On expressive rule-based logics

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Abstract
We investigate a family of rule-based logics. The focus is on very expressive languages. We provide a range of characterization results for the expressive powers of the logics and relate them with corresponding game systems.

1 Introduction
In this article we introduce and investigate very powerful logics based on rules in the style of Datalog (see, e.g., [3], [8], [2]) and Prolog. The point is then to couple the related languages with the framework developed in [7]. We also investigate the Turing-complete logic defined in [4], [7] and based on game-theoretic semantics. In fact, many of the results obtained below have counterparts in the setting of the Turing-complete logic of [4], [7].

We begin the story by a recap of systems as defined in [7]. We then define a rule-based logic RLO which is tailor-made for classifying finite ordered structures. We show how to capture RE with RLO. The logic RLO is rule-based, with, inter alia, Datalog-style rules and beyond. Computations with RLO are deterministic. We then lift the restriction to ordered structures and investigate RL which we show to capture RE without the assumption of models having a distinguished linear order. The next step is to consider systems with nondeterministic rules. To this end, we define NRL. As an extension of RL it trivially captures RE, but we show a somewhat stronger result relating to model constructions. We also establish an analogous result for a version of the Turing-complete logic from [7]. In fact, a rather similar result has already been established in [7]. We then investigate GRL which is tailor-made for systems as defined in [7].

Concerning the logic RL and its many variants we study, there exist various languages with essentially the same model recognition capacity. These include, inter alia, the while languages discussed in [1] and the Turing complete logic of [4]. However, RL and its variants have quite nice qualities, relating especially to simplicity and flexibility.
of use. Notably, the logics NRL and GRL offer various interesting features for modeling scenarios, thereby having useful properties that go beyond mere recognition. Furthermore, compared to the Turing-complete logic of [4], the variants of RL are inductive whereas the logic of [4] is coinductive.

2 Preliminaries

We denote models by letters of type $A$, $B$, $M$, and so on. The domain of $A$ is denoted by $A$ and similarly for the other letters. For simplicity, we sometimes write $R$ to indicate both the relation $R^A$ and the relation symbol $R$.

For simplicity, models are assumed to have a relational vocabulary (no function or constant symbols). Also, they are assumed finite (with also a finite vocabulary), although it will be easy to see that many of the results below do not really depend upon this assumption. Also, the exclusion of function and constant symbols could be easily avoided by considering partial function symbols. We omit this option indeed for the sake of simplicity. We assume there exists a canonical linear ordering $<_{symb}$ of the full infinite set of all relation symbols. This enables unique binary encodings of models.

The encoding of a model $M$ with respect to a linear ordering $<_{symb}$ is the binary string $\text{enc}_{<_{symb}}(M)$ defined such that it begins with $|M|$ bits 1 followed by a single 0, and after this are the encodings of the relations as follows.

1. The relations are encoded as a concatenation of all the relation encodings, one relation at a time, in the order indicated by $<_{symb}$.
2. For a $k$-ary relation $R^M$, we simply list a bit string of length $M^k$ where the $n$th bit is 1 iff the $n$th tuple (with respect to the standard lexicographic order of $M^k$ defined with respect to the linear ordering $<_{symb}$) is in the relation $R^M$.

We note that this encoding scheme is similar to the one defined in [8]. We also note that obviously $<_{symb}$ does not necessarily need to be in the vocabulary of the model to be encoded. We may write $\text{enc}(M)$ when the linear ordering $<_{symb}$ is known from the context of irrelevant. It is of course obvious that different linear orderings of the model domain are bound to give different encodings.

The logic $\mathcal{L}_{RE}$ [6] consists of sentences of the form $IY \exists X_1 \ldots \exists X_n \psi$ where the part $\exists X_1 \ldots \exists X_n \psi$ is a formula of existential second order logic and $IY$ a new operator (where $Y$ is a unary second-order relation variable). Here $\psi$ is the first-order part. We have $M \models IY \exists X_1 \ldots \exists X_n \psi$ if we can expand the domain of $M$ by a finite set $S$ of new elements such that

$$(M + S, Y \mapsto S) \models \exists X_1 \ldots \exists X_n \psi$$

where $(M + S, Y \mapsto S)$ is the model obtained from $M$ by the following operations.

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3 However, it is not difficult to simulate inductive computations rather directly in the logics of [4],[7] simply by considering the corresponding computation tables (or game graphs of the semantic game). And also the reverse simulation of the logics in [4],[7] is possible.
1. We first extend the domain of $\mathcal{M}$ by the set $S$ of new elements. The relations are kept as they are.

2. We then expand the so obtained model to interpret the new unary symbol $Y$ as the set $S$. (That is, $Y$ names the fresh elements in the domain.)

Let $\tau$ be a vocabulary and consider a class $C$ of finite $\tau$-models. We say that a Turing machine $TM$ defines a semi decision procedure for $C$ if $TM$ accepts a bit string $s$ iff $s = enc_<(\mathcal{M})$ for some $\mathcal{M} \in C$ and some linear ordering $<$ of the domain of $\mathcal{M}$. When not accepting, the machine does not have to halt. A model class is in RE (recursively enumerable) iff there is a Turing machine that defines a semi decision procedure for it.

When considering ordered models, i.e., models where some distinguished predicate $<$ in the vocabulary is always a linear ordering of the domain, we can—even then—use the above definition for semi decision procedures for model classes. However, we can then also use the following clearly equivalent definition: a Turing machine $TM$ defines a semi decision procedure for $C$ if $TM$ accepts a bit string $s$ iff $s = enc_<(\mathcal{M})$ for some $\mathcal{M} \in C$ and the distinguished ordering $<$ of the domain of $\mathcal{M}$. Both definitions result in the same class of semi decidable classes of ordered models.

3 On systems

3.1 Elements of systems

We now consider systems as defined in [7]. Let $\sigma$ be a signature and $A$ a set of actions. Let $I$ be a set of agents. (Technically $A$ and $I$ are simply sets.) Consider a set $S$ of structures over the vocabulary $\sigma$. We note that in one interesting and significant case, $\sigma$ has only unary relation symbols and $S$ is simply a set of states (or points with some local information based on unary predicates, i.e., propositional valuations). Then we will ultimately end up with just a slight generalization of Kripke models. However, it is also instructive to think of the structures in $S$ simply as relational first-order $\sigma$-models in the usual sense of model theory.

