ASYMPTOTIC ANALYSIS OF
A NONSIMPLE THERMOELASTIC ROD

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Abstract. The asymptotic analysis of a one-dimensional nonsimple thermoelectric problem is considered in this paper. By a detailed spectral analysis, the asymptotic expressions for eigenvalues and eigenfunctions of the considered system are developed. It is shown that the eigenfunctions form a Riesz basis on the Hilbert space and the eigenvalues asymptotically fall on two branches. One branch is along the negative horizontal axis in the complex plane and the other branch is asymptotic to a vertical line that is parallel to the imaginary axis. This gives the spectrum-determined growth condition for the $C_0$ -semigroup associated to the system, and consequently, the asymptotic and the exponential stability of the solutions are deduced. The approach developed in this paper confirms the already-existing results; furthermore, it can be extended to a larger field of applications such as coupled system of rod or beam with diffusion equation. The method will be illustrated by an example of thermoelastic beam equations with Dirichlet boundary conditions.

1. Introduction. In general, the one-dimensional linear thermoelasticity problem for a homogeneous rod of uniform cross section with Dirichlet-Dirichlet boundary condition can be formulated as (see [12, 18])

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} - c^2 \frac{\partial \theta}{\partial x} & 0 < x < 1, \ t > 0, \\
\frac{\partial \theta}{\partial t} &= \frac{\partial^2 \theta}{\partial x^2} - \gamma \frac{\partial^2 u}{\partial x \partial t} & 0 < x < 1, \ t > 0, \\
u(x, t) = \theta(x, t) &= 0 & x = 0, 1, \ t \geq 0,
\end{align*}
\]

where $u = u(x, t)$ represents the displacement of the solid elastic material and $\theta = \theta(x, t)$ represents the absolute temperature. The coupling constant $\gamma$ is a measure of the mechanical-thermal coupling present in the system. The constant $c$ can be viewed as the small amplitude wave speed about a uniform temperature.

Many important qualitative properties of system (1) have been published in recent years. By using a spectral approach, it is shown in [12] that the semigroup associated to system (1) is uniformly exponentially stable. Later the same result has been obtained by Liu and Zheng [18] by frequency domain multiplier method. By different spectral approaches (see for example [12] and [10]) it was shown that

2010 Mathematics Subject Classification. 65E99, 45M10.
Key words and phrases. Nonsimple thermoelasticity, asymptotic analysis, exponential stability.
there are two branches of eigenvalues for the system (1), which have the following asymptotic expansions

$$\lambda_k = -(k\pi)^2 + (\gamma c)^2 + O(k^{-2}),$$

$$\sigma_k = -\frac{(\gamma c)^2}{2} + ic\left(k\pi + \frac{(\gamma c)^2}{2k\pi}(1 - \frac{\gamma^2}{2})\right) + O(k^{-1}),$$

(2)

where $k$ is a large positive integer. It is seen from (2) that the first branch of eigenvalues is produced by the heat equation while the second one is associated with the elastic vibrations. In [19], numerical simulations show that the eigenfrequency of system (1) approaches asymptotically to a vertical line that is parallel to the imaginary axis and contained in the open left half of the complex plane. Guo and Yung [10] proved this numerical conjecture theoretically.

A significant result of Guo and Chen [9] showing that there is a real eigenvalue for the system (1) that is greater than the dominant eigenvalue of “pure” heat equation. A more generalized result was proved in [8] that there is a set of generalized eigenfunctions of the system (1), which forms a Riesz basis for the state space. By Riesz basis property, the dynamic behavior of the system (1) can be expressed in terms of its eigenfrequencies. Moreover, the Riesz basis property concludes the spectrum-determined growth condition which implies automatically the exponential stability. This is one of the hard and important problems in the stability analysis of infinite-dimensional systems. From the well-known fact that $\omega(A) = \max\{s(A), \omega_{ess}(A)\}$ and (2), where $\omega(A)$, $s(A)$ and $\omega_{ess}(A)$ denote the growth order of the semigroup $e^{At}$, the spectral bound and essential bound of $A$, respectively, we see that the spectrum-determined-growth condition

$$\omega(A) = s(A)$$

is always true for system (1). In [14], it is proved that the asymptote of the complex eigenvalues given in (2) is also the spectrum-determined-growth condition of $A$ (the linear operator of the system), i.e.,

$$\omega(A) = s(A) = -\frac{(\gamma c)^2}{2}.$$

(3)

