THE STRONG SPECTOR-GANDY THEOREM FOR THE HIGHER ANALYTICAL POINTCLASSES

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There is little doubt that the single, most important elementary result of effective descriptive set theory is the

**Restricted Quantification Theorem** (Kleene [1959], DST(4D.3)).

For any two spaces \( X, Y \) of type \( \leq 1 \), if \( Q \subseteq X \times Y \) is \( \Pi^1_1 \) and

\[
P(x) \iff (\exists y \in \Delta^1_1(x)) Q(x, y),
\]

then \( P \) is also \( \Pi^1_1 \).

It yields the \( \Delta \)-Uniformization Criterion DST(4D4), which is the key tool in the effective proofs of several results about Borel uniformizations of Borel sets, cf. Section 4F of DST. It also motivated Spector to prove the following basic fact which provides a converse to it:

**(SG) Spector-Gandy Theorem** (Spector [1960], also Gandy [1960]). For every space \( X \) of type \( \leq 1 \) and every \( \Pi^1_1 \) set \( P \subseteq X \), there is some \( \Pi^0_1 \) set \( Q \subseteq X \times N \) such that

\[
(1) \quad P(x) \iff (\exists ! \alpha \in \Delta^1_1(x)) Q(x, \alpha);
\]

and so

\[
(2) \quad P \in \Pi^1_1(X) \iff (1) \text{ holds with some } Q \in \Pi^0_1(X \times N).
\]

Spector actually proved the following stronger form of this theorem:

**(SSG) Strong Spector-Gandy Theorem** (Spector [1960]). For every space \( X \) of type \( \leq 1 \) and every \( \Pi^1_1 \) set \( P \subseteq X \), there is some \( \Pi^0_1 \) set \( Q \subseteq X \times N \) such that

\[
(3) \quad P(x) \iff (\exists \alpha) Q(x, \alpha) \iff (\exists \alpha \in \Delta^1_1(x)) Q(x, \alpha).
\]

In DST(6E.7), ynm gave a proof of (SG) for the odd levels \( \Pi^1_{2n+1} \) of the analytical hierarchy under the hypothesis of Projective Determinacy and our aim here is to prove a similar extension of (SSG) as follows:

*The results in this note will be included in Part III of Arant, Gregoriades, and Moschovakis [?], the advanced part for which we assume familiarity with the material (and notation) of Chapters 1 – 7 of Moschovakis [2009] (DST).

1A space of type \( \leq 1 \) is a finite product of copies of \( \mathbb{N} \) and \( \mathcal{N} \).
Strong Spector-Gandy Theorem for $\Pi_{2n+1}^1$. Assume $\text{Det}(\Sigma_{2n}^1)$. For every space $X$ of type $\leq 1$ and every $\Pi_{2n+1}^1$ set $P \subseteq X$, there is a $\Pi_{2n}^1$ set $Q \subseteq X \times \mathbb{N}$ such that

$$P(x) \iff (\exists \alpha)Q(x, \alpha) \iff (\exists \alpha \in \Delta_{2n+1}^1(x))Q(x, \alpha).$$

There were many reasons why Gandy’s weaker result (without the $(\exists ! \alpha)$) was generally adopted in the 70’s as the standard version of this basic fact: it is quite easy to prove; it suffices to prove (2); and (most significantly), it can be established in a similar way for the analog of $\Pi_{1}^1$ in many theories of effective definability, including normal (Kleene) recursion in type 3 (Theorem 9 in Moschovakis [1967]) and inductive definability in every acceptable structure $\mathfrak{A}$, Theorem 7D.2 in Moschovakis [1974]. What was missed in the 70’s is that (SSG) has important applications in effective descriptive set theory, cf. Chapter 5 of Gregoriades [2016]; it remains to be seen if the same is true of its extension to the higher analytical pointclasses under projective determinacy.

