Log adjunction: moduli part

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November 3, 2021

Abstract

Upper moduli part of adjunction is introduced and its basic property are discussed. The moduli part satisfies the BP in the case of rational multiplicities and is nef in the maximal case.

Usually we assume that the base field $k$ is algebraically closed of characteristic 0, e.g., $k = \mathbb{C}$.

We start from the divisorial part.

Definition 1. There are two equivalent approaches to define the divisorial part of adjunction or discriminant: due to Kawamata [Kaw] and due to Ambro [Am]. The definition of the divisorial part of adjunction $D_{\text{div}}$ on $Z$ supposes, that $X/Z$ is surjective (in codimension 1 over $Z$), and the following generic lc property

GLC: the pair $(X/Z, D)$ is lc over the generic points of the base $Z$.

The definition of its b-divisorial version $D_{\text{div}}$ of $Z$ supposes additionally that $X/Z$ is proper and $(X, D)$ is a log pair [PSh, 7.1 and Remark 7.7]. Notice also that $(D_{\text{div}})^{Z} = D_{\text{div}}$ and $D_{\text{div}} = \sum d_{W} W$, where $d_{W}$ are defined by [PSh, Construction 7.2 and Remark 7.7].

Recall the definition of the divisorial pull-back $f^{\circ}$ [PSh, 2.4]. Let $f: X \to Z$ be a surjective morphism of normal irreducible varieties (or algebraic spaces). Then the homomorphism of Weil $\mathbb{R}$-divisors

$$f^{\circ}: \text{Div}_{\mathbb{R}} Z \to \text{Div}_{\mathbb{R}} X$$

*Partially supported by NSF (grant DMS-0701465). 2010 Mathematical Subject Classification 14E30.
is uniquely linearly extended from the prime Weil divisors. For a prime Weil divisor $D$ on $Z$, put $f^o D \overset{\text{def}}{=} f^* D$ over the generic point of $D$ and 0 outside. Note that $D$ is Cartier near its generic point.

The homomorphism $f^o$ has a natural and unique extension on the $b\mathbb{R}$-divisors:

$$f^o : \text{Div}_\mathbb{R} Z \to \text{Div}_\mathbb{R} X,$$

such that for every morphism $f' : X' \to Z'$, birationally equivalent to $f$ over $Z$, and for every $b\mathbb{R}$-divisor $D \in \text{Div}_\mathbb{R} Z$, $f'^o (D_{Z'}) = (f^o D)_{X'}$ holds in codimension 1 over $Z'$. In other words, the difference $f'^o (D_{Z'}) - (f^o D)_{X'}$ is truly exceptional over $Z'$ [Sh00, Definition 3.2]. In general $f'^o (D_{Z'}) = (f^o D)_{X'}$ does not hold everywhere even for a Cartier $b$-divisor $D$, e.g., on prime truly exceptional divisors with respect to $f'$.

On the $\mathbb{R}$-Cartier $b\mathbb{R}$-divisors the pull-back $f^o$ is exactly well-known $f^*$ [Sh00, 7.2]:

$$f^o D = f^* D.$$

Indeed, by definition, for every birational base change $Z'$, $f'^o D = f'^* D$ in codimension 1 over $Z'$. Additionally, $b\mathbb{R}$-divisors $f'^o D$ and $f'^* D$ are vertical, and, for every vertical prime $b$-divisor $D$ of $X$, there exists a morphism $f' : X' \to Z'$, birationally equivalent to $f$ over $Z$ such that $D$ and $f'D$ are prime divisors on $X'$ and $Z'$ respectively.

Recall also the following properties.

**General properties of divisorial part of adjunction** (cf. [PSH] Lemma 7.4]).

Let $(X/Z, D)$ be a log pair as in Definition 4 and $W$ be a prime divisor on $Z$. The $\mathbb{R}$-divisor $D_{\text{div}}$ and $b\mathbb{R}$-divisor $D'_{\text{div}}$ satisfy the following properties:

1. birationality: $d_W = \text{mult}_W D_{\text{div}} = \text{mult}_W D'_{\text{div}}$ is independent of a crepant model of $(X, D)$ over $Z$ and $D_{\text{div}}$ is also independent of a model of $Z$;

2. semiadditivity: for any $\mathbb{R}$-Cartier divisor $\Delta$ on $Z$ the log pair $(X/Z, D')$ with $D' = D + f^* \Delta$ satisfies GLC and equalities $D'_{\text{div}} = D_{\text{div}} + \Delta, D'_{\text{div}} = D_{\text{div}} + \overline{\Delta}$ hold;

3. $(X, D)$ is lc (respectively klt) over the generic point of $W$ if and only if $d_W \leq 1$ (respectively $< 1$);

4. effectiveness: if $D$ is effective over the generic point of $W$ then $d_W \geq 0$; so, $D_{\text{div}} \geq 0$ if $D \geq 0$;
(5) rationality: if $D$ is a $\mathbb{Q}$-divisor then $D_{\text{div}}, D_{\text{div}}$ are respectively $\mathbb{Q}$-, $b$-$\mathbb{Q}$-divisors;

(6) boundary: if $D$ is an $\mathbb{R}$-(respectively $\mathbb{Q}$-)boundary then $D_{\text{div}}$ is an $\mathbb{R}$-(respectively $\mathbb{Q}$-)boundary; a similar statement holds for subboundaries (cf. (3) above).

Notice that in (3) the relative klt property over the generic point of $W$ means the klt property in prime $b$-divisors of $X$ with the center $W$.

**Proof.** Immediate by definition.

**Corollary 1.** Let $(X/Z, D)$ be a log pair under the adjunction assumption GLC. Suppose that BP holds for the divisorial part of adjunction $D_{\text{div}}$. Then the divisorial part of adjunction exactly preserves lc singularities:

1. $(X, D)$ is lc if and only if so does $(Z, D_{\text{div}})$;
2. $(X, D)$ is klt over $Z$ if and only if so does $(Z, D_{\text{div}})$.

We can omit BP. Then lc and klt properties on $Z$ are determined directly by the $b$-$\mathbb{R}$-divisor $D_{\text{div}}$.

**Proof.** The last phrase of the statement means that the pair $(Z, D_{\text{div}})$ is lc, if $\text{mult}_P D_{\text{div}} \leq 1$ for every prime $b$-divisor $P$ of $Z$, that is, $D_{\text{div}}$ is a $b$-subboundary. Respectively, the klt property over $Z$ means: $\text{mult}_P D < 1$ for every prime $b$-divisor $P$ of $X$, vertical over $Z$, where $D = B(X, D)$. So, if $D_{\text{div}}$ satisfies BP, stable over $Z$, then the lc property holds for $(Z, D_Z)$ with the trace $D_Z = (D_{\text{div}})_Z$ and the klt property holds for $(X, D)$ over $Z$.

The proof follows immediately from General property (3).

**Definition 2.** Let $(X/Z, D)$ be a log pair with proper surjective $X/Z$ and under GLC. Then a moduli part $\mathcal{M}^X$ is easily defined as a $b$-$\mathbb{R}$-divisor

$$\mathcal{M} \overset{\text{def}}{=} \mathcal{M}^X \overset{\text{def}}{=} D_{\text{mod}} \overset{\text{def}}{=} K + D - f^o(K_Z + D_{\text{div}}),$$

where $D$ is the codiscrepancy $b$-$\mathbb{R}$-divisor of $(X, D)$ and $K = K_X, K_Z$ are canonical $b$-divisors of $X$ and of $Z$ respectively. As a canonical $b$-divisor $K$ the moduli part $D_{\text{mod}}$ is defined up to a linear equivalence on $X$. 

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So, the log adjunction holds:

\[ K + D = f^o(K_Z + D_{\text{div}}) + D_{\text{mod}}. \]

Recall, that the codiscrepancy b-R-divisor \( D = B(X, D) = -A(X, D) \) in the sum \( K + D \) corresponds to the log pair \((X, D)\) and is \( K + D - K \), that is, \( K + D = K + D \). For a crepant pair \((X', D')\) of \((X, D)\), \( D' = D \).

Bold vs script denotes very special behaviour of a b-codiscrepancy and (anti)similar for a canonical b-divisor. All other b-R-divisors, including, b-R-Cartier ones will be denoted by calligraphic capital letters.

