The boundary $N = 2$ supersymmetric sine-Gordon model

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Abstract

We construct an action for the $N = 2$ supersymmetric sine-Gordon model on the half-line, which we argue is both supersymmetric and integrable. The boundary interaction depends on three continuous boundary parameters, as well as the bulk mass parameter.
1 Introduction

Many important results have been obtained for two-dimensional bulk $N = 2$ superconformal field theories [1] and their integrable perturbations [2]. Some of these results have already played an important role in closed string theory. (See, e.g., [3, 4, 5], and references therein.) Soon after the pioneering work of Ghoshal and Zamolodchikov [6] on boundary integrable field theories, Warner [7] initiated a program to generalize the bulk $N = 2$ results to the boundary case, which is evidently relevant for open strings.

We consider here the $N = 2$ supersymmetric sine-Gordon (SSG) model [8] on the half-line $x \leq 0$, which is the prototypical boundary integrable model with extended supersymmetry, as is the $N = 0$ (ordinary) boundary sine-Gordon model in the absence of supersymmetry. Specifically, we construct an action which we argue is both supersymmetric and integrable. This action has two further notable features. First, the boundary interaction has three continuous boundary parameters – one more than in the cases of $N = 0$ [6] or $N = 1$ [9, 10]. We expect that this additional parameter affords the possibility of interpolating continuously between Neveu-Schwarz and Ramond boundary conditions. Secondly, the boundary interaction depends also on the bulk mass parameter $g$. Our action is exact for the bulk massless case; but in the massive case, it is only an approximation to first order in $g$. While the possibility of such bulk-boundary mixing was already anticipated in [6], it does not occur for $N = 0$, and first appears for $N = 1$.

The outline of this Letter is as follows. In Section 2, we present the action of the boundary $N = 2$ SSG model in terms of several unknown functions. We then proceed to determine these functions from the requirements of supersymmetry and integrability. In Section 3 we treat the simpler massless bulk case, and then in Section 4 we consider the more difficult massive bulk case. We briefly discuss our results in Section 5. In the Appendix we construct, using results from [8], the bulk conserved currents which we need for our analysis.

2 The action

The Euclidean-space action of the boundary $N = 2$ supersymmetric sine-Gordon model is given by

$$ S = \int_{-\infty}^{\infty} dy \int_{-\infty}^{0} dx \, \mathcal{L}_0 + \int_{-\infty}^{\infty} dy \, \mathcal{L}_b, $$

where the bulk Lagrangian density is given by [8]

$$ \mathcal{L}_0 = \frac{1}{2} \left( -\partial_\bar{z} \varphi^- \partial_z \varphi^+ - \partial_z \varphi^- \partial_\bar{z} \varphi^+ + \bar{\psi}^- \partial_\bar{z} \bar{\psi}^+ + \psi^- \partial_z \psi^+ + \bar{\psi}^+ \partial_\bar{z} \bar{\psi}^- + \psi^+ \partial_z \psi^- \right) $$

$$ + \ g \cos \varphi^+ \bar{\psi}^- \psi^- + g \cos \varphi^- \bar{\psi}^+ \psi^+ + g^2 \sin \varphi^+ \sin \varphi^-, $$

where $\varphi^\pm$ form a complex scalar field; $\psi^\pm$ and $\bar{\psi}^\pm$ are the components of a complex Dirac Fermion field; $g$ is the bulk mass parameter; and $z = \frac{1}{2}(y + ix), \bar{z} = \frac{1}{2}(y - ix)$. (The standard
dimensionless bulk coupling constant $\beta$ is absent, since it can be made to appear in an overall factor $\frac{1}{\beta}$ in front of the action by appropriate field rescalings. We consider the classical limit $\beta \to 0$.) Motivated in part by our $N = 1$ result \cite{10}, we propose the following boundary Lagrangian at $x = 0$:

$$L_b = -\frac{i}{2}(\psi^+\psi^- + \psi^+\psi^+) - \frac{1}{2}a^-\partial_ya^+ - B(\varphi^+, \varphi^-)$$

$$+ \frac{1}{2} \left( f^+(\varphi^+)a^+ + g\tilde{f}^+(\varphi^+)a^- \right) (\psi^- + \bar{\psi}^-) + \frac{1}{2} \left( f^-(\varphi^-)a^- + g\tilde{f}^-(\varphi^-)a^+ \right) (\psi^+ + \bar{\psi}^+),$$

