Fast Robust Subspace Tracking via PCA in Sparse Data-dependent Noise

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Abstract—This work studies the robust subspace tracking (ST) problem. Robust ST can be simply understood as a (slow) time-varying subspace extension of robust PCA. It assumes that the true data lies in a low-dimensional subspace that is either fixed or changes slowly with time. The goal is to track the changing subspaces over time in the presence of additive sparse outliers and to do this quickly (with a short delay). We introduce a “fast” mini-batch robust ST solution that is provably correct under mild assumptions. Here “fast” means two things: (i) the subspace changes can be detected and the subspaces can be tracked with near-optimal delay, and (ii) the time complexity of doing this is the same as that of simple (non-robust) PCA. Our main result assumes piecewise constant subspaces (needed for identifiability), but we also provide a corollary for the case when there is a little change at each time.

A second contribution is a novel non-asymptotic guarantee for PCA in linearly data-dependent noise. An important setting where this is useful is for linearly data dependent noise that is sparse with support that changes enough over time. The analysis of the subspace update step of our proposed robust ST solution uses this result.

I. INTRODUCTION

Principal Components Analysis (PCA) is one of the most widely used and well studied dimension reduction techniques. It is solved via singular value decomposition (SVD) following by retaining the top $r$ singular vectors for getting an $r$-dimensional subspace approximation. Robust PCA (RPCA) refers to PCA in the presence of outliers. According to [4], it can be defined as the problem of decomposing a given data matrix into the sum of a low-rank matrix (true data) and a sparse matrix (outliers). The column space of the low-rank matrix then gives the desired principal subspace (PCA solution). A common application of RPCA is in video analytics in separating a video into a slow-changing background image sequence (modeled as a low-rank matrix) and a foreground image sequence consisting of moving objects or people (modeled as a sparse matrix) [4]. The RPCA problem has been extensively studied in the last decade since [4], [5] introduced the principal components pursuit solution and obtained the first guarantees for it. Follow-up work by Hsu et al [6] studied it further. Later work [7]–[9] has developed provable non-convex solutions that are much faster. Alternating Projections or AltProj was the first such approach [7].

Contributions. This work has two contributions. (1) First, we introduce a “fast” mini-batch robust ST solution that is provably correct under mild assumptions. Here “fast” means two things: (i) the subspace changes can be detected and the subspaces can be tracked with near-optimal delay (the number of data samples required to track an $r$-dimensional subspace of $\mathbb{R}^n$ to $\varepsilon$ accuracy is within log factors of $r$); and (ii) the time complexity of doing this is just $O(ndr \log(1/\varepsilon))$, which is, order-wise, the same as that of solving the basic (non-robust) PCA problem for an $n \times d$ matrix. Our main result assumes piecewise constant subspaces (needed for identifiability), but we also provide a corollary for the case when there is a little change at each time. (2) Our second contribution is a novel

Robust Subspace Tracking (ST) can be simply understood as a (slow) time-varying subspace extension of RPCA. It assumes that the true data lies in a low-dimensional subspace that is either fixed or changes slowly with time. We focus on slow changing subspaces because it is not clear how to distinguish the effect of a sudden subspace change from that of an outlier. The goal is to track the changing subspaces over time in the presence of additive sparse outliers and to do this quickly (with a short delay). Time-varying subspaces is a more appropriate model for long data sequences, e.g., long surveillance videos, since if a single subspace model is used, the resulting matrix may not be sufficiently low-rank. Moreover the tracking setting (short tracking delay) is needed for applications where near real-time estimates are needed, e.g., video-based surveillance (object tracking), monitoring seismological activity, or detection of anomalous behavior in dynamic social networks. While many heuristics exist for robust ST, e.g., [10]–[16], there has been little work on provably correct solutions [17], [18]. The first result [17] needed many restrictive assumptions (most importantly it required assumptions on intermediate algorithm estimates) and a large tracking delay (the delay was proportional to $1/\varepsilon^2$ to get a $\varepsilon$ accurate estimate). The second one [18] significantly improved upon [17], but still required a very specific model on subspace change, needed an $\varepsilon$-accurate initial subspace estimate in order to guarantee $\varepsilon$-accurate recovery at later time instants, and its tracking delay was $r$-times sub-optimal. Our work builds on [18] and removes these, and two other more technical, limitations that we explained later.

Parts of this manuscript have appeared in ICML 2018 [1], ISIT 2018 [2] and Allerton 2017 [3].
non-asymptotic guarantee for PCA in data-dependent noise that satisfies certain simple assumptions. An important setting where these hold is for linearly data dependent noise that is sparse with enough support changes over time. This problem occurs in the subspace update step of our proposed robust ST solution. The PCA result is also of independent interest. As an example, it is useful for analyzing PCA and subspace tracking with missing data [19].

Organization. We first summarize our notation and then provide a brief discussion of the significance of our PCA guarantee and how it is used in analyzing our robust ST solution next. In Sec. II, we present the result for PCA in data-dependent noise and its corollary for the sparse data-dependent noise case. In Sec. III, we define the robust ST problem, state the assumptions required to ensure its identifiability, and provide and discuss the correctness guarantee for it. Related work is discussed in (NORST) algorithm for solving it, and provide and discuss the non-asymptotic guarantee for PCA in data-dependent noise (PCA-SDDN). This problem occurs when studying the SVD solution for solving (i) PCA with missing data, (ii) ST with missing data, and (iii) robust ST (with outliers and with and without missing data). We briefly explain the technical novelty of our result here. Let \( W \) denote the sparse linearly data-dependent noise matrix corrupting a true low rank \( r \) data matrix \( L \). We observe \( y_t = \ell_t + w_t \), \( t = 1, 2, \ldots, \alpha \) with \( \ell_t = P a_t \). \( P \) is an \( n \times r \) matrix with orthonormal columns and \( r \ll n \). Since \( w_t \) is linearly data-dependent and sparse, without loss of generality, we can express it as \( w_t = I_T M_{s,t} \ell_t \) with \( T_t = \text{support}(w_t) \) and \( M_{s,t} \) being the data-dependency matrix at time/column \( t \). Let \( b \) denote the maximum of the fraction of nonzero entries in any row of \( W \). We compute the PCA estimate, \( \hat{P} \), as the \( r \)-SVD of \( Y := [y_1, y_2, \ldots, y_\alpha] = L + W \).

(1) The sparsity of the noise along with a careful application of the Cauchy-Schwarz inequality implies that \( \| \mathbb{E}[\frac{1}{\alpha} \sum_t \ell_t w_t'] \| \leq \sqrt{b} \max_t \| \mathbb{E}[\ell_t w_t'] \| \), i.e., the time-averaged noise power is at most \( \sqrt{b} \) times its maximum instantaneous value. Thus, if \( b \) is small enough (noise support changes sufficiently across columns), the former is much smaller than the latter. (2) Since \( w_t \) depends on \( \ell_t \), this means that the data-noise correlation \( \mathbb{E}[\ell_t w_t'] \) is not zero and, its time-averaged value, \( \| \mathbb{E}[\frac{1}{\alpha} \sum_t \ell_t w_t'] \| \leq \sqrt{b} \max_t \| \mathbb{E}[\ell_t w_t'] \| \). Thus, even though signal-noise correlation is not zero (and is, in fact, proportional to signal power), its time-averaged value is \( \sqrt{b} \) times smaller. Since \( \text{SE}(\hat{P}, P) \leq \frac{\| Y Y' - LL' \|}{\lambda_r(\Sigma)^2} \) when the numerator is small enough (by Davis-Kahan sin \( \theta \) theorem), the above two facts imply that

\[
\text{SE}(\hat{P}, P) \leq \sqrt{b} \left( 2 \max_t \| M_{s,t} \Sigma \| + \max_t \| M_{s,t} M_{s,t}' \| \right) \leq \sqrt{b} (2q + q^2) f
\]

Here \( \Sigma := \mathbb{E}[\ell_t \ell_t'] \text{ EVD} = PA P' \), \( q := \max_t \| M_{s,t} P \| \), and \( f \) denotes the condition number of \( \Lambda \). \( q \) can be understood as the noise-to-signal ratio and is thus a measure of the noise level. (3) Suppose that the \( a_t \)'s are i.i.d. and bounded, i.e., \( \| a_t \|^2 \leq \mu r \lambda_{\max}(\Sigma) \). Since the noise is data-dependent, and since we assume that our data \( \ell_t \) is generated from a low \( (r) \) dimensional subspace, we can use the above facts and matrix-Bernstein [20] to show that \( \text{SE}(\hat{P}, P) \leq \epsilon \) with high probability, \( 1 - 3n^{-10} \), if the sample complexity \( \alpha \) is \( \Omega(\frac{\epsilon^2}{\mu^2 r^2} \log n) \). Thus, in order to achieve a recovery error \( \epsilon \) that is a fraction of the noise level, \( q \), the required sample complexity is near optimal (is within log factors of \( r \)).
In the above discussion we have assumed zero correlated noise, but our actual result also handles that. This can model the fact that the true data is only approximately low rank. Moreover it provides a guarantee for a more general setting than PCA-SDDN.

Use to analyze Robust ST. For solving the robust ST problem (recover $\ell_t$ and its subspace from $y_t := \ell_t + x_t$ where $x_t$ denotes the sparse outlier at time $t$), we develop a mini-batch algorithm that (a) processes the observed data to return an estimate of $\ell_t$, $\hat{\ell}_t$, at each time $t$; and (b) uses $\alpha$-mini-batches of $\hat{\ell}_t$ to compute a new estimate of the current subspace. This process is repeated $K$ times with $K$ new $\alpha$-length mini-batches for the current subspace; after this time, the algorithm enters a “subspace change detect” phase. Denote the estimate from the $k$-th iteration by $\hat{\ell}_k$. Suppose that the processing is such that (i) $\hat{\ell}_k = \ell_t + w_t$ where $w_t$ is sparse and data-dependent noise whose support equals the set of outlier entries at time $t$; and (ii) $q_k := \max_t ||M_q \| P\||$ is proportional to the subspace recovery error from iteration $k-1$, i.e., $q_k = C \cdot \text{SE}(\hat{\ell}_{k-1}, P)$ and $q_k < 2$. In defining $q_k$, the max is taken over the mini-batch used in iteration $k$. We can use our PCA result to show that $\text{SE}(\hat{\ell}_k, P) \leq \sqrt{b(2q_k + \frac{q_k^2}{2})} \lambda_k \leq \sqrt{2b}q_k$. Thus, if $b$ is small enough, $b < c/\kappa^2$, then $q_{k+1}$, and hence, $\text{SE}(\hat{\ell}_k, P)$, decreases by a constant fraction in each new iteration (the decay is geometric). Since the error in recovering $\ell_t$ satisfies $||\hat{\ell}_t - \ell_t||/||\ell_t|| \leq q_k$, this also decays geometrically with each iteration.

II. PCA in DATA-DEPENDENT NOISE

A. Problem Setting

For $t = 1, 2, \ldots, \alpha$ we are given $y_t \in \mathbb{R}^n$ that satisfies

$$y_t := \ell_t + w_t + v_t, \quad \ell_t = P a_t, \quad w_t = M_t \ell_t, \quad a_t \in \mathbb{R}^n$$

$P$ is an $n \times r$ basis matrix with $r \ll n$; $\ell_t$ is the true data vectors that lies in an $r$-dimensional subspace of $\mathbb{R}^n$, span($P$); $a_t$’s are the projections of $\ell_t$’s onto this subspace; $w_t$ is data-dependent noise with $M_t$ being the data-dependency matrix at time $t$; and $v_t$ is uncorrelated noise. This means that $\mathbb{E}[\ell_t v_t'] = 0$ for all times $t$. Here $a_t$ and $v_t$ are treated as random variables (r.v.), while everything else is deterministic. The goal is to estimate $\text{span}(P)$ from the observed data stream $y_t$, $t = 1, 2, \ldots, \alpha$.

B. SVD solution and guarantee for it

SVD Solution. We compute the subspace estimate $\hat{P}$ as the matrix of top $r$ left singular vectors of $Y := [y_1, y_2, \ldots, y_\alpha]$. Equivalently it is the matrix of top $r$ eigenvectors of $\frac{1}{\alpha} \sum t y_t y_t'$. We make the following assumptions on the subspace coefficients, $a_t$, and the uncorrelated noise, $v_t$.

**Assumption 2.1** (Statistical Assumption on $a_t$). Assume that the $a_t$’s are zero mean; mutually independent; have identical diagonal covariance matrix $\Lambda$, i.e., $\mathbb{E}[a_t a_t'] = \Lambda$; and are bounded: $\max_t ||a_t||^2 \leq \mu r \lambda_{\max}(\Lambda)$. Define $\lambda^+ := \lambda_{\max}(\Lambda)$, $\lambda^- := \lambda_{\min}(\Lambda)$, $f := \frac{\lambda^-}{\lambda^+}$.

As we explain in Sec. III, this assumption is almost equivalent to assuming $\mu$-incoherence of the right singular vectors of the matrix $L := [\ell_1, \ell_2, \ldots, \ell_\alpha]$. We call it $\mu$ statistical right incoherence there.

**Assumption 2.2** (Statistical Assumption on $v_t$). Assume that $v_t$ is uncorrelated with $\ell_t$, i.e., $\mathbb{E}[\ell_t v_t'] = 0$, and $v_t$’s are zero-mean, independent and identically distributed (i.i.d.) with covariance $\Sigma_v := \mathbb{E}[v_t v_t']$, and are bounded. Let $\lambda_\nu := ||\Sigma_v||$ be the noise power and let $r_v := \max ||v_t||^2/\lambda_\nu^2$ be the effective noise dimension.

For a decomposition of the data-dependency matrix $M_t$ as $M_t = M_{2,t} M_{1,t}$ with $||M_{2,t}|| = 1$, let

$$q := \max_t ||M_{1,t}||, \quad b := \left\| 1/\alpha \sum_{t=1}^\alpha M_{2,t} M_{2,t}' \right\|.$$ (2)

Observe that $b \leq \max_t ||M_{2,t}||^2 = 1$. In many settings, for example, when $w_t$ is sparse with changing support, $b$ is much smaller than one. Our result given below exploits this fact.

**Theorem 2.3** (PCA in Data-Dependent Noise). Consider the data $y_t$ defined by (1); and assume that Assumptions 2.1 and 2.2 hold. Also assume that $w_t = M_t \ell_t$ with the parameters $b,q$ satisfying $b < 1$, $q < 2$, and $4/\sqrt{b q} + \frac{\lambda^+}{\lambda^-} + H(\alpha) + H_{\text{denom}}(\alpha) < 1$. Here,

$$H(\alpha) := C \sqrt{\mu} \sqrt{\frac{r \log n}{\alpha}} + C \sqrt{\mu} \sqrt{\frac{\lambda^+}{\lambda^-}} \sqrt{\frac{r \log n}{\alpha}},$$

$$H_{\text{denom}}(\alpha) := C \sqrt{\mu} \sqrt{\frac{r \log n}{\alpha}}.$$ (4)

Then, with probability at least $1 - 10n^{-10}$, the matrix of top $r$ eigenvectors of $\frac{1}{\alpha} \sum t y_t y_t'$, $\hat{P}$, satisfies

$$\text{SE}(\hat{P}, P) \leq \frac{4/\sqrt{b q} + \frac{\lambda^+}{\lambda^-} + H(\alpha)}{1 - 4/\sqrt{b q} - \frac{\lambda^-}{\lambda^+} - H(\alpha) - H_{\text{denom}}(\alpha)}$$

Theorem 2.3 is proved in Appendix I. It uses the Davis-Kahan sin $\theta$ theorem [21] followed by matrix Bernstein [20] to bound each term. To understand Theorem 2.3 simply, first assume that $v_t = 0$ and $H(\alpha), H_{\text{denom}}(\alpha)$ are small enough ($\alpha$ is large enough). From the definition of $q$, the instantaneous signal-noise correlation $||\mathbb{E}[\ell_t v_t']|| \leq q \lambda^+$ and the instantaneous data-dependent noise power $||\mathbb{E}[w_t v_t']|| \leq q^2 \lambda^-$. Thus $q^2$ is the data-dependent noise-to-signal ratio. Also, $\lambda^+$ and $\lambda^-$ quantify the maximum and the minimum signal power respectively. The PCA subspace recovery error depends on
the ratio between the sum of (time-averaged values of) signal-
noise correlation and noise power and the minimum signal
space eigenvalue $\lambda^-$. By Cauchy-Schwarz, it is not hard to see
that the time-averaged values of both these quantities satisfies
$\frac{1}{\alpha t} \sum_{t=1}^{\alpha t} E[w_i w_j'] || \leq \sqrt{b} f^3 \lambda^- + \frac{1}{\alpha t} \sum_{t=1}^{\alpha t} E[|\ell_i w_j'|] || \leq \sqrt{b} f \lambda^-$. Thus, if $b < 1$, the time-averaged values are sig-
ificantly smaller than the instantaneous ones and this is what
helps us get a small bound on the subspace recovery error. For a
constant $c_1 < 1$, by assuming $b < (c_1/4)^2$, we can ensure that
$\text{SE}(\hat{P}, P) \leq c_1 q$, i.e., the subspace recovery error is a
fraction of $g$.

In the general case when $v_t \neq 0$, we can guarantee that
$\text{SE}(\hat{P}, P)$ is at most $c_1 \max(q, \lambda^+ / \lambda^-)$.

C. Application to PCA in Sparse Data-Dependent Noise
(PCA-SDDN)

An important application of the above result is for data-
dependent noise, $w_t$, that is sparse. In this work we will
show how a guarantee for PCA in sparse data-dependent noise
(PCA-SDDN) helps obtain a fast and delay-optimal robust ST
algorithm. If we set $M_{s,t} := I_{T_t}$ then $w_t$ is sparse with support
$T_t$. Thus for $t = 1, 2, \cdots, \alpha$

$$y_t := \ell_t + w_t + v_t, \quad \text{where } \ell_t = P a_t, \quad w_t = I_{T_t} M_{s,t} \ell_t$$

The assumption on $b$ is now equivalent to a bound on the 
maximum fraction of non-zero entries in any row of $W := [w_1, \cdots, w_{\alpha t}]$. To see why this is true, notice that
$b = \frac{1}{\alpha} \sum_{t=1}^{\alpha t} I_{T_t} I_{T_t}'$. The matrix $\sum_{t=1}^{\alpha t} I_{T_t} I_{T_t}'$ is a diagonal matrix with $(i, i)$-th entry equal to the number of times $t$ for which $i \in T_t$. This is the same as the number of non-zero entries in the $i$-th row of $W$. Using this fact we get the following corollary.

Corollary 2.4 (PCA in Sparse Data-Dependent Noise). Assume
that $y_t$’s satisfy (5), Assumptions 2.1, 2.2 hold, and
$q := \max_t ||M_{s,t} P|| \leq 2$. Let $b$ denote the maximum fraction of nonzeros in any row of the noise matrix $[w_1, w_2, \cdots, w_{\alpha}]$, and let $g := \frac{\sqrt{b}}{\sqrt{\alpha}}$. For an $\epsilon_{\text{SE}} > 0$, if

$$4\sqrt{b} g + g < 0.4 \epsilon_{\text{SE}},$$

and if

$$\alpha \geq \alpha^* := C \max(\frac{q^2 f^2}{\epsilon_{\text{SE}}} r \log n, \frac{q f}{\epsilon_{\text{SE}}} \max(r_v, r) \log n),$$

then w.p. at least $1 - 10 r^{-10}$, $\text{SE}(\hat{P}, P) \leq \epsilon_{\text{SE}}$.

This corollary follows from Theorem 2.3 by picking $\alpha$ large
enough so that $H(\alpha) < \epsilon_{\text{SE}}/10$ and $H_{\text{denom}}(\alpha) < 1/10$ (since this term appears in the denominator, we do not need it to be smaller than $\epsilon_{\text{SE}}$, just a constant upper bound suffices).

