Gravitational geons in (1+1) dimensions

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Received 17 March 2008
Published 15 August 2008
Online at stacks.iop.org/CQG/25/175004

Abstract

It is well known that general relativity does not admit gravitational geons that are stationary, asymptotically flat, singularity free and topologically trivial. However, it is likely that general relativity will receive corrections at large curvatures and the modified field equations may admit solutions corresponding to this type of geon. If geons are produced in the early universe and survive until today they could account for some of the dark matter that has been ‘observed’ in galaxies and galactic clusters. In this paper I consider gravitational geons in (1+1)-dimensional theories of gravity. I show that the Jackiw–Teitelboim theory with corrections proportional to $R^2$ and $\Box R$ admits gravitational geons. I also show that gravitational geons exist in a class of theories that includes Lagrangians proportional to $R^{2/3}$.

PACS numbers: 04.20.−q, 04.50.kd

1. Introduction

A gravitational geon is a nonsingular configuration of the gravitational field, without horizons, that persists for a long period of time [1, 2] (see, also, [3]). An interesting class of geons consists of nonsingular, asymptotically flat, topologically trivial vacuum spacetimes without horizons. It has been shown [4–6] that general relativity does not admit such geons. However, it is widely believed that general relativity will receive corrections in regions of large spacetime curvature. These corrections may be either classical and/or quantum mechanical in nature (both types occur in string theory). Quantum effects are expected to become important by the time the Planck scale is reached, but it is possible that classical corrections may appear long before the Planck scale. It is also possible that the modified classical field equations admit gravitational geons of the type discussed above. Such geons should have masses and sizes of the order of the scale at which the corrections become important. If geons are produced in the early universe and survive until today they could account for some of the dark matter that has
been ‘observed’ in galaxies and galactic clusters (this possibility has also been discussed by Sones [3] for quantum geons with a Klein–Gordon field).

Static spherically symmetric gravitational geons have been found [7, 8] in (3+1)-dimensional theories with a cosmological constant that depends on the radial coordinate and is different in the radial and angular directions. However, it is difficult to find exact solutions to most generalizations of Einstein’s equations in (3+1) dimensions. To simplify the problem I will look for geons in two types of (1+1)-dimensional theories of gravity. The first theory that I will consider is a modified Jackiw–Teitelboim theory [9, 10]. The modifications involve adding terms proportional to $R^2$ and $\Box R$ to the field equations. Such terms, involving higher order polynomials and derivatives of the curvature are expected to occur in quantum theories of gravity. It is shown that this theory admits gravitational geons. The second theory considered is based on the Lagrangian

$$L = \sqrt{-g} \left[ \frac{1}{\phi} R + V(\phi) \right],$$

which has been shown to admit nonsingular black holes, for a particular potential [13]. Here I show that the theory also admits geons for certain choices of $V(\phi)$. For a particular choice of $V(\phi)$ I also show that $\phi$ can be eliminated from the action and the Lagrangian, written in terms of the Ricci scalar, is proportional to $R^{2/3}$.

2. Field equations and geons

First consider the (1+1)-dimensional theory proposed by Jackiw [9] and Teitelboim [10]

$$R = \Lambda + 8\pi G T,$$

where $R$ is the Ricci scalar, $\Lambda$ is a cosmological constant, $G$ is Newton’s constant and $T$ is the trace of the energy–momentum tensor. It has been shown [11] that this theory possesses many features in common with (3+1)-dimensional general relativity. At first sight this theory does not seem to follow from the Einstein field equations, $G_{\mu\nu} = \lambda g_{\mu\nu} + 8\pi T_{\mu\nu}$, due to the fact that $G_{\mu\nu}$ is identically equal to zero in (1+1) dimensions. However, in (1+1) dimensions there exists a conformal anomaly $\langle T \rangle \propto R$ and Sanchez [12] has used this to show that (2) does follow from Einstein’s equations.

