NECESSARY AND SUFFICIENT CONDITIONS FOR ERGODICITY OF CIR MODEL DRIVEN BY STABLE PROCESSES WITH MARKOV SWITCHING

ZHENZHONG ZHANG, ENHUA ZHANG AND JINYING TONG*

Department of Applied Mathematics, Donghua University
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Abstract. In this paper, we consider long time behavior of the Cox-Ingersoll-Ross (CIR) interest rate model driven by stable processes with Markov switching. Under some assumptions, we prove an ergodicity-transience dichotomy, namely, the interest rate process is either ergodic or transient. The sufficient and necessary conditions for ergodicity and transience of such interest model are given under some assumptions. Finally, an application to interval estimation of the interest rate processes is presented to illustrate our results.

1. Introduction. In 1985, Cox, Ingersoll and Ross (CIR, for short) [7] first established that the instantaneous interest rate satisfies the following stochastic differential equation (SDE)

\[ dX_t = (\delta - \beta X_t)dt + \sigma \sqrt{X_t} dW_t, \tag{1} \]

where \( W_t \) is a standard Brownian motion and \( \delta, \beta, \sigma \) are positive constants. Since then, the SDE (1) is called CIR model which has almost perfect properties: non-negativity, ergodicity, with explicit transition densities. However, as a one factor model, a problem with the model (1) is that it describes interest rate movements driven by only one source of market risk. Some empirical analyses show that processes with Markov switching can better capture the reality data. For example, Zhou and Mamon [24] show that the CIR model with Markov switching provides better fits for the three-month zero-coupon yields of Canada from January 1986 to December 1995 than the CIR model (1). In addition, Wu [22] finds that diffusion processes with Markov switching can well capture the monthly exchange rates of six major Asia-Pacific currencies from January 2000 to December 2011. Moreover, Smith [20] finds that the Markov-switching model is better than the stochastic-volatility model in predicting interest rate volatility according to monthly observations on 30-day treasury bills from June 1964 to December 1996 in U.S. Therefore, many researchers extend CIR model to CIR model with Markov switching. For instance,
Zhang, Tong and Hu [23] consider the long-term behavior of the CIR model with Markov switching, that is

\[
dX_t = (2\beta(r_t)X_t + \delta(r_t))dt + \sigma(r_t)\sqrt{X_t}dW_t,
\]

(2)

where \((r_t)_{t \geq 0}\) is a Markov chain taking the state space \(S = \{1, 2, ..., n\}\). They show that under some conditions, the model (2) has a unique stationary distribution.

Furthermore, Tong and Zhang [21] find that the transition semigroup corresponding to the SDE (2) converges to the stationary distribution at an exponential rate in the Wasserstein distance.

Although SDE (2) has modeled effects of exogenous noise on interest rate, this model can not characterize some classes of noise distributions with infinite variance, which extensively exist in many economic phenomena [6]. For example, Mandelbrot [14] finds that the distributions of the changes of monthly wool prices from 1890 to 1937 follow \(\alpha\) stable distribution with \(\alpha = 1.7\). Stable distributions and power distributions are frequently found in analyses of critical behavior and financial data. According to the generalized central limit theorem, summation of a sequence of independent and identically distributed variables with infinite variance converges weakly to a stable distribution. Hence, it is natural to replace the Brownian motion with an \(\alpha\)-stable process in the model (2).

The main aim of this paper is to study the CIR model driven by symmetric \(\alpha\)-stable processes with Markov switching. To be precise, we consider the following SDE

\[
dX_t = (2\beta(r_t)X_t + \delta(r_t))dt + \sigma(r_t)\sqrt{X_t}dZ_t,
\]

(3)

where \((Z_t)_{t \geq 0}\) is a symmetric \(\alpha\)-stable process with index \(1 < \alpha < 2\) and its Lévy measure

\[
\nu_\alpha(dz) := \frac{C_\alpha}{|z|^{1+\alpha}}dz.
\]

(4)

Here \(C_\alpha = \frac{\alpha^{2-\alpha}\Gamma((1+\alpha)/2)}{\pi^{1/2}\Gamma(1-\alpha/2)}\) and \(\Gamma(\cdot)\) is the Gamma function defined by

\[
\Gamma(s) = \int_0^{+\infty} t^{s-1}e^{-t}dt, \quad s \in (0, +\infty).
\]

The Markov chain \((r_t)_{t \geq 0}\) is the same as the model (2). Then, given \(r_t \equiv i, i \in S\), the SDE (3) reduces to

\[
dX_t^{(i)} = \left(2\beta(i)X_t^{(i)} + \delta(i)\right)dt + \sigma(i)\sqrt{X_t^{(i)}}dZ_t.
\]

(5)

Roughly speaking, this system (3) is operated as follows: Assume \(r_0 = i_0\) and let \(\tau_1\) be the jump time when the Markov chain \(r_t\) jumps from \(i_0\) to \(i_1\). During the time period \([0, \tau_1]\), the system evolves according to Eq. (5) with \(i = i_0\). Then in \([\tau_1, \tau_2]\), the system behaves like Eq. (5) with \(i = i_1\). The system will continue to switch as long as the Markov chain jumps. In other words, Eq. (3) can be regarded as Eqs. (5) switching from one to another according to the state of the Markov chain.

If the number of states of the Markov chain is only one, then the model (3) reduces to CIR driven by stable processes without Markov switching, which have been extensively considered by many researchers. For instance, Li and Ma [13] give some asymptotic properties of estimators in a CIR model driven by a spectrally positive stable-Lévy process. Handa [9] considers ergodic properties of the CIR model driven by stable processes without Markov switching. Very recently, Jiao, Ma and Scotti [10] obtain an explicit formula for the bond price and some distributions...
for large jumps of the model (3) with \(n = 1\). Besides, they also show the ergodicity of this model and present the Laplace transform of the stationary distribution.

Compared model (3) with (2), at first glance, there is only one noise change. One can not help asking whether there are some essential differences? Yes, from the perspective of generator of the Markov process, the main difference is that the second order differential operator corresponding to the SDE (3) is replaced by the fractional Laplace operator. The former is a local operator, whereas, the latter is non-local. To be precise, the infinitesimal generator \(A\) of the Markov process \((X_t, \mathcal{F}_t)_{t \geq 0}\) is given by for all \(V(\cdot, \cdot) \in C^2(\mathbb{R} \times \mathbb{S}) : \mathbb{R} \times \mathbb{S} \mapsto \mathbb{R}_{\geq 0}\)

\[
AV(x, i) = \sigma^2(i)x \int_{\mathbb{R}\setminus\{0\}} \left(V(x + y, i) - V(x, i) - V_x(x, i)y1_{\{|y| \leq 1\}}(y)\right) \frac{C_\alpha}{|y|^\alpha + 1} dy + V_x(x, i)(2\beta(i)x + \delta(i)) + QV(x, \cdot)(i),
\]

where \(QV(x, \cdot)(i) = \sum_{j=1}^n q_{ij}V(x, j)\) and \(\mathbb{R}_{\geq 0} := [0, +\infty)\). Here \(V(\cdot, \cdot) \in C^2(\mathbb{R} \times \mathbb{S})\) means for all \(i \in \mathbb{S}, V(\cdot, i) \in C^2(\mathbb{R})\).

Moreover, one can see from (6) that there is an integral term, which is a non-local operator. This leads to the essential difficulty. Besides, note that the coefficients \(\sigma(i) \sqrt{X_t}\) are H"older continuous with exponent \(\frac{1}{n} \in (\frac{1}{2}, 1)\), which are non-uniformly elliptic. To the best of our knowledge, all the existing literature is out of work to infer the recurrence and transience of the model (3) with \(n \geq 2\). With help of the Gauss hypergeometric function, the Khasminskii Lemma and the theory of nonsingular \(M\)-matrix, we overcome these difficulties and give an almost sufficient and necessary condition for ergodicity of the model (3).

Compared with existing literature, our contributions of this manuscript are the following.

- We prove the nonnegativity of the SDE model (3).
- An almost necessary and sufficient condition of ergodicity of the SDE model (3) is given.
- We construct a class of Lyapunov functions to deduce the recurrence and transience of the SDE model (3).
- A new method is presented to prove the following limit

\[
\lim_{x \to +\infty} |x|^\alpha - \theta \int_{\{|y| \geq 1\}} (|x + y|^\theta - |x|^\theta) \frac{1}{|y|^\alpha + 1} dy = \int_{|z| > 0} [1 + z]^\theta - 1 \frac{dz}{|z|^\alpha + 1},
\]

where \(1 < \alpha < 2\) and \(\theta \in (0, 1)\).

