On uniformly recurrent subgroups of finitely generated groups

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August 12, 2018

Abstract

We prove that if $G$ is a finitely generated group and $Z$ is a uniformly recurrent subgroup of $G$ then there exists a minimal system $(X, G)$ with $Z$ as its stability system. This answers a query of Glasner and Weiss [7] in the case of finitely generated groups. Using the same method (introduced by Alon, Grytczuk, Haluszczak and Riordan [2]) we will prove that finitely generated sofic groups have free Bernoulli-subshifts admitting an invariant probability measure.

Keywords. uniformly recurrent subgroups, sofic groups

1 Introduction

Let $\Gamma$ be a countable group and $\text{Sub}(\Gamma)$ be the compact space of all subgroups of $\Gamma$. The group $\Gamma$ acts on $\text{Sub}(\Gamma)$ by conjugation. Uniformly recurrent subgroups (URS) were defined by Glasner and Weiss [7] as closed, invariant subsets $Z \subset \text{Sub}(\Gamma)$ such that the action of $\Gamma$ on $Z$ is minimal (every orbits are dense).

Now let $(X, \Gamma, \alpha)$ be a $\Gamma$-system (that is, $X$ is a compact metric space and $\Gamma \rightarrow \text{Homeo}(X)$ is a homomorphism). For each point $x \in X$ one can define the topological stabilizer subgroup $\text{Stab}_\alpha^0(x)$ by

$$\text{Stab}_\alpha^0(x) = \{ \gamma \in \Gamma \mid \gamma \text{ fixes some neighborhood of } x \}.$$

Let us consider the $\Gamma$-invariant subset $X^0 \subset X$ such that $x \in X^0$ if and only if $\text{Stab}_\alpha(x) = \text{Stab}_\alpha^0(x)$. Then $X^0$ is a dense $G_\delta$-set and we have a $\Gamma$-equivariant map $S_\alpha : X^0 \rightarrow \text{Sub}(\Gamma)$ such that if $y \in X^0$ then $S_\alpha(y) = \text{Stab}_\alpha(y)$. The closure of the invariant subset $S_\alpha(X^0) \subset \text{Sub}(\Gamma)$ is called the stability system of $(X, \Gamma, \alpha)$ (see also [12], [10]). If the action is minimal, then the stability system of $(X, \Gamma, \alpha)$ is an URS. Glasner and Weiss proved (Proposition 6.1, [7]) that for every URS $Z \subset \text{Sub}(G)$ there exists a topologically transitive (that is there is a dense orbit) system $(X, \Gamma, \alpha)$ with $Z$ as its stability system. They asked (Problem

*AMS Subject Classification: 37B05, 20E99
whether for any URS $Z$ there exists a minimal system $(X, \Gamma, \alpha)$ with $Z$ as its stability system. Recently, Kawabe [12] gave an affirmative answer for this question in the case of amenable groups. We will prove the following result.

**Theorem 1.** If $\Gamma$ is a finitely generated group and $Z \subset \text{Sub}(\Gamma)$ is an URS, then there exists a minimal system $(X, \Gamma, \alpha)$ with $Z$ as its stability system.

In the proof we will use the Lovász Local Lemma technique of Alon, Grytczuk, Haluszcak and Riordan [2] to construct a minimal action on the space of rooted colored $\Gamma$-Schreier graphs. This approach has already been used to construct free $\Gamma$-Bernoulli subshifts by Aubrun, Barbieri and Thomassé [1]. The other result of the paper is about free $\Gamma$-Bernoulli-subshifts, that is closed $\Gamma$-invariant subsets $M$ of $K^\Gamma$, where $K$ is some finite alphabet and the action of $\Gamma$ on $M$ is free. For a long time all finitely generated groups that had been known to have free Bernoulli-subshifts were residually-finite. Then Dranishnikov and Schroeder [11] constructed a free Bernoulli-subshift for any torsion-free hyperbolic group. Somewhat later Gao, Jackson and Seward proved that for any countable group $\Gamma$ there exists a free continuous action of $\Gamma$ on a Cantor set admitting an invariant measure. It seems that so far all groups $\Gamma$ for which free Bernoulli-shifts with an invariant probability measure proved to exist were either residually-finite (Toeplitz-shifts) or amenable (when the existence of invariant measure is obvious). Using the coloring technique of Alon, Grytczuk, Haluszcak and Riordan we prove a combination of these results for finitely generated sofic groups.

