Some properties of Kaehler submanifolds with recurrent tensor fields

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Abstract

The properties of Kaehler submanifolds with recurrent the second fundamental form in spaces of constant holomorphic sectional curvature are being studied in this article.

Introduction

Let $M^{2m+2l}$ be a Kaehler manifold of complex dimension $m+l$ ($m \geq 1, l \geq 1$) with almost complex structure $J$ and a Riemannian metric $\tilde{g}$, $\nabla$ be the Riemannian connection coordinated with $\tilde{g}$, $\tilde{R}$ be the Riemannian curvature tensor of manifold $M^{2m+2l}$. Let $F^{2m}$ be a Kaehler submanifold of complex dimension $m$ in $M^{2m+2l}$ with induced Riemannian metric $g$. The restriction $J$ to $F^{2m}$ defines induced almost complex structure on $F^{2m}$, which we will denote by the same symbol $J$. Let $\nabla$ be the Riemannian connection coordinated with $g$, $D$ be the normal connection, $b$ be the second fundamental form, $R^\perp$ be the tensor of normal curvature of submanifold $F^{2m}$, $\nabla = \nabla \oplus D$ be the connection of van der Waerden — Bortolotti. $b$ is called parallel if $\nabla b \equiv 0$. A tensor of normal curvature $R^\perp$ is called parallel if $\nabla R^\perp \equiv 0$.

According to the definition of recurrent tensor field (see [1], note 8), nonzero form $b \neq 0$ is called recurrent if there exists 1-form $\mu$ on $F^{2m}$ such that $\nabla b = \mu \otimes b$.

Theorem 1. Let $F^{2m}$ be a Kaehler submanifold of complex dimension $m$ in a Kaehler manifold $M^{2m+2l}(c)$ of complex dimension $m+l$ and constant holomorphic sectional curvature $c$. If $F^{2m}$ has recurrent the second fundamental form $b$ then the tensor of normal curvature $R^\perp \neq 0$ is parallel.

It is known (see [1], note 8, theorem 3), that for a Riemannian manifold $M$ with recurrent tensor of Riemannian curvature $\tilde{R}$ and irreducible narrowed linear group of holonomy, it is necessary that the tensor of Riemannian curvature $\tilde{R}$ be parallel (i.e. $\nabla \tilde{R} \equiv 0$) with the condition $\dim M \geq 3$. A Riemannian manifold $M$ is called locally symmetric if $\nabla \tilde{R} \equiv 0$.

Theorem 2. Let $F^{2m}$ be a Kaehler submanifold of complex dimension $m$ in a Kaehler manifold $M^{2m+2l}(c)$ of complex dimension $m+l$ and constant holomorphic sectional curvature $c$. If $F^{2m}$ has recurrent the second fundamental form $b$ then $F^{2m}$ is locally symmetric submanifold.

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1 Main notations and formulas.

Let $M^{n+p}$ be $(n+p)$-dimensional $(n \geq 2, p \geq 2)$ smooth Riemannian manifold, $\tilde{g}$ be a Riemannian metric on $M^{n+p}$, $\nabla$ be the Riemannian connection coordinated with $\tilde{g}$, $F^n$ be $n$-dimensional smooth submanifold in $M^{n+p}$, $g$ be the induced Riemannian metric on $F^n$, $\nabla$ be the Riemannian connection on $F^n$ coordinated with $g$, $TF^n$ and $T^\perp F^n$ be tangent and normal bundles on $F^n$, respectively, $R$ and $R_i$ be the tensors of Riemannian and Ricci curvature of connection $\nabla$, respectively, $b$ be the second fundamental form $F^n$, $D$ be the normal connection, $R^\perp$ be the tensor of normal curvature, $\nabla$ be the connection of Van der Waerden -- Bortolotti.

The formulas of Gauss and Weingarten have, respectively, the following form [2] :

$$\nabla_X Y = \nabla_X Y + b(X, Y),$$

for any vector fields $X, Y$, tangent to $F^n$, and vector field $\xi$ normal to $F^n$.

The equations of Gauss, Peterson -- Codacci and Ricci have, respectively, the following form [2] :

$$\check{R}(X, Y, Z, W) = R(X, Y, Z, W) + \tilde{g}(b(X, Z), b(Y, W)) - \tilde{g}(b(X, W), b(Y, Z)),$$

$$\check{R}(X, Y) = (\nabla_X b)(Y, Z) - (\nabla_Y b)(X, Z),$$

for any vector fields $X, Y, Z, W$, tangent to $F^n$, and vector fields $\xi, \eta$, normal to $F^n$.

