CONVERGENCE OF MOMENTS OF TWISTED COE MATRICES

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Abstract. We investigate eigenvalue moments of matrices from Circular Orthogonal Ensemble multiplicatively perturbed by a permutation matrix. More precisely we investigate variance of the sum of the eigenvalues raised to power $k$, for arbitrary but fixed $k$ and in the limit of large matrix size. We find that when the permutation defining the perturbed ensemble has only long cycles, the answer is universal and approaches the corresponding moment of the Circular Unitary Ensemble with a particularly fast rate: the error is of order $1/N^3$ and the terms of orders $1/N$ and $1/N^2$ disappear due to cancellations. We prove this rate of convergence using Weingarten calculus and classifying the contributing Weingarten functions first in terms of a graph model and then algebraically.

1. Introduction

Since their introduction by Dyson [Dys62a, Dys62b], the so-called circular ensembles of random matrices have been used successfully to model various physical processes, such as the time evolution of complex quantum systems [Meh04, Haa10] or scattering from a potential of unknown or complex structure [BS90, Bee97], or even the local statistics of the zeros of Riemann zeta and other $L$-functions [MS05]. The three classical circular ensembles — unitary, orthogonal and symplectic — are used to model systems with different basic symmetries. In particular, the Circular Orthogonal Ensemble (COE) corresponds to a system which is invariant with respect to time reversal.

As well as modeling an entire system, a random matrix from COE can be used to model scattering from a part of a composite system. A particular inspiration for the present paper is the work of Joyner, Müller and Sieber [JMS14] where a small number of random scatterers were combined with a carefully chosen magnetic fluxes to produce a spectrum following Circular Symplectic (CSE) statistics even though the system had no spin which is normally associated with CSE. This model was later realized experimentally using microwave networks [RAJ+16]. For other examples of “composite” random matrix ensembles and their applications, see [PZK98].

Mathematical analysis of such models would require understanding the spectral statistics of products of random matrices. In this paper we make a step in this direction by studying the moments of the matrices from COE multiplicatively perturbed by a fixed permutation matrix. Intuitively, without the time-reversal symmetry which is destroyed by the permutation, the result should follow the prediction of the Circular Unitary Ensemble (CUE), even though the perturbed COE is but a tiny submanifold\(^1\) of the CUE. We seek to confirm this intuition with rigorous quantitative results and to find the conditions on the permutation matrix sufficient for the convergence.

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1 More precisely, the probability measure we consider is supported on a high codimension submanifold of $U(N)$, the compact manifold which is the support of the uniform probability measure defining the CUE.
As another source of inspiration, we would like to cite the idea of Degli Esposti and Knauf [DEK04], who proposed that periodic orbit expansions in quantum chaos should be modeled on similar expansions in suitable random matrix ensembles. It should be noted that due to abundance of invariance and averaging available within a classical random matrix model, there are normally more direct methods to arrive at an answer for any particular quantity (such as the spectral correlation functions or their Fourier transform, the form factor), than to expand it into products of matrix elements and do careful combinatorial accounting. What Degli Esposti and Knauf suggested is that doing the computation the hard combinatorial way in random matrices would shed light on similar computations for quantum chaotic systems where such expansions are often the best way to proceed. After the publication of [DEK04], a lot of progress was made in understanding the origin of the correct combinatorial contributions [SR01, Sie02, BSW03, MHB+04, HMA+07, MHA+09, BK13a, BK13b], however a rigorous mathematical derivation with error control and convergence analysis is still missing for any quantum chaotic model.

In the present work, we are firmly within the realm of random matrices. However, due to the perturbation considered, the invariance of the random matrix ensemble (in the case of COE, invariance under the conjugation $U \mapsto V^T U V$, with $V$ unitary) is broken and expanding the moment into a product of matrix elements followed by careful combinatorial accounting mentioned above becomes a necessity. In addition to establishing convergence as mentioned above, we develop a complete algebraic characterization of the permutations contributing to the three leading orders of the $k$-th moment expansion.

2. Summary of the main results

Let $P_N$ be a fixed $N \times N$ permutation matrix. We are interested in the properties of the “$P_N$-twisted COE” ensemble, the set of matrices of the form $P_N U$, where $U$ is a random unitary symmetric matrix distributed according to Circular Orthogonal Ensemble measure. Circular Orthogonal Ensemble (COE) is the classical compact symmetric space $U(N)/O(N)$ identified with the set of all unitary symmetric matrices endowed with the unique probability measure invariant under the action by the unitary group

\begin{equation}
U \mapsto V U V^T, \quad V \in U(N).
\end{equation}

Defining Circular Unitary Ensemble (CUE) as the compact group $U(N)$ with uniform (Haar) measure, we can represent COE as the image of CUE under the mapping $V \mapsto V \bar{V}^{-1} = V V^T$; here the bar denotes complex conjugation which plays the role of Cartan involution in the case of $U(N)/O(N)$. Practical aspects of integrating over the CUE and COE are discussed in Appendix A.

We will investigate the $P_N$-twisted ensemble by studying the moments

\begin{equation}
M_k(N) := \left< \left| \text{Tr}(P_N U)^k \right|^2 \right>_{\text{COE}(N)}.
\end{equation}

We will study the asymptotics of $M_k(N)$ for an arbitrary fixed $k$ and for $N \to \infty$. When changing the size of the matrices we have to specify a new permutation matrix $P_N$ for every $N$. It turns out the answers depend mostly on the lengths of the cycles of $P_N$; our results will be valid for arbitrary sequences of $P_N$ provided some minimal cycle condition is satisfied.
For comparison purposes, we now state the moment formulas of the classical Circular Unitary and Circular Orthogonal ensembles. When \( k < N \), we have

\[
\mathbb{E} \left| \text{Tr} U^k \right|^2_{\text{CUE}(N)} = k,
\]

\[
\mathbb{E} \left| \text{Tr} U^k \right|^2_{\text{COE}(N)} = 2k - k \sum_{m=1}^{k} \frac{1}{m + (N - 1)/2} = 2k + O \left( \frac{1}{N} \right). \tag{4}
\]

Both expressions can be derived from two-point correlation functions calculated in [Meh04]; explicitly they were calculated in, for example, [HKS+96, Eqs. (45) and (48)]. In addition, we refer to [DEK04, Section III] for an evaluation of (3) in terms of Weingarten functions, which is close to the methods of the present work. If the permutation matrix \( P_N \) is an involution, i.e. \( P_N^2 = I \) or all cycles of \( P_N \) are of length 1 or 2, the moments \( M_k(N) \) are identical to the \( \text{COE}(N) \) moments (see Lemma 3.1 below).

In order to get a sense of what happens for \( P_N \) with long cycles, we investigated a particular permutation matrix \( P_N \) corresponding to the grand cycle \((1 \ 2 \ldots N)\). This \( P_N \) acts on a matrix \( U \) with \( N \) rows \( r_j \) by cyclically shifting the rows as follows,

\[
P_N \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{pmatrix} = \begin{pmatrix} r_2 \\ r_3 \\ \vdots \\ r_1 \end{pmatrix}.
\]

For low values of \( k \) it is possible to use Weingarten calculus (see a short review in Appendix A) and carefully enumerate all contributing permutations to get exact results. The enumeration method behind this calculation will be reported elsewhere; here we just list the results,

\[
M_2(N) = 2,
\]

\[
M_3(N) = \frac{3(N^3 + 3N^2 - N - 2)}{(N-1)(N+1)(N+3)} = 3 + \frac{3}{N^3} + \ldots,
\]

\[
M_4(N) = 4,
\]

\[
M_5(N) = \frac{5(N^8 + 21N^7 + 129N^6 - 2103N^5 - 55545N^4 - 3653N^3 + 14463N^2 - 4572N - 9072)}{(N-3)(N-2)(N-1)(N+1)(N+2)(N+3)(N+5)(N+7)(N+9)} = 5 - \frac{1200}{N^3} + \ldots
\]

Three features stand out. First, there is a convergence to the CUE answer for the moments, cf. equation (3). Second, the convergence is unexpectedly fast: the terms of order \( \frac{1}{N} \) and \( \frac{1}{N^2} \) are conspicuously absent. Third, the answer matches CUE exactly for \( k = 2, 4 \). We can now confirm that the first two observations are valid for all \( k \) and a wide class of permutations \( P_N \).

\textbf{Theorem 2.1.} Let \( P_N \) be an \( N \times N \) permutation matrix such that all cycles of \( P_N \) are longer than \( 2k \), \( k \in \mathbb{N} \). Then the \( k \)-th moment \( M_k(N) \) defined by (2) is a rational function of \( N \).

\textsuperscript{2}We cannot yet provide any insight for the other natural question: is \( M_k(N) = k \) for all even \( k \)?
independent of the particular choice of $P_N$ and satisfying the estimate

$$M_k(N) = k + O\left(N^{-3}\right).$$

We remark here that the constant, implicit in the right-hand side of equation (10), is naturally independent of $N$ and the particular choice of $P_N$ (since the function $M_k(N)$ is independent of the latter), but is, in principle, dependent on $k$. Extending this theorem to cover the possibility of $k$ increasing together with $N$ remains a challenging question.

2.1. Related results. Of related works known to the authors, we would like to mention a study by Poźniak, Życzkowski and Kuś [PZK98], who surveyed several composite ensembles obtained from the classical one by multiplying independently sampled matrices. While they did not consider the twisted COE ensemble we study here, their physical motivation was similar.

In mathematical literature, there are several results concerning the distribution of $\text{Tr}(P_N M)$, which is shown to converge to normal as $N \to \infty$ when $M$ is uniformly distributed in the orthogonal group $O(N)$ [DDN03, Mec08]. The difference in the ensemble (COE versus the orthogonal group) is probably irrelevant. However in this work we address the so-called linear moments — expectations of $|\text{Tr}(P_N U)^k|^2$ with $k$ potentially large — whereas studying the distribution of $\text{Tr}(P_N U)$ is equivalent to understanding the nonlinear moments — expectations of $|\text{Tr}(P_N U)|^{2n}$. To put it another way, we are addressing the distribution of eigenvalues, whereas the results of [DDN03, Mec08] concern the normality of individual entries of the matrix (this idea goes back to Borel [Bor06]).

2.2. Notation. We now briefly review the notation used in the rest of the paper. We will use notation $\{N\}$ and $\{k\}$ to denote the set of first $N$ (correspondingly, $k$) natural numbers. As additional indexing symbols we will use the natural numbers with bars on top and will denote

$$\{k\} = \{1, 2, \ldots, k\}$$

We will write $S_k$ for the symmetric group on $k$ elements. We will most actively use the symmetric group $S_{2k}$ which will be viewed as the group of permutations of the ordered set

$$\mathcal{K} := \{k\} \cup \{k\} = [1, 1, 2, 2, \ldots, k, k].$$

To each permutation $\omega \in S_n$ we put in correspondence its “cycle type”, a sequence of non-negative integers $(\alpha_1, \alpha_2, \ldots)$ such that $\alpha_j$ is the number of cycles of $\omega$ of length $j$. Naturally, the sequence is identically zero after some point and $\sum_{j=1}^{\infty} j\alpha_j = n$. We denote by $\ell(\omega)$ the number of cycles of $\omega$, i.e.

$$\ell(\omega) = \sum_{j=1}^{\infty} \alpha_j.$$  

We will normally record the cycle type as $1^{\alpha_1}2^{\alpha_2}3^{\alpha_3}\cdots$, omitting all terms with $\alpha_j = 0$ and occasionally omitting the $1^{\alpha_1}$ term.

