Resilient Multi-Dimensional Consensus and Optimization in Adversarial Environment

Jiaqi Yan, Xiuxian Li, Yilin Mo†, and Changyun Wen

Abstract—This paper considers the multi-dimensional consensus and optimization in networked systems, where some of the agents might be misbehaving (or faulty). Despite the influence of these misbehaviors, the healthy agents aim to reach an agreement within the convex hull of their initial states in the consensus setting. To this end, we develop a “safe kernel” updating scheme, where each healthy agent computes a “safe kernel” based on the information from its neighbors, and modifies its state towards this kernel at every step. Assuming that the number of malicious agents is upper bounded, sufficient conditions on the network topology are presented to guarantee the achievement of resilient consensus. Given that the consensus serves as a fundamental objective of many distributed coordination problems, we next investigate the application of the proposed algorithm in distributed optimization. A resilient subgradient descent algorithm is subsequently developed, which combines the aforementioned safe kernel approach with the standard subgradient method. We show that, with appropriate stepizes, the states of benign agents converge to a subset of the convex hull formed by their local minimizers. Some numerical examples are finally provided to verify the theoretical results.

Index Terms—Resilient algorithms, Cyber-security, Robust graphs, Multi-dimensional systems.

I. INTRODUCTION

The past decades have witnessed remarkable research interest in networked systems. One of its fundamental focuses would be the canonical consensus problem, which has been widely investigated, and which can be illustrated in various applications including formation control of mobile robots [1]–[3], data fusion in sensor networks [4], [5], etc. Given a set of autonomous agents (such as vehicles, sensors), such problem seeks a distributed protocol that the agents can utilize to reach a common decision/agreement on the average of their initial opinions [6], [7].

In these years, considerable attention has been paid to the development of consensus algorithms [1], [7], [8]. Such protocols are normally based on the hypothesis that every computing agent is trustworthy and cooperate to follow the algorithms throughout their execution. Nevertheless, as the scale of the network increases, it becomes more difficult to secure every agent. On one hand, autonomous agents will communicate with each other to make control decisions. This opens the system to malicious attacks [9]. On the other hand, some agents may not be willing to follow the given rules if they weigh their private interests more than the public ones. It is reported that such misbehaving agents can either dictate the final consensus value, or prevent the network from reaching an agreement [10]. Given the wide applications of consensus algorithm in safety-critical systems, the need for resilient protocols has received much research interest.

In fact, the issue related to resilient consensus has been addressed in the literature over decades. Most approaches adopt the idea of simply ignoring the suspicious values. For example, Dolev et al. consider the scenario in a complete network, where an approximate agreement is desired in the presence of misbehaving agents [11]. In order to overrule the effects of malicious nodes, a secure updating strategy is proposed. The essential idea is that each normal agent discards the most extreme values in its neighborhood and updates the state based on the remaining ones at any time. Such protocol has then inspired a family of algorithms, namely Mean-Subsequence Reduced (MSR) algorithm [12]–[14]. In a recent work [15], Leblanc et al. modify MSR and present the Weighted Mean-Subsequence-Reduced (W-MSR) algorithm. Different from that in MSR, a normal agent only removes the extreme values that are strictly larger or smaller than its current state. This mechanism results in its own state being kept in the update law and turns out to keep more useful information than MSR. Furthermore, instead of the complete graphs, they analyze W-MSR in more general topologies. A novel property named network robustness is hereby introduced, which characterizes the resilience of W-MSR in terms of the graph structure. Later, Dibaji et al. generalize the above results to second-order systems [16]. The aforementioned strategies ensure the resilient consensus even in adversarial environment. Namely, the final agreement is guaranteed to be within the range limited by the minimum and maximum values of the networks’ initial states.

Observe that most of the existing research on resilient consensus focuses on the scenario where agents’ states are assumed to be scalar variables. This implication however produces crucial limitations in various practical applications such as vehicle formation control on a 2D-plane. A naive way to generalize the results on a scalar system to a multi-dimensional system is to apply MSR or W-MSR to each entry of the state vectors. The region that the final value converges to can be immediately identified as a multi-dimensional “box”. Particularly, each edge of this “box” is limited by the minimum and maximum values of benign agents’ initial state in one dimension. A question thus arises naturally: is this result too conservative? Or are there any alternatives that can provide more accurate convergence results? This paper is devoted to answering these questions.

As reported in [17], in the presence of misbehaviors, no distributed rule can facilitate the exact average consensus of the benign agents’ initial states. As a compromise, in this
paper, we aim to develop a resilient algorithm in multi-dimensional spaces such that it guarantees the agreement on a convex combination of these states. To limit the influence of network misbehaviors on normal agents, a “safe kernel” protocol is developed, where each benign agent creates a “safe kernel” and modifies its state towards this kernel. Under certain topological conditions, the proposed strategy guarantees the healthy agents to reach an agreement within the convex hull of their initial values. As a result, it improves the accuracy of that by simply applying the existing algorithms to each dimension.

Moreover, since the idea behind the canonical consensus serves as a fundamental principle in many distributed coordination settings, the “safe kernel” rule provides a powerful tool in handling misfunctioning components in multi-dimensional spaces. To see this, the resilient distributed optimization is also studied in this work. With the combination of standard subgradient method, the safe kernel technique is accommodated in this scenario and secure the system as well. The theoretical analysis shows the final solution is ensured to be within a subset of the convex hull formed by normal agents’ local minimizers.

Some preliminaries results are contained in our former work [18]. The main differences between this paper and [18] are: 1) This work provides rigorous proof regarding resilient distributed optimization; 2) In [18], each benign agent is required to calculate the vertices of safe kernel at every step, which is usually with high computational cost. Compared with it, the algorithm proposed in this paper avoids the vertex enumeration problem for a polytope and instead only looks for two “critical points” in the safe kernel. Hence, it turns out to be much more lightweight.

Notations:
For a vector $a$, $a_i$ denotes its $i$-th component. For a point set $S \subset \mathbb{R}^d$, let $\text{Conv}(S)$ be its convex hull, namely the set of all convex combinations of the points in $S$. Furthermore, given two positive numbers $x$ and $y$, $x \mod y$ represents the remainder of the Euclidean division of $x$ by $y$.

II. Preliminaries
We start by introducing some technical preliminaries, which would be useful in our further analysis.

A. Robust network
Consider the graph $G = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V}$ is the set of agents, and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges. An edge between agents $i$ and $j$ is denoted by $e_{ij} \in \mathcal{E}$, indicating these two agents can communicate directly with each other. The neighborhood of an agent $i \in \mathcal{V}$ is then defined as

$$
\mathcal{N}_i = \{ j \in \mathcal{V} | e_{ij} \in \mathcal{E} \}.
$$

As one might imagine, there is a close coupling between the network topology and maximum number of tolerable faulty agents. For the resilient algorithm employed in this paper, we characterize its efficiency in terms of network robustness. It was first introduced in [15] and is formally defined below:

**Definition 1.** $(r\text{-robustness})$: A network $G = \{\mathcal{V}, \mathcal{E}\}$ is said to be $r$-robust, if for any pair of disjoint and nonempty subsets $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{V}$, at least one of the following statements hold:

1) There exists an agent in $\mathcal{V}_1$, such that it has at least $r$ neighbors outside $\mathcal{V}_1$;
2) There exists an agent in $\mathcal{V}_2$, such that it has at least $r$ neighbors outside $\mathcal{V}_2$.

**Definition 2.** $(r, s)$-robustness): A network $G = \{\mathcal{V}, \mathcal{E}\}$ is said to be $(r, s)$-robust, if for any pair of disjoint and nonempty subsets $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{V}$, at least one of the following statements holds:

1) Any agent in $\mathcal{V}_1$ has at least $r$ neighbors outside $\mathcal{V}_1$;
2) Any agent in $\mathcal{V}_2$ has at least $s$ neighbors outside $\mathcal{V}_2$;
3) There are no less than $s$ agents in $\mathcal{V}_1 \cup \mathcal{V}_2$, such that each of them has at least $r$ neighbors outside the set it belongs to ($\mathcal{V}_1$ or $\mathcal{V}_2$).

