Scars in strongly driven Floquet matter: resonance vs emergent conservation laws

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We consider a clean quantum system subject to strong periodic driving. The existence of a dominant energy scale, $h\gamma$, can generate considerable structure in an effective description of a system which, in the absence of the drive, is non-integrable, interacting, and does not host localization. In particular, we uncover points of freezing in the space of drive parameters (frequency and amplitude). At those points, the dynamics is severely constrained due to the emergence of a local conserved quantity, which prevents the system from heating up ergodically, starting from any generic state, even though it delocalizes over an appropriate subspace. At large drive frequencies, where a naive Magnus expansion would predict a vanishing effective (average) drive, we devise instead a strong-drive Magnus expansion in a moving frame. There, the emergent conservation law is reflected in the appearance of an ‘integrability’ of a vanishing effective Hamiltonian. These results hold for a wide variety of Hamiltonians, including the Ising model in a transverse field in any dimension and for any form of Ising interactions. Further, we construct a real-time perturbation theory which captures resonance phenomena where the conservation breaks down giving way to unbounded heating. This opens a window on the low-frequency regime where the Magnus expansion fails.

I. INTRODUCTION

For closed systems with time-independent Hamiltonians, the notion of ergodicity has been formulated at the level of eigenstates in the eigenstate thermalization hypothesis (ETH)\cite{BEE02}. According to ETH, the expectation value of a local observable in a single energy eigenstate of a complex (disorder-free) many-body quantum system is equal to the thermal expectation value of the observable at a temperature corresponding to the energy density of that eigenstate. The implication of the ergodicity hypothesis in the context of time-dependent (‘driven’) closed quantum systems is an open question of fundamental importance.

Relatively recent progress in this line has occurred for systems subjected to a periodic drive (namely, Floquet systems)\cite{KHA16}, which are perhaps conceptually closest to a static system. These studies indicate that a quantum system that satisfies ETH, when subjected to a periodic drive, approaches a state which locally looks like an ‘infinite-temperature’ state. This is in accordance with the ergodicity hypothesis – in systems which satisfy ETH (we will call them generic), energy is the only local conserved quantity, and any time-dependence breaks this conservation, allowing the system to explore the entire Hilbert space.

The breaking down of ETH in interacting systems due to the presence of localized states – either due to disorder (many-body localization)\cite{BEE12} or other mechanisms (like many-body Wannier-Stark localization)\cite{BE13} is well-known within the equilibrium set-up, and their persistence under periodic perturbations has also been observed\cite{RBM98, RBE99}, but the common intuition is that a translationally invariant, interacting, non-integrable many-body system will be ergodic. However, this intuition has encountered a number of remarkable counterexamples recently within the static setting. It has been shown that in such systems there can be highly excited energy eigenstates, dubbed as scars, which do not satisfy ETH\cite{KMA08, RLS08}. Most of these examples (see, however, \cite{DSM14}) indicate the non-trivial (weak) breaking of ergodicity by certain eigenstates.

On the non-equilibrium side, stable Floquet states are seen in finite-size closed interacting Floquet systems which are not localized in the absence of a drive\cite{GHA18, XLH18}. In particular, it has recently been shown that ergodicity is broken in disorder-free generic systems under a periodic drive if the drive strength is greater than a threshold value (compared with the interaction strength) – a KAM like scenario\cite{KSS18}.

The emergence of constraints on dynamics under strong periodic driving is known for non-interacting systems – for strongly driven free fermions, there exist special points in the space of the drive parameters, where freezing for all time are observed for any arbitrary initial state, for any (including infinite) system-size\cite{GHA18, XLH18}. This is surprising since the appropriate description for such a system is a periodic generalized Gibbs’ ensemble\cite{KSS18}. Such an ensemble, though much less ergodic than a thermal one due to presence of an extensive number of (periodically) conserved quantities, leaves ample space for substantial dynamics of the response in general. Hence, in addition to the integrability, other constraints emerge at those special freezing points. Those freezing points can be thought of as “scars” in the space of drive Hamiltonians. Here we uncover and similar scar phenomenology in interacting Floquet systems, and provide an analytical understanding of the phenomenon.

Here we demonstrate that a generic, interacting, translationally invariant Ising system can exhibit non-ergodic...
behavior under a strong periodic drive. For certain isolated sets of values of the drive parameters – the scars on the drive parameter space – a local quasi-conserved quantity (that exhibits only small fluctuations about its initial value with time) emerges. The Floquet Hamiltonian is then no longer ergodic, i.e., its eigenstates (Floquet states) do not look like the otherwise expected infinite temperature states, but instead are characterized by eigenvalues of the quasi-conserved quantity. This is because the dynamics does not mix different eigenstates with different eigenvalues of the quasi-conserved quantity. This however, does not mean that there is no dynamics. Indeed, even at the scar points, we see pronounced dynamics evidenced by substantial growth in sub-system entanglement entropy as delocalization within a sector takes place. A finite size analysis of the numerical results indicates the stability of the scars under an increase in the system size.

At high driving frequencies, the conventional Magnus expansion – controlled in the driving frequency as the largest energy scale – fails, as the average Hamiltonian generally does not exhibit the conservation law in question. To remedy this, we present a strong-drive Magnus expansion, constructed in a ‘moving’ frame incorporating the strong driving term. Here, the conservation law is manifest at low order in the expansion. For a general class of Hamiltonians, including the Ising model in a transverse field in any dimension and any form of the Ising interactions, we find that the effective Hamiltonian vanishes exactly up to two leading orders for our example, capturing the freezing (observed from exact numerics) to a good approximation away from the resonances. This suggests that the expansion is either convergent or asymptotic, in this setting.

For lower drive frequencies, no controlled approximation scheme for Floquet systems is available. Here, we formulate a novel perturbation theory, Floquet-Dyson perturbation theory (FDPT), which again uses the fact that the drive amplitude is large. This we find works best in the low-frequency regime, where we benchmark it for simple systems against an exact solution, and against exact numerics. This enables us to account for isolated first-order resonances, which are of particular interest as their sparseness implies stable non-thermal states to first order. The stability is maintained in the thermodynamic limit if our perturbation expansion is an asymptotic one, which is indicated by the finite-size analysis of our numerical results – the freezing is insensitive to increase in system-size (see the finite-size result App. A). In particular, the FDPT is remarkably accurate in predicting the resonances (obtained from exact numerics) close to integrability, and hence at the scars. This opens up a recipe to construct stable Floquet state with desired properties by choosing suitable drive terms.

We organize this paper as follows. After a brief introduction to Floquet and our notation, we first present the phenomenology of scarring. We then present the high-field Magnus expansion, and finally the FDPT. We conclude with a summary and outlook.

