Stability conditions for the Horndeski scalar field gravity model

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Abstract. We constrain the viable models of Horndeski gravity, written in its equivalent Generalised Galileon version, by resorting to the Witten positive energy theorem. We find that the free function $G_3(\phi, X)$ in the Lagrangian is constrained to be a function solely of the scalar field, $G_3(\phi)$, and relations among the free functions are found. Other criterion for stability are also analysed, such as the attractiveness of gravity, and the Dolgov-Kawasaki instability. Some applications for cosmology are discussed.

Keywords: gravity, modified gravity

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1 Introduction

General Relativity (GR) is the most established theory of gravity to date. It leads to second order metric field equations, which are free from ghost instabilities, being compatible with several observational tests [1, 2]. However, in order to match astrophysical and cosmological data it requires the existence of two dark components, which comprise around 95% of the energy content of the Universe and have not been directly detected so far. Furthermore, it lacks a consistent quantum version. Moreover, GR is not the most general theory which leads to second order field equations, hence higher dimensions theories such as Lovelock gravity [3], or scalar-tensor theories as Horndeski gravity constitute robust alternatives [6, 7]. In particular, Horndeski gravity is the most general extension of GR, in four-dimensional spacetime, involving a scalar field. Later on, Horndeski himself extended the model relying on an Abelian vector field whose action is invariant under $U(1)$ transformations, instead of the scalar field [8]. This model reduces in flat spacetime to the Einstein-Maxwell action [9], and if gauge invariance is relaxed one is left with the Proca action [10, 11]. However the Horndeski scalar field work was ignored for decades until its was rediscovered in a different formulation, the Galileon action [12, 13]. A classical Galileon in flat spacetime is a field, $\pi$, which obeys to a Galilean symmetry, $\pi \rightarrow \pi + b_\mu x^\mu + c$, with $b_\mu$ and $c$ being a constant four-vector and a constant scalar, respectively. Its covariant generalisation breaks the Galilean symmetry, but gives origin to field equations of order non higher than two in the spacetime derivatives [14], in such way that both classical and quantum pathologies are absent. Hence, Horndeski scalar gravity and Generalised Galileon Gravity are equivalent to each other at least in four dimensions [13]. Moreover, Horndeski gravity encompasses General relativity, Brans-Dicke theory, Quintessence, Dilaton, Chameleon gravity models, or even $f(R)$ theories upon a suitable conformal transformation.

These scalar models have a rich lore of theoretical and cosmological implications (see, for instance, refs. [13, 15]), as well as the Horndeski vector field model [16, 17].
More recently, gravitational data from the collision of two neutron stars [18] imposed stringent restrictions on alternative gravity models [19–23], namely from the constraint on the speed of gravitational waves, \(-3 \times 10^{-15} \leq c_g/c - 1 \leq 7 \times 10^{-16}\). For instance, by inspecting Horndeski gravity models which also account for late time acceleration of the Universe, two of the Horndeski free functions in the Lagrangian densities were shown to be restricted to \(G_4 = G_4(\phi)\) and \(G_5 \approx \text{const.} := G_5^{(0)}\) [22]. Some constraints on the full Lagrangian also arise from an effective field approach on this and other modified gravity models as in ref. [24]. Surprisingly, Horndeski analogue in the teleparallel framework evades some of these stringent constraints and is compatible with PPN expansion [25–27].

However, even if classical and quantum pathologies could be avoided in the Horndeski scalar gravity, provided the action functional is not degenerated, the corresponding Hamiltonian may not be bounded from below [28]. Moreover, an important criteria for gravitational theories is that they obey the positive energy theorem proposed in refs. [29, 30] for general relativity, and which got an alternative elegant proof due to Witten [31]. In broad terms, this theorem states that the total gravitational Hamiltonian of an isolated system is non-negative. Witten’s work led to improvements and generalisations of the positive energy theorem for non supersymmetric gravity theories [32–36], and for extended supergravity theories [37, 38].

The goal of the present work is to further constrain the viable Horndeski models in light of Witten’s positive energy theorem. To this end, we shall employ Nester’s version for technical reasons [32]. Furthermore, the metric signature used throughout this work is \((+−−−)\).