Intuitively, the network robustness is a connectivity measure for graphs. It claims that for any two disjoint and nonempty subsets of agents, there are “many” agents within these sets that have a sufficient number of neighbors outside.

B. Sarymsakov matrix
Another important tool would be the Sarymsakov matrix. Given a row stochastic matrix $A = (a_{ij})$, define the directed graph $G(A)$ associated with it as $G(A) = \{\mathcal{V}, \mathcal{E}\}$, where $(i, j) \in \mathcal{E}$ if and only if $a_{ij} > 0$. For a set $\mathcal{V}' \subset \mathcal{V}$, its one-stage consequent indice [19] is defined by

$$
F_A(\mathcal{V}') = \{ j : a_{ij} > 0 \text{ for some } i \in \mathcal{V}' \}.
$$
Namely, $F_A(\mathcal{V}')$ is the set of nodes who have influence on the ones in $\mathcal{V}'$.

Based on the one-stage consequent indices, the Sarymsakov matrix is formally defined below [20]:

**Definition 3.** (Sarymsakov matrix): A row stochastic matrix $A$ is said to be Sarymsakov, if for any disjoint nonempty sets $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{V}$, one of following statements hold:

1) $F_A(\mathcal{V}_1) \cap F_A(\mathcal{V}_2) \neq \emptyset$;
2) $F_A(\mathcal{V}_1) \cap F_A(\mathcal{V}_2) = \emptyset$ and $|F_A(\mathcal{V}_1) \cup F_A(\mathcal{V}_2)| > |\mathcal{V}_1 \cup \mathcal{V}_2|$.

Regarding the Sarymsakov matrix, it means either that sets $\mathcal{V}_1$ and $\mathcal{V}_2$ have some influencing nodes in common, or they have no common influencing nodes but the number of influencers is greater than that of being influenced.

III. Problem Formulation
Consider a group of $N$ agents who cooperate over the undirected graph $G = \{\mathcal{V}, \mathcal{E}\}$. At any time $k \geq 0$, let $x_i(k) \in \mathbb{R}^d$ denote the current state of agent $i$. The agents are said to reach a (distributed) consensus if and only if there exists a constant $\bar{x}$, such that $\lim_{k \to \infty} x_i(k) = \bar{x}$ holds for every agent $i$. In particular, if $\bar{x} = 1/N \sum_{i=1}^{N} x_i(0)$, an average consensus is achieved.

Observe that many practical applications fit into the framework of average consensus (e.g., [1], [21]). Various strategies
have been developed to facilitate it in the literatures (see [22] and [7] for examples), the details of which are omitted here due to the space limitation.

A. Resilient consensus problem

It is worth noticing that an implicit assumption for the effectiveness of the existing approaches is that all agents are reliable throughout the execution, and cooperate to achieve the desired value. However, as the number of local agents increases, certain concerns arise that make this assumption to be violated. As discussed before, the strong dependence of distributed algorithms on the communication infrastructures creates lots of vulnerabilities for cyber attacks, where the transmitted information might be manipulated by external adversaries. Additionally, “non-participant” agent may exist, who deviates from the normal update rule and sends out self-designed information to its neighbors for its own benefits. Clearly, such misbehaviors would degrade the performance of consensus protocols: they can either prevent the benign agents from reaching a consensus, or manipulate the final agreement to be false. In fact, as shown in [10], a single “stubborn” agent can cause all agents to agree on an arbitrary value, by simply keeping this value constant.

These security concerns lead to the study of resilient consensus protocols. That is, we intend to present a secure strategy to achieve the agreement among healthy agents while raising its resilience so as to avoid being influenced by the network misbehaviors too much. By saying “resilient”, we aim to achieve the following objectives, regardless of the choice of initial states and even in the adversarial environment:

1) Agreement: As k goes to infinity, it holds that \( x^i(k) = \bar{x} \) with some \( \bar{x} \in \mathbb{R}^d \), for any benign agent \( i \);
2) Validity: At any time and for any benign agent, its state remains in the convex hull of all benign agents’ initial values.

We elucidate these conditions as below. Firstly, the states of the benign agents should converge to the same constant value even in the presence of misbehaving ones. In addition, they are not allowed to leave the convex hull of their initial states throughout the procedure. It is observed that if 1D problem is considered, the validity condition would be equivalent to the standard one adopted in the existing literatures [11]–[16]. That is, the states of benign agents should always remain in the interval formed by the minimum and maximum of their initial values. There has been much work proved to be effective in this simple case (e.g., MSR in [11] and W-MSR in [15]). However, few research efforts have been devoted to the more general multi-dimensional systems.

A naive way to tackle this problem is by simply applying the existing scalar protocols to each component of the state vectors. Nevertheless, we should note that the region that benign agents converge to is only guaranteed as a multi-dimensional “box”, each edge of which is limited by the minimum and maximum values of their initial states at one dimension. The validity condition thus fails to be ensured. To see this, we present a 2-dimensional illustration in Fig. 1 indicating this naive way cannot guarantee the convergence to a point inside the convex hull of normal agents’ initial states. Therefore, this paper intends to address this problem and come up with an alternate method satisfying both Conditions 1) and 2).

![Fig. 1: A 2D illustration with agents marked with circles. The location of the node indicates its initial value. With the direct application of existing algorithms to each dimension, the final agreement is ensured to be within the rectangle represented by oblique lines. However, a better solution satisfying the validity condition of converging to the solid triangle is expected.](image)

B. Attack model

We define \( \mathcal{F} \) as the set of malicious/faulty agents. Any agent \( i \in \mathcal{F} \) could either be the adversarial one with the value being manipulated by the attacker, or the non-participant agent who does not follow the standard updating rule. On the other hand, \( \mathcal{B} \) is the collection of benign agents who will always follow the predefined updating strategy and compute the desired function. It is clear that \( \mathcal{B} \cap \mathcal{F} = \emptyset \) and \( \mathcal{B} \cup \mathcal{F} = \mathcal{V} \).

The network misbehaviors could be characterized by the scope of threats:

1) (\( F \)-total attack model) There are at most \( F \) misbehaving agents in the network. That is, \( |\mathcal{F}| \leq F \).
2) (\( F \)-local attack model) There are at most \( F \) misbehaving agents in the neighborhood of any agent. That is, \( |\mathcal{F} \cap \mathcal{N}_i| \leq F \), for any agent \( i \in \mathcal{V} \).

This paper focuses on the worst-case situation, where no restrictions are imposed on the transmitted information of agent \( i \in \mathcal{F} \). Namely both adversarial and non-participant agents are allowed to send out arbitrary and different data to their neighbors. Furthermore, the faulty agents could also collude among themselves to decide on the deceptive values to be communicated.

In this paper, we impose the following assumption on network topology:

**Assumption 1.** For any \( i \in \mathcal{V} \), it is held that \( |\mathcal{N}_i| \geq (d + 1)F + 1 \).

IV. A RESILIENT MULTI-DIMENSIONAL CONSENSUS STRATEGY

To simplify notations, the following definitions are given beforehand:

**Definition 4.** Consider a set \( \mathcal{A} \subset \mathbb{R}^d \) with cardinality \( m \).

For some \( n \in \mathbb{Z}_{\geq 0} \) and \( n \leq m \), let \( S(\mathcal{A}, n) \) be the set of all its subset with cardinality \( m - n \).

\(^1\)To be more precise, \( \mathcal{A} \) should be defined as a multi-set since we allow duplicate elements in the set, e.g., the states of \( m \) agents shall be counted as \( m \) points even if some of them may be identical.
It is clear that the set $S(A, n)$ contains $\binom{m}{n}$ elements, and each of them is associated with a convex hull. The intersection of all these convex hulls plays a crucial role in our algorithm, which is formally defined below:

**Definition 5.** Consider a set $A \subseteq \mathbb{R}^d$ with cardinality $m$. For some $n \in \mathbb{Z}_{\geq 0}$ and $n \leq m$, we define $\Psi(A, n)$ as

$$\Psi(A, n) \triangleq \bigcap_{S \in S(A, n)} \text{Conv}(S).$$

In view of Definition 5, $\Psi(A, n)$ is a subset of convex hulls formed by any $m - n$ points in $A$.

