TWO REPRESENTATIONS OF THE FUNDAMENTAL GROUP
AND INVARIANTS OF LENS SPACES

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Abstract. This article is a continuation of work on construction and calculation
various of modifications of invariant based on the use Euclidean metric
values attributed to elements of manifold triangulation. We again address
the well investigated lens spaces as a standard tool for checking the nontriviality
of topological invariants.

1. Introduction

Let us shortly remind the main constructions of works [1], [2], [4].

Let there be given the universal covering of an oriented three-dimensional manifold,
considered as a simplicial complex, in which every simplex is mapped into the
three-dimensional Euclidean space (so that the intersections of their images in \( \mathbb{R}^3 \)
are possible). To each edge is assigned thus a Euclidean length, and each tetrahe-
dron has a sign “plus” or “minus” depending on orientation of its image in \( \mathbb{R}^3 \); the
defect angle (minus algebraic sum of dihedral angles) at each edge is equal to zero
modulo \( 2\pi \). Then we consider infinitesimal translations of vertices of the complex
which lead to infinitesimal changes of edge lengths, while these latter lead, in their
turn, to the infinitesimal changes of defect angles. Thus, the following sequence of
vector spaces arises:

\[
0 \rightarrow \mathfrak{e}_3 \xrightarrow{C} (dx) \xrightarrow{B} (dl) \xrightarrow{A} (d\omega) \xrightarrow{B^T} (\ldots) \xrightarrow{C^T} (\ldots) \rightarrow 0.
\]

(1)

Here \( \mathfrak{e}_3 \) is the Lie algebra of motions of three-dimensional Euclidean space; \((dx)\)
is the space of column vectors of vertex coordinates differentials; \((dl)\) is the space
of column vectors of edge length differentials; \((d\omega)\) is the space of column vectors
of differentials of defect angles. The superscript \( T \) means matrix transposition.
Matrices \( A \), \( B \) and \( C \) consist of partial derivatives. One can see [4] that the
sequence (1) is an algebraic complex (superposition of two next matrices is equal to
zero). Besides, it is possible to show [1] that for manifolds with finite fundamental
group this complex is acyclic.

For lens spaces we consider two sets of representations of fundamental group
(recall that \( \pi_1(L(p, q)) = \mathbb{Z}_p \)):

\[
f_k : \mathbb{Z}_p \rightarrow \{ \text{group of motions of 3-dimensional Euclidean space} \}, \quad (2a)
\]

\[
g_j : \mathbb{Z}_p \rightarrow \{ \text{group of automorphisms of vector spaces entering in (1)} \}, \quad (2b)
\]

\[ j, k = 0, \ldots, p - 1. \]

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Figure 1. Simplicial complex for lens space (all indexes are taken modulo $p$)

The realization of representations $f_k$ was described in detail in [2] (there they were denoted $\varphi_k$). As to representations $g_j$, we will describe them below when we consider the structure of spaces entering in complex (4).

2. Calculation of invariants for $L(p, q)$

In figure 1 we show the simplicial complex (bipyramid) corresponding to the lens space $L(p, q)$. Let index $m$ (in figure) vary from 0 to $p - 1$. Then gluing together $p$ bipyramids numbered by $m$ by identical vertices, edges and 2-simplices we obtain an image of simplicial complex for $\tilde{L}(p, q) = S^3$ in $\mathbb{R}^3$.

Every element of group $\mathbb{Z}_p$ can be represented in the group of motions of three-dimensional Euclidean space as a rotation through angle $2\pi k/p$ around some axis which we denote $x_0$. The integer parameter $k$ numbers the characters of these representations. Thus, in the right Cartesian coordinate system $Ox_1x_{p-1}x_0$ the coordinates of vertices of simplicial complex look like (cf. [2]):

$$B_m \left( \rho \cos \frac{2\pi mk}{p}, \rho \sin \frac{2\pi mk}{p}, 0 \right),$$

$$C_m \left( \sigma \cos \left( \alpha + \frac{2\pi qmk}{p} \right), \sigma \sin \left( \alpha + \frac{2\pi qmk}{p} \right), s \right),$$

where $\alpha$, $\rho$, $\sigma$, $s$ are some parameters that fix, to within the motions of $\mathbb{R}^3$, the location of vertices and hence all geometrical values of the complex such as edge lengths, volumes of tetrahedra, dihedral angles, etc. For instance, a simple calculation shows that the oriented volumes of tetrahedra are determined in this case by
the formula:

\[ V_{C_0 C_1 B_{m+1} B_m} = \frac{4}{\rho \sigma s \sin \frac{\pi k}{p} \sin \frac{\pi q k}{p}} \sin \left( \alpha + \frac{\pi (q - 1 - 2m)k}{p} \right). \]  \hfill (5)

