ON A PARTIALLY ORDERED SET ASSOCIATED TO RING MORPHISMS

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Abstract. We associate to any ring $R$ with identity a partially ordered set $\text{Hom}(R)$, whose elements are all pairs $(a,M)$, where $a = \ker \varphi$ and $M = \varphi^{-1}(U(S))$ for some ring morphism $\varphi$ of $R$ into an arbitrary ring $S$. Here $U(S)$ denotes the group of units of $S$. The assignment $R \mapsto \text{Hom}(R)$ turns out to be a contravariant functor of the category $\text{Ring}$ of associative rings with identity to the category $\text{ParOrd}$ of partially ordered sets. The maximal elements of $\text{Hom}(R)$ constitute a subset $\text{Max}(R)$ which, for commutative rings $R$, can be identified with the Zariski spectrum $\text{Spec}(R)$ of $R$. Every pair $(a,M)$ in $\text{Hom}(R)$ has a canonical representative, that is, there is a universal ring morphism $\psi: R \to S[R/a,M/a]$ corresponding to the pair $(a,M)$, where the ring $S[R/a,M/a]$ is constructed as a universal inverting $R/a$-ring in the sense of Cohn. Several properties of the sets $\text{Hom}(R)$ and $\text{Max}(R)$ are studied.

1. Introduction

In this paper, we study a contravariant functor $\text{Hom}(-): \text{Ring} \to \text{ParOrd}$ from the category $\text{Ring}$ of all associative rings with identity to the category $\text{ParOrd}$ of partially ordered sets. This functor associates to every ring $R$ the set of all pairs $(a,M)$, where $a = \ker \varphi$ and $M = \varphi^{-1}(U(S))$ for some ring morphism $\varphi: R \to S$. Here $S$ is any other ring, that is, any object of $\text{Ring}$, and $U(S)$ denotes the group of units (= invertible elements) of $S$. With respect to a suitable partial order, the set $\text{Hom}(R)$ turns out to be a meet-semilattice (Lemma 2.6). The idea is to measure and classify, via the study of the partially ordered set $\text{Hom}(R)$, all ring morphisms from the fixed ring $R$ to any other ring $S$.

We have at least five motivations to study our functor $\text{Hom}(-)$:

1. We want to generalize the theory developed by Bavula for left Ore localizations [3,4,5] to arbitrary ring morphisms. In those papers, Bavula discovered the importance of maximal left denominator sets. Therefore here we want to extend his idea from ring morphisms $R \to [S^{-1}]R$ that arise as left Ore localizations to arbitrary ring morphisms $\varphi: R \to S$. In view of Bavula’s results, we pay a particular attention to the maximal elements of the partially ordered set $\text{Hom}(R)$. For every ring $R$, the subset $\text{Max}(R)$ of all maximal elements of $\text{Hom}(R)$ is always non-empty (Theorem 5.6).

2. For a commutative ring $R$, the set $\text{Max}(R)$ is in one-to-one correspondence with the Zarisky spectrum $\text{Spec}(R)$ of $R$ (Proposition 5.3). Thus $\text{Max}(R)$ could

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1
be used as a good substitute for the spectrum of a possibly non-commutative
ring $R$. Unluckily, the assignment $R \mapsto \text{Max}(R)$ is not a contravariant
functor (Theorem 5.7). This is not quite surprising, because, in the commutative case, the
maximal spectrum, i.e., the topological subspace of $\text{Spec}(R)$ whose elements are all
maximal ideals of the commutative ring $R$, is not a functor either. All this is related
to the paper [21] by Manuel Reyes. Notice that the category $\text{ParOrd}$ of partially
ordered sets is isomorphic to the category of all Alexandrov $T_0$-spaces, which is a
full subcategory of the category $\text{Top}$ of topological spaces. Thus our contravariant
functor $\text{Hom}(-)$ can also be viewed as a functor of $\text{Ring}$ into $\text{Top}$.

(3) The $\text{Hom}$ of a direct limit of rings $R_i$ is the inverse limit of the corresponding
partially ordered sets $\text{Hom}(R_i)$ (Theorem 4.11). We are motivated to the study of
the (good) behavior of our functor $\text{Hom}(-)$ with respect to direct limits of rings,
because spectra of commutative monoids has a similar behavior [20 Corollary 2.2].
Notice that Reyes’ universal contravariant functor $p\text{-Spec}: \text{Ring} \rightarrow \text{Set}$ can be
defined as the inverse limit of the spectra of the commutative subrings of $R$ [21
Proposition 2.14].

(4) An approach similar to ours appears in the paper [24] by Vale. He also
considers a contravariant functor from the category $\text{Ring}$, but to the category of
ringed spaces. When the ring $R$ is commutative, he also gets a sort of “completion”
of $\text{Spec}(R)$.

(5) Finally, the partially ordered set $\text{Hom}(R)$ always has a least element, the pair
$(0, U(R))$, which corresponds to the identity morphism $R \rightarrow R$. More generally, like
in Bavula’s case, the set $\text{Hom}(R)$ has a natural partition into subsets $\text{Hom}(R, a)$
(Section 2). The least elements of these subsets $\text{Hom}(R, a)$, with $a$ contained in
the Jacobson radical $J(R)$ of $R$, correspond to local morphisms (Proposition 7.6),
that is, to the ring morphisms $\varphi: R \rightarrow S$ such that, for every $r \in R$, $\varphi(r)$ invertible
in $S$ implies $r$ invertible in $R$. Thus our interest in the functor $\text{Hom}(-)$ is also
motivated by the several applications of local morphisms [9, 12]. Notice that the
subset $\text{Hom}(R, 0)$ classifies all ring extensions $\varphi: R \hookrightarrow S$.

Every pair $(a, M)$ in $\text{Hom}(R)$ has a canonical representative, that is, a universal
ring morphism $\psi: R \rightarrow S_{(R/a, M/a)}$ corresponding to the pair $(a, M)$ (Theorem 6.10).
The ring $S_{(R/a, M/a)}$ is constructed as a universal inverting $R/a$-ring in the sense of
Cohn [10]. Any other ring morphism $\varphi: R \rightarrow S$ corresponding to $(a, M)$ has a
canonical factorization through $\psi$ (Theorem 7.3). One of the mappings appearing in
this factorization of $\varphi$ is a ring epimorphism $\varphi^\sharp: R \rightarrow T$, which still corresponds to
the pair $(a, M)$. Ring epimorphisms, that is, epimorphisms in the category $\text{Ring}$,
currently play a predominant role in Homological Algebra [11, 9, 14, 15, 16], in
particular left flat morphism, that is, when the codomain is a flat left $R$-module.
The functor $\text{Hom}(-)$ is not representable (Section 2).

The meet-semilattice $\text{Hom}(R)$ has a smallest element $(0, U(R))$, but does not
have a greatest element in general. Hence, for some results, instead of $\text{Hom}(R)$, it is
more convenient to enlarge the partially ordered set $\overline{\text{Hom}(R)}$ adjoining to it a
further element, a new greatest element 1, setting $\overline{\text{Hom}(R)} := \text{Hom}(R) \cup \{1\}$. In
some sense, this new greatest element 1 corresponds to the zero morphism $R \rightarrow S$
for any ring $S$. This enlarged partially ordered set $\overline{\text{Hom}(R)}$ is a bounded lattice
(Theorem 6.8).

Finally, we specialize some of our results to Bavula’s case of left ring of fractions.
In Bavula’s case, the ring morphism $\varphi: R \rightarrow S$ is the canonical mapping of $R$ into
the right ring of fractions $S$ of $R$ with respect to some right denominator set. Such a $\varphi$ is clearly a ring epimorphism.

Throughout, all rings are associative, with identity $1 \neq 0$, and all ring morphisms send $1$ to $1$. The group of (right and left) invertible elements of $R$ will be denoted by $U(R)$, and the Jacobson radical of $R$ will be denoted by $J(R)$.

2. The partially ordered set $\text{Hom}(R)$

Let $R$ be a ring. We associate to each ring morphism $\varphi: R \to S$ into any other ring $S$ the pair $(a, M)$, where $a := \ker(\varphi)$ is the kernel of $\varphi$ and $M := \varphi^{-1}(U(S))$ is the inverse image of the group of units $U(S)$ of $S$. In the next lemmas, we collect the basic properties of these pairs $(a, M)$. Recall that a monoid $S$ is cancellative if, for every $x, y, z \in S$, $xz = yz$ implies $x = y$ and $zx = zy$ implies $x = y$. An element $x$ of a ring $R$ is regular if, for all $r \in R$, $rx = 0$ implies $r = 0$ and $xr = 0$ implies $r = 0$.

Lemma 2.1. Let $\varphi: R \to S$ be a ring morphism and $(a, M)$ its associated pair. Then:

1. $M$ is a submonoid of the multiplicative monoid $R$.
2. $U(R) \subseteq M$.
3. $M = M + a = M + a + J(R)$ and $a \cap M = \emptyset$.
4. $M/a := \{ m + a \mid m \in M \}$ consists of regular elements of $R/a$. In particular, $M/a$ is a cancellative submonoid of the multiplicative monoid $R/a$.

Proof. (1), (2) and (4) are easy. (3) The inclusions $M \subseteq M + a \subseteq M + a + J(R)$ are trivial. In order to prove that $M + a + J(R) \subseteq M$, notice that $M + a \subseteq M$ and $1 + J(R) \subseteq U(R) \subseteq M$. Since $M$ is multiplicatively closed and contains $1$, it follows that $M \supseteq (M + a)(1 + J(R)) = M + a + M J(R) + a J(R) \supseteq M + a + 1 R J(R) = M + a + J(R)$.

Remark 2.2. The monoid $M$ is not cancellative in general. As an example consider $R = \mathbb{Z}/6\mathbb{Z}$, $S = \mathbb{Z}/2\mathbb{Z}$, $\varphi: \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ the canonical projection, $x = 1 + 6\mathbb{Z}$ and $y = z = 3 + 6\mathbb{Z}$. Then $x, y, z \in M$ and $xz = yz$, but $x \neq y$.

Recall that a multiplicatively closed subset $M$ of a ring $R$ is saturated if, for every $x, y \in R$, $xy \in M$ implies $x \in M$ and $y \in M$. A ring $R$ is directly finite if, for every $x, y \in R$, $xy = 1$ implies $yx = 1$.

Lemma 2.3. Let $\varphi: R \to S$ be a ring morphism and $(a, M)$ its associated pair. Then:

1. If $S$ is a (not-necessarily commutative) integral domain, then the ideal $a$ is completely prime.
2. If $S$ is a division ring, then $R$ is the disjoint union of $a$ and $M$, i.e., $\{a, M\}$ is a partition of the set $R$.
3. If $S$ is a directly finite ring, e.g., if $S$ is an integral domain, then $M$ is a saturated multiplicatively closed subset of $R$.

Proof. (3) Suppose $S$ directly finite, $x, y \in R$ and $xy \in M = \varphi^{-1}(U(S))$. Then $\varphi(x) \varphi(y) = \varphi(xy) \in U(S)$. Hence there exists $s \in S$ such that $\varphi(x) \varphi(y) s = 1$ and $s \varphi(x) \varphi(y) = 1$. Thus $\varphi(x)$ is right invertible and $\varphi(y)$ is left invertible. Since $S$ is directly finite, we have that $\varphi(x)$ and $\varphi(y)$ are both invertible in $S$, so that $x \in M$ and $y \in M$. 

