A well posed acoustic analogy based on a moving acoustic medium

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Abstract

For flows of a lossless gas, the stagnation enthalpy obeys a linear convected wave equation with coefficients which depend on the flow variables. This equation is self-adjoint and one has a reciprocity relation between source and observer. It fulfills for subsonic flow a quadratic conservation equation implying stability. It is taken as basis for an acoustic analogy and is applied to the sound generation by the collision of a convected vortex and a rigid cylinder.

1 Introduction

The concept of an acoustic analogy was introduced by Lighthill [1]. This idea offered him the possibility to derive important results on the generation of aerodynamic noise without relying on expansions or perturbation schemes. This is especially important, as the acoustic energy generated by an unsteady flow field represents usually only a minute part of the energy flux occurring in the flow and errors which are small compared to the flow quantities may be very large if one compares them with the sound quantities. The conditions under which perturbation schemes based on small sound level or on small flow Mach number may be used, have been clarified in later researches [2, 3, 4], although there still remain open questions.

Lighthill derived without any approximations an exactly valid equation, which admits an acoustic interpretation. He showed that every compressible flow fulfills an inhomogeneous wave equation with a quadrupole type source distribution. As the inhomogeneous wave equation describes the generation

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and propagation of sound in an ideal acoustic medium at rest, he had related aerodynamic noise to sound waves in the ideal acoustic medium. This relation is exactly true, but it relates the effects of convection and refraction by a steady basic flow to a source distribution in an acoustic medium without flow and this is difficult to visualize. One is therefore inclined to look for a relation to an ideal acoustic medium in motion.

Here we will identify the operator which describes the propagation of sound in irrotational, isentropic flow and we will base the analogy on that operator. This of course does not mean, that vortices or entropy inhomogeneities are excluded. It is very similar to the situation found in Lighthill’s analogy. There, flows were excluded from the medium and they occurred as sources. Here, vortices and entropy inhomogeneities are excluded and they occur as sources. The equation is therefore very well suited to study the sound radiation from vortices which are convected in an irrotational flow.

The acoustic variable which we use is the stagnation enthalpy. This variable has been used with aeroacoustic applications in mind before [5, 6]. The equations considered before differ however from that derived here, but compare [7]. The operator which replaces the wave operator here is a self-adjoint one, even if the flow field is completely arbitrary. It is not necessary that it is irrotational or that it fulfills the basic equations of fluid dynamics. The self-adjointness leads to a reciprocity principle which is valid for an exchange of source and observer. It also implies the existence of a variational principle from which the basic equation of the analogy can be derived. This principle is the simplest generalization of the variational principle for the wave equation, partial derivatives with respect to time are replaced by material derivatives, which seems natural if one requires Galilei-invariance. From the variational principle one can then conclude the validity of an energy theorem. The main drawback is of course the lack of a simple explicit expression for the Green’s function. Notice however, that for certain important situations a Green’s function to first order has been determined in [8]. An alternative would of course be a numerical solution. To check its feasibility, a general purpose PDE-solver for a PC has been applied to calculate the sound generated by a two-dimensional vortex convected along a circular cylinder.

2. The acoustic analogy

2.1 Preliminaries

Lighthill based his theory of aerodynamic sound generation on an equation, which he obtained by cross-differentiation from the Euler equations and from
the continuity equation
\[ \frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v}) + \nabla p = 0 \]
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \]
namely
\[ \Delta p - \frac{1}{a^2} \frac{\partial^2 p}{\partial t^2} + \nabla \cdot \nabla \cdot (\rho \mathbf{v}) \mathbf{v} = 0. \] (1)

