A Converse Bound on Wyner-Ahlswede-Körner Network via Gray-Wyner Network

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Abstract—We show a reduction method to construct a code for the Gray-Wyner (GW) network from a given code for the Wyner-Ahlswede-Körner (WAK) network. By combining this reduction with a converse bound on the GW network, we derive a converse bound on the WAK network. The derived bound gives an alternative proof of the strong converse theorem for the WAK network.

I. INTRODUCTION

We revisit the coding problem over the Wyner-Ahlswede-Körner network, which is also known as the lossless source coding with one-helper. The achievable rate region of this network was characterized in [10], [12]. This network is regarded as one of typical problems of the network information theory in the sense that it contains some basic difficulties that arise in multiuser problems; in particular, the characterization of the achievable rate region involves an auxiliary random variable and Markov chain structure, which makes it difficult to derive converse bounds of this network. The strong converse theorem for this network was proved by Ahlswede-Gács-Körner in [11] with a technique called the blowing-up lemma. The exponential strong converse was recently shown by Oohama in [7] with some new techniques in the information-spectrum method.

The coding problem over the Gray-Wyner (GW) network is another basic problem of the network information theory introduced in [3]. The characterization of the achievable rate region of this network also involves an auxiliary random variable; however, it does not involve Markov chain structure. The strong converse theorem for this network was shown by Gu-Effros in [4]. By a type based refinement of their approach, the second-order rate region of the GW network was shown in [8].

A motivation of this work is to develop an alternative converse approach to the WAK network. Since the approach in [4], [8] is based on centralized encoding nature of the GW network, it is not applicable to the WAK network directly. However, we derive a converse bound on the WAK network by showing a reduction from the GW network to the WAK network and then by applying the converse approach [4], [8] of the GW network. In order to explain an overview of our approach, let us formally introduce each network below.

A. Gray-Wyner Network

The coding system of the GW network consists of three encoders

\[
\varphi^{(n)}_i : \mathcal{X}^n \times \mathcal{Y}^n \to \mathcal{M}_i^{(n)}, \quad i = 0, 1, 2
\]

and two decoders

\[
\psi^{(n)}_1 : \mathcal{M}_0^{(n)} \times \mathcal{M}_1^{(n)} \to \mathcal{X}^n, \\
\psi^{(n)}_2 : \mathcal{M}_0^{(n)} \times \mathcal{M}_2^{(n)} \to \mathcal{Y}^n.
\]

We omit the blocklength \(n\) when it is obvious from the context. For \((X^n, Y^n) \sim P\), the error probability \(P_{GW}(\Phi_n | P)\) of code \(\Phi_n = (\varphi_0, \varphi_1, \varphi_2, \psi_1, \psi_2)\) is defined as the probability such that

\[
(\psi_1(\varphi_0(X^n, Y^n)), \varphi_1(X^n, Y^n)) \neq (X^n, Y^n).
\]

A rate triplet \((r_0, r_1, r_2)\) is defined to be achievable if there exists a sequence of code \(\{\Phi_n\}_{n=1}\) such that

\[
\limsup_{n \to \infty} \frac{1}{n} \log |\mathcal{M}_i^{(n)}| \leq r_i, \quad i = 0, 1, 2
\]

and

\[
\lim_{n \to \infty} P_{GW}(\Phi_n | P_{XY}^n) = 0,
\]

where \(P_{XY}^n\) is the distribution of all i.i.d. of \(P_{XY}\). Then, the achievable rate region \(\mathcal{R}_{GW}(P_{XY})\) is defined as the set of all achievable rate triplets.

Let \(\mathcal{R}_{GW}(P_{XY})\) be the set of all rate triplets \((r_0, r_1, r_2)\) such that there exists a test channel \(P_{W|XY}\) with \(|W| \leq |X||Y| + 2\) satisfying

\[
r_0 \geq I(W \land X, Y), \quad r_1 \geq H(X|W), \quad r_2 \geq H(Y|W).
\]

It is known that the achievable region of the GW network is characterized as \(\mathcal{R}_{GW}(P_{XY}) = \mathcal{R}_{GW}(P_{XY})\).

