Topology and Fragility in Cosmology

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Abstract

We introduce the notion of topological fragility and briefly discuss some examples from the literature. An important example of this type of fragility is the way globally anisotropic Bianchi V generalisations of the FLRW $k = -1$ model result in a radical restriction on the allowed topology of spatial sections, thereby excluding compact cosmological models with negatively curved three-sections with anisotropy. An outcome of this is to exclude chaotic mixing in such models, which may be relevant, given the many recent attempts at employing compact FLRW $k = -1$ models to produce chaotic mixing in the cosmic microwave background radiation, if the Universe turns out to be globally anisotropic.

key words: cosmology, topology, chaotic mixing, fragility.
1 Introduction

It is well known that general relativity (GR) is a local metrical theory and therefore the corresponding Einstein field equations do not fix the global topology of spacetime. Given this freedom in the topology of the spacetime manifold \( \mathcal{M} \), a question arises as to how free the choice of these topologies may be and how one may hope to determine them, which in turn is intimately related to the observational consequences of the spacetime possessing non-trivial topologies.

These questions have motivated two sets of work: (i) those attempting to tabulate mathematically the set of possible topologies for the spacetime, given certain symmetry constraints, such as homogeneity \([1] - [3]\), and (ii) those relating to the possible observational (or physical) consequences of adopting particular topologies for the spacetime \([4] - [22]\).

To determine the actual topology of the spacetime, one would have to ultimately rely on the observations–dynamics–topology correspondences, in the sense of looking at those observational or dynamical features of the universe (or cosmological models) which are dependent on the topology of spacetime. These correspondences can take various forms, such as for example the existence of (i) dynamical solutions with physically identifiable properties which can only arise with certain choices of topology \([13]\), (ii) identifiable images of galaxies implying closed 3-spaces \([4, 7, 9]\), and (iii) features, such as potential mixing of the cosmic microwave background radiation (CMWBR), which could possibly be identified, through the studies of the corresponding spectra, as being induced by non-trivial topology of the spacetime \([23, 24, 14, 18]\).

To use these correspondences effectively, however, it is important to study their nature, by examining whether they are sensitive in the sense that (i) changes in spacetime topology produce observable dynamical consequences, and (or) (ii) changes in the assumptions

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1Here, in line with its usage in the literature, by the topology of spacetime we mean the topology of the \( t = \text{const} \) spacelike section \( \mathcal{M}_3 \) of the spacetime manifold \( \mathcal{M}_4 \).
underlying cosmological models (such as symmetry) can have severe constraining effects on the allowed spacetime topologies.

We shall refer to such sensitivity as topological fragility\(^2\). If present, such fragility could have important consequences: it could either facilitate or hinder the task of inferring the topology of the spacetime, depending upon its precise nature.

Our aim in this work is to point out, with the help of examples, that such topological fragilities can arise naturally in usual cosmological modelling and could therefore be consequential in practice.

## 2 Topology and geometry in cosmology

Even though the metrical structure of a space does not generally fix its topology, the geometry can in certain settings severely constrain the possible set of allowed topologies. For example, in the case of compact 2-manifolds, there is a well known relationship between the topology and the geometry \([1, 27]\). However, in the case of 3-manifolds, \(M_3\), the situation is much more involved. In particular, for a general spacetime geometry very little can be said about the underlying topology of the spacetime manifold \(M_4\).

General asymmetric spacetime geometries are, however, rarely the object of study in cosmological modelling. What is usually done in practice is to impose simplifying assumptions (such as those involving symmetry) in order to reduce the resulting field equations to a manageable form. In particular, to reduce the complicated nonlinear set of Einstein’s partial differential equations to a manageable set of ordinary differential equations, it is usual to assume spatial homogeneity \([28]\).

In what follows we shall assume that the spacetime manifold \(M_4\) is decomposable into the form \(M_4 = \mathcal{R} \times M_3\), where the spacelike 3-manifolds \(M_3\) are orientable, connected and complete Riemannian manifolds. These are the main topological properties

\(^2\)As a counterpart to the concept of dynamical fragility introduced elsewhere \([25, 26]\).
one might expect in any reasonable model of the universe [29]. Further we shall assume
the spacetime to be expanding and homogeneous, which would include the Friedmann-
Lemaître-Robertson-Walker (FLRW) and the Bianchi models.