Now, as discussed earlier, it is believed that there will be corrections to the Einstein field equations in regions of large spacetime curvature. These corrections may be expected to contain higher powers and derivatives of the Riemann tensor and its contractions. In (1+1) dimensions the Riemann and Ricci tensors can be written in terms of the Ricci scalar, so the corrections will involve higher powers and derivatives of the Ricci scalar. Here I consider modifications to the Jackiw–Teitelboim theory of the form

$$R + \alpha R^2 + \beta \Box R - \Lambda = 8\pi G T,$$

where $\alpha$ and $\beta$ are constants and $\Box = \nabla_\mu \nabla^\mu$. These are the most general corrections involving polynomials and derivatives of $R$ that involve constants with dimensions of (length)$^2$. For the remaining paper I will set $\Lambda = 0$ for simplicity.

In vacuum spacetimes with

$$ds^2 = -f(r)\,dt^2 + f^{-1}(r)\,dr^2,$$

the Ricci scalar is $R = -f'''$ and the field equation is given by

$$f''' - \alpha(f'')^2 + \beta \frac{d}{dr}(ff'') = 0. \tag{5}$$
Here I take \( r \) to be a radial-like coordinate, so that \( r \geq 0 \). The general vacuum solution of the original Jackiw–Teitelboim theory (i.e. \( \alpha = \beta = 0 \)) with \( \Lambda = 0 \) is
\[
f_0(r) = ar + b, \tag{6}
\]
where \( a \) and \( b \) are constants. Now consider solutions to (5) of the form
\[
f(r) = f_0(r) + \epsilon(r) \tag{7}
\]
with \(|\epsilon(r)| \ll |f_0(r)|\).

First consider solutions with \( a = 0 \) and \( b = 1 \). The linearization of (5) gives
\[
\epsilon'' + \beta \epsilon'''' = 0. \tag{8}
\]
If \( \beta < 0 \) set \( \sigma = 1/\sqrt{-\beta} \) and the general solution is
\[
\epsilon(r) = A_1 + A_2 r + B_1 e^{-\sigma r} + B_2 e^{\sigma r}. \tag{9}
\]
Imposing the condition \(|\epsilon| \ll 1\) gives \( A_2 = B_2 = 0, |B_1| \ll 1 \) and we can set \( A_1 = 0 \) since it just modifies the constant term in \( f \). In order to be able to neglect the \( \alpha(\epsilon'')^2 \) term relative to the \( \epsilon'' \) term the additional constraint
\[
\left| \frac{\alpha}{\beta} A \right| \ll 1 \tag{10}
\]
must be satisfied. If \( \alpha \simeq \beta \) this reduces to \(|A| \ll 1\). Thus,
\[
f(r) = 1 + A e^{-\sigma r}, \quad |A| \ll 1 \quad \text{and} \quad \left| \frac{\alpha}{\beta} A \right| \ll 1, \tag{11}
\]
is an approximate solution to the field equations if \( \beta < 0 \). In fact, if \( \alpha = 2\beta \) this is an exact solution for arbitrary \( A \). The Ricci scalar is given by
\[
R = \frac{A}{\beta} e^{-\sigma r} \tag{12}
\]
and is bounded everywhere. Note that \( R \) does not have to be small at the origin even when \(|A| \ll 1\). For example, if \(|A| \simeq 10^{-3}\) then \( R \) is about three orders of magnitude less than the scale set by \( \beta \), which could be quite large. For \(|A| \ll 1\) there are no horizons and this solution then describes a static gravitational geon. If \( \alpha = 2\beta \) we require that \( A > -1 \) to avoid the presence of a horizon.

If \( \beta > 0 \) set \( \sigma = 1/\sqrt{\beta} \) and the relevant solution is
\[
\epsilon(r) = A \cos(\sigma r + B), \quad |A| \ll 1 \quad \text{and} \quad \left| \frac{\alpha}{\beta} A \right| \ll 1, \tag{13}
\]
where \( A \) and \( B \) are constants. This solution can be thought of as an infinite sequence of geons.