In regards to the permutation matrix $P_N$, we will abuse notation slightly by also denoting the corresponding permutation by $P_N$, occasionally dropping the subscript $N$ to reduce notational clutter.
Among the permutations in $S_{2k}$ we distinguish three permutations we will use repeatedly,
\begin{align}
T &:= (1\bar{1})(2\bar{2})\cdots(k\bar{k}), \\
Q &:= (1\bar{k})(2\bar{1})(3\bar{2})\cdots(k\bar{k} - 1), \\
s &:= (12\ldots k)(1\bar{2}\ldots k).
\end{align}
Note that $T$ and $Q$ are involutions, i.e. $Q^2 = T^2 = \text{id}$. We will often need to refer to the cycle length of the commutators $[\omega, T] := \omega T \omega^{-1} T$ and $[\omega, Q] := \omega Q \omega^{-1} Q^{-1}$, for some $\omega \in S_{2k}$. In this case we will drop the parentheses and write simply $\ell[\omega, T]$ and $\ell[\omega, Q]$.

3. Deriving the moments of twisted COE

3.1. The case of the involution. We start with a simple result suggesting that short cycles in $P_N$ fail to change the eigenvalue statistics.

Lemma 3.1. Let $P$ be an orthogonal matrix such that $P^2 = I$. Then
\begin{equation}
\left\langle |\text{Tr}(PU)^k|^2 \right\rangle_{\text{COE}(N)} = \left\langle |\text{Tr}U^k|^2 \right\rangle_{\text{COE}(N)}.
\end{equation}

Proof. The conditions on the matrix $P$ imply that $P^T = P^{-1} = P$, therefore $P$ is real symmetric and can be diagonalized as
\[P = SAS^T,\]
where $S$ is an orthogonal matrix and $\Lambda$ is diagonal. We can further represent $\Lambda = AA^T$, where $A$ is unitary (this representation is not unique). Substituting this into the trace, we get
\[\text{Tr}(PU)^k = \text{Tr}(SAA^T S^T U \cdots SAA^T S^T U) = \text{Tr}(A^T S^T U \cdots SAA^T S^T USA) = \text{Tr}(A^T S^T USA)^k.\]

But due to the invariance COE with respect to the action of the unitary group defined by (1),
\[\left\langle |\text{Tr}(A^T S^T USA)^k|^2 \right\rangle_{\text{COE}(N)} = \left\langle |\text{Tr}U^k|^2 \right\rangle_{\text{COE}(N)},\]
since the integration measure is invariant with respect to change of variables from $U$ to $(SA)^T USA$. \qed

3.2. Trace expansion. We begin our march towards the proof of Theorem 2.1 by expanding the trace of the power in the definition of moment function $M_k(N)$,
\begin{align}
\text{Tr}(PU)^k &= \sum_{n_1, n_2, \ldots, n_k = 1}^N (PU)_{n_1, n_2} (PU)_{n_2, n_3} \cdots (PU)_{n_k, n_1} \\
&= \sum_{n_1, n_2, \ldots, n_k = 1}^N U_{P(n_1), n_2} U_{P(n_2), n_3} \cdots U_{P(n_k), n_1} \\
&= \sum_{I \in F} U_{i_1, i_T} U_{i_2, i_1} \cdots U_{i_k, i_{k-1}},
\end{align}
where in (19) we assigned a different letter to each index of $U$ and introduced the set $F$
\begin{equation}
F_N := \left\{ (i_1, i_T, i_2, \ldots, i_k, i_{\bar{k}}) \in [N]^{2k} : P_N(i_\bar{b}) = i_{\bar{b} + 1} \quad \forall b \in [k] \right\},
\end{equation}
with \( b + 1 \) understood to be modulo \( k \). The ordered \( 2k \)-tuple \((i_1, i_\bar{T}, i_2, \ldots, i_k, i_\bar{T})\) is denoted by \( I \) and can be viewed as a function from \( K \) to \([N]\) (refer to Section 2.2 for notation).

Expanding the square in the definition of \( M_k(N) \), we obtain
\[
M_k(N) = \sum_{I,J \in \mathcal{F}_N} \langle U_{i_1,i_\bar{T}} U_{i_2,i_\bar{T}} \cdots U_{i_k,i_\bar{T}} \bar{U}_{j_1,j_\bar{T}} \bar{U}_{j_2,j_\bar{T}} \cdots \bar{U}_{j_k,j_\bar{T}} \rangle_{\text{COE}(N)},
\]
where the bar over \( U \) denotes the complex conjugation. The average of the product of COE matrix elements is given by the so-called Weingarten calculus \([BB96, \text{Mat}12, \text{CM17}]\), which we review in Appendix A.2 (see also Appendix A.1 for its CUE counterpart). In particular, it is zero unless index values \( J \) are a permutation of the index values \( I \). More precisely,
\[
M_k(N) = \sum_{I,J \in \mathcal{F}_N} \sum_{\omega \in S_{2k} \setminus \{I \circ \omega = J\}} W^{\text{COE}}_{g,N,k}(\omega).
\]

where \( W^{\text{COE}}_{g,N,k}(\omega) \) is the Weingarten function of COE. The function \( W^{\text{COE}}_{g,N,k} \) depends only on the conjugacy class (equivalently, “cycle type” or the vector of cycle lengths) of the permutation \([\omega, T] := \omega T \omega^{-1} T^{-1}\), where \( T \in S_{2k} \) is the fixed permutation defined in equation (14).

Switching the order of summation we arrive to
\[
M_k(N) = \sum_{\omega \in S_{2k}} W^{\text{COE}}_{g,N,k}(\omega) \sum_{I \in \mathcal{F}_N : I \circ \omega \in \mathcal{F}_N} 1 =: \sum_{\omega \in S_{2k}} W^{\text{COE}}_{g,N,k}(\omega) \Phi(\omega).
\]

The following theorem gives a simple description of the function \( \Phi(\omega) \) introduced in (23) and is a major step towards proving the main result.

**Theorem 3.2.** Recall the definition of the set \( \mathcal{F}_N \), equation (20). For any \( \omega \in S_{2k} \),
\[
\Phi(\omega) := \# \{ I \in \mathcal{F}_N : I \circ \omega \in \mathcal{F}_N \}
\]
\[= \chi_\omega N^\frac{1}{2} \ell[\omega, Q],\]
where the factor \( \chi_\omega \) is the indicator function of the set
\[
\Omega_{k,N} := \{ \omega \in S_{2k} : \exists I \in [N]^{2k} \text{ s.t. } I \circ \omega \in \mathcal{F}_N \},
\]
and \( \ell[\omega, Q] \) denotes the number of cycles in the commutator \( \omega Q \omega^{-1} Q^{-1} \) with \( Q \) defined in equation (15). The set \( \Omega_{k,N} \), and therefore its indicator \( \chi_\omega \) and the function \( \Phi(\omega) \), are independent of \( N \) and permutation \( P_N \) as long as all of cycles of \( P_N \) are longer than \( 2k \).

We will prove Theorem 3.2 in the next section. At this point we would like to point out that Theorem 3.2 effectively establishes the first claim of our main result, Theorem 2.1, that \( M_k(N) \) is independent of \( P_N \): equation (23) represents \( M_k(N) \) as a finite sum of terms depending on \( \omega \) only.

### 3.3. Graph model of a permutation
To each permutation \( \omega \in S_{2k} \) we now associate a multigraph \( G_{\omega} \) which will enable us to easily access all the data needed to evaluate the contribution of a permutation \( \omega \) to the sum in (23).

**Definition 3.3.** The graph model of \( \omega \in S_{2k} \) is a multigraph\(^3\) \( G_{\omega} \) with the vertex set \( K \), see (12), and the following labeled edges: for every \( b \in [k] \) there is
- an undirected solid edge between \( b \) and \( \bar{b} \),

\(^3\)A multigraph can have more than one edge between a given pair of vertices.
Figure 1. Examples of graph models with $k = 6$. Undirected edges are drawn in violet. When there is more than one edge between a pair of vertices (one dashed and one solid), they are drawn on top of each other; if they are directed in the same way only one arrow is drawn.

- a directed solid edge from $b$ to $b + 1$ (modulo $k$),
- an undirected dashed edge between $\omega(b)$ and $\omega(b)$,
- a directed dashed edge from $\omega(b)$ to $\omega(b + 1)$.

Some examples of graph models are shown in Figure 1. Each vertex of $G_\omega$ has degree 4, being incident to one each of the four types of edges (directed edges may be incoming or outgoing). More specifically, each vertex $z \in \mathcal{K}$ is incident to the solid undirected edge connecting $z$ and $\overline{z}$, which we will denote by $(z - \overline{z})$, and is incident to the dashed undirected edge $(z - \omega T \omega^{-1}(z))$. The directed solid edge will be $(Q(z) \rightarrow z)$ if $z \in [k]$ and $(z \rightarrow Q(z))$ if $z \in \overline{[k]}$. The directed dashed edge will be $(\omega Q \omega^{-1}(z) \rightarrow z)$ if $\omega^{-1}(z) \in [k]$ or $(z \rightarrow \omega Q \omega^{-1}(z))$ if $\omega^{-1}(z) \in \overline{[k]}$.

The solid edges of $G_\omega$ form a cycle graph $C_s$ with the same vertex set $\mathcal{K}$ and the edges
\begin{equation}
1 - \overline{1} \rightarrow 2 - \overline{2} \rightarrow \ldots - \overline{k} \rightarrow .
\end{equation}
The dashed edges similarly form a cycle graph $C_d$ with the edges
\begin{equation}
\omega(1) - \omega(\overline{1}) \rightarrow \omega(2) - \omega(\overline{2}) - \ldots - \omega(\overline{k}) \rightarrow ,
\end{equation}
and $\omega$ provides an isomorphism between $C_s$ and $C_d$. Note that all directed edges point in the same direction in cycle graphs $C_s$ and $C_d$.

In a slight abuse of usual graph terminology, we will use the term directed cycle to refer to a cycle of $G_\omega$ that is made entirely of directed edges, but without regard for their direction. A directed cycle will be called balanced if there is equal number of edges going into each direction. See Figure 2 for two examples of graphs with balanced cycles and Figure 1b for an unbalanced example (namely, cycles $\{2, 3\}$ and $\{\overline{4}, 5\}$).

**Lemma 3.4.** Recall $T = (1 \overline{1}) \ldots (k \overline{k})$, $Q = (\overline{1} 2)(\overline{2} 3) \ldots (\overline{k} 1)$ and $s = (1 2 \ldots k)(\overline{1} \overline{2} \ldots \overline{k})$. For each $\omega \in S_{2k}$ the graph $G_\omega$ has the following properties.
(a) \( \omega = (2\,4)(3\,5)(\overline{2}\,4)(\overline{3}\,5) \)

(b) \( \omega = (2\,4)(3\,5)(\overline{1}\,3)(\overline{2}\,4) \)

Figure 2. Some graph models of permutations with balanced cycles, \( k = 6 \).

