TOWERS OF $MU$-ALGEBRAS AND THE GENERALIZED HOPKINS-MILLER THEOREM

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Abstract. Our results are of three types. First we describe a general procedure of adjoining polynomial variables to $A_{\infty}$-ring spectra whose coefficient rings satisfy certain restrictions. A host of examples of such spectra is provided by killing a regular ideal in the coefficient ring of $MU$, the complex cobordism spectrum. Second, we show that the algebraic procedure of adjoining roots of unity carries over in the topological context for such spectra. Third, we use the developed technology to compute the homotopy types of spaces of strictly multiplicative maps between suitable $K(n)$-localizations of such spectra. This generalizes the famous Hopkins-Miller theorem and gives strengthened versions of various splitting theorems.

Key words: $S$-algebras, topological derivations, Morava $K$-theories, Witt vectors.

1. Introduction

Our goals in this paper are three-fold. First, we generalize and extend the methods of of constructing $A_{\infty}$ ring spectra by adjoining ‘polynomial variables’. More precisely, we work with the categories of strict rings and modules constructed and adopt the terminology of the cited reference. In particular we consider $S$-algebras and commutative $S$-algebras rather than the equivalent notions of $A_{\infty}$ and $E_{\infty}$ ring spectra. So, suppose that $R$ is a commutative $S$-algebra such that its coefficient ring $R_*$ has no elements of odd degree and the graded ideal $I_*$ in $R_*$ is generated by a regular sequence $u_1, u_2, \ldots$. Then it is known from work of Strickland, that the $R$-module $R/I$ obtained by killing the ideal $I$ has a structure of an $R$-ring spectrum. We assume that $R/I$ is actually an $R$-algebra. Then it turns out that, informally speaking, all $R$-modules ‘between’ $R/I$ and $R$ also possess structures of $R$-algebras. More precisely, denote by $R/I(u_k^l)$ the $R$-module obtained from $R$ by killing the sequence $(u_1, u_2, \ldots, u_{k-1}, u_k^l, u_{k+1}, \ldots)$ in $R_*$. Then we show that there are no obstructions for the existence of strictly associative products on $R(u_k^l)$. The natural ‘reduction maps’ between these $R$-algebras also admit strictly multiplicative liftings.

The strictly multiplicative products thus constructed are typically not unique. In fact (unless there are some additional assumptions such as the sparseness of the coefficient ring) they are not

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unique even up to homotopy as shown by Strickland. There is an infinite tower of obstructions to uniqueness in which the first term is precisely Strickland’s obstruction.

The basic example for our theory is provided by taking $R = MU$, the complex cobordism $S$-algebra and $MU/I$ be the Eilenberg-MacLane $S$-algebra $H\mathbb{Z}/p$. It follows by induction that all $MU$-algebras obtained by killing any sequence of polynomial generators and/or a prime $p$ have structures of $MU$-modules.

Second, we consider the question of adjoining roots of unity to an $S$-algebra. This problem was also treated in the recent work [14] of Schwanzl, Vogt and Waldhausen. Their definition of a topological extension has better formal properties than ours but it applies in a far less general situation. We show that one can adjoin roots of unity to such spectra as Morava $K$-theories $K(n)$, Johnson-Wilson theories $E(n)$ and many other algebras over the complex cobordism $S$-algebra $MU$.

Third, we address the problem of computing $S$-algebra maps between a certain completion $\hat{E}(n)$ of $E(n)$ and an $MU$-algebra $E$ which is assumed to be ‘strongly $K(n)$-complete’ in some precise sense explained later on in the paper. Examples of such $MU$-algebras include $\hat{E}(n)$, $K(n)$ and the Artinian completion of the $v_n$-localization of the Brown-Peterson spectrum $BP$. It turns out that the space of $S$-algebra maps between $\hat{E}(n)$ and a strongly $K(n)$-complete $MU$-algebra is homotopically discrete with the set of connected components being equal to the set of multiplicative cohomology operations $\hat{E}(n) \rightarrow E$. We call the results of this type the generalized Hopkins-Miller theorem because the original Hopkins-Miller theorem (cf.[12]) asserts that the space of $S$-algebra self-maps of the spectrum $E_n$ is homotopically equivalent to the (discrete) space of multiplicative operations from $E_n$ to itself. Here $E_n$ is a 2-periodic version of the completed Johnson-Wilson theory $\hat{E}(n)$ which came to be popularly known as the Morava $E$-theory.

We chose to work with the $2(p^n - 1)$-periodic theory $\hat{E}(n)$ rather than with $E_n$. The relation of $\hat{E}(n)$ to $E_n$ is the same as the relation of the Adams summand of $p$-completed complex $K$-theory spectrum $\hat{KU}$ to $\hat{KU}$ itself. The advantage of $\hat{E}(n)$ is that it is smaller than $E_n$, however it does not admit the action of the full Morava stabilizer group.

One consequence of our generalized Hopkins-Miller theorem is that $\hat{E}(n)$ admits a unique $S$-algebra structure, the result previously obtained in [4].

Another consequence is that $\hat{E}(n)$ splits off $E$ as an $S$-algebra for a certain class of strongly $K(n)$-complete $MU$-algebras $E$. Such splittings were previously known to be multiplicative only up to homotopy.

Throughout the paper the symbol $\mathbb{F}_p$ will denote the prime field with $p$ elements, $L/\mathbb{F}_p$ an arbitrary (but fixed) finite separable extension of $\mathbb{F}_p$. Further $W(L)$ and $W_n(L)$ denote the ring of
Witt vectors and the ring of Witt vectors of length $n$ respectively. For an $R$-algebra $A$ we denote by $A^e$ its enveloping $R$-algebra $A \wedge_R A^{op}$ where $A^{op}$ is the algebra $A$ with opposite multiplication.

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2. Adjoining polynomial variables to $S$-algebras

In this section we assume that $R$ is a fixed commutative $S$-algebra such that $R_* = \pi_* R$ is a graded commutative ring concentrated in even degrees. We will also assume without loss of generality that $R$ is $q$-cofibrant in the sense of [6]. All objects under consideration will be $R$-modules or $R$-algebras and smash products and homotopy classes of maps will be understood to be taken in the category of $R$-modules.

We begin by reminding the reader the notions of topological derivations and topological singular extensions of $R$-algebras. A detailed account can be found in [8]. Let $A$ be an $R$-algebra and $M$ an $A$-bimodule. Then the $R$-module $A \vee M$ has the obvious structure of an $R$-algebra (‘square-zero extension’ of $A$). Consider the set $[A, A \vee M]_{R-\text{alg}/A}$ of homotopy classes of $R$-algebra maps from $A$ to $A \vee M$ which commute with the projection onto $A$. Then there exists an $A$-bimodule $\Omega_A$ and a natural in $M$ isomorphism

$$[A, A \vee M]_{R-\text{alg}/A} \cong [\Omega_A, M]_{A-\text{bimod}}$$

where the right hand side denotes the homotopy classes of maps in the category of $A$-bimodules.

Definition 2.1. The topological derivations $R$-module of $A$ with values in $M$ is the function $R$-module $F_{A \wedge A^{op}}(\Omega_A, M)$. We denote it by $\text{Der}(A, M)$ and its $i$th homotopy group by $\text{Der}^{-i}(A, M)$.

The $A$-bimodule $\Omega_A$ is constructed as the homotopy fibre of the multiplication map $A \wedge A \to A$. There exists the following homotopy fibre sequence of $R$-modules:

$$\text{THH}(A, M) \to M \to \text{Der}(A, M)$$

(2.1)

Here $\text{THH}(A, M)$ is the topological Hochschild cohomology spectrum of $A$ with values in $M$, $\text{THH}(A, M) := F_{A \wedge A^{op}}(A, M)$.

We will frequently use the notion of a primitive operation from $A$ to $M$. Denote by $m : A \wedge A \to A$ the multiplication map and by $m_l : A \wedge M \to M$ and $m_r : M \wedge A \to M$ the left and right actions of $A$ in $M$ respectively. Then a map $p : A \to M$ is called primitive if $p$ is a ‘derivation up to
homotopy’, i.e. the following diagram is homotopy commutative:

\[
\begin{array}{ccc}
A \wedge A & \xrightarrow{m} & A \\
\downarrow & & \downarrow \\
A \wedge M \vee M & \xrightarrow{m \vee m_r} & M
\end{array}
\]

There is a forgetful map \( l : \text{Der}^*(A, M) \to [A, M]^* \) defined as follows. For any topological derivation \( d : A \to A \vee M \) let \( l(d) \) be the composite map \( A \to A \vee M \to M \) where the last map is just the projection onto the wedge summand. Then it is easy to see that the image of \( l \) is contained in the set of primitive operations from \( A \) to \( M \).

Suppose we are given a topological derivation \( \xi : A \to A \vee M \). Consider the following homotopy pullback diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\xi} & A \\
\downarrow & & \downarrow \\
A & \xrightarrow{\xi} & A \vee M
\end{array}
\]

Here the rightmost downward arrow is the canonical inclusion of a retract. Then we have the following homotopy fibre sequence of \( R \)-modules:

\[(2.2) \quad \Sigma^{-1} M \to X \to A \]

**Definition 2.2.** The homotopy fibre sequence \((2.2)\) is called the topological singular extension associated with the topological derivation \( \xi : A \to A \vee M \).

For an element \( x \in R_* \) we will denote by \( R/x \) the cofibre of the map \( R \xrightarrow{x} R \). Let \( I_* \) be a graded ideal generated by (possibly infinite) regular sequence of homogeneous elements \((u_1, u_2, \ldots) \in R_* \). We assume in addition that each \( u_k \) is a nonzero divisor in the ring \( R_* \). (This assumption is satisfied in all cases of interest). Then we can form the \( R \)-module \( R/I \) as the infinite smash product of \( R/u_k \). By [15], Proposition 4.8 there is a structure of an \( R \)-ring spectrum on \( A \). Clearly the coefficient ring of \( R/I \) is isomorphic to \( R_*/I_* \) where \( R_*/I_* \) is understood to be the direct limit of \( R_*/(u_k, u_2, \ldots, u_k) \).

Let us denote the \( R \)-algebra \( R/I \) by \( A \). Our standing assumption is that \( A \) has a structure of an \( R \)-algebra (i.e. strictly associative). This may seem a rather strong condition but, as we see shortly such a situation is quite typical. In fact P.Goerss proved in [8] that any spectrum obtained by killing a regular sequence in \( MU \), the complex cobordism \( S \)-algebra, has a structure of an \( MU \)-algebra.