**Theorem 2.** Let $\Gamma$ be a finitely generated sofic group. Then there exists a free Bernoulli-subshift for $\Gamma$.

## 2 The space of colored rooted $\Gamma$-Schreier graphs

Let $\Gamma$ be a finitely generated group with a minimal symmetric generating system $Q = \{\gamma_i\}_{i=1}^n$. Let $H \in \text{Sub}(\Gamma)$. Then the Schreier graph of $H$, $S^Q_\Gamma(H)$ is constructed as follows.

- The vertex set of $S^Q_\Gamma(H)$, $V(S^Q_\Gamma(H)) = \Gamma/H$ (that is $\Gamma$ acts on the vertex set of $S^Q_\Gamma(H)$ on the left).
- The vertices corresponding to the cosets $aH$ and $bH$ are connected by a directed edge labeled by the generator $\gamma_i$ if $\gamma_i aH = bH$.

The coset class of $H$ is called the root of the graph $S^Q_\Gamma(H)$. We will consider the usual shortest path distance on $S^Q_\Gamma(H)$ and denote the ball of radius $r$ around the root $H$ by $B_r(S^Q_\Gamma(H), H)$. Note that $B_r(S^Q_\Gamma(H), H)$ is a rooted edge-labeled graph. The space of all Schreier graphs $S^Q_\Gamma$ is a compact metric space, where $d_{S^Q_\Gamma}(S^Q_\Gamma(H_1), S^Q_\Gamma(H_2)) = 2^{-r}$. 


if \( r \) is the largest integer for which the \( r \)-balls \( B_r(S^Q_t(H_1), H_1) \) and \( B_r(S^Q_t(H_2), H_2) \) are rooted-labeled isomorphic. Clearly, \( s : \text{Sub}(\Gamma) \to S^Q_t \), \( s(H) = S^Q_t(H) \) is a homeomorphism commuting with the \( \Gamma \)-actions. Note that if \( \gamma \in \Gamma \) and \( H \in \text{Sub}(\Gamma) \), then

\[
\gamma(S^Q_t(H)) = S^Q_t(\gamma H \gamma^{-1}),
\]

where the underlying labeled graphs of \( S^Q_t(H) \) and \( S^Q_t(\gamma H \gamma^{-1}) \) are isomorphic. The graph \( S^Q_t(\gamma H \gamma^{-1}) \) can be regarded as the same graph as \( S^Q_t(H) \) with the new root \( \gamma(\text{root}(S^Q_t(H))) \). We will use the root-change picture of the \( \Gamma \)-action on \( S^Q_t \) later in the paper.

Now let \( K \) be a finite alphabet. A rooted \( K \)-colored Schreier graph is a rooted Schreier graph \( S^Q_H \) equipped with a vertex-coloring \( c : \Gamma/H \to K \). Let \( S^K,Q_t \) be the set of all rooted \( K \)-colored Schreier-graphs. Again, we have a compact, metric topology on \( S^K,Q_t \):

\[
d_{S^K,Q_t}(S, T) = 2^{-r},
\]

if \( r \) is the largest integer such that the \( r \)-balls around the roots of the graphs \( S \) and \( T \) are rooted-colored-labeled isomorphic. We define \( d_{S^K,Q_t}(S, T) = 2 \) if the 1-balls around the roots are nonisomorphic and even the colors of the roots are different. Again, \( \Gamma \) acts on the compact space \( S^K,Q_t \) by the root-changing map. Hence, we have a natural color-forgetting map \( F : S^K,Q_t \to S^Q_t \) that commutes with the \( \Gamma \)-actions. Notice that if a sequence \( \{S_n\}_{n=1}^{\infty} \subset S^K,Q_t \) converges to \( S \in S^K,Q_t \), then for any \( r \geq 1 \) there exists some integer \( N_r \geq 1 \) such that if \( n \geq N_r \) then the \( r \)-balls around the roots of the graph \( S_n \) and the graph \( S \) are rooted-colored-labeled isomorphic. Let \( H \in \text{Sub}(\Gamma) \) and \( c : \Gamma/H \to K \) be a vertex coloring that defines the element \( S_{H,c} \in S^K,Q_t \). Then of course, \( \gamma(S_{H,c}) = S_{H,c} \) if \( \gamma \in H \). On the other hand, if \( \gamma(S_{H,c}) = S_{H,c} \) and \( \gamma \notin H \) then we have the following lemma that is immediately follows from the definitions of the \( \Gamma \)-actions.

