RESONANCES AS VISCOSITY LIMITS FOR EXTERIOR DILATION ANALYTIC POTENTIALS

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Abstract. For exterior dilation analytic potential, $V$, we use the method of complex scaling to show that the resonances of $-\Delta + V$, in a conic neighbourhood of the real axis, are limits of eigenvalues of $-\Delta + V - i\epsilon x^2$ as $\epsilon \to 0+$, if $V$ can be analytically extended from $\mathbb{R}^n$ to a truncated cone in $\mathbb{C}^n$.

1. Introduction and statement of results

We extend the results of [Z2], when $V \in L^\infty_{\text{comp}}$, to the case of exterior dilation analytic potentials. For motivation and pointers to the literature we refer to [Z2].

Thus, we consider

$$H := -\Delta + V,$$

where $V$ is a real-valued potential which can be analytically extended from $\{x \in \mathbb{R}^n : |x| > R\}$, for some $R > 0$, to a truncated cone

$$C^R_{\beta_0} := \{z \in \mathbb{C}^n : |\text{Im} z| < \tan \beta_0 |\text{Re} z| \text{ and } |\text{Re} z| > R\}, \quad \beta_0 \leq \pi/8.$$

We still denote the analytic extension by $V$ and assume that

$$\lim_{C^R_{\beta_0} \ni |z| \to \infty} V(z) = 0. \quad (1.1)$$

The resonances of $H$ are defined by the Aguilar-Balslev-Combes-Simon theory, see [HS, §16, §18], [DyZ2, §4.5] and a review in §3.

We now introduce a regularized operator,

$$H_\epsilon := -\Delta - i\epsilon x^2 + V, \quad \epsilon > 0. \quad (1.2)$$

(We write $x^2 := x_1^2 + \cdots + x_n^2$.) It is easy to see, with details reviewed in §4, that $H_\epsilon$ is a non-normal unbounded operator on $L^2(\mathbb{R}^n)$ with a discrete spectrum. We have

\textbf{Theorem 1.} Suppose that $\{z_j(\epsilon)\}^\infty_{j=1}$ are the eigenvalues of $H_\epsilon$. Then, uniformly on any compact subsets of $\{z : -2\beta_0 < \arg z < 3\pi/2 + 2\beta_0\}$,

$$z_j(\epsilon) \to z_j, \quad \epsilon \to 0+,$$

where $z_j$ are the resonances of $H$. 

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**Notation.** We use the following notation: \( f = \mathcal{O}_\ell(g)_H \) means that \( \|f\|_H \leq C_\ell g \) where the norm (or any seminorm) is in the space \( H \), and the constant \( C_\ell \) depends on \( \ell \). When either \( \ell \) or \( H \) are absent then the constant is universal or the estimate is scalar, respectively. When \( G = \mathcal{O}_\ell(g) : H_1 \to H_2 \) then the operator \( G : H_1 \to H_2 \) has its norm bounded by \( C_\ell g \). Also when no confusion is likely to result, we denote the operator \( f \mapsto gf \) where \( g \) is a function by \( g \).

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### 2. Spectral Deformation and Analytic Vectors

We will review several basic concepts in the Aguilar-Balslev-Combes-Simon theory, such as spectral deformation and analytic vectors. For a detailed introduction, we refer to [HS, §17] and the references given there.

Let \( h \in C_\infty(\mathbb{R}) \) be a non-decreasing function which satisfies

\[
\begin{cases}
  h(t) = 0, & t < 2R, \\
  h(t) = 1, & t > 8R.
\end{cases}
\]  

(2.1)

Moreover, we assume that

\[
\sup_{t \in \mathbb{R}} h(t) + th'(t) \leq 3/2.
\]  

(2.2)

We define \( g : \mathbb{R}^n \to \mathbb{R}^n \) as a smooth mapping by

\[
g(x) := h(|x|) x = \begin{cases}
  0, & |x| < 2R, \\
  x, & |x| > 8R,
\end{cases}
\]  

(2.3)

and consider, for \( \theta \in \mathbb{R} \), the related family of maps \( \phi_\theta : \mathbb{R}^n \to \mathbb{R}^n \) defined by

\[
\phi_\theta(x) = x + \theta g(x)
\]  

(2.4)

We let \( Df \) denote the derivative of a map \( f : \mathbb{R}^n \to \mathbb{R}^n \), then

\[
Dg(x) = h(|x|)I + |x|^{-1}h'(|x|)x \cdot x^T.
\]

Using diagonalization, It is easy to see that

\[
0 \leq h(|x|)I \leq Dg(x) \leq (h(|x|) + |x|h'(|x|))I \leq 3/2 I
\]  

(2.5)

where \( A \leq B \) means \( B - A \) is positive semi-definite and the last inequality is implied by (2.2). Hence \( \sup_{x \in \mathbb{R}^n} \|Dg(x)\| \leq 3/2 \), where \( \| \cdot \| \) denotes the operator norm on the
set of linear transformation on \( \mathbb{R}^n \). We note that \( D\phi_\theta(x) = I + \theta(Dg)(x) \), if \( |\theta| < 2/3 \), then \( D\phi_\theta \) is invertible by a Neumann series argument,

\[
(D\phi_\theta)^{-1} = \sum_{j=0}^{\infty} (-1)^j \theta^j (Dg)^j.
\]

Hence \( \phi_\theta \) is a diffeomorphism of \( \mathbb{R}^n \) for \( |\theta| < 2/3 \) by the inverse function theorem.

We should remark that all the above argument is valid when we extend the definition (2.4) of \( \phi_\theta \) to \( \theta \in \mathbb{C} \). We have \( \phi_\theta : \mathbb{R}^n \rightarrow \mathbb{C}^n \) is a diffeomorphism provided \( |\theta| < 2/3 \).

We now introduce the behavior of functions under the action of the maps \( \phi_\theta \). We first define \( U_\theta \) for \( \theta \in \mathbb{R} \) by

\[
(U_\theta f)(x) = J_\theta(x)^{1/2} f(\phi_\theta(x))
\]

(2.6)

where \( J_\theta(x) \) is the Jacobian of \( \phi_\theta \),

\[
J_\theta(x) = \det D\phi_\theta(x) = \det(I + \theta(Dg)(x)).
\]

(2.7)

It is easy to see that \( U_\theta \), \( \theta \in \mathbb{R} \) is unitary on \( L^2(\mathbb{R}^n) \) with the inverse \( U_\theta^{-1} \) given by

\[
(U_\theta^{-1} f)(x) = J_\theta(\phi_\theta^{-1}(x))^{-1/2} f(\phi_\theta^{-1}(x)).
\]

(2.8)

(2.5) and (2.7) show that \( J_\theta(x)^{1/2} \) extends analytically to complex \( \theta \) provided \( \theta < 2/3 \). Hence, to extend the operators \( U_\theta \) from \( \theta \in \mathbb{R} \) to \( \theta \in \mathbb{C} \), at least for small \( |\theta| \), we need to find a dense set of functions \( f \) in \( L^2(\mathbb{R}^n) \) that can be analytically extended on a small complex neighborhood of \( \mathbb{R}^n \) in \( \mathbb{C}^n \) such that \( f \circ \phi_\theta \in L^2(\mathbb{R}^n) \). For that we introduce the set of analytic vectors in \( L^2(\mathbb{R}^n) \).

**Definition 1.** Let \( A \) be the linear space of all entire functions \( f(z) \) having the property that in any conical region \( C_\varepsilon \),

\[
C_\varepsilon := \{ z \in \mathbb{C}^n : |\text{Im } z| \leq (1 - \varepsilon) \text{Re } z \},
\]

for any \( \varepsilon > 0 \), we have for any \( k \in \mathbb{N} \),

\[
\lim_{z \in C_\varepsilon \rightarrow \infty} |z|^k |f(z)| = 0.
\]

The set of analytic vectors in \( L^2(\mathbb{R}^n) \) are the restrictions to \( \mathbb{R}^n \) of \( A \), which is also denoted by \( A \).

