Fermions tunnelling from black holes

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Abstract

We investigate the tunnelling of spin-1/2 particles through event horizons. We first apply the tunnelling method to Rindler spacetime and obtain the Unruh temperature. We then apply fermion tunnelling to a general non-rotating black hole metric and show that the Hawking temperature is recovered.

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1. Introduction

In recent years, a semi-classical method of modelling Hawking radiation as a tunnelling effect has been developed and has garnered a lot of interest [1–20]. The earliest work with black hole tunnelling was done by Kraus and Wilczek [1], an approach that was subsequently refined by various researchers [2–4]. From this emerged an alternative way of understanding black hole radiation. In particular, one can calculate the Hawking temperature in a manner independent of traditional Wick rotation methods or Hawking’s original method of modelling gravitational collapse [21]. Tunnelling provides not only a useful verification of thermodynamic properties of black holes but also an alternate conceptual means for understanding the underlying physical process of black hole radiation. It has been shown to be very robust, having been successfully applied to a wide variety of exotic spacetimes such as Kerr and Kerr–Newman cases [8, 9, 12], black rings [10], the three-dimensional BTZ black hole [5, 11], Vaidya [16], other dynamical black holes [17], Taub–NUT spacetimes [12] and Gödel spacetimes [20]. Tunnelling methods have even been applied to horizons that are not black hole horizons, such as Rindler spacetimes [4, 12] and it has been shown that the Unruh temperature [22] is in fact recovered.

In general the tunnelling methods involve calculating the imaginary part of the action for the (classically forbidden) process of s-wave emission across the horizon, which in turn is related to the Boltzmann factor for emission at the Hawking temperature. There are two different approaches that are used to calculate the imaginary part of the action for the emitted particle. The first black hole tunnelling method developed was the null geodesic method used by Parikh and Wilczek [3] which followed from the work of Kraus and Wilczek [1]. The other approach to black hole tunnelling is the Hamilton–Jacobi ansatz used by Agheben et al
which is an extension of the complex path analysis of Padmanabhan et al [4]. Both of these approaches to tunnelling use the fact that the WKB approximation of the tunnelling probability for the classically forbidden trajectory from inside to outside the horizon is given by

$$\Gamma \propto \exp(-2 \text{Im } I),$$

(1)

where $I$ is the classical action of the trajectory to leading order in $\hbar$ (here set equal to unity). Where these two methods differ is in how the action is calculated. For the null geodesic method the only part of the action that contributes an imaginary term is $\int_{r_{\text{in}}}^{r_{\text{out}}} p_r \, dr$, where $p_r$ is the momentum of the emitted null s-wave. Then by using Hamilton’s equation and knowledge of the null geodesics it is possible to calculate the imaginary part of the action. For the Hamilton–Jacobi ansatz it is assumed that the action of the emitted scalar particle satisfies the relativistic Hamilton–Jacobi equation. From the symmetries of the metric one picks an appropriate ansatz for the form of the action and plugs it into the relativistic Hamilton–Jacobi equation to solve. (For a detailed comparison of the Hamilton–Jacobi ansatz and null geodesic methods see [12].)

Since a black hole has a well-defined temperature it should radiate all types of particles like a black body at that temperature (ignoring grey body effects). The emission spectrum therefore is expected to contain particles of all spins; the implications of this expectation were studied 30 years ago [23]. However, application of tunnelling methods themselves to date has only involved scalar particles. Specifically, there is no other black hole tunnelling calculation (to the best of our knowledge) that models fermions tunnelling from the black hole. In fact comparatively little has been done for fermion radiation for black holes. The Hawking temperature for fermion radiation has been calculated for 2D black holes [24] using the Bogoliubov transformation and more recently was calculated for evaporating black holes using a technique called the generalized tortoise coordinate transformation (GTCT) [25–27]. The latter result [27] is interesting because there is a contribution to the fermion emission probability due to a coupling effect between the spin of the emitted fermion and the acceleration of the Kinnersley black hole. From this one may infer that when fermions are emitted from rotating black holes there might be a coupling term between the fermion spin and the black hole’s angular momentum present in the tunnelling probability.

In this paper, we extend the tunnelling method to model spin-1/2 particle emission from non-rotating black holes. In order to do this we will follow an analogous approach to the original approach used by Padmanabhan et al [4]. The Hamilton–Jacobi ansatz emerged from an application of the WKB approximation to the Klein–Gordon equation. We will start by reviewing this general calculation, and then apply a WKB approximation to the Dirac equation. We consider Rindler spacetime first and confirm that the Unruh temperature is recovered. Insofar as fermionic vacua are distinct from bosonic vacua and can lead to distinct physical results [28], this result is non-trivial. We then extend this technique to general 4D black hole metric and show the Hawking temperature is recovered. We illustrate this result in several coordinate systems—Schwarzschild, Painlevé and Kruskal—to demonstrate that the result is independent of this choice. This last system is particularly interesting since it has no coordinate singularities at the horizon. That we obtain the expected Hawking temperature indicates that tunnelling can be understood as a bona-fide physical phenomenon.