This work is organised as follows: in section 2, one briefly introduces the Horndeski gravity as constrained by gravitational data; in section 3, we study some stability criteria such as the requirement of attractive gravity, Dolgov-Kawasaki instabilities, Witten’s theorem, and the zero energy states. In section 4, we analyse some cosmological implications of our results. We present our conclusions in section 5.

2 Horndeski gravity after GW170817

Taking into account the stringent constraints from gravitational data [18], the action functional of Horndeski gravity, written in the form of Generalised Galileon theories, reads:

\[
S = \int d^4x \sqrt{-g} \left[ \sum_{i=2}^{5} \mathcal{L}_i + \mathcal{L}_M \right],
\]

where \(g\) stands for the metric determinant, \(\mathcal{L}_M\) is the matter Lagrangian density and \(\mathcal{L}_i\) are the Horndeski Lagrangian densities defined as:

\[
\mathcal{L}_2 := G_2(\phi, X),
\]

\[
\mathcal{L}_3 := G_3(\phi, X) \Box \phi,
\]

\[
\mathcal{L}_4 := -G_4(\phi) R,
\]

\[
\mathcal{L}_5 := G_5^{(0)} G_{\mu\nu} \nabla^\mu \nabla^\nu \phi,
\]

where \(G_i(\phi, X)\) are arbitrary functions of the scalar field, \(\phi\), and its kinetic term, \(X := \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi\), \(R\) is the scalar curvature, and \(G_{\mu\nu}\) is the Einstein tensor.

Varying the action with respect to the metric field yields [13]:

\[
G_{\mu\nu} = \kappa \left[ T_{\mu\nu}^M + T_{\mu\nu}^T \right],
\]
where the effective gravitational coupling is given by \( \hat{\kappa} = \hat{\kappa}(\phi) := \frac{1}{2G_4(\phi)} \), and the effective energy-momentum tensor reads:

\[
\hat{T}_{\mu\nu} := G_2X \nabla_\mu \phi \nabla_\nu \phi - G_2 g_{\mu\nu} + G_3X \Box \phi \nabla_\mu \phi \nabla_\nu \phi - 2 \nabla_\mu (G_3 \nabla_\nu \phi) + \nabla_\mu G_3 \nabla_\nu \phi,
\]

where \( G_{iX} := \frac{\partial G_i}{\partial X} \), \( G_{i\phi} := \frac{\partial G_i}{\partial \phi} \), and \( T^M_{\mu\nu} := \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \) is the energy-momentum tensor of matter.

In fact, casting metric field equations in the form of eq. (2.6) is not only a mathematical artifice, but also makes some calculations much easier and directly comparable with General Relativity. The physics of the system is not altered by this redefinition in terms of an effective energy-momentum tensor. Thus, the problem of testing the gravity model translates into having suitable properties of the Einstein tensor and its curvature effect on the manifold, and requiring some properties to be satisfied by the higher-order corrections encoded in the right hand side of the field equations.

In light of these results from gravitational waves constraints, some stability criteria should be established, as we shall pursue in the next section.

3 Stability criteria

3.1 Attractive gravity

An important stability criterion concerns the avoidance of ghost and gradient instabilities, which translates into the demand on the positiveness of both numerator and denominator of the speed of sound of gravitational waves. For the Horndeski models, after GW170817, this implies that \( G_4(\phi) > 0 \). This matches precisely the requirement of an attractive gravity model, \( \hat{\kappa} > 0 \).

3.2 Dolgov-Kawasacki instabilities

The Dolgov-Kawasacki instabilities [39] arise when a gravity theory allows for a dynamical equation for the scalar curvature through the trace of the metric field equations and the associated “squared mass” is non-positive. Since the trace of the viable Horndeski models renders an algebraic equation for the scalar curvature, no Dolgov-Kawasacki occurs:

\[
R = -\frac{1}{2G_4} \left[ T + \hat{T} \right],
\]

where \( T = T^\Lambda M_\Lambda \) is the trace of the energy-momentum tensor of matter fields and \( \hat{T} := 2G_2X - 4G_2 + 2G_3X \Box \phi X + 2\nabla_\Lambda G_3 \nabla^\Lambda \phi - 6 \left[ G_{4\phi} \Box \phi + 2G_{4\phi\phi}X \right] \).

