PARABOLIC VECTOR BUNDLES OVER ORBIFOLD KLEIN SURFACES

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Abstract. Real and quaternionic parabolic vector bundles over a \( n \)-pointed Riemann surface equipped with an anti-holomorphic involution are studied.

1. Introduction

Holomorphic vector bundles on compact Riemann surfaces is a rich topic that started with the papers of Weil and Atiyah [We], [At] and continues to be a very active area of research. The crucial notion of stability was introduced by Mumford [Mu]. Inspired by it Narasimhan and Seshadri proved that stable vector bundles of degree zero of a compact Riemann surface correspond to irreducible unitary representations of the fundamental group of the surface [NS].

Mehta and Seshadri introduced the notion of parabolic vector bundles, [MS], and proved an analog of the above mentioned theorem of [NS] for the unitary representations of the fundamental group of a punctured Riemann surface. More precisely, they proved that given a finite subset \( S \) of a compact Riemann surface \( X \), irreducible representations of \( \pi_1(X \setminus S) \) in \( U(r) \) correspond to the stable parabolic vector bundles on \( X \) of rank \( r \) and parabolic degree zero with parabolic structure over \( S \).

A Klein surface is a compact Riemann surface equipped with an anti-holomorphic involution. Just as compact Riemann surfaces correspond to smooth complex projective curves, Klein surfaces correspond to geometrically irreducible complex projective curves defined over the field of real numbers. Let \( X \) be compact connected Riemann surface equipped with an anti-holomorphic involution \( \sigma \). A real vector bundle on \((X, \sigma)\) is a holomorphic vector bundle on \( X \) equipped with an anti-holomorphic lift \( \tau \) of \( \sigma \) such that \( \tau^2 =\text{Id} \). A quaternionic vector bundle on \((X, \sigma)\) is a holomorphic vector bundle on \( X \) equipped with an anti-holomorphic lift \( \tau \) of \( \sigma \) such that \( \tau^2 =-\text{Id} \).

Real and quaternionic vector bundles on Klein surfaces are extensively investigated. Our aim here is to study the parabolic analog of them.

Given \((X, \sigma)\) as above, and a finite subset \( S \subset X \) with \( \sigma(S) = S \), set \( X' \in X \setminus S \), and fix a point \( x_0 \in X' \) not preserved by \( \sigma \). Then there is a natural extension \( \Gamma(X', x_0) \) of \( \pi_1(X', x_0) \) with cokernel \( \mathbb{Z}/2\mathbb{Z} \). There is also an extension \( \tilde{U}(r) \) of \( U(r) \) with cokernel \( \mathbb{Z}/2\mathbb{Z} \). A \( \sigma \)-homomorphism from \( \Gamma(X', x_0) \) to \( \tilde{U}(r) \) is by definition a group homomorphism \( \Gamma(X', x_0) \to \tilde{U}(r) \) that induces the identity on \( \mathbb{Z}/2\mathbb{Z} \). Quaternionic-homomorphisms can be defined similarly (see Remark 3.1).

We prove the following:

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Theorem 1.1. There is a natural bijective correspondence between the equivalence classes of $\sigma$-homomorphisms $\Gamma(X', x_0) \rightarrow \tilde{U}(r)$ and polystable real parabolic vector bundles of rank $r$ and parabolic degree zero. Moreover, this correspondence takes the irreducible $\sigma$-homomorphisms surjectively to the stable real parabolic bundles.

There is a natural bijective correspondence between the equivalence classes of quaternionic-homomorphisms $\Gamma(X', x_0) \rightarrow \tilde{U}(r)$ and polystable quaternionic parabolic vector bundles of rank $r$ and parabolic degree zero. Moreover, this correspondence takes the irreducible quaternionic-homomorphisms surjectively to the stable quaternionic parabolic bundles.

A similar theorem is proved for compact type real, as well as quaternionic, parabolic vector bundles (see Theorem 3.3).

2. Parabolic vector bundles and stability

2.1. Real and quaternionic parabolic vector bundles. Let $X$ be a compact connected Riemann surface; the almost complex structure on $X$ will be denoted by $J_X$. Let

$$\sigma : X \rightarrow X$$

be a diffeomorphism satisfying the following two conditions:

1. $\sigma$ is anti-holomorphic, meaning it takes $J_X$ to $-J_X$, or equivalently, $J_X \circ d\sigma = -(d\sigma) \circ J_X$, and
2. $\sigma \circ \sigma = \text{Id}_X$.

Fix a finite subset

$$S = \{x_1, \cdots, x_d\} \subset X$$

such that $\sigma(S) = S$. Note that $\sigma$ need not fix the points of $S$ individually.

