SINGULAR INTEGRALS
UNSUITABLE FOR THE CURVATURE METHOD
WHOSE $L^2$-BOUNDEDNESS STILL IMPLIES RECTIFIABILITY

By

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Abstract. The well-known curvature method initiated in works of Melnikov and Verdera is now commonly used to relate the $L^2(\mu)$-boundedness of certain singular integral operators to the geometric properties of the support of measure $\mu$, e.g., rectifiability. It can be applied, however, only if Menger curvature-like permutations directly associated with the kernel of the operator are non-negative. We give an example of an operator in the plane whose corresponding permutations change sign but the $L^2(\mu)$-boundedness of the operator still implies that the support of $\mu$ is rectifiable. To the best of our knowledge, it is the first example of this type. We also obtain several related results with Ahlfors–David regularity conditions.

1 Introduction

We start with necessary notation and background facts. Note that we work only in the complex plane and therefore usually skip dimension markers in definitions.

Let $E \subset \mathbb{C}$ be a Borel set and $B(z, r)$ be an open disc with center $z \in \mathbb{C}$ and radius $r > 0$. We denote by $\mathcal{H}^1(E)$ the 1-dimensional Hausdorff measure of $E$. A set $E$ is called rectifiable if it is contained in a countable union of Lipschitz graphs, up to a set of $\mathcal{H}^1$-measure zero. A set $E$ with $\mathcal{H}^1(E) < \infty$ is called purely unrectifiable if it intersects any Lipschitz graph in a set of $\mathcal{H}^1$-measure zero.

We consider Calderón–Zygmund kernels $K : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ with the following properties: there exist constants $C > 0$ and $\eta \in (0, 1]$ such that $|K(z)| \leq C|z|^{-1}$ for all $z \in \mathbb{C} \setminus \{0\}$, and moreover

$$|K(z) - K(z + \zeta)| \leq C \frac{|\zeta|^\eta}{|z|^{1+\eta}} \quad \text{if} \quad |\zeta| \leq \frac{1}{2}|z|, \ z, \zeta \in \mathbb{C}. $$

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Given a fixed positive Radon measure $\mu$ on $\mathbb{C}$, a kernel $K$ and an $f \in L^1(\mu)$, we define the truncated Calderón–Zygmund operator (CZO) as

$$
T_{K,\varepsilon}f(z) := \int_{E \setminus B(z, \varepsilon)} f(\zeta)K(z - \zeta)d\mu(\zeta), \quad E = \text{spt} \mu, \varepsilon > 0.
$$

We do not define the CZO $T_K$ explicitly because several delicate problems such as the existence of the principal value might arise. On the contrary, the integral in (1) always converges absolutely and thus the principal value problem can be avoided. Nevertheless, we say that $T_K$ is $L^2(\mu)$-bounded if the operators $T_{K,\varepsilon}$ are $L^2(\mu)$-bounded uniformly on $\varepsilon$.

How to relate the $L^2(\mu)$-boundedness of a certain CZO to the geometric properties of the support of $\mu$ is an old problem in harmonic analysis. It stems from Calderón’s paper [1] where it is proved that the Cauchy transform, i.e., the CZO with $K(z) = 1/z$, is $L^2(\mathcal{H}^1|E)$-bounded if $E$ is a Lipschitz graphs with small slope. Later on, Coifman, McIntosh and Meyer [5] removed the small Lipschitz constant assumption. In [6] David fully characterized rectifiable curves $\Gamma$, for which the Cauchy transform is $L^2(\mathcal{H}^1|\Gamma)$-bounded. These results led to further development of tools for understanding the above-mentioned problem.

A new quantitative characterization of rectifiability in terms of the so-called $\beta$-numbers introduced by Jones [12] and the concept of uniform rectifiability proposed by David and Semmes [9, 8] are among these tools. Several related definitions for the plane are in order. (We refer the reader to [9, 8] for definitions and results in the multidimensional case.) A Radon measure $\mu$ on $\mathbb{C}$ is called (1-dimensional) Ahlfors–David regular (or AD-regular, for short) if it satisfies the inequalities

$$
C^{-1}r \leq \mu(B(z, r)) \leq Cr, \quad \text{where} \ z \in \text{spt} \mu, \ 0 < r < \text{diam}(\text{spt} \mu)
$$

and $C > 1$ is some fixed constant. A measure $\mu$ is called uniformly rectifiable if it is AD-regular and spt $\mu$ is contained in an AD-regular curve.

The well-known David–Semmes problem is stated in the plane as follows: is the $L^2(\mu)$-boundedness of the Cauchy transform sufficient for the uniform rectifiability of the AD-regular measure $\mu$? This problem was settled by Mattila, Melnikov and Verdera in [14]:

**Theorem A** ([14]). Let $\mu$ be an AD-regular measure on $\mathbb{C}$. The measure $\mu$ is uniformly rectifiable if and only if the Cauchy transform is $L^2(\mu)$-bounded.

Note that an analogous problem in higher dimensions in the codimension 1 was more recently solved by Nazarov, Tolsa and Volberg in [17].
The proof of Theorem A relied on the so-called curvature method that was new at that time but soon became very influential in solving many long-standing problems (see [19] and especially historical remarks there). Let us describe the heart of the method. Given pairwise distinct points $z_1, z_2, z_3 \in \mathbb{C}$, their Menger curvature is

$$c(z_1, z_2, z_3) = \frac{1}{R(z_1, z_2, z_3)},$$

where $R(z_1, z_2, z_3)$ is the radius of the circle passing through $z_1, z_2$ and $z_3$ (with $R(z_1, z_2, z_3) = \infty$ and $c(z_1, z_2, z_3) = 0$ if the points are collinear). This geometric characteristic is closely related to the Cauchy kernel as shown by Melnikov [15]:

$$c(z_1, z_2, z_3)^2 = \sum_{s \in S_3} \frac{1}{(z_{s_2} - z_{s_1})(z_{s_3} - z_{s_1})},$$

where $S_3$ is the group of permutations of three elements. Moreover, Melnikov also introduced a notation of the curvature of a Borel measure $\mu$:

$$c^2(\mu) = \iint\int c(z_1, z_2, z_3)^2 \, d\mu(z_1) \, d\mu(z_2) \, d\mu(z_3).$$

One can consider $c^2_\varepsilon(\mu)$, a truncated version of $c^2(\mu)$, which is defined via the triple integral in (3) but over the set

$$\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_k - z_j| \geq \varepsilon > 0, \ 1 \leq k, j \leq 3, \ j \neq k\}.$$

If $\mu$ also has linear growth, i.e., $\mu(B(z, r)) \leq Cr$ for all $z \in \text{spt} \, \mu$ and some $C > 0$, then the relation between the curvature and the $L^2(\mu)$-norm of the Cauchy transform (of measure) is specified by the following identity due to Melnikov and Verdera [16]:

$$\iint \left| \int_{\mathbb{C}\setminus B(z, \varepsilon)} \frac{d\mu(\zeta)}{\zeta - z} \right|^2 d\mu(z) = \frac{1}{6} c^2(\mu) + R_\varepsilon(\mu), \quad |R_\varepsilon(\mu)| \leq C \mu(\mathbb{C}).$$

The formulas (2) and (4), generating the curvature method, are remarkable in the sense that they relate an analytic notion (the Cauchy transform) with a metric-geometric one (the curvature). It is, however, very important here that the permutations in (2) are always non-negative.

Later on, Theorem A was pushed even further by David and Léger [13, 7]. They essentially used the non-negativity of (2) to prove the following assertion.

**Theorem B** ([13]). *Let $E \subset \mathbb{C}$ be a Borel set such that $0 < \mathcal{H}^1(E) < \infty$. If the Cauchy transform is $L^2(\mathcal{H}^1|E)$-bounded, then $E$ is rectifiable.*
Note that the $L^2(\mathcal{H}^1 | E)$-boundedness of the Cauchy transform and the identity (4) imply that $c^2(\mathcal{H}^1 | E) < \infty$. Consequently, to prove Theorem B it is enough to show that $c^2(\mathcal{H}^1 | E) < \infty$ and this was actually done in [13].

