Periodic homogenization of a pseudo-parabolic equation via a spatial-temporal decomposition

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Abstract Pseudo-parabolic equations have been used to model unsaturated fluid flow in porous media. In this paper it is shown how a pseudo-parabolic equation can be upscaled when using a spatio-temporal decomposition employed in the Peszynska-Showalter-Yi paper [8]. The spatial-temporal decomposition transforms the pseudo-parabolic equation into a system containing an elliptic partial differential equation and a temporal ordinary differential equation. To strengthen our argument, the pseudo-parabolic equation has been given advection/convection/drift terms. The upscaling is done with the technique of periodic homogenization via two-scale convergence. The well-posedness of the extended pseudo-parabolic equation is shown as well. Moreover, we argue that under certain conditions, a non-local-in-time term arises from the elimination of an unknown.

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1 Introduction

Groundwater recharge and pollution prediction for aquifers need models for describing unsaturated fluid flow in porous media. Pseudo-parabolic equations were found to be adequate models, see eqn. 25 in [3]. In [8] a spatial-temporal decomposition
of a pseudo-parabolic system was introduced. It was shown that this decomposition made upscaling of this system rather straightforward in several classical situations such as vanishing time-delay and double-porosity systems. In [8] a toy pseudo-parabolic model was derived from a balance equation describing flow through a partially saturated porous medium. In our framework, a convective term that was dropped in [8], is retained in order to show that this term yields no additional problems for upscaling with the spatial-temporal decomposition. We want to convey the message that this decomposition can be applied not only to the physical system in [8] but also to other physical systems with pseudo-parabolic equations, such as the concrete corrosion reaction model introduced in [9]. Both these pseudo-parabolic systems are physical systems on a spatial micro scale with an intrinsic microscopic periodicity of size $\varepsilon \ll 1$. Similar intrinsic microscopic periodic behaviors are found in highly active research fields using composite structures or nano-structures.

In this paper, we use this spatial-temporal decomposition to upscale our pseudo-parabolic equation by using the concept of periodic homogenization via two-scale convergence, which leads to a homogenized system that retains the spatial-temporal decomposition. We start in Section 2 with formulating our pseudo-parabolic system $(Q^{\varepsilon})$, the decomposition system $(P^{\varepsilon})$ and stating our assumptions. In Section 3, an existence and uniqueness result for weak solutions to our problem $(P^{\varepsilon})$ is derived. In Section 4, we apply the idea of two-scale convergence to a weak version of problem $(P^{\varepsilon})$, denoted $(P^{\varepsilon}_{w})$, that contains the microscopic information at the $\varepsilon$-level. Furthermore in this section, an upscaled system $(P^{0}_{w})$ of the weak system $(P^{\varepsilon}_{w})$ is derived in the limit $\varepsilon \downarrow 0$, and, under certain conditions, an upscaled strong system $(P^{0}_{s})$ is obtained after eliminating several variables. This upscaled strong system contains a non-local-in-time term, but the system has lost the partial differential equation framework as a consequence. Contrary, the upscaled weak system $(P^{0}_{w})$ keeps the partial differential equation framework due to the spatial-temporal decomposition.

2 Basic system and assumptions

Our pseudo-parabolic system $(Q^{\varepsilon})$ consists of a family of $N$ partial differential equations for the variable vector $U^{\varepsilon}(t, x, x/\varepsilon) = (U^{\varepsilon}_{1}, \ldots, U^{\varepsilon}_{\alpha}, \ldots, U^{\varepsilon}_{N})$ with $t > 0$ and $x = (x_{1}, \ldots, x_{i}, \ldots, x_{d}) \in \Omega \subset \mathbb{R}^{d}$. For $\varepsilon \in (0, \varepsilon_{0})$ with $\varepsilon_{0} > 0$, system $(Q^{\varepsilon})$ is formulated as

\[
(Q^{\varepsilon}) \begin{cases}
    M^{\varepsilon}G^{-1}\partial_{t}U^{\varepsilon} - \nabla \cdot ((E^{\varepsilon} \cdot \nabla + D^{\varepsilon})G^{-1}(\partial_{t}U^{\varepsilon} + LU^{\varepsilon})) = H^{\varepsilon} + (K^{\varepsilon} - M^{\varepsilon}G^{-1}L)U^{\varepsilon} + J^{\varepsilon} \cdot \nabla U^{\varepsilon} & \text{on } \mathbb{R}^{+} \times \Omega, \\
    U^{\varepsilon} = U_{*} & \text{on } \{0\} \times \Omega, \\
    \partial_{t}U^{\varepsilon} + LU^{\varepsilon} = 0 & \text{on } \mathbb{R}^{+} \times \partial \Omega.
\end{cases}
\]
The vectors $V^e$ and $U^e$ are both functions of the time coordinate $t$, the global or macro position coordinate $x$, and also periodic functions of the micro (or nano) coordinate $y \in Y$, where $y = x/\varepsilon$, where the size of the micro domain $Y$ is $\mathcal{O}(\varepsilon)$ of the size of the macro domain $\Omega$.

