COEFFICIENT BOUNDS FOR A SUBCLASS OF BI-PRESTARLIKE FUNCTIONS ASSOCIATED WITH THE CHEBYSHEV POLYNOMIALS

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ABSTRACT. In this paper, we introduce and investigate a new subclass of bi-pre starlike functions defined in the open unit disk, associated with Chebyshev Polynomials. Furthermore, we find estimates of first two coefficients of functions in these classes, making use of the Chebyshev polynomials. Also, we obtain the Fekete-Szegő inequalities for function in these classes. Several consequences of the results are also pointed out as corollaries.

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1. INTRODUCTION

Let $A$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

normalized by the conditions $f(0) = 0 = f'(0) - 1$ defined in the open unit disk

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}.$$

Let $S$ be the subclass of $A$ consisting of functions of the form (1.1) which are also univalent in $\Delta$. Let $S^*(\alpha)$ and $K(\alpha)$ denote the well-known subclasses of $S$, consisting of starlike and convex functions of order $\alpha$, $0 \leq \alpha < 1$, respectively. The function

$$s(z) = \frac{z}{(1-z)^{2(1-\alpha)}} = z + \sum_{n=2}^{\infty} \Psi_n(\alpha) z^n$$
where
\[ \Psi_n(\alpha) = \left( \prod_{k=2}^{n} \frac{(k-2\alpha)}{(n-1)!} \right) \]  

is the well-known extremal function for the class \( S^*(\alpha) \). Also \( f \in \mathcal{A} \) is said to be prestar-like functions of order \( \alpha (0 \leq \alpha < 1) \), denoted by \( \mathcal{R}(\alpha) \) if \( f \ast s(z) \in S^*(\alpha) \). We note that \( \mathcal{R}(1/2) = S^*(1/2) \) and \( \mathcal{R}(0) = \mathcal{K}(0) \). Using the convolution techniques, Ruscheweyh [15] introduced and studied the class of prestarlike functions of order \( \alpha \). For functions \( f \in \mathcal{S} \), we have \( f \in \mathcal{K}(0) \iff zf' \in S^*(0) \). The Koebe one quarter theorem [6] ensures that the image of \( \Delta \) under every univalent function \( f \in \mathcal{A} \) contains a disk of radius \( \frac{1}{4} \). Thus every univalent function \( f \) has an inverse \( f^{-1} \) satisfying
\[ f^{-1}(f(z)) = z, \quad (z \in \Delta) \quad \text{and} \quad f(f^{-1}(w)) = w \quad (|w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}). \]

A function \( f \in \mathcal{A} \) is said to be bi-univalent in \( \Delta \) if both \( f \) and \( f^{-1} \) are univalent in \( \Delta \). Let \( \Sigma \) denote the class of bi-univalent functions defined in the unit disk \( \Delta \). Since \( f \in \Sigma \) has the Maclaurian series given by (1.1), a computation shows that its inverse \( g = f^{-1} \) has the expansion
\[ g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 + \cdots. \]

An analytic function \( f \) is subordinate to an analytic function \( g \), written \( f(z) \prec g(z) \), provided there is an analytic function \( w \) defined on \( \Delta \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) satisfying \( f(z) = g(w(z)) \).

Chebyshev polynomials, which is used by us in this paper, play a considerable act in numerical analysis. We know that the Chebyshev polynomials are four kinds. The most of books and research articles related to specific orthogonal polynomials of Chebyshev family, contain essentially results of Chebyshev polynomials of first and second kinds \( T_n(x) \) and \( U_n(x) \) and their numerous uses in different applications, see Doha [5] and Mason [11].

The well-known kinds of the Chebyshev polynomials are the first and second kinds. In the case of real variable \( x \) on \((-1, 1)\), the first and second kinds are defined by
\[ T_n(x) = \cos n\theta, \]
\[ U_n(x) = \frac{\sin(n+1)\theta}{\sin \theta} \]
where the subscript \( n \) denotes the polynomial degree and where \( x = \cos \theta \). We consider the function
\[ \Phi(z, t) = \frac{1}{1 - 2tz + z^2}. \]
We note that if \( t = \cos \alpha, \quad \alpha \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \), then for all \( z \in \Delta \)
\[ \Phi(z, t) = \frac{1}{1 - 2tz + z^2} = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\alpha}{\sin \alpha} z^n} = 1 + 2 \cos \alpha z + (3 \cos^2 \alpha - \sin^2 \alpha)z^2 + \cdots. \]

