A global observer for attitude and gyro biases from vector measurements

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Abstract: We consider the classical problem of estimating the attitude and gyro biases of a rigid body from vector measurements and a triaxial rate gyro. We propose a simple “geometry-free” nonlinear observer with guaranteed uniform global asymptotic convergence and local exponential convergence; the stability analysis, which relies on a strict Lyapunov function, is rather simple. The excellent behavior of the observer is illustrated through a detailed numerical simulation.

Keywords: Attitude estimation; nonlinear observer; guidance; navigation systems.

1. INTRODUCTION

Estimating the attitude of a rigid body from vector measurements has been for decades a problem of interest, because of its importance for a variety of technological applications such as satellites or unmanned aerial vehicles.

The attitude of the body can be described by the rotation matrix $R \in \text{SO}(3)$ from inertial to body axes. The measurement vectors $u_1, \ldots, u_n \in \mathbb{R}^3$ correspond to the expression in body axes of known vectors $U_1, \ldots, U_n \in \mathbb{R}^3$ which are constant in inertial axes, i.e., $u_k(t) = R^T(t)U_k$.

The goal is to reconstruct the attitude at time $t$ using only the knowledge of the measurement vectors until $t$. The problem would be very easy if the measurements were perfect: indeed, using for instance only the two vectors $u_1(t)$ and $u_2(t)$ and noticing $R^T(x \times y) = R^T x \times R^T y$ since $R$ is a rotation matrix, we readily find

$$R^T(t) = R^T(t) \cdot (U_1 U_2 U_1 \times U_2) \cdot (U_1 U_2 U_1 \times U_2)^{-1} = (u_1(t) u_2(t) u_1(t) \times u_2(t)) \cdot (U_1 U_2 U_1 \times U_2)^{-1}.$$

But in real situations, the measurement vectors are always corrupted at least by noise. Moreover, the $U_k$’s may possibly be not strictly constant: for instance a triaxial magnetometer measures the (locally) constant Earth magnetic field, but is easily perturbed by ferromagnetic masses and electromagnetic perturbations; similarly, a triaxial accelerometer can be considered as measuring the direction of gravity provided it is not undergoing a substantial acceleration (see e.g. Martin and Salaün (2010b) for a detailed discussion of this assumption and its consequences in the framework of quadrotor UAVs). That is why, despite the additional cost, it may be interesting to use a triaxial rate gyro to supplement the possibly not so good vector measurements.

The literature on attitude estimation from vector measurements can be broadly divided into three categories:

i) optimization-based methods; ii) stochastic filtering; iii) nonlinear observers. Details on the various approaches can be found e.g. in the surveys Crassidis et al. (2007); Zamani et al. (2015) and the references therein.

The first category is the minimization of a cost function, and is usually referred to as Wahba’s problem. The attitude is algebraically recovered at time $t$ using only the measurements at time $t$. No filtering is performed, and possibly available velocity information from rate gyros is not exploited.

The second category mainly hinges on Kalman filtering and its variants. Despite their many qualities, the drawback of those designs is that convergence cannot in general be guaranteed except for mild trajectories. Moreover the tuning is not completely obvious, and the computational cost may be too high for small embedded processors.

The third, and more recent, approach proposes nonlinear observers with a large guaranteed domain of convergence and a rather simple tuning through a few constant gains. These observers can be designed: a) directly on $\text{SO}(3)$ (or the unit quaternion space), see e.g. Mahony et al. (2008); Martin and Salaün (2010a); Vasconcelos et al. (2008); Grip et al. (2012); b) or more recently, on $\mathbb{R}^{3 \times 3}$, i.e., deliberately “forgetting” the underlying geometry Batista et al. (2012b,a); Grip et al. (2015). Probably the best-known design is the so-called nonlinear complementary filter of Mahony et al. (2008); as noticed in Martin and Salaün (2007), it is a special case of so-called invariant observers Bonnabel et al. (2008).

