A Vizing-type result for semi-total domination

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Abstract

A set of vertices $S$ in a simple isolate-free graph $G$ is a semi-total dominating set of $G$ if it is a dominating set of $G$ and every vertex of $S$ is within distance 2 or less with another vertex of $S$. The semi-total domination number of $G$, denoted by $\gamma_{t2}(G)$, is the minimum cardinality of a semi-total dominating set of $G$. In this paper we study semi-total domination of Cartesian products of graphs. Our main result establishes that for any graphs $G$ and $H$, $\gamma_{t2}(G \square H) \geq \frac{1}{3} \gamma_{t2}(G) \gamma_{t2}(H)$.

Keywords: Cartesian products, total domination number, semi-total domination number.

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1 Introduction

In this paper we study bounds on a recently introduced domination invariant applied to Cartesian products of graphs. At its core, our work is motivated by the longstanding conjecture of V.G. Vizing [17] on the domination of product graphs, which states that for any graphs $G$ and $H$, $\gamma(G \square H) \geq \gamma(G)\gamma(H)$. Here, $\gamma(G)$ is the domination number of $G$, which is the minimum size of a set $D$ of vertices so that every vertex not in $D$ is adjacent to some vertex in $D$, and $\square$ is the Cartesian product of graphs. The breakthrough “double-projection” result of Clark and Suen [5] gave the first Vizing-type bound of $\gamma(G \square H) \geq \frac{1}{2}\gamma(G)\gamma(H)$. Recently, Brešar [1] improved this bound to $\gamma(G \square H) \geq \frac{(2\gamma(G) - \rho(G))\gamma(H)}{3}$, where $\rho(G)$ is the two-packing number of $G$. For more on attempts to solve Vizing’s conjecture over more than five decades since it was stated, see the survey [2].

Over the years, due to the unyielding nature of the conjecture, devotees have used offshoots of the domination number to attempt Vizing-type inequalities, in hopes of better understanding the difficulties of the original problem. For example, Brešar, Henning, and Rall [12] defined the paired and rainbow domination numbers, and Henning and Rall [12] conjectured a Vizing-type inequality for total domination. This last conjecture was proved by Ho [14], who showed that for any graphs $G$ and $H$, $\gamma_t(G \square H) \geq \frac{1}{2}\gamma_t(G)\gamma_t(H)$. In this result, $\gamma_t(G)$ is the total domination number of $G$, which is the minimum size of a set $T$ of vertices so that every vertex of $G$ is adjacent to some vertex in $T$. A sharp example was given in [12] and the characterization of pairs of graphs attaining equality is an active problem, see [3] and [15].

Since the difference between a totally dominating set and a dominating set is that every vertex in a totally dominating set must be adjacent to some other vertex in that set, while this rule does not have to hold in a dominating set, we find it instructive to consider Vizing-type inequalities for domination invariants that share properties with both domination and total domination. That is, we want to consider some domination function in between domination and total domination. Such a function, first investigated by Goddard, Henning, and McPillan [6], is the semi-total domination number of $G$, $\gamma_{t2}(G)$, which is the minimum size of a set of vertices $S$ in $G$, so that every vertex of $S$ is of distance at most 2 to some other vertex of $S$, and every vertex not in $S$ is adjacent to a vertex in $S$. Although introduced only a few years ago, this function has seen much recent attention, see [8, 9, 10, 11, 16, 18].

Although we cannot prove it, we believe that $\gamma_{t2}(G \square H) \geq \frac{1}{2}\gamma_{t2}(G)\gamma_{t2}(H)$ for any graphs $G$ and $H$. Our result depends on the method of Clark and Suen [5] and requires more careful analysis of semi-total dominating sets. We show that for any graphs $G$ and $H$, $\gamma_{t2}(G \square H) \geq \frac{1}{3}\gamma_{t2}(G)\gamma_{t2}(H)$.