One of the assumptions of our semi-classical calculation is to neglect any change of angular momentum of the black hole due to the spin of the emitted particle. For zero-angular momentum black holes with mass much larger than the Planck mass this is a good approximation. Furthermore, statistically particles of opposite spin will be emitted in equal numbers, yielding no net change in the angular momentum of the black hole (although second-order statistical fluctuations will be present). We confirm that spin-1/2 fermions are
also emitted at the Hawking temperature. This final result, while not surprising, furnishes an important confirmation of the robustness of the tunnelling approach.

2. Review of the Hamilton–Jacobi ansatz

We will consider a general (non-extremal) black hole metric of the form

\[ ds^2 = -f(r) \, dt^2 + \frac{dr^2}{g(r)} + C(r) h_{ij} \, dx^i \, dx^j. \] (2)

The Klein–Gordon equation for a scalar field \( \phi \) is

\[ g^\mu\nu \partial_\mu \partial_\nu \phi - \frac{m^2}{\hbar^2} \phi = 0. \]

Applying the WKB approximation by assuming an ansatz of the form

\[ \phi(t, r, x^i) = \exp \left[ \frac{i}{\hbar} I(t, r, x^i) + I_1(t, r, x^i) + O(\hbar) \right] \]

and then inserting this back into the Klein–Gordon equation we get the usual result of the Hamilton–Jacobi equation to the lowest order in \( \hbar \):

\[-[g^{\mu\nu} \partial_\mu I \partial_\nu I + m^2] + O(\hbar) = 0 \]

(obtained after dividing by the exponential term and multiplying by \( \hbar^2 \)).

For our metric (2) the Hamilton–Jacobi equation is explicitly

\[-\left( \frac{\partial_t I}{f(r)} \right)^2 + g(r) \left( \frac{\partial_r I}{g(r)} \right)^2 + \frac{h_{ij} J_i J_j}{C(r)} \partial_i I \partial_j I + m^2 = 0 \] (3)

where we neglect the effects of the self-gravitation of the particle. There exists a solution of the form

\[ I = -Et + W(r) + J(x^i) + K, \] (4)

where

\[ \partial_t I = -E, \quad \partial_r I = W'(r), \quad \partial_i I = J_i, \]

and \( K \) and the \( J_i \)'s are constant (\( K \) can be complex). Since \( \partial_t \) is the timelike Killing vector for this coordinate system, \( E \) is the energy. Solving for \( W(r) \) yields

\[ W_{\pm}(r) = \pm \int \frac{dr}{\sqrt{f(r)g(r)}} \sqrt{E^2 - f(r) \left( m^2 + \frac{h_{ij} J_i J_j}{C(r)} \right)} \] (5)

since the equation was quadratic in terms of \( W(r) \). One solution corresponds to scalar particles moving away from the black hole (i.e. + outgoing) and the other solution corresponds to particles moving towards the black hole (i.e. - incoming). Imaginary parts of the action can only come due to the pole at the horizon or from the imaginary part of \( K \). The probabilities of crossing the horizon each way are proportional to

\[ \text{Prob}[\text{out}] \propto \exp \left[ -\frac{2}{\hbar} \text{Im} I \right] = \exp \left[ -\frac{2}{\hbar} (\text{Im} W_+ + \text{Im} K) \right] \] (6)

\[ \text{Prob}[\text{in}] \propto \exp \left[ -\frac{2}{\hbar} \text{Im} I \right] = \exp \left[-\frac{2}{\hbar}(\text{Im} W_- + \text{Im} K) \right]. \] (7)

To ensure that the probability is normalized so that any incoming particles crossing the horizon have a 100% chance of entering the black hole we set \( \text{Im} K = -\text{Im} W_- \) and since \( W_+ = -W_- \) this implies that the probability of a particle tunnelling from inside to outside the
horizon is
\[ \Gamma \propto \exp \left[ -\frac{4}{\hbar} \text{Im} \ W_+ \right]. \]  
(8)

Henceforth we set \( \hbar \) to unity and also drop the ‘+’ subscript from \( W \). Integrating around the pole at the horizon leads to the result [12]
\[ W = \frac{\pi i E}{\sqrt{g'(r_0)g(r_0)}}, \]  
(9)
where the imaginary part of \( W \) is now manifest. This leads to a tunnelling probability of
\[ \Gamma = \exp \left[ -\frac{4\pi}{\sqrt{f'(r_0)g'(r_0)}} E \right] \]  
and implies the usual Hawking temperature of
\[ T_H = \frac{\sqrt{f'(r_0)g'(r_0)}}{4\pi}. \]  
(10)

It can be shown [19] that the proper Hawking temperature is recovered for multiple choices of the form of the metric for the same black hole.

3. Spin-1/2 particles and Rindler space

We first consider the Rindler spacetime, for which the tunnelling calculation of a scalar field has shown [4, 12] that the Unruh temperature [22] is recovered.

We will only show the calculation explicitly for the spin-up case; the final result is also the same for the spin-down case as can be easily shown using the methods described below. Due to the statistical nature of the heat bath we assume that no angular momentum is imparted to the accelerating detector (i.e. on average there are as many spin-up particles as spin-down particles detected). The fermionic heat bath as seen by accelerated observers has many applications, such as understanding the effects of acceleration on entanglement [28].