Corollary 2.4 shows that it is possible to achieve re-
covery error that is a fraction of $q$, i.e, $\epsilon_{\text{SE}} = c_1 q$, if
(i) $4\sqrt{b} f \leq 0.8 c_1$ (the data-dependent noise support
changes enough over time so that $b$ is small), (ii) $\lambda^+ \leq
0.8 c_1 \epsilon_{\text{SE}} \lambda^-$ (the uncorrelated noise power is small enough), and (iii) $\alpha \geq C \max(\frac{f^2 \log n}{r_v}, \frac{1}{\epsilon_{\text{SE}}} \max(r_v, r) \log n)$. Notice that the sample complexity $\alpha$ increases with $1/\epsilon_{\text{SE}} = 1/(c_1 q)$. However, if we can make a stronger assumption that $\lambda^+ \leq 0.8 c_1 \epsilon_{\text{SE}} \lambda^-$, then we only need $\alpha \geq C \max(\frac{f^2 \log n}{r_v}, \max(r_v, r) \log n)$. Furthermore if $r_v \leq C r$, then just $\alpha \geq C f^2 \log n$ suffices. Treating $f$ as a numerical constant, observe that this sample complexity is order-wise near-optimal: $r$ is the minimum number of samples needed to even define a subspace.

In particular, in the setting when $v_t = 0$, if the noise support
changes enough so that $b$ is small enough, we can estimate the
subspace to a fraction of the square root of the noise level, $q$, using just order $r \log n$ samples. The reason this is possible is because the $a_t$’s are bounded and $w_t = M_t P a_t$ and so the “randomness” in $w_t$ is only $r$-dimensional (this has implications for what matrix Bernstein returns for the required sample complexity). When $v_t \neq 0$, we have a similar result: if $v_t$ has effective dimension that is of order $r$, we can still track to $\epsilon_{\text{SE}} = c \max(q, \sqrt{g})$; here $g$ is the square root of uncorrelated noise level.

D. Generalizations of Theorem 2.3

For notational simplicity, in Theorem 2.3, we have provided
a simple result that suffices for the correctness proof of our
robust ST algorithm. We state and prove a much more general
result in the Supplement given in the ArXiv version of this
work [22, Appendix IV] that relaxes this result in three ways.
First, it replaces the identically distributed assumption on
$a_t$ and $v_t$ by the following: let $A := \sum_t A_t / \alpha$, $\lambda_{\text{avg}} := \lambda_{\min}(A)$, $\lambda_{\max} := \max_t \lambda_{\max}(A_t)$ and $\lambda_{v,\max} := \max_t \lambda_{\max}(\Sigma_{v,t})$. It
requires that the distributions are “similar” enough so that $f := \lambda_{\max} / \lambda_{\text{avg}}$ is bounded by a numerical constant and $\lambda_{v,\max}$ replaces $\lambda^+$ in $H(\alpha)$ and $H_{\text{denom}}(\alpha)$ expressions.

Secondly, it replaces $\lambda^+$ by $\|P^t \Sigma_{v} P\|$ in the numerator, while $-\lambda_{v}^+$ in the denominator gets replaced by $-\left(\lambda_{\max}(\Sigma_{v} - P^t \Sigma_{v} P P^t) - \lambda_{\min}(P^t \Sigma_{v} P)\right)$. Here again, in case of time-

For all situations, the minimum eigenvalues get replaced by the
minimum eigenvalue of the average covariance matrix while
the maximum ones get replaced by the maximum eigenvalue
over all times $t$. Thirdly, we also provide a guarantee for the
case when $a_t$’s and $v_t$’s are sub-Gaussian random vectors. In
this case, the required sample complexity increases to order $n$ instead of $\max(r_v, r_v) \log n$ that we have for the bounded case result given above.

These last two changes allow us to recover the well known
result for PCA under the Gaussian spiked covariance model
(uncorrelated isotropic noise) [23] as a special case of our most
general result. Spiked covariance means $w_t = 0$ and $\Sigma_{v} = \lambda^+ I$. Thus, $q = 0$, $\|P^t \Sigma_{v} P\| = 0$ and $\|\Sigma_{v} - P^t \Sigma_{v} P\| = 0$ and so we get the following corollary.
Corollary 2.5 (Spiked Covariance Model, Gaussian noise [23]). In the setting of Theorem 2.3, if \( w_t = 0 \) (no data-dependent noise), \( \Sigma_v = \lambda_v^2 I \), and \( a_t, v_t \) are Gaussian, then, w.p. at least \( 1 - 5\exp(-cn) \), \( \mathbb{P}(E, P) \leq \frac{\lambda_v}{\text{H}(\alpha)} \cdot \frac{H_{\text{denom}}(\alpha)}{\alpha} \cdot \text{H}(\alpha) = C\sqrt{\beta}f/\sqrt{\gamma} \), \( H_{\text{denom}}(\alpha) = C\sqrt{\beta}f/\sqrt{\gamma} \) and \( g = \frac{\lambda_v}{\alpha^2} \).

If \( a_t, v_t \) are bounded then \( H(\alpha), H_{\text{denom}}(\alpha) \) are as given in Theorem 2.3.

Notice that, under the spiked covariance model, as long as we let the sample complexity \( \alpha \) grow with the noise level \( \sigma \), we do not need any bound on noise power. For example, the noise power \( \lambda_v^2 \) could even be larger than \( \lambda^2 \). This is possible because, under this model, \( \mathbb{E}[(\sum y_t y_t^T)/\alpha] = \mathcal{P}(\mathcal{P}) + \lambda_v^2 I \).

Thus, its matrix of top \( r \) eigenvectors equals \( \mathcal{P} \). As a result, the error between \( \mathcal{P} \) and \( \mathcal{P} \) is only due to the fact that we are using a finite \( \alpha \) to approximate the expected value. In other words, we only have statistical error. The “bias” terms are zero.

III. NEARLY OPTIMAL ROBUST SUBSPACE TRACKING (NORST)

In this section, we define the robust ST problem, explain the assumptions needed to make it identifiable, and then explain our proposed mini-batch solution and its guarantee.

A. Problem setting and algorithm design constraints

At each time \( t \), we observe a data vector \( y_t \in \mathbb{R}^n \) that satisfies

\[
y_t := \ell_t + x_t + \nu_t, \quad \text{for } t = 1, 2, \ldots, d
\]

where \( \nu_t \) is small unstructured noise, \( x_t \) is the sparse outlier vector, and \( \ell_t \) is the true data vector that lies in a fixed or slowly changing low-dimensional subspace of \( \mathbb{R}^n \), i.e.,

\[
\ell_t = P(t) a_t
\]

where \( P(t) \) is an \( n \times r \) basis matrix with \( r \ll n \) and with \( \| (I - P(t-1)P(t-1)^T)P(t) \| \) small compared to \( \| P(t) \| = 1 \). We use \( T_t \) to denote the support set of \( x_t \). As an example, in the video application, \( y_t \) is the video image at time/frame \( t \), \( \ell_t \) is the background at time \( t \), \( T_t \) is the support of the foreground at \( t \), and \( x_t \) equals the difference between foreground and background images on \( T_t \) while being zero everywhere else.

Slow subspace change is typically a valid assumption for background images of videos taken using a static camera. Given a good initial subspace estimate, \( \hat{P}_0 \), the goal is to develop a mini-batch algorithm to track \( \text{span}(P(t)) \) and \( \ell_t \) either immediately or within a short delay. A by-product is that \( x_t \) and \( T_t \) can also be tracked accurately. The initial subspace estimate, \( \hat{P}_0 \), can be computed by applying a few iterations of any existing RPCA solutions, e.g., PCP [4] or AltProj [7], on the first order \( r \) data points, i.e., on \( Y_{[1,t_{\text{train}}]} \), with \( t_{\text{train}} = Cr \).

Dynamic RPCA. This is the offline version of the above problem. Define matrices \( L, X, V, Y \) with \( L = [\ell_1, \ell_2, \ldots, \ell_d] \) and \( Y, X, V \) similarly defined. The goal is to recover \( L \) and its column space with accuracy \( \varepsilon \). We use \( r \) to denote the rank of \( L \). The maximum fraction of nonzeros in any row (column) of the outlier matrix \( X \) is denoted by max-outlier-frac-row (max-outlier-frac-col).

Algorithm constraints. We will develop a nearly real-time tracking algorithm that (i) computes an online estimate of \( x_t \) and its support \( T_t \), and of \( \ell_t \) immediately at each time \( t \) using the previous subspace estimate, \( \hat{P}_{(t-1)} \), and observed data \( y_t \); (ii) it updates the subspace estimates in a mini-batch fashion; and (iii) it provides improved smoothing estimates of all quantities after a delay that is within log factors of \( r \). As we explain in Sec. III-E, recovering \( x_t, T_t, \) and \( \ell_t \), one at a time is the only way to obtain improved row-wise outlier tolerance compared to standard RPCA. However with doing this, correct recovery requires one extra assumption: slow enough subspace change compared to the minimum outlier magnitude.

B. Nearly Optimal Robust ST (NORST) via Recursive Projected Compressive Sensing (CS): main idea

The algorithm begins with an initial subspace estimate \( \hat{P}_0 \). At each time \( t \), we use \( \hat{P}_{(t-1)} \) and \( y_t \) to solve a noisy projected compressive sensing (CS) problem to estimate \( x_t \) and its support \( T_t \) from \( y_t = \Psi x_t + b_t \). Here \( \Psi = I - \hat{P}_{(t-1)}P_{(t-1)}^T, y_t = \Psi y_t, \) and \( b_t = \Psi \ell_t + \Psi \nu_t \) (is small under the slow subspace change assumption). This step uses \( l_t \) minimization followed by thresholding to estimate \( T_t \), and Least Squares (LS) on \( T_t \) to get \( \hat{x}_t \). We compute \( \hat{\ell}_t \) by subtraction as \( \hat{\ell}_t = y_t - \hat{x}_t \). Every \( \alpha \) time instants, we update the subspace estimate by solving the PCA problem using the previous \( \alpha \) \( \hat{\ell}_t \)'s as observed data, i.e., by \( r \)-SVD on \( \hat{L}_{t:\alpha} \). This is repeated \( K \) times, each time with a new set of \( \alpha \) \( \hat{\ell}_t \)'s. At this point, the algorithm enters the subspace change detect phase. The complete algorithm is specified in Algorithm 1, and explained in detail in Sec III-G. Besides \( \alpha \) and \( K \), it has two other parameters: \( \xi \) (assumed upper bound on \( \| b_t \| \)) and \( \omega_{\text{supp}} \) (threshold used for support recovery).

C. Identifiability and other assumptions

For this discussion assume that \( \nu_t = 0 \). At each time \( t \) we have just one \( n \)-length observed data vector \( y_t \) but the subspace \( P(t) \) is specified by \( nr \) scalars (it is an \( r \)-dimensional subspace of \( \mathbb{R}^n \)). Thus, even if we had perfect data \( y_t = \ell_t \) available, it would be impossible to exactly recover each different \( P(t) \). One way to address this is by assuming that the \( P(t) \)’s do not change for at least \( r \) time instants.

Assumption 3.6 (Piecewise Constant Subspace Change). Let \( t_1, \ldots, t_j, \ldots, t_J \) denote the subspace change times. Let \( t_0 = 1 \) and \( t_{j+1} = d \). Assume that

\[
P(t) = P_j \text{ for all } t \in [t_j, t_{j+1}), \quad j = 1, 2, \ldots, J
\]
with \( t_{j+1} - t_j > r \). Since \( y_i = \ell_i + x_i \) (is imperfect), our guarantee needs a larger bound than \( r \).

Even with the above assumption, a sparse \( x_t \) and its support \( T_t \) cannot be correctly distinguished from \( \ell_t = P_j a_t \) without more assumptions. Correct recovery of \( x_t \) and \( T_t \) requires that (i) the \( x_t \)'s are sparse enough (ensured by bounding the maximum allowed outlier fractions per column), (ii) the columns of \( P_j \) are not sparse (ensured by the standard incoherence/denseness assumption from the RPCA literature \[4], [7], [24]\), and (iii) the \( a_t \)'s are bounded. (iv) Correct support recovery also requires subspace change that is slow enough compared to the minimum nonzero entry of \( x_t \) (minimum outlier magnitude), denoted \( x_{\min} \). Correct subspace update requires that (v) the \( r \times \alpha \) sub-matrices formed by a mini-batch of \( x_t \)'s are well-conditioned, and (vi) the outlier support \( T_t \) changes enough over time so that there is at least one outlier-free observation of each scalar entry of \( \ell_t \) in each mini-batch of \( y_i \)'s. One way to ensure (v) is to assume that the \( a_t \)'s are i.i.d. while (vi) can be ensured by bounding the maximum fraction of outliers in any row of any \( \alpha \)-mini-batch sub-matrix of \( X \). We use max-outlier-frac-row(\( \alpha \)) to denote this quantity. We summarize the above assumptions on \( P_j \)'s and \( a_t \)'s in Assumption 3.7, those on the outlier fractions in Assumption 3.8, and slow subspace change compared to \( x_{\min} \) in Assumption 3.9.

**Assumption 3.7 (\( \mu \)-Incoherence).** Assume the following.

1) (Left Incoherence) Assume that \( P_j \)'s are \( \mu \)-incoherent with \( \mu \) being a constant random variable. This means that \( \max_{i=1,2,\ldots,n} \| (P_j)_{(i)} \|_2 \leq \mu r/n \). Here \( P^{(i)} \) denotes the \( i \)-th row of \( P \).

2) (Statistical Right Incoherence) Assume Assumption 2.1, i.e., the subspace coefficients \( a_t \) are zero mean, mutually independent, have identical diagonal covariance matrix \( \Lambda := \mathbb{E}[a_t a_t^\top] \), and are bounded: \( \max_i \| a_t \|_2 \leq \mu \sqrt{\lambda_{\text{max}}(\Lambda)} \). Let \( \lambda^+ (\lambda^-) \), \( f := \lambda^+/\lambda^- \) denote the maximum (minimum) eigenvalue and condition number of \( \Lambda \).

The second assumption above allows us to obtain high probability upper bounds on the tracking delay of our approach. As we explain later in Sec. III-F, it can be interpreted as a statistical version of right singular vectors' incoherence. The incoherence assumption on \( P_j \) is nearly equivalent to left singular vectors' incoherence. It is exactly equivalent if we consider the sub-matrices \( L_j := [\ell_{t_j}, \ell_{t_{j+1}}, \ldots, \ell_{t_{j+1}-1}] \).

**Assumption 3.8 (Outliers are spread out).** Let max-outlier-frac-col := \( \max_i \| T_t \|_2/n \); let max-outlier-frac-row(\( \alpha \)) be the maximum fraction of nonzeros per row of any sub-matrix of \( X_{[t_{\text{train}},d]} \) with \( \alpha \) consecutive columns, and let max-outlier-frac-row\( \alpha \)\(_{\text{init}} \) be the maximum fraction of outliers per row of any sub-matrix of \( X_{[1,t_{\text{train}}]} \). Assume that max-outlier-frac-col \( \leq \frac{c_1}{\sqrt{n}}, \) max-outlier-frac-row(\( \alpha \)) \( \leq \frac{c_2}{\sqrt{n}}, \) and max-outlier-frac-row\( \alpha \)\(_{\text{init}} \) \( \leq \frac{c_3}{\sqrt{n}} \).

**Assumption 3.9 (Slow subspace change).** Let \( x_{\min} := \min_{\alpha \in \ell_t} \| x_i \|_2 \) and let \( SE_j := SE(P_{j-1}, P_j) \). Assume that \( SE_j \leq 0.8 \) and \( SE_j \leq \frac{c_4}{\sqrt{n}} \).

The order notation used here and below assumes that \( f, \mu \) are constants.

### D. Guarantees

Before stating our main result, we define a few terms next.

**Definition 3.10.** Let the mini-batch size \( \alpha := C \sqrt{r \log n} \), the number of subspace update iterations needed to get an \( \varepsilon \) accurate estimate, \( K = K(\varepsilon) := C \log(1/\Delta) \), where \( \Delta := \max_j SE_j, \) noise power \( \lambda^+ := \max \| E[v_t v_t^\top] \|_2 \), and effective noise dimension, \( r_v := \frac{\max \| x_i \|_2^2}{\lambda^+} \). Recall from Algorithm 1 that \( t_j \) denotes the time at which the \( j \)-th subspace change is detected.

We have the following result.

**Theorem 3.11.** Assume that Assumptions 3.6, 3.7, 3.8, and 3.9 hold. Assume that the noise \( v_t \) is bounded, i.i.d. over time, independent of \( T_t \), uncorrelated with \( \ell_t \), i.e., \( \mathbb{E}[v_t v_t^\top] = 0 \), and with \( r_v \leq C r \), and \( \sqrt{\lambda^+/\lambda^-} < 0.01 \). Also, assume that \( \ell_t \)'s and \( T_t \)'s are independent.

Pick an \( \varepsilon \) that satisfies \( c_1 \sqrt{\lambda^+/\lambda^-} \leq \varepsilon \leq \min \left( c_4 \sqrt{\frac{x_{\min}}{\lambda^+}}, 0.01 \right) \). Consider Algorithm 1 with \( K = K(\varepsilon) \) as defined above, \( \alpha = C \sqrt{r \log n} \), \( \rho_{\text{evals}} = 2\varepsilon^2 \lambda^+ \), \( \zeta = x_{\min}/15 \) and \( \omega_{\text{supp}} = x_{\min}/2 \). If

1) \( \max_k \| v_{t} \|_2 \leq c_5 x_{\min} \),
2) \( t_{j+1} - t_j > (K + 2) \alpha \), and \( SE_j > 9 \sqrt{\varepsilon} \),
3) initialization\(^3\): \( SE(P_0, P_0) \leq \min \left( c_6 \sqrt{\frac{x_{\min}}{\lambda^+}}, 0.25 \right) \),
then, w.p. at least \( 1 - 10dn^{-10} \),

\[
\begin{align*}
&1) \ t_j \leq \ell_t \leq t_j + 2\alpha, \\
&\text{SE}(\hat{P}_t), P_t(\|) \leq \\
&\left( \begin{array}{c}
(\varepsilon + SE_j) \\
(0.3)k^{-1} (\varepsilon + SE_j)
\end{array} \right) \\
&\left( \begin{array}{c}
\varepsilon \\
\varepsilon
\end{array} \right)
\begin{array}{c}
\text{if } t \in [t_j, \ell_t + j \alpha), \\
\text{if } t \in [\ell_t + (k - 1) \alpha, \ell_t + k \alpha), \\
\text{if } t \in [\ell_t + K \alpha + \alpha, \ell_t + 1),
\end{array}
\end{align*}
\]

and \( \| \hat{\ell}_t - \ell_t \| \leq 1.2SE(\hat{P}_t), P_t(\|)\|\ell_t\| + \|v_t\| \).

The time complexity is \( O(ndr \log(1/\varepsilon)) \) and memory complexity is \( O(n\alpha) \approx O(f^2nr \log n) \).

**Proof.** We prove this in Sec. VI.

\(^3\)This can be satisfied by using \( C \log r \) iterations of AltProj \[7\] on the first \( t_{\text{train}} = Cr \) data samples.
Corollary 3.12. [Smoothing NORST for dynamic RPCA] Under the assumptions of Theorem 3.11, the following also hold: \( \text{SE}(P_{(t)}^{\text{smoothing}}, \ell(t)) \leq \varepsilon, \| \ell(t) - \ell_{t} \| \leq \varepsilon \| \ell_{t} \| + \| \nu_{t} \| \) at all times \( t \). Its time complexity is \( \mathcal{O}(n dr \log(1/\varepsilon)) \) and memory complexity is \( \mathcal{O}(K n \alpha) = \mathcal{O}(dr \log n \log(1/\varepsilon)) \). All these quantities are computed within a delay of at most \( (K + 2)\alpha \).

The above result guarantees that NORST can detect subspace changes in delay at most \( \alpha = Cr \log n \) and track them to \( \varepsilon \) accuracy in delay at most \( (K + 2)\alpha = Cr \log n \log(\Delta/\varepsilon) \). The corollary for smoothing NORST guarantees that, with this delay, each column of \( L \), \( \ell_{t} \), is recovered to \( \varepsilon \) relative accuracy.

The minimum delay needed to compute an \( r \)-dimensional subspace even with perfect data \( y_{t} = \ell_{t} \) is \( r \). Thus, our result guarantees near optimal detection and tracking delay ("near optimal" means that it is within log factors of the minimum delay). Moreover, the required lower bound on the delay between subspace change times is also near optimal. Quick and reliable change detection is an important feature, e.g., this feature has been used in [25] to detect structural changes in a dynamic social network.