Here \( \rho \) denotes the density, \( p \) the pressure, \( a \) the speed of sound, and \( \mathbf{v} \) the particle velocity. This equation is valid for isentropic flow if losses and temporal variations of the speed of sound can be ignored. Temporal averages can be subtracted, therefore one may assume that only fluctuating quantities are contained in eq. (1). No linearization has been performed in the derivation. Therefore eq. (1) is valid under very general conditions. Often, especially for low Mach number flows, one neglects the acoustic contributions in the double divergence in eq. (1) and considers this term as a known source term for the sound generation. Then one describes the sound field as a wave obeying the wave equation, i.e. as a sound field propagating in a non-moving medium. Convection and refraction effects are then neglected. Eq. (1) is however generally true and these effects are in principle contained in eq. (1). Attempts to extract them have been made by separating the velocity into a mean and fluctuating part and thereby to obtain these effects. Although it should be possible to describe mean flow effects in this way, it has been felt that a more appropriate description should be obtained through a modification of eq. (1). Instead of the wave operator, a ”convected wave operator” which describes the propagation of sound in a moving medium seems more appropriate. One then rewrites all fluid quantities as a superposition
\[ \mathbf{v} \rightarrow \mathbf{v}_0 + \mathbf{v}, p \rightarrow p_0 + p, \ldots \]
of ”nonacoustic” and ”acoustic” variables and rewrites the basic equations in terms of these variables. If one neglects contributions which are nonlinear in the non-subscribed variables, one obtains the acoustic equations. It is well known, that one can derive from this system one ”convected wave equation” for unidirectional (in \( x \)-direction) shear flow or for potential flows, namely
\[ \mathcal{L}_{\text{shear}} p = \frac{1}{a_0^2} \frac{D}{D t} \nabla \cdot a_0^2 \nabla p - \frac{1}{a_0^2} \frac{D^3 p}{D t^3} - 2(\nabla u_0) \cdot \nabla \frac{\partial p}{\partial x} = 0 \quad \text{with} \quad \frac{D}{D t} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \]
for unidirectional and with an acoustic potential \( \phi \)
\[ \mathcal{L}_{\text{pot}} \phi = \nabla \cdot (\rho_0 \nabla \phi - \frac{\rho_0}{a_0^2} \frac{D \phi}{D t} \mathbf{v}_0) - \frac{\partial}{\partial t} \frac{\rho_0}{a_0^2} \frac{D \phi}{D t} = 0 \]
for irrotational flows. For aerodynamic noise one should not neglect the nonlinear terms. It is however possible, very similar to Lighthill’s approach, to collect the linear and nonlinear terms and to derive an inhomogeneous wave equation. It is even possible – and it has been done e.g. by Tam and Auriault [9] – to follow this approach in the full continuity and Euler equations and derive an inhomogeneous linear system of equations. Then one could use for \( v_0, p_0 \), etc. the temporal mean values. The equations which are obtained by this method are much more complicated than the wave equation and are usually solvable only numerically. Furthermore they are rather different from the equations usually studied in mathematical physics and little is known about existence and uniqueness of solutions. It is however known, that the linear parts of the equations agree with the stability equations and as many flows are unstable, one has to expect, that the equations will be unstable. Of course, exponentially growing solutions are physically excluded – at least for longer times – and therefore they do not occur in the correct solutions. Small errors will however produce these instabilities and provisions have to be made to limit their growth. How these provisions influence the sound obtained from these calculations is difficult to assess.

Here we follow a different strategy which is related to the propagation of sound waves in potential flows, but differs from the above described method significantly. We do not separate the velocity into an irrotational part and a remainder but follow a path used by Howe [5]. He observed that Bernoulli’s equation states that the stagnation enthalpy differs from the potential only by a sign and by a time derivative. For irrotational sound waves one could therefore use the stagnation enthalpy instead of the potential. The stagnation enthalpy is however defined also for rotational flows and could be used as a variable of an acoustic analogy for arbitrary flows. This is what we will do. We will derive a convected wave equation for the stagnation enthalpy. We will discuss its main properties and show that it possesses many of the formal properties of the ordinary wave operator. An important example is an energy conservation theorem with an energy density which is positive for subsonic flows. This excludes instabilities, the influence of small errors remains small. The provisions which are necessary in many other analogies to limit the troublesome growth of instabilities are not necessary here.

### 2.2 Basic Relations

To obtain the equations for the acoustic analogy including convection effects, one starts from the Euler equation for compressible flow. Crocco’s form of
these equations reads

\[
\frac{\partial \mathbf{v}}{\partial t} + \nabla B = -\mathbf{L}, \quad \mathbf{L} = \omega \times \mathbf{v} - T \nabla s, \quad \omega = \text{curl } \mathbf{v}.
\]  

(2)

\(B\) denotes the total enthalpy \(B = h + \frac{1}{2}v^2\) with the enthalpy \(h\) and the velocity \(v\). \(T\) is the temperature and \(s\) the entropy. From the energy theorem one finds for the total enthalpy

\[
\frac{DB}{Dt} = \frac{1}{\rho} \frac{\partial p}{\partial t}
\]

