and therefore \( \mu(H) \) also includes the edge from \( H_i \) to the vertex \( x_{i,1} \) of the grid and a path \( L_{i} \) from \( x_{i,1} \) to \( \hat{T}_j \).

Now suppose \( \mu' \) is a second model of \( H \) in \( A \), which again we assume to be minimal and tree-like. By the same argument, \( \mu'(H) \) must contain an entire column \( Q_{j'} \) and path \( L'_{i'} \) from some vertex \( x_{i',1} \) to \( Q_{j'} \). But then, if \( j' < j \), then \( Q_{j'} \) has a non-empty
intersection with \( i' \) and if \( j < j' \) then \( Q_j \) has a non-empty intersection with \( L_{i'} \). Hence, \( \mu \) and \( \mu' \) are not disjoint.

This concludes the case where \( H \) contains a component \( C \) with two outgoing edges to two distinct other components. The case where there is a component \( C \) with incoming edges from two other distinct components is analogous, using the right acyclic attachment instead of the left acyclic attachment.

**Proof of Item 2.** Let \( C \) and \( C' \) be as in the statement of the Item 2 chosen so that from \( C' \) no component \( X \) of \( H \) is reachable such that \( X \) has two edges to another component \( Y \). Let \( e_1 = (s_1, t_1) \) and \( e_2 = (s_2, t_2) \) be two distinct edges with tail in \( C \) and head in \( C' \).

By Item 1 we can assume that the block graph of \( H \) is a directed path with parallel edges between components.

Let \( k > f(2) \) and let \( A_{2k} = \{(P_1, \ldots, P_{2k}), (Q_1, \ldots, Q_{2k})\} \) be the acyclic grid of order \( 2k \). Again, \( V(P_i) \cap V(Q_j) = \{x_{i,j}\} \). Let \( G_k \) be the graph obtained from \( A_{2k} \) by adding \( k \) disjoint copies \( H_1, \ldots, H_k \) of \( H \). For \( v \in V(H) \) or \( e \in E(H) \) let \( v^i \) and \( e^i \) be the vertex or edge corresponding to \( v \) and \( e \) in the copy \( H_i \), respectively. For all \( 1 \leq i \leq k \) we delete the edges \( e_1^i \) and \( e_2^i \) and add edges \( (s_1, x_{2i-1,1}), (s_2, x_{2i,1}) \) and \( (x_{2i,2i-1,1}, t_1^i), (x_{2i,2i}, t_2^i) \). See Figure 5.2 for an illustration.

As \( A_{2k} \) contains two disjoint paths \( P_1 \) linking \( x_{2i-1,1} \) to \( x_{2i,2i-1} \) and \( P_2 \) linking \( x_{2i,1} \) to \( x_{2i,2k} \), \( G_k \) contains \( H \) as a butterfly minor. Furthermore, it is easily seen that \( H \preceq_b G - S \) for every set \( S \subseteq G_k \) of order \( < k \).

Hence, we only need to show that \( G_k \) does not contain two distinct butterfly models of \( H \). Let \( \mu \) be a minimal tree-like butterfly model of \( H \) in \( A \), i.e. a tree-like model such that no proper subgraph of \( \mu(H) \) contains a model of \( H \). As \( \mu \) is tree-like, every \( \mu(v) \) is the union of an in-branching and an out-branching which only share their root \( r_v \). As \( \mu \) is minimal, if \( X \subseteq H \) is strongly connected, then \( \mu(X) \) contains a maximal strongly connected subgraph \( \rho(X) \) which contains every root \( r_v \) for \( v \in V(X) \). Let \( X_1, \ldots, X_l \) be the components of \( H \) reachable from \( C' \) in topological order. By the choice of \( C \) and \( C' \), between \( C' \) and \( X_1 \) and between \( X_i \) and \( X_{i+1} \), for all \( i < l \), there is exactly one edge.

---

**Fig. 5.2.** Illustration of the construction in the proof of Item 2 in Lemma 5.5.

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15
A construction very similar to the construction in the proof of Theo-

Proof of Item 4:

not distinct.

contain a column

V

unique vertex in

must contain

ˆ µ

other hand, in

A

the edge

copy of

strongly connected subgraph

ρ

X

like and minimal, every

or edge

e

∈

H

attachment as defined in Definition 5.4. As before,

C

s

case is analogous using right acyclic attachments instead.

Let

models are not disjoint.

P

there must be an index

j

P

and two disjoint paths

P_1

linking

x_{2i-1,1}

to

x_{2k,2i-1}

and

P_2

linking

x_{2i,1}

to

x_{2i,2k}.

Now let

µ'

be another minimal tree-like model of

H

in

G_k.

