Singular terms of helicity amplitudes at one-loop in QCD and the soft limit of the cross sections of multi-parton processes

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Abstract
We describe a general method that enables us to obtain all the singular terms of helicity amplitudes of n-parton processes at one loop. The algorithm uses helicity amplitudes at tree level and simple color algebra. We illustrate the method by calculating the singular part of the one loop helicity amplitudes of all $2 \rightarrow 3$ parton subprocesses. The results are used to derive the soft gluon limit of the cross sections of all $2 \rightarrow 4$ parton scattering subprocesses which provide a useful initial condition for the angular ordering approximation to coherent multiple soft gluon emission, incorporated in existing Monte Carlo simulation programs.

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1 Introduction

The technical developments achieved recently in perturbative QCD calculations (helicity method [1, 3, 2, 4], string theory based derivations [5, 6]) enable us to calculate one-loop corrections to helicity amplitudes up to five-parton and perhaps also to six-parton processes. Some partial results for the five parton amplitudes are by now available.

In this development the utilization of the helicity method was decisively important. It was used both in the string theory based derivation of the one-loop corrections to the four-gluon [5] and five-gluon [6] helicity amplitudes, as well as in the recent conventional diagrammatic calculation of the one loop corrections of the helicity amplitudes for all $2 \rightarrow 2$ parton processes [7].

The helicity method which was originally proposed for calculating tree amplitudes [4] can be extended to the calculation of loop corrections by observing that the difficulties given by the appearance of $\gamma_5$ in the definition of the helicity states can be circumvented if instead of conventional dimensional regularization the ’t Hooft-Veltman regularization [8, 9] or dimensional reduction [10, 11, 12, 13] are utilized. While employing these regularizations requires some care in calculating two- or more-loop contributions, their use is rather straightforward in the case of one-loop corrections. The regularization dependent terms in the loop contributions obtained with conventional dimensional regularization, dimensional reduction or ’t Hooft-Veltman regularization are determined by the universal process independent structure of the singular contributions. The process independent transition rules have been worked out recently [7].

An important technical issue in the evaluation of next-to-leading QCD corrections to physical cross-section is the analytic cancelation of the soft and collinear singularities between the loop and the Bremsstrahlung contributions. The singular terms in the cross-sections coming from Bremsstrahlung contributions have universal, process independent structure. They depend only on the color representation of the external legs and each leg contributes additively. It was recently shown that these terms can easily be obtained with the help of a Born-level algorithm [14] (see also [15]). Since the virtual corrections are cancelled by the Bremsstrahlung contributions, one expects that a simple Born-level algorithm should also be sufficient to calculate directly the singular terms of the virtual corrections as well. The aim of this paper is to show that indeed such an algorithm exists.

In section 2 we give a brief description of our algorithm for calculating the singular terms of one-loop helicity amplitudes. In section 3 we illustrate the power of the method by calculating the singular terms of one-loop helicity amplitudes of all $2 \rightarrow 3$ subprocesses. The soft singular terms of the one-loop helicity amplitudes of the $2 \rightarrow n - 2$ parton processes determine also the soft limit of the cross sections of the $2 \rightarrow n - 1$ parton scattering processes via the cancelation theorem. In view of its phenomenological usefulness, we summarize our results for the soft limit of the Born cross sections of all $2 \rightarrow 4$ subprocesses in Section 4.

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1 A comprehensive treatment of the soft limit in QCD is presented in [15]. For a recent paper in the large $N_c$ limit see [16].
2 Singular terms in one-loop helicity amplitudes

In \( D \)-dimensional regularisation the virtual corrections to helicity amplitudes at one loop have soft \( 1/\epsilon^2 \) and \( 1/\epsilon \), collinear \( 1/\epsilon \) and ultraviolet \( 1/\epsilon \) singularities. The ultraviolet singular term is trivially cancelled by the \( \overline{\text{MS}} \) coupling constant counter term

\[
\mathcal{A}^{\text{UV}} = -\frac{(n-2)(4\pi)^\epsilon}{2\epsilon \Gamma(1-\epsilon)} \left( \frac{g}{4\pi} \right)^2 \beta_0 \mathcal{A}^{\text{tree}}
\]

where \( n \) denotes the number of the external legs, \( \beta_0 \) is the first term in the \( \beta \)-function

\[
\beta_0 = \frac{(11N_c - 2N_f)}{3},
\]

\( g \) denotes the renormalized coupling constant and \( D = 4 - 2\epsilon \). Color and helicity labels are supressed. In the following we shall discuss only renormalized helicity amplitudes.

In axial gauge the collinear singularities come from the self energy corrections to the external lines \[17\]. For each helicities and color subamplitudes they are proportional to the Born term since the Altarelli-Parisi functions, \( P_{a/a}(z) \), for diagonal splitting preserve helicity in the \( z \to 1 \) limit. Therefore, the collinear singularities have the form

\[
\mathcal{A}^{\text{loop \ col}} = - \left( \frac{g}{4\pi} \right)^2 \sum_a \frac{\gamma(a)}{\epsilon} \mathcal{A}^{\text{tree}}.
\]

The constant \( \gamma(a) \) represents the contribution from virtual graphs to the Altarelli-Parisi kernel \( P_{a/a}(\xi) \) or equivalently it is defined by the behavior of \( P_{a/a}(\xi) \) near \( \xi = 1 \)

\[
\int_1^\infty d\xi P_{a/a}(\xi) = 2C(a) \ln(1-z) + \gamma(a) + O(1-z),
\]

where \( C(a) \) is the color charge of parton \( a \). Specifically, \( C(a) \) and \( \gamma(a) \) are

\[
C(g) = N_c, \quad \gamma(g) = \frac{11N_c - 2N_f}{6}, \quad \text{for \ gluons,}
\]

\[
C(q) = \frac{N_f^2 - 1}{2N_c}, \quad \gamma(q) = \frac{3(N_f^2 - 1)}{4N}, \quad \text{for \ quarks.}
\]

The structure of multiple soft emission from hard processes in QED was investigated by Gammer and Yennie \[18\]. They have shown that the energetic electrons participating in a hard process receive an eikonal phase factor. In quantum chromodynamics, the situation is very similar except that the eikonal factor is a matrix equal to the path order product of the matrix-valued gluon field \[19, 15, 20\].