(1) The subgraph of \( G_\omega \) consisting of all undirected edges is a disjoint union of cycle graphs; each cycle is of even length and alternates solid and dashed edges. The number of cycles of length \( 2n \) is half the number of \( n \)-cycles of the permutation \( [\omega, T] := \omega T \omega^{-1} T^{-1} \).

(2) The subgraph of \( G_\omega \) consisting of all directed edges is a disjoint union of cycle graphs (ignoring the direction of the edges); each cycle is of even length and alternates solid and dashed directed edges. The number of cycles of length \( 2n \) is half the number of \( n \)-cycles of the permutation \( [\omega, Q] := \omega Q \omega^{-1} Q^{-1} \).

(3) The graphs of \( G_\omega \) and \( G_{\omega^n} \) are identical if and only if \( \omega' = \omega s^n \) for any \( n \in \mathbb{Z} \); the graphs \( G_\omega \) and \( G_{\omega'} \) are isomorphic if and only if \( \omega' = s^n \omega s^m \) for some \( n, m \in \mathbb{Z} \).

(4) If all cycles of \( P_N \) are longer than \( 2k \), there exist \( I \in \mathcal{F}_N \) such that \( I \circ \omega \in \mathcal{F}_N \) if and only if all directed cycles of \( G_\omega \) are balanced.

Proof. Since each vertex of \( G_\omega \) is incident to exactly one each of the four types of edges, removing all directed edges we obtain a graph with each vertex of degree 2, incident to one solid and one dashed edge. Such a graph must be a union of cycles. It has edge coloring with two colors so it must be bipartite and therefore all cycles are even.

Consider a cycle of length \( 2n \) and let \( (b, \overline{b}) \) be one of its solid edges. Starting at vertex \( b \), we will traverse the cycle by applying permutations. We know that \( T^{-1}(b) = \overline{b} \), so we have traversed the first solid edge. Next by applying \( \omega T \omega^{-1} \) to \( \overline{b} \) we traverse the undirected dashed edge incident to \( \overline{b} \). Hence, by applying \( [\omega, T] = \omega T \omega^{-1} T^{-1} \) to a vertex \( b \), we traverse two edges of its undirected cycle. Since \( b \) is in a graph cycle of length \( 2n \), we must have \( [\omega, T]^n(b) = b \). By applying \( [\omega, T] \) to \( \overline{b} \) \( n \) times, we will traverse the other \( n \) vertices of our graph cycle, again returning to \( \overline{b} \). Thus we have found two \( n \)-cycles of \( [\omega, T] \) that directly correspond to the vertices of a \( 2n \)-cycle in \( G_\omega \).

We may traverse directed \( 2n \)-cycles of \( G_\omega \) in a similar manner using \( [\omega, Q] \), again finding two corresponding \( n \)-cycles of \( [\omega, Q] \).

Consider the graphs \( G_\omega \) and \( G_{\omega s^n} \) for some permutation \( \omega \). The solid subgraph \( C_s \) will be the same for both graphs by definition. The dashed subgraph \( C_d \) in \( G_{\omega s^n} \) can then be
and this choice uniquely determines the values of $\omega$ and $m$ only if $\omega(s) = 0$. Observe that $\omega(s - 1)$ can only occur if there are exactly $|n, \omega|$ and for some $k$. Thus $\omega^n = \omega'$. The isomorphism between $G_\omega$ and $G_{\omega^n}$ is given by $s^n$. For example, a solid edge between $b$ and $\bar{b}$ in $G_\omega$ is mapped to the solid edge between $(b + n)$ and $\bar{b} + n$ in $G_{\omega^n, \omega}$. A dashed directed edge from $\omega(b)$ to $\omega(b + 1)$ in $G_\omega$ is mapped to a dashed edge from $s^n, \omega(b)$ to $s^n, \omega(b + 1)$ in $G_{\omega^n, \omega}$ and similarly for the other types of edges.

Conversely, if two graphs $G_\omega$ and $G_{\omega'}$ are isomorphic, the solid subgraph $C_s$ must remain unchanged and the only possible isomorphisms are rotations $s^n$. Thus for some $s^n$, $G_{\omega^n, \omega}$ is the same as $G_{\omega'}$. Hence we know that $\omega' = s^n \omega m$ for some $n, m \in [k]$.

To establish part (4) of the Lemma, we discuss the meaning of directed edges. Comparing the definition of a solid edge and the definition of $F_N$, equation (20), we see that $I \in F_N$ if and only if $P(i_u) = i_v$ for every solid directed edge $(u \rightarrow v)$. Similarly, $I \circ \omega \in F_N$ if and only if $P(i_u) = i_v$ for every dashed directed edge $(u \rightarrow v)$. To see the latter, observe that a dashed directed edge $(u \rightarrow v)$ means there exists a $b \in [k]$ such that $u = \omega(b)$ and $v = \omega(b + 1)$ and, on the other hand, $I \circ \omega \in F_N$ means that $P_N(i_{\omega(b)}) = i_{\omega(b + 1)}$.

Let us now consider a directed cycle. Labeling the vertices in the cycle $v_1, \ldots, v_{2n}$, we obtain $2n$ equations of the form $i_{v_j+1} = P^{\epsilon_j}(i_{v_j})$, where $\epsilon_j = \pm 1$ depending on the direction of the edge, $+1$ for $(v_j \rightarrow v_{j+1})$ and $-1$ for $(v_j \leftarrow v_{j+1})$. Continuing the substitution process all the way around the cycle, we will eventually obtain an equation relating $i_{v_1}$ to itself of the form $i_{v_1} = P^m(i_{v_1})$ where $m = \sum_{j=1}^{2n} \epsilon_j$. Because $|m| \leq 2n \leq 2k$ and we are are only allowing $P$ to have cycles strictly longer than $2k$, one can make an assignment to $i_{v_1}$ if and only if $m = 0$. Observe that $m = 0$ can only occur if there are exactly $n$ edges directed along the cycle and $n$ edges pointing in the other direction, i.e. the cycle is balanced.

For the later use, we also note that if the cycle is balanced, $i_{v_1}$ can be chosen arbitrarily and this choice uniquely determines the values of $i_{v_2}, \ldots, i_{v_{2n}}$ through the recursion $i_{v_{j+1}} = P^{\epsilon_j}(i_{v_j})$.

**Remark 3.5.** It is not hard to describe the set of all “allowable” graphs that are graph models of some permutation $\omega \in S_{2k}$ and to show that the mapping between $S_{2k}$ and allowable graphs is $k$-to-1, but we will not need it in this paper.

**Remark 3.6.** There is also an algebraic characterization of $\chi(\omega)$ that we present here without proof since we will not use it. For any $z \in K := [k] \cup \bar{K}$ denote by $O_z$ the orbit of $z$ under the action of $[\omega, Q] := \omega Q \omega^{-1} Q^{-1}$. Then $\chi(\omega) = 1$ if and only if for all $z \in K$ the sets $O_z$ and $\omega^{-1}(O_z)$ contain the same number of symbols from $[\bar{K}]$. 

represented by

$$
(29) \quad \omega s^n(1) - \omega s^n(\bar{1}) \rightarrow \omega s^n(2) - \ldots - \omega s^n(\bar{k}) \rightarrow \omega(1 + n) - \omega(\bar{1} + n) \rightarrow \omega(2 + n) - \ldots - \omega(\bar{k} + n) \rightarrow .
$$

$$
(30) \quad \omega s^n(1) - \omega s^n(\bar{1}) \rightarrow \omega(1 + n) - \omega(\bar{1} + n) \rightarrow \omega(2 + n) - \ldots - \omega(\bar{k} + n) \rightarrow .
$$

Since addition is performed modulo $k$, we see that (30) is identical to (28). Thus the graphs $G_\omega$ and $G_{\omega s^n}$ are the same as unions of identical pairs of graphs.

Conversely, consider two graphs $G_\omega$ and $G_{\omega'}$ that are the same. Then their dashed subgraphs will be the same, that is:

$$
(31) \quad \omega(1) - \omega(\bar{1}) \rightarrow \omega(2) - \ldots - \omega(\bar{k}) \rightarrow \omega'(1) - \omega'(\bar{1}) \rightarrow \omega'(2) - \ldots - \omega'(\bar{k}) \rightarrow .
$$

This implies that $\omega(i + n) = \omega'(i)$ for all $i \in [k]$ and for some $n$. Thus $\omega^n = \omega'$.
We recall now that in expansion (23) of $M_k(N)$, the Weingarten function depends on the cycle type of the permutation $[\omega, T]$, whereas the factor $\Phi(\omega)$ depends on the cycle type of the permutation $[\omega, Q]$ as well as the function $\chi_\omega$. Now that this information is represented readily in the graph model $G_\omega$, we are ready to prove Theorem 3.2.

**Proof of Theorem 3.2.** As long as $P_N$ has cycles longer than $2k$, a permutation $\omega \in S_{2k}$ belongs to the set $\Omega_{k,N}$ if and only if the graph $G_\omega$ has balanced directed cycles. The latter property is independent of $N$ and the particular choice of $P_N$.

To count the number of ordered $2k$-tuples $I$ appearing in the definition of the function $\Phi(\omega)$, equation (24), we note that for each directed cycle we have the freedom to chose $i_{v_i}$ arbitrarily from $[N]$; all other $i_v$ participating in the cycle are determined recursively (see the proof of Lemma 3.4). By Lemma 3.4, part (2), the number of directed cycles is $1/2\ell[\omega, Q]$. Since each directed cycle gives $N$ choices independently of other cycles, the total number of possible choices of $I$ is $N^{1/2\ell[\omega, Q]}$. □

### 3.4. Contributing permutations.

We are now getting ready to use expansion (23) to evaluate the $k$-th moment $M_k(N)$. We review the properties of Weingarten COE function $W_{k_N,k}$ in Appendix A.2. At this point we just need the observation that

$$W_{k_N,k}(\omega) = O \left( \frac{1}{N^{2k-1/2\ell[\omega, T]}} \right),$$

where $\ell[\omega, T]$ is the length (number of cycles) of the permutation $[\omega, T]$.

Combining this with Theorem 3.2, we obtain that the contribution of a permutation $\omega$ has the leading order of $1/N$ to power $2k - \frac{1}{2}\ell[\omega, T] - \frac{1}{2}\ell[\omega, Q]$.

**Example 3.7.** If $\omega = s^n$, then $\frac{1}{2}\ell[\omega, T] = \frac{1}{2}\ell[\omega, Q] = k$ producing a constant term contribution. We will show that all other cycles give contribution with a higher power of $1/N$.

To establish Theorem 2.1 we will need to investigate graphs with large $\frac{1}{2}\ell[\omega, T] + \frac{1}{2}\ell[\omega, Q]$. This means that vast majority of directed and undirected cycles in the graph models $G_\omega$ ought to have length 2 (to maximize the number of cycles that can be made from a fixed number of edges); other cycles we will call “long”. In fact we will only need to consider graph models with

- no long cycles — leading order
- one long cycle (of either type) of length 4 — order $1/N$
- two cycles of length 4 or one cycle of length 6 — order $1/N^2$.

