BIFURCATION ANALYSIS OF THE DAMPED KURAMOTO-SIVASHINSKY EQUATION WITH RESPECT TO THE PERIOD

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Abstract. In this paper, we study bifurcation of the damped Kuramoto-Sivashinsky equation on an odd periodic interval of period $2\lambda$. We fix the control parameter $\alpha \in (0, 1)$ and study how the equation bifurcates to attractors as $\lambda$ varies. Using the center manifold analysis, we prove that the bifurcated attractors are homeomorphic to $S^1$ and consist of four or eight singular points and their connecting orbits. We verify the structure of the bifurcated attractors by investigating the stability of each singular point.

1. Introduction. In this paper, we are interested in the damped Kuramoto-Sivashinsky equation (DKSE):

$$u_t = \alpha u - (1 + \partial_x x)^2 u - uu_x.$$  

Here, $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ and $\alpha$ is a control parameter related to the driving force of the system. The damped Kuramoto-Sivashinsky equation has emerged as a fundamental tool for understanding the onset and evolution of secondary instabilities in many driven nonequilibrium systems [1, 3, 4, 7, 14, 15, 18]. For example, it provides a crude model of directional solidification [4, 5]. In this case, $u$ can be considered as the interfacial position and $x$ is the distance along the interface. In case $\alpha = 1$, (1.1) is reduced to the usual Kuramoto-Sivashinsky equation (KSE) which has been known as an important model in the pattern formation arising from different contexts of hydrodynamics and moving interfaces.

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In this paper, we study the dynamical bifurcation of the DKSE (1.1) under the odd periodic boundary condition:
\[
\begin{aligned}
& u \text{ is periodic on } \Omega = [-\lambda, \lambda], \text{ i.e. } u(-\lambda, t) = u(\lambda, t) \text{ for all } t \geq 0, \\
& u(-x, t) = -u(x, t) \text{ for all } x \in \Omega \text{ and } t \geq 0.
\end{aligned}
\] (1.2)

We shall treat the DKSE as a dynamical system in an appropriate phase space. In this setting, both the half period \( \lambda \) (domain size) and the control parameter \( \alpha \) are regarded as bifurcation parameters. The Swift-Hohenberg equation (SHE) also enjoys a similar bifurcation scenario since both the KSE and the SHE share the same linear part. For the SHE, there have been intensive studies on the bifurcation phenomena according to the variations of the domain size ([6, 8, 9, 16, 17, 19, 20]). In [2], the authors studied the dynamical bifurcation of the DKSE when \( \lambda \) is fixed and \( \alpha \) varies. It turns out that as \( \alpha \) passes through a critical number the DKSE bifurcates from the trivial solution to an attractor. Moreover, the structure of the bifurcated attractor was analyzed in detail.

In this paper, we study the dynamical bifurcation of the DKSE by fixing \( \alpha \) and varying \( \lambda \). This setting suggests more abundant structure of bifurcation. The study of bifurcation analysis for varying domain size was initiated in [16, 17] for the SHE, where the authors reduced the SHE on the center manifold for the bifurcation analysis. However, for the DKSE, due to the nonlinearity in (1.1), the bifurcation analysis of (1.1) is more complicated than that of the SHE in [16, 17]. We take care of this nonlinear analysis by employing the so-called second approximation formula of the center manifold function introduced in [10, 11]. By the study of the reduced bifurcation equations on the center manifold, the structure of the bifurcated solutions is clearly analyzed. For readers’ convenience, we briefly review the second approximation formula of the center manifold function in the Appendix.

To analyze the bifurcation phenomenon due to the variation of the domain size for the DKSE, we scale the spatial variable \( x \) to a fixed interval \([-1, 1]\) by introducing a new variable \((\tilde{x}, \tilde{t}) = (\lambda^{-1}x, \lambda^{-4}t)\) and a new function \( \tilde{u}(\tilde{x}, \tilde{t}) = u(x, t) \). In what follows we write \( u(x, t) \) instead of \( \tilde{u}(\tilde{x}, \tilde{t}) \) for simplicity if there is no confusion. In terms of the new variable, we rewrite the DKSE (1.1) as
\[
\frac{\partial u}{\partial \tilde{t}} = -(\lambda^2 + \partial_{xx})^2 u + \alpha \lambda^4 u - \lambda^3 uu_x, \quad x \in [-1, 1]
\] (1.3)
such that \( u \) is odd periodic on \([-1, 1]\) and \( u(-1) = u(1) \). We shall focus on (1.3) hereafter.

For the functional setting of the odd periodic DKSE, we define
\[
H = \left\{ u \in L^2(\Omega; \mathbb{R}) : u(-1) = u(1) = 0, \ u(-x) = -u(x) \text{ for all } x \in \Omega, \right. \\
\left. \text{and } \int_{-1}^1 u(x)dx = 0 \right\},
\]
\[
H^4_{\text{per}}(\Omega; \mathbb{R}) = \left\{ u \in H^4(\Omega; \mathbb{R}) : \frac{\partial^j u}{\partial x^j}(-1) = \frac{\partial^j u}{\partial x^j}(1) \text{ for } j = 0, 1, 2, 3 \right\},
\]
\[
H_1 = H^4_{\text{per}}(\Omega; \mathbb{R}) \cap H.
\]

We reformulate (1.3) in the following abstract form
\[
\begin{aligned}
\frac{du}{dt} & = \mathcal{L}_{\alpha, \lambda} u + \lambda^3 G(u), \\
u(0) & = u_0,
\end{aligned}
\] (1.4)
where \( L_{\alpha, \lambda} u = -(\lambda^2 + \partial_{xx})^u + \alpha \lambda^4 u \) and \( G(u) = -uu_x \). It is easy to check that \( L_{\alpha, \lambda} \) and \( G : H_1 \to H \) are well defined. We often write \( G(u) = G(u, u) \), where \( G(\cdot, \cdot) : H_1 \times H_1 \to H \) is defined by \( G(u, v) = -uv_x \).

Throughout this paper, we assume that \( \alpha \in (0, 1) \). The eigenvalues of \( L_{\alpha, \lambda} \) are \( \{ \lambda^2 \beta_n(\alpha, \lambda) \}_{n=1}^{\infty} \), where

\[
\beta_n(\alpha, \lambda) = \alpha - \sigma_n(\lambda) \quad \text{and} \quad \sigma_n(\lambda) = P\left( \frac{n\pi}{\lambda} \right)
\]

for \( n = 1, 2, \ldots \). Here, \( P(x) = (1 - x^2)^2 \) for \( x \in \mathbb{R} \). We often write \( \beta_n \) or \( \beta_n(\lambda) \) instead of \( \beta_n(\alpha, \lambda) \) if there is no confusion. The corresponding eigenvectors are \( \phi_n(x) = \sin(n\pi x) \) for \( n \geq 1 \). We note that \( \| \phi_n \|_H = 1 \) for all \( n \geq 1 \). Meanwhile, since \( \lambda > 0 \), the equation \( \beta_1(\alpha, \lambda) = 0 \) has two roots

\[
\lambda_1(\alpha) = \frac{\pi}{\sqrt{1 + \sqrt{\alpha}}} \quad \text{and} \quad \lambda_2(\alpha) = \frac{\pi}{\sqrt{1 - \sqrt{\alpha}}}. \tag{1.5}
\]

In general, the equation \( \beta_n(\alpha, \lambda) = 0 \) admits two solutions \( n\lambda_1(\alpha) \) and \( n\lambda_2(\alpha) \). We see that \( \beta_n > 0 \) if and only if \( \lambda \in I_n(\alpha) := (n\lambda_1(\alpha), n\lambda_2(\alpha)) \). We notice that \( n\lambda_2 < (n + 1)\lambda_1 \) for \( n \geq 1 \) if and only if \( \sqrt{\alpha} < q(n) \), where

\[
q(x) = \frac{2x + 1}{2x^2 + 2x + 1}.
\]

Note that \( q \) is decreasing for \( x \geq 1 \). Hence, if \( \sqrt{\alpha} < q(n) \) for some \( n \in \mathbb{N} \), then \( \{ I_k(\alpha) \}_{k=1}^{n+1} \) is a disjoint collection. In this case, let

\[
J_n(\alpha) = (0, (n + 1)\lambda_1) \setminus \bigcup_{i=1}^n I_i(\alpha)
\]

or

\[
J_n(\alpha) = (0, \lambda_1] \cup [\lambda_2, 2\lambda_1] \cup [2\lambda_2, 3\lambda_1] \cup \cdots \cup [n\lambda_2, (n + 1)\lambda_1].
\]

Then, \( \beta_k(\alpha, \lambda) \leq 0 \) for any \( k \geq 1 \) and \( \lambda \in J_n(\alpha) \). Following [16], we will refer to each component of the set \( J_n(\alpha) \) as a gap corresponding to \( n \) and \( \alpha \).

In the following, we provide a brief outline of what follows in the remaining part of this paper. Since \( \alpha \in (0, 1) \) and \( q(n) \in (0, 1] \) is a decreasing function of \( n \in \mathbb{N} \cup \{ 0 \} \), there is \( N \geq 1 \) such that

\[
0 < q(N) \leq \sqrt{\alpha} < q(N - 1) \leq q(0) = 1. \tag{1.6}
\]

Under the condition (1.6), we will prove that the trivial solution \( u = 0 \) of the DKSE (1.3) is globally asymptotically stable for all \( \lambda \in J_{N+1}(\alpha) \). Since \( \beta_n(\alpha, \lambda) > 0 \) on \( I_n(\alpha) \) for \( n \leq N - 1 \), one may expect that there is a bifurcation as \( \lambda \) passes through the end points of \( I_n(\alpha) \). On the other hand, if we set \( \alpha_0 := q(N)^2 \), it holds that \( \lambda_0 := N\lambda_2(\alpha_0) = (N + 1)\lambda_1(\alpha_0) \) and \( \beta_N(\alpha_0, \lambda_0) = \beta_{N+1}(\alpha_0, \lambda_0) = 0 \). If \( \alpha \) passes through the number \( \alpha_0 \) to the right, then \( N\lambda_2(\alpha) > (N + 1)\lambda_1(\alpha) \) such that \( I_n(\alpha) \) and \( I_{N+1}(\alpha) \) are overlapped slightly. On this overlapped interval of \( \lambda \), \( \beta_N(\alpha, \lambda) > 0 \) and \( \beta_{N+1}(\alpha, \lambda) > 0 \). Thus, a secondary bifurcation may occur on this overlapped region. This kind of reasoning was first introduced in [16, 17] for the Swift-Hohenberg equation, where by means of the center manifold analysis the authors studied the qualitative properties of the structures around this secondary bifurcation. They also suggested a problem of bifurcation analysis for other phase transition equations which have similar structures as the SHE. This motivates our study on the DKSE in this paper. We will study the bifurcation phenomena for the DKSE (1.3) in a similar scenario. Using the attractor bifurcation theory of [10, 11], we will prove that there happen several kinds of bifurcation inside the overlapped interval from which attractors appear. We analyze the final pattern of solutions by studying the stability of these attractors on the center manifold.
It is interesting to compare our result for the DKSE with that for the SHE in [16, 17]. We recall that by the relation (1.6), \( \alpha \) is closely related to the number \( N \) and \( \Lambda_N(\alpha) := I_N(\alpha) \cap I_{N+1}(\alpha) \neq \emptyset \). For each \( \lambda \in \Lambda_N(\alpha) \), the bifurcated attractor \( \mathcal{A}_N(\alpha, \lambda) \) is homeomorphic to \( S^3 \) and consists of singular points and their connecting orbits. The singular points come from the combination of the eigenvectors \( \phi_N \) and \( \phi_{N+1} \) and their stability determines the structure of \( \mathcal{A}_N(\alpha, \lambda) \). The stability of each singular points has quite different structure in the DKSE and the SHE.

