Diffusion approximation for self-similarity of stochastic advection in Burgers’ equation

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Abstract

Self-similarity of Burgers’ equation with stochastic advection is studied. In self-similar variables a stationary solution is constructed which establishes the existence of a stochastically self-similar solution for the stochastic Burgers’ equation. The analysis assumes that the stochastic coefficient of advection is transformed to a white noise in the self-similar variables. Furthermore, by a diffusion approximation, the long time convergence to the self-similar solution is proved in the sense of distribution.

Keywords Self similarity; stochastic Burgers equation; diffusion approximation

1 Introduction

The deterministic Burgers’ partial differential equation for a field \(w(t, x)\) is

\[
\frac{\partial w}{\partial t} = \nu \frac{\partial^2 w}{\partial x^2} - w \frac{\partial w}{\partial x} \tag{1}
\]

and was proposed by Burgers [6] to help understand the statistical theory of turbulent fluid motion. Here \(w(t, x)\) is analogous to the velocity field and \(\nu\) represents the dissipative viscosity. To better model the randomness inherent in the presumed chaos of turbulence, the following stochastic Burgers’
equation has been suggested \[7, 8, 19, 21, 23, 37, 38\] and studied recently by many people \[3, 4, 11, 12, 20, 34, 35, 32\]:

\[
w_t = \nu w_{xx} - w w_x + h(t, x, w, w_x) \tag{2}
\]

where \(h(t, x, w, w_x)\) represents stochastic effects defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\). On bounded domains the existence and uniqueness of global solution was studied by Da Prato et al. \[34\] when the noise term \(h(t, x, w, w_x)\) is white in time, and by Holden et al. \[20\] using a white noise calculus. On an unbounded domain the existence of a solution was studied through a Cole–Hopf transformation by Bertini et al. \[3\] with \(h(t, x, w, w_x)\) an additive space-time white noise.

We consider a family of solutions with special spatial-temporal form, namely the family of self-similar solutions, of the stochastic Burgers’ equation \(2\) on the unbounded real line with the particular stochastic advection \(h(t, x, w, w_x) = (w\zeta)_x\) for some special space-time noise \(\zeta(t, x)\) to be defined later. Here for the stochastic system, the self-similarity is in the sense of distribution which is defined later. The existence of self-similar solutions and the asymptotic emergence of self-similar solutions comprises the self-similarity of the stochastic Burgers’ equation. Importantly for applications, the form of the stochastic advection, \((w\zeta)_x\), is appropriate for globally conserved quantities \(w\). Such stochastic advection is potentially of great interest in applications as it potentially illuminates some of the stochastic nature of chaotic turbulence in fluid flows. A thorough understanding of turbulence remains an outstanding challenge and researchers are increasingly invoking stochastic terms to model its effects in important environmental applications \[22, 16, 43, \text{e.g.}\]. We need to know how stochastic advection affects long term dynamics.

Self-similarity is an important property of systems of physical interest, of which Burgers’ equation is a special case. Many researchers have studied the existence of self-similar solutions of deterministic systems \[5, 15, 28, 36, 44, \text{e.g.}\], and described the asymptotic behavior of self-similar solutions \[2, 15, 30, 44, \text{e.g.}\]. But very little is known about self-similarity in stochastic spatio-temporal systems. We prove the existence and emergence of self-similar solutions, in the sense of distribution, for the stochastic Burgers’ equation \(2\). The stochastic advection \(h = (w\zeta)_x\) for the stochastic Burgers’ equation \(2\) transforms to a multiplicative white noise in the following self-similar variables. As in earlier research \[2, 36, 44, \text{e.g.}\] we introduce log-time and stretched space,

\[
\tau = \log t , \quad \xi = \frac{x}{\sqrt{t}} , \quad \text{for } t \geq 1 ,
\]
and then define the stochastic field
\[ u(\tau, \xi, \omega) = \sqrt{t} w(t, x, \omega), \quad \omega \in \Omega. \]

Straightforward algebra derives that the SPDE (2) transforms in the similarity variables to
\[ du = \left[ \nu u_{\xi\xi} + \frac{1}{2} \xi u_{\xi} + \frac{1}{2} u - uu_{\xi} \right] d\tau + (udW)_{\xi}. \tag{3} \]

We call a solution \( w(t, x, \omega) \) to the stochastic Burgers’ equation (2) a stochastically self-similar solution if the distribution of \( \sqrt{t} w(t, x, \omega) \) just depends on the self-similar variable \( \xi = x/\sqrt{t} \). By this definition, any stationary solution \( \bar{u}(\xi, \omega) \) to equation (3) is a stochastically self-similar solution of stochastic Burgers’ equation (2). In order to construct a self-similar solution of the stochastic Burgers’ equation (2) we assume that \( W(\tau, \xi, \omega) \) is an \( L^2(\mathbb{R}) \)-valued Wiener process defined on \( (\Omega, \mathcal{F}, \mathbb{P}) \) with covariance operator \( Q \) which is detailed later.

To construct a stationary solution of the transformed SPDE (3), we consider the system in a weighted space \( L^2(K) \) which is defined in the next section. First, by using energy estimates and the compact embedding results of the weighted space, we show the tightness of solutions with initial value in the space \( L^2(K) \). Then the classical Bogolyubov–Krylov method \cite{1} implies the existence of a stationary solution of the SPDE (3). Due to the multiplicative noise, the method showing the attraction of the stationary solution in the case of additive noise \cite{41} fails here due to the appearance of the unbounded term \( \dot{W} \). Instead we apply a diffusion approximation to this stationary solution. For this we introduce the following random equation
\[ u^{*\epsilon}_\tau = L u^{*\epsilon} - u^{*\epsilon} u^{*\epsilon}_{\xi} - \frac{1}{2}(u^{*\epsilon} q)_{\xi\xi} + \frac{1}{2}(u^{*\epsilon} q')_{\xi} + \frac{1}{\sqrt{\epsilon}}(u^{*\epsilon} \bar{\eta}^{\epsilon})_{\xi} \tag{4} \]
which is a Burgers’ type equation with a singular random perturbation. Here \( \bar{\eta}^{\epsilon} \) is a stationary process solving \( (14) \), and \( q \) and \( q' \) are introduced in Assumption 2. Then attraction of the stationary solutions is derived by the method used for the case of additive noise, and by the approximation of \( (4) \) to \( (3) \), the attraction is passed to the stationary solution of \( (3) \). Here the most difficult part is to show the effectiveness of the approximation. We follow a martingale method to show the tightness of the family of stationary solutions of the approximating model. Then passing to the limit derives the attraction of stationary solutions of the stochastic advection Burgers’ equation \( (3) \).

However, for simplicity and intuition of the discussion in our approach we consider the following Burgers’ type equation
\[ u^{*\epsilon}_\tau = L u^{*\epsilon} - u^{*\epsilon} u^{*\epsilon}_{\xi} + \frac{1}{\sqrt{\epsilon}}(u^{*\epsilon} \bar{\eta}^{\epsilon})_{\xi}. \tag{5} \]
The limit of the above equation is shown to be the following SPDE

$$u_\tau = \mathcal{L}u - uu_\xi + \frac{1}{2}(uq)_\xi - \frac{1}{2}(uq')_\xi + (u\dot{W})_\xi$$  

(6)

for some new Wiener process $\dot{W}$ distributed as $W$. By the assumptions on $q$, all results for equation (5) hold for equation (4) by exactly same discussion. Then we have approximation of (4) to (3).

To show the approximation of the stationary solutions, in our approach we consider the statistical solution of the equations (3) and (4).

2 Preliminary

We consider the stochastic PDE (3) in the self-similarity variables. For brevity we introduce the linear operator

$$\mathcal{L}u = \nu u_\xi + \frac{1}{2}u_\xi + \frac{1}{2}u$$

Denoting the weight function by $K(\xi) = \exp\{-\xi^2/4\nu\}$, introduce the following weighted functional space for exponent $p > 0$

$$L^p(K) = \left\{ u \in L^p(\mathbb{R}) : \|u\|_{L^p(K)}^p = \int_\mathbb{R} |u(\xi)|^p K(\xi) d\xi < \infty \right\},$$

and for positive integer exponent $k$

$$H^k(K) = \left\{ u \in L^2(K) : \|u\|_{H^k(K)}^2 = \sum_{0 \leq \alpha \leq k} \|D^\alpha u\|_{L^2(K)}^2 < \infty \right\}.$$

For $p = 2$, denote by $\langle \cdot, \cdot \rangle$ the inner product in space $L^2(K)$. Then the linear operator $\mathcal{L}$ is self-adjoint and generates an analytic semigroup $S(\tau)$ on the space $L^2(K)$ with the domain $D(\mathcal{L}) = H^2(K)$ [24]. Further, the eigenvalues of the operator $\mathcal{L}$ are

$$\lambda_k = -\frac{k}{2}, \quad k = 0, 1, 2, \ldots,$$

with the corresponding eigenfunctions

$$e_0(\xi) = \frac{1}{\sqrt{4\pi \nu}} \exp\{-\xi^2/4\nu\}, \quad e_k(\xi) = c_k \partial^k_\xi e_0(\xi), \quad k = 1, 2, \ldots,$$

which forms a standard orthonormal basis of $L^2(K)$ when we choose $c_k$ as the constants such that $\|e_k\|_{L^2(K)} = 1$. 

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In the following we denote by \( E_c = \text{span}\{e_1(\xi)\} \) and
\[
E_s = E_c^\perp = \left\{ u \in L^2(K) : \int_{\mathbb{R}} u(\xi) \, d\xi = 0 \right\}.
\]
We also denote by \( \Pi_s \) the linear projection from \( L^2(K) \) to \( E_s \).

The following are some important basic properties on these weighted spaces \cite{24}.

**Lemma 1.**

1. The embedding \( H^1(K) \subset L^2(K) \) is compact.
2. There exists \( C > 0 \) such that for any \( u \in H^1(K) \)
\[
\int_{\mathbb{R}} |u(\xi)|^2 K(\xi) \, d\xi \leq C \int_{\mathbb{R}} |\nabla u(\xi)|^2 K(\xi) \, d\xi.
\]
3. For any \( u \in H^1(K) \),
\[
\frac{1}{2} \int_{\mathbb{R}} |u(\xi)|^2 K(\xi) \, d\xi \leq \int_{\mathbb{R}} |\nabla u(\xi)|^2 K(\xi) \, d\xi.
\]
4. For any \( u \in E_s \),
\[
(\mathcal{L}u, u) \leq -\frac{1}{2} \|u\|_{H^1(K)}^2.
\]
5. If \( u \in H^1(K) \), then \( K^{1/2}u \in L^\infty(\mathbb{R}) \).
6. For any \( q > 2 \) and \( \epsilon > 0 \) there exists constants \( C_{\epsilon,q} > 0 \) and \( R > 0 \), such that for any \( u \in H^1(K) \cap L^q_{\text{loc}}(\mathbb{R}) \)
\[
\|u\|_{L^2(K)}^2 \leq \epsilon \|u\|_{L^q(\mathbb{R})}^2 + C_{\epsilon,q} \|u\|_{L^q(B(0,R))}^2.
\]

**Remark 1.** By item 3 in the above lemma, in the space \( H^1(K) \) we define the norm \( \|\nabla u\|_{L^2(K)} \) which is equivalent to \( \|u\|_{H^1(K)} \).

Further, by the spectral properties of the linear operator \( \mathcal{L} \), we define \((-\mathcal{L} + 1/2)^\gamma\) for any \( \gamma \in \mathbb{R} \) \cite{12}. Then define the Sobolev space \( H^\gamma(K) \), for any \( \gamma \in \mathbb{R} \), as \( \mathcal{D}((-\mathcal{L} + 1/2)^{\gamma/2}) \), the domain of \((-\mathcal{L} + 1/2)^{\gamma/2}\). By the embedding theorem \cite{12}, \( H^{\gamma_1}(K) \) is compactly embedding into \( H^{\gamma_2}(K) \) for \( \gamma_1 > \gamma_2 \).

We make the following assumptions on the stochastic force.

**Assumption 2.**

1. The stochastic force \( t^{1/2}\eta = t^{1/2}(u\zeta)_x \) is written, in the self-similar variables, as \( (uW)_\xi \). Here \( W \) is an \( L^2(K) \)-valued Wiener process with covariance operator \( Q \) such that
\[
Q\varphi(\xi) = \int_{\mathbb{R}} q(\xi,\zeta)\varphi(\zeta)K(\zeta) \, d\zeta \quad \text{for any } \varphi \in L^2(K),
\]
with \( q(\xi, \zeta) = q(\zeta, \xi) \) positive, and
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} q(\xi, \zeta) K_\xi(\xi) K_\zeta(\zeta) \, d\xi \, d\zeta < \infty.
\]
The covariance \( Q \) shares the same eigenbasis as that of the operator \( L \).

2. \( W_\xi(\tau, \xi) \) is an \( L^2(K) \)-valued Wiener process with covariance operator \( Q' \) such that
\[
Q' \phi(\xi) = \int_{\mathbb{R}} q'(\xi, \zeta) \phi(\zeta) \, d\zeta
\]
with \( q'(\xi, \zeta) = q'(\zeta, \xi) \) positive, and
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} q'(\xi, \zeta) K(\xi) K(\zeta) \, d\xi \, d\zeta < \infty.
\]
Furthermore,
\[
\text{Tr} Q < \infty \quad \text{and} \quad \text{Tr} Q' < \infty,
\]
and \( q(\xi) := q(\xi, \xi) \in H^2(K) \),
\[
\|q\|_{L^\infty(\mathbb{R} \times \mathbb{R})} \quad \text{and} \quad \|q'\|_{L^\infty(\mathbb{R} \times \mathbb{R})} \quad \text{are small.}
\]

From the above assumptions, \( q'(\xi, \zeta) = \partial_\xi \partial_\zeta q(\xi, \zeta) \) and the Wiener process \( W \) has the series representation
\[
W(\tau, \xi) = \sum_{k=0}^{\infty} \sqrt{q_k} e_k(\xi) \beta_k(\tau),
\]
where \( \{\beta_k\}_k \) are independent standard Brownian motions.