We define a domain \( D_{\beta_0} \) in \( \mathbb{C} \) by

\[
D_{\beta_0} = \{ \theta \in \mathbb{C} : |\text{Re } \theta| + |\text{Im } \theta| < \tan \beta_0 \}.
\]

(2.9)

Note that \( D_{\beta_0} \subset \{ z \in \mathbb{C} : |z| < 1/2 \} \) since \( \beta_0 \leq \pi/8 \), (2.5) and (2.7) guarantee that the Jacobian \( J_\theta \) is uniformly bounded for \( \theta \in D_{\beta_0} \). Then, we recall the following results in [HS, Proposition 17.10]:
Proposition 2. Let $\mathcal{U} \equiv \{U_\theta : \theta \in D_{\beta_0}\}$ be a spectral deformation family associated with vector field $g$ defined by (2.3). Then,

- the map $(\theta, f) \in D_{\beta_0} \times \mathcal{A} \rightarrow U_\theta f$ is an $L^2$-analytic map;
- for any $\theta \in D_{\beta_0}$, $U_\theta \mathcal{A}$ is dense in $L^2(\mathbb{R}^n)$.

We conclude this section with some properties about the deformation of $\mathbb{R}^n \subset \mathbb{C}^n$ under the map $\phi_\theta$ provided $\theta \in D_{\beta_0}$. We recall that $\phi_\theta : \mathbb{R}^n \rightarrow \mathbb{C}^n$ is injective with the Jacobian $J_\theta \neq 0$ provided $|\theta| < 2/3$. Hence $\phi_\theta(\mathbb{R}^n) \subset \mathbb{C}^n$ is an $n$-dimensional totally real submanifolds, see [DyZ2, §4.5]. Let $\Gamma_{a(\theta)} = \phi_\theta(\mathbb{R}^n)$, where $a(\theta) \in (-\pi/2, \pi/2)$ is defined by

$$a(\theta) = \arg(1 + \theta). \quad (2.10)$$

In the literature about complex scaling, one can define $L^2(\mathbb{R}^n)$ with volume element $|dw| = |J_\theta(x)| \, dx$ where $w = \phi_\theta(x)$ are the coordinates on $\Gamma_{a(\theta)}$, see [DyZ2, §2.7, §4.5] for details. Then we have the following:

Proposition 3. For any $\theta \in D_{\beta_0}$, $\Gamma_{a(\theta)}$ satisfies

$$\begin{align*}
\Gamma_{a(\theta)} \cap B_{\mathbb{C}^n}(0, 2R) &= B_{\mathbb{R}^n}(0, 2R), \\
\Gamma_{a(\theta)} \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n}(0, 12R) &= e^{ia(\theta)R} \mathbb{R}^n \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n}(0, 12R), \\
\Gamma_{a(\theta)} &\subset \mathbb{R}^n \cup C_{\beta_0}^R.
\end{align*} \quad (2.11)$$

Furthermore, the spectral deformation operator $U_\theta$ extends to an isometry:

$$U_\theta : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n).$$

Proof. In view of (2.3) and (2.4), it is easy to see that $\Gamma_{a(\theta)} = \phi_\theta(\mathbb{R}^n)$ satisfies the first two equations of (2.11). For $\theta \in D_{\beta_0}$, we have

$$\frac{|\text{Im } \phi_\theta(x)|}{|\text{Re } \phi_\theta(x)|} = \frac{|\text{Im } \theta||\chi(|x|)|}{|1 + \text{Re } \theta \chi(|x|)|} \leq \frac{|\text{Im } \theta|}{|1 - |\text{Re } \theta||} < \tan \beta_0,$$

where the last inequality is implied by (2.9). Moreover, $\phi_\theta(x) = x$ for $|x| < 2R$, and $|\text{Re } \phi_\theta(x)| \geq (1 - |\text{Re } \theta||x| > (1 - \tan \beta_0)|x| > |x|/2 \geq R$ provided $|x| \geq 2R$, since $\beta_0 \leq \pi/8$. Hence $\Gamma_{a(\theta)} \subset \mathbb{R}^n \cup C_{\beta_0}^R$.

Now we assume that $\theta \in D_{\beta_0}$, for any $f \in L^2(\Gamma_{a(\theta)})$, we can define $U_\theta f$ on $\mathbb{R}^n$ by (2.6). To see $U_\theta f \in L^2(\mathbb{R}^n)$, we compute directly:

$$\int_{\mathbb{R}^n} |U_\theta f(x)|^2 \, dx = \int_{\mathbb{R}^n} |J_\theta(x)^{1/2} f(\phi_\theta(x))|^2 \, dx \quad (2.12)$$

$$\begin{align*}
&= \int_{\mathbb{R}^n} |f(\phi_\theta(x))|^2 |J_\theta(x)| \, dx \\
&= \int_{\Gamma_{a(\theta)}} |f(w)|^2 |dw| < \infty.
\end{align*}$$
which also shows that \( \|U_\theta f\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)} \) and \( U_\theta \) is one-to-one. It remains to show that \( U_\theta \) is onto. For \( g \in L^2(\mathbb{R}^n) \), let \( G(w) = J_\theta(\phi_\theta^{-1}(w))^{-1/2}g(\phi_\theta^{-1}(w)), w \in \Gamma_a(\theta). \) We can follow (2.12) to derive

\[
\int_{\Gamma_a(\theta)} |G(w)|^2 |dw| = \int_{\mathbb{R}^n} |g(x)|^2 dx.
\]

Hence \( G \in L^2(\Gamma_a(\theta)) \), then we conclude that \( U_\theta \) is onto since \( U_\theta G = g \). \( \square \)

### 3. Resonances

We will follow Aguilar-Balslev-Combes-Simon theory to define the resonances of \( H \equiv -\Delta + V \), see [HS, §16, §18] and those resonances in a conic neighborhood of the real axis can be identified with the eigenvalues of certain non-self-adjoint operators associated with \( H \). Using the analytic vectors \( \mathcal{A} \), we recall the definition:

**Definition 4.** The resonances of \( H \) associated with analytic vectors \( \mathcal{A} \) are the poles of the meromorphic continuations of all matrix elements \( \langle f, R_H(z)g \rangle \) \( (R_H(z) \) denotes the resolvent of \( H \), \( f, g \in \mathcal{A} \) from \( \{z \in \mathbb{C} : \text{Im } z > 0\} \) to \( \{z \in \mathbb{C} : \text{Im } z \leq 0\} \).

First, we introduce the spectral deformed Schrödinger operators \( H(\theta) \) of \( H \) associated with the spectral deformation family \( \mathcal{U} = \{U_\theta : \theta \in D_{\beta_0}\} \). Consider, for \( \theta \in D_{\beta_0} \cap \mathbb{R} \),

\[
H(\theta) := U_\theta HU_\theta^{-1} = p_\theta^2 + V(\phi_\theta(x)),
\]

where

\[
p_\theta^2 = U_\theta p^2 U_\theta^{-1}, \quad p_j \equiv \frac{1}{i} \frac{\partial}{\partial x_j}.
\]

In view of Proposition 3, we can extend \( H(\theta) \) to \( \theta \in D_{\beta_0} \). We recall the following basic facts about \( p_\theta^2, \theta \in D_{\beta_0} \) in [HS, §18]:

**Proposition 5.** Let \( p_\theta^2 \) be as defined in (3.2), then \( p_\theta^2, \theta \in D_{\beta_0} \) is an analytic family of operators with domain \( D(p_\theta^2) = H^2(\mathbb{R}^n) \). For the spectrum, we have \( \sigma(p_\theta^2) = \sigma_{\text{ess}}(p_\theta^2) = e^{-2\alpha(\theta)}[0, \infty) \).