3.3 Witten’s positive energy theorem

We can recast the Horndeski energy-momentum tensor in another way by rearranging terms:

\[
\hat{T}_{\alpha}^\sigma = \left[ G_2X + 3G_3X \Box \phi - 2G_{4\phi} + 2G_{4\phi\phi} \right] \nabla^\sigma \phi \nabla_\alpha \phi + \left[ \delta^\sigma_\alpha (-G_2 - 2G_{4\phi} \Box \phi) + 2G_{4\phi} \nabla^\sigma \nabla_\alpha \phi \right] + 
\left[ -2G_3X \nabla^{(\sigma} \nabla_{\alpha)} X + G_{4\phi} \nabla^\sigma \nabla_\alpha \phi + \delta^\sigma_\alpha G_3X \nabla_\Lambda X \nabla^\Lambda \phi \right] + 2\delta^\sigma_\alpha (G_{3\phi} - 2G_{4\phi\phi}) X.
\]
For technical reasons, we shall use Nester’s version [32] of the positive energy theorem of Witten [31]. We start by considering the total momentum four-vector $p^\mu$ for an asymptotically flat spacetime which is given by the surface integral of the difference of two connections [32]:

$$16\pi GV^\mu p_\mu = -\frac{1}{2} \int_{S=\partial \Sigma} V^\mu \delta^{\sigma\alpha \beta} g^{\rho\delta} \Delta \Gamma^{\sigma}_{\rho \delta} dS_{\sigma \alpha},$$  \hspace{1cm} (3.3)

where $V^\mu := \epsilon_0 \gamma^\mu \epsilon_0$, with $\epsilon_0$ being the value of the Dirac spinor $\epsilon$ at spatial infinity up to corrections of $O(1/r)$, $\Sigma$ is an arbitrary 3-surface whose boundary at spatial infinity is $\partial \Sigma = S$, and

$$\Delta \Gamma^{\sigma}_{\alpha \beta} = \Gamma^{\mu}_{\alpha \beta} - \Gamma^{\mu}_{\alpha \beta} (flat) = O(1/r^2).$$  \hspace{1cm} (3.4)

The total momentum four-vector is related with the integral of the two form $E^{\sigma \alpha} := 2 (\pi \Gamma^\sigma \gamma^\Delta \epsilon - \gamma^\beta \Gamma^\sigma \epsilon) \Sigma \epsilon^\sigma$ by [31]:

$$16\pi GV^\mu p_\mu = \frac{1}{2} \int_{S=\partial \Sigma} E^{\sigma \alpha} dS_{\sigma \alpha} = \int_{\Sigma} \nabla_\alpha E^{\sigma \alpha} d\Sigma\sigma,$$  \hspace{1cm} (3.5)

where we have used the following notations $\tilde{\epsilon} = \epsilon^1 \gamma_0$, and $\gamma^\mu$ are the Dirac’s $\gamma$ matrices.

It can be shown that in a supersymmetric version of the positive energy theorem [33, 34] the divergence of the previous two-form can be cast as:

$$\hat{\nabla}_\alpha E^{\sigma \alpha} = 2G^{\alpha \epsilon^i} \gamma^\alpha \epsilon^i + 4\overline{\nabla} \epsilon^i \Gamma^{\sigma \alpha \beta} \hat{\nabla}_\beta \epsilon^i + \delta \chi^a \gamma^\alpha \delta \chi^a,$$  \hspace{1cm} (3.6)

where $\hat{\nabla}$ is the supersymmetric extension of the covariant derivative associated to the change of the gravitino field under a supersymmetric transformation, $i = 1, \ldots, N$ is the total number of supersymmetries, and $\delta \chi^a$ represents the change of a spin-$\frac{1}{2}$ field under supersymmetric transformation. Notice that supersymmetry arises as its gauge version, supergravity, whose ground state is a sum of quadratic charges and hence strictly positive. This is the main motivation for Witten’s positive energy theorem as General Relativity is the limit of supergravity in the absence of supersymmetry.