B. Wyner-Ahlswede-Körner Network

The coding system of the WAK network consists of two encoders

\[
\tilde{\varphi}^{(n)}_0 : \mathcal{X}^n \to \tilde{\mathcal{M}}_0^{(n)}, \\
\tilde{\varphi}^{(n)}_2 : \mathcal{Y}^n \to \tilde{\mathcal{M}}_2^{(n)}
\]

and one decoder

\[
\tilde{\mathcal{M}}_0^{(n)} \times \tilde{\mathcal{M}}_2^{(n)} \to \mathcal{Y}^n.
\]

For \((X^n, Y^n) \sim P\), the error probability \(P_{WAK}(\tilde{\Phi}_n | P)\) of code \(\tilde{\Phi}_n = (\tilde{\varphi}_0, \tilde{\varphi}_2, \psi)\) is defined as the probability such that

\[
\psi(\tilde{\varphi}_0(X^n), \tilde{\varphi}_2(Y^n)) \neq Y^n.
\]

A rate pair \((r_0, r_2)\) is defined for later convenience of relating the WAK network with the GW network, we use unconventional notations; the helper’s encoder is \(\tilde{\varphi}_0\) and the main encoder is \(\tilde{\varphi}_2\).
to be achievable if there exists a sequence of code \( \{ \hat{\Phi}_n \}_{n=1}^\infty \)
such that
\[
\limsup_{n \to \infty} \frac{1}{n} \log |\mathcal{N}^{(n)}_i| \leq r_i, \quad i = 0, 2
\]
and
\[
\lim_{n \to \infty} P_{W\text{AK}}(\hat{\Phi}_n | P^n_{XY}) = 0.
\]

Then, the achievable region \( \mathcal{R}_{W\text{AK}}(P_{XY}) \) is defined as the set of all achievable rate pairs.

Let \( \mathcal{R}^*_{W\text{AK}}(P_{XY}) \) be the set of all rate pair \((r_0, r_2)\) such that there exists a test channel \( P_{W|X} \) with \(|W| \leq |X||Y| + 2\) satisfying
\[
\begin{align*}
    r_0 &\geq I(W \land X), \\
r_2 &\geq H(Y|W).
\end{align*}
\]
It is known that the achievable region of the WAK network is characterized as \( \mathcal{R}_{W\text{AK}}(P_{XY}) = \mathcal{R}^*_{W\text{AK}}(P_{XY}) \).

### C. Overview of Approach

Although the GW network and the WAK network appear to be completely different problems (the former is centralized encoding while the latter is distributed encoding), it is known that the achievable rate regions of these networks have the following intimate connection [3]:
\[
\begin{align*}
\{ (r_0, r_1, r_2) &\in \mathcal{R}^*_{GW}(P_{XY}) : r_0 + r_1 = H(X) \} \\
&= \{ (r_0, r_1, r_2) : (r_0, r_2) \in \mathcal{R}^*_{W\text{AK}}(P_{XY}), r_0 + r_1 = H(X) \}.
\end{align*}
\]

In fact, by noting the identity \( H(X) + I(W \land Y|X) = I(W \land X, Y) + H(X|W) \), we can verify that the condition \( I(W \land X, Y) + H(X|W) = H(X) \) enforces the Markov chain condition \( W \rightarrow X \rightarrow Y \).

Inspired by the connection in (1), we shall show a converse bound on the WAK network by using the reduction argument. For a given WAK code \( \hat{\Phi}_n = (\hat{\varphi}_0, \hat{\varphi}_2, \hat{\psi}) \) with rates \((\hat{r}_0, \hat{r}_2)\), we construct a GW code \( \hat{\Phi}_n = (\varphi_0, \varphi_1, \varphi_2, \psi_1, \psi_2) \) with rates \((r_0, r_1, r_2)\) such that \( \hat{r}_0 \simeq r_0, \hat{r}_2 \simeq r_2, \) and \( r_1 \simeq H(X) - r_0 \); we also show that the error probability of the constructed GW code \( \hat{\Phi}_n \) is as small as that of the original WAK code \( \hat{\Phi}_n \). Then, we apply a converse bound on the GW network, which gives a converse bound on the WAK network via the above reduction argument. Our approach gives an alternative proof of the strong converse for the WAK network without using the blowing-up lemma nor Oohama’s method.