Now consider solutions of the form (7) with \( a = 1 \) and \( b = 0 \). The linearized field equation is
\[
\beta \epsilon'''' + \beta \epsilon''' + \epsilon'' = 0. \tag{14}
\]
First consider \( \beta > 0 \) with \( \sigma = 1/\sqrt{\beta} \) and let \( x = 2\sigma \sqrt{r} \) and \( y = \epsilon'' \). The field equation is given by
\[
x^2 y'' + xy' + x^2 y = 0, \tag{15}
\]
which is a Bessel equation of order zero. The solution, which is nonsingular at the origin, is
\[
y(x) = \bar{A} J_0(x), \tag{16}
\]
where \( \bar{A} \) is a constant and \( J_0(x) \) is the Bessel function of the first kind of order zero. This can be written as
\[
\epsilon''(r) = \bar{A} J_0(2\sigma \sqrt{r}) \tag{17}
\]
and can be integrated twice using
\[ \int x^n J_{n-1}(x) \, dx = x^n J_n(x) \]  
(18)
to obtain
\[ \epsilon(r) = Ar J_2(2\sigma \sqrt{r}), \]
where \( A = \beta \tilde{A} \) and integration constants have been set to zero. The function \( f(r) \) can then be written as
\[ f(r) = r[1 + AJ_2(2\sigma \sqrt{r})], \quad |A| \ll 1 \quad \text{and} \quad \left| \frac{\alpha}{\beta} A \right| \ll 1, \]
(20)
where I have imposed \( |\epsilon| \ll 1 \) and \( |\alpha(\epsilon'')^2| \ll |\epsilon'| \). Since
\[ J_n(x) \simeq \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{n\pi}{2} - \frac{\pi}{4} \right) \quad \text{as} \quad x \to \infty \]
(21)
we see that \( f(r) \to r \) as \( r \to \infty \). The Ricci scalar is given by
\[ R = -\frac{A}{\beta} J_0(2\sigma \sqrt{r}) \]
(22)
and is therefore bounded and goes to zero as \( r \to \infty \).

Now consider \( \beta < 0 \) with \( \sigma = 1/\sqrt{-\beta} \) and define \( x = 2\sigma \sqrt{r} \) and \( y = \epsilon'' \). The field equation becomes
\[ x^2 y'' + xy' - x^2 y = 0. \]
(23)
This is a modified Bessel equation of order zero. The general solution is
\[ y(x) = AI_0(x) + BK_0(x), \]
(24)
where \( A \) and \( B \) are constants, \( I_0 \) is the modified Bessel function of the first kind of order zero and \( K_0 \) is the modified Bessel function of the second kind of order zero. Unfortunately, neither of these functions is bounded on \( x > 0 \) and we do not find geons in the linearized approximation for \( \beta < 0 \).

Next I will consider geons in gravitational theories based on the Lagrangian
\[ L = \sqrt{-g} \left[ \frac{1}{\phi} R + V(\phi) \right]. \]
(25)
which has been used in [13] to produce nonsingular black holes in (1+1) dimensions. The field equations that follow from the Lagrangian (4) are [13]
\[ \phi^3 V(\phi) + 2f(\phi')^2 \phi' + f' \phi' = 0, \]
(26)
\[ \phi^2 V(\phi) + f' \phi' = 0, \]
(27)
and
\[ \frac{dV(\phi)}{d\phi} = -\frac{1}{\phi^2} f''. \]
(28)
Solving for \( V(\phi) \) in (27) and substituting it into (26) gives
\[ \phi \phi'' - 2(\phi')^2 = 0. \]
(29)
The solution to this equation is
\[ \phi(r) = \frac{1}{Ar + B}. \]
(30)
where $A$ and $B$ are constants. The constant $B$ can be eliminated by redefining $r$ giving
\[ \phi(r) = \frac{1}{Ar}, \] (31)

The field equations can now taken to be (27) and (28) with $\phi$ given by (31). Substituting (31) into the two field equations gives
\[ V(\phi) = Af' \quad \text{and} \quad \frac{dV(\phi)}{d\phi} = -A^2 r^2 f''. \] (32)

It is easy to show that the first equation along with $\phi = 1/Ar$ implies the second equation. The remaining field equation is then given by
\[ Af' = V(\phi) \quad \text{with} \quad \phi = \frac{1}{Ar}. \] (33)

Thus, given a function $f(r)$ it is easy to solve for $V(\phi)$.