In General Relativity, $G_{\mu \nu} = 8\pi GT_{\mu \nu}$, and substituting the previous equation into eq. (3.5), the positiveness of the integrand is ensured provided the energy-momentum tensor of matter fields satisfies the dominant energy condition, $\epsilon_0^0 \gamma^0 \epsilon_0^1$ is non-spacelike, and the Witten condition $\gamma^\lambda \overline{\nabla} \chi^i = 0$ is chosen. We note that the value of $\overline{\nabla} \chi^i$ and $\delta \chi^a$ are set by supersymmetry [33, 34].

This procedure can be considered for non-supersymmetric theories [33, 35], resulting in the introduction of three functions in the definitions of the gradient of the Dirac spinor and the change of the gravitino-like terms, to be later determined, in order to ensure the positiveness of the generalised version of the integrand of eq. (3.5):

$$\nabla_\alpha \epsilon^i := \nabla_\alpha \epsilon^i + \frac{i}{2} \gamma_\alpha f_2^{ij} \epsilon^j,$$  \hspace{1cm} (3.7)

$$\delta \chi^a = i\gamma^\lambda \nabla_\lambda \phi f_3^{ai} \epsilon^i + f_3^{ai} \epsilon^i,$$  \hspace{1cm} (3.8)

where the functions, $f_1$, $f_2$, $f_3$, relax the constraints set by supersymmetry [33, 35] and further introduce a scalar field through a contraction of the covariant derivative with the $\gamma$ matrix.

In fact, the problem of a given gravity theory to have a non-negative energy in the sense of Witten’s theorem translates into obtaining solutions for the functions $f_1$, $f_2$ and $f_3$ which shall depend on the model’s free functions that may appear in the metric field equations.
In the case of Horndeski theories, the above free functions should depend on the scalar field and its kinetic term, i.e., $f_i = f_i(\phi, X)$, $i = 1, \ldots, 3$. By inserting eq. (2.6), together with the definitions of eqs. (3.7) and (3.8), the divergence of the two-form $E^{\sigma\alpha}$ reads:

$$\nabla_\alpha E^{\sigma\alpha} = 2\hat{k}(\phi) T_\alpha^\sigma \epsilon^\alpha \epsilon^i + 4\nabla_\alpha \epsilon^\sigma \Gamma_\sigma^\alpha \beta \nabla_\beta \epsilon^i + \delta \nabla^\sigma \gamma^\sigma \delta \chi^a +$$

$$+ \left[ -2f_2(\phi, X)^{ai} f_2(\phi, X)^{aj} + 2\hat{k}(\phi) (G_{2\phi}(\phi, X) + G_{3\phi}(\phi, X) \square \phi - 2G_{3\phi}(\phi, X)) \right] \nabla^\sigma \phi \nabla_\alpha \phi \epsilon^i \gamma^a \epsilon^i +$$

$$+ 2G_{4\phi}(\phi) \delta^i \delta^j \left( \hat{k}(\phi) f_1(\phi, X)^{ij} \right)^2 - \delta^i \delta^j f_3(\phi, X)^{ai} f_3(\phi, X)^{aj}$$

$$+ 2\hat{k}(\phi) (G_{2\phi}(\phi, X) \delta^\sigma_{\alpha} - 2G_{3\phi}(\phi, X) \square \phi \delta^\sigma_{\alpha} + 2G_{4\phi}(\phi) \nabla^\sigma \nabla_\alpha \phi) \delta^i \gamma^a \epsilon^i +$$

$$+ 2i \left[ f_2(\phi, X)^{ai} f_3(\phi, X)^{aj} - f_2(\phi, X)^{ai} f_3(\phi, X)^{aj} f_3(\phi, X)^{ai} \right] \nabla_\alpha \phi \epsilon^i \gamma^a \epsilon^j +$$

$$- 2f_2(\phi, X)^{ai} f_2(\phi, X)^{aj} + 4\hat{k}(\phi) (G_{3\phi}(\phi, X) - 2G_{4\phi}(\phi)) \delta^i \gamma^a \epsilon^i +$$

$$+ 2\hat{k}(\phi) \left[ -2G_{3\phi}(\phi, X) \nabla^i (\phi \nabla^\sigma \phi) + \delta^i \delta^j G_{3\phi}(\phi, X) \nabla^\lambda \phi \nabla_\lambda X \right] \epsilon^i \gamma^a \epsilon^j . \quad (3.9)$$