For any vector field $\xi$ normal to $F^n$, we denote as $A_\xi$ the second fundamental tensor relatively to $\xi$. For $A_\xi$ the relation holds

$$\check{g}(b(X, Y), \xi) = \check{g}(A_\xi X, Y),$$

for any vector fields $X, Y$, tangent to $F^n$.

A normal vector field $\xi$ is called nondegenerate if $\det A_\xi \neq 0$.

Covariant derivatives $\nabla b$, $(\nabla A)_\xi$ and $\nabla R^\perp$, are defined by the following equalities, respectively ([2] ):

$$(\nabla_X b)(Y, Z) = D_X (b(Y, Z)) - b(\nabla_X Y, Z) - b(Y, \nabla_X Z),$$

$$(\nabla_X A)_\xi Y = \nabla_X (A_\xi Y) - A_\xi (\nabla_X Y) - A_{D_X \xi} Y,$$

$$(\nabla_X R^\perp)(Y, Z) = D_X (R^\perp(Y, Z) \xi) - R^\perp(\nabla_X Y, Z) \xi - R^\perp(Y, \nabla_X Z) \xi - R^\perp(Y, Z) D_X \xi,$$

for any vector fields $X, Y, Z$, tangent to $F^n$, and vector field $\xi$ normal to $F^n$.

Let indices, in this article, take the following values: $i, j, k, s, t = 1, \ldots, n$, $\alpha, \beta, \gamma = 1, \ldots, p$. We will use the Einstein rule.

Let $x$ be an arbitrary point $F^n$, $T_x F^n$ and $T_x^\perp F^n$ be the tangent and normal spaces $F^n$ at point $x$, respectively, $U(x)$ be some neighborhood of point $x$, $(u^1, \ldots, u^n)$ be local
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coordinates on $F^n$ in $U(x)$, $\{\partial/\partial u^i\}$ be a local basis in $TF^n$, $\{n_{\alpha i}\}$ be a field of bases of normal vectors in $T^\bot F^n$ in $U(x)$. We may always choose the basis $\{n_{\alpha i}\}$ orthonormalized and assume that $\tilde{g}(n_{\alpha i}, n_{\beta j}) = \delta_{\alpha\beta}$, where $\delta_{\alpha\beta}$ is the Kronecker symbol. We introduce the following designations: 

$$g_{ij} = g \left( \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right), \quad b_{ij} = b \left( \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right) = b_{ij}^\alpha n_{\alpha i}, \quad \Gamma_{ij,k} = g \left( \nabla_{\alpha a^\alpha} \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right),$$

$$\Gamma_{ij}^k = g^k l \Gamma_{ij,l}, \quad \nabla_i b_{jk}^\alpha = \frac{\partial b_{jk}^\alpha}{\partial u^i} - \Gamma_{ij}^t b_{tk}^\alpha - \Gamma_{ik}^t b_{jt}^\alpha, \quad \Gamma_{ij}^k = g \left( n_{\alpha i}, \nabla_{\alpha a^\alpha} n_{\beta j} \right),$$

$$\Gamma_{ij}^k = \delta_{ij} \Gamma_{ij}^k, \quad \Gamma_{ij}^k = \delta_{\alpha\beta} \Gamma_{ij}^k, \quad \Gamma_{ij}^k + \Gamma_{ji}^k = 0, \quad \nabla_i b_{jk}^\alpha = (\nabla_i b_{jk}^\alpha + \Gamma_{ij}^k b_{jk}^\alpha),$$

$$\nabla_i a_{ij}^k = \partial_\alpha^k a_{ij}^\alpha - \Gamma_{ij}^l a_{ij}^\alpha + \Gamma_{\alpha ij}^l a_{ij}^\alpha, \quad a_{ij}^k = \frac{\partial}{\partial u^i} A_{n_{\alpha i}} \left( \frac{\partial}{\partial u^j} \right),$$

$$\nabla_i a_{ij}^k = \nabla_i a_{ij}^k - \frac{1}{2} \Gamma_{\alpha ij}^k a_{ij}^\alpha, \quad \nabla_i a_{ij}^k \left( \frac{\partial}{\partial u^j} \right) = \nabla_i a_{ij}^k \left( \frac{\partial}{\partial u^j} \right),$$

where $\|g^{ij}\|$ and $\|\delta^{\alpha\beta}\|$ are inverse matrixes to $\|g_{ij}\|$ and $\|\delta_{\alpha\beta}\|$, respectively.