By the same argument

there must be an index

j

such that

µ'(H)

contains the edges

(s_1', x_{2i-1,1}), (s_2', x_{2i,1})

and

two disjoint paths

P_1'

linking

x_{2j-1,1}

to

x_{2k,2j-1}

and

P_2'

linking

x_{2j,1}

to

x_{2j,2k}.

But clearly,

(P_1 ∪ P_2) ∩ (P_1' ∪ P_2') ≠ ∅

and hence the models are not disjoint.

Proof of Item 3

Let

H

and

C',

we can assume that the block graph of

H

is a simple directed path without parallel edges.

Choose

C

and

C'

such that

C
does not embed into

C'

with respect to butterfly embeddings or vice versa and among all such pairs choose

C'

so that it is the latest such component in the block graph of

H,

i.e. no component

C''

which is part of such a pair is reachable from

C'.

We assume that

C
has no butterfly embedding into

C'

as defined above. The other case is analogous using right acyclic attachments instead.

Let

S ≠ C'
be the component of

H
such that

H
contains an edge

e = (s, t)
with

s ∈ V(S)
and
t ∈ V(C').
Let

e' = (w, t)
be any edge in

C'
with head

t,
which must exist as

C'
is not trivial.
Now let

k > f(2)
and

A = A_k^{e, e', H, S, C'}
be the left acyclic attachment as defined in Definition 5.4.
As before,

H ≤_b A − D
for any set

D
of
<k
vertices.
We will show that

H
has no two disjoint butterfly models in

A.

Let

µ
be a minimal tree-like butterfly model of

H
in

A.
Let

H_1, ..., H_k
be the disjoint copies of

H
in

A
and as before we write

v^i, e^i
for the copy of a vertex

v ∈ V(H)
and an edge

e ∈ E(H)
in the

i-th
copy. Furthermore, as in the previous proofs, as

µ
is tree-like and minimal, every

µ(v)
is the union of two branchings sharing only their root

r_v
and for every strongly connected subgraph

X ⊆ H
the model

µ(X)
contains a maximal strongly connected subgraph

ρ(X)
which contains all roots of

v ∈ X
Let

C^i
be the copy of

C'
i
in

H_i
with the edge

e^i
removed and let

C'^i
be the copy of

C'
i
where the edge

e^i
is replaced by the column

Q_i
of the grid

A_k
= (P_1, P_2, Q_1, ..., Q_k)
used to construct

A.
As

C
has no butterfly embedding in

C',

ρ(C)
cannot be contained in

C'^i
for any

1 ≤ i ≤ k
and therefore

ρ(C) ⊆ H_i − V(C'^0 ∪ X_1 ∪ ... ∪ X_i)
for some

1 ≤ i ≤ k,
where

X_1, ..., X_i
are the components of

H
reachable from

C'.

On the other hand,

ρ(H) ⊆ H_j − V(C'^0 ∪ X_1 ∪ ... ∪ X_j)
for any

1 ≤ j ≤ k.
Hence,

ρ(C)
must contain

C'^j
for some

j
and a path

L_j
from

x_{i,j}
onto a vertex on

Q_j,
where

x_{i,j}
is the unique vertex in

V(P_j) ∩ V(Q_j).
for all

1 ≤ i,j ≤ k.

Now let

µ'
be another butterfly model of

H
in

A.
By the same argument,

µ'(H)
must contain a column

Q_{j'}
and a path

L_{j'}
from

x_{i',1}
onto a vertex on

Q_{j'}.
But then

µ
and

µ'
are not distinct.

Proof of Item 4

A construction very similar to the construction in the proof of Theorem 4.4 shows that this case holds and we omit the details.

Proof of Item 5

Let

C_1, C_2, C_3
be a triple of strongly connected components as stated in Item 5.
We prove Item 5 in the case that

|C_2| < |C_3|
the other case is analogous. By
we can assume that the block graph of \( H \) is a simple directed path without parallel edges.

We can assume that the distance between \( C_1, C_2 \) and \( C_3 \) in the block graph is minimized among all triples of components satisfying conditions of Item 5.

As the distance is minimized one can easily show by a simple case distinction that \( C_1, C_2 \) and \( C_3 \) are reachable from \( \rho \) of \( H \) in the maximal strongly connected subgraph \( \rho \) with a root \( v \) such that every \( \rho \) in \( H \) is the union of an in-branching and an out-branching which only share their roots \( r_v \) and for every strongly connected subgraph \( S \subseteq V(A) \) of size at most \( f(2) \) we have \( H \nsubseteq A - S \). In the rest we show that there are no two distinct models of \( H \) in \( A \) as a butterfly minor.