We further classify the permutations $\omega$ by whether or not they mix the elements of $K$ of different types, with and without bars.

**Definition 3.8.** A permutation $\omega \in S_{2k}$ is called regular if, for every $b \in [k] \subset K$, $\omega(b) \in [k]$. The set of all regular permutations can be naturally represented as $S_k \times S_k \subset S_{2k}$, with two halves acting on $[k]$ and $[\bar{k}]$ respectively. A permutation $\omega \in S_{2k}$ that is not regular (i.e. there exist $b_1, b_2 \in [k]$ such that $\omega(b_1) = \bar{b}_2$) is called irregular.

We will denote by $R_{\alpha}^3$ (corresp. $T_{\alpha}^3$) the set of regular (corresp. irregular) permutations $\omega \in S_{2k}$ with $\chi_\omega = 1$, with cycle half-type of $[\omega, T]$ equal to $\alpha$ and with cycle half-type of $[\omega, Q]$ equal to $\beta$.

**Example 3.9.** The regular permutation $\omega = (2 4)(3 5)(\overline{2 4})(\overline{3 5})$ contributes $R_{id}^3$, since $G_\omega$ has a directed cycle of length 6, see Figure 2a. The regular permutation $\omega = (2 4)(3 5)(\overline{1 3})(\overline{2 4})$...
contributes to $R_{id}^{id}$ since its cycle of length 6 is undirected, see Figure 2b. Both contribute to $M_k(N)$ at order $1/N^2$.

Permutations shown in Figure 1 are irregular. The permutation $\omega = (1 \, \overline{4})$ in Figure 1a contributes to $T_{21}^{id}$ while the permutation in Figure 1b has unbalanced cycles and hence does not contribute.

**Remark 3.10.** It can be easily seen from the definition of the graph model $G_\omega$ that if $\omega$ is regular then both types of directed edges point from a vertex in $[k]$ to a vertex in $[k]$. Therefore, along any directed cycle the edge directions must alternate and, by Lemma 3.4 part (4), $\chi_\omega = 1$ for every regular $\omega$.

We will classify all irregular permutations contributing to the remaining relevant sets, such as $I_{id}^{id}$, $I_{21}^{id}$, $I_{21}^{21}$ etc. We will not, however, classify regular permutations directly. Instead, in the course of proving Theorem 2.1 we will establish a pair of identities which turns out to be sufficient for evaluating their contribution.

We now list the necessary information about permutations contributing to the expansion of $M_k(N)$ to three leading orders. The proofs are highly technical and are deferred to Appendices B and C correspondingly.

**Lemma 3.11.** We have
\begin{equation}
R_{id}^{id} = \{s^n\}_{n \in [k]} \quad |R_{id}^{id}| = k^n,
\end{equation}
\begin{equation}
R_{id}^{21} = \emptyset.
\end{equation}

**Lemma 3.12.** We have
\begin{equation}
I_{\alpha}^{id} = \emptyset \quad \text{for any } \alpha,
\end{equation}
\begin{equation}
I_{id}^{21} = \{s^n(1 \, \overline{4})s^m\}_{n,m \in [k]} \quad |I_{id}^{21}| = k^{2},
\end{equation}
\begin{equation}
I_{id}^{21} = \{s^n(b \, \overline{1})s^m\}_{n,m \in [k], \, 3 \leq b \leq k} \quad |I_{id}^{21}| = k^{2}(k - 2),
\end{equation}
\begin{equation}
I_{id}^{21} = \{s^n(1 \, \overline{b})s^m \mid s^n(1 \, \overline{1})(b \, \overline{b})s^m\}_{n,m \in [k], \, 3 \leq b \leq k - 1} \quad |I_{id}^{21}| = k^{2}(k - 3),
\end{equation}
\begin{equation}
I_{id}^{21} = \{s^n(1 \, \overline{b})(2 \, \overline{b})s^m\}_{n,m \in [k]} \quad |I_{id}^{21}| = k^{2}
\end{equation}

**Remark 3.13.** The restriction $b \geq 3$ in equations (37) and (38) suggest a natural question about which set do $\omega = (2 \, \overline{1})$ and $\omega = (1 \, \overline{2})(2 \, \overline{1})$ belong to. It turns out they do not contribute at all due to having unbalanced directed cycles.

**Remark 3.14.** Counting directly from our set notation, it would appear that $|I_{id}^{21}| = 2k^2(k - 3)$. However, there is a symmetry in this notation causing each permutation to be counted exactly twice. The details are described in the proof of (37), see Lemma C.4.

3.5. **Proof of the main result.** The last ingredient we need for proving our main result is the expansion of the relevant Weingarten functions in inverse powers of $N$. These expansions are obtained using orthogonality relations in Appendix A.2.

**Proof of Theorem 2.1.** Recall that through a combination of equation (23) and Theorem 3.2 we have represented $M_k(N)$ as
\[ M_k(N) = \sum_{\omega \in S_{2k}} \chi_\omega W_{C,\omega}^{S_{N,k}}(\omega) N^{\frac{1}{2}} \ell[\omega, Q], \]
where the indicator function $\chi_\omega$ was shown to be independent of the permutation $P_N$ under the conditions of the present theorem. We thus have a finite sum of rational functions of $N$ with no dependence on $P_N$ in any contributing term. Therefore $M_k(N)$ is a rational function of $N$ and has a convergent asymptotic expansion in $\frac{1}{N}$. We are therefore justified (for a fixed $k$) to evaluate this expansion term-by-term.

We start by evaluating the corresponding moments for the CUE matrices. This will allow us to evaluate the contribution of the regular permutation to $M_k(N)$ from the limited information provided by Lemma 3.11; in addition, it is a warm-up for the more lengthy calculation of the full COE expansion. Define

$$M_{CUE}^k(N) := \langle |\Tr(PU)^k|^2 \rangle_{CUE(N)}.$$  

On one hand, since the measure on $CUE(N)$ is by definition invariant with respect to multiplication by a unitary matrix $P$, we have

$$M_{CUE}^k(N) = \langle |\Tr(UP)^k|^2 \rangle_{CUE(N)} = \langle |\Tr(U)^k|^2 \rangle_{CUE(N)} = k,$$

see equation (3).

On the other hand, expanding $\Tr(UP)^k$ as in (19), we obtain

$$M_{CUE}^k(N) = \sum_{I, J \in F_N} \langle U_{i_1,j_1} \cdots U_{i_k,j_k} \cdots U_{j_1,j_1} \cdots U_{j_k,j_k} \rangle_{CUE(N)},$$

where the definition of $F_N$ remains identical to COE case, equation (20). Averages of products of elements of CUE matrices is evaluated using Weingarten functions $W_{CUE}^{N,k}$, see Appendix A.1. The result is usually written using two permutations $\sigma$ and $\pi$ acting on indices with and without bars correspondingly. We will view them as two halves of a regular permutation $\omega$, see Definition 3.8, and will write $\omega = (\sigma, \pi) \in S_k \times S_k \subset S_{2k}$. Applying (57), we get

$$M_k^{CUE}(N) = \sum_{\lambda \vdash k} W_{CUE}^{N,k} (\lambda) \sum_{\omega \in S_{2k}} 1 \Phi(\omega) =: \sum_{\lambda \vdash k} W_{CUE}^{N,k} (\lambda) P_{\lambda}^R,$$

where $\Phi(\omega)$ is defined by (24) and therefore obeys Theorem 3.2.

Since $W_{CUE}^{N,k} (\sigma^{-1}\pi)$ only depends on the cycle structure of $\sigma^{-1}\pi$, for the sake of creating the following sum, we will write $W_{CUE}^{N,k} (\lambda)$, where $\lambda$ is the cycle structure of $\sigma^{-1}\pi$. It is important to note that for $\omega = (\sigma, \pi)$,

$$[\omega, T] = \omega T \omega^{-1} T^{-1} = (\sigma \pi^{-1}, \pi \sigma^{-1}),$$

and therefore $\lambda$ is half the cycle structure of $[\omega, T]$, analogously to the COE case. Defining

$$S_{CUE}^\lambda := \{\omega = (\sigma, \pi) \in S_k \times S_k : \sigma^{-1}\pi \in S_k \text{ has cycle type } \lambda\},$$

we may write

$$M_k(N) = \sum_{\lambda \vdash k} W_{CUE}^{N,k} (\lambda) \sum_{\omega \in S_{CUE}^\lambda} \Phi(\omega) =: \sum_{\lambda \vdash k} W_{CUE}^{N,k} (\lambda) P_{\lambda}^R.$$
From Theorem 3.2 we conclude that $P^R_N$ is a polynomial in $N$. More precisely, it can be expanded as

\begin{equation}
(44) \quad P^R_N = |\mathcal{R}^\text{id}_N| N^k + \left( |\mathcal{R}^2_1| + \left( N^k - 1 + \left( \left( N^k + 1 \right) + \left( N^k + 2 \right) \right) \right) \right) N^k - \ldots
\end{equation}

Combining equations (41) and (43) we get

\begin{equation}
(45) \quad k = P^R_N \mathcal{W}^{\text{CUE}}_{N,k}(\text{id}) + P^R_N \mathcal{W}^{\text{CUE}}_{N,k}(2^1) + P^R_N \mathcal{W}^{\text{CUE}}_{N,k}(2^2) + P^R_N \mathcal{W}^{\text{CUE}}_{N,k}(3^1) + \ldots,
\end{equation}

and no other Weingarten functions have terms of orders $N^{-k}$, $N^{-k-1}$ or $N^{-k-2}$. The relevant terms of these relevant functions are (see Appendix A.1)

\begin{align}
(46) & \quad \mathcal{W}^{\text{CUE}}_{N,k}(\text{id}) = \frac{1}{N^k} + \frac{0}{N^{k+1}} + \frac{k(k-1)/2}{N^{k+2}} + \ldots \\
(47) & \quad \mathcal{W}^{\text{CUE}}_{N,k}(2^1) = -\frac{1}{N^{k+1}} + \frac{0}{N^{k+2}} + \ldots, \\
(48) & \quad \mathcal{W}^{\text{CUE}}_{N,k}(3^1) = \frac{2}{N^{k+2}} + \ldots, \\
(49) & \quad \mathcal{W}^{\text{CUE}}_{N,k}(2^2) = \frac{1}{N^{k+2}} + \ldots
\end{align}

Substituting these expansions together with (44) into (45) and collecting the terms contributing to the first three orders of a $1/N$ expansion, we obtain

\begin{equation}
\begin{split}
k = |\mathcal{R}^\text{id}| + \left( \left( N^k - 1 + \left( \left( N^k + 1 \right) + \left( N^k + 2 \right) \right) \right) \right) \frac{1}{N} \\
+ \left( \frac{k(k-1)}{2} + \left( N^k - 1 + \left( \left( N^k + 1 \right) + \left( N^k + 2 \right) \right) \right) \right) \frac{1}{N^2} + \ldots
\end{split}
\end{equation}

We already know that $|\mathcal{R}^\text{id}| = k$ from (34) and the $\frac{1}{N}$ term we get $|\mathcal{R}^\text{id}| = |\mathcal{R}^2| = 0$. Finally, from the $\frac{1}{N^2}$ term we get

\begin{equation}
(50) \quad \frac{k^2(k-1)}{2} + |\mathcal{R}^2| + |\mathcal{R}^3| - 2|\mathcal{R}^\text{id}| + |\mathcal{R}^\text{id}| = 0.
\end{equation}

