Solar Neutrinos and Grand Unification

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Abstract

We consider the Grand Unification (GU) scenario for neutrino masses which is based on the see-saw mechanism with the mass of the heaviest right handed (RH) neutrino at the GU-scale: $M_3 \sim \Lambda_{\text{GU}}$, and on the quark-lepton symmetry for fermions from the third generation. The scenario predicts for the light neutrinos: $m_3 \sim (2-4) \cdot 10^{-3} \text{ eV}$ and $m_2 \sim (0.3-3) \cdot 10^{-5} \text{ eV}$ (in the case of a linear mass hierarchy of the RH neutrinos or/and in presence of the Planck scale suppressed non-renormalizable operators). It also predicts large $\nu_e - \nu_\mu$ mixing: $\sin^2 2\theta_{e\mu} \gtrsim 0.2$. In this scenario the solar neutrinos ($\nu_\odot$) undergo both the $\nu_e \rightarrow \nu_\tau$ resonance conversion in the Sun and substantial $\nu_e \rightarrow \nu_\mu$ vacuum oscillations on the way from the Sun to the Earth. The interplay of both effects enlarges the range of neutrino parameters which solve the $\nu_\odot$-problem. In particular, $\nu_e - \nu_\tau$ mixing angle can be as small as the corresponding quark mixing: $\sin^2 2\theta_{e\tau} \geq (2 - 5) \cdot 10^{-4}$. The scenario predicts peculiar (oscillatory) distortion of the boron neutrino energy spectrum and seasonal variations of signals. Manifestations of these effects in the Super-Kamiokande and SNO experiments are studied.
1. Introduction

Small neutrino masses, in particular, the ones implied by the solar neutrino data, may be considered as a possible indication of Grand Unification (GU). This statement holds in the following context (which we will call the Grand Unification scenario for neutrino masses).

1. Neutrino masses are generated by the see-saw mechanism \[1][2]:

\[ m = -\frac{m_D^2}{M} = -\frac{h_D^2 v_u^2}{M}, \]

where \( m_D \) is the Dirac neutrino mass, \( m_D = h_D v_u \), \( h_D \) is the neutrino Yukawa coupling, \( v_u \) is the VEV of the Higgs doublet \( H_u \), and \( M \) is the Majorana mass of the right handed (RH) neutrino. A direct mass term for the left handed components may be generated by the induced VEV of the Higgs triplet of \( SU(2)_L \) [2]. We assume that this term, if it exists, is much smaller than that in (1), at least for the heaviest neutrino.

2. Quark-lepton symmetry is realized for the third generation of fermions at the Grand Unification scale, \( \Lambda_{GU} = 2 \cdot 10^{16} \) GeV, which is defined as the scale of the gauge coupling unification in the Minimal Supersymmetric Standard Model. The quark-lepton symmetry relates the (Dirac type) Yukawa coupling of the neutrino with the Yukawa coupling of the top quark at \( \Lambda_{GU} \):

\[ h_{3D}(\Lambda_{GU}) = h_t(\Lambda_{GU}). \]

At the electroweak scale \( h_t(M_{EW}) = m_t/v_u \), where \( m_t \) is the top quark mass. The existence of quark-lepton symmetry and the equality (2) for the third generation are supported by the asymptotic mass relation \( m_b(\Lambda_{GU}) = m_\tau(\Lambda_{GU}) \) which after renormalization effects results in a successful prediction of the \( b \)-quark mass at low energies in supersymmetric GUT’s [3].

The boundary condition (2) is satisfied, in particular, in GUT’s based on \( SO_{10} \). Furthermore, in \( SO_{10} \)-type unification all four Yukawa couplings in the third generation can be equal: \( h_b = h_\tau = h_t = h_{3D} \).

Exact quark-lepton symmetry is, however, broken for the light generations. This follows from the failure of the asymptotic mass relations \( m_s(\Lambda_{GU}) = m_\mu(\Lambda_{GU}) \) and \( m_d(\Lambda_{GU}) = \)
which are inconsistent with low energy determinations. One possible origin of this breakdown is non-renormalizable operators which are suppressed by $\Lambda_{GU}/M_P$, where $M_P$ is the Planck scale. These operators can contribute significantly to the lighter generation masses while leaving the third generation relation essentially uncorrected. Therefore for the lighter generations one can expect relations of the type (2) to hold only as an order of magnitude. One would also expect the leptonic mixing angles involving the lighter generations to be different from their quark analogs.

3. The Majorana mass of the RH neutrino from the third generation is at the $GU$ scale: $M_3 \sim \Lambda_{GU}$. Using this mass, the boundary condition (2) and the known value of the top quark mass one finds from the see-saw formula the mass of the third neutrino ($\approx \nu_\tau$) $m_3 \sim \text{several } \times 10^{-3}$ eV. The mass $m_3$ is in the range suggested by the Mikheyev-Smirnov-Wolfenstein (MSW) solution of the solar ($\nu_\odot$-) neutrino problem:

$$m = (2 - 4) \cdot 10^{-3} \text{ eV}$$

(barring any degeneracy in the spectrum). Thus in the $GU$-scenario the $\nu_\odot$-problem can be solved by the $\nu_e \rightarrow \nu_\tau$ resonance conversion in the Sun provided the mixing between the first and the third generations is sufficiently large. *Vice versa*, using the value of the light neutrino mass (3) and the top quark mass at the electroweak scale along with the boundary condition (2) one finds from the see-saw formula $M_3 \sim 10^{16}$ GeV. The approximate equality of the mass scales $\Lambda_{GU}$ and $M_3$ may be considered as a hint for Grand Unification in addition to the gauge coupling unification and the $b - \tau$ unification.

An alternative solution to the solar neutrino problem is via $\nu_e \rightarrow \nu_\mu$ resonance conversion. In this case the boundary condition for the second generation, $m_{2D}(\Lambda_{GU}) \approx m_c(\Lambda_{GU})$ ($m_c$ is the charm quark mass), leads to $M_2 \sim (10^{10} - 10^{11})$ GeV in the intermediate mass scale (see, e.g., [7]).

Note that in the $2\nu$-case, it is impossible to distinguish $\nu_e \rightarrow \nu_\mu$ and $\nu_e \rightarrow \nu_\tau$ transitions using the $\nu_\odot$-data only. Both $\nu_\mu$ and $\nu_\tau$ are detected by the neutral current interactions which are the same (up to higher order corrections) for both neutrinos. The situation can
be different if the mixing of all three neutrinos is taken into account. We will show that solar neutrino data alone may disentangle the two possibilities (GU-scale scenario and the intermediate scale scenario).

A possible discovery of $\nu$-oscillations by CHORUS/NOMAD \cite{8} or new experiments like E803 (COSMOS) \cite{9} would imply $m_3 \sim O(1 \text{ eV})$, favoring the intermediate scale physics. Also confirmations of the LSND result \cite{10} and oscillation interpretation of the atmospheric neutrino anomaly will exclude the simplest version of the GU-scenario.

In this paper we reconsider the Grand Unification scenario and study its signatures in the solar neutrinos. We calculate $M_3$ implied by the $\nu_e \rightarrow \nu_{\tau}$ solution of the $\nu_\odot$-problem using the known mass of the top quark and discuss possible relations of $M_3$ to $\Lambda_{GU}$ (see sect. 2). The $\nu_e - \nu_\tau$ mixing is considered in sect. 3. We show that it is quite plausible that $\nu_e \rightarrow \nu_{\tau}$ conversion will be accompanied by sizable $\nu_e \rightarrow \nu_\mu$ vacuum oscillations of solar neutrinos (sect. 4). The interplay of both transitions (sect. 5) has a number of consequences: it enlarges the region of neutrino parameters in which one can get a correct description of the $\nu_\odot$-data (sect. 6), it leads to peculiar distortion of the boron neutrino spectrum (sect. 7) and to seasonal variations of signals (sect. 8). Possible manifestations of these effects which can be considered as signatures of the GU-scenario in the Super-Kamiokande \cite{11,12} and SNO \cite{13} experiments are studied in sects. 7, 8. In sect. 9 we summarize our main results and also comment on possible modifications of the GU-scenario which would allow one to explain other neutrino data.