The rest of the paper is organized as follows. In the next section, we state our main results. All the proofs are given in Section IV.

We close the paper with some discussions in Section IV.

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1In fact, we can show the cardinality bound \(|W| \leq |X| + 1\). However, for later convenience of relating the WAK network with the GW network, we apply a slightly loose bound; there is no harm in enlarging the cardinality of the auxiliary random variable.

II. MAIN RESULT

For a joint type \( P_{XY} \in \mathcal{P}_r(X \times Y) \), let \( P_{T_{XY}} \) be the uniform distribution on the joint type class \( \mathcal{T}_{XY}^{(r)} \). The main result of this paper is the following reduction theorem claiming that we can construct a GW code from a given WAK code.

**Theorem 1:** For a given WAK code \( \hat{\Phi}_n = (\hat{\varphi}_0, \hat{\varphi}_2, \hat{\psi}) \) and a joint type \( P_{T_{XY}} \) satisfying
\[
\log |T_{XY}^{(r)}| \geq \log |\tilde{M}_0|,
\]
where \( T_{XY}^{(r)} \) is the type class of the marginal type \( P_X \), there exists a GW code \( \Phi_n = (\varphi_0, \varphi_1, \varphi_2, \psi_1, \psi_2) \) such that
\[
\begin{align*}
    \log |M_0| &\leq \log |\tilde{M}_0| + \log n + n \log \log |X| + 2, \\
    \log |M_1| &\leq \log |T_{XY}^{(r)}| + \log n + n \log |X| + 2, \\
    \log |M_2| &\leq \log |M_2|,
\end{align*}
\]
and
\[
P_{GW}(\hat{\Phi}_n | P_{T_{XY}}) \leq P_{W\text{AK}}(\hat{\Phi}_n | P_{T_{XY}}).
\]

Next, we shall derive a converse bound on the WAK network by combining Theorem 1 with a converse bound on the GW network. For that purpose, let us introduce a slightly relaxed version \( R^*_{W\text{AK}}(\delta | P_{XY}) \) of \( R_{W\text{AK}}(P_{XY}) \) as follows. For \( \delta > 0 \), let \( \mathcal{R}^*_{W\text{AK}}(\delta | P_{XY}) \) be the set of all rate pairs \((r_0, r_2)\) such that there exists a test channel \( P_{W|X} \) with \(|W| \leq |X||Y| + 2\) satisfying
\[
\begin{align*}
    r_0 &\geq I(W \land X, Y), \\
r_2 &\geq H(Y|W), \\
\delta &\geq I(W \land Y|X).
\end{align*}
\]
Note that \( \mathcal{R}^*_{W\text{AK}}(0 | P_{XY}) = \mathcal{R}_{W\text{AK}}(P_{XY}) \).

**Corollary 2:** For a given WAK code \( \hat{\Phi}_n = (\hat{\varphi}_0, \hat{\varphi}_2, \hat{\psi}) \), it hold that
\[
P_{W\text{AK}}(\hat{\Phi}_n | P^n_{XY}) \geq P \left( (\hat{r}_0, n, \hat{r}_2, n) \notin \mathcal{R}^*_{W\text{AK}}(\delta_n | t_{X^n|Y^n}), t_{X^n \in E_n} \right) \left( 1 - \frac{1}{n} \right),
\]
where \( t_{X^n|Y^n} \) is the joint type of \((X^n, Y^n)\),
\[
\begin{align*}
    \hat{r}_0, n &:= \frac{1}{n} \log |\tilde{M}_0| + \Delta_n + \frac{\log \log |X| + 2}{n}, \\
    \hat{r}_2, n &:= \frac{1}{n} \log |\tilde{M}_2| + \frac{1 + \log |Y|}{n}, \\
    \delta_n &:= \Delta_n + \frac{\log \log |X| + 3 + \log |X|}{n},
\end{align*}
\]
\[
\Delta_n = \frac{(|X||(Y|+1)+3) \log(n+1)}{n}, \quad \text{and} \quad E_n \text{ is the set of types defined by}
\]
\[
E_n := \left\{ P_X : H(\tilde{X}) \geq \frac{1}{n} \log |\tilde{M}_0| + \frac{|X| \log(n+1)}{n} \right\}.
\]

By noting the continuity of region \( \mathcal{R}^*_{W\text{AK}}(\delta | P_{XY}) \) at \( \delta = 0 \), we can show the following strong converse theorem for the WAK network.