Until recently, very few things were known in this direction beyond the CZO associated with the Cauchy kernel and its coordinate parts $\text{Re} \, z/|z|^2$ and $\text{Im} \, z/|z|^2$; see [14, 2]. But recently Chousionis, Mateu, Prat and Tolsa [2] (see also [3]) extended Theorems A and B to the CZOs associated with the kernels (5)

$$\kappa_n(z) := \frac{(\text{Re} \, z)^{2n-1}}{|z|^{2n}}, \quad n \in \mathbb{N},$$

thus providing for $n \geq 2$ the first non-trivial example of CZOs with the above-mentioned properties but not directly related to the Cauchy transform (for $n = 1$ one gets $\text{Re} \, z/|z|^2 = \text{Re} (1/z)$). Note that the results in [2] require a slightly different notation than in Theorems A and B. Namely, given a real-valued Calderón–Zygmund kernel $K$, one has to consider the following permutations that substitute the curvature (2):

$$p_K(z_1, z_2, z_3) := K(z_1 - z_2)K(z_1 - z_3) + K(z_2 - z_1)K(z_2 - z_3) + K(z_3 - z_1)K(z_3 - z_2).$$

Analogously to (3), for any Borel measure $\mu$ set

$$p_K(\mu) = \int\int\int p_K(z_1, z_2, z_3) \, d\mu(z_1) \, d\mu(z_2) \, d\mu(z_3).$$

One can also define $p_{\varepsilon, K}(\mu)$, the truncated version of $p_K(\mu)$, in an obvious way.

In the case of kernels (5) as in [2] one puts $K(z) = \kappa_n(z)$ in (6) and (7). It is shown in [2] that the permutations $p_{\kappa_n}(z_1, z_2, z_3)$ are non-negative for all triples $(z_1, z_3, z_3) \in \mathbb{C}^3$ and this is appreciably used in a curvature-like method in [2].

In [4], kernels of the form

$$K_t(z) := \kappa_N(z) + t \cdot \kappa_n(z), \quad n \leq N, \ n, N \in \mathbb{N}, \ t \in \mathbb{R},$$

i.e., linear combinations of the kernels (5) of different order were introduced. Clearly, one obtains a kernel of the form (5) from (8) when $n = N$ (and $t \neq -1$) or $t = 0$. It turns out that this slight modification of the kernel leads to a diverse behaviour of the corresponding CZO depending on the parameter $t$. For example, it is shown in [4] that if $t$ belongs to the set

$$\Omega(n, N) := \begin{cases} 
\{0\} \cup \mathbb{R} \setminus (-\frac{1}{2} (3 + \sqrt{9 - 4 \frac{N}{n}}); 2 - \frac{N}{n}) & \text{if } n < N \leq 2n, \\
\{0\} \cup \mathbb{R} \setminus (-\frac{1}{2} (\sigma_{n,M} + \sqrt{\sigma_{n,M}^2 - 4 \frac{N}{n}}); \sigma_{n,M} - 3) & \text{if } N \geq 2n,
\end{cases}$$

where \( \sigma_{n,M} := 3 + \left(\frac{N}{n} - 2\right)\sqrt{N - 2n} \), then

\[
(10) \quad p_{K_t}(z_1, z_2, z_3) \geq 0 \quad \text{for all } (z_1, z_2, z_3) \in \mathbb{C}^3.
\]

Moreover, taking into account this property and using a curvature-like method, the following Theorem B type result is proved in [4].

**Theorem C** ([4]). Let \( E \subset \mathbb{C} \) be a Borel set such that \( 0 < H^1(E) < \infty \). If the CZOTK with \( t \in \Omega(n, N) \) is \( L^2(\mathcal{H}^1|E|) \)-bounded, then \( E \) is rectifiable.

From the other side, it is shown in [4] that there exist triples \((z_1, z_2, z_3)\) such that \( p_{K_t}(z_1, z_2, z_3) \) change sign if \( t \) belongs to the interval

\[
(11) \quad \omega(n, N) := (-N/n; 0).
\]

Obviously, \( \omega(n, N) \subset \mathbb{R} \setminus \Omega(n, N) \). Note also that

\[
\omega(n, 2n) = (-2; 0) = \mathbb{R} \setminus \Omega(n, 2n).
\]

For this reason, a curvature-like method cannot be applied directly for \( t \in \omega(n, N) \). Moreover, it follows from Huovinen’s result in [10] that Theorem C fails for \( t = -1 \in \omega(n, N) \) in the sense that there exists an AD-regular purely unrectifiable set \( E \) with \( 0 < \mathcal{H}^1(E) < \infty \) such that the operator \( T_{K_t} \) with \( t = -1 \) is \( L^2(\mathcal{H}^1|E|) \) -bounded. In the case \((n, N) = (1, 2)\), i.e., for the kernels

\[
(12) \quad k_t(z) := \frac{(\text{Re} \, z)^3}{|z|^4} + t \cdot \frac{\text{Re} \, z}{|z|^2}, \quad t \in \mathbb{R},
\]

even more is known due to Jaye and Nazarov [11]. Namely, for \( t = -3/4 \in \omega(1, 2) \) there also exists a purely unrectifiable (but not AD-regular) set \( E \) such that \( T_{K_t} \) is \( L^2(\mathcal{H}^1|E|) \)-bounded. For the details see also [4, Remark 2].

Thus we come to the question of what happens when \( t \in \omega(n, N) \), i.e., the permutations \( p_{K_t}(z_1, z_2, z_3) \) change sign, and curvature-like methods as in [14, 13, 2, 4] do not work. In this paper a partial answer is given in the case of kernels (12). Namely, we show that for \( t \in (-2; -\sqrt{2}) \subset \omega(1, 2) \) the analogues of Theorems A and B are still valid (a plausible conjecture for the kernels (8) with \( t \in \omega(n, N) \) is also stated). To the best of our knowledge, this is the first example of kernels with this property in the plane. We also establish an analogue of Theorem A for the kernels (8) with \( t \in \Omega(n, N) \). The corresponding results are given in the next section.
2 Main results

The following two theorems are analogues of Theorems A and B for the kernels (12) with \( t \in (-2; -\sqrt{2}) \), whose corresponding permutations change sign and a curvature-like method cannot be applied directly. We will prove them in Section 3 by exploiting sharp estimates for permutations related to the kernels (5) but not to the ones in (12). Recall that \( \omega(1, 2) = (-2; 0) \), see (11).

**Theorem 1.** Let \( \mu \) be an AD-regular measure on \( \mathbb{C} \) and \( T_k \) the CZO associated with the kernel (12), where \( t \in (-2; -\sqrt{2}) \subset \omega(1, 2) \). The measure \( \mu \) is uniformly rectifiable if and only if \( T_k \) is \( L^2(\mu) \)-bounded.

Note that this theorem fails if \( t = -1 \in \omega(1, 2) \). It follows from the aforementioned Huovinen’s result [10].

**Theorem 2.** Let \( E \subset \mathbb{C} \) be a Borel set such that \( 0 < \mathcal{H}^1(E) < \infty \), and \( T_k \), the CZO associated with the kernel (12), where \( t \in (-2; -\sqrt{2}) \subset \omega(1, 2) \). If \( T_k \) is \( L^2(\mathcal{H}^1|E) \)-bounded, then \( E \) is rectifiable.

This theorem supplements Theorem B type results about CZO associated with the kernels \( k_t \) (see Figure 1). By [4], if \( t \notin (-2; 0) \), then the permutations \( p_k \) are non-negative and the \( L^2(\mathcal{H}^1|E) \)-boundedness of \( T_k \) implies that \( E \) is rectifiable by a curvature-like method. According to [4], the permutations \( p_k \) for \( t \in (-2; 0) \) change sign, and by [10, 11], if \( t = -1 \) or \( t = -3/4 \), then the operator \( T_k \) is \( L^2(\mathcal{H}^1|E) \)-bounded but \( E \) is not rectifiable. The interval \((-2; -\sqrt{2})\) corresponds to Theorem 2 of this paper.
Remark 1. As we will see at the end of Section 3, it is plausible that analogues of Theorems 1 and 2 are valid for the kernels (8) with $|t| > \sqrt{N/n}$. Note that in particular $(-N/n; -\sqrt{N/n}) \subset o(n, N)$, i.e., for $t$ from this interval the corresponding permutations change sign.

We now formulate a Theorem A type result for the kernels (8) in the case where $t \in \Omega(n, N)$, i.e., the corresponding permutations are non-negative (see (9) and (10)).

**Theorem 3.** Let $\mu$ be an AD-regular measure on $\mathbb{C}$ and $T_K$, the CZO associated with the kernel (8), where $t \in \Omega(n, N)$. The measure $\mu$ is uniformly rectifiable if and only if $T_K$ is $L^2(\mu)$-bounded.

Since the permutations are non-negative here, we can use a curvature-like method. The proof that will be given in Section 4 is more or less analogous to the one used for the kernels (5) in [2, Section 8].

Let us say a few words about the notation in this paper. As usual, $C$ stands for a positive constant which may change its value at different places. Sometimes $C$ may depend on some parameters and then we indicate it by writing, for instance, $C(\varepsilon)$ or $C_\varepsilon$, where $\varepsilon$ is a parameter. On the other hand, constants with subscripts, such as $\eta_1$ or $\theta_0$, retain their values throughout the paper. The notation $A \lesssim B$ for positive $A$ and $B$ means that there is a positive constant $C$ such that $A \leq CB$. If this $C$ depends on a parameter, say $\varepsilon$, we write $A \lesssim \varepsilon B$. Also, $A \approx B$ is equivalent to $A \lesssim B \lesssim A$.