The construction we are about to describe allows one to construct new \( R \)-algebras by ‘adjoining’ the indeterminates \( u_k \) to the \( R \)-algebra \( A \). The basic idea is the same as in [8] where Morava
$K$-theories at an odd prime were shown to possess $MU$-algebra structures. However the arguments we use here are considerably more general, in particular we make no assumption that the prime 2 is invertible. Let us introduce the notation $A(u^l_k)_*$ for the $R_*$-algebra

$$\lim_{n \to \infty} R_*/(u_1, u_2, \ldots, u_{k-1}, u_k^l, u_{k+1}, \ldots).$$

For each $l$ reduction modulo $u_k^l$ determines a map of $R_*$-algebras

$$A(u^l_{k+1})_* \to A(u^l_k)_*.$$ 

Now we can formulate our main theorem in this section.

**Theorem 2.3.** For each $l$ there exist $R$-algebras $A(u^l_k)$ with coefficient rings $A(u^l_k)_*$ and $R$-algebra maps

$$R_{l,k} : A(u^l_{k+1}) \to A(u^l_k)$$

which give the reductions mod $u_k^l$ on the level of coefficient rings.

**Proof.** Set $d := |u_k|$. Suppose by induction that the $R$-algebras $A(u^l_k)$ with the required properties were constructed for $l \leq i$. We will show that there exists an appropriate Bokstein operation from $A(u^i_k)$ to $\Sigma^{dl+1}A$ which allow us to build the next stage.

Consider the cofibre sequence

$$\Sigma^{dl} R \xrightarrow{u^i_k} R \xrightarrow{\rho_{l,k}} R/ u^l_k \xrightarrow{\beta_{l,k}} \Sigma^{dl+1} R.$$ 

According to [15] the $R$-module $R/u^l_k$ admits an associative product $\varphi : R/ u^l_k \wedge R/ u^l_k \to R/ u^l_k$ and for any other product $\varphi'$ there exists a unique element $u \in \pi_{2dl+2}R/ u^l_k$ for which $\varphi' = \varphi + u \circ (\beta_{l,k} \wedge \beta_{l,k})$.

**Lemma 2.4.** There exists a map of $R$-ring spectra $r_{i,k} : R/ u^{i+1}_k \to R/ u^i_k$ realizing the reduction map on coefficient rings. Moreover in the cofibre sequence

$$(2.3) \quad R/ u^{i+1}_k \xrightarrow{r_{i,k}} R/ u^i_k \xrightarrow{\beta_{i,k}} \Sigma^{di+1} R/ u_k$$

the second map $\beta_{i,k} : R/ u^i_k \to \Sigma^{di+1} R/ u_k$ is a primitive operation.
Proof. Use induction on $i$ (in fact we only need the inductive assumption in order that $R/u_k$ be an algebra spectrum over $R/u_k^i$.) Consider the following diagram of $R$-modules:

\[
\begin{array}{ccccccc}
\Sigma^d i+1 R & \xrightarrow{u_k} & \Sigma^d R & \xrightarrow{u_k} & \Sigma^d i R/u_k \\
\downarrow & & \downarrow & & \downarrow \\
R & \xrightarrow{i+1} & R & \xrightarrow{i} & R & \xrightarrow{pt} & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
R/u_k^{i+1} & \xrightarrow{r_{i,k}} & R/u_k^i & \xrightarrow{\rho_{i,k}} & \Sigma^d i R/u_k & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^d R & \xrightarrow{u_k} & \Sigma^d i R/u_k \\
\end{array}
\]

Here the rows and columns are homotopy cofibre sequences of $R$-modules. The map $r_{i,k} : R/u_k^{i+1} \to R/u_k^i$ is determined uniquely by the requirement that the diagram commute in the homotopy category of $R$-modules.

We now show that it is possible to choose a product on $R/u_k^{i+1}$ so that $r_{i,k}$ becomes an $R$-ring spectrum map. First take any associative product $\varphi : R/u_k^{i+1} \wedge R/u_k^{i+1} \to R/u_k^{i+1}$ which exists by [15], Proposition 3.1. As in [15], Proposition 3.15 there is an obstruction $d(\varphi) \in \pi_{2d(i+1)+2}(R/u_k^i)$ for the map $r_{i,k}$ to be homotopy multiplicative. If the product $\varphi$ is changed the obstruction changes according to the formula

\[
d(\varphi + u \circ (\beta_{i,k} \wedge \beta_{i,k})) = d(\varphi) + r_{i,k} \ast u.
\]

Since the map $r_{i,k} : R_*/u_k^{i+1} \to R_*/u_k^i$ is just the reduction map it is surjective and we conclude that there exists a product on $R/u_k^{i+1}$ for which the obstruction vanishes.

Further the map $\beta_{i,k} : R/u_k^i \to \Sigma^{d+1} R/u_k$ coincides with the composition

\[
R/u_k^i \xrightarrow{\beta_{i,k}} \Sigma^{d+1} R \xrightarrow{\rho_{i,k}} \Sigma^{d+1} R/u_k \to \Sigma^{d+1} R/u_k.
\]

Here the last map is (a suspension of) the canonical $R$-ring map $R/u_k^i \to R/u_k^i$ which exists by the inductive assumption. The composition $\rho_{i,k} \circ \beta_{i,k}$ is the Bokstein operation which is primitive by [15], Proposition 3.14. It follows that $\beta_{i,k}$ is also primitive and our lemma is proved.

Now taking the smash product of (2.3) with $R/u_1 \wedge R/u_2 \wedge \ldots \wedge R/u_{k-1} \wedge R/u_{k+1}$ we get the following homotopy cofibre sequence of $R$-modules:

\[
A(u_k^{i+1}) \xrightarrow{R_{i,k}} A(u_k^i) \xrightarrow{Q_{i,k}} \Sigma^{d+1} A.
\]
It follows that $Q_{i,k}$ is a primitive operation and $A(u_{k}^{i+1})$ has an $R$-ring spectrum structure such that $R_{i,k}$ is an $R$-ring map. (Notice our abuse of notations here in using the symbol $R_{i,k}$ even though $A(u_{k}^{i+1})$ is not yet proved to be an $R$-algebra).

Next we will describe the set of all primitive operations $A(u_{k}^{i}) \to \Sigma^{*}A$. Consider the cofibre sequence

$$\Sigma^{[u_{i}]} R \xrightarrow{u_{i}} R \xrightarrow{\rho_{i}} R/\rho_{i} \xrightarrow{\delta_{i}} \Sigma^{[u_{i}]+1} R.$$  

For $l \neq k$ we introduce the operation

$$Q_{l}: A(u_{k}^{i}) = R/u_{1} \wedge R/u_{2} \wedge \ldots \wedge R/u_{k-1} \wedge R/u_{k} \wedge R/u_{k+1} \wedge \ldots \to \Sigma^{[u_{i}]+1} A$$

obtained by smashing $\rho_{l} \circ \beta_{l}: R/\rho_{l} \to \Sigma^{[u_{i}]+1} R/\rho_{l}$ with the identity map on the remaining smash factors. The operations $Q_{l}$ are primitive, again by [3], Proposition 3.14. The next result shows that $Q_{l}$ and $Q_{i,k}$ are essentially all primitive operations from $A(u_{k}^{i})$ into (suspensions of) $A$.

**Lemma 2.5.** Any primitive operation $A(u_{k}^{i}) \to \Sigma^{*}A$ can be written uniquely as an infinite sum $a_{k}Q_{i,k} + \Sigma_{i \neq k}a_{i}Q_{i}$ for $a_{i} \in A_{*}$.

**Proof.** We will only give a sketch since the arguments of Strickland ([3], Proposition 4.17) carry over almost verbatim. Let $\eta: R \to A(u_{k}^{i})$ be the unit map and

$$J_{s} = (u_{1}, u_{2}, \ldots, u_{k-1}, u_{k}^{i}, u_{k+1}, \ldots)$$

be the kernel of the map $\eta_{s}: R_{s} \to A(u_{k}^{i})_{s}$. Given a primitive operation $Q: A(u_{k}^{i}) \to \Sigma^{*}A$ define the function $p(Q): J_{s} \to A(u_{k}^{i})_{s}$ as follows. For any $x \in J_{s}$ we have a cofibre sequence

$$\Sigma^{[x]} R \xrightarrow{x} R \xrightarrow{\rho_{x}} R/\rho_{x} \xrightarrow{\beta_{x}} \Sigma^{[x]+1} R.$$  

Then there is a unique map $f_{x}: R/x \to A(u_{k}^{i})$ such that $f_{x} \circ \rho_{x} = \eta$. Further there is a unique map $y: \Sigma^{[x]+1} R \to \Sigma^{*}A$ such that $Q \circ f_{x} = y \circ \beta_{x}$. We define $p(Q)(x) := y \in \pi_{[x]+1-s}A_{s}$.

It follows that the function $p$ actually embeds the set of primitive operations $A(u_{k}^{i}) \to \Sigma^{*}A$ into $\text{Hom}_{R_{s}}^{*}(J/\Sigma^{2}, A_{s})$. Further it is straightforward to check that

$$p(Q_{s})(u_{l}) = \delta_{st} \text{ for } s, t \neq k, \text{ } p(Q_{s})(u_{k}^{i}) = 0;$$  

$$p(Q_{i,k})(u_{j}) = 0 \text{ for } j \neq k, \text{ } p(Q_{i,k})(u_{k}^{i}) = 1.$$  

Therefore the elements $Q_{i,k}, Q_{s}, s \neq k$ form a basis in $\text{Hom}_{R_{s}}(J_{s}/\Sigma^{2}, A_{s})$ dual to the basis $(u_{1}, u_{2}, \ldots, u_{k-1}, u_{k}^{i}, u_{k+1}, \ldots)$ in $J_{s}/\Sigma^{2}$ and our lemma is proved.
Now we come to the crucial part of the proof. We are going to show that the Bokstein operation $Q_{i,k}$ can be improved to a topological derivation from which it would follow that $A(u_{i+1}^j)$ is an $R$-algebra and $R_{i,k}: A(u_{i+1}^j) \to A(u_k^j)$ lifts to an $R$-algebra map.

The following lemma (interesting in its own right) is preparatory for computing the topological Hochschild cohomology of $A(u_{i+1}^j)$ with coefficients in $A$.

**Lemma 2.6.** For any regular ideal $J_*= (x_1, x_2, \ldots)$ in $R_*$ and any product on $R/J$ there is a multiplicative isomorphism

$$\pi_*(R/J)^e \cong \Lambda_{R_*/J_*} (\tau_1, \tau_2, \ldots)$$

where $|\tau_i| = |x_i| + 1$.

**Proof.** We have the universal coefficients spectral sequence

$$E^2_{**} = Tor^{R_*}_{**} (R_*/J_*, (R_*/J_*)^{op}) \implies \pi_*(R/J)^e.$$ 

This is a standard right half-plane spectral sequence of homological type with $E^\infty$ term associated to an increasing filtration

$$R_*/J_* = F_0 \subseteq F_1 \subseteq \ldots \subseteq \pi_*(R/J).$$

This spectral sequence is multiplicative as shown in [3], Lemma 1.3. Therefore using the standard Koszul resolution for the $R_*$-module $R_*/J_*$ we see that $E^2_{**} = \Lambda_{R_*} (\tau_1, \tau_2, \ldots)$. The differentials of this spectral sequence vanish on the generators $\tau_i$ and we conclude that $E^2_{**} = E^\infty_{**}$.

