Application of the physical theory of diffraction in ultrasonic non-destructive evaluation

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Abstract. We study two canonical problems of interest in non-destructive evaluation, diffraction of a plane elastic wave by a thin crack and diffraction of a plane elastic wave by a wedge, both in the high-frequency regime. In applications this regime is usually treated using the so-called Kirchhoff approximation. It is very easy to implement but there are situations when it is known to give distorted results. We discuss an easy correction procedure, which is applicable not only in geometrical regions but inside penumbrae as well. It involves a version of the Physical Theory of Diffraction that relies on the Geometrical Theory of Diffraction rather than the full solution of the corresponding canonical problem.

1. Introduction

The fields scattered by straight edged obstacles have a complicated structure, particularly in the so-called radiating near field, the first Fresnel zone outside the classical near field zone. In optics, acoustics, electrodynamics and elastodynamics such fields can be modelled well using the high-frequency asymptotics known as the Geometrical Optics/Acoustics/Electrodynamics/Elastodynamics (GO/GA/GE) \cite{1} and Geometrical Theory of Diffraction (GTD) \cite{2}, all employing the concept of a ray. The former theories describe waves carried by incident and reflected rays and the latter, waves diffracted by edges, corners and vertices as well as waves carried by rays which are grazing obstacles. The regions that support different types of waves are classified as either geometrical or transition zones. In a geometrical region, each point is reached by an isolated ray or else by a finite number of isolated rays carrying the same type of wave (the rays are considered coalescing—as opposed to isolated—if the difference between the phases they carry is less than say $\pi$). A transition zone is a boundary layer that lies between an irradiated
geometrical region and the related geometrical shadow. When elastic waves are scattered by straight edged cracks the only types of geometrical regions are those irradiated by longitudinal or transverse incident or reflected waves or else regions irradiated by head waves. Therefore, the only transition zones are the associated penumbras which surround incident and reflected shadow boundaries; neighbourhoods of rays of grazing incidence and reflection; and finally, neighbourhoods of critical rays that mark the boundaries of the head wave regions. GO/GA/GE and GTD combine to give a perfectly adequate description of geometrical regions but are inapplicable inside transition zones. A more sophisticated, Uniform, GTD is required to complete the description. The radiating near field of obstacles with curved edges require additional asymptotics to describe caustic and focal zones (e.g. [3], [4], [5], [6]).

The near radiating field of the scatterer is of practical importance in applications, such as Non-Destructive Evaluation (NDE) of industrial components and structures. When dealing with obstacles of a complicated shape, instead of combining GO/GA/GE and GTD, NDE modellers often simulate the high-frequency scattered fields using the so-called Kirchhoff approximation. This is also referred to as the physical optics approximation and relies on a numerical evaluation of the Kirchhoff-Rayleigh integral (e.g. [7], [8], [9]). Unlike the classical GTD, the Kirchhoff approximation works in the far field and in the near radiating field, it is regular inside penumbras. The resulting leading terms describe the reflected waves, and the terms which describe diffracted waves have the same phases as the GTD asymptotics. However, away from the directions normal to the obstacle edge the Kirchhoff amplitudes of the diffracted waves are quite different to those predicted by GTD [10], [11], particularly in elastodynamics. The elastodynamic Kirchhoff approximation does not describe the contribution of the Rayleigh waves or head waves either.

The Physical Theory of Diffraction (PTD) has been developed to overcome some of these limitations [11]. In PTD, the total field on a scatterer is described similarly to the Kirchhoff approximation, as an GO/GA/GE field, while near the scatterer edge the total field is corrected by adding to it the difference between the solution of the relevant canonical problem and GO/GA/GE. A procedure similar to PTD was developed by McMaken [12] to describe diffraction of elastic waves by embedded planar cracks. However, the canonical displacements on the planar cracks and other elastic wedges are particularly difficult to compute [12], [13], [14]. To overcome this difficulty we offer a modification of PTD and correct the Kirchhoff approximation by employing GTD rather than the full solution of the corresponding canonical problem. The GTD diffraction coefficients can be computed in an efficient manner using algorithms described in [15] and [16]. In this paper we show how this approach can be employed to simulate diffraction by a planar crack and elastic wedge. The proposed method is applicable inside geometrical regions and penumbras but not other transition zones.
2. Scatter by a semi-infinite planar crack

Let us assume that a uniform elastic solid contains a semi-infinite planar crack $S$. Let us introduce the associated Cartesian coordinate system $(x_1, x_2, x_3)$ (see Fig. 1) and cylindrical polar coordinates $(r, \theta, x_3)$, such that we have $x_1 = r \cos \theta, \quad x_2 = r \sin \theta$. Then the crack can be described as $S = \{x : x_1 \geq 0, x_2 = 0\}$, with $x = (x_1, x_2, x_3)$—a point in space and the upper and lower faces of the crack designated, respectively, by $S^+$ and $S^-$. 