When the extra unstructured noise \( \nu_{t} = 0 \), we can track to any \( \varepsilon > 0 \) otherwise we can track to \( \varepsilon \geq \sqrt{\lambda_{\nu}^{-}\lambda^{-}} \) (square root of the noise level). It is possible to slightly relax this requirement to \( \varepsilon \geq \lambda_{\nu}^{\alpha}/\lambda^{-} \) by picking a larger \( \alpha \), \( \alpha = C(r \log n)(\lambda^{-}/\lambda_{\nu}^{\alpha}) \), but it cannot be eliminated. The reason is that at each time \( t \), we have an under-determined set of equations corrupted by unstructured noise \( \nu_{t} \). Even assuming the subspace is known or has been perfectly estimated, it is under-determined: we have \( n + r \) unknowns at each time \( t \) but only \( n \) observed scalars. This is also true for any other under-determined problem as well, e.g., standard RPCA or CS.

Notice also that we have assumed the "effective noise dimension", \( r_{e} \in \mathcal{O}(r) \). This requirement can be eliminated if we set \( \alpha = C f_{t}^{2} \max(r, r_{e}) \log n \).

From the perspective of recovering the true data \( \ell_{t} \), both \( \nu_{t} \) and \( x_{t} \) are noise or perturbations. The difference is that \( \nu_{t} \) is a vector of small disturbances or modeling errors, while \( x_{t} \) is a sparse outlier vector with few nonzero entries. By definition, an outlier is an infrequent but large disturbance. Our result tolerates what can be called "bi-level perturbations": the small perturbation \( \nu_{t} \) needs to be small enough and the minimum outlier magnitude \( x_{\min} \) needs to be large enough so that \( \| x_{\min} \| \geq 0.2 x_{\min} \) (minimum outlier magnitude). Moreover, \( x_{\min} \) also needs to be large enough to satisfy Assumption 3.9.

The need for both these assumptions is explained in Sec. III-E. Assuming that \( x_{\min}^{2} \) is of order \( \lambda^{+} \) (signal power), Assumption 3.9 requires that \( \text{SE}_{v} \) be \( O(1/\sqrt{T}) \). However this is not as restrictive as it may seem. The reason is that \( \text{SE}(\cdot) \) is only measuring the sine of the largest principal angle. If all principal angles are roughly equal, then, this still allows the chordal subspace distance (\( l_{2} \) norm of the vector of sines of all \( r \) principal angles) [26] to be \( O(1) \).

Our result assumes a minor lower bound on \( \text{SE}_{v} \). This is needed to guarantee reliable subspace change detection. Changes that are smaller than order \( \varepsilon \) cannot be detected when the previous subspace is only tracked to accuracy \( \varepsilon \). However, such changes also increase the tracking error only by an extra factor of \( \varepsilon \) and hence can be treated as noise. If change detection is not important, then, as we explain in Sec. V, we can use a simpler NORST algorithm that does not need the lower bound.

Consider the piecewise constant subspace change assumption. In practice, e.g., in the video application, typically the subspaces change by a little at each time. This can be modeled as piecewise constant subspaces plus modeling error \( \nu_{t} \). We explain this point in Sec. V where we also provide a corollary for this setting. This corollary explains why the NORST algorithm "works" (gives good, but not perfect, subspace estimates and estimates of \( \ell_{t} \)) for real videos or for simulated data generated so that \( P_{(t)} \) changes a little at each \( t \); see Sec. VII and more detailed experiments in [27].

To keep the theorem statement simple, we have used tighter bounds than required. Define the intervals \( J_{j,1} = [t_{j}, t_{j} + \alpha) \), \( J_{j,k} = [t_{j} + (k - 1)\alpha, t_{j} + k\alpha) \) for \( k = 2, 3, \ldots, K \), and \( J_{j,K+1} = [t_{j} + (K + 1)\alpha, t_{j+1}) \). For \( t \in J_{j,k} \), for \( k = 1, 2, \ldots, K \), we only need \( 0.3^{k-1} (\varepsilon + \text{SE}_{v}) \sqrt{r\lambda_{t}} \leq c_{\min} \Delta_{t} \), i.e., the required lower bound on the minimum outlier magnitude at time \( t \) decreases as the subspaces get estimated better. For the outliers \( x_{t} \), for \( t \in J_{j,K+1} \), we do not require any lower bound. Secondly, if the outlier vector is such that some entries are very small while the others are large enough, then we can treat the smaller entries as "noise" \( \nu_{t} \).

This will work as long as these small entries are small enough so that the sum of their squares is sufficiently smaller than the square of the magnitude of the larger entries, i.e., for \( t \in J_{j,k} \), we can split \( x_{t} \) as \( x_{t} = (x_{t})_{\text{small}} + (x_{t})_{\text{large}} \) with the two components being such that \( c(x_{t})_{\text{large}} \geq ||(x_{t})_{\text{small}}|| \) and \( c(x_{t})_{\text{large}} \geq 0.3^{k-1} (\varepsilon + \max_{t} \text{SE}_{t}) \sqrt{r\lambda_{t}} \). Finally, if we also state the PCA-SDDN result in its most general form, the subspace error decay rate of 0.3 can be replaced by \( (6\sqrt{b_{0}}f) \) with \( b_{0} := \max_{t} \text{outlier-frac-row} \), so this requirement becomes \( c(x_{t})_{\text{large}} \geq (6\sqrt{b_{0}}f)^{k-1} (\varepsilon + \max_{t} \text{SE}_{t}) \sqrt{r\lambda_{t}} \). With this change, the expression for \( K \) becomes \( K = \left\lceil \frac{\log(\Delta)}{-\log(6\nu b_{0} f)} \right\rceil \).

Thus, a smaller \( b_{0} \) means that the subspace error decays faster. This, in turn, means that a smaller \( K \) suffices (faster tracking
and a smaller required lower bound on \( t_{j+1} - t_j \). It also means a smaller lower bound is needed on the outlier magnitudes at most times.

E. How slow subspace change (Assumption 3.9) enables improved outlier tolerance

We explain here how the use of Assumption 3.9 enables improved outlier tolerance. Briefly, the reason is we recover each outlier \( x_t \) and its support \( \mathcal{T}_t \) individually. To understand things simply, assume \( \nu_t = 0 \).

Given a good previous subspace estimate, \( \hat{P}_{(t-1)} \), slow subspace change implies that \( \text{SE}(\hat{P}_{(t-1)}, P_{(t)}) \) is small. Consider an \( \alpha \) length interval \( J \) during with \( \hat{P}_{(t-1)} = \hat{P} \) (computed in the previous \( \alpha \) interval). To exploit slow subspace change, we project each \( y_t \) orthogonal to \( P \) to get \( \hat{y}_t := \Psi x_t + b_t \) where \( b_t := \Psi \hat{e}_t \) is small because of above. Here \( \Psi := I - \hat{P} \hat{P}' \). Now \( b_t \) itself does not have any structure. But, the matrix \( B_J \) formed by the \( b_t \)'s for \( t \in J \), is low rank with rank \( r \). Accurately recovering \( X_J \) from \( \hat{Y}_J := \Psi X_J + B_J \) when \( B_J \) has rank \( r \) is impossible if the fraction of outliers in any row or in any column of \( X_J \) is more than \( c/r \). The reasoning is the same as that used for standard RPCA [7]: we can construct a sparse matrix \( X_J \) with rank \( 1/\max(\text{max-outlier-frac-row}, \text{max-outlier-frac-col}) \). Thus if \( \text{max-outlier-frac-row} = c \), we can construct a sparse \( X_J \) with rank \( 1/c = C \ll r \). If the rank of \( X_J \) is less than \( r \), that of \( \Psi X_J \) will also be less than \( r \), making the recovery problem un-identifiable: if we try to find a matrix \( \hat{B}_J \) of rank at most \( r \) and a matrix \( \hat{X}_J \) that is the sparsest and both satisfy \( \hat{Y}_J := \Psi \hat{X}_J + \hat{B}_J \), it is possible that we get the solution \( \hat{B}_J = B_J + \Psi \hat{X}_J \) and \( \hat{X}_J = 0 \). Because rank of \( \Psi \hat{X}_J \) is less than \( r \) and that of \( \hat{B}_J \) is \( r \), it is possible that the sum still has rank \( r \).

Thus, if we would like to improve row-wise outlier tolerance to \( O(1) \), we cannot jointly recover all columns of \( X_J \) by exploiting the low rank structure of \( B_J \). The only other way to proceed is as we do: recover them one \( x_t \) at a time from \( y_t \). Here we can only use the fact that \( \| b_t \| \) is small due to slow subspace change. The problem of recovering a single \( x_t \) from \( y_t \) is a standard noisy CS problem [28], with small noise \( b_t \). To our best knowledge, there are no entry-wise recovery guarantees for CS. One can only bound \( \| \hat{x}_{t,cs} - x_t \| \) by a constant (that depends on the restricted isometry constant of \( \Psi \)), \( C_t \), times \( \| b_t \| \). Here \( \hat{x}_{t,cs} \) is the output of the CS step (line 7 of Algorithm 1). With this, correct support recovery, \( \hat{T}_t = \mathcal{T}_t \), is ensured only if \( x_{\min} > 2C \| b_t \| \). The worst case bound on \( \| b_t \| \) comes from when the subspace has changed but the change has not been detected so that \( \hat{P}_{(t-1)} = \hat{P}_{(t-1)} = \hat{P}_J \). At this time, \( \| b_t \| \leq \max_{t, J} \text{SE}(\hat{P}_{(t-1)}, P_J) \sqrt{\lambda^r} \). Also, we can show that \( \text{SE}(\hat{P}_{(t-1)}, P_J) \leq \text{SE}_J + \varepsilon \). Thus, exact support recovery is guaranteed if Assumption 3.9 holds and \( \varepsilon \) is chosen as specified in the theorem. When \( \nu_t \neq 0 \), the bound on \( \| b_t \| \) contains a \( \| \nu_t \| \) term. In this case, exact support recovery also needs \( \| \nu_t \| \leq C_5 x_{\min} \).

Exact support recovery followed by LS on the recovered support and then subtraction to get \( \hat{t}_t \) implies that \( \hat{t}_t \) satisfies \( \hat{t}_t = t_t + e_t \) with \( e_t := -I_{\mathcal{T}_t} (I_{\mathcal{T}_t} \hat{\Psi} I_{\mathcal{T}_t})^{-1} I_{\mathcal{T}_t} \hat{\Psi} t_t \). Notice that \( e_t \) is sparse and linearly data-dependent and, conditioned on \( \hat{P} \) and the support set \( \mathcal{T}_t \), the matrix \( M_{s,t} \) is deterministic. So we can apply the PCA-SDDN result from the previous section. It also needs statistical right incoherence, \( q := \max || M_{s,t} P_J || \leq C \text{SE}(\hat{P}, P_J) \) (holds by left incoherence and max-outlier-frac-col \( c/r \)), and max-outlier-frac-row(\( c \)) (constant row-wise outlier fraction bound). If the support recovery were incorrect, the estimated support \( \hat{T}_t \) would depend on \( b_t \) and hence on \( t_t \). This would mean that, even conditioned on \( \hat{P} \) and \( \mathcal{T}_t \), the matrices \( M_{s,t} \) are not deterministic making the PCA-SDDN result inapplicable.

F. Understanding Statistical Right Incoherence

Let \( L_j := L_{[t_j, t_{j+1}]} \). From our assumptions, \( L_j = P_j A_j \) with \( A_j := [a_{ij}, a_{ij+1}, \ldots a_{ij+1}] \), the columns of \( A_j \) are zero mean, mutually independent, have identical covariance \( \Lambda \), \( \Lambda \) is diagonal, and bounded. Let \( d_j := t_{j+1} - t_j \). Define a diagonal matrix \( \Sigma \) with \( (i,i) \)-th entry \( \sigma_i \) satisfying \( \sigma_i^2 := \sum_i (a_{ij})^2/d_j \). Define a \( d_j \times r \) matrix \( V \) with the \( t \)-th entry of the \( i \)-th column being \( \{i,j\} := (a_{ij})/(\sigma_i \sqrt{d_j}) \). Clearly, \( L_j = P_j \Sigma V' \) and each column of \( V \) is unit 2-norm. This can be interpreted as an approximation to the SVD of \( L_j \); we say approximation because the columns of \( V \) are not necessarily exactly mutually orthogonal. However, if \( d_j \) is large enough, one can argue using scalar Hoeffding inequality (applicable because \( a_i \)'s are bounded), that, whp, (i) the columns of \( V \) are approximately mutually orthogonal, i.e. \( ||v_i'\nu_j|| \leq \varepsilon \) for all \( i \neq j \); and (ii) 0.99\( \lambda_i \leq \sigma_i^2 \leq 1.01\lambda_i \) for all \( i = 1, 2, \ldots, r \). Thus, by the boundedness assumption on the \( a_i \)'s, the \( t \)-th row of \( V \) satisfies \( \sum_{i=1} ||v_i'\nu_j||^2 \leq (1/d_j)(1/\min_j \sigma_i^2)^2 ||a_{ij}||^2 \leq (1/d_j)(1/\lambda_j)^2 \nu_j^2 \). This is the standard incoherence assumption with parameter \( f \). Thus, whp, the approximate right singular vectors’ matrix \( V \) of \( L_j \) satisfies the standard incoherence assumption.

G. Nearly Optimal Robust ST via ReProCS (NORST-ReProCS): details

Algorithm 1 uses the Recursive Projected Compressive Sensing framework introduced in [11]. It starts with a “good” estimate of the initial subspace. This can be obtained by
Algorithm 1 NORST Algorithm. We obtain $\hat{P}_0$ by $C(\log r)$ iterations of AltProj on $Y_{[1, t_{train}]}$. $t_{train} = Cr$. 

1: **Input:** $\hat{P}_0$, $y_t$; **Output:** $\hat{x}_t$, $\hat{\ell}_t$, $\hat{P}_{t}$; 
2: **Parameters:** $\omega_{supp}$, $\xi$, $\alpha$, $K$, $\omega_{evals}$; 
3: $\hat{P}_{t_{train}} \leftarrow \hat{P}_0$; $j \leftarrow 1$, $k \leftarrow 1$; 
4: **phase update; $\hat{\ell}_0 := t_{train}$;**
5: **for** $t > t_{train}$ **do**
6: $\Psi := I - \hat{P}_{(t-1)}\hat{P}_{(t-1)}'$; $y_{t} \leftarrow \Psi y_t$;
7: $\hat{x}_{t,cs} := \arg \min_{\hat{x}} \| \hat{x} \|_1$ s.t. $\| \hat{y}_t - \Psi \hat{x} \| \leq \xi$.
8: $\hat{T}_t := \{ j : \| \hat{x}_{t,cs} \| > \omega_{supp} \}$
9: $\hat{x}_t := I_{\hat{T}_t} (\Psi_{\hat{T}_t} \Psi_{\hat{T}_t}^{-1} \hat{x}_{t,cs})$.
10: $\hat{\ell}_t \leftarrow y_t - \hat{x}_t$;
11: **if** phase $= \text{detect}$ and $t = \hat{t}_{j-1, f \_in} + u \alpha$ **then**
12: $\Phi := (I - \hat{P}_{j-1}\hat{P}_{j-1})'$;
13: $B := \Phi L_{t, \alpha}$ with $\hat{L}_{t, \alpha} := \{ \hat{\ell}_{t-\alpha+1, \alpha}^t, \hat{\ell}_{t-\alpha+2, \alpha}^t, \ldots, \hat{\ell}_t ^t \}$.
14: **if** $\max(BB^\top) \geq \alpha \omega_{evals}$ **then**
15: **phase update; $\hat{\ell}_j := t$;**
16: **end if**
17: **end if**
18: **if** phase $= \text{update}$ **then**
19: if $t = \hat{t}_j + u \alpha - 1$ for $u = 1, 2, \ldots$, then $\hat{P}_{j, k} \leftarrow \text{SVD}_r(L_{t, \alpha})$, $\hat{P}_{j, k} \leftarrow \hat{P}_{j, k}, k \leftarrow k + 1$.
20: **else** $\hat{P}_{j} := \hat{P}_{(t-1)}$ 
21: **end if**
22: **end if**
23: **if** $t = \hat{t}_j + K \alpha - 1$ **then** 
24: $\hat{t}_{j, f \_in} := t$, $\hat{P}_{j} := \hat{P}_{(t)}$ 
25: $k \leftarrow 1$, $j \leftarrow j + 1$, phase $=$ detect. 
26: **end if**
27: **end if**
28: **end if**
29: **end for**
30: **Smoothing NORST:** At $t = \hat{t}_j + K \alpha$, for all $t \in [\hat{t}_{j-1} + K \alpha, \hat{t}_j + K \alpha - 1]$, 
31: $\hat{P}_{t}^{(smoothing)} := \text{basis}(\hat{P}_{j}^{(smoothing)}, \hat{P}_{j})$, where basis$(M)$ refers to a basis matrix that has span equal to span$(M)$.
32: $\Psi := I - \hat{P}_{t}^{(smoothing)}$, $\hat{x}_t^{(smoothing)} := I_{\hat{T}_t} (\hat{x}_{t,cs} - \hat{x}_t)$, $\hat{x}_t^{(smoothing)} := y_t - \hat{x}_t^{(smoothing)}$.

using a few iterations of AltProj applied to $Y_{[1, t_{train}]}$ with $t_{train} = Cr$. It then iterates between (a) Projected Compressive Sensing (CS) / Robust Regression\(^7\) in order to estimate the sparse outliers, $x_t$’s, and hence the $\ell_t$’s, and (b) Subspace Update to update the estimates $\hat{P}_{(t)}$. Projected CS proceeds as follows. At time $t$, if the previous subspace estimate, $\hat{P}_{(t-1)}$, is accurate enough, because of slow subspace change, projecting $y_t$ onto its orthogonal complement will nullify most of $\ell_t$. We compute $\hat{y}_t := \Psi y_t$, where $\Psi := I - \hat{P}_{(t-1)} \hat{P}_{(t-1)}'$. Clearly $\hat{y}_t = \Psi x_t + \Psi (\ell_t + \nu_t)$ and $\| \Psi (\ell_t + \nu_t) \|$ is small due to slow subspace change and small $\nu_t$. Recovering $x_t$ from $\hat{y}_t$ is now a CS / sparse recovery problem in small noise [28]. We compute $\hat{x}_{t, cs}$ using noisy $l_1$ minimization followed by thresholding based support estimation to obtain $\hat{T}_t$. A Least Squares (LS) based debiasing step on $\hat{T}_t$ returns the final $\hat{x}_t$. We then estimate $\ell_t$ as $\hat{\ell}_t := y_t - \hat{x}_t$.

The $\hat{\ell}_t$’s are then used for the Subspace Update step which toggles between the “detect” phase and the “update” phase. It starts in the “update” phase with $\hat{t}_0 = t_{train}$. We then perform $K$ r-SVD steps with the $k$-th one done at $t = \hat{t}_0 + k \alpha - 1$. Each such step uses the last $\alpha$ estimates, i.e., uses $L_{t, \alpha}$. Thus at $t = \hat{t}_0 + K \alpha - 1$, the subspace update of $\hat{P}_0$ is complete. At this point, the algorithm enters the “detect” phase. For any $j$, if the $j$-th subspace change is detected at time $t$, we set $\hat{t}_j = t$. At this time, the algorithm enters the “update” (subspace update) phase. We then perform $K$ r-SVD steps with the $k$-th r-SVD step done at $t = \hat{t}_j + k \alpha - 1$ on $L_{t, \alpha}$. Thus, at $t = \hat{t}_{j, f \_in} := \hat{t}_j + K \alpha - 1$, the update is complete. At this time, the algorithm enters the “detect” phase.