For one soft gluon, the main result is very simple: it states that the singular contributions come from a configuration where the soft gluons are attached to external legs of the graphs. Therefore, the soft contribution can easily be calculated in terms of the Born amplitude. For example, let us consider the contribution of a virtual soft gluon to the matrix element of a process with \( n \) external legs, \( 2 \to (n-2) \).

The insertion of a soft gluon which connects the external legs \( i \) and \( j \) has a twofold effect: First, the Born amplitude gets rotated in the color space by the insertion of the color matrices appearing in the two vertices of the soft line connecting the hard lines \( i \) and \( j \)

\[
\mathcal{A}(2 \to n-2)_{c_1c_2...c_n} \to \sum_{a, c_i, c_j} t_{c_i}^a t_{c_j}^a \mathcal{A}(2 \to n-2)_{c_1'c_2'...c_n}
\]

(2.6)
where the labels of the initial partons are \((n - 1)\) and \(n\); \(c_i\)'s denote color indices for the external partons and \(t^a_{c_i c'_i}\) is the SU(3) generator matrix for the color representation of line \(i\). That is, \(t^a_{ij}\) is \((1/2)\lambda^a_{ij}\) for an outgoing quark, \(-(1/2)\lambda^{*a}_{ij}\) for an outgoing antiquark, and \(if_{aij}\) for an outgoing gluon. For an incoming parton, one can use the same formula as long as one uses the conjugate color representations, \(-(1/2)\lambda^{*a}_{ij}\) for a quark, \((1/2)\lambda^a_{ij}\) for an antiquark, and \(if_{aij}\) for a gluon.

Secondly, after carrying out the loop integral and dropping singular terms corresponding to collinear configurations, we pick up the same eikonal factor as in QED, i.e.

\[
\left(-\frac{g^4}{4\pi}\right)^2 c_\Gamma \frac{1}{c^2} \left(-\frac{\mu^2}{s_{ij}}\right)^\epsilon,
\]

where we introduced the short hand notation

\[
c_\Gamma = (4\pi)^\epsilon \frac{\Gamma^2(1 - \epsilon)\Gamma(1 + \epsilon)}{\Gamma(1 - 2\epsilon)}.
\]

Using unitarity this form of the soft factor can be confirmed by integrating the gluon momenta over the Bremsstrahlung eikonal factor [14].

To obtain the complete soft singular contributions, one should consider all configurations where a gluon connects two external legs. So the soft singular terms of loop helicity amplitudes can be written as

\[
A^{\text{loop}}_{\text{soft}}(2 \rightarrow n - 2)_{c_1 c_2, \ldots c_n} = -\sum_{i<j} \left(\frac{g}{4\pi}\right)^2 c_\Gamma \frac{1}{c^2} \left(-\frac{\mu^2}{s_{ij}}\right)^\epsilon \sum_{a_i c_i a'_{i'} c'_{i'}} t^a_{ci c'_i} t^{a'}_{c'_i c'_j} A^{\text{tree}}(2 \rightarrow n - 2)_{c_1 c'_i, \ldots c'_j, \ldots c_n}
\]

The color matrix of the Born term is an invariant tensor, therefore the sum of insertions to all external lines must vanish (soft identity [14]). The soft structure given by (2.9) can easily be decomposed into color subamplitudes. As one possible simple test of the correctness of this algorithm, one can use it to reproduce the singular terms of the one loop helicity amplitudes of all \(2 \rightarrow 2\) subprocesses [14].

### 3 Singular terms of the one loop color subamplitudes

In this section we give the singular soft and collinear terms of the one loop color subamplitudes of the \(2 \rightarrow 3\) subprocesses \(qQ \rightarrow qQg\) and \(qg \rightarrow ggg\). One can obtain the result for the \(qq \rightarrow qgg\) equal flavor subprocess from the unequal flavor process with simple permutations. We do not give the result for the subprocess \(gg \rightarrow ggg\) since it was obtained earlier in [14] and we found agreement. We note that we calculated the one loop helicity amplitudes of the subprocess \(qQ \rightarrow qQg\) with standard diagrammatic method [21] and could confirm the correctness of the Born level algorithm for the singular terms also in this case.

#### 3.1 Subprocess class \(0 \rightarrow \bar{q}\bar{Q}Qgg\)

Here we present the singular terms of the one loop virtual corrections of the helicity color subamplitudes of the process \(0 \rightarrow \bar{q}\bar{Q}Qgg\). The momenta of the partons are labeled as

\[
0 \rightarrow \text{antiquark}_1(\bar{q}) + \text{antiquark}_2(Q) + \text{quark}_2(Q) + \text{quark}_1(q) + \text{gluon}(g).
\]

3
The color structure of the amplitudes is the same at one loop as at tree level:

\[
A^{\text{tree}}(\bar{q}, Q; Q, q, g) = g^3 \left[ \sum_{(q_1 \neq q_2) \in \{q,Q\}} (T^g)_{q_1 q_2} \delta_{q_2 q_1} a_{q_1 q_2}^{(0)}(h_q, h_Q; h_Q, h_q, h_5) \right. \\
- \left. \sum_{(q_1 \neq q_2) \in \{q,Q\}} \frac{1}{N_c} (T^g)_{q_1 q_2} \delta_{q_2 q_1} a_{q_1 q_2}^{(0)}(h_q, h_Q; h_Q, h_q, h_5) \right]
\]

\[
A^{\text{loop}}(\bar{q}, Q; Q, q, g) = g^3 \left( \frac{g}{4\pi} \right)^2 \left[ \sum_{(q_1 \neq q_2) \in \{q,Q\}} (T^g)_{q_1 q_2} \delta_{q_2 q_1} a_{q_1 q_2}^{(1)}(h_q, h_Q; h_Q, h_q, h_5) \right. \\
- \left. \sum_{(q_1 \neq q_2) \in \{q,Q\}} \frac{1}{N_c} (T^g)_{q_1 q_2} \delta_{q_2 q_1} a_{q_1 q_2}^{(1)}(h_q, h_Q; h_Q, h_q, h_5) \right]
\]

