KRONECKER QUIVERS AND AMENABILITY

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Abstract. We apply the notion of hyperfinite families of modules to the wild path algebras of extended Kronecker quivers \( k\Theta(d) \). While the preprojective and postinjective component are hyperfinite, we show the existence of a family of non-hyperfinite modules in the regular component for some \( d \). Making use of dimension expanders to achieve this, our construction is more explicit than previous results.

1. Introduction

The notions of hyperfiniteness for countable sets of modules and amenable representation type for algebras have been introduced by [Ele17]. We will work with the definitions as follow.

Definition 1. Let \( k \) be a field, \( A \) be a finite dimensional \( k \)-algebra and let \( \mathcal{M} \) be a set of finite dimensional \( A \)-modules. One says that \( \mathcal{M} \) is hyperfinite provided for every \( \varepsilon > 0 \) there exists a number \( L_\varepsilon > 0 \) such that for every \( M \in \mathcal{M} \) there exist both, a submodule \( N \subseteq M \) such that
\[
\dim_k N \geq (1 - \varepsilon) \dim_k M,
\]
and modules \( N_1, N_2, \ldots, N_t \in \text{mod } A \), with \( \dim_k N_i \leq L_\varepsilon \), such that \( N \cong \bigoplus_{i=1}^t N_i \).

The \( k \)-algebra \( A \) is said to be of amenable representation type provided the set of all finite dimensional \( A \)-modules (or more precisely, a set which meets every isomorphism class of finite dimensional \( A \)-modules) is hyperfinite.

Previously, the author has shown that tame quiver algebras are of amenable representation type (giving a new proof and not using a previous result on string algebras) while wild quiver algebras are not (using results of E1, thus working towards [Ele17] Conjecture 1], presuming that finite dimensional algebras are of tame type if and only if they are of amenable representation type.

In this note we will focus on hyperfinite families for (wild) Kronecker algebras while also ascertaining that the wild Kronecker algebras are not of amenable representation type.

We further use the following facts from [Eck19].

Proposition 2. Let \( \mathcal{M} \) be a family of \( A \)-modules. If \( \mathcal{M} \) is hyperfinite, so is the family of all finite direct sums of modules in \( \mathcal{M} \).

Proposition 3. Let \( A \) be a finite dimensional \( k \)-algebra. Let \( \mathcal{M}, \mathcal{N} \subseteq \text{mod } A \) where \( \mathcal{N} \) is hyperfinite. If there is some \( L \geq 0 \) such that for all \( M \in \mathcal{M} \), there exists a submodule \( P \subseteq M \) of codimension less than or equal to \( L \), and \( P \in \mathcal{N} \), then \( \mathcal{M} \) is also hyperfinite.

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Proposition 4. Let \( k \) be a field and \( A, B \) be two finite dimensional \( k \)-algebras. Let \( F : \text{mod} \, A \to \text{mod} \, B \) be an additive, left-exact functor such that there exists \( K_1, K_2 > 0 \) with
\[
(2) \quad K_1 \dim X \leq \dim F(X) \leq K_2 \dim X,
\]
for all \( X \in \text{mod} \, A \). If \( \mathcal{N} \subseteq \text{mod} \, A \) is a hyperfinite family, then the family \( \{ F(X) : X \in \mathcal{N} \} \subseteq \text{mod} \, B \) is also hyperfinite.

2. THE SPECIAL CASE OF THE 2-KRONECKER QUIVER

Let us first recall the situation for the tame 2-Kronecker quiver. It follows from the results on string algebras in [Ele17, Proposition 10.1], but we give a direct and independent proof here for illustration purposes and to the convenience of the reader.

Lemma 5. Let \( X = \tau rI(i) \) be some indecomposable postinjective \( kQ \)-module of defect \( \partial(X) = d \). Then there is an injective module \( I(j) \) such that there exists a non-zero morphism \( \theta : X \to I(j) \). Moreover, for any direct summand \( Z \) of \( \ker \theta \), we have \( \partial(Z) < d \).

Theorem 6. Let \( k \) be any field. Then the path algebra of the 2-Kronecker quiver \( \Theta(2) \) is of amenable representation type.

Proof. We fix a notation for the the vertices and arrows as follows.

\[
\begin{array}{ccc}
1 & \rightarrow & a \rightarrow 2 \\
& b & \\
\end{array}
\]

It is well-known (see e.g. [Ben98, Theorem 4.3.2] or [Bur86]) that the indecomposable preprojective and postinjective \( k \)-representations of \( Q \) are given by
\[
P_i : \quad k^i \xrightarrow{[\begin{smallmatrix} \text{id} \\ 0 \end{smallmatrix}]} k^{i+1}, \quad Q_i : \quad k^{i+1} \xrightarrow{[\begin{smallmatrix} \text{id} & 0 \\ 0 & \text{id} \end{smallmatrix}]} k^i,
\]
respectively, while the indecomposable regular representations are given by
\[
R_n(\phi, \psi) : \quad k^n \xrightarrow{\phi} k^n, \quad \psi
\]
where either \( \phi \) is the identity and \( \psi \) is given by the companion matrix of a power of a monic irreducible polynomial over \( k \), or \( \phi \) is the identity and \( \psi \) is given by the companion matrix of a polynomial of the form \( \lambda^n \).

We will show that the preprojective, the regular and the postinjective component are each hyperfinite families to conclude the amenability of \( \text{mod} \, kQ \). We will give an argument for the indecomposable objects in each component and then apply Proposition 2 to extend the result.