For the SHE, the interval \( \Lambda_N(\alpha) \) is divided into three parts: \( K_1 = ((N+1)\lambda_1, \delta_1) \), \( K_2 = (\delta_1, \delta_2) \), and \( K_3 = (\delta_2, N\lambda_2) \), where \( \delta_i \) depends on \( \alpha \) for \( i = 1, 2 \). On \( K_1 \cup K_3 \), the singular points are the perturbations of \( \phi_N \) or \( \phi_{N+1} \) and stable modes are \( \phi_N \) on \( K_1 \) and \( \phi_{N+1} \) on \( K_3 \). On \( K_2 \), mixed states of \( \phi_N \) and \( \phi_{N+1} \) also emerge, and they are unstable while the single states \( \phi_N \) and \( \phi_{N+1} \) are stable. This phenomena reflects the transition on the overlapped interval \( \Lambda_N(\alpha) \) and the bifurcated attractor has similar structure with that illustrated in Figure 3 in Section 3. As a consequence one can say that the bifurcation structure on the interval \( \Lambda_N(\alpha) \) is the same regardless of different values of \( \alpha \in (0, 1) \).

Unlike the SHE, the bifurcation phenomena of the DKSE are dependent on \( \alpha \in (0, 1) \) as well as \( \lambda \in \Lambda_N(\alpha) \). It turns out that there are three different types of transitions on \( \Lambda_N(\alpha) \) according to the values of \( N \): (i) \( N = 2, 3, 4 \), (ii) \( N = 5 \), and (iii) \( N \geq 6 \). If \( N = 2, 3, 4 \), the interval \( \Lambda_N \) is divided into two parts \((N+1)\lambda_1, \delta_1) \) and \((\delta_0, N\lambda_2) \). In the first interval the only singular points are the perturbations of \( \phi_N \) or \( \phi_{N+1} \) and the stable modes are the perturbations of \( \phi_N \). In the second interval, there appear stable mixed states of \( \phi_N \) and \( \phi_{N+1} \). The single states \( \phi_N \) and \( \phi_{N+1} \) become unstable. On the other hand, for \( N \geq 6 \) the interval \( \Lambda_N \) is divided into three parts as in the SHE and shares the same transition structure. In particular, on the interval \((\delta_1, \delta_2) \) the single states of \( \phi_N \) and \( \phi_{N+1} \) are stable and their mixtures are unstable. The case \( N = 5 \) connects these two different cases \( N = 2, 3, 4 \) and \( N \geq 6 \) in the sense that \( \Lambda_N = 5 \) is split into three parts but the stable singular points on \((\delta_1, \delta_2) \) are mixed states. As a consequence, when the singular points arise from mixed states of \( \phi_N \) and \( \phi_{N+1} \) on \( \Lambda_N(\alpha) \), (i) if \( \alpha \) is large in \((0, 1) \), then the mixed states are stable; (ii) if \( \alpha \) is small in \((0, 1) \), then the single state \( \phi_N \) or \( \phi_{N+1} \) is stable and the mixed states are unstable. The structures of transition in each case are depicted in Figure 1-3 in Section 3. These results are given in Theorem 3.1 and Theorem 3.5.

The organization of this paper is as the following. In Section 2, we show that under the condition (1.6) the solutions of DKSE bifurcates to an attractor when \( \lambda \) passes through \( n\lambda_1 \) to the right for \( 1 \leq n \leq N \) or \( n\lambda_2 \) to the left for \( 1 \leq n \leq N - 1 \). The main theorems of this section are Theorem 2.2 and Theorem 2.3. In Section 3, we study bifurcation when \( \alpha \) is slightly bigger than \( \alpha_0 \). We prove that if \( \lambda = \lambda_0 \) is fixed, then the DKSE bifurcates to an attractor as \( \alpha \) crosses \( \alpha_0 \) to the right. Then, we fix \( \alpha \) which is slightly bigger than \( \alpha_0 \) and study bifurcation on the overlapped interval \( I_N(\alpha) \cap I_{N+1}(\alpha) \). In Theorem 3.5, we show that there happen bifurcations mixing up two bifurcation branches emanating from the end points of the overlapped interval. In Section 4, we give a brief conclusion. Finally, in Section 5 we provide a summary of attractor bifurcation theory used in this paper.

2. Bifurcation near \( n\lambda_1 \) and \( n\lambda_2 \) for \( n < N \) when \( N \geq 2 \). In this section, we prove that (1.3) has a pitchfork bifurcation at the end points of a gap. Throughout this section, let \( N \geq 2 \) be a fixed integer and assume (1.6). We note that
\[
\sqrt{\alpha} < q(N - 1) \leq q(1) = \frac{3}{5},
\]
(2.1)

Since \( q(N) \leq \sqrt{\alpha} < q(N - 1) \), there are \( N \) gaps such that
\[
J_{N-1}(\alpha) = (0, \lambda_1] \cup [\lambda_2, 2\lambda_1] \cup \cdots \cup [(N - 1)\lambda_2, N\lambda_1).
\]

The next lemma shows that if \( \lambda \in J_{N-1}(\alpha) \), the trivial solution of (1.3) is asymptotically stable.

**Lemma 2.1.** Assume (1.6) holds. If \( \lambda \in J_{N-1}(\alpha) \), then \( u = 0 \) is a globally asymptotically stable solution for (1.3).

**Proof.** For \( u \in H \setminus \{0\} \), we express \( u = \sum_{k=1}^{\infty} y_k(t)\phi_k(x) \), where \( \phi_k(x) = \sin(k\pi x) \).

We note that
(1) \( L_{\alpha, \lambda} \) is a symmetric operator;
(2) \[ \langle G(u), u \rangle = -\frac{1}{3} \int_{-1}^{1} u^3(x) dx = 0; \]
(3) since \( \lambda \in J_{N-1}(\alpha) \), we have \( \beta_n(\alpha, \lambda) \leq 0 \) for all \( n \geq 1 \) and \( \beta_n(\alpha, \lambda) < 0 \) for all \( n \geq N \).

Hence, multiplying (1.4) by \( u \) and integrating the result over the interval \([-1, 1] \) with respect to the variable \( x \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|u\|^2_{L^2} = \langle L_{\alpha, \lambda} u, u \rangle = \int_{-1}^{1} \left( \sum_n \lambda^4 \beta_n y_n \phi_n \right) \left( \sum_m y_m \phi_m \right) dx = \lambda^4 \left( \sum_{n=1}^{\infty} \beta_n y_n^2(t) \right) \leq 0.
\]

We then deduce that as \( \lambda \in J_{N-1}(\alpha) \) the only possible invariant sets of the equation (1.3) \( u \) lies in the span of \( \{\phi_k(x)\} \) as \( \lambda = k\lambda_1 \) or \( \lambda = k\lambda_2 \) for \( k = 1, 2, 3, \ldots N - 1 \). However, it is easy to see that the equation (1.3) does not have a solution in form of \( u = y_k(t)\sin k\pi x \). By Theorem 3.16 of [10], this implies that \( u = 0 \) is globally asymptotically stable in \( H \).

**Theorem 2.2.** Let \( N \geq 2 \) be fixed and assume (1.6). If \( 1 \leq n \leq N \), we have the following.

(a) As \( \lambda \) crosses \( n\lambda_1(\alpha) \) to the right and stays near \( n\lambda_1(\alpha) \), the solutions of (1.3) bifurcate from basic states to an attractor \( A_n(\alpha, \lambda) \). There exists \( \varepsilon > 0 \) such that for any bounded open set \( U \subset H \) with \( 0 \in U \) and \( \lambda \in (n\lambda_1, n\lambda_1 + \varepsilon) \), \( A_n(\alpha, \lambda) \) attracts \( U \setminus \Gamma \) in \( H \), where \( \Gamma \) is the stable manifold of \( u = 0 \) with codimension 1.

(b) The bifurcated attractor \( A_n(\alpha, \lambda) \) consists of two steady state solutions which can be approximated as
\[
u^\pm(x) = \pm \gamma_n \sin(n\pi x) + O(\gamma_n^2),
\]
where \( \gamma_n(\lambda) \) is defined by (2.6) below.
Proof. Let $n \in \{1, 2, \ldots, N\}$ be fixed. We recall that the eigenvalues of $\mathcal{L}_{a,\lambda}$ are $\{\lambda^j \beta_k \}_{k=1}^\infty$. If $|\lambda - n\lambda_1|$ is small enough, then it follows that

$$
\beta_n(a, \lambda) = \begin{cases} < 0, & \text{if } \lambda < n\lambda_1, \\ = 0, & \text{if } \lambda = n\lambda_1, \\ > 0, & \text{if } \lambda > n\lambda_1. 
\end{cases}
$$

Moreover, since $\sqrt{\alpha} < q(N-1)$, it holds that $\beta_k(a, n\lambda_1) < 0$ for $k \neq n$. Using this fact together with Lemma 2.1, we obtain the assertion (a) by Theorem 5.1.

To show the part (b), we define $E_1 = \text{span}\{\phi_n\}$ and $E_2 = E_1^\perp \subset H$. Let $P_j : H \rightarrow E_j$ be the canonical projections and $\mathcal{L}_{a,\lambda} = \mathcal{L}_{a,\lambda|E_j}$, for $j = 1, 2$. If $\Phi_{\lambda} : E_1 \rightarrow E_2$ is a center manifold function and $v = P_1 u$, then the reduced bifurcation equation of (1.3) on the center manifold reads

$$
\frac{dv}{dt} = \mathcal{L}_{a,\lambda}^1 v + \lambda^3 P_1 G(y_n \phi_n + \Phi_{\lambda}(y_n)),
$$

which is equivalent to

$$
\frac{dy_n}{dt} = \lambda^4 \beta_n y_n + \lambda^3 \langle G(y_n \phi_n + \Phi_{\lambda}(y_n)), \phi_n \rangle. \tag{2.2}
$$

According to the second approximation formula (5.10) on the center manifold, this equation can be written as

$$
\frac{dy_n}{dt} = \lambda^4 \beta_n(a) y_n + \lambda^3 \langle G(y_n \phi_n, y_n \phi_n), \phi_n \rangle + a \lambda^2 y_n^3 + o(|y_n|^3), \tag{2.3}
$$

where

$$
a = \sum_{k \neq n} \frac{1}{2 \beta_n(a) - \beta_k(a)} \langle G(\phi_n, \phi_n), \phi_k \rangle \times \langle G(\phi_n, \phi_k) + G(\phi_k, \phi_n), \phi_n \rangle.
$$

By direct calculation, we have

$$
- \langle G(\phi_n, \phi_n), \phi_k \rangle = - \langle G(\phi_k, \phi_n), \phi_n \rangle
$$

$$
= \frac{1}{2} \langle G(\phi_n, \phi_k), \phi_n \rangle = \begin{cases} \frac{n\pi}{2}, & k = 2n, \\ 0, & k \neq 2n. \end{cases} \tag{2.4}
$$

Thus, (2.3) becomes

$$
\frac{dy_n}{dt} = \lambda^4 (\alpha - \sigma_n) y_n - \frac{\lambda^2 n^2 \pi^2}{4(\alpha + \sigma_{2n} - 2\sigma_n)} y_n^3 + o(|y_n|^3). \tag{2.5}
$$

Let $\varepsilon_n = \lambda - n\lambda_1$ be small. Then,

$$
\frac{n\pi}{\lambda} = \frac{n\pi}{n\lambda_1 + \varepsilon_n} = \frac{\pi}{\lambda_1} - \frac{\pi \varepsilon_n}{n\lambda_1^2} + O(\varepsilon_n^2),
$$

from which we obtain from (1.5) that

$$
\alpha - \sigma_n = \alpha - 1 + \left[ \frac{2\pi^2}{\lambda_1^2} - \frac{\pi^4}{\lambda_1^4} \right] - 4 \left[ \frac{\pi^2 \varepsilon_n}{n\lambda_1^2} - \frac{\pi^4 \varepsilon_n}{n\lambda_1^2} \right] + O(\varepsilon_n^2)
$$

$$
= \frac{4\sqrt{\alpha}(1 + \sqrt{\alpha}) \varepsilon_n}{n\lambda_1} + O(\varepsilon_n^2),
$$
Thus, \( \alpha \sigma_2 - \sigma_n = -6\left(\frac{n\pi}{\lambda}\right)^2 + 15\left(\frac{n\pi}{\lambda}\right)^4 \)

\[
= -6\left[\frac{\pi^2}{\lambda_1^4} - \frac{2\pi^2\varepsilon_n}{n\lambda_1^3} + O(\varepsilon_n^2)\right] + 15\left[\frac{\pi^4}{n\lambda_1^4} - \frac{4\pi^4\varepsilon_n}{n\lambda_1^3} + O(\varepsilon_n^2)\right]
\]

\[
= 3\pi^4\left[n\lambda_1(3 + 5\sqrt{\alpha}) - (16 + 20\sqrt{\alpha})\varepsilon_n\right] + O(\varepsilon_n^2).
\]

Thus, \( \alpha + \sigma_2 - 2\sigma_n > 0 \) for small \( \varepsilon_n > 0 \). Moreover, (2.5) has two asymptotically stable equilibrium points: \( y_n = \pm \gamma_n(\lambda) \), where

\[
\gamma_n^2(\lambda) = \frac{4(\alpha + \sigma_2 - 2\sigma_n)(\alpha - \sigma_n)\lambda^2}{n^2\pi^2}
\]

\[
= \frac{48(3 + 5\sqrt{\alpha})(\sqrt{\alpha} + \alpha)}{n\lambda_1}\varepsilon_n + O(\varepsilon_n^2).
\]

Thus, we have a pitchfork bifurcation and the proof is complete. \( \square \)

By the same argument as in the proof of Theorem 2.2, we obtain the following theorem.

