A COMPACT $G_2$-CALIBRATED MANIFOLD WITH FIRST BETTI NUMBER $b_1 = 1$

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Abstract. We construct a compact, formal 7-manifold with a closed $G_2$-structure and with first Betti number $b_1 = 1$, which does not admit any torsion-free $G_2$-structure, that is, it does not admit any $G_2$-structure such that the holonomy group of the associated metric is a subgroup of $G_2$. We also construct associative calibrated (hence volume-minimizing) 3-tori with respect to this closed $G_2$-structure and, for each of those 3-tori, we show a 3-dimensional family of non-trivial associative deformations. We also construct a fibration of our 7-manifold over $S^2 \times S^1$ with generic fiber a (non-calibrated) coassociative 4-torus and some singular fibers.

1. Introduction

A 7-manifold $M$ is said to admit a $G_2$-structure if there is a reduction of the structure group of its frame bundle from the linear group $GL(7, \mathbb{R})$ to the exceptional Lie group $G_2$. A $G_2$-structure is equivalent to the existence of a certain type of a non-degenerate 3-form $\varphi$ (the $G_2$ form) on the manifold. Indeed, by [20] a manifold $M$ with a $G_2$-structure comes equipped with a Riemannian metric $g$, a cross product $P$, a 3-form $\varphi$, and orientation, which satisfy the relation

$$\varphi(X,Y,Z) = g(P(X,Y),Z),$$

for all vector fields $X, Y, Z$ on $M$.

If the 3-form $\varphi$ is covariantly constant with respect to the Levi-Civita connection of the metric $g$ or, equivalently, the intrinsic torsion of the $G_2$-structure vanishes [12], then the holonomy group of $g$ is contained in $G_2$, and the 3-form $\varphi$ is closed and coclosed [20]. In this case, the $G_2$-structure is said to be torsion-free. The first complete examples of metrics with holonomy $G_2$ were obtained by Bryant and Salamon in [6], while compact examples of Riemannian manifolds with holonomy $G_2$ were constructed first by Joyce [33], and then by Kovalev [36], Kovalev and Lee [37], and Corti, Haskins, Nordström, Pacini [11]. More recently, a new construction of compact manifolds with holonomy $G_2$ has been given in [35] by gluing families of Eguchi–Hanson spaces.

A $G_2$-structure is called closed, or calibrated, if the 3-form $\varphi$ is closed [20], and a $G_2$-structure is said to be coclosed, or cocalibrated, if the 3-form $\varphi$ is coclosed. These two classes of $G_2$-structures are very different in nature, the closed condition of the $G_2$ form being much more restrictive; for example, Crowley and Nordström in [12] prove that coclosed $G_2$-structures always exist on closed spin manifolds and satisfy the parametric $h$-principle.

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Recently, for a compact 7-manifold $M$ endowed with a closed non-parallel $G_2$-structure, Podestà and Raffero in [41] have proved that the identity component of the automorphism group of $M$ is Abelian with dimension bounded by $\min\{6, b_2(M)\}$.

Compact $G_2$-calibrated manifolds have interesting curvature properties. It is well known that a $G_2$ holonomy manifold is Ricci-flat or, equivalently, both Einstein and scalar-flat. On a compact $G_2$-calibrated manifold, both the Einstein condition [9] and scalar-flatness [5] are equivalent to the holonomy being contained in $G_2$. In fact, Bryant in [5] shows that the scalar curvature is always non-positive.

All the previously known examples in the literature of compact 7-manifolds which are not a product of $S^1$ and a symplectic half-flat 6-manifold in the sense of [8] admitting a closed $G_2$ form, which is not also coclosed, have first Betti number strictly bigger than one. The first example of a compact $G_2$-calibrated manifold that does not have holonomy contained in $G_2$ was obtained in [18]. This example is a nilmanifold, that is a compact quotient of a simply connected nilpotent Lie group by a lattice, endowed with an invariant closed $G_2$-structure. In [10] Conti and the first author classified the nilpotent 7-dimensional Lie algebras that admit a closed $G_2$-structure. All those examples are non-formal. Other examples were given in [19]. They are formal compact solvable manifolds with first Betti number $b_1 = 3$.

In this paper, we construct a compact formal 7-manifold with a closed $G_2$-structure and with first Betti number $b_1 = 1$ not admitting any torsion-free $G_2$-structure. To our knowledge, this manifold is the first example of compact $G_2$-calibrated manifold that satisfies all these properties and it is not a product.

To construct such a manifold, we start with a compact 7-manifold $M$ equipped with a closed $G_2$ form $\varphi$ and with first Betti number $b_1(M) = 3$. Then we quotient $M$ by a finite group preserving $\varphi$ to obtain an orbifold $\tilde{M}$ with a closed orbifold $G_2$ form $\tilde{\varphi}$ and with first Betti number $b_1(\tilde{M}) = 1$ (Proposition 15). We resolve the singularities of the 7-orbifold $\tilde{M}$ to produce a smooth 7-manifold $\tilde{\tilde{M}}$ with a closed $G_2$ form $\tilde{\tilde{\varphi}}$, with first Betti number $b_1(\tilde{\tilde{M}}) = 1$ and such that $(\tilde{M}, \tilde{\varphi})$ is isomorphic to $(\tilde{\tilde{M}}, \tilde{\tilde{\varphi}})$ outside the singular locus of $\tilde{M}$ (Theorem 21). The idea of this construction stems from our study of Joyce’s original techniques of “$G_2$-orbifold resolutions” [33, 34] that enabled him to construct compact Riemannian manifolds with holonomy $G_2$. (There “$G_2$-orbifold” means an orbifold with a closed and coclosed orbifold $G_2$ form.)

Next, we prove that $\tilde{\tilde{M}}$ has the aforementioned properties. More precisely, using the concept of 3-formal minimal model, introduced in [21] as an extension of formality [14] (see Section 8 for details) we prove that the 7-manifold $\tilde{\tilde{M}}$ is formal (Proposition 24). On the other hand, we show that $\tilde{\tilde{M}}$ has fundamental group $\pi_1(\tilde{\tilde{M}}) = \mathbb{Z}$ (Proposition 23), this resulting from the careful choice of the action of the finite group acting on $M$. Finally, using this last result and that $b_1(\tilde{\tilde{M}}) = 1$, we prove that if $\tilde{\tilde{M}}$ carries a $G_2$ form such that the holonomy group of the associated metric is a subgroup of $G_2$, then $\tilde{\tilde{M}}$ has a finite covering which is a product of a 6-dimensional simply connected Calabi–Yau manifold and a circle, and so there exist a closed 2-form $\omega$ and a closed 1-form $\eta$ on $\tilde{\tilde{M}}$ such that $\omega^3 \wedge \eta \neq 0$ at every point of $\tilde{\tilde{M}}$. But we see that this is not possible by the cohomology
of \( \widetilde{M} \) determined in Proposition 22. This shows that \( \widetilde{M} \) does not admit any torsion-free \( G_{2} \)-structure (Theorem 25).

Now, let us recall that for each 7-manifold \( N \) with a \( G_{2} \)-structure \( \phi \), one may define a special class of 3-dimensional orientable submanifolds of \( N \) called associative 3-folds (see section 8 for details). Their tangent spaces are subalgebras of the cross-product algebras induced by \( \phi \) on the tangent spaces of \( N \); in fact, these latter subalgebras are isomorphic to \( \mathbb{R}^{3} \) with the standard vector product. If the \( G_{2} \)-structure \( \phi \) is closed, then \( \phi \) is a calibration and every associative 3-fold is a minimal submanifold of \( N \) (moreover, locally volume-minimizing in its homology class [34, Proposition 3.7.2]).

For the compact 7-manifold \( M \) with the closed \( G_{2} \) form \( \varphi \) mentioned above, we consider a non-trivial involution of \( M \) preserving \( \varphi \), and we construct an example of a 3-dimensional family of associative volume-minimizing 3-tori in \( \widetilde{M} \) (Proposition 30). This deformation family is “maximal” (Corollary 32). On the other hand, we show in Proposition 33 that each associative 3-torus fixed by the above involution becomes rigid and isolated after an arbitrary small closed perturbation of the ambient \( G_{2} \)-structure.

For a \( G_{2} \)-structure \( \phi \) (not necessarily closed or coclosed) on a 7-manifold \( N \), we have another natural class of orientable submanifolds of \( N \): the so-called coassociative 4-folds. Such a submanifold may be defined by the vanishing of \( \phi \) (see section 9). When the \( G_{2} \)-structure \( \phi \) is closed, the space of deformations of a coassociative 4-fold \( X \) is a smooth manifold of dimension equal to the positive part of the second Betti number \( b_{2}^{+}(X) \). If also \( b_{2}^{+}(X) = 3 \), then the deformations of \( X \) may ‘fill out’ an open set in the ambient \( G_{2} \)-manifold. We construct a smooth fibration map \( \widetilde{M} \rightarrow S^{2} \times S^{1} \) with generic fiber a coassociative torus and some singular fibers, with both smooth and singular fibers forming maximal deformation families (Proposition 35).

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2. Orbifolds

In this section we collect some basic facts and definitions concerning \( G_{2} \) forms on smooth manifolds and on orbifolds (see [11, 14, 31, 15, 20, 29, 30, 31, 32, 34, 42] for details).

Let us consider the space \( \mathcal{O} \) of the Cayley numbers (or octonions) which is a non-associative algebra over \( \mathbb{R} \) of dimension 8. We can identify \( \mathbb{R}^{7} \) with the subspace of \( \mathcal{O} \) consisting of pure imaginary Cayley numbers. Then, the product on \( \mathcal{O} \) defines on \( \mathbb{R}^{7} \) the 3-form \( \varphi_{0} \) given by

\[ \varphi_{0} = e^{127} + e^{347} + e^{567} + e^{135} - e^{236} - e^{146} - e^{245}, \]  

where \( \{ e^{1}, \ldots, e^{7} \} \) is the standard basis of \( (\mathbb{R}^{7})^{*} \). Here, \( e^{127} \) stands for \( e^{1} \wedge e^{2} \wedge e^{7} \), and so on. The stabilizer of \( \varphi_{0} \) under the standard action of \( \text{GL}(7, \mathbb{R}) \) on \( \Lambda^{3}(\mathbb{R}^{7})^{*} \) is the Lie
group $G_2$, which is one of the exceptional Lie groups, and it is a compact, connected, simply connected, simple Lie subgroup of $SO(7)$ of dimension 14.

Note that $G_2$ acts irreducibly on $\mathbb{R}^7$ and preserves the standard metric and orientation for which $\{e_1, \ldots, e_7\}$ is an oriented orthonormal basis. The $GL(7, \mathbb{R})$-orbit of $\varphi_0$ is open in $\Lambda^3(\mathbb{R}^7)^*$, so $\varphi_0$ is a stable 3-form on $\mathbb{R}^7$ [30].

Definition 1. Let $V$ be a real vector space of dimension 7. A 3-form $\varphi \in \Lambda^3(V^*)$ on $V$ is a $G_2$ form (or $G_2$-structure) on $V$ if there is a linear isomorphism $u: (V, \varphi) \longrightarrow (\mathbb{R}^7, \varphi_0)$ such that $u^*\varphi_0 = \varphi$, where $\varphi_0$ is given by (1).

A $G_2$-structure on a 7-dimensional smooth manifold $M$ is a reduction of the structure group of its frame bundle from $GL(7, \mathbb{R})$ to the exceptional Lie group $G_2$. Gray in [20] proved that a smooth 7-manifold $M$ carries $G_2$-structures if and only if $M$ is orientable and spin.

The presence of a $G_2$-structure is equivalent to the existence of a differential 3-form $\varphi$ (the $G_2$ form) on $M$, which can be defined as follows. Denote by $T_p(M)$ the tangent space to $M$ at $p \in M$, and by $\Omega^3(M)$ the algebra of the differential forms on $M$.

Definition 2. Let $M$ be a smooth manifold of dimension 7. A $G_2$ form on $M$ is a differential 3-form $\varphi \in \Omega^3(M)$ such that, for each point $p \in M$, $\varphi_p$ is a $G_2$ form on $T_p(M)$ (in the sense of Definition 1) that is, for each $p \in M$, there is a linear isomorphism $u_p: (T_p(M), \varphi_p) \longrightarrow (\mathbb{R}^7, \varphi_0)$ satisfying $u_p^*\varphi_0 = \varphi_p$, where $\varphi_0$ is given by (1).

Therefore, if $\varphi$ is a $G_2$ form on $M$, then $\varphi$ can be locally written as (1) with respect to some (local) basis $\{e^1, \ldots, e^7\}$ of the (local) 1-forms on $M$.

Note that there is a 1-1 correspondence between $G_2$-structures and $G_2$ forms on $M$. In fact, if $\varphi \in \Omega^3(M)$ is a $G_2$ form on $M$, the subbundle of the frame bundle whose fiber at $p \in M$ consists of the isomorphisms $u_p: (T_p(M), \varphi_p) \longrightarrow (\mathbb{R}^7, \varphi_0)$, such that $u_p^*\varphi_0 = \varphi_p$, defines a principal subbundle with fiber $G_2$, that is a $G_2$-structure on $M$.

Since $G_2 \subset SO(7)$, a $G_2$ form on $M$ determines a Riemannian metric and an orientation on $M$. Let $\varphi$ be a $G_2$ form on $M$. Denote by $g_\varphi$ the Riemannian metric induced by $\varphi$, and by $\nabla_\varphi$ the Levi-Civita connection of $g_\varphi$. Let $*_{\varphi}$ be the Hodge star operator determined by $g_\varphi$ and the orientation induced by $\varphi$.

Definition 3. We say that a manifold $M$ has a closed $G_2$-structure if there is a $G_2$ form $\varphi$ on $M$ such that $\varphi$ is closed, that is $d\varphi = 0$. A manifold $M$ has a coclosed $G_2$-structure if there is a $G_2$ form $\varphi$ on $M$ such that $\varphi$ is coclosed, i.e. $d(*_{\varphi}\varphi) = 0$. A $G_2$ form $\varphi$ on $M$ is torsion-free if $\nabla_\varphi\varphi = 0$ (equivalently if the $G_2$-structure is closed and coclosed [20]).

Orbifold $G_2$ forms.

Definition 4. A (smooth) $n$-dimensional orbifold is a Hausdorff, paracompact topological space $X$ endowed with an atlas $\{(U_p, \tilde{U}_p, \Gamma_p, f_p)\}$ of orbifold charts, that is $U_p \subset X$ is a neighbourhood of $p \in X$, $\tilde{U}_p \subset \mathbb{R}^n$ an open set, $\Gamma_p \subset GL(n, \mathbb{R})$ a finite group acting on $\tilde{U}_p$, and $f_p: \tilde{U}_p \rightarrow U_p$ is a $\Gamma_p$-invariant map with $f_p(0) = p$, inducing a homeomorphism $\tilde{U}_p/\Gamma_p \cong U_p$. Moreover, the charts are compatible in the following sense:

If $q \in U_q \cap U_p$, then there exist a connected neighbourhood $V \subset U_q \cap U_p$ and a diffeomorphism $F: f_p^{-1}(V)_0 \rightarrow f_q^{-1}(V)$, where $f_p^{-1}(V)_0$ is a connected component of $f_p^{-1}(V)$,
such that $F(\sigma(x)) = \rho(\sigma)(F(x))$, for any $x$, and $\sigma \in \text{Stab}_\Gamma(p)$, where $\rho: \text{Stab}_\Gamma(p) \to \Gamma$ is a group isomorphism.