We assume that a Riemannian manifold $M^{n+p}$ is almost Hermitian manifold with almost complex structure $J$ (see [3], chapter 6, section 6.1). Then $M^{n+p}$ has even dimension: $n + p = 2(m + l)$, where a number $m + l$ is called complex dimension of $M^{n+p}$; the Riemannian metric $\tilde{g}$ is almost Hermitian, i.e. for any vector fields $\tilde{X}, \tilde{Y}$, tangent to $M^{n+p}$, the following condition holds:

$$\tilde{g}(J\tilde{X}, \tilde{J}\tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y}). \quad (1.10)$$

Almost Hermitian manifold $M^{n+p}$ is called Kaehler manifold ( [3] ) if almost complex structure $J$ is parallel, i.e. for any vector fields $\tilde{X}, \tilde{Y}$, tangent to $M^{n+p}$, the following condition holds:

$$\nabla_i \tilde{J}\tilde{Y} = \tilde{J}\nabla_i \tilde{X}. \quad (1.11)$$

A submanifold $F^n$ of a Kaehler manifold $M^{n+p}$ is called Kaehler submanifold if for any vector field $X \in TF^n$, vector field $JX \in TF^n$. $F^n$ is Kaehler manifold relative to induced almost complex structure $J$ and induced almost Hermitian metric $g$ (see [3], chapter 6, par. 6.7). Kaehler submanifold $F^n$ in Kaehler manifold $M^{n+p}$, has even dimension $n = 2m$ and codimension $p = 2l$. Number $m$ is called complex dimension, and number $l$ is called complex codimension of Kaehler submanifold $F^n$.

We denote by $M^{2m+2l}(c)$, a Kaehler manifold of complex dimension $m + l$ of constant holomorphic sectional curvature $c$. The tensor of Riemannian curvature $\tilde{R}$ of space $M^{2m+2l}(c)$ complies with the formula [1]:

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = \frac{c}{4} \left( \tilde{g}(\tilde{Z}, \tilde{Y})\tilde{X} - \tilde{g}(\tilde{X}, \tilde{Z})\tilde{Y} + \tilde{g}(J\tilde{Y}, \tilde{Z})J\tilde{X} - \tilde{g}(J\tilde{X}, \tilde{Z})J\tilde{Y} + 2\tilde{g}(\tilde{X}, J\tilde{Y})J\tilde{Z} \right), \quad (1.12)$$

for any vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}$, tangent to $M^{2m+2l}(c)$. 

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2 The properties of covariant derivative $\nabla$.

**Lemma 1**. Let $F^n$ be a submanifold in a Riemannian manifold $M^{n+p}$. Then the following equality holds:

$$g((\nabla Z A)\xi X, Y) = g((\nabla Z b)(X, Y), \xi) \quad \forall X, Y, Z \in TF^n, \quad \forall \xi \in T^\perp F^n. \quad (2.1)$$

**Proof.** We will find the expressions of the left and the right parts of the equality (2.1), in local coordinates. We assume

$$Z = Z^i \frac{\partial}{\partial u^i}, \quad X = X^j \frac{\partial}{\partial u^j}, \quad Y = Y^k \frac{\partial}{\partial u^k}, \quad \xi = \xi^\alpha n_\alpha. \quad (2.2)$$

We have:

$$g((\nabla Z A)\xi X, Y) = Z^i X^j Y^k \xi^\alpha g_{sk} \nabla_i a^s_{\alpha |j} = Z^i X^j Y^k \left( \xi^\alpha g_{sk} \nabla_i a^s_{\alpha |j} - \xi^\alpha g_{sk} \Gamma^{\perp}_{\alpha |i} a^s_{\beta |j} \right) =$$

$$= Z^i X^j Y^k \left( \xi^\alpha \nabla_i (g_{sk} A^s_{\alpha |j}) - \xi^\alpha \Gamma^{\perp}_{\alpha |i} (g_{sk} A^s_{\beta |j}) \right) = Z^i X^j Y^k \left( \xi^\alpha \nabla_i b_{\alpha |j} - \xi^\alpha \Gamma^{\perp}_{\alpha |i} b_{\beta |j} \right) =$$

$$= Z^i X^j Y^k \left( \xi^\alpha \delta_{\alpha \beta} \nabla_i b^\beta_{\gamma |j} - \xi^\alpha \Gamma^{\perp}_{\gamma |i} b^\gamma_{\beta |j} \right) =$$

$$= Z^i X^j Y^k \left( \xi^\alpha \delta_{\alpha \beta} \nabla_i b^\beta_{\gamma |j} - \xi^\alpha \Gamma^{\perp}_{\gamma |i} b^\gamma_{\beta |j} \right) =$$

Lemma is proved.