On the other hand, it was shown in [12] that for the system (1) with Dirichlet-Neumann or Neumann-Dirichlet boundary conditions, there is a sequence of generalized eigenfunctions which forms a Riesz basis for the Hilbert state space. The success in obtaining this result lies in the simplicity of the corresponding characteristic equation as well as the explicit structure of the eigenfunctions. However, for the Dirichlet-Dirichlet boundary condition, the characteristic equations become a complicated transcend equation, and the eigenfunctions satisfy a fourth-order ordinary differential equation. For this end, Guo in [10] and [8] proposed another way to analyze the asymptotic behavior.

This paper concerns the appropriate spectral eigenvalues estimates, showing that the eigenfunctions forms a Riesz basis, proving that the spectrum-determined growth condition holds and deducing the exponential stability for a nonsimple thermoelastic rod (see system (8) below). Our main tool in proving this, is a result due to Hansen [12] (see Proposition 3.1 below). More precisely, we treat the case where classic thermoelasticity involves higher order gradients of displacement. This makes the spectral analysis of the corresponding system more difficult and the proofs of basic theorems, as presented in [12], need further elaboration. Hence, the elaborated method in this paper allows the study of spectral properties of linear operators containing higher order gradients of displacement. The high order derivatives clarify
the possible configurations of the materials more and more finely by the values of the successive higher gradients \[3, 7, 15\]. Moreover, in recent years, the spectral analysis becomes an important tool in the study of control problems (see for example \[17\]) and \(L^p - L^q\) decay estimations (see for example \[24\]) for classic thermoelastic materials. In other words, our study can be useful in the study of control and \(L^p - L^q\) decay estimations problems of nonsimple materials which is a topic that has not yet been addressed in the literature despite its various applications in applied sciences. This work continues the task developed in several recent papers where it has been tried to clarify the time decay of solutions for nonsimple thermoelastic problems \[1, 2, 4, 5, 16, 21, 23\].

This paper is organized as follows. In Section 2, we state the basic equations and semigroup setting of the nonsimple thermoelastic problem. In Section 3, it is shown that the corresponding eigenfunctions forms a Riesz basis for the Hilbert state space. In Section 4, we shall prove the existence of eigenvalues and show that there are two branches of eigenvalues for our system. One branch is along the negative horizontal axis in the complex plane and the other branch is asymptotic to a vertical line that is parallel to the imaginary axis. In section 5, we deduce the spectrum-determined growth condition, the asymptotic and the exponential stability. In the last section we conclude by confirming the already-existing results and by illustrating the method on a example of thermoelastic beam equations with Dirichlet boundary conditions.

2. Basic equations and semigroup setting. We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers \(1, 2, 3\), summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding cartesian coordinate. When supply terms are not present, the linear evolution equations of the general three-dimensional theory of nonsimple thermoelastic solids are given by \[3, 7, 15\]

\[
\rho \frac{\partial^2 u_i}{\partial t^2} = (\sigma_{ji} - T_{kji,k}, j),
\]

\[
\rho T_0 \frac{\partial S}{\partial t} = -q_i, i,
\]

and the constitutive equations for isotropic and centrosymmetric solids are

\[
\sigma_{ij} = 2\mu e_{ij} + \delta_{ij} (\lambda e_{kk} + \beta T),
\]

\[
\rho T_0 S = \beta T_0 e_{kk} + \rho c_E T,
\]

\[
T_{ijk} = \frac{\omega_1}{2} (u_{i,rr} \delta_{jk} + 2u_{r, rj} \delta_{ij} + u_{j, rr} \delta_{ij}) + \omega_2 (u_{r, ri} \delta_{jk} + u_{r, rj} \delta_{ik})
+ 2 \omega_3 u_{k, rr} \delta_{ij} + 2 \omega_4 u_{k, ji} + \omega_5 (u_{i, jk} + u_{j, ik}),
\]

\[
e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}),
\]

\[
q_i = -k T_{i},
\]