(3)

where $a^\pm$ are Fermionic boundary degrees of freedom which anticommute with $\psi^\pm$ and $\bar{\psi}^\pm$. Such boundary degrees of freedom were introduced by Ghoshal and Zamolodchikov \cite{6} to describe the Ising model in a boundary magnetic field, and were further exploited in \cite{6, 11, 12}. Moreover, $f^\pm(\varphi^\pm)$, $\tilde{f}^\pm(\varphi^\pm)$ and $B(\varphi^+, \varphi^-)$ are potentials (functions of the scalar fields $\varphi^\pm$, but not of their derivatives), which are still to be determined. Note that, as mentioned in the Introduction, the boundary Lagrangian depends on the bulk mass parameter $g$.

Variation of the action gives the classical equations of motion. For the bulk, the equations are

$$\partial_z\partial_{\bar{z}}\varphi^\pm = -g \sin \varphi^\mp \bar{\psi}^\mp - g^2 \sin \varphi^\mp \cos \varphi^\mp,$$

$$\partial_{\bar{z}}\psi^\mp = -g \cos \varphi^\mp \psi^\pm,$$

$$\partial_z\psi^\mp = g \cos \varphi^\mp \bar{\psi}^\pm.$$  

(4)

The boundary conditions at $x = 0$ are

$$\partial_x\varphi^\pm = -\frac{\partial B}{\partial \varphi^\mp} + \frac{1}{2} \left( \partial f^\mp(\varphi^\mp)a^\mp + g\frac{\partial \tilde{f}^\mp}{\partial \varphi^\mp}a^\pm \right) (\psi^\pm + \bar{\psi}^\mp),$$

$$\psi^\pm - \bar{\psi}^\mp = if^\pm a^\pm + ig\tilde{f}^\pm a^\mp,$$

$$\partial_ya^\pm = f^\mp(\psi^\pm + \bar{\psi}^\mp) + g\tilde{f}^\pm(\psi^\mp + \bar{\psi}^\mp).$$  

(5)

Following a similar strategy as in \cite{6, 11}, we now proceed to determine the functions $f^\pm(\varphi^\pm)$, $\tilde{f}^\pm(\varphi^\pm)$ and $B(\varphi^+, \varphi^-)$ which appear in the boundary action by demanding both $N = 2$ supersymmetry and integrability.

That the bulk $N = 2$ SSG model is integrable implies that it has an infinite number of classical integrals of motion constructed from the densities $T_{s+1}$, $\bar{T}_{s+1}$, $\Theta_{s-1}$, $\bar{\Theta}_{s-1}$, which obey

$$\partial_zT_{s+1} = \partial_{\bar{z}}\Theta_{s-1}, \quad \partial_{\bar{z}}\bar{T}_{s+1} = \partial_z\bar{\Theta}_{s-1}.$$  

(6)

The densities for $s = \frac{1}{2}, 1, 3$ are given in the Appendix A.

As observed in \cite{6}, it follows from the continuity Eqs. (3) that the boundary model has the integral of motion

$$P_s = \int_{-\infty}^{0} dx \ (T_{s+1} + \bar{T}_{s+1} - \Theta_{s-1} - \bar{\Theta}_{s-1}) + i\Sigma_s(y),$$  

(7)
provided that the following condition holds at \( x = 0 \)

\[
T_{s+1} - \mathbf{T}_{s+1} + \Theta_{s-1} - \overline{\Theta}_{s-1} = \partial_y \Sigma_s(y),
\]  

(8)

where \( \Sigma_s(y) \) is a local boundary term. Hence, our task reduces to investigating the constraints (8) for \( s = \frac{1}{2} \) (to have supersymmetry) and \( s = 3 \) (to have integrability). Since this remains a formidable undertaking, it is convenient to treat separately the bulk massless and bulk massive cases.