A. Description of the resilient algorithm

In this part, we shall provide a resilient solution to the multi-dimensional consensus. Each normal agent $i \in B$ starts with an initial state $x_i(0) \in \mathbb{R}^d$. At any instant $k > 0$, it makes updates as outlined in Algorithm 1.

**Algorithm 1** Resilient multi-dimensional consensus algorithm

1: Receive the states from all neighboring agents $j \in N_i$, and collect these values in $X_i(k)$.
2: Compute the set $\Psi(X_i(k), F)$. For brevity, let us denote this set to be $S^i(k)$.
3: Let $p = k \mod d$. For each neighboring state $x \in X_i(k)$, calculate $l_p(x) = e_p^T x$, where $e_p$ is the $p$-th canonical basis vector in $\mathbb{R}^d$.
4: Among all $l_p(x)$'s, let $m^i$ be $F + 1$-th largest value and $l^i$ be the $dF + 1$ smallest value. Find $y^i(k) \in S^i(k)$, such that

$$l_p(y^i(k)) \leq m^i. \quad (2)$$

5: Similarly, among all $l_p(x)$'s, let $M^i$ be $F + 1$-th smallest value and $M^i$ be the $dF + 1$ largest value. Find $z^i(k) \in S^i(k)$, such that

$$l_p(z^i(k)) \geq M^i. \quad (3)$$

6: Agent $i$ updates its local state as:

$$x_i(k + 1) = \frac{x_i(k) + y^i(k) + z^i(k)}{3}. \quad (4)$$

7: Transmit the new state $x_i(k + 1)$ to all neighbors $j \in N_i$.

From the definition of $l_p(x)$, clearly it returns the $p$-th entry of $x$. Hence at each time step, the normal agents search on one dimension and sort the received values at this dimension. We shall show later (in Remark 1) that the calculation of $y^i(k)$ and $z^i(k)$ only involves exactly $d(F + 1) + 1$ points in $X_i(k)$ and thus the proposed protocol is much more lightweight than the ones proposed in [13]. Finally, as every fault-free agent is only required to access the information in its neighborhood, Algorithm 1 can be implemented in a distributive manner.

B. Relationship with the existing algorithms in scalar systems

Algorithm 1 adopts a similar idea to that in most resilient protocols. At every step, the normal agent $i$ obtains the states in its neighborhood, whereas up to $F$ of them might be faulty. To ensure its state updated in a safe manner, agent $i$ hopes to exclude the misleading information. Yet as it has no knowledge on the identities of these values, it intends to ignore the most extreme ones. In scalar systems, it is natural to refer “extreme” as the largest or smallest numbers. However, as the vector space is associated with a partial order, how to define "extreme" is one of the most challenges in resilient multi-dimensional algorithms.

To address this issue, we develop the aforementioned technique. $S^i(k)$ can be interpreted as a “safe kernel” illustrated in Fig. 2. By Definitions 4 and 5, it intuitively ignores the combination of any $F$ values. At any time, the healthy agent computes and moves its state toward this kernel. The impact of malicious agents on the benign ones is thus limited, with formal proof given in Theorem 1.

![Fig. 2: A 2D illustration of “safe kernel”. Suppose agent $i \in B$ has 5 neighbors and each of their states is represented by the location of a circle. Let $F = 1$. The green region denotes $S^i(k) = \Psi(X_i(k), 1)$, namely the safe kernel.](image)

Finally, as that in the standard W-MSR, the update law (4) always involves the normal agent’s own state. As claimed in [15], this mechanism helps to keep more useful information at each step.

V. ALGORITHM ANALYSIS

This section is devoted to the theoretical analysis of Algorithm 1. We shall show that the proposed algorithm is both realizable and effective.

A. Realizability

In order to demonstrate its realizability, we first prove that $S^i(k)$ is nonempty. To this end, it is helpful to introduce Helly’s Theorem as below, which is a key supporting technique of this paper:

**Helly’s Theorem** [24]. Let $X_1, \cdots, X_p$ be a finite collection of convex subsets in $\mathbb{R}^d$, with $p > d$. If the intersection of every $d + 1$ of these sets is nonempty, then the whole collection has a nonempty intersection. That is,

$$\bigcap_{j=1}^{p} X_j \neq \emptyset.$$

Below is an immediate result of Helly’s Theorem:
Corollary 1. Let $A$ be a set with cardinality $m$ in $\mathbb{R}^d$. For any $n \in \mathbb{Z}_{\geq 0}$, if $m \geq (n+d+1)$, then the following relation holds

$$\Psi(A, n) \neq \emptyset.$$  

Proof. See appendix.  

Invoking Corollary 1, Assumption 1 guarantees $S_i^r(k) \neq \emptyset$ at any step.

Next we need to show the existence of $y^r_i(k)$ and $z^r_i(k)$. To see this, the below results would be helpful, the proof of which are given in the appendix:

Lemma 1. Consider two collections of sets $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$. If for any $j \in J$, there exists an $i^* \in I$ such that $A_{i^*} \subset B_j$, then

$$\bigcap_{i \in I} A_i \subset \bigcap_{j \in J} B_j.$$  

Lemma 2. Consider any set $A_1$ with cardinality $m_1$ and $A_2$ with cardinality $m_2$. If $A_1 \subset A_2$, then for any $n \leq m_1$, the following statement holds:

$$\Psi(A_1, n) \subset \Psi(A_2, n).$$  

Lemma 3. Let $A$ be a set with $|A| \geq (d+1)n + 1$. The following relations hold for an arbitrary linear function $l(x)$:

1) If there exists at least $dn + 1$ points $\bar{x}$ in $A$, such that $l(\bar{x}) \leq m$, then there exists a point $y \in \Psi(A, n)$, such that $l(y) \leq m$;

2) If there exists at least $dn + 1$ points $\bar{x}$ in $A$, such that $l(\bar{x}) \geq M$, then there exists a point $z \in \Psi(A, n)$, such that $l(z) \geq M$.

Proof. By Corollary 1 $\Psi(A, n) \neq \emptyset$. We then show the rationale of the statements as follows:

1) Define a set $B \subset A$ with $|B| = (d+1)n + 1$, such that it includes $dn + 1$ points $\bar{x}$ with $l(\bar{x}) \leq m$. We then consider the convex hull formed by these $\bar{x}$’s. Clearly, any element $x$ in this convex hull also has $l(x) \leq m$. We could infer from Definition 5 that $\Psi(B, n)$ is a subset of this convex hull, and thus the first statement holds by invoking Lemma 2.

2) The second statement is proved in a similar manner as above.  

As a result of Lemma 3, one can always find the points $y^r_i(k)$ and $z^r_i(k)$ such that (2) and (3) hold.

Remark 1. The constructive proof of Lemma 3 also sheds light on the calculations of $y^r_i(k)$ and $z^r_i(k)$. Namely, the normal agent $i$ could pick $(d+1)F + 1$ points in $X^r_i(k)$ with $dF + 1$ of them having $p$-th component upper bounded by $y^r_i(k)$ (this can be achieved by sorting only). If these points are collected in a new set $Y^r_i(k)$, then any point in $\Psi(Y^r_i(k), F)$ can be $y^r_i(k)$. A similar procedure would also be conducted to compute $z^r_i(k)$.

B. Resiliency

Now we are ready to establish the main results. For simplicity, we denote the convex hull formed by the benign agents’ states at time $k$ as $\Omega(k)$. The following proposition presents the non-expansion property of $\Omega(k)$:

**Proposition 1** (Validity). Consider the network $G = (V, E)$. Suppose the misbehaving agents follow either $F$-local or $F$-total attack model. With Algorithm 1 the following relation holds at any $k \geq 0$:

$$\Omega(k + 1) \subset \Omega(k).$$  

Proof. Consider the scenario under either $F$-local or $F$-total attacks. For a benign agent $i$, there exist no less than $|X^r_i(k)| - F$ benign ones in its neighborhood. By definitions, one obtains that $S^r_i(k)$ is included in the convex hull formed by any $|X^r_i(k)| - F$ neighboring values. Hence, it is trivial to derive that $S^r_i(k)$ is a subset of the convex hull of the benign neighbors’ states, that is, $S^r_i(k) \subset \Omega(k)$. Therefore, one directly has that $x^r_i(k + 1) \in \Omega(k)$ as it is a convex combination of some points in $\Omega(k)$.

