Abstract

In calculating the energy corrections to the hydrogen levels we can identify two different types of modifications of the Coulomb potential $V_C$, with one of them being the standard quantum electrodynamics corrections, $\delta V$, satisfying $|\delta V| \ll |V_C|$ over the whole range of the radial variable $r$. The other possible addition to $V_C$ is a potential arising due to the finite size of the atomic nucleus and as a matter of fact, can be larger than $V_C$ in a very short range. We focus here on the latter and show that the electric potential of the proton displays some undesirable features. Among others, the energy content of the electric field associated with this potential is very close to the threshold of $e^+e^-$ pair production. We contrast this large electric field of the Maxwell theory with one emerging from the non-linear Euler-Heisenberg theory and show how in this theory the short range electric field becomes smaller and is well below the pair production threshold.
I. INTRODUCTION

The hydrogen atom was one of the stepping stones of quantum mechanics. Today it has become a laboratory for precision physics (especially from the experimental point of view) where several areas of physics are combined to explain the intricacies of this rather simple system. The inputs required are taken from particle physics \[1\] (as such the hydrogen atom constitutes also a testing ground for fundamental theories like Quantum Electrodynamics (QED) and its electroweak extension \[2\]), quantum field theories which provide the necessary formalism (the two body Breit equation \[3, 4\], the one-body Dirac equation or the Bethe-Salpeter equation \[5\]) and nuclear (or hadronic) physics contributing to the physics of the finite extension of the central nucleus \[6, 7\]. Indeed, latest efforts in the field tend to extract the static properties of the proton from hydrogen transitions \[6, 8, 9\]. Finally, we can imagine the hydrogen atom as a testing ground for new ideas. From the theoretical point of view, the electroweak corrections to the Coulomb potential \(V_C\) play a crucial role in understanding the simple two body system of the hydrogen atom. The high precision of experimental confirmation of such corrections underpins the Standard Model of particle physics or, in case of disagreement, reveals new physics. These electroweak corrections comprised in \(\delta V\) satisfy usually the inequality \(|\delta V| \ll |V_C|\) over the full range of the radial variable \(r\). Life would be relatively easy up to this point, were it not for hadronic corrections due to the finite size of the proton. This second class of corrections lacks the high precision of the electroweak corrections and therefore blurs the extraction of the latter. Such a situation is also encountered in other areas of precision physics (see, e.g., the problem of the anomalous magnetic moment of the muon \[10\]). One can circumvent the problem of the hadronic uncertainties by trying to determine experimentally the finite size corrections (FSC) relying on the correctness and accuracy of the other corrections. For instance, in its simplest version, the FSC to the ground states can be parametrized by the proton charge radius \(<r^2>=\int \rho_p(r) r^2 d^3r\) where \(\rho_p(r)\) is the charge density of the proton. The energy correction (also in its simplest form) is given by \(\delta E = (e^2/6)|\Psi(0)|^2 <r^2>\) \[11\]. Comparing the transition energies, \(\Delta E_{\text{exp}}\), with the theoretical ones given as \(\Delta E_{\text{point}}(\text{theory}) + \delta E_{\text{FSC}}\), one can extract the proton radius from the measured transition energies. By using one static property of the proton (the radius), one of course loses some insight into the full structure of the extended proton. Indeed, the finite proton size is characterized by all moments and not only the second one.
which is the radius. The charge distribution $\rho_p$ determines the electric field $E_p$, the electric potential $V_p$ and partly the scalar interaction potential $V_{ep}$ between the electron and the proton. However, by extracting $<r^2>$ alone, we do not obtain the full information of the electromagnetic structure of the proton. In the present work we show that it is worthwhile to consider the role of the full proton structure in the hydrogen atom as it exhibits novel and interesting features. First of all, $V_p$ is not a correction to the Coulomb potential in the short range. At the same time the fact that the deviation is of a short range character allows us to treat it as a correction in calculating corrections to the hydrogen energies by means of time-independent perturbation theory. Secondly, as we will show, the energy content of the field $E_p$ is very close to the pair production threshold. Given the uncertainties in the electromagnetic form factors on which $E_p$ is based this is a rather disturbing fact.

Finally, the scalar interaction potential $V_{ep}$ which includes now the electron Darwin term has a large repulsive core close to the center. By reversing the sign of the electron charge we arrive at the interaction Hamiltonian of the positron-proton system. The repulsive core becomes a deep potential well. The consequence of such a well is a possible bound state or resonance not observed in any experiment. Taking all this together, it appears that the proton electric field inside the hydrogen atom might be too strong to be realistic. Of course, even though the parametrizations of the electromagnetic form factors lack a high precision, there is no doubt on the correct magnitude of these form factors or on the method to calculate $E_p$ by using them within the framework of the Maxwell’s equations. If we want to lower $E_p$ we will have to go one step ahead and modify the Maxwell’s equations. Such a modification cannot be arbitrary, but must be a consequence of a deeper principle. Indeed, it is well known that quantum mechanics modifies the Maxwell’s equations by the existence of a four-photon vertex due to light-light scattering. This leads us to the Euler-Heisenberg theory, which is a nonlinear version of electrodynamics. In its electrostatic limit it is possible to calculate the modified electric field $E_{p\gamma\gamma}$ once the Maxwellian result $E_p$ is given, i.e. we have $E_{p\gamma\gamma}[E_p]$. One can show that $E_{p\gamma\gamma}$ is a correction to $E_p$ for large $r$, but in the short range region the deviations are more significant. Indeed, $E_{p\gamma\gamma}$ turns out to be much smaller than $E_p$ in this region which, in principle, is the effect we are looking for. There is a priori no reason to discard this short range region as its only observable manifestation will be in the energy corrections which will be small due to the short range character (the situation is similar to the case of $E_p$ in comparison to the Coulomb potential). However, the
magnitude of the field will exceed the limits of validity of the Euler-Heisenberg theory valid for weak fields. This can be traced back partly to the large values of $E_p$ close to the center. We will show that the inclusion of higher orders improves the situation. We speculate that already a third order result might be sufficient to reduce the Maxwellian result of $E_p$ and at the same time obeying the weak field condition imposed in the Euler-Heisenberg theory. In the second step we show how the light-light formalism can be extended to the whole scalar interaction in the hydrogen atom, this is to say including the electron Darwin term.

The notation convention in the present work consistently follows [3]. This implies, $e^2 = \alpha$ and the Breit equation as well as the Euler-Heisenberg Lagrangian are as in [3].

The article is organized as follows. In Section II we discuss the derivation of the scalar interaction potential in the hydrogen atom via the Breit equation with form factors. We identify here also the part which is due to the electric potential of the proton and compare it with the point-like Coulomb result. In the Section III, we give an overview of the Euler-Heisenberg theory, specializing on the electrostatic case. In the fourth section we apply the results of the Euler-Heisenberg theory to the hydrogen atom giving a new perspective to the short range interaction. In Section V, we comment on the truncation of the expansion which we use in the article. In the last section we draw our conclusions.