### 3 Bulks massless case

We begin by considering the simpler case of no bulk mass, \( g = 0 \). The supersymmetry constraints are

\[
T^\pm_3 - \mathbf{T}^\pm_3 = \partial_x \varphi^\pm \psi^\mp - \partial_x \varphi^\pm \bar{\psi}^\pm = -i \partial_x \varphi^\pm (\psi^\pm + \bar{\psi}^\pm) + \partial_y \varphi^\pm (\psi^\pm - \bar{\psi}^\pm) = \partial_y \Sigma^\pm_3.
\]

(9)

Let us assume that the boundary terms have the form

\[
\Sigma^\pm_3 = i g^\pm (\varphi^\pm) a^\pm.
\]

(10)

We find, with the help of the boundary conditions (5), that the constraints (9) are satisfied provided that

\[
f^\pm(\varphi^\pm) = \frac{\partial g^\pm (\varphi^\pm)}{\partial \varphi^\pm}, \quad g^\pm(\varphi^\pm) f^\mp(\varphi^\mp) = \frac{\partial B}{\partial \varphi^\mp}. \tag{11}
\]

These equations imply the relation

\[
f^+(\varphi^+) f^-(\varphi^-) = \frac{\partial^2 B}{\partial \varphi^+ \partial \varphi^-}. \tag{12}
\]

We next turn to the integrability constraint

\[
T_4 - \mathbf{T}_4 = \partial_y \Sigma_3.
\]

(13)

Eliminating \( a^\pm \) from the boundary conditions (5), we obtain

\[
\partial_x \varphi^\pm = -\frac{\partial B}{\partial \varphi^\mp} - \frac{i}{2} \frac{\partial \ln f^\mp}{\partial \varphi^\mp} (\psi^\pm - \bar{\psi}^\pm)(\psi^\pm + \bar{\psi}^\pm), \]

\[
\partial_y \psi^\pm - \partial_y \bar{\psi}^\pm = \frac{\partial \ln f^\pm}{\partial \varphi^\mp} \partial_y \varphi^\pm (\psi^\pm - \bar{\psi}^\pm) + if^+ f^-(\psi^\pm + \bar{\psi}^\pm). \tag{14}
\]

We substitute these boundary conditions, as well as the bulk equations of motion (4) with \( g = 0 \), into the LHS of the constraint (13). An examination of the pure Bosonic terms reveals that this constraint can be satisfied only if the potential \( B \) obeys

\[
B = -4 \frac{\partial^2 B}{\partial \varphi^2} = -4 \frac{\partial^2 B}{\partial \varphi^{+2}}. \tag{15}
\]
We conclude that $B$ is given by
\begin{equation}
B = \alpha \cos\left(\frac{1}{2} (\varphi^+ - \varphi^+_0)\right) \cos\left(\frac{1}{2} (\varphi^- - \varphi^-_0)\right),
\end{equation}
(16)
where $\alpha$, $\varphi^\pm_0$ represent three independent real parameters. It follows from the relation (12) that $f^\pm(\varphi^\pm)$ are given by
\begin{equation}
f^\pm(\varphi^\pm) = \sqrt{\frac{\alpha}{2}} \sin\left(\frac{1}{2} (\varphi^± - \varphi^±_0)\right),
\end{equation}
(17)
Remarkably, with these choices of $B$ and $f^\pm$, the remaining terms (i.e., those which are not pure Bosonic) on the LHS of (13) can also be expressed as a total derivative. The computation, which is rather arduous, follows the general lines of the simpler $N = 1$ case [10]. However, there is a final “twist”, requiring use of the amusing identity
\begin{equation}
\partial_y \left( \psi^+ \psi^- + \bar{\psi}^+ \bar{\psi}^- - \psi^\dagger \psi^- - \psi^+ \bar{\psi}^- \right) = (A + B) \psi^+ \psi^- + (-A + B) \bar{\psi}^+ \bar{\psi}^- - B \bar{\psi}^+ \psi^- - B \psi^+ \bar{\psi}^-,
\end{equation}
(18)
where
\begin{equation}
A = 2if^+ f^-, \quad B = \frac{\partial \ln f^+}{\partial \psi^+} \partial_y \psi^+ + \frac{\partial \ln f^-}{\partial \phi^-} \partial_y \phi^-.
\end{equation}
(19)