Thus, we write
\[ \Phi(z, t) = 1 + U_1(t)z + U_2(t)z^2 + \ldots \quad (z \in \Delta, t \in (-1, 1)) \]
Also, it is known that
\[ U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t), \]
and
\[ U_1(t) = 2t; \quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \cdots. \]

The Chebyshev polynomials of the first kind have the generating function of the form
\[ T_n(t) = \frac{1 - t}{1 - 2tz + z^2} \quad (z \in \triangle). \]

All the same, the Chebyshev polynomials of the first kind and the second kind are well connected by the following relationship
\[ \frac{dT_n(t)}{dt} = nU_{n-1}(t), \]
\[ T_n(t) = U_n(t) - tU_{n-1}(t), \]
\[ 2T_n(t) = U_n(t) - U_{n-2}(t). \]

Several authors have introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients (see [2, 3, 10, 17, 19, 21]). Recently, Jahangiri and Hamidi [9] introduced and studied certain subclasses of bi-prestarlike functions mentioned as below:

The expansion of \( s(z) = \frac{z}{(1 - z)^2(1 - \alpha)} \) is given by
\[ s(z) = z + \frac{(2 - 2\alpha)}{1!} z^2 + \frac{(2 - 2\alpha)(3 - 2\alpha)}{2!} z^3 + \frac{(2 - 2\alpha)(3 - 2\alpha)(4 - 2\alpha)}{3!} z^4 + \cdots. \]

So, by the definition of hadamard product, we have
\[ F(z) = \frac{z(z)}{(1 - z)^2(1 - \alpha)} * f(z) = s(z) * f(z) \]
\[ F(z) = z + \frac{(2 - 2\alpha)a_2}{1!} z^2 + \frac{(2 - 2\alpha)(3 - 2\alpha)a_3}{2!} z^3 + \frac{(2 - 2\alpha)(3 - 2\alpha)(4 - 2\alpha)a_4}{3!} z^4 + \cdots \]
equivalently
\[ F(z) = z + \Psi_2(\alpha)a_2z^2 + \Psi_3(\alpha)a_3z^3 + \Psi_4(\alpha)a_4z^4 + \cdots \]

Similarly, for the inverse function \( g = f^{-1} \), we obtain
\[ G(w) = \frac{w}{(1 - w)^2(1 - \alpha)} * g(w) = s(w) * g(w) \]
\[ G(w) = w - \frac{(2 - 2\alpha)a_2}{1!} w^2 + \frac{(4(2 - 2\alpha)^2a_2^2 - (2 - 2\alpha)(3 - 2\alpha)a_3)}{2!} w^3 + \cdots \]
equivalently
\[ G(w) = w - \Psi_2(\alpha)a_2w^2 + (2\Psi_2(\alpha)a_2^2 - \Psi_3(\alpha)a_3) w^3 + \cdots \]

We define bi-prestarlike functions in the open unit disk, associated with Chebyshev Polynomials as below:
Definition 1.1. For $0 \leq \lambda \leq 1$, and $t \in (0, 1)$ a function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{R}_\Sigma(\lambda, \alpha, \Phi(z, t))$ if the following subordination hold:

\[ (1 - \lambda) \frac{zF''(z)}{F(z)} + \lambda \left( 1 + \frac{zF''(z)}{F'(z)} \right) \prec \Phi(z, t) \quad (1.7) \]

and

\[ (1 - \lambda) \frac{wG''(w)}{G(w)} + \lambda \left( 1 + \frac{wG''(w)}{G'(w)} \right) \prec \Phi(w, t) \quad (1.8) \]

where $z, w \in \Delta$ and $F$ and $G$ is given by (1.5) and (1.6), respectively.