Let \( H' = \mu(H) \) be a minimal tree-like butterfly model of \( H \) in \( A \), i.e. a tree-like model such that no proper subgraph of \( \mu(H) \) contains a model of \( H \). As \( \mu \) is tree-like, every \( \mu(v) \) is the union of an in-branching and an out-branching which only share their root \( r_v \) and for every strongly connected subgraph \( X \subseteq H \) the model \( \mu(X) \) contains a maximal strongly connected subgraph \( \rho(X) \) which contains all roots \( r_v \) of \( v \in X \).

Let \( C_1', C_2', C_3' \) be the copies of \( C_1, C_2, C_3 \) in \( H \) used to construct \( A \).

As \( |C_2'| < |C_3'| \), \( \rho(C_3) \) cannot be contained in \( C_3' \) for any \( 1 \leq i \leq k \). Also for all \( j \in [k] \), \( \rho(C_3) \) cannot appear in any \( X'_1, \ldots, X'_j \) where \( X'_1, \ldots, X'_j \) are the components of \( H' \) such that \( C'_j \) is reachable from them (w.r.t. reachability in \( C_2 \) in \( H \)). Therefore \( \rho(C_3) \subseteq H_i - V(X'_1 \cup \cdots \cup X'_j) \), for some \( 1 \leq i \leq k \). On the other hand, \( \rho(C_2') \subseteq \bigcup_{j \in [k]} H_j - V(Y'_1 \cup \cdots \cup Y'_j) \), where \( Y'_1, \ldots, Y'_j \) are strong components of \( H' \) which are reachable from \( C_2' \). Note that the difference between the two cases is that model of \( C_2 \) can be obtained by going back and forth through arbitrarily many \( C'_2 \)'s as there are directed paths which connects them together.

Analogously we have \( \rho(C_1') \subseteq \bigcup_{j \in [k]} H_j - V(C'_j) \cup Y'_j \cup \cdots \cup Y'_{j} \cup Y'. \) There is a directed path \( P_1 \) in \( H' \) which connects \( \rho(C_1) \) to \( \rho(C_2) \) and there is a directed path \( P_2 \) in

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**Fig. 5.3. Illustration of the construction in the proof of Item 5 in Lemma 5.**
We close the section by giving a positive result, i.e. we provide a vertex-cyclic ultra-

Theorem 5.6. Before providing a proof for the theorem we need some lemmas and

There is an algorithm which for a given

5.1 Positive Instance for Erdős-Pósa Property in Vertex Cyclic Graphs

We close the section by giving a positive result, i.e. we provide a vertex-cyclic ultra-

H’ which connects \( \rho(C_2) \) to \( \rho(C_3) \). Considering the structures of \( \rho(C_1), \rho(C_2), \rho(C_3) \)
as explained, \( P_1, P_2 \) will go through the acyclic grid in \( A \) (maybe they go through some of

\( C_2 \)'s as well) and they will cut the acyclic grid into different regions. The path \( P_2 \) has

a subpath \( P_{1}^{H'} \) which starts at some vertex \( v_{i,1} \) and ends at some vertex \( v_{2k,j_2} \) for \( 2 \leq i \leq 2k \)

and \( j_2 \in [2k] \). Also \( P_1 \) has a subpath \( P_{1}^{H'} \) which starts at some vertex \( v_{j_1,1} \) and ends at

the vertex \( v_{j_1,2k} \) for some \( j_1 \in [2k] \). Let \( H'' \) be another model of \( H \) in \( A \). Similar to \( H' \)

we have subpaths \( P_{1}^{H''}, P_{2}^{H''} \). We have \( P_{1}^{H'} \cap P_{2}^{H''} \neq \emptyset \) or \( P_{2}^{H'} \cap P_{1}^{H''} \neq \emptyset \). Hence
every two models of \( H \) in \( A \) will intersect.

□

Proof of Theorem 5.2 follows from Lemma 5.5 and 5.3.

Theorem 5.6. Let \( H \) be a digraph consisting of two disjoint cycles joined by a single edge.

There is a function \( h : \mathbb{N} \rightarrow \mathbb{N} \) such that for every integer \( k \) and every graph \( G \)
either there are \( k \) distinct topological models of \( H \) in \( G \) or there is a set \( S \subseteq V(G) \) such that

\( |S| \leq h(|H| + k) \) and \( H \not\preceq G - S \).

Furthermore, for every \( H \) and \( k \) there is a polynomial-time algorithm which either

finds \( k \) distinct topological models of \( H \) in \( G \) or finds a set \( S \subseteq G \) of vertices of size at most \( h(|H| + k) \) which hits

every topological model of \( H \) in \( G \).

