ON HIGHER ORDER K-BONACCI MATRICES

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ABSTRACT: In this paper, we have constructed the higher order k-bonacci matrices and studied some of their basic properties. We have also shown that these matrices satisfying some new and interesting relations in k-bonacci recurrence. This is the interesting generalization of the work of Z. Cheng-Heng [1, 2].

1. INTRODUCTION

Fibonacci numbers have many applications, including appearance in the nature, such as the branching of trees, the patterns on a pineapple, the florets of a sunflower, the spirals of a pine cone, the family trees of cows and bees, and the placement of leaves on the stems of many plants, etc. These numbers were known in ancient India as Hemachandra numbers. The key property of the Fibonacci numbers is that every number equals to the sum of the two preceding numbers, therefore we obtain a numerical sequence \( f_0, f_1, \ldots, f_n, f_{n+1} \ldots \) such that \( f_{n+1} = f_n + f_{n-1} \) with \( f_0 = 1, f_1 = 1 \) of numbers. The sequence is called the recurrent sequence and the relation is known as the recurrence relation [15]. One can generalize the idea to higher order recurrence such as Tribonacci sequence, where every number equals to the sum of the previous three numbers and with the sum of previous four numbers one can make tetrabonacci sequence and so forth. Similarly, in a more general settings, for the K-bonacci sequence every number equals to the sum of the previous k numbers [14].

The idea of the sequence can be generalized to matrices. Many researchers have studied such interesting matrices. Fibonacci numbers have been generalized to obtain a square Fibonacci matrix [6]. Further, it is generalized to Fibonacci Q-matrix [5] and sequence of matrices of order \( 2^r \) by Cheng-Heng [1, 2]. In this paper, we obtain a generalization of the matrices of order \( 2^r \) to the k-bonacci matrices of order \( k^r \) and obtain some interesting results. k-bonacci numbers and its matrices have various applications in tournament sequences, graph theory, modular form, dynamic interpretation and memory allocation schemes [11, 12].

The paper is organized as follows. Section 2 contains some preliminaries of the k-bonacci numbers. Section 3 defines the k-bonacci matrices of order \( k^r \). In section 4, we obtain some of its interesting properties. Section 5 contains some new results of Fibonacci matrices of order \( 2^r \).

2. PRELIMINARIES AND NOTATIONS

For an integer \( k \geq 2 \), the k-bonacci numbers \( f_{j,k} \) are defined as [14]

\[
f_{0,k} = 0, f_{1,k} = 0, f_{2,k} = 0 \ldots f_{k-2,k} = 0, f_{k-1,k} = 1 \text{ and } f_{j,k} = \sum_{n=1}^{k} f_{j-n,k} \text{ for all } j \geq k.
\]  

(2.1)
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It is easy to obtain some values for \( f_{j,k} \), as given below:

\[
  f_{j,k} = \begin{cases} 
    2^{j-k}, & k \leq j \leq 2k - 1 \\
    2^{j-k} - (2^{j-2k+1} - 1), & 2k \leq j \leq 3k - 2.
  \end{cases}
\]

When \( k = 2 \), the equation (2.1) reduces to the usual Fibonacci numbers \( f_{j,2} \) or \( f_j \) (for simple notation),

\[
  f_0 = 0 \quad \text{for} \quad j = 0, f_1 = 1 \quad \text{for} \quad j = 1 \quad \text{and} \quad f_j = f_{j-1} + f_{j-2} \quad \text{for all} \quad j, \quad j \geq 2.
\]

The backward k-bonacci numbers are given by the following recurrence relation

\[
  f_{j-k,k} = f_{j,k} - \sum_{n=1}^{k-1} f_{j-n,k} \quad \text{for all} \quad j, \quad j \leq k - 1.
\]

From backward recurrence relation, one can easily obtain some \( k \)-bonacci backward values:

\[
  f_{j,k} = \begin{cases} 
    1, -1, 0, \ldots, 0 & 0 \leq j \leq k - 1 \\
    2, -3, 1, 0, \ldots, 0 & -k \leq j \leq -1 \\
    4, -8, 5, -1, 0, \ldots, 0 & -2(k + 1) \leq j \leq -(k + 1) \\
    8, -20, 18, -7, 1, 0, \ldots, 0 & -(3k + 2) \leq j \leq -(2k + 2) \\
    16, -48, 56, -32, 9, -1, 0, \ldots, 0 & -(4k + 3) \leq j \leq -(3k + 3)
  \end{cases}
\]

