Exact boundary integral solution for the Stokes traction of an active particle

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The mechanics and statistical mechanics of a suspension of active particles are determined by the traction (force per unit area) on their surfaces. Here we present an exact solution of the boundary integral equation of a spherical active particle in an imposed slow viscous flow. Both single- and double-layer integral operators can be simultaneously diagonalised in a basis of irreducible tensorial harmonics and the solution, thus, can be presented as an infinite number of linear relations between the harmonic coefficients of the traction and the boundary velocity. These generalise Stokes laws for the force and torque. Using these relations we show that while the suspension stress in an active suspension can vanish, the dissipated power always remains positive. Thus, any analogy of this dissipative system to superfluidity must be made with caution.

I. INTRODUCTION

Active particles are a class of colloids that create fluid flow around them even when they are stationary. Examples include microorganisms [1] and autophoretic particles [2]. The exterior flow of active particles is due to local non-equilibrium processes such as ciliary motion (in the case of microorganisms) and osmotic flows (in the case autophoretic particles). These non-equilibrium processes in a thin layer at the surface of the colloids can be modeled by adding a surface slip \( v^A \) to the commonly used no-slip boundary condition on colloidal surfaces [3]. In slow viscous flow, the resulting traction, a key dynamical quantity in the mechanics and statistical mechanics of active suspensions [4], is a linear functional of the boundary condition. An exact solution for this linear functional has remained elusive.

Here we show that the linear functional relationship between traction and boundary velocity can be obtained exactly for a spherical active particle using the boundary integral formulation of the Stokes equation [5–7]. The resulting direct boundary integral equation has both a single-layer contribution from the traction and a double-layer contribution from surface slip. Our key result is the simultaneous diagonalisation of both the single- and double-layer integral operators in a basis of tensorial spherical harmonics. The boundary integral equation is reduced, thereby, to an infinite-dimensional diagonal linear system that can be solved trivially. We obtain compact, closed-form relations between the harmonic coefficients of the traction and the velocity boundary condition. The first two terms of these are the familiar coefficients \( 6\pi \eta b \) and \( 8\pi \eta b^3 \) in Stokes laws for the force and torque of a spherical particle of radius \( b \) in rigid body motion in an unbounded fluid of dynamic viscosity \( \eta \) [8]. Our exact solution will be useful in iterative numerical solutions of the boundary integral equation for many particles, where the diagonal analytical solution can be used to initialise iterations that converge to the diagonally-dominant numerical solutions [4].

In what follows, we present our solution and some implications thereof in detail. In Section II we provide the boundary integral representation of Stokes flow for three distinct contributions to the traction on the surface of an active particle in an imposed flow. These are (a) rigid body motion, (b) imposed flow, and (c) active slip. Exact expressions for the traction have been derived in Section III using spectral expansions and Ritz-Galerkin discretisation of the boundary integral equations [9–11]. The solution has been obtained in terms of generalised friction coefficients. We emphasise that these friction coefficients are distinct for imposed flow and activity. In Section IV, we obtain an exact expression for the power dissipated by an active particle. We show that while the net suspension stress in an active suspension may vanish, the system continues to have a positive-definite power dissipation. We conclude in Section V by summarising our results and suggesting directions for future work.

II. BOUNDARY INTEGRAL FORMULATION OF STOKES EQUATION

In this section we present the boundary integral equation to obtain the traction on an active particle due to the most general form of surface velocity and an arbitrary imposed flow \( \mathbf{v}^\infty (\mathbf{r}) \). We consider a spherical active particle
of radius $b$ in an incompressible fluid of viscosity $\eta$. The boundary condition at the surface of the particle is
\[ v(R + \rho) = V + \Omega \times \rho + v^A(\rho) = v^D(\rho) + v^A(\rho). \]

The rigid body motion $v^D$ is specified by the translational velocity $V$ and angular velocity $\Omega$ of the particle. Here, $\rho$ is the radius vector, $R$ is the centre of the colloid and $v^A$ is its active slip velocity. The only restriction on the active slip is that its integral over the surface of the sphere is zero. This ensures conservation of mass in the fluid.

