On the Solvability of Some Nonlinear Functional Integral Equations on $L^p(\mathbb{R}_+)$

Mahmoud Bousselsal

$^a$Laboratory of Nonlinear Partial Differential Equations and History of Mathematics, Department of Mathematics, ENS Mohamed El Bachir El Ibrahimi,16050, Vieux-Kouba, Algiers, (Algeria)

Abstract. In this paper, we prove theorems on the existence of solutions in $L^p(\mathbb{R}_+)$, $1 \leq p < \infty$, for some functional integral equations. The basic tool used in the proof is the fixed point theorem due to Darbo with respect to so called measure of noncompactness. The obtained results generalize and extend several ones obtained earlier in many papers and monographs. An example which shows the applicability of our results is also included.

1. Introduction

Integral equation have a lot of applications in many branches of mathematical physics, engineering, mechanics, biology and economics see [24] and references therein. Several different techniques were proposed to study the existence of solutions of the functional integral equations in appropriate function spaces. Although all of these techniques have the same goal, they differ in the function spaces and the fixed point theorems to be applied.

Many papers in the field of functional integral equations give different sets of conditions for the existence of solutions of such equations, see for instance [2, 7, 10, 13, 16, 18] . Apart from that, integral equations are often investigated in research papers and monographs (cf. [6, 8, 11, 12, 15, 17]) and the references cited therein.

Agarwal and O’Regan [4] in 2004, proved the existence of the solutions for the nonlinear integral equation

$$x(t) = \int_0^{+\infty} k(t,s)f(t,x(s))ds, \; t \in \mathbb{R}_+$$

in $C([0, +\infty))$, where $C([0, +\infty))$, denotes the space of bounded and continuous functions on $\mathbb{R}_+$ which have limit at infinity.

2010 Mathematics Subject Classification. 45G10, 45M99, 47H09
Keywords. Existence, the space of Lebesgue integrable functions, measure of noncompactness.

Received: 07 May 2020; Revised: 20 January 2021; Accepted: 13 March 2021
Communicated by Snežana Živković-Zlatanović
Email address: bousselsal55@gmail.com, mahmoud.bousselsal@g.ens-kouba.dz (Mahmoud Bousselsal)
In [23], the author gave the existence of an integrable solutions of the following functional integral equation
\[ x(t) = f(t, x(t)) + g\left(t, \int_0^{\infty} k(t, s) f(t, x(s)) ds\right), \quad t \in \mathbb{R}_+. \]

In [20], the authors discussed the solvability the functional integral equation of convolution type
\[ x(t) = f(t, x(t)) + \int_0^{+\infty} k(t - s)(Qx)(s) ds \]
using a new construction of a measure of noncompactness in \( L^p(\mathbb{R}_+). \)

Next, the authors in [3] study the existence of solutions to the following general functional integral equation
\[ x(t) = f(t, x(t)) + g\left(t, \int_0^{+\infty} k(t - s)(Qx)(s) ds\right) \]
using the same new construction of a measure of noncompactness in \( L^p(\mathbb{R}_+). \)

In this paper, we consider the following more general integral equation
\[ x(t) = f_1(t, x(t)) + f_2\left(t, (Q_1 x)(t), \int_0^{+\infty} u(t, s, (Q_2 x)(s)) ds\right). \quad (1) \]

This equation includes many important integral and functional equations that arise in nonlinear analysis and its applications. We look for solutions to (1) in \( L^p(\mathbb{R}_+), \ 1 \leq p < \infty. \) The main tool used in our considerations is the conjunction of the techniques of measure of noncompactness with Darbo fixed point theorem. An example is presented to show the importance and the applicability of our results.

2. Notation, Definitions and Auxiliary Facts

**Definition 2.1.** The function \( f(t, x, y) = f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is said to have the Carathéodory property if \( f \) is measurable in \( t \) for any \((x, y) \in \mathbb{R}_+ \times \mathbb{R}\) and continuous in \( x, y \) for almost all \( t \in \mathbb{R}_+. \)

Now, we are going to recall some notion about the continuity of the linear integral operator on the space \( L^p = L^p(\mathbb{R}_+). \) Let \( \Delta = \{(t, s) : 0 \leq s \leq t \} \) and \( K : \Delta \to \mathbb{R} \) be the linear Fredholm operator \( K : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+) \) defined by \( (Kx)(t) = \int_0^{+\infty} k(t, s)x(s) ds. \) It is a continuous operator, and \( \|Kx\|_p \leq \|K\| \|x\|_p. \) The norm of the operator is majorized by
\[ \|K\| = \sup\left\{ \|Kx\|_{L^p(\mathbb{R}_+)}, \|x\|_{L^p(\mathbb{R}_+)} \leq 1 \right\} \text{ and hence } \|K\| < \infty. \]

**Remark 2.2.** Observe that if \( \Omega \) is a nonempty and measurable subset of \( \mathbb{R}_+, \) then we can also consider the linear Volterra integral operator \( (Kx)(t) = \int_0^{t} k(t, s)x(s) ds \) associated with the Lebesque space \( L^p(\Omega), \ 1 \leq p \leq \infty. \) Namely, if \( x \in L^p(\Omega), \ 1 \leq p \leq \infty, \) then we can extend \( x \) to be the whole half axis \( \mathbb{R}_+ \) by putting \( x(t) = 0 \) for \( t \in \mathbb{R}_+ \setminus \Omega. \) Then we can treat the operator \( K \) in the usual way (see [21]).