We will now show that there are no nontrivial multiplicative extensions in $E^2_{**}$. This is not immediate since it is not a spectral sequence of commutative algebras. Take a representative of the generator $\tau_i$ in $\pi_*(R/J)^e$. The indeterminacy in choosing this representative is an odd degree element in $R_*/J_*$. Since the ring $R_*/J_*$ is even this representative is in fact determined canonically and we will still denote it by $\tau_i$. Since the associated graded ring to $\pi_*(R/J)^e$ is graded commutative we conclude that $\tau_i^2$ belongs to the filtration component $F_1$. Since $F_1/F_2$ is spanned by $\tau_i$’s which are odd-dimensional it follows that in fact $\tau_i \in F_0 = R_*/J_*$. 

Next consider the map of $R$-ring spectra $f : (R/J)^e \to F_R(R/J, R/J)$ which is induced by the structure of an $R/J$-bimodule spectrum on $R/J$. Then $f$ induces a map of $R_*/J_*$-algebras $f_* : \pi_*(R/J)^e \to \pi_* F_R(R/J, R/J)$. By [15], Proposition 4.15 the ring $\pi_* F_R(R/J, R/J)$ is a completed exterior algebra over $R_*/J_*$. Therefore $f_*(\tau_i^2) = 0$. Since $\tau^2 \in R_*/J_*$ and the map $f$ is $R_*/J_*$-linear it implies that $\tau_i^2 = 0$. Similarly all graded commutators of the elements $\tau_i$ vanish in the ring $\pi_*(R/J)^e$. This finishes the proof of Lemma 2.6.
Remark 2.7. The $R_s/J_s$-algebra $\pi_s(R/J \wedge_R R/J)$ need not be an exterior algebra in general. For example take $R/J = MU/2$, the reduction of $MU$ modulo 2. Then it can be shown using the methods of [10] that $\pi_s MU/2 \wedge_{MU} MU/2$ is an $MU/2_s$-algebra on one generator in degree 1 whose square is equal to $x_1 \in MU/2_s$. We will discuss this and related phenomena elsewhere.

Lemma 2.8. There is the following isomorphism of graded $R_s$-modules:

$$\text{gr} \text{THH}^* (A(u_k^1), A) = A_s[\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k, \ldots];$$

$$\text{gr} \text{Der}^{s-1} (A(u_k^1), A) = A_s[\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k, \ldots]/(A_s).$$

Here $\text{gr}(\cdot)$ denotes the associated graded module. Moreover the image of the forgetful map

$$\text{Der}^* (A(u_k^1), A) \to [A(u_k^1), A]^*$$

is the set of all primitive operations from $A(u_k^1)$ to $\Sigma^* A$.

**Proof.** Denote by $J_s$ the ideal in $R$ generated by $(u_1, u_2, \ldots, u_{k-1}, u_k^1, u_{k+1}, \ldots)$. Then we have the following spectral sequence

$$\text{Ext}^{**}_{\pi_s(A(u_k^1))} (A(u_k^1), A_s) = \text{Ext}^{**}_{\pi_s(R/J)} (R/J, A_s) \Rightarrow \text{THH}^* (A(u_k^1), A).$$

By Lemma 2.6 $\pi_s(R/J)^e = \Lambda_{R/J}(\tau_1, \tau_2 \ldots)$. Therefore

$$\text{Ext}^{**}_{\pi_s(R/J)} (R/J, A) = A_s[\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k, \ldots],$$

where $|\bar{u}_l| = -|u_l| + 2$ for $l \neq k$ and $|\bar{u}_k| = -d_k + 2$. Since our spectral sequence is even it collapses and we obtain the desired isomorphism

$$\text{gr} \text{THH}^* (A(u_k^1), A) = A_s[\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k, \ldots].$$

The isomorphism involving $\text{Der}^* (A(u_k^1), A)$ is obtained similarly.

Now consider the spectral sequence

$$\text{Ext}^{**}_{R_s} (A(u_k^1), A_s) = \text{Hom}^* (\Lambda(z_1, z_2, \ldots), A_s) \Rightarrow [A(u_k^1), A]^*.$$

Here $|z_l| = |u_l| + 1$ for $l \neq k$ and $|z_k| = d_k + 1$. This spectral sequence is easily seen to collapse and denoting by $\bar{z}_l$ the elements in the dual basis in $\Lambda(z_1, z_2, \ldots)$ we identify its $E_2 = E_\infty$-term with $\Lambda_{A_s}(\bar{z}_1, \bar{z}_2, \ldots)$. (Of course this is an identification only as $A_s$-modules, not as rings.) It is clear that the elements $z_l$ correspond to the primitive operations $Q_l : A(u_k^1) \to \Sigma^{|u_l|+1} A$ for $l \neq k$ whilst $z_k$ corresponds to $Q_{i,k} : A(u_k^1) \to \Sigma^{|d_k|+1} A$.

Furthermore the forgetful map $l$ (operating on the level of $E_2$-terms) sends the element $\bar{u}_l \in \text{gr} \text{Der}^* (A(u_k^1), A)$ to $\bar{z}_l$. In other words all primitive operations $A(u_k^1) \to \Sigma^* A$ are covered by $l$ up
to higher filtration terms. Since the image of $l : \text{Der}^*(A(u_k^i), A) \to [A(u_k^i), A]^*$ is contained in the subspace of the primitive operations no higher filtration terms are present and we conclude that all Bokstein operations $Q_l$ and $Q_{t,k}$ are in the image of $l$. With this Lemma 2.8 is proved.

**Remark 2.9.** Part of the Lemma 2.8 could be reformulated more canonically as follows: the associated graded to the filtered ring $\text{THH}^*(A(u_k^i), A)$ is isomorphic to the symmetric algebra over $A^*$ generated by the $A^*$-module $J/IJ$ (with degrees shifted by one).

So we proved that there exists a topological derivation

$$\tilde{Q}_{t,k} : A(u_k^i) \to A(u_k^i) \lor \Sigma^{d+1} A$$

such that its composition with the projection onto the wedge summand

$$A(u_k^i) \lor \Sigma^{d+1} A \to \Sigma^{d+1} A$$

is the Bokstein operation $Q_{t,k}$. Associated with $\tilde{Q}_{t,k}$ is a topological singular extension

$$\Sigma^{d+1} A \to X \to A(u_k^i)$$

such that $X$ is weakly equivalent to $A(u_k^{i+1})$. In other words the $R$-module $A(u_k^{i+1})$ admits a structure of an $R$-algebra so that the reduction map $A(u_k^{i+1}) \to A(u_k^i)$ is an $R$-algebra map. The inductive step is completed and Theorem 2.3 is proved.

The singular extension of $R$-algebras

$$A(u_k^{i+1}) \xrightarrow{R_{t,k}} A(u_k^i) \xrightarrow{Q_{t,k}} \Sigma^{d+1} A$$

is of fundamental importance to us. We will call the $R$-algebra $A(u_k^{i+1})$ an elementary extension of $A(u_k^i)$. We will denote by $A(u_k^\infty)$ the $R$-algebra $\text{holim}_{l \to \infty} A(u_k^l)$. In the case of adjoining several, or perhaps, infinitely many indeterminates $u_{k_1}, u_{k_2}, \ldots$ we will write $A(u_{k_1}^\infty, u_{k_2}^\infty, \ldots)$ for the $R$-algebra

$$\text{holim}_{l \to \infty} A(u_{k_1}^\infty(u_{k_2}^\infty, \ldots(u_{k_l}^\infty)).$$

Now let $R = MU$, the complex cobordism spectrum. It is well-known that $MU$ is a commutative $S$-algebra and $MU_* = \mathbb{Z}[x_1, x_2, \ldots]$, the polynomial algebra on infinitely many generators in even degrees.

**Corollary 2.10.** Let $J_*$ be a regular ideal in $MU_* = \mathbb{Z}[x_1, x_2, \ldots]$ generated by any subsequence of the regular sequence $p^{v_0}, x_1^{v_1}, x_2^{v_2}, \ldots$ where $v_l \geq 1$ for all $l$. Then $MU/J$ admits a structure of an $MU$-algebra.
Proof. First assume that the element $p^{\nu_0}$ does not belong to $J_\ast$. Denote by $I_\ast$ the ideal in $MU_\ast$ generated by all polynomial generators $(x_1, x_2, \ldots)$. Then $MU/I$ is the integral Eilenberg-MacLane spectrum $HZ$ which possesses a canonical structure of an $MU$-algebra (even a commutative $MU$-algebra).

Taking successive elementary extensions corresponding to elements $x_{i_k}^{\nu_k}$ with $\nu_k > 1$ we construct an $MU$-algebra

$$\widetilde{MU}/J = HZ(x_{i_1}^{\nu_1}, x_{i_2}^{\nu_2}, \ldots).$$

Similarly taking elementary extensions corresponding to those polynomial generators $\{u_1, u_2, \ldots\}$ whose powers are not in $J_\ast$ and passing to the (inverse) limit we construct the $MU$-algebra $\widetilde{MU}/J(u_1^\infty, u_2^\infty, \ldots)$ whose underlying $MU$-module is $MU/J$.

If the element $p^{\nu_0}$ does belong to $J_\ast$ the only difference is that we start our induction with the Eilenberg-MacLane $MU$-algebra $HZ/p^{\nu_0}$ instead of $HZ$. With this Corollary 2.10 is proved.

Remark 2.11. Notice that our method also shows that the $MU$-algebra structures on $MU/J$ for different $J$ are compatible in the sense that various reduction maps given by killing elements $x_i^{\nu_i}$ are actually $MU$-algebra maps.

Remark 2.12. The statement of Corollary 2.10 remains true with $MU$ replaced with $MU_{(p)}$, the localization of $MU$ at any prime $p$. The proof is the same verbatim.

Recall that the Brown-Peterson spectrum $BP$ is obtained from $MU_{(p)}$ by killing all polynomial generators in $MU_{(p)\ast}$ except for $v_i = x_2(p^i - 1)$ provided that we use Hazewinkel generators for $MU_{(p)\ast} = \mathbb{Z}[x_1, x_2, \ldots]$. Therefore $BP_{\ast} = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$. It follows that $BP$ possesses an $MU$-algebra structure. We define

$$BP\langle n \rangle = BP/(v_i, i > n) = HZ_{(p)}(v_1^\infty, v_2^\infty, \ldots, v_n^\infty)$$

$$P(n) = BP/(v_i, i < n) = HF_p(v_1^\infty, v_2^\infty, \ldots)$$

$$B(n) = v_n^{-1}P(n) = v_n^{-1}[HF_p(v_1^\infty, v_2^\infty, \ldots)]$$

$$k(n) = BP/(v_i, i \neq n) = HF_p(v_n^\infty)$$

$$K(n) = v_n^{-1}BP/(v_i, i \neq n) = v_n^{-1}[HF_p(v_n^\infty)]$$

$$E(n) = v_n^{-1}BP/(v_i, i > n) = v_n^{-1}[HZ_{(p)}(v_1^\infty, v_2^\infty, \ldots, v_n^\infty)]$$

Note that inverting an element in the coefficient ring of an $MU_{(p)}$-ring spectrum is an instance of Bousfield localization in the homotopy category of $MU_{(p)}$-modules. Further Bousfield localization
preserves algebra structures we conclude that all spectra listed above admit structures of $MU_{(p)}$-algebras (and therefore $MU$-algebras).