In this paper, we propose a new observer of attitude and gyro biases from gyro measurements and (at least) two measurement vectors. It also “forgets” the geometry of $\text{SO}(3)$, which yields a very simple structure with a straightforward proof of its uniform global asymptotic convergence (notice the observer of Mahony et al. (2008) is only quasi-globally convergent, with moreover a much more involved proof). It is easy to tune and implement, and demonstrates excellent performance in simulation. This observer can be seen as a modification and simplifi-
cation of the linear cascaded observer proposed in Batista et al. (2012b), along with a much simpler proof of convergence based on a strict Lyapunov function. Compared to the global exponential stability result of Batista et al. (2012b), we only prove uniform global asymptotic stability plus local exponential stability. However, in the case where our observer reduces to the observer of Batista et al. (2012b) by using directly non-filtered measurements in certain terms, our Lyapunov function again serves as a strict Lyapunov function, which provides a simpler and more direct alternative to the non-trivial stability analysis of Batista et al. (2012b).

The paper runs as follows: the model used to design the observer is described in section 2; the observer is presented in section 3, and its convergence is proved; finally, section 4 illustrates the excellent behavior of the observer on a detailed numerical simulation.

2. THE DESIGN MODEL

We consider a moving rigid body subjected to the angular velocity $\omega$ (in body axes). Its orientation (from inertial to body axes) matrix $R \in \text{SO}(3)$ is related to $\omega$ by

$$R = R_\omega x,$$  

where the skew-symmetric matrix $\omega \times$ is defined by $\omega \times x := \omega \times x$ whatever the vector $x$.

The rigid body is equipped with a triaxial rate gyro measuring the angular velocity $\omega$, and two additional triaxial sensors (for example accelerometers, magnetometers or sun sensors) providing the measurements of two vectors $\alpha$ and $\beta$. These two vectors correspond to the expression in body axes of two known independent vectors $\alpha_i$ and $\beta_i$, which are constant in inertial axes. In other words,

$$\alpha := R^T \alpha_i, \quad \beta := R^T \beta_i.$$  

Since $\alpha_i$, $\beta_i$ are constant, we readily find

$$\dot{\alpha} = \alpha \times \omega, \quad \dot{\beta} = \beta \times \omega.$$  

As any sensor, the rate gyro is biased, and rather provides the measurement

$$\omega_m := \omega + b,$$  

where $b$ is a slowly-varying (for instance with temperature) unknown bias. The effect of this bias on attitude estimation may be important when the observer gains are small, hence it is worth determining its value. But being not exactly constant, it cannot be calibrated offline and must be estimated online together with the attitude.

Our objective is to design an estimation scheme that can reconstruct online the orientation matrix $R(t)$ and the bias $b(t)$, using i) the measurements of the gyro and of the two vector sensors; ii) the knowledge of the constant vectors $\alpha_i$ and $\beta_i$. The model on which the design will be based therefore consists of the dynamics

$$\dot{\alpha} = \alpha \times \omega, \quad \dot{\beta} = \beta \times \omega, \quad \dot{b} = 0,$$  

together with the measurements

$$\omega_m := \omega + b, \quad \alpha_m := \alpha, \quad \beta_m := \beta.$$  

3. THE OBSERVER

We show the state of the design system (2)–(7) can be estimated by the observer

$$\dot{\hat{\alpha}} = \hat{\alpha} \times (\omega_m - \hat{b}) - k_\alpha (\hat{\alpha} - \alpha_m)$$  

$$\dot{\hat{\beta}} = \hat{\beta} \times (\omega_m - \hat{b}) - k_\beta (\hat{\beta} - \beta_m)$$  

$$\dot{\hat{b}} = l_\alpha \hat{\alpha} \times \alpha_m + l_\beta \hat{\beta} \times \beta_m,$$  

where $k_\alpha, k_\beta, l_\alpha, l_\beta > 0$ are strictly positive constants. Notice this observer is very similar to the so-called bias observer of Batista et al. (2012b). The main difference is the use of the filtered terms $\hat{\alpha} \times (\omega_m - \hat{b})$ and $\hat{\beta} \times (\omega_m - \hat{b})$ instead of their unfiltered versions, which avoids injecting too much measurement noise when the gains are small; as a consequence the error system below is no longer linear in the measured quantities $\omega_m, \alpha_m, \beta_m$, and the technique of proof of Batista et al. (2012b) is no longer applicable.