To understand the change detection strategy, consider the $j$-th subspace change. Assume that the previous subspace $\hat{P}_{j-1}$ has been accurately estimated by $t = \hat{t}_{j-1, f \_in} = \hat{t}_{j-1} + K \alpha - 1$ and that $t_{j-1, f \_in} < t_j$. Let $\hat{P}_{j-1}$ denote this estimate. At this time, the algorithm enters the “detect” phase in order to detect the next ($j$-th) change. Let $B_t := (I - \hat{P}_{j-1} \hat{P}_{j-1}) L_{t, \alpha}$. At every $t = \hat{t}_{j-1, f \_in} + u \alpha - 1$, $u = 1, 2, \ldots$, we detect change by checking if the maximum singular value of $B_t$ is above a pre-set threshold, $\sqrt{\omega_{evals} \alpha}$, or not. We claim that, with high probability (whp), under assumptions of Theorem 3.11, this strategy has no “false subspace detections” and correctly detects change within a delay of at most $2 \alpha$ samples. The former is true because, for any time for which $[t - \alpha + 1, t] \subseteq [\hat{t}_{j-1, f \_in}, \hat{t}_j]$, all singular values of the matrix $B_t$ will be close to zero (will be of order $\varepsilon \sqrt{\lambda}$) and hence its maximum singular value will be below $\sqrt{\omega_{evals} \alpha}$. Thus, whp, $\hat{t}_j \geq t_j$.

To understand why the change is correctly detected within $2 \alpha$ samples, first consider $t = \hat{t}_{j-1, f \_in} + \big\lceil \frac{t - \hat{t}_{j-1, f \_in} + \alpha - 1}{\alpha} \big\rceil \alpha := t_{j, s}$. Since we assumed that $\hat{t}_{j-1, f \_in} < t_j$ (the previous subspace update is complete before the next change), $t_j$ lies in the interval $[t_{j, s} - \alpha + 1, t_{j, s}]$. Thus, not all of the $\ell_t$’s in this interval satisfy $\ell_t = \hat{P}_t a_t$. Depending on where in the interval $t_j$ lies, the algorithm may or may not detect the change at this time. However, in the next interval, i.e., for $t \in [t_{j, s} + 1, t_{j, s} + \alpha]$, all of the $\ell_t$’s satisfy $\ell_t = \hat{P}_t a_t$. We can prove that, whp, $B_t$ for this time $t$ will have maximum singular value that is above the threshold. Thus, if the change is not detected at $t_{j, s}$, whp, it will get detected at $t_{j, s} + \alpha$. Hence, whp, either $\hat{t}_j = t_{j, s}$, or $\hat{t}_j = t_{j, s} + \alpha$, i.e., $t_j \leq \hat{t}_j \leq t_j + 2 \alpha$.

Algorithm parameters. Algorithm 1 assumes knowledge of 4

\(^7\)Robust Regression (with a sparsity model on the outliers) assumes that observed data vector $y$ satisfies $y = P a + x + b$ where $P$ is a tall matrix (given), $a$ is the vector of (unknown) regression coefficients, $x$ is the (unknown) sparse outliers, $b$ is (unknown) small noise/modeling error. An obvious way to solve this is by solving $\min_{a, x} \| a \|_1 + \| y - P a - x \|_2^2$. In this, one can solve for $a$ in closed form to get $a = P'(y - x)$. Substituting this, the minimization simplifies to $\min_{a, x} \| a \|_1 + \| (I - PP') (y - x) \|_2^2$. This is equivalent to the Lagrangian version of the projected CS problem that NORST solves (given in line 7 of Algorithm 1).
model parameters: $r$, $\lambda^+$, $\lambda^-$ and $x_{\min}$ to set the algorithm parameters. The initial dataset used for estimating $P_0$ (using AltProj) can be used to get an accurate estimate of $r$, $\lambda^-$ and $\lambda^+$ using standard techniques. Thus one really only needs to set $x_{\min}$. If continuity over time is assumed, we can let it be time-varying and set it as $\min_{t \in \mathcal{T}_{t-1}} |(\hat{x}_{t-1})_i|$ at $t$.

**Time complexity.** The time complexity is $O(rdr \log (1/\epsilon))$. We explain this in Appendix III-A.

IV. RELATED WORK

We first briefly discuss related work on PCA and then discuss robust PCA and subspace tracking papers. While there has been a large amount of work in the last decade on finite-sample guarantees for PCA [23], [29] and related problems, such as sparse PCA [30], [31] and kernel PCA [32], [33] most of these assume either the spiked covariance model (noise is modeled as being isotropic) [23], [33] or that the observed data $y_i$ is i.i.d. [23], [29] or consider noiseless settings [30], [31] (typical in sparse PCA). The setting that we study involves linearly data dependent noise $w_t = M_t \ell_t$ with the dependency matrix $M_t$ being time-varying. Thus, the noise is clearly not isotropic. Moreover, this also means that the observed data $y_t = \ell_t + w_t + v_t$ cannot be identically distributed over time. In fact, our guarantee is interesting only in the setting where $M_t$ changes enough over time so that the time-averaged expected value of signal-noise correlation and of noise power is sufficiently smaller than their respective instantaneous values.

We should mention also that, while many sophisticated eigenvector perturbation bounds exist in the literature [34]–[36], these are designed for different settings than the one we are interested in. For our setting, only the classical Davis-Kahan sin theta theorem [21] applies. In our analysis, we need to bound the sine of the largest principal angle between the true and estimated subspaces, because this helps us get a bound on the “noise”error seen by the projected compressed sensing step at the next time instant. Thus, [34], which only provides coordinate-wise bounds, cannot be used. The perturbation seen by our sample covariance matrix is additive and our observed data $y_t$ is not identically distributed, and thus the results of [35], [36] do not apply either.

The robust PCA (RPCA) problem has been extensively studied since the first two papers by Candès et al and Chandrasekharan at al [4], [5] and follow-up work by Hsu et al [6] all of which studied a convex optimization solution, called Principal Components Pursuit or PCP. A faster non-convex solution, called Alternating Projections or AltProj, was introduced in [7]. Later work has studied a projected gradient descent based approach, RPCA-GD [8]. The problem of RPCA with partial support knowledge was studied in [37]. All RPCA guarantees assume $\mu$-incoherence of left and right singular vectors of $L$ (needed to ensure that $L$ is not sparse). One way to ensure that $X$ is not low rank is to assume that an entry of $X$ is nonzero with probability $\rho$ independent of all others (Bernoulli model) and to assume a bound on $\rho$. This was assumed in [4]. This can sometimes be a strong assumption, e.g., in the video setting, it requires that foreground objects are one pixel wide and jump around completely randomly over time. But, if it holds, and if another stronger left-right incoherence assumption holds, then $\rho \in O(1)$ (linear sparsity) can be tolerated while also allowing the rank of $L$, $r$, to grow nearly linearly with $\min(n, d)$ [4]. The other approach to ensure that $X$ is low rank is to assume a bound of $O(1/r)$ on the maximum fraction of nonzeros (outliers) in any row or in any column of $X$. This is assumed in most of the later works [5]–[9].

Our work provides a fast mini-batch solution to the related problem of robust subspace tracking (RPCA with explicitly assuming slowly changing subspaces). Because we replace right incoherence by its statistical version, we are able to obtain guarantees on detection and tracking delay of our approach and show that both are nearly optimal (are within log factors of the minimum required delay $r$). This also means that the memory complexity of NORST is also near optimal: we only need to store $\alpha n$-length vectors in memory with $\alpha = Cr \log n$. Of course, any RPCA approach could also be applied in a mini-batch fashion on $\alpha$-consecutive column sub-matrices, and then it will also have the same memory complexity. We assume this here in our discussion.

With this assumption, max-outlier-frac-row gets replaced by max-outlier-frac-row($\alpha$) and $r$ gets replaced by $r$ for the RPCA guarantees as well.

Because we assume a lower bound on the minimum outlier magnitudes that is proportional to SE$_j$, we obtain the following improvement in outlier tolerance (explained in Sec. III-E). Treating $f$ as a constant, for any mini-batch after $t_{\text{train}}$, we only need max-outlier-frac-row($\alpha$) $\in O(1)$. For standard RPCA, unless a random model on outlier support is assumed, max(max-outlier-frac-col, max-outlier-frac-row($\alpha$)) $\in O(1/r)$ is needed [7]. For the video application, this implies that NORST tolerates slow moving and occasionally static foreground objects much better than standard RPCA methods that do not assume slow subspace change. This is also corroborated by our experiments on real videos, e.g., see Fig 4 in Sec. VII and also see a more detailed and quantitative evaluation on real data provided in [27]. Since our algorithm needs to be initialized with a standard batch RPCA approach such as AltProj [7] applied to the first $t_{\text{train}} = Cr$ data points, for this initial short batch, we do need AltProj assumptions to hold and this is why we need max-outlier-frac-row$_{\text{init}} \leq \frac{\alpha}{r}$. For the per column fraction, we also need max-outlier-frac-col $\in O(1/r)$. Thus, the overall fraction of outliers allowed in a given matrix is still

\[\max_{i,j} |\mathbf{U} \mathbf{V}^\top|_{i,j} \leq \sqrt{\frac{\alpha}{rn}}\] where $\mathbf{U}, \mathbf{V}$ are the matrices of left and right singular vectors of $L$.\]
O(1/r), which is the same as standard RPCA, but these can be less spread out row-wise (some rows could have many more outliers than others).

Moreover, we are able to guarantee that each column of $L$, $\ell_i$, is recovered to $\varepsilon$ relative accuracy and that the support of outliers can be recovered exactly. Neither is guaranteed by existing RPCA results, these only guarantee $\|L - \hat{L}\|_F \leq \varepsilon$.

Finally, in terms of time complexity, the NORST complexity of $O(nar \log(1/\varepsilon))$ per mini-batch is comparable to that of simple (non-robust) PCA. In comparison to RPCA solutions, this is much faster than PCP [4]–[6] which needs $O(na^{2+1/2})$ and $r$-times faster than AltProj [7] which needs $O(nar^2 \log(1/\varepsilon))$. RPCA-GD [8] is as fast as NORST but requires an even tighter outlier fractions’ bound than other RPCA solutions: $\max(\max-outlier-frac-row, \max-outlier-frac-col) \in O(1/r^3/2)$.

Our work builds upon the simple-ReProCS (s-ReProCS) solution and guarantee [18] and removes many of its limitations. S-ReProCS assumes a specific model of slow subspace change: only one subspace direction can change at each change time, and the amount of change needs to be bounded. Even with this assumption, its tracking delay is of order $r \log n \log(1/\varepsilon)$. Since only one direction is changing, this delay is $r$-times sub-optimal. The same is true for its required lower bound on subspace change times. A second limitation of s-ReProCS is that, in order to track subspaces to $\varepsilon$ accuracy, it requires the initial subspace estimate to also be $\varepsilon$ accurate. This, in turn, implies that one needs to run the AltProj or PCP algorithm on the initial mini-batch to convergence. Instead, our approach only requires the initial subspace error to be $O(1/\sqrt{r})$. Thus, only order $\log r$ iterations of AltProj suffice to initialize our algorithm.

Thirdly, the s-ReProCS guarantee needs a stronger statistical right incoherence assumption than ours: it needs an entrywise bound of $\max_i \max_j (|a_{ij}|)^2 \leq \mu \lambda_j$. Lastly, we develop important extensions of our main result for (i) only tracking subspace changes (without detecting the change), and (ii) for subspace changes by a little at each time $t$.

An earlier version of Theorem 3.11 appeared in ICML 2018 [1]. The results of the current manuscript improve upon the ICML result in various ways: we need a weaker statistical right incoherence assumption, a weaker lower bound on SE$_J$, and we develop two important extensions of our main result for subspace changes at each time and for applications not requiring change detection. Moreover, [1] did not prove the result for PCA in data-dependent noise, but only used the result proved in our older ISIT paper [2]. The problem of ST with missing data is a special case of robust ST, while ST with missing data and outliers is a simple generalization of robust ST. Interesting guarantees for both of these follow as easy corollaries of either our current result or of its earlier version from [1]. A corollary of the result of [1] for ST-miss is presented in [38]. In comparison to the result of [38], a similarly derived ST-miss corollary of our current result has all the advantages mentioned earlier in this paragraph.

V. EXTENSIONS: SUBSPACE CHANGE AT EACH TIME, SUBSPACE TRACKING WITHOUT DETECTION

A. Subspace changing at each time

Suppose $y_t = \hat{\ell}_t + x_t$ where $\hat{\ell}_t = P(t)\hat{a}_t$, $P(t)$ changes by a little at each time $t$, but has more significant changes at certain times $t_j$. We show here how this case can be handled by treating the error generated by changes at each time $t$ as extra unstructured noise $\nu_t$. Assume that $\hat{a}_t$’s are zero mean, bounded, and i.i.d. with diagonal covariance matrix $\Lambda$. Let $\lambda^+$ be its maximum eigenvalue and $f$ the condition number. Define $P_j$ as the matrix of top $r$ left singular vectors of the matrix $\tilde{L}_j := [\hat{\ell}_{t_j}, \hat{\ell}_{t_{j+1}}, \ldots, \hat{\ell}_{t_{j+1-1}}]$, or equivalently of $[P(t_j), P(t_{j+1}), \ldots, P(t_{j+1-1})]$. Let $a_t := P_j \ell_t$, $\xi_{t_j} := P_j \hat{a}_t$ and $\nu_t := \hat{\ell}_t - \ell_t = P_j a_t$.

Another way to understand the above is that we are expressing $\tilde{L}_j = L_j + V_j$ where $L_j$ is the rank-$r$ SVD of $\tilde{L}$, while $V_j$ is the rest. While $L_j V_j' = 0$, we cannot say anything about individual vectors $\ell_t, \nu_t$ or their expected value. In general, $\mathbb{E}[\ell_t, \nu_t] \neq 0$. But even then, we can always use Cauchy-Schwarz to get the bound $\|\mathbb{E}[\ell_t, \nu_t]\| \leq \sqrt{\lambda^+ \lambda^-}$. Thus, to analyze this case, we need to modify Corollary 2.4 for PCASDDN as follows: we now need $4\sqrt{\beta} f + \lambda^+ \Lambda^{-1/2} \sqrt{\lambda^-} f < 0.4\varepsilon_{SE}$. There is no change to the required lower bound on $\alpha$. From our definition of $\nu_t$, $\lambda^+ \Lambda^{-1/2} \leq \mathbb{E}(P_j, P(t)) \lambda^+$. Using $\lambda^+ \leq \lambda^+, \lambda^- \leq \lambda^-$, a simple sufficient condition to ensure that the third term is small $(\lambda^+ \Lambda^{-1/2} \leq 0.01\varepsilon_{SE}/f)$ is $\mathbb{E}(P_j, P(t))^2 \lambda^+ \leq 0.01\varepsilon_{SE}/f^2$.

Corollary 5.13 (Subspace changing at each time). Consider the setting defined in the first paragraph above. If $\mathbb{E}(P_j, P(t))^2 < 0.01\varepsilon_{SE}/f^2$, Theorem 3.11 applies with $P_j, \xi_t$ and $\nu_t$ as defined above.

B. NORST-NoDet: NORST without subspace change detection

A simpler version of the NORST algorithm that does not detect change is as follows. The robust regression (projected CS) step is exactly as explained earlier. The subspace update step is much simpler; it just updates $\tilde{P}(t)$ as the top $r$ left singular vectors of $\tilde{L}_t \xi_t$ once every $\alpha$ frames. We refer to it as NORST-NoDet. We have the following guarantee for it.

Theorem 5.14. Consider Algorithm 2 with parameters set as $\alpha = C f^2 \mu r \log n$, $\zeta = x_{min}/15$ and $\omega_{supp} = x_{min}/2$. Assume everything stated in Theorem 3.11 except the lower bound on SE$_J$. Then, w.p. at least $1 - 10dn^{-10}$,

$$\mathbb{E}(\tilde{P}(t), P(t)) \leq \begin{cases} \min(4f SE_{J_1}, 1) & \text{if } t \in J_1, \\ (0.3)^{-1} \min(4f SE_{J_2}, 1) & \text{if } t \in J_k, \\ \varepsilon := c \sqrt{\lambda^+ / \lambda^-} & \text{if } t \in J_K, \end{cases}$$
where \( J_1 = \{ [t_j/\alpha] \alpha, (t_j/\alpha + 1) \alpha \}, J_k = \{ [t_j/\alpha] + (k-1) \alpha + (k-1)(k+1) \alpha \} \) for \( k = 2, 3, \cdots, K-1 \) and \( J_K = \{ [t_j/\alpha] + (K+1) \alpha \}. \) The time complexity is \( O(ndr \log(1/\varepsilon)) \) and memory complexity is \( O(n \alpha) = O(f^2 n r \log n \log(1/\varepsilon)). \)

The advantage of NORST-NoDet is that it does not require a lower bound on the amount of change, \( SE_j \), and it needs fewer algorithm parameters (does not need \( K \) or \( \omega_{\text{evals}} \)). The disadvantage is it does not detect subspace change, we cannot obtain a “smoothing” version of it that solves the dynamic RPCA problem to \( \varepsilon \) accuracy at all times, and its subspace error bound is larger for the intervals during which the subspace changes, \( [t_j/\alpha], (t_j/\alpha + 1)\alpha \). For times \( t \) in this interval, the bound is \( \min(4/S \varepsilon, 1) \). Assuming small enough \( \varepsilon \), this is larger than \( (\varepsilon + SE_j) \) which is the NORST bound for this interval. The reason is NORST stops tracking after the current subspace has been estimated accurately enough and until the next change is detected. During this period, it uses \( \hat{P}_{-1} \) as the estimate. But NORST-NoDet updates the subspace in every interval. For the change interval, the rank of \( L_{t;\alpha} \) is more than \( r \). It can be \( 2r \) in general. This is why it is not possible to guarantee a better bound for the \( r \)-SVD estimate. At the same time, without extra assumptions, it is not possible to obtain a guarantee for \( 2r \)-SVD estimate either.

For analyzing the change interval we use the following modification of PCA-DDNN. Its proof is in Appendix I. The proof of Theorem 5.14 is given in Appendix III.

**Corollary 5.15.** Assume that \( y_t = \xi_t + \omega_t + \nu_t \) with \( \omega_t = M_{11}^t \xi_t \) with \( \xi_t = P_0 \alpha_t \) for \( t \in [0, \alpha_0] \) and \( \xi_t = P_0 \alpha_t \) for \( t \in [0, \alpha_0] \), and \( SE(P_0, P) \leq \Delta \). Assume also that Assumptions 2.1, 2.2 hold, \( \max_t \max(||M_{11}^t P_0||, ||M_{11}^t P||) \leq q < 1 \), and the fraction of nonzeros in any row of the noise matrix \( [\omega_1, \omega_2, \ldots, \omega_\alpha] \) is equal to \( b \). Let \( g := \frac{\alpha^*}{\alpha} \). If \( \Delta < c/f \), and if \( \alpha \geq \alpha^* = C \max \left( \frac{2f}{\alpha^*}, \frac{\alpha^*}{\alpha}, \frac{\alpha^*}{\alpha} \right) \) then w.p. at least \( 1 - 10n^{-10} \),

\[
SE(\hat{P}, P) \leq 1.1 \left( \left( \alpha/\alpha^* \right) \Delta + 4v_b \|q\| + \frac{\alpha^*}{\alpha^*} \max(r_v, r) \right) \leq 3.3 \Delta f + 4.4v_b f + 1.1 \frac{\lambda^*}{\lambda^*}.
\]

**VI. PROOF OF CORRECTNESS OF THE NORST ALGORITHM**

In this section we state the three main lemmas and explain how they help prove Theorem 3.11. After this, we prove the three lemmas.

**A. Main Lemmas**

We define or recall a few things first.

1) Recall \( \Delta = \max_j SE(P_{j-1}, P) \), let \( \Delta_0 = SE(\hat{P}_0, P_0) \); recall \( c \sqrt{\lambda^*/\lambda^*} < \varepsilon \leq 0.01 < 0.2 \)

**Algorithm 2 NORST-NoDet**

1: Input: \( \hat{P}_0, y_t \); Output: \( \hat{x}_t, \hat{t}_t, \hat{P}(t) \); Parameters: \( \omega_{\text{supp}}, \xi_t, \alpha \)

2: \( \hat{P}(_{t_{\text{train}}}) \leftarrow \hat{P}_0 \)

3: for \( t > t_{\text{train}} \) do

4: Lines 6-11 of Algorithm 1

5: if \( t = t_{\text{train}} + w_{\text{out}} - 1 \) for \( u = 1, 2, \cdots \), then

6: \( P_u \leftarrow \text{SVd} [L_{t;\alpha}], \hat{P}(t) \leftarrow \hat{P}_u \)

7: else

8: \( \hat{P}(t) \leftarrow \hat{P}(t_{-1}) \)

9: end if

10: end for

2) Let \( \hat{P}_{0,T} = \hat{P}_{T-1} \) and recall (from Algorithm) that \( \hat{P}_{T-1} = \hat{P}_{T-1,K} \).

