Chaos from linear systems: Implications for communicating with chaos, and the nature of determinism and randomness

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Abstract. A method is developed for producing deterministic chaotic motion from the linear superposition of a bi-infinite sequence of randomly polarized basis functions. The resultant waveform is also formally a random process in the usual sense. In the example given, a three-dimensional embedding produces an idealized version of Lorenz motion. The one-dimensional approximate return map is piecewise linear; a tent or shift, depending on the Poincaré section. The results are presented in an informal style so that they are accessible to a wide audience interested in both theory and applications of symbolic dynamics communication.

1. Introduction
Chaotic dynamics is usually modeled by the action of a set of coupled nonlinear differential equations. The motion in state space is therefore deterministic, although the symbolic dynamics of the system [1] can be a discrete-alphabet random process. In the classical theory of random processes [2], a function constructed from the linear superposition of basis functions weighted by a random variable is a continuous-time random process. The relationship between dynamical systems and random processes is most apparent in an area of mathematics known as ergodic theory [3]. In this field, a measure-preserving map creates random symbols by mapping a measure space into itself, and the symbols are labels on elements of a coarse-grained partition. It is clear then, that the classical theory of random processes and chaotic dynamics have much in common, but how far does this connection go? I present an argument that the connection is deeper than previously suspected. A basis function is first developed that has properties that are specified by self-similarity considerations. This basis function or pulse that is developed (figure 1) is time-localized (but not strictly time limited), oscillatory and decaying in the negative time direction, and decaying without oscillation in positive time. When a bi-infinite sequence of these basis functions, spaced by unit time and with polarity specified by a random bit sequence (positive = 1, negative = 0) is linearly superposed, the resulting waveform is deterministic in a three-dimensional embedding space. A time-delay embedding [4] produces a structure in state space that is an idealized version of the Lorenz [5] invariant set. (I will use the term “attractor” loosely; there is really no attractor in the usual sense because the dynamics is not defined off the invariant set unless further formalism is developed.) In this viewpoint, it is a linear superposition of randomly polarized continuous-time pulses that creates the deterministic flow.

In addition to theoretical interest, this construction allows for analysis of the system and its signal using methods of linear analysis that are not possible with other chaotic systems. The power spectral density of a signal with purely random bits, for example is the same as the square of the Fourier transform of the basis function [2]. Linear filters for the signal can also be constructed: In additive white Gaussian noise, for example, the filter that maximizes the probability of correct bit detection has
the usual matched-filter impulse response from linear signal theory [6]. (One can do better by considering adjacent bits, but this filter maximizes the probability of detecting a single bit correctly.) Furthermore, the construction bridges the gap between traditional communication theory [6] and symbolic dynamics communication [7] in other ways. The determinism inherent in the signal can be viewed as a result of a special type of intersymbol interference [6]: The oscillatory tails of the function overlap to build a shift map dynamics, and the decaying heads produce the Cantor-set structure along the compressing direction. With this view, the symbolic dynamics signal produced by linear synthesis is a special type of traditional communication waveform.

Perhaps the most interesting physical implication is that linear synthesis of randomly weighted functions can produce deterministic chaotic motion. This interpretation may have implications for many areas of physics in which linearity is assumed as a basic physical attribute. In this paper I make a plausible argument that such linear synthesis of chaos is indeed possible. The Lorenz system is used as a template for the derivation, and in this process the unique characteristics that appear in the function shown in figure 1 become clear.

2. Extraction of an approximate basis function for the Lorenz system
In this section, a basis function that has the properties required to produce chaotic motion in the manner desired is investigated heuristically using the Lorenz [5] system. The result is that there exists an approximate basis function for the Lorenz system, even with the standard parameter values, that can be used to closely mimic Lorenz behavior.

2.1. Producing a regular timing interval

Figure 1. Idealized pulse shape has the basic characteristics necessary to produce the attributes of chaotic motion. Pulse is constructed as described in steps 1–4 in the figure. Key characteristics are the exponentially decaying pulse “head,” and an exponentially rising oscillatory “tail.” For this figure, the parameters are \( A = 0.16, R_s = 1, \alpha = 1.7, t_o = -0.6, t_1 = 0.25, \) and \( \lambda = 13. \) These values can be varied to change the pulse characteristics, and other approaches to the pulse definition are possible.

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The Lorenz system with the standard parameter values can be interpreted as a binary waveform source. One problem with the Lorenz signal for communication applications is that the zero crossings are irregular, as are the return times to a Poincaré surface. This timing irregularity introduces a difficulty in detection of the signal: There is no regular timing grid for setting a sampling clock [6].

Imagine that for each trajectory of the Lorenz system, the $x(t)$ signal is sampled from peak to peak (positive or negative) at a regular time interval, small enough that there is little error. Some trajectories will contain more sample points than others. Now, the sampled $x(t)$ signal is regenerated, but the output of the source (a D/A converter, for example) is speeded up by an amount proportional to the number of points in the sampled signal between return times. The result is that each peak will occur in equal time. (In practice, a technique is used whereby segments of the trajectory between Poincaré crossings are stored and indexed according to their symbolic future, so that a regular output sample clock can be used, but this is not important for the argument here.) The $x(t)$ signal for the Lorenz system with standard parameter values is shown in figure 2 with this timing regularization procedure applied.

![Figure 2. A Lorenz signal with regularized timing.](image)

With a regular timing interval, the first simplification of Lorenz behavior has been obtained. There is no longer any dependence of cycle times on the symbol sequence. Furthermore, as a communication signal, bit detection is much easier: A regular sample clock entrained (via a phase locked loop or other methods [6]) to the regular time spacing will be more immune to noise in the signal than a peak detector alone.