So, on every model of \( X' \) of \( X \) over \( Z \)

\[ M' \overset{\text{def}}{=} (M)_{X'} = D'_{\text{mod}} \overset{\text{def}}{=} (D'_{\text{mod}})_{X'} = K_{X'} + D' - f^o(K_Z + D_{\text{div}})_{X'}, \]

where \( D' = D_{X'} \) is the trace of \( D \) on \( X' \) and is a crepant divisor of \( D \) on \( X' \) if \((X', D')\) is a crepant log pair of \((X, D)\). Moreover, if a morphism \( f': X' \to Z' \) is birationally equivalent to \( f \) over \( Z \), then

\[ M' = K_{X'} + D' - f'^o(K_Z + D'_{\text{div}}) = K_{X'} + D' - f'^*(K_{Z'} + D'_{\text{div}}) \]

in codimension 1 over \( Z' \). This allows to calculate the moduli part \( M' \) immediately in codimension 1 over \( Z' \). In general, for truly exceptional divisors with respect to \( f' \), none of pull-backs \( f'^o, f'^* \) give an exact formula everywhere. However, subsequent blowups of the variety \( X' \) and of the base \( Z' \) allows to determine the moduli part. That is the meaning of b-(R-)divisors.

**Proposition 1.** b-R-Divisor \( D_{\text{div}} \) satisfies BP, or, equivalently, \( K_Z + D_{\text{div}} \) is an R-Cartier b-R-divisor if and only if the moduli part \( M \) is an R-Cartier b-R-divisor.

More precisely, let \( f': X' \to Z' \) be a proper morphism birationally equivalent to \( f \) over \( Z \) such that \( K_X + D \) is stable over \( X' \). Then if \( K_Z + D_{\text{div}} \) is an R-Cartier b-R-divisor, stable over \( Z' \), then \( M \) is an R-Cartier b-R-divisor, stable over model \( X' \), and \( M = \overline{M'} \), where

\[ M' = K_{X'} + D' - f'^*(K_{Z'} + D'_{\text{div}}). \]

Conversely, if \( M \) is an R-Cartier b-R-divisor, stable over the model \( X' \) and additionally \( f' \) has equidimensional fibers, then BP holds for \( D_{\text{div}} \) with stability over \( Z' \).
In the proposition we have two kinds of stabilities. For an $\mathbb{R}$-Cartier b-$\mathbb{R}$-divisor $D$ of $X$, the *stability over* a model $X'$ of $X$ means that $D_{X'}$ is $\mathbb{R}$-Cartier and $D = \overline{D_{X'}}$. Respectively, the *stability* of BP, for an b-$\mathbb{R}$-divisor $D$ over $X'$, means that $(X', D_{X'})$ is a log pair and $D = \mathbb{B}(X', D_{X'})$. Notice that either of stabilities over $X'$ implies the same over every model of $X$ over $X'$. Equivalently, each stability holds over every sufficiently high model of $X$ if the stability holds on over some of models. So, by Hironaka if a proper morphisms $f': X' \to Z'$ is birationally equivalent to $f$ over $Z$ (or $Z'$) then replacing $X'$ by a higher model of $X$ over $Z$ we can suppose that $(X', D')$ is a crepant model of $(X, D)$ and satisfies BP with stability over $X'$.

**Proof.** According to the last remark, required $f'$ exists for every model $Z'$ of $Z$. For the converse statement, a flattering due to Hironaka allows to construct $f'$ with equidimensional fibers for some sufficiently high models $Z'$ over $Z$.

If $K_Z + D_{\text{div}}$ is $\mathbb{R}$-Cartier, stable over $Z'$, then immediately by definition and our assumptions $M = K + D - f^*(K_Z + D_{\text{div}}) = K + D - f^*(K_{Z'} + D'_{\text{div}}) = (K_{X'} + D' - f'^*(K_{Z'} + D'_{\text{div}}))$. Conversely, let $M$ be $\mathbb{R}$-Cartier, stable over $X'$, and $f'$ be with equidimensional fibers. Then the $\mathbb{R}$-Cartier property of the b-$\mathbb{R}$-divisor $K + D_{\text{div}}$ and its stability over $Z'$ follows from the following two statements.

For a proper morphism $f: X \to Z$ of normal varieties, with equidimensional fibers, the equality of pull-backs $f^o = f^*$ holds on $\mathbb{R}$-Cartier divisors, wherein $D$ on $Z$ is $\mathbb{R}$-Cartier if and only if $f^o D$ is $\mathbb{R}$-Cartier on $X$. General hyperplane sections of $X$ reduce the proof to the case of a finite morphism $f$.

Let $f: X \to Z$ be a composition of surjective morphisms of normal varieties $g: X \to Y$ and $h: Y \to Z$, and $D, E$ be $\mathbb{R}$-divisors on $Y$ and $Z$ respectively such that $E$ is $\mathbb{R}$-Cartier and $g^o D = f^* E$ in codimension 1 over $Y$. Then $D = h^* E$, in particular, $D$ is also $\mathbb{R}$-Cartier and $g^* D = f^* E$. Reduce to the case $E = 0$ for $D := D - h^* E$. Then $D = 0$ too.

If there exists an $\mathbb{R}$-Cartier b-$\mathbb{R}$-divisor $\mathcal{L}$ such that $K + D \sim_{\mathbb{R}} f^* \mathcal{L}$ then a *descent* of the moduli part $M^X$ on the base $Z$ is well-defined. It is a moduli
part on base:
\[ \mathcal{M}^Z \eqdef \mathcal{D}_\text{mod} \eqdef \mathcal{L} - \mathcal{K}_Z - \mathcal{D}_{\text{div}}. \]

So,
\[ f^* \mathcal{M}^Z = f^* \mathcal{L} - f^* (\mathcal{K}_Z + \mathcal{D}_{\text{div}}) \sim_R \mathcal{K} + \mathcal{D} - f^* (\mathcal{K}_Z + \mathcal{D}_{\text{div}}) = \mathcal{M}^X, \]
that gives log adjunction
\[ \mathcal{K} + \mathcal{D} \sim_R f^*(\mathcal{K}_Z + \mathcal{D}_{\text{div}} + \mathcal{D}_{\text{mod}}). \]

The last equivalence also known as the canonical class formula.

Of course, the pull-back \( f^* \mathcal{L} \) in the definition of \( \mathcal{M}^Z \) can be replaced on more general \( f^* \mathcal{L} \). However, this does not give anything new. Indeed, if \( f^* \mathcal{L} \) is an \( \mathbb{R} \)-Cartier b-\( \mathbb{R} \)-divisor then \( \mathcal{L} \) is also an \( \mathbb{R} \)-Cartier b-\( \mathbb{R} \)-divisor and \( f^* \mathcal{L} = f^* \mathcal{L} \). Additionally, according to our assumptions, the b-\( \mathbb{R} \)-divisor \( \mathcal{K} + \mathcal{D} \) is \( \mathbb{R} \)-Cartier and the \( \mathbb{R} \)-linear equivalence preserves the last property.

(The \( \mathbb{R} \)-Cartier property is preserved also for the numerical equivalence \( \equiv \) over \( \mathbb{Z} \) that allows to define a numerical version of log adjunction.)

**Proposition 2.** *The moduli part \( \mathcal{D}_{\text{mod}} \) is an \( \mathbb{R} \)-Cartier b-\( \mathbb{R} \)-divisor if and only if the divisorial part \( \mathcal{D}_{\text{div}} \) satisfies BP.*

*Proof.* By definition \( \mathcal{D}_{\text{mod}} \) is an \( \mathbb{R} \)-Cartier b-\( \mathbb{R} \)-divisor if and only if \( \mathcal{K}_Z + \mathcal{D}_{\text{div}} \) is \( \mathbb{R} \)-Cartier. The last property is equivalent to BP for the b-\( \mathbb{R} \)-divisor \( \mathcal{D}_{\text{div}} \). \( \square \)

Unlike \( \mathcal{M}^X \), the moduli part \( \mathcal{M}^Z \) on the base is defined up to an \( \mathbb{R} \)-linear equivalence. However, \( \mathcal{M}^Z \) is defined up to a \( \mathbb{Q} \)-linear equivalence if \( \mathcal{K} + \mathcal{D} \sim_\mathbb{Q} f^* \mathcal{L} \) for an \( \mathbb{R} \)-Cartier b-\( \mathbb{R} \)-divisor \( \mathcal{L} \) (possibly, with nonrational multiplicities). More precisely, \( \mathcal{M}^Z \) is defined up to an \( n \)-linear equivalence if \( \mathcal{K} + \mathcal{D} \sim_n f^* \mathcal{L} \). Respectively in the log adjunction, the equivalence \( \sim_\mathbb{R} \) can be replaced by \( \sim_\mathbb{Q} \) or, more precisely, by \( \sim_n \):

\[ \mathcal{K} + \mathcal{D} \sim_n f^*(\mathcal{K}_Z + \mathcal{D}_{\text{div}} + \mathcal{D}_{\text{mod}}). \]