We shall show in the following sections that changes in the symmetry properties of
the universe can have significant constraining effects on the allowed topologies. In this
sense such models are topologically fragile.

3 The FLRW setting

Standard cosmological models, almost universally employed for the purpose of interpreta-
tion of observations, are the spatially homogeneous and isotropic Friedmann–Lemaître–
Robertson–Walker (FLRW) spacetime manifolds $\mathcal{M}_4$, which can be split into $\mathcal{R} \times \mathcal{M}_3$, possessing three-dimensional spacelike $t = \text{const}$ hypersurfaces of homogeneity $\mathcal{M}_3$.

If these 3-surfaces are assumed to be *globally* homogeneous-and-isotropic, i.e. to pos-
sess a continuous six-parameter isometry group acting transitively on the whole 3-spaces
$\mathcal{M}_3$, then the correspondence between the geometry and topology of the 3-spaces is very
tight and results in: $\mathcal{R}^3$ (the Euclidean 3-space) for $k = 0$, the 3-sphere $S^3$ and the pro-
jective 3-space $\mathcal{P}^3$ for the $k = +1$ case, and the hyperbolic 3-space $\mathcal{H}^3$ for the $k = -1$
case.

The assumption of global homogeneity-and-isotropy of 3-spaces $\mathcal{M}_3$ is, however, too
restrictive and not necessarily demanded by cosmological observations [29, 5, 22]. As
a result, it is customary to adopt a less restrictive setting of *local* homogeneity-and-
isotropy [29, 5, 22].

A word of clarification is in order here: corresponding to each 3-manifold $(\mathcal{M}_3, g)$,
there exists a simply connected covering manifold $(\tilde{\mathcal{M}}_3, \tilde{g})$ such that $(\mathcal{M}_3, g)$ is obtained
from $(\tilde{\mathcal{M}}_3, \tilde{g})$ by identifying points in $\tilde{\mathcal{M}}_3$ which are equivalent under a discrete group

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3With their flat, elliptic or hyperbolic constant spatial curvatures being specified by the curvature
parameter $k = 0, \pm 1$, respectively.
of isometries of $\tilde{M}_3$. In other words, $M_3$ is obtained by forming the quotient space $M_3 = \tilde{M}_3/\Gamma$, where $\Gamma$ is a discrete group of isometries of $\tilde{M}_3$ without fixed points, acting properly discontinuously $[29]$. By construction $(\tilde{M}_3, \tilde{g})$ is locally indistinguishable from $(M_3, g)$. The global features $\Gamma$ can, however, be quite different. The identification of points in $\tilde{M}_3$ via $\Gamma$ produces 3-manifolds $M_3$ which are multi-connected, and usually admit a lower-dimensional group of isometries. So for the FLRW cases, for example, one usually obtains quotient manifolds $M_3 = \tilde{M}_3/\Gamma$ which do not admit the full six-dimensional group of isometries ($G_6$), i.e., the quotient manifolds $M_3$ are not maximally symmetric. This amounts to saying that one or more of the linearly independent Killing vector fields associated with the isotropies alone, and defined on $\tilde{M}_3$ by the FLRW metric, are excluded by the topological identification, since they cannot be globally defined on $M_3$. And yet $M_3$ is still locally homogeneous-and-isotropic. In other words, the metric tensor of $(M_3, g)$ is the same at every point, but $M_3 = \tilde{M}_3/\Gamma$ is not globally isotropic since it does not permit global maximal symmetry. It should be stressed that in general topological identifications lower the dimension of the group of isotropies, breaking the global isotropy of the 3-spaces $\tilde{M}_3$. Clearly, the breaking of the global isotropy is apparent in many cases, since the identifications define preferred directions.