We will use the following metric for Rindler spacetime:
\[ ds^2 = -f(z) \, dt^2 + dx^2 + dy^2 + \frac{dz^2}{g(z)}, \quad f(z) = a^2 z^2 - 1, \quad g(z) = a^2 z^2 - 1, \]
so chosen for its convenience in extending the technique to normal black holes. The Dirac equation is
\[ i\gamma^\mu D_\mu \psi + \frac{m}{\hbar} \psi = 0, \]  
(11)
where
\[ D_\mu = \partial_\mu + \Omega_\mu, \quad \Omega_\mu = \frac{i}{2} \Gamma_\mu^{\alpha\beta} \Sigma_{\alpha\beta}, \quad \Sigma_{\alpha\beta} = \frac{1}{2} [\gamma^\alpha, \gamma^\beta]. \]
The \( \gamma^\mu \) matrices satisfy \( \{ \gamma^\mu, \gamma^\nu \} = 2g^{\mu\nu} \times 1 \). There are many different ways to choose the \( \gamma^\mu \) matrices and we will use the following chiral form:
\[ \gamma^t = \frac{1}{\sqrt{f(z)}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^x = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \]
\[ \gamma^y = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^z = \sqrt{g(z)} \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \]
where the \( \sigma^i \)'s are simply the Pauli sigma matrices:
\[ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]
and $\xi_{\uparrow/\downarrow}$ are the eigenvectors of $\sigma^3$. Note that
\[ \gamma^5 = i\gamma^\tau \gamma^y \gamma^z = \begin{pmatrix} g(z) & -1 \\ -1 & 0 \end{pmatrix} \]
is the resulting $\gamma^5$ matrix.

Measuring spin in the $z$-direction (i.e. the direction of the accelerating observer) we employ the following ansatz for the Dirac field, respectively corresponding to the spin-up and spin-down cases:
\[
\psi_{\uparrow}(t, x, y, z) = \begin{bmatrix} A(t, x, y, z) & 0 \\ B(t, x, y, z) & 0 \end{bmatrix} \exp \left[ i \frac{\bar{\hbar}}{\hbar} I_{\uparrow}(t, x, y, z) \right]
\]
\[
\psi_{\downarrow}(t, x, y, z) = \begin{bmatrix} 0 & C(t, x, y, z) \\ 0 & D(t, x, y, z) \end{bmatrix} \exp \left[ i \frac{\bar{\hbar}}{\hbar} I_{\downarrow}(t, x, y, z) \right].
\]

In order to apply the WKB approximation we insert ansatz (12) for spin-up particles into the Dirac equation. Dividing by the exponential term and multiplying by $\hbar$ the resulting equations to leading order in $\bar{\hbar}$ are
\[
-B \left( \frac{1}{\sqrt{f(z)}} \partial_t I_{\uparrow} + \sqrt{g(z)} \partial_z I_{\uparrow} \right) + Am = 0 \quad (14)
\]
\[
-B(\partial_x I_{\uparrow} + i\partial_y I_{\uparrow}) = 0 \quad (15)
\]
\[
A \left( \frac{1}{\sqrt{f(z)}} \partial_t I_{\uparrow} - \sqrt{g(z)} \partial_z I_{\uparrow} \right) + Bm = 0 \quad (16)
\]
\[
-A(\partial_x I_{\uparrow} + i\partial_y I_{\uparrow}) = 0. \quad (17)
\]

Note that although $A, B$ are not constant, their derivatives—and the components $\Omega_{\mu}$—are all of order $O(\bar{\hbar})$ and so can be neglected to lowest order in WKB.

When $m \neq 0$ equations (14) and (16) couple whereas when $m = 0$ they decouple. We employ the ansatz
\[
I_{\uparrow} = -Et + W(z) + P(x, y)
\]
and insert it into equations (14)–(17),
\[
-B \left( -\frac{E}{\sqrt{f(z)}} + \sqrt{g(z)W'(z)} \right) + mA = 0 \quad (19)
\]
\[
-B(P_x + iP_y) = 0 \quad (20)
\]
\[
-A \left( \frac{E}{\sqrt{f(z)}} + \sqrt{g(z)W'(z)} \right) + mB = 0 \quad (21)
\]
where we consider only the positive frequency contributions without loss of generality. Equations (20) and (22) both yield \((P_x + iP_y) = 0\) regardless of \(A\) or \(B\), implying
\[
P(x, y) = h(x + iy),
\]
where \(h\) is some arbitrary function.

Consider first \(m = 0\). Equations (19) and (21) then have two possible solutions
\[
A = 0 \quad \text{and} \quad W'(z) = W'_+(z) = \frac{E}{\sqrt{f(z)g(z)}},
\]
\[
B = 0 \quad \text{and} \quad W'(z) = W'_-(z) = -\frac{E}{\sqrt{f(z)g(z)}},
\]
corresponding to motion away from (+) and towards (−) the horizon, chosen to be at \(z = 1/a\).

Since the solution \([A, 0, 0, 0]\) is an eigenvector of \(\gamma^5\) and has a negative eigenvalue its spin and momentum vectors are opposite, which is consistent with the fact that the particle is moving towards the horizon and the spin is up. The solution \([0, 0, B, 0]\) is also an eigenvector of \(\gamma^5\) with positive eigenvalue; its spin and momentum vectors are therefore in the same direction, consistent with the particle being spin up and moving away from the horizon.

