Matrix Bispectrality of Full Rank One Algebras

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Abstract

We study algebraic properties of full rank 1 algebras in a general framework and derive a method to verify if one such matrix polynomial sub-algebra is bispectral. We give two examples illustrating the method. In the first one, we consider the eigenvalue to be scalar-valued, whereas in the second one, we assume it to be matrix-valued. In the former example we put forth a Pierce decomposition of that algebra.

As a byproduct, we answer positively a conjecture of F. A. Grünbaum concerning certain noncommutative matrix algebras associated to the bispectral problem.

Key words: full rank 1 algebras, bispectral algebras, bispectral triple.

1 Introduction

Classical orthogonal polynomials as many important special functions satisfy remarkable relations both in the physical as well as in the spectral variables [DG04]. More precisely, they are eigenfunctions of an operator in the physical variable (say $x$) with eigenvalues depending on the spectral variable (say $z$) as well as the other way around, eigenfunctions of an operator in $z$ with $x$-dependent eigenvalues. Such bispectral property was explored in the scalar case in the work of J. J. Duistermaat and F. A. Grünbaum [DG86]. It turned out to have deep connections with many problems in Mathematical Physics. Indeed, it could be arranged in suitable manifolds which were naturally parameterized by the flows of the Korteweg de-Vries (KdV) hierarchy or its master-symmetries [ZM91, ZVS00]. It led to generalizations associated to the Kadomtsev-Petviashvili (KP) hierarchy [Zub92b, Wil93, Hor02, Ili99].

This article concerns a promising generalization of the original bispectral problem. More precisely, we consider the triples $(L, \psi, B)$ satisfying systems of equations

\[ L\psi(x, z) = \psi(x, z)F(z) \quad (\psi B)(x, z) = \theta(x)\psi(x, z) \]  \hspace{1cm} (1)

with $L = L(x, \partial_x), B = B(z, \partial_z)$ linear matrix differential operators, i.e., $L\psi = \sum_{i=0}^{l} a_i(x) \cdot \partial_i x \psi, \psi B = \sum_{j=0}^{m} \partial_j z \psi \cdot b_j(z)$. The functions $a_i, b_j, F, \theta$ and the nontrivial common eigenfunction $\psi$ are in principle compatible sized matrix valued functions.

A triple $(L, \psi, B)$ satisfying (2) is called a bispectral triple. We remark that all the differential operators are considered in a neighborhood of an arbitrary given point and following [DG86] we assume that the functions are smooth enough so that all the derivatives considered make sense.

Now we fix the normalized \footnote{If $L = L(x, \partial_x), L = \sum_{i=0}^{l} a_i(x) \partial_i x$ with $a_l$ constant and scalar, $a_{l-1} = 0$, then $L$ is called normalized.} operator $L$ and the eigenfunctions $\psi(x, z)$. We are interested in the bispectral pairs associated to $L = L(x, \partial_x)$, i.e., operators $B = B(z, \partial_z)$ such that $(\psi B)(x, z) = \theta(x)\psi(x, z)$ for some
function $\theta = \theta(x)$. It is not hard to verify that the set of operators $B = B(z, \partial_z)$ satisfying (2) generates a noncommutative algebra of operators.

The main goal of this article is to establish a method to verify whether an algebra of matrix polynomials is bispectral or not. For a given scalar eigenvalue function the corresponding algebra of matrix eigenvalues is characterized. Moreover, the isomorphism between the matrix eigenvalues and the corresponding operator is given explicitly. These results may be used to give positive answers to the three conjectures in [Grü14].

The noncommutative (or matrix) version of the bispectral problem was first studied in [Zub90, Zub92a, Zub92c] for the situation where both the physical and spectral operators were acting on the same side of the eigenfunction and the eigenvalues are scalar valued. Later on, several generalizations were considered. See [SZ01, Kas15, BL08, GI03, Grü14, GHY15] and references therein. This article follows up on the possibility of having the physical and the spectral operators acting on different sides. We also follow the suggestion in [Grü14] of considering both eigenvalues as matrix valued.

The plan of this article is as follows: In Section 2, we develop some preliminary theoretical results which allow us to determine if an algebra is bispectral and a subalgebra of the matrix polynomial algebra. In Section 3 we apply these results to the three algebras presented in [Grü14] to prove that they are bispectral. In Subsection 3.1 the function $F(z)$ is scalar valued and $\theta(x)$ is matrix valued, while in the Subsection 3.2 both are matrix valued.

In the Appendix, we prove the noncommutative Ad-Condition which implies that the bispectral algebras are subalgebras of the matrix polynomial algebra. Furthermore, some computations were performed with the software Singular and Maxima.

## 2 General Results

We consider the triples $(L, \psi, B)$ satisfying systems of equations

\[
L\psi(x, z) = \psi(x, z)F(z) \quad (\psi B)(x, z) = \theta(x)\psi(x, z)
\]  

(2)

with $L = L(x, \partial_x)$, $B = B(z, \partial_z)$ linear matrix differential operators, i.e., $L\psi = \sum_{i=0}^{m} a_i(x) \cdot \partial_x^i \psi$, $\psi B = \sum_{j=0}^{n} \partial_z^j \psi \cdot b_j(z)$. The functions $a_i : U \subset \mathbb{C} \rightarrow M_N(\mathbb{C}), b_j : V \subset \mathbb{C} \rightarrow M_N(\mathbb{C}), F : V \subset \mathbb{C} \rightarrow M_N(\mathbb{C}), \theta : U \subset \mathbb{C} \rightarrow M_N(\mathbb{C})$ and the nontrivial common eigenfunction $\psi : U \times V \subset \mathbb{C}^2 \rightarrow M_N(\mathbb{C})$ are in principle compatible sized meromorphic matrix valued functions defined in suitable open subsets $U, V \subset \mathbb{C}$.

A triple $(L, \psi, B)$ satisfying (2) is called a bispectral triple.

Now we fix the normalized operator $L$ and the eigenfunctions $\psi(\cdot, z)$. We are interested in the bispectral pairs associated to $L = L(x, \partial_x)$, i.e., operators $B = B(z, \partial_z)$ such that $(\psi B)(x, z) = \theta(x)\psi(x, z)$ for some function $\theta = \theta(x)$. It is not hard to verify that the set of operators $B = B(z, \partial_z)$ satisfying (2) generates a noncommutative algebra of operators.

We first note that $\theta$ satisfying Equation (2) has to be an element of the algebra of polynomials with $N \times N$ matrix coefficients, which we denote by $M_N(\mathbb{C})[x]$. The proof follows closely an argument in the original paper of [DG86]. See the Appendix A.

Clearly the set

\[
\mathcal{A} = \{ \theta \in M_N(\mathbb{C})[x] \mid \exists B = B(z, \partial_z), (\psi B)(x, z) = \theta(x)\psi(x, z) \}
\]  

(3)

is a noncommutative $\mathbb{C}$-algebra.

We shall start with some theoretical results.

**Definition 1.** Define $Bp(z, L)$ to be the set of bispectral partners to $L$, i.e.,

\[
Bp(z, L) = \{ B = B(z, \partial_z) \mid \exists \theta \in M_N(\mathbb{C}[x]), (\psi B)(x, z) = \theta(x)\psi(x, z) \}.
\]
A straightforward consequence of the definition is the following.

**Lemma 1.** The set $B_p(z, L)$ is a $C$-algebra.

However, much more can be said about the properties of the algebra $B_p(z, L)$ in the case that will be studied in the sequel. For that we have to consider the following important class of algebras:

**Definition 2.** Let $K$ be a field, $C$ be a graded $K$-algebra, we define a full rank 1 algebra to be a subalgebra $A \subset C$ such that

$$A = E \oplus \bigoplus_{j=k_0}^\infty C_j$$

for some finite dimensional $K$-vector space $E$ and $k_0 \in \mathbb{N}$. Furthermore, we denote by $\{e_i^{[j]}\}_{1 \leq i \leq N, j \geq 0}$ some basis for $C_j$. See [Cas17].

**Remark 1.** Note that for a full rank 1 algebra $A \subset C$, we consider the smallest positive integer such that $C_j \subset A$, for all $j \geq k_0$. For this $k_0$ we can write $E = \left( \bigoplus_{j=0}^{k_0-1} C_j \right) \cap A$.

The results in Theorems 1, 2 and 3 will be used in the sequel to provide a positive answer to the conjectures of Grunbaum [Grü14]. They are of interest on their own.

**Theorem 1.** Let $C$ be a graded $K$-algebra where $\dim_K C_j = N < \infty$, $C_j = \sum_{i=1}^N K \cdot e_i^{[j]}$. Suppose that for every $t$, $1 \leq t \leq N$, $i, j \in \mathbb{N}$, there exist $1 \leq r, s \leq N$ such that $e_i^{[j+t]} = e_i^{[j]} e_r^{[j]} e_s^{[j]}$ and $A \subset C$ is a full rank 1 algebra. Then, $A$ is a finitely generated $K$-algebra.

**Proof.** We write

$$A = E \oplus \bigoplus_{j=k_0}^\infty C_j.$$

Since $E$ is a finite dimensional $K$-vector space $E$, we can consider a basis $\{\alpha_1, ..., \alpha_m\}$ for $E$ and write $E = \sum_{s=1}^m K \cdot \alpha_s$. Define

$$A_0 := K \cdot \langle e_i^{[k]}, \alpha_s \mid 1 \leq i \leq N, k_0 \leq k \leq 2k_0 - 1, 1 \leq s \leq m \rangle.$$

We claim that $A = A_0$.

First of all, we prove that for every $q \geq 2$, $(q - 1)k_0 \leq k \leq qk_0 - 1$, $e_i^{[k]} \in A_0$. The initial step is clear for $q = 2$. Assume that $e_i^{[p]} \in A_0$ for $(q - 1)k_0 \leq p \leq qk_0 - 1$, $1 \leq i \leq N$ and note that $qk_0 \leq k \leq (q + 1)k_0 - 1$ implies $(q - 1)k_0 \leq k - k_0 \leq qk_0 - 1$ and $e_i^{[k-k_0]} \in A_0$ for $1 \leq i \leq N$. Consider $1 \leq i \leq N$, by hypothesis there exists $1 \leq r, s \leq N$ such that $e_i^{[k]} = e_r^{[k-k_0]} \cdot e_s^{k_0} \in A_0$. This proves the inductive step. The assertion follows by induction.
Since $k_0 + \mathbb{N} = \bigcup_{q=2}^{\infty} \{ k \in \mathbb{N} \mid (q-1)k_0 \leq k \leq qk_0 - 1 \}$. We have that $e_i^{[k]} \in A_0$ for $1 \leq i \leq N$, $k \geq k_0$ then $\bigoplus_{j=k_0}^{\infty} C_j \subset A_0$. But $E = \sum_{s=1}^{m} K \cdot a_s \subset A_0$. Thus, $A = A_0$ and $A$ is a finitely generated $k$-algebra.

**Remark 2.** The converse is not true. Consider for example the graded algebra $C = M_N(\mathbb{K}[x])$ and $A = \mathbb{K}[x]$, then $A$ is a finitely generated $\mathbb{K}$-algebra which is not of full rank 1.

Now we use the following theorem whose proof may be found in [Sta85].

**Theorem 2 (Stafford).** Let $R \subset S$ be algebras over a central field $\mathbb{K}$ such that $S$ is Noetherian and $S/R$ is a finite dimensional $\mathbb{K}$-vector space. Then, $R$ is Noetherian.

**Corollary 1.** Let $C$ be a Noetherian graded algebra and $A \subset C$ be a full rank 1 $\mathbb{K}$-algebra over a central field $\mathbb{K}$. Then, $A$ is Noetherian.

**Proof.** Since $A = E \oplus \bigoplus_{j=k_0}^{\infty} C_j$ for some finite dimensional vector space $E$, we can consider the complement of any subspace $F$ with respect to $\bigoplus_{j=0}^{k_0-1} C_j$ and obtain $C = F \oplus A$ then $\dim_{\mathbb{K}}(C/A) = \dim_{\mathbb{K}}(F) < \infty$. Since $C$ is Noetherian the previous theorem implies the assertion. $\Box$

**Definition 3.** Let $\mathbb{K}$ be a field, $C$ be a $\mathbb{K}$-algebra and $S \subset C$. We define

$$\mathbb{K} \cdot <S> = \text{span} \left\{ \prod_{j=1}^{n} s_j \mid s_1, ..., s_n \in S, n \in \mathbb{N} \right\}.$$

The following theorem connects the bispectral property to full rank 1 algebras.

**Theorem 3 (Full Rank One Algebras).** Let $C$ be a graded $\mathbb{K}$-algebra, $\Gamma \subset C$ and $A \subset C$ two $\mathbb{K}$-algebras with the following properties:

1. $\Gamma$ is a full rank 1 algebra, with decomposition $\Gamma = E \oplus \bigoplus_{j=k_0}^{\infty} C_j$, for some $k_0 \in \mathbb{N}$.
2. $\mathbb{K} \cdot \langle E \rangle = \Gamma$.
3. $A \cap \bigoplus_{k=0}^{k_0-1} C_k = E$.

Then, $\Gamma = A$.

**Proof.** We shall break the proof in 2 steps.

Step 1: The inclusion $\Gamma \subset A$.

Using (2) and (3) we have $E \subset A$ and $\Gamma$ is the algebra generated by $E$, since $A$ is an algebra we obtain the
Proof. Since $e \leq 1$ therefore Not that $2 \leq 1$. Then, $x \in N$. We recall that for $S$ in particular $\theta = \theta_1 + \theta_2 = A \cap \bigoplus_{k=0}^{\infty} C_k$ we have that $\theta_2 \in \Gamma \subseteq A$. In particular $\theta = \theta_1 + \theta_2 \in A \cap \bigoplus_{k=0}^{\infty} C_k = E \subseteq \Gamma$, then $\theta = \theta_1 + \theta_2 \in \Gamma$. ♦

Definition 4. The shift operator $S_N \in M_N(K[x])$ is defined by

$$S_N = \sum_{s=1}^{N-1} e_{s,s+1}$$

for $N \geq 2$, where as usual $e_{r,s}$ denotes the matrix with 1 at entry $(r,s)$ and zeros elsewhere.

We recall that for $N \geq 2$

$$S_N^j = \begin{cases} \sum_{s=1}^{N-j} e_{s,s+j} & \text{if } 0 \leq j \leq N-1, \\ 0 & \text{if } j \geq N. \end{cases}$$

In particular $S_N$ is nilpotent of degree $N$.