Notice that every integral domain is directly finite, because if \( x, y \) are element of an integral domain \( S \) and \( xy = 1 \), then \( yxy = y \), so \((yx − 1)y = 0\), hence \( xy = 1 \). □

We will now deal with preorders on a set \( X \), that is, reflexive and transitive relations on \( X \). Recall that, if \( X \) is a set, or more generally a class, and \( \rho \) is a preorder on \( X \), then it is possible to associate to \( \rho \) an equivalence relation \( \sim_\rho \) on \( X \) and a partial order \( \leq_\rho \) on the quotient set \( X/\sim_\rho \). The equivalence relation \( \sim_\rho \) on \( X \) is defined, for every \( x, y \in X \), by \( x \sim_\rho y \) if \( xpy \) and \( ypx \). The partial order \( \leq_\rho \) on the quotient set \( X/\sim_\rho := \{ [x]_{\sim_\rho} \mid x \in X \} \) is defined by \( [x]_{\sim_\rho} \leq_\rho [y]_{\sim_\rho} \) if \( xpy \).

On the class \( \mathcal{H}(R) \) of all morphisms \( \varphi: R \to S \) of \( R \) into arbitrary rings \( S \), there are two natural preorders. If \( \varphi: R \to S \), \( \varphi': R \to S' \) are two ring morphisms, we have a first preorder \( \rho \) on \( \mathcal{H}(R) \), defined setting \( \varphi\rho\varphi' \) if \( \text{ker}(\varphi) \subseteq \text{ker}(\varphi') \) and \( \varphi^{-1}(U(S)) \subseteq \varphi'^{-1}(U(S)) \). A second preorder \( \sigma \) on \( \mathcal{H}(R) \) is defined setting \( \varphi\sigma\varphi' \) if there exists a ring morphism \( \psi: S \to S' \) such that \( \psi\varphi = \varphi' \).

Correspondingly, there is a first equivalence relation \( \sim \) on the class \( \mathcal{H}(R) \), defined, for all ring morphisms \( \varphi: R \to S \), \( \varphi': R \to S' \) with associated pairs \( (a, M), (a', M') \) respectively, by \( \varphi \sim \varphi' \) if \( (a, M) = (a', M') \). That is, \( \varphi \sim \varphi' \) if and only if \( \text{ker}(\varphi) = \text{ker}(\varphi') \) and \( \varphi^{-1}(U(S)) = \varphi'^{-1}(U(S)) \). Let \( \text{Hom}(R) := \mathcal{H}(R)/\sim \) denote the set (class) of all equivalence classes \([\varphi]_{\sim}\) modulo \( \sim \), that is, equivalently, the set of all pairs \( (\text{ker}(\varphi), \varphi^{-1}(U(S))) \). The partial order \( \leq \) on \( \text{Hom}(R) = \mathcal{H}(R)/\sim \) associated to the preorder \( \rho \) on \( \mathcal{H}(R) \) is defined by setting \( (a, M) \leq (a', M') \) if \( a \subseteq a' \) and \( M \subseteq M' \).

As far as the second natural preorder \( \sigma \) on \( \mathcal{H}(R) \) is concerned, the equivalence relation \( \equiv \) on \( \mathcal{H}(R) \) associated to \( \sigma \) is defined, for every \( \varphi: R \to S \), \( \varphi': R \to S' \) in \( \mathcal{H}(R) \), by \( \varphi \equiv \varphi' \) if \( (a, M) = (a', M') \). That is, \( \varphi \equiv \varphi' \) if \( \text{ker}(\varphi) = \text{ker}(\varphi') \) and \( \varphi^{-1}(U(S)) = \varphi'^{-1}(U(S)) \). Let \( \text{Hom}(R) := \mathcal{H}(R)/\equiv \) denote the set (class) of all equivalence classes \([\varphi]_{\equiv}\) modulo \( \equiv \), that is, equivalently, the set of all pairs \( (\text{ker}(\varphi), \varphi^{-1}(U(S))) \). The partial order \( \leq \) on the quotient class \( \mathcal{H}(R)/\equiv \), associated to the preorder \( \sigma \) on \( \mathcal{H}(R) \), is defined by setting \( [\varphi]_{\equiv} \leq [\varphi']_{\equiv} \) if \( \varphi\sigma\varphi' \).

**Remark 2.4.** If there exists a ring morphism \( \psi: S \to S' \) such that \( \psi\varphi = \varphi' \), then \( (a, M) \leq (a', M') \). Equivalently, for all \( \varphi: R \to S \), \( \varphi': R \to S' \) in \( \mathcal{H}(R) \), \( \varphi\sigma\varphi' \) implies \( \varphi\rho\varphi' \).

Thus, for all \( \varphi: R \to S \), \( \varphi': R \to S' \) in \( \mathcal{H}(R) \), \( \varphi \equiv \varphi' \) implies \( (a, M) = (a', M') \), i.e., \( \varphi \sim \varphi' \). Equivalently, the identity mapping \( \mathcal{H}(R) \to \mathcal{H}(R) \) is a preorder morphism of \( (\mathcal{H}(R), \sigma) \) onto \( (\mathcal{H}(R), \rho) \). Similarly, there is an induced surjective morphism of factor classes

\[
\mathcal{H}(R)/\equiv \to \mathcal{H}(R)/\sim = \text{Hom}(R), \quad [\varphi]_{\equiv} \mapsto [\varphi]_{\sim}.
\]

The implication \( \varphi\sigma\varphi' \) implies \( \varphi\rho\varphi' \) cannot be reversed in general, that is, there are morphisms \( \varphi: R \to S \) and \( \varphi': R \to S' \) with \( (a, M) \leq (a', M') \), but for which there does not exist a ring morphism \( \psi: S \to S' \) with \( \psi\varphi = \varphi' \). For instance, let \( k \) be a finite field, \( \overline{k} \) its algebraic closure, \( M_2(k) \) the ring of \( 2 \times 2 \) matrices with entries in \( k \), and \( \varphi: k \to \overline{k} \) and \( \varphi': k \to M_2(k) \) the canonical embeddings. Then \( \overline{k} \) and \( M_2(k) \) are simple rings, so that all ring morphisms \( \psi: \overline{k} \to M_2(k) \) are injective. But \( k \) is finite and \( \overline{k} \) is infinite, so that there is no ring morphism \( \psi: k \to M_2(k) \).

The implication \( \varphi\sigma\varphi' \) implies \( (a, M) \leq (a', M') \) can be reversed in some special cases, for instance when we restrict our attention to localizations at left denominator sets. See Remark [7.5].
Proposition 2.5. Let \( \text{Ring} \) be the category of rings with identity and \( \text{ParOrd} \) the category of partially ordered sets. Then \( \text{Hom}(\cdot) : \text{Ring} \to \text{ParOrd} \) is a contravariant functor.

Proof. The functor \( \text{Hom} \) assigns to each ring \( R \) the set \( \text{Hom}(R) \) of all pairs \((a, M)\), where \( a := \ker(\varphi) \) and \( M := \varphi^{-1}(U(S)) \) for some ring morphism \( \varphi : R \to S \), partially ordered by \( \leq \), where \((a, M) \leq (b, N)\) if \( a \subseteq b \) and \( M \subseteq N \). Moreover, it assigns to each ring morphism \( f : R \to R' \) the increasing mapping

\[
\text{Hom}(f) : \text{Hom}(R') \to \text{Hom}(R), \quad (a', M') \in \text{Hom}(R') \mapsto (f^{-1}(a'), f^{-1}(M')).
\]

Notice that if \( \varphi' : R' \to S \) is a ring morphism, \( a' := \ker(\varphi') \) and \( M' := \varphi'^{-1}(U(S)) \), then \( \varphi' f : R \to S \) is a ring morphism,

\[
f^{-1}(a') = \ker(\varphi' f)
\]

and

\[
f^{-1}(M') = (\varphi' f)^{-1}(U(S)).
\]

\[\square\]

The functor \( \text{Hom}(\cdot) \) is not representable. Namely, suppose the contravariant functor \( \text{Hom}(\cdot) : \text{Ring} \to \text{Set} \) representable, i.e., that there exists a ring \( A \) with \( \text{Hom}(\cdot) \) naturally isomorphic to the contravariant functor \( \text{Hom}_{\text{Ring}}(\cdot, A) : \text{Ring} \to \text{Set} \). Now, for every ring \( A \) there always exists a ring \( R \) with \( \text{Hom}_{\text{Ring}}(R, A) = \emptyset \) (If \( A \) has characteristic 0, take for \( R \) any ring of characteristic \( \neq 0 \). If \( A \) has characteristic \( n \geq 2 \), take for \( R \) any ring of characteristic \( p \) prime with \( p \neq n \).) Our functor \( \text{Hom}(\cdot) \) is such that \( \text{Hom}(R) \neq \emptyset \) for every ring \( R \). Hence the functors \( \text{Hom}(\cdot) \) and \( \text{Hom}_{\text{Ring}}(\cdot, A) \) can never be isomorphic.

For any fixed proper ideal \( a \) of \( R \), set

\[
\text{Hom}(R, a) := \{ (\ker(\varphi), \varphi^{-1}(U(S))) \mid \varphi : R \to S, \ker(\varphi) = a \}.
\]

Clearly, \( \text{Hom}(R) \) is the disjoint union of the sets \( \text{Hom}(R, a) \):

\[
\text{Hom}(R) = \bigcup_{a \in R} \text{Hom}(R, a).
\]

In particular, the partial order \( \leq \) on \( \text{Hom}(R) \) induces a partial order on each subset \( \text{Hom}(R, a) \).

The following lemma has an easy proof.

Lemma 2.6. Let \((a, M), (a', M')\) be the elements of \( \text{Hom}(R) \) corresponding to two morphisms \( \varphi : R \to S \) and \( \varphi' : R \to S' \). Then the element of \( \text{Hom}(R) \) corresponding to the product morphism \( \varphi \times \varphi' : R \to S \times S' \) is \((a \cap a', M \cap M')\).

As a consequence, the partially ordered set \( \text{Hom}(R) \) turns out to be a meet-semilattice. In particular, with respect to the operation \( \wedge \), \( \text{Hom}(R) \) is a commutative semigroup in which every element is idempotent and which has a zero element (= the least element \((0, U(R))\) of \( \text{Hom}(R) \), which corresponds to the identity morphism \( R \to R \)). We will see in Theorem 5.8 that the partially ordered set \( \text{Hom}(R) \) always has maximal elements, but does not have a greatest element in general, so the semigroup \( \text{Hom}(R) \) does not have an identity in general.
3. A universal construction

Let $R$ be any ring and $N$ be any fixed subset of $R$. Let $X := \{x_n \mid n \in N\}$ be a set of non-commuting indeterminates in one-to-one correspondence with the set $N$. Let $R[X]$ be the free $R$-ring over $X$ ([7] and [22, Example 1.9.20 on Page 124]). Then there are a canonical ring morphism $\varphi : R \to R[X]$ and a mapping $\varepsilon : X \to R[X]$ such that for every ring $S$, every ring morphism $\psi : R \to S$ and every mapping $\zeta : X \to S$ there is a unique ring morphism $\tilde{\psi} : R[X] \to S$ such that $\psi = \tilde{\psi}\varphi$ and $\zeta = \tilde{\psi}\varepsilon$.

