Temporal switching to extend the bandwidth of thin absorbers: supplement

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I. First-principle derivation of conventional coupled-mode theory (CMT)

In this section, we present a first-principle derivation of the conventional CMT using the input-output formulation Eq. (7) of the main text. We first note that the QNMs appear in pairs [1] and their complex wave characteristics obey \( \omega_n = -\omega_n^* \), \( k_n = -k_n^* \), and \( |c_n|^2 = |c_n^*|^2 \) with the label \( n = \pm 1, \pm 2, \pm 3, \ldots \). Therefore, to ensure the real-valued nature of the outgoing radiation \( |\Psi_{out}(z_0, t)| \) and the internal fields \( |\Psi(z, t)| \), these QNM pairs \( n = \pm 1, \pm 2, \pm 3, \ldots \) should be employed as units for the QNM expansion and thus the summation in Eq. (7) of the main text. A specific label scheme for the QNMs is fixed when setting \( \tilde{\omega}_n = \tilde{\gamma}_n c_0 / (z_0 \sqrt{\varepsilon_r}) \) and assuming \( 0 \leq \text{Re}(\tilde{\gamma}_1) < \text{Re}(\tilde{\gamma}_2) < \ldots \). As mentioned in the main text, we consider one QNM-pair approximation associated with the complex-frequency pair \( \tilde{\omega}_{\pm 1} \), such that the internal fields \( |\Psi(z, t)| \approx \sum_{n=\pm 1} \psi_n(t)|\Phi_n(z)| \). This approximation is valid for our thin absorber where \( \varepsilon_r \gg 1 \) and when the operating frequency \( \omega_0 \sim \text{Re}(\tilde{\omega}_{-1}) > 0 \). As we know, the conventional CMT deals with the energy-normalized positive-frequency component of the mode amplitude [2]. Here, it can be
identified to be simply $\psi_{-1}(t)$ since, in the weak coupling limit $\varepsilon_r \to \infty$ and the lossless scenario $\sigma = 0$, its complex frequency $\tilde{\omega}_{-1}$ approaches to the (positive) resonant frequency and the total stored energy inside the layer, i.e., $E_{\text{tot}}(t) = \int_{z_0}^{0} \frac{1}{2} \varepsilon_0 \langle \Psi | (\varepsilon_r 0 0) | \Psi \rangle dz$, is shared by the QNM pair $n = \pm 1$ as $|\psi_{-1}(t)|^2$. As for excitations, we follow the CMT convention to adopt the power-normalized complex amplitude $a_{+}^{in}(z_0, t)$ of positive frequencies, i.e., $a_{+}^{in}(z_0, t) = \sqrt{2/\eta_0} \frac{1}{2\pi} \int_{0}^{\infty} a(\omega) e^{i\omega(t-z_0/c_0)} d\omega$. Under the slow-envelope approximation, i.e., $a_{+}^{in}(z_0, t) = A_{+}^{in}(t) e^{i\omega_0 t}$ with $A_{+}^{in}(t + \Delta t) \approx A_{+}^{in}(t)$ when $\Delta t \sim O(2\pi/\omega_0)$, the time-averaged incoming power with respect to $2\pi/\omega_0$ is $\langle P^{in}(z_0, t) \rangle = |a_{+}^{in}(z_0, t)|^2$ as it should. For the outgoing radiation, we have the power-normalized complex amplitude $b_{+}^{out}(z_0, t) = \sqrt{2/\eta_0} \frac{1}{2\pi} \int_{0}^{\infty} b(\omega) e^{j\omega(t+z_0/c_0)} d\omega$, satisfying similarly $\langle P^{out}(z_0, t) \rangle = |b_{+}^{out}(z_0, t)|^2$. Finally, decoupling the positive frequency with the negative ones in Eq. (7) of the main text in the spirit of the rotating-wave approximation and employing the slow-envelope approximation for the complex excitation $a_{+}^{in}(z_0, t)$, we can reach the conventional CMT under one QNM-pair approximation, i.e., Eq. (8) of the main text.