4 Bulk massive case

We now consider the general case $g \neq 0$. The supersymmetry constraints now read
\begin{equation}
T^∓_± - \bar{T}^∓_± + Θ^∓_± - \bar{Θ}^∓_± = -i\partial_x \varphi^±(\psi^± + \bar{\psi}^±) + \partial_y \varphi^±(\psi^± - \bar{\psi}^±) + g \sin \varphi^±(\psi^± + \bar{\psi}^±) = \partial_y \Sigma^±_{±}.
\end{equation}
(20)
Given the potential $B$ (16), the only way that we have found to satisfy this constraint is to introduce the $g$-dependent interactions given in the boundary conditions (5). Indeed, this is how we first arrived at those $g$-dependent corrections. Making the Ansatz
\begin{equation}
\Sigma^±_{±} = ig^±(\varphi^±) a^± + ig \tilde{g}^±(\varphi^±) a^±,
\end{equation}
(21)
we derive the relations
\begin{equation}
f^±(\varphi^±) = \frac{\partial g^±(\varphi^±)}{\partial \varphi^±}, \quad \tilde{f}^±(\varphi^±) = \frac{\partial \tilde{g}^±(\varphi^±)}{\partial \varphi^±},
\end{equation}
(22)
\footnote{Since we regard the fields $\varphi^±$ as a complex-conjugate pair, then so are $\varphi^±_0$. That is, $\varphi^±_0$ count as one complex, or two real, parameters.}
\footnote{As discussed below, the potential $B$ does not change when a bulk mass is turned on.}
and
\[ \sin \varphi^\pm = i \left( g^\pm \tilde{f}^\pm + \tilde{g}^\pm f^\pm \right), \quad g^\pm f^\mp + g^2 \tilde{g}^\pm \tilde{f}^\mp = \frac{\partial B}{\partial \varphi^\pm}. \] (23)

It follows that
\[ g^\pm \tilde{g}^\pm = i \cos \varphi^\pm + C^\pm, \] (24)
\[ f^+ f^- + g^2 \tilde{f}^+ \tilde{f}^- = \frac{\partial^2 B}{\partial \varphi^+ \partial \varphi^-}, \] (25)

where \( C^\pm \) are arbitrary integration constants.

The relations (22), (24) and (25) are sufficient to ensure \( N = 2 \) supersymmetry. Further restrictions come from implementing the integrability constraint
\[ T_4 - \overline{T}_4 + \Theta_2 - \overline{\Theta}_2 = \partial_y \Sigma_3(y), \] (26)

Due to the difficulty of this computation, we now restrict our analysis to first order in \( g \). Elimination of \( a^\pm \) from the boundary conditions (3) now gives (dropping terms of order \( g^2 \) and higher)
\[ \partial_x \varphi^\pm = \text{massless} + i g M^\pm \tilde{\psi}^\pm \tilde{\bar{\psi}}^\pm, \]
\[ \partial_y \psi^\pm - \partial_y \bar{\psi}^\pm = \text{massless} - g M^\pm \partial_y \varphi^\pm (\psi^\mp - \bar{\psi}^\pm) + 2ig f^\pm \tilde{f}^\pm (\psi^\mp + \bar{\psi}^\pm), \] (27)

where
\[ M^\pm = \frac{\tilde{f}^\pm}{f^\pm} \left( \frac{\partial \ln f^\pm}{\partial \varphi^\pm} - \frac{1}{f^\pm} \frac{\partial \tilde{f}^\pm}{\partial \varphi^\pm} \right), \] (28)

and “massless” represents the corresponding \( g = 0 \) result (14). Examination of the pure Bosonic terms in (26) leads to the same result (14) for the potential \( B \). To first order in \( g \), the relation (25) reduces to the corresponding massless one (12); hence, we are lead to the same result (17) for \( f^\pm(\varphi^\pm) \). This determines \( g^\pm(\varphi^\pm) \), up to an integration constant. We can then use (24) to determine \( \tilde{g}^\pm(\varphi^\pm) \). The choice \( C^\pm = i \cos \varphi_0^\pm \) gives the simple result
\[ \tilde{f}^\pm(\varphi^\pm) = \frac{i}{\sqrt{\alpha}} \sin \left( \frac{1}{2} (\varphi^\pm + \varphi_0^\pm) \right). \] (29)