(3)

If one writes \(d\rho = a^{-2}d\rho + \rho_s ds\) where \(\rho_s\) denotes the derivative of the density with respect to the entropy, one gets from the continuity equation

\[
\frac{\rho}{a^2} \left( \frac{DB}{Dt} \right) + \text{div } \mathbf{w} = -\rho_s \frac{\partial s}{\partial t} = q_s,
\]

(4)

where \(\mathbf{w}\) denotes the mass flux. If one multiplies Crocco’s vortex theorem (2) with the density \(\rho\) one obtains for the mass flux \(\mathbf{w}\) the equation

\[
\frac{\partial \mathbf{w}}{\partial t} - \frac{\rho \mathbf{v} DB}{a^2 Dt} + \rho \nabla B = -\rho \mathbf{L} + q_s \mathbf{v} = \mathbf{K}.
\]

(5)

It is easy to eliminate the mass flux \(\mathbf{w}\) from these equations. One then obtains an equation, which is linear in \(B\)

\[
\mathcal{L} B = \nabla \cdot (\rho \nabla B - \frac{\rho \mathbf{v} DB}{a^2 Dt}) - \frac{\partial}{\partial t} \frac{\rho DB}{a^2 Dt} = -\text{div } \rho \mathbf{L} + \frac{\partial q_s}{\partial t} + \text{div } q_s \mathbf{v} = q_{tot}.
\]

(6)

The operator \(\mathcal{L}\) obtained here agrees completely with the operator \(\mathcal{L}_{\text{pot}}\) given in the previous section for the propagation of irrotational sound waves.

If one inserts the sources from the equations (4) und (5), one finds

\[
q_{tot} = \left( \frac{\partial}{\partial t} \rho_s \frac{\partial}{\partial t} + \text{div } \rho_s \mathbf{v} \frac{\partial}{\partial t} + \text{div } \rho T \nabla \right) s + \text{div } \rho \mathbf{v} \times \omega.
\]

(7)

The sources are linear expressions in the vorticity vector and the entropy. In this analogy one may think of the sound as being generated from vorticity and entropy inhomogeneities.

With acoustical applications in mind, the total enthalpy was first used by Howe [5] and recently proposed also by Doak [6]. A comparison with their equations shows, that the equation for \(B\) is not uniquely determined. Howe’s convected wave operator is

\[
\mathcal{L}_{\text{Howe}} B = \Delta B - \frac{1}{a^2} \frac{DV}{Dt} \cdot \nabla B - \frac{D}{Dt} \frac{1}{a^2} \frac{DB}{Dt}
\]

\(\Delta B\) reads
and Doak's

\[ \mathcal{L}_{\text{Doak}} B = \Delta B - \frac{1}{a^2} \left[ \frac{\partial^2 B}{\partial t^2} + \left( 2 \mathbf{v} \frac{\partial}{\partial t} + \omega \times \mathbf{v} + T \nabla s - 2 \nabla h \right) \nabla B + \mathbf{v} \cdot \nabla \cdot \nabla B \right]. \]

The right hand sides of these equations differ from the right hand side of eq. (6). We will restrict ourselves to eq. (6). Notice however that the principle parts – i.e. those terms which contain second derivatives of \( B \) – of the three convected wave operators agree. This means that they agree in the high frequency limit of geometric acoustics and agree with the well known geometric acoustic theory.

A certain simplification of eq. (6) is possible if one requires that \( \rho \) and \( \mathbf{v} \) fulfill the continuity equation. One then gets

\[ \mathcal{L} B = \nabla \cdot (\rho \nabla B) - \rho \frac{D}{D t} \frac{1}{a^2} \frac{D B}{D t}. \]  

The equation (6) or (8) is a generalization of the wave equation. It reduces to the wave equation if one assumes in (6) \( \mathbf{v} = 0 \) and if one assumes further, that \( \rho \) and \( a^2 \) are constant. The equation agrees then with Lighthill’s equation in the form of Powell [10]. It seems to be a rather complicated equation. Considering a variational principle we will however see, that equation (6,8) is actually the simplest equation which contains a flow velocity. A simplification of the source is possible for an ideal gas. For an ideal gas \( \rho \) is the product of two functions which depend only on \( p \) and on \( s \). Then one has \( \rho_s = \rho f(s) \) with some function \( f(s) \) and one obtains for the sources of eq. (6)

\[ q_{\text{tot}} = - \text{div} \rho \mathbf{L} + \rho \frac{D}{D t} f(s) \frac{\partial s}{\partial t} = - \text{div} \rho \mathbf{L} - \rho \frac{\partial \mathbf{v}}{\partial t} \cdot f(s) \nabla s \]

if one makes use of the entropy conservation law.