Let $E$ be a holomorphic vector bundle on $X$. A quasiparabolic structure on $S$ is a strictly decreasing filtration of subspaces

$$E_{x_j} = E^1_{x_j} \supseteq E^2_{x_j} \supseteq \cdots \supseteq E^{n_j}_{x_j} \supseteq E^{n_j+1}_{x_j} = 0$$

for every $1 \leq j \leq d$. A parabolic structure on $E$ over $S$ is a quasiparabolic structure as above together with a decreasing sequence of real numbers

$$0 \leq \alpha^1_j < \alpha^2_j < \cdots < \alpha^{n_j}_j < 1, \quad 1 \leq j \leq d;$$

the real number $\alpha^i_j$ is called the parabolic weight of the subspace $E^i_{x_j}$ in the quasiparabolic filtration. The multiplicity of a parabolic weight $\alpha^i_j$ at $x_j$ is defined to be the dimension of the complex vector space $E^i_{x_j}/E^{i+1}_{x_j}$. A parabolic vector bundle is a vector bundle with a parabolic structure. (See [MS, Definition 1.5].)

For notational convenience, a parabolic vector bundle $(E, \{E^i_j\}, \{\alpha^i_j\})$ as above will also be denoted by $E_s$.

Let $E$ be a holomorphic vector bundle on $X$ of rank $r$. Let $\bar{E}$ be the $C^\infty$ complex vector bundle of rank $r$ on $X$ whose underlying real vector bundle of rank $2r$ is the real vector bundle underlying $E$, while multiplication by $\sqrt{-1}$ on the fibers of $\bar{E}$ coincides with the multiplication by $-\sqrt{-1}$ on the fibers of $E$. The holomorphic structure on $\bar{E}$ induces a holomorphic structure on $\sigma^*\bar{E}$. This holomorphic structure on $\sigma^*\bar{E}$ uniquely determined by the condition that a section of $\bar{E}$ defined over an open subset $U \subset X$ is holomorphic if and only if the corresponding section of $\bar{E}$ over $\sigma(U)$ is holomorphic.

Let $E_s = (E, \{E^i_j\}, \{\alpha^i_j\})$ be a parabolic bundle. Then for any $x_j \in S$, the filtration $\{E^i_j\}$ of $E_{x_j}$ induces a filtration of the fiber $(\sigma^*\bar{E})_{\sigma(x_j)}$ using the conjugate linear identification of $E_{x_j}$ with $(\sigma^*\bar{E})_{\sigma(x_j)}$. For any $j$, the parabolic weights $\{\alpha^i_j\}_{i=1}^{n_j}$ associated to $\{E^i_j\}_{i=1}^{n_j}$ can be considered as parabolic weights associated to...
this filtration of \((\sigma^*E)_x\). The new parabolic vector bundle obtained this way will be denoted by \(\sigma^*E\).

A real parabolic vector bundle on \(X\) is a parabolic bundle \(E_\sigma = (E, \{E_j\}, \{\alpha_j\})\) together with a holomorphic isomorphism

\[\tau : E \rightarrow \sigma^*E\]

such that

1. \(\tau\) produces an isomorphism of the parabolic vector bundle \(E_\sigma\) with the parabolic vector bundle \(\sigma^*E_\sigma\), and
2. \(\tau \circ \tau = \text{Id}_E\).

A quaternionic parabolic vector bundle on \(X\) is a parabolic bundle \(E_\sigma\) with a holomorphic isomorphism \(\tau : E \rightarrow \sigma^*E\) such that

1. \(\tau\) produces an isomorphism of the parabolic vector bundle \(E_\sigma\) with the parabolic vector bundle \(\sigma^*E_\sigma\), and
2. \(\tau \circ \tau = -\text{Id}_E\).

Let \(E_\sigma^*\) be the parabolic dual of \(E_\sigma\) \([\text{Yo}, \text{Section 3}], \text{[Bis, p. 309]}\). The holomorphic vector bundle underlying the parabolic vector bundle \(E_\sigma\) will be denoted by \(E_\sigma^*\). Note that \(E_\sigma^0\) is a subsheaf of \(E_\sigma^*\), and the inclusion map \(E_\sigma^0 \hookrightarrow E_\sigma^*\) is an isomorphism over \(X \setminus S\). This inclusion fails to be an isomorphism over \(x_j \in S\) if there is any nonzero parabolic weight of \(E_\sigma^*\) at \(x_j\).