**Theorem 2.3.** Let \( N \geq 2 \) be fixed and suppose (1.6) holds. If \( 1 \leq n \leq N - 1 \), we have the following.

(a) As \( \lambda \) crosses \( n\lambda_2(\alpha) \) to the left and stays near \( n\lambda_2(\alpha) \), the solutions of (1.3) bifurcates from the basic states to an attractor \( \tilde{A}_n(\alpha, \lambda) \). There exists \( \varepsilon > 0 \) such that for any bounded open set \( U \subset H \) with \( 0 \in U \) and \( \lambda \in (n\lambda_2 - \varepsilon, n\lambda_2) \), \( \tilde{A}_n(\alpha, \lambda) \) attracts \( U \setminus \Gamma \) in \( H \), where \( \Gamma \) is the stable manifold of \( u = 0 \) with codimension 1.

(b) The bifurcated attractor \( \tilde{A}_n(\alpha, \lambda) \) consists of two steady state solutions which can be approximated as

\[
u^\pm(x) = \pm \tilde{\gamma}_n \sin(n\pi x) + O(\gamma_n^2),
\]

where \( \tilde{\gamma}_n(\lambda) \) is defined by (2.7).

**Proof.** Let \( n \in \{1, 2, \ldots, N - 1\} \) be fixed. If \( |\lambda - n\lambda_2| \) is small enough, then it follows that

\[
\beta_n(\alpha, \lambda) = \begin{cases} < 0, & \text{if } \lambda > n\lambda_2, \\ 0, & \text{if } \lambda = n\lambda_2, \\ > 0, & \text{if } \lambda < n\lambda_2, \end{cases}
\]

and \( \beta_k(\alpha, n\lambda_2) < 0 \) for \( k \neq n \). Thus, by Theorem 5.1, the solutions of (1.3) bifurcates to an attractor \( \tilde{A}_n(\alpha, \lambda) \) as \( \lambda \) crosses \( n\lambda_2(\alpha) \) to the left and stay near \( n\lambda_2(\alpha) \).

As in the previous theorem, we obtain the same reduced bifurcation equation (2.5) on the center manifold. Let \( \varepsilon_n = n\lambda_2 - \lambda > 0 \) be small. We see that

\[
\frac{n\pi}{\lambda} = \frac{\pi}{\lambda_2} + \frac{\pi\varepsilon_n}{n\lambda_2} + O(\varepsilon_n^2),
\]

\[
\alpha - \sigma_n = \frac{4\sqrt{\alpha}(1 - \sqrt{\alpha})\varepsilon_n}{n\lambda_2} + O(\varepsilon_n^2),
\]

\[
\sigma_2 - \sigma_n = \frac{3\pi^4\left[n\lambda_2(3 - 5\sqrt{\alpha}) + (16 - 20\sqrt{\alpha})\varepsilon_n\right]}{n\lambda_2(1 - \sqrt{\alpha})} + O(\varepsilon_n^2).
\]
By (2.1), $\alpha + \sigma_2n - 2\sigma_n > 0$ for small $\tilde{\varepsilon}_n > 0$. Moreover, (2.5) has two asymptotically stable equilibrium points: $y_n = \pm \tilde{\varepsilon}_n(\lambda)$, where

$$\tilde{\varepsilon}_n^2(\lambda) = \frac{48(3 - 5\sqrt{\alpha})(\sqrt{\alpha} - \alpha)}{n\lambda_2} \tilde{\varepsilon}_n + O(\tilde{\varepsilon}_n^2).$$

(2.7)

This finishes the proof. \qed

3. Bifurcation near $\alpha = q(N)^2$ for $N \geq 2$. Throughout this section, let $N \geq 2$ and define

$$\sqrt{\alpha_0} = q(N), \quad \lambda_0 = N\lambda_2(\alpha_0) = (N + 1)\lambda_1(\alpha_0).$$

(3.1)

In the previous section, by considering the condition $\sqrt{\alpha} < q(N - 1)$, we have proved that (1.3) bifurcates from the trivial solution to an attractor as $\lambda$ passes through the boundary of $J_{N-1}(\alpha)$ from the inside of $J_{N-1}(\alpha)$ to the outside of $J_{N-1}(\alpha)$. On the other hand, it holds that for $\sqrt{\alpha} > q(N)$,

$$\Lambda_N(\alpha) := \{(N + 1)\lambda_1(\alpha), N\lambda_2(\alpha)\} = I_N(\alpha) \cap I_{N+1}(\alpha) \neq \emptyset.$$ In this case, there are two nontrivial branches of bifurcation in the interval $\Lambda_N$: the branch emanating from $(N + 1)\lambda_1(\alpha)$ to the right and the branch emanating from $N\lambda_2(\alpha)$ to the left. These two branches are mixed up in the interval $\Lambda_N$. It is very interesting to study this mixed structure of $\Lambda_N(\alpha, \lambda)$. We will study the bifurcation phenomena of the DKSE when the pair $(\alpha, \lambda)$ is close enough to $(\alpha_0, \lambda_0)$, in particular, $\sqrt{\alpha}$ is slightly bigger than $q(N)$. The bifurcation analysis is given in two ways. First, we fix $\lambda = \lambda_0$ and show that an attractor bifurcates for $\alpha > \alpha_0$. Second, we fix $\alpha$ which is sufficiently close to $\alpha_0$ and show that there are several mixed bifurcation on the interval $\Lambda_N$.

Throughout this section, we assume that $q(N) = \sqrt{\alpha_0} \leq \sqrt{\alpha} < q(N - 1)$ for some fixed $N \geq 2$. Then, it follows that $\beta_N(\alpha, \lambda), \beta_{N+1}(\alpha, \lambda) \geq 0$ for $\lambda \in \Lambda_N(\alpha)$ and $\beta_k(\alpha, \lambda) < 0$ for all $k \neq N, N + 1$. By regarding the DKSE as a suspended two bifurcation parameters $\alpha$ and $\lambda$, we can employ the classical Center Manifold Theorem (e.g. see Theorem 2.12 of [10] or Theorem 7.1.2 of [21]). Indeed, if $E_1 = \text{span}\{\phi_N, \phi_{N+1}\}$ and $E_2 = E_1^\perp$ in $H$, there exist a small $\eta_0 > 0$ and a center manifold function $\Phi(\cdot, \alpha, \lambda) : E_1 \to E_2$ such that $\Phi(0, \alpha, \lambda) = 0$, $D_v\Phi(0, \alpha, \lambda) = 0$, and the corresponding center manifold

$$\mathcal{M} = \{v + \Phi(v, \alpha, \lambda) \in E_1 \oplus E_2 : \|v\|_H + |\alpha - \alpha_0| + |\lambda - \lambda_0| < \eta_0\}$$

is locally invariant and attracts all local flows near the trivial solution $u = 0$. Since $\lambda_0 \in \Lambda_N(\alpha)$, if $\alpha > \alpha_0$ and $\alpha - \alpha_0$ is sufficiently small, it follows that

$$|\alpha - \alpha_0| + |\lambda - \lambda_0| < \eta_0, \quad \forall \lambda \in \Lambda_N(\alpha).$$

(3.2)

To see this, suppose that $\alpha > \alpha_0$ and let $\varepsilon = \varepsilon(\alpha) = \sqrt{\alpha} - \sqrt{\alpha_0}$. We note that

$$N\lambda_2(\alpha) - (N + 1)\lambda_1(\alpha) = \frac{\pi}{\sqrt{1 - \alpha}} \left\{N\sqrt{1 + \sqrt{\alpha}} - (N + 1)\sqrt{1 - \sqrt{\alpha}}\right\}
= \frac{\pi(2N^2 + 2N + 1)}{\sqrt{1 - \alpha}(N\sqrt{1 + \sqrt{\alpha}} + (N + 1)\sqrt{1 - \sqrt{\alpha}})} \varepsilon.$$

(3.3)

Hence, if $0 < \varepsilon \ll 1$, then $0 < N\lambda_2(\alpha) - (N + 1)\lambda_1(\alpha) \ll 1$ which leads us to (3.2). Moreover, since $\beta_1(\alpha, \lambda_1(\alpha)) = 0$, we obtain
\[ \beta_n(\alpha, \lambda) = \left\{ \frac{n\pi}{\lambda} \right\} - \left( \frac{\pi}{\lambda_1} \right) \left\{ 2 - \left( \frac{\pi}{\lambda} \right)^2 - \left( \frac{n\pi}{\lambda_1} \right)^2 \right\} \]

\[ = \left\{ \left( \frac{\pi}{\lambda} \right)^2 - \left( \frac{n\pi}{\lambda_1} \right)^2 \right\} \left\{ 2\sqrt{\alpha} + \left( \frac{\pi}{\lambda} \right)^2 - \left( \frac{n\pi}{\lambda_1} \right)^2 \right\}. \tag{3.4} \]

Similarly, \( \beta_1(\alpha, \lambda_2(\alpha)) = 0 \) implies that

\[ \beta_n(\alpha, \lambda) = \left\{ \left( \frac{n\pi}{\lambda} \right)^2 - \left( \frac{\pi}{\lambda_2} \right)^2 \right\} \left\{ 2\sqrt{\alpha} + \left( \frac{\pi}{\lambda} \right)^2 - \left( \frac{n\pi}{\lambda_2} \right)^2 \right\}. \tag{3.5} \]

In particular, we deduce from (3.3), (3.4), and (3.5) that

\[ \beta_N(\alpha, \lambda), \beta_{N+1}(\alpha, \lambda) = O(\varepsilon), \forall \lambda \in \Lambda_N. \tag{3.6} \]

In the following, we always assume (3.2).

The first main result of this section is the following.

**Theorem 3.1.** Let \( N \geq 2, \alpha_0 = q(N)^2 \) and fix \( \lambda = \lambda_0 = (N+1)\lambda_1(\alpha_0) = N\lambda_2(\alpha_0) \). Then as \( \alpha \) crosses \( \alpha_0 \) to the right and stays near \( \alpha_0 \), the equation (1.3) bifurcates from the trivial solution to an attractor \( \mathcal{A}_N(\alpha, \lambda) \) which is homeomorphic to \( S^1 \).

For any bounded open set \( U \subset H \) with \( 0 \in U \), \( \mathcal{A}_N(\alpha, \lambda) \) attracts \( U \setminus \Gamma \) in \( H \), where \( \Gamma \) is the stable manifold of \( u = 0 \) with codimension 2. Furthermore, the attractor \( \mathcal{A}_N(\alpha, \lambda) \) has the following structure.

(a) For \( 2 \leq N < 8 \), the attractor \( \mathcal{A}_N(\alpha, \lambda) \) consists of 4 singular points and their connecting orbits. The singular points can be expressed as

\[ u_1^+(x) = \pm a_1 \sin(N\pi x) + o(\sqrt{\alpha - \alpha_0}), \]
\[ u_2^+(x) = \pm a_2 \sin((N + 1)\pi x) + o(\sqrt{\alpha - \alpha_0}), \tag{3.7} \]

where \( u_1^+ \) are stable nodes and \( u_2^+ \) are saddles. Here, \( a_1 \) and \( a_2 \) are defined in (3.19) such that \( a_1 = O(\sqrt{\alpha - \alpha_0}) \).

