Weighted energy estimates for wave equation with space-dependent damping term for slowly decaying initial data

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Abstract. This paper is concerned with weighted energy estimates for solutions to wave equation $\partial_t^2 u - \Delta u + a(x)\partial_t u = 0$ with space-dependent damping term $a(x) = |x|^\alpha$ ($\alpha \in [0, 1)$) in an exterior domain $\Omega$ having a smooth boundary. The main result asserts that the weighted energy estimates with weight function like polynomials are given and these decay rate are almost sharp, even when the initial data do not have compact support in $\Omega$. The crucial idea is to use special solution of $\partial_t u = |x|^\alpha \Delta u$ including Kummer’s confluent hypergeometric functions.

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1 Introduction

In this paper we consider the wave equation with space-dependent damping term

$$\begin{cases}
\partial_t^2 u - \Delta u + a(x)\partial_t u = 0, & x \in \Omega, \ t > 0, \\
u(x, t) = 0, & x \in \partial \Omega, \ t > 0, \\
u(x, 0) = u_0(x), \quad \partial_t u(x, 0) = u_1(x), & x \in \Omega,
\end{cases}$$

(1.1)

where $a(x) = |x|^\alpha$ with a parameter $\alpha \geq 0$, $\Omega$ is an exterior domain in $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary and satisfies $0 \not\in \overline{\Omega}$ and the initial data $(u_0, u_1)$ satisfy the compatibility condition of order 1, that is,

$$(u_0, u_1) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega).$$

(1.2)

If $\alpha = 0$, then (1.1) becomes the usual damped wave equation. Although we can take $\Omega = \mathbb{R}^N$, we skip the case of whole space for the simplicity of terminology. The term $a(x)\partial_t u$ describes the damping effect, which plays a role in reducing the energy of the wave. We remark that the coefficient $a(x)$ of the damping term is uniformly bounded in $\Omega$ and therefore it is well-known that (1.1) has a unique solution $u$ in the following class:

$$u \in C^2([0, \infty); L^2(\Omega)) \cap C^1([0, \infty); H^1_0(\Omega)) \cap C([0, \infty); H^2(\Omega) \cap H^1_0(\Omega)).$$

(1.3)

(see Ikawa [5] Theorem 2).

Our purpose of this paper is to establish the weighted energy estimates and the asymptotic behavior of solutions to (1.1) without assuming the compactness of the support of initial data.

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Matsumura [13] proved that if $\Omega = \mathbb{R}^N$ and $a(x) \equiv 1$, then the solution $u$ of (1.1) satisfies the energy decay estimate
\[
\int_{\mathbb{R}^N} \left( |\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2 \right) \, dx \leq C (1 + t)^{\frac{N-1}{2}} \|(u_0, u_1)\|_{(H^1 \cap L^4) \times (L^2 \cap L^4)}^2.
\]
Later on, it is shown in [29, 10, 12, 18, 4, 17] that if $\Omega$ is an exterior domain, then cases with a symmetric. Similar weighted energy estimates for higher derivatives of solutions are also shown in [7, 3].

For decay estimate energy estimates obtained in the first part.

The study of asymptotic behavior of solutions to (1.1) as a solution of (2.1) as an application of weighted energy estimates is quite few; note that Ikehata gave one of weighted energy estimates in [14, 16] that they actually treated the case the damping coefficient also depends on $t$. These works clarify that the threshold of diffusion phenomena is $\alpha = 1$.

We would summarize that if the initial data are not compactly supported, then a kind of weighted energy estimates is also dealt with the energy decay of solutions to (1.1) for general cases with $a(x) \geq \langle x \rangle^{-\alpha} = (1 + |x|^2)^{-\alpha} 0 \leq \alpha \leq 1$. On the other hand, Mochizuki [16] showed that if $0 \leq a(x) \leq C(\langle x \rangle)^{-\alpha}$ for some $\alpha > 1$, then in general, the energy of the solution to (1.1) does not vanish as $t \to \infty$. Moreover, the solution is asymptotically equivalent with the free wave equation. We remark that they actually treated the case the damping coefficient also depends on $t$. These works clarify that the threshold of diffusion phenomena is $\alpha = 1$.

After that certain decay estimates for weighted energy
\[
\int_{\Omega} \left( |\nabla u|^2 + |\partial_t u|^2 \right) \exp \left( \frac{c_\delta |x|^{2-\alpha}}{1 + t} \right) \, dx \leq C_\delta (1 + t)^{\frac{N-1}{2}} \int_{\Omega} \left( |u_0|^2 + |\nabla u|^2 + |u_1|^2 \right) \exp \left( c_\delta |x|^{2-\alpha} \right) \, dx
\]
(with any $\delta > 0$ and some $c_\delta, C_\delta > 0$) have been proved by Ikehata [8] (without compactness of support of initial data) and Todorova–Yordanov [20] and Radu–Todorova–Yordanov [21] when $a(x)$ is radially symmetric. Similar weighted energy estimates for higher derivatives of solutions are also shown in Radu–Todorova–Yordanov [20]. In the case $\Omega = \mathbb{R}^N$ and $a(x) = \langle x \rangle^{-\alpha}$, the second author proved in [28] that the solution $u$ of (1.1) has the same asymptotic behavior as the one of the following parabolic problem
\[
\begin{aligned}
\partial_t v - a(x)^{-1} \Delta v &= 0, \quad x \in \mathbb{R}^N, \ t > 0, \\
v(x, 0) &= u_0(x), \quad x \in \mathbb{R}^N.
\end{aligned}
\]
Then in [23, 24] the problem (1.1) in an exterior domain with non-radially symmetric damping terms satisfying
\[
\lim_{|x| \to \infty} \left( |x|^\alpha a(x) \right) = a_0 > 0
\]
could be considered and it is shown that the asymptotic behavior of solutions to (1.1) can be also given by the solution of (2.1), however, only when the initial data are compactly supported.

We would summarize that if the initial data are not compactly supported, then a kind of weighted energy estimates is quite few; note that Ikehata gave one of weighted energy estimates in [8] but the initial data are required to have an exponential decay. The study of asymptotic behavior of solutions seems difficult to treat without weighted energy estimates.

The first purpose of this paper is to establish a weighted energy estimates for solutions to (1.1) with a typical damping $a(x) = |x|^{-\alpha}$, which can be applied to initial data with polynomial decay. The second is to find the asymptotic behavior of solutions to (1.1) as a solution of (2.1) as an application of weighted energy estimates obtained in the first part.

Now we are in a position to state our first result.
Remark proved that if \( \Phi \) this means that if \( \alpha \in C \). In Ikehata [1], [2] essentially used functions similar to Gaussian function \( \Phi(x, t) = (1 + t)^{-N/2a} e^{-|x|^2/(2a)} \).

**Theorem 1.1.** Let \( u \) be a unique solution of \( (1.1) \) with initial data \((u_0, u_1) \in (H^2 \cap H^{1}_{0}(\Omega)) \times H^{1}_{0}(\Omega) \). Assume that \( N \geq 2 \) and there exists \( \gamma \in [\alpha, N + 2 - 2\alpha] \) such that

\[
\int_{\Omega} ((|\nabla u_0(x)|^2 + (u_1(x))^2)|x|^\gamma \, dx < \infty. \tag{1.6}
\]

Then there exists a constant \( C > 0 \) such that

\[
\begin{align*}
\int_{\Omega} ((|\nabla u(x, t)|^2 + (\partial_{t} u(x, t))^2)(t + |x|^{2-\alpha})^\frac{\alpha}{2} \, dx &+ \int_{0}^{t} \left( \int_{\Omega} (|\partial_{t} u(x, s)|^2 |u(x, s)|^2 (s + |x|^{2-\alpha})^\frac{\alpha}{2} \, dx \right) \, ds \\
&\leq \left\{ C \int_{\Omega} ((|\nabla u_0(x)|^2 + (u_1(x))^2)|x|^\gamma \, dx + \int_{\Omega} |u_0(x)|^2 |x|^{-\alpha} \, dx \right. \text{ if } \gamma < 2 - \alpha, \tag{1.7} \\
&\left. C \int_{\Omega} ((|\nabla u_0(x)|^2 + (u_1(x))^2)|x|^\gamma \, dx \right.ight. \text{ otherwise.}
\end{align*}
\]

**Remark 1.1.** In the case \( \gamma = 2 - \alpha \), we can use the weighted Hardy inequality

\[
\left( \frac{N - \alpha}{2} \right)^2 \int_{\Omega} |u|^2 |x|^{-\alpha} \, dx \leq \int_{\Omega} |\nabla u|^2 |x|^{2-\alpha} \, dx
\]

which implies that \((1.7)\) can be regarded as an estimate continuously depending on \( \alpha \) (see the proof of Proposition 3.4 in page 14).

**Remark 1.2.** If \( \alpha = 0 \) and \( \gamma \leq 2 \), then the assertion of Theorem 1.1 does not have novelty. Indeed, [22] proved that if \( \alpha = 0 \), then

\[
\int_{\Omega} ((|\nabla u(x, t)|^2 + |\partial_{t} u(x, t)|^2) \, dx = O(t^{-2})
\]

as \( t \to \infty \) by only assuming \( \nabla u_0, |x|^2 (u_0 + u_1) \in L^2(\Omega) \). Therefore Theorem 1.1 is meaningful when either \( \alpha \in (0, 1) \) or \( \gamma > 2 \) is satisfied.

**Remark 1.3.** Here we point out our basic idea of the choice of weight functions in the energy functional. If \( \Phi \in C^2(\mathbb{R}^N \times [0, \infty)) \) is a positive function, then by putting \( w = \Phi^{-1}_v \), we can formally compute the following weighted estimate for \((2.1)\):

\[
\frac{d}{dt} \int_{\Omega} |v|^2 a(x) \Phi^{-1} \, dx = \frac{2}{\alpha} \int_{\partial \Omega} v \partial_{\nu} v a(x) \Phi^{-1} \, d\Sigma - \frac{1}{\alpha} \int_{\Omega} |v|^2 a(x) \Phi^{-2} \partial_{\nu} \Phi \, dx \\
= \frac{2}{\alpha} \int_{\partial \Omega} v \Delta v \Phi^{-1} \, d\Sigma - \frac{1}{\alpha} \int_{\Omega} |v|^2 \Phi^{-2} \Phi \, dx - \frac{1}{\alpha} \int_{\Omega} |v|^2 \Phi^{-2} \left( a(x) \partial_{\nu} \Phi - \Delta \Phi \right) \, dx \\
= \frac{2}{\alpha} \int_{\partial \Omega} w \Delta (\Phi w) \, d\Sigma - \frac{1}{\alpha} \int_{\Omega} |w|^2 \Delta \Phi \, dx - \frac{1}{\alpha} \int_{\Omega} |w|^2 \left( a(x) \partial_{\nu} \Phi - \Delta \Phi \right) \, dx.
\]

Integration by parts we have

\[
\frac{d}{dt} \int_{\Omega} |v|^2 a(x) \Phi^{-1} \, dx \leq -2 \int_{\Omega} |\nabla w|^2 \Phi \, dx - \int_{\Omega} |w|^2 \left( a(x) \partial_{\nu} \Phi - \Delta \Phi \right) \, dx.
\]

This means if \( \Phi \) is a positive (super-)solution of \((2.1)\), then the value \( \int_{\Omega} |v|^2 a(x) \Phi^{-1} \, dx \) is decreasing. Therefore an \( L^2 \)-estimate with weighted measure \( a(x) \, dx \) can be proved directly from an \( L^{\infty} \)-estimate for \( \Phi \). In Ikeda [8], Todorova–Yordanov [26], [28], [23] and [24] essentially used functions similar to Gaussian function

\[
\Phi(x, t) = (1 + t)^{-N/2a} e^{-|x|^2/(2a)}
\]
Theorem 1.2. Let $u$ be a solution to the heat equation (2.1) for initial data for a certain class which also contains functions behave like polynomials.