II. FLOQUET IN A NUTSHELL

The Floquet states $|\mu_n\rangle$ are elements of a complete orthonormal set of eigenstates of the time-evolution operator $U(T,0)$ for time evolution from $t = 0$ to $t = T$, for a system governed by a time-periodic Hamiltonian with a period $T = 2\pi/\omega$. The Floquet formalism is particularly useful for following the dynamics stroboscopically at discrete time instants $t = nT$. From the above definition it follows that

$$U(T,0)|\mu_n\rangle = e^{-i\mu_n T}|\mu_n\rangle,$$

where the $\mu_n$’s are real. It is customary to define an effective Floquet Hamiltonian $H_{eff}$ as

$$U(T,0) = e^{-iH_{eff} T}.$$

(We will set $\hbar = 1$ in this paper). When observed stroboscopically at times $t = nT$, the dynamics can be thought of as being governed by the time-independent Hamiltonian $H_{eff}$, which has eigenvalues $\mu_n/T$ (modulo integer multiples of $2\pi/T$) and eigenvectors $|\mu_n\rangle$. In the infinite time limit, the expectation values of a local operator $\mathcal{O}$ can be written in terms of the expectation values in the Floquet eigenstates as

$$\lim_{N \to \infty} \langle \psi(NT)|\mathcal{O}|\psi(NT)\rangle = \sum_n |c_n|^2 \langle \mu_n | \mathcal{O} | \mu_n \rangle = \mathcal{O}_{DE}.$$

(3)

where $|\psi(0)\rangle = \sum_n c_n |\mu_n\rangle$, and the subscript “DE” denotes the Diagonal Ensemble average defined as above. This is equivalent to taking a “classical” average over the properties of the Floquet eigenstates $\{|\mu_n\rangle\}$. The diagonal ensemble average of $m^x$, given by

$$m^x_{DE} = \sum_n |c_n|^2 \langle \mu_n | m^x | \mu_n \rangle.$$

(4)

The absence of interference between the Floquet states in a DE average ensures that it is sufficient to study the properties of individual Floquet states (and their spectrum average) in order to characterize the gross behavior of the driven system in the infinite time limit. In the following we will therefore mostly concentrate on DE averages and the properties of the Floquet states.

III. THE SCAR PHENOMENOLOGY

A. Freezing and Quasi-Conservation

This section discusses the scar phenomenology for a periodically driven, interacting, non-integrable Ising chain
described by
\[
H(t) = H_0(t) + V, \quad \text{where}
\]
\[
H_0(t) = H_0^x + \text{sgn}(\sin(\omega t)) \cdot H_D, \quad \text{with}
\]
\[
H_0^x = - \sum_{n=1}^{L} J \sigma_n^x \sigma_{n+1}^x + \sum_{n=1}^{L} \kappa \sigma_n^x \sigma_{n+2}^x - h_0^x \sum_{n=1}^{L} \sigma_n^x,
\]
\[
H_D = - h_D^x \sum_{n=1}^{L} \sigma_n^x,
\]
\[
V = - h^z \sum_{n=1}^{L} \sigma_n^z,
\]
where \(\sigma_n^{x/y/z}\) are the Pauli matrices.

The main result is that at large drive amplitude \(h_D^x\), the longitudinal magnetization
\[
m^x = \frac{1}{L} \sum_{i} \sigma_i^x
\]
emerges as a quasi-conserved quantity under the drive condition (‘scar points’ in the drive parameter space) given by
\[
h_D^x = k \omega,
\]
where \(k\) are integers. Fig. 1 (a), main frame, shows that at those scar points (marked with arrows), the diagonal ensemble average \(m^x(0)\) is equal to its initial value \(m^x(0)\), to very high accuracy, indicating that \(m^x\) remains frozen at its initial value for arbitrarily long times. As seen from the figure, this happens for a very broad range of \(\omega\). The scar/freezing appears above a sharp threshold value of \(h_D^x\) (Fig. 1 (a), inset). The phenomenon is reminiscent of the non-monotonic peak-valley structure of freezing observed in integrable Floquet systems in the thermodynamic limit.

The figure shows that freezing happens for two very different kind of initial states, namely, the highly polarized initial ground state of \(H(0)\) as well as a high-temperature thermal state. The initial thermal density matrix we chose is of the form
\[
\rho_{Th}(t = 0) = \frac{2^L}{Z} \sum_{j} e^{-\beta \varepsilon_j} |\varepsilon_j>(\varepsilon_j),
\]
where |\(\varepsilon_j\)\rangle is the \(j\)-th eigenstate of an initial Hamiltonian \(H_I\), with \(\varepsilon_j\). We have chosen \(H_I = H(t = 0, h_D^x = 5.0, h_0^x = 0.1, J = 1, \kappa = 0.7)\), with \(H(t)\) from (Eq. [5]), and \(Z = \sum_{j} e^{-\beta \varepsilon_j}\) is the partition function. This is a mixture of eigenstates |\(\varepsilon\)\rangle. Hence we obtain the final diagonal ensemble density matrix by taking the diagonal ensemble density matrix for each |\(\varepsilon\)\rangle, weighted by its
Boltzmann weight in $\rho_{Th}(0)$, i.e.,
\[
\rho_{DE}(t \to \infty) = \sum_j e^{-\beta \epsilon_j} \left( \sum_k |\langle \epsilon_j | \mu_k \rangle|^2 |\mu_k \rangle \langle \mu_k | \right)
\]
\[
= \sum_k \left( \sum_j e^{-\beta \epsilon_j} |\langle \epsilon_j | \mu_k \rangle|^2 \right) |\mu_k \rangle \langle \mu_k |
\]

The quasi-conservation of $m^x$ for a generic thermal state suggests that all the Floquet states must be organized according to the emergent conservation law. This is shown to be true in Fig. 1(b), which shows the expectation value $\langle m^x \rangle$ in the Floquet eigenstates (corresponding to the drive in Fig. 1(b)), plotted against their serial number (normalized by the dimension $D_H$ of the Hilbert space), arranged in decreasing order of their $\langle m^x \rangle$ values.

For the scar points given by $h_D^\beta = 40$ and $\omega = 10, 20, 40$, the values of $\langle m^x \rangle$ of the Floquet states coincide with the eigenvalues of $m^x$, indicating that all the eigenstates of $m^x$ which participate in constituting a given Floquet state have the same $m^x$ eigenvalues. This explains conservation/freezing of $m^x$ for dynamics starting with any generic initial state. As we will see later, the condition of the scar (Eq. 7) can be deduced both from the FDPT and a Magnus expansion in a time-dependent frame, and the latter confirms the effect over the entire spectrum and explains the steps to the leading orders.

IV. STRONG-DRIVE MAGNUS EXPANSION

We next provide a modified Magnus expansion which incorporates the large size of the drive from the start, using the (inverse of the) driving field as a small parameter. This makes the emergence of a conserved quantity manifest, for a wide range of Hamiltonians – the terms in the time-independent part of the Hamiltonian that commutes with the time-dependent part of the Hamiltonian ($H_0^\beta$ here) can have any form. This is because the factor pre-multiplying the terms involving it, vanishes to second order regardless of its form. For example, it applies to transverse field Ising models in any dimension, with any Ising interaction. From this, one can immediately read off the scars found above.

The conventional Magnus expansion uses the inverse of a large frequency as a small parameter (see, e.g. 23) for obtaining the Floquet Hamiltonian $H_{eff}$ (Eq. 2) as given below.

\[
H_{eff} = \sum_{n=0}^\infty H_F^{(n)}, \quad \text{where}
\]
\[
H_F^{(0)} = \frac{1}{T} \int_0^T dt \, H(t),
\]
\[
H_F^{(1)} = \frac{1}{2T} \int_0^T dt_1 \int_0^{t_1} dt_2 [H(t_1), H(t_2)], \quad (11)
\]

and so on. In our case, we have $h_D^\beta > \omega$, making the series non-convergent even when $\omega$ is greater than all other couplings in the Hamiltonian, so that it is qualitatively wrong even at leading order: the first-order term $H^{(0)}$ is the time average over one period of $H(t)$ (Eq. 5), an interacting generic Hamiltonian which does not conserve $m^x$. Hence we would have no hint of the scars from even the first-order term.