From above values, we observe the following interesting properties in every block of \( k \) terms:

1. Numbers are alternative positive and negative.
2. Sum of the numbers are zero.
3. First number belongs to \( k \)-bonacci numbers, for \( j = nk \) where \( n \geq 0 \). From forward and backward \( k \)-bonacci numbers, one can see that
   \[
   f_{k,k} = f_{-1} = 1 \\
   f_{k+1,k} = f_{-(k+1)} = 2 \\
   f_{k+2,k} = f_{-(2k+1)} = 4 \\
   \ldots = \ldots = \ldots
   \]

   Therefore, we find an interesting relation.

   \[
   f_{k+j,k} = f_{-(jk+1),k} = 2^j, j \geq 0.
   \]

4. The last non zero number is \(-1\) or \(1\) for \( j = n(k + 1) + 1 \) where \( n \geq 0 \).
5. The second last non zero number is subsequent odd number for \( j = n(k+1) \) where \( n \geq 0 \). Every number before that is even except for the first block (there are only two terms).

For \( k = 2 \), equation (2.2) becomes the backward Fibonacci relation \[20\]

\[
  f_{j-2} = f_j - f_{j-1} \quad \text{for all} \quad j, \quad j > 1.
\]

Now we focus our attention to the matrices. The \( k \)-bonacci matrix [14] of order \( k \) is defined as

\[
  F_{j,k}^{(k)} = \begin{pmatrix}
    f_{j+k-1,k} & \cdots & f_{j,k} \\
    f_{j+k-2,k} & \cdots & f_{j-1,k} \\
    \vdots & \vdots & \vdots \\
    f_{j,k} & \cdots & f_{j-k+1,k}
  \end{pmatrix} = (f_{j+k-\lambda-\mu+1,k}) = f_{\lambda \mu}, \quad 1 \leq \lambda, \mu \leq k. \quad (2.3)
\]

where \( j \geq 0 \).

By taking \( k = 2 \), the above matrix \( F_{j,k}^{(k)} \) reduces to Fibonacci matrix of order 2, which is considered in [4].
In view of equation (2.2) we construct the backward k-bonacci matrix of order \( k \) as
\[
F_{-j,k}^{(k)} = (f_{-(j+k-\lambda-\mu+1)}, j \geq 0.
\]

3. Matrices of Order \( k \)

We have constructed the \( k \)-th order \( k \)-bonacci matrices with the help of \( k \)-bonacci matrix of order \( k \) [see (2.3)] as follows.
\[
\sum_{n=0}^{k-1} F_{j+n,k}^{(k)} = \sum_{n=0}^{k-1} (f_{\lambda\mu+n,k}) \quad \text{(from equation (2.3))}
\]
\[
= \left( \sum_{n=0}^{k-1} f_{\lambda \mu+n,k} \right)
\]
\[
= F_{\lambda \mu+k,k}^{(k)}
\]

Therefore we have the \( k \)-bonacci matrix of order \( k \), we can construct the \( k \)-bonacci matrix of order \( k^2 \)
\[
\sum_{n=0}^{k-1} F_{j+n,k}^{(k^2)} = \sum_{n=0}^{k-1} \left( F_{\lambda \mu+n,k}^{(k)} \right) \quad \text{(from (2.3))}
\]
\[
= \left( \sum_{n=0}^{k-1} F_{\lambda \mu+n,k}^{(k)} \right)
\]
\[
= F_{\lambda \mu+k,k}^{(k^2)}
\]

Similarly the \( k \)-bonacci matrix of order \( k^r \) can be written as
\[
\sum_{n=0}^{k-1} F_{j+n,k}^{(k^r)} = \sum_{n=0}^{k-1} \left( F_{\lambda \mu+n,k}^{(k^r-1)} \right) \quad \text{(from (2.3))}
\]
\[
= \left( \sum_{n=0}^{k-1} F_{\lambda \mu+n,k}^{(k^r-1)} \right)
\]
\[
= F_{\lambda \mu+k,k}^{(k^r)}
\]

More precisely, the \( k \)-bonacci matrix of order \( k^r \) is given as
\[
F_{j+k,k}^{(k^r)} = \sum_{n=0}^{k-1} F_{j+n,k}^{(k^r)} \quad \text{for all } j \text{ and } r, \quad \text{where } j \geq 0, r \geq 1.
\]

4. Properties of Higher Order Matrices

Lemma 4.1 can be obtained by using equation (2.1) and Theorem 4 of [4].