Now, we will collect some definitions and basic results which will be used further on throughout the paper.

First, we denote by \( L^p(\mathbb{R}_+) \) the space of Lebesgue \( p- \) integrable functions on \( \mathbb{R}_+ \) equipped with the standard norm, \( x \in L^p(\mathbb{R}_+), \|x\|_p = \left( \int_0^{+\infty} |x(t)|^p dt \right)^{1/p}. \]

Next, we recall some basic facts concerning measure of noncompactness. Assume that \((E, \|\|)\) is a real Banach space with zero element \( \theta. \) Let \( B(x, r) \) denote the closed ball centered at \( x \) and with radius \( r. \) The symbol \( B \) stands for the ball \( B(\theta, r). \) If \( X \) is a subset of \( E, \) then \( \overline{X} \) and \( \text{Conv}X \) denote the closure and convex
Then every compact continuous map $T : \Omega \to \Omega$ has at least one fixed point.

In the following, we state a fixed point theorem of Darbo type proved by Banas and Goebel [9]

**Theorem 2.5.** (See [15], [9]) Let $\Omega$ be a nonempty, bounded, closed, and convex subset of a Banach space $E$, and let $T : \Omega \to \Omega$ be a continuous mapping such that a constant $k \in [0, 1)$ exists with the property

$$\mu(TX) \leq k\mu(X)$$

for any nonempty $X$ of $\Omega$. Then $T$ has a fixed point in the set $\Omega$. 

---

**Definition 2.3.** [9] A mapping $\mu : M_E \to [0, +\infty]$ is said to be a measure of noncompactness in $E$ if it satisfies following conditions

1. The family $\ker \mu = \{X \in M_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset N_E$.
2. $X \subset Y \implies \mu(X) \leq \mu(Y)$
3. $\mu(X) = \mu(\text{Conv}X) = \mu(X)$
4. $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)Y$, for $\lambda \in [0, 1]$
5. If $(X_n)$ is a sequence of nonempty, bounded, closed subsets of $E$ such that $X_{n+1} \subset X_n$, ($n = 1, 2, \ldots$) and $\lim_{n \to \infty} \mu(X_n) = 0$, then the set $X_\infty = \cap_{n=1}^{\infty} X_n$ is nonempty.

Observe that the intersection set $X_\infty$ belongs to $\ker \mu$. Indeed, since $\mu(X_\infty) \leq \mu(X_n)$ for any $n$, then we infer $\mu(X_\infty) = 0$, so $X_\infty \in \ker \mu$. For other facts concerning measures of noncompactness we refer to [9], [19].

In the following, we give a nonempty $X \subset L^p(\mathbb{R}_+)$ bounded, $\varepsilon > 0$, and $T > 0$. For arbitrary function $x \in X$, we let

$$\omega(x, \varepsilon) = \sup \left\{ \left( \int_0^\infty \left| x(t) - x(t + h) \right|^p \, dt \right)^{\frac{1}{p}} : |h| < \varepsilon \right\}$$

and

$$\omega_0(X, \varepsilon) = \lim_{\varepsilon \to 0} \omega(X, \varepsilon).$$

Also, let

$$d_T(X) = \sup \left\{ \left( \int_0^\infty \left| x(t) \right|^p \, dt \right)^{\frac{1}{p}} : x \in X \right\}$$

and

$$d(X) = \lim_{T \to \infty} d_T(X).$$

Then, the function $\mu : M_{L^p(\mathbb{R}_+)} \to [0, +\infty]$ given by $\mu(X) = \omega_0(X) + d(X)$ is a measure of noncompactness on $L^p(\mathbb{R}_+)$, see ([20]).
Now, we need to characterize the compact subsets of $L^p(\mathbb{R}_+)$.

**Theorem 2.6.** [20] Let $\mathcal{F}$ be a bounded set in $L^p(\mathbb{R}^n)$ with $1 \leq p < +\infty$. Then, $\mathcal{F}$ has a compact closure in $L^p(\mathbb{R}^n)$ if and only if $\lim_{b \to 0} \left\| \tau_b f - f \right\|_p = 0$ uniformly in $f \in \mathcal{F}$, where $\tau_b f(x) = f(x + b)$ for all $x \in \mathbb{R}^n$. In addition, if $\varepsilon > 0$, there is a bounded and measurable subset $\Omega$ of $\mathbb{R}^n$ such that $\left\| f \right\|_{L^p(\mathbb{R}^n \setminus \Omega)} < \varepsilon$ for all $f \in \mathcal{F}$.