2.2. Subtracting Lorenz signals to obtain an approximate basis

The question that arises for the Lorenz signal, is how much can we think of the signals as being linear compositions of pulse shapes? One way to answer this question is to assume that the Lorenz signal is such a composition, and then find out what the pulse shape is. This only works for the timing-regularized signal, because the composition must consist of fixed pulse shapes at a fixed time interval. The construction of a non-time-equalized signal may be possible, but this would require the pulse length and timing interval to depend on the present symbolic state, which depends on the future bit sequence for the Lorenz system. This makes the current pulse shape and temporal positioning depend on the pulses coming up, which defeats the concept of pulse synthesis in its simplest form. Shown in Fig. 3 are two Lorenz $x(t)$ signals differing only in the center bit.
Figure 3. Two Lorenz signals only differing in the center bit are subtracted to extract the approximate pulse basis shown in red.

When the blue signal is subtracted from the black, and the result divided by two, the approximate pulse shape shown in red is obtained. If the Lorenz signal really were actually a linear superposition of pulses, the pulse shape thus extracted would not vary at all for any possible bit sequence surrounding the center bits that are always inverted relative to each other. In other words, with this extraction technique, the extracted pulse basis would be exactly the same for all bit sequences that match in all positions except for opposite bits at time zero. Each time a new signal is formed, and the pulse basis is extracted, the basis changes. This means that there really is no exact linear pulse basis for Lorenz, a result that is not unexpected. The Lorenz system is a nonlinear dynamical system and it would be surprising if its dynamics could be constructed by a linear superposition of randomly signed pulses. Shown in figure 4 is a density image for a series of pulse basis extractions as shown in figure 3. Compare this image with the single extracted basis (the red signal) in figure 3. The brighter regions are where the extracted basis is more likely to trace. This image can be considered to be similar to what would be obtained on an oscilloscope if the extraction were performed in real time. The image shows that there are roughly two very likely paths that a basis can take in the past (before the large time zero pulse) and yet there are also possible pseudo-basis functions that differ markedly from the most probable ones. This splitting of paths is understandable in terms of the symbolic dynamics. There are two high-density paths in the subpulse preceding the main pulse lobe, and four in the one preceding that, and so on. This splitting corresponds to whether the trajectory switched lobes or not in the previous cycle, which is the main cause of differences in the trajectory. The future dependence of the pulse shape is not nearly as obvious, because two different trajectories on the same lobe are close together in state space independent of whether a lobe switch is about to occur.
Figure 4. Density image for multiple pulse basis extractions using the method illustrated in figure 3. The brighter regions correspond to the higher probability extracted basis functions.

Now in figure 4, it is obvious that there is a large deviation from the assumption of a linear basis function for the Lorenz system, but by averaging a large number of the extracted functions, one can arrive at something that approximates a basis function. A motivation for the usefulness of this approach is that if symbolic trajectories are used that are more likely, the result shown in Fig. 4 indicate that the approximation may not be too bad. By averaging over 1000 extractions with randomly chosen symbolic trajectories, the basis function in figure 5 is obtained.

Figure 5. The pulse basis extracted from the Lorenz system by creating two signals differing in only one bit position. This pulse was produced by averaging over 1000 pulses produced this way.
2.3. Synthesis of an approximate Lorenz signal using the extracted basis

The synthesis of an approximate Lorenz signal is now easy. Linear superposition of the approximate basis function in Figure 5 is used with weights given by a time-regularized Lorenz signal so that the linear synthesis can be compared to the time-regularized signal. It is important to remember that time-regularization simply compresses each cycle of the true Lorenz signal into a fixed time window, so that with time rescaling equal to the inverse of the regularization procedure, a true Lorenz signal would result. For a given pulse basis $u(t)$, the amplitude of a signal consisting of a linear superposition of randomly polarized pulses will simply have the amplitude $a(t) = \sum_{k=1}^{\infty} f_k u(t-k) + \sum_{k=1}^{\infty} p_k u(t+k)$ where the $f_k$ are the future bits and the $p_k$ are the past.

2.3.1. The signal

A comparison of a time-regularized Lorenz signal and the approximate linear synthesis are shown in Figure 6.

![Figure 6. Lorenz-type signals generated by (black) time regularization, essentially the real Lorenz signal, (blue) using the average extracted pulse basis, and (red) using the analytically described basis.](image)

For comparison and future reference, a signal generated using the analytically-defined basis of Figure 1 is also shown. The approximate basis function generates a signal that is nearly identical to the time-regularized signal. The analytically-defined function differs more than the approximate basis, but as will be shown in sections 3 and 6, it is a true linear basis, whereas the approximate Lorenz basis is not. Two points about Figure 6 are in order: (1) The symbol sequence that was used has no more than two 0’s or 1’s in a row, so that the outlying signal trajectories do not cause much deviation from the approximate pulse. If longer runs of the same symbol are allowed, the deviation is greater. (2) The use of the approximate Lorenz basis does yield a better match than the analytically-described basis, so if accuracy of the signal is of greater interest than a true deterministic dynamics, the experimental extraction procedure is superior. I have not attempted to construct a basis function that scales in a way that is similar to Figure 1, but has the overall shape closer to Figure 5, but it is likely that by making a tradeoff between fitting accuracy and true deterministic structure, a better fit could be made to the Lorenz $x(t)$ signal than with the pulse of Figure 1.
2.3.2. The dynamics
A delay embedding of the signal in figure 6 generated by linear synthesis using the approximate basis is shown in figure 7.