The remaining of this manuscript is organized as follows. In Section 2, we give some notation and prove the existence of a unique solution of the CIR model (3). The sufficient conditions for existence of a unique stationary distribution are presented in Section 3. Conditions for transience of the CIR model (3) are obtained in Section 4. Finally, two examples are given to illustrate our results in Section 5.

2. Preliminaries. Throughout this paper, let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a complete probability space with filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual conditions (i.e. it is right continuous and increasing while \(\mathcal{F}_0\) contains all \(\mathbb{P}\)-null sets). We denote by \(X_t^{x,i}\) the interest rate process \(X_t\) with initial value \(X_0 = x, \tau_0 = i\). For any bounded open interval (or bounded left closed right open interval) \(D \subset \mathbb{R}_{\geq 0}, i \in \mathbb{S}\), define

\[
\tau_D = \inf \{t : X_t^{x,i} \in D\}.
\]
Let \((r_t)_{t \geq 0}\) be a continuous time Markov chain on the state space \(S = \{1, 2, ..., n\}\) with \(n < \infty\) satisfying
\[
P\{r_{t+\Delta} = j \mid r_t = i\} = \begin{cases} q_{ij}\Delta + o(\Delta) & \text{if } i \neq j \\ 1 + q_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}
\]
where \(\Delta > 0\). The Q-matrix \(Q = (q_{ij})_{n \times n}\) is assumed to be irreducible and conservative. Hence \(q_{ij} > 0\) if \(i \neq j\) and
\[
q_{ii} = -\sum_{j \neq i} q_{ij}.
\]
Moreover, the Markov chain \((r_t)_{t \geq 0}\) and the stable process \((Z_t)_{t \geq 0}\) are assumed to be independent. Since the Markov chain \((r_t)_{t \geq 0}\) is irreducible and conservative, it has a unique stationary distribution \(\pi = (\pi(1), \pi(2), ..., \pi(n))\) which can be obtained by solving the following system of equations
\[
\pi Q = 0 \quad \text{and} \quad \sum_{i=1}^{n} \pi(i) = 1.
\]
To guarantee the existence of a unique solution, we impose the following assumptions (A):
\[
1 < \alpha < 2, \sigma(i) > 0, \delta(i) > 0, \beta(i) \in \mathbb{R}, i \in S.
\]
Since the CIR model (3) describes the evolution of interest rates, \(X_t\) should be pathwise unique. Next, we prove the pathwise uniqueness of the solution defined by (3) in the following Lemma. However, if \(\alpha \neq 2\), the process \((X_t)_{t \geq 0}\) can take the negative value, which is different from the CIR model (2).

**Lemma 2.1.** Let assumptions (A) hold. The SDE (3) has a pathwise unique solution.

**Proof.** Given \(r_t \equiv i\), according to Theorem 4 of Fournier [8], there is a pathwise unique solution to the equation
\[
Y_t = y + \int_0^t (2\beta(i)Y_s + \delta(i))ds + \int_0^t \sigma(i) \sqrt{Y_s}dZ_s, \quad Y_0 = y.
\]

The idea of this proof, roughly speaking, is that the process \((X_t)_{t \geq 0}\) can be regarded as Eqs. (9) switching from one to another according to the state of Markov chain. Based on the pathwise uniqueness of (9), we show the CIR model (3) has a pathwise unique solution.

Let \(N_T\) be the number of jumps for the Markov chain on the interval \([0, T]\), \(\tau_n\) be the \(n\)-th jump time of the Markov chain \(r_t\) on the interval \([0, T]\) and \(\tau_0 = 0\). For simplicity, we denote by \(Y_t^{y, r_0}\) the solution of \((Y_t)_{t \geq 0}\) with initial value \(Y_0 = y, r_0 = r_0\). Now, we construct the sample path of \(X_t\) as follows.

(i) Given initial value \(X_0 = x_0, r_0 = i_0\), for all \(t \in [0, \tau_1]\), let \(X_t = Y_t^{x_0, i_0}\).

(ii) For \(t \in [\tau_1, \tau_2]\), let \(X_t = Y_t^{X_{\tau_1}, r_{\tau_1}}\).

By induction, for \(t \in [\tau_k, \tau_{k+1}]\), we have
\[
X_t = Y_t^{X_{\tau_k}, r_{\tau_k}}.
\]
Consequently, we get for $t \in [0, T]$

$$X_t = \sum_{k=0}^{N_{T}-1} Y_t^{X_{\tau_k}, r_{\tau_k}} 1_{\{\tau_k \leq t < \tau_{k+1}\}} + Y_t^{X_{\tau_{N_{T}}}, r_{\tau_{N_{T}}}} 1_{\{\tau_{N_{T}} \leq t \leq T\}}$$

$$= \int_0^t (2\beta(r_s)X_s + \delta(r_s))ds + \int_0^t \sigma(r_s) \sqrt{|X_s|}dZ_s.$$ 

Recalling that the process $Y_t^{X_{\tau_k}, r_{\tau_k}}$ has a pathwise unique solution on $t \in [\tau_k, \tau_{k+1}]$ for $k = 0, 1, ..., N_T - 1$, we immediately obtain that the process $(X_t, r_t)_{t \geq 0}$ has a pathwise unique solution.

**Lemma 2.2.** Let assumptions (A) hold. Then for any $t > 0$, $P(X_s = 0, 0 < s < t) = 0$.

**Proof.** If there exists some $t_0$ such that $X_{t_0} = 0$, by the positivity of $\delta(i)$, we have $X_{t_0} \neq 0$. One can obtain the desired result.

### 3. Stationary distribution

In this section, we mainly find sufficient conditions such that the CIR model (3) has a unique stationary distribution based on theory of $M$-matrix and Foster-Lyapunov criterion. For convenience for readers, we cite some equivalence conditions about nonsingular $M$-matrix. For more information on theory of $M$-matrix, the readers can see Berman and Plemmons [4].

**Proposition 1.** The following statements are equivalent.

1. $A$ is a nonsingular $M$-matrix.
2. $A$ is semipositive; that is, there exists $x \gg 0$ in $\mathbb{R}^n$ such that $Ax \gg 0$.
3. All the leading principal minors of $A$ are positive; that is
$$\left|\begin{array}{ccc}
  a_{11} & \cdots & a_{1k} \\
  \vdots & \ddots & \vdots \\
  a_{k1} & \cdots & a_{kk}
\end{array}\right| > 0 \quad \text{for every } k = 1, 2, ..., n.$$

The proof for the existence of a unique stationary distribution is rather technical, so we first present a useful lemma.

**Lemma 3.1.** If $\sum_{i=1}^{n} \pi(i)\beta(i) < 0$, then there exists a constant $\theta_1 > 0$ such that for all $\theta \in (0, \theta_1)$, the matrix

$$A(\theta) = -2\theta \text{diag}(\beta(1), \beta(2), ..., \beta(n)) - Q$$

is a nonsingular $M$-matrix.

**Proof.** The proof is similar to that of Theorem 4.6 of Mao [15] and Theorem 3.4 in reference [16], we omit it here.

To prove our main results, we introduce two special functions and two Lemmas. First, the Gauss hypergeometric function is given by

$$2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}, \quad (10)$$

for $a, b, c, z \in \mathbb{C}, c \notin \mathbb{Z}^-$. For $\omega \in \mathbb{C}$ and $n \in \mathbb{Z}^+$, $(\omega)_n$ is defined by

$$(\omega)_0 = 1 \quad \text{and} \quad (\omega)_n = \omega(\omega + 1) \cdots (\omega + n - 1).$$
If \( \text{Re}(c) > \text{Re}(b) > 0 \), the Gauss hypergeometric function will be analytically continuous on \( \mathbb{C} \setminus (1, \infty) \) as
\[
_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt,
\]
where \( \Gamma(\cdot) \) is the Gamma function. Here \( \mathbb{Z}^+, \mathbb{Z}^- , \mathbb{C} \) represent non-negative integers, non-positive integers and complex numbers respectively. For further properties about the hypergeometric functions, please see [1].