Hence with the Rindler horizon at \(z = 1/a\) the \((±)\) cases correspond to outgoing/incoming solutions of the same spin. Note that neither of these cases is an antiparticle solution since we assumed positive frequency modes as a part of the ansatz. In computing the imaginary part of the action we note that \(P(x, y)\) must be complex (other than the trivial solution of \(P = 0\)), and so will yield a contribution. However, it is the same for both incoming and outgoing solutions, and so will cancel out in computing the emission probability
\[
\Gamma \propto \frac{\text{Prob}[\text{out}]}{\text{Prob}[\text{in}]} = \frac{\exp[-2(\text{Im} W_+ + \text{Im} h)]}{\exp[-2(\text{Im} W_- + \text{Im} h)]} = \exp[-2(\text{Im} W_+ - \text{Im} W_-)] = \exp[-4\text{Im} W_+]
\]
using reasoning similar to the scalar case. We obtain
\[
W_+(z) = \int \frac{E \, dz}{\sqrt{f(z)g(z)}},
\]
and after integrating around the pole (and dropping the + subscript)
\[
W = -\frac{\pi i E}{\sqrt{g'(z_0)f'(z_0)}} = \frac{\pi i E}{2a},
\]
The resulting tunnelling probability is
\[
\Gamma = \exp \left[ -\frac{2\pi}{a} E \right]
\]
recovering
\[
T_H = \frac{a}{2\pi}
\]
which is the Unruh temperature.

In the massive case equations (19) and (21) no longer decouple. We will start by eliminating the function \(W'(z)\) from the two equations so we can find an equation relating \(A\) and \(B\) in terms of known values. Subtracting \(B \times (21)\) from \(A \times (19)\) gives
\[
\frac{2ABE}{\sqrt{f(z)}} + mA^2 - mB^2 = 0
\]
\[
m \sqrt{f(z)} \left( \frac{A}{B} \right)^2 + 2E \left( \frac{A}{B} \right) - m \sqrt{f(z)} = 0
\]
and so
\[
\frac{A}{B} = \frac{-E \pm \sqrt{E^2 + m^2 f(z)}}{m \sqrt{f(z)}},
\]
where
\[
\lim_{z \to 0} \left( \frac{-E + \sqrt{E^2 + m^2 f(z)}}{m \sqrt{f(z)}} \right) = 0
\]
and
\[
\lim_{z \to 0} \left( \frac{-E - \sqrt{E^2 + m^2 f(z)}}{m \sqrt{f(z)}} \right) = -\infty.
\]

Consequently at the Rindler horizon either \( A \to 0 \) or \( B \to 0 \). For \( A \to 0 \) at the horizon, we solve (21) in terms of \( m \) and insert into (19)
\[
-B \left( \frac{E}{\sqrt{f(z)}} + \sqrt{g(z)} W'(z) \right) + \frac{A^2}{B} \left( \frac{-E}{\sqrt{f(z)}} + \sqrt{g(z)} W'(z) \right) = 0
\]
\[
\frac{E B}{\sqrt{f(z)}} \left( 1 + \frac{A^2}{B^2} \right) - B \sqrt{g(z)} W'(z) \left( 1 - \frac{A^2}{B^2} \right) = 0
\]
\[
W'(z) = W'(z) = \frac{E}{\sqrt{f(z)g(z)}} \left( 1 + \frac{A^2}{B^2} \right) \left( 1 - \frac{A^2}{B^2} \right),
\]
whereas for \( B \to 0 \) at the horizon we solve (19) in terms of \( m \) and insert into (21) to get
\[
-A \left( \frac{E}{\sqrt{f(z)}} + \sqrt{g(z)} W'(z) \right) + \frac{B^2}{A} \left( -E \right) + \sqrt{g(z)} W'(z) \right) = 0
\]
\[
\frac{-E A}{\sqrt{f(z)}} \left( 1 + \frac{B^2}{A^2} \right) - A \sqrt{g(z)} W'(z) \left( 1 - \frac{B^2}{A^2} \right) = 0
\]
\[
W'(z) = W'(z) = -\frac{E}{\sqrt{f(z)g(z)}} \left( 1 + \frac{B^2}{A^2} \right) \left( 1 - \frac{B^2}{A^2} \right).
\]

Since the extra contributions vanish at the horizon, the result of integrating around the pole for \( W \) in the massive case is the same as the massless case and we recover the Unruh temperature for the fermionic Rindler vacuum.

The spin-down case proceeds in a manner fully analogous to the spin-up case discussed above. Other than some changes of sign the equations are of the same form as the spin-up case. For both the massive and massless cases the Unruh temperature (27) is obtained, implying that both spin-up and spin-down particles are emitted at the same rate.

4. Black hole Fermion emission

We turn next to a general static spherically symmetric black hole. As stated in the introduction, we will ignore any change in the angular momentum of the black hole due to the spin of the emitted particle. This is a good approximation for black holes of sufficient mass. The zero angular momentum state is maintained because statistically as many particles with spin in one direction will be emitted as particles with spin in the opposite direction.