Let $I$ be the two-sided ideal of $R[X]$ generated by the subset $\{nx_n - 1 \mid n \in N\} \cup \{nx_n - 1 \mid n \in N\}$ and $S_{(R,N)} := R[X]/I$. Clearly, $I$ could be the improper ideal of $R[X]$ and $S_{(R,N)}$ could be the zero ring. There is a canonical mapping $\chi_{(R,N)} : R \to S_{(R,N)}$, composite mapping of $\varphi : R \to R[X]$ and the canonical projection $R[X] \to R[X]/I$. The $R$-ring $R[X]/I$ is the universal $N$-inverting $R$-ring in the sense of [10, Proposition 1.3.1].

Lemma 3.1. If $(0,M) \in \text{Hom}(R)$ for a ring $R$, then the canonical ring morphism $\chi_{(R,M)} : R \to S_{(R,M)}$ is injective and $\chi_{(R,M)}^{-1}(U(S_{(R,M)})) = M$.

Proof. If $(0,M) \in \text{Hom}(R)$, there are a ring $S$ and ring morphism $f : R \to S$ such that $(0,M)$ is associated to $f$. In particular, $f$ is an injective mapping. The morphism $f$ clearly factors through $\chi_{(R,M)}$, that is, there is a ring morphism $g : S_{(R,M)} \to S$ with $g\chi_{(R,M)} = f$. As $f$ is injective, $\chi_{(R,M)}$ is also injective.

Moreover, $g\chi_{(R,M)} = f$ implies that $M = f^{-1}(U(S)) \supseteq \chi_{(R,M)}^{-1}(U(S_{(R,R)}))$. Finally, the elements of $M$ are clearly mapped to invertible elements of $S_{(R,M)}$ via $\chi_{(R,M)}$, by construction, and so $\chi_{(R,M)}^{-1}(U(S_{(R,M)})) = M$. \hfill $\square$

The proof of the following lemma is immediate.

Lemma 3.2. If $(a,M) \in \text{Hom}(R)$, then $(a/a, M/a) \in \text{Hom}(R/a)$.

Theorem 3.3. Let $R$ be a ring and $(a,M)$ be an element of $\text{Hom}(R)$. Then $S_{(R/a,M/a)}$ is a non-zero ring, and if $\psi : R \to S_{(R/a,M/a)}$ denotes the composite mapping of the canonical projection $\pi : R \to R/a$ and $\chi_{(R/a,M/a)} : R/a \to S_{(R/a,M/a)}$, then $\ker(\psi) = a$ and $\psi^{-1}(U(S_{(R/a,M/a)})) = M$. Moreover, for any ring morphism $f : R \to S$ such that $\ker(f) \supseteq a$ and $f^{-1}(U(S)) \supseteq M$, there is a unique ring morphism $g : S_{(R/a,M/a)} \to S$ such that $g\psi = f$.

Proof. Since $(a,M) \in \text{Hom}(R)$, there are ring morphisms $\varphi : R \to S$ such that $\ker(\varphi) = a$ and $\varphi^{-1}(U(S)) = M$. More generally, let $f : R \to S$ be any ring morphism with $\ker(f) \supseteq a$ and $f^{-1}(U(S)) \supseteq M$. Then $f$ factors as the composite mapping of the canonical projection $\pi : R \to R/a$ and a unique morphism $\overline{f} : R/a \to S$. Now construct the ring $S_{(R/a,M/a)} : = (R/a)[\overline{X}]/I$, where $\overline{X} := \{x_{\overline{m}} \mid \overline{m} \in M/a\}$. By the universal property of the free $R/a$-ring $(R/a)[\overline{X}]$, there is a unique ring morphism $\overline{f} : (R/a)[\overline{X}] \to S$ such that $\overline{f} = \tilde{f}\psi'$ and $\zeta = \tilde{f}\varepsilon$, where $\psi' : R/a \to (R/a)[\overline{X}]$ and $\varepsilon : \overline{X} \to (R/a)[\overline{X}]$ are the canonical mapping and $\zeta : \overline{X} \to S$ is defined by $\zeta(x_{\overline{m}}) = ((f(m))^{-1}) \overline{m}$ for every $\overline{m} \in M/a$. See the diagram below. From $\overline{f} = \tilde{f}\psi'$, we get that $f(\overline{m}) = \overline{f}(\overline{m}) = f(m)$, and, from $\zeta = \tilde{f}\varepsilon$, we have that $\overline{f}(x_{\overline{m}}) = \overline{f}(x_{\overline{m}}) = \zeta(\overline{m}) = (f(m))^{-1}$ and $\overline{m}x_{\overline{m}} - 1$ and $\overline{m}x_{\overline{m}} - 1$ of the two-sided ideal $I$ of $(R/a)[\overline{X}]$ are mapped to zero via $\tilde{f}$, so that $\tilde{f}$ factors in a unique way through a ring morphism $g : S_{(R/a,M/a)} = (R/a)[\overline{X}]/I \to \overline{X}$.
S, that is, \( f = g\pi' \), where \( \pi': (R/a)(\overline{X}) \to (R/a)\{\overline{X}\}/I = S_{(R/a,M/a)} \) denotes the canonical projection. This, applied to any ring morphisms \( \varphi: R \to S \) such that \( \ker(\varphi) = a \) and \( \varphi^{-1}(U(S)) = M \), shows that \( S_{(R/a,M/a)} \) is a non-zero ring. Moreover, set \( \psi = \chi(R/a,M/a)\pi = \pi'\psi'\pi \) and \( f = \overline{f}\pi = \overline{f}\psi'\pi = g\pi'\psi'\pi \). Then \( f = g\psi \). This proves the existence of \( g \) in the last part of the statement of the theorem.

\[
\begin{array}{c}
R \\
\downarrow \pi \\
R/a \\
\downarrow \psi' \\
(R/a)(\overline{X}) \\
\downarrow \overline{f} \\
S \\
\downarrow f \\
\overline{S}_{(R/a,M/a)} \\
\downarrow \pi \\
\end{array}
\]

Now we apply again the previous results to any ring morphism \( \varphi: R \to S \). Since \( (a,a,M/a) \in \text{Hom}(R/a) \) by Lemma 3.2, we now have that \( \chi(R/a,M/a): R/a \to \overline{S}_{(R/a,M/a)} \) is an injective mapping by Lemma 3.1. Thus the kernel \( \ker(\psi) \) of \( \psi = \chi(R/a,M/a)\pi \) is \( \{0\} \). Also,

\[
\psi^{-1}(U(S_{(R/a,M/a)})) = (\pi'\psi'\pi)^{-1}(U(S_{(R/a,M/a)})) = \\
= (\chi(R/a,M/a)\pi)^{-1}(U(S_{(R/a,M/a)})) = \\
= \pi^{-1}(\chi(R/a,M/a)U(S_{(R/a,M/a)})) = \pi^{-1}(M/a) = M.
\]

It remains to prove the uniqueness of \( g \), that is, if \( g': S_{(R/a,M/a)} \to S \) is another ring morphism such that \( \pi'\psi'\pi = f \), then \( g = g' \). Now \( S_{(R/a,M/a)} \) is generated, as a ring, by the image of \( R \) via \( \psi = \pi'\psi'\pi \) and the inverses of the elements of \( \psi(M) \). Since \( g\psi = g'\psi \), both mappings \( g \) and \( g' \) send each \( \psi(r) \) to \( f(r) \) and each \( \psi(m) \) to \( f(m) \). It follows that \( g = g' \), as desired. \qed

Theorem 4.3 shows that, for any pair \( (a,M) \) in \( \text{Hom}(R) \), there is a canonical ring morphism \( \psi: R \to S_{(R/a,M/a)} \) that realizes that pair. Moreover, the universal property described in the last part of the statement of the theorem shows that the canonical morphism \( \psi: R \to S_{(R/a,M/a)} \) is one of the least elements in the class \( \mathcal{H}(R,a) \) of all morphisms \( f: R \to S \) such that \( a \subseteq \ker(f) \) and \( M \subseteq f^{-1}(U(S)) \) with respect to the preorder \( \sigma \), in the sense that \( \psi \psi f \) for every morphism \( f: R \to S \) with \( a \subseteq \ker(f) \) and \( M \subseteq f^{-1}(U(S)) \).

4. Direct limits

Now let \( (R_i)_{i \in I} \) be a direct system of rings indexed on a directed set \( (I,\leq) \). Hence, for every \( i, j \in I, i \leq j \), we have compatible connecting ring morphisms

\[
\mu_{ij}: R_i \to R_j.
\]

Applying our functor \( \text{Hom}(\cdot) \), we get an inverse system \( (\text{Hom}(R_i))_{i \in I} \) of partially ordered sets, with connecting partially ordered set morphisms

\[
\text{Hom}(\mu_{ij}): \text{Hom}(R_j) \to \text{Hom}(R_i).
\]

**Theorem 4.1.**

\[
\text{Hom}(\varprojlim R_i) \cong \varprojlim \text{Hom}(R_i).
\]
Proof. Let \( \mu_j: R_j \to \lim R_i \) be the canonical ring morphisms, for every \( j \in I \). These morphisms induce partially ordered set morphisms
\[
\text{Hom}(\mu_j): \text{Hom}(\lim R_i) \to \text{Hom}(R_j).
\]
Let \( H := \lim \text{Hom}(R_i) \subseteq \prod_{j \in I} \text{Hom}(R_j) \) be the inverse limit of the inverse system \((\text{Hom}(R_i))_{i \in I}\) of partially ordered sets, and \( h_j: H \to \text{Hom}(R_j) \) the canonical mapping. By the universal property of inverse limit, there exists a unique partially order set morphism \( \Psi: \text{Hom}(\lim R_i) \to H \) such that \( h_j \Psi = \text{Hom}(\mu_j) \) for every \( j \in I \). Thus \( \Psi(a, M) = (\mu_i^{-1}(a), \mu_i^{-1}(M))_{i \in I} \).

We will show that \( \Psi \) is a bijection and \( \Psi^{-1} \) is a morphism of partially order sets.

First we prove that the mapping \( \Psi \) is injective. Let \((a, M), (a', M') \) be two elements of \( \text{Hom}(\lim R_i) \) with \( \Psi(a, M) = \Psi(a', M') \). Then \( (\mu_j^{-1}(a), \mu_j^{-1}(M)) = (\mu_j^{-1}(a'), \mu_j^{-1}(M')) \) for every \( j \in I \). We claim that, for any two subset \( X \) and \( Y \) of \( \lim R_i \), if \( \mu_j^{-1}(X) = \mu_j^{-1}(Y) \) for every \( j \in I \), then \( X = Y \). If we prove the claim, then \( (\mu_j^{-1}(a), \mu_j^{-1}(M)) = (\mu_j^{-1}(a'), \mu_j^{-1}(M')) \) for every \( j \in I \) implies \((a, M) = (a', M')\), which proves that \( \Psi \) is injective.