At the colloidal scale, the fluid satisfies Stokes equations for slow viscous flow, $\nabla \cdot \sigma = 0$ and $\nabla \cdot v = 0$, with the Cauchy stress tensor $\sigma_{\alpha\beta} = -p\delta_{\alpha\beta} + \eta(\partial_\alpha v_\beta + \partial_\beta v_\alpha)$, where $p$ is the fluid pressure. The traction $f$ is the normal component of the Cauchy stress tensor evaluated at the surface of the colloid. In the following, we obtain an exact expression for the traction from the solution of the Stokes equation in terms of the boundary condition (1). It is convenient to express the traction as a sum of three distinct contributions
\[ f = f^D + f^\infty + f^A. \]

Here, $f^D$ is the traction due to the colloid’s rigid body motion $v^D$ alone, $f^\infty$ represents the traction on a no-slip particle when held stationary in an imposed flow $v^\infty$, and $f^A$ is the contribution from active surface slip $v^A$. We will now use the boundary integral formulation of the Stokes equation to obtain the explicit forms of the above contributions to the traction. By linearity of Stokes flow, the three parts of the traction satisfy independent boundary integral equations. These are

\[ v^D_\alpha(r) = -\int G_{\alpha\beta}(r, r') f^D_\beta(r') \, dS, \quad \text{(rigid body)} \]
\[ v^\infty_\alpha(r) = \int G_{\alpha\beta}(r, r') f^\infty_\beta(r') \, dS, \quad \text{(external flow)} \]
\[ \frac{1}{2} v^A_\alpha(r) = -\int \left[ G_{\alpha\beta}(r, r') f^A_\beta(r') - K_{\beta\alpha\nu}(r', r) \partial_\nu v^A_\beta(r') \right] \, dS, \quad \text{(active slip)} \]

Here, and throughout the paper, Einstein summation convention is used, $S$ is the surface of the colloid, $r, r' \in S$, and $\hat{n}$ is the unit normal vector to the surface of the colloid, pointing into the surrounding fluid. The integral kernels for the fluid velocity are the Green’s function $G$ of Stokes flow and the stress tensor $K$ associated with it. Together with the pressure field $P$ they satisfy [12]
\[ \nabla_\alpha G_{\alpha\beta}(r, r') = 0, \quad -\nabla_\alpha P_{\beta}(r, r') + \eta \nabla^2 G_{\alpha\beta}(r, r') = -\delta(\rho - r') \delta_{\alpha\beta}, \]
\[ K_{\alpha\beta\mu}(r, r') = -\delta_{\alpha\mu} P_{\beta}(r, r') + \eta \left( \nabla_\nu G_{\alpha\beta}(r, r') + \nabla_\alpha G_{\nu\beta}(r, r') \right), \]

where the derivative is taken with respect to the first argument, i.e. here $\nabla = \nabla_r$. Furthermore, the Green’s function satisfies the symmetry $G_{\alpha\beta}(r, r') = G_{\beta\alpha}(r', r)$. By analogy with potential theory the terms in (3) containing the Green’s function $G$ and the stress tensor $K$ are referred to as ‘single-layer’ integral and ‘double-layer’ integral, respectively [13]. In writing the rigid body part of (3), we have used the well-known result that rigid body motion is an eigenfunction of the double-layer integral operator with eigenvalue $-1/2$ [14] (see equations (11) for a proof). In the following, we shall solve these integral equations, finding an exact solution for the Stokes traction of an active particle in an arbitrary imposed flow.