The following theorem gives us a concrete example of a nontrivial full rank 1 algebra.

Theorem 4. Let $N \in \mathbb{Z}_+$ and the following elements in $M_N(K[x])$:

$$a_0 = S_N,$$

$$a_1 = Ix + (-1)^N e_{N1} x^N,$$

$$a_k = e_{1N} x^k, \text{if } 2 \leq k \leq N-1,$$

$$\beta_k = e_{kk} x^N + (-1)^N e_{N1} x^{2N-1}, \text{if } 1 \leq k \leq N.$$

Then, $x^{2N} M_N(K[x])$ is contained in the subalgebra $A$ of $M_N(K[x])$ that is generated by $a_j, \beta_k$, $0 \leq j \leq N-1$, $1 \leq k \leq N$.

Proof. Since $\beta_k^2 = e_{kk} x^{2N}$ for $2 \leq k \leq N-1$ and $a_n^2 = Ix^n + (-1)^n e_{N1} x^{n+N-1}$ for $n \geq 1$ we have $\beta_k a_1^n = e_{kk} x^{n+2N} \in A$ for $n \geq 1$, in other words $e_{kk} x^n \in A$ for $n \geq 2N$. On the other hand, $a_0^{N-1} a_1 a_0^{N-1} = (-1)^N e_{N1} x^{n+N-1} A$ for $n \geq 1$, hence $e_{1N} x^n \in A$ for $n \geq N$. However, $e_{1N} x^k \in A$ for $2 \leq k \leq N-1$, therefore $e_{1N} x^n \in A$ for $n \geq 2$.

Note that $a_2 a_1^n = e_{1N} x^n + (-1)^N e_{11} x^{n+N-1} \in A$, $a_1^n a_2 = e_{1N} x^n + (-1)^N e_{N1} x^{N+n-1} \in A$. Then $e_{11} x^n \in A$ and $e_{NN} x^n \in A$ for $n \geq N + 1$. This implies that $\beta_1 a_1^n = (e_{11} x^n + (-1)^N e_{N1} x^{2N-1})(Ix^n + (-1)^N e_{N1} x^{N+n-1}) = e_{11} x^n + (-1)^N e_{N1} x^{2N+n-1} \in A$ for $n \geq 1$. Thus, $e_{N1} x^n \in A$ for $n \geq 2N$. 

The previous proposition implies \( a_j^i = \sum_{s=1}^{N-j} e_{s,s+j} \) for \( 0 \leq j \leq N - 1 \) then \( a_0^{N-j}(e_{N1}x^n)a_0^{-1} = e_{ij}x^n \in A \) for \( 1 \leq i, j \leq N, n \geq 2N \) and this proves the assertion. \( \Box \)

**Corollary 2.** The algebra \( A \) is full rank 1

**Proof.** Note that \( A = E \oplus x^{2N}M_N(\mathbb{K}[x]) \) with \( E = A \cap \oplus_{j=0}^{2N-1}M_N(\mathbb{K}[x]). \)

In the next section we give a family of algebras whose bispectrality can be obtained using the Theorem 3.

### 3 The Examples

We begin with the example of the matrix algebra given in the paper [Grü14]. There the algebra considered is the set of polynomials of the form

\[
\theta(x) = \left( \begin{array}{cc} r_{01} & r_{02} \\ 0 & r_{11} \end{array} \right) + \left( \begin{array}{cc} r_{11} & r_{12} \\ 0 & r_{11} \end{array} \right) x + \left( \begin{array}{cc} r_{11} & r_{12} \\ r_{22} & r_{22} \end{array} \right) x^2 + \left( \begin{array}{cc} r_{11} & r_{12} \\ r_{22} & r_{22} \end{array} \right) x^3 + x^4 p(x)
\]

(4)

where \( p \in M_2(\mathbb{C})[x] \) and all the variables \( r_{01}, r_{02}, r_{11}, r_{12}, r_{22}, r_{22}, r_{11}, r_{12} \in \mathbb{C} \). Note that this algebra is full rank one and the relations that must be determined to obtain the complete description of the algebra are in the monomials of degree less than or equal to three. In the following subsection, we generalize this algebra to an arbitrary size of matrix \( N \) and find the relations that determine them.

#### 3.1 Family of Algebras Linked to a Nilpotent Element in \( M_N(\mathbb{K}) \)

As a particular example, we consider a nilpotent element \( S \in M_N(\mathbb{K}) \) of degree \( D \geq 2 \), consider the matrix valued function

\[
\psi(x, z) = e^{xz} \left( I_z + \sum_{m=1}^{D} (-1)^m S^{m-1} x^{-m} \right),
\]

and note that \( L\psi(x, z) = -z^2\psi(x, z) \) for the ordinary differential operator

\[
L = -\partial_x^2 + 2 \sum_{m=1}^{D} (-1)^{m+1} m S^{m-1} x^{-m-1}.
\]

Moreover, if \( m \) is even we have:

\[
(\text{ad}L)^m(\theta) = (-1)^{m/2} 2^m (z^2)^{m/2} \psi b_m = (-1)^{m/2} m S^{m-1} L^{m/2} \psi b_m = \left( (-1)^{m/2} 2^m (L^{m/2} \cdot b_m) \right) \psi.
\]

Therefore,

\[
\left( (\text{ad}L)^m(\theta) - \left( (-1)^{m/2} m (L^{m/2} \cdot b_m) \right) \psi \right) = 0.
\]

However, the operator \( (\text{ad}L)^m(\theta) - \left( (-1)^{m/2} m (L^{m/2} \cdot b_m) \right) \) is independent of \( z \) and its kernel contains the infinite dimensional linearly independent set \( \{ \psi(\cdot, z) \} \in \mathbb{C} \). Thus, the operator is zero and \( (\text{ad}L)^m(\theta) = (-1)^{m/2} m (L^{m/2} \cdot b_m) \).

Now we characterize the algebra \( \mathcal{A} = \{ \theta \in M_N(\mathbb{K}[x]): \exists B(z, \partial_z), (\psi B)(x, z) = \theta(x) \psi(x, z) \} \) for this particular example. We begin with the definition of the family \( \mathcal{P} = \{ P_k \}_{k \in \mathbb{N}} \) which will be used to describe the map \( \theta \mapsto \mathcal{B} \) such that \( (\psi B)(x, z) = \theta(x) \psi(x, z) \).
Definition 5. For \( k \in \mathbb{N} \) and \( \theta \in M_N(\mathbb{K}[x]) \), we define

\[
P_k(\theta) = \frac{\theta^{(k+1)}(0)}{k!} - \sum_{j=k+2}^{k+D} (-1)^k S^{k-j} \left[ \frac{\theta^{(j)}(0)}{j!}, S^{j-k-1} \right]. \tag{5}
\]

The family \( \mathcal{P} = \{P_k\}_{k \in \mathbb{N}} \) can be used to describe the algebra prescribed by Equation \((4)\). If \( \theta(x) = \sum_{k=0}^{\infty} a_k x^k \in M_2(\mathbb{C}[x]) \) for \( m \geq 4 \) satisfies \( P_0(x \theta(x)) = P_0(\theta(0)x), P_0(x^2 \theta(x)) = 0 \) for \( j \geq 2 \),

\[
\sum_{k=0}^{q} (-1)^k S^{k-q+1} P_k(\theta) = 0, \text{ for } 0 \leq q \leq 1.
\]

Then, \([S_2, a_0] = 0, [S_2, a_1] = [S_2, a_0], S_2 a_2 S_2 = S_2 a_1 \), and \( S_2 a_3 S_2 = a_2 S_2 + S_2 a_2 - a_1 \). Writing

\[
a_k = \begin{pmatrix} r_1^{(1)} & r_2^{(1)} \\ r_1^{(2)} & r_2^{(2)} \end{pmatrix},
\]

we have that \( r_0^{(2)} = 0, r_0^{(1)} = r_1^{(1)}, r_2^{(2)} = 0, r_2^{(1)} = r_1^{(1)} = r_2^{(1)} \) and \( r_3^{(1)} = r_2^{(2)} + r_1^{(2)} - r_2^{(1)} \). These equations are exactly those that describe the algebra of the form of Equation \((4)\) as a sub-algebra of \( M_2(\mathbb{C}[x]) \).

We show now some properties of the family \( \mathcal{P} := \{P_k\}_{k \in \mathbb{N}} \).

Lemma 2. For every \( \theta \in M_N(\mathbb{K}[x]) \),

\[
P_k(\theta) = P_0 \left( \frac{\theta^{(k+1)}(0)}{k!} x - \sum_{r=2}^{D} \frac{\theta^{(r+k)}(0)}{(r+k)!} x^r \right). \tag{6}
\]

Proof. In fact,

\[
P_0 \left( \frac{\theta^{(k+1)}(0)}{k!} x - \sum_{r=2}^{D} \frac{\theta^{(r+k)}(0)}{(r+k)!} x^r \right) = \frac{\theta^{(k+1)}(0)}{k!} - \sum_{r=2}^{D} (-1)^r \left[ \frac{\theta^{(r+k)}(0)}{(r+k)!}, S^{r-1} \right]
\]

\[
= \frac{\theta^{(k+1)}(0)}{k!} - \sum_{j=k+2}^{k+D} (-1)^j \left[ \frac{\theta^{(j)}(0)}{j!}, S^{j-k-1} \right] = P_k(\theta).
\]

The previous lemma allows us to study the properties of the family \( \mathcal{P} = \{P_k\}_{k \in \mathbb{N}} \) through \( P_0 \).

Lemma 3 (Product Formula for \( P_0 \)). If \( \theta_1, \theta_2 \in M_N(\mathbb{K}[x]) \) then,

\[
P_0(\theta_1 \theta_2) = \sum_{s=0}^{D} \left\{ P_0(x^s \theta_1(x)) \frac{\theta_2^{(s)}(0)}{s!} + \frac{\theta_1^{(s)}(0)}{s!} P_0(x^s \theta_2(x)) \right\} - (\theta_1 \theta_2)'(0).
\]

Proof. By the definition of \( P_0 \),

\[
P_0(\theta_1 \theta_2) = (\theta_1 \theta_2)'(0) - \sum_{r=2}^{D} (-1)^r \left[ \frac{\theta_1 \theta_2}'(0)}{r!}, S^{r-1} \right]
\]

\[
= \theta_1'(0) \theta_2(0) + \theta_1(0) \theta_2'(0) - \sum_{r=2}^{D} (-1)^r \left[ \sum_{t=0}^{r} \frac{\theta_1^{(t)}(0)}{t!} \frac{\theta_2^{(r-t)}(0)}{(r-t)!}, S^{r-1} \right]
\]

\[
= \theta_1'(0) \theta_2(0) + \theta_1(0) \theta_2'(0) - \sum_{r=2}^{D} \sum_{t=0}^{r} (-1)^r \left[ \frac{\theta_1^{(t)}(0)}{t!}, S^{t-1} \right] \frac{\theta_2^{(r-t)}(0)}{(r-t)!}
\]

\[
= \sum_{s=0}^{D} \left\{ P_0(x^s \theta_1(x)) \frac{\theta_2^{(s)}(0)}{s!} + \frac{\theta_1^{(s)}(0)}{s!} P_0(x^s \theta_2(x)) \right\} - (\theta_1 \theta_2)'(0).
\]
On the other hand,

\[-\sum_{r=2}^{\infty} \sum_{t=0}^{r-1} (-1)^r \frac{\theta_1^{(r-t)}(0)}{(r-t)!} \left[ \frac{\theta_1^{(t)}(0)}{t!} , S^{t-1} \right] = \theta_2'(0) \theta_2(0) + \theta_1(0) \theta_2'(0) - \sum_{r=2}^{\infty} (-1)^r \left[ \frac{\theta_1^{(r)}(0)}{r!} , S^{r-1} \right] \theta_2(0)\]

\[-\sum_{r=2}^{\infty} (-1)^r \theta_1(0) \left[ \frac{\theta_2^{(r)}(0)}{r!} , S^{r-1} \right] = -\sum_{r=2}^{\infty} (-1)^r \theta_1(0) \left[ \frac{\theta_2^{(r-1)}(0)}{(r-1)!} , S^{r-1} \right] \theta_2(0)\]

\[= \left( \theta_1(0) - \sum_{r=2}^{\infty} (-1)^r \theta_1(0) \left[ \frac{\theta_2^{(r)}(0)}{r!} , S^{r-1} \right] \right) \theta_2(0) + \theta_1(0) \left( \theta_2'(0) - \sum_{r=2}^{\infty} (-1)^r \left[ \frac{\theta_1^{(r)}(0)}{r!} , S^{r-1} \right] \right)\]

\[-\sum_{r=2}^{\infty} (-1)^r \theta_2^{(r-t)}(0) \left( \frac{\theta_2^{(t)}(0)}{t!} , S^{t-1} \right) = -\sum_{r=2}^{\infty} (-1)^r \theta_2^{(r-t)}(0) \left( \frac{\theta_2^{(t)}(0)}{t!} , S^{t-1} \right)\]

\[= P_0(\theta_1) \theta_2(0) + \theta_1(0) P_0(\theta_2) - \sum_{r=2}^{\infty} (-1)^r \left( \theta_1(0) , S^{r-1} \right) \frac{\theta_2^{(r)}(0)}{r!} + \frac{\theta_1^{(r)}(0)}{r!} \left[ \theta_2(0) , S^{r-1} \right]\]

However,

\[\sum_{r=2}^{\infty} (-1)^r \left( \frac{\theta_1^{(r)}(0)}{r!} , S^{r-1} \right) = \frac{1}{(r-1)!} \frac{\theta_2^{(r-1)}(0)}{(r-1)!} + \frac{\theta_2^{(r-1)}(0)}{(r-1)!} \left[ \frac{\theta_2^{(0)}(0)}{0!} , S^{0-1} \right]\]

\[= P_0(\theta_1) \theta_2(0) + \theta_1(0) P_0(\theta_2) - \sum_{r=2}^{\infty} (-1)^r \left( \theta_1(0) , S^{r-1} \right) \frac{\theta_2^{(r)}(0)}{r!} + \frac{\theta_1^{(r)}(0)}{r!} \left[ \theta_2(0) , S^{r-1} \right]\]

On the other hand,

\[\sum_{r=2}^{\infty} (-1)^r \left( \frac{\theta_1^{(r)}(0)}{r!} , S^{r-1} \right) = (-1)^D \left( \theta_1(0) , S^{D-1} \right) \frac{\theta_2^{(D)}(0)}{D!} + \frac{\theta_1^{(D)}(0)}{D!} \left[ \theta_2(0) , S^{D-1} \right]\]

\[+ \sum_{r=2}^{\infty} (-1)^r \left( \theta_1(0) , S^{r-1} \right) \frac{\theta_2^{(r)}(0)}{r!} + \frac{\theta_1^{(r)}(0)}{r!} \left[ \theta_2(0) , S^{r-1} \right]\]
Lemma 4. Then, if $\theta_1(0), S^{D-1} \sum_{s=1}^{D-1} \left( P_0(\theta_1(0)x^s) \frac{\theta_2(s)(0)}{s!} + \frac{\theta_2(s)(0)}{s!} P_0(\theta_2(0)x^s) \right) + (\theta_1\theta_2)'(0)$.