II. BREIT EQUATION WITH FORM FACTORS

In accordance with scattering theory in quantum mechanics, the Born approximation of the scattering amplitude is proportional to the Fourier transform of the potential. Vice versa, given a scattering amplitude from the Feynman diagrams suitably expanded in powers of $1/c$ ($c$ is the velocity of light) we can derive a potential in $r$ space. This well known procedure has been applied many times in physics to get some insight into old and new dynamics. One of the application of this principle is the Breit equation which starting from the elastic electron proton scattering amplitude gives us some well known interaction terms (like the Coulomb, electron Darwin, fine and hyperfine structure [15]) and some new input in the non-relativistic picture (like the proton Darwin and retardation terms [6,15]). Finally, it is possible to include in the Breit equation, the finite size corrections in an elegant way by
using the most general current of the proton and replacing

\[ e\gamma_{\mu} \rightarrow e \left( F_1(q^2)\gamma_{\mu} + \frac{F_2(q^2)}{2m_p}\sigma_{\mu\nu}q^{\nu} \right) \]  

(1)

In case of a point-like object, \( F_1 = 1 \) and \( F_2 = 0 \). The following Breit equation is thus quite general:

\[
\hat{U}(p_X, p_p, q) = 4\pi e^2 \left[ F_1^X F_1^p \left( -\frac{1}{q^2} + \frac{1}{8m_X^2c^2} + \frac{1}{8m_p^2c^2} + \frac{i\sigma_{\mu}(q \times p_p)}{4m_p^2c^2q^2} - \frac{i\sigma_{\mu}(q \times p_X)}{4m_X^2c^2q^2} \right) 
\right. 
\left. + \frac{p_X \cdot p_p}{m Xm_p c^2 q^2} \times \frac{\left(p_X \cdot q \right) \left(p_p \cdot q \right)}{m Xm_p c^2 q^4} - \frac{i\sigma_{\mu}(q \times p_p)}{2m_p^2c^2q^2} + \frac{i\sigma_{\mu}(q \times p_X)}{2m_X^2m_p c^2q^2} + \frac{\sigma_{\mu} \cdot \sigma_{\mu}}{4m_Xm_p c^2} \right. 
\left. - \frac{(\sigma_{\mu} \cdot q)(\sigma_{\mu} \cdot q)}{4m_Xm_p c^2q^2} \right) 
+ \frac{F_2^X F_2^p \left( \frac{1}{4m_p^2c^2} - \frac{i\sigma_{\mu}(q \times p_p)}{2m_p^2c^2q^2} + \frac{i\sigma_{\mu}(q \times p_X)}{2m_X^2m_p c^2q^2} - \frac{(\sigma_{\mu} \cdot q)(\sigma_{\mu} \cdot p)}{4m_Xm_p c^2q^2} \right) 
\right. 
\left. + \frac{\sigma_{\mu} \cdot \sigma_{\mu}}{4m_Xm_p c^2} \right) 
\left. + \frac{F_2^X F_2^p \left( \frac{\sigma_{\mu} \cdot \sigma_{\mu}}{4m_Xm_p c^2} - \frac{(\sigma_{\mu} \cdot q)(\sigma_{\mu} \cdot q)}{4m_Xm_p c^2q^2} \right) \right],
\]  

(2)

where \( X \) stands for electron or muon. In the rest of the article we will focus on the scalar part of this equation neglecting the structure of the point-like lepton. This implies that we will not take into account terms with spins or momentum dependent interactions. We are then left with the \( ep \) interaction potential given by [6],

\[
V_{ep}(q) = -4\pi e^2 \left[ \frac{\tilde{G}_E(q)}{q^2} - \frac{1}{8m_p^2c^2} \tilde{G}_E(q) - \frac{1}{8m_p^2c^2} \tilde{G}_E(q) \right],
\]  

(3)

where \( \tilde{G}_E(q) = G_E(q)(1 - q^2/8m_p^2) \). The above, in principle, is valid at higher order in the \( 1/c \) expansion. At the lowest order of the non-relativistic expansion we have

\[
\tilde{G}_E \simeq G_E = F_1 - \frac{q^2}{4m_p^2c^2} F_2
\]  

\[
\tilde{G}_M \simeq G_M = F_1 + F_2
\]  

(4)

where \( G_E \) and \( G_M \) are the electric and magnetic Sachs form factors. With this approximation, the first term in the square brackets in (3), called the Coulomb term, will become \( G_E/q^2 \) whereas the electron Darwin term is well approximated by \( V_{eD} = 4\pi e^2(1/8m_e^2c^2)F_1 \).
(and similarly the proton Darwin term $V_{pD} = 4\pi e^2(1/8m_p^2c^2)F_1$). The Fourier transform of (3) gives

$$V_{ep}(\mathbf{r}) = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q} \cdot \mathbf{r}} V_{ep}(\mathbf{q})$$

$$= -e[V_C(\mathbf{r}) + V_{pD}(\mathbf{r}) + V_{eD}(\mathbf{r})] = -e[V_C(\mathbf{r}) + V_{p}(\mathbf{r}) + V_{eD}(\mathbf{r})]$$

(5)

and we can identify the electric potential of the proton $V_p$ from the Breit equation to be given by the parts which, apart from being scalar (no spin and momentum dependence) are also independent of the properties of the probe (in this case the lepton). By this token the proton Darwin term, $V_{pD}$, is part of the electric potential, but the electron Darwin, $V_{eD}$, not. This has some consequences for the electric field at small distances inside the proton and also for the charge distribution, but is of no further importance here. Using the Poisson equation we can also say more about both the Darwin terms in $\mathbf{r}$ space, i.e., we have

$$V_{eD}(\mathbf{r}) \propto V_{pD}(\mathbf{r}) \propto \rho(\mathbf{r}) \propto \int d^3q e^{i\mathbf{q} \cdot \mathbf{r}} \tilde{G}_E(\mathbf{q})$$

(6)

Expanding the Dirac Hamiltonian (Ref. [3], p.125) up to order $1/c^2$ gives the well known result

$$\hat{H}_{Dirac} \simeq \frac{\mathbf{p}^2}{2m_e} - \frac{\mathbf{p}^4}{4m_e^2c^2} + eV_p - \frac{e}{4m_e^2c^2} \mathbf{\sigma} \cdot \mathbf{E} \times \mathbf{\hat{p}} - \frac{e}{8m_e^2c^2} \nabla \cdot \mathbf{E}.$$

(7)

The last term is the electron Darwin term which using the Maxwell’s equations is, of course, proportional to $\rho$. If we go from the Maxwell theory to the Euler-Heisenberg theory, the Gauss law is not valid anymore and it therefore matters if we treat the electron Darwin term in the Breit equation formalism or the Dirac one. Apart from this, we note that the fact that no proton Darwin term appears in the Dirac equation is based on the difference between the Breit and the Dirac equation. The Breit equation is a true two-body equation whereas the Dirac equation is a one particle equation. We will come back to this point later.

### A. Scalar potentials of the extended proton

It is sometimes useful to resort to simplifying assumptions in order to obtain analytical formulae providing a physical insight into the problem, rather than using more precise inputs which do not lead to analytical expressions and are difficult to handle numerically. Take, for instance, the energy correction $\delta E$ due to the finite size effects in the hydrogen atom. The calculation is, in principle, a two scale problem where one encounters the nuclear form
factors which are important at distances of only a couple of fermis combined with the atomic wave function which extends to about $10^4$ fm. The combination of scales makes a numerical calculation difficult and hence it is convenient to use the dipole parametrization for the proton form factor which allows analytical calculations. Hence we choose,

$$G_E(q^2) = \frac{1}{(1 + q^2/m^2)^2}. \quad (8)$$

This leads to the electric potentials,

$$V_p(r) = V_C + V_{pD}, \quad \text{with}$$

$$V_C(r) = \frac{e}{r} \left(1 - e^{-mr} \left(1 + \frac{mr}{2}\right)\right)$$

$$V_{pD}(r) = -\frac{e}{8m_p^2} \left(1 + \frac{\kappa_p(1 - 2\kappa^2)}{1 - \kappa^2} \right) m^3 \frac{e^{-mr}}{2} - \frac{\kappa^2 \kappa_p m^2}{(1 - \kappa^2)^2} \frac{e^{-mr}}{r} + \frac{\kappa^2 \kappa_p m^2}{(1 - \kappa^2)^2} \frac{e^{-\kappa mr}}{r}, \quad (9)$$

where $\kappa = 2m_p/m$ and $\kappa_p$ the anomalous magnetic moment of the proton. To arrive at the interaction potential it suffices to multiply the above expressions with $-e$ and add the electron Darwin term. The main conclusions of the paper will remain mostly insensitive to the parametrization of the electric form-factors as we will explain below.