**Corollary 2.7.** Let $\mathcal{F}$ be a bounded set in $L^p(\mathbb{R}^n)$ with $1 \leq p < +\infty$. The closure of $\mathcal{F}$ in $L^p(\mathbb{R}^n)$ is compact if and only if $\lim_{b \to 0} \left\| \int_{-b}^b \left[ f(x) - f(x + h) \right]^p dx \right\|^{\frac{1}{p}} = 0$ uniformly in $f \in \mathcal{F}$. Also, for $\varepsilon > 0$, there is a constant $T > 0$ such that $\left\| f(x) \right\|_p < \varepsilon$ for all $f \in \mathcal{F}$.

**Lemma 2.8.** [14]. Let $\Omega$ be a Lebesgue measurable subset of $\mathbb{R}^n$ and $1 \leq p \leq \infty$. If $\{f_n\}$ is a sequence in $L^p(\Omega)$ convergent to $f \in L^p(\Omega)$ in norm, then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which converges to $f$ a.e. in $\Omega$ and a function $g \in L^p(\Omega)$, such that

$$|f_{n_k}(x)| \leq g(x), \text{ for all } k \geq 1, \text{ a.e. } x \in \Omega.$$

Also, we need the following result which is a classical result in Topology.

**Lemma 2.9.** Let $E$ be a metric space and $(x_n)$ a sequence in $E$. If there exists $x \in E$ such that any subsequence $(x_{n_k})$ of $(x_n)$ converges to $x$, then $x_n \to x$ in $E$, as $n \to \infty$.

We shall study the existence of the solutions of eq.(1) assuming some conditions are satisfied.

3. Main Results

**Theorem 3.1.** Assume that the following conditions are satisfied.

1. The function $f_1: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions, and there exist constant $\lambda_1 \in [0, 1)$ and $a_1 \in L^p(\mathbb{R}_+)$ such that

$$|f_1(t, x) - f_1(s, y)| \leq |a_1(t) - a_1(s)| + \lambda_1 |x - y|$$

for any $x, y \in \mathbb{R}$ and almost all $s, t \in \mathbb{R}$ with $f_1(\cdot, 0) \in L^p(\mathbb{R}_+)$.

2. The functions $u: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $k: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy Carathéodory conditions, and there exist $g_1, g_2 \in L^p(\mathbb{R}_+)$ and $g \in L^p(\mathbb{R}_+) (\frac{1}{p} + \frac{1}{q} = 1)$ such that

$$|u(t, s, x)| \leq k(t, s) |x|,$n
$$|u(t_1, s, x) - u(t_2, s, x)| \leq g(s) \left| g_2(t_1) - g_2(t_2) \right|, \quad k(t, s) \leq g_1(t) g(s) \forall t, s \in \mathbb{R}_+, x \in \mathbb{R}.$$

3. The function $f_2: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions, and there exist constants $\lambda_2, \lambda_3 \geq 0$ and $a_2 \in L^p(\mathbb{R}_+)$ such that

$$|f_2(t, x, y) - f_2(s, z, w)| \leq |a_2(t) - a_2(s)| + \lambda_2 |x - z| + \lambda_3 |y - w|$$

for any $x, y, z, w \in \mathbb{R}$ and almost all $s, t \in \mathbb{R}_+$. $f_2(\cdot, 0) \in L^p(\mathbb{R}_+)$. Then, $f_2(\cdot, 0) \to f_2$ in $L^p(\mathbb{R}_+)$.

4. The operators $Q_i, i = 1, 2$ act continuously from $L^p(\mathbb{R}_+)$ into itself and constants $b_i \in \mathbb{R}_+, i = 1, 2$ exist such that

$$\|Q_i x\|_p \leq b_i \|x\|_{L^p(\mathbb{R}_+)}$$

for any $x \in L^p(\mathbb{R}_+)$ and $T \in \mathbb{R}_+$.

5. There exists the nonnegative constant $q_0$ such that the inequality $\omega_0(Q_1 X) \leq q_0 \omega_0(X)$ holds for all nonempty and bounded subset $X$ of the norm ball $B_{r_0}$ where

$$r_0 = \frac{\|f_1(\cdot, 0)\|_p + \|f_2(\cdot, 0, 0)\|_p}{1 - (\lambda_1 + \lambda_2 b_1 + \lambda_3 b_2 \|K\|)}.$$
6. $M = \max \{\lambda_1 + \lambda_2 b_1 + \lambda_3 b_2 \|K\|, \lambda_1 + \lambda_2 q_n\} < 1$.