Now it is only with the assumption of local homogeneity-and-isotropy, that many other topological alternatives become possible for $M_3$ $[23, 30]$. So, for example, in the case of FLRW $k = 0$ there are six orientable and compact 3-manifolds, whereas the $k = \pm 1$ cases allow an infinite number of orientable and compact topological alternatives for the $t = \text{const}$ 3-manifolds.

As an example of topological fragility in the FLRW setting, we recall the way the

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$^4$Such as, for example, whether or not there is unicity of geodesics between two fixed points.

$^5$The only exception is the orientable compact projective 3-space $\mathbb{P}^3$, whose covering space is the 3-sphere $S^3$. This follows because the isotropy group $H_p$ of any point $p \in S^3$ leaves invariant precisely that point and the antipodal point, which in turn is the identification one uses to build the quotient manifold $\mathbb{P}^3 \equiv S^3/\Gamma$. 

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assumption of global homogeneity-and-isotropy radically restricts the choice of possible topologies. 

As other examples of topological fragility we recall that one cannot have an isolated electric charge in any orientable compact 3-space (by using Gauss’s law). This amounts to saying that the overall electric charge in such spaces is related to the topology, which in turn may explain one feature of the observed universe that would otherwise be an arbitrary initial condition. Further, it has recently been shown that a discrete change in the topology of $\mathcal{M}_3$ from the $S^3$ to the quaternionic manifold $Q^3$ can exclude certain solutions of the Maxwell’s equations. In this way, changes in topology can induce important dynamical (physical) consequences.

4 Bianchi models and topology

In the FLRW setting, the complexity of the topological structures and the set of alternatives contrast strikingly with the simplicity of the local metric properties. The next more general setting usually considered in cosmological modelling is that of the homogeneous anisotropic Bianchi models. Now the introduction of local anisotropy greatly reduces the richness of allowed topologies, since in this case the so called space-form problem is simpler than in the FLRW case, which in turn is due to the fact that even though one still has similar discrete translations, there are fewer reflections and rotations which could be combined with these.

For the case of locally homogeneous Bianchi types, a correspondence between the eight

6 Of course we are assuming that observationally (i.e. within the present observational range and accuracy) local and global homogeneities cannot be distinguished - i.e. up to this level of resolution, observations are stable with respect to such changes. If this turns out not to be the case, then this will become an example of resolution induced topological fragility. This is interesting as it highlights how the nature of fragility might also depend on the scope and accuracy (and hence the epoch) of observations.

7 For the Kantowski-Sachs cases the covering space is $S^2 \times R^1$. For all the other such locally homogeneous Bianchi cases the covering space $\tilde{\mathcal{M}}_3$ has the topology $R^3$, except for the Bianchi type IX where $\tilde{\mathcal{M}}_3 = S^3$. 
Thurston types of homogeneous geometries [31, 32] and the Bianchi types can be set up. This has been recently discussed by Fagundes [2] and Fujiwara et al. [3].

An important outcome of the latter work [3] has been to show that no anisotropic expansion is allowed for the Bianchi V model with a closed (compact without boundary) spatial section. This result, which we shall use in the next section, can be understood from the following argument. Let \((\mathcal{M}_3, g)\) be a locally homogeneous spacelike section of a Bianchi V spacetime. According to Milnor [33], \((\mathcal{M}_3, g)\) is necessarily locally isometric to a maximally symmetric 3-space of negative constant curvature, i.e. it locally admits the hyperbolic geometry. On the other hand, Mostow’s [34] (see also [32]) rigidity theorem ensures that if two closed hyperbolic 3-manifolds are homeomorphic then they are isometric. This amounts to saying that Bianchi V spacetimes are rigid, permitting only isotropic expansion, i.e. they allow only an overall change in the scale factor.

