Regular geodesic languages and the falsification by fellow traveler property

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Abstract We furnish an example of a finite generating set for a group that does not enjoy the falsification by fellow traveler property, while the full language of geodesics is regular.

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1 Introduction

In this short note we answer the following question of Neumann and Shapiro from [5]:

Question Can one find a monoid generating set \( A \) of a group \( G \) so that the language of geodesics is regular but \( A \) does not have the falsification by fellow traveler property?

The converse to this statement is Proposition 4.1 of their paper, which states that if \( A \) has the falsification by fellow traveler property then the full language of geodesics on \( A \) is regular, and this fact is the reason for the property’s existence. Several authors have used the falsification by fellow traveler property as a route to finding other (geometric) properties of groups; Rebbechi uses a version of the property to prove that relatively hyperbolic groups are biautomatic [6], and the author exploits the property to prove that a certain class of groups is almost convex [4]. The author discusses various attributes and extensions of the property in [1, 2, 3].

In this article we answer the question via an example first given by Cannon to demonstrate that a group may have a regular language of geodesics with respect to one generating set but not another. Neumann and Shapiro include it in [5] to prove that the falsification by fellow traveler property is generating set dependent.
2 Definitions

Definition 2.1 (Finite state automaton; regular language) Let $A$ be a finite set of letters, and let $A^*$ be the set of all finite strings, including the empty string, that can be formed from the letters of $A$. A finite state automaton is a quintuple $(S, A, \tau, Y, s_0)$, where $S$ is a finite set of states, $\tau$ is a map $\tau : S \times A \to S$, $Y \subseteq S$ are the accept states, and $s_0 \in S$ is the start state. A finite string $w \in A^*$ is accepted by the finite state automaton if starting in the state $s_0$ and changing states according to the letters of $w$ and the map $\tau$, the final state is in $Y$. The set of all finite strings that are accepted by a finite state automaton is called the language of the automaton. A language $L \subseteq A^*$ is regular if it is the language of a finite state automaton.

Suppose $G$ is a group with finite generating set $A$. A word in $A^*$ represents a path in the Cayley graph based at any vertex. Define $d(a, b)$ to be the distance between two points $a$ and $b$ in the Cayley graph with respect to the path metric. Paths can be parameterized by non-negative $t \in \mathbb{R}$ by defining $w(t)$ as the point at distance $t$ along the path $w$ if $t$ is between 0 and the length of $w$, and the endpoint of $w$ otherwise.

Definition 2.2 (The (asynchronous) fellow traveler property) Paths $u$ and $v$ are said to $k$-fellow travel if $d(u(t), v(t)) \leq k$ for all $t \geq 0$. They asynchronously $k$-fellow travel if there is a non-decreasing proper continuous function $\phi : [0, \infty) \to [0, \infty)$ such that $d(u(t), v(\phi(t))) \leq k$. A language $L \subseteq A^*$ enjoys the (asynchronous) fellow traveler property if there is a constant $k$ such that for each $u, v \in L$ that start at the identity and end at distance 0 or 1 apart in the Cayley graph, $u$ and $v$ (asynchronously) $k$-fellow travel.

Definition 2.3 (The (asynchronous) falsification by fellow traveler property) A finite generating set $A$ for a group $G$ has the (asynchronous) falsification by fellow traveler property if there is a constant $k$ such that every non-geodesic word in the Cayley graph of $G$ with respect to $A$ is (asynchronously) $k$-fellow traveled by a shorter word.

The property arises naturally in the context of geodesic regular languages, and the proof of Proposition 4.1 in [5] uses the property to build an appropriate
finite state automaton. The author proves in [1] that the synchronous and asynchronous versions of the falsification by fellow traveler property are equivalent.

3 The Example

Let $G$ be the split extension of $\mathbb{Z}^2$, generated by $\{a, b\}$, by $\mathbb{Z}_2$, generated by $\{t\}$, such that $t$ conjugates $a$ to $b$ and $b$ to $a$, with presentation

$$\langle a, b, t \mid t^2 = 1, ab = ba, tat = b \rangle.$$ 

Performing one Tietze transformation (removing $b = tat$) we obtain

$$\langle a, t \mid t^2 = 1, atat = tata \rangle.$$ 

Let $A = \{a^{\pm 1}, t^{\pm 1}\}$ be the inverse-closed generating set corresponding to this presentation. The Cayley graph for $G$ with respect to $A$ is shown in Figure 1. We can consider the vertices of the graph as either being in the top or the bottom layer, where edges labeled $t$ link top and bottom layers. We declare the identity vertex to lie in the bottom. Each vertex can also be given a coordinate $(x, y)$ where $x$ is the distance in the East-West direction from the identity, and

![Cayley Graph Diagram](image-url)
y is the distance in the North-South direction. In this way each vertex (group element) is uniquely specified by the triple \((x, y, \text{bottom})\) or \((x, y, \text{top})\).

