Asymptotic Theories of Classes Defined by Forbidden Homomorphisms

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Abstract

We study the first-order almost-sure theories for classes of finite structures that are specified by homomorphically forbidding a set $\mathcal{F}$ of finite structures. If $\mathcal{F}$ consists of undirected graphs, a full description of these theories can be derived from the Kolaitis-Prömel-Rothschild theorem, which treats the special case where $\mathcal{F} = \{K_n\}$. The corresponding question for finite sets $\mathcal{F}$ of finite directed graphs is wide open.

We present a full description of the almost-sure theories of classes described by homomorphically forbidding finite sets $\mathcal{F}$ of oriented trees; all of them are $\omega$-categorical. In our proof, we establish a result of independent interest, namely that every constraint satisfaction problem for a finite digraph has first-order convergence, and that the corresponding asymptotic theory can be described as a finite linear combination of $\omega$-categorical theories.

Keywords. Almost-sure theory, random digraph, graph homomorphism, oriented tree, constraint satisfaction problem

1 Introduction

A famous result of Glebskii, Kogan, Ligons’kii, and Talanov [12] and Fagin [9] states that the almost-sure theory $T_G$ of the class $\mathcal{G}$ of all finite undirected graphs is complete, i.e., every first-order property of graphs holds asymptotically almost surely, or its negation holds asymptotically almost surely (this is sometimes referred to as a first-order 0-1 law). Moreover, $T_G$ is $\omega$-categorical, i.e., it has exactly one countable model up to isomorphism, the so-called Rado graph [23], which is of fundamental importance in model theory and can be described as the Fraïssé-limit of the class of all finite graphs [14].

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If instead of $G$ we consider the class $G_3$ of all finite triangle-free graphs, i.e., finite graphs that do not contain a $K_3$ subgraph, we still get a first-order 0-1 law, and the resulting almost-sure theory again has a single countably infinite model $M_3$. However, somewhat surprisingly, this model is not the Fraïssé-limit of the class of all finite undirected triangle-free graphs: Erdős, Kleitman, and Rothschild [7] proved that if $n$ tends to infinity, almost all triangle-free graphs with $n$ vertices are bipartite, i.e., have a homomorphism to $K_2$. Clearly, there are triangle-free graphs $G$ without a homomorphism to $K_2$ (e.g., the cycle with five vertices, $C_5$). The structure $M_3$ does not contain copies of $G$, and hence $M_3$ is not the Fraïssé-limit of the class of all finite triangle-free graphs.

The mentioned result about triangle-free graphs has been generalised by Kolaitis, Prömel, and Rothschild [16] to the class of finite graphs without a copy of $K_k$, for every $k \geq 3$.

**Theorem 1** (Kolaitis, Prömel, and Rothschild [16]). For every $k \geq 3$, a random $K_k$-free graph is asymptotically almost surely $(k - 1)$-partite.

In fact, Kolaitis, Prömel, and Rothschild determined the almost-sure theory of the class of $K_k$-free graphs (see Theorem 6 below). In subsequent work, Prömel and Steger [22] studied classes of finite graphs that are given by forbidding some graph $H$ as a (weak) subgraph. In this article, we study graph classes that are specified by forbidding a set $\mathcal{F}$ of finite graphs homomorphically, rather than forbidding them as (induced or weak) subgraphs. In particular, we will be interested in the almost-sure theory of the class

$$\text{Hom-Forb}(\mathcal{F}) := \{ G \in \mathcal{G} \mid \text{no } F \in \mathcal{F} \text{ has a homomorphism to } G \}.$$  

The difference between subgraphs and homomorphisms is irrelevant if $\mathcal{F} = \{K_n\}$, because $K_n$ has a homomorphism to a graph $G$ if and only if $K_n$ is isomorphic to an (induced) subgraph of $G$. Quite remarkably, one can obtain a complete description of the almost-sure theory of $\text{Hom-Forb}(\mathcal{F})$ for every set $\mathcal{F}$ of finite graphs (even if $\mathcal{F}$ is infinite; see Theorem 7). Let $\chi(G)$ be the chromatic number of $G$, i.e., the smallest $n$ such that $G$ has a homomorphism to $K_n$. If $\mathcal{F} = \emptyset$ then $\text{Hom-Forb}(\mathcal{F}) = \mathcal{G}$ and the almost-sure theory is the theory of the Rado graph [23]. For non-empty $\mathcal{F}$, define $k_\mathcal{F} := \min_{G \in \mathcal{F}} \chi(G)$. We show in Section 3 that the graphs in $\text{Hom-Forb}(\mathcal{F})$ are asymptotically almost surely $(k_\mathcal{F} - 1)$-partite, that $\text{Hom-Forb}(\mathcal{F})$ satisfies a first-order 0-1 law, and that the almost-sure theory of $\text{Hom-Forb}(\mathcal{F})$ is $\omega$-categorical, i.e., has an up to isomorphism unique countable model.

The class $\text{Hom-Forb}(\mathcal{F})$ has also been studied in model theory. The *age* of a relational structure $H$ is the class of all finite structures that embed into $H$. Cherlin, Shelah, and Shi [3] showed that there exists an (up to isomorphism unique) countable model-complete structure $M_\mathcal{F}$ whose age equals $\text{Hom-Forb}(\mathcal{F})$. However, if $k_\mathcal{F} > 2$ then $\text{Hom-Forb}(\mathcal{F})$ contains graphs that are not $(k_\mathcal{F} - 1)$-colorable, and hence this model does not satisfy the almost-sure theory of $\text{Hom-Forb}(\mathcal{F})$ (unless $\mathcal{F} = \emptyset$).

If $\mathcal{F}$ is a set of finite directed graphs, then much less is known about the almost-sure theory of $\text{Hom-Forb}(\mathcal{F})$, even if $\mathcal{F}$ is finite. Some results have been obtained recently by
Kühn, Osthus, Townsend, and Zhao [18] for Hom-Forb($T_3$) and Hom-Forb($C_3$), where $T_3$ is the transitive tournament with three vertices and $\vec{C}_3$ is the directed cycle with three vertices, answering questions that have been raised in model theory [5]. In particular, they proved that asymptotically almost surely the graphs in Hom-Forb($T_3$) are bipartite. It is easy to come up with infinite sets $\mathcal{F}$ of finite digraphs such that Hom-Forb($\mathcal{F}$) does not have a 0-1 law (see Example 3).

1.1 Our Results

Our proof that every class of undirected graphs defined by forbidden homomorphisms have a 0-1 law and an $\omega$-categorical almost-sure theory can be found in Section 3 (Theorem 7). For classes of directed graphs, we present in Section 3 a full description of the almost-sure theories of classes of digraphs described by forbidding finitely many orientations of finite trees; again, all of these classes have a 0-1 law and an $\omega$-categorical almost-sure theory (Theorem 8). To prove this, we show a result that is of independent interest: For every finite digraph $G$ the CSP of $G$ satisfies a first-order convergence law, and the asymptotic theory of the CSP is linearly composed from finitely many $\omega$-categorical theories in a sense that will be made precise. This result and some others generalise from digraphs to general relational structures. For concreteness and clarity of presentation, we focus on the case of graphs and digraphs in this article.

1.2 Connection with Random Graphs

With the same proof technique as used for oriented graphs to prove our result, we can also prove a result about random (simple undirected) graphs with a homomorphism to a fixed finite graph, which we believe is of independent interest.

**Theorem 2.** Let $H$ be a finite simple undirected graph. Then asymptotically almost surely, a graph with vertices $\{1, \ldots, n\}$ and a homomorphism to $H$ is $\ell$-partite, where $\ell$ is the size of the largest clique contained in $H$, and each color class asymptotically almost surely has at least $\lfloor n/\ell \rfloor$ many elements.

A weaker result (with $\lfloor n/\ell \rfloor$ replaced by $n/2\ell$) for the special case where $H = K_\ell$ has been shown by Turner [25]. Our strengthening relies on the proof technique that we use for studying the almost sure theory of CSP($D$) for oriented graphs $D$ (see Remark 2). Our results also imply that for a graph $G$ with vertices $\{1, \ldots, n\}$ and a homomorphism to another fixed finite graph $H$ asymptotically almost surely there exists a unique partition of the vertices into $\ell$ parts such that all edges of $G$ are between distinct parts of the partition; here, $\ell$ is the size of the largest clique in $H$. This generalises a result of Dyer and Frieze [6, Theorem 3.1] from $H = K_\ell$ to arbitrary finite graphs $H$, and already follows from our results for undirected graphs (see Remark 1).
1.3 Connection with Model Theory

The \( \omega \)-categorical theories that arise in our main result, Theorem \[8\], satisfy a property that is of interest in model theory: they are pseudo finite, i.e., every finite subset of the theory has a finite model. An example of a pseudo-finite theory is the theory of the Rado graph, which follows from the fact that its first-order theory equals the almost-sure theory of finite undirected graphs. A class of theories whose pseudo-finiteness can be shown similarly was isolated by Kruckman \[17\]. Showing that a theory is pseudo finite can be quite challenging. It is a major open question whether the Fraïssé-limit of the class of all finite triangle-free graphs has a pseudo-finite theory (see, e.g., \[8\]). The mentioned result of Erdős, Kleitman, and Rothschild shows that this problem cannot be approached via almost sure theories of finite structures in the same way as for the Rado graph.