Corollary 3 (II): If \((r_0, r_2) \notin R_x^{\text{str}}(P_{XY})\) and \(r_0 < H(X)\) then for any sequence of WAK codes \(\{\hat{\Phi}_n\}_{n=1}^\infty\) satisfying
\[
\limsup_{n \to \infty} \frac{1}{n} \log |\hat{M}_0^{(n)}| \leq r_0, \tag{7}
\]
\[
\limsup_{n \to \infty} \frac{1}{n} \log |\hat{M}_2^{(n)}| \leq r_2, \tag{8}
\]
it holds that
\[
\lim_{n \to \infty} P_{\text{WAK}}(\hat{\Phi}_n|P_{XY}) = 1. \tag{9}
\]

III. PROOFS

A. Proof of Theorem I

For a given WAK code \(\hat{\Phi}_n = (\hat{\varphi}_0, \hat{\varphi}_2, \hat{\psi})\), the encoder \(\hat{\varphi}_0\) induces a partition \(\hat{\varphi}_0^{-1}(m) \cap T_X^n\), \(m \in \hat{M}_0\) of the type class \(T_X^n\). Basic strategy to construct encoder \(\varphi_1\) is to assign distinct codewords to each element in \(\hat{\varphi}_0^{-1}(m) \cap T_X^n\); however, some partitions may have much larger cardinality than others. The following lemma states that, with a negligible penalty rate, we can construct a modified WAK code having “balanced” property, from which Theorem I follows immediately.

Lemma 4 (Balanced Code): For a given WAK code \(\hat{\Phi}_n = (\hat{\varphi}_0, \hat{\varphi}_2, \hat{\psi})\) and a joint type \(P_{XY}\) satisfying \(\psi\), there exists another WAK code \(\hat{\Phi}_n = (\varphi_0, \varphi_2, \psi)\) such that
\[
\log |\hat{M}_0| \leq \log |M_0| + n \log |X| + 2, \tag{10}
\]
\[
\log |\hat{M}_2| = \log |M_2|, \tag{11}
\]
\[
P_{\text{WAK}}(\hat{\Phi}_n|P_{\hat{X}Y}) \leq P_{\text{WAK}}(\hat{\Phi}_n|P_{\hat{X}Y}), \tag{12}
\]
and
\[
\log |\varphi_0^{-1}(m) \cap T_X^n| \leq \log \frac{|T_X^n|}{|M_0|} \tag{13}
\]
for every \(m \in \hat{M}_0\).

Proof: Let
\[
L_n := \log |\hat{M}_0| \leq n \log |X|.
\]
Let
\[
\hat{M}_0 = \bigcup_{i=0}^{L_n} \hat{M}_0(i)
\]
be the partition of \(\hat{M}_0\), where
\[
\hat{M}_0(i) = \left\{ m : \frac{|T_X^n|}{|M_0|} \cdot 2^{(i-1)} < |\varphi_0^{-1}(m) \cap T_X^n| \leq \frac{|T_X^n|}{|M_0|} \cdot 2^i \right\}
\]
for \(1 \leq i \leq L_n\) and
\[
\hat{M}_0(0) = \left\{ m : |\varphi_0^{-1}(m) \cap T_X^n| \leq \frac{|T_X^n|}{|M_0^{(n)}|} \right\}.
\]