Our dimensionless decomposition system $(P^e)$ consists of a family of $N$ partial differential equations (PDEs) and a family of $N$ ordinary differential equations (ODEs) for the two variable vectors $V^e(t, x, x/\varepsilon) = (V^e_t, \ldots, V^e_N)$ and $U^e(t, x, x/\varepsilon)$. For $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 > 0$, it is formulated as

\[
\begin{align*}
\begin{cases}
M^e V^e - \nabla \cdot (E^e \cdot \nabla V^e + D^e V^e) = H^e + K^e U^e + J^e \cdot \nabla U^e & \text{on } \mathbb{R}^+ \times \Omega, \\
\partial_t U^e + LU^e = GV^e & \text{on } \mathbb{R}^+ \times \Omega, \\
U^e = U_0 & \text{on } \{0\} \times \Omega, \\
V^e = 0 & \text{on } \mathbb{R}^+ \times \partial \Omega.
\end{cases}
\end{align*}
\]

Above, the $\varepsilon$-dependent notation $e^e(t, x) = c(t, x, x/\varepsilon)$ is used for the $\varepsilon$-independent 1-, 2- and 3-tensors of assumption (A1).

(A1) For all $\alpha, \beta \in \{1, \ldots, N\}$ and for all $i, j \in \{1, \ldots, d\}$, we have

\[
M_{\alpha\beta}, E_{ij}, D_{i\alpha\beta}, H_{\alpha}, K_{i\alpha\beta}, J_{i\alpha\beta} \in L^\infty(\mathbb{R}^+ \times \Omega; C_0(\gamma)), \]

\[
L_{\alpha\beta}, G_{\alpha\beta} \in L^\infty(\mathbb{R}^+; W^{1,\infty}(\Omega)), \
U_0 \in C^1(\Omega)^N,
\]

with $G$ invertible.

(A2) Let the tensors $M^e$ and $E^e$ be in diagonal form\(^1\) with elements $m^e_{\alpha} > 0$ and $e^e_{\alpha} > 0$, respectively, satisfying $1/m^e_{\alpha}, 1/e^e_{\alpha} \in L^\infty(\mathbb{R}^+ \times \Omega; C_0(\gamma))$.

(A3) The inequality

\[
\|D^e_{ij\alpha\beta}\|_{L^\infty(\mathbb{R}^+ \times \Omega; C_0(\gamma))} \lesssim \frac{4}{d N^2 \|1/m^e_{\alpha}\|_{L^\infty(\mathbb{R}^+ \times \Omega; C_0(\gamma))} \|1/e^e_{\alpha}\|_{L^\infty(\mathbb{R}^+ \times \Omega; C_0(\gamma))}}
\]

holds for all $\alpha, \beta \in \{1, \ldots, N\}$, for all $i \in \{1, \ldots, n\}$, and for all $\varepsilon \in (0, \varepsilon_0)$.

Remark, inequality (2) holds for the $Y$-averaged functions $\overline{D^e}_{ij\alpha\beta}, \overline{M^e}_{\alpha\beta}$, and $\overline{E^e}_{ij}$ in $L^\infty(\mathbb{R}^+ \times \Omega)$, using $|Y| \overline{f}(t, x) = \int_Y f(t, x, y) dy$.

### 3 Existence and uniqueness of weak solutions to $(P^e)$

In this section, we show the existence and uniqueness of a weak solution $(U, V)$ to $(P^e)$. We define a weak solution to $(P^e)$ for $\varepsilon \in (0, \varepsilon_0)$ and $T \in \mathbb{R}^+$ as a pair of

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\(^{1}\) Due to the Theorem of Jacobi about quadratic forms (cf. [4]) in combination with the coercivity of both $M^e$ and $E^e$, we are allowed to assume diagonal forms of $M^e$ and $E^e$ as the orthogonal transformations, necessary to put their quadratic forms in diagonal form, modify the domain $\Omega^e$ and the coefficients of $D^e, H^e, K^e$ and $J^e$ without changing their regularity.
sequences \((U^e, V^e) \in H^1((0, T) \times \Omega)^N \times L^\infty((0, T), H^1_0(\Omega))^N\) satisfying

\[
\begin{align*}
\int_\Omega \phi^\top (\varepsilon \chi^e U^e - H^e - K^e U^e - f^e \cdot \nabla u^e + (\nabla \phi)^\top (\varepsilon E^e \nabla U^e + D^e V^e)) \, dx &= 0, \\
\int_\Omega \psi^\top (\partial_t U^e + L^e U^e - G^e V^e) \, dx &= 0,
\end{align*}
\]

for a.e. \(t \in (0, T)\), for all test-functions \(\phi \in H^1_0(\Omega)^N\) and \(\psi \in L^2(\Omega)^N\).