This problem can be remedied when the strong drive is constituted of modulating the strength of a fixed field/potential (the most natural way of employing a periodic drive). The largest coupling ($h_D^\beta$ here) can be eliminated from the Hamiltonian by switching to a time-dependent frame as follows23. We introduce a unitary transformation

\[
|\psi_{mov}(t)\rangle = W(t)|\psi(t)\rangle,
\]
\[
\hat{O}_{mov} = W(t)\hat{O}W(t)^\dagger, \quad (12)
\]

where $|\psi(t)\rangle$ is the wave function and $\hat{O}$ is any predefined
FIG. 2. (De)localization of the wave function over the $x$-basis (simultaneous eigenstates of all of the $\sigma^x_i$’s) as evidenced by the half-chain entanglement entropy ($E_{1/2}$) vs system size $L$, for different driving strengths $h_D'$ (rows) and initial states (columns: left maximally $m^x$ polarized; middle: $L/2$-domain-pair state with vanishing total $m^z$; right: Néel state). Top row (small $h_D' = 5$): $E_{1/2}$ entropy grows linearly with system size for all initial states, signaling ergodicity. For stronger drives ($h_D' = 20, 40$ in middle, bottom row, respectively), scars appear, and $E_{1/2}$ depends strongly on the initial states, reflecting the size of the emergent magnetization sectors: for the fully polarized initial states (left column), $E_{1/2}$ does not grow at all for the freezing/scar points (marked as (F) in the figure legends and represented by almost indistinguishably coincidental black and violet triangles), while for the Néel and the $L/2$-domain-pair initial states, there is considerable growth in $E_{1/2}$ even at the scar points, reflecting (at least partial) delocalization over the large concomitant magnetization sectors. The results are for $J = 1, \kappa = 0.7, h_0^x = e/10, L = 14$, averaged over $10^4$ cycles after driving for $10^{10}$ cycles.

operator (the subscript $\text{mov}$ marks the quantities in the moving frame).

The crux of the expansion is then evident for a $W(t)$ of the following form,

$$W(t) = \exp \left[ -i \int_0^t dt' H_D \times r(t') \right], \tag{13}$$

where $r(t)$ is $T$-periodic parameter. If the total Hamiltonian were constant up to the time-dependent prefactor $r(t)$, i.e. $H(t) = r(t) \times H(0)$, the above would just give the solution of the static Schrödinger equation, but with a rate of phase accumulation for each (time-independent) eigenstate given by the integrand of the variable prefactor. In particular, any conservation law of $H(0)$ would be bequeathed to the time-dependent problem. Now, if the drive is not the only, but still the dominant part, of the Hamiltonian, there will be corrections to this picture, but it suggests the eigenbasis of the drive and its conservation law(s) should remain perturbatively useful starting points.

Given the form of $H_D(t)$ in Eq. 5 – the transformed Hamiltonian reads

$$H_{\text{mov}} = W(t)^\dagger H(t) W(t) - i W(t)^\dagger \partial_t W, \tag{14}$$

where the second term exactly cancels the part from the
first term which has $h_D^2$ as its coupling, and hence $H_{\text{mov}}$ is free from any coupling of order $h_D^2$ (see App. C for details).

A. Scars in the Driven Interacting Ising Chain

In the case of Eq. (3), we have

$$H(t) = H_0^x + V - \text{sgn}(\sin(\omega t))h_D^x \sum_i \sigma_i^z.$$  \hspace{1cm} (15)

Switching to the moving frame employing the transformation in Eq. (13) gives

$$H_{\text{mov}} = H_0^x - h^x \sum_i [\cos(2\theta)\sigma_i^x + \sin(2\theta)\sigma_i^z], \text{ where}$$

$$\theta(t) = -h_D^x \int_0^t dt' \text{sgn}(\sin(\omega t')).$$  \hspace{1cm} (16)

After some algebra, we find the Magnus expansion of $H_{\text{mov}}$ to have the following leading terms:

$$H_F^{(0)} = H_0^x - \frac{h^x}{2h_D^x T} \left[ \sin(2h_D^x T) \sum_i \sigma_i^z \right.$$

$$+ (1 + \cos(2h_D^x T) - 2\cos(h_D^x T)) \sum_i \sigma_i^y \bigg].$$  \hspace{1cm} (17)

Note that this is useful for $h_D^x \gg 1/T$, the regime we are interested in. The next-order term is yet more complex (see App. IV for a derivation of each term):

$$H_F^{(1)} = \frac{1}{2T} (\Sigma_1 + \Sigma_2 + \Sigma_3),$$  \hspace{1cm} (18)

where the $\Sigma_i$'s are obtained by integrating the $K_i$'s, and are given by

$$\Sigma_1 = -h^x [H_0^x, S_z] \left\{ \frac{1}{4(h_D^x)^2} (2\cos(2h_D^x T) - \cos(h_D^x T) - 1)$$

$$+ \frac{3T}{h_D^x} \sin(h_D^x T) + \frac{T}{4h_D^x} \sin(h_D^x T) \right\}. \hspace{1cm} (19)$$

$$\Sigma_2 = -h^x [H_0^x, S_y] \left\{ \frac{1}{2(h_D^x)^2} + \frac{T^2}{4} \right\} \sin(h_D^x T) + \frac{T}{4h_D^x} [1 - \cos(h_D^x T)] \right\}. \hspace{1cm} (19)$$

V. FLOQUET-DYSON PERTURBATION THEORY

Here we develop a theory which opens up a window on the otherwise difficult-to-access low-frequency regime. We first test it for an exactly soluble problem, and then apply it to the Ising chain studied in the previous section.

We find the theory provide valuable insights for both systems. In particular, it identifies a resonance condition corresponding to the dips, as well as a freezing condition corresponding to the maxima in the response plotted in Figs. 2 and 4 respectively. A coincidence of the two accounts for the varying dip depths in that figure. While a comprehensive treatment of the many-body problem is not yet possible, we believe that these items capture ingredients central for its understanding.

We first present the general formulation of the FDPT. The goal is to construct the Floquet states $|\mu_n\rangle$. The central ingredient is that the driven Hamiltonian

$$H(t) = H_0(t) + V.$$  \hspace{1cm} (20)

consists of a large time dependent term with a time-independent set of eigenvalues. This is appropriate for the case of a strong driving field. The theory then treats a small perturbation $V$ which is time-independent.
We thus work in the basis of eigenstates of \( H_0(t) \), denoted as \(|n\rangle\), so that
\[
H_0(t)|n\rangle = E_n(t)|n\rangle,
\] (21)
and \( \langle m|n \rangle = \delta_{mn} \).

Next, we assume without loss of generality that \( V \) is completely off-diagonal in this basis, namely,
\[
\langle n|V|n \rangle = 0
\]
(22)
for all \( n \). We will now find solutions of the time-dependent Schrödinger equation
\[
\frac{i}{\hbar}\frac{\partial|\psi_n(t)\rangle}{\partial t} = H(t)|\psi_n(t)\rangle
\]
(23)
which satisfy
\[
|\psi_n(T)\rangle = e^{-i\mu_n} |\psi_n(0)\rangle.
\]
(24)

For \( V = 0 \), each eigenstate \(|n\rangle\) of \( H_0(t) \) is a Floquet state, with Floquet quasienergy \( \mu_n^{(0)} = \int_0^T dt E_n(t) \) (unique up to the addition of \( 2\pi p \), where \( p \) is an integer).

