The Geometric Origin of Electric Force

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Abstract. Every textbook on quantum field theory points out the formal, mathematical parallels between gauge theory and general relativity. In this paper, we make these parallels visual. The differential geometry behind general relativity can be visualized using an embedding space. We will use similar embedding techniques to show the spatial geometry associated with an electric field. As one might expect from the Lagrangian, electric fields have time-changing geometry. This paper focuses on exposing the geometrical origin of electric force by using both a near-exact solution and a more simple, physically insightful, approximate solution.

1. Introduction
Before Einstein showed that a gravitational field could be related to geometry (i.e. space-time curvature), gravity was visualized as a vector field. In much the same way, electromagnetic forces are also visualized using vector fields. Picking up where Einstein left off, we incorporate electromagnetism into the geometric structure of space-time.

Our paper builds on previous work explained in reference [1] and the references therein. We further develop the idea of a geometric origin to electric force by working out an example of an electron (without spin) in an electric field both in a near-exact approach and with a simple, physically insightful approximation. This geometric interpretation has several surprises including a unique charge and a dependence on the absolute energy ($E = \omega$).

2. The near-exact solution
We begin by defining the geometry of a background electric field as shown in figure 1 using a three-dimensional gauge-embedding fiber which contains a two-dimensional subspace called the gauge fiber. The vector spaces associated with these bundles are placed at every point in the space-time manifold. The gauge-embedding fiber allows one to compare vectors at different space-time points using ordinary parallel transport. In the language of differential geometry, the structure is built by inserting an $\mathbb{R}^2$ vector bundle into a trivial $\mathbb{R}^3$ vector bundle. Using this geometric interpretation, the wave function is a vector in the $\mathbb{R}^2$ gauge fiber. The two orthogonal and normalized basis vectors $\vec{e}_a$ that span the gauge fiber are identified with the real and imaginary directions where the index $a$ runs from 1 to 2 or equivalently $R, I$ for real and imaginary.

We use the Schrödinger equation with the mass-energy term,

$$iD_t \psi = -\frac{1}{2m}D^2 \psi + m \psi,$$

(1)

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Figure 1. The gauge fiber is a 2D subspace of a gauge-embedding space.

where \( h = c = 1 \) as a non-relativistic approximation to the complex Klein-Gordon equation. To couple the wave function to electromagnetism, the partial derivatives \( \partial_{\mu} \) are replaced by covariant derivatives \( D_{\mu} \) where \( \mu \) is a space-time index. In the covariant derivative

\[
D_{\mu} = \partial_{\mu} - i A_{\mu},
\]

(2)

we absorb the charge into the definition of \( A_{\mu} \) to allow a geometric interpretation of \( A_{\mu} \) as a correction to the choice of basis vectors. The geometric interpretation follows from

\[
\bar{e}^b \cdot \partial_{\mu} (\psi^a \bar{e}_a) = \partial_{\mu} \psi^b + (\bar{e}^b \cdot \partial_{\mu} \bar{e}_a) \psi^a
\]

(3)

where we associate the real and imaginary components of the wave function with basis vectors \( \psi = \psi^a \bar{e}_a = \psi^R \bar{e}_R + \psi^I \bar{e}_I \), we use the product rule while taking the derivative, and we use the dual-basis vectors \( \bar{e}^a \) to project the result back onto the gauge fiber. We identify the electromagnetic vector potential via

\[
(A_{\mu})^a_b = i \bar{e}^a \cdot \partial_{\mu} \bar{e}_b
\]

(4)

as described in references [1, 2, 3, 4]. Because we choose orthogonal and normalized basis vectors \( \bar{e}_a \), the vector potential \( (A_{\mu})^a_b \) is proportional to the rotation generator \( (A_{\mu})^a_b = A_{\mu} \left( \begin{array}{cc} 0 & i \\ -i & 0 \end{array} \right) \) where \( A_{\mu} \) is the traditional electromagnetic vector potential.