These choices for \( f^\pm \) and \( \tilde{f}^\pm \) satisfy the entire set of relations (22), (24) and (25) to first order in \( g \), thereby ensuring supersymmetry to that order. One must now consider the contributions of order \( g \) to the LHS of the constraint (26) that are not pure Bosonic. Remarkably, after another long computation, we find that all such contributions can be expressed as a total \( y \) derivative. We conclude that, to first order in \( g \), both \( P_{1/2}^3 \) and \( P_3 \) are integrals of motion.
5 Discussion

We have demonstrated that the boundary $N = 2$ SSG action (2), (3), with $B$, $f^\pm$ and $\tilde{f}^\pm$ given by Eqs. (16), (17) and (29), respectively, has the integrals of motion $P_1^\pm$ and $P_3$ (7), to first order in the bulk mass parameter $g$. The conservation of $P_1^\pm$ means that the model has on-shell $N = 2$ supersymmetry. The conservation of $P_3$ provides strong evidence that the model is integrable. We therefore conjecture that the quantized model (with appropriate boundary corrections that are higher order in $g$) has $N = 2$ supersymmetry and is integrable.

Our action bears some similarity to the so-called Landau-Ginzburg model discussed in Section 7 of [7] with bulk superpotential $W = \cos \varphi$. However, there is an essential difference: the boundary (Bosonic) potential $|W|^2$ proposed in [7], which differs from ours (16), is not integrable. In [7], the boundary potential is selected solely on the basis of supersymmetry. While that choice does achieve supersymmetry, it is not the unique such choice. Indeed, our action also maintains supersymmetry, albeit in a more intricate manner through $g$-dependent boundary interactions, which evidently is the price to be paid for integrability. As a bonus, our boundary action contains more boundary parameters (a total of three, in addition to the bulk mass parameter) than the boundary action considered previously [7].

It remains a challenge to work out corrections to the boundary action that are higher order in $g$. A naive guess for the case $\varphi_0^\pm = 0$ is

$$f^\pm(\varphi^\pm) = \gamma \sin \frac{\varphi^\pm}{2}, \quad \tilde{f}^\pm(\varphi^\pm) = \frac{i}{2\gamma} \sin \frac{\varphi^\pm}{2}, \quad \gamma = \sqrt{\frac{\alpha}{8} + \sqrt{\frac{\alpha^2}{64} + \frac{g^2}{4}}}.$$  (30)

Indeed, these expressions have the correct $g \to 0$ limit, and satisfy the relations (22), (24) and (25) which ensure supersymmetry. However, we have not attempted to check the conservation of $P_3$ with this choice.

It would clearly be advantageous to have superfield formulations of the boundary $N = 1, 2$ SSG models. Presumably, this would entail making more precise the notion of the “boundary of a superspace”. Some progress in this direction has already been made in [13, 14].

We intend to present exact boundary scattering matrices and thermodynamics of the model in forthcoming publications.

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A Bulk conserved currents

We explicitly construct here the first few bulk classically-conserved currents for the $N = 2$ supersymmetric sine-Gordon model. For this task, a superfield approach is essential. (For an
exposition of $N = 2$ superfield formalism in two dimensions, see Chapter 23 of [3].) Following Kobayashi and Uematsu [8], we introduce the covariant derivatives

$$D_\pm = \frac{\partial}{\partial \theta^\pm} + \frac{1}{2} \theta^\mp \partial_z, \quad \overline{D}_\pm = \frac{\partial}{\partial \bar{\theta}^\pm} + \frac{1}{2} \bar{\theta}^\mp \partial_{\bar{z}},$$

which obey

$$\{D_+, D_-\} = \partial_z, \quad \{\overline{D}_+, \overline{D}_-\} = \partial_{\bar{z}},$$

$$\{D_+, \overline{D}_\pm\} = 0 = \{D_-, \overline{D}_\pm\}, \quad D^2_\pm = 0 = \overline{D}^2_\pm,$$

$$[D_\pm, \partial_z] = 0 = [\overline{D}_\pm, \partial_{\bar{z}}],$$

and the chiral superfields $\phi^\pm$ obeying

$$D_\mp \phi^\pm = \overline{D}_\mp \phi^\pm = 0.$$  \hspace{1cm} (33)