3) Constants for Theorem 3.11: \( c_1 = c_2 = 0.01 \) (bounds on max-outlier-frag-col, max-outlier-frag-row(\( \alpha \))), \( \Delta = 1/(30 \sqrt{n}) \). Use \( b_0 = c_2/f^2 \) to denote the bound on max-outlier-frag-row(\( \alpha \)).

4) Let \( q_0 := 1.2(\varepsilon + SE_j) \), \( q_k := 1.2 \max(q_{k-1} - 1/\varepsilon) \).

Clearly \( q_k = \max(0.2k \varepsilon, 0.1 \varepsilon) \).

First consider the simpler case when \( t_j \)'s are known. In this case \( t_j = t_j \). Define the events

- \( \Gamma_{j,0} := \{ \text{assumed bound on } SE(P_0, P_0) \} \).
- \( \Gamma_{j,k} := \{ \text{SE}(P_{j-k}, P_{j-k}) \leq SE(P_0, P_0) \} \).
- \( \Gamma_{j,0} := \{ \Gamma_{j-1,K} \cap \{ \text{SE}(P_{j-k}, P_{j-k}) \leq q_{k-1}/4 \} \} \) for \( j = 1, 2, \ldots, K \) and \( K = 1, 2, \ldots, K \).

Using the expression for \( K \) given in the theorem, and since \( \hat{P}_j = \hat{P}_{j,k} \) (from the Algorithm), it follows that \( \Gamma_{j,K} \) implies \( SE(P_j, P_j) = SE(P_{j,K}, P_{j,K}) \leq \varepsilon \).

Observe that, if we can show that \( \operatorname{Pr}(\Gamma_{j,K}^c | \Gamma_{0,0}) \geq 1 - d_{\text{in}}^{-10} \), we have obtained all the subspace recovery bounds of Theorem 3.11. The next two lemmas, Lemmas 6.16 and 6.17, applied sequentially help show that this is true. The first one proves that \( \operatorname{Pr}(\Gamma_{1,1} | \Gamma_{0,0}) \geq 1 - 10n^{-10} \), the second one proves that \( \operatorname{Pr}(\Gamma_{j,k} | \Gamma_{j,k-1}) \geq 1 - 10n^{-10} \) for \( k = 1, 2, \ldots, K \). The bounds on \( ||e_t - \hat{e}_t|| \) follow easily.

To prove the actual result with \( t_j \) unknown, we also need Corollary 6.19 and Lemma 6.20 which proves that the change detection step works as desired. Moreover, we will need a different definition of \( \Gamma_{j,0} \); we cannot set it equal to \( \Gamma_{j-1,K} \). The proof is given in Appendix II.

**Lemma 6.16 (first update interval).** Under the conditions of Theorem 3.11, conditioned on \( \Gamma_{0,j} \),

1) for all \( t \in [t_j^u, t_j^u + \alpha] \), \( \| \Psi(\xi_t + v_t) \| < (\varepsilon + \Delta) \sqrt{\mu \lambda^+} + \sqrt{\varepsilon \lambda^+} < \| \hat{x}_t - x_t \| \leq 7 \| x_{min} / 15, T_t = T_t, \} \), the error \( e_t := \hat{x}_t - x_t \) satisfies

\[
e_t = I_{T_t} (\Psi_{T_t} \Psi_{T_t}^{-1} I_{T_t} \Psi(\xi_t + v_t))
\]

and \( \| e_t \| < 1.2(\varepsilon + \Delta) \sqrt{\mu \lambda^+} + \sqrt{\varepsilon \lambda^+}) \). Here \( \Psi = I - \hat{P}_{0,T} \hat{P}_{0,T}^t \). Recall we let \( \hat{P}_{0,T} = \hat{P}_{j-1} \).
2) w.p. at least $1 - 10n^{-10}$, $\hat{P}_{j,1}$ satisfies $\text{SE}(\hat{P}_{j,1}, P_j) \leq \max(q_{k-1}/4, \varepsilon)$, i.e., $\Gamma_{j,k}$ holds.

**Lemma 6.17** (k-th update interval). Under the conditions of Theorem 3.11, conditioned on $\Gamma_{j,k-1}$,

1) for all $t \in [\hat{t}_j + (k - 1)\alpha, \hat{t}_j + k\alpha - 1]$, all claims of the first part of Lemma 6.16 holds, $\| \Psi(\ell_t + \nu_t) \| \leq \max(0, 3^{k-1}(\varepsilon + \Delta_2), \varepsilon) \| \mu \lambda \| + \sqrt{\mu \lambda \| \Delta_2 \|}$, and $\| e_t \| \leq \max((0.3)^{k-1}1.2(\varepsilon + \Delta_2), \varepsilon) \| \mu \lambda \| + \sqrt{\mu \lambda \| \Delta_2 \|}$. Here $\Psi = I - \hat{P}_{j,k-1} P_{j,k-1}'$.

2) w.p. at least $1 - 10n^{-10}$, $\hat{P}_{j,k}$ satisfies $\text{SE}(\hat{P}_{j,k}, P_j) \leq \max(q_{k-1}/4, \varepsilon)$, i.e., $\Gamma_{j,k}$ hold.

**Remark 6.18.** For the case of $j = 0$, in both the lemmas above, $\Delta$ gets replaced by $\text{SE}(\hat{P}_0, P_0)$.

**Corollary 6.19.** Under the conditions of Theorem 3.11, the following also hold.

1) For all $t \in [\hat{t}_j, \hat{t}_j + \alpha]$, conditioned on $\Gamma_{j-1,K}$, all claims of the first item of Lemma 6.16 hold.

2) For all $t \in [\hat{t}_j + K\alpha, t_{j+1}]$, conditioned on $\Gamma_{j,K}$, the first item of Lemma 6.17 holds with $k = K$.

Thus, for all times $t$, under appropriate conditioning, $e_t$ satisfies (7).

The following lemma shows that, w.h.p., we can detect subspace change within $2\sigma$ time instants without any false detections. Recall that the detection threshold $\omega_{evals} = 2\sigma^2 \lambda^+$. 

**Lemma 6.20** (Subspace Change Detection). Assume that the conditions of Theorem 3.11 hold.

1) Consider an $\alpha$-length time interval $\mathcal{J}^\alpha \subset [t_j, t_{j+1}]$ during which $\hat{P}_{t-1} = \hat{P}_{j-1}$ so that $\Psi = I - \hat{P}_{j-1} P_{j-1}'$. Let $\Phi = \hat{P}_{j-1} P_{j-1}'$. Assume that $\text{SE}(\hat{P}_{j-1}, P_{j-1}) \leq \varepsilon$ and $e_t$ satisfies (7). Then, w.p. at least $1 - 10n^{-10}$,

$$\lambda_{\max} \left( \frac{1}{\alpha} \sum_{t \in \mathcal{J}^\alpha} \Phi \ell_t \ell_t' \Phi \right) \geq 0.5 \lambda^+ \text{SE}_j (\text{SE}_{j-8} \varepsilon) > \omega_{evals}$$

since $\text{SE}_j > 9\sqrt{T}\varepsilon$.

2) Consider an $\alpha$-length time interval $\mathcal{J}^\alpha \subset [t_j, t_{j+1}]$ during which $\hat{P}_{t-1} = \hat{P}_j$ so that $\Psi = I - \hat{P}_j P_j'$. Let $\Phi = \hat{P}_j P_j'$. Assume that $\text{SE}(\hat{P}_j, P_j) \leq \varepsilon$ and $e_t$ satisfies (7). Then, w.p. at least $1 - 10n^{-10}$,

$$\lambda_{\max} \left( \frac{1}{\alpha} \sum_{t \in \mathcal{J}^\alpha} \Phi \ell_t \ell_t' \Phi \right) \leq 1.37 \varepsilon^2 \lambda^+ < \omega_{evals}$$

**B. Proof of the first two lemmas**

The projected CS proof (item one of the first two lemmas) uses the following lemma from [11] that relates the s-Restricted Isometry Constant (RIC), $\delta_s(\epsilon)$ [28] of $I - PP'$ to incoherence of $P$.

**Lemma 6.21.** [11] For an $n \times r$ basis matrix $P$, (1) $\delta_s(I - PP') = \max_{\|T\| \leq s} \| I_T P \|^2$; and (2) $\max_{\|T\| \leq s} \| I_T P \|^2 \leq s \max_{i=1,2,\ldots,n} \| I_T P \|^2 \leq s \mu \sqrt{n}$. 

The last bound of the above lemma used the definition of the incoherence parameter $\mu$. We will apply this lemma with $s = \max(\text{outlier-frac-col} \cdot n)$. The subspace update step proof (item 2 of the first two lemmas) uses Corollary 2.4 for PCA-SDDN and the following simple lemma proved in the Supplement given in the Arxiv version of this work [22, Appendix V].

**Lemma 6.22.** Let $Q_1$, $Q_2$ and $Q_3$ be $r$-dimensional subspaces in $\mathbb{R}^n$ such that $\text{SE}(Q_1, Q_2) = \Delta_1$ and $\text{SE}(Q_2, Q_3) = \Delta_2$. Then, $\Delta_1 - 2\Delta_2 \leq \text{SE}(Q_1, Q_3) \leq \Delta_1 + \Delta_2$.

**Proof of Lemma 6.16.** Proof of item 1. First consider $j > 0$. We have conditioned on the event $\Gamma_{j,0} := \Gamma_{j-1,K}$. This implies that $\text{SE}(\hat{P}_{j-1}, P_{j-1}) \leq \varepsilon$.

For the interval $t \in [\hat{t}_j, \hat{t}_j + \alpha]$, $\hat{P}_{t-1} = \hat{P}_{j-1}$ and thus $\Psi = I - \hat{P}_{j-1} P_{j-1}'$ (from Algorithm). Let $s = \max(\text{outlier-frac-col} \cdot n)$. For the sparse recovery step, we need to bound the 2s-RIC of $\Psi$. To do this, we obtain bound on $\max_{\|T\| \leq 2s} \| I_T P_{j-1}' \|$. As follows. Consider any set $T$ such that $|T| \leq 2s$. Then,

$$\| I_T P_{j-1}' \| \leq \| I_T(I - P_{j-1}' P_{j-1}) P_{j-1}' \| + \| I_T P_{j-1}' P_{j-1}' P_{j-1}' \|$$

$$\leq \text{SE}(P_{j-1}, \hat{P}_{j-1}) + \| I_T P_{j-1}' \|$$

$$= \text{SE}(P_{j-1}, P_{j-1}) + \| I_T P_{j-1}' \|$$

Using Lemma 6.21, and the bound on max-outlier-frac-col from Theorem 3.11,

$$\max_{\|T\| \leq 2s} \| I_T P_{j-1}' \| \leq \| I_T P_{j-1}' \| \leq 2s \max_i \| I_T P_{j-1}' \|^2 \leq 2s \mu \sqrt{n} \leq 0.01$$

Thus, using $\text{SE}(\hat{P}_{j-1}, P_{j-1}) \leq \varepsilon$, (where $c \sqrt{\lambda^+ / \lambda^-} \leq \varepsilon \leq 0.01$),

$$\max_{\|T\| \leq 2s} \| I_T P_{j-1}' \| \leq \varepsilon + \max_{\|T\| \leq 2s} \| I_T P_{j-1}' \| \leq \varepsilon + 0.1$$

Finally, using Lemma 6.21, $\delta_{2s}(\Psi) \leq 0.11^2 < 0.15$. Hence

$$\| (\Psi_T' \Psi_T')^{-1} \| \leq \frac{1}{1 - \delta_1(\Psi)} \leq \frac{1}{1 - \delta_2(\Psi)} \leq \frac{1}{1 - 0.15} < 1.2.$$ 

When $j = 0$, there are some minor changes. From the initialization assumption, we have $\text{SE}(\hat{P}_0, P_0) \leq 0.25$. Thus, $\max_{\|T\| \leq 2s} \| I_T P_0 \| \leq 0.25 + 0.1 = 0.35$. Thus, using Lemma 6.21, $\delta_{2s}(\Psi_0) \leq 0.35^2 < 0.15$. The rest of the proof given below is the same for $j = 0$ and $j > 0$.

Next we bound norm of $b_i := \Psi(\ell_i + \nu_i)$.

$$\| b_i \| = \| \Psi(\ell_i + \nu_i) \| \leq \| (I - \hat{P}_{j-1} \hat{P}_{j-1}') P_j a_i \| + \| \nu_i \| \leq \text{SE}(\hat{P}_{j-1}, P_{j-1})$$

$$\leq (\varepsilon + \text{SE}(P_{j-1}, P_{j-1})) \mu \sqrt{\lambda^+} + \sqrt{\lambda^-}$$

where $a_i$ follows from Lemma 6.22 with $Q_1 = \hat{P}_{j-1}, Q_2 = \hat{P}_{j-1}$ and $Q_3 = P_j$. Under the assumptions of Theorem 3.11, the RHS of (a) is bounded by $\varepsilon_{\min}/15$. This is why we have set $\xi = \varepsilon_{\min}/15$ in the Algorithm. Using these facts, and
$\delta_{2s}(\Psi) \leq 0.15$, the CS guarantee from [28, Theorem 1.3] implies that

$$\|\hat{x}_{t,cs} - x_t\| \leq 7\xi = 7x_{min}/15 \leq x_{min}/2$$

Consider support recovery. From above,

$$|(|\hat{x}_{t,cs} - x_t|)| \leq ||\hat{x}_{t,cs} - x_t|| \leq 7x_{min}/15 \leq x_{min}/2$$

The Algorithm sets $\omega_{supp} = x_{min}/2$. Consider an index $i \in \mathcal{T}_t$. Since $|(|\hat{x}_t|)| \geq x_{min}$,

$$x_{min} - |(|\hat{x}_{t,cs} - x_t|)| \leq \frac{1}{2} x_{min} \leq |(|\hat{x}_t|)| - |(|\hat{x}_{t,cs} - x_t|)| \leq \frac{x_{min}}{2}$$

Thus, $|(|\hat{x}_{t,cs} - x_t|)| > \frac{x_{min}}{2} = \omega_{supp}$ which means $i \in \hat{T}_t$. Hence $\mathcal{T}_i \subseteq \hat{T}_t$. Next, consider any $j \notin \hat{T}_t$. Then, $|(|\hat{x}_t|)| = 0$ and so

$$|(|\hat{x}_{t,cs} - x_t|)| = |(|\hat{x}_{t,cs} - x_t|)| - |(|\hat{x}_t|)| \leq |(|\hat{x}_{t,cs} - x_t|)| - |(|\hat{x}_t|)| \leq \frac{x_{min}}{2}$$

which implies $j \notin \hat{T}_t$ and $\hat{T}_i \subseteq \hat{T}_t$ implying that $\hat{T}_t = \hat{T}_t$.

With $\hat{T}_t = \mathcal{T}_t$ and since $\mathcal{T}_t$ is the support of $x_t$, $x_t = I_{\mathcal{T}_t}I_{\mathcal{T}_t}^\top x_t$, and so

$$\hat{x}_t = I_{\mathcal{T}_t} (\Psi_{\mathcal{T}_t})^{-1} \Psi_{\mathcal{T}_t}^\top \Psi_{\mathcal{T}_t}^\top x_t + \mathcal{X}_{\mathcal{T}_t}$$

since $\Psi_{\mathcal{T}_t}^\top \Psi_{\mathcal{T}_t}^\top \Psi_{\mathcal{T}_t} = I_{\mathcal{T}_t}^\top I_{\mathcal{T}_t}$. Thus $e_t = \hat{x}_t - x_t$ satisfies

$$e_t = I_{\mathcal{T}_t} (\Psi_{\mathcal{T}_t})^{-1} \Psi_{\mathcal{T}_t}^\top (\Psi_{\mathcal{T}_t}^\top x_t + \mathcal{X}_{\mathcal{T}_t})$$

This implies that $e_t$ is a problem of PCA in sparse data-dependent noise (SDDN). To analyze this, we use Corollary 2.4.

Define $(e_{\ell})_{\ell} = I_{\mathcal{T}_t} (\Psi_{\mathcal{T}_t})^{-1} \Psi_{\mathcal{T}_t}^\top (\Psi_{\mathcal{T}_t}^\top x_t + \mathcal{X}_{\mathcal{T}_t})$ and $(e_{\nu})_{\nu} = I_{\mathcal{T}_t} (\Psi_{\mathcal{T}_t})^{-1} \Psi_{\mathcal{T}_t}^\top \Psi_{\mathcal{T}_t}^\top x_t + \mathcal{X}_{\mathcal{T}_t}$. Recall from the Algorithm that we compute $P_{\mathcal{T}_t}$ as the top $r$ eigenvectors of $\frac{1}{n} \sum_{i=1}^n \hat{\ell}_i^\top x_i$. In the notation of Corollary 2.4, $\hat{y}_t \equiv \hat{\ell}_t$, $v_i \equiv (e_{\ell})_{\ell}$, $v_i \equiv \mathcal{X}_{\mathcal{V}_t}$, and $y_t \equiv (e_{\nu})_{\nu}$. Also, $\lambda^\top_{\mathcal{T}_t}$ is the $r$-th largest singular value of $\hat{\Psi}_{\mathcal{T}_t}$. Applying Corollary 2.4 with $q \equiv q_0$, $b \equiv b_0$ and using $\xi_{SE} = max(q_0/4, \varepsilon)$, observe that we require

$$4\sqrt{b_0q_0f} + (2.2)^2 \lambda^\top_{\mathcal{V}_t} / \lambda^- \leq 0.4 \max(q_0/4, \varepsilon)$$

From above, $\max(q_0/4, \varepsilon) = q_0/4$ (if the max is $\varepsilon$ we stop the tracking). The required bound holds since $q_0/4 \geq \varepsilon > c\sqrt{\lambda^\top_{\mathcal{V}_t} / \lambda^-}$ (from Theorem) and $\sqrt{\lambda_{-1}} = 0.01/f$. Corollary 2.4 also requires $\alpha \geq \alpha_s$ which is defined in it. Our choice of $\alpha = C_{\mathcal{F}}^2/\mu / \log n$ satisfies this since $q_0^2 / \varepsilon_{SE}^2 = 4^2$ and $(\lambda^\top_{\mathcal{T}_t} / \lambda^-)^2 / \varepsilon_{SE}^2 < C$. Thus, by Corollary 2.4, with probability at least $1 - 10n^{-10}$, $\text{SE}(\hat{P}_{\mathcal{T}_t}, P_j) \leq \max(q_0/4, \varepsilon) \boxdot$

**Remark 6.23** (Clarification about conditioning). In the proof above we have used Corollary 2.4 for $\hat{l}_t$’s for $t \in \mathcal{J}^o := [\hat{t}_j, \hat{t}_j + \alpha]$. This corollary assumes that, for $t \in \mathcal{J}^o$, $\alpha_t$’s are mutually independent and $M_{\mathcal{S}_t}$’s are deterministic matrices. Let $y_{\text{old}} := \{y_1, y_2, \ldots, y_{t_j-1}\}$. We apply Corollary 2.4 conditioned on $y_{\text{old}}$, for a $y_{\text{old}} \in \mathcal{Y}_{\text{old}}$. Conditioned on $y_{\text{old}}$, clearly, the matrices $M_{\mathcal{S}_t}$ used in the proof above are deterministic. Also $y_{\text{old}}$ is independent of the $\alpha_t$’s for $t \in \mathcal{J}^o$ and thus, even conditioned on $y_{\text{old}}$, the $\alpha_t$’s for $t \in \mathcal{J}^o$ are mutually independent. Corollary 2.4 tells us that, for any $y_{\text{old}} \in \mathcal{Y}_{\text{old}}$, conditioned on $y_{\text{old}}$, w.p. at least $1 - 10n^{-10}$, $\text{SE}(\hat{P}_{\mathcal{T}_t}, P_j) \leq \max(q_0/4, \varepsilon)$. Since this holds with the same probability for all $y_{\text{old}} \in \mathcal{Y}_{\text{old}}$, it also holds with the same probability when we condition on $\mathcal{Y}_{\text{old}}$. Thus, conditioned on $\mathcal{Y}_{\text{old}}$, with this probability, $\mathcal{Y}_{\text{old}}$ holds. An analogous argument also applies for the next proof.