In general, according to Examples 1 (5-6) and Proposition b-\( \mathbb{R} \)-divisors \( \mathcal{D}_{\text{mod}}, \mathcal{D}_{\text{mod}} \) are not \( \mathbb{R} \)-Cartier.
Example 1. (1) Let $(X/Z, D)$ be a log pair with a birational contraction $f: X \to Z$, that is, $f$ is a proper birational map of normal varieties. Then $D_{\text{div}} = f_* D$ and $D_{\text{mod}} = 0$. Moreover, $D_{\text{mod}} = 0$ and is a b-divisor and the divisorial part $D_{\text{div}}$ satisfies BP. The naive moduli part $K + D - f^*(K_Z + D)$ is the discrepancy $R$-divisor of $(Z, D)$ on $X$, if $(Z, D)$ is a log pair and $f_* D = D$ as b-$R$-divisors, equivalently, the $R$-divisor $D$ does not have exceptional components and is a proper birational preimage of $f_* D$ with respect to $f$. This situation corresponds to the definition of discrepancies for the pair $(Z, D)$ and the discrepancies are determined for prime divisors on the blowup $X/Z$. In general, however, more natural to present the codiscrepancy b-$R$-divisor $D = \mathbb{B}(X, D) = -\mathbb{A}(X, D)$ as the divisorial part of adjunction for $(X/Z, D)$:

$$D = D_{\text{div}}.$$  

In particular, BP for $D_{\text{div}}$ is stable over $Z$, that is, $K_Z + D_{\text{div}}$ is a $R$-Cartier b-$R$-divisor, stable over $Z$, if $K + D \equiv 0$ over $Z$ and $K_Z + f_* D$ is an $R$-Cartier $R$-divisor on $Z$, equivalently, $(X/Z, D)$ is a 0-pair or $(X, D)$ and $(Z, f_* D)$ are crepant over $Z$. In this situation $D_{\text{div}} = \mathbb{B}(Z, f_* D) = -\mathbb{A}(Z, f_* D)$. Notice that the birational map $f$ should be proper to define a moduli part of adjunction. Indeed, a birational map $f$ is surjective (at least in codimension 1) for any base change if and only if the map is proper.

(2) Let $(C/C', D)$ be a log pair with a surjective map $f: C \to C'$ of normal curves, separable, if the characteristic of the base field $k$ is positive. Then for every point $p \in C'$ the geometric fiber $f^* p$ can be identified with the divisor of the fiber $f^* p = \sum m_i p_i$, where the sum runs over all point $p_i$ of the fiber $f^{-1} p$ and $m_i = \text{mult}_{p_i} f$ denotes the multiplicity of $f$ in $p_i$. Denote respectively by $r_i$ the ramification index of $f$ in $p_i$. (If the characteristic of $k$ does not divide $m_i$ then $r_i = m_i - 1$.) Let $d_i = \text{mult}_{p_i} D$ be the multiplicity of the $R$-divisor $D$ in $p_i$. Then the multiplicity $d' = \text{mult}_p D_{\text{div}}$ of the divisorial part of adjunction can be determined by the formula

$$d' = \max \left\{ \frac{r_i + d_i}{m_i} \right\}.$$  

In characteristic 0, if $D$ has the same multiplicities $d_i = d \leq 1$ over $p$, the multiplicities of divisorial and moduli parts are

$$d' = \frac{m - 1 + d}{m} \quad \text{and} \quad \text{mult}_{p_i} D_{\text{mod}} = (d - 1)(1 - \frac{m_i}{m}).$$
respectively, where \( m = \max\{m_i\} \). Since the moduli part is defined only up to an \( \mathbb{R} \)-linear equivalence, the last formula is meaningful only for a complete curve. In particular, if \( D = 0 \) then \( d' = (m - 1)/m \).

\[ D^{\text{mod}} \leq 0 \]

and is equal to 0 if and only if all \( m_i = m \), for instance: \( C/C' \) is a Galois covering. Next Example (3) generalizes the last statement.

Notice that \( D^{\text{mod}} = 0 \) also if all \( d_i = 1 \) for \( m \geq 2 \) or all \( d_i = d \) for \( m = 1 \) (cf. the maximal property in Proposition-Definition 1).

(3) Let \( f : X \rightarrow Z \) be a (finite) Galois covering and \((X/Z, D)\) be a log pair with an invariant \( \mathbb{R} \)-divisor \( D \). Then \( D^{\text{mod}} = 0 \) by Example (2). Indeed, on one hand, hyperplane sections reduce the determination of the moduli part in prime divisors to a curve case. On the other hand, we can blow up every given prime divisor preserving the assumptions.

(4) Let \( f : X \rightarrow Z \) be an unramified double covering of surfaces and \( C \) be a nonsingular curve on the base \( Z \) which splits on the the covering \( X \), that is, \( f^*C = C_1 + C_2 \), where \( C_1, C_2 \) are nonsingular curves on \( X \). First consider a log pair \((X/Z, D)\) with a divisor \( D = C_1 \). Then by Example (2) \( D_{\text{div}} = C \), the \( b\)-divisor \( D_{\text{div}} = \mathcal{B}(Z, C) \) satisfies BP and stable over \( Z \).

Remove now a closed point \( p \in C_1 \) on \( X \). Then for the log pair \((X'/Z, D)\) with \( X' = X \setminus p \), the map \( X' \rightarrow Z \) is surjective and surjective for every birational base change. So, the \( b\)-divisor \( D_{\text{div}} \) is well-defined but does not satisfies BP. More precisely, \( D_{\text{div}} = \mathcal{B}(Z, 0) \) over \( f(p) \) and \( = \mathcal{B}(Z, C) = \mathcal{B}(Z, 0) + C \) over the other points of the base \( Z \). This is why for log adjunction we usually assume that \( f \) is proper.

(5) Let \( f : X \rightarrow Z \) be an unramified double covering of surfaces and \( C_1, C_2 \) be nonsingular curves on the base \( Z \), which intersect transversally in a single point \( p \in Z \) and split on \( X \), that is, \( f^*C_1 = D_1 + D'_1 \) and \( f^*C_2 = D_2 + D'_2 \), where \( D_1, D'_1, D_2, D'_2 \) are nonsingular curves on \( X \). We suppose also that \( D_1 \cap D_2 = D'_1 \cap D'_2 = \emptyset \). Take a log pair \((X/Z, D)\) with an \( \mathbb{R} \)-divisor \( D = d_1D_1 + d_2D_2 + D'_1 + D'_2 \), where \( d_1, d_2 \) are real numbers < 1 and linearly independent over the rational numbers, that is, the equality \( a_1d_1 + a_2d_2 = a \) \( a_1, a_2, a \in \mathbb{Q} \), is possible only for \( a_1 = a_2 = a = 0 \). Then the \( b\)-\( \mathbb{R} \)-divisor \( D_{\text{div}} \) does not satisfy BP.

More precisely, as in Example (4) above the \( b\)-\( \mathbb{R} \)-divisor \( D_{\text{div}} \) does not satisfy BP over \( p \in Z \). Indeed, for the blowup \( Z' \rightarrow Z \) of the nonsingular point \( p \), the multiplicity of \( D_{\text{div}} \) in the exceptional divisor \( E \) is \( d = \max\{d_1, d_2\} \).
and \(d < 1\) (but \(D_{\text{div}} = C_1 + C_2\)). Denote by \(X' = X \otimes \mathbb{Z} Z'\) the corresponding blowup of \(X\). On the covering \(X'\) of \(Z'\), \(E\) splits into two exceptional curves of the 1st kind \(E_1, E_2\), where \(E_1\) intersects the proper transform of \(D_1\). Then by semiadditivity (2) in General properties [PSH, Lemma 7.4] the pair \((X/\mathbb{Z}, D)\) can be replaced by the log pair \((X'/\mathbb{Z}', D')\) with the \(\mathbb{R}\)-divisor \(D' = D + E_1 + (1 + d_2 - d_1)E_2\), where \(D\) denotes the proper birational preimage of the \(\mathbb{R}\)-divisor \(D\) on \(X'\). We suppose also that \(d = d_1 > d_2\). The new divisorial part of adjunction stabilizes on same blowups if such exist. The multiplicities \(d_1, 1 + d_2 - d_1\) of the new pair are linearly independent over the rational numbers. The prove concludes the induction on the number of blowups required to get a stability of BP. The process will never stops. This contradicts BP.

The same argument shows BP for every \(\mathbb{Q}\)-divisor \(D\) (cf. Theorem 1 below).

Similarly, one can construct an example with a \(b\)-\(\mathbb{Q}\)-divisor \(D \neq B(X, D)\) instead of \(D\) without BP, the divisorial part of adjunction of which \(D_{\text{div}}\) satisfies BP.