### 3 The proof of Theorems 1 and 2

Recall that
\[
\kappa_1(z) = \frac{\text{Re} z}{|z|^2}, \quad \kappa_2(z) = \frac{(\text{Re} z)^3}{|z|^4} \quad \text{and} \quad k_t(z) = \kappa_2(z) + t \cdot \kappa_1(z).
\]

The following result from [2] will be necessary below.

**Lemma 1** (Proof of Proposition 2.1 in [2]). Given $u = (x, y)$ and $v = (a, b)$ in $\mathbb{C}$,
\[
p_{\kappa_m}(0, u, v) = \sum_{k=1}^{m} \binom{m}{k} \frac{(ax - a)^{2(m-k)}}{|u|^{2m}|v|^{2m}|u - v|^{2m}} h_k(u, v),
\]
where $h_k(u, v) := (ax)^{2k-1}(y-b)^{2k} + (x(x-a))^{2k-1}b^{2k} + (a(a-x))^{2k-1}y^{2k} \geq 0$.

To prove Theorems 1 and 2 we first obtain sharp pointwise estimates for the permutations related to the kernels (5).
Lemma 2. It holds that 

\begin{equation}
 p_{k_2}(z_1, z_2, z_3) \leq 2p_{k_1}(z_1, z_2, z_3) \quad \text{for all } (z_1, z_2, z_3) \in \mathbb{C}^3. 
\end{equation}

**Proof.** It is enough to prove (14) for $(z_1, z_2, z_3) = (0, u, v)$ as the permutations of the form (6) are invariant under translations. Given $u = (x, y)$ and $v = (a, b)$, by (13) we get 

\[
2p_{k_1}(0, u, v) - p_{k_2}(0, u, v) = \frac{2h_1(u, v)}{|u|^2|v|^2|u - v|^2} - \frac{2x^2a^2(x - a)^2h_1(u, v) + h_2(u, v)}{|u|^4|v|^4|u - v|^4} 
\]

Now we obtain a lower estimate of the expression in the square brackets before $h_1(u, v)$. Expanding $|u|^2|v|^2|u - v|^2$ gives 

\[
(x^2 + y^2)(a^2 + b^2)((x - a)^2 + (y - b)^2) - x^2a^2(x - a)^2 
\]

\[
= x^2a^2(y - b)^2 + (x^2b^2 + a^2y^2 + b^2y^2)((x - a)^2 + (y - b)^2) 
\]

\[
\geq x^2a^2(y - b)^2 + (x^2b^2 + a^2y^2)(x - a)^2. 
\]

Thus, 

\[
2p_{k_1}(0, u, v) - p_{k_2}(0, u, v) \geq \frac{G(x, y, a, b)}{|u|^4|v|^4|u - v|^4}, 
\]

where 

\[
G(x, y, a, b) := 2(x^2a^2(y - b)^2 + (x^2b^2 + a^2y^2)(x - a)^2)h_1(u, v) - h_2(u, v). 
\]

Notice that by Lemma 1, 

\[
h_1(u, v) = ax(y - b)^2 + x(x - a)b^2 + a(a - x)y^2 
\]

\[
h_2(u, v) = (ax)^3(y - b)^4 + (x(x - a))b^4 + (a(a - x))^3y^4. 
\]

Consequently, to prove the required inequality it is enough to show that 

\[
G(x, y, a, b) \geq 0. 
\]

We separate the discussion into three cases.

(1) Let $a = 0$. Then 

\[
G(x, y, 0, b) = 2x^4b^2 \cdot x^2b^2 - x^6b^4 = x^6b^4 \geq 0. 
\]
(2) Let \(b = 0\). Then
\[
G(x, y, a, 0) = 2(a^2 x^2 y^2 + a^2 y^2 (x - a)^2) (axy^2 + a(a - x)y^2) - (a^3 x^3 y^4 + a^3 (a - x)^3 y^4) \\
= 2a^3 y^4 (x^2 + (x - a)^2) (x + (a - x)) - a^3 y^4 (x^3 - (x - a)^3) \\
= a^4 y^4 (2(x^2 + (x - a)^2) - (x^2 + x(x - a) + (x - a)^2)) \\
= a^4 y^4 (x^2 - x(a - x) + (x - a)^2) \\
= a^4 y^4 ((x - \frac{1}{2} a)^2 - x(a - x) + (x - a)^2) \\
= a^4 y^4 \left( \left( \frac{1}{2} \right)^2 + \frac{3}{4} a^2 \right) \geq 0.
\]

(3) Let \(a \neq 0\) and \(b \neq 0\). We divide \(G(x, y, a, b)\) by \(a^6 b^4\), put \(\alpha = x/a\) and \(\beta = y/b\) and take into account that by Lemma 1 in these settings one has
\[
\frac{h_2(u, v)}{a^4 k - 2 b^2 k} = a^{2k-1} (\beta - 1)^{2^k} + a^{2k-1} (a - 1)^{2^{k-1}} - (a - 1)^{2^{k-1}} \beta^{2^k}, \quad k = 1, 2.
\]

Therefore
\[
G(x, y, a, b) = 2 (a^2 (\beta - 1)^2 + (\alpha^2 + \beta^2) (a - 1)^2 (\alpha (\beta - 1)^2 + a(a - 1) - (a - 1)\beta^2) \\
- (a^3 (\beta - 1)^4 + a^3 (a - 1)^3 - (a - 1)\beta^4).
\]

Removing brackets and further collecting terms give
\[
G(x, y, a, b) = (a^2 - a + 1)(\beta^4 - 4a\beta^3 + 6a^2\beta^2 - 4a^3\beta + a^4) \\
= \left( \left( \frac{1}{2} \right)^2 + \frac{3}{4} \right) (a - \beta)^4 \geq 0.
\]

Thus \(G(x, y, a, b)\) is non-negative in all the cases and so we are done. \(\square\)

**Remark 2.** The inequality (14) is sharp as it is known from [4, Lemma 3] that
\[
2 \left[ \frac{\text{Re} (z_1 - z_2)}{|z_1 - z_2|} \frac{\text{Re} (z_1 - z_3)}{|z_1 - z_3|} \frac{\text{Re} (z_2 - z_3)}{|z_2 - z_3|} \right]^2 p_{k_1}(z_1, z_2, z_3) \leq p_{k_2}(z_1, z_2, z_3).
\]

Indeed, when all sides of the triangle \((z_1, z_2, z_3)\) make a small angle with the horizontal, the multiplier in the square brackets is close to 1 in modulus.

The estimate (14) allows us to obtain an inequality for \(L^2\)-norms.

**Lemma 3.** Let \(\mu\) have linear growth and \(\varepsilon > 0\) be fixed. Then
\[
\|T_{x_2, \varepsilon} f\|_{L^2(\mu)} \leq \sqrt{2} \|T_{x_1, \varepsilon} f\|_{L^2(\mu)} + C \sqrt{\mu(\mathbb{C})}, \quad C > 0.
\]
Proof. From Lemma 2 and the definition (7) of $p_K(\mu)$ we immediately get that
\begin{equation}
 p_{k_2,\varepsilon}(\mu) \leq 2p_{k_1,\varepsilon}(\mu).
\end{equation}

Now we use the identity
\begin{equation}
 \|T_{K,\varepsilon}1\|_{L^2(\mu)}^2 = \frac{1}{3}p_{K,\varepsilon}(\mu) + \mathcal{R}_{K,\varepsilon}(\mu), \quad |\mathcal{R}_{K,\varepsilon}(\mu)| \leq C_K\mu(\mathbb{C}), \quad C_K > 0,
\end{equation}
where $K$ is any real antisymmetric Calderón–Zygmund kernel with non-negative permutations. This identity is a generalization of (4) and is contained in [2, Lemma 3.3] (see also [4, Section 5]). In these terms the inequality (16) gives
\begin{equation}
 \frac{1}{3}p_{k_2,\varepsilon}(\mu) + \mathcal{R}_{k_2,\varepsilon}(\mu) \leq 2\left(\frac{1}{3}p_{k_1,\varepsilon}(\mu) + \mathcal{R}_{k_1,\varepsilon}(\mu)\right) + \mathcal{R}_{k_2,\varepsilon}(\mu) - 2\mathcal{R}_{k_1,\varepsilon}(\mu),
\end{equation}
and, consequently,
\begin{equation}
 \|T_{k_2,\varepsilon}1\|_{L^2(\mu)}^2 \leq 2\|T_{k_1,\varepsilon}1\|_{L^2(\mu)}^2 + C\mu(\mathbb{C}), \quad C > 0.
\end{equation}
Applying the inequality $\sqrt{ax^2 + b} \leq \sqrt{ax} + \sqrt{b}$ valid for $a, b, x \geq 0$, we get (15). \qed

Remark 3. Lemmas 2 and 3 are particular cases of [3, Lemma 7] and [18, Main Lemma], correspondingly, but with an explicit constant. The explicitness of the constant is very important here and actually enables us to obtain the result.