The basis vectors used to represent a constant electric field \( E_o \) in the \( x \) direction are:

\[
\bar{e}_R^j = \begin{pmatrix} \cos(\alpha t) \\ \sin(\alpha t) \\ 0 \end{pmatrix} \quad \text{and} \quad \bar{e}_I^j = \begin{pmatrix} -E_o x \\ E_o x \cos(\alpha t) \\ \sqrt{1 - (E_o x / \alpha)^2} \end{pmatrix}
\]

(5)

corresponding to \( A_t = E_o x \) where \( R \) and \( I \) represent the real and imaginary directions, \( \alpha \) is an arbitrary real constant parameterizing a set of geometries with equal curvature, and \( E_o \) also represents the magnitude of the curvature.
Figure 2. The curved path is the expectation value of the wave packet on a space-time chart. The straight line represents the future light cone for the same starting point.

By substituting these specific basis vectors into the Schrödinger equation, we arrive at our solution:

$$\psi(x, t) = A(t)e^{i(E+mE_0)x - E_0}t$$ \text{ where } \quad z = \frac{E_0x - E}{E_0} \left(2mE_0\right)^{\frac{1}{2}}.  

Equation 6 represents the contravariant matter vector (complex wave function) of a particle in a constant electric field in the $x$ direction with an energy eigenvalue of $E$. The energy eigenstates were found and normalized for a box 0.033 nm long. Using the various energy levels of the Airy function, we formed a Gaussian wave-packet initially centered about $x_0$. The expectation value for the particle in the simulation is shown in figure 2. The resulting wave function is then mapped onto the gauge fiber using the basis vectors in equation 5 and can be seen in figure 3.

Figure 3 shows an electric field $E_x = 90000$ nm$^{-2}$ in geometric units defined by equations 4 and 5. Using the trajectory of the expectation value from figure 2, we calibrate the apparent electric field magnitude in conventional units. The path is fit to an acceleration where $a = \frac{qE_x}{m}$, $m$ is the electron’s mass, and $q$ is the electron’s charge. Converting to standard units yields an acceleration of $3.1 \times 10^{27}$ m/s$^2$. Substituting $q$ and $m$, we find the electric field in the simulation has a magnitude of $1.77 \times 10^{16}$ Volts/m. These parameters are chosen so that the geometric features described in this paper can be seen to scale in figure 3.

The computer simulation shown in figure 3 allows us to see how the matter wave propagates in time. As seen in the figure, the matter vectors rotate counter clockwise as $t$ increases, and the gauge fibers change orientation within the gauge-embedding space both in $t$ and in $x$.

The wave’s propagation direction can be determined from the figure by finding the wave front, which is the surface where the wave function’s vectors are related by parallel transport in the gauge-embedding space. The wave propagates in the space-time direction perpendicular to the plane of the wave front. Figure 3 shows that the wave function begins with $v = 0$ at $t = 0$ because the wave front is parallel to the $x$-axis. When the wave function evolves to $t \approx 0.002$ nm, the geometric curvature of the gauge fiber gives a counter-clockwise advantage to the right side. This means the vectors on the upper right side of the figure are rotated counter clockwise relative to the vectors in the upper left side of the figure. Therefore, the new wave front has a slight negative slope in the $t$ vs $x$ plane, and the wave propagation will be to the left ($-\hat{x}$ direction). Figure 4 summarizes this paragraph’s observations.
Figure 3. The geometry of an electric field with an evolving wave function. The two short, perpendicular vectors form the basis which spans the gauge fiber. The longer vectors are the wave function.

Figure 4. Space-time diagram associated with figure 3 showing wave fronts and corresponding propagation vectors.
3. The Geometric Origin of Electric Force: An Insightful, Simple Calculation

A geometric understanding of electric force requires that the mass energy be included in the calculation. The energy, $\omega$, is defined as the rate of rotation of the mass vector on the gauge fiber (the disks pictured in figure 3). A rather simple yet powerful calculation explains why.