Then,

$$P_0(\theta_1\theta_2) = P_0(\theta_1\theta_2(0)) + \theta_1(0)P_0(\theta_2) - (\theta_1\theta_2)'(0)$$

Corollary 4. If $\theta_1(0) = 0$, then

$$P_0(\theta_1\theta_2) = P_0(\theta_1\theta_2(0)) - \theta_1'(0)P_0(\theta_2(0)x) + \sum_{s=1}^{D} \left( P_0(x^s\theta_1(x)) \frac{\theta_2(s)(0)}{s!} + \frac{\theta_2(s)(0)}{s!} P_0(x^s\theta_2(x)) \right).$$

Corollary 5. If $\theta_1(0) = \theta_2(0) = 0$, then

$$P_0(\theta_1\theta_2) = \sum_{s=1}^{D} \left( P_0(x^s\theta_1(x)) \frac{\theta_2(s)(0)}{s!} + \frac{\theta_2(s)(0)}{s!} P_0(x^s\theta_2(x)) \right).$$

Theorem 5 (Product Formula for $P_k$). If $\theta_1, \theta_2 \in M_N(\mathbb{K}[x])$, then

$$P_k(\theta_1\theta_2) = \sum_{l=0}^{k+D} \left( P_k(x^l\theta_1(x)) \frac{\theta_2(l)(0)}{t!} + \frac{\theta_2(l)(0)}{t!} P_k(x^l\theta_2(x)) \right) - \frac{(\theta_1\theta_2)^{(k+1)}(0)}{k!}.$$
Lemma 5 (Translation). For every \( \theta \in M_N(\mathbb{K}[x]) \), \( k \geq 0 \), we have that

\[
P_k(\theta) = \begin{cases} 
  P_k(\theta^{-1+t}(0)x^{k+1}) + P_{k-t}(\theta) - \frac{\theta^{(k+1-t)}(0)}{(k-t)!} & \text{if } 0 \leq t \leq k, \\
  P_k(\theta(0)x^{k+1}) + P_0(x(\theta(x) - \theta(0))) & \text{if } t = k + 1, \\
  P_0(x^{t-k}\theta(x)) & \text{if } t \geq k + 2.
\end{cases}
\] (7)

Proof. The proof is a straightforward computation.

We shall now provide the aforementioned description of the Algebra \( \mathcal{A} \).

Theorem 6. Let

\[ \Gamma := \left\{ \theta \in M_N(\mathbb{K}[x]) \mid P_0(x\theta(x)) = P_0(\theta(0)x), P_0(x^j\theta(x)) = 0, j \geq 2, \right\} \]

Then, \( \Gamma = \mathcal{A} \).

Moreover, for each \( \theta \) we have an explicit expression for the operator \( B \).

Before proving the theorem, we study the relations defining the algebra \( \Gamma \). They are given in the following result:

Proposition 1. The algebra \( \Gamma \) is the subset of \( \theta \in M_N(\mathbb{K}[x]) \) such that

\[
\sum_{k=0}^{q} (-1)^k S^{k+D-q-1} P_k(\theta) = 0, 0 \leq q \leq D - 1.
\]

(8)

(9)

for \( 0 \leq q \leq D - 1 \).

Proof. We notice that \( \Gamma \) is defined by two relations:

\[
P_0(x\theta(x)) = P_0(\theta(0)x), P_0(x^j\theta(x)) = 0, j \geq 2
\]

(10)

and Equation (9) (after a trivial change of the summation variable). Equation (10) is equivalent to

\[
\sum_{r=D-q}^{D} (-1)^r S^{r-1} \frac{\theta^{(r-D+q)}(0)}{(r-D+q)!} = 0,
\]

for \( 0 \leq q \leq D - 1 \). If \( q = D - 1 \), then

\[
0 = \sum_{r=2}^{D} (-1)^r S^{r-1} \frac{\theta^{(r-1)}(0)}{(r-1)!} = P_0 \left( \sum_{r=2}^{D} \frac{\theta^{(r-1)}(0)}{(r-1)!} x^r \right)
\]
= P_0 \left( \sum_{r=1}^{D-1} \frac{\theta^{(r)}(0)}{r!} x^{r+1} \right) = P_0(x(\theta(x) - \theta(0))).

In other words, P_0(x\theta(x)) = P_0(x\theta(0)).

If 0 \leq q \leq D - 2, then 2 \leq D - q and

\begin{align*}
0 &= \sum_{r=D-q}^{D} (-1)^{r} \left[ \sum_{j=0}^{q} \frac{\theta^{(j)}(0)}{j!} x^{j+D-q} \right] \\
&= P_0 \left( \sum_{j=0}^{q} \frac{\theta^{(j)}(0)}{j!} x^{j+D-q} \right) = P_0 \left( x^{D-q} \sum_{j=0}^{q} \frac{\theta^{(j)}(0)}{j!} x^j \right) = P_0(x^{D-q}\theta(x))
\end{align*}

for 0 \leq q \leq D - 2. Thus, P_0(x^j\theta(x)) = 0 for j \geq 2.

Now let us prove the theorem.

**Proof.** We shall break the proof in different steps.

**Step 1:** The set \( \Gamma \) is an algebra.

Clearly, \( \Gamma \) is a vector space since \( P_k \) is linear for all \( 0 \leq k \leq D - 1 \).

If \( \theta_1, \theta_2 \in \Gamma \), then

\[ P_0(x\theta_1(x)) = P_0(\theta_i(0)x), \quad P_0(x^j\theta_i(x)) = 0, \quad \sum_{k=0}^{q} (-1)^k s^{k+D-q-1} P_k(\theta_i) = 0, \]

for \( j \geq 2, \quad 0 \leq q \leq D - 1, \quad i = 1, 2. \)

Note that, using Corollary 4 and \( P_0(x\theta_1(x))(0) = 0 \), we obtain

\[ P_0(x\theta_1(x)\theta_2(x)) = P_0((x\theta_1(x))\theta_2(x)) = P_0(\theta_1(0)x)\theta_2(0) - \theta_1(0)P_0(\theta_2(0)x) + (x\theta_1)'(0)P_0(\theta_2(0)) \]

\[ = P_0(\theta_1(0)x)\theta_2(0) - \theta_1(0)P_0(\theta_2(0)x) + \theta_1(0)P_0(\theta_2(0)x) = \theta_1(0)\theta_2(0) = P_0(\theta_1(0)\theta_2(0)x). \]

If \( j \geq 2 \), then \( P_0(x^{j-1}\theta_1(x))(0) = 0, \quad P_0(x\theta_2(x))(0) = 0 \). Using Corollary 5 we obtain

\[ P_0(x^j\theta_1(x)\theta_2(x)) = P_0((x^{j-1}\theta_1(x))(x\theta_2(x))) \]

\[ = \sum_{s=1}^{D} \left\{ P_0(x^j s^{-1}\theta_1(x)) \frac{(x\theta_2)'(0)}{s!} + \frac{\theta_1(0)}{s!} P_0(x^{s+1}\theta_2(x)) \right\} = 0 \]

Lemma 5 (Translation) implies

\[ P_k(x^{k+1}\theta_i(x)) = P_k(\theta_i(0)x^{k+1}), \quad P_k(x^j\theta_i(x)) = P_0(x^{j-k}\theta_i(x)) = 0, \]

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for \( t \geq k + 2, \ i = 1, 2 \).

Using Theorem 5 (Product Formula for \( P_k \)) we have:

\[
P_k(\theta_1 \theta_2) = \sum_{t=0}^{k+D} \left\{ P_k(x^t \theta_1(x)) \frac{\theta_2^{(t)}(0)}{t!} + \frac{\theta_1^{(t)}(0)}{t!} P_k(x^t \theta_2(x)) \right\} - \frac{(\theta_1 \theta_2)^{(k+1)}(0)}{k!} \\
= \sum_{t=0}^{k} \left\{ P_k(x^t \theta_1(x)) \frac{\theta_2^{(t)}(0)}{t!} + \frac{\theta_1^{(t)}(0)}{t!} P_k(x^t \theta_2(x)) \right\} + P_k(\theta_1(0)x^{k+1} \theta_2^{(k+1)}(0)/(k+1)!) \\
+ \frac{\theta_1^{(k+1)}(0)}{(k+1)!} P_k(\theta_2(0)x^{k+1}) - \frac{(\theta_1 \theta_2)^{(k+1)}(0)}{k!} \\
= P_k(\theta_1) \theta_2(0) + \theta_2(0) P_k(\theta_2) \\
+ \sum_{t=0}^{k} \left\{ P_k(x^t \theta_1(x)) \frac{\theta_2^{(t)}(0)}{t!} + \frac{\theta_1^{(t)}(0)}{t!} P_k(x^t \theta_2(x)) - (k+1) \frac{\theta_1^{(t)}(0)}{t!} \frac{\theta_2^{(k+1-t)}(0)}{(k+1-t)!} \right\}.
\]

Thus, for \( 0 \leq q \leq D - 1 \) we have

\[
\sum_{k=0}^{q} (-1)^k S^{k+D-q-1} P_k(\theta_1 \theta_2) = \sum_{k=0}^{q} (-1)^k S^{k+D-q-1} \left[ P_k(\theta_1) \theta_2(0) + \theta_2(0) P_k(\theta_2) \\
+ \sum_{t=0}^{k} \left\{ P_k(x^t \theta_1(x)) \frac{\theta_2^{(t)}(0)}{t!} + \frac{\theta_1^{(t)}(0)}{t!} P_k(x^t \theta_2(x)) - (k+1) \frac{\theta_1^{(t)}(0)}{t!} \frac{\theta_2^{(k+1-t)}(0)}{(k+1-t)!} \right\} \right] \\
= \sum_{k=0}^{q} (-1)^k \left[ S^{k+D-q-1}, \theta_1(0) \right] P_k(\theta_2) \\
+ \sum_{k=0}^{q} \sum_{t=1}^{k} (-1)^k S^{k+D-q-1} \left\{ (k+1) \frac{\theta_1^{(k+1-t)}(0)}{(k+1-t)!} + \frac{\theta_1^{(t)}(0)}{t!} \theta_2^{(k+1-t)}(0)/(k+1-t)! \right\} \frac{\theta_2^{(t)}(0)}{t!} \\
+ \sum_{k=0}^{q} \sum_{t=1}^{k} (-1)^k S^{k+D-q-1} \frac{\theta_1^{(t)}(0)}{t!} \left\{ (k+1) \frac{\theta_2^{(k+1-t)}(0)}{(k+1-t)!} + \frac{\theta_1^{(t)}(0)}{t!} \theta_2^{(k+1-t)}(0)/(k+1-t)! \right\} \\
+ \sum_{k=0}^{q} \sum_{t=1}^{k} (-1)^k S^{k+D-q-1} \frac{\theta_1^{(t)}(0)}{t!} \frac{\theta_2^{(k+1-t)}(0)}{(k+1-t)!} \\
+ \sum_{k=0}^{q} \sum_{t=1}^{k} \frac{\theta_1^{(t)}(0)}{t!} \theta_2^{(k+1-t)}(0)/(k+1-t)! \right] \\
+ \sum_{k=0}^{q} \sum_{t=1}^{k} \frac{\theta_1^{(t)}(0)}{t!} \theta_2^{(k+1-t)}(0)/(k+1-t)!.}
\]
However,

\[
\sum_{k=0}^{q} \sum_{t=1}^{k} (-1)^k S^{k+D-q-1} \frac{\theta_1^{(t)}(0)}{t!} \left\{ P_{k-t}(\theta_1) - \frac{\theta_2^{(k+t-1)}(0)}{(k-t)!} \right\}
\]

\[
= \sum_{s=0}^{q-1} \sum_{k=s+1}^{q} (-1)^k S^{k+D-q-1} \frac{\theta_1^{(s)}(0)}{(k-s)!} \left\{ P_s(\theta_1) - \frac{\theta_2^{(s+1)}(0)}{s!} \right\}
\]

\[
= \sum_{s=0}^{q-1} (-1)^s \sum_{j=1}^{q-s} \frac{\theta_1^{(j)}(0)}{j!} \left\{ P_s(\theta_1) - \frac{\theta_2^{(s+1)}(0)}{s!} \right\}
\]

\[
= \sum_{s=0}^{q-1} (-1)^s \left\{ P_s(\theta_1) - \frac{\theta_2^{(s+1)}(0)}{s!} \right\}. \]

After a few simple calculations this term is equal to:

\[
- \sum_{s=0}^{q-1} (-1)^s \left[ S^{D-q+s-1}, \theta_1(0) \right] P_s(\theta_2) - \sum_{k=0}^{q-1} \sum_{s=0}^{k-1} (-1)^k S^{k+D-q-1} \frac{\theta_1^{(s+1)}(0) \theta_2^{(k-s)}(0)}{(s+1)! (k-s)!}
\]

Similarly, we see that

\[
\sum_{k=0}^{q} \sum_{t=1}^{k} (-1)^k S^{k+D-q-1} \frac{\theta_1^{(t)}(0)}{t!} \left\{ P_{k-t}(\theta_2) - \frac{\theta_1^{(k+t-1)}(0)}{(k-t)!} \right\}
\]

\[
= \sum_{k=0}^{q} \sum_{s=0}^{k-1} (-1)^k S^{k+D-q-1} \frac{\theta_2^{(s+1)}(0) \theta_1^{(k-s)}(0)}{(s+1)! (k-s)!}
\]