2. $\nu_e - \nu_\tau$ conversion of solar neutrinos and the scale of Grand Unification

Let us focus on the mass of the third neutrino $m_3$ which determines $\Delta m^2_{13}$ responsible for the $\nu_e - \nu_\tau$ conversion. We perform first a calculation of $m_3$ neglecting $\nu_\tau$ mixing with lighter generations. The procedure adopted is as follows. We use (1), (2) and the two–loop renormalization group equations (RGE) with the particle content of the Minimal Supersymmetric Standard Model. We fix the QCD gauge coupling $\alpha_3(M_Z) \equiv g^2_3(M_Z)/4\pi = 0.118$ at the $Z^0$ boson mass scale, $M_Z$, as an input and set the effective supersymmetry threshold at
The renormalized neutrino mass at a scale $\mu < \Lambda_{GU}$ with the boundary condition (2) can be written as

$$m_3(\mu) = \frac{m_3^2(\mu)}{M_{03}} \left[ \frac{\kappa(\mu)}{\kappa(\Lambda_{GU})} \right] \left[ \frac{h_t^2(\Lambda_{GU})}{h_t^2(\mu)} \right], \quad (4)$$

where $\kappa$ is the coefficient of the effective dimension-5 neutrino mass operator, $L_{eff} = \kappa_{ij} L_i L_j H_u H_u$, $M_{03}$ is the mass of the RH neutrino in the absence of mixing between generations. The RGE for $\kappa$ (which has been worked out only to one loop) is given by

$$\frac{d\kappa}{dt} = \frac{1}{16\pi^2} \left[ 6h_t^2 + 2h_r^2 - \frac{6}{5}g_1^2 - 6g_2^2 \right]. \quad (5)$$

Here $g_1$ and $g_2$ are the $U(1)$ and $SU(2)$ gauge couplings, and $h_r$ is the Yukawa couplings of the tau lepton. The one–loop RGE for the top Yukawa coupling $h_t$ is

$$\frac{dh_t^2}{dt} = \frac{h_t^2}{8\pi^2} \left[ 6h_t^2 + h_b^2 - \frac{13}{15}g_1^2 - 3g_2^2 - \frac{16}{3}g_3^2 \right], \quad (6)$$

where $h_b$ is the bottom Yukawa coupling.

In our numerical calculations of $m_3$ we kept the full two–loop RGE equations for $h_t$, $h_b$, $h_\tau$. The results are summarized in Table I, where the values of $m_3$ are shown for different values of $\tan \beta \equiv v_u/v_d$ and three possible values of $m_t$; the Majorana mass is fixed to be $M_{03} = 10^{16}$ GeV.

As follows from Table I the $\nu_\tau$–mass is insensitive to the value of $\tan \beta$ except for $\tan \beta \lesssim 2$. This feature can be understood from the one-loop semi–analytic expression for $m_3$. Indeed, the neutrino mass at a lower scale $\mu$ can be written as

$$m_3(\mu) = \frac{m_3^2(\mu)}{M_{03}} \left( \frac{\alpha_1(\mu)}{\alpha_G} \right)^{4/99} \left( \frac{\alpha_3(\mu)}{\alpha_G} \right)^{-16/9} \times \exp \left[ \frac{1}{8\pi^2} \int_{\ln \mu}^{\ln \Lambda_{GU}} dt \left( 3h_t^2 + h_b^2 - h_\tau^2 \right) \right], \quad (7)$$

where $\alpha_G \simeq 1/25$ is the unified gauge coupling constant and $\alpha_1 \equiv g_1^2/4\pi$. The neutrino mass depends on $\tan \beta$ via the Yukawa coupling $h_t = m_t\sqrt{1 + \tan^2 \beta/(v\tan \beta)}$. Obviously, for $\tan \beta \geq 2$, the coupling $h_t$ is essentially independent of $\tan \beta$.

Using the results of Table I and the mass of the light neutrinos suggested by the solar neutrino data we can estimate the RH-neutrino mass $M_{03}$. For $m_t = 175$ GeV and
\( \tan \beta = 1.7 \) we find \( M_{03} = (0.5 - 0.7) \Lambda_{GU} \). For \( \tan \beta \lesssim 2 \) the top Yukawa coupling is near its fixed point value. Such a low value may be preferable from the point of view of successful \( b - \tau \) unification. On the other hand, for \( \tan \beta \geq 2 \), we get \( M_{03} = (0.18 - 0.25) \Lambda_{GU} \). Thus for the whole range of \( \tan \beta \) the \( \Delta m^2 \) required for solution of the \( \nu_{\odot} \)-problem corresponds to \( M_{03} \) in the interval

\[ M_{03} \lesssim (0.2 - 0.7) \Lambda_{GU}. \] (8)

One remark is in order. If \( M_{03} \) is substantially below \( \Lambda_{GU} \), one should take into account the effect of the Dirac neutrino Yukawa coupling \( h_{3D} \) on the evolution of \( h_t, h_b, h_\tau \) and \( h_{3D} \) itself in the interval between \( M_{03} \) and \( \Lambda_{GU} \), since in this interval loops involving \( \nu_{3R} \) will contribute. Although the momentum range for the running is rather small the effect can be substantial since both \( h_t \) and \( h_{3D} \) are large (near the fixed point value of \( h_t \)). The boundary condition at \( M_{03} \) on the Dirac Yukawa coupling is modified to

\[
\frac{h^2_{3D}(M_{03})}{h^2_t(M_{03})} = \frac{h^2_{3D}(\Lambda_{GU})}{h^2_t(\Lambda_{GU})} \left( \frac{\alpha_1(M_{03})}{\alpha_G} \right)^{4/99} \left( \frac{\alpha_3(M_{03})}{\alpha_G} \right)^{-16/9} \times 
\exp \left[ \frac{1}{8\pi^2} \int_{\ln \Lambda_{GU}}^{\ln M_{03}} dt \left( 3h^2_{3D} - 3h^2_t + h^2_\tau - h^2_b \right) \right].
\]

In the relatively small momentum range between the scales \( M_{03} \) and \( \Lambda_{GU} \) the exponential factor above is almost unity because \( h_{3D} = h_t \) and \( h_b = h_\tau \) at \( \Lambda_{GU} \). A difference between \( h_t \) and \( h_{3D} \) develops because of the difference in the gauge contribution. The effect of the gauge couplings is to diminish the Dirac mass \( m_{3D} \), and consequently, the mass \( m_3 \) at low scale. Indeed, for \( M_{03}/\Lambda_{GU} = 1/10 \), \( m_t = 175 \) GeV and the set of values \( \tan \beta = (1.7, 3, 10, 30, 60) \) we find the neutrino mass \( m_3 = (2.15, 0.95, 0.82, 0.84, 0.96) \cdot 10^{-3}(10^{16}\text{GeV}/M_3) \) eV. Comparing these values with the numbers in Table I we conclude that the threshold effect related to \( M_{03} \) decreases the predicted value of \( m_3 \) by 5 - 25%. Equivalently, the predicted value of \( M_{03} \) for a fixed \( m_3 \) decreases by as much as 25% for \( \tan \beta = 1.7 \) (fixed point value), and by 6% for moderate and large \( \tan \beta \) (\( M_{03}/\Lambda_{GU} = 1/10 \)). So, the values of \( M_{03} \) are not changed significantly from those in (8).
Let us now take into account mixing of the third generation with the light generations of fermions. For simplicity we will consider mixing between two (say second and third) generations. Suppose $M_{02}$ is the Majorana mass of the RH neutrino component which leads via the see-saw mechanism to a certain mass $m_2$ in the absence of mixing between generations. For fixed values of $m_2$ and $m_3$ as well as the Dirac masses $m_{2D}$, $m_{3D}$, the mixing in the Majorana mass matrix of the RH neutrinos, $\hat{M}$, will change values of the corresponding RH neutrino masses: $M_2 \neq M_{02}$ and $M_3 \neq M_{03}$, where $M_2$ and $M_3$ are the masses in presence of mixing. It can be shown, however, that the product of masses does not depend on the mixing, that is, $M_2 \cdot M_3 = M_{02} \cdot M_{03}$ (up to small renormalization group corrections) $[7]$. From this equality we get a relation between hierarchies of masses in the presence of mixing, $\epsilon \equiv M_2/M_3$, and without mixing, $\epsilon_0 \equiv M_{02}/M_{03}$:

$$\epsilon = \epsilon_0 \left(\frac{M_{03}}{M_3}\right)^2. \quad (9)$$

The change $M_{03} \rightarrow M_3$ is related to the mixing angle $\theta_M$ in $\hat{M}$ in the following way $[7]$:

$$\sin^2 \theta_M \approx \epsilon \left[\frac{M_3}{M_{03}} - 1\right]. \quad (10)$$