The tree-level \( a_{ij}^{(0)} \) subamplitudes are:

\[
a_{ij}^{(0)}(h_q, h_Q; h_Q, h_q, +) = p_a(h_q, h_Q; h_Q, h_q, +) \frac{\langle ij \rangle}{\langle ig \rangle \langle gj \rangle}, \quad (3.5)
\]

\[
a_{ij}^{(0)}(h_q, h_Q; h_Q, h_q, -) = p_a(h_q, h_Q; h_Q, h_q, -) \frac{[ij]}{[ig][gj]}, \quad (3.6)
\]

The helicity dependence is completely absorbed in the factor \( p_a \) which is given by

\[
p_a(h_q, h_Q; h_Q, h_q, +) = \mp \frac{\langle mn \rangle^2}{\langle qq \rangle \langle QQ \rangle}, \quad (3.7)
\]

\[
p_a(h_q, h_Q; h_Q, h_q, -) = \pm \frac{\langle mn \rangle^2}{\langle qq \rangle \langle QQ \rangle}, \quad (3.8)
\]

where \( m \) and \( n \) labels those (anti)quarks which have opposite helicity to the gluon and the upper signs apply when \( m \) and \( n \) are both quarks or antiquarks, while the lower signs apply when one of them is a quark and the other is an antiquark.

Using these expressions, the singular parts of the renormalized one-loop subamplitudes can be written in a simple form:

\[
a_{Qq}^{(1),\text{sing}} = -\frac{c_T}{\varepsilon^2} \left\{ N_c a_{Qq}^{(0)} \left[ \left( -\frac{m^2}{s_{Qq}} \right)^{\varepsilon} + \left( -\frac{m^2}{s_{Qg}} \right)^{\varepsilon} \right] + \left( -\frac{m^2}{s_{qQ}} \right)^{\varepsilon} \right\} \\
- \frac{1}{N_c} a_{Qq}^{(0)} \left[ \left( -\frac{m^2}{s_{Qq}} \right)^{\varepsilon} + \left( -\frac{m^2}{s_{Qg}} \right)^{\varepsilon} \right] \left[ \left( -\frac{m^2}{s_{qQ}} \right)^{\varepsilon} \right] + \left( -\frac{m^2}{s_{qQ}} \right)^{\varepsilon} \right] \\
- \frac{1}{N_c} a_{QQ}^{(0)} \left[ \left( -\frac{m^2}{s_{Qq}} \right)^{\varepsilon} + \left( -\frac{m^2}{s_{Qg}} \right)^{\varepsilon} \right] + \left( -\frac{m^2}{s_{Qq}} \right)^{\varepsilon} \right] \\
- \frac{1}{N_c} a_{QQ}^{(0)} \left[ \left( -\frac{m^2}{s_{Qg}} \right)^{\varepsilon} + \left( -\frac{m^2}{s_{Qq}} \right)^{\varepsilon} \right] \right\} \\
- \frac{1}{\varepsilon} (4\gamma(q) + \gamma(g)) a_{Qq}^{(0)}
\]

\[
(3.9)
\]
We use the momentum labels as follows:

\[ a^{(1),\text{sing}}_{QQ} = - \frac{ct}{\varepsilon^2} \left\{ N_c a^{(0)}_{QQ} \left[ \left( - \frac{\mu^2}{s_{qg}} \right) + \left( - \frac{\mu^2}{s_{qg}} \right) + \left( - \frac{\mu^2}{s_{qg}} \right) \right] \right. \]

\[ - \frac{1}{N_c} a^{(0)}_{QQ} \left[ - \left( - \frac{\mu^2}{s_{qg}} \right) + \left( - \frac{\mu^2}{s_{qg}} \right) + \left( - \frac{\mu^2}{s_{qg}} \right) \right] \]

\[ + N_c a^{(0)}_{Qq} \left[ - \left( - \frac{\mu^2}{s_{qg}} \right) - \left( - \frac{\mu^2}{s_{qg}} \right) - \left( - \frac{\mu^2}{s_{qg}} \right) \right] \]

\[ + N_c a^{(0)}_{qg} \left[ - \left( - \frac{\mu^2}{s_{qg}} \right) + \left( - \frac{\mu^2}{s_{qg}} \right) + \left( - \frac{\mu^2}{s_{qg}} \right) \right] \]

\[ \left. \frac{-1}{\varepsilon} (4\gamma(q) + \gamma(g)) a^{(0)}_{QQ} \right. \]

(3.10)

The remaining subamplitudes can be obtained by simple label permutations

\[ a^{(1),\text{sing}}_{qQ} = a^{(1),\text{sing}}_{Qq} \mid_{q \rightarrow \bar{Q}, Q \rightarrow q} \quad a^{(1),\text{sing}}_{qq} = a^{(1),\text{sing}}_{QQ} \mid_{q \rightarrow \bar{Q}, Q \rightarrow q} \quad \] (3.11)

3.2 Subprocess class 0 → qggg\(\bar{q}\)

In this section we present the singular terms of the one-loop virtual corrections of the helicity color subamplitudes of the process 0 → qggg\(\bar{q}\).