Let the crack be irradiated by an incident time-harmonic plane wave. This means that the incident field and all scattered fields contain the common factor $\exp(-i\omega t)$, where $\omega$ denotes the circular frequency, $t$ is time and $i$ is the imaginary unity. For simplicity of presentation we omit this factor throughout.

It is well known that the amplitude $\mathbf{U}(x)$ of the time-harmonic displacement vector field satisfies the reduced equation of motion

$$\nabla \times (\nabla \times \mathbf{U}(x)) + \kappa^2 \nabla \cdot (\nabla \cdot \mathbf{U}(x)) + k_T^2 \mathbf{U}(x) = 0, \quad (1)$$

where $\kappa = c_L/c_T$, $c_L$ and $c_T$ are the respective longitudinal and transverse wave speeds; and $k_L = \omega/c_L$ and $k_T = \omega/c_T$ are the respective longitudinal and transverse wave numbers.

Let us assume that the faces of the crack satisfy the standard traction free boundary conditions

$$\sigma_{12}(x)|_S = \sigma_{22}(x)|_S = \sigma_{23}(x)|_S = 0. \quad (2)$$

Above, $\sigma(x) = \{\sigma_{ij}(x)\}_{i,j=1}^3$ is the stress tensor, which according to Hooke’s law is related to the displacement $\mathbf{U}(x)$ via

$$\sigma_{ij}(x) = \lambda \delta_{ij} \nabla \cdot \mathbf{U}(x) + \mu \left( U_{ij}(x) + U_{ji}(x) \right), \quad i, j = 1, 2, 3, \quad (3)$$

with $\lambda = \rho (c_L^2 - 2c_T^2)$ and $\mu = \rho c_T^2$—the Lamé constants and $\rho$—solid density.

![Figure 1. The geometry of the plane crack.](image)

The incident time-harmonic plane wave can be described by

$$\mathbf{U}^{Inc}(x) = d^p e^{ik_{\alpha}x} p^x. \quad (4)$$
Here we follow the tradition and omit the dependence of $U(x)$ on $p$ and $k_\alpha$. The index $\alpha = L, TV$ or $TH$ refers to the longitudinal, transverse-vertical or transverse-horizontal polarisation of incident wave, respectively. Note that it is a subscript when describing scalars, and superscript otherwise. The vector
denotes the direction of wave propagation and is specified by two angles, $\phi_\alpha$ and $\theta_\alpha$ (Fig. 1). It defines the polarisation vectors
\begin{equation}
d^L = p, \quad d^{TH} = \frac{e_2 \times p}{|e_2 \times p|}, \quad d^{TV} = d^{TH} \times p.
\end{equation}

### 2.1. The high-frequency approximations

Using the well known Green’s theorem the above problem can be reformulated as an integral equation which relates the unknown scattered field $U^{\text{Scat}}(x)$ at an arbitrary point outside the crack to the total field $U(x) = U^{\text{Scat}}(x) + U^{\text{Inc}}(x)$ on its surface which, of course, is also unknown. The component representation of this relationship is given by
\begin{equation}
U_k^{\text{Scat}}(x) = - \int_S \tau^G_{ij,k}(x - x')U_i(x')ds(x'), \quad i, k = 1, 2, 3.
\end{equation}

Above, $x'$ is an arbitrary point on the surface of the crack, $ds(x')$ is an elementary area surrounding $x'$ and $\tau^G$ is the Green’s stress tensor [17]. The integral on the right of (7) is known as the Rayleigh integral.