The equation (4) and (5) rsp. (6) are now considered as the basic equations of the acoustic analogy. They agree formally with the linearized equations which describe small perturbations of a steady potential flow. The zeroth order equations are then given by \( B_0 = 0 \) (because of the Bernoulli equation) and \( \text{div} \mathbf{w}_0 = 0 \), i.e. the zeroth order versions of the eqs. (1) and (5). The first order eqs. of (4) and (5) are then obtained, if the fields \( \rho, a, \) and \( \mathbf{v} \) are replaced by their zero order values. Notice that \( \rho \mathbf{v} \) in eq. (5) then becomes \( \rho_0 \mathbf{v}_0 \) and it differs from \( \mathbf{w} \) which becomes \( \mathbf{w}_1 \). In that sense the left hand side of the eqs. (1) and (5) rsp. (6) are considered as equations which describe sound propagation in potential flows. This is very similar to Lighthill’s interpretation of the wave equation as an equation which describes sound propagation in a medium at rest. This interpretation is valid only if
\( \rho, a, \) and \( v \) are steady fields which fulfill the equations for compressible irrotational flow. We will however not require this, as \( \mathcal{L} \) is also well defined without this assumption and there are no advantages in assuming it. The eqs. (4) and (5) resp. (6) are exactly valid identities. They are considered in the following as a system of linear partial differential equations for the variables \( B \) and \( w \). This implies also that the right hand sides of the equations (4) and (5) resp. (6) are considered as the sources of the sound. They are related to vortices and entropy inhomogeneities. This seems reasonable if one wants to study the generation and propagation of sound in a potential flow. Then the flow consists of a superposition of an irrotational steady part and an unsteady part. Often one will superpose these two contributions linearly. We will not require that. For the steady flow one has a constant value of the total enthalpy \( B \). One may then assume \( B \) to be zero. \( B \) is then solely related to the unsteady part and is small if this part is small. In irrotational flow there is a potential \( \Phi \). This obeys the Bernoulli equation

\[
\Phi_t + B = 0
\]

(9)

\( B \) then differs in regions where the flow is irrotational from the temporal derivative of the potential only by its sign, it is however – contrary to the acoustic potential defined everywhere.

This equation (6) was originally derived in [7]. Let us now derive its main properties. The first important point is that \( \mathcal{L} \) is formally self-adjoint. This follows from the fact that one has for arbitrary functions \( B \) and \( \tilde{B} \) the identity

\[
\tilde{B} \mathcal{L} B - B \mathcal{L} \tilde{B} = \frac{\partial l_0}{\partial t} + \frac{\partial l_i}{\partial x^i}
\]

(10)

with

\[
l_0 = -\frac{\rho}{a^2} \left( \tilde{B} \frac{D B}{D t} - B \frac{D \tilde{B}}{D t} \right), \quad l_i = \rho \left( \tilde{B} \frac{\partial B}{\partial x^i} - B \frac{\partial \tilde{B}}{\partial x^i} \right) + l_0 v_i.
\]

(11)

This equation is easy to check. It implies also that one has for a scalar product \( (f, g) \) defined by \( (f, g) = \int f g d^3x dt \) the relation

\[
(\tilde{B}, \mathcal{L}B) = (B, \mathcal{L}\tilde{B})
\]

(12)

if \( B \) and \( \tilde{B} \) vanish on the boundary of the integration region or decay sufficiently rapidly at infinity.
2.3 Reciprocity

One may derive from the symmetry in eq. (12) a reciprocity relation. To be specific let $G(x, t, y, t')$ be the Green’s function associated with $L$, i.e.

$$L G(x, t, y, t') = -\delta(x - y) \delta(t - t'), \quad \text{with} \quad G(x, t, y, t') = 0 \quad \text{for} \quad t < t',$$