In the rest of this section whenever we refer to \( H \), it means the \( H \) as stated in

Theorem 5.6. Before providing a proof for the theorem we need some lemmas and
definitions. An \( l \)-cycle is a cycle of length at least \( l \). An \( l \)-cluster in a graph \( G \) is a

maximal subgraph of \( G \) consisting of \( l \)-cycles such that every two of them intersect each

other. For a set \( C \) of \( l \)-clusters in \( G \) we write \( G[C] \) to denote the subgraph of \( G \) induced

by the set of vertices occurring in an \( l \)-cluster in \( C \).

For three disjoint \( l \)-cycles \( C_1, C_2, C_3 \subseteq G \), the cycle \( C_1 \) is an \( l \)-transit cycle of

\( C_2, C_3 \) if there is a path \( P \) in \( G \) which connects \( C_2 \) to \( C_3 \) and \( P \cap C_1 \neq \emptyset \). A cycle \( C \)
is an \( l \)-transit cycle if it is an \( l \)-transit cycle for some pair \( C_2, C_3 \) of disjoint \( l \)-cycles. A

set \( C \) of \( l \)-clusters in \( G \) is bipartite if all \( C \in C \) are pairwise vertex disjoint and there is no \( l \)-transit cycle in \( G[C] \). A graph \( G \) is an \( l \)-cluster bipartite graph if all of its \( l \)-clusters together form a bipartite set of \( l \)-clusters. By Theorem 4.3 a single \( l \)-cycle has the

Erdős-Pósa property, witnessed by some function \( j_1 : \mathbb{N} \rightarrow \mathbb{N} \). In particular \( j_1(2) \) means that in a given graph \( Z \) either there are two disjoint cycles of length at least \( l \) or there is a set \( S_1 \) of size at most \( j_1(2) \) such that there is no cycle of length at least \( l \) in \( Z - S_1 \).

The following lemma is required for some algorithmic aspects of Theorem 5.6.

Lemma 5.7. There is an algorithm which for a given \( l \)-cluster bipartite graph \( G \) of
directed tree-width at most \( w \), finds all of its \( l \)-clusters in time and space \( |G|^{O(w+1)} \).

Proof. We first observe that we can check whether a vertex \( v \in V(G) \) lies in an \( l \)-cycle or not, and if it is in some \( l \)-cycle find at least one of those cycles, namely, its corresponding \( l \)-cycle. To see this for a vertex \( v \in V(G) \) we guess \( l - 1 \) distinct other vertices which
together with \( v \) form a model of \( l \)-cycle. By Theorem 3.7 we can check if they form an \( l \)-cycle in time and space \( |G|^{O(w+1)} \). So we can find corresponding \( l \)-cycle of each vertex \( v \in V(G) \). We put two vertices \( u, v \in V(G) \) in one \( l \)-cluster if their corresponding \( l \)-cycles intersect. Recall that in an \( l \)-cluster bipartite graph it is impossible to have three \( l \)-cycles \( C_1, C_2, C_3 \) such that \( C_1 \cap C_2 \neq \emptyset \) and \( C_2 \cap C_3 \neq \emptyset \) and \( C_1 \cap C_3 = \emptyset \), because then \( C_2 \) is an \( l \)-transit cycle. \( \square \)

We use the following essential lemma in the rest of this section.

**Lemma 5.8.** Given an integer \( k \) and \( l \)-cluster bipartite graph \( G \). Either there are \( k \) disjoint minors of \( H \) in \( G \) or there is a set \( S_k \subseteq V(G) \) such that it hits every model of \( H \) in \( G \). Furthermore \( |S_k| \leq 2(k-1) \cdot f_1(2) + (l-1)k(\max\{f_1(2), l\}) + k - 1 \).

**Proof.** We break the proof into three steps. First we either find \( k \) disjoint models of \( H \) which have 2 cycles of length at least \( l \) or a set \( S_k \subseteq G \) of size at most \( k-1+(k-1) \cdot f_1(2) \) which hits every model of \( H \) in \( G \) which has two \( l \)-cycles. We know every such model has its cycles in two different \( l \)-clusters.

Let \( C \) be the set of all \( l \)-clusters in \( G \). If there is an element in \( C \) which does not have a path to (or from) any other element of \( C \) in \( G \) then it does not participate in any minor of \( H \) as required in the above so we can ignore them. We partition the rest of \( C \) into two partitions \( C_1, C_2 \) such that for any element in \( C_1 \) there is a path to some element in \( C_2 \). As non of the elements has an \( l \)-transit cycle, we always have this bi-partition.

We add a vertex \( v_1 \) to \( G \) and for each \( C' \subseteq C_1 \) an edge from \( v_1 \) to a vertex in \( C' \). Similarly, add a vertex \( v_2 \) and for each \( C'' \subseteq C_2 \) an edge from one vertex of \( C'' \) to \( v_2 \).