Now we similarly expand our primary target, the moment function $M_k(N)$ for COE. Combining equation (23) with Theorem 3.2 and Definition 3.8, and using the fact that $\mathcal{R}^\text{id}$, $\mathcal{R}^2$ and $\mathcal{T}^\text{id}$ are empty to reduce the number of terms, we get

\begin{equation}
M_k(N) = \mathcal{W}^{\text{COE}}_{N,k}(\text{id}) \left( \left| \mathcal{R}^\text{id}_N \right| N^k + \left| \mathcal{I}^2_1 \right| N^{k-1} + \left( \left| \mathcal{I}^2_1 \right| + \left| \mathcal{I}^3_1 \right| + \left| \mathcal{I}^3_1 \right| \right) N^{k-2} \right) \\
+ \mathcal{W}^{\text{COE}}_{N,k}(2^1) \left( \left| \mathcal{R}^2_1 \right| + \left| \mathcal{I}^3_1 \right| \right) N^{k-1} \\
+ \mathcal{W}^{\text{COE}}_{N,k}(2^2) \left| \mathcal{R}^\text{id}_N \right| N^k + \mathcal{W}^{\text{COE}}_{N,k}(3^1) \left| \mathcal{R}^\text{id}_N \right| N^k + O\left( \frac{1}{N^3} \right).
\end{equation}
The necessary Weingarten functions are (see Appendix A.2)

\begin{align}
W_{g,\text{COE}}(\text{id}) &= \frac{1}{N^k} - \frac{k}{N^{k+1}} + \frac{k(3k - 1)/2}{N^{k+2}} + \ldots \tag{52} \\
W_{g,\text{COE}}(2^1) &= -\frac{1}{N^{k+1}} + \frac{k + 2}{N^{k+2}} + \ldots \tag{53} \\
W_{g,\text{COE}}(3^1) &= \frac{2}{N^{k+2}} + \ldots \tag{54} \\
W_{g,\text{COE}}(2^2) &= \frac{1}{N^{k+2}} + \ldots \tag{55}
\end{align}

Substituting these together with irregular counts given in Lemma 3.12, we get

\begin{align*}
M_k(N) &= \left(1 - \frac{k}{N} + \frac{3k^2 - k}{2N^2} \right) \left( k + \frac{k^2}{N} + \frac{k^3 - 2k^2 + \left| R_{id}^{2^1} \right| + \left| R_{id}^{3^1} \right|}{N^2} \right) \\
&\quad - \frac{1}{N^2} \left( \left| R_{2^1}^{2^1} \right| + k^2(k - 2) \right) + \frac{1}{N^2} \left| R_{2^2}^{id} \right| + \frac{2}{N^2} \left| R_{3^1}^{id} \right| + O\left(\frac{1}{N^3}\right).
\end{align*}

Expanding and collecting terms, we use (50) to get

\begin{align*}
M_k(N) &= k + \frac{0}{N} + \frac{0}{N^2} + O\left(\frac{1}{N^3}\right) = k + O\left(\frac{1}{N^3}\right),
\end{align*}

which is the desired result. \qed

\textbf{Acknowledgment}

This material is based upon work supported by the National Science Foundation under Grant No. 1815075 and by the Binational Science Foundation under Grant No. 2016281. The idea for this research project arose during discussions with Chris Joyner and we are profoundly grateful for his help and support. We would like to thank Jon Keating and the interest he took in our results and for bringing the work [Mec08] to our attention. We are grateful to the anonymous referee for several improving suggestions.

\textbf{Data Availability}

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

\textbf{Appendix A. Weingarten calculus}

\textbf{A.1. Circular Unitary Ensemble (the unitary group $U(N)$).} Unitary Weingarten functions are building blocks for integration of products of matrix elements over the unitary group. Let $k \leq N$ and let $i = (i_1, \ldots, i_k) \in [N]^k$ and $\bar{i} = (i_\tau, \ldots, i_\bar{\tau}) \in [N]^k$ be two arbitrary sequences of indices from $[N] := \{1, \ldots, N\}$ with distinct entries. The defining property of the Weingarten function of $\sigma \in S_k$ is

\begin{align}
W_{g,\text{CUE}}(\sigma) &= \int_{U(N)} U_{i_11}U_{i_22} \cdots U_{i_kk} \bar{U}_{i_{\sigma(1)}1} \bar{U}_{i_{\sigma(2)}2} \cdots \bar{U}_{i_{\sigma(k)}k} du,
\end{align}

where $du$ is the uniform (Haar) measure on the unitary group $U(N)$. Due to invariance properties of the measure, the value of the function is independent of the choice of the
sequences $\mathbf{i}$ and $\mathbf{i}^\prime$; both are often taken to be equal to $(1, 2, \ldots, k)$, although that tends to obscure the meaning of $\sigma$ as a permutation acting on the order of indices $i$ rather than their values. Furthermore, Weingarten functions depend only on the equivalence class of $\sigma$, which is in turn determined by its cycle structure.

Knowledge of Weingarten functions is enough to evaluate the average of a general product of the elements of a $U \in U(N)$ via the following formula. Let $\mathbf{i}, \mathbf{i}^\prime \in [N]^k$ and $\mathbf{j}, \mathbf{j}^\prime \in [N]^{k'}$. Then

$$
\int_{U(N)} U_{i_1\tau} \cdots U_{i_k\tau} U_{j_{1\prime}\tau} \cdots U_{j_{k\prime}\tau} \, du = \delta_{k,k'} \sum_{\sigma \in S_k : \sigma(i) = j, \pi \in S_{k'} : \pi(i) = j} W_{N,k}^{\text{CUE}}(\sigma^{-1}\pi),
$$

where $\delta_{k,k'}$ is the Kronecker delta function.

One of the possible way to compute Weingarten functions is via the orthogonality relations

$$
NW_{N,k}^{\text{CUE}}(\omega) + \sum_{i=1}^{k-1} W_{N,k}^{\text{CUE}}((i\,k)\omega) = \delta_{\omega(k),k} W_{N,k-1}^{\text{CUE}}(\omega^i),
$$

where $(i\,k)$ is the transposition between $i$ and $k$, and where $\omega^i$ is the restriction of $\omega$ from $S_k$ to $S_{k-1}$, which is well defined here due to condition $\omega(k) = k$ enforced by the Kronecker delta. Relations (58) together with the initial condition $W_{N,0}^{\text{CUE}}(\emptyset) = 1$ (or, one step up, $W_{N,1}^{\text{CUE}}((1)) = 1/N$) fully determine the Weingarten functions for all $N$ and $k \leq N$.

We will denote the cycle structure of $\sigma$ as $1^{\alpha_1}2^{\alpha_2}\ldots k^{\alpha_k}$, where $\alpha_j$ is the number of cycles of $\sigma$ of length $j$. We note that $\alpha_1 + \alpha_2 + \ldots + \alpha_k = \ell(\sigma)$ and $\alpha_1 + 2\alpha_2 + \ldots + k\alpha_k = k$. Since a Weingarten function depends only on the cycle structure of $\sigma$, we will abuse notation slightly and write $W_{N,k}^{\text{CUE}}(1^{\alpha_1}2^{\alpha_2}\ldots k^{\alpha_k})$ for $W_{N,k}^{\text{CUE}}(\sigma)$ when convenient.

The first term in the asymptotic expansion of a Weingarten function as $N \to \infty$ is

$$
W_{N,k}^{\text{CUE}}(1^{\alpha_1}2^{\alpha_2}\ldots k^{\alpha_k}) = \prod_{j=1}^{k} \left(W_{N,j}^{\text{CUE}}(j^1)\right)^{\alpha_j} + \mathcal{O}(N^{\ell-2k-2}), \quad \ell = \ell(\sigma),
$$

where

$$
W_{N,j}^{\text{CUE}}(j^1) = (-1)^{j-1} C_{j-1} N^{1-2j} + \mathcal{O}(N^{1-2j}), \quad C_{j-1} = \frac{1}{j} \left(2j - 2 \atop j - 1\right),
$$

where $C_n$ are the Catalan numbers. We note that the product $\prod_{j=1}^{k} \left(W_{N,j}^{\text{CUE}}(j^1)\right)^{\alpha_j}$ is of order $N^{\ell-2k}$, meaning the asymptotic expansion of the CUE Weingarten function does not have a term of the order $N^{\ell-2k-1}$. It should be emphasized that $j^1$ refers to cycle structure of a permutation with $j$ elements.

At this point we provide a historical information on Weingarten calculus, to the best of our knowledge. Weingarten functions were first defined and systematically studied by Samuel [Sam80] who obtained expansion (57), orthogonality relations (58) as well as an expression for $W_{N,k}^{\text{CUE}}$ in terms of characters of $S_k$ (the proof of the expression is attributed in [Sam80] to Fritz Beukers). The function is named after Weingarten who in an earlier work [Wei78] obtained asymptotic results equivalent to (59). Averages over unitary group were used extensively in physics (see [BB96] for one of many applications, to quantum transport) and were eventually rediscovered in the mathematical literature by Collins [Col03]. A beautiful interpretation of asymptotic coefficients of the Weingarten function as the number of monotone factorizations by Matsumoto and Novak [MN13] allowed one of the present authors...
with Kuipers [BK13a] to put the use of random matrix theory in quantum chaotic transport on a more solid mathematical basis. Notation in the present section is kept in line with the mathematical sources such as [Mat13, CM17].

In the present work we use the first few terms of Weingarten functions for specific cycle structures, whose calculation we will now discuss. More precisely, we need all Weingarten functions whose expansion has terms of order $N^{-k}$, $N^{-k-1}$, and $N^{-k-2}$. From (59)-(60), these Weingarten functions are $W_{gn,k}^{\text{CUE}}(\text{id})$, $W_{gn,k}^{\text{CUE}}(2^1)$, $W_{gn,k}^{\text{CUE}}(3^1)$ and $W_{gn,k}^{\text{CUE}}(2^2)$. In our notation we have omitted $1^{\alpha_1}$ and any factor where $\alpha_j = 0$, for the sake of brevity. The leading order term of each function can be obtained from the product in (59), while the next order term is 0 because the error bound in (59). The resulting expansions are given by (46)-(49), but we have yet to determine the third term in the expansion of

\begin{equation}
W_{gn,k}^{\text{CUE}}(\text{id}) = \frac{1}{N^k} + \frac{0}{N^{k+1}} + \frac{t_k}{N^{k+2}} + \ldots
\end{equation}

In order to do so, we write out equation (58) as it applies to $\text{id} \in S_k$.

\begin{equation}
NW_{gn,k}^{\text{CUE}}(\text{id}) + (k - 1)W_{gn,k}^{\text{CUE}}(2^1) = W_{gn,k}^{\text{CUE}}(\text{id}),
\end{equation}

or, extracting the coefficients of the term $1/N^{k+1}$ on both sides,

\begin{equation}
t_k = k - 1 + t_{k-1}.
\end{equation}

Since $W_{gn,1}^{\text{CUE}}(\text{id}) = 1/N$, we have $t_1 = 0$ and therefore $t_k = k(k - 1)/2$. We conclude that

\begin{equation}
W_{gn,k}^{\text{CUE}}(\text{id}) = \frac{1}{N^k} + \frac{0}{N^{k+1}} + \frac{k(k - 1)/2}{N^{k+2}} + \ldots
\end{equation}

We remark that the coefficient $k(k - 1)/2$ can also be obtained as the number of primitive (monotone) factorizations of the identity into two transpositions [MN13].