**Figure 7.** The “attractor” structure consists of layered bands of trajectories. If the shift property is to be obeyed dynamically across the whole attractor, certain scaling properties must apply to the pulse dynamics, otherwise there can be overlap of trajectories.

A peak magnitude return map for the signal of figure 6 is shown in figure 8. The map is similar to a tent map, which is expected if the peak amplitude (including sign) return map were a shift map. The deviation from a tent map is obvious in the figure, and is expected since there is no exact linear basis pulse for the Lorenz time-regularized system. If the symbolic dynamics is controlled to avoid the peak in the map, however, the deviation from a deterministic map will not be as severe. This could be achieved by using a runlength constraint, for example, allowing no more than two 0’s or 1’s in a row in the symbol sequence.
3. Synthesis of true chaotic dynamics

3.1. A Comment on discrete time

The assertion that a signal generated by the random superposition of unit pulses can be precisely the same as a signal generated by a deterministic dynamical system is most easily demonstrated in a discrete-time continuous-amplitude system. In a discrete-time system, time is an integer index. A continuous-amplitude system is usually thought of as a system where the output ranges on some continuous subset of $\mathbb{R}^N$, but in chaotic dynamics the subset is often a Cantor set. The output of a discrete time system is easier to deal with than a flow, since the output is an integer indexed sequence of points in the real number system.

The demonstration of the linear synthesis procedure in discrete time is surprisingly simple. The key observation is that the shift map in its symbolic form can be decomposed into the sum of an infinite number of shift systems operating in parallel. (A more formal approach will be used later to allow for the derivation of waveform synthesis.) In symbolic form, the shift is simply viewed as the left shift of the binary fraction representing the current state: $x_{n+1} = S(b_3b_2b_1...,)$, where the bits represent the current state in binary fraction. If the output of several shift systems is added in real time, then in general the output will not be constrained to the unit interval and thus the signal cannot be a shift signal. Thus, linear superposition in general does not hold. But consider the following: If one considers shift systems that start on a point whose binary fraction contains only a single nonzero bit, then linear superposition of these shifts causes no interference between bits:

![Figure 8. Return map from linear superposition of pulses when the shift condition is violated. Note in particular the deviation near the peak where the 1-1 onto nature of the Lorenz system is violated.](image-url)
The pulse amplitude thus builds up exponentially with time, and then cuts off abruptly after $t = 0$. Any signal that can be produced by a shift map can also be produced by an infinite sum of unit pulses weighted by the appropriate bits.

3.2. On to Continuous Time

The picture developed in section 3.1 can be carried over to continuous time with the insight gained from discrete time and some “intuitive” modifications. There are certain restrictions on the pulse shape that describes a dynamical system imposed by the condition that the resulting signal should have the basic properties of deterministic chaos. These conditions are determinism, exponential expansion, folding, and oscillatory behavior in a bounded region. As shown in section 2.2, by producing two signals whose symbolic dynamics differ only in one bit position, one can extract an approximate pulse basis for the Lorenz system. The two signals are subtracted, and if they were actually the linear superposition of pulses, only twice the basis pulse would remain. Since the Lorenz system is not exactly describable in this way, however, there is some deviation for different symbol sequences, so the average many trials is used to get the basis. It is also possible to develop an analytical description of a pulse that has the desired properties based on the constraint conditions. This pulse basis is exactly related to its signal, unlike the pseudo-Lorenz pulse, so that there is an exact correspondence between pulse characteristics and dynamics.

The pulse basis has several properties that are crucial to maintaining deterministic dynamical behavior that is Lorenz-like. The key required properties are:
1. The pulse head must decay without oscillating, and approach an exponential decay with increasing time. This is necessary to obtain the exponential compression along the stable direction of the resulting attractor.

2. The pulse tail must rise exponentially in base 2 with increasing time, and oscillate in order to produce the oscillatory nature of the Lorenz-type signal.

3. There is an offset pulse or pedestal function during the time 0 interval that is responsible for the two-lobed nature of the attractor.

4. The negative-time lobes of the pulse must scale, that is, the pulse amplitude \( p(t-\delta t) = 2^n p(t+n\delta t) \) where \( \delta t \) is the symbol interval. This scaling property is obvious for integral times and is discussed further below.

The requirement for the discrete-time structure of the pulse to decay as \( 2^n \) in the past (negative \( n \)) is essential so that the amplitude dynamics of the system at integral times be a shift map. This is a result of the discrete-time shift map being describable by \( x(n+1) = 2^n n(n) \mod 1 \) in the limit of infinite forward-time Lyapunov exponent, and is a basic topological property of the pulse and the resulting attractor. The pulse must also scale at all other times, so that the shift property is maintained all over the attractor. Otherwise there could be aliasing of trajectories. This aliasing is caused by overlap of the generating sequence for the real number system. If for example, the second past pulse lobe did not scale, the center point in the generating sequence would be offset. If then, the rest of the subpulses scaled properly, there would be an overlap of the resulting continuum intervals for the remaining pulse sequences. Thus there is a simple requirement if the system is to obey the shift condition as it should for all times: The underlying continuous time structure is repeated at each symbol interval after the zero-time interval. Thus the sub-pulses are replicas of the first one if the offset lobe or pedestal function at \( t = 0 \) is subtracted out. The result of not following this scaling requirement was shown in the map of figure 8 for the attractor in figure 7 produced by the pseudo-Lorenz pulse.