Let $T$ denote the set of $(S,A,I)$-sequences; as defined in [7], an $(S,A,I)$-sequence is a finite tuple

$$ (\mathcal{M}_0,a_0,\ldots,\mathcal{M}_n,a_n) $$

where each $\mathcal{M}_i$ is a structure in $S$ and $a_i$ is a tuple of actions in $A^I$. We note that the following generalizations could be possible (but we omit considering them here explicitly).

1. Instead of letting $a_i \in A^I$ be a tuple of actions involving any individual actions from $A$, we can define a function that limits the available actions based on the earlier sequence, meaning that a set $A[i] \subseteq A$ can be determined by $(\mathcal{M}_0,a_0,\ldots,\mathcal{M}_i)$ and then we must have $a_i \in (A[i])^I$.

2. Furthermore, we can let the set of active agents be $I[i] \subseteq I$, similarly determined by $(\mathcal{M}_0,a_0,\ldots,\mathcal{M}_i)$. Then we must have $a_i \in A[i][I[i]]$. 

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3. Yet further, we can make the actions available to each individual agent depend also on the earlier sequence. Formalizing all this is a triviaility. Indeed, we must then have \( a_i \in A[i][i] \) with the additional condition that the \( k \)-th member of \( a_i \) (the action of the \( k \)-th agent in \( a_i \)) must be chosen from \( A[i][k] \subseteq A \) of actions available to agent \( k \) in round \( i \), with \( A[i][k] \) being determined by \( (M_0, a_0, \ldots, M_k) \).

Note that also the empty sequence is a \((S, A, I)\)-sequence.

A **system frame base**, as given in Definition 3.1 of [7], is a pair \((S, F)\) where \( S \) is a set of \( \sigma \)-structures and \( F \) is a function \( F : T \to \mathcal{P}(S) \) where \( T \) is some subset of the set of all sequences in \( T \) (where \( T \) denotes the set of all \((S, A, I)\)-sequences).

According to Definition 3.2 in [7], a **system frame** is a triple \((S, F, G)\) where \((S, F)\) is a system frame base and \( G \) is a function mapping from some set \( E \subseteq T \times \mathcal{P}(S) \) into \( S \cup \{\text{end}\} \) such that when \( G((x, U)) \neq \text{end} \), the condition \( G((x, U)) \in U \) holds. Thus \( G \) is essentially a choice function that chooses the actual next world from the set of possible future worlds chosen by \( F \). It can be interpreted, e.g., as chance or some kind of a grand controller of the system.

Finally, a **system**, as given in Definition 3.3 of [7], is defined as a tuple
\[
(S, F, G, (f_i)_{i \in I})
\]
where \((S, F, G)\) is a system frame and each \( f_i \) is a function \( f : V_i \to A \) with \( V_i \) being some subset of the set \( T' \) of tuples
\[
(M_0, a_0, \ldots, M_k),
\]
that is, tuples that are like \((S, A, I)\)-sequences but with the last tuple of actions (which in the above tuple would be \( a_k \)) removed. Intuitively, \( f_i \) is a strategy that gives an action for the agent \( i \) based on sequences of the above mentioned type, i.e., the type
\[
(M_0, a_0, \ldots, M_k).
\]
In fact, as in [7], we can even identify \( f_i \) with the agent. The agent is the strategy the agent follows. For conveniences, let us call sequences of type
\[
(M_0, a_0, \ldots, M_k)
\]
**structure ended \((S, A, I)\)-sequences**, or simply **structure ended sequences** when \((S, A, I)\) is clear from the context or irrelevant. We note that \( S \) is called the domain of the system, and it should not be confused by the domains of individual models in \( S \).

Systems evolve as given in [7]. However, the next section defines some scenarios with a closer look at constraints on system evolution.

\[\text{Note here that } G \text{ is undefined or outputs } \text{end} \text{ when the set } U \text{ is the empty set.}\]
3.2 Controlling systems

As discussed in [7], it is interesting to consider a framework where the agents do not see the structures \( M \in S \) directly, but a perception of them. In that article, this is realized by defining two functions \( p_i \) and \( d_i \) for each agent \( i \in I \). There are many ways to define \( p_i \) and \( d_i \). Here we define the function \( p_i \) so that it maps from the set of nonempty structure ended sequences in the underlying system to a new set \( P_i \) of models; the models in \( P_i \) can be considered, e.g., as perceived (or perceivable) models for the agent \( i \). Thus \( p_i \) is a some kind of a perception function that gives the current perceived model to an agent. The signature of the models in \( P_i \) need not be similar to the signature of the models in the domain \( S \) of the underlying system.\(^3\)

The function \( d_i \) maps from a set \( M_i \times \langle P_i \rangle \) into the set \( A_i \times M_i \). Here \( M_i \) is simply a set that contains the mental states of the agent \( i \) and \( \langle P_i \rangle \) is the set of finite nonempty sequences

\[
(\mathcal{P}_0, a_0, \ldots, \mathcal{P}_{k-1}, a_{k-1}, \mathcal{P}_k)
\]

that contain perceived models \( \mathcal{P}_j \in P_i \) of the agent \( i \) and action tuples \( a_j \in A^f \) by all the agents (so \( P_i \) is indeed the set of perceivable models of agent \( i \)). And \( A_i \) is the set of actions available to agent \( i \), so in the general case, \( A_i = A \). Note that these sequences end with a structure, so they are quite similar to the structure ended sequences

\[
(\mathcal{M}_0, a_0, \ldots, \mathcal{M}_{k-1}, a_{k-1}, \mathcal{M}_k)
\]

of the system itself. However, in the perception sequences we only have the perceived rather than real models (the real models are the models in the system domain \( S \)). Note that it is very natural to limit \( d_i \) so that it does not depend on the actions of agents other than the agent \( i \). This means that each \( a_j \) in a perception sequence is replaced by the single action \( a_j(i) \) by agent \( i \) in the tuple \( a_j \in A^f \). Then information (of varying quality) about the actions of other agents in the perception sequences can be considered to be encoded, for example, in (if anywhere) the next perceived model \( \mathcal{P}_{j+1} \).