For each $p \in X$, let $n_p = \#\Gamma_p$ be the order of the orbifold point (if $n_p = 1$ the point is smooth, also called a non-orbifold point). The singular locus of the orbifold is the set $S = \{ p \in X \mid n_p > 1 \}$. Therefore $M - S$ is a smooth $n$-dimensional manifold. The singular locus $S$ is stratified: if we write $S_k = \{ p \mid n_p = k \}$, and consider its closure $\overline{S_k}$, then $\overline{S_k}$ inherits the structure of an orbifold. In particular $S_k$ is a smooth manifold, and the closure consists of some points of $S_l$, $l \geq 2$.

We say that the orbifold is \textit{locally oriented} if $\Gamma_p \subset \text{GL}_+(n, \mathbb{R})$ for any $p \in X$. As $\Gamma_p$ is finite, we can choose a metric on $\tilde{U}_p$ such that $\Gamma_p \subset \text{SO}(n)$. An element $\sigma \in \Gamma_p$ admits a basis in which it is written as

$$
\sigma = \text{diag} \left( \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{pmatrix}, \ldots, \begin{pmatrix} \cos \theta_r & -\sin \theta_r \\ \sin \theta_r & \cos \theta_r \end{pmatrix}, 1, \ldots, 1 \right),
$$

for $\theta_1, \ldots, \theta_r \in (0, 2\pi)$. In particular, the set of points fixed by $\sigma$ is of codimension $2r$. Therefore the set of singular points $S \cap U_p$ is of codimension $\geq 2$, and hence $X - S$ is connected (if $X$ is connected). Also we say that the orbifold $X$ is \textit{oriented} if it is locally oriented and $X - S$ is oriented.

A natural example of orbifold appears when we take a smooth manifold $M$ and a finite group $\Gamma$ acting smoothly and effectively on $M$, then $\tilde{M} = M/\Gamma$ is an orbifold. If $M$ is oriented and the action of $\Gamma$ preserves the orientation, then $\tilde{M}$ is an oriented orbifold. Note that for every $\tilde{p} \in \tilde{M}$, the group $\Gamma_{\tilde{p}}$ is the stabilizer of $p \in M$, with $\tilde{p} = \tilde{\pi}(p)$ under the natural projection $\tilde{\pi}: M \to \tilde{M}$.

Let $X$ be an orbifold of dimension $n$. An \textit{orbifold $k$-form} $\alpha$ on $X$ consists of a collection of differential $k$-forms $\alpha_p$ ($p \in X$) on each open $\tilde{U}_p$ which are $\Gamma_p$-invariant and that match under the compatibility maps between different charts.

The space of orbifold $k$-forms on $X$ is denoted by $\Omega^k_{\text{orb}}(X)$. The wedge product of orbifold forms and the exterior differential $d$ on $X$ are well defined. Thus, we have

$$
d: \Omega^k_{\text{orb}}(X) \to \Omega^{k+1}_{\text{orb}}(X).
$$

The cohomology of $(\Omega^k_{\text{orb}}(X), d)$ is the cohomology of the topological space $X$ with real coefficients, $H^*(X)$ (see [7] Proposition 2.13).

\textbf{Remark 5.} Suppose that $X = M/\Gamma$ is an orbifold, where $M$ is a smooth manifold and $\Gamma$ is a finite group acting smoothly and effectively on $M$. Then, the definition of orbifold forms implies that any $\Gamma$-invariant differential $k$-form $\alpha$ on $M$ defines an orbifold $k$-form $\hat{\alpha}$ on $X$, and vice-versa. Moreover, it is straightforward to check that the exterior derivative on $M$ preserves $\Gamma$-invariance. Thus, if $(\Omega^k(M))^\Gamma$ denotes the space of the $\Gamma$-invariant differential $k$-forms on $M$, and $H^k(M)^\Gamma \subset H^k(M)$ is the subspace of the de Rham cohomology classes of degree $k$ on $M$ such that each of these classes has a representative that is a $\Gamma$-invariant differential $k$-form, then we have

$$
\Omega^k_{\text{orb}}(X) = (\Omega^k(M))^\Gamma, \quad H^k(X) = H^k(M)^\Gamma.
$$

\textbf{Definition 6.} Let $X$ be a 7-dimensional orbifold. We call $\varphi \in \Omega^3_{\text{orb}}(X)$ an \textit{orbifold $G_2$ form} on $X$ if, for each $p \in X$, $\varphi_p$ is a $G_2$ form (in the sense of Definition 2) on the open
$\tilde{U}_p \subset \mathbb{R}^7$ of the orbifold chart $(U_p, \tilde{U}_p, \Gamma_p, f_p)$. If in addition $\varphi$ is also closed ($d\varphi = 0$) we call $\varphi$ an \textit{closed orbifold $G_2$ form}.

An orbifold $G_2$-structure can also be defined as a reduction of the orbifold frame bundle from $GL(7, \mathbb{R})$ to $G_2$, as in the case of smooth manifolds.

If $M$ is a smooth 7-manifold with a closed $G_2$ form $\varphi$, and $\Gamma$ is a finite group acting effectively on $M$ and preserving $\varphi$, then $\varphi$ induces an orbifold closed $G_2$ form on the 7-orbifold $\hat{M} = M/\Gamma$.

**Definition 7.** Let $X$ be a 7-dimensional orbifold with an orbifold closed $G_2$ form $\varphi$. A \textit{closed $G_2$ resolution} of $(X, \varphi)$ consists of a smooth manifold $\tilde{X}$ with a closed $G_2$ form $\tilde{\varphi}$ and a map $\pi: \tilde{X} \to X$ such that:

- $\pi$ is a diffeomorphism $\tilde{X} - E \to X - S$, where $S \subset X$ is the singular locus and $E = \pi^{-1}(S)$ is the \textit{exceptional locus}. Also, $\tilde{X} - E$ is open and dense in $\tilde{X}$.
- $\tilde{\varphi}$ and $\pi^*\varphi$ agree in the complement of a small neighbourhood of $E$.

### 3. Formality of manifolds and orbifolds

In this section we review some definitions and results about formal manifolds and formal orbifolds (see [3, 14, 17, 21] for more details).

We work with \textit{differential graded commutative algebras}, or DGAs, over the field $\mathbb{R}$ of real numbers. The degree of an element $a$ of a DGA is denoted by $|a|$. A DGA $(A, d)$ is said to be \textit{minimal} if:

1. $A$ is free as an algebra, that is $A$ is the free algebra $\bigwedge V$ over a graded vector space $V = \bigoplus_i V^i$, and
2. there is a collection of generators $\{a_\tau\}_{\tau \in I}$ indexed by some well ordered set $I$, such that $|a_\mu| \leq |a_\tau|$ if $\mu < \tau$, and each $da_\tau$ is expressed in terms of the previous $a_\mu$, $\mu < \tau$. This implies that $da_\tau$ does not have a linear part.

Morphisms between DGAs are required to preserve the degree and to commute with the differential. In our context, the main example of DGA is the de Rham complex $(\Omega^*(M), d)$ of a smooth manifold $M$, where $d$ is the exterior differential.

The cohomology of a differential graded commutative algebra $(\mathcal{A}, d)$ is denoted by $H^*(\mathcal{A})$. This space is naturally a DGA with the product inherited from that on $\mathcal{A}$ while the differential on $H^*(\mathcal{A})$ is identically zero. A DGA $(\mathcal{A}, d)$ is connected if $H^0(\mathcal{A}) = \mathbb{R}$, and it is 1-connected if, in addition, $H^1(\mathcal{A}) = 0$.

We say that $(\bigwedge V, d)$ is a \textit{minimal model} of a differential graded commutative algebra $(\mathcal{A}, d)$ if $(\bigwedge V, d)$ is minimal and there exists a morphism of differential graded algebras 

$$\psi: (\bigwedge V, d) \longrightarrow (\mathcal{A}, d)$$

inducing an isomorphism $\psi^*: H^*(\bigwedge V) \simeq H^*(\mathcal{A})$ on cohomology. In [28], Halperin proved that any connected differential graded algebra $(\mathcal{A}, d)$ has a minimal model unique up to isomorphism. For 1-connected differential algebras, a similar result was proved by Deligne, Griffiths, Morgan and Sullivan [14, 27, 44].
A minimal model of a connected smooth manifold $M$ is a minimal model $(\bigwedge V, d)$ for the de Rham complex $(\Omega^*(M), d)$ of differential forms on $M$. If $M$ is a simply connected manifold, then the dual of the real homotopy vector space $\pi_1(M) \otimes \mathbb{R}$ is isomorphic to the space $\bigwedge V^i$ of generators in degree $i$, for any $i$. The latter also happens when $i > 1$ and $M$ is nilpotent, that is, the fundamental group $\pi_1(M)$ is nilpotent and its action on $\pi_j(M)$ is nilpotent for all $j > 1$ (see [14]).

We say that a DGA $(A, d)$ is a model of a manifold $M$ if $(A, d)$ and $M$ have the same minimal model. In this case, if $(\bigwedge V, d)$ is the minimal model of $M$, we have

$$
(A, d) \leftrightarrow (\bigwedge V, d) \xrightarrow{\psi} (\Omega^*(M), d),
$$

where $\psi$ and $\nu$ induce isomorphisms on cohomology, i.e. these are quasi-isomorphisms.

A minimal algebra $(\bigwedge V, d)$ is formal if there exists a morphism of differential algebras $\psi : (\bigwedge V, d) \rightarrow (H^*(\bigwedge V), 0)$ inducing the identity map on cohomology. A DGA $(A, d)$ is formal if its minimal model is formal. A smooth manifold $M$ is formal if its minimal model is formal. Many examples of formal manifolds are known: spheres, projective spaces, compact Lie groups, symmetric spaces, flag manifolds, and compact Kähler manifolds.

The formality property of a minimal algebra is characterized as follows.

**Theorem 8** ([14]). A minimal algebra $(\bigwedge V, d)$ is formal if and only if the space $V$ can be decomposed into a direct sum $V = C \oplus N$ with $d(C) = 0$, $d$ is injective on $N$ and such that every closed element in the ideal $I(N)$ generated by $N$ in $\bigwedge V$ is exact.

This characterization of formality can be weakened using the concept of $s$-formality introduced in [21].

**Definition 9.** A minimal algebra $(\bigwedge V, d)$ is $s$-formal $(s > 0)$ if for each $i \leq s$ the space $V^i$ of generators of degree $i$ decomposes as a direct sum $V^i = C^i \oplus N^i$, where the spaces $C^i$ and $N^i$ satisfy the following conditions:

1. $d(C^i) = 0$,
2. the differential map $d : N^i \rightarrow \bigwedge V$ is injective, and
3. any closed element in the ideal $I_s = I(\bigwedge_{i \leq s} V^i)$, generated by the space $\bigwedge_{i \leq s} N^i$ in the free algebra $\bigwedge (\bigwedge_{i \leq s} V^i)$, is exact in $\bigwedge V$.

A smooth manifold $M$ is $s$-formal if its minimal model is $s$-formal. Clearly, if $M$ is formal then $M$ is $s$-formal for every $s > 0$. The main result of [21] shows that sometimes the weaker condition of $s$-formality implies formality.

**Theorem 10** ([21]). Let $M$ be a connected and orientable compact differentiable manifold of dimension $2n$ or $(2n - 1)$. Then $M$ is formal if and only if it is $(n - 1)$-formal.

One can check that any simply connected compact manifold is 2-formal. Therefore, Theorem 10 implies that any simply connected compact manifold of dimension at most six is formal. (This result was proved earlier in [39].)

Note that Crowley and Nordström in [12] have introduced the Bianchi–Massey tensor on a manifold $M$, and they prove that if $M$ is a closed $(n - 1)$-connected $(4n - 1)$-manifold, with $n \geq 2$, then $M$ is formal if and only if the Bianchi–Massey tensor vanishes.
For later use, we recall here the following characterization of the $s$-formality of a manifold.

**Lemma 11** ([22]). Let $M$ be a manifold with minimal model $(\bigwedge V, d)$. Then $M$ is $s$-formal if and only if there is a map of differential algebras

$$\vartheta : (\bigwedge V^{\leq s}, d) \longrightarrow (H^*(M), d = 0),$$

such that the map $\vartheta^* : H^*(\bigwedge V^{\leq s}, d) \longrightarrow H^*(M)$ induced on cohomology is equal to the map $\imath^* : H^*(\bigwedge V, d) = H^*(M)$ induced by the inclusion $\imath : (\bigwedge V^{\leq s}, d) \hookrightarrow (\bigwedge V, d)$.

In particular, $\vartheta^* : H^i(\bigwedge V^{\leq s}) \longrightarrow H^i(M)$ is an isomorphism for $i \leq s$, and a monomorphism for $i = s + 1$.

**Definition 12.** Let $X$ be an orbifold. A *minimal model* for $X$ is a minimal model $(\bigwedge V, d)$ for the DGA $(\Omega^*_\orb(X), d)$. The orbifold $X$ is *formal* if its minimal model is formal.

For a simply connected orbifold $X$, the dual of the real homotopy vector space $\pi_*(X) \otimes \mathbb{R}$ is isomorphic to the space $V^i$ of generators in degree $i$, for any $i$, where $\pi_*(X)$ is the homotopy group of order $i$ of the underlying topological space in $X$. In fact, the proof given in [14] for simply connected manifolds, also works for simply connected orbifolds (that is, orbifolds for which the topological space $X$ is simply connected).

Moreover, the proof of Theorem [10] given in [21] only uses that the cohomology $H^*(M)$ is a Poincaré duality algebra. By [13], we know that the singular cohomology of an orbifold also satisfies a Poincaré duality. Thus, Theorem [10] also holds for compact connected orientable orbifolds. That is, we have

**Proposition 13.** Let $X$ be a connected and orientable compact orbifold of dimension $2n$ or $(2n-1)$. Then $X$ is formal if and only if it is $(n-1)$-formal. In particular, any simply connected compact orientable orbifold of dimension at most $6$ is formal.

4. A 7-ORBIFOLD WITH AN ORBIFOLD CLOSED $G_2$ FORM

Let $G$ be the connected nilpotent Lie group of dimension 7 consisting of real matrices of the form

$$a = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where $A_1$ and $A_2$ are the matrices

$$A_1 = \begin{pmatrix} 1 & -x_2 & x_1 & x_4 & -x_1x_2 & x_6 \\ 0 & 1 & 0 & -x_1 & x_1 & \frac{1}{2}x_1^2 \\ 0 & 0 & 1 & 0 & -x_2 & -x_4 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & -x_3 & x_1 & x_5 & -x_1x_3 & x_7 \\ 0 & 1 & 0 & -x_1 & x_1 & \frac{1}{2}x_1^2 \\ 0 & 0 & 1 & 0 & -x_3 & -x_5 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $x_i \in \mathbb{R}$, for any $i \in \{1, \ldots, 7\}$. Then, a global system of coordinate functions $\{x_1, \ldots, x_7\}$ for $G$ is given by $x_i(a) = x_i$, with $i \in \{1, \ldots, 7\}$. Note that if a matrix
A ∈ G has coordinates \(a_i\), then the change of coordinates of \(a \in G\) by the left translation \(L_A\) are given by

\[
\begin{align*}
L'_A(x_i) &= x_i \circ L_A = x_i + a_i, \quad i = 1, 2, 3, \\
L'_A(x_4) &= x_4 + a_2 x_1 + a_4, \\
L'_A(x_5) &= x_5 + a_3 x_1 + a_5, \\
L'_A(x_6) &= x_6 - \frac{1}{2} a_2 x_1^2 - a_1 x_4 - a_1 a_2 x_1 + a_6, \\
L'_A(x_7) &= x_7 - \frac{1}{2} a_3 x_1^2 - a_1 x_5 - a_1 a_3 x_1 + a_7.
\end{align*}
\]

A standard calculation shows that a basis for the left invariant 1–forms on \(G\) consists of

\[
\{dx_1, dx_2, dx_3, dx_4 - x_2 dx_1, dx_5 - x_3 dx_1, dx_6 + x_1 dx_4, dx_7 + x_1 dx_5\}. \tag{3}
\]

Let \(\Gamma\) be the discrete subgroup of \(G\) consisting of matrices whose entries \((x_1, x_2, \ldots, x_7) \in 2\mathbb{Z} \times \mathbb{Z}^6\), that is \(x_i\) are integers and \(x_1\) is even. It is easy to see that \(\Gamma\) is a subgroup of \(G\). So the quotient space of right cosets

\[
M = \Gamma \backslash G \tag{4}
\]

is a compact 7-manifold. Hence the forms \(dx_1, dx_2, dx_3, dx_4 - x_2 dx_1, dx_5 - x_3 dx_1, dx_6 + x_1 dx_4, dx_7 + x_1 dx_5\) descend to 1–forms \(e^1, e^2, e^3, e^4, e^5, e^6, e^7\) on \(M\) such that

\[
de^i = 0, \quad i = 1, 2, 3, \quad de^4 = e^{12}, \quad de^5 = e^{13}, \quad de^6 = e^{14}, \quad de^7 = e^{15}, \tag{5}
\]

and such that at each point of \(M\), \(\{e^1, e^2, e^3, e^4, e^5, e^6, e^7\}\) is a basis for the 1–forms on \(M\). Here, \(e^{12}\) stands for \(e^1 \wedge e^2\), and so on.