(6) Let \(\mathcal{M}_5\) denote the moduli space of stable rational curves with 5 marked points, \(\mathcal{U}_5 \rightarrow \mathcal{M}_5\) be the corresponding universally family and \(\mathcal{P}_1, \ldots, \mathcal{P}_5\) be sections corresponding to the marked points. The family is a smooth three dimensional (nonstandard) conic bundle over a nonsingular surface \(\mathcal{M}_5\). Let \(D_1, D'_1\) be divisors on \(\mathcal{U}_5\) sweeping respectively by components \(C, C'\) of stable curves \(C \cup C'\) with points \(p_1, p_2, p_3 \in C, p_4, p_5 \in C'\), where \(p_i = \mathcal{P}_i \cap (C \cup C')\). Similarly, divisors \(D'_2, D_2\) on \(\mathcal{U}_5\) are sweeping respectively by components \(C, C'\) of stable curves \(C \cup C'\) with points \(p_1, p_2 \in C', p_3, p_4, p_5 \in C\). As in Example (5) take a log pair \((\mathcal{U}_5/\mathcal{M}_5, D)\) with an \(\mathbb{R}\)-divisor \(D = d_1D_1 + d_2D_2 + D'_1 + D'_2\), where \(d_1, d_2\) are real numbers < 1 and linearly independent over the rational numbers. Then the divisorial part of adjunction \(D_{\text{div}}\) does not satisfy BP over the (closed) point \(p\) of the base \(\mathcal{M}_5\), corresponding to the stable curve \(C_1 \cup C_2 \cup C_3\) with points \(p_1, p_3 \in C_1, p_4, p_5 \in C_3\). The \(b\)-\(\mathbb{R}\)-divisor \(D_{\text{div}}\) has the same multiplicities over \(p\) as the corresponding \(b\)-\(\mathbb{R}\)-divisor in Example (5). Adding to \(D\) sections \(\mathcal{P}_i\) with arbitrary multiplicities \(\leq 1\), one can suppose that \(D\) is \(\mathbb{R}\)-ample over \(\mathcal{M}_5\) with the same \(b\)-\(\mathbb{R}\)-divisor \(D_{\text{div}}\).

Similarly one can construct an example with a \(\mathbb{R}\)-divisor \(D\) such that \(K_{\mathcal{U}_6} + D \sim_\mathbb{R} 0\) over \(\mathcal{M}_5\) and \(D_{\text{div}}\) does not satisfy BP. (According to [PSH]...
Theorem 8.1] the horizontal part of $D$ should be not effective, cf. also Example 3 below.) It is sufficient to find such multiplicities $a_1, \ldots, a_5 \leq 1$ that $D := D + \sum a_i P_i \equiv 0$ on every irreducible component of the curve over the point $p$. This gives an example over a neighborhood of $p \in \mathcal{M}_5$. Adding vertical prime divisors with appropriate multiplicities one can find an example over $\mathcal{M}_5$. More precisely, one can easily find required multiplicities $a_i$, when the multiplicities $d_1, d_2$ are close to 1. In this case one can pick up multiplicities $a_i$ close to $1/2$ for $i \neq 3$ and to 0 for $i = 3$ (actually $a_3 < 0$).

Again for rational $D$ BP holds.

**Theorem 1.** Let $(X/Z, D)$ be a log pair with a $\mathbb{Q}$-divisor $D$, proper $X/Z$ and under GLC of Definition 1. Then the divisorial part of adjunction $D_{\text{div}}$ is well-defined and always satisfies BP.

A proof below uses the following fact.

**Proposition 3** (Transitivity of divisorial part of adjunction). Let

$$f : X \twoheadrightarrow Y \twoheadrightarrow Z$$

a composition of two proper surjective morphisms and $(X/Z, D)$ be a log pair under the adjunction assumption GLC over $Z$. Then the pair $(X/Y, D)$ also satisfies GLC. Suppose that the divisorial part of adjunction $D_Y$ of $(X/Y, D)$ satisfies BP, stable over the base $Y$. Then the transitivity holds: $(Y/Z, D_Y)$ is also a log pair, satisfying the adjunction assumption GLC, where $D_Y = (D_Y)_Y$ the trace of the b-divisor $D_Y$ on $Y$, and

$$D_Z = (D_Y)_Z$$

holds, where $D_Z, (D_Y)_Z$ denote respectively divisorial parts of adjunction of pairs $(X/Z, D), (Y/Z, D_Y)$.

The usage of the base $Y$ with stability of BP can be omitted. Then, in the transitivity formula, the divisor $D_Y$ should be replaced by the trace $D_Y' = (D_Y)_Y$ on a model $Y'$ of $Y$ over $Z$, over which the stability of BP holds.

**Proof.** If $(X, D)$ is lc over an open subset $U \subset Z$ then the same holds over the open set $g^{-1}U \subset Y$. By surjectivity of $g$ the preimage $g^{-1}U$ is not empty if $U$ is not empty.

The transitivity follows from more precise result – General property (3). It is sufficient to verify the transitivity in each prime b-divisor $W$ of $Z$. By
General properties (1-2) we can suppose that \( W \) is a prime divisor on \( Z \) and 
\[ d_W = \text{mult}_W \mathbb{D}_Z = 1. \]
Then by General property (3), \((X, D)\) is lc but not klt over the generic point of \( W \). After blowing up of \( X \) and changing of \( D \) on its crepant \( \mathbb{R} \)-divisor, we can suppose that \( \text{mult}_V D = 1 \) for some prime divisor \( V \) on \( X \) over \( W \), that is, \( f(V) = W \). We can suppose also that \( h(V) \) is a prime divisor on \( Y \). Again by General property (3), 
\[ d_{f(V)} = \text{mult}_{f(V)} D_Y = 1 \]
and \((Y, D_Y)\) is not klt over the generic point of \( W = g(h(V)) \). On the other hand, \((Y, D_Y)\) is lc over the generic point of \( W \) by Corollary [1] (and the open property of lc). Thus \( \text{mult}_W (D_Y)_Z = 1 \) too.

In the proof of Theorem [1] we use also the following construction.

**Example 2 (Crepant pull-back).** Let \((Z, D_Z)\) be a log pair and \( f : X \to Z \)
be a morphism of normal varieties, separable, finite and surjective over the generic point (separable alteration). Then there exits a natural and unique divisor \( D \) on \( X \), converting the variety \( X \) into a log pair \((X, D)\) crepant to \((Z, D_Z)\), that is, for every canonical divisor \( K_Z = (\omega_Z) \) on \( Z \), where \( \omega_Z \) is a nonzero rational differential form of the top degree on \( Z \),
\[ K + D = f^*(K_Z + D_Z), \]
holds, where \( K = (f^*\omega) \) is a canonical divisor on \( X \). In this case by Example [1] (2) \( D_{\text{div}} = D_Z \) and \( D_{\text{mod}} = D_{\text{mod}} = 0 \). If the map \( f \) is proper and surjective then \( \mathbb{D}_{\text{div}} = \mathbb{B}(Z, D_Z) \) and \( D_{\text{mod}} = 0 \). In other words, \( \mathbb{D}_{\text{div}} \) satisfies BP and stable over \( Z \).

**Corollary 2.**
\[ \mathcal{M}_{X/Z} \sim \mathcal{M}_{X/Y} + f^*\mathcal{M}_{Y/Z} \]
(actually, \( = \) for appropriate canonical \( b \)-divisors of \( X, Y, Z \)), where \( \mathcal{M}_{X/Z}, \mathcal{M}_{X/Y}, \mathcal{M}_{Y/Z} \) are respectively the moduli part of adjunction for \((X/Z, D), (X/Y, D), (Y/Z, \mathbb{D}_Y)\).

**Proof.**
\[ \mathcal{M}_{X/Z} = K + D - f^*(g^*(K_Z + \mathbb{D}_Z)) = K + D - f^*(K_Y + \mathbb{D}_Y - \mathcal{M}_{Y/Z}) = K + D - f^*(K_Y + \mathbb{D}_Y) + f^*\mathcal{M}_{Y/Z} = \mathcal{M}_{X/Y} + f^*\mathcal{M}_{Y/Z}. \]
Since the moduli parts are defined up to a linear equivalence we can choose required canonical \( b \)-divisors to have \( = \) instead of \( \sim \).
Proof of Theorem 1. $\mathbb{D}_{\text{div}}$ is well-defined by [PSh] 7.1-2 (cf. also Definition 1 above). Using the birational nature of $\mathbb{D}_{\text{div}}$, namely, General properties (1) and (5), and by Proposition 3 one can reduce the proof to two cases: morphism $f : X \to Z$ is finite or is a family of curves.