We use the momentum labels as follows:

\[ 0 \rightarrow \text{quark}(q) + \text{gluon}(1) + \text{gluon}(2) + \text{gluon}(3) + \text{antiquark}(\bar{q}) \] (3.12)

The color structure of the amplitudes at tree level is given by

\[ \mathcal{A}^{\text{tree}}(q, g, g, g, \bar{q}) = g^3 \sum_{\text{perm (1,2,3)}} (T^{g_1 T^{g_2} T^{g_3}})_{qq} b^{(0)}_{123}(h_q, h_1, h_2, h_3, h_{\bar{q}}), \] (3.13)

while at one loop one finds

\[ \mathcal{A}^{\text{loop}}(q, g, g, g, \bar{q}) = g^3 \left( \frac{g}{4\pi} \right)^2 \left[ \sum_{\text{perm (1,2,3)}} (T^{g_1 T^{g_2} T^{g_3}})_{qq} b^{(1)}_{123}(h_q, h_1, h_2, h_3, h_{\bar{q}}) \right. \]

\[ + \sum_{i=1,2,3} (T^{g_i})_{qq} \delta_{g_{ki}} b^{(1)}_i(h_q, h_1, h_2, h_3, h_{\bar{q}}) \]

\[ + \delta_{q\bar{q}} \text{Tr}(T^{g_1 T^{g_2} T^{g_3}}) b_{q}^{(1)}(h_q, h_1, h_2, h_3, h_{\bar{q}}) \]

\[ + \delta_{q\bar{q}} \text{Tr}(T^{g_3 T^{g_2} T^{g_1}}) b_{q}^{(1)}(h_q, h_1, h_2, h_3, h_{\bar{q}}) \]. \] (3.14)
In this equation $k$ and $l$ are the two indices from the set $\{1, 2, 3\} \setminus i$ for a given $i \in \{1, 2, 3\}$.

The tree-level $b_{123}^{(0)}$ subamplitudes vanish for helicity configurations $(\pm, h_1, h_2, h_3, \pm)$ and $(\mp, h, h, h, \pm)$. When there is one gluon with negative helicity (denoted by $I$) and two with positive helicity, then

$$b_{123}^{(0)}(-, h_1, h_2, h_3, +) = \frac{p_b(-, h_1, h_2, h_3, +)}{\langle q_1 \rangle \langle 12 \rangle \langle 23 \rangle \langle 3q \rangle \langle qq \rangle},$$

$$b_{123}^{(0)}(+, h_1, h_2, h_3, -) = \frac{p_b(+, h_1, h_2, h_3, -)}{\langle q_1 \rangle \langle 12 \rangle \langle 23 \rangle \langle 3q \rangle \langle qq \rangle},$$

where

$$p_b(-, h_1, h_2, h_3, +) = i(qI)^3 \langle \bar{q}I \rangle,$$

$$p_b(+, h_1, h_2, h_3, -) = -i(qI)\langle \bar{q}I \rangle^3.$$  

The subamplitudes for processes with opposite helicities can be obtained from (3.15 - 3.18) by replacing $\langle \rangle$ with $[\ ]$.

Using these expressions, the singular parts of the renormalized one-loop subamplitudes can again be written in a simple form. For the $b_{ijk}^{(1)\text{, sing}}$ amplitudes one finds

$$b_{123}^{(1)\text{, sing}} = \frac{-C_T}{\varepsilon^2} \left\{ N_c \left[ \left( \frac{-\mu^2}{s_{1q}} \right)^\varepsilon + \left( \frac{-\mu^2}{s_{12}} \right)^\varepsilon + \left( \frac{-\mu^2}{s_{23}} \right)^\varepsilon + \left( \frac{-\mu^2}{s_{3q}} \right)^\varepsilon \right] \right\} b_{123}^{(0)}$$

$$- \frac{1}{\varepsilon} \left\{ 2\gamma(q) + 3\gamma(g) \right\} b_{123}^{(0)},$$

and the other permutations are obtained by simple rearrangement of indices:

$$b_{132}^{(1)\text{, sing}} = b_{123}^{(1)\text{, sing}} \mid_{2 \leftrightarrow 3}, \quad b_{231}^{(1)\text{, sing}} = b_{123}^{(1)\text{, sing}} \mid_{1 \leftrightarrow 2, 2 \leftrightarrow 3, 3 \leftrightarrow 1},$$

$$b_{213}^{(1)\text{, sing}} = b_{123}^{(1)\text{, sing}} \mid_{1 \leftrightarrow 2}, \quad b_{312}^{(1)\text{, sing}} = b_{123}^{(1)\text{, sing}} \mid_{1 \leftrightarrow 3, 2 \leftrightarrow 3}, \quad b_{321}^{(1)\text{, sing}} = b_{123}^{(1)\text{, sing}} \mid_{1 \leftrightarrow 3}.$$  

For the $b_{i}^{(1)\text{, sing}}$ amplitudes one finds

$$b_{1}^{(1)\text{, sing}} = \frac{-C_T}{\varepsilon^2} \left\{ \left[ -\left( \frac{-\mu^2}{s_{13}} \right)^\varepsilon + \left( \frac{-\mu^2}{s_{1q}} \right)^\varepsilon + \left( \frac{-\mu^2}{s_{23}} \right)^\varepsilon + \left( \frac{-\mu^2}{s_{2q}} \right)^\varepsilon \right] b_{123}^{(0)} \right\}$$

$$+ \left[ 2 \leftrightarrow 3 \right] b_{132}^{(0)} + \left[ q \leftrightarrow \bar{q} \right] b_{231}^{(0)} + \left[ 2 \leftrightarrow 3, q \leftrightarrow \bar{q} \right] b_{213}^{(0)},$$

$$b_{2}^{(1)\text{, sing}} = \frac{-C_T}{\varepsilon^2} \left\{ \left[ \left( \frac{-\mu^2}{s_{13}} \right)^\varepsilon - \left( \frac{-\mu^2}{s_{1q}} \right)^\varepsilon - \left( \frac{-\mu^2}{s_{23}} \right)^\varepsilon + \left( \frac{-\mu^2}{s_{2q}} \right)^\varepsilon \right] b_{132}^{(0)} \right\}$$