By Menger’s theorem, either there are \( k \) disjoint paths from \( v_1 \) to \( v_2 \) or there is a set of vertices of size at most \( k-1 \) which disconnects \( v_1 \) from \( v_2 \).

First we claim that if there are \( k \) disjoint paths from \( v_1 \) to \( v_2 \), then we have \( k \) disjoint copies of \( H \) as required. It is clear that any path from \( v_1 \) to \( v_2 \) corresponds to a model of \( H \) in \( G \) with both cycles in \( G[C] \). On the other hand, two models from two disjoint paths may intersect only if they go through each other components in \( C_1 \) or \( C_2 \). But this cannot happen, as otherwise we have an \( l \)-transit cycle in \( C \).

Similarly, if there is a vertex set \( W \) of size at most \( k-1 \) that disconnects \( v_1 \) and \( v_2 \), then for every \( v \in W \cap V(G[C_1 \cup C_2]) \) let \( S_v \) be the set of vertices of size at most \( f_1(2) \) which hits every cycle of length at least \( l \) in the \( l \)-cluster that \( v \) belongs to. Let \( S_1 = W \cup \bigcup_{v \in W} S_v \). Then \( S_1 \) is a hitting set of every model of \( H \) in \( G \) which obtained from 2 disjoint \( l \)-cycles. But size of \( S_1 \) is at most \( k-1+ (k-1) \cdot f_1(2) \) as claimed.

Now in the second step we consider each \( l \)-cluster in \( G-S_1 \). Each \( l \)-cluster is strongly connected. Suppose there are \( t \) disjoint \( l \) clusters \( C_1, \ldots, C_t \) such that for all \( i \in [t] \) : \( H \not\preceq C_i \). If \( t \geq k \) then we have \( k \) disjoint models of \( H \) in \( G \). Otherwise for all \( i \in [t] \) we can choose a set \( S_i' \subseteq V(C_i) \) of size at most \( f_1(2) \) vertices such that \( C_i - S_i' \) has no \( l \)-cycle. Let \( S_2 = \bigcup_{i=1}^t S_i' \). We have \( |S_2| \leq (k-1) f_1(2) \). In \( G-S_1-S_2 \) there is no model of \( H \) which has both of its cycles in one \( l \)-cluster.

In the third step we proceed on \( G-S_1-S_2 \). Take a set \( C' \) of all \( l \)-clusters in \( G-S_1-S_2 \). Take a set of corresponding small cycles \( C'' \) of maximum size which consisting of disjoint cycles of size at least \( s \) in \( G-S_1-S_2-C' \). By our choice of \( S_1, S_2, C' \) it is clear that every corresponding small cycle has length at most \( l-1 \). Like a first step, add a vertex \( v_1 \) to \( G-S_1-S_2 \) and for each \( C' \subseteq C' \) an edge from \( v_1 \) to a
vertex in \( C' \). Add a vertex \( v_2 \) and for each \( C'' \in C' \) an edge from one vertex of \( C'' \) to \( v_2 \). By Menger theorem either there are \((l - 1)k\) internally vertex disjoint paths \( P \) from \( v_1 \) to \( v_2 \) or there is a hitting set of size at most \( k(l - 1) - 1 \) which hits every path from \( v_1 \) to \( v_2 \).

In the first case we can find \( k \) disjoint models of \( H \) as follows. For \( P = \{v_1, u_1, \ldots, u_n, v_2\} \in P \), we say \( u_1 \) is the start point and \( u_n \) is the end point of the path \( P \). We know that each path in \( P \) denotes a model of \( H \). Furthermore, by the first step (choice of \( S_1 \)) start point of each two paths are on two disjoint \( l \)-cycles \( c_1, c_2 \) and there is no path between \( c_1, c_2 \).

Each corresponding small cycle can route at most \( l - 1 \) paths. We give the following recursive algorithm to find a set \( H \) of at least \( k \) disjoint models of \( H \) in \( G - S_1 - S_2 \).

Take a path \( P \in P \) and let \( c \) be its endpoint corresponding small cycle. Suppose \( P_1', \ldots, P_{t'} \in P \) intersecting \( c \). We know that \( t \leq l - 1 \) as the size of \( c \) is at most \( l - 1 \). Put the corresponding model of \( H \) w.r.t. \( P \in \mathcal{H} \). Set \( \mathcal{P} := \mathcal{P} \setminus \{P_1', \ldots, P_{t'}\} \) and recurse.

In each step, the algorithm finds a model of \( H \) which is disjoint from any other model which are already in \( \mathcal{H} \), so at the end \( \mathcal{H} \) consists of disjoint models of \( H \). Furthermore in each step algorithm eliminates at most \( l - 1 \) paths from \( P \). So algorithm will run for at least \( k \) steps, that follows \( \mathcal{H} \) has at least \( k \) disjoint models of \( H \).