A compact type real parabolic vector bundle on \(X\) is a parabolic bundle \(E_\sigma\) together with a holomorphic homomorphism \(\tau : E \rightarrow \sigma^*E\) such that

1. \(\tau\) produces an isomorphism of the parabolic vector bundle \(E_\sigma\) with the parabolic vector bundle \(\sigma^*E_\sigma\), and
2. \((\sigma^*\tau^*)_\circ \tau = \text{Id}_E\).

Since \(\sigma^*E_\sigma^0\) is in general a proper subsheaf of \(\sigma^*E_\sigma\), the homomorphism \(\tau\) is not an isomorphism in general. Note that \(\tau\) is an isomorphism over \(X \setminus S\) because the vector bundle underlying \(\sigma^*E_\sigma\) is identified with \(\sigma^*E\) over \(X \setminus S\).

A compact type quaternionic parabolic vector bundle on \(X\) is a parabolic bundle \(E_\sigma\) with a holomorphic homomorphism \(\tau : E \rightarrow \sigma^*E\) such that

1. \(\tau\) produces an isomorphism of the parabolic vector bundle \(E_\sigma\) with the parabolic vector bundle \(\sigma^*E_\sigma\), and
2. \((\sigma^*\tau^*)_\circ \tau = -\text{Id}_E\).

### 2.2. Parabolic semistability

The parabolic degree of a parabolic vector bundle \(E_\sigma = (E, \{E_j\}, \{\alpha_j\})\) is defined to be

\[
\text{par-deg}(E_\sigma) = \text{degree}(E) + \sum_{j=1}^d \sum_{i=1}^{n_j} \alpha_j \cdot \dim(E_j/E_{j+1})
\]

\([\text{MS, Definition 1.11}]\). Take any holomorphic subbundle \(F \subset E\). For each \(x_j \in S\), the fiber \(F_{x_j}\) has a filtration obtained by intersecting the quasiparabolic filtration
of $E_{x_j}$ with the subspace $F_{x_j}$. The parabolic weight of a subspace $V \subset F_{x_j}$ in this filtration is the maximum of the numbers
\[ \{ \alpha_j^i \mid V \subset E_j^i \cap F_{x_j} \}. \]
This parabolic structure on $F$ will be denoted by $F_\ast$.

A parabolic vector bundle $E_\ast = (E, \{ E_j^i \}, \{ \alpha_j^i \})$ is called stable (respectively, semistable) if for all subbundles $F \subseteq E$ of positive rank, the inequality
\[ \frac{\text{par-deg}(F_\ast)}{\text{rank}(F)} < \frac{\text{par-deg}(E_\ast)}{\text{rank}(E)} \quad \left( \text{respectively,} \quad \frac{\text{par-deg}(F_\ast)}{\text{rank}(F)} \leq \frac{\text{par-deg}(E_\ast)}{\text{rank}(E)} \right) \]
holds [MS, Definition 1.13]. A parabolic vector bundle $E_\ast$ is called polystable if
(1) it is parabolic semistable, and
(2) it is a direct sum of parabolic stable vector bundles (see [Yo], [Bis] for direct sum of parabolic vector bundles).

A stable parabolic bundle is simple, meaning any holomorphic automorphism of it is a constant nonzero scalar multiplication.

A real or quaternionic parabolic vector bundle $(E_\ast, \tau) = ((E, \{ E_j^i \}, \{ \alpha_j^i \}), \tau)$ is called stable (respectively, semistable) if for all subbundles $F \subsetneq E$ of positive rank with $\tau(F) = \sigma^*F$, the inequality
\[ \frac{\text{par-deg}(F_\ast)}{\text{rank}(F)} < \frac{\text{par-deg}(E_\ast)}{\text{rank}(E)} \quad \left( \text{respectively,} \quad \frac{\text{par-deg}(F_\ast)}{\text{rank}(F)} \leq \frac{\text{par-deg}(E_\ast)}{\text{rank}(E)} \right) \]
holds. A real (respectively, quaternionic) parabolic vector bundle $(E_\ast, \tau)$ is called polystable if it is parabolic semistable and it is a direct sum of real (respectively, quaternionic) parabolic stable vector bundles. Any automorphism of a stable real or quaternionic parabolic bundle is multiplication by a constant nonzero real number.

A compact type real or quaternionic parabolic vector bundle
\[ (E_\ast, \tau) = ((E, \{ E_j^i \}, \{ \alpha_j^i \}), \tau) \]
is called stable (respectively, semistable) if for all subbundles $F \subseteq E$ of positive rank such that $\tau(F)$ annihilates $F$, the inequality
\[ \frac{\text{par-deg}(F_\ast)}{\text{rank}(F)} < \frac{\text{par-deg}(E_\ast)}{\text{rank}(E)} \quad \left( \text{respectively,} \quad \frac{\text{par-deg}(F_\ast)}{\text{rank}(F)} \leq \frac{\text{par-deg}(E_\ast)}{\text{rank}(E)} \right) \]
holds. A compact type real (respectively, quaternionic) parabolic vector bundle $(E_\ast, \tau)$ is called polystable if it is parabolic semistable and it is a direct sum of compact type real (respectively, quaternionic) parabolic stable vector bundles.