First we verify BP for the finite morphisms. In this part of the proof, the assumption, that the base field $k$ has the characteristic 0, is essentially important. For simplicity one can suppose that $k = \mathbb{C}$ is the field of complex numbers. According to Hironaka after an appropriate blowup of the base and replacing $X$ by an induced normal base change one can assume that $Z$ is nonsingular with reduced (1 are the only nonzero multiplicities) divisor $\Delta$ with only simple normal crossing and such that $f$ is only ramified and $D$ is supported over $\text{Supp} \Delta$. General hyperplane sections and dimensional induction reduce the problem to a verification of BP over a neighborhood of a finite set of isolated closed points $z \in Z$. Over a sufficiently small neighborhood of every such point $z$ in the classical topology, the covering $f$ is a union of simple branches. Again by the transitivity of Proposition 3 it is sufficient to verify BP in the simple mapping case when the fiber $f^{-1}z = X_z$ consists of a single point, and in the case of unramified map $f$. In the first case, up to an analytic isomorphism, the pair $(X/Z,D)$ is toric, that is, $X \to Z$ is a toric finite map with an invariant divisor $D$ on $X$. The invariant divisor on the base $Z$ is $\Delta$. By General properties (2-3) one can suppose that the divisorial part of adjunction is reduced: $D_{\text{div}} = \Delta$. Then $D$ is also reduced and BP holds over $Z$. (Actually, in this toric situation, the rationality of $D$ does not matter.)

The second case, with an unramified map, is a bit harder. Again by the transitivity of Proposition 3 we consider the case of two sheets: $X = X_1 \cup X_2, X_1 \cap X_2 = \emptyset$ and the map $X \to Z$ consists of two isomorphisms $X_1, X_2 \to Z$. General properties (2-3) allow to suppose that $D_{\text{div}} = \Delta$. However, this time we know only that the multiplicities of the divisor $D$ are not exceeding 1 over $\text{Supp} \Delta$ and are equal to 0 otherwise. If all multiplicities of $D$ over $\text{Supp} \Delta$ are equal to 1, BP holds. Otherwise there exists a multiplicity $d_i < 1$ over a prime divisor of $\text{Supp} \Delta$. By construction and our assumptions every such multiplicity is rational and they form a finite set $\{d_i\}$. Hence there exists a positive integer $m$ such that all numbers $md_i$ are integral. The following algorithm allows to find a model of $Z$ over which $D_{\text{div}}$ is stable and BP holds.

Put $d = \min\{d_i\}$. Let $D_i$ be a prime divisor on $X$ over $\text{Supp} \Delta$ with the multiplicity $\text{mult}_{D_i} D = d_i = d$. By construction and our assumptions $d < 1$. If $D_i$ intersects on $X$ all prime divisors $D_j$ over $\text{Supp} \Delta$ with multiplicities
$d_j < 1$ and with the image $f(D_j)$ intersecting $f(D_i)$ on $Z$ then BP holds over a neighborhood of $f(D_i)$ (to verify this one can use Example 1 (2)). (Here as usually a prime divisor means closed irreducible subvariety of codimension 1, but not only its generic point.) Moreover, the multiplicity $d_i$ can be replaced by 1, not changing $\mathbb{D}_{\text{div}}$. After finitely many steps either there are no prime divisors $D_i$ over $\text{Supp} \Delta$ with the multiplicity $d_i$, or there is another prime divisor $D_j$ over $\text{Supp} \Delta$ with multiplicity $d_j < 1$, with $D_i \cap D_j = \emptyset$, but $f(D_i) \cap f(D_j) \neq \emptyset$. The last intersection is a nonsingular subvariety of the base $Z$ of codimension 2. In the first case take new $d = \min \{d_i\}$. The algorithm terminates because $d < 1$ and $md \in \mathbb{Z}$. In the second case blow up the intersection $f(D_i) \cap f(D_j)$ and respectively the covering $X$. Let $X' \to Z'$ be the induced unramified covering with blown up divisors $E_1, E_2, E$ on $X'_1, X'_2, Z'$ respectively, where $X'_1, X'_2, Z'$ are blowups of varieties $X_1, X_2, Z$. To be more precise suppose that $D_i \subset X_1$. Let $D, D_i, D_j$ denote proper birational transforms of those divisors on $X'$ and $D'$ denote the divisor on $X'$, corresponding to the crepant transform of the pair $(X, D)$. Then $D_i \subset X'_1, D_j \subset X'_2$ and

$$\text{mult}_{E_1} D' = d \leq \text{mult}_{E_2} D' = d_j < 1.$$ 

Again by Example 1 (2) $\text{mult}_E D'_{\text{div}} = \max \{d, d_j\} = d_j$. Now we replace the pair $(X/\mathbb{Z}, D)$ by $(X'/Z', D' + (1 + d - d_j)E_1 + E_2)$ and return to the beginning of the algorithm. The validity of BP this does not change. (The new divisor $\Delta' = \Delta + E$.) We contend that the algorithm stops when $\{d_i\} = \emptyset$ that gives BP with stability over the constructed base. Indeed, if $d_j = d$ then a new prime divisor over $\text{Supp} \Delta$ with $d_i < 1$ will not appear and one intersection $f(D_i) \cap f(D_j) \neq \emptyset$ disappear. If $d_j > d$ then exactly one new prime divisor $E_1$ over $\text{Supp} \Delta$ will appear with the multiplicity $1 > 1 + d - d_j > d$. Notice that again $m(1 + d - d_j) \in \mathbb{Z}$. Hence a new prime divisor $D_j$ with the multiplicity $d$ will not appear and after finitely many steps $D_i$ intersects all prime divisors $D_j$ over $\text{Supp} \Delta$ with multiplicities $d_j < 1$ for which $f(D_i) \cap f(D_j) \neq \emptyset$. In this case, as we did it before we increase the multiplicity $d_i = d$ up to 1.

In conclusion consider the case with the map $f$ being a family of curves. After an appropriate surjective proper base change $Z' \to Z$ with $\dim Z' = \dim Z$ (alteration) one can suppose that the family is maximally good. More precisely, there exists a commutative diagram

$$
\begin{array}{ccc}
X' & \to & X \\
/ & f & \downarrow f' \\
Z' & \to & Z
\end{array}
$$

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such that $f'$ is a projective family of (semi)stable curves with the smooth generic fiber and the preimage of horizontal prime divisors of $\text{Supp}D$ is a union of sections $S_i$ of the family $f'$ [dJ, Theorem 2.4]. (Sections $S_i$ are prime divisors too.) Maps $X' \to X$ and $Z' \to Z$ are finite generically. Let $D'$ be the crepant pull-back of Example 2. By construction $D'$ is a $\mathbb{Q}$-divisor. Then by the transitivity of Proposition 3 for the composition, $X' \to X \to Z$ gives the same divisorial part of adjunction for the pair $(X'/Z, D')$ as for $(X, D) \to Z$, that is, gives $D_{\text{div}}$. On the other hand, the transitivity of Proposition 3 for the composition $X' \to Z' \to Z$ and already established BP for the map $Z' \to Z$, BP for $D_{\text{div}}$ follows from BP for the divisorial part of adjunction of the pair $(X'/Z', D')$. We contend that BP holds for $(X'/Z', D')$ over $Z'$. By [dJ, Theorem 2.4, (vii)(b) and the property 2.2.2], $X' \to X$ is unramified generically over $Z$ or $Z'$. The horizontal part of the divisor $D'$ has the same multiplicities in the sections $S_i$ as in the corresponding prime divisors of $X$ in $D$. In our situation the assumption GLC means that all $\text{mult}_{S_i} D' \leq 1$ for all sections $S_i$. This follows from GLC for $(X/Z, D)$ by construction and [dJ, Remark 2.5]. Again by [dJ, Remark 2.5 and Theorem 2.4], after an additional blowup of $Z'$ and a normal base change of $f'$, we can suppose that $X'/Z'$ is toroidal with invariant $\text{Supp}D'$. Thus $D'_{\text{div}}$ can be computed by an lc threshold with a vertical divisorial lc center in $\text{Supp}D'$. More precisely, to determine $D'_{\text{div}}$ one can suppose that the base $Z'$ is a curve. By construction all sections $S_i$ and vertical divisors, including vertical prime components of $\text{Supp}D'$, form a divisor with toroidal crossings. Thus in the determination one can reduce all horizontal prime components of $\text{Supp}D'$ in $S_i$, that is, to replace their multiplicities by 1. Adding locally (in classical or etale topology) (multi)sections $S_i$ with $\text{mult}_{S_i} D' = 0$, preserving mutually disjoint and toroidal property of (multi)sections $S_i$, one can suppose that $S = \sum S_i$ is ample over $Z'$ and is in the reduced part of $D'$. (The ample divisor intersects any vertical divisor.) By Example (2) this reduce the determination of $D'_{\text{div}}$ to that of the divisorial adjunction of the log pair $(S/Z', D'_S)$, where $S = \cup S_i \to Z'$ is a toroidal covering and $D'_S$ is the adjunction of $D'$ on $S$ supported in the invariant divisor. Under the base change the crepant divisor change of $D'_S$ will be the adjunction of the crepand change of $D'$ on a modification of $S'$. Thus this case also follows from already known one for finite (toroidal) morphisms.