1.4 Connection with Database Theory

Note that sets \( F \) of finite structures that are homomorphically forbidden can be interpreted from a database perspective: each structure \( \mathcal{F} \in F \) can be viewed as a so-called conjunctive query \( \phi \), by turning the elements of the structure into existentially quantified variables and the relations of the structure into a conjunction of atomic formulas. Conjunctive queries are the most important class of queries encountered in database theory. It is well-known and easy to see that a finite structure \( B \) (i.e., a database) satisfies \( \phi \) if and only if \( \mathcal{F} \) has a homomorphism into \( B \); see \[3\]. Results about Hom-Forb(\( F \)) can then be interpreted as statements about the uniform distribution on finite structures that do not satisfy the given set of conjunctive queries. An important subclass of conjunctive queries are those that are Berge acyclic; this is precisely the notion of acyclicity that we need for our results. For other acyclicity notions, see \[10\].

2 Preliminaries

A digraph \( D \) is a pair \((V, E)\) consisting of a set of vertices \( V(D) = V \) and a set \( E(D) = E \subseteq V^2 \) of edges. We write

- \( T_k \) for the transitive tournament on \( k \) vertices, i.e., the digraph \( \{1, \ldots, k\}; < \) where \(<\) denotes the usual strict linear order on \( \{1, \ldots, k\} \).
- \( P_k \) for the directed path of length \( k \), i.e., the digraph
  \[ \{(1, 2), (2, 3), \ldots, (k-1, k)\}. \]
- \( C_k \) for the directed cycle of length \( k \), i.e., the digraph
  \[ \{(0, 1), \ldots, (k-1), ((u, v) \mid u = v + 1 \mod k}\}. \]
A digraph is symmetric if \((x, y) \in E\) implies \((y, x) \in E\) for all \(x, y \in V\) and irreflexive if \((x, x) \notin E\) for every \(x \in V\). If \(G\) is a digraph and \(v \in V(G)\), then we write \(G - v\) for the digraph \((V(G) \setminus \{v\}, E(G) \cap (V(G) \setminus \{v\})^2)\).

An undirected graph \(G\) is a pair \((V, E)\) where \(V\) is a set of vertices and \(E\) is a set of 2-element subsets of \(V\). There is an obvious bijection between undirected graphs with vertices \(V\) and irreflexive symmetric digraphs with vertices \(V\). We write \(K_n\) for the undirected graph with vertex set \(V = \{1, \ldots, n\}\) whose edge set consists of all 2-element subsets of \(V\). If \(G\) is an undirected graph, then an orientation of \(G\) is a digraph \(O\) such that \((x, y) \in E(O)\) implies \(\{x, y\} \in E(G)\) and for every \(\{x, y\} \in E(G)\) either \((x, y) \in E(O)\) or \((y, x) \in E(O)\), but not both. An oriented graph is an orientation of some undirected graph (i.e., an irreflexive digraph whose edge relation is antisymmetric).

### 2.1 Almost Sure Theories

Let \(\mathcal{C}\) be an isomorphism-closed class of finite (directed or undirected) graphs and let \(\mathcal{C}_n\) be the set of graphs from \(\mathcal{C}\) with vertices \(\{1, \ldots, n\}\). Graphs with vertex set \(\{1, \ldots, n\}\) for some \(n \in \mathbb{N}\) will also be called labelled.

We consider first-order logic over graphs. An atomic formula is an expression of the form \(E(x, y)\), of the form \(x = y\), or of the form \(\bot\) (for false) for some variables \(x, y\). First-order formulas (over graphs) are built from atomic formulas by the usual Boolean connectives and universal and existential quantification. Let \(\phi\) be a first-order sentence. We write \(G \models \phi\) if \(G\) satisfies \(\phi\) in the usual sense. Define

\[
\phi^C_n := \frac{|\{G \in \mathcal{C}_n \mid G \models \phi\}|}{|\mathcal{C}_n|}.
\]

If \(\lim_{n \to \infty} \phi^C_n\) exists, we denote the limit by \(\phi^C\). If this limit exists for every first-order sentence \(\phi\), then we say that \(\mathcal{C}\) has first-order convergence.

If \(\phi^C = 1\) we say that \(\phi\) holds in \(\mathcal{C}\) asymptotically almost surely. More generally, a property \(P\) holds asymptotically almost surely in \(\mathcal{C}\) if the fraction of structures in \(\mathcal{C}_n\) that has the property \(P\) tends to one as \(n\) tends to infinity.

A theory is a set of first-order sentences; a model of a theory \(T\) is a structure that satisfies all sentences in \(T\). A theory \(T\) is complete if it has a model and for every first-order sentence \(\phi\) either \(\phi \in T\) or \(\neg \phi \in T\). The almost sure theory of \(\mathcal{C}\) is the set of first-order sentences \(\phi\) such that \(\phi^C = 1\) and denoted by \(T^C\). We say that \(\mathcal{C}\) has a first-order 0-1 law if \(T^C\) is a complete theory.

### 2.2 Graph Homomorphisms Background

If \(G\) and \(H\) are digraphs, then a homomorphism from \(G\) to \(H\) is a map \(h : V(G) \to V(H)\) such that for all \((x, y) \in E(G)\) we have \((h(x), h(y)) \in E(H)\). We write \(G \to H\) if there exists a homomorphism from \(G\) to \(H\), and otherwise we write \(G \not\to H\). Two graphs \(G\) and
$H$ are called homomorphically equivalent if $G \rightarrow H$ and $H \rightarrow G$. We write $\text{CSP}(H)$ (for constraint satisfaction problem) for the class of all finite digraphs $G$ with a homomorphism to $H$. If $\mathcal{F}$ is a set of graphs, then we write $\text{Hom-Forb}(\mathcal{F})$ for the class of finite digraphs $G$ such that no digraph in $\mathcal{F}$ maps homomorphically to $G$. It is easy to see that for every digraph $H$ there exists a set of finite digraphs $\mathcal{F}$ such that $\text{CSP}(H) = \text{Hom-Forb}(\mathcal{F})$. We write

- $G_1 \times G_2$ for the digraph with vertices $V(G_1) \times V(G_2)$ and edges
  \[
  \{(a_1, b_1), (a_2, b_2) \mid (a_1, a_2) \in E(G_1), (b_1, b_2) \in E(G_2)\}.
  \]
- $G^2$ instead of $G \times G$.
- $G_1 \uplus G_2$ for the disjoint union of $G_1$ and $G_2$, i.e., for the digraph with vertices $V(G_1) \cup V(G_2)$ and edges $E(G_1) \cup E(G_2)$ where we assume that $V(G_1) \cap V(G_2) = \emptyset$ (otherwise, rename the elements of $V(G_1)$ and $V(G_2)$).

A homomorphism from $H$ to $H$ is called an endomorphism of $H$. Note that if $H$ is finite and if $h$ is a bijective endomorphism of $H$, then the inverse $h^{-1}$ of $h$ is also an endomorphism, and $h$ is called an automorphism of $H$. The set of all automorphisms of $H$ forms a group with respect to composition and is denoted by $\text{Aut}(H)$. An orbit under the action of $\text{Aut}(H)$ is a set of the form $\{\alpha(a) \mid \alpha \in \text{Aut}(H)\}$ for some $a \in V(H)$. A finite digraph $H$ is called a core if all endomorphisms of $H$ are automorphisms. It is easy to see that every finite digraph $G$ is homomorphically equivalent to a finite core digraph $H$, and that this core digraph is unique up to isomorphism [13], and will be called the core of $H$.

All these definitions work analogously also for undirected graphs instead of digraphs. Note that for every $k \geq 1$, an undirected graph $G$ is $k$-colourable if and only if $G \rightarrow K_k$. The definition for undirected graphs and for the corresponding symmetric and irreflexive digraphs is usually equivalent, via the mentioned bijection. However, note that if $H$ is a symmetric and irreflexive digraph, then $\text{CSP}(H)$ might contain digraphs that are not symmetric, whereas if $H$ is an undirected graph, then $\text{CSP}(H)$ by definition only contains undirected graphs.