It is worth pointing out that the separation of terms, eq. (3.9), provides a sufficient set of conditions to straightforwardly satisfy the required constraints. Of course, it is possible, for instance, to combine the second set of terms with the third set of terms and the fifth set of terms with the sixth set of terms, but these combinations would lead to a quite complex set of constraints, which, most likely, would be too hard to solve and too difficult to interpret in terms of restrictions to the Horndeski model. Therefore, in order to have the identification with the general expression of eq. (3.5), so to solve for the theorem functions $f_1, f_2, f_3$, the solutions of a set of equations need to be found, namely:

$$
\begin{align*}
2f_2(\phi, X)^{ai} f_2(\phi, X)^{aj} &= 2\hat{k}(\phi) (G_{2\phi}(\phi, X) + G_{3\phi}(\phi, X) \square \phi - 2G_{3\phi}(\phi, X) + 2G_{4\phi}(\phi)) \delta^i \delta^j \\
24 \left( \hat{k}(\phi) f_1(\phi, X)^{ij} \right)^2 \delta^a &= f_3(\phi, X)^{ai} f_3(\phi, X)^{aj} \delta^a + 2\hat{k}(\phi) (G_{2\phi}(\phi, X) \delta^a + 2G_{4\phi}(\phi) \square \phi \delta^a) \\
&- 2G_{4\phi}(\phi) \nabla^\sigma \nabla_\alpha \phi) \delta^i \delta^j \\
4 \left( \hat{k}(\phi) f_1(\phi, X)^{ij} \right)^2 &= f_2(\phi, X)^{ai} f_3(\phi, X)^{aj} + f_2(\phi, X)^{ai} f_3(\phi, X)^{aj} \\
2f_2(\phi, X)^{ai} f_2(\phi, X)^{aj} &= 4\hat{k}(\phi) (G_{3\phi}(\phi, X) - 2G_{4\phi}(\phi)) \delta^i \gamma^a \epsilon^i +$$

From this set, we can easily verify that:

- the fourth equality is always satisfied since $\epsilon^i \epsilon^j = -\epsilon^j \epsilon^i$;
- the fifth condition can be inserted in the first one, resulting a simpler condition, namely $G_{2\phi} + G_{3\phi} \square \phi - 4G_{3\phi} + 6G_{4\phi} = 0$;
- the last equation implies that $G_{3\phi} = 0 \implies G_{3} = G_{3}(\phi)$. This is a quite relevant result: the Witten positive energy theorem implies that $G_{3}$ can only be a function of $\phi$. This narrows down the viable models within Horndeski theory even further alongside with the recent gravitational data.
Thus, the set of equations becomes:

\[
\begin{align*}
G_{2X}(\phi, X)\delta^{ij} &= (4G_{4\phi}(\phi) - 6G_{4\phi\phi}(\phi)) \delta^{ij} \\
24 \left( \kappa(\phi)f_1(\phi, X) \right)^2 \delta_{\alpha} &= f_3(\phi, X) a_i f_3(\phi, X)^{a_j} \delta_{\alpha} + 2\kappa(\phi)(G_2(\phi, X) \delta^{\alpha}_\alpha + 2G_{4\phi}(\phi) \Box \delta^{\alpha}_\alpha - 2G_{4\phi}(\phi) \nabla^\alpha \nabla_{\alpha} \phi) \delta^{ij} \\
4 \left( \kappa(\phi)f_1(\phi, X)^{ij} \right) \phi &= f_2(\phi, X) a_i f_3(\phi, X)^{a_j} + f_2(\phi, X)^{a_j} f_3(\phi, X)^{a_i} \\
f_2(\phi, X)^{a_i} f_2(\phi, X)^{a_j} &= 2\kappa(\phi) (-G_{3\phi}(\phi) + 2G_{4\phi\phi}(\phi)) \delta^{ij}
\end{align*}
\]

provided that \( \kappa(\phi) \neq 0 \). This set also constrains the relations among the different functions \( f_i(\phi, X) \) for each model with \( G_2(\phi, X) \), \( G_3(\phi) \), \( G_4(\phi) \), \( G_5^{(0)} \).