\footnote{F. Ambro informed the author that he has an alternative proof for toroidal morphisms with invariant $\text{Supp}D'$.}
Theorem 1 and Examples (5-6) show that, in general BP is unstable with respect to multiplicities. However, BP holds as a certain good limit, that is, \( D_{\text{div}} \) is always pseudo-BP. Examples (5-6) also shows that conditions on vertical component multiplicities of \( D \) are important. From the birational point of view over \( Z \), this should not be so important that shows the next result. Its proof is also a preparation to the proof of our main construction in Proposition-Definition 1.

**Theorem 2.** Let \((X/Z, D)\) be a log pair under GLC. Then there exists a log pair \((X'/Z', D')\) such that

1. morphisms \( X \to Z, X' \to Z' \) are birationally equivalent; and
2. the pair \((X'/Z', D')\) is crepant to \((X/Z, D)\) over the generic point of \( Z \);
3. \((X'/Z', D')\) satisfies GLC and
4. \( D'_{\text{div}} \) satisfies BP.

Hence \( D_{\text{div}} \) satisfies BP over the generic point of \( Z \).

The last property is not surprising (cf. Proposition-Definition 1 below).

**Proof.** By Proposition 3, Theorem 1 and Stein factorization we can suppose that \( X' \to Z' \) is a contraction and to construct rational \( D'_{\text{div}} \) or, moreover, integral.

We can replace \((X/Z, D)\) by a log pair \((X'/Z', D')\) which satisfies (1-3); and the crepant property (3) holds everywhere. Moreover, the morphisms \( f': X' \to Z' \) is toroidal and the divisor \( D' \) does not intersect toroidal embedding, that is, is supported in the invariant divisor. We can use for this a toroidalization of \( X \to Z \) with the closed subset \( \text{Supp} \ D \) [AK, Theorem 2.1].

Recall that by definition the toroidal embedding is nonsingular on \( X' \) and on \( Z' \). We contend that if we replace all vertical multiplicities of divisor \( D' \) outside of the toroidal embedding by 1, then BP (4) will hold and is stable over \( Z' \). Notice that all horizontal prime components of \( \text{Supp} \ D' \) are in the invariant divisor and have multiplicities \( \leq 1 \) in \( D' \). Hence, in particular, \( D'_{\text{div}} = \Delta \) is the complement to the toroidal embedding on \( Z' \). Notice that the last condition is preserved under the modifications of the pair \((X'/Z', D')\) below.
Verify BP, stable over $Z'$, that is, for every prime exceptional divisor $W$ of $Z'$, the equality $d_W = \text{mult}_W D'_\text{div} = b_W = \text{mult}_W B(Z', \Delta)$ holds. We can establish this by induction on $1 - b_W$. Since $(Z', \Delta)$ is lc, $b_W \leq 1$ holds and by construction multiplicities $b_W$ are integers. We will use also induction on dimension of center $Z'_W$. General hyperplane sections of the base allow to reduce the dimension of such a center to 0, that is, to the case with a closed point center $Z'_W$. Suppose first that $b_W = 1$ and use induction on the number of blowups, that is, we can suppose that $W$ is the blowup of a closed point on $Z'$. Then there exists a toroidal blowup of $X'$ with a center over $W$. Actually, we can toroidally blow up both varieties $X', Z'$ simultaneously (use subsequent toroidal resolution of $X'$ [AK, Proposition 4.4]). As above $D'_\text{div} = \Delta$ and $d_W = 1$.

Suppose now that $b_W \leq 0$. If center $Z'_W$ is an lc center of $(Z', \Delta)$ then we blow up it as above. Thus we can suppose that center $Z'_W$ is not an lc center. Then we can add a general hyperplane section $H$ through the center and preserve the toroidal property, including the horizontal part of the divisor $D'$. Replace $D'$ by $D' + f^*H$. This increases $b_W$ and completes induction by General property (2).

By the crepant assumption (2) and General property (1), $D'_\text{div} = D'_\text{div}$ holds over the generic point of the base. This gives the last statement. \(\Box\)

In general the behaviour of multiplicities $d_W$ is unpredictable, except for general estimations (see General properties (3-6)). However, in one important situation, a good behaviour of multiplicities of divisorial part of adjunction is expected (and already known in some cases). See for details [PSH, Proposition 9.3, (i)] and [Sh20, 6.8]. Here we briefly recall only one main result about hyperstandard multiplicities.

**Corollary 3.** Let $\mathcal{R} \subset [0, 1]$ be a finite subset of rational numbers and $d$ be a natural number. Then there exists a finite subset of rational numbers $\mathcal{R}' \subset [0, 1]$ such that, for every 0-pair $(X/Z, B)$ with $X/Z$ of weak Fano type, $\dim X \leq d$ and a boundary $B \in \Phi(\mathcal{R})$,

$$B_{\text{div}} \in \Phi(\mathcal{R}')$$

holds. In particular, the divisorial part of adjunction $B_{\text{div}}$ is also a boundary and there exists a real number $\varepsilon > 0$ such that every nonzero multiplicity of $B_{\text{div}}$ is $\geq \varepsilon$. The number $\varepsilon$ depends only on $\mathcal{R}$ and $d$.  

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The same follows for any proper $X/Z$ from Index conjecture \cite{Sh20} Conjecture 2).

**Proof.** The last statement follows from the dcc property of $\Phi(\mathcal{R})$. \hfill $\Box$

With the effective b-semiampleness of the moduli part of adjunction $B_{\text{mod}}$ \cite[(7.13.3)]{PSH} \cite[Conjecture 3]{Sh20} this reduces the birational effectiveness of an Iitaka map \cite[Conjecture 1.1]{BZ} to the same problem for a big log canonical divisor (\cite[Theorem 1.3]{HMX} and cf. \cite[Theorem 1.3]{BZ}). However, Index problem remains the main missing point to complete the proof.

**Proposition 4.** For a log pair $(X/Z, D)$ under GLC, the following properties are equivalent.

1. $(X/Z, D)$ is a log stable pair and $(Z, D_{\text{div}})$ is a log pair;

2. $D_{\text{div}}$ satisfies BP stable over $Z$, in particular, $(Z, D_{\text{div}})$ is a log pair; and if additionally $X \to Z$ has equidimensional fibers then (1-2) are equivalent to

3. $D_{\text{mod}}$ is $\mathbb{R}$-Cartier over $X$.

The log stable property in (1) means that, for every $\mathbb{R}$-Cartier divisor $\Delta$ on $Z$, $(X, D + f^*\Delta)$ is lc if and only if $(Z, D_{\text{div}} + \Delta)$ is lc. The global version of this property is equivalent to the local one over a neighborhood of every point of $Z$. Notice also that, by the log adjunction and the negativity lemma, the b-nef over $Z$ property of $D_{\text{mod}}$ (cf. Conjecture 1 below) implies that it is enough to assume the if part of the log stability.

**Proof.** $(1) \Rightarrow (2)$: We need to verify that, for every prime exceptional divisor $W$ on $Z$, $d_W = \text{mult}_W D_{\text{div}} = b_W = \text{mult}_W \mathbb{B}(Z, D_{\text{div}})$ holds. Use a toroidalization $X' \to Z'$ as in the proof of Theorem 2 after a reduction to a contraction $X' \to Z'$. Suppose additionally that the invariant part $\Delta \subset Z'$ contains $W$ and $g^{-1}\text{Supp} D_{\text{div}}$ with all exceptional divisors of a birational projective base change $g: Z' \to Z$. Let $D_{Z'} = \mathbb{B}(Z, D_{\text{div}})_{Z'}$ be a codiscrepancy. General property (2) allows to replace $D'$ by $D' + F' F'$, where $F'$ is such an $\mathbb{R}$-divisor on $Z'$ supported on $\Delta$ that $\text{mult}_W (D_{Z'} + F') = 1$ and $(Z', D_{Z'} + F')$ is lc. After an extension of $\Delta$ we can suppose that $F' \sim_\mathbb{R} 0/Z$ and $F' = g^* F$ for an $\mathbb{R}$-Cartier divisor $F$ on $Z$. The pair $(Z, D_{\text{div}} + F)$ is
also lc. Thus by the log stability (1) and construction \((X, D + f^*F)\) and its crepant pair \((X', D' + f'^*F')\) are lc. Now make a change \(D := D + f^*F\) and \(F := F' := 0\). Then \(d_W \leq b_W = 1\) by General property (3). (In general \(d_W > b_W\) is possible.) Moreover, \(d_W = b_W = 1\). Indeed, in our construction we can assume that that \(\mathcal{B}(Z, D_{\text{div}})\) has only one prime b-divisor \(W\) over its center in \(Z\) with \(b_W = 1\). Thus if \(d_W < 1\) then \((X, D)\) is klt over the center. Increasing singularity in the center we get a contradiction with log stability.