Remark 1.2. Suppose $f \in \Sigma$. Then $\mathcal{R}_\Sigma(0, \alpha, \Phi(z, t)) \equiv \mathcal{P}\mathcal{S}_\Sigma^*(\alpha, \Phi(z, t))$ : thus $f \in \mathcal{P}\mathcal{S}_\Sigma^*(\alpha, \Phi(z, t))$ if the following subordination holds:

\[ \frac{zF'(z)}{F(z)} \prec \Phi(z, t) \quad \text{and} \quad \frac{wG'(w)}{G(w)} \prec \Phi(w, t) \]

where $z, w \in \Delta$ and $G$ is given by (1.6).

Remark 1.3. Suppose $f \in \Sigma$. Then $\mathcal{R}_\Sigma(1, \alpha, \Phi(z, t)) \equiv \mathcal{K}_\Sigma^*(\alpha, \Phi(z, t))$ : thus $f \in \mathcal{K}_\Sigma^*(\alpha, \Phi(z, t))$ if the following subordination holds:

\[ 1 + \frac{zF''(z)}{F'(z)} \prec \Phi(z, t) \quad \text{and} \quad 1 + \frac{wG''(w)}{G'(w)} \prec \Phi(w, t) \]

where $z, w \in \Delta$ and $g$ is given by (1.6).

In this paper, motivated by recent works of Altınkaya and Yalçın [1] we introduce a subclass bi-prestarlike function class associated with Chebyshev polynomials and obtain the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the functions $f \in \mathcal{R}_\Sigma(\lambda, \alpha, \Phi(z, t))$ by subordination.

2. Initial Taylor Coefficients $f \in \mathcal{R}_\Sigma(\lambda, \alpha, \Phi(z, t))$

Theorem 2.1. Let $f$ given by (1.1) be in the class $\mathcal{R}_\Sigma(\lambda, \alpha, \Phi(z, t))$ and $t \in (0, 1)$. Then

\[ |a_2| \leq \frac{2t \sqrt{2t}}{\sqrt{[2(1 + 2\lambda)\Psi_3(\alpha) - (\lambda^2 + 5\lambda + 2)\Psi_2^2(\alpha)][4t^2 + (1 + \lambda)^2\Psi_2^2(\alpha)]}} \quad (2.1) \]

and

\[ |a_3| \leq \frac{4t^2}{(1 + \lambda)^2\Psi_2^2(\alpha)} + \frac{t}{(1 + 2\lambda)\Psi_3(\alpha)} \quad (2.2) \]

where $0 \leq \lambda \leq 1$ and $t \neq \frac{(1 + \lambda)\Psi_2(\alpha)}{\sqrt{(\lambda^2 + 5\lambda + 2)\Psi_2^2(\alpha) - 2(1 + 2\lambda)\Psi_3(\alpha)}}$.

Proof. Let $f \in \mathcal{R}_\Sigma(\lambda, \alpha, \Phi(z, t))$ and $g = f^{-1}$. Considering (1.7) and (1.8), we have

\[ (1 - \lambda) \frac{zF''(z)}{F(z)} + \lambda \left( 1 + \frac{zF''(z)}{F'(z)} \right) = \Phi(z, t) \quad (2.3) \]

and

\[ (1 - \lambda) \frac{wG''(w)}{G(w)} + \lambda \left( 1 + \frac{wG''(w)}{G'(w)} \right) = \Phi(w, t) \quad (2.4) \]

Define the functions $u(z)$ and $v(w)$ by

\[ u(z) = c_1z + c_2z^2 + \cdots \quad (2.5) \]
and
\[ v(w) = d_1 w + d_2 w^2 + \cdots \] (2.6)

are analytic in \( \triangle \) with \( u(0) = 0 = v(0) \) and \( |u(z)| < 1, |v(w)| < 1, \) for all \( z, w \in \triangle. \) It is well-known that

\[ |u(z)| = |c_1 z + c_2 z^2 + \cdots| < 1 \quad \text{and} \quad |v(w)| = |d_1 w + d_2 w^2 + \cdots| < 1, \quad z, w \in \triangle, \quad (2.7) \]

then

\[ |c_j| \leq 1 \quad \text{and} \quad |d_j| \leq 1 \quad \text{for all} \quad j \in \mathbb{N}. \quad (2.8) \]