Remark 2. An example of such stochastic force is
\[
\zeta(x, t, \omega) = \sqrt{t} \sigma \left( \frac{x}{\sqrt{t}} \right) \frac{d}{dt} \beta(t, \omega)
\]
where \( \frac{d}{dt} \beta(t, \omega) \) is some random process such that \( \frac{d}{d\tau} \beta(e^\tau, \omega) \) is white in log-time \( \tau \). The special assumptions on \( \zeta(x, t, \omega) \) do not exclude the existence of self-similar solutions for other cases.

Recall that a random process \( \{u(\tau)\}_{\tau \geq 0} \) is said to be stationary if its joint probability distribution does not change when shifted in time \( \tau \). For the SPDE \( (3) \), to construct a stationary solution it is convenient to consider the transition semigroup associated to equation \( (3) \). We define \( \{P_\tau\}_{\tau \geq 0} \) on the space consisting of bounded continuous functions \( \psi : L^2(K) \cap L^\infty(\mathbb{R}) \rightarrow \mathbb{R} \) by
\[
(P_\tau \psi)(u^0) = \mathbb{E} \psi(u(\tau; u^0)),
\]
where \(u(\tau; u^0)\) is the solution of equation (3) with initial value \(u^0 \in L^2(K) \cap L^\infty(\mathbb{R})\). Denote by \(\mathcal{M}\) the space consisting all probability measures on the space \(L^2(K) \cap L^\infty(\mathbb{R})\) and endow \(\mathcal{M}\) with the topology of weak convergence. Define the dual semigroup \(\{P^*_\tau\}_{\tau \geq 0}\) acting on \(\mathcal{M}\) as

\[
\int_{L^2(K) \cap L^\infty(\mathbb{R})} \psi(u)(P^*_\tau \mu)(du) = \int_{L^2(K) \cap L^\infty(\mathbb{R})} (P^*_\tau \psi)(u) \mu(du)
\]

for any \(\mu \in \mathcal{M}\) and bounded continuous function \(\psi : L^2(K) \cap L^\infty(\mathbb{R}) \to \mathbb{R}\). If \(\mathcal{L}(u^0)\), the distribution of initial values \(u^0\), equals \(\mu\), then \(P^*_\tau \mu\) is the distribution of the solution \(u(\tau; u^0)\) [33, Proposition 11.1]. Sometimes \(\mathcal{M}\) is too large, so we need the smaller space

\[
\mathcal{M}_2 = \left\{ \mu \in \mathcal{M} : \int_{L^2(K) \cap L^\infty(\mathbb{R})} \|u\|_{L^2(K)}^2 \mu(du) < \infty \right\}.
\]

A probability space \(\mu \in \mathcal{M}\) is said to be a stationary measure for the stochastic Burgers’ equation (3) if

\[
P^*_\tau \mu = \mu, \quad \text{for all} \quad \tau > 0.
\]

The following property of stationary measure is useful [33, Proposition 11.5].

**Lemma 3.** If \(\mu \in \mathcal{M}\) is a stationary measure for (3) and the initial value \(u^0\) is \(\mathcal{F}_0\) measurable with \(\mathcal{L}(u^0) = \mu\), then the solution process \(\bar{u}(\tau; u^0)\) is a stationary solution to the stochastic Burgers’ equation (3).

### 3 Existence of self-similar solutions

By definition, a stationary solution to the SPDE (3) is a stochastically self-similar solution to the stochastic Burgers’ equation (2). Next we construct a stationary solution to the SPDE (3) from any initial value \(u^0 \in L^2(K) \cap L^\infty(\mathbb{R})\).

For any \(\tau > 0\), in the mild sense, the transformed stochastic Burgers’ SPDE (3) is written as

\[
u(\tau) = S(\tau)u_0 + \int_0^\tau S(\tau-s)u(s)\xi(s)ds + \int_0^\tau S(\tau-s)(u(s)dW(s))_\xi. \quad (11)
\]

Then by the standard method for the existence of mild solutions to SPDEs [33] we obtain the following theorem whose proof is given in Appendix A.

**Theorem 4.** Assume Assumption 2 holds. For any \(T > 0\) and initial value \(u_0 \in L^2(K) \cap L^\infty(\mathbb{R})\), there is a unique mild solution \(u(\tau, \xi)\) to SPDE (2) in \(L^2(\Omega, C(0, T; L^2(K))) \cap L^2(0, T; H^1(K)))\). Moreover this mild solution is also the unique weak solution.
We construct a stationary solution by the Bogolyubov–Krylov method. For this we need some estimates in the spaces $L^\infty(\mathbb{R})$ and $L^2(K)$.

### 3.1 $L^\infty(\mathbb{R})$ estimates

We follow the approach for a scalar convection-diffusion equation [44] which was also applied to characterise solutions to a stochastic Burgers’ equation with additive noise [41].

We introduce

$$
\text{sgn}(u)^+ = \begin{cases} 
1, & u > 0, \\
0, & u \leq 0;
\end{cases}
$$

and

$$
\text{sgn}(u)^- = \begin{cases} 
1, & u < 0, \\
0, & u \geq 0.
\end{cases}
$$

Then for $u \in L^2(\mathbb{R})$ with $u_\xi(t) \in L^2(\mathbb{R})$, the integral

$$
\int_\mathbb{R} u_\xi \phi(u) d\xi = -\int_\mathbb{R} u^2 \phi'(u) d\xi \leq 0 \ (\geq 0)
$$

for any nondecreasing (nonincreasing) $\phi \in C^1(\mathbb{R})$. By a density discussion the integral $\int_\mathbb{R} u_\xi \text{sgn}(u)^+ d\xi \leq 0 \ (\geq 0)$. Moreover, the integrals $\int_\mathbb{R} uu_\xi \text{sgn}(u)^\pm d\xi = 0$ and $\int_\mathbb{R} (\xi u_\xi + u) \text{sgn}(u)^\pm d\xi = 0$. Denote by $u^\pm = \text{sgn}(u)^\pm u$. Let $m = \|u_0\|_{L^\infty(\mathbb{R})}$. Then multiplying $\text{sgn}(u - m)^+$ and $\text{sgn}(u^* - m)^+$ on both sides of (3) and integrating on $\mathbb{R} \times [0, \tau]$ with $\tau > 0$, the integral

$$
\int_\mathbb{R} (u(\tau, \xi) - m)^+ d\xi \leq 0.
$$

Therefore, $u(\tau, \xi) \leq m$ for any $\tau > 0$. Similarly $u(\tau, \xi) \geq -m$ for $\tau > 0$. Then $\|u(\tau)\|_{L^\infty(\mathbb{R})} \leq m$ for all $\tau > 0$.

### 3.2 Estimates in the space $H^1(K)$

We first give a uniform estimate in the space $L^2(K)$.

Let $u(\tau, \xi) = u_c(\tau, \xi) + u_s(\tau, \xi)$ with $u_c \in E_c$ and $u_s \in E_s$. Then

$$
du_c = 0,
$$

$$
du_s = [Lu_s - \Pi_s(uu_\xi)] \ d\tau + \Pi_s(u_\xi dW). 
$$

So

$$
u(\tau, \xi) = u_c(0, \xi) = \langle u_0, e_0 \rangle e_0(\xi) = \int_\mathbb{R} u_0(\xi) d\xi e_0(\xi)
$$

which is totally determined by the mass of the initial value, namely $M := \int_\mathbb{R} u_0(\xi) d\xi$. 

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Notice that \( \| \cdot \|_{L^2(K)} \) is continuous on space \( L^2(K) \). Now applying Itô’s formula \([9]\) to \( \| u_\epsilon(\tau) \|_{L^2(K)}^2 \), we obtain

\[
\frac{1}{2} \frac{d}{d\tau} \| u_\epsilon(\tau) \|_{L^2(K)}^2 \leq -\frac{1}{2} \| u_\epsilon \|_{H^1(K)}^2 - \langle uu_\epsilon, u \rangle + \frac{1}{2} \left[ \| u_\epsilon \|_{L^2_2}^2 + \| u \|_{L^4_2}^2 + \| (u_\epsilon)_\xi \|_2^2 + \langle (uW)_\xi, u \rangle \right].
\]

Here \( \| \cdot \|_{L^2_2} \) and \( \| \cdot \|_{L^4_2} \) are the norms defined on the Hilbert–Schmidt spaces \( \mathcal{L}_2(Q^{1/2}L^2(K), L^2(K)) \) and \( \mathcal{L}_2(Q^{1/2}L^2(K), L^2(K)) \) respectively \([33]\).

Noticing that, by Assumption \([2]\) \( \| q \|_{L^\infty(R \times \mathbb{R})} \) and \( \| q' \|_{L^\infty(R \times \mathbb{R})} \) are small enough \([5]\), there exists some positive constant \( c \) such that

\[
\frac{1}{2} \frac{d}{d\tau} \| u_\epsilon(\tau) \|_{L^2(K)}^2 \leq -c \| u_\epsilon \|_{H^1(K)}^2 + c \| u_\epsilon \|_{H^1(K)}^2 - \langle uu_\epsilon, u \rangle + \langle (uW)_\xi, u \rangle.
\]

Integrating by parts yields

\[
\langle uu_\epsilon, u \rangle = -\frac{1}{3} \int_R u^3 K_\xi \, d\xi.
\]

By property \([6]\) in Lemma \([11]\) for any \( \epsilon, \epsilon' > 0 \) and \( q > 2 \), there exist positive constants \( C_\epsilon, C_{\epsilon', q} \) and \( R \) such that

\[
\left| \int_R (u)^3 K_\xi \, d\xi \right| = \frac{1}{2} \left| \int_R u_\xi K^{1/2}(u)^2 K^{1/2} \, d\xi \right|
\]

\[
\leq \frac{1}{2} \left[ \int_R (u)^2 \xi^2 K \, d\xi \right]^{1/2} \left[ \int_R (u)^4 K \, d\xi \right]^{1/2}
\]

\[
\leq C \| u_\xi \|_{L^2(K)} \| u \|_{L^2(K)} \| u \|_{L^\infty(R)}
\]

\[
\leq 3\epsilon C \| u_\xi \|_{L^2(K)}^2 + C_{\epsilon' q} \left[ \epsilon' \| u_\xi \|_{L^2(K)}^2 + C_{\epsilon' q} \| u \|_{L^\infty(B(0, R))}^2 \right] \| u \|_{L^\infty(R)}^2
\]

\[
\leq 3 \left[ \epsilon C + \epsilon' C_{\epsilon} \| u \|_{L^\infty(R)}^2 \right] \| u_\xi \|_{L^2(K)}^2 + 3C_{\epsilon' q} \| u \|_{L^\infty(R)}^2 \| u_\xi \|_{L^2(K)}^2.
\]

with some positive constant \( C_{\epsilon' q, R} \). Then

\[
|\langle uu_\epsilon, u \rangle| \leq \left[ \epsilon C + \epsilon' C_{\epsilon} \| u \|_{L^\infty(R)}^2 \right] \| u_\xi \|_{L^2(K)}^2 + C_{\epsilon} C_{\epsilon' q, R} \| u \|_{L^\infty(R)}^2.
\]

(12)

Then for any \( \epsilon \) and \( \epsilon' > 0 \), there are positive constants that we still denote
by $C_\varepsilon$ and $C_{\varepsilon', q, R}$ for some positive $q$ and $R$ such that

$$
\frac{1}{2} \frac{d}{d\tau} \|u_s(\tau)\|_{L^2(K)}^2 \\
\leq -c\|u_s\|_{H^1(K)}^2 + c\|u_c\|_{L^\infty(\mathbb{R})} + \left[\varepsilon C + \varepsilon' C_\varepsilon\|u\|_{L^\infty(\mathbb{R})}\right] \|u_s\|_{L^2(K)}^2 \\
+ C_\varepsilon C_{\varepsilon', q, R} \|u\|_{L^\infty(\mathbb{R})}^4 + \langle (u\dot{W})_\xi, u \rangle \\
\leq \left[-c + \varepsilon C + \varepsilon' C_\varepsilon \|u\|_{L^\infty(\mathbb{R})}\right] \|u_s\|_{H^1(K)}^2 \\
+ \left[c + \varepsilon C + \varepsilon' C_\varepsilon \|u\|_{L^\infty(\mathbb{R})}\right] \|u_c\|_{H^1(K)}^2 \\
+ C_\varepsilon C_{\varepsilon', q, R} \|u\|_{L^\infty(\mathbb{R})}^4 + \langle (u\dot{W})_\xi, u \rangle.
$$

(13)

Now choose $\varepsilon$ and $\varepsilon' > 0$ small enough, and since $u_c = M_0(\xi)$ and $\|u_c\|_{H^1(K)} \leq CM$ for some $C > 0$, then by the Gronwall lemma

$$
E\|u(\tau)\|^2_{L^2(K) \cap L^\infty(\mathbb{R})} \leq R_1, \quad \text{for all } \tau \geq 0,
$$

with some positive random variable $R_1$.