Lemma 4.1.
\[
\sum_{j=0}^{n-1} f_{j,k} = \frac{1}{k-1} \left( f_{n+k-1,k} - f_{k-1,k} - \sum_{i=1}^{k-2} i f_{n+k-2-i,k} \right).
\]

The Sum Formula for the higher order \( k \)-bonacci matrices is given by the following Theorem.

Theorem 4.2. If \( n \geq 1, j \geq 0 \) and \( r \geq 1 \), then
\[
\sum_{j=0}^{n-1} F_{j,k}^{(k^r)} = \frac{1}{k-1} \left( F_{n+k-1,k}^{(k^r)} - F_{k-1,k}^{(k^r)} - \sum_{i=1}^{k-2} i F_{n+k-2-i,k}^{(k^r)} \right).
\]
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**Proof.** We have the matrix from equation (2.3)
\[
\sum_{j=0}^{n-1} F_{j,k}^{(k^r)} = \sum_{j=0}^{n-1} \left( F_{j+k-\lambda-\mu+1,k}^{(k^r-1)} \right)
\]

The summation on the right hand side of the above expression can be further simplified using Lemma 4.1
\[
\frac{1}{k-1} \left( F_{n+k-1+k-\lambda-\mu+1,k}^{(k^r-1)} - F_{n+k-1+k-\lambda-\mu+1,k}^{(k^r)} - \sum_{i=1}^{k-2} iF_{n+k-2-i+k-\lambda-\mu+1,k}^{(k^r-1)} \right)
\]
\[
\frac{1}{k-1} \left( F_{n+2k-\lambda-\mu,k}^{(k^r-1)} - F_{2k-\lambda-\mu,k}^{(k^r-1)} - \sum_{i=1}^{k-2} iF_{n+2k-1-i-\lambda-\mu,k}^{(k^r-1)} \right)
\]
\[
\frac{1}{k-1} \left( F_{n+k-1,k}^{(k^r)} - F_{n+k-1,k}^{(k^r-1)} - \sum_{i=1}^{k-2} iF_{n+k-2-i,k}^{(k^r)} \right).
\]

Thus, Theorem 4.2 holds true for \( n \geq 1, j \geq 0 \) and \( r \geq 1 \).

Lemma 4.3 is obtained by equation (2.1) and Theorem 1 of [3].

**Lemma 4.3.** If \( j \geq 2 \), then
\[
f_{j+k-1,k} = 2f_{j+k-2,k} - f_{j-2,k}.
\]

**Theorem 4.4.** If \( j \geq 2 \), then
\[
F_{j+k-1,k}^{(k^r)} = 2F_{j+k-2,k}^{(k^r)} - F_{j-2,k}^{(k^r)}.
\]

**Proof.** Right hand side of Theorem 4.4 can be written in the following form by equation (2.3),
\[
2F_{j+k-2,k}^{(k^r)} - F_{j-2,k}^{(k^r)} = 2 \left( F_{j+k-2+k-\lambda-\mu+1,k}^{(k^r-1)} \right) - \left( F_{j-2+k-\lambda-\mu+1,k}^{(k^r-1)} \right)
\]
\[
= 2 \left( F_{j+k-2-k-1-\lambda-\mu,k}^{(k^r-1)} \right) - \left( F_{j-2-k-1-\lambda-\mu,k}^{(k^r-1)} \right).
\]

In particular, using Lemma 4.3, we obtain
\[
\frac{1}{k-1} \left( F_{j+2k-\lambda-\mu,k}^{(k^r-1)} \right) = F_{j+k-1,k}^{(k^r)}.
\]

**Lemma 4.5.** If \( n \geq 1 \), then
\[
\sum_{j=1}^{n} \frac{f_{j-1,k}}{2^j} = 1 - \frac{f_{k+n,k}}{2^n}.
\]

**Proof.** (Proof by Mathematical Induction) In the initial step, we must verify that the above expression is true for \( n = 1 \),
\[
\frac{f_{0,k}}{2} = \frac{f_{k,k} - \left( \sum_{r=1}^{k-1} f_{r,k} \right)}{2} = \frac{2f_{k,k} - f_{k+1,k}}{2} = 1 - \frac{f_{k+1,k}}{2}.
\]