Second, for \( \alpha \in (1, 2), \theta \in (0, \alpha) \), let
\[
E(\alpha, \theta) = \frac{\alpha [2F_1(-\theta, \alpha - \theta, \alpha - \theta + 1; 1) + 2F_1(-\theta, \alpha - \theta, \alpha - \theta + 1; -1)]}{\theta(\alpha - \theta)} + \frac{\alpha}{\theta} \sum_{j=1}^{\infty} C^2_{\alpha j} \frac{2}{2j - \alpha} - \frac{2}{\theta},
\]
where \( C^2_{\alpha j} = \frac{\theta(\theta-1) \cdots (\theta-j+1)}{j!} \). According to Chen and Wang [5], the series of (12) can be rewritten as
\[
E(\alpha, \theta) = \frac{2}{\alpha - \theta} + 2\alpha \sum_{j=1}^{\infty} \frac{(\theta - 1) \cdots (\theta - 2j + 1)}{(2j)!} \left( \frac{1}{2j - \alpha} + \frac{1}{2j + \alpha - \theta} \right)
\]
and the series appearing in (13) is absolutely convergent. Letting \( \theta \to 0 \) yields
\[
\lim_{\theta \to 0} E(\alpha, \theta) = \frac{2}{\alpha} + 2\alpha \sum_{j=1}^{\infty} \frac{(-1) \cdots (-2j + 1)}{(2j)!} \left( \frac{1}{2j - \alpha} + \frac{1}{2j + \alpha} \right)
\]
\[
= \frac{2}{\alpha} - \sum_{j=1}^{\infty} \frac{4\alpha}{4j^2 - \alpha^2}
\]
\[
= \pi \cot \left( \frac{\alpha \pi}{2} \right).
\]

Since \( \alpha \in (1, 2) \), we have \( \pi \cot \left( \frac{\alpha \pi}{2} \right) < 0 \).

**Lemma 3.2.** For \( \alpha \in (1, 2), r \in [0, +\infty) \), then
\[
\lim_{x \to \infty} \frac{1}{1 - \alpha} \left( 1 - \left( \frac{x}{x + r} \right)^{1-\alpha} \right) = 0.
\]

To show the existence of a unique stationary distribution, we follow the main idea of Tong and Zhang [21] and Zhang et al. [23]. Let us present a similar result from Khasminskii [11] (pp. 107-109) for stable processes with Markov switching.

**Lemma 3.3.** If there exists a bounded open domain \( U \subset \mathbb{R}_{\geq 0} \) with a regular (i.e. smooth) boundary such that its closure \( \bar{U} \subset \mathbb{R}_{\geq 0} \), having the following conditions:

\begin{enumerate}[(B.1)]
\item \( \inf_{(x,i) \in U \times \mathbb{S}} 2^{\alpha^2} < \infty \);
\item \( \sup_{(x_0, r_0) \in (G-U) \times \mathbb{S}} \mathbb{E}_{\bar{U}} < \infty \) for every compact subset \( G \) of \( \mathbb{R} \) such that \( U \subset G \), where \( \tau_U = \inf \{ t \geq 0, X^x_{t, r_0} \in U \} \);
\end{enumerate}

Then the SDE model (3) has a unique stationary distribution.

**Proof.** The proof is similar to that of Theorem 4.1 of Khasminskii [11] and Theorem 5.1 of Arapostathis, Biswas and Caffarelli [3]. We omit it here. \( \square \)

**Theorem 3.4.** Let assumptions (A) hold. If \( \sum_{i=1}^{n} \pi(i) \beta(i) < 0 \), then the CIR model (3) has a unique stationary distribution.
Proof. To prove the theorem, according to Lemma 3.3, it suffices to verify the conditions (B.1) and (B.2). Let $N$ be a positive real number. Set
\[ U = \left\{ x \in \mathbb{R} : \frac{1}{N} < |x| < N \right\}. \]

Since $\sigma(i) > 0, i \in S$, it is clear that
\[ \inf_{(x,i) \in U \times S} \sigma^2(i)|x|^2 \geq \frac{\sigma_{\min}^2}{N^2} > 0. \]

So, the condition (B.1) in Lemma 3.3 holds.

Next, we will verify the condition (B.2) in Lemma 3.3. To be precise, we shall show that there exists a sufficiently large $N > 0$ such that for all $(x_0,i_0) \in [N,N^{-1}] \times S \cup \{[N,\infty) \times S\}$
\[ E(\tau_U) := E(\tau_u|X_0 = x_0, r_0 = i_0) < +\infty, \quad (14) \]
where $\tau_U = \inf \{t : X_r^{x_0,i_0} \in U\}$. The proof of (14) is divided into two cases.

Case 1. We will show that
\[ E(\tau_U) < +\infty \quad \text{for all} \quad (x_0,i_0) \in [N,\infty) \times S. \]

First, we explain our strategy of the proof in this case. By Itô’s formula, for all $(x_0,i_0) \in [N,\infty) \times S$, we have
\[ EV(X_t,r_t) = V(x_0,r_0) + E \int_0^t AV(X_s,r_s)ds, \quad (15) \]
where $V : \mathbb{R} \times S \mapsto \mathbb{R}_+ \geq 0$ is a Lyapunov function and $AV(x,i)$ is defined in (6). If we can show
\[ AV(x,i) < -1, \quad x \in [N,\infty), i \in S, \]
then we have
\[ 0 \leq V(x_0,i_0) - E(t \wedge \tau_U) \quad \text{for all} \quad t \geq 0. \]

Letting $t \to +\infty$ yields
\[ E(\tau_U) \leq V(x_0,i_0) < +\infty \quad \text{for all} \quad (x_0,i_0) \in [N,\infty) \times S. \quad (16) \]

Next, we define the Lyapunov function $V$. By Lemma 3.1, there exists a constant $\theta_1 > 0$ such that for all $\theta \in (0,\theta_1)$, the matrix $A(\theta)$ is a nonsingular $M$-matrix. So Proposition 1 implies that there exists a vector $\xi \gg 0$ (a vector $\xi \gg 0$ means all elements of $\xi$ are positive) such that
\[ A(\theta)\xi \gg 0, \]
i.e.
\[ -2\theta^2 \xi \beta(i) - \sum_{j=1}^n q_{ij} \xi_j > 0 \quad \text{for all} \quad i \in S. \quad (17) \]

To proceed, we first consider the case the process $(X_t)$ never hits the origin, that is $X_t \neq 0$, for all $t > 0$. Then the Lyapunov function $V : \mathbb{R} \times S \mapsto \mathbb{R}_+$ is defined by
\[ V(x,i) = \xi|x|^\theta, \quad (18) \]
where $\theta \in (0,\min\{1,\theta_1\})$. Furthermore, we will show in the Appendix that $V \in D(A)$, where $D(A)$ denotes the domain of the generator $A$. 
Applying (6) gives

\[ AV(x, i) = \sigma^\alpha(i)x \int_{0<|y|\leq 1} (V(x + y, i) - V(x, i) - V_z(x, i)y) \frac{C_\alpha}{|y|^\alpha+1} dy \]
\[ + \sigma^\alpha(i)x \int_{|y|>1} (V(x + y, i) - V(x, i)) \frac{C_\alpha}{|y|^\alpha+1} dy \]
\[ + (2\beta(i)x + \delta(i))V_z(x, i) + \sum_{j=1}^{n} q_{ij} \xi_j |x|^{\theta}. \]

(19)

Then, we will prove that there exists a sufficiently large \( N > 1 \) and for all \( x > N, i \in S \)

\[ AV(x, i) < -1. \]

(20)

To this end, it suffices to prove

\[ \lim_{x \to +\infty} \alpha x^{\alpha-\theta-1} \frac{\partial}{\partial C_\alpha \sigma^\alpha(i)} AV(x, i) = -\infty. \]

(21)

We divided the proof of (21) into three steps.