We start with the preprojectives, and let \( \varepsilon > 0 \). Set \( K_\varepsilon := \left\lceil \frac{1}{2\varepsilon} \right\rceil + 1 \) and \( L_\varepsilon = \frac{1}{\varepsilon} + 3 \). Let \( X = P_i \) be some indecomposable preprojective. If \( \dim X \leq L_\varepsilon \), there is nothing to show. We may thus assume that \( \dim X > L_\varepsilon \), implying \( i \geq K_\varepsilon \),
and write \( i = j \cdot K_\varepsilon + r \), where \( 0 \leq r < K_\varepsilon \). Now consider the standard basis \( \{e_1, e_2, \ldots, e_i\} \) of \( k^i \). Let \( U \) be the submodule of \( X \) generated by the subset \[
\{e_1, \ldots, e_{K_\varepsilon-1}\} \cup \{e_{K_\varepsilon+1}, \ldots, e_{2K_\varepsilon-1}\} \cup \ldots \\
\cup \{e_{(j-1)K_\varepsilon+1}, \ldots, e_{jK_\varepsilon-1}\} \cup \{e_{jK_\varepsilon+1}, \ldots, e_i\},
\]
dropping every \( K_\varepsilon \)-th basis vector. Then \( U \) decomposes into \( j \) direct summands of type \( P_{K_\varepsilon-1} \) and a smaller rest term. All summands will thus be of \( K \)-dimension smaller than \( 2(K_\varepsilon - 1) + 1 < L_\varepsilon \). Moreover,

\[
\dim U = \dim X - j = \dim X - \frac{i - r}{K_\varepsilon} = \dim X - \frac{\dim X - 1}{2K_\varepsilon} + \frac{r}{K_\varepsilon} \\
\geq \dim X - \varepsilon (\dim X - 1) > (1 - \varepsilon) \dim X.
\]

This shows that the family of indecomposable preprojective modules \( \mathcal{P}(kQ) \) is hyperfinite.

If \( X = R_n(\phi, \psi) \) is an indecomposable regular module, we may consider the submodule \( Y \) generated by the basis vectors \( \{e_1, \ldots, e_{n-1}\} \) of the vector space at vertex 1. Note that we assume that \( \psi \) corresponds to the Frobenius companion matrix of a monic polynomial. Then \( Y \cong P_{n-1} \), so by the above it belongs to the hyperfinite family \( \mathcal{P}(kQ) \). We have that \( \dim Y = \dim X - 1 \). Thus, Proposition\( \ref{prop:hyperfiniteness-regular-modules} \) implies the hyperfiniteness of the indecomposable regular modules.

We are left to deal with the postinjective case. By Lemma\( \ref{lem:postinjective-hyperfiniteness} \) for each indecomposable postinjective \( X \), we can find a submodule \( Y := \ker \theta \) of strictly smaller defect. Moreover, if \( Y \) had a postinjective summand \( Z \), it must have defect \( \partial(Z) < \partial(X) \). In this situation, all indecomposable postinjective modules have defect \( d = 1 \). Choose the hyperfinite family \( \mathcal{N}_0 = \mathcal{P}(kQ) \cup \mathcal{R}(kQ) \) of all preprojective and regular modules. For all postinjective indecomposables, the submodule \( Y \) must be in \( \text{add}\mathcal{N}_0 \), since there are no non-zero postinjective modules \( Z \) with defect \( \partial(Z) < 1 \). This family is hyperfinite by the above. Moreover, the codimension of \( Y \) is bounded by the dimension of the indecomposable injectives, of which there are only two. Hence, we can use Proposition\( \ref{prop:hyperfiniteness-injective-modules} \) to prove the hyperfiniteness of the indecomposable postinjectives.

Now apply Proposition\( \ref{prop:hyperfiniteness-injective-modules} \) to \( \mathcal{P}(kQ) \cup \mathcal{R}(kQ) \cup \mathcal{Q}(kQ) \) to see that \( \text{mod}\ kQ \) is hyperfinite, and thus \( kQ \) is amenable. \( \square \)

3. Hyperfiniteness from Fragmentability and for Exceptional Modules

We will continue by considering the path algebra of a (wild) extended Kronecker algebra and show that the indecomposable preprojective and postinjective modules for these algebras form hyperfinite families. We start with a result connecting to the notion of fragmentability from graph theory and making use of the tree structure of coefficient quivers of certain modules.

**Definition 7** [EF01], [EM94]. Let \( \varepsilon \) be a non-negative real number, and \( C \) an integer. We say that a graph \( G = (V, E) \) is \((C, \varepsilon)\)-fragmentable provided there is a set \( X \subseteq V \), called the fragmenting set, such that

(1) \( |X| \leq \varepsilon |V| \), and

(2) every component of \( G \setminus X \) has at most \( C \) vertices.

Now consider a class \( \Gamma \) of graphs. We will say that \( \Gamma \) is \( \varepsilon \)-fragmentable provided there is an integer \( c \) such that for all \( G \in \Gamma \), \( G \) is \((c, \varepsilon)\)-fragmentable. Moreover, a class \( \Gamma \) of graphs is called fragmentable if

\[
\varepsilon_f(\Gamma) := \inf\{ \varepsilon : \Gamma \text{ is } \varepsilon\text{-fragmentable} \} = 0.
\]
Remark. We may relax the definition to say that a class $\Gamma$ of graphs is fragmentable if for any $\varepsilon > 0$, there are positive integers $n_0, c(\varepsilon)$ such that if $G \in \Gamma$ is a graph with $n \geq n_0$ non-isolated vertices, then there is a set $X$ of vertices, with $|X| \leq \varepsilon n$, such that each component of $G \setminus X$ has $\leq c(\varepsilon)$ vertices.

