Singular spacetime
and
quantum probe

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Abstract

We examine the dependence of quantization on global properties of a classical system. Quantization based on local properties may lead to ambiguities and inconsistency between local and global symmetries of a quantum system. Our quantization method based on global characteristics has sound foundation. Presented results give insight into the nature of removable type singularity of spacetime at the quantum level.

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1 Introduction

Finding a theory of quantum gravity is not only an intellectual adventure but also the present day need, since the number of cosmological data on the very early Universe increases and they call for theoretical description. It seems that understanding of the nature of spacetime singularities in quantum context is the core of the problem. The insight can be achieved by studying some suitable ‘toy models’ which include both singular spacetimes and quantum rules.

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The toy model considered here concerns a spacetime with removable type singularities. We analyze classical and quantum dynamics of a test particle in singular and corresponding regular spacetimes. To quantize classical dynamics of the systems we apply the group quantization method \[1\]. We show that quantization of the system with regular spacetime can be carried out without problems. In the singular case not all classical observables are well defined globally since spacetime includes incomplete geodesics. Therefore, smaller number of classical observables can be mapped into quantum observables. As the result, the classical dynamics of a test particle in a spacetime with removable type singularities can be quantized, but the local symmetry of the quantum singular system cannot be as rich as in the corresponding regular case.

2 Classical dynamics

We present dynamics of a test particle in two de Sitter’s type spacetimes. For simplicity we restrict ourselves to two dimensional spacetimes. In conclusions we make comments concerning the four dimensional cases.

The considered spacetimes are defined to be

\[ V_p = (R^1 \times R^1, \hat{g}), \quad V_h = (R^1 \times S^1, \hat{g}). \]  

(1)

In both cases the metric tensor \( g_{\mu\nu} := (\hat{g})_{\mu\nu} \ (\mu, \nu = 0, 1) \) is defined by the line-element

\[ ds^2 = dt^2 - \exp(2t/r) \, dx^2, \]  

(2)

where \( r \) is a positive real constant.

Eq.(1) presents all possible topologies of the de Sitter spacetime in two dimensions. \( V_p \) is a plane with global \((t,x) \in R^2\) coordinates. \( V_h \) is defined to be a one-sheet hyperboloid embedded in 3d Minkowski space. There exists [3] an isometric immersion map \( f \) of \( V_p \) into \( V_h \) defined by

\[ V_p \ni (t,x) \longrightarrow f(t,x) := (y^0,y^1,y^2) \in V_h, \]  

(3)

where

\[ y^0 = r \sinh(t/r) + \frac{x^2}{2r} \exp(t/r), \]

\[ y^1 = -r \cosh(t/r) + \frac{x^2}{2r} \exp(t/r), \quad y^2 = -x \exp(t/r), \]
and where
\[(y^2)^2 + (y^1)^2 - (y^0)^2 = r^2.\] (4)

One can check that \(f\) maps \(V_p\) onto a non-compact half of \(V_h\) and that the induced metric on \(V_h\) is identical to the metric defined by (2).

Let us emphasize that \(f\) cannot be extended to an isometry \(F : V_p \to V_h\) such that \(F(V_p) = V_h - \{x(R^1)\} =: V_x\), where \(x : R^1 \to V_h\) is defined to be a complete smooth curve on \(V_h\), and such that \(V_x\) is simply connected [4]. In other words, \(V_p\) and \(V_h\) are not almost globally isometric spacetimes having only different boundary conditions.

It is known [5] that \(V_p\) is geodesically incomplete. But all incomplete geodesics in \(V_p\) can be extended to complete ones in \(V_h\), which means that \(V_p\) has removable type singularities. The singularity type of \(V_p\) is not as severe as in the case of spacetimes with essential type singularities [6]. The latter includes both incomplete geodesics and blowing up curvature scalars. In our case both \(V_p\) and \(V_h\) have constant curvatures \(R = -2r^{-2}\).

The action integral, \(S\), describing a relativistic test particle of mass \(m\) in gravitational field \(g_{\mu\nu}\ (\mu, \nu = 0, 1)\) is proportional to the length of a particle world-line
\[S = \int L(\tau) \, d\tau, \quad L(\tau) := -m\sqrt{g_{\mu\nu}(x^0(\tau), x^1(\tau)) \dot{x}^\mu(\tau) \dot{x}^\nu(\tau)},\] (5)
where \(\tau\) is an evolution parameter, \(x^\mu\) are spacetime coordinates and \(\dot{x}^\mu := dx^\mu/d\tau\). It is assumed that \(\dot{x}^0 > 0\), i.e. \(x^0\) has interpretation of time monotonically increasing with \(\tau\).