**Lemma 14.** The nilmanifold \(M\) defined by (1) is diffeomorphic to the mapping torus \(M_\nu\) of the diffeomorphism of the 6-torus \(\nu : T^6 = \mathbb{R}^6 / \mathbb{Z}^6 \to T^6 = \mathbb{R}^6 / \mathbb{Z}^6\), induced by the linear automorphism of \(\mathbb{R}^6\) associated to the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 2 & -2 & 1 & 0
\end{pmatrix}.
\]

**Proof.** Consider the projection

\[
p : M \to S^1 = \mathbb{R} / 2\mathbb{Z} \\
[(x_1, \ldots, x_7)] \mapsto (x_1 + 2\mathbb{Z}). \tag{6}
\]

The fiber over \(x_1 + 2\mathbb{Z} \in S^1\) is the set of equivalence classes of \(\mathbb{R}^6\) by the equivalence relation

\[
(x_2, \ldots, x_7) \sim (x_2 + a_2, x_3 + a_3, x_4 + a_2 x_1 + a_4, x_5 + a_3 x_1 + a_5, x_6 + \frac{1}{2} a_2 x_1^2 + a_6, x_7 - \frac{1}{2} a_3 x_1^2 + a_7),
\]
where \( a_i \in \mathbb{Z} \), for \( i = 2, \ldots, 7 \). The quotient \( \mathbb{R}^6/\sim \) is then the 6-torus \( \mathbb{R}^6/\Lambda(x_1) \) with lattice \( \Lambda(x_1) \subset \mathbb{R}^6 \) given by the span over \( \mathbb{Z} \) of the columns of the matrix

\[
B(x_1) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
x_1 & 0 & 1 & 0 & 0 & 0 \\
0 & x_1 & 0 & 1 & 0 & 0 \\
-\frac{1}{2}x_1^2 & 0 & 0 & 0 & 1 & 0 \\
0 & -\frac{1}{2}x_1^2 & 0 & 0 & 0 & 1 
\end{pmatrix}.
\]

The fiber \( p^{-1}(x_1 + 2\mathbb{Z}) = \mathbb{R}^6/\Lambda(x_1) \) can be identified with the standard torus \( T^6 = \mathbb{R}^6/Z^6 \), by the diffeomorphism

\[
f_{x_1}: \mathbb{R}^6/\Lambda(x_1) \rightarrow \mathbb{R}^6/Z^6
\]

\[
[(x_2, \ldots, x_7)] \mapsto [B(x_1)^{-1}(x_2, \ldots, x_7)].
\]

Therefore, \( p^{-1}([0,2]/2\mathbb{Z}) \cong ([0,2] \times T^6)/\nu \), for an appropriate diffeomorphism \( \nu : \{0\} \times T^6 \cong \{2\} \times T^6 \), that we describe next.

The manifold \( M \) is obtained by identifying the two presentations \( \{0\} \times T^6 \) and \( \{2\} \times T^6 \) of the fiber over \( 0 + 2\mathbb{Z} = 2 + 2\mathbb{Z} \) via the map

\[
h: p^{-1}(0 + 2\mathbb{Z}) \rightarrow p^{-1}(2 + 2\mathbb{Z}),
\]

\[
[(x_2, x_3, x_4, x_5, x_6, x_7)] \in \mathbb{R}^6/\Lambda(0) \mapsto [(x_2, x_3, x_4, x_5, x_6 - 2x_4, x_7 - 2x_5)] \in \mathbb{R}^6/\Lambda(2),
\]

that corresponds to the matrix

\[
C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -2 & 0 & 1 & 0 \\
0 & 0 & 0 & -2 & 0 & 1 
\end{pmatrix}.
\]

Thus \( M \) is the manifold obtained from \( [0,2] \times T^6 \) by identifying the ends \( \{0\} \times T^6 \cong \{2\} \times T^6 \) by the diffeomorphism \( \nu \) of \( T^6 \) induced by the linear automorphism of \( \mathbb{R}^6 \)

\[
(x_2, \ldots, x_7) \rightarrow E(x_2, \ldots, x_7)^T,
\]

where

\[
E = B(2)^{-1}C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
-2 & 0 & 1 & 0 & 0 & 0 \\
0 & -2 & 0 & 1 & 0 & 0 \\
2 & 0 & -2 & 0 & 1 & 0 \\
0 & 2 & 0 & -2 & 0 & 1 
\end{pmatrix}.
\]

Swapping the coordinates \( (x_2, \ldots, x_7) \) to the order \( (x_2, x_4, x_6, x_3, x_5, x_7) \), we get the matrix in the statement.

Now we consider the action of the finite group \( \mathbb{Z}_2 \) on \( G \) given by

\[
\rho: G \rightarrow G
\]

\[
(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (-x_1, -x_2, x_3, x_4, -x_5, -x_6, x_7),
\]

(7)
where $\rho$ is the generator of $\mathbb{Z}_2$. This action satisfies the condition $\rho(a \cdot b) = \rho(a) \cdot \rho(b)$, for $a, b \in G$, where $\cdot$ denotes the natural group structure of $G$. This follows since $\rho$ is conjugation by the matrix

$$
J = \begin{pmatrix} J_1 & 0 \\
0 & J_2 \end{pmatrix}, \quad J_1 = \text{diag}(1, -1, -1, 1, 1), \quad J_2 = \text{diag}(1, 1, -1, -1, 1),
$$
i.e. $\rho(a) = j a j^{-1}$. Moreover, $\rho(\Gamma) = \Gamma$. Thus, $\rho$ induces an action on the quotient $M = \Gamma \backslash G$. Denote by $\rho: M \rightarrow M$ the $\mathbb{Z}_2$-action. Then, the induced action on the 1-forms $e^i$ is given by

$$
\rho^* e^i = -e^i, \quad i = 1, 2, 5, 6, \quad \rho^* e^j = e^j, \quad j = 3, 4, 7. \quad (8)
$$

**Proposition 15.** The quotient space $\hat{M} = M/\mathbb{Z}_2$ is a compact 7-orbifold with first Betti number $b_1(\hat{M}) = 1$, and with an orbifold closed $G_2$ form.

**Proof.** Since the $\mathbb{Z}_2$-action on $M$ is smooth and effective, the quotient space $\hat{M} = M/\mathbb{Z}_2$ is a 7-orbifold, which is compact because $M$ is compact. Moreover, using Nomizu’s theorem [10], from [3] we have that the first de Rham cohomology group of $M$ is $H^1(M) = \langle [e^1], [e^2], [e^3] \rangle$. Then, as a consequence of (2) and from (8), the first cohomology group of $\hat{M}$ is

$$
H^1(\hat{M}) = H^1(M)^{\mathbb{Z}_2} = \langle [e^1], [e^2], [e^3] \rangle^{\mathbb{Z}_2} = \langle [e^3] \rangle.
$$

So the first Betti number of $\hat{M}$ is $b_1 = 1$.

We define the 3-form $\varphi$ on $M$ given by

$$
\varphi = e^{123} + e^{145} + e^{167} - e^{246} + e^{257} + e^{347} + e^{356}. \quad (9)
$$

It is clear that $\varphi$ is a $\mathbb{Z}_2$-invariant $G_2$ form on $M$. We claim that $\varphi$ is also closed. Indeed, on the right-hand side of (9) all the terms, except the last 3 terms, are closed. But $d(e^{257} + e^{347} + e^{356}) = 0$ by the equation (3). Thus $\varphi$ induces an orbifold closed $G_2$ form $\hat{\varphi}$ on $\hat{M}$. □

**Remark 16.** As we shall see, in Proposition [31] below, the fibers $p^{-1}(x)$ of the map defined by Lemma [14] are complex tori $\mathbb{C}^3/\Lambda(x_i)$, moreover these have a Calabi–Yau structure induced by $\varphi$.

Denote by $\pi: \hat{M} \rightarrow \hat{M}$ the natural projection. The singular locus $S$ of $\hat{M}$ is the image by $\pi$ of the set $S'$ of points in $M$ that are fixed by the $\mathbb{Z}_2$-action defined by (7). So $S$ consists of all the 3-dimensional spaces $S_a = \pi(S'_a) = S'_a/\mathbb{Z}_2$, where

$$
S'_a = \begin{cases}
\{ \Gamma \cdot (a_1, a_2, x_3, x_4, a_5, a_6, x_7) \mid x_3, x_4, x_7 \in \mathbb{R} \} \subset M, & \text{if } a_1 = 0 \\
\{ \Gamma \cdot (a_1, a_2, x_3, x_4, a_5, \frac{3}{2} a_2 + a_6 - x_4, x_7) \mid x_3, x_4, x_7 \in \mathbb{R} \} \subset M, & \text{if } a_1 = 1
\end{cases} \quad (10)
$$

and $a = (a_1, a_2, a_5, a_6) \in A = \{0, 1\} \times \{0, 1/2\}^3$. Therefore, there are $2^4 = 16$ components of the singular locus of the orbifold.
The set $S'_a$ is included in the fiber $p^{-1}(0 + 2\mathbb{Z})$ or $p^{-1}(1 + 2\mathbb{Z})$ of the projection $p$ defined by (9). For $a = (0,0,0,0)$, $S'_a$ is a Lie subgroup of $T^6$, hence it is abelian and so isomorphic to a 3-torus $T^3$. As we shall see in the next section, $S$ is a disjoint union of 16 copies of $T^3$.

5. LOCAL MODEL AROUND THE SINGULAR LOCUS

To desingularize the orbifold $\hat{M} = M/\mathbb{Z}_2$ considered in Proposition 15, we study here each of the 16 connected components $S'_a$ (defined before) of the singular locus $S$ of $\hat{M}$.

The situation here has a partial similarity to a desingularization of $G_2$-orbifolds previously worked out by Joyce [34, Chap. 11,12], cf. also [35]. We show that a neighbourhood of each component of the singular locus of the orbifold $\hat{M}$ is diffeomorphic to the product $T^3 \times (B/\mathbb{Z}_2)$ of a 3-torus $T^3$ and the quotient $B/\pm 1$ of an open ball around zero in $\mathbb{C}^2$, and we replace $B/\pm 1$ with an appropriate neighbourhood of the Eguchi–Hanson space.

For each $a = (a_1,a_2,a_3,a_4,a_5,a_6) \in A = \{0,1\} \times \{(0,1/2)\}^3$, consider the element $a = (a_1,a_2,0,0,a_5,a_6,0) \in G$. If $a \in A$ and $a_1 = 0$, then

$$aga^{-1} \in \Gamma, \forall g \in \Gamma,$$

and the left translation $L_a : G \to G$ acts (diffeomorphically) on the right cosets of $\Gamma$ (noting also that $L_a(gx) = aga^{-1}L_a(x)$). The induced diffeomorphism $L_a : M \to M$ preserves the $G_2$ form $\varphi$ on $M$ defined by (9) and satisfies

$$L_a(\rho(b)) = a\rho(b) = \rho(a)\rho(b) = \rho(ab) = \rho(L_a(b)),$$

for every $b \in M$. Here we used, in the second equality in (12), that for each $a \in A$, $a^{-1}\rho(a) = (-2a_1,-2a_2,0,2a_1a_2,-2a_3,-2a_6+2a_3^2a_2,2a_1a_3)$ which is in $\Gamma$. So, if $a_1 = 0$, $L_a : M \to M$ defines an orbifold diffeomorphism $L_a : \hat{M} \to \hat{M}$ sending $S_0$ to $S_a$, where $0 = (0,0,0,0) \in A$. For each $a = (1,a_2,a_3,a_4,a_5,a_6) \in A$, consider $a' = (0,a_2,0,0,a_5,a_2+a_6,0) \in G$. The corresponding orbifold diffeomorphism $L_{a'} : \hat{M} \to \hat{M}$ preserving $\varphi$ is well-defined, as above, and sends $S_{(1,0,0,0)}$ to $S_{(1,a_2,a_3,a_4)}$. Therefore, it suffices to do the desingularization around $S_0$ and $S_{(1,0,0,0)}$ as we can translate it to the other singularities $S_a$.

From now on, we focus on $S_0 = \{(0,0,x_3,x_4,0,0,x_7)\} \subset \hat{M}$. The desingularization around $S_{(1,0,0,0)}$ can be obtained in a similar way (see Remark 15). We consider the corresponding set

$$S' = S'_0 = \{(0,0,x_3,x_4,0,0,x_7)\} \subset M,$$

which is a fixed locus of the $\mathbb{Z}_2$-action (given by (7)) and is isomorphic to a 3-torus $T^3$.

The following proposition allows us to show an appropriate local model around $S_0$ that we will use in the next section to desingularize $S_0$.

**Proposition 17.** There exist neighbourhoods $U'$ and $U''$ of $S'$ in the manifold $M$ with $U'' \subset U'$, and there are closed $G_2$ forms $\phi$ and $\psi$ on $M$ and $U'$, respectively which are invariant by the $\mathbb{Z}_2$-action given by (7), and such that $\phi$ is equal to the $G_2$ form $\varphi$, defined
by \([9]\), outside \(U' \cong T^3 \times B^4_\epsilon\) and \(\phi = \psi\) is the standard \(G_2\) form on \(U'' \cong T^3 \times B^4_{\epsilon/2}\) (given by \([15]\) below).

**Proof.** We define a small neighbourhood \(U'\) of \(S'\) in \(M\) as follows. A point in \(U'\) is given by \((x_1, \ldots, x_7)\), with \((x_1, x_2, x_5, x_6)\) small and such that, under the equivalence relation given by the action of \(\Gamma\) on the points of \(U'\),

\[(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \sim (x_1, x_2, x_3 + a_3, x_4 + a_4, x_5 + a_3 x_1, x_6, x_7 + a_7 - \frac{1}{2} a_3 x_1^2).