Next, we consider lossless screens and analytically examine the derived conventional CMT. We start our discussion by rewriting Eq. (8) [see main text] in the conventional form [3]

$$\frac{d\psi_{-1}(t)}{dt} = (j\Omega - \Gamma)\psi_{-1}(t) + K^T a_{+}^{in}(z_0, t), \quad (S1)$$

$$b_{+}^{out}(z_0, t) = C a_{+}^{in}(z_0, t) + D \psi_{-1}(t),$$

where the resonant (angular) frequency of the single mode $\Omega = \text{Re}(\tilde{\omega}_{-1})$, its decay rate $\Gamma = \text{Im}(\tilde{\omega}_{-1})$, the coupling between the mode and the port $K = D = -c_{-1} \sin(k_{-1}z_0) \sqrt{\frac{2}{\eta_0}}$ and the direct path between the incoming and outgoing wave in the port $C \approx e^{2|\omega_0 z_0/c_0 +}$
\[ \int_{-2z_0/c_0}^{0} \tilde{K}_{-1}(\tau) e^{-j\omega_0 \tau} c_0/z_0 d\tau. \]

In the scenario when \( \sigma = 0 \) as focused here, we have the following relationships \( \tilde{k}_n z_0 = \tilde{y}_n \), \( \tan \tilde{y}_n = -j\varepsilon_{r} \) and \( c_n^2 \varepsilon_0 z_0 = 1/\varepsilon_{r} \). In addition, the (normalized) complex frequency \( \tilde{y}_n \equiv \tilde{\omega}_n z_0 \sqrt{\varepsilon_{r}}/c_0 \) of QNMs can be explicitly given as \( -\tilde{y}_n^* = \tilde{y}_n = (2n-1)\pi / 2 - j \text{Re}(\text{ArcTanh}(\sqrt{\varepsilon_{r}})), n = 1,2,3,\cdots \). Furthermore, the propagator \( \tilde{K}_n(\tau) \) due to the nth QNM for the direct path (see the main text) can be simplified greatly to be
\[ \tilde{K}_n(\tau) = -\cos \left[ \tilde{y}_n \left( 2 + \frac{c_0}{z_0} \tau \right) \right]. \]

Therefore, we can easily calculate the (approximated) direct path \( C \) in Eq. (S1), which turns out to be
\[ C = 1 + j \left( \frac{8}{\pi} - \pi \right) \frac{1}{\sqrt{\varepsilon_{r}}} + O \left( \frac{1}{\varepsilon_{r}} \right) \]
in the weak-coupling limit when \( \varepsilon_{r} \rightarrow \infty \). We point out that in the process of decoupling the positive and negative frequencies for the derivation of the conventional CMT in Eq. (S1), we exclude the contribution of QNM \( n = 1 \) for the direct path \( C \). This contribution is
\[ \int_{-2z_0/c_0}^{0} \tilde{K}_1(\tau) e^{-j\omega_0 \tau} c_0/z_0 d\tau, \omega_0 = \tilde{\omega}_1 \] and supposed to be of higher order since the operation frequency \( \omega_0 \) is far away from the complex frequency \( \tilde{\omega}_1 \) of the QNM \( n = 1 \). Indeed, in the scenario here, it approaches to \( (2 - 16/\pi^2)/\varepsilon_{r} \) when \( \varepsilon_{r} \rightarrow \infty \).