Thus,

\[
\sum_{k=0}^{q} (-1)^k S^{k+D-q-1} P_k(\theta_1 \theta_2) = \sum_{s=0}^{q-1} (-1)^s \left[ S^{D-q+s-1}, \theta_1(0) \right] P_s(\theta_2) - \sum_{k=0}^{q-1} \sum_{s=0}^{k-1} (-1)^k S^{k+D-q-1} \frac{\theta_1^{(s+1)}(0) \theta_2^{(k-s)}(0)}{(s+1)! (k-s)!}
\]

\[
- \sum_{s=0}^{q-1} (-1)^s \left[ S^{D-q+s-1}, \theta_1(0) \right] P_s(\theta_2) - \sum_{k=0}^{q-1} \sum_{s=0}^{k-1} (-1)^k S^{k+D-q-1} \frac{\theta_1^{(s+1)}(0) \theta_2^{(k-s)}(0)}{(s+1)! (k-s)!}
\]

\[
= (-1)^q \left[ S^{D-1}, \theta_1(0) \right] P_k(\theta_2) - \sum_{k=0}^{q-1} \sum_{s=0}^{k-1} (-1)^k S^{k+D-q-1} \frac{\theta_1^{(s+1)}(0) \theta_2^{(k-s)}(0)}{(s+1)! (k-s)!}
\]

\[
- \sum_{k=0}^{q-1} \sum_{s=0}^{k-1} (-1)^k S^{k+D-q-1} \frac{\theta_1^{(s+1)}(0) \theta_2^{(k-s)}(0)}{(s+1)! (k-s)!}
\]

\[
+ \sum_{k=0}^{q-1} \sum_{s=0}^{k-1} (-1)^k S^{k+D-q-1} \frac{\theta_1^{(s+1)}(0) \theta_2^{(k-s)}(0)}{(s+1)! (k-s)!}
\]

\[
= (-1)^q \left[ S^{D-1}, \theta_1(0) \right] P_k(\theta_2).
\]
Nevertheless, \([S^{D-1}, \theta_1(0)] = 0\) and we obtain \(\sum_{k=0}^{q} (-1)^k S^{k+D-q-1} P_k(\theta_1 \theta_2) = 0\) for \(0 \leq q \leq D - 1\). Therefore, \(\theta_1 \theta_2 \in \Gamma\). Thus, \(\Gamma\) is an algebra.

Step 2: Since \(\Gamma\) contains the ideal \(x^{2D} M_N(\mathbb{K}[x])\), if we define \(E = \bigoplus_{j=0}^{2D-1} M_N(\mathbb{K}[x]) j \cap \Gamma\), then we have this step.

Step 3: The algebra \(\Gamma\) is generated by \(E\), i.e., \(\mathbb{K} \cdot \langle E \rangle = \Gamma\).

This step follows applying Theorem 4 and since \(E\) contains the elements mentioned in that theorem.

Step 4: The inclusion \(A \cap \bigoplus_{j=0}^{2D-1} M_N(\mathbb{K}[x]) j \subset E\).

Let \(\theta \in A \cap \bigoplus_{j=0}^{2D-1} M_N(\mathbb{K}[x]) j\) then there exists \(B = B(z, \partial_z)\) such that \((\psi B)(x, z) = \theta(x) \psi(x, z)\). We write \(\theta(x) = \sum_{j=0}^{2D-1} a_j x^j\). After a few simple computations we obtain that:

\[
B = 2^{D-1} \sum_{j=0}^{2D-1} \partial_z^j \left( a_k + \sum_{l=1}^{2D-1} \frac{(-1)^l}{2^l} \sum_{r=j+l-1}^{2D-1} (\mu^{l-1} r S^{r-j-l+1} P_r(\theta)) \right). \tag{11}
\]

With \(\mu \in M_{2D}(\mathbb{K}[x])\) given by

\[
\mu_{rj} = \begin{cases} 
(-1)^{r-j} & \text{if } r + 2 \leq j \leq \min \{r + D, 2D - 1\}, \\
r & \text{if } j = r + 1, \\
0 & \text{if otherwise.} 
\end{cases}
\]

Furthermore,

\[
e^{-xz}(\psi B - \theta \psi) = x^{-D} \left( \sum_{q=0}^{D-1} \left\{ \sum_{j=0}^{q} (-1)^{q-j-D} \left[ S^{D-q+j-1} a_j \right] + \sum_{l=1}^{2D-1-j} \frac{(-1)^l}{2^l} \sum_{r=j+l-1}^{2D-1} (\mu^{l-1} r S^{r-j-l+1} P_r(\theta)) \right\} x^q \right).
\]

\[
= x^{-D} \left( \sum_{q=0}^{D-1} \left\{ \sum_{j=0}^{q} (-1)^{q-j-D} \left[ S^{D-q+j-1} a_j \right] + \sum_{l=1}^{2D-2-q} \left( \sum_{j=0}^{q} (-1)^{q-j-D+l-1} \sum_{r=j+l-1}^{2D-1} (\mu^{l-1} r S^{r-j-l+1} P_r(\theta)) \right) \frac{1}{2^l} \right\} x^q \right).
\]

However,

\[
\sum_{j=0}^{q} (-1)^{q-j-D+l} \sum_{r=j+l-1}^{2D-1} (\mu^{l-1} r S^{r-j-l+1} P_r(\theta))
\]
Therefore,\[D\sum_{r=l}^{2D-l-1} (-1)^{q-j-D+1} \sum_{j=0}^{q} (-1)^j (\mu^{l-1})_{jr} S^{r+D-q-l} P_{r}(\theta),\]
for \(1 \leq l \leq 2D - 2 - q\), and
\[= \sum_{r=l}^{2D-2} (-1)^{q-j-D+1} \left( \sum_{j=0}^{q} (-1)^j (\mu^{l-1})_{jr} S^{r+D-q-l} P_{r}(\theta) \right),\]
for \(2D - 1 - q \leq l \leq 2D - 1\).

Note that \(r\)-th component of \(v\mu\) is given by \((v\mu)_r = \sum_{j=0}^{2D-1} (-1)^j \mu_{jr} = \sum_{j=0}^{2D-1} (-1)^j \mu_{jr} = (-1)^{r-1}(r - 1) + \sum_{j=0}^{r-2} (-1)^j (-1)^{r-j} = 0\) for \(v \in K^2D\) defined by \(v_j = (-1)^j\) and \(0 \leq j, r \leq 2D - 1\). Therefore, \(v\mu = 0\). Clearly this implies \(\sum_{j=0}^{r-1} (-1)^j (\mu^{l-1})_{jr} = (v\mu^{l-1})_r = 0\) for \(l \geq 2\).

Therefore,
\[
e^{-x\xi} (\psi B - \theta \psi) = x^{-D} \left[ \sum_{q=0}^{q} (-1)^q (-1)^{-D} \left[ S^{q+D} a_j \right] + \frac{1}{z} \sum_{j=0}^{q} (-1)^q (-1)^{-D+1} S^{j+D-q-1} P_j(\theta) \right] x^j.\]

Since \(\theta \in A\) we have
\[
\sum_{j=0}^{q} (-1)^q (-1)^{-D} \left[ S^{q+D+1} \frac{\theta(j)(0)}{j!} \right] = 0,
\]
\[
\sum_{j=0}^{q} (-1)^q (-1)^{-D+1} S^{j+D-q-1} P_j(\theta) = 0,
\]
for \(0 \leq q \leq D - 1\). Thus, \(\theta \in E\).

Step 5: The inclusion \(E \subset A \cap \oplus_{j=0}^{2D-1} M_N(K[x])_j\).

By the previous step we have Equation (11) valid for every \(\theta \in E\) and using Proposition 1 we obtain that \((\psi B)(x, z) = \theta(x) \psi(x, z)\). Then, \(\theta \in A \cap \oplus_{j=0}^{2D-1} M_N(K[x])_j\).

Furthermore, we have an explicit expression for the operator \(B\).

If \(\theta(x) = \sum_{j=0}^{M} a_j x^j \in \Gamma\), then
\[
B = \sum_{j=0}^{M} \partial_j \left( a_k + \sum_{l=1}^{M-k} \frac{(-1)^l}{z^l} \sum_{r=j+l-1}^{M} (\mu^{l-1})_{jr} S^{r+j-l+1} P_r(\theta) \right).
\]
with \( \mu \in M_{M+1}(\mathbb{K}[x]) \) given by
\[
\mu_{rj} = \begin{cases} 
(-1)^{r-j} & \text{if } r + 2 \leq j \leq \min \{ r + D, M \}, \\
r & \text{if } j = r + 1, \\
0 & \text{if otherwise.}
\end{cases}
\] (13)

satisfies \( (\psi B)(x, z) = \theta(x)\psi(x, z) \). ◊

In particular, Theorem 6 implies that the \( \mathbb{K} \)-algebra \( \Gamma \) is not trivial.

A remarkable property of this family of algebras is the existence of a Pierce decomposition whose definition we shall now recall.

**Definition 6.** Let \( R \) be a noncommutative ring with unit. We say that a set of elements \( r_1, \ldots, r_n \in R \) is Pierce decomposition of \( R \) if \( 1 = \sum_{j=1}^{n} r_j \) and \( r_j r_i = \delta_{ij} \) for all \( 1 \leq i, j \leq n \).

See [ABvW17] for more information on the Pierce decomposition. The next definition presents a Pierce decomposition of the algebra \( A \).

**Definition 7.** Define \( a_k(x) = e_{kk} + \sum_{j=1}^{N-1} a_{kj} x^j \in M_N(\mathbb{K}[x]) \) for \( 2 \leq k \leq N - 1, N \geq 3 \) with
\[
a_{kj} = (-1)^{i+1}(\delta_{k,j+1} e_{k1} + \delta_{k,N-j} e_{Nk}) + (-1)^j \delta_{N,j+1} e_{N1} = (-1)^{i+1}(\delta_{j,k-1} e_{k1} + \delta_{j,N-k} e_{Nk}) + (-1)^j \delta_{j,N-1} e_{N1}
\]
for \( 1 \leq j \leq N - 1 \).

In the previous definition we have two cases:
- If \( N \) is even and the numbers \( k - 1, N - k, N - 1 \) are different, then
\[
a_{kj} = \begin{cases} 
(-1)^{k} e_{k1} & \text{if } j = k - 1, \\
(-1)^{N-k+1} e_{Nk} & \text{if } j = N - k, \\
(-1)^{N-1} e_{N1} & \text{if } j = N - 1, \\
e_{kk} & \text{if } j = 0, \\
0 & \text{otherwise.}
\end{cases}
\] (14)

- If \( N \) is odd
  - If \( k \neq \frac{N+1}{2} \), then \( k - 1, N - k, N - 1 \) are different:
\[
a_{kj} = \begin{cases} 
(-1)^{k} e_{k1} & \text{if } j = k - 1, \\
(-1)^{N-k+1} e_{Nk} & \text{if } j = N - k, \\
(-1)^{N-1} e_{N1} & \text{if } j = N - 1, \\
e_{kk} & \text{if } j = 0, \\
0 & \text{otherwise.}
\end{cases}
\] (15)

  - If \( k = \frac{N+1}{2} \), then
\[
a_{\frac{N+1}{2}j} = \begin{cases} 
(-1)^{\frac{N+1}{2}} (e_{\frac{N+1}{2}+1} + e_{N \frac{N+1}{2}}) & \text{if } j = \frac{N-1}{2}, \\
(-1)^{N-1} e_{N1} & \text{if } j = N - 1, \\
e_{kk} & \text{if } j = 0, \\
0 & \text{otherwise.}
\end{cases}
\] (16)
The following lemma relates these elements with the family \( \{ P_k \}_{k \in \mathbb{N}} \). Its importance is that it shall be used to prove that the elements \( a'_k s \) satisfy the second family of relations that defines \( A \).

**Lemma 6.**

- If \( k < \frac{N+1}{2} \), then \( k - 1 < N - k < N - 1 \) and

\[
P_l(a_k) = \begin{cases} 
(-1)^l (e_{k, l+1} - e_{k+l+1, k}) & \text{if } 0 \leq l \leq k - 3, \\
(-1)^k e_{k+1} + (-1)^{k+1} e_{2k-1, k} & \text{if } l = k - 2, \\
(-1)^l (e_{l+2, 1} - e_{k+l+1, k}) & \text{if } k - 1 \leq l \leq N - k - 2, \\
(N - k - 1)(-1)^{N-k-1} e_{Nk} + (-1)^{N-k+1} e_{N-k+1, 1} & \text{if } l = N - k - 1, \\
(-1)^{l+1} (e_{N, N-l+1} - e_{l+2, 1}) & \text{if } N - k \leq l \leq N - 3, \\
(N - 1)(-1)^{N-1} e_{N1} & \text{if } N - k \leq l \leq N - 3.
\end{cases}
\](17)

- If \( k = \frac{N+1}{2} \), then \( N - k = k - 1 = \frac{N-1}{2} < N - 1 \) and

\[
P_l(a_k) = \begin{cases} 
(-1)^l \left( e_{\frac{N+1}{2}, \frac{N+1}{2}-l} - e_{\frac{N+1}{2}+l, \frac{N+1}{2}} \right) & \text{if } 0 \leq l \leq k - 3 = N - k - 2 = \frac{N-5}{2}, \\
\frac{N-3}{2}(-1)^{N+1} e_{N, N+1} + (-1)^{N+1} e_{N+1, 1} & \text{if } l = k - 2 = N - k - 1 = \frac{N-3}{2}, \\
(-1)^{l+1} (e_{N, N-l+1} - e_{l+2, 1}) & \text{if } N - k \leq l \leq N - 3, \\
(N - 1)(-1)^{N-1} e_{N1} & \text{if } l = N - 2.
\end{cases}
\](18)

- If \( k > \frac{N+1}{2} \), then \( N - k < k - 1 < N - 1 \) and

\[
P_l(a_k) = \begin{cases} 
(-1)^l (e_{k, l+1} - e_{k+l+1, k}) & \text{if } 0 \leq l \leq N - k - 2, \\
(N - k - 1)(-1)^{N-k-1} e_{Nk} + (-1)^{N-k+1} e_{2k-N} & \text{if } l = N - k - 1, \\
(-1)^l (e_{k, l+1} - e_{N, N-l+1}) & \text{if } N - k \leq l \leq k - 3, \\
k(-1)^k e_{k+1} + (-1)^{k+1} e_{N, N-k+1} & \text{if } l = k - 2, \\
(-1)^{l+1} (e_{N, N-l+1} - e_{l+2, 1}) & \text{if } k - 1 \leq l \leq N - 3, \\
(N - 1)(-1)^{N-1} e_{N1} & \text{if } l = N - 2.
\end{cases}
\](19)

**Proof.** The proof is a straightforward. \( \diamond \)

Now we prove that this family is contained in \( A \).