The mixing in $\hat{M}$ can raise the mass $M_3$ to the $GU$-scale: $M_{03} \rightarrow M_3 \sim \Lambda_{GU}$, for fixed masses of the light neutrinos and for fixed Dirac masses. This implies that the second mass, $M_2$, decreases by factor $M_{03}/M_3$, and according to $[7]$ the hierarchy of masses is enhanced by a factor $(M_{03}/M_3)^2$. If $M_3$ increases, e.g., by a factor 3, the hierarchy of masses is enhanced by one order of magnitude. Similar results can be obtained for mixing between the third and the first generations.

From (8) we see that the mass of $\nu_3$ in the absence of mixing, $M_{03}$, is (1.5 - 5) times smaller than $\Lambda_{GU}$. As follows from (10) small mixing between generations $\theta_M \sim O(\sqrt{\epsilon})$ is enough to get the equality $M_3 = \Lambda_{GU}$. Another consequence of the increase in $M_3$ and the strengthening of the hierarchy is the enhancement of mixing of the light neutrinos which will be discussed in the sect. 3.

The proximity of $\Lambda_{GU}$ and $M_3$ suggested by the solar neutrino data (or even possible
coincidence of these two values) can be considered as a hint for Grand Unification. Let us comment on possible relations between these two scales.

Since the neutrino mass term $M\nu^T_R \nu_R$ is a singlet of the standard model symmetry group, there is no direct relation between the neutrino mass $M$ and the scale of the gauge coupling unification. Some additional assumptions are needed to connect $M_3$ and $\Lambda_{GU}$ and, in general, one should not expect a coincidence of these scales. Possible relations between $M_{03}$ and $\Lambda_{GU}$ depend on the mechanism of $\nu_R$ mass generation. Two simple mechanisms are worth mentioning:

(i) The Majorana mass $M_3$ is generated directly by the Yukawa couplings with a Higgs multiplet $\Phi$

$$h_3M\nu^T_R \nu_R \Phi \, ,$$

so that $M_3 = h_3 \langle \Phi \rangle_0$. A relation between the interaction (11) and physics of Grand Unification can arise in $SO_{10}$ models, where $\Phi$ is the $\mathbf{126}$-plet. If $\Phi$ is responsible for the symmetry breaking $SO_{10} \rightarrow SU_5$ at the $GU$-scale, then $\langle \Phi \rangle_0 \sim \Lambda_{GU}$. Still $h_3M$ should be fixed. Note that $h_t(\Lambda_{GU}) \simeq (0.4 - 1.5)$ for a wide range of $\tan\beta$. Therefore, to get $M_{03}$ in the region (7) one should take $h_3M \sim (0.13 - 1.4) \cdot h_t$ in the absence of mixing. This allows for the possibility, especially if there is some mixing in the mass matrix $\hat{M}$, that all the Yukawa couplings of fermions from the third generation are the same, in particular, $h_{3M} \sim h_t$.

(ii) The mass $M_3$ can be generated by the effective operator

$$\frac{h^2}{M_S}\nu^T_R \nu_R \Phi \Phi \, ,$$

where $M_S$ is some mass at or above the $GU$-scale. In this case $M_3 \simeq h^2_R\langle \Phi \rangle^2_0/M_S$. In $SO_{10}$ models $\Phi$ is $\mathbf{16}_H$-plet, and the operator (12) is generated by terms in the superpotential

$$h_R \mathbf{16}_H \mathbf{16} \, + \, M_S SS \, ,$$

which mix $\nu_R$ with an $SO_{10}$-singlet or an adjoint, $S$, having the Majorana mass $M_S$. The $\mathbf{16}_H$-plet breaks $SO_{10}$ to $SU_5$ and it is possible to identify its VEV with $\Lambda_{GU}$. For
$M_S \sim (1 - 2) \cdot \Lambda_{GU}$ one can take $h_{3M} \sim h_t$ in agreement with equality of all Yukawa couplings in the third generation.

3. $\nu_e - \nu_\tau$ mixing

In the two neutrino case the mixing angle needed to solve the $\nu_\odot$-problem should be in the range determined by \[14\]:

$$\sin^2 2\theta = (3 - 10) \cdot 10^{-3}. \quad (14)$$

As we will see in sect. 4, the effect of mixing between $\nu_e$ and $\nu_\mu$ enlarges the region of solutions \[14\] to:

$$\sin^2 2\theta_{e\tau} = (0.2 - 10) \cdot 10^{-3}. \quad (15)$$

Let us first compare \[14\] with the corresponding mixing in the quark sector. Taking $\theta_{e\tau} \sim V_{td} \sim (4 - 11) \cdot 10^{-3}$, where $V_{td}$ is the element of the CKM quark mixing matrix, we get

$$\sin^2 2\theta_{e\tau} = (0.06 - 0.5) \cdot 10^{-3}. \quad \text{These values cover the lower part of the range (15).}$$

Thus, the $\nu_e - \nu_\tau$ mixing similar to the quark mixing is enough to solve the $\nu_\odot$- problem provided the $\nu_e - \nu_\mu$ mixing is sufficiently large: $\sin^2 2\theta > 0.3$ (see sect. 6). For $\sin^2 2\theta_{e\mu} < 0.3$ the values $\theta_{e\tau} \sim V_{td}$ are too small. In this connection let us consider the possibility of enhancing the mixing, so that $\theta_{e\tau} \sim (2 - 3) V_{td}$.

In view of the violation of quark-lepton symmetry among lighter generations there is no reason to expect exact equality of $\theta_{e\tau}$ and $V_{td}$. A difference in quark and lepton mixing can come both from difference in the Dirac mass matrices: $m_D \neq m_q$ (violation of the quark-lepton symmetry) and from the Majorana mass matrix (see-saw enhancement of lepton mixing \[17\]). Let us consider these two possibilities in order.

1) The quark-lepton symmetry for light generations can be broken by the Yukawa couplings with additional Higgs multiplets. The contributions from these couplings to masses correct the bad asymptotic mass relations $m_d(\Lambda_{GU}) = m_e(\Lambda_{GU})$ and $m_s(\Lambda_{GU}) = m_\mu(\Lambda_{GU})$. Such corrections tend to enhance the mixing in the lepton sector. An explicit example has
been given in the context of $SO_{10}$ \[18\]. There a minimal set of Higgs multiplets was introduced that couples to fermions: namely, a single $10$-plet and a single $\overline{126}$-plet. The Standard Model singlet from the $\overline{126}$-plet generates the Majorana mass term for the RH neutrinos. The same $\overline{126}$-plet also contains a pair of SM doublets which receive vacuum expectation values, contributing to the quark and lepton masses. As a consequence, the structure of the Majorana mass matrix gets related to the quark and lepton mass matrices. The Yukawa couplings of $\overline{126}$-plet correct the bad asymptotic mass relations $m_d(\Lambda_{GU}) = m_e(\Lambda_{GU})$ and $m_s(\Lambda_{GU}) = m_\mu(\Lambda_{GU})$. This led to the prediction $\theta_{e\tau} \sim 3V_{td}$, where the factor 3 is the same Clebsch-Gordon coefficient that corrects the mass relations for light generations \[19\]. In this case one gets $\sin^2 2\theta_{e\tau} = (0.5 - 4) \cdot 10^{-3}$ – within the range \[15\].