§ 1. Background. The proof outlined in the next section uses the methods and results of Sections 6D and 6E of DST about the game quantifier $\forall_\alpha$ defined in the beginning of 6D,

$$(\forall \alpha)Q(x, \alpha) \iff (\exists \alpha_0)(\forall \alpha_1)(\exists \alpha_2)\cdots Q(x, (\alpha_0, \alpha_1, \alpha_2, \ldots)).$$

In particular, we will use systematically the (trivial) characterizations

$$\Pi_1^1 = \forall \Sigma_0^0, \Sigma_2^1 = \forall \Pi_1^1, \Pi_3^1 = \forall \Sigma_2^1, \ldots$$

rather than the classical definitions of these pointclasses in terms of the quantifiers $\exists \alpha, \forall \alpha$. We will work only with spaces $X$ of type $\leq 1$, we fix good parametrizations in $\mathcal{N}$ (cf. DST(3H.1)) in some standard way on $\Sigma_0^1 = \Sigma_0^0$ and we extend them to all the analytical pointclasses by the recursion

$$G_{2n+1}^{\Pi_1^1}(\varepsilon, x) \iff (\forall \alpha)(G_{2n}^{\Sigma_1^1}(\varepsilon, x, \alpha),$$

$$G_{2n+2}^{\Sigma_1^1}(\varepsilon, x) \iff (\forall \alpha)(G_{2n+1}^{\Pi_1^1}(\varepsilon, x, \alpha),$$

$$G_{2n+1}^{\Sigma_1^1}(\varepsilon, x) \iff (\exists \alpha)(G_{2n+1}^{\Pi_1^1}(\varepsilon, x), G_{2n}^{\Pi_1^1}(\varepsilon, x) \iff (\exists \alpha)(G_{2n}^{\Pi_1^1}(\varepsilon, x));$$

$\varepsilon$ is a $\Delta_k^1$ code of some $Q \subseteq X$ if

$$Q(x) \iff G_{k}^{\Sigma_1^1}(\varepsilon_0, x) \iff G_{k}^{\Pi_1^1}(\varepsilon_1, x).$$

The first, basic theorem we need is (perhaps) the main fact about the game quantifier:
**Third Periodicity Theorem** (DST(6E.1), ynm). Assume $\text{Det}(\Sigma^1_{2n})$. If a pointset $A \subseteq \mathcal{N}$ is in $\Sigma^1_{2n}$ with code $\zeta$ and player I wins $A$, then I has a $\Delta^1_{2n+1}(\zeta)$ winning strategy.

The proof of DST(6E.1) gives a great deal of information about $\Pi^1_{2n+1}$ beyond its statement, and we need to make explicit and use some of this here.

A *partial strategy* (for player I in a game on $\mathbb{N}$) is a partial function $p : \mathbb{N} \rightarrow \mathbb{N}$ such that $\neg(\text{Seq}(u) \& \text{lh}(h)$ is even $) \implies p(u) = 0$; if $p$ is total, it then defines a strategy for player I by which he plays

$$a_0 = p(\langle \rangle), \ a_2 = p(\langle a_0, a_1 \rangle), a_4 = p(\langle a_0, a_1, a_2, a_3 \rangle) \ldots$$

when Player II plays $a_1, a_3, \ldots$ As usual, for any $p : \mathbb{N} \rightarrow \mathbb{N}$,

$$\text{Graph}(p) = \{(u, t) : p(u) = t\}$$

and we typically identify $p$ with its graph.