### III. GENERALISED STOKES LAWS

To solve the integral equations (3) for the unknown surface tractions, we expand the velocity and the traction at the colloid’s surface in tensorial spherical harmonics $Y^{(l)}(\hat{\rho})$ as
\[ v^\lambda(R + \rho) = \sum_1^\infty \hat{w}_l V^{(l)}(\hat{\rho}) \odot Y^{(l-1)}(\hat{\rho}), \quad f^\lambda(R + \rho) = \sum_1^\infty \hat{w}_l F^{(l)}(\hat{\rho}) \odot Y^{(l-1)}(\hat{\rho}), \]
where $\lambda = \{D, \infty, A\}$, $w_l = 1/[(l - 1)!(2l - 3)!]$, and $\hat{w}_l = (2l - 1)/((4\pi l^2))$. The product $\odot$ represents a maximal contraction of indices between two tensors. The tensorial spherical harmonics are defined as
\[ Y^{(l)}_{\alpha_1...\alpha_l}(\hat{\rho}) = (2l - 1)!! \Delta^{(l)}_{\alpha_1...\alpha_l, \beta_1...\beta_l} \hat{\rho}_{\beta_1} \cdots \hat{\rho}_{\beta_l} = \rho^{l+1}(-1)^l \nabla_{\alpha_1} \cdots \nabla_{\alpha_l} \rho^{-1}, \]
with $\rho = \|\rho\|_2$, where $\|\cdot\|_2$ is the Euclidean norm, and $\Delta^{(l)}$ is a rank $2l$ tensor, which projects a tensor of rank $l$ onto its symmetric and traceless part. An excellent summary of properties of and identities involving the $\Delta$-tensor
can be found in [15]. We use the orthogonality of the basis functions, \( w_{l+1} \tilde{w}_{l+1} \int Y(l)Y(l')dS = \delta_{ll'}\Delta(l) \), to obtain the expansion coefficients

\[
V^\lambda(l) = \tilde{w}_l \int v^\lambda(R + \rho)Y(l-1)(\rho)dS, \quad F^\lambda(l) = w_l \int f^\lambda(R + \rho)Y(l-1)(\rho)dS.
\]

By definition, \( F^\lambda(l) \) and \( V^\lambda(l) \) are symmetric-irreducible in their last \( l-1 \) indices, and thus, can each be expressed as the sum of three irreducible tensors, \( F^{\lambda(l)} \) and \( V^{\lambda(l)} \), with the index \( \sigma = \{s, a, t\} \) labelling the symmetric irreducible (rank \( l \)), the antisymmetric (rank \( l-1 \)) and trace (rank \( l-2 \)) parts of the reducible tensors, respectively [15]. This decomposition, and the projection of the expansion coefficients onto their irreducible spaces are given by

\[
F^\lambda(l) = D^{(l)}(\sigma) \otimes F^{\lambda(\sigma)}, \quad V^\lambda(l) = P^{(l)}(s) \otimes F^{\lambda(l)},
\]

where we have defined the decomposition operators \( D^{(l)}(\sigma) \) and the projection operators \( P^{(l)}(\sigma) \), whose explicit forms are available in Appendix A. Repeated mode indices \( (l') \) are summed over implicitly for the decomposition operator. These irreducible tensors are naturally parametrised in terms of the tensorial spherical harmonics as follows

\[
V^{(ls)} = V^{(ls)}_{ls} Y(l)(e), \quad V^{(la)} = V^{(la)}_{la} Y(l-1)(e), \quad V^{(lt)} = V^{(lt)}_{lt} Y(l-2)(e),
\]

where the uniaxial parameterisation are defined in terms of the orientation vector \( e \) of the active particle and \( V^{(ls)}_{ls} \) are the scalar strengths of the modes. From this parametrisation, it follows that the \( V^{(ls)} \) are either even (apolar) or odd (polar) under inversion symmetry \( e \to -e \) with respect to the orientation of the particle. A plot of the vector fields \( v^{A} \) and \( f^{A} \) due to the leading modes of apolar (2s), polar (3t), achiral (3a) and chiral (4a) symmetry can be found in figure 1, which uses the above parametrisation. This plot is useful to visualise and gain an intuitive understanding of the higher modes of the surface slip velocity.