We are ready now to prove Theorems 1 and 2.

By (15) and the triangle inequality,
\begin{align*}
 \|T_{k,\varepsilon}1\|_{L^2(\mu)} &= \|(T_{k_2,\varepsilon} + t \cdot T_{k_1,\varepsilon})1\|_{L^2(\mu)} \\
 &\geq |t|\|T_{k_1,\varepsilon}1\|_{L^2(\mu)} - \|T_{k_2,\varepsilon}1\|_{L^2(\mu)} \\
 &\geq (|t| - \sqrt{2})\|T_{k_1,\varepsilon}1\|_{L^2(\mu)} - C\sqrt{\mu(\mathbb{C})}.
\end{align*}
Consequently,
\begin{equation}
 \|T_{k_1,\varepsilon}1\|_{L^2(\mu)} \leq \frac{\|T_{k_1,\varepsilon}1\|_{L^2(\mu)} + C\sqrt{\mu(\mathbb{C})}}{|t| - \sqrt{2}}, \quad |t| > \sqrt{2},
\end{equation}
and therefore for any cube $Q \subset \mathbb{C}$,
\begin{equation}
 \|T_{k_1,\varepsilon}\chi_Q\|_{L^2(\mu;Q)} \leq \frac{\|T_{k_1,\varepsilon}\chi_Q\|_{L^2(\mu;\mathbb{C})} + C\sqrt{\mu(Q)}}{|t| - \sqrt{2}}, \quad |t| > \sqrt{2}.
\end{equation}
Applying a variant of the $T1$ Theorem of Nazarov, Treil and Volberg from [19, Theorem 9.40], we infer that the $L^2(\mu)$-boundedness of $T_{k_1}$ with $|t| > \sqrt{2}$ implies that $T_{k_1}$ (and hence the Cauchy transform) is $L^2(\mu)$-bounded. Therefore, by Theorems A and B, we get the desired result. Note that the “only if” part of Theorem 1 follows from [6].
**Remark 4.** Computer experiments suggest that the following inequality holds:

\[ p_{K_n}(z_1, z_2, z_3) \leq \frac{N}{n} p_{k_n}(z_1, z_2, z_3). \]  

(Lemma 2 corresponds to the case \((n, N) = (1, 2)\).) Moreover, if \(u = -\gamma + i\), \(v = \gamma + i\) and \(\gamma > 0\), then (see [4, Example 1])

\[ p_{k_n}(0, u, v) = \frac{\gamma^{2m-2}((\gamma^2 + 1)m - \gamma^2m)}{(\gamma^2 + 1)^{2m}}, \quad m \in \mathbb{N}, \]

and therefore

\[ \lim_{\gamma \to \infty} \frac{p_{k_n}(0, u, v)}{p_{k_n}(0, u, v)} = \lim_{\gamma \to \infty} \frac{1 - (\gamma^2/(\gamma^2 + 1))^N}{1 - (\gamma^2/(\gamma^2 + 1))^n} = \frac{N}{n}. \]

It means that the constant \(N/n\) is sharp if (19) is true.

It would follow from (19) in the same manner as above that the \(L^2(\mu)\)-boundedness of \(T_{K_t}\) with \(|t| > \sqrt{N/n}\) implies that \(T_{k_n}\) is \(L^2(\mu)\)-bounded. This would give the analogues of Theorems 1 and 2 for the more general case of kernels (8) via theorems in [2] instead of Theorems A and B.

### 4 The proof of Theorem 3

We now consider the kernels (8) with \(t \in \Omega(n, N)\) (see (9)). As mentioned above, the corresponding permutations are non-negative and hence a curvature-like method can be used directly. Namely, we will adapt the arguments from [2, Section 8], which in turn stem from [8], to our settings. Note that the “only if” part of Theorem 3 follows from [6] (even for all \(t \in \mathbb{R}\)). Thus we only need to prove the “if” part.

Suppose that \(\mu\) is an AD-regular measure on \(\mathbb{C}\) and \(T_{K_t}\), the CZO associated with the kernels (8), \(t \in \Omega(n, N)\). It is proved in [4, Lemmas 5 and 6] that if

\[
\begin{align*}
t &\in \mathbb{R} \setminus \left[ -\frac{1}{2} \left( 3 + \sqrt{9 - 4 \frac{N}{n}} \right); 2 - \frac{N}{n} \right], \quad n < N \leq 2n, \\
&t \in \mathbb{R} \setminus \left[ -\frac{1}{2} \left( \sigma_{n,M} + \sqrt{\sigma_{n,M}^2 - 4 \frac{N}{n}} \right); \sigma_{n,M} - 3 \right], \quad n \geq 2n,
\end{align*}
\]

where \(\sigma_{n,M} = 3 + (\frac{N}{n}) - 2\sqrt{N - 2n}\) as above, then

\[ p_{K_t}(z_1, z_2, z_3) \geq C(t) \cdot p_{k_n}(z_1, z_2, z_3), \quad C(t) > 0, \quad (z_1, z_2, z_3) \in \mathbb{C}^3. \]

Consequently, \(p_{k_n,\varepsilon}(\mu) \geq C(t) \cdot p_{k_n,\varepsilon}(\mu)\) and hence from (17) we conclude that for \(t\) as in (20) and (21) and any cube \(Q \subset \mathbb{C},\)

\[
\|T_{k_n,\varepsilon}Q\|_{L^2(\mu|Q)} \leq C(t)(\|T_{K_t,\varepsilon}Q\|_{L^2(\mu|Q)} + C\sqrt{\mu(Q)}).
\]
Therefore, by the $T1$ Theorem from [19, Theorem 9.40] and [2, Theorem 1.3], the measure $\mu$ is uniformly rectifiable.

What is left, according to (9), is to prove Theorem 3 for

\[(23) \quad t = 2 - \frac{N}{n}, \quad n < N \leq 2n,\]

\[(24) \quad t = \sigma_{n,M} - 3, \quad N \geq 2n,\]

\[(25) \quad t = -\frac{1}{2} \left(3 + \sqrt{9 - 4\frac{N}{n}}\right), \quad n < N \leq 2n,\]

\[(26) \quad t = -\frac{1}{2} \left(\sigma_{n,M} + \sqrt{\sigma_{n,M}^2 - 4\frac{N}{n}}\right), \quad N \geq 2n.\]

To manage these cases, we introduce additional notation. Given two distinct points $z, w \in \mathbb{C}$, we denote by $L_{z,w}$ the line passing through $z$ and $w$. Given three pairwise distinct points $z_1, z_2, z_3 \in \mathbb{C}$, we denote by $\angle(z_1, z_2, z_3)$ the smallest angle (belonging to $[0; \pi/2]$) formed by the lines $L_{z_1,z_2}$ and $L_{z_1,z_3}$. If $L$ and $L'$ are lines, then $\angle(L, L')$ is the smallest angle (belonging to $[0; \pi/2]$) between them. Also, $\theta_V(L) := \angle(L, V)$, where $V$ is the vertical. Furthermore, for a fixed constant $\tau \geq 1$ and complex numbers $z_1, z_2$ and $z_3$, set

\[(27) \quad \mathcal{O}_\tau := \left\{(z_1, z_2, z_3) : \frac{|z_i - z_j|}{|z_i - z_k|} \leq \tau \text{ for pairwise distinct } i, j, k \in \{1, 2, 3\}\right\},\]

so that all triangles with vertexes $z_1, z_2$ and $z_3$ in $\mathcal{O}_\tau$ have comparable sides.

Given $a_0 \in (0, \pi/2)$ and $(z_1, z_2, z_3) \in \mathbb{C}^3$, in what follows we will sometimes need the conditions

\[(28) \quad \theta_V(L_{z_1,z_2}) + \theta_V(L_{z_2,z_3}) + \theta_V(L_{z_1,z_3}) \geq a_0\]

and

\[(29) \quad \theta_V(L_{z_1,z_2}) + \theta_V(L_{z_2,z_3}) + \theta_V(L_{z_1,z_3}) \leq \frac{3}{2} \pi - a_0.\]

We will also use the following result.