The problem under consideration can be solved analytically [17]. The high-frequency asymptotics of the resulting scattered field $U^{\text{Scat}}(x)$, which are valid outside penumbral, critical and grazing regions have been shown to be [17]
\begin{equation}
U^{\text{Scat}}(x) = U^{\text{GE}}(x) + U^{\text{Rayleigh}}(x) + U^{\text{GTD}}(x) + O(|kT\pi|^{-1}),
\end{equation}
where the geometrical elastodynamics field $U^{\text{GE}}(x)$ and Rayleigh field $U^{\text{Rayleigh}}(x)$ comprise reflected waves and surface waves, respectively, and the GTD field is
\begin{equation}
U^{\text{GTD}}(x) = \sum_{\beta=L, TV, TH} D^{\text{GTD}}_{\alpha\beta}(\theta, \phi_\alpha, \phi_\beta)\frac{e^{ik_\beta(x_3 \cos \phi_\beta + r \sin \phi_\beta)}}{\sqrt{k_\beta r}}d^\beta(p^\beta).
\end{equation}

Above, $D^{\text{GTD}}_{\alpha\beta}$ are the so-called GTD diffraction coefficients and $p^\beta$ are the directions of diffracted waves. The recipes for calculating $U^{\text{GE}}(x)$, $U^{\text{Rayleigh}}(x)$ and $D^{\text{GTD}}_{\alpha\beta}$ can be found in [17] but need not be reproduced here.

As mentioned in the Introduction, many numerical high-frequency NDE models employ another high-frequency approximation: assuming the boundary $S^+$ to be irradiated and boundary $S^-$ to lie in the geometric shadow ($-\pi < \theta_\alpha < 0$), the total field $U(x)$ in (7) is assumed to be
\begin{equation}
U(x')|_{S^+} \approx A^\alpha e^{ik_\alpha(\sin \phi_\alpha \cos \theta_\alpha x'_3 + \cos \phi_\alpha x'_3)}, \quad U(x')|_{S^-} \approx 0.
\end{equation}
where the vector amplitudes $\mathbf{A}^\alpha$ are approximations to the crack opening displacements given by (A.3) and (A.5). Substituting then the known fields (10) into (7) the scattered fields $\mathbf{U}^{\text{Scat}}(\mathbf{x})$ are evaluated in the Kirchhoff approximation,

$$U_{k}^{\text{Kir}}(\mathbf{x}) = \int_{S^+} \tau_{02,k}(\mathbf{x} - \mathbf{x}')A^\alpha_i e^{ik_\alpha(x'_1 \sin \phi_\alpha + x'_2 \cos \phi_\alpha)} ds(\mathbf{x}').$$

(11)

The Kirchhoff-Rayleigh integrals on the right of (11) are usually evaluated numerically.

As also mentioned in the Introduction, there are geometrical regions where the Kirchhoff approximation deteriorates and GTD performs better. We will now show that the Kirchhoff approximation can be improved by evaluating the Kirchhoff-Rayleigh integral asymptotically and comparing the result to (8). Let us introduce auxiliary matrices

$$B_{ij}^H(\hat{x}) = 2k^{-2}\hat{x}_i\hat{x}_j + (1 - 2k^{-2})\delta_{ij},$$

$$B_{ij}^{TV}(\hat{x}) = \hat{x}_i d_j^{TV} + \hat{x}_j d_i^{TV},$$

(12)

where $\mathbf{x} = |\mathbf{x}|$ and $\hat{x}$ is a unit vector in the direction of $\mathbf{x}$, $\hat{x} = \mathbf{x}/|\mathbf{x}|$. It is convenient to represent the phase of the incident $\alpha$ wave on $S$ in terms of the scattered $\beta$ wave,

$$k_\alpha(x'_1 \sin \phi_\alpha + x'_2 \cos \phi_\alpha) = k_\beta(x'_1 \cos \psi_{\beta}^{\text{GE}} + x'_2 \sin \phi_\beta),$$

(13)

where $\alpha, \beta = L, TV, TH$ and the projections of the reflected $\beta$ ray onto the edge and inner normal to this edge are, respectively,

$$\cos \phi_\beta = k_\alpha \cos \phi_\alpha, \quad \cos \psi_{\beta}^{\text{GE}} = k_\alpha \sin \phi_\alpha \cos \theta_\alpha.$$  

(14)

When the observation point lies outside all penumbrae, the Kirchhoff-Rayleigh integral (11) can be evaluated asymptotically using the standard stationary phase method [4] to give

$$\mathbf{U}^{\text{Kir}}(\mathbf{x}) = \mathbf{U}^{\text{GE}}(\mathbf{x}) + \sum_{\beta=L,TV,TH} D_{\alpha \beta}^{K^2}(\theta, \phi_\alpha, \theta_\alpha) d^3 \frac{e^{ik_\beta(x_3 \cos \phi_\beta + r \sin \phi_\beta)}}{\sqrt{k_\beta r}} + O(|k_{TR}|^{-1}),$$

