INTRODUCTION

The issue of the orientation of hydraulic fractures has been investigated since the 1950s. Hubbert and Willis found that the state of stress underground is not hydrostatic but depends on tectonic conditions. The least stress is approximately horizontal in tectonically relaxed areas, while in areas of tectonic compression, the least stress is approximately vertical and is close to the overburden pressure in value. Hydraulic fractures are usually formed perpendicular to the least principal stress, therefore, in tectonically relaxed areas hydraulic fractures are vertical, while in tectonically compressed areas, hydraulic fractures are horizontal.
they are horizontal. Lamont and Jessen\textsuperscript{2} concluded that restricted fractures also propagate in a plane normal to the least principal compressive stress. The least principal stress is usually the overburden stress in some shallow formations, and thus, the hydraulic fracture propagates in a horizontal plane. The least principal stress tends to be horizontal in reservoirs deeper than approximately 1000 ft, and thus, the hydraulic fracture is vertical.\textsuperscript{3-4} Nowadays, it is generally believed that the least principal stress is vertical in formations shallower than approximately 2000 ft, and horizontal in formations deeper than 4000 ft. This implies that horizontal fractures form at a depth shallower than 2000 ft and vertical fractures form at a depth deeper than 4000, during hydraulic fracturing. However, different values of these threshold depths are found by researchers. Lu et al.\textsuperscript{5} reported that the maximum principal stress in horizontal orientations fell between 3045 and 3360 m deep. It is therefore important to evaluate in situ stresses case by case to predict the orientation of hydraulic fractures. Additionally, the azimuth of the vertical fracture depends on the azimuth of the minimum and maximum horizontal stresses.\textsuperscript{6-7} A review of the subject is given by Huang et al.\textsuperscript{8} The generation of horizontal fractures as we discussed above contributes to a useful stimulation and completion technique for modern reservoir development—Frac-packing.

Frac-packing is a special hydraulic fracturing technique developed in the early 1990s.\textsuperscript{9} It combines the stimulation advantages of highly conductive hydraulic fractures with sand control using gravel pack to improve sand control capacity and well productivity in unconsolidated reservoirs. An overview of the technology was presented by Ellis.\textsuperscript{10} A full technical description of frac-packing was given by Ghalambor et al.\textsuperscript{11} Since frac-packing is mainly used for fracturing unconsolidated reservoirs in shallow depth where the overburden stress is normally the least stress, horizontal fractures are believed to be created. The orientation of the fracture can be informed by the pressure-time data plot.\textsuperscript{12} A straight line in the plot of the bottom hole pressure (BHP) versus square-root-time ($t^{0.5}$) indicates a vertical fracture.\textsuperscript{13} A non-linear behavior was shown by Morales et al.\textsuperscript{14} for non-vertical fractures. A unit slope in a log-log diagnostic plot should indicate pressure depletion in the region covered by a horizontal fracture.\textsuperscript{15}

A big challenge in evaluating the productivity of frac-packed wells with horizontal fractures is the unknown fracture conductivity. This problem arises because formation sands invade into the fracture and reduce the permeability of the gravel/propellant pack in the fracture. Berg\textsuperscript{16} presented a correlation for estimating the permeability of particle packs. It requires the knowledge of particle size distribution and frac-pack porosity. Similar correlations were presented by Van Baaren\textsuperscript{17} and Nelson\textsuperscript{18} for consolidated sedimentary rocks. These correlations do not apply to the prediction of fracture permeability due to the unknown size distribution of the gravel/propellant pack with the invasion of formation sands.

A practical technique to determine fracture permeability is the analysis of pressure-time data obtained from pressure transient testing on fractured wells. Numerous analytical models have been developed to analyze pressure transient data for wells with vertical fractures. These models are used to identify multiple flow regimes including reservoir linear flow, fracture-reservoir bilinear flow, and pseudo-steady flow. Mathematical models for reservoir linear flow were given by several investigators including Gringarten et al.\textsuperscript{19} and Cinco et al.\textsuperscript{20} A fracture-reservoir bilinear flow model was presented by Cinco and Samaniego.\textsuperscript{21} The pseudo-steady flow model for multi-fractured horizontal wells was developed by Li et al.\textsuperscript{22} A pseudo-steady flow model for high-energy gas-fractured wells was given by Li et al.\textsuperscript{23} A pseudo-steady flow model for re-fractured wells was presented by Shan et al.\textsuperscript{24} However, none of these models applies to wells with horizontal fractures.