A.2. Circular Orthogonal Ensemble (compact symmetric space $U(N)/O(N)$). Circular Orthogonal Ensemble was introduced in the seminal article [Dys62a] (see also its "prequel" published later the same year [Dys62b]). The rest of the references is given at the end of this section, after the relevant facts are stated.

For a given $\omega \in S_{2k}$, considered as a permutation of $\mathcal{K}$, the COE Weingarten function is defined by

\begin{equation}
W_{gn,k}^{\text{COE}}(\omega) = \int_{\text{COE}(N)} U_{i_1i_{\tau}} U_{i_2i_{\tau}} \ldots U_{i_{k_{\tau}} \overline{U}_{i_{\omega(1)i_{\omega(2)}} \overline{U}_{i_{\omega(2)i_{\omega(3)}}} \overline{U}_{i_{\omega(3)i_{\omega(4)}}} \ldots \overline{U}_{i_{\omega(k-1)i_{\omega(k)}}} du},
\end{equation}

where $du$ is the COE probability measure and $(i_1, i_{\tau}, \ldots, i_{k_{\tau}}, i_{\overline{\tau}}) =: I$ is an arbitrary sequence of distinct values from $[N]$. The average of any product of matrix elements can now be calculated as follows: for any $I: K \to [N]$ and $J: K' \to [N]$

\begin{equation}
\int_{\text{COE}(N)} U_{i_1i_{\tau}} \ldots U_{i_{k_{\tau}} \overline{U}_j \ldots \overline{U}_{i_{k_{\tau}} \overline{U}}} du = \delta_{k,k'} \sum_{\omega \in S_{2k}: I_0\omega = J} W_{gn,k}^{\text{COE}}(\omega).
\end{equation}

Orthogonality relations for $W_{gn,k}^{\text{COE}}$ take the following form:

\begin{equation}(N + 1)W_{gn,k}^{\text{COE}}(\omega) + \sum_{z=1}^{k-1} W_{gn,k}^{\text{COE}}((z \ast k)\omega) + \sum_{z=1}^{k-1} W_{gn,k}^{\text{COE}}((\overline{z} \ast k)\omega) = \delta_{\{\omega(k), \omega(z)\}, \{k_{\tau}, k_{\overline{\tau}}\}} W_{gn,k-1}^{\text{COE}}(\omega^k),\end{equation}
where the right-hand side is non-zero if and only if \( \omega \) leaves the set \( \{ k, \bar{k} \} \) invariant, in which case \( \omega' \) is the natural projection of \( \omega \) to \( S_{2(k-1)} \).

From invariance properties of COE one deduces that \( W_{g, N,k}^{\text{COE}} \) depends only on the cycle structure of the permutation \( [\omega, T] = \omega T \omega^{-1}T^{-1} \). The latter permutation has an even number of cycles of any length. We will therefore use \( W_{g, N,k}^{\text{COE}}(1^{\alpha_1}2^{\alpha_2} \ldots k^{\alpha_k}) \) to denote the Weingarten function of a permutation \( \omega \) such that \( [\omega, T] \) has \( 2 \alpha_1 \) cycles of length 1, \( 2 \alpha_2 \) cycles of length 2 and so on. To give an example, the cycle structure corresponding to \( \omega = (1, b) \) is \( 1^k \) if \( b = 1 \) and \( 1^{k-2}2^1 \) if \( b \neq 1 \). As before, we have \( \sum \alpha_j = k \) and we let \( h = \frac{1}{2} \ell[\omega, T] := \alpha_1 + \alpha_2 + \ldots + \alpha_k \). We also note that since \( T \) is an involution, \( [\omega^{-1}, T] \) has the same cycle structure as \( [\omega, T] \) and in some sources formulas such as (65) are written in terms of \( \omega^{-1} \) instead.

For a fixed \( k \), a Weingarten function is a rational function of \( N \) with the asymptotic expansion in \( 1/N \) given by [BB96, Sec. IV]

\[
W_{g, N,k}^{\text{COE}}(1^{\alpha_1}2^{\alpha_2} \ldots k^{\alpha_k}) = \prod_{j=1}^{k} (W_{g, N,j}^{\text{COE}}(j^1))^{\alpha_j} + O(N^{h-2k-2}),
\]

with

\[
W_{g, N,j}^{\text{COE}}(j^1) = (-1)^{j-1}C_{j-1}N^{1-2j} - (4)^{j-1}N^{-2j} + O(N^{-1-2j}), \quad C_{j-1} = \frac{1}{j} \binom{2j-2}{j-1}.
\]

We note that unlike (59), expansion (67) does contain terms of order \( N^{h-2k-1} \) but they can be obtained from expanding the first product.

We will need all functions \( W_{g, N,k}^{\text{COE}} \) which have terms of order \( N^{-k-2} \) or above. According to (67)-(68) those are \( W_{g, N,k}^{\text{COE}}(\text{id}) \), \( W_{g, N,k}^{\text{COE}}(2^1) \), \( W_{g, N,k}^{\text{COE}}(2^2) \) and \( W_{g, N,k}^{\text{COE}}(3^1) \). The first two terms of each expansion can be obtained from the product in (67). For the factors one can either use (68) or explicit formulas [BB96, Table II and IV],

\[
W_{g, N,1}^{\text{COE}}(1^1) = \frac{1}{N+1},
W_{g, N,2}^{\text{COE}}(2^1) = \frac{-1}{N(N+1)(N+3)},
W_{g, N,3}^{\text{COE}}(3^1) = \frac{2}{(N-1)N(N+1)(N+3)(N+5)}.
\]

The results are given in (52)-(55).

We now employ (66) with \( \omega = \text{id} \) to determine the last term in the expansion

\[
W_{g, N,k}^{\text{COE}}(\text{id}) = \frac{1}{N^k} - \frac{k}{N^{k+1}} + \frac{t_k}{N^{k+2}} + \ldots
\]

Since \( \omega' = (z \ k) \) gives rise to cycle structure \( 2^1 \) for any \( z \neq k, \bar{k} \), we have

\[
(N+1)W_{g, N,k}^{\text{COE}}(\text{id}) + 2(k-1)W_{g, N,k}^{\text{COE}}(2^1) = W_{g, N,k-1}^{\text{COE}}(\text{id}).
\]

Substituting expansions (52) and (53) and extracting the coefficient of \( N^{-k-1} \) we get

\[
t_k = t_{k-1} + 3k - 2.
\]
From (69) we have $t_1 = 1$ resulting in

\begin{equation}
    t_k = \sum_{j=1}^{k} (3j - 2) = \frac{3k^2 - k}{2}.
\end{equation}

Much of the material of this section has first appeared in [BB96] albeit without derivation. Careful mathematical treatment was done more recently by Matsumoto and co-authors [Mat12, Mat13, CM17], whose notation we mostly follow. In particular, relation (66) first appears in [BB96, Eq. (4.3)] in terms of a different notation (using the list of cycle lengths in $[\omega, T]$) but an elegant proof is given in [CM17, Lemma 5.3].

Some other possibly relevant results that we do not use here are as follows. The coefficients of the asymptotic expansion of COE Weingarten function as $N \to \infty$ were shown [BK13a, Proof of Thm. 6.4] to count palindromic monotone (or primitive) factorizations, by analogy with the monotone factorizations for CUE [MN13]. This can be used for alternative derivation of (52)-(55) together with (72). COE Weingarten functions can be calculated as pseudo-inverse of power functions [Mat13, Eq. (2.5)] through their simple relation with Weingarten function for the orthogonal group $O(N)$, see [Mat12] and [CM17, Thm. 5.4]. Uniform bound in terms of $k$ obtained in [CM17] could be crucial to considering the simultaneous limit $k,N \to \infty$ and accessing the spectral form factor.

**Appendix B. Classifying regular permutations**

In this section we prove Lemma 3.11, in several steps. It is also a warm up for the more involved proof of Lemma 3.12 reported in Appendix C.

**Lemma B.1.** If $[\omega, T] = \text{id}$, then for $z_1, z_2 \in K$, $\omega(z_1) = z_2$ if and only if $\omega(z_1) = z_2$.

**Proof.** Let $z_1, z_2 \in K$ such that $\omega(z_1) = z_2$. Note that $\omega(z_1) = z_2$. Observe that since $[\omega, T] = \text{id}$,

\begin{equation}
    z_2 = [\omega, T](z_2) = \omega T \omega^{-1} T^{-1}(z_2) = \omega T \omega^{-1}(z_2) = \omega T(z_1) = \omega(z_1).
\end{equation}

Hence $\omega(z_1) = z_2$. The other direction comes immediately from replacing $z_1$ with $z_2$ in the calculation above. \qed

**Lemma B.2.** If $[\omega, Q] = \text{id}$, then for $m, p \in [k]$, $\omega(m) = \bar{p}$ if and only if $\omega(m+1) = p+1$.

**Proof.** Let $m, p \in [k]$ such that $\omega(m) = \bar{p}$. Observe that since $[\omega, Q] = \text{id}$,

\begin{equation}
    p + 1 = [\omega, Q](p + 1) = \omega Q \omega^{-1} Q^{-1}(p + 1) = \omega Q \omega^{-1}(\bar{p}) = \omega Q(m) = \omega(m + 1).
\end{equation}

Hence $\omega(m+1) = p+1$. The reverse direction is calculated similarly. \qed

**Lemma B.3.** $\mathcal{R}_{\text{id}}^{21} = \{s^n : 0 \leq n \leq k-1\}$.

**Proof.** Since all regular permutations $\omega$ map $[k]$ to $[k]$ and $[\overline{k}]$ to $[\overline{k}]$, we know for some $m, p \in [k]$, $\omega(m) = p$. Lemma B.1 implies that $\omega(m) = \bar{p}$, which by Lemma B.2 implies $\omega(m+1) = p+1$, etc. This means that $\omega = s^{p-m}$.

\qed

**Lemma B.4.** $\mathcal{R}_{\text{id}}^{21} = \emptyset$.

**Proof.** Suppose there is an $\omega \in \mathcal{R}_{\text{id}}^{21}$, which by the definition must contain exactly one directed cycle of length 4. Denote the solid edges involved in this cycle by $l \to l+1$ and $m-1 \to m$. Since $\omega$ is regular, and the dashed directed edges in the graph must go from an element of $[\overline{k}]$ to an element of $[k]$, the dashed edges involved in the cycle of length 4 must be $l \to m$ and
$m - 1 \rightarrow l + 1$. All other cycles, both directed and undirected, have length 2 and thus every other dashed directed (corresponding undirected) edge coincides with a solid directed (corresponding undirected) edge. The graph therefore fits the shape in Figure 3 which is not permissible, since it is not possible for the dashed edges to form a single cycle. Thus the set $R_{\text{id}}^{21}$ is empty.