To show the existence of an invariant measure we further need the estimates of $u$ in space $H^1(K)$. From (13) we have for some constant $C_1, C_2 > 0$ such that

$$
E \int_0^\tau \|u(s)\|_{H^1(K)} ds \leq C_1 \tau + C_2.
$$

By the Chebyshev inequality

$$
\frac{1}{T} \int_0^T \mathbb{P}(\|u(\tau)\|_{H^1(K)} > K) \, d\tau \\
\leq \frac{1}{K^2 T} \int_0^T E\|u(\tau)\|^2_{H^1(K)} \, d\tau \\
\leq \frac{1}{K^2 T} (C_1 T + C_2) \to 0, \quad K \to \infty.
$$

Then by the Krylov–Bogoliubov method [1] $P_\tau$ has at least one stationary measure denoted by $\mu$. Let $\bar{u}(\tau, \xi)$ be the solution with initial value distributing as $\mu$, then $\bar{u}(\tau, \xi)$ is a stationary solution to the transformed SPDE [3] with distribution $\mu$ (Lemma [3]), and by the construction of the stationary solution, we have $\bar{u} \in H^1(K)$. Moreover,

$$
\int_{\mathbb{R}} \bar{u} \, d\xi = \int_{\mathbb{R}} u_0 \, d\xi \quad \text{and} \quad \|\bar{u}\|_{L^\infty(\mathbb{R})} \leq m.
$$

**Theorem 5.** Assume Assumption 2 holds. For any initial $u_0 \in L^2(K) \cap L^\infty(\mathbb{R})$, there is a stationary solution, denoted by $\bar{u}$, such that $\bar{u} \in H^1(K)$. Further, there is a sequence $\tau_n$, with $\tau_n \to \infty$ as $n \to \infty$, such that

$u(\tau_n)$ converges in distribution to $\bar{u}$ in $L^2(K)$.
as $n \to \infty$. Here $u(\tau)$ is the solution to the SPDE (3) with initial value $u_0$.

Next we show that the above convergence holds for any sequence $\tau_n$ with $\tau_n \to \infty$ as $n \to \infty$; that is, $u(\tau)$ converges in distribution to $\bar{u}$ as $\tau \to \infty$. It is impractical to follow the approach used for the stochastic Burgers’ equation with additive noise [41]. As Section 1 states, we consider in the next section a Burgers’ equation with random fast fluctuating advection which is an approximation to the SPDE (3).

4 Burgers’ equation with random fast fluctuations: self-similar solution and stability

We consider the following randomly fluctuating advection in a Burgers’ type equation (5). The random force $\tilde{\eta}(\tau, \xi) = \tilde{\eta}(\tau/\epsilon, \xi)$ in which $\tilde{\eta}(\tau, \xi)$ is the stationary Ornstein–Uhlenbeck process solving

$$d\eta = -\eta \, d\tau + dW(\tau, \xi).$$

Then

$$\mathbb{E}[\tilde{\eta}(\tau, \xi)\tilde{\eta}(s, \zeta)] = \frac{1}{2} q(\xi, \zeta) \exp \left( -\frac{|\tau - s|}{\epsilon} \right),$$

and for $0 < \tau \leq s$

$$\mathbb{E}[\tilde{\eta}(\tau, \xi) \mid \mathcal{F}_s] = \tilde{\eta}(\tau, \xi) \exp \left( -\frac{\tau - s}{\epsilon} \right).$$

Moreover, $\tilde{\eta}(\tau) \in H^1(K)$ for any $\tau > 0$. By the assumption on $W_\xi(\tau, \xi)$, the process $\tilde{\eta}_\xi(\tau, \xi) = \tilde{\eta}(\tau/\epsilon, \xi)$ which solves

$$d\eta = -\eta \, d\tau + dW_\xi(\tau, \xi)$$

and also $\tilde{\eta}_\xi(\tau) \in H^1(K)$ for any $\tau > 0$ with

$$\mathbb{E}[\tilde{\eta}_\xi(\tau, \xi)\tilde{\eta}_\xi(s, \zeta)] = \frac{1}{2} q'(\xi, \zeta) \exp \left( -\frac{|\tau - s|}{\epsilon} \right).$$

For initial value we assume $u_\epsilon^0 \in L^2(\Omega, L^2(K) \cap L^\infty(\mathbb{R}))$ and

$$\int_{\mathbb{R}} u_\epsilon^0 \, d\xi = M, \quad \|u_\epsilon^0\|_{L^\infty(\mathbb{R})} = m$$

for some deterministic positive constants $m$ and $M$.  

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We study the dynamics of the random differential equation (RDE) \((5)\) for fixed \(\epsilon > 0\). First, for any \(\tau > 0\), in the mild sense the RDE \((5)\) is written as

\[
u'(\tau) = S(\tau)u'_0 + \int_0^\tau S(\tau-s)\nu'(s)u'_\xi(s) \, ds + \frac{1}{\sqrt{\epsilon}} \int_0^\tau S(\tau-s)(\nu'(s)\bar{\eta}'_\xi(s))_\xi \, ds.
\]

Then by the same discussion for equation \((3)\) this theorem follows.

**Theorem 6.** Assume Assumption \([2]\) holds. For any \(T > 0\) and initial \(u'_0 \in L^2(\Omega, L^2(K) \cap L^\infty(\mathbb{R}))\) satisfying \((19)\), the RDE \((5)\) has a unique mild solution

\[u' \in L^2(\Omega, L^2(0, T; H^1(K)) \cap C(0, T; L^2(K)) \} .
\]

We next construct a stationary solution for the RDE \((5)\), which is attractive for any \(\epsilon > 0\). We follow the discussion in section \([3]\)

First, by the same discussion as in section \([3.1]\),

\[\|u'(\tau)\|_{L^\infty(\mathbb{R})} \leq m \quad \text{for all } \tau > 0 ,
\]

with \(m = \|u'_0\|_{L^\infty(\mathbb{R})}\). Next we give a uniform estimate in \(\tau\) in the space \(H^1(K)\) from the estimate in space \(L^2(K)\) for \(u'\) with any fixed \(\epsilon > 0\). We follow the same discussion as in section \([3.2]\).

Let \(u'(\tau, \xi) = u'_c(\tau, \xi) + u'_s(\tau, \xi)\) with \(u_c \in E_c\) and \(u_s \in E_s\). Then

\[
\begin{align}
du'_c &= 0, \\
du'_s &= \left[Lu'_s - \Pi_s(u'\eta'_\xi)\right] d\tau + \frac{1}{\sqrt{\epsilon}} (u'\bar{\eta}'_\xi) \, d\xi .
\end{align}
\]

So

\[u'_c(\tau, \xi) = u'_c(0, \xi) = \langle u'_0, e_0 \rangle e_0(\xi) = \int_\mathbb{R} u'_0(\xi) \, d\xi e_0(\xi)
\]

which is totally determined by the mass of the initial value \(M := \int_\mathbb{R} u'_0(\xi) \, d\xi\).

Now by multiplying \(u'\) in the space \(L^2(K)\) on both sides of the RDE \((5)\),

\[
\frac{1}{2} \frac{d}{d\tau} \|u'_s\|^2_{L^2(K)} \leq -\frac{1}{2} \|u'_s\|^2_{H^1(K)} - \langle u'\xi, u' \rangle + \frac{1}{\sqrt{\epsilon}} \langle (u'\bar{\eta}')_\xi, u' \rangle .
\]

Integrating by parts,

\[\langle u'\xi, u' \rangle = -\frac{1}{3} \int_\mathbb{R} (u')^3 \eta\xi \, d\xi .
\]

Then by the same discussion as for \((12)\) we have that for any \(\epsilon, \epsilon' > 0\) and \(q > 2\), there exist positive constants \(C_\epsilon, R\) and \(C_{\epsilon', q, R}\) such that

\[|\langle u'\xi, u' \rangle| \leq \left[\epsilon C + \epsilon' C_\xi ||u'||_{L^\infty(\mathbb{R})}^2 \right] ||u'_\xi||^2_{L^2(K)} + C_\epsilon C_{\epsilon', q, R} ||u'||_{L^\infty(\mathbb{R})}^4 \|. 
\]
Moreover, for any $\varepsilon > 0$ there is some positive constant $C_\varepsilon$ such that
\[ |\langle u' \hat{\eta}' \rangle_\xi, u' \rangle | \leq C_\varepsilon \| \hat{\eta}' \|^2_{L^2(K)} \| u' \|^2_{L^\infty(\mathbb{R})} + \varepsilon \| u'_\xi \|^2_{L^2(K)} . \]
Then for any $\varepsilon$ and $\varepsilon' > 0$, there are positive constants that we still denote by $C_\varepsilon$ and $C_{\varepsilon', q, R}$ for some positive $q$ and $R$ such that
\[
\frac{1}{2} \frac{d}{d\tau} \| u'_s(\tau) \|^2_{L^2(K)} \\
\leq -\frac{1}{2} \| u'_s \|^2_{H^1(K)} + \left[ \varepsilon C + \varepsilon' C_\varepsilon \| u' \|^2_{L^\infty(\mathbb{R})} \right] \| u'_s \|^2_{L^2(K)} \\
+ C_\varepsilon C_{\varepsilon', q, R} \| u' \|^4_{L^\infty(\mathbb{R})} \\
+ \frac{1}{\sqrt{\varepsilon}} \left[ C_\varepsilon \| \hat{\eta}' \|^2_{L^2(K)} \| u' \|^2_{L^\infty(\mathbb{R})} + \varepsilon \| u'_\xi \|^2_{L^2(K)} \right] \\
+ \left[ \varepsilon C + \varepsilon' C_\varepsilon \| u' \|^2_{L^\infty(\mathbb{R})} + \frac{1}{\sqrt{\varepsilon}} \right] \| u'_s \|^2_{H^1(K)} \\
+ C_\varepsilon C_{\varepsilon', q, R} \| u' \|^4_{L^\infty(\mathbb{R})} + \frac{1}{\sqrt{\varepsilon}} C_\varepsilon \| \hat{\eta}' \|^2_{L^2(K)} \| u' \|^2_{L^\infty(\mathbb{R})} .
\]
Now choose $\varepsilon$ and $\varepsilon' > 0$ small enough and since $u'_\varepsilon = Me_0(\xi)$, $\| u' \|^2_{L^\infty} \leq m$ and $\| u'_\varepsilon \|^2_{H^1(K)} \leq CM$ for some $C > 0$, then by the Gronwall lemma and the properties of $\hat{\eta}'$, 
\[
\mathbb{E} \| u'(\tau, \omega) \|^2_{L^2(K) \cap L^\infty(\mathbb{R})} \leq R^*_1 \quad \text{for all } \tau \geq 0,
\]
with some positive constant $R^*_1$.

Similarly to the discussion in subsection 3.2,
\[
\mathbb{E} \int_0^\tau \| u'(s) \|^2_{H^1(K)} ds \leq R^*_2 \tau + R^*_3
\]
for some constants $R^*_2$ and $R^*_3$. Then we show the existence of a stationary measure. Notice that $u'$ is not a Markov process, we consider the process $(u', \hat{\eta}')$ which is a Markov process on the space $(L^2(K) \cap L^\infty(\mathbb{R})) \times H^1(K)$.

For this we introduce the space $\mathfrak{M}$ consisting all probability measures on the space $(L^2(K) \cap L^\infty(\mathbb{R})) \times H^1(K)$ and endow the space $\mathfrak{M}$ with the topology of weak convergence. Denote by $\{ \mathcal{P}^\varepsilon_t \}_{t \geq 0}$ the continuous Markov semigroup associating with $(u', \hat{\eta}')$ on $\mathfrak{M}$ and the dual semigroup as
\[
\mathcal{P}^\varepsilon_t \eta(A) = \mathbb{P}\{(u'(\tau, \cdot), \hat{\eta}'(\tau)) \in A \},
\]
for any Borel set $A \subset (L^2(K) \cap L^\infty(\mathbb{R})) \times H^1(K)$ and $\eta \in \mathcal{M}$ with form $\eta = \mu * \tilde{\nu}$. Here, $u'(\tau, \cdot)$ is the solution to equations (5) with initial value distributing as $\mu$, and $\tilde{\nu}$ is the distribution of $\hat{\eta}'$. 

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Now by the compact embedding $H^1(K) \subset L^2(K)$ and the classical Bogolyubov–Krylov method [11], a stationary measure exists, denoted by $\bar{\eta}^\epsilon = \bar{\mu}^\epsilon * \bar{\nu}$.

Next we show the emergence of the stationary measure $\bar{\eta}^\epsilon$ by showing it is attractive. We follow the discussion for deterministic systems [44, 25] which was also applied to the case of additive noise in a stochastic Burgers’ equation [11]. The following lemma is a key step.

**Lemma 7.** For any $u_{1,0}, u_{2,0} \in L^2(K) \cap L^\infty(\mathbb{R})$ with

$$\int_{\mathbb{R}} u_{1,0}(\xi) \, d\xi = \int_{\mathbb{R}} u_{2,0}(\xi) \, d\xi.$$ 

Let $u_1^\epsilon(\tau, \xi)$ and $u_2^\epsilon(\tau, \xi)$ be the solutions to RDE (5) with initial value $u_{1,0}$ and $u_{2,0}$ respectively. Then the function

$$\phi^\epsilon(\tau) = \int_{\mathbb{R}} |u_1^\epsilon(\tau, \xi) - u_2^\epsilon(\tau, \xi)| \, d\xi$$

is strictly decreasing almost surely.