For \( V \) non-zero but small, we develop a Dyson series for the wave function to first order in \( V \). The drive amplitude \( h_D^x \) is the largest scale in \( H(t) \), and hence when we say \( V \) is small, we mean \( |V/h_D^x| \ll 1 \). \( V \) can otherwise be comparable to the other couplings of the undriven Hamiltonian. In our ansatz, the \( n \)-th eigenstate is written as
\[
|\psi_n(t)\rangle = \sum_m c_m(t) e^{-i \int_0^t dt' E_m(t')} |m\rangle,
\]
(25)
where \( c_m(t) \approx 1 \) for all \( t \) while \( c_m(t) \) is of order \( V \) (and therefore small) for all \( m \neq n \) and all \( t \).

We find (for details of the algebra, see App. C)
\[
c_m(0) = -i \langle m|V|n\rangle \frac{\int_0^T dt e^{i \int_0^t dt' [E_m(t') - E_n(t')]} - 1}{e^{i \int_0^t dt' [E_m(t') - E_n(t')]} - 1}.
\](26)

We see that \( c_m(t) \) is indeed of order \( V \) provided that the denominator on the right hand side of Eq. (25) does not vanish; we will call this case non-degenerate. If
\[
e^{i \int_0^T dt' [E_m(t') - E_n(t')]} = 1,
\]
(27)
we have a resonance between states \(|m\rangle\) and \(|n\rangle\), and the above analysis breaks down. Now, if there are several states which are connected to \(|n\rangle\) by the perturbation \( V \), Eq. (26) describes the amplitude to go to each of them from \(|n\rangle\). Up to order \( V^2 \), the total probability of excitation away from \(|n\rangle\) is given by \( \sum_{m \neq n} |c_m(0)|^2 \) at time \( t = 0 \).

A. Single Large Spin: An Exactly Soluble Test-bed

As an illustration of the FDPT, we discuss a system with a single spin governed by a time-dependent Hamiltonian. We will briefly discuss some results obtained from the FDPT (which give the conditions for perfect freezing and resonances), numerical results, and exact results for the Floquet operator. The details are presented in App. D.

Model: We consider a single spin \( \hat{S} \), with \( S^2 = S(S+1) \), which is governed by a Hamiltonian of the form
\[
H(t) = -\hbar^x S^x - \hbar^z S^z - \hbar_D \text{sgn}(\sin(\omega t)) S^x.
\]
(28)
The time period is \( T = 2\pi/\omega \). Since \( \sin(\omega t) \) is positive for \( 0 < t < T/2 \) and negative for \( T/2 < t < T \), the Floquet operator is given by
\[
U = e^{i T/2} [(\hbar^x - h_D^z) S^x + \hbar^z S^z]
\times e^{i T/2} [(\hbar^x + h_D^z) S^x + \hbar^z S^z].
\]
(29)
It is clear from the group properties of matrices of the form \( e^{i T^d \hat{S}} \), that \( U \) in Eq. (29) must be of the same form and can be written as
\[
U = e^{i \hat{k} \cdot \hat{S}},
\]
where \( \hat{k} = (\cos \theta \sin \phi, \sin \theta \cos \phi, \sin \theta \sin \phi) \).
(30)

We will work in the basis in which \( S^z \) is diagonal. Since the eigenstates of \( U \) in Eq. (29) are the same as the eigenstates of the matrix \( M = \hat{k} \cdot \hat{S} \), the expectation values of \( S^x \) in the different eigenstates take the values \( \cos \theta \) times \( S, S - 1, \ldots, -S \). The maximum expectation value is given by \( m_{\text{max}}^z = S \cos \theta \).

Analytical results from FDPT: We can use the FDPT to derive the correction to \( m_{\text{max}}^x \) to first order in the small parameter \( \hbar^x / h_D^z \). Namely, we find how the state given by \(|0\rangle \equiv |S^z = S\rangle \) mixes with the state \(|1\rangle \equiv |S^z = -1\rangle \). We discover that
\[
c_1(0) = \frac{\sqrt{2S}}{h_D^z} \frac{e^{i \hbar^x T/2} [e^{i h_D^x T/2} - \cos(\hbar^x T/2)]}{e^{i \hbar^x T} - 1},
\] (31)
Three possibilities arise at this stage.
(i) The denominator of Eq. (31) is not zero. Then the expectation value of \( S^x \) in this state will be close to \( S \) since \( \hbar^x / h_D^z \) is small. In addition, if the numerator of Eq. (31) vanishes, we get perfect freezing, namely, \( \langle S^x \rangle = S \).
(ii) The denominator of Eq. (31) vanishes, i.e., \( \hbar^x \) is an integer multiple of \( 2\pi / T \), but the numerator does not vanish. This is called the resonance condition. Clearly, the perturbative result for \( c_1(0) \) breaks down in this case, and we have to either develop a degenerate perturbation theory or do an exact calculation.
(iii) Both the numerator and the denominator of Eq. (31) vanish. Once again the perturbative result breaks down and we have to do a more careful calculation.

We would like to make a comment on the dependence of the result in Eq. (31) on the value of \( S \). At \( t = 0 \), the probability of state \(|1\rangle \) is \( |c_1(0)|^2 \) and the probability of
state \( |0 \rangle \) is \( 1 - |c_1(0)|^2 \). Hence the expectation value of \( S^z / S \) is given by

\[
\frac{m_{\text{max}}^x}{S} = \frac{1}{S} \left[ S \left( 1 - |c_1(0)|^2 \right) + (S - 1) |c_1(0)|^2 \right]
\]

\[
= 1 - 2 \left( \frac{h^x}{h_D^x} \right)^2 \times \frac{1 + \cos^2 (h^z T/2) - 2 \cos (h^z T/2) \cos (h_D^x T/2)}{4 \sin^2 (h^z T/2)}.
\]

(32)

We expect Eq. (31) to break down at a sufficiently large value of \( S \) since it was derived using first-order perturbation theory which is accurate only if \( |c_1(0)| \ll 1 \). However, we observe that the value of \( m_{\text{max}}^x / S \) in Eq. (32) is independent of \( S \). We therefore have the striking result in this model that we can use first-order perturbation theory for values of \( S \) which are not large to derive an expression like Eq. (32) which is then found to hold for arbitrarily large values of \( S \).

**Numerical results:** Given the values of the parameters \( S, T, h^x, h^z \), and \( h_D^z \), we can numerically compute \( U \) and its eigenstates. From the eigenstates, we can calculate \( m_{\text{max}}^x \) which is the maximum value of the expectation value of \( \langle S^x \rangle \). In Fig. 3, we plot \( m_{\text{max}}^x \) versus \( h^x \), for \( S = 20 \), \( T = 10 \), \( h^z = 1 \), and (a) \( h_D^z = 40 \) and (b) \( h_D^z = 12.8 \pi \approx 40.212 \). In Fig. 3 (a), we see large dips for \( h^x \) equal to all integer multiples of \( 2 \pi / T \). In Fig. 3 (b), we see large dips for \( h^x \) equal to odd integer multiples of \( 2 \pi / T \), but the dips are much smaller for \( h^x \) equal to even integer multiples of \( 2 \pi / T \).

We can understand these results using the FDPT. In Fig. 3 (a), we have \( h_D^z = 40 \); hence \( \cos(h_D^z T/2) \approx 1 \), and the numerator of Eq. (31) can never vanish. We therefore obtain large dips for \( h^x \) equal to all integer multiples of \( 2 \pi / T \) where the denominator of Eq. (31) vanishes (case (ii)). However, in Fig. 3 (b), \( h_D^z = 12.8 \pi \) so that \( \cos(h_D^z T/2) = 1 \). Hence the numerator of Eq. (31) vanish when \( h^x \) is equal to even integer multiples of \( 2 \pi / T \) (case (iii)). This explains why the dips in \( m_{\text{max}}^x \) are much smaller for \( h^x \) equal to even integer multiples of \( 2 \pi / T \), but they continue to be large for \( h^x \) equal to odd integer multiples of \( 2 \pi / T \).