The expression is clearly true. We assume that expression (4.1) is true for \( n = m \), i.e,
\[
\sum_{j=1}^{m} \frac{f_{j-1,k}}{2^j} = 1 - \frac{f_{k+m,k}}{2^m}
\]
In the next step, we will show that it holds for \( n = m + 1 \),
\[
\sum_{j=1}^{m+1} \frac{f_{j-1,k}}{2^j} = 1 - \frac{f_{k+m,k} - f_{m,k}}{2^{m+1}} + \frac{f_{m,k}}{2^{m+1} - \left(\sum_{j=1}^{m+1} f_{m+1,k}\right)}
\]
\[
= 1 - \frac{f_{k+m,k} - f_{m,k}}{2^{m+1}}
\]
This is the required result. \( \square \)

**Theorem 4.6.**
\[
\sum_{j=1}^{n} \frac{F^{(k'r)}}{2^j} = 1 - \frac{F^{(k'r)}}{2^n}
\]

**Proof.** Theorem 4.6 can be proved by using equation (2) and Lemma 4.5.

\[
\sum_{j=1}^{n} \frac{F^{(k'r)}}{2^j} = \sum_{j=1}^{n} \left(\frac{F^{(k'r-1)}}{2^j} - \frac{(F^{(k'r)})^2}{2^n}\right)
\]

The following Lemma is obtained by equation (2.1) and theorem 3(a) of [4].

**Lemma 4.7.**
\[
\sum_{n=0}^{m} f_{kn+j+1,k} = \sum_{n=-mk}^{k-1} f_{j-n,k}
\]

**Theorem 4.8.** If \( j \geq 0 \) and \( r \geq 1 \), then
\[
\sum_{n=0}^{m} F^{(k'r)}_{kn+j+1,k} = \sum_{n=-mk}^{k-1} F^{(k'r)}_{j-n,k}
\]

**Proof.** From the matrix of (2.3), we have
\[
\sum_{n=0}^{m} F^{(k'r)}_{kn+j+1,k} = \sum_{n=0}^{m} \left(\frac{F^{(k'r-1)}}{k(n+1)+j-\lambda-\mu+2,k}\right)
\]
Which gives, using lemma 4.7, the following.
\[
= \sum_{n=-mk}^{k-1} \left(\frac{F^{(k'r-1)}}{j-n+k-\lambda-\mu+1,k}\right)
\]

**Lemma 4.9** is obtained by equation (2.1) and Theorem 3(b) of [4].

**Lemma 4.9.** If \( m \geq 1 \), then
\[
\sum_{n=1}^{m} f_{kn,k} = \sum_{n=k(1-m)}^{k-1} f_{k-1-n,k}
\]

**Theorem 4.10.** If \( m \) is greater than 1, then
\[
\sum_{n=1}^{m} F^{(k'r)}_{kn,k} = \sum_{n=k(1-m)}^{k-1} F^{(k'r)}_{k-1-n,k}
\]
Further, by Lemma 4.11.

Lemma 4.11. It states that if

If

Theorem 4.12. If \( m \geq 1 \), then

Proof. The above theorem immediately follows from the Theorem 4.10.

By Theorem 5 of \([4]\) and by equation (2.1), we can get the following Lemma.

Lemma 4.13. It states that if \( n \geq 0 \), then

Theorem 4.14. If \( n \geq 0 \), then

Proof. By equation (2.3), right hand side of the theorem can be written as

From Lemma 4.13, we have

Thus, the theorem is proved.

Theorem 4.15. If \( n \geq 1 \), then

\[
f_{j+n,k} = 2^n \sum_{i=1}^{k-n} f_{j-i,k} + (2^n - 1)f_{j-k+(n-1),k} + (2^n - 2)f_{j-k+(n-2),k} + (2^n - 2^2)f_{j-k+(n-3),k} + \ldots + 2^{n-1}f_{j-k,k}.
\]
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Proof. In the initial step, proof is carried out by induction on \( n \). Clearly it is true for \( n = 1 \),

\[
= 2 \sum_{i=1}^{k-1} f_{j-i,k} + f_{j-k,k}
\]

\[
= 2(f_{j-1,k} + \ldots + f_{j-k+1,k}) + f_{j-k,k}
\]

Thus we assume that the expression (4.2) is true for \( n = m \), so far it should hold true for