The components of these superfields are given by

$$\varphi^\pm = \phi^\pm \bigg|_{\theta^\pm=\bar{\theta}^\pm=0}, \quad F^\pm = \overline{D}_\pm D_\pm \phi^\pm,$$

$$\psi^\pm = D_\mp \phi^\mp, \quad \bar{\psi}^\pm = \overline{D}_\mp \phi^\mp,$$

where $\bigg|_{\theta^\pm=\bar{\theta}^\pm=0}$ is the standard shorthand for the projection $\varphi^\pm = \phi^\pm \bigg|_{\theta^\pm=\bar{\theta}^\pm=0}$. The bulk action is given by

$$S = \int d^2z d^4\theta \phi^+ \phi^- + g \int d^2z d^2\theta^+ \cos \phi^+ + g \int d^2z d^2\theta^- \cos \phi^-, \hspace{1cm} (35)$$

from which follow the bulk equations of motion

$$\overline{D}_\pm D^\pm \phi = g \sin \phi^\mp.$$  \hspace{1cm} (36)

The corresponding component expressions for the action (up to total derivatives) and equations of motion (after eliminating the auxiliary fields $F^\pm$ by means of their field equations) are given by Eqs. (2) and (4), respectively.

We are now in position to work out the classically-conserved currents. With the help of the equations of motion, one can show that the quantity $X = D_\mp \phi^+ D_- \phi^-$ obeys

$$\overline{D}_\pm X = D_\mp A^\pm,$$  \hspace{1cm} (37)

with $A^\pm = \mp g \cos \phi^\mp$. Acting on both sides of the first of these equations with $D_\mp \overline{D}_-$, we obtain

$$\partial_{\bar{z}} (D_\mp X) = -\partial_z (\overline{D}_\mp A^\mp).$$  \hspace{1cm} (38)

We therefore identify the supercurrents

$$\pm T^\pm_\mp = D_\mp X = \pm \partial_z \varphi^\mp \psi^\pm, \quad \mp \Theta^\pm_\mp = \overline{D}_\mp A^\mp = \mp g \sin \varphi^\mp \bar{\psi}^\mp.$$  \hspace{1cm} (39)
Moreover, acting on both sides of the first equation in (34) with $D_- D_+ \overline{D}_-$, we obtain
\[
\partial_z (D_- D_+ X) = \partial_z (\overline{D}_- D_- A^+) ,
\]
and we therefore identify the energy-momentum tensor
\[
-T_2 = D_- D_+ X = -\partial_z \varphi^+ \partial_z \varphi^- + \psi^- \partial_z \psi^+ ,
\]
\[
-\Theta_0 = \overline{D}_- D_- A^+ = g \cos \varphi^- \bar{\psi}^+ \psi^+ + g^2 \sin \varphi^- \sin \varphi^+ .
\]

Similarly, the quantity $X = \overline{D}_+ \phi^+ D_- \phi^-$ obeys
\[
D_{\pm} X = \overline{D}_\mp \overline{A}^\pm ,
\]
with $\overline{A}^\pm = -A^\pm$. It follows that
\[
\partial_z (\overline{D}_\mp X) = \partial_z (D_{\pm} \overline{A}^\mp) .
\]

We therefore identify
\[
\pm \overline{T}_2^\pm = \overline{D}_\mp \overline{X} = \pm \partial_z \varphi^+ \bar{\psi}^+ , \quad \pm \overline{\Theta}_0^\pm = D_{\pm} \overline{A}^\mp = \mp g \sin \varphi^+ \psi^+ .
\]

Moreover,
\[
\partial_z (\overline{D}_- D_+ X) = \partial_z (\overline{D}_- D_- A^+) ,
\]
and we therefore identify
\[
-T_2 = \overline{D}_- D_+ X = -\partial_z \varphi^+ \partial_z \varphi^- + \bar{\psi}^- \partial_z \psi^+ , \quad \overline{\Theta}_0 = \Theta_0 .
\]