**Lemma 2.1.** Let \( \gamma \notin H \) and \( \gamma(S_{H,c}) = S_{H,c} \). Then there exists a colored-labeled graph-automorphism of the \( K \)-colored labeled graph \( S_{H,c} \) moving the vertex representing \( H \) to the vertex representing \( \gamma(H) \neq H \).

Note that we have a continuous \( \Gamma \)-equivariant map \( \pi : S^K,Q_t \to \text{Sub}(\Gamma) \), where \( \pi(t) = s^{-1} \circ F(t) \). Let \( Z \) be an URS of \( \Gamma \). Let \( H \in Z \) and let \( t \in S^K,Q_t \) be corresponding to a vertex coloring of the Schreier graph \( S^Q_t(H) \). We say that the element \( t \in S^K,Q_t \) is \( Z \)-proper if \( \text{Stab}_o(t) = H \), where \( o \) is the right action of \( \Gamma \) on \( S^K,Q_t \). Note that if \( H \in Z \) and \( t \) is representing the Schreier graph \( S^Q_t \), then by Lemma 2.1, \( t \) is \( Z \)-proper if and only if there is no non-trivial colored-labeled automorphism of \( t \).
Proposition 2.1. Let $Y \subset S^K_{\Gamma} \Gamma$ be a closed $\Gamma$-invariant subset consisting of $Z$-proper elements. Let $(M, \Gamma, \alpha) \subset (Y, G, \alpha)$ be a minimal $\Gamma$-subsystem. Then for any $m \in M$, $\operatorname{Stab}_\alpha^0(m) = \operatorname{Stab}_\alpha(m) \in Z$. Also, $\pi(M) = Z$.

Proof. Let $h \in \operatorname{Stab}_\alpha(m)$. Then $h \in Z$, that is, $h$ fixes the root of $m$. Therefore, $h$ fixes the root of $m'$ provided that $d_{S^K_{\Gamma}, \alpha}(m, m')$ is small enough. Thus, $h \in \operatorname{Stab}_\alpha^0(m)$. Since $\pi$ is a $\Gamma$-equivariant continuous map and $M$ is a closed $\Gamma$-invariant subset, $\pi(M) = Z$.

3 The proof of Theorem 1

Let $Z$ be an URS of $\Gamma$. By Proposition 2.1 it is enough to construct a closed $\Gamma$-invariant subset $Y \subset S^K_{\Gamma} \Gamma$ for some alphabet $K$ such that all the elements of $Y$ are $Z$-proper. Let $H \in Z$ and consider the Schreier graph $S = S^K_{\Gamma}(H)$. Following [1] and [2] we call a coloring $c : \Gamma \to K$ nonrepetitive if for any path $(x_1, x_2, \ldots, x_n)$ in $S$ there exists some $1 \leq i \leq n$ such that $c(x_i) \neq c(x_{n+i})$. We call all the other colorings repetitive.

Theorem 3. [Theorem 1 [2]] For any $d \geq 1$ there exists a constant $C(d) > 0$ such that any graph $G$ (finite or infinite) with vertex degree bound $d$ has a nonrepetitive coloring with an alphabet $K$, provided that $|K| \geq C(d)$.

Proof. Since the proof in [2] is about edge-colorings and the proof in [1] is in slightly different setting, for completeness we give a proof using Lovász’s Local Lemma, that closely follows the proof in [2]. Now, let us state the Local Lemma.