Let us write out now more explicitly some of the vector spaces entering in \( \mathbf{1} \) (due to the symmetry of the complex, it is enough for us to deal only with the first three of them):

\[
\epsilon_3 = \begin{pmatrix}
\frac{d\varphi_1}{dx_1} \\
\frac{d\varphi_{p-1}}{dx_1} \\
\frac{d\varphi_0}{dx_1} \\
\frac{dx_{p-1}}{dx_1}
\end{pmatrix}, \quad (dx) = \begin{pmatrix}
(dx)_0 \\
(dx)_1 \\
\vdots \\
(dx)_{p-1}
\end{pmatrix}, \quad (dl) = \begin{pmatrix}
(dl)_0 \\
(dl)_1 \\
\vdots \\
(dl)_{p-1}
\end{pmatrix}. \hfill (6)
\]

where \( dx_{...} \) and \( d\varphi_{...} \) are infinitesimal translations and rotations in the described above coordinate system \( O x_1 x_{p-1} x_0 \).

Every element of group \( \mathbb{Z}_p \) acts on spaces \( (dx) \) and \( (dl) \) by cyclic shifts as follows: it sends subspaces \( (dx)_m \) and \( (dl)_m \) to \( (dx)_{j m} \) and \( (dl)_{j m} \) respectively. Index \( j \) changes from 0 to \( p - 1 \). So, these actions completely determine representations \( g_j \).

Subspaces \( (dx)_m \) and \( (dl)_m \) look like (all indexes are taken modulo \( p \)):

\[
(dx)_m = \begin{pmatrix}
\frac{dx_B_m}{dx_1} \\
\frac{dy_B_m}{dx_1} \\
\frac{dz_B_m}{dx_1} \\
\frac{dx_{C_m/q}}{dx_1} \\
\frac{dy_{C_m/q}}{dx_1} \\
\frac{dz_{C_m/q}}{dx_1}
\end{pmatrix}, \quad (dl)_m = \begin{pmatrix}
\frac{dl_B m}{dx_1} \\
\frac{dl_B m+1}{dx_1} \\
\vdots \\
\frac{dl_B m+q}{dx_1} \\
\frac{dl_B m+q+1}{dx_1} \\
\frac{dl_B m+q+2}{dx_1}
\end{pmatrix}. \hfill (7)
\]

In general, matrices \( A, B \) and \( C \) specifying the mappings of vector spaces in complex \( \mathbf{1} \) are not block diagonal. In order to make them such we perform unitary change-of-basis transformations in all spaces entering in the complex. Our goal is to split \( \mathbf{1} \) into the direct sum of subcomplexes each of which would correspond to the character of some representation \( g_j \) (see (2)). The unitarity of transformations is necessary to preserve the symmetric form \( \mathbf{1} \) of the complex. Let \( \varepsilon \) denote the primitive root of unity of degree \( p \). Then

- the change-of-basis matrix in space \( \epsilon_3 \) is

\[
U_1 = \begin{pmatrix}
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\
\end{pmatrix}; \hfill (8)
\]
• the change-of-basis matrix in space \((dx)\) is
\[
U_2 = \frac{1}{\sqrt{p}} \begin{pmatrix}
1H_0 & 1H_0 & \cdots & 1H_0 \\
1H_1 & \varepsilon^kH_1 & \cdots & \varepsilon^{(p-1)k}H_1 \\
\vdots & \vdots & \ddots & \vdots \\
1H_{p-1} & \varepsilon^{(p-1)k}H_{p-1} & \cdots & \varepsilon^{(p-1)2k}H_{p-1}
\end{pmatrix},
\]
(9)
where
\[
H_m = U_1 \begin{pmatrix}
\varepsilon^{-mk} & 0 & 0 \\
0 & \varepsilon^{mk} & 0 \\
0 & 0 & 1
\end{pmatrix};
\]
(10)
• the change-of-basis matrix in space \((dl)\) is
\[
U_3 = \frac{1}{\sqrt{p}} \begin{pmatrix}
1I & 1I & \cdots & 1I \\
1I & \varepsilon^kI & \cdots & \varepsilon^{(p-1)k}I \\
\vdots & \vdots & \ddots & \vdots \\
1I & \varepsilon^{(p-1)k}I & \cdots & \varepsilon^{(p-1)2k}I
\end{pmatrix},
\]
(11)
where \(I\) is the identical matrix of size \((p + 2) \times (p + 2)\).