Thus, by induction and the virtue of Lemma 4.3 we have the desired result. \( \square \)

Theorem 4.16.

\[
F_{j+n,k}^{(k')} = 2^n \sum_{i=1}^{k-n} F_{j-i,k}^{(k')} + (2^n - 1) F_{j-k+(n-1),k}^{(k')} + (2^n - 2) F_{j-k+(n-2),k}^{(k')} + \ldots + 2^{n-1} F_{j-k,k}^{(k')}.
\] (4.3)

Proof. Proof of the Theorem 4.16 is carried out by induction on \( n \). For \( n = 1 \), the expression

(4.3) takes the form

\[
= 2 \sum_{i=1}^{k-1} F_{j-i+k-k-\mu+1,k}^{(k'-1)} + F_{j-k-\mu+1,k}^{(k'-1)}
\]

By equation (2.3), it takes the form

\[
= 2(F_{j+k-\mu,k}^{(k'-1)} + F_{j-1+k-\mu,k}^{(k'-1)} + \ldots + F_{j-\mu+2,k}^{(k'-1)}) + F_{j-k-\mu+1,k}^{(k'-1)}
\]

Thus result is true for \( n = 1 \). We assume that expression (4.3) is true for \( n = m \). For \( n = m+1 \) it directly follows from the proof of Theorem 4.15 \( \square \)

Theorem 4.17. If \( r \geq 1 \), then

\[
F_{j,k}^{(k')} = F_{1,k}^{(k')} \begin{pmatrix} I_{k'-1} & I_{k'-1} & O_{k'-1} & \ldots & O_{k'-1} \\ I_{k'-1} & O_{k'-1} & I_{k'-1} & \ldots & O_{k'-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_{k'-1} & O_{k'-1} & O_{k'-1} & \ldots & I_{k'-1} \\ I_{k'-1} & O_{k'-1} & O_{k'-1} & \ldots & O_{k'-1} \end{pmatrix} j^{-1} = F_{1,k}^{(k')} (Q_k^r) j^{-1}.
\]

Where \( I_{k'-1} \) is the identity matrix of order \( k'-1 \) and \( O_{k'-1} \) is the null matrix of order \( k'-1 \). Matrix \( Q \) is the combined representation of both matrices.

Proof. We have the matrix from (2.3)

\[
F_{j,k}^{(k')} = F_{j+k-\mu+1,k}^{(k'-1)}
\]

Above expression can be rewritten as

\[
= F_{j+k-\mu,k}^{(k')} (Q_k^r)
\]

\[
= F_{j+k-\mu-1,k}^{(k'-1)} (Q_k^r)^2
\]

\[
= F_{k+2-\mu-1,k}^{(k'-1)} (Q_k^r)^j
\]
Hence, the theorem is proved. □

5. Some Fibonacci Results of Matrices of Order $2^r$

In this section, we obtain some new Fibonacci identities for matrices of order $2^r$. In order to get these identities, we have used Fibonacci \cite{1} and Lucas matrices \cite{2}.

For the proof of Lemmas 5.1, 5.4, 5.6 and 5.8 see the paper \cite{20}. In particular, Lemma 5.1 contains the well-defined identities of Fibonacci and Lucas numbers.

\textbf{Lemma 5.1.}  (i) $F_{m+n} = F_{m-1}F_n + F_mF_{n+1}$ (ii) $F_{m+n} + F_{m+n+2} = L_{m+n+1}$ (iii) $L_{n+1} + L_{n+3} = 5F_{n+2}$ for all integers $n$

\textbf{Theorem 5.2.}

\begin{align*}
(i) & L^{(2^r)}_{m+n} + L^{(2^r)}_{m+n+2} = 5F^{(2^r)}_{m+n+1} \\
(ii) & F^{(2^r)}_{m+n} + F^{(2^r)}_{m+n+2} = L^{(2^r)}_{n+m+1}
\end{align*}

\textbf{Proof.} First identity of the Theorem 5.2 can be proved by applying induction on $r$, therefor for $r = 1$, we have

$$L^{(2)}_{n+m} + L^{(2)}_{n+m+2} = \begin{pmatrix} L_{n+m+1} & L_{n+m} \\ L_{n+m} & F_{n+m-1} \end{pmatrix} + \begin{pmatrix} L_{n+m+3} & L_{n+m+2} \\ L_{n+m+2} & L_{n+m+1} \end{pmatrix} = \begin{pmatrix} 5F_{n+m+2} & 5F_{n+m+1} \\ 5F_{n+m+1} & 5F_{n+m} \end{pmatrix} = 5F^{(2)}_{n+m+1}$$

