Large-\(N\) transitions of the connectivity index

Francesco Aprile, Vasilis Niarchos

Crete Center for Theoretical Physics and
Crete Center for Quantum Complexity and Nanotechnology,
Department of Physics, University of Crete, 71303, Greece

E-mail: aprile@physics.uoc.gr, niarchos@physics.uoc.gr

ABSTRACT: The connectivity index, defined as the number of decoupled components of a quantum system, can change under deformations of the Hamiltonian or during the dynamical change of the system under renormalization group flow. Such changes signal a rearrangement of correlations of different degrees of freedom across spacetime and field theory space. In this paper we quantify such processes by studying the behavior of entanglement entropy, relative quantum entropy and quantum mutual information in a specific example: the RG flow in the Coulomb branch of large-\(N\) superconformal field theories. We argue that in this context there is an interesting sharp large-\(N\) transition in the middle of the RG flow from a non-separable phase of the Higgsed UV gauge theory to a separable phase of deformed decoupled CFTs in the IR. The entanglement entropy on a sphere with radius \(\ell\) detects this transition via the formation of a separatrix on the co-dimension-two Ryu-Takayanagi surface in multi-centered brane geometries above a critical value of \(\ell\). Other measures of entanglement and separability based on the relative quantum entropy detect a transition to a phase where they vanish identically. From the IR point of view the effect is closely related to the resummation of an infinite set of irrelevant multi-trace interactions.

KEYWORDS: Entanglement entropy, relative quantum entropy, holography, Coulomb branch, conformal field theory
1 Introduction

In any quantum system we can arbitrarily partition the total Hilbert space \( \mathcal{H} \) into two subspaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \). For a given configuration we can ask to what extent states in \( \mathcal{H}_1 \) are entangled with states in \( \mathcal{H}_2 \), or how strongly observables computed in \( \mathcal{H}_1 \) are correlated with observables in \( \mathcal{H}_2 \). This is an interesting question that can reveal useful information about the state of the system and its dynamical properties.
A well studied example is to take a system defined in \( p \) spatial dimensions and separate the degrees of freedom inside a spatial region \( A \) from the degrees of freedom in the complement \( A^c \). A natural measure of the entanglement of the two sets of degrees of freedom (in Hilbert spaces \( \mathcal{H}_A \) and \( \mathcal{H}_{A^c} \)) is the entanglement entropy defined as the von-Neumann entropy of the reduced density matrix \( \rho_A \)

\[
S = -\text{Tr}_{\mathcal{H}_A} [\rho_A \log \rho_A] .
\]  

(1.1)

\( \rho_A \) is obtained by tracing the total density matrix \( \rho \) over the states of the outside Hilbert space \( \mathcal{H}_{A^c} \)

\[
\rho_A = \text{Tr}_{\mathcal{H}_{A^c}} \rho .
\]  

(1.2)

\( S \) is an interesting quantity that has played a central role in many recent developments. For example, when applied to \((p + 1)\)-dimensional relativistic conformal field theories its dependence on the characteristic size of the region \( A \) holds information about basic constants of the theory, e.g. the central charge \( c \) in \((1 + 1)\) dimensions [1] (see e.g. [2] for a review), or the \( F \)-function in \((2 + 1)\) dimensions [3] etc.

Another possibility is to partition the system in field theory space, namely split the degrees of freedom at each point of spacetime into two subsets. This type of partitioning arises naturally, for example, when we have two distinct quantum systems with Hamiltonians \( H_1 \) and \( H_2 \) interacting weakly via an interaction Hamiltonian \( H_{\text{int}} \), but it can also be considered more generally without reference to a specific type of dynamics.

The first question we want to ask in this paper is the following. Given an arbitrary split of the degrees of freedom of a quantum system, e.g. a quantum field theory, in spacetime and/or in field theory space, can we define a meaningful measure of the entanglement or strength of correlation between the subsystems? Several well known measures from quantum information theory that quantify the notion of a distance from separability, e.g. measures based on the relative quantum entropy, turn out to be very well suited for this purpose. We will review the relevant concepts, and give specific definitions, in section 2.

The second question we want to raise concerns the behavior of such measures under deformations of the theory, or under the dynamical change of the parameters of the system under the renormalization group (RG) flow.

For example, there are many cases where by tuning the parameters of a theory, or by looking at the system at different energies, the interaction coupling in \( H_{\text{int}} \) between two subsystems becomes weak or even turns off. In the latter case the subsystem Hamiltonians \( H_1 \) and \( H_2 \) decouple completely. Any observable computed in this product theory (e.g. an arbitrary correlation function) factorizes in a product of observables of theory 1 and theory 2. It is useful to define quite generally a connectivity index that characterizes how many independent parts a quantum system possesses. Specifically, if the arbitrary correlation function of a system factorizes in a product of correlation functions of \( n \) independent subsystems we define the connectivity index to be \( n \). An analogous definition was considered previously in [4]\(^1\) (see

\(^1\)In [4] the connectivity index was defined in a different (not necessarily equivalent) manner in terms of the independent gauge groups of a gauge theory.
also [5] for very closely related work). Identifying the connectivity index of a generic theory is not a straightforward task. Notice that a theory with \( n \) decoupled components will have in general \( n \) independently conserved energy-momentum tensors. This suggests the possibility that the connectivity index counts the number of conserved energy-momentum tensors. This relation is discussed further in section 8.

The connectivity index can decrease when \( H_{\text{int}} \) turns on, or increase when \( H_{\text{int}} \) turns off. The measures of entanglement mentioned above will behave accordingly. It is possible, however, to encounter more subtle situations where the effects mediated by \( H_{\text{int}} \) are suppressed until a finite value of the interaction coupling. RG flows in the Coulomb branch of large-\( N \) superconformal field theories (SCFTs) provide concrete examples.

For definiteness, let us consider the Coulomb branch of the four-dimensional \( SU(N) \) \( \mathcal{N} = 4 \) super-Yang-Mills (SYM) theory. In the ultraviolet (UV) we have an \( SU(N) \) gauge theory with connectivity index 1. Turning on the vacuum expectation values of the adjoint scalars we move away from the origin of the Coulomb branch, the gauge group is Higgsed, say to \( SU(N_1) \times SU(N_2) \times U(1) \), and there is an RG flow to the infrared (IR) where an \( SU(N_1) \) gauge theory decouples from an \( SU(N_2) \) gauge theory. The connectivity index in the far IR is 2. At low energies the leading order interaction between the two IR CFTs is mediated by an irrelevant double-trace dimension 8 operator \([6,7]\) (see section 4.3 for specific expressions). This operator mediates interaction and turns off only at the extreme IR.

The interest in the large-\( N \) limit stems from the following observation. At leading order in the \( 1/N \) expansion, the multi-trace operators that mediate the interaction between the IR CFTs do not contribute to the anomalous dimension of any combination of the two IR energy-momentum tensors and despite the deformation both remain independently conserved. In this situation, we argue that observables that measure entanglement between the \( SU(N_1) \) and \( SU(N_2) \) degrees of freedom detect a connectivity index 2 in a vicinity of the IR fixed point. Since the connectivity index is 1 in a vicinity of the UV fixed point we conclude that the change of the connectivity index does not happen infinitesimally away from the IR, but it happens in the middle of the RG flow at energies comparable to the mass of the \( W \)-bosons set by the expectation values of the adjoint scalars on the Coulomb branch. It is interesting to ask how different observables detect this ‘large-\( N \) transition’.

One observable that we consider in the main text, in order to verify the above picture, is the entanglement entropy \([1.1]\) for a spherical geometry \( A \) with radius \( \ell \). As \( \ell \) changes from 0 to \( +\infty \) we probe physics from the UV to the IR. In the large-\( N \) limit we can evaluate the entanglement entropy with the generalized Ryu-Takayanagi prescription \([8–10]\) by determining a minimal co-dimension-2 hypersurface in the multi-throat geometry of separated stacks of branes. We perform this analysis quite generally for the 4d \( \mathcal{N} = 4 \) SYM theory on D3 branes, the 3d ABJM theory on M2 branes and for the 2d CFT on D1-D5 branes. In all cases we find that the Ryu-Takayanagi minimal hypersurface exhibits a separatrix at a radius \( \ell_c \) where it shows signs of critical behavior. We argue that this is the entanglement entropy description of the large-\( N \) transition of the connectivity index.

In section 2 we define other measures of entanglement based on the concept of relative
quantum entropy. Although we do not know how to compute these measures holographically from gravity in the large-\(N\) limit we argue that they must be zero along the RG flow in a vicinity of the IR fixed point and non-zero in a vicinity of the UV fixed point. Specific arguments in favor of this proposal are presented in section 8. We discuss possible behaviors in the middle of the RG flow in section 9. The overall quantum field theory dynamics that we are considering is summarized in section 3.

The main computational results of the paper are presented in sections 4-7. Section 5 contains a description of the qualitative features of the Ryu-Takayanagi surface in multi-centered geometries. The reader can consult this section for a quick overview of the results that arise by studying the holographic entanglement entropy in this context. Concrete quantitative results based on the analysis of the equations of the Ryu-Takayanagi minimal surface are presented in sections 6, 7. For instance, in section 6 we notice that the UV expansion of the holographic entanglement entropy does not receive contributions from the lowest order harmonics. This is a gravity prediction for a corresponding statement about entanglement entropy in large-\(N\) superconformal field theories.

Interesting aspects of our story and other open issues are summarized and further discussed in the concluding section 9. Useful technical details are relegated to appendix A.

2 Separability, relative quantum entropy and other useful concepts

Assume that we have a \((p + 1)\)-dimensional quantum system with Hilbert space \(\mathcal{H}\) and we partition \(\mathcal{H}\) both in spacetime and field theory space. In spacetime we separate states supported inside a spatial region \(A\) from states in the complement \(A^c\). In field theory space, we separate (at each point of spacetime) degrees of freedom of a subsystem 1 from degrees of freedom of a subsystem 2. Then, the reduced density matrix \(\rho_A\) (1.2) is a matrix that lives in the product Hilbert space \(\mathcal{H}_{A,1} \otimes \mathcal{H}_{A,2}\). We are interested in a measure that quantifies the entanglement of the states of the two subsystems 1 and 2. We will focus on the properties of the density matrix \(\rho_A\) keeping the additional dependence on the size of the region \(A\) as a useful way to keep track of the entanglement across different length (or energy) scales.

A standard definition in quantum information theory (see [11] for a review) postulates that the state represented by \(\rho_A\) is separable if it can be written as a sum of product states in the form

\[
\rho_A = \sum_k p_k \rho_{A,1}^{(k)} \otimes \rho_{A,2}^{(k)},
\]

with \(p_k \geq 0, \sum_k p_k = 1\). If not, \(\rho_A\) is called entangled. In the special case with a single probability coefficient \(p_k\) non-zero, i.e. when

\[
\rho_A = \rho_{A,1} \otimes \rho_{A,2}
\]

the state is called simply separable. This is the case mentioned in the introduction where no correlations between subsystems 1 and 2 exist.
Testing for separability is in general a very hard problem. However, it is possible to formulate a measure that quantifies how far from separability a quantum system is by using the concept of relative quantum entropy. For any two density matrices $\rho, \sigma$ the relative quantum entropy of $\rho$ with respect to $\sigma$ is defined as

$$S(\rho \| \sigma) = \text{Tr} [\rho \log \rho] - \text{Tr} [\rho \log \sigma]. \tag{2.3}$$

One can prove the Klein inequality (see e.g. [11]), which states that $S(\rho \| \sigma)$ is a positive-definite quantity that vanishes only when $\rho = \sigma$, i.e. when the states $\rho$ and $\sigma$ are indistinguishable. On the other extreme, the relative quantum entropy $S(\rho \| \sigma)$ is infinite when the two states are perfectly distinguishable. This fact played a useful role in the recent work [12].

One can use the relative quantum entropy to define a measure of how far a system is from separability. The usual approach defines the following quantity

$$D_{\text{REE}}(\rho) = \min_{\sigma = \text{separable}} S(\rho \| \sigma), \tag{2.4}$$

which is called relative entropy of entanglement. The minimum is obtained by sampling over the whole space of separable states. $D_{\text{REE}}(\rho)$ is zero if and only if $\rho$ is a separable state.

Since we are interested in simply separable states we can modify this definition in an obvious way by taking the minimum over the simply separable states. In what follows, however, we consider instead a related quantity that we define as follows. Concentrating on the specific context of our density matrix $\rho_A$, and a partitioning into two subsystems 1 and 2, we consider the relative quantum entropy

$$S_{12}(\rho_A) \equiv S(\rho_A \| \rho_{A,1} \otimes \rho_{A,2}) \tag{2.5}$$

where $\rho_{A,1} \otimes \rho_{A,2}$ is defined as the tensor product of the reduced density matrices

$$\rho_{A,1} = \text{Tr}_{H_{A,2}} [\rho_A] , \quad \rho_{A,2} = \text{Tr}_{H_{A,1}} [\rho_A]. \tag{2.6}$$

This quantity vanishes if and only if our system is completely decoupled into the two subsystems 1 and 2. In fact, one can show that the definition (2.5) is simply the quantum mutual information

$$S_{12}(\rho_A) = S(\rho_{A,1}) + S(\rho_{A,2}) - S(\rho_A) \tag{2.7}$$

where $S(\rho)$ is the standard entanglement entropy (1.1) and $S(\rho_{A,1}), S(\rho_{A,2})$ are ‘inter-system’ entanglement entropies. A version of the latter with $A^c = \emptyset$ was studied recently in the context of holography in [13].