(13)

where we have assumed that $G$ is causal, i.e. it vanishes for all times before the source is switched on, which occurs at $t = t'$. There exists also an advanced Green’s function $G_{\text{adv}}$ with

$$L G_{\text{adv}}(x, t, y, t') = -\delta(x - y) \delta(t - t'), \quad \text{with} \quad G_{\text{adv}}(x, t, y, t') = 0 \quad \text{for} \quad t > t'.$$

(14)

It is easy to see that eq. (13) is invariant with respect to time reversal $t \to -t$ if at the same time the sign of the velocity is reversed. Therefore time reversal transforms a Green’s function into a Green’s function. As the time reversal interchanges the inequalities $t > t'$ and $t > t'$ one has

$$G(x, -t, y, -t'; -v(x, -t)) = G_{\text{adv}}(x, t, y, t'; v(x, t))$$

(15)

where we have added for clarity the function $v(x, t)$ to the list of arguments of the Green’s function. In addition the functions $\rho(x, t)$ and $a(x, t)$ have to be replaced by $\rho(x, -t)$ and $a(x, -t)$.

One may now apply eq. (12) with $B = G(x, t, y, t')$ and $\tilde{B} = G_{\text{adv}}(x, t, z, t'')$ and one obtains

$$(G_{\text{adv}}(x, t, z, t''), L G(x, t, y, t')) = (L G_{\text{adv}}(x, t, z, t''), G(x, t, y, t'))$$

(16)

Let us indicate briefly that there are no contributions from the right hand side of eq. (10). If the integration in eq. (16) is performed over a large cylinder in the $x,t$-space which extends over a large sphere in $x$-space and over all $t$ with $T_0 < t < T_1$, one has surface contributions which are to be evaluated over the large sphere at $t = T_0$ and at $t = T_1$ and over the surface of the large sphere in $x$-space for all $t$ with $T_0 < t < T_1$. If $T_0$ is before $t'$ and $t''$ and $T_1$ after $t'$ and $t''$ there are no contributions from the space integrals at $t = T_0$ and at $t = T_1$, as at least one factor vanishes in $l_0$ and in the $l_i$, namely the factors containing $G_{\text{adv}}$ at $t = T_1$ and those containing $G$ at $t = T_0$. There is also no contribution from the large surface in $x$-space if it is selected so large that no signal which was generated at $t = t'$ and at $x = y$ has reached this surface. Therefore eq. (16) is true, and one can evaluate
the scalar products with the $\delta$-functions in the eqs. (13) and (14) easily, and one obtains the equation

$$- G_{\text{adv}}(y, t', z, t'') = -G(z, t'', y, t')$$

which can with eq. (15) be rewritten as

$$G(y, -t', z, -t''); -v(y, -t)) = G(z, t'', y, t'; v(z, t)).$$

This is the reciprocity principle with reversed flow.

### 2.4 Variational Principle and Energy Conservation

From the self-adjointness on can conclude the existence of a variational principle from which eq. (6) can be derived. One has

$$\delta L = 0 \quad \text{mit} \quad L = \frac{1}{2}(B, \mathcal{L}B) - (B, q_{\text{tot}})$$

as

$$\delta L = \frac{1}{2}(\delta B, \mathcal{L}B) + \frac{1}{2}(B, \mathcal{L}\delta B) - (\delta B, q_{\text{tot}}) = (\delta B, \mathcal{L}B - q_{\text{tot}}).$$

The Lagrangian from (18) can be simplified somewhat, if the second derivatives in $\mathcal{L}$ are eliminated with integration by parts. One finds then

$$L = \int \int \left[ \rho \frac{2}{a^2} \left( \frac{DB}{Dt} \right)^2 - \rho \left( \nabla B \right)^2 - q_{\text{tot}}B \right] d^3x \, dt.$$  

(20)

It seems very remarkable that this variational principle seems to be the simplest possible extension of the well known principle for the wave equation which is invariant with respect to Galilei-transformations. The Lagrangian in eq. (18) differs from the Lagrangian of the wave equation only by the fact, that partial derivatives with respect to time are replaced by material derivatives formed with the velocity field $v$.