**Proposition 8.** Let $d, \ell \in \mathbb{N}$, $t \in \mathbb{N}$, and $a_0, a_1, a_{t+1}$ be as in [Rin98, Section 8]. Let $A$ be the path algebra of a quiver $Q$. Let $M$ be a class of indecomposable tree modules for $A$, i.e. of modules $M$ such that there exists bases of $(M_i)_{i \in Q_0}$ such that the corresponding coefficient quiver is a tree, and additionally assume that the maximal indegree is $d$ and the maximal path length is $\ell$. Then $M$ is hyperfinite.

**Proof.** Let $M \in M$. By [EM94, Lemma 3.6], it is enough to show that the removal of at most $d^\ell$ basis elements decomposes the coefficient quiver into components of size at most half that of $M$ which are a member of $M$. Since $M$ is a tree module, there is a vertex $v$ (one of the central points of the underlying tree graph) in the coefficient quiver whose removal will result in splitting the quiver into (non-connected) subtrees of size at most half that of $M$. If this vertex $v$ is not a source in the coefficient quiver, it can be removed, and the induced subtrees are submodules of $M$, which are themselves tree modules in $M$ (pick the bases given by restriction). If $v$ is a source in the coefficient quiver whose removal will result in splitting the quiver into (non-connected) subtrees of size at most half that of $M$. Then the family of preprojective $A$-modules is hyperfinite.

**Lemma 10.** Fix $m \geq 3$. Let $\alpha_1, \alpha_{t+1}$ be as in [Rin98, Section 8]. That is $a_0 = 0, a_1 = 1$ and $a_{t+1} = m \alpha_t - a_{t-1}$. Then the closed-form solution of this recurrence is given by

$$a_t = \frac{\varphi^t - \psi^t}{\sqrt{m^2 - 4}} \text{ where } \varphi = \frac{m + \sqrt{m^2 - 4}}{2} \text{ and } \psi = \frac{m - \sqrt{m^2 - 4}}{2}.$$ 

Moreover, the quotient $a_t/a_{t+1}$ of consecutive terms converges to $\varphi^{-1}$.

**Proof.** Routine exercise.

**Lemma 11.** Fix $m \geq 2$. Let $Q(t)$ be an indecomposable postinjective module as described in [Rin98, Section 8], with coefficient quiver $\Gamma$ given there. Then the outdegree of the vertices of $\Gamma$ is bounded by two, and the indegree is bounded by $(t - 1)(m - 2) + m$.

**Proof.** By the description of the arrow maps for the postinjective indecomposable module $Q(t)$ in the dual of [Rin98, Proposition 3], the first $m - 1$ matrices have no common non-zero columns, so the outdegree of each source with respect to the arrows $\alpha_i$, $1 \leq i \leq m - 1$ is at most one. On the other hand, each row of one of these arrow matrices contains exactly one one. Moreover, as the matrix for the last arrow $\alpha_m$ is constructed by concatenating zero matrices or column block
matrices containing a single identity matrix block, at most one arrow \(\alpha_m\) starts at each source. Indeed, the concatenation involves \(t - 1\) matrices – the \(C(a_{j-1}, a_j)\) – containing \(m - 2\) identity matrices, the \(E(a_{j-1})\), of varying size \(a_{j-1}\) each, and one additional identity matrix \(E(a_t)\). Combining this information yields the desired result.

**Lemma 12.** Fix \(m \geq 3\). Let \(M\) be a module of dimension vector \((a_{t+1}, a_t)\). Then we can express

\[
t = \log_\varphi \left( \dim M + \sqrt{\dim M^2 + \frac{4}{m-2}} \right) - \log_\varphi \left( 2 \frac{1 + \varphi}{\sqrt{m^2 - 4}} \right).
\]

Moreover, for \(\dim M \geq 3\), it holds that \(t \leq c \sqrt{\dim M}\) for some constant \(c\).

**Proof.** Clearly, \(\dim M = a_{t+1} + a_t\). Now using the closed form of Lemma 10 we have that

\[
\dim M = \frac{\varphi^{t+1} - \psi^{t+1}}{\sqrt{m^2 - 4}} + \frac{\varphi^t - \psi^t}{\sqrt{m^2 - 4}} = \frac{\varphi^{2t+1} - (\varphi \psi)^t \psi}{\varphi^t \sqrt{m^2 - 4}} + \frac{\varphi^{2t} - (\varphi \psi)^t}{\varphi^t \sqrt{m^2 - 4}} = \frac{\varphi^{2t+1} - \psi + \varphi^{2t} - 1}{\varphi^t \sqrt{m^2 - 4}} = \varphi^t \frac{1 + \varphi}{\sqrt{m^2 - 4}} - \varphi^{-t} \frac{1 + \psi}{\sqrt{m^2 - 4}}.
\]

By substitution, noting that real powers of positive numbers are positive and using \((1 + \varphi)(1 + \psi) = m + 2\), we get

\[
t = \log_\varphi \left( \dim M + \sqrt{\dim M^2 + \frac{4}{m-2}} \right) - \log_\varphi \left( 2 \frac{1 + \varphi}{\sqrt{m^2 - 4}} \right).
\]

Now it remains to show the estimate. We first note that for \(\varphi > 1\) and

\[
\frac{2 + 2\sqrt{2}}{\sqrt{m^2 - 4}} > \frac{m + \sqrt{m^2 - 4}}{\sqrt{m^2 - 4}} > 1,
\]

the subtrahend is always positive, resulting in its omittance leaving an upper bound.