This study presents an analytical model based on reservoir-fracture crossflow to describe the behavior of a vertical well with a horizontal fracture. The model focuses on the transient flow period when the flow in the fracture dominates, which enables us to characterize reservoir properties such as fracture permeability. Particle invasion can plug the pores in the fracture and reduce its permeability. This model can be used as a diagnosis tool to calculate the effective fracture permeability and fracture conductivity, which is one of the potential ways that this model can be used for. It provides petroleum engineers with an analytical approach to obtaining fracture properties from well test data.

\section{Mathematical Model}

Figure 1 illustrates a reservoir section around a frac-packed vertical well. The fracture is assumed to be horizontal at the
mid-depth of the pay zone where the vertical in situ stress is the least \( \sigma_{\text{min}} \). The following assumptions were made in model formulation.

1. The reservoir is homogeneous and isotropic.
2. Darcy’s law applies.
3. Fracture propagates in the radial direction with a constant width \( w \).
4. Single-phase liquid flow prevails in the reservoir and fracture.
5. The radius of the fracture is much greater than the pay zone thickness.

If assumption 5 is valid, it is expected that vertical flow prevails in the region above and below the fracture, while radial flow dominates in the fracture. An analytical solution for early time is derived in Appendix A and outlined in this section. Darcy unit system is used for the derivation.

Based on the principle of conservation of mass, the governing equation takes the following form:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p_d}{\partial r} \right) = F \frac{\partial p_d}{\partial t} + \frac{1}{M^2} p_d
\]  

(1)

where the parameters are defined as

\[
p_d = p_e - p_f
\]

(2)

where \( p_d \) is fracture pressure, \( p_e \) is the fracture pressure at radial distance \( r \) and time \( t \), \( \phi_f \) is fracture porosity, \( c_f \) is fracture compressibility, \( \mu \) is fluid viscosity, \( k_f \) is fracture permeability, \( h \) is pay zone thickness, \( w \) is the average fracture width, and \( k_m \) is the matrix permeability in the zone.

The initial condition for the fracture is set as:

\[
p_d = 0 \text{ at } t = 0, \quad \text{for all } r
\]

(5)

The outer boundary condition for the fracture is expressed as:

\[
p_d = 0 \text{ at } r = \infty, \quad \text{for all } t
\]

(6)

The inner boundary condition for the fracture can be expressed using Darcy’s law applied to the assumed line-source wellbore:

\[
r0 \lim \left( r \frac{\partial p_d}{\partial r} \right) = \frac{q\mu}{2\pi wk_f} \quad \text{for } t > 0
\]

(7)

The upper and lower boundary conditions for the reservoir matrix are no-flow boundaries expressed by:

\[
\frac{dp}{dy} = 0 \text{ at the upper and lower boundaries}
\]

(8)

The inner boundary conditions for the reservoir matrix are expressed as:

\[
p = p_f \text{ at fracture faces}
\]

(9)

The analytical solution in Darcy units takes the following form (see Appendix A for derivation in Darcy’s units):

\[
\begin{aligned}
\ln \left( \frac{\gamma F r^2}{4t} \right) + \text{Ei} \left( \frac{Fr^2}{4t} \right) - \frac{Fr^2}{4t} + \frac{Fr^2}{4t} \left( \frac{x}{M} \right) - 1 \\
\text{where } I_0 \text{ is the modified Bessel function of the first kind of zero-order, } K_0 \text{ is the modified Bessel function of the second kind of zero-order, } Ei \text{ is exponential integral, and } \gamma \text{ is the exponential function of Euler’s constant (} \gamma = e^{0.5572} = 1.78\). Considering a skin factor \( S \) near the wellbore, Equation (10) degenerates to an expression for the pressure at the wellbore where \( r = r_w \): \end{aligned}
\]

(10)

where \( I_0 \) is the modified Bessel function of the first kind of zero-order, \( K_0 \) is the modified Bessel function of the second kind of zero-order, \( Ei \) is exponential integral, and \( \gamma \) is the exponential function of Euler’s constant (\( \gamma = e^{0.5572} = 1.78 \)). Considering a skin factor \( S \) near the wellbore, Equation (10) degenerates to an expression for the pressure at the wellbore where \( r = r_w \):

\[
\begin{aligned}
\ln \left( \frac{\gamma F r_w^2}{4t} \right) + \text{Ei} \left( \frac{Fr_w^2}{4t} \right) - \frac{Fr_w^2}{4t} + \frac{Fr_w^2}{4t} \left( \frac{x}{M} \right) - 1 \end{aligned}
\]