where \(\beta = (3\lambda + 2\mu) \varsigma\), \(\varsigma\) is the coefficient of linear thermal expansion, \(\lambda\) and \(\mu\) are Lamé’s constants. \(T\) is the absolute temperature of the medium, \(T_0\) is the reference uniform temperature of the body chosen such that \(|(T - T_0)/T_0| \ll 1\). \(q_i\) is the heat conduction vector, \(k\) is the coefficient of thermal conductivity, \(c_E\) is the specific heat at constant strain. \(\sigma_{ij}, T_{ijk}\) are the stress and the hyper-stress tensors, \(u_i\) is
the displacement vector, $e_{ij}$ are the components of the strain tensor, $\rho$ is the mass density and $\varpi_i$ ($i = 1, \cdots, 5$) are constitutive coefficients.

In the one-dimensional case, the evolution equations (4) become

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x} - \frac{\partial^2 T}{\partial x^2},$$  
$$\rho T_0 \frac{\partial S}{\partial t} = -\frac{\partial q}{\partial x},$$

and the constitutive equations (5) take the form

$$\sigma = (\lambda + 2\mu) \frac{\partial u}{\partial x} - \beta T,$$
$$T = \varpi \frac{\partial^2 u}{\partial x^2},$$
$$\rho T_0 S = \beta T_0 \frac{\partial u}{\partial x} + \rho c E T,$$
$$q = -k \frac{\partial T}{\partial x}.$$

If we substitute the above constitutive equations into the evolution equations (6), we obtain the system of field equations in $(0, \ell) \times \mathbb{R}^+$

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} - \varpi \frac{\partial^4 u}{\partial x^4} - \beta \frac{\partial T}{\partial x},$$  
$$\rho c E \frac{\partial T}{\partial t} = -\beta T_0 \frac{\partial^2 u}{\partial x \partial t} + k \frac{\partial^2 T}{\partial x^2}.$$

By the change of variables

$$x' = \frac{x}{\ell}, \ t' = \frac{kt}{\rho c E \ell^2}, \ \theta = \frac{T - T_0}{T_0}, \ u' = \frac{u}{\ell} \sqrt{\frac{\lambda + 2\mu}{\rho c E T_0}},$$

the equations (7) become in $(0, 1) \times \mathbb{R}^+$ (dropping the asterisks for convenience)

$$\frac{\partial^2 u}{\partial t^2} = \frac{c^2}{\ell^2} \frac{\partial^2 u}{\partial x^2} - \alpha \frac{\partial^4 u}{\partial x^4} - \gamma \frac{\partial \theta}{\partial x},$$  
$$\frac{\partial \theta}{\partial t} = \frac{c^2}{\ell^2} \frac{\partial^2 \theta}{\partial x^2} - \gamma \frac{\partial^2 u}{\partial x \partial t},$$

where

$$\gamma = \frac{\beta}{\rho c E \sqrt{\frac{\lambda + 2\mu}{\rho c E T_0}}}, \ \ c^2 = \frac{(\lambda + 2\mu)\rho c_E^2 \ell^2}{k^2}, \ \ \alpha = \frac{\varpi \rho c_E^2}{k^2}.$$  

The coupling constants $\gamma$ and $c$ are generally small in comparison to 1. The constant $\gamma$ is a measure of the mechanical-thermal coupling present in the system, while constant $c$ can be viewed as the small amplitude wave speed about a uniform temperature. The positive constant $\alpha$ is the coefficient of the fourth order spatial derivative characterizing the nonsimple elastic solid. If $\alpha = 0$, the system (8) reduces to the classic thermoelastic system (1).

In terms of the new variables

$$y = (u_x, u_t, \theta) = (y_1, y_2, y_3),$$
The initial-boundary value problem (9)-(11) is equivalent to the problem

where $D_i = \frac{d^i}{dx^i}$ and $\gamma_i = \frac{2n_i}{\sigma_i}$, $i = 1, 2, 3$. We are going to study the system (9) with the following initial conditions

and boundary conditions

We denote by $\mathcal{H}$ the complex Hilbert space

equipped with the inner product

for $z = (z_1, z_2, z_3)$. In view of (13), the corresponding norm in $\mathcal{H}$ is given by

The initial-boundary value problem (9)-(11) is equivalent to the problem

where $\mathcal{D}(A) \subset \mathcal{H}$ is the linear operator defined by

The adjoint operator of $A$ is easily calculated and given by

for $z \in \mathcal{D}(A^*) = \mathcal{D}(A)$.