And for the resolvent \( R_\theta(z) := (p_\theta^2 - z)^{-1} \) we have:

**Proposition 6.** For \( \delta > 0 \), we have

\[
R_\theta(z) = \mathcal{O}_\delta(1/|z|) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n), \quad -2\alpha(\theta) + \delta < \arg z < 2\pi - 2\alpha(\theta) - \delta. \quad (3.3)
\]

**Proof.** We note that in the notation of Proposition 3,

\[
p_\theta^2 = U_\theta(-\Delta_{\alpha(\theta)})U_\theta^{-1}, \quad (3.4)
\]
where \(-\Delta_{a(\theta)} : H^2(\Gamma_{a(\theta)}) \to L^2(\Gamma_{a(\theta)})\) is defined as the restriction of \(\Delta\), to the totally real submanifold \(\Gamma_{a(\theta)}\), see [DyZ2, §4.5]. Since \(U_\theta : L^2(\Gamma_{a(\theta)}) \to L^2(\mathbb{R}^n)\) and \(U_\theta^{-1} : L^2(\mathbb{R}^n) \to L^2(\Gamma_{a(\theta)})\) are both isometries, we have

\[
\|(p_\theta^2 - z)^{-1}\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} = \|(-\Delta_{a(\theta)} - z)^{-1}\|_{L^2(\Gamma_{a(\theta)}) \to L^2(\Gamma_{a(\theta)})}, \quad z \notin e^{-2ia(\theta)}[0, \infty),
\]

and thus (3.3) is a direct consequence of [DyZ2, Theorem 4.35].

Then we introduce some preliminary properties of the spectrum of \(H(\theta)\):

**Proposition 7.** There exists \(R > 0\) such that for any \(\theta \in D_{\beta_0}\), we have

\[
\sigma(H(\theta)) \cap i(R, \infty) = \emptyset.
\]

As for the essential spectrum \(\sigma_{ess}(H(\theta))\), we have more precisely,

\[
\sigma_{ess}(H(\theta)) = e^{-2ia(\theta)}[0, \infty).
\]

**Remark:** In fact, \(\sigma(H(\theta)) \cap \{z : 0 < \arg z < 2\pi - 2a(\theta)\}\) is discrete and lies in \((-\infty, 0)\), which is a consequence of the following Lemma 1.

**Proof.** For \(\theta \in D_{\beta_0}\), we have

\[
(H(\theta) - z)R_\theta(z) = I + V(\phi_\theta(x))R_\theta(z).
\]

Now assume \(z \in i(R, \infty)\), note that \(\theta \in D_{\beta_0} \implies -\beta_0 < a(\theta) < \beta_0\), we have

\[
-2a(\theta) + \pi/4 < \arg z < 2\pi - 2a(\theta) - \pi/4, \quad \text{for all } \theta \in D_{\beta_0}.
\]

Using (3.3), we see that \(R_\theta(z) = \mathcal{O}(1/|z|) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\) for all \(z \in i(R, \infty)\) and \(\theta \in D_{\beta_0}\). Recalling \(\phi_\theta(\mathbb{R}^n) \subset \mathbb{R}^n \cup C^R_{\beta_0}\) and \(V \in L^\infty(\mathbb{R}^n \cup C^R_{\beta_0})\), we conclude that

\[
\sup_{z \in i(R, \infty)} \|V(\phi_\theta(x))R_\theta(z)\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} = \mathcal{O}(R^{-1}), \quad \text{uniformly for } \theta \in D_{\beta_0}.
\]

Then for \(R \gg 1\), \(I + V(\phi_\theta(x))R_\theta(z)\) is invertible using the Neumann series:

\[
(I + V(\phi_\theta(x))R_\theta(z))^{-1} = \sum_{j=0}^{\infty} (V(\phi_\theta(x))R_\theta(z))^j.
\]

Hence \(H(\theta) - z\) is invertible by (3.5), for all \(z \in i(R, \infty)\).

For the essential spectrum \(\sigma_{ess}(H(\theta))\), note that \(\sigma_{ess}(p_\theta^2) = e^{-2ia(\theta)}[0, \infty)\) in Proposition 5, by the invariance under compact perturbations, it suffices to show that \(V(\phi_\theta(x))\) is \(p_\theta^2\)-compact, i.e. \(V(\phi_\theta(x)) : D(p_\theta^2) = H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\) is compact. Since \(H^2(B_{\mathbb{R}^n}(0, R)) \subset L^2(B_{\mathbb{R}^n}(0, R)), \forall R > 0\), and \(V \circ \phi_\theta \in L^\infty(\mathbb{R}^n), \; V(\phi_\theta(x)) \to 0, \; x \to \infty\) by (1.1), it is easy to see the compactness of \(V(\phi_\theta(x)) : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\).

Now we state the main result in this section, in which we identify the resonances defined in Definition 4 as the eigenvalues of certain spectral deformed operators \(H(\theta)\).
Lemma 1. Let $H = -\Delta + V$ be a self-adjoint Schrödinger operator with a real-valued potential $V$ satisfying our assumptions as in §1. Then for any $\theta \in D_{\beta_0} \cap \mathbb{C}^+$, we have:

- For $f, g \in \mathcal{A}$, the function
  \[ F_{f,g}(z) \equiv \langle f, R_H(z)g \rangle, \]  
  defined for $\text{Im} \, z > 0$, has a meromorphic continuation across $[0, \infty)$ into $S_\theta^- \equiv \mathbb{C} \setminus e^{-2a(\theta)}[0, \infty)$.

- The poles of the meromorphic continuations of all matrix elements $F_{f,g}(z)$ into $S_\theta^-$ are eigenvalues of the operator $H(\theta)$.

Proof. With $F_{f,g}(z)$ defined in (3.6), the assumption on $V$ implies that $F_{f,g}$ is analytic on $\mathbb{C}^+ \equiv \{ z \in \mathbb{C} : \text{Im} \, z > 0 \}$. Fix $z \in \mathbb{C}^+$. For $\theta \in D_{\beta_0} \cap \mathbb{R}$, $U_\theta$ is unitary and thus we can write
  \[ F_{f,g}(z) = \langle U_\theta f, (U_\theta R_H(z)U_\theta^-1)U_\theta g \rangle = \langle U_\theta f, R_{H(\theta)}(z)U_\theta g \rangle. \]  
Proposition 7 implies that $\theta \in D_{\beta_0} \rightarrow R_{H(\theta)}(z)$ is an analytic map provided $z \in i(R, \infty)$. Since we can write $U_\theta f$ instead of $U_\theta f$ in (3.7), we have
  \[ \theta \in D_{\beta_0} \rightarrow \theta = F_{f,g}(z; \theta) \equiv \langle U_\theta f, R_{H(\theta)}(z)U_\theta g \rangle \]  
is an analytic map provided $z \in i(R, \infty)$. Hence for any $z \in i(R, \infty)$, we have
  \[ F_{f,g}(z; \theta) = F_{f,g}(z), \quad \forall \theta \in D_{\beta_0}, \]  
since this is true for all $\theta \in D_{\beta_0} \cap \mathbb{R}$. Now fix any $\theta \in D_{\beta_0} \cap \mathbb{C}^+$, Proposition 7 guarantees that $F_{f,g}(z; \theta)$ can be meromorphically continued from $i(R, \infty)$ to $S_\theta^-$ since $\sigma_{\text{ess}}(H(\theta)) \cap S_\theta^- = \emptyset$. We have shown that $F_{f,g}(z; \theta) = F_{f,g}(z), \, z \in i(R, \infty)$, then by the identity principle for meromorphic functions, we conclude that $F_{f,g}(z; \theta)$ is a meromorphic continuation of $F_{f,g}(z)$ from $\mathbb{C}^+$ to $S_\theta^-$. 