Now, it is highly natural to make \( d_i \) depend only on the current mental state and the last perceived model of \( i \). We call such a \( d_i \) mapping from \( M_i \times P_i \) into \( A_i \times M_i \) a simple \( d_i \). Whether we use a simple \( d_i \) or not, the system evolves as follows.

First we have a system and the functions \( d_i \) and \( p_i \) for each agent \( i \). We also have an initial mental state \( m_{\text{initial}}(i) \in M_i \) for each agent \( i \). The system itself determines an initial structure \( \mathcal{M}_0 \in S \) (or alternatively, we arbitrarily just appoint the structure \( \mathcal{M}_0 \)). Inductively, from any structure ended sequence

\[
(\mathcal{M}_0, a_0, \ldots, \mathcal{M}_k)
\]

we obtain, using the function \( p_i \) for each agent \( i \), the next perceived model \( \mathcal{P}_i \) for the agent \( i \). From there, we use \( d_i \) to determine the new mental state \( m \in M_i \) of the agent

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\(^3\)Often the models \( P_i \) can be just propositional valuations. However, the same also applies to the domain \( S \) of the system, and the scenario where both perceived models and the models in \( S \) are propositional valuations is of course important.
and the action $a_i' \in A_i$ by the agent. For simple $d_i$, we have $d_i : M_i \times P_i \rightarrow A_i \times M_i$ and for a non-simple one $d_i : M_i \times \langle P_i \rangle \rightarrow A_i \times M_i$. (There are of course many relevant variants between simple and general non-simple, e.g., taking into account the full tuple of immediately previous actions by the agents.) With the mental state of each agent $i$ updated, and with an action $a_i' \in A_i$ for each agent determined, we do the following. We build the tuple $a_k \in A^I$ of actions by all agents from the (at this stage known) individual actions $a_i'$ for each agent $i$. Then we add this tuple $a_k$ to the structure ended sequence $(\mathcal{M}_0, a_0, \ldots, \mathcal{M}_k)$ from where we started the description of the inductive step. Then, based on 

$$(\mathcal{M}_0, a_0, \ldots, \mathcal{M}_k, a_k),$$

the system determines (using $F$ and $G$) the new model $\mathcal{M}_{k+1}$. Then we of course repeat the step similarly from the new structure ended sequence

$$(\mathcal{M}_0, a_0, \ldots, \mathcal{M}_{k+1}).$$

A particularly interesting setting (let us call it $\textbf{M-finitary}$) is the one where each $M_i$ is finite. Also the case (call it $\textbf{MP-finitary}$) where each $M_i$ and each $P_i$ is finite is interesting. The case where each $M_i$ and $P_i$ and also the domain $S$ of the system is finite can be called $\textbf{MPS-finitary}$. The $p\textbf{-quasi-finitary}$ case is the one where the range of each $p_i$ is finite (while $S$ need not be); the range of $p_i$ being finite means that $p_i$ is intuitively finitary in the sense that it sees only finitely many (intuitively different) cases that it maps differently. That is, the domain of $p_i$ partitions into finitely many equivalence classes (each class sharing an output) and $i$ sees the inputs in each class as being similar to each other or even indistinguishable from each other. We can even redefine the domain of each $p_i$ to consist of, e.g., finitely many isomorphism classes of some relation $R_i$ interpreted by the models in $S$. The idea is that $R_i$ represents the sphere of (menta, physical, or a combination of those) perception of the agent $i$. The further requirement of $R_i$ having only finitely many tuples could also perhaps be forced.

All in all, at least the following cases should be specially distinguished. The first one is the $\textbf{MP-finitary}$ case where each $d_i$ is simple (in the formal sense defined above), and furthermore, the set $I$ of agents is also finite. Let us call the case $\textbf{1-elementary}$. Note that in the 1-elementary case, we can always assume that $A$ is finite (as the union of the ranges of the functions $d_i$ is finite). The 1-elementary which is also $p\textbf{-quasi-finitary}$ is highly interesting (let us call it $\textbf{2-elementary}$). In the 2-elementary case, if each $p_i$ furthermore depends only on (i.e., the output is always determined by) the current model $\mathfrak{M}$ in the system domain $S$, we call the case $\textbf{elementary}$ and each $p_i$ is called an $\textbf{elementary perception function}$. Obviously then we can simply regard $S$ as the domain of the functions $p_i$. Finally, the elementary case where $S$ is finite can be called $\textbf{strongly elementary}$. The $a\textbf{-elementary}$ case is the 2-elementary case where each $p_i$ depends on the current model and the previous tuple of actions.\(^4\)

\(^4\)Of course in the very beginning, there is no previous tuple of actions, but in the subsequent rounds there is.
strongly a-elementary case is the a-elementary case with the domain $S$ being finite. The following sections contain logics for many different scenarios of system simulation with ideas visioned already in [5].

4 Rule-based logics

In this section we consider rule-based logics. While there are similarities to systems such as, e.g., Datalog variants, there are also various notable differences. We begin by considering ordered models.

4.1 Ordered models

Let $\tau$ be a relational vocabulary that contains a distinguished binary relation symbol $<$ which will always be considered a linear order over the domain of the model investigated. We exclude constant symbols and function symbols from the vocabulary for the sake of simplicity. They could be added however, especially if considering partial function symbols (noting also that partial constant symbols would be interpreted as constants that have at most one reference point in the model domain). Nevertheless, we indeed let $\tau$ be a relational vocabulary here to streamline the exposition. We note that $\tau$ is assumed to be finite (although it will be trivial to see which results would go through for infinite $\tau$). The vocabulary $\tau$ can contain nullary relation symbols. Recall that a nullary symbol $Q$ is interpreted either as $\top$ or $\bot$ (true or false) by a model with $Q$ in the vocabulary.

Let $\tau^+$ be an extended vocabulary, $\tau^+ \supseteq \tau$, where the part $\tau^+ \setminus \tau$ contains “relation symbols” dubbed tape predicates. These are exactly as relation symbols but they are not considered to be part of the underlying vocabulary $\tau$. Tape predicates can be nullary. In the beginning of computation, tape predicates are interpreted as the empty relation (and each nullary tape predicate as $\bot$). On the technical level, we shall mostly try to reserve the terms relation symbol and tape predicate for different and disjoint sets of symbols. Tape predicates are auxiliary and relation symbols part of the input model. However, we can of course define models that interpret tape predicates as if they were relation symbols.