If $\mathcal{F}$ is a set of undirected graphs then $\text{Hom-Forb}(\mathcal{F})$ (defined in the introduction) denotes a class of undirected graphs. A tree is a connected undirected graph without cycles. We later need the following result of Nešetřil and Tardif ([21]; see Theorem 2.4.4 in [11]).

**Theorem 3.** For every finite non-empty set of orientations of finite trees $\mathcal{F}$ there exists a finite digraph $D = (V, E)$ such that $\text{Hom-Forb}(\mathcal{F}) = \text{CSP}(D)$.

Clearly, we may additionally require that the finite digraph $D$ from Theorem 3 is a core, and with this assumption the structure $D$ is unique up to isomorphism, and called the dual of $\mathcal{F}$.
Example 1. For \( k \geq 1 \), if \( \mathcal{F} = \{ \bar{P}_{k+1} \} \), then we have \( \text{Hom-Forb}(\mathcal{F}) = \text{CSP}(T_k) \).

A first-order formula \( \phi(x_1, \ldots, x_k) \) is called primitive positive if it has the form

\[
\exists y_1, \ldots, y_n(\psi_1 \land \cdots \land \psi_m)
\]

for some atomic formulas \( \psi_1, \ldots, \psi_m \) over the variables \( x_1, \ldots, x_k, y_1, \ldots, y_n \). Existential positive formulas are disjunctions of primitive positive formulas. A first-order formula is called existential (universal) if it does not contain universal (existential) quantifiers and all negation symbols appear in front of atomic formulas; note that every existential positive formula is existential.

Lemma 1. Let \( D \) be a finite core digraph and \( u \in V(D) \). Then the orbit of \( u \) in \( \text{Aut}(D) \) has a primitive positive definition in \( D \).

The conjunctive query of a digraph \( D \) is the (primitive positive and quantifier-free) formula \( \phi \) defined as follows: we have a variable \( x_a \) for every vertex \( a \) of \( D \), and for every edge \( (a, b) \in E(D) \) we add a conjunct \( E(x_a, x_b) \) to \( \phi \).

2.3 Model Theory Background

Many results about the first-order logic of graphs hold more generally also for first-order logic over general relational structures; and in fact, this more general setting of relational structures will also be convenient in some of our proofs in Section 4.5. For an introduction to model theory of general relational structures we refer to [15].

An embedding of \( G \) into \( H \) is an isomorphism of \( G \) with a substructure of \( H \); a bijective embedding is called an isomorphism. A reduct of a structure \( H \) is a structure \( G \) obtained from \( H \) by dropping some of the relations of \( H \); in this case, \( H \) is called an expansion of \( G \). A theory is simply a set of first-order sentences. A theory is satisfiable if it has a model. A satisfiable theory \( T \) is called \( \omega \)-categorical if all countable models of \( T \) are isomorphic.

A first-order theory \( T \) is model complete if for every embedding \( e \) between models \( G \) and \( H \) of \( T \) preserves all first-order formulas \( \phi \), i.e., if \( G \) satisfies \( \phi(a_1, \ldots, a_n) \), then \( H \) satisfies \( \phi(e(a_1), \ldots, e(a_n)) \). A structure is called model complete or \( \omega \)-categorical if its first-order theory is. Reducts of \( \omega \)-categorical structures are \( \omega \)-categorical as well (see, e.g., [15]).

Note that all finite structures are model-complete and \( \omega \)-categorical. Further examples of structures that are both model-complete and \( \omega \)-categorical are provided by structures that are homogeneous in a finite relational signature; a structure \( S \) is called homogeneous if all isomorphisms between finite substructures can be extended to an automorphism of \( S \).

A class of finite relational structures \( \mathcal{C} \) has the amalgamation property if for all structures \( A, B_1, B_2 \in \mathcal{C} \) such that \( A \) is a substructure of both \( B_1 \) and \( B_2 \) such that \( V(A) = \)
V(B_1) \cap V(B_2) there exists C \in \mathcal{C} and embeddings f_1: B_1 \rightarrow C and f_2: B_2 \rightarrow C such that f_1(a) = f_2(a) for all a \in A. A class of finite structures which is closed under substructures, isomorphism, has countably many isomorphism classes, and has the amalgamation property is called an \textit{amalgamation class}.

\textbf{Theorem 4} (Fraïssé’s theorem; see, e.g. [14]). Let \mathcal{C} be an amalgamation class. Then there exists an up to isomorphism unique countable homogeneous structure whose age is \mathcal{C}, which is called the \textit{Fraïssé-limit} of \mathcal{C}.

Two theories are called \textit{equivalent} if they have the same models. Two theories are called \textit{companions} if they have the same universal consequences. A companion of a theory T which is model-complete is called a \textit{model companion} of T; a model companion might not exist, but if it exists, then it is unique up to equivalence (see, e.g., [15]).

\textbf{Theorem 5} (Theorem 4.5.8 in [2]). A countable \(\omega\)-categorical structure H is model complete if and only if it has a homogeneous expansion by relations that are both existentially and universally definable in H.

An \textit{extension axiom} is a first-order sentence of the form \(\forall x_1, \ldots, x_n \exists y. \psi\) where \(\psi\) is quantifier-free. The following well-known fact can be shown by a back-and-forth argument.

\textbf{Lemma 2.} Every first-order theory of a homogeneous structure in a finite relational language is equivalent to a set of extension axioms.

\textbf{Lemma 3.} Let D be a finite digraph (or graph) and let

\[ T := \{\neg \phi \mid \phi \text{ is primitive positive and } D \models \neg \phi\}. \]

Then T has an \(\omega\)-categorical model companion. Its countable model U(D) is universal in the sense has the property that a countable digraph (or graph) C has an embedding into U(D) if and only if C maps homomorphically to D.

\textit{Proof.} The first part of the statement follows from Saracino’s theorem [23]. The second statement follows from a standard compactness argument; see, e.g., [2].

For the special case that \(D = K_k\), for \(k \geq 2\), Kolaitis, Prömel, and Rothschild [16] also proved the following.

\textbf{Theorem 6} ([16]). For \(k \geq 2\), the almost-sure theory of CSP(K_k) equals the first-order theory of U(K_k).

The graph U(K_k) will be called the \textit{generic k-partite graph}. Note that for \(k \geq 2\) this graph is not homogeneous.
3 Forbidden Undirected Graphs

The results of Kolaitis, Prömel, and Rothschild (Theorem 1 and Theorem 6) have a remarkably general consequence which provides a complete description of the almost-sure theory of Hom-Forb(F) for all sets of finite undirected graphs F.

**Theorem 7.** Let F be a non-empty set of finite undirected graphs. Then the almost-sure theory of Hom-Forb(F) equals the first-order theory of the generic (k_F - 1)-partite graph.

**Proof.** Let k := k_F. We claim that no graph F ∈ F is (k − 1)-partite. By the definition of k we have that k ≤ χ(F). It follows that there is no homomorphism from F to K_{k−1}, which proves the claim. The claim implies that every (k − 1)-partite graph is contained in Hom-Forb(F).

Next, observe that Hom-Forb(F) ⊆ Hom-Forb(K_k). To see this, let F ∈ F be such that k = min χ(F). Suppose for contradiction that there is a graph H ∈ Hom-Forb(F) \ Hom-Forb(K_k). Then there exists a homomorphism from K_k to H. But then note that F has a homomorphism to K_k since χ(F) ≤ k, and hence F has a homomorphism to H, a contradiction to the assumption that H ∈ Hom-Forb(F).

By Theorem 1 the graphs in Hom-Forb(K_k) are asymptotically almost surely (k − 1)-partite, and hence the graphs in Hom-Forb(F) are asymptotically almost surely (k − 1)-partite as well. Since Hom-Forb(F) contains the class of all (k − 1)-partite graphs, the almost-sure theory T of Hom-Forb(F) equals the (complete) almost-sure theory of the class of (k − 1)-partite graphs: if ϕ is a first-order sentence that holds asymptotically almost surely in the class of (k − 1)-partite graphs, then it holds asymptotically almost surely in the class of Hom-Forb(K_k), and hence a fortiori in the class Hom-Forb(F). Therefore, T is the theory of the generic (k − 1)-partite graph by Theorem 6.

**Definition 1.** Let H be a finite undirected graph. Then the co-chromatic number of H is the smallest k ∈ N such that there exists a graph G such that G ̸→ H and G → K_k; note that K_{χ(H)+1} ̸→ H and K_{χ(H)+1} → K_{χ(H)+1} so k ≤ χ(H) + 1.