In the weak-coupling regime where \( \varepsilon_{r} \rightarrow \infty \) and when \( \sigma = 0 \) for the lossless screen, we can also expand the other system parameters in Eq. (S1). Specifically, we have
\[ K = D = \sqrt{\frac{2c_0}{z_0} \left[ \frac{1}{\sqrt{\varepsilon_{r}}} + \frac{1}{2c_0\varepsilon_{r}} + O \left( \frac{1}{c_0^2\varepsilon_{r}} \right) \right]} \]
and
\[ \Gamma = -\frac{c_0}{z_0\varepsilon_{r}} \left[ 1 + \frac{1}{3\varepsilon_{r}} + O \left( \frac{1}{c_0\varepsilon_{r}} \right) \right]. \]
We can easily see that in the leading order the system parameters as derived satisfy all the requirements imposed by the time-reversal symmetry and energy conservation principles, which are \( D^+D = 2\Gamma, K = D \) and \( CD^* = -D \) [3]. Finally, we remark that the performance of the conventional CMT as Eq. (S1) will not be improved when we incorporate higher-order terms in system parameters, since in its derivation the crucial (rotating-wave) approximation has been already made by decoupling the positive and
negative frequencies. Typically, the approximated equation under the constraints of general 
principles will behave relatively better.

II. Formation of a QNM EP

We analyze the formation of the QNM EP $\tilde{y}(\varepsilon_r^{EP}) = -jp, p > 0$ as seen in Fig. 1(c) of the main 
text. This EP arising as the merging of the $n = \pm 1$ QNMs is of the second order. Therefore, around 
the EP, we expand the complex frequency $\tilde{y}_{\pm 1}(\varepsilon_r)$ of the QNMs as $\tilde{y}_{\pm 1}(\varepsilon_r) = -jp + x_1\sqrt{\Delta \varepsilon} + x_2\Delta \varepsilon + \cdots, \Delta \varepsilon = \varepsilon_r - \varepsilon_r^{EP}$, and intend to determine the parameters $p$ and $x_1$ explicitly. To this 
end, we substitute this ansatz for $\tilde{y}_{\pm 1}(\varepsilon_r)$ into the transcendental equation $\tan(k_n z_0) = 
\sqrt{\frac{\varepsilon_r}{c_0}} \tilde{\omega}_n \sqrt{1 - j\sigma/(\varepsilon_0 \varepsilon_r \tilde{\omega}_n)}$ for the complex frequencies $\tilde{\omega}_n$, or 
equivalently

$$\tan \left( \tilde{y}_n \sqrt{\frac{1 + j\frac{\tilde{\omega}}{\tilde{y}_n \varepsilon_r^{EP} + \Delta \varepsilon}}{1 + j\frac{\tilde{\omega}}{\tilde{y}_n \varepsilon_r^{EP} + \Delta \varepsilon}}} \right) = -j \sqrt{\frac{1 + j\frac{\tilde{\omega}}{\tilde{y}_n \varepsilon_r^{EP} + \Delta \varepsilon}}{1 + j\frac{\tilde{\omega}}{\tilde{y}_n \varepsilon_r^{EP} + \Delta \varepsilon}}} \sqrt{\frac{\varepsilon_r^{EP} + \Delta \varepsilon}{\varepsilon_r^{EP} + \Delta \varepsilon}}$$

(S2)

where $\tilde{y}_n = \tilde{\omega}_n z_0 \sqrt{\varepsilon_r/c_0}, z_0 = -d$ and $\tilde{\sigma} = \sigma d \sqrt{\mu_0/\varepsilon_0}$. Then, expanding Eq. (S2) order by order 
with respect to $\Delta \varepsilon$ and equating the resulting coefficient of each power of $\sqrt{\Delta \varepsilon}$ to zero, we obtain 
up to $O(\sqrt{\Delta \varepsilon})$

$$\tanh \left( p \sqrt{1 - \frac{\tilde{\omega}}{p \sqrt{\varepsilon_r^{EP}}}} \right) = \sqrt{\varepsilon_r^{EP}} \sqrt{1 - \frac{\tilde{\omega}}{p \sqrt{\varepsilon_r^{EP}}}}$$