**Step 1.** Observing that it is difficult to show the limit (21) directly, we first rewrite the expression for \( \frac{\partial}{\partial C_\alpha \sigma^\alpha(i)} AV(x, i) \). By the definition of \( V(x, i) \), we have \( V(x, i) = \xi_i x^\theta \) for \( x > 1 \). Then

\[ \frac{\alpha x^{\alpha-\theta-1} \partial}{\partial C_\alpha \sigma^\alpha(i)} AV(x, i) = \frac{\alpha x^{\alpha-\theta-1}}{\partial C_\alpha \sigma^\alpha(i)} \left( \xi_i (2\beta(i)x + \delta(i))^x \theta - \theta x^{\theta-1}y \right) \frac{1}{|y|^\alpha+1} dy \]
\[ + \frac{\xi_i \alpha x^{\alpha-\theta}}{\theta} \int_{\{0<|y|\leq 1\}} ((x + y)^\theta - x^\theta - \theta x^{\theta-1}y) \frac{1}{|y|^\alpha+1} dy \]
\[ + \frac{\xi_i \alpha x^{\alpha-\theta}}{\theta} \int_{\{|y|\geq 1\}} (|x + y|^\theta - |x|^\theta) \frac{1}{|y|^\alpha+1} dy \]
\[ = A(x, i) + B(x, i) + C(x, i), \]

where

\[ A(x, i) = \alpha x^{\alpha-\theta-1} \left( \xi_i (2\beta(i)x + \delta(i))^x \theta - \theta x^{\theta-1}y \right) \frac{1}{|y|^\alpha+1} dy, \]
\[ B(x, i) = \xi_i \alpha x^{\alpha-\theta-1} \int_{\{0<|y|\leq 1\}} ((x + y)^\theta - x^\theta - \theta x^{\theta-1}y) \frac{1}{|y|^\alpha+1} dy, \]
\[ C(x, i) = \xi_i \alpha x^{\alpha-\theta-1} \int_{\{|y|\geq 1\}} (|x + y|^\theta - |x|^\theta) \frac{1}{|y|^\alpha+1} dy. \]

**Step 2.** We will deal with \( C(x, i) \) and show that

\[ \lim_{x \to +\infty} C(x, i) = \xi_i E(\alpha, \theta). \]

(22)

Let \( z = y/x \). Then

\[ C(x, i) = \frac{\xi_i \alpha |x|^\alpha}{\theta} \int_{\{|y|\geq 1\}} \left( 1 + \frac{y}{x} \right)^\theta - 1 \frac{1}{|y|^\alpha+1} dy \]
where

\[ C_1(x, i) = \frac{\xi_i \alpha}{\theta} \int_{\{z \geq \frac{1}{2}\}} (|1 + z|^\theta - 1) \frac{1}{|z|^\alpha} \, dz, \]

\[ C_2(x, i) = \frac{\xi_i \alpha}{\theta} \int_{\{z \geq 1\}} (|1 + z|^\theta - 1) \frac{1}{|z|^\alpha} \, dz. \]

As for \( C_1(x, i) \), let \( x = \frac{1}{2}, \epsilon > 0 \). Then

\[ C_1(x, i) = h(\epsilon, i) = \frac{\xi_i \alpha}{\theta} \int_{\{z \leq |z| \leq 1\}} (|1 + z|^\theta - 1) \frac{1}{|z|^\alpha} \, dz. \]

Let \( z = r \cos \theta_1, r \geq 0, \theta_1 \in [0, \pi] \). Then

\[ h(\epsilon, i) = \int_0^\pi \int_{\{r \leq |r| \leq 1\}} (|1 + 2r \cos \theta_1 + r^2|^\theta - 1) \frac{1}{r^\alpha} \, d\theta_1 \, dr. \]

Clearly, \( \lim_{x \to +\infty} C_1(x, i) = \lim_{x \to -\infty} h(\epsilon, i) \). In order to prove \( \lim_{x \to +\infty} h(\epsilon, i) \) exists, it suffices to prove

\[ \lim_{\epsilon_2 > \epsilon_1 > 0, \epsilon \to 0^+} |h(\epsilon_2, i) - h(\epsilon_1, i)| = 0. \]

Since

\[ \frac{|1 + 2r \cos \theta_1 + r^2|^\theta - 1}{r^\alpha} = \frac{\cos \theta_1}{r^\alpha} + \frac{1}{2r^{\alpha-1}} + O(r^{2-\alpha}), \] as \( r \to 0 \),

and \( \alpha \in (1, 2) \), we get

\[ |h(\epsilon_2, i) - h(\epsilon_1, i)| \leq C \int_{\epsilon_1}^{\epsilon_2} \int_0^\pi \left( \frac{\cos \theta_1}{r^\alpha} + \frac{1}{2r^{\alpha-1}} \right) \, d\theta_1 \, dr 
   + \int_{\epsilon_1}^{\epsilon_2} \int_0^\pi O(r^{2-\alpha}) \, d\theta_1 \, dr 
   \leq C \int_0^\pi \theta_1 \int_{\epsilon_1}^{\epsilon_2} \frac{1}{2r^{\alpha-1}} \, dr + \int_{\epsilon_1}^{\epsilon_2} \int_0^\pi O(r^{2-\alpha}) \, d\theta_1 \, dr \to 0 \]

for some \( C \). Hence, letting \( x \to +\infty \), we have

\[ \lim_{x \to +\infty} C(x, i) = \frac{\xi_i \alpha}{\theta} \int_{|z| > 1} (|1 + z|^\theta - 1) \frac{dz}{|z|^\alpha} + \frac{\xi_i \alpha}{\theta} \int_{|z| < 1} (|1 + z|^\theta - 1) \frac{dz}{|z|^\alpha} 
   + \xi_i \alpha \int_{-\infty}^{-1} (|1 + z|^\theta - 1) \frac{dz}{|z|^\alpha} + \xi_i \alpha \int_{1}^{+\infty} (|1 + z|^\theta - 1) \frac{dz}{|z|^\alpha} 
   = C_3(x, i) + C_4(x, i) + C_5(x, i), \]
where
\[ C_3(x,i) = \frac{\xi_i \alpha}{\theta} \int_{-1}^{1} \frac{(1 + z)^\theta - 1}{|z|^{\alpha+1}} \, dz \]
\[ C_4(x,i) = \frac{\xi_i \alpha}{\theta} \int_{1}^{+\infty} \frac{(1 + z)^\theta - 1}{|z|^{\alpha+1}} \, dz \]
\[ C_5(x,i) = \frac{\xi_i \alpha}{\theta} \int_{-\infty}^{-1} \frac{(1 + z)^\theta - 1}{|z|^{\alpha+1}} \, dz. \]

For \(|z| < 1\), we have
\[(1 + z)^\theta - 1 = \sum_{j=1}^{\infty} C_\theta^j z^j.\]

Then
\[ C_3(x,i) = \frac{\xi_i \alpha}{\theta} \sum_{j=1}^{\infty} C_\theta^j \int_{-1}^{1} \frac{z^j}{|z|^{\alpha+1}} \, dz \]
\[ = \frac{\xi_i \alpha}{\theta} \sum_{j=1}^{\infty} C_\theta^{2j} \int_{-1}^{1} \frac{z^{2j-1} \text{sgn}(z)}{|z|^{\alpha}} \, dz + \frac{\xi_i \alpha}{\theta} \sum_{j=1}^{\infty} C_\theta^{2j-1} \int_{-1}^{1} \frac{z^{2j-1}}{|z|^{\alpha+1}} \, dz. \]

Note that
\[ \int_{-1}^{1} \frac{z^j}{|z|^{\alpha+1}} \, dz = 0, \quad \text{for all } j \in \mathbb{N}+. \]

We have
\[ C_3(x,i) = \frac{\xi_i \alpha}{\theta} \sum_{j=1}^{\infty} C_\theta^{2j} \int_{-1}^{1} \frac{z^{2j-1} \text{sgn}(z)}{|z|^{\alpha}} \, dz \]
\[ = \frac{2\xi_i \alpha}{\theta} \sum_{j=1}^{\infty} C_\theta^{2j} \int_{0}^{1} \frac{z^{2j-1-\alpha}}{2j-\alpha} \, dz \]
\[ = \frac{2\xi_i \alpha}{\theta} \sum_{j=1}^{\infty} C_\theta^{2j} \frac{1}{2j-\alpha}. \]

As for the integral \(C_4(x,i)\), we have
\[ C_4(x,i) = \frac{\xi_i \alpha}{\theta} \int_{1}^{+\infty} \left( z^\theta \left( 1 + \frac{1}{z} \right)^\theta - 1 \right) \, \frac{dz}{z^{\alpha+1}} \]
\[ = \frac{\xi_i \alpha}{\theta} \int_{1}^{+\infty} \left( \sum_{j=0}^{+\infty} C_\theta^j z^{-j+\theta} - 1 \right) \, \frac{dz}{z^{\alpha+1}} \]
\[ = \frac{\xi_i \alpha}{\theta} \left( \sum_{j=0}^{+\infty} C_\theta^j \frac{1}{j+\alpha-\theta} - \frac{1}{\alpha} \right). \]