As a concept, separability is very well adapted to describe properties related to the connectivity index and its behavior under changes of the system, e.g. under renormalization group flows that lead to Hilbert space fragmentation. We will soon examine these properties in a specific context of large-$N$ quantum field theories.
3 Hilbert space fragmentation in quantum field theory

There are several common mechanisms in quantum field theory that change the connectivity index. For example, in strongly coupled gauge theories an operator will frequently hit the unitarity bound and decouple from the remaining degrees of freedom as a free field.\(^2\) Another common example, involves gauge theories whose gauge group \(G\) is Higgsed under the RG flow from \(G\) in the UV to a product \(G_1 \times G_2 \times \cdots G_n\) in the IR. In both cases the Hilbert space fragments and the connectivity index increases. It should be noted that there are also situations where the connectivity index decreases under RG running. This occurs naturally in RG flows where a mass gap develops in the IR, e.g. a massive degree of freedom is removed from the spectrum in the far IR or a gauge group confines.

In this paper we will examine closely the case of gauge group Hissing in the Coulomb branch of large-\(N\) superconformal field theories. As we explained in the introduction this situation exhibits an interesting phenomenon. The general setup has the following ingredients. The UV conformal field theory (CFT\(_{\text{UV}}\) with gauge group \(G_{\text{UV}}\)) flows to an IR conformal field theory which is a product of decoupled theories, e.g. CFT\(_{\text{IR}}\) = CFT\(_1\)×CFT\(_2\) (with gauge group \(G_1 \times G_2\)). At small energies, slightly above the extreme IR, CFT\(_1\) and CFT\(_2\) are deformed by irrelevant interactions of the form

\[
\int d^{p+1}x \left( g_1 V_1 + g_2 V_2 + h_{12} O_1 O_2 + \ldots \right) .
\]  

(3.1)

The operators \(V_1\) and \(V_2\) are single-trace operators in CFT\(_1\) and CFT\(_2\) respectively (with irrelevant couplings \(g_1, g_2\) of order \(N\)). As such, they do not mediate any interaction between the two IR theories. The inter-CFT interaction is provided solely by the third term that involves a double-trace operator of the form \(O_1 O_2\), where \(O_1\) and \(O_2\) are single-trace in CFT\(_1\) and CFT\(_2\).

\(^3\)Explicit examples of such operators will be provided in the next section for \(\mathcal{N} = 4\) SYM theory. The dots in (3.1) indicate interactions of higher scaling dimension, i.e. more irrelevant operators, that become increasingly important as we increase the energy. We stress that in this expansion the interaction between CFT\(_1\) and CFT\(_2\) is always mediated by multi-trace operators as dictated by gauge invariance (a point emphasized in [15]).

This picture applies independently of whether we take a large-\(N\) limit or not. The fact that makes the large-\(N\) limit interesting is that many of the effects of the multi-trace operators are subleading in the 1/\(N\)-expansion (in particular, effects associated to the separability of the IR theories). Hence, at leading order we expect a finite range of energies above the extreme IR where the interaction between CFT\(_1\) and CFT\(_2\) works without spoiling separability. As we move up in energy the irrelevant couplings \(g_1, g_2, h_{12}, \ldots\) increase and the expansion in (3.1) resums. At some characteristic energy scale —comparable to the scale set by the vacuum expectation value that Higgsed the UV gauge group— the system is expected to pass

\(^2\)There are many well known examples of this type of decoupling. For instance, a class of three-dimensional superconformal field theories with a rich pattern of such features at strong coupling was studied in [14].

\(^3\)The double-trace coupling \(h_{12}\) is \(\mathcal{O}(N^0)\). The overall Lagrangian is normalized so that all terms are \(\mathcal{O}(N^2)\).
eventually (somewhere in the middle of the RG flow) from the IR separable phase to the UV non-separable phase. This transition smoothens out once we start including $1/N$ corrections, but the main point about the competition of the suppressing effects of the large-$N$ limit and the effects of inverse RG running remains.

The main purpose of this paper is to quantify this transition using the measures of entanglement presented in the previous section 2.

**Entanglement entropy.** The entanglement entropy of large-$N$ conformal field theories with a weakly curved gravitational dual can be computed efficiently using the Ryu-Takayanagi prescription in the AdS/CFT correspondence. This computation, which will be performed in the next four sections, involves the analysis of a minimal co-dimension-2 surface in multi-centered brane geometries in ten- or eleven-dimensional supergravities. The non-standard feature of this computation is that the minimal surface embeds non-trivially along the compact manifolds transverse to AdS. We will see that the above transition is closely related to the formation of a separatrix in the geometry of the Ryu-Takayanagi minimal surface.

**Relative entropy of entanglement and quantum mutual information.** In section 2 we presented two measures of separability, the relative entropy of entanglement $D_{\text{REE}}(\rho)$ (2.4) and the quantum mutual information $S_{12}(\rho_A)$ (2.7). Currently, we are not aware of an efficient computational method for such quantities in interacting quantum field theories, either directly in quantum field theory or holographically. Nevertheless, the above discussion indicates that we should anticipate the following features.

In a system where the connectivity index changes along the RG flow, as in the gauge theory example above from $n = 1$ in the UV to $n = 2$ in the IR, the relative quantum entropy on a sphere of radius $\ell$ will be a function of $\ell$, $S_{12}(\ell)$, that vanishes at $\ell = \infty$ (the IR) and is non-zero at $\ell = 0$ (the UV). At leading order in the $1/N$ expansion, $S_{12}(\ell)$ is expected to remain zero in a finite interval around $\ell = \infty$, because the interaction-mediating operators in the IR are multi-trace. In this interval the entanglement entropy $S(\rho)$ is directly related to the entanglement entropies of the reduced density matrices $\rho_{A,1}$, $\rho_{A,2}$

$$S(\rho_A) = S(\rho_{A,1}) + S(\rho_{A,2}).$$

(3.2)

Below a certain critical size, this relation ceases to exist and $S_{12}(\ell)$ becomes strictly positive, i.e.

$$S(\rho_A) > S(\rho_{A,1}) + S(\rho_{A,2}).$$

(3.3)

Formally, the maximum value of $S_{12}(\ell)$ is expected to be $S_{12}(0)$, i.e. the maximum is expected to occur at the extreme UV where the IR CFTs are maximally entangled or the least separable. It is of interest to understand precisely how this transition to a non-zero UV value occurs. A discussion of possible behaviors is provided in section 9.

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4The statement $\rho_A = \rho_{A,1} \otimes \rho_{A,2}$ holds with an appropriate definition of the Hilbert spaces $H_{A,1}$ and $H_{A,2}$ over which we trace in (2.6). This point is further clarified in section 8.
Notice that the definition of $S_{12}$ in (2.5)-(2.6) is most suitable in the IR description where the massive $W$-boson degrees of freedom have been integrated out and the theory is formulated in terms of the degrees of freedom of the IR CFTs. Since this is a less natural description in the UV, a more suitable definition of $S_{12}$ may be useful for the whole RG flow. Such issues, are not expected, however, from other measures of separability, e.g. the relative entropy of entanglement $D_{\text{REE}}(\rho)$. We anticipate these measures to exhibit the same behavior as the one described above.

We elaborate further on these features with additional supporting evidence in section 8.

4 Coulomb branch of SCFTs and multi-centered geometries

In this preparatory section we collect useful facts and notation for the geometries involved in the holographic computation of the entanglement entropy in the Coulomb branch of superconformal field theories.

4.1 Notation and main features of multi-centered brane geometries

We focus on supersymmetric conformal field theories with a weakly curved gravitational dual in string/M-theory. The gravitational description of the Coulomb branch of these theories is directly related to the geometry of a discrete collection of flat parallel D/M-branes in 10 or 11-dimensional supergravity. This geometry is uniquely specified by a single harmonic function $H = H(\vec{y})$, where $\vec{y}$ are the coordinates transverse to the brane volume. The supergravity solution also carries charge under the corresponding $(p+1)$-form gauge fields and generically sources the dilaton $\Phi$.\footnote{The specific well known expressions for these fields can be found in the literature. Here we will concentrate on the metric, which is the only field that participates directly in our computation.}

In this paper, we will focus on multi-centered geometries given by:

- D3 branes in $D = 10$ dimensions, relevant for the $d = 4 \mathcal{N} = 4$ SYM theory,
- M2 branes in $D = 11$ dimensions, relevant for the $d = 3 \mathcal{N} = 8$ ABJM Chern-Simons-Matter theories [16],
- D1-D5 bound states in $\mathbb{R}^{1,5} \times \mathcal{M}^4$. The D5 branes wrap the compact manifold $\mathcal{M}^4$ (usually taken as $T^4$ or $K3$) and give rise at low energies to an interacting $(1 + 1)$-dimensional superconformal field theory.

The corresponding geometries in asymptotically flat space\footnote{The metric of the D3 and D1-D5 systems is given here in the string frame of type IIB string theory.} are given by the metrics

\begin{equation}
\text{D3 : } ds^2 = H_3^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H_3^{1/2} \delta_{ij} dy^i dy^j,
\end{equation}

\begin{equation}
\text{M2 : } ds^2 = H_2^{-2/3} \eta_{\mu\nu} dx^\mu dx^\nu + H_2^{1/3} \delta_{ij} dy^i dy^j,
\end{equation}
D1D5 : 
\[ ds^2 = (H_1 H_5)^{-1/2} \eta_{\mu \nu} dx^\mu dx^\nu + (H_1 H_5)^{1/2} \delta_{ij} dy^i dy^j + \left( \frac{H_1}{H_5} \right)^{1/2} ds^2(M^4). \] 

(4.3)

The harmonic functions \( H_3, H_2 \) are

\[
H_3(\vec{y}) = 1 + \sum_{I=1}^{K} \frac{N_I \rho_3}{|\vec{y} - \vec{y}_I|^3}, \quad \rho_3 = 4\pi g_s \alpha'^2
\]

(4.4)

\[
H_2(\vec{y}) = 1 + \sum_{I=1}^{K} \frac{M_I \rho_2}{|\vec{y} - \vec{y}_I|^5}, \quad \rho_2 = 2^5 \pi^2 \ell_P^6
\]

(4.5)

The vectors \( \vec{y}_I \) locate the position of the different stacks of branes in the transverse space. We are considering \( K \) stacks of D3 (M2) branes, each one made out of \( N_I \) D3 branes (\( M_I \) M2 branes). \( g_s \) and \( \alpha' \) are the string coupling constant and string Regge slope. \( \ell_P \) is the eleven-dimensional Planck length.

For the D1-D5 system, the two harmonic functions \( H_1 \) and \( H_5 \) are:

\[
H_1(\vec{y}) = 1 + \sum_{I=1}^{K} \frac{Q_1^{(1)} \rho_1}{|\vec{y} - \vec{y}_I|^2}, \quad \rho_1 = \frac{g_s \alpha'}{v}
\]

(4.6)

\[
H_5(\vec{y}) = 1 + \sum_{I=1}^{K} \frac{Q_5^{(5)} \rho_5}{|\vec{y} - \vec{y}_I|^2}, \quad \rho_5 = g_s \alpha'
\]

(4.7)

where \( v \) is essentially the volume of \( M_4 \), i.e. \( v = V_4/(2\pi)^4 \alpha'^2 \). It will be technically convenient to focus on D1-D5 bound states with parameters that obey the relation

\[
\frac{Q_J^{(1)}}{Q_1^{(1)}} = \frac{Q_J^{(5)}}{Q_1^{(5)}} \quad \forall \ 1 < J \leq K.
\]

(4.8)

This restriction guarantees that the dilaton \( \Phi \), given by the relation \( e^{2\Phi} = H_1/H_5 \), will be constant in the near-horizon limit.

**Near-horizon limit.** For the D3 and D1-D5 branes, the decoupling limit [17] is defined by sending \( \alpha' \to 0 \), and keeping the ratios \( \vec{u} = \vec{y}/\alpha' \) and \( \vec{u}_I = \vec{y}_I/\alpha' \) fixed. As a result, the 1 in the harmonic functions drops out, and the geometry remains finite in units of \( \alpha' \). Under the assumption (4.8), the product \( H_1 H_5 \) simplifies

\[
H_{1,5} \equiv (H_1 H_5)^{1/2} = \sum_{I=1}^{K} \frac{Q_I \rho}{|\vec{u} - \vec{u}_I|^2}, \quad Q_I = \sqrt{Q_1^{(1)} Q_5^{(5)}}, \quad \rho = \frac{g_s^2 \alpha'^2}{v}.
\]

(4.9)

For the M2 branes the decoupling limit is obtained by sending \( \ell_P \to 0 \) and keeping \( \vec{u} = \vec{y}/\ell_P^{3/2} \) and \( \vec{u}_I = \vec{y}_I/\ell_P^{3/2} \) fixed.
In summary, the D1-D5 system is now described by the function $H_{1\cup 5}$, and the D3 and M2 backgrounds are described by

$$H_3 \to \sum_{I=1}^{K} \frac{N_I \rho_3}{|\vec{u} - \vec{u}_I|^4}, \quad H_2 \to \sum_{I=1}^{K} \frac{M_I \rho_2}{|\vec{u} - \vec{u}_I|^6}. \quad (4.10)$$

The resulting geometry interpolates between an $AdS_{p+2} \times S^q$ space at $|\vec{u}| \to \infty$, that captures the UV fixed point with connectivity index 1, and a decoupled product of $K AdS_{p+2} \times S^q$ spaces as $\vec{u}$ gets scaled towards the centers $\vec{u}_I$. The latter describes the extreme IR fixed point with connectivity index $K$.