It is natural to introduce on \(U'\) the coordinates \((x'_1, \ldots, x'_7)\) defined by

\[
x'_5 = x_5 - x_1 x_3, \\
x'_7 = x_7 + \frac{1}{2} x_3 x_1^2, \\
x'_j = x_j, \quad j \neq 5, 7.
\]

(13)

Therefore, if \(B^3_1\) is the open ball of radius \(\epsilon\) in \(\mathbb{R}^4\) centered at 0, \(U'\) is determined by \((x'_1, \ldots, x'_7)\) with \((x'_1, x'_2, x'_5, x'_6) \in B^4_\epsilon\), for some small \(\epsilon > 0\), and

\[(x'_1, x'_2, x'_3, x'_4, x'_5, x'_6, x'_7) \sim (x'_1, x'_2, x'_3 + a_3, x'_4 + a_4, x'_5, x'_6, x'_7 + a_7).

That is,

\[U' \cong T^3 \times B^4_\epsilon, \]

(14)

where \(T^3 = \mathbb{R}^3/\mathbb{Z}^3\) has coordinates \((x'_3, x'_4, x'_5)\), and \(B^4_\epsilon \subset \mathbb{R}^4\) has coordinates \((x'_1, x'_2, x'_5, x'_6)\).

Note that for \(\epsilon < \frac{1}{7}\), the neighbourhoods \(L_a(U') \cap L_b(U') = \emptyset\), for any \(a, b \in A\) distinct.

If we restrict the 1-forms \(e^1, \ldots, e^7\) to \(S'\), by setting \(x'_1 = x'_2 = x'_5 = x'_6 = 0\), we get

\[e^5|_{S'} = dx_5 - x_3 dx_1 = dx'_5, \]

\[e^7|_{S'} = dx_7 = dx'_7, \]

\[e^j|_{S'} = dx_j = dx'_j, \quad j \neq 5, 7.
\]

since \(dx'_7 = dx_7 + \frac{1}{2} x_3^2 dx_3 + x_3 x_3 dx_1\) and \(dx'_5 = dx_5 - x_1 dx_3 - x_3 dx_1\).

Thus, \(e^j|_{S'} = dx'_j, 1 \leq j \leq 7\), and the restriction \(\varphi|_{S'}\) to \(S' \subset U'\) of the closed \(G_2\) form \(\varphi\) on \(M\) given by \([9]\), that is

\[
\varphi = e^{123} + e^{145} + e^{167} - e^{246} + e^{257} + e^{347} + e^{356}
\]

\[= e^{347} + e^3 (e^{12} + e^{56}) - e^4 (e^{15} - e^{26}) + e^7 (e^{16} + e^{25})
\]

coincides with the restriction \(\psi|_{S'}\) to \(S'\) of the standard \(G_2\) form \(\psi\) on \(U' \cong T^3 \times B^4_\epsilon\) given by

\[
\psi = dx'_{347} + dx'_3 \wedge (dx'_{12} + dx'_{56}) - dx'_4 \wedge (dx'_{15} - dx'_{26}) + dx'_7 \wedge (dx'_{16} + dx'_{25}),
\]

(15)

that is, we have \(\psi|_{S'} = \varphi|_{S'}\). The notation \(\psi|_{S'}\) in the previous sentence and later is used in the sense of a section of the bundle \(\Lambda^3 T^*M\) restricted to a subset of \(M\). Here, \(dx'_{12}\) stands for \(dx'_1 \wedge dx'_2\), and so on, with the coordinates \(x'_i\) as defined in \([13]\). Moreover, using \([7]\) and \([13]\), one can check that the \(G_2\) form \(\psi\) on \(U' \cong T^3 \times B^4_\epsilon\) is invariant by the \(\mathbb{Z}_2\)-action.

Now let us modify the \(G_2\)-structure \(\varphi\) on \(M\) inside \(U' \cong T^3 \times B^4_\epsilon\) so that it is equal to the 3-form \(\psi\) given by \([15]\) on a smaller neighbourhood \(U''\) of \(S'\). The 3-form \(\psi - \varphi\) is closed on \(U'\), and it satisfies the condition \((\psi - \varphi)|_{T^3 \times \{0\}} = 0\), hence it defines the zero de
Rham cohomology class on $U'$. So $\psi - \varphi = d\alpha$, for some 2-form $\alpha$ on $U'$. Moreover, as $|\psi - \varphi| \leq C r$, where $r$ is the radial coordinate of $B_\epsilon^4 \subset \mathbb{R}^4$, we can take $|\alpha| \leq C r^2$. Indeed, following the standard procedure of [25, p. 542], we can use the homotopy operator to determine $\alpha$. Write the 3-form $\varphi - \psi$ as

$$\psi - \varphi = \beta_0 \wedge dr + \beta_1,$$

for some forms $\beta_0$ and $\beta_1$ on $U'$ and this latter domain can be written as $T^3 \times (0, \varepsilon) \times S^3$. Now $\beta_0$ and $\beta_1$ can be regarded as forms on the 6-manifold $T^3 \times S^3$ depending on the parameter $r \in (0, \varepsilon)$. Then $d\beta_0 = \partial \beta_1 / \partial r$ and $d\beta_1 = 0$.

The 2-form $\alpha = \int_0^r \beta_0 \, dr$ is smooth and satisfies $d\alpha = \psi - \varphi$.

On $B_\epsilon^4$ consider a bump function $\rho(r)$ such that $\rho(r) = 1$ for $r \leq \varepsilon / 2$, and $\rho(r) = 0$ for $\varepsilon \geq r \geq 3\varepsilon / 4$. Define the 3-form $\phi$ on $M$ by

$$\phi = \varphi + d(\rho \alpha).$$

Then $\phi = \varphi$ outside $U'$ and $\phi = \psi$ in

$$U'' \cong T^3 \times B_{\varepsilon / 2}^4.$$

Moreover, $|d\phi| \leq C / \varepsilon$ for a uniform constant, so $|d(\rho \alpha)| \leq C \varepsilon$. For $\varepsilon > 0$ small enough, $\phi$ is non-degenerate, hence it defines a closed $G_2$ form on $M$. Now, using (16), one can check that the $G_2$ form $\phi$ is still $\mathbb{Z}_2$-invariant.

**Remark 18.** In the case when $S' = S_{(1,0,0,0)}$, a neighbourhood $U'$ of $S'$ is given by $(x_1, \ldots, x_7)$, with $x_1 - 1, x_2, x_5, x_6 + x_4$ small and with the equivalence relation

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \sim (x_1, x_2, x_3 + a_3, x_4 + a_4, x_5 + a_3(x_1 - 1), x_6 - a_4, x_7 + a_7 - \frac{1}{2} a_3 x_1^3)$$

defined by the action of the subgroup $\{(0, 0, a_3, -a_3, -a_4, a_7) \in \Gamma | a_3, a_4, a_7 \in \mathbb{Z}\}$ on $U'$. In place of (13) we use the following change of coordinates

$$x_1' = x_1 - 1, \quad x_5' = x_5 - (x_1 - 1) x_3,$$

$$x_6' = x_6 + x_4, \quad x_7' = x_7 + x_5 + \frac{1}{2} x_3(x_1 - 1)^2, \quad x_j' = x_j, \quad j = 2, 3, 4.$$

so (13) holds with

$$(x_1', x_2', x_3', x_4', x_5', x_6', x_7') \sim (x_1', x_2', x_3' + a_3, x_4' + a_4, x_5', x_6', x_7' + a_7 - a_3 / 2)$$

and $c^j |_{S'} = dx_j'$.

As a consequence of Proposition 17 we have the following corollary.

**Corollary 19.** There exist neighbourhoods $U$ and $V$ of $S_0$ in the orbifold $\widehat{M} = M / \mathbb{Z}_2$ with $V \subset U$, and there are orbifold closed $G_2$ forms $\widehat{\phi}$ and $\widehat{\psi}$ on $\widehat{M} = M / \mathbb{Z}_2$ and $U$, respectively such that $\widehat{\phi} = \widehat{\psi}$ outside $U$, and $\widehat{\phi} = \widehat{\psi}$ in the neighbourhood $V$ of $S_0$. Moreover, the singular locus $S$ of $\widehat{M}$ is covered by the disjoint union $\bigcup_{a \in A} L_a(U)$.

**Proof.** We define the neighbourhoods $U$ and $V$ of $S_0$ by

$$U = U' / \mathbb{Z}_2 \cong T^3 \times (B_\varepsilon^4 / \mathbb{Z}_2), \quad V = U'' / \mathbb{Z}_2 \cong T^3 \times (B_{\varepsilon / 2}^4 / \mathbb{Z}_2),$$

where $U'$ and $U''$ are given by (14) and (17), respectively. Consider the closed $G_2$ forms $\psi$ and $\phi$ defined by (15) and (16), respectively. By Proposition 17 both these forms are...
$\mathbb{Z}_2$-invariant, and hence they descend to orbifold closed $G_2$ forms $\hat{\psi}$ and $\hat{\phi}$ on $U$ and $\hat{M}$, respectively and they satisfy the required conditions.

As we have noticed in the proof of Proposition 17 we have $L_a(U') \cap L_b(U') = \emptyset$, for any $a, b \in \Lambda$ distinct. So, $S \subset \bigcup_{a \in \Lambda} L_a(U)$. \hfill \square

Remark 20. Note that the $G_2$ form $\psi$ given by (15) can be defined as the restriction to $U'$ of the $G_2$ form $\Psi$ on $T^3 \times \mathbb{C}^2$ defined by (21) (see below). Firstly, we see that in the coordinates $(x'_1, \ldots, x'_7)$, defined by (13) the action of $G_2$ on $U'$ is given by

$$(x'_1, x'_2, x'_5, x'_6) \mapsto (-x'_1, -x'_2, -x'_5, -x'_6),$$

and is fixing $(x'_3, x'_4, x'_7)$. Introduce now the complex coordinates

$$z_1 = x'_1 + ix'_2,$$
$$z_2 = x'_5 + ix'_6,$$

so that $U' \cong T^3 \times B^4$, where $B^4 \subset \mathbb{C}^2$, and the action of $\mathbb{Z}_2$ on $\mathbb{C}^2$ is given by

$$\rho : \mathbb{C}^2 \to \mathbb{C}^2$$

$$(z_1, z_2) \mapsto (-z_1, -z_2).$$

The natural $\text{SU}(2)$-structure on $\mathbb{C}^2$ is given by the Kähler form $\omega$ and the $(2,0)$-form $\Omega$ defined, respectively, by

$$\omega = \frac{i}{2} (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2) = dx'_{12} + dx'_{56},$$
$$\Omega = dz_1 \wedge dz_2 = (dx'_{15} - dx'_{26}) + i(dx'_{25} + dx'_{16}).$$

The action of $\mathbb{Z}_2$ on $\mathbb{C}^2$ given by (19) preserves both these forms. The standard closed $G_2$-structure on $T^3 \times \mathbb{C}^2$ is given by

$$\Psi = dx'_{347} + dx'_{3} \wedge \omega - dx'_{4} \wedge \Re \Omega + dx'_{7} \wedge \Im \Omega.$$

So the restriction $\Psi|_{U'}$ to $U'$ coincides with the 3-form $\psi$ defined by (15). Then, Corollary 10 implies that

$$\hat{\Psi}|_V = \hat{\phi} = \hat{\psi}$$

in the neighbourhood $V$ of $S_0$, where $\hat{\Psi}$ is the orbifold closed $G_2$ form induced by $\Psi$ on $T^3 \times (\mathbb{C}^2/\mathbb{Z}_2)$, and $\hat{\Psi}|_V$ is the restriction of $\hat{\Psi}$ to $V$.

6. Resolving the singular locus

In this section we desingularize the singular locus $S$ of $\hat{M}$ to get a smooth compact 7-manifold $\tilde{M}$ diffeomorphic to $\hat{M}$ outside $S$, and such that $\tilde{M}$ has the required properties, i.e. with first Betti number $b_1(\tilde{M}) = 1$, with no torsion-free $G_2$-structures and with a closed $G_2$ form $\tilde{\phi}$ such that $\tilde{\phi} = \hat{\phi}$ outside a neighbourhood of $S$, where $\hat{\phi}$ is the orbifold closed $G_2$ form on $\hat{M}$ given in the proof of Proposition 15.

Theorem 21. A closed $G_2$-resolution of $(\hat{M}, \hat{\phi})$ (in the sense of Definition 7) induced by the blow up of the origin in $\mathbb{C}^2$ via the local model defined by Corollary 10 produces a smooth compact manifold $\tilde{M}$ with a closed $G_2$ form $\tilde{\phi}$ and with first Betti number $b_1(\tilde{M}) = 1$.
Proof. We know that doing the desingularization around the component $S_0$ of $S$, we can translate it to the other components $S_a$ of the singular locus $S$ via the diffeomorphism $L_a: \hat{M} \rightarrow \hat{M}$ defined in section 5.

Let $V$ be the neighbourhood of $S_0$ given by (18) with the orbifold closed $G_2$ form $\hat{\Psi}|_V$ induced on $V$ by the $G_2$ form $\Psi$ defined by (21). In order to desingularize $S_0$, we shall replace the factor $B^4_{t/2}/\mathbb{Z}_2$ of $V$ by a smooth 4-manifold that agrees with $B^4_{t/2}/\mathbb{Z}_2$ in a neighbourhood of its boundary.

Firstly we consider the complex orbifold $X = \mathbb{C}^2/\mathbb{Z}_2$. By Remark 20 we know that the action of $\mathbb{Z}_2$ on $\mathbb{C}^2$ defined by (19) preserves the natural integrable $SU(2)$-structure $(\omega, \Omega)$ on $\mathbb{C}^2$ given by (20). (Thus $\mathbb{Z}_2$ is a finite subgroup of $SU(2)$.) We resolve the singularity of $X = \mathbb{C}^2/\mathbb{Z}_2$ to get a smooth manifold $\hat{X}$ with a (non-torsion-free) $SU(2)$-structure. This goes as follows: take the blow-up $\hat{\mathbb{C}}^2$ of $\mathbb{C}^2$ at the origin. This is given by

$$\hat{\mathbb{C}}^2 = \{(z_1, z_2, [w_1, w_2]) \in \mathbb{C}^2 \times \mathbb{CP}^1 \mid w_1 z_2 = w_2 z_1\}.$$ 

Now we quotient $\hat{\mathbb{C}}^2$ by $\mathbb{Z}_2$ in order to get a smooth manifold

$$\hat{X} = \hat{\mathbb{C}}^2/\mathbb{Z}_2,$$

and a map $\pi: \hat{X} \rightarrow X$ such that $\pi$ is a diffeomorphism $\hat{X} - \pi^{-1}(0) \rightarrow X - \{0\}$.

It is known that the $(2,0)$-form $\Omega = dz_1 \wedge dz_2$ on $X$ extends to a nowhere vanishing $(2,0)$-form on $\hat{X}$, that we call $\Omega$ again (see [13]). This can be easily checked as follows, using the two affine charts. For the first one, we take $w_1 = 1$, $w_2 = w$, $z_1 = z$, $z_2 = wz$, so the chart of $\hat{\mathbb{C}}^2$ is parameterized by $(z, w) \in \hat{\mathbb{C}}^2$. The quotient by $\mathbb{Z}_2$ is given by $(z, w) \mapsto (-z, w)$, so $\hat{X}$ is parameterized by $(u, w) \in \mathbb{C}^2$, with $u = z^2$. The form $\Omega$ in these coordinates $(u, w)$ has the following expression

$$\Omega = dz_1 \wedge dz_2 = dz \wedge d(wz) = z dz \wedge dw = \frac{1}{2} du \wedge dw.$$

Thus $\Omega$ is non-zero and is defined on the whole chart. The computations for other chart are similar.