The Lagrangian (5) is invariant under the reparametrization \(\tau \to f(\tau)\). This gauge symmetry leads to the constraint
\[G := g^{\mu\nu}p_\mu p_\nu - m^2 = 0,\] (6)
where \(g^{\mu\nu}\) is an inverse of \(g_{\mu\nu}\) and \(p_\mu := \partial L/\partial \dot{x}^\mu\) are canonical momenta.

Since a test particle does not modify the geometry of spacetime, the local symmetry of the system is defined by the set of all Killing vectors of spacetime. Each Killing vector field \(X^\mu\ (\mu = 0, 1)\) has corresponding dynamical integral [7]
\[D = p_\mu X^\mu.\] (7)

The dynamical integrals restricted to the constraint surface (6) and specifying particle trajectories admissible by the dynamics define the physical phase-space \(\Gamma\).
Since the hyperboloid (4) is invariant under the proper Lorentz transformations, the symmetry group of $V_h$ is $SO^\uparrow(1,2)$. In the standard parametrization [2] the infinitesimal transformations of $SO^\uparrow(1,2)$ group read

\[
\begin{align*}
(p, \theta) &\rightarrow (p, \theta + a_0 r), \\
(p, \theta) &\rightarrow (p - a_1 r \sin \rho/r \sin \theta/r, \theta + a_1 r \cos \rho/r \cos \theta/r), \\
(p, \theta) &\rightarrow (p + a_2 r \sin \rho/r \cos \theta/r, \theta + a_2 r \cos \rho/r \sin \theta/r),
\end{align*}
\]

(8)

where $(a_0, a_1, a_2) \in \mathbb{R}^3$ are parameters.

The corresponding dynamical integrals (7)

\[
J_0 = p_\theta r, \quad J_1 = -p_\rho r \sin \rho/r \sin \theta/r + p_\theta r \cos \rho/r \cos \theta/r, \quad J_2 = p_\rho r \sin \rho/r \cos \theta/r + p_\theta r \cos \rho/r \sin \theta/r,
\]

(9)

where $p_\theta := \partial L/\partial \dot{\theta}$, $p_\rho := \partial L/\partial \dot{\rho}$ are canonical momenta) satisfy the commutation relations of $sl(2, \mathbb{R})$ algebra

\[
\{J_a, J_b\} = \varepsilon_{abc} \eta^{cd} J_d,
\]

(10)

where $\varepsilon_{abc}$ is the anti-symmetric tensor with $\varepsilon_{012} = 1$ and $\eta^{cd}$ is the Minkowski metric tensor.

The constraint (6) in terms of (9) reads

\[
J_0^2 - J_1^2 - J_2^2 = -\kappa^2, \quad \kappa = mr.
\]

(11)

The particle trajectories are found [2] to be defined by

\[
J_0 y^a = 0, \quad J_2 y^1 - J_1 y^2 = r^2 p_\rho.
\]

(12)

Each point point $(J_0, J_1, J_2)$ of the one-sheet hyperboloid (11) defines a particle trajectory (12) available for dynamics. Therefore (11) defines the physical phase-space $\Gamma_h$ with $SO^\uparrow(1,2)$ as the symmetry group. The spacetime and phase-space of $V_h$ system have the same symmetry group.

The infinitesimal symmetry transformations of $V_\rho$ system read [2]

\[
\begin{align*}
(t, x) &\rightarrow (t, x + b_0), \quad (t, x) \rightarrow (t - rb_1, x + xb_1), \\
(t, x) &\rightarrow (t - 2rxb_2, x + (x^2 + r^2e^{-2t/r})b_2),
\end{align*}
\]

(13)

(14)

where $(b_0, b_1, b_2) \in \mathbb{R}^3$ are parameters.
The corresponding dynamical integrals (7) are

\[ P = p_x, \quad K = -rp_t + xp_x, \quad M = -2rxp_t + (x^2 + r^2e^{-2t/r})p_x, \quad (15) \]

where \( p_x = \partial L/\partial \dot{x}, \) \( p_t = \partial L/\partial \dot{t}. \)

One can check that the integrals (15) satisfy the commutation relations of \( sl(2, R) \) algebra in the form

\[ \{P, K\} = P, \quad (16) \]
\[ \{K, M\} = M, \quad \{P, M\} = 2K. \quad (17) \]

The constraint (6) leads to

\[ K^2 - PM = \kappa^2. \quad (18) \]

In case of \( V_p \) system, contrary to the case of \( V_h \) system, some points \((P, K, M)\) of (18) cannot describe the trajectories available for a particle:

For \( P = 0 \) there are two lines \( K = \pm \kappa \) on the hyperboloid (18). Since by assumption \( \dot{t} > 0 \), we have that \( p_t = \partial L/\partial \dot{t} = -mt \left( \dot{t} - \dot{x}\exp(2t/r) \right)^{-1/2} < 0. \)