### 4.2 UV physics

For the cases we analyze the asymptotic $|\vec{u}| \to \infty$ geometry is an $AdS_{p+2} \times S^q$ space with $(p, q) = (1, 3), (2, 7), (3, 5)$ for the D1-D5, M2 and D3 brane systems respectively. The radius of each $AdS_{p+2}$ space is

- **D3**: $R_{UV}^2 = \left(4\pi g_s \sum_I N_I\right)^{1/2}$ \quad (4.11)
- **M2**: $R_{UV}^2 = \frac{1}{4} \left(2^5 \pi^2 \ell_6^6 \sum_I M_I\right)^{1/3}$ \quad (4.12)
- **D1D5**: $R_{UV}^2 = \left(\frac{g_s^2}{v} \sum_I Q_I\right)$ \quad (4.13)

These UV $AdS_{p+2} \times S^q$ geometries are dual to the microscopic $(p+1)$-dimensional superconformal field theories mentioned in the beginning of the previous subsection. For concreteness, let us focus for the moment on the most emblematic case, i.e. the duality between string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4 SU(N)$ SYM theory.

The multi-centered D3 brane solutions are dual to a configuration in $\mathcal{N} = 4$ SYM in which the $SU(N)$ gauge group has been Higgsed down to $SU(N_1) \times \ldots \times SU(N_K) \times U(1)$ ($N = N_1 + \ldots + N_K$). Conformal invariance, as well as the $SO(6)$ R-symmetry of the theory, are broken by the non-vanishing vacuum expectation value of the gauge-invariant chiral operators

$$\mathcal{O}^{(n)} \propto C_{i_1,\ldots,i_n}^{(n)} \text{Tr}[X^{i_1} \ldots X^{i_n}], \quad (4.14)$$

where $C_{i_1,\ldots,i_n}^{(n)}$ are totally symmetric traceless rank $n$ tensors of the $SO(6)$-charged real adjoint scalars $X^i$ of the theory. These modes arise in the gravity dual from a Kaluza-Klein decomposition of the transverse $S^5$ space. By analyzing the asymptotic, large $|\vec{u}|$, behavior of these modes in the multi-centered geometry one can determine the vacuum expectation value of the operators (4.14) [18–20].

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4.3 IR physics

The decoupled product of gauge theories that arises in the extreme infrared of the Coulomb branch translates, in the dual multi-centered geometry, into a decoupled product of $K$ string theories on the $AdS_{p+2}^{(I)} \times S^q_{(I)}$ spacetimes. Each of these spacetimes arises from the full multi-centered geometry by taking the limit $\vec{u} \to \vec{u}_I$ that isolates the gravitational dynamics near the $I$-th center. The radius of the $I$-th $AdS$ spacetime is weighted by the single coefficient $N_I$, $M_I$ or $Q_I$, respectively.

As explained in section 3, in the IR description of the RG flow the IR CFTs interact off criticality via an infinite set of irrelevant multi-trace interactions. For example, in the case of $\mathcal{N} = 4$ SYM theory the leading single-trace operator $V_I$ for the $I$-th IR theory (see equation (3.1)) is a dimension 8 operator of the form \cite{6,7,21}

$$V = \text{Tr} \left[ F_{\mu\nu} F^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} - \frac{1}{4} (F_{\mu\nu} F^{\mu\nu})^2 \right] + \ldots . \quad (4.15)$$

The coefficient $g_I$, which is proportional to the sum

$$g_I \propto \sum_{J \neq I} \frac{N_J \rho^3}{|\vec{u}_J - \vec{u}_I|^4} , \quad (4.16)$$

is the only single-trace datum that knows about the interaction between the IR CFTs. (4.15) is also the type of interaction that appears in the small field strength expansion of the Dirac-Born-Infeld action that describes the exit from the near-horizon throat. In the current context the single-trace interaction (4.15) describes how the throat in question connects with the rest of the geometry.

Besides the single-trace operator (4.15) there are also double-trace dimension 8 operators of the form \cite{6}

$$\text{Tr}_I \left[ F_{\mu\nu} F^{\mu\nu} \right] \text{Tr}_J \left[ F_{\mu\nu} F^{\mu\nu} \right] + \ldots \quad (4.17)$$

which mediate the direct inter-CFT interactions mentioned in equation (3.1).

5 Holographic Entanglement entropy

In a field theory in $p+1$ dimensions, the static entanglement entropy of a space-like region $A$ is defined as the von-Neumann entropy of the density matrix $\rho_A$ which is obtained by tracing out the degrees of freedom in the complement of $A$ (see equations (1.1), (1.2)).

For conformal theories living on the boundary of $AdS_{p+2}$, the Ryu-Takayanagi prescription (RT) computes the holographic entanglement entropy (HEE) by considering the area of a $p$-dimensional minimal surface in $AdS_{p+2}$, whose boundary is $\partial A$. We will refer to this surface as $\gamma_{RT}$ \cite{8}. There is a beautiful derivation of the correctness of this prescription for spherical entangling surfaces. By conformally mapping the density matrix $\rho_A$ to a thermal density matrix, the authors of \cite{22} showed that the thermal entropy of the dual hyperbolic
black hole coincides with the HEE computed à la Ryu-Takayanagi. The relation between the entropy of $\rho_A$ and the minimal area condition was further investigated and clarified in [23].

For non-conformal theories with a gravity dual, a natural extension of the Ryu-Takayanagi prescription was given in [9,10]. These authors considered the functional

$$S[\partial A] = \frac{1}{4G_N^D} \int d^{D-2}\xi e^{-\Phi} \sqrt{\det g_{ind}}$$  \hspace{1cm} (5.1)

where $g_{ind}$ is the induced metric of a minimal co-dimension-2 surface $\gamma$ in the full string theory or M-theory background. The surface $\gamma$ is again specified to have $\partial A$ as its boundary. This generalized prescription is the prescription we will apply in the computation that follows. In our setup, the dilaton field $\Phi$ is a constant for all the cases we will consider; the D3, M2 and D1-D5 branes. This statement is obvious for D3 and M2 branes, and follows from the assumption (4.8) in the case of the D1-D5 bound states.

It is clear that for $AdS_{p+2} \times S^q$ spaces, the Ryu-Takayanagi prescription is in perfect agreement with (5.1). When there is no dependence on the transverse sphere, the problem of a minimal surface $\gamma$ that wraps $S^q$ reduces to the problem of finding $\gamma_{RT}$ in $AdS_{p+2}$. The Newton constant in $AdS_{p+2}$ is related to $G_N^D$ through the formula

$$G_N^{p+2} = G_N^D / \text{Vol}(S^q).$$  \hspace{1cm} (5.2)

A typical class of examples in which the prescription (5.1) is non-trivial are the confining backgrounds of [24,25], for which the entanglement entropy was studied in [10]. These backgrounds are of the type $\mathcal{M}_{p+1} \times \mathcal{C}_{D-p-1}$, where $\mathcal{C}$ is a cone over a certain compact manifold $\mathcal{S}$. The volume of $\mathcal{S}$ may shrink along the radial coordinate of the cone, and since $\gamma$ wraps $\mathcal{S}$, it will be sensitive to the dynamics of these extra dimensions along the RG flow.

Similarly, the multi-centered geometries of interest in this paper are not product spaces globally. They become locally $AdS_{p+2} \times S^q$ spacetimes only in certain asymptotic regions. If the dimension of $\gamma$ was different from $D-2$, other data would be needed to determine it, and the surface would not be unique for a given $\partial A$. An example appears in the holographic computation of the Wilson loop in [26].

Multi-centered geometries

The remainder of this section provides a qualitative description of the surface $\gamma$ in the multi-centered backgrounds described previously. We consider spherical entangling surfaces when $p = 2, 3$, and intervals when $p = 1$. It is useful to choose space-like coordinates adapted to these geometries. In dimensions $p = 2, 3$, we choose spherical coordinates: $\vec{x} = (\sigma, \phi_1, \ldots, \phi_{p-1})$, where $\sigma > 0$ is the radius of the sphere and $\phi_i$ are angles. In one dimension we use a similar notation: $\sigma$ is the spatial field theory coordinate that runs along the real line. The entangling region $A$ is described by the equation $\sigma^2 < \ell^2$. This means that $\sigma \in \mathcal{I}_\ell$ where $\mathcal{I}_\ell = (-\ell, \ell)$ for $p = 1$, and $\mathcal{I}_\ell = [0, \ell)$ for $p = 2, 3$.

The main example we will consider in detail is the case of the two-centered geometry. The two-centered geometries are conveniently described by hyper-cylindrical coordinates in
the transverse space. The branes are separated along a direction \( z \), and the space orthogonal to \( z \) is described by hyper-spherical coordinates \((y, \Omega_1, \ldots, \Omega_{q-1})\). In this setting, the functions \( H_i \) of the previous section will depend both on \( z \in \mathbb{R} \) and \( y > 0 \). The origin \( z = 0 \) is taken to be the center of mass. We can also introduce polar coordinate in the \((z, y)\) plane,

\[ z = r \cos \theta, \quad y = r \sin \theta \]

with \( r > 0 \) and \( \theta \in [0, \pi] \). For coincident branes, \( K = 1 \), the coordinate \( r \) becomes the radial coordinate of \( \text{AdS}_{p+2} \), and \( \theta \) becomes the polar angle of the \( q \)-sphere.

The minimal surface is static with Dirichlet boundary conditions in the time direction, which will not play any further role. The coordinates describing the co-dimension-2 surface are chosen as follows

\[ \xi_i = \phi_i, \quad i = 1, \ldots, p-1, \]
\[ \xi_{j+p-1} = \Omega_j, \quad j = 1, \ldots, q-1, \]
\[ \xi_{D-3} = \theta, \]
\[ \xi_{D-2} = \sigma. \]  

(5.3)

The embedding in the \( D \)-dimensional background is specified by the function \( r(\sigma, \theta) \), where \( \sigma \in A \) and \( \theta \in [0, \pi] \). This function is an interesting object because it mixes the evolution along a field theory direction, \( \sigma \), with the change of the geometry along the transverse space direction \( \theta \). The non-trivial dependence on \( \theta \) originates from \( H_i \) which are explicit functions of \( \theta \).

The behavior of \( r(\sigma, \theta) \) can be understood qualitatively by regarding \( r(\sigma, \theta) \) as a map from \( \mathcal{I}_\ell \times [0, \pi] \) to the plane \((z, y)\). We imagine foliating the surface \( r(\sigma, \theta) \) by fixing a certain \( \sigma_0 \), drawing the curve \( r_{\sigma_0}(\theta) = r(\sigma_0, \theta) \) in the plane \((z, y)\), and moving \( \sigma_0 \) in the interval \( \mathcal{I}_\ell \). For example, in \( \text{AdS}_{p+2} \times S^q \), the solution is given by the Ryu-Takayanagi surface which is \( \theta \) independent, therefore \( r(\sigma, \theta) = r(\sigma) \), and the map \( r_{\sigma_0}(\theta) \) draws circles of radius \( r(\sigma_0) \). From this simple analysis we are able to infer three out of the four boundary conditions that fix a generic \( \theta \)-dependent solution on \( \mathcal{I}_\ell \times [0, \pi] \):

\[ r(\sigma, \theta) \bigg|_{\sigma = \ell} = \infty, \quad \partial_\theta r(\sigma, \theta) \bigg|_{\theta = 0} = 0, \quad \partial_\theta r(\sigma, \theta) \bigg|_{\theta = \pi} = 0. \]

(5.4)

We will discuss the boundary condition at \( \sigma = 0 \) in a moment.

To start thinking about \( r(\sigma, \theta) \) in two-centered solutions, it is useful to first consider the limit \( \ell \to \infty \). In this limit the surface probes the physics of the deep IR of the field theory where the UV gauge group has been Higgsed and the energy scales of interest are well below the mass of the massive \( W \) bosons. In the gravity dual this limit zooms into the vicinity of the two centers which can be regarded as decoupled. The surface \( \gamma \) is then given by the union \( \gamma_1 \cup \gamma_2 \), where \( \gamma_i = \gamma_{RT} \times S^q \). At this point, it is important to recall that \( \gamma_{RT} \) has a turning point at \( \sigma = 0 \), i.e. \( r(\sigma) > r(0) \) for any \( \sigma \in \mathcal{I}_\ell \). The fact that \( \sigma = 0 \) is the turning point follows from the symmetries of the entangling surface and from the assumption that \( \gamma_{RT} \) is convex.
Figure 1: Qualitative behavior of the map $r_{\sigma_0}(\theta)$ as a function of $\sigma_0$, and for large values of $\ell$. The red line represents the separatrix. Below the separatrix, a suitable set of variables that describe the surface will be given in Section 7.1.

When $\ell$ is finite, but large enough for $\gamma$ to probe the IR throats, the picture we have just described will be approximately valid only locally close to each of the two centers. In a neighborhood of $\sigma_0 = 0$ the map $r_{\sigma_0}(\theta)$ draws approximately small disconnected circles around the position of each stack of branes (points $z_1$ and $z_2$ in the $(z,y)$ plane in Figure 1). The curve $r_m(\theta) \equiv r(\sigma = 0, \theta)$ generalizes the notion of turning point in the AdS Ryu-Takayanagi surface and obeys the boundary conditions

$$\partial_\sigma r(\sigma, \theta) = 0 \quad \text{at} \quad \sigma = 0 \quad \text{for any} \ \theta.$$  

The overall picture in the IR is summarized by the brown curves in Figure 1.