By comparing our results with the \( N = 1 \) Supergravity, where \( f_3 = 0 \), [33, 35] a boundary condition is found, namely:

\[
f_{ij}(\phi_0) = \sqrt{\frac{G_2(\phi_0)}{6G_4(\phi_0)}} \delta_{ij}, \quad (3.10)
\]

where \( \phi_0 \) is the value at spatial infinity of the stationary point of the potential encoded in the function \( G_2 \). In fact, as we shall see in the section 4, the effective potential will be of the form of the quotient \( G_2/G_4 \), where \( G_2 \) is the potential part of the function \( G_2 \).

Furthermore, we can state that such configuration is stable in the sense of the Witten’s theorem and refs. [31, 33, 35, 36], provided the set of eqs. (3.3) can be solved together with the above boundary condition.

These results are obtained from the metric field equations. However, we should realise that if we vary the action relatively to the scalar field, we will get further equations as we shall see in the next subsection.

### 3.4 Scalar field equations

Varying the action functional relatively to the scalar field, and taking into consideration both gravitational waves data and our results, we get:

\[
G_{2\phi} - 2G_{2X\phi} X - G_{2XX} \nabla^\mu X \nabla_\mu \phi - G_{2X} \Box \phi + 2G_{3\phi} \Box \phi + 2G_{3\phi\phi} X - G_{4\phi} R = 0. \quad (3.11)
\]

Solving this equation relatively to the Ricci scalar, and inserting into eq. (3.1), we get:

\[
G_{2\phi} - 2G_{2X\phi} X - G_{2XX} \nabla^\mu X \nabla_\mu \phi - G_{2X} \Box \phi + 2G_{3\phi} \Box \phi + 2G_{3\phi\phi} X = -\frac{1}{2}T - G_{2X} X + 2G_{2} - 2G_{3\phi} X + 3G_{4\phi} \Box \phi + 6G_{4\phi\phi} X. \quad (3.12)
\]

### 3.5 Zero energy states

A comparison with the work of ref. [33], a state of zero energy corresponds to the case where, in addition to obeying the Witten condition and \( \delta \chi^a \) for all \( c_i \), we have both \( f_3 = 0 \) and either \( f_2 = 0 \) or \( \phi = \phi_0 \). This leads to the following solutions:

\[
f_3 = 0, \quad (3.13)
\]

\[
f_1 = f_1(\phi) = (c_1 \phi + c_2) 2G_4(\phi) = \pm \sqrt{\frac{G_2}{6G_4}}, \quad (3.14)
\]

\[
G_{3\phi} = \frac{3}{2} G_{4\phi\phi}, \quad (3.15)
\]

\[
f_2^2 = f_2^2(\phi) = \frac{G_{3\phi}}{3G_4}. \quad (3.16)
\]
Hence, zero energy states are found to be stable for $f_3 = 0$ and $\phi = \phi_0$, provided we can solve the above system of equations (once one of the $G_i$ functions are given, the system can, in principle, be fully solved).

In order to get further physical knowledge, we shall analyse in the next section the main implications of our results for cosmology.

4 Cosmological implications

In cosmology, we may look for solutions which satisfy the cosmological principle, i.e., a homogeneous and isotropic Universe, hence the metric field of such scenario is given by the Robertson-Walker metric:

$$ds^2 = dt^2 - a(t)^2 dx^i dx_i,$$

where $a(t)$ is the scale factor and we are considering a spacetime with no spatial curvature, as data suggest.

Moreover, the scalar field should, in principle, obey the same symmetry as spacetime itself, hence $\phi = \phi(t)$.