According to (15) \( K - xP = -rp_t, \) thus \( K - xP > 0, \) i.e. \( K > 0 \) for \( P = 0. \)

Therefore, the line \((P = 0, K = -\kappa)\) is not available for the dynamics. The hyperboloid (18) without this line defines the physical phase-space \( \Gamma_p. \) The particle trajectories are \([2]\]

\[ x = M/2K, \quad \text{for} \quad P = 0 \quad (19) \]

and

\[ xP = K - \sqrt{\kappa^2 + (rP)^2 \exp(-2t/r)}, \quad \text{for} \quad P \neq 0. \quad (20) \]

Since \( \Gamma_p \) is topologically equivalent to a plane \( R^2 \) we can parametrize \( \Gamma_p \) as follows \([2]\]

\[ P = p, \quad K = pq - \kappa, \quad M = pq^2 - 2\kappa q, \quad (21) \]

where \((q, p) \in R^2. \)

The local symmetry of both \( V_p \) and \( V_h \) systems is defined by \( sl(2, R) \) algebra. However, the symmetry groups are different. The Lie group \( SO^+ (1, 2) \) cannot be the global symmetry of \( V_p \) system, since \( V_p \) is only a subspace of \( V_h \) due to (3). Since the Killing vector field generated by (14) is not complete (see, App.A of \([2]\) ), the symmetry group of \( V_p \) is the Lie group with the Lie algebra defined by (16) only. Therefore, in case of \( V_p \) system the Lie algebra
corresponding to the symmetry group is different from \( sl(2, R) \) algebra of all the Killing vector fields.

The classical observables are defined to have the following properties:
(i) they specify particle trajectories available for dynamics \( (V_h \text{ and } V_p \text{ are integrable systems}) \), (ii) they are gauge invariant (have vanishing Poisson’s brackets with the constraint \( G \), Eq.(6)), (iii) they satisfy the algebra corresponding to the symmetry group of \( V_h \) or \( V_p \) system or (iii) they satisfy the algebra corresponding to the local symmetry of \( V_h \) or \( V_p \) system.

3 Quantum dynamics

In case of \( V_h \) system the classical observables are \( J_0, J_1 \) and \( J_2 \). We choose the following parametrization [2]
\[
J_0 = J, \quad J_1 = J \cos \beta - \kappa \sin \beta, \quad J_2 = -J \sin \beta - \kappa \cos \beta, \tag{22}
\]
where \( J \in R^1 \) and \( \beta \in S^1 \) are the canonical coordinates.

The observables satisfy both sets of conditions \( A \) := \{\( (i), (ii), (iii) \)\} and \( \tilde{A} \) := \{\( (i), (ii), (\tilde{iii}) \)\}, since \( sl(2, R) \) is the algebra of \( SO^\uparrow(1, 2) \) group.

The quantum observables corresponding to (22) read [2]
\[
\hat{J}_0 = \frac{\hbar}{i} \frac{d}{d\beta}, \quad \hat{J}_1 = \cos \beta \hat{J}_0 - \left( \kappa - \frac{i\hbar}{2} \right) \sin \beta, \quad \hat{J}_2 = -\sin \beta \hat{J}_0 - \left( \kappa - \frac{i\hbar}{2} \right) \cos \beta. \tag{23}
\]

Since they are unbounded, they are defined [2] on a dense subspace \( \Omega \) of the Hilbert space \( L^2[0, 2\pi] \)
\[
\Omega := \{ \psi \in L^2[0, 2\pi] \mid \psi \in C^\infty[0, 2\pi], \psi^{(n)}(0) = \psi^{(n)}(2\pi), n = 0, 1, 2... \}. \tag{24}
\]

Eqs. (23) and (24) define an essentially self-adjoint representation of \( sl(2, R) \) algebra (see, App.C of [2]). It can be further examined for its integrability to the unitary representation of \( SO^\uparrow(1, 2) \) group.

In case of \( V_p \) system the two sets of conditions \( A \) and \( \tilde{A} \) lead to different sets of observables, because \( SO^\uparrow(1, 2) \) is no longer the symmetry group of the system. We examine the consequences of each choice separately.

With the choice \( \tilde{A} \) the classical observables are \( P, K \) and \( M \). The symplectic transformation \( (q, p) \rightarrow (I, \sigma) \) defined by
\[
q := -\cot \frac{\sigma}{2}, \quad p := (1 - \cos \sigma)(I + \kappa \cot \frac{\sigma}{2}) \tag{25}
\]
leads to
\[ I_0 := \frac{1}{2}(M + P) = I, \]
\[ I_1 := \frac{1}{2}(M - P) = I \cos \sigma - \kappa \sin \sigma, \quad I_2 := K = -I \sin \sigma - \kappa \cos \sigma. \]  
(26)

where \( I \in \mathbb{R}^1 \) and \( 0 < \sigma < 2\pi \) are the canonical coordinates.