The co-chromatic number of K_n is n + 1. The co-chromatic number of C_{2n+1}, for n ≥ 2, is 3 because C_{2n−1} ̸→ H and C_{2n−1} → K_3.

**Corollary 1.** Let H be a finite undirected graph with co-chromatic number k. Then the almost-sure theory of CSP(H) equals the almost-sure theory of CSP(K_{k−1}).

**Proof.** Let F be the class of all finite undirected graphs that do not have a homomorphism to H. Recall that CSP(H) = Hom-Forb(F). Let G be a graph such that G ̸→ H and G → K_k, which exists by the definition of the co-chromatic number k. So G ∈ F. Also note that k_F = k by the definition of k. By Theorem 7 the almost-sure theory of Hom-Forb(F) equals the theory of the generic (k − 1)-partite graph, which equals the almost-sure theory of CSP(K_{k−1}) by Theorem 6. □
Example 2. Let $H$ be the Gr"otzsch graph, which is the (up to isomorphism unique) triangle-free graph of chromatic number four with the smallest number of vertices. The co-chromatic number of $H$ is 3 because $K_3 \nrightarrow H$ and $K_3 \rightarrow K_3$. By Corollary 1 the almost-sure theory of $\text{CSP}(H)$ equals the almost-sure theory of $\text{CSP}(K_2)$, i.e., the almost sure theory of the class of all finite bipartite graphs.

Remark 1. Note that Corollary 1 implies that the generalisation of the result of Dyer and Frieze mentioned in Section 1.2 follows from their result for $H = K_\ell$.

4 Forbidden Digraphs

There are sets of digraphs $F$ where $\text{Hom-Forb}(F)$ does not have a first-order 0-1 law, as the following example illustrates. If $D = (V, E)$ is a digraph, we also write $x \rightarrow y$ instead of $(x, y) \in E$ (and instead of $E(x, y)$ in logical formulas). We write $x_1 \rightarrow x_2 \rightarrow x_3$ as a shortcut for $x_1 \rightarrow x_2 \wedge x_2 \rightarrow x_3$ and we write $x_1 \nrightarrow x_2$ as a shortcut for $(x, y) \notin E$ (and instead of $\neg E(x, y)$ in logical formulas).

Example 3. Let $F$ be the class of disjoint unions of the directed path with four vertices $P_4$ and a $\vec{C}_k$ where $k$ is not a multiple of 3. It is easy to see that $\text{Hom-Forb}(F) = \text{CSP}(D)$ where $D$ is the disjoint union of $\vec{C}_3$ and $T_3$. If $\phi$ is the primitive positive sentence

$$\exists x_1, x_2, x_3(x_1 \rightarrow x_2 \rightarrow x_3 \wedge x_1 \rightarrow x_3)$$

then $\phi^\omega$ equals $1/2$ as we will see in Example 4.

In the following we focus on the special case where $F$ is a finite set of orientations of trees.

4.1 Forbidden Orientations of Trees

We will provide an explicit description of the almost-sure theory of $\text{Hom-Forb}(F)$ for finite sets $F$ of orientations of trees.

Theorem 8. Let $F$ be a finite set of finite oriented trees. Then the almost-sure theory of $\text{Hom-Forb}(F)$ is (pseudo-finite and) $\omega$-categorical; In particular, $\text{Hom-Forb}(F)$ has a first-order 0-1 law.

Our proof uses Theorem 3 which asserts the existence of a digraph $D$ such that $\text{CSP}(D) = \text{Hom-Forb}(F)$. The following properties of the digraph $D$ will be useful.

Proposition 1 ([19]). For every finite non-empty set of orientations of finite trees $F$ the dual digraph $D$ of $F$ (see Theorem 3) has no directed cycles, and has no non-trivial automorphisms.

Proof. If $D$ contains a directed cycle, then all orientations of trees homomorphically map to $D$, a contradiction. The fact that it has no non-trivial automorphisms is stated in Lemma 4.1 in [19].
4.2 Density

The key to studying random digraphs with a homomorphism to a finite digraph $D$ is a certain notion of density, which has already been studied for undirected graphs [20] and which we also use for directed graphs.

**Definition 2.** Let $D$ be a digraph. Then the density of $D$ is defined to be the maximum of

$$\sum_{(u,v) \in E(D)} \delta(u)\delta(v)$$

over all probability distributions $\delta: V(D) \to [0,1]$ (satisfying $\sum_{v \in V(D)} d(v) = 1$). If $\delta$ realizes this maximum, we call it a density function of $D$, and the support $\{v \in V(D) \mid \delta(v) > 0\}$ of $\delta$ is called a densest subgraph of $D$.

The following is a reformulation of a result of Motzkin and Straus [20] for oriented graphs.\footnote{The authors thank Oleg Pikhurko for pointing this out to us.}

**Lemma 4 ([20]).** Let $D$ be an oriented graph. Then the density of $D$ equals

$$\frac{1}{2}(1 - 1/k) = \frac{k - 1}{2k}$$

where $k$ is the largest integer such that $D$ has a subgraph which is an orientation of a $K_k$.

**Definition 3** (Blow-up graphs and digraphs). Let $G$ be an undirected graph. A blow-up of $G$ is a graph obtained from $G$ by adding for each vertex $u \in V(G)$ arbitrarily many ‘copies’ of $u$, i.e., vertices with the same neighbourhood as $u$; among the different copies of $u$ there are no edges. Blow-ups of digraphs $D$ are defined analogously.

**Lemma 5.** Let $D$ be an oriented graph and let $\delta$ be a density function of $D$. Then the subgraph of $D$ induced by the support of $\delta$ is an orientation of a blow-up of $K_k$, for some $k \geq 2$, such that the sum over $\delta(u)$ for all the copies $u$ of the same vertex of $K_k$ equals $1/k$.

**Proof.** We prove the statement by induction on $n := |V(D)|$. If $n = 1$ then the statement is trivial. If $n > 1$ we distinguish three cases.

- for some $u \in V(D)$ we have $\delta(u) = 0$; in this case, we apply the inductive assumption for the oriented graph $D - u$ and are done.

- for all $u \in V(D)$ we have that $\delta(u) > 0$, and $D$ is not an orientation of $K_n$. Let $v_1, \ldots, v_n$ be the vertices of $D$ and assume without loss of generality that neither
$(v_1, v_2)$ nor $(v_2, v_1)$ is an edge in $D$. We will write $v_1, \ldots, v_n$ for the vertices of $D$. For $x_1, \ldots, x_n \in \mathbb{R}$, define

$$f(x_1, \ldots, x_n) := \sum_{(v_i, v_j) \in E(D)} x_i x_j.$$  

Suppose that $x_1, \ldots, x_n \geq 0$ are chosen such that $x_1 + \cdots + x_n = 1$ and $f(x_1, \ldots, x_n)$ is maximal. Let $E$ be $E(D) \cup \{(v, u) \mid (u, v) \in E(D)\}$. Note that if $\sum_{(v_i, v_j) \in E} x_i \neq \sum_{(v_i, v_j) \in E} x_i$ then we do not have a maximum of $f$ at $(x_1, \ldots, x_n)$. Indeed, if $\sum_{(v_i, v_j) \in E} x_i > \sum_{(v_i, v_j) \in E} x_i$, then

$$f(x_1 + t, x_2 - t, x_3, \ldots, x_n) > f(x_1, \ldots, x_n)$$

for all $t \in [0, x_2]$, which contradicts our assumption. Therefore, we have

$$\sum_{(v_i, v_j) \in E} x_i = \sum_{(v_i, v_j) \in E} x_i$$

and for all $t \in [0, x_2]$

$$f(x_1 + t, x_2 - t, x_3, \ldots, x_n) = f(x_1, x_2, x_3, \ldots, x_n).$$

Applied to $x_i := \delta(v_i)$ for $i \in \{1, \ldots, n\}$, this means that

$$\delta(v_1) + \delta(v_2), 0, \delta(v_3), \ldots, \delta(v_n)$$

also maximises $f$, and hence defines another density function $\delta'$ of $D$. We may apply the inductive assumption to the graph $D - v_2$ and the density function $\delta'$, and obtain that $D - v_2$ has a subgraph $H$ which is an orientation of a blow-up of $K_k$, for some $k \geq 2$, such that the sum over $\delta'(u)$ for all the copies $u$ of a vertex of $K_k$ equals $1/k$. We claim that the subgraph $H'$ of $D$ with vertices $V(H) \cup \{v_2\}$ is a blow-up of $K_k$ as well. We first notice that for any $v_3 \in V(H)$, if $(v_1, v_3) \notin E$ then $(v_2, v_3) \notin E$, because otherwise $\sum_{(v_1, v_i) \in E} \delta(v_i) = \sum_{(v_2, v_i) \in E} \delta(v_i)$, so $\sum_{(v_2, v_i) \in E} \delta(v_i) < \sum_{(v_2, v_i) \in E} \delta(v_i)$, in contradiction to $\delta'$ being a density function of $D$. Moreover, since $\sum_{(v_1, v_i) \in E} \delta(v_i) = \sum_{(v_2, v_i) \in E} \delta(v_i)$, necessarily $v_2$ has an edge in $E$ to every other vertex in $V(H)$ connected to $v_1$ in $E$: otherwise, $\sum_{(v_2, v_i) \in E} \delta(x_i) < \sum_{(v_2, v_i) \in E} \delta(x_i)$ which again leads to a contradiction to the maximality of $\delta$.