1. **ep Interaction Potentials**

Let us have an unbiased look at the interaction potentials depicted in Figures 1-3.
FIG. 2: The short range $ep$ interaction potential including the electron Darwin term. The dashed line represents the point-like Coulomb potential $-\alpha/r$. The short distance behavior is mostly due to the electron Darwin term. This is shown in the inlay.

FIG. 3: Comparison of the different finite size corrections to the electron-proton interaction potential.

The first one in Fig. 1 resembles a Wood-Saxon potential with a depth of $-3$ MeV. The second one (Fig. 2) has a minimum at $-0.3$ MeV, but also a repulsive core with a height of $2 \times 10^3$ MeV. The potentials are, of course, not nuclear potentials (although they have to do
with hardionic properties), but reflect the finite size of the proton. It is interesting to infer on the effect of these potentials from an angle which is different from the time independent perturbation theory. To this end we choose the variational principle (see next sub-section) which reveals that the ground state is close to $-13$ eV. In spite of the modifications of the electron proton potential at short distances the resulting energy eigenvalue is still like in the standard hydrogen atom (plus corrections, of course). On the other hand, by replacing in $V_{ep}$, $e \rightarrow -e$, one obtains the positron proton potential with a potential depth of $10^6$ MeV. The question we can ask here is if such a deep but short range potential will lead to a bound state or a resonance.

2. Proton electric potential

Before proceeding, let us address two issues connected with the electric potential of the proton. The first one is the inclusion of the proton Darwin term into the definition of the electric potential. Apart from being suppressed by the squared proton mass, the effects are mild as shown in Fig. 4 for the charge distribution (evaluated using $\nabla^2 V_p(r)$). However, this term is large at short distances and we shall come to the effects of it later. Due to

$$\nabla \cdot \mathbf{E} = \frac{2}{r} \mathbf{E} + \frac{dE}{dr} \propto \rho,$$

the charge distribution is non-zero at the center when we include

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{proton_charge_dist}
\caption{The charge distribution of the proton.}
\end{figure}
the proton Darwin term in the potential. The second issue involves testing the sensitivity of the results to the choice of the parametrization of the electromagnetic form factors. This is shown in Fig. 5. The main observation of the large value of the short range potentials does not change qualitatively for the three parametrizations. Notice also that all three cases display a local minimum albeit displaced by one fermi. This displacement is not of much significance for our main points. Therefore, we continue working with the dipole parametrization in the rest of the work.

FIG. 5: The electron-proton interaction potential in different parametrizations.

It is not common to see results similar to those presented here (and in the subsequent sub-section) very often since the simplest estimate of the correction to the energy level(s) of the hydrogen atom is

$$\delta E = \int d^3x |\Psi(x)|^2 \left[ eV_{p}(r) + \frac{e^2}{r} \right]$$

which after integration by parts becomes

$$\frac{e}{6} |\Psi(0)|^2 \int d^3x r^2 \nabla^2 V_{p}.$$  \hspace{1cm} (11)

Using the Poisson equation and the definition of the radius as the second moment of the
charge distribution, i.e.,

\[ < r^2 > = \int d^3 x r^2, \tag{12} \]

one finally gets

\[ \delta E = \frac{e}{6} |\Psi(0)|^2 Z e < r^2 >. \tag{13} \]

Seemingly one does not need to know anything about the electric fields inside the proton as everything is encoded in the proton radius. However, a calculation of the electric field corresponding to this potential (which will be presented in the next section) shows that it is extremely large towards the center. One can calculate the energy content in the electric field:

\[ \mathcal{E}[E_p] = \frac{1}{8\pi} \int E^2_p d^3 x \simeq 1 \text{ MeV}, \tag{14} \]

which is very close to \(2m_e\) and given the uncertainty of the form-factors at the border of the \(e^+e^-\) pair production threshold. One is justified to ask if the field inside the hydrogen atom can be so large and dangerously close to the pair production threshold. Finally, is there a way to reduce it?

### 3. Some important perspectives

Before embarking on the variational calculations it is illuminating to give the finite size corrections or, in other words the potential at short distances, some other perspectives. Even at the cost that this is obvious let us remark that the field/potential of the proton at small \(r\) is large, but still finite as compared to \(1/r\). Therefore, it is not a correction, but its short range allows us to treat it as a correction in \(\delta E\). We shall discuss this further in the next subsection.

Classically, by Gauss law, the electron will not “feel” the finite size of the proton (assuming its charge distribution to be spherically symmetric). Hence, the fact that we can calculate it and measure it is indeed a quantum effect which would not be present in the classical treatment as long as we treat the proton also classically, i.e., as an object with a sharp radius (hard sphere). The classical electron will only see the Coulomb potential as long as it is outside the proton. If we relax the classical picture and replace the sharp proton by a “medium” with a charge distribution without border then the classical electron will also get affected by the \(V_{ep}\) depicted in figures 1-3. For example, let us consider the potential
(solid line) in Fig. 2 with a depth of around 350 keV. In this case, a classical ground state
which is a few hundred keV deep and with an orbit at a radius of about 4 fm is possible.
Classically, the electron in spite of being accelerated cannot radiate losing energy as \( E = V_{ep} \)
\((l = 0)\) is classically the lowest possible energy. This means that the local minimum around
350 keV would be the real minimum of the electron in the hydrogen atom. A quantum
mechanical calculation with the same potential, however, reproduces the realistic picture of
a ground state close to 13.6 eV and with a radius which is about 4 orders of magnitude
larger and in agreement with the atomic radius. The effect of the modified potential (as
compared to \(1/r\)) is only a tiny correction to the ground state energy.

The quantum effect mentioned above is based on the fact that we need to define the wave
function everywhere in space and as a result also the potential. Hence, it is expected that
we will find more examples in nature manifesting such a quantum effect. Indeed, an electron
around a spherically symmetric mass distribution (formally, this has to do with the Dirac
equation in the Schwarzschild metric) will know if it “moves” around a black hole or a star
\([16]\). It can be shown that in the case of a black hole there are no bound states which is
partly a macroscopic quantum effect related to the same fact that quantum mechanically
the electron “knows” that the proton is extended.

B. Variational approach to finite size corrections

The usual separation of variables in the Schrödinger equation gives the following equation
for the radial function \(R(r)\),

\[
-\frac{1}{2\mu} \left( \frac{2}{r} \frac{dR}{dr} + \frac{d^2R}{dr^2} \right) + V(r)R = ER, \tag{15}
\]

which, using the definition: \(R = \chi(r)/r\), becomes

\[
-\frac{1}{2\mu} \frac{d^2\chi}{dr^2} + V(r)\chi = E\chi. \tag{16}
\]

with the normalization condition \(\int_0^\infty dr \chi^2(r) = 1\).