**Proof of Lemma 6.17.** We first present the proof for the $k = 2$ case and then generalize it for an arbitrary $k$. Consider $k = 2$. We have conditioned on $\mathcal{Y}_{\text{old}}$. This implies that $\text{SE}(\hat{P}_{\mathcal{T}_t}, P_j) \leq q_0/4$. We consider the interval $t \in [\hat{t}_j + \alpha, \hat{t}_j + 2\alpha]$. For this interval, $\hat{P}_{(t-1)} = \hat{P}_{\hat{t}_j}$ and thus $\Psi = I - \hat{P}_{\hat{t}_j}\hat{P}_{\hat{t}_j}$. Consider any set $\mathcal{T}$ such that $|\mathcal{T}| \leq 2s$. We have

$$\|I_{\mathcal{T}}' \hat{P}_{\hat{t}_j}\| \leq \|I_{\mathcal{T}}'(I - P_j)\hat{P}_{\hat{t}_j}\| + \|I_{\mathcal{T}}' P_j \hat{P}_{\hat{t}_j}\| \leq \text{SE}(\hat{P}_{\hat{t}_j}, P_j) + \|I_{\mathcal{T}}' P_j\|$$

The equality holds since SE is symmetric for subspaces of the same dimension. Using $\text{SE}(\hat{P}_{\hat{t}_j}, P_j) \leq \max(q_0/4, \varepsilon)$, (8),

$$\max_{|\mathcal{T}| \leq 2s} \|I_{\mathcal{T}}' \hat{P}_{\hat{t}_j}\| \leq \max(q_0/4, \varepsilon) + \max_{|\mathcal{T}| \leq 2s} \|I_{\mathcal{T}}' P_j\| \leq \max(q_0/4, \varepsilon) + 0.1.$$}

By the assumptions of Theorem 3.11, $q_0 \leq 0.96$ and $\varepsilon \leq 0.2$. Using this and Lemma 6.21.

$$\delta_{2s}(\Psi) = \max_{|\mathcal{T}| \leq 2s} \|I_{\mathcal{T}}' \hat{P}_{\hat{t}_j}\| \leq 0.35^2 < 0.15$$

Finally,

$$\|b_j\| \leq \|\Psi(\ell_t + \nu_t)\| \leq \|I - \hat{P}_{\hat{t}_j}\hat{P}_{\hat{t}_j}\| \|P_j\nu_t\| \leq \max(q_0/4, \varepsilon) \sqrt{\mu / \lambda^\top_{\mathcal{V}_t} + \sqrt{\lambda^\top_{\mathcal{V}_t}}}$$

The rest of the proof is the same and this ensures exact
support recovery and the expression for $e_t$.

*Proof of Item 2:* Again, updating $\hat{P}_i$ using $\hat{e}_i$’s is a PCA-SDDN problem. We use Corollary 2.4. We compute $\hat{P}_{j,2}$ as the top $r$ eigenvectors of $\frac{1}{n} \sum_{t=\ell_j+\kappa}^{\ell_j+k+1} \hat{e}_i \hat{e}_i'$. From item 1, $e_t$ satisfies (7) for this interval. In the notation of Corollary 2.4, $y_t \equiv \hat{e}_t$, $w_t \equiv (e_t)_t$, $\ell_t \equiv \ell_t$, $v_t \equiv (e_t)_t + \nu_t$, $P = \hat{P}_j$, $\hat{P} \equiv \hat{P}_{j,2}$, and $M_{s,t} = - (\Psi_t \Psi_t')^{-1} \Psi_t \hat{e}_i$. So $\|M_{s,t}P\| = \| (\Psi_t \Psi_t')^{-1} \Psi_t \hat{e}_i \| \leq 1.2 \max(\kappa_0/4, \varepsilon) := q_1$. Applying Corollary 2.4 with $q = q_1$, $b = b_0$ ($b_0$ bounds max-outlier-frac-row($\alpha$)), and setting $\varepsilon_{SE} = \max(q_1/4, \varepsilon)$, observe that we require

$$4 \sqrt{b_0 q_1 f} + (2.2)^2 \lambda_1^+ / \lambda^- \leq 0.4 \max(q_1/4, \varepsilon)$$

Once again recall that the max is $q_1/4$. The above bound holds since $\sqrt{b_0 f} \leq 0.01$ and $q_1/4 > \varepsilon > \sqrt{\lambda_1^+ / \lambda^-}$. Corollary 2.4 also requires $\alpha \geq \alpha_0$. Our choice of $\alpha = C f^2 \mu_r \log n$ satisfies this requirement since $\gamma_1^2 / \varepsilon_{SE}^2 = 4^2$ and $(\lambda_1^+ / \lambda^-) / \varepsilon_{SE} < C$. Thus, from Corollary 2.4, with probability at least $1 - 10n^{-10}$, $\| \hat{P}_{j,2} - P \| \leq \max(q_1/4, \varepsilon)$.

**B) General k:** We have conditioned on $\Gamma_{j,k-1}$. This implies that $\| \hat{P}_{j,k-1} - P \| \leq \max(k_1-1,4/\varepsilon)$. Consider the interval $[\ell_j + (k-1)\alpha, \ell_j + k\alpha]$. In this interval, $P(\ell) \equiv \hat{P}_{j,k-1}$ and thus $\Psi = I - \hat{P}_{j,k-1} \hat{P}_{j,k-1}'$. Using the same idea as for the $k = 2$ case, we have that for the $k$-th interval, $q_k-1 = \max(0.3^{k-1} q_0, \varepsilon)$. Pick $\varepsilon_{SE} = \max(q_{k-1} - 1, 4/\varepsilon)$. From this it is easy to see that

$$\delta_{2a}(\Psi) \leq \left( \max_{|T| \leq 2^a} \|I_T' \hat{P}_{j,k-1}\|^2 \right)$$

$$\leq \left( \text{SE}(\hat{P}_{j,k-1} - P) + \max_{|T| \leq 2^a} \|I_T' P\|^2 \right)$$

$$\leq \left( \text{SE}(\hat{P}_{j,k-1} - P) + 0.1 \right)^2$$

$$\leq \left[ \max(0.3^{k-1}(\varepsilon + \text{SE}(\hat{P}_{j,k-1} - P), \varepsilon) + 0.1 \right]^2 < 0.15$$

where (a) follows from (8). Also, before,

$$\| \Psi(e_t + \nu_t) \| \leq \text{SE}(\hat{P}_{j,k-1} - P) \|a_i\| + \|\nu_i\|$$

$$\leq \max(0.3^{k-1}(\varepsilon + \text{SE}(\hat{P}_{j,k-1} - P), \varepsilon) \sqrt{\mu_r \lambda_1^+} + \sqrt{\tau_r \lambda_1^+}$$

$$\leq \max(0.3^{k-1}(\varepsilon + \text{SE}(\hat{P}_{j,k-1} - P), \varepsilon) \sqrt{\mu_r \lambda_1^+} + \sqrt{\tau_r \lambda_1^+}$$

*Proof of Item 2:* Again, updating $\hat{P}_i$ from $\hat{e}_i$’s is a problem of PCA in sparse data-dependent noise given in Corollary 2.4. From Item 1 of this lemma we know that, for $t \in [\ell_j + (k-1)\alpha, \ell_j + k\alpha]$, $e_t$ satisfies (7). We update the subspace, $\hat{P}_j$ as the top $r$ eigenvectors of $\frac{1}{n} \sum_{t=\ell_j+\kappa}^{\ell_j+k+1} \hat{e}_i \hat{e}_i'$. In the setting above $y_t \equiv \hat{e}_t$, $w_t \equiv (e_t)_t$, $\ell_t \equiv \ell_t$, $v_t \equiv (e_t)_t + \nu_t$, and $M_{s,t} = - (\Psi_t \Psi_t')^{-1} \Psi_t \hat{e}_i$, and so $\|M_{s,t}P\| = \| (\Psi_t \Psi_t')^{-1} \Psi_t \hat{e}_i \| \leq 1.2 \max(k_0/4, \varepsilon) := q_1$. Applying Corollary 2.4 with $q = q_1$, $b = b_0$ ($b_0$ bounds max-outlier-frac-row($\alpha$)), and setting $\varepsilon_{SE} = \max(q_1/4, \varepsilon)$, we require $4 \sqrt{b_0 q_1 f} + \lambda_1^+ / \lambda^- \leq 0.4 \max(q_1/4, \varepsilon)$. This holds as explained earlier and hence, by Corollary 2.4, the result follows.

*Proof of Lemma 6.20*

*Proof of Item 1:* We are considering an $\alpha$-consecutive frames interval $J_\alpha$ in $[t_j, t_{j+1}]$ during which $P(t-1) = \hat{P}_{j-1}$. Thus $\Psi = \Phi = I - \hat{P}_{j-1} \hat{P}_{j-1}'$. Recall from earlier that at all times $t$, $\hat{e}_t = \hat{e}_t - e_t + \nu_t$, where $e_t = (e_t)_t + (e_\nu)_t$, $w_t \equiv (e_t)_t = \Psi_t \Psi_t^{-1} \hat{e}_t$ is sparse and data-dependent noise, and $\nu_t \equiv (e_\nu)_t + \nu_t$ is small unstructured noise. As in the earlier proofs, $w_t = (e_t)_t$, can be expressed as $w_t = I_{T_t} M_{s,t} \ell_t$ where $M_{s,t} = (\Psi_t \Psi_t^{-1})^{-1} \hat{e}_t \Psi_t$. Thus, $q = q_0 = 1.2 \text{SE}(\hat{P}_{j-1} - P_j) \leq 1.2(\varepsilon + \text{SE}_j) + b = b_0$. Let

$$\frac{1}{\alpha} \sum_t \hat{P}_t e_t \hat{e}_t' \Phi' = \frac{1}{\alpha} \sum_t \hat{P}_t e_t \hat{e}_t' \Phi' + \Phi_{\text{noise}} \Phi + \Phi_{\text{cross}} \Phi$$

where $\text{noise} = \frac{1}{\alpha} \sum_t w_t w_t' + \frac{1}{\alpha} \sum_t v_t v_t'$ and cross contains the cross terms. By Weyl’s inequality,

$$\lambda_{\max} \left( \frac{1}{\alpha} \sum_t \hat{P}_t e_t \hat{e}_t' \Phi' \right) \geq \lambda_{\max} \left( \frac{1}{\alpha} \sum_t \hat{P}_t e_t \hat{e}_t' \Phi \right) - \| \Phi_{\text{cross}} \Phi \|$$

Using Corollary I.2 from Appendix I, w.p. at least $1 - 10n^{-10}$, if $\alpha$ is as given in our Theorem,

$$\| \Phi_{\text{cross}} \Phi \| \leq 2.02 \sqrt{b} \| \Phi P_j \| q_0 \lambda^+$$

Since $\| \Phi P_j \| \leq q = 1.2(\varepsilon + \text{SE}_j)$, using the above two inequalities,

$$\lambda_{\max} \left( \frac{1}{\alpha} \sum_t \hat{P}_t e_t \hat{e}_t' \Phi \right) \geq \lambda_{\max} \left( \frac{1}{\alpha} \sum_t \hat{P}_t e_t \hat{e}_t' \Phi \right) - 2.02 \sqrt{b} (1.2(\varepsilon + \text{SE}_j))^2 \lambda^+$$

We bound the first term of (11), Term1, as follows. Let $\Phi P_j^{QR} = E_j R_j$ be its reduced QR decomposition. Thus $E_j$ is an $n \times r$ matrix with orthonormal columns and $R_j$ is an $r \times r$ upper triangular matrix. Let

$$A := R_j \left( \frac{1}{\alpha} \sum_{t \in J_\alpha} a_t a_t' \right) R_j'$$

Observe that Term1 can also be written as

$$\text{Term1} = [E_j E_j'] A [E_j' E_j']$$

and thus $\lambda_{\max}(A) = \lambda_{\max}(\text{Term1})$. We work with $\lambda_{\max}(A)$ in the sequel. We will use the following simple claim.

**Claim 6.24:** If $X \succeq 0$ (i.e., $X$ is a p.s.d matrix), where $X \in \mathbb{R}^{r \times r}$, then $RXR' \succeq 0$ for all $R \in \mathbb{R}^{r \times r}$. 




Proof. Since \( X \) is p.s.d., \( y'Xy \geq 0 \) for any vector \( y \). Use this with \( y = R'z \) for any \( z \in \mathbb{R}^r \). We get \( z'RXX'Rz \geq 0 \). Since this holds for all \( z \), \( RXX'R \geq 0 \). □

By Lemma 1.3 from Appendix I, with \( \epsilon_0 = 0.01\lambda^- \),
\[
\Pr \left( \frac{1}{\alpha} \sum_t a_t a_t' - (\lambda^- - \epsilon_0)I \geq 0 \right) \geq 1 - 2n^{-10}
\]
By Claim 6.24 from above, with probability \( 1 - 2n^{-10} \),
\[
R_j \left( \frac{1}{\alpha} \sum_t a_t a_t' - (\lambda^- - \epsilon_0)I \right) R_j' \geq 0
\]
\[
\Rightarrow \lambda_{\min} \left( R_j \left( \frac{1}{\alpha} \sum_t a_t a_t' - (\lambda^- - \epsilon_0)I \right) R_j' \right) \geq 0
\]
Using Weyl’s inequality, with the same probability,
\[
\lambda_{\min} \left( R_j \left( \frac{1}{\alpha} \sum_t a_t a_t' - (\lambda^- - \epsilon_0)I \right) R_j' \right) \leq \lambda_{\max} \left( R_j \left( \frac{1}{\alpha} \sum_t a_t a_t' \right) R_j' \right) - (\lambda^- - \epsilon_0)\lambda_{\max} (R_j R_j')
\]
and so,
\[
\lambda_{\max}(A) \geq \lambda^- - \epsilon_0)\lambda_{\max} (R_j R_j'). \tag{13}
\]
Using Lemma 6.22 and since \( SE(\tilde{P}_{j-1}, P_{j-1}) \leq \epsilon \) we get
\[
\lambda_{\max}(R_j R_j') = ||R_j||^2 = SE^2(\tilde{P}_{j-1}, P_{j-1}) \geq (SE_j - 2\epsilon)^2 \tag{14}
\]
Thus, combining (11), (12), (13), (14), w.p. at least \( 1 - 10n^{-10} \),
\[
\lambda_{\max} \left( \frac{1}{\alpha} \sum_{t \in J} \Phi \ell_t \ell_t' \Phi \right) \geq 0.99\lambda^- (SE_j - 2\epsilon)^2 - 2.02\sqrt{b_0}(1.2(\epsilon + SE_j)^2)\lambda^+ \\
\geq 0.99\lambda^- SE_j(0.6SE_j - 8\epsilon) = 0.59\lambda^- SE_j(SE_j - 8\epsilon)
\]
In the above, we used \( \sqrt{b_0}f = 0.1 \). Since \( SE_j > 9\sqrt{f}\epsilon, 0.59\lambda^- SE_j(SE_j - 8\epsilon) > 5\lambda^+ \epsilon^2 > \omega_{\text{evals}} \).

Proof of Item 2: We proceed as in the proof of item 1 except that now \( \Phi = \Psi = I - \tilde{P}_j \tilde{P}_j' \). Thus, \( q = q_K = \epsilon \) and \( \|\Phi F_P\| \leq q_K \). Using Weyl’s inequality and Corollary 1.2 from Appendix I, w.p. at least \( 1 - 10n^{-10} \),
\[
\lambda_{\max} \left( \frac{1}{\alpha} \sum_{t \in J} \Phi \ell_t \ell_t' \Phi \right) \leq \lambda_{\max} \left( \frac{1}{\alpha} \sum_t \Phi \ell_t \ell_t' \Phi + ||\Phi \text{cross} \Phi| + \lambda_{\max}(\Phi \text{noise} \Phi) \right) \\
\leq \lambda_{\max} \left( \frac{1}{\alpha} \sum_{t \in J} \Phi P_j a_t a_t' P_j' \right) + 2.02\sqrt{b_0}\|\Phi F_P\|q_K\lambda^+ + 1.01\sqrt{b_0}q_K\lambda^+ + \epsilon^2\lambda^-
\]
Proving as before to bound \( \lambda_{\max}(\text{Term1}) \), define \( \Phi F_p \leq QR \), define \( A \) as before, we know
\[
\lambda_{\max}(\text{Term1}) = \lambda_{\max}(E_j' (\text{Term1}) E_j) = \lambda_{\max}(A).
\]
Further,
\[
\lambda_{\max}(A) = \lambda_{\max} \left( R_j \left( \frac{1}{\alpha} \sum_{t \in J} a_t a_t' \right) R_j' \right) \\
(a) \leq \lambda_{\max} \left( \frac{1}{\alpha} \sum_{t \in J} a_t a_t' \right) \lambda_{\max} (R_j R_j')
\]
where (a) uses Ostrowski’s theorem [39, Theorem 5.4.9]. We have
\[
\lambda_{\max}(R_j R_j') = \sigma_{\max}^2 (R_j) = \|\Phi F_P\|^2 \leq \epsilon^2
\]
and we can bound \( \lambda_{\max}(\frac{1}{\alpha} \sum_{t \in J} a_t a_t') \) using the first item of Lemma I.3. Combining all of the above, and using \( \|\Phi F_P\| \leq q_K \leq \epsilon \) and \( b_0 f^2 = 0.01 \), w.p. at least \( 1 - 10n^{-10} \),
\[
\lambda_{\max} \left( \frac{1}{\alpha} \sum_{t \in J} \Phi \ell_t \ell_t' \Phi \right) \leq 1.37\epsilon^2 \lambda^+
\]
Recall that \( \omega_{\text{evals}} = 2\epsilon^2 \lambda^+ \) and thus, \( 1.37\epsilon^2 \lambda^+ < \omega_{\text{evals}} \). □

VII. EMPIRICAL EVALUATION

In this section we present numerical experiments on synthetic and real data to validate our theoretical claims. Extra experimental details are presented in the Supplementary Material.

Synthetic Data. First we compare the results of NORST and smoothing-NORST with RST, Online RPCA, and static RPCA methods. We generate the changing subspaces, \( P_j = e^{\gamma_j B_j} P_{j-1} \) as done in [12] where \( \gamma_j \) controls the amount of subspace change and \( B_j \)’s are skew-symmetric matrices. In the first experiment we used the following parameters. \( n = 1000, d = 12000, J = 2, t_1 = 3000, t_2 = 8000, r = 30, \gamma_1 = 0.001, \gamma_2 = \gamma_1 \). We set \( \alpha = 300 \). Next, we generate the coefficients \( a_t \in \mathbb{R}^r \) as independent zero-mean, bounded random variables. They are \( (a_t)_{i,j} i,j \sim \text{unif}[-q_i, q_i] \) where \( q_i = \sqrt{f} - \sqrt{f(i-1)/2r} \) for \( i = 1, 2, \ldots, r-1 \) and \( q_r = 1 \). Thus the condition number is \( f \) and we selected \( f = 50 \). For the sparse supports, we considered two models according to which the supports are generated. First we use Model G.24 [18] which simulates a moving object pacing in the video. For the first \( t_{train} = 100 \) frames, we used a smaller fraction of outliers with parameters \( s/n = 0.01, b_0 = 0.01 \). For \( t > t_{train} \) we used \( s/n = 0.05 \) and \( b_0 = 0.3 \). Secondly, we used the Bernoulli model to simulate sampling uniformly at random, i.e., each entry of the matrix, is independently selected with probability \( p = 0.01 \) for the first \( t_{train} \) frames and with probability \( p = 0.3 \) for subsequent frames. The sparse outlier magnitudes for both support models are generated uniformly at random from the interval \([x_{\min}, x_{\max}]\) with \( x_{\min} = 10 \) and \( x_{\max} = 20 \).
We initialized the s-ReProCS and NORST algorithms using \( \text{AltProj} \) applied to \( Y_{[1,t_{\text{train}}]} \) with \( t_{\text{train}} = 100 \). For the parameters to \( \text{AltProj} \) we used the true value of \( r \), 15 iterations and a threshold of 0.01. This, and the choice of \( \gamma_1 \) and \( \gamma_2 \) ensure that \( \text{SE}(\hat{P}_{\text{init}}, P_0) \approx \text{SE}(P_1, P_0) \approx \text{SE}(P_2, P_0) \approx 0.01 \). The other algorithm parameters are set as mentioned in the theorem, i.e., \( K = \lceil \log(c/\varepsilon) \rceil = 8 \), \( \alpha = C \gamma \log n = 300 \), \( \omega = x_{\text{min}}/2 = 5 \) and \( \xi = tx_{\text{min}}/15 = 0.67 \), \( \omega_{\text{evals}} = 2\pi^2\lambda^+ = 7.5 \times 10^{-4} \). For the other online methods we implement the algorithms without modifications. The regularization parameter for ORRPCA was set as with \( \lambda_1 = 1/\sqrt{n} \) and \( \lambda_2 = 1/\sqrt{d} \) according to [13]. Wherever possible we set the tolerance as \( 10^{-6} \) and 100 iterations to match that of our algorithm. As shown in Fig. 1, NORST is significantly better than all the RST methods - s-ReProCS [18], and two popular heuristics - ORRPCA [13] and GRASTA [12].