(b) For \( N \geq 8 \), the attractor \( \mathcal{A}_N(\alpha, \lambda) \) consists of 8 singular points and their connecting orbits. The singular points can be expressed as

\[ u_1^+(x) = \pm a_1 \sin(N\pi x) + o(\sqrt{\alpha - \alpha_0}), \]
\[ u_2^+(x) = \pm a_2 \sin((N + 1)\pi x) + o(\sqrt{\alpha - \alpha_0}), \]
\[ u_3^+(x) = b_1 \sin(N\pi x) \pm b_2 \sin((N + 1)\pi x) + o(\sqrt{\alpha - \alpha_0}), \]
\[ u_4^+(x) = -b_1 \sin(N\pi x) \pm b_2 \sin((N + 1)\pi x) + o(\sqrt{\alpha - \alpha_0}), \tag{3.8} \]

where \( u_1^+, u_2^+, u_3^+, u_4^+ \) are stable nodes and \( u_3^+, u_4^+ \) are saddles. Here, \( a_1, a_2, b_1, \) and \( b_2 \) are defined in (3.19) below such that \( a_1, b_1 = O(\sqrt{\alpha - \alpha_0}) \).

We need the following lemma for the proof of Theorem 3.1.

**Lemma 3.2.** Let \( N \geq 2 \) and \( q(N) < \sqrt{\alpha} < q(N - 1) \). Then, for any \( \lambda \in \Lambda_N(\alpha) \), it holds that

\[ \beta_N(\alpha, \lambda) > 0, \quad \beta_{N+1}(\alpha, \lambda) > 0, \]
\[ \beta_n(\alpha, \lambda) < 0, \quad \forall n \neq N, N + 1. \tag{3.9} \]

**Proof.** Since \( \lambda \in I_N \cap I_{N+1} \), we see that \( \beta_N(\alpha, \lambda) > 0 \) and \( \beta_{N+1}(\alpha, \lambda) > 0 \). Since \( q(n) \) is decreasing, \( \sqrt{\alpha} < q(n) \) for all \( n = 1, \cdots, N - 1 \). Hence, \( I_n(\alpha) \cap \Lambda_N(\alpha) = \emptyset \) for each \( n = 1, \cdots, N - 1 \) such that

\[ \beta_n(\alpha, \lambda) < 0, \quad n = 1, \cdots, N - 1. \tag{3.10} \]
On the other hand, we note that for $N \geq 2$

$$(N + 2)\lambda_1(\alpha) - N\lambda_2(\alpha) = \frac{\pi}{\sqrt{1 - \alpha}} \left\{ (N + 2)\sqrt{1 - \sqrt{\alpha}} - N\sqrt{1 + \sqrt{\alpha}} \right\}$$

and

$$\left\{ (N + 2)\sqrt{1 - \sqrt{\alpha}} \right\}^2 - \left\{ N\sqrt{1 + \sqrt{\alpha}} \right\}^2 = (4N + 4) - \sqrt{\alpha}(2N^2 + 4N + 4)$$

$$> (4N + 4) - q(N - 1)(2N^2 + 4N + 4)$$

$$= \frac{2(2N^3 - 3N^2 - 4N + 4)}{2N^2 - 2N + 1} \geq 0.$$}

Here, the equality holds only when $N = 2$. This implies that $I_n(0) \cap \Lambda_N(0) = \emptyset$ for each $n \geq N + 2$. Hence, $\beta_n(\alpha, \lambda) < 0$ for $\lambda \in \Lambda_N(0)$ and $n \geq N + 2.$

Based on the above observation, we prove Theorem 3.1 in the following. The proof is divided into several steps.

**Step 1. Existence of bifurcated attractor.** By Lemma 3.2, if $|\alpha - \alpha_0|$ is small enough, then

$$\beta_n(\alpha, \lambda_0) \begin{cases} < 0, & \text{if } \alpha < \alpha_0, \\ = 0, & \text{if } \alpha = \alpha_0, \\ > 0, & \text{if } \alpha > \alpha_0, \end{cases}$$

where $n = N$ or $N + 1$. Hence, Theorem 5.1 together with Lemma 2.1 and Lemma 3.2 says that as $\alpha$ passes through $\alpha_0$ to the right, (1.3) with $\lambda = \lambda_0$ bifurcates to an attractor $A_N(\alpha, \lambda)$. Moreover, for any bounded open set $U \subset H$ with $0 \in U$, $A_N(\alpha, \lambda)$ attracts $U \setminus \Gamma$ in $H$, where $\Gamma$ is the stable manifold of $u = 0$ with codimension 2.

**Step 2. Reduction on the center manifold.** Let $E_1 = \text{span}\{\phi_N, \phi_{N+1}\}$ and $E_2 = E_1^\perp$ in $H$. For $u \in H$, we write $u = \sum_{k=1}^{\infty} v_k \phi_k$. For simplicity, we write $\psi_1 = \phi_N, \psi_2 = \phi_{N+1}, z_1 = y_N$, and $z_2 = y_{N+1}$. For $u \in H$ a solution of (1.3), we denote

$$u = z_1 \psi_1 + z_2 \psi_2 + \Phi(z, \alpha, \lambda) =: v + \Phi(z, \alpha, \lambda).$$

Here, $\Phi : E_1 \to E_2$ is a center manifold function such that $\Phi(z, \alpha, \lambda) = o(|z|)$, where $z = (z_1, z_2)$. Let $P_j : H \to E_j$ be the canonical projections and $L_{\alpha, \lambda} = L_{\alpha, \lambda}|_{E_j}$, for $j = 1, 2$. Then, the reduced system of (1.3) on the center manifold is given by

$$\frac{dv}{dt} = L_{\alpha, \lambda}^1 v + \lambda^3 P_1 G(z_1 \psi_1 + z_2 \psi_2 + \Phi),$$

or equivalently

$$\begin{cases} \frac{dz_1}{dt} = \lambda^4 \beta_N z_1 + \lambda^3 h_1(z_1, z_2) + \lambda^2 \sum_{i,j,l=1,2} a_{ijl}^1 z_i z_j z_l + o(|z|^3), \\ \frac{dz_2}{dt} = \lambda^4 \beta_N z_2 + \lambda^3 h_2(z_1, z_2) + \lambda^2 \sum_{i,j,l=1,2} a_{ijl}^2 z_i z_j z_l + o(|z|^3), \end{cases}$$

(3.12)

where $z = (z_1, z_2)$,

$$h_1(z_1, z_2) = \langle G(z_1 \psi_1 + z_2 \psi_2, z_1 \psi_1 + z_2 \psi_2), \psi_1 \rangle,$$

$$h_2(z_1, z_2) = \langle G(z_1 \psi_1 + z_2 \psi_2, z_1 \psi_1 + z_2 \psi_2), \psi_2 \rangle.$$
\[ a_{ijl}^1 = \sum_{n \neq N, N+1} \frac{1}{2\beta_n - \beta} \left\langle G(n, \psi_1), \phi_n \right\rangle \times \left\langle G(\psi_1, \phi_n) + G(\phi_n, \psi_1), \psi_1 \right\rangle, \]
\[ a_{ijl}^2 = \sum_{n \neq N, N+1} \frac{1}{2\beta_n - \beta} \left\langle G(n, \psi_1), \phi_n \right\rangle \times \left\langle G(\psi_1, \phi_n) + G(\phi_n, \psi_1), \psi_2 \right\rangle. \]

Since \( N \geq 2 \), we derive from (2.4) that \( h_1(z_1, z_2) = h_2(z_1, z_2) = 0 \).

To compute \( a_{ijl}^k \), we need the following:

\[ \langle G(\psi_1, \phi_n), \psi_1 \rangle = \left\{\begin{array}{ll} -\frac{N\pi}{2}, & n = 2N, \\ 0, & n \neq 2N, \\ \frac{(N+1)\pi}{2}, & n = 1, \end{array}\right. \]
\[ \langle G(\psi_1, \phi_1), \psi_1 \rangle = \left\{\begin{array}{ll} \frac{N\pi}{2}, & n = 2N, \\ 0, & n \neq 2N, \end{array}\right. \]
\[ \langle G(\psi_1, \phi_n), \psi_2 \rangle = \left\{\begin{array}{ll} -\frac{\pi}{2}, & n = 1, \\ \frac{(N+1)\pi}{2}, & n = 2N + 1, \\ 0, & n \neq 1, 2N + 1, \end{array}\right. \]
\[ \langle G(\psi_2, \phi_n), \psi_1 \rangle = \left\{\begin{array}{ll} \frac{N\pi}{2}, & n = 2N + 2, \\ 0, & n \neq 2N + 2, \end{array}\right. \]
\[ \langle G(\psi_2, \phi_n), \psi_2 \rangle = \left\{\begin{array}{ll} \frac{(N+1)\pi}{2}, & n = 2N + 1, \\ 0, & n \neq 1, 2N + 1, \end{array}\right. \]

Using these, we obtain

\[ a_{111}^1 = -\frac{N^2\pi^2}{4(2\beta_N - \beta_{2N})}, \]
\[ a_{212}^1 = \frac{N(N+1)\pi^2}{4} \left( \frac{1}{2\beta_N - \beta_1} - \frac{1}{2\beta_N - \beta_{2N+1}} \right), \]
\[ a_{221}^1 = -\frac{N^2\pi^2}{4} \left( \frac{1}{2\beta_N - \beta_1} + \frac{1}{2\beta_N - \beta_{2N+1}} \right), \]
\[ a_{112}^2 = -\frac{(N+1)^2\pi^2}{4} \left( \frac{1}{2\beta_{N+1} - \beta_1} - \frac{1}{2\beta_{N+1} - \beta_{2N+1}} \right), \]
\[ a_{121}^2 = \frac{N(N+1)\pi^2}{4} \left( \frac{1}{2\beta_{N+1} - \beta_1} + \frac{1}{2\beta_{N+1} - \beta_{2N+1}} \right), \]
\[ a_{222}^2 = -\frac{(2\beta_{N+1} - \beta_{2N+2})}{4}, \]
\[ a_{112}^1 = a_{122}^1 = a_{211}^1 = a_{111}^2 = a_{122}^2 = a_{121}^2 = a_{211}^2 = a_{212}^2 = a_{221}^2 = 0. \]
In the sequel, (3.12) becomes

\[
\begin{align*}
\frac{dz_1}{dt} &= \lambda^4 \beta_N z_1 - \frac{\pi^2 \lambda^2}{4} \tilde{F}_1(z_1, z_2) + o(|z|^3), \\
\frac{dz_2}{dt} &= \lambda^4 \beta_{N+1} z_2 - \frac{\pi^2 \lambda^2}{4} \tilde{F}_2(z_1, z_2) + o(|z|^3).
\end{align*}
\]  

(3.13)

Here,

\[
\tilde{F}_1(y) = d_{11}(\alpha, \lambda) z_1^3 + d_{12}(\alpha, \lambda) z_1 z_2^2,
\]

\[
\tilde{F}_2(y) = d_{21}(\alpha, \lambda) z_1^2 z_2 + d_{22}(\alpha, \lambda) z_2^3,
\]

and

\[
\begin{align*}
d_{11}(\alpha, \lambda) &= \frac{N^2}{2 \beta_N - \beta_{2N}} = \frac{N^2}{\sigma_{2N} - \sigma_N} + O(\varepsilon), \\
d_{12}(\alpha, \lambda) &= -\frac{N}{2 \beta_N - \beta_1} + \frac{N(2N+1)}{2 \beta_{N+1} - \beta_2} = -\frac{N}{\sigma_1 - \sigma_N} + \frac{N(2N+1)}{\sigma_{2N+1} - \sigma_N} + O(\varepsilon), \\
d_{21}(\alpha, \lambda) &= \frac{N+1}{2 \beta_{N+1} - \beta_1} + \frac{(N+1)(2N+1)}{2 \beta_{N+1} - \beta_{2N+1}} = \frac{N+1}{\sigma_1 - \sigma_{2N+1}} + \frac{(N+1)(2N+1)}{\sigma_{2N+1} - \sigma_{N+1}} + O(\varepsilon), \\
d_{22}(\alpha, \lambda) &= \frac{(N+1)^2}{2 \beta_{N+1} - \beta_{2N+2}} = \frac{(N+1)^2}{\sigma_{2N+2} - \sigma_{N+1}} + O(\varepsilon).
\end{align*}
\]

Here, we used (3.6). This completes the step 2.

