Discussion of Three Examples to Recent Results of Finite- and Fixed-Time Convergent Algorithms

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Abstract. This note discusses three examples given in the recent technical correspondence paper [1], which addresses the results presented in [2,3,4]. It is shown that the first example ([1], Section 3) is irrelevant to the results of [2]. The second example ([1], Section 4) establishes a well-known fact about the result of [3] that a continuous differentiator can exactly differentiate a signal, only if its second derivative is equal zero. This note provides a method to extend the algorithms presented in [3] to the general case. Finally, the third example ([1], Section 5) presents a particular case related to Theorem 1 of [4]. Theorem 1 of [4] remains, however, valid in the most practical case of selecting control gains or after imposing an additional condition represented by a strict Raleigh’s inequality. The result of Theorem 2 in [4] estimating the fixed convergence time holds as well.

1. Introduction

The recently published technical correspondence paper [1] addresses the results presented in [2,3,4]. Three examples questioning validity of the obtained results are provided.

This note discusses the examples given in [1]. It is shown that the first example ([1], Section 3) is irrelevant to the results of [2]. The second example ([1], Section 4) establishes the well-known fact about the result of [3] that a continuous differentiator can exactly differentiate a signal, only if its second derivative is equal zero. This note provides a method to extend the algorithms presented in [3] to the general case. It is shown that if the signal second derivative is not equal to zero, the differentiator can be modified by including discontinuous terms to achieve the goal. Finally, the third example ([1], Section 5) presents a special case of control gains and initial conditions.

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where the result of Theorem 1 in [4] on estimating the finite convergence time does not hold. However, this note demonstrates that the result of Theorem 1 in [4] still remains valid in the most practical case of selecting control gains. The general result of Theorem 1 in [4] also remains valid, if an additional condition represented by a strict Raleigh’s inequality is imposed. The result of Theorem 2 in [4] on estimating the fixed convergence time holds as well.

This note is organized as follows. Section 2–4 subsequently discuss the examples given in Sections 3, 4, and 5 of [1]. Section 6 summarizes the discussions.

2. Discussion of Example of Section 3 in [1]

Following the notation of Lemma 1 in [1], note that the paper [2] considers only systems with initial conditions in the form \(x_0, x_{20}, x_{30}, 0\). Therefore, Lemma 1 of [1] is applicable to the systems studied in [2], only if \(a = b = 0\), that is, \(x(t_0) = [0, 0, 0, 0]\) is the origin. However, in this case, the system (2) of Section 3 in [1] has only the zero solution, \(x(t) = 0\) for \(t \geq 0\), which is finite-time convergent to the origin. Remark 2 and Fig. 1 of Section 3 in [1] are not relevant to the result of [2], since \(k_2 = 1 < 2\) and the conditions of Lemma 4 in [2] do not hold. Proposition 3 of Section 3 in [1] is not relevant to the result of [2] as well, since the paper [2] studies only attractivity (convergence) problems but not finite-time stability ones. The difference between finite-time stability and finite-time attractivity concepts can be consulted in Section 4 of [5].

3. Discussion of Example of Section 4 in [1]

This is a well-known fact that the finite- and fixed-time convergent differentiators proposed in Theorems 1 and 2 of [3] converge to the real system states exactly, only if the output \(n\)-th derivative is equal to zero, \(y^{(n)}(t) = 0\), for all \(t \geq T\), where \(T\) is a certain finite time and \(n\) is the dimension of the differentiator. Note that in the series of papers mentioned in [1] the finite- or fixed-time convergent differentiators are used as parts of finite- or fixed-time convergent controllers, whose setpoints are represented by equilibria, that is, the condition \(y^{(n)}(t) = 0\) holds after a certain finite \(T\).

Furthermore, those differentiators can be modified to achieve finite- or fixed-time convergence in the general situation by adding the term \(-\lambda \text{sign}(z_1(t) - y(t))\) to the last differentiator equations, where \(\lambda > |y^{(n)}(t)|\) is a uniform bound for the output \(n\)-th derivative. For example, in case of the fixed-time convergent differentiator proposed in Theorem 2 of [3], the corresponding equations take the form

\[
\dot{z}_1(t) = z_2(t) - k_1 |z_1(t) - y(t)|^{\alpha_1} \text{sign}(z_1(t) - y(t)) \\
- \kappa_1 |z_1(t) - y(t)|^{\beta_1} \text{sign}(z_1(t) - y(t)) \\
\vdots \\
\dot{z}_i(t) = z_{i+1}(t) - k_i |z_1(t) - y(t)|^{\alpha_i} \text{sign}(z_1(t) - y(t)) \\
- \kappa_i |z_1(t) - y(t)|^{\beta_i} \text{sign}(z_1(t) - y(t)),
\]