Using (2.5) and (2.6) in (2.3) and (2.4) respectively, we have

\[ (1 - \lambda) \frac{zF'(z)}{F(z)} + \lambda \left( 1 + \frac{zF''(z)}{F'(z)} \right) = 1 + U_1(t)u(z) + U_2(t)u^2(z) + \cdots, \] (2.9)

and

\[ (1 - \lambda) \frac{wG'(w)}{G(w)} + \lambda \left( 1 + \frac{wG''(w)}{G'(w)} \right) = 1 + U_1(t)v(w) + U_2(t)v^2(w) + \cdots. \] (2.10)

In light of (1.1) - (1.3), and from (2.9) and (2.10), we have

\[ 1 + (1 + \lambda) \Psi_2(\alpha) a_2 z + [2(1 + 2\lambda) \Psi_3(\alpha) a_3 - (1 + 3\lambda) \Psi_2^2(\alpha) a_2^2] z^2 + \cdots \]
\[ = 1 + U_1(t)c_1 z + [U_1(t)c_2 + U_2(t)c_1^2] z^2 + \cdots, \]

and

\[ 1 - (1 + \lambda) \Psi_2(\alpha) a_2 w + \{(8\lambda + 4) \Psi_3(\alpha) - (3\lambda + 1) \Psi_2^2(\alpha)\} a_2^2 - 2(1 + 2\lambda) \Psi_3(\alpha) a_3 \}
\[ = 1 + U_1(t)d_1 w + [U_1(t)d_2 + U_2(t)d_1^2] w^2 + \cdots. \]

which yields the following relations:

\[ (1 + \lambda) \Psi_2(\alpha) a_2 = U_1(t)c_1, \] (2.11)
\[ -(1 + 3\lambda) \Psi_2^2(\alpha) a_2^2 + 2(1 + 2\lambda) \Psi_3(\alpha) a_3 = U_1(t)c_2 + U_2(t)c_1^2 \] (2.12)

and

\[ -(1 + \lambda) \Psi_2(\alpha) a_2 = U_1(t)d_1, \] (2.13)
\[ (4(1 + 2\lambda) \Psi_3(\alpha) - (1 + 3\lambda) \Psi_2^2(\alpha)) a_2^2 - 2(1 + 2\lambda) \Psi_3(\alpha) a_3 = U_1(t)d_2 + U_2(t)d_1^2. \] (2.14)

From (2.11) and (2.13) it follows that

\[ c_1 = -d_1 \] (2.15)

and

\[ 2(1 + \lambda)^2 \Psi_2^2(\alpha) a_2^2 = U_1^2(t)(c_1^2 + d_1^2). \] (2.16)

Adding (2.12) to (2.14) and using (2.16), we obtain

\[ a_2^2 = \frac{U_1^3(t)(c_2 + d_2)}{2 \left[ (2(1 + 2\lambda) \Psi_3(\alpha) - (1 + 3\lambda) \Psi_2^2(\alpha)) U_1^2(t) - (1 + \lambda)^2 \Psi_2^2(\alpha) U_2(t) \right]} \] (2.17)

Applying (2.8) to the coefficients \( c_2 \) and \( d_2, \) and using (1.4) we have

\[ |a_2| \leq \frac{2t \sqrt{2t}}{\sqrt{\left| (2(1 + 2\lambda) \Psi_3(\alpha) - (1 + 3\lambda) \Psi_2^2(\alpha)) U_1^2(t) - (1 + \lambda)^2 \Psi_2^2(\alpha) U_2(t) \right|}}. \] (2.17)
By subtracting (2.14) from (2.12) and using (2.15) and (2.16), we get

$$a_3 = \frac{U_1^2(t)(c_1^2 + d_1^2)}{2(1 + \lambda)^2 \Psi_2^2(\alpha)} + \frac{U_1(c_2 - d_2)}{4(1 + 2\lambda)\Psi_3(\alpha)}.$$ 

Using (1.4), once again applying (2.8) to the coefficients $c_1, c_2, d_1$ and $d_2$, we get

$$|a_3| \leq \frac{4t^2}{(1 + \lambda)^2 \Psi_2^2(\alpha)} + \frac{t}{(1 + 2\lambda)\Psi_3(\alpha)}.$$ \hspace{1cm} (2.18)

By taking $\lambda = 0$ or $\lambda = 1$ and $t \in (0, 1)$, one can easily state the estimates $|a_2|$ and $|a_3|$ for the function classes $R_\Sigma(0, \alpha, \Phi(z, t)) = PS^*_\Sigma(\alpha, \Phi(z, t))$ and $R_\Sigma(1, \alpha, \Phi(z, t)) = K^*_\Sigma(\alpha, \Phi(z, t))$ respectively.