**Lemma 2.1.** A real or quaternionic parabolic vector bundle $((E, \{ E_j^i \}, \{ \alpha_j^i \}), \tau)$ is semistable (respectively, polystable) if and only if the parabolic vector bundle $(E, \{ E_j^i \}, \{ \alpha_j^i \})$ is semistable (respectively, polystable).

A compact type real or quaternionic parabolic vector bundle $((E, \{ E_j^i \}, \{ \alpha_j^i \}), \tau)$ is semistable (respectively, polystable) if and only if the parabolic vector bundle $(E, \{ E_j^i \}, \{ \alpha_j^i \})$ is semistable (respectively, polystable).

**Proof.** Given a real or quaternionic parabolic vector bundle
\[ ((E, \{ E_j^i \}, \{ \alpha_j^i \}), \tau), \]
if $(E, \{ E_j^i \}, \{ \alpha_j^i \})$ is semistable (respectively, polystable), then $(E, \{ E_j^i \}, \{ \alpha_j^i \})$, $\tau)$ is evidently semistable (respectively, polystable). If $((E, \{ E_j^i \}, \{ \alpha_j^i \}), \tau)$ is semistable, then from the uniqueness of the Harder–Narasimhan filtration of the parabolic vector bundle $(E, \{ E_j^i \}, \{ \alpha_j^i \})$ it follows that the Harder–Narasimhan filtration is actually preserved by $\tau$. Consequently, $(E, \{ E_j^i \}, \{ \alpha_j^i \})$ is semistable.
If \( (E, \{E_j\}, \{\alpha^j_i\}) \) is polystable, then we know that \( (E, \{E_j\}, \{\alpha^j_i\}) \) is semistable because \( (E, \{E_j\}, \{\alpha^j_i\}), \tau \) is semistable. So in that case \( (E, \{E_j\}, \{\alpha^j_i\}) \) has a unique maximal polystable parabolic subbundle \( F \) with parabolic slope same as that of \( (E, \{E_j\}, \{\alpha^j_i\}) \) \[MS\] page 23, Lemma 1.5.5]. Now from the uniqueness of this maximal polystable parabolic subbundle \( F \) it follows that it is invariant under \( \tau \). Therefore, this subbundle \( F \) has a \( \tau \) invariant complement \( F' \), because \( ((E, \{E_j\}, \{\alpha^j_i\}), \tau) \) is polystable. Note that if \( F' \) is nonzero, then the maximality of \( F \) is contradicted, because \( F' \) also has a unique maximal polystable parabolic subbundle \( F \) with parabolic slope same as that of \( (E, \{E_j\}, \{\alpha^j_i\}) \). Therefore, we conclude that \( F' = 0 \). This implies that \( (E, \{E_j\}, \{\alpha^j_i\}) \) is polystable.

Very similar arguments apply for the compact real or quaternionic cases. □

3. Unitary connections and parabolic bundles

For notational convenience, the complement \( X \setminus S \) will be denoted by \( X' \). Fix a base point \( x_0 \in X' \). Let

\[ \varrho : \pi_1(X', x_0) \longrightarrow U(r) \]

be a homomorphism. It produces a complex vector bundle \( E(\varrho) \longrightarrow X' \) of rank \( r \) equipped with a flat unitary connection. In particular, \( E(\varrho) \) has a holomorphic structure. The Deligne extension of \( E(\varrho) \) to \( X \) is a holomorphic vector bundle equipped with a logarithmic connection \[DS, MS\] Section 1]. This holomorphic vector bundle on \( X \) has a natural parabolic structure. The parabolic weights at \( x_j \) and their multiplicities are given by the eigenvalues of the image under \( \varrho \) of an element of \( \pi_1(X', x_0) \) produced by an oriented loop around \( x_j \); this element of \( \pi_1(X', x_0) \) is not unique, but its conjugacy class is so, and hence the eigenvalues and their multiplicities are also uniquely determined. More precisely, the quasiparabolic filtration at \( x_j \) and the parabolic weights at \( x_j \) are given by the residue at \( x_j \) of the logarithmic connection.

This parabolic vector bundle given by \( \varrho \), which we will denote by \( E^0(\varrho) \), is polystable of parabolic degree zero; moreover, the parabolic vector bundle \( E^0(\varrho) \) is stable if and only if the representation \( \varrho \) is irreducible \[MS\] Proposition 1.12.