Moreover, we compute the integral \(C_5(x,i)\). It follows that
\[ C_5(x,i) = \frac{\xi_i \alpha}{\theta} \int_{-\infty}^{-1} \frac{(1 + z)^\theta - 1}{|z|^{\alpha+1}} \, dz \]
\[ = \frac{\xi_i \alpha}{\theta} \int_{1}^{+\infty} \left( z^\theta \left( 1 - \frac{1}{z} \right)^\theta - 1 \right) \, \frac{dz}{z^{\alpha+1}}. \]
\[ \lim_{x \to \infty} C(x, i) = \xi_i E(\alpha, \theta). \]
For the term $A(x, i)$, by direct computation we have

$$A(x, i) = \frac{\alpha x^{\alpha-1}}{\theta C_\alpha \sigma^\alpha(i)} \left( \xi_i \beta(2\beta(i)x + \delta(i)) x^{\theta-1} + \sum_{j=1}^{n} q_{ij} \xi_j x^\theta \right)$$

$$= \frac{\alpha}{\theta C_\alpha \sigma^\alpha(i)} \left( 2\theta \xi_i \beta(i) + \sum_{j=1}^{n} q_{ij} \xi_j \right) x^{\alpha-1} + \frac{\alpha \xi_i \delta(i)}{\theta C_\alpha \sigma^\alpha(i)} x^{\alpha-2}. \quad (25)$$

Now, since

$$-2\theta \xi_i \beta(i) - \sum_{j=1}^{n} q_{ij} \xi_j > 0 \quad \text{for all } i \in \mathbb{S},$$

by (23)-(25) we get

$$\lim_{x \to +\infty} \frac{\alpha x^{\alpha-1}}{\theta C_\alpha \sigma^\alpha(i)} AV(x, i) = \lim_{x \to +\infty} \frac{\alpha}{\theta C_\alpha \sigma^\alpha(i)} \left( 2\theta \xi_i \beta(i) + \sum_{j=1}^{n} q_{ij} \xi_j \right) x^{\alpha-1}$$

$$+ \lim_{x \to +\infty} \frac{\alpha \xi_i \delta(i)}{\theta C_\alpha \sigma^\alpha(i)} x^{\alpha-2} + \xi_i E(\alpha, \theta)$$

$$= \lim_{x \to +\infty} \frac{\alpha x^{\alpha-1}}{\theta C_\alpha \sigma^\alpha(i)} \left( 2\theta \xi_i \beta(i) + \sum_{j=1}^{n} q_{ij} \xi_j \right) + \xi_i E(\alpha, \theta)$$

$$= -\infty.$$

Then, there exists a sufficiently large $N_1 > 0$ such that for all $x > N_1, i \in \mathbb{S}$

$$AV(x, i) < -1. \quad (26)$$

Hence, for all $(x_0, i_0) \in [N_1, +\infty) \times \mathbb{S}$

$$E(\tau_U) < +\infty. \quad (27)$$

Next, we consider the process $(X_t)$ can hit the origin. In this case, define the Lyapunov function

$$V(x, i) = \xi_i (1 + \varphi(x))^\theta.$$

where $\theta \in (0, \min\{1, \theta_1\})$. Here $\varphi(x) \in C^2(R)$ is a nondecreasing, radial function satisfying $\varphi(x) = |x|$ for $|x| > 1$ and $0 \leq \varphi(x) \leq |x|$ for $|x| \leq 1$. By the similar discussion of the case that the process $(X_t)_{t \geq 0}$ never hits the origin, one can obtain the result (27).

**Case 2.** We show that there exists a sufficiently large $N_2 > 0$ such that for any $N > N_2$

$$\sup_{(x_0, i_0) \in [0, \frac{1}{N}] \times \mathbb{S}} E(\tau_U) < +\infty \quad \text{for all } (x_0, i_0) \in \left[0, \frac{1}{N}\right] \times \mathbb{S}. \quad (28)$$

Suppose that the statement is false. Then, for any $N > 0$, there exists $(x_0, i_0) \in [0, \frac{1}{N}] \times \mathbb{S}$ such that

$$E(\tau_U) = +\infty.$$

In other words, for any $\varepsilon \in [0, \frac{1}{N}], (x_0, i_0) \in [0, \varepsilon] \times \mathbb{S}$, the process $X_t^{x_0, i_0}$ can never hit the interval $U = (\frac{1}{N}, N)$. Hence, the state $0$ is absorbing, i.e. $P(X_t^{x, i} = 0, 0 \leq$
This is in contraction to the result of lemma 2.2. Consequently, (28) holds. Then, by (27) and (28), we have for $N = \max\{N_1, N_2\}$

$$E(n_U) < +\infty \quad \text{for all } (x_0, i_0) \in \{R_\geq 0 - U\} \times S. \quad (29)$$

The case when $x < 0$ is treated in the similar way. Hence, we verifies the condition (B.2) in Lemma 3.3. The proof is therefore complete. \hfill \Box

4. Transience. In the previous section, we have shown that under the condition $\sum_{i=1}^n \pi(i)\beta(i) < 0$, the CIR model (3) has a unique stationary distribution. The natural question is: what happens if $\sum_{i=1}^n \pi(i)\beta(i) > 0$ ? In this section, we will answer this question.

**Lemma 4.1.** If $\sum_{i=1}^n \pi(i)\beta(i) > 0$, then there exists a constant $\theta_2 > 0$ such that for all $\theta \in (0, \theta_2)$, the matrix

$$B(\theta) = 2\theta \text{diag}(\beta(1), \beta(2), \ldots, \beta(n)) - Q$$

is a nonsingular $M$-matrix.

**Proof.** The proof is similar to that of Theorem 4.6 of Mao [15] or Lemma 3.4 of Li et al. [12]. We omit it here. \hfill \Box

**Theorem 4.2.** Let assumptions (A) hold. If $\sum_{i=1}^n \pi(i)\beta(i) > 0$, then the CIR model (3) is transient.

**Proof.** By Lemma 4.1, there exists a constant $\theta_2 > 0$ such that for all $\theta \in (0, \theta_2)$, $B(\theta)$ is a nonsingular $M$-matrix. According to Proposition 1, $B(\theta)$ is semipositive. So there exists a vector $\eta \gg 0$ such that

$$B(\theta)\eta \gg 0,$$

namely, for all $i \in S$

$$2\theta \eta_i \beta(i) - \sum_{j=1}^n q_{ij} \eta_j > 0. \quad (30)$$

Let $\theta \in (0, \min\{1, \theta_2\})$ and define the function $V : R \times S \to R_\geq 0$ by

$$V(x, i) = \eta_i (1 + \varphi(x))^{-\theta},$$

where $\varphi(x)$ is defined in the proof of Theorem 3.4. Clearly, $V \in C^2(R \times S)$. Moreover, since $V$ is bounded, we have $V \in D(A)$. Our first goal is to show that there exists a sufficiently large $N > 0$ such that for all $x > N$, $i \in S$

$$AV(x, i) < 0 \quad (31)$$

and the proof of (31) is divided into three steps.