We now consider a family of Kähler $(1,1)$-forms on $\mathbb{C}^2 - \{0\}$ that extend to $\hat{\mathbb{C}}^2/\mathbb{Z}_2$. These determine the Eguchi–Hanson metric (see [16], [34, p.153]). Define $\omega_t = \frac{1}{2\rho} \partial \overline{\partial} f_t$, where $r = ||(z_1, z_2)||$ and

$$f_t(r) = (r^4 + t^4)^{1/2} + 2t^2 \log r - t^2 \log((r^4 + t^4)^{1/2} + 2).$$

Also note that $f_t - r^2 = t^2 h(t, r)$, where $h(t, r) = t^2((r^4 + t^4)^{1/2} + r^2)^{-1} + 2 \log r - \log((r^4 + t^4)^{1/2} + 2)$, which is smooth on $\mathbb{R}^2 - \{0\}$. Take $\rho$ a bump function such that $\rho \equiv 1$ if $r \leq \epsilon/4$ and $\rho \equiv 0$ if $r \geq \epsilon/2$. Define $\tilde{\omega}_r = \omega_{e^2} + \frac{1}{2\rho} \partial \overline{\partial} (\rho(r^2 - f_t))$. We claim that there exists $t > 0$ such that $\tilde{\omega}_r$ is non-degenerate on $B^4_{t/2}$. In order to check it on the neck $B^4_{t/2} - B^4_{t/4}$, let $m > 0$ be such that any $\omega$ with $||\omega - \omega_{e^2}|| < m$ is symplectic. As $h$ is bounded in $C^2$ on $[0, 1] \times [\epsilon/4, \epsilon/2]$, we can choose $t > 0$ so that $||\partial \overline{\partial} (\rho(r^2 - f_t))|| < m$. Then $\tilde{\omega}_r$ is symplectic.

We define the $G_2$ form $\hat{\Psi}$ on $T^3 \times (B^4_{t/2}/\mathbb{Z}_2)$ by

$$\hat{\Psi} = dx_{347} + dx_3 \wedge \tilde{\omega}_t - dx_4 \wedge \Re \Omega + dx_7 \wedge \Im \Omega.$$
Thus, for \((\epsilon/2) - \eta \leq r < \epsilon/2\), we have \(\tilde{\Psi} = \hat{\Psi}\) on \(T^3 \times (B^4_{t}/\mathbb{Z}_2)\), and hence \(\hat{\Psi} = \hat{\phi}\) on \(T^3 \times (B^4_{t}/\mathbb{Z}_2)\) by \((22)\). Now we glue \(T^3 \times (B^4_{t}/\mathbb{Z}_2)\) endowed with this \(G_2\) form \(\hat{\Psi}\) to \(\hat{M} = (T^3 \times (B^4_{t}/\mathbb{Z}_2))\) with the \(G_2\) form \(\hat{\phi}\) given in Corollary \([19]\). These two glue nicely to give a \(G_2\) form \(\tilde{\varphi}\) on the resulting smooth manifold \(\hat{M}\).

The map \(\pi: \tilde{X} \to X\) defines a map that we denote by the same symbol \(\pi: \tilde{M} \to \hat{M}\), which satisfies the conditions of Definition \([7]\). Thus, \((\hat{M}, \tilde{\varphi})\) is a closed \(G_2\)-resolution of \((\hat{M}, \tilde{\varphi})\).

Finally, that \(b_1(\hat{M}) = 1\) follows from Proposition \([22]\) below. \(\square\)

### 7. Topology of the constructed manifold

The next proposition gives some topological invariants of \(\hat{M}\) (cf. \([34]\) \S12.1). In fact, we shall be able to further determine the de Rham cohomology ring of \(\hat{M}\) later in this section.

**Proposition 22.** There is an isomorphism

\[H^*(\hat{M}) \cong H^*(\hat{M}) \oplus \left(\bigoplus_{i=1}^{16} H^*(T^3) \otimes [E_i]\right),\]

where \([E_i] \in H^2(\hat{M})\) is the class of the exceptional divisor \(E_i \subset \tilde{X} = \mathbb{C}^2/\mathbb{Z}_2\) with \(1 \leq i \leq 16\).

**Proof.** Let \(V \subset \hat{M}\) be a neighbourhood of the exceptional divisors, that is \(V = \bigcup_i V_i\), where \(V_i \cong T^3 \times (B^4_{t}/\mathbb{Z}_2) \sim T^3\), where \(\sim\) means homotopy equivalence. Let \(U\) be the complement \(\hat{M} - \bigcup_i E_i\). Then \(U \cap V = \bigcup_i (U \cap V_i)\), where \(U \cap V_i \sim T^3 \times (S^3/\mathbb{Z}_2)\).

Let \(V \subset \hat{M}\) be the preimage of \(V\) under \(\pi: \tilde{M} \to \hat{M}\). Then \(\tilde{V} = \bigcup_i \tilde{V}_i\) with \(\tilde{V}_i \cong T^3 \times (B^4_{t}/\mathbb{Z}_2) \sim T^3 \times E_i\) and \(E_i \cong \mathbb{C}P^1 \cong S^2\).

The map \(\pi^*: H^k(\hat{M}) \to H^k(\hat{M})\) is injective. In fact, let \(\alpha \in H^k(\hat{M})\) be a non-zero element. As the cohomology of \(\hat{M}\) is a Poincaré duality algebra, there is some \(\beta \in H^{7-k}(\hat{M})\) such that \(\alpha \cdot \beta = [\hat{M}]\). Applying \(\pi^*\), and noting that \(\pi: \tilde{M} \to \hat{M}\) is a degree 1 map, we have that \(\pi^*\alpha \cdot \pi^*\beta = [\hat{M}]\). Then \(\pi^*\alpha \neq 0\).

We write the Mayer–Vietoris sequences associated to \(\tilde{M} = U \cup V\) and \(\hat{M} = U \cup \tilde{V}\) as

\[
\begin{array}{c}
\xymatrix{
H^{k-1}(U \cap V) \ar[r]^-{\delta^{k-1}} & H^k(\hat{M}) \ar[d]^-{\pi^*} \ar[r]^-{\delta^k} & H^k(U \cup V) \ar[d] \ar[r]^-{\delta^k} & H^k(U \cap V) \ar[d] \ar[r]^-{\delta^k} & \oplus_{i=1}^{16} H^{k-2}(T^3) \otimes [E_i] \\
H^{k-1}(U \cap \tilde{V}) \ar[r]^-{\delta^{k-1}} & H^k(\tilde{M}) \ar[d] & H^k(U \cup \tilde{V}) \ar[d] & H^k(U \cap \tilde{V}) \ar[d] & \end{array}
\]

where \(Q\) is the cokernel of \(\pi^*\). It is clear that \(\text{im} \delta^{k-1} = \text{im} \delta^k\). This happens for all \(k\). So \(\ker \delta^k = \ker \delta^{k-1}\). Applying the snake lemma to the vertical exact sequences in the
second and third columns of the diagram above, we have an exact sequence
\[ 0 \to \text{im} \delta_{k-1} \to \text{im} \delta_k \to \ker f \to \ker \delta_k \to \delta_k \to 0. \]
This concludes that \( f \) is an isomorphism. Therefore there is an exact sequence
\[ 0 \to H^*(\tilde{\mathcal{M}}) \to H^*(\tilde{\mathcal{M}}) \to \bigoplus_{i=1}^{16} H^*(T^3) \otimes [E_i] \to 0, \]
where \([E_i] \in H^2(\tilde{\mathcal{M}})\) is the class of the exceptional divisor \(E_i \subset \tilde{\mathbb{C}}^2/\mathbb{Z}_2\) (1 \(\leq i \leq 16\)).

Now let us construct a splitting of the above exact sequence. For this, we take the Thom form \(\eta_i\) of each of the exceptional divisors \(E_i \subset \tilde{\mathbb{C}}^2/\mathbb{Z}_2\). Let \(E\) be one of these exceptional divisors. The Thom form of \(E\) is a compactly supported 2-form \(\eta\) on a neighbourhood of \(E\) such that \([\eta] = [E]\). Moreover \(\eta^2\) represents a 4-form such that \(\int_F \eta^2 = [F] \cdot [E]^2 = -2\) for each fiber \(F = \{p\} \times \tilde{\mathbb{C}}^2/\mathbb{Z}_2\) of \(\tilde{V} = T^3 \times \tilde{\mathbb{C}}^2/\mathbb{Z}_2\). If \(\lambda\) is the bump 4-form on the origin of \(\mathbb{C}^2\), pulled-back to \(\tilde{\mathbb{C}}^2/\mathbb{Z}_2\), then \([\eta^2] = -2[\lambda]\). Pulling-back to \(\tilde{V} = T^3 \times \mathbb{C}^2/\mathbb{Z}_2\), we have that \([\eta^2] = -2[T^3]\). With this we construct the compactly supported cohomology of \(\tilde{V_i} = T^3 \times \mathbb{C}^2/\mathbb{Z}_2\) as the forms \(\bigwedge (e^3, e^4, e^7) \wedge [\eta_i]\). This gives the splitting.

The algebra structure of \(H^*(\tilde{\mathcal{M}})\) can be described explicitly as follows. Under the isomorphism given in Proposition 22, i.e.
\[ H^*(\tilde{\mathcal{M}}) \cong H^*(\tilde{\mathcal{M}}) \oplus \left( \bigoplus_{i=1}^{16} H^*(T^3) \otimes [E_i] \right), \]
the elements of \(H^*(\tilde{\mathcal{M}})\) multiply following its algebra structure. Moreover, an element \(\alpha \in H^*(\tilde{\mathcal{M}})\) and \(\beta \otimes [E_j]\) multiply as \(\alpha \cdot (\beta \otimes [E_j]) = (i^*_j \alpha \wedge \beta) \otimes [E_j]\), where \(i_j : S_j \subset \tilde{\mathcal{M}}\) is the inclusion of the \(j\)-th component \(S_j\) of the singular locus. Finally,
\[ [E_j] \cdot [E_j] = -2[\lambda] = -2e_{1256}, \quad [E_i] \cdot [E_j] = 0 \text{ if } i \neq j, \]

since it is the Poincaré dual of the \(T^3\) given by coordinates \((x_3, x_4, x_7)\). So \((\beta \otimes [E_j]) \cdot (\gamma \otimes [E_j]) = -2\beta \wedge \gamma \wedge [\lambda] \in H^*(\tilde{\mathcal{M}})\). In summary,
\[ (\alpha_1, \sum_j \beta_{1j} \otimes [E_j]) \cdot (\alpha_2, \sum_j \beta_{2j} \otimes [E_j]) = \]
\[ = \left( \alpha_1 \wedge \alpha_2 - 2 \sum_j \beta_{1j} \wedge \beta_{2j} \wedge \lambda, \sum_j (\beta_{1j} \wedge \alpha_2 + \alpha_1 \wedge \beta_{2j}) \otimes [E_j] \right). \]

To complete the proof of Theorem 21, we compute the Betti numbers of \(\tilde{\mathcal{M}}\). Recall that according to Nomizu’s theorem 40, the de Rham cohomology of the nilmanifold \(\Gamma \backslash G\) is isomorphic to Chevalley–Eilenberg cohomology of the Lie algebra of \(G\). We easily find that the de Rham cohomology groups \(H^2(M)\) and \(H^3(M)\) of the nilmanifold \(M\) are
\[ H^2(M) = \langle [e_{16}], [e_{17}], [e_{23}], [e_{24}], [e_{25} + e_{34}], [e_{35}], [e_{27} - e_{45} - e_{36}] \rangle, \]
\[ H^3(M) = \langle [e_{136}], [e_{146}], [e_{147}], [e_{157}], [e_{167}], [e_{234}], [e_{235}], [e_{236} + e_{245}], [e_{237} + e_{345}], [e_{246}], [e_{257}], [e_{247} + e_{256} + e_{346}], [e_{257} + e_{347} + e_{356}] \rangle, \]
and thus
\[ H^2(\wt{M}) = H^2(M)^{\mathbb{Z}_2} = \langle [e^{16}], [e^{35} + e^{34}] \rangle, \]
\[ H^3(\wt{M}) = H^3(M)^{\mathbb{Z}_2} = \langle [e^{136}], [e^{146}], [e^{157}], [e^{167}], [e^{235}], \\
\quad [e^{236} + e^{245}], [e^{246}], [e^{257} + e^{347} + e^{356}] \rangle. \]

Then, Proposition 15 and Proposition 22 imply that the Betti numbers of \( \wt{M} \) are as follows:
\[ b_1(\wt{M}) = b_1(\wt{M}) = 1, \]
\[ b_2(\wt{M}) = b_2(\wt{M}) + 16 = 18, \]
\[ b_3(\wt{M}) = b_3(\wt{M}) + 16 b_1(T^3) = 56. \] (23)

**Proposition 23.** The compact manifold \( \wt{M} \) has fundamental group \( \pi_1(\wt{M}) = \mathbb{Z} \).

**Proof.** Let \( \wh{\pi} : M \to \wt{M} \) be the quotient map. Fix \( p_0 \in M \) to be the point with coordinates \((0, \ldots, 0)\), and let \( q_0 = \wh{\pi}(p_0) \) be the image of \( p_0 \) under the projection \( \wh{\pi} \). Let \( \gamma_1, \ldots, \gamma_7 \) be the loops on \( M \), where \( \gamma_i \) is the image under \( \wh{\pi} \) of the path from \( p_0 \) to \( e_i = (0, \ldots, 1, \ldots, 0) \). These are generators of the fundamental group \( \pi_1(M, p_0) \) subject to the relations
\[ [\gamma_1, \gamma_2] = \gamma_4, [\gamma_1, \gamma_3] = \gamma_5, [\gamma_1, \gamma_4] = \gamma_6, [\gamma_1, \gamma_5] = \gamma_7, \] (24)
and the fact that the other commutators are zero, i.e. \( \gamma_2, \gamma_4 \) commute, etc.

We claim that any loop \( \alpha \) on \( \wt{M} \) lifts to \( M \) (non-uniquely). The (closed) portions of \( \alpha \) that lie in the orbifold locus lift uniquely. The (open) part of \( \alpha \) that lies off the orbifold locus lift to two possible paths (since over there \( \wh{\pi} \) is a double covering). Take any of those lifts. The result is a continuous path \( \alpha \) on \( M \) such that \( \wh{\alpha} = \wh{\pi} \circ \alpha \). This is a well-defined loop, because the end-point lifts uniquely to the base point. This concludes that \( \pi_1(\wt{M}, q_0) \) is generated by the images \( \wh{\gamma}_i = \wh{\pi} \circ \gamma_i \), \( 1 \leq i \leq 7 \).

Now recall that \( \mathbb{Z}_2 \) acts by (7). Under it, the image of \( \gamma_1 \) is the same as the path from \((0, 0, \ldots, 0)\) to \((\frac{1}{2}, 0, \ldots, 0)\) followed by the same path in the reversed direction. This is contractible, hence \( \wh{\gamma}_1 = 0 \). The same happens with \( \gamma_2 \), so \( \wh{\gamma}_2 = 0 \). Using the relations (24), we conclude that \( \pi_1(\wt{M}, q_0) = \langle \gamma_3 \rangle \). Therefore \( \pi_1(\wt{M}) \cong \mathbb{Z} \), since \( b_1(\wt{M}) = 1 \).