We denote the expectation value of the kinetic energy operator \(\frac{\hbar^2}{2m_e} = -\frac{\nabla^2}{2m_e}\) by \(E_{\text{kin}}\) which
in terms of \(\chi\) is given as

\[
E_{\text{kin}} = \frac{1}{2m_e} \int_0^\infty dr \left( \frac{d\chi}{dr} \right)^2. \tag{17}
\]
With the inclusion of both the Darwin terms, it is convenient to write the potential as

\[ V_{ep}(r) = \frac{-\alpha}{r} + C_1 e^{-\kappa r} + C_2 \frac{e^{-m r}}{r} + C_3 \frac{e^{-m r}}{r} \]  
(18)

with

\[ C_1 = \alpha \left[ A \frac{m^3}{2} + A \frac{m^3 \kappa_p}{2(1 - \kappa^2)} + \frac{m}{2} \right] \]

\[ C_2 = \alpha \left[ 1 + A \frac{m^2 \kappa^2 \kappa_p^2}{(1 - \kappa^2)^2} \right] \]

\[ C_3 = -C_2 + \alpha \]  
(19)

with \( A = 1/(8m_p^2) + 1/(8m_e^2) \), \( \kappa = \frac{2m_e}{m} \) and \( m^2 = 0.71 \text{ GeV}^2 \) is the parameter from the dipole parametrization.

All possible trial radial wave functions are subjected to the following condition resulting from the Schrödinger equation at the origin:

\[ -\frac{1}{\mu r} (R'(0) + rR''(0) + \ldots) - \frac{1}{2m_e} R''(0) + R(0) \lim_{r \to 0} V(r) = ER(0). \]  
(20)

Since we have \( C_2 + C_3 = \alpha \), the condition which ensures the regularity at the origin is simply \( R'(0) = 0 \). One trial wave function which satisfies this condition is

\[ R(r) = N (e^{-\eta r} - \frac{1}{d} e^{-d \eta r}) , \]  
(21)

where \( d \) is a parameter which we choose later to be 2. The normalization constant is easily calculated to be

\[ N = \sqrt{\frac{4\eta^3}{\left( \frac{1}{d^2} - \frac{16}{(d+1)^2} + 1 \right)}} . \]  
(22)

The kinetic part of the expectation value of energy is then \( E_k = \frac{1}{2m_e} \frac{92n^2}{127} \). The potential term is obtained from the expression \( E_{pot} = \int_0^\infty dr V(r) R^2(r) \) which gives,

\[ E_{pot} = -\alpha N^2 \left( \frac{1}{(2\eta)^2} - \frac{1}{(3\eta)^2} + \frac{1}{4(4\eta)^2} \right) \]

\[ + \quad C_1 N^2 \left( \frac{2}{(2\eta + m)^2} - \frac{2}{(3\eta + m)^2} + \frac{2}{4(4\eta + m)^2} \right) \]

\[ + \quad C_2 N^2 \left( \frac{1}{(2\eta + m)^2} - \frac{1}{(3\eta + m)^2} + \frac{1}{4(4\eta + m)^2} \right) \]

\[ + \quad C_3 N^2 \left( \frac{1}{(2\eta + km)^2} - \frac{1}{(3\eta + km)^2} + \frac{1}{4(4\eta + km)^2} \right) . \]  
(23)
Minimizing $E(\eta) = E_{\text{pot}}(\eta) + E_{\text{kin}}(\eta)$ with respect to the variational parameter $\eta$ we find $\eta = 0.00432$ MeV. This corresponds to $E = -13.274$ eV and $\sqrt{\langle r^2 \rangle} = 0.862555$ fm where $\langle r^2 \rangle$ is the expectation value with respect to the trial wave function. Coming back to the large deviation of (18) as compared to the Coulomb potential at short distances and noticing the local minimum at roughly four fermi (where classically we would find bound solutions), it is a priori not excluded to find a new bound state close to the proton which we could be overlooked by directly applying the perturbation method to calculate the energy correction. The variational approach convinces us that this is not the case and we are dealing still with the standard hydrogen atom (plus corrections). Having done this exercise we can perform a similar (variational) calculation for the positron-proton system by simply reversing the sign of our potential. Again, what would be a senseless undertaking if we had only the Coulomb potential, appears now in a different light by contemplating the deep minimum at the center of $-V_{ep}$. For (negative) binding energies we pay attention to the fact that the mass of the atom, i.e., $m_e + m_p + E$ must be positive. The results are extremely sensitive (the sensitivity is not due to the variational method, but to the form of the potential) to the choice of the electron mass. We therefore replace $m_e$ by $m_x$ and vary this parameter. We find a very narrow range of $m_x$, namely $(0.48429 \text{ MeV}, 0.48843 \text{ MeV})$, where a negative binding energy is possible.

The second variational ansatz we use will be:

$$R(r) = Ne^{-\eta r} (1 + \eta r).$$

(24)

For this with a negative charge ($-e$) for the probe particle we obtain for the electron mass an $\eta$ and for this value of $\eta$ an energy expectation value of $-13.1123$ eV. So the variational principle seems to work fine. The RMS radius at this point is: 0.849069 fm.

It is worth noting that any value of the probe mass for a charge of $-e$ gives a bound state according to this variational principle. The same happens for the ansatz in the preceding paragraphs.

Now for the opposite charge $+e$ for the probe particle, almost the same as in the variational ansatz 21 occurs: there is a narrow interval in which the energy expectation value is negative suggesting a bound state appearance. This interval is: $(0.491837 \text{ MeV}, 0.495918 \text{ MeV})$. Evidently there are approximations and inaccuracies involved in these calculations. We took the anomalous magnetic moment of the probe equal to zero in all cases. We use the
dipole parametrization for the form factor and another parametrization could change the variational calculation results as it is sensitive to the choice of the mass.

Apart from the bound state with negative binding energy there is a clear possibility of a Gamow state for the positron-proton case as can be seen from the potential. This positive energy state (resembling an $\alpha$-nucleus case) would be located between zero and the peak of the positive bump of the potential which the particle would traverse by the quantum mechanical tunneling mechanism. In this case we would talk about a resonance.

III. OVERVIEW OF THE EULER-HEISENBERG THEORY

We give a brief account and the basic equations of the Euler-Heisenberg Theory \cite{13}. For a detailed exposition, we refer the reader to \cite{12}. In quantum electrodynamics, the one loop exchange of electrons allows the photon-photon interaction. Due to Furry’s theorem (based on charge conjugation invariance), only an even number of external photons are allowed in a Feynman diagram. The Euler-Heisenberg theory considers the $\gamma\gamma \rightarrow \gamma\gamma$ case. This self-interaction of photons, which is not present in classical electrodynamics, is a quantum mechanical effect that can be effectively translated into the configuration space as an additional term in the Lagrangian. In doing so, and under certain conditions (see below), we can use the full Lagrangian (the classical one from Maxwell’s theory and the one resulting from the $\gamma\gamma \rightarrow \gamma\gamma$ interaction) to construct a non-linear classical field theory. In the regimes where applicable, this non-linear theory can be used to calculate quantum effects that are outside the scope of the linear Maxwell’s theory.

Any Lagrangian describing electromagnetic fields must be gauge and relativistic invariant. Hence, the Lagrangian is necessarily a function of the invariants

\[ \mathcal{F} \equiv -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}(E^2 - B^2), \]

\[ \mathcal{G} \equiv \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} = E \cdot B. \]

Assuming constant and uniform fields, the one loop effective Lagrangian of QED encoding the light-light interaction is \cite{17,18}

\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{EH}, \]

where the Maxwell’s Lagrangian is
\[ \mathcal{L}_0 = \frac{1}{4\pi} \mathcal{F} \]  

and the Euler-Heisenberg term is

\[ \mathcal{L}_{EH} = -\frac{1}{8\pi^2} \int_0^\infty ds s^{-3} e^{-ms} \times \left[ (es)^2 (ab) \coth (esb) \cot (esa) - 1 - \frac{2}{3} (es)^2 \mathcal{F} \right], \]  

with \( a = |e|E/m_e^2 \) and \( b = |e|B/m_e^2 \). For weak fields the Lagrangian (28) can be expanded as an asymptotic series [19]. The first term in the expansion correcting the Maxwell’s Lagrangian is

\[ \mathcal{L}_{EH}^{(1)} = \eta \left( (E^2 - B^2)^2 + 7(E \cdot B)^2 \right) \]  

with

\[ \eta \equiv \frac{\alpha^2}{360\pi^2 m_e^4}, \]  

where we have set \( \hbar = c = 1 \).