If there is a hitting set \( S_3 \), then for every \( v \in S \) we create a set \( S_v \) as follows. We set \( S_v := \{v\} \). If \( v \in S \cap C' \) set the \( S_v \subseteq V(G) \) of size at most \( f_1(2) \) which hits every \( l \)-cycle in strongly connected component of \( v \). If a vertex \( v \in S \cap C' \) then \( v \in C' \) for some \( C'' \in C' \) and we set \( S_v := V(C) \), in this case we have \( |S_v| \leq l - 1 \). Set \( S_3 := \bigcup_{v \in S} S_v \).

We claim \( S_3 \) hits every model of \( H \) in \( G - S_1 - S_2 \).

Suppose there is a model \( H' \) of \( H \) in \( G - S_1 - S_2 - S_3 \) consisting of cycles \( c_1, c_2 \) with a path from \( c_1 \) to \( c_2 \). In our construction, there is no path between \( v_1, v_2 \) by the choice of \( S_3 \), so either the \( c_1 \) has no incoming edge from \( v_1 \) or the \( c_2 \) has no edge to \( v_2 \).

We claim either \( c_1 \) or \( c_2 \) does not exist so there is no such \( H' \) at all. Suppose \( c_1 \) exists. We know that \( c_2 \) is not in any \( l \)-cluster of \( G - S_1 - S_2 \) (recall the choice of \( S_3 \)). So \( c_2 \) is a cycle disjoint from any \( l \)-cluster and therefore either is in \( C'' \) or intersects \( C' \). As \( c_1 \) exists, it means we did not take any vertex from its \( l \)-cluster into \( S_3 \), so there is a path from \( v_1 \) to \( c_1 \) and therefore to \( c_2 \). In order to destroy connections from \( v_1 \) to \( v_2 \) we chose a vertex \( v \in C' \) by Menger algorithm and therefore \( V(C') \cap S_3 = V(C') \), but then \( c_2 \cap S_3 \neq \emptyset \), so \( c_2 \) does not exist.

The size of \( S_3 \) is at most \((l - 1)(k - 1)(\max\{f_1(2), l\}) \). There is no model of \( H \) in \( G - S_1 - S_2 - S_3 \), we set \( S_k = S_1 \cup S_2 \cup S_3 \). The size of \( S_k \) is at most \( 2(k - 1) \cdot f_1(2) + (l - 1)(k - 1)(\max\{f_1(2), l\}) + k - 1 \) as claimed.

\[ \square \]

**Lemma 5.9.** There is a function \( f(k, w) : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) such that for every \( k, w \geq 0 \), every digraph \( G \) of directed tree-width at most \( w \) either contains \( k \) disjoint topological models of \( H \) or a set of at most \( f(k, w) \) vertices hitting every model of \( H \).

**Proof.** For the proof of the lemma we need a special form of directed tree decompositions. A directed tree-decomposition \((T, \beta, \gamma)\) is special, if

1. for all \( e = (s, t) \in E(T) \) the set \( \beta(T_e) \) is a strong component of \( G - \gamma(e) \) and
2. \( \bigcup_{t \in V(T)} \beta(t') \cap \bigcup_{e \in V(T)} \gamma(e) = \emptyset \) for every \( t \in V(T) \).

It was shown in [6] that every digraph of directed tree width \( w' \) has a special directed tree decomposition of width at most \( 5w' + 10 \).
We set \( f(0, w) = f(1, w) = 0 \) and for \( k > 1 \) we set \( f(k, w) := 5w + 10 + 2f_1(2) + f(k - 1, w) + 3|S_k| \), where \( S_k \) is as provided in the Lemma 5.8.

Let \( G \) be a digraph of directed tree-width at most \( w \) and let \((T, \beta, \gamma)\) be a directed tree-decomposition of \( G \) of width \( w \). For \( t \in T \) let \( G_t := G[\beta(T_t)] \). We prove the lemma by induction on \( k \). Clearly, for \( k = 0 \) or \( k = 1 \) there is nothing to show. So let \( k > 1 \). If \( H \not\preceq G \), then again there is nothing to show. Otherwise, let \( t \) be a node in \( T \) of minimal height such that \( H \preceq G_t \). By definition of special directed tree-decompositions, for every successor \( c \) of \( t \) the digraph \( G_c \) is strongly connected. So if \( c \) is a successor of \( t \) then \( G_c \) does not contain two disjoint cycles of length at least \( l \) as otherwise \( H \preceq G_c \) contradicting the choice of \( t \). So there is a hitting set \( S_t \) of size at most \( f_1(2) \) such that \( G_c - S_t \) has no cycle of length at least \( l \).