Having defined the spectral expansions on the boundary (5), we now discretise the integral equation (3), using the procedure of Ritz and Galerkin (see Appendix A for a detailed derivation). The result is a self-adjoint linear system in the irreducible expansion coefficients;

\[
\begin{align*}
V^{D(l)} &= -G^{(l, s', s''}) \otimes F^{D(l')}, & \text{(rigid body)} \\
V^{\infty(l)} &= G^{(l, s', s'')} \otimes F^{\infty(l')}, & \text{(external flow)} \\
\frac{1}{2} V^{A(l)} &= -G^{(l, s', s'')} \otimes F^{A(l')}, & \text{(active slip)}
\end{align*}
\]
The matrix elements $G^{(l\sigma, l')\sigma'}$ and $K^{(l\sigma, l')\sigma'}$ are due to the single-layer and double-layer respectively in the direct formulation of the boundary integral representations. As we now show, this leads to distinct friction coefficients for imposed flow and active slip. In Appendix A we derive explicit forms of the matrix elements of the linear system of equations. The result is

$$
G^{(l\sigma, l)\sigma} \odot F^{\lambda(l)\sigma} = g_{ls} F^{\lambda(l)\sigma},
$$

$$
G^{(l, l')\sigma} \odot F^{\lambda(l')\sigma} = g_{ls} F^{\lambda(l)\sigma},
$$

$$
G^{(l\sigma, l)\sigma} \odot F^{\lambda(l')\sigma} = g_{ls} F^{\lambda(l)\sigma},
$$

$$
K^{(l\sigma, l)\sigma} \odot V^{\lambda(l)\sigma} = k_{ls} V^{\lambda(l)\sigma},
$$

$$
K^{(l, l')\sigma} \odot V^{\lambda(l')\sigma} = k_{ls} V^{\lambda(l)\sigma},
$$

$$
K^{(l\sigma, l)\sigma} \odot V^{\lambda(l')\sigma} = k_{ls} V^{\lambda(l)\sigma},
$$

(11)

Here the scalar $l$-dependent factors are: $g_{ls} = g_{3l}(l + 1)/(2l + 1)$, $g_{la} = 1/(4\pi b w_l)$, $g_{lt} = g_{la}(l - 2)/(2l - 5)$, $k_{ls} = k_{ls}(l + 1)$, $k_{la} = -3/[2(2l - 1)]$, $k_{lt} = -k_{ls}(2l - 3)$. Thus, we have shown that the linear system arising from (3) is diagonal not only in $(l)$, but also in all its irreducible subspaces labelled by $(l\sigma)$. Using the above solution, we can write down the generalised Stokes laws for an isolated active particle in an unbounded domain as

$$
F^{D(l\sigma)} = -\gamma_{ls} V^{D(l\sigma)}, \quad F^{\infty(l\sigma)} = \gamma_{ls}^\infty V^{\infty(l\sigma)}, \quad F^{A(l\sigma)} = -\gamma_{la} V^{A(l\sigma)},
$$

(12)

for which, using (11), we can give the scalar generalised friction coefficients exactly to arbitrary order in $(l)$ as

$$
\gamma_{ls}^\infty = 4\pi b \left(\frac{2l + 1}{l + 1} \frac{(l - 1)!}{(2l - 3)!!}\right),
$$

$$
\gamma_{ls} = 4\pi b \left(\frac{2l + 1}{l + 1} \frac{(l - 1)!}{(2l - 3)!!}\right),
$$

$$
\gamma_{la} = 4\pi b \left(\frac{1}{l + 1} \frac{(l - 1)!}{(2l - 5)!!}\right),
$$

$$
\gamma_{lt} = 4\pi b \left(\frac{1}{l + 1} \frac{(l - 1)!}{(2l - 5)!!}\right).
$$

(13)

It should be noted that the friction coefficients due to imposed fluid flow ($\gamma_{ls}^\infty$), are distinct from those due to active surface slip ($\gamma_{ls}$). The difference is due to the double-layer integral in (3). The friction coefficients $\gamma_{ls}^\infty$, $\gamma_{ls}^\infty$, and $\gamma_{ls}^\infty$ are available in the literature in terms of a Taylor expansion of the imposed flow about the centre of the particle and referred to as the FaxÅ©n relations [16–18]. On the other hand, our results have been obtained in terms of expansion coefficients of the imposed flow for arbitrary $(l\sigma)$. A Taylor expansion of the imposed flow [11] and its expansion coefficients are related as

$$
V^{\infty(l\sigma)} = P^{(l\sigma)} \odot \left[ b^{l-1} \Delta^{(l-1)} \left( 1 + \frac{\rho^2}{4l + 2} \nabla^2 \right) \nabla^{(l-1)} \right] R.
$$

(14)

Here $[\ldots]_R$ denotes that the function inside the bracket is evaluated at the centre $R$ of the particle. In this paper, we have used the approach to expand the boundary fields in tensorial spherical harmonics for both imposed flow and active surface slip. It should be noted that a corresponding Taylor expansion about the centre of the particle is not possible for the active slip which is only defined at the surface of the particle.