We conclude this section with a few remarks about the commutativity of the products on the $MU$-algebras considered. First in the case of the odd prime $p$ BP and other spectra derived from it have coefficient rings concentrated in degrees congruent to 0 mod 4 and therefore all products are unique up to homotopy and automatically commutative. If $p=2$ then Strickland shows that $BP$ still admits a structure of a commutative $MU$-ring spectrum, but this does not follow directly from our construction. We conjecture that in the context of Corollary 2.10 one can choose an $MU$-algebra structure on $MU/J$ compatible with any given structure of an $MU$-ring spectrum on $MU/J$.

The situation with $E(n)$ is different. The spectrum $E(n)$ is Landweber exact regardless of the prime $p$. Therefore it possesses a unique homotopy associative and commutative multiplication compatible with its structure (in the classical sense) of an $MU$-algebra spectrum. Our results show that this multiplication can be improved to an $MU$-algebra structure which in turn gives an $A_\infty$-structure. On the other hand Strickland shows in [15] that at the prime 2 there is no homotopy commutative multiplication on $E(n)$ in the category of $MU$-modules (to get a homotopy commutative product one has to choose a different set of generators). Therefore, quite curiously, $E(n)$ is a homotopy commutative $A_\infty$-ring spectrum and simultaneously a $MU$-algebra that is not commutative even up to homotopy.

3. Separable extensions of $R$-algebras

We keep our convention that $R$ is a commutative $S$-algebra with coefficient ring $R_*$ concentrated in even degrees. Let $A$ be an $R$-algebra with coefficient ring $A_*$ and $A_* \subseteq \overline{A}_*$ a ring extension in the usual algebraic sense. In this section we consider the problem of finding an $R$-algebra $\overline{A}$ together with an $R$-algebra map $A \rightarrow \overline{A}$ which realizes the given extension $A_* \subseteq \overline{A}_*$ in homotopy. If such an algebra $\overline{A}$ exists we say that the algebraic extension $A_* \subseteq \overline{A}_*$ admits a topological lifting.

We now describe a general framework in which it is possible to prove that topological liftings exist. Suppose that $R_* \rightarrow \overline{R}_*$ is a separable extension of $R$. That means that $R_*$ is a subring of a graded commutative ring $\overline{R}_*$ and $\overline{R}_*$ is a separable algebra over $R_*$, i.e. $\overline{R}_*$ is a projective module over $\overline{R}_* \otimes_{R_*} \overline{R}_*$. In addition we assume that $\overline{R}_*$ is a projective $R_*$-module. Then for any ideal $I_*$ in $R_*$ the $R_*$-algebra $\overline{R}_*/I_* := R_*/I_* \otimes_{R_*} \overline{R}_*$ is a separable $R_*/I_*$-algebra so $R_*/I_* \rightarrow \overline{R}_*/I_*$ is also a separable extension.
Further assume that the ideal $I_*$ is generated by a regular sequence $(u_1, u_2, \ldots)$ (possibly infinite) of nonzero divisors, the $R$-module $A = R/I$ is supplied with a structure of an $R$-algebra and the algebraic extension $R_*/I_* \to R_*/I_*$ admits a topological lifting. In other words there exists an $R$-algebra $\overline{A}$ and an $R$-algebra map $A \to \overline{A}$ realizing the given separable extension on the level of coefficient rings.

Recall that in the previous section we constructed an $R$-algebra structure on the $R$-module $A(u_1^r) := R/u_1 \wedge R/u_2 \wedge \ldots \wedge R/u_k \wedge R/u_{k+1} \wedge \ldots$.

Then we have the following

**Theorem 3.1.** There exist $R$-algebras $\overline{A}(u_1^r)$ realizing in homotopy the $R_*$-module $\overline{A}(u_1^r)_* := A(u_1^r)_* \otimes_{R_*} \overline{R}$ and $R$-algebra maps $\overline{R}_{l,k} : \overline{A}(u_1^{l+1}) \to \overline{A}(u_1^r)$, which give the reduction maps on the level of coefficient rings. Moreover the algebraic extension of rings $\overline{A}(u_1^r)_* \to \overline{A}(u_1^r)_*$ admits a topological lifting so that the following diagram of $R$-algebras is homotopy commutative:

$$
\begin{array}{ccc}
A(u_1^{l+1}) & \xrightarrow{\overline{R}_{l,k}} & A(u_1^r) \\
\downarrow & & \downarrow \\
A(u_1^{l+1}) & \xrightarrow{\overline{R}_{l,k}} & A(u_1^r)
\end{array}
$$

Before embarking on the proof of the theorem we will formulate and prove a useful lemma which will be called ‘flat base change’. It will be used several times in this paper. Though the lemma is really a standard exercise in homological algebra we have been unable to find a proper reference and will therefore give a complete proof.

**Lemma 3.2.** Let $k$ be a graded commutative ring and $B, C$ be graded $k$-algebras such that $C$ is flat as a $k$-module. Let $M$ and be a graded left $C$-module which is flat as a $k$-module and $N$ be a left graded $C \otimes_k B$-module. Then there is the following ‘flat base change’ isomorphism:

$$
\Ext^*_C(M, N) \cong \Ext^*_C(B \otimes_k C, M).
$$

**Proof.** We have the following change of rings spectral sequence corresponding to the algebra map $id \otimes 1 : C \to C \otimes_k B$ (cf. [3], Ch. XVI, 6):

$$
\Ext^*_C(B \otimes_k C, M, N) \Rightarrow \Ext^*_C(M, N)
$$

In fact, this spectral sequence was deduced in the cited reference for ungraded rings and modules, but it does not affect the arguments. Since $M$ and $C$ are flat $k$-modules we obtain

$$
\Tor^*_C(C \otimes_k B, M) \cong \Tor^*_C(B \otimes_k C, M) \cong \Tor^*_C(B, \Tor^*_C(C, M))
$$
Here the first isomorphism is an instance of the isomorphism Corollary 3.3. Let $k \hookrightarrow \overline{k}$ be a separable extension of a graded commutative ring $k$. Let $C$ be a graded ring and $M, N$ be graded left $\overline{k} \otimes C$-modules. Then there is the following isomorphism:

$$\text{Ext}_{\overline{k} \otimes C}(M \otimes \overline{k}, N) \cong \text{Ext}_{\overline{k}}(M, N).$$

**Proof.** Since the $\overline{k} \otimes \overline{k}$-module $\overline{k}$ is projective it is flat. Moreover we obviously have $\overline{k} \otimes \overline{k} \otimes \overline{k} \cong \overline{k}$. Therefore by Lemma 3.1

$$\text{Ext}_{\overline{k} \otimes C}(\overline{k} \otimes M, N) \cong \text{Ext}_{\overline{k}}(\overline{k} \otimes M, N) \cong \text{Ext}_{\overline{k}}(M \otimes \overline{k}, N) \cong \text{Ext}_{\overline{k}}(M, N).$$

**Proof** of Theorem 3.1. Proceeding by induction suppose that the $R$-algebras $\overline{A(u^i_1)}$ with required properties were constructed for $l \leq i$. Consider the spectral sequence

$$T_{\alpha} R_{\ast} (A(u^i_1)_*, \overline{A(u^i_1)}_{\ast}) \cong R_\ast \otimes_{R_\ast} \overline{R_\ast} \otimes_{R_\ast} \text{Ext}_{R_\ast} (A(u^i_1)_*, A(u^i_1)_{\ast})$$

$$\cong \overline{R_\ast} \otimes_{R_\ast} \overline{R_\ast} \otimes_{R_\ast} \Lambda_{A(u^i_1)} (\tau_1, \tau_2, \ldots) \cong \overline{R_\ast} \otimes_{R_\ast} \overline{A(u^i_1)}_{\ast} \otimes \Lambda (\tau_1, \tau_2, \ldots) \Rightarrow \pi_* \overline{A(u^i_1)}_{\ast}.$$

Here the first isomorphism is an instance of the isomorphism

$$T_{\alpha} R_{\ast} (P \otimes_R M, Q \otimes_R N) \cong P \otimes_R Q \otimes_R T_{\alpha} R_{\ast} (M, N)$$

for projective $R$-modules $M$ and $N$. The second isomorphism follows from the standard calculation with the Koszul complex and the third isomorphism is obvious.

Since by Lemma 2.6 $\pi_* A(u^i_1)^c = \Lambda_{A(u^i_1)} (\tau_1, \tau_2, \ldots)$ we see that the exterior generators $\tau_i$ are permanent cycles and it follows that our spectral sequence collapses multiplicatively so the ring $\pi_* \overline{A(u^i_1)}_{\ast}$ is isomorphic to the exterior algebra on $\tau_1, \tau_2, \ldots$ with coefficients in $R_\ast \otimes_{R_\ast} \overline{A(u^i_1)}_{\ast}$. Next notice that the $R$-algebra map $A(u^1_1) \to \overline{A(u^i_1)}$ induces a map of $R$-modules

$$\text{THH}(A(u^1_1), A(u^1_1)) \to \text{THH}(A(u^1_1), \overline{A(u^1_1)})$$

(3.2)

We claim that (3.2) is a weak equivalence. To see this consider the spectral sequences

$$\text{Ext}^{\ast}_{A(u^1_1)} (A(u^1_1)_*, \overline{A(u^1_1)}_{\ast}) \Rightarrow \text{THH}^{\ast}(A(u^1_1), \overline{A(u^1_1)})$$

(3.3)

and

$$\text{Ext}^{\ast}_{A(u^1_1)} (\overline{A(u^1_1)}_{\ast}, \overline{A(u^1_1)}_{\ast}) \Rightarrow \text{THH}^{\ast} (\overline{A(u^1_1)}_{\ast}, \overline{A(u^1_1)}_{\ast})$$

(3.4)
The map (3.4) determines a map between spectral sequences (3.3) and (3.4). From Corollary 3.3 we deduce that these spectral sequences are isomorphic and therefore (3.2) is a weak equivalence.

Finally we conclude that the map \( \text{Der}(A(u_k^i), A) \to \text{Der}(A(u_k^i), \overline{A}) \) induced by the \( R \)-algebra map \( A(u_k^i) \to A(u_k^i) \) is weak equivalence.