Conversely, any polystable parabolic vector bundle of rank \( r \) and parabolic degree zero is given by a homomorphism from \( \pi_1(X', x_0) \) to \( U(r) \) \[MS, Biq\]. Therefore, a parabolic vector bundle \( E_* = (E, \{E_j\}, \{\alpha^j_i\}) \) of parabolic degree zero is polystable if and only if \( E|_{X \setminus S} \) has a unitary flat connection \( \nabla \) compatible with the parabolic structure over \( S \). The compatibility condition in question means that the Deligne extension for \( \nabla \) is \( E \), and the parabolic structure on \( E \) over any \( x_j \in S \) is given by the residue at \( x_j \) of the logarithmic connection.

The unitary structure is not unique. Indeed, if \( h \) is a flat Hermitian structure on \( E|_{X \setminus S} \) compatible with the parabolic structure of \( E_* \), then the Hermitian structure \( c \cdot h \) also has this property for every positive real number \( c \). On the other hand, any two Hermitian structures on \( E|_{X \setminus S} \) satisfying the above conditions differ by a holomorphic automorphism of the parabolic vector bundle \( E_* \). In particular, if \( E_* \) is stable, then any two Hermitian structures on \( E|_{X \setminus S} \) satisfying the above conditions differ by multiplication with a constant positive real number (recall that a stable parabolic bundle is simple).

Although, the flat Hermitian structure on a polystable parabolic vector bundle \( E_* \) of parabolic degree zero is not unique, the flat unitary connection is unique. In fact, if \( E_* \) and \( E'_* \) are two polystable parabolic vector bundles of parabolic degree zero (they may be of different ranks), and

\[ \phi : E_* \longrightarrow E'_* \]
is a parabolic homomorphism, then \( \phi \) is flat with respect to the unitary flat connections for \( E \) and \( E' \); in particular, kernel(\( \phi \)) (respectively, image(\( \phi \))) is preserved by the unitary flat connection for \( E \) (respectively, \( E' \)).

Now take \( x_0 \in X' \) such that \( \sigma(x_0) \neq x_0 \). The homotopy classes of paths starting from \( x_0 \) and ending in the two point set \( \{ x_0, \sigma(x_0) \} \) has a natural structure of a group [BHHS Section 5.1]; we will denote this group by \( \Gamma(X', x_0) \). This group \( \Gamma(X', x_0) \) fits in a short exact sequence
\[
1 \rightarrow \pi_1(X', x_0) \rightarrow \Gamma(X', x_0) \xrightarrow{\beta} \mathbb{Z}/2\mathbb{Z} \rightarrow 1
\]  
(3.1)
(see [BHHS p. 211, (5.2)]). We recall that \( \beta(z) \) is 0 (respectively, 1) if \( z \) ends at \( x_0 \) (respectively, \( \sigma(x_0) \)).

Consider \( C^r \) with the standard Hermitian structure \( h_0 \). So \( U(r) \) is the group of all linear automorphisms of \( C^r \) that preserve \( h_0 \). Let \( U'(r) \) be the space of conjugate linear isomorphisms \( A \) of \( C^r \) such that \( h_0(A(x), A(y)) = h_0(x, y) \) for all \( v \in C^r \).

The \( (\text{disjoint}) \) union
\[
\hat{U}(r) = U(r) \cup U'(r)
\]
is a group under composition. We note that \( \hat{U}(r) \) has a homomorphism to \( \mathbb{Z}/2\mathbb{Z} \) that sends \( U(r) \) to 0 and \( U'(r) \) to 1. The resulting short exact sequence of groups
\[
1 \rightarrow U(r) \rightarrow \hat{U}(r) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1
\]
is right-split.

A \( \sigma \)-homomorphism is a group homomorphism
\[
\varrho : \Gamma(X', x_0) \rightarrow \hat{U}(r)
\]
such that \( \varrho(\pi_1(X', x_0)) \subset U(r) \) and \( \varrho(\beta^{-1}(1)) \subset U'(r) \), where \( \beta \) is the homomorphism in (3.1). A \( \sigma \)-homomorphism \( \varrho \) is called irreducible if the action of \( \varrho(\Gamma(X', x_0)) \) on \( C^r \) does not preserve any nonzero proper complex subspace of \( C^r \). Two \( \sigma \)-homomorphisms \( \varrho_1 \) and \( \varrho_2 \) from \( \Gamma(X', x_0) \) to \( \hat{U}(r) \) are called equivalent if there is an element \( B \in U(r) \) such that \( \varrho_1(z) = B^{-1} \varrho_2(z)B \) for all \( z \in \Gamma(X', x_0) \).