(15)

where a Kirchhoff diffraction coefficient describing a diffracted $\beta$ wave generated by an incident $\alpha$ waves is

$$D_{\alpha \beta}^{K^2} = -\frac{e^{i\pi}}{2\sqrt{2\pi \sin \phi_\beta}} \frac{B_{ij}^{\beta}(\mathbf{p}^\beta)A^\alpha_i}{\cos \psi_{\beta}^{\text{GE}} - \cos \psi_{\beta}^{\text{Diff}}},$$

(16)

with $\psi_{\beta}^{\text{Diff}}$—the angle between the diffracted $\beta$ ray and inner normal to the edge. Comparing (8) with (15), outside the transition zones we have

$$\mathbf{U}^{\text{Scat}}(\mathbf{x}) = \mathbf{U}^{\text{Kir}}(\mathbf{x}) + \mathbf{U}^{\text{Rayleigh}}(\mathbf{x}) +$$

$$\sum_{\beta=L,TV,TH} D_{\alpha \beta}^{\text{Corr}}(\theta, \phi_\alpha, \theta_\alpha) d^3 \frac{e^{ik_\beta(x_3 \cos \phi_\beta + r \sin \phi_\beta)}}{\sqrt{k_\beta r}} + O(|k_{TR}|^{-1}),$$

(17)
where the correction to the Kirchhoff diffraction coefficient is
\[ D_{\alpha\beta}^{\text{Corr}}(\theta, \phi_\alpha, \phi_\alpha) = D_{\alpha\beta}^{\text{GTD}}(\theta, \phi_\alpha, \phi_\alpha) - D_{\alpha\beta}^{\text{Kir}}(\theta, \phi_\alpha, \phi_\alpha). \] (18)

It is possible to show that this correction applies inside all penumbras as well as inside geometrical regions.

3. The scatter by a two-dimensional elastic wedge
Let us now consider scatter of a plane wave by a two-dimensional infinite wedge of angle $2\Omega$ (Fig. 2). Let us introduce the associated Cartesian coordinate system $(x_1, x_2)$ and polar coordinates $(r, \theta)$, such that we have $x_1 = r \cos \theta$ and $x_2 = r \sin \theta$. Then we can say that the elastic wedge occupies the region \( \{(r, \theta) : -\Omega \leq \theta < \Omega\} \) and designate its boundary by $S = S^+ \cup S^-$, with $S^+$ and $S^+$—its upper face and lower face, respectively. As above, we assume that the wedge faces satisfy the traction free boundary conditions and let the time-harmonic incident wave be of the form (4). This problem can be reduced to an integral equation, which can be solved numerically [13], [14], [16] and then the integral representing the scattered field $U_{\Omega}^{\text{Scat}}(x)$ can be evaluated asymptotically. When the observation point lies outside all transition zones, the evaluation can be carried out using the standard high-frequency asymptotics [4] to give
\[ U_{\Omega}^{\text{Scat}}(x) = U_{\Omega}^{\text{GE}}(x) + U_{\Omega}^{\text{Rayleigh}}(x) + U_{\Omega}^{\text{GTD}}(x) + O([k_T r]^{-1}), \] (19)
where the geometrical elastodynamics term $U_{\Omega}^{\text{GE}}(x)$ is the sum of all multiply reflected plane waves arriving at the observation point, $U_{\Omega}^{\text{Rayleigh}}(x)$ is the sum of two scattered Rayleigh waves and the edge diffraction term $U_{\Omega}^{\text{GTD}}(x)$ is
\[ U_{\Omega}^{\text{GTD}}(x) = \sum_{\beta=L,T,V} D_{\alpha\beta,\Omega}^{\text{GTD}}(\theta, \phi_\alpha) \frac{e^{ik_{\beta} r}}{\sqrt{k_{\beta} r}} d_{\beta}, \] (20)
with the polarisation vectors are $d_L = (\cos \theta, \sin \theta)$ and $d^{TV} = (-\sin \theta, \cos \theta)$.