By the Poincare-Bendixon Theorem, the bifurcated attractor \( A_N(\alpha, \lambda) \) is homeomorphic to \( S^1 \) and consists of singular points and their connecting orbits. In the following, we investigate the stability of equilibrium points which describes a complete diagram of the attractor \( A_N(\alpha, \lambda) \). This analysis is heavily dependent on the range of \( \lambda \) as we shall see.

**Step 3. Finding singular points.** We note from (3.1) and (3.6) that

\[
\sigma_{2N} - \sigma_N = \frac{3N^2 \pi^2}{\lambda^2} \left( -2 + \frac{5N^2 \pi^2}{\lambda^2} \right) = \frac{3 \pi^2}{\lambda^2} \left( -2 + \frac{5 \pi^2}{\lambda^2} \right) + O(\varepsilon)
\]

\[
= 3(1 - \sqrt{\alpha_0})(3 - 5 \sqrt{\alpha_0}) + O(\varepsilon)
\]

\[
= \frac{12N^2(3N^2 - 2N - 1)}{(2N^2 + 2N + 1)^2} + O(\varepsilon).
\]

Hence, for \( N \geq 2 \)

\[
d_{11} = \frac{(2N^2 + 2N + 1)^2}{12(N - 1)(3N + 1)} + O(\varepsilon) > 0. 
\]

(3.14)

Similarly, we obtain

\[
d_{22} = \frac{(N+1)^2}{3(1 + \sqrt{\alpha_0})(3 + 5 \sqrt{\alpha_0})} + O(\varepsilon) = \frac{(2N^2 + 2N + 1)^2}{12(N + 2)(3N + 2)} + O(\varepsilon) > 0.
\]

(3.15)
In the sequel, we obtain
\[
\sigma_N = \left\{ 1 - \frac{\pi^2}{\lambda_0^2} + O(\varepsilon) \right\}^2 = \alpha_0 + O(\varepsilon) = \left(\frac{(2N + 1)^2}{(2N^2 + 2N + 1)^2}\right) + O(\varepsilon),
\]
\[
\sigma_{N+1} = \left\{ 1 - \frac{\pi^2}{\lambda_0^2} + O(\varepsilon) \right\}^2 = \alpha_0 + O(\varepsilon) = \left(\frac{(2N + 1)^2}{(2N^2 + 2N + 1)^2}\right) + O(\varepsilon),
\]
\[
\sigma_1 = \left\{ 1 - \frac{1}{N^2}(1 - \sqrt{\alpha_0}) + O(\varepsilon) \right\}^2 = \frac{(2N^2 + 2N - 1)^2}{(2N^2 + 2N + 1)^2} + O(\varepsilon),
\]
\[
\sigma_{2N+1} = \left\{ 1 - \frac{(2N + 1)^2}{N^2}(1 - \sqrt{\alpha_0}) + O(\varepsilon) \right\}^2 = \frac{(6N^2 + 6N + 1)^2}{(2N^2 + 2N + 1)^2} + O(\varepsilon),
\]
which imply that
\[
\sigma_1 - \sigma_N, \sigma_1 - \sigma_{N+1} = \frac{4(N - 1)(N + 1)(N + 2)}{4N(N + 1)(N + 2)} + O(\varepsilon),
\]
\[
\sigma_{2N+1} - \sigma_N, \sigma_{2N+1} - \sigma_{N+1} = \frac{4N(N + 1)(3N + 1)(3N + 2)}{4N(N + 1)(3N + 1)(3N + 2)} + O(\varepsilon).
\]

In the sequel, we obtain
\[
d_{12}(\alpha, \lambda) = \frac{(N^2 - 4N - 2)(2N^2 + 2N + 1)^2}{2(N - 1)(N + 2)(3N + 1)(3N + 2)} + O(\varepsilon),
\]
\[
d_{21}(\alpha, \lambda) = \frac{(N^2 + 6N + 3)(2N^2 + 2N + 1)^2}{2(N - 1)(N + 2)(3N + 1)(3N + 2)} + O(\varepsilon).
\]

Formally, the truncated system of (3.13) has the following singular points:
\[
(y_1, y_2) = (\pm a_1, 0), (0, \pm a_2), (b_1, \pm b_2), (-b_1, \pm b_2),
\]
where
\[
\begin{align*}
\begin{cases}
    a_1(\alpha, \lambda) = \frac{2\lambda}{\pi} \sqrt{\frac{\beta_0}{d_{11}}}, \\
    a_2(\alpha, \lambda) = \frac{2\lambda}{\pi} \sqrt{\frac{\beta_{N+1}}{d_{22}}}, \\
    b_1(\alpha, \lambda) = \frac{2\lambda}{\pi} \sqrt{\frac{d_{22}\beta_0 - d_{12}\beta_{N+1}}{B}}, \\
    b_2(\alpha, \lambda) = \frac{2\lambda}{\pi} \sqrt{\frac{d_{11}\beta_{N+1} - d_{21}\beta_0}{B}}
\end{cases}
\end{align*}
\]
with \(B(\alpha, \lambda) = d_{11}d_{22} - d_{12}d_{21}\). It follows from (3.16) that \(a_i, b_i = O(\sqrt{\alpha - \alpha_0})\) for \(i = 1, 2\). By noting \(\sigma_N(\lambda_0) = \sigma_{N+1}(\lambda_0) = \alpha_0\), we see that
\[
\begin{align*}
\begin{cases}
    a_1(\alpha, \lambda) = \frac{2\lambda}{\pi} \sqrt{\frac{\alpha - \alpha_0}{d_{11}}}, \\
    a_2(\alpha, \lambda) = \frac{2\lambda}{\pi} \sqrt{\frac{\alpha - \alpha_0}{d_{22}}}, \\
    b_1(\alpha, \lambda) = \frac{2\lambda}{\pi} \sqrt{\frac{(d_{22} - d_{12})(\alpha - \alpha_0)}{B}}, \\
    b_2(\alpha, \lambda) = \frac{2\lambda}{\pi} \sqrt{\frac{(d_{11} - d_{21})(\alpha - \alpha_0)}{B}}
\end{cases}
\end{align*}
\]
The points \(b_1\) and \(b_2\) may not exist according to the signs inside the square roots which depends on the range of \(\lambda\). We shall examine the possibilities of the existence of \((b_1, \pm b_2)\) and \((-b_1, \pm b_2)\) and study the stabilities of all the singular points. The next lemma provides a useful information to do it.
Lemma 3.3. (a) \( \frac{d_{21}}{d_{11}} > 1 \) for all \( N \geq 2 \).
(b) \[
\begin{align*}
\begin{cases}
\frac{d_{12}}{d_{22}} < 0 & \text{for } N = 2, 3, 4, \\
0 < \frac{d_{12}}{d_{22}} < 1 & \text{for } N = 5, 6, 7, \\
\frac{d_{12}}{d_{22}} > 1 & \text{for } N \geq 8.
\end{cases}
\end{align*}
\]
(c) \( B = d_{11}d_{22} - d_{12}d_{21} \begin{cases}
> 0 & \text{for } N = 2, 3, 4, 5, \\
< 0 & \text{for } N \geq 6.
\end{cases} \)

Proof. The assertions (a) and (b) come from the following identity:
\[
\begin{align*}
\frac{d_{21}}{d_{11}} &= \frac{6(N^2 + 6N + 3)}{(N + 2)(3N + 2)} + O(\varepsilon), \\
\frac{d_{12}}{d_{22}} &= \frac{6(N^2 - 4N - 2)}{(N - 1)(3N + 1)} + O(\varepsilon).
\end{align*}
\]
(3.20)

We note that \( d_{11}d_{22} - d_{12}d_{21} > 0 \) if and only if
\[
\frac{d_{21}}{d_{11}} \frac{d_{12}}{d_{22}} < 1,
\]
which is equivalent to
\[
36(N^2 + 6N + 3)(N^2 - 4N - 2) - (N - 1)(N + 2)(3N + 1)(3N + 2)
= 27N^4 + 54N^3 - 821N^2 - 848N - 212 < 0.
\]
This is true for only \( N = 2, 3, 4, 5 \).

Step 4. Stability of singular points. Using the facts \( \beta_N(\alpha_0, \lambda_0) = \beta_{N+1}(\alpha_0, \lambda_0) = 0 \), by Lemma 3.3, we have
\[
\beta_N(\alpha, \lambda_0) - \frac{d_{12}}{d_{22}} \beta_{N+1}(\alpha, \lambda_0) = (\alpha - \alpha_0)(1 - \frac{d_{12}}{d_{22}}) \begin{cases}
> 0 & \text{if } N < 8, \\
< 0 & \text{if } N \geq 8.
\end{cases}
\]
(3.21)

Similarly, we have
\[
\beta_{N+1}(\alpha, \lambda_0) - \frac{d_{21}}{d_{11}} \beta_N(\alpha, \lambda_0) = (\alpha - \alpha_0)(1 - \frac{d_{21}}{d_{11}}) < 0 \quad \text{for } N \geq 2.
\]
(3.22)

By (3.18), (3.21), (3.22) and the assertion (c) of Lemma 3.3, we see that \( (\pm b_1, \pm b_2) \) are equilibrium points of the truncated equation of (3.13) if and only if \( N \geq 8 \).

To study the stabilities of the equilibrium points of the truncated equation of (3.13), let \( F = (F_1, F_2) \) with
\[
F_1(z_1, z_2) = \lambda^4 \beta_N z_1 - \frac{\pi^2 \lambda^2}{4} \tilde{F}_1(z_1, z_2),
\]
\[
F_2(z_1, z_2) = \lambda^4 \beta_N z_2 - \frac{\pi^2 \lambda^2}{4} \tilde{F}_2(z_1, z_2).
\]

Then we have the following:
\[
DF = \begin{pmatrix}
\lambda^4 \beta_N - \frac{\pi^2 \lambda^2}{4} (3d_{11}z_1^2 + d_{12}z_2^2) & -\frac{\pi^2 \lambda^2}{2} d_{12}z_1z_2 \\
-\frac{\pi^2 \lambda^2}{2} d_{21}z_1z_2 & \lambda^4 \beta_{N+1} - \frac{\pi^2 \lambda^2}{4} (d_{21}z_2^2 + 3d_{22}z_2^2)
\end{pmatrix}.
\]
First, suppose that \( z_1^2 = a_1^2 \) and \( z_2 = 0 \). Then, \( DF \) has two eigenvalues

\[
\begin{cases}
\tau_1(\alpha, \lambda) = -2\lambda^4 \beta_N < 0, \\
\tau_2(\alpha, \lambda) = \lambda^4 \left( \beta_{N+1} - \frac{d_{21}}{d_{11}} \beta_N \right) < 0.
\end{cases}
\]  
(3.23)

Similarly, if \( z_1 = 0 \) and \( z_2 = a_2^2 \), then \( DF \) has two eigenvalues

\[
\begin{cases}
\tau_3(\alpha, \lambda) = -2\lambda^4 \beta_{N+1} < 0, \\
\tau_4(\alpha, \lambda) = \lambda^4 \left( \beta_N - \frac{d_{12}}{d_{22}} \beta_{N+1} \right) \left\{ \\
> 0 \quad \text{if } N < 8, \\
< 0 \quad \text{if } N \geq 8.
\end{cases}
\]  
(3.24)

Hence, we obtain the following properties.

(i) \((\pm a_1, 0)\) are stable for \( N \geq 2 \).

(ii) \((0, \pm a_2)\) are saddles for \( 2 \leq N < 8 \).

(iii) \((0, \pm a_2)\) are stable for \( N \geq 8 \).