The above description refers to the IR patch of the surface $\gamma$ associated to a space-like region of large enough radius $\ell$. In the opposite regime, we can ask what happens at $\sigma_0 = \ell - \epsilon$ ($\epsilon \ll \ell$), when the curve $r(\ell - \epsilon, \theta)$ is close to the UV boundary. Because the boundary is $AdS_{UV} \times S^q$, this curve is again approximately $\theta$-independent and the associated map $r_{\sigma_0}(\theta)$ draws a large circle in the $(z,y)$ plane (captured by the blue curves in Figure 1).

The inevitable conclusion of the above analysis is that, although the surface $\gamma$ is always simply connected, the topology of the curves $\{r_{\sigma_0}(\theta)\}_{\sigma_0 \in I_\ell}$ may change as we vary $\sigma_0$. When $\ell$ is large enough, the minimal surface will have a UV patch where $r_{\sigma_0}(\theta)$ is topologically $S^1$, and an IR patch where $r_{\sigma_0}(\theta)$ is topologically $S^1 \times S^1$. For such a surface $\gamma$ there is necessarily a branch point. The curve $r_b(\theta) \equiv r(\sigma_b, \theta)$ at which this branch point belongs will be referred to from now on as the separatrix. This is sketched as the red line in Figure 1.

The topology change that we described above does not occur for surfaces with small enough $\ell$ that can only probe the UV part of the full geometry. For such surfaces the curves $r_{\sigma_0}(\theta)$ are topologically $S^1$ for any $\sigma_0 \in I_\ell$. It is clear that the discriminating quantity
between the existence of the topology change or not, for a given \( \ell \), is the turning point curve \( r_m(\theta) \). Accordingly, we will distinguish between the following two phases:

- **Phase A**, for \( \ell < \ell_c \), where the topology of \( r_m(\theta) \) is \( S^1 \). In this case we can describe \( \gamma \) with single-valued coordinates.

- **Phase B**, for \( \ell > \ell_c \), where the topology of \( r_m(\theta) \) is \( S^1 \times S^1 \). In that case a separatrix exists and when \( r_{\sigma_0}(\theta) \) moves below the separatrix, \( r(\sigma, \theta) \) becomes double-valued.

The counterpart of the transition between these phases in field theory is a transition of the behavior of the entanglement entropy as a function of \( \ell \) at \( \ell_c \).

The qualitative behavior of \( \gamma \) for multi-centered geometries can be deduced by following the same logic as in the two-centered solution. However, in the general case it will not be possible to restrict the discussion to a certain plane \((z, y)\), and one has to consider the full transverse space.

### 6 UV expansion of the entanglement entropy

In this section we will study more explicitly the HEE of phase A. The equation of motion of \( r(\sigma, \theta) \) is a non-linear, quite challenging, PDE. Yet, we are able to obtain a series expansion of the solution by expanding in a small dimensionless parameter that combines the mass scale of symmetry breaking (equivalently the center separation in the geometry) and the sphere radius \( \ell \). Our perturbative solution is analytic in the variables \( \sigma \) and \( \theta \), and at zeroth order coincides with the AdS \( \gamma_{RT} \) solution. The perturbative solution does not allow us to detect analytically the formation of the separatrix as we approach \( \ell_c \), but it confirms the qualitative description of the previous section.

By direct integration of the generalized HEE functional we obtain a series of finite corrections to the \( AdS_{p+2} \) entanglement entropy. Perhaps suprisingly, the translation of the result to field theory language suggests that the lowest chiral primary operators do not contribute to these corrections.

#### 6.1 Minimal surface action and its equations of motion

In phase A the variable \( r(\sigma, \theta) \) is single-valued as a function of \( \theta \), thus we can write the induced metric on \( \gamma \) by using the coordinates (5.3). Referring to the components of the background metrics (4.1), (4.2) and (4.3), with the generic notation,

\[
d s^2 = g_{\mu \nu} dx^\mu dx^\nu, \quad \vec{x} = \left(t, \sigma, \phi_1, \ldots, \phi_{p-1}, r, \theta, \Omega_1, \ldots, \Omega_{q-1}\right),
\]

the induced metric on \( \gamma \), in the coordinates (5.3), is given by

\[
 d s_{\text{ind}}^2 = d s_{\text{ind}}^2 \bigg|_{(\sigma, \theta)} + d s_{\text{ind}}^2 \bigg|_{(\phi, \Omega)} , \tag{6.1}
\]
where
\[ ds^2 |_{(\sigma, \theta)} = \left( g_{\sigma \sigma} + g_{rr} \left( \frac{\partial r}{\partial \sigma} \right)^2 \right) d\sigma^2 + 2 g_{rr} \frac{\partial r}{\partial \sigma} \frac{\partial r}{\partial \theta} d\sigma d\theta + \left( g_{\theta \theta} + g_{rr} \left( \frac{\partial r}{\partial \theta} \right)^2 \right) d\theta^2 , \]
\[ ds^2 |_{(\phi, \Omega)} = g_{ij} d\phi^i d\phi^j + g_{ab} d\Omega^a d\Omega^b . \]
(6.2)

The HEE functional is then
\[ S_p = \frac{1}{4 G_D N} \int d\vec{\Omega} \sqrt{g_{ab}} \int d\vec{\phi} \sqrt{g_{ij}} \int d\sigma d\theta L_p [\theta, r(\sigma, \theta)] \]
(6.3)

where the Lagrangian \( L_p \) can be put into a form valid for all cases of interest here (the D3, M2 and D1-D5 branes),
\[ L_p = \sigma^{p-1} K[\theta, r] H[\theta, r] \sqrt{1 + \frac{\partial r^2}{r^2} + (H[\theta, r])^2 \partial_\sigma r^2} . \]
(6.4)

In (6.4) we defined the functions
\[
\begin{align*}
D3 : & \quad H^2 = H_3 , \quad K = r^5 \sin^4 \theta \\
M2 : & \quad H^2 = H_2 , \quad K = r^7 \sin^6 \theta \\
D1D5 : & \quad H^2 = H^2_{1,15} , \quad K = r^5 \sin^2 \theta .
\end{align*}
\]
(6.5)

In the two-centered geometries we fix the origin of the \( z \) axis at the center of mass of the system, namely we set
\[ z_1 N_1 + z_2 N_2 = 0 . \]
(6.6)

After the implementation of the condition (6.6), the Euler-Lagrange equation following from (6.4) depends only on a single dimensionful parameter, \( z_1 \) for example. Schematically, the single PDE that we need to solve is the equation of motion of \( r \)
\[ \text{Eq} [r(\sigma, \theta), z_1] = 0 . \]
(6.7)

The explicit form of this equation is provided in Appendix A.

6.2 Perturbative UV Solution

Before entering the details of the calculation, we review the \( AdS \) solution making manifest the underlying scale invariance. This is our starting point towards a perturbative solution of the non-linear PDE (6.7) that follows from (6.4).

It is convenient to work with the variable \( \zeta = 1/r^2 \), in the cases of D3 and D1-D5 branes, and \( \zeta = 16 R^2 / r^4 \) for the M2 branes. The UV boundary is now at \( \zeta = 0 \). In our conventions, the metric of \( AdS_{p+2} \) is written as
\[ ds^2 = \frac{1}{R^2} \frac{1}{\zeta} \left( -dt^2 + d\sigma^2 + \sigma^2 d\vec{\phi}_{p-1}^2 + R^4 \frac{d\zeta^2}{4} \right) , \]
(6.8)
where $R = R_{UV}$ is the $AdS$ UV radius defined case by case in (4.11)-(4.13). The Ryu-Takayanagi surface is obtained from the embedding function $\zeta(\sigma)$. Its equation of motion and the corresponding solution are,

$$\text{Eq} \left[ \zeta(\sigma), z_1 = 0 \right] = \zeta'' + \frac{p - 1}{2} \frac{\zeta'^2}{\zeta} + \frac{p - 1}{x} \zeta' \left( 1 + \frac{R^4}{4} \frac{\zeta'^2}{\zeta} \right) + \frac{2p}{R^4} = 0, \quad (6.9)$$

$$\zeta(\sigma) = \frac{\ell^2}{R^4} \left( 1 - \frac{\sigma^2}{\ell^2} \right) \equiv \frac{\ell^2}{R^4} F \left( \frac{\sigma}{\ell} \right). \quad (6.10)$$

It should be noted that with our choice of spherical entangling surfaces, the embedding function is independent of $p$. In the r.h.s. of (6.10) we wrote $\zeta(\sigma)$ in a conformal fashion: we isolated the pre-factor $\ell^2$, and defined the function $F(\hat{\sigma})$ that depends only on the dimensionless combination $\hat{\sigma} = \sigma/\ell$. The pre-factor captures the weight of $\zeta(\sigma)$ under rescaling of $\ell$. We also notice that the equation (6.9) has weight zero; in particular, the corresponding equation for $F(\hat{\sigma})$ has no $\ell$ dependence.

Now the idea is to consider a UV ansatz for $\zeta(\sigma, \theta)$ of the type,

$$\zeta(\sigma, \theta) = \frac{\ell^2}{R^4} F(\hat{\sigma}, \theta). \quad (6.11)$$

As expected, by plugging (6.11) into the equation of motion we obtain an equation for $F(\hat{\sigma}, \theta)$ which depends only on the dimensionless parameter $\varepsilon = \frac{\Delta}{R^2}$ for D3 and D1-D5 branes with $\Delta \equiv z_1 \ell$, and $\varepsilon = \frac{\Delta}{R^{3/2}}$ for M2 branes with $\Delta = z_1 \sqrt{\ell}$. The limit $\varepsilon \to 0$ is well defined and gives back (6.9). Around it we can solve the equation for $F(\hat{\sigma}, \theta)$ in perturbation theory. Schematically, our problem becomes

$$\text{Eq} \left[ F_p(\hat{\sigma}, \theta), \Delta \right] = 0,$n

$$F_p(\hat{\sigma}, \theta) = (1 - \hat{\sigma}^2) + \sum_{k=1}^{\infty} \Delta^k f_p^{(k)}(\hat{\sigma}, \theta). \quad (6.12)$$

In (6.12) we restored the label $p$ to stress that the perturbative solution depends on the number of dimensions. The functions $f_p^{(k)}$ capture the two-center deformation of the UV $AdS$ solution. Solving for $f_p^{(k)}$ still requires finding the solution of a set of PDEs. However, this problem is tractable and analytic solutions can be obtained.

Perturbative equations

For D3 and D1-D5 branes it is possible to write down simple explicit formulae. Results for the M2 branes are more involved due to the fact that the UV $AdS$ comes in horospherical coordinates. However, the algorithm to find the perturbative solution is valid for generic $p$.

For $p = 1, 3$, the functions $f_p^{(k)}$ solve a PDE of the form,

$$\partial_{\hat{\sigma}}^2 f_p^{(k)} + \frac{p - 1}{\hat{\sigma}(1 - \hat{\sigma}^2)} \partial_{\hat{\sigma}} f_p^{(k)} + \frac{1}{(1 - \hat{\sigma}^2)^2} \left( \partial_{\hat{\theta}}^2 f_p^{(k)} + (p + 1) \cot \theta \partial_{\theta} f_p^{(k)} \right) = F^{(k)}(\hat{\sigma}, \cos \theta) \quad (6.13)$$
where $\mathcal{F}^{(k)}$ are forcing terms whose explicit $\theta$ dependence is inherited from $\mathcal{H} = \mathcal{H}(\zeta, \cos \theta)$. At fixed $k$, the forcing term $\mathcal{F}^{(k)}$ is determined by the lower order solutions $f_p^{(m)}$ for $m < k$. We find the first non-trivial $\mathcal{F}^{(k)}$, and solve for $f_p^{(k)}$. Then we proceed to compute $\mathcal{F}^{(k+1)}$, solve for $f_p^{(k+1)}$, and continue by iteration. An important observation is that upon the change of variable $v = \cos \theta$, the forcing terms $\mathcal{F}^{(k)}$ become polynomials in $v$ with $\sigma$-dependent coefficients. Therefore, the ansatz

$$f_p^{(k)} = g_p^{(k,k)}(\sigma)v^k + g_p^{(k,k-1)}(\sigma)v^{k-1} + \ldots + g_p^{(k,0)},$$

(6.14)

which is compatible with the boundary conditions $\partial_{\sigma} f_p^{(k)} = 0$ at $\theta = 0, \pi$, solves the $\theta$ dependence in (6.13). The set of functions $\{v^m\}_{m=0}^\infty$ is just a rewriting of the standard Fourier basis in a way that is compatible with our boundary conditions. For any $f_p^{(k)}$ of the form (6.14), the PDE (6.13) generates a set of $k$ ODEs for the functions $\{g_p^{(k,n)}\}_{n=0}^k$. The boundary conditions that uniquely specify the solution of each $g_p^{(k,n)}(\sigma)$ are

$$g_p^{(k,n)}(\sigma = 1) = 0, \quad \partial_{\sigma} g_p^{(k,n)}(\sigma = 0) = 0.$$  

(6.15)

The use of the coordinate $\zeta$ makes manifest the fact that in order to have a perturbative solution which is consistent with the UV $AdS$ asymptotics, the functions $g_p^{(k,n)}$ have to vanish like $(1 - \hat{\sigma}^2)^\alpha$ with $\alpha \geq 1$. When $\alpha > 1$, corrections will be sub-leading at the boundary.