(11)

Since the analytical solution was derived for early time flow from the matrix to fracture in the fractured area, the solution is expected to be valid for flow time up to the moment when the radius of investigation reaches the radius of fracture expressed as:
\[ t_{\text{max}} = \frac{\varphi_f \mu c_f R^2}{k_f} \] (12)

where \( R \) is the radial extension (radius) of fracture.

### 3 | MODEL VALIDATION

Although no clean data set has been found to validate the derived analytical solution yet, the degenerated form of the solution was verified for the special case of pure radial flow in the future. In fact, for practical values of \( r_w \) and \( M \), the value of \( \frac{r_w}{M} \) is very close to zero.

Since
\[
\frac{r}{M} \lim_{r \to 0} I_0 \left( \frac{r}{M} \right) = 1
\]
Equation (11) becomes
\[
p_w = p_e - \frac{q\mu}{2\pi wk_f} \left\{ \ln \left( \frac{2M}{\gamma r_w} \right) - \frac{1}{2} Ei \left( \frac{t}{FM^2} \right) + \frac{1}{2} e^{-\frac{t}{FM^2}} \left[ \ln \left( \frac{\gamma Fr_w^2}{4t} \right) + Ei \left( \frac{Fr_w^2}{4t} \right) - \frac{Fr_w^2}{4t} \right] + S \right\}
\] (15)

Substituting Equation (3) into Equation (18) results in
\[
p_w = p_e - \frac{q\mu}{4\pi wk_f} \left[ \ln \left( \frac{4k_f t}{\gamma \varphi_f \mu c_f r_w^2} \right) + 2S \right]
\] (19)

which is the classic solution for radial flow, proving that the analytical solution is valid. Classic solution for radial flow can be used as ground evidence to verify the correctness of newly proposed models, as it is established result and has been tested true in field applications for years. Readers could refer to *Well Productivity Handbook*, 2nd edition, Elsevier, Cambridge (2019) by Dr Boyun Guo for more details.

### 4 | APPLICATIONS

#### 4.1 | Type curves

Equations (10) and (11) are difficult to use in applications owing to their involvements of the modified Bessel function of the first kind of zero-order \( I_0 \) and the modified Bessel function of the second kind of zero-order \( K_0 \) and the exponential integral \( Ei \). Type curves were generated for easy applications. Equations (10) and (11) are rearranged to get \( p_d \) and \( p_w \) respectively, in the following forms:

\[
p_d = \frac{q\mu}{2\pi wk_f} \Phi (r, t)
\] (20) and
\[
p_w = p_e - \frac{q\mu}{2\pi wk_f} W (r_w, t)
\] (21)

where the fracture function \( \Phi (r, t) \) and well function \( W (r_w, t) \) are defined respectively as:

\[
\Phi (r, t) = K_0 \left( \frac{r}{M} \right) - \frac{1}{2} I_0 \left( \frac{r}{M} \right) Ei \left( \frac{t}{FM^2} \right) + \frac{1}{2} e^{-\frac{t}{FM^2}} \left[ \ln \left( \frac{\gamma Fr_w^2}{4t} \right) + Ei \left( \frac{Fr_w^2}{4t} \right) - \frac{Fr_w^2}{4t} \right] - \frac{Fr_w^2}{4t} I_0 \left( \frac{r}{M} \right) - 1
\] (22)

and

\[
W (r_w, t) = K_0 \left( \frac{r_w}{M} \right) - \frac{1}{2} I_0 \left( \frac{r_w}{M} \right) Ei \left( \frac{t}{FM^2} \right) + \frac{1}{2} e^{-\frac{t}{FM^2}} \left[ \ln \left( \frac{\gamma Fr_w^2}{4t} \right) + Ei \left( \frac{Fr_w^2}{4t} \right) - \frac{Fr_w^2}{4t} \right] - \frac{Fr_w^2}{4t} I_0 \left( \frac{r_w}{M} \right) - 1 + S
\] (23)
or approximately

\[ W(r_w, t) = \ln \left( \frac{2M}{r_w^2} \right) - \frac{1}{2} Ei \left( \frac{t}{FM^2} \right) + S \] (24)

Notice that when the radius \( r \) is changed to the wellbore radius, Equation (20) becomes Equation (21). Equation (22) is the mathematical solution to the governing Equation (1). The result can be conveniently verified by taking the partial derivative of Equation (22) with respect to radius \( r \).