2). An enhancement of lepton mixing can follow from mixing in the Majorana mass matrix of the RH neutrinos \[17\]. Indeed, the total lepton mixing angle (for two generation case) can be written as

$$\theta = \theta_D + \theta_s,$$  \hspace{1cm} (16)

where $\theta_D$ follows from the Dirac mass matrices of the neutrinos and charge leptons ($\theta = \theta_D$, if $\hat{M} = \hat{I}$) and $\theta_s$ specifies the effect of the see-saw mechanism itself \[17\]. Let us consider mixing between the first and the third generation. The see-saw angle $\theta_s$ can be expressed in terms of the mass hierarchies $\epsilon_D \equiv m_{1D}/m_{3D}$, $\epsilon \equiv M_1/M_3$ and $\epsilon_0 \equiv M_{01}/M_{03}$ \[7\]:

$$\sin^2 \theta_s \approx \frac{\epsilon^2_D}{\epsilon} \left[ \sqrt{\epsilon_0} - 1 \right] = \frac{\epsilon^2_D}{\epsilon} \left[ \frac{M_3}{M_{03}} - 1 \right].$$  \hspace{1cm} (17)

For linear mass hierarchy of the RH neutrinos, $\epsilon_0 \sim \epsilon_D$, we get

$$\sin^2 \theta_s \approx \epsilon_D \frac{M_3}{M_{03}}.$$  \hspace{1cm} (18)

Therefore $\epsilon_D = 10^{-5}$ and $M_3/M_{03} \sim 2$ lead to $\theta_s \sim 3 \cdot 10^{-3}$, which is comparable with $V_{td}$. If the hierarchy of masses is strengthened by one order of magnitude ($\epsilon = 0.1\epsilon_0$), then the angle $\theta_s$ becomes $10^{-2}$. In this case $\theta_s$ gives the main contribution to the lepton mixing, and $\sin^2 2\theta_{e\tau}$ is of the order $10^{-3}$.
According to (10) for the linear hierarchy one gets \( \theta_M \sim \sqrt{\epsilon} \sim \sqrt{\epsilon_D} \). That is, the mixing in \( \hat{M} \) similar to mixing in the Dirac mass matrices can lead to enhancement of the lepton mixing up to the value implied by the \( \nu_\odot \)-problem. Moreover, as we discussed in sect. 2, this enhancement can be related to increase of \( M_3 \) to \( \Lambda_{GU} \).

There is an alternative mechanism for generating enhanced lepton mixing. Mixings in the lepton sector can differ substantially from those in the quark sector as a consequence of the nature of Grand Unification symmetry breaking. The stronger hierarchy observed in the up–quark masses relative to the hierarchies in the down quarks and charged lepton masses gives some indication towards such a possibility. One example was suggested in [20]. The main idea is that in \( SU_5 \) (or in the \( SU_5 \) decomposition of \( SO_{10} \)) \( u_i \) and \( u'_i \) quarks lie in the same \( 10_i \) representation, while the \( d_i \) and \( d'_i \) (and similarly leptons \( l_i \) and \( l'_i \) ) are in separate representations: \( 10_i \) and \( 5_i \) respectively. If a hierarchy factor \( \epsilon_D \) is associated with the \( 10 \)-plets, but not with the \( 5 \)-plets, the down quark and charged lepton masses (\( 5 \cdot 10 \) ) will scale as \( \epsilon_D \), whereas the up-quark masses (\( 10 \cdot 10 \) ) will scale as \( \epsilon_D^2 \). The left–handed CKM angles will be small, as they scale as \( \epsilon_D \). The right–handed CKM mixing angles for quarks (unphysical in the Standard Model) will be of order one, since the \( 5_i \) have no hierarchy factor. Since the left–handed lepton doublets are in \( 5_i \), their mixing angles will be large. The specific model constructed in Ref. [20] leads to \( \theta_{e\tau} \approx 0.032 \) which is within the range (15).

4. Parameters of \( \nu_e - \nu_\mu \) system

If the mixing angles in the Majorana matrix \( \hat{M} \) are small, the mass of the second neutrino can be estimated as

\[
{m_2} \approx - \frac{(h_c v_u)^2}{M_{02}},
\]

where \( h_c \) is the Yukawa coupling of the charm quark. For \( M_{02} \approx M_{03} \approx \Lambda_{GU} \) we find \( m_2 \leq 10^{-7} \text{ eV} \) and \( \Delta m_{12}^2 \approx 10^{-14} \text{ eV}^2 \) which is far below the sensitivity of the \( \nu_\odot \)-data. Let us assume a \textit{linear} mass hierarchy in the RH neutrino mass matrix, according to which
the eigenvalues of $\hat{M}$ have the same hierarchy as the eigenvalues of the Dirac mass matrix $m_D$: $M_i \propto m_{iD}$, or $\epsilon \sim \epsilon_D$. In this case $M_{02} \sim 10^{14}$ GeV, and for the light neutrino we get from the see-saw formula (1) $m_2 \approx (0.3 - 3) \cdot 10^{-5}$ eV. Consequently,

$$\Delta m^2_{12} \approx (10^{-11} - 10^{-9}) \, \text{eV}^2,$$

which is in the range where vacuum oscillations on the way from the Sun to the Earth are important.

Let us comment on the possible origin of the linear mass hierarchy of $\hat{M}$. If in $SO_{10}$ models the same $126$-plet contributes both to the RH neutrino masses and to the charged fermion masses, the hierarchy is naturally linear [21].

The linear hierarchy may be due to family symmetry. Let us consider a $U(1)$ symmetry whose breaking is characterized by a single small parameter $\lambda$ (which may be the VEV of a singlet Higgs scalar divided by the Planck mass) having the $U(1)$-charge $-1$. We assume that the fermionic $16_i$-plets of $SO(10)$ carry family $U(1)$ numbers $q_i$. Then as a consequence of the $U(1)$ invariance, the mass term $16_i 16_j$ will be suppressed by a factor $\lambda^{q_i+q_j}$. Since the Majorana mass terms ((11) or (12)) have the same flavor structure as the Dirac mass terms they will have similar hierarchy in the eigenvalues provided that the only source of $U(1)$-breaking is $\lambda$.

Note that in the case of linear mass hierarchy the lightest RH neutrino has the mass $(10^{-6} - 10^{-5}) \Lambda_{GU} \sim (10^{10} - 10^{11})$ GeV which is in the correct range to explain baryogenesis via leptogenesis [22].

The $\nu_e - \nu_\mu$ mixing angle is expected to be of the order the Cabibbo angle, $\theta_c$. It can be larger than $\theta_c$ due to violation of exact quark–lepton symmetry in the light generations. Furthermore, an additional enhancement may follow from the see-saw mechanism itself, as was discussed in sect. 3. Therefore $\theta_{e\mu} \sim (1 - 2) \theta_c$, and consequently, $\sin^2 2\theta_{e\mu} \sim 0.2 - 0.7$ are quite plausible without any additional assumptions. This mixing can lead to observable effects in the solar neutrinos.