**Form of the Floquet operator in different cases:** We now present expressions for the Floquet operator \( U \) in Eq. (30) based on the exact results derived in App. D 1a. The purpose of this exercise is to show that the form of \( U \) is quite different in cases (i-iii).

Assuming that \( h_D^z \) is positive and much larger than \( |h^x| \) and \( |h^z| \), we find, to zero-th order in \( h^x / h_D^z \), that

\[
\cos \left( \frac{\gamma}{2} \right) = \cos \left( \frac{h^x T}{2} \right), \quad \text{and} \quad \hat{k} = \hat{x}.
\]

(33)

provided that \( e^{i h^x T} \neq 1 \) (case (i)). Eq. (33) implies that the Floquet operator corresponds to a rotation about the \( \hat{x} \) axis by an angle \( \gamma \).

If \( e^{i h^x T} = 1 \), i.e., \( \cos(h^z T/2) = \pm 1 \), but \( \cos(h^z T/2) \neq e^{i h_D^z T/2} \), the denominator of Eq. (31) vanishes but the numerator does not (case (ii), called the resonance condition). It turns out that we then have to expand up to second order in \( h^x / h_D^z \). This gives

\[
\hat{k} = \cos \left( \frac{h_D^x T}{4} \right) \hat{x} - \sin \left( \frac{h_D^x T}{4} \right) \hat{y},
\]

\[
\text{if } \cos \left( \frac{h_D^x T}{2} \right) = 1,
\]

\[
= \sin \left( \frac{h_D^x T}{4} \right) \hat{x} + \cos \left( \frac{h_D^x T}{4} \right) \hat{y},
\]

\[
\text{if } \cos \left( \frac{h_D^x T}{2} \right) = -1.
\]

(34)

This implies that the Floquet operator corresponds to a rotation about an axis lying in the \( y - z \) plane. This implies that the expectation value of \( S^y \) will be zero in all the eigenstates of the Floquet operator.

Finally, if \( e^{i h^x T} = 1 \) and \( \cos(h^z T/2) = e^{i h_D^z T/2} \), both the numerator and denominator of Eq. (31) vanish (case (iii)). We then discover that

\[
\hat{k} = \frac{h^x \hat{x} - h^z \hat{z}}{\sqrt{(h^x)^2 + (h^z)^2}},
\]

(35)

Hence, the Floquet operator corresponds to a rotation about an axis lying in the \( x - z \) plane.

To summarize, assuming that \( h^x / h_D^z \) is small, we obtain quite different results depending on which of the three cases (i-iii) arise. We see these differences both in the numerical results for \( m_{\text{max}}^x \) shown in Fig. 3 and in the forms of the Floquet operator in Eqs. (33) and (35) which are obtained by an exact calculation.

![Fig. 3](image-url)

FIG. 3. Plots of the maximum expectation value of \( S^x \) versus \( h^x \), for \( S = 20 \), \( T = 10 \), \( h^z = 1 \), and (a) \( h_D^z = 40 \) and (b) \( h_D^z = 12.8 \pi \approx 40.212 \). In figure (a) we see pronounced dips for \( h^x \) equal to all integer multiples of \( 2 \pi / T \), while in figure (b) we see pronounced dips only when \( h^x \) is equal to odd integer multiples of \( 2 \pi / T \), as predicted by the FDPT result, Eq. (31).

**B. FDPT for the Interacting Ising Chain**

Now we apply FDPT to our interacting Ising chain (Eq. 5) studied numerically above. We set \( h^x \ll h_D^z \),
FIG. 4. Freezing and resonances in the magnetization ratio $m_x^e/m_0^z$ versus drive strength $h_x^z$. Observable, initial states at zero (panels a, b) and high temperature (inverse temperature $\beta = 10^{-2}$) (panel c) and drive parameters as described in Fig. 1 a). Results shown for slow (a: $\omega = 0.4$) and very slow (b, c: $\omega = 0.04$) drives (green line-points). The resonances obtained from first-order FDPT, Eq. (40), (purple vertical lines) show a remarkable match with the numerical values of dips in $m_x^e$. (Some higher order resonances are also visible at $\omega = 0.4$ in frame (a)).

and treat $V$ as the perturbation. We use periodic boundary conditions.

The eigenstates $|n\rangle$ of $H_0(t)$ are diagonal in the basis of the operators $\sigma_n^x$. In particular, the state in which all spins $\sigma_n^x = +1$, will be denoted as $|0\rangle$, and we start by calculating the Floquet state $|m_{\text{max}}^z\rangle$ (maximally polarized Floquet state) obtained by perturbing this state to first order in $h_z^z/h_x^z$. While calculating $m_x^e$ from the perturbation theory we use this Floquet state.

The rationale for this is as follows. Firstly, if we start with a fully polarized state in the $+x$ direction (as is done, for example, in the experiments by Monroe(41), or, with the ground state of $H(0)$, with $h_x^z \gg h_z^z$, $\kappa$, then the initial state is expected to have a strong overlap with this particular Floquet state; hence at very long times, the expectation values of the observables in the wave function will be well approximated by the expectation value over this Floquet state.

Secondly, in this setting, the insights from the single-spin problem studied above are most directly transferable - in particular, we again encounter the ideas of resonances and scars. With these in hand, we can then identify a number of features present in the data more generally, in particular for high-temperature states (which are in turn of interest in the context of the NMR experiments by Rovny(42)). We find that the perturbation theory works best in the vicinity of the scars with their emergent integrability (see below), and present a limited exploration of the performance of FDPT away from these in App. D.

For the expansion of the Floquet state to leading order, the computation proceeds entirely along the lines of that presented for the single spin. We denote the state in which all spins $\sigma_n^x = +1$ except for the site $m$ where $\sigma_m^x = -1$ as $|m\rangle$. In the limit in which $h_x^z$ is much larger than $J$, $\kappa$ and $h_0^z$, we find that, to leading order in $h_z^z/h_x^z$, Eq. (D24) takes the form

$$c_m(0) \approx \frac{h_z^z e^{iAT/2} [e^{ih_x^z T} - \cos(AT/2)]}{h_x^z e^{iAT} - 1}, \quad A = 4(J - \kappa) + 2h_0^z. \quad (36)$$

The magnetization of this maximally polarized Floquet state is given as follows. The expectation value of $\sum_{n=1}^L \sigma_n^x$ in each of the $m$ states is $L - 2$ and in the state $|0\rangle$ is $L$. This gives

$$m_x^e = 1 - 2 \frac{4}{L} \sum_{m=1}^L |c_m(0)|^2. \quad (37)$$

1. Resonances and stability of the scars

The resonance condition, Eq. (27), (36),

$$e^{iAT} = 1 \quad \text{where} \quad A = 4(J - \kappa) + 2h_0^z, \quad (38)$$
signals the singularities of our expansion, where $c_m(0)$ naively diverges. For our Hamiltonian this occurs for

$$h_0^x = -2J + 2\kappa + \frac{p\omega}{2}, \quad (39)$$

where $p$ is an integer.