A quaternionic-homomorphism is a map
\[
\varrho : \Gamma(X', x_0) \rightarrow \hat{U}(r)
\]
such that
1. \( \varrho(\pi_1(X', x_0)) \subset U(r) \).
2. \( \varrho(\beta^{-1}(1)) \subset U'(r) \), and
3. \( \varrho(yz) = (-1)^{\varrho(y)\varrho(z)} \varrho(y)\varrho(z) \) for all \( y, z \in \Gamma(X', x_0) \).

Note that \( \varrho \) is not a homomorphism of groups. Exactly as in the case of \( \sigma \)-homomorphisms, a quaternionic-homomorphism \( \varrho \) as above is called irreducible if no nonzero proper complex subspace of \( C^r \) is preserved by \( \varrho(\Gamma(X', x_0)) \). Similarly, two quaternionic-homomorphism are called equivalent if they differ by conjugation by some element of \( U(r) \).

**Remark 3.1.** It is possible, as in [Sch], to replace quaternionic homomorphism \( \varrho : \Gamma(X', x_0) \rightarrow \hat{U}(r) \) by \( \sigma \)-homomorphisms \( \varrho : \Gamma(X', x_0) \rightarrow W(r) \), where \( W(r) \) fits in the (unique up to isomorphism) non-splittable short exact sequence
\[
1 \rightarrow U(r) \rightarrow W(r) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1.
\]

**Theorem 3.2.** There is a natural bijective correspondence between the equivalence classes of \( \sigma \)-homomorphisms \( \Gamma(X', x_0) \rightarrow \hat{U}(r) \) and polystable real parabolic vector bundles of rank \( r \) and parabolic degree zero. Moreover, this correspondence takes the irreducible \( \sigma \)-homomorphisms surjectively to the stable real parabolic bundles.
There is a natural bijective correspondence between the equivalence classes of quaternionic-homomorphisms \( \Gamma(X', x_0) \to \hat{U}(r) \) and polystable quaternionic parabolic vector bundles of rank \( r \) and parabolic degree zero. Moreover, this correspondence takes the irreducible quaternionic-homomorphisms surjectively to the stable quaternionic parabolic bundles.

Proof. Take a polystable real parabolic vector bundle
\[
(E, \tau) = ((E, \{E_i\}, \{\alpha_i\}), \tau)
\]
of rank \( r \) and parabolic degree zero. From Lemma 2.1, we know that the parabolic vector bundle \( E \) is polystable. Therefore, the restriction \( E|_{X \setminus S} \) has a unique unitary flat connection \( \nabla \) compatible with the parabolic structure. So \( \sigma^*\nabla \) is the unitary flat connection on \( \sigma^*E \). Therefore, this isomorphism \( \tau \) between parabolic vector bundles \( E \) and \( \sigma^*E \) preserves the connection \( \nabla \). Since \( \tau \) is also an involution, from this it follows that the conjugate linear map
\[
\tau(y) : E_y \to E_{\sigma(y)}
\]
is an isometry for all \( y \in X \setminus S \).

Next, for any \( \gamma \in \pi_1(X', x_0) \), by taking parallel translation along \( \gamma \) with respect to \( \nabla \), we get the monodromy homomorphism
\[
g' : \pi_1(X', x_0) \to U(E_{x_0}),
\]
where \( U(E_{x_0}) \) is the group of \( \mathbb{C} \)-linear unitary automorphisms of the fiber \( E_{x_0} \).

Now, for any \( \gamma \in \beta^{-1}(1) \subset \Gamma(X', x_0) \), where \( \beta \) is the projection in (3.2), by taking parallel translation along \( \gamma \) with respect to \( \nabla \), we get a \( \mathbb{C} \)-linear isometry
\[
T_\gamma : E_{x_0} \to E_{\sigma(x_0)}.
\]
Therefore, \( \tau(\sigma(x_0)) \circ T_\gamma \) is a conjugate linear isometry of \( E_{x_0} \), because \( \tau(\sigma(x_0)) \) in (3.2) is a conjugate linear isometry.

Combining these, we have a map
\[
g : \Gamma(X', x_0) \to \hat{U}(E_{\sigma(x_0)})
\]
that sends any \( \gamma \in \pi_1(X', x_0) \) to \( g'(\gamma) \), and sends any \( \gamma \in \beta^{-1}(1) \) to \( \tau(\sigma(x_0)) \circ T_\gamma \). It is straightforward to check that \( g \) is a \( \sigma \)-homomorphism; we may identify \( \hat{U}(E_{x_0}) \) with \( \hat{U}(r) \) by choosing an orthonormal basis of \( E_{x_0} \). A different choice of an orthonormal basis of \( E_{x_0} \) would give a new \( \sigma \)-homomorphism which is equivalent to this \( g \). As noted earlier, any automorphism of \( E \) preserves the unitary flat connection \( \nabla \).