In order to prove the claim, assume \( X, Y \subseteq \lim R_i \) and \( \mu_j^{-1}(X) = \mu_j^{-1}(Y) \) for all \( j \in I \). If \( x \in X \), then there exists \( i \in J \) and \( r_i \in R_i \) with \( \mu_i(r_i) = x \). Hence \( r_i \in \mu_i^{-1}(X) = \mu_i^{-1}(Y) \), so that \( x = \mu_i(r_i) \in Y \). Thus \( X \subseteq Y \). Similarly \( Y \subseteq X \). This concludes the proof of the claim, which shows that \( \Psi \) is injective.

Let us prove that \( \Psi \) is surjective. Let \((a_i, M_i)_{i \in I}\) be an element of \( H \), so that \( (\mu_j^{-1}(a_i), \mu_j^{-1}(M_i)) = (a_i, M_i) \) for every \( i \in J \). Here \((a_i, M_i)\) is an element of \( \text{Hom}(R_i) \), so that \( (a_i, M_i) \) corresponds to the ring morphism
\[
\psi_i: R_i \to S_{(R_i/a_i, M_i/a_i)}
\]
(Theorem 5.6). We now show that \( (S_{(R_i/a_i, M_i/a_i)})_{i \in I} \), with suitable canonical connecting maps, form a direct system of rings. Since \( \mu_j^{-1}(a_i) = a_i \), the morphisms \( \mu_{ij} \) induce monomorphisms \( \mu_{ij}: R_i/a_i \to R_j/a_j \), and \( \mu_{ij}(M_j/a_i) \subseteq M_j/a_j \). Thus \( \mu_{ij} \) extends to a ring monomorphism \( R_i/a_i \{ x_m + a_i \mid m \in M_i \} \to R_j/a_j \{ x_m + a_j \mid m \in M_j \} \) that maps \( x_m + a_i \) to \( x_{\mu_{ij}(m)} + a_j \). These canonical ring monomorphisms induce ring morphisms \( \nu_{ij}: S_{(R_i/a_i, M_i/a_i)} \to S_{(R_j/a_j, M_j/a_j)} \), for all \( i \leq j \). The diagrams
\[
\begin{array}{ccc}
R_i & \xrightarrow{\psi_i} & S_{(R_i/a_i, M_i/a_i)} \\
\mu_{ij} & & \nu_{ij} \\
R_j & \xrightarrow{\psi_j} & S_{(R_j/a_j, M_j/a_j)}
\end{array}
\]
clearly commute for every \( i \leq j \). Hence we have a morphism of direct systems of rings, and, taking the direct limit, we get a ring morphism
\[
\psi: \lim R_i \to \lim S_{(R_i/a_i, M_i/a_i)}.
\]
Let \((a, M) \in \text{Hom}(\lim R_i) \) be the pair corresponding to this ring morphism \( \psi \).

We claim that \( \Psi(a, M) = ((a_i, M_i))_{i \in I} \), that is, the \( \mu_i^{-1}(a) = a_i \) and \( \mu_i^{-1}(M) = M_i \) for each \( i \in I \). Let us prove that \( \mu_i^{-1}(a) = a_i \).
An element \( r_i \in R_i \) belongs to \( \mu_i^{-1}(a) \) if and only if \( \mu_i(r_i) \in a = \ker \psi \), that is, if and only if \( \psi \mu_i(r_i) = 0 \). Now we have commutative diagrams

\[
\begin{array}{ccc}
R_i & \xrightarrow{\psi_i} & S_{(R_i/a_i,M_i/a_i)} \\
\downarrow & & \downarrow \\
\lim R_i & \xrightarrow{\psi} & \lim S_{(R_i/a_i,M_i/a_i)},
\end{array}
\]

so that \( \psi \mu_i(r_i) = 0 \) if and only if \( \nu_i \psi_i(r_i) = 0 \), which occurs if and only if there exists \( j \geq i \) such that \( \psi_{ij} \psi_i(r_i) = 0 \), that is, \( \psi_{ij} \mu_{ij}(r_i) = 0 \), i.e., if and only if there exists \( j \geq i \) such that \( \mu_{ij}(r_i) \in a_j \). Equivalently, if and only if \( r_i \in a_i \). This proves that \( \mu_i^{-1}(a) = a_i \) for every \( i \).

We will now prove that \( \mu_i^{-1}(M) = M_i \). Set \( S := \lim\rightarrow S_{(R_i/a_i,M_i/a_i)} \). An element \( r_i \in R_i \) belongs to \( \mu_i^{-1}(M) \) if and only if \( \mu_i(r_i) \in M_i \), that is, if and only if \( \psi \mu_i(r_i) \in U(S) \), i.e., if and only if \( \psi \mu_i(r_i) \in U(S) \). This occurs if and only if there exists \( s \in S \) such that \( \psi \mu_i(r_i) = 1 \) and \( \nu_i \psi_i(r_i)s = 1 \). Now any element \( s \) of \( S \) is of the form \( \psi_j(s_j) \) for some \( j \geq i \) and \( s_j \in S_{(R_j/a_j,M_j/a_j)} \). Also, \( \nu_j(s_j)\nu_i \psi_i(r_i) = 1 \) and \( \nu_i \psi_i(r_i)\nu_j(s_j) = 1 \) in \( S \) if and only if there exists \( k \geq i, j \) such that \( \nu_{jk}(s_j)\nu_k \psi_i(r_i) = 1 \) and \( \nu_k \psi_i(r_i)\nu_{jk}(s_j) = 1 \) in \( S_{(R_k/a_k,M_k/a_k)} \). This occurs if and only if \( \mu_{ik}(r_i) \) is invertible in \( S_{(R_k/a_k,M_k/a_k)} \), that is, if and only if \( \psi_k \mu_{ik}(r_i) \) is invertible in \( S_{(R_k/a_k,M_k/a_k)} \), i.e., \( \mu_{ik}(r_i) \in M_k \), that is, \( r_i \in \mu_i^{-1}(M_k) = M_i \). This shows that \( \mu_i^{-1}(M) = M_i \), and concludes the proof of the claim. Thus \( \Psi \) is surjective.

Finally, let us prove that \( \Psi^{-1} \) is a morphism of partially order sets. Let \( ((a_i,M_i))_{i \in I}, ((a'_i,M'_i))_{i \in I} \) be two elements in \( H \) with \( ((a_i,M_i))_{i \in I} \subseteq ((a'_i,M'_i))_{i \in I} \), that is, with \( a_i \subseteq a'_i \) and \( M_i \subseteq M'_i \) for every \( i \in I \). We have direct systems of rings \( S_{(R_i/a_i,M_i/a_i)}, i \in I \), and \( S_{(R'_i/a'_i,M'_i/a'_i)}, i \in I \), and canonical projections \( \pi_i : R_i/a_i \to R'_i/a'_i \), which extend to the free \( R/a \)-ring \( R/a_i \{X_{M_i/a_i}\} \) (to the free \( R/a'_i \)-ring \( R/a'_i \{X_{M'_i/a'_i}\} \)), sending each indeterminate \( x_m \in X_{M_i/a_i} \) to the indeterminate \( x_m \in X_{M'_i/a'_i} \). In this way, we get a canonical ring morphism \( R/a_i \{X_{M_i/a_i}\} \to R/a'_i \{X_{M'_i/a'_i}\} \), which induces, factoring out the corresponding ideals \( I_i \) and \( I'_i \), a ring morphism \( S_{(R/a_i,M/a_i)} \to S_{(R'/a'_i,M'/a'_i)} \) and commutative squares

\[
\begin{array}{ccc}
R_i & \xrightarrow{\psi} & R'_i \\
\downarrow & & \downarrow \\
S_{(R_i/a_i,M_i/a_i)} & \xrightarrow{\pi} & S_{(R'_i/a'_i,M'_i/a'_i)}.
\end{array}
\]

Taking the direct limit, we get a commutative triangle

\[
\begin{array}{ccc}
\lim R_i & \xrightarrow{\psi} & \lim S_{(R_i/a_i,M_i/a_i)} \\
\downarrow & & \downarrow \\
S & \xrightarrow{\psi'} & S'.
\end{array}
\]

Thus we have \( \psi \sigma \psi' \) with respect to the preorder \( \sigma \) on \( H(R) \), so that \( (a,M) \leq (a',M') \).
For the last paragraph of this section, we have been inspired by \[24\]. Any preordered set \((X, \leq)\) can be viewed as a category whose objects are the elements of \(X\) and, for every pair \(x, y \in X\) of objects of the category, there is exactly one morphism \(x \to y\) if \(x \leq y\), and no morphism \(x \to y\) otherwise. This applies in particular to our partially ordered set \(\text{Hom}(R)\), for any ring \(R\). There is a covariant functor \(F_R: \text{Hom}(R) \to \text{Ring}\). It associates to any object \((a, M)\) of \(\text{Hom}(R)\) the ring \(S_{(R/a, M/a)}\). Like in the proof of the previous theorem, where we show that \(\Psi^{-1}\) is a partially ordered set morphism, we have that if \((a, M) \leq (a', M')\), then there is a canonical morphism \(S_{(R/a, M/a)} \to S_{(R/a', M'/a')}\). So we have, for every ring \(R\), a covariant functor \(F_R: \text{Hom}(R) \to \text{Ring}\). This can be expressed by means of diagrams in the category \(\text{Ring}\). Formally, a diagram of shape \(J\) in a category \(\mathcal{C}\) is a functor \(F\) from \(J\) to \(\mathcal{C}\). Here we are considering only the case in which the category \(J\) is a partially ordered set. Thus we have, for every ring \(R\), a diagram of shape \(\text{Hom}(R)\) in the category \(\text{Ring}\).

5. Maximal elements in \(\text{Hom}(R)\)

We now recall a classification due to Bokut (see \[8\] and \[11\] pp. 515-516). Let \(\mathcal{D}_0\) be the class of integral domains, \(\mathcal{D}_2\) the class of invertible rings, that is, rings \(R\) such that the universal mapping inverting all non-zero elements of \(R\) is injective, and \(\mathcal{E}\) be the class of rings embeddable in division rings. Then \(\mathcal{D}_0 \supset \mathcal{D}_2 \supset \mathcal{E}\). Notice that a ring \(R \in \mathcal{D}_0\) is in \(\mathcal{D}_2\) if and only if the mapping \(\chi_{(R, R \setminus \{0\})}: R \to S_{(R, R \setminus \{0\})}\) is injective, if and only if \((0, R \setminus \{0\}) \in \text{Hom}(R)\).

**Proposition 5.1.** Let \(a\) be an ideal of a ring \(R\) such that \((a, R \setminus a) \in \text{Hom}(R)\). Then \(a\) is a completely prime ideal of \(R\), the ring \(R/a\) is invertible, and \((a, R \setminus a) \in \text{Hom}(R)\) is a maximal element of \(\text{Hom}(R)\).