Next we consider diffusion phenomena. To state the result, we introduce the heat semigroup corresponding to (2.1) as follows:

\[
L^p_{d\mu} = \left\{ f \in L^p_{\text{loc}}(\Omega) \mid \|f\|_{L^p_{\mu}} := \left( \int_{\Omega} |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} < \infty \right\},
\]

and $L_{\mu} u = a(x)^{-1} \Delta u$, with domain $D(L_{\mu}) = \{ u \in C^2(\mathbb{R}^N) \, \mu = 0 \}$; note that the operator $-L_{\mu}$ is nonnegative and symmetric in $L^2_{d\mu}$. Define $-L_{\alpha}$ as the Friedrichs extension of $-L_{\mu}$ (for the precise definition see Lemma 2.6).

Then the statement of diffusion phenomena is the following:

**Theorem 1.2.** Let $(u_0, u_1)$ satisfy the compatibility condition of order 1. Assume that there exists $\gamma \in (2 - \alpha, N + 2 - 2\alpha)$ such that

\[
\mathcal{E}_0 := \int_{\Omega} (|\nabla u_0|^2 + |u_1|^2) |x|^{\gamma} \, dx < \infty, \quad \mathcal{E}_1 := \int_{\Omega} (|\nabla u_1|^2 + |u_2|^2) |x|^{\gamma+2} \, dx < \infty
\]

with $u_2 = -\Delta u_0 + a(x) u_1$. Then $u_0 + |x|^{\alpha} u_1 \in L^2_{d\mu}$ and there exists a constant $C > 0$ such that

\[
\|u(t) - e^{tL}[u_0 + |x|^{\alpha} u_1]\|_{L^2_{d\mu}} \leq C(1 + t)^{-\frac{\gamma}{2N - 2\alpha}} (\mathcal{E}_0 + \mathcal{E}_1)^{\frac{1}{2}}.
\]

**Remark 1.4.** Under the assumption in Theorem 1.2 we have $u_0, |x|^{\alpha} u_1 \in L^p_{d\mu}$ for $p \in \left( \frac{2(N-\alpha)}{N+\gamma-2}, 2 \right]$ by the simple calculation with Hölder’s inequality

\[
\| |x|^{\alpha} u_1 \|_{L^p_{d\mu}} \leq \left( \int_{\Omega} |u_1|^2 |x|^{\gamma} \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |x|^{\gamma \frac{N + \gamma - 2}{2N - 2\alpha} (p - \frac{2(N-\alpha)}{N+\gamma-2})} \, dx \right)^{\frac{1}{2}} < \infty.
\]

This gives a decay estimate for $e^{tL}[u_0 + |x|^{\alpha} u_1]$ as

\[
\|e^{tL}[u_0 + |x|^{\alpha} u_1]\|_{L^2_{d\mu}} \leq C(1 + t)^{-\frac{\gamma}{2N - 2\alpha} + \frac{\gamma}{2N - 2\alpha} + \epsilon}\|u_0 + |x|^{\alpha} u_1\|_{L^p_{d\mu}}
\]

for sufficiently small $\epsilon > 0$ (see also Remark 2.5 in page 9). Therefore the estimate (1.8) enables us to determine the asymptotic behavior of solution to the problem (1.1) with non-compactly supported initial data.

Finally, we give a corollary of Theorems 1.1 and 1.2 with decay estimates similar to corresponding heat equation (2.1) for initial data for a certain class which also contains functions behave like polynomials.

**Corollary 1.3.** Let $(u_0, u_1)$ satisfy the compatibility condition of order 1. Assume that

\[
\int_{\Omega} (|\nabla u_0|^2 + |u_1|^2) |x|^{N+2-2\alpha} \, dx < \infty.
\]
Then for every \( \varepsilon > 0 \),
\[
\int_{\Omega} \left( |\nabla u(x, t)|^2 + (\partial_t u(x, t))^2 \right) \left( t + |x|^{2-\alpha} \right)^{\frac{\alpha}{2+1-\frac{\alpha}{2}}} \, dx = O(1)
\]
as \( t \to \infty \). Moreover, further assume that
\[
\int_{\Omega} \left( |\nabla u_1|^2 + |\Delta u_0 - a(x) u_1|^2 \right) |x|^{N-2\alpha} \, dx < \infty.
\]
Then for every \( \varepsilon > 0 \),
\[
\left\| u(t) - e^{tL} [u_0 + |x|^{\alpha} u_1] \right\|_{L^2_{\mu}} = O(t^{-\frac{N}{2+1-\alpha} + \frac{1}{2}}} + \varepsilon^{\frac{1}{2}}),
\]
\[
\left\| e^{tL} [u_0 + |x|^{\alpha} u_1] \right\|_{L^2_{\mu}} = O(t^{-\frac{N}{2+1-\alpha} + \varepsilon})
\]
as \( t \to \infty \).

This paper is organized as follows. In Section 2, we construct a suitable weight function from a self-similar solution of (2.1) with a parameter, which is quite different from that in [8], [26], [21], [28], [23] and [24] as mentioned before. We also mention the properties of the semigroup generated by \( L = a(x) \Delta \).

In Section 3, the weighted energy estimates for solutions to (1.1) are proved. Section 4 is devoted to show ones for higher derivatives. Finally, the asymptotic behavior of solutions to (1.1) is given in Section 5.

2 Preliminaries

2.1 Self-similar solution and Kummer’s confluent hypergeometric function

To construct a suitable weight function, we start with a construction of radially symmetric self-similar solutions of the following heat equation
\[
\partial_t \Phi = |x|^\alpha \Delta \Phi, \quad x \in \mathbb{R}^N, \ t > 0.
\]
If \( \Phi \) is a solution of (2.1). Then we easily see that
\[
\lambda^{-\tilde{\beta}} \Phi(\lambda x, \lambda^{2-\alpha} t)
\]
with \( \tilde{\beta} \in \mathbb{R} \) is also a solution of (2.1). The following is the characterization of radially symmetric self-similar solutions.

Lemma 2.1. Let \( \tilde{\beta} \in \mathbb{R} \). A radial solution \( \Phi \) of (2.1) satisfies
\[
\Phi(x, t) = \lambda^{\tilde{\beta}} \Phi(\lambda x, \lambda^{2-\alpha} t), \quad x \in \mathbb{R}^N \setminus \{0\}, \ t > 0
\]
for every \( \lambda > 0 \) if and only if
\[
\Phi(x, t) = \frac{1}{t^{\frac{\alpha}{2}}} \varphi \left( \frac{|x|^{2-\alpha}}{2(2-\alpha)^2 t} \right), \quad x \in \mathbb{R}^N \setminus \{0\}, \ t > 0
\]
where \( \varphi \in C^\infty((0, \infty)) \) satisfies
\[
\varphi''(s) + \left( \frac{N-\alpha}{2-\alpha} + s \right) \varphi'(s) - \frac{\tilde{\beta}}{2-\alpha} \varphi(s) = 0, \quad s > 0.
\]
Proof. By direct calculation, we can verify that the function \( t^{\frac{2}{2-\alpha}} \varphi \left( \frac{2s}{(2-\alpha)^2t} \right) \) satisfies both (2.1) and (2.2) if \( \varphi \) satisfies (2.4). Conversely, let \( \Phi \) satisfy (2.1) and (2.2). Then choosing \( \lambda = \frac{(2-\alpha)^2}{2-\alpha} \), we see by (2.2) that

\[
\Phi(x, t) = \left( -\frac{2}{2-\alpha} \right) t^{\frac{2}{2-\alpha}} \Phi \left( \frac{x}{t} \right), \quad x \in \mathbb{R}^N \setminus \{0\}, t > 0.
\]

Therefore noting that \( \Phi \) is radial, by taking

\[
\varphi(s) = (2-\alpha) \frac{2s}{2-\alpha} \Phi \left( \frac{s}{2-\alpha} \right), \quad s \in \mathbb{R}^N \setminus \{0\}, t > 0,
\]

we have (2.3) and (2.4). \( \square \)

**Lemma 2.2.** Let \( \beta \in \mathbb{R} \). Assume that \( \varphi \) satisfies

\[
s \varphi''(s) + \left( \frac{N-\alpha}{2-\alpha} + s \right) \varphi'(s) + \beta \varphi(s) = 0, \quad s > 0
\]

with \( \lim_{s \to 0} \varphi(s) = 1 \). Then

\[
\varphi(s) = e^{-s} M \left( \frac{N-\alpha}{2-\alpha} - \beta, \frac{N-\alpha}{2-\alpha}; s \right),
\]

where \( M(\cdot, \cdot; \cdot) \) is the Kummer’s confluent hypergeometric function (see Definition [A.1]).

**Proof.** Taking \( \psi(s) = e^s \varphi(s) \), we see from the direct calculation that \( \psi \) satisfies Kummer’s confluent hypergeometric differential equation

\[
s \psi''(s) + \left( \frac{N-\alpha}{2-\alpha} - s \right) \psi'(s) - \left( \frac{N-\alpha}{2-\alpha} - \beta \right) \psi(s) = 0, \quad s > 0.
\]

Therefore Lemma [A.1](i) yields that \( \psi(s) \) can be given by

\[
\psi(s) = k_1 M \left( \frac{N-\alpha}{2-\alpha} - \beta, \frac{N-\alpha}{2-\alpha}; s \right) + k_2 U \left( \frac{N-\alpha}{2-\alpha} - \beta, \frac{N-\alpha}{2-\alpha}; s \right), \quad s > 0.
\]

For some \( k_1, k_2 \in \mathbb{R} \). Since \( \psi(s) \) and \( M(b, c; s) \) converge to 1 as \( s \to 0 \) and \( U(b, c; s) \) is unbounded at \( s = 0 \), we have \( k_2 = 0 \) and then

\[
1 = \lim_{s \to 0} \psi(s) = k_1 \lim_{s \to 0} M \left( \frac{N-\alpha}{2-\alpha} - \beta, \frac{N-\alpha}{2-\alpha}; s \right) = k_1.
\]

By the definition of \( \psi \), the proof is complete. \( \square \)

Let us fix the notation of concrete functions which we use later.