Consider now the quantity $X$
\[
X = D_+ \phi^+ D_- \phi^- \left[ (\partial_z \phi^+)^2 + (\partial_z \phi^-)^2 \right] - 2 \partial_z D_+ \phi^+ \partial_z D_- \phi^- .
\]
It obeys the conservation equations (34) with
\[
A^\pm = \mp g \cos \phi^+ \left[ 3 (\partial_z \phi^+)^2 + (\partial_z \phi^-)^2 \right] - 2 g \sin \phi^+ D_+ \phi^+ D_- \phi^- \partial_z \phi^\pm .
\]

It follows that
\[
T_4 = D_- D_+ X = - (\partial_z \varphi^+)^2 \partial_z \varphi^- - (\partial_z \varphi^-)^3 \partial_z \varphi^+ + \psi^- \partial_z \psi^+ \left[ (\partial_z \varphi^+)^2 + 3 (\partial_z \varphi^-)^2 \right] + 2 \psi^+ \partial_z \psi^- (\partial_z \varphi^+)^2 + 2 \psi^- \partial_z \varphi^+ \partial_z^2 \varphi^+ + 2 \partial_z^2 \varphi^+ \partial_z^2 \varphi^- - 2 \partial_z \psi^- \partial_z^2 \psi^+ ,
\]
\[
\Theta_2 = \overline{D}_- D_- A^+ = g^2 \sin \varphi^- \sin \varphi^+ \left[ (\partial_z \varphi^+)^2 + (\partial_z \varphi^-)^2 \right] - 2 g^2 \cos \varphi^- \cos \varphi^+ \partial_z \varphi^- \partial_z \varphi^+ + g \cos \varphi^- \bar{\psi}^+ \psi^+ \left[ (\partial_z \varphi^+)^2 + (\partial_z \varphi^-)^2 \right] + 2 g \sin \varphi^- \partial_z \varphi^- \left( -\psi^+ \partial_z \bar{\psi}^+ + \bar{\psi}^+ \partial_z \psi^+ \right) - 2 g \cos \varphi^- \partial_z \bar{\psi}^+ \partial_z \psi^+ .
\]
Similarly, using

$$\mathcal{X} = D_+ \phi^+ D_- \phi^- \left[ (\partial_\xi \phi^+)^2 + (\partial_\xi \phi^-)^2 \right] - 2 \partial_\xi D_+ \phi^+ \partial_\xi D_- \phi^-,$$

we obtain

$$\mathcal{T}_4 = \mathcal{D}_- D_+ \mathcal{X} = -(\partial_\xi \varphi^+)^3 \partial_\xi \varphi^- -(\partial_\xi \varphi^-)^3 \partial_\xi \varphi^+$$

$$+ \bar{\psi}^- \partial_\xi \bar{\psi}^+ \left[ (\partial_\xi \varphi^+)^2 + 3(\partial_\xi \varphi^-)^2 \right] + 2 \bar{\psi}^+ \partial_\xi \bar{\psi}^-(\partial_\xi \varphi^+)^2$$

$$+ 2 \bar{\psi}^- \bar{\psi}^+ \partial_\xi \varphi^+ \partial_\xi \varphi^- + 2 \partial_\xi \varphi^+ \partial_\xi \varphi^- - 2 \partial_\xi \bar{\psi}^+ \partial_\xi \bar{\psi}^-,$$

$$\mathcal{O}_2 = - \mathcal{D}_- D_+ \mathcal{A}^+ = g^2 \sin \varphi^- \sin \varphi^+ \left[ (\partial_\xi \varphi^+)^2 + (\partial_\xi \varphi^-)^2 \right]$$

$$- 2 g^2 \cos \varphi^- \cos \varphi^+ \partial_\xi \varphi^- \partial_\xi \varphi^+ + g \cos \varphi^- \bar{\psi}^+ \psi^+ \left[ (\partial_\xi \varphi^+)^2 + (\partial_\xi \varphi^-)^2 \right]$$

$$+ 2 g \sin \varphi^- \partial_\xi \varphi^- (- \psi^+ \partial_\xi \bar{\psi}^+ + \bar{\psi}^+ \partial_\xi \psi^+) - 2 g \cos \varphi^- \partial_\xi \bar{\psi}^+ \partial_\xi \psi^+. \quad (51)$$

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