**Proof.** Since \((a, R \setminus a) \in \text{Hom}(R)\), the set \(R \setminus a\) is multiplicatively closed, that is, \(a\) is completely prime. Moreover, \((a, R \setminus a) \in \text{Hom}(R)\) implies that there exists a morphism \(\varphi: R \to S\) with kernel \(a\) for which all elements of \(\varphi(R \setminus a)\) are invertible. The induced mapping \(\overline{\varphi}: R/a \to S\) is an injective morphisms for which the image of every non-zero element of \(R/a\) is invertible in \(S\). The injective morphism \(\overline{\varphi}\) factors through the universal inverting mapping \(\psi: R/a \to S_{(R/a, M/a)}\) by Theorem 3.3. Thus \(\overline{\varphi}\) injective implies \(\psi\) injective, i.e., \(R/a\) is an invertible ring. Finally, \((a, R \setminus a)\) is a maximal element of \(\text{Hom}(R)\), because if \((a, R \setminus a) \leq (a', M')\), then \(a \leq a'\) and \(R \setminus a \subseteq M'\). Hence \(a' \cap M' = \emptyset\) implies \(a' = a\) and \(M' = R \setminus a\). \(\square\)

In the following example, we show that not all maximal elements of \(\text{Hom}(R)\) are of the form \((a, R \setminus a)\) for some completely prime ideal \(a\).

**Example 5.2.** Let \(R\) be the ring of \(n \times n\) matrices with entries in a division ring \(D\), \(n \geq 1\). Then any homomorphism \(\varphi: R \to S\), \(S\) any ring, is injective because \(R\) is simple. Every element of \(M := \varphi^{-1}(U(S))\) is regular by Lemma 2.1. But regular elements in \(R\) are invertible. This proves that \(\text{Hom}(R)\) has exactly one element, the pair \((0, U(R))\). Thus, clearly, \(\text{Hom}(R)\) has a greatest element, which is not of the form \((a, R \setminus a)\) because \(R\) is simple, but not a domain, and \(R\) has no completely prime ideals.

**Proposition 5.3.** For any commutative ring \(R\), the maximal elements of \(\text{Hom}(R)\) are the pairs \((P, R \setminus P)\), where \(P\) is a prime ideal.
Proposition 5.4. Let \( R \) be a commutative ring. Then \( \text{Hom}(R) \) has a greatest element if and only if \( R \) has a unique prime ideal.

Proof. If \( \text{Hom}(R) \) has a greatest element \((a, M)\), then \((a, M)\) is the unique maximal element of \( \text{Hom}(R) \), so that \( R \) has a unique prime ideal by Proposition 5.3.

Conversely, let \( R \) be a commutative ring with a unique prime ideal \( P \). Then \( R \) is a local ring with maximal ideal \( P \). Clearly, the pair \((P, R \setminus P)\) belongs to \( \text{Hom}(R) \), because it is associated to the canonical morphism of \( R \) onto the field \( R/P \). For any other ring morphism \( \varphi: R \to S \), one has \( \ker \varphi \subseteq P \) because \( \ker \varphi \) is a proper ideal of \( R \). In order to show that \((P, R \setminus P)\) is the greatest element of \( \text{Hom}(R) \), it suffices to show that \( \varphi^{-1}(U(S)) \subseteq R \setminus P \). We claim that \( \varphi^{-1}(U(S)) \), which is clearly a multiplicatively closed subset of \( R \), is saturated, that is, if \( r, r' \in R \) and \( \varphi(rr') \in U(S) \), then \( \varphi(r) \in U(S) \) (this is sufficient, because \( R \) is commutative). If \( r, r' \in R \) and \( \varphi(rr') \in U(S) \), then there exists \( s \in S \) such that \( \varphi(r)\varphi(r')s = 1 \). Thus \( \varphi(r) \) is invertible in \( S \). This proves the claim. The complement of a saturated multiplicatively closed subset of a commutative ring is a union of prime ideals [2, p. 44, exercise 7]. Since \( R \) has a unique prime ideal, the saturated multiplicatively closed subsets of \( R \) are only \( R \setminus P \), the improper subset \( R \) of \( R \), and the empty set \( \emptyset \). It follows that \( \varphi^{-1}(U(S)) = R \setminus P \). This concludes the proof. \( \square \)

Example 5.5. As an example, we now describe the structure of the partially ordered set \( \text{Hom}(\mathbb{Z}) \), where \( \mathbb{Z} \) is the ring of integers.

For an arbitrary element \((a, M)\) of \( \text{Hom}(\mathbb{Z}) \), we have that \( a = n\mathbb{Z} \) for some non-negative integer \( n \neq 1 \). For \( n = 0 \), the set \( M \) must be a saturated subset of \( \mathbb{Z} \). Hence \( \mathbb{Z} \setminus M \) is a union of prime ideals [2, p. 44, exercise 7]. Thus there exists a subset \( P \) of the set \( \mathbb{P} := \{ p \mid p \text{ is prime number} \} \) such that \( M \) is the set \( M_P \) of all \( z \in \mathbb{Z} \), \( z \neq 0 \), with \( p \mid z \) for every \( p \in P \). For any such subset \( P \) of \( \mathbb{P} \), the pair \((0, M_P)\) corresponds to the embedding of \( \mathbb{Z} \) into its ring of fractions with denominators in the multiplicatively closed subset \( M_P \) of \( \mathbb{Z} \).

Now assume that \( a = n\mathbb{Z} \) for some \( n \geq 2 \) and that \((a, M)\) corresponds to some ring morphism \( \varphi: \mathbb{Z} \to S \). Then \( \varphi \) induces an injective ring morphism \( \varphi\mathbb{Z}; \mathbb{Z}/n\mathbb{Z} \to S \), and \( M/n\mathbb{Z} \) is a multiplicatively closed subset of \( \mathbb{Z}/n\mathbb{Z} \) that consists of regular elements and contains \( U(\mathbb{Z}/n\mathbb{Z}) \). Since in a finite ring all regular elements are invertible, it follows that \( M/n\mathbb{Z} = U(\mathbb{Z}/n\mathbb{Z}) \), so that \( M = M_{\text{div}(n)} \), where \( \text{div}(n) := \{ p \in \mathbb{P} \mid p \mid n \} \). Thus

\[
\text{Hom}(\mathbb{Z}) = \{ (0, M_P) \mid P \text{ is a subset of } \mathbb{P} \} \cup \{ (n\mathbb{Z}, M_{\text{div}(n)}) \mid n \in \mathbb{Z}, \ n \geq 2 \}.
\]

Notice that for any \( P, P' \subseteq \mathbb{P} \) and \( n, n' \geq 2 \):

1. \((0, M_P) \leq (0, M_{P'})\) if and only if \( M_P \subseteq M_{P'} \), if and only if \( P' \subseteq P \).
2. \((n\mathbb{Z}, M_{\text{div}(n)}) \leq (n'\mathbb{Z}, M_{\text{div}(n')})\) if and only if \( n\mathbb{Z} \subseteq n'\mathbb{Z} \), if and only if \( n' \mid n \) (because \( n' \mid n \) implies \( \text{div}(n') \subseteq \text{div}(n) \), from which \( M_{\text{div}(n)} \subseteq M_{\text{div}(n')} \)).
3. \((0, M_P) \leq (n\mathbb{Z}, M_{\text{div}(n)})\) if and only if \( \text{div}(n) \subseteq P \).
(4) \( (n\mathbb{Z}, M_{\text{div}(n)}) \leq (0, M_P) \) never occurs.

In order to better describe the partially ordered set \( \text{Hom}(\mathbb{Z}) \), we will now present an order-reversing injective mapping \( \rho : \text{Hom}(\mathbb{Z}) \to (\mathbb{N}_0)^\mathbb{P}^* \), where \( \mathbb{N}_0 \) denotes the set of non-negative integers with its usual order, \( \mathbb{N}_0 := \mathbb{N}_0 \cup \{+\infty\} \) with \( n \leq \infty \) for all \( n \in \mathbb{N}_0 \), \( \mathbb{P}^* := \mathbb{P} \cup \{0\} \) and the order on the product \( (\mathbb{N}_0)^\mathbb{P}^* \) is the component-wise order. Via this \( \rho \), the partially ordered set \( \text{Hom}(\mathbb{Z}) \) can be identified as a partially ordered subset of the opposite partially ordered set of \( (\mathbb{N}_0)^\mathbb{P}^* \). Notice that every positive integer \( n \) can be written uniquely as a product of primes, \( n = p_1^{e_1} \cdots p_t^{e_t} \) for suitable distinct primes \( p_i \in \mathbb{P} \) and positive integers \( e_i \), so that there is an order-preserving injective mapping \( \rho' : \mathbb{N} \to \mathbb{N}_0^\mathbb{P} \), where the set \( \mathbb{N} \) of positive integers is ordered by the relation \( \mid \) (divides). Here

\[
\rho'(n)(p) = \begin{cases} e_i & \text{if } p = p_i \text{ for some } i, \\ 0 & \text{if } p \in \mathbb{P} \setminus \{p_1, \ldots, p_t\} \end{cases}
\]

for every \( n \in \mathbb{N} \) and \( p \in \mathbb{P} \). Similarly, there are characteristic functions of subsets \( P \) of \( \mathbb{P} \), so that there is an order-preserving bijection \( \chi : \mathcal{P}(\mathbb{P}) \to \{0, +\infty\}^{\mathbb{P}} \), defined by \( \chi(P) = \chi_P \) for every \( P \) in the power set \( \mathcal{P}(\mathbb{P}) \) of all subsets of \( \mathbb{P} \), where \( \chi_P : \mathbb{P} \to \{0, +\infty\} \) is such that

\[
\chi_P(p) = \begin{cases} +\infty & \text{if } p \in P, \\ 0 & \text{if } p \in \mathbb{P} \setminus P \end{cases}
\]

for every \( P \subseteq \mathbb{P} \) and \( p \in \mathbb{P} \).

Our mapping \( \rho \) will extend both the order-preserving injective mapping \( \rho' \) and the order isomorphism \( \chi \). Define \( \rho : \text{Hom}(\mathbb{Z}) \to (\mathbb{N}_0)^\mathbb{P}^* \) by

\[
\rho(n\mathbb{Z}, M_{\text{div}(n)})(p) = \begin{cases} e_i & \text{if } p = p_i \text{ for some } i, \\ 0 & \text{if } p \in \mathbb{P} \setminus \{p_1, \ldots, p_t\}, \\ 0 & \text{if } p = 0 \end{cases}
\]

for every \( n \geq 2 \) and \( p \in \mathbb{P}^* := \mathbb{P} \cup \{0\} \), and

\[
\rho(0, M_P)(p) = \begin{cases} +\infty & \text{if } p \in P, \\ 0 & \text{if } p \in \mathbb{P} \setminus P, \\ 1 & \text{if } p = 0 \end{cases}
\]

for every \( P \subseteq \mathbb{P} \).