Which is true. Thus, we assume that it is true for $r = p$, i.e.

$$L^{(2^p)}_{n+m} + L^{(2^p)}_{n+m+2} = 5F^{(2^p)}_{n+m+1}$$

Therefore, we should prove it for $r = p + 1$,

$$L^{(2^{p+1})}_{n+m} + L^{(2^{p+1})}_{n+m+2} = \begin{pmatrix} L^{(2^p)}_{n+m+1} & L^{(2^p)}_{n+m} \\ L^{(2^p)}_{n+m} & L^{(2^p)}_{n+m-1} \end{pmatrix} + \begin{pmatrix} L^{(2^p)}_{n+m+3} & L^{(2^p)}_{n+m+2} \\ L^{(2^p)}_{n+m+2} & L^{(2^p)}_{n+m+1} \end{pmatrix} = \begin{pmatrix} 5F^{(2^p)}_{n+m+2} & 5F^{(2^p)}_{n+m+1} \\ 5F^{(2^p)}_{n+m+1} & 5F^{(2^p)}_{n+m} \end{pmatrix} = 5F^{(2^{p+1})}_{n+m+1}$$

Hence, we have the result. □

(ii) Second identity of the theorem can be proved in a similar way.

\textbf{Theorem 5.3.}  If $r = 2t$ and $t \geq 1$, then

\begin{align*}
(i) & 5^tF^{(2^t)}_{m+n} = F^{(2^t)}_{m-1}F^{(2^t)}_n + F^{(2^t)}_mF^{(2^t)}_{n+1} \\
(ii) & 5^{t-1}F^{(2^t)}_{m+n} = F^{(2^t)}_{m-1}F^{(2^t)}_n + F^{(2^t)}_mF^{(2^t)}_{n+1}
\end{align*}

If $r = 2s + 1$ and $s \geq 0$, then

\begin{align*}
(i) & 5^{-t}F^{(2^t)}_{m+n} = F^{(2^t)}_{m-1}F^{(2^t)}_n + F^{(2^t)}_mF^{(2^t)}_{n+1} \\
(ii) & 5^{-t-1}F^{(2^t)}_{m+n} = F^{(2^t)}_{m-1}F^{(2^t)}_n + F^{(2^t)}_mF^{(2^t)}_{n+1}
\end{align*}

\textbf{Proof.} Even $r$ can be obtained from odd $r$ by applying induction on $t$. On taking $t = 1$, $r$ becomes $r = 2$ in the identity (i) of the Theorem 5.3, therefore we have

$$F^{(2^2)}_{m-1}F^{(2^2)}_n + F^{(2^2)}_mF^{(2^2)}_{n+1} = \begin{pmatrix} F^{(2)}_m & F^{(2)}_m \\ F^{(2)}_{m-1} & F^{(2)}_{m-2} \end{pmatrix} \begin{pmatrix} F^{(2)}_{n+1} & F^{(2)}_n \\ F^{(2)}_n & F^{(2)}_{n-1} \end{pmatrix} + \begin{pmatrix} F^{(2)}_{m+1} & F^{(2)}_m \\ F^{(2)}_m & F^{(2)}_{m-1} \end{pmatrix} \begin{pmatrix} F^{(2)}_{n+2} & F^{(2)}_n \\ F^{(2)}_n & F^{(2)}_{n+1} \end{pmatrix}$$
By multiplying and simplifying these matrices, we get (see Theorem 5.2)

\[
\begin{pmatrix}
L_{n+m}^{(2)} & L_{n+m-1}^{(2)} \\
L_{n+m-1}^{(2)} & L_{n+m-2}^{(2)}
\end{pmatrix} + \begin{pmatrix}
L_{n+m}^{(2)} & L_{n+m+1}^{(2)} \\
L_{n+m+1}^{(2)} & L_{n+m}^{(2)}
\end{pmatrix} = 5F_{n+m}^{(2)}
\]