Now it is possible to obtain from a variational principle an energy theorem, resp. an energy conservation law, if the Lagrangian density does not depend explicitly from the time. If $l$ denotes the Lagrangian density from eq. (20), i.e.

$$l = \frac{\rho}{2a^2} \left( \frac{DB}{Dt} \right)^2 - \frac{\rho}{2} \left( \nabla B \right)^2 - q_{\text{tot}}B,$$

(21)

one has for the energy theorem

$$\frac{\partial}{\partial t} \left( \dot{B} \frac{\partial l}{\partial B} - l \right) + \frac{\partial}{\partial x^i} \dot{B} \frac{\partial l}{\partial B_{x^i}} = \frac{\partial l}{\partial t}.$$  

(22)
where the time derivative on the right hand side acts only on the explicit
time dependance in $l$, i.e. here in $\rho$, $a$, $\mathbf{v}$ and $q_{\text{tot}}$. The variable $B$ and its
derivatives are to be kept constant. The time derivative on the left hand
side of eq. (22) acts also on the implicit dependance in $B$ and its derivatives
$\dot{B}$ und $B_{x^i}$. Only the spatial coordinates $x^i$ are to be kept constant. An
energy conservation law is obtained from (22) if the Lagrangian density does
not contain the time explicitly. In general one obtains for the energy flux
$U_i = \dot{B} \frac{\partial l}{\partial B_{x^i}}$ and the energy density $e = \dot{B} \frac{\partial l}{\partial B} - l$ the explicit expressions

$$U = \rho \dot{B} \left( \frac{1}{a^2} \frac{D B}{D t} \mathbf{v} - \nabla B \right)$$

(23)

and

$$e = \frac{\rho}{a^2} \dot{B} \left( \frac{D B}{D t} \right)^2 + \frac{\rho}{2} \left( \nabla B \right)^2 - \frac{\rho}{a^2} \dot{B} \left( \nabla B \right)^2 - \frac{\rho}{2} \left( \mathbf{v} \cdot \nabla B \right)^2$$

which shows that the energy density $e$ is positive for subsonic $\mathbf{v}$. The energy
theorem is then of the form

$$\frac{\partial e}{\partial t} + \text{div} \mathbf{U} = q_{\text{En}}$$

with an energy density $e$ and an energy source density $q_{\text{En}}$. A useful relation
is obtained if one integrates this equation over the time for finite time events
or if one averages this equation for the case of steady phenomena. If one
denotes the resulting quantities by an overbar, one obtains

$$\text{div} \bar{\mathbf{U}} = \bar{q}_{\text{En}}.$$  

(24)

Another important conclusion can be drawn from the energy theorem if
one applies it to the solution of an initial value problem with vanishing right
hand side $q_{\text{tot}}$. If one assumes that the solution vanishes for large $|\mathbf{x}|$, one obtains

$$\frac{\partial}{\partial t} \int e \, d^3 x = 0,$$

i.e. the toatal energy in the sound field remains constant. As it is a sum of
positive contributions, none of these – e.g. $\dot{B}$ – can grow exponentially in
time, i.e. instabilities cannot occur.

The physical meaning of this energy flux becomes clearer if one considers
an irrotational isentropic region. There on may write (23) with the eqs. (3)
and (9) as

$$\mathbf{U} = \dot{B} \left( \frac{1}{a^2} \rho \mathbf{v} + \rho \nabla \Phi_t \right).$$

(25)
One may compare this energy flux with the flux from the Blokhintzev energy theorem which is valid for the propagation of irrotational sound waves in an irrotational mean flow in linear approximation. One has neglected quantities which are quadratically in the acoustical quantities. Notice that no linearization has been assumed in the derivation of the energy theorem (21,22). It is insofar an exact identity, only dissipative effects have been ignored. If one denotes in the Blokhintzev energy flux with $U_{Bl}$, the density, the speed of sound and the velocity of the irrotational mean flow with $\rho_0$, $a_0$ and $v_0$ and with $p'$ and $\phi'$ the acoustic pressure and the acoustic potential, one can write

$$U_{Bl} = \left(\frac{1}{\rho_0} p' + v_0 \cdot \nabla \phi'\right) \left(\frac{1}{a_0^2} p' v_0 + \rho_0 \nabla \phi'\right).$$

(26)

A comparison of eq. (26) with eq. (25) shows, that both energy fluxes are products of two factors, where the factors of eq. (25) are just the time derivatives of the factors of the Blokhintzev energy flux (24). If one thinks of the sound field as a superposition of temporal Fourier modes, one finds that the average energy flux consists of a superposition from fluxes of the modes. In the energy flux of eq. (23) all contributions contain an additional factor $\omega^2$ if $\omega$ denotes the angular frequency of the Fourier modes.