Consider the following homotopy commutative diagram of \( R \)-algebras:

\[
\begin{array}{ccc}
A(u_k^i) & \xrightarrow{\xi} & A(u_k^i) \\
\downarrow & & \downarrow \\
\overline{A} \vee \Sigma^{d_i+1}A & \xleftarrow{Q_{i,k}} & \overline{A} \vee \Sigma^{d_i+1}A
\end{array}
\]

Here the vertical arrows are topological derivations. The right downward arrow is the Bokstein operation \( Q_{i,k} \) constructed in the previous section, the middle arrow is the only one that makes the right square commute. Since derivations of \( A(u_k^i) \) with values in \( \overline{A} \) are in one-to-one correspondence with derivations of \( A(u_k^i) \) with values in \( \overline{A} \) the left downward arrow \( \xi \) making the left square commute exists and is unique.

Now the derivation \( \xi \) gives rise to a topological singular extension

\[
\Sigma^{d_i} \overline{A} \to ? \to A(u_k^i)
\]

which on the level of coefficient rings reduces to the algebraic singular extension

\[
\Sigma^{d_i} \overline{A}_* \to A(u_k^{i+1})_* \to A(u_k^i)_*
\]

Therefore we can denote \( ? \) by \( A(u_k^{i+1}) \) and because of the diagram (3.3) we have a map of topological singular extensions

\[
\begin{array}{ccc}
\Sigma^{d_i} A & \xrightarrow{\xi} & A(u_k^{i+1}) \\
\downarrow & & \downarrow \\
\Sigma^{d_i} A & \xleftarrow{Q_{i,k}} & A(u_k^i)
\end{array}
\]

With this the inductive step is completed and Theorem 3.1 is proved.

We now show that Theorem 3.1 gives a way of adjoining roots of unity to various \( MU \)-algebras. Let \( \mathbb{F}_p \hookrightarrow L \) be a finite separable extension of the field \( \mathbb{F}_p \) (typically obtained by adjoining roots of an irreducible factor of the cyclotomic polynomial \( x^{p^n-1} - 1 \) for some \( n \)). Let \( R = \widehat{MU}_p \), the \( p \)-completion of the complex cobordism spectrum \( MU \) and \( \overline{R}_* = W(L) \otimes MU_* \) where \( W(L) \) is the ring of Witt vectors of \( L \). Since \( L \) is a separable extension of \( \mathbb{F}_p \), the algebra \( W(L) \) is separable over the \( p \)-adic integers \( \widehat{\mathbb{Z}}_p \), and it follows that \( W(L) \otimes MU_* \) is a separable algebra over \( \widehat{MU}_p = \widehat{\mathbb{Z}}_p \otimes MU_* \).
Corollary 3.4. Let $J_*$ be the ideal in $\widehat{MU}_{p*} = \widehat{\mathbb{Z}}_p[x_1, x_2, \ldots]$ generated by $(p, x_{i_1}, x_{i_2}, \ldots)$ where $x_{i_k}$ are polynomial generators in $\widehat{MU}_{p*}$. Then for any finite separable extension $\mathbb{F}_p \hookrightarrow L$ there exist $\widehat{MU}_p$-algebras $MU/J$, $MUL/J$, $\widehat{MUL}/J$ and $\widehat{MU}/J$ together with the commutative diagram in the homotopy category of $\widehat{MU}_p$-algebras

$$
\begin{array}{ccc}
\widehat{MU}/J & \longrightarrow & MUL/J \\
\downarrow & & \downarrow \\
MU/J & \longrightarrow & MUL/J
\end{array}
$$

which reduces in homotopy to the following diagram of $MU_*$-algebras:

$$
\begin{array}{ccc}
\widehat{\mathbb{Z}}_p \otimes MU_*/(x_{i_1}, x_{i_2} \ldots) & \longrightarrow & W(L) \otimes MU_*/(x_{i_1}, x_{i_2} \ldots) \\
\mod p & & \mod p \\
MU_*/J_* & \longrightarrow & L \otimes MU_*/J_*
\end{array}
$$

**Proof.** Denote by $u_1, u_2, \ldots$ the collection of polynomial generators of $\widehat{MU}_{p*}$ which are not in $J_*$. There is an isomorphism of rings $MU_*/J_* \cong \widehat{MU}_{p*}/J_* \cong \mathbb{F}_p[u_1, u_2, \ldots]$. Let $I$ be the maximal ideal $(p, x_1, x_2, \ldots)$ in $\widehat{MU}_{p*} = \widehat{\mathbb{Z}}_p[x_1, x_2, \ldots]$. Then $\widehat{MU}/I$ is the Eilenberg-MacLane spectrum $H\mathbb{F}_p$ and the canonical map $\widehat{MU}/I \to MU/I$ is a commutative $S$-algebra map. Therefore $MU/I$ is an $\widehat{MU}_p$-algebra (even a commutative $\widehat{MU}_p$-algebra). Further the extension $\mathbb{F}_p \hookrightarrow L$ determines a map of commutative $S$-algebras $H\mathbb{F}_p \to HL$ which is also a map of commutative $\widehat{MU}_p$-algebras. Using Theorem 3.1 we can adjoin the variable $u_i$ to the topological extension $H\mathbb{F}_p \to HL$ and obtain the tower of topological extensions

$$
\{H\mathbb{F}_p(u_{i_k}^j) \to HL(u_{i_k}^j)\}.
$$

Passing to the limit we get the topological extension $H\mathbb{F}_p(u_{i_k}^\infty) \to HL(u_{i_k}^\infty)$.

This procedure could be repeated so we can adjoin another polynomial generator to the extension $H\mathbb{F}_p(u_{i_k}^\infty) \to HL(u_{i_k}^\infty)$ (or any number of polynomial generators). In this way we construct the extension

$$
H\mathbb{F}_p(u_{1}^\infty, u_{2}^\infty, \ldots) \to HL(u_{1}^\infty, u_{2}^\infty, \ldots)
$$

which is the same as the extension $MU/J \to MUL/J$ for the ideal $J_* = (p, x_{i_1}, x_{i_2}, \ldots)$ in $\widehat{MU}_{p*}$. At this point we adjoin the indeterminate corresponding to the prime $p$. We then get the following
diagram of $\hat{\text{MU}}_p$-algebras:

$$
\begin{array}{cccccccc}
\text{MU}/J & \leftarrow & \text{MU}/J(p^2) & \leftarrow & \cdots & \leftarrow & \text{MU}/J(p^n) & \leftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{MUL}/J & \leftarrow & \text{MUL}/J(p^2) & \leftarrow & \cdots & \leftarrow & \text{MUL}/J(p^n) & \leftarrow & \cdots \\
\end{array}
$$

where the $\hat{\text{MU}}_p$-algebras $\text{MU}/J(p^n)$ and $\text{MUL}/J(p^n)$ realize the $\hat{\text{MU}}_p$-algebras $W_n(F_p) \otimes \text{MU}/(x_{i_1}, x_{i_2}, \ldots)$ and $W_n(L) \otimes \text{MU}/(x_{i_1}, x_{i_2}, \ldots)$ respectively.

Finally taking the (homotopy) inverse limit we get an $\hat{\text{MU}}_p$-algebra map

$$
\text{MU}/J(p^\infty) = \hat{\text{MU}}/J \to \hat{\text{MUL}}/J = \text{MUL}/J(p^\infty)
$$

which realizes in homotopy the extension of rings $\hat{\mathbb{Z}}_p \otimes \text{MU}/J \to W(L) \otimes \text{MU}/J$. Corollary 3.4 is then proved.

**Remark 3.5.** It is not hard to prove that the diagram (3.6) is equivariant with respect to the Galois group $\text{Gal}(L/F_p)$ in a suitable sense but we save this observation for future work.

## 4. Localized towers of $\text{MU}$-algebras

In this section we investigate the behaviour of the towers of $\text{MU}$-algebras constructed in the previous section under Bousfield localization. We start with an almost trivial

**Theorem 4.1.** Let

(4.1) \[ I \to A \to B \]

be a singular extension of $R$-algebras for a commutative $S$-algebra $R$. Then for any $R$-module $M$ the cofibre sequence

(4.2) \[ I_M \to A_M \to B_M \]

is also a singular extension of $R$-algebras (here $?_M$ denotes Bousfield localization with respect to $M$)

**Proof.** The singular extension (4.1) is associated with a certain topological derivation $\xi: B \to B \vee \Sigma M$ so that there is a homotopy pullback square of $R$-algebras

$$
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
B & \xi & \to & B \vee \Sigma I
\end{array}
$$
where the vertical map $B \to B \vee \Sigma M$ is the canonical inclusion of the wedge summand. Localizing this diagram with respect to $M$ we get the homotopy pullback diagram

$$
\begin{array}{ccc}
A_M & \longrightarrow & B_M \\
\downarrow & & \downarrow \\
B_M & \longrightarrow & B_M \vee \Sigma I_M \\
\end{array}
$$

Therefore (1.2) is a singular extension associated with the topological derivation $B_M \xrightarrow{\xi_M} B_M \vee \Sigma I_M$ and the theorem is proved.

Now let $I_*$ be a regular ideal $(u_1, u_2, \ldots)$ in $R_*$ and $A = R/I$ be an $R$-algebra. Consider the $R$-algebra $A(u_1^\infty, u_2^\infty, \ldots, u_k^\infty)_{u_1^{-1}[A(u_1^\infty)]}$, the Bousfield localization of the $R$-algebra $A(u_1^\infty, u_2^\infty, \ldots, u_k^\infty)$ with respect to $u_1^{-1}[A(u_1^\infty)]$. Then we have

**Proposition 4.2.** (i) There is the following weak equivalence of $R$-algebras:

$$A(u_1^\infty, u_2^\infty, \ldots, u_k^\infty)_{u_1^{-1}[A(u_1^\infty)]} \simeq (u_1^{-1}[A(u_1^\infty)])(u_2^\infty, u_3^\infty, \ldots, u_k^\infty).$$

(ii) If $R$ is connective and the elements $u_1, u_2, \ldots, u_k$ have positive degrees then the coefficient ring of the $R$-algebra $A(u_1^\infty, u_2^\infty, \ldots, u_k^\infty)_{u_1^{-1}[A(u_1^\infty)]}$ is the $R_*$-algebra $A_*[u_1^\pm 1][u_2, u_3, \ldots, u_k]$.