We also provide a comparison of smoothing techniques in Fig 1. To ensure a valid time comparison, we implement the static RPCA methods on the entire data matrix \( Y \). Although, we could implement the static techniques on disjoint batches of size \( \alpha \), we observed that this did not yield significant improvement in terms of reconstruction accuracy, while being considerably slower, and thus we report only the latter setting. As can be seen, smoothing NORST outperforms all static RPCA methods, both for the moving object and the Bernoulli models. For the batch comparison we used PCP, AltProj and RPCA-GD. We set the regularization parameter for PCP \( 1/\sqrt{n} \) in accordance with [4]. The other known parameters, \( r \) for Alt-Proj, outlier-fraction for RPCA-GD, are set the ground truth. For all algorithms we set the threshold as \( 10^{-6} \) and the number of iterations to 100. All results are averaged over 100 independent runs.

In Fig. 2 we validate our claim of NORST admitting a higher fraction of outliers per row. We only compare with AltProj since it is has the highest tolerance among other methods. We chose 10 different values of each of \( r \) and \( b_0 \) (we slightly misuse notation here to let \( b_0 := \text{max-outlier-frac-row} \) for this section only). For each pair of \( b_0 \) and \( r \) we implemented NORST and AltProj over 100 independent trials and computed the relative error, \( \|\hat{L} - L\|_F/\|L\|_F \) for each run. We illustrate the fraction of times the error seen by each algorithm is less than a threshold, 0.5. We chose this threshold since for smaller values, AltProj consistently failed. As can be seen, NORST is able to tolerate a much larger fraction of outlier-per-row as compared to AltProj.

In the third experiment we analyze the effect of the lower bound on the outlier magnitude \( x_{\text{min}} \) with the performance of NORST and AltProj. We show the results in Fig. 3. The only change in data generation is that we now choose three
different values of $x_{\min} = \{0.5, 5, 10\}$, and we set all the non-zero entries of the sparse matrix to be equal to $x_{\min}$. This is actually harder than allowing the sparse outliers to take on any value since for a moderately low value of $x_{\min}$ the outlier-lower magnitude bound of Theorem 3.11 is violated. This is indeed confirmed by the numerical results presented in Fig. 3. (i) When $x_{\min} = 0.5$, NORST works well since now all the outliers get classified as the small unstructured noise $\nu_t$. (ii) When $x_{\min} = 10$, NORST still works well because now $x_{\min}$ is large enough so that the outlier support is mostly correctly recovered. (iii) But when $x_{\min} = 5$ the NORST reconstruction error stagnates around $10^{-3}$. All AltProj errors are much worse than those of NORST because the outlier fraction per row is the same as in the first experiment and thus the effect of varying $x_{\min}$ is not pronounced.

Real Data. We also evaluate our algorithm for the task of Background Subtraction. For the AltProj algorithm we set $r = 40$. The remaining parameters were used with default setting. For NORST, we set $\alpha = 60$, $K = 3$, $\xi_t = \|\Psi \hat{\Psi}_{t-1}\|_2$. We found that these parameters work for most videos that we verified our algorithm on. For RPCA-GD we set the “corruption fraction” $\alpha = 0.2$ as described in their paper.

We use two standard datasets, the Meeting Room (MR) and the Lobby (LB) sequences. LB is a relatively easy sequence since the background is static for the most part, and the foreground occlusions are small in size. As can be seen from Fig. 4 (first two rows), most algorithms perform well on this dataset. MR is a challenging data set since the color of the foreground (person) is very similar to the background curtains, and the size of the object is very large. Thus, NORST is able to outperform all methods, while being fast.

VIII. CONCLUSIONS AND FUTURE DIRECTIONS

In this work we developed a fast and (nearly) delay optimal robust subspace tracking solution that we called NORST. NORST is a mini-batch algorithm with memory complexity that is also nearly optimal. It detects subspace changes and tracks them to $\varepsilon$ accuracy with a delay that is more than the subspace dimension $r$ by only log factors: the delay is order $r \log n \log(1/\varepsilon)$. The memory complexity is $n$ times this number while $nr$ is the amount of memory required to store the output subspace estimate. Our guarantee for NORST needs assumptions similar to those needed by standard robust PCA solutions. Different from standard robust PCA, we need slow subspace change, we replace right singular vectors’ incoherence by a statistical version of it, but we need a weaker bound on outlier fractions per row.

Slow subspace change is a natural assumption for background images of static camera videos (with no sudden scene changes). Our statistical assumptions on $a_t$ are mild and can be relaxed further. As already explained, the identically distributed requirement can be relaxed. In the video application, the zero mean assumption can be approximately satisfied if we estimate the mean background image by computing the empirical average of the first $t_{\text{train}}$ frames, $\hat{L}_{[1:t_{\text{train}}]}$, obtained using AltProj. Mutual independence of $a_t$’s models the fact that the changes in each background image w.r.t. a “mean” background are independent, when conditioned on their subspace. This is valid, for example, if the background changes are due to illumination variations or due to moving curtains (see Fig. 4). Mutual independence can be relaxed to instead assuming an autoregressive model on the $a_t$’s: this will require using the matrix Azuma inequality [20] to replace matrix Bernstein. We believe the zero mean requirement can also be eliminated.

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Appendix I
Proofs for Sec. II

A. Proof of Theorem 2.3

Proof of Theorem 2.3. This uses the Davis-Kahan sin theta theorem [21]:

Lemma 1.1 (Davis-Kahan sin $\theta$ theorem). Let $D_0$ be a Hermitian matrix whose span of top $r$ eigenvectors equals $\text{span}(P)$. Let $D$ be the Hermitian matrix with top $r$ eigenvectors $P$. Then,

$$
\text{SE}(\hat{P}, P) \leq \frac{\| (D - D_0) P \|}{\lambda_r(D_0) - \lambda_{r+1}(D)} \|D - D_0\|
$$

as long as the denominator is positive. The second inequality follows from the first using Weyl's inequality.

For our proof, set $D_0 = \frac{1}{r} \sum_{t} l_t l_t'$ and $D = \frac{1}{r} \sum_{t} y_t y_t'$. Then, $\hat{P}$ is its matrix of top $r$ eigenvectors. Observe

$$
D - D_0 = \frac{1}{r} \sum_{t} (y_t y_t' - l_t l_t')
$$

$$
= \frac{1}{r} \sum_{t} (w_t w_t' + v_t v_t' + l_t w_t' + v_t l_t')
$$

$$
= \text{noise}_{w} + \text{noise}_{v} + \text{cross}_{l, w} + \text{cross}_{l, v}
$$

$$
= \text{noise} + \text{cross}
$$

Also notice that $\lambda_{r+1}(D_0) = 0$, $\lambda_r(D_0) = \lambda_{\min} \left( \frac{1}{r} \sum_{t} a_t a_t' \right)$. Now, applying Theorem 1.1,

$$
\text{SE}(\hat{P}, P) \leq \frac{2 \|\text{cross}\| + \|\text{noise}\|}{\lambda_{\min} \left( \frac{1}{r} \sum_{t} a_t a_t' \right) - \text{numerator}}
$$

Now, we can bound $\|\text{cross}\| \leq \|\text{E}[\text{cross}]\| + \|\text{cross} - \text{E}[\text{cross}]\|$ and similarly for the noise term. We use the Cauchy-Schwartz inequality for bounding the expected values of both. Recall that $M_t = M_{2,t} M_{1,t}$ with $b := \|\frac{1}{r} \sum_{t} M_{2,t} M_{2,t}'\|$ and $q := \max_t \|M_{1,t} P\|$ with $q < 2$. Thus,

$$
\|\text{E}[\text{cross}]\| \leq \left\| \frac{1}{r} \sum_{t} M_{t} P A P' M_{1,t}' M_{2,t}' \right\| + \|\text{noise}\|
$$

$$
\leq \left\| \frac{1}{r} \sum_{t} M_{t} P A P' M_{1,t}' (\cdot) \right\| \left\| \frac{1}{r} \sum_{t} M_{2,t} M_{2,t}' \right\| + \lambda^+ + \lambda^+
$$

$$
\leq \|\text{max}_t \|M_{t} P A P' M_{1,t}'\|^2 b + \lambda^+ \leq \sqrt{b q^2} \lambda^+ + \lambda^+
$$

Similarly,

$$
\|\text{E}[\text{cross}]\| \leq \left\| \frac{1}{r} \sum_{t} M_{2,t} M_{1,t} P A P' \right\| \left\| \frac{1}{r} \sum_{t} M_{2,t} M_{2,t}' \right\|
$$

$$
\leq \left\| \frac{1}{r} \sum_{t} P A P' M_{1,t}' M_{1,t} P A P' \right\| \left\| \frac{1}{r} \sum_{t} M_{2,t} M_{2,t}' \right\|
$$

$$
\leq \|\text{max}_t \|M_{1,t} P A P'\|^2 b \leq \sqrt{b q} \lambda^+
$$

Since $v_t$ is uncorrelated noise, $\text{E}[\text{cross}] = 0$ and $\text{E}[\text{cross}] = 0$. We now lower bound $\lambda_{\min} \left( \frac{1}{r} \sum_{t} a_t a_t' \right)$ as

$$
\lambda_{\min} \left( \frac{1}{r} \sum_{t} a_t a_t' \right) = \lambda_{\min} \left( \Lambda - \left( \frac{1}{r} \sum_{t} a_t a_t' - \Lambda \right) \right)
$$

$$
\geq \lambda_{\min}(\Lambda) - \lambda_{\max} \left( \frac{1}{r} \sum_{t} a_t a_t' - \Lambda \right)
$$

$$
\geq \lambda - \left\| \frac{1}{r} \sum_{t} a_t a_t' - \Lambda \right\|
$$

and thus we have

$$
\text{SE}(\hat{P}, P) \leq \frac{4 \sqrt{b q} \lambda^+ + \lambda^+ + 2 \|\text{cross} - \text{E}[\text{cross}]\| + \|\text{noise} - \text{E}[\text{noise}]\|}{\lambda - \left\| \frac{1}{r} \sum_{t} a_t a_t' - \Lambda \right\| - \text{numerator}}
$$

Concentration bounds. Now we only need to bound $\|\text{noise} - \text{E}[\text{noise}]\|$ and $\|\text{cross} - \text{E}[\text{cross}]\|$. These are often referred to as “statistical error”, while the error due to nonzero $\|\text{E}[\text{cross}]\|$ or $\|\text{E}[\text{noise}]\|$ is called the “bias”. We use concentration bounds...
from Lemma 1.3.

\[
\|\text{noise} - \mathbb{E}[\text{noise}]\| + 2\|\text{cross} - \mathbb{E}[\text{cross}]\|
\]

\[
\leq \left\| \frac{1}{\alpha} \sum_t (w_t w_t' - \mathbb{E}[w_t w_t']) \right\| + \left\| \frac{1}{\alpha} \sum_t (v_t v_t' - \mathbb{E}[v_t v_t']) \right\|
\]

\[
+ 2 \left\| \frac{1}{\alpha} \sum_t (\ell_t v_t' - \mathbb{E}[\ell_t v_t']) \right\| + 2 \left\| \frac{1}{\alpha} \sum_t w_t v_t' \right\|
\]

\[
\leq C \sqrt{\mu q^2 f} \sqrt{\frac{r \log n}{\alpha}} \lambda^{-} + C \sqrt{\mu} \frac{\lambda^{+}}{\lambda^{-}} \sqrt{\frac{r \log n}{\alpha}} \lambda^{-} + C \sqrt{\mu q f} \sqrt{\frac{r \log n}{\alpha}} \lambda^{-}
\]

\[
+ C \sqrt{\mu q f} \sqrt{\frac{r \log n}{\alpha}} \lambda^{-} + C \sqrt{\mu} \frac{\lambda^{+}}{\lambda^{-}} \sqrt{\frac{r \log n}{\alpha}} \lambda^{-}
\]

\[
\leq C \sqrt{\mu q f} \sqrt{\frac{r \log n}{\alpha}} \lambda^{-} + C \sqrt{\mu} \frac{\lambda^{+}}{\lambda^{-}} \sqrt{\frac{r \log n}{\alpha}} \lambda^{-} := H(\alpha) \lambda^{-}
\]

where the last line follows from using \( q^2 \leq 2q \) and \( \lambda^{+} \leq \lambda^{-} \).

In case we only need to bound \( \|\text{noise} - \mathbb{E}[\text{noise}]\| \), we can get a tighter bound that contains only the first two terms and not all five. Clearly, we have

\[
\|\text{noise} - \mathbb{E}[\text{noise}]\|
\]

\[
\leq C \sqrt{\mu q^2 f} \sqrt{\frac{r \log n}{\alpha}} \lambda^{-} + C \sqrt{\mu} \frac{\lambda^{+}}{\lambda^{-}} \sqrt{\frac{r \log n}{\alpha}} \lambda^{-} +
\]

\[
:= H\text{noise}(\alpha)
\]

The bound on \( \frac{1}{\alpha} \sum_t a_t a_t' - \Lambda \|_2 \) follows directly from the first item of Lemma 1.3.

\[\square\]

B. A useful corollary that follows from above proof

From the above proof, we can write out a bound for \( \|\Phi \text{cross} \Phi'\| \) for a projection matrix \( \Phi \) by noticing that each term of cross is of the form \( \sum_t \ell_t \ell_t' = P \sum_t a_t \ell_t \). Thus \( \|P'\text{cross}\| = \|\text{cross}\|. \) Thus, \( \|\Phi \text{cross} \Phi'\| \leq \|\Phi P\|\|\text{cross}\| \leq \|\Phi P\|\|\mathbb{E}[\text{cross}]\| + \|\text{cross} - \mathbb{E}[\text{cross}]\|\). Similarly, we can also get a bound on \( \lambda_{\max} (\text{noise}) = \|\text{noise}\| \).

Assume \( b = 0.01/f^2, q > \varepsilon > \sqrt{\mu} \). Consider cross. If \( \alpha \geq C \max (\sqrt{\mu} \log n, 2 \sqrt{q} \max (r_v, r) \log n) \), then \( H(\alpha) \leq \varepsilon_1 \lambda^{-} \). If we set \( \varepsilon_1 = 0.002 \max (\sqrt{\mu} \sqrt{\varepsilon}, 1) \) and \( b = 0.01/f^2 \) (bound on max-outlier-frac-row(\( \alpha \))), then, since \( \varepsilon > \sqrt{\mu} \), \( \alpha = C f^2 \max (r_v, r) \log n \) suffices. Since \( q \geq \varepsilon \), then, \( \varepsilon_1 < 0.002 \sqrt{\mu} b \).

\[
\|\Phi \text{cross} \Phi'\| \leq \|\Phi P\| (2 \sqrt{\sqrt{\mu} \lambda^{-} + H(\alpha) \lambda^{-}) \leq 2.02 \sqrt{\mu} \|\Phi P\| q^2 \lambda^{-}
\]

Consider noise. We will use \( H\text{noise}(\alpha) \) for this. If \( \alpha > C \max (\sqrt{\mu} \log n, 2 \sqrt{q} \max (r_v, r) \log n) \), then \( H\text{noise}(\alpha) \leq \varepsilon_2 \lambda^{-} \). If we set \( \varepsilon_2 = 0.002 \sqrt{\mu} \max (q^2, \varepsilon^2) \), then since \( \varepsilon > 2 \sqrt{q} \), thus, \( \alpha = C f^2 \max (r_v, r) \log n \) suffices. Since \( q \geq \varepsilon \), \( \varepsilon_2 = 0.002 \sqrt{\mu} b^2 \). We have the following corollary.

\[\text{Corollary 1.2.} If \alpha = C f^2 \max (r_v, r) \log n, and if } q \geq \varepsilon > \sqrt{\mu}, \text{ then, w.p. } 1 - 10n^{-10}, \]

\[
\|\Phi \text{cross} \Phi'\| \leq \|\Phi P\| (2 \sqrt{b q} \lambda^{+} + H(\alpha) \lambda^{-}) \leq 2.02 \sqrt{b} \|\Phi P\| q^2 \lambda^{+}
\]

\[
\lambda_{\max} (\Phi \text{noise} \Phi) \leq \|\text{noise}\| \leq \sqrt{\mu} b^2 \lambda^{+} + \lambda^{+} + H(\alpha) \lambda^{-}
\]

\[
\leq 1.01 \sqrt{b^2} \lambda^{+} + \lambda^{+} \leq 1.01 \sqrt{b^2} \lambda^{+} + \varepsilon^2 \lambda^{-}
\]

\[C. \text{ Main idea of the proof of Corollary 5.15} \]

The key difference in this proof is our choice of \( D_0 \). Since we want to bound \( \text{SE}(\hat{P}, P) \), we need to pick it in such a way that its matrix of top \( r \) singular vectors equals \( \text{span}(P) \).