A transformation rule of the first kind is a construct of the form

$$X(x_1, \ldots, x_k) : - \varphi(x_1, \ldots, x_k)$$

where $X \in \tau^+$ is a $k$-ary symbol and $\varphi(x_1, \ldots, x_k)$ is a first-order $\tau^+$-formula whose set of free variables is precisely $\{x_1, \ldots, x_k\}$. We note, especially with the reader familiar with Datalog and similar languages in mind, that $X$ can but does not have to be a tape predicate, it can be any symbol in $\tau^+$. We note also, concerning the variables $x_1, \ldots, x_k$, that the variables do not have to be pairwise distinct. The symbol $X$ is called the head symbol of the rule, and the left-hand side formula $X(x_1, \ldots, x_1)$ simply the head of the

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5The set of relation symbols and tape predicates in a $\tau^+$-formula is required to be some subset of $\tau^+$. A $\tau^+$-formula may also be called a formula in the vocabulary $\tau^+$.  

rule. The right-hand side formula \( \varphi(x_1, \ldots, x_k) \) is the **body** of the rule. Transformation rules of the first kind will also be called **1-transformers**. We stress that the formula \( \varphi(x_1, \ldots, x_k) \) can indeed be any first-order formula within the given constraints: it does not have to be free or negations or quantifiers or anything like that.

Now, let \( \mathfrak{M} \) be a \( \tau^+ \)-model. Consider a 1-transformer \( F \) of the form

\[
X(x_1, \ldots, x_k) : - \varphi(x_1, \ldots, x_k).
\]

We let \( F \) be the operator such that

\[
F(F, \mathfrak{A}) = \{ (a_1, \ldots, a_k) \in A^k \mid \mathfrak{A}, (x_1 \mapsto a_1, \ldots, x_k \mapsto a_k) \models \varphi(x_1, \ldots, x_k) \}
\]

where \( (x_1 \mapsto a_1, \ldots, x_k \mapsto a_k) \) is the assignment that maps \( x_i \) to \( a_i \) for each \( i \in \{1, \ldots, k\} \).

Therefore, \( F \) is the operator that takes any 1-transformer \( F \) and model \( \mathfrak{A} \) (where the symbols in \( F \) are in the vocabulary) as an input and gives the relation determined by the rule body as the output. To put this shortly, the operator \( F \) evaluates the rule \( F \) on the input model; the evaluation process is similar to the one in, e.g., Datalog. As in Datalog, we can use \( F \) to update \( \mathfrak{A} \) by replacing the relation \( X^{\mathfrak{A}} \) corresponding to the head symbol by the relation \( F(F, \mathfrak{A}) \).

A **transformation rule of the second kind**, or a **2-transformer**, is the construct denoted by

\[
I
\]

which simply adds a single domain element to the current model and extends the relation < so that the new element becomes the last element in the order. Other relations are kept as they are. A **conditional 2-transformer** is a rule of the form

\[
I : - \varphi
\]

where \( \varphi \) is a first-order sentence in the vocabulary \( \tau^+ \). The interpretation is that if \( \varphi \) holds, then we extend the domain (in the same way as the rule \( I \) does), and otherwise we do not extend the domain, we just move on. A **transformation rule of the third kind**, or a **3-transformer**, is a rule of the form

\[
D : - \psi(x)
\]

where \( \psi(x) \) is a first-order formula in the vocabulary \( \tau^+ \) and with a single free variable, \( x \). The rule deletes from the model \( \mathfrak{A} \) precisely all elements \( a_0 \) such that we have \( \mathfrak{A}, (x \mapsto a_0) \models \psi(x) \). One can also easily define natural conditional 1-transformers and 3-transformers, with the idea that whether or not they are executed depends on an additional first-order sentence. If the first-order sentence holds, we execute the rule, and if the first-order sentence fails to hold, we just move on without changing the model.

A **control rule** is a rule of the form

\[
k
\]

\[^6\text{The set of relation symbols and tape predicates that a } \tau^+ \text{-model interpretes is precisely } \tau^+.\]
where \( k \) is a positive integer written in binary. The rule asserts that we should go execute the rule number \( k \) if such rule exists (i.e., we jump to the rule \( k \) if possible). If such rule does not exist, the computation halts. We shall define later on how rule numbers are used exactly. A conditional control rule is a rule of the form

\[ k : -\chi \]

where \( k \) is a positive integer written in binary and \( \chi \) is a first-order sentence in the vocabulary \( \tau^+ \). The rule states that if \( \chi \) holds, then we should go to execute the rule number \( k \). If \( \chi \) does not hold, we move on to the next rule (and if there is no next rule, the computation halts). If there is no rule \( k \) at all but \( \chi \) holds, then the computation halts.

A program is a finite sequence of rules, i.e., a list of the form

\[ 1 : F_1 \\
\vdots \\
k : F_k \]

where each \( F_i \) is a rule (please see an example in the proof of Theorem 4.1 below). We begin each line with a number (the rule number) and a colon. The rule number—officially written in binary—helps in using control rules. However, the rule number and colon can of course be dropped, as they increase in the obvious way, beginning with 1. The program is executed one rule at a time (unlike Datalog), starting from rule 1 and proceeding from there: if we are at rule \( F_i \), and it is not a control rule, we first execute the rule and then move on to the next rule below the current one. Also, if the last rule is executed (and it is not a control rule leading to a jump to some existing rule), then the computation ends after that. Control rules allow for jumps that do not necessarily proceed in the way indicated by the rule numbers. If a control rule leads to a rule number (i.e., line number) for which there is no rule, the computation ends. Recall that the transformer rules transform the model in the way described above, so the model typically changes as the computation progresses.

Consider a program \( \Pi \) where the set of relation symbols is \( \sigma_0 \) and tape predicates \( \sigma_1 \) (obviously \( \sigma_0 \cap \sigma_1 = \emptyset \)). Let \( \mathcal{M} \) be a model whose vocabulary is \( \tau \supseteq \sigma_0 \). (Note that \( \tau \cap \sigma_1 = \emptyset \).) Computation with the input \( \mathcal{M} \) then proceeds as described above, starting with the expansion of \( \mathcal{M} \) to the vocabulary \( \tau^+ = \tau \cup \sigma_1 \) such that tape predicates are interpreted as empty relations (and \( \perp \) for nullary tape predicates). We call this expansion the \( \Pi \)-expansion of \( \mathcal{M} \).