The equations for the functions $g_p^{(k,n)}$ are linear ODEs with forcing terms induced by $\mathcal{F}^{(k)}$. The highest mode $g_p^{(k,k)}$ has no forcing term. At fixed $n < k$, the equation for $g_p^{(k,n)}$ has forcing terms induced by the functions $g_p^{(k,m)}$ with $n < m \leq k$. Starting from $n = k$ and solving for $g_p^{(k)}$ it is possible to generate the forcing term for $g_p^{(k-1)}$ and solve its equation. At the next step we generate the forcing terms for $g_p^{(k-2)}$ and solve its equation. Repeating this algorithm it is possible to calculate the full tower of $\{g_p^{(k,n)}\}_{n=0}^k$ modes.

We conclude this subsection with one relevant comment: there is no $f_p^{(1)}$ contribution to the perturbative solution. This statement follows from: 1) the fact that the equation of motion depends just on $\mathcal{H}^2$, 2) the expansion of $\mathcal{H}$ in terms of $\Delta$ is given by the UV expansion of the harmonic functions (4.9) and (4.10), and 3) in the latter, the contribution at order $\Delta$ is proportional to the center of mass condition and therefore vanishes.

### 6.3 Two-centered D3 geometries

We are now in position to carry out the perturbative calculation in the two-centered D3 brane solution more explicitly. The analytic result for $F_3(\hat{\sigma}, \theta)$ can be written in a compact form by defining the variable $X = (1 - \hat{\sigma}^2)$. The first non-trivial corrections to $\gamma_{RT}$ are

$$F_3(\hat{\sigma}, \theta) - X = \frac{2N_1}{3N_2} \left( 6 \cos^2 \theta - 1 \right) X^2 \left( \frac{\Delta}{R^2_{UV}} \right)^2 + \frac{N_1 - N_2}{N_2} \left( 8 \cos^3 \theta - 3 \cos \theta \right) X^{5/2} \left( \frac{\Delta}{R^2_{UV}} \right)^3 + \left( g^{(4,4)} \cos^4 \theta + g^{(4,2)} \cos^2 \theta + g^{(4,0)} \right) \left( \frac{\Delta}{R^2_{UV}} \right)^4 + \ldots$$

(6.16)
where

\[ g^{(4,4)} = -\frac{16N_1 N_2^2 - 3N_1 N_2 + N_2^3 X^3}{N_2} , \tag{6.17} \]

\[ g^{(4,2)} = \frac{16N_1 9N_1^2 - 17N_1 N_2 + 9N_2^2 X^3}{N_2} , \tag{6.18} \]

\[ g^{(4,0)} = \frac{4 N_1^2}{9 N_2^2} X^2 - \frac{N_1}{N_2} \frac{27 N_1^2 - 71 N_1 N_2 + 27 N_2^2 X^3}{45 N_2^2} . \tag{6.19} \]

It is intriguing that \( f^{(2)}(X, \theta) \) and \( f^{(3)}(X, \theta) \) are separable, whereas \( f^{(4)}(X, \theta) \) is not. In general, higher modes \( f^{(k)} \) with \( k \geq 4 \) are also not separable. We will come back to this aspect of the solution later on. Finally, we could have guessed from the beginning that when \( N_1 = N_2 \) a symmetry argument implies that \( f^{(k)} \) with \( k \) odd will be vanishing.

Plugging the solution (6.16) into the HEE functional given by equations (6.3)-(6.4) we obtain

\[ S_3 = \frac{1}{4G^{(5)}_N} \left( 4\pi R_{UV}^5 \right) \left( I_3(\ell) - \frac{4}{9} \frac{N_1^2}{N_2^2} \left( \frac{\Delta}{R_{UV}^2} \right)^4 + \ldots \right) , \tag{6.20} \]

\[ I_3(\ell) = \int_0^1 d\hat{\sigma} \frac{\hat{\sigma}^2}{(1-\hat{\sigma}^2)^2} = \int_{a/\ell}^1 ds \frac{\sqrt{1-s^2}}{s^3} . \tag{6.21} \]

In (6.20) we used the relation

\[ G^{(5)}_N = \frac{G^{(10)}_N}{\pi^3 R_{UV}^5} . \tag{6.22} \]

The integral \( I_3(\ell) \) is the \( AdS \) Ryu-Takayanagi result \[8\] with \( a/\ell \) their UV cutoff. Surprisingly, even though the profile of the surface gets corrections at order \( \Delta^2 \) and \( \Delta^3 \), the first non-vanishing contribution to the entanglement entropy comes at fourth order. Higher order correction are also non-trivial but their expression is too cumbersome and not sufficiently illuminating to repeat here. In agreement with the expectation that the renormalized entanglement entropy decreases along the RG flow, the first non-trivial correction to \( I_3(\ell) \) in (6.20) comes with a negative sign.

Geometrically, the reason why there are nor \( \Delta^2 \) neither \( \Delta^3 \) corrections to the HEE can be seen as follows. We first observe that

\[ (6 \cos^2 \theta - 1) \propto Y_{\tilde{0},2}^{(5)}(\theta) , \quad (8 \cos^3 \theta - 3 \cos \theta) \propto Y_{\tilde{0},3}^{(5)}(\theta) , \tag{6.23} \]

where \( Y_{\tilde{0},l}^{(5)} \) are the \( S^4 \)-invariant 5-dimensional spherical harmonics. Then, we notice that the expression of the integrand of \( S_3 \), at order \( \Delta^2 \) or \( \Delta^3 \), takes the form of a scalar product\(^7\) between the harmonics (6.23) and the identity. In particular we find,

\[ S_3 \sim I_3(\ell) + \langle 1|Y_{\tilde{0},2} \rangle \left( \frac{\Delta}{R_{UV}^2} \right)^2 + \langle 1|Y_{\tilde{0},3} \rangle \int_0^1 d\hat{\sigma} \hat{\sigma}^2 \sqrt{1-\hat{\sigma}^2} \left( \frac{\Delta}{R_{UV}^2} \right)^3 + \ldots \tag{6.24} \]

\(^7\)The measure in the scalar product is \( \sqrt{g_{ij}} \) on the \( S^5 \).
The result (6.20) follows from the orthonormality condition \(\langle Y_m|Y_n \rangle = \delta_{mn} \). We would like to stress that the decomposition (6.24) is not immediately obvious, and it comes out from the interplay between the UV expansion of the metric and the form of the solution.

The use of the scalar product between harmonics may be a useful way of packaging the expansion of the HEE. It also suggests that in order to have non-vanishing corrections, we should find at least terms of the type \(Y^2\). The only way to generate such contributions is through the non-linearity of the background metric, and indeed multi-centered geometries are non-linear solutions.

As we briefly reviewed in section 4.2 the field theory description of the two-centered D3 solution is well understood at the UV. By splitting the stack of coincident branes along the \(z\) direction, we give an expectation value to one of the real adjoint scalar fields of \(\mathcal{N} = 4\) SYM. Therefore, the 1-point function of the gauge invariant chiral operators \(\mathcal{O}^{(n)}\), defined in (4.14), will be non-trivial. Given the relation between these operators and the harmonics of \(S^5\), it is possible to show that the AdS/CFT correspondence correctly reproduces the 1-point function of the operators \(\mathcal{O}^{(n)}\) unambiguously [18]. This fact invites us to think of the result (6.24) as the statement that at small distances corrections to the entanglement entropy associated to \(\mathcal{O}^{(2)}\) and \(\mathcal{O}^{(3)}\) vanish. It would be interesting to examine this possibility directly in field theory.

### 6.4 Two-centered D1-D5 geometries

In this section we repeat the perturbative computation of the entanglement entropy in two-centered D1-D5 geometries producing a prediction for the corresponding two-dimensional conformal field theories.

Keeping the notation \(X = (1 - \hat{\sigma}^2)\), the analytic form of \(F_{1\cup 5}\) up to fourth order is

\[
F_{1\cup 5}(\hat{\sigma}, \theta) - X = -\frac{2}{3} \frac{Q_1}{Q_2} (4 \cos^2 \theta - 1) X^2 \left( \frac{\Delta}{R_{\text{UV}}^2} \right)^2 \\
+ \frac{Q_1}{Q_2} \frac{Q_2}{Q_2} (4 \cos^3 \theta - 4 \cos \theta) X^{5/2} \left( \frac{\Delta}{R_{\text{UV}}^2} \right)^3 \\
+ \left( g^{(4,4)} \cos^4 \theta + g^{(4,2)} \cos^2 \theta + g^{(4,0)} \right) \left( \frac{\Delta}{R_{\text{UV}}^2} \right)^4,
\]

where

\[
g^{(4,4)} = -\frac{16 Q_1}{Q_2} \frac{18 Q_1^2 - 49 Q_1 Q_2 + 18 Q_2^2}{45 Q_2^2} X^3,
\]

\[
g^{(4,2)} = -\frac{16 Q_1^2}{135 Q_2^2} X^2 + \frac{24}{5} \frac{Q_1}{Q_2} (Q_1 - Q_2)^2 X^3,
\]

\[
g^{(4,0)} = \frac{8}{15} Q_1^2 \frac{Q_2}{Q_2} X + \frac{22}{135} Q_1^2 Q_1 - \frac{Q_1}{Q_2} 18 Q_1^2 - 53 Q_1 Q_2 + 18 Q_2^2 \frac{45 Q_2^2}{X^3}.
\]
Figure 2: In the upper figure we plot the transverse scalar function $F_{1U5}$ in the $(z,y)$ plane for $\Delta = 2$ and $z_1 = -z_2 = 1$. We are using the polar coordinates $z = \zeta \cos \theta$ and $y = \zeta \sin \theta$. The location of the branes is indicated by a red dot. The UV boundary is at the origin. The foliation corresponds to equally spaced intervals in $(0,\ell)$, and is approximately made by circles. For this value of $\Delta$ the equation of motion is satisfied with a minimum accuracy of $10^{-5}$. In the inset we show an extrapolation to a value of $\Delta$ which comes closer to the formation of the separatrix. In the lower figure we use coordinates $z = r \cos \theta$ and $y = r \sin \theta$ with $r = 1/\sqrt{\zeta}$. In this specific plot the function $F_{1U5}$ is extrapolated to $\Delta = 2.755$. The r.h.s. part of the plot, where the solution is less reliable, has been excised. The qualitative features of this solution agree with the features anticipated in the general discussion in Section 5. We see how the surface deforms around the centers and how the turning point of the surface approaches a separatrix.

As we found in the case of the D3 brane solution, the $\hat{\sigma}$ and $\theta$ dependence of $f^{(2)}_1$ and $f^{(3)}_1$ factorizes and we can write

$$f^{(2)}_1 \propto Y^{(3)}_{0,2} X^2, \quad f^{(3)}_1 \propto Y^{(3)}_{0,3} X^{5/2},$$

where $Y^{(3)}_{0,l}$ are harmonics of $S^3$ symmetric with respect to the $\vec{\Omega}$ angles.

From the series expansion of $F_{1U5}$, we obtain a series expansion for the HEE. At lower
Figure 3: We plot the finite part of the HEE defined by subtracting $I_1(\ell)$. In units of $R_{UV}/4G_N^{(3)}$, the blue (red) curve represents the series expansion up to order $\Delta^{18}$ ($\Delta^{16}$). The embedding function $\zeta$ and the HEE have different sensibility with respect to $\Delta$.

orders we find

$$S_1 = \frac{R_{UV}}{4G_N^{(3)}} \left( I_1(\ell) - \frac{1}{20} \frac{Q_1^2}{Q_2^2} \left( \frac{\Delta}{R_{UV}^2} \right)^4 + \ldots \right) \quad (6.30)$$

$$I_1(\ell) = \int_0^1 d\tilde{\sigma} \frac{2}{1 - \tilde{\sigma}^2} = 2 \int_{2a/\ell}^{\pi/2} ds / \sin s \quad (6.31)$$

where we used the relation

$$G_N^{(3)} = \frac{G_N^{(6)}}{2\pi^2 R_{UV}^4}. \quad (6.32)$$

The integral $I_1(\ell)$ is the Ryu-Takayanagi result [8], and the first non-trivial correction comes at fourth order, as in the case of the D3 brane system. Along the lines of (6.24), we find that the vanishing of $\Delta^2$ and $\Delta^3$ corrections can be interpreted as the vanishing of the scalar product between different harmonics.

In addition, we computed $F_{1,5}$ for a D1-D5 system with $Q_1 = Q_2$ up to order $\Delta^{18}$, and studied the convergence of the series. We checked explicitly that at orders $k > 3$, separation of variables does not occur for any $f^{(k)}$. Because our perturbative expansion makes use of a spectral decomposition, it works quite well in a certain range of $\Delta$. An example is given in Figure 2, where we observe that the qualitative features of the solution agree with the features anticipated in section 5.