**Proof.** Let $U^\epsilon(\tau, \xi) = u_1^\epsilon(\tau, \xi) - u_2^\epsilon(\tau, \xi)$, then it satisfies the following linear equation

$$U^\epsilon_{\tau} = U^\epsilon_{\xi\xi} - \frac{1}{2}[(u_1^\epsilon + u_2^\epsilon - \xi - \frac{1}{\sqrt{\epsilon}}\bar{\eta}^\epsilon)U^\epsilon]_{\xi}. \quad (23)$$

Notice that for any solution $u^\epsilon$ of the SPDE (5), let $v^\epsilon = u_1^\epsilon$, then

$$v^\epsilon_{\tau} = L v^\epsilon + \frac{1}{2} v^\epsilon - (v^\epsilon)^2 - u^\epsilon v^\epsilon_{\xi} + \frac{1}{\sqrt{\epsilon}}(u^\epsilon \bar{\eta}^\epsilon)_{\xi\xi}.$$ 

By the same discussion as in section 3.1 and the construction of $\bar{\eta}^\epsilon$,

$$(u_1^\epsilon(\tau, \xi) + u_2^\epsilon(\tau, \xi) - \xi - \frac{1}{\sqrt{\epsilon}}\bar{\eta}^\epsilon(\tau, \xi))_{\xi}$$

is bounded by a random constant for any $\tau > 0$. Then for almost all fixed $\omega \in \Omega$, the equation (23) is a linear equation with bounded coefficient, then the result follows by the discussion for the corresponding deterministic system [44, 25].

Now we study the attracting property of any stationary measure. Further, we introduce the following subspace of $\mathcal{M}$; define

$$\mathcal{M}_2 = \left\{ \eta \in \mathcal{M} : \int_{(L^2(K) \cap L^\infty(\mathbb{R})) \times H^1(K)} (\|u\|^2_{L^2(K)} + \|\eta\|^2_{H^1(K)}) \eta(d(u, \eta)) < \infty \right\}.$$ 

We next show that for any $\eta \in \mathcal{M}_2$, with form $\eta = \mu * \bar{\nu}$, there is a stationary measure $\bar{\eta}^\epsilon = \bar{\mu}^\epsilon * \bar{\nu} \in \mathcal{M}_2$ such that $\mathcal{P}_{\tau}^\epsilon \eta$ converges weakly to $\bar{\eta}^\epsilon$ as $\tau \to \infty$.
Associated with the solution to the RDE (5) we choose \( \mu \in \mathcal{M}_2 \) which has the form

\[
\mu = \delta_M \ast \mu_s,
\]

where \( \delta_M \) is some Dirac measure on \( E_c \) and \( \mu_s \) is supported on \( E_s \). Then consider the limit of \( \mathcal{P}_{\tau_n}^\mu \eta \) as \( \tau \to \infty \) with \( \eta = \mu \ast \nu \). First by the approach of the Bogolyubov–Krylov method, we have a probability measure \( \bar{\mu}^\epsilon \), and subsequence \( \tau_n \) with \( \tau_n \to \infty , n \to \infty \), such that

\[
\mathcal{P}_{\tau_n}^\mu \eta \to \bar{\eta}^\epsilon := \bar{\mu}^\epsilon \ast \bar{\nu}, \ n \to \infty .
\]

Suppose \( \bar{\mu}'^\epsilon \) is another probability measure such that for some \( \tau_n' \to \infty , n \to \infty \),

\[
\mathcal{P}_{\tau_n}^\mu \eta \to \bar{\eta}'^\epsilon := \bar{\mu}'^\epsilon \ast \bar{\nu}, \ n \to \infty .
\]

Denote by \( \bar{u}^\epsilon (\tau, \xi) \) and \( \bar{u}'^\epsilon (\tau, \xi) \) the two solutions of RDE (5) with initial value \( \bar{u}^1(\xi) \) and \( \bar{u}^2(\xi) \), distributed as \( \bar{\mu}^\epsilon \) and \( \bar{\mu}'^\epsilon \) respectively. Then

\[
\int_{\mathbb{R}} \bar{u}^1(\xi) \, d\xi = \int_{\mathbb{R}} \bar{u}^2(\xi) \, d\xi .
\]

By Lemma 7 the function

\[
\int_{\mathbb{R}} |\bar{u}^\epsilon (\tau, \xi) - \bar{u}'^\epsilon (\tau, \xi)| \, d\xi
\]

is almost surely strictly decreasing in \( \tau \) which contradicts the stationarity of \( \bar{u}^\epsilon \) and \( \bar{u}'^\epsilon \). Hence we deduce the following theorem.

**Theorem 8.** Assume Assumption 2 holds. For any initial \( u_0^\epsilon \in L^2(\Omega, L^2(K) \cap L^\infty(\mathbb{R})) \) satisfying (19), the solution \( u^\epsilon (\tau, \xi) \) to the RDE (5), converges in distribution, as \( \tau \to \infty \), to \( \bar{u}^\epsilon \) in the space \( L^2(K) \) which is the unique solution to RDE (5) with

\[
\int_{\mathbb{R}} \bar{u}^\epsilon (\tau, \xi) \, d\xi = \int_{\mathbb{R}} u_0^\epsilon (\xi) \, d\xi .
\]

**Remark 3.** By the construction of \( \bar{u}^\epsilon \),

\[
\|\bar{u}^\epsilon (\tau)\|_{L^\infty(\mathbb{R})} \leq m, \ E\|\bar{u}^\epsilon (\tau)\|_{H^1(K)}^2 \leq C \quad \text{for all } \tau \geq 0 ,
\]

for some constant \( C > 0 \).

We want to pass the above convergence property to the stochastic Burgers’ equation (4); that is, we want to pass to the limit \( \epsilon \to 0 \) in \( u^\epsilon \) in the space \( C([0, \infty), L^2(K)) \). We give some estimates uniform in \( \epsilon \) in the next section.
5 Some a priori estimates on finite time intervals

This section shows the tightness of $\{u^\epsilon\}_{0<\epsilon\leq1}$ in the space $C(0, T; L^2(K))$ for any $T > 0$. We assume $u^\epsilon_0 \in L^2(\Omega, L^2(K) \cap L^\infty(\mathbb{R}))$ and converges in distribution to $u^\epsilon_0 \in L^2(\Omega, L^2(K) \cap L^\infty(\mathbb{R}))$.

From equation (22), by the chain rule,

$$\frac{1}{2} \frac{d}{dt} \|u^\epsilon_s(\tau)\|^2_{L^2(K)} \leq -\frac{1}{2} \|u^\epsilon(\tau)\|^2_{H^1(K)} + \langle u^\epsilon(\tau)u^\epsilon_\xi(\tau), u^\epsilon(\tau) \rangle + \frac{1}{\sqrt{\epsilon}} \bar{\eta}(\tau)u^\epsilon_\xi(\tau) + \frac{1}{\sqrt{\epsilon}} \bar{\eta}_\xi(\tau)u^\epsilon(\tau), u^\epsilon(\tau) \rangle.$$  \hspace{1cm} (28)

Define the two integrals

$$I^1_\epsilon(\tau) = \frac{1}{\sqrt{\epsilon}} \int_0^\tau \langle u^\epsilon_\xi(s)\bar{\eta}(s), u^\epsilon(s) \rangle \, ds,$$

$$I^2_\epsilon(\tau) = \frac{1}{\sqrt{\epsilon}} \int_0^\tau \langle u^\epsilon(s)\bar{\eta}_\xi(s), u^\epsilon(s) \rangle \, ds.$$

Now by the factorization method, for some $0 < \alpha < 1$,

$$\frac{1}{\sqrt{\epsilon}} \int_0^\tau \langle u^\epsilon_\xi(s)\bar{\eta}(s), u^\epsilon(s) \rangle \, ds = \frac{\sin \pi \alpha}{\alpha} \int_0^\tau (\tau - s)^{\alpha-1}Y^\epsilon(s) \, ds$$

where

$$Y^\epsilon(s) = \frac{1}{\sqrt{\epsilon}} \int_0^s (s - r)^{-\alpha} \langle u^\epsilon_\xi(r)\bar{\eta}(r), u^\epsilon(r) \rangle \, dr$$

$$= \frac{1}{2\sqrt{\epsilon}} \int_0^s (s - r)^{\alpha} \langle (u^\epsilon(\tau))^2 \bar{\eta}_\xi(r), \bar{\eta}(r) \rangle \, dr$$

$$= -\frac{1}{2\sqrt{\epsilon}} \int_0^s (s - r)^{-\alpha} \int_{\mathbb{R}} (u^\epsilon(\tau))^2 \bar{\eta}(r)K \, d\xi \, dr$$

$$- \frac{1}{2\sqrt{\epsilon}} \int_0^s (s - r)^{-\alpha} \int_{\mathbb{R}} (u^\epsilon(\tau))^2 \bar{\eta}_\xi(r)K \, d\xi \, dr$$

$$= Y^\epsilon_1(s) + Y^\epsilon_2(s).$$

Then for any $T > 0$, there is some positive constant $C_T$ such that

$$\sup_{0 \leq \tau \leq T} |I^1_\epsilon(\tau)|^2 \leq C_T \int_0^T |Y^\epsilon_1(s)|^2 \, ds + C_T \int_0^T |Y^\epsilon_2(s)|^2 \, ds.$$
We first consider $Y_1^\epsilon$. By the $L^\infty(\mathbb{R})$ estimates on $u^\epsilon$, and the construction of $\bar{\eta}^\epsilon$, 

$$
\mathbb{E}|Y_1^\epsilon(s)|^2
= \frac{1}{\mathbb{E}} \left| \mathbb{E} \int_0^s \int_\rho^s \left( (s-r)^{-\alpha} (s-\rho)^{-\alpha} \int_\mathbb{R} (u^\epsilon(r,\xi))^2 \bar{\eta}^\epsilon(r,\xi)K_\xi d\xi \right.ight.
\left. \times \int_\mathbb{R} (u^\epsilon(\rho,\zeta))^2 \bar{\eta}^\epsilon(\rho,\zeta)K_\zeta d\zeta \right) dr d\rho \left| \right.
\leq \frac{m^4}{\epsilon} \left| \int_0^s \int_\rho^s \left( (s-r)^{-\alpha} (s-\rho)^{-\alpha} \int_\mathbb{R} \mathbb{E}\bar{\eta}^\epsilon(r,\xi)\bar{\eta}^\epsilon(\rho,\zeta)K_\xi K_\zeta d\xi d\zeta dr d\rho \right| \right.
\leq C_{1,T},
$$

and similarly

$$
\mathbb{E}|Y_2^\epsilon(s)|^2
= \frac{1}{\mathbb{E}} \left| \mathbb{E} \int_0^s \int_\rho^s \left( (s-r)^{-\alpha} (s-\rho)^{-\alpha} \int_\mathbb{R} (u^\epsilon(r,\xi))^2 \bar{\eta}^\epsilon(r,\xi)K_\xi d\xi \right.ight.
\left. \times \int_\mathbb{R} (u^\epsilon(\rho,\zeta))^2 \bar{\eta}^\epsilon(\rho,\zeta)K_\zeta d\zeta \right) dr d\rho \left| \right.
\leq C_{1,T},
$$

for some positive constant $C_{1,T}$. Then

$$
\mathbb{E} \sup_{0 \leq \tau \leq T} |I_1^\epsilon(\tau)| \leq C_T C_{1,T}.
$$

By the same discussion for $I_2^\epsilon$, a similar expectation holds:

$$
\mathbb{E} \sup_{0 \leq \tau \leq T} |I_2^\epsilon(\tau)| \leq C_T C_{2,T},
$$

for some positive constant $C_{2,T}$. Then by the same discussion for (12) and the Gronwall lemma,

$$
\mathbb{E} \sup_{0 \leq \tau \leq T} \|u^\epsilon(\tau)\|^2_{L^2(K)} + \mathbb{E} \int_0^T \|u^\epsilon(s)\|^2_{H^1(K)} ds \leq C_T
$$

for some positive constant $C_T$. Notice that in the mild sense

$$
\begin{align*}
u^\epsilon(\tau) &= S(\tau)u_0^\epsilon + \int_0^\tau S(\tau-\sigma)u^\epsilon(\sigma)u_\xi^\epsilon(\sigma) d\sigma \\
&\quad + \frac{1}{\sqrt{\epsilon}} \int_0^\tau S(\tau-\sigma)(u^\epsilon(\sigma)\bar{\eta}^\epsilon(\sigma)) d\sigma.
\end{align*}
$$
Then for any $T > \tau > \delta > 0$, 

$$
\|u^\epsilon(\tau) - u^\epsilon(\delta)\|_{L^2(K)} \leq \left\| (S(\tau) - S(\delta)) u_0 \right\|_{L^2(K)} + \left\| \int_{\tau}^{\delta} S(\tau - \sigma) u^\epsilon(\sigma) u^\epsilon_\xi(\sigma) d\sigma \right\|_{L^2(K)} + \frac{1}{\sqrt{\epsilon}} \left\| \int_{\tau}^{\delta} S(\tau - \sigma)(u^\epsilon(\sigma) \bar{\eta}^\epsilon(\sigma))_\xi d\sigma \right\|_{L^2(K)} + \left\| \int_{\delta}^{\tau} [S(\tau - \sigma) - S(\delta - \sigma)] u^\epsilon(\sigma) u^\epsilon_\xi(\sigma) d\sigma \right\|_{L^2(K)} + \frac{1}{\sqrt{\epsilon}} \left\| \int_{\delta}^{\tau} [S(\tau - \sigma) - S(\delta - \sigma)](u^\epsilon(\sigma) \bar{\eta}^\epsilon(\sigma))_\xi d\sigma \right\|_{L^2(K)}.
$$