**Lemma 2**. Let $F^{2m}$ be a Kähler submanifold in a Kähler manifold $M^{2m+2}$. Then for any $X \in TF^{2m}$ and for any $\xi \in T^\perp F^{2m}$ the following equality holds:

$$\left( \nabla X A \right)_{\partial \xi} = J \left( \nabla X A \right)_{\xi} \quad (2.3)$$

**Proof.** From (1.1), because of (1.11), we obtain the following equalities (see, for example, [3], chapter 6, section 6.1, lemma 6.26):

$$\nabla X JY = J \nabla X Y, \quad Jb(X, Y) = b(X, JY), \quad \forall X, Y \in TF^{2m}. \quad (2.4)$$

From (1.2) we have:

$$\nabla X J \xi = -A_{J \xi} X + D_X J \xi, \quad J \nabla X \xi = J(-A_{\xi} X + D_X \xi).$$
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Hence, because of (1.11), we obtain:

\[-A_{J\xi}X + D_X J\xi = J(-A_{\xi}X + D_X \xi).\]

Therefore,

\[-A_{J\xi}X + JA_{\xi}X = JD_X \xi - D_X J\xi.\]

Since $F^{2m}$ is a Kaehler submanifold, then, from here, we have

\[A_{J\xi}X = JA_{\xi}X, \quad D_X (J\xi) = JD_X \xi, \quad \forall X, Y \in TF^{2m}. \quad (2.5)\]

From (1.7) we have

\[(\nabla_X A)_{J\xi}Y = \nabla_X (A_{J\xi}Y) - A_{J\xi}(\nabla_X Y) - A_{D_X (J\xi)}Y.\]

Hence, using (2.4) and (2.5), we have:

\[(\nabla_X A)_{J\xi}Y = \nabla_X J(A_{\xi}Y) - JA_{\xi}(\nabla_X Y) - A_{D_X (J\xi)}Y =

= J\nabla_X (A_{\xi}Y) - JA_{\xi}(\nabla_X Y) - JA_{D_X (J\xi)}Y = J(\nabla_X A)_{J\xi}Y.\]

Lemma is proved.

**Lemma 3.** Let $F^{2m}$ be a Kaehler submanifold in a Kaehler manifold $M^{2m+2}(c)$ of constant holomorphic sectional curvature $c$. Then for any $X, Y, Z \in TF^{2m}$ and for any $\xi \in T^\perp F^{2m}$ the following equalities hold:

\[(\nabla_{JZ} b)(X, Y) = J \left( (\nabla_Z b)(X, Y) \right), \quad (2.6)\]

\[(\nabla_{JZ} A)_{J\xi} = -J(\nabla_Z A)_{J\xi}, \quad (2.7)\]

\[JA_{\xi} = -A_{\xi} J, \quad (2.8)\]

\[J(\nabla_Z A)_{J\xi} = -(\nabla_Z A)_{J\xi} J. \quad (2.9)\]

**Proof.** 1. Because of (1.12), the equation (1.4) takes the following form:

\[(\nabla_X b)(Y, Z) = (\nabla_Y b)(X, Z), \quad \forall X, Y, Z \in TF^{2m}. \quad (2.10)\]

Using (2.10), from (1.7) we obtain:

\[(\nabla_{JZ} b)(X, Y) = (\nabla_X b)(JZ, Y) = D_X (b(JZ, Y)) - b(\nabla_X (JZ), Y) - b(JZ, \nabla_X Y).\]

Hence, using (2.4) and (2.5), we have:

\[(\nabla_{JZ} b)(X, Y) = D_X (b(JZ, Y)) - b(J\nabla_X Z, Y) - b(JZ, \nabla_X Y) =

= J(D_X (b(Z, Y))) - J(b(\nabla_X Z, Y)) - J(b(Z, \nabla_X Y)) =

= J(D_X (b(Z, Y))) - J(b(\nabla_X Z, Y)) - J(b(Z, \nabla_X Y)) =

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The equality (2.6) is proved.

2. Using (2.6), from (2.1) we obtain:

\[ \tilde{g}(\tilde{\nabla}JZA)_\xi X, Y = \tilde{g}(\tilde{\nabla}b)(X, Y, \xi) = \tilde{g}(J((\tilde{\nabla}b)(X, Y)), \xi). \]

Hence, because of (1.10) and equality \( J^2 = -I \), we have:

\[ \tilde{g}(\tilde{\nabla}JZA)_\xi X, Y = -\tilde{g}(\tilde{\nabla}b)(X, Y, J\xi) = -\tilde{g}(((\tilde{\nabla}ZA)_\xi X, Y) = -\tilde{g}(J((\tilde{\nabla}ZA)_\xi X), Y). \]

From here we get (2.7).

3. From (1.6), using (2.4), we obtain:

\[ \tilde{g}(J\xi X, Y) = -\tilde{g}(A_\xi X, JY) = -\tilde{g}(b(X, JY), \xi) = -\tilde{g}(b(JX, Y), \xi) = -\tilde{g}(A_\xi JX, Y). \]

Thus,

\[ \tilde{g}(J\xi X, Y) = -\tilde{g}(A_\xi JX, Y) \quad \forall X, Y \in TF^{2m}, \quad \forall \xi \in T^\perp F^{2m}. \]

The derived equality is equivalent to (2.8).