In the $GU$-scenario the masses of $\nu_1$ and $\nu_2$ are so small even in the case of the linear mass
hierarchy of the RH components that possible contributions from the Planck scale physics can be important. In general, an expression for neutrino masses is the sum

\[ m \approx m_s + m_{nr} , \]  

(21)

where \( m_s \) is the see-saw mass and \( m_{nr} \) is the contribution from possible effective non-renormalizable interactions associated to the Planck scale physics [23]:

\[ \frac{\alpha_{ij}}{M_p} L_i L_i H_u H_u . \]  

(22)

Here \( \alpha_{ij} \) are constants expected to be of order 1, \( L_i \) are the leptonic doublets and \( M_p \) is the Planck mass. The interaction (22) may follow immediately from string compactification or from renormalizable interactions via the exchange of particles with mass \( \sim M_p \).

The interaction (22) gives the mass \( m_{nr} \sim \frac{\alpha_{ij}}{M_p} v_u^2 \sim 10^{-5} \) eV. The contribution \( m_{nr} \) to the mass of the third neutrino can be neglected, whereas \( m_{nr} \) is important for \( m_2 \) and it dominates in \( m_1 \). Again, \( \Delta m_{12}^2 \sim m_{nr}^2 \sim 10^{-10} \) eV\(^2\) is in the range (20). Also the mixing is modified. The interactions (22) determine the mixing between the first and second neutrinos. If \( \alpha_{ij} \sim O(1) \), this mixing angle can be large: \( \sin^2 2\theta_{e\mu} \sim O(1) \). The mixing of light generations with the third generation remains small: e.g., \( \theta_{e\tau} \sim m_{nr}/m_3 \sim 0.01 \). Nevertheless this contribution is comparable to \( V_{td} \). Therefore the total \( \nu_e - \nu_\tau \) mixing can be enhanced by interactions (22), so that \( \sin^2 2\theta_{e\tau} \) will be in the interval (14).

5. The interplay of resonance conversion and vacuum oscillations

As we have established in sects. 3, 4 the GU- scenario with linear mass hierarchy of the RH neutrino masses or/and with additional effects from the non-renormalizable Planck scale interactions can naturally provide the pattern of the neutrino masses and mixing with \( m_3 \sim (2 - 3) \cdot 10^{-3} \) eV, \( m_2 \sim (0.3 - 3) \cdot 10^{-5} \) eV, \( m_1 < m_2 \), \( \sin^2 2\theta_{e\tau} = (0.2 - 3) \cdot 10^{-3} \) and \( \sin^2 2\theta_{e\mu} \sim 0.2 - 0.7 \). For these values of parameters both the \( \nu_e \rightarrow \nu_\tau \) resonance flavor conversion and \( \nu_e - \nu_\mu \) vacuum oscillations on the way from the Sun to the Earth become important. Moreover, non-trivial interplay of these two effects takes place. We will call such
a possibility the *hybrid* (resonance conversion plus vacuum oscillation) solution of the solar neutrino problem.

Due to the mass hierarchy and the smallness of $\theta_{e\tau}$ the dynamics of the three neutrino system is reduced to a two neutrino task. Indeed, the electron neutrino state can be written in terms of the mass eigenstates $\nu_i$ ($i = 1, 2, 3$) as

$$\nu_e = \cos \theta_{e\tau} \cdot \tilde{\nu} + \sin \theta_{e\tau} \cdot \nu_3,$$

where

$$\tilde{\nu} \equiv \cos \theta_{e\mu} \nu_1 + \sin \theta_{e\mu} \nu_2.$$  

Inside the Sun due to the smallness of $\Delta m^2_{12}$, the system $\nu_1 - \nu_2$ is “frozen”. The evolution of the whole system consists of the $\nu_e$ resonance conversion to the state $\nu' = \cos \theta_{e\tau} \nu_3 - \sin \theta_{e\tau} \tilde{\nu}$ (orthogonal to $\nu_e$). On the way from the surface of the Sun to the Earth the state $\nu_3$ decouples from the system: large mass difference $\Delta m^2_{13}$ leads to averaged oscillation effect or/and to loss of coherence. On the way to the Earth the $\nu_e - \nu_\mu$ vacuum oscillations occur due to mass splitting $\Delta m^2_{12}$ between $\nu_1$ and $\nu_2$. Taking this into account it is easy to write the $\nu_e$ survival probability [24]:

$$P = PV(\Delta m^2_{12}, \theta_{e\mu}) \cdot PR(\Delta m^2_{13}, \theta_{e\tau}) + O(\sin^2 \theta_{e\tau}).$$

Here $PV(\Delta m^2_{12}, \theta_{e\mu})$ is the $2\nu$ - vacuum oscillation probability characterized by parameters $\Delta m^2_{12}$, $\theta_{e\mu}$, and $PR$ is the $2\nu$ - averaged survival probability of the resonance conversion characterized by $\Delta m^2_{13}$ and $\theta_{e\tau}$ ($PR$ is averaged over the production region). For $\sin^2 2\theta_{e\tau} \leq 3 \cdot 10^{-3}$ the $O(\sin^2 \theta_{e\tau})$ - corrections in (24) can be safely neglected and the total probability is factorized. The exact formula has been derived in [25]. In fig. 1 we show typical dependence of the survival probability on the neutrino energy.

The survival probability (24) satisfies the following inequalities:

$$\left(1 - \sin^2 2\theta_{e\mu}\right) \cdot PR \leq P \leq PR \cdot \Delta P = \sin^2 2\theta_{e\mu} \cdot PR.$$ 

That is, $P$ is an oscillatory function of the neutrino energy inscribed in the band (25) (fig. 1). The width of the band equals $\Delta P = \sin^2 2\theta_{e\mu} \cdot PR$. 

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If vacuum oscillations are averaged out, we get from (24)

\[ P = (1 - \frac{1}{2} \sin^2 2\theta_{e\mu}) \cdot P_R. \]  

(26)

The properties (25, 26) allow one to derive immediately several consequences of the interplay of the vacuum oscillations and resonance conversion.

- The solar \( \nu_e \) are transformed both to \( \nu_\mu \) and \( \nu_\tau \): the corresponding transition probabilities equal \( P(\nu_e \rightarrow \nu_\tau) \approx 1 - P_R \) and \( P(\nu_e \rightarrow \nu_\mu) \approx P_R - P \), where \( P \) and \( P_R \) are defined in (24).

- According to (25), vacuum oscillations lead to additional suppression of the \( \nu_e \)-flux in comparison with the pure resonance conversion. As a consequence, new regions of neutrino parameters appear in which one can describe the \( \nu_\odot \)-data. In particular, \( \sin^2 2\theta_{e\tau} < 10^{-3} \) are allowed now (sect. 6, fig. 2).

- For \( \Delta m_{12}^2 \leq 10^{-9} \text{eV}^2 \) the vacuum oscillations are not averaged and lead to additional oscillatory distortion of the \( ^8B \)-or/and \( pp \)-neutrino spectra (sect. 7).

- One expects additional seasonal variations of neutrino flux due to the dependence of the vacuum oscillation probability on the distance from the Sun to the Earth (sect. 8).

- For large \( \Delta m_{12}^2 \) the \( pp \)-neutrino flux is suppressed by the averaged vacuum oscillation effect. For small \( \Delta m_{12}^2 < 10^{-11} \text{eV}^2 \) the suppression depends on the neutrino energy, thus leading to distortion of the \( pp \)-neutrino energy spectrum.

Thus, a detailed study of the energy spectra of the \( ^8B \)-neutrinos, and (in future) of \( pp \)-neutrinos as well as measurements of seasonal variations of the signals will allow one to test the \( GU \)-scenario.

6. New regions of parameters
The interplay of the resonance conversion and vacuum oscillations opens new possibilities in description of the $\nu_\odot$-data [24]. Some of these possibilities were discussed in [25]. In particular, for $\sin^2 2\theta_{e\tau} \sim 10^{-3}$ and $\Delta m_{12}^2 \sim 10^{-4}$ eV$^2$ the high energy part of the boron neutrino spectrum lies in the nonadiabatic edge of the two neutrino suppression pit (due to the resonance conversion). For these values of parameters the resonance conversion does not change neutrino fluxes at low and intermediate energies. The suppression of the $^7$Be-neutrino flux implied by the $\nu_\odot$-data can be due to vacuum oscillations, if e.g., the first high energy dip of the oscillation probability is at $E \sim 1$ MeV. For this to occur, $\Delta m_{12}^2$ should be in the range $(7 - 9) \cdot 10^{-12}$ eV$^2$. Thus, the vacuum oscillation suppression pit is at low energies, and the resonance conversion pit is at high energies. Note that $\Delta m_{13}^2 \sim 10^{-4}$ eV$^2$ correspond to values of $M_3$ substantially below the $GU$-scale: $M_3 \sim 5 \cdot 10^{14}$ GeV.