![Figure 3](image_url)

**Figure 3.** A candidate graph of $\omega \in R_{\text{id}}^{21}$.

For comparison, we also provide an algebraic proof. Notice that for all $\omega \in S_{2k}$, $\omega QT\omega^{-1} \in S_{2k}$ consists of two cycles of length $k$, namely

$$
(\omega(1) \omega(2) \ldots \omega(k))(\omega(k) \ldots \omega(\overline{k}) \omega(\overline{1})).
$$

(75)

Since $\alpha = \text{id}$, $\omega T = T\omega$. Since $\omega$ is regular, $\omega QT\omega^{-1}Q^{-1}$ leaves $[k]$ invariant. We know that $\omega QT\omega^{-1}Q^{-1}$ has only one nontrivial cycle in $[k]$ and that this cycle has length 2. Let $(\overline{q}_1 \overline{q}_2)$ be this cycle. Then we have

$$
\omega Q(\overline{q}_1) = Q\omega(\overline{q}_2), \quad \omega Q(\overline{q}_2) = Q\omega(\overline{q}_1),
$$

$$
\omega Q(b) = Q\omega(b) \quad \text{for all } b \in [k] \setminus \{\overline{q}_1, \overline{q}_2\}.
$$

Introduce the notation $r_1 = \omega(q_1) = \omega T(\overline{q}_1)$ and $r_2 = \omega(q_2) = \omega T(\overline{q}_2)$ and assume, without loss of generality, that $r_1 < r_2$. We will next understand the action of $\omega QT\omega^{-1}$ on $r \neq r_1, r_2$. We have $T\omega^{-1}(r) \neq \overline{q}_1, \overline{q}_2$ and therefore

$$
\omega QT\omega^{-1}(r) = Q\omega T\omega^{-1}(r) = QT\omega\omega^{-1}(r) = QT(r) = r + 1.
$$

(76)

On the other hand,

$$
\omega QT\omega^{-1}(r_2) = \omega Q(\overline{q}_2) = Q\omega(\overline{q}_1) = Q\omega T(q_1) = QT\omega(q_1) = QT(r_1) = r_1 + 1.
$$

(77)

Then repeated compositions of $\omega QT\omega^{-1}$ will produce the cycle:

$$
(r_2 \ (r_1 + 1) \ (r_1 + 2) \ldots \ (r_2 - 1)).
$$

(78)

We have just obtained a cycle of $\omega QT\omega^{-1}$ that does not contain $r_1$, but according to (75), a single cycle must contain all elements of $[k]$. This is a contradiction and therefore $R_{\text{id}}^{21} = \emptyset$.

**Remark B.5.** The algebraic proof of Lemma B.4 has certain advantages (it is easier to check that every eventuality is considered), but it is certainly longer and harder to construct. In fact, it was constructed in the first place by mapping the “graph-based” proof into algebraic properties: for example, the “dashed edges form a single cycle” corresponds to the cycle structure of $\omega QT\omega^{-1}$, equation (75).

For this reason, we will use “graph-based” arguments for the even more sophisticated proofs of Appendix C.
Appendix C. Classifying irregular permutations

In this section we will prove the assertions made in Lemma 3.12. In the computations in this section, note that all addition and subtraction performed on elements of \( \mathcal{K} := [k] \cup \overline{[k]} \) is done modulo \( k \), even though not explicitly stated each time.

We will be classifying the irregular permutations into the sets \( \mathcal{I}_\alpha^\beta \) based on the cycle structures of \([\omega, T]\) and \([\omega, Q]\), see Definition 3.8. Since the cycle structures of these commutators are determined by the types of cycles in \( G_\omega \), if \( G_\omega \) and \( G_{\omega'} \) are isomorphic, then \( \omega \) and \( \omega' \) are in the same set \( \mathcal{I}_\alpha^\beta \). Thus we may further dissect our problem into finding the representative permutation \( \omega \) corresponding to each isomorphism class of graphs coming from \( \mathcal{I}_\alpha^\beta \), and expressing the rest of them as \( s^n \omega s^m \) for \( n, m \in [k] \) in accordance with Lemma 3.4 part (3).

Our task is to classify graphs with the majority of directed and undirected cycles having length two; those are formed by dashed edges coinciding with solid edges. The direction must also coincide (if the edges are directed) to make the cycle balanced. Dashed edges not coinciding with a solid edge will be called chords (see Figure 2 for the visual reason behind the term chord). To put it another way, chords are the dashed edges belonging to a cycle of length greater than 2.

We will also frequently appeal to another property of \( G_\omega \) discussed in Section 3.3: the dashed subgraph forms a single cycle with all directed edges pointing in the same direction along the cycle.

Lemma C.1. \( \mathcal{I}_\alpha^{id} = \emptyset \) for any \( \alpha \).

Proof. Since \( \beta = id \) we have no directed chords. An irregular permutation \( \omega \), by definition, satisfies \( \omega(q) = \overline{l} \) for some \( q \in [k], \overline{l} \in \overline{[k]} \). Consider the directed dashed edge \((\omega(q-1) \rightarrow \omega(q)) = (\omega(q-T) \rightarrow \overline{l})\). In order to not create a chord, it would have to match an existing solid directed edge, but the only solid directed edge incident to \( \overline{l} \) is \((\overline{l} \rightarrow \overline{l}+1)\), which points in the opposite direction. This would make an unbalanced 2-cycle, so it is not allowed. Hence, when considering irregular permutations, we will never have \([\omega, Q] = id\). \( \square \)

Lemma C.2. \( \mathcal{I}_id^{21} = \{ s^n(1 \overline{1})s^m : n, m \in [k] \} \), \( |\mathcal{I}_id^{21}| = k^2 \).

Proof. If \( \omega \in \mathcal{I}_id^{21} \), the graph model \( G_\omega \) has exactly one directed 4-cycle and no undirected cycles larger than 2. We know from the proof of Lemma C.1 that we have at least the directed chord \((\omega(q-T) \rightarrow \overline{l})\). Since we have only one directed 4-cycle, this chord must be part of it. Certainly \( \omega(q-T) \) must equal either \( m \) or \( m-1 \) for some \( m \in [k] \). We find the former case cannot work because we would have have 3 arrows pointing in the same direction in a cycle of length 4, making a balanced cycle impossible, as shown in Figure 4a. For the latter case, the diagram looks like Figure 4b, an arrangement which can be completed to become a balanced cycle.

The unique way to complete Figure 4b to produce a balanced 4-cycle is shown in Figure 5. Note that we exclude the case \( m = l + 1 \) since it produces loops and makes no sense in our graph model.

Recall that \( G_{\omega} \) must have a closed path traversing all dashed edges that alternates between directed and undirected edges where all directed edges point in the same direction along the path. Let us now consider the consequences of \([\omega, T] = id\). Since \( \omega(q) = \overline{l} \), there is an undirected dashed edge \((\overline{l} - \omega(\overline{q}))\). In order to produce no undirected chords, \((\overline{l}, \omega(\overline{q}))\) must match with the existing undirected edge attached to \( \overline{l} \), that is \((l, \overline{l})\). In other words, we must
have \( \omega(\bar{q}) = l \). Since a directed dashed edge goes into \( l \), a directed dashed edge must also leave \( l \). If this dashed directed edge follows the solid directed edge connected to \( l \), the two edges will be directionally imbalanced, as in Figure 6a. Thus there must be a directed chord leaving \( l \), but since we already have the maximum allowed number of chords, we must have \( m = l \), as shown in Figure 6b.

From this we see there can be only one isomorphism class of graphs in this set of permutations. Notice that the graph above can be produced by \( \omega = (l \bar{l}) = s^l(1 \bar{1})s^{-l} \). Thus we will characterize our set of permutations as \( \mathcal{I}_{id}^{2^1} = \{ s^n(1 \bar{1})s^m : n, m \in [k] \} \). Since the permutations \( s^n(1 \bar{1})s^m \) are different for distinct values of \( m \) and \( n \), we may conclude that \( |\mathcal{I}_{id}^{2^1}| = k^2 \).

\[ \square \]

**Lemma C.3.** \( \mathcal{I}_{2^1}^{2^1} = \{ s^n(b \bar{1})s^m : n, m \in [k], 3 \leq b \leq k \} \), \( |\mathcal{I}_{2^1}^{2^1}| = k^2(k - 2) \).
Proof. We have already shown that having $\beta = 2^1$ will give us a directed cycle of the type shown in Figure 5. Since we now have $\alpha = 2^1$, we will also have an undirected 4-cycle.

We will first consider the possibilities when we start with the specific directed 4-cycle type given in Figure 6b, that is when $m = l$. Is it possible to have an undirected 4-cycle somewhere else in the diagram without creating more directed chords? Figures 7a and 7b show the only two possible types of undirected 4-cycles that could appear. We observe that in the first, a closed walk on dashed edges would not have all directed edges pointing the same way along the path and that in the second, we cannot have a closed walk on the dashed edges at all. Hence for a permutation from $\mathcal{T}_2^b$, in Figure 5 there is at least one directed edge between $m$ and $l$. In other words, $m \neq l$.

Let us consider the edges to the left of the directed cycle, beginning with the dashed undirected edge incident to $l$. If this edge is not a chord, we would have Figure 6a since no more directed edges can be chords either. We have already noted that the dashed path is impossible because of the unbalanced 2-cycle. Similarly, when assuming the undirected dashed edge incident to $m$ is not a chord, we obtain Figure 8 which also contains an illegal path. Thus, in order to have a permissible dashed path, we need to have undirected dashed chords attached to both $m$ and $l$.

We now have two options: one is to have the dashed chord $(l, m)$, and the other is to have two different chords proceeding from $l$ and $m$. The first option is shown in Figure 9a. Notice that we have only one way to finish the undirected 4-cycle, but this configuration admits no way of having only one closed dashed path without creating more chords, which we cannot have. Thus the only possible completion of the undirected 4-cycle is the structure shown in Figure 9b, which corresponds to the permutation $\omega = (m \overrightarrow{l})$ for $m \neq l$ and $m \neq l + 1$.

The different isomorphism classes can be represented by the graphs produced by $(b \overrightarrow{T})$ for $3 \leq b \leq k$. The permutations $s^n(b \overrightarrow{T})s^m$ are different for distinct values of $b$, $m$ and $n$, we may conclude that $|\mathcal{T}_2^b| = k^2(k - 2)$. \qed
Lemma C.4. $I_{id}^{2} = \{s^{n}(1 \bar{b})(b \bar{T})s^{m}, s^{n}(1 \bar{T})(b \bar{b})s^{m} : n, m \in [k], 3 \leq b \leq k - 1\}$, $|I_{id}^{2}| = k^{2}(k - 3)$.

