Effective coupling between two Brownian particles

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We use the system-plus-reservoir approach to study the dynamics of a system composed of two independent Brownian particles. We present an extension of the well-known model of a bath of oscillators which is capable of inducing an effective coupling between the two particles depending on the choice made for the spectral function of the bath oscillators. The coupling is non-linear in the variables of interest and an exponential dependence on these variables is imposed in order to guarantee the translational invariance of the model if the two particles are not subject to any external potential. The effective equations of motion for the particles are obtained by the Laplace transform method and besides recovering all the local dynamical properties for each particle we end up with an effective interaction potential between them. We explicitly analyze one of its possible forms.

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Usually the Brownian motion of a given dynamical variable is modelled by considering the system it describes - the system of interest - coupled to a thermal bath responsible for its energy loss. Assuming that any degree of freedom of this environment is only weakly perturbed by the system of interest, we can describe it as a set of independent harmonic oscillators with a coupling which is bilinear in the reservoir and system variables and endowed with a particular spectral function. This model has been successfully used to describe general properties, classical or quantum mechanical, of dissipative systems with only one degree of freedom subject to arbitrary potentials. Indeed, it has been extensively shown in the literature that, within the range of interest, other approaches to dealing with dissipative systems described by a single dynamical variable always furnish us with the same results as those obtained by the bath of oscillators with a properly chosen spectral function. It is the case, for example, of the application of the collective coordinate method to describing the damped motion of quantum solitons or microscopic attempts to describe more realistic systems such as the electron gas of a metallic environment.

However, despite all its success there are certain dissipative systems for which the usual model of the bath of oscillators can be shown to be inappropriate to account for the physics we expect from them. Here it should be stressed that by the usual model we mean independent oscillators coupled bilinearly in coordinates to the system of interest.

Suppose one immerses two independent particles in the same medium where each of them would separately behave as a Brownian particle. Since for each individual particle we could mimic the effect of the medium by the bath of oscillators it would be very natural to try to generalize the model to cope with the presence of those two particles. This generalization is quite straightforward and the only point where one should be a bit cautious is when introducing the well-known counter-term in the generalized model. In order to do that in an unambiguous way all one has to do is employ the equivalent model for the system-bath Lagrangian where the interaction is described by a velocity-coordinate coupling and perform the usual canonical transformations to achieve the desired coordinate-coordinate coupling with the appropriate counter-term (see below). It is then a simple matter to obtain the equations of motion for each particle under the influence of the environment using, for example, the Laplace transform method. These are coupled Langevin equations that when written in terms of the center of mass and relative coordinates, \( q \) and \( u \), of the two particles present a somewhat bizarre result; namely, the latter obeys a free particle equation of motion. Therefore, if we give an initial velocity to one of those particles the other must instantaneously reply in such a way that the relative coordinate evolves linearly in time ignoring completely the presence of the environment. This is a very unexpected result to say the least.

It is our intention in this letter to propose an extension of the usual model of a bath of oscillators in order to fix this deficiency. As we will see, on top of succeeding in so doing, we will also be able to describe an effective coupling between the two particles mediated by the presence of the bath and the resulting interaction potential depends on the specific form of the spectral function of the environment oscillators. This effect reminds us of the formation of Cooper pairs or bipolarons in material systems due to the electron-phonon interaction.

For the sake of completeness we shall start our procedure by briefly reviewing some general aspects concerning the usual model for the bath of oscillators that will be useful later on.

The Lagrangian for the complete system is given by

\[
L_S + L_R + L_I. \tag{1}
\]

\( L_S \) is the Lagrangian of the system of interest, \( L_R \) is the Lagrangian of the reservoir, which in the case of the bath
of oscillators is
\[ L_R = \sum_{k=1}^{N} \left( \frac{m_k R_k^2}{2} - \frac{m_k \omega_k^2 R_k^2}{2} \right), \tag{2} \]
and \( L_I \) is the interaction Lagrangian which can be written in two equivalent forms \[1\] 2. For example, in an electromagnetic-like fashion we will have a coordinate-velocity coupling \( L_I = -\sum_k \tilde{C}_k R_k x \). Switching to the Hamiltonian formulation and performing the following canonical transformation for the reservoir dynamical variables, \( P_k \rightarrow m_k \omega_k R_k \) and \( R_k \rightarrow \frac{P_k}{m_k \omega_k} \), we can show that the coupling term transforms into
\[ L_I = -\sum_{k=1}^{N} \left( R_k C_k x + \frac{C_k^2}{2m_k \omega_k^2} x^2 \right), \tag{3} \]
where the coefficients have been redefined as \( C_k = \tilde{C}_k \omega_k \). Therefore, the canonical transformation changes the coordinate-velocity coupling into a coordinate-coordinate coupling plus a new quadratic term which is necessary to preserve the translational invariance of \[1\] when the system of interest is not acted by any external force. We refer the reader to \[10\] for a discussion on the invariant form of the latter.