**Step 1.** First, we rewrite the expression of $AV(x, i)$. With help of (6), we have

$$AV(x, i) = \eta_i \sigma^a(i) x \int_{|y| \leq 1} \left( (1 + \varphi(x + y))^{-\theta} - (1 + \varphi(x))^{-\theta} \right. $$

$$+ \theta \varphi_x(x)(1 + \varphi(x))^{-\theta-1}y \frac{C_\alpha}{|y|^\alpha+1} dy$$

$$+ \eta_i \sigma^a(i) x \int_{|y| > 1} \left( (1 + \varphi(x + y))^{-\theta} - (1 + \varphi(x))^{-\theta} \right) \frac{C_\alpha}{|y|^\alpha+1} dy$$

$$- \theta(2\beta(i)x + \delta(i))\eta_i \varphi_x(x)(1 + \varphi(x))^{-\theta-1} + \sum_{j=1}^n q_{ij} \eta_j (1 + \varphi(x))^{-\theta}. $$
Furthermore, recalling the definition of $\varphi(x)$, we have for $x > 1$ large enough

$$V(x, i) = \eta_i (1 + x)^{-\theta}$$

and

$$\frac{\alpha(1 + x)^{\alpha+\theta}}{C_\alpha \theta \sigma^{\alpha(i)} x} AV(x, i)$$

$$= \frac{\alpha(1 + x)^{\alpha+\theta}}{C_\alpha \theta \sigma^{\alpha(i)} x} \left( (2\beta(i)x + \delta(i))V_x(x, i) + \sum_{j=1}^{n} q_{ij} \eta_j (1 + x)^{-\theta} \right)$$

$$+ \frac{\eta_i \alpha(1 + x)^{\alpha+\theta}}{\theta} \int_{|y| > 1} \left( (1 + \varphi(x + y))^{-\theta} - (1 + x)^{-\theta} \right) \frac{dy}{|y|^\alpha + 1}$$

$$+ \frac{\eta_i \alpha(1 + x)^{\alpha+\theta}}{\theta} \int_{0 < |y| \leq 1} \left( (1 + x + y)^{-\theta} - (1 + x)^{-\theta} \right) \frac{dy}{|y|^{\alpha+1}}$$

$$- \theta (1 + x)^{-\theta - 1} y \right) \frac{dy}{|y|^{\alpha+1}}.$$ 

Hence, in order to prove (31), it suffices to show

$$\lim_{x \to +\infty} \frac{\alpha(1 + x)^{\alpha+\theta}}{C_\alpha \theta \sigma^{\alpha(i)} x} AV(x, i) = -\infty. \quad (32)$$

To prove the limit (32), we define

$$A(x, i) = \frac{\alpha(1 + x)^{\alpha+\theta}}{C_\alpha \theta \sigma^{\alpha(i)} x} \left( (2\beta(i)x + \delta(i))V_x(x, i) + \sum_{j=1}^{n} q_{ij} \eta_j (1 + x)^{-\theta} \right),$$

$$B(x, i) = \frac{\eta_i \alpha(1 + x)^{\alpha+\theta}}{\theta} \int_{0 < |y| \leq 1} \left( (1 + x + y)^{-\theta} - (1 + x)^{-\theta} \right) \frac{dy}{|y|^{\alpha+1}},$$

$$C(x, i) = \frac{\eta_i \alpha(1 + x)^{\alpha+\theta}}{\theta} \int_{|y| > 1} \left( (1 + \varphi(x + y))^{-\theta} - (1 + x)^{-\theta} \right) \frac{dy}{|y|^{\alpha+1}}.$$ 

Thus

$$\frac{\alpha(1 + x)^{\alpha+\theta}}{C_\alpha \theta \sigma^{\alpha(i)} x} AV(x, i) = A(x, i) + B(x, i) + C(x, i).$$

Then, we will deal with $\frac{\alpha(1 + x)^{\alpha+\theta}}{C_\alpha \theta \sigma^{\alpha(i)} x} AV(x, i)$ by considering $A(x, i), B(x, i)$ and $C(x, i)$ respectively.

**Step 2.** We first consider $C(x, i)$. In this step, we aim to show that

$$\lim_{x \to +\infty} C(x, i) = -\eta_i E(\alpha, \theta).$$
To this end, we divide the interval \( \{ y : |y| > 1 \} = \{ y : y \in (-\infty, -x - 1) \cup (-x - 1, -x + 1) \cup (x + 1, \infty) \} \). Then

\[
C(x, i) = \frac{\eta \alpha (1 + x)^{\alpha + \theta}}{\theta} \int_{\{ |y| > 1 \}} \left( (1 + \varphi(x + y))^{-\theta} - (1 + x)^{-\theta} \right) \frac{dy}{|y|^\alpha + 1}
\]

\[
= \frac{\eta \alpha (1 + x)^{\alpha + \theta}}{\theta} \int_{-\infty}^{-x} \left( (1 - x - y)^{-\theta} - (1 + x)^{-\theta} \right) \frac{dy}{|y|^\alpha + 1}
\]

\[
+ \frac{\eta \alpha (1 + x)^{\alpha + \theta}}{\theta} \int_{-x}^{x} \left( (1 + \varphi(x + y))^{-\theta} - (1 + x)^{-\theta} \right) \frac{dy}{|y|^\alpha + 1}
\]

\[
+ \frac{\eta \alpha (1 + x)^{\alpha + \theta}}{\theta} \int_{x}^{\infty} \left( (1 + \varphi(x + y))^{-\theta} - (1 + x)^{-\theta} \right) \frac{dy}{|y|^\alpha + 1}
\]

By changing variables, we have

\[
C(x, i) = \frac{\eta \alpha (1 + x)^{\alpha}}{\theta} \int_{x + 1}^{\infty} \left( \left( \frac{1 - x + y}{1 + x} \right)^{-\theta} - 1 \right) \frac{dy}{y^{\alpha + 1}}
\]

\[
+ \frac{\eta \alpha (1 + x)^{\alpha}}{\theta} \int_{x - 1}^{x} \left( \left( \frac{1 - x + y}{1 + x} \right)^{-\theta} - 1 \right) \frac{dy}{y^{\alpha + 1}}
\]

\[
+ \frac{\eta \alpha (1 + x)^{\alpha}}{\theta} \int_{1}^{x - 1} \left( \left( 1 + x - y \right)^{-\theta} - 1 \right) \frac{dy}{y^{\alpha + 1}}
\]

\[
+ \frac{\eta \alpha (1 + x)^{\alpha}}{\theta} \int_{1}^{\infty} \left( \left( 1 + x + y \right)^{-\theta} - 1 \right) \frac{dy}{y^{\alpha + 1}}.
\]

Put

\[
C_1(x, i) = \frac{\eta \alpha (1 + x)^{\alpha}}{\theta} \int_{x + 1}^{\infty} \left( \left( 1 - x + y \right)^{-\theta} \right) \frac{dy}{y^{\alpha + 1}}
\]

\[
C_2(x, i) = \frac{\eta \alpha (1 + x)^{\alpha}}{\theta} \int_{x - 1}^{x} \left( \left( 1 + x + y \right)^{-\theta} - 1 \right) \frac{dy}{y^{\alpha + 1}}
\]

\[
C_3(x, i) = \frac{\eta \alpha (1 + x)^{\alpha}}{\theta} \int_{1}^{x - 1} \left( \left( 1 + x - y \right)^{-\theta} - 1 \right) \frac{dy}{y^{\alpha + 1}}
\]

Next, using the Gauss hypergeometric function, we will show

\[
\lim_{x \to +\infty} C_1(x, i) = \frac{\eta \alpha_2 F_1(\theta, \alpha + \theta, \alpha + \theta + 1, 1)}{\theta (\alpha + \theta)} + \frac{\eta \alpha_2 F_1(\theta, \alpha + \theta, \alpha + \theta + 1, -1)}{\theta (\alpha + \theta)} - \frac{2 \eta}{\theta}.
\]
To this end, put
\[ D_1(x, i) = \frac{\eta_i \alpha (1 + x)^\alpha}{\theta} \int_{x+1}^{+\infty} \left( \frac{1 - x + y}{1 + x} \right)^{-\theta} \frac{dy}{y^{\alpha + 1}} \]
\[ + \frac{\eta_i \alpha (1 + x)^\alpha}{\theta} \int_{x+1}^{+\infty} \left( \frac{1 + x + y}{1 + x} \right)^{-\theta} \frac{dy}{y^{\alpha + 1}}. \]
\[ D_2(x, i) = \frac{2\eta_i \alpha (1 + x)^\alpha}{\theta} \int_{x+1}^{+\infty} y^{-\alpha -1} dy. \]

Hence, we have
\[ C_1(x, i) = D_1(x, i) - D_2(x, i). \tag{34} \]

To estimate \( D_1(x, i) \), by changing variables \( y = \frac{x+1}{t}, dy = \frac{-(x+1)}{t^2} dt \), one has
\[ D_1(x, i) = \frac{\eta_i \alpha (1 + x)^\alpha}{\theta} \int_0^1 \left( \frac{1 - x + \frac{1+x}{t}}{1 + x} \right)^{-\theta} t^{\alpha + 1} \frac{1}{(x+1)^{\alpha + 1}} \left( \frac{-(x+1)}{t^2} \right) dt \]
\[ + \frac{\eta_i \alpha (1 + x)^\alpha}{\theta} \int_0^1 \left( \frac{1 + \frac{1}{t}}{1 + x} \right)^{-\theta} t^{\alpha + 1} \frac{1}{(x+1)^{\alpha + 1}} \left( \frac{-(x+1)}{t^2} \right) dt \]
\[ = \frac{\eta_i \alpha}{\theta} \left[ \int_0^1 \left( 1 + \frac{1}{1 + x} \right)^{-\theta} t^{\alpha + \theta - 1} dt + \int_0^1 (1 + t)^{-\theta} t^{\alpha + \theta - 1} dt \right]. \]