4. From (1.8), using (2.4) and (2.8), for any \( X, Y \in TF^{2m} \) and for any \( \xi \in T^\perp F^{2m} \), we have:

\[ J\left((\nabla_X A)_\xi Y\right) = J(\nabla_X (A_\xi Y) - A_\xi (\nabla_X Y) - A_{D_\xi X} Y) = \]

\[ = \nabla_X J(A_\xi Y) + A_\xi J(\nabla_X Y) + A_{D_\xi X} (JY) = \]

\[ = -\nabla_X (A_\xi JY) + A_\xi (\nabla_X JY) + A_{D_\xi X} (JY) = -\tilde{\nabla}X (A_\xi JY). \]

Thus,

\[ J\left((\nabla_X A)_\xi Y\right) = -\tilde{\nabla}X (A_\xi JY), \quad \forall X, Y \in TF^{2m}, \quad \forall \xi \in T^\perp F^{2m}. \]

The obtained equality is equivalent to (2.9). Lemma is proved.

**Lemma 4.** Let \( F^{2m} \) be a Kaehler submanifold in a Kaehler manifold \( M^{2m+2l} \). Then the following equality holds

\[ \nabla_Z (\tilde{g}(X, JY) J\xi) = 0 \quad \forall X, Y, Z \in TF^{2m}, \quad \forall \xi \in T^\perp F^{2m}. \quad (2.11) \]

**Proof.** By the definition of covariant derivative \( \nabla \), we have:

\[ \nabla_Z (\tilde{g}(X, JY) J\xi) = \]

\[ = D_Z (\tilde{g}(X, JY) J\xi) - \tilde{g}(\nabla_Z X, JY) J\xi - \tilde{g}(X, \nabla_Z (JY)) J\xi - \tilde{g}(X, JY) D_Z (J\xi). \]

We transform the right part of the last equality, writing it in local coordinates and using the designations (2.2):

\[ \left( \frac{\partial (g_{kl} X^k (JY)^l (J\xi)^r)}{\partial u^i} + \Gamma^r_{\alpha i} g_{kl} X^k (JY)^l (J\xi)^\alpha \right) Z^i n_r = \]
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\[-g_{kl} \left( \frac{\partial X^k}{\partial u^i} + \Gamma^k_{im} X^m \right) (JY)^l (J\xi)^r Z^i n_\tau - g_{lk} \left( \frac{\partial (JY)^l}{\partial u^i} + \Gamma^l_{im} (JY)^m \right) X^k (J\xi)^r Z^i n_\tau -
\]

\[-g_{kl} X^k (JY)^l \left( \frac{\partial (J\xi)^r}{\partial u^i} + \Gamma^r_{\sigma i} (J\xi)^\sigma \right) Z^i n_\tau =
\]

\[= \left( \frac{\partial g_{kl}}{\partial u^i} X^k (JY)^l (J\xi)^r - g_{lk} \Gamma^k_{im} X^m (JY)^l (J\xi)^r - g_{lk} \Gamma^l_{im} X^k (JY)^m (J\xi)^r \right) Z^i n_\tau =
\]

\[= \left( \frac{\partial g_{kl}}{\partial u^i} - g_{ml} \Gamma^m_{ik} - g_{mk} \Gamma^m_{il} \right) X^k (JY)^l (J\xi)^r Z^i n_\tau = 0.
\]

Lemma is proved.

**Lemma 5.** Let $F^{2m}$ be a Kaehler submanifold in a Kaehler manifold $M^{2m+2l}(c)$ of constant holomorphic sectional curvature $c$. Then the following equality holds

\[R^\perp(X, Y)\xi = \frac{c}{2} \tilde{g}(X, JY) J\xi + b(X, A_\xi Y) - b(Y, A_\xi X),
\]

\[\forall X, Y \in TF^{2m}, \quad \forall \xi \in T^\perp F^{2m}.
\]  \tag{2.12}

**Proof.** Because of (1.12), we have:

\[\tilde{R}(X, Y, \xi, \eta) = \tilde{g}(\tilde{R}(X, Y)\xi, \eta) = \frac{c}{2} \tilde{g}(X, JY) \tilde{g}(J\xi, \eta).
\]

Then the equation (1.5) takes the form:

\[R^\perp(X, Y, \xi, \eta) = \frac{c}{2} \tilde{g}(X, JY) \tilde{g}(J\xi, \eta) + \tilde{g}([A_\xi, A_\eta]X, Y).
\]