Recalling that $F_{f,g}(z; \theta) = \langle U_\theta f, (H(\theta) - z)^{-1}U_\theta g \rangle$ and that $U_\theta \mathcal{A}, \, U_\theta \mathcal{A}$ are both dense in $L^2(\mathbb{R}^n)$, thus if $H(\theta)$ has an eigenvalue at $\lambda_\theta \in S_\theta^-$, there must exist $f, g \in \mathcal{A}$ such that $\lambda_\theta$ is a pole of $F_{f,g}(z; \theta)$. Conversely, if $F_{f,g}(z; \theta)$ has a pole $\lambda_\theta \in S_\theta^-$, then it must be an eigenvalue of $H(\theta)$. \qed

Remark: For nonzero resonance $\lambda$ of $H$, we can define its multiplicity as the (algebraic) multiplicity of $\lambda$ as an eigenvalue of some $H(\theta)$. More precisely, let $\lambda \in \{ z : -2\beta_0 < \arg z < 3\pi/2 + 2\beta_0 \}$ be a resonance of $H$, there exists $\theta \in D_{\beta_0} \cap \mathbb{C}^+$ such that $-2a(\theta) < \arg \lambda$. Lemma 1 implies that $\lambda$ is also an eigenvalue of $H(\theta)$, then we define the multiplicity of resonance $\lambda$ as follows:

\[ m(\lambda) := m_{\theta}(\lambda) \equiv -\frac{1}{2\pi i} \text{tr} \oint_{\lambda} (H(\theta) - z)^{-1}dz, \]  

where the integral is over a positively oriented circle enclosing $\lambda$ and containing no eigenvalues of $H(\theta)$ other than $\lambda$. To see that the multiplicity $m(\lambda)$ is well-defined,
we need to show that \( m(\lambda) \) does not depend on the choice of \( \theta \). Assume \( \theta_0, \theta_1 \in D_{\beta_0} \) satisfy \(-2a(\theta_0) \leq -2a(\theta_1) < \arg \lambda \), let \( \theta_t = (1-t)\theta_0 + t\theta_1 \) then \(-2a(\theta_t) < \arg \lambda \) for all \( t \in [0, 1] \). Let \( C_\lambda \) be a positively oriented circle enclosing \( \lambda \) with sufficiently small radius such that \( C_\lambda \subset \{ z : \arg z > -2a(\theta_1) \} \) and contains no resonances of \( H \) other than \( \lambda \). Therefore, \( C_\lambda \) contains no eigenvalues of \( H(\theta_t) \) other than \( \lambda \) for all \( t \in [0, 1] \) as a consequence of Lemma 1. Now we have

\[
m_{\theta_t}(\lambda) = -\frac{1}{2\pi i} \text{tr} \int_{C_\lambda} (H(\theta_t) - z)^{-1} dz, \quad t \in [0, 1].
\]

Hence \( m_{\theta_t}(\lambda) \) depends continuously on \( t \) which implies that \( m_{\theta_1}(\lambda) \) must be a constant as it is integer-valued. In particular, we have \( m_{\theta_0}(\lambda) = m_{\theta_1}(\lambda) \), thus \( m(\lambda) \) is well-defined.

4. Eigenvalues and complex scaling

In this section we will show that the eigenvalues of \( H_\varepsilon \equiv -\Delta - \varepsilon x^2 + V \) are invariant under complex scaling, in other words, these eigenvalues are the same as the eigenvalues of

\[
H_\varepsilon(\theta) := U_\theta H_\varepsilon U_\theta^{-1} = p_\theta^2 - \varepsilon \phi_\theta(x)^2 + V(\phi_\theta(x)), \quad \theta \in D_{\beta_0} \cap \mathbb{C}^+.
\]

First we recall some basic properties about the Davies harmonic oscillator and its deformation, see [Z2, §3] for details. The operator \( H_{\varepsilon, \gamma} := -\Delta + e^{-i\gamma} \varepsilon x^2, \varepsilon > 0, 0 \leq \gamma < \pi, \) was used by Davies [Da1] to illustrate properties of non-normal differential operators. We are more interested in the deformations of \( H_{\varepsilon, \gamma} \) under complex scaling. Let

\[
Q_{\varepsilon, \theta} = -\Delta_\theta - \varepsilon x_\theta^2, \quad \text{where } x_\theta = z|_{\Gamma_\theta}
\]

be a deformed operator on \( \Gamma_\theta \) as in [Z2, §3]. In view of (3.4), we have

\[
p_\theta^2 - \varepsilon \phi_\theta(x)^2 = U_\theta Q_{\varepsilon, \alpha(\theta)} U_\theta^{-1}, \quad \theta \in D_{\beta_0}.
\]

Hence we can study the spectrum and the resolvents of \( p_\theta^2 - \varepsilon \phi_\theta(x)^2 \) using the relevant results about \( Q_{\varepsilon, \alpha(\theta)} \). We recall [Z2, Lemma 4.] that \( \sigma(Q_{\varepsilon, \alpha(\theta)}) = \sqrt{\varepsilon} e^{-i\pi/4(n + 2|N_0^\alpha|)} \) for \( \theta \in D_{\beta_0} \), then by (4.1) we have

**Proposition 8.** For \( \theta \in D_{\beta_0}, \varepsilon > 0 \), the spectrum of \( p_\theta^2 - \varepsilon \phi_\theta(x)^2 \) is independent of \( \theta \) and given by \( \sqrt{\varepsilon} e^{-i\pi/4(n + 2|N_0^\alpha|)} \).

For the resolvents of \( p_\theta^2 - \varepsilon \phi_\theta(x)^2 \):

\[
R_{\varepsilon, \theta}(z) := (p_\theta^2 - \varepsilon \phi_\theta(x)^2 - z)^{-1}, \quad \theta \in D_{\beta_0},
\]

we recall [Z2, Lemma 5.] that for \( \delta > 0, -\pi/8 < \theta < \pi/8 \), we have

\[
(Q_{\varepsilon, \theta} - z)^{-1} = O_\delta(1/|z|) : L^2(\Gamma_\theta) \rightarrow L^2(\Gamma_\theta), \quad -2\theta + \delta < \arg z < 3\pi/2 + 2\theta - \delta,
\]

uniformly for \( 0 < \varepsilon < \varepsilon_0 \), where \( \varepsilon_0 > 0 \) is a constant. Using (4.1), we have
Proposition 9. Let $\theta \in D_{\beta_0}$, $\delta > 0$, then uniformly for $0 < \varepsilon < \varepsilon_0$, we have
\[ R_{\varepsilon,\theta}(z) = O_\delta(1/|z|) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad -2a(\theta) + \delta < \arg z < 3\pi/2 + 2a(\theta) - \delta. \] (4.3)

Now we state the main result about the eigenvalues of $H_\varepsilon$:

Lemma 2. For any $\theta \in D_{\beta_0}$, $0 < \varepsilon < \varepsilon_0$,
\[ z \mapsto R_{H_\varepsilon(\theta)}(z) = (H_\varepsilon(\theta) - z)^{-1}, \quad -\pi/4 < \arg z < 7\pi/4, \]
is a meromorphic family of operators on $L^2(\mathbb{R}^n)$ with poles of finite rank. Furthermore, the poles of $(H_\varepsilon(\theta) - z)^{-1}$ do not depend on $\theta \in D_{\beta_0}$ and coincide, with agreement of multiplicities, with the poles of $(H_\varepsilon - z)^{-1}$.

Proof. For fixed $\theta \in D_{\beta_0}$, one can compute
\[ (H_\varepsilon(\theta) - z)R_{\varepsilon,\theta}(z) = I + V(\phi_\theta(x))R_{\varepsilon,\theta}(z), \]
then we obtain from (4.3) that
\[ R_{H_\varepsilon(\theta)}(z) = R_{\varepsilon,\theta}(z)(I + V(\phi_\theta(x))R_{\varepsilon,\theta}(z))^{-1}, \]
\[ -2a(\theta) + \delta < \arg z < 3\pi/2 + 2a(\theta) - \delta, \quad |z| \gg 1, \] (4.4)
where for large $|z|$, $I + V(\phi_\theta(x))R_{\varepsilon,\theta}(z)$ is invertible by a Neumann series argument. Note that $R_{\varepsilon,\theta}(z) : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$, $\arg z \neq -\pi/4$ by Proposition 8, recalling that $V(\phi_\theta(x)) : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact (see the proof of Proposition 7), we conclude that $z \mapsto V(\phi_\theta(x))R_{\varepsilon,\theta}(z)$ is an analytic family of compact operators for $-\pi/4 < z < 7\pi/4$. Hence $z \mapsto (I + V(\phi_\theta(x))R_{\varepsilon,\theta}(z))^{-1}$ is a meromorphic family of operators in the same range of $z$. In particular $z \mapsto R_{H_\varepsilon(\theta)}(z)$, $-\pi/4 < z < 7\pi/4$ is a meromorphic family of operators on $L^2(\mathbb{R}^n)$ with poles of finite rank.

The poles and their multiplicities are independent of $\theta$. For that we modify the proof of Lemma 1 and define matrix elements:
\[ G_{f,g}(z) = \langle f, (H_\varepsilon - z)^{-1}g \rangle, \]
and
\[ G_{f,g}(z; \theta) = \langle U_\theta f, (H_\varepsilon(\theta) - z)^{-1}U_\theta g \rangle, \]
for all $f, g \in \mathcal{A}$.