**Then** the nonlinear integral equation (1) have at least one solution in the space $L^p(\mathbb{R}_+)$.

**Proof.** First, we define the operator $F : L^p(\mathbb{R}_+) \rightarrow L^p(\mathbb{R}_+)$ by

$$(Fx)(t) = f_1(t,x(t)) + f_2(t,(Q_1x)(t), \int_0^\infty u(t,s,(Q_2x(s)))ds).$$

Setting $(F_1x)(t) = f_1(t,x(t))$ and $(F_2x)(t) = f_2(t,(Q_1x)(t), \int_0^\infty u(t,s,(Q_2x(s)))ds)$.

Further considering the Carathéodory conditions, we infer that $Fx$ is measurable for any $x \in L^p(\mathbb{R}_+)$. Now, we prove that $Fx$ is measurable for any $x \in L^p(\mathbb{R}_+)$. We have

$$\|Fx\| \leq \left|f_1(t,x) - f_1(t,0)\right| + \left|f_1(t,0)\right| + \left|f_2(t,(Q_1x)(t), \int_0^\infty u(t,s,(Q_2x(s)))ds) - f_2(t,0,0)\right| + \left|f_2(t,0,0)\right|.$$

By using the Minkowski inequality, we get

$$\left(\int_0^{t_0^+} |(Fx)(t)|^p dt\right)^\frac{1}{p} \leq \left(\int_0^{t_0^+} |f_1(t,x) - f_1(t,0)|^p dt\right)^\frac{1}{p} + \left(\int_0^{t_0^+} |f_1(t,0)|^p dt\right)^\frac{1}{p} + \left(\int_0^{t_0^+} |f_2(t,(Q_1x)(t))|^p dt\right)^\frac{1}{p} + \left(\int_0^{t_0^+} |f_2(t,0,0)|^p dt\right)^\frac{1}{p}.$$

Then,

$$\left(\int_0^{t_0^+} |(Fx)(t)|^p dt\right)^\frac{1}{p} \leq \left(\int_0^{t_0^+} |f_1(t,x) - f_1(t,0)|^p dt\right)^\frac{1}{p} + \left(\int_0^{t_0^+} |f_1(t,0)|^p dt\right)^\frac{1}{p} + \left(\int_0^{t_0^+} |f_2(t,(Q_1x)(t))|^p dt\right)^\frac{1}{p} + \left(\int_0^{t_0^+} |f_2(t,0,0)|^p dt\right)^\frac{1}{p}.$$

So, By using assumptions (1), (2), (6) we obtain

$$\|Fx\|_p \leq \lambda_1 \|x\|_p + \left\|f_1(.,0,0)\right\|_p + \left\|f_2(.,0,0)\right\|_p + \lambda_2 b_1 \|x\|_p + \lambda_3 b_2 \|K\| \|x\|_p.$$

Therefore,

$$\|Fx\|_p \leq \left\|f_1(.,0,0)\right\|_p + \left\|f_2(.,0,0)\right\|_p + \lambda_1 b_1 \|x\|_p + \lambda_3 b_2 \|K\| \|x\|_p.$$ (2)

Hence, $F(x) \in L^p(\mathbb{R}_+)$ and $F$ is well defined and also from (2), we have $F(B_{r_0}) \subseteq B_{r_0}$, where $r_0$ is

$$r_0 = \frac{\left\|f_1(.,0,0)\right\|_p + \left\|f_2(.,0,0)\right\|_p}{1 - (\lambda_1 + \lambda_2 b_1 + \lambda_3 b_2 \|K\|).}$$

Now, we prove that $F$ is continuous in $L^p(\mathbb{R}_+)$. It is enough to prove that $F_2$ is continuous. Indeed, Let $(x_n)$ be a sequence in $L^p(\mathbb{R}_+)$ which converges to $x \in L^p(\mathbb{R}_+)$, since $Q_i, i = 1,2$ are continuous for a.e. $t \in \mathbb{R}_+$ and from lemma 2.8, it follows that up a subsequence that

$$x_n \rightarrow x, \quad Q_i x_n \rightarrow Q_i x, \text{ for } i = 1,2$$

$$\exists \varphi \geq 0, \quad \varphi \in L^p(\mathbb{R}_+) : \max \left\|x_n(s)\right\|, \left|Q_i x_n(s)\right\| \leq \varphi(s) \text{ a.e. on } \mathbb{R}_+.$$
Since $u$ satisfies the Carathéodory conditions, $Q_{x_n} \to Q_x$, almost everywhere on $\mathbb{R}_+$. It follows from assumption (2) that
\[ u(t, s, Q_{x_n}) \to u(t, x, Q_{x(t)}) \text{ for almost } t, s \in \mathbb{R}_+ \] (3)
and
\[ |u(t, s, Q_{x_n})| \leq k(t, s) \varphi(s). \] (4)

Then we have by using the Lebesgue’s Dominated Convergence Theorem
\[ \int_0^{+\infty} u(t, s, Q_{x_n}(s))ds \to \int_0^{+\infty} u(t, s, Q_{x(t)}(s))ds. \]