Now, consider a scenario where \( \sigma_1 \) contains the nullary tape predicate \( X_{\text{true}} \). We consider this a special tape predicate and write \( \mathcal{M} \models \Pi \) for a model of vocabulary \( \tau \) and a program of vocabulary \( \tau^+ = \tau \cup \sigma_1 \) if the computation beginning with the the \( \Pi \)-expansion of \( \mathcal{M} \) ultimately halts such that \( X_{\text{true}} \) holds (is equal to \( \top \)) in the final model at halting.

We call RLO (for rule logic with order) the system consisting of programs with 1-transformers, 2-transformers and conditional control rules, as described above. Conditional transformer rules are not included. Note that non-conditional control rules can
of course be simulated with conditional control rules. We say that RLO defines a class of \(\tau\)-models \(C\) if there is a program \(\Pi\) such that for all \(\tau\)-models \(\mathfrak{M}\), we have \(\mathfrak{M} \models \Pi\) iff \(\mathfrak{M} \in C\). Note here that we of course restrict attention to finite models only. The same definition of definability also applies to other logics we shall consider in this article.

**Theorem 4.1.** Let \(\tau\) be a vocabulary with \(<\) and limit attention to ordered \(\tau\)-models. Then, RLO can define a class \(C\) of \(\tau\)-models iff \(C\) is in \(\text{RE}\).

**Proof.** It is clear that computations with RLO can be simulated by a Turing machine. For the other direction, suppose a Turing machine \(TM\) recognizes some class \(C\) of ordered \(\tau\)-models. Now recall from the preliminaries the logic \(L_{\text{RE}}\) that can define precisely the semi-decidable classes of models. Therefore there is a sentence \(IY\exists X_1 \ldots \exists X_n \varphi\) of \(L_{\text{RE}}\) that defines \(C\) with respect to the class of all finite ordered \(\tau\)-models. Now notice that for any formula \(\exists Z_1 \ldots \exists Z_m \psi\) of existential second-order logic (with \(\psi\) being first-order), there clearly exists an equivalent formula \(\exists Z^\beta\) (with \(\beta\) being first-order) where the arity of \(Z\) is the sum of the arities of the predicates \(Z_1, \ldots, Z_m\). Now, let \(\exists Z\psi'\) (with \(\psi'\) being first-order) be an existential second-order the formula equivalent to \(\exists X_1 \ldots \exists X_n \varphi\), and suppose that the sum of the arities of the relation variables \(X_1, \ldots, X_n\) is \(k\). Thereby the arity of \(Z\) is \(k\). Note that now \(IY\exists Z\psi'\) is equivalent to the formula \(IY\exists X_1 \ldots \exists X_n \varphi\) of \(L_{\text{RE}}\) that defines \(C\). We will next write a program that is equivalent to \(IY\exists Z\psi'\). Note that \(Y\) and \(Z\) will of course be tape predicates, and the program will also use some other tape predicates.

Let \(\text{step}_Z(x_1, \ldots, x_k)\) be the first-order formula whose interpretation over any ordered model \(\mathfrak{M}\) (which interprets \(Z\)) is the relation \(R \subseteq N^k\) such that the binary encoding of \(R\) is the string that points to the integer \(z\) that is one larger than the integer \(z'\) that \(Z^\mathfrak{M}\) points to. If \(z'\) is already the maximum integer, then \(R\) will simply be equal to \(Z^\mathfrak{M}\). The binary encoding here for relations is of course the one described in the preliminaries. The formula \(\text{step}_Z(x_1, \ldots, x_k)\) is routine to write using \(<\).

Next, let \(\text{max}(Z)\) be the first-order formula which states that \(Z\) is the total relation over the current domain. Note that this is equivalent to the binary encoding of \(Z\) being the maximum string (containing only bits 1) with respect to all bit strings of length \(d^k\) where \(d\) is the size of the current domain. The required program is as follows. Note that for readability, none of the indices are in binary.

\[
\begin{align*}
1 & : X_{\text{true}} : - \psi' \\
2 & : 365 : - X_{\text{true}} \\
3 & : 6 : - \text{max}(Z) \\
4 & : Z(x_1, \ldots, x_k) : - \text{step}_Z(x_1, \ldots, x_k) \\
5 & : 1 \\
6 & : I \\
7 & : Y(x) : - Y(x) \lor \neg \exists y(x < y) \\
8 & : Z(x_1, \ldots, x_k) : - \bot \\
9 & : 1 \\
\end{align*}
\]

The program tests if \(\psi'\) holds, and if not, it modifies \(Z\) to be the next relation with respect to the lexicographic ordering of \(k\)-tuples defined with respect to \(<\). Once all
relations $Z$ have been tested, the domain is extended and $Z$ is set to be the empty relation. The procedure is then repeated.

Note that we can clearly add all the conditional rules and 3-transformers (and also a conditional 3-transformer), and the resulting system will still define precisely the classes of ordered models in RE. Indeed, we only need to prove that the stronger system can be simulated by a Turing machine, and this is clear. And we can do even more, of course.

A conditional rule tuple is a construct of the form

$$(\text{If } \varphi_1 \text{ then } F_1, \text{ else if } \varphi_2 \text{ then } F_2, \ldots, \text{ else if } \varphi_{k-1} \text{ then } F_{k-1}, \text{ else } F_k)$$

where each $\varphi_i$ is a first-order $\tau^+$-sentence and $F_i$ is a rule (of any kind discussed above), and we have $k \geq 1$, so singleton tuples are allowed. A conditional rule tuple will occupy a line in a program just like the rules above. For example, if $C$ is a conditional rule tuple, then for example the line 6 in the program could be of type

6 : $C$

A conditional rule tuple is interpreted in the most obvious way as follows: take the first precondition formula $\varphi_i$ that holds and then execute rule $F_i$. If no precondition rule holds, execute the last rule $F_k$. After executing that one rule $F_i$, do the following.

1. If the rule $F_i$ instructed to jump to line $m$, continue from that line. If the line $m$ does not exist, the computation stops.