$$+ \left[ 1 \leftrightarrow 3 \right] b_{312}^{(0)} + \left[ q \leftrightarrow \bar{q} \right] b_{231}^{(0)} + \left[ 1 \leftrightarrow 3, q \leftrightarrow \bar{q} \right] b_{213}^{(0)}.$$
\[ b_{3}^{(1),\text{sing}} = -\frac{c}{\varepsilon^2} \left\{ \left[ -\left( -\frac{\mu^2}{s_{q2}} \right) \varepsilon + \left( -\frac{\mu^2}{s_{q3}} \right) \varepsilon + \left( -\frac{\mu^2}{s_{12}} \right) \varepsilon - \left( -\frac{\mu^2}{s_{13}} \right) \varepsilon \right] b_{123}^{(0)} \right\} (3.24) \]

\[ +[1 \leftrightarrow 2] b_{213}^{(0)} + [q \leftrightarrow \bar{q}] b_{321}^{(0)} + [1 \leftrightarrow 2, q \leftrightarrow \bar{q}] b_{312}^{(0)} \}, \]

\[ b_{q}^{(1),\text{sing}} = -\frac{c}{\varepsilon^2} \left\{ \left[ -\left( -\frac{\mu^2}{s_{q3}} \right) \varepsilon + \left( -\frac{\mu^2}{s_{q4}} \right) \varepsilon + \left( -\frac{\mu^2}{s_{12}} \right) \varepsilon - \left( -\frac{\mu^2}{s_{1q}} \right) \varepsilon \right] b_{123}^{(0)} \right\} (3.25) \]

\[ +(1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1) + (1 \rightarrow 3, 2 \rightarrow 1, 3 \rightarrow 2) \}, \]

\[ b_{\bar{q}}^{(1),\text{sing}} = -\frac{c}{\varepsilon^2} \left\{ \left[ -\left( -\frac{\mu^2}{s_{q2}} \right) \varepsilon + \left( -\frac{\mu^2}{s_{q4}} \right) \varepsilon + \left( -\frac{\mu^2}{s_{12}} \right) \varepsilon - \left( -\frac{\mu^2}{s_{1\bar{q}}} \right) \varepsilon \right] b_{132}^{(0)} \right\} (3.26) \]

\[ + (1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1) + (1 \rightarrow 3, 2 \rightarrow 1, 3 \rightarrow 2) \}, \]

4 Soft limits of the cross sections of 2 → 4 scattering processes

The soft singular terms in the one loop radiative corrections of the 2 → 3 hard processes determine the soft momentum dependence of a soft gluon emission from the hard partons that is the soft gluon momentum limit of general 2 → 4 processes

\[ \text{parton}(p_1) + \text{parton}(p_2) \rightarrow \text{parton}(p_3) + \text{parton}(p_4) + \text{parton}(p_5) + \text{gluon}(k). \] (4.1)

The soft momentum dependence factorizes into the eikonal factor

\[ e(m, n) = g^2 \frac{p_n \cdot p_m}{p_n \cdot k p_m \cdot k} \] (4.2)

so that we may write in the soft limit [14, 15]

\[ \sum_{\text{color}} |M(p_1, p_2, p_3, p_4, p_5, k)|^2 \to g^4 \sum_{n<m} e(m, n) \psi_{mn}(p_i, h_i). \] (4.3)

Integrating the d-dimensional phase space over the gluon momenta we obtain the same singular factor as given by eq. (2.7). Since the soft singularities cancel we can obtain the spin dependent \( \psi_{mn}(p_i, h_i) \) coefficient functions of the eikonal factors by calculating the singular contributions of the virtual corrections to the colored summed cross section of the 2 → 3 hard scattering process.

On the other hand, the calculation of the spin dependent \( \psi_{mn}(p_i, h_i) \) functions can also be carried out directly from the soft Bremsstrahlung equation (see ref. [14]). We carried out the calculation using both methods, in order to test the correctness of our results. It is an interesting feature of the soft limit that the spin dependence of the \( \psi_{mn} \) functions is factorizable: it can be absorbed into a simple factor \( p \) (see eqs. (3.7, 3.8, 3.17, 3.18) ).
is independent of the color structure. The helicity dependence of the \( \psi_{mn}(p_i, h_i) \) functions can be given as

\[
\psi_{mn}(p_i, h_i) = |p(p_i, h_i)|^2 \psi_{mn}^{si}(p_i),
\]

where \( \psi_{mn}^{si} \) is the spin independent part of \( \psi_{mn}(p_i, h_i) \).

The angular ordering approximation to the soft gluon distribution in the hard process is derived by the azimuthal averaging over the eikonal factor. Shower Monte Carlo programs use this piece of information, therefore, it is of interest to know the soft coefficient functions. In particular one can develop an initial condition for the Monte Carlo programs where for a given \( 2 \to 3 \) hard parton process a cone for the first gluon emission from each external line is chosen according to the probabilities defined with the help of soft emission. Therefore to obtain analytic expressions for the \( \psi_{mn} \) functions is of phenomenological interest. We note that 2-jet and 3-jet initial conditions are worked out for \( e^+e^- \) shower Monte Carlo programs but the 3-jet initial conditions are not yet developed for hadron-hadron and electron-proton collisions.