The existence and uniqueness can only hold when the first equation of \((P^e_w)\) satisfies all the conditions of Lax-Milgram. The next lemma provides the coercivity condition, while the continuity condition is trivially satisfied.

**Lemma 1.** Assume assumptions (A1) - (A3) hold, then there exist positive constants \(\bar{m}_\alpha, \bar{e}_i, \bar{H}, \bar{K}_\alpha, \bar{f}_i\) for \(\alpha \in \{1, \ldots, N\}\) and \(i \in \{1, \ldots, d\}\) such that the following a-priori estimate holds for a.e. \(t \in (0, T)\).

\[
\sum_{\alpha=1}^N \bar{m}_\alpha \|U^e_\alpha\|_{L^2(\Omega)}^2 + \sum_{i=1}^d \sum_{\alpha=1}^N \bar{e}_i \|\partial_i V^e_\alpha\|_{L^2(\Omega)}^2 \\
\leq \bar{H} + \sum_{\alpha=1}^N \bar{K}_\alpha \|U^e_\alpha\|_{L^2(\Omega)}^2 + \sum_{i=1}^d \sum_{\alpha=1}^N \bar{f}_i \|\partial_i U^e_\alpha\|_{L^2(\Omega)}^2 \tag{1}
\]

**Proof.** See pages 92, 93 in [9] for proof and relation with parameters of \((P^e_w)\). \(\square\)

**Theorem 1.** Assume assumptions (A1) - (A3) hold, then there exists a unique pair \((U^e, V^e) \in H^1((0, T) \times \Omega)^N \times L^\infty((0, T), H^1_0(\Omega))^N\) such that \((U^e, V^e)\) is a weak solution to \((P^e_w)\).

**Proof.** Use \(\phi = V^e\) and apply Lemma 1. Then use \(\psi \in \{U^e, \partial_t U^e\}\). Moreover, apply a gradient to the second equation of \((P^e)\) and test that equation with \(\nabla U^e\) and \(\partial_t \nabla U^e\). Application of Young’s inequality, use of (1) and application of Gronwall’s inequality, see [2, Thm. 1], yields the existence for \(U^e\). Then Lax-Milgram yields the existence for \(V^e\). Uniqueness follows from the bilinearity of \((P^e_w)\). For more details, see pages 93 and 94 in [9]. \(\square\)

4 Upscaling the system \((P^e_w)\) via two-scale convergence

Based on two-scale convergence, see [1], [5], [7] for details, we obtain the following Lemma ensuring that the weak solution to problem \((P^e_w)\) has two-scale limits in the limit \(\varepsilon \downarrow 0\).

**Lemma 2.** Assume assumptions (A0), (A1), (A2) to hold. For each \(\varepsilon \in (0, \varepsilon_0]\), let the pair of sequences \((U^\varepsilon, V^\varepsilon) \in H^1((0, T) \times \Omega) \times L^\infty((0, T), H^1_0(\Omega))\) be the unique weak solution to \((P^e_w)\). Then this sequence of weak solutions satisfies the estimate

\[
\sum_{\alpha=1}^N \bar{m}_\alpha \|U^\varepsilon_\alpha\|_{L^2(\Omega)}^2 + \sum_{i=1}^d \sum_{\alpha=1}^N \bar{e}_i \|\partial_i V^\varepsilon_\alpha\|_{L^2(\Omega)}^2 \\
\leq \bar{H} + \sum_{\alpha=1}^N \bar{K}_\alpha \|U^\varepsilon_\alpha\|_{L^2(\Omega)}^2 + \sum_{i=1}^d \sum_{\alpha=1}^N \bar{f}_i \|\partial_i U^\varepsilon_\alpha\|_{L^2(\Omega)}^2
\]
Using Lemma 2, we upscale \((P^u_ε)\) to \((P^u_0)\) via two-scale convergence.