**Theorem 1.** If $\text{Det}(\Sigma^1_{2n})$ and $X$ is of type $\leq 1$, then there is a recursive function

$$u_1(\varepsilon, x) = (u^\Sigma_1(\varepsilon, x), u^\Pi_1(\varepsilon, x))$$

and for each $\varepsilon$ and each $x$, a partial function $p_{\varepsilon, x}$ such that

(5) $p_{\varepsilon, x}(u) = t \iff G^{\Pi,1}_{2n+1}(u^\Pi_1(\varepsilon, x), u, t)$,

and if

(6) $P_\varepsilon(x) \iff \text{if } G^{\Pi,1}_{2n+1}(\varepsilon, x) \iff (\exists \alpha)G^{\Sigma,1}_{2n}(\varepsilon, x, \alpha)$, then

(7) $\iff (\exists! \beta)(\forall u)G^{\Pi,1}_{2n+1}(u^\Pi_1(\varepsilon, x), u, \beta(u))$

(8) $\iff \text{Graph}(p_{\varepsilon, x})$ is $\Delta^1_{2n+1}$ with code $u^1(\varepsilon, x)$

(9) $\iff (\exists \beta \in \Delta^1_{2n+1}(\varepsilon, x))(\forall u)G^{\Pi,1}_{2n+1}(u^\Pi_1(\varepsilon, x), u, \beta(u))$.

The construction of $p_{\varepsilon, x}$ is quite complex but effective, and so the definition of $u^\Pi_1(\varepsilon, x)$ so that (5) holds is routine. The same construction proves (6), which is trivially equivalent with (7). To define $u^\Sigma_1(\varepsilon, x)$ and prove the crucial (8), we use the closure of $\Sigma^1_{2n+1}$ under number quantification and the characteristic property of a good parametrization, which supplies $u^\Sigma_1(\varepsilon, x)$ so that for all $\varepsilon, x, u, t$,

(10) $G^{\Sigma,1}_{2n+1}(u^\Sigma_1(\varepsilon, x), u, t) \iff (\forall s)(G^{\Pi,1}_{2n+1}(u^\Pi_1(\varepsilon, x), u, s) \implies t = s)$,

and this implies easily that

(11) $P_\varepsilon(x) \implies p_{\varepsilon, x}$ is total

$$\implies (\forall u)(\forall t)(G^{\Sigma,1}_{2n+1}(u^\Sigma_1(\varepsilon, x), u, t) \iff G^{\Pi,1}_{2n+1}(u^\Pi_1(\varepsilon, x), u, t)).$$
To prove the converse of (11) and complete the proof of (8), assume the right-hand-side holds, but (towards a contradiction), \( p_{\varepsilon,x} \) is not total, so for some \( \overline{u} \), \( p_{\varepsilon,x}(\overline{u}) \uparrow \); but then
\[
(\forall s) \neg G^{\Sigma,1}_{2n+1}(u^\Sigma_1(\varepsilon, x), \overline{u}, s) \text{ but } G^{\Sigma,1}_{2n+1}(u^\Sigma_1(\varepsilon, x), \overline{u}, 0),
\]
which is absurd.

Finally, (8) with (6) trivially imply (9).

The second theorem we need is a basic fact about the structure of the analytical hierarchy under determinacy hypotheses:

**Theorem 2** (DST(6E.14), ynm). If \( \text{Det}(\Sigma^1_{2n}) \) and \( X \) is of type \( \leq 1 \), then there is a recursive function \( u_2(\zeta) \), such that for all \( R \subseteq X \) and all \( \zeta \):

if \( R \) is \( \Delta^1_{2n+1} \) with code \( \zeta \), then
\[
R(x) \iff (\exists!\gamma)G^{\Pi,1}_{2n}(u_2(\zeta), x, \gamma) \iff (\exists\gamma)G^{\Pi,1}_{2n}(u_2(\zeta), x, \gamma)
\]
\[
\iff (\exists\gamma \in \Delta^1_{2n+1}(\zeta, x))G^{\Pi,1}_{2n}(u_2(\zeta), x, \gamma).
\]

Theorem 2 is an effective version of the extension to \( \Delta^1_{2n+1} \) of Lusin’s favorite characterization of the Borel sets, DST(2E.8):

\[
R \subseteq X \text{ is Borel} \iff R \text{ is the continuous, injective image of a closed } F \subseteq N.
\]

Its proof uses most everything preceding it in Section 6 of DST, including the version of the 3rd Periodicity Theorem DST(6E.1) in Theorem 1 and Solovay’s trick, as in the proof of DST(6E.12). However, the computation of codes we need for this uniform version is quite routine.