**Definition 2.1.** For \( \beta \in \mathbb{R} \), define

\[
\varphi_\beta(s) = e^{-s} M \left( \frac{N-\alpha}{2-\alpha} - \beta, \frac{N-\alpha}{2-\alpha}; s \right), \quad s \geq 0.
\]

**Lemma 2.3 (Properties of \( \varphi_\beta \)).** The following assertions hold:

- (i) For every \( \beta \in \mathbb{R} \),

\[
s \varphi_\beta''(s) + \left( \frac{N-\alpha}{2-\alpha} + s \right) \varphi_\beta'(s) + \beta \varphi_\beta(s) = 0, \quad s > 0.
\]
\[ \textbf{Proof.} \, \textbf{(i)} \text{ The assertion is directly verified by Definition 2.1 and Lemma 2.2.} \\
\textbf{(ii)} \text{ Set } \bar{\varphi}(s) := s\varphi'_\beta(s) + \beta\varphi(s) \text{ for } s \geq 0. \text{ Then we see from (2.6) that } \\
\bar{\varphi}'(s) = s\varphi''_\beta(s) + \beta + 1)\varphi'_\beta(s) \\
\quad = -\left(\frac{N-\alpha}{2-\alpha} s\right)\varphi'_\beta(s) - \beta\varphi(s) + (\beta + 1)\varphi'_\beta(s) \\
\quad = -\left(\frac{N-\alpha}{2-\alpha} - \beta - 1\right)\varphi'_\beta(s) - \bar{\varphi}(s) \\
\text{ and therefore } \\
s\varphi''_\beta(s) + s\bar{\varphi}(s) = -\left(\frac{N-\alpha}{2-\alpha} - \beta - 1\right)s\varphi'_\beta(s) \\
\quad = \left(\frac{N-\alpha}{2-\alpha} - \beta - 1\right)\left(\frac{N-\alpha}{2-\alpha} s\right)\varphi'_\beta(s) + \beta\varphi(s) \\
\quad = \frac{N-\alpha}{2-\alpha} \left(\frac{N-\alpha}{2-\alpha} - \beta - 1\right)\varphi'_\beta(s) + \left(\frac{N-\alpha}{2-\alpha} - \beta - 1\right)\bar{\varphi}(s) \\
\quad = -\frac{N-\alpha}{2-\alpha} \bar{\varphi}'(s) + \bar{\varphi}(s) + \left(\frac{N-\alpha}{2-\alpha} - \beta - 1\right)\bar{\varphi}(s) \\
\quad = -\frac{N-\alpha}{2-\alpha} \bar{\varphi}'(s) - (\beta + 1)\bar{\varphi}(s). \\
\text{This means that } \bar{\varphi}_\beta \text{ satisfies with (2.6) with } \beta \text{ replaced with } \beta + 1. \text{ Noting that } \varphi'_\beta(0) = 1 - \frac{(2-\alpha)\beta}{N-\alpha}, \text{ we have } \lim_{s \to 0} \bar{\varphi}(s) = \beta \text{ and therefore by Lemma 2.2 we deduce } \bar{\varphi}(s) = \beta\varphi(s) \text{ for all } s \geq 0. \\
\textbf{(iii).} \text{ By Lemma A.1 \textbf{(ii)}, we see that there exists } R_\beta > 0 \text{ such that for every } s \geq R_\beta, \\
\text{ or equivalently, for every } s \geq R_\beta, \\
1 \leq \frac{\Gamma\left(\frac{N-\alpha}{2-\alpha} - \beta\right)}{\Gamma\left(\frac{N-\alpha}{2-\alpha} - \beta\right)} \leq M\left(\frac{N-\alpha}{2-\alpha} - \beta, \frac{N-\alpha}{2-\alpha}, s\right) \leq \frac{3}{2} \frac{\Gamma\left(\frac{N-\alpha}{2-\alpha} - \beta\right)}{\Gamma\left(\frac{N-\alpha}{2-\alpha} - \beta\right)} \leq s^{-\beta}. \quad (2.7) \\
\text{Since } \varphi_\beta \text{ is continuous in } [0, R_\beta], \text{ we verify the assertion.} \\
\textbf{(iv)} \text{ Since } \beta < \frac{N-\alpha}{2-\alpha} \text{ is satisfied, it follows from the definition of } M(b, c, s) \text{ in Appendix (see page 23) that } \\
\varphi_\beta(s) > 0 \text{ for all } s \geq 0. \text{ From (2.7) and the positivity of } \varphi_\beta, \text{ we have also lower bounds with the same power } s^{-\beta}. \quad \square \]
2.2 Construction of weight functions via $\varphi_\beta$

Here we define the suitable weight functions for weighted energy estimates for solutions of (1.1).

**Definition 2.2.** For $\beta \in \mathbb{R}$, define

$$\Phi_\beta(x, t) = t^{-\beta} \varphi_\beta \left( \frac{|x|^{2-a}}{(2-a)^2 t} \right), \quad x \in \mathbb{R}^N, \ t > 0.$$ 

To state the properties of $\Phi_\beta$, we also introduce

$$\Psi_\beta(x, t) = \left( t + \frac{|x|^{2-a}}{(2-a)^2 t} \right)^\beta, \quad x \in \mathbb{R}^N, \ t > 0.$$ 

**Lemma 2.4 (Properties of $\Phi_\beta$).** The following assertions hold:

(i) For every $\beta \in \mathbb{R}$,

$$\partial_t \Phi_\beta(x, t) = |x|^a \Delta \Phi_\beta(x, t), \quad x \in \mathbb{R}^N, \ t > 0.$$ 

(ii) For every $\beta \in \mathbb{R}$,

$$\partial_t \Phi_\beta(x, t) = -\beta \Phi_{\beta+1}(x, t) \quad x \in \mathbb{R}^N, \ t > 0.$$ 

(iii) For every $\beta \in \mathbb{R}$,

$$|\Phi_\beta(x, t)| \leq C_\beta \Psi^{-\beta}(x, t), \quad x \in \mathbb{R}^N, \ t > 0.$$ 

(iv) For every $\beta < \frac{N-a}{2-a}$,

$$\Phi_\beta(x, t) \geq c_\beta \Psi^{-\beta}(x, t), \quad x \in \mathbb{R}^N, \ t > 0.$$ 

**Proof.** (iii) and (iv) directly follow from the corresponding assertions in Lemma 2.3. (i) is a consequence of Lemmas 2.1, 2.2 and Lemma 2.3(i). The equality in (ii) can be proved by Lemma 2.3(ii) as follows:

$$\partial_t \Phi_\beta(x, t) = -\beta t^{-\beta-1} \varphi_\beta \left( \frac{|x|^{2-a}}{(2-a)^2 t} \right) - t^{-\beta} \varphi_\beta' \left( \frac{|x|^{2-a}}{(2-a)^2 t} \right) \frac{|x|^{2-a}}{(2-a)^2 t^2}.$$ 

$$= -t^{-\beta-1} \left[ \beta \varphi_\beta \left( \frac{|x|^{2-a}}{(2-a)^2 t} \right) + \varphi_\beta' \left( \frac{|x|^{2-a}}{(2-a)^2 t} \right) \frac{|x|^{2-a}}{(2-a)^2 t} \right]$$

$$= -t^{-\beta-1} \varphi_{\beta+1} \left( \frac{|x|^{2-a}}{(2-a)^2 t} \right)$$

$$= -\beta \Phi_{\beta+1}(x, t).$$

The proof is complete. \(\square\)

**Remark 2.1.** Since the asymptotic behavior of $\varphi_\beta(s)$ at $s = \infty$ is explicitly given via Lemma A.1(ii), we can deduce that for every $x \in \mathbb{R}^N \setminus \{0\}$,

$$\lim_{t \downarrow 0} \Phi_\beta(x, t) = \frac{\Gamma(\frac{N-a}{2-a})}{\Gamma\left( \frac{N-a}{2-a} - \beta \right)} |x|^{(2-a)\beta}.$$ 

This means that $\Phi_\beta$ is a solution of (2.1) with initial value $\frac{\Gamma(\frac{N-a}{2-a})}{\Gamma\left( \frac{N-a}{2-a} - \beta \right)} |x|^{(2-a)\beta}$.

**Remark 2.2.** It follows from Lemma 2.4(iii) and (iv) that if $\beta < \frac{N-a}{2-a}$, then $\Phi_{\beta+1}$ and $\Psi_\beta$ are equivalent in the sense of weighted functions:

$$c_\beta^{-1} \Psi_\beta(x, t) \leq \Phi_\beta(x, t)^{-1} \leq C_\beta^{-1} \Psi_\beta(x, t).$$
2.3 Semigroup generated by $|x|^a \Delta$ with Dirichlet boundary condition

Here we collect the statements of the properties of the semigroup generated by $|x|^a \Delta$ with Dirichlet boundary condition in a weighted $L^2$-space $L^2_{dp}$.

First we introduce the bilinear form
\[
\begin{cases}
   a(u,v) := \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx, \\
   D(a) := \{ u \in C_c^{\infty}(\Omega) : u(x) = 0 \quad \forall x \in \partial \Omega \}
\end{cases}
\]

in a Hilbert space $L^2_{dp}$. Then the form $a$ is closable, and therefore, we denote $a_*$ as a closure of $a$. Then we remark the following four lemmas stated in [23].

Lemma 2.5 ([23] Lemma 2.1). The bilinear form $a_*$ can be characterized as follows:
\[
D(a_*) = \left\{ u \in L^2_{dp} \cap H^1(\Omega) : \int_{\Omega} \frac{\partial u}{\partial x_j} \varphi \, dx = - \int_{\Omega} u \frac{\partial \varphi}{\partial x_j} \, dx \quad \forall \varphi \in C_c^{\infty}(\mathbb{R}^N) \right\},
\]
\[
a_*(u,v) = \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx.
\]

Lemma 2.6 ([23] Lemma 2.2). The operator $-L_*$ in $L^2_{dp}$ defined by

\[
D(L_*) := \left\{ u \in D(a_*) : \exists f \in L^2_{dp} \text{ s.t. } a_*(u,v) = (f,v)_{L^2_{dp}} \quad \forall v \in D(a_*) \right\},
\]

\[-L_* u := f\]

is nonnegative and selfadjoint in $L^2_{dp}$. Therefore $L_*$ generates an analytic semigroup $e^{L_* t}$ on $L^2_{dp}$ and satisfies
\[
\|e^{L_* t} f\|_{L^2_{dp}} \leq \|f\|_{L^2_{dp}}, \quad \|L_* e^{L_* t} f\|_{L^2_{dp}} \leq \frac{1}{t} \|f\|_{L^2_{dp}}, \quad \forall f \in L^2_{dp}.
\]

Furthermore, $L_*$ is an extension of $L$ defined on $C_c^{\infty}(\Omega)$ with Dirichlet boundary condition.

Lemma 2.7 ([23] Lemma 2.3). We have
\[
\{ u \in H^2(\Omega) \cap H^1_0(\Omega) : a(x)^{-\frac{1}{2}} \Delta u \in L^2(\Omega) \} \subset D(L_*)
\]

and its inclusion is continuous.

Lemma 2.8 ([23] Proposition 2.6). Let $e^{L_* t}$ be given in Lemma 2.6. For every $f \in L^2_{dp}$, we have
\[
\|e^{L_* t} f\|_{L^\infty} \leq C t^{-\frac{N(a)}{2(a-1)^2}},
\]
(2.10)

Moreover, for every $f \in L^1_{dp} \cap L^2_{dp}$ we have
\[
\|e^{L_* t} f\|_{L^2_{dp}} \leq C t^{-\frac{N(a)}{2(a-1)^2}} \|f\|_{L^2_{dp}}.
\]
(2.11)

and
\[
\|L_* e^{L_* t} f\|_{L^2_{dp}} \leq C t^{-\frac{N(a)}{2(a-1)^2}} \|f\|_{L^1_{dp}}.
\]
(2.12)

Remark 2.3. Applying the Riesz–Thorin theorem, we deduce from (2.11) and Lemma 2.6 that the following $L^2_{dp} \cap L^p_{dp}$ estimates with $1 \leq p \leq 2$ also hold:
\[
\|L_* e^{L_* t} f\|_{L^2_{dp}} \leq C t^{-\frac{N(a)}{2(a-1)^2}} \|f\|_{L^p_{dp}}. \quad f \in L^p_{dp} \cap L^2_{dp}.
\]
(2.13)
3 Weighted energy estimates

In this section we consider the weighted energy estimates for solutions of (1.1). First we construct them for compactly supported initial data. Then by the standard approximation argument we establish them for all reasonable initial data. The crucial point is to derive several estimates which are uniform for the size of the support of initial data.

3.1 Weighted energy estimates with compactly supported initial data

For simplicity we will use

\[ \mathcal{H}_c = \{(f, g) \in (H^2 \cap H^1_0(\Omega)) \times H^1_0(\Omega) : f \text{ and } g \text{ are compactly supported in } \mathbb{R}^N\}. \]

Then the finite propagation property gives the following lemma.

Lemma 3.1. Let \( u \) be a solution of (1.1) with \((u_0, u_1) \in \mathcal{H}_c\). Then \((u(t), \partial_t u(t)) \in \mathcal{H}_c\) for every \( t \geq 0 \).

3.1.1 Estimates for \( \nabla u \) and \( \partial_t u \) with weight function \( \Psi^\beta \)

Here we define the weighted energy functionals which are useful in the present paper.