We transform the second term in the right part of the obtained equality, using self-adjointness of operator $A_\xi$:

\[\tilde{g}([A_\xi, A_\eta]X, Y) = \tilde{g}((A_\xi A_\eta - A_\eta A_\xi)X, Y) = \tilde{g}(A_\xi (A_\eta X), Y) - \tilde{g}(A_\eta (A_\xi X), Y) =
\]

\[= \tilde{g}(A_\eta X, A_\xi Y) - \tilde{g}(A_\xi X, A_\eta Y) = \tilde{g}(b(A_\xi Y, X), \eta) - \tilde{g}(b(A_\xi X, Y), \eta).
\]

Then for any $\eta \in T^\perp F^{2m}$ we have:

\[R^\perp(X, Y, \xi, \eta) \equiv \tilde{g}(R^\perp(X, Y)\xi, \eta) =
\]

\[= \tilde{g}(\frac{c}{2} \tilde{g}(X, JY) J\xi, \eta) + \tilde{g}(b(A_\xi Y, X), \eta) - \tilde{g}(b(A_\xi X, Y), \eta).
\]

From here we obtain the equality (2.12). Lemma is proved.
Lemma 6. Let $F^{2m}$ be a Kaehler submanifold in a Kaehler manifold $M^{2m+2l}$ of constant holomorphic sectional curvature $c$. Then the following equality holds

$$(\nabla_Z R^\perp)(X, Y)\xi =$$

$$= (\nabla_Zb)(X, A_\xi Y) + b(X, (\nabla_Z A)\xi Y) - (\nabla_Zb)(Y, A_\xi X) - b(Y, (\nabla_Z A)\xi X),$$

$$\forall X, Y, Z \in TF^{2m}, \quad \forall \xi \in T^\perp F^{2m}. \quad (2.13)$$

**Proof.** From formula (1.9), using (2.12), we obtain:

$$(\nabla_Z R^\perp)(X, Y)\xi = D_Z\left(\frac{c}{2}\tilde{g}(X, JY)J\xi + b(X, A_\xi Y) - b(Y, A_\xi X)\right) -$$

$$- \left(\frac{c}{2}\tilde{g}(\nabla_Z X, JY)J\xi + b(\nabla_Z X, A_\xi Y) - b(\nabla_Z Y, A_\xi X)\right) -$$

$$- \left(\frac{c}{2}\tilde{g}(X, J(\nabla_Z Y))J\xi + b(X, A_\xi (\nabla_Z Y)) - b(\nabla_Z Y, A_\xi X)\right) -$$

$$- \left(\frac{c}{2}\tilde{g}(X, JY)J(D_Z\xi) + b(X, A_{D_\xi \xi} X) - b(Y, A_{D_\xi \xi} X)\right) =$$

$$= \frac{c}{2}\left(D_Z(\tilde{g}(X, JY)J\xi) - \tilde{g}(\nabla_Z X, JY)J\xi - \tilde{g}(X, J(\nabla_Z Y))J\xi - \tilde{g}(X, JY)J(D_Z\xi) +$$

$$+ D_Z(b(X, A_\xi Y)) - D_Z(b(Y, A_\xi X)) - b(\nabla_Z X, A_\xi Y) + b(Y, A_\xi (\nabla_Z X)) -$$

$$- b(X, A_\xi (\nabla_Z Y)) + b(\nabla_Z Y, A_\xi X) - b(X, A_{D_\xi \xi} X) + b(Y, A_{D_\xi \xi} X).$$

Hence, using (2.3) and (2.4), we have

$$(\nabla_Z R^\perp)(X, Y)\xi =$$

$$= \frac{c}{2}\tilde{g}(X, JY)J\xi + D_Z(b(X, A_\xi Y)) - D_Z(b(Y, A_\xi X)) - b(\nabla_Z X, A_\xi Y) +$$

$$+ b(Y, A_\xi (\nabla_Z X)) - b(X, A_\xi (\nabla_Z Y)) + b(\nabla_Z Y, A_\xi X) - b(X, A_{D_\xi \xi} Y) + b(Y, A_{D_\xi \xi} X).$$

Therefore, because of (2.11), we obtain:

$$(\nabla_Z R^\perp)(X, Y)\xi = \left(D_Z(b(X, A_\xi Y)) - b(\nabla_Z X, A_\xi Y) - b(X, A_\xi (\nabla_Z Y)) - b(X, A_{D_\xi \xi} Y)\right) -$$

$$- \left(D_Z(b(Y, A_\xi X)) - b(\nabla_Z Y, A_\xi X) - b(Y, A_\xi (\nabla_Z X)) - b(Y, A_{D_\xi \xi} X)\right).$$