Next, to consider the stability of the points \((b_1, \pm b_2)\) and \((-b_1, \pm b_2)\), we assume that \( N \geq 8 \). For \((z_1, z_2) = (b_1, \pm b_2), (-b_1, \pm b_2)\), we have

\[
DF(z_1, z_2) = \begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{pmatrix},
\]

where

\[
\begin{align*}
\zeta_{11}(\alpha, \lambda) &= -\frac{2\lambda^4 d_{11} d_{22}}{B} \left( \beta_N - \frac{d_{12}}{d_{22}} \beta_{N+1} \right), \\
\zeta_{12}(\alpha, \lambda) &= \mp \frac{2\lambda^4 d_{12}}{B} \sqrt{d_{22} \beta_N - d_{12} \beta_{N+1}} \sqrt{d_{11} \beta_{N+1} - d_{21} \beta_N}, \\
\zeta_{21}(\alpha, \lambda) &= \mp \frac{2\lambda^4 d_{21}}{B} \sqrt{d_{22} \beta_N - d_{12} \beta_{N+1}} \sqrt{d_{11} \beta_{N+1} - d_{21} \beta_N}, \\
\zeta_{22}(\alpha, \lambda) &= -\frac{2\lambda^4 d_{11} d_{22}}{B} \left( \beta_{N+1} - \frac{d_{21}}{d_{11}} \beta_N \right).
\end{align*}
\]

Here the sign of \( \zeta_{12} \) and \( \zeta_{21} \) are for \((b_1, \pm b_2)\), while in the case of \((-b_1, \pm b_2)\) they must be \( \pm \). Since

\[
\zeta_{11} + \zeta_{22} = -\frac{2\lambda^4 d_{11} d_{22}}{B} \left( \beta_N - \frac{d_{12}}{d_{22}} \beta_{N+1} \right) + \left( \beta_{N+1} - \frac{d_{21}}{d_{11}} \beta_N \right) < 0,
\]

\[
\zeta_{11} \zeta_{22} - \zeta_{12} \zeta_{21} = \frac{4\lambda^8 d_{11} d_{22}}{B} \left( \beta_N - \frac{d_{12}}{d_{22}} \beta_{N+1} \right) \left( \beta_{N+1} - \frac{d_{21}}{d_{11}} \beta_N \right) < 0,
\]  
(3.25)

\( DF \) has one positive eigenvalue and one negative eigenvalue which implies that each of \((b_1, \pm b_2)\) and \((-b_1, \pm b_2)\) is a saddle point. This completes the proof of Theorem 3.1.

Now, let us fix \( \alpha > \alpha_0 \) with \( 0 < \alpha - \alpha_0 \ll 1 \) so that (3.2) is valid. Since \( I_N(\alpha) \cap I_{N+1}(\alpha) \neq \emptyset \), inside the interval \( \Lambda_N(\alpha) \) we may expect a new bifurcation on the center manifold which comes from the mixture of two bifurcation branches emanating from \( \lambda = (N+1)\lambda_1(\alpha) \) and \( \lambda = N\lambda_2(\alpha) \). To see this, we begin with the following lemma.
Lemma 3.4. Let $N \geq 2$ and $q(N) < \sqrt{\alpha} < q(N - 1)$. Then, for all $\lambda \in \Lambda_N(\alpha)$,
\[
\begin{cases}
\frac{\partial \beta_N}{\partial \lambda} < 0 & \text{for } N \geq 2, \\
\frac{\partial \beta_{N+1}}{\partial \lambda} > 0 & \text{for } N \geq 3.
\end{cases}
\] (3.26)

Proof. A simple computation gives
\[
\frac{\partial \beta_n}{\partial \lambda}(\alpha, \lambda) = \frac{4n^2 \pi^2}{\lambda^3} \left\{ \left( \frac{n \pi}{\lambda} \right)^2 - 1 \right\}.
\]
Since $(N + 1)\lambda_1 < \lambda < N\lambda_2$, we infer that for $N \geq 2$,
\[
\left( \frac{N \pi}{\lambda} \right)^2 < \frac{N^2}{(N + 1)^2} (1 + \sqrt{\alpha}) < \frac{N^2}{(N + 1)^2} \left[ 1 + q(N - 1) \right] = \frac{N^2}{(N + 1)^2} \cdot \frac{2N^2}{N^2 + (N - 1)^2} < 1.
\]
Furthermore, if $N \geq 3$, then
\[
\left[ \left( \frac{N + 1}{\lambda} \right)^2 \right] > \frac{(N + 1)^2}{N^2} (1 - \sqrt{\alpha}) > \frac{(N + 1)^2}{N^2} \left[ 1 - q(N - 1) \right] = \frac{(N + 1)^2}{N^2} \cdot \frac{2(N - 1)^2}{N^2 + (N - 1)^2} > 1.
\]
Thus, (3.26) is proved. \hfill \Box

The second result of this section is the following.

Theorem 3.5. Let $N \geq 2$. As $\alpha$ crosses $\alpha_0$ to the right and stays near $\alpha_0$, for each $\lambda \in \Lambda_N$ with the condition (3.2) and the equation (1.3) bifurcates from the trivial solution to an attractor $A_N(\alpha, \lambda)$ which is homeomorphic to $S^1$. For any bounded open set $U \subset H$ with $0 \in U$, $A_N(\alpha, \lambda)$ attracts $U \setminus \Gamma$ in $H$, where $\Gamma$ is the stable manifold of $u = 0$ with codimension 2. Let $a_i$ and $b_i$ be numbers defined by (3.18) for $i = 1, 2$; then the attractor $A_N(\alpha, \lambda)$ has the following structure.

(a) $a_i, b_i = O(\sqrt{\alpha - \alpha_0})$ for $i = 1, 2$.

(b) Let $2 \leq N \leq 4$. Then, there exists a number $\delta_0 = \delta_0(\alpha, N) \in \Lambda_N$ such that for $(N + 1)\lambda_1 < \lambda < \delta_0$, $A_N(\alpha, \lambda)$ consists of 4 singular points and their connecting orbits. The singular points can be expressed as
\[
u^+(x) = \pm \alpha_1 \sin(N \pi x) + o(\sqrt{\alpha - \alpha_0}),
\]
\[
u^- (x) = \pm \alpha_2 \sin((N + 1) \pi x) + o(\sqrt{\alpha - \alpha_0}),
\] (3.27)
where $u_1^+$ are stable nodes and $u_1^-$ are saddles.

If $\delta_0 < \lambda < N\lambda_2$, $A_N(\alpha, \lambda)$ consists of 8 singular points and their connecting orbits. The singular points can be expressed as $u_1^+,$ $u_2^+,$ and
\[
u^1 (x) = b_1 \sin(N \pi x) + b_2 \sin((N + 1) \pi x) + o(\sqrt{\alpha - \alpha_0}),
\]
\[
u^2 (x) = -b_1 \sin(N \pi x) + b_2 \sin((N + 1) \pi x) + o(\sqrt{\alpha - \alpha_0}),
\] (3.28)
where $u_1$ are saddle and $u_2^+$ are stable nodes.

(c) Let $N \geq 5$ and $u_i^+$ be an expression given by (3.27) and (3.28) for $i = 1, 2, 3, 4$. Then, there exists a number $\delta_1 = \delta_1(\alpha, N), \delta_2 = \delta_2(\alpha, N) \in \Lambda_N$ such that for $(N + 1)\lambda_1 < \lambda < \delta_1$ or $\delta_2 < \lambda < N\lambda_2$, $A_N(\alpha, \lambda)$ consists of 4 singular points and their connecting orbits. The singular points can be expressed as $u_1^+$ and
$u_{2}^{\pm}$. If $(N + 1)\lambda_{1} < \lambda < \delta_{1}$, $u_{1}^{\pm}$ are stable nodes and $u_{2}^{\pm}$ are saddles. If 
$\delta_{2} < \lambda < N\lambda_{2}$, $u_{1}^{\pm}$ are saddles and $u_{2}^{\pm}$ are stable nodes.

If $\delta_{1} < \lambda < \delta_{2}$, $A_{N}(\alpha, \lambda)$ consists of 8 singular points and their connecting 
orbits. The singular points can be expressed as $u_{1}^{\pm}, u_{2}^{\pm}, u_{3}^{\pm}$, and $u_{4}^{\pm}$. If $N = 5$, 
then $u_{1}^{\pm}, u_{2}^{\pm}$ are saddles and $u_{3}^{\pm}, u_{4}^{\pm}$ are stable nodes. If $N \geq 6$, then $u_{1}^{\pm}, u_{2}^{\pm}$ 
are stable nodes and $u_{3}^{\pm}, u_{4}^{\pm}$ are saddles.

(d) We have
\[
\begin{cases}
(N + 1)\lambda_{1} < \lambda_{0} < \delta_{0}, & \text{for } 2 \leq N \leq 4, \\
(N + 1)\lambda_{1} < \lambda_{0} < \delta_{1}, & \text{for } 5 \leq N \leq 7, \\
\delta_{1} < \lambda_{0} < \delta_{2}, & \text{for } N \geq 8.
\end{cases}
\]

Proof. (a) We fix $\alpha$ which is slightly bigger than $q(N)^{2}$ such that (3.2) holds true. 
The reduced equation of (1.3) on the center manifold is given by (3.13) which bifurcates 
an invariant set $A_{N}(\alpha, \lambda)$. By the Poincare-Bendixon Theorem, $A_{N}(\alpha, \lambda)$ is 
homeomorphic to $S^{1}$ and consists of equilibrium points and their connecting orbits. 
Moreover, by the local attractiveness of the center manifold $A_{N}(\alpha, \lambda)$ becomes an 
attractor. The truncated system of (3.13) has the following equilibrium points:
\[
(y_{1}, y_{2}) = (\pm a_{1}, 0), (0, \pm a_{2}), (b_{1}, \pm b_{2}), (-b_{1}, \pm b_{2}),
\]
where $a_{i}$ and $b_{i}$ are given by (3.18) for $i = 1, 2$. It follows from (3.6) that $a_{i}, b_{i} = 
O(\sqrt{\alpha - \alpha_{0}})$ for $i = 1, 2$.

(b) and (c) Since $\lambda \neq \lambda_{0}$, (3.19) is no longer true. We will prove that each of 
(3.30) is a nondegenerate singular point of the truncated system in certain suitable 
parameter regimes. We keep the notations for the function $F = (F_{1}, F_{2})$ as in the 
proof of Theorem 3.1. For the eigenvalues of $DF$, we also use the same notations 
$\tau_{i}$ defined by (3.23) and (3.24) for $i = 1, 2, 3, 4$. We divide the bifurcation analysis 
into several cases according to the values of $N$.

**Case I.** $N = 2, 3, 4$. By Lemma 3.3, we have
\[
\beta_{N} - \frac{d_{12}}{d_{22}} \beta_{N+1} > 0 \quad \text{and} \quad B = d_{11}d_{22} - d_{12}d_{21} > 0.
\]
We also recall from Lemma 3.3 that $d_{21}/d_{11} > 0$. If $N = 3, 4$, then by Lemma 
3.4 $\beta_{N}$ is decreasing and $\beta_{N+1}$ is increasing with respect to $\lambda \in \Lambda_{N}$. 
Hence, there exists a number $\delta_{0} = \delta_{0}(\alpha, N) \in \Lambda_{N}$ such that 
\[
\beta_{N+1} - \frac{d_{21}}{d_{11}} \beta_{N} > 0 \iff \delta_{0} < \lambda < N\lambda_{2}.
\]
For $N = 2$, we note that $\beta_{N}$ is decreasing with respect to $\lambda$ and $\beta_{N+1}(\alpha, \cdot)$ has a 
unique local maximum in $\Lambda_{N}(\alpha)$. Since $\beta_{N+1}(\alpha, N\lambda_{2}) > 0$, (3.31) is still valid for 
$N = 2$. As a consequence, if $N = 2, 3, 4$, then $b_{1}$ and $b_{2}$ exist only for $\delta_{0} < \lambda < N\lambda_{2}$.

If $z_{1}^{2} = a_{1}^{2}$ and $z_{2} = 0$, then $DF(z_{1}, z_{2})$ has two eigenvalues $\tau_{1}(\alpha, \lambda)$ and $\tau_{2}(\alpha, \lambda)$. 
If $z_{1} = 0$ and $z_{2} = a_{2}^{2}$, then $DF(z_{1}, z_{2})$ has two eigenvalues $\tau_{3}(\alpha, \lambda)$ and $\tau_{4}(\alpha, \lambda)$. 
It is obvious that $\tau_{1}, \tau_{3} < 0$ and $\tau_{4} > 0$. Hence, $(0, \pm a_{2})$ are saddle points. 
On the other hand, $\tau_{2} < 0$ for $(N + 1)\lambda_{1} < \lambda < \delta_{0}$ and $\tau_{2} > 0$ for $\delta_{0} < \lambda < N\lambda_{2}$.
Moreover, $(\pm a_{1}, 0)$ are stable points for $(N + 1)\lambda_{1} < \lambda < \delta_{0}$ and saddle points for 
$\delta_{0} < \lambda < N\lambda_{2}$.