After subtracting $I_1(\ell)$ the HEE of the D1-D5 system is expressed as a series expansion in $\Delta$ with coefficients that can be determined analytically. The resulting series is alternating. For example, the coefficient of $\Delta^{2k}$ for $k = 1, \ldots, 6$ are,

$$\left\{ \frac{1}{20}, \frac{8}{567}, -\frac{1,567}{170,100}, \frac{40,729}{7,016,625}, -\frac{101,669,532}{23,308,883,125}, \frac{30,609,041,679}{9,050,920,001,125}, \ldots \right\}. \quad (6.33)$$

The corresponding curve is also plotted in Figure 3.
6.5 Two-centered M2 geometries

We conclude this section by analyzing the perturbative solution of $F_2$ for two-centered M2 brane geometries. The notation is unchanged, $X = (1 - \hat{\sigma}^2)$. The leading contributions to the embedding function are given by

$$F_2(\hat{\sigma}, \theta) - X = -3 \frac{M_1}{M_2} \left( 8 \cos^2 \theta - 1 \right) X^{3/2} \left( \frac{\Delta}{R_{UV}^{3/2}} \right)^2$$

$$+ 64 \sqrt{2} \frac{M_1 - M_2}{M_2} \frac{M_1}{M_2} \left( \cos^3 \theta - \frac{3}{10} \cos \theta \right) X^{7/4} \left( \frac{\Delta}{R_{UV}^{3/2}} \right)^3$$

$$+ \left( g^{(4,4)} \cos^4 \theta + g^{(4,2)} \cos^2 \theta + g^{(4,0)} \right) \left( \frac{\Delta}{R_{UV}^{3/2}} \right)^4 + \ldots$$

where $R_{UV}$ is the radius of the UV AdS and,

$$g^{(4,4)} = -\frac{32 M_1}{M_2} \frac{10 M_2^2 - 37 M_1 M_2 + 10 M_2^2}{M_2^2} X^2$$

$$g^{(4,2)} = \frac{20 M_1}{M_2} \frac{8 M_1^2 - 17 M_1 M_2 + 8 M_2^2}{M_2^2} X^2$$

$$g^{(4,0)} = -\frac{21 M_1^2}{M_2^2} \left( \sqrt{X} - \log \left( 1 + \sqrt{X} \right) - \frac{X}{2} \right) - \frac{M_1}{M_2} \frac{32 M_1^2 - 89 M_1 M_2 + 32 M_2^2}{4 M_2^2} X^2 .$$

Certain features of $F_2$ are similar to the previous cases. In particular, we find for any $p$ that the corrections $f_p^{(2)}$ and $f_p^{(3)}$ are solved by separation of variables. The origin of this feature is unclear. It is possible that supersymmetry is related to this effect (recall that we are studying BPS configurations).

The HEE expanded at lower orders is

$$S_2 = \frac{1}{4 G_N^{(4)}} (2 \pi R_{UV}^2) \left( I_2(\ell) - \frac{35}{4} \frac{M_1^2}{M_2^2} \left( \frac{\Delta}{R_{UV}^{3/2}} \right)^4 + \ldots \right)$$

$$I_2(\ell) = \int_0^1 d\hat{\sigma} \frac{\hat{\sigma}}{(1 - \hat{\sigma}^2)^{3/2}} = \int_{a/\ell}^1 \frac{ds}{s^2}$$

where the lower-dimensional Newton constant is,

$$G_N^{(4)} = \frac{G_N^{(11)}}{\pi^4/3 R_{S7}^4} .$$

In defining $G_N^{(4)}$ we made use of the relation $R_{S7} = 2 R_{UV}$. The expression (6.40) again shows that the first non-trivial correction to the Ryu-Takayanagi result $I_2(\ell)$ [8] comes at fourth order.
7 IR expansion of the entanglement entropy

As we increase the radius \( \ell \) of the entangling surface, the bulk minimal surface \( \gamma \) starts to probe the interior of the \( D \)-dimensional bulk geometry. For a given \( \ell_c \), the surface hits the branching point, and for \( \ell \geq \ell_c \) the topology of \( \gamma \) is that of a pant with two legs. This phenomenon is the bulk manifestation of the decoupling occurring in the Coulomb branch of large-\( N \) theories. Geometrically, for \( \ell \geq \ell_c \), the surface is “attracted” towards the position of the branes. The qualitative picture to keep in mind is given by Figure 1.

Target space coordinates adapted to the center-of-mass become problematic if we want to describe \( \gamma \) in Phase B. Below the separatrix \( r(\sigma, \theta) \) is double-valued as a function of \( \theta \) for fixed \( \sigma \) in a neighborhood of \( \sigma = 0 \). To overcome this problem we will use a different system of coordinates. This is also motivated by the following field theory observation. The end-point of the RG flow is a collection of decoupled theories, therefore the leading contribution to the entanglement entropy in the deep IR has to be the sum of the entanglement entropies of each individual throat. This expectation implies that as \( \ell \to \infty \), the contribution to the area of \( \gamma \) coming from the patch outside the separatrix has to become subleading. We will see how the new coordinate system clarifies the role of the separatrix as we take the deep IR limit.

7.1 Adapted coordinates

We first focus on two-centered geometries with \( \mathbb{Z}_2 \) symmetry, namely \( z_1 = -z_2 \equiv \bar{z} \). The change of coordinates relevant for this case is constructed as follows. Starting from the hyper-cylindrical coordinates \((z, y)\) we introduce

1) polar coordinates \( z = r \cos \theta \) and \( y = r \sin \theta \),

2) we define the \((u, v)\) variables by means of the relation,

\[
u + iv = \left(\sqrt{(z + iy)^2 - \bar{z}^2}\right)^2,
\]

which is equivalent to

\[
r^2 = \sqrt{(u + \bar{z}^2)^2 + v^2}, \quad \theta = \frac{1}{2} \arctan\left(\frac{v}{u + \bar{z}^2}\right),
\]

and finally,

3) we consider polar coordinates \( u = \eta \cos \psi \) and \( v = \eta \sin \psi \).

The geometry in the \((u, v)\) plane is such that the two stacks of branes are both located at the origin, \( u = 0 \) and \( v = 0^\pm \), one in the upper half plane and the other in the lower half plane. The \( \mathbb{Z}_2 \) symmetry has become a reflection symmetry between these two planes. From the relation (7.1), it is simple to see that the interval \( \{ |z| \leq \bar{z}, y = 0 \} \) has been mapped to \( \{-\bar{z}^2 < u < 0, v = 0^\pm\} \), whereas the \( y \)-axis \( \{y > 0, z = 0\} \) and semi-infinite lines
Figure 4: Circles in the $(u,v)$ plane (l.h.s. picture) are mapped to closed curves in the $(z,y)$ plane (r.h.s. picture). The red dots indicate the position of the branes. The black dot in the $(u,v)$ plane is mapped to the origin in the $(z,y)$ plane.

$\{|z| \geq \bar{z}, y = 0\}$ have been mapped to $\{u < -\bar{z}^2, v = 0\}$ and $\{u > 0, v = 0^\pm\}$, respectively. See Figure 4 for an illustration. Geodesics can cross the line $\{u < -\bar{z}^2, v = 0\}$, and go from the upper to the lower half plane. The lines $\{u > -\bar{z}^2, v = 0^\pm\}$, instead, are a boundary. The change of variables (7.1) is borrowed from 2d complex analysis [27].

UV and IR limits. As an example, the two-centered D1-D5 metric with $Q_1 = Q_2 \equiv Q$ has the following translation in the new coordinates

\[
\begin{align*}
H_{1,5} (dz^2 + dy^2) &\rightarrow \frac{R_{UV}^2}{4\eta^2} \left( 1 + \frac{\bar{z}^2}{\sqrt{\bar{z}^4 + \eta^2 + 2\bar{z}^2\eta\cos \psi}} \right) (d\eta^2 + \eta^2 d\psi^2) , \\
H_{-1,5}^{-1} dz^2 &\rightarrow \frac{\eta^2}{R_{UV}^2} \left( \frac{dx^2}{\bar{z}^2 + \sqrt{\bar{z}^4 + \eta^2 + 2\bar{z}^2\eta\cos \psi}} \right).
\end{align*}
\]  

Formulas (7.3)-(7.4) are useful as concrete reference for the subsequent calculations. However, the discussion that follows is general, and it holds for any $p$, i.e. for D3 and M2 branes as well.

Describing the Coulomb branch in this coordinate system is advantageous because the UV and the IR limits of the geometry can be formally explored by sending $\eta \rightarrow \infty$ and $\eta \rightarrow 0$, as in the case of coincident branes. In the limit $\eta \rightarrow \infty$ we recover the UV $AdS \times S$ geometry with radius $R_{UV}$,

\[
\begin{align*}
ds_{UV}^2 &= \frac{\eta}{R_{UV}^2} dx^2 + \frac{R_{UV}^2}{4\eta^2} d\eta^2 + R_{UV}^2 \left( \frac{d\psi^2}{4} + \sin^2 \frac{\psi}{2} d\Omega^2 \right) \\
&\rightarrow R_{UV}^2 \left( \frac{\rho^2 dx^2 + d\rho^2}{\rho^2} + \frac{d\psi^2}{4} + \sin^2 \frac{\psi}{2} d\Omega^2 \right) \quad \text{with} \quad \eta = R_{UV}^2 \rho^2 .
\end{align*}
\]

\footnote{The determination of the arctan in (7.2) has to be chosen correctly when $u > -\bar{z}^2$ and $u < -\bar{z}^2$.}
In the $\eta \to 0$ limit we obtain the metric
\[ ds^2_{IR} = \frac{1}{4 R^2_{IR}} \eta^2 \frac{dx^2}{\bar{z}^2} + R^2_{IR} \frac{d\eta^2}{\eta^2} + R^2_{IR} \left( d\psi^2 + \sin^2 \psi d\vec{\Omega}^2 \right) . \] (7.7)

It is important to point out two facts about (7.5) and (7.7). The first is that the metric (7.7) is described by a radial coordinate which is essentially a double covering of the UV AdS. The second is that the metric (7.7) still depends on $\bar{z}$ and therefore we need to properly define the IR limit. In fact, from the field theory side we know that in the limit $\bar{z} \to \infty$ the theory is decoupled at all energy scales and consists of two independent SCFTs. However, taking the limit $\bar{z} \to \infty$ in (7.7) does not return an AdS solution. This issue is simply solved by defining
\[ \eta_{IR} = \eta / \bar{z} . \] (7.8)

The correct IR limit is then obtained by keeping the variable $\eta_{IR}$ fixed, while taking the limit $\bar{z} \to \infty$. This prescription gives the IR AdS as the zeroth order metric of a $1/\bar{z}$ expansion,
\[ ds^2 = \frac{\eta^2_{IR}}{4 R^2_{IR}} dx^2 + R^2_{IR} \frac{d\eta^2_{IR}}{\eta^2_{IR}} + R^2_{IR} \left( d\psi^2 + \sin^2 \psi d\vec{\Omega}^2 \right) + O \left( \frac{1}{\bar{z}} \right) . \] (7.9)

All corrections vanish in the limit $\bar{z} \to \infty$ and we recover the expected decoupling of the full geometry.

At this point, it is also useful to write down the expression for the $\gamma_{RT}$ surface embedded in the metric (7.9). The equation of motion and the solution of $\eta(\sigma)$ are,
\[ \eta''_{IR} + \left[ \frac{p-1}{x} - (p+2) \frac{\eta'_{IR}}{\eta_{IR}} + 4 R^4_{IR} \frac{p-1}{x} \frac{\eta^2_{IR}}{\eta^3_{IR}} \right] \eta'_{IR} - \frac{p}{4 R^4_{IR}} \eta^3_{IR} = 0 \] (7.10)
\[ \eta_{IR}(\hat{\sigma}) = \frac{1}{\ell} \frac{2 R^2_{IR}}{\sqrt{1 - \hat{\sigma}^2}} \ \text{with} \ \hat{\sigma} = \sigma / \ell . \] (7.11)

On the other hand, the embedding function for $\gamma_{RT}$ in the UV AdS is easily obtained from the solution (6.10) by noticing that (7.5) gives the AdS metric (6.8) after the change of variables $\eta = 1/\zeta$. We thus find the relation
\[ \eta_{UV} = \frac{R^4_{UV}}{\ell^2 (1 - \hat{\sigma}^2)} , \ \ \eta_{IR} = \frac{2 R^2_{IR}}{R^2_{UV} \sqrt{\eta_{UV}}} . \] (7.12)

The property (7.12) fits naturally with the observation that (7.7) is a double covering of (7.5).

7.2 Details of the IR expansion

The original embedding function $r(\sigma, \theta)$ described in Section 5 becomes in the new coordinates $\eta = \eta(\sigma, \psi)$. This function is always single-valued as a function of $\psi$. Exploiting the symmetry of the $\mathbb{Z}_2$ symmetric solution we can restrict $\psi \in (0, \pi]$ and impose appropriate boundary conditions at $\psi = \pi$. 

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The minimal surface is governed by the Euler-Lagrange equations of a Lagrangian with the structure of (6.4). For quick reference we repeat here the specifics of the D1-D5 case,

\[ L_{1,5} = \mathcal{K}[\psi, \eta] \mathcal{H}[\psi, \eta] \sqrt{1 + \frac{\partial_\psi \eta^2}{\eta^2} + \left( \mathcal{H}[\psi, \eta] \right)^2 \frac{\partial_\eta \eta^2}{\eta^4}}, \] (7.13)

\[ \mathcal{H}[\psi, \eta] = 2\bar{\varepsilon} \cosh \left[ \frac{1}{4} \log \left( 1 + \frac{\eta^2}{\bar{\varepsilon}^4} + \frac{2\eta}{\bar{\varepsilon}^2} \cos \psi \right) \right], \] (7.14)

\[ \mathcal{K}[\psi, \eta] = \frac{1}{\eta} \left( \sqrt{1 + \bar{\varepsilon}^4 \eta^2 + 2\bar{\varepsilon}^2 \eta \cos \psi} \right). \] (7.15)

The reader can find the Lagrangian for the D3 case in Appendix A. Details about the equation of motion are not important, and numerical studies of the solution will be presented elsewhere. In this section, we focus mainly on the role of the separatrix, and discuss how to describe (globally) the Ryu-Takayanagi surface.