### 4.1 Subprocess class \( 0 \to \bar{q}Qqg(g) \)

The momenta of the partons are labeled as in (3.1) Then for the spin independent part of the soft coefficients we obtain the expressions

\[
\psi_{\bar{q}Q}(h_i) = \frac{V}{N_c} \left( s_{\bar{q}q} s_{Qg} s_{\bar{q}g} s_{Qg} \right)^{-1} \times \left( \begin{array}{c}
(N_c^2 + 3) s_{\bar{q}q} s_{Qg} s_{Qg} - 2 s_{\bar{q}g} s_{qg} s_{Qg} - 2 s_{qg} s_{Qg} s_{Qg} \\
+ (N_c^2 - 3) \left( s_{\bar{q}g} s_{Qg} s_{qg} + s_{qg} s_{Qg} s_{qg} - s_{qg} s_{qg} s_{Qg} \right) \end{array} \right)
\]

\[
\psi_{\bar{q}q}(h_i) = -\frac{V}{N_c} \left( s_{\bar{q}q} s_{Qg} s_{\bar{q}g} s_{Qg} \right)^{-1} \times \left( \begin{array}{c}
2 s_{\bar{q}g} s_{qg} s_{Qg} + 2 s_{qg} s_{Qg} s_{qg} - (N_c^2 + 1) s_{Qg} s_{qg} s_{Qg} \\
- s_{Qg} s_{qg} s_{qg} + (N_c^2 - 2) \left( s_{qg} s_{Qg} s_{qg} + s_{gq} s_{Qg} s_{gq} \right) \end{array} \right)
\]

\[
\psi_{\bar{q}g}(h_i) = \frac{V}{N_c} \left( s_{\bar{q}g} s_{Qg} s_{\bar{q}g} s_{Qg} \right)^{-1} \times \left( \begin{array}{c}
(N_c^2 - 3) \left( s_{\bar{q}g} s_{qg} s_{Qg} + s_{qg} s_{Qg} s_{qg} - s_{qg} s_{Qg} s_{qg} \right) \\
+ (2 - N_c^2) \left( s_{qg} s_{Qg} s_{qg} + s_{Qg} s_{qg} s_{qg} \right) \\
+ (N_c^2 - 3) s_{qg} s_{qg} s_{Qg} s_{Qg} \end{array} \right)
\]

Out of these four \( \psi_{mn}^{si} \) functions one can obtain all the ten functions by changing the labels correspondingly

- \( \psi_{\bar{q}Q}^{si} = \psi_{\bar{q}Q}^{si} \) with \( q \leftrightarrow \bar{q}, Q \leftrightarrow \bar{Q} \)

---

8
The spin summed coefficient functions are 

- \( \psi_{qQ}^s = \psi_{qQ}^s \) with \( q \leftrightarrow Q, \bar{q} \leftrightarrow \bar{Q} \)
- \( \psi_{qg}^s = \psi_{qg}^s \) with \( q \leftrightarrow \bar{q}, Q \leftrightarrow Q \)
- \( \psi_{Q\bar{Q}}^s = \psi_{q\bar{q}}^s \) with \( q \leftrightarrow Q, \bar{q} \leftrightarrow \bar{Q} \)
- \( \psi_{Qg}^s = \psi_{qg}^s \) with \( q \leftrightarrow \bar{q} \)
- \( \psi_{Qg}^s = \psi_{qg}^s \) with \( q \leftrightarrow Q, \bar{q} \leftrightarrow \bar{Q} \)

The spin summed coefficient functions are

\[
\psi_{mn} = \frac{2}{s_{Q\bar{Q}}s_{qg}} \left( s_{Qg}^2 + s_{Q\bar{Q}}^2 + s_{Qg}^2 + s_{Q\bar{Q}}^2 \right) \psi_{mn}^s
\]  

(4.9)

### 4.2 Subprocess class 0 → q\bar{q}q\bar{q}g(g)

In the case of equal flavor, we should use the same amplitudes as in the case of unequal flavor but we should antisymmetrize it in the momenta \( \bar{Q} \leftrightarrow \bar{g} \) and in the corresponding color and helicity labels.

The \( \psi_{mn}(h_i) \) functions then can be written in the form

\[
\psi_{mn}(h_i) = -\frac{V}{N} \left( |p_a(q, \bar{Q}; Q, q; g)|^2 A_{mn} + |p_a(\bar{Q}, q; Q, q; g)|^2 B_{mn} - 2\text{Re}(p_a(q, \bar{Q}; Q, q; g)p_a(\bar{Q}, q; Q, q; g)^*C_{mn}) \right).
\]

(4.10)

To simplify the final expressions we use the notation

\[
\{ij\} \equiv \frac{2s_{ij}}{s_{ig}s_{jg}}, \quad f_{ij} \equiv \frac{\langle ij \rangle}{\langle ig \rangle \langle gj \rangle}
\]

but here \( g \) denotes the hard gluon momentum, and \( i, j \) can run over values \( q, Q, \bar{q} \) or \( \bar{Q} \). Then the auxiliary functions \( A, B, C \) appearing in (4.10) can be written in the simple form

\[
A_{q\bar{Q}} = \left( \frac{3}{2N_c} - \frac{N_c}{2} \right) \left( \{q\bar{Q}\} + \{\bar{q}Q\} - \{\bar{q}\bar{Q}\} \right) + \frac{1}{N_c} \{q\bar{q}\} - \left( \frac{3}{2N_c} + \frac{N_c}{2} \right) \{q\bar{Q}\} + \frac{1}{N_c} \{Q\bar{Q}\}
\]

\[
B_{q\bar{Q}} = A_{q\bar{Q}}|_{\bar{q} \leftrightarrow Q}
\]

\[
C_{q\bar{Q}} = \frac{V}{N_c^2} \left( f_{q\bar{Q}} f_{\bar{q}Q}^* + f_{q\bar{q}} f_{q\bar{Q}}^* \right) + V \left( f_{q\bar{Q}} f_{Qq}^* + f_{q\bar{q}} f_{Q\bar{q}}^* \right) - \frac{1}{N_c^2} \left( f_{q\bar{Q}} f_{Qq}^* + f_{Q\bar{q}} f_{q\bar{Q}}^* \right) - \left( f_{q\bar{Q}} f_{Qq}^* + f_{q\bar{q}} f_{Q\bar{q}}^* + f_{q\bar{Q}} f_{Q\bar{q}}^* + f_{Q\bar{q}} f_{q\bar{Q}}^* + f_{q\bar{Q}} f_{q\bar{Q}}^* + f_{Q\bar{q}} f_{Q\bar{q}}^* \right)
\]

\[
A_{qg} = \left( \frac{N_c}{2} - \frac{1}{N_c} \right) \left( \{q\bar{Q}\} + \{q\bar{Q}\} \right) - \left( \frac{1}{2N_c} + \frac{N_c}{2} \right) \{q\bar{q}\}
\]
We use the momentum labels as in (3.12) and we introduce the shorthand notation.