Definition 3.1. For \( \beta \in \mathbb{R} \) and for the solution \( w \) of (1.1) with initial data \((f, g) \in (H^2 \cap H^1_0(\Omega)) \times H^1_0(\Omega)\), we define

\[
E^\beta_{\alpha x}[t_0, w](t) := \int_\Omega |\nabla w(x, t)|^2 \Psi^\beta(x, t_0 + t) \, dx,
\]

\[
E^\beta_{\alpha t}[t_0, w](t) := \int_\Omega |\partial_t w(x, t)|^2 \Psi^\beta(x, t_0 + t) \, dx,
\]

\[
E^\beta_{\alpha}[t_0, w](t) := \int_\Omega |w(x, t)|^2 a(x) \Psi^\beta(x, t_0 + t) \, dx.
\]

Note that these are finite in particular if \((f, g) \in \mathcal{H}_c\) (see Lemma 3.1).

Throughout this paper, we will use the notation \( \Psi^\beta = \Psi^\beta(x, t_0 + t) \), for simplicity.

Lemma 3.2. Let \( u \) be a solution of (1.1) with initial data \((u_0, u_1) \in \mathcal{H}_c\) and let \( \beta \geq 0 \). Then there exist constants \( t_1 = t_1(\alpha, \beta) \geq 1 \) and \( K_1 = K_1(\beta) > 0 \) such that if \( t_0 \geq t_1 \), then

\[
\frac{d}{dt} E^\beta_{\alpha x}[t_0, u](t) + E^\beta_{\alpha t}[t_0, u](t) \leq -\frac{1}{2} E^\beta_{\alpha x}[t_0, \partial_t u](t) + K_1 E^{\beta-1}_{\alpha x}[t_0, u](t), \quad t \geq 0.
\]

Proof. Firstly we easily see that

\[
\frac{d}{dt} E^\beta_{\alpha x}[t_0, u](t) = 2 \int_\Omega \partial_t u(\partial_t^\beta u) \Psi^\beta \, dx + \beta \int_\Omega |\partial_t u|^2 \Psi^{\beta-1} \, dx.
\]

Secondly by integration by parts we deduce

\[
\frac{d}{dt} E^\beta_{\alpha x}[t_0, u](t) = 2 \int_\Omega \nabla(\partial_t u) \cdot \nabla \Psi^\beta \, dx + \beta \int_\Omega |\nabla u|^2 \Psi^{\beta-1} \, dx
\]

\[
= -2 \int_\Omega (\partial_t u) \nabla \Psi^\beta \, dx - 2\beta \int_\Omega (\partial_t u) \nabla u \cdot \frac{x|u|^{-\alpha}}{2 - \alpha} \Psi^{\beta-1} \, dx + \beta \int_\Omega |\nabla u|^2 \Psi^{\beta-1} \, dx
\]
Then the Schwarz inequality yields that
\[
\frac{d}{dt} E_{\alpha}^\beta[u_0, u](t) \leq -2 \int_{\Omega} (\partial_t u) \Delta u \Psi^\beta dx + \int_{\Omega} |\partial_t u|^2 |x|^{-\alpha} \Psi^\beta dx \\
+ \beta^2 \int_{\Omega} |\nabla u|^2 \frac{|x|^{2-\alpha}}{(2 - \alpha)^2} \Psi^{\beta-2} dx + \beta \int_{\Omega} |\nabla u|^2 \Psi^{\beta-1} dx
\]
\[
\leq -2 \int_{\Omega} (\partial_t u) \Delta u \Psi^\beta dx + \int_{\Omega} |\partial_t u|^2 |x|^{-\alpha} \Psi^\beta dx + \beta(1 + \beta) \int_{\Omega} |\nabla u|^2 \Psi^{\beta-1} dx.
\]
Therefore combining the above estimates with (1.1) implies that
\[
\frac{d}{dt} \left[ E_{\alpha}^\beta[u_0, u](t) + E_{\alpha}^\beta_0[t_0, u](t) \right] \\
= 2 \int_{\Omega} (\partial_t u)(\partial_t^2 u - \Delta u) \Psi^\beta dx + \beta \int_{\Omega} |\partial_t u|^2 \Psi^{\beta-1} dx \\
+ \int_{\Omega} |\partial_t u|^2 |x|^{-\alpha} \Psi^\beta dx + \beta(1 + \beta) \int_{\Omega} |\nabla u|^2 \Psi^{\beta-1} dx
\]
\[
\leq - \int_{\Omega} |\partial_t u|^2 |x|^{-\alpha} \Psi^\beta dx + \beta \int_{\Omega} |\partial_t u|^2 \Psi^{\beta-1} dx + \beta(1 + \beta) \int_{\Omega} |\nabla u|^2 \Psi^{\beta-1} dx.
\]
Thus by noticing that \(\alpha \geq 0\) and therefore \(\Psi^{-\frac{\alpha}{2\alpha}} \leq (2 - \alpha)^{\frac{\alpha}{2\alpha}} |x|^{-\alpha}\) we obtain
\[
\frac{d}{dt} \left[ E_{\alpha}^\beta[u_0, u](t) + E_{\alpha}^\beta_0[t_0, u](t) \right] \leq \int_{\Omega} |\partial_t u|^2 \left( -1 + \beta(2 - \alpha)^{\frac{\alpha}{2\alpha}} \right) |x|^{-\alpha} \Psi^\beta dx \\
+ \beta(1 + \beta) \int_{\Omega} |\nabla u|^2 \Psi^{\beta-1} dx.
\]
If \(t_0 \geq t_1 := 1 + (2\beta)^{\frac{\alpha}{2\alpha}}(2 - \alpha)^{\frac{\alpha}{2\alpha}}\), we obtain the desired inequality. \(\square\)

3.1.2 Weighted energy estimates for the case \(\beta \leq 1\)

**Lemma 3.3.** Let \(u\) be a solution of (1.1) with initial data \((u_0, u_1) \in \mathcal{H}_c\). Then for every \(t_0 \geq 1\) and \(t \geq 0\),
\[
\frac{d}{dt} \left[ 2 \int_{\Omega} u \partial_t u dx + E_{\alpha}^0[t_0, u](t) \right] = 2E_{\alpha}^0[t_0, u](t) - 2E_{\alpha}^0[t_0, u](t).
\]

**Proof.** By (1.1) we have
\[
\frac{d}{dt} \left[ 2 \int_{\Omega} u \partial_t u dx + \int_{\Omega} |x|^{-\alpha} |u|^2 dx \right] = 2 \int_{\Omega} (\partial_t u)^2 dx + 2 \int_{\Omega} u (\partial_t^2 u + |x|^{-\alpha}\partial_t u) dx \\
= 2 \int_{\Omega} (\partial_t u)^2 dx + 2 \int_{\Omega} u \Delta u dx.
\]
Integration by parts implies the desired assertion. \(\square\)

**Proposition 3.4.** If \(\frac{\alpha}{2 - \alpha} \leq \beta \leq 1\), then there exists a constant \(M_1 = M_1(\alpha, \beta, N) > 0\) such that for every \(t_0 \geq 1\) and \(t \geq 0\),
\[
E_{\alpha}^\beta[t_0, u](t) + E_{\alpha}^\beta_0[t_0, u](t) + E_{\alpha}^\beta_0[t_0, u](t) + \int_{0}^{t} E_{\alpha}^\beta[t_0, \partial_t u](s) ds \\
\leq M_1 \left( \int_{\Omega} (|\nabla u|^2 + |u|^2)^{(2-\alpha)\beta} dx + \int_{\Omega} |u|^2 |x|^{-\alpha} dx \right).
\]
In particular, if \( \beta = 1 \), then one has
\[
E_{\partial\beta}^1[0, u](t) + E_{\partial\alpha}^1[0, u](t) + 2\int_0^t E_{\partial\alpha}^1[0, \partial_t u](s) \, ds \leq M_1 \int_\Omega (|\nabla u_0|^2 + |u_1|^2) |x|^{2-\alpha} \, dx.
\]

Proof. In the case \( \frac{\alpha}{2-\alpha} < \beta \leq 1 \), by Lemmas 3.2 and 3.3 we have
\[
\frac{d}{dt} \left[ E_{\partial\beta}^\alpha[0, u](t) + E_{\partial\alpha}^\beta[0, u](t) + \frac{K_1}{2} \left( 2 \int_\Omega u \partial_t u \, dx + E_{\partial\alpha}^0[0, u](t) \right) \right]
\leq -\frac{1}{2} E_{\partial\alpha}^\beta[0, \partial_t u](t) + K_1 E_{\partial\alpha}^0[0, \partial_t u](t) + K_1 E_{\partial\alpha}^{\beta-1}[0, u](t) - K_1 E_{\partial\alpha}^0[0, u](t).
\]
Taking \( t_0' \geq t_1 \) such that \( 4K_1(2-\alpha)^{\frac{2\alpha}{2-\alpha}} \leq \frac{\beta-\frac{\alpha}{2-\alpha}}{4K_1} \), we deduce
\[
\frac{d}{dt} \left[ E_{\partial\alpha}^\beta[0, u](t) + E_{\partial\beta}^\alpha[0, u](t) + \frac{K_1}{2} \left( 2 \int_\Omega u \partial_t u \, dx + E_{\partial\alpha}^0[0, u](t) \right) \right] \leq -\frac{1}{4} E_{\partial\alpha}^\beta[0, \partial_t u](t).
\]
Noting that
\[
\left| 2 \int_\Omega u \partial_t u \, dx \right| \leq \frac{1}{2} E_{\partial\alpha}^0[0, u](t) + 2 \int_\Omega |\partial_t u|^2 |x|^{2-\alpha} \, dx
\leq \frac{1}{2} E_{\partial\alpha}^0[0, u](t) + 2(2-\alpha)^{\frac{2\alpha}{2-\alpha}} E_{\partial\alpha}^\beta[0, u](t)
\leq \frac{1}{2} E_{\partial\alpha}^0[0, u](t) + \frac{1}{2K_1} E_{\partial\alpha}^\beta[0, u](t),
\]
we have
\[
E_{\partial\alpha}^\beta[0, u](t) + \frac{3}{4} E_{\partial\beta}^\alpha[0, u](t) + \frac{K_1}{4} E_{\partial\alpha}^0[0, u](t) + \frac{1}{2} \int_0^t E_{\partial\alpha}^0[0, \partial_t u](s) \, ds
\leq E_{\partial\alpha}^\beta[0, u](0) + \frac{5}{4} E_{\partial\beta}^\alpha[0, u](0) + \frac{3}{4} \frac{K_1}{4} E_{\partial\alpha}^0[0, u](0).
\]
If \( \beta = 1 \), then a weighted Hardy inequality
\[
\left( \frac{N-\alpha}{2} \right)^2 \int_\Omega |u|^2 |x|^{-\alpha} \, dx \leq \int_\Omega |\nabla u_0|^2 |x|^{2-\alpha} \, dx
\]
implies the desired estimate (see e.g., Mitidieri [15] and also Arendt–Goldstein-Goldstein [11]). On the other hand, if \( \beta = \frac{\alpha}{2-\alpha} \), then
\[
\frac{d}{dt} \left[ E_{\partial\alpha}^\frac{\alpha}{2-\alpha}[0, u](t) + E_{\partial\alpha}^\frac{\alpha}{2-\alpha}[0, u](t) + 2 \int_\Omega u \partial_t u \, dx + E_{\partial\alpha}^0[0, u](t) \right]
\leq -\frac{1}{2} E_{\partial\alpha}^\frac{\alpha}{2-\alpha}[0, \partial_t u](t) + 2 \nu E_{\partial\alpha}^0[0, \partial_t u](t) + K_1 E_{\partial\alpha}^{\frac{\alpha}{2-\alpha}}[0, u](t) - 2 \nu E_{\partial\alpha}^0[0, u](t)
\leq \left( \frac{1}{2} + 2(2-\alpha)^{\frac{2\alpha}{2-\alpha}} \right) E_{\partial\alpha}^\frac{\alpha}{2-\alpha}[0, \partial_t u](t) + (K_1 t_0^\frac{\alpha}{2-\alpha} - 2 \nu) E_{\partial\alpha}^0[0, u](t).
\]
Therefore taking \( \nu = 8^{-1}(2-\alpha)^{\frac{2\alpha}{2-\alpha}} \) and \( t_0 \geq t_1 \) such that \( t_0^\frac{\alpha}{2-\alpha} \geq 4K_1(2-\alpha)^{\frac{2\alpha}{2-\alpha}} \), we obtain the same inequality as the previous case. \( \square \)
3.1.3 Estimates for $u$ with weight function $\Phi_1$

Here we define new weighted energy functional via Kummer’s confluent geometric functions, which plays a crucial role in this paper.