Now, when \(\dim M \geq 3\), we have

\[
t \leq \log_\varphi (1 + \sqrt{2}) + \log_\varphi (\dim M) < \log_\varphi (3) + \log_\varphi (\dim M) \leq 2 \log_\varphi (\dim M).
\]

Now, for \(\varphi > 1\), it is enough to further consider \(\ln(\dim M)\). Clearly,

\[
\exp(2\sqrt{\dim M}) = \frac{(2\sqrt{\dim M})^2}{2!} = 2 \dim M > \dim M,
\]

so \(2\sqrt{\dim M} > \ln \dim M\). All in all, this combines to the desired inequality

\[
t \leq 2 \log_\varphi \dim M = 2 \frac{1}{\ln \varphi} \ln \dim M < \frac{4}{\ln \varphi} \sqrt{\dim M}.
\]

**Proposition 13.** The family of indecomposable postinjective \(k\Theta(m)\)-modules is hyperfinite.

**Proof.** We want to give a proof similar to that of [EM94, Lemma 3.6]. Let \(\varepsilon > 0\). For shortness, we write \(\alpha = \frac{1}{4} + \delta\) for some \(0 < \delta < \frac{1}{4}\). Let \(Q(t)\) be the indecomposable postinjective module of dimension vector \((a_{t+1}, a_t)\). Let \(n = \dim Q(t)\) and assume \(n > 5\). We show how to split this module into small components by a sequence of stages. Before each stage \(i\), all components are isomorphic to indecomposable postinjective modules, having no more than \(\alpha^{-1}n\) vertices in their coefficient quivers, while the number of components with more than \(\alpha'n\) vertices is at most \(\alpha^{-1}\). Since the coefficient quiver \(\Gamma\) of \(Q(s)\) is a tree, there exists one vertex, whose removal creates subtrees of size at most \(\frac{\dim Q(t)}{2}\). Note that we can
assume that the vertex to remove is a sink, since all sources have outdegree at most two by Lemma 11 and all their neighbours are sinks, and the size of $\alpha$ allows for this modification. But a removal of a sink corresponds to passing to the cokernel of an inclusion of $S(1) \hookrightarrow Q(s)$. Yet, this cokernel must have smaller dimension than $Q(s)$, and since $Q(s)$ is postinjective, must also be postinjective. This implies that the cokernel is the direct sum of indecomposable postinjective modules for smaller $s$, as the dimension of the indecomposable postinjectives strictly increases for growing $s$. This proves that after stage $i$, all components are indecomposable postinjective modules with no more than $\alpha n$ vertices in their coefficient quivers.

The number of stages is the least $k$ such that $\alpha^k n \leq L$, for an $L$ to be determined later. Hence $\alpha^k - 1 \geq n > L$, so that $\alpha^{k-1} n > L$, so that $\alpha^{k-1} < \frac{n}{L}$.

Unfortunately, this process does not create a submodule of $Q(t)$, but a factor module given by the direct sum of many smaller indecomposable postinjective modules. To attain a submodule, we must delete further vertices in each stage. Note that in each stage, we only deal with the components $M$ with $\alpha^i n < \dim M \leq \alpha^{i-1} n$. Since $\frac{n}{L} > a_{i-1}$, not in every stage a reduction takes place. But when a reduction does take place, we create submodules from $Q(s)$ by removing all the vertices adjacent to the deleted sink. By the structure of the canonical coefficient quivers of $Q(s)$, there are at most $s + 2$ such vertices, and $s \leq c\sqrt{\dim Q(s)}$ by Lemma 12. Note that while the dimension of the submodules left before stage $i$ are smaller than $\dim Q(s)$, as we have removed at least one more source, the operand in stage $i$ is still $\alpha^{i-1} n \geq \dim Q(s)$. This implies that in stage $i$, we remove at most $A\sqrt{\alpha^{i-1} n}$ vertices, for $A = 2 + c$, requiring $\alpha^{k-1} n > 2$. Thus, choose $\lambda = \frac{1}{2}$.

Now, the total number $r_i$ of vertices removed in stage $i$ is at most

$$r_i < \frac{A n^\lambda}{\alpha} (n/L)^{(1-\lambda)} \alpha^{(k-i)(1-\lambda)}$$

and since $\alpha^{k-i} < n/L$, we have $\alpha^{1-i} < (n/L)\alpha^{k-i}$. Hence

$$r_i < \frac{A n^\lambda}{\alpha} (n/L)^{(1-\lambda)} \alpha^{(k-i)(1-\lambda)}$$

where $\beta = \alpha^{1-\lambda}$ (so $0 < \beta < 1$). Then the total number $R$ of vertices removed from the coefficient quiver of $Q(t)$ is

$$\sum_{i=1}^{k} r_i < \frac{A n^\lambda}{\alpha L^{(1-\lambda)}} \sum_{i=1}^{k} \beta^{k-i} < \frac{A n^\lambda}{\alpha L^{(1-\lambda)}} \sum_{i=0}^{\infty} \beta^{i} = \frac{A n^\lambda}{\alpha L^{(1-\lambda)}} \frac{1}{1 - \beta}.$$ 

Since $1 - \lambda > 0$, it follows that we can choose $L = L_\epsilon$ independent of $n$ such that $R \leq \epsilon n$. This then shows the hyperfiniteness of $\{Q(t) : t \geq 0\}$ and thus of the postinjective component.