In our generalization of the previous model, the system of interest with a single degree of freedom will be represented by the free particle Lagrangian
\[ L_S = \frac{1}{2} M \dot{x}^2. \tag{4} \]

The heat bath will be described as a symmetrized collection of independent harmonic modes,
\[ L_R = \frac{1}{2} M \sum_{k} \left( \dot{R}_k \dot{R}_{-k} - \omega_k^2 R_k R_{-k} \right). \tag{5} \]

For the coupling term we initially assume the interaction Lagrangian
\[ L_I = -\sum_{k} \tilde{C}_k(x) \dot{R}_k. \tag{6} \]

We have just shown above that this kind of coupling is equivalent to the coordinate-coordinate coupling once an adequate potential renormalization is introduced. Then, following the same procedure that led us to \[13\] it is easy to see the equivalence between \[13\] and
\[ L_I = -\frac{1}{2} \sum_{k} (C_{-k}(x) R_k + C_k(x) R_{-k}) \]
\[ -\sum_{k} \frac{C_k(x) C_{-k}(x)}{2m_k \omega_k^2}. \tag{7} \]

where \( C_k(x) = \tilde{C}_k(x) \omega_k \). In order to represent the effect of a local interaction of the particle with a spatially homogeneous environment we choose
\[ C_k(x) = \kappa_k e^{i k x}. \tag{8} \]

With this choice it is easy to show that the entire system is translationally invariant and, moreover, the coupling to the environment is homogeneous. If the particle is displaced by a distance \( d \), the coupling term transforms into \( C_{-k}(x + d) \dot{R}_k = C_{-k}(x) e^{-i k d} \dot{R}_k \), which suggests the definition of a new set of canonical variables \( \tilde{R}_k = e^{-i k d} \dot{R}_k \) that renders the total Lagrangian invariant. It is important to notice that with a coupling like \[8\], the potential renormalization in \[8\] is a constant and therefore does not contribute to the particle dynamics. A coupling like \[8\] appears, for example, in the interaction of a particle with the density operator of a fermionic bath \[8\].

Now, we write the Euler-Lagrange equations for all the variables involved in the problem, take their Laplace transforms and eliminate the bath variables, in order to find the following equation of motion for the system of interest
\[ M \ddot{x} + \int_0^t K(x(t) - x(t'), t - t') \dot{x}(t') dt' = F(t). \tag{9} \]

Here the nonlinear dissipation kernel, given by
\[ K(r, \tau) = \sum_{k} \frac{k^2 \kappa_k \kappa_{-k}}{m_k \omega_k^2} \cos k r \cos \omega_k \tau, \tag{10} \]
shows that the interaction with the thermal bath induces a systematic influence on the system which is non-local and non-instantaneous. The function \( F(t) \) can be interpreted as a fluctuating force
\[ F(t) = -\frac{\partial}{\partial x} \sum_{k} \left\{ \left( C_{-k}(x) \dot{R}_{k}(0) + C_k(x) \dot{R}_{-k}(0) \right) \frac{\cos \omega_k t}{2} + \left( C_{-k}(x) \dot{R}_{k}(0) + C_k(x) \dot{R}_{-k}(0) \right) \frac{\sin \omega_k t}{2\omega_k} \right\}, \tag{11} \]
where \( \dot{R}_k = R_k + \frac{C_k(x)}{m_k \omega_k} \). \( F(t) \) depends explicitly on the initial conditions of the bath variables and its statistical properties are obtained from the initial state of the total system. Now we need to choose a suitable distribution of oscillators, in the continuum limit, which leads us to the Brownian motion. The kernel in \[10\] can be written as
\[ K = \sum_{k} \int_{0}^{\infty} d\omega 2k^2 \kappa_k \kappa_{-k} \frac{\text{Im} \chi_k^{(0)}(\omega)}{\pi \omega} \cos k(x(t) - x(t')) \cos \omega(t - t'), \tag{12} \]
where \( \text{Im} \chi_k^{(0)}(\omega) = \frac{1}{2 \pi m_k \omega_k} \delta(\omega - \omega_k) \) is the imaginary part of the dynamical response \( \chi_k^{(0)}(\omega) \) of a non-interacting
oscillator with wave number $k$. If we now assume that the bath oscillators are in fact only approximately non-interacting and replace only their response functions by those of damped oscillators, $\chi_k(\omega)$, one has
\[
\text{Im} \chi_k(\omega) = \frac{\gamma_k \omega}{m_k [\omega^2 - \omega_k^2]^2 + \omega^2 \gamma_k^2},
\]
where $\gamma_k$ is the relaxation frequency of that oscillator.