Applying (11) then gives,
\[ D_1(x, i) = \frac{\eta_i \alpha \, {}_2F_1(\theta, \alpha + \theta, \alpha + \theta + 1, \frac{x+1}{x+1}) + \eta_i \alpha \, {}_2F_1(\theta, \alpha + \theta, \alpha + \theta + 1, -1)}{\theta(\alpha + \theta)}. \tag{35} \]

Furthermore, it is straightforward by elementary computation that
\[ D_2(x, i) = \frac{2\eta_i}{\theta}. \tag{36} \]

By (34)-(36), we get
\[ C_1(x, i) = \frac{\eta_i \alpha \, {}_2F_1(\theta, \alpha + \theta, \alpha + \theta + 1, \frac{x-1}{x+1})}{\theta(\alpha + \theta)} \]
\[ + \frac{\eta_i \alpha \, {}_2F_1(\theta, \alpha + \theta, \alpha + \theta + 1, -1)}{\theta(\alpha + \theta)} - \frac{2\eta_i}{\theta}. \]

Letting \( x \to +\infty \) yields (33).

To estimate \( C_2(x, i) \), note that \( 0 \leq \varphi(x - y) \leq 1, y \in [x-1, x+1] \). Then, by Lemma 3.2, we have
\[ \lim_{x \to +\infty} C_2(x, i) = 0. \tag{37} \]

By restricting the function \((1 + t)^{-\theta} - 1\) to intervals \((-1, 1)\), and using its Taylor expansion, that is
\[ (1 + t)^{-\theta} - 1 = \sum_{j=1}^{\infty} C_j \theta^j, \]
we get
\[
C_3(x, i) = \frac{\eta_i\alpha(1 + x)^\alpha}{\theta} \int_1^{x-1} \sum_{j=1}^{\infty} C_{-\theta}^j \frac{(-1)^j}{(1 + x)^j} y^{j+\alpha-1} dy \\
+ \frac{\eta_i\alpha(1 + x)^\alpha}{\theta} \int_1^{x+1} \sum_{j=1}^{\infty} C_{-\theta}^j \frac{1}{(1 + x)^j} y^{j+\alpha-1} dy \\
= \frac{\eta_i\alpha(1 + x)^\alpha}{\theta} \int_1^{x-1} \sum_{j=1}^{\infty} C_{-\theta}^j \frac{(-1)^j}{(1 + x)^j} y^{j+\alpha-1} dy \\
+ \frac{\eta_i\alpha(1 + x)^\alpha}{\theta} \int_1^{x+1} \sum_{j=1}^{\infty} C_{-\theta}^j \frac{1}{(1 + x)^j} y^{j+\alpha-1} dy \\
+ \frac{\alpha}{1-\alpha} \left( \left( \frac{x-1}{x+1} \right)^{1-\alpha} - 1 \right)
= D_3(x, i) + D_4(x, i),
\]
where
\[
D_3(x, i) = \frac{\eta_i\alpha(1 + x)^\alpha}{\theta} \int_1^{x-1} \sum_{j=2}^{\infty} C_{-\theta}^j \frac{(-1)^j}{j - \alpha} \left( (x - 1)^{j-\alpha} - 1 \right)
\]
\[
+ \frac{\eta_i\alpha(1 + x)^\alpha}{\theta} \int_1^{x+1} \sum_{j=2}^{\infty} C_{-\theta}^j \frac{1}{j - \alpha} \left( (x + 1)^{j-\alpha} - 1 \right),
\]
\[
D_4(x, i) = \frac{\alpha}{1-\alpha} \left( \left( \frac{x-1}{x+1} \right)^{1-\alpha} - 1 \right).
\]
By Lemma 3.2, we have
\[
\lim_{x \to +\infty} D_4(x, i) = 0. \tag{38}
\]
For the term \(D_3(x, i)\), by direct computation, we have
\[
D_3(x, i) = \frac{\eta_i\alpha(1 + x)^{\alpha-j}}{\theta} \sum_{j=2}^{\infty} C_{-\theta}^j \frac{(-1)^j}{j - \alpha} \left( (x - 1)^{j-\alpha} - 1 \right)
\]
\[
+ \frac{\eta_i\alpha(1 + x)^{\alpha-j}}{\theta} \sum_{j=2}^{\infty} C_{-\theta}^j \frac{1}{j - \alpha} \left( (x + 1)^{j-\alpha} - 1 \right)
\]
\[
= \frac{\eta_i\alpha}{\theta} \sum_{j=2}^{\infty} C_{-\theta}^j \frac{(-1)^j}{j - \alpha} \left( \left( \frac{x-1}{x+1} \right)^{j-\alpha} - 1 \right) + \frac{\eta_i\alpha}{\theta} \sum_{j=2}^{\infty} C_{-\theta}^j \frac{1}{j - \alpha} (1 + x)^{\alpha-j}
\]
\[
- \frac{\eta_i\alpha}{\theta} \sum_{j=2}^{\infty} C_{-\theta}^j \frac{(-1)^j}{j - \alpha} (1 + x)^{\alpha-j} - \frac{\eta_i\alpha}{\theta} \sum_{j=2}^{\infty} C_{-\theta}^j \frac{1}{j - \alpha} (1 + x)^{\alpha-j}
\]
\[
+ \frac{\eta_i\alpha}{\theta} \sum_{j=2}^{\infty} C_{-\theta}^j \frac{1}{j - \alpha}.
\]
Note that if $j$ is odd, $(-1)^j = -1$, while if $j$ is even, $(-1)^j = 1$. Then

$$D_3(x, i) = \eta_i^\alpha \theta \sum_{j=2}^{\infty} C_\theta^j \left( (-1)^j \left( \frac{x-1}{x+1} \right)^{j-\alpha} - 1 \right)$$

$$- \frac{\eta_i^\alpha}{\theta} \sum_{j=1}^{\infty} C_{-\theta}^{2j} \frac{2(1+x)^{\alpha-2j}}{2j-\alpha} + \frac{\eta_i^\alpha}{\theta} \sum_{j=1}^{\infty} C_{-\theta}^{2j} \frac{2}{2j-\alpha}.$$

Letting $x \to +\infty$ yields that

$$\lim_{x \to +\infty} D_3(x, i) = \frac{\eta_i^\alpha}{\theta} \sum_{j=1}^{\infty} C_{-\theta}^{2j} \frac{2}{2j-\alpha}. \quad (39)$$

Combining (38) with (39), we get

$$\lim_{x \to +\infty} C_3(x, i) = \frac{\eta_i^\alpha}{\theta} \sum_{j=1}^{\infty} C_{-\theta}^{2j} \frac{2}{2j-\alpha}. \quad (40)$$

By (33)-(40) and applying (12), we have

$$\lim_{x \to +\infty} C(x, i) = \frac{\eta_i a_2 F_1(\theta, \alpha + \theta, \alpha + \theta + 1, 1) + \eta_i a_2 F_1(\theta, \alpha + \theta, \alpha + \theta + 1, -1)}{\theta(\alpha + \theta)}$$

$$+ \frac{\eta_i}{\theta} \sum_{j=1}^{\infty} C_{-\theta}^{2j} \frac{2}{2j-\alpha} - \frac{2\eta_i}{\theta}$$

$$= - \eta_i E(\alpha, \theta). \quad (41)$$

**Step 3.** We shall use the theory of $M$-matrix to show

$$\lim_{x \to +\infty} A(x, i) = -\infty \quad \text{and} \quad \lim_{x \to +\infty} B(x, i) = 0.$$