Remark 2.2. Let $f$ given by (1.1) be in the class $PS^*_\Sigma(\alpha, \Phi(z, t))$. Then

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|\Psi_3(\alpha) - \Psi_2^2(\alpha)|}}$$

and

$$|a_3| \leq \frac{4t^2}{\Psi_2^2(\alpha)} + \frac{t}{\Psi_3(\alpha)}.$$ 

where $t \neq \frac{\Psi_2(\alpha)}{2\sqrt{2\Psi_2^2(\alpha) - 2\Psi_3(\alpha)}}$.

Remark 2.3. Let $f$ given by (1.1) be in the class $K^*_\Sigma(\alpha, \Phi(z, t))$. Then

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|3\Psi_3(\alpha) - 4\Psi_2^2(\alpha)|}}$$ \hspace{1cm} (2.19)

and

$$|a_3| \leq \frac{t^2}{\Psi_2^2(\alpha)} + \frac{t}{3\Psi_3(\alpha)}.$$ \hspace{1cm} (2.20)

where $t \neq \frac{\Psi_2(\alpha)}{\sqrt{8\Psi_2^2(\alpha) - 6\Psi_3(\alpha)}}$.

For $\alpha = 0$, Theorem 2.1 yields the following corollary.

Corollary 2.4. Let $f$ given by (1.1) be in the class $R_\Sigma(\lambda, 0, \Phi(z, t))$. Then

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|(1 + \lambda)^2 - 2(2\lambda^2 + 4\lambda + 1)t^2|}}$$

and

$$|a_3| \leq \frac{t^2}{(1 + \lambda)^2} + \frac{t}{3(1 + 2\lambda)}$$

where $0 \leq \lambda \leq 1$ and $t \neq \frac{1 + \lambda}{2\sqrt{\lambda}}$ for $\lambda \neq 0$.

By taking $\alpha = 0$ in the above remarks we get the estimates $|a_2|$ and $|a_3|$ for the function classes $S^*_\Sigma(\frac{1}{2}, \Phi(z, t))$ and $K^*_\Sigma(\frac{1}{2}, \Phi(z, t))$. 

Remark 2.5. Let \( f \) given by (1.1) be in the class \( \mathcal{S}^{*}_{\Sigma}(\frac{1}{2}, \Phi(z,t)) \). Then
\[
|a_2| \leq 2t\sqrt{2t}
\]
and
\[
|a_3| \leq 4t^2 + t.
\]

Remark 2.6. Let \( f \) given by (1.1) be in the class \( \mathcal{K}^{*}_{\Sigma}(\frac{1}{2}, \Phi(z,t)) \). Then for \( t \neq \frac{1}{\sqrt{2}} \),
\[
|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|4 - 8t^2|}}
\]
and
\[
|a_3| \leq t^2 + \frac{t}{3}.
\]

3. Fekete-Szegö Inequality For The Function Class \( \mathcal{R}_{\Sigma}(\lambda, \alpha, \Phi(z,t)) \)

Due to Zaprawa [24], in this section we obtain the Fekete-Szegö inequality for the function classes \( \mathcal{R}_{\Sigma}(\lambda, \alpha, \Phi(z,t)) \).

Theorem 3.1. Let \( f \) given by (1.1) be in the class \( \mathcal{R}_{\Sigma}(\lambda, \alpha, \Phi(z,t)) \) and \( \mu \in \mathbb{R} \). Then we have
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{t}{(1+2\lambda)\Psi_3(\alpha)} & |\mu - 1| \leq \frac{(1+\lambda)^2\Psi_{22}^2(\alpha)}{4t^2 + 2(1+2\lambda)\Psi_{33}(\alpha) - (\lambda^2 + 5\lambda + 2)\Psi_2^2(\alpha)} \\
\frac{8(1-\mu)t^3}{(2(1+2\lambda)\Psi_3(\alpha) - (\lambda^2 + 5\lambda + 2)\Psi_2^2(\alpha))4t^2 + (1+\lambda)^2\Psi_2^2(\alpha)} & |\mu - 1| \geq \frac{(1+\lambda)^2\Psi_{22}^2(\alpha)}{4t^2 + 2(1+2\lambda)\Psi_{33}(\alpha) - (\lambda^2 + 5\lambda + 2)\Psi_2^2(\alpha)}
\end{cases}
\]