Note that $-2a(\theta) + \pi/4 < \pi/2 < 3\pi/2 + 2a(\theta) - \pi/4$ since $-\beta_0 < a(\theta) < \beta_0$, for $\theta \in D_{\beta_0}$, using (4.3) and Neumann series argument, $H_\varepsilon(\theta) - z$ is invertible at $z = i\rho$, $\rho \gg 1$ for each $\theta \in D_{\beta_0}$. Like (3.8), we have
\[ \theta \in D_{\beta_0} \to G_{f,g}(z; \theta) = \langle U_\theta f, R_{H(\theta)}(z)U_\theta g \rangle \]
is an analytic map provided $z = i\rho$, $\rho \gg 1$. Hence we have
\[ G_{f,g}(z; \theta) = G_{f,g}(z), \quad \forall \theta \in D_{\beta_0}, \quad z = i\rho, \quad \rho \gg 1, \] (4.5)
since this is true for all $\theta \in D_{\beta_1} \cap \mathbb{R}$. Now fix any $\theta \in D_{\beta_0}$, note that $G_{f,g}(z)$ and $G_{f,g}(z; \theta)$ are both meromorphic in $-\pi/4 < z < 7\pi/4$, we conclude that

$$G_{f,g}(z; \theta) = G_{f,g}(z), \quad -\pi/4 < z < 7\pi/4,$$ 

by (4.5) and the identity principle of meromorphic functions.

Now argue as in the end of the proof of Lemma 1: if $(H_\varepsilon - z)^{-1}$ has a pole at $\lambda_\theta \in \mathbb{C} \setminus e^{-i\pi/4}[0, \infty)$, then there must exist $f, g \in \mathcal{A}$ such that $\lambda_\theta$ is a pole of $G_{f,g}(z; \theta)$, by (4.6), $\lambda_\theta$ is also a pole of $G_{f,g}(z)$ thus $(H_\varepsilon(\theta) - z)^{-1}$ must have a pole at $\lambda_\theta$ and vice versa. Hence for any $\theta \in D_{\beta_0}$, the poles of $(H_\varepsilon(\theta) - z)^{-1}$ in $\mathbb{C} \setminus e^{-i\pi/4}[0, \infty)$ coincide the poles of $(H_\varepsilon - z)^{-1}$ in $\mathbb{C} \setminus e^{-i\pi/4}[0, \infty)$.

To show the agreement of multiplicities, for any pole $\lambda$ of $(H_\varepsilon(\theta) - z)^{-1}$, the multiplicity of $\lambda$ is defined by

$$m_{\varepsilon,\theta}(\lambda) = -\frac{1}{2\pi i} \int_\lambda (H_\varepsilon(\theta) - z)^{-1}dz,$$

where the integral is over a positively oriented circle independent of $\theta$ enclosing $\lambda$ and containing no poles other than $\lambda$. Since $m_{\varepsilon,\theta}(\lambda)$ is continuous on $\theta \in D_{\beta_0}$ and integer-valued, it must be independent of $\theta \in D_{\beta_0}$. Hence we have

$$m_{\varepsilon,\theta}(\lambda) = m_{\varepsilon,0}(\lambda) = -\frac{1}{2\pi i} \int_\lambda (H_\varepsilon - z)^{-1}dz$$

which is the multiplicity of $\lambda$ as a pole of $(H_\varepsilon - z)^{-1}$. \hfill $\square$

5. Meromorphic continuation

In this section we will introduce a new way to express the meromorphic continuations of resolvents $R_{H(\theta)}(z)$ and $R_{H_\varepsilon(\theta)}(z)$ in a given region $\Omega \in \{z : -2a(\theta) < \arg z < 3\pi/2 + 2a(\theta)\}$, which is crucial in the proof of Theorem 1. For that we will first review some properties about $R_{\theta}(z)$ and the weighted $L^2$ space, $\langle x \rangle^{-2}L^2(\mathbb{R}^n)$.

**Lemma 3.** Let $\langle x \rangle^{-2}L^2(\mathbb{R}^n)$ be a weighted $L^2$ space with the norm

$$\|u\|_{\langle x \rangle^{-2}L^2(\mathbb{R}^n)} = \|\langle x \rangle^2u\|_{L^2(\mathbb{R}^n)}.$$ 

Then $H^2(\mathbb{R}^n) \cap \langle x \rangle^{-2}L^2(\mathbb{R}^n)$ is compactly embedded in $L^2(\mathbb{R}^n)$.

**Proof.** Let $u_n \in H^2(\mathbb{R}^n) \cap \langle x \rangle^{-2}L^2(\mathbb{R}^n)$ with $\|u_n\|_{H^2(\mathbb{R}^n)} \leq 1$ and $\|\langle x \rangle^2u_n\|_{L^2(\mathbb{R}^n)} \leq 1$. For some $r > 0$ to be decided, we have

$$\int_{|x| \geq r} |u_n(x)|^2dx \leq \langle r \rangle^{-4} \int_{|x| \geq r} \langle x \rangle^4|u_n(x)|^2dx \leq \langle r \rangle^{-4}\|\langle x \rangle^2u_n\|_{L^2(\mathbb{R}^n)}^2 = \langle r \rangle^{-4}. $$
Then we choose $r$ sufficiently large such that $\int_{|x|\geq r} |u_n(x)|^2 \, dx < 1/8$ for all $n$. Since $H^2(B(0,r)) \subset L^2(B(0,r))$, there exists subsequence $\{u_n^{(1)}\} \subset \{u_n\}$ satisfying

$$\int_{B(0,r)} |u_n^{(1)}(x) - u_m^{(1)}(x)|^2 \, dx < 1/2, \quad \text{for all } n, m.$$

Hence we have

$$\|u_n^{(1)} - u_m^{(1)}\|_{L^2(\mathbb{R}^n)}^2 = \int_{B(0,r)} |u_n^{(1)}(x) - u_m^{(1)}(x)|^2 \, dx + \int_{|x|\geq r} |u_n^{(1)}(x) - u_m^{(1)}(x)|^2 \, dx$$

$$< 1/2 + \int_{|x|\geq r} (2|u_n^{(1)}(x)|^2 + 2|u_m^{(1)}(x)|^2) \, dx$$

$$< 1/2 + 2/8 + 2/8 = 1.$$

By the same argument, we can find $\{u_n^{(1)}\} \supset \cdots \supset \{u_n^{(j)}\} \supset \cdots$ with

$$\|u_n^{(j)} - u_m^{(j)}\|_{L^2(\mathbb{R}^n)} < 1/j, \quad \text{for all } n, m.$$

Then the subsequence $\{u_j^{(j)}\} \subset \{u_n\}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$. \hfill \Box

Lemma 4. Fix $\theta \in D_{\beta_0} \cap \mathbb{C}^+$, $R_\theta(z)$ is an analytic family of operators $\langle x \rangle^{-2} L^2 \to \langle x \rangle^{-2} L^2$ for $-2a(\theta) < \arg z < 2\pi - 2a(\theta)$. Furthermore, if $\Omega \subset \{ z : -2a(\theta) < \arg z < 2\pi - 2a(\theta) \}$ then there exists $C = C_{\Omega, \theta}$ such that

$$\| R_\theta(z) \|_{\langle x \rangle^{-2} L^2 \to \langle x \rangle^{-2} L^2} \leq C, \quad z \in \Omega.$$

Proof. In view of (2.1) and (2.9), we have

$$|x|/2 < |\phi_\theta(x)| = |1 + \theta h(|x|)/|x| < 3|x|/2 \quad \Rightarrow \quad \langle x \rangle/2 < \langle \phi_\theta(x) \rangle < 3\langle x \rangle/2.$$