Hence for almost all $t \in \mathbb{R}_+$
\[ f_2(t, Q_{x_n}(t), \int_0^{+\infty} u(t, s, Q_{x_n}(s))ds) \to f_2(t, Q_{x(t)}(t), \int_0^{+\infty} u(t, s, Q_{x(t)}(s))ds). \] (5)

We have for almost everywhere in $\mathbb{R}_+$ the following estimate
\[ \left| f_2(t, Q_{x_n}(t), \int_0^{+\infty} u(t, s, Q_{x_n}(s))ds) \right| \leq \lambda_1 g(t) + \lambda_2 g_1(t) \int_0^{+\infty} g(s)\varphi(s)ds + |f(t, 0, 0)| \] (6)

Regarding the assumptions on $g, g_1$ and $|f(t, 0, 0)|$ we get
\[ \lambda_1 g(t) + \lambda_2 g_1(t) \int_0^{+\infty} g(s)\varphi(s)ds + |f(t, 0, 0)| \in L^p(\mathbb{R}_+). \] (7)

Then from (5), (6), (7) and by using the Lebesgue’s Dominated Convergence Theorem, we get
\[ \|F_{x_n} - F_{x(t)}\|_{L^p} \to 0. \]

Since any sequence $\{x_n\}$ converging to $x$ in $L^p$ has a subsequence $\{x_{n_k}\}$ such that $\|F_{x_{n_k}} - F_{x(t)}\|_{L^p} \to 0$, we can conclude that $F_2$ is a continuous operator. Further, we will show that
\[ \omega_0(FX) \leq \lambda_1 + \lambda_2 b_1 + \lambda_3 b_2 \|K\| \omega_0(X) \]
for any nonempty set $X \subset B_{\varepsilon}$. To this end, we fix an arbitrary $\varepsilon > 0$. Let us choose $x \in X$ and $t, h \in \mathbb{R}_+$ with $|h| \leq \varepsilon$. We have
\[
\begin{align*}
(Fx)(t) - (Fx)(t + h) & \leq \left| f_1(t, x(t)) - f_1(t + h, x(t)) \right|
+ \left| f_1(t, x(t)) - f_1(t + h, x(t + h)) \right|
+ \left| f_2(t, Q_{x(t)}(t), \int_0^{+\infty} u(t, s, Q_{x(t)}(s))ds) \right|
- f_2(t + h, Q_{x(t)}(t), \int_0^{+\infty} u(t, s, Q_{x(t)}(s))ds)
- f_2(t + h, Q_{x(t)}(t + h), \int_0^{+\infty} u(t, s, Q_{x(t)}(s))ds)
- f_2(t + h, Q_{x(t)}(t + h), \int_0^{+\infty} u(t + h, s, Q_{x(t)}(s))ds)
+ f_2(t + h, Q_{x(t)}(t + h), \int_0^{+\infty} u(t + h, s, Q_{x(t)}(s))ds).
\end{align*}
\]
Therefore
\[
(Fx(t) - (Fx)(t + h) | x(t) - x(t + h) | ≤ |a_1(t) - a_1(t + h)| + |x(t) - x(t + h)|
+ \lambda_2 [(Q_1x(t) + h) - (Q_1x)(t)] + |a_2(t) - a_2(t + h)|
+ \lambda_3 \int_0^\infty u(t + h, s, (Q_2x)(s)) - u(t, s, (Q_2x)(s)) ds dt.
\]

By Minkowski’s inequality, we get
\[
\left( \int_0^\infty |(Fx)(t) - (Fx)(t + h)|^p dt \right)^{\frac{1}{p}} ≤ \left( \int_0^\infty |a_1(t) - a_1(t + h)|^p dt \right)^{\frac{1}{p}} + \lambda_1 \left( \int_0^\infty |x(t) - x(t + h)|^p dt \right)^{\frac{1}{p}}
+ \lambda_2 \left( \int_0^\infty |Q_1x(t) - (Q_1x)(t + h)|^p dt \right)^{\frac{1}{p}}
+ \lambda_3 \left( \int_0^\infty \int_0^\infty \int_0^\infty u(t + h, s, (Q_2x)(s)) - u(t, s, (Q_2x)(s)) ds dt \right)^{\frac{1}{p}}.
\]

Consequently we get
\[
\lambda_1 \left( \int_0^\infty |Q_1x(t) - (Q_1x)(t + h)|^p dt \right)^{\frac{1}{p}} ≤ \lambda_1 \left( \int_0^\infty |Q_2x(t) - (Q_2x)(t + h)|^p dt \right)^{\frac{1}{p}}
+ \lambda_2 \left( \int_0^\infty |Q_2x(t) - (Q_2x)(t + h)|^p dt \right)^{\frac{1}{p}}
+ \lambda_3 \left( \int_0^\infty \int_0^\infty \int_0^\infty u(t + h, s, (Q_2x)(s)) - u(t, s, (Q_2x)(s)) ds dt \right)^{\frac{1}{p}}.
\]