2. If the rule $F_i$ did not instruct to jump, go to the next line after the current conditional rule tuple (having of course executed $F_i$ already). If that next line does not exist (meaning we are at the last line of the program), the computation stops.

A parallel rule is a tuple of the form $(G_1, \ldots, G_k)$ where each $G_i$ is a conditional rule tuple. A parallel rule is executed as follows (where we—at first—assume no deadlocks arise).

First note that each conditional rule tuple $G_i$ determines one rule $F_i$ to be executed. These rules $F_1, \ldots, F_k$ (one rule for each $G_i$) are executed in parallel as follows. We first execute the transformation rules $F_i$ (simultaneously, based on the current $\tau^+$-model $\mathcal{M}_{\text{current}}$) that do not involve deleting or adding domain points (so no $I$ or $D$ in the syntax). Being transformation rules, these rules do not involve jumps either. This way we obtain a model $\mathcal{M}_1$. Then we do the rules $F_i$ with deletions ($D$ appears in the rule), but not based on $\mathcal{M}_1$ but the model $\mathcal{M}_{\text{current}}$ instead. We end up with the variant of $\mathcal{M}_1$ that has the points to be deleted indeed removed. Note that we remove the union of the points that the successful deletion rules instruct to be deleted, and if there are conditional deletion rules, the condition is evaluated against $\mathcal{M}_{\text{current}}$. Then we do the additions (one point per successful addition rule; again if the rule is conditional, the condition is evaluated with respect to $\mathcal{M}_{\text{current}}$). The model after the additions is the

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7A singleton tuple is just a rule $F_1$. 

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new model $\mathcal{M}_{\text{new}}$ that the parallel rule constructs. If deadlocks arise at the above stages (meaning that at least two rules would treat some head predicate $X$ differently), the computation simply halts without the model being modified at all, i.e., with $\mathcal{M}_{\text{current}}$. Note that deletion and addition rules cannot lead to deadlocks. Finally, after the now described modification step (if the computation did not lead to a deadlock), we check for control rules in $(F_1, \ldots, F_k)$. If there are no control rules in $(F_1, \ldots, F_k)$, we continue from the next line after the parallel rule; if there is no next line, the computation ends with $\mathcal{M}_{\text{new}}$ being the final model. If there are control rules among $(F_1, \ldots, F_k)$, we first compile a list $L$ of all line numbers where we should jump, with conditional control rules evaluated based on $\mathcal{M}_{\text{current}}$. If there are different numbers in $L$, this is a deadlock, and the computation halts (in this case without the modifications, i.e., with $\mathcal{M}_{\text{current}}$ being the final model). If there is a single jump instruction in $L$, we jump to the corresponding line and continue from there. If that line does not exist, the computation ends with the modified model $\mathcal{M}_{\text{new}}$.

Note that we can very naturally define parallel rules based on transformation rules only. Deadlocks can always be avoided by using different head symbols in each collection of possibly parallel actions. By this we mean forbidding the use of the same symbol as a head symbol in different simultaneous conditional rule tuples $G_i$. By using different nullary head symbols in the parallel actions, we can even directly simulate jump rules; such a parallel rule is then followed by jump rules specifying how to jump based on the nullary predicates.

All the above rules can be added to RLO and we can still simulate the resulting logic with a deterministic Turing machine.

We next turn to the case without order. Most of the ideas and notions will be carried over to the following subsection relatively directly.

### 4.2 Without order

Above we investigated the case of models with an order. However, an order is not really required. Consider the syntax of RLO (without the assumption of order). Redefine RLO such that where we previously had a first-order formula, we can now use a formula of existential second-order logic. Call the resulting logic RL (rule logic). The programs are run as those of RLO. However, models do not have any distinguished order predicate in them. Note that we can still investigate ordered models with RL, but then $<$ does not automatically update itself to a linear order when the model domain is extended. Instead, now $<$ is treated as other predicates. Indeed, $I$ just adds a point, and no order relation is extended.

**Theorem 4.2.** For any $\tau$, RL can define a class $C$ of $\tau$-models iff $C$ is in RE.

**Proof.** Simulating RL with a Turing machine is easy. For the other direction, we again use $\mathcal{L}_{\text{RE}}$ which defines precisely all RE-classes of models (whether or not a distinguished order is present). We need to find a program equivalent to a sentence $IY\exists X_1 \cdots \exists X_n \psi$.
where $\psi$ is first-order. The following does that.

\begin{align*}
1 & : X_{\text{true}} & : - \exists X_1 \ldots \exists X_n \psi \\
2 & : 365 & : - X_{\text{true}} \\
3 & : X_{\text{domain}}(x) & : - x = x \\
4 & : I & \\
5 & : Y(x) & : - Y(x) \lor \neg X_{\text{domain}}(x) \\
6 & : 1 &
\end{align*}

To add nondeterminism to RL, we introduce the following rules.

\[ \exists X \]

where $X$ is a relation symbol or tape predicate (of any arity) and

\[ \exists (k_1, \ldots, k_n) \]

where $(k_1, \ldots, k_n)$ is a tuple of positive integers. The rule $\exists X$ is executed such that we nondeterministically choose an interpretation for $X$. (The old interpretation of $X$ is overridden.) The rule $\exists (k_1, \ldots, k_n)$ is executed such that we nondeterministically jump to one of the rule numbers $k_1, \ldots, k_n$ (that is, we jump to a line with the chosen number). More rigorously, we nondeterministically choose one of $k_1, \ldots, k_n$ and then attempt to jump to the line with that number. If such a rule (i.e., line) exists, we continue from there. If not, the computation ends. We can also allow for the case $\exists ()$ where () is the empty tuple. This simply terminates the computation.