This suggests considering all possible first-order resonances based on Eq. (27), by considering the resonance condition more generally; evaluating the change $E_m - E_n$ due to the flip of only a single spin, $\sigma_0$, with $n$-th nearest neighbor spins on the right/left denoted by $\sigma_{\pm n}$ yields the first-order resonance condition

$$h_0^x \sigma_0 + J\sigma_0(\sigma_{-1} + \sigma_1) - \kappa\sigma_0(\sigma_{-2} + \sigma_2) = \frac{p\omega}{2}. \quad (40)$$

Here $p \in \mathbb{Z}$ denotes the number of photons involved in the resonance. Of course, individual resonances may be absent if there are no (net) matrix elements between the states in question.
This approach can be rather successful at identifying the location of the numerically observed isolated resonances, as displayed in Fig. 4. There, the strength of the freezing is displayed as a function of driving strength, for both slow, and very slow, drives, $\omega = 0.4, 0.04$, respectively.

The right panel of Fig. 4 emphasizes the generality of this result: the considerations of the first-order resonances obtained above yield the response even for the initially weakly-polarized ($m^x = 0.05$) high-temperature initial state.

Considering the expression for the magnetization, obtained from substituting the expression for $c_m(0)$ (Eq. (36) into the expression of $m^x$ (Eq. (37)),

$$1 - m^x = 2 \left( \frac{h^z}{h^z_D} \right)^2 \times$$

$$1 + cos^2(\frac{AT}{2}) - 2 \cos(\frac{AT}{2}) \cos(h^z_D T) \frac{\sin^2(\frac{AT}{2})}{4 \sin^2(\frac{AT}{2})},$$

we would like to make the following observations.

Firstly, Eq. (41) indicates that $m^x$ should keep oscillating with $h^z_D$ with a period $\omega$ (except when $\cos(\omega T/2)$ is close to zero), as indeed observed in Fig. 5. Notice, therefore, that the 'high-field limit' is not entirely simple but still endowed with a fine-structured periodicity.

Secondly, when $\omega = 2\pi/T$ is large, we can approximate $\cos(\omega T/2) \simeq 1 - (\omega T)^2/8$ and $\sin(\omega T/2) \simeq \omega T/2$ in Eq. (41):

$$1 - m^x = 2 \left( \frac{h^z}{h^z_D} \right)^2 \frac{4(1 - A^2 T^2/8) \sin^2(h^z_D T/2)}{A^2 T^2}.$$ 

This shows that freezing becomes weaker with increasing $\omega$. An exception to this occurs when the numerator in Eq. (42) vanishes, namely, when $\omega = h^z_D/k$, where $k$ is an integer. At these points, we have $m^x/m^x(0) = 1$, i.e., perfect freezing. Those are precisely the 'scar' points given by Eq. (7), where the peaks of freezing are obtained numerically (Fig. 1).

As encountered in the single spin example, there is an interesting interplay between the scars – where no dynamics takes place – and the resonances, where heating is hugely amplified. When the two coincide, this can destroy the inertness of the scar point. This is manifested as sharp dips in $m^x_{DE}$ in the numerical results discussed above, and for intermediate values of $h^z_D$ in the inset of Fig. 1(a). The FDPT predicts isolated resonances in parameter space and provides a guideline for choosing the Hamiltonian parameters to avoid resonances and observe stable scars. Our choice of parameters for Fig. 1 is guided by the theory (Eq. (40)), and we indeed observe resonance-free strong freezing at the scar points.

VI. CONCLUSIONS AND OUTLOOK

In conclusion, we have demonstrated that generic interacting Floquet systems subjected to a strong periodic drive can exhibit scars, i.e., points in the drive parameter space at which the system becomes non-ergodic due to the emergence of constraints in the form of a quasi-conservation law. This is captured by our strong-field Magnus expansion in a time-dependent frame. For low drive frequencies, we formulate a novel perturbation theory (Floquet-Dyson perturbation theory) which works, even at first order, very accurately at or near integrability of the scar points. In particular, the resonances predicted by the theory accurately coincide with the sharp dips in the quasi-conserved quantity. At the resonances, the system absorbs energy without bound from the drive, and hence a scar ‘competes’ with the resonance. The resonances predicted by the theory appear to be isolated in parameter space, and thus, the theory provides a guideline for choosing parameters for observing resonance-free stable scars, as we demonstrate here. These results hold in particular for Ising systems in any dimension and with any form of the Ising interactions.

Our work also touches on various Floquet experiments. In the original experimental work on Floquet many-body localization, the interest of a large drive was already noted. In the context of the studies of Floquet time crystals, the two kinds of states studied above have also played a central role: the trapped ion experiment used a fully polarized starting state, while the NMR experiment employed a high temperature state.

Our work points towards the important role in non-equilibrium settings played by the generation of emergent conservation laws and constraints, in contrast to only focusing on those existing in the static (undriven) system, and their demise under an external drive. Our work also opens a door for stable Floquet engineering in interacting systems, and indicates a recipe for tailoring interesting states and structured Hilbert spaces by choosing suitable drive Hamiltonians.

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Appendix A: Finite-Size Analysis

FIG. 6. The plot shows that $m_{DE}^x$ is showing no perceptible $L$-dependence for two very different values of $\omega$ when the drive amplitude is large. The plot corresponds to drive Hamiltonian $\mathcal{H}$ in the main text, with parameters $J = 1$, $\kappa = 0.7\pi/3$, $h_0^x = e/10$, $h_D^x = 40$, $L = 14$.

Here, we show that the numerical results exhibit no discernible finite size effect in $m^x$ (Fig. 6).

Appendix B: Robustness of emergent conservation law with respect to variation of drive field

From Fig. 2 we note that the fully-polarized state is quite special—at the scar points not only its magnetization remains strongly frozen closed to unity, its entanglement entropy also does not grow. This is in stark contrast with other $x$-basis states for which, though $m^x$ remains conserved, entanglement entropy experiences substantial growth. This can be understood from the step-like structure (Fig. 1) appearing at the scar points. We expect this phenomenology to be present for other strong drives which divide up Hilbert space into sectors which are at most weakly mixed as long as these sectors are separated by finite gaps.

We illustrate this by arresting the entangle dynamics of the $L/2$-domain-pair state, which sees substantial growth of $E_{\perp}$ under the drive with uniform longitudinal field (Fig. 2, middle column). Instead of a uniform field, we choose the following drive Hamiltonian

$$H_D = -h_D^e \sum_{i=1}^{L/2} \sigma_i^x + h_D^x \sum_{i=L/2+1}^{L} \sigma_i^x, \quad (B1)$$

keeping the rest of the set-up same as given by Eq. (3). For $H_D$ of above form, $L/2$-domain-pair state is in an

FIG. 7. Freezing the entanglement growth of the $L/2$-domain-pair state under half-up half-down field drive (Eq. B1). Fate of a fully polarized state under the same drive is also shown for comparison. The main frame shows $E_{\perp}$, while the inset shows $m^x$, averaged over $10^{10}$ cycles, after driving for $10^{10}$ cycles. The results are for $J = 1$, $\kappa = 0.7$, $h_0^x = e/10$, $h_D^x = 40$, $L = 14$. 

FIG. 5. Periodicity in drive strength, $h_D^e$, of magnetization response (diagonal ensemble average $m_{DE}^x$, Eq. (3)). Top row shows periodicity for both off-resonance (left, $h_0^e = -0.2$) and on-resonance (right, $h_0^e = -0.21$) drives. Other parameters, and initial low-temperature state, as in Fig. 2b. In both cases the leading frequency of oscillations is $\Omega \approx 157.08 \approx 2\pi/\omega$, visible in the bottom panel, as predicted by Eq. (41).
Finally, we note that the entanglement growth is strongly suppressed for the \(L/2\)-domain-pair state, especially, at \(\omega = 8, 10\) and \(13.33\cdots\) which are the scar points corresponding to the applied drive amplitude \(h_D^* = 40\), while substantial growth of entanglement is observed for the fully polarized initial state. This is in stark contrast with the results for the uniform drive (left and middle columns of Fig. 3).