In order to show that this mapping \( \rho \) is an order-reversing embedding of \( \text{Hom}(\mathbb{Z}) \) into \( (\mathbb{N}_0)^\mathbb{P}^* \), we must prove that \( \rho \) satisfies the following four properties, corresponding to the four properties (1)-(4) above:

1. \( \rho(0, M_{P'})(p) \leq \rho(0, M_P)(p) \) for every \( p \in \mathbb{P}^* \) if and only if \( P' \subseteq P \).
2. \( \rho(n\mathbb{Z}, M_{\text{div}(n')})(p) \leq \rho(n\mathbb{Z}, M_{\text{div}(n)})(p) \) for every \( p \in \mathbb{P}^* \) if and only if \( n' \mid n \).
3. \( \rho(n\mathbb{Z}, M_{\text{div}(n)})(p) \leq \rho(0, M_P)(p) \) for every \( p \in \mathbb{P}^* \) if and only if \( \text{div}(n) \subseteq P \).
4. For every \( P \subseteq \mathbb{P} \) and every \( n \geq 2 \), there exists \( p \in \mathbb{P}^* \) such that

\[
\rho(n\mathbb{Z}, M_{\text{div}(n)})(p) < \rho(0, M_P)(p)
\]

Let \( P, P' \) be subsets of \( \mathbb{P} \). In order to prove (1'), notice that \( \rho(0, M_{P'})(p) \leq \rho(0, M_P)(p) \) for every \( p \in \mathbb{P}^* \) if and only if, for every \( p \in \mathbb{P} \), \( \rho(0, M_{P'})(p) = +\infty \) implies \( \rho(0, M_P)(p) = +\infty \). This is equivalent to \( p \in P' \) implies \( p \in P \) for every \( p \in \mathbb{P} \), that is, if and only if \( P' \subseteq P \).

Now let \( n, n' \geq 2 \) be integers. Then \( \rho(n\mathbb{Z}, M_{\text{div}(n')})(p) \leq \rho(n\mathbb{Z}, M_{\text{div}(n)})(p) \) for every \( p \in \mathbb{P}^* \) if and only if, for every prime \( p \), \( p|n' \) implies \( p|n \) and the exponent of
Reg of Hom(R, for left localizations at left Ore sets in [4, Theorem 2.1.2]. The greatest element has a greatest element. This would correspond with what has been done by Bavula Theorem 3.3, M_
. Since the monoids M_
 there is no contravariant functor from the category of rings to Theorem 5.7. Apply Zorn’s Lemma, which concludes the proof of the theorem. □

The elements of M_
 are mapped to invertible elements of Z_n, hence we can apply Zorn’s Lemma, which concludes the proof of the theorem.

Propositions 5.1 and 5.3 show that the set Max(R) of all maximal elements of Hom(R) could be used as a good substitute for the spectrum of a non-commutative ring R. Let us show that the set of all maximal elements is never empty.

**Theorem 5.6.** For every ring R, the partially ordered set Hom(R) has maximal elements.

**Proof.** Let R be a ring. It is known that R always has maximal two-sided ideals, that is, maximal elements in the set of all proper two-sided ideals (this is a very standard application of Zorn’s Lemma). Let m be a maximal two-sided ideal of R. Set \( \mathcal{F} := \{ M \mid M = \varphi^{-1}(U(S)), S \text{ is any ring and } \varphi: R \to S \text{ is a ring morphism with } \ker(\varphi) = m \} \). Then \( \mathcal{F} \) is non-empty (consider the canonical projection \( \varphi: R \to R/m \)). Partially order \( \mathcal{F} \) by set inclusion. Let \( M_\lambda (\lambda \in \Lambda) \) be a chain in \( \mathcal{F} \). By Theorem 5.3 \( M_\lambda = \psi_\lambda^{-1}(U(S(R/m,M_\lambda/m))) \), where \( \psi_\lambda: R \to S(R/m,M_\lambda/m) \) is the canonical mapping. Since the monoids \( M_\lambda (\lambda \in \Lambda) \) are linearly ordered by set inclusion, the rings \( S(R/m,M_\lambda/m) \) form a direct system of rings over a linearly ordered set, and there is a ring morphism

\[
\psi = \lim_\to \psi_\lambda: R \to S = \lim_\to S(R/m,M_\lambda/m).
\]

The elements of \( M_\lambda \) are mapped to invertible elements of \( S_\lambda \) via \( \psi_\lambda \), so that they are mapped to invertible elements of \( S \) via \( \psi \). Thus \( \psi^{-1}(U(S)) \supseteq \bigcup_{\lambda \in \Lambda} M_\lambda \), i.e., \( \psi^{-1}(U(S)) \in \mathcal{F} \) is an upper bound of the chain of the monoids \( M_\lambda \). Hence we can apply Zorn’s Lemma, which concludes the proof of the theorem.

From Proposition 2.5, Theorem 5.6 and [21, Theorem 1.1], we get that:

**Theorem 5.7.** There is no contravariant functor from the category of rings to the category of sets that assigns to each ring R the set of all maximal elements of Hom(R).

It would be very interesting to determine, for any ring R, if the set \( \text{Hom}(R,0) \) has a greatest element. This would correspond with what has been done by Bavula for left localizations at left Ore sets in [11, Theorem 2.1.2]. The greatest element of \( \text{Hom}(R,0) \) corresponds to a suitable submonoid \( M \) of the multiplicative monoid \( \text{Reg}_R \) of all regular elements of R. Notice that:

1. When R is commutative, the greatest element of \( \text{Hom}(R,0) \) is clearly \( (0,\text{Reg}_R) \).
2. More generally, if \( \text{Reg}_R \) is a right Ore set or a left Ore set, then the greatest element of \( \text{Hom}(R,0) \) is \( (0,\text{Reg}_R) \).
(3) If the ring $R$ is contained in a division ring, the greatest element of $\text{Hom}(R, 0)$ is $(0, R \setminus \{0\})$.

(4) More generally, suppose that the canonical morphism $\chi_{(R, \text{Reg}_R)}: R \to S_{(R, \text{Reg}_R)}$ is injective, or, equivalently, $R$ is contained in a ring in which all regular elements of $R$ are invertible. For example, $R$ could be an invertible ring (see the definition in the first paragraph of this Section). Then the greatest element of $\text{Hom}(R, 0)$ is $(0, \text{Reg}_R)$.

(5) If $M$ is a submonoid of $R$, the pair $(0, M)$ is a maximal element of $\text{Hom}(R, 0)$ if and only if the canonical morphism $\chi_{(R, M)}: R \to S_{(R, M)}$ is injective and, for every regular element $x \in R$, $x \notin M$, the canonical mapping $\chi_{(R, \text{M}(x))}: R \to S_{(R, \text{M}(x))}$ is not injective.

Remark 5.8. For any ring $R$, consider the three sets

$$\text{Div}(R) := \{(a, R \setminus a) \mid a = \ker \varphi \text{ for a morphism } \varphi: R \to D \text{ into some division ring } D\}$$

$$\text{Cpr}(R) := \{(P, R \setminus P) \in \text{Hom}(R) \mid P \text{ is a completely prime ideal of } R\}$$

$$\text{Max}(R) \text{ of all maximal elements of } \text{Hom}(R).$$

In general, we have $\text{Div}(R) \subseteq \text{Cpr}(R) \subseteq \text{Max}(R) \subseteq \text{Hom}(R)$ (Proposition 5.1), and all these inclusions can be proper. When $R$ is commutative, $\text{Div}(R) = \text{Cpr}(R) = \text{Max}(R) \cong \text{Spec}(R) \subseteq \text{Hom}(R)$ (Proposition 5.3).

Related to this, we can consider Cohn’s spectrum $\mathbf{X}(R)$ of the ring $R$, that is, the topological space $\mathbf{X}(R)$ of all epic $R$-fields, up to isomorphism. Recall that a ring morphism $f: R \to D$ is an epic $R$-field in the sense of [10, p. 154] if $D$ is a division ring and there is no division ring different from $D$ between $f(R)$ and $D$. Notice that there are rings $R$ for which there is no epic $R$-field $R \to D$. For instance, if $R$ is a ring that is not IBN, there is no ring morphism $R \to D$, for any division ring $D$. Clearly, there is an onto mapping $\mathbf{X}(R) \to \text{Div}(R)$.

6. The partially ordered set $\overline{\text{Hom}(R)}$

As we have said in Section 2, the partially ordered set $\text{Hom}(R)$ is a meet-semilattice, hence a commutative semigroup in which every element is idempotent, has a smallest element $(0, U(R))$, but does not have a greatest element in general. Hence we now enlarge the partially ordered set $\text{Hom}(R)$ adjoining to it a further element, a new greatest element $1$, setting $\text{Hom}(R) := \text{Hom}(R) \cup \{1\}$. Here $(a, M) \leq 1$ for every element $(a, M) \in \text{Hom}(R)$. This new element $1$ of $\text{Hom}(R)$ represents in some sense the zero morphism $R \to 0$, where $0$ is the zero ring with one element. The zero ring is not a ring in our sense strictly, because we have supposed in the Introduction that all our rings have an identity $1 \neq 0$. This is the reason why the zero morphism $R \to 0$ does not appear in the definition of $\text{Hom}(R)$. Moreover, the pair $(a, M)$ corresponding to the zero morphism $R \to 0$ would clearly have $a = R$, but it is not clear what $M$ should be.

The contravariant functor $\text{Hom}(\_): \text{Ring} \to \text{ParOrd}$ extends to a contravariant functor $\overline{\text{Hom}(\_)}: \text{Ring} \to \overline{\text{ParOrd}}$ simply extending, for every ring morphism $\varphi: R \to S$, the mapping $\text{Hom}(\varphi): \text{Hom}(S) \to \text{Hom}(R)$ to the mapping $\overline{\text{Hom}(\varphi)}: \overline{\text{Hom}(S)} \to \overline{\text{Hom}(R)}$, where $\overline{\text{Hom}(\varphi)}(1) = 1$.

First of all, we will now show that $\overline{\text{Hom}(R_1 \times R_2)}$, where $R_1 \times R_2$ denotes the ring direct product, is canonically isomorphic to the cartesian product $\overline{\text{Hom}(R_1)} \times \overline{\text{Hom}(R_2)}$. 

\[ \overline{\text{Hom}(R_1 \times R_2)} \cong \overline{\text{Hom}(R_1)} \times \overline{\text{Hom}(R_2)} \]
\( \text{Hom}(R_2) \). We first need an elementary proposition. For every pair \( R, S \) of rings, we will denote the set of all ring morphisms \( R \to S \), including the zero morphism \( R \to 0 \), by \( \text{Hom}_{\text{Ring}}(R, S) \).

**Proposition 6.1.** Let \( R_1, R_2, S \) be rings. Then there is a bijection between
\[
\text{Hom}_{\text{Ring}}(R_1 \times R_2, S)
\]
and the set of all triples \((e, \psi_1, \psi_2)\), where \( e \in S \) is an idempotent element and \( \psi_1: R_1 \to eS, \psi_2: R_2 \to (1 - e)S(1 - e) \) are ring morphisms (possibly zero, when \( e = 0 \) or \( e = 1 \)).

**Proof.** Let \( \mathcal{T} \) denote the set of all the triples \((e, \psi_1, \psi_2)\) in the statement. Let \( \Phi: \text{Hom}_{\text{Ring}}(R_1 \times R_2, S) \to \mathcal{T} \) be defined by \( \Phi(\varphi) = (\varphi(1_{R_1}, 0_{R_2}), \varphi|_{R_1}, \varphi|_{R_2}) \). Here \( \varphi: R_1 \times R_2 \to S \) is any ring morphism, so that \( e := \varphi(1_{R_1}, 0_{R_2}) \) is an idempotent element of \( S \), and \( \varphi|_{R_1}: R_1 \to eS, \varphi|_{R_2}: R_2 \to (1 - e)S(1 - e) \) denote the restrictions of \( \varphi \) to \( R_1, R_2 \) respectively (or, more precisely, to the subsets \( R_1 \times \{0\} \) and \( \{0\} \times R_2 \) of \( R_1 \times R_2 \)). We leave to the reader to check that \( \Phi \) is a well-defined surjective mapping. As far as injectivity is concerned, notice that if \( \varphi: R_1 \times R_2 \to S \) and \( \varphi': R_1 \times R_2 \to S \) are ring morphisms and \( \Phi(\varphi) = \Phi(\varphi') \), then \( \varphi = \varphi' \) because \( R_1 \times R_2 \) is the direct sum of \( R_1 \) and \( R_2 \) as additive abelian groups, and therefore \( \varphi = \varphi' \) are completely determined by their restrictions to the direct summands \( R_1 \) and \( R_2 \) of \( R_1 \times R_2 \).