Then, for \(1 \leq i \leq L_n\), we have
\[
|\hat{M}_0(i)| \leq \frac{|M_0|}{2^{(i-1)}}; \tag{14}
\]
otherwise, we have
\[
\left| \bigcup_{m \in \hat{M}_0(i)} \varphi_0^{-1}(m) \cap T_X^n \right| > |T_X^n|,
\]
which is a contradiction. To construct \(\varphi_0\), for each \(1 \leq i \leq L_n\) and \(m \in \hat{M}_0(i)\), we further partition \(\varphi_0^{-1}(m)\) into \(2^i\) subsets so that
\[
|\varphi_0^{-1}(\hat{m}) \cap T_X^n| \leq \frac{|T_X^n|}{|M_0|}
\]
for every \(\hat{m} \in \hat{M}_0(i)\), where \(\hat{M}_0(i)\) is the set of indices induced by such a partition. Then, we have
\[
|\hat{M}_0(i)| = 2^i |\hat{M}_0(i)| \leq 2|\hat{M}_0|,
\]
where the last inequality follows from \(\psi\). For \(m \in \hat{M}_0(0)\), we keep \(\varphi_0^{-1}(m)\) unchanged from \(\varphi_0^{-1}(m)\), and thus \(\hat{M}_0(0) = M_0(0)\). On the other hand, we set \(\varphi_2 = \varphi_2\). By noting
\[
|\hat{M}_0| = \sum_{i=0}^{L_n} |\hat{M}_0(i)| \leq (2L_n + 1)|\hat{M}_0| \leq 4L_n|\hat{M}_0|,
\]
we can verify that the encoders constructed in this manner satisfy \(\psi\). Furthermore, since \(\varphi_0\) is finer than \(\hat{\varphi}_0\), we can construct a decoder \(\hat{\psi}\) satisfying \(\psi\).

Now, we prove Theorem I For a given WAK code \(\hat{\Phi}_n = (\hat{\varphi}_0, \hat{\varphi}_2, \hat{\psi})\), by Lemma 3 we can construct a WAK code \(\hat{\Phi}_n = (\varphi_0, \varphi_2, \psi)\) satisfying \(\psi\). We set \(\varphi_0 = \hat{\varphi}_0\) and \(\varphi_2 = \hat{\varphi}_2\). We take \(M_1\) so that
\[
|M_1| = \max_{m \in M_0} |\varphi_0^{-1}(m) \cap T_X^n|,
\]
and we construct \(\varphi_1\) so that distinct numbers are assigned to the elements in \(\varphi_0^{-1}(m) \cap T_X^n\) for each \(m \in M_0\). By \(\psi\) and \(\psi\), the encoders constructed in this manner satisfy \(\psi\). Furthermore, since \(\varphi_0(x, \varphi_1(x')) \neq (\varphi_0(x'), \varphi_1(x'))\) for any \(x \neq x' \in T_X^n\), there exists a decoder \(\psi_1\) that can reconstruct \(X^n\) without an error under the distribution \(P_{\hat{X}Y}\). Thus, by using \(\psi\) for \(\varphi_2\), \(\psi\) is also satisfied.

B. Proof of Corollary 2

To prove Corollary 2 we combine Theorem I with the following converse bound on the GW network, which is a type based refinement of the strong converse of the GW network derived in \(\psi\).

Lemma 5: (II) For a given GW code \(\hat{\Phi}_n\), suppose that the probability of error satisfies
\[
1 - P_{\text{WAK}}(\hat{\Phi}_n|P_{\hat{X}Y}) \geq 2^{-n\alpha_n}
\]

\footnote{Our approach only gives the strong converse theorem under the condition \(r_0 < H(X)\); however, the strong converse theorem for the WAK network is known to hold without this condition in \(\psi\). In fact, for \(r_0 \geq H(X)\), it can be shown as the strong converse theorem for the Slepian-Wolf network with full side-information.}

\footnote{For simplicity, we assume \(\log |\hat{M}_0^{(n)}|\) is an integer.}

\footnote{This step is inspired by the information-spectrum slicing \(\psi\).}
for some positive $\alpha_n$. Let $\beta_n$ be another positive number. Then there exists $P_{\bar{W}|\bar{X},\bar{Y}}$ with $|V| \leq |X||Y| + 2$ such that

$$\frac{1}{n} \log |\mathcal{M}_0(n)| \geq I(\bar{W} \wedge \bar{X}, \bar{Y}) - |X||Y| \log(n+1) - (\alpha_n + \beta_n),$$

$$\frac{1}{n} \log |\mathcal{M}_1(n)| \geq H(\bar{X}|\bar{W}) - \frac{1}{n} 2^{-n\beta_n} \log |X|,$n

$$\frac{1}{n} \log |\mathcal{M}_2(n)| \geq H(\bar{Y}|\bar{W}) - \frac{1}{n} 2^{-n\beta_n} \log |Y|,$n

where $(\bar{X}, \bar{Y}) \sim P_{\bar{X}\bar{Y}}$.