Our main interest is to study the system for times longer than the typical time scale of the reservoir. In other words, we wish to study the low-frequency limit of eq. (13) in which $\text{Im} \chi_k(\omega) \propto \omega$ and, therefore, we can assume
\[
\text{Im} \chi_k(\omega) \approx f(k) \omega \theta(\Omega - \omega),
\]
where $f(k) \approx \frac{\gamma_k}{m_k \omega_k}$. Here we have, as usual, introduced a high frequency cutoff $\Omega$ as the characteristic frequency of the bath. A functional dependence like (14) for the dynamical response of the bath has been employed in the references [3, 4] for fermionic environments. The particular choice of the dynamical susceptibility of the bath allows us to separate its time and length scales and to obtain a Markov dynamics when we replace (14) in (12) and integrate with respect to $\omega$ taking the limit $\Omega \to \infty$. With these considerations the equation of motion (9) reads
\[
M \ddot{x}(t) + \eta \dot{x}(t) = F(t),
\]
where we have defined $\eta = \sum_k k^2 \kappa_k \kappa_{-k} f(k)$. Notice that with this modification we obtain a relation between the damping constant and some microscopic parameters of the oscillating bath. With the current prescription and supposing that the bath is initially in thermal equilibrium it is easy to show that, for high temperatures, the fluctuating force $F(t)$ satisfies the relations $\langle F(t) \rangle = 0$ and $\langle F(t) F(t') \rangle = 2 \eta k_B T \delta(t - t')$, which are characteristics of white noise. Notice that this is valid only if we assume a classical distribution of oscillators as the initial state of the bath.

In conclusion, the system-reservoir model with non-linear coupling presented here allows us to reproduce the result one would have obtained by coupling the particle of interest bilinearly to a bath of non-interacting harmonic oscillators with the spectral function $J(\omega) = \eta \omega$ [1, 2].

Now we are going to study the dynamics of a system with two degrees of freedom immersed in a dissipative environment. In this case the Lagrangian of the system of interest is
\[
L_S = \frac{1}{2} M \dot{x}_1^2 + \frac{1}{2} M \dot{x}_2^2 + \frac{1}{2} \sum_k k^2 R_k,
\]
and the coupling term
\[
L_I = -\frac{1}{2} \sum_k [(C_{-k}(x_1) + C_{-k}(x_2)) R_k + (C_k(x_1) + C_k(x_2)) R_{-k}].
\]
Notice that we have not included any counter-term in (17) since our system is translationally invariant. The equations of motion for this Lagrangian are then
\[
\dot{M} \ddot{x}_i + \int_0^t K(x_i(t) - x_i(t'), t - t') \dot{x}_i(t') dt' + \int_0^t K(x_i(t) - x_j(t'), t - t') \dot{x}_j(t') dt' + \frac{\partial}{\partial x_i} V(x_i(t) - x_j(t)) = F_i(t),
\]
where $i \neq j = 1, 2$, the fluctuating force $F_i(t)$ has the form given by (11) replacing $x(t)$ by $x_i(t)$ and
\[
V(r(t)) = -\sum_k \frac{\kappa_k \kappa_{-k}}{m_k \omega_k^2} \cos kr(t).
\]

In the long-time limit, the second term in (15) can be written as $\eta \dot{x}_i$ providing us with the usual dissipative term. The third term represents a cross-dissipative term that depends on the velocity of the second particle and the relative distance between them. The last term is clearly an effective interaction induced by the coupling with the thermal reservoir. To see the explicit form of this interaction and the non-local influence in the cross-dissipative term, we need to evaluate the sum in (14) with $\text{Im} \chi_k^{(0)}(\omega)$ replaced by (14). For this we had better define the function $g(k)$ as
\[
\eta g(k) = \frac{L}{2 \pi} \kappa_k \kappa_{-k} f(k),
\]
where we have assumed a one-dimensional environment of length $L$.