**Theorem 2.** Assume the conditions of Lemma 2 are met. Then the two-scale limits \(u \in H^1((0,T) \times \Omega)^N\), \(\psi \in H^1((0,T);L^2(\Omega;H^1_0(Y)/\mathbb{R}))\) and \(v \in L^\infty((0,T);H^1_0(\Omega))^N\) introduced in Lemma 2 form the weak solution triple to

\[
(P^0_\Psi)
\]

\[
\begin{aligned}
\int_\Omega \phi^\top \left[ Mv - \bar{H} - \bar{K}u - \bar{J} \cdot \nabla u - \frac{1}{|Y|} \int_Y \nabla \cdot \nabla \psi \right] d\Omega \\
+ (\nabla \phi)^\top \cdot (E^* \cdot \nabla v + D^* v) d\Omega = 0,
\end{aligned}
\]

\[
\begin{aligned}
\int_\Omega \psi^\top [\partial_t u + Lu - Gv] d\Omega = 0,
\end{aligned}
\]

\[
\begin{aligned}
\int_Y \xi^\top \cdot \nabla_y \left[ \partial_t \psi + L \psi - \delta v - \delta \cdot \nabla v \right] d\Omega = 0,
\end{aligned}
\]

\[
\begin{aligned}
\xi \cdot \nabla_y \psi(0,x) = 0, & \quad \text{on } \Omega, \\
\psi = 0, & \quad \text{on } \Omega \times Y,
\end{aligned}
\]

for a.e. \(t \in (0,T)\), for all test-functions \(\phi \in H^1_0(\Omega)^N\), \(\psi \in L^2(\Omega)^N\), and \(\xi \in H^1_0(Y)^d \times \mathbb{R}^N\), where the effective coefficients \(E^*\) and \(D^*\) are given by

\[
E^* = \frac{1}{|Y|} \int_Y E \cdot (1 + \nabla_y W) d\Omega,
\]

\[
D^* = \frac{1}{|Y|} \int_Y D + E \cdot \nabla_y \delta d\Omega,
\]

\[
\delta = \nabla_y (G \delta),
\]

\[
\omega = \nabla_y W \otimes \delta,
\]

and the tensor \(\delta_{\alpha\beta} \in L^\infty((0,T) \times \Omega;H^1_0(Y)/\mathbb{R})\) and vector \(W = L^\infty((0,T) \times \Omega;H^1_0(Y)/\mathbb{R})\) satisfy the cell problems

\[
0 = \int_Y \Phi^\top \cdot (\nabla_y (E \cdot (1 + \nabla_y W))) d\Omega, \quad 0 = \int_Y \Psi^\top (\nabla_y \cdot (D + E \cdot \nabla_y \delta)) d\Omega
\]

for all \(\Phi \in C^\infty_0(Y)^d\), \(\Psi \in C^\infty_0(Y)^{N \times N}\).
Proof. In \( (P_\epsilon) \), we choose \( \phi = \phi_\epsilon = \Phi(t, x, \frac{x}{\epsilon}) \) and \( \psi = \psi_\epsilon = \Psi(t, x) + \epsilon \varphi(t, x, \frac{x}{\epsilon}) \) for the test-functions \( \Phi \in L^2((0,T); C^0_c(\Omega)) \), \( \Psi \in L^2((0,T); C^0(\Omega)) \), and \( \varphi \in L^2((0,T); C^0(\Omega)) \). Two-scale convergence limits, see [1], [5], [7], and cell-function arguments, see [6], give \( (P_0) \). Details in pages 97, 98 of [9]. ⊓⊔

We have shown that upscaling system \( (P_\epsilon) \) yields system \( (P_0) \). This system contains only PDEs with respect to \((t, x)\). However, an extra variable \( \nabla_y U \) was needed. Removing \( \nabla_y U \) needs the use of continuous semi-group theory, see papers 10 and 14 of [10], for solving the third equation of system \( (P_0) \). This leads to a non-local-in-time term as a consequence of removing \( \nabla_y U \).

5 Conclusion

Our main goal of this paper is to show that the spatial-temporal decomposition, as employed in [8], allows for the straightforward upscaling of pseudo-parabolic equations, in specific for system \( (Q_\epsilon) \). The upscaling procedure is here performed using the concept of two-scale convergence as reported in Section 4. Moreover, the decomposition is retained in the upscaled limit. A non-local-in-time term arose when an extra variable was eliminated. The spatial-temporal decoupling showed why this non-local term is non-local in time.

In future research we intend to investigate the applicability of the spatial-temporal decomposition of our pseudo-parabolic system to perforated periodic domains, corrector estimates (convergence speed estimate) and high-contrast situations.

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