By the $L^\infty(\mathbb{R})$ estimate of $u^\epsilon$, and by estimate (29), the expectation 

$$
\mathbb{E} \left\| \int_{\delta}^{\tau} S(\tau - \sigma) u^\epsilon(\sigma) u^\epsilon_\xi(\sigma) d\sigma \right\|_{L^2(K)} \leq \mathbb{E} \int_{\delta}^{\tau} \|S(\tau - \sigma) u^\epsilon(\sigma) u^\epsilon_\xi(\sigma)\|_{L^2(K)} d\sigma \leq \mathbb{E} \int_{\delta}^{\tau} \|u^\epsilon(\sigma) u^\epsilon_\xi(\sigma)\|_{L^2(K)} d\sigma \leq m \mathbb{E} \int_{\delta}^{\tau} \|u^\epsilon_\xi(\sigma)\|_{L^2(K)} d\sigma \leq mC_T \sqrt{\tau - \delta}.
$$

Expanding by $\{e_k\}_k$ and by (15), 

$$
\frac{1}{\epsilon} \mathbb{E} \left\| \int_{\delta}^{\tau} S(\tau - \sigma) (u^\epsilon(\sigma) \bar{\eta}^\epsilon(\sigma))_\xi d\sigma \right\|_{L^2(K)}^2 \leq \frac{1}{\epsilon} \mathbb{E} \sum_k \int_{\delta}^{\tau} \int_{\delta}^{\tau} \left[ e^{-\lambda_k(\tau - \sigma)} \int_{\mathbb{R}} u^\epsilon(\sigma, \xi) \bar{\eta}^\epsilon(\sigma, \xi)(e_k K)_\xi d\xi \right. \\
\times e^{-\lambda_k(\tau - \lambda)} \int_{\mathbb{R}} u^\epsilon(\lambda, \zeta) \bar{\eta}^\epsilon(\lambda, \zeta)(e_k K)_\zeta d\zeta \] \, d\sigma \, d\lambda \leq \frac{m^2}{\epsilon} \sum_k \int_{\delta}^{\tau} \int_{\delta}^{\tau} \left[ e^{-\lambda_k(\tau - \sigma)} e^{-\lambda_k(\tau - \lambda)} \int_{\mathbb{R}} \int_{\mathbb{R}} \bar{\eta}^\epsilon(\sigma, \xi) \bar{\eta}^\epsilon(\lambda, \zeta) \\
\times (e_k K)_\xi (e_k K)_\zeta d\xi \, d\zeta \right] \, d\sigma \, d\lambda \leq C_T (\tau - \delta),
$$
for some positive constant $C_T$. By the strong continuity of the semigroup $S(\tau)$ and a similar discussion, from (30) the expectation
\[
\mathbb{E}\|u(\tau) - u(\delta)\|_{L^2(K)} \leq C_T \sqrt{\tau - \delta}.
\]

(31)

Now we need the following lemma [29]. Suppose $X_1$ and $X_2$ are two Banach spaces. Let $T > 0$, $1 \leq p \leq \infty$, and $B$ be a compact operator from $X_1$ to $X_2$; that is, $B$ maps bounded sets of $X_1$ to relatively compact subsets of $X_2$.

**Lemma 9** [29]. Let $H$ be a bounded subset of $L^1(0,T;X_1)$ such that $G = BH$ is a subset of $L^p(0,T;X_2)$ bounded in $L^r(0,T;X_2)$ with $r > 1$. If
\[
\lim_{\sigma \to 0} \|u(\cdot + \sigma) - u(\cdot)\|_{L^p(0,T;X_2)} = 0 \quad \text{uniformly for } u \in G,
\]
then $G$ is relatively compact in $L^p(0,T;X_2)$ (and in $C(0,T;X_2)$ if $p = +\infty$).

Let $X_1 = H^1(K)$, $X_2 = L^2(K)$ and $B$ be the embedding from $X_1$ to $X_2$, by Lemma 9 from estimates (29) and (31) we obtain the following main theorem of this section.

**Theorem 10.** Assume Assumption 2 holds. For any $T > 0$, and $u^{\epsilon}_0 \in L^2(\Omega;L^2(K) \cap L^\infty(\mathbb{R}))$ satisfying (19), the distribution of $\{u^\epsilon\}_{0<\epsilon \leq 1}$ is tight in the space $C(0,T;L^2(K))$.

6 Diffusion approximation

This section determines the limit of $u^\epsilon$, the solutions of the rde (5), as $\epsilon \to 0$. We first show the tightness of $u^\epsilon$ in the space $C([0,\infty),L^2(K))$ by Theorem 10. Then we determine the limit of $u^\epsilon$ in the space $C([0,\infty),L^2(K))$ by a martingale approach.

6.1 Tightness in space $C([0,\infty),L^2(K))$

We need the following result on the tightness of a family processes [13, Theorem 3.9.1].

**Lemma 11.** Let $\mathcal{X}$ be a Polish space and let $\{X^\epsilon\}_{0<\epsilon \leq 1}$ be a family of processes with sample paths in the space $C([0,\infty),\mathcal{X})$. Suppose that for any $\delta > 0$ and $T > 0$ there exists a compact set $\Gamma_{\delta,T} \subset \mathcal{X}$ such that for all $0 < \epsilon \leq 1$
\[
\mathbb{P}\{X^\epsilon(t) \in \Gamma_{\delta,T} \text{ for } 0 \leq t \leq T\} \geq 1 - \delta.
\]

(32)

Then $\{X^\epsilon\}$ is tight in the space $C([0,\infty),\mathcal{X})$ if and only if $\{F(X^\epsilon)\}_{0<\epsilon \leq 1}$ is tight in the space $C([0,\infty),\mathbb{R})$ for any $F \in C_b(\mathcal{X})$, where $C_b(\mathcal{X})$ is the space consisting of all continuous and bounded functions on $\mathcal{X}$.
Remark 4. We do not need to verify the tightness of \( \{F(X')\}_{0<\epsilon \leq 1} \) for all \( F \in C_b(\mathcal{X}) \). One just needs to verify the tightness for all \( F \) in a dense subset of \( C_b(\mathcal{X}) \) in the topology of uniform convergence on compact sets [13, Theorem 3.9.1].

By Theorem 10, the pre-condition (32) in Lemma 11 holds. Next we show the tightness of \( \{F(u')\}_{0<\epsilon \leq 1} \) in the space \( C([0,\infty)) \) for any \( F \in C_b(L^2(K)) \). We follow a martingale approach. Continue to let \( \mathcal{X} \) be a Polish space and \( \{X^\epsilon\}_{0<\epsilon \leq 1} \) be a family of processes valued in the space \( C([0,\infty), \mathcal{X}) \) adapted to the filtration \( \mathcal{F}^\tau_\epsilon \). Let \( \mathcal{L}^\epsilon = \bigcup_{\tau \geq 0} \mathbb{E} Y(\tau) \). Let

\[
\mathcal{M}^\epsilon = \left\{ (Y,Z) \in \mathcal{L}^\epsilon \times \mathcal{L}^\epsilon : Y(\tau) - \int_0^\tau Z(s) \, ds \text{ is } \mathcal{F}^\tau_\epsilon \text{-martingale} \right\}.
\] (33)

Then the following lemma applies [13, Theorem 3.9.4].

**Lemma 12.** For any bounded continuous function \( F \) on \( \mathcal{X} \) with bounded support, and for any \( \delta > 0 \) and \( T > 0 \), there is \((Y^\epsilon, Z^\epsilon) \in \mathcal{M}^\epsilon \) such that

\[
\limsup_{\epsilon \to 0} \mathbb{E} \left[ \sup_{\tau \in [0,T]} |Y^\epsilon(\tau) - F(X^\epsilon(\tau))| \right] < \delta
\] (34)

and

\[
\sup_{\epsilon} \mathbb{E} \left[ \|Z^\epsilon\|_{L^p([0,T])} \right] < \infty \quad \text{for some } p \in (1, \infty].
\] (35)

Then \( \{F(X^\epsilon)\}_{0<\epsilon \leq 1} \) is tight in \( C([0,\infty), \mathbb{R}) \).

By Remark 4 we just need to show the tightness of the following family of real valued processes [20]:

\[
\{f((u^\epsilon, \varphi))\}_{0<\epsilon \leq 1}
\]

for any \( \varphi \in \mathcal{D}(\mathbb{R}) \) and twice differentiable compactly supported functions \( f \). From the RDE 15

\[
f((u^\epsilon(\tau), \varphi)) - f((u_0^\epsilon, \varphi)) = \int_0^\tau f'((u^\epsilon(s), \varphi))(u^\epsilon(s), \mathcal{L}^\epsilon) \, ds - \int_0^\tau f'((u^\epsilon(s), \varphi))\langle u^\epsilon(s)u^\xi(s), \varphi \rangle \, ds
\]

\[
+ \frac{1}{\sqrt{\epsilon}} \int_0^\tau f'((u^\epsilon(s), \varphi))((u^\epsilon(s)\bar{\eta}^\epsilon(s))_\xi, \varphi) \, ds.
\] (36)

One can see that the singular term in the above equation is difficult to treat. To treat this term we follow a perturbation approach developed by Kushner [20]. Let \( \mathcal{F}^\tau_\epsilon \) be the \( \sigma \)-algebra generated by \( \{\bar{\eta}^\epsilon(s) : 0 \leq s \leq \tau\} \). Then
introduce the process

\[ F_1^\epsilon(\tau) = \frac{1}{\sqrt{\epsilon}} \mathbb{E} \left[ \int_\tau^\infty f'(\langle u^\epsilon(\tau), \varphi \rangle) \langle (u^\epsilon(\tau) \bar{\eta}^\epsilon(s)) \rangle_\xi, \varphi \rangle \, ds \mid F_\tau^\epsilon \right]. \tag{37} \]

For the process \( F_1^\epsilon(\tau) \) the following lemma holds.

**Lemma 13.** Assume Assumption 2 holds. Then

\[ F_1^\epsilon(\tau) = \sqrt{\epsilon} f'(\langle u^\epsilon(\tau), \varphi \rangle) \langle (u^\epsilon(\tau) \bar{\eta}^\epsilon(\tau)) \rangle_\xi, \varphi \rangle. \tag{38} \]

Furthermore,

\[ \mathbb{E}|F_1^\epsilon(\tau)| \leq C \sqrt{\epsilon} \]

for some positive constant \( C \), and for any \( T > 0 \)

\[ \mathbb{E} \sup_{0 \leq \tau \leq T} |F_1^\epsilon(\tau)| \to 0 \quad \text{as} \quad \epsilon \to 0. \]

**Proof.** The equality (38) is implied by (16) and the property of conditional expectation. Then by the \( L^\infty \) bound on \( u^\epsilon \) and the estimates on \( \bar{\eta}^\epsilon \),

\[ \mathbb{E}|F_1^\epsilon(\tau)| \leq \sqrt{\epsilon} \|f'(\tau)\|_{L^\infty(\mathbb{R})} \|u^\epsilon(\tau)\|_{L^\infty(\mathbb{R})} \|\bar{\eta}^\epsilon(\tau)\|_{L^2(K)} \|\varphi_\xi + \frac{1}{2} \xi \varphi\|_{L^2(K)} \]

for some positive constant \( C \). Further, by the maximal estimate on stochastic integral [33, Lemma 7.2], for any \( T > 0 \)

\[ \mathbb{E} \sup_{0 \leq \tau \leq T} \|\bar{\eta}^\epsilon(\tau)\|_{L^2(K)}^2 \leq C_T \]

for some positive constant \( C_T \). Then by (38)

\[ \mathbb{E} \sup_{0 \leq \tau \leq T} |F_1^\epsilon(\tau)| \to 0 \quad \text{as} \quad \epsilon \to 0. \]

The proof is complete. \( \square \)

To apply Lemma 12, we first construct \((Y^\epsilon, Z^\epsilon) \in \mathcal{F}^\epsilon\). For this we introduce the operator \( A^\epsilon \) defined by

\[ A^\epsilon f(\tau) = \mathbb{P} - \lim_{\delta \to 0} \frac{1}{\delta} \mathbb{E} \left[ f(\tau + \delta) - f(\tau) \mid F_\tau^\epsilon \right] \tag{39} \]

for any \( F_\tau^\epsilon \) measurable function \( f \) with \( \sup_{\tau} \mathbb{E}|f(\tau)| < \infty \). Then Ethier and Kurtz’s proposition [13, Proposition 2.7.6] yields that

\[ f(\tau) - \int_0^\tau A^\epsilon f(s) \, ds \]

is a martingale with respect to \( F_\tau^\epsilon \). Now define \((Y^\epsilon, Z^\epsilon) \) as

\[ Y^\epsilon(\tau) = f(\langle u^\epsilon(\tau), \varphi \rangle) + F_1^\epsilon(\tau), \quad Z^\epsilon(\tau) = A^\epsilon Y^\epsilon(\tau). \]

Then we establish the following lemma.
Lemma 14.