Defining the error vectors, $e_\alpha := \hat{\alpha} - \alpha$, $e_\beta := \hat{\beta} - \beta$ and $e_b := b - \hat{b}$, the error system reads

$$e_\alpha = e_\alpha \times \omega - (e_\alpha + e_b) \times e_b - k_\alpha e_\alpha$$  

$$e_\beta = e_\beta \times \omega - (e_\beta + e_b) \times e_b - k_\beta e_\beta$$  

$$e_b = l_\alpha e_\alpha \times \alpha_m + l_\beta e_\beta \times \beta_m.$$  

It will be convenient to use the rotated variables $\bar{E}_\alpha := R e_\alpha, \bar{E}_\beta := R e_\beta, \bar{E}_b := R e_b, \Omega := R \omega$. In terms of the rotated variables, the error system reads

$$\dot{\bar{E}}_\alpha = R (e_\alpha \times \omega) - (R \alpha + R e_\alpha) \times R e_b - k_\alpha R e_\alpha$$  

$$\dot{\bar{E}}_\beta = R (e_\beta \times \omega) - (R \beta + R e_\beta) \times R e_b - k_\beta R e_\beta$$  

$$\dot{\bar{E}}_b = \Omega \times \bar{E}_b + l_\alpha \bar{E}_\alpha \times \alpha_m + l_\beta \bar{E}_\beta \times \beta_m.$$  

Indeed, we find e.g. (11) by writing

$$\dot{\bar{E}}_\alpha = 2 \bar{R} e_\alpha + R e_\alpha$$  

$$= R (\omega \times e_\alpha)$$  

$$+ R(e_\alpha \times \omega) - (R \alpha + R e_\alpha) \times R e_b - k_\alpha R e_\alpha,$$

where we have used (1), (11) and $R(x \times y) = R x \times R y$ since $R$ is a rotation matrix. It is of course equivalent to work with this rotated error system; notice also $|\Omega| = |\omega|$.

**Theorem 1.** Assume $k_\alpha, k_\beta, l_\alpha, l_\beta > 0$ and $\omega$ bounded. Then the equilibrium point $(\bar{E}_\alpha, \bar{E}_\beta, \bar{E}_b) := (0, 0, 0)$ of the error system (11)–(13) is uniformly globally asymptotically stable and locally exponentially stable.

**Proof.** Consider the candidate Lyapunov function

$$V := \sigma_1 V_1 + \sigma_2 V_2^2 + V_3,$$

where the coefficients $\sigma_1, \sigma_2 > 0$ are yet to be defined, and

$$V_1(E_\alpha, E_\beta, E_b) := \frac{1}{2} \left( l_\alpha |E_\alpha|^2 + l_\beta |E_\beta|^2 + |E_b|^2 \right)$$  

$$V_2(E_\alpha, E_\beta, E_b) := \frac{1}{2} \left| E_b - \frac{l_\alpha}{k_\alpha} E_\alpha \times E_\alpha - \frac{l_\beta}{k_\beta} E_\beta \times E_\beta \right|^2.$$  

Clearly, $V$ is positive definite and radially unbounded.

We now compute the time derivatives of its pieces along the trajectories of the error system (11)–(13).
\[ \dot{V}_i = l_\alpha(E_\alpha, E_\beta \times \alpha_i + E_\beta \times E_\alpha - k_\alpha E_\alpha) \\
+ l_\beta(E_\beta, E_\beta \times \beta_i + E_\beta \times E_\beta - k_\beta E_\beta) \\
+ (E_\alpha \times E_\beta + l_\alpha E_\alpha \times \alpha_i + l_\beta E_\beta \times \beta_i) \\
= -k_{\alpha i} |E_\alpha|^2 - k_{\beta i} |E_\beta|^2, \]

where we have used \(\langle x, x \times y \rangle = 0\).