**Definition 3.2.** For $\lambda \in [0, \frac{N-\alpha}{2-\alpha})$, choose $\varepsilon_* \in (0, \frac{1}{\lambda})$ such that $\lambda = (1 - 3\varepsilon_*)\frac{N-\alpha}{2-\alpha}$. Define

$$\lambda_* = \lambda(1 - 2\varepsilon_*)^{-1} < \frac{N-\alpha}{2-\alpha}$$

and

$$E^1_\Phi[t_0, u](t) = \int_\Omega (2u(x,t)\partial_t u(x,t) + a(x)|u|^2) [\Phi_\lambda(x,t_0 + t)]^{-1} dx.$$ 

**Lemma 3.5.** For every $w \in H^1_0(\Omega)$ having a compact support on $\mathbb{R}^N$ and $\lambda > -\frac{N-\alpha}{2-\alpha}$

$$\int_\Omega |w|^2 |x|^{-\alpha} \Psi^{1-1} dx \leq 4 \min\left\{\frac{N-\alpha}{2-\alpha}, \frac{N-2}{2-\alpha} + \lambda \right\} \int_\Omega |\nabla w|^2 \Psi^{1-1} dx. \quad (3.1)$$

**Proof.** Observe that

$$\text{div} (\Psi^{1-1} \nabla \Psi) = \Delta \Psi^{1-1} + (\lambda - 1) \Psi^{1-2} |\nabla \Psi|^2$$

$$= \left[\frac{N-\alpha}{2-\alpha} (t_0 + t) + \left(\frac{N-\alpha}{2-\alpha} + \lambda - 1\right) \frac{|x|^{2-\alpha}}{(2-\alpha)^2}\right] |x|^{-\alpha} \Psi^{1-2}$$

$$\geq \min\left\{\frac{N-\alpha}{2-\alpha}, \frac{N-2}{2-\alpha} + \lambda \right\} |x|^{-\alpha} \Psi^{1-1}.$$ 

On the other hand, integration by parts and Schwarz’s inequality yield that

$$\int_\Omega |w|^2 \text{div} (\Psi^{1-1} \nabla \Psi) dx = 2 \int_\Omega w \nabla w : \nabla \Psi^{1-1} dx$$

$$\leq 2 \left(\int_\Omega |w|^2 |x|^{-\alpha} \Psi^{1-1} dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla w|^2 |x|^{\alpha} |\nabla \Psi^{2} \Psi^{1-1} dx\right)^{\frac{1}{2}}$$

$$\leq 2 \left(\int_\Omega |w|^2 |x|^{-\alpha} \Psi^{1-1} dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla w|^2 \Psi^{1-1} dx\right)^{\frac{1}{2}}.$$ 

Combining the estimates above, we obtain (3.1). $\Box$

**Lemma 3.6.** For $w \in H^1_0(\Omega)$ having a compact support in $\mathbb{R}^N$, one has

$$(1 - \varepsilon_*) \int_\Omega |w|^2 |\nabla \Phi_\lambda|^2 \Phi_\lambda^{3+2\varepsilon_*} dx \leq \frac{1}{1 - \varepsilon_*} \int_\Omega |\nabla w|^2 \Phi_\lambda^{1+2\varepsilon_*} dx + \int_\Omega |w|^2 \Delta \Phi_\lambda \Phi_\lambda^{2+2\varepsilon_*} dx. \quad (3.2)$$

**Proof.** Putting $v = \Phi_\lambda^{1+\varepsilon_*} w$ (that is, $w = \Phi_\lambda^{-\varepsilon_*} v$) and using integration by parts, we have

$$\int_\Omega |\nabla w|^2 \Phi_\lambda^{1+2\varepsilon_*} dx = \int_\Omega |\nabla (\Phi_\lambda^{1+\varepsilon_*} v)|^2 \Phi_\lambda^{1+2\varepsilon_*} dx$$

$$\geq (1 - \varepsilon_*) \left[ 2 \int_\Omega v \nabla v : \nabla \Phi_\lambda dx + (1 - \varepsilon_*) \int_\Omega v^2 |\nabla \Phi_\lambda|^2 \Phi_\lambda^{1-\varepsilon_*} dx \right]$$

$$= (1 - \varepsilon_*) \left[ - \int_\Omega v^2 \Delta \Phi_\lambda dx + (1 - \varepsilon_*) \int_\Omega v^2 |\nabla \Phi_\lambda|^2 \Phi_\lambda^{1-\varepsilon_*} dx \right].$$

Rewriting $v$ in terms of $w$, we deduce (3.2). $\Box$
Lemma 3.7. There exist constants $t_2 = t_2(\lambda, \alpha, N, \eta = \eta(\lambda))$ and $K_2 = K_2(\lambda, \alpha, N) > 0$ such that for every $t_0 \geq t_2$,

$$
\frac{d}{dt} \left[ E_{\Phi}^{1}[t_0, u](t) \right] \leq -\eta E_{\Phi}^{1}[t_0, u](t) + K_2 E_{\Phi}^{1}[t_0, \partial_t u](t), \quad t \geq 0.
$$

Proof. We see from Lemma 3.4 (ii) and (i) that

$$
\begin{align*}
\frac{d}{dt} & \int_{\Omega} \left[ 2u(\partial_t u)\Phi_{\lambda_1}^{-1+2\varepsilon} + d x \right] \\
& = 2 \int_{\Omega} u(\partial^2_t u)\Phi_{\lambda_1}^{-1+2\varepsilon} + d x + 2 \int_{\Omega} \left| \partial(u)^2 \Phi_{\lambda_1}^{-1+2\varepsilon} + d x - (1 - 2\varepsilon) \int_{\Omega} u(\partial_t u) \left[ \partial_t \Phi_{\lambda_1} \right] \Phi_{\lambda_1}^{-2+2\varepsilon} + d x \\
& = 2 \int_{\Omega} u(\partial^2_t u)\Phi_{\lambda_1}^{-1+2\varepsilon} + d x + 2 \int_{\Omega} \left| \partial(u)^2 \Phi_{\lambda_1}^{-1+2\varepsilon} + d x + (1 - 2\varepsilon) \lambda \int_{\Omega} u(\partial_t u) \Phi_{\lambda_1+1} \Phi_{\lambda_1}^{-2+2\varepsilon} + d x \\
\end{align*}
$$

and

$$
\frac{d}{dt} \int_{\Omega} \left[ |u|^2 |x|^{-\alpha} \Phi_{\lambda_1}^{-1+2\varepsilon} + d x \right] = 2 \int_{\Omega} u(\partial^2_t u) + |x|^{-\alpha} \partial_t u) \Phi_{\lambda_1}^{-1+2\varepsilon} + d x + 2 \int_{\Omega} \left| \partial(u)^2 \Phi_{\lambda_1}^{-1+2\varepsilon} + d x - (1 - 2\varepsilon) \int_{\Omega} u(\partial_t u) \left[ \partial_t \Phi_{\lambda_1} \right] \Phi_{\lambda_1}^{-2+2\varepsilon} + d x \\
= 2 \int_{\Omega} u(\partial^2_t u) + |x|^{-\alpha} \partial_t u) \Phi_{\lambda_1}^{-1+2\varepsilon} + d x + 2 \int_{\Omega} \left| \partial(u)^2 \Phi_{\lambda_1}^{-1+2\varepsilon} + d x + (1 - 2\varepsilon) \int_{\Omega} u(\partial_t u) \Phi_{\lambda_1+1} \Phi_{\lambda_1}^{-2+2\varepsilon} + d x.
$$

Combining the equalities above we have

$$
\frac{d}{dt} \left[ E_{\Phi}^{1}[t_0, u](t) \right] = 2 \int_{\Omega} u(\partial^2_t u + |x|^{-\alpha} \partial_t u) \Phi_{\lambda_1}^{-1+2\varepsilon} + d x + 2 \int_{\Omega} \left| \partial(u)^2 \Phi_{\lambda_1}^{-1+2\varepsilon} + d x - (1 - 2\varepsilon) \int_{\Omega} u(\partial_t u) \left[ \partial_t \Phi_{\lambda_1} \right] \Phi_{\lambda_1}^{-2+2\varepsilon} + d x + \lambda \int_{\Omega} u(\partial_t u) \Phi_{\lambda_1+1} \Phi_{\lambda_1}^{-2+2\varepsilon} + d x.
$$

On the other hand, we see from integration by parts and Lemma 3.6 that

$$
2 \int_{\Omega} u(\Delta u) \Phi_{\lambda_1}^{-1+2\varepsilon} + d x
$$

$$
= 2(1 - 2\varepsilon) \int_{\Omega} u(\nabla u \cdot \nabla \Phi_{\lambda_1}) \Phi_{\lambda_1}^{-2+2\varepsilon} + d x - 2 \int_{\Omega} |\nabla u|^2 \Phi_{\lambda_1}^{-1+2\varepsilon} + d x,
$$

which simplifies to

$$
= (1 - 2\varepsilon) \left[ - \int_{\Omega} |u|^2 (\Delta u)^2 \Phi_{\lambda_1}^{-2+2\varepsilon} + d x + 2 (1 - \varepsilon) \int_{\Omega} |u|^2 |\nabla \Phi_{\lambda_1}|^2 \Phi_{\lambda_1}^{-3+2\varepsilon} + d x \right] - 2 \int_{\Omega} |\nabla u|^2 \Phi_{\lambda_1}^{-1+2\varepsilon} + d x.
$$

Noting that $\partial_t^2 u + |x|^{-\alpha} \partial_t u = \Delta u$, using above two estimates, we have

$$
\frac{d}{dt} \left[ E_{\Phi}^{1}[t_0, u](t) \right] \leq -\frac{2\varepsilon}{1 - \varepsilon} \int_{\Omega} |\nabla u|^2 \Phi_{\lambda_1}^{-1+2\varepsilon} + d x + 2 \int_{\Omega} |\partial_t u|^2 \Phi_{\lambda_1}^{-1+2\varepsilon} + d x + \lambda \int_{\Omega} u(\partial_t u) \Phi_{\lambda_1+1} \Phi_{\lambda_1}^{-2+2\varepsilon} + d x.
$$

Then the assertions (iii) and (iv) in Lemma 3.4 imply that

$$
\frac{d}{dt} \left[ E_{\Phi}^{1}[t_0, u](t) \right] \leq -\frac{2\varepsilon}{1 - \varepsilon} (1 - 2\varepsilon) \int_{\Omega} |\nabla u|^2 \Psi^4 + d x + \frac{2}{1 - 2\varepsilon} \int_{\Omega} |\partial_t u|^2 \Psi^4 + d x + \frac{\lambda C_{\lambda_1+1}}{1 - 2\varepsilon} \int_{\Omega} |u| |\partial_t u| \Psi^{4-1} + d x.
$$