Because the above relation holds for any normal node, one has $\Omega(k + 1) \subset \Omega(k)$.

Hence, Algorithm 1 guarantees the validity condition of resilient consensus. That is, the healthy agents would never be out of the convex hull of their initial values, namely $\Omega(0)$, despite the influence of the misbehaving ones. In what follows, we will provide sufficient conditions on network topology, under which the agreement condition will also be satisfied.

**Proposition 2** (Agreement). Consider the network $G = (V, E)$. Suppose the misbehaving agents follow an $F$-local attack model. If the network is with $(d+1)F + 1)$-robustness, then with Algorithm 1 all the benign agents are guaranteed to achieve consensus exponentially, regardless of the actions of misbehaving agents.

Proof. To proceed, at any $p \in \{1, 2, ..., d\}$, let us respectively denote $m_p(k)$ and $M_p(k)$ as the minimum and maximum value among the $p$-th components of normal agents’ states at time $k$. That is,

$$m_p(k) \triangleq \min_{i \in B} x^r_{i,p}(k),$$

$$M_p(k) \triangleq \max_{i \in B} x^r_{i,p}(k).$$  

To establish the achievement of consensus, it is equivalent to prove that the benign agents reach an agreement at any dimension. Due to the symmetry between different dimensions, without loss of generality, we would only focus on the first component of local states. The temporal difference is thereby defined as $\Delta_1(k) = M_1(k) - m_1(k)$. We attempt to show that $\Delta_1(k)$ asymptotically approaches 0.

For notation convenience, the following definitions are further imposed for any $k \geq k_0$ and any $\epsilon \in \mathbb{R}$:

$$\mathcal{V}^M(k, \bar{k}, \epsilon) \triangleq \{i \in \mathcal{V} : x^r_{i,1}(\bar{k}) > M_1(k) - \epsilon\},$$

$$\mathcal{V}^m(k, \bar{k}, \epsilon) \triangleq \{i \in \mathcal{V} : x^r_{i,1}(\bar{k}) < m_1(k) + \epsilon\}.$$  

Note that the subscript is dropped in the above notations for the sake of brevity. Clearly, $\mathcal{V}^M(k, \bar{k}, \epsilon)$ [resp. $\mathcal{V}^m(k, \bar{k}, \epsilon)$] includes all agents whose state’s first component is greater
[resp. less] than $M(k) - \epsilon \leq M(k) - \epsilon$ at time $\tilde{k}$. We then define

$$B^M(k, \tilde{k}, \epsilon) \equiv Y^M(k, \tilde{k}, \epsilon) \cap B,$$
$$B^n(k, \tilde{k}, \epsilon) \equiv Y^n(k, \tilde{k}, \epsilon) \cap B,$$

which contains only benign agents in $Y^M(k, \tilde{k}, \epsilon)$ and $Y^n(k, \tilde{k}, \epsilon)$, respectively.

Suppose that $M_1(k) \neq m_1(k)$, i.e., $\Delta_1(k) > 0$ at some time step $k$ with $k \mod d = 1$. Define $\epsilon_0 = \Delta_1(k)/2$. It is easy to know that $B^M(k, k, \epsilon_0)$ and $B^n(k, k, \epsilon_0)$ are disjoint. Furthermore, since each of these sets contains a benign agent with the first component being $M_1(k)$ or $m_1(k)$, both of them are nonempty. As the network is $((d + 1)F + 1)$-robust, there exists one benign node in either $B^M(k, k, \epsilon_0)$ or $B^n(k, k, \epsilon_0)$ that has at least $(d + 1)F + 1$ neighboring agents outside its set.

Without loss of generality, let $i \in B^M(k, k, \epsilon_0)$ be such an agent who has no less than $(d + 1)F + 1$ neighbors in $Y^M(k, k, \epsilon_0)$. Moreover, under the $F$-local attack model, no less than $dF + 1$ points in agent $i$'s neighborhood have their first components upper bounded by $M_1(k) - \epsilon_0$. Therefore one has that $m_i^1 \leq M_1(k) - \epsilon_0$.

As $y_i^1(k) \leq m_i^1 \leq M_1(k) - \epsilon_0$, we obtain that

$$x_i^1(k + 1) = \frac{1}{3}(x_i^1(k) + y_i^1(k) + z_i^1(k))$$
$$\leq \frac{2}{3}M_1(k) + \frac{1}{3}(M_1(k) - \epsilon_0)$$
$$= M_1(k) - \frac{1}{3}\epsilon_0.$$ (9)

It is pointed out that this upper bound also applies to any benign agent in $Y^M(k, k, \epsilon_0)$, as it will apply its own value for updates.

Similarly, if the benign agent $j \in B^n(k, k, \epsilon_0)$ has at least $(d + 1)F + 1$ neighbors outside its set, we know that $z_j^1(k) \geq M^j \geq m_1(k) + \epsilon_0$ and shall have an analogous result that $x_j^1(k + 1) \geq m_1(k) + \epsilon_0/3$, which again is the lower bound for every benign agent in $Y^M(k, k, \epsilon_0)$.

Define $\epsilon_1 = \epsilon_0/3$. From former discussions, one knows that at least one benign agent in $B^n(k, k, \epsilon_0)$ has its first component decreased to below $M_1(k) - \epsilon_1$, or one benign agent in $B^M(k, k, \epsilon_0)$ has its first component increased to above $m_1(k) + \epsilon_1$, or both. As a result, it must be either $B^M(k, k + 1, \epsilon_1) \subseteq B^M(k, k, \epsilon_0)$, or $B^n(k, k + 1, \epsilon_1) \subseteq B^n(k, k, \epsilon_0)$, or both.

Then consider the update at $k + 2$. For any normal agent in $Y \setminus Y^M(k, k + 1, \epsilon_1)$, it is trivial to see that

$$x_i^1(k + 2) = \frac{1}{3}(x_i^1(k + 1) + y_i^1(k + 1) + z_i^1(k + 1))$$
$$\leq \frac{1}{3}(M_1(k) - \epsilon_1) + \frac{2}{3}M_1(k + 1)$$
$$\leq \frac{1}{3}(M_1(k) - \epsilon_1) + \frac{2}{3}M_1(k)$$
$$= M_1(k) - \epsilon_2,$$ (10)

with $\epsilon_2 = \epsilon_1/3$. Thereby, $B^M(k, k + 2, \epsilon_2) \subseteq B^M(k, k + 1, \epsilon_1)$. Hence, for each $1 \leq t \leq d$, we can recursively define $\epsilon_t = \epsilon_0/3^t$ and obtain that $B^M(k, k + d, \epsilon_d) \subseteq B^M(k, k + 1, \epsilon_1) \subseteq B^M(k, k + d, \epsilon_d)$.

Similarly, $B^n(k, k + d + \epsilon_d) \subseteq B^n(k, k, \epsilon_0)$. Note that $\epsilon_d < \epsilon_0$, and therefore the sets $B^M(k, k + d, \epsilon_d)$ and $B^n(k, k + d, \epsilon_d)$ are still disjoint. If both sets are nonempty, as above, one can conclude that at least one of the following statements is true: $B^M(k, k + 2d, \epsilon_{2d}) \subseteq B^M(k, k + d, \epsilon_d)$, or $B^n(k, k + 2d, \epsilon_{2d}) \subseteq B^n(k, k + d, \epsilon_d)$.