The exact form of this function can be obtained from a microscopic description of the bath and the details of the interaction between the system of interest and the bath degrees of freedom. As we are not interested in such a detailed description of the many-body reservoir, we are going to choose a form for $g(k)$ that satisfies the condition $\int_0^\infty g(k) k^2 dk = 1$ derived from the definition of the damping constant $\eta$. Notice that we have replaced $\sum_k \to \frac{L}{2 \pi} \int dk$. A reasonable choice for this function is
\[
g(k) = A e^{-k/k_0},
\]
where $A = 1/(2 k_0^3)$ is the normalization constant and $k_0$ determines the characteristic length scale of the environment. For example, in the case of fermionic environments $k_0$ is of the order of $k_F$ [3]. One should bear in mind that the choice of $g(k)$ is guided either by the knowledge of the microscopic details of the environment or by some phenomenological input about the effective interaction between the two particles.

With the form chosen for $g(k)$ the kernel of the third term in (17) is, again only for high temperatures,
\[
K(x_1(t) - x_2(t'), t - t') = 2 \delta(t - t') \eta(u(t)),
\]
where \( u(t) = x_1(t) - x_2(t) \) and
\[
\eta(u(t)) = \eta \left( \frac{1}{(k_0^2 u^2 + 1)^2} - \frac{4u^2 k_0^2}{(k_0^2 u^2 + 1)^3} \right),
\]
(23)
and the effective potential reads
\[
V(u(t)) = -\frac{2\Omega \eta}{\pi k_0^2 (k_0^2 u^2(t) + 1)}.
\]
(24)

The strength of the effective potential depends on the characteristic length and time scales. Therefore the contribution of this coupling to the dynamics of the Brownian particles is only important for times longer than the typical time scale of the reservoir and distances of the order of (or less than) the characteristic length \( k_0^{-1} \). The fluctuating forces still satisfy the white noise properties but now they present an additional property associated with the distance between the particles. The forces \( F_1(t) \) and \( F_2(t) \) are also spatially correlated, that is
\[
\langle F_1(t)F_2(t') \rangle = 2\eta(u(t))k_B T \delta(t-t').
\]
(25)

In figure 1 we see the noise correlation strength as a function of the distance between the particles. For short distances the noise correlation has the standard Brownian behavior. However, for longer distances the correlation function becomes negative and this anti-correlation induces an anomalous diffusive process in the system which will ultimately tend to normal diffusion once the particles are infinitely far apart.

In terms of the relative and center-of-mass coordinates the equations of motions read
\[
M\ddot{u}(t) + \eta \dot{u}(t) - \eta(u(t))\dot{u} + V'(u(t)) = F_u(t)
\]
(26)
and
\[
M\ddot{q}(t) + \eta \dot{q}(t) + \eta(u(t))\dot{q} = F_q(t),
\]
(27)
where \( F_u(t) = F_1(t) - F_2(t) \) and \( F_q(t) = (F_1(t) + F_2(t))/2 \).

From the form of \( V(u(t)) \) and \( \eta(u(t)) \) and the statistical properties of the fluctuating forces, \( \langle F_u(t) \rangle = 0, \langle F_u(t)F_u(t') \rangle = 4kT(\eta - \eta(u))\delta(t-t') \) and \( \langle F_q(t)F_q(t') \rangle = kT(\eta + \eta(u))\delta(t-t') \), it is evident that at large distances the equations of motion for the relative and center of mass coordinates represent the motion of Brownian particles with reduced mass \( M/2 \) and total mass \( 2M \), respectively. In general the dissipation depends on the relative distance between the particles and for distances such that \( uk_0 \ll 1 \), we have up to first order in \( u(t) \), \( V'(u) \propto -u(t) \). In this approximation both dissipation and fluctuations are negligible and then we have an undamped oscillatory motion for \( u(t) \).

In conclusion, we presented a system-plus-reservoir model with a coupling which is non-linear in the system coordinates and, in the adequate limit, allows us to reproduce the phenomenological results known for the dynamics of a dissipative system with only one degree of freedom. In our model the dissipation coefficient is directly expressed in terms of a few parameters of the thermal bath that can be measured experimentally. Moreover, our model is capable of inducing an effective coupling between the two Brownian particles which arises from the non-local effects generated by the bath and depends on the choice of the dynamical susceptibility of the reservoir. In the single particle case, the non-local effects will be relevant only in a non-markovian approximation.

We should finally stress that we have only reported our findings for the classical or high temperature regime. The quantum mechanical analysis of this system requires the evaluation of the reduced density operator for the two particles of interest down to very low temperatures and will be presented elsewhere [1].

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