14
Finally, Lemma 3.5 implies

$$\frac{d}{dt} \left[ E_{\alpha}^\beta [t_0, u](t) \right] \leq -\eta \left( \int_\Omega |\nabla u|^2 \Psi^4 \, dx + \int_\Omega |u|^2 |x|^{-\alpha} \Psi^{4-1} \, dx \right) + K_2 \left( \int_\Omega |\partial_t u|^2 \Psi^4 \, dx + \int_\Omega |u|^2 \Psi^{4-2} \, dx \right),$$

with the constants $\eta = \eta(\lambda) > 0$, $K_2 = K_2(\alpha, \lambda, N) > 0$. Observe that

$$\Psi^{-1} = \left( t_0 + t + \frac{|x|^{2-\alpha}}{(2-\alpha)^2} \right)^{-\frac{2(1-\alpha)}{2-\alpha}} \leq t_0^{-\frac{2(1-\alpha)}{2-\alpha}} (2-\alpha)^\frac{2}{2-\alpha} |x|^{-\alpha}.$$

By choosing $t_0 \geq t_2$ with $t_2$ sufficiently large, we obtain the desired inequality. \(\square\)

3.1.4 Weighted energy estimates for the case $1 < \beta < \frac{N-\alpha}{2-\alpha} + 1$

**Lemma 3.8.** If $\lambda \leq \beta - \frac{\alpha}{2-\alpha}$, then for every $\delta > 0$, there exists a constant $K_3 = K_3(\alpha, \lambda, \delta)$ such that for every $t_0 \geq 1$ and $t \geq 0$,

$$\int_\Omega 2u \partial_t u \Phi_{\lambda}^{1+2\epsilon} \, dx \leq K_3 E_{\alpha}^\beta [t_0, u](t) + \delta E_{\alpha}^\lambda [t_0, u](t). \quad (3.3)$$

In particular,

$$-K_3 t_0^{(\beta-1-\frac{\alpha}{2-\alpha})} E_{\alpha}^\beta [t_0, u](t) + \delta_0 E_{\alpha}^\lambda [t_0, u](t) \leq E_{\alpha}^\lambda [t_0, u](t) \leq K_3 t_0^{(\beta-1-\frac{\alpha}{2-\alpha})} E_{\alpha}^\beta [t_0, u](t) + 2E_{\alpha}^\lambda [t_0, u](t).$$

**Proof.** Young’s inequality yields that

$$\left| \int_\Omega 2u \partial_t u \Phi_{\lambda}^{1+2\epsilon} \, dx \right| \leq 2K' \int_\Omega |u| |\partial_t u| \Psi^4 \, dx$$

$$\leq \frac{(K')^2}{\delta} \int_\Omega \left| \partial_t u \right|^2 |\partial_t u|^\alpha \Psi^4 \, dx + \delta \int_\Omega |u|^2 |x|^{-\alpha} \Psi^4 \, dx$$

$$\leq \frac{(K')^2}{\delta} \int_\Omega \left| \partial_t u \right|^2 |\partial_t u|^\alpha \Psi^4 \, dx + \delta \int_\Omega |u|^2 |x|^{-\alpha} \Psi^4 \, dx$$

$$\leq \frac{(K')^2}{\delta} \int_\Omega \left| \partial_t u \right|^2 |\partial_t u|^\alpha \Psi^4 \, dx + \delta \int_\Omega |u|^2 |x|^{-\alpha} \Psi^4 \, dx.$$
Proof. We recall that Lemma 3.2 and Lemma 3.7 with \( \lambda = \beta = 1 < \frac{N-\alpha}{2-\alpha} \) assert that

\[
\frac{d}{dt} \left[ E^\beta_{\alpha_x}[t_0, u(t)] + E^\beta_{\alpha_t}[t_0, u(t)] \right] \leq -\frac{1}{2} E^\beta_{\alpha_x}[t_0, \partial_t u(t)] + \frac{K_1 K_2}{\eta} E^\beta_{\alpha_x}[t_0, \partial_x u(t)] \tag{3.6}
\]

Therefore we have

\[
\frac{d}{dt} \left[ E^\beta_{\alpha_x}[t_0, u(t)] + E^\beta_{\alpha_t}[t_0, u(t)] + \frac{K_1}{\eta} E^\beta_{\alpha_x}[t_0, u(t)] \right] \leq -\frac{1}{2} E^\beta_{\alpha_x}[t_0, \partial_t u(t)] + \frac{K_1 K_2}{\eta} E^\beta_{\alpha_x}[t_0, \partial_x u(t)]
\]

Therefore we have

\[
\frac{d}{dt} \left[ E^\beta_{\alpha_x}[t_0, u(t)] + E^\beta_{\alpha_t}[t_0, u(t)] + \frac{K_1}{\eta} E^\beta_{\alpha_x}[t_0, u(t)] \right] \leq -\frac{1}{2} E^\beta_{\alpha_x}[t_0, \partial_t u(t)] + \frac{K_1 K_2}{\eta} E^\beta_{\alpha_x}[t_0, \partial_x u(t)]
\]

Observe that

\[
E^\beta_{\alpha_x}[t_0, u(t)] + E^\beta_{\alpha_t}[t_0, u(t)] + \frac{K_1}{\eta} E^\beta_{\alpha_x}[t_0, u(t)] \geq E^\beta_{\alpha_x}[t_0, u(t)] + \left( 1 - \frac{K_1 K_3}{\eta} t_0^{-\frac{2(1-\alpha)}{2-\alpha}} \right) E^\beta_{\alpha_t}[t_0, u(t)] + \frac{K_1 \delta_0}{\eta} E^\beta_{\alpha_x}[t_0, u(t)]
\]

and

\[
E^\beta_{\alpha_x}[t_0, u(t)] + E^\beta_{\alpha_t}[t_0, u(t)] + \frac{K_1}{\eta} E^\beta_{\alpha_x}[t_0, u(t)] \leq E^\beta_{\alpha_x}[t_0, u(t)] + \left( 1 + \frac{K_1 K_3}{\eta} t_0^{-\frac{2(1-\alpha)}{2-\alpha}} \right) E^\beta_{\alpha_x}[t_0, u(t)] + \frac{2 K_1}{\eta} E^\beta_{\alpha_x}[t_0, u(t)]
\]

In this case by choosing \( t_3 \geq t_2 \) such that \( t_3^{-\frac{2(1-\alpha)}{2-\alpha}} \leq \frac{\eta}{4 K_1 K_2} \), we obtain for \( t_0 \geq t_3 \),

\[
\frac{d}{dt} \left[ E^\beta_{\alpha_x}[t_0, u(t)] + E^\beta_{\alpha_t}[t_0, u(t)] + \frac{K_1}{\eta} E^\beta_{\alpha_x}[t_0, u(t)] \right] \leq -\frac{1}{4} E^\beta_{\alpha_x}[t_0, \partial_t u(t)].
\]

Proceeding the same argument as in the previous case, we obtain \[3.4\]. \( \square \)

### 3.2 Weighted energy estimates for decaying initial data

**Proof of Theorem [7]** Let \( (u_0, u_1) \) satisfy (1.6). Fix \( \eta \in C^\infty_c(\mathbb{R}^N, [0, 1]) \) satisfying \( \eta(x) = 1 \) for \( x \in B(0, 1) \) and \( \eta(x) = 0 \) for \( x \in \mathbb{R}^N \setminus B(0, 2) \). Set for each \( n \in \mathbb{N} \),

\[
u(n^{-1})(x) := \eta(n^{-1} x) u_0(x), \quad u_{1n}(x) := \eta(n^{-1} x) u_1(x).
\]

Then clearly we have \( (u_{0n}, u_{1n}) \in \mathcal{H}_c \) for every \( n \in \mathbb{N} \). Let \( u_n \) be a solution of (1.1) with initial data \( (u_{0n}, u_{1n}) \). Moreover, noting \( \|u_{1n}(x)\| \leq \|u_1(x)\| \) and

\[
\|\nabla u_{0n}(x)\| \leq \|\nabla u_0(x)\| + n^{-1} [\nabla \eta(n^{-1} x)] [u_0(x)] \\
\leq \|\nabla u_0(x)\| + |x|^{-1} \left( \sup_{y \in \mathbb{R}^N} |\nabla \eta(y)| \right) |u_0(x)|,
\]

16
by Lemma 3.5 we have

\[
\int_{\mathbb{R}^N} |\nabla u_0|^2 |x|^\gamma \, dx \leq 2 \left( \int_{\mathbb{R}^N} |\nabla u_0|^2 |x|^{\gamma} \, dx + \int_{\mathbb{R}^N} |u_0|^2 |x|^{\gamma - 2} \, dx \right) \leq 2 \left( 1 + \frac{4}{(N + \gamma - 2)^2} \right) \int_{\mathbb{R}^N} |\nabla u_0|^2 |x|^\gamma \, dx.
\]

On the other hand, we can check that \((u_{0n}, u_{1n}) \to (u_0, u_1)\) in \(H_0^1(\Omega) \times L^2(\Omega)\) as \(n \to \infty\). This means that the strong continuity of the semigroup \((u_0, u_1) \mapsto (u, \partial_t u)\) in \(H_0^1(\Omega) \times L^2(\Omega)\) implies

\[(u_n, \partial_t u_n) \to (u, \partial_t u) \text{ in } H_0^1(\Omega) \times L^2(\Omega) \text{ as } n \to \infty.\]

Consequently, applying Proposition 3.4 (\(\gamma \leq 2 - \alpha\)) or Proposition 3.9 (\(\gamma > 2 - \alpha\)) with \(\beta = \frac{\alpha}{2 - \alpha}\) and letting \(n \to \infty\), we obtain the desired weighted energy estimates (1.7).

\[\square\]

### 4 Weight energy estimates for higher order derivatives

We prove weighted energy estimates for higher order derivatives. To state the assertion, we use the compatibility condition of order \(k\), which is defined as follows: The initial data \((u_0, u_1)\) satisfy the compatibility condition of order \(k\) if \(u_{j+1} = -\Delta u_{j-1} + |x|^{-\alpha} u_j \in L^2(\Omega)\) \((j = 1, \ldots, k)\) can be successively defined with \((u_{j-1}, u_j) \in (H^2 \cap H_0^1(\Omega)) \times H_0^1(\Omega)\) for \(j = 1, \ldots, k\).

**Theorem 4.1.** Assume that \((u_0, u_1)\) satisfies the compatibility condition of order \(k\) greater than 1. Put \(u_\ell = -\Delta u_{\ell-2} + a(x)u_{\ell-1}\) \((\ell = 2, \ldots, k + 1)\). Suppose that

\[
\int_{\Omega} \left( |\nabla u_\ell|^2 + |u_{\ell+1}|^2 \right) |x|^{\gamma + 2\ell} \, dx < \infty
\]

for all \(\ell = 0, \ldots, k\). Then there exists \(M_\ell = M_\ell(\alpha, \gamma, N, \ell)\) such that

\[
\left( E_{a, \ell}^{\gamma + 2\ell} [t_0, \partial _t^\ell u](t) + E_{a, \ell}^{\gamma + 2\ell} [t_0, \partial _t^{\ell+1} u](t) \right) + \int_{t_0}^{t} E_{a, \ell}^{\gamma + 2\ell} [t_0, \partial _t^{\ell+1} u](s) \, ds \leq M_\ell \sum_{j=0}^{\ell} \int_{\Omega} \left( |\nabla u_j|^2 + |u_{j+1}|^2 \right) |x|^{\gamma + 2j} \, dx
\]

(4.1)

**Lemma 4.2.** Under the assumption in Theorem 4.1 one has for \(\ell = 0, \ldots, k\),

\[
E_{a, \ell}^{\gamma + 2\ell} [t_0, \partial _t^\ell u](0) + E_{a, \ell}^{\gamma + 2\ell} [t_0, \partial _t^{\ell+1} u](0) + E_{a, \ell}^{\gamma + 2\ell - 1} [t_0, \partial _t^\ell u](0) < \infty.
\]

Since the same approximation argument as in the proof of Theorem 1.1 is also applicable in this case, we focus only on the case where the initial data \((u_0, u_1)\) have compact supports.