□

Remark. Note that the logarithm can be bounded above by any radical power: We have $\ln x \leq n \sqrt[n]{x}$. This implies that we can prove an adaptation of Proposition 5 in the case of coefficient quivers that are graphs of genus at most $\gamma$ for fixed $\gamma \geq 0$ or for rectangular lattices of dimension $d$ for a fixed integer $d$, provided the indegree has a logarithmic bound with respect to the dimension.
4. A Family of Non-Hyperfinite Modules

In the previous section we have seen that both the preprojective and the postinjective component of extended Kronecker algebras are hyperfinite. Yet, Elek has shown that any wild Kronecker algebra is not of amenable representation type by showing that there are non-hyperfinite families of modules over the free algebras \(k(x_1, \ldots, x_r)\) with \(r \geq 2\) generators. Thus, the regular component must contain a non-hyperfinite family. We are interested in understanding and providing such a concrete counterexample of a non-hyperfinite family of modules for algebras of non-amenable representation type.

To this end, we recall the following definition of [LZ08; DS11; Bou09; DW10].

**Definition 14.** Let \(k\) be a field, \(d \in \mathbb{N}\) and \(\alpha > 0\). For a vector space \(V\) and a set \(\{T_1, \ldots, T_d\}\) of endomorphisms of \(V\), the pair \((V, \{T_i\}_{i=1}^d)\) is called an \(\alpha\)-dimension expander of degree \(d\) provided for every subspace \(W \subset V\) of dimension less than or equal to \(\frac{\dim_V W}{d}\), we have that

\[
\dim_k \left(W + \sum_{i=1}^d T_i(W)\right) \geq (1 + \alpha) \dim_k W.
\]

The following (weaker) notion is also found in the literature.

**Definition 15.** Let \(k\) be a field, \(d \in \mathbb{N}\), \(0 < \eta \leq 1\) and \(\alpha > 0\). Given a vector space \(V\) and a set \(\{T_1, \ldots, T_d\}\) of endomorphisms of \(V\), the pair \((V, \{T_i\}_{i=1}^d)\) is called an \((\eta, \alpha)\)-dimension quasi-expander of degree \(d\) provided for every subspace \(W \subset V\) of dimension at most \(\eta \dim_k V\), we have that

\[
\dim_k \left(\sum_{i=1}^d T_i(W)\right) \geq (1 + \alpha) \dim_k W.
\]

**Remark.** Every \(\alpha\)-dimension expander of degree \(d\) along with the identity map \(id_V\) is a \((\frac{1}{2}, \alpha)\)-dimension quasi-expander of degree \(d+1\).

Now, a sequence of dimension quasi-expanders of degree \(d\) of increasing dimension gives rise to a non-hyperfinite family for the \(d\)-Kronecker algebra \(k\Theta(d)\):  

**Proposition 16.** Let \(k\) be a field, \(d \in \mathbb{N}\) and \(\eta, \alpha > 0\). If \(\{(V_i, \{T_i^{(1)} \}, \ldots, T_i^{(d)}\})\) \(i \in I\) is a sequence of \((\eta, \alpha)\)-dimension quasi-expanders of degree \(d\) such that \(\dim V_i\) is unbounded, then the induced sequence of \(k\Theta(d)\)-modules \(V_i \rightarrow \cdots \rightarrow V_i\) is not hyperfinite.

**Proof.** Let \(\alpha > 0\) and \(\{(V_i, \{T_i^{(1)} \}, \ldots, T_i^{(d)}\})\) be a sequence of \(\alpha\)-dimension expanders degree \(d\) and of unbounded dimension \(\dim V_i\). Consider the sequence

\[
\left\{M_i = \left((V_i, V_i), (T_i^{(1)} \ldots, T_i^{(d)})\right)\right\}_{i \in I} \in \text{mod } k\Theta(d).
\]

If this sequence is hyperfinite, for each \(\varepsilon > 0\), there exists an \(L_\varepsilon > 0\) and we can find some \(M \in \{M_i\}_{i \in I}\) - given by an \((\eta, \alpha)\)-dimension quasi-expander space \(V\) - such that \(\dim M = 2 \dim V > 2 \frac{2\varepsilon}{\eta}\) with a suitable submodule \(P\) exhibiting hyperfiniteness. We will denote the vector space of \(P_j\) at vertex \(v \in Q_0\) by \(P_j(v)\). We have that

\[
\dim P_j(1) + \dim P_j(2) = \dim P_j \leq L_\varepsilon < \eta \dim V,
\]
also noting that each \( P_j(v) \) is a subspace of the \( (\eta, \alpha) \)-dimension quasi-expander space \( V \). As each \( P_j \) is a \( k \Theta(d) \)-module, thus \( T_1(P_j(1)) + \cdots + T_d(P_j(2)) \subseteq P_j(2) \), this implies that

\[
\dim P_j(2) \geq (1 + \alpha) \dim P_j(1).
\]