**Lemma 4** (Lemma 10 in [4]). Fix $a_0 \in (0, \pi/2)$. Given $K_t$ and $(z_1, z_2, z_3) \in \mathcal{O}_\tau$,

(i) if the condition (28) is satisfied and $t$ is as in (23) or (24), or

(ii) if both the conditions (28) and (29) are satisfied and $t$ is as in (25) or (26),

then the following inequality holds:

\[(30) \quad p_{K_t}(z_1, z_2, z_3) \geq C(a_0, \tau) \cdot c(z_1, z_2, z_3)^2, \quad C(a_0, \tau) > 0.\]

On the one hand, if we are in the clause (i) of Lemma 4, i.e., in the same settings as in [2], then we can undeviatingly follow the scheme from [2, Section 8] (exchanging $p_{K_n}$ for $p_{K_t}$) in order to get our result for $t$ as in (23) or (24).
On the other hand, by clause (ii) of Lemma 4, we can ensure that the inequality (30) is true for \( t \) as in (25) or (26) if the sides of the triangles \((z_1, z_2, z_3)\) are far from both the vertical and horizontal. Consequently, the scheme from [2, Section 8] cannot be applied directly for such \( t \). Nevertheless, as we show below, it works after several modifications (besides the exchange of \( p_{\kappa_n} \) for \( p_{K_t} \)) connected basically with adapting geometrical arguments to both the conditions (28) and (29). Note that some of the arguments in [2, Section 8] are very sketchy and so, for the sake of completeness, we give a proof that is more detailed than the corresponding one in [2, Section 8].

The fact that the \( L^2(\mu) \)-boundedness of \( T_{K_t} \) implies that \( \mu \) is uniformly rectifiable will be proved by means of a corona-type decomposition. We now recall how such a decomposition is defined in [8, Chapter 2] for a given AD-regular Borel measure \( \mu \).

The elements \( Q \) playing the role of dyadic cubes are usually called \( \mu \)-cubes.

Given a 1-dimensional AD regular Borel measure \( \mu \) on \( \mathbb{C} \), for each \( j \in \mathbb{Z} \) (or \( j \geq j_0 \) if \( \mu(\mathbb{C}) < \infty \)) there exists a family \( D_j \) of Borel subsets of \( \text{spt} \mu \), i.e., \( \mu \)-cubes \( Q \) of the \( j \)th generation, such that

- each \( D_j \) is a disjoint partition of \( \text{spt} \mu \), i.e., if \( Q, Q' \in D_j \) and \( Q \neq Q' \), then \( \text{spt} \mu = \bigcup_{Q \in D_j} Q \) and \( Q \cap Q' = \emptyset \);
- if \( Q \in D_j \) and \( Q' \in D_k \) with \( k \leq j \), then either \( Q \subseteq Q' \) or \( Q \cap Q' = \emptyset \);
- for all \( j \in \mathbb{Z} \) and \( Q \in D_j \), we have \( 2^{-j} \lesssim \text{diam} (Q) \lesssim 2^{-j} \) and \( \mu(Q) \approx 2^{-j} \).

In what follows, \( D := \bigcup_{j \in \mathbb{Z}} D_j \). Moreover, given \( Q \in D_j \), we define the side length of \( Q \) as \( \ell(Q) = 2^{-j} \), which actually indicates the generation of \( Q \). Obviously, \( \ell(Q) \approx \text{diam} (Q) \). The value of \( \ell(Q) \) is not well-defined if the \( \mu \)-cube \( Q \) belongs to \( D_j \cap D_k \) with \( j \neq k \). To avoid this, one may consider a \( Q \in D_j \) as a couple \((Q, j)\).

Given \( \lambda > 1 \) and \( Q \in D \), set

\[
\lambda Q := \{ x \in \text{spt} \mu : \text{dist}(x, Q) \leq (\lambda - 1)\ell(Q) \}.
\]

We will also need the following version of P. Jones’ \( \beta \)-numbers for \( \mu \)-cubes (see [9]):

\[
\beta_q(Q) = \inf_L \left( \frac{1}{\ell(Q)} \int_{\eta_1 Q} \left( \frac{\text{dist}(x, L)}{\ell(Q)} \right)^q d\mu(x) \right)^{1/q}, \quad 1 \leq q \leq \infty,
\]

where \( \eta_1 > 4 \) is some constant to be fixed later and the infimum is taken over all affine lines \( L \). We will mostly use \( \beta_1(Q) \) and denote by \( L_Q \) the best approximating line for \( \beta_1(Q) \).
Given $Q \in \mathcal{D}_j$, the sons of $Q$, forming the collection $\text{Sons}(Q)$, are the $\mu$-cubes $Q' \in \mathcal{D}_{j+1}$ such that $Q' \subseteq Q$.

By [8, Chapter 2], one says that $\mu$ admits a corona decomposition if there are parameters $\eta, \delta > 0$ and a triple $(\mathcal{B}, \mathcal{G}, \text{Tree})$, where $\mathcal{B}$ and $\mathcal{G}$ are subsets of $\mathcal{D}$ and $\text{Tree}$ is a family of subsets $S$ of $\mathcal{G}$, such that the following conditions are satisfied:

1. $\mathcal{D} = \mathcal{B} \cup \mathcal{G}$ and $\mathcal{B} \cap \mathcal{G} = \emptyset$.
2. $\mathcal{B}$ satisfies a Carleson packing condition, i.e.,

\[
\sum_{Q \in \mathcal{B} : Q \subseteq R} \mu(Q) \lesssim_\eta \mu(R) \quad \text{for all } R \in \mathcal{D}.
\]

3. $\mathcal{G} = \bigcup_{S \in \text{Tree}} S$ and the union is disjoint.
4. Each $S \in \text{Tree}$ is called a tree and is coherent: each $S$ has a unique maximal element $Q_S$, which contains all other elements of $S$ as subsets, i.e.

   - a $\mu$-cube $Q' \in \mathcal{D}$ belongs to $S$ if $Q \subseteq Q' \subseteq Q_S$ for some $Q \in S$;
   - if $Q \in S$, then either all elements of $\text{Sons}(Q)$ lie in $S$ or none of them do.
5. For each $S \in \text{Tree}$, there exists a (possibly rotated) Lipschitz graph $\Gamma_S$ with constant smaller than $\eta$ such that $\text{dist}(x, \Gamma_S) \leq \delta \text{ diam}(Q)$ whenever $x \in 2Q$ and $Q \in S$.
6. The maximal $\mu$-cubes $Q_S$, for $S \in \text{Tree}$, satisfy the Carleson packing condition

\[
\sum_{S \in \text{Tree} : Q_S \subseteq R} \mu(Q_S) \lesssim \mu(R) \quad \text{for all } R \in \mathcal{D}.
\]

According to [8, Section 1, (C4) and (C6)], if $\mu$ is uniformly rectifiable, then it admits a corona decomposition for all $\eta, \delta > 0$. Conversely, the existence of a corona decomposition for a single set of $\eta$ and $\delta$ implies that $\mu$ is uniformly rectifiable.

We now turn to constructing our corona decomposition. Let $\varepsilon > 0$ be some small constant to be chosen at the very end of the construction. From now on, $\mathcal{B}_0(\varepsilon)$ stands for the family of cubes $Q \in \mathcal{D}$ such that $\beta_1(Q) \geq \varepsilon$. Furthermore, $\mathcal{G}_0(\varepsilon) := \mathcal{D} \setminus \mathcal{B}_0(\varepsilon)$. The aim is to show that $\mathcal{B}_0(\varepsilon)$ satisfies a Carleson packing condition.

Note that constants in some inequalities below depend also on the constants $\eta_1, \eta_2$ and $\eta_3$ appearing in definitions of and estimates for $\beta$-numbers. Since these $\eta$ will be fixed at some point, we do not indicate this dependence explicitly. On the contrary, we emphasize the dependence of some forthcoming inequalities on the essential parameter $\varepsilon$. 
We start by observing that for any $y, z \in \eta_1 Q$, using that $\ell(Q) \approx \mu(Q)$, we get

$$\beta_2(Q)^2 \leq \frac{1}{\ell(Q)} \int_{\eta_1 Q} \left( \frac{\text{dist}(x, L_{y,z})}{\ell(Q)} \right)^2 d\mu(x)$$

$$= \frac{1}{\ell(Q)} \left( \int_{x \in \eta_1 Q, \frac{\text{dist}(x, L_{y,z})}{\ell(Q)} < \varepsilon^2} \left( \frac{\text{dist}(x, L_{y,z})}{\ell(Q)} \right)^2 d\mu(x) \right)$$

$$+ \int_{x \in \eta_1 Q, \frac{\text{dist}(x, L_{y,z})}{\ell(Q)} \geq \varepsilon^2} \left( \frac{\text{dist}(x, L_{y,z})}{\ell(Q)} \right)^2 d\mu(x) \right) \lesssim \varepsilon^4 + \frac{1}{\ell(Q)} \int_{\eta_1 Q, \frac{\text{dist}(x, L_{y,z})}{\ell(Q)} \geq \varepsilon^2} \left( \frac{\text{dist}(x, L_{y,z})}{\ell(Q)} \right)^2 d\mu(x).$$

**Lemma 5.** Let $B_1 = B(\zeta_1, r_1)$ and $B_2 = B(\zeta_2, r_2)$ be two balls such that $B_1 \cap \text{spt} \mu \subset \eta_1 Q$, $B_2 \cap \text{spt} \mu \subset \eta_1 Q$, $\text{dist}(B_1, B_2) \approx \ell(Q)$ and $r_1 \approx r_2 \approx \ell(Q)$. If $y \in B_1$ and $z \in B_2$, then for $\varepsilon$ small enough,

$$\int_{\eta_1 Q, \frac{\text{dist}(x, L_{y,z})}{\ell(Q)} \geq \varepsilon^2} \text{dist}(x, L_{y,z})^2 d\mu(x) \lesssim \ell(Q)^2 p_{K_i}^{(\varepsilon; Q)}(\mu),$$

where

$$p_{K_i}^{(\varepsilon; Q)}(\mu) := \iint_{\frac{|x-y|}{\ell(Q)} \geq \varepsilon^2} \iint_{\frac{|y-z|}{\ell(Q)} \geq \varepsilon^2} p_{K_i}(x, y, z) d\mu(x)d\mu(y)d\mu(z).$$

Note that the existence of the above-mentioned balls $B_1$ and $B_2$ is guaranteed in the AD-regular case.