Here we introduce another energy functional \(E_{\Phi}^d\) in stead of \(E_\Phi^d\).

**Definition 4.1.** For \(\beta \in \mathbb{R}\) and for the solution \(w\) of (1.1) with initial data \((f, g) \in \mathcal{H}\),

\[
E_{\Phi}^d [t_0, w](t) := 2 \int_{\Omega} w(x, t) \partial_t w(x, t) \Psi^d(x, t_0 + t) \, dx.
\]

**Remark 4.1.** \(E_{\Phi}^d\) is meaningful if \(\lambda < \frac{N - \alpha}{2 - \alpha}\), that is, \(\Phi_{\lambda}\) is positive for all \(x \in \mathbb{R}^N\).
Lemma 4.3. Under the assumption in Theorem 4.1 one has

\[ \frac{d}{dt} \left[ E^\alpha_{\alpha}(t_0, u)(t) + E^\beta_{\beta}(t_0, u)(t) \right] \leq -E^\gamma_{\alpha}(t_0, u)(t) + K_4(\beta)E^\delta_{\delta}(t_0, u)(t) \]

\[ + K_5(\alpha, \lambda)E^{\lambda-1}_{\alpha}(t_0, u)(t) \]

(4.2)

with \( K_4(\lambda) = 2 + \lambda \) and \( K_5(\alpha, \lambda) = \lambda \left( \lambda + (2 - \alpha)^\frac{2}{\alpha} + 1 \right) \).

Proof. By a simple calculation we have

\[ \frac{d}{dt} \int_{\Omega} \left( 2u\partial_t u + |x|^{-\alpha}|u|^2 \right) \Psi^4 \, dx = 2 \int_{\Omega} |\partial_t u|^2 \Psi^4 \, dx + 2 \int_{\Omega} u \left( \partial^2_t u + |x|^{-\alpha} u \right) \Psi^4 \, dx \]

\[ + \lambda \int_{\Omega} \left( 2u\partial_t u + |x|^{-\alpha}|u|^2 \right) \Psi^{\lambda-1} \, dx. \]

By using the equation in (1.1) we see from integration by parts twice that

\[ 2 \int_{\Omega} u \left( \partial^2_t u + |x|^{-\alpha} u \right) \Psi^4 \, dx = 2 \int_{\Omega} u(\Delta u) \Psi^4 \, dx \]

\[ = -2 \int_{\Omega} |\nabla u|^2 \Psi^4 \, dx - 2\lambda \int_{\Omega} u \nabla u \cdot \frac{|x|^{-\alpha} u}{2 - \alpha} \Psi^{\lambda-1} \, dx \]

\[ \leq - \int_{\Omega} |\nabla u|^2 \Psi^4 \, dx + \lambda^2 \int_{\Omega} |u|^2 \frac{|x|^{2-2\alpha}}{(2 - \alpha)^2} \Psi^{\lambda-2} \, dx \]

\[ \leq - \int_{\Omega} |\nabla u|^2 \Psi^4 \, dx + \lambda^2 \int_{\Omega} |u|^2 |x|^{-\alpha} \Psi^{\lambda-1} \, dx. \]

On the other hand, noting that \( \Psi^\frac{2}{\alpha} \leq (2 - \alpha)^\frac{2}{\alpha} |x|^{-\alpha} \), we have

\[ \lambda \int_{\Omega} \left( 2u\partial_t u + |x|^{-\alpha}|u|^2 \right) \Psi^{\lambda-1} \, dx \leq \lambda \int_{\Omega} |\partial_t u|^2 \Psi^4 \, dx + \lambda \int_{\Omega} |u|^2 \Psi^{\lambda-2} \, dx + \lambda \int_{\Omega} |u|^2 |x|^{-\alpha} \Psi^{\lambda-1} \, dx \]

\[ \leq \lambda \int_{\Omega} |\partial_t u|^2 \Psi^4 \, dx + \lambda \left( (2 - \alpha)^\frac{2}{\alpha} + 1 \right) \int_{\Omega} |u|^2 |x|^{-\alpha} \Psi^{\lambda-1} \, dx. \]

And hence we have (4.2). \( \square \)

Lemma 4.4. Under the assumption in Theorem 4.1 one has

\[ \frac{d}{dt} \left[ E^\alpha_{\alpha}(t_0, u)(t) + E^\beta_{\beta}(t_0, u)(t) \right] \leq -E^\gamma_{\alpha}(t_0, u)(t) + K_6(\alpha, \beta)E^\delta_{\delta}(t_0, u)(t). \]

Remark 4.2. If \((u_0, u_1)\) has a compact support, then we can use the following estimate

\[ \frac{d}{dt} \left[ E^\alpha_{\alpha}(t_0, u)(t) + E^\beta_{\beta}(t_0, u)(t) \right] \leq -E^\gamma_{\alpha}(t_0, u)(t) + K_6(\alpha, \beta)E^{\delta-1}_{\alpha}(t_0, u)(t). \]
Proof of Lemma 4.4 Let $\nu > 0$ be determined later. By using Lemmas 3.2 and 4.3 we see that

$$
\frac{d}{dt}\left[ E_{\bar{a}}^{\frac{\nu}{2}}[t_0, u(t)] + E_{\bar{a}}^{\alpha}[t_0, u(t)] + \nu\left( E_{\bar{a}}^{\frac{\nu}{2}}[t_0, u(t)] + E_{\bar{a}}^{\alpha}[t_0, u(t)] \right) \right] 
\leq -\frac{1}{2} E_{\bar{a}}^{\alpha}[t_0, \partial_t u(t)] + K_1 E_{\bar{a}}^{\alpha}[t_0, u(t)] - \nu E_{\bar{a}}^{\alpha}[t_0, u(t)] 
+ \nu K_4 \left( \beta - \frac{\alpha}{2} \right) E_{\bar{a}}^{\frac{\nu}{2}}[t_0, u(t)] + \nu K_5 \left( \alpha, \beta - \frac{\alpha}{2} \right) E_{\bar{a}}^{\frac{\nu}{2}}[t_0, u(t)] 
\leq \left( \nu K_4(2 - \alpha) \right)^{\frac{\nu}{2} - 1} - \frac{1}{2} E_{\bar{a}}^{\alpha}[t_0, \partial_t u(t)] + \left( K_1 t_4^{\frac{\nu}{2} - 1} - \nu \right) E_{\bar{a}}^{\alpha}[t_0, u(t)] + \nu K_3 E_{\bar{a}}^{\frac{\nu}{2}}[t_0, u(t)].
$$

Taking $\nu_4^{\frac{\nu}{2}} = 4^{-1}(2 - \alpha)^{-\frac{\nu}{2} - 1}$ and $t_4 \geq t_3$ such that $K_1 t_4^{\frac{\nu}{2} - 1} \leq \nu_4^{\frac{\nu}{2}}$, we have the desired estimate. \qed

Proof of Theorem 4.7 First we note that by Theorem 1.1 we have

$$
\left( E_{\bar{a}}^{\frac{\nu}{2}}[t_0, u(t)] + E_{\bar{a}}^{\alpha}[t_0, u(t)] \right) + \int_0^\nu E_{\bar{a}}^{\frac{\nu}{2}}[t_0, \partial_t u(s)] ds \leq M_1 \int_\Omega \left( |\nabla u_0| + |u_1| \right) dx. \quad (4.3)
$$

Then applying Lemma 4.4 with $\beta = \frac{\nu + 2j}{2 - \alpha}$ and $w = \partial_t^j u$, we have

$$
\frac{d}{dt}\left[ E_{\bar{a}}^{\frac{\nu + 2j}{2 - \alpha}}[t_0, \partial_t^j u(t)] + E_{\bar{a}}^{\alpha}[t_0, \partial_t^j u(t)] + K_1 E_{\bar{a}}^{\frac{\nu + 2j}{2 - \alpha}}[t_0, \partial_t^j u(t)] + E_{\bar{a}}^{\alpha}[t_0, \partial_t^j u(t)] \right] 
\leq -\frac{1}{4} E_{\bar{a}}^{\frac{\nu + 2j}{2 - \alpha}}[t_0, \partial_t^{j+1} u(t)] + K_6 \left( \alpha, \frac{\nu + 2j}{2 - \alpha} \right) E_{\bar{a}}^{\frac{\nu + 2j}{2 - \alpha}}[t_0, \partial_t^j u(t)] 
\leq -\frac{1}{4} E_{\bar{a}}^{\frac{\nu + 2j}{2 - \alpha}}[t_0, \partial_t^{j+1} u(t)] + K_6 \left( \alpha, \frac{\nu + 2j}{2 - \alpha} \right) E_{\bar{a}}^{\frac{\nu + 2j}{2 - \alpha}}[t_0, \partial_t^j u(t)].
$$

This gives that there exists a constant $\tilde{M}_j$ such that

$$
E_{\bar{a}}^{\frac{\nu + 2j}{2 - \alpha}}[t_0, \partial_t^j u(t)] + E_{\bar{a}}^{\frac{\nu + 2j}{2 - \alpha}}[t_0, \partial_t^j u(t)] + E_{\bar{a}}^{\frac{\nu + 2j}{2 - \alpha}}[t_0, \partial_t^{j+1} u(t)] + \int_0^\nu E_{\bar{a}}^{\frac{\nu + 2j}{2 - \alpha}}[t_0, \partial_t^{j+1} u(t)] ds 
\leq \tilde{M}_j \left( E_{\bar{a}}^{\frac{\nu + 2j}{2 - \alpha}}[t_0, \partial_t^j u(0)] + E_{\bar{a}}^{\frac{\nu + 2j}{2 - \alpha}}[t_0, \partial_t^j u(0)] + E_{\bar{a}}^{\frac{\nu + 2j}{2 - \alpha}}[t_0, \partial_t^{j+1} u(0)] + \int_0^\nu E_{\bar{a}}^{\frac{\nu + 2j}{2 - \alpha}}[t_0, \partial_t^{j+1} u(t)] ds \right). \quad (4.4)
$$

Combining (4.3) and (4.4), we obtain the desired inequality. \qed

Remark 4.3. If $(u_0, u_1)$ has compact supports or decays fast enough, then we also can prove that

$$
\left( E_{\bar{a}}^{\frac{\nu + 2j}{2 - \alpha}}[t_0, \partial_t^j u(t)] + E_{\bar{a}}^{\frac{\nu + 2j}{2 - \alpha}}[t_0, \partial_t^j u(t)] \right) + \int_0^\nu E_{\bar{a}}^{\frac{\nu + 2j}{2 - \alpha}}[t_0, \partial_t^{j+1} u(t)] ds \leq M
$$

which is much better than the estimate in Theorem 4.1. In other words, the energy decay estimates for higher order derivatives heavily depend on the behavior of initial data near spacial infinity. We would omit the proof of the estimate mentioned above.
5 Diffusion phenomena

To finish this paper, we prove Theorem 1.2 which deals with diffusion phenomena for the solution of (1.1) with the initial data \((u_0, u_1)\) satisfying a compatibility condition of order 1.