Definitions and Notation. For notation and graph terminology, we will typically follow [13]. Throughout this paper, all graphs will be considered undirected, simple, connected, and finite. Specifically, let $G$ be a graph with vertex set $V = V(G)$ and edge
set \( E = E(G) \). Two vertices \( v, w \in V \) are neighbors, or adjacent, if \( vw \in E \). The open neighborhood of \( v \in V \) is the set of neighbors of \( v \), denoted \( N_G(v) \), whereas the closed neighborhood is \( N_G[v] = N_G(v) \cup \{v\} \). The open neighborhood of \( S \subseteq V \) is the set of all neighbors of vertices in \( S \), denoted \( N_G(S) \), whereas the closed neighborhood of \( S \) is \( N_G[S] = N_G(S) \cup S \). The distance between two vertices \( v, w \in V \) is the length of a shortest \((v, w)\)-path in \( G \), and is denoted by \( d_G(v, w) \). The Cartesian product of two graphs \( G(V_1, E_1) \) and \( H(V_2, E_2) \), denoted by \( G \square H \), is a graph with vertex set \( V_1 \times V_2 \) and edge set \( E(G \square H) = \{((u_1, v_1), (u_2, v_2)) : v_1 = v_2 \text{ and } (u_1, u_2) \in E_1, \text{ or } u_1 = u_2 \text{ and } (v_1, v_2) \in E_2\} \).

A subset of vertices \( S \subseteq V(G) \) is called a semi-total dominating set if \( N[S] = V(G) \) and for any vertex \( u \in S \), there exists a vertex \( v \in S \) so that \( d(u, v) \leq 2 \). The semi-total domination number of \( G \), written \( \gamma_{t2}(G) \), is the size of a minimum semi-total dominating set of \( G \). A 2-packing is a subset of vertices \( T \) of \( G \) so that every pair of vertices in \( T \) is of distance at least 3. The size of a maximum 2-packing of \( G \) is called the 2-packing number, and is written \( \rho(G) \).

We will also make use the standard notation \([k] = \{1, \ldots, k\}\), and for two vertices \( u, v \), we write \( u \sim v \) to indicate that \( u \) is adjacent to \( v \).

## 2 Main Results

In this section we provide our main results. We begin by establishing a Vizing’s-type result which makes use of the 2-packing number.

**Theorem 1** For any isolate-free graphs \( G \) and \( H \),

\[
\gamma_{t2}(G \square H) \geq \rho(G)\gamma_{t2}(H).
\]

**Proof.** Without loss of generality, we assume \( G \) is a \((\rho, \gamma)\)-graph (where \( \rho = \gamma \)), and let \( \{v_1, \ldots, v_{\rho(G)}\} \) be a maximum 2-packing of \( G \). Since each vertex from our packing is distance at least 3 from any other vertex of our packing, we observe that for \( i = 1, \ldots, \rho(G) \), the closed neighborhoods \( N[v_i] \) are pairwise disjoint. Let \( \{V_1, \ldots, V_{\rho(G)}\} \) be a partition of \( V(G) \) such that \( N[v_i] \subseteq V_i \). Let \( D \) be a \( \gamma_{t2}(G \square H) \)-set. For \( i = 1, \ldots, \rho(G) \), let \( D_i = D \cap (V_i \times V(H)) \), and let \( H_i = \{v_i\} \times V(H) \). Further, let \( S_i \) be a minimum set of vertices in \( G \square H \) that semi-totally dominates \( H_i \), and contains as many vertices in \( H_i \) as possible. Then, \( S_i \subseteq V_i \times V(H) \). Next suppose that \( S_i \) contains a vertex \( x \) such that \( x \) is not in \( H_i \). Then, \( x \) is the unique vertex which semi-totally dominates \( x' \), for some \( x' \in H_i \). Since \( x' \) has neighbors, all of which are dominates by vertices in \( S_i \), if we replace \( x \) by \( x' \) in \( S_i \), we see that \( S_i \) is still semi-total dominating (Since \( x' \) is at distance at least 2 from a vertex which dominates one of its neighbors). Moreover, we have found a set of vertices from \( G \square H \) that semi-totally dominates \( H_i \) and contains more vertices in \( H_i \) than does \( S_i \), a contradiction. Hence, we have \( S_i \subseteq H_i \), and so \( S_i \) is
a semi-total dominating set of the copy of $H$ in $G \Box H$ induced by the set $H_i$. Since $D_i$ semi-totally dominates $\{v_i\} \times V(H)$, $|D_i| \geq |S_i|$. Thus,