As previously mentioned, the new Lagrangian can be used to construct a classical theory of fields. The Lagrangian (29) gives rise to new Maxwell’s equations (for the moment we will restrict ourselves to the lowest order in the fine structure constant \( \alpha \) and discuss higher orders below) which due to the self-interaction resemble the standard Maxwell equation in matter [3]. In this theory, the homogeneous Maxwell’s equations remain unchanged (since they define the electromagnetic potentials):

\[ \nabla \cdot B = 0, \]

\[ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0. \]

The inhomogeneous new Maxwell’s equations follow from a variation of the Euler-Heisenberg Lagrangian. Let us define four auxiliary vectors: the displacement and the polarization vector

\[ \mathbf{D} \equiv \mathbf{E} + 4\pi \mathbf{P}, \]

\[ \mathbf{P} \equiv \frac{\partial \mathcal{L}_{EH}^{(1)}}{\partial \mathbf{E}} = \eta \left[ 4\mathbf{E}(E^2 - B^2) + 14\mathbf{B}(E \cdot B) \right]; \]

and the magnetic intensity and the magnetization vector

\[ \mathbf{H} \equiv \mathbf{B} - 4\pi \mathbf{M}, \]

\[ \mathbf{M} \equiv \frac{\partial \mathcal{L}_{EH}^{(1)}}{\partial \mathbf{B}} = \eta \left[ 4\mathbf{B}(E^2 - B^2) - 14\mathbf{E}(E \cdot B) \right]. \]
In terms of these vectors, the new inhomogeneous Maxwell’s equations read

\[ \nabla \cdot \mathbf{D} = \rho, \]
\[ \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{j}. \]  

(34)

As in classical electrodynamics, given the charge density and the current distribution, the electric and magnetic fields can be calculated by solving the new Maxwell’s equations.

We take the point of view that the external sources \( \rho \) and \( \mathbf{j} \) are the same whether we consider the Maxwell or the Euler-Heisenberg theory. Of course, the resulting fields will differ. We mention that this natural point of view leaves untouched many standard definitions and results. For instance, the charge radius of the proton (or any other particle), defined by

\[ < r^2 > = \int d^3 x \, r^2 \rho, \]  

(35)

remains the same in both cases.

A. Ranges of validity of the theory

The Euler-Heisenberg theory is an approximation. In order for the approximation to be valid, the fields have to obey certain conditions. Firstly, the electric field should never surpass the so called critical field

\[ E \ll E_c \equiv \frac{m^2_e}{\sqrt{\alpha}} \approx 3 \text{ MeV}^2. \]  

(36)

The reason for this requirement is to keep small the probability of the electron pair production

\[ P_{e^+e^-} \propto \exp \left( -\frac{\pi m_e^2}{|e|E} \right), \]  

(37)

as it should be if we want to describe the electromagnetic interaction using classical fields. If we wish to use the weak field expansion, then the magnetic field has to obey a similar condition:

\[ B \ll B_c \equiv \frac{m^2_e}{\sqrt{\alpha}}. \]  

(38)

Secondly, as mentioned in the previous section, the Euler-Heisenberg Lagrangian is derived assuming the fields to be uniform in the whole space. The energy associated with such
fields is then infinite. However, as long as the fields obey
\[
\frac{1}{m_e} |\nabla E| \ll E, \quad (39)
\]
\[
\frac{1}{m_e} |\nabla B| \ll B, \quad (40)
\]
the Euler-Heisenberg theory should give good results when applied to non-uniform fields.

A word of caution is in order here. For a Coulomb field, we obtain from (39) a restriction for the range of applicability of the Euler-Heisenberg theory in the form, \( r > 772 \text{ fm} \). Taken at face value it would prevent us from using the Euler-Heisenberg theory for fields inside the proton. However, we note that (i) the numerical restriction is derived from a Coulomb field, whereas the electric field inside the proton is not Coulomb-like (due to form factors as can be seen in Fig. 7) and (ii) we assume that it is sufficient that Eqs (36) and (39) are satisfied after the field is calculated from the Euler-Heisenberg theory. Furthermore, we mention that (39) is satisfied in a small region around 0.34 fm when the electric field is evaluated from \( V_C(r) \) given in (9). We shall come back to this point later.

A necessary condition for pair production in non-uniform fields is that the energy content of the fields has to be large enough to reach the mass of the particles that could be produced. The question which arises is if we should consider the pair production threshold before or after the Euler-Heisenberg corrections to the fields. This is a crucial point since we will show that the Euler-Heisenberg corrections can reduce the field strength below the pair production limit (36). Assuming \( B = 0 \), the energy of the field is given by [20, 21],
\[
\mathcal{E}[E] = \int \left\{ \frac{E^2}{8\pi} + 4\eta E^4 + ... \right\} d^3x < 2m_e, \quad (41)
\]
where the first term in (41) is the energy of the field as given by classical electrodynamics and the second term is a correction due to the non-linearity of the new Maxwell’s equations given by the Euler-Heisenberg term (29). A sufficient condition for the absence of pair creation is then given by,
\[
\mathcal{E}[E] < 2m_e. \quad (42)
\]
We will come back to these conditions after solving for particular electric fields.

A valid question is whether we should keep our results at the most quadratic in \( e \) since our Lagrangian is up to this order. Answering this question we should not forget that the theory is a self-interacting one and bears a strong formal similarity to electrodynamics in
matter. In analogy to the Euler-Heisenberg theory let us consider within Maxwell theory (whose Lagrangian is linear in $e$) the electromagnetic waves in conducting matter. Ohm’s law $j = \sigma E$ together with the Maxwell equations leads to the telegraph equation whose solution is $E = E_0 e^{ikr-\omega t}$. The dispersion relation reads $k = |k| = k(\sigma(\omega), \omega)$ in which one includes the frequency dependence of the conductivity $\sigma = \sigma_0/(1-i\omega\tau)$. The damping time $\tau$ depends on the charge via, $\tau = m\sigma_0/ne^2$. The conductivity $\omega$ enters $k$ in a non-linear way which in turn enters the solution. We do not expand neither the conductivity nor the solution in the charge albeit the Lagrangian (and the damping equations which model the damping) are linear in $e$. If it is not stringent to expand our results in $e$, it means that we can retain the full non-linearity of the theory \[22\]. Doing it we should re-consider the meaning of perturbation for the following reason. In the static case of the Euler-Heisenberg theory the deviations from the Maxwellian field $E_0$ at small distances can be large. Certainly, these large deviations are not corrections to $E_0$. But this is the case also for electric potential of the proton where at short distances the proton potential is not even approximately Coulomb and nonetheless this finite size correction is considered a perturbation as far as physical observables, such as the energy level, are concerned. Basically, the nature of perturbation in such a case is due to the very short range of the new effect. In the Euler-Heisenberg theory happens exactly the same: the large deviations are restricted to a very small range and therefore appear as perturbations in the observables. By keeping the full dependence in $r$ at the short range we follow here the same procedure used also in \[20, 21\].

**B. The electrostatic case**

In contrast to the Maxwell case, where in the static case the equations for the electric and magnetic field decouple, in the Euler-Heisenberg theory we have to set $B = 0$ by hand if we want a pure electrostatic case. So, let us now set $B = 0$ and neglect time derivatives in the new Maxwell’s equations.