So $H'$ is indeed a blow-up of $K_k$; note that by the induction hypothesis the sum of $\delta(u)$ over all the copies $u$ of the same vertex of $K_k$ equals $1/k$, which proves the statement of the lemma.

- $D$ is an orientation of $K_n$. Note that in this case the probability distribution $\delta$ on $\{v_1, \ldots, v_n\}$ maximises

$$\frac{(\delta(v_1) + \cdots + \delta(v_n))^2 - \delta(v_1)^2 - \cdots - \delta(v_n)^2}{2} = \frac{1 - (\delta(v_1)^2 + \cdots + \delta(v_n)^2)}{2}. \quad (2)$$
The expression $\delta(v_1)^2 + \cdots + \delta(v_n)^2$ is minimised if $\delta(v_1) = \cdots = \delta(v_n) = 1/n$. To see this, suppose that $\delta(v_1) > 1/n > \delta(v_2)$. Then the family
\[(y_1, \ldots, y_n) := (\delta(v_1) - 1/n + \delta(v_2), 1/n, \delta(v_3), \ldots, \delta(v_n))\]
is such that $\sum_i y_i = 1$ and $y_1^2 + \cdots + y_n^2 < \delta(v_1)^2 + \cdots + \delta(v_n)^2$. Indeed, for all $x > y \in [0, 1]$, $x^2 + y^2 \geq (x - \varepsilon)^2 + (y + \varepsilon)^2$ for all $\varepsilon \in (0, x - y)$. We can reason analogously if $\delta(v_i) > 1/n > \delta(v_j)$, for $i, j \in \{1, \ldots, n\}$, and therefore we can modify the family finitely many times to obtain a family $x_i$ satisfying $x_1 = \cdots = x_n = 1/n$. Hence, the maximum of (2) is attained if all the $\delta(v_i)$ are equal.

This concludes the inductive step and the lemma is proven.

If $D$ does not contain directed cycles, we can refine the statement of Lemma 5 even further.

**Definition 4** (Butterfly). A $k$-butterfly, for $k \geq 1$, is a blow-up of
\[T'_k := (\{0, \ldots, k\}, \{(i, j) \mid i < j\} \setminus \{(0, k)\}).\]

For an illustration, see Figure 1. Note that both a blow-up of $T_k$ and a $k$-butterfly have a subgraph isomorphic to $T_k$. However, note that $T'_k$ is a core and does not have a homomorphism to $T_k$, and $T_{k+1}$ does not have a homomorphism to $T'_k$.

**Lemma 6.** Let $D$ be an oriented graph without directed cycles and let $\delta$ be a density function of $D$. Then the subgraph $S$ of $D$ induced by $\{v \in V(D) \mid \delta(v) > 0\}$ is either a blow-up of $T_k$ or a $k$-butterfly, for some integer $k \geq 1$. In the first case, the sum over $\delta(u)$ for all the copies $u$ of a vertex of $T_k$ equals $1/k$. In the second case, the sum over $\delta(u)$ for all the copies $u$ of a vertex from $2, \ldots, k-1$ equals $1/k$.

**Proof.** By Lemma 5 we have that $S$ induces an orientation of a blow-up of $K_k$, for some $k$. If $S$ is the orientation of a blow-up of $T_k$ then we are done. Otherwise, note that a
blow-up of $K_k$ contains at least one copy of $K_k$. By the assumption that $D$ does not contain directed cycles, any orientation of this copy must be isomorphic to $T_k$. So there must be a vertex $u$ of $K_k$ such that not all copies of $u$ are oriented as in a blow-up of $T_k$. Again by the assumption that $D$ does not contain directed cycles, this vertex can only be the vertex 1 or the vertex $k$ of $T_k$. The statement about $\delta$ then follows from Lemma 5.

4.3 Random digraphs with a homomorphism to $D$

In this section we show that a random digraph with a homomorphism to a fixed finite digraph $D$ asymptotically almost surely has a homomorphism to $D$ whose image is contained in a subgraph $S$ of $D$ which is an orientation of a blow-up of $K_\ell$. In fact, we show a stronger statement about the uniform distribution on all pairs $(G, f)$ where $G$ is a graph with vertex set $\{1, \ldots, n\}$ and $f$ is a homomorphism from $G$ to $D$; we refer to these pairs as $D$-coloured digraphs.

**Theorem 9.** Let $D$ be a finite oriented graph. Let $\ell$ be the largest integer so that $D$ contains a copy of an orientation of a blow-up of $K_\ell$. Then asymptotically almost surely a $D$-colored digraph $(G, f)$ is such that the image of $f$ induces a copy $S$ of an orientation of a blow-up of $K_\ell$, and $|f^{-1}(v)| \geq \lceil n/k \rceil$ for every $v \in V(S)$.

The proof of Theorem 9 below requires some tools from convergence of measures, for example from [1]. We recall the results we need. For a compact Polish space $S$, we denote by $\mathcal{P}(S)$ the space of probability measures on $S$. If $\mu \in \mathcal{P}(S)$ and $f$ is a continuous real function on $S$, then we write $\mu f$ for $\int_S f d\mu$. In the weak convergence topology a sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\mathcal{P}(S)$ converges to $\mu$ if and only if $\mu_n f \to \mu f$ for every bounded, continuous real function $f$ on $S$ (see, e.g., [1, Section 1]). This topology makes $\mathcal{P}(S)$ a Polish space.

A family $\Pi$ of probability measures is called

- **tight** if for every $\epsilon > 0$ there exists a compact set $K$ such that $\mu(K) > 1 - \epsilon$ for every $\mu \in \Pi$.
- **relatively compact** if every sequence of elements of $\Pi$ contains a weakly convergent subsequence.

Since $\mu(S) = 1$ and $S$ is compact, we have that $\mathcal{P}(S)$ is tight. Prokhorov’s theorem states that if $\Pi$ is tight, then it is relatively compact (see [1, Theorem 5.1]).

**Corollary 2.** $\mathcal{P}(S)$ is relatively compact.

**Proof of Theorem 9.** Let $Z$ be the subset of $[0,1]^{V(D)}$ of tuples summing to 1. For a homomorphism $f$ of $G$ to $D$, the vector $(g_i)_{i \in V(D)} \in Z$ where $g_i = \frac{|f^{-1}(i)|}{n}$ will be called the repartition of $f$. We consider the uniform distribution on $D$-coloured digraphs, i.e., every pair $(G, f)$ where $G$ is a digraph with vertex set $\{1, \ldots, n\}$ and $f$ is a homomorphism from $G$ to $D$ is equally likely. Let $\mu_n$ be the distribution of the random variable that maps
(\(G, f\)) to the repartition of \(f\). Then \(\mu_n\) is a probability measure on \(Z\). Note that \(Z\) is compact and Polish so we can apply the results from the first paragraph of the proof.

Let us denote by \(O\) the set of tuples \((x_i)_{i \in V(D)} \in Z\) maximizing \(\sum_{i,j \in E(D)} x_i x_j\). By Lemma 5, the set \(O\) consists of tuples whose non-zero entries are all in an orientation of a blow-up of \(K_\ell\). We will show that for any convergent subsequence of \((\mu_n)_{n \in \mathbb{N}}\) we have that \(\mu_n(Z \setminus O)\) goes to zero when \(n\) goes to \(\infty\). Since the space of measures on \(Z\) is sequentially compact, this means that with probability going to 1, for an element \((G, f)\) the repartition of \(f\) is in \(O\). This also implies that asymptotically almost surely, \(G\) has a homomorphism to \(D\) whose image is contained in a subgraph of \(D\) which is an orientation of a blow-up of \(K_\ell\).