$$
\gamma_{t2}(G \Box H) \geq \sum_{i=1}^{\rho(G)} |S_i| \geq \sum_{i=1}^{\rho(G)} \gamma_{t2}(H) = \rho(G)\gamma_{t2}(H).
$$

\[ \square \]

Next, we prove a Vizing’s type result which relies only on the semi-total domination number. We do this by partitioning minimum semi-total dominating sets into parts that are and are not totally dominating. Notice that for any graph $G$, if $U = \{u_1, \ldots, u_k\}$ is a minimum semi-total dominating set of $G$, then $U$ can be separated into two sets, $X$ and $Y$, where $X$ is the set of vertices of $U$ which are adjacent to at least one other vertex of $U$, and $Y = U \setminus X$. We call such sets $X$, allied and such sets $Y$, free.

For any graph $G$, consider the set of minimum semi-total dominating sets of vertices, $\{U_1, \ldots, U_k\}$, and for $1 \leq i \leq k$ let $X_i$ and $Y_i$ be partitions of $U_i$ into allied and free sets, respectively. We call $U_i$ so that $|X_i|$ is of maximum size for $1 \leq i \leq k$ a maximum allied semi-total dominating set of $G$, the partition $\{X_i, Y_i\}$ a maximum allied partition of $G$, the set $X_i$ a maximum allied set of $G$, and the set $Y_i$ a minimum free set of $G$.

For any maximum allied partition of $G$, $\{X, Y\}$, let $x(G) = |X|$ and $y(G) = |Y|$.

**Theorem 2** For any isolate-free graphs $G$ and $H$,

$$
\gamma_{t2}(G \Box H) \geq \frac{1}{3} \gamma_{t2}(G)\gamma_{t2}(H).
$$

**Proof.** Let $D$ be a minimum semi-total dominating set of $G \Box H$. Let $k = \gamma_{t2}(G)$ and $U = \{u_1, \ldots, u_k\}$ be a maximum allied semi-total dominating set of $G$ with maximum allied partition $\{X, Y\}$. Suppose $X = \{u_1, \ldots, u_\ell\}$ and $Y = \{u_{\ell+1}, \ldots, u_{\ell+m}\}$.

Form a partition $\{\pi_1, \ldots, \pi_\ell, \pi_{\ell+1}, \ldots, \pi_{\ell+m}\}$ of $V(G)$ where $\pi_i \subseteq N(u_i)$ and $x \in \pi_i$ implies $x$ is adjacent to $u_i$ for $1 \leq i \leq \ell$, $\pi_j \subseteq N[u_j]$ and $x \in \pi_j$ implies $x$ is adjacent to $u_j$ for $\ell+1 \leq j \leq \ell+m$. Furthermore, we define this partition to have the property that if $u_i \in X$ and $u_j \in Y$ so that $d(u_i, u_j) = 2$, then $N(u_i) \cap N(u_j) \cap \pi_j = \emptyset$. That is, for any vertex $u_j$ of $Y$ which is of distance 2 to some vertex of $X$, there exists a vertex $u$ which is adjacent to $u_j$ and to a vertex of $X$, and $u$ belongs to $\pi_i$ for some $i \in [\ell]$.