**Proposition 6.2.** Let \( R_1 \) and \( R_2 \) be rings. Then there is a canonical bijection between \( \text{Hom}(R_1 \times R_2) \) and the cartesian product \( \text{Hom}(R_1) \times \text{Hom}(R_2) \).

**Proof.** First of all, we show that, for any ring morphism \( \varphi: R_1 \times R_2 \to S \), we have
\[
(1) \quad \ker(\varphi) = \ker(\varphi|_{R_1}) \times \ker(\varphi|_{R_2})
\]
and
\[
(2) \quad \varphi^{-1}(U(S)) = (\varphi|_{R_1})^{-1}(U(eS)) \times (\varphi|_{R_2})^{-1}(U((1 - e)S(1 - e))).
\]
Here, like in the proof of Proposition 6.1, \( e \) is the image via \( \varphi \) of the idempotent element \((1_{R_1}, 0_{R_2})\) of \( R_1 \times R_2 \), and \( \varphi|_{R_1}: R_1 \to eS, \varphi|_{R_2}: R_2 \to (1 - e)S(1 - e) \) are the restrictions of \( \varphi \) to \( R_1, R_2 \). We leave the easy proof of (1) to the reader. As far as (2) is concerned, notice that this formula makes no sense when one of the morphisms \( \varphi, \varphi|_{R_1}, \varphi|_{R_2} \) is zero. In these three cases, either \( e = 0 \) or \( e = 1 \), and the morphisms \( \varphi, \varphi|_{R_1}, \varphi|_{R_2} \) correspond to the greatest element \( 1 \) of \( \text{Hom}(R_1 \times R_2), \text{Hom}(R_1) \) or \( \text{Hom}(R_2) \), respectively. Also remark that if \((r_1, r_2) \in R_1 \times R_2 \), then \((r_1, r_2) \in \varphi^{-1}(U(S)) \) if and only if \( \varphi(r_1, r_2) \in U(S) \). Now \( \varphi(r_1, r_2) \in eS \times (1 - e)S(1 - e) \) is invertible in \( S \) if and only if it is invertible in \( eS \times (1 - e)S(1 - e) \), that is, if and only if \( (\varphi(r_1))(r_1) \) is invertible in \( eS \) and \( (\varphi(r_2))(r_2) \) is invertible in \( (1 - e)S(1 - e) \). This concludes the proof of (2).

From (1) and (2), it follows that the mapping
\[
\text{Hom}(R_1 \times R_2) \to \text{Hom}(R_1) \times \text{Hom}(R_2),
\]
\[
(\ker(\varphi), \varphi^{-1}(U(S))) \mapsto ((\ker(\varphi|_{R_1}), (\varphi|_{R_1})^{-1}(U(eS))), (\ker(\varphi|_{R_2}), (\varphi|_{R_2})^{-1}(U((1 - e)S(1 - e))))
\]
is a well-defined injective mapping. Its surjectivity is proved considering, for any pair of ring morphisms \( \varphi_1: R_1 \to S_1, \varphi_2: R_2 \to S_2 \), the ring morphism \( \varphi_1 \times \varphi_2: R_1 \times R_2 \to S_1 \times S_2 \).
We saw in Lemma 2.4 and the paragraph following it, that the partially ordered set \( \text{Hom}(R) \) is a meet-semilattice, so that \( \text{Hom}(R) \), with respect to the operation \( \wedge \), is a commutative semigroup in which every element is idempotent. Now \( \text{Hom}(R) \) is also a meet-semilattice, but with a greatest element 1, so \( \text{Hom}(R) \), with respect to the operation \( \wedge \), turns out to be a commutative monoid in which every element is idempotent. Hence we can view the functor \( \text{Hom}(-) \) as a functor of \( \text{Ring} \) into the category \( \text{CMon} \) of commutative monoids. Now, commutative monoids have a spectrum, set of its prime ideals, i.e., there is a contravariant functor \( \text{Spec} \) from the category \( \text{CMon} \) to the category \( \text{Top} \) of topological spaces [20]. For every commutative monoid \( A \), \( \text{Spec}(A) \) is a spectral space in the sense of Hochster. Hence the composite functor \( \text{Spec} \circ \text{Hom}(-) : \text{Ring} \to \text{Top} \) associates to every ring a spectral topological space.

**Theorem 6.3.** For every ring \( R \), the partially ordered set \( \text{Hom}(R) \) is a bounded lattice.

**Proof.** It is clear that \( \text{Hom}(R) \) is a partially ordered set with a least element \( (0, U(R)) \) and a greatest element 1. Since we already know that \( \text{Hom}(R) \) is a meet-semilattice, we only have to show that any pair of elements \( (a, M), (a', M') \) of \( \text{Hom}(R) \) has a least upper bound in \( \text{Hom}(R) \) (it is clear that the least upper bound exists when one of the two elements is the greatest element 1 of \( \text{Hom}(R) \)). Suppose \( (a, M), (a', M') \in \text{Hom}(R) \). Let \( \psi : R \to S_{(R/a, M/a)} \) be the ring morphism corresponding to the pair \( (a, M) \) as in the statement of Theorem 3.3. Similarly for \( \psi' : R \to S_{(R/a', M'/a')} \).

We will now prove that \( (a'', M'') = (a, M) \lor (a', M') \) in the partially ordered set \( \text{Hom}(R) \). Since \( \omega \) factors through \( \psi \), we have that \( (a'', M'') \geq (a, M) \). Similarly, \( (a'', M'') \geq (a', M') \). Conversely, let \( \chi : R \to T \) be any morphism with associated pair \( (b, N) \) and with \( (b, N) \geq (a, M), (a', M') \). By the universal property of Theorem 3.3 there is a unique ring morphism \( g : S_{(R/a, M/a)} \to T \) such that \( g\psi = \chi \).

By the universal property of pushout, there exists a unique morphism \( h : P \to T \) such that \( h\varepsilon = g \) and \( h\varepsilon' = g' \), where \( \varepsilon : S_{(R/a, M/a)} \to P \) and \( \varepsilon' : S_{(R/a', M'/a')} \to P \) are the canonical mappings into the pushout. Then \( h\omega = h\varepsilon\psi = g\psi = \chi \), so \( \chi \) factors through \( \omega \), hence \( (b, N) \geq (a'', M'') \).

\[
\begin{array}{c}
R \xrightarrow{\psi} S_{(R/a, M/a)} \\
\psi' \\
S_{(R/a', M'/a')} \xrightarrow{\varepsilon} P \xrightarrow{g} T \\
\omega \\
\end{array}
\]

\[\square\]
7. Ring epimorphisms

Recall that a ring morphism \( \varphi: R \to S \) is an epimorphism if, for all ring morphisms \( \psi, \psi': S \to T \), \( \psi \varphi = \psi' \varphi \) implies \( \psi = \psi' \).

**Proposition 7.1.** ([17, 10] Proposition 4.1.1), [28] Proposition XI.1.2] The following conditions on a ring morphism \( \varphi: R \to S \) are equivalent:

(a) \( \varphi \) is an epimorphism,
(b) \( s \otimes 1 = 1 \otimes s \) in the \( S \)-\( S \)-bimodule \( S \otimes_R S \) for all \( s \in S \).
(c) The \( R \)-\( R \)-bimodule \( S \otimes_R S \) is isomorphic to the \( R \)-\( R \)-bimodule \( S \) via the canonical isomorphism induced by the multiplication \( \cdot: S \times S \to S \) of the ring \( R \).
(d) The pushout \( R \to S \ast_R S \) of \( \varphi \) with itself is naturally isomorphic to \( R \to S \).
(e) \( S \otimes_R (S/\varphi(R)) = 0 \).

**Proposition 7.2.** Let \( \varphi: R \to S \) be a ring morphism and \( (a, M) \) be its corresponding pair in \( \text{Hom}(R) \). Let \( T \) be the subring of \( S \) generated by \( \varphi(R) \) and all the elements \( \varphi(m)^{-1} \) (\( m \in M \)). Then the corestriction \( \varphi^T: R \to T \) is a ring epimorphism and its corresponding pair in \( \text{Hom}(R) \) is \( (a, M) \).

**Proof.** Let \( T' \) be the subset of \( T \) consisting of all elements \( a \in T \) with \( a \otimes 1 = 1 \otimes a \) in the \( S \)-\( S \)-bimodule \( T \otimes_R T \). The subset \( T' \) of \( T \) is a subring of \( T \) that contains \( \varphi(R) \), because \( T \otimes_R T \) is a \( T \)-\( T \)-bimodule in which multiplication by elements of \( T \) is defined by \( t(t' \otimes t'') = (tt') \otimes t'' \) and \( (t' \otimes t'')t = t' \otimes (tt') \), so that \( a \otimes 1 = 1 \otimes a \) and \( b \otimes 1 = 1 \otimes b \) imply \( (ab) \otimes 1 = a(b \otimes 1) = a(1 \otimes b) = a \otimes b = (a \otimes 1)b = (1 \otimes a)b = 1 \otimes (ab) \). This shows that \( T' \) is a subring of \( T \). Moreover if \( m \in M \), then in \( T \otimes_R T \) we have that \( \varphi(m)^{-1} \otimes 1 = \varphi(m)^{-1} \otimes \varphi(m) \varphi(m)^{-1} = \varphi(m)^{-1} \varphi(m) \otimes \varphi(m)^{-1} = 1 \otimes \varphi(m)^{-1} \). It follows that \( T \subseteq T' \), hence \( T = T' \). It follows that the corestriction \( \varphi^T: R \to T \) is an epimorphism. Finally \( M = \varphi^{-1}(U(S)) \) and \( \varphi(m) \in U(T) \) for every \( m \in M \). Thus \( M \subseteq \varphi^{-1}(U(T)) \subseteq \varphi^{-1}(U(S)) = M \).

Now consider the universal construction of Theorem 3.3. Let \( \varphi: R \to S \) be a ring morphism and \( (a, M) \) be its corresponding pair in \( \text{Hom}(R) \). Via the canonical ring morphism \( \psi: R \to S_{(R/a, M/a)} \), the subring \( T \) of \( S_{(R/a, M/a)} \) generated by \( \psi(R) \) and the inverses of the images of the elements of \( M \) is the whole ring \( S_{(R/a, M/a)} \). It follows that the canonical ring morphism \( \psi: R \to S_{(R/a, M/a)} \) is a ring epimorphism.