**Proof.** Note that the condition $\text{dist}(x, L_{y,z}) \geq \varepsilon^2 \ell(Q)$ implies that

$$|x - y| \geq \varepsilon^2 \ell(Q) \quad \text{and} \quad |x - z| \geq \varepsilon^2 \ell(Q).$$

Consequently, since $x \in \eta_1 Q$, $y \in B_1$ and $z \in B_2$,

$$|x - z| \approx |x - y| \approx |y - z|,$$

where the comparability constants depend on $\eta_1$ and $\varepsilon$. We now separate two cases.

1. Suppose that

$$\varepsilon^{10} \leq \theta_V(L_{x,y}) + \theta_V(L_{y,z}) + \theta_V(L_{x,z}) \leq \frac{3}{2} \pi - \varepsilon^{10}.$$

Then by the clause (ii) of Lemma 4, where we put $a_0 = \varepsilon^{10}$ and $x = \tau(\varepsilon, \eta)$ chosen from (32), we have $c(x, y, z)^2 \lesssim \varepsilon p_{K_i}(x, y, z)$.

2. Now let

$$\theta_V(L_{x,y}) + \theta_V(L_{y,z}) + \theta_V(L_{x,z}) < \varepsilon^{10}$$
or
\[
\theta_V(L_{x,y}) + \theta_V(L_{y,z}) + \theta_V(L_{x,z}) > \frac{3}{2} \pi - \varepsilon^{10}.
\]

In this case \( \text{dist}(x, L_{y,z}) \lesssim \varepsilon^{10} \ell(Q) \). Thus for \( \varepsilon \) small enough we get a contradiction with the assumption \( \text{dist}(x, L_{y,z}) \geq \varepsilon^2 \ell(Q) \).

Summarizing,
\[
\int_{\frac{\text{dist}(x, L_{y,z})^2}{\ell(Q)^2} \geq \varepsilon^2} \frac{\ell(Q)^4}{\mu(B_1)\mu(B_2)} \int_{B_1} \int_{B_2} \int_{\frac{\text{dist}(x, L_{y,z})}{\ell(Q)}} (\frac{\text{dist}(x, L_{y,z})}{|x-y||x-z|})^2 d\mu(x)d\mu(y)d\mu(z)
\leq \varepsilon \ell(Q)^2 \int_{\eta_1} \int_{\eta_1} \int_{\frac{\text{dist}(x, L_{y,z})}{\ell(Q)}} c(x, y, z)^2 d\mu(x)d\mu(y)d\mu(z)
\leq \varepsilon \ell(Q)^2 p^{(c;Q)}_{K_i}(\mu).
\]

We used the well-known identity \( c(x, y, z) = \text{dist}(x, L_{y,z})/(|x-y||x-z|) \).

The estimate for \( \beta_2(Q)^2 \) that we obtained above and Lemma 5 give
\[
\beta_2(Q)^2 \lesssim \varepsilon^4 + \frac{C(\varepsilon)}{\ell(Q)} p^{(c;Q)}_{K_i}(\mu), \quad Q \in \mathcal{D}.
\]

We now take into account that \( \beta_1(Q) \lesssim \beta_2(Q) \) by Hölder’s inequality and, consequently, if \( Q \in B_0(\varepsilon) \), i.e., \( \beta_1(Q) \geq \varepsilon \), then \( \beta_2(Q) \gtrsim \varepsilon \). From this we deduce for sufficiently small \( \varepsilon \) that
\[
\mu(Q) \lesssim \varepsilon p^{(c;Q)}_{K_i}(\mu) \quad \text{for any } Q \in B_0(\varepsilon).
\]

From this we immediately get that
\[
\sum_{Q \in B_0(\varepsilon): Q \subseteq R} \mu(Q) \lesssim \varepsilon \sum_{Q \in B_0(\varepsilon): Q \subseteq R} p^{(c;Q)}_{K_i}(\mu).
\]

To estimate the latter sum, we will use the notation
\[
A_j(\varepsilon) := \{ x : \varepsilon^2 \ell(Q) \leq |x-y| \leq C \ell(Q) \}, \quad Q \in \mathcal{D}_j, \ C > 0.
\]

These are the concentric annuli \( B(y, C \ell(Q)) \setminus B(y, \varepsilon^2 \ell(Q)) \), contained in the ball \( B(y, C \ell(R)) \), where \( \ell(R) = 2^{-j_0} \). They have bounded overlap depending on \( \varepsilon \) and thus the sum \( \sum_{j \geq j_0} \int_{R \cap A_j(\varepsilon)} \) is less than \( C(\varepsilon) \int_R \) with some \( C(\varepsilon) > 0 \). These
observations lead to the following:

\[
\sum_{Q \in B_0(\varepsilon) : Q \subseteq R} p_{K_\varepsilon}(\mu) \\
\leq \int_{\eta R} \int_{\eta R} \left( \sum_{j \geq j_0} \sum_{Q \subseteq B_0(\varepsilon) \cap \|R\|} \int_{\eta R} p_{K_\varepsilon}(x, y, z) d\mu(x) \right) d\mu(y) d\mu(z) \\
\lesssim \int_{\eta R} \int_{\eta R} \left( \sum_{j \geq j_0} \int_{\eta R} p_{K_\varepsilon}(x, y, z) d\mu(x) \right) d\mu(y) d\mu(z) \\
\lesssim \varepsilon p_{K_\varepsilon}(\mu \lfloor (\eta R)).
\]

Since \( T_{K_\varepsilon} \) is \( L^2(\mu) \)-bounded, we get \( p_{K_\varepsilon}(\mu \lfloor F) \lesssim \mu(F) \) for any \( F \subset \mathbb{C} \). Consequently,

\[
p_{K_\varepsilon}(\mu \lfloor (\eta R)) \lesssim \mu(R) \quad \text{for all } R \in \mathbb{D},
\]

and therefore we reach the desired inequality

\[
\sum_{Q \in B_0(\varepsilon) : Q \subseteq R} \mu(Q) \lesssim \varepsilon \mu(R) \quad \text{for all } R \in \mathbb{D}.
\]

Thus, for any \( \varepsilon > 0 \), there exists the decomposition

(33) \quad \mathbb{D} = B_0(\varepsilon) \cup \mathcal{S}_0(\varepsilon),

where \( B_0(\varepsilon) \) satisfies a Carleson packing condition and for any cube \( Q \in \mathcal{S}_0(\varepsilon) \) there exists a line \( L_Q \) such that \( \text{dist}(x, L_Q) \lesssim \sqrt{\varepsilon} \ell(Q) \) for all \( x \in \frac{1}{2} \eta_1 Q \) (since \( \beta_1(Q) < \varepsilon \) for such cubes and \( \beta_\infty(Q) \lesssim \sqrt{\beta_1(2Q)} \)). More details can be found in [8, Ch. 6].

Using the decomposition (33), we now can apply [8, Lemma 7.1] in order to obtain a new decomposition (still depending on \( \varepsilon \)) but already with a family of stopping times regions. Suppose that \( \theta_0 \) is small enough and \( \varepsilon \ll \theta_0 \) to prove the following assertion.

**Lemma 6.** For all sufficiently small \( \varepsilon > 0 \), there exists a decomposition \( \mathbb{D} = B \cup \mathcal{S} \), where \( B = B(\varepsilon) \) satisfies a Carleson packing condition (with a constant depending on \( \varepsilon \)) and \( \mathcal{S} = \mathcal{S}(\varepsilon) \) can be partitioned into a family Tree of coherent regions \( S \), satisfying the following. For each \( S \in \text{Tree} \) denote

\[
\alpha(S) := \frac{1}{10} \theta_0 \quad \text{if } \theta_0 \leq \theta_\gamma(L_{Q_S}) \leq \pi/2 - \theta_0
\]

and

\[
\alpha(S) := 10 \theta_0 \quad \text{if } \theta_\gamma(L_{Q_S}) < \theta_0 \text{ or } \theta_\gamma(L_{Q_S}) > \pi/2 - \theta_0.
\]
Then we have

- if \( Q \in S \), then \( \angle(L_Q, L_{Q_s}) \leq \alpha(S) \);
- if \( Q \) is a minimal cube of \( S \), then either at least one element of \( \text{Sons}(Q) \) lies in \( B \) or else \( \angle(L_Q, L_{Q_s}) \geq \frac{1}{2}\alpha(S) \).