Let $D_i = (\pi_i \times V(H)) \cap D$. Let $P_i = \{v : (u, v) \in D_i \text{ for some } u \in \pi_i\}$, which are the projections of $D_i$ onto $H$. We call vertices of $V(H)$ missing, if they are not dominated from $P_i$ and write $M_i = V(H) - N_H[P_i]$. Vertices of $P_i$ which are of distance at most 2 to some other vertex of $P_i$ or $M_i$ we call covered and write $Q_i = \{v \in P_i : \exists w \in P_i \cup M_i \text{ such that } 0 < d(v, w) \leq 2\}$. Vertices of $P_i$ of distance at least 3 to other vertices of $P_i$ or $M_i$ we call uncovered and write $R_i = \{v \in P_i : \forall w \in (P_i \cup M_i) \setminus \{v\}, d(v, w) \geq 3\}$.

For $v \in V(H)$, let

$$
D^v = D \cap (V(G) \times \{v\}) = \{(u, v) \in D : u \in V(G)\}
$$

For any isolate-free graphs $G$ and $H$,
and $C$ be a subset of $\{1, \ldots, k\} \times V(H)$ given by
\[
C = \{(i, v) : \pi_i \times \{v\} \subseteq N_{G \square H}(D^v) \text{ or } v \in R_i\}.
\]

Let $N = |C|$. We will bound $N$ from above by considering the following.
\[
\mathcal{L}_i = \{(i, v) \in C : v \in V(H)\},
\]
\[
\mathcal{R}^v = \{(i, v) \in C : 1 \leq i \leq k\}.
\]

These definitions are well-known as they appeared in the seminal work [5], nonetheless, we would like to remind the reader of their interpretation. The set $C$ is a double indexing set, which indicates where you have cells that are either horizontally dominated or dominated by vertices of $R_i$. A cell is just a copy of $\pi_i$ for some $i$, at some height $v \in V(H)$. We represent $G$ along the horizontal axis of the Cartesian product and $H$ along the vertical. Thus, horizontally dominated cells are precisely, $\pi_i \times \{v\}$ which is contained in $N_{G \square H}(D^v)$. Now, $L_i$ are elements of $C$ with a fixed $i$ and $R^v$ are elements of $C$ along a fixed $v$.

Since counting vertices vertically and horizontally produces the same amount, we have
\[
N = \sum_{i=1}^{k} |\mathcal{L}_i| = \sum_{v \in V(H)} |\mathcal{R}^v|.
\]

Notice that if $v \in M_i$, then the vertices in $\pi_i \times \{v\}$ which are not in $D^v$ must be adjacent to the vertices in $D^v$ since $D$ is a semi-total dominating set of $G \square H$. Furthermore, the vertices of $R_i$ are counted in $\mathcal{L}_i$. This means that $|\mathcal{L}_i| \geq |M_i| + |R_i|$. Hence we obtain the following lower bound for $N$,
\[
N \geq \sum_{i=1}^{k} (|M_i| + |R_i|) \tag{1}
\]

To find an upper bound on the above quantity, we bound the size of $\mathcal{R}^v$.

**Claim 1** For any $v \in V(H)$, $|\mathcal{R}^v| \leq 2|D^v|$.

**Proof.** Suppose $|\mathcal{R}^v| > 2|D^v|$ for some $v \in V(H)$. For $(i, v) \in \mathcal{R}^v$, by definition, $\pi_i \times \{v\} \subseteq N_{G \square H}(D^v)$ or $v \in R_i$.

In what follows, we construct a semi-total dominating set $T$ of $G$.