Theorem 4 (The Local Lemma). Let $X$ be a finite set and $\Pr$ be a probability distribution on the subsets of $X$. For $1 \leq i \leq r$ let $A_i$ be a set of events, where an “event” is just a subset of $X$. Suppose that for all $A \in A_i$, $\Pr(A_i) = p_i$. Let $\mathcal{A} = \bigcup_{i=1}^r A_i$. Suppose that there are real numbers $0 \leq a_1, a_2, \ldots, a_r < 1$ and $\Delta_{ij} \geq 0$, $i, j = 1, 2, \ldots, r$ such that the following conditions hold:

- for any event $A \in A_i$ there exists a set $D_A \subset \mathcal{A}$ with $|D_A \cap A_i| \leq \Delta_{ij}$ for all $1 \leq j \leq r$ such that $A$ is independent of $\mathcal{A} \setminus (D_A \cup \{A_i\})$,
- $p_i \leq a_i \prod_{j=1}^r (1 - a_j)^{\Delta_{ij}}$ for all $1 \leq i \leq r$.

Then $\Pr(\cap_{A \in \mathcal{A}} A) > 0$.

Let $G$ be a finite graph with maximum degree $d$. It is enough to prove our theorem for finite graphs. Indeed, if $G'$ is a connected infinite graph with vertex degree bound $d$, then for each ball around a given vertex $p$ we have a nonrepetitive coloring. Picking a pointwise convergent subsequence of the colorings we obtain a nonrepetitive coloring of our infinite graph $G'$.

Let $C$ be a large enough number, its exact value will be given later. Let $X$ be the set of all random $\{1, 2, \ldots, C\}$-colorings of $G$. Let $r = \operatorname{diam}(G)$ and for
1 \leq i \leq r \) and for any path \( P \) of length \( 2i - 1 \) let \( A(P) \) be the event that \( P \) is repetitive. Set

\[ A_i = \{ A(P) : P \text{ is a path of length } 2i - 1 \text{ in } G \}. \]

Then \( p_i = C^{-i} \). The number of paths of length \( 2j - 1 \) that intersects a given path of length \( 2i - 1 \) is less or equal than \( 4ijd^2j \). So, we can set \( \Delta_{ij} = 4ij\Delta d^2j \). Let \( a_i = \frac{1}{2d^2} \). Since \( a_i \leq \frac{1}{2} \), we have that \( (1 - a_i) \geq \exp(-2a_i) \). In order to be able to apply the Local Lemma, we need that for any \( 1 \leq i \leq r \)

\[ p_i \leq a_i \prod_{j=1}^{r} \exp(-2a_j\Delta_{ij}). \]

That is

\[ C^{-i} \leq a^{-i} \prod \exp(-8ija^{-j}d^{2j}), \]

or equivalently

\[ C \geq a \exp\left(8 \sum_{j=1}^{r} \frac{j}{2d^2}\right). \]

Since the infinite series \( \sum_{j=1}^{\infty} \frac{j}{d^2} \) converges to 2, we obtain that for large enough \( C \), the conditions of the Local Lemma are satisfied independently on the size of our finite graph \( G \). This ends the proof of Theorem 3.

Let \( |K| = C(|Q|) \) and let \( c: \Gamma/H \to K \) be a nonrepetitive \( K \)-coloring that gives rise to an element \( y \in S_{K,Q}^{\Gamma} \). The following proposition finishes the proof of Theorem 1.

**Proposition 3.1.** All elements of the orbit closure \( Y \) of \( y \) in \( S_{K,Q}^{\Gamma} \) are \( Z \)-proper.

**Proof.** Let \( x \in Y \) with underlying Schreier graph \( H' \) and coloring \( c': \Gamma/H \to K \). Since \( Z \) is an URS, \( H' \in Z \). Indeed, \( \pi^{-1}(Z) \) is a closed \( \Gamma \)-invariant set and \( y \in \pi^{-1}(Z) \). Clearly, \( \alpha(y)(x) = x \) if \( \gamma \in H' \). Now suppose that \( \alpha(y)(x) = x \) and \( \gamma \notin H' \) (that is \( x \) is not \( Z \)-proper). By Lemma 2, there exists a colored-labeled automorphism \( \theta \) of the graph \( x \) moving \( \text{root}(x) \) to \( \gamma(\text{root}(x)) \neq \text{root}(x) \).