**Theorem 7.** For \( N \geq 3 \) we have \( \{ a_k \}_{1 \leq k \leq N-1} \subset A \).

Before proving this theorem we have a handy remark.

**Remark 3.** We shall adopt the convenient convention that \( \sum_{i \in \emptyset} x_i = 0 \). Define \( e_{ij} = 0 \) if \( i \) or \( j \) is outside the set \( \{1, ..., N\} \).
Proof. We first verify the first family of relations. Pick 2 ≤ k ≤ N − 1, then

\[ P_0(x^k(x)) = e_{kk} - (-1)^k \left( (-1)^ke_{k1}, S_N^{k-1} \right) - (-1)^{N-k+1} \left( (-1)^{N-k+1}e_{Nk}, S_N^{N-k} \right) \]

\[ -(-1)^N \left[ (-1)^{N-1}e_{N1}, S_N^{N-1} \right] = e_{kk} - \left[ e_{k1}, S_N^{k-1} \right] - \left[ e_{Nk}, S_N^{N-k} \right] - \left[ e_{N1}, S_N^{N-1} \right] = e_{kk} - (e_{kk} - e_{11}) \]

\[ -(e_{NN} - e_{kk}) + (e_{NN} - e_{11}) = e_{kk} = P_0(a_k(0)x). \]

If N is odd

\[ P_0 \left( x^\alpha \right) = e_{N+1, \frac{N+1}{2}} - (-1)^{\frac{N+1}{2}} \left[ (-1)^{\frac{N+1}{2}} \left( e_{N+1,1} + e_{N, \frac{N+1}{2}} \right), S_N^{\frac{N+1}{2}} \right] - (-1)^N \left[ (-1)^{N-1}e_{N1}, S_N^{N-1} \right] \]

\[ = e_{N+1, \frac{N+1}{2}} - \left( e_{N+1, \frac{N+1}{2}} + e_{NN} - \left( e_{11} + e_{N, \frac{N+1}{2}} \right) \right) + e_{NN} - e_{11} = e_{N+1, \frac{N+1}{2}} = P_0 \left( x^\alpha \right). \]

If r ≥ 2, then

\[ P_r(x^\alpha) = (-1)^r \left[ e_{kk}, S_N^{r-1} \right] - (-1)^{k+r-1} \left[ (-1)^ke_{k1}, S_N^{k+r-2} \right] - (-1)^{N-k+r} \left[ (-1)^{N-k+1}e_{N1}, S_N^{N-k+r-1} \right] \]

\[ -(-1)^r (e_{kk+r-1} - e_{k-r+1,k}) - (-1)^{r-1} (e_{kk+r-1} - 0) - (-1)^{r+1} (0 - e_{k-r+1,k}) = 0. \]

If N is odd

\[ P_0 \left( x^\alpha \right) = -(-1)^r \left[ e_{N+1, \frac{N+1}{2}}, S_N^{r-1} \right] - (-1)^{\frac{N+1}{2}} \left[ (-1)^{\frac{N+1}{2}} \left( e_{N+1,1} + e_{N, \frac{N+1}{2}} \right), S_N^{\frac{N+2}{2}} \right] \]

\[ = -(-1)^r \left( e_{N+1, \frac{N+1}{2}} - e_{N+2-r, \frac{N+1}{2}} \right) + (-1)^r \left( e_{N+1, \frac{N+1}{2}} - e_{N+2-r, \frac{N+1}{2}} \right) = 0. \]

The second family of relations has a number of cases which we will check.

Using Lemma 6:

- If \( q < \frac{N-3}{2} \),

  - If \( 2 \leq k < q + 2 \), then \( q < N - k - 1 \). Since, \( k - N + q + 1 < 0 \) and \( q + 3 < N \) we have

    \[ \sum_{j=0}^{q} (-1)^j S_N^{N+j-q-1} = \sum_{j=0}^{k-3} (-1)^j S_N^{N+j-q-1} (1) \left( e_{k,k-j-1} - e_{k+j+1,k} \right) \]

    \[ + (-1)^k S_N^{N-q-k-3} (1) e_{k+1} + (-1)^{k+1} e_{2k-1} + \sum_{j=k-1}^{q} (-1)^j S_N^{N+j-q-1} (1) \left( e_{j+2,1} - e_{k+j+1,k} \right) \]

    \[ = \sum_{j=0}^{k-3} e_{k-N+j+q-1,k-j-1} - \sum_{j=0}^{k-3} e_{k-N+j+q+2,k} + k e_{q+3-N,1} - e_{k-N+q+2,k} \]

    \[ + \sum_{j=k-1}^{q} e_{q+3-N,1} - \sum_{j=k-1}^{q} e_{k-N+j+q+2,k} = 0. \]
If \( qj + N - 2 < 0 \). Since, \( 2q - N + 3 < 0 \).

\[
\sum_{j=0}^{q} (-1)^j S^{N+j-q-1} p_j(\alpha_k) = \sum_{j=0}^{q-1} (-1)^j S^{N+j-q-1} (-1)^j (e_{q+2,q+1-j} - e_{q+j+3,q+2})
\]

\[+(-1)^q e_{1N} \left((-1)^q (q+2)e_{q+2} + (1)^q e_{2q+1,3,q+2}\right) = \sum_{j=0}^{q-1} (-1)^j S^{N+j-q-1} (-1)^j e_{2q-N-j+3,q+1-j}
\]

\[\sum_{j=0}^{q-1} (-1)^j S^{N+j-q-1} (-1)^j e_{2q-N+4,q+2} = 0.\]

If \( q + 2 < k < N - q - 1 \), then \( q < k + 2 \). Since, \( k - N + q + 1 < 0 \).

\[
\sum_{j=0}^{q} (-1)^j S^{N+j-q-1} p_j(\alpha_k) = \sum_{j=0}^{q} (-1)^j S^{N+j-q-1} (-1)^j (\alpha_{k-j-1} - \alpha_{k+1})
\]

\[= \sum_{j=0}^{q} (-1)^j S^{N+j-q-1} (\alpha_{k-j-1} - \alpha_{k+1})
\]

\[+(-1)^q e_{1N} \left((N-k)(1)^N e_{N,k} + (-1)^q (\alpha_{k-q-1} - \alpha_{N,q-1})\right)\]

\[= \sum_{j=0}^{q-1} e_{N-j+q+1,k-j-1} - \sum \alpha_{k-N+q+2,k} + (N-k) e_{1k} - (N-k) e_{1k} = 0.\]

If \( k = N - q - 1 \), then \( N - k = q + 1 \). Since, \( k - N + q + 1 < 0 \).

\[
\sum_{j=0}^{q} (-1)^j S^{N+j-q-1} p_j(\alpha_k) = \sum_{j=0}^{q-1} (-1)^j S^{N+j-q-1} (-1)^j (\alpha_{k-j-1} - \alpha_{k+1})
\]

\[+(-1)^q e_{1N} \left((N-k)(1)^N e_{N,k} + (-1)^q (\alpha_{k-q-1} - \alpha_{N,q-1})\right)\]

\[= \sum_{j=0}^{q-1} e_{N-j+q+1,k-j-1} - \sum \alpha_{k-N+q+2,k} + (N-k) e_{1k} - (N-k) e_{1k} = 0.\]

If \( k > N - q - 1 \), then \( q + 1 > k \). Since, \( j \geq k - N + q + 1 \) implies \( k - N + j + q + 1 \leq 0 < 1 \).

\[
\sum_{j=0}^{q} (-1)^j S^{N+j-q-1} p_j(\alpha_k) = \sum_{j=0}^{N-k-2} (-1)^j S^{N+j-q-1} (-1)^j (\alpha_{k-j-1} - \alpha_{k+1})
\]

\[+(-1)^{N-k+1} S^{N-k-q-1} \left((N-k)(1)^N e_{N,k} + (-1)^{N-k+1} (\alpha_{k+1})\right)\]

\[= \sum_{j=0}^{q-1} e_{N-j+q+1,k-j-1} - \sum \alpha_{k-N+q+2,k} + (N-k) e_{1k} - (N-k) e_{1k} = 0.\]
• If \( q = \frac{N-3}{2} \) (for \( N \) odd) then,
  
  - The case \( 2 \leq k < q + 2 = \frac{N+1}{2} \) is similar to the case \( q < \frac{N-3}{2} \) and \( 2 \leq k < q + 2 \).
  
  - If \( k = q + 2 = \frac{N+1}{2} \) then,
    
    \[
    \sum_{j=0}^{q} (-1)^j S_{N}^{N+j-q-1} P_j(a_k) = \sum_{j=0}^{q-1} (-1)^j S_{N}^{N+j-q-1} (-1)^j \left( e_{N+j-1} + e_{N+j+1} \right)
    \]
    
    \[
    + (-1)^q S_{N}^{N-q-3} \left( (-1)^k e_{k+1} + (-1)^{k+1} e_{2k-1} \right) + \sum_{j=k-1}^{q} (-1)^j S_{N}^{N+j-q-1} (-1)^j \left( e_{j+2,1} - e_{k+j+1,1} \right)
    \]
    
    \[
    = \sum_{j=0}^{q-1} e_{j-1} + e_{N-j} - j - \sum_{j=0}^{q-1} e_{j+1} + e_{N+j+1} = -q e_{1} + e_{N+1} + 0.
    \]
  
  - The case \( q + 2 = \frac{N+1}{2} = N - q - 1 < k \) is similar to the case \( q < \frac{N-3}{2} \) and \( k > N - q - 1 \).

• If \( \frac{N-3}{2} < q \leq N - 3 \),
  
  - If \( 0 \leq k \leq N - q - 2 \), then \( q < N - k - 1 \). Since, \( k - N + q + 1 < 0 \) and \( q + 3 \leq N \). Therefore,
    
    \[
    \sum_{j=0}^{q} (-1)^j S_{N}^{N+j-q-1} P_j(a_k) = \sum_{j=0}^{k-3} (-1)^j S_{N}^{N+j-q-1} (-1)^j \left( e_{k-1} - e_{k+j+1,1} \right)
    \]
    
    \[
    + (-1)^k S_{N}^{N-q-3} \left( (-1)^k e_{k+1} + (-1)^{k+1} e_{2k-1} \right) + \sum_{j=k-1}^{q} (-1)^j S_{N}^{N+j-q-1} (-1)^j \left( e_{j+2,1} - e_{k+j+1,1} \right)
    \]
    
    \[
    = \sum_{j=0}^{q-1} \left( e_{j-1} + e_{N-j} - j - \sum_{j=0}^{q-1} e_{j+1} + e_{N+j+1} = -q e_{1} + e_{N+1} + 0.\right.
    \]
  
  - If \( N - q - 1 \leq k < q + 2 \), then \( N - k - 1 < q, k - 2 < q \). Since \( q < N - k - 1 \) and \( q + 3 \leq N \). Therefore,
    
    \[
    \sum_{j=0}^{q} (-1)^j S_{N}^{N+j-q-1} P_j(a_k) = \sum_{j=0}^{k-3} (-1)^j S_{N}^{N+j-q-1} (-1)^j \left( e_{k-1} - e_{k+j+1,1} \right)
    \]
    
    \[
    + (-1)^k S_{N}^{N-q-3} \left( (-1)^k e_{k+1} + (-1)^{k+1} e_{2k-1} \right) + \sum_{j=k-1}^{q} (-1)^j S_{N}^{N+j-q-1} (-1)^j \left( e_{j+2,1} - e_{k+j+1,1} \right)
    \]
    
    \[
    = \sum_{j=0}^{q-1} \left( e_{j-1} + e_{N-j} - j - \sum_{j=0}^{q-1} e_{j+1} + e_{N+j+1} = -q e_{1} + e_{N+1} + 0.\right.
    \]

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\[-\sum_{j=k-1}^{q} e_{k-N+q+2k} + (-1)^{N-k-1} S_{N}^{2N-k-2q+2} \left((-1)^{N-k-1}(N-k-1)e_{Nk} + (-1)^{N-k+1}e_{N-k+1,1}\right) \]

\[+ \sum_{j=N-k}^{q} (-1)^{j} s_{N}^{q+j-q-1}(-1)^{j} \left(e_{N,N-j-1} - e_{j+2,1}\right) \]

\[= \sum_{j=0}^{k-3} e_{k-N-j+q+1,k-j-1} - \sum_{j=0}^{k-3} e_{k-N-j+q+2k} + k e_{q+3-N,1} - e_{k-N+q+2k} + \sum_{j=k-1}^{N-k-2} e_{q+3-N,1} \]

\[= \sum_{j=k-1}^{q} e_{k-N+q+2k} + (N-k-1)e_{k-N+q+2k} + e_{q+3-N,1} + \sum_{j=N-k}^{q} e_{q+1-j,N-j-1} - \sum_{j=N-k}^{q} e_{q+3-N,1} \]

\[= \sum_{j=0}^{k-3} e_{k-N-j+q+1,k-j-1} + \sum_{j=N-k}^{q} e_{q+1-j,N-j-1} = 0. \]

- If \( k = N - q - 1 \), then \( N-k-1 = q \). Since \( q + 3 \leq N \).

\[\sum_{j=0}^{q} (-1)^{j} s_{N}^{q+j-q-1} p_{j}(a_{k}) = \sum_{j=0}^{k-3} (-1)^{j} s_{N}^{q+j-q-1}(-1)^{j} \left(e_{k,k-j-1} - e_{k+j+1,k}\right) \]

\[+ (-1)^{k} s_{N}^{q-k-3} \left((-1)^{k} ke_{k1} + (-1)^{k+1} e_{2k-1,k}\right) + \sum_{j=k-1}^{N-k-2} (-1)^{j} s_{N}^{q+j-q-1}(-1)^{j} \left(e_{j+2,1} - e_{k+j+1,k}\right) \]

\[+ (-1)^{q} s_{N}^{N-1} \left((N-k-1)(-1)^{N-k-1}e_{Nk} + (-1)^{N-k+1}e_{N-k+1,1}\right) \]

\[= \sum_{j=0}^{k-3} e_{k-N-j+q+1,k-j-1} - \sum_{j=0}^{k-3} e_{k-N-j+q+2k} + k e_{q+3-N,1} - e_{k-N+q+2k} \]

\[+ \sum_{j=k-1}^{N-k-2} e_{q+3-N,1} - \sum_{j=k-1}^{N-k-2} e_{k-N+q+2k} + (N-k-1)e_{1k} \]

\[= \sum_{j=0}^{k-3} e_{j-k-j-1} - \sum_{j=0}^{k-3} e_{1k} - \sum_{j=0}^{k-3} e_{1k} - \sum_{j=k-1}^{N-k-2} e_{1k} + (N-k-1)e_{1k} = 0. \]