**Proof.** For (i) consider the following tower of $R$-algebras:

$$A(u_1^\infty) \leftarrow A(u_1^\infty, u_2^\infty) \leftarrow \cdots \leftarrow A(u_1^\infty, u_2^\infty, u_3^\infty) \leftarrow \cdots$$

Bousfield-localizing it with respect to $u_1^{-1}A(u_1^\infty)$ we get the tower

$$u_1^{-1}[A(u_1^\infty)] \leftarrow u_1^{-1}[A(u_1^\infty, u_2^\infty)] \leftarrow \cdots \leftarrow u_1^{-1}[A(u_1^\infty, u_2^\infty, u_3^\infty)] \leftarrow \cdots$$

Notice that the first term of the localized tower as well as its successive subquotients (isomorphic to suspensions of $u_1^{-1}[A(u_1^\infty)]$) are $u_1^{-1}[A(u_1^\infty)]$-local and therefore so is its homotopy inverse limit. Clearly the canonical map

$$\operatorname{holim}_{l \to \infty} A(u_1^\infty, u_2^l) \to \operatorname{holim}_{l \to \infty} u_1^{-1}A(u_1^\infty, u_2^l) = (u_1^{-1}[A(u_1^\infty)])(u_2^\infty)$$

is $u_1^{-1}[A(u_1^\infty)]$-equivalence and we conclude that

$$(u_1^{-1}[A(u_1^\infty)])(u_2^\infty) \simeq A(u_1^\infty, u_2^\infty)_{u_1^{-1}[A(u_1^\infty)]}$$

We then proceed by adjoining $u_i$. There is a tower of $R$-algebras

$$A(u_1^\infty)(u_2^\infty) \leftarrow A(u_1^\infty)(u_2^\infty, u_3^\infty) \leftarrow \cdots \leftarrow A(u_1^\infty)(u_2^\infty, u_3^\infty, u_4^\infty) \leftarrow \cdots$$
where the successive stages

\[ \Sigma^{|u^3|} A(u^\infty_1)(u^\infty_2) \to A(u^\infty_1)(u^\infty_2, l_3^{l+1}) \to A(u^\infty_1)(u^\infty_2, u_3^l) \]

are singular extensions of \( R \)-algebras. Localizing this tower with respect to \( u_1^{-1}[A(u^\infty_1)] \) and using (4.3) and Theorem 4.3 we get the tower

\[
(u_1^{-1}[A(u^\infty_1)])(u_2^\infty) \leftrightarrow (u_1^{-1}[A(u^\infty_1)])(u_2^\infty, u_3^2) \leftrightarrow \ldots \leftrightarrow (u_1^{-1}[A(u^\infty_1)])(u_2^\infty, u_3^l) \leftrightarrow \ldots
\]

whose homotopy limit

\[
\text{holim}_{l \to \infty}(u_1^{-1}[A(u^\infty_1)])(u_2^\infty, u_3^l) = (u_1^{-1}[A(u^\infty_1)])(u_2^\infty, u_3^\infty)
\]

is clearly weakly equivalent to the localization of \( A(u^\infty_1)(u_2^\infty, u_3^\infty) \) with respect to \( u_1^{-1}A(u^\infty_1) \). Repeating this process for \( u_4, u_5, \ldots, u_k \) we obtain the desired isomorphism

\[
A(u^\infty_1, u_2^\infty, \ldots, u_k^\infty)_{u_1^{-1}[A(u^\infty_1)]} \simeq (u_1^{-1}[A(u^\infty_1)])(u_2^\infty, u_3^\infty, \ldots, u_k^\infty).
\]

For (ii) notice that since \( u_1 \) has positive degree the ring \( A(u^\infty_1)_* \) is a polynomial ring over \( A \) on one variable \( u_1 \). Furthermore \( u_1^{-1}[A(u^\infty_1)]_* = A_*[u_1^{+1}] \). Denote by \( B \) the ring

\[
(u_1^{-1}[A(u^\infty_1)])(u_2^\infty, u_3^\infty, \ldots, u_k^\infty)_* \simeq A(u^\infty_1, u_2^\infty, \ldots, u_k^\infty)_{u_1^{-1}[A(u^\infty_1)]}_*
\]

Then \( B \) is by definition the completion of the ring \( u_1^{-1}R_*/(u_{k+1}, u_{k+2}, \ldots) \) at the ideal \( (u_2, u_3, \ldots, u_k) \).

Since this ideal is regular the associated graded ring is polynomial:

\[
gr B = A_*[u_1^{+1}][u_2, u_3, \ldots, u_k].
\]

Since the elements \( u_i, l = 1, \ldots, k \) have positive degrees the \( R_* \)-algebra \( A(u^\infty_1, u_2^\infty, \ldots, u_k^\infty)_* \) is in fact an \( A_* \)-algebra and therefore, so is \( B \). It follows that \( B \) is isomorphic to the completion of its associated graded ring and our proposition is proved.

Our main application of the developed localization techniques is to the Johnson-Wilson theories \( E(n) \). Recall that the \( MU \)-algebra spectrum \( E(n) \) has the coefficient ring \( E(n)_* = \mathbb{Z}(p)[v_1, v_2, \ldots, v_{n-1}][v_{n-1}^{1}] \).

The \( MU_* \)-algebra structure on \( E(n)_* \) is defined by the correspondence \( x_i \to 0 \) for \( i \neq 2(p^n - 1) \) and \( x_{2(p^n-1)} \to v_n \) where \( x_i \) are the Hazewinkel generators of \( MU_{(p)_*} \). We have the following

**Corollary 4.3.** There exist \( MU \)-algebras \( \widehat{E}(n), \widehat{EL}(n), K(n), KL(n) \), and the homotopy commutative diagram of \( MU \)-algebras

\[
\begin{align*}
\widehat{E}(n) \longrightarrow \widehat{EL}(n) \\
\downarrow \quad \downarrow \\
K(n) \longrightarrow KL(n)
\end{align*}
\]
which realizes in homotopy the diagram of $MU$-algebras

$$\hat{\mathbb{Z}}_p[v_n][v_1, v_2, \ldots, v_{n-1}] \longrightarrow W(L)[v_n][v_1, v_2, \ldots, v_{n-1}]$$

$$\mathbb{F}[v_n^{\pm 1}] \longrightarrow L[v_n]$$

**Proof.** We first construct the map of $MU(p)$-algebras $k(n) \to kL(n)$ by adjoining $v_n$ to the extension $H\mathbb{F}_p \to HL$. Here $k(n)$ and $kL(n)$ are the connective Morava $K$-theories with coefficient rings $k(n)_* = \mathbb{F}_p[v_n]$ and $kL(n)_* = L[v_n]$. Adjoining indeterminates $p, v_1, v_2, \ldots, v_{n-1}$ we construct the homotopy commutative diagram of $MU$-algebras

$$\hat{\mathbb{Z}}_p[v_n][v_1, v_2, \ldots, v_{n-1}] \longrightarrow W(L)[v_n][v_1, v_2, \ldots, v_{n-1}]$$

$$\mathbb{F}[v_n^{\pm 1}] \longrightarrow L[v_n]$$

which realizes in homotopy the diagram

$$\hat{\mathbb{Z}}_p[v_1, v_2, \ldots, v_{n-1}] \longrightarrow W(L)[v_1, v_2, \ldots, v_{n-1}]$$

$$\mathbb{F}_p \longrightarrow L[v_n]$$

According to Proposition 4.2 Bousfield localization of $\hat{\mathbb{B}}P(n)$ with respect to $K(n)$ on the level of coefficient rings amounts to inverting $v_n$ and completing at the ideal $(v_1, v_2, \ldots, v_{n-1})$. Further notice that since $\pi_*\hat{\mathbb{B}}PL(n)$ is a free $\pi_*\hat{\mathbb{B}}P(n)$-module of finite rank (which is equal to the degree of the field extension $L/\mathbb{F}_p$) the $MU$-algebra $\hat{\mathbb{B}}PL(n)$ is a finite cell $\hat{\mathbb{B}}P(n)$-module and therefore its $K(n)$-localization is equivalent to $\hat{\mathbb{B}}P(n)_{K(n)} \wedge_{\hat{\mathbb{B}}P(n)} \hat{\mathbb{B}}PL(n)$ and its coefficient ring is

$$W(L) \otimes \pi_*\hat{\mathbb{B}}P(n)_{K(n)} = W(L)[v_n^{\pm 1}][[v_1, v_2, \ldots, v_{n-1}]].$$

Therefore we obtain the diagram (4.4) by localizing (4.5) with respect to $K(n)$ in the category of $MU$-algebras. Corollary 4.3 is proved.

5. **Computing homotopy groups of mapping spaces**

In this section we investigate spaces of strictly multiplicative maps from $\hat{\mathbb{E}}L(n)$ to certain $MU$-algebras which we call 'strongly $KL(n)$-complete'. As a consequence we obtain versions of the Hopkins-Miller theorem as well as splitting theorems for such spectra. Such theorems (in a weaker, up to homotopy form) were previously obtained by methods of formal group theory,
Recall that \( \hat{EL}(n) \) and \( KL(n) \) denote topological extensions of spectra \( \hat{E}(n) \) and \( K(n) \) corresponding to the separable field extension \( \mathbb{F}_p \hookrightarrow L \). All our results remain valid and are still nontrivial for \( L = \mathbb{F}_p \) in which case \( \hat{EL}(n) \) and \( KL(n) \) specialize to the completed Johnson-Wilson theory \( \hat{E}(n) \) and Morava \( K \)-theory \( K(n) \) respectively.

Let \( A, B \) be \( S \)-algebras, which we will assume without loss of generality to be \( q \)-cofibrant in the sense of \[\ref{cofibrant}\. That means in particular, that the topological space of \( S \)-algebra maps \( B \to A \) has the ‘correct’ homotopy type (i.e. the one that depends only on the homotopy type of \( A \) and \( B \) as \( S \)-algebras). Denote this topological space by \( F_{S-alg}(B, A) \). Also denote the set of multiplicative up to homotopy maps from \( A \) to \( B \) by \( \text{Mult}(A, B) \).

We recall one result from \[\ref{unstable} which will be needed later on.

**Theorem 5.1.** Let \( \Sigma^{-1}M \to X \to A \) be a singular extension of \( S \)-algebras associated with a derivation \( d : A \to A \vee M \) and \( f : B \to A \) a map of \( S \)-algebras. Then \( f \) lifts to an \( S \)-algebra map \( B \to X \) iff a certain element in \( \text{Der}^0(B, M) \) is zero. Assuming that a lifting exists the homotopy fibre of the map

\[
F_{S-alg}(B, X) \to F_{S-alg}(B, A)
\]

over the point \( f \in F_{S-alg}(B, A) \) is weakly equivalent to \( \Omega^\infty \text{Der}(B, \Sigma^{-1}M) \), the 0th space of the spectrum \( \text{Der}(X, \Sigma^{-1}M) \). In particular if the spectrum \( \text{Der}(X, \Sigma^{-1}M) \) is contractible then (5.1) is a weak equivalence.

Recall that the spectrum \( \hat{EL}(n) \) with coefficient rings \( W(L)[v_1, v_2, \ldots, v_{n-1}] \) has a structure of an \( S \)-algebra.