For any \(A \subseteq Z\), denote by \(d_A\) the maximum of \(\sum_{(i,j) \in E(D)} x_i x_j\) over \((x_i)_{i \in V(D)} \in A\).

**Fact 1.** For any \(A \subseteq Z\), we have

\[
\mu_n(A) \leq \frac{|D|^n 2^{n^2 d_A}}{2^{(t-1)*X^2}} \sim \frac{|D|^n 2^{n^2 d_A}}{2^{n^2 d}},
\]

where for two sequences \((f_n)_n\) and \((g_n)_n\) of \(\mathbb{R}^\ast\), \(f_n \sim g_n\) means \(f_n/g_n \to 1\).

**Proof of the fact.** We have

\[
\mu_n(A) = \frac{|\{(G, f) \mid G \in \text{CSP}(D)_n, f : G \to D \text{ with repartition in } A\}|}{|\{(G, f) \mid G \in \text{CSP}(D)_n, f : G \to D\}|}.
\]

Therefore, it is enough to prove that the number of \(D\)-coloured digraphs whose repartition is in \(A\) is at most \(|D|^n 2^{n^2 d_A}\) and that \(|\text{CSP}(D)_n| \geq 2^{(t-1)*X^2} \sim 2^{n^2d}\).

If \(\delta = (\delta_i)_{i \in V(D)}\), we write \(\binom{n}{n \delta}\) to be the multinomial coefficient \(\binom{n}{(n \delta_i)_{i \in V(D)}}\) if for all \(i \in V(D)\) we have \(\delta_i = k_i/n\) for some \(k_i \in \mathbb{N}\), and 0 otherwise. The number of \(D\)-coloured digraphs \((G, f)\) with \(G \in \text{CSP}(D)_n\) and the repartition of \(f : G \to D\) equal to \(\delta \in A\) is

\[
\binom{n}{n \delta} 2^{n^2 \sum_{(i,j) \in E(D)} \delta_i \delta_j},
\]

Therefore, the number of \(D\)-coloured digraphs \((G, f)\) such that the repartition of \(f\) is in \(A\) is

\[
\sum_{(\delta_i)_{i \in V(D)} \in A} \binom{n}{n \delta} 2^{n^2 \sum_{(i,j) \in E(D)} \delta_i \delta_j} \leq 2^{n^2 d_A} \sum_{(\delta_i)_{i \in V(D)} \in A} \binom{n}{n \delta} \leq |D|^n 2^{n^2 d_A}.
\]

For the other inequality, for every \(o \in O\) the number of graphs admitting a homomorphism with repartition \(o\) is at least \(2^{(t-1)\frac{n^2}{2}}\). \(\Box\)

**Fact 2.** Suppose that \(A \subseteq Z \setminus O\) is compact. Then \(\mu_n(A) \to 0\).
Proof of Fact 2. Since $A$ does not contain a point $x$ such that $\sum_{(i,j) \in E(D)} x_i x_j = d$, we have $d_A < d$ where $d$ is the density of $D$. In particular,

$$\frac{|D|^n 2^n d_A}{2^{(\ell-1)[\frac{n}{\ell}]}^2} \leq |D|^n 2^n (d_A - d) \to 0.$$ 

Since by Fact 1 we have $\mu_n(A) \leq \frac{|D|^n 2^n d_A}{2^{(\ell-1)[\frac{n}{\ell}]}^2}$, we also get $\mu_n(A) \to 0$.

Since $\mathcal{P}(\mathcal{Z})$ is relatively compact (Corollary 2), the sequence $(\mu_n)_{n \in \mathbb{N}}$ has a weakly convergent subsequence $(\mu_{n_k})_{k \in \mathbb{N}}$ with a limit $\mu$. Then $\mu$ must be inner regular (it is in fact regular, see Theorem 1.1), i.e., for any measurable set $\mathcal{X}$, we have

$$\mu(\mathcal{X}) = \sup\{\mu(A) : A \text{ is a measurable compact subset of } \mathcal{X}\}.$$ 

In particular,

$$\mu(\mathcal{Z} \setminus \mathcal{O}) = \sup\{\mu(A) : A \text{ is a compact subset of } \mathcal{Z} \setminus \mathcal{O}\}$$

$$= \sup\{\lim_{k} \mu_{n_k}(A) : A \text{ is a compact subset of } \mathcal{Z} \setminus \mathcal{O}\}$$

$$= 0 \text{ by Fact 2}$$

Since this is true for the limit $\mu$ of any subsequence of $(\mu_n)_{n \in \mathbb{N}}$, this implies that $\mu_n(\mathcal{Z} \setminus \mathcal{O}) \to 0$.

Example 4. We revisit the digraph $D$ from Example 3 which was the disjoint union of $\tilde{C}_3$ and $T_3$. We still have to prove that the probability of the sentence $\phi$ which states that a given graph admits a homomorphism from $T_3$ asymptotically holds in $\text{CSP}(D)$ with probability exactly $1/2$. By Theorem 9 asymptotically almost surely a $D$-coloured digraph $(G, f)$ is such that the image of $f$ induces a copy $S$ of $\tilde{C}_3$ or a copy $S$ of $T_3$. In both cases, a.a.s. the pre-images of the vertices of $S$ under $f$ have size at least $[n/3]$. In the second case $G$ a.a.s. satisfies $\phi$, and in the first case, $G$ a.a.s. does not satisfy $\phi$, because if it would, then $\tilde{C}_3$ must contain a subgraph of $T_3$, which it does not. Note that the number $c_n$ of digraphs with vertex set $\{1, \ldots, n\}$ that map homomorphically to $T_3$ equals the number of digraphs that map homomorphically to $\tilde{C}_3$, so both cases are equally likely.

4.4 Blow-ups of $K_\ell$

In the previous section we have seen that orientations of blow-ups of $K_\ell$ play a particular role when studying the almost-sure theory of $\text{CSP}(D)$ for finite oriented graphs $D$. We will now investigate which orientations $D$ of blow-ups of $K_\ell$ can arise as the dual of a finite set of forbidden orientations of trees.
Our arguments require some elements from [19]. Let $A$ be a directed graph and $B$ a subgraph of $A$. We say that $A$ dismantles into $B$ if there is an enumeration $y_1, \ldots, y_r$ of $V(A) \setminus V(B)$ such that for every $i \leq r$ there is a homomorphism $\phi_i$ from the induced subgraph of $B$ with vertex set $B \cup \{y_i, \ldots, y_r\}$ to the induced subgraph of $B$ with vertex set $B \cup \{y_i, \ldots, y_r\}$ which extends the identity map.

**Lemma 7.** Let $D$ be the dual of a finite set of oriented trees. Assume that there are $x_1, x_2, x_3 \in V(D)$ such that $x_1 \to x_2 \to x_3$ and $x_1 \to x_3$ and $D$ contains subgraphs $L_n$ and $L_m$ that are oriented cliques of size $n$ and $m$, respectively, such that for all $l_1 \in L_n$, $l_2 \in L_m$, and $i \in \{1,2,3\}$ we have $l_1 \to x_i \to l_2$. Suppose $L_n$ and $L_m$ are chosen so that $n + m$ is largest possible. Then there cannot be $x_4 \in V(D)$ such that $x_4 \to x_1$, $x_4 \to x_3$, $x_4 \to x_2$ and $x_4 \to l$ for all $l \in L_n \cup L_m$.

**Proof.** Let us assume $A$ dismantles to $B$ and that $y_1, \ldots, y_r$ and $(\phi_1, \ldots, \phi_r)$ are witnesses of this dismantlement. In this case, say that $y_i$ is “sent to” $\phi_i(y_i)$ and, since vertices of $A \setminus B$ are sent in an one by one, we say that $y_i$ is “sent before” $y_j$ when $j > i$. Remark that, under those assumptions, up to modifying $(\phi_i)_{i \leq r}$, we can always assume that $y_i$ is sent to a vertex in $B$. Indeed, for $i \leq r$ set $\psi(y_i) = \phi_r(\phi_{r-1}(\ldots \phi_1(y_i))) \in B$, we can replace $\phi_1$ by the extension of the identity on $B \cup \{y_{i+1}, \ldots, y_r\}$ which sends $y_i$ to $\psi(y_i)$ and we still get a dismantlement from $A$ to $B$. Let us now state the result from [19] that we need: if $D$ is the dual of a finite set of oriented trees, then $D^2$ dismantles into $\Delta_{D^2}$, where $\Delta_{D^2}$ denotes the subgraph of $D^2$ induced on the diagonal $\{(a,a) \mid a \in V(D)\}$. This follows from the proof of Theorem 5.7 in [19].