5 Fragility of mixing in closed FLRW \(k = -1\) models

Recently compact FLRW \(k = -1\) models have been considered as examples of relativistic cosmological models which possess rigorous (chaotic) mixing properties [23, 24, 14, 10, 18]. From a general point of view, the geodesic flows on compact negative curvature manifolds have been known to result in \(K\)-flows [35] with the corresponding Kolmogorov entropy given by

\[
K \propto \frac{1}{V^{1/D}},
\]

where \(V\) is the volume of the closed manifold and \(D\) is its dimension [35]. This clearly shows that \(K \to 0\) as \(V \to \infty\), as for example in the case of flows on open \(\mathcal{H}^3\). This, however, is an all time result which as it stands is not very informative for the case of the universe with a finite lifetime. In such cases, one can still derive useful information. In particular, it has been shown that for such models the deviation of neighbouring geodesics is sensitively dependent on the cosmological density parameter \(\Omega_0\) and the redshift \(z\). For
example the maximum distance apart of such geodesics initially making an angle $\alpha$ at the
surface of decoupling $z = z_d$ is given by [14]

$$\delta(z_d) = \frac{\alpha R_0 \sinh(\lambda(z_d))}{1 + z_d}, \quad (5.2)$$

where $\lambda(z)$ is an analytic function of $z$ and $\Omega_0$ given in eq. (10) of ref. [14], and $R_0$ is the
value of the scale parameter at $t = t_0$. For significant mixing of null geodesics to occur
one would require [14]

$$f = \frac{\delta(z_d)}{L_c(t_d)} \alpha >> 1, \quad (5.3)$$

where $L_c(t_d)$ is the topological compactification scale calculated at $t = t_d$. In this way a
measure of the effective mixing can be obtained once an estimate of $L_c$ and $\Omega_0$ is given.

Now despite the enormous success of FLRW models, they are nevertheless approxi-
mate, with the real universe unlikely to be truly isotropic and homogeneous. The question
then arises as to whether such mixing can still occur if, for example, one of these sym-
metry restrictions is removed. Here as a first step, we look at the effects of including
anisotropies.

As natural anisotropic generalisation of the FLRW $k = -1$ isotropic models, we con-
sider the Bianchi type V anisotropic models which also possess negative curvature every-
where on their three-spatial sections. To see this explicitly, we may consider the example
of the locally rotationally symmetric anisotropic Bianchi V model given by the metric [36]

$$ds^2 = dt^2 - a^2(t) e^{2z} (dx^2 + dy^2) - b^2(t) dz^2, \quad (5.4)$$

where $a$ and $b$ are differentiable functions of $t$. Clearly as $a \to b$ the model tends towards
the $k = -1$ FLRW isotropic model and hence it can be treated as its simplest anisotropic
generalisation.

Now as was pointed out in the previous section, FLRW $k = -1$ isotropic models can
possess compact spatial three-surfaces and therefore give rise to mixing, as quantified
by (5.2) and (5.3). On the other hand, recalling that Bianchi V homogeneous compact
universes allow only a change of the overall scale factor \([3]\), it is immediately clear that the presence of non-zero (Bianchi V type) anisotropy destroys the possibility of having closed space sections, which in turn using (5.1) inhibits the possibility of mixing in such models.

As a result global (Bianchi V type) anisotropic expansion and closed space sections are not compatible and therefore the mixing property of the compact \(k = -1\) models is not stable with respect to such anisotropic generalisations. Consequently, smoothing of the microwave background radiation by such a method becomes questionable.

6 Conclusion

We have introduced the concept of topological fragility and pointed out cosmological examples where it is present. Such fragility can be of potential significance, as is demonstrated by the fact that the globally anisotropic Bianchi V generalisations of the compact isotropic and homogeneous FLRW \(k = -1\) models prohibit the closure of their spatial sections, thereby destroying their mixing capability. This may be important for mixing scenarios based on compact \(k = -1\) models, if the anisotropies are global, since small anisotropies are bound to be present in the real universe.

What remains to be done is to find out the extent to which this type of fragility is also present in anisotropic and inhomogeneous generalisations of the FLRW \(k = -1\) models.

Finally, we should stress again that our results do not forbid the universe to be locally anisotropic-and-inhomogeneous and have compact spatial section with the topology corresponding to a compact hyperbolic space. They do, however, give an example of topological fragility that might occur in other settings.
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