For example, the identity has the coordinate \((0, 0, \text{bottom})\), the word \(a^3ta^4\) has the coordinate \((3, 4, \text{top})\), the word \(a^3ta^4t\) has the coordinate \((3, 4, \text{bottom})\), and the word \(ta^3ta^4\) has the coordinate \((4, 3, \text{bottom})\).

**Lemma 3.1** Each word from 1 to a vertex in the top layer has an odd number of \(t\) letters, and each word from 1 to a vertex in the bottom layer has an even number of \(t\) letters.

**Proof** Suppose a vertex has the coordinate \((x, y, \text{top})\). Then \(a^xta^y\) is a word to this vertex. Suppose \(w\) is any other word to this vertex. Then \(wa^{-y}ta^{-x} =_G 1\) so under the map which sends \(t\) to \(t\) and \(a\) to 1 this word must be sent to an even power of \(t\), so \(w\) has an odd number of \(t\) letters. Similarly if a vertex lies in the bottom layer there is a word \(a^xta^y\) to it from 1. If \(w\) is any other word to this vertex then \(wta^{-y}ta^{-x} =_G 1\) gets sent to an even power of \(t\) so \(w\) has an even number of \(t\) letters.

**Lemma 3.2** The word \(ta^nta^mt\) for any \(m, n \in \Z\) is not geodesic.

**Proof** The word \(ta^nta^mt\) can be written as \(a^mta^n\) which is shorter.

**Lemma 3.3** Each vertex in the top layer has a unique geodesic to it from the identity of the form \(a^xta^y\), where \((x, y, \text{top})\) is the coordinate of the vertex.

**Proof** By Lemma 3.1 any geodesic to the vertex with coordinate \((x, y, \text{top})\) has an odd number of \(t\) letters. If a word has three or more \(t\) letters then it has a subword of the form \(ta^nta^mt\) which is not geodesic by Lemma 3.2. So any geodesic to this vertex has exactly one \(t\) letter, so is of the form \(a^i ta^j\). This path has coordinate \((i, j, \text{top})\) so it must be that \(i = x\) and \(j = y\).

**Proposition 3.4** A does not have the falsification by fellow traveler property.

**Proof** Suppose by way of contradiction that \(A\) has the falsification by fellow traveler property with positive constant \(k\), and choose \(n >> k\). Consider the word \(w = ta^nta^n\) which ends at the coordinate \((n, n, \text{top})\). Any word that ends at this vertex must move East at least \(n\) units and North at least \(n\) units, and by Lemma 3.1 must have an odd number of \(t\) letters. If it has just one \(t\) letter then it must be the unique geodesic \(a^nta^n\) which clearly does not \(k\)-fellow travel \(w\). Otherwise it has three or more \(t\) letters, so has length at least \(2n + 3\) so is not shorter than \(w\), so we are done.
Theorem 3.5  There is a group and finite generating set such that the language of all geodesics is regular but fails to have the falsification by fellow traveler property.

Proof  By Proposition 3.4 the group $G$ with generating set $A$ fails the falsification by fellow traveler property.

Consider the language $L = \{a^x, a^xta^y, a^xta^ytta^{x_2} : x, x_1, x_2, y \in \mathbb{Z}, x_1 \cdot x_2 \geq 0\}$. $L$ is the language of the finite state automaton in Figure 2. All states are accept states.

![Figure 2: A finite state automaton accepting the language $L$](image)

We now show that $L$ is the language of all geodesics on $A$ for $G$. By Lemma 3.3 every group element corresponding to a vertex in the top layer has a unique geodesic representative of the form $a^xta^y$. Otherwise the group element corresponds to a vertex in the bottom, so has an even number of $t$ letters. If a word has more than two $t$ letters then it has a subword of the form $ta^mta^nt$ which is not geodesic by Lemma 3.2, so a geodesic word for a bottom element has either zero $t$ letters, so is of the form $a^x$, or has two $t$ letters, so is of the form $a^{x_1}ta^yta^{x_2}$. If $x_1$ and $x_2$ don’t have the same sign then we can find a shorter word $a^{(x_1+x_2)}ta^yt$. Otherwise $a^{x_1}ta^yta^{x_2}$ is a geodesic to a vertex with coordinate $(x_1 + x_2, y, \text{bottom})$. Notice that this gives a family of $x_1 + x_2 + 1$ geodesics to this vertex.

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