- If \( k = q + 2 \), then \( k-2 = q \). Since \( q < N-k-1 \).

\[\sum_{j=0}^{q} (-1)^{j} s_{N}^{q+j-q-1} p_{j}(a_{k}) = \sum_{j=0}^{k} (-1)^{j} s_{N}^{q+j-q-1}(-1)^{j} \left(e_{k,k-j-1} - e_{k+j+1,k}\right) \]

\[+ (-1)^{q} s_{N}^{q-k-3} \left((-1)^{k} ke_{k1} + (-1)^{k+1} e_{2k-1,k}\right) = \sum_{j=0}^{k-3} e_{k-N-j+q+1,k-j-1} - \sum_{j=0}^{k-3} e_{k-N-j+q+2k} + k e_{q+3-N,1} - e_{k-N+q+2k} = 0. \]

- If \( k > q + 2 \), then \( k > q > N-q-1 \) and \( N-k-1 < q \).

\[\sum_{j=0}^{q} (-1)^{j} s_{N}^{q+j-q-1} p_{j}(a_{k}) = \sum_{j=0}^{N-k-2} (-1)^{j} s_{N}^{q+j-q-1}(-1)^{j} \left(e_{k,k-j-1} - e_{k+j+1,k}\right) \]

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\[ +(-1)^{N-k-1}S_N^{2N-k-q-2}\left((-1)^{N-k+1}(N-k)e_N^k + (-1)^{N-k+1}(e_{2k-N} - e_{N-k})\right) \]
\[ + \sum_{j=N-k}^q (-1)^j S_N^{N+j-q-1}(-1)^j \left(e_{k-j-1} - e_{N,N-j-1}\right) \]
\[ = \sum_{j=0}^{N-k-2} e_{N-j+q+1,k-j-1} - \sum_{j=0}^{N-k-2} e_{k-N+j+q+2,k} + (N - k)e_{k-N+j+q+2,k} + e_{2k-N+j+q+2,k-N} - e_{k-N+j+q+2,k} + \sum_{j=N-k}^q e_{j+1,N-j-1} - \sum_{j=0}^{q} e_{k-N+j+q+1,k-j-1} - \sum_{j=N-k}^q e_{q+1,j,N-j-1} = \sum_{j=0}^{q} e_{k-N+j+q+1,k-j-1} = 0. \]

- If \( q = N - 2 \) and \( k < \frac{N+1}{2} \) then \( k - 1 < N - k \leq N - 1 \). Therefore,
\[ \sum_{j=0}^{N-2} (-1)^j S_N^{j+q-1}p_j(a_k) = \sum_{j=0}^{N-2} (-1)^j S_N^{j+1}p_j(a_k) = \sum_{j=0}^{k-3} (-1)^j S_N^{j+1}(-1)^j(e_{k-j-1} - e_{k+j+1,k}) \]
\[ +(-1)^k S_N^{k-1}\left((-1)^k ke_k 1 + (-1)^{k+1}e_{2k-1,k}\right) + \sum_{j=k-1}^{N-k-2} (-1)^j S_N^{j+1}(-1)^j(e_{j+1,k} - e_{k+j+1,k}) \]
\[ +(-1)^{N-k+1}S_N^{N-k}\left((N - k - 1)(-1)^{N-k-1}e_N^k + (-1)^{N-k+1}e_{N-N-k+1}\right) \]
\[ + \sum_{j=N-k}^{N-3} (-1)^j S_N^{j+1}(-1)^{j+1}(e_{N,N-j-1} - e_{j+1,k}) + (-1)^N S_N^{N-1}(N - 1)(-1)^N(-1)e_{N-1} \]
\[ = \sum_{j=0}^{k-3} e_{k-j-1,k-j-1} - \sum_{j=0}^{k-3} e_{k,k} + ke_{11} - e_{kk} + \sum_{j=k-1}^{N-k-2} e_{j+1,k} - \sum_{j=k-1}^{N-k-2} e_{kk} + (N - k - 1)e_{kk} \]
\[ + e_{11} - \sum_{j=N-k}^{N-3} e_{N-j+1,N-j-1} + \sum_{j=N-k}^{N-3} e_{11} - (N - 1)e_{11} \]
\[ = \sum_{j=0}^{k-3} e_{k-j-1,k-j-1} - (k - 2)e_{kk} + ke_{11} - e_{kk} + (N - k)e_{11} - (N - 2k)e_{kk} + e_{11} \]
\[ - \sum_{r=0}^{k-3} e_{k-r-1,k-r-1} + (k - 2)e_{11} - (N - 1)e_{11} = 0. \]

- If \( q = N - 1 \) and \( k < \frac{N+1}{2} \) then
\[ \sum_{j=0}^{N-1} (-1)^j S_N^{j+q-1}p_j(a_k) = \sum_{j=0}^{N-1} (-1)^j S_N^{j}p_j(a_k) = \sum_{j=0}^{k-2} (-1)^j S_N^{j}(e_{k-j-1} - e_{k+j+1,k}) \]
\[ +(-1)^k S_N^{k-2}\left((-1)^k ke_k 1 + (-1)^{k+1}e_{2k-1,k}\right) + \sum_{j=k-1}^{N-k-2} (-1)^j S_N^{j}(-1)^j(e_{j+2,1} - e_{k+j+1,k}) \]
\[ +(-1)^{N-k+1}S_N^{N-k-1}\left((N - k - 1)(-1)^{N-k-1}e_N^k + (-1)^{N-k+1}e_{N-N-k+1}\right) \]
If \( q = N - 2 \) and \( K > \frac{N+1}{2} \) then \( N - k < k - 1 \leq N - 1 \) and

\[
\sum_{j=0}^{N-2} (-1)^j S_N^j (-1)^{j+1} (e_{N,N-j-1} - e_{j+2,1}) + (-1)^N S_N^{N-2} (N-1)(-1)^{N-1} e_{N1}
\]

\[
= \sum_{j=0}^{k-3} e_{k-j,k-j-1} - \sum_{j=0}^{k-3} e_{k+1,k} + k e_{21} - e_{k+1,k} + \sum_{j=k+1}^{N-k-2} e_{21} + \sum_{j=k+1}^{N-k-2} e_{k+1,k} + (N - k - 1)e_{k+1,k}
\]

\[
= \sum_{j=0}^{k-3} e_{k-j,k-j-1} - (k-2)e_{k+1,k} + k e_{21} - e_{k+1,k} + (N - 2k)e_{21} - (N - 2k)e_{k+1,k} + (N - k - 1)e_{k+1,k}
\]

\[
+ e_{21} - \sum_{r=0}^{k-3} e_{k-r,k-r-1} + (k-2)e_{21} - (N-1)e_{21} = 0.
\]

If \( q = N - 1 \) and \( k > \frac{N+1}{2} \) then \( N - k < k - 1 \leq N - 1 \) and

\[
\sum_{j=0}^{N-1} (-1)^j S_N^j (-1)^{j+1} (e_{N,N-j-1} - e_{j+2,1}) + (-1)^N S_N^{N-2} (N-1)(-1)^{N-1} e_{N1}
\]

\[
= \sum_{j=0}^{k-3} e_{k-j,k-j-1} - \sum_{j=0}^{k-3} e_{k+1,k} + k e_{21} - e_{k+1,k} + \sum_{j=k+1}^{N-k-2} e_{21} + \sum_{j=k+1}^{N-k-2} e_{k+1,k} + (N - k - 1)e_{k+1,k}
\]

\[
= \sum_{j=0}^{k-3} e_{k-j,k-j-1} - (N - k - 1)e_{k+1,k} + (N - k - 1)e_{21} - (N - k - 1)e_{k+1,k} - \sum_{j=0}^{N-k} e_{k-j,k-j} - \sum_{r=0}^{k-3} e_{k-r,k-r-1} = 0.
\]
\begin{align*}
&+(-1)^k S_N^{k-2} \left\{ k(-1)^k e_{k1} + (-1)^{k+1} e_{N,N-k+1} \right\} + \sum_{j=k-1}^{N-3} (-1)^j S_N^{j+1}(e_{N,N-j-1} - e_{j+2,1}) \\
&+(-1)^N S_N^{N-2}(N-1)(-1)^{N-1} e_{N1} = \sum_{j=0}^{N-k-2} e_{k-j,k-j-1} - \sum_{j=0}^{N-k-2} e_{k+1,k} + (N-k-1) e_{k+1,k} \\
&+e_{2k-N+1,2k-N} + \sum_{j=N-k}^{k-3} e_{k-j,k-j-1} - \sum_{j=N-k}^{k-3} e_{N-j,N-j-1} + k e_{21} - e_{N-k+2,N-k+1} - \sum_{j=k-1}^{N-3} e_{N-j,N-j-1} \\
&+ \sum_{j=k-1}^{N-3} e_{21} - (N-1) e_{21} = \sum_{j=0}^{k-3} e_{k-j,k-j-1} - \sum_{j=0}^{k-3} e_{k-r,k-r-1} = 0.
\end{align*}

- If \( q = N-2 \) and \( k = \frac{N+1}{2} \) for \( N \) odd we have \( N-k = k-1 < N-1 \), then

\begin{align*}
&\sum_{j=0}^{N-2} (-1)^j S_N^{j+q-1} P_j(\alpha_k) = \sum_{j=0}^{N-2} (-1)^j S_N^{j+1} P_j(\alpha_k) = \sum_{j=0}^{N-3} (-1)^j S_N^{j+1}(-1)^j(e_{N+j,N+j-1} - e_{N+3+j,N+2,j}) \\
&+(-1)^{\frac{N-3}{2}} \sum_{j=0}^{\frac{N-3}{2}} (-1)^{j+1} e_{N+j,N+j} + (-1)^{\frac{N+1}{2}} \left( \frac{N+1}{2} \right) e_{N+1,1} \\
&+ \sum_{j=\frac{N-3}{2}}^{N-3} e_{21} - (N-1) e_{21} = \sum_{j=0}^{\frac{N-3}{2}} e_{N-1,j,N-1,j} - \left( \frac{N-3}{2} \right) e_{N+1,N+1} + \left( \frac{N}{2} \right) e_{N+1,N+1} \\
&+ \left( \frac{N+1}{2} \right) e_{11} - \sum_{r=0}^{\frac{N-3}{2}} e_{N-1,r,N-1,r} + \left( \frac{N-3}{2} \right) e_{11} - (N-1) e_{11} = 0.
\end{align*}

- If \( q = N-1 \) and \( k = \frac{N+1}{2} \) then \( N-k = k-1 < N-1 \) and

\begin{align*}
&\sum_{j=0}^{N-1} (-1)^j S_N^{j+q-1} P_j(\alpha_k) = \sum_{j=0}^{N-1} (-1)^j S_N^{j} P_j(\alpha_k) = \sum_{j=0}^{\frac{N-3}{2}} (-1)^j S_N^{j}(-1)^j(e_{N+j,N+j-1} - e_{N+3+j,N+2,j}) \\
&+(-1)^{\frac{N-3}{2}} S_N^{\frac{N-3}{2}} \left\{ \left( \frac{N-3}{2} \right) (-1)^{\frac{N+1}{2}} e_{N+1,N+1} + (-1)^{\frac{N+1}{2}} \left( \frac{N+1}{2} \right) e_{N+1,1} \right\} \\
&+ \sum_{j=\frac{N-3}{2}}^{N-3} (-1)^j S_N^{j}(e_{N,N-j-1} - e_{j+2,1}) + (-1)^{N-2} S_N^{N-2}(N-1)(-1)^{N+1} e_{N1}
\end{align*}
Recall that a Pierce decomposition of a noncommutative ring $R$ with unit 1 is a finite set of elements $r_1, ..., r_n \in R$ such that $1 = \sum_{j=1}^n r_j$ and $r_i r_j = \delta_{ij}$ for all $1 \leq i, j \leq n$. Now we state the Pierce decomposition of $A$.

**Corollary 6.** If $\alpha_1 = I - \sum_{k=2}^{N-1} \alpha_k$, then $\{\alpha_k\}_{1 \leq k \leq N-1}$ is a Pierce decomposition of $A$.

**Proof.** In fact, if $2 \leq k < l \leq N - 1$, then

$$\alpha_k \alpha_l = \left( e_{kk} + \sum_{j=1}^{N-1} a_{kj} x^j \right) \left( e_{ll} + \sum_{r=1}^{N-1} a_{lr} x^r \right) = \sum_{j,r=1}^{N-1} a_{kj} a_{lr} x^{j+r}.$$

However,

$$a_{kj} a_{lr} = \left( (-1)^{j+1} \delta_{k,j+1} e_{k1} + \delta_{k,N-j} e_{Nk} \right) \left( (-1)^{r+1} \delta_{l,r+1} e_{l1} + \delta_{l,N-r} e_{Nl} \right) + (-1)^{l+1} \delta_{N,j+1} e_{N1} = 0$$

implies $a_k \alpha_l = 0$.

On the other hand,

$$a_k^2 = \left( e_{kk} + \sum_{j=1}^{N-1} a_{kj} x^j \right) \left( e_{kk} + \sum_{j=1}^{N-1} a_{kj} x^j \right) = e_{kk} + \sum_{j=1}^{N-1} (e_{kk} a_{kj} + a_{kj} e_{kk}) x^j + \sum_{j,r=1}^{N-1} a_{kj} a_{kr} x^{j+r}.$$

However,

$$e_{kk} a_{kj} + a_{kj} e_{kk} = (-1)^{j+1} \left( \delta_{k,j+1} e_{k1} + \delta_{k,N-j} e_{Nk} \right)$$

and

$$a_{kj} a_{kr} = (-1)^{j+r} \delta_{k,N-j} \delta_{k,r+1} e_{N1} = (-1)^{N-1} \delta_{k,N-k} \delta_{r,N-k} e_{N1}$$

imply

$$a_k^2 = e_{kk} + \sum_{j=1}^{N-1} (-1)^{j+1} \left( \delta_{k,j+1} e_{k1} + \delta_{k,N-j} e_{Nk} \right) x^j + (-1)^{N-1} e_{N1} x^{N-1} = e_{kk} + \sum_{j=1}^{N-1} a_{kj} x^j = a_k.$$

Thus, $a_k \alpha_l = \delta_{kl} a_k$ for $2 \leq k, l \leq N - 1$. By the definition of $\alpha_1$ and the previous properties we can extend this to $a_k \alpha_l = \delta_{kl} a_k$ for $1 \leq k, l \leq N - 1$. The assertion follows by the Theorem 7.
Corollaries 7, 8, and Proposition 8 give an answer to the Conjectures 1, 2 and 3 of [Grü14] about three bispectral full rank 1 algebras as we shall describe in the following sections. Moreover, these algebras are Noetherian and finitely generated because they are contained in the $N \times N$ matrix polynomial ring $M_N(\mathbb{K}[x])$.