### 2.5 Solutions

As a first application one may consider the case of constant values of the velocity $v$, density $\rho$ and speed of sound $a$. Then vorticity and entropy inhomogeneities are convected with the velocity $v$. These inhomogeneities and also the total enthalpy $B$ are then functions of $x - vt$ only, i.e. $B = B(x - vt)$, $s = F_s(x - vt)$ and $\omega = F_\omega(x - vt)$. Then eq. (1) leads to

$$\rho \Delta B = q_{tot}, \quad q_{tot} = \left(\frac{\partial}{\partial t} \left(\rho_s \frac{\partial}{\partial t} + \text{div} \rho_s v \frac{\partial}{\partial t}\right) + \text{div} \rho T \nabla\right) F_s + \text{div} \rho \nabla \times F_\omega.$$

One may then introduce $B_\omega$ and $B_s$ by

$$\rho \Delta B_\omega = F_\omega \quad \text{und} \quad \rho \Delta B_s = F_s$$

and one obtains

$$B = \left(\frac{\partial}{\partial t} \left(\rho_s \frac{\partial}{\partial t} + \text{div} \rho_s v \frac{\partial}{\partial t}\right) + \text{div} \rho T \nabla\right) B_s + \text{div} \rho \nabla \times B_\omega.$$

(27)

This shows – as one might have expected – that passively convected entropy and vorticity inhomogeneities do not radiate sound. In the general case these
quantities will not be passively convected and one needs for its determination extra equations. For the entropy one may use the equation of entropy conservation

\[
\frac{D s}{D t} = 0
\]

and for the vorticity the Beltrami vortex theorem

\[
\frac{D \omega}{D t}\rho = \frac{\omega}{\rho} \cdot \nabla \mathbf{v}.
\]

Here we consider especially the two-dimensional case. Then the right hand side of the Beltrami vortex theorem, which is related to the stretching of vortex lines, vanishes.

In addition to the differential equations one needs boundary conditions. If one is interested in cases where the vorticity vanishes at solid walls in the flow region, one may use the relation (9) between total enthalpy \(B\) and potential \(\Phi\) and one finds that the normal component of the velocity vanishes at a fixed surface if the normal derivative of \(B\) vanishes there. With eq. (23) one notices that the normal component of the energy flux (23) vanishes at rigid walls if the normal component of the velocity \(\mathbf{v}\) vanishes there. One would expect this of course.

In addition one needs conditions of no-reflexion at the boundary of the computation region. We will here assume the simplest quasi-onedimensional condition and require there

\[
\frac{\partial B}{\partial t} = -(\mathbf{v} + \mathbf{n}a) \cdot \nabla B,
\]

where \(\mathbf{n}\) denotes the outer normal of the computation region.

As a numerical example, we consider a localized vortex of radius 1 and of vanishing total vorticity which is convected in a flow around a circular cylinder of radius \(1/2\) and situated at \(x = 0\) and \(y = 0\). The initial azimuthal velocity \(w\) around the center of the vortex, which is situated initially at \(x_0 = -3, y_0 = 0.5\), is assumed to be

\[
w = (1 - 4r^2)(1 - r^2)^2,
\]

\(r\) denotes the distance from the vortex center. For the velocity, we assume an incompressible potential flow of velocity 1 in \(x\)-direction at \(x = -\infty\). The density is chosen as 1, the speed of sound as 2. Initial values for \(B\) are obtained from eq.(27) and are given by

\[
B = 0.5(y - y_0)(1 - r^2)^3 \quad \text{for} \quad r < 1.
\]
This problem is treated with the general purpose PDE-solver PDEase/2 which runs on a PC. A grayplot of the $B$-field at two different times is shown in figure 1.

The left half shows a very early stage with the vortex to the left of the cylinder. As the vortex is inserted in an inhomogeneous velocity field, sound radiation begins immediately. The right half shows a later stage, where the vortex has approached the cylinder. As the flow Mach number is not small, one notices significant deviations from a dipole character. Hydrodynamic instabilities are not observed, but numerical small-scale errors are noticeable. A reliable numerical solution of eq. (6) requires obviously more efforts.

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