More generally, for any ring morphism \( \varphi: R \to S \), we have the canonical factorization described in the next Theorem:

**Theorem 7.3.** Let \( \varphi: R \to S \) be any ring morphism and \( (a, M) \) its corresponding element in \( \text{Hom}(R) \). Then \( \varphi \) is the composite ring morphism of the mappings

\[
R \xrightarrow{\pi} R/a \xrightarrow{\chi} S_{(R/a, M/a)} \xrightarrow{g} T \xrightarrow{\varepsilon} S
\]

where \( T \) is the subring of \( S \) generated by \( \varphi(R) \) and the inverses \( \varphi(m)^{-1} \) of the images of the elements of \( M \), \( g: S_{(R/a, M/a)} \to T \) is a surjective ring epimorphism and \( \varepsilon: T \to S \) is the ring embedding.

This theorem shows that any ring morphisms \( \varphi: R \to S \) can be factorized as:

1. a canonical mapping \( R \to S_{(R/a, M/a)} \), which only depends on the pair \( (a, M) \in \text{Hom}(R) \) associated to \( \varphi \).
2. A ring morphism \( S_{(R/a, M/a)} \to T \), which is a surjective mapping and is an epimorphism in the category of rings.
3. A ring embedding \( \varepsilon: T \to S \).
Proof. Apply the universal property of Theorem 8.3 to the corestriction \( \varphi|^{T'} : R \to T' \), getting a factorization \( R \xrightarrow{\pi} R/a \xrightarrow{\chi} S_{(R/a,M/a)} \xrightarrow{g} T \) of \( \varphi|^{T'} \). The mapping \( g \) is surjective, because \( T \) is generated by the images of the elements of \( R \) and the inverses of the elements of \( M \), like \( S_{(R/a,M/a)} \). Moreover, \( g \) is a ring epimorphism, because \( \varphi|^{T'} : R \to T \) is a ring epimorphism by Proposition 7.2, so \( \psi g = \psi' g \) implies \( \psi g \chi = \psi' g \chi \), i.e., \( \psi \varphi|^{T'} = \psi' \varphi|^{T'} \), from which \( \psi = \psi' \). \( \Box \)

For any other ring morphism \( f : R \to S' \) such that \( \ker(f) = a \) and \( f^{-1}(U(S')) = M \), there is a unique ring morphism \( g : S_{(R/a,M/a)} \to S' \) such that \( g \varphi = f \). It follows that the subring \( T' \) of \( S' \) generated by \( f(R) \) and the elements \( f(m)^{-1} \) is the image \( g(S_{(R/a,M/a)}) \) of \( S_{(R/a,M/a)} \). Hence the corestriction \( f|^{T'} : R \to T' \) is the composite mapping of \( \varphi : R \to S_{(R/a,M/a)} \) and the corestriction \( g|^{T'} : S_{(R/a,M/a)} \to T' \).

Recall that a subset \( T \) is a left Ore subset of \( R \) if it is a submonoid of \( R \) such that \( Tr \cap Rt \neq \emptyset \) for every \( r \in R \) and \( t \in T \). A subset \( T \) of the ring \( R \) is called a \textit{left denominator set} if it is a left Ore subset and, for every \( r \in R \), \( t \in T \), if \( rt = 0 \), then there exists \( t' \in T \) with \( t'r = 0 \). A left ring of fractions \( \varphi : R \to [T^{-1}]R \) exists if and only if \( T \) is a left denominator set in \( R \).

Compare Lemma 4.14 with the fact that a left quotient ring \( [T^{-1}]R \) of a ring \( R \) with respect to a multiplicatively closed subset \( T \) of \( R \) exists if and only if \( T \) is a left Ore set and the set \( T = \{ t + \text{ass}(T) \in R/\text{ass}(T) \mid t \in T \} \) consists of regular elements (see 19, 2.1.12 and 18). Here \( \text{ass}(T) \) denotes the set of all elements \( r \in R \) for which there exists an element \( t \in T \) with \( tr = 0 \). That is, \( \text{ass}(T) \) is the kernel \( a \) of the canonical morphism \( R \to [T^{-1}]R \).

**Lemma 7.4.** Let \( T \) be a left denominator set in \( R \) and \( \varphi : R \to S = [T^{-1}]R \) the canonical mapping into the left ring of fractions. Then \( M := \varphi^{-1}(U(S)) \) is a left denominator set in \( R \) containing \( T \), \( a = \text{ass}(T) = \text{ass}(M) \) and \( S = [M^{-1}]R \).

Proof. It is well known that \( a = \ker(\varphi) = \text{ass}(T) \). Moreover, \( M \supseteq T \). Let us prove that \( M \) is a left Ore subset of \( R \). Fix \( r \in R \) and \( m \in M \). We must show that \( Mr \cap Mr \neq \emptyset \). Now \( \varphi(r)\varphi(m)^{-1} \in S = [T^{-1}]R \), so that there exist \( r' \in R \) and \( t \in T \) such that \( \varphi(r)\varphi(m)^{-1} = \varphi(t)^{-1}\varphi(r') \). Then \( \varphi(t)\varphi(r) = \varphi(r')\varphi(m) \) in \( S \), so that there exists \( t' \in T \) with \( rtt' = mr't' \in rT \cap mR \subseteq rM \cap mR \). \( t'tr = t'r'm \in Tr \cap Rm \subseteq Mr \cap Rm \). This proves that \( M \) is a left Ore subset of \( R \).

In order to see that \( M \) is a left denominator set, notice that if \( r \in R \), \( m \in M \) and \( rm = 0 \), then \( \varphi(r)\varphi(m) = 0 \), so \( \varphi(r) = 0 \). Hence \( r \in \ker(\varphi) = \text{ass}(T) \), so that \( tr = 0 \) for some \( t \in T \). But \( T \subseteq M \). This proves that \( M \) is a left denominator set. It is now clear that \( S = [T^{-1}]R = [M^{-1}]R \), and thus \( a = \ker(\varphi) = \text{ass}(M) \). \( \Box \)

Clearly, Lemma 7.4 holds not only for left denominator sets and left rings of fractions, but also for right denominator sets and right rings of fractions, because associating the pair \( (a,M) \) to a ring morphism \( \varphi \) is left/right symmetric.

**Remark 7.5.** We have already noticed in Remark 2.4 that if there exists a ring morphism \( \psi : S \to S' \) such that \( \psi \varphi = \varphi' \), then \( (a,M) \leq (a',M') \). This can be inverted for left localizations, i.e., if \( S = [T^{-1}]R \) and \( S' = [T'^{-1}]R \) for suitable left denominator sets \( T, T' \), \( \varphi : R \to S \), \( \varphi' : R \to S' \) are the canonical mappings, and \( (a,M) \leq (a',M') \), then there exists a ring morphism \( \psi : S \to S' \) such that \( \psi \varphi = \varphi' \).
To prove it, suppose that $S = [T^{-1}]R$ and $S' = [T'^{-1}]R$ for left denominator sets $T, T'$, that $\varphi: R \to S$, $\varphi': R \to S'$ are the canonical mappings and $(a, M) \leq (a', M')$. Since $M \subseteq M'$, so $T \subseteq M \subseteq M'$, the elements of $T$ are mapped to invertible elements of $S' = [M'^{-1}]R$ via the canonical mapping $\varphi': R \to S'$. By the universal property of the mapping $\varphi: R \to S = [T^{-1}]R$, there exists a unique ring morphism $\psi: S \to S'$ such that $\psi \varphi = \varphi'$.

Similarly for the equivalence relation $\sigma$: If $S = [T^{-1}]R$ and $S' = [T'^{-1}]R$ for left denominator sets $T, T'$, and $\varphi: R \to S$, $\varphi': R \to S'$ are the canonical mappings, then $\varphi \sigma \varphi'$ if and only if there exists a ring isomorphism $\psi: S \to S'$ such that $\psi \varphi = \varphi'$.

Finally, we have already remarked in the Introduction that a ring morphism $\varphi: R \to S$ is local if and only if $M = U(R)$. Moreover, $\ker(\varphi) \subseteq J(R)$ for every local morphism $\varphi: R \to S$ [13, Lemma 3.1]. It follows that local morphisms correspond to the least elements of $\text{Hom}(R, a)$ with respect to the partial order $\leq$. More precisely:

**Proposition 7.6.** Let $\varphi: R \to S$ be a ring morphism and $(a, M)$ its corresponding pair in $\text{Hom}(R)$. Then $\varphi$ is a local morphism if and only if $(a, M)$ is the least element of $\text{Hom}(R, a)$ for some ideal $a \subseteq J(R)$.

**Proof.** Suppose that $(a, M)$ is the least element of $\text{Hom}(R, a)$ for some ideal $a \subseteq J(R)$. By Proposition 7.7, the least element of $\text{Hom}(R, a)$ is $(a, \pi^{-1}(U(R/a)))$, where $\pi: R \to R/a$ is the canonical projection. Thus $M = \pi^{-1}(U(R/a))$. Let us prove that $\varphi$ is local. If $r \in R$ and $\varphi(r)$ is invertible in $S$, then $r \in M$, so that $r \in \pi^{-1}(U(R/a))$. Hence $r + a$ is invertible in $R/a$. Hence $r + J(R)$ is invertible in $R/J(R)$, so $r$ is invertible in $R$, as desired. This proves that $\varphi$ is a local morphism. The inverse implication is trivial. \hfill $\square$

More generally, for an arbitrary proper ideal $a$ of $R$, not-necessarily contained in $J(R)$, we have that:

**Proposition 7.7.** For every proper ideal $a$ of a ring $R$, the partially ordered set $\text{Hom}(R, a)$ always has a least element, which is the pair $(a, M)$ corresponding to the canonical projection $\pi: R \to R/a$, that is, the pair $(a, M)$ with $M = \pi^{-1}(U(R/a))$.

**Proof.** We must show that, for every ring morphism $\varphi: R \to S$ with $\ker(\varphi) = a$, we have $\pi^{-1}(U(R/a)) \subseteq \varphi^{-1}(U(S))$. Now, given $\varphi: R \to S$ with $\ker(\varphi) = a$, let $\pi: R \to R/a$ denote the canonical projection. By the first isomorphism theorem for rings, there exists a unique injective ring morphism $\tilde{\varphi}: R/a \to S$ such that $\varphi = \tilde{\varphi} \pi$. It is now easily checked that $\pi^{-1}(U(R/a)) \subseteq \varphi^{-1}(U(S))$. \hfill $\square$

We conclude the paper indicating a further possible generalization of our the results in this paper. In Remark 5.8, we have already mentioned Cohn’s spectrum $X(R)$ of a ring $R$, consisting of all epic $R$-fields, up to isomorphism. P. M. Cohn has shown that any epic $R$-field $R \to D$ is characterized up to isomorphism by the collection of square matrices with entries in $R$ which are carried to singular matrices with entries in the division ring $D$. He has also given the conditions under which a collection of square matrices over $R$ is of this type, calling such a collection a “prime matrix ideal” of $R$. The natural ideal is therefore to refine the theory developed in the previous sections, classifying all morphisms $\varphi: R \to S$, not only via our pairs $(a, M)$, where $M$ is the set of all elements of $R$ mapped to invertible
elements of $R$, but also via the collection of all $n \times m$ matrices with entries in $R$ which are carried to invertible $n \times m$ matrices with entries in the ring $S$.

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