The Gauss law for $D$ allows us to make a direct connection to the Maxwellian field $E_0$ via

$$\nabla \cdot D = \rho = \nabla \cdot E_0 \quad (43)$$

Hence we end up with an algebraic equation of the form

$$E + 4\pi P = E + \tilde{\eta}_1 E^2 E = E_0 \quad (44)$$
where
\[ \tilde{\eta}_1 = 16\pi\eta = \frac{2\alpha^2}{45\pi m_e^4}. \] (45)
For the spherically symmetric case we obtain a third order polynomial equation for the magnitude of the electric field \( E \) given the “standard” Maxwellian field \( E_0 \):
\[ E + \tilde{\eta}_1 E^3 = E_0 \] (46)
The solution of the third order algebraic equation (45) is always dependent on \( E_0 \), i.e., \( E[E_0] \).
The only real solution to (46) is given by Cardano’s formula
\[ E = \left( \frac{E_0}{2\tilde{\eta}_1} \right)^{1/3} \times \left( 1 + \sqrt{1 + \frac{4}{27\tilde{\eta}_1 E_0^2}} + \sqrt{1 - \frac{4}{27\tilde{\eta}_1 E_0^2}} \right). \] (47)
As advertised above, for strong fields \( E_0 \), the Euler-Heisenberg field \( E \) given by (46) behaves as \( E \propto (E_0)^{1/3} \) (this amounts to retaining the cubic term in the polynomial equation (46)), i.e., the field becomes weaker as compared to the classical Maxwellian counterpart. The above actually is true no matter the value of \( E_0 \), but the reduction of the field strength is more pronounced the bigger the value of \( E_0 \) is. Do note that, in accordance with the last paragraph of the previous section, the result (47) is not a perturbative series in the fine structure constant.

The case of a point charge, where \( E_0 = \sqrt{\frac{\alpha}{r^2}} \), is worked out in References [20, 21]. It is found there that the energy of a point charge is \( \mathcal{E}[E] = 2.09m_e \) after the \( \gamma - \gamma \) corrections are included: a finite result that contrasts with the classical electrodynamics case. Do note that the energy of the point charge barely surpasses two times the electron mass. We will see that for our model for the proton the resulting electrostatic energy is below the pair production threshold.

A valid question is what will happen if we include terms of higher orders in \( \alpha \) in the Lagrangian. To this end we can do two things. First, we can choose to keep more terms in the weak field expansion of (28) (see appendix). Secondly, we can take the contribution to the effective Lagrangian due to higher loops, specifically, the two loops Euler-Heisenberg Lagrangian [23]. The contribution of the second loop to the Euler-Heisenberg Lagrangian reads [19, 23],
\[ \mathcal{L}^{(2)}_{EH} = \frac{\alpha^3}{64\pi^4 m_e^4} \left( \frac{16(E^2 - B^2)^2}{81} + \frac{263(E \cdot B)^2}{162} \right) \]
\[ - \frac{\alpha^4}{256\pi^4 m_e^5} \left( \frac{1219(E^2 - B^2)^3}{2025} + \frac{8656(E^2 - B^2)(E \cdot B)^2}{2025} \right). \] (48)
Specializing, as before, to the case of \( B = 0 \), the Lagrangian (48) reduces to

\[
\mathcal{L}_{EH}^{(1+2)} = \mathcal{L}_{EH}^{(1)} + \mathcal{L}_{EH}^{(2)} = \eta_1 \mathbf{E}^4 + \eta_2 \mathbf{E}^6
\]

with

\[
\begin{align*}
\eta_1 &= \frac{\alpha^2}{360\pi^2 m_e^4} + \frac{\alpha^3}{324\pi^4 m_e^4} \\
\eta_2 &= \frac{\alpha^3}{630\pi^2 m_e^8} - \frac{1219\alpha^4}{518400\pi^4 m_e^8}
\end{align*}
\]

(50)

The algebraic version of the Gauss’s law is now a quintic polynomial

\[
E + \tilde{\eta}_1 E^3 + \tilde{\eta}_2 E^5 = E_0,
\]

(51)

where

\[
\begin{align*}
\tilde{\eta}_1 &= 16\pi \eta_1, \\
\tilde{\eta}_2 &= 24\pi \eta_2.
\end{align*}
\]

(52)

We shall refer to the expansion of the polynomial equation as order (e.g. the third order polynomial, \( E + \eta_1 E^3 = E_0 \) is called first order since it is the first correction of the Euler-Heisenberg theory). We also distinguish the loop contribution and for instance, Eq. (51) would be second order with two loops. Let us note that the contribution from the two loops Lagrangian amounts to a small correction to the one loop coefficients, hence it can be safely ignored in the first approximation. In any case, for strong classical fields \( E_0 \), the Euler-Heisenberg field \( E \) given by (51) behaves as \( E \propto (E_0)^{1/5} \). The result is that by taking more and more terms in the weak field expansion of the Euler-Heisenberg Lagrangian, the resulting field strength gets smaller and smaller compared to the classical field given by the Maxwell’s equations. We can continue taking more and more terms from the weak field expansion. For example, taking four terms the equation for the electric fields becomes

\[
E + \tilde{\eta}_1 E^3 + \tilde{\eta}_2 E^5 + \tilde{\eta}_3 E^7 + \tilde{\eta}_4 E^9 + \ldots = \sum_{i=0}^{\infty} \tilde{\eta}_i E^{2i+1} = E_0
\]

(53)

where \( \tilde{\eta}_0 = 1 \) and, taking into account only the one-loop Lagrangian, the third and fourth coefficients are (see appendix)

\[
\begin{align*}
\tilde{\eta}_3 &= \frac{32\alpha^4}{315\pi m_e^{12}}, \\
\tilde{\eta}_4 &= \frac{160\alpha^5}{297\pi m_e^{16}}.
\end{align*}
\]

(54)
In the case of the proton, the maximal value of $E_0$ is $5 \times 10^3 \, MeV$. In this case the field will behave as $E \propto (E_0)^{1/9}$, and in general, taking $n$ terms in the weak field expansion of the Lagrangian results in a field behaving as $E \propto (E_0)^{1/2n+1}$. However, as the weak field expansion for the Euler-Heisenberg Lagrangian is an asymptotic series, it does not make much sense to try to take much more terms in (53). We will address this point in more detail in a subsequent section. However, already here we can exemplify the effect of taking more terms in the Lagrangian expansion for a Coulomb-like field. Eq. (47) applied to the classical Coulomb field $E_0 \simeq r^{-2}$ gives at short distances a field that behaves like $E \simeq r^{-2/3}$, which, together with the condition (39) results in the range of validity of $r > 257$ fm. If we take into account the next correction to the Gauss law given by Eq. (51), the resulting electric field at short distances behaves like $E \simeq r^{-2/5}$ which then leads to $r > 154$ fm. This shows how taking into account the corrections from the Euler-Heisenberg expansion results in electric fields that behave better than the classical ones.

IV. RESULTS

Below we will present the results in the form of figures since most of the equations cannot be handled analytically. Let us start by showing the effect of the $\gamma\gamma$ interaction (Euler-Heisenberg) on the electric field of the proton. In Fig. 6 we have plotted $E_p$ with the finite size corrections (i.e., with the inclusion of the form-factors (FF)) in the Maxwell theory and in the Euler-Heisenberg theory (two loops). The Darwin term refers here to the proton Darwin term. The effect is clear: a reduction of the field strength at small distances.

In Fig. 7 (left panel) we show a comparison between the electric field of the point-like proton and the extended proton after the $\gamma\gamma$ corrections at one loop are included. We also compare the effect of higher orders by the inclusion of the hypothetical (see text) third order result (right panel). The saturation of the result is visible: the reduction of the field strength is less with higher order. The inlay demonstrates the short range behavior. It is also obvious that the fluctuations at distances around 1 fm become much less prominent with higher orders of the expansion included. The inclusion of every higher order brings us closer to satisfying the condition in (39).