Now we proceed similarly as in the proof of Lemma 2 or in the proof of Theorem 2. Let \( a \in V(x) \) be a vertex such that there is no \( b \in X \) such that \( \text{dist}_x(b, \theta(b)) < \text{dist}_x(a, \theta(a)) \). Let \( a = a_1, a_2, \ldots, a_{n+1} = \theta(a) \) be a shortest path between \( a \) and \( \theta(a) \). For \( 1 \leq i \leq n \), let \( \gamma_k(a_i) = a_{i+1} \). Then let \( a_{n+2} = \gamma_k(a_{n+1}), a_{n+3} = \gamma_k(a_{n+2}), \ldots, a_{2n} = \gamma_k(a_{2n-1}) \). Since \( \theta \) is a colored-labeled automorphism, for any \( 1 \leq i \leq n \)

\[ c(a_i) = c(a_{i+n}). \] (1)

**Lemma 3.1.** The walk \( (a_1, a_2, \ldots, a_{2n}) \) is a path.
Proof. Suppose that the walk above crosses itself, that is for some \( i, j, a_j = a_{n+i} \). If \((n+1) - j \geq (n+i) - (n+1) = i - 1, \) then \( \text{dist}(a_2, \theta(a_2)) = \text{dist}(a_2, a_{n+2}) < \text{dist}(a, \theta(a)) \). On the other hand, if \((n+1) - j \leq (n+i) - (n+1) = i - 1, \) then
\[
\text{dist}(a_n, \theta(a_n)) = \text{dist}(a_n, a_{2n-1}) < \text{dist}(a, \theta(a)).
\]
Therefore, \((a_1, a_2, \ldots, a_{2n})\) is a path. \( \square \)

By (1) and the previous lemma, the \( K \)-colored Schreier-graph \( x \) contains a repetitive path. Since \( x \) is in the orbit closure of \( y \), this implies that \( y \) contains a repetitive path as well, in contradiction with our assumption. \( \square \)