![Figure 2. Geometry of the wedge problem.](image)

Similarly to the previous section, in NDE, the most widely used approach to calculating the scattered field in the high-frequency regime is the Kirchhoff
Approximation. As above, this means that the problem is reformulated using the Green’s Theorem and the wedge surface displacement in the Rayleigh integral is approximated as the geometrical elastodynamics displacement $U_{GE}^Ω(x)$, that is, as the sum of a finite number of plane waves multiply reflected by the wedge faces $S^±$. These waves can be evaluated by using ray tracing and amplitude calculations to give

$$C_{ij}^{β}d^{β}e^{ikβr\cos(\theta−\theta_{ij}^{β})}, \quad i = 1, 2, \quad β = L, TV, \quad j = 1, 2, ..., N_{β}$$ \hspace{1cm} (21)

where the index $β$ describes the wave polarisation, the first subscript $i$ indicates whether the plane wave is impinging on the irradiated wedge face $S^+$ ($i = 1$) or the other face $S^−$ ($i = 2$) and the second subscript $j$ specifies the number of underlying reflections, with $N_{β}^1$ ($N_{β}^2$) being the maximum number of multiply reflected $β$-plane waves which impinge on the face $S^+$ ($S^−$). Each of these waves is characterised by its polar angle $θ_{ij}^{β}$ and constant amplitude $C_{ij}^{β}$. This implies that the wedge Kirchhoff approximation $U_{Kir}^Ω(x)$ can be represented as a sum of the Kirchhoff-Rayleigh integrals of type (11) evaluated over the wedge faces. Using the non-uniform expansion (15), outside all transition zones, we can write

$$U_{Kir}^Ω(x) = U_{GE}^Ω(x) + \sum_{β=L, TV} D_{αβ,Ω}^{Kir}(θ_1, θ_α) d^{β}e^{ikβr\sqrt{kβr} + O([kr]−1)}, \hspace{1cm} (22)$$

where the Kirchhoff diffraction coefficients are

$$D_{αβ,Ω}^{Kir}(θ_1, θ_α) = 2\sum_{i=1}^{N_{β}^1} \sum_{j=1}^{N_{β}^2} C_{ij}^{β}D_{αβ,Ω}^{Kir}(θ_1, θ_{ij}^{β}). \hspace{1cm} (23)$$

Again, there are geometrical regions where the Kirchhoff approximation deteriorates and GTD performs better. The Kirchhoff approximation can be improved by comparing (22) to (20). This comparison shows that outside all transition regions we have

$$U_{Kir}^Scat(Ω) = U_{Kir}^Ω(x) + U_{Rayleigh}^Ω(x) + \sum_{β=L, TV} D_{αβ,Ω}^{Corr}(θ_1, θ_α) e^{ikβr\sqrt{kβr} + O([kr]−1)}, \hspace{1cm} (24)$$

where the corrected Kirchhoff diffraction coefficients are

$$D_{αβ,Ω}^{Corr}(θ_1, θ_α) = D_{αβ,Ω}^{GTD}(θ_1, θ_α) − D_{αβ,Ω}^{Kir}(θ_1, θ_α).$$

It is possible to show that this correction applies inside all penumbras as well as inside geometrical regions.
4. Numerical results
The GTD diffraction coefficients and corrected Kirchhoff diffraction coefficients for the planar crack and 110° wedge are presented in Figs. 3 and 4, respectively. The wave speeds are assumed to be those of steel, $c_L = 5890\text{ m/s}$ and $c_T = 3210\text{ m/s}$.

As can be seen from the formulae in main text, inside penumbras both Kirchhoff and GTD coefficients are irregular. Figs. 3–5 show that the corrected Kirchhoff diffraction coefficient which is the difference between the two is regular there, that is, the singularities cancel. However, these figures also show that the corrected coefficients for the diffracted transverse waves exhibit unrealistic spikes in regions surrounding critical rays. These could be smoothed
Figure 6. SV45 pulse echo configuration. a) Orientation of the crack (thick black line) is specified by the angle $\alpha$. b) Maximum displacements calculated using GTD (blue line), Kirchhoff approximation (red line) and corrected Kirchhoff approximation (green line).

by developing a fully uniform version of GTD. The problem is more challenging than the one discussed here.

Numerical results presented in Figs. 3–5 confirm that the Kirchhoff-Rayleigh integral gives a good approximation when simulating longitudinal diffracted waves but, as expected, the quality of approximation deteriorates in the regions surrounding critical rays and when the incidence is near grazing (Fig. 5b).

In Fig. 6 we study an SV45 pulse echo configuration. The underlying geometry is shown in Fig. 6a). A transducer beam (indicated by arrows) irradiates a finite sized crack (presented as a thick black line) whose orientation is specified by angle $\alpha$. In Fig. 6b) we plot maximum amplitudes of the received signal (in dB) against the crack orientation angle $\alpha$. We compare three different approximations, GTD (blue line), Kirchhoff (red line) and corrected Kirchhoff (green line). As expected, the corrected Kirchhoff approximation coincides with the Kirchhoff approximation at the angles of specular reflection and with GTD at the angles of oblique incidence.