The picture of what happens when a pulse (see figure 1) is used that obeys the basic conditions for true deterministic chaos is shown in figure 9. Not only does the attractor have Lorenz-like structure, but the one-sided peak absolute value return map is approximately a piecewise linear tent. Because of the exponentially decaying heads of past pulses, however, there is a Cantor set structure along the compressing direction. This structure is necessary for the system to be equivalent to a bi-infinite shift, and not just a one-sided shift. In this case, there is strict determinism both forward and backward in time.

More formal statements will be left for the addendum (section 6), but it is worth mentioning now that there are a few properties of this embedding that are of physical importance. First, the entropy rate is exactly one bit per unit time, unlike the Lorenz system. (The uncontrolled Lorenz system with standard parameters has conditional probabilities on transitions, so that the entropy is less than one bit per Poincaré cycle, and the cycle time is irregular.) Therefore the per-cycle expanding positive Lyapunov

![Figure 9](image_url). Embedded signal. Insets show the one-sided Poincaré map (upper left), and the fractal structure of the invariant set (lower right).
exponent is exactly 1 in base 2. The compressing Lyapunov exponent is given by the exponential in the pulse basis function. Thus, if the limit function for decay is $2^a$, the compressing exponent is $a$. Since the pulse heads split the future states according to $2^a$ with unit return time, the dimension along the compressing direction is $1/a$. All these quantities can be expressed exactly because of the simple method by which the signal is produced. (See section 6, addendum.)

4. Engineering
The results presented so far are enough to proceed to some engineering-oriented discussion. It has been said that the Lorenz signal is an example of a binary symmetric antipodal communication signal [7], and that this is not merely an interpretation, but that it is fundamental to the nature of the signal. In this section, this claim will be made clearer. An obvious observation is that the Lorenz signal, when controlled to carry a sequence of bits representing information, has a clear set of spikes in the positive and negative direction, and these spikes are naturally the bit sequence when threshold detection is performed.

4.1. Filter driven by impulse noise
With the formalism of linear synthesis, further analogies can be made. Because the signal is the linear superposition of pulses with regular spacing, it can be produced by a linear filter excited by impulse noise, or by a sequence of impulses that represent binary information. The impulse response of the filter is simply the basis pulse shape. It is true that it is formally an infinite impulse response (IIR) a causal (the input starts before the output) filter, but a common practice in engineering is to approximate such a filter with a finite impulse response (FIR) filter with its output shifted in time. This is shown in figure 10.

![Figure 10. Randomly polarized impulse stream $x$ excites filter with impulse response $h$. Output $y = x^*h$ tracks the symbolic dynamics of $x$, but delayed in time. (This is an FIR approximation, so the output bit sequence is not correlated to the input in the time window here.)](image)

In this picture, the impulse stream does not correspond to the output signal, because the input is started at zero time and the filter takes time to “ring up” to the correct time response.

4.2. Comparison to traditional signals
In the traditional approach to communication theory, signals containing binary data are composed by the linear superposition of pulses separated by a fixed timing interval. One of the simplest of these waveforms is the binary polar return-to-zero square pulse train shown in figure 11. There is an obvious correspondence between bits and pulses in this signal. Compare this to the signal in figure 2, where the amplitudes of the peaks vary depending on the bit sequence, not just the current bit. In the case of the square pulse train, there is no intersymbol interference [6].

4.2.1. Pulse waveform shapes
This pulse train is called a binary polar antipodal return-to-zero waveform, because a binary 0 is represented by a negative pulse amplitude, a binary 1 by a positive amplitude, and the waveform returns to zero amplitude between pulses. The waveform is produced by the linear combination of square pulses centered at regular timing points weighted by $\pm 1$ according to the current bit. These timing points where the pulses are centered are called the sample times, the timing instants, or the timing grid, among other things.

The requirement for signals to be limited in bandwidth leads to more complex pulse shapes such as sinc pulses and the family of Nyquist pulse shapes. Signals composed of the linear superposition of these pulse shapes are bandlimited with the same power spectral density as the squared magnitude of the pulse’s Fourier transform. They are usually constructed so that there is no intersymbol interference, that is, the signal amplitude at the sampling times is equal to the amplitude of the individual pulse. There may, however, be no obvious visual correspondence between the peaks in the signal and the sampling times at which the amplitude is unity. The signal shown in figure 12 is constructed by the linear superposition of sinc pulses, defined by $\text{sinc}(t) = \sin(\pi t) / \pi t$. 

**Figure 11.** Square pulse train containing binary data.
Figure 12. Bipolar pulse waveform composed from sinc functions. The sinc pulse basis is shown in blue, and the sample times are shown as red unit samples.

The sinc pulse signal has some unusual properties, some good and some bad. First, the bandwidth is the minimum necessary for Nyquist-limited sampling [6], that is the frequency bandwidth is half the bit rate, corresponding to a sample rate of twice the highest frequency component, the Nyquist sampling criterion. The sinc basis pulse shape at a given sample time places zeros at all the other sample times, and thus there is no intersymbol interference and the amplitude is ±1 at each sample time. The sinc basis pulse is very broad in time, and many cycles of its sidelobes are required to accurately create the superposition. The most unusual and detrimental property is that the signal amplitude can swing much wider than the unit sample distance, and although the amplitude at the sample instants is still ±1, there is no obvious relationship between pulse peaks and sample times. What looks like a pulse peak and thus a 1 may actually be two 1’s in succession.