This allows for an identification of the energy-momentum tensor for the scalar field with a perfect fluid energy-momentum tensor:

$$\hat{\rho} := \hat{T}_{00} = (G_2X - G_{3\phi})\dot{\phi}^2 - G_2,$$

$$\hat{p} := -\frac{1}{3}\hat{T}_{ii} = -\left[(-G_2 - 2G_2\ddot{\phi}) + (G_{3\phi} - 2G_{4\phi})\dot{\phi}^2\right].$$

4.1 Inflation

The hot Big Bang model, although consistent with data, had some conundrums, such as the homogeneity and isotropy, and spatial flatness of the Universe, and the absence of topological defects such as magnetic monopoles, which require inflation as an explanation. The latter but also provides the seeds for large scale structure formation. Several models of inflation agree with data with impressive precision [40–42].

A very common approach in this scenario is to consider a scalar field slow-rolling down its potential, $V$. During this phase, the potential energy dominates the kinetic term, and for the viable Horndeski models, we can see from eq. (4.2) that $\hat{\rho} \approx -G_2 = -\hat{p}$. If we write $G_2(\phi, X) = X - V \approx -V$, we retrieve the usual description of scalar field inflation in General Relativity.

Furthermore, the Friedmann equation, obtained from the time-time component of the metric field equations, becomes:

$$3H^2 = \frac{1}{3G_4(\phi)} \left[\rho_m + \hat{\rho}\right],$$

which is of the standard form $H^2 \approx \frac{8\pi G}{3} V(\phi)$ for $G_4(\phi) \neq 0$ and in the absence of other matter fields.

This means that during slow-roll inflation, Horndeski viable models should behave as the standard GR case although with $V(\phi)^{\text{eff}} \sim G_2(\phi)/G_4(\phi)$.

Another interesting cosmological scenario is the inclusion of a dominating cosmological constant, as it will be seen in the next subsection.
4.2 Cosmological constant

Comparing the definition of eq. (3.7) with $N = 1$ Supergravity leads to the following identification [33]:

$$\frac{1}{2} \kappa(\phi)f_1(\phi)^{ij} = \left( \pm \frac{1}{12} \Lambda \right)^{1/2},$$

(4.5)

where the $+$ sign occurs when $\Lambda > 0$, and the $-$ sign when $\Lambda < 0$ to ensure that the energy is positive definite. In addition, one has $f_3 = 0$.

For simplicity, we can assume that $i, j$ are single valued. This leads to:

$$f_1 = f_1(\phi) = 2 \sqrt{\frac{\Lambda}{3}} G_4(\phi),$$

(4.6)

$$G_2 = G_2(\phi) = 2\Lambda G_4(\phi),$$

(4.7)

$$G_3(\phi) = \frac{3}{2} G_{4\phi},$$

(4.8)

$$f_2 = f_2(\phi) = \frac{G_{3\phi}}{3G_4}.$$  

(4.9)

Hence, once again the quotient $G_2(\phi)/G_4(\phi)$ appears related to the effective potential, in this case leading to the cosmological constant. In this sense, the cosmological constant is the quotient of two cosmological functions of the Lagrangian.

5 Conclusions

In this work we have analysed the Horndeski gravity in its Generalised Galileon version using well known criteria such as the attractiveness of gravity, and the absence of Dolgov-Kawasacki instabilities to constrain the functions $G_i(\phi, X)$ of theory. In addition to these requirements, we have considered the positiveness of the Hamiltonian as discussed by Witten and shown that interesting constraints arise. For instance, function $G_3$ is shown to depend only on the scalar field and that there are relations involving the other functions.

In fact, in a cosmological context, for instance, the ensemble of these constraints can be selective enough to allow for a quite small set of viable models.

Furthermore, our approach is very general and contains previous results found in the literature. Indeed, choosing $G_2(\phi, X) = \frac{1}{2} f(\phi)X - V(\phi), G_4(\phi) = \kappa + g(\phi)$, where $\kappa = c^4/16\pi G$ with the speed of light $c$, the Newton’s gravitational constant $G$, and $f(\phi), g(\phi)$ are arbitrary functions of the scalar field, and $G_3(\phi) = 0$ corresponds to the situation of a scalar field nonminimally coupled to gravity discussed in ref. [35].

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