The electric field $E_x = F_{xt}$, when thought of as a curvature tensor, is defined by the phase difference $\theta$ associated with parallel transporting a vector (our wave function) along a closed path enclosing a space-time area element $\Delta x \Delta t$ given by $\Delta x \Delta t F_{xt} = \theta$ or equivalently

$$\theta = E_x \Delta x \Delta t.$$  \hfill (7)

This curvature affects the wave front. We will show this effect is an acceleration.

Our analysis is again guided by figures 3 and 4. Recall, the wave front is defined by the set of connected space-time points where the matter field vectors are related by parallel transport. When the wave is stationary at $t = 0$, the wave front is parallel to the $x$-axis. The wave advances perpendicular to the wave front. Near $t \approx 0.002$ nm, this new wave front forms an angle $\delta$ with the $x$ axis. This wave front corresponds to a wave moving with velocity $v$ to the left. Notice that the angle $\delta$ in radians is an approximation for the velocity at non relativistic speeds:

$$v = \left| \frac{dx}{dt} \right| = \tan(\delta) \approx \delta.$$  \hfill (8)

Near $t \approx 0.002$ nm, the wave front is tilted because the curvature of the gauge fiber leads the wave at two different spatial positions shown in figure 4 to appear to rotate at different rates when compared to each other. The curvature will give the right side a relative counter clockwise advantage. Over a period of time $\Delta t$, the left side will rotate in a counter-clockwise direction by $\omega \Delta t$, but the right side will rotate $\omega \Delta t + \theta$. The time at which the right side is along the same wave front as the left side is a little earlier than $\Delta t$ by an amount $s$ as expressed in

$$\omega \Delta t = \omega (\Delta t - s) + \theta.$$  \hfill (9)

We relate $s$ to the new velocity by using the triangle in figure 4 with the angle $\delta$ opposite to the side of length $s$

$$v = \delta = \frac{s}{\Delta x}.$$  \hfill (10)

where we again use the small-angle approximation. Substituting equation 7 for $\theta$ and equation 10 for $s$, we find

$$E_x = \omega \frac{v}{\Delta t}.$$  \hfill (11)

This indicates that over a time interval approximately equal to $\Delta t$, the velocity changed from 0 to $v$. Writing $\frac{v}{\Delta t}$ as the acceleration $a$,

$$E_x = \omega a.$$  \hfill (12)

We recognize this as Newton’s second law, $F = ma$, only if we include the rest mass in the frequency $\omega$ and recognize $F = E_x$. The force is equal to the electric field because we absorbed the charge into the definition of $A_\mu$ back in equation 2.

If our wave function was rotating clockwise, then the counter-clockwise advantage on the right side from the curvature would have been opposite the direction of rotation. Here we would have the left side more advanced than the right side, and the particle would accelerate to the right. Thus, in this non-relativistic, geometric optics approach, we understand the electric force for both positive and negative charges in a purely geometric manner.
4. Discussion
This paper studies the geometry of gauge theory involving classical fields. All of our examples use a background electric field and a test wave function propagating in that background. Although not second-quantized, there are several surprises even at this classical level.

The first surprise is charge uniqueness. If the electric force is geometrical in origin as described by equations 4 and 12, then all particles that propagate as rotating vector fields on the curved gauge fiber would have to share the same charge. Particles that rotate in opposite directions on the gauge fiber will accelerate in opposite directions. We are able to completely avoid the traditional concept of charge, yet we are still able to describe electromagnetic force. A second surprise is the sensitivity of the physical trajectory to the magnitude of the frequency. As seen in equation 12 from section 3, the acceleration in space-time depends on the actual magnitude of the frequency, not just the relative frequency.

In our work, the acceleration is due to a spatially dependent phase lag. The same logic holds in general relativity; the geodesic can be understood as wave propagation through a medium with a varying effective index of refraction [5]. However, not all spatially dependent phase lags lead to a deflection [6].

At the conference, we also presented research that made explicit contact with the Yang-Mills hidden spatial geometry described by Ken Johnson et al. in references [7, 8, 9]. Due to space limitations, this research will be presented in a future publication. We also plan to extend the geometry described in this presentation to incorporate spin.

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