In the first case, if $\pi_i \times \{v\} \subseteq N_{G \square H}(D^v)$, we note that if some vertex $x \in \pi_i$, then $x$ is adjacent to vertices in $B^v$ where $B^v$ is the projection of $D^v$ onto $G$. 
Subcase 1. Suppose \( u \in B^v \). If \( u \in \pi_i \) such that \((i, v) \notin R^v, u \neq u_i \) and \( 1 \leq i \leq \ell + m \), then \( u \in N(u_i) \). If \( u \in \pi_i \) such that \((i, v) \in R^v \), then there exists \((u', v) \in B^v \) such that \( u \in N(u') \). If \( u \in \pi_i \) such that \((i, v) \notin R^v, u = u_i \) for some \( \ell + 1 \leq i \leq \ell + m \), then notice that we can find a vertex \( x_i \) which is a neighbor of \( u_i \) in \( \pi_i \). Note that \( x_i \) need not be a member of \( B^v \), but simply a neighbor of \( u_i \). Select one such vertex \( x_i \) for every such \( u \), and let \( A \) be the set of these vertices \( x_i \). Thus, \( B^v \subseteq T, A := \{ u_i : (i, v) \notin R^v, u_i \notin B^v, 1 \leq i \leq \ell + m \} \subseteq T \), and \( \{ x_i : (i, v) \notin R^v, x_i \sim u \) for some \( u \in U \cap B^v \} \subseteq T \).

Subcase 2. Suppose \( u \in \{ u_i : (i, v) \notin R^v, 1 \leq i \leq \ell \} \). If \( u \in \pi_j \) such that \((j, v) \notin R^v \), then \( u \in N(u_j) \). If \( u \in \pi_j \) such that \((j, v) \in R^v \), then there exists \((u', v) \in B^v \) such that \( u \in N(u') \). Thus, in this subcase, \( u \) is adjacent either to a vertex of \( B^v \) or a vertex \( u_j \). There are no new vertices that need to be added to \( T \).

Subcase 3. Suppose \( u \in \{ u_i : (i, v) \notin R^v, \ell + 1 \leq i \leq \ell + m \} \). Suppose \( u \) is of distance 2 to some vertex \( u_j \in X \). By the definition of the partition, there exists some vertex \( w \) adjacent to \( u \) and \( u_j \), so that \( w \in \pi_{j'} \) for some \( j' \in \llbracket \ell \rrbracket \). If \((j', v) \in R^v \), then there exists \( u' \in B^v \) so that \( u' \sim w \sim u \), which means that \( u \) is of distance at least 2 to some vertex of \( B^v \). Since \( T \) contains \( B^v \), these vertices are already distance 2 from another vertex in \( T \).

We are left to consider the case when \( u \) is of distance at least 3 to any vertex of \( X \). Since \( U \) is a minimum semi-total dominating set of \( G \), there exists some vertex \( u_j \in Y \), so that \( d(u, u_j) = 2 \). If \((j, v) \notin R^v \), these vertices are already in \( T \) so no action needs to be taken.

If \((j, v) \in R^v \), then there exists some vertex \( u' \in B^v \) so that \( u' \sim u_j \). We will select \( u_j \) and place it in \( T \) to make \( T \) a semi-total dominating set of \( G \). Notice that in this case, the number of such vertices \( u_j \) is at most equal to \( |D^v| \). Let \( S \) be the set of such vertices \( u_j \), which are of distance 2 to a vertex \( u \in Y \) and at least of distance 3 to any vertex of \( X \). Then \( S \) will be a subset of the set \( T \). This finishes Subcase 3.

In the second case, if \( v \in R_i \), then since \( D \) is a semi-total dominating set, there is some vertex \( (u, v) \in (\pi_i \times \{v\}) \cap D^v \) and \( (w, v) \in (\pi_j \times \{v\}) \cap D^v \), for some \( j \in [k] \), so that \((u, v) \) is at most distance 2 from \((w, v) \).