The problems with the sinc pulse signal can be alleviated by use of the Nyquist shaped pulse basis. This basis is a parametrized class defined as $p(t) = \frac{\cos(2\pi ft)}{1-(4f t)^2} \text{sinc}(rt)$ where $r$ is the sample rate and $t$ is a parameter controlling the pulse shape. For $t = 0.5$ the waveform shape of figure 13 results. Although the correct sample times still do not lie exactly at the signal peaks, there is now a much clearer correspondence between peaks and the encoded bits.
The Nyquist pulse shown in the center in blue has a narrower main pulse lobe, and the sidelobes drop much more quickly. The signal also has less excursion past the unit sample amplitude, and there is some correspondence between peaks and samples. The price is that the absolute bandwidth is doubled over the sinc pulse signal. For $\alpha = 1.5$ in figure 14, the signal becomes even more defined, and looks like a shaped version of the simple return to zero waveform in figure 11.

There is now a definite correspondence between the samples and the pulse peaks, looking much more intuitively like a sequence of pulses representing bits than either figure 12 or even figure 13. This pulse waveform more closely resembles the simple square pulse waveform in figure 11 except that the pulse peaks are smoother.
One aspect of the linear synthesis approach that allows for comparison to traditional waveforms is the variable height of the offset pedestal in the interval about $t = 0$. Compare the signal in figure 15 to the Nyquist shaped pulse signal of figure 14. This height, given by $R_0$, (see figure 1) can be varied to change the band thickness on the attractor. In figure 1, a height of $R_0 = 1$ was chosen to produce a signal that is similar to the traditional Lorenz signal. (The entire construction method in this case was chosen to produce an idealized signal that still had a similar appearance to Lorenz.) Larger values of $R_0$ will produce less intersymbol interference between pulses, and thus less variation in the amplitudes of signal peaks. This concept is shown in figure 15 for different choices of $R_0$. With $R_0 = 1000$, the amplitude of the central pedestal is so much larger than the negative time oscillations that they are practically nonexistent in relation to the height of the pulse. In this case, the signal looks very much like the Nyquist shaped pulse signal in figure 14. Both are essentially return-to-zero waveforms, and the peak amplitudes are independent of surrounding peaks.

This change in $R_0$ thus adjusts the height of the central pedestal offset function $\Pi(t)$ as described in the pulse definition in figure 1. The signals in figure 15 were then produced by linear superposition of a stream of these pulses, and presented on the same scale. The exact amplitudes are not important, the basic shape and band thickness is the relevant issue. One can think of the signals for large values of $R_0$ to be signals where the negative time oscillations are diminished, and ultimately become irrelevant as $R_0$ is increased, so that the basis pulses consist of only a central pedestal and exponential forward

**Figure 15.** A sequence of Lorenz-type signals in delay embedding shows how the limit as the band thickness on the approaches zero produces the traditional binary bipolar signal without memory at right. This represents the projection of the $R_0 = 1000$ signal on one axis. There is almost no intersymbol dependence.
decay. As the height of the central pedestal is increased, the exponential decay in forward time also becomes of less importance, as it must be fitted at lower values of the pulse amplitude relative to the peak.

Now there is the further question of similarity between classical pulse shapes and the Lorenz pulses, both analytically defined and extracted. One obvious difference is that the “chaos pulses” only oscillate in negative time, and decay exponentially in forward time. The common classical communication pulses discussed here oscillate in both directions. There are examples of traditional communication pulses that have one-sided time oscillations, but they are not considered here for brevity. Figure 16 compares the pulse shapes discussed so far for common parameter values.

![Figure 16. Comparison of pulse basis functions. Sinc (black), Nyquist =0.5 (red) and =1 (green), average extracted Lorenz (blue) and analytical Lorenz (purple).](image)

Note how the extracted basis and the Lorenz averaged basis have humps where the sinc and Nyquist have zeros. This is because the Lorenz introduces what may be considered to be deliberate intersymbol interference. This is why the pulse amplitudes vary, it is also why Lorenz chaos is predictable. The Lorenz pulses are also time-limited in the future, not strictly, but practically since they decay exponentially at a high rate. The tails of pulses in the future at a given bit position overlap, and this is why the future information is contained in the present state. Finally figure 17 shows the spectra of all these pulse shapes.
Figure 17. Pulse spectra. Sinc (black), Nyquist =0.5 (red), Nyquist =1 (green), Lorenz average extracted (blue), and analytical Lorenz (purple). The slightly noisy red curve almost exactly matching the blue curve is a spectrum obtained from a periodogram spectral estimator for a real Lorenz signal, thus demonstrating that the Fourier transform of the Lorenz extracted pulse is practically identical to the square root of the Power Spectral Density for a real Lorenz signal. The analytical Lorenz pulse also has remarkably similar spectral features.

5. Discussion and conclusion
The material presented here forms the basic signal theory needed to understand chaos signals using symbolic dynamics encodings in terms of well-developed principles of communication theory. These results represent the first step in the development of a comprehensive signal theory for chaos-generated waveforms. The most important and surprising result is that a signal representing a chaos signal can be constructed accurately from the superposition of a linear pulse basis. The implications of this development will be explored more fully in future publications. Furthermore, this approach is pivotal in our development of systematic methods for designing chaotic oscillators that produce desirable waveforms for communication. Perhaps the most important aspect of such a theory, however, is the suggestion that electronic devices may be designed for conventional pulse trains, opening up application areas in conventional communication systems.