We say that the generalized Hopkins-Miller theorem holds for an \( S \)-algebra \( E \) if the following two conditions hold:

1. \( \pi_0F_{S-alg}(\hat{EL}(n), E) = \text{Mult}(\hat{EL}(n), E) \)
2. \( \pi_iF_{S-alg}(\hat{EL}(n), E) = 0 \) for \( i > 0 \).

**Proposition 5.2.** The generalized Hopkins-Miller theorem holds for the \( S \)-algebra \( KL(n) \).

**Proof.** Consider the Bousfield-Kan spectral sequence (cf. \[\ref{unstable} for the mapping space \( F_{S-alg}(\hat{EL}, E) \). The identification of the \( E_2 \)-term is standard and we refer the reader to \[\ref{Kunneth} for necessary details. Here we only mention that the key ingredient in this identification is the existence of the K"{u}nneth formula

\[
KL(n)_*(\hat{E}(n) \wedge \hat{EL}(n)) = KL(n)_*\hat{E}(n) \otimes_{KL(n)_*} KL(n)_*\hat{EL}(n).
\]
This is the result we need:

\[ E_2^{st} = \begin{cases} 
\text{Der}^{st}_{KL(n)}(\widehat{E}(n), K(n), KL(n)) & \text{for } (s, t) \neq (0, 0), \\
\text{Mult}(\widehat{E}(n), KL(n)) & \text{for } (s, t) = (0, 0). 
\end{cases} \]

Here \( \text{Der}_{k}^{*,*}(?, ?) \) is defined for a graded \( k \)-algebra \( R_* \) and a graded \( R_* \)-bimodule \( M_* \) as the shifted Hochschild cohomology:

\[ \text{Der}_{k}^{*,*}(A_*, M_*) := HH_{k}^{*,*}(A_*, M_*) \]

(the second grading reflects the fact that \( A_* \) and \( M_* \) are graded objects).

Further

\[ \widehat{E}(n)_*K(n) = \widehat{E}(n)_*K(n) \otimes W(L) \otimes L = E(n)_*K(n) \otimes L \otimes L \]

and computations in [11], Chapter VI show that

\[ E(n)_*K(n) = \Sigma_* = \mathbb{F}_p[u_n^{\pm 1}|t_k > 0]/(t_k^{n_p} - v^{p^k - 1}t_k) \]

where the degree of \( t_k \) is \( 2(p^k - 1) \). We have:

\[ HH^{**}_{KL(n)}(\Sigma_* \otimes L \otimes L, KL(n)_*) = HH^{**}_{KL(n)}(\Sigma_* \otimes L, K(n)_* \otimes L) \]

\[ = HH^{**}_{K(n)_*}(\Sigma_*, K(n)_*) \otimes HH^{**}_{F_p}(L, L) = HH^{**}_{K(n)_*}(\Sigma_*, K(n)_*) \otimes L. \]

An easy cohomological calculation (due to A.Robinson, cf. [13]) shows that \( HH^{**}_{K(n)_*}(\Sigma_*, K(n)_*) = K(n)_* \) and therefore

\[ HH^{**}_{KL(n)}(\widehat{E}(n)_*, KL(n)) = K(n) \otimes L = KL(n)_* \]

Thus the Bousfield-Kan spectral sequence reduces to its corner term and Proposition 5.2 is proved.

**Remark 5.3.** Notice that even though in Proposition 5.2 we used the results of the previous section that spectra \( KL(n) \), and \( \widehat{E}(n) \) admit \( S \)-algebra structures we did not specify which structures are used. In other words Proposition 5.2 is valid with *any* choice of \( S \)-algebra structures on the spectra in question.

We can now regard \( KL(n) \) as a bimodule over \( \widehat{E}(n) \) by choosing an \( S \)-algebra map \( \widehat{E}(n) \rightarrow KL(n) \) (which is supplied by Proposition 5.2). Therefore it makes sense to consider spectra of topological Hochschild cohomology and topological derivations of \( \widehat{E}(n) \) with values in \( KL(n) \). It turns out that these spectra do not depend up to weak equivalence which bimodule structure we choose.
Proposition 5.4. The canonical map of $S$-modules

$$\text{THH}(\widehat{EL}(n), KL(n)) \to KL(n)$$

is a weak equivalence. Furthermore the $S$-module $\text{Der}(\widehat{EL}(n), KL(n))$ is contractible.

**Proof.** Since there is a homotopy cofibre sequence of $S$-modules

$$\Sigma^{-1} \text{Der}(\widehat{EL}(n), KL(n)) \to \text{THH}(\widehat{EL}(n), KL(n)) \to KL(n)$$

it suffices to prove the statement about $\text{THH}$. Consider the following spectral sequence

$$\text{Ext}^{**}_{\widehat{EL}(n), \widehat{EL}(n)}(\widehat{EL}(n)_*, KL(n)_*) \Rightarrow \text{THH}^*(\widehat{EL}(n), KL(n))$$

We have the following isomorphisms:

$$\text{Ext}^{**}_{\widehat{EL}(n), \widehat{EL}(n)}(\widehat{EL}(n)_*, KL(n)_*) \cong \text{Ext}^{**}_{\widehat{EL}(n), \widehat{EL}(n) \otimes \widehat{EL}(n)}(\widehat{EL}(n)_*, KL(n)_*)$$

$$\cong \text{Ext}^{**}_{\widehat{EL}(n), \widehat{EL}(n)}(KL(n)_*, KL(n)_*) \cong HH_{KL(n)}^*(\widehat{EL}(n)_*, KL(n)_*)$$

The first isomorphism follows from the fact that $E(n)_*$-module $\widehat{EL}(n)_* \widehat{EL}(n)$ is flat ([12], Proposition 15.1) and the second and third isomorphisms are obvious.

The isomorphism (5.2) shows that the spectral sequence (5.3) collapses giving the weak equivalence

$$\text{THH}(\widehat{EL}(n), KL(n)) \simeq KL(n)$$

and Proposition 5.4 is proved.

We now describe a particular class of $MU$-algebras $E$ which will be called strongly $KL(n)$-complete and for which the set of multiplicative maps $\widehat{EL}(n) \to E$ has an especially simple form.

**Definition 5.5.** The category $\text{SC}(KL(n))$ of strongly $KL(n)$-complete $MU$-algebras is the smallest subcategory of the homotopy category of $MU$-algebras over $KL(n)$ which contains $KL(n)$ itself and closed under singular extensions with fibre $KL(n)$ and inverse limits of towers of $MU$-algebras over $KL(n)$.

**Remark 5.6.** Note that the $MU_*$-module $\pi_*, KL(n)$ is linearly compact and the tower of linearly compact modules has vanishing $\lim^1$. Since linearly compact modules are closed under inverse limits and singular extensions we conclude that for any $E \in \text{SC}(KL(n))$ the $MU_*$-module $E_*$ is linearly compact. In particular we never have to worry about $\lim^1$-problems. Furthermore it follows that the coefficient rings of strongly $KL(n)$-complete $MU$-algebras are concentrated in even degrees.
Remark 5.7. Examples of strongly $KL(n)$-complete $MU$-algebras include $KL(n)$ and $\widehat{EL}(n)$. To extend the number of examples recall the notion of the Artinian completion of a spectrum due to A.Baker and U.Würgler, cf. [9]. Let $R$ be a commutative $S$-algebra and $I$ be a graded maximal ideal in the coefficient ring of $R$ generated by a regular sequence $u_1, u_2, \ldots$, possibly infinite. Then for an $R$-module $M$ define the Artinian completion $\widehat{M}$ of $M$ as the homotopy inverse limit

$$\widehat{M} := \text{holim}_R R/(u_{i_1}^{i_1}, u_{i_2}^{i_2+1}, \ldots)$$

where the inverse limit is taken over the collections of positive integers $n, i_n, i_{n+1}, \ldots$. In fact Baker and Würgler have a more general definition of the Artinian completion but the one just given is sufficient for our purposes.

Now taking $R = MU(\nu)$ and $M = v_n^{-1}BPL := v_n^{-1}[HL(v_1^\infty, v_2^\infty \ldots)]$ or $M = v_n^{-1}PL(n) := v_n^{-1}BPL/(v_1, v_2, \ldots, v_{n-1})$ we could form Artinian completions $v_n^{-1}BPL$ and $v_n^{-1}PL(n)$. It follows easily that $v_n^{-1}BPL$ and $v_n^{-1}PL(n)$ are strongly $KL(n)$-complete $MU$-algebras.

Let $E$ be a strongly $KL(n)$-complete $MU$-algebra. The canonical $S$-algebra map (which is also an $MU$-algebra map) $E \rightarrow KL(n)$ determines via composition the morphism of topological spaces (unbased)

$$(5.4) \quad F_{S-alg}(EL(n), E) \rightarrow F_{S-alg}(EL(n), KL(n))$$

Proposition 5.8. The map (5.4) is a weak equivalence. In particular $F_{S-alg}(EL(n), E)$ is a homotopically discrete space. Moreover the $S$-module $\text{Der}(EL(n), E)$ is contractible.

Proof. We filter the class $SC(KL(n))$ by subclasses $\{SC_\nu(KL(n))\}$ where $\nu$ is an ordinal. The lowest level $SC_0(KL(n))$ consists of one copy of $KL(n)$. Suppose that the class $\{SC_\nu(KL(n))\}$ has been defined for $\nu < \mu$. We say that an $MU$-algebra $E$ belongs to the class $SC_\mu(KL(n))$ if and only if

1. $E$ weakly equivalent to an inverse limit of the tower of $MU$-algebras over $KL(n)$ with each term belonging to some $SC_\nu(KL(n))$, $\nu < \mu$ or
2. there is a singular extension $KL(n) \rightarrow E \rightarrow F$ where $F \in SC_\nu(KL(n))$ and $\nu < \mu$.

To prove that the map (5.4) is a weak equivalence we will use transfinite induction up the filtration on $SC(KL(n))$. The base case $\nu = 0$ is implied by Propositions 5.2 and 5.3. Suppose that our statement is true for $MU$-algebras belonging to $SC_\nu(KL(n))$ for $\nu < \mu$ and consider $E \in SC_\mu(KL(n))$. If $E$ can be included in the singular extension as in the condition (2) above then applying Theorem 5.1 and Proposition 5.4 we get the desired conclusion.

Finally suppose that $E$ is the inverse limit of the tower $\{E_\ell\}$ where $E_\ell \in SC_{\nu_\ell}(KL(n))$ and $\nu_\ell < \mu$. Applying the functor $F_{S-alg}(\widehat{EL}(n), ?)$ to $\{E_\ell\}$ we get a tower of topological spaces
\[ F_{S-alg}(\hat{EL}(n), E_i) \] By inductive assumption all spaces \( F_{S-alg}(\hat{EL}(n), E_i) \) are weakly equivalent to \( F_{S-alg}(E, KL(n)) \) and the maps in the tower \( \{ F_{S-alg}(\hat{EL}(n), E_i) \} \) respect this equivalence. We conclude that the space \( F_{S-alg}(\hat{EL}(n), E) \) is also weakly equivalent to \( F_{S-alg}(\hat{EL}(n), KL(n)) \). Similar arguments show that \( \text{Der}(\hat{EL}(n), KL(n)) \) is contractible and Proposition 5.8 is proved.