\[ Z^\epsilon(\tau) = f'(\langle u^\epsilon(\tau), \varphi \rangle)\langle u^\epsilon(\tau), L\varphi \rangle - f'(\langle u^\epsilon(\tau), \varphi \rangle)\langle \frac{1}{2}(u^\epsilon(\tau))^2, \varphi \rangle + f''(\langle u^\epsilon(\tau), \varphi \rangle)\langle u^\epsilon(\tau), \eta^\epsilon(\tau) \rangle, \varphi, \xi + \frac{1}{2}\xi \varphi \rangle^2 \]

\[ - f'(\langle u^\epsilon(\tau), \varphi \rangle)\langle \langle u^\epsilon(\tau) \eta^\epsilon(\tau) \rangle \xi, (\varphi, \xi + \frac{1}{2}\xi \varphi) \rangle \eta^\epsilon(\tau) \rangle + \frac{1}{\sqrt{\epsilon}} f''(\langle u^\epsilon(\tau), \varphi \rangle) \langle \langle u^\epsilon(\tau), L\varphi \rangle - \langle \frac{1}{2}(u^\epsilon(\tau))^2, \varphi \rangle \rangle \langle (u^\epsilon(\tau))_\xi, \varphi \rangle \right) \]

\[ - \frac{1}{\sqrt{\epsilon}} f'(\langle u^\epsilon(\tau), \varphi \rangle) \langle \langle u^\epsilon(\tau) \eta^\epsilon(\tau) \rangle \xi, \varphi \rangle \left[ \langle u^\epsilon(\tau), L((\varphi, \xi + \frac{1}{2}\xi \varphi) \eta^\epsilon(\tau)) \rangle \right] . \]

Proof. By (36),

\[ A^\epsilon f(\langle u^\epsilon(\tau), \varphi \rangle) = f'(\langle u^\epsilon(\tau), \varphi \rangle)\langle u^\epsilon(\tau), L\varphi \rangle - f'(\langle u^\epsilon(\tau), \varphi \rangle)\langle u^\epsilon(\tau)u_\xi, \varphi \rangle + \frac{1}{\sqrt{\epsilon}} f'(\langle u^\epsilon(\tau), \varphi \rangle) \langle (u^\epsilon(\tau)\eta^\epsilon(\tau))_\xi, \varphi \rangle . \]

Now consider \( A^\epsilon F^\epsilon_1 \). By (38) and the construction of \( \eta^\epsilon \),

\[ \mathbb{E}[F^\epsilon_1(\tau + \delta) | \mathcal{F}_\tau] \]

\[ = -\sqrt{\epsilon}\mathbb{E}\left\{ \left[ f'(\langle u^\epsilon(\tau + \delta), \varphi \rangle) - f'(\langle u^\epsilon(\tau), \varphi \rangle) \right] \times \langle u^\epsilon(\tau + \delta) \eta^\epsilon(\tau + \delta), \varphi \xi + \frac{1}{2}\xi \varphi \rangle | \mathcal{F}_\tau \right\} \]

\[ - \sqrt{\epsilon} f'(\langle u^\epsilon(\tau), \varphi \rangle) \mathbb{E}\left[ \left[ \langle u^\epsilon(\tau + \delta) - u^\epsilon(\tau) \rangle \eta^\epsilon(\tau + \delta), \varphi \xi + \frac{1}{2}\xi \varphi \rangle | \mathcal{F}_\tau \right] \right] \]

\[ - \sqrt{\epsilon} f'(\langle u^\epsilon(\tau), \varphi \rangle) \langle u^\epsilon(\tau), (\varphi, \xi + \frac{1}{2}\xi \varphi) \rangle \langle \eta^\epsilon(\tau), e^{-\delta/\epsilon}, \varphi \xi + \frac{1}{2}\xi \varphi \rangle . \]

Then

\[ A^\epsilon F^\epsilon_1(\tau) = f''(\langle u^\epsilon(\tau), \varphi \rangle)\langle u^\epsilon(\tau), \eta^\epsilon(\tau) \rangle, \varphi \xi + \frac{1}{2}\xi \varphi \rangle^2 \]

\[ + \sqrt{\epsilon} f''(\langle u^\epsilon(\tau), \varphi \rangle) \langle u^\epsilon(\tau), L\varphi \rangle - \langle \frac{1}{2}(u^\epsilon(\tau))^2, \varphi \rangle \rangle \langle (u^\epsilon(\tau)\eta^\epsilon(\tau))_\xi, \varphi \rangle \]

\[ - \frac{1}{\sqrt{\epsilon}} f'(\langle u^\epsilon(\tau), \varphi \rangle) \langle \langle u^\epsilon(\tau) \eta^\epsilon(\tau) \rangle \xi, \varphi \rangle \left[ \langle u^\epsilon(\tau), L((\varphi, \xi + \frac{1}{2}\xi \varphi) \eta^\epsilon(\tau)) \rangle \right] . \]

This completes the proof. \( \square \)

Now by the above construction of \( (Y^\epsilon, Z^\epsilon) \),

\[ Y^\epsilon(\tau) - f(\langle u^\epsilon(\tau), \varphi \rangle) = -F^\epsilon_1(\tau) \]

Then by Lemma (39)

\[ \lim_{\epsilon \to 0} \mathbb{E} \sup_{0 \leq \tau \leq T} |F^\epsilon_1(\tau)| = 0 . \]
Furthermore, by the $L^\infty(\mathbb{R})$ estimates on $u^\epsilon(\tau)$,
\[ \sup_{\epsilon} \mathbb{E}\|Z^\epsilon(\tau)\|_{L^2(0,T)} < \infty. \] (40)

By Lemma 12, we thus deduce the following theorem.

**Theorem 15.** Assume Assumption 2 holds, and $u_0^\epsilon \in L^2(\Omega, L^2(K) \cap L^\infty(\mathbb{R}))$ satisfies (19), the family of distributions of processes $\{u^\epsilon\}_{0<\epsilon\leq 1}$ is tight in the space $C([0,\infty), L^2(K))$.

**Remark.** By (27), all discussions in this section and section 5 hold for $\bar{u}^\epsilon$. So the above tightness result also holds for $\{\bar{u}^\epsilon\}_{0<\epsilon\leq 1}$.

### 6.2 Pass to the limit $\epsilon \to 0$

We show the convergence of $u^\epsilon$ as $\epsilon \to 0$ and determine the limit. For this we introduce a limit martingale problem and show any accumulation point of $\{u^\epsilon\}$ is a solution to this martingale problem. By the convergence result of Walsh [40, Theorem 6.15], we just need to consider finite dimensional distributions of $\{\langle u^\epsilon, \varphi_1 \rangle, \ldots, \langle u^\epsilon, \varphi_n \rangle\}$ for any $\varphi_1, \ldots, \varphi_n \in D(\mathbb{R})$; that is, we just pass to the limit $\epsilon \to 0$ in the following equality
\[ \mathbb{E}\left\{ Y^\epsilon(\tau) - Y^\epsilon(s) - \int_s^\tau Z^\epsilon(r) \, dr \right\} = 0 \] (41)
for any bounded continuous function $h$ and $0 < r_1 < \cdots < r_n < T$ with any $T > 0$. Denote by $u$ one limit point in the sense of distribution of $u^\epsilon$ as $\epsilon \to 0$ in the space $C([0,\infty), L^2(K))$. Notice that we can not have the limit $\lim_{\epsilon \to 0} f(\langle u^\epsilon(\tau), \varphi \rangle) = f(\langle u(\tau), \varphi \rangle)$ just with the convergence of $u^\epsilon$ to $u$ in distribution. However, by the Skorohod theorem we construct new probability space and new variables without changing distributions in $C([0,\infty), L^2(K))$ (for simplicity we do not introduce new notation) such that $u^\epsilon$ almost surely converges to $u$ in the space $C([0,\infty); L^2(K))$.

Then by the estimates in Lemma 12 and the construction of $Y^\epsilon$, the limit of $Y^\epsilon(\tau) - Y^\epsilon(s)$ is $f(\langle u(\tau), \varphi \rangle) - f(\langle u(s), \varphi \rangle)$.

Now we treat the integral term. First denote by $Z_k^\epsilon(\cdot)$, $k = 1, 2, 3, 4$, the first four terms of $Z^\epsilon(\cdot)$ and by $Z_5^\epsilon(\cdot)$ the last two terms in $Z^\epsilon(\cdot)$. Then, in distribution as $\epsilon \to 0$,
\[ \int_s^\tau Z_1^\epsilon(r) \, dr \to \int_s^\tau f'(\langle u(r), \varphi \rangle) \langle u(r), \mathcal{L}\varphi \rangle \, dr, \]
and by the estimates on $u^\epsilon$ in section 5 and estimates on $\tilde{\eta}^\epsilon$ in section 2
\[ \mathbb{E}\int_s^\tau |Z_5^\epsilon(r)| \, dr \to 0. \]
Notice that \((u')^2\) is bounded in \(L^2(0,T; L^2(K))\) for any \(T > 0\) and by the tightness of \(u'\) in the space \(C(0,T; L^2(K))\), \(u'\) converges almost everywhere to \(u\) on \([0,T] \times \mathbb{R}\), then by the \(L^\infty(\mathbb{R})\) bound on \(u'\) and \(u\), we have in distribution for any \(T > 0\)

\[
\langle (u')^2, \varphi \rangle \to \langle u^2, \varphi \rangle \quad \text{as} \quad \epsilon \to 0.
\]  

(42)

So in distribution as \(\epsilon \to 0\)

\[
\int_s^T Z^2_\epsilon(r)dr \to \int_s^T f'((u(r), \varphi))(u(r)u_\epsilon(r), \varphi)dr.
\]

Next we treat terms \(Z^3_\epsilon\) and \(Z^4_\epsilon\). For any \(u \in L^2(K)\) define the bilinear operator \(\Sigma(u)\) such that

\[
\langle \Sigma(u), \varphi, \varphi \rangle = \int_R \int_R u(\xi)u(\zeta)q(\xi, \zeta)(\varphi(\xi)K(\xi))_\xi(\varphi(\zeta)K(\zeta))_\zeta d\xi d\zeta
\]

(43)

for any \(\varphi \in \mathcal{D}(\mathbb{R})\), and define the linear operator

\[
\langle A(u), \varphi \rangle = \frac{1}{2} \int_R u(\xi)q(\xi, \xi)(\varphi(\xi)K(\xi))_\xi d\xi
\]

(44)

\[
+ \frac{1}{2} \int_R u(\xi)q'(\xi, \xi)(\varphi(\xi)K(\xi))_\xi d\xi.
\]

For this operator \(\Sigma\) the following lemma holds.

**Lemma 16.** For any \(u \in H^1(K)\), let \(B(\tau, \xi) = (u(\xi)W(\tau, \xi))_\xi\), then \(B\) is an \(L^2(K)\) valued Wiener process with the covariation operator \(\Sigma(u)\).

**Proof.** The proof is direct. By (9),

\[
W_\xi = \sum_{k=1}^\infty \sqrt{q_k}e_k(\xi)\beta_k(\tau).
\]

Then by the representation of \(q(\xi, \zeta)\),

\[
\mathbb{E}B(\tau, \xi)B(\tau, \zeta)
\]

\[
= \mathbb{E} \left( u(\xi) \sum_k \sqrt{q_k}e_k(\xi)\beta_k(\tau) \right) \left( u(\zeta) \sum_l \sqrt{q_l}e_l(\zeta)\beta_l(\tau) \right)_{\xi, \zeta}
\]

\[
= u_\xi(\xi)u_\zeta(\zeta)q_\xi(\xi, \zeta) + u_\xi(\xi)u_\zeta(\zeta)q(\xi, \zeta) + u(\xi)u_\zeta(\zeta)q_\xi(\xi, \zeta)
\]

\[
+ u(\xi)u(\zeta)q_{\xi, \zeta}(\xi, \zeta).
\]

By the definition of \(\Sigma(u)\) this proves the lemma. \(\square\)
To pass to the limit $\epsilon \to 0$ in $Z^3_\delta$ and $Z^4_\delta$ we apply again the perturbation method \cite{26}. Let

$$F^\epsilon_3(\tau) = f''((u^\epsilon(\tau), \varphi)) \int_\tau^\infty E \left[ \langle u^\epsilon(\tau)\tilde{\eta}(s), \varphi_\xi + \frac{1}{2} \xi \varphi \rangle^2 \right] \ ds$$

and

$$F^\epsilon_4(\tau) = f'((u^\epsilon(\tau), \varphi)) \int_\tau^\infty E \left[ \langle (u^\epsilon(\tau)\tilde{\eta}(s))_\xi, (\varphi_\xi + \frac{1}{2} \xi \varphi)(\tilde{\eta}(s)) \rangle \right] \ ds.$$

By the construction of $\tilde{\eta}$ and $\tilde{\eta}_\xi$,

$$E [\tilde{\eta}(s, \xi)\tilde{\eta}(s, \zeta) | \mathcal{F}^\epsilon_\tau] = e^{-2(s-\tau)/\epsilon}q(\tau, \xi)\tilde{\eta}(\tau, \xi) + \frac{1}{2}q(\xi, \zeta)(1 - e^{-2(s-\tau)/\epsilon}), \quad (45)$$

$$E [\tilde{\eta}(s, \xi)\tilde{\eta}_\xi(s, \xi) | \mathcal{F}^\epsilon_\tau] = e^{-2(s-\tau)/\epsilon}q(\tau, \xi)\tilde{\eta}_\xi(\tau, \xi) + \frac{1}{2}q(\xi, \zeta)(1 - e^{-2(s-\tau)/\epsilon}). \quad (46)$$

Then

$$F^\epsilon_3(\tau) = \frac{\epsilon}{2} f''((u^\epsilon(\tau), \varphi)) \left[ \langle u^\epsilon(\tau)\tilde{\eta}(\tau), \varphi_\xi + \frac{1}{2} \xi \varphi \rangle^2 - \frac{1}{2} \langle \Sigma(u^\epsilon(\tau)) \varphi, \varphi \rangle \right] \quad (47)$$

and

$$F^\epsilon_4(\tau) = \frac{\epsilon}{2} f'((u^\epsilon(\tau), \varphi)) \left[ \langle (u^\epsilon(\tau)\tilde{\eta}(\tau))_\xi, (\varphi_\xi + \frac{1}{2} \xi \varphi)(\tilde{\eta}(\tau)) \rangle - \frac{1}{2} \langle A(u^\epsilon(\tau)), \varphi \rangle \right]. \quad (48)$$

Then by the estimates on $u^\epsilon(\tau)$, $\tilde{\eta}(\tau)$ and $\tilde{\eta}_\xi(\tau)$, direct computation yields this lemma.