\[
\frac{d}{dt} V_i = -k_{\alpha i} l_\alpha^2 |E_\alpha|^4 - k_{\beta i} l_\beta^2 |E_\beta|^4 - k_{\alpha i} l_{\alpha i} |E_\alpha|^2 |E_\beta|^2 \\
- k_{\beta i} l_{\beta i} |E_\beta|^2 |E_\beta|^2 - (k_{\alpha i} + k_{\beta i}) l_{\alpha i} |E_\alpha|^2 |E_\beta|^2.
\]

Setting \(u_{\alpha i} := \frac{l_\alpha}{k_{\alpha i}} \alpha_i \times E_\alpha + \frac{l_{\beta i}}{k_{\beta i}} \beta_i \times E_\beta, \)

\[
\langle E_\beta, \dot{\alpha}_i - \dot{\beta}_i \rangle \\
= \langle E_\beta, E_\beta \times \alpha_i \rangle \times \langle E_\beta, E_\beta \times \beta_i \rangle \\
- \langle E_\beta, E_\beta \alpha_i \rangle \times \langle E_\beta, E_\beta \times \beta_i \rangle + \langle E_\beta, E_\beta \beta_i \rangle \times \langle E_\beta, E_\beta \times \beta_i \rangle \\
= \langle E_\beta, \frac{l_{\alpha i}}{k_{\alpha i}} \alpha_i \times E_\beta \times \frac{l_{\beta i}}{k_{\beta i}} \beta_i \times E_\beta \rangle \\
- \langle E_\beta, \frac{l_{\alpha i}}{k_{\alpha i}} \alpha_i \times E_\beta \times \frac{l_{\beta i}}{k_{\beta i}} \beta_i \times E_\beta \rangle \\
= \langle E_\beta, -\frac{l_{\alpha i}}{k_{\alpha i}} \alpha_i \times E_\beta \times \frac{l_{\beta i}}{k_{\beta i}} \beta_i \times E_\beta \rangle,
\]

where we have used \(\langle x, y \times z \rangle = 0\) and \(\langle x, y \times z \rangle = \langle x, y \rangle \times z\) several times.

We next bound the last two expressions. Since \(\alpha_i, \beta_i\) are independent and \(k_{\alpha i}, l_{\alpha i}, k_{\beta i}, l_{\beta i} > 0\), the matrix

\[
-\left(\frac{l_{\alpha i}}{k_{\alpha i}} a_{\alpha i}^2 + \frac{l_{\beta i}}{k_{\beta i}} b_{\beta i}^2 \right)
\]

is positive definite, see (Tayebi et al., 2013, Lemma 2). As a consequence, there exists \(\mu > 0\) such that

\[
\langle E_\beta, \frac{l_{\alpha i}}{k_{\alpha i}} a_{\alpha i}^2 + \frac{l_{\beta i}}{k_{\beta i}} b_{\beta i}^2 \rangle E_\beta \leq -\mu |E_\beta|^2.
\]

Using then Young’s inequality \(\langle x, y \rangle \leq \frac{|x|^2}{2} + \frac{|y|^2}{2} \) and \(\sum_{i=1}^n |x_i| \leq n \sum_{i=1}^n |x_i| \) yields

\[
\langle E_\beta, \dot{\alpha}_i - \dot{\beta}_i \rangle \\
\leq -\mu |E_\beta|^2 + \frac{\varepsilon}{2} |E_\beta|^2 \\
+ \frac{1}{\varepsilon} \frac{l_{\alpha i}}{k_{\alpha i}} \alpha_i \times (E_\beta \times E_\alpha) |E_\beta|^2 \\
+ \frac{1}{\varepsilon} \frac{l_{\beta i}}{k_{\beta i}} \beta_i \times (E_\beta \times E_\beta) |E_\beta|^2 \\
\leq \left(\frac{\varepsilon}{2} - \mu\right) |E_\beta|^2 \\
+ \frac{l_{\alpha i}^2}{\varepsilon k_{\alpha i}^2} |E_\alpha|^2 |E_\beta|^2 + \frac{l_{\beta i}^2}{\varepsilon k_{\beta i}^2} |E_\beta|^2 |E_\beta|^2.
\]

Similarly,
Remark 2. $V_l$ alone is a (non strict) Lyapunov function for the error system (11)–(13); using repeatedly Barbalat’s lemma, (non uniform) global asymptotic stability can be very easily established, see Martin and Sarras (2016). On the other hand, the more complicated Lyapunov function $V$ used in the proof is strict, hence yields uniform stability.