We will now extend the fermion tunnelling approach to a general black hole with spherical symmetry. The metric is
\[
d s^2 = - f(r) \, dt^2 + \frac{dr^2}{g(f)} + r^2 (d\theta^2 + \sin^2(\theta) \, d\phi^2)
\]
where for this case we will pick for the $\gamma$ matrices
\[
\gamma^t = \frac{1}{\sqrt{f(r)}} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma^r = \sqrt{g(r)} \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix},
\]
\[
\gamma^\theta = \frac{1}{r} \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \quad \gamma^\phi = \frac{1}{r \sin \theta} \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix},
\]
where we measure spin in terms of the $r$-direction. The matrix for $\gamma^5$ is
\[
\gamma^5 = i \gamma^t \gamma^r \gamma^\theta \gamma^\phi = i \sqrt{g(r) f(r)} \frac{1}{r \sin \theta} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

The spin-up (i.e. $+ve$ $r$-direction) and spin-down (i.e. $-ve$ $r$-direction) solutions have the form
\[
\psi_\uparrow (t, r, \theta, \phi) = \begin{bmatrix} A(t, r, \theta, \phi) \xi_\uparrow \\ B(t, r, \theta, \phi) \xi_\uparrow \end{bmatrix} \exp \left[ \frac{i}{\hbar} I_\uparrow (t, r, \theta, \phi) \right] = \begin{bmatrix} A(t, r, \theta, \phi) \\ B(t, r, \theta, \phi) \end{bmatrix} \exp \left[ \frac{i}{\hbar} I_\uparrow (t, r, \theta, \phi) \right], \tag{29}
\]
\[
\psi_\downarrow (t, x, y, z) = \begin{bmatrix} C(t, r, \theta, \phi) \xi_\downarrow \\ D(t, r, \theta, \phi) \xi_\downarrow \end{bmatrix} \exp \left[ \frac{i}{\hbar} I_\downarrow (t, r, \theta, \phi) \right] = \begin{bmatrix} 0 \\ C(t, r, \theta, \phi) \end{bmatrix} \exp \left[ \frac{i}{\hbar} I_\downarrow (t, r, \theta, \phi) \right]. \tag{30}
\]

We will only solve the spin-up case explicitly since the spin-down case is fully analogous. Employing ansatz (29) into the Dirac equation results in
\[
- \left( \frac{iA}{\sqrt{f(r)}} \partial_t I_\uparrow + B \sqrt{g(r)} \partial_r I_\uparrow \right) + Am = 0 \tag{31}
\]
\[
- \frac{B}{r} \left( \partial_\theta I_\uparrow + \frac{1}{\sin \theta} i \partial_\phi I_\uparrow \right) = 0 \tag{32}
\]
\[
\left( \frac{iB}{\sqrt{f(r)}} \partial_t I_\uparrow - A \sqrt{g(r)} \partial_r I_\uparrow \right) + Bm = 0 \tag{33}
\]
\[
- \frac{A}{r} \left( \partial_\theta I_\uparrow + \frac{1}{\sin \theta} i \partial_\phi I_\uparrow \right) = 0 \tag{34}
\]
to leading order in $\hbar$. We assume the action takes the form
\[
I_\uparrow = -Et + W(r) + J(\theta, \phi), \tag{35}
\]
where we only concern ourselves with positive frequency contributions as before. This yields
\[
\left( \frac{iAE}{\sqrt{f(r)}} - B \sqrt{g(r)} W'(r) \right) + mA = 0 \tag{36}
\]
\[
- \frac{B}{r} \left( J_\theta + \frac{1}{\sin \theta} i J_\phi \right) = 0 \tag{37}
\]
\[- \left( \frac{iBE}{\sqrt{f(r)}} + A\sqrt{g(r)}W'(r) \right) + Bm = 0 \quad (38)\]
\[- \frac{A}{r} \left( J_\theta + \frac{1}{\sin \theta} iJ_\phi \right) = 0. \quad (39)\]

Note that (37) and (39) result in the same equation regardless of \(A\) or \(B\) (i.e. \(J_\theta + \frac{1}{\sin \theta} iJ_\phi = 0\) must be true), implying that \(J(\theta, \phi)\) must be a complex function. As with the Rindler case, the same solution for \(J\) is obtained for both the outgoing and incoming cases. Consequently, the contribution from \(J\) cancels out upon dividing the outgoing probability by the incoming probability as in equation (7). We therefore can ignore \(J\) from this point (or else pick the trivial \(J = 0\) solution).

Equations (36) and (38) (for \(m = 0\)) have two possible solutions:
\[A = -iB \quad \text{and} \quad W'(r) = W'_+(r) = \frac{E}{\sqrt{f(r)g(r)}} \]
\[A = iB \quad \text{and} \quad W'(r) = W'_-(r) = -\frac{E}{\sqrt{f(r)g(r)}} \]
where \(W'_+\) corresponds to outward solutions and \(W'_-\) corresponds to the incoming solutions. The overall tunnelling probability is
\[
\Gamma = \frac{\text{Prob[out]}}{\text{Prob[in]}} = \frac{\exp[-2(\text{Im } W_+)]}{\exp[-2(\text{Im } W_-)]} = \exp[-4\text{ Im } W_+], \quad (40)
\]
with
\[W_+(r) = \int \frac{E \, dr}{\sqrt{f(r)g(r)}}. \]
After integrating around the pole (and dropping the + subscript) we find
\[W = \frac{\pi iE}{\sqrt{g'(r_0)f'(r_0)}} \quad (41)\]
giving
\[
\Gamma = \exp \left[ -\frac{4\pi}{\sqrt{g'(r_0)f'(r_0)}} E \right] \quad (42)
\]
for the resultant tunnelling probability to leading order in \(\hbar\).