We can do a similar comparison for the electron-proton interaction potential $eV_p$ on the one hand and $eV_p^{\gamma\gamma}$ on the other hand. This calculation (at one loop) can be found in Fig. 22.
FIG. 6: Effect of the Euler-Heisenberg theory (right panel) on the electric field of the proton (with form factors included). By two loops we mean, second order ($5^{th}$ order polynomial as in Eq. (51)) with two loop corrections.

8 without the inclusion of the electron or proton Darwin term. To understand the effect of the latter, let us recall that the end result will matter depending on whether we handle the electron Darwin term within the Dirac theory or the Breit formalism. In the Dirac theory the full interaction potential after the Euler-Heisenberg corrections would be given by

$$V^{\gamma\gamma}_{ep} (\text{Dirac}) = -e(V_C + V_{pD})^{\gamma\gamma} + \frac{e}{8m_e^2} \nabla \cdot E^{\gamma\gamma}$$  \hspace{1cm} (55)

This is to be contrasted with the result within the Breit formalism which gives

$$V^{\gamma\gamma}_{ep} (\text{Breit}) = -e(V_C + V_{pD})^{\gamma\gamma} + \frac{e}{8m_e^2} \rho_C$$  \hspace{1cm} (56)

We recall that in the Breit convention we use $\rho_C = e\rho$. Fig. 9 summarizes the effect of the $\gamma\gamma$ correction within the Dirac formalism as given in Eq. (55), where, $V_{pD}$ corresponds to the proton Darwin term and $\frac{e}{8m_e^2} \nabla \cdot E^{\gamma\gamma}$ corresponds to the electron Darwin term. In the Breit formalism (Eq. (56)), $\frac{e}{8m_e^2} \rho_C$ corresponds to the electron Darwin term.

Fig. 10 does the same for the Breit Hamiltonian (56) where in addition we display the effects of the electron Darwin term proportional to $\rho$. Note that the effects of the proton Darwin become visible only in a very small range of $r$ as will be seen in Fig. 12. Finally, Fig. 11 compares the Dirac (55) and the Breit Hamiltonian (56).
FIG. 7: Comparison of the effect of the Euler-Heisenberg theory on the electric field of the proton. We compare with the point-like proton (left panel) and demonstrate the effects of higher orders (right panel). The Darwin term refers to proton Darwin. Again, by two loop we mean 2\textsuperscript{nd} order with two loops (Eq.(51)). This will be the same in the subsequent figures.

Coming back to the Dirac formalism we can also examine the effect of the proton Darwin term at small distances. In spite of being suppressed by the proton mass squared this term has an effect at small distances. Fig. 12 summarizes our findings here. Fig. 13 does the same for the Breit formalism. In both figures the electron Darwin term is included.

The difference between the the potential with and without electron Darwin term in the Dirac theory is displayed in Fig. 14. In order to see this difference, we have not included the proton Darwin terms in this plot.

Finally, we recall that the energy content of the electric field is $\mathcal{E}[E] = \frac{1}{8\pi} \int E^2 d^3x$. Applying it to the proton field $E_p$ without the Euler-Heisenberg corrections we obtained $\mathcal{E}[E_p] \approx 1\text{MeV}$. After the corrections this number reduces considerably

\[
\mathcal{E}[E_{\gamma\gamma,1\text{-loop}}] \approx 0.46 \text{ MeV} \\
\mathcal{E}[E_{\gamma\gamma,2\text{-loop}}] \approx 0.19 \text{ MeV}
\]  (57)

This includes corrections to the energy-momentum tensor (see appendix for details). One can see that the loop results are well below the pair production threshold field.
FIG. 8: The electron proton interaction potential with and without the $\gamma\gamma$ correction at one loop. The Darwin terms are not included here.

V. STATIC ELECTRIC FIELDS IN THE EULER-HEISENBERG THEORY

The full Euler-Heisenberg Lagrangian, in principle, contains information also on the strong fields. For static electric fields generated by a single source, the weak field expansion of this Lagrangian gives a polynomial equation for the electric field $E$ whose order $N$ indicates the truncation of the expansion. In general, we have

$$E + \sum_{i=1}^{N} \tilde{\eta}_i E^{2i+1} = E_0$$

where $E_0$ is the result of computing the electric field in the Maxwell theory assuming static charge distribution. The coefficients $\tilde{\eta}_i$ are listed in the appendix where one can see that they have the structure $\tilde{\eta}_i = \frac{n_i \alpha^{i+1}}{\pi m_e^2}$, with $n_i$ being purely numerical factors. Effects of higher loops should, in principle, enter the coefficients $\tilde{\eta}_i$, but this will only correct these coefficients as was evident from the two-loops example. We have already shown that the effect of the polynomial equation is to reduce the value $E_0$ when starting with a large electric field. This implies that the Euler-Heisenberg reduces the electric field strength automatically below the pair production threshold. In this context arises the obvious question, up to which order $N$ should one continue the expansion. Since we deal here with non-convergent series a reasonable starting assumption is to insist on a reduction such that equation (58) is satisfied.
FIG. 9: Effect of the Euler-Heisenberg theory on the electron proton interaction potential within the Dirac theory (see Eq. (55) and text).

FIG. 10: Effect of the Euler-Heisenberg theory on the electron proton interaction potential within the Breit theory (see Eq. (56) and text).

To probe more into this matter we re-write equation (58) as

\[ \xi + \sum_{i=1}^{N} \tilde{n}_i \xi^{2i+1} = \xi_0 \equiv \frac{E_0 \sqrt{\alpha}}{m_c^2} \]  

(59)
with the dimensionless variable

\[ \xi \equiv \frac{E\sqrt{\alpha}}{m^2_e} \]  

(60)

and \( \tilde{n}_i = (n_i\alpha)/\pi \). In general, \( \xi \) will, of course depend on \( r \) if \( \xi_0 \) is space dependent. What we have in mind is to choose a constant \( \xi_0 \) corresponding to the maximal field strength. In the case of the proton this value is \( 5 \times 10^3 \text{MeV} \) which implies that we have \( \xi_0 = 1700 \). We look numerically for \( \tilde{N} \) such that \( \xi < 1 \). The result can be found in Fig. 15. For \( \tilde{N} = 9 \) we obtain \( \xi = 0.99 \). Choosing another \( \xi_0 > 1 \) gives globally the same result. It is only if we start with \( \xi_0 < 1 \) can we treat equation (58) fully perturbatively. For strong fields we should expand up to order \( \tilde{N} \) such that the next orders starting with \( \tilde{N} + 1 \) are small and can be treated perturbatively. Indeed, for an observable like the energy-momentum tensor we can write

\[ T_{00} = \sum_i \frac{k_i}{\pi^2} m^4_e \xi^{2i} \]  

(61)

where \( k_i \) are purely numerical coefficients of the order 1 (see appendix). In such a case the above sum, albeit not convergent, makes sense if we have \( \xi < 1 \).
VI. CONCLUSIONS

It is not often that we encounter in physics extreme strong electric fields, fields such that $\xi = E\sqrt{\alpha}/m_e^2 \gg 1$ and with an energy content which would, in principle, allow pair production. Using Maxwell’s electrodynamics, such an electric field felt by the electron in the hydrogen atom seems possible due to the finite size effects of the proton. Not only are we at the threshold of pair production (the exact number of the energy content will depend on the parametrization of the form factors), but other strange phenomena like a positron-proton bound state (or, in general, a positive energy Gamow state) seem possible. A valid question arises whether such strong fields are indeed present at the short distances in the hydrogen atom. Strong electromagnetic fields have been of interest for quite some time (see e.g. [24–26]) and can be applied in different contexts [27]. We have demonstrated that going from Maxwell’s electrodynamics to the Euler-Heisenberg theory, the strong electric field will be reduced automatically far below the pair production threshold, at least as far as the energy content of the field is concerned. At the same time, the potential as seen by a positron will be too flat to allow bound states. To reach this effect we have to expand the original
FIG. 13: Effect of the proton Darwin term on the electron proton interaction potential within the Breit theory (see text) using the Euler-Heisenberg at one loop.