Moreover,

\[
2(1 - \varepsilon) \dim V \leq \sum_{j=1}^{t} (\dim P_j(1) + \dim P_j(2)) \leq \sum_{j=1}^{t} \dim P_j(1) + \dim V
\]

\[
\Leftrightarrow (1 - 2\varepsilon) \dim V \leq \sum_{j=1}^{t} \frac{\dim P_j(2)}{1 + \alpha} \leq \frac{\dim V}{1 + \alpha}
\]

which in light of inequality (3) yields that

\[
(1 - 2\varepsilon) \dim V \leq \sum_{j=1}^{t} \frac{\dim P_j(2)}{1 + \alpha} \leq \frac{\dim V}{1 + \alpha}
\]

\[
\Leftrightarrow \quad \varepsilon \geq \frac{\alpha}{2(1 + \alpha)},
\]

contradicting the hyperfiniteness of the sequence \( \{M_i : i \in I\} \). \( \square \)

**Remark.** If all \( T_i \) are such that \( T_i \circ T_j = 0 \) for any combination, then in general \( \{V, \{T_i\}_{i=1}^{d}\} \) is neither a dimension expander nor a dimension quasi-expander: We must have \( \im T_i \subseteq \bigcap_{i=1}^{d} \ker T_i \) for all \( 1 \leq j \leq d \). Without loss of generality, we consider \( 0 \neq v \in \im T_i \) (if all \( T_i \)'s were zero, the claim is obviously true). Let \( W = \langle v \rangle \). Then \( \sum_{i=1}^{d} T_i(W) = 0 \), so the dimension property cannot hold for some non-trivial subspace of dimension one. Thus – unless \( \eta \dim V < 1 - (V, \{T_i\}_{i=1}^{d}) \) cannot be a dimension (quasi-)expander.

Proposition \[16\] reduces the problem of exhibiting a non-hyperfinite family to finding families of dimension expanders for fixed \( d \) and \( \alpha \) such that the dimension of the vector spaces is unbounded. We will make use of a proposition and a theorem of Lubotzky-Zelmanov \[LZ08\] to achieve this. Their result provides a variety of \( \alpha \)-dimension expanders of degree two over the complex numbers and generalizes to every field of characteristic zero. We first need another

**Definition 17.** Consider a group \( \Gamma \) generated by a finite set \( S \). Given a Hilbert space \( H \) and a unitary representation \( \rho : \Gamma \to U(H) \), where \( U(H) \) denotes the unitary endomorphisms of \( H \), the **Kazhdan constant** is defined as

\[
K^{S}_{\Gamma}(H, \rho) = \inf_{0 \neq v \in H} \max_{s \in S} \left\{ \frac{\|\rho(s)v - v\|}{\|v\|} \right\}.
\]

Further, the group \( \Gamma \) has the **property (T)** if

\[
K^{S}_{\Gamma} = \inf_{(H, \rho) \in \mathcal{R}_{\alpha}(\Gamma)} \{K^{S}_{\Gamma}(H, \rho)\} > 0,
\]

where \( \mathcal{R}_{\alpha}(\Gamma) \) is the family of all unitary representations of \( \Gamma \) which have no non-trivial \( \Gamma \)-fixed vector. In this case, \( K^{S}_{\Gamma} \) is called the **Kazhdan constant of \( \Gamma \) with respect to \( S \)**.

This Kazhdan constant is now relevant in the following Proposition determining the expansion rate \( \alpha \):

**Proposition 18.** \[LZ08, Proposition 2.1\] If \( \rho : \Gamma \to U_n(\mathbb{C}) \) is an irreducible unitary representation, then \( (\mathbb{C}^n, \rho(S)) \) is an \( \alpha \)-dimension expander of degree \( |S| \) where

\[
\alpha = \frac{\kappa}{12}, \quad \kappa = K^{S}_{\Gamma}(\text{Sl}_n(\mathbb{C}), \text{adj} \rho), \quad \text{where Sl}_n(\mathbb{C}) \text{ denotes the subspace of all linear}
\]
transformations of zero trace, and \(\text{adj} \rho\) is the adjoint representation on \(\text{End}(\mathbb{C}^n)\) induced by conjugation.

Remarks. (1) The endomorphism space \(\text{End}(\mathbb{C}^n) \cong M_n(\mathbb{C})\) and its subspace \(S \ell_n(\mathbb{C})\) become Hilbert spaces via \(\langle S, T \rangle = \text{tr}(ST^*)\).

(2) The induced representation \(\text{adj} \rho\) on \(M_n(\mathbb{C})\), given as

\[
\gamma \mapsto (T \mapsto \rho(\gamma)T\rho(\gamma)^{-1}),
\]

is unitary, as \(\text{adj}(\rho)(\gamma)\) is surjective and preserves the inner product for each \(\gamma \in \Gamma\).

(3) The subspace \(S \ell_n(\mathbb{C})\) of trace zero matrices is invariant under \(\text{adj} \rho\), since conjugation by invertible matrices preserves the trace. Thus, \((S \ell_n(\mathbb{C}), \text{adj} \rho)\) is a unitary representation.

(4) Note that if \(\rho\) is irreducible, then by Schur’s Lemma, \(S \ell_n(\mathbb{C})\) does not have any non-trivial \(\text{adj} \rho(\Gamma)\)-fixed vector: If \(T \in S \ell_n(\mathbb{C})\) is fixed by \(\text{adj} \rho\), then \(\ker T\) is an invariant subspace of \(\rho\), as \(T(v) = 0\) along with \(\rho(\gamma)T = T\rho(\gamma)\) implies that \(T(\rho(\gamma)(v)) = 0\). By the irreducibility of \(\rho\), \(\ker T\) must be a trivial subspace. If \(\ker T = 0\), we have that \(T\) is invertible, even \(T = \lambda I\) for some eigenvalue \(\lambda\) of \(T\). But \(0 = \text{tr} T = n\lambda\), a contradiction. Thus \(\ker T = \mathbb{C}^n\), so \(T\) is trivial.