It is intuitive that in the unbounded domain the generalised Stokes laws, expressing the linear relation between the irreducible modes of traction and the corresponding modes of boundary velocity, must be scalar relations due to symmetry considerations. A visualisation of this in terms of the active slip velocity and the resulting hydrodynamic traction due to a given $(l\sigma)$ mode is shown in figure 1. To summarise, the main results of this section are exact expressions for the friction coefficients $\gamma_{ls}^\infty$ (and $\gamma_{ls}$) due to imposed flow (and active surface slip) obtained using the direct formulation of the boundary integral equation. Equations (10-14) are the central results of this paper.

IV. SUSPENSION STRESS AND POWER DISSIPATION

We now apply the above general solution to a selection of physically relevant cases. With (7) and (8) the irreducible modes are readily expressed in terms of commonly used physical quantities. The rigid body motion velocity expansion coefficients $V^{D(l\sigma)}$ have only two non-vanishing modes corresponding to translational velocity $V = V^{D(1s)}$ and rotational velocity $\Omega = V^{D(2a)}/2b$. Similarly, the first two modes of the imposed flow are: $V^{\infty(1s)} = V^{\infty}$, and $V^{\infty(2a)} = 2b\Omega^{\infty}$, while the first two modes of activity are $V^{A(1s)} = -V^{A}$ and $V^{A(2a)} = -2b\Omega^{A}$. Here, the active translational velocity $V^{A}$ and the active angular velocity $\Omega^{A}$ of a spherical active particle [19–21] are given by

$$
V^{A} = -\frac{1}{4\pi b^2} \int v^{A}(\rho)dS, \quad \Omega^{A} = -\frac{3}{8\pi b^4} \int \rho \times v^{A}(\rho)dS.
$$

(15)
We also define the rate of strain dyadic $E^\lambda = V^{\lambda(2s)}/b$, due to activity or imposed flow, as

$$E^\lambda_{\alpha\beta} = \frac{3}{8\pi b} \int (\hat{\rho}_\alpha v^\lambda_\beta + v^\lambda_\alpha \hat{\rho}_\beta) \, dS.$$  

Analogously, we identify the most commonly used traction tensors produced by the corresponding velocity fields

$$F^{\lambda(1s)} = F^\lambda, \quad F^{\lambda(2s)} = \frac{1}{b} T^\lambda, \quad F^{\lambda(2s)} = \frac{1}{b} S^\lambda,$$

where $F$, $T$ and $S$ are the familiar hydrodynamic force, torque and stresslet. The latter is

$$S^\lambda_{\alpha\beta} = \int \left[ \frac{1}{2} (f^\lambda_{\alpha\beta} + f^\lambda_{\beta\alpha}) - \frac{\delta_{\alpha\beta}}{3} f^\lambda_{\gamma\gamma} \right] dS.$$  

We note that this definition is different from the combination of traction and velocity. To the best of our knowledge, this difference in the stresslets due to imposed flow and active slip has not been mentioned before in the literature. However, as one can see from (18) the combined dissipation of imposed flow and slip velocity will still be greater than this phenomenon has previously been linked to a ‘superfluidlike’ transition to a dissipation-free macroscopic flow [24].

Here, we have used the generalised Stokes laws (12). It follows that $\dot{E}^A$, $\dot{E}^\infty \geq 0$, i.e. the power dissipation is always positive definite. For rates of strain due to both, imposed flow $E^\infty$ and slip velocity $E^A$, it is easy to see from (17) that the active slip can be tuned such that the response to shear vanishes, i.e $S^\infty + S^A = 0$. In active suspensions this phenomenon has previously been linked to a ‘superfluidlike’ transition to a dissipation-free macroscopic flow [24]. However, as one can see from (18) the combined dissipation of imposed flow and slip velocity will still be greater than zero and so the analogy to superfluidity must be made with caution.