Figure 2 presents fracture function curves for \( F = 0.0003 \text{ s/cm}^2 \) and \( M = 1200 \text{ cm} \) at three values of time. The function tends to stabilize after 1 hour of flow time. Since the pressure profile inside the fracture is not measurable with today’s technologies, the fracture function curve has limited applications.

Figures 3-5 show well function curves for \( F = 0.00015 \text{ s/cm}^2 \), \( 0.000075 \text{ s/cm}^2 \), and \( 0.00005 \text{ s/cm}^2 \) with \( S = 0 \). Since the wellbore pressure is measurable with today’s technologies, the well function curves can be used for revealing fracture and reservoir properties. Rearranging Equation (21) gives:

\[ \frac{2\pi w_k_f (p_e - p_w)}{q\mu} = W(r_w, t) \] (25)

The left-hand side (LHS) of this equation is called Dimensionless Pressure (DP) which can be evaluated with well test data and assumed fracture conductivity \( w_k_f \). By matching the \( W(r, t) \) to the DP through tuning \( F \) and \( M \) values, it is possible to reveal fracture permeability, average fracture width, and matrix permeability.

4.2 Field case study

A vertical well was completed with frac-packing against an oil pay zone from 833 m (2732 ft) to 849 m (2787 ft). Known reservoir and fluid properties are summarized in Table 1. The well was pumped at a constant flow rate of 8.75 m³/day (101 STB/day). Bottom hole pressure (BHP) was recorded with a
pressure gauge during pumping. Averaged early time BHP data are plotted in Figure 6.

Figure 7 shows a tuned match of $W(r, t)$ to dimensionless pressure data. This match was obtained using the following parameter values:

$$F = 0.0001 \frac{s}{cm^2}$$

$$M = 1100 \text{ cm}$$

$$wk_f = 12.4 \text{ cm} - \text{Darcy}$$

which gives:

$$k_f = 2.88 \text{Darcy}$$

$$w = 4.3 \text{ cm}$$

$$k_m = 0.0021 \text{Darcy}$$

This result was checked by the radius of investigation through Equation (12):

$$R = \sqrt{\frac{k_f t_{max}}{\phi_f \mu c_f}} = \sqrt{\frac{(2.88)(150)}{(0.35)(7)(0.000162)}} = 2100 \text{ cm}$$

which is consistent with the proppant volume consumed in this frac-packing operation.

5 | DISCUSSION ON TYPE CURVE MATCHING

The value of the parameter of interest remains unchanged regardless of the way it is expressed. By this principle, we can express the same parameter, which is the Dimensionless Pressure in this case, in two different ways and make them equal to reveal the value of certain parameters. In this paper, one way is to use the real transient pressure/rate test data as expressed by Equation (25), and the other is to use the result of the analytical model as expressed by Equation (24). By tuning the F and M values in the analytical model to match the

| Parameters                        | Value in Darcy units | Value in Field units |
|-----------------------------------|----------------------|----------------------|
| Net pay thickness                 | 1677 cm              | 55 ft                |
| Porosity                          | 0.21                 | 21 %                 |
| Reservoir pressure                | 80.9 Atm             | 1189 psia            |
| Oil formation volume factor       | 1.13 cc/stcc         | 1.13 rb/stb          |
| Oil viscosity                      | 7 cp                 | 7 cp                 |
| Formation volume factor           | 1.1 cm$^3$/scm$^3$    | 1.1 rb/stb           |
| Total compressibility of matrix   | 0.000162 l/Atm       | 0.000011 l/psi       |
| Fracture porosity                 | 0.35                 | 35 %                 |
| Total compressibility of fracture | 0.000118 l/Atm       | 0.000008 l/psi       |
| Wellbore radius                   | 10 cm                | 0.328 ft             |

TABLE 1 Reservoir and fracture properties
Dimensionless Pressure, the reservoir and fracture properties within the F and M coefficients can therefore be obtained.