Hence, for any $k \geq 1$, as long as both $B^M(k, k + kd, \epsilon_{kd})$ and $B^n(k, k + kd, \epsilon_{kd})$ are nonempty, we can repeat the above analysis and conclude that at least one of these two sets will shrink at the next time step. Since $|B^M(k, k, \epsilon_0)| + |B^n(k, k, \epsilon_0)| \leq |B|$, there must be the case that either $B^M(k, k + d|B|, \epsilon_{d|B|}) = \emptyset$, or $B^n(k, k + d|B|, \epsilon_{d|B|}) = \emptyset$, or both. We assume the former statement holds. From (7), at time step $k + d|B|$, all the fault-free agents have their first elements being at most $M_1(k) - \epsilon_{d|B|}$, i.e., $M_1(k + d|B|) \leq M_1(k) - \epsilon_{d|B|}$. On the other hand, from Proposition 1 we have $m_1(k + d|B|) \geq m_1(k)$. As a result,

$$\Delta_1(k + d|B|) \leq \left(1 - \frac{1}{2 \cdot 3^{d|B|}}\right)\Delta_1(k).$$ (11)

Therefore we conclude that $\Delta_1(k)$ vanishes exponentially.

The next result elaborates a different condition for the proposed algorithm to succeed in $F$-total threats:

**Proposition 3** (Agreement). Consider the network $G = (\mathcal{V}, \mathcal{E})$. Suppose the misbehaving agents follow an $F$-total attack model. If the network is with $(dF + 1, F + 1)$-robust, then with Algorithm 1 all the benign agents are guaranteed to achieve consensus exponentially, regardless of the actions of misbehaving agents.

**Proof.** See appendix.

**Remark 2.** By definitions, we note that a $((d + 1)F + 1)$-robust graph is $(dF + 1, F + 1)$-robust as well, but not vice versa. That is to say, the network which is able to tolerate $F$-local attacks could also survive the $F$-total ones, while the converse is not true. This observation is consistent with the fact that the $F$-globally bounded threats are special versions of locally bounded ones.

Given the above results, one thus immediately concludes that the proposed algorithm facilitates the resilient consensus, as stated below:

**Theorem 1.** Consider the network $G = (\mathcal{V}, \mathcal{E})$. Suppose the network satisfies one of the following conditions:
1) under $F$-local attack model, and is $((d + 1)F + 1)$-robust,
2) under $F$-total attack model, and is $(dF + 1, F + 1)$-robust.

With Algorithm 1 benign agents exponentially achieve the resilient consensus, regardless of the actions of misbehaving ones. That is, as $k \to \infty$,

$$x^i(k) = x^\tilde{i}(k) = \hat{x} \text{ for any } i, j \in B,$$ (12)

where $\hat{x} \in \Omega(0)$.

**Proof.** The theorem is immediately achieved as both the validity and agreement conditions have been established in Propositions 1, 3.
Theorem 1 indicates that under certain topological conditions, the safe kernel approach guarantees all benign agents reach an agreement on a weighted average of their initial states. It protects the local states of benign agents from being driven to arbitrary values, and thus could withstand the compromise of partial agents while providing a desired level of security. Furthermore, since its convergence does not depend on the actions of misbehaving agents, it works effectively even in the worst-case scenario, where the faulty agents could have full knowledge of graph topology, updating rules, etc. and could also be Byzantine agents that are able to send different information to different neighbors. Finally, let us consider the scenario when there is no faulty agents at all, i.e., $|\mathcal{F}| = 0$. Theorem 1 shows that the proposed strategy only guarantees a ‘decent’ solution (i.e., within $\Omega(1)$) instead of the exact average value. This implies that in order to increase the system’s resilience, we sacrifice its performance during normal operations. Hence, there exists a trade-off between security and optimality.

### C. Discussions on the network failing to meet sufficient conditions

Observe that in Assumption 1, the network is assumed to have large connectivity. Furthermore, the resiliency analysis shows the required network robustness increases linearly with the dimension of state. Thus, one might ask: what if the network is not “connected” or “robust” enough to meet these assumptions?

A naive way to handle this problem works as follows. Suppose the given network can tolerate $F$ (locally/globally) faulty agents only when the system dimensionality is no more than $d’(<d)$. Then one could group every $d’$ of the $d$ components together (if $d$ is not divisible by $d’$, then there is a single group whose cardinality is strictly less than $d’$). At any time step, Algorithm 1 is applied within each group. The updated results will then be rejoined in order as a $d$-dimensional vector to be broadcast to its neighbors. It is worth pointing out that this approach fails to guarantee the validity condition but instead only restricts the achieved agreement within the convex hull on every $d’$ (or less) dimensions in a group. Particularly, if $d’ = 1$, its performance will degrade to that of directly applying W-MSR.

Another possible solution is to add some “trusted agents” in the network, which always follow the prescribed rules and cannot be compromised by any attacker. Under the scenario where the network holds a subset of such agents, the procedure of creating the safe kernel could be much simplified. That is, for any normal agent, if it has at least one trusted neighbor, then the safe kernel is the convex hull formed only by these trusted neighboring nodes. Assumption 1 can be further relaxed as: For any $i \in \mathcal{V}$, it either contains at least one trusted neighbor or holds that $|\mathcal{N}_i| \geq (d + 1)F + 1$.

### VI. DISTRIBUTED OPTIMIZATION IN ADVERSARIAL ENVIRONMENT

We note that the idea behind consensus is a fundamental principle for the design of many distributed coordination algorithms, such as distributed optimization, distributed formation, etc. Hence the “safe kernel” technique can be accommodated in these consensus-based scenarios while increasing their security and resilience as well. In this sense, it serves as a tool in working against the security breaches in multi-dimensional spaces.

To see this, let us take the following optimization problem as an example:

$$\min_x \frac{1}{N} \sum_{i=1}^{N} f^i(x),$$

(13)

where $f^i: \mathbb{R}^d \to \mathbb{R}$, only known by agent $i$, is the multi-dimensional convex function. The agents aim to cooperatively solve the above problem with only local information.

As that in the standard consensus settings, the traditional optimization algorithms are very sensitive to network misbehaviors. Actually, it is always possible for an adversary to arbitrarily affect the outcome of optimization while avoid being detected.

#### A. Description of resilient subgradient algorithm

To cope with the security issues, a resilient solution is outlined in Algorithm 2 which combines the “safe kernel” rule with the standard subgradient method (see [22], [26]).

**Algorithm 2** Resilient subgradient descent algorithm

1-5: Steps 1 to 5 are the same as those in Algorithm 1.
6: Agent $i$ updates its local state as:

$$x^i(k + 1) = \frac{x^i(k) + y^i(k) + z^i(k)}{3} - \beta_k d^i(k),$$

(14)

where $\beta_k > 0$ and $d^i(k)$ stands for the subgradient of $f^i$ at $(x^i(k) + y^i(k) + z^i(k))/3$.
7: Transmit the updated state $x^i(k + 1)$ to all neighbors $j \in \mathcal{N}_i$.

Form a point set as

$$\mathcal{M} \triangleq \{x^i | x^i \in \text{argmin}_{x} f^i(x), i \in \mathcal{B}\},$$

which contains the minimizers of all benign agents’ local functions. We further adopt the following standard assumptions:

**Assumption 2.** Each local function $f^i$ is with bounded subgradient. That is, there exists a scalar $L > 0$, such that the following relation holds for any $d_i(x) \in \partial f^i(x)$, and all $i \in \mathcal{V}$:

$$||d_i(x)|| \leq L.$$  

(15)

**Assumption 3.** $\mathcal{M}$ is bounded. Furthermore, the stepsize $\{\beta_k\}$ is nonincreasing, satisfying that $\lim_{k \to \infty} \beta_k = 0$ and $\sum_{k=0}^{\infty} \beta_k = \infty$.

#### B. Algorithm analysis

Then we shall investigate the results of the proposed solution. Given the subgradient term in (14), identifying the region where the final states converge to is one of the main focuses.

As before, this section also considers the scenarios under the below conditions:
C1) under $F$-local attack model, and is $(d+1)F+1)$-robust, C2) under $F$-total attack model, and is $(dF+1, F+1)$-robust.