Lemma 5.1. Under the assumption in Theorem 1.2, we have \(u(t) \in D(L_\alpha)\) for all \(t \geq 0\) and

\[
|\Omega|^\frac{\alpha}{2} \partial_\Omega u \in L^\infty(0, \infty; L^2_{\partial_\Omega}).
\]

Proof. We note that \(u(t) \in D(a_\alpha)\) by the inclusion \(H^1_\alpha(\Omega) \subset D(a_\alpha)\). Applying Theorems 1.1 and 4.1 with \(\ell = 1\), we have for \(t \geq 0\),

\[
\int_\Omega \left( |\nabla u|^2 + |\partial_\Omega u|^2 \right) \Psi_{2,2}^\alpha dx + \int_\Omega \left( |\nabla \partial_\Omega u|^2 + |\partial_\Omega^2 u|^2 \right) \Psi_{1,2}^\alpha dx \leq M
\]

and therefore the assumption \(\gamma \geq 2 - \alpha\) yields that for \(t \geq 0\),

\[
\int_\Omega \left( |\nabla u|^2 + |\partial_\Omega u|^2 \right) \sum_{\alpha} dx + \int_\Omega \left( |\nabla \partial_\Omega u|^2 + |\partial_\Omega^2 u|^2 \right) \sum_{\alpha} dx \leq M. \tag{5.1}
\]

This inequality implies

\[
\|\partial_\Omega u\|_{L^2(\Omega)}^2 = \int_\Omega |\Delta u|_{\Omega} dx
\]

\[
\leq \int_\Omega |\Delta u|_{\Omega} dx + \int_\Omega |\partial_\Omega u|^2 \sum_{\alpha} dx
\]

\[
\leq \int_\Omega |\partial_\Omega^2 u|^2 \sum_{\alpha} dx + \int_\Omega |\partial_\Omega^3 u|^2 \sum_{\alpha} dx
\]

\[
\leq M'.
\]

Since \(u(t)\) satisfies the Dirichlet boundary condition on \(\partial \Omega\), by Lemma 2.7 we deduce \(u(t) \in D(L_\alpha)\). Moreover, by (5.1) we see that

\[
\left\| \sum_{\alpha} \right\| \sum_{\alpha}^2 dx \leq M''.
\]

This completes the proof. \(\square\)

Proof of Theorem 1.2. Applying Theorem 4.1 with \(k = 1\), we have

\[
\int_0^t E_\alpha(s, \partial_\Omega u) ds \leq M_0 \varepsilon_0, \quad t \geq 0. \tag{5.2}
\]

\[
E_\alpha(s, \partial_\Omega u) \leq M_1 (\varepsilon_0 + \varepsilon_1), \quad t \geq 0. \tag{5.3}
\]

\[
\int_0^t E_\alpha(s, \partial_\Omega u) ds \leq M_1 (\varepsilon_0 + \varepsilon_1), \quad t \geq 0. \tag{5.4}
\]

Here we rewrite (1.1) as the problem

\[
\partial_\Omega u - |\Omega|^\frac{\alpha}{2} \Delta u = -|\Omega|^\frac{\alpha}{2} \partial_\Omega^2 u, \quad t \geq 0.
\]
with \( u(0) = u_0 \in L^2_{\mathcal{M}^*} \), where we regard \(-|x|^\alpha \partial_t^2 u\) as a inhomogeneous term for the heat equation \( \partial_t u - |x|^\alpha \Delta u = 0 \) in \( L^2_{\mathcal{M}^*} \). By virtue of Lemma \([5.1]\) we see from the standard semigroup theory that

\[
    u(t) = e^{tL^*} u_0 - \int_0^t e^{(t-s)L^*} [ |x|^\alpha \partial_t^2 u(s) ] \, ds.
\]

As in \([28]\) (see also \([23]\) and \([24]\)), we have

\[
    u(t) - e^{tL^*}[u_0 + |x|^\alpha u_1] = - \int_0^t e^{(t-s)L^*} [ |x|^\alpha \partial_t^2 u(s) ] \, ds - e^{2L^*} [ |x|^\alpha \partial_t u(t/2)] \\
    - \int_0^t L_s e^{(t-s)L^*} [ |x|^\alpha \partial_t u(s) ] \, ds.
\]

Taking the \( L^2_{\mathcal{M}^*} \)-norm, we see

\[
    \left\| u(t) - e^{tL^*}[u_0 + |x|^\alpha u_1] \right\|_{L^2_{\mathcal{M}^*}} \leq \int_0^t \left\| |x|^\alpha \partial_t^2 u(s) \right\|_{L^2_{\mathcal{M}^*}} \, ds + \left\| |x|^\alpha \partial_t^2 u(t/2) \right\|_{L^2_{\mathcal{M}^*}} \\
    + \int_0^t \left\| L_s e^{(t-s)L^*} [ |x|^\alpha \partial_t u(s) ] \right\|_{L^2_{\mathcal{M}^*}} \, ds.
\]

Schwarz’s inequality and the definition of \( \Psi^\alpha \) yield

\[
    J_1 \leq \frac{t}{2} \int_0^t \left\| |x|^\alpha \partial_t^2 u(s) \right\|_{L^2_{\mathcal{M}^*}}^2 \, ds \\
    \leq \frac{t}{2} (2 - \alpha)^{\frac{\alpha}{2}} \int_0^t \left( \int_\Omega |x|^{-\alpha} |\partial_t^2 u(s)|^2 \Psi^{\frac{\alpha}{2}} \, dx \right) \, ds \\
    \leq \frac{t}{2} (2 - \alpha)^{\frac{\alpha}{2}} \int_0^t (t_0 + s)^{\frac{\alpha}{2}} \, \left( \int_\Omega |x|^{-\alpha} |\partial_t^2 u(s)|^2 \Psi^{\frac{\alpha}{2}} \, dx \right) \, ds \\
    \leq \frac{t}{2} (2 - \alpha)^{\frac{\alpha}{2}} \left( t_0 + \frac{t}{2} \right)^{\frac{\alpha}{2} - 1} \int_0^t \int_\Omega E_a^{\frac{\alpha}{2}} [t_0, \partial_t^2 u(s)] \, ds.
\]

By \([5.4]\), we have \( J_1 \leq K_1(1 + t)^{-\frac{\alpha}{2 \alpha - \alpha}} (E_0 + E_1)^{\frac{1}{2}} \). By a computation similar to the one for \( J_1 \), we deduce from \([5.3]\) that

\[
    J_2 = \int_\Omega |x|^\alpha |\partial_t u(t/2)|^2 \, dx \\
    \leq (2 - \alpha)^{\frac{\alpha}{2}} \int_\Omega |x|^{-\alpha} |\partial_t u(t/2)|^2 \Psi^{\frac{\alpha}{2}} \, dx \\
    \leq (2 - \alpha)^{\frac{\alpha}{2}} \left( t_0 + \frac{t}{2} \right)^{\frac{\alpha}{2} - 1} \int_\Omega |x|^{-\alpha} |\partial_t u(t/2)|^2 \Psi^{\frac{\alpha}{2}} \, dx \\
    \leq (2 - \alpha)^{\frac{\alpha}{2}} \left( t_0 + \frac{t}{2} \right)^{\frac{\alpha}{2} - 1} \int_\Omega E_a^{\frac{\alpha}{2}} [t_0, \partial_t u(t/2)] \, dx \\
    \leq K_2^\alpha (1 + t)^{-\frac{\alpha}{2 \alpha - \alpha}} (E_0 + E_1).
\]
For $\mathcal{F}_3$, we divide the proof into two cases $2 - \alpha \leq \gamma \leq N + \alpha$ and $N + \alpha < \gamma < N + 2 - \alpha$. In the former case, we set $1 < p(\theta) \leq 2$ ($\theta > 0$) as

$$p(\theta) = \frac{2(N - \alpha + \theta)}{N + \gamma - 3\alpha + \theta} > 1.$$  

that is,

$$p(\theta)\alpha - \alpha = (\gamma - \alpha)\frac{p(\theta)}{2} - (N + \theta)\left(1 - \frac{p(\theta)}{2}\right).$$

Then we have

$$\left\| |x|^\alpha \partial_t u(s) \right\|_{L^p_{x,\theta}}^2 = \left(\int_{\Omega} |x|^{p(\theta)u - \sigma} |\partial_t u(s)|^{p(\theta)} \, dx \right)^{\frac{2}{p(\theta)}}$$

$$\leq \left(\int_{\Omega} |x|^{N - \theta} \, dx \right)^{\frac{2}{p(\theta)} - 1} \int_{\Omega} |x|^{\gamma - \alpha} |\partial_t u(s)|^2 \, dx$$

$$\leq (2 - \alpha)^{\frac{2}{\gamma - \alpha}} \left(\int_{\Omega} |x|^{N - \theta} \, dx \right)^{\frac{2}{p(\theta)} - 1} \int_{\Omega} |\partial_t u(s)|^2 |x|^{\gamma - \alpha} \Psi_{\theta} \, dx$$

$$\leq (2 - \alpha)^{2\gamma - \alpha} \left(\int_{\Omega} |x|^{N - \theta} \, dx \right)^{\frac{2}{p(\theta)} - 1} E_{\theta}^{\frac{2\gamma}{\gamma - \alpha}}[0, \partial_t u(s)].$$

And then by [5.2],

$$\mathcal{F}_3 \leq \int_0^\gamma \left(\int_{\Omega} |x|^\alpha |\partial_t u(s)|^2 \, dx \right)^{\frac{2}{p(\theta)}} \, ds$$

$$\leq \int_0^\gamma (t - s)^{-\frac{N - \alpha}{\gamma - \alpha} - 1} \left(\int_{\Omega} |x|^\alpha |\partial_t u(s)|^2 \, dx \right)^{\frac{2}{p(\theta)}} \, ds$$

$$\leq \left(\frac{1}{2}\right)^{-\frac{N - \alpha}{\gamma - \alpha} - 1} \left(\int_0^\gamma \left\| |x|^\alpha \partial_t u(s) \right\|_{L^p_{x,\theta}}^2 \, ds \right)^{\frac{1}{2}}$$

$$\leq \mathcal{K}_{3,\theta}(1 + t)^{-\frac{N - \alpha}{\gamma - \alpha} - 1} E_{\theta}^{\frac{\gamma}{\gamma - \alpha}}.$$  

Noting that by choosing $\theta$ small enough,

$$\frac{N - \alpha}{2 - \alpha} \left(1 - \frac{1}{p(\theta)} - \frac{1}{2}\right) + 1 = \frac{(\gamma - 2\alpha)(N - \alpha)}{2(N + \theta - \alpha)(2 - \alpha)} + 1$$

$$= \frac{\gamma - \alpha}{2(2 - \alpha)} + \frac{1}{2} - \frac{1}{2} - \frac{(\gamma - 2\alpha)\theta}{2(N + \theta - \alpha)(2 - \alpha)}$$

$$\geq \frac{\gamma - \alpha}{2(2 - \alpha)},$$

we have $\mathcal{F}_3 \leq \mathcal{K}_{3,\theta}(1 + t)^{-\frac{N - \alpha}{\gamma - \alpha}}$. In the latter case, we choose $p = 1$ and then we deduce

$$\left\| |x|^\alpha \partial_t u(s) \right\|_{L^p_{x,\theta}}^2 = \left(\int_{\Omega} |\partial_t u(s)| \, dx \right)^2$$

$$\leq \left(\int_{\Omega} |x|^\alpha \, dx \right) \int_{\Omega} |x|^{\gamma - \alpha} |\partial_t u(s)|^2 \, dx$$

$$\leq (2 - \alpha)^{\frac{2\gamma - \alpha}{\gamma - \alpha}} \left(\int_{\Omega} |x|^\alpha \, dx \right) E_{\theta}^{\frac{2\gamma}{\gamma - \alpha}}[0, \partial_t u(s)].$$
Lemma A.1. The following three assertions hold:

(i) The pair \((M(b, c; \cdot), U(b, c; \cdot))\) is a fundamental system of the equation (A.1).
(ii) If \( c > b > 0 \), then \( M(b, c; s) \) and satisfies \( M(b, c; s) \sim \frac{\Gamma(c)}{\Gamma(b)} s^{b-c} e^s \) as \( s \to \infty \), more precisely,

\[
\lim_{s \to \infty} \left( \frac{M(b, c; s)}{s^{b-c} e^s} \right) = \frac{\Gamma(c)}{\Gamma(b)}.
\]

(iii) If \( b > 0 \), then \( U(b, c; s) \) satisfies \( U(b, c; s) \sim s^b \) as \( s \to \infty \), more precisely,

\[
\lim_{s \to \infty} \left( \frac{U(b, c; s)}{s^b} \right) = 1.
\]

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