Hence, we obtain
\[
\left( \int_0^\infty |(Fx)(t) - (Fx)(t + h)|^p dt \right)^{\frac{1}{p}} ≤ \lambda_1 \left( \int_0^\infty |Q_1x(t) - (Q_1x)(t + h)|^p dt \right)^{\frac{1}{p}}
+ \lambda_2 \left( \int_0^\infty |Q_2x(t) - (Q_2x)(t + h)|^p dt \right)^{\frac{1}{p}}
+ \lambda_3 \left( \int_0^\infty \int_0^\infty \int_0^\infty u(t + h, s, (Q_2x)(s)) - u(t, s, (Q_2x)(s)) ds dt \right)^{\frac{1}{p}}.
\]

Therefore, we obtain
\[
\omega(FX, \epsilon) ≤ \omega(a_1, \epsilon) + \lambda_1 \omega(X, \epsilon) + \omega(a_2, \epsilon)
+ \lambda_2 \omega(Q_1X, \epsilon) + \lambda_3 \omega(Q_2X, \epsilon).
\]

Since \(a_1\), \(a_2\), \(Q_2\) are compacts set in \(L^p(\mathbb{R})\), we have \(\omega(a_1, \epsilon) \rightarrow 0, \omega(a_2, \epsilon) \rightarrow 0\) and \(\omega(Q_2, \epsilon) \rightarrow 0\) as \(\epsilon \rightarrow 0\). Then, by going to the limit in (8) as \(\epsilon \rightarrow 0\) and from assumption (5), we obtain
\[
\omega_0(FX) ≤ (\lambda_1 + \lambda_2 q_0) \omega_0(X).
\]

In the following, we fix an arbitrary number \(T > 0\). Then, for an arbitrary function \(x \in X\), we have
\[
\left( \int_T^\infty |F(x)(t)|^p dt \right)^{\frac{1}{p}} ≤ \left( \int_T^\infty |f_1(t, x) - f_1(t, 0)|^p dt \right)^{\frac{1}{p}} + \left( \int_T^\infty |f_2(t, 0)|^p dt \right)^{\frac{1}{p}}
+ \left( \int_T^\infty |f_2(t, 0)|^p dt \right)^{\frac{1}{p}} + \left( \int_T^\infty |f_2(t, 0, 0)|^p dt \right)^{\frac{1}{p}}.
\]
Therefore

\[
\left(\int_0^\infty |F(x)(t)|^p \, dt\right)^{\frac{1}{p}} \leq \lambda_1 \left(\int_0^\infty |x(t)|^p \, dt\right)^{\frac{1}{p}} + \left(\int_0^\infty |f_1(t, 0)|^p \, dt\right)^{\frac{1}{p}} + \lambda_2 \left(\int_0^\infty |Q_1(x)(t)|^p \, dt\right)^{\frac{1}{p}} + \lambda_3 \left(\int_0^\infty k(t, s)(Q_2(x)(s))ds\right)^{\frac{1}{p}} \leq \lambda_1 \left(\int_0^\infty |x(t)|^p \, dt\right)^{\frac{1}{p}} + \lambda_2 \left(\int_0^\infty |f_1(t, 0)|^p \, dt\right)^{\frac{1}{p}}
\]

Then we have

\[
\left(\int_0^\infty |f(t)|^p \, dt\right)^{\frac{1}{p}} \leq \lambda_1 \left(\int_0^\infty |x(t)|^p \, dt\right)^{\frac{1}{p}} + \lambda_2 b_1 \left(\int_0^\infty |x(t)|^p \, dt\right)^{\frac{1}{p}} + \lambda_3 b_2 \|K\| \left(\int_0^\infty |x(t)|^p \, dt\right)^{\frac{1}{p}} + \left(\int_0^\infty |f_2(t, 0)|^p \, dt\right)^{\frac{1}{p}}.
\]

Since \(|f_1(t, 0)|\) and \(|f_2(t, 0, 0)|\) are compacts in \(L^p(\mathbb{R}_+)^\), then, as \(T\) goes to \(+\infty\), we obtain \(\left(\int_0^\infty |f_1(t, 0)|^p \, dt\right)^{\frac{1}{p}}\) and \(\left(\int_0^\infty |f_2(t, 0, 0)|^p \, dt\right)^{\frac{1}{p}}\) go to 0.