Hence, using (1.7), we obtain:

$$(\nabla_Z R^\perp)(X, Y)\xi = \left((\nabla_Z b)(X, A_\xi Y) + b(X, \nabla_Z (A_\xi Y)) - b(X, A_\xi (\nabla_Z Y)) - b(X, A_{D_\xi \xi} Y)\right) -$$
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\[-\left(\nabla_Z b)(Y, A_\xi X) + b(Y, \nabla_Z (A_\xi X)) - b(Y, A_\xi (\nabla_Z X)) - b(Y, A_DZ_\xi X)\right].\]

Now, using (1.8), we obtain:

\[(\nabla_Z R^\perp)(X, Y)\xi = \]

\[= \left(\nabla_Z b)(X, A_\xi Y) + b(X, (\nabla_Z A)\xi Y)\right) - \left(\nabla_Z b)(Y, A_\xi X) + b(Y, (\nabla_Z A)\xi X)\right).\]

Lemma is proved.

**Lemma 7.** Let \(F^{2m}\) be a Kaehler submanifold in a Kaehler manifold \(M^{2m+2l}(c)\) of constant holomorphic sectional curvature \(c\). Then the following equality holds

\[(\nabla_Z R^\perp)(X, Y, \xi, \eta) = \tilde{g}([\nabla_Z A]_\xi, X, Y, \tilde{g}([A_\xi, (\nabla_Z A)]_\eta, X, Y), \forall X, Y, Z \in TF^{2m}, \forall \xi, \eta \in T^\perp F^{2m}. \quad (2.14)\]

**Proof.** Because of (2.13), we have:

\[(\nabla_Z R^\perp)(X, Y, \xi, \eta) \equiv \tilde{g}\left(\left(\nabla_Z R^\perp\right)(X, Y)\xi, \eta\right) = \tilde{g}\left(\left(\nabla_Z b\right)(X, A_\xi Y), \eta\right) - \]

\[\tilde{g}\left(\left(\nabla_Z b\right)(Y, A_\xi X), \eta\right) + \tilde{g}\left(b(X, (\nabla_Z A)_\xi Y), \eta\right) - \tilde{g}\left(b(Y, (\nabla_Z A)_\xi X), \eta\right).\]

In the derived equality, we transform the first and the second terms using (2.1), the third and the fourth using (1.6):

\[(\nabla_Z R^\perp)(X, Y, \xi, \eta) = \tilde{g}\left((\nabla_Z A)_\eta X, A_\xi Y\right) - \tilde{g}\left((\nabla_Z A)_\eta Y, A_\xi X\right) + \]

\[+ \tilde{g}\left(A_\eta X, (\nabla_Z A)_\xi Y\right) - \tilde{g}\left(A_\eta Y, (\nabla_Z A)_\xi X\right).\]

Hence, because of self-adjointness of operators \(A_\xi\) and \((\nabla A)_\xi\), we obtain:

\[(\nabla_Z R^\perp)(X, Y, \xi, \eta) = \tilde{g}\left(A_\xi (\nabla_Z A)_\eta X, Y\right) - \tilde{g}\left(Y, (\nabla_Z A)_\eta A_\xi X\right) + \]

\[+ \tilde{g}\left((\nabla_Z A)_\xi A_\eta X, Y\right) - \tilde{g}\left(Y, A_\eta (\nabla_Z A)_\xi X\right) = \]

\[= \tilde{g}\left([A_\xi, (\nabla_Z A)_\eta]X, Y\right) + \tilde{g}\left(([\nabla_Z A]_\xi, A_\eta]X, Y\right).\]

Lemma is proved.

**Lemma 8.** Let \(F^{2m}\) be a Kaehler submanifold in a Kaehler manifold \(M^{2m+2l}(c)\) of constant holomorphic sectional curvature \(c\). Then the following equality holds

\[(\nabla_{JZ} R^\perp)(X, Y, J\xi, \eta) = (\nabla_Z R^\perp)(X, Y, J\xi, \eta) - 2\tilde{g}([\nabla_Z A]_J\xi, A_\eta]X, Y), \forall X, Y, Z \in TF^{2m}, \forall \xi, \eta \in T^\perp F^{2m}. \quad (2.15)\]
Proof. From (2.14) we obtain:

\[
(\nabla_{JZ} R^\perp)(X, Y, \xi, \eta) = \tilde{g}([([\nabla_J Z A]_\xi, A_\eta]X, Y) + \tilde{g}([A_\xi, (\nabla_J Z A)_\eta]X, Y).
\]

Hence, using (2.7), we have:

\[
(\nabla_{JZ} R^\perp)(X, Y, \xi, \eta) = \tilde{g}([J(\nabla_J Z A)_\xi, A_\eta]X, Y) + \tilde{g}([J A_\xi, (\nabla_J Z A)_\eta]X, Y).
\]