By the mean value theorem, there exists $\gamma \in \{x + t : t \in (-1, 1)\}$ such that

$$B(x, i) = \frac{\eta_i a(1+x)^{\alpha+\theta}}{\theta} \int_{\{0<|y|\leq 1\}} \frac{1}{2} y^2 V_{xx}(\gamma, i) \frac{1}{|y|^{\alpha+1}} dy$$

$$= \eta_i a(1+x)^{\alpha+\theta} \int_{\{0<|y|\leq 1\}} \frac{\theta + 1}{2} y^2 (1+\gamma)^{-\theta-2} \frac{1}{|y|^{\alpha+1}} dy.$$

To estimate $B(x, i)$, noticing that $V_{xx}(x, i)$ is decreasing in $x$, we have

$$0 \leq B(x, i) \leq \frac{\eta_i a(\theta + 1)}{2} (1+x)^{\alpha+\theta} x^{-\theta-2} \int_{\{0<|y|\leq 1\}} y^2 \frac{1}{|y|^{\alpha+1}} dy.$$

Letting $x \to +\infty$ yields that

$$\lim_{x \to +\infty} B(x, i) = 0. \quad (42)$$
For the term $A(x, i)$, straightforward calculations show that

$$A(x, i) = -\frac{\alpha(1 + x)^{\alpha+\theta} C(i)x}{\theta \sigma^2(i)x} (2\beta(i)(x + 1) + \delta(i) - 2\beta(i))\theta \eta_i (1 + x)^{-\theta-1}$$

$$+ \frac{\alpha(1 + x)^{\alpha+\theta} n}{C(i)x} \sum_{j=1}^{n} q_{ij} \eta_j (1 + x)^{-\theta}$$

$$= \frac{\alpha(1 + x)^{\alpha}}{C(i)x} \left( -2\theta \eta_i \beta(i) + \sum_{j=1}^{n} q_{ij} \eta_j \right) - \frac{\eta_i \alpha(1 + x)^{\alpha-1}}{C(i)x} (\delta(i) - 2\beta(i)).$$

Recalling that

$$2\theta \eta_i \beta(i) - \sum_{i=1}^{n} q_{ij} \eta_j > 0, \quad \forall i \in S.$$

we have

$$\lim_{x \to +\infty} A(x, i) = -\infty. \quad (43)$$

It then follows from (33) to (43) that

$$\lim_{x \to +\infty} A(x, i) = \lim_{x \to +\infty} A(x, i) - \eta_i E(\alpha, \theta) = -\infty.$$

Then, there exists a sufficiently large $N > 0$ such that for all $x > N, i \in S$

$$AV(x, i) < 0.$$

Note that $\lim_{|x| \to +\infty} V(x, i) = 0$ and the infinitesimal operator corresponding to the CIR model (3) is uniformly elliptic in domain $D = (-\infty, -\varepsilon] \cup [\varepsilon, +\infty)$, where $\varepsilon > 0$ is a sufficiently small positive number, according to the Foster-Lyapunov drift conditions (cf. [17], Section 2, p.443), we obtain that $X_t$ is transient in whole $D$.

Consequently, to prove the transience of the $X_t$ in the whole space in this case, we only need to check the transience of $X_t$ in the domain $D^c = (-\varepsilon, \varepsilon)$. If the process $X_t$ is recurrent in interval $(-\varepsilon, \varepsilon)$, by the arbitrary of $\varepsilon$, we must have $P(\lim_{t \to +\infty} X_t = 0) = 1$, which is in contraction to the result of lemma 2.2. This completes the proof.

5. **An application to interval estimation.** In this section, as an application of our theoretic results, we present an interval estimation of the interest rate process $X_t$ defined by (3).

**Example 5.1.** Consider a stable CIR model with two-state Markov switching and the coefficients are $\beta(1) = 0.005, \beta(2) = -0.5, \delta(1) = 0.1, \delta(2) = 0.2, \sigma(1) = 0.5, \sigma(2) = 0.3$.

Let the generator of the Markov chain $r_t$ be

$$Q = \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}.$$ 

Then, by (8), we get

$$\pi(1) = \frac{1}{4} \quad \text{and} \quad \pi(2) = \frac{3}{4}.$$

To verify the theoretical result in Theorem 3.4, we perform a computer simulation of 30000 iterations of the single path of $X_t$ with initial value $X_0 = 0.3, r_0 = 1$. The sample path is shown in Fig. 1. In Theorem 3.4, we give the condition
\[ \sum_{i=1}^{n} \pi(i) \beta(i) < 0 \] for the existence of a unique stationary distribution. It is easy to verify that the coefficients satisfy the condition. Then we compute the average values of the last 20000, 10000 iterations of \( X_t \) when \( \alpha = 1.75 \), which are 0.1628 and 0.1581 respectively. Besides, we compute the variance of the last 20000, 10000 iterations of \( X_t \), which are 0.0072 and 0.0076 respectively. We can see that the differences of the two average values and the two variance are very small, which means that the process \( X_t \) is stationary and this verifies our theoretical result in Theorem 3.4. Furthermore, it is well known that if \( \alpha \) becomes bigger, the jump sizes become smaller. So we choose \( \alpha = 1.25 \) and \( \alpha = 1.75 \) to see the effects that the increase of \( \alpha \) causes. The first picture in Fig. 1 is the path with \( \alpha = 1.25 \) and the second is \( \alpha = 1.75 \). As we can see from Fig. 1, the fluctuations in the first picture of Fig. 1 on some time intervals are much larger than that of the second one. Moreover, if we sort the iterations of \( X_t (\alpha = 1.75) \) into sorted data from the smallest to the largest, then the 150th and 29850th value in the sorted data are -0.0914 and 0.4676, respectively. Thus, we get a confidence interval of \( X_t \), namely, for any sufficiently large \( t \)

\[ P(-0.0914 < X_t < 0.4676) \approx 0.99. \]

Similarly, when \( \alpha = 1.25 \), for any sufficiently large \( t \), one has

\[ P(-0.0682 < X_t < 0.5689) \approx 0.99. \]

Example 5.2. Continuation of Example 5.1, we choose \( \alpha = 1.75 \) and change the value of \( \beta(i), i = 1, 2 \) to \( \beta(1) = -0.005, \beta(2) = 0.5 \).

We perform a computer simulation of 30000 iterations of the single path of \( X_t \) with initial value \( x_0 = 0.3, r_0 = 1 \). The sample path is shown in Fig. 2. The coefficients satisfy the condition \( \sum_{i=1}^{n} \pi(i) \beta(i) > 0 \) in Theorem 4.2. Then, it is obvious in Fig. 2 that the CIR model (3) is transient, which supports our theoretical result.

6. Proof of the Lyapunov function \( V \in D(A) \). In this appendix we shall show that the Lyapunov function \( V \) in the proof of Theorem 3.4 is in the domain \( D(A) \) of the generator \( A \). First, we define the domain \( D(A) \) by

\[ D(A) = \left\{ f(x, i) \text{ is Borel measurable : there is a Borel measurable function } g(x, i) \text{ such that } f(X_t, r_t) - f(X_0, r_0) - \int_0^t g(X_s, r_s) ds \text{ is a local martingale} \right\}. \]

For \( f \in D(A) \) we define

\[ Af = \left\{ g(x, i) \text{ is Borel measurable : } \right. \]

\[ \int_0^t g(X_s, r_s) ds \text{ is a local martingale} \} , \]

where \( A \) is the generator of \( (X_t, r_t)_{t \geq 0} \). It is obvious

\[ \left\{ f \in C^2(\mathbb{R} \times S) : \int_{\{|y| > 1\}} (f(x + y, i) - f(x, i)) \frac{C_{\alpha}}{|y|^{\alpha+1}} dy < \infty \text{ for } x \in \mathbb{R} \right\} \subseteq D(A) \]
Then, by a result of Sandrić [19], we have
\[ \left| \int_{\{ |y| \geq 1 \}} (V(x + y, i) - V(x, i)) \frac{C_\alpha}{|y|^\alpha + 1} \, dy \right| < \infty \]
for all \( x \in \mathbb{R} \), where \( V \) is defined in the proof of Theorem 3.4. Hence, by relation (44), we have \( V \in \mathcal{D}(A) \).

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Figure 2. Computer simulation of a single path of $X_t$ with initial value $X_0 = 0.3$, $r_0 = 1$ and $\alpha = 1.75$.

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*E-mail address: zzzhang@dhu.edu.cn (Zhenzhong Zhang)*
*E-mail address: enhuazhang1993@163.com (Enhua Zhang)*
*E-mail address: jytong@dhu.edu.cn (Jinying Tong)*