The Type Curve Matching approach can be used as a reliable technique to obtain reservoir and fracture properties. It should be noted that the well-testing data that are used as model input for this purpose should contain the transient pressure/rate test data recorded at the very beginning of the test. This upper limit of time that makes the testing data usable is expressed as \( t_{\text{max}} \) shown in Equation (12).

Substituting the transient pressure/rate data into Equation (25), we can obtain a series of DP points, then adjusting the F and M values in Equation (24) to match the DP series. The resultant F and M values are used to calculate the fundamental reservoir or fracture properties such as fracture permeability.

## 6 CONCLUSIONS

An analytical solution was developed in this study to describe fluid flow through a horizontal fracture to a vertical well. The solution applies to frac-packed vertical wells in shallow depths where the vertical formation stress is the least in situ stress. The following conclusions are drawn.

1. The analytical solution degenerates to the classic solution for radial flow when the mass-feeding of the reservoir matrix to the fracture is negligible. This partially validates the correctness of the solution.
2. The analytical solution can be used to analyze transient pressure data to reveal fracture and reservoir properties where horizontal fractures are created. Tuning model parameters to match the measured bottom hole pressure data allows for the estimation of fracture conductivity, fracture permeability, average fracture width, and reservoir permeability.
3. Field case analysis with the analytical solution shows that type curves generated by the solution can provide a quick and reasonable result of fracture and reservoir properties when a data match is used as a solution technique.
4. The Type Curve in this paper has demonstrated a very promising match on Dimensionless Pressure from field data and analytical model. It is desirable however to further verify the complete solution using field data in the future.

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## Nomenclature

| Symbol | Description |
|--------|-------------|
| \( Q_m \) | Mass flow rate from the matrix to fracture |
| \( V \) | Volume element contribution from the matrix to fracture |
| \( c_m \) | Compression coefficient at one side of fracture expressed |
| \( \Phi_m \) | Matrix permeability |
| \( k_m \) | Matrix permeability |
| \( p_{\text{d}} \) | Pressure drawdown |
| \( h \) | Pay zone thickness |
| \( Q(r) \) | Flow rate from volume V |
| \( \rho \) | Fluid density |
| \( \phi_f \) | Fracture porosity |
| \( \phi_e \) | Pressure at the no-flow boundary |
| \( c_f \) | Compressibility coefficient of fracture |
| \( \phi_r \) | Porosity of fracture |
| \( k_f \) | Fracture permeability |
| \( \mu \) | Fluid viscosity |
| \( t \) | Time since fluid flow is initiated |
| \( r_w \) | Wellbore radius |
| \( S \) | Skin factor |
| \( p_{\text{w}} \) | Bottomhole pressure |
| \( r \) | Distance from the wellbore |
| \( w \) | Fracture width |
| \( q \) | Liquids flow rate at reservoir conditions |
| \( F \) | Coefficient group |
| \( M \) | Coefficient group |
| \( R \) | Radius or extension of the fracture |
| \( \Phi(r,t) \) | Fracture function |
| \( W(r_w,t) \) | Well function |
| \( I_0 \) | Modified Bessel function of the first kind of zero-order |
| \( K_0 \) | Modified Bessel function of the second kind of zero-order |
| \( Ei(x) \) | Exponential integration function |
| \( \gamma \) | Exponential function of Euler’s constant, \( \gamma = e^{0.5572} = 1.78 \) |

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APPENDIX A

Mathematical modeling of fluid flow in reservoir-radial fracture systems

Assumptions

The following simplifying assumptions are made when formulating the general governing equation:

1. The fluid pay zone is homogeneous in all properties and isotropic concerning vertical permeability.
2. The zone is completely saturated with a liquid phase between an upper boundary and a lower boundary.
3. Radial fracture is placed at the mid-depth of the pay zone.
4. Radial fracture has a constant width.
5. Fluid flow from non-fractured areas is negligible.