This study has been motivated by the need to refine the Kirchhoff approximation as used in ultrasonic NDE [7]. The refinement is now implemented in CIVA, a commercial NDE simulation package produced by CEA-LIST. It allows for benchmarking and modelling a wider class of realistic NDE configurations, thus increasing the scope for CIVA’s experimental validation.

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Similarly, when the incident wave is transverse the constant vector amplitude $A$ can be written as

$$A^L = 2\kappa^2 \cos \vartheta_L \left( \frac{\sin 2\vartheta_T \cos \varphi_\alpha - \cos 2\vartheta_T}{\sin 2\vartheta_T \sin \varphi_\alpha} \right),$$

where the Rayleigh function $R(\vartheta_L, \vartheta_T)$ is given by

$$R(\vartheta_L, \vartheta_T) = \sin 2\vartheta_T \sin 2\vartheta_L + \kappa^2 \cos^2 2\vartheta_T.$$

Similarly, when the incident wave is transverse the constant vector amplitudes $A^{SV}$ and $A^{SH}$ can be represented, respectively, by

$$A^{SV} = \begin{pmatrix} -2 \cos \vartheta_T & \kappa^2 \cos 2\vartheta_T \cos \varphi_\alpha \\ \kappa^2 \cos 2\vartheta_T \sin \varphi_\alpha \end{pmatrix}, \quad A^{SH} = \begin{pmatrix} -2 \sin \varphi_\alpha \\ 0 \\ 2 \cos \varphi_\alpha \end{pmatrix}.$$

**Appendix A. The Geometrical Elastodynamics**

On the illuminated surface $S^+$ the geometrical elastodynamics term can be represented as a product of the constant vector $A^\alpha$ and rapidly oscillating exponent (10). Let us introduce a new Cartesian coordinate system $(y_1, y_2, y_3)$, where the $y_2$-axis coincides with the $x_2$-axis, the $(y_1y_2)$-plane is the incidence plane and the $y_3$-axis lies in the $(x_1x_3)$-plane. Let $\vartheta_\alpha$ be the angle which the incident wave vector makes with the $y_2$-axis and let $\varphi_\alpha$ denote the angle between the $y_1$-axis and $x_1$-axis. The angles $(\vartheta_\alpha, \varphi_\alpha)$ are related to the angles $(\varphi_\alpha, \theta_\alpha)$ via

$$\cos \vartheta_\alpha = |\sin \varphi_\alpha \sin \theta_\alpha|, \quad \cos \varphi_\alpha = \frac{\sin \varphi_\alpha \cos \theta_\alpha}{\sqrt{1 + \sin^2 \varphi_\alpha \sin^2 \theta_\alpha}}.$$  \hfill (A.1)

Let us denote by $\vartheta_\beta, \beta = L, T$ the angles which the reflected $\beta$ wave vector makes with the $y_2$-axis, so that we have

$$\sin \vartheta_L = \frac{\kappa_T}{\kappa_L} \sin \vartheta_\alpha, \quad \sin \vartheta_T = \frac{\kappa_T}{\kappa_T} \sin \vartheta_\alpha.$$  \hfill (A.2)

When the incident wave type is longitudinal the constant vector amplitude $A^L$ can be written as

$$A^L = 2\kappa^2 \cos \vartheta_L \left( \frac{\sin 2\vartheta_T \cos \varphi_\alpha - \cos 2\vartheta_T}{\sin 2\vartheta_T \sin \varphi_\alpha} \right),$$

where the Rayleigh function $R(\vartheta_L, \vartheta_T)$ is given by

$$R(\vartheta_L, \vartheta_T) = \sin 2\vartheta_T \sin 2\vartheta_L + \kappa^2 \cos^2 2\vartheta_T.$$  \hfill (A.3)

Similarly, when the incident wave is transverse the constant vector amplitudes $A^{SV}$ and $A^{SH}$ can be represented, respectively, by

$$A^{SV} = \begin{pmatrix} -2 \cos \vartheta_T & \kappa^2 \cos 2\vartheta_T \cos \varphi_\alpha \\ \kappa^2 \cos 2\vartheta_T \sin \varphi_\alpha \end{pmatrix}, \quad A^{SH} = \begin{pmatrix} -2 \sin \varphi_\alpha \\ 0 \\ 2 \cos \varphi_\alpha \end{pmatrix}.$$  \hfill (A.5)