The starting point is similar to that of Section 6.2. We know that the equation of motion of \( \eta(\sigma, \psi) \) depends on the dimensionful parameter \( \bar{\varepsilon} \), and we want to exploit the scale invariance of the IR fixed point by writing a suitable ansatz for the solution. The idea is to recover the IR solution (7.11) in the limit \( \bar{\varepsilon} \to \infty \), therefore we consider,\(^9\)

\[ \eta(\sigma, \psi) = \frac{\bar{\varepsilon}}{\ell} F(\hat{\sigma}, \psi). \] (7.16)

The equation of motion for the field \( F(\hat{\sigma}, \psi) \) depends on a single dimensionless parameter \( \frac{\Delta}{R^2} = \frac{\bar{\varepsilon}}{\ell} R^2 \). The limit \( \frac{\Delta}{R^2} \to \infty \) is well defined and gives back the equation (7.10). It is then possible to set up a perturbative calculation in inverse powers of \( \Delta \) whose form is

\[ F(\hat{\sigma}, \psi) = \frac{2R^2 H}{\sqrt{1 - \hat{\sigma}^2}} + \sum_{k=1}^{\infty} \frac{1}{\Delta^k} f^{(k)}(\hat{\sigma}, \theta). \] (7.17)

The functions \( f^{(k)} \) would be determined at each order in perturbation theory. However, unlike the UV expansion, now the perturbative series breaks down in some range of \( \hat{\sigma} \). We can understand this point in two ways. One way is to realize that the expansion in inverse powers of \( \Delta \) that we are using involves, for example, expressions like

\[ \sqrt{1 + \frac{F^2(\hat{\sigma}, \psi)}{\Delta^2}} + \frac{2F(\hat{\sigma}, \psi)}{\Delta} \cos \psi = 1 + \sum_i c_i(\psi) \left( \frac{F(\hat{\sigma}, \psi)}{\Delta} \right)^i \] (7.18)

(see e.g. (7.15)). Therefore, it would be strictly valid as long as \( F(\hat{\sigma}, \psi) < \Delta \) for any \( \hat{\sigma}, \psi \). Problems arise with this requirement when \( \hat{\sigma} \to 1 \) because the surface is approaching the UV boundary and \( F \) diverges.

\(^9\)Notice that this is a different ansatz compared to the UV (6.11).
The second argument relies on the observation that the functions \( f^{(k)} \) will generically diverge faster than \( \eta_{UV} \sim 1/(1 - \sigma^2) \), thus violating the known UV AdS asymptotics. For example, in our D1-D5 system the first \( f^{(k)} \) that we find are\(^{10}\)

\[
F(\hat{\sigma}, \psi) = \frac{2R_{IR}^2}{\sqrt{1 - \hat{\sigma}^2}} + \frac{v}{1 - \hat{\sigma}^2} \left( \frac{R_{IR}^2}{\Delta} \right) + \frac{3}{4} \frac{v^2}{(1 - \hat{\sigma}^2)^{3/2}} \left( \frac{R_{IR}^2}{\Delta} \right)^2 + \frac{v(v^2 - 3(1 - \hat{\sigma}^2))}{(1 - \hat{\sigma}^2)^2} \left( \frac{R_{IR}^2}{\Delta} \right)^3 + \ldots
\]

(7.19)

where \( v = \cos \psi \). The second line involves powers higher than \((1 - \hat{\sigma}^2)^{-1}\).

From these observations we conclude that the perturbative expansion (7.17) is a good approximation of the solution only below a certain \( \hat{\sigma}_s \), potentially related to the existence of the separatrix. The right way to recover the UV solution is to make use of a matched expansion.

Before discussing the matching procedure at the UV boundary, we would like to make the following comment. In the limit \( \Delta \to \infty \), it is clear that the separatrix becomes a UV cut-off and the full geometry breaks into the sum of two disconnected throats. Such fragmentation is nicely understood in the \((u, v)\) plane as the process of zipping the upper from the lower half plane (the dashed line on the left plot of Figure 4 on the u-axis moves off to infinity). However, for \( \Delta \gg 1 \) but finite, the IR geometry is still connected all the way up to the UV and the separatrix is the natural short distance cut-off from the deep IR perspective. The resummation of the series (7.9) seems to be in direct relation with the resummation of an infinite set of irrelevant interactions that one has to perform in the effective IR field theory to reconstruct the whole RG flow.

The matching expansion is based on the assumption that as we zoom into the boundary region \( \hat{\sigma} \to 1 \) we effectively look into the UV AdS. In order to do so, it is standard to define both a rescaled variable \( \tilde{\sigma} = (1 - \hat{\sigma})/\epsilon \) and a rescaled function \( F = \epsilon^\alpha \tilde{F} \), and take the limit \( \epsilon \to 0 \) in the equation of motion. In such a limit, the new variable \( \tilde{\sigma} \) and new function \( \tilde{F} \) are kept fixed. Because \( F \) diverges at the boundary \( \alpha \) has to be negative. In our case we know that \( \alpha = -2 \) because we are taking a limit in which the theory is conformal and we know the scalings. As a result, the matching procedure gives back \( \eta_{UV} \) with an overall constant that we need to determine. By inspection of the equation of motion we find that

\[
F = \frac{1}{\Delta} \frac{R_{UV}^4}{1 - \hat{\sigma}^2}.
\]

(7.20)

The matched expansion leads to an expression of the form

\[
F(\hat{\sigma}, \psi) = \frac{2R_{IR}^2}{\sqrt{1 - \hat{\sigma}^2}} + \frac{1}{\Delta} \frac{R_{UV}^4}{1 - \hat{\sigma}^2} + \left[ \ldots \text{matched expansion corrections} \ldots \right].
\]

(7.21)

\(^{10}\)In writing these \( f^{(k)} \) we are imposing one boundary condition, \( \partial_\hat{\sigma} f^{(k)} = 0 \) at \( \hat{\sigma} = 0 \), and we are fixing the remaining integration constant to some value. In principle we should keep this integration constant and use it as a matching parameter. However, the argument we want to make here does not depend on this choice.
Returning to the original embedding field $\eta(\hat{\sigma}, \psi)$ we find:

$$\eta(\hat{\sigma}, \psi) = \frac{2 R_{IR}^2}{\ell} \sqrt{1 - \hat{\sigma}^2} + \frac{1}{\ell^2} \frac{R_{UV}^4}{1 - \hat{\sigma}^2} + \left[ \ldots \text{matched expansion corrections} \ldots \right], \quad (7.22)$$

### 7.3 Entanglement fragmentation

Inserting the solution (7.21) into the entropy functional we can calculate the leading large-$\ell$ behavior of the holographic entanglement entropy in the Coulomb branch RG flow. The resulting expression will give the correct expectation: the HEE receives one contribution from the UV $AdS_{p+2}$ (with radius $R_{UV}$), and another one from the two disconnect IR $AdS_{p+2}$ (with radius $R_{IR}$). In the following, we will make this statement more precise by splitting the integration over $\hat{\sigma} \in [0, \ell]$ into an IR and a UV contribution.

It is useful to define the integral

$$I_p[a_\ell, 1] = \int^{s_{\text{max}}}_{s_{\text{min}}} ds \left( \frac{1 - s^2}{s^p} \right)^{(p-2)/2}. \quad (7.23)$$

We already encountered $I_p$ in Section 6. In particular, $I_p[a_\ell, 1]$ calculates the $\ell$ dependence of the HEE of spherical entangling surfaces for pure $AdS_{p+2}$.

In the limit $\Delta \to \infty$, the form of the solution (7.21) implies that the HEE is that of two $AdS_{p+2}$ with radius $R_{IR}$, as expected

$$\frac{S_p}{A_p} (\ell \to \infty) = 2 C_p^{IR} R_{IR}^p I_p[a_\ell, 1] \quad (7.24)$$

where $A_p$ is the area of the entangling surface and

$$C_p^{IR} = \frac{1}{4 G_N} \text{Vol}(S^{D-p-2}) R_{IR}^{D-p-2}. \quad (7.25)$$

The factor of 2 in (7.24) counts the two disconnected $AdS$ throats, and comes from the integration over the angle $\psi$. For a generic multi-centered configuration with $K$ IR throats the result will be given in terms of the sum of $K$ contributions. The integration over $s$ needs the UV regulator $a/\ell$, as usual in $AdS$. Notice that this cut-off is the one that regulates the volume of the IR $AdS$ after taking the decoupling limit.

At $\Delta \gg 1$ the exact solution of $\eta(\hat{\sigma}, \psi)$ will exhibit a separatrix and thus we need to consider the matched expansion. We can estimate roughly that the IR solution becomes sub-leading compared to the UV at

$$X_c \approx \frac{R_{UV}^4}{2 R_{IR}^2} \frac{1}{\Delta} = \frac{R_{UV}^4}{2 \bar{z} R_{IR}^2} \frac{1}{\ell} \equiv \frac{\bar{a}}{\ell}, \quad (7.26)$$

where $X = 1 - \hat{\sigma}^2$. Therefore it is useful to separate the integration over $\hat{\sigma}$ in a UV contribution, in which we can use $\eta \approx \eta_{UV}$, and an IR contribution, in which we can use $\eta \approx \eta_{IR}$. In our approximation, this way of splitting the integral over $\hat{\sigma}$ isolates the contributions coming
from below and above the separatrix. This is a natural thing to do because in the limit $\Delta \to \infty$ the separatrix will become the UV cut-off. The final result for the HEE is

$$\frac{S_p}{A_p} = 2 C_p^{IR} R_{IR}^p I_p \left[ \frac{\bar{a}}{\bar{L}}, 1 \right] + C_p^{UV} R_{UV}^p I_p \left[ \frac{a}{\bar{L}}, \bar{a} \right] + \ldots$$

(7.27)

where $a$ is a UV cut-off, $\bar{a}$ can be read from (7.26), and finally

$$C_p^{UV} = \frac{1}{4 G_{DN}^{D-p-2}} \text{Vol}(S^{D-p-2}) R_{UV}^{D-p-2}.$$  

(7.28)

The result (7.27) agrees with the general expectations for the HEE along RG flows [28,29].

8 More about the connectivity index in the IR effective theory

The behavior of the entanglement entropy that we studied in previous sections suggests that the change of the connectivity index along the RG flow, say between 1 in the UV and 2 in the IR, is a process with a sharp large-$N$ transition at intermediate energies. In sections 1 and 3 we proposed that the origins of this transition can be traced to the qualitatively different properties of the theory at large and small energies. In particular, any direct interaction between the IR CFTs at small energies is mediated by multi-trace operators that cannot change the IR connectivity index. This statement implied that the mutual quantum information $S_{12}$, or the relative entropy of entanglement $D_{REE}$, have to vanish identically in a vicinity of the IR fixed point. Since this is a crucial point for the proposed picture we would like to summarize here some well known facts that support its validity.

**Energy-momentum conservation, bi-gravity and the connectivity index.** As described in section 3, in quantum field theory the infrared effective description of our setup involves two IR CFTs deformed separately by single- and multi-trace interactions denoted schematically by $V_I$ in (3.1). The only direct interaction between these theories comes from multi-trace interactions of the schematic form $O_1 O_2$ in (3.1). All the interactions are IR-irrelevant, which means that one has to work with an explicit UV cutoff.

Refs. [4,5] demonstrated that the multi-trace interactions do not introduce any anomalous dimensions to the two energy-momentum tensors of the deformed IR product theory at leading order in $1/N$ in the large-$N$ limit. In particular, even after the deformation, the theory continues to have two separately conserved energy-momentum tensors. This is our first sign that the connectivity index cannot be modified as we increase the energy and as long as the IR effective field theory expansion is valid. The subleading $1/N$ corrections introduce an anomalous dimension to a linear combination of the energy-momentum tensors and the connectivity index will necessarily get reduced to 1.

On the holographically dual side the IR effective description involves a bi-gravity (bi-string) theory [4,5,15] with the following features. The spacetime of each graviton asymptotes towards the UV to a deformed $AdS \times S$ space. The UV deformation introduces the ‘1’ in the harmonic function of each throat as we expand the full harmonic function of the double-center
solution around each center. This deformation captures the irrelevant single-trace part of the deformations $V_I$ in each theory mentioned previously.\footnote{From the UV point of view the IR bi-gravity description arises as we localize the wavefunction of the single graviton in the multi-center geometry in the vicinity of each center.}

At leading order in $1/N$, the multi-trace deformations impose modified boundary conditions for the fields in the bulk $[30,31]$. It was shown in $[4,5]$ that the bulk gravitons remain massless at leading order and the bi-gravity theory is trivial (namely, besides the modified boundary conditions, the theory in the bulk is a decoupled product of string theories living on separate spacetimes with separate Lagrangians). Subleading $1/N$ corrections make a linear combination of the bulk gravitons massive (i.e. modify the gravity Lagrangians) and reduce the connectivity index.

**Entanglement entropy as a sum of two separate terms.** Our statement for a connectivity index 2 in the vicinity of the IR fixed point implied that the mutual quantum information $S_{12} (\rho_A)$ is vanishing at small enough energies. This, in turn, implied that the entanglement entropy is a sum of two separate terms (see equation (3.2)).

Indeed, the computation of the entanglement entropy à la Ryu-Takayanagi in the IR bi-gravity description receives two separate contributions from two minimal co-dimension-2 surfaces that are embedded in the spacetime of each throat. $1/N$ corrections would again spoil this decoupling by introducing several types of new effects in this computation.

**Factorizability of correlation functions.** We mentioned in the introduction that one of the signs of separability is complete factorization in all correlation functions. The presence of the double-trace inter-CFT interactions in (3.1) modifies the correlation functions already at leading order in the $1/N$ expansion. In particular, the correlation function $\langle O_1(x_1)O_2(x_2) \rangle$ (recall $O_i$, $i = 1, 2$, is an operator in the IR CFT) receives $h_{12}$ contributions and is no longer vanishing. That seems to spoil the extreme IR factorizability, so one may wonder how this effect is consistent with the proposed separability in the vicinity of the IR fixed point.