Hence,

\[
d(FX) \leq (\lambda_1 + \lambda_2 b_1 + \lambda_3 b_2 \|K\|) d(X).
\]

So, from (9) and (10) it follows

\[
\mu(FX) \leq \max \{\lambda_1 + \lambda_2 b_1 + \lambda_3 b_2 \|K\|, \lambda_1 + \lambda_2 q_n\} \mu(X).
\]

By (11), assumption (6) and Theorem 2.5, we deduce that the operator \(F\) has a fixed point \(x\) in \(B_\varepsilon\) and consequently, eq.(1) has at least one solution in \(L^p(\mathbb{R}_+)\). \(\square\)

4. Example

Consider the functional integral equation

\[
x(t) = \frac{\cos x(t)}{t + 2} + \frac{|x(t)|}{21(1 + |x(t)|)} e^{-t} + \frac{1}{10} \int_0^{+\infty} \sin \left(|x(s)| e^{-|x(s)|}\right) ds + \frac{1}{10} \int_0^{+\infty} e^{-t} (s + 2 t) ds.
\]

Eq. (12) is a special case of Eq. (1) with

\[
f_1(t, x, y) = \frac{\cos x(t)}{t + 2}, \ f_2(t, x, y, z) = \frac{1}{t + 2} x + \frac{1}{t + 2} y, \ Q_1(x)(s) = \frac{|x(s)|}{1 + |x(s)|} e^{-s},
\]

\[
k(t, s) = \frac{1}{\sqrt{\pi (t + 2)(s + 2)}} e^{-s}, \ (Q_2(x)(s) = e^{-|x(s)|}|x(s)|, \ |(Q_2(x)(s)| \leq \frac{1}{2} |x(s)|).
\]

In this example, hypothesis (1) holds with \(a_1(t) = \frac{1}{t + 2}\) and \(\lambda_1 = \frac{1}{2}\), indeed, we have

\[
|f_1(t, x) - f_1(t, y)| = \frac{\cos x(t) - \cos y}{t + 2} \leq \frac{|x(t) - y(t)|}{t + 2}.
\]

In addition, \(f_1(t, 0) = \frac{1}{t + 2} \in L^p(\mathbb{R}_+),\) indeed, \(\|f_1(t, 0)\|_{L^p(\mathbb{R}_+)} = \int_0^{+\infty} \frac{ds}{(1 + s)^p} = \frac{1}{p-1}\) for all \(p > 1\). Thus, we have \(\|f_1(t, 0)\|_{L^p(\mathbb{R}_+)} = \left(\frac{1}{p-1}\right)^{\frac{1}{p}}\). Further we have

\[
|f_2(t, x, y) - f_2(s, z, w)| \leq \frac{1}{3} |x - z| + \frac{1}{10} |y - w|.
\]
Thus, according to assumption (5) we may put

\[ a_2(t) = 0, \lambda_2 = \frac{1}{7} \text{ and } \lambda_3 = \frac{1}{10}, \, f_2(t, 0, 0) = 0 \in L^p(\mathbb{R}_+). \]

We have

\[
\begin{align*}
    u(t, s, x) &= k(t, s) \sin x \\
    |u(t, s, x)| &\leq k(t, s) |x| \\
    |u(t_1, s, x) - u(t_2, s, x)| &\leq |k(t_1, s) - k(t_2, s)| g(s)
\end{align*}
\]

and

\[
\begin{align*}
    k(t, s) &\leq \frac{1}{\alpha(t+3)^2} \times \frac{1}{(s+2)^2} \\
    g_1(t) &= \frac{1}{\alpha(t+3)^2} \\
    g_2(s) &= g(s) = \frac{1}{\alpha(t+3)^2} \\
    |k(t_1, s) - k(t_2, s)| &\leq \frac{1}{\alpha(t+3)^2} - \frac{1}{\alpha(t+3)^2} \left( g(s) \right)
\end{align*}
\]

Q_1 and Q_2 satisfied assumption (4) of theorem 3.1 with \( b_2 = 1, b_1 = \frac{1}{7} \). By using theorem 3.4 in [1], we have \( ||K|| \leq \frac{1}{7} \).

Further, for \( \varepsilon \geq 0, ||x|| \leq r_0, ||h|| < \varepsilon \) and by the Mean theorem, we get

\[
\begin{align*}
    &\left( \int_0^{\infty} |(Q_1 x)(t+h) - (Q_1 x)(t+h)|^p \, dt \right)^\frac{1}{p} \\
    \leq &\frac{1}{7} \left( \int_0^{\infty} |x(t+h) - x(t+h)|^p \, dt \right)^\frac{1}{p} + \frac{\varepsilon}{7} e^{-\theta h} \left( \int_0^{\infty} e^{-\theta t} \, dt \right)^\frac{1}{p}, \, 0 < \theta < 1 \\
    \leq &\frac{1}{7} \left( \int_0^{\infty} |x(t+h) - x(t+h)|^p \, dt \right)^\frac{1}{p} + \frac{\varepsilon}{7} e^{-\theta h} \left( \int_0^{\infty} e^{-\theta t} \, dt \right)^\frac{1}{p} + \frac{\varepsilon}{7} M
\end{align*}
\]

where \( M = \left( \int_0^{\infty} e^{-\theta t} \, dt \right)^\frac{1}{p} \). Hence, from the last estimate, we get as \( \varepsilon \) goes to 0

\[ \omega_0(Q_1 X) \leq \frac{1}{7} \omega_0(X). \] (13)

Thus, according to assumption (5) we may put \( q_0 = \frac{1}{7} \). Further we get \( \lambda_1 + \lambda_2 b_1 + \lambda_3 b_2 ||K|| \leq \frac{1}{2} + \frac{1}{21} + \frac{1}{10} < 1 \) and \( \lambda_1 + \lambda_2 q_0 = \frac{1}{2} + \frac{1}{30} = 0.52 < 1 \). The inequality of assumption (6) is satisfied with the constant \( M < 1 \). Since all of the assumptions of Theorem 3.1 are fullfilled, we deduce that the functional integral equation (12) has at least one solution belonging to the ball \( B_{q_0} \) of the space \( L^p(\mathbb{R}_+) \).