It can be readily established from Proposition that the safe kernel $S_l(k)$ is contained in the convex hull formed by only normal agents’ states. Hence (14) is equivalent to

$$x^i(k+1) = \sum_{j \in B} \bar{a}_{ij}^i(k)x^j(k) - \beta_k d^i(k),$$

(16)

where $\bar{a}_{ij}^i(k) \geq 0$ and $\sum_{j \in B} \bar{a}_{ij}^i(k) = 1$. For any $k \geq 0$, let us denote $\bar{A}(k) = (\bar{a}_{ij}^i(k)) \in \mathbb{R}^{[B] \times [B]}$, which is row-stochastic. The following result on $\bar{A}(k)$ can be further obtained:

**Lemma 4.** Consider the network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ satisfying Condition C1) or C2). With Algorithm 2 let $\bar{A}(k)$ be the one established by (16) at time $k$. Then each $\bar{A}(k)$ is a Sarymsakov matrix.

**Proof.** See appendix.

To proceed, it is also helpful to introduce the following definition [27].

**Definition 6.** Given a sequence of row-stochastic matrices $\{R(k)\}$, a sequence of stochastic vectors $\{r(k)\}$ is called an absolute probability sequence for $\{R(k)\}$ if

$$r^T(k+1)R(k) = r^T(k), \forall k \geq 0.$$

It has been reported in [27] that for any sequence of row-stochastic matrices, the absolute probability sequence always exists. Consequently, there exists a sequence of stochastic vectors $\{\pi(k)\}$ such that

$$\pi^T(k+1)\bar{A}(k) = \pi^T(k), \forall k \geq 0.$$

(17)

Multiplying $\bar{A}(k), \bar{A}(k-1), \cdots, \bar{A}(0)$ successively on the right hand of (17) yields

$$\pi^T(k+1)\bar{A}(k, 0) = \pi^T(0),$$

(18)

where $\bar{A}(k, s) \triangleq \bar{A}(k)\bar{A}(k-1)\cdots\bar{A}(s)$ for any $k \geq s$.

**Lemma 5.** Consider the network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Suppose the network satisfies Condition C1) or C2). With Algorithm 2 the limit of $\pi(k)$ exists.

**Proof.** See appendix.

Thereby, there exists some $\pi_\infty = [\pi_{\infty, 1}, \cdots, \pi_{\infty, |B|}]^T$ such that $\lim_{k \to \infty} \pi(k)$. Note that $\{\pi(k)\}$ in (17) is in general not unique, which implies its limit $\pi_\infty$ is not unique as well. Let the set of optimizers of minimizing $\sum_{i \in B} \pi_{\infty, i}f^i(x)$ be $\mathcal{X}_\pi^*$. Define $\mathcal{X}^*$ as the union of all possible $\mathcal{X}_\pi^*$, i.e.,

$$\mathcal{X}^* \triangleq \bigcup_{\pi_\infty} \mathcal{X}_\pi^* \subset \text{Conv} (\mathcal{M}).$$

It is now ready to provide the main result. In particular, a resilient solution within $\mathcal{X}^*$ is guaranteed to be achieved, as formally stated below.

**Theorem 2.** Consider the network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Suppose the network satisfies Condition C1) or C2). With Algorithm 2 all the benign agents finally achieve an agreement and converge to $\mathcal{X}^*$, regardless of the actions of misbehaving agents. That is, the follows hold for any $i, j \in B$:

$$\lim_{k \to \infty} x^i(k) - x^j(k) = 0,$$

(19)

and

$$\lim_{k \to \infty} x^i(k) \in \mathcal{X}^*.$$

(20)

**Proof.** Firstly we shall show the agreement among normal agents. To this end, let us denote $\delta(k) \triangleq \beta_kL$. It can be obtained by applying the same arguments as in Propositions 2 and recursively defining $\epsilon_k = 1/3^k \epsilon_0 - t \delta(k)$ that

$$\Delta_1(k + |B|) \leq \left(1 - \frac{1}{2 \cdot 3^d |B|}\right) \Delta_1(k) + 2|B|\delta(k).$$

(21)

The consensus is thus established by invoking [10] Theorem 6.4.

Next we focus on the proof of (20). For any $x$, define

$$d_S(x) \triangleq \min_{y \in \mathcal{S}} ||x - y||,$$

i.e., the distance from $x$ to the set $\mathcal{S}$. Moreover, let $v^i(k) = \sum_{j \in B} \bar{a}_{ij}^i(k)x^j(k)$ and $\bar{x}(k) = 1/|B| \sum_{j \in B} x^j(k)$. For any $i \in B$ Denote $x^* = \sum_{j \in B} \bar{a}_{ij}^i(k)\mathcal{P}_{\mathcal{X}^*}(x^j(k)) \in \mathcal{X}^*_\pi$, where $\mathcal{P}_{\mathcal{X}^*}()$ stands for the projection operator. In view of (16), the following relation holds for agent $i$:

$$||x^i(k) - x^*||^2 = ||v^i(k) - \beta_k d^i(k) - x^*||^2$$

$$= ||v^i(k) - x^*||^2 + \beta_k^2 d^i(k)|^2 + 2\beta_k (d^i(k), x^* - v^i(k))$$

$$\leq \sum_{j \in B} \bar{a}_{ij}^i(k)||x^j(k) - \mathcal{P}_{\mathcal{X}^*}(x^j(k))|^2 + \beta_k^2 L^2$$

$$+ 2\beta_k (f^i(x^*), f^i(v^i(k)))$$

$$= \sum_{j \in B} \bar{a}_{ij}^i(k)d^2_{\mathcal{X}^*_{\pi}}(x^j(k)) + \beta_k^2 L^2$$

$$+ 2\beta_k (f^i(x^*), f^i(\bar{x}(k)))$$

$$+ 2\beta_k (f^i(\bar{x}(k)) - f^i(v^i(k)))$$

$$\leq \sum_{j \in B} \bar{a}_{ij}^i(k)d^2_{\mathcal{X}^*_{\pi}}(x^j(k)) + \beta_k^2 L^2$$

$$+ 2\beta_k (f^i(x^*), f^i(\bar{x}(k))) + 2\beta_k L||\bar{x}(k) - v^i(k)||,$$

(22)

where the first inequality is due to the convexity of the squared norm $|| \cdot ||^2$ and the function $f^i$. One thus has that

$$d^2_{\mathcal{X}^*_{\pi}}(x^i(k+1)) \leq \sum_{j \in B} \bar{a}_{ij}^i(k)d^2_{\mathcal{X}^*_{\pi}}(x^j(k)) + \beta_k^2 L^2$$

$$+ 2\beta_k (f^i(x^*), f^i(\bar{x}(k))) + 2\beta_k L||\bar{x}(k) - v^i(k)||.$$

(23)

Multiplying $\pi_i(k + 1)$ (i.e., the $i$th component of $\pi(k + 1)$) and summing the above relation over $i \in B$ yields

$$\sum_{i \in B} \pi_i(k + 1)d^2_{\mathcal{X}^*_{\pi}}(x^i(k+1)) \leq \sum_{i \in B} \pi_i(k)d^2_{\mathcal{X}^*_{\pi}}(x^i(k)) + \beta_k^2 L^2$$

$$+ 2\beta_k \left[\sum_{i \in B} \pi_i(k + 1)f^i(x^*) - \sum_{i \in B} \pi_i(k + 1)f^i(\bar{x}(k))\right]$$

$$+ 2\beta_k L \sum_{i \in B} \pi_i(k + 1)||\bar{x}(k) - v^i(k)||.$$

(24)
Denote $d_k^2 = \sum_{i \in B} \pi_i(k)d_{X^*_i}(x^i(k))$. It follows that
\[
\begin{aligned}
&d_{k+1}^2 \leq d_k^2 + \beta_k L^2 + 2\beta_k \left[ \sum_{i \in B} \pi_i(k+1) f_i^\ast(x^\ast) \right. \\
&- \left. \sum_{i \in B} \pi_i(k+1) f_i(x^i(k)) \right] \\
&+ 2\beta_k L \sum_{i \in B} \pi_i(k+1) ||x^i(k) - x^i(k)||.
\end{aligned}
\] (25)

Since the consensus can be achieved among healthy agents, for any $\epsilon > 0$, one can always find some $k_\epsilon$, such that the follows are true for any $k \geq k_\epsilon$:
\[
\beta_k \leq \epsilon;
\] (26)
\[
||x^i(k) - x^i(k)|| \leq \epsilon;
\] (27)
\[
\sum_{i \in B} \pi_i(k+1) f_i(x^\ast) \leq \sum_{i \in B} \pi_i(k+1) f_i(x^i(k)) \leq \epsilon.
\] (28)

Notice that (28) holds because $\lim_{k \to \infty} \pi(k+1) = \pi_\infty$ and $\sum_{i \in B} \pi_\infty, f_i(x^\ast) - \sum_{i \in B} \pi_\infty, f_i(x^i(k)) \leq 0$.