**V. DISCUSSION**

By using the direct boundary integral formulation of the Stokes equation we have obtained exact scalar relations between the coefficients of the traction and boundary velocity of an active particle in an unbounded fluid flow. We call these linear relations generalised Stokes laws. In future work, we will extend our calculations to obtain the traction on an active particle near surfaces such as an infinite plane no-slip wall. This exact solution will also be useful in obtaining efficient iterative numerical solutions of the boundary integral equation for many particles. In this case, the exact one-body solution can be used to initialise iterations that converge to the diagonally dominant numerical solutions. The complete set of modes of the traction derived here can also be used to study the rheology of active suspensions. All of these directions present exciting avenues for future work on the mechanics and statistical mechanics of active colloidal suspensions.

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Appendix A: Derivation of the matrix elements of the linear system of equations

In this section, we show how to arrive at the irreducible linear system (10) starting from the integral equations of (3) in an unbounded fluid flow. In an unbounded fluid, the Green’s function $G$ and the corresponding stress tensor $\mathbf{K}$ of the Stokes equation are given as

$$
G_{\alpha\beta}(s) = \frac{1}{8\pi s} \left( \frac{\delta_{\alpha\beta}}{s} + \frac{s_{\alpha} s_{\beta}}{s^3} \right), \quad \mathbf{K}_{\alpha\beta\nu}(\mathbf{s}) = -\frac{3}{4\pi} \frac{s_{\alpha} s_{\beta} s_{\nu}}{s^3}
$$

(A1)

with $s = |\mathbf{r} - \mathbf{r}'|$ and $s = \|\mathbf{s}\|_2$. Using translational invariance of these tensors, the singular boundary integrals defining the matrix elements can be solved in Fourier space. We define the Fourier transform of a function $\varphi(\mathbf{s})$ as $\hat{\varphi}(\mathbf{k}) = \int \varphi(\mathbf{s})e^{-i\mathbf{k} \cdot \mathbf{s}}d\mathbf{s}$. This gives the Fourier transform of the Green’s function and the corresponding stress tensor

$$
\hat{G}_{\alpha\beta}(\mathbf{k}) = \frac{\delta_{\alpha\beta} - \hat{k}_{\alpha}\hat{k}_{\beta}}{\eta k^2}, \quad \hat{K}_{\alpha\beta\nu}(\mathbf{k}) = i \frac{\hat{k}_{\alpha}\delta_{\beta\nu} + \hat{k}_{\beta}\delta_{\alpha\nu} - 2\hat{k}_{\beta}\hat{k}_{\nu}}{k},
$$

(A2)

where $k = \|\mathbf{k}\|_2$ and $i = \sqrt{-1}$ is the imaginary unit.

The irreducible matrix elements of (10), $G^{(l,l')}(\sigma,\sigma') = D^{(l)}(\sigma) \otimes G^{(l')}(\sigma') \otimes D^{(l')}(\sigma')$ and $\mathbf{K}^{(l,l')}(\sigma,\sigma') = \mathbf{P}^{(l)}(\sigma) \otimes \mathbf{K}^{(l')}(\sigma') \otimes D^{(l')}(\sigma')$ are defined in terms of single-layer matrix element $G^{(l,l')}$, see (A5), and double-layer matrix element $\mathbf{K}^{(l,l')}$, see (A8), along with the decomposition operators $D^{(l)}$ and the corresponding projection operators $\mathbf{P}^{(l)}$.