For example if
\[ f(r) = 1 - \frac{2m}{r + \ell}, \] (34)
where $m$ and $\ell$ are positive constants, then
\[ V(\phi) = \frac{2mA^3 \phi^2}{(1 + A\ell\phi)^2}. \] (35)

The Ricci scalar is given by
\[ R = \frac{4m}{(r + \ell)^3} \] (36)
and is finite for all $r \geq 0$. If $\ell > 2m$ there are no horizons and the solution is a static gravitational geon. It is easy to see that other theories with different potentials can be constructed that admit gravitational geons. One simply chooses a function $f(r)$ that describes a gravitational geon and solves for $V(\phi)$.

It is interesting to note that $\phi$ can be eliminated in favor of $R$ in the action. From (28) we see that $R = \phi^2 \frac{dV}{d\phi}$. Using this and (35) it is easy to show that
\[ A\phi = \frac{x}{1 - \ell x} \] (37)
where
\[ x = (R/4m)^{1/3}. \] (38)

Substituting this into the Lagrangian (25) gives
\[ L = \sqrt{-g} \left[ 6mA \left( \frac{R}{4m} \right)^{2/3} - A\ell R \right]. \] (39)

The last term can be dropped, since $R$ does not contribute to the field equations in (1+1) dimensions. The vacuum field equations that follow from (39) and $R_{\mu\nu} = 1/2 R g_{\mu\nu}$ are
\[ R g_{\mu\nu} = 4R^{1/3} [g_{\mu\nu} \Box R^{-1/3} - \nabla_\mu \nabla_\nu R^{-1/3}] \] (40)
and it is easy to show that (34) is a solution.
3. Conclusion

In this paper I examined two gravitational theories in (1+1) dimensions to see if they admit geons. The first theory was a modification of the Jackiw–Teitelboim theory of the form

\[ R + \alpha R^2 + \beta \square R - \Lambda = 8\pi GT, \]  

where \( \alpha \) and \( \beta \) are constants. I showed that there are vacuum solutions with \( \Lambda = 0 \) describing gravitational geons.

I also examined theories that follow from the Lagrangian

\[ L = \sqrt{-g} \left[ \frac{1}{\phi} R + V(\phi) \right]. \]  

I showed that there exist potentials that lead to theories that admit gravitational geon solutions. For a particular choice of \( V(\phi) \) I also showed that \( \phi \) can be eliminated from the action and the Lagrangian, written in terms of the Ricci scalar, is proportional to \( R^{2/3} \).

Acknowledgment

This research was supported by the Natural Sciences and Engineering Research Council of Canada.

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ADDENDUM

Addendum to ‘Gravitational geons in 1+1 dimensions’

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Received 24 February 2010
Published 7 July 2010
Online at stacks.iop.org/CQG/27/169701

Abstract

In a recent paper (Vollick D N 2008 Class. Quantum Grav. 25 175004), I found gravitational geons in two classes of (1+1)-dimensional theories of gravity. In this paper, I examine these theories, with the possibility of a cosmological constant, and find strong field gravitational geons. In the spacetimes in Vollick (2008 Class. Quantum. Grav. 25 175004) a test particle that is reflected from the origin suffers a discontinuity in $\frac{d^2t}{d\tau^2}$. The geons found in this paper do not suffer from this problem.