Let $L$ be the subgraph of $D$ induced on $L_m \cup L_n \cup \{x_1, x_2, x_3\}$. We assume for contradiction that there exists $x_4 \in V(D)$ as in the statement of the lemma, and use the fact that $D^2$ dismantles into $\Delta_{D^2}$. In particular, we need to send $(x_4, x_1)$ and $(l, x_2)$ for each $l \in L$ to some vertex in $\Delta_{D^2}$. We argue that the first of those vertices that is sent to $\Delta_{D^2}$ cannot be sent to a vertex of $\Delta_{L^2}$. Indeed, if $(x_4, x_1)$ is sent to a vertex of $\Delta_{L^2}$ before $(l, x_2)$ for every $l \in L$, it needs to be sent to $(u, u) \in \Delta_{L^2}$ such that $(u, u) \to (x_1, x_2)$ and $u \to x_1$, meaning that $u \in L_n$. Since we also have $(x_4, x_1) \to (u, x_2)$ and $(u, x_2)$ has not been sent to $\Delta_{L^2}$, this would imply $(u, u) \to (u, x_2)$ which is impossible. On the other hand, if some $l \in L_n$ is such that $(l, x_2)$ is sent to $(u, u) \in \Delta_{L^2}$ first, then we would have $(x_4, x_1) \to (u, u)$ and thus $x_1 \to u$ and $x_4 \to u$, implying that $u \in L \setminus (L_n \cup \{x_2\})$. In this case $(l, x_2) \to (u, u)$ which is impossible. Therefore, whichever vertex of $\{(x_4, x_1)\} \cup \{(l, x_2) \mid l \in L_n\}$ is sent first, it must be sent to $(x_5, x_5)$ for some $x_5 \in D(V(L))$.

We have two cases: either $(x_4, x_1)$ is sent first, or $(l, x_2)$ for some $l \in L_n$ is sent first. If $(x_4, x_1)$ is sent first, then since $(x_4, x_1) \to (l, x_2)$ for all $l \in V(L)$, necessarily $x_5 \to l$, therefore $V(L) \cup \{x_5\}$ forms an oriented clique that is larger than $L$. If there is $l$ such that $(l, x_2)$ is sent first, then $(x_4, k) \to (x_5, k')$ for $k \in L_n \cup \{x_1\}$ and $x_5 \to k'$ for $k' \in L_m$. Therefore, $V(L) \cup \{x_5\}$ forms an oriented clique that is larger than $L$. In both cases we have reached a contradiction to the assumption that $n + m$ is largest possible. \qed
Lemma 8. Let $D$ be the dual of a finite set of oriented trees. Assume that there are $x_1, x_2, x_3 \in V(D)$ such that $x_1 \rightarrow x_2 \rightarrow x_3$ and $x_1 \rightarrow x_3$ and $D$ has a subgraph $L_n$ that is an oriented clique of size $n$ such that for all $l \in L_n$ and $i \in \{1, 2, 3\}$ we have $l \rightarrow x_i$. Choose $L_n$ so that $n$ is maximal. Then there cannot be $x_4 \in V(D)$ such that $x_4 \rightarrow x_1$, $x_4 \rightarrow x_2$, $x_4 \rightarrow x_3$ and $x_4 \rightarrow l$ for all $l \in V(L_n)$.

Proof. Assume for contradiction that there exists $x_4 \in V(D)$ as in the lemma. Let $L$ be the subgraph of $D$ induced on $L_n \cup \{x_1, x_2, x_3\}$. We use the fact that $D^2$ dismantles into $\Delta_{D^2}$ as in the proof of the previous lemma. In particular, we need to send $(x_4, x_2)$ and $(l, x_3)$ for each $l \in L_n \cup \{x_1, x_2\}$ somewhere. Looking at which one needs to be sent first, we can argue that they cannot be sent to a vertex from $\Delta_{L^2}$. Indeed, if $(x_4, x_2)$ is sent to a vertex from $\Delta_{L^2}$ before $(l, x_3)$ for each $l \in L_n \cup \{x_1, x_2\}$, then it needs to be sent to $(u, u) \in \Delta_{L^2}$ such that $(u, u) \rightarrow (x_1, x_3)$, so $u \rightarrow x_1$, meaning that $u \in V(L_n)$. Since we also have $(x_4, x_2) \rightarrow (u, x_3)$ and $(u, x_3)$ has not been sent to the diagonal, this is impossible. But if for some $l \in V(L_n)$ the vertex $(l, x_3)$ is sent to $(u, u) \in \Delta_{L^2}$ first, then we would have $(x_4, x_2) \rightarrow (u, u)$. The only possible $u$ is $x_3$, however if that is the case, $(l, x_3)$ cannot be sent to $(u, u)$, we have a contradiction.

Therefore, there must be $x_5, x_6 \in D \setminus V(L_{n+3})$ such that $(x_4, x_1)$ is sent to $(x_4, x_1)$ and $(x_1, x_2)$ is sent to $(x_1, x_2)$. We have two cases: either $(x_4, x_2)$ is sent first, or one of the $(l, x_3)$ is sent first. If $(x_4, x_2)$ is sent first, then since $(x_4, x_2) \rightarrow (l, x_2)$ for all $l \in V(L_n)$, necessarily $x_5 \rightarrow l$, Therefore, $V(L) \cup \{x_5\}$ forms a larger oriented clique than $L$. If there $l$ such that $(l, x_2)$ is sent first, then $(x_4, k) \rightarrow (x_6, x_6)$ for $k \in L_n \cup \{x_1\}$. Therefore, $V(L) \cup \{x_6\}$ forms a larger oriented clique than $L$. Both cases are in contradiction to the maximal choice of $n$.

Lemma 9. Let $D$ be the dual of a finite set of oriented trees and let $\ell$ be maximal such that $D$ contains the orientation $S$ of a blow-up of $K_\ell$. Then $S$ is isomorphic to $T_k$.

Proof. Assume for contradiction that $S$ is a proper blow-up of $K_\ell$, meaning it strictly contains a copy of $K_\ell$, which we denote by $Q$. In particular, there are $u \in V(Q)$ and $v \in V(S) \setminus V(Q)$ such that $u$ and $v$ have the same edges and non-edges with the other vertices of $S$. This, by the maximality of $k$, contradicts either Lemma 7 or Lemma 8 respectively, depending on whether $u$ both sends and receives edges from $Q$ or whether it only receives edges from the other vertices of $Q$. Therefore, $S$ can only be an orientation of $K_\ell$ with no vertices blown-up. Hence, by the acyclicity of $D$ (Proposition 1), we obtain that $S$ is isomorphic to $T_k$.

4.5 Model theory of $T_\ell$-coloured digraphs

In the particular case that $D$ equals $T_\ell$, we can use the results from the previous section to fully determine the almost-sure theory of $D$-coloured digraphs. In fact, asymptotically almost surely the graphs from $\text{CSP}(T_\ell)$ have exactly one homomorphism to $T_\ell$, and thus
we can also determine the almost-sure theory of CSP($T_\ell$). This can then be combined with the results in the previous section to determine the almost-sure theory for all duals $D$ of finite sets of oriented trees.

It will be convenient to treat $D$-coloured digraphs as relational structures.

**Definition 5.** Let $D$ be a finite digraph. If $(G, f)$ is a $D$-coloured digraph, then we denote by $G_f$ the relational structure which consists of the digraph $G$ equipped with unary predicates $P_u := f^{-1}(u)$ for each $u \in V(D)$.

**Lemma 10.** For every finite digraph $D$ there exists an (up to isomorphism unique) countable digraph $C(D)$ with a homomorphism $g: C(D) \to D$ such that the structure $C(D)_g$ is homogeneous and the age of $C(D)_g$ consists of all structures of the form $G_f$ for a finite $D$-coloured digraph $(G, f)$.

**Proof.** Note that the class of all structures of the form $G_f$ for some finite $D$-coloured digraph $(G, f)$ is an amalgamation class. The statement therefore follows from Fraïssé’s theorem (Theorem 4). \qed

**Lemma 11.** Suppose that $D$ is a finite digraph without non-trivial automorphisms. Then $C(D)$ is isomorphic to the structure $U(D)$ introduced in Lemma 3.