$$
\left[ D^{(l)} \otimes F^{\lambda(l)} \right]_{\alpha_1 \cdots \nu_1 \cdots \lambda_1 \cdots \kappa_1 \cdots \kappa_l \cdots \lambda_{l-1}}^{\sigma_1 \cdots \mu_1 \cdots \mu_1 \cdots \sigma_{l-1} \cdots \sigma_{l-1}} = \Delta^{(l)}_{\alpha_1 \cdots \nu_1 \cdots \beta_{\kappa_1 \cdots \kappa_l \cdots \lambda_{l-1}} \cdots \beta_{\kappa_1 \cdots \kappa_l \cdots \lambda_{l-1}}} F^{\lambda(l)}_{\sigma_1 \cdots \mu_1 \cdots \mu_1 \cdots \sigma_{l-1} \cdots \sigma_{l-1}}.
$$

D^{(l)}(\sigma) \otimes F^{\lambda(l)} = \frac{\delta_{\sigma_1 \cdots \mu_1 \cdots \mu_{l-1}}}{\Delta^{(l-1)}} \epsilon_{\sigma_1 \cdots \mu_1 \cdots \mu_{l-1}} \epsilon_{\beta_{\kappa_1 \cdots \kappa_l \cdots \lambda_{l-1}}} F^{\lambda(l)}_{\sigma_1 \cdots \mu_1 \cdots \mu_{l-1}}.

(A3)

and the corresponding projection operators $\mathbf{P}^{(l)}$.

$$
\left[ P^{(l)} \otimes F^{\lambda(l)} \right]_{\beta_{\kappa_1 \cdots \kappa_l \cdots \lambda_{l-1}}}^{\sigma_{l-1} \cdots \mu_{l-1} \cdots \sigma_{l-1} \cdots \sigma_{l-1}} = \Delta^{(l)}_{\beta_{\kappa_1 \cdots \kappa_l \cdots \lambda_{l-1}} \cdots \sigma_{l-1} \cdots \mu_{l-1} \cdots \sigma_{l-1} \cdots \sigma_{l-1}} F^{\lambda(l)}_{\sigma_{l-1} \cdots \mu_{l-1} \cdots \sigma_{l-1} \cdots \sigma_{l-1}}.
$$

(A4)

Here, $\epsilon$ is the Levi-Civita tensor and $\delta$ is the Kronecker delta.

The matrix elements of the linear system are obtained from (3) by expansion of the boundary fields in tensorial spherical harmonics. The single-layer matrix element $G^{(l,l')}(\sigma,\sigma')$ is

$$
G^{(l,l')} = \bar{w}_l \bar{w}_l' \int Y^{(l-1)}(\mathbf{\hat{r}}) G(\mathbf{r} - \mathbf{r}') Y^{(l'-1)}(\mathbf{\hat{r}}') d\mathbf{s} d\mathbf{s}'.
$$

(A5)

We solve the above integral, using (A2) along with the plane wave expansion

$$
e^{ik\cdot\rho} = q_m j_{m-1}(k\rho) Y^{(m-1)}(\mathbf{k}) \otimes Y^{(m-1)}(\mathbf{k}),
$$

(A6)

in terms of the spherical Bessel functions $j_m(k\rho)$, where $\rho = \|\mathbf{\rho}\|_2 = b$ and $q_m = i^{m-1}4\pi b^2 w_m \bar{w}_m$. The single-layer matrix element is then

$$
G^{(l,l')}_{\alpha_1 \cdots \nu_1 \cdots \lambda_1 \cdots \kappa_1 \cdots \lambda_{l-1} \cdots \lambda_{l-1}} = \tau^{G}_{l' \mu_{l-1} m_1 \cdots m_{l-1}} \int_{dS} dS Y^{(l-1)}(\mathbf{\hat{r}}) Y^{(m-1)}(\mathbf{k}) j_{m-1}(k\rho) j_{m-1}(k\rho) \int_{dS} dS' Y^{(l'-1)}(\mathbf{\hat{r}}') Y^{(m'-1)}(\mathbf{k}) Y^{(m'-1)}(\mathbf{k}) \int_{d\Omega} d\Omega' Y^{(m'-1)}(\mathbf{k}) k^2 \hat{G}_{\alpha\beta}(\mathbf{k}) Y^{(m'-1)}(\mathbf{k}),
$$