**Lemma 17.** As $\epsilon \to 0$,

$$\sup_{\tau \geq 0} E F^\epsilon_3(\tau) = \mathcal{O}(\epsilon), \quad \text{and} \quad \sup_{\tau \geq 0} E F^\epsilon_4(\tau) = \mathcal{O}(\epsilon).$$

Now following the same discussion as in Lemma 13 and (45–46), we have the following lemma.

**Lemma 18.**

$$A^\epsilon F^\epsilon_3(\tau) = f''((u^\epsilon(\tau), \varphi))\left[ \frac{1}{2} \Sigma(u^\epsilon(\tau)) \varphi, \varphi \right] - \langle u^\epsilon(\tau)\tilde{\eta}(\tau), \varphi_\xi + \frac{1}{2} \xi \varphi \rangle + R^\epsilon_3(\tau)$$

and

$$A^\epsilon F^\epsilon_4(\tau) = f'((u^\epsilon(\tau), \varphi))\left[ \langle A(u^\epsilon(\tau)), \varphi \rangle - \langle u^\epsilon(\tau)\tilde{\eta}(\tau), (\varphi_\xi + \frac{1}{2} \xi \varphi)(\tilde{\eta}(\tau)) \rangle + R^\epsilon_3(\tau) \right]$$

25
with

\[
\sup_{\tau \geq 0} \mathbb{E}|R_3^\epsilon(\tau)| = O(\epsilon) \quad \text{and} \quad \sup_{\tau \geq 0} \mathbb{E}|R_4^\epsilon(\tau)| = O(\epsilon)
\]
as \epsilon \to 0.

Proof. This is similar to the discussion in the proof of Lemma 14. First we have for any \( \delta > 0 \)

\[
E[F_3^\epsilon(\tau + \delta) \mid F_\delta^\epsilon] = \frac{\epsilon}{2} E \left[ \left( f''(\langle u^\epsilon(\tau + \delta), \varphi \rangle) - f''(\langle u^\epsilon(\tau), \varphi \rangle) \right) \times \left( \langle u^\epsilon(\tau + \delta) \tilde{\eta}^\epsilon(\tau + \delta) \varphi + \frac{1}{2} \xi \varphi \rangle^2 - \langle \Sigma(u^\epsilon(\tau + \delta)) \varphi, \varphi \rangle \right) \right]
\]

and

\[
E[F_4^\epsilon(\tau + \delta) \mid F_\delta^\epsilon] = \frac{\epsilon}{2} E \left[ \left( f'(\langle u^\epsilon(\tau + \delta), \varphi \rangle) - f'(\langle u^\epsilon(\tau), \varphi \rangle) \right) \times \left( \langle u^\epsilon(\tau + \delta) \tilde{\eta}^\epsilon(\tau + \delta) \varphi + \frac{1}{2} \xi \varphi \rangle - \langle \Sigma(u^\epsilon(\tau)) \varphi, \varphi \rangle \right) \right]
\]

Then by the definition of \( A^\epsilon \) and (45)–(46), direct computation yields the result. The proof is complete. \( \Box \)
Now we have the following \( F^t_s \) martingale

\[
\mathcal{M}^t(\tau) = f(\langle u^t(\tau), \varphi \rangle) - f(\langle u^0, \varphi \rangle) - \int_0^\tau f'(\langle u^t(r), \varphi \rangle) \left[ \langle u^t(r), \mathcal{L} \varphi \rangle + \frac{1}{2} \langle u^t(r)^2, \varphi_\xi + \frac{1}{2} \xi \varphi \rangle + \langle A(u^t(r)), \varphi \rangle \right] dr
- \frac{1}{2} \int_0^\tau f''(\langle u^t(r), \varphi \rangle) \langle \Sigma(u^t(r)) \varphi, \varphi \rangle dr
\]

where

\[
R^t(\tau) = \int_0^\tau [Z_3^\varsigma(s) + R_3(s) + R_4(s)] ds
\]

with \( \mathbb{E}[R^t(\tau)] = \mathcal{O}(\epsilon) \) as \( \epsilon \to 0 \). Now passing to the limit \( \epsilon \to 0 \), the distribution of the limit \( u \) solves the martingale problem

\[
\mathcal{M}(\tau) = f(\langle u(\tau), \varphi \rangle) - f(\langle u_0, \varphi \rangle) - \int_0^\tau f'(\langle u(r), \varphi \rangle) \left[ \langle u(r), \mathcal{L} \varphi \rangle + \frac{1}{2} \langle u(r)^2, \varphi_\xi + \frac{1}{2} \xi \varphi \rangle + \langle A(u(r)), \varphi \rangle \right] dr
- \frac{1}{2} \int_0^\tau f''(\langle u(r), \varphi \rangle) \langle \Sigma(u(r)) \varphi, \varphi \rangle dr
\]

which, by Lemma 16 is equivalent to the martingale solution to the SPDE (6) for some new Wiener process \( \bar{W} \) with the same distribution as that of \( W \). By the general theory of SPDES, the martingale solution to the SPDE (6) is unique in the space \( L^2(0, T; H^1(K)) \cap C(0, T; L^2(K)) \) for any \( T > 0 \). Then we deduce the following theorem.

**Theorem 19.** Assume Assumption 2 holds and that the initial data \( u^0 \in L^2(\Omega, L^2(K) \cap L^\infty(\mathbb{R})) \) satisfies (14). Moreover, \( u^0 \) converges in distribution to \( u_0 \). The solution of RDE (5), \( u^\epsilon \), converges in distribution in the space \( C([0, \infty), L^2(K)) \) to \( u \) which solves the SPDE (6) with initial data \( u_0 \).

Notice that the SPDE (6) is different from the stochastic Burgers’ equation (3). Now we consider random Burgers’ type equation (4). By the assumption that \( q(\xi) \in H^2(K) \) and (3), the extra terms \( \frac{1}{2}(u'q')_\xi \) and \( \frac{1}{2}(u'q')_\xi \) do not change the estimates in sections (3) and (3). Then we derive this theorem.

**Theorem 20.** Assume Assumption 2 holds and \( u^0 \in L^2(\Omega, L^2(K) \cap L^\infty(\mathbb{R})) \) converges in distribution to \( u_0 \in L^2(\Omega, L^2(K) \cap L^\infty(\mathbb{R})) \) as \( \epsilon \to 0 \). Moreover, assume (12) holds. The solution of RDE (4) converges, as \( \epsilon \to 0 \), in distribution in the space \( C([0, \infty), L^2(K)) \) to the solution of SPDE (2) with initial data \( u_0 \).
By the above theorem and Remark 5 we have the following result on the convergence of stationary statistical solutions. Denote by $\mathbb{P}^\epsilon = \mathcal{D}(\hat{u}^\epsilon, \hat{\eta}^\epsilon)$, a stationary statistical solution to the system (3) coupled with $\tilde{\eta}^\epsilon$. Let $\mathbb{P}^\epsilon_n = \mathcal{D}(\tilde{u}^\epsilon)$. Then we have the following corollary.

**Corollary 21.** For $\epsilon \to 0$, there is a sequence $\epsilon_n \to 0$ as $n \to \infty$, such that

$$\mathbb{P}^\epsilon_n \to \mathbb{P} \text{ weakly as } n \to \infty,$$

where $\mathbb{P}$ is a probability space on $C([0, \infty); L^2(K))$, which is a stationary statistical solution to stochastic Burgers equation (3).

We have shown that for any $\eta = \mu * \tilde{\nu} \in \mathcal{M}_2$, there is a stationary measure $\mu^\epsilon = \tilde{\mu}^\epsilon * \tilde{\nu}$ such that $P_\nu \eta$ weakly converges to $\mu^\epsilon$ as $\tau \to \infty$. Denote by $\tilde{u}^\epsilon$ the solution to (3) with initial data $\tilde{u}^\epsilon(0)$. Then $\mathbb{P}^\epsilon = \mathcal{D}(\tilde{u}^\epsilon, \tilde{\eta}^\epsilon)$ is a stationary statistical solution to the system (3) coupled with (14). We are concerned with the marginal distribution $\mathbb{P}^\epsilon_n = \mathcal{D}(\tilde{u}^\epsilon)$.

For any statistical solution $\mathbb{P}^\epsilon$ to (3) coupled with (14), denote by $\mathbb{P}^\epsilon_n$ the marginal distribution, then $\mathbb{P}^\epsilon_n$ converges weakly to $\mathbb{P}^\epsilon$ as $\tau \to \infty$. Let $\mathbb{P}^\epsilon = \mathcal{D}(\tilde{u}^\epsilon, \tilde{\eta}^\epsilon)$ with initial data $\tilde{u}^\epsilon(0)$ solving (3) coupled with (14) with new Wiener process $\tilde{W}$ distributing the same as $W$. Then $\mathbb{P}^\epsilon_n = \mathcal{D}(\tilde{u}^\epsilon(\cdot + \tau))$. By the continuous dependence on initial data of the solution, as

$$\mathcal{D}(\tilde{u}^\epsilon(\tau)) \to \mathcal{D}(\tilde{u}^\epsilon(0)) \text{ weakly as } \tau \to \infty,$$

we have

$$\mathcal{D}(\tilde{u}^\epsilon(\cdot + \tau)) \to \mathcal{D}(\tilde{u}^\epsilon) \text{ weakly as } \tau \to \infty.$$  

That is $\mathbb{P}^\epsilon_n \to \mathbb{P}^\epsilon$ weakly as $\tau \to \infty$. Furthermore by the convergence result Corollary 21 there is $\epsilon_n \to 0$ as $n \to \infty$, $\mathbb{P}^\epsilon_n$ converges weakly to some probability measure $\mathbb{P}$ which is a stationary statistical solution of SPDE (3) in the space $C([0, \infty), L^2(K))$.

Now consider any solution $u \in C([0, \infty); L^2(K))$ to the SPDE (3). Then $\mathbb{P} = \mathcal{D}(u)$ is a statistical solution to the SPDE (3) and, by Theorem 20 there is a statistical solution $\mathbb{P}^\epsilon_n$ to the system (3) coupled with $\tilde{\eta}^\epsilon$, such that the marginal distribution $\mathbb{P}^\epsilon_n$ converges to $\mathbb{P}$ weakly as $\epsilon \to 0$. Moreover $\mathbb{P}^\epsilon_n$ converges weakly to $\mathbb{P}_\tau$ as $\epsilon \to 0$. One can choose $u_0 = u_0$ in Theorem 20.

Then for any $\delta > 0$, there is $T > 0$ such that for $\tau > T$ and for sufficiently large $n$

$$d_p(\mathbb{P}_\tau, \overline{\mathbb{P}}) \leq d_p(\mathbb{P}_\tau, \mathbb{P}^\epsilon_n) + d_p(\mathbb{P}^\epsilon_n, \overline{\mathbb{P}}) + d_p(\overline{\mathbb{P}}, \mathbb{P}) \leq \delta.$$  

That is $d_p(\mathbb{P}_\tau, \overline{\mathbb{P}}) \to 0$ as $\tau \to \infty$. Assume $\mathbb{P} = \mathcal{D}(\tilde{u})$ and $\overline{\mathbb{P}} = \mathcal{D}(\tilde{u})$, then $\tilde{u}$ is a stationary process and

$$\mathcal{D}(\tilde{u}(\tau)) \to \mathcal{D}(\tilde{u}(0)) \text{ weakly as } \tau \to \infty.$$
Then we conclude with the following main theorem.