Acknowledgment
The author would like to thank the referee for carefully reading the paper.

References
[1] A. Aghajani, D. O’Regan and A. Shole Haghighi, Measure of noncompactness on \( L^p(\mathbb{R}^n) \) and Applications, CUBO A Math. J. vol. 17, No 01, (85-97), 2015.
[2] A. Aghajani, J. Banas, Y. Jafari, Existence of solution for a class nonlinear Volterra singular integral, Comput. Math. Appl. 62 (2011), 1215-1227.
[3] W. Al Sayed and M.A. Darwish, On the existence of solutions of a perturbed functional integral equation in the space of Lebesgue Integrable functions on \( \mathbb{R}_+ \), Journal of Math. and Appl. JMA No 41, pp. 19-27 (2018).
[4] R. P. Argawal and D. O’Regan, Fredholm and Volterra integral equations with inghajani, tegrable singularieties, Hokkaido Math. J. 33 (2004), no. 2, 443-456.
[5] R. P. Argawal and D. O’Regan, Fixed point theory and applications, Cambridge University Press (2004).
[6] J. Banas, Z. Knap, On measures of weak noncompactness and nonlinear integral equations of convolution type, J. Math. Anal. 146 (2) (1990) 353-362.
[7] J. Banas, Z. Knap, Integrable solutions of a functional-integral equation, Revista Mat. Univ. Complutense de Madrid 2 (1989) 31-38.
[8] J. Banas, J. Rivero, On measures of weak noncompactness, Ann. Mat. Pure Appl. 151, (1988) 213-224.
[9] J. Banas and K. Goebel, Measures of Noncompactness In Banach spaces, Lecture Notes in Pure and Applied Mathematics 60, Marcel Dekker, New York, 1980.
[10] J. Banas, R. Rzepka, An application of a measure of noncompactness in the study of asymptotic stability. Appl. Math. Lett. 16 (2003), 1-6
[11] M. Bousselsal, A. bellour and M.A. Taoudi, On the solvability of a nonlinear integro-diﬀerential equation on the half axis, Meddit. J. Math. 2016, 13, 5, pp. 2887-2896
[12] M. Bousselsal, A. Bellour and M.A Taoudi, Integrable solutions of a nonlinear integral equation related to some epidemic models, Glasnick Mat. 49, (69), 3 95406, (2014)
[13] M. Bousselsal and Jah Sidi, Integrable Solutions of a nonlinear integral equation via noncompactness measure and Krasnoselskii’s fixed point theorem, International J. of Analysis, vol. 2014, article ID 280, 709, 10 pages.
[14] H. Brezis, Analyse fonctionnelle, théorie et applications, edition, Masson, (1983).
[15] G. Darbo, Punti uniti in trasformazioni a codominio non compatto, Rend. Sem. Mat. Univ. Padova, 24 (1955), 84-92.
[16] M. Darwish, On a perturbed functional integral equation of Urysohn type, Appl. Math. Comput. 218 (2012), 8800-8805.
[17] B. Folland, Real Analysis, A Wiley-Interscience Publication, 1999.
[18] W. Goma El-Sayed, Nonlinear functional integral equations of convolution type, Port. Math. 54 (1997), 449-456.
[19] K. Kuratowski, Sur les espaces complets. Fund. Math. 15 (1930), 301-309.
[20] H; Khosravi, R. Allahyari, A. S. Haghighi, Existence of solutions of functional integral equations of convolution type using a new construction of a measure of noncompactness on $L^p(R_+).$ Appl. Math. and Comp. 260 (2015) 140-147.
[21] M.M.A. Metwali, The solvability of functional quadratic Volterra-Urysohn integral equations on the half line, Sc. Fas. Math. 61, 2018.
[22] W. Rudin, Real and complex analysis, Mc Graw-Hill, New York, 1987.
[23] N. Salhi and M.A. Taoudi, Existence of Integrable solutions of an integral equation of Hammerstein type on an unbounded interval, Meditter. J. math. 9, (2012), 729-739.
[24] P.P. Zabrejko, A. I. Koshelev, M.A. Krasnosel’skii, S.G. Mikhlin, L.S. Rakovshchik, V.J. Stetsenko, Integral Equations, Nauka Moscow, 1968.