### 4.3 Subprocess class

\[ \psi_{A,B} \]

procedure as in the previous process now with the functions can be obtained by

\[
\begin{align*}
B_{qq} &= A_{qq}|_{q\rightarrow Q} \\
C_{qq} &= -\frac{V}{N_c^2} \left( f_{QQ}f_{qQ}^* + f_{qq}f_{qQ}^* \right) + \frac{1}{N_c^2} \left( f_{qq}f_{qQ}^* + f_{qq}f_{qQ}^* \right) \\
&\quad - f_{qq}f_{qQ}^* + V f_{qq}f_{qQ}^* - N_c^2 \left( f_{qq}f_{qQ}^* + f_{qq}f_{qQ}^* \right) \\
&\quad + \left( f_{qQ}f_{qQ}^* + f_{qq}f_{qQ}^* + f_{qq}f_{qQ}^* + f_{qq}f_{qQ}^* + f_{qq}f_{qQ}^* + f_{qq}f_{qQ}^* \right)
\end{align*}
\]

\[
\begin{align*}
A_{qq} &= \left( \frac{3}{2N_c} - \frac{N_c}{2} \right) \left( \{\bar{q}Q\} + \{qQ\} - \{\bar{q}Q\} \right) \\
&\quad + \left( \frac{3N_c}{2} - N_c - \frac{N_c^3}{2} \right) \{qQ\} + \left( \frac{N_c}{2} - \frac{1}{N_c} \right) \{\bar{q}Q\} \\
B_{qq} &= A_{qq}|_{q\rightarrow Q} \\
C_{qq} &= N_c^2 \left( f_{QQ}f_{qQ}^* + f_{qq}f_{qQ}^* + f_{qq}f_{qQ}^* - f_{qq}f_{qQ}^* - f_{qq}f_{qQ}^* \right) \\
&\quad + \left( f_{QQ}f_{qQ}^* - f_{qq}f_{qQ}^* - f_{QQ}f_{qQ}^* - f_{qq}f_{qQ}^* \right)
\end{align*}
\]

In order to get the expressions for the other \( \psi_{mn} \) functions, we have to do the same relabeling procedure as in the previous process now with the \( A, B \) and \( C \) functions. The spin summed \( \psi_{mn} \) functions can be obtained by

\[
\begin{align*}
|p_a(\bar{q}, \bar{Q}; Q, q; g)|^2 &\rightarrow \frac{2}{s_{QQ}s_{qq}} \left( s_{QQ}^2 + s_{qQ}^2 + s_{qQ}^2 + s_{qQ}^2 \right) \\
|p_a(Q, \bar{q}; Q, q; g)|^2 &\rightarrow \frac{2}{s_{qQ}s_{qq}} \left( s_{QQ}^2 + s_{qQ}^2 + s_{qQ}^2 + s_{qQ}^2 \right) \\
\text{Re}(p_a(\bar{q}, \bar{Q}; Q, q; g)p_a(Q, \bar{q}; Q, q; g)^*C_{mn}) &\rightarrow \frac{s_{qq}^2 + s_{qQ}^2}{s_{qq}s_{qq}s_{qQ}s_{qQ}} \left( s_{qq}s_{QQ} + s_{qQ}s_{qQ} - s_{qq}s_{qQ} \right) \text{Re}(C_{mn})
\end{align*}
\]

### 4.3 Subprocess class 0 \( \rightarrow \bar{q}\bar{q}ggg(g) \)

We use the the momentum labels as in (3.12) and we introduce the short hand notation

\[
\mathcal{D} \equiv s_{12}s_{13}s_{23}s_{1q}s_{1q}s_{2q}s_{2q}s_{3q}s_{3q}s_{qq}.
\]
Then for the soft color coefficient functions we obtain
\[
\psi_{q_1}^{x_1} = \frac{V}{N_c} \frac{s_{qq}}{s_{1q} s_{1q} s_{2q} s_{3q} s_{3q}} (4.12)
\]
\[
+ \left( N_c^5 - N_c \right) \frac{1}{s_{23} s_{1q} s_{qq}} \left( \frac{1}{s_{12} s_{3q}} + \frac{1}{s_{13} s_{2q}} \right)
\]
\[
- \frac{N_c V}{D} \left\{ -3 \cdot s_{23} s_{1q} s_{2q} s_{3q} - s_{23} s_{1q} s_{1q} s_{qq} + s_{23} s_{1q} s_{2q} s_{3q} + s_{13} s_{1q} s_{2q} s_{qq} + s_{13} s_{2q} s_{3q} s_{qq} - s_{13} s_{1q} s_{3q} s_{2q} + (2 \leftrightarrow 3) \right\}.
\]
From the above expression we get five other \(\psi_{q_1}^{x_i}\) functions by changing the labels of the momenta in the spin independent part

- \(\psi_{q_2}^{x_1} : (q, \bar{q}, 1, 2, 3) \rightarrow (q, \bar{q}, 2, 3, 1)\)
- \(\psi_{q_3}^{x_1} : (q, \bar{q}, 1, 2, 3) \rightarrow (q, q, 1, 2)\)
- \(\psi_{q_4}^{x_1} : (q, \bar{q}, 1, 2, 3) \rightarrow (\bar{q}, q, 1, 2, 3)\)
- \(\psi_{q_5}^{x_1} : (q, \bar{q}, 1, 2, 3) \rightarrow (\bar{q}, q, 2, 3, 1)\)
- \(\psi_{q_6}^{x_1} : (q, \bar{q}, 1, 2, 3) \rightarrow (\bar{q}, q, 1, 2)\)