where
\[ i = 1, \ldots, n - 1 \]

\[ \dot{z}_n(t) = -k_n \left| z_1(t) - y(t) \right|^{a_n} \text{sign}(z_1(t) - y(t)) \]

\[ -\kappa_n \left| z_1(t) - y(t) \right|^{b_n} \text{sign}(z_1(t) - y(t)) \]

\[ -\lambda \text{sign}(z_1(t) - y(t)) \],

where the gains \( k_1, \ldots, k_n \) and \( \kappa_1, \ldots, \kappa_n \) satisfy the conditions of Theorem 2 of [3] and \( \lambda > |y(n)(t)| \). This modification keeps the convergence fixed time estimates given in Theorem 2 of [3] and the convergence finite time estimates given in Theorem 1 of [3] in the most practical case of selecting the control gains, as noted in the next section.

The differentiator (1) is not smooth; however, a smooth differentiator for \((n - 1)\)-th derivative of the output can be constructed by increasing the dimension of the differentiator (1) by one, i.e., adding the equation for \( z_{n+1} \) and moving the term \(-\lambda \text{sign}(z_1(t) - y(t))\) to this equation, provided that the condition \( \lambda > |y(n+1)(t)| \) holds.

4. Discussion of Example of Section 5 in [1]

4.1. The result of Theorem 1 in [4] remains valid in the most practical case or after imposing an additional condition

Indeed, the result of Theorem 1 in [4] remains valid in the most practical case of selecting the control gains \( k_1, k_2, \ldots, k_n \) by assigning the eigenvalues of the matrix \( A \) as the multiple roots of its characteristic polynomial in the form \((\lambda - \mu)^n = 0\), where all \( \mu_i = -\mu \) and \( \mu > 0 \) is a positive real number. This is the assignment scheme mostly used by control scientists and engineers, which is commonly implemented due to its simplicity and the fact that increasing the absolute value of \( \mu \) leads to accelerating the convergence of a linear system state towards the origin.

To see this, consider the example given in Section 5 of [1]. Then, \( k_1 = \mu^2, k_2 = 2\mu \), and the condition required by Remark 9 and Proposition 10 of [1] does not hold, since \( k_2^2 - 4k_1 = 4\mu^2 - 4\mu^2 = 0 \). Further calculations yield that the corresponding matrix \( P \) is given by

\[
P = \begin{pmatrix}
\frac{1}{\mu^2} + \frac{\mu^2 + 1}{4\mu^2} & \frac{1}{2\mu^2} & \frac{1}{4\mu^2} \\
\frac{1}{4\mu^2} & \frac{1}{2\mu^2} & \frac{\mu^2 + 1}{4\mu^2}
\end{pmatrix}.
\]

The right-hand side of the formula (22) in [1] takes the form \( \mu \lambda_{\text{max}}(P) \). Assuming \( \lambda_{\text{min}}(Q) = 1 \), the inequality \( \mu \lambda_{\text{max}}(P) > 1 \) holds for any \( \mu > 0 \), which is verified directly calculating the maximum eigenvalue of the matrix \( P \) as a function of \( \mu \). For instance, \( \mu \lambda_{\text{max}}(P) = 1 + \frac{\sqrt{\pi}}{2}, \) if \( \mu = 1 \).

Thus, Theorem 1 in [4] still provides a method to estimate the finite convergence time for Bhat and Bernstein algorithm [6] in the most practical and broadly employed case of selecting its control gains \( k_1, k_2, \ldots, k_n \).

Actually, a more general result takes place.
Proposition 1. Let the conditions of Theorem 1 in [4] be valid and, in addition, the Raleigh’s inequality
\[ x^T Px < \lambda_{\max}(P) \|x\|^2, \]  
strictly holds for all \( x \neq 0 \). Then, the formula (6) of Theorem 1 in [4] for estimating the convergence time holds.