The possibility of constructing strictly deterministic communication signals that carry an arbitrary symbol sequence perhaps seems strange. Mathematically, however, it is indeed possible, and this idealization is no more unrealistic than the construction of pulse waveforms that can never be realized exactly in classical signal theory. In the Lorenz system, for example, the choice of initial condition determines the sequence of 0’s and 1’s that will evolve in the dynamics. By choosing the initial condition with absolute precision, one can cause the trajectory to follow a desired symbol sequence
forever. This is clearly impossible in practice, but mathematically it is possible. In this manner, the starting point maps to a symbol sequence, or if the symbol sequence is reduced to its binary fraction, the initial point has its image point in symbol space. Since the mapping from symbol space to the trajectory that corresponds to the point in symbol space is done all at once, this type of encoder is a symbol sequence encoder, not an individual symbol to pulse waveform encoder. In theory, the whole waveform is determined by the whole infinite symbol sequence. Of course noise, control signal jitter, and finite measurement accuracy will affect the period of predictability in a physical implementation, but it is possible mathematically to construct a signal carrying an arbitrary sequence of bits, and having the property of strict determinism. Also, it will of course be impossible to know the whole future symbol sequence, and it is practical to terminate the target sequence at some finite value, say 8 or 10 bits. The truncation of the symbol sequence will also limit predictability.

It may seem even stranger that it is possible to construct strictly deterministic communication signals from a randomly polarized linear superposition of basis pulses. By carefully constructing the pulses, however, this is indeed possible. One can consider the infinite-time limit of the output of the random waveform source as defining the deterministic dynamics. The oscillatory tails of the pulses in map to the continuum of real numbers in a linear fashion, and this is the origin of the linear shift dynamics. The damped heads map to a Cantor set of low dimensionality, this being determined by the rate of damping. All these concepts are being formalized for future publication.

There has been some controversy concerning whether symbolic dynamics signals can perform better than classical signals, and the following discussion addresses this point. First, for technological applications, we are focusing on the electronics, not the performance of the signals. It can also be shown that these signals can be generated by classical methods. Ornstein [3] has described how a chaotic process can be duplicated to any desired degree of accuracy by a sequence of purely random choices intermediated by purely deterministic moves in state space. An example of this is the well-known shift map, \( x_{n+1} = 2x_n \mod 1 \), which can be approximated by the shift register in binary, \( 0.b_1 b_2 b_3 \ldots \), where the \( 0.b_k \) are bits being shifted in from the left. This signal looks like a shift map signal except for a small amount of jitter caused by truncation, which becomes smaller if the register length is increased further. Now if the bits being fed into the register are some binary sequence representing encoded information, the shift map signal will produce the bit sequence corresponding to the sequence being put in. The least significant bit eventually appears in the most significant position, and this determines whether the value of \( x \) is greater than (symbolic 1) or less than (symbolic 0) the value \( \frac{1}{2} \).

The Lorenz system can be mimicked similarly, by fitting finite pieces of trajectory together to span the distance from Poincaré surface to Poincaré surface. Each point on the Poincaré surface corresponds to a binary symbol sequence, so the next piece to add to the signal is chosen to closely match the desired sequence. As with the shift map, a shift register is used to store these target sequences, and it is left-shifted on each Poincaré crossing. This mapping is thus a sliding-block pulse encoder; the sequence of bits in the finite register is used to index a trajectory segment corresponding to the pattern in the register. The signal is therefore a rather unusual case of a signal that falls into the category of sliding-block coded pulse waveforms, where each pulse is a finite-time section of Lorenz trajectory, and the pulse shape is determined by the pattern in a sliding block register. (One can think of this sliding block coder as a machine that accepts binary sequences and produces symbols of a higher cardinality alphabet, each of which corresponds to a waveform section.) There are just a lot more pulse shapes than the basic alphabet (binary), each corresponding to long sequences of bits. With this viewpoint, it is clear that such a chaos signal is really a special type of waveform that can be described by classical signal theory, and thus is not really new at all. It is the means by which it can be produced that makes it unusual.

The existence of deterministic digital communication signals with known time-evolution equations suggests an entirely new approach to the design of signal processing electronics. This includes signal generation, transmission, amplification, transformation, and filtering, and probably extends to all types of signal “handling.” The most direct mathematical basis for the construction of these signals is to
view them as being produced by the action of a deterministic chaotic system, instead of an explicit random process. They can also be generated, however, by piecing together stored pieces of trajectory in a sequence defined by the output of a sliding block coder. Although in theory these two views are the same, the signal generated by a chaotic system is a random process, and a random process can be the product of a dynamical system acting on a measure space, it is by considering the deterministic systems description of the signals that we are led to consider new approaches for processing them. It is therefore not surprising that by considering this viewpoint, we have also been led towards electronic architectures that are radically different from the ones used for traditional digital signaling.

6. Addendum

Previous sections have discussed in an informal way the construction of a Lorenz-like dynamics from the linear superposition of pulses of a specific type. This pulse structure was shown in figure 1. This pulse has a central pedestal function, an oscillatory decaying tail function such that the lobes of this function are self – similar, and an exponentially decaying non – oscillatory head function. Linear superposition of these pulses produces a signal that looks very much like the Lorenz \( x(t) \) signal. Embedding this signal produces a structure that is an idealized version of the Lorenz attractor. That is, there are two distinct attractor lobes, the underlying map is the binary Bernoulli shift map. Furthermore the attractor has a Cantor set structure along the contracting direction of low dimension corresponding to a highly – compressive flow.

6.1. Derivation of deterministic solution

What must be shown for the result to be useful is somewhat subtle. No matter what the embedding, if there exists a three – dimensional point set upon which a vector field is defined, and this vector field, when integrated produces (in any projection) the signal that was produced by linear superposition, then the signal can be generated by a deterministic dynamical system. This is a simple truism, but not so easily shown in practice. What is sufficient, however, is to show that there is an embedding of the signal that produces an embedded structure, and that each point on this structure produces a unique signal trajectory. Any function of the embedding coordinates will suffice to produce the signal, but a single coordinate linear projection is preferable.