In other words, the lc property can be replaced by the relative klt property in the definition of the log stability.

(1) \(\iff\) (2) By General property (2) it is sufficient to verify the implication assuming BP stable over \(Z\), that is, \((X, D)\) is lc exactly when \((Z, D_{\text{div}})\) is lc. If \((X, D)\) is lc then the b-divisor \(D_{\text{div}}\) is lc, that is, it is a subboundary, by General property (3). Hence \((Z, D_{\text{div}})\) is lc by BP stable over \(Z\). Conversely, if \((X, D)\) is not lc, then \(\mathcal{B}(X, D)\) is not lc, that is, it is not a subboundary. Hence the b-divisor \(D_{\text{div}}\) also is not lc by General property (3) and because a prime b-divisor \(W\) on \(X\) with the multiplicity \(\text{mult}_B(X, D) > 1\) dominates a prime divisor on a blowup of the base \(Z\). Thus \((Z, D_{\text{div}})\) is not lc by BP stable over \(Z\).

(2) \(\iff\) (3) by Proposition \(\square\).

Let \((X/Z, D)\) be a log pair the generic fiber of which is a weakly lc pair. Recall that weakly lc means in this situation that \((X_\eta, D_\eta)\) is lc with a boundary \(D_\eta\) and \(K_{X_\eta} + D_\eta\) is nef where \(\eta\) is the general point of \(Z\). Then in the class of weakly lc pairs birationally equivalent to \((X/Z, D)\) with respect to the base \(Z\), that is, isomorphic or crepant generically over \(Z\), the upper moduli part of adjunction has a largest value \(D_{\text{mm}}\). It is largest modulo \(\mathbb{R}\)-linear equivalence on \(X\), attained on log pairs naturally related to moduli spaces of the generic fiber, and will be constructed in Proposition-Definition \(\square\) below modulo LMMP. This allows to define a birationally invariant with respect to the base \(Z\) moduli part of adjunction \(\mathcal{D}_{\text{mm}} = \mathcal{D}_{\text{mm}}(D) = \mathcal{D}_{\text{mm}}(X/Z, D)\) which satisfies certain remarkable properties (e.g., Conjecture \(\square\)). However, as usually in mathematics we prefer the adjective maximal instead of largest.

**Proposition-Definition 1** (Maximal log pair). Let \((X/Z, D)\) be a weakly lc pair with a boundary \(D\) over the generic point of \(Z\). Suppose that LMMP holds indimensions \(\leq \dim X\). Then there exists a maximal log pair \((X_m/Z_m, B_m)\), a weakly lc pair birationally equivalent to \((X/Z, D)\) with respect to the base \(Z\). The maximal property means the inequality \(B_m^{\mod} \geq B^{\mod}\) modulo lin-
ear equivalence for every weakly lc pair \((X'/Z', B')\), birationally equivalent to \((X/Z, D)\) with respect to \(Z\).

Moreover, \(D^{\text{mm}} = B_{m_{\text{mod}}}\) is a b-nef \(\mathbb{R}\)-Cartier b-\(\mathbb{R}\)-divisor.

For appropriate canonical divisors on \(X\) and \(Z\), \(\geq\) holds literally without the linear equivalence.

**Example 3.** Let \((X/Z, D)\) be a pair under GLC and generically be a 0-pair over \(Z\). Then it is a maximal log pair exactly when \((X/Z, D)\) is a 0-pair everywhere over \(Z\) with a boundary \(D\). The last condition can be omitted keeping the same maximal moduli part of adjunction (cf. [Sh20, Proposition 13, (1)]).

In this situation \(D^{\text{mm}} = f^*D_{\text{mod}}\). Thus \(D^{\text{mm}}\) is a generalization of \(f^*D_{\text{mod}}\) with consequent conjectures (cf. Conjecture 1 below).

Moduli construction of \((X_m/Z_m, B_m)\) and a proof of the proposition-definition actually need a weaker assumption on LMMP: in \(\dim X/Z + 1\), and will be explained in [Sh13]. The moduli approach does not use also the b-nef property of \(D^{\text{mm}}\) (cf. Conjecture 1 below).

**Proof of Proposition-Definition 1.** The nef property of \(M_X = \mathcal{M}^{\text{mm}}_X\) was established in [ACShS, Theorem 1.1] using the theory of foliations for contractions \(X/Z\). The b-nef property of \(\mathcal{M}^{\text{mm}}\) follows from stability of \(\mathcal{M}^{\text{mm}}\) below. Thus, for contractions \(X/Z\), we can suppose the b-nef property. Actually, we need the b-nef property in \(\dim X - 1\). An alternative approach to the b-nef property will be discussed after Conjecture 1 below.

Fix the birational class of \((X/Z, D)\), that is, the class of pair \((X'/Z', B')\) which are birationally equivalent to \((X/Z, D)\) with respect to the base \(Z\). More precisely, we consider the subclass of pairs \((X'/Z', B')\) which are weakly lc with a boundary \(B'\) and \((X'/Z', B')\) is crepant to \((X/Z, D)\) generically over \(Z\) or \(Z'\).

**Construction of** \((X_m/Z_m, B_m)\). Suppose first that \(X/Z\) is a contraction. By [AK, Theorem 2.1] we can suppose that \((X/Z, B)\) is toroidal with a nonsingular projective base \(Z\) and with a boundary \(B\) such that

\[\text{Supp} B\] is in the invariant divisor;

with the same multiplicities as \(D\) generically over \(Z\), that is, in all those prime divisors nonexceptional on original \(X\); and

with multiplicities 1 in all other invariant prime divisors.
By our assumptions we can construct a weakly lc pair \((X_m/Z_m, B_m)\) over \(Z = Z_m\). This is a required maximal model. By construction \((X_m/Z_m, B_m)\) belongs to the considered class weakly lc models. Moreover, \(B_{m, \text{div}, Z_m} = \Delta_m\) is the invariant divisor on \(Z_m\).

In general, we do not know the existence of toroidilization (why not). So, we use the following construction. Take a Stein factorization \(X \xrightarrow{f} Y \xrightarrow{g} Z\) of \(X/Z\) where \(f, g\) are respectively a contraction and a finite morphism. For an appropriate model of \(X/Z\), we can suppose that \(Z\) is nonsingular, \(g\) is toroidal and there is a morphism \(\varphi: Y \to Y_m\) for a maximal log pair \((X_m/Y_m, B_m)\) constructed above such that the divisorial part \((B_{m, \text{div}})_Y\) is supported in the invariant divisor \(D\) of the toroidal finite morphism \(g\). We suppose also that \((X/Y, B)\) is isomorphic to \((X_m/Y_m, B_m)\) over \(Y \setminus D\) isomorphic to \(Y_m \setminus \varphi(D)\). The crepant model \((X/Y, (B_{m, \text{div}})_X)\) of \((X_m/Y_m, B_m)\) is a weakly lc pair if the trace \((B_{m, \text{div}})_X\) is effective. Otherwise, we replace \((B_{m, \text{div}})_X\) by \((B_{m, \text{div}})_X + f^*E\), where \(E = D - (B_{m, \text{div}})_Y\). By construction \(E\) is effective, \((B_{m, \text{div}})_Y + E\) is the invariant divisor \(D\). By General property (2), \(D\) is the divisorial part of adjunction for \((X/Y, (B_{m, \text{div}})_X + f^*E)\). By construction, the moduli part of adjunction for \((X/Y, (B_{m, \text{div}})_X + f^*E)\) is the same \(\mathcal{D}^{\text{mm}}\) and maximal that will be established below (already known for contractions).

Then we can reconstruct \((X/Y, (B_{m, \text{div}})_X + f^*E)\) into a weakly lc pair using LMMP over \(Y\): take log resolution and replace multiplicities of all exceptional divisors over \(D\) by 1 and by 0 otherwise, and apply LMMP over \(Y\). The maximal property below for contractions warrants that the constructed model is crepant to \((X/Y, (B_{m, \text{div}})_X + f^*E)\) and is maximal over \(Y\).