We call NRL (where $N$ stands for nondeterminism) the extension of RL with non-conditional 3-transformers and the rules of the two kinds above, that is, rules of type $\exists X$ and $\exists (k_1, \ldots, k_n)$. As already done in [7], we also study constructions. Consider the class $\mathcal{C}$ of all finite $\tau$-models. We say that $\mathcal{R} \subseteq \mathcal{C} \times \mathcal{C}$ is an RE-construction if there exists a possibly nondeterministic Turing machine $TM$ such that the following holds: we have $(\mathfrak{M}, \mathfrak{N}) \in \mathcal{R}$ iff with some input $\text{enc}_{<}(\mathfrak{M})$ for some $<$ there exists some computation such that $TM$ halts in an accepting state such that we have $\text{enc}_{<'}(\mathfrak{N})$ for some $<'$ on the output tape at halting. Note that there are many natural equivalent formulations of the notion. We say that NRL can compute $\mathcal{R} \subseteq \mathcal{C} \times \mathcal{C}$ if there exists a program $\Pi$ such that the following holds: the program $\Pi$ can halt on the input $\mathfrak{M} \in \mathcal{C}$ with $X_{\text{true}}$ holding and the current $\tau$-model at halting being $\mathfrak{N} \in \mathcal{C}$ iff we have $(\mathfrak{M}, \mathfrak{N}) \in \mathcal{R}$. Note that “can halt” here of course means that there exists a favourable computation under the different possibilities allowed by the available nondeterminism. Note also that we do not care about tape predicates when considering what the input and output models are: the models $\mathfrak{M}$ and $\mathfrak{N}$ are $\tau$-models, even though during computation we modify $\tau^+$-models that take into account tape predicates.

\footnote{Of course similar rules $\forall X$ and $\forall (k_1, \ldots, k_n)$ could be defined to allow for an obvious way to include alternation into the picture.}
The following is a rather trivial variant of Theorem 4.2, now with the nondetermi-
nistic logic NRL. In the following, $C$ is the class of all finite $\tau$-models where $\tau$ is any finite relational vocabulary.

**Theorem 4.3.** NRL can compute $R \subseteq C \times C$ iff $R$ is an RE-construction.

**Proof.** This is a trivial variant of Theorem 4.2. Consider $\tau$-models. Note that $R$ is an RE-construction iff there is a Turing machine that recognizes (in RE) the class $D$ of $(\tau \cup \{\sim\} \cup P)$-models defined as follows.

1. All models in $D$ consist of a disjoint union of two models $A \in C$ and $B \in C$. The additional binary relation $\sim$ is an equivalence relation with two equivalence classes, one class consisting of the domain of $A$ and the other one of the domain of $B$. The unary predicate $P$ is true on precisely the elements belonging to one (but not the other) equivalence class. (Intuitively, the unary predicate $P$ denotes which model is the input model.) Thus $D$ encodes the pairs $(C, D)$ where $P$ denotes the first member $C$ of the pair.

2. We have $(A, B) \in R$ iff $D$ contains a model that represents $(A, B)$.

To compute $R$ with NRL, we write a program that first creates, when the input model is $A$, some model that encodes a pair $(A, B)$ (see the above description). This construction is done non-deterministically, with the possibility to construct any finite $\tau$-model whatsoever to represent $B$. Then we use Theorem 4.2 and the fragment RL to recognize the model class $D$ as described above, that is, our program then makes true the predicate $X_{true}$ and “halts” iff the model encoding $(A, B)$ is in $D$. Note that “halts” here means that we first make an auxiliary predicate $X_{halt}$ true and then continue the computation as follows. When $X_{halt}$ has become true, we make sure that only $B$ remains as an output model (the tape predicates used in the computation do not count, only the relation symbols in $\tau$). For this we use the deletion operator as one of the constructs. After this we halt.

It is also easy to show that RL with 3-transformers added can compute a partial function $R \subseteq C \times C$ iff $R$ is a partial function that is recursively enumerable such that there is Turing machine for the partial function $R$. This latter condition means that there is a deterministic Turing machine $TM$ such that given any $A \in C$ and any $<$, the machine $TM$ halts on the input $enc_<(A)$ iff there exists a model $B$ such that $(A, B) \in R$, and furthermore, the output on halting is then $enc_{<'}(B)$ for some $<'$.

Now, the statement corresponding to Theorem 4.3 holds also for the Turing-complete logic $L[;]$ as defined in [7]. We write $(M, \varphi, N)$ if Eloise has a winning strategy in the semantic game beginning with $M$ and $\varphi$ and with Eloise being the verifier and with the general assignment function being empty, and furthermore, that winning strategy always leads to Eloise winning so that the current model at the time of winning is $N$. Note that tape predicates (encoded in the general assignment) and first-order variables

\[\text{We could even require that } X_{true} \text{ is } X_{halt}.\]
(also in the general assignment) do not count towards what the final model \( \mathcal{M} \) looks like, only the relations in the signature \( \tau \) of the models \( \mathcal{M} \) and \( \mathcal{N} \) count. The winning strategy can be assumed positional (this clearly makes no difference due to the positional determinacy of reachability games even on infinite arenas). The winning strategy could even be assumed finite (which is typical in the logic \( \mathcal{L}[;] \) and its relatives in [4]), as by König’s lemma, in every semantic game where Eloise has a winning strategy, we can find a bound on how many rounds a play can last before a win occurs. This follows due to the game-tree being finitely branching.

We say \( \mathcal{L}[;] \) can compute \( \mathcal{R} \) if there is a formula \( \varphi \) of \( \mathcal{L}[;] \) such that we have \((\mathcal{M}, \varphi, \mathcal{N}) \) iff \((\mathcal{M}, \mathcal{N}) \in \mathcal{R}\).

The following has a similar counterpart proven already in [7]. Here \( \mathcal{C} \) is the class of all finite \( \tau \)-models for any finite relational vocabulary \( \tau \).

**Theorem 4.4.** \( \mathcal{L}[;] \) can compute \( \mathcal{R} \subseteq \mathcal{C} \times \mathcal{C} \) iff \( \mathcal{R} \) is an RE-construction.

**Proof.** The proof is almost identical to the proof of the above Theorem. Moreover, the technicalities of the argument have essentially already been given in [7].

So, firstly, simulating \( \mathcal{L}[;] \) by an alternating Turing-machine is straightforward, and we can turn this simulation so that it runs with a nondeterministic machine of course. For the other direction, simulating a Turing-machine with \( \mathcal{L}[;] \), we write a formula \( \varphi \) of \( \mathcal{L}[;] \) to do as described in the proof of Theorem 4.3. Indeed, suppose an input model \( \mathfrak{A} \) is given. The formula lets Eloise construct the model \( \mathfrak{S} \) corresponding to \((\mathfrak{A}, \mathfrak{B})\) (here anything can potentially be constructed as model \( \mathfrak{B} \) by Eloise). Then the formula allows Eloise to enter to play the game with Abelard to check whether \( \mathfrak{S} \) belongs to \( \mathcal{D} \). This part can be done by the Turing-completeness of \( \mathcal{L}[;] \). After winning this game, Eloise simply should still make sure, using deletion operators, that the output model is \( \mathfrak{B} \). Tape predicates and first-order variables do not count towards what the output (or input) model is.