§2. The main result. We assume the notation of §1 and prove the following, uniform version of the Strong Spector-Gandy Theorem for \( \Pi^1_{2n+1} \) on page 2:

**Main Theorem.** If \( \text{Det}(\Sigma^1_{2n}) \) and \( X \) is of type \( \leq 1 \), then there is a recursive function \( v(\varepsilon) \) such that for all \( \varepsilon \) and all \( x \),

\[
G^{\Pi,1}_{2n+1}(\varepsilon, x) \iff (\exists!\alpha)G^{\Pi,1}_{2n}(v(\varepsilon), x, \alpha)
\]
\[
\iff (\exists\alpha \in \Delta^1_{2n+1}(\varepsilon, x))G^{\Pi,1}_{2n}(v(\varepsilon), x, \alpha).
\]

Crucial to the proof will be the following equivalence which holds for every \( R \subseteq N \times N \) (jrm):

\[
(\exists!\beta)(\exists!\gamma)R(\beta, \gamma) & (\exists\beta)(\exists!\gamma)R(\beta, \gamma) \iff (\exists!(\beta, \gamma))R(\beta, \gamma),
\]
where the binary quantifier on the right is defined in the natural way,

\[(\exists! (\beta, \gamma)) R(\beta, \gamma) \iff (\exists \beta)(\exists! \gamma) R(\beta, \gamma)\]

\[\& (\forall \beta)(\forall! \gamma)(\forall \beta')(\forall! \gamma') \left( [R(\beta, \gamma) \& R(\beta', \gamma')] \implies [\beta = \beta' \& \gamma = \gamma'] \right)\].

Proof of (13). (\(\implies\) Let \(\beta\) be the unique \(\beta\) such that \((\exists! \gamma) R(\beta, \gamma)\); if \(\beta'\) is such that \((\exists! \gamma) R(\beta', \gamma)\), then \((\exists! \gamma) R(\beta', \gamma)\), so \(\beta' = \beta\). So the left-hand-side gives unique \(\beta\) and \(\gamma\) for which \(R(\beta, \gamma)\) and then the right-hand-side holds with the pair \((\beta, \gamma)\).

(\(\impliedby\)) Let \((\beta, \gamma)\) be the unique pair \((\beta, \gamma)\) such that \(R(\beta, \gamma)\). This implies that \(\beta\) is the unique \(\beta\) such that for some \(\gamma\), \(R(\beta, \gamma)\), which implies that there is such a \(\beta\), which gives the first conjunct on the left-hand-side. In the same way, there is a \(\beta\) (namely \(\beta\)) such that \((\exists! \gamma) R(\beta, \gamma)\), which proves the second conjunct on the left.

To interpret the binary quantifier \((\exists! (\beta, \gamma))\) in terms of the unary \((\exists! \alpha)\), we need a recursive isomorphism of \(\mathcal{N}\) with \(\mathcal{N} \times \mathcal{N}\), and the simplest of these is the “interweaving” of two sequences

\[\alpha \mapsto ((\alpha(0), \alpha(2), \ldots), (\alpha(1), \alpha(3), \ldots)) = (\lambda(t)\alpha(2t), \lambda(t)\alpha(2t + 1))\];

using this, (13) takes the form

\[(14) \quad (\exists! \beta)(\exists! \gamma) R(\beta, \gamma) \& (\exists! \beta)(\exists! \gamma) R(\beta, \gamma)
\]

\[\iff (\exists! \alpha) R(\lambda(t)\alpha(2t), \lambda(t)\alpha(2t + 1)),\]

which is the form of this equivalence we will use in the proof that follows.