Now we prove that the resolution process does not alter the fundamental group. Let us treat the case of the orbifold locus \( S_0 \cong T^3 \subset \wt{M} \). Let \( \pi : \wt{M} \to \wt{M} \) be the resolution map. Take \( U \) a neighbourhood of \( S_0 \), and \( V = \wt{M} - S_0 \). Consider \( \wh{\pi} : \wt{U} = \pi^{-1}(U) \) and \( \wh{\pi} : \wt{V} = \pi^{-1}(V) \). Then by Seifert–van Kampen, \( \pi_1(\wt{M}) \) is the amalgamated sum of \( \pi_1(U) \) and \( \pi_1(V) \) over \( \pi_1(U \cap V) \). And \( \pi_1(\wt{M}) \) is the amalgamated sum of \( \pi_1(\wt{U}) \) and \( \pi_1(\wt{V}) \) over \( \pi_1(\wt{U} \cap \wt{V}) \). Note that \( \wt{V} \cong V, \wt{U} \cap \wt{V} \cong U \cap V, \text{ and } U \sim T^3, \wt{U} \sim T^3 \times \mathbb{CP}^1 \), so that \( \pi_1(\wt{U}) \cong \pi_1(U) \). Therefore \( \pi_1(\wt{M}) \cong \pi_1(\wt{M}) \cong \mathbb{Z} \).

Next, we complete the properties of \( \wt{M} \) proving that it is formal, and that it does not admit any torsion-free \( G_2 \)-structure.

**Proposition 24.** The compact manifold \( \wt{M} \) is formal.
Proof. We are going to first check that the orbifold $\hat{M}$ is formal. Note that the cohomology group $H^3(\hat{M})$ of $\hat{M}$ decomposes as

$$H^3(\hat{M}) = A \oplus B,$$

where $A = \langle [e^{136}], [e^{235}] \rangle$ and $B = \langle [e^{146}], [e^{157}], [e^{167}], [e^{246}], [e^{236} + e^{245}], [e^{257} + e^{347} + e^{356}] \rangle$. Then, the multiplication by $[e^3]$ vanishes on $A$, and it defines an isomorphism $[e^3] : H^2(\hat{M}) \to A$. Moreover, the multiplication by $[e^3]$ is injective on $B \to H^1(\hat{M})$. For this just check that the map $H^3(\hat{M}) \times H^3(\hat{M}) \to \mathbb{R}$, $(\alpha, \beta) \mapsto \int \alpha \wedge \beta \wedge e^3$ has matrix (on the given basis of $H^3(\hat{M})$) of the form

$$\begin{pmatrix}
0 & \ldots & 0 & 1 \\
0 & \ldots & 1 & * \\
\vdots & & \vdots & \\
1 & * & \ldots & *
\end{pmatrix}.
$$

On the other hand, with respect to the basis $\alpha_1 = [e^{16}]$, $\alpha_2 = [e^{25} + e^{34}]$ of $H^2(\hat{M})$, we have $\alpha_1^2 = 0$ and $\alpha_2^2 = 2[e^{2345}] = -2[e^3] \wedge [e^{236} + e^{245}]$, but $\alpha_1 \wedge \alpha_2 = 2[e^{1256}] \neq 0$ as $e^{1256} - e^{1346} = de^{456}$ from $\mathbb{[10]}$.

Since $M$ is a compact nilmanifold, the minimal model of $M$ is the minimal DGA $((\bigwedge V, d), V = \langle e^1, \ldots, e^7 \rangle)$ and the differential $d$ is defined by $\mathbb{[15]}$. Let $F = \mathbb{Z}_2$ be the finite group acting on $M$, and on the minimal model. So $((\bigwedge V)^F, d)$ is a model (not minimal) of $\hat{M} = M/F$. Let $\psi : (\bigwedge V, d) \to ((\bigwedge V)^F, d)$ be a minimal model of $\hat{M}$. Using notation of Definition $\mathbb{[9]}$ we write $W^i = C^i + N^i$, $i \leq 3$. We shall write $a_j, b_j, c_j, n_j$ for the generators of, respectively $C^i$, $i \leq 3$, and $N^3$. Then,

$$W^1 = C^1 = \langle a_1 \rangle,$$

$$W^2 = C^2 = \langle b_1, b_2 \rangle,$$

$$W^3 = C^3 \oplus N^3,$$

where $C^3 = \langle c_1, c_2, c_3, c_4, c_5, c_6 \rangle$ and $N^3 = \langle n_1, n_2 \rangle$,

the differential $d$ is given by $d(C^i) = 0$, $dn_1 = b_1^2$, $dn_2 = b_2^2 + 2a_1 c_5$, and the morphism $\psi : (\bigwedge V, d) \to ((\bigwedge V)^F, d)$ of differential algebras is defined by

$$\psi(a_1) = e^3, \quad \psi(b_1) = e^{16}, \quad \psi(b_2) = e^{25} + e^{34},$$

$$\psi(c_1) = e^{146}, \quad \psi(c_2) = e^{157}, \quad \psi(c_3) = e^{167},$$

$$\psi(c_4) = e^{246}, \quad \psi(c_5) = e^{236} + e^{245}, \quad \psi(c_6) = e^{257} + e^{347} + e^{356},$$

$$\psi(n_j) = 0, \; j = 1, 2.$$

Now we can prove that $\hat{M}$ is 3-formal, and so it is formal by Proposition $\mathbb{[13]}$. For this we have to look at the closed elements of $I(N^3) \subset \bigwedge W^{\leq 3}$, and check that the image through $\psi$ is exact. However this is clear since $\psi(N^3) = 0$.

To check the formality of $\hat{M}$, now we have to work out the 3-minimal model of it, with the algebra structure of $H^*(\hat{M})$ given above. Note that there is a Thom form $\eta_i$ such that $[\eta_i] = [E_i]$. It is clear that $\eta_i \wedge \eta_j = 0$ for $i < j$. For $1 \leq k \leq 16$ we take 3-forms $\theta_k'$ such that $d\theta_k' = \eta_k^2 - 2e^{1256}$. As the exceptional divisors $E_k$ lie over the 3-tori $\mathbb{[10]}$, the Thom forms $\eta_k$ can be defined only with the coordinates of the fibers, that is,

$$L_{e_a}(\eta_k) = i_{e_a}(\eta_k) = 0, \; \text{for} \; a = 3, 4, 7$$

(25)
Therefore, the same property can be arranged for \( \theta_k^i \). Any 5-form with that property vanishes, so
\[
\theta_k^i \wedge \eta_i = 0, \tag{26}
\]
We also note that \([\epsilon^{16}]|_{E_i} = 0\), hence there are 3-forms \( \theta_i'' \) such that \( d\theta_i'' = \epsilon^{16} \wedge \eta_i \). The forms \( \theta_i'' \) are arranged to satisfy \( \varpi \). Thus
\[
\theta_i'' \wedge \eta_j = 0. \tag{27}
\]

Therefore, the minimal model of \( \tilde{M} \) must be a differential graded \( (\bigwedge \tilde{W}, \tilde{d}) \) where \( \tilde{W} \) is the graded vector space \( \tilde{W} = \bigoplus_i \tilde{W}^i \) with
\[
\tilde{W}^1 = W^1, \\
\tilde{W}^2 = W^2 \oplus S^2, \quad S^2 = \{B_i \mid 1 \leq i \leq 16\}, \\
\tilde{W}^3 = W^3 \oplus S^3 \oplus R^3, \quad S^3 = \{C_i^4, C_i^7 \mid 1 \leq i \leq 16\}, \\
R^3 = \{(D_{ij} \mid 1 \leq i < j \leq 16) \oplus (D_k', D_k'' \mid 1 \leq k \leq 16)\},
\]
and the differential \( \tilde{d} \) is given by \( \tilde{d}|_{W^i} = d, \tilde{d}(B_i) = \tilde{d}(C_i^4) = \tilde{d}(C_i^7) = 0 \), and
\[
\tilde{d}(D_{ij}) = B_i B_j, \quad \tilde{d}(D_k') = B_k^2 + 2(b_1 b_2 + a_1 c_1), \quad \tilde{d}(D_k'') = b_1 B_k.
\]
Now, we define the map of differential algebras \( \vartheta : (\bigwedge \tilde{W}^{\leq 3}, \tilde{d}) \longrightarrow (\Omega^*(\tilde{M}), \tilde{d}) \), by \( \vartheta|_{W} = \psi \) and
\[
\vartheta(B_i) = \eta_i, \quad \vartheta(C_i^4) = e^4 \wedge \eta_i, \quad \vartheta(C_i^7) = e^7 \wedge \eta_i, \quad \vartheta(D_{ij}) = 0, \quad \vartheta(D_k') = \theta_k', \quad \vartheta(D_k'') = \theta_k'',
\]
where \( 1 \leq i \leq 16 \) and \( 1 \leq i < j \leq 16 \) and \( 1 \leq k \leq 16 \). This is a 3-minimal model of \( \tilde{M} \).

To check the 3-formality, observe that \( \tilde{N}^3 = N^3 \oplus R^3 \). We have to see that the closed elements of degree \( \leq 7 \) in \( I(\tilde{N}) \) are exact. In degree 4 there are no closed elements. In degree 5, we have the elements
\[
b_1 D_{jk} - B_j D_{k}'', \quad B_j (D_{ij}' - D_{k}') - (D_{ij} B_j - D_{ik} B_k).
\]

The image via \( \vartheta \) is zero by using \( \varpi \) and \( \vartheta \), so the elements are exact. In degree 6, we only have \( a_1(D_{ij} B_j + 2b_2 D_{ij}'' - B_k D_{jk}) \). This lies in \( H^6(\tilde{M}) \), and multiplying by \( a_1 \), it vanishes. By Poincaré duality, it defines the zero cohomology class. In degree 7, the closed elements are those in \( \varpi \) times \( b_1 \) or \( b_i \), and the elements
\[
B_j B_k(D_{ij}' - D_{m}') - D_{jk}(B_i^2 - B_m^2), \quad b_1^2 D_{jk} - B_j B_k n_1, \\
(b_2^2 + 2a_1 c_3) D_{jk} - B_j B_k n_2, \quad B_i b_1 D_{k}'' - B_i B_j n_1.
\]
All of them are clearly exact.

**Theorem 25.** The compact manifold \( \tilde{M} \) does not admit any torsion-free G\(_2\)-structure.

**Proof.** We prove the theorem by contradiction. Suppose that \( \tilde{M} \) admits a torsion-free G\(_2\)-structure with associated metric \( g \). Then, the restricted holonomy group of \( g \) is a subgroup of G\(_2\). By [34, Theorem 10.2.1] the only connected Lie subgroups of G\(_2\) that can arise as restricted holonomy of the Riemannian metric \( g \) are G\(_2\), SU(3), SU(2) and \{1\}. Since \( b_1(\tilde{M}) = 1 \) and \( \pi_1(\tilde{M}) = \mathbb{Z} \), the restricted holonomy group of \( g \) must be SU(3).
Therefore, $\widetilde{M}$ has a finite covering $N \times S^1$ with $N$ being a 6-dimensional simply connected Calabi–Yau manifold. Indeed, by Proposition 1.1.1 of [33] we know that $(\widetilde{M}, g)$ must admit as Riemannian finite cover a product $N \times S^1$, for some compact, simply connected 6-manifold $N$. Since the holonomy group of the induced metric on the finite cover is the product of the holonomy group of $N$ and the trivial group, the induced metric on $N$ is Ricci-flat and its holonomy group is SU(3). That is $N$ is a Calabi–Yau manifold.

The deck transformations group of the covering map $N \times S^1 \to \widetilde{M}$ consists of maps which are products of an isometry of the Calabi–Yau manifold $N$ and a rotation of finite order on $S^1$. This is due to the fact that the deck transformations are isometries and that $N \times S^1$ is a Riemannian product. Furthermore, the above deck transformations are homotopic to the identity. Therefore, $H^*(N \times S^1) \cong H^*(\widetilde{M})$ and the minimal models are the same. Thus, on $N \times S^1$ and, consequently on $\widetilde{M}$ there exist a closed 2-form $\omega$ and a closed 1-form $\eta$, such that $[\omega]^3 \sim [\eta] \neq 0$ in the cohomology of $\widetilde{M}$. But this is not possible in the algebra $H^*(\widetilde{M})$. First, we see that it is not possible in $H^*(\widetilde{M})$ since we must have $\eta = e^3$, and we know that $H^2(\widetilde{M}) = \langle e^{16}, e^{25} + e^{34} \rangle$. Then we use Proposition 22. □

Remark 26. The proof of Theorem 25 also shows that $\widetilde{M}$ cannot be a product of $S^1$ and a 6-manifold.

Moreover, by [23], the Poincaré polynomial of $\widetilde{M}$ is

$$P(t) = (1 - 2t + 21t^2 - 2t^3 + t^4)(1 + t)^3.$$ 

It can be checked that the factor of degree 4 is irreducible over the rationals and the polynomial $P(t)$ does not factorize as the product of two polynomials of degree greater than 1 with non-negative integer coefficients. Therefore, $\widetilde{M}$ cannot be a product of two manifolds of dimension greater than 1 too.

8. Associative 3-folds in $\widetilde{M}$

The closed $G_2$ form $\tilde{\varphi}$ constructed on $\widetilde{M}$ defines an associative calibration on $\widetilde{M}$. This means that, for any $p \in \widetilde{M}$, we have that every oriented 3-dimensional subspace $V$ of the tangent space $T_p \widetilde{M}$ satisfies $\tilde{\varphi}(p)\mid_V = \lambda \text{vol}_V$, for some $\lambda \leq 1$, where the volume form $\text{vol}_V$ is induced from the restriction to $V$ of the inner product $g_\widetilde{M}$ at $p$ (see [29] and [34 §3.7]). The 3-dimensional orientable submanifolds $Y \subset \widetilde{M}$ calibrated by the $G_2$ form $\tilde{\varphi}$, i.e. those submanifolds $Y \subset \widetilde{M}$ that satisfy $\tilde{\varphi}(p)\mid_{T_pY} = \text{vol}_Y(p)$, for each $p \in Y$ and for some unique orientation of $Y$, are often called associative 3-folds. Every compact calibrated submanifold $Y$ is volume-minimizing in its homology class, in particular $Y$ is minimal [34 Proposition 3.7.2].

We shall produce examples of associative 3-folds in $\widetilde{M}$ from the fixed locus of a $G_2$-involution of the compact manifold $M = \Gamma \backslash G$ defined in [4], applying the following.

Proposition 27 ([34 Proposition 10.8.1]). Let $N$ be a 7-manifold with a closed $G_2$ form $\phi$, and let $\sigma : N \to N$ be an involution of $N$ satisfying $\sigma^* \phi = \phi$ and such that $\sigma$ is not the identity map. Then the fixed point set $P = \{p \in N \mid \sigma(p) = p\}$ is an embedded associative 3-fold. Furthermore, if $N$ is compact then so is $P$. 


Remark 28. Note that Proposition 10.8.1 in [34] is stated for the $G_2$-structures that are closed and coclosed, but the coclosed condition is not used in the proof.

Recall from section [4] the 7-dimensional Lie group $G$, and consider on $G$ the involution given by

$$\sigma: (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (-x_1, -x_2, x_3, x_4, -x_5, \frac{x_7}{2}, -x_6, x_7). \quad (29)$$

The involution $\sigma$ is equivariant with respect to the left multiplications by elements of the subgroup $\Gamma \subset G$. Indeed, for each $a \in G$ and $A \in \Gamma$ we may write, noting the properties of the $\mathbb{Z}_2$-action $\rho$ on $G$ defined by (7),

$$L_A(\sigma(a)) = L_A(L_{A'}(\rho(a))) = L_{A'}(L_A(\rho(a))) = L_{A'}(\rho(A') \cdot \rho(a)) = L_{A'}(\rho(L_A(a))) = \sigma(L_A(a)),$$

where $A' = \rho(A)$ and $L_{A'}$ denotes the left translation by an element with coordinates $(x_i) = (0, 0, 0, 0, 0, \frac{x_7}{2}, 0)$ in $G$. Therefore, $\sigma$ descends to the quotient manifold $M = \Gamma \backslash G$. The induced map on $M$, still denoted by $\sigma$, commutes with $\rho$ and so $\sigma$ descends to the orbifold $\hat{M} = M/\mathbb{Z}_2$. From now on, we denote by $\hat{\sigma}$ the involution of $\hat{M}$ induced by $\sigma$.