Here \( \mathcal{G} \subseteq \mathcal{G}_0(\varepsilon) \) and therefore for any \( Q \in \mathcal{G} \) one has \( \beta_1(Q) \leq \varepsilon \).

Lemma 6 is an analogue of [2, Lemma 8.1] which comes from [8, Lemma 7.1]. The main difference between [2, Lemma 8.1] and [8, Lemma 7.1] is that two different values of the parameter \( \alpha(S) \) have to be chosen, according to the angle \( \theta_V(L_{Q_s}) \). In our case the situations where the angle \( \theta_V(L_{Q_s}) \) is close to zero and \( \pi/2 \) have to be also distinguished.

To obtain the required Lipschitz graph, one can follow the proof of [8, Proposition 8.2]. This leads to the following statement.

**Lemma 7.** For each \( S \in \text{Tree} \) from Lemma 6, there exists a Lipschitz function \( A_S : L_{Q_s} \rightarrow L_{\perp_Q} \) with norm \( \lesssim \alpha(S) \) such that, denoting by \( \Gamma_S \) the graph of \( A_S \),

\[
\text{dist}(x, \Gamma_S) \lesssim \sqrt{\varepsilon \ell(Q)}
\]

for all \( x \in 2Q \) with \( Q \in S \).

The proof will be completed if we show that the maximal \( \mu \)-cubes \( Q_S, S \in \text{Tree} \), satisfy the Carleson packing condition

\[
\sum_{S \in \text{Tree}: Q \subseteq R} \mu(Q_S) \lesssim \mu(R) \quad \text{for all } R \in \mathcal{D}.
\]

To do so, we will distinguish several types of trees.

Here and subsequently, \( \text{Stop}(S) \) denotes the family of the minimal \( \mu \)-cubes of \( S \in \text{Tree} \), which may be empty. By Lemma 6, we can split \( \text{Stop}(S) \) as follows:

\[(34) \quad \text{Stop}(S) = \text{Stop}_\alpha(S) \cup \text{Stop}_\beta(S), \quad \text{Stop}_\alpha(S) \cap \text{Stop}_\beta(S) = \emptyset,
\]

where \( \text{Stop}_\beta(S) \) contains all minimal \( \mu \)-cubes \( Q \) such that at least one element of \( \text{Sons}(Q) \) belongs to \( B \), and \( \text{Stop}_\alpha(S) \) contains all minimal \( Q \) such that

\[
\angle(L_Q, L_{Q_s}) \geq \frac{1}{2}\alpha(S).
\]

The first set that we will consider is

\[
\Delta_1 := \left\{ S \in \text{Tree} : \mu(Q_S \setminus \bigcup_{Q \in \text{Stop}(S)} Q) \geq \frac{1}{2} \mu(Q_S) \right\}.
\]

Clearly, if \( S \in \text{Tree} \setminus \Delta_1 \), then by (34),

\[(35) \quad \frac{1}{2} \mu(Q_S) < \mu(\bigcup_{Q \in \text{Stop}(S)} Q) = \mu(\bigcup_{Q \in \text{Stop}_\alpha(S)} Q) + \mu(\bigcup_{Q \in \text{Stop}_\beta(S)} Q).
\]
Now let
\[ \Delta_2 := \{ S \in \text{Tree} \setminus \Delta_1 : \mu(\bigcup_{Q \in \text{Stop}_\beta(S)} Q) \geq \frac{1}{4} \mu(Q_S) \}, \]
i.e., not less than \( \frac{1}{4} \mu(Q_S) \) of the measure of the minimal cubes for these trees have sons in \( B \). The rest of the trees are in
\[ \Delta_3 := \{ S \in \text{Tree} \setminus (\Delta_1 \cup \Delta_2) : \mu(\bigcup_{Q \in \text{Stop}_\beta(S)} Q) \geq \frac{1}{4} \mu(Q_S) \}. \]
Indeed, if \( S \in \text{Tree} \setminus (\Delta_1 \cup \Delta_2 \cup \Delta_3) \), then (35) is valid and moreover
\[ \mu(\bigcup_{Q \in \text{Stop}_\alpha(S)} Q) < \frac{1}{4} \mu(Q_S) \text{ and } \mu(\bigcup_{Q \in \text{Stop}_\beta(S)} Q) < \frac{1}{4} \mu(Q_S). \]
This means that \( \text{Tree} \setminus (\Delta_1 \cup \Delta_2 \cup \Delta_3) = \emptyset \).

We also split \( \Delta_3 \) in the three disjoint sets:
\[ \begin{align*}
\Delta_3' & := \{ S \in \Delta_3 : \theta_0 \leq \theta_V(L_{Q_S}) \leq \pi/2 - \theta_0 \}, \\
\Delta_3'' & := \{ S \in \Delta_3 : \theta_V(L_{Q_S}) < \theta_0 \}, \\
\Delta_3''' & := \{ S \in \Delta_3 : \theta_V(L_{Q_S}) > \pi/2 - \theta_0 \}. 
\end{align*} \]
So we have the disjoint union
\[ \text{Tree} = \Delta_1 \cup \Delta_2 \cup \Delta'_3 \cup \Delta''_3 \cup \Delta'''_3. \]
The procedure now is to check the required Carleson packing condition for all components of this union.

For all \( S \in \text{Tree} \) the sets \( Q_S \setminus \bigcup_{Q \in \text{Stop}(S)} Q \) are pairwise disjoint and hence for \( S \in \Delta_1 \) we get
\[ \sum_{S \in \Delta_1 : Q_S \subseteq R} \mu(Q_S) \leq 2 \sum_{S \in \text{Tree} : Q_S \subseteq R} \mu(Q_S \setminus \bigcup_{Q \in \text{Stop}(S)} Q) \leq 2 \mu(R). \]
If \( S \in \Delta_2 \), then by definition and the fact that \( \mu(Q) \approx \mu(Q') \) for \( Q' \in \text{Sons}(Q) \),
\[ \mu(Q_S) \leq 4 \mu(\bigcup_{Q \in \text{Stop}_\beta(S)} Q) \leq \sum_{Q \in \text{Stop}(S) \setminus \text{B'} \cup \text{Sons}(Q)} \mu(Q') \]
and consequently, by Lemma 6,
\[ \sum_{S \in \Delta_2 : Q_S \subseteq R} \mu(Q_S) \leq \sum_{S \in \Delta_2 : Q_S \subseteq R} \sum_{Q \in \text{Stop}(S) \setminus \text{B'} \cup \text{Sons}(Q)} \mu(Q') \leq \sum_{Q \in \text{B'} : Q_S \subseteq R} \mu(Q) \leq \epsilon \mu(R). \]
Let us consider the case $S \in \Delta_3'$. We will need $\beta$-numbers defined for balls $B(x, r)$:

$$\beta_q(x, r) := \inf_{L} \left( \frac{1}{r} \int_{B(x, 2r)} \left( \frac{\text{dist}(x, L)}{r} \right)^q d\mu(x) \right)^{1/q}, \quad 1 \leq q \leq \infty,$$

where the infimum is taken over all affine lines $L$.