Proof. The formula (6) of Theorem 1 in [4] follows from the relation similar to the Raleigh’s inequality
\[ (x^T Px)^{1+m} \leq \lambda_{\max}(P) \|x\|^2, \]
which is established for some \( m = (\gamma - 1)/\gamma < 0 \), where \( \gamma \in (1-\varepsilon, 1) \) and \( \varepsilon > 0 \) is a sufficiently small positive number. If the Raleigh’s inequality (2) strictly holds for all \( x \neq 0 \), such an \( m \) always exists for any \( x \neq 0 \) in view of the strict inequality. Furthermore, the relation (3) holds for \( x = 0 \), turning to an equality. Thus, the non-strict relation (3) is valid for any \( x \in \mathbb{R}^n \) and, therefore, the formula (6) of Theorem 1 in [4] holds. ■

Note that the Raleigh’s inequality (2) does not strictly hold for all \( x \neq 0 \) in the example given in Section 5 of [4]. On the other hand, it does hold strictly in the considered case, if the eigenvalues of the matrix \( A \) are assigned as the multiple roots of its characteristic polynomial in the form \((\lambda - \mu_i)^n = 0\), where all \( \mu_i = -\mu \) and \( \mu > 0 \) is a positive real number. Validity of the formula (6) of Theorem 1 in [4] in this case is illustrated by the following simulations.

The \( n \)-dimensional chain of integrators
\[ \dot{x}_1(t) = x_2(t), \quad x_1(t_0) = x_{10}, \]
\[ \dot{x}_2(t) = x_3(t), \quad x_2(t_0) = x_{20}, \]
\[ \ldots \]
\[ \dot{x}_n(t) = u(t), \quad x_n(t_0) = x_{n0}, \]
is simulated for \( n = 2, 3, 4, 5 \). The scalar control input \( u(t) \) is assigned according to Bhat and Bernstein algorithm [6]
\[ u(t) = v_1(t) + v_2(t) + \ldots + v_n(t), \]
where \( v_i(t) = -k_i \mid x_i(t) \mid^\gamma \text{sign}(x_i(t)) \) and the exponents \( \gamma_i \), \( i = 1, \ldots, n \), are defined by \( \gamma_{i-1} = \gamma \gamma_{i+1}/(2\gamma_{i+1} - \gamma) \), \( i = 2, \ldots, n \), \( \gamma_{n+1} = 1 \), and \( \gamma_n = \gamma \). The control gains \( k_i \), \( i = 1, \ldots, n \), are assigned such that all multiple roots of its characteristic polynomial \((\lambda - \mu_i)^n = 0\) are equal to \( \mu_i = -1 \). Namely, \( k_1 = 1, k_2 = 2 \) for \( n = 2 \); \( k_1 = 1, k_2 = 3, k_3 = 3 \) for \( n = 3 \); \( k_1 = 1, k_2 = 4, k_3 = 6, k_4 = 4 \) for \( n = 4 \); and \( k_1 = 1, k_2 = 5, k_3 = 10, k_4 = 10, k_5 = 5 \) for \( n = 5 \). The parameter \( \gamma \) is set to \( \gamma = 10/11 \) in all simulations. The convergence time estimated is computed according to the formula (6) of Theorem 1 in [4].

The simulation results are given in the following tables, which confirm validity of the formula (6) of Theorem 1 in [4].
The authors thank the author of [1] for the example given in Section 5 of [1] as the really relevant and insightful one.

4.2. The result of Theorem 2 in [4] remains valid

It is argued in Subsection 5.3 of [1] that the inequality (23) there is not valid for all $x \in \mathbb{R}^n$, since both parts of the inequality (23) tend to zero as $x$ tends to zero. Following this logic, the example of Subsection 5.2 could be constructed only for initial values $x_0$ sufficiently close to zero. Furthermore, the result of Theorem 2 in [4] providing an upper estimate for fixed convergence time would remain valid, since it takes into account initial values arbitrarily distant from zero.

Indeed, consider the example given in Section 5 of [1]. Let the gains $\kappa_1, \kappa_2$ in Theorem 2 in [4] are selected the same as $k_1, k_2$: $k_1 = \kappa_1 = 1$, $k_2 = \kappa_2 = 6$. Then, assuming $P_1 = P$ and setting $Q_1 = Q$ to the $2 \times 2$ identity matrix, the right-hand side of the formula (22) in the fixed-time convergence case is equal to

$$2 - \frac{k_2 - \sqrt{k_2^2 - 4k_1}}{2} \frac{\lambda_{\text{max}}(P)}{\lambda_{\text{min}}(Q)} = (3 - \sqrt{8})(\frac{10}{3} + \sqrt{10}) \approx 1.14447 > 1.$$
Thus, the result of Theorem 2 in [4] remains valid and, in addition, provides a practically useful upper estimate for fixed convergence time in the example given in Section 5 of [1]. It should be noted that convergence time estimates based on Lyapunov functions proposed in [7] are too conservative and cannot be used for practical estimation of fixed convergence time.

5. Conclusions

This note discussed the examples given in [1]. It has been shown that the results opposed in [1] remain valid in most practical cases or can be successfully modified or are irrelevant to the given examples.

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