Proof. Graphs for the permutations $\omega \in I_{id}^{2}$ will have exactly two directed 4-cycles and no undirected chords. Since each $\omega$ is irregular, our previous work shows that at least one of the 4-cycles is of the form shown in Figure 5. Let us assume for now that we do not have $m = l$, i.e. we do not have the case shown in Figure 6b. Consider a dashed path going in the forward direction and starting with the directed edge $(m - 1 \rightarrow l)$. Since we have no undirected chords, so the next edge in the path must be $(l, l)$. As shown in Figure 6a, we cannot continue on to the directed edge $(l - 1 \rightarrow l)$ without creating an unbalanced cycle. Since $m \neq l$, we must have a third directed chord proceeding from $l$ that was not in our original 4-cycle. Similarly, following the dashed path backward from $(m \rightarrow l + 1)$, we find there must be a fourth directed chord entering $m$. This leaves only two possible configurations for what happens to the left of our original 4-cycle, as shown in Figure 10 and Figure 11. However, upon attempting to complete the second directed 4-cycle we see the case in Figure 10 would have an unbalanced directed cycle, making Figure 11 the only possible configuration, a graph corresponding to the permutation $(m \bar{l})(\bar{T} \bar{m})$. Thus the permutations corresponding to graphs of this type are $\{s^{n}(1 \bar{b})(b \bar{T})s^{m} : n, m \in [k], 3 \leq b \leq k - 1\}$.

Now consider the case where $m = l$ in our first 4-cycle. Then it would have the form shown in Figure 6b. The other 4-cycle can only take three forms, as shown in Figure 12. Notice that in Figure 12a, we are unable to complete a closed walk along the dashed edges. In Figure 12b it is possible to complete a closed walk on the dashed edges, but not all directed edges will be pointing the same way. (Note that if the directions of the dashed edges in the second 4-cycle are inverted, the same problem occurs.) Thus the only remaining option is 13, where both 4-cycles have the same form.
We already know that the individual cycles can be produced by \((l\tilde{1})\), so the complete set of permutations corresponding to graphs of this form will be \(\{s^n(1\tilde{1})(b\tilde{b})s^m: n, m \in [k], 3 \leq b \leq k - 1\}\).

Counting directly from our set notation, it would appear that \(|\mathcal{I}_{id}^{22}| = 2k^2(k - 3)\). However, we would be overcounting due to symmetry. Notice that the transformation
\[
\begin{align*}
b &\mapsto k + 2 - b', \\
n &\mapsto n' = n + b' - 1, \\
m &\mapsto m' + b' - 1
\end{align*}
\]
leaves the permutations \(s^n(1\tilde{b})(b\tilde{1})s^m\) and \(s^n(1\tilde{1})(b\tilde{b})s^m\) invariant. Thus in the calculations above we are counting each permutation exactly twice, and the actual cardinality of the set is \(k^2(k - 3)\). \(\square\)

**Lemma C.5.** \(\mathcal{I}_{id}^{31} = \{s^n(1\tilde{1})(2\tilde{2})s^m: n, m \in [k]\}, \ |\mathcal{I}_{id}^{31}| = k^2.\)

**Proof.** We are now considering the case of having exactly one directed 6-cycle and no undirected cycles longer than 2. Since we are using irregular permutations \(\omega\), we have \(\omega(q) = \bar{l}\) for some \(q, l \in [k]\). By Lemma B.2 we also have \(\omega(\bar{l}) = l\). These imply we have a chord going into \(\bar{l}\) and a chord coming out of \(l\), as shown in Figure 14a.

These two chords must belong to our single directed 6-cycle, so we will examine the connection possibilities. Recall that our 6-cycle must alternate dashed and solid directed cycles.
edges. One connection possibility is that $\omega(q + 1)$ is connected to $\omega(q-1)$ by a single solid edge. However, no matter what direction we choose for this edge, we will have four edges pointing in the same direction with only two edges left with which to balance the directed cycle. We conclude this connection type is impossible. The same problem occurs if we attempt to connect $l-1$ and $l+1$ with a dashed edge. Therefore, we are left with two cases: (1) $l-1 = \omega(q-1)$ or (2) $\omega(q+1) = l+1$. The proofs for these cases are very similar, so we will only discuss the proof for (1) here. Observe our chosen connection for this case in Figure 14b.

To finish the 6-cycle we need two more directed edges connecting $\omega(q+1)$ and $l+1$, one dashed and one solid. Now $\omega(q+1)$ must be either $r$ or $\bar{r}$ for some $r \in [k]$.

The diagrams representing the two possibilities are as follows: Figure 15a for $\omega(q+1) = r$ and Figure 15b for $\omega(q+1) = \bar{r}$. Notice that in the first case, the only way to finish the directed 6-cycle is with the dashed edge ($r-1 \rightarrow l+1$). However, this creates two disjoint dashed paths, so this will not work.

In the second case, the only way to complete the directed 6-cycle is with the dashed edge ($l+1 \rightarrow r+1$). Now consider the dashed path on this graph. We already have the maximum number of chords allowed. Since any dashed directed edges between $r$ and $l+1$ must coincide with solid directed edges, the only way to ensure all the directed edges on the dashed path point in the same direction is to eliminate the possibility of such edges and make $r = l+1$. This gives us the configuration shown in Figure 16. Note that this structure is produced by $\omega = (l1)(l+1 \bar{r}+1)$. Hence the set of all permutations that produce this structure will be $\{s^n(1 \bar{r})(2 \bar{r})s^m \mid n, m \in [k] \}$. Since there is no rotational symmetry, we may conclude that the cardinality of the set is $k^2$.

\[\square\]

**References**

[BB96] P. W. Brouwer and C. W. J. Beenakker, *Diagrammatic method of integration over the unitary group, with applications to quantum transport in mesoscopic systems*, J. Math. Phys. 37 (1996), 4904–4934.

[Bee97] C. W. J. Beenakker, *Random-matrix theory of quantum transport*, Rev. Mod. Phys. 69 (1997), 731–808.

[BK13a] G. Berkolaiko and J. Kuipers, *Combinatorial theory of the semiclassical evaluation of transport moments I: Equivalence with the random matrix approach*, J. Math. Phys. 54 (2013), 112103, also arXiv:1305.4875.

[BK13b] G. Berkolaiko and J. Kuipers, *Combinatorial theory of the semiclassical evaluation of transport moments II: Algorithmic approach for moment generating functions*, J. Math. Phys. 54 (2013), 123505, 32.
Figure 15. Completing the 6-cycle.

Figure 16. The valid configuration for Lemma C.5.

[Bor06] E. Borel, *Sur les principes de la théorie cinétique des gaz*, Ann. Sci. École Norm. Sup. (3) 23 (1906), 9–32.

[BS90] R. Blümel and U. Smilansky, *Random-matrix description of chaotic scattering: Semiclassical approach*, Phys. Rev. Lett. 64 (1990), 241–244.

[BSW03] G. Berkolaiko, H. Schanz, and R. S. Whitney, *Form factor for a family of quantum graphs: an expansion to third order*, J. Phys. A 36 (2003), 8373–8392.

[CM17] B. Collins and S. Matsumoto, *Weingarten calculus via orthogonality relations: new applications*, ALEA Lat. Am. J. Probab. Math. Stat. 14 (2017), 631–656.

[Col03] B. Collins, *Moments and cumulants of polynomial random variables on unitary groups, the Itzykson-Zuber integral, and free probability*, Int. Math. Res. Not. (2003), 953–982.

[DDN03] A. D’Aristotile, P. Diaconis, and C. M. Newman, *Brownian motion and the classical groups*, Probability, statistics and their applications: papers in honor of Rabi Bhattacharya, IMS Lecture Notes Monogr. Ser., vol. 41, Inst. Math. Statist., Beachwood, OH, 2003, pp. 97–116.

[DEK04] M. Degli Esposti and A. Knauf, *On the form factor for the unitary group*, J. Math. Phys. 45 (2004), 4956–4979.

[Dys62a] F. J. Dyson, *Statistical theory of the energy levels of complex systems, I*, J. Mathematical Phys. 3 (1962), 140–156.

[Dys62b] F. J. Dyson, *The threefold way. Algebraic structure of symmetry groups and ensembles in quantum mechanics*, J. Mathematical Phys. 3 (1962), 1199–1215.

[Haa10] F. Haake, *Quantum signatures of chaos*, enlarged ed., Springer Series in Synergetics, Springer-Verlag, Berlin, 2010, With a foreword by H. Haken.

[HKS+96] F. Haake, M. Kuś, H.-J. Sommers, H. Schomerus, and K. Życzkowski, *Secular determinants of random unitary matrices*, J. Phys. A 29 (1996), 3641–3658.

[HMA+07] S. Heusler, S. Müller, A. Altland, P. Braun, and F. Haake, *Periodic-orbit theory of level correlations*, Phys. Rev. Lett. 98 (2007), 044103.

[JMS14] C. H. Joyner, S. Müller, and M. Sieber, *GSE statistics without spin*, EPL (Europhysics Letters) 107 (2014), 50004.

[Mat12] S. Matsumoto, *General moments of matrix elements from circular orthogonal ensembles*, Random Matrices Theory Appl. 1 (2012), 1250005, 18.
[Mat13] S. Matsumoto, *Weingarten calculus for matrix ensembles associated with compact symmetric spaces*, Random Matrices Theory Appl. **2** (2013), 1350001, 26.

[Mec08] E. Meckes, *Linear functions on the classical matrix groups*, Trans. Amer. Math. Soc. **360** (2008), 5355–5366.

[Meh04] M. L. Mehta, *Random matrices*, third ed., Pure and Applied Mathematics (Amsterdam), vol. 142, Elsevier/Academic Press, Amsterdam, 2004.

[MHA+09] S. Müller, S. Heusler, A. Altland, P. Braun, and F. Haake, *Periodic-orbit theory of universal level correlations in quantum chaos*, New J. Phys. **11** (2009), 103025.

[MHB+04] S. Müller, S. Heusler, P. Braun, F. Haake, and A. Altland, *Semiclassical foundation of universality in quantum chaos*, Phys. Rev. Lett. **93** (2004), 014103.

[MN13] S. Matsumoto and J. Novak, *Jacys-Murphy elements and unitary matrix integrals*, Int. Math. Res. Not. IMRN (2013), 362–397, also arXiv:0905.1992 [math.CO].

[MS05] F. Mezzadri and N. C. Snaith (eds.), *Recent perspectives in random matrix theory and number theory*, London Mathematical Society Lecture Note Series, vol. 322, Cambridge University Press, Cambridge, 2005.

[PZK98] M. Poźniak, K. Życzkowski, and M. Kuś, *Composed ensembles of random unitary matrices*, J. Phys. A **31** (1998), 1059–1071.

[RAJ+16] A. Rehemanjiang, M. Allgaier, C. H. Joyner, S. Müller, M. Sieber, U. Kuhl, and H.-J. Stöckmann, *Microwave realization of the gaussian symplectic ensemble*, Phys. Rev. Lett. **117** (2016), 064101.

[Sam80] S. Samuel, *U(N) integrals, 1/N, and the De Wit-’t Hooft anomalies*, J. Math. Phys. **21** (1980), 2695–2703.

[Sie02] M. Sieber, *Leading off-diagonal approximation for the spectral form factor for uniformly hyperbolic systems*, J. Phys. A **35** (2002), L613–L619.

[SR01] M. Sieber and K. Richter, *Correlations between periodic orbits and their rôle in spectral statistics*, Phys. Scr. **T90** (2001), 128–133.

[Wei78] D. Weingarten, *Asymptotic behavior of group integrals in the limit of infinite rank*, J. Mathematical Phys. **19** (1978), 999–1001.

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