In the following, we will exhibit an explicit example, coming from representations of \(\text{SL}(2, \mathbb{Z})\). To this end, we first consider representations of the special linear group \(\text{SL}(2, \mathbb{P})\) of \(2 \times 2\)-matrices over the finite field of characteristic \(p\), \(\mathbb{F}_p\).

**Lemma 19.** For each prime \(p \in \mathbb{P}\), there is an irreducible, complex \(p\)-dimensional representation of \(\text{SL}(2, \mathbb{P})\).

**Proof.** Let \(p\) be a prime number. Then \(G = \text{SL}(2, \mathbb{P})\) acts on \(\mathbb{P}(\mathbb{F}_p) = \{0, 1, \ldots, p - 1, \infty\}\) by

\[
\pi: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left( z \mapsto \frac{az + b}{cz + d} \right),
\]

with the usual conventions that \(\frac{a}{b} = \infty\) for \(x \neq 0\) and \(\frac{a\infty + b}{c\infty + d} = \frac{a}{b}\). This permutation action extends to a permutation representation \(\rho: G \to \text{GL}_{p+1}(\mathbb{C})\), with

\[
g \mapsto \left( \sum_{z \in \mathbb{P}(\mathbb{F}_p)} \lambda_z e_z \mapsto \sum_{z \in \mathbb{P}(\mathbb{F}_p)} \lambda_{\pi(z)} e_{\pi(z)} \right),
\]

identifying \(\mathbb{C}^{p+1}\) with the free, complex vector space on \(\mathbb{P}(\mathbb{F}_p)\) via \(e_1 \leftrightarrow e_0, \ldots, e_{p-1} \leftrightarrow e_p\). The character values of \(\chi_\rho\) can be calculated via the number of fixed points of \(\pi\) on representatives of the conjugacy classes of \(\text{SL}(2, \mathbb{P})\). Consider the subspace \(W = \{ v \in \mathbb{C}^{p+1} : \sum_{i=1}^{p+1} v_i = 0 \}\) of dimension \(p\). It is \(\rho\)-invariant and the restriction of \(\rho\) to \(W\) is the complement of the trivial representation in \(\rho\). Using character theory, this is sufficient information to show that \(\rho|_W\) is an irreducible complex representation of \(\text{SL}(2, \mathbb{P})\) (see also [FH91, Section 5.2]). \(\square\)

**Corollary 20.** The group \(\text{SL}_2(\mathbb{Z})\) has irreducible, unitary representations of unbounded dimension.

**Proof.** Consider the natural maps \(\pi_p: \text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/p\mathbb{Z}) = \text{SL}(2, p)\) mapping each matrix to the matrix of the residue classes of its entries modulo \(p\). Let \(\rho: \text{SL}(2, p) \to \text{GL}(V)\) be an irreducible \(p\)-dimensional representation. As \(\text{SL}(2, p)\) is a discrete group, we can endow \(V\) with an inner product in such a way to assume that \(\rho\) is unitary. Now consider \(\rho \circ \pi_p\). This is certainly a group homomorphism. Moreover, a subspace \(W \subseteq V\) is \(\text{SL}(2, p)\)-invariant if and only if \(W\) is \(\text{SL}_2(\mathbb{Z})\)-invariant, showing that \(\rho \circ \pi_p\) is irreducible since \(\rho\) is. \(\square\)
Remark. The subgroups $\Gamma(p) := \ker(SL_2(\mathbb{Z}) \to SL(2, \mathbb{Z}/p\mathbb{Z}))$ are sometimes called principal congruence subgroups. We have
\[ \Gamma(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{array}{c} a \equiv d \equiv 1 \mod p \\ b \equiv c \equiv 0 \mod p \end{array} \right\}. \]
As the projections are surjective, the subgroups have finite index $p^2 - p$ in $SL_2(\mathbb{Z})$.

**Definition 21** ([Lub94]). Let $G$ be a finitely generated group generated by a finite symmetric set of generators $S$. Given a family $\{N_i\}_{i \in I}$ of finite index normal subgroups, $G$ is said to have property $\tau$ with respect to the family $\{N_i\}_{i \in I}$ if for each $i$, $N_i$ contains $\tau$ and if $H, \rho$ is a non-trivial irreducible representation of $G$ factoring through $G/N_i$ for some $i \in I$ are bounded away from the trivial representation.

Remark. This definition is equivalent to requiring that the trivial representation is isolated in the set of all unitary representations of $G$ whose kernel contains some $N_i$ or to requiring that the nontrivial irreducible representations of $G$ factoring through $G/N_i$ for some $i \in I$ are bounded away from the trivial representation. Further note that a finitely generated group having property $(T)$ has property $\tau$ with respect to all finite index normal subgroups.

**Theorem 22.** The group $SL_2(\mathbb{Z})$ has property $\tau$ with respect to $\{\Gamma(p)\}_{p \in \mathbb{Z}^+}$.

Proof. By Selberg’s $3 \overline{1}$ theorem, given a congruence subgroup $\Gamma(p)$ of $SL_2(\mathbb{Z})$, the smallest positive eigenvalue $\lambda_1(\Gamma(p)\backslash \mathbb{H})$ of the Laplacian on the principal modular curve $\Gamma(p)\backslash \mathbb{H}$ is at least $1 \overline{p}$. Here, $\mathbb{H}$ denotes the hyperbolic plane endowed with the structure of a Riemannian manifold as in the Poincaré half-plane model. Yet, by [Lub94] Theorem 4.3.2, having $\lambda_1$ bound away from zero is equivalent to $SL_2(\mathbb{Z})$ having property $\tau$ with respect to $\{\Gamma(p)\}_{p \in \mathbb{Z}^+}$.