In contrast, for the $GU$-scenario with $\Delta m_{13}^2 = (4 - 10) \cdot 10^{-6}$ eV$^2$ the $^7$Be-flux can be suppressed by the resonance conversion. Since $\sin^2 2\theta_{e\tau} \lesssim 10^{-3}$, the suppression pit is narrow and the high energy (observable) part of the boron neutrino spectrum is suppressed insufficiently. An additional suppression can be due to vacuum oscillations, especially if the observable part of the boron neutrino flux is situated in the first high energy dip of the oscillatory curve (see fig. 1). Thus we get the configuration with the resonance conversion pit at low energies and the vacuum oscillation pit at high energies. The region of parameters, where this configuration gives a good fit of the data is shown in fig. 2. (In the $\chi^2$-analysis we considered the Super-Kamiokande and Kamiokande as separate experiments. Thus, there is one degree of freedom for the hybrid solution and three degrees of freedom for $2\nu$ solution of the solar neutrino problem.) As follows from fig. 2 values of mixing angles as small as $\sin^2 2\theta_{e\tau} = 2 \cdot 10^{-4}$ are now allowed.

Further diminishing of $\theta_{e\tau}$ implies an increase of $\theta_{e\mu}$ in such a way that the hybrid solution converges to pure vacuum oscillation solution. Furthermore, it is impossible to diminish $\Delta m_{13}^2$ and therefore to increase $M_3$ in the hybrid solution. The minimal value, $\Delta m_{13}^2 \approx 4 \cdot 10^{-6}$ eV$^2$, is defined by the condition that $^7$Be-neutrino line is at the adiabatic edge of the resonance conversion pit.
For the $pp$-neutrinos one gets the averaged oscillation effect: $P \approx 1 - 0.5 \sin^2 2\theta_{e\mu} \sim 0.7 - 0.9$. Therefore, the Germanium production rate in the Gallium experiments is typically at the lower border of the allowed region ($\sim 60$ SNU’s). This suppression is weaker than the one in the pure vacuum oscillation solution.

Since the mixing angle responsible for the resonance conversion is very small, $\sin^2 2\theta_{e\tau} < 3 \cdot 10^{-3}$, the day-night effect is negligible.

7. Distortion of the boron neutrino spectrum and signals in SuperKamiokande and SNO

An interplay of vacuum oscillations and resonance conversion can lead to peculiar distortion of the boron neutrino energy spectrum. In particular, for $\Delta m^2_{12} > 10^{-11}$ eV$^2$ one expects an additional oscillatory modulation of the spectrum due to vacuum oscillations (fig. 1).

Let us consider a manifestation of such an oscillatory distortion in the energy spectrum of the recoil electrons measured by the Super-Kamiokande as well as (in future) SNO experiments.

In presence of the neutrino conversion described by the survival probability $P(E_{\nu})$ the number of recoil electrons, $N(E_{\text{vis}})$, with a given visible energy, $E_{\text{vis}}$, equals:

$$N(E_{\text{vis}}) = \int dE_e \cdot f(E_{\text{vis}}, E_e) \cdot \int_{E_e - \frac{m_{
u_e}}{2}} dE_{\nu} \cdot \Phi(E_{\nu}) \cdot P(E_{\nu}) \frac{d\sigma_{\nu_{e}}(E_e, E_{\nu})}{dE_e} + (1 - P(E_{\nu})) \frac{d\sigma_{\mu}(E_e, E_{\nu})}{dE_e},$$

(27)

where $E_e$ is the total energy of the recoil electron, $\Phi(E_{\nu})$ is the original boron neutrino flux, $f(E_{\text{vis}}, E_e)$ is the energy resolution function which can be parameterized as

$$f(E_{\text{vis}}, E_e) = \frac{1}{\sqrt{2\pi} E_e \sigma(E_e)} \cdot \exp \left[ - \left( \frac{E_{\text{vis}} - E_e}{\sqrt{2E_e \sigma(E_e)}} \right)^2 \right].$$

(28)

We use the value of $\sigma$ determined in the Super-Kamiokande calibration experiment $[1,2]$. It has an approximate dependence on the neutrino energy $\sigma \propto \frac{1}{\sqrt{E_e}}$. 

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Similar number of events $N_0(E_{vis}) = N(E_{vis}, P = 1)$ has been found in the absence of conversion. Using $N$ and $N_0$ we calculate ratios, $R^i_e$, of events with and without oscillations in the $\Delta E = 0.5$ MeV energy bins corresponding to the Super-Kamiokande presentation:

$$R^i_e = \frac{\int_{E_{vis}}^{E_i+\Delta E} dE_{vis'} N'(E_{vis}')}{\int_{E_{vis}}^{E_i+\Delta E} dE_{vis'} N_0(E_{vis}')}.$$  

(29)

In figs. 3, 4 we show (by histograms) the ratios $R^i_e$ expected for different values of the neutrino parameters as well as the ratios measured by the Super-Kamiokande experiment during 306 days [12]. For comparison we present also the distortions expected from the pure MSW-solution (fig. 3a) and pure vacuum oscillation solution (fig. 3b).

As follows from fig. 4, the integrations over the neutrino energy and the electron energy convoluted with the resolution function lead to strong averaging of the oscillatory behavior. Indeed, for present water Cherenkov detectors, like the Super-Kamiokande, the width of the resolution function (on semi-height), $2\sigma$, is $\sim 4$ MeV at 10 MeV which is comparable with the width of the largest dip of the oscillatory curve (in the energy scale). To illustrate the effect of smoothing we show the ratios for the ideal resolution $f(E_{vis}, E_e) = \delta(E_{vis} - E_e)$.

The most profound effect follows from the first (the widest) dip of the oscillatory curve.

Let us consider manifestations of the oscillatory behavior in details. For this we fix parameters of the $\nu_e - \nu_\tau$ system and find the modification of the dependence of $R_e$ on $E_{vis}$ for different values $\Delta m^2_{12}$. We will fix $\sin^2 2\theta_{e\mu} = 0.5$. An increase of the angle $\theta_{e\mu}$ leads to enhancement of distortion.

With increase of $\Delta m^2_{12}$ the oscillatory curve which corresponds to the vacuum oscillation probability shifts to high energies (see fig. 1). In such a way different parts of the oscillatory curve turn out to be in the observable range of the boron neutrino ($\nu_B^-$) spectrum: $E \sim (5 - 15)$ MeV.

We find the following distortion picture for different intervals of $\Delta m^2$:

- $\Delta m^2_{12} < 10^{-11}$ eV$^2$: The oscillatory curve is below the observable range $(5 - 15)$ MeV.

Therefore the distortion of $\nu_B$-spectrum is close to that produced by the resonance conversion alone.
• $\Delta m_{12}^2 < 6 \cdot 10^{-11}$ eV$^2$: the $\nu_B$-spectrum is on the rising (with energy) part of the first (high energy) dip of the oscillatory curve. In this case the vacuum oscillations enhance the distortion produced by the resonance conversion. The slope of the $R_e(E_{\text{vis}})$ is bigger than in the pure MSW-case (fig. 4 a).

• $\Delta m_{12}^2 = (6 - 10) \cdot 10^{-11}$ eV$^2$: the minimum of the first dip of the oscillatory curve is in the range $(5 - 15)$ MeV. This leads to the dip in the $R_e(E_{\text{vis}})$ dependence (fig. 4 b, c).