Let suppose that Theorem 5.3(i) is true for \( t = p \). We need to show it true for \( t = p + 1 \),

\[
F_{m-1}^{(2p+2)}F_{m}^{(2p+2)} + F_{m}^{(2p+2)}F_{n+1}^{(2p+2)} = \begin{pmatrix}
F_{m}^{(2p+1)} & F_{m-1}^{(2p+1)} \\
F_{m-1}^{(2p+1)} & F_{m-2}^{(2p+1)}
\end{pmatrix} \begin{pmatrix}
F_{n+1}^{(2p+1)} & F_{n}^{(2p+1)} \\
F_{n}^{(2p+1)} & F_{n-1}^{(2p+1)}
\end{pmatrix} + \begin{pmatrix}
F_{m}^{(2p+1)} & F_{m-1}^{(2p+1)} \\
F_{m-1}^{(2p+1)} & F_{m-2}^{(2p+1)}
\end{pmatrix} \begin{pmatrix}
F_{m+1}^{(2p+1)} & F_{m}^{(2p+1)} \\
F_{m}^{(2p+1)} & F_{m-1}^{(2p+1)}
\end{pmatrix}
\]

\[
= 5^{p+1}F_{n+m}^{(2p+2)}
\]

We showed that it is true \( t = p + 1 \).

Identity(ii) of Theorem 5.3 can be proved by induction on \( s \), therefore when \( s = 0 \), \( r \) becomes \( r = 1 \), identity can be written as

\[
F_{m-1}^{(2)}F_{n}^{(2)} + F_{m}^{(2)}F_{n+1}^{(2)}
\]

Then, we have

\[
\begin{pmatrix}
F_{n+m} & F_{n+m-1} \\
F_{n+m-1} & F_{n+m-2}
\end{pmatrix} + \begin{pmatrix}
F_{n+m+2} & F_{n+m+1} \\
F_{n+m+1} & F_{n+m}
\end{pmatrix}
\]

which becomes by lemma 5.1

\[
\begin{pmatrix}
L_{n+m+1} & L_{n+m} \\
L_{n+m} & L_{n+m-1}
\end{pmatrix} = L_{n+m}^{(2)}
\]

Which is absolutely true. For \( s = 1 \), it is also true

\[
F_{m-1}^{(2)}F_{n}^{(2)} + F_{m}^{(2)}F_{n+1}^{(2)} = 5(F_{m+n+1}^{(2)} + F_{m+n+1}^{(2)}) = 5F_{n+m}^{(2)}
\]

Therefore we suppose that Theorem (12(ii)) is true for \( s = q - 1 \). We need to show that it is also true for \( s = q \)

\[
F_{m-1}^{(2q+1)}F_{n}^{(2q+1)} + F_{m}^{(2q+1)}F_{n+1}^{(2q+1)} = F_{m-1}^{(2q)}F_{n}^{(2q)} + F_{m}^{(2q)}F_{m+1}^{(2q)}F_{n+1}^{(2q)}
\]

After simplifying, we have the desire result.

\[
= (F_{n+m+1}^{(2q)}5F_{n+m+1}^{(2q)} + F_{n+m+1}^{(2q)}5^{q}F_{n+m+1}^{(2q)}) = 5^{q}L_{n+m}^{(2q+1)}
\]

\[
\Box
\]

Lemma 5.4.

\[
F_{n} + L_{n} = 2F_{n+1}
\]

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Theorem 5.5.

\[ F_n^{(2^r)} + L_n^{(2^r)} = 2F_{n+1}^{(2^r)} \]

Proof. (Proof by Mathematical Induction on \( r \)) For \( r = 1 \), we have

\[
F_n^{(2)} + L_n^{(2)} = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} + \begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix} = \begin{pmatrix} 2F_{n+2} & 2F_{n+1} \\ 2F_{n+1} & 2F_n \end{pmatrix} = 2F_{n+1}^{(2)}
\]

Which is clearly true. Assume that the identity is true for \( r = n \), i.e.

\[ F_n^{(2^r)} + L_n^{(2^r)} = 2F_{n+1}^{(2^r)} \quad (5.1) \]

Therefore, for \( r = n + 1 \), we need to show that it is true

\[
F_n^{(2^{n+1})} + L_n^{(2^{n+1})} = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} + \begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix} = \begin{pmatrix} 2F_{n+2} & 2F_{n+1} \\ 2F_{n+1} & 2F_n \end{pmatrix} = 2F_{n+1}^{(2^{n+1})}
\]

Hence, we have the result. \( \square \)

Lemma 5.6.