There are still two additional independent coefficient functions which cannot be obtained by symmetry properties

\[
\psi_{q_7}^{x_1} = \frac{1 - N_c^3}{N_c^2} \frac{s_{qq}}{s_{1q} s_{1q} s_{2q} s_{3q} s_{3q}}
\]
\[
+ \frac{N_c V}{D} \sum_{Z(123)} \left\{ -2 \cdot s_{12} s_{3q} s_{1q} s_{2q} - 2 \cdot s_{12} s_{3q} s_{1q} s_{2q} \right\}
\]
\[
- \frac{V}{D} \sum_{Z(123)} \left[ s_{12} s_{3q} s_{1q} s_{2q} s_{3q} + s_{13} s_{2q} s_{1q} s_{2q} s_{3q} - 2 \cdot s_{23} s_{1q} s_{1q} s_{2q} s_{3q} \right]
\]
\[
- 2 \cdot s_{2q} s_{2q} s_{3q} s_{1q} + s_{12} s_{13} s_{23} s_{1q} s_{1q} + (q \leftrightarrow \bar{q}) \right\},
\]
where \(Z(123)\) denotes the cyclic permutations and

\[
\psi_{x_1}^{x_1} = \frac{N_c^2 V}{s_{qq}} \left( \frac{1}{s_{12} s_{13} s_{2q} s_{3q}} + \frac{1}{s_{13} s_{23} s_{1q} s_{3q}} \right) \right) + (q \leftrightarrow \bar{q}) \right) \right)
\]
\[
- \frac{N_c V}{D} \left\{ -2 \cdot s_{12} s_{13} s_{23} s_{1q} s_{2q} s_{3q} - s_{23} s_{1q} s_{1q} s_{2q} s_{3q} + 2 \cdot s_{23} s_{1q} s_{1q} s_{2q} s_{3q} \right\}
\]
\[
+ 2 \cdot s_{23} s_{2q} s_{3q} \right] \left[ s_{1q} s_{2q} s_{3q} + s_{1q} s_{2q} s_{3q} \right]
\]
\[
+ 2 \cdot s_{23} s_{2q} s_{3q} \right] \left[ s_{1q} s_{2q} s_{3q} + s_{1q} s_{2q} s_{3q} \right]
\]
Spin summation results in

\[
\psi_{mn} = 2 \sum_{i \in \{1, 2, 3\}} \left( s_{iq}^3 s_{iq} + s_{iq}^3 s_{iq} \right) \psi_{mn} \right)
\]
4.4 Subprocess class $0 \rightarrow ggggg(g)$

We introduce the simple notation for labeling the quarks and gluons as

$$0 \rightarrow \text{gluon}(1) + \text{gluon}(2) + \text{gluon}(3) + \text{gluon}(4) + \text{gluon}(5)$$

(4.16)

Due to Bose symmetry it is sufficient to calculate only one of the ten $\psi_{mn}$ functions

$$\psi_{13} = 2N_c^4 V \sum_{P(2,4,5)} \frac{1}{s_{13} s_{32} s_{24} s_{45} s_{51}}$$

$$- 24 N_c^2 V \left( \frac{1}{s_{12} s_{14} s_{15} s_{24} s_{25} s_{34}} + \frac{1}{s_{14} s_{15} s_{23} s_{24} s_{25} s_{34}} \right.$$  

$$+ \frac{1}{s_{14} s_{15} s_{23} s_{24} s_{25} s_{34}} - \frac{1}{s_{12} s_{14} s_{15} s_{34} s_{35}}$$

$$- \frac{1}{s_{12} s_{14} s_{23} s_{24} s_{25} s_{34} s_{45}} + \frac{1}{s_{12} s_{14} s_{15} s_{23} s_{34} s_{45}}$$

$$+ \frac{1}{s_{12} s_{14} s_{23} s_{24} s_{25} s_{34} s_{45}} - \frac{1}{s_{12} s_{14} s_{15} s_{23} s_{34} s_{35}}$$

$$- \frac{1}{s_{12} s_{14} s_{23} s_{24} s_{25} s_{34} s_{45}} + \frac{2s_{13}}{s_{24}}$$

$$- \frac{1}{s_{12} s_{14} s_{23} s_{24} s_{25} s_{34} s_{45}} - \frac{s_{25}}{s_{12} s_{15} s_{23} s_{24} s_{34} s_{45}} \right) .$$

Although in the expression for $\psi_{13}^{si}$ the symmetry $1 \leftrightarrow 3$ is not manifest, using momentum conservation we can check that it has the symmetry $1 \leftrightarrow 3$. We get the other $\psi_{mn}^{si}$ functions now simply by the usual replacements of the labels of the momenta, i.e. $\psi_{13}^{si} \rightarrow \psi_{12}^{si}$ by $3 \leftrightarrow 2$.

The spin dependent part of the born amplitude is given by

$$p_d = i(mn)^4$$

if $m$ and $n$ are the only negative helicity gluons and by

$$p_d = i[mn]^4$$

if $m$ and $n$ are the only positive helicity gluons. So, for the helicity summed $\psi_{mn}$ functions we get

$$\psi_{mn} = 2 \sum_{i<j} s_{ij}^4 \psi_{mn}^{si} .$$

(4.18)

5 Conclusions

We presented an algorithm of general validity, which enables us to calculate the singular terms of any one-loop helicity amplitude in QCD. The correctness of the algorithm was tested by explicitly demonstrating the cancellation of the soft singularities between the loop and Bremsstrahlung contributions for spin dependent $2 \rightarrow 3$, $2 \rightarrow 4$ processes. We also emphasized the universality of the collinear singularities (see [2.3]). Furtheron, we obtained short formulæ for the soft limit of the spin dependent cross sections of all $2 \rightarrow 4$ processes. These results represent a necessary input for shower Monte Carlo programs for the production of three well-separated jets.
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