We therefore recover the expected Hawking temperature
\[T_H = \frac{\sqrt{f'(r_0)g'(r_0)}}{4\pi \hbar} \quad (43)\]
in the massless case.

Solving equations (36) and (38) for \(A\) and \(B\) in the case that \(m \neq 0\) leads to the result
\[
\left( \frac{A}{B} \right)^2 = \frac{-iE + \sqrt{f(r)m}}{iE + \sqrt{f(r)m}} \]
and approaching the horizon we see that \(\lim_{r \to r_0} \left( \frac{A}{B} \right)^2 = -1\). Following a procedure similar to what was done above, we obtain the same result for the Hawking temperature as in the massless case.

The spin-down calculation is very similar to the spin-up case discussed above. Other than some changes of sign, the equations are of the same form as the spin-up case. For both the massive and massless spin-down cases the Hawking temperature (43) is obtained, implying that both spin-up and spin-down particles are emitted at the same rate. This is consistent with our initial assumption that there are as many spin-up as spin-down fermions emitted.
4.1. Painlevé coordinates

In this section we demonstrate that Painlevé coordinates can be used to recover the results of the preceding section, albeit by a somewhat different computational route.

Using the transformation

\[ t \rightarrow t - \int \frac{1 - g(r)}{f(r) g(r)} \, dr \]  

we obtain from the metric (28)

\[ ds^2 = -f(r) \, dt^2 + 2 \sqrt{f(r)} \sqrt{1 - g(r)} \, dr + dr^2 + r^2 d\Omega^2 \]  

which is the Painlevé form of a spherically symmetric metric.

This coordinate system has a number of interesting features. At any fixed time the spatial geometry is flat. At any fixed radius the boundary geometry for the Painlevé metric is exactly the same as that of the unaltered black hole metric. Also, this form of the Painlevé metric is a very convenient metric to use for black hole tunnelling since the imaginary part of the action for the incoming solution is zero which means \( \text{Prob} \left[ \text{in} \right] = 1 \) [19]. This property also holds for fermion tunnelling.

We choose the representation for the \( \gamma \) matrices to be

\[
\gamma^t = \frac{1}{\sqrt{f(r)}} \begin{pmatrix} 1 & 0 \\ 0 & 1 + \sqrt{1 - g(r)} \sigma_3 \end{pmatrix}, \quad \gamma^r = \frac{1}{\sqrt{g(r)}} \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \\
\gamma^\theta = \frac{1}{r} \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad \gamma^\phi = \frac{1}{r \sin \theta} \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}.
\]

The matrix for \( \gamma^5 \) for this case is

\[
\gamma^5 = i \gamma^t \gamma^r \gamma^\theta \gamma^\phi = \frac{\sqrt{g(r)}}{f(r) r^2 \sin \theta} \begin{pmatrix} 1 & 0 \\ 0 & -1 - \sqrt{1 - g(r)} \sigma_3 \\ 0 & 1 + \sqrt{1 - g(r)} \sigma_3 \end{pmatrix}.
\]

Measuring spin in the \( r \)-direction we have, as before, the two following ansatz for the spin-1/2 Dirac field which correspond to the spin-up (i.e. +ve \( r \)-direction) and spin-down (i.e. -ve \( r \)-direction) cases:

\[
\psi_\uparrow(t, r, \theta, \phi) = \begin{bmatrix} A(t, r, \theta, \phi) \\ 0 \end{bmatrix} \exp \left[ i \frac{\bar{\hbar}}{\hbar} I_1(t, r, \theta, \phi) \right],
\]

\[
\psi_\downarrow(t, x, y, z) = \begin{bmatrix} C(t, r, \theta, \phi) \\ 0 \end{bmatrix} \exp \left[ i \frac{\bar{\hbar}}{\hbar} I_1(t, r, \theta, \phi) \right].
\]

Once again we will only solve the spin-up case explicitly. Insertion of the ansatz into the Dirac equation results in the following equations to the leading order in \( \hbar \),
To solve these equations we pick the ansatz (35) for the action, again working only with positive frequency contributions. The equations for $J$ are the same as in the last section, and we can dispense with this function for the same reasons as before. We obtain

$$-B \left( \frac{1}{\sqrt{f(r)}} (1 + \sqrt{1 - g(r)}) \partial_t I_t + \sqrt{g(r)} \partial_r I_r \right) + Am = 0 \quad (48)$$

$$-A \left( \frac{1}{\sqrt{f(r)}} (1 - \sqrt{1 - g(r)}) \partial_t I_t - \sqrt{g(r)} \partial_r I_r \right) + Bm = 0 \quad (50)$$

Equations (52) and (53) (for $m = 0$) have two possible solutions:

$$A = 0 \quad \text{and} \quad W'(r) = W_+(r) = \frac{E(1 + \sqrt{1 - g(r)})}{\sqrt{f(r)}g(r)}$$

$$B = 0 \quad \text{and} \quad W'(r) = W_-(r) = \frac{-E(1 - \sqrt{1 - g(r)})}{\sqrt{f(r)}g(r)}.$$  