Next, suppose that $\delta_{0} < \lambda < N\lambda_{2}$. If $(z_{1}, z_{2}) = (b_{1}, \pm b_{2})$ or $(-b_{1}, \pm b_{2})$, then it 
follows from the formula (3.25) that $\zeta_{11} + \zeta_{22} < 0$ and $\zeta_{11}\zeta_{22} - \zeta_{12}\zeta_{21} > 0$. Hence, $DF$ 
has two negative eigenvalues which implies that each of $(b_{1}, \pm b_{2})$ and $(-b_{1}, \pm b_{2})$ is
stable. The following diagram (Figure 1) depicts the bifurcated attractor $A_N(\alpha, \lambda)$ according to the values of $\lambda$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram.png}
\caption{Structure of the bifurcated attractor for $N = 2, 3, 4$.}
\end{figure}

**Case II.** $N = 5$. Let us fix $\alpha$ and write $\beta_N(\lambda)$ instead of $\beta_N(\alpha, \lambda)$ and so on. By (3.14)–(3.16), we have

$$
\frac{d_{21}}{d_{11}} \sim \frac{348}{119} > 1, \quad 0 < \frac{d_{12}}{d_{22}} \sim \frac{9}{32} < 1, \quad \text{and} \quad B = d_{11}d_{22} - d_{12}d_{21} > 0.
$$

Then, by Lemma 3.4 there exist two numbers $\delta_1 = \delta_1(\alpha, N), \delta_2 = \delta_2(\alpha, N) \in \Lambda_N$ such that

$$
\beta_{N+1} = \frac{d_{21}}{d_{11}} \beta_N \quad \text{at} \quad \lambda = \delta_1,
$$

$$
\beta_N = \frac{d_{12}}{d_{22}} \beta_{N+1} \quad \text{at} \quad \lambda = \delta_2.
$$

(3.32)

Suppose that $\delta_2 \leq \delta_1$. Then, by Lemma 3.4

$$
\beta_N(\delta_2) \geq \beta_N(\delta_1) \quad \text{and} \quad \beta_{N+1}(\delta_2) \leq \beta_{N+1}(\delta_1).
$$

This implies, by (3.32),

$$
1 \leq \frac{\beta_N(\delta_2)}{\beta_N(\delta_1)} \cdot \frac{\beta_{N+1}(\delta_1)}{\beta_{N+1}(\delta_2)} = \frac{d_{21}}{d_{11}} \cdot \frac{d_{12}}{d_{22}} \sim \frac{783}{952} < 1,
$$

which is a contradiction. Hence, $\delta_1 < \delta_2$ such that for all $\lambda \in (\delta_1, \delta_2)$

$$
\beta_{N+1} > \frac{d_{21}}{d_{11}} \beta_N \quad \text{and} \quad \beta_N > \frac{d_{12}}{d_{22}} \beta_{N+1},
$$

which implies that both $b_1(\alpha, \lambda)$ and $b_2(\alpha, \lambda)$ exist only for $\lambda \in (\delta_1, \delta_2)$.

Now, we study stability of static solutions of (3.13). Suppose that $\lambda \in ((N + 1)\lambda_1, \delta_1) \cup (\delta_2, N\lambda_2)$. Then, we have four equilibrium points $(\pm a_1, 0)$ and $(0, \pm a_2)$ and it holds that

$$
\beta_{N+1} < \frac{d_{21}}{d_{11}} \beta_N \quad \text{and} \quad \beta_N > \frac{d_{12}}{d_{22}} \beta_{N+1} \quad \text{for} \quad \lambda \in ((N + 1)\lambda_1, \delta_1),
$$

$$
\beta_{N+1} > \frac{d_{21}}{d_{11}} \beta_N \quad \text{and} \quad \beta_N < \frac{d_{12}}{d_{22}} \beta_{N+1} \quad \text{for} \quad \lambda \in (\delta_2, N\lambda_2).$$
Hence, proceeding as in the Case I, we see from (3.23) and (3.24) that if \( \lambda \in ((N + 1)\lambda_1, \delta_1) \), then \((\pm a_1, 0)\) are stable points and \((0, \pm a_2)\) are saddle points; if \( \lambda \in (\delta_2, N\lambda_2) \), then \((\pm a_1, 0)\) are saddle points and \((0, \pm a_2)\) are stable points.

Suppose that \( \delta_1 < \lambda < \delta_2 \). Then, we have eight equilibrium points \((\pm a_1, 0)\), \((0, \pm a_2)\), \((b_1, \pm b_2)\), and \((-b_1, \pm b_2)\). We also see that

\[
\beta_{N+1} > \frac{d_{21}}{d_{11}} \beta_N \quad \text{and} \quad \beta_N > \frac{d_{12}}{d_{22}} \beta_{N+1}.
\]

Hence, \((\pm a_1, 0)\) and \((0, \pm a_2)\) are saddle points. Moreover, it follows from (3.25) that \(DF\) has two negative eigenvalues. Thus, each of \((b_1, \pm b_2)\) and \((-b_1, \pm b_2)\) is stable. Figure 2 depicts the bifurcated attractor \(A_N(\alpha, \lambda)\) according to the values of \(\lambda\).

![Bifurcation Diagrams](image)

(a) \((N + 1)\lambda_1 < \lambda < \delta_1\)
(b) \(\delta_1 < \lambda < \delta_2\)
(c) \(\delta_2 < \lambda < N\lambda_2\)

**Figure 2.** Structure of the bifurcated attractor for \(N = 5\).

**Case III.** \(N = 6, 7\). By Lemma 3.3, \(B = d_{11}d_{22} - d_{12}d_{21} < 0\). For \(N = 6\), we derive from (3.14)–(3.16) that

\[
\frac{d_{21}}{d_{11}} \sim \frac{45}{16} > 1 \quad \text{and} \quad 0 < \frac{d_{12}}{d_{22}} \sim \frac{12}{19} < 1.
\]
For $N = 7$, we have
\[
\frac{d_{21}}{d_{11}} \sim \frac{188}{69} > 1 \quad \text{and} \quad 0 < \frac{d_{12}}{d_{22}} \sim \frac{19}{22} < 1.
\]
Then, by Lemma 3.4 there exist two numbers $\delta_1 = \delta_1(\alpha, N), \delta_2 = \delta_2(\alpha, N) \in \Lambda_N$ such that
\[
\beta_N = \frac{d_{12}}{d_{22}} \beta_{N+1} \quad \text{at} \quad \lambda = \delta_1,
\]
\[
\beta_{N+1} = \frac{d_{21}}{d_{11}} \beta_N \quad \text{at} \quad \lambda = \delta_2.
\] (3.33)

Suppose that $\delta_2 \leq \delta_1$. Then, by Lemma 3.4
\[
\beta_N(\delta_2) \geq \beta_N(\delta_1) \quad \text{and} \quad \beta_{N+1}(\delta_2) \leq \beta_{N+1}(\delta_1).
\]
This implies by (3.33)
\[
1 \leq \frac{\beta_N(\delta_2)}{\beta_N(\delta_1)} \cdot \frac{\beta_{N+1}(\delta_1)}{\beta_{N+1}(\delta_2)} = \frac{d_{11}}{d_{21}} \cdot \frac{d_{22}}{d_{12}} < 1,
\]
a contradiction. Hence, $\delta_1 < \delta_2$ such that for all $\lambda \in (\delta_1, \delta_2)$
\[
\beta_{N+1} < \frac{d_{21}}{d_{11}} \beta_N \quad \text{and} \quad \beta_N < \frac{d_{12}}{d_{22}} \beta_{N+1},
\] (3.34)
which implies that both $b_1(\alpha, \lambda)$ and $b_2(\alpha, \lambda)$ exist only for $\lambda \in (\delta_1, \delta_2)$.

If $\lambda \in ((N + 1)\lambda_1, \delta_1) \cup (\delta_2, N\lambda_2)$, then we have four equilibrium points $(\pm a_1, 0)$ and $(0, \pm a_2)$. As previous cases, we deduce from (3.23) and (3.24) that if $\lambda \in ((N + 1)\lambda_1, \delta_1)$, then $(\pm a_1, 0)$ are stable points and $(0, \pm a_2)$ are saddle points; if $\lambda \in (\delta_2, N\lambda_2)$, then $(\pm a_1, 0)$ are saddle points and $(0, \pm a_2)$ are stable points.

Suppose that $\delta_1 < \lambda < \delta_2$. Then, we have eight equilibrium points $(\pm a_1, 0), (0, \pm a_2), (b_1, \pm b_2)$, and $(-b_1, \pm b_2)$. By (3.34), $(\pm a_1, 0)$ and $(0, \pm a_2)$ are stable points. For $(b_1, \pm b_2)$ and $(-b_1, \pm b_2)$, we infer from (3.25) and (3.34) that $\zeta_{11}\zeta_{22} - \zeta_{12}\zeta_{21} < 0$, which implies that $DF$ has a positive and a negative eigenvalues. Thus, each of $(b_1, \pm b_2)$ and $(-b_1, \pm b_2)$ is a saddle point. The diagram (Figure 3) depicts the bifurcated attractor $A_N(\alpha, \lambda)$ according to the values of $\lambda$.

Case IV. $N \geq 8$. By Lemma 3.3, we have
\[
\frac{d_{21}}{d_{11}} > 1, \quad \frac{d_{12}}{d_{22}} > 1, \quad \text{and} \quad B = d_{11}d_{22} - d_{12}d_{21} < 0.
\]
Then, by Lemma 3.4 there exist two numbers $\delta_1 = \delta_1(\alpha, N), \delta_2 = \delta_2(\alpha, N) \in \Lambda_N$ such that
\[
\beta_N = \frac{d_{12}}{d_{22}} \beta_{N+1} \quad \text{at} \quad \lambda = \delta_1,
\]
\[
\beta_{N+1} = \frac{d_{21}}{d_{11}} \beta_N \quad \text{at} \quad \lambda = \delta_2.
\] (3.35)

It is obvious that $\delta_1 < \delta_2$. We note that
\[
\beta_{N+1} < \frac{d_{21}}{d_{11}} \beta_N \quad \text{and} \quad \beta_N < \frac{d_{12}}{d_{22}} \beta_{N+1}
\]
if and only if $\lambda \in (\delta_1, \delta_2)$. Hence, both $b_1(\alpha, \lambda)$ and $b_2(\alpha, \lambda)$ exist only for $\lambda \in (\delta_1, \delta_2)$. Then, proceeding as in the previous cases we conclude the following. For $\lambda \in ((N + 1)\lambda_1, \delta_1) \cup (\delta_2, N\lambda_2)$, we have four equilibrium points $(\pm a_1, 0)$ and $(0, \pm a_2)$ such that if $\lambda \in ((N + 1)\lambda_1, \delta_1)$, then $(\pm a_1, 0)$ are stable points and $(0, \pm a_2)$ are saddle points; if $\lambda \in (\delta_2, N\lambda_2)$, then $(\pm a_1, 0)$ are saddle points and $(0, \pm a_2)$ are stable points; if $\lambda \in (\delta_1, \delta_2)$, then $(\pm a_1, 0)$ and $(0, \pm a_2)$ are saddle points.
stable points. For \( \delta_1 < \lambda < \delta_2 \), we have eight equilibrium points \((\pm a_1, 0), (0, \pm a_2), (b_1, \pm b_2), \) \((-b_1, \pm b_2)\). The points \((\pm a_1, 0)\) and \((0, \pm a_2)\) are stable, \((b_1, \pm b_2)\) and \((-b_1, \pm b_2)\) are saddle.

(d) The relation (3.29) follows immediately from the definition of \(\delta_i (i = 0, 1, 2)\) and the inequalities in (3.23) and (3.24) which occur at \(\lambda = \lambda_0\).

We note that (3.29) provides an information about the locations of \(\delta_0, \delta_1, \delta_2\).