Since the signal is produced directly by the linear superposition of the basis pulses according to a given random bit sequence, it is clear that if each point on the embedded “attractor” allows one to solve for the symbol sequence uniquely then determinism is assured. This does not mean that, in the abstract, we must think of the attractor as being a symbol space representation, since this is simply a tool used in the argument. Since each point corresponds to a unique symbol sequence, then the structure is deterministic, but the preferable view is that each point maps to a given signal, and this corresponds to the embedding time. The mechanism of symbolic dynamics is used to construct the argument for determinism, but the continuous nature of the embedding assures the existence of a continuous flow.

The problem then, is to show that the signal thus produced by the superposition of an infinite number of randomly polarized pulses is deterministic. A completely random (a Bernoulli process or coin toss) polarization is not necessary, but it is this case that yields a continuous structure across the unstable direction. (A simple alternating sequence would also produce a deterministic dynamic – a closed period 1 orbit of the return map.) Determinism in this case means that the signal itself has an underlying dynamic that is deterministic. If the dynamic is Lorenz, then a three-dimensional embedding of the signal should be sufficient to describe this dynamic.

One possibly confusing aspect of the requirement for determinism is the relationship between symbol space and state space. Since there is a 1 – 1 correspondence between symbol sequences and signal states in a deterministic system, is it sufficient to prove that the system is deterministic in its symbolic state space? It is obvious that if one knows the symbolic state of the system, that is, the whole bi – infinite symbol sequence, then one can specify the signal for all time, and in this sense, full determinism is an immediate consequence. This however, is not a sufficient requirement for determinism, since the symbolic state in this case already contains information about the infinite
future. In the case of the shift map, the one – sided symbol sequence also contains the infinite future, but since this state is exactly the same as the statepoint of the map, knowledge of the statepoint will yield the symbolic state. In the continuous – time case, this is not so obvious, and is, in fact, what needs to be shown. If it can be shown that knowledge of the coordinates of a point in a three – dimensional space will allow prediction, then determinism has been proved. This is because it is the existence of a three – dimensional flow field that is sought, a dynamical system described by continuous differential equations. The subtle point here is that without this structure, the 1 – 1 relationship between symbol space and infinite signal trajectories is not a given, so it can’t be assumed that knowing the symbol sequence will allow prediction.

The most obvious choice for an embedding is the usual time–delay embedding. In this case one would use the triplet \((x(t_1), x(t_2), x(t_3))\) embedded in a three – dimensional Cartesian space to represent the statepoint of the system. The problem with this approach is as follows: To obtain a closed – form derivation of determinism, one needs to be able to essentially solve a set of linear equations. There is no choice of pure time – delay embedding that will produce a factorable set of equations, though, there is always interference from the decaying heads of the pulses that makes it impossible to form the solution.

The embedding that is used instead is the triple \((\phi, x_0, x_1)\), where \(\phi\) is the phase angle of the signal and \(x_0\) and \(x_1\) are delay coordinates. In particular, choosing the delay between points to be one full Poincaré cycle yields a particularly simple formulation of the problem. In this case, the two delay points are never in the same timing interval – thus the formulation does not require multiple cases to be considered. Furthermore, with this formulation, the exponential scaling terms can be easily factored. To see this, consider the two delay points in the embedding in terms of the phase angle \(\phi\)

\[
\begin{align*}
x_0 &= b_0 f_0(\phi) + \sum_{k=1}^{\infty} b_k u(-k + \phi) + \sum_{k=0}^{\infty} b_k u(k + \phi) \quad \text{and} \\
x_1 &= b_1 f_1(\phi) + \sum_{k=0}^{\infty} b_k u(-k + \phi) + \sum_{k=0}^{\infty} b_k u(1 + k + \phi).
\end{align*}
\]

Here \(f_0\) is the central pedestal function that exists only in a single phase interval, \(u\) is the unit pulse function, and the \(b_k\) are the bits in the bi – infinite symbol sequence. These equations can be further reduced to the following form by pulling out the past and future bits as

\[
\begin{align*}
x_0 &= b_0 U(\phi) + \sum_{k=0}^{\infty} f_k 2^{-k} T(\phi) + \sum_{k=0}^{\infty} p_k e^{-i(k-1/2)} e^{-i\phi} \\
x_1 &= b_1 U(\phi) + \sum_{k=2}^{\infty} f_k 2^{-(k-1)} T(\phi) + \sum_{k=0}^{\infty} p_k e^{-i(k+1/2)} e^{-i\phi}
\end{align*}
\]

where the past bits are \(p_k = b_{-k}\) for and \(f_k = b_k\) for all \(k \geq 0\).

Factoring out the exponential terms and denoting the functional dependence on the phase angle \(\phi\) as a subscript yields

\[
\begin{align*}
x_0 &= b_0 U_\phi + \frac{1}{2} b_1 T_\phi + T_\phi \sum_{k=2}^{\infty} f_k 2^{-k} + e^{-i(\phi-1/2)} \sum_{k=1}^{\infty} p_k e^{-i\phi} \\
x_1 &= b_1 U_\phi + \frac{1}{2} e^{i/2} b_0 e^{-i\phi} + 2 T_\phi \sum_{k=2}^{\infty} f_k 2^{-k} + e^{-i(\phi+1/2)} \sum_{k=0}^{\infty} p_k e^{-i\phi}.
\end{align*}
\]

Now there are two equations in terms of the phase angle \(\phi\), the two summations over past and future bit sequences, and terms including the present bit pair, \((b_0, b_1)\). The crucial realization at this point is that if the pair \((b_0, b_1)\) can be found, then with a known phase angle \(\phi\), the only unknowns in the equations are the summations over past and future bit sequences. Furthermore, the summations
represent numbers, one in the real continuum and one on a Cantor set. Since these are both real numbers, the problem is reduced to two equations in two unknowns, allowing for a linear solution.