**Proof.** A finite digraph embeds into $C(D)$ if and only if it maps homomorphically to $D$ (by Lemma 10) if and only if it embeds into $U_D$. Thus, the theory $T$ of $C(D)$ is a companion of the theory of $U(D)$, since structures with the same age satisfy the same universal consequences. We claim that $C(D)$ is model complete. By Theorem 4, it suffices to verify that each of the unary predicates $P_v$ for $v \in V(D)$ has an existential and a universal definition in $C(D)$. Let $\phi_v$ be a primitive positive formula that defines the orbit of $v$ in $D$ in $\text{Aut}(D)$, which exists by Lemma 4. Since $D$ has no non-trivial automorphisms, the only element that satisfies $\phi_v$ in $D$ is the vertex $v$. The expansion of $D$ in the signature of $C(D)_g$ where $P_u$ denotes $\{u\}$, for each $u \in V(D)$, has an embedding $e$ into $C(D)_g$ by the definition of $C(D)_g$. Note that $e(v)$ satisfies $\phi_v$ in $C(D)$. Moreover, every vertex of $C(D)$ in $P_v$ satisfies $\phi_v$ by the homogeneity of $C(D)_g$. Vertices of $C(D)$ outside $P_v$, however, do not satisfy $\phi_v$. To see this, note that every element $x$ of $C(D)$ must be in $P_u$ for some $u \in V(D)$, which in turn implies by the homogeneity of $C(D)_g$ that $x$ satisfies $\phi_u$ in $D$. But if $x$ also satisfies $\phi_v$, then this implies that $u = v$. So indeed, $P_v$ has the existential (even primitive positive) definition $\phi_v$. The negation of $P_v$ then has the existential definition $\bigvee_{u \neq v} \phi_u$. Since the model companion is unique up to equivalence of theories, and both $C(D)$ and $U(D)$ are model-complete and companions of each other, this implies that $C(D)$ and $U(D)$ have the same first-order theory. Since $C(D)_g$ is homogeneous in a finite relational signature and therefore $\omega$-categorical, so is its reduct $C(D)$; this implies that $C(D)$ and $U(D)$ are even isomorphic. \qed

**Proposition 2.** Let $\ell \geq 1$. The almost-sure theory of the class of all structures $G_f$ for $T_\ell$-coloured digraphs $(G, f)$ equals the theory of $C(T_\ell)_g$. 19
Proof. To prove the statement of the lemma, it suffices to check that every extension axiom that holds in \( U(T_\ell) \) holds asymptotically almost surely in \( G_f \) for random \( T_\ell \)-coloured digraphs \((G,f)\) (Lemma 2). Let \( \phi \) be such an extension axiom of the form 
\[ \forall x_1, \ldots, x_m \exists y. \psi(x_1, \ldots, x_m, y), \] 
and let \((G,f)\) be a random \( T_\ell \)-coloured digraph with vertices \( \{1, \ldots, n\} \) drawn from the uniform distribution. Let \( a_1, \ldots, a_m \) be elements of \( G \). Then the probabilities that a given element \( b \in V(G) \setminus \{a_1, \ldots, a_m\} \) satisfies \( \psi(a_1, \ldots, a_m, b) \) are independent and larger than some positive constant, because \( \phi \) holds in \( U(T_\ell) \). The probability that such an element \( b \) exists tends to 1 exponentially fast if \( |V(G)| \) tends to infinity, because \( |f^{-1}(u)| \geq \lceil n/\ell \rceil \) for every \( u \in V(T_\ell) \) by Theorem 9. We may then use the union bound to derive that \( G_f \) satisfies \( \phi \) asymptotically almost surely.

Corollary 3. Let \( \ell \geq 1 \). The almost-sure theory of \( \text{CSP}(T_\ell) \) equals the theory of \( C(T_\ell) \) (and, by Lemma 11, of \( U(T_\ell) \)).

Proof. In the distribution on \( \text{CSP}(T_\ell) \) obtained from the uniform distribution on \( T_\ell \)-coloured digraphs by forgetting the colouring, the graphs that have more than one homomorphism to \( T_\ell \) are more likely than in the uniform distribution on \( \text{CSP}(T_\ell) \). However, note that if \((G,f)\) is a \( T_\ell \)-coloured digraph, and \( u \in V(G) \), then \( G_f \) asymptotically almost surely satisfies 
\[ \exists u_1, \ldots, u_k, u_{k+1}, \ldots, u_{\ell}(E(u_1, u_2) \land \cdots \land E(u_{k-1}, u) \land E(u, u_{k+1}) \land \cdots \land E(u_{\ell-1}, u_\ell)) \]
for some \( k \in \{1, \ldots, \ell\} \). Hence, asymptotically almost surely a \( T_\ell \)-coloured digraph \((G,f)\) is such that \( G \) has exactly one homomorphism to \( T_\ell \). This implies the statement.

Remark 2. With the same techniques as used in the proof of Theorem 9 and Corollary 3 one can show Theorem 7. (If \( \ell \) is the size of the largest clique in an undirected graph \( H \), then asymptotically almost surely an undirected graph with a homomorphism to \( H \) has exactly \( \ell! \) many homomorphisms to \( H \).)

Theorem 10. Let \( D \) be the dual of a finite set of oriented trees. Then there exists an \( \ell \in \mathbb{N} \) such that the almost-sure theory of \( \text{CSP}(D) \) equals the almost-sure theory of \( \text{CSP}(T_\ell) \).

Proof. Theorem 9 implies that asymptotically almost surely a random \( D \)-coloured graph is such that all colours come from a copy \( H \) of a blow-up of an orientation of \( K_\ell \), for some \( \ell \in \mathbb{N} \). By Lemma 9, \( H \) is in fact isomorphic to \( T_\ell \). Now the same argument as in Corollary 3 shows that a graph from \( \text{CSP}(D) \) asymptotically almost surely has exactly one homomorphism to \( T_\ell \), and that the almost sure theory of \( \text{CSP}(D) \) equals the almost sure theory of \( \text{CSP}(T_\ell) \).

4.6 Proofs of the main results

We can finally prove Theorem 8 if \( \mathcal{F} \) is a finite set of orientations of finite trees, the almost-sure theory of \( \text{Hom-Forb}(\mathcal{F}) \) is \( \omega \)-categorical.
Proof of Theorem 8. Let $F$ be a finite set of orientations of finite trees. By Theorem 3 there is a finite digraph $D$ such that $\text{Hom-Forb}(F) = \text{CSP}(D)$. Theorem 10 implies that the almost-sure theory $T$ of $\text{CSP}(D)$ equals the almost-sure theory of $\text{CSP}(T_\ell)$ for some $\ell \geq 1$. By Corollary 3 the theory $T$ equals the theory of $U(T_\ell)$, and $U(T_\ell)$ is $\omega$-categorical by Lemma 3.

As we mentioned in the introduction, we believe that our results about classes of the form $\text{CSP}(D)$ for finite digraphs $D$ are of independent interest. As demonstrated in Example 1 these classes do not always have a first-order 0-1 law. However, our proofs show that their asymptotic behaviour is composed linearly from finitely many $\omega$-categorical first-order theories in a sense that we introduce now.

A class $C$ of finite digraphs has a finite linear combination of 0-1 laws if there exists $k \in \mathbb{N}$ and classes $D_1, \ldots, D_k$ with first-order 0-1 laws and $\lambda_1, \ldots, \lambda_k \in [0,1]$ such that $\sum_i \lambda_i = 1$ and for all first-order $\tau$-sentences $\phi$

$$\phi^C = \sum_{i=1}^k \lambda_i \phi^{D_i}.$$  

The following result can be shown by adapting the proof of Theorem 9 to the setting where we allow double edges, and by performing our arguments for the densest subgraphs. The main argument is that we are able to define a similar notion of density, where double edges are counted twice, and use it in the same way. It is to be noted that a double edge has higher density that any orientation of $K_\ell$. The results stating that $\text{CSP}(D)$ concentrates on its densest parts remains true and so we obtain the following.

**Theorem 11.** For every finite digraph $D$, the class $\text{CSP}(D)$ has a finite linear combination of 0-1 laws.

5 Concluding Remarks and Open Problems

Many of our results for digraphs (e.g., Theorem 11) can be generalised to general relational structures with essentially the same proof. For the appropriate notion of acyclicity for general structures, see [21]. Theorem 9 has been generalized to general structures [21] and therefore, the same approach as we have taken here can be applied to study $\text{Hom-Forb}(F)$ where $F$ is a finite set of acyclic structures. However, we do not know of any elegant characterisation of the densest substructure for general structures (in the style of Lemma 5). We conclude with other open questions.

**Question 1.** Determine the almost-sure theory of $\text{Hom-Forb}(\{\vec{C}_3\})$. Is it complete? Is it $\omega$-categorical?

**Question 2.** Let $\tau$ be a finite relational signature and let $F$ be a finite set of finite $\tau$-structures. Does $\text{Hom-Forb}(F)$ have first-order convergence?
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