To prove Corollary 2, we first decompose the error probability by type $P_n(\mathcal{X} \times \mathcal{Y})$ as

$$P_{\text{Wark}}(\Phi_n|P^\mu_{\bar{X}\bar{Y}}) = \sum_{P_{\bar{X}\bar{Y}} \in P_n(\mathcal{X} \times \mathcal{Y})} P^\mu_{\bar{X}\bar{Y}}(T^n_{\bar{X}\bar{Y}}) P_{\text{Wark}}(\Phi_n|P^\mu_{\bar{X}\bar{Y}}) \geq \sum_{P_{\bar{X}\bar{Y}} \in P_n(\mathcal{X} \times \mathcal{Y})} P^\mu_{\bar{X}\bar{Y}}(T^n_{\bar{X}\bar{Y}}) P_{\text{Wark}}(\Phi_n|P^\mu_{\bar{X}\bar{Y}}).$$

For each joint type $P_{\bar{X}\bar{Y}}$ satisfying $P_{\bar{X}} \in \mathcal{E}_n$, there exists (possibly different codes for different joint types) a GW code $\Phi_n = (\Phi_0, \Phi_1, \Phi_2, \psi_1, \psi_2)$ satisfying (3)-6 of Theorem 1. By Lemma 5 with $\alpha_n = \beta_n = \frac{\log n}{n}$, if

$$r_{0,n} := \frac{1}{n} \log |\mathcal{M}_0(n)| + \frac{|X||Y| \log(n+1)}{n} + (\alpha_n + \beta_n),$$

$$r_{1,n} := \frac{1}{n} \log |\mathcal{M}_1(n)| + \frac{1}{n} 2^{-n\beta_n} \log |X|,$n

$$r_{2,n} := \frac{1}{n} \log |\mathcal{M}_2(n)| + \frac{1}{n} 2^{-n\beta_n} \log |Y|,$n

are such that $(r_{0,n}, r_{1,n}, r_{2,n}) \notin R^*_{\text{Wark}}(P_{\bar{X}\bar{Y}})$, then

$$P_{\text{Wark}}(\Phi_n|P^\mu_{\bar{X}\bar{Y}}) > 1 - 2^{-n\alpha_n}.$$  \hfill (15)

We claim that $(r_{0,n}, r_{1,n}, r_{2,n}) \in R^*_{\text{Wark}}(P_{\bar{X}\bar{Y}})$ implies $(\bar{r}_{0,n}, \bar{r}_{2,n}) \in R^*_{\text{Wark}}(P_{\bar{X}\bar{Y}})$. In fact, when $(r_{0,n}, r_{1,n}, r_{2,n}) \in R^*_{\text{Wark}}(P_{\bar{X}\bar{Y}})$, then there exists $P_{\bar{W}|\bar{X},\bar{Y}}$ such that

$$r_{0,n} \geq I(\bar{W} \wedge \bar{X}, \bar{Y}),$$

$$r_{1,n} \geq H(\bar{X}|\bar{W}),$$

$$r_{2,n} \geq H(\bar{Y}|\bar{W}).$$

From (3) and (16), we have

$$\bar{r}_{0,n} \geq I(\bar{W} \wedge \bar{X}, \bar{Y}).$$

From (5) and (18), we have

$$\bar{r}_{2,n} \geq H(\bar{Y}|\bar{W}).$$

From (4), (16), and (17), we have

$$H(\bar{X}) + I(\bar{W} \wedge \bar{Y}|\bar{X}) = I(\bar{W} \wedge \bar{X}, \bar{Y}) + H(\bar{X}|\bar{W}) \leq r_{0,n} + r_{1,n} \leq \frac{1}{n} \log |T^n_{\bar{X}\bar{Y}}| + 3 \log n + \log \log |X| + \log |X| + 3 + |X||Y| \log(n+1) \leq \frac{n}{|X||Y|} .$$