where $\int_{dS}$ implies the integral over the surface of a sphere with radius $b$, $\int_{d\Omega}$ the integral over the surface of a unit-sphere, and $\int_{d\Omega'}$ a scalar definite integral from 0 to $\infty$, and with $\tau^{G}_{l' \mu_{l-1} m_1 \cdots m_{l-1}} = i^{m+3m'}2\eta^{l'} \bar{w}_l \bar{w}_l' w_m w_m' \bar{w}_m \bar{w}_m' / \pi$. The integral over the pair of spherical Bessel functions can be found in [25]. With this, the results for integrals over
outer products of multiple tensorial spherical harmonics in [26], and the properties of the irreducible tensor $\Delta$, we obtain the result for the single-layer matrix element

$$G_{\alpha l_1 \ldots l_{n-1} \beta k_1 \ldots k_{n-1}'} = \delta_{l l'} G_0^{(l)} \left[ \delta_{\alpha \beta} K_{\nu}^{(l-1)} - \frac{1}{l+1} \frac{2l+1}{2l+2} \Lambda_{\nu}^{(l)} \right].$$

(A7)

Here, $G_0^{(l)} = (l+1)^2/(2\pi \eta b w_{l-1})$ and $\Lambda_{\nu}^{(l)} = \Delta_{\nu}^{(l)} - \Delta_{\nu}^{(l)}$. The double-layer matrix element is

$$K^{(l,l')}_{\alpha l_1 \ldots l_{n-1} \beta k_1 \ldots k_{n-1}'} = \tilde{w}_l w_{l'} \int \bar{Y}^{(l-1)}(\bar{r}) K(r' - r) \cdot \bar{r}^{l'-1}(\bar{r}') dS dS'. \quad (A8)$$

The above integral can be solved by using (A2) and (A6) to obtain

$$K^{(l,l')}_{\alpha l_1 \ldots l_{n-1} \beta k_1 \ldots k_{n-1}'} = \tau^{K}_{l l'} \int dS Y_{\nu_1 \ldots \nu_{m-1}}^{(l)}(\bar{r}) Y_{\mu_1 \ldots \mu_{m-1}}^{(l)}(\bar{r}) \int dk k j_{m-1}(kb) j_{m'-1}(kb)$$

$$\times \int dS' \rho_{\bar{r}} Y_{l_1 \ldots l_{n-1}'}^{(l'-1)}(\bar{r}') Y_{\kappa_1 \ldots \kappa_{n-1}'}^{(l-1)}(\bar{r}') \int d\Omega_k Y_{\mu_1 \ldots \mu_{m-1}}^{(l)}(\hat{k}) \hat{K}_{\beta \alpha \eta}(k) Y_{\nu_1 \ldots \nu_{m-1}'}(\hat{k})$$

with $\tau^{K}_{l l'} = \tau^{m'+3m+2b^2 \tilde{w}_l w_{l'} w_{m} w_{m'} \tilde{w}_m \tilde{w}_{m'}/\pi$. The relevant integral over spherical Bessel functions can again be found in [25]. The full expression of the double-layer matrix element is

$$K^{(l,l')}_{\alpha l_1 \ldots l_{n-1} \beta k_1 \ldots k_{n-1}'} = \delta_{l l'} K_0^{(l)} \left[ \delta_{\alpha \beta} K_{\nu}^{(l-1)} - \frac{2l+1}{2l+2} \Lambda_{\nu}^{(l)} \right], \quad (A9)$$

where $K_0^{(l)} = 3/(4l - 6)$. As is evident from the single-layer (A7) and double-layer (A9) matrix elements, the linear system (10) naturally diagonalises in the modes (l) of the expansion coefficients. It is worth noting that both, single- and double-layer irreducible matrix elements vanish identically for non-diagonal combinations of modes (lt, l't'), ie when $l \neq l'$, apart from (lt, lt') = (l1, l'1). However, upon contraction with an irreducible tensor these too vanish, ie $G^{(l1,l'1)} \otimes F^{(la)} = 0$ and $K^{(l1,l'1)} \otimes V^{(la)} = 0$. Thus, the linear system arising from (3) is diagonal not only in (l), but also in all its irreducible subspaces labelled by (l1).
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