Appendix C: Strong-field Floquet expansion

Here, we provide the details of the derivation of the effective Hamiltonian, Eqs. [17 - 19]. Carrying out the Pauli algebra gives

\[
H_{\text{mov}} = H_0^* - h^2 \sum_i \left[ \cos(2\theta)\sigma_i^x + \sin(2\theta)\sigma_i^y \right],
\]

where

\[
\theta(t) = - hD^* \int_0^t dt' \ \text{sgn}(\sin \omega t'). \tag{C1}
\]

We note that the frame change does not affect \(m\), since it commutes with \(W(t)\). Now we do the Magnus expansion of \(H_{\text{mov}}\). We then find that

\[
H^{(1)}_F = \frac{1}{2i(\hat{\imath})T} \int_0^T dt_1 \int_0^{t_1} dt_2 [H(t_1), H(t_2)]. \tag{C2}
\]

Arranging the terms in the commutator we get,

\[
[H(t_1), H(t_2)] = K_1 + K_2 + K_3,
\]

where

\[
K_1 = - h^2 \{ \cos(\theta(t_2)) - \cos(\theta(t_1)) \},
\]

\[
K_2 = - h^2 \{ \sin(\theta(t_2)) - \sin(\theta(t_1)) \},
\]

\[
K_3 = \{ h^2 \sin(\theta(t_2) - \theta(t_1)\} [S_x, S_y]. \tag{C3}
\]

Next we note that the integral in Eq. (C2) can be broken in the following way.

\[
I[f(\theta(t_1), \theta(t_2))] = \int_0^T dt_1 \int_0^{t_1} dt_2 [f(\theta(t_1), \theta(t_2))]
\]

\[
= I_1[f(\theta(t_1), \theta(t_2))] + I_2[f(\theta(t_1), \theta(t_2))]
\]

\[
+ I_3[f(\theta(t_1), \theta(t_2))],
\]

where

\[
I_1[f(\theta(t_1), \theta(t_2))] = \int_0^{T/2} dt_1 \int_0^{t_1} dt_2 [f(\theta(t_1), \theta(t_2))],
\]

\[
I_2[f(\theta(t_1), \theta(t_2))] = \int_0^{T/2} dt_1 \int_0^{T/2} dt_2 [f(\theta(t_1), \theta(t_2))],
\]

\[
I_3[f(\theta(t_1), \theta(t_2))] = \int_0^{T/2} dt_1 \int_0^{T/2} dt_2 [f(\theta(t_1), \theta(t_2))].
\]

Finally, we note that

For \(I_1, \theta(t_1) = - hD^* t_1, \theta(t_2) = - hD^* t_2,\)

For \(I_2, \theta(t_1) = - hD^* (t_1 - T), \theta(t_2) = - hD^* t_2,\)

For \(I_3, \theta(t_1) = hD^* t_1, \theta(t_2) = hD^* t_2.\) \tag{C4}

Using Eqs. (C2), (C3), (C4) and evaluating the integrals, we finally get the results.

Appendix D: Floquet-Dyson Perturbation Theory

We start from Eq. (23), which implies that

\[
i \sum m c_m(t) e^{-i \int_0^T dt' E_m(t')} |m\rangle = V \sum m c_m(t) e^{-i \int_0^T dt' E_m(t')} |m\rangle, \tag{D1}
\]

where the dot over \(c_m\) denotes \(d/dt\). Taking the inner product of Eq. (D1) with \(\langle n |\) and using Eq. (22), we find, to first order in \(V\), that

\[
\dot{c}_n = 0. \tag{D2}
\]

We can therefore choose

\[
c_n(t) = 1 \tag{D3}
\]

for all \(t\). We thus have

\[
|\psi_n(t)\rangle = e^{-i \int_0^T dt' E_n(t')} |n\rangle + \sum_{m \neq n} c_m(t) e^{-i \int_0^T dt' E_m(t')} |m\rangle. \tag{D4}
\]

Hence Eq. (24) implies that the Floquet eigenvalue is still given by \(\mu_n = \int_0^T dt E_n(t)\) up to first order in \(V\).

Next, taking the inner product of Eq. (D1) with \(|m\rangle\), where \(m \neq n\), we find, to first order in \(V\), that

\[
\dot{c}_m = -i \langle m | V | n \rangle e^{i \int_0^T dt' [E_n(t') - E_m(t')]}, \tag{D5}
\]

so that

\[
c_m(T) = c_m(0) - i \langle m | V | n \rangle \times \int_0^T dt e^{i \int_0^T dt' [E_n(t') - E_m(t')]} \tag{D6}
\]

We now impose the condition on \(|\psi_n(T)\rangle\) of Eq. (25) such that \(\psi_n(0))\) turns out to be a Floquet state, i.e., (from Eq. (D4)) we must have

\[
\psi_n(T) = e^{-i \int_0^T dt E_n(t)} \psi_n(0), \tag{D7}
\]

namely, we must have

\[
c_m(T) = e^{i \int_0^T dt [E_m(t) - E_n(t)]} c_m(0) \tag{D8}
\]

for all \(m \neq n\). Clearly, \(|\psi_n(0)\rangle\) satisfying this condition can be identified as the Floquet state \(|\mu_n\rangle\).
1. Single spin model

a. Model

We consider a single spin-$S$ object which evolves according to the time-dependent Hamiltonian

$$H(t) = -h^z S^z - h^z S^z - h_D^z \operatorname{sgn}(\sin(\omega t)) S^z. \quad (D9)$$

Since $\sin(\omega t)$ is positive for $0 < t < T/2$ and negative for $T/2 < t < T$, where $T = 2\pi/\omega$, the Floquet operator is given by

$$U = e^{i(T/2) [(h^z - h_D^z)S^z + h^z S^z]} \times e^{i(T/2) [(h^z + h_D^z)S^z + h^z S^z]}. \quad (D10)$$

The group properties of matrices of the form $e^{i\alpha S}$ imply that $U$ in Eq. (D10) must be of the same form and can therefore be written as

$$U = e^{i\tilde{k} S},$$

where

$$\tilde{k} = (\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi) \quad (D11)$$

is a unit vector. We will work in the basis in which $S^z$ is diagonal; hence we choose the polar angles in such a way that the $x$-component of $\tilde{k}$ is equal to $\cos \theta$. The eigenstates of $U$ in Eq. (D11) are the same as the eigenstates of the matrix $M = k \cdot S$. It is then clear that the expectation values of $S^z$ in the different eigenstates take the values $\cos \theta$ times $S, S - 1, \cdots, -S$. The maximum expectation value is given by $s_{\max} = S \cos \theta$.