(S3)

and
The emergence of the EP requires that \( x_1 \neq 0 \), and then combining Eqs. (S3) and (S4) allows us to evaluate the EP \( \hat{y}(\varepsilon_{r}^{EP}) = -jp \) with

\[
p = \frac{(3\varepsilon_{r}^{EP} - 1)\hat{\sigma} \pm \sqrt{\hat{\sigma}(\varepsilon_{r}^{EP})^2(\hat{\sigma} - 8) + 2\varepsilon_{r}^{EP}(4 + \hat{\sigma})}}{4(\varepsilon_{r}^{EP} - 1)\sqrt{\varepsilon_{r}^{EP}}}. \tag{S5}
\]

In turn, with a given \( \hat{\sigma} \) value for the normalized conductivity, substituting the expression for \( p \) in Eq. (S5) into Eq. (S3) allows us to determine the parameter \( \varepsilon_{r}^{EP} \). To obtain the expansion coefficient \( x_1 \) of \( \hat{y}_{\pm 1}(\varepsilon_{r}) \) around the EP with respect to \( \sqrt{\Delta \varepsilon} \), we need to resort to the next order \( O(\Delta \varepsilon) \), which after considering Eqs. (S3) and (S4) can be simplified to be

\[
j \left\{ 4p^3(p\sqrt{\varepsilon_{r}^{EP}} - \hat{\sigma}) + x_1^2\sqrt{\varepsilon_{r}^{EP}}\hat{\sigma}\left[\left(4p\sqrt{\varepsilon_{r}^{EP}} - \hat{\sigma}\right)\left(1 + \hat{\sigma}\right) - 4p^2\varepsilon_{r}^{EP}\right]\right\} = 0. \tag{S6}
\]

Interestingly, the expansion coefficient \( x_2 \) is absence in Eq. (S6) due to the relationships in Eqs. (S3) and (S4), which enables an analytical result for the expansion coefficient \( x_1 \):
Fig. S1. Parameters $\varepsilon_r^{EP}$, $p$ and $x_1$ for the EP characteristics $\hat{y}_{\pm 1}(\varepsilon_r) = -jp \pm x_1 \sqrt{\varepsilon_r - \varepsilon_r^{EP}} + \cdots$ as a function of $\hat{\sigma}$. Their values $\varepsilon_r^{EP} \approx 2.1$, $p \approx 1.84$ and $x_1 \approx 1.53$ when $\hat{\sigma} = \hat{\sigma}^{opt} = 1.998$ are indicated by the red circle, green square and blue diamond symbols, respectively.

By combining Eqs. (S3), (S5) and (S7), we can determine, for an arbitrary $\hat{\sigma} > 0$, the values of the parameters $\varepsilon_r^{EP}$, $p$ and $x_1$, see Fig. S1, entering the expression $\hat{y}_{\pm 1}(\varepsilon_r) = -jp \pm x_1 \sqrt{\varepsilon_r - \varepsilon_r^{EP}} + \cdots$. We point out that, for each specific value of $\hat{\sigma}$, there is only one solution for $p$ in Eq. (S5) by choosing either plus or minus sign in its numerator such that the corresponding $x_1$ from Eq. (S7) is nonzero in order for the existence of the EP. The other solution for $p$ in Eq. (S5), which leads to $x_1 = 0$, is associated with the trivial QNM mentioned in [1]. In the example in the main text when $\hat{\sigma} = \hat{\sigma}^{opt} = 1.998$, we get $\varepsilon_r^{EP} \approx 2.1$, $p \approx 1.84$ and $x_1 \approx 1.53$ (see the red circle, green square and blue diamond symbols respectively in Fig. S1).
References

[1] Rigorously speaking, there exists one additional trivial QNM for our lossy absorber associated with complex frequency $\tilde{\omega}_0 = j\sigma/(\varepsilon_0\varepsilon_r)$ and wavenumber $\tilde{k}_0 = 0$, which, however, should not be included in the QNM expansion since in this case the mode profile $|\tilde{\Psi}_0(z)\rangle$ is a zero vector and non-normalizable, see Eq. (4) of the main text.

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