In the above construction, the composition connective \( ; \) is used to make sure Abelard cannot end the game by losing at some early stage before the constructions are ready. To put this shortly, Abelard cannot stop the constructions by losing the game intentionally too early in the play. \( \Box \)

### 4.3 A general setting

Consider the following transform rules (with \( \varphi \) and \( \psi(x) \) allowed to be written in existential second-order logic).

1. \( \lambda x_1, \ldots, x_k : \neg \varphi \)
2. \( I \)
3. \( \lambda : \neg \psi(x) \)
4. \( \exists \lambda \)

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A conditional transformer tuple is a conditional rule tuple

\[(\text{If } \varphi_1 \text{ then } F_1, \text{ else if } \varphi_2 \text{ then } F_2, \ldots, \text{ else if } \varphi_{k-1} \text{ then } F_{k-1}, \text{ else } F_k)\]

where each rule $F_i$ is a rule of the above four kinds listed, that is, of type

1. $X(x_1, \ldots, x_k) : - \varphi$
2. $I$
3. $D : - \psi(x)$
4. $\exists X$

where we allow the formulae $\varphi$ and $\psi(x)$ to be formulae of existential second-order logic. Furthermore, any of the sentences $\varphi_i$ in the conditional rule tuple can be a sentence of existential second-order logic. The case $k = 1$ is allowed, and then the conditional transformer tuple is just a rule $F_1$. A non-deterministic transformer is a tuple

\[(C_1, \ldots, C_m)\]

where each $C_i$ is a conditional transformer tuple. The idea is to consider $\langle C_1, \ldots, C_m \rangle$ as a rule such that when executed, we nondeterministically pick one $C_i$ and execute it. A parallel transformer is a tuple $\langle T_1, \ldots, T_n \rangle$ where each $T_i$ is a non-deterministic transformer. These are executed as follows. Beginning from

\[\langle T_1, \ldots, T_n \rangle = \langle (C_{1,1}, \ldots, C_{1,m_1}), \ldots, (C_{n,1}, \ldots, C_{n,m_n}) \rangle,\]

we get as output, using non-determinism (which can be guided by $n$ different agents or $n$ deterministic strategies, possibly encoded by $n$ automata), a tuple $\langle C_{1,j_1}, \ldots, C_{n,j_n} \rangle$ of conditional transformer tuples. The tuple is obtained such that each agent $k \in \{1, \ldots, n\}$ chooses $C_{k,j_k}$ from $\langle C_{k,1}, \ldots, C_{k,m_k} \rangle$. The output tuple $\langle C_{1,j_1}, \ldots, C_{n,j_n} \rangle$ turns into a tuple

\[(F_1, \ldots, F_n)\]

of rules, where $F_k$ is determined based on $C_{k,j_k}$ and its internal structure. Note that we are thus playing a game where each agent $k$ makes the nondeterministic choice from $\langle C_{k,1}, \ldots, C_{k,m_k} \rangle$. The obtained rule tuple $\langle F_1, \ldots, F_k \rangle$ is then executed in parallel as described above when defining the way parallel rules are treated. Note that if a deadlock is obtained, the computation ends without the current model being modified. A parallel transformer is considered a single line of code. It relates to the agents making a parallel choice.\[^{[10]}\]

\[^{[10]}\]Note that we can guide the computation line flow with parallel transformers as well if we (1) encode nullary predicates to be modified in the parallel transformer and then (2) write further rules (in subsequent lines) that choose the outcome line to be executed based on the results of the transformation. Of course we can even define a general parallel rule where we can directly obtain also control rules, not only transform rules.
Let a **conditional flow control rule** be a tuple of the form

\[(\text{If } \varphi_1 \text{ then } p_1, \text{ else if } \varphi_2 \text{ then } p_2, \ldots, \text{ else if } \varphi_{k-1} \text{ then } p_{k-1}, \text{ else } p_k)\]

where each \(p_i\) is a nondeterministic control rule of type \(\exists(k_{i1}, \ldots, k_{i\ell_i})\) as defined above. The case \(k = 1\) is of course allowed, being the case where the flow control rule is just a single rule \(\exists(k_{i1}, \ldots, k_{i\ell_i})\).

Let GRL denote the logic where we have all parallel transformers and conditional flow control rules. Let **sorted** GRL be the logic where each line in each program has one of two labels: \(A\) or \(G\), for agents and general controller. These lines are also called \(A\)-lines and \(G\)-lines. The \(A\)-lines are parallel transformers \((T_1, \ldots, T_m)\) where \(m\) is the same for each \(A\)-line of the program (the number of agents). \(G\)-lines are conditional flow control rules or lines of the following types.

1. \(X(x_1, \ldots, x_k) : \neg \varphi\)
2. \(I\)
3. \(D : \neg \psi(x)\)
4. \(\exists X\)

where all the formulae can be in existential second-order logic.

The point of sorted GRL is that we guide systems as defined in the beginning of the article. The parallel transformers are guided by agents, \((T_1, \ldots, T_m)\) being a tuple for \(m\) agents. The other rules are controlled by the general controller \(G\). What the agents are trying to achieve can be specified in many ways, depending on the modelling purpose. However, one scenario is that the agents are jointly trying to make the system halt with \(X_{true}\) holding. We say that the agents have a winning strategy with the input model \(M\) if there exist functions \(f_1, \ldots, f_m\) that give the choices for nondeterminism in parallel transform rules in computations beginning with \(M\). When the functions \(f_1, \ldots, f_m\) are followed, then every computation leads to the system halting with \(X_{true}\) holding. However, this is just a reachability game. Many other settings are interesting. It is also of utmost interest to limit the domains of \(f_1, \ldots, f_m\). For example, we could make each \(f_i\) depend only on some single predicate \(R_i\), conceived as the range of (physical or even perhaps mental) perception (or horizon) of agent \(i\).

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