**Stability of \(\mathcal{D}^{\text{mm}}\).** Since in the last construction the divisorial part of adjunction for \((X/Y, (B_{m, \text{div}})_X + f^*E)\) is an integral invariant divisor \(D\) then the stability for \(X/Z\) is equivalent to the stability for \(X/Y\) by Theorem \(\square\) and Proposition \(\square\) So, we can suppose that \(X/Z\) is a contraction. By definition \(\mathcal{D}^{\text{mm}} = B_{m, \text{mod}}\). So, by Proposition \(\square\) it is enough to verify that \(B_{m, \text{div}}\) satisfies BP stable over \(Z_m\). Using General properties, dimensional induction and induction on the number of monoidal transformations we can consider only one such transformation in the following situation. Let \(P\) be an invariant prime cycle and \(Z \to Z_m\) be a monoidal transformation in \(P\). The transformation is toroidal and the invariant divisor \(\Delta\) on \(Z\) is the birational transform of \(\Delta_m\) plus the exceptional divisor \(E\) (over \(P\)) of the
transformation. To establish required stability it is enough to verify that \( \text{mult}_E \mathcal{B}_{m, \text{div}} = 1 \), that is, \( \Delta = \mathcal{B}_{m, \text{div}, Z} \). Taking hyperplane sections we can suppose that \( P \) is a closed point. We can suppose also that \( \Delta_m \) is sufficiently large and \((X_m, B_m)\) itself (not only over \( Z_m \)) is a weakly lc model.

To compute the last divisorial part of adjunction we can take a toroidal resolution of \( X_m \) over \( Z \) and then construct a model \((X/Z, B)\) as in above Construction using LMMP. By construction \( \Delta = \mathcal{B}_{\text{div}, Z} \). Thus to verify that \( B_{m, \text{div}, Z} = \mathcal{B}_{m, \text{div}, Z} \), it is enough (and actually necessary) to verify that \((X, B)\) is a crepant model of \((X_m, B_m)\). (As usually we consider such models over pt.) Indeed, \((Z, \Delta) \to (Z_m, \Delta_m)\) is crepant.

On the other hand, the weakly lc pair \((X_m, B_m)\) can be constructed from \((X, B)\) by LMMP and a crepant transformation. If \( \Delta \) is sufficiently large the only possible curves negative with respect to \( K + B \) are curves \( C \) over \( E \). These curves are on reduced prime divisors of invariant part \( V \). Moreover, by above Construction and adjunction \((C.K + B) = (C.K_V + B_V) = (C.B_V^{\text{mm}}) \) and \( \geq 0 \) by the b-nef property of \( B_V^{\text{mm}} \) for \((V/E, B_V)\) by [ACShS Theorem 1.1] and we need this property in dimension \( \leq \dim X - 1 \). Hence there are no negative curves and \((X, B)\) is crepant to \((X_m, B_m)\).

The maximal property of \( D^{\text{mm}} \). In particular, \( D^{\text{mm}} \) modulo linear equivalence is independent of construction. Moreover, \( D^{\text{mm}} \) is unique for fixed canonical divisors on \( X \) and \( Z \). This can be verified directly using General properties. Notice for this especially General property (2) which implies that \( D'^{\text{mm}} = D^{\text{mm}} \) for \( D' = D + f^*\Delta \).

If \( X \to Z \) is a contraction, adding effective divisors we can suppose that \( B'_{\text{div}} = B_{m, \text{div}} = \mathcal{B}_{m, \text{div}, Z'} \). If additionally \( \mathcal{B}_{m, \text{div}} \) is BP stable over \( Z' \) then the required inequality follows from the property that, for a larger divisor, its positive part in the Zariski decomposition is larger too. Perhaps, the same holds even if \( \mathcal{B}'_{\text{div}} \) does not satisfies BP.

In general, we use the above construction of \((X/Y/Z, B_m)\) with \( B_m := (\mathcal{B}_m)_X + f^*E \), where \( \mathcal{B}_{m, \text{div}} = D \) is BP stable over \( Z \). The maximal moduli part \( \mathcal{M}^{\text{mm}} \) over \( Z \) is the same as over \( Y \):

\[
D^{\text{mm}} = K + \mathcal{B}_m - (f \circ g)^\circ(K_Z + D_Z) = K + \mathcal{B}_m - f^\circ(g^\circ(K_Z + D_Z)) = K + \mathcal{B}_m - f^\circ(K_Y + D),
\]

where \( D_Z \) is the invariant divisor for the toroidal morphism \( g \). Notice also that any weakly lc model over \( Z' \) is a weakly lc model over \( Y'/Z' \) with a birational morphism \( Y \to Y' \) for sufficiently high model \( Y \) because any curve over \( Y' \) is a curve over \( Z' \).
We do not need to suppose that the divisorial part of adjunction $B'_{\text{div}}$ of $(X'/Y', B')$ is BP stable over $Y$. The required maximal property follows from Corollary 2. Indeed, the corollary

$$B'_{X'/Z'} = B'_{X'/Y'} + M_{Y/Z},$$

where $B'_{X'/Z'}, B'_{X'/Y'}, M_{Y/Z}$ denotes respectively the moduli part of adjunction for $(X'/Z', B'), (X'/Y', B'), (Y/Z, B'_{\text{div}})$. However, in nonstable situation we consider a moduli part and divisorial part of adjunction as b-divisors. Since $B'_{X'/Y'} \leq D_{\text{mm}}$, it is enough to verify that $M_{Y/Z} \leq 0$ for appropriate canonical b-divisors of $Y, Z$ (cf. Example (2)).

By definition, construction and the lc property of $B'_{\text{div}}$ we can suppose that $B'_{\text{div}} \leq D$ and $= \overline{D}$ over $Y \setminus D$ (cf. Theorem 2). By General property (2) after increasing we can suppose that the divisorial part of adjunction $D'$ for $(Y/Z, B'_{\text{div}})$ is $D_Z$. Since $Y/Z$ is toroidal

$$M_{Y/Z} = K_Y + B'_{\text{div}} - g^\circ (K_Z + D_Z) = K_Y + B'_{\text{div}} - K_Y - D = B'_{\text{div}} - D \leq 0$$

holds. This concludes the proof of maximal property.

In other words, $D_{\text{mm}}$ is the positive b-divisor of a relative lc b-divisor.

\[\square\]

**Conjecture 1.** For every positive $a \in \mathbb{R}$, $D_{\text{mm}} + a\mathcal{P}$ is b-semiample and effectively b-semiample if the moduli type of irreducible components of generic fiber is bounded where $\mathcal{P}$ is a divisor of canonical polarization for moduli of irreducible components of generic fiber. Moreover, the Iitaka dimension of $D_{\text{mm}}$ is at least the Kodaira dimension of the general fiber of $(X/Z, D)$ plus the variation of $(X/Z, D)$.

The effective b-semiample property of $D_{\text{mm}} + a\mathcal{P}$ is meaningful even if $D$ and $a$ are not rational. This can be explained in terms of geography of log models (see Conjecture 2 below and [Sh20, Bounded affine span and index of divisor in Section 12]).

A geometrical interpretation of the conjecture: the corresponding contraction gives a family of log canonical models of components of fibers. So, in addition to a rational morphism from $Y$ in the Stein factorization $X \to Y \to Z$ to the coarse moduli of irreducible components of fibers, we have a rational morphism of $X$ to the family with log canonical models in fibers whereas $\mathcal{P}$ is
ample on the base, $D^{\text{mm}}$ is nef on the family and ample on its fibers. In particular, the coarse moduli can be extended to a fibration with corresponding log canonical models.

Notice that the conjecture implies the Kodaira additivity in the strong form (with variation). The semiampleness of $D^{\text{mm}}$ does not hold in general according to [K99, Section 3].

The conjecture implies also that $D^{\text{mm}}$ is b-nef. This property naturally related to the Viehweg positivity for the direct image of a relative dualizing sheaf and the polarization $P$. We already used the b-nef property in the proof of Proposition-Definition 1. The author thinks that this property follows from its special klt case with a 1-dimensional base by [Am03, Theorem 0.1] and adjunction.

Finally, we expect the following strong form of stability of $D^{\text{mm}}$ with respect to horizontal part of $D$ which is supposed to be a boundary.

**Conjecture 2** (Geography of log adjunction). Let $X/Z$ be a proper morphism, $\eta$ be the general point of $Z$ and $S_i$ be distinct prime b-divisors of $X$. Consider a polyhedron $\mathfrak{P}$ in the wlc geography of $\mathfrak{M}_{S_\eta}$ of $(X_{\eta}/\eta, \sum S_{i,\eta})$ [ShCh, Section 3]. Then $D^{\text{mm}}(D)$ is linear on $\mathfrak{P}$: for every two (b-)divisors $D_1, D_2$ such that $D_{1,\eta}, D_{2,\eta} \in \mathfrak{P}$ and any two real numbers $w_1, w_2 \in [0,1]$ with $w_1 + w_2 = 1$,

$$D^{\text{mm}}(w_1 D_1 + w_2 D_2) = w_1 D^{\text{mm}}(D_1) + w_2 D^{\text{mm}}(D_2).$$

In other words $D^{\text{mm}}$ is a piecewise linear function of the horizontal part $D_\eta$ of $D$.

Since $\mathfrak{P}$ is rational it is enough to verify Conjecture 2 for $\mathbb{Q}$-divisors $D$.

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