**Remark 5.9.** We see that for any strongly \( KL(n) \)-complete \( MU \)-algebra \( E \) the canonical \( S \)-algebra map \( \hat{EL}(n) \to KL(n) \) lifts uniquely to a map \( \hat{EL}(n) \to E \). Therefore \( E \) is naturally a \((\text{bi})\)module over \( \hat{EL}(n) \).

The following lemma describes the structure of cohomology operations from \( \hat{EL}(n) \) to a strongly \( KL(n) \)-complete \( MU \)-algebra \( E \).

**Lemma 5.10.** Let \( E \) be a strongly \( KL(n) \)-complete \( MU \)-algebra. Then the evaluation map

\[
E^* \hat{EL}(n) \to \text{Hom}_{\hat{EL}(n)*}(\hat{EL}(n)* \hat{EL}(n), E_*)
\]

is an isomorphism.

**Proof.** We have the following natural isomorphism of \( S \)-modules

\[
F_S(\hat{EL}(n), E) \cong F_{\hat{EL}(n)}(\hat{EL}(n) \wedge \hat{EL}(n), E)
\]

Consider further the spectral sequence

\[
\text{Ext}^{**}_{\hat{EL}(n)*}(\hat{EL}(n)* \hat{EL}(n), E_*) \Rightarrow [\hat{EL}(n) \wedge \hat{EL}(n), E]^*_\hat{EL}(n) = [\hat{EL}(n), E]^*
\]

We claim that all higher \( \text{Ext} \) groups vanish so that our spectral sequence reduces to its first column which is \( \text{Ext}^{0*}_{\hat{EL}(n)*}((\hat{EL}(n)*) \hat{EL}(n), E_*) \). (This would clearly give us the statement of the lemma).

To see this first assume that \( E = KL(n) \)

Since the \( \hat{EL}(n)* \)-module \( \hat{EL}(n)* \hat{EL}(n) \) is flat we have by flat base change:

\[
\text{Ext}^{i*}_{\hat{EL}(n)*}(\hat{EL}(n)* \hat{EL}(n), KL(n)*) = \text{Ext}^{i*}_{KL(n)*}(\hat{EL}(n)* \hat{EL}(n) \otimes_{\hat{EL}(n)*} KL(n)*, KL(n)*)
\]

and the last \( \text{Ext} \)-group is zero for \( i > 0 \) since \( KL(n)* \) is injective as a graded module over itself.

Therefore our claim about the vanishing of higher \( \text{Ext} \)-groups in (5.5) is proved for \( E = KL(n) \).

To get the general case we will use transfinite induction as in Proposition 5.8.

Suppose that \( E \) can be included in a singular extension

(5.5) \[ KL(n) \to E \to F \]
where $F \in SC(KL(n))$ and such that $Ext^i_{\hat{E}(n)}(\hat{EL}(n)_*, \hat{EL}(n)_*, F_*) = 0$ for $i > 0$. Since the coefficient rings of spectra $KL(n), E, F$ are even the homotopy long exact sequence corresponding to the cofibre sequence (5.3) is in fact a short exact sequence of coefficient rings:

\[
KL(n)_* \to E_* \to F_*
\]

Associated to (5.6) is a long exact sequence

\[
Ext^1_{\hat{E}(n)}(\hat{EL}(n)_*, \hat{EL}(n)_*, F_*) \to Ext^2_{\hat{E}(n)}(\hat{EL}(n)_*, \hat{EL}(n)_*, KL(n)_*) \to \ldots
\]

from which it follows that $Ext^i_{\hat{E}(n)}(\hat{EL}(n)_*, \hat{EL}(n)_*, E_*) = 0$.

Further let $E$ be the homotopy inverse limit of strongly $KL(n)$-complete $MU$-algebras $E_l$ for which higher $Ext$’s do vanish. Then

\[
Ext^i_{\hat{E}(n)}(\hat{EL}(n)_*, \hat{EL}(n)_*, KL(n)_*) = \lim_{l \to \infty} Ext^i_{\hat{E}(n)}(\hat{EL}(n)_*, \hat{EL}(n)_*, E_l)_* = 0
\]

for $i > 0$. This concludes the proof of Lemma 5.10.

We can now formulate our main

**Theorem 5.11.** The generalized Hopkins-Miller theorem holds for any strongly $KL(n)$-complete $MU$-algebra $E$.

**Proof.** By Proposition 5.8 we know that the space $F_{S-alg}(\hat{EL}(n), E)$ is homotopically discrete. We need to check that the map

\[
\pi_0F_{S-alg}(\hat{EL}(n), E) \to Mult(\hat{EL}(n), E)
\]

is a bijection. Consider the following commutative diagram

\[
\begin{array}{ccc}
\pi_0F_{S-alg}(\hat{EL}(n), E) & \longrightarrow & \pi_0F_{S-alg}(\hat{EL}(n), KL(n)) \\
\downarrow & & \downarrow \\
Mult(\hat{EL}(n), E) & \longrightarrow & Mult(\hat{EL}(n), KL(n))
\end{array}
\]

By Propositions 5.2 and 5.8 the upper horizontal and the right vertical arrows are bijections. It follows that the map (5.7) is injective. So it remains to show that an arbitrary multiplicative
operation $\hat{EL}(n) \to E$ lifts to an $S$-algebra map. Since any such operation determines an $\hat{EL}(n)_*$-algebra map $\hat{EL}(n)_*\hat{EL}(n) \to E_*$ we conclude by Lemma 5.10 that there is an injective map

\begin{equation}
Mult(\hat{EL}(n), E) \to Hom_{\hat{EL}(n)_* - alg}(\hat{EL}(n)_*\hat{EL}(n), E_*)
\end{equation}

We claim that any $\hat{EL}(n)_*$-algebra map $\hat{EL}(n)_*\hat{EL}(n) \to KL(n)_*$ lifts uniquely to a map $\hat{EL}(n)_*\hat{EL}(n) \to E_*$ so that there is an isomorphism

$$Hom_{\hat{EL}(n)_* - alg}(\hat{EL}(n)_*\hat{EL}(n), E_*) \cong Hom_{\hat{EL}(n)_* - alg}(\hat{EL}(n)_*\hat{EL}(n), KL(n)_*).$$

Using inductive arguments as in the previous lemma we see that our claim is equivalent to the vanishing of certain Hochschild cohomology classes in $HH^{**}_{\hat{EL}(n)_*}(\hat{EL}(n)_*\hat{EL}(n), KL(n)_*)$. Flat base change gives an isomorphism

$$HH^{**}_{\hat{EL}(n)_*}(\hat{EL}(n)_*\hat{EL}(n), KL(n)_*)$$

$$= HH^{**}_{KL(n)_*}(\hat{EL}(n)_*\hat{EL}(n) \otimes_{\hat{EL}(n)_*} KL(n)_*, KL(n)_*)$$

$$= HH^{**}_{KL(n)_*}(\hat{EL}(n)_* KL(n), KL(n)_*) = KL(n)_*.$$ 

We see that all higher Hochschild cohomology groups are zero and therefore our claim is proved.

Further we have an obvious change of rings isomorphism

$$Hom_{\hat{EL}(n)_* - alg}(\hat{EL}(n)_*\hat{EL}(n), KL(n)_*) \cong Hom_{KL(n)_* - alg}(\hat{EL}(n)_* KL(n), KL(n)_*)$$

The last term corresponds bijectively by Proposition 5.8 to homotopy classes of $S$-algebra maps from $\hat{EL}(n)$ to $E$. This shows that any multiplicative operation $\hat{EL}(n) \to E$ lifts to an $S$-algebra map and Theorem 5.11 is proved.

**Remark 5.12.** The class of strongly $KL(n)$-complete $MU$-algebras is probably not the most general one for which the conclusion of Theorem 5.11 holds. It is plausible that one has the generalized Hopkins-Miller theorem for any $KL(n)$-local $MU$-algebra but such a result seems out of reach at present.

Tracing the proofs of Proposition 5.2 and Theorem 5.11 we see that they don’t depend on the choice of the $S$-algebra structure on $\hat{E}L(n)$ as long as the ring spectrum structure is fixed. (But they do depend on the choice of the $S$-algebra structure on $E_*$.) This gives the following

**Corollary 5.13.** The spectrum $\hat{EL}(n)$ has a unique (up to a noncanonical isomorphism) structure of an $S$-algebra compatible with its structure of a ring spectrum.
Proof. Let \( \widehat{EL}'(n) \) denote the \( S \)-algebra whose underlying ring spectrum is equivalent to \( \widehat{EL}(n) \) but the \( S \)-algebra structure is possibly different. Then the given multiplicative up to homotopy weak equivalence \( \widehat{EL}'(n) \to \widehat{EL}(n) \) can be lifted to an \( S \)-algebra map which shows that the would-be exotic \( S \)-algebra structure on \( \widehat{EL}'(n) \) is actually isomorphic to the standard one.

Remark 5.14. A similar statement about the uniqueness of the \( A_\infty \) structure on \( \widehat{E}(n) \) was proved by A.Baker in [1]. However the proof in the cited reference was based on the obstruction theory of A.Robinson and it is unclear at this time to what extent this theory carries over in the present context.

Theorem 5.11 allows one to obtain splittings of various strongly \( KL(n) \)-complete \( MU \)-algebras. For example consider \( \widehat{v}^{-1}BP \), the Artinian completion of the \( v \)-localization of the Brown-Peterson spectrum \( BP \). Then \( \widehat{v}^{-1}BP \) is a strongly \( KL(n) \)-complete \( MU \)-algebra obtained from \( KL(n) \) by adjoining the collection of indeterminates corresponding to the regular sequence \( \{p,v_1,v_2,\ldots,v_{n-1},v_n,\ldots\} \). There is a canonical map \( \widehat{v}^{-1}BP \to \widehat{EL}(n) \) which on the level of coefficient ring reduces to killing the ideal \( (v_{n+1},v_{n+2},\ldots) \) in \( \widehat{v}^{-1}BP \). From Theorem 5.11 we conclude that any \( S \)-algebra map \( \widehat{EL}(n) \to \widehat{EL}(n) \) lifts uniquely to an \( S \)-algebra map \( \widehat{EL}(n) \to \widehat{v}^{-1}BP \). In particular the identity map \( id : \widehat{EL}(n) \to \widehat{EL}(n) \) can be so lifted and we obtain

**Theorem 5.15.** The \( S \)-algebra map \( \widehat{v}^{-1}BP \to \widehat{EL}(n) \) admits a unique \( S \)-algebra splitting \( \widehat{EL}(n) \to \widehat{v}^{-1}BP \)

An analogous theorem was proved (in a weaker, up to homotopy form) in [3].

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