$W_+$ corresponds to outward solutions and $W_-$ corresponds to the incoming solutions. Note that $W_+$ has a pole at the horizon but $W_-$ has a well-defined limit at the horizon and does not have a pole (i.e. $\lim_{r \to r_0} W_-(r) = -\frac{E}{2} \sqrt{\frac{g''(r_0)}{f'(r_0)}}$). This implies that the imaginary part $W_-$ is zero and confirms that Prob [in] = 1. So the overall tunnelling probability is

$$\Gamma \propto \text{Prob}[\text{out}] \quad \Gamma \propto \exp[-2 \text{Im } W_+]$$

Therefore,

$$W_+(r) = \int \frac{E(1 + \sqrt{1 - g(r)})}{\sqrt{f(r)}g(r)} dr$$

and after integrating around the pole (and dropping the + subscript):

$$W = \frac{2\pi i E}{\sqrt{g'(r_0)f'(r_0)}}.$$  

(54)

So the resulting tunnelling probability is once again

$$\Gamma = \exp \left[ -\frac{4\pi}{\sqrt{g'(r_0)f'(r_0)E}} \right]$$

and the normal Hawking temperature is also recovered for the Painlevé massless case

$$T_H = \frac{\sqrt{f'(r_0)g'(r_0)}}{4\pi}.$$  

(55)

Solving equations (52) and (53) for $A$ and $B$ in the case that $m \neq 0$ leads to the results that $A \to 0$ as $r \to r_0$ or $B \to 0$ as $r \to r_0$. So the same final result will be recovered in the massive case.
4.2. Kruskal–Szekers metric

In the preceding subsections we employed metrics that had coordinate singularities at the horizon. One might be concerned that the tunnelling effect is dependent upon this. Here we demonstrate that this is not the case, by investigating fermion tunnelling in the Kruskal–Szekers metric

\[ ds^2 = f(r)(-dT^2 + dX^2) + r^2 d\Omega^2, \tag{56} \]

where

\[ f(r) = \frac{32M^3 e^{-\frac{\pi r}{2M}}}{r} \left( \frac{r}{2M} - 1 \right) e^{\frac{r}{2M}} = X^2 - T^2. \]

The metric (56) is well behaved at both the future and past horizons \( X = \pm T \) (corresponding to \( r = 2M \)). Note that the metric has a timelike Killing vector \( X\partial_T + T\partial_X \) (and not \( \partial_T \)).

For this calculation we will employ the following representation for the \( \gamma \) matrices:

\[
\gamma^T = \frac{1}{\sqrt{f(r)}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^X = \frac{1}{\sqrt{f(r)}} \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \\
\gamma^\theta = \frac{1}{r} \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \quad \gamma^\phi = \frac{1}{r \sin \theta} \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix},
\]

where we measure the spin referenced to the \( X \)-direction. The matrix for \( \gamma^5 \) is

\[ \gamma^5 = i\gamma^r \gamma^\theta \gamma^\phi = \frac{1}{f(r) r^2 \sin \theta} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \]

The spin-up (i.e. +ve \( X \)-direction) and spin-down (i.e. −ve \( X \)-direction) solutions have the form

\[
\psi^\uparrow(T, X, \theta, \phi) = \begin{bmatrix} A(T, X, \theta, \phi) \xi^\uparrow \\ B(T, X, \theta, \phi) \xi^\uparrow \end{bmatrix} \exp \left[ \frac{i}{\hbar} I^\uparrow(T, X, \theta, \phi) \right],
\]

\[
\psi^\downarrow(T, X, y, z) = \begin{bmatrix} C(T, X, \theta, \phi) \xi^\downarrow \\ D(T, X, \theta, \phi) \xi^\downarrow \end{bmatrix} \exp \left[ \frac{i}{\hbar} I^\downarrow(T, X, \theta, \phi) \right]. \tag{58} \]

Once again inserting the spin-up ansatz (57) (the spin-down case being similar) into the Dirac equation yields the following equations:

\[
-\frac{B}{\sqrt{f(r)}} (\partial_T I^\uparrow + \partial_X I^\uparrow) + Am = 0 \tag{59}
\]

\[
-\frac{B}{r} \left( \partial_y I^\uparrow + \frac{1}{\sin \theta} i\partial_y I^\uparrow \right) = 0 \tag{60}
\]

\[
\frac{A}{\sqrt{f(r)}} (\partial_T I^\uparrow - \partial_X I^\uparrow) + Bm = 0 \tag{61}
\]
to leading order in $\hbar$. This time we can infer only that the action takes the form

$$I = -I(X, T) + J(\theta, \phi).$$

(63)

The equations for $J$ are unchanged from previous calculations. We thus ignore these equations since they do not affect the final result and only concern ourselves with solving for $I(X, T)$.

In order to solve the equations we need a definition of the energy of the wave. We will define energy via the timelike Killing vector

$$\partial_\chi = N(X\partial_T + T\partial_X),$$

where $N$ is a normalization constant chosen so that the norm of the Killing vector is equal to 1 at infinity. This yields

$$\partial_\chi = \frac{1}{4M}(X\partial_T + T\partial_X)$$

(64)

and so

$$\partial_\chi I = -E.$$  

(65)

Using (65) with (59) and (61) we shall solve the equations.