Thus, we have $(\bar{r}_{0,n}, \bar{r}_{2,n}) \notin R^*_{\text{Wark}}(\delta_n|P_{\bar{X}\bar{Y}})$. By taking the contraposition and by (15), if $(\tilde{r}_{0,n}, \tilde{r}_{2,n}) \notin R^*_{\text{Wark}}(\delta_n|P_{\bar{X}\bar{Y}})$, then we have

$$P_{\text{Wark}}(\Phi_n|P^\mu_{\bar{X}\bar{Y}}) \geq P_{\text{Wark}}(\Phi_n|P^\mu_{\bar{X}\bar{Y}}) > 1 - 2^{-n\alpha_n}.$$}

Thus, we have

$$P_{\text{Wark}}(\Phi_n|P^\mu_{\bar{X}\bar{Y}}) \geq \sum_{P_{\bar{X}\bar{Y}} \in P_n(\mathcal{X} \times \mathcal{Y})} P^\mu_{\bar{X}\bar{Y}}(T^n_{\bar{X}\bar{Y}}) P_{\text{Wark}}(\Phi_n|P^\mu_{\bar{X}\bar{Y}}) \geq \sum_{P_{\bar{X}\bar{Y}} \in P_n(\mathcal{X} \times \mathcal{Y})} P^\mu_{\bar{X}\bar{Y}}(T^n_{\bar{X}\bar{Y}})(1 - 2^{-n\alpha_n}) = P(\tilde{r}_{0,n}, \tilde{r}_{2,n}) \notin R^*_{\text{Wark}}(\delta_n|t_{X^nY^n}), \ t_{X^n} \in \mathcal{E}_n) \left(1 - \frac{1}{n}\right). \hfill \blacksquare$$

C. Proof of Corollary 2

To discuss the continuity of region $R^*_{\text{Wark}}(\delta|P_{XY})$ at $\delta = 0$, let us consider the following supporting line of the region:

$$R_\mu(\delta|P_{XY}) := \min\{r_0 + \mu r_2 : (r_0, r_2) \in R^*_{\text{Wark}}(\delta|P_{XY})\}$$

for $\mu \geq 0$. For brevity, we write $R_\mu(P_{XY}) = R_\mu(0|P_{XY})$.

Lemma 6: For a given $P_{XY}$ and $\mu \geq 0$, we have

$$\lim_{\delta \to 0} R_\mu(\delta|P_{XY}) = R_\mu(P_{XY}).$$

Proof: By definition, $R_\mu(\delta|P_{XY}) \leq R_\mu(P_{XY})$ for any $\delta > 0$. Let $P_{W|XY}$ be a test channel such that

$$I(W \wedge X, Y) + \mu H(Y|W) = R_\mu(\delta|P_{XY})$$

and $I(W \wedge X|Y) \leq \delta$. Let $P_{\bar{W}X\bar{Y}} = P_{W|XY}$. Note that $P_{\bar{X}\bar{Y}} = P_{XY}$ and $\bar{W} \wedge \bar{X} \wedge \bar{Y}$. By noting that $D(P_{WXY}\|P_{\bar{W}X\bar{Y}}) = I(W \wedge X|Y) \leq \delta$ and by the Pinsker inequality, we have

$$\|P_{WXY} - P_{\bar{W}X\bar{Y}}\|_1 \leq \sqrt{\delta/2}.$$

Thus, by the continuity of the entropy, there exists $\delta'$ such that $\delta' \to 0$ as $\delta \to 0$ and

$$R_\mu(P_{XY}) \leq I(\bar{W} \wedge \bar{X}) + \mu H(\bar{Y}|\bar{W}) = I(\bar{W} \wedge \bar{X}, \bar{Y}) + \mu H(\bar{Y}|\bar{W}) \leq I(W \wedge X, Y) + \mu H(Y|W) + \delta' = R_\mu(\delta|P_{XY}) + \delta',$$
Similarly, we can show $R_n(P_{XY})$ is continuous with respect to $P_{XY}$.