The fixed locus $\hat{P}$ of $\hat{\sigma}$ is the image by the natural projection $\hat{\pi}: M \to \hat{M}$ of the set $P$ of points in $M$ that are fixed by the involution $\sigma: M \to M$ or by $\sigma \circ \rho = L_{A'}: M \to M$. Thus, $\hat{P}$ consists of all the 3-dimensional spaces $\hat{P}_b = \hat{\pi}(P_b) = P_b/\mathbb{Z}_2$, where

$$P_b = \begin{cases} \{ \Gamma \cdot (b_1, b_2, x_3, x_4, b_5, b_6, x_7) \mid x_3, x_4, x_7 \in \mathbb{R} \} \subset M, & \text{if } b_1 = 0 \\ \{ \Gamma \cdot (b_1, b_2, x_3, x_4, b_5, \frac{b_7}{2} b_2 + b_6 - x_4, x_7) \mid x_3, x_4, x_7 \in \mathbb{R} \} \subset M, & \text{if } b_1 = 1 \end{cases}$$

and

$$b = (b_1, b_2, b_5, b_6) \in \mathbb{B} = \{0, 1\} \times \{0, 1/2\} \times \{0, 1/2\} \times \{1/4, 3/4\}.$$ 

Hence, $P$ is a disjoint union of 16 copies of a 3-torus $T^3$. Now one can check that the fixed locus $\hat{P}$ of $\hat{\sigma}$ consists of 8 disjoint copies of $T^3$ since in the orbifold $\hat{M}$ the points of coordinates $(b_1, b_2, x_3, x_4, b_5, 1/4, x_7)$ and $(b_1, b_2, x_3, x_4, b_5, 3/4, x_7)$ are the same. Observe that the fixed loci $P$ of $\sigma$ and $S'$ of $\rho$ do not intersect, and hence the fixed locus $\hat{P}$ of $\hat{\sigma}$ and the singular locus $S$ of the orbifold $\hat{M}$ also do not intersect.

Proposition 29. Each of the eight disjoint copies of 3-tori in $\hat{M}$, which are the fixed locus $\hat{P}$ of $\hat{\sigma}$, define eight embedded, associative (calibrated by $\hat{\varphi}$), minimal 3-tori in $\hat{M}$.

Proof. Since the $G_2$ form $\varphi$ on $M$ defined in [4] is preserved by the involution $\sigma$ of $M$, each of the 16 tori $P_b$ in $M$ fixed by $\sigma$ is an associative 3-fold in $(M, \varphi)$ by Proposition [27]. Now we know that the $\mathbb{Z}_2$-action $\rho$ on $M$ preserves the $G_2$ form $\varphi$ on $M$, and induces the $G_2$ form $\hat{\varphi}$ on $\hat{M}$ (see section [4]), so that the pull-back of $\hat{\varphi}$ sends $\hat{\varphi}$ to $\varphi$. Thus, the 2-to-1 projection map $\hat{\pi}: M \to \hat{M}$ outside the set $S'$ of points in $M$ fixed by $\rho$ is a local isomorphism of the closed $G_2$-structures and hence also a local isometry of the induced metrics. Consequently, $\hat{\pi}$ preserves the associative calibrated property of submanifolds, and so each of the eight copies of $T^3$ is an associative (and minimal) 3-fold in $\hat{M}$. Furthermore, as we mentioned above, these 3-tori do not meet the singular locus $S$ of $\hat{M}$. 
To complete the proof, let us recall that the $G_2$-structure $\tilde{\varphi}$ on $\tilde{M}$ agrees, away from a neighbourhood $U$ of $S$, with the $G_2$-structure $\hat{\varphi}$ induced on $\hat{M}$ from $M$. It follows that the above 3-tori lift diffeomorphically to the resolution $\hat{M}$ and define $8$ embedded, associative (calibrated by $\hat{\varphi}$), minimal 3-tori in $\hat{M}$.

McLean [38] studied the deformation problem for several types of calibrated submanifolds. For compact associative 3-folds, the problem may be expressed as a non-linear elliptic PDE, with index zero, if the $G_2$ form is closed and coclosed. This result was generalized by Akbulut and Salur to arbitrary, not necessarily closed or coclosed, $G_2$ forms [2 Theorem 6]. It follows that any compact associative 3-fold in $\hat{M}$ is either rigid or, otherwise, has infinitesimal associative deformations which in general need not arise from the actual deformations (as the linear part of the deformation problem may have a nontrivial cokernel).

As we now show, the 3-tori in the present example do have associative deformations.

**Proposition 30.** Each of the eight associative 3-tori in $\tilde{M}$ arising from the fixed locus of $\sigma$ has a smooth 3-dimensional family of non-trivial associative deformations.

**Proof.** As in the previous sections, in light of the symmetry by left translations, it suffices to consider just one component $Y_0$ of the fixed locus of $\sigma$. A tubular neighbourhood of $Y_0$ in $\hat{M}$ is isometric to a tubular neighbourhood of the image of $Y_0$ in the smooth locus $\hat{M} \setminus S$. As the projection $M \to \hat{M}$ is a local isometry away from the preimage of $S$ we may work on $\hat{M}$ with the $G_2$-structure $\varphi$ and consider a component of the preimage of $Y_0$ which by abuse of notation we continue to denote by $Y_0 \subset M$. We may choose $Y_0$ to be defined by $x_1 = x_2 = x_5 = 0$, $x_6 = \frac{1}{7}$, then the associative 3-torus $Y_0$ is contained in the fiber $p^{-1}(0 + 2\mathbb{Z})$ of the projection $p : M \to \mathbb{R}/2\mathbb{Z}$ (see (5)).

Every fiber $p^{-1}(x_1)$ has a natural structure of a complex 3-torus $\mathbb{C}^3/\Lambda(x_1)$, where the complex coordinates on $\mathbb{C}^3$ are given by $x_2 + ix_3$, $x_4 + ix_5$, $x_6 + ix_7$. Moreover, these latter 3-tori are biholomorphic to the standard 3-torus $\mathbb{C}^3/\mathbb{Z}^6$ because the linear isomorphisms $B(x_1)$ and $C$ from Lemma 14 are contained in the image of $\text{SL}(3, \mathbb{R})$ under the chain of natural embeddings of groups $\text{SL}(3, \mathbb{R}) \subset \text{SL}(3, \mathbb{C}) \subset \text{SL}(6, \mathbb{R})$. The complex 3-form $\Omega = (e^2 + ie^3) \wedge (e^4 + ie^5) \wedge (e^6 + ie^7)$ induces, via the pull-back, on each complex torus $p^{-1}(x_1)$ a holomorphic trivialization of the canonical bundle of $(3, 0)$-forms. The (pull-back of the) closed 2-form $\omega = e^2 \wedge e^3 + e^4 \wedge e^5 + e^6 \wedge e^7$ induces on $p^{-1}(x_1)$ a Ricci-flat Kähler metric which depends non-trivially on $x_1$ and when $x_1 = 0$ coincides with the ‘usual’ Kähler metric on $\mathbb{C}^3/\mathbb{Z}^6$. Thus each fiber $p^{-1}(x_1)$ has a torsion-free $\text{SU}(3)$ (Calabi–Yau) structure compatible with the closed $G_2$-structure $\varphi = e^1 \wedge \omega - \Re(\Omega)$ on $M$, in the sense that $\iota^*_x \omega = \iota^*_x (dx_1 \varphi)$ and $\iota^*_x \Re(\Omega) = \iota^*_x \varphi$, where $\iota_x : p^{-1}(x_1) \to M$ denotes the embedding.

It is not difficult to check that $Y_0$ is a special Lagrangian 3-torus in the Calabi–Yau threefold $Z_0 = p^{-1}(0)$. Furthermore, the special Lagrangian tori

$$Y(a, b, c) = \{(a, y_1, y_2, b, \frac{1}{4} a, c, y_3) + \mathbb{Z}^6 \mid (y_1, y_2, y_3) \in \mathbb{R}^3\}$$

in $p^{-1}(0)$ are associative in $(M, \varphi)$ as $\varphi|_{Y(a,b,c)} = dx_3 \wedge dx_4 \wedge dx_7|_{Y(a,b,c)} = e^3 \wedge e^4 \wedge e^7|_{Y(a,b,c)}$. For small $a, b, c$, the $Y(a, b, c)$ induce well-defined non-trivial associative deformations of $Y_0$ in $(\tilde{M}, \tilde{\varphi})$. \hfill $\square$
We next show that the result of Proposition \ref{prop:deformation} is optimal. For this, we require some foundational results about the deformations of associative 3-folds.

Let \( N \) be a 7-manifold with a \( G_2 \)-structure \( \phi \). Denote by \( \chi_\phi \) the 3-form on \( N \) with values in \( TN \) determined by
\[
\langle \chi_\phi(u, v, w), a \rangle = \ast_\phi \phi(u, v, w, a),
\]
for all \( u, v, w, a \in TN \). The 3-form \( \chi_\phi \) may also be locally expressed as
\[
\chi_\phi = \sum_{k=1}^{7} (e_{j,k} \ast_\phi \phi) \otimes e_j,
\]
for any local positive-oriented orthonormal frame field \( \{e_j\} \) on \( N \) \cite[p. 1217]{29}.

For \( P \) an oriented 3-dimensional submanifold of \( N \), let \( \omega_P \) denote a global section of \( \Lambda^3 TP \) given by \( f_1 \wedge f_2 \wedge f_3 \), for any local positive orthonormal frame field \( \{f_k\} \) on \( P \). It can be checked that then \( \chi_\phi(\omega) \) is a section of the normal vector bundle \( \mathcal{N}_{P/N} \) of \( P \) in \( N \). Furthermore, the submanifold \( P \) will be associative with respect to \( \phi \) (and calibrated when \( \phi \) is closed) if and only if \( \chi_\phi(\omega_P) = 0 \) (cf. \cite{29} or \cite{38}).

Now, let us consider \( P \) a compact associative 3-fold with respect to \( \phi \). It is by now a standard consequence of the tubular neighborhood theorem that smooth local deformations of \( P \) may be given by \( P(v) = \exp_\phi(P) \) for smooth sections \( v \) of the normal vector bundle \( \mathcal{N}_{P/N} \) of \( P \) in \( N \) with \( \|v\|_{C^0} \) small, where the exponential map and the \( C^0 \) norm are defined using the metric \( g_\phi \). For every \( C^0 \)-small normal vector field \( v \in \Gamma(\mathcal{N}_{P/N}) \) and every closed \( G_2 \)-structure \( \phi \) on \( N \), define the ‘deformation map’
\[
F(v, \phi) = (\exp_\phi, \phi)(\omega_P) \in \Gamma(\mathcal{N}_{P/N}, \phi),
\]
where the normal bundle \( \mathcal{N}_{P/N}, \phi \) is defined using the metric \( g_\phi \). Then \( P(v) \) will be associative calibrated by \( \phi \) precisely when \( F(v, \phi) = 0 \).

**Proposition 31.** In the case when \( P = Y \) is one of the eight associative 3-tori given in Proposition \ref{prop:torus}, the kernel of the derivative \( D_1 F|_{(0, \mathcal{Z})} \) of the map (31) in the first argument has dimension 3.

**Corollary 32.** The family (30) of associative local deformations of \( Y \) is maximal (that is, it is not contained as a proper subset in another associative local deformation family).

**Proof of Proposition 31.** For the same reason as in the proof of Proposition \ref{prop:deformation} we may take \( P \) to be the associative 3-torus \( Y_0 \) in \( M \) with the closed \( G_2 \)-structure \( \varphi \). Thus \( Y_0 \) is defined by \( x_1 = x_2 = x_5 = 0, x_6 = \frac{1}{4} \) and \( x_3, x_4, x_7 \in \mathbb{R}/\mathbb{Z} \) define the local coordinates on \( Y_0 \).

It is easy to check that the frame field \( e_i \) dual to \( e^i \) on \( M \) (see (5)) is given, in the local coordinates \( x_i \) induced from \( G \), by
\[
e_1 = \partial_t + x_2 \partial_4 + x_3 \partial_5 - x_1 x_2 \partial_6 - x_1 x_3 \partial_7, \quad e_2 = \partial_2, \quad e_3 = \partial_3,
e_4 = \partial_4 - x_1 \partial_6, \quad e_5 = \partial_5 - x_1 \partial_7, \quad e_6 = \partial_6, \quad e_7 = \partial_7,
\]
(32)
where \( \partial_i = \frac{\partial}{\partial x_i} \) denote the local coordinate vector fields.

The restrictions to \( Y_0 \) of the vector fields \( e_i, i = 1, 2, 5, 6 \), give an orthonormal frame field inducing a trivialization of the normal bundle \( \mathcal{N}_{Y_0/M}, \varphi \) and the restrictions of \( e_k, k = 3, 4, 7 \) define an orthonormal frame field on \( Y_0 \) trivializing the tangent bundle \( TY_0 \).
The linear operator in question acting on the sections of $N_{Y_0/M, \varphi}$ and may be expressed (see [2], [23, Theorem 2.1]) as
\[
D_1(v) = D_1 F|_{(0, \tilde{\varphi})}(v) = \sum_{k=3,4,7} e_k \times \nabla_{e_k}^\perp v + \sum_{i=1,2,5,6} (\nabla v^* \varphi)(e_i, \omega_Y) \otimes e_i,
\]
where $\times$ denoted the octonionic cross-product corresponding to the $G_2$-structure $\varphi$, $\nabla^\perp$ is the connection on $N_{Y_0/M}$ induced by the Levi-Civita connection $\nabla$ of $g_\varphi$ and $\omega_Y = e_3 \wedge e_4 \wedge e_7 \in \Gamma(\Lambda^3 T Y_0)$. The second sum in (33) contains the terms arising from the failure of the $G_2$-structure to be torsion-free and does not contain derivatives of $v$. On the other hand, the first sum in (33) is a Dirac-type operator arising in McLean’s results [38, §5].

In the present case, we may consider $D_1$ as a first order differential operator acting on functions $v = (v_1, v_2, v_5, v_6)$, where each $v_i(x_3, x_4, x_7)$ is periodic with period 1 in each variable $x_k$. It is not difficult to check that the first order terms in $D_1$ are equivalent to the standard ‘flat space’ Dirac operator given in terms of the Pauli spin matrices. The zero order terms may be determined by a straightforward, albeit lengthy computation; the following table gives some check-points for the readers convenience.

**Table 1.** The values of $2 \nabla_{e_i} e_i$ on $Y_0$, for $i = 1, 2, 5, 6$, $k = 3, 4, 7$.

|   | 1 | 2 | 5 | 6 |
|---|---|---|---|---|
| 3 | $e_5$ | 0 | $-e_1$ | 0 |
| 4 | $e_2 + e_6$ | $e_1$ | 0 | $-e_1$ |
| 7 | $e_5$ | 0 | $-e_1$ | 0 |

**Table 2.** The values of $2 \nabla e_i e^j$ on $Y_0$, for $i = 1, 2, 5, 6$, $j = 1, \ldots, 7$.

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | $0$ | $-e^4$ | $-e^5$ | $-e^6$ | $e^3 - e^4$ | $e^4$ | $e^5$ |
| 2 | $-e^4$ | 0 | 0 | $-2e^1$ | 0 | 0 | 0 |
| 3 | $e^3 + e^7$ | 0 | $-e^1$ | 0 | 0 | 0 | $-e^1$ |
| 4 | $e^4$ | 0 | 0 | $-e^1$ | 0 | 0 | 0 |

We then obtain
\[
D_1 : v = \begin{pmatrix} v_1 \\ v_2 \\ v_5 \\ v_6 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -\partial_3 & \partial_4 & -\partial_7 \\ \partial_3 - 1 & 0 & -\partial_7 & -\partial_4 \\ -\partial_4 & \partial_7 & 0 & -\partial_3 \\ \partial_7 + 1 & \partial_4 & \partial_3 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_5 \\ v_6 \end{pmatrix}
\]

By considering the Fourier expansions of $v$, we find that the kernel of $D_1$ consists of constant vectors and is spanned by $e_i|_{Y_0}$, for $i = 2, 5, 6$. The latter corresponds to the tangent space at $Y_0$ to the 3-dimensional family of associative deformations given in [30]. □
We also show, as a direct consequence of the next result, that the associative 3-tori $Y$ in Proposition 29 become rigid after a suitable arbitrary small perturbation of the closed $G_2$-structure $\tilde{\phi}$ on an arbitrary small neighbourhood of $Y$ in $\tilde{M}$.