• $\Delta m_{12}^2 = (1.0 - 1.2) \cdot 10^{-10}$ eV$^2$: the $\nu_B$-spectrum is in the decreasing part of the first dip which manifests itself as the negative slope in the distortion of $R_e(E_{\text{vis}})$ (fig. 4d).

• $\Delta m_{12}^2 = (1.2 - 2) \cdot 10^{-10}$ eV$^2$: the first (high energy) bump of the oscillatory curve is in the range $(5 - 15)$ MeV which produces the bump in the recoil electron spectrum (fig. 4e, 4f).

• $\Delta m_{12}^2 = (2 - 5) \cdot 10^{-10}$ eV$^2$: the second dip of the oscillatory curve is in the range $(5 - 15)$ MeV. Again with increase of $\Delta m_{12}^2$ first an enhancement of the slope occurs (fig. 4g), then a dip appears, and finally the slope becomes negative (fig. 4h).

• $\Delta m_{12}^2 > 5 \cdot 10^{-10}$ eV$^2$: the second dip shifts to $E > 15$ MeV. There are several narrow dips in the observable part of spectrum. Some non-averaged effect still exists in the high energy part.

• $\Delta m_{12}^2 > 10^{-9}$ eV$^2$: The periods of the oscillatory curve in the observable interval are substantially smaller than $2\sigma$. Strong averaging of the oscillation effect takes place. The ratio $R_e$ approaches the asymptotic dependence which can be obtained from (27) by substitution $P_V \rightarrow \bar{P}_V = 1 - 0.5 \sin^2 2\theta_{\mu\mu}$:

$$ R_e \approx R_{e}^{R} \cdot \bar{P}_V + (1 - \bar{P}_V) \frac{N_{\mu}}{N_0} = (1 - 0.5 \sin^2 2\theta) \cdot R_{e}^{R} + 0.5 \sin^2 2\theta \frac{N_{\mu}}{N_0}, $$

where $R_{e}^{R}$ is the ratio for the case of pure resonance conversion, $N_{\mu}$ is the effect of muon (tau)-neutrino:
\[ N_\mu = \int dE_e \int dE_\nu f \cdot \Phi \cdot \frac{d\sigma_{\nu_\mu}}{dE_e}. \quad (31) \]

Note that this contribution does not depend on the probabilities of transitions. \( N_0 \) is the number of events in absence of oscillations. The last term in (30) is relatively small: \( \sim 0.045 \).

According to (30) for large \( \Delta m_{12}^2 \), vacuum oscillations result in flattening of the distortion stipulated by the resonance conversion alone (decrease of the slope).

In certain intervals of \( \Delta m_{12}^2 \) an additional effect of vacuum oscillations leads to distortion which contradicts observations. Therefore, in these intervals large mixing angles (typically \( \sin^2 2\theta_{e\mu} > 0.5 \)) can be already excluded.

The smoothing effect is weaker in the SNO experiment. The integration over the neutrino energy gives weaker averaging and energy resolution is expected to be slightly better. In our calculation we use the cross-sections of reaction \( \nu d \rightarrow e p p \) from [26] and the energy resolution function in the form (28) with \( \sigma(E_e) = 14\%/\sqrt{10\text{MeV}/E_e} \). One needs experiments with at least two times better energy resolution (like HELLAZ [27]) to measure the oscillatory modulation of the energy spectrum.

Deviations of the spectrum distortion from the simple form predicted by the pure MSW solution or vacuum oscillation solution will indicate the effect of the third neutrino considered above.

8. Seasonal variations of signals

The vacuum oscillation probability depends explicitly on the distance between the Sun and the Earth \( L \): \( P_V = P_V(L) \). Therefore seasonal variations due to pure geometrical effect, \( 1/L^2 \), related to the eccentricity of the Earth orbit will be modified [28]. Depending on values of the neutrino parameters, the geometrical effect can be enhanced or suppressed by the \( L \)-dependence in the probability. Moreover, for experiments which are sensitive to the high energy part of the boron neutrino spectrum the modification is strongly correlated to the distortion of the recoil electron energy spectrum [28].
Let us consider the seasonal variations in the case of hybrid solution. Using the probability (24) we can rewrite the expression for the number of events in the Super-Kamiokande experiment (27) in the following way:

\[ N = \int dE_e \int dE_\nu f \cdot \Phi \cdot P_R \cdot P_V \cdot \left[ \frac{d\sigma_{\nu_e}}{dE_e} - \frac{d\sigma_{\nu_\mu}}{dE_e} \right] + N_\mu, \]  

(32)

where \( N_\mu \) is defined in (31). Note that \( N_\mu \) changes with time only due to the geometrical dependence of the flux. If the resonance conversion probability varies weakly in the observable energy interval, we can substitute it by certain average value \( \bar{P}_R \). Then the expression for the number of event becomes

\[ N \approx \bar{P}_R \cdot N_{\text{vac}} + \left( 1 - \bar{P}_R \right) \cdot N_\mu, \]  

(33)

where \( N_{\text{vac}} \) is the number of events expected in the case of pure vacuum oscillations. Notice that for the configuration shown in the fig.1 the probability \( P_R \) indeed changes in a small interval from 0.7 to 0.9. According to (33) the vacuum oscillation effect is attenuated by the factor \( \bar{P}_R \). This means that to get the same overall suppression of flux one needs smaller value of \( \sin^2 2\theta_{e\mu} \), and therefore the seasonal variations become weaker. Moreover, there is an additional damping term in (33).

Let us introduce the seasonal asymmetry \( A \) as

\[ A = 2 \frac{N^W - N^S}{N^W + N^S}, \]  

(34)

where \( N^W = \int_W dt N(t) \) and \( N^S = \int_S dt N(t) \) are the integral numbers of events during the winter and summer time correspondingly. Using (33) we get

\[ A = A_V \cdot \left[ 1 - \left( 1 - \bar{P}_R \right) \cdot \frac{N_\mu}{N} \right]. \]  

(35)

Here \( A_V \) is the asymmetry in the case of pure vacuum oscillations and \( \bar{N} \) is the number of events averaged over the year. The last term in (35) is typically smaller than 10%. As we mentioned above, in presence of the resonance conversion the value \( \sin^2 2\theta_{e\mu} \) which leads to a good fit of the data is smaller than in the pure vacuum oscillation case: \( \sin^2 2\theta_{e\mu} < 0.8. \)
Then from (35) we find that seasonal variations related to the probability are attenuated by some factor $< 0.7$ in comparison with variations in the pure vacuum oscillations solution.

9. Discussion and conclusions

Let us first summarize our main results.

1. We have considered the Grand Unification scenario which is based on the see-saw mechanism with the Majorana mass of the RH neutrino $M_3 \sim \Lambda_{GU}$ and on the quark-lepton symmetry for fermions from the third generation. The $GU$-scenario leads to the mass of the third neutrino, $m_3 \sim (2 - 4) \cdot 10^{-3}$ eV, in the range of masses implied by the resonance conversion of the solar neutrinos.

2. We have found that in the $GU$-scenario the MSW - solution of the $\nu_\odot$-problem implies the mass $M_3 \sim (0.2 - 0.7)\Lambda_{GU}$ (depending on the value of $\tan \beta$) in the absence of mixing of the RH neutrinos and $M_3$ can coincide with $\Lambda_{GU}$ if small ($\theta_M \sim \sqrt{M_1/M_3}$) mixing in the Majorana mass matrix of the RH neutrinos is taken into account.

The proximity of the mass $M_3$ to $\Lambda_{GU}$ can be considered as an indication of the Grand Unification.