Governing equation

The volume between the two shaded areas in Figure A.1 illustrates a fracture volume element at a radial distance \( r \) from the wellbore centerline. The volume has a height \( w \), a length in radial direction \( dr \), and a width in the tangential direction \( rd\theta \). The principle of mass conservation is stated as

Mass flow rate \( IN - Mass \ flow \ rate \ OUT = Rate \ of \ change \ of \ mass \ in \ the \ volume \ element \)

or

\[
q_r \rho |_{r+dr} + Q_m - q_r \rho |_r = q_f wrd\theta \frac{dp}{dt} \tag{A.1}
\]

where \( \rho \) is the fluid density, \( q_f \) is fracture porosity, and \( Q_m \) is the mass flow rate contribution from the matrix to fracture. For the infinitesimal element, Eq. (A.1) can be expanded as

\[
q_r \rho |_r + \frac{\partial (q_r \rho)}{\partial r} dr - q_r \rho |_r + Q_m = q_f wrd\theta \frac{dp}{dt} \tag{A.2}
\]

which is simplified to

\[
\frac{\partial (q_r \rho)}{\partial r} dr + Q_m = q_f wrd\theta \frac{dp}{dt} \tag{A.3}
\]

Darcy’s law gives the following expressions for the flow rate \( q_r \) in Eq. (A.3):

\[
q_r = \frac{k_f wrd\theta}{\mu} \frac{dp}{dr} \tag{A.4}
\]

where \( p_f \) is the pressure in the fracture at point \( r \). The mass flow rate \( Q_m \) into the fracture is due to the expansion of fluid in the matrix. Consider the fluid in a volume element \( V \) in one side of fracture expressed as

\[
V = \varphi \frac{h}{2} r d\theta dr \tag{A.5}
\]

Based on the definition of compression coefficient

\[
c_m = \frac{1}{V} \left( \frac{\partial V}{\partial p} \right) \tag{A.6}
\]

Differentiation of Eq. (A.5) concerning time gives an expression of the flow rate from the volume \( V \):

\[
c_m V \frac{dp}{dt} = - \frac{\partial V}{\partial t} = - Q(r) \tag{A.7}
\]

Using Eq. (A.5) the pressure decline rate \( \frac{dp}{dt} \) in Eq. (A.7) is then expressed as

\[
\frac{dp}{dt} = - \frac{Q(r)}{c_m V} = - \frac{2Q(r)}{c_m \varphi h rd\theta dr} \tag{A.8}
\]

The following equation governs linear flow in the vertical direction to the fracture (Dake, 1978):

\[
\frac{\partial^2 p}{\partial y^2} = \frac{q_f \mu c_m}{k_m} \frac{dp}{dt} \tag{A.9}
\]

Substituting Eq. (A.8) into Eq. (A.9) yields:

\[
\frac{\partial^2 p}{\partial y^2} = - \frac{2\mu Q(r)}{k_m wrd\theta dr} \tag{A.10}
\]

Integrating Eq. (A.10) one-time yields:

\[
\frac{\partial p}{\partial y} = - \frac{2\mu Q(r)}{k_m wrd\theta dr} y + C_1 \tag{A.11}
\]

where \( C_1 \) is integration constant and can be determined using the no-flow boundary condition

\[
\left( \frac{\partial p}{\partial y} \right)_{y=\frac{h}{2}} = 0 \tag{A.12}
\]

Applying Eq. (A.12) to Eq. (A.11) gives:

\[
C_1 = \frac{2\mu Q(r)}{k_m wrd\theta dr} \tag{A.13}
\]

Substituting Eq. (A.13) into Eq. (A.11) yields

\[
\frac{\partial p}{\partial y} = \frac{2\mu Q(r)}{k_m wrd\theta dr} \left( \frac{1}{2} \frac{y}{h} \right) \tag{A.14}
\]
Separating variables and integrating Eq. (A.14) gives

\[ p = \frac{2\mu Q(r)}{k_m r d \theta d r} \left( \frac{y}{2} - \frac{y^2}{2h} \right) + C_2 \quad (A.15) \]

where the integration constant \( C_2 \) can be determined using the boundary condition at the fracture face

\[ p|_{r=0} = p_f \quad (A.16) \]

Applying Eq. (A.16) to Eq. (A.15) gives

\[ C_2 = p_f \quad (A.17) \]

Substituting Eq. (A.17) into Eq. (A.15) results in

\[ p = \frac{2\mu Q(r)}{k_m r d \theta d r} \left( \frac{y}{2} - \frac{y^2}{2h} \right) + p_f \quad (A.18) \]

Along the no-flow boundary \( y = \frac{b}{2} \) where the pressure is \( p_e \), Eq. (A.18) demands

\[ Q(r) = \frac{4k_m r d \theta d r}{\mu h} (p_e - p_f) \quad (A.19) \]