4. Conclusion. We have studied the dynamic bifurcation of the DKSE on an interval with odd periodic condition. The DKSE (1.1) has two bifurcation parameters: the control parameter \(\alpha\) and the period \(\lambda\) (domain size). In this paper, we consider the dynamic bifurcation of DKSE when \(\alpha\) is a fixed number satisfying (1.6) and \(\lambda\) varies. A similar scenario of bifurcation analysis has been carried out in [16, 17] for the Swift-Hohenberg equation. In particular, it was suggested in [17] to see the bifurcation phenomena due to the variation of period (domain size) for different phase transition equations. This paper answers this question for the DKSE. Using the center manifold analysis based on the second approximation formula, we classified the attractor bifurcating from the trivial solution. First, we proved that there happens a pitchfork bifurcation at the end points of each gap of type \((0, \lambda_1),\)
The parameterized operator $L_{\lambda}$ is supercritical at $\lambda = k\lambda_1$ for $1 \leq k \leq N$. The bifurcation is subcritical at $k\lambda_2$ for $1 \leq k \leq N - 1$.

The situation is quite interesting if $\alpha > \alpha_0$, then we have some mixed states in the overlapped interval $\Lambda_N = ((N + 1)\lambda_1, N\lambda_2)$. The structure of the bifurcated attractor $\mathcal{A}(\alpha, \lambda)$, which consists of singular points and their connecting orbits, is heavily dependent on the number $N$ and the value of $\lambda \in \Lambda_N$. The singular points are either stable nodes or saddles. The possible stability of singular points is the following:

1. **(A1)** The stable singular points are perturbations of the eigenvector of low frequency $N$.
2. **(A2)** The stable singular points are perturbations of the eigenvector of high frequency $N + 1$.
3. **(A3)** The stable singular points are perturbations of the eigenvectors of both low and high frequencies.
4. **(A4)** The stable singular points are perturbations of superpositions of eigenvectors.

If $2 \leq N \leq 4$, there exists a number $\delta_0 \in \Lambda_N$ such that
   - (i) for $\lambda < \delta_0$, we have four singular points and (A1) is valid;
   - (ii) for $\lambda > \delta_0$, we have eight singular points and (A4) is valid.

If $N \geq 5$, there exist two numbers $\delta_1, \delta_2$ in $\Lambda_N$ such that $\delta_1 < \delta_2$ and
   - (iii) for $\lambda < \delta_1$, we have four singular points and (A1) is valid;
   - (iv) for $\lambda > \delta_2$, we have four singular points and (A2) is valid;
   - (v) for $\delta_1 < \lambda < \delta_2$ and $N = 5$, we have eight singular points and (A4) is valid;
   - (vi) for $\delta_1 < \lambda < \delta_2$ and $N \geq 6$, we have eight singular points and (A3) is valid.

We also estimated $\delta_i$ for $i = 0, 1, 2$ in terms of the number $\lambda_0$.

The structures of the bifurcated attractors are illustrated in Figures 1-3. An interesting point in those pictures is that the structures for $N = 2, 3, 4$ and $N \geq 6$ are completely different. For $N = 2, 3, 4$, the interval $\Lambda_N$ is divided into two parts and the stable singular points are perturbations of superpositions of eigenvectors $\phi_N$ and $\phi_{N+1}$. On the other hand, for $N \geq 6$ the interval $\Lambda_N$ is divided into three parts and the stable singular points are perturbations of eigenvectors. The case $N = 5$ connects these two cases in the sense that $\Lambda_N$ is split into three parts but the stable singular points are perturbations of superpositions of eigenvectors. Thus, one may say that a transition occurs at $N = 5$ if we regard the number $N$ as another transition parameter. This phenomena does not happen for the SHE as shown in [16, 17] and discussed in the Introduction.

5. **Review of attractor bifurcation theory.** In this section, we briefly review the attractor bifurcation theory developed by Ma and Wang in [10, 11].

Let $H_1$ and $H$ be two Hilbert spaces with a dense inclusion $H_1 \hookrightarrow H$. Let us consider the nonlinear evolution equation

$$\begin{cases}
\frac{du}{dt} = L_\lambda u + G(u, \lambda), \\
u(0) = u_0,
\end{cases}$$

(5.1)

where $u : [0, \infty) \to H$ is the unknown function and $\lambda \in \mathbb{R}$ is the system parameter. The parameterized operator $L_\lambda : H_1 \to H$ are linear completely continuous fields.
depending continuously on $\lambda$ and satisfy

\[
\begin{align*}
L_\lambda &= -A + B_\lambda, \quad \text{a sectorial operator,} \\
A : H_1 &\to H, \quad \text{a linear homeomorphism,} \\
B_\lambda : H_1 &\to H, \quad \text{parameterized linear compact operator.}
\end{align*}
\]

Then, $L_\lambda$ generates an analytic semigroup $\{S_\lambda(t) = e^{-tL_\lambda}\}_{t \geq 0}$ and we can define fractional power operators $L_\lambda^\alpha$ for any $0 \leq \alpha \leq 1$ with domain $H_\alpha = D(L_\lambda^\alpha)$. Moreover, $H_0 = H$ and if $\alpha_1 > \alpha_2$, then $H_{\alpha_1} \subset H_{\alpha_2}$. We assume that $G(\cdot, \lambda) : H_\alpha \to H$ are parameterized $C^r$ bounded operators for some $0 \leq \alpha < 1$ and $r \geq 0$, and depend continuously on $\lambda$ such that

\[
G(u, \lambda) = o(\|u\|_{H_\alpha}), \quad \forall \lambda \in \mathbb{R}.
\]

In this paper, we are interested in the case that there exists an eigenvalue sequence $\{\rho_k\} \subset \mathbb{C}$ and eigenvector sequence $\{\epsilon_k, h_k\} \subset H_1$ of $A$ satisfying

\[
\begin{align*}
Az_k &= \rho_kz_k, \\
\{\epsilon_k, h_k\} &\text{is a basis of } H, \\
\Re\rho_k &\to \infty \quad \text{as } k \to \infty,
\end{align*}
\]

for some constants $b, C > 0$. The condition (5.4) implies that $A$ is a sectorial operator. Hence, we can define fractional power operators $A^\alpha$ with domain $H_\alpha = D(A^\alpha)$ for any $0 \leq \alpha \leq 1$. For the compact operator $B_\lambda : H_1 \to H$, we assume that there exists a constant $0 \leq \theta < 1$ such that

\[
B_\lambda : H_\theta \to H \quad \text{is bounded for all } \lambda \in \mathbb{R}.
\]

It is known that $L_\lambda = -A + B_\lambda$ is a sectorial operator if (5.4) and (5.5) hold.

Let $\beta_1(\lambda), \cdots, \beta_k(\lambda), \cdots \in \mathbb{C}$ be the eigenvalues of $L_\lambda$ counting multiplicities. Suppose that

\[
\Re\beta_j(\lambda) = \begin{cases} 
< 0, & \text{if } \lambda < \lambda_0 \\
= 0, & \text{if } \lambda = \lambda_0 \\
> 0, & \text{if } \lambda > \lambda_0
\end{cases} \quad (1 \leq j \leq m)
\]

and

\[
\Re\beta_j(\lambda_0) < 0, \quad \forall \ j \geq m + 1.
\]

We define the eigenspace of $L_\lambda$ at $\lambda_0$ by

\[
E_0 = \bigcup_{j=1}^{m} \bigcup_{k=1}^{\infty} \{u \in H_1 : (L_{\lambda_0} - \beta_j(\lambda_0))^k u = 0\}.
\]

Then, it is known that $\dim E_0 = m$.

We now introduce the notion of attractor bifurcation. We say that (5.1) bifurcates from $(u, \lambda) = (0, \lambda_0)$ to an attractor $\Omega_{\lambda}$ if there exists a sequence of attractors $\{\Omega_{\lambda_n}\}$ of (5.1) with $0 \not\in \Omega_{\lambda_n}$ such that

\[
\lim_{n \to \infty} \max_{u \in \Omega_{\lambda_n}} |u| = 0, \quad \lim_{n \to \infty} \lambda_n = \lambda_0.
\]

If the invariant sets $\Omega_{\lambda}$ are attractors and are homotopy equivalent to an $m$-dimensional sphere $S^m$, then the bifurcation is called an $S^m$-attractor bifurcation. The following dynamic bifurcation theorem for (5.1), which comes from theorem
For any second approximation formula. To see this, we assume that

\[ (u, \lambda) = (0, \lambda_0) \]

and \( u = 0 \) is a locally asymptotically stable equilibrium point of (5.1) at \( \lambda = \lambda_0 \). Then, we have the following:

(a) The equation (5.1) bifurcates from \((u, \lambda) = (0, \lambda_0)\) to an attractor \( A_\alpha \) for \( \lambda > \lambda_0 \), with \( m - 1 \leq \dim A_\alpha \leq m \), which is connected if \( m > 1 \).

(b) For any \( u_\lambda \in A_\alpha \), \( u_\lambda \) can be expressed as

\[ u_\lambda = v_\lambda + o(\|v_\lambda\|_{H^1}), \quad v_\lambda \in E_0. \]

(c) If \( u = 0 \) is globally asymptotically stable for (5.1) at \( \lambda = \lambda_0 \), then for any bounded open set \( U \subset H \) with \( 0 \in U \), there exists \( \varepsilon > 0 \) such that as \( \lambda_0 < \lambda < \lambda_0 + \varepsilon \), \( A_\alpha \) attracts \( U \setminus \Gamma \) in \( H \), where \( \Gamma \) is the stable manifold of \( u = 0 \) with codimension \( m \). In particular, if (5.1) has a global attractor for all \( \lambda \) near \( \lambda_0 \), then \( \varepsilon \) can be chosen independent of \( U \).

To obtain information about the bifurcated attractor \( A_\alpha \) obtained by Theorem 5.1, we reduce equation (5.1) on the center manifold. The center manifold at the bifurcation point \( \alpha_0 \) persists for \( \alpha \in (\alpha_0, \alpha_0 + \varepsilon) \), and equation (5.1) is reduced on this manifold to the following finite dimensional system:

\[ \frac{dv}{dt} = L_1^\alpha v + P_L G(v + \Phi(v, \alpha), \alpha) \quad (5.9) \]

in a neighborhood of \( 0 \) in \( E_0 \). Here, \( P_L : H \rightarrow E_0 \) is the canonical projection, \( L_1^\alpha = L_1|_{E_0} \), and \( v = P_L u \). Generally, it is not easy to compute (5.9) exactly. In [10], the authors suggest an efficient way to calculate (5.9) which is called the second approximation formula. To see this, we assume that \( G(\cdot, \alpha) : H_1 \rightarrow H \) is decomposed as

\[ G(u, \alpha) = \sum_{k=2}^\infty G_k(u, \alpha), \]

where \( G_k(\cdot, \alpha) : H_1 \times \cdots \times H_1 \rightarrow H \) is an \( n \)-multilinear mapping and \( G_k(u, \alpha) = G_k(u, \cdots, u, \alpha) \). Under the conditions (5.6)–(5.8), we set \( E_0 = \text{span}\{\phi_1, \cdots, \phi_m\} \) and \( v = \sum_{k=1}^m y_k \phi_k \). Then, the second approximation of (5.9) is given by

\[ \frac{dy_k}{dt} = \beta_k(\alpha)y_k + g_{2k}(y, \alpha) + g_{3k}(y, \alpha) + g_k(y, \alpha) + o(|y|^3), \quad k = 1, \cdots, m, \quad (5.10) \]

where \( y = (y_1, \cdots, y_m) \) and

\[ g_{jk}(y, \alpha) = \frac{1}{\|\phi_k\|_H} \left\langle G_j \left( \sum_{i=1}^m y_i \phi_i \right), \phi_k \right\rangle, \quad j = 2, 3, \quad k = 1, \cdots, m, \]

\[ g_k(y, \alpha) = \frac{1}{\|\phi_k\|_H} \sum_{i,j,l=1}^m a_{ijl}^k y_i y_j y_l, \quad k = 1, \cdots, m, \]

\[ a_{ijl}^k = \sum_{n=m+1}^\infty \frac{(G_2(\phi_i, \phi_l, \alpha), \phi_n) \cdot (G_2(\phi_i, \phi_l, \alpha), \phi_k)}{(2\beta_k(\alpha) - \beta_n(\alpha))\|\phi_n\|_H}. \]

The second approximation formula (5.10) is very useful for calculating the reduced equation (5.9) when the inner product of the nonlinear term with the elements in \( E_0 \) becomes zero. Earlier research provides a good example [13]. We use this formula when we consider the DKSE (1.1) on the center manifolds.
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