If $P$ is a compact associative 3-fold with respect to a closed $G_2$-structure $\phi$ on a 7-manifold $N$, denote by $M_{P,\phi}$ the set of smooth associative 3-folds calibrated by $\phi$ and isotopic to $P$. The next proposition is a rather general result and possibly of independent interest; it is not specific to the particular construction in this paper.

**Proposition 33.** Let $\phi$ be a closed $G_2$-form on a 7-manifold $N$ and $P \subset N$ a compact associative 3-fold calibrated by $\phi$. Suppose that the kernel of $D_3F|_{(0,\phi)}$ is spanned by the normal vector fields $f_1, \ldots, f_m$, where $1 \leq m \leq 4$ and $f_j$ are linearly independent at each point of $P$. Then there is a neighbourhood $U$ of $P$ and a closed $G_2$-structure $\psi$ with arbitrary small $\|\psi - \phi\|_{C^0}$ (defined using the metric $g_\phi$), such that the only element of $M_{P,\psi}$ contained in $U$ is $P$.

**Sketch-proof.** We claim that the argument of Gayet in [23, Proposition 2.6] adapts to the present situation to give a proof of Proposition 33. The only difference from the hypotheses of Gayet’s result is that in the present case the kernel of $D_1$ is spanned by up to four, rather than one, non-vanishing vector fields.

We give a brief review of the proof in [23] with a modification for a $m$-dimensional kernel of $D_1$. The normal bundle of an associative 3-fold is always trivial (e.g. [35, Remark 2.14]) and a tubular neighbourhood of $P$ may be chosen diffeomorphic to $P \times \mathbb{R}^4$ with $u_i$, $i = 1, 2, 3, 4$, the coordinates on $\mathbb{R}^4$ such that $\partial/\partial u_j = f_j$ for $j = 1, \ldots, m$. For each $j$, we may write $f_j \cdot \phi = \sum_{i \neq j} u_i \wedge \beta_{ji}$, for some 2-forms $\beta_{ji}$ and define $\psi_j = d(u_j \sum_{i \neq j} u_i \beta_{ji})$. For every $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$ close to zero, the 3-form $\phi_\lambda = \phi + \sum_{j=1}^m \lambda_j \psi_j$ gives a well-defined $G_2$-structure such that $\phi_\lambda|_P = \phi|_P$. Thus $P$ is associative with respect to $\phi_\lambda$ for each $\lambda$. Let $D_3^\lambda$ be the derivative in the first variable of the deformation map (31), associated with $\phi_\lambda$. It suffices to prove that $D_3^\lambda$ is injective and then Proposition 33 will follow from McLean’s theory by application of the implicit functions theorem in Banach spaces as $D_3^\lambda$ is a self-adjoint elliptic operator of index zero (cf. [38, §5], [23, Proposition 2.2]).

Let $v = v_0 + \sum_{i=m+1}^4 \frac{\partial}{\partial u_i}$ and $v_0 = \sum_{j=1}^m v_j f_j$. It follows from the proof of (33) in [23] that the operator $D_3^\lambda$ admits an expansion of the form

$$D_3^\lambda v = D_3 v + \sum_{j=1}^m \lambda_j v_j f_j + O(|\lambda|^2 v) + O(|\lambda|^2 v)$$

for small $\lambda$ and $v$. The key point in the argument of [23, Proposition 2.6] is an application of the elliptic theory to show that if $D_3^\lambda v = 0$, then $\nabla v_0 = O(|\lambda| v)$ and the norms of $v$ and $v_0$ are Lipschitz equivalent and

$$D_3 v = -\sum_{j=1}^m \lambda_j v_j f_j + O(|\lambda|^2 v).$$  \hspace{1cm} (34)

On the other hand, by considering an elliptic estimate for $(D_3)^2$ one can deduce that $D_3 v = O(|\lambda|^2 v)$ thus obtaining a contradiction with (34) for any small non-zero $\lambda_j$, unless $v = 0$. \hspace{1cm} \qed
9. A coassociative torus fibration

In this section, we shall consider a special class of 4-dimensional submanifolds, which are defined on each 7-manifold $N$ with a $G_2$-structure defined by a 3-form $\phi$. We may write $\phi = e^{123} + e^{145} + e^{167} - e^{246} + e^{257} + e^{347} + e^{356}$ (cf. (1)) and then the local co-frame field \{ $e_1, \ldots, e_7$ \} is orthonormal in the metric $g_\phi$ (induced by $\phi$) and also positively oriented. The Hodge dual of $\phi$ is therefore

$$\theta = *\phi = e^{4567} + e^{2367} + e^{2345} - e^{1357} + e^{1346} + e^{1256} + e^{1247}.$$  

The 4-form $\theta$ satisfies $\theta(p)|_W = \lambda \operatorname{vol}_W$ with some $\lambda \leq 1$, for each $p \in N$ and every oriented 4-dimensional subspace $W$ of the tangent space $T_p N$. Here the volume form $\operatorname{vol}_W$ is induced from the restriction to $W$ of the inner product $g_\phi$ at $p$ (cf. section 8). The orientable 4-dimensional submanifolds $X \subset N$ satisfying $\theta(p)|_{T_p X} = \operatorname{vol}_X(p)$, for all $p \in X$ and for some unique orientation of $X$, are called coassociative 4-folds. The latter condition on $X$ is equivalent to $\phi|_X = 0$. Note that if the 4-form $\theta$ is not closed, then $\theta$ is not a calibration and the coassociative submanifolds of $N$ need not be minimal.

Once again, we start with the 7-manifold $M$ with the $G_2$-structure $\varphi$ defined in section 4 (see (9) and Lemma 14) and also use the orthonormal frame field $e^i$ dual to $e^i$ on $M$,

$$e_1 = \partial_1 + x_2 \partial_4 + x_3 \partial_5 - x_1 x_2 \partial_6 - x_1 x_3 \partial_7, \quad e_2 = \partial_2, \quad e_3 = \partial_3,$$

$$e_4 = \partial_4 - x_1 \partial_5, \quad e_5 = \partial_5 - x_1 \partial_7, \quad e_6 = \partial_6, \quad e_7 = \partial_7. \quad (35)$$

Observe that, in particular, the vectors $e_4, e_5, e_6, e_7$ span the same 4-dimensional subspace as $\partial_4, \partial_5, \partial_6, \partial_7$, and this subspace is coassociative at each point of $M$. Furthermore, this latter subspace is invariant under the action of the linear isomorphisms $B(x_1), C$ and $E$ in Lemma 14.

Recall from section 8 that the fibers $p^{-1}(x_1)$ of the map $p : M \to S^1$ have the structure of Calabi–Yau complex 3-tori $\mathbb{C}^3/\Lambda(x_1) \cong \mathbb{C}^3/\mathbb{Z}^3$ compatible with the $G_2$-structure on $M$. We find that for each $x_1 \in \mathbb{R}/2\mathbb{Z}$ and $w \in \mathbb{C}/\mathbb{Z}$ the complex 2-torus

$$X_{x_1,w} = \{(w, z_2, z_3) \in p^{-1}(x_1) : (z_2, z_3) \in \mathbb{C}^2/\mathbb{Z}^2\}$$

is a well-defined complex submanifold of $p^{-1}(x_1)$ and a coassociative 4-fold in $M$. (Here we used $w = x_2 + i x_3, z_2 = x_4 + i x_5, z_3 = x_6 + i x_7$ to denote the complex coordinates.) The tori $X_{x_1,w}$ are the fibers of a coassociative fibration map

$$q : M \to T^3 = (\mathbb{R}/2\mathbb{Z}) \times (\mathbb{R}^2/\mathbb{Z}^2),$$

$$[(x_1, \ldots, x_7)] \mapsto (x_1 + 2\mathbb{Z}, x_2 + \mathbb{Z}, x_3 + \mathbb{Z}) \quad (36)$$

Note that in the definition of $q$ we use the local coordinates $\{x_i\}$ on the nilmanifold $M$ defined by (4).

The fibers of $q$ may be considered as a deformation family. By McLean’s theorem the local deformations of a compact coassociative 4-fold $X$ in a 7-manifold with a closed $G_2$-structure form a smooth manifold of dimension $b_+^2(X)$ (cf. section 8). (McLean stated this result for torsion-free $G_2$-structures but his argument only uses the closed condition, as was subsequently observed by Goldstein [24].) A 4-torus has $b_+^2 = 3$, therefore the fibers of $q$ form a maximal deformation family of coassociative 4-folds.
The map $q$ is $\rho$-equivariant (with a natural involution induced by $\rho$ on the image of $q$) and induces a coassociative fibration of the orbifold

$$\tilde{q}: \tilde{M} \to (T^2/\pm 1) \times S^1 \simeq S^2 \times S^1$$

with the $S^2$ factor understood as an orbifold (sometimes referred to as the ‘pillowcase’) homeomorphic to the standard 2-sphere. When $x_1 \in \{0,1\}$ and $x_2 \in \{0, \frac{1}{2}\}$ the fiber of $\tilde{q}$ is a singular orbifold homeomorphic to $S^2 \times T^2$ (with $S^2$ again understood as the ‘pillowcase’).

We next show that the map $\tilde{q}$ lifts to $\tilde{M}$ and induces a coassociative fibration $\tilde{q}$ on $\tilde{M}$, so that there is a commutative diagram

$$\begin{array}{ccc}
\tilde{M} & \xrightarrow{\pi} & \tilde{M} \\
\tilde{q} \downarrow & & \tilde{q} \downarrow \\
S^2 \times S^1 & \xrightarrow{\sim} & (T^2/\pm 1) \times S^1
\end{array}$$

where the horizontal arrows are, respectively, the resolution (blow-up) $\tilde{M} \to \tilde{M}$ and the ‘pillowcase homeomorphism’.

In order to construct the desired $\tilde{q}$, we first deduce from the construction of $\tilde{q}$ that every fiber passing through a singular locus of $\tilde{M}$ has singular points. A neighbourhood of each singular point of this singular fiber is diffeomorphic (in the orbifold sense) to a neighbourhood of $(T^2/\pm 1) \times T^2$ in $(\mathbb{R} \times T^3/\pm 1) \times T^3$, for suitable embeddings $T^2 \to \mathbb{R} \times T^3$ and $T^2 \to T^3$. For example, near the (equivalence class of) the zero vector in $\mathbb{R}^2$ the embeddings are induced by $(x_5, x_6) \mapsto (x_1, x_2, x_5, x_6)$ and $(x_4, x_7) \mapsto (x_3, x_4, x_7)$, where as usual the local coordinates $x_i$ on $\tilde{M}$ correspond to the local coordinates on the compact manifold $M$.

As $\pi: \tilde{M} \to \tilde{M}$ is a diffeomorphism away from the preimage of the singular locus of $\tilde{M}$, it is easy to see that a generic fiber of $\tilde{q}$ will be diffeomorphic to the 4-torus. But there will also be singular fibers.

We can understand the singular fibers of $\tilde{q}$ via the local model of the complex surface $S \to \mathbb{C}^2/\pm 1$ defined by blowing up the singular point of the cone $\mathbb{C}^2/\pm 1$. Here the complex coordinates on $\mathbb{C}^2$ correspond to $\zeta_1 = x_4 + ix_7$ and $\zeta_2 = x_5 + ix_6$ in the above notation. Consider the cone $\mathbb{C}/\pm 1 \subset \mathbb{C}^2/\pm 1$, where $\mathbb{C}$ is understood as a complex line with coordinate $\zeta_2$ in $\mathbb{C}^2$ passing through the origin. The proper transform of this cone is a non-singular complex curve passing through the exceptional divisor on $S$. The inverse image of $\mathbb{C}/\pm 1$ in $S$ is the union of the latter complex curve and the exceptional divisor (which is a copy of $\mathbb{CP}^1$) over the singular point. We find that the lifted fiber in $\tilde{M}$ has a singularity, locally modeled on the intersection of two copies of $\mathbb{R}^4$ along $\mathbb{R}^2$.

We claim

**Lemma 34.** The fibers $X$ of $\tilde{q}$ are coassociative in the $G_2$-structure $\tilde{\varphi}$ on $\tilde{M}$.

**Proof.** Firstly, observe that the defining condition $\tilde{\varphi}|_X = 0$ for a 4-dimensional submanifold $X$ to be coassociative is point-wise and linear in the 3-form $\tilde{\varphi}$. Now recall that at each point in the resolution region in $\tilde{M}$ the $G_2$-structure $\tilde{\varphi}$ is a linear combination of $\tilde{\varphi}$ induced from $\tilde{M}$ and the $G_2$-structure corresponding to the Riemannian product of the 3-torus and
a Ricci-flat Kähler complex surface. The 3-form of former $G_2$-structure vanishes on the fibers of $\tilde{q}$ by the above discussion, and the 3-form on the latter $G_2$-structure vanishes on the fibers because each relevant fiber is a Riemannian product with a special Lagrangian factor (or a complex factor, depending on which complex structure is considered) in the latter complex surface.

□

The structure of a neighbourhood of a singular fiber in $\tilde{M}$ is in fact a suspension over the familiar elliptic fibration of a Kummer $K3$ surface. When one blows up the singular points of $T^4/\pm 1 = (\mathbb{C}^2/(\mathbb{Z} + i\mathbb{Z}))^2/\pm 1$, the proper transform of $T^2/\pm 1 = (\mathbb{C}/(\mathbb{Z} + i\mathbb{Z}))/\pm 1$ is a non-singular complex curve, whereas the inverse image of $T^2/\pm 1$ is a singular complex curve which is an image of a (non-bijective) immersion of the Riemann sphere $S^2$. This immersion takes two distinct points to the same point in the image and is one-to-one elsewhere on $S^2$. We find that, respectively, each singular fiber in $\tilde{M}$ is the image of a non-singular 4-manifold $T^2 \times S^2$ under an immersion, intersecting itself along $T^2$. The singular fibers occur in one-dimensional families parameterized by $S^1$, with coordinate $x_1$ (as in the model example above).

Results of the deformation theory in [38] remain valid for coassociative immersions of compact smooth 4-manifolds. Notice that, as $b_2^+(T^2 \times S^2) = 1$, the latter $S^1$-families of singular fibers are maximal deformation families.

To summarise, we obtain from the above.

**Proposition 35.** The map $\tilde{q}$ defined by the commutative diagram (37) is smooth and its fibers are coassociative 4-folds in $(\tilde{M}, \tilde{\phi})$. Every smooth fiber of $\tilde{q}$ is diffeomorphic to $T^4$. The singular fibers occur in 1-parameter family, of codimension 2 in $\tilde{M}$. The family of the smooth fibers of $q$ and the family of the singular fibers of $q$ are each a maximal family of coassociative deformations.

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