Then it is equivalent to prove the lemma with $\langle \phi_\theta(x) \rangle$ replacing $\langle x \rangle$. We recall Proposition 3 to write

$$\langle \phi_\theta(x) \rangle^2 R_\theta(z) \langle \phi_\theta(x) \rangle^{-2} = U_{a(\theta)} \langle x_{a(\theta)} \rangle^2 \langle -\Delta_{a(\theta)} - z \rangle^{-1} \langle x_{a(\theta)} \rangle^{-2} U_{a(\theta)}^{-1},$$

where $x_{a(\theta)}$ is the coordinate on $\Gamma_{a(\theta)}$. Then it suffices to show that, for any $0 < \alpha < \beta_0$,

$$\langle w \rangle^2 \langle -\Delta_{\alpha} - \lambda^2 \rangle^{-1} \langle w \rangle^{-2} : L^2(\Gamma_{\alpha}) \to L^2(\Gamma_{\alpha}), \quad \text{Im}(e^{i\alpha \lambda}) > 0, \quad (5.2)$$

is analytic with uniformly bounded norm provided $\lambda$ in any compact subset of $\lambda \in \mathbb{C} : \text{Im}(e^{i\alpha \lambda})$, where $w$ denotes the coordinate on $\Gamma_{\alpha}$. To prove (5.2), consider the integral kernel of that operator:

$$K(\lambda, w_1, w_2) = \langle w_1 \rangle^2 R_0(\lambda, w_1, w_2) \langle w_2 \rangle^{-2}, \quad w_1, w_2 \in \Gamma_{\alpha}, \quad (5.3)$$
where $R_0(\lambda, w_1, w_2)$ is the integral kernel of $(-\Delta_\alpha - \lambda^2)^{-1} : L^2(\Gamma_\alpha) \to L^2(\Gamma_\alpha)$. It is easy to see that
\[
|K(\lambda, w_1, w_2)| \leq (1 + |w_1|^2) |R_0(\lambda, w_1, w_2)| \langle w_2 \rangle^2 \\
\leq 2(1 + |w_1 - w_2|^2 + |w_2|^2) (1 + |w_2|^2)^{-1} |R_0(\lambda, w_1, w_2)| (5.4) \\
\leq 2(1 + |w_1 - w_2|^2) |R_0(\lambda, w_1, w_2)|.
\]

To introduce the explicit formula of $R_0(\lambda, w_1, w_2)$, we recall that one can define $((w_1 - w_2) \cdot (w_1 - w_2))^{1/2}$ for $w_1, w_2 \in \Gamma_\alpha$, see [DyZ2, §4.5]. Then we can write
\[
R_0(\lambda, w_1, w_2) = C_n \lambda^{n-2} (w_1 - w_2) \cdot (w_1 - w_2)^{1/2} - \frac{n-2}{2} \alpha \Gamma_{\frac{1}{2} - 1} (\lambda, (w_1 - w_2) \cdot (w_1 - w_2))^{1/2}
\]
where $H_{\frac{1}{2}}^{(1)}$ denote the Hankel functions of the first kind, and we can estimate $|R_0(\lambda, w_1, w_2)|$ as follows:
\[
|R_0(\lambda, w_1, w_2)| \leq P_n(\lambda ((w_1 - w_2) \cdot (w_1 - w_2))^{1/2}) \frac{\lambda^{n-2}}{((w_1 - w_2) \cdot (w_1 - w_2))^{1/2}} e^{-\text{Im} \lambda ((w_1 - w_2) \cdot (w_1 - w_2))^{1/2}}
\]
where $P_n$ is a polynomial of degree $(n - 3)/2$, see [GaSm, §2.2] and [DyZ2, §4.5] for details. Using (2.11), it is easy to see that for any $\delta$ small, there exists $C_\delta > 0$ such that $|\arg((w_1 - w_2) \cdot (w_1 - w_2))^{1/2} - \alpha| < \delta$ provided $|w_1 - w_2| > C_\delta$. Note that $0 < \arg \lambda + \alpha < \pi$, for every $\lambda$, we can choose $\delta = \delta_\lambda$ such that $2\delta < \arg \lambda + \alpha < \pi - 2\delta$, then for $|z - w| > C_\lambda$, we have
\[
\delta < \arg((w_1 - w_2) \cdot (w_1 - w_2))^{1/2} < \pi - \delta,
\]
and thus
\[
e^{-\text{Im} \lambda ((w_1 - w_2) \cdot (w_1 - w_2))^{1/2}} < e^{-c_\lambda |w_1 - w_2|}, c_\lambda > 0, \text{ if } |w_1 - w_2| > C_\lambda. (5.6)
\]

Then using (5.4), (5.5) and (5.6), we conclude that
\[
\sup_{w_1 \in \Gamma_\alpha} \int_{\Gamma_\alpha} |K(\lambda, w_1, w_2)| dw_1 < M_\lambda, \sup_{w_2 \in \Gamma_\alpha} \int_{\Gamma_\alpha} |K(\lambda, w_1, w_2)| dz < M_\lambda.
\]

By Schur criterion, we proved (5.2), the analyticity in $\lambda$ is easy to see using the explicit formula of $R_0(\lambda, w_1, w_2)$. If $\lambda \in K \subset \{ \lambda \in \mathbb{C} : \text{Im}(e^{i\alpha} \lambda) \}$, then there exist $c_K$ and $C_K$ such that
\[
e^{-\text{Im} \lambda ((w_1 - w_2) \cdot (w_1 - w_2))^{1/2}} < e^{-c_K |w_1 - w_2|}, c_K > 0, \text{ if } |w_1 - w_2| > C_K.
\]

Follow the above argument, there exists $M = M_K > 0$ such that
\[
\|\langle w \rangle^2 (-\Delta_\alpha - \lambda^2)^{-1} \langle w \rangle^{-2} \|_{L^2(\Gamma_\alpha) \to L^2(\Gamma_\alpha)} < M_K, \text{ for all } \lambda \in K,
\]
which completes the proof. \qed

Now we state the main result of this section:
Lemma 5. Fix any $\theta \in D_{b_0} \cap \mathbb{C}^+$ and $\Omega \subseteq \{ z : -2a(\theta) < \arg z < 3\pi/2 + 2a(\theta) \}$, there exists $\chi \in C_c^\infty(\mathbb{R}^n)$, $\chi \equiv 1$ on $B(0, T)$ for some $T > 0$ such that for $0 \leq \varepsilon < \varepsilon_0$, $H_{\varepsilon}(\theta) - \chi V - z$ is invertible in $\Omega$ and

$$z \mapsto (I + R_{H_{\varepsilon}(\theta) - \chi V}(z))^{-1}, \quad z \in \Omega,$$

is a meromorphic family of operators on $L^2(\mathbb{R}^n)$ with poles of finite rank, where we write $R_{H_{\varepsilon}(\theta) - \chi V}(z) = (H_{\varepsilon}(\theta) - \chi V - z)^{-1}$ for simplicity. Moreover,

$$m_{\varepsilon, \theta}(z) := \frac{1}{2\pi i} \text{tr} \oint_z (I + R_{H_{\varepsilon}(\theta) - \chi V}(w))^{-1} \partial_w (R_{H_{\varepsilon}(\theta) - \chi V}(w) \chi V) dw,$$

where the integral is over a positively oriented circle enclosing $z$ and containing no poles other than possibly $z$, satisfies

$$m_{\varepsilon, \theta}(z) = \frac{1}{2\pi i} \text{tr} \oint_z (w - H_{\varepsilon}(\theta))^{-1} dw, \quad 0 \leq \varepsilon < \varepsilon_0,$$

where $H_0(\theta) = H(\theta)$.