An important point to note is that if the parameters $h^x, h^z, h_D^z$ and $T$ are fixed and only the spin $S$ is varied, the values of $\gamma$ and $\tilde{k}$ in Eqs. (D11) do not change. This means that if we can calculate these quantities for one particular value of $S$, the results will hold for all $S$. In particular, $m_{\max} = s_{\max}/S = \cos \theta$ will not depend on $S$. We have confirmed this numerically for a variety of parameter values.

b. Results from FDPT

Next, we apply the perturbation theory developed in Sec. V. Writing the Hamiltonian as $H = H_0(t) + V$, where

$$H_0(t) = -h^x S^x - h^z S^z - h_D^z \operatorname{sgn}(\sin(\omega t)) S^z,$$

$$V = -h^z S^z,$$

we can do perturbation theory to study how the state given by $|0\rangle \equiv |S^z = S\rangle$ mixes with the state $|1\rangle \equiv |S^z = S - 1\rangle$. Following the steps leading up to Eq. (26), and using the fact that $\langle 0 | S^z | 1 \rangle = \sqrt{S/2}$, we find that

$$c_1(0) = \frac{\sqrt{2S}}{h_D^z} e^{ih^x T/2} |e^{ih^z T/2} - \cos(h^z T/2)| e^{ih^z T} - 1. \quad (D13)$$

c. Exact results

It is instructive to look at the form of the Floquet operator $U$ in different cases. We first derive an exact expression for $U$ using the identity that if

$$e^{i\alpha S} e^{i\gamma S} = e^{i\tilde{k} S}, \quad (D14)$$

then

$$\cos \left( \frac{\gamma}{2} \right) = \cos \left( \frac{\alpha}{2} \right) \cos \left( \frac{\chi}{2} \right) - \tilde{m} \cdot \tilde{n} \sin \left( \frac{\alpha}{2} \right) \sin \left( \frac{\chi}{2} \right),$$

$$\tilde{k} = \frac{1}{\sin \left( \gamma/2 \right)} \left[ \tilde{m} \sin \left( \frac{\alpha}{2} \right) \cos \left( \frac{\chi}{2} \right) + \tilde{n} \sin \left( \frac{\alpha}{2} \right) \sin \left( \frac{\chi}{2} \right) - \tilde{m} \times \tilde{n} \sin \left( \frac{\alpha}{2} \right) \sin \left( \frac{\chi}{2} \right) \right]. \quad (D15)$$

We can derive Eq. (D15) from Eq. (D14) for the case $S = 1/2$ when $\tilde{S} = \tilde{s}/2$. Eq. (D15) then follows for any value of $S$ due to the group properties of the matrices given in Eq. (D14).

We will now use Eqs. (D14-D15) along with Eq. (D11) which can be written in the form

$$\alpha = \frac{T}{2} \sqrt{(h^z_D - h^z)^2 + (h^x)^2},$$

$$\tilde{m} = -\frac{(h^z_D - h^z) \hat{x} - h^z \hat{z}}{\sqrt{(h^z_D - h^z)^2 + (h^x)^2}},$$

$$\chi = \frac{T}{2} \sqrt{(h^z_D + h^z)^2 + (h^x)^2},$$

$$\tilde{n} = \frac{(h^z_D + h^z) \hat{x} + h^z \hat{z}}{\sqrt{(h^z_D - h^z)^2 + (h^x)^2}}, \quad (D16)$$

where we have assumed that $h^z_D$ is positive and much larger than $|h^x|$ and $|h^z|$. If $e^{i\hbar^x T} \neq 1$, we can write the expressions in Eqs. (D16) to zero-th order in the small parameter $h^x/h^z_D$ to obtain

$$\alpha = \frac{T}{2} (h^z_D - h^z), \quad \tilde{m} = -\hat{x},$$

$$\chi = \frac{T}{2} (h^z_D + h^z), \quad \tilde{n} = \hat{x}. \quad (D17)$$

Eqs. (D14-D15) then imply that

$$\cos \left( \frac{\gamma}{2} \right) = \cos \left( \frac{h^x T}{2} \right), \quad \text{and} \quad \tilde{k} = \hat{x}. \quad (D18)$$

We thus find that the Floquet operator for the time period $T$ corresponds to a rotation about the $\hat{x}$ axis.

If $e^{i\hbar^x T} = 1$, i.e., $\cos(h^x T/2) = \pm 1$, the denominator of Eq. (D13) vanishes. If $e^{i\hbar^x T/2} \neq \cos(h^x T/2)$, we have to expand the expressions in Eqs. (D16) up to second
order in $\hbar^2/\hbar_D^2$ to find that
\[ \hat{k} = \cos\left(\frac{\hbar_D^2 T}{4}\right) \hat{z} - \sin\left(\frac{\hbar_D^2 T}{4}\right) \hat{y} \]
if $\cos\left(\frac{\hbar^x T}{2}\right) = 1,$
\[ = \sin\left(\frac{\hbar_D^2 T}{4}\right) \hat{z} + \cos\left(\frac{\hbar_D^2 T}{4}\right) \hat{y} \]
if $\cos\left(\frac{\hbar^x T}{2}\right) = -1.$  
\[ (D19) \]
Hence the Floquet operator corresponds to a rotation about an axis lying in the $y-z$ plane. This implies that the expectation value of $S^y$ will be zero in all the eigenstates of the Floquet operator.

If $e^{i\hbar^x T} = 1$ and $e^{i\hbar^y T/2} = \cos(\hbar^x T/2) = \pm 1,$ both the numerator and denominator of Eq. (D13) vanish. We then discover that
\[ \hat{k} = \frac{\hbar^x \hat{x} - \hbar^z \hat{z}}{\sqrt{(\hbar^x)^2 + (\hbar^z)^2}} \]  
\[ (D20) \]
In this case, the Floquet operator corresponds to a rotation about an axis lying in the $x-z$ plane.

2. **FDPT for the Ising chain**

\[ E_m(t) - E_0(t) = 4(J-\kappa) + 2\hbar^x_0 + 2\hbar^y_0 \text{sgn}(\sin(\omega t)). \]  
\[ (D21) \]
We now use the notations and results from Sec. V to construct the Floquet state $|\psi(0)\rangle$ obtained by perturbing the unperturbed (Floquet) eigenstate $|0\rangle$ to first order in $V$ given by
\[ \psi(0) = c_0|0\rangle + \sum_{m \neq 0} \sqrt{L} c_m|0\rangle|m\rangle \]
\[ = c_0|0\rangle + \sqrt{L} c_m|0\rangle|L - 2\rangle, \]  
\[ (D22) \]
where
\[ |L - 2\rangle \equiv \frac{1}{\sqrt{L}} \sum_{m = 1}^L |m\rangle \]  
\[ (D23) \]
is a translation invariant and normalized state in which $\sum_m \sigma_m^z = L - 2.$ Taking $c_0(t) = 1$ for all $t$ (just changes the normalization) and using $\langle m|V|0\rangle = -\hbar^x$ in Eq. (26), we get
\[ c_m(0) = \frac{\hbar^x}{\sqrt{2}} \int_0^T dt \int_0^T \left[ E_m(t') - E_0(t') \right] \]  
\[ (D24) \]
where $E_m(t) - E_0(t)$ is given in Eq. (D21).

**Fig. 8.** The result shows mismatches between the numerical dips in $m_{FDPT}^x$ and the first order FDPT prediction (vertical lines obtained from Eq. (40)). The parameters are chosen such that the condition for scar is not satisfied, i.e., $\hbar_D^2 \neq n\omega.$

3. **Failure of FDPT and Emergent Integrability at the Scars**

FDPT always works fine in integrable systems (e.g., the single large spin case discussed here, and also other studied examples not reported here). However, FDPT seems to lose accuracy away from integrability, and hence from the scar points. This is an interesting indirect indication of the fact that integrability emerges at the scar points. In contrast to very accurate prediction of resonances in the Fig. 4 (main text), here Fig. 8 shows substantial mismatch between the FDPT predictions and the true numerical resonances (dips) away from the scars.

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