4 Sofic groups and invariant measures

First, let us recall the notion of a finitely generated sofic group. Let \( \Gamma \) be a finitely generated infinite group with a minimal, symmetric generating system \( Q = \{ \gamma_i \}_{i=1}^\infty \) and a surjective homomorphism \( \kappa : F_n \to \Gamma \) from the free group \( F_n \) with generating system \( \overline{Q} = \{ r_i \}_{i=1}^n \) mapping \( r_i \) to \( \gamma_i \). Let Cay\( _\Gamma ^Q \) be the Cayley graph of \( \Gamma \) with respect to the generating system \( Q \), that is the Schreier graph corresponding to the subgroup \( H = \{ 1 \} \). Let \( \{ G_k \}_{k=1}^\infty \) be a sequence of finite \( F_n \)-Schreier graphs. We call a vertex \( p \in V(G_k) \) a \((\Gamma, r)\)-vertex if there exists a rooted isomorphism
\[
\Psi : B_r(G_k, p) \to B_r(\text{Cay}_\Gamma ^Q, 1_\Gamma)
\]
such that if \( e \) is a directed edge in the ball \( B_r(G_k, p) \) labeled by \( r_i \), then the edge \( \Phi(e) \) is labeled by \( \gamma_i \). We say that \( \{ G_k \}_{k=1}^\infty \) is a sofic approximation of Cay\( _\Gamma ^Q \), if for any \( r \geq 1 \) and a real number \( \varepsilon > 0 \) there exists \( N_{r,\varepsilon} \geq 1 \) such that if \( k \geq N_{r,\varepsilon} \) then there exists a subset \( V_k \subset V(G_k) \) consisting of \((\Gamma, r)\)-vertices such that \( |V_k| \geq (1-\varepsilon)|V(G_k)| \). A finitely generated group \( \Gamma \) is called sofic if the Cayley-graphs of \( \Gamma \) admit sofic approximations. Sofic groups were introduced by Gromov in [8] under the name of initially subamenable groups, the word “sofic” was coined by Weiss in [13]. It is important to note that all the amenable, residually-finite and residually amenable groups are sofic, but there exist finitely generated sofic groups that are not residually amenable (see the book of Capraro and Lupini [4] on sofic groups). It is still an open question whether all groups are sofic. Now let \( \Gamma \) be a finitely generated sofic group with generating system \( Q = \{ \gamma_i \}_{i=1}^n \) and a sofic approximation \( \{ G_k \}_{k=1}^\infty \). Using Theorem [8] for each \( k \geq 1 \) let us choose a nonrepetitive coloring \( c_k : V(G_k) \to K \), where \( |K| \geq C(|Q|) \). We can associate a probability measure \( \mu_k \) on the space of \( K \)-colored \( F_n \)-Schreier graphs \( S_{F_n}^{G_k} \). Note that the origin of this construction can be traced back to the paper of Benjamini and Schramm [3]. For a vertex \( p \in V(G_k) \) we consider the rooted \( K \)-colored Schreier graph \( (G_k^{c_k}, p) \). The measure \( \mu_k \) is defined as
\[
\mu_k = \frac{1}{|V(G_k)|} \sum_{p \in V(G_k)} \delta(G_k^{c_k}, p),
\]
where $\delta(G^c_k, p)$ is the Dirac-measure on $S_{\mathbb{F}_n}^{K^c}$ concentrated on the rooted $K$-colored Schreier graph $(G^c_k, p)$. Clearly, $\mu_k$ is invariant under the action of $\mathbb{F}_n$. Since the space of $\mathbb{F}_n$-invariant probability measures on the compact space $S_{\mathbb{F}_n}^{K^c}$ is compact with respect to the weak-topology, we have a convergent subsequence $\{\mu_{n_k}\}_{k=1}^{\infty}$ converging weakly to some probability measure $\mu$. Let $C_d^{\mathbb{F}_n}(N)$ be the Schreier graph corresponding to the normal subgroup $N = \text{Ker}(\kappa)$. This means that we have a natural graph isomorphism from $C_d^{\mathbb{F}_n}(N)$ to $\text{Cay}_Q^G$ that changes the labels $r_i$ to $\gamma_i$.

**Proposition 4.1.** The probability measure $\mu$ is concentrated on the $\mathbb{F}_n$-invariant closed set $\Omega$ of nonrepetitive $K$-colorings on $C_d^{\mathbb{F}_n}(N)$.

**Proof.** Let $U_r \subset S_{\mathbb{F}_n}^{K^c}$ be the clopen set of $K$-colored Schreier graphs $G$ such that the ball $B_r(G, \text{root}(G))$ is not rooted-labeled isomorphic to $B_r(C_d^{\mathbb{F}_n}(N), 1\Gamma)$. By our assumptions on the sofic approximations, $\lim_{k \to \infty} \mu_k(U_r) = 0$, hence $\mu(U_r) = 0$. Now let $V_r \subset S_{\mathbb{F}_n}^{K^c}$ be the clopen set of $K$-colored Schreier graphs $G$ such that the ball $B_r(G, \text{root}(G))$ contains a repetitive path. By our assumptions on the colorings $c_k$, $\mu_k(V_r) = 0$ for any $k \geq 1$. Hence $\mu(V_r) = 0$. Therefore $\mu$ is concentrated on $\Omega$.

Now we prove Theorem 2. Observe that we have an $\mathbb{F}_n$-equivariant continuous map $\Sigma : \Omega \to K^\Gamma$, where $\mathbb{F}_n$ acts on the Bernoulli space $K^\Gamma$ on the right by $\rho(f)(\gamma) = f(\gamma \kappa(\rho))$ for $\rho \in \mathbb{F}_n, \gamma \in \Gamma$. Then the image of $F$ is a closed $\Gamma$-invariant subset in $K^\Gamma$, that is a Bernoulli subshift consisting of elements that are given by nonrepetitive $K$-colorings. The pushforward of $\mu$ under $\Sigma$ is a $\Gamma$-invariant probability measure concentrated on $Y$. By Proposition 3.1, $\Gamma$ acts freely on $Y$, hence Theorem 2 follows.

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