Let the number \( F = \sum_{k=1}^{\infty} f_k 2^{-k} \), and \( P = \sum_{k=1}^{\infty} p_k e^{-\lambda k} \). The fact that \( F \) exists in the real continuum makes it a little more easily understood as a simple number. \( P \), however, exists on a Cantor set of lesser dimension, but is just as "real" a number as \( F \). In the solution of the linear equations, \( F \) can only take on certain values, and these values are always in the constrained Cantor set. Now we have the set of linear equations,

\[
\begin{align*}
  x_0 &= A_0 F + B_0 P + C_0 \\
  x_1 &= A_1 F + B_1 P + C_0
\end{align*}
\]

which can be expressed in matrix form as

\[
\begin{pmatrix}
  x_0 \\
  x_1
\end{pmatrix} = \begin{pmatrix}
  A_0 & B_0 \\
  A_1 & B_1
\end{pmatrix} \begin{pmatrix}
  F \\
  P
\end{pmatrix} + \begin{pmatrix}
  C_0 \\
  C_1
\end{pmatrix}.
\]

This matrix equation has a solution if the determinant

\[
\det(M) = 2 T_p e^{\lambda/2} - T_p e^{-\lambda/2} = T_p \left( 2 e^{\lambda/2} - e^{-\lambda/2} \right) \neq 0.
\]

The determinant here is never zero unless \( T_p = 0 \), which occurs when both sample points fall in the zeros of the pulse basis function. In this case, it is not possible to solve for the past and future symbol sequences, because the equation is degenerate. The degeneracy, however, is in the choice of coordinates, not in the signal itself, since the choice of embedding points is separated by exactly one bit time, so that it is possible for both to fall in the pulse zeros at the same time. A different choice of timing will remove the degeneracy at these instants, but will not produce an exactly solvable set of equations.

This concludes a condensed version of a derivation of determinism. The material included here is sufficient for the desired result, but will be expanded upon in a future paper.

6.2. Properties of the invariant measure and polar embedding

In this section I briefly discuss some of the physical quantities of importance to chaotic dynamics that can be related to the characteristics of the pulse basis. This is not an in-depth discussion of theoretical considerations, but is intended as a summary that is useful for applications.

6.2.1. Entropy and Lyapunov exponents

The properties of the type of chaos that results are a direct consequence of the pulse basis formulation. For example, the entropy must be exactly one bit per Poincaré cycle, since the original random process that is embedded is a binary pulse train encoded with a Bernoulli process. This makes the expansion base exactly 2 (the numerical base in which the Lyapunov exponent is unity, or the expanding Lyapunov exponent in base 2 is unity) and the forward Lyapunov exponent is identified with the original choice of \( \lambda \) as the exponent in the decaying head function.

6.2.2. State splitting and dimension

In the sense that the heads of all the past pulses interfere with the state at a given time, we can think of the addition of the exponentially decaying heads as splitting the state for a given future symbol sequence into an infinite number of points lying on a Cantor set. Each pulse in the past splits the state in two, and this occurs an infinite number of times effectively producing a Cantor set. Without much difficulty, the dimension of a Cantor set that is constructed by binary splitting by a distance of \( \beta \) is given by

\[
D_c = \frac{\log 2}{\log 1/\beta}.
\]

For the middle third erasing Cantor set, for example, one could construct the
same set by splitting a single point by a distance of 1/3 in an infinite binary tree. The dimension thus computed is \( D_c = \frac{\log 2}{\log 3} \) which is of course the known value for this set. In the case of the signal with unit pulse having decay \( e^{-\lambda} \) for each successive splitting, the dimension (using the natural logarithm in this case) is \( D_c = \frac{\log 2}{\lambda} \), the expected Kaplan-Yorke [8] result for a compressing Lyapunov exponent of \( \lambda \) and expanding exponent of 2 (in binary). The interesting thing here is that this result comes from a completely different place – the 2 in the numerator comes from the simple entropy consideration that there are 2 possible states for each past pulse, and the \( \lambda \) comes from the decay function in the unit pulse. Note that this dimension is the dimension of the Cantor set structure, the dimension of the whole “attractor” is \( 2 + D_c \). Also, the Kaplan-Yorke conjecture strictly relates the Lyapunov dimension to the information dimension, but for this simple Cantor set and a Bernoulli process for the symbolic dynamics, the information dimension is the same as the box-counting dimension. (Roughly, each branch of the set is visited with equal likelihood, since all symbolic trajectories are equally probable.)

6.2.3. Polar embedding
It is possible to embed the triple as given before in the derivation in polar coordinates to obtain a picture of this type of embedding, but the negative excursions of using \( \rho = x_0 \) as the radial coordinate in cylindrical gives an ambiguity – for a given phase angle, there are two possible points in space depending on whether the signal is positive or negative. It is therefore useful to embed using \( \rho = x_0 + \text{offset} \). Figure 18 use the following values of \( \text{offset} \) for cylindrical embeddings.

![Figure 18. Two polar embeddings of the signal for different values of the offset parameter.](image-url)
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