**Theorem 22.** Assume Assumption 2 holds. For any solution of the stochastic Burgers’ equation (2) with \( w(1, x) \in L^2(K) \cap L^\infty(\mathbb{R}) \), there is a unique self-similar solution \( \tilde{w}(t, x) \) such that

\[
\sqrt{t}w(t, x) - \sqrt{t}\tilde{w}(t, x) \to 0 \quad \text{as} \ t \to \infty \quad \text{in distribution in} \ L^2(\mathbb{R}).
\]

**A Existence result for stochastic Burgers’ type equation**

We prove Theorem 2. Noticing that the nonlinearity is non-Lipschitz, for any \( R > 0 \) we introduce the following smooth cut-off function \( \chi_R : [0, \infty) \to \mathbb{R} \) defined by

\[
\chi_R(x) = \begin{cases} 1, & 0 \leq x \leq R, \\ 0, & x \geq 2R. \end{cases}
\]

Consider the following system

\[
du^R = [\mathcal{L}u^R - \chi_R(\|u^R\|_{H^1(K)})u^R\bar{u}_\xi^R]d\tau + (u^RdW)\xi, \quad u^R(0) = u_0.
\]

Denote by \( B_R(u^R) := B_R(u^R, u^R) := \chi_R(\|u^R\|_{H^1(K)})u^R\bar{u}_\xi^R \), then \( B_R : H^1(K) \to L^2(K) \) is Lipschitz in the following sense

\[
\|B_R(u) - B_R(v)\|_{L^2(K)} \leq L_R\|u - v\|_{H^1(K)}
\]

for \( u, v \in H^1(K) \). Now for each \( u_0 \in L^2(K) \) define the nonlinear operator

\[
\mathcal{T}(u^R)(\tau) = S(\tau)u_0 + \int_0^\tau S(\tau - s)B_R(u^R(s))ds + \int_0^\tau S(\tau - s)(u^R(s)dW(s))\xi.
\]

Then \( \mathcal{T} \) maps \( L^2(\Omega, C(0, T; L^2(K)) \cap L^2(0, T; H^1(K))) \) to itself. For any \( u^R \in L^2(\Omega, C(0, T; L^2(K)) \cap L^2(0, T; H^1(K))) \), by the properties of the semigroup \( S(\tau) \),

\[
\|(\mathcal{T}u^R)(\tau)\|_{L^2(K)} \leq \|u_0\|_{L^2(K)} + \int_0^\tau \|u^R\|_{L^2(K)}d\tau + \left\|\int_0^\tau S(\tau - s)(u^RdW)\xi\right\|_{L^2(K)}
\]

\[
\leq \|u_0\|_{L^2(K)} + \int_0^\tau \|u^R\|_{L^\infty(\mathbb{R})}\|u^R\|_{L^2(K)}d\tau
\]

\[
+ 2\left\|\int_0^\tau S(\tau - s)(u^RdW)\xi\right\|^2_{L^2(K)} + 2
\]

\[
\leq \|u_0\|_{L^2(K)} + \int_0^\tau \|u^R\|_{H^1(K)}^2d\tau + 2\left\|\int_0^\tau S(\tau - s)(u^RdW)\xi\right\|^2_{L^2(K)} + 2
\]
For the third term, by Assumption \[2\] and the properties of the stochastic integral \[33\], for some constant \(C > 0\)

\[
\mathbb{E} \max_{0 \leq \tau \leq T} \left\| \int_0^\tau S(\tau - s)(u^R dW)_\xi \right\|_{L^2(K)}^2 \\
\leq 2\mathbb{E} \max_{0 \leq \tau \leq T} \left\| \int_0^\tau S(\tau - s)u^R dW \right\|_{L^2(K)}^2 + 2\mathbb{E} \max_{0 \leq \tau \leq T} \left\| \int_0^\tau S(\tau - s)u^R dW_\xi \right\|_{L^2(K)}^2 \\
\leq C \int_0^T \|u^R(s)\|_{H^1(K)}^2 ds + C \int_0^T \|u^R(s)\|_{L^2(K)}^2 ds.
\]

Then for some constant \(C > 0\)

\[
\mathbb{E}\|T u^R\|^2_{C(0,T;L^2(K))} \leq \|u_0\|_{L^2(K)}^2 + C \int_0^T \|u^R(s)\|_{H^1(K)}^2 ds.
\]

On the other hand

\[
\|\langle T u^R(\tau) \rangle_{H^1(K)}\|_{H^1(K)} \\
\leq \|S(\tau)u_0\|_{H^1(K)} + \left\| \int_0^\tau S(\tau - s) \chi_R(\|u^R\|_{H^1(K)}) u^R u^R_\xi ds \right\|_{H^1(K)} \\
+ \left\| \int_0^\tau S(\tau - s)(u^R dW)_\xi \right\|_{H^1(K)}.
\]

By the properties of the semigroup \(S(\tau)\), for \(T > 0\)

\[
\int_0^T \|S(\tau)u_0\|_{H^1(K)}^2 d\tau \leq \frac{1}{2} \|u_0\|_{L^2(K)}^2,
\]

and for some constant \(C_R > 0\)

\[
\int_0^T \left\| \int_0^\tau S(\tau - s) \chi_R(\|u^R\|_{H^1(K)}) u^R u^R_\xi ds \right\|_{H^1(K)}^2 d\tau \\
\leq C_R \int_0^T \|u^R(\tau)\|_{H^1(K)}^2 d\tau.
\]

For the stochastic term, first we have

\[
\left\| \int_0^\tau S(\tau - s)(udW)_\xi \right\|_{H^1(K)} \\
\leq \left\| \int_0^\tau S(\tau - s)u_\xi dW \right\|_{H^1(K)} + \left\| \int_0^\tau S(\tau - s)u(dW)_\xi \right\|_{H^1(K)}.
\]

30
By the properties of $S(\tau)$, and direct calculation by the expanding $u^R_\xi$ and $W$ in terms of the orthonormal basis, for some constant $C > 0$

$$
E \int_0^T \left\| \int_0^\tau S(\tau - s)u^R_\xi dW(s) \right\|_{H^1(K)}^2 \, d\tau \leq C \int_0^T \|u^R\|_{H^1(K)}^2 \, d\tau.
$$

Similarly

$$
E \int_0^T \left\| \int_0^\tau S(\tau - s)u^R dW_\xi(s) \right\|_{H^1(K)}^2 \, d\tau \leq C \int_0^T \|u^R\|_{H^1(K)}^2 \, d\tau.
$$

Then for some constants $C > 0$ and $C_R > 0$, such that

$$
E \int_0^T \|T u^R(\tau)\|_{H^1(K)}^2 \, d\tau \leq C \|u_0\|_{L^2(K)}^2 + C_R \int_0^T \|u^R(\tau)\|_{H^1(K)}^2 \, d\tau.
$$

That is, $T$ maps $L^2(\Omega, C(0,T; L^2(K)) \cap L^2(0,T; H^1(K)))$ to itself. Then by the classical method for the wellposedness of SPDEs, a unique solution $u^R$ exists for the cut-off system (50) in $L^2(\Omega, C(0,T; L^2(K)) \cap L^2(0,T; H^1(K)))$.

Now define an increasing sequence of stopping times $\{\tau_n\}$:

$$
\tau_R := \inf\{\tau > 0 : \|u^R(\tau)\|_{H^1(K)} \geq R\}
$$

when it exits, and $\tau_R = \infty$ otherwise. Let $\tau_\infty = \lim_{R \to \infty} \tau_R$, a.s., and set $u^{\tau_R}(\tau) = u^R(\tau \wedge \tau_R)$. Then $u^{\tau_R}$ is a local solution of equation (3) for $0 \leq \tau \leq \tau_R$. To show the limit $R \to \infty$ of $u^{\tau_R}$ we need an estimate on $\|u\|_{H^1(K)}$.

Notice the functional $\| \cdot \|_{H^1(K)}$ is continuous and differentiable on $H^1(K)$.

Applying Itô’s formula (9) to $\|u_s(\tau)\|_{H^1(K)}^2$ yields

$$
\frac{1}{2} \frac{d}{dt} \|u_s\|_{H^1(K)}^2 = \langle Lu_s, -Lu_s \rangle - \langle uu_\xi, Lu_s \rangle + \langle (uW)_\xi, Lu_s \rangle + \frac{1}{2} \left[ \|L^{1/2} u_\xi\|_{L^2}^2 + \|L^{1/2} u\|_{L^2}^2 \right].
$$

By the Cauchy inequality, for some small $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$
\langle uu_\xi, Lu \rangle \leq \varepsilon \|u\|_{H^3(K)}^2 + C_\varepsilon \|u\|_{H^1(K)}^2.
$$

Then, by the assumption (8) on $q$ and $q'$,

$$
\frac{1}{2} \frac{d}{dt} \mathbb{E}[\|u_s\|_{H^1(K)}^2] \leq -c_{\varepsilon} \mathbb{E}[\|u_s\|_{H^2(K)}^2] + c_{\varepsilon} \mathbb{E}[\|u_c\|_{H^2(K)}^2] + C_{\varepsilon'} \mathbb{E}[\|u_s\|_{H^1(K)}^2] + C_{\varepsilon'} \mathbb{E}[\|u_c\|_{H^1(K)}^2].
$$
for some constants $c_\varepsilon > 0$ and $C_\varepsilon > 0$. Then by Gronwall lemma,
\[
\mathbb{E}\|u(\tau)\|_{H^1(K)}^2 \leq K(\tau)
\]
for some function $K : (0, \infty) \to [0, +\infty)$.

Now we show $u(\tau) := \lim_{R \to \infty} u^R$ is the global solution to (3). By the above estimates, for $\tau \in [0, T \wedge \tau_R]$
\[
\mathbb{E}\|u^{\tau_R}(\tau)\|_{H^1(K)} \leq K(\tau).
\]
Also
\[
\mathbb{E}\|u^{\tau_R}(T)\|_{H^1(K)}^2 = \mathbb{E}\|u^R(T \wedge \tau_R)\|_{H^1(K)}^2 \geq \mathbb{E}\left\{1_{\{\tau_R \leq T\}}\|u^R(T \wedge \tau_R)\|_{H^1(K)}^2\right\} \geq \mathbb{P}\{\tau_R \leq T\} R^2.
\]
Then
\[
\mathbb{P}\{\tau_R \leq T\} \leq \frac{K(T)}{R^2}
\]
which, by the Borel–Cantelli Lemma, implies
\[
\mathbb{P}\{\tau_\infty > T\} = 1.
\]
Then we have the global well-posedness of the equation (3). By the property of $S(\tau)$, this mild solution is also a weak solution [33],

**B Statistical solution**

We introduce the statistical solution of the random Burgers equation (5) coupled with $\eta^\varepsilon$. The statistical solution was introduced to study universal properties of turbulent flows [17, 18, 39, e.g.],

The system of random equations (1) coupled with (14) is said to have a statistical solution in the space $C([0, \infty); L^2(K))$ if there is a probability measure $\mathcal{Q}_\varepsilon$ supported on $C([0, \infty); L^2(K) \times H^1(K))$, and processes $(\hat{u}^\varepsilon, \hat{\eta}^\varepsilon) \in C([0, \infty); L^2(K))$, $\hat{W}$ defined on a new probability space, such that

1. $\mathcal{D}(\hat{u}^\varepsilon, \hat{\eta}^\varepsilon) = \mathcal{Q}_\varepsilon$;

2. $\hat{W}$ is Wiener processes distributed the same as $W$;

3. $\mathcal{D}(\hat{u}^\varepsilon(0)) = \mathcal{D}(u_0)$, $\mathcal{D}(\hat{\eta}^\varepsilon) = \mathcal{D}(\eta^\varepsilon)$ and $\hat{u}^\varepsilon(0)$ are independent from $\hat{W}$;
4. the process \( \hat{u}^\varepsilon \) is a weak solution of (14) with \( \hat{\eta}^\varepsilon \) replaced by \( \check{\eta}^\varepsilon \). Here \( \check{\eta}^\varepsilon \) is a stationary process solving (14) with \( \hat{W} \) replaced by \( \check{W} \).

A stationary statistical solution is a statistical solution, a Borel measure \( \check{\mathbb{P}}^\varepsilon \), which is invariant under the following translation on \( C([0, \infty); L^2(K) \times H^1(K)) \)

\[
(u(\cdot), \eta(\cdot)) \mapsto (u(\cdot + \tau), \eta(\cdot + \tau)), \quad \tau \geq 0
\]

for \( (u, \eta) \in C([0, \infty); L^2(K) \times H^1(K)) \). For a statistical solution of the system of the random equation (4) coupled with (14), we denote by \( \check{\mathbb{P}}^\varepsilon = \mathcal{D}(\hat{u}(\cdot + \tau), \check{\eta}(\cdot + \tau)) \), which is also a statistical solution of the random equation (4) coupled with (14). For a stationary statistical solution \( \check{\mathbb{P}}^\varepsilon \),

\[
\check{\mathbb{P}}^\varepsilon = \check{\mathbb{P}}^\tau, \quad \tau \geq 0.
\]

The following result establishes a relation between the stationary measure and stationary statistical solution.

**Lemma 23.** If the random equation (4) coupled with (14) has a stationary measure supported on \( L^2(K) \times H^1(K) \), then there is a stationary statistical solution in \( C([0, \infty); L^2(K) \times H^1(K)) \).

**Proof.** The proof is direct by the following observation [10]: Let \( \check{\mathbb{P}}^\varepsilon = \mathcal{D}(\check{u}^\varepsilon, \check{\eta}^\varepsilon) \) be a stationary statistical solution to the random equation (4) coupled with (14), then

\[
\check{\eta}^\varepsilon = \mathcal{D}(\check{u}^\varepsilon(0), \check{\eta}^\varepsilon(0))
\]

is a stationary measure for the Markov process defined by the random equation (4) coupled with (14); Conversely, assume \( \check{\eta}^\varepsilon \) is a stationary measure of the random equation (4) coupled with (14), let \( (\check{u}^\varepsilon, \check{\eta}^\varepsilon) \) be a solution of the random equation (4) coupled with (14), with \( \mathcal{D}(\check{u}^\varepsilon(0), \check{\eta}^\varepsilon(0)) = \check{\eta}^\varepsilon \), then \( \check{\mathbb{P}}^\varepsilon = \mathcal{D}(\check{u}^\varepsilon, \check{\eta}^\varepsilon) \) is a stationary statistical solution of the random equation (4) coupled with (14).

For the stochastic Burgers type equation (3), a statistical solution in the space \( C([0, \infty; L^2(K)) \) is a probability measure \( \mathbb{P} \) supported on \( C([0, \infty; L^2(K)) \) and there are processes \( \check{u} \in C([0, \infty; L^2(K)) \) and \( \check{W} \) defined on a new probability space such that

i \( \mathcal{D}(\check{u}) = \mathbb{P} \);

ii \( \check{W} \) is Wiener processes distribute same as \( W \);

iii \( \mathcal{D}(\check{u}(0)) = \mathcal{D}(u_0) \) and \( \check{u}(0) \) are independent from \( \check{W} \);
the process \( \hat{u} \) is a weak solution of stochastic Burgers type equation (3) with \( W \) replaced by \( \hat{W} \).

Notice that the above definition of statistical solution is a solution to a martingale problem [31, Chapter V].

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