We pick \( D_0 = \frac{1}{\alpha} P ((\alpha - \alpha_0) A + \alpha_0 P' P_0 A P_0' P') P' \)

Clearly, \( \lambda_{r+1}(D_0) = 0 \). With this choice of \( D_0 \),

\[\bar{D} - D_0 = \text{cross} + \text{cross} + \text{noise}
\]

\[
+ \left( \frac{1}{\alpha} \sum_t \ell_t \ell_t' - \mathbb{E}[\frac{1}{\alpha} \sum_t \ell_t \ell_t'] \right) + \left( E[\frac{1}{\alpha} \sum_t \ell_t \ell_t' - D_0] \right)
\]

where cross, noise are as defined earlier with the change that \( \ell_t \) is now defined differently. Thus, the only thing that changes when bounding these is our definition of \( q \). The last term in the expression above equals \( \varepsilon_0 P' P_0 A P_0' P' + \varepsilon_0 P_0 A P_0' P' \). This is what generates the extra \( 4f A \) term in our SE bound. A complete proof is provided in the Supplement given in the ArXiv version of this work [22, Appendix IV-A1].

\[D. \text{ Concentration Bounds} \]

We state the lemma below so that it can also be used in proving the most general PCA result given in the Supplement given in the ArXiv version of this work [22, Appendix IV-A1]. Let \( \Lambda_t = \mathbb{E}[a_t a_t'] \), \( \Lambda = \frac{1}{n} \sum_t \Lambda_t \), \( \lambda_{\max} = \max_t \|\Lambda_t\| \), \( \lambda_{\avg} = \lambda_{\min} (\Lambda) \), \( f = \lambda_{\max} / \lambda_{\avg} \), \( \lambda_{\max, \avg} = \max_t \|\mathbb{E}[v_t v_t']\| \) and \( g = \lambda_{\max, \avg} / \lambda^{-} \)

To use the lemma under the simpler i.i.d. assumption used in the main paper, remove the \( \max, \avg \) subscripts from all terms, e.g., replace \( \lambda_{\max} \) by \( \lambda^{+} \), \( \lambda_{\avg} \) by \( \lambda^{-} \) and so on.
Lemma 1.3. With probability at least $1 - 10n^{-10}$,
\[
\begin{align*}
\left\| \frac{1}{\alpha} \sum_{t} a_t a'_t - \Lambda \right\| & \leq C \sqrt{\mu f} \sqrt{\frac{r \log n}{\alpha}} \lambda_{\text{avg}}, \\
\left\| \frac{1}{\alpha} \sum_{t} \ell_t w'_t - \frac{1}{\alpha} \mathbb{E} \left[ \sum_{t} \ell_t w'_t \right] \right\| & \leq C \sqrt{\mu q f} \sqrt{\frac{\log n}{\alpha}} \lambda_{\text{avg}}, \\
\left\| \frac{1}{\alpha} \sum_{t} w_t w'_t - \frac{1}{\alpha} \mathbb{E} \left[ \sum_{t} w_t w'_t \right] \right\| & \leq C \sqrt{\mu q^2 f} \sqrt{\frac{\log n}{\alpha}} \lambda_{\text{avg}}, \\
\left\| \frac{1}{\alpha} \sum_{t} \ell_t v'_t - \frac{1}{\alpha} \mathbb{E} \left[ \sum_{t} \ell_t v'_t \right] \right\| & \leq C \sqrt{\mu q^2 f} \sqrt{\frac{\log n}{\alpha}} \lambda_{\text{avg}}, \\
\left\| \frac{1}{\alpha} \sum_{t} v_t v'_t - \frac{1}{\alpha} \mathbb{E} \left[ \sum_{t} v_t v'_t \right] \right\| & \leq C \sqrt{\mu q^2 f} \sqrt{\frac{\log n}{\alpha}} \lambda_{\text{avg}},
\end{align*}
\]

Proof of Lemma 1.3. $a_t a'_t$ term. This and all other terms use Matrix Bernstein for rectangular matrices, Theorem 1.6 of [40]. This says the following. For a finite sequence of $d_1 \times d_2$ zero mean independent matrices $Z_k$ with
\[
\left\| Z_k \right\|_2 \leq R, \quad \text{and} \quad \max \left( \left\| \sum_k \mathbb{E}[Z_k' Z_k] \right\|_2, \left\| \sum_k \mathbb{E}[Z_k Z'_k] \right\|_2 \right) \leq \sigma^2,
\]
we have $\mathbb{P}(\left\| \sum_k Z_k \right\|_2 \geq s) \leq (d_1 + d_2) \exp \left( - \frac{s^2}{2R^2} \right) \leq (d_1 + d_2) \exp \left( - \frac{s^2}{2 \sigma^2 R}{\frac{\log n}{\alpha}} \right)$. Let $Z_t := a_t a'_t$ and we apply the above result to $Z_t = \tilde{Z}_t - \mathbb{E}[\tilde{Z}_t]$ with $s = c\alpha$. Now it is easy to see that $\left\| Z_t \right\|_2 \leq 2\|a_t a'_t\|_2 \leq 2\|a_t\|_2$, i.e., $\|Z_t\|_2 \leq 2\|a_t\|_2 \leq 2\mu \lambda_{\max} := R$ and similarly, $\|\mathbb{E}[Z_t']\| = \|\mathbb{E}[a'_t]Z_t\| \leq 2\alpha \cdot \max \|a_t\|_2 \|a'_t\| \leq 2\alpha \cdot \max \|a_t\|_2 \|a'_t\| \leq 2\alpha \cdot \max \|a_t\|_2 \|a'_t\| \leq 2\alpha \cdot \max \|a_t\|_2 \|a'_t\| \leq 2\alpha \cdot \max \|a_t\|_2 \|a'_t\| \leq 2\alpha \cdot \max \|a_t\|_2 \|a'_t\| \leq 2\alpha \cdot \max \|a_t\|_2 \|a'_t\| \leq 2\alpha \cdot \max \|a_t\|_2 \|a'_t\|$, and thus, w.p. at most $2 \exp \left( - \frac{c^2}{\sqrt{\lambda_{\max}}} \right)$.

Thus, the overall structure of this proof is similar to that in [17], [41].

Proof of Theorem 3.11. The overall structure of this proof is similar to that in [17], [41]. Define
\[
\hat{t}_{j-1,\text{fin}} := \hat{t}_{j-1} + K\alpha, \quad t_{j,\text{s}} = \hat{t}_{j-1,\text{fin}} + \left[ \frac{t_j - \hat{t}_{j-1,\text{fin}}}{\alpha} \right] \alpha
\]

Thus, $\hat{t}_{j-1,\text{fin}}$ is the time at which the $(j-1)$-th subspace update is complete; w.h.p., this occurs before $t_j$. With this assumption, $t_{j,\text{s}}$ is such that $t_j$ lies in the interval $[t_{j,\text{s}} - \alpha + 1, t_{j,\text{s}}]$. Recall from the algorithm that we increment $j$ to $j + 1$ at $t = \hat{t}_j + K\alpha = \hat{t}_{j,\text{fin}}$. Define the events
\[
\begin{align*}
1) & \quad \text{Det0} := \{ \hat{t}_j = t_{j,\text{s}} \} = \{ \lambda_{\max} \left( \frac{1}{\alpha} \sum_{t=\hat{t}_{j-1,\text{fin}}+1}^{t_{j,\text{s}}} \Phi \ell_t \Phi \right) > \omega_{\text{evals}} \}, \\
2) & \quad \text{Det1} := \{ \hat{t}_j = t_{j,\text{s}} + \alpha \} = \{ \lambda_{\max} \left( \frac{1}{\alpha} \sum_{t=\hat{t}_{j-1,\text{fin}}+1}^{t_{j,\text{s}}+\alpha} \Phi \ell_t \Phi \right) > \omega_{\text{evals}} \}, \\
3) & \quad \text{NoFalseDets} := \{ \text{FalseDets} = 0 \}, \\
4) & \quad \text{Det0} := \{ \text{SE}(P_0, P_0) \leq 0.25 \}, \\
5) & \quad \text{Det1} := \{ \text{Det1} \cap \text{Det0} \cap \text{NoFalseDets} \},
\end{align*}
\]

Let $p_0$ denote the probability that, conditioned on $\Gamma_{j-1,\text{end}}$, the change got detected at $t = t_{j,\text{s}}$, i.e., let
\[
p_0 := \Pr(\text{Det0} | \Gamma_{j-1,\text{end}}).
\]

Thus, $\Pr(\text{Det0} \cap \Gamma_{j-1,\text{end}}) = 1 - p_0$. It is not easy to bound $p_0$. However, as we will see, this will not be needed. Assume that $\Gamma_{j-1,\text{end}} \cap \text{Det0} \cap \text{Det1}$ holds. Consider the interval $J_{\alpha} := [t_{j,\text{s}}, t_{j,\text{s}} + \alpha]$. This interval starts at or after $t_j$, so, for all $t$ in this interval, the subspace has changed. For this interval, $\Phi = I - P_{j-1} P_j^{-1}$. Applying the first item of Lemma 6.20, w.p. at least $1 - 10n^{-10}$,
\[
\lambda_{\max} \left( \frac{1}{\alpha} \sum_{t \in J_{\alpha}} \Phi \ell_t \Phi \right) \geq \omega_{\text{evals}}
\]

Thus, we conclude that, $\| \sum_{t} \ell_t w'_t - \mathbb{E}[\sum_{t} \ell_t w'_t] \|_2 \geq \epsilon \alpha \cdot \frac{\sigma}{\sqrt{\mu q^2 \lambda_{\max}}}$. Thus, $\sigma^2 = \alpha \mu q^2 \lambda_{\max}^2$.
replaced by $\Gamma_{j,\text{end}} \cap \text{Det0} \cap \text{Det1} \cap \text{SubUpd}_1 \cap \cdots \cap \text{SubUpd}_{k-1}$ and with the k-th SVD interval being $J_k := [\tilde{t}_j + (k-1)\alpha, \tilde{t}_j + k\alpha)$. Applying Lemmas 6.16, and 6.17 for each $k$, we get

$$\Pr(\text{SubUpd}|\Gamma_{j,\text{end}} \cap \text{Det0} \cap \text{Det1}) \geq (1 - 10^{-10}n^{-10})^{K+1}.$$ 

We can also do a similar thing for the case when the change is detected at $t_j, +$, i.e. when Det$0$ holds. In this case, we replace $\Gamma_{j,0}$ by $\Gamma_{j,\text{end}} \cap \text{Det0}$ and $\Gamma_{j,\text{end}} \cap \text{Det0} \cap \text{Det1} \cap \text{SubUpd}_1 \cap \cdots \cap \text{SubUpd}_{k-1}$ and conclude that

$$\Pr(\text{SubUpd}|\Gamma_{j,\text{end}} \cap \text{Det0}) \geq (1 - 10^{-10}n^{-10})^{K}.$$ 

Finally consider the NoFalseDets event. First, assume that $\Gamma_{j,\text{end}} \cap \text{Det0} \cap \text{SubUpd}$ holds. Consider any interval $J^\alpha \subseteq [\tilde{t}_j, \tilde{t}_j + 1)$. In this interval, $P(t) = \hat{P}_j\Phi = I - P_j'\hat{P}_j'$ and $\text{SE}(P_j, P_j') \leq \varepsilon$. Using the second part of Lemma 6.20 we conclude that w.p. at least $1 - 10^{-10}$,

$$\lambda_{\text{evals}}\left(\sum_{\alpha} \left| \Phi \hat{e}_\alpha \hat{e}_\alpha^\top \Phi \right| \leq \omega_{\text{evals}}\right.$$ 

Since Det$0$ holds, $\tilde{t}_j = t_j, +$. Thus, we have a total of $\left\lfloor \tilde{t}_j + t_j, + - (K + \alpha) \right\rfloor$ intervals $J^\alpha$ that are subsets of $[\tilde{t}_j, \tilde{t}_j + 1)$. Moreover, $\left\lfloor \tilde{t}_j + t_j, + - (K + \alpha) \right\rfloor \leq \left\lfloor \tilde{t}_j + t_j - (K + 1) \right\rfloor$ since $\alpha \leq \alpha$. Thus,

$$\Pr(\text{NoFalseDets}|\Gamma_{j,\text{end}} \cap \text{Det0} \cap \text{SubUpd}) \geq (1 - 10^{-10})^{\left\lfloor \tilde{t}_j + t_j - (K + 1) \right\rfloor}.$$ 

On the other hand, if we condition on $\Gamma_{j,\text{end}} \cap \text{Det0} \cap \text{Det1} \cap \text{SubUpd}$, then $\tilde{t}_j = t_j, + \alpha$. Thus,

$$\Pr(\text{NoFalseDets}|\Gamma_{j,\text{end}} \cap \text{Det0} \cap \text{Det1} \cap \text{SubUpd}) \geq (1 - 10^{-10})^{\left\lfloor \tilde{t}_j + t_j - (K + 1) \right\rfloor}.$$ 

We can now combine the above facts to bound $\Pr(\Gamma_{j,\text{end}}|\Gamma_{j,\text{end}}^{\text{end}})$: Recall that $p_0 := \Pr(\text{Det0}|\Gamma_{j,\text{end}})$. Clearly, the events $(\text{Det0} \cap \text{SubUpd} \cap \text{NoFalseDets})$ and $(\text{Det0} \cap \text{Det1} \cap \text{SubUpd} \cap \text{NoFalseDets})$ are disjoint. Thus,

$$\Pr(\Gamma_{j,\text{end}}|\Gamma_{j,\text{end}}^{\text{end}}) = p_0 \Pr(\text{SubUpd} \cap \text{NoFalseDets}|\Gamma_{j,\text{end}} \cap \text{Det0})$$

$$+ (1 - p_0) \Pr(\text{Det1}|\Gamma_{j,\text{end}} \cap \text{Det0}).$$

$$\Pr(\text{SubUpd} \cap \text{NoFalseDets}|\Gamma_{j,\text{end}} \cap \text{Det0} \cap \text{Det1}) \geq p_0 (1 - 10^{-10})^{\left\lfloor \tilde{t}_j + t_j - (K + 1) \right\rfloor}$$

$$+ (1 - p_0)(1 - 10^{-10}).$$

$$= (1 - 10^{-10})^{\left\lfloor \tilde{t}_j + t_j - (K + 1) \right\rfloor} \geq (1 - 10^{-10})^{\left\lfloor \tilde{t}_j + t_j - (K + 1) \right\rfloor}.$$ 

Thus, since the events $\Gamma_{j,\text{end}}$ are nested, $\Pr(\Gamma_{j,\text{end}}|\Gamma_{0,\text{end}}) = \prod_j \Pr(\Gamma_{j,\text{end}}|\Gamma_{j-1,\text{end}}) \geq \prod_j (1 - 10^{-10})^{\tilde{t}_j + t_j - (K + 1)} = (1 - 10^{-10})^{d} \geq 1 - 100dn^{-10}.$

Proof of Corollary 3.12. It should be noted that basis($M$) is not a unique matrix, it refers to any matrix $P$ that has orthonormal columns and whose span equals the span of $M$. Thus basis($P_{j-1}, P_j$) ≡ basis($P_{j-1}, P_{j-1}, P_j$) ≡ basis($P_j, P_{j+1} P_{j-1}$). Let us denote any of these matrices by $\hat{P}_{j-1,j}$. For $t \in [\tilde{t}_j + K\alpha, t_j]$, $P(t) = P_{j-1}$ while for $t \in [t_j, \tilde{t}_j + K\alpha - 1]$, $P(t) = P_{j}$. For all $t$ in these two intervals $P(t) = \hat{P}_{j-1,j}$. The proof of this corollary is an easy consequence of this fact and the fact that, for two basis matrices $P_1, P_2$ that are mutually orthonormal, i.e., for which $P_1P_2' = 0$,

$$(I - P_1'P_1 - P_2'P_2) = (I - P_1'P_1)(I - P_2'P_2).$$

Thus, $SE(P_{j-1,j}, P_{j}) \leq SE(P_{j-1,j}, P_{j-1}) \leq \varepsilon$ and $SE(P_{j-1,j}, P_{j}) \leq SE(P_{j}, P_{j}) \leq \varepsilon$. □

**Appendix III**

**PROOFS FOR SECTION III: TIME COMPLEXITY DERIVATION AND PROOF OF THEOREM 5.14**

A. Time complexity derivation

Consider initialization. To ensure that $SE(P_0, P_0) \in O(1/\sqrt{\tau})$, we need to use $C\log r$ iterations of AltProj. Since there is no lower bound in the AltProj guarantee on the required number of matrix columns (except the trivial lower bound of rank) [7], we can use $t_{\text{train}} = Cr$ frames for initialization. Thus the initialization complexity is $O(n(t_{\text{train}})^2 \log(\sqrt{\tau}) = O(nr^3 \log r)$ [7]. The projected-CS step complexity is equal to the cost of a matrix vector multiplication with the measurement matrix times negative logarithm of the desired accuracy in solving the $l_1$ minimization problem. Since the measurement matrix for the CS step is $I - P(t_{\text{train}})P(t_{\text{train}})'$, the cost per CS step (per frame) is $O(nr \log(1/\varepsilon))$ [42] and so the total cost is $O((d - t_{\text{train}})nr \log(1/\varepsilon))$. The subspace update involves at most $(d - t_{\text{train}})/\alpha$ rank-r SVDs on $n \times \alpha$ matrices all of which have constant eigen-gap (this is indirectly proved in the proofs of the second item of Lemmas 6.16 and 6.17). Thus the total time for subspace update steps is at most $((d - t_{\text{train}})/\alpha)*O(nr \log(1/\varepsilon)) = O((d - t_{\text{train}})nr \log(1/\varepsilon))$ [43]. Thus the running time of the complete algorithm is $O(nrd \log(1/\varepsilon) + nr^3 \log r)$. As long as $r^2 \log r \leq d \log(1/\varepsilon)$, the time complexity of the entire algorithm is $O(nrd \log(1/\varepsilon))$.

B. Proof of Theorem 5.14 for NORST-NoDet

In this algorithm we do not detect change. We just keep updating the subspace by r-SVD applied every $\alpha$ time instants on the last $\alpha$ $\hat{t}$’s. $L(t\alpha)$. For $\alpha$-intervals $J$ for which $P(t) = P_j$ for all $t \in J$, there is no change to the analysis. We start at $t = t_0 = t_0 = 1$ with initial subspace estimate $P_0$ available. Let $\Delta_0 = SE(P_0, P_0)$. The first subspace update is done at $t = \alpha$, the second at $t = 2\alpha$, and so on. By Lemma 6.16 with $\hat{P}_{j,0} = P_0$, we can show that after one update, the error reduces to $1.2^2 \Delta_0 / 4$. After this, by applying Lemma...
Lemma 6.17 \[ K - 1 \text{ times, we can show that, after at most} \ K = \log(\Delta_0/\epsilon) \text{, the error reduces to} \ 1.2\epsilon. \] Beyond this time, the error does not decrease further. We know that \( P(t) = P_0 \) for \( t \in [t_0, (K+2)\alpha] \), but can change after that.

Consider the \( \alpha \)-interval \( \mathcal{J} \) that contains the change time \( t_1 \).

The projected CS analysis for this interval remains exactly the same as above. But to analyze the subspace update for this interval we need to use Corollary 5.15. More generally consider the \( j \)-th change, and the interval \( \mathcal{J} = [(t_j/\alpha) + 1, (t_j/\alpha) + \alpha] \), which is the \( \alpha \)-frame interval that contains \( t_j \).

For \( t \in \mathcal{J} \), we have \( \ell_t = y_t - \hat{x}_t = \ell_t + e_t + \nu_t \) where

\[
e_t = I - \hat{P}_{j-1}'\hat{P}_{j-1}, \ell_t = P_{j-1}a_t \text{ for } t \in [(t_j/\alpha), t_j) \text{ and } \ell_t = P_ja_t \text{ for } t \in [t_j, (t_j/\alpha) + \alpha].
\]

Let \( \hat{P}_{j,0} \) denote the subspace estimate \( \hat{P}(t) \) computed for this interval. We apply Corollary 5.15 with \( y_t \equiv \ell_t \), \( w_t \equiv (e_t)_t \), \( v_t \equiv (e_\nu)_t + \nu_t \), \( \ell_t \equiv \ell_t \), \( M_{1,t} = -(\Psi_{T_t}'\Psi_{T_t})^{-1}\Psi_{T_t}' \), \( \hat{P} = \hat{P}_{j,0} \), \( P = P_j \), \( P_0 = P_{j-1} \). Since \( \|M_{1,t}P\| \leq 1.2\epsilon \), \( \|M_{1,t}P\| \leq \|P_{j,0}'\Psi_{T_t}'\Psi_{T_t}'P_{j,0}\| \leq 1.2\epsilon + \text{SE}(P_{j-1}'P_j) \), thus \( q_0 = 1.2\epsilon + \text{SE}(P_{j-1}'P_j) \). Also, \( b \equiv b_0 = 0.01/f^2 \) which is the upper bound on max-outlier-frac-row(\( \alpha \)), \( \|E[(e_\nu)'(e_\nu)']/(\ell_t)\| \leq (1.2)^2\lambda_t^+ \). Thus, with probability at least \( 1 - 10n^{-10} \),

\[
\text{SE}(\hat{P}_{j,0}, P_j) \leq 2.5(3\Delta + 4 \cdot 0.1 \cdot 1.2(\epsilon + \Delta) + \frac{\lambda_t^+}{\lambda_t^+}) \leq 10\Delta
\]

Here we used \( \frac{\lambda_t^+}{\lambda_t^+} = \epsilon^2 < \Delta \).

Redefine \( \hat{t}_j = [t_j/\alpha] + \alpha \) and \( \hat{P}_{j,0} \) to denote the estimate from the change interval. To analyze the next \( \alpha \)-interval for new-NORST, we apply Lemma 6.16 with above re-definitions. Thus, \( q_0 = 1.2 \cdot 10\Delta \). We can conclude that \( \text{SE}(\hat{P}_{j,1}, P_j) \leq \max(0.3q_0, \epsilon) = q_1 \). For the next \( K - 1 \) intervals, we apply Lemma 6.17 \( K - 1 \) times with \( q_k = 1.2 \max(0.25q_{k-1}, \epsilon) \).

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