Let \( \sqsubseteq \) be a linearisation of the topological order of the children of \( t \). Let \( F = (c_1, \ldots, c_m) \) be the tuple of children of \( t \) satisfying the following conditions:

1. \( H \preceq F(t) := \Gamma(t) \cup \bigcup_{c \in F} G_c \).
2. Subject to 1, \( F \) is the lexicographically smallest tuple w.r.t. \( \sqsubseteq \).

It is easy to see that there are no 3 distinct nodes \( c^1, c^2, c^3 \in F \) such that there is a cycle of length at least \( l \) in \( G_{c^1} - \Gamma(t), G_{c^2} - \Gamma(t), G_{c^3} - \Gamma(t) \), as otherwise we could choose a smaller set \( F \) satisfying the conditions, contradicting the fact that \( F \) satisfies the second condition.

So suppose there are at most two nodes \( c^1, c^2 \in F \) containing a cycle of length at least \( l \) in \( G_{c^1} - \Gamma(t) \) and \( G_{c^2} - \Gamma(t) \), respectively. Let \( S(t) := \Gamma(t) \cup S_{c^1} \cup S_{c^2} \).

By construction, \( S(t) \) hits every cycle of length at least \( l \) in \( F(t) \). Hence, in \( G_0 := F(t) - S(t) \) there is no minor of \( H \) but there is a minor of \( H \) in \( F(t) \).

If \( G - F(t) \) contains \( k - 1 \) disjoint topological models of \( H \) then this implies that \( G \) has \( k \) disjoint models of \( H \) and we are done. Otherwise, by induction hypothesis, there is a set \( S \subseteq V(G - F(t)) \) of order at most \( f(k - 1, w) \) such that \( H \not\preceq G - F(t) - S \).

Note that every model of \( H \) in \( G - S - S(t) \) must contain vertices of \( G_0 \) and also vertices of \( G - S - S(t) - F(t) \). Let \( G_1 := (G - S - S(t) - F(t)) \cap G[\beta(T_t)] \) and \( G_2 := (G - S - S(t) - F(t)) - G_1 \).

In the rest of the proof, we will first construct a hitting set for every model of \( H \) in \( G_{0,1} := G_0 \cup G_1 \), then construct a hitting set of models of \( H \) which have both of their cycles in \( G_2 \) connected by a path containing vertices of \( G_1 \cup G_0 \) and finally find a hitting set of models of \( H \) which have one cycle in \( G_2 \) and the other in \( G_1 \). In any of the three cases, if we fail to find the required hitting set, we output \( k \) disjoint models of \( H \). As no other choice of any model of \( H \) remains, we are done with the proof.

By construction there is no cycle of length at least \( l \) in \( G_0 \). Also by construction there is no minor of \( H \) in each of \( G_0, G_1, G_2 \). By Lemma 5.6 there is no path \( P \) in \( G(G_0 \cup G_1 \cup G_2) \) with start and end point in \( G_1 \) such that \( P \cap G_2 \neq \emptyset \).

If \( H \preceq G_{0,1} \), then let \( C_{G_{0,1}} \) be the set of all \( l \)-clusters in \( G_{0,1} \). As \( G_0 \) does not contain any cycle of length at least \( l \), the clusters in \( C_{G_{0,1}} \) are all contained in \( G_1 \). Furthermore, no two distinct clusters can share a vertex as otherwise there would be a minor of \( H \) in \( G_1 \). Finally, in \( G_{0,1} \) there cannot be an \( l \)-transit cycle as otherwise the choice of \( F \) would not have been minimal w.r.t. \( \sqsubseteq \). For, suppose there was an \( l \)-transit cycle \( C_1 \) in \( G_{0,1} \), i.e. there are \( l \)-cycles \( C_1, C_2, C_3 \) in \( G_{0,1} \) and a path from \( C_2 \) to \( C_3 \) containing a vertex of \( C_1 \). As \( G_0 \) does not contain any \( l \)-cycle, \( C_1, C_2, C_3 \) are all in \( G_1 \).
But as $G_1$ does not contain $H$ as a topological minor, the subpath of $P$ from $C_2$ to $C_1$ and also the subpath of $P$ from $C_1$ to $C_3$ must contain a vertex of $G_0$. But this implies that $F$ does not satisfy the second condition.

So $G_{0,1}$ is a $l$-cluster bipartite graph. The following Lemma 5.8 shows that in any $l$-cluster bipartite graph either there are $k$ disjoint models of $H$ or a small set of vertices are a hitting set for all models of $H$. So in $G_{0,1}$ either we find $k$ disjoint minors of $H$ or there is a set $S_{G_{0,1}}$ that hits every minor of $H$ in $G_{0,1}$. In the first case we are done, so suppose we have the set $S_{G_{0,1}}$. Now we have to consider all $l$-clusters in $G_2$.