Proof of the Main Theorem. Fix \(\varepsilon\), let \(P_\varepsilon(x) \iff G_{2n+1}^{H,1}(\varepsilon, x)\) and set \(u(\varepsilon, x) = u_2(u_1(\varepsilon, x))\). By (9) and (5) in Theorem 1,

\[P_\varepsilon(x) \iff \{(u, t) : G_{2n+1}^{H,1}(u_1^{H}(\varepsilon, x), u, t)\}\]

is \(\Delta^1_{2n+1}\) with code \(u_1(\varepsilon, x)\);

and then by Theorem 2 with \(\zeta := u_1(\varepsilon, x)\) and for all \(u\) and \(t\),

\[(15) \quad G_{2n+1}^{H,1}(u_1^{H}(\varepsilon, x), u, t) \iff (\exists! \gamma) G_{2n}^{H,1}(u(\varepsilon, x), u, t, \gamma)
\]

\[\iff (\exists! \gamma) G_{2n}^{H,1}(u(\varepsilon, x), u, t, \gamma);\]

and so for all \(u\) and \(\beta\),

\[(16) \quad G_{2n+1}^{H,1}(u_1^{H}(\varepsilon, x), u, \beta(u)) \iff (\exists! \gamma) G_{2n}^{H,1}(u(\varepsilon, x), u, \beta(u), \gamma)
\]

\[\iff (\exists! \gamma) G_{2n}^{H,1}(u(\varepsilon, x), u, \beta(u), \gamma).\]
Using the first equivalence in (16) with (7) in Theorem 1 we get

\[ P_\varepsilon(x) \iff (\exists! \beta)(\forall u)(\exists \gamma)G_{2n}^{\Pi,1}(u, \beta(u), \gamma) \]

\[ \iff (\exists! \beta)(\exists \gamma)(\forall u)G_{2n}^{\Pi,1}(u, \beta(u), (\gamma)_u); \]

and using (6) of Theorem 1 with the second one, we get

\[ P_\varepsilon(x) \iff (\exists \beta)(\forall u)(\exists! \gamma)G_{2n}^{\Pi,1}(u, \beta(u), \gamma) \]

\[ \iff (\exists \beta)(\exists! \gamma)(\forall u)G_{2n}^{\Pi,1}(u, \beta(u), (\gamma)_u). \]

If we now set

\[ R_\varepsilon(x, \beta, \gamma) \iff (\forall u)G_{2n}^{\Pi,1}(u, \beta(u), (\gamma)_u) \]

and put (17) and (18) together, we get

\[ P_\varepsilon(x) \iff (\exists! \alpha)R_\varepsilon(x, \lambda(t)\alpha(2t), \lambda(t)\alpha(t+1)) \]

\[ \iff (\exists! \alpha)\left[(\forall u)G_{2n}^{\Pi,1}(u, \lambda(t)\alpha(2t)u, (\lambda(t)\alpha(t+1))u)\right]. \]

Finally, the relation within the braces \([\quad]\) is \(\Pi_2\), and so the good parametrization property gives us a recursive \(v(\varepsilon)\) such that

\[ P_\varepsilon(x) \iff (\exists! \alpha)G_{2n}^{\Pi,1}(x, \alpha, \varepsilon), \]

which is the first claim in the theorem.

The second claim is proved in the same way, with a couple of the steps justified by different arguments.

By (8) and (5) in Theorem 1

\[ P_\varepsilon(x) \iff \{(u, t) : G_{2n+1}^{\Pi,1}(u_1^\Pi(\varepsilon, x), u, t)\} \]

is \(\Delta_{2n+1}^1\) with code \(u_1(\varepsilon, x)\);

and then by Theorem 2 setting \(\zeta := u_1(\varepsilon, x)\), for all \(u\) and \(t\),

\[ G_{2n+1}^{\Pi,1}(u_1^\Pi(\varepsilon, x), u, t) \implies (\exists \gamma \in \Delta_{2n+1}^1(u_1(\varepsilon, x)))G_{2n}^{\Pi,1}(u, \beta(u), (\gamma)_u) \]