The total mass flow rate from the matrix in both sides of fracture is expressed as

\[ Q_m = 2\rho Q(r) = \frac{8\rho k_m r d \theta d r}{\mu h} (p_e - p_f) \quad (A.20) \]

The change of density in the right side of Eq. (A.3) can be expressed in terms of change of pressure through compressibility:

\[ \frac{\partial \rho}{\partial t} = c_f \rho \frac{\partial p_f}{\partial t} \quad (A.21) \]

where \( c_f \) is fracture compressibility. Substituting Eqs. (A.4), (A.20), and (A.21) into Eq. (A.3) and rearranging the latter result in a general governing equation:

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p_f}{\partial r} \right) + \frac{8k_m}{hk_f w} (p_e - p_f) = \frac{\rho \mu c_f}{k_f} \frac{\partial p_f}{\partial t} \quad (A.22) \]

We define pressure drawdown \( p_d \) as

\[ p_d = (p_e - p_f) \quad (A.23) \]

Substituting Eq. (A.23) into Eq. (A.22) yields

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p_d}{\partial r} \right) = \frac{\rho c_f}{k_f} \frac{\partial p_d}{\partial t} + \frac{8k_m}{hk_f w} p_d \quad (A.24) \]

If the coefficient group \( \frac{k_m c_f}{k_f} \) for the fracture capacity is denoted by \( F \)

\[ F = \frac{\rho c_f}{k_f} \quad (A.25) \]

and the coefficient group \( \frac{8k_m}{hk_f w} \) for the matrix contribution is denoted by \( \frac{1}{M} \), and

\[ M = \sqrt{\frac{hk_f w}{8k_m}} \quad (A.26) \]

Eq. (A.24) can be simplified to give

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p_d}{\partial r} \right) = F \frac{\partial p_d}{\partial t} + \frac{1}{M^2} p_d \quad (A.27) \]

Initial and boundary conditions

The initial condition is no pressure drawdown anywhere before flow:

\[ p_d = 0 \text{ at } t = 0, \quad \text{for all } r \quad (A.28) \]

The outer boundary condition is no pressure drawdown far away from the wellbore, which is expressed as:

\[ p_d = 0 \text{ at } r = \infty, \quad \text{for all } t \quad (A.29) \]

The inner boundary condition is Darcy’s law applied to assumed line-source wellbore:

\[ r_0 \lim_{r \to 0} \left( r \frac{\partial p_d}{\partial r} \right) = \frac{q \mu}{2\pi w k_f}, \quad \text{for } t > 0 \quad (A.30) \]

**Solution.** A solution to Eq. (A.27) with conditions defined by Eqs. (A.28) through (A.30) is

\[
 p_d = \frac{q \mu}{2\pi w k_f} \left[ K_0 \left( \frac{r}{M} \right) - \frac{1}{2} I_0 \left( \frac{r}{M} \right) Ei\left( \frac{r}{FM^2} \right) + \frac{1}{2} e^{-r/M^2} \ln \left( \frac{\gamma F_r^2}{4t} \right) + Ei \left( \frac{F_r^2}{4t} \right) - \frac{F_r^2}{4t} I_0 \left( \frac{r}{M} \right) - \frac{1}{r/2M^2} \right] \quad (A.31)
\]
where $I_0$ is the modified Bessel function of the first kind of zero-order, $K_0$ is the modified Bessel function of the second kind of zero-order, $Ei$ is exponential integral, and $\gamma$ is an exponential function of Euler's constant ($e^{0.5572} = 1.78$). Considering a skin factor $S$ near the wellbore, Eq. (A.31) degenerates to the expression for the pressure drawdown at the wellbore where $r = r_w$:

$$p^*_d = \frac{q\mu}{2\pi wk_f} \left\{ K_0 \left( \frac{r_w}{M} \right) - \frac{1}{2} I_0 \left( \frac{r_w}{M} \right) Ei \left( \frac{1}{FM^2} \right) + \frac{1}{2} e^{-\frac{\gamma}{M^2}} \left[ \ln \left( \frac{\gamma F r_w^2}{4t} \right) + Ei \left( \frac{F r_w^2}{4t} \right) - \frac{F r_w^2}{4t} I_0 \left( \frac{r_w}{M} \right) - 1 \right] + S \right\} (A.32)$$

where $p^*_d = p_e - p_w$.

**Figure A1** A reservoir body with a horizontal radial fracture at the mid-depth