PACS number: 04.50.kd

In a recent paper [1], I examined geons [2] in (1+1)-dimensional theories of gravity with a spacetime metric given by

$$ds^2 = -f(r) \, dt^2 + f(r)^{-1} \, dr^2,$$

(1)

where $r$ is a radial-like coordinate with $r \geq 0$. Imposing a reflecting boundary condition at $r = 0$ leads to the problem that $\frac{d^2t}{d\tau^2}$ suffers a jump discontinuity upon reflection unless $f'(0) = 0$. This follows from the geodesic equation

$$\frac{d^2t}{d\tau^2} = -\left(\frac{f'}{f}\right) \frac{dr}{d\tau} \frac{dt}{d\tau},$$

(2)

with $dr/d\tau$ changing sign upon reflection. None of the solutions in [1] satisfied this condition, except (13) with $B = 0$. In this paper, I find additional strong field geon solutions that either satisfy $f'(0) = 0$ or that can be extended to negative $r$ (i.e. with $-\infty < r < \infty$).

Consider the field equation (see equation (3) in [1] with $\alpha = 0$)

$$R + \beta \Box R = \Lambda,$$

(3)

In terms of the metric function $f(r)$, this equation can be written as

$$f'' + \beta \frac{d}{dr} (ff'') = -\Lambda.$$

(4)

Integrating twice it gives

$$f + \beta \left[ ff'' - \frac{1}{2} (f')^2 \right] = C_1 + C_2 r - \frac{1}{4} \Lambda r^2,$$

(5)
where $C_1$ and $C_2$ are the integration constants. In terms of $h = \sqrt{T} \,( f > 0)$, the above equation can be written as

$$h^2 + 2\beta h^3 h'' = C_1 + C_2 r - \frac{1}{2} \Lambda r^2.$$  \hspace{1cm} (6)

First consider the case $C_2 = \Lambda = 0$. Integrating once it gives

$$\frac{1}{2} v^2 + V(h) = E$$ \hspace{1cm} (7)

where $v = dh/dr$, $E$ is an integration constant and

$$V(h) = \frac{1}{2\beta} \left[ \ln|h| + C_1 \frac{2}{2h^2} \right].$$ \hspace{1cm} (8)

From this equation it is easy to see that $R = -f'' \simeq \frac{1}{\beta} \ln f$ for large $f$ (the solution will approach a large value of $f$ only if $\beta < 0$). Thus, we are only interested in solutions in which $f$ remains finite. The only nontrivial solutions with $h'(0) = 0$ and which satisfy $h > 0$ occur when $\beta$ and $C_1$ are positive. In this case, $V$ has one local minimum and goes to $\infty$ as $h$ goes to zero and $\infty$. Therefore, $f$ will undergo periodic oscillations and these solutions are similar to solutions (13) in [1], except that they are not restricted to weak fields. Note that we can extend these solutions to negative values of $r$. These solutions therefore correspond to infinite sequences of geons.

Now consider the case with $C_1 = C_2 = 0$. In this case $f = -\frac{1}{2} \Lambda r^2 \,( h = \sqrt{\frac{1}{2} \Lambda |r|} )$ is a solution if $\Lambda < 0$, so that $f > 0$. The behavior of the solutions can be analyzed by letting $h = rZ(r)$ and $x = \ln|\sigma|$. The field equation in these variables is given by

$$\frac{d^2Z}{dx^2} = -\frac{dZ}{dx} - \frac{dU(Z)}{dZ},$$ \hspace{1cm} (9)

where the potential is given by

$$U(Z) = \frac{1}{2\beta} \left[ \ln|Z| - \frac{\Lambda}{4Z^2} \right].$$ \hspace{1cm} (10)