Remark 3. If in the observer (8)-(10) the filtered terms $\hat{\alpha} \times (\omega_m - \hat{\alpha})$ and $\hat{\beta} \times (\omega_m - \hat{\alpha})$ are replaced by their filtered versions $\hat{\alpha} \times (\omega_m - \hat{\omega}_m) \times \hat{\beta} \times (\omega_m - \hat{\omega}_m)$, we recover the first part of the linear cascaded observer of Batista et al. (2012b). In this case, the strict Lyapunov function $V$ used in the proof can also be used to establish uniform global asymptotic stability, which provides a simpler and more constructive proof than the “abstract” proof of Batista et al. (2012b) based on uniform observability arguments.

Remark 4. The result obviously also holds if the scalar gains $k_\alpha, k_\beta, l_\alpha, l_\beta$ are replaced by 3×3 symmetric definite positive matrices; these added degrees of freedom for the tuning might be useful in practice if the components of the measurement vectors are produced by sensors with very different characteristics.

It is also clear that more than two vectors can be used with an obvious generalization of the proposed structure.

Remark 5. The observer does not use the knowledge of the constant vectors $\alpha$ and $\beta$. This may be an interesting feature in some applications when those vectors for example are not precisely known and/or (slowly) vary.

We then have the obvious but important following corollary, which gives an estimate of the true orientation matrix $R$ by using the knowledge of $\alpha$ and $\beta$. Notice it is considerably simpler than the approach proposed in Batista et al. (2012b), where the estimated orientation matrix is obtained through an additional observer of dimension 9.

Corollary 6. Under the assumptions of theorem 1, the matrix $\tilde{R}$ defined by

$$\tilde{R} := \left( \begin{array}{cc} \hat{\alpha} \times \hat{\beta} \\ \hat{\alpha} \times \hat{\beta} \times \hat{\alpha} \end{array} \right) \cdot R_i^T$$

uniformly globally asymptotically converges to $R$.

Proof. By Theorem 1, $\hat{\alpha} \to \alpha$ and $\hat{\beta} \to \beta$. Hence,

$$\tilde{R} = R_t$$

$$R_t := \left( \begin{array}{cc} \hat{\alpha} \times \hat{\beta} \\ \hat{\alpha} \times \hat{\beta} \times \hat{\alpha} \end{array} \right) \cdot R_i^T$$

uniformly globally asymptotically converges to $R$.

Proposition 7. Consider the polar decomposition of $\hat{R}^T$

$$\hat{R}^T = \hat{R}^T (\hat{R} \hat{R}^T)^{\frac{1}{2}}.$$

$\hat{R}$, which is by construction the best approximation of $\hat{R}$ among all orthogonal matrices, is a rotation matrix that uniformly globally asymptotically converges to $R$. When $\hat{\alpha}$ and $\hat{\beta}$ are not collinear, $\hat{R}$ is uniquely defined by

$$\hat{R}^T := \left( \begin{array}{cc} \hat{\alpha} \times \hat{\beta} \\ \hat{\alpha} \times \hat{\beta} \times \hat{\alpha} \end{array} \right) \cdot R_i^T.$$

Proof. Since $\hat{R}$ is the product of a matrix with orthogonal columns by a rotation matrix,

$$(\hat{R} \hat{R}^T)^{\frac{1}{2}} = R_t \left( \begin{array}{cc} \frac{\hat{\alpha}}{|\hat{\alpha}|} \\ \frac{\hat{\beta}}{|\hat{\beta}|} \\ \frac{\hat{\alpha} \times \hat{\beta}}{|\hat{\alpha} \times \hat{\beta}|} \end{array} \right) \cdot R_i^T.$$