Combining these, we get a map from the isomorphism classes of polystable real parabolic vector bundles, of rank \( r \) and parabolic degree zero, to the equivalence classes of \( \sigma \)-homomorphisms \( \Gamma(X', x_0) \to \hat{U}(r) \). This map is a bijection. It’s inverse can be constructed as follows.

Take any \( \sigma \)-homomorphisms
\[
g : \Gamma(X', x_0) \to \hat{U}(r).
\]
The restriction of \( g \) to the subgroup \( \pi_1(X', x_0) \) produces polystable parabolic vector bundle \( (E, \{E_i\}, \{\alpha_i\}) \), of rank \( r \) and parabolic degree zero, equipped with a flat unitary structure over \( X' \). The fiber \( E_{x_0} \) equipped with the Hermitian structure is identified with \( \mathbb{C}^r \) with the standard Hermitian structure. Take any \( \gamma \in \beta^{-1}(1) \). Let
\[
T_\gamma : E_{x_0} \to E_{\sigma(x_0)}
\]
be the parallel transport, along \( \gamma \), with respect to the unitary flat connection on \( E|_{X'} \). Now consider the isomorphism \( T_\gamma \circ g(\gamma)^{-1} \) of \( E_{x_0} \) with \( E_{\sigma(x_0)} \) (since the
Hilbert space $E_{x_0}$ is identified with $\mathbb{C}^r$ with the standard inner product, we consider $g(\gamma)$ as a self-map of $E_{x_0}$). It is straight-forward to check that

- $T_\gamma \circ g(\gamma)^{-1}$ is conjugate linear,
- it is an isometry, and
- it is independent of the choice of $\gamma$.

For any point $x \in X'$, choose a path $\phi_x$ in $X'$ from $x$ to $x_0$. Let

$$\tilde{\phi}_x : E_x \to E_{x_0}$$

be the parallel transport, along $\phi_x$, for the flat unitary connection on $E|_{X'}$. Similarly, let

$$\tilde{\sigma}(\phi)_x : E_{\sigma(x)} \to E_{\sigma(x_0)}$$

be the parallel transport, along $\sigma(\phi_x)$, for the flat unitary connection on $E|_{X'}$. Now define

$$\tau(x) : E_x \to E_{\sigma(x)}, \quad v \mapsto (\tilde{\sigma}(\phi)_x)^{-1} \circ (T_\gamma \circ g(\gamma)^{-1}) \circ \tilde{\phi}_x(v).$$

It is straight-forward to check that

- $\tau(x)$ is a conjugate linear isometry,
- it is independent of the choice of the path $\phi_x$, and
- $\tau$ extends to an isomorphism of the parabolic vector bundle $E_* := (E, \{E_j\}, \{\alpha_j^i\})$ with the parabolic vector bundle $\sigma^* E_*$. In fact, $\tau$ produces a real parabolic structure on $E_*$. This way we get the inverse of the previously constructed map from the isomorphism classes of polystable real parabolic vector bundles, of rank $r$ and parabolic degree zero, to the equivalence classes of $\sigma$-homomorphisms $\Gamma(X', x_0) \to \hat{U}(r)$.

A polystable real parabolic bundle is stable if and only if every automorphism of it is the multiplication by a nonzero scalar. On the other hand, a $\sigma$-homomorphism $\varrho : \Gamma(X', x_0) \to \hat{U}(E_{\sigma(x_0)})$ is irreducible if and only if any automorphism of $\mathbb{C}^r$ commuting with $\varrho(\Gamma(X', x_0))$ is a scalar multiplication. From these it follows that the above bijection takes stable real parabolic vector bundles of rank $r$ and parabolic degree zero surjectively to the equivalence classes of irreducible $\sigma$-homomorphisms $\Gamma(X', x_0) \to \hat{U}(r)$.