Euler-Heisenberg Lagrangian in higher order in $\alpha$. We can consistently use some terms in the expansion by the requirement $\xi < 1$ even though the expansion is only asymptotic. We emphasize that fields resulting from the Euler-Heisenberg corrections cannot be viewed as small corrections to fields calculated with the Maxwell theory. However, due to the short range in which the Euler-Heisenberg results differ significantly from the Maxwell’s ones, it is possible to treat the change in the observables perturbatively. Similarly, before we reach $\xi < 1$, every order in the expansion in the Euler-Heisenberg Lagrangian is not a correction to the original Maxwellian field (as far as its strength is concerned), however, this happens at a very short range. However, whenever $\xi < 1$, i.e., from an order on at which this condition is satisfied we can talk about perturbations in the expansion of the observables like the energy-momentum tensor.

Author Contribution Statement

All authors have contributed equally to the discussions, calculations and preparation of the manuscript.
FIG. 14: Effect of the electron Darwin term on the electron proton interaction potential within the Euler Heisenberg theory with the Dirac formalism (see text). Note that the proton Darwin is not included.

Appendix

In a purely electric field the Euler-Heisenberg Lagrangian reduces to

\[ \mathcal{L}_{EH} = -\frac{m_e^4}{8\pi^2} \int_0^\infty ds s^{-3} e^{sm_e^2} \frac{e^{(eEs)\cot(eEs)} - 1}{3} + \frac{1}{3} e^2 E^2 s^2. \] (62)

With the help of the Taylor expansion

\[ (eEs)\cot(eEs) = 1 - \frac{1}{3} (e^2 s^2) E^2 - \frac{1}{45} (e^4 s^4) E^4 - \frac{2}{945} (e^6 s^6) E^6 - \frac{1}{4725} e^8 s^8 - \frac{2}{93555} e^{10} s^{10} E^{10} + \ldots, \] (63)

the Lagrangian (62) can be expanded as

\[ \mathcal{L}_{EH} = -\frac{m_e^4}{8\pi^2} \sum_{n=0}^\infty \frac{(-1)^n B_{2n+4}}{(2n+4)(2n+3)(2n+2)} \left( \frac{2eE}{m_e^2} \right)^{2n+4} = \sum_{n=0}^\infty a_n E^{2n+4} \] (64)
FIG. 15: Effects of higher orders on the dimensionless variable \( \xi \equiv E \sqrt{\alpha / m_e^2} \).

where \( B_{2n+4} \) are Bernoulli numbers.

The series (64) is divergent, non-alternating and is not Borel summable. This divergence is related to the possibility of pair production due to an electric field. However, for low field strengths, we can take the first terms of the expansion as an approximation. The first four terms of (64) are

\[
\mathcal{L}_{EH} = \frac{\alpha^2}{360 \pi^2 m_e^4} E^4 + \frac{\alpha^3}{630 \pi^2 m_e^8} E^6 + \frac{\alpha^4}{315 \pi^2 m_e^{12}} E^8 + \frac{4\alpha^5}{297 \pi^2 m_e^{16}} E^{10} + \ldots
\]  
(65)

\[
4\pi \frac{\partial \mathcal{L}^{(1)}}{\partial E} = \frac{2\alpha^2}{45 \pi m_e^3} E^3 + \frac{\alpha^3}{630 \pi m_e^7} E^5 + \frac{32\alpha^4}{315 \pi m_e^{11}} E^7 + \frac{160\alpha^5}{297 \pi m_e^{15}} E^9.
\]  
(66)
We list the coefficients in the Lagrangian expansion:

\[
\begin{align*}
    a_0 &= \frac{\alpha^2}{360\pi^2 m^4} \\
    a_1 &= \frac{\alpha^3}{630\pi^2 m^8} \\
    a_2 &= \frac{\alpha^4}{315\pi^2 m^{12}} \\
    a_3 &= \frac{4\alpha^5}{297\pi^2 m^{16}} \\
    a_4 &= \frac{225225\pi^2 m^{20}}{22112\alpha^6} \\
    a_5 &= \frac{128\alpha^7}{117\pi^2 m^{24}} \\
    a_6 &= \frac{462976\alpha^8}{26775\pi^2 m^{28}} \\
    a_7 &= \frac{22459904\alpha^9}{61047\pi^2 m^{32}} \\
    a_8 &= \frac{1430413312\alpha^{10}}{141075\pi^2 m^{36}} \\
    a_9 &= \frac{20364132352\alpha^{11}}{8 \times 7245\pi^2 m^{40}} \\
    a_{10} &= \frac{123922856542208\alpha^{12}}{8 \times 1036035\pi^2 m^{44}} \\
    a_{11} &= \frac{1379781312512\alpha^{13}}{8 \times 225\pi^2 m^{48}} \\
    a_{12} &= \frac{56921405366730752\alpha^{14}}{8 \times 152685\pi^2 m^{52}}
\end{align*}
\]

Note that the coefficients in Eq. (50) are approximately related to the coefficients above, i.e., \( a_1 \approx \eta_1 \) and \( a_2 \approx \eta_2 \). From this we get the following coefficients in the polynomial
equation for the electric field \((\sum_0 \tilde{\eta}_i E^{2i+1} = E_0 \text{ with } \tilde{\eta}_0 = 1)\)

\[
\tilde{\eta}_1 = 16\pi \times a_0 \\
\tilde{\eta}_2 = 24\pi \times a_1 \\
\tilde{\eta}_3 = 32\pi \times a_2 \\
\tilde{\eta}_4 = 40\pi \times a_3 \\
\tilde{\eta}_5 = 48\pi \times a_4 \\
\tilde{\eta}_6 = 56\pi \times a_5 \\
\tilde{\eta}_7 = 64\pi \times a_6 \\
\tilde{\eta}_8 = 72\pi \times a_7 \\
\tilde{\eta}_9 = 80\pi \times a_8 \\
\tilde{\eta}_{10} = 88\pi \times a_9 \\
\tilde{\eta}_{11} = 96\pi \times a_{10} \\
\tilde{\eta}_{12} = 104\pi \times a_{11} \\
\tilde{\eta}_{13} = 112\pi \times a_{12}
\]

From the Lagrangian (65), the symmetric energy-momentum tensor can be calculated using the expression

\[
T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial F F_{\mu\nu}} - g_{\mu\nu} \mathcal{L}. \tag{67}
\]

For the 00 component we have

\[
T_{00} = \frac{b_1}{2} E^2 + \frac{3}{4} b_2 E^4 + \frac{5}{8} b_3 E^6 + \frac{7}{16} b_4 E^8 + \frac{9}{32} b_5 E^{10} + \ldots. \tag{68}
\]
with

\[ b_1 = \frac{1}{4\pi} \]
\[ b_2 = \frac{\alpha}{90\pi^2 m_e^4} \]
\[ b_3 = \frac{4\alpha^3}{315\pi^2 m_e^8} \]
\[ b_4 = \frac{16\alpha^4}{315\pi^2 m_e^{12}} \]
\[ b_5 = \frac{128\alpha^5}{297\pi^2 m_e^{16}} \]
\[ b_6 = \frac{1415168\alpha^6}{225225\pi m_e^{20}} \]

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