Proof. From (2.12) and (2.14)
\[
a_3 - \mu a_2^2 = \frac{U_1^3(t)(c_2 + d_2)}{(4(1+2\lambda)\Psi_3(\alpha) - 2(1+3\lambda)\Psi_2^2(\alpha))U_1^3(t) - 2U_2(t)(1+\lambda)^2\Psi_2^2(\alpha)} + \frac{U_1(t)(c_2 - d_2)}{4(1+2\lambda)\Psi_3(\alpha)}
\]
\[
= U_1(t)\left[ h(\mu) + \frac{1}{4(1+2\lambda)\Psi_3(\alpha)} \right] c_2 + \left[ h(\mu) - \frac{1}{4(1+2\lambda)\Psi_3(\alpha)} \right] d_2 \] (3.1)
where
\[
h(\mu) = \frac{(1-\mu)U_1^2(t)}{2[2(1+2\lambda)\Psi_3(\alpha) - (1+3\lambda)\Psi_2^2(\alpha))U_1^2(t) - (1+\lambda)^2\Psi_2^2(\alpha)U_2(t)]. \] (3.2)

Then, in view of (1.4), we conclude that
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{t}{(1+2\lambda)\Psi_3(\alpha)}, & 0 \leq |h(\mu)| \leq \frac{1}{4t|h(\mu)|}, \\
\frac{1}{4(1+2\lambda)\Psi_3(\alpha)}, & |h(\mu)| \geq \frac{1}{4(1+2\lambda)\Psi_3(\alpha)}
\end{cases}
\]
Taking \( \mu = 1 \), we have the following corollary.
Corollary 3.2. If \( f \in \mathcal{R}_\Sigma(\lambda, \alpha, \Phi(z, t)) \), then
\[
|a_3 - a_2^2| \leq \frac{t}{(1 + 2\lambda)\Psi_3(\alpha)}.
\] (3.3)

Corollary 3.3. Let \( f \) given by (1.1) be in the class \( \mathcal{S}_\Sigma^*(\alpha, \Phi(z, t)) \) and \( \mu \in \mathbb{R} \). Then we have
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{t}{\Psi_3(\alpha)}, & |\mu - 1| \leq \frac{\Psi_3(\alpha)}{8(1 - \mu)t^3} \\
\frac{8|1 - \mu|t^3}{((3\Psi_2(\alpha) - 4\Psi_2(\alpha)^2t^2 + \Psi_2(\alpha)t)^1)} & |\mu - 1| \geq \frac{\Psi_3(\alpha)}{8(1 - \mu)t^3}.
\end{cases}
\]

Especially, for \( \mu = 1 \) if \( f \in \mathcal{S}_\Sigma^*(\frac{1}{2}, \Phi(z, t)) \) we obtain
\[
|a_3 - a_2^2| \leq t.
\] (3.4)

Corollary 3.4. Let \( f \) given by (1.1) be in the class \( \mathcal{K}_\Sigma^*(\alpha, \Phi(z, t)) \) and \( \mu \in \mathbb{R} \). Then we have
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{t}{3\Psi_3(\alpha)}, & |\mu - 1| \leq \frac{\Psi_3(\alpha)}{2t^2 + 3\Psi_3(\alpha) - 4\Psi_2(\alpha)} \\
\frac{2|1 - \mu|t^3}{|(3\Psi_3(\alpha) - 4\Psi_2(\alpha)^2t^2 + \Psi_2(\alpha)t)^1|} & |\mu - 1| \geq \frac{\Psi_3(\alpha)}{2t^2 + 3\Psi_3(\alpha) - 4\Psi_2(\alpha)}.
\end{cases}
\]

Especially, for \( \mu = 1 \) if \( f \in \mathcal{K}_\Sigma^*(\frac{1}{2}, \Phi(z, t)) \) we obtain
\[
|a_3 - a_2^2| \leq \frac{t}{3}.
\] (3.5)

4. Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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