It is perhaps simpler to describe the resolution of this question in AdS/CFT language along the following lines. For concreteness, let us focus on the single-trace (scalar) operators $O_1$, $O_2$ and also assume for clarity that the total effective field theory action is

$$S_{\text{total}} = S_1 + S_2 + \int d^{p+1} x \ h_{12} O_1 O_2.$$  \hspace{1cm} (8.1)

$S_1, S_2$ are actions solely in QFT$_1$ and QFT$_2$. To simplify the argument, let us also ignore the irrelevant single-trace deformations of the IR CFTs. In the bulk bi-gravity theory there are two scalar fields, $\phi_1$ and $\phi_2$, corresponding to $O_1$, $O_2$. With the boundary of each $AdS$ spacetime at large radius $r_i$ ($i = 1, 2$), each of these fields will asymptote to

$$\phi_i = \frac{\beta_i}{r_i^{\Delta_i}} + \ldots + \frac{\alpha_i}{r_i^{p+1-\Delta_i}} + \ldots.$$  \hspace{1cm} (8.2)
\( \Delta_i \) is the scaling dimension of the operator \( O_i \). Assuming \( \Delta_i > \frac{p+1}{2} \), the double-trace deformation on the r.h.s. of equation (8.1) is irrelevant. Also, with this assumption the \( \beta_i \) term in (8.2) is the leading term as \( ri \to \infty \).

The generating functional of the theory (8.1) is obtained by adding sources for \( O_i \),

\[
\delta S = \int d^{p+1}x \ (J_1 O_1 + J_2 O_2) ,
\]

and computing the quantum path integral of the full theory

\[
Z = e^{-W[J_1, J_2]}
\]
as a function of the sources \( J_i \). Then, connected correlation functions of \( O_1, O_2 \) are computed by functional derivatives of \( W \) with respect to \( J_i \).

In gravity one computes the on-shell gravity action \( I_{GR} \) as a function of the asymptotic coefficients \( \beta_i \) in (8.2). In the case at hand, these obey the boundary conditions

\[
\beta_1 = J_1 + h_{12} \alpha_2 , \quad \beta_2 = J_2 + h_{12} \alpha_1 .
\]

Using standard designer (bi)gravity techniques [32, 33] one can fix a second pair of relations between \( \beta_1 \) and \( \alpha_1 \) on the one hand, and \( \beta_2 \) and \( \alpha_2 \) on the other. This allows us to re-express the bulk solution in terms of \( J_i \), and the corresponding on-shell gravity action also in terms of \( J_i \). Since the bulk theory is a direct product of two gravity theories

\[
I_{GR}[J_1, J_2] = I_{GR,1}[J_1, J_2] + I_{GR,2}[J_1, J_2] .
\]

The basic relation of the AdS/CFT correspondence is

\[
W[J_1, J_2] = I_{GR}[J_1, J_2] .
\]

Because of (8.6), (8.7) we see, for example, that the correlation functions \( \langle O_1(x_1)O_2(x_2) \rangle \) are non-vanishing and factorizability is seemingly lost. However, the above procedure reveals that the main effect of the double-trace deformation is to mix the sources \( J_i \). Denoting the new combinations as \( \tilde{J}_i \), so that

\[
W[\tilde{J}_1, \tilde{J}_2] = I_{GR,1}[\tilde{J}_1] + I_{GR,2}[\tilde{J}_2] ,
\]

we see that there is a new basis of operators (dual to \( \tilde{J}_i \)) where factorization of correlation functions reappears. The new basis is non-trivially related to the old one.

What we have described is consistent with our statement \( S_{12}(\rho_A) = 0 \), or equivalently, with the statement that the density matrices obey the relation

\[
\rho_A = \rho_{A,1} \otimes \rho_{A,2} ,
\]

once we define the Hilbert spaces \( H_{A,1} \) and \( H_{A,2} \) (over which we trace) appropriately in order to account for the new basis of operators identified in (8.8).

An extension of this argument to the more complicated case of single- and multi-trace interactions in the context of the Coulomb branch RG flow (3.1) is envisioned with appropriate modifications.
9 Discussion

Generic processes rearrange the interactions and correlations between different degrees of freedom in a quantum system. In some cases the Hilbert space experiences a fragmentation where the interaction between degrees of freedom in different parts of the system becomes weak or even disappears.\footnote{The inverse is also possible. The interactions between different parts of a fragmented Hilbert space may turn on and grow.} When the latter happens all correlation functions factorize and we say that the process changed the connectivity index of the system.

In this paper we pointed out that there are instances where such processes do not occur smoothly and proportionally to an interaction coupling that turns on or off, but abruptly at finite interaction coupling. We examined a particular class of examples that occur in the Coulomb branch of large-$N$ superconformal field theories. In that class the effect was a consequence of a competition between large-$N$ effects and effects associated to the specifics of the renormalization group flow. It would be interesting to learn if there are other classes of quantum systems that exhibit this kind of behavior.

We discussed two major probes of this transition. The first one is entanglement entropy on a spatial region $A$ and the second one are quantum information measures of separability, e.g. relative entropy of entanglement and quantum mutual information. The main lessons and emerging open questions of our study can be summarized as follows.

**Entanglement entropy.** For spherical regions the entanglement entropy $S$ is a function of the radius $\ell$ of the sphere. We computed this function in the Coulomb branch of large-$N$ gauge theories and noticed that the transition manifests itself through the formation of a separatrix in the Ryu-Takayanagi surface. The separatrix is absent for $\ell < \ell_c$ and present for any $\ell \geq \ell_c$, where $\ell_c$ is a critical radius. The presence of $\ell_c$ signals a change in the behavior of the entanglement above $\ell_c$, but since we lack an analytic solution of the Ryu-Takayanagi surface in all regimes, it has been hard to determine the precise nature of this change. It would be very interesting to learn if the entanglement entropy is a $C^\infty$ function at $\ell_c$, or whether some derivative of $S$ diverges.

It would also be important to understand better why the perturbative UV results (6.20), (6.30), and (6.40), do not depend on $\Delta^2$ and $\Delta^3$ corrections to the RT surface. In field theory, it is natural to associate those contributions to operators of dimension 2 and 3. The perturbative holographic computation would be reliable for small entangling regions and one way to proceed would be to develop a small length OPE expansion for the twist fields. Because of supersymmetry some coefficients in the OPE may be directly vanishing, or may vanish when the limit $n \to 1$ in the replica trick is taken. This would also provide a non-trivial check of the RT prescription out of conformality.

**Entanglement measures of separability.** We pointed out that the quantum information notion of separability is a very suitable probe of physics in processes that change the connectivity index. In our examples we argued that the quantum mutual information $S_{12}(\rho_A)$
(2.7) (say, for a split in two systems in the IR) is identically zero above a radius $\ell^*_c$ of the spherical region $A$. Below $\ell^*_c$ the system enters the non-separable UV phase where the $S_{12}(\ell)$ is expected to grow as $\ell$ decreases and the interaction between the IR CFTs (mediated by the light $W$-bosons) increases.

It would be interesting to know:

(a) the precise relation between the critical radii $\ell_c$ and $\ell^*_c$, e.g. whether $\ell_c = \ell^*_c$. From the point of view of the IR physics it is clear that the critical behavior appears at some scale in the middle of the RG flow, because as we go backwards on the RG flow towards the UV we need to resum an infinite series of IR-irrelevant growing multi-trace interactions. At leading order in the large-$N$ limit the passage to a UV description with running single-trace data happens abruptly. Since all observables capture this specific underlying property of the RG flow, it is natural to expect that all of them will record the transition at the same scale. For that reason, it is natural to expect that the critical radii $\ell_c$, $\ell^*_c$ are the same. It would be interesting to verify this by an explicit computation. At the moment we do not know how to compute efficiently the precise $\ell^*_c$ directly in field theory, or in holography. On the other hand, $\ell_c$ is in principle computable by solving the PDEs of the Ryu-Takayanagi minimal surface.

(b) we would like to determine the precise behavior of $S_{12}(\ell)$ at $\ell^*_c$, e.g. in order to verify whether it is continuous at that point, or whether some derivative diverges.

(c) it would be useful for many general purposes to know how to compute $S_{12}(\ell)$ efficiently, for instance in the AdS/CFT correspondence. Notice that the definition of $S_{12}(\ell)$ involves the entanglement entropy $S(\ell)$, that can be computed holographically à la Ryu-Takayanagi, and the entanglement entropies of the reduced density matrices $\rho_{A,1}$, $\rho_{A,2}$. The authors of the recent paper [13] argued that the latter entropies for $A^c = \emptyset$ are computed in $AdS \times S^4$ spacetimes by a co-dimension-2 surface that goes through the equator of the transverse sphere $S$. It would be interesting to know if there is a generalization of this statement for $A^c \neq \emptyset$. Related questions and quantities have been discussed in the recent condensed matter literature in [34–37].

(d) as explained near the end of section 3 the splitting in the Hilbert spaces $H_1$ and $H_2$ of the IR theories (and the definition (2.5), (2.6)) is less natural from the UV point of view. It is therefore interesting to examine alternative definitions of $S_{12}(\ell)$ along the whole RG flow. It would also be useful to explore definitions that are applicable more generally for an arbitrary partitioning of a system in field theory space irrespective of whether there is a natural split in some corner of parameter space. Such definitions would be a useful probe of the strength of interactions and correlations between two arbitrarily chosen parts of the system across spacetime and field theory space.

Similar observations and questions can be made for other measures of separability, for example the relative entropy of entanglement $D_{\text{REE}}$ (2.4). $D_{\text{REE}}$ is not plagued by the issues of item (d), but most likely it is a much harder quantity to compute explicitly.
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A Minimal surface equations

In all cases analyzed in the main text, the Lagrangian of the Ryu-Takayanagi minimal surface can be put into the form

\[ \mathcal{L} = \sigma^{p-1} \mathcal{K}[\theta, \eta] \mathcal{H}[\theta, \eta] \sqrt{1 + \frac{1}{\alpha} \frac{\partial \eta}{\eta^2} + (\mathcal{H}[\theta, \eta])^2 \frac{\partial \sigma}{\eta^2}} \]  \tag{A.1}

where \( \alpha \) is a constant. We will use the notation \( \partial_\sigma \eta = \eta^{(1,0)} \) and \( \partial_\theta \eta = \eta^{(0,1)} \). For example, the case of D3 branes in the coordinates of section 7 is

\[ \mathcal{H}^2 = \frac{1}{\eta^4} \frac{1}{4 \sqrt{\bar{z}^4 + \eta^2 + 2 \bar{z}^2 \eta \cos \psi}} \left[ 2 \left( \bar{z}^2 + \sqrt{\bar{z}^4 + \eta^2 + 2 \bar{z}^2 \eta \cos \psi} \right)^2 - \eta^2 \right], \]  \tag{A.2}

\[ \mathcal{K} = \eta^3 \left[ \sqrt{1 + \frac{\bar{z}^4}{\eta^2} + \frac{2 \bar{z}^2}{\eta} \cos \psi + \left( \frac{\bar{z}^2}{\eta} + \cos \psi \right)} \right]^2. \]  \tag{A.3}

The equation of motion is quite complicated and can be expressed as the sum of different pieces. We found convenient to write it as

\[ D_0 + D_1 + D_2 + D_3 = 0. \]  \tag{A.4}

The first operator, \( D_0 \), is a generalization of the flat space minimal surface equation, namely

\[ D_0 = -d_{(2,0)} \eta^{(2,0)} - d_{(1,1)} \eta^{(1,1)} - d_{(0,2)} \eta^{(0,2)} + \frac{1}{\alpha \eta} \left[ \eta^{(0,1)} \right]^2 \]  \tag{A.5}

with

\[ d_{(2,0)} = 1 + \left( \frac{\eta^{(0,1)}}{\eta} \right)^2, \quad d_{(1,1)} = -2 \frac{\eta^{(0,1)} \eta^{(1,0)}}{\eta^2}, \quad d_{(0,2)} = \frac{1}{\eta^2 \mathcal{F}^2} \left[ \frac{\eta^{(1,0)}}{\eta} \right]^2. \]  \tag{A.6}

The remaining terms are

\[ D_1 = \frac{1}{\mathcal{F}^2} \left[ 1 + \frac{\left( \eta^{(0,1)} \right)^2}{\alpha \eta^2} + \left( \eta^{(1,0)} \right)^2 \mathcal{F}^2 \right] \left( \frac{\mathcal{K}^{(0,1)}}{\mathcal{K}} - \frac{\eta^{(0,1)} \mathcal{K}^{(1,0)}}{\alpha \eta^2 \mathcal{K}} \right), \]  \tag{A.7}
\[ D_2 = -\frac{1}{2} \left( 1 + \left( \frac{\eta^{(0,1)}}{\alpha \eta^2} \right)^2 \right) \left[ \partial_{\eta} \left( \frac{1}{F^2} \right) - \frac{\eta^{(0,1)}}{\alpha \eta^2} \partial_{\eta} \left( \frac{1}{F^2} \right) \right], \quad \text{(A.8)} \]

\[ D_3 = 1 - \frac{d}{x} \eta^{(1,0)} \left[ 1 + \left( \frac{\eta^{(0,1)}}{\alpha \eta^2} \right)^2 + \left( \eta^{(1,0)} \right)^2 \right]. \quad \text{(A.9)} \]

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