The proof for the quaternionic parabolic vector bundles and quaternionic-homomorphisms is very similar. We omit the details. \qed

Consider the product group $U(r) \times (\mathbb{Z}/2\mathbb{Z})$. A homomorphism

$$\varrho : \Gamma(X', x_0) \to U(r) \times (\mathbb{Z}/2\mathbb{Z})$$

will be called of compact type if $\varrho(\pi_1(X', x_0)) \subseteq U(r)$, and $\varrho$ fits in the commutative diagram

$$
\begin{array}{cccccc}
0 & \to & \pi_1(X', x_0) & \to & \Gamma(X', x_0) & \to & \mathbb{Z}/2\mathbb{Z} & \to & 0 \\
\uparrow{\varrho} & & \uparrow{\beta} & & \uparrow{\varrho} & & \uparrow{\alpha} & & \uparrow{1} \\
0 & \to & U(r) & \to & U(r) \times (\mathbb{Z}/2\mathbb{Z}) & \to & \mathbb{Z}/2\mathbb{Z} & \to & 0
\end{array}
$$

of homomorphisms.

A map

$$\varrho : \Gamma(X', x_0) \to U(r) \times (\mathbb{Z}/2\mathbb{Z})$$

will be called a compact quaternionic homomorphism if

1. $\varrho(\pi_1(X', x_0)) \subseteq U(r) = U(r) \times \{1\}$,
2. $\varrho(\beta^{-1}(1)) \subseteq U(r) \times \{1\}$, and
3. $\varrho(yz) = (-1)^{\beta(yz)} \varrho(y) \varrho(z)$ for all $y, z \in \Gamma(X', x_0)$. 

A compact type homomorphism or a compact quaternionic type homomorphism \( \varrho : \Gamma(X', x_0) \to U(r) \times \mathbb{Z}/2\mathbb{Z} \) will be called irreducible, if the standard action of \( \varrho(\Gamma(X', x_0)) \) on \( C' \) does not preserve any nonzero proper complex subspace. Two such homomorphisms \( \varrho_1 \) and \( \varrho_2 \) will be called equivalent, if there is an element \( B \in U(r) \) such that \( \varrho_1(z) = B^{-1}\varrho_2(z)B \) for all \( z \in \Gamma(X', x_0) \).

Following is the compact analog of Theorem 3.2:

**Theorem 3.3.** There is a natural bijective correspondence between the equivalence classes of compact homomorphisms \( \Gamma(X', x_0) \to U(r) \times \mathbb{Z}/2\mathbb{Z} \) and polystable compact type real parabolic vector bundles of rank \( r \) and parabolic degree zero. Moreover, this correspondence takes the irreducible compact homomorphisms surjectively to the stable compact type real parabolic bundles.

There is a natural bijective correspondence between the equivalence classes of compact quaternionic homomorphisms \( \Gamma(X', x_0) \to \tilde{U}(r) \) and polystable compact type quaternionic parabolic vector bundles of rank \( r \) and parabolic degree zero. Moreover, this correspondence takes the irreducible compact quaternionic homomorphisms surjectively to the stable compact type quaternionic parabolic bundles.

**Proof.** A Hermitian form on a finite dimensional complex vector space \( V \) identifies \( V \) with \( V^* \). Hence a Hermitian structure on a complex vector bundle \( W \) identifies \( W \) with \( W^* \). Incorporating this fact, the line of proof of Theorem 3.2 works here as well. \( \square \)

**References**

[At] M. F. Atiyah, Vector bundles over an elliptic curve, *Proc. London Math. Soc.* 7 (1957), 414–452.

[Biq] O. Biquard, Fibrés paraboliques stables et connexions singulières plates, *Bull. Soc. Math. France* 119 (1991), 231–257.

[Bis] I. Biswas, On the principal bundles with parabolic structure, *Jour. Math. Kyoto Univ.* 43 (2003), 305–332.

[BHH] I. Biswas, J. Huismann and J. Hurtubise, The moduli space of stable vector bundles over a real algebraic curve, *Math. Ann.* 347 (2010), 201–233.

[De] P. Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, 163, Springer-Verlag.

[HL] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, E31, Friedr. Vieweg & Sohn, Braunschweig, 1997.

[MS] V. B. Mehta and C. S. Seshadri, Moduli of vector bundles on curves with parabolic structures, *Math. Ann.* 248 (1980), 205–239.

[Mu] D. Mumford, *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 34, Springer-Verlag, Berlin-New York, 1965.

[NS] M. S. Narasimhan and C. S. Seshadri, Stable and unitary vector bundles on a compact Riemann surface, *Ann. of Math.* 82 (1965), 540–567.

[Sch] P. Schaffhauser. On the Narasimhan-Seshadri correspondence for Real and Quaternionic vector bundles. *J. Differential Geom.*, 105(1):119–162, 2017.

[Yo] K. Yokogawa, Infinitesimal deformation of parabolic Higgs sheaves, *Int. Jour. Math.* 6 (1995), 125–148.

[We] A. Weil, Généralisation des fonctions abéliennes, *Jour. Math. Pure Appl.* 17 (1938), 47–87.

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