\[ \implies (\exists \gamma \in \Delta_{2n+1}^1(\varepsilon, x))G_{2n}^{\Pi,1}(u, \beta(u), (\gamma)_u) \]

\[ \implies (\exists \gamma)G_{2n}^{\Pi,1}(u, \beta(u), (\gamma)_u) \]

\[ \implies G_{2n+1}^{\Pi,1}(u_1^\Pi(\varepsilon, x), u, t), \]

where the step introducing the underlined proposition is justified because, as a general fact,

\[ \Delta_{2n+1}^1(u_1(\varepsilon, x)) \subseteq \Delta_{2n+1}^1(\varepsilon, x). \]
It follows that for all \( u \) and \( t \),
\[
G_{2n+1}^{\Pi_1}(u^I(\varepsilon, x), u, t, \gamma) \iff (\exists \gamma \in \Delta^1_{2n+1}(\varepsilon, x))G_{2n}^{\Pi_1}(u(\varepsilon, x), u, t, \gamma),
\]
and hence for all \( u \) and \( \beta \),
\[
G_{2n+1}^{\Pi_1}(u^I(\varepsilon, x), u, \beta(u)) \iff (\exists \gamma \in \Delta^1_{2n+1}(\varepsilon, x))G_{2n}^{\Pi_1}(u(\varepsilon, x), u, \beta(u), \gamma).
\]

Using this with (9) in Theorem 1, we get
\[
P_\varepsilon(x) \iff (\exists \beta \in \Delta^1_{2n+1}(\varepsilon, x))G_{2n}^{\Pi_1}(u(\varepsilon, x), u, \beta(u), \gamma);
\]
and from this, it is a standard exercise in restricted quantification to get
\[
P_\varepsilon(x) \iff (\exists \beta \in \Delta^1_{2n+1}(\varepsilon, x))\forall u (\exists \gamma \in \Delta^1_{2n+1}(\varepsilon, x))G_{2n}^{\Pi_1}(u(\varepsilon, x), u, \beta(u), \gamma);
\]
where \( v(\varepsilon) \) is introduced in (21), and this is the second claim in the theorem.

\section{Comments.}
The cumbersome computation of “uniformities” like \( u^1_1(\varepsilon, x), u^2_2(\zeta), \ldots, v(\varepsilon) \) is rarely needed to discover, prove and communicate results in effective descriptive set theory.

Consider, for example, a proposition of the form
\[
(\forall x)(\exists y \in \Delta^1_1(x)) P(x, y) \quad (P \subseteq \mathcal{X} \times \mathcal{Y}).
\]

When one is primarily interested in applications, one replaces it by the weaker
\[
(\exists f : \mathcal{X} \rightarrow \mathcal{Y})(\forall x) P(x, f(x))
\]
which (typically) suffices for the application we are considering. A proof of (9) often yields a classical proof of (**) which can be understood without direct reference to the effective notions. For a good example of this, see Corollary (39.20) in Kechris [1995], which is a “classical version” of the Third Periodicity Theorem DST(6E.1) as we quoted it on page 8.

In the effective theory, where we want to keep track of the parameters involved, the key, intuitive notion is that of a constructive (or effective, or uniform) proof. For example, to prove the first claim in the Main
Theorem, one tries to prove constructively that

\[ P(x) \text{ is } \mathbf{\Pi}_{2n+1} \iff \text{there is a set } R(x, \alpha) \text{ in } \mathbf{\Pi}_{2n} \text{ such that} \]

\[ P(x) \iff (\exists! \alpha) R(x, \alpha) \]

which is not very simple in this case because it requires constructive proofs of Theorems 1 and 2, but once thought out, these proofs yield (practically) routinely the “uniformities” \( u_1(\varepsilon, x) \), \( u_2(\zeta) \) etc., and the precise statements and rigorous proofs of these results and the Main Theorem.

A rigorous definition of “constructive proof” which justifies this mode of reasoning is given in Moschovakis [2010], but it is not practically useful except for mathematicians who are familiar with intuitionistic logic.

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