**Corollary 7.** Let $\Gamma$ be the sub-algebra of $M_2(\mathbb{C})[x]$ of the form

$$
\begin{pmatrix}
    r_0^{11} & r_0^{12} \\
    r_0^{12} & r_0^{13}
\end{pmatrix} + 
\begin{pmatrix}
    r_1^{11} & r_1^{12} \\
    r_1^{12} & r_1^{13}
\end{pmatrix} x + 
\begin{pmatrix}
    r_2^{11} & r_2^{12} \\
    r_2^{12} & r_2^{13}
\end{pmatrix} x^2 + 
\begin{pmatrix}
    r_3^{11} & r_3^{12} \\
    r_3^{12} & r_3^{13}
\end{pmatrix} x^3 + x^4 p(x),
$$

where $p \in M_2(\mathbb{C})[x]$ and all the variables $r_0^{11}, r_0^{12}, r_1^{11}, r_1^{12}, r_2^{11}, r_2^{12}, r_3^{11}, r_3^{12}, r_3^{13} \in \mathbb{C}$. Then $\Gamma = \mathcal{A}$. Moreover, for each $\theta$ we have an explicit expression for the operator $\mathcal{B}$.

**Proof.** The proof is given by the Theorem 6 with $N = 2$. \hfill \qed

**Corollary 8.** Let $\Gamma$ the sub-algebra of $M_3(\mathbb{C})[x]$ of the form

$$
\begin{pmatrix}
    r_0^{11} & r_0^{12} & r_0^{13} \\
    r_2^{11} & r_2^{12} & r_2^{13} \\
    r_2^{22} & r_2^{23} & r_2^{33}
\end{pmatrix} + 
\begin{pmatrix}
    r_1^{11} & r_1^{12} & r_1^{13} \\
    r_1^{12} & r_1^{13} & r_1^{23} \\
    r_1^{22} & r_1^{23} & r_1^{33}
\end{pmatrix} x + 
\begin{pmatrix}
    r_2^{11} & r_2^{12} & r_2^{13} \\
    r_2^{12} & r_2^{13} & r_2^{23} \\
    r_2^{22} & r_2^{23} & r_2^{33}
\end{pmatrix} x^2 + 
\begin{pmatrix}
    r_3^{11} & r_3^{12} & r_3^{13} \\
    r_3^{12} & r_3^{13} & r_3^{23} \\
    r_3^{22} & r_3^{23} & r_3^{33}
\end{pmatrix} x^3
$$

\begin{equation}
\begin{pmatrix}
    r_4^{11} & r_4^{12} & r_4^{13} \\
    r_4^{12} & r_4^{13} & r_4^{23} \\
    r_4^{22} & r_4^{23} & r_4^{33}
\end{pmatrix} x^4
\end{equation}

\begin{equation}
\begin{pmatrix}
    r_5^{11} & r_5^{12} & r_5^{13} \\
    r_5^{12} & r_5^{13} & r_5^{23} \\
    r_5^{22} & r_5^{23} & r_5^{33}
\end{pmatrix} x^5 + x^6 p(x),
\end{equation}

where $p \in M_3(\mathbb{C})[x]$ and all the variables $r_0^{11}, r_0^{12}, ..., r_3^{33} \in \mathbb{C}$ are arbitrary.

Then, $\Gamma = \mathcal{A}$ and for each $\theta$ we have an explicit expression for the operator $\mathcal{B}$.

**Proof.** The proof is given by the Theorem 6 with $N = 3$. \hfill \qed
3.2 An Example linked to the Spin Calogero Systems

This example is linked to the spin Calogero systems whose relation with bispectrality can be found in [BGK08]. We consider the case when both "eigenvalues" $F$ and $\theta$ are matrix valued. Let

$$\psi(x, z) = \frac{e^{xz}}{(x - 2)xz} \left( \frac{x^2z^2 - 2x^2z^2 - 2z^2 + 3xz + 2xz - 2}{x^2z^2} \right)$$

and

$$\mathcal{L} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \partial_x^2 + \begin{pmatrix} 0 & 1 \\ -\frac{1}{x - 2} & 0 \end{pmatrix} \partial_z + \begin{pmatrix} -\frac{1}{x^2(x - 2)} & 0 \\ \frac{x - 1}{2x(x - 2)} & -\frac{x - 1}{2x^2(x - 2)} \end{pmatrix},$$

then $\mathcal{L} \psi = \psi F$ with

$$F(z) = \begin{pmatrix} 0 & 0 \\ 0 & z^2 \end{pmatrix}.$$ 

On the other hand, it is easy to check that $\psi \mathcal{B} = \theta \psi$ for

$$\mathcal{B} = \partial_z^3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \partial_z^2 \begin{pmatrix} 0 & 1 \\ -\frac{1}{2z + 1} & 0 \end{pmatrix} + \partial_z \begin{pmatrix} 1 \\ \frac{1}{z^2(z - 1)} \end{pmatrix} + \begin{pmatrix} \frac{1}{z^2} \\ 6z^{-3} \end{pmatrix}$$

and

$$\theta(x) = \begin{pmatrix} x \\ x^2(x - 2) \end{pmatrix}.$$ 

The following proposition characterizes the algebra $\mathcal{A}$ of all polynomial $F$ such that there exist $\mathcal{L} = \mathcal{L}(x, \partial_x)$ with $\mathcal{L} \psi = \psi F$.

**Theorem 8.** Let $\Gamma$ be the sub-algebra of $M_2(\mathbb{C})[z]$ of the form

$$\begin{pmatrix} a & 0 \\ b - a & b \end{pmatrix} + \begin{pmatrix} c & c \\ a - b - c & -c \end{pmatrix} z + \begin{pmatrix} a - b - c & c + a - b \\ d & e \end{pmatrix} \frac{z^2}{2} + z^3 p(z),$$

where $p \in M_2(\mathbb{C})[z]$ and all the variables $a, b, c, d, e$ are arbitrary. Then $\Gamma = \mathcal{A}$.

**Proof.** We shall break the proof in different steps.

Step 1: The set $\Gamma$ is an algebra. Clearly if $F_1, F_2 \in \Gamma$, then $F_1 + F_2 \in \Gamma$ and $aF_1 \in \Gamma$ if $a \in \mathbb{C}$,

$$F_1(z) = \begin{pmatrix} a_1 & 0 \\ b_1 - a_1 & b_1 \end{pmatrix} + \begin{pmatrix} c_1 & c_1 \\ a_1 - b_1 - c_1 & -c_1 \end{pmatrix} z + \begin{pmatrix} a_1 - b_1 - c_1 & c_1 + a_1 - b_1 \\ d_1 & e_1 \end{pmatrix} \frac{z^2}{2} + z^3 p_1(z),$$

$$F_2(z) = \begin{pmatrix} a_2 & 0 \\ b_2 - a_2 & b_2 \end{pmatrix} + \begin{pmatrix} c_2 & c_2 \\ a_2 - b_2 - c_2 & -c_2 \end{pmatrix} z + \begin{pmatrix} a_2 - b_2 - c_2 & c_2 + a_2 - b_2 \\ d_2 & e_2 \end{pmatrix} \frac{z^2}{2} + z^3 p_2(z).$$

Thus,

$$F_1(z)F_2(z) = \begin{pmatrix} a_1a_2 & 0 \\ b_1b_2 - a_1a_2 & b_1b_2 \end{pmatrix} + \begin{pmatrix} a_1c_2 + b_2c_1 & a_1c_2 + b_2c_1 \\ a_1c_2 + b_2c_1 & a_1c_2 + b_2c_1 \end{pmatrix} z +$$

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\[
\begin{pmatrix}
  a_1a_2 - b_1b_2 - a_1c_2 - b_2c_1 \\
  b_2e_1 - a_2e_1 + b_1d_2 + a_2d_1 - 3b_1c_2 + 3a_1c_2 + 2b_2c_1 - 2a_2c_1 - b_1b_2 + a_1b_2 + a_2b_1 - a_1a_2 \\
  a_1c_2 + b_2e_1 + a_1a_2 - b_1b_2 \\
  b_1e_2 + b_2e_1 - b_1c_2 + a_1c_2 - b_1b_2 + a_1b_2 + a_2b_1 - a_1a_2
\end{pmatrix}
\left(\frac{z^2}{2} + z^3p(z)\right).
\]

For some polynomial \( p \in M_2(C)[z] \). In particular \( F_1F_2 \in \Gamma \). Since \( M_2(C)[z] \) is an algebra and \( \Gamma \) is closed for operations induced by \( M_2(C)[z] \) we have that \( \Gamma \) is an algebra.

**Step 2**: There exists a finite dimensional vector space \( E \) such that \( \Gamma = E \oplus z^3M_2(C)[z] \).

Consider \( F \in \Gamma \), then

\[
F(z) = \left(\begin{array}{cc}
  a & 0 \\
  b - a & b
\end{array}\right) + \left(\begin{array}{cc}
  c & c \\
  a - b - c & -c
\end{array}\right)z + \left(\begin{array}{cc}
  a - b - c & c + a - b \\
  d & e
\end{array}\right)\left(\frac{z^2}{2} + z^3p(z)\right)
\]

where

\[
\begin{align*}
  a_1 &= \left(\begin{array}{cc}
  1 & 0 \\
  -1 & 0
\end{array}\right) + \left(\begin{array}{cc}
  0 & 0 \\
  1 & 0
\end{array}\right)z + \left(\begin{array}{cc}
  1 & 1 \\
  0 & 0
\end{array}\right)\frac{z^2}{2}, \\
  a_2 &= \left(\begin{array}{cc}
  0 & 0 \\
  1 & 1
\end{array}\right) + \left(\begin{array}{cc}
  0 & 0 \\
  -1 & 0
\end{array}\right)z + \left(\begin{array}{cc}
  -1 & -1 \\
  0 & 0
\end{array}\right)\frac{z^2}{2}, \\
  a_3 &= \left(\begin{array}{cc}
  1 & 1 \\
  -1 & -1
\end{array}\right)z + \left(\begin{array}{cc}
  -1 & 1 \\
  0 & 0
\end{array}\right)\frac{z^3}{2}, \\
  a_4 &= \left(\begin{array}{cc}
  0 & 0 \\
  1 & 0
\end{array}\right)\frac{z^3}{2},
\end{align*}
\]

If \( E = \text{span}\{a_i | 1 \leq i \leq 5\} \) we obtain this step.

**Step 3**: The algebra \( \Gamma \) is generated by \( E, C \cdot \langle E \rangle = \Gamma \).

Since \( a_1 + a_2 = I \) we have that \( E = \text{span}\{I, a_1, a_3, a_4, a_5\} \). On the other hand,

\[
\begin{align*}
  a_1^2 &= \frac{4 + 2x^2 + 2x^3 + z^4}{4}e_{11} + \frac{2x^2 + z^4}{4}e_{12} + \frac{2x^2 - z^2 + z^3}{2}e_{21} + \frac{z^3 - z^2}{2}e_{22} \\
  a_1a_3 &= -\frac{4z + 2x^2 + z^4}{4}e_{11} + \frac{4x + 2x^3 + z^4}{4}e_{12} + \frac{2z - 3x^2 + z^3}{2}e_{21} + \frac{-2z + z^2 + z^3}{2}e_{22} \\
  a_1a_4 &= \frac{e_{11}z^4}{4}, \quad a_1a_5 = \frac{e_{12}z^4}{4}, \quad a_3a_1 = -\frac{z^4 - 4x^3}{4}e_{11} - \frac{z^4 - 2x^3}{4}e_{12} + \frac{2x^2 + z^3 - z^2}{2}e_{21} - \frac{z^3}{2}e_{22} \\
  a_3^2 &= \frac{z^4 - 6x^3}{4}e_{11} - \frac{2x^3 + z^4}{4}e_{12} + \frac{z^3}{2}e_{21} - \frac{z^3}{2}e_{22} \\
  a_3a_4 &= \frac{2x^3 + z^4}{4}e_{11} + \frac{z^3}{2}e_{21}, \quad a_3a_5 = \frac{2x^3 + z^4}{4}e_{12} - \frac{z^3}{2}e_{22} \\
  a_4a_1 &= \frac{2x^2 + z^4}{4}e_{21} + \frac{z^4}{4}e_{22}, \quad a_4a_3 = -\frac{z^4 - 2x^3}{4}e_{21} + \frac{2x^2 + z^4}{4}e_{22} \\
  a_4^2 &= 0, \quad a_4a_5 = 0, \quad a_5a_1 = \frac{z^3 - z^2}{2}e_{21}, \quad a_5a_3 = -\frac{z^3}{2}e_{21} - \frac{z^3}{2}e_{22}, \quad a_5a_4 = \frac{e_{21}z^4}{4}, \quad a_5^2 = \frac{e_{22}z^4}{4}.
\end{align*}
\]
Therefore, $e_{ij}z^3 \in \mathbf{C} \cdot \langle E \rangle$ and $e_{ij}z^4 \in \mathbf{C} \cdot \langle E \rangle$ for $1 \leq i, j \leq 2$ and using $\alpha_4, \alpha_5$ we obtain that $e_{ij}z^k \in \mathbf{C} \cdot \langle E \rangle$ for $1 \leq i, j \leq 2$ and $k \geq 3$. In particular $\mathbf{C} \cdot \langle E \rangle = \Gamma$.

Step 4: The inclusion $\mathbf{A} \cap \bigoplus_{k=0}^2 M_2(\mathbb{C})[z]_k \subset E$.