3. Assuming a linear hierarchy of the RH neutrino masses or/and the presence of contributions from non-renormalizable operators related to Planck scale physics we get $m_2 \sim (0.3 - 3) \cdot 10^{-5}$ eV$^2$ and $\Delta m^2_{12} \sim (10^{-11} - 10^{-9})$ eV$^2$. Furthermore, the mixing between the first and the second generations can be rather large: $\sin^2 2\theta_{e\mu} \gtrsim 0.2$. In this case the $\nu_e \to \nu_\mu$ vacuum oscillations on the way from the Sun to the Earth give rise to observable effects.

4. The $GU$-scenario leads to non-trivial interplay of the $\nu_e \to \nu_\tau$ resonance conversion and $\nu_e \to \nu_\mu$ vacuum oscillations of the solar neutrinos. An additional vacuum oscillation effect enlarges possible range of neutrino mixing, so that $\sin^2 2\theta_{e\tau}$ can be as small as $2 \cdot 10^{-4}$ which is of the order of corresponding quark mixing.

5. The interplay of the resonance conversion and vacuum oscillations leads to new type of distortion of the boron neutrino spectrum, namely, to oscillatory modulation of the energy
dependence produced by the resonance conversion alone.

The integrations over the neutrino energy and the electron energy convoluted with the energy resolution function in the Super-Kamiokande experiment (and similarly SNO) result in strong smoothing of the oscillatory behavior of the recoil electron energy spectrum. Still depending on $\Delta m^2_{12}$ one can observe an enhancement of the positive slope, or appearance of negative slope as well as dip or bump in the dependence of $R_e$ on $E_{\text{vis}}$. Two times better energy resolution is needed to observe real oscillatory behaviour.

Certain ranges of parameters are already excluded by existing data.

6. Seasonal variations of signals are expected due to dependence of the vacuum oscillation probability on distance from the Sun to the Earth. Typically these variations are expected to be weaker than in the pure vacuum oscillation case.

The $GU$-scenario considered in this paper does not allow one to explain the LSND result or to solve the atmospheric neutrino problem in terms of neutrino oscillations. It predicts “null” results in the CHORUS and NOMAD experiments as well as in future short and long baseline oscillation experiments. Null result is also expected in the neutrinoless double beta decay searches etc. Therefore a positive result in at least one of these experiments will exclude the simplest version of the $GU$-scenario. Also the scenario does not supply any appreciable contribution to the hot dark matter (HDM) in the Universe.

It is possible to modify the $GU$-scenario discussed here so that the second neutrino mass is $m_2 \sim (0.1 – 1)$ eV. For this the mass of the corresponding RH neutrino should be $M_{02} \sim 10^{10}$ GeV in the absence of mixing (or for small mixing). Such a modification can accommodate the LSND result and give a sizable contribution to HDM. An explanation of the atmospheric neutrino anomaly requires also large mixing between $\nu_\mu$ and $\nu_\tau$. This is more difficult to achieve, since strong mass hierarchy accompanied by large mixing angle implies further tuning of parameters.

Additional freedom in explaining of the data arises if new (sterile) neutrino states are introduced.
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Note added.

While this work was revised the paper [30] appeared in which renormalization effects for neutrinos in a similar context have been studied. Our results are in agreement with those in [30].
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TABLE I. The mass of third neutrino $m_3$ as the function of $\tan \beta$ for $M_R = 10^{16}$ GeV and three possible values of physical top quark mass: $m_t = 175, 170, 180$ GeV.

| $\tan \beta$ | $\nu_\tau$ -mass ($10^{-3}$ eV) |
|--------------|---------------------------------|
|              | $m_t = 170$ GeV | $m_t = 175$ GeV | $m_t = 180$ GeV |
| 1.7          | 1.47              | 2.85            | —              |
| 1.9          | 1.16              | 1.64            | 2.71           |
| 3            | 0.83              | 0.99            | 1.17           |
| 5            | 0.76              | 0.88            | 1.00           |
| 10           | 0.73              | 0.84            | 0.95           |
| 30           | 0.74              | 0.85            | 0.96           |
| 50           | 0.76              | 0.92            | 1.05           |
| 60           | 0.83              | 0.99            | 1.17           |
FIG. 1. The averaged (over production area) survival probability $P(\nu_e \rightarrow \nu_e)$ (solid line) as a function of the neutrino energy $E$ for the following values of parameters: $\sin^2 2\theta_{e\mu} = 0.77$, $\Delta m^2_{21} = 6.0 \cdot 10^{-11}\text{eV}^2$, $\sin^2 2\theta_{e\tau} = 8.0 \cdot 10^{-4}$, $\Delta m^2_{31} = 1.1 \cdot 10^{-5}\text{eV}^2$. The dependence of the averaged MSW probability $P_R(\nu_e \rightarrow \nu_e)$ alone on $E$ is also shown (dashed line).
FIG. 2. The 95% C.L. regions of solution of the $\nu_\odot$-problem in $\Delta m_{31}^2$ and $\sin^2 2\theta_{e\mu}$ plot (306 days Super-Kamiokande data are taken into account) for different vacuum oscillation parameters $\sin^2 2\theta_{e\mu}$, $\Delta m_{21}^2$, and the original boron neutrino flux (in units of the standard solar model) $f_B \equiv \Phi(\delta B)/\Phi_{SSM}(\delta B)$. The thin solid lines correspond to $\sin^2 2\theta_{e\mu} = 0.48$, $\Delta m_{21}^2 = 1.4 \cdot 10^{-10}$ eV$^2$, and the values of $f_B$ indicated at the curves. The thick lines correspond to (a) $\sin^2 2\theta_{e\mu} = 0.66$, $\Delta m_{21}^2 = 8.3 \cdot 10^{-11}$ eV$^2$ and $f_B=1.0$; (b) the same as (a) but $f_B=0.9$; (c) $\sin^2 2\theta_{e\mu} = 0.62$, $\Delta m_{21}^2 = 9.5 \cdot 10^{-11}$ eV$^2$, $f_B = 0.8$; (d) $\sin^2 2\theta_{e\mu} = 0.50$, $\Delta m_{21}^2 = 7.8 \cdot 10^{-11}$ eV$^2$, $f_B = 0.7$. 
FIG. 3. The expected spectrum deformations of the recoil electrons in the Super-Kamiokande (solid lines) and SNO (long dashed lines) experiments for (3a) two neutrino conversion with $\sin^2 2\theta = 8.8 \cdot 10^{-3}$, $\Delta m^2 = 5.0 \cdot 10^{-6}$ eV$^2$, (3b) vacuum oscillation with $\sin^2 2\theta = 0.82$, $\Delta m^2 = 6.4 \cdot 10^{-11}$ eV$^2$. The curves are normalized so that $R_e(10\text{MeV})=0.368$ for the Super-Kamiokande and $R_e(10\text{MeV})=0.256$ for the SNO experiment. Thin lines correspond to ideal energy resolution; thick lines correspond to real energy resolution. The Super-Kamiokande 306 days data are shown (statistical errors only).
$\Delta m_{21} = 3.0 \times 10^{-11} \text{ eV}^2$

(4a)

$\Delta m_{21} = 6.0 \times 10^{-11} \text{ eV}^2$

(4b)
$\Delta m_{21}^2 = 8.0 \times 10^{-11} \text{ eV}^2$

$\Delta m_{21}^2 = 1.1 \times 10^{-10} \text{ eV}^2$
$\Delta m_{31}^2 = 1.4 \times 10^{-10} \text{ eV}^2$

$\Delta m_{31}^2 = 1.6 \times 10^{-10} \text{ eV}^2$
\[ \Delta m^2_{12} = 2.0 \times 10^{-10} \text{ eV}^2 \]

\[ \Delta m^2_{12} = 3.0 \times 10^{-10} \text{ eV}^2 \]
FIG. 4. The same as in figs. 3 but for the hybrid solution with parameters $\sin^2 2\theta_{e\mu} = 0.5$, $\sin^2 2\theta_{e\tau} = 6.0 \cdot 10^{-4}$, $\Delta m_{31}^2 = 8.0 \cdot 10^{-6} \text{eV}^2$ and different values of $\Delta m_{21}^2$ indicated in the figures.