In the derived equality, we transform the second term using (2.8) and (2.9):

\[
[\nabla_J Z A]_{\xi}J(A_\eta)\xi = A_\xi J(\nabla_J Z A)_\eta A_\xi = -J A_\xi (\nabla_J Z A)_\eta + J(\nabla_J Z A)_\eta A_\xi.
\]

Therefore,

\[
(\nabla_{JZ} R^\perp)(X, Y, \xi, \eta) = -\tilde{g}([J(\nabla_J Z A)_\xi, A_\eta]X, Y) + \tilde{g}([J A_\xi, (\nabla_J Z A)_\eta]X, Y).
\]

Hence, because of (2.3) and (2.5), and using (2.14), we obtain:

\[
(\nabla_{JZ} R^\perp)(X, Y, \xi, \eta) = \tilde{g}([J(A_\xi), (\nabla_J Z A)_\eta]X, Y) = (\nabla_J R^\perp)(X, Y, J_\xi, \eta) - 2\tilde{g}([[\nabla_J Z A]_\xi, A_\eta]X, Y).
\]

Lemma is proved.

3 Proofs of theorems 1, 2.

Proof theorem 1. Let for some 1-form \( \mu \) on \( F^{2m} \), the following condition holds

\[
(\nabla_X b)(Y, Z) = \mu(X)b(Y, Z) \quad \forall X, Y, Z \in TF^{2m}.
\]  

Then for any vector field \( \xi \in T^\perp F^{2m} \), we have:

\[
\tilde{g}((\nabla_X b)(Y, Z), \xi) = \tilde{g}(\mu(X)b(Y, Z), \xi).
\]

Hence, using (2.1) and (1.6), we obtain:

\[
\tilde{g}((\nabla_X A)_\xi Y, Z) = \tilde{g}(\mu(X)A_\xi Y, Z) \quad \forall X, Y, Z \in TF^{2m}, \quad \forall \xi \in T^\perp F^{2m}.
\]

Thus, the condition (3.1) is equivalent to the condition

\[
(\nabla_X A)_\xi = \mu(X)A_\xi, \quad \forall X \in TF^{2m}, \quad \forall \xi \in T^\perp F^{2m}. \tag{3.2}
\]

From (3.2) we obtain the equality:

\[
(\nabla_{JX} A)_\xi = \mu(JX)A_\xi \quad \forall X \in TF^{2m}, \quad \forall \xi \in T^\perp F^{2m}. \tag{3.3}
\]
Some properties of Kaehler submanifolds with recurrent tensor fields.

On the other hand, from (3.2), because of (2.7), we have:

\[
(\nabla_{JX}A)_\xi = -J(\mu(X)A_\xi), \quad \forall X \in TF^{2m}, \quad \forall \xi \in T^\perp F^{2m}.
\] (3.4)

From (3.3) and (3.4), we obtain:

\[
\mu(JX)A_\xi = -J(\mu(X)A_\xi), \quad \forall X \in TF^{2m}, \quad \forall \xi \in T^\perp F^{2m}.
\]

Hence, for any \(Y \in TF^{2m}\), we have:

\[
\mu(JX)A_\xi Y = -\mu(X)J(A_\xi Y), \quad \forall X \in TF^{2m}, \quad \forall \xi \in T^\perp F^{2m}.
\] (3.5)

Using (3.5), we obtain:

\[
\mu(JX)\tilde{g}(A_\xi Y, A_\xi Y) = -\mu(X)\tilde{g}(J(A_\xi Y), A_\xi Y) = 0,
\]

\[
\forall X, Y \in TF^{2m}, \quad \forall \xi \in T^\perp F^{2m}.
\] (3.6)

Since \(b \neq 0\) then there exists nondegenerate vector field \(\xi \in T^\perp F^{2m}\), and from (3.6) we come to the equality:

\[
\mu(X) = 0 \quad \forall X \in TF^{2m}.
\]

Then 1-form \(\mu \equiv 0\) and, therefore,

\[
(\nabla_X A)_\xi = 0, \quad \forall X \in TF^{2m}, \quad \forall \xi \in T^\perp F^{2m}.
\] (3.7)

Hence, because of (2.14), we obtain the conclusion of the theorem.

Proof of theorem 2.

Form (1.3) we obtain:

\[
\nabla_W R(X, Y, Z, V) = \tilde{g}(\nabla_W b)(X, V), b(Y, Z)) + \tilde{g}(b(X, V), (\nabla_W b)(Y, Z)) -
\]

\[-\tilde{g}((\nabla_W b)(X, Z), b(Y, V)) - \tilde{g}(b(X, Z), (\nabla_W b)(Y, V)) \quad \forall X, Y, Z, V, W \in TF^{2m}.
\]

Therefore, because of (3.7), \(\nabla R \equiv 0\). Theorem is proved.

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