**Proof:** Let $P_{XY}$ and $P_{X^*Y}$ be such that $\|P_{XY} - P_{X^*Y}\|_1 \leq \epsilon$. Let $P_{W|X}$ be a test channel such that

$$I(W \land X, Y) + \mu H(Y)I(W) = R_n(P_{XY}).$$

Let $P_{WX^*Y} = P_{W|X}P_{X^*Y}$. Then,

$$\|P_{WX^*Y} - P_{WX^*Y}\|_1 = \|P_{X^*Y} - P_{XY}\|_1 \leq \epsilon.$$

Thus, by the continuity of the entropy, there exists $\epsilon'$ such that $\epsilon' \to 0$ as $\epsilon \to 0$ and

$$R_n(P_{X^*Y}) \leq I(W \land X) + \mu H(Y)I(W) + \epsilon'.$$

Similarly, we can show $R_n(P_{XY}) \leq R_n(P_{X^*Y}) + \epsilon'.

Now, we prove Corollary 3 by using Corollary 2. In the following, we use the same notations $(\tilde{r}_{0,n}, \tilde{r}_{2,n}, \delta_n)$ as Corollary 2.

Let $K_n \subseteq P_n(X \times Y)$ be the set of all joint types $P_{X^*Y}$ such that

$$|P_{X^*Y}(x, y) - P_{XY}(x, y)| \leq \frac{\log n}{n^{1/2}}$$

for every $(x, y) \in X \times Y$. By the Hoeffding inequality, we have

$$P_{\tilde{r}_{X^*Y} \in K_n} \geq 1 - \frac{2|X||Y|}{n^2}.$$

Since $r_0 < H(X)$, (7) implies that there exists $\nu > 0$ such that

$$\frac{1}{n} \log |\tilde{M}(n)| \leq H(X) - \nu$$

for sufficiently large $n$. Thus, by the continuity of the entropy, $t_{X^*Y} \in K_n$ implies $t_{X^*Y} \in E_n$.

Since $(\tilde{r}_{0,n}, \tilde{r}_{2,n}) \not\in R_{\text{วก}}^n(P_{XY})$, there exists $\mu \geq 0$ and $\nu > 0$ such that

$$r_0 + \mu r_2 \leq R_n(P_{XY}) - (3 + \mu)\nu.$$

Also, (7) and (8) imply

$$\frac{1}{n} \log |\tilde{M}(n)| \leq r_i + \nu, \quad i = 0, 2$$

for sufficiently large $n$. By Lemma 6 and Lemma 7 $t_{X^*Y} \in K_n$ imply

$$R_n(P_{XY}) \leq R_n(\delta_n t_{X^*Y}) + \nu$$

for sufficiently large $n$, which implies

$$\frac{1}{n} \log |\tilde{M}(n)| + \frac{\mu}{n} \log |\tilde{M}(n)| \leq r_0 + \mu r_2 + (1 + \mu)\nu \leq R_n(P_{XY}) - 2\nu \leq R_n(\delta_n t_{X^*Y}) - \nu.$$

Thus, $t_{X^*Y} \in K_n$ implies $\tilde{r}_{0,n} + \mu \tilde{r}_{2,n} < R_n(\delta_n t_{X^*Y})$, i.e., $(\tilde{r}_{0,n}, \tilde{r}_{2,n}) \not\in R_{\text{วก}}^n(\delta_n t_{X^*Y})$ for sufficiently large $n$.

Consequently, by Corollary 2, we have

$$P_{\text{วก}}(\delta_n t_{X^*Y}) \geq P_{\tilde{r}_{0,n}, \tilde{r}_{2,n}}(\tilde{r}_{0,n}, \tilde{r}_{2,n}) \not\in R_{\text{วก}}^n(\delta_n t_{X^*Y}), \quad t_{X^*Y} \in E_n \left(1 - \frac{1}{n}\right)$$

which implies (9).

**IV. DISCUSSIONS**

In this paper, in order to derive a converse bound on the WAK network from a converse bound on the GW network, we showed a reduction method to construct a GW code from a given WAK code. Since the WAK network is distributed coding and the GW network is centralized coding, an opposite reduction, i.e., constructing a WAK code from a given GW code, is not possible in general.

Since the residual terms in Corollary 2 are $O((\log n)/n)$, it may give an outer bound for the second-order region of the WAK network (cf. (9)). However, $\delta = 0$ could be singular points of the region $R_{\text{วก}}^n(\delta|P_{XY})$ though this region is continuous at $\delta = 0$. Thus, some careful treatment is needed to investigate a second-order outer bound, which is an interesting future research problem.

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