When $\hat{\alpha}$ and $\hat{\beta}$ are not collinear, the expression for $\hat{R}$ follows at once from $\hat{R}^T = \hat{R}^T (\hat{R} \hat{R}^T)^{\frac{1}{2}}$: when $\hat{\alpha} \neq 0$ but $\hat{\beta} = 0$, one may choose $\hat{R}^T := R_t^T$; when $\hat{\beta} \neq 0$ but $\hat{\alpha} = 0$, one may choose $\hat{R}^T := (\frac{\hat{\beta}}{|\hat{\beta}|}, \hat{E}_2, \hat{E}_3) \cdot R_i^T$, where $\frac{\hat{\alpha}}{|\hat{\alpha}|}$, $\hat{E}_2$ and $\hat{E}_3$ form a direct orthonormal frame. □

Fig. 1. Components of true $\omega$ (red) and measured $\omega_m$ (blue).

4. SIMULATIONS

The good behavior of the observer is now illustrated in simulation. The system starts in the initial state $R(0) := 1$, i.e. $\phi(0), \theta(0), \psi(0)) := (0, 0, 0)$, and then undergoes the angular velocity $\omega(t)$ displayed in Fig. 1. The constant vectors $\alpha$ and $\beta$ are respectively set to the nominal values $(0, 0, 1)^T$ and $(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})^T$, which mimics the gravity and
magnetic vectors. Moreover, $\beta_i$ is subjected to violent disturbances for $t \in [500, 700]$, see Fig. 2; of course only the nominal values of $\alpha_i$ and $\beta_i$ are known to the observer.

The observer is fed with the measured signals $\omega_m, \alpha_m$ and $\beta_m$, see Fig. 1-3-4; the measured velocity is affected by the (unknown) slowly drifting bias $\bar{b}$, see Fig. 1-5. All the measurement signals are corrupted by band-limited independent gaussian white noises with sample time $10^{-3}$ and rather large noise powers ($2 \times 10^{-6}$ for the components of $\alpha_m, \beta_m$, and $2 \times 10^{-5}$ for those of $\omega_m$). The tuning gains are set to $(k_\alpha, k_\beta, \eta_\alpha, \eta_\beta) = (10, 0.15, 0.15)$. The observer is initialized with no error, but suddenly reinitialized to zero at $t = 100$. The convergence of the estimated state $\hat{\alpha}, \hat{\beta}$ and $\bar{b}$ is as anticipated excellent after the reinitialization, very fast for $\hat{\alpha}, \hat{\beta}$ and slower for $\bar{b}$, in accordance with the choice of gains, see Fig. 3-4-5. The convergence is also very good for $t \in [500, 700]$ when $\beta_i$ is violently disturbed: the error $\beta - \hat{\beta}$ exhibits only small spikes, $\bar{b}$ is only slightly affected, and $\hat{\alpha}$ not affected at all at this scale; this illustrates the (desirable) independence between $\hat{\alpha}$ and $\hat{\beta}$, which are only slightly coupled through $\bar{b}$. Notice also that the disturbances on $\beta_i$ are interpreted by the observer as a variation of $\bar{b}$.

We insist that the observer does not need the knowledge of the “constant” vectors $\alpha_i$ and $\beta_i$, which is why it can converge to the true state even when $\alpha_i, \beta_i$ vary not too fast (with respect to the gains $k_\alpha, k_\beta$). These vectors are needed only when the rotation matrix $R$ (or Euler angles, or quaternions) must be reconstructed. Fig. 6-7-8 show the reconstruction of the Euler angles $\phi, \theta, \psi$ (in radians) from $\hat{\alpha}, \hat{\beta}$ using the nominal values of $\alpha_i, \beta_i$. Notice the pitch angle $\theta$ and roll angle $\phi$ (which are paramount for the control of for instance UAVs) are perfectly estimated even during the period where $\beta_i$ is violently disturbed (the fast oscillations of $\theta$ just after $t = 100$ are due to the angle determinacy which switches between 0 and $\pi$, and have nothing to do with the observer); inevitably the estimated yaw angle $\phi$ is severely affected, since a completely wrong value of $\beta_i$ is used. This scenario illustrates an interesting practical feature of the observer, not enjoyed by e.g. the observer in Mahony et al. (2008): even if the magnetic sensor (i.e., $\beta_m$) is easily disturbed, it is nevertheless worth using to help the accelerometer (i.e., $\alpha_m$) in the estimation of the gyro biases, hence in the reconstruction of the pitch and roll angles, whatever the excitation (or absence of) provide by the angular velocity.