It is claimed in [8, Section 12, Inequality (12.2)] that for all $S \in \Delta_3$ there exists $\eta_2 > 1$ such that

$$\mu(Q_S) \lesssim \int_{X_S} \beta_1(x, \eta_2 r)^2 \frac{d\mu(x) dr}{r},$$

where

$$X_S := \{(x, r) \in \text{spt } \mu \times \mathbb{R}^+: x \in \eta_2 Q_S, \frac{1}{\eta_2} d(x) \leq r \leq \eta_2 \text{ diam}(Q_S)\}$$

and

$$d(x) := \inf_{Q \in S} \{\text{dist}(x, Q) + \text{diam}(Q)\}.$$ 

By Holder’s inequality, $\beta_1(x, \eta_2 r) \lesssim \beta_2(x, \eta_2 r)$. Moreover, it follows from [13, Lemma 2.5 and Proof of Proposition 2.4] (or by the arguments analogous to the ones in the proof of Lemma 5) that if $\mu$ is AD-regular, then there exists $\eta_3 \geq 2$ such that for any $x \in \text{spt } \mu$,

$$\beta_2(x, \eta_2 r)^2 \lesssim \frac{1}{\eta_2^2 r} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} c(u, v, w)^2 d\mu(u) d\mu(v) d\mu(w),$$

where

$$\mathcal{O}_{\eta_3}(x, \rho) := \{(u, v, w) \in (B(x, \eta_3 \rho))^3 : |u-v| \geq \frac{\rho}{\eta_3}, |v-w| \geq \frac{\rho}{\eta_3}, |u-w| \geq \frac{\rho}{\eta_3}\}.$$ 

Note also that for any $(u, v, w) \in \mathcal{O}_{\eta_3}(x, \rho)$ we have $|u-v| \leq 2 \eta_3 \rho, |v-w| \leq 2 \eta_3 \rho$ and $|u-w| \leq 2 \eta_3 \rho$, and thus for a fixed $\eta_3$,

$$|u-v| \approx |v-w| \approx |u-w| \approx \rho.$$ 

Therefore if a triple $(u, v, w) \in \mathcal{O}_{\eta_3}(x, \eta_2 r)$ with $(x, r) \in X_S$, then at least one side of the triangle $(u, v, w)$ makes a big angle with the vertical and horizontal. Indeed, by construction, if $\eta_1$ is chosen much bigger than $\eta_2$, then $\beta_2(x, \eta_2 r) \lesssim \sqrt{\varepsilon}$, and consequently the angle between one side of $(u, v, w)$ and the best approximating line $L_{Q_S}$ is less than $C(\eta_3) \sqrt{\varepsilon}$ with some $C(\eta_3) > 0$. Furthermore, $\theta_0 \leq \theta_v(L_{Q_S}) \leq \pi/2 - \theta_0$ and thus the angle that one side of $(u, v, w)$ makes with the vertical and horizontal belongs to

$$\left(\frac{9}{10} \theta_0 - C(\eta_3) \sqrt{\varepsilon}; \frac{9}{10} \theta_0 + C(\eta_3) \sqrt{\varepsilon}\right) \supseteq \left(\frac{1}{2} \theta_0; \frac{\pi}{2} - \frac{1}{2} \theta_0\right),$$
where $\varepsilon$ is chosen sufficiently small. This fact enables us to use the clause ii of Lemma 4 and exchange the curvature for our permutation $p_{K_t}$:

$$
\beta_2(x, \eta_2 r)^2 \lesssim_{\theta_0} \frac{1}{\eta_2 r} \int_{\mathbb{S}(x, \eta_2 r)} \frac{1}{n} p_{K_t}(u, v, w) \, d\mu(u)d\mu(v)d\mu(w), \quad (x, r) \in X_S.
$$

Summarizing and taking into account that $\theta_0$ is fixed, we get

$$
\mu(Q_S) \lesssim \int_{X_S} \int_{\mathbb{S}(x, \eta_2 r)} p_{K_t}(u, v, w) \, d\mu(u)d\mu(v)d\mu(w) \frac{d\mu(x)dr}{(\eta_2 r)^2}.
$$

What is more, it is shown after [8, Lemma 7.9] that the regions $X_S$ (see (36)) with $S \in \Delta_3'$ have bounded overlap. By this reason,

$$
\sum_{S \in \Delta_3', Q_S \subseteq R} \mu(Q_S) \lesssim \int_{0}^{2\eta_2 R} \int_{2\eta_2 R} \int_{\mathbb{S}(x, \eta_2 r)} p_{K_t}(u, v, w) \, d\mu(u)d\mu(v)d\mu(w) \frac{d\mu(x)dr}{(\eta_2 r)^2}
$$

$$
\lesssim \mu(R).
$$

The third inequality is by Fubini’s theorem. See the definition of $\mathbb{S}_\tau$ in (27).

Finally, by the $L^2$-boundedness of $T_{K_t}$, we get

$$
\sum_{S \in \Delta_3', Q_S \subseteq R} \mu(Q_S) \lesssim \mu(R).
$$

Suppose now that $S \in \Delta_3''$. If $Q \in \text{Stop}_\alpha(S)$, then $\text{Sons}(Q) \cap \mathbb{B} = \emptyset$ and by Lemma 6,

$$
\angle(L_Q, L_{Q_S}) \leq \alpha(S), \quad \angle(L_Q, L_{Q_S}) \geq \frac{1}{2} \alpha(S), \quad \alpha(S) = 10\theta_0,
$$

and thus

$$
\theta_V(L_Q) \leq \angle(L_Q, L_{Q_S}) + \theta_V(L_{Q_S}) < 10\theta_0 + \theta_0 = 11\theta_0,
$$

$$
\theta_V(L_Q) \geq \angle(L_Q, L_{Q_S}) - \theta_V(L_{Q_S}) > 5\theta_0 - \theta_0 = 4\theta_0.
$$

Since $\beta_1(Q) < \varepsilon$, we can choose $\varepsilon$ small enough in order that $\angle(L_Q, L_{Q'}) \leq \theta_0$, $Q' \in \text{Sons}(Q)$, and hence

$$
3\theta_0 < \theta_V(L_{Q'}) < 12\theta_0, \quad Q' \in \text{Sons}(Q).
$$
Consequently, any element of \( \text{Sons}(Q) \) is the maximal \( \mu \)-cube of a tree belonging either to \( \Delta_1, \Delta_2 \) or \( \Delta'_1 \). Additionally, from the definition of \( \Delta_3 \) and the fact that minimal cubes for a single tree are pairwise disjoint it follows that

\[
\mu(Q_S) \leq 4\mu(\bigcup_{Q \in \text{Stop}_\alpha(S)} Q) = 4 \sum_{Q \in \text{Stop}_\alpha(S)} \mu(Q) = 4 \sum_{Q \in \text{Stop}_\alpha(S)} \sum_{Q' \in \text{Sons}(Q)} \mu(Q').
\]

From the above-mentioned we deduce that

\[
\sum_{S \in \Delta''_1: Q_S \subseteq R} \mu(Q_S) \leq 4 \sum_{S \in \Delta'_1: Q_S \subseteq R} \sum_{Q' \in \text{Sons}(Q)} \mu(Q').
\]

Take into account that the maximal cubes of all trees from \( \Delta_1 \cup \Delta_2 \cup \Delta'_3 \) satisfy a Carleson packing condition (with a constant depending on \( \varepsilon \)). By this reason,

\[
\sum_{S \in \Delta^''_3: Q_S \subseteq R} \mu(Q_S) \lesssim \varepsilon \mu(R).
\]

One can argue for \( S \in \Delta''''_3 \) in the same manner as for \( S \in \Delta''_3 \). Indeed, if \( \varepsilon \) is appropriately chosen and \( Q \in \text{Stop}_\alpha(S) \), then

\[
\pi/2 - 12\theta_0 < \theta_V(L_{Q'}) < \pi/2 - 3\theta_0, \quad Q' \in \text{Sons}(Q),
\]

and hence any element of \( \text{Sons}(Q) \) is the maximal \( \mu \)-cube of a tree belonging either to \( \Delta_1, \Delta_2 \) or \( \Delta'_3 \).

Summarizing, we proved that maximal cubes of all types of trees satisfy a Carleson packing condition and so the triple \( (B, \mathcal{G}, \text{Tree}) \) is a corona decomposition as required.

5 Additional remarks

To finish, we would like to mention a corollary of the results presented in the previous sections. Let \( \mu \) be a Radon measure on \( \mathbb{C} \) with linear growth. If the CZO associated with the kernel \( k_t \) where

\[
t \in (-\infty; -\sqrt{2}) \cup (0; \infty),
\]

is \( L^2(\mu) \)-bounded, then all 1-dimensional CZOs associated with odd and sufficiently smooth kernels are also \( L^2(\mu) \)-bounded. We refer the reader to [18, Sections 1 and 12] for the more precise description of what we mean by “sufficiently smooth kernels”.

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Indeed, it follows from (18) and (22) with \((n, N) = (1, 2)\) that for any \(t\) as in (37) and any cube \(Q \subset \mathbb{C}\), one has
\[
\|T_{k_1, \epsilon} \chi_Q\|_{L^2(\mu|Q)} \leq C(t)(\|T_{k_1, \epsilon} \chi_Q\|_{L^2(\mu|Q)} + \sqrt{\mu(Q)}), \quad C(t) > 0,
\]
where \(T_{k_1}\), as we have already mentioned before, is the CZO associated with the real part of the Cauchy kernel, i.e., with the Cauchy kernel, up to a constant. Using the \(T1\) Theorem from [19, Theorem 9.40], we conclude that the \(L^2(\mu)\)-boundedness of \(T_{k_1}\) with \(t\) as in (37) implies that the Cauchy transform is \(L^2(\mu)\)-bounded. Furthermore, as proved in [18], if the Cauchy transform is \(L^2(\mu)\)-bounded, then all 1-dimensional CZOs associated with odd and sufficiently smooth kernels are also \(L^2(\mu)\)-bounded.

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