\[ F_{n+1}^2 + F_n^2 = F_{2n+1} \]

Theorem 5.7. If \( r = 2t \) and \( t \geq 1 \), then

\[ (i) 5^r F_{2n+1}^{(2^r)} = (F_{n+1}^{(2^r)})^2 + (F_n^{(2^r)})^2 \]

If \( r = 2s + 1 \) and \( s \geq 0 \), then

\[ (ii) 5^{r-1} F_{2n+1}^{(2^r)} = (F_{n+1}^{(2^r)})^2 + (F_n^{(2^r)})^2 \]

Proof. It is similar to the proof of Theorem 5.3. \( \square \)

Lemma 5.8.

\[ F_{n+1}^2 - F_n^2 = F_{n+2}F_{n-1} \]

Theorem 5.9. If \( r = 2t \) and \( t \geq 1 \), then

\[ (i) 5^r L_{2n+1}^{(2^r)} = (F_{n+1}^{(2^r)})^2 - (F_n^{(2^r)})^2 \]

If \( r = 2s + 1 \) and \( s \geq 0 \), then

\[ (ii) 5^{r-1} L_{2n+1}^{(2^r)} = (F_{n+1}^{(2^r)})^2 - (F_n^{(2^r)})^2 \]

Proof. It follows easily from the proof of Theorem 5.3. \( \square \)

Lemma 5.10.

\[ \sum_{i=1}^{n} F_i^2 = F_n F_{n+1} \]
Theorem 5.11. If \( r = 2s + 1, \ s \geq 0 \) and \( n \geq 1 \), then

\[
(i) \sum_{i=1}^{n} \left( F^{(2r)}_{i} \right)^{2} = 5^{\frac{r-1}{2}} \left( F^{(2r)}_{2n+1} - F^{(2r)}_{1} \right)
\]

If \( r = 2t, \ t \geq 1 \) and \( n \geq 1 \), then

\[
(ii) \sum_{i=1}^{n} \left( F^{(2r)}_{i} \right)^{2} = 5^{\frac{r-2}{2}} \left( L^{(2r)}_{2n+1} - L^{(2r)}_{1} \right)
\]

\[\text{Proof. Since} \]

\[
\left( F^{(2)}_{i} \right)^{2} = \begin{pmatrix} F^{2}_{i+1} + F^{2}_{i} & F_{i}(F^{2}_{i+1} + F^{2}_{i-1}) \\ F^{2}_{i} & F_{i}^{2} + F_{i-1} \end{pmatrix} = \begin{pmatrix} F_{2i+1} & F_{2i} \\ F_{2i} & F_{2i-1} \end{pmatrix}
\]

By the formula 13 on the page 177 of [20], we get

\[
= \begin{pmatrix} F_{2i+1} & F_{2i} \\ F_{2i} & F_{2i-1} \end{pmatrix}
\]

By mathematical induction on \( s \), by taking \( s = 0 \), \( r \) becomes \( r = 1 \), we have

\[
\sum_{i=1}^{n} \left( F^{(2)}_{i} \right)^{2} = \sum_{i=1}^{n} \begin{pmatrix} F^{2}_{2i+1} & F_{2i} \\ F_{2i} & F_{2i-1} \end{pmatrix}
\]

\[
= \sum_{i=1}^{n} F^{(2)}_{2i} = \sum_{i=2}^{2n} F^{(2)}_{i} = F^{(2)}_{2n+1} - F^{(2)}_{1}
\]

It is directly followed by Theorem 5 of \[1\].

Since above expression is true for \( s = 0 \). We assume that it is true for \( s = p - 1 \). We need to prove it for \( s = p \),

\[
\sum_{i=1}^{n} \left( F^{(2p+1)}_{i} \right)^{2} = \sum_{i=1}^{n} \left( F^{(2p)}_{i} \right)^{2}
\]

\[
= \sum_{i=1}^{n} 5^{p} F^{(2p)}_{2i} F^{(2)}_{2i} = 5^{p} \sum_{i=1}^{n} F^{(2p+1)}_{i} = 5^{p} \sum_{i=2}^{2n} F^{(2p+1)}_{i}
\]

\[
= 5^{p} \left( F^{(2p+1)}_{2n+1} - F^{(2p+1)}_{1} \right)
\]

We have proved the first identity of Theorem 5.11.

(ii) Second identity of the Theorem can be prove in a similar way.

\[\square\]
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