5. CONCLUSION

We have presented a simple nonlinear “geometry-free” observer for attitude and gyro bias estimation, with guaranteed uniform global convergence and local exponential convergence. Simulations demonstrate that it performs very well, even with noisy measurements and not so constant inertial vectors $\alpha_i$ and $\beta_i$ and bias $\bar{b}$. It can be seen as an interesting alternative to the SO(3)-based observer of Mahony et al. (2008) or of the more complicated “geometry-free” observer of Batista et al. (2012b).
Fig. 4. Components of true $\beta$ (red), measured $\beta_m$ (blue) and estimated $\hat{\beta}$ (orange).

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REFERENCES

Batista, P., Silvestre, C., and Oliveira, P. (2012a). A GES attitude observer with single vector observations. *Automatica*, 48(2), 388–395.

Batista, P., Silvestre, C., and Oliveira, P. (2012b). Globally exponentially stable cascade observers for attitude estimation. *Control Engineering Practice*, 20(2), 148 – 155.

Bonnabel, S., Martin, P., and Rouchon, P. (2008). Symmetry-preserving observers. *IEEE Transactions on Automatic Control*, 53(11), 2514–2526.

Crassidis, J.L., Markley, F.L., and Cheng, Y. (2007). Survey of nonlinear attitude estimation methods. *Journal of Guidance, Control, and Dynamics*, 30(1), 12–28.

Grip, H.F., Fossen, T.I., Johansen, T.A., and Saberi, A. (2012). Attitude estimation using biased gyro and vector measurements with time-varying reference vectors. *IEEE Transactions on Automatic Control*, 57(5), 1332–1338.

Grip, H.F., Fossen, T.I., Johansen, T.A., and Saberi, A. (2015). Globally exponentially stable attitude and gyro bias estimation with application to GNSS/INS integration. *Automatica*, 51, 158–166.
Martin, P. and Sarras, I. (2016). A simple global observer for attitude and gyro biases. ArXiv e-prints, arXiv:1604.03714v1 [math.OC].
Tayebi, A., Roberts, A., and Benallegue, A. (2013). Inertial vector measurements based velocity-free attitude stabilization. IEEE Transactions on Automatic Control, 58(11), 2893–2898.
Vasconcelos, J., Silvestre, C., and Oliveira, P. (2008). A nonlinear observer for rigid body attitude estimation using vector observations. IFAC Proceedings Volumes, 41(2), 8599 – 8604.
Zamani, M., Trumpf, J., and Mahony, R. (2015). Nonlinear Attitude Filtering: A Comparison Study. ArXiv e-prints, arXiv:1502.03990 [cs.SY].

Fig. 7. True $\theta$ (red) and estimated $\hat{\theta}$ (blue).

Fig. 8. True $\psi$ (red) and estimated $\hat{\psi}$ (blue).

Higham, N.J. (2008). Functions of Matrices: Theory and Computation. SIAM.
Khalil, H. (2002). Nonlinear Systems. Prentice Hall.
Mahony, R., Hamel, T., and Pflimlin, J.M. (2008). Nonlinear complementary filters on the special orthogonal group. IEEE Transactions on Automatic Control, 53(5), 1203–1218.
Martin, P. and Salaün, E. (2007). Invariant observers for attitude and heading estimation from low-cost inertial and magnetic sensors. In IEEE Conference on Decision and Control, 1039–1045.
Martin, P. and Salaün, E. (2010a). Design and implementation of a low-cost observer-based attitude and heading reference system. Control Engineering Practice, 18(7), 712–722.
Martin, P. and Salaün, E. (2010b). The true role of accelerometer feedback in quadrotor control. In IEEE International Conference on Robotics and Automation, 1623–1629.