In view of (26) and (27), one obtains that for all $k \geq k_\epsilon$,
\[
d_{k+1}^2 \leq d_k^2 + 2\beta_k \left[ \sum_{i \in B} \pi_i(k+1) f_i^\ast(x^\ast) \right. \\
- \left. \sum_{i \in B} \pi_i(k+1) f_i(x^i(k)) \right] + \beta_k \epsilon c,
\] (29)
for some $c$.

Then we shall study the limit of $d_k^2$. Some similar arguments as that after (22) in [28] Theorem 6] would be applied. However, we still briefly present it here for the sake of completeness.

For brevity, let $\theta = \epsilon c$. Firstly, we claim that there exists some $k_0 \geq k_\epsilon$ such that
\[
\sum_{i \in B} \pi_i(k_0+1) f_i^\ast(x^i(k_0)) - \sum_{i \in B} \pi_i(k_0+1) f_i^\ast(x^i(x^\ast)) \leq \theta.
\] (30)

We shall prove by contradiction. Suppose that at any $k \geq k_0$,
\[
\sum_{i \in B} \pi_i(k+1) f_i(x^i(k)) - \sum_{i \in B} \pi_i(k+1) f_i(x^i(x^\ast)) > \theta.
\]
Then it follows from (29) that
\[
d_{k+1}^2 \leq d_k^2 - \theta \beta_k.
\] (31)

Hence $\lim_{N \to \infty} d_{k+N}^2 \leq d_k^2 - \theta \sum_{t=k}^N \beta_t < 0$ given that $\lim_{N \to \infty} \sum_{t=1}^N \beta_t = \infty$. Therefore a contradiction is concluded and thus (30) holds.

By the continuity of $f^\ast$ (due to it being convex over $\mathbb{R}^d$) and boundedness of $X^\ast$, one has $d_{X^\ast}^\ast(x^i(k_0)) \leq \eta(\theta)$ under (28) and (30). We then choose $k_0$ in the sense that for any $k \geq k_0$, if (30) holds at $k$, then $d_{X^\ast}^\ast(x^i(k)) \leq \eta$ and $\max_{i \in B} ||x^i(k) - x^i(x^\ast)||^2 \leq \eta$. Then one has that $d_{k_0}^2 \leq \eta$. We thus obtain that
\[
d_{k_0+1}^2 \leq d_{k_0}^2 + 2\epsilon \beta_{k_0} + \theta \beta_{k_0} + 4\eta + (1 + \frac{2}{c})\theta \beta_{k_0}.
\] (32)

If (30) also holds at $k_0 + 1$, similarly we shall have $d_{k_0+2}^2 \leq 4\eta + (1 + \frac{2}{c})\theta \beta_{k_0+1}$. Otherwise invoking (31) one has that $d_{k_0+2}^2 \leq d_{k_0+1}^2 - \theta \beta_{k_0+1} \leq 4\eta + (1 + \frac{2}{c})\theta \beta_{k_0}$. Then it is always held that $d_{k_0+2}^2 \leq 4\eta + (1 + \frac{2}{c})\theta \beta_{k_0}$. By induction, we can establish that $d_k^2 \leq 4\eta + (1 + \frac{2}{c})\theta \beta_{k_0}$ for any $k \geq k_0$. Hence $\lim_{k \to \infty} d_k^2 \leq 4\eta$. Now invoking the arbitrariness of $\theta$ (due to the arbitrariness of $\epsilon$) and $\lim_{\theta \to 0} \eta = 0$, one concludes that
\[
\lim_{k \to \infty} d_k^2 = 0.
\] (33)

Therefore, for any benign agent $i$, its state $x^i(k)$ will finally converge to $X^\ast$, which completes the proof. \qed

Remark 3. In contrast to [10], where all agents converge to the convex hull spanned by the local optimizers of benign agents in scalar systems, the result here is in multi-dimensional spaces and the convergent optimal set is sharper, i.e., $X^\ast \subset Conv(M)$. Moreover, the convergence analysis is intrinsically different from that in [10] as the method therein is not applicable in the high dimensional case and hence we have resorted to the absolute probability sequence for row-stochastic matrices.

Note that unlike that in the canonical consensus problems, the benign agents may not converge to a constant value in resilient distributed optimization settings (although they could always reach an agreement within $X^\ast$). This phenomenon is also discussed in the literature [10].

VII. NUMERICAL EXAMPLE

In this section, we provide numerical examples to verify the theoretical results established in the previous sections.

Example 1: The communication network is given by Fig. 3 where the node set is $\mathcal{V} = \{1, 2, ..., 6\}$. It is verified that the graph is $(3, 2)$-robust, and Proposition 3 indicates that it can tolerate a single misbehaving node in a 2-dimensional problem. Suppose that agent 2 is compromised. It intends to prevent others from reaching a correct consensus by violating the rule in Algorithm 1 and setting its states as $x_2^0(k) = 1.5 + \sin(k/5)$ and $x_2^0(k) = k/25 + 1$ at any time $k > 0$. On the other hand, the benign agents are initialized with $x^1(0) = (0, 0), x^2(0) = (2, 0), x^3(0) = (1, 3), x^5(0) = (2, 4), x^6(0) = (3, 3)$, and always follow (4) for updates.

![Fig. 3: Communication network.](image-url)
functions of normal agents are provided as $f^2(x) = x_1^2 + x_2^2$, $f^3(x) = (x_1 - 1)^2 + (x_2 - 0.5)^2$, $f^4(x) = (x_1 - 0.5)^2 + (x_2 - 1)^2$, $f^5(x) = (x_1 + 1)^2 + (x_2 - 1)^2$ and $f^6(x) = (x_1 + 0.3)^2 + (x_2 - 0.2)^2$ (we regard the magnitude of their gradients to be capped by some sufficiently large number). The stepsize in the algorithm is settled as $\beta_k = \frac{1}{3k}$. From Fig. 5, it is observed that the states of benign agents achieve consensus, which can be checked within the convex hull of their local minimizers.

Fig. 5: The trajectory of local states under Algorithm 2 where the area surrounded by the dashed lines is the convex hull of the initial states of benign agents.

According to Definition 5, $\Psi(A, n)$ is an intersection of $\binom{m}{n}$ convex hulls. Since $m \geq n(d+1)+1$, it is trivial to prove that $\binom{m}{n} > d$ holds.

On the other hand, each of these convex hulls is created by excluding $n$ elements of $A$. Then consider any $d+1$ of them, they discard at most $n(d+1)$ points in all. Since $m \geq n(d+1)+1$, it must be the case that at least one element in $A$ is retained by all of them. This indicates that any $d+1$ convex hulls must have a nonempty intersection. By applying Helly’s Theorem, the proof is completed.

B. Proof of Lemma 7

Denote a subset of $\{A_i\}_{i \in I}$ as $\{A_i\}_{i \in I}$, such that $\{A_i\}_{i \in I}$ contains all $A_i$ which has a superset in $\{B_j\}_{j \in J}$. The proof is then completed by noticing that

$$\bigcap_{i \in I} A_i \subset \bigcap_{i \in I} A_i \subset \bigcap_{j \in J} B_j,$$

C. Proof of Lemma 2

We shall prove that every set $S_2$ in $\mathcal{S}(A_2, n)$ is a superset for some set $S_1$ in $\mathcal{S}(A_1, n)$. To see this, notice that

$S_2 = A_2 \setminus S_2 \supset (A_2 \setminus S_2) \cap A_1 = A_1 \setminus (S_2 \cap A_1),$

where $S_2 = A_2 \setminus S_2$ is a set with cardinality $n$. Notice that $S_2 \cap A_1$ has cardinality no greater than $n$, which means that $A_1 \setminus (S_2 \cap A_1)$ is a superset of some set in $\mathcal{S}(A_1, n)$. The proof is thus finished by invoking Lemma 1.