For $\Lambda < 0$ and $\beta > 0$, $U \to \infty$ as $Z \to 0, \infty$ and there is one minimum at $Z = \sqrt{|\Lambda|}/2$. The damping term $-dZ/dx$ will cause the motion to decay to the solution $Z = \sqrt{|\Lambda|}/2$ at large $r$. Thus, at large $r$, the function $f(r)$ will approach $\frac{1}{2} |\Lambda| r^2$. Note that this solution can be extended to negative values of $r$ since $Z \to \sqrt{|\Lambda|}/2$ for large $x$ where $x = \ln|\sigma|$. This solution therefore corresponds to a geon with $-\infty < r < \infty$ or with $r \geq 0$ and a reflective boundary condition at $r = 0$ ($f' = 2r Z^2 + 2r^2 ZZ'$, so that $f'(0) = 0$ if $Z(0)$ and $Z'(0)$ are finite). At large $r$, the Ricci scalar approaches $-\Lambda$. There are no geon solutions for $\beta < 0$.

Now consider the case $C_1 = \Lambda = 0$. A solution of this equation is $f = C_2 r + \frac{1}{2} \beta C_2^3$. I have been unable to show analytically that, for $\beta > 0$ and $C_2 > 0$, all solutions approach this solution for large $r$. However, it is easy to show that this solution is stable in the sense that all nearby solutions do converge to it for large $r$. For simplicity, I will take $C_2 = 1$. Now let

$$f(r) = \left[ r + \frac{1}{2} \beta \right] (1 + g(r)).$$ \hspace{1cm} (11)

The function $g(r)$ satisfies the linearized equation

$$\beta \left[ x^2 \frac{d^2 g}{dx^2} + \frac{d}{dx} \left( \frac{dg}{dx} - g \right) \right] + x g = 0,$$ \hspace{1cm} (12)

where $x = r + \beta$. Now define $y = 2\sigma \sqrt{x}$, where $\sigma = \beta^{-1/2}$. In terms of $y$, the equation is

$$y^2 \frac{d^2 g}{dy^2} + y \frac{dg}{dy} + (y^2 - 4) g = 0.$$ \hspace{1cm} (13)
This is a Bessel equation of order two, so the general solution is
\[ g(x) = A J_2(2\sigma \sqrt{x}) + B Y_2(2\sigma \sqrt{x}), \]  
(14)
where \( A \) and \( B \) are constants. Here I have included \( Y_2 \), which diverges at the origin, since I am considering only large \( x \). Note that
\[ J_2(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{5}{4} \pi \right) \quad \text{as} \quad x \to \infty \]  
(15)
(a similar expression holds for \( Y_2 \)). Thus, \( g \) goes to zero and \( f \to r + \frac{1}{2} \beta \) for large \( r \). In this case, \( R = -f'' \) goes to zero at large \( r \). I have studied the differential equation (5) with (11) numerically (\( C_1 = 0, C_2 = 1, \Lambda = 0 \)), using Maple, for the initial values \( g(0) \in [-0.9, 3] \) and with \( \beta = 1 \). The value of \( g'(0) \) is determined by setting \( f'(0) = 0 \). This gives \( g'(0) = -2[1 + g(0)] \). The solutions oscillate and decay slowly for large \( r \) in a similar fashion to the linearized solutions.

The second class of theories examined in [1] was based on the Lagrangian [3]
\[ L = -\sqrt{g} \left( \frac{1}{\phi} R + V(\phi) \right). \]  
(16)
It was shown that the field equations could be integrated giving
\[ Af' = V(\phi) \quad \text{with} \quad \phi = \frac{1}{Ar}, \]  
(17)
where \( A \) is a constant. Thus, for a given \( f(r) \), it is easy to solve for \( V(\phi) \). A simple function that satisfies \( f'(0) = 0 \) and has Schwarzschild behavior at large \( r \) is
\[ f(r) = 1 - \frac{2mr^2}{r^3 + 2m\ell^2}, \]  
(18)
where \( m \) and \( \ell \) are constants. The potential is given by
\[ V(\phi) = \frac{2mA^2\phi^2(1 - 4m\ell^2A^3\phi^3)}{(1 + 2m\ell^2A^3\phi^3)}. \]  
(19)

Acknowledgment

This research was supported by the Natural Sciences and Engineering Research Council of Canada.

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