Remark. For more details and a background on the geometry, see [Lub94] Chapter 4 or [Lao15] Section 3.3 respectively.

**Theorem 23.** Let $k$ be a field of characteristic zero. Then the wild Kronecker algebra $k\Theta(3)$ is not of amenable representation type.

Proof. By Proposition [16], it is enough to find a sequence of $\alpha$-dimension expanders of degree two and of unbounded dimension for some $\alpha > 0$. Now, by an application of Proposition [18], it suffices to exhibit a sequence of irreducible, unitary representations $\rho : \Gamma \to U_n(\mathbb{C})$ of unbounded dimension for some group $\Gamma$ with a generating set $S$ of cardinality two, such that the Kazhdan constants $K^\mathbb{R}(S\ell_n(\mathbb{C}), \text{adj } \rho)$ are uniformly bounded from below by a constant $\kappa > 0$.

We let $\Gamma = SL_2(\mathbb{Z})$ with generating set $S = \{ (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) \}$. For now, we specialize to $k = \mathbb{C}$. By Corollary 20 there is a sequence $\rho_p : \Gamma \to U_p(\mathbb{C})$ of non-trivial irreducible, unitary representations of unbounded dimension. Moreover, by Theorem 22 $SL_2(\mathbb{Z})$ has property $\tau$ with respect to $\{\Gamma(p)\}$, i.e. there is a constant $\kappa > 0$ such that if $(H, \sigma)$ is a non-trivial unitary irreducible representation of $SL_2(\mathbb{Z})$ whose kernel contains $\Gamma(p)$ for some $p \in \mathbb{P}$, then the Kazhdan constant $K^\mathbb{R}(H, \sigma) > \kappa$. Yet, by the remarks following Proposition 18 the $(S\ell_p(\mathbb{C}), \text{adj } \rho_p)$ are unitary representations factoring through $SL(2, \mathbb{C})$, i.e. their kernels contain $\Gamma(p)$, and they do not contain non-trivial fixed vectors, so are irreducible. Thus, for their Kazhdan constants we have that $K^\mathbb{R}(S\ell_p(\mathbb{C}), \text{adj } \rho_p) > \kappa$.

The case for general $k$ follows as in [LZ08] comments after Example 3.4: As $\text{char } k = 0$, $k$ contains $\mathbb{Q}$ and the representations of Corollary 20 are all defined over $\mathbb{Q}$, say $\rho_p : \Gamma \to \text{GL}_p(\mathbb{Q})$. If $|k| \leq N$, then $k$ can be embedded into $\mathbb{C}$ and so can $\text{GL}_p(k) \subset \text{GL}_p(\mathbb{C})$. As the $\rho_p$ factor through a finite group, they can be unitarized over $\mathbb{C}$. We have $\mathbb{C}^p = \mathbb{C} \otimes_k k^p$, thus every $k$-subspace $W \subset k^p$
Remark. This proof shows that it is not necessary for us to use the fact that the group $\text{SL}_2(\mathbb{Z})$ has property $(\tau)$ with respect to all principal congruence subgroups, let alone all congruence subgroups. Our result follows from property $(\tau)$ with respect to infinitely many $\Gamma(p)$ such that $p$ is unbounded. Thus, weaker versions of Selberg’s Theorem should suffice in proving this. For these, see e.g. [Tao15, Section 3.3]. Also see [DSV03, Theorem 4.4.4], where by the use of only elementary methods it is shown that the corresponding graphs are expanders.

Remark. Put $s = (\frac{1}{3}, \frac{1}{3})$ and $t = (\frac{0}{1}, \frac{1}{3})$. Then the desired (counter)example for $k\Theta(3)$ is given by the family $\{(k^p, k^p), (\text{id}, \rho_p(s), \rho_p(t))\}_{p \in \mathbb{P}}$.

Note that we do have

$$\rho_p(s) = \begin{pmatrix} 0 & \ldots & 0 & -1 & -1 \\ 1 & \ldots & 1 & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & 1 \end{pmatrix} \in \text{GL}_p(\mathbb{Q}), \quad \rho_3(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\rho_5(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho_7(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\rho_{11}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

While the latter matrices $\rho_p(t)$ share a pattern, the strict rule to construct them ad-hoc is unclear. Note that we do have a certain symmetry $a_{i,j} = a_{p+1-i,p+1-j}$.

**Theorem 24.** Let $k$ be any field. Then there exists some $d \geq 3$ such that $k\Theta(d)$ is not of amenable representation type.

**Proof.** Similarly to the proof of Theorem 23 we must find a family of dimension expanders of unbounded dimension. From [DW10, Theorem 1.2], we know that to give an explicit construction of degree-$d$ dimension quasi-expanders, it is sufficient to have an explicit construction of so-called $d$-monotone expander graphs. Note that the authors attribute this to implicit work in the initial publication of [DS11].
An explicit construction of a constant-degree (discrete) monotone expander graph was suggested and outlined in [Bou09] and presented in [BY13, Corollary 2].

Now, given any field $k$, this construction allows us to find degree-$d$ dimension quasi-expanders of arbitrarily large dimension, thus showing that the wild $d$-Kronecker algebras $k\Theta(d)$ are not of amenable representation type. □

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