One can easily check that the line \( (P = 0, K = -\kappa) \) of the hyperboloid (18) turns into the generatrix \( (I_0 = I_1, I_2 = -\kappa) \) of the hyperboloid (11) (with \( J_a \) replaced by \( I_a \)).

The observables \( I_a \ (a = 0, 1, 2) \) satisfy \( sl(2, \mathbb{R}) \) algebra, have the same functional form as \( J_a \ (a = 0, 1, 2) \) observables, but are defined on different domains. Therefore, the functional form of corresponding quantum observables \( \hat{I}_a \) is again defined by (23) (with \( \hat{J}_a \) replaced by \( \hat{I}_a \) and \( \beta \) replaced by \( \sigma \)). However, the carrier space of \( \hat{I}_a \) is different. Now, it is defined to be [2]
\[
\Omega_\alpha := \{ \psi \in L^2[0, 2\pi] \mid \psi \in C^\infty[0, 2\pi], \ \psi^{(n)}(0) = e^{i\alpha} \psi^{(n)}(2\pi), \ n = 0, 1, 2, \ldots \},
\]  
(27)

where \( 0 \leq \alpha < 2\pi \).

It can be easily proved (see, App.C of [2]) that Eq. (23), with \( \hat{J}_a \) replaced by \( \hat{I}_a \) and \( \beta \) replaced by \( \sigma \), and Eq. (27) define essentially self-adjoint representations of \( sl(2, \mathbb{R}) \) algebra. However, the choice \( \tilde{A} \) leads to ambiguities. Since the end points of the range of \( \sigma \) in (26) do not coincide, there is no reason to choose any specific value for \( \alpha \). Thus, there are infinitely many unitarily nonequivalent quantum systems corresponding to a single classical \( V_p \) system. Such a quantum theory has no predictability. It can be useful as a phenomenological model, but we are interested in finding a fundamental description. Obviously, these representations cannot be lifted to the unitary representation of the symmetry group, because the symmetry group has the algebra defined by Eq.(16) which is only a subalgebra of \( sl(2, \mathbb{R}) \) algebra.

Now, we consider the choice \( A \). The set of observables consists of \( P \) and \( K \) which satisfy the algebra (16). The corresponding quantum observables read [2]
\[
\hat{P} = \frac{\hbar}{i} \frac{d}{dq}, \quad \hat{K} = q \hat{P} + \frac{\hbar}{2i} - \kappa.
\]  
(28)

The common invariant dense domain \( \Lambda \) for \( \hat{P} \) and \( \hat{K} \) is defined to be [2]
\[
\Lambda := \{ \psi \in L^2(\mathbb{R}) \mid \psi \in C^\infty_0(\mathbb{R}) \}.
\]  
(29)
The representation defined by (28) and (29) is essentially self-adjoint (see, App.D of [2]). Next step would be lifting this representation to the unitary representation of the symmetry group of $V_p$ system.

4 Conclusions

It is interesting that at the phase-space level one has $\Gamma_h = \Gamma_p \cup \{ \text{generatrix} \}$, which is quite different from the correspondence between $V_h$ and $V_p$ systems at the spacetime level. This subtle phase-space difference between $V_p$ and $V_h$ systems leads to quite different quantum systems. The problem of incomplete geodesics of $V_p$ spacetime translates into the boundary condition problem at the quantum level. The latter leads to ambiguities. Uniqueness of the quantization procedure can be achieved by favouring the global symmetries of the singular $V_p$ system. No quantization problem occurs in case of the regular spacetime $V_h$. We can see that quantization is very sensitive to the choice of spacetime topology.

Generalization of our results to the four dimensional de Sitter spacetimes is straightforward. The quantum dynamics of a particle on hyperboloid is presented in [8]. The de Sitter spacetime with the topology $R^1 \times R^3$, the four dimensional analog of $V_p$, is geodesically incomplete and it can be embedded isometrically [3] into the four dimensional analog of $V_h$ by generalization of the map (3). This is why a direct application of our method should lead, after tedious calculations, to the results similar in its essence to the results presented here.

We believe that one can generalize our results further to any spacetime with topology admitting removable type singularities. Quantization of dynamics of a test particle in such singular spacetime should be feasible, unless the system has no globally well defined observables.

Further analyses of the quantum dynamics can be carried out at the level of the unitary representations of the symmetry groups. The results will be published elsewhere [9].

Our paper concerns removable type singularities. It is possible that analyses for spacetimes with essential type singularities [6] may bring unexpected results.

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