Proof. We modify the argument in [Z2, §4] to our setting. First there exists $\delta = \delta_\Omega$ such that $\Omega \subset C_\delta := \{ z : -2a(\theta) + \delta < \arg z < 3\pi/2 + 2a(\theta) - \delta, \ |z| > \delta \}$, we recall (3.3) and (4.3) that uniformly for $0 \leq \varepsilon < \varepsilon_0$, we have

$$R_{\varepsilon, \theta}(z) = O_\delta(1/|z|) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n), \quad z \in C_\delta.$$  

(5.9)

Hence $\| R_{\varepsilon, \theta}(z) \|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} < C_\delta$, $\forall z \in C_\delta$, for some $C_\delta > 0$. In view of (1.1), for $T$ sufficiently large, we have $\|(1 - \chi)V\|_{L^\infty} < 1/2C_\delta$ and thus

$$\| R_{\varepsilon, \theta}(z)(1 - \chi)V \|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} < 1/2, \quad \text{for all } z \in C_\delta.$$  

(5.10)

Then $(I + R_{\varepsilon, \theta}(z)(1 - \chi)V)$ is invertible by the Neumann series argument, which implies that $H_{\varepsilon}(\theta) - \chi V - z$ is invertible and

$$R_{H_{\varepsilon}(\theta) - \chi V}(z) = (H_{\varepsilon}(\theta) - \chi V - z)^{-1} = (I + R_{\varepsilon, \theta}(z)(1 - \chi)V)^{-1} R_{\varepsilon, \theta}(z), \quad \forall z \in C_\delta.$$  

(5.11)

Since $\chi V \in L^\infty(\mathbb{R}^n)$, (5.9) and (5.11) imply that for $z \in C_\delta, |z| \gg 1$, both $I + \chi V R_{H_{\varepsilon}(\theta) - \chi V}(z)$ and $I + R_{H_{\varepsilon}(\theta) - \chi V}(z) \chi V$ are invertible by the Neumann series argument. Hence we have

$$R_{H_{\varepsilon}(\theta)}(z) = R_{H_{\varepsilon}(\theta) - \chi V}(z)(I + \chi V R_{H_{\varepsilon}(\theta) - \chi V}(z))^{-1}$$

$$= R_{H_{\varepsilon}(\theta) - \chi V}(z) \sum_{j=0}^{\infty} (-1)^j (\chi V R_{H_{\varepsilon}(\theta) - \chi V}(z))^j$$

$$= R_{H_{\varepsilon}(\theta) - \chi V}(z) \left( I - \chi V \sum_{j=0}^{\infty} (-1)^j (R_{H_{\varepsilon}(\theta) - \chi V}(z) \chi V)^j R_{H_{\varepsilon}(\theta) - \chi V}(z) \right)$$

$$= R_{H_{\varepsilon}(\theta) - \chi V}(z) [I - \chi V (I + R_{H_{\varepsilon}(\theta) - \chi V}(z) \chi V)^{-1} R_{H_{\varepsilon}(\theta) - \chi V}(z)],$$

(5.12)
Using (5.11), we have

\[ R_{H_\varepsilon(\theta) - \chi V}(z)\chi V = (I + R_{\varepsilon,\theta}(z)(1 - \chi)\chi V)^{-1}R_{\varepsilon,\theta}(z)\chi V. \]

For \( \varepsilon > 0 \), \( R_{\varepsilon,\theta}(z) : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n) \cap \langle \phi_\theta(x) \rangle^{-2}L^2(\mathbb{R}^n) \), then Lemma 3 implies that \( R_{\varepsilon,\theta}(z) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) is compact. For \( \varepsilon = 0 \), note that \( \chi V : L^2(\mathbb{R}^n) \to \langle x \rangle^{-2}L^2(\mathbb{R}^n) \), by Lemma 4 we have \( R_\theta(z)\chi V : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n) \cap \langle x \rangle^{-2}L^2(\mathbb{R}^n) \), then Lemma 3 implies that \( R_\theta(z)\chi V : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) is compact. Hence we can conclude that \( z \mapsto R_{H_\varepsilon(\theta) - \chi V}(z)\chi V \) is an analytic family of compact operators for \( z \in C_\delta, \ 0 \leq \varepsilon < \varepsilon_0 \), and thus \( z \mapsto (I + R_{H_\varepsilon(\theta) - \chi V}(z)\chi V)^{-1} \) is a meromorphic family of operators in the same range of \( z \).

Then we recall Lemma 1 and 2 that \( R_{H_\varepsilon(\theta)}(z) \) is meromorphic in \(-2\alpha(\theta) < \arg z < 3\pi/2 + 2\alpha(\theta)\), by the identity principle of meromorphic operators, we conclude that (5.12) holds for all \( z \in C_\delta \) in the sense of meromorphic family of operators.

To obtain the multiplicity formula, we assume that \( z \in \Omega \), then there exists a neighborhood \( z \in U \subset \Omega \) and finite rank operators \( A_j, \ 1 \leq j \leq J \) such that

\[ (I + R_{H_\varepsilon(\theta) - \chi V}(w)\chi V)^{-1} - \sum_{j=1}^J \frac{A_j}{(w - z)^j} \text{ is holomorphic in } w \in U. \]

Let \( C_z \subset U \) be a positively oriented circle enclosing \( z \) and containing no poles of \((I + R_{H_\varepsilon(\theta) - \chi V}(w)\chi V)^{-1}\) other than possibly \( z \), thus it also contains no poles of \((w - H_\varepsilon(\theta))^{-1}\) other than possibly \( z \) as a consequence of (5.12). On the one hand, we can compute

\[
m_{\varepsilon,\theta}(z) = \frac{1}{2\pi i} \text{tr} \int_{C_z} \frac{1}{(w - z)^j} \partial_z^{j-1} (R_{\varepsilon,\theta}(z)\chi V)^{-1} \partial_z \partial^{j-1} (R_{\varepsilon,\theta}(z)\chi V)^{-1} \partial_z \partial^{j-1}(R_{\varepsilon,\theta}(z)\chi V) dw \]

\[
= \frac{1}{2\pi i} \text{tr} \int_{C_z} \frac{1}{(w - z)^j} \partial_z^{j-1} (A_j R_{\varepsilon,\theta}(z)\chi V)^{-1} \partial_z \partial^{j-1}(R_{\varepsilon,\theta}(z)\chi V) dw \]

\[
= \frac{1}{2\pi i} \text{tr} \int_{C_z} \frac{1}{(w - z)^j} \partial_z^{j-1} (A_j R_{\varepsilon,\theta}(z)\chi V)^{-1} \partial_z \partial^{j-1}(A_j R_{\varepsilon,\theta}(z)\chi V) dw \]

\[
= \sum_{j=1}^J \frac{1}{(j - 1)!} \partial_z^{j-1} (A_j R_{\varepsilon,\theta}(z)\chi V)^{-1} \partial_z \partial^{j-1}(A_j R_{\varepsilon,\theta}(z)\chi V) \]

\[
= \sum_{j=1}^J \sum_{k=0}^{j-1} \frac{1}{k!(j - 1 - k)!} \text{tr} A_j \partial_z^k R_{\varepsilon,\theta}(z)\chi V \partial_z^{j-1-k} R_{\varepsilon,\theta}(z)\chi V. \]
On the other hand, by (5.12) we have

\[
\frac{1}{2\pi i} \int_{C_z} (w - H_\varepsilon(\theta))^{-1} dw
\]

\[
= \frac{1}{2\pi i} \int_{C_z} R_{H_\varepsilon(\theta) - \chi V}(w) \chi V(I + R_{H_\varepsilon(\theta) - \chi V}(w))^{-1} R_{H_\varepsilon(\theta) - \chi V}(w) dw
\]

\[
= \frac{1}{2\pi i} \int_{C_z} \sum_{j=1}^{J} \frac{R_{H_\varepsilon(\theta) - \chi V}(w) \chi V A_j R_{H_\varepsilon(\theta) - \chi V}(w)}{(w - z)^j} dw
\]

(5.14)

\[
= \sum_{j=1}^{J} \frac{1}{(j - 1)!} \text{tr} \partial_z^{j-1}(R_{H_\varepsilon(\theta) - \chi V}(z) \chi V A_j R_{H_\varepsilon(\theta) - \chi V}(z))
\]

\[
= \sum_{j=1}^{J} \sum_{k=0}^{j-1} \frac{1}{k!(j - 1 - k)!} \text{tr} \partial_z^{j-1-k} R_{H_\varepsilon(\theta) - \chi V}(z) \chi V A_j \partial_z^k R_{H_\varepsilon(\theta) - \chi V}(z).
\]

Now we compare (5.13) and (5.14). Since \(A_j\) factors have finite rank, we can apply cyclicity of the trace to obtain the multiplicity formula (5.8).

\[\square\]

6. Proof of convergence

The proof of convergence is based on Lemma 1, Lemma 2, Lemma 5 and the following lemma:

**Lemma 6.** Fix any \(\theta \in D_{\beta_0} \cap C^+\) and \(\Omega \supseteq \{z : -2a(\theta) < \arg z < 3\pi/2 + 2a(\theta)\}\), there exists \(\chi \in C_c^\infty(\mathbb{R}^n)\), \(\chi \equiv 1\) on \(B(0,T)\) for some \(T > 0\) such that for \(0 < \varepsilon < \varepsilon_0\),

\[T_{\varepsilon,\theta}(z) := (H_\varepsilon(\theta) - \chi V - z)^{-1} \phi_\theta(x)^2(H(\theta) - \chi V - z)^{-1} \chi V\]

is an analytic family of operators: \(L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)\). Furthermore, there exists \(C = C_{\Omega,\theta}\) such that

\[\|T_{\varepsilon,\theta}(z)\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \leq C, \quad z \in \Omega, \quad \text{uniformly for } 0 < \varepsilon < \varepsilon_0.\]

(6.1)

**Proof.** We recall the proof of Lemma 5 that for \(T\) sufficiently large, \(H_\varepsilon(\theta) - \chi V - z\) is invertible, then (5.10) and (5.11) imply that

\[\|(H_\varepsilon(\theta) - \chi V - z)^{-1}\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \leq C_\Omega, \quad z \in \Omega,
\]

for some \(C_\Omega > 0\). Hence it suffices to prove

\[\|\phi_\theta(x)^2(H(\theta) - \chi V - z)^{-1} \chi V\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \leq C_\Omega, \quad z \in \Omega.\]

(6.2)

By Lemma 4, we have \(\|R_\theta(z)\|_{(x)^{-2} L^2 \rightarrow (x)^{-2} L^2} \leq C\). We can choose \(T\) sufficiently large such that (5.10) still holds and \(\|(1 - \chi) V\|_{L^\infty} < 1/2C\), then we have

\[\|R_\theta(z)(1 - \chi V)\|_{(x)^{-2} L^2 \rightarrow (x)^{-2} L^2} < 1/2, \quad z \in \Omega.\]
Hence \((I + R_\theta(z)(1 - \chi)V)^{-1} : L^2 \to L^2\) defined by the Neumann series in the proof of Lemma 5 also maps \(\langle x \rangle^{-2}L^2\) to \(\langle x \rangle^{-2}L^2\) by the same Neumann series and we have
\[
\|(I + R_\theta(z)(1 - \chi)V)^{-1}\|_{\langle x \rangle^{-2}L^2 : \langle x \rangle^{-2}L^2} < 2, \quad z \in \Omega. \tag{6.3}
\]
Since \(\chi V : L^2 \to \langle x \rangle^{-2}L^2\) with the operator norm bounded by \(\|\langle x \rangle^2\chi V\|_{L^\infty} = C_\Omega\), by Lemma 4, (5.11) and (6.3) we conclude that
\[
\|(H(\theta) - \chi V - z)^{-1}\chi V\|_{L^2 : \langle x \rangle^{-2}L^2} = \|(I + R_\theta(z)(1 - \chi)V)^{-1}R_\theta(z)\chi V\|_{L^2 : \langle x \rangle^{-2}L^2} \leq \|(I + R_\theta(z)(1 - \chi)V)^{-1}R_\theta(z)\|_{\langle x \rangle^{-2}L^2 : \langle x \rangle^{-2}L^2} \|\chi V\|_{L^2 : \langle x \rangle^{-2}L^2} \leq C_\Omega,
\]
which implies (6.2). \qed

Now we state the result about the convergence of eigenvalues of the deformed operator \(H_\varepsilon(\theta)\):

**Theorem 2.** Fix \(\theta \in D_{\beta_0} \cap \mathbb{C}^+\) and \(\Omega \in \{z : -2a(\theta) < \arg z < 3\pi/2 + 2a(\theta)\}\), there exists \(\delta_0 = \delta_0(\Omega)\) satisfying the following:

For any \(0 < \delta < \delta_0\) there exists \(\varepsilon' > 0\) such that for any \(z \in \Omega\) with \(m_\theta(z) > 0\) and \(0 < \varepsilon < \varepsilon'\), \(H_\varepsilon(\theta)\) has \(m_\theta(z)\) eigenvalues in \(B(z, \delta)\), where \(m_\theta(z)\) is the multiplicity of the eigenvalue of \(H(\theta)\) at \(z\) - see (3.9).

**Proof.** Since the eigenvalues of \(H(\theta)\) are isolated and \(\bar{\Omega}\) is compact, there are finite many \(z \in \Omega\) with \(m_\theta(z) > 0\), we denote them by \(z_1, \ldots, z_J\). Then we can choose \(\delta_0\) such that \(B(z_j, \delta_0), j = 1, \ldots, J\) are disjoint.

Now we fix \(\delta < \delta_0\), by Lemma 5, \(I + R_{H(\theta) - \chi V}(w)\chi V\) is invertible in \(\Omega \setminus \{z_1, \ldots, z_J\}\), thus we have

\[
\|(I + R_{H(\theta) - \chi V}(w)\chi V)^{-1}\|_{L^2 : L^2} < C(\delta), \quad w \in \partial B(z, \delta), \text{ for all } z \in \{z_1, \ldots, z_J\},
\]

for some \(C(\delta) > 0\). We note that in the notation of Lemma 6,

\[
I + R_{H_{\varepsilon}(\theta) - \chi V}(w)\chi V - (I + R_{H(\theta) - \chi V}(w)\chi V) = i\varepsilon T_{\varepsilon, \theta}(w).
\]

Hence there exists \(0 < \varepsilon' < \varepsilon_0\) such that for any \(\varepsilon < \varepsilon'\),

\[
\|(I + R_{H(\theta) - \chi V}(w)\chi V)^{-1} (I + R_{H_{\varepsilon}(\theta) - \chi V}(w)\chi V) - (I + R_{H(\theta) - \chi V}(w)\chi V))\| < 1
\]
on \(\partial B(z, \delta)\). Now we apply Gohberg-Sigal-Rouché theorem, see [GS] and [DyZ2, Appendix C.] to obtain that

\[
\frac{1}{2\pi i} \text{tr} \int_{\partial B(z, \delta)} (I + R_{H_{\varepsilon}(\theta) - \chi V}(w)\chi V)^{-1} \partial_w (R_{H_{\varepsilon}(\theta) - \chi V}(w)\chi V) dw = \frac{1}{2\pi i} \text{tr} \int_{\partial B(z, \delta)} (I + R_{H(\theta) - \chi V}(w)\chi V)^{-1} \partial_w (R_{H(\theta) - \chi V}(w)\chi V) dw.
\]
Then we recall (5.8) to conclude that
\[
\frac{1}{2\pi i} \text{tr} \int_{\partial B(z,\delta)} (w - H_\varepsilon(\theta))^{-1} dw = m_\theta(z),
\]
which implies that \( H_\varepsilon(\theta) \) has \( m_\theta(z) \) eigenvalues in \( B(0, \delta) \). \( \square \)

Finally, we can give the proof of Theorem 1.

**Proof.** We assume from now on that \( \varepsilon < \varepsilon_0 \). Fix any \( \Omega \subset \{ z : -2\beta_0 < \arg z < 3\pi/2 + 2\beta_0 \} \), we can choose \( \theta \in D_{\beta_0} \cap \mathbb{C}^+ \) such that
\[
\Omega \subset \{ z : -2a(\theta) < \arg z < 3\pi/2 + 2a(\theta) \}.
\]
In view of Lemma 1, we see that \( \{ z_j \}_{j=1}^\infty \) the resonances of \( H \) in \( \Omega \), can be identified as the eigenvalues of \( H(\theta) \), denoted by \( \{ z_{\theta,j} \}_{j=1}^\infty \). Similarly, Lemma 2 guarantees that \( \{ z_j(\varepsilon) \}_{j=1}^\infty \), the eigenvalues of \( H_\varepsilon \) in \( \{ z : -2a(\theta) < \arg z < 3\pi/2 + 2a(\theta) \} \), are the eigenvalues of \( H_\varepsilon(\theta) \), denoted by \( \{ z_{\theta,j}(\varepsilon) \}_{j=1}^\infty \). Hence it suffices to show
\[
z_{\theta,j}(\varepsilon) \to z_{\theta,j}, \quad \varepsilon \to 0^+, \quad \text{uniformly on } \Omega,
\]
which is a direct result of Theorem 2. \( \square \)

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