Consider first the massless case. Here either $A = 0$ or $B = 0$. For $A = 0$ (outgoing case):

$$\partial_T I + \partial_X I = 0$$

$$\frac{1}{4M}(X\partial_T I + T\partial_X I) = -E.$$  

The first equation implies the general solution of $I = h(X - T)$ and the second in turn leads to

$$4ME = (X - T)h'(X - T)$$

$$h'(X - T) = \frac{4ME}{(X - T)}$$

which has a simple pole at the black hole horizon $X = T$. Setting $\eta = X - T$ we have

$$h'(\eta) = \frac{4ME}{\eta}.$$  

(66)

Integrating (66) around the pole at the horizon (doing a half circle contour) implies

$$\text{Im } I_{\text{out}} = 4\pi ME$$

for outgoing particles.

For the incoming case, $B = 0$:

$$\partial_T I - \partial_X I = 0$$

$$\frac{1}{4M}(X\partial_T I + T\partial_X I) = -E.$$  

The first equation implies the general solution $I = k(X + T)$ and so the second leads to

$$-4ME = (X + T)k'(X + T)$$

$$k'(X + T) = \frac{-4ME}{(X + T)}.$$
Note that this equation does not have a pole at the black hole horizon $X = T$. Hence for incoming particles

$$\text{Im} I_{in} = 0$$

and so $\text{Prob}[\text{in}] = 1$ like in the Painlevé case. The final result for the tunnelling probability is

$$\Gamma = \frac{\text{Prob}[\text{out}]}{\text{Prob}[\text{in}]} = \exp[-2 \text{Im} I_{out}] = \exp[-8\pi ME]$$

and we see that the Hawking temperature $T_H = \frac{1}{8\pi M}$ is recovered in the massless case.

In the massive case we must use equations (65), (59) and (61) to solve for $\frac{A}{B}$. A straightforward calculation yields

$$\frac{A}{B} = -\frac{4ME \pm \sqrt{16M^2 E^2 + m^2 f(r)(X^2 - T^2)}}{\sqrt{f(r)m(X + T)}}$$

where we note as the black hole horizon $(X = T)$ is approached that either $\frac{A}{B} \to 0$ or $\frac{A}{B} \to \frac{-4ME}{\sqrt{f(2M)mT}}$. Subtracting (59)/$A$ from (61)/$B$ leads to

$$\partial_T I = -\partial_X I \left( \frac{1 - \left(\frac{A}{B}\right)^2}{1 + \left(\frac{A}{B}\right)^2} \right)$$

and so from (65) we obtain

$$\partial_X I = \frac{4ME(1 + \left(\frac{A}{B}\right)^2)}{X(1 - \left(\frac{A}{B}\right)^2) - T(1 + \left(\frac{A}{B}\right)^2)}$$

where $\frac{A}{B} \to 0$ at $X = T$.

From equation (67) we find that

$$\lim_{X \to T} \left[ X \left( 1 - \left(\frac{A}{B}\right)^2 \right) - T \left( 1 + \left(\frac{A}{B}\right)^2 \right) \right] = 0$$

and

$$\lim_{X \to T} \frac{\partial}{\partial X} \left[ X \left( 1 - \left(\frac{A}{B}\right)^2 \right) - T \left( 1 + \left(\frac{A}{B}\right)^2 \right) \right] = \lim_{X \to T} \left[ \left( 1 - \left(\frac{A}{B}\right)^2 \right) - 2(X + T) \left(\frac{A}{B}\right) \frac{\partial}{\partial X} \left(\frac{A}{B}\right) \right] = 1.$$
for fermion emission using tunnelling methods is relatively simple and straightforward to compute.

For accelerated observers using Rindler coordinates we found the expected Unruh temperature. We also applied fermion tunnelling to a general static spherically symmetric black hole metric in both Schwarzschild and Painlevé forms, and found that the usual Hawking temperature is recovered. That this situation does not depend on coordinate singularities was demonstrated by showing the same results hold for the Kruskal–Szekers metric. Our results indicate not only that the tunnelling method is robust, but that it can indeed be understood as a physical phenomenon.

Extending fermion tunnelling to rotating spacetimes in which the emitted particles have orbital angular momentum would be a natural next step. Computing fermion tunnelling in the background of the Kinnersley black hole is a natural step. Based on the emission probability results from the Kinnersley black hole [27], we expect that the final tunnelling probability should be of the form \[ \exp \left( -\frac{1}{\kappa} (E - \Omega_\mu J_\mu + C) \right) \], where \( C \) parametrizes the coupling between the spin of the field and the angular momentum of the black hole. Extending fermion tunnelling to dynamical black holes such as Vaidya or those used in [17] would be a logical next step. Computing corrections to the tunnelling probability by fully taking into account conservation of energy will yield corrections to the fermion emission temperature. In various scalar field cases this is inherent in the Parikh/Wilczek tunnelling method [3, 7–18] and can be incorporated into the Hamilton–Jacobi tunnelling approach [6]. Another avenue of research is to perform tunnelling calculations to higher order in WKB (in both the scalar field and fermionic cases) in order to calculate grey body effects. The possibility of calculating a density matrix for the emitted particles via the tunnelling approach in order to calculate correlations between particles is another interesting line of research. Work on these areas is in progress.

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