Now, in an obvious way,
\[
C \rightarrow U_2^\dagger CU_1,
\]
\[
B \rightarrow U_3^\dagger BU_2,
\]
\[
A \rightarrow U_3^\dagger AU_3,
\]
and each of matrices \(A\), \(B\) and \(C\) becomes block-diagonal (the rearrangement of rows or/and columns can be required). Thus, the acyclic complex \(1\) splits into the following direct sum of \(p\) acyclic subcomplexes:
\[
0 \rightarrow \left( \frac{d\varphi_j}{dx_j} \right) \xrightarrow{\sqrt{p}C_j} \left( dx_j \right)_j \xrightarrow{B_j} \left( dl_j \right)_j \xrightarrow{A_j} \left( d\omega_j \right)_j \xrightarrow{B_j^\dagger} \left( .. \right)_j \rightarrow 0, \quad (12a)
\]
\[
j = 0, \pm 1 \mod p,
\]
\[
0 \rightarrow \left( dx_j \right)_j \xrightarrow{B_j} \left( dl_j \right)_j \xrightarrow{A_j} \left( d\omega_j \right)_j \xrightarrow{B_j^\dagger} \left( .. \right)_j \rightarrow 0, \quad (12b)
\]
\[
j = 2, \ldots, p - 2.
\]
The superscript \(\dagger\) means operation of Hermitian conjugation.

We define the torsions of complexes \(12\) by formulas:
\[
\mathcal{T}_j = \begin{cases}
p^{-2} |(\det C_j | \overline{\mathcal{C}_j})|^{-2} |(\det D_j | B_j | \overline{\mathcal{C}_j})|^2 (\det c_j | A_j)^{-1}, & j = 0, \pm 1 \mod p, \\
| (\det D_j | B_j | \overline{\mathcal{C}_j})|^2 (\det c_j | A_j)^{-1}, & j = 2, \ldots, p - 2.
\end{cases}
\]
(13)
Here \(C_j\) is a maximal subset of edges such that the restriction of matrix \(A_j\) onto its corresponding subspace of \((dl)\) (we denote it as \(c_j | A_j\)) is nondegenerate. \(\overline{\mathcal{C}_j}\) is the complement of \(C_j\). Sets \(D_j\) and \(\overline{\mathcal{D}_j}\) consist of vertices and are defined similarly.
On the base of torsions (13), one can construct invariant values not changing under the Pachner moves $2 \leftrightarrow 3$ and $1 \leftrightarrow 4$. In our case, these invariants take the form:

$$I_{j,1}(L(p,q)) = \begin{cases} \frac{l_{B_k}^2 B_0 l_{B_1}^2 C_1 \prod_{m=0}^{p-1} l_{B_m}^2 C_0}{\prod_{m=0}^{p-1} 6V_{C_0}C_1B_{m+1}B_m} : k = 1, \ldots, \left\lfloor \frac{p}{2} \right\rfloor, \\ j = 0, \ldots, p-1. \end{cases}$$

(14)

Here $l_{\ldots}$ are usual Euclidean edge lengths, $V_{\ldots}$ are volumes of tetrahedra given by (5). We use definition $I_{j,1}$ to emphasize that, in contrast with work [3], here we deal with nontrivial representations $f_k$. The calculations implied by formula (14) are as follows. Using acyclicity of complexes, we choose subsets $C_j$ and $D_j$. Then we calculate the corresponding minors of matrices $A_j$, $B_j$ and $C_j$. In particular, we use the known expressions for partial derivatives $\frac{\partial \omega}{\partial l}$ (see [2] for details). After this, we find the torsions according to (13) and substitute them into (14).

The computer calculations which have been carried out for first several $p$’s allow us to suppose that the general formula for invariants of lens spaces (14) looks as follows:

$$I_{0,1}(L(p,q)) = \left\{ (-1)^{p-1} \frac{\Delta_4^4}{p^4} : k = 1, \ldots, \left\lfloor \frac{p}{2} \right\rfloor \right\},$$

(15a)

$$I_{\pm 1,1}(L(p,q)) = \left\{ (-1)^{p-1} \frac{\Delta_2^2 \Delta_2^2}{p^4} : k = 1, \ldots, \left\lfloor \frac{p}{2} \right\rfloor \right\},$$

(15b)

$$I_{j,1}(L(p,q)) = \left\{ (-1)^p \Delta_{(j-1)k}^2 \Delta_{j+1}^2 : k = 1, \ldots, \left\lfloor \frac{p}{2} \right\rfloor \right\},$$

(15c)

where we have denoted

$$\Delta_m = 4 \sin \frac{\tau m}{p} \sin \frac{\tau q m}{p}.$$  

(16)

3. Discussion

Let us note once again that the value $\Delta_m$ determined in (16) is nothing but the module of $m$th component of the Reidemeister torsion of $L(p,q)$. It is of course very intriguing to calculate invariants of such kind for other 3- and 4-manifolds.

In the future we plan to investigate similar invariants based on $SL(2)$-solution of the pentagon equation [6], [7]. After that, we plan to move on from lens spaces to manifolds with noncommutative or/infinite fundamental group.

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