D. Proof of Proposition 3

Proposition 3 is proved in a similar manner to that of Proposition 2. The essential point is that if $\mathcal{V}^M(k, k+\kappa d, \epsilon_{kad})$ and $\mathcal{V}^m(k, k+\kappa d, \epsilon_{kad})$ are nonempty and disjoint, and if both of these sets contain some benign agents, then under $(dF+1, F+1)$-robust graph, there exists at least one benign agent in either $\mathcal{V}^M(k, k+\kappa d, \epsilon_{kad})$ or $\mathcal{V}^m(k, k+\kappa d, \epsilon_{kad})$ that has no less than $dF+1$ neighboring agents outside its set. Suppose the benign agent $i \in \mathcal{V}^M(k, k+\kappa d, \epsilon_{kad})$ is such a node. Following a similar proof procedure of Proposition 2, we know that agent $i$ will always apply a state who has its first entry being no more than $M_1(k) - \epsilon_{kad}$ for updating, under Algorithm 1. The result can be finally concluded by applying the proof techniques as before.

E. Proof of Lemma 3

As implied in the proof of Propositions 2 and 3 under Condition C1 or C2), for any disjoint and nonempty subsets $B_1, B_2 \subseteq B$, there must exist an agent $i$ in $B_1 \cup B_2$, whose update will be influenced by at least one outside normal neighbor $j$. Without loss of generality, let $i \in B_1$. Therefore, it is clear that $j \in F_{\mathcal{A}(k)}(B_1)$. Hence one of the following cases must hold:

1) $j \in F_{\mathcal{A}(k)}(B_2)$, resulting in $F_{\mathcal{A}(k)}(B_1) \cap F_{\mathcal{A}(k)}(B_2) \neq \emptyset$;
2) $j \notin F_{\mathcal{A}(k)}(B_2)$. Hence $F_{\mathcal{A}(k)}(B_1) \cap F_{\mathcal{A}(k)}(B_2) = \emptyset$.

Note that any normal agent also applies its own state in

VIII. Conclusion

Due to their wide applications, the distributed coordinations in networked systems have attracted much research interest. In this paper, we are interested in the achievement of consensus under malicious agents. A resilient algorithm is proposed in this work, which is also applicable in the high dimensional spaces. Under certain network topology, this strategy guarantees the benign agents exponentially reach an agreement within the convex hull of their initial states. Finally, we make a connection with distributed optimization showing how our protocol can be accommodated to increase the system resiliency and security in that context. This result indicates the main ideas in this paper could also be applied to other consensus-based problems with different system settings as well.

APPENDIX

A. Proof of Corollary 7

Corollary 7 is obvious when $n = 0$. Thus we only focus on the scenario when $n \geq 1$. 

Fig. 4: The trajectory of local states under Algorithm 1 where the area surrounded by the dashed lines is the convex hull of the initial states of benign agents.
Recalling Definition, any $\bar{A}(k)$ is Sarymsakov.

F. Proof of Lemma 5

In view of Lemma 4, each is $\bar{A}(k)$ is Sarymsakov. Since the set of Sarymsakov matrices is closed under matrix multiplication, any $\bar{A}(0)\bar{A}(1) \cdots \bar{A}(k)$ belongs to the Sarymsakov class as well. Invoking [20, Theorem 1], there exists a column vector $\nu = [\nu_1, \nu_2, \ldots, \nu_N]^T$ with each $\nu_k \geq 0$ and $\sum_{i=1}^N \nu_i = 1$, such that

$$\lim_{k \to \infty} \bar{A}(k, 0) = \mathbf{1} \nu^T. \tag{34}$$

Combining it with (18), the proof is completed.

REFERENCES

[1] W. Ren, R. W. Beard, and E. M. Atkins, “Information consensus in multivehicle cooperative control,” IEEE Control Systems Magazine, vol. 27, no. 2, pp. 71–82, 2007.
[2] ——, “A survey of consensus problems in multi-agent coordination,” in Proceedings of the 2005, American Control Conference, 2005. IEEE, 2005, pp. 1859–1864.
[3] R. L. Raffard, C. J. Tomlin, and S. P. Boyd, “Distributed optimization for cooperative agents: Application to formation flight,” in Decision and Control, 2004. CDC. 43rd IEEE Conference on, vol. 3. IEEE, 2004, pp. 2453–2459.
[4] D. P. Spanos, R. Olfati-Saber, and R. M. Murray, “Distributed sensor fusion using dynamic consensus,” in IFAC World Congress. Citeseer, 2005.
[5] C. C. Moallemi and B. Van Roy, “Consensus propagation,” IEEE Transactions on Information Theory, vol. 52, no. 11, pp. 4753–4766, 2006.
[6] N. A. Lynch, Distributed algorithms. Elsevier, 1996.
[7] R. Olfati-Saber, J. A. Fax, and R. M. Murray, “Consensus and cooperation in networked multi-agent systems,” Proceedings of the IEEE, vol. 95, no. 1, pp. 215–233, 2007.
[8] E. Wei and A. Ozdaglar, “Distributed alternating direction method of multipliers,” in Decision and Control (CDC), 2012 IEEE 51st Annual Conference on. IEEE, 2012, pp. 5445–5450.
[9] Y. Mo, T. H.-J. Kim, K. Brancik, D. Dickinson, H. Lee, A. Perrig, and B. Sinopoli, “Cyber–physical security of a smart grid infrastructure,” Proceedings of the IEEE, vol. 100, no. 1, pp. 195–209, 2012.
[10] S. Sundaram and B. Gharesifard, “Distributed optimization under adversarial nodes,” IEEE Transactions on Automatic Control, 2018.
[11] D. Dolesh, N. A. Lynch, S. S. Pinter, E. W. Stark, and W. E. Weihl, “Reaching approximate agreement in the presence of faults,” Journal of the ACM (JACM), vol. 33, no. 3, pp. 499–516, 1986.
[12] R. M. Kieckhafer and M. H. Azadmanesh, “Reaching approximate agreement with mixed-mode faults,” IEEE Transactions on Parallel and Distributed Systems, vol. 5, no. 1, pp. 53–63, 1994.
[13] N. H. Vaidya, L. Tseng, and G. Liang, “Iterative approximate byzantine consensus in arbitrary directed graphs,” in Proceedings of the 2012 ACM Symposium on Principles of Distributed Computing. ACM, 2012, pp. 365–374.
[14] M. M. de Azevedo and D. M. Blough, “Multistep interactive convergence: An efficient approach to the fault-tolerant clock synchronization of large multicomputers,” IEEE Transactions on Parallel and Distributed Systems, vol. 9, no. 12, pp. 1195–1212, 1998.
[15] H. J. LeBlanc, H. Zhang, X. Koutsoukos, and S. Sundaram, “Resilient asymptotic consensus in robust networks,” IEEE Journal on Selected Areas in Communications, vol. 31, no. 4, pp. 766–781, 2013.
[16] S. M. Dibaji and H. Ishii, “Consensus of second-order multi-agent systems in the presence of locally bounded faults,” Systems & Control Letters, vol. 79, pp. 23–29, 2015.
[17] L. Su and N. H. Vaidya, “Fault-tolerant distributed optimization (part iv): Constrained optimization with arbitrary directed networks,” arXiv preprint arXiv:1511.0821, 2015.
[18] J. Yan, Y. Mo, X. Li, and C. Wen, “A “safe kernel” approach for resilient multi-dimensional consensus,” arXiv preprint arXiv:1911.10836, 2019.
[19] E. Seneta, “Coefficients of ergodicity: Structure and applications,” Advances in Applied Probability, vol. 11, no. 3, pp. 576–590, 1979.