Putting these cases together, we have the following disjoint union of sets

\[
T = B^v \cup \{ u_i : (i, v) \notin R^v, 1 \leq i \leq \ell \} \cup \{ u_i : (i, v) \notin R^v, u_i \notin B^v, \ell + 1 \leq i \leq \ell + m \} \\
\cup A \cup S
\]

(2)

To show \( T \) is a semi-total dominating set of \( G \), it is enough to show that \( T \) is a dominating set, since we showed in each subcase of the first case, and in the second case, that every vertex of \( T \) is of distance at most 2 to some other vertex of \( T \). If a vertex \( u \) is contained in \( \pi_i \) for \((i, v) \in R^v \), then \( u \) is dominated by some vertex of \( B^v \). If \((i, v) \notin R^v \), then \( u \) is dominated either by \( \{ u_i : (i, v) \notin R^v, 1 \leq i \leq \ell \} \), or \( \{ u_i : (i, v) \notin R^v, u_i \notin B^v, \ell + 1 \leq i \leq \ell + m \} \), or \( A \).
Furthermore,

\[ |T| = |B^v| + (\gamma_2(G) - |R^v| + |S|) = 2|D^v| + (\gamma_2(G) - |R^v|) < \gamma_2(G) \]

which is a contradiction. (\ covenant)

Thus, by claim [1]

\[ N = \sum_{v \in V(H)} |R^v| \leq \sum_{v \in V(H)} 2|D^v| = 2|D| \quad (3) \]

We now show a semi-total dominating set of \( H \) in terms of \( M_i \).

**Claim 2** For any \( i \in [k] \), there exists a set \( X_i \) of at most \( |R_i| - 1 \) vertices of \( V(H) \) so that \( M_i \cup P_i \cup X_i \) is a semi-total dominating set of \( H \).

**Proof.** We first observe that \( P_i \cup M_i \) is a dominating set of \( H \) with the additional property that the vertices of \( M_i \) dominate only themselves, not their neighbors. Thus, every vertex \( x \in R_i \) must be either of distance 3 to some vertex \( y \in R_i \) or every vertex of distance 2 from \( x \) is a vertex of \( M_i \). This holds since otherwise some vertex of distance 2 from \( x \) is not dominated by \( P_i \cup M_i \). Furthermore, if \( x \in R_i \) which is of distance 3 to some vertex \( y \in R_i \), then we may select one vertex \( z \) on a path from \( x \) to \( y \) such that \( z \) is of distance at most 2 to both \( x \) and \( y \).

We now construct a semi-total dominating set of \( H \), \( T_i \), by including the vertices of \( M_i \), the vertices of \( P_i \) and vertices \( X_i \) which are of distance at most 2 to two vertices of \( R_i \) which are themselves of distance three to each other. The minimum number of such vertices is at most \( |R_i| - 1 \), which can be easily verified by induction on \( |R_i| \), and the result follows. (\ covenant)

By Claim [2] for each \( i \), we can construct a semi-total dominating set of \( H \), \( T_i = M_i \cup R_i \cup Q_i \cup X_i \). This gives \( |M_i| + |R_i| \geq \gamma_2(H) - |X_i| - |Q_i| \). However, note that \( X_i \cap Q_i = \emptyset \) and \( |X_i| + |Q_i| \leq |P_i| \). This implies that \( |M_i| + |R_i| \geq \gamma_2(H) - |P_i| \). Thus, we have

\[ \sum_{i=1}^{k} \left( |M_i| + |R_i| \right) \geq \sum_{i=1}^{k} \left( \gamma_2(H) - |P_i| \right) = \gamma_2(G)\gamma_2(H) - |D| \quad (4) \]

Combining equations (1), (3), and (4) we obtain

\[ |D| \geq \frac{1}{3} \gamma_2(G)\gamma_2(H) \]

\( \square \)
3 Conclusion

In this paper we have proven two Vizing’s like results on the semi-total domination number. Our main result shown in Theorem 2 shows that for isolate-free graphs $G$ and $H$, it must be the case that $\gamma_{t2}(G \square H) \geq \frac{1}{3}\gamma_{t2}(G)\gamma_{t2}(H)$. However, we do not believe this bound is sharp, and conjecture a stronger result.

Conjecture 1 For any isolate-free graphs $G$ and $H$,

$$\gamma_{t2}(G \square H) \geq \frac{1}{2}\gamma_{t2}(G)\gamma_{t2}(H).$$

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