Let $F \in \mathbf{A} \cap \bigoplus_{k=0}^2 M_2(\mathbb{C})[z]_k$ then there exists $\mathcal{L} = \mathcal{L}(x, \partial_x)$ such that $\mathcal{L}\psi = \psi F$. We write

$$F(z) = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} + \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} z + \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} z^2.$$

After a computation we obtain that

$$\mathcal{L} = \begin{pmatrix} \frac{s_0^1}{s_0^2} x^3 + (s_0^1 - 2s_0^2 - 2s_0^3) x^2 + s_0^1 \frac{x}{x^2 - 2x^2} & \frac{s_0^1}{s_0^2} x^2 + (s_0^1 - 2s_0^2 - 2s_0^3) x + s_0^1 \frac{x}{x^2 - 2x^2} \\ s_0^1 \frac{x^2}{x^2 - 2x^2} x^3 + (s_0^1 - 2s_0^2 - 2s_0^3) x^2 + s_0^1 \frac{x}{x^2 - 2x^2} & s_0^1 \frac{x^2}{x^2 - 2x^2} x^2 + (s_0^1 - 2s_0^2 - 2s_0^3) x + s_0^1 \frac{x}{x^2 - 2x^2} \end{pmatrix}$$

and

$$s_{20}^1 = s_{00}^2 - s_{10}^2 = 0, s_{10}^2 = s_{01}^2, s_{20}^2 = s_{11}^2, s_{21}^2 = s_{12}^2, s_{22}^2 = s_{13}^2, s_{11}^2 = s_{02}^2 - s_{12}^2 - s_{22}^2.$$

then $F \in E$.

Step 5: The inclusion $E \subset \mathbf{A} \cap \bigoplus_{k=0}^2 M_2(\mathbb{C})[z]_k$.

By the previous step we have Equation 20 valid for every $F \in E$ and $(\mathcal{L}\psi)(x, z) = \psi(x, z)F(z)$, then $F \in \mathbf{A} \cap \bigoplus_{k=0}^2 M_2(\mathbb{C})[z]_k$.

We are under the hypothesis of Theorem 3, this concludes the proof of the assertion.
Conclusion and Final Comments

In this article we characterized the bispectral triples associated to a certain class of matrix-valued eigenfunctions. Furthermore, we established important properties of the full rank 1 algebras as a model of some bispectral algebras. These properties include the fact that they are Noetherian and finitely generated. An important role was played by the Ad-condition due to the fact that the matrix-valued operators were acting from opposite directions.

Natural questions arise when searching for a characterization of the bispectral partners to a given operator. Indeed, there is a family of maps \( \{ P_k \}_{k \in \mathbb{N}} \) with the translation and product properties of Theorem 5 and Lemma 5 which generate the algebra in Example 3.1. Is it possible to do this for the general case? Or, would this be possible at least for the bispectral partners of a given Schrödinger operator?

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A The Ad-Condition and Polynomial Eigenvalues

We close this article with a generalization of a key lemma from the work of [DG86] to the noncommutative case.

**Proposition 2.** Let \( \mathcal{L} = \mathcal{L}(x, \partial_x) = \sum_{j=0}^{l} L_j(x) \partial_x^j, \Theta = \Theta(x, \partial_x) = \sum_{s=0}^{m} \theta_s(x) \partial_x^s \) then \([L_l, \theta_m] = 0\) implies \(\deg_{\theta_s}(ad(\mathcal{L})(\Theta)) \leq m + l - 1\) and \([L_l, \theta_m] \neq 0\) implies \(\deg_{\theta_s}(ad(\mathcal{L})(\Theta)) = m + l\).

**Proof.** By definition \(ad(\mathcal{L})(\Theta) = [\mathcal{L}, \Theta] = \mathcal{L}\Theta - \Theta\mathcal{L}\) then

\[
(ad\mathcal{L})(\Theta) = \mathcal{L}\Theta - \Theta\mathcal{L} = \left( \sum_{j=0}^{l} L_j(x) \partial_x^j \right) \left( \sum_{s=0}^{m} \theta_s(x) \partial_x^s \right) - \left( \sum_{s=0}^{m} \theta_s(x) \partial_x^s \right) \left( \sum_{j=0}^{l} L_j(x) \partial_x^j \right) \\
= \sum_{j=0}^{l} \sum_{s=0}^{m} L_j \partial_x^j (\theta_s \partial_x^s) - \sum_{s=0}^{m} \sum_{j=0}^{l} \theta_s \partial_x^s (L_j \partial_x^j) = \sum_{j=0}^{l} \sum_{k=0}^{j} \sum_{s=0}^{m} L_j \left( \begin{array}{c} j \\ k \end{array} \right) \theta_s^{(k)} \partial_x^{j-k+s} - \sum_{s=0}^{m} \sum_{j=0}^{l} \theta_s \sum_{r=0}^{s} \left( \begin{array}{c} s \\ r \end{array} \right) L_j^{(r)} \partial_x^{s-r+j}
\]

\[
= \sum_{j=0}^{l} \sum_{s=0}^{m} \sum_{k=0}^{j} \sum_{j=0}^{l} L_j \left( \begin{array}{c} j \\ k \end{array} \right) \theta_s^{(j-k)} \partial_x^{k+s} - \sum_{s=0}^{m} \sum_{j=0}^{l} \theta_s \sum_{j=0}^{l} \left( \begin{array}{c} j \\ s \end{array} \right) L_j^{(j-s)} \partial_x^{k+s}
\]

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we obtain:

The claim is clear for 

Now we prove by induction that

If we assume the system of equations 

In particular if 

The previous proposition implies that if 

(1) Assume the condition for 

or 

\[ L_l \theta_m - \theta_m L_l = [L_l, \theta_m], \] hence 

\[ \[L_k, \theta_m\] = 0 \implies \deg_{\partial_x}(ad(\mathcal{L})(\theta)) \leq m + l - 1 \quad \text{and} \quad [L_k, \theta_m] \neq 0 \implies \deg_{\partial_x}(ad(\mathcal{L})(\theta)) = m + 1. \]

The previous proposition implies that if 

or 

then 

\[ (k, s) = (l, m - 1), \] therefore

\[
a_{m+l-1} = \sum_{j=1-1}^1 L_j \left( \frac{j}{j-1} \right) \theta_m^{(j-1)} - \theta_m L_{m-1} - L_l \theta_m - m \theta_m L_l = [L_{l-1}, \theta_m] + [L_l, \theta_{m-1}] + LL_j \theta_m - m \theta_m L_l.
\]

If 

\[ [L_{l-1}, \theta_m] = 0, [L_l, \theta_{m-1}] = 0, \]

then

\[ a_{m+l-1} = LL_j \theta_m - m \theta_m L_l. \]

In particular if 

\[ m = 0, \theta_0 = 0 \quad \text{and} \quad [L_{l-1}, \theta_0] = 0, \]

then

\[ a_{l-1} = LL_j \theta_0. \]

If we assume the system of equations 2 we obtain:

\[
(ad \mathcal{L})(\theta) = (\partial_x \mathcal{L})(\theta) = (\partial_x \mathcal{E}) - \partial_x \mathcal{L} \psi = \mathcal{L}(\psi B) - \psi F = (\psi F) B - \psi B F = \psi [F, B] = \psi (ad F)(B).
\]

Now we prove by induction that

\[
(ad \mathcal{L})^r(\theta) \psi = \psi (ad F)^r(B),
\]

for all \( r \in \mathbb{Z}_+ \).

The claim is clear for \( r = 1 \). Assume the condition for \( r \) and consider the case \( r + 1 \), then

\[
\psi (ad F)^{r+1}(B) = \psi (ad F)(ad F)^r(B) = (F \psi)(ad F)^r(B) - F(\psi)(ad F)^r(B)
\]

\[
= (F \psi)(ad F)^r(B) - (ad \mathcal{L})^r(\theta) \psi = (\mathcal{L} \psi)(ad F)^r(B) - (ad \mathcal{B})^r(\theta) \psi
\]

\[
= (\mathcal{L} \psi)(ad F)^r(B) - (ad \mathcal{L})^r(\theta)(\mathcal{L} \psi) = (\mathcal{L} \psi)(ad F)^r(B) - (ad \mathcal{L})^r(\theta)(\mathcal{L} \psi)
\]

\[
= (\mathcal{L}(ad \mathcal{L})^r(\theta) - (ad \mathcal{L})^r(\theta) \mathcal{L} \psi = (ad \mathcal{L})(ad \mathcal{L})^r(\theta) \psi = (ad \mathcal{L})^{r+1}(\theta) \psi.
\]
If deg$_{\theta_0}$ $\mathcal{B} = m$ and $F$ is scalar we use the Proposition 2 to conclude $(ad \mathcal{L})^{m+1}(\theta)\psi = (\psi(ad F))^{m+1}(\mathcal{B}) = 0$, similarly if deg $\mathcal{L} = l$ then deg$_{\theta_i} (ad \mathcal{L})^{m+1}(\theta) \leq (m+1)(l-1)$, in our case deg$_{\theta_z} (ad \mathcal{L})^{m+1}(\theta) \leq m + 1 < \infty$ since $\psi(\cdot, z) \in \ker((ad \mathcal{L})^{m+1}(\theta))$ for every $z \in \mathbb{C}$ and $\{\psi(\cdot, z)\}_{z \in \mathbb{C}}$ is a linearly independent set and dim $\ker((ad \mathcal{L})^{m+1}(\theta)) \leq$ deg$_{\theta_z} (ad \mathcal{L})^{m+1}(\theta)$ if $(ad \mathcal{L})^{m+1}(\theta) \neq 0$ we have that $(ad \mathcal{L})^{m+1}(\theta) = 0$.

Finally we claim that if $\mathcal{L} = \sum_{i=0}^{l} L_i \partial_x^i$ with $L_i \in \mathbb{C} \setminus \{0\}$ and $L_{l-1} = 0$, then $\text{coeff}((ad \mathcal{L})^{k+1}(\theta), \partial_x, (k+1)(l-1)) = (IL_i)^{k+1} \theta^{(k+1)}(0)$ for every $k \in \mathbb{N}$.

The claim is obvious for $k = 0$. If we assume that the claim is valid for $k$, then

$$\text{coeff}((ad \mathcal{L})^{k+2}(\theta), \partial_x, (k+2)(l-1)) = \text{coeff}((ad \mathcal{L})(ad \mathcal{L})^{k+1}(\theta), \partial_x, (k+2)(l-1)) = ILL_i \partial_x ((IL_i)^{k+1} \theta^{(k+1)}) = (IL_i)(IL_i)^{k+1} \theta^{(k+2)} = (IL_i)^{k+2} \theta^{(k+2)}$$

because $L_i$ is constant and scalar.

Since $(ad \mathcal{L})^{m+1}(\theta) = 0$ we have that $\theta^{(m+1)} = 0$ and $\theta$ has to be a polynomial with deg $\theta \leq m$.

References

[ABvW17] PN Anh, GF Birkenmeier, and L van Wyk. Peirce decompositions, idempotents and rings. arXiv preprint arXiv:1702.05261, 2017.

[BGK08] Maarten Bergvelt, Michael Gekhtman, and Alex Kasman. Spin calogero particles and bispectral solutions of the matrix KP hierarchy. Mathematical Physics Analysis and Geometry, 12, 07 2008.

[BL08] Carina Boyallian and Jose Liberati. Matrix-valued bispectral operators and quasideterminants. Journal of Physics A: Mathematical and Theoretical, 41:365209, 08 2008.

[Cas17] William R. Casper. Bispectral Operator Algebras. University of Washington, 2017. Thesis (Ph.D.)–University of Washington, 2017-06.

[DG86] J.J. Duistermaat and F.A. Grünbaum. Differential equations in the spectral parameter. Communications in Mathematical Physics, 103(2):177–240, 1986.

[DG04] A. J. Durán and F.A. Grünbaum. Orthogonal matrix polynomials satisfying second-order differential equations. Int. Math. Res. Not., 2004(10), 2004.

[GHY15] Joel Geiger, E. Horozov, and Milen Yakimov. Noncommutative bispectral darboux transformations. Transactions of the American Mathematical Society, 369, 08 2015.

[GI03] Alberto Grünbaum and Plamen Iliev. A noncommutative version of the bispectral problem. J. Comput. Appl. Math., 161:99–118, 12 2003.
[Grü14] F. Alberto Grünbaum. Some noncommutative matrix algebras arising in the bispectral problem. SIGMA Symmetry Integrability Geom. Methods Appl., 10:078, 2014.

[Hor02] E. Horozov. Bispectral operators of prime order. Communications in Mathematical Physics, 231(2):287–308, 2002.

[Ili99] P. Iliev. Discrete versions of the Kadomtsev-Petviashvili hierarchy and the bispectral problem. PhD thesis, Dissertation, 1999.

[Kas15] Alex Kasman. Bispectrality of n-component KP wave functions: A study in non-commutativity. Symmetry, Integrability and Geometry: Methods and Applications, 11, 05 2015.

[Sta85] J. T. Stafford. On the ideals of a noetherian ring. Transactions of the American Mathematical Society, 289(1):381–392, 1985.

[SZ01] Alexander Sakhnovich and Jorge Zubelli. Bundle bispectrality for matrix differential equations. Integral Equations Operator Theory, 41(4):472, 2001.

[Wil93] George Wilson. Bispectral commutative ordinary differential operators. J. Reine Angew. Math., 442:177–204, 1993.

[ZM91] J.P. Zubelli and F. Magri. Differential equations in the spectral parameter, Darboux transformations and a hierarchy of master symmetries for KdV. Communications in Mathematical Physics, 141(2):329–351, 1991.

[Zub90] J.P. Zubelli. Differential equations in the spectral parameter for matrix differential operators. Physica D: Nonlinear Phenomena, 43(2-3):269–287, 1990.

[Zub92a] Jorge P. Zubelli. On a zero curvature problem related to the ZS-AKNS operator. J. Math. Phys., 33(11):3666–3675, 1992.

[Zub92b] Jorge P. Zubelli. On the polynomial τ-functions for the KP hierarchy and the bispectral property. Lett. Math. Phys., 24(1):41–48, 1992.

[Zub92c] Jorge P. Zubelli. Rational solutions of nonlinear evolution equations, vertex operators, and bispectrality. J. Differential Equations, 97(1):71–98, 1992.

[ZVS00] Jorge P. Zubelli and D. S. Valerio Silva. Rational solutions of the master symmetries of the KdV equation. Communications in Mathematical Physics, 211(1):85–109, 2000.