Claim 1. There are no 3 cycles $c_1, c_2, c_3$ of length at least $l$ in $G_2$ such that there is a path $P_1$ from $c_1$ to $c_2$ and a path $P_2$ from $c_2$ to $c_3$ in $G - S - (S(t))$.

Proof. We know that there is no minor of $H$ in $G_2$. If there are 3 cycles as stated in the claim, then both paths $P_1, P_2$ must contain a vertex of $G_0 \cup G_1$. But then there is a path between two vertices of $G[\beta(T_i)] - \Gamma(t)$ which does not go through $\Gamma(t)$ but intersects $G - G[\beta(T_i)]$. But this is impossible by Lemma 3.6.

By Claim 1 and Lemma 5.8 and the fact that all $l$-clusters in $G_2$ are vertex disjoint, either there are $k$ disjoint models of $H$ in $G - S - S(t) - S_{G_{0,1}}$ such that both of their cycles are in $G_2$ or there is a set of vertices $S_{G_2} \subseteq V(G - S - S(t))$ such that there is no model of $H$ in $G' = G - S - S(t) - S_{G_{0,1}} - S_{G_2}$ with both of its cycles in $G_2 - S_{G_{0,1}} - S_{G_2}$. In the first case we are done. So suppose we have $S_{G_2}$ as in Lemma 5.8.

Any model of $H$ in $G'$ must map one cycle of $H$ to $G_{0,1} - S_{G_{0,1}} - S_{G_2}$ and the other to $G_2 - S_{G_{0,1}} - S_{G_2}$.

Let $C$ be the set of clusters in $G - S - S(t) - S_{G_{0,1}} - S_{G_2}$. All $l$-clusters in $C$ are pairwise vertex disjoint.

There is no path between two clusters $c_1, c_2 \in G[C] \cap (G_{0,1} - S_{G_{0,1}} - S_{G_2})$ because there is no such path fully in $G_{0,1}$ and it cannot go through a vertex $v \in G_2 - S_{G_{0,1}} - S_{G_2}$ by Lemma 3.6. There is no cluster $c$ in $G_{0,1} - S_{G_{0,1}} - S_{G_2}$ such that it has a path to a cluster $c' \in G_2 - S_{G_{0,1}} - S_{G_2}$ and a path from a cluster $c'' \in G_2 - S_{G_{0,1}} - S_{G_2}$, as otherwise there is a path between $c', c''$ in $G'$. So $C$ is a bipartition and by Lemma 5.8 either there are $k$ disjoint minors of $H$ in $G'$ or there is a hitting set $S_{G'}$ in $G'$. So either there are $k$ disjoint minors of $H$ in $G$ or a set $S_H = (G - G') \cup S_{G'}$ of size at most $w + 2f_1(2) + f(k - 1, w) + 3|S_t|$ which hits every minors of $H$ in $G$. 

Proof of Theorem 5.6. Let $H$ be as in the statement of the theorem with two cycles $C_1, C_2$. Let $l, s$ be the length of $C_1, C_2$ resp. such that $l \geq s$. W.l.o.g suppose there is an edge from $C_1$ to $C_2$.

Let $g : \mathbb{N} \to \mathbb{N}$ be the function as defined in Theorem 5.10. We claim that $h : \mathbb{N} \to \mathbb{N}$ defined by $h(k) := f(k, g((k + 1) \cdot l))$ witnesses the Erdős-Pósa property of $H$. To see this, let $G$ be any digraph and let $k \geq 1$. If the directed tree-width of $G$ is at least $g((k + 1) \cdot l)$, then by Theorem 5.10 $G$ contains the cylindrical wall $W_{(k+1)\cdot l}$ of order $(k+1)\cdot l$ as topological minor, which contains $k$ disjoint copies of $H$ as topological minor. Otherwise, i.e. if the directed tree-width of $G$ is $< g((k + 1) \cdot l)$, then by Lemma 5.9
$G$ contains $k$ disjoint topological models of $H$ or a set $S$ of at most $f(k, g((k + 1) \cdot l))$ vertices such that $H \not\subseteq G - S$. □

6 Conclusion

In this paper we studied the generalised Erdős-Pósa property for directed graphs with respect to topological and butterfly minors. We provided an exact generalisation of Robertson and Seymour’s classification of undirected graphs with the Erdős-Pósa property to strongly connected digraphs. Furthermore, for the natural and much larger class of vertex-cyclic digraphs we obtained an almost exact characterisation. We also provide a novel approach to prove the Erdős-Pósa property holds in the special case of vertex cyclic graphs.

We believe that the techniques developed here will provide the tools to give a complete characterisation of the vertex-cyclic digraphs with the Erdős-Pósa property but we leave this to future research.

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