Following [1, 2, 4], one can easily prove the following result.

Lemma 2.1. Let $A$ and $\mathcal{H}$ be defined as before. Then

1) $\mathcal{D}(A)$ is dense in $\mathcal{H}$.
2) $A$ and $A^*$ are dissipative in $\mathcal{H}$.
3) $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of $A$.
4) The operator $A^{-1} : \mathcal{H} \to \mathcal{H}$ is compact on $\mathcal{H}$. 
As a direct result, the spectrum of $A$ consists only of the isolated eigenvalues with finite multiplicity, i.e., $\sigma(A) = \sigma_p(A)$, and satisfies $\Re(\lambda) \leq 0$, $\forall \lambda \in \sigma(A)$.

Then by Lemma 2.1, the Lumer-Phillips theorem (see [22]) asserts the following result.

**Proposition 2.1.** Let $A$ and $H$ be defined as before. Then $A$ generates a $C_0$-semigroup of contractions $S(t)$ on $H$. Hence, the system (15) is well-posed, i.e., for any $y^0 = (y^0_1, y^0_2, y^0_3) \in H$, system (15) has a unique weak solution $y(t) = S(t)y^0 \in C(\mathbb{R}^+, H)$. Furthermore, if $y^0 \in \mathcal{D}(A)$, $y(t) = S(t)y^0 \in C^1(\mathbb{R}^+, H) \cap C(\mathbb{R}^+, \mathcal{D}(A))$ is the classical solution to (15).

3. **Riesz basis property.** In this section we will show that the eigenfunctions of $A$ form a Riesz basis. Let us first recall that a set of vectors $\{f_k\}$ are said to be a Riesz basis for the Hilbert space $H$ if there exists a bounded and invertible operator $L$ from $H$ to $H$ such that $f_k = Le_k$, where $\{e_k\}$ is an orthogonal basis for $H$. We refer the reader to [25] for details. Our main tool in proving this is the following result [12].

**Proposition 3.1.** Let $\{E_j^k\}$, $j, k \in \mathbb{N}^*$, be a Riesz basis for Hilbert space $H$. Further assume there are $n$ by $n$ matrices $M(k)$ for which

$$
\begin{pmatrix}
\Phi_1^k \\
\Phi_2^k \\
\vdots \\
\Phi_n^k
\end{pmatrix}
= M(k)
\begin{pmatrix}
E_1^k \\
E_2^k \\
\vdots \\
E_n^k
\end{pmatrix}
$$

with

(i) $\inf_k |\det M(k)| > 0$

(ii) $\sup_{k,j,l} |M_{ij}(k)| < \infty$.

Then $\{\Phi_j^k\}$ also is a Riesz basis for $H$.

We can now state the main result of this section.

**Proposition 3.2.** Let $A$ (resp. $A^*$) be the operator defined as in (16) (resp. 17). Then the eigenfunctions of $A$ (resp. $A^*$) form a Riesz basis for $H$.

**Proof.** For $k \in \mathbb{N}^*$ we define

$$
\mu_k = k\pi, \quad s_k = \sqrt{c^2 + \alpha \mu_k^2}, \quad \delta = (\gamma c)^2
$$

and

$$
E_1^k = \begin{pmatrix}
\cos(\mu_k x) & 0 \\
0 & 0 \\
0 & \cos(\mu_k x)
\end{pmatrix}, \quad E_2^k = \begin{pmatrix}
0 & -\sin(\mu_k x) \\
0 & 0
\end{pmatrix}, \quad E_3^k = \begin{pmatrix}
0 & 0 \\
0 & \cos(\mu_k x)
\end{pmatrix}.
$$

By duality it will be enough to show that the eigenfunctions of $A^*$ form a Riesz basis. One can easily verify the following

$$
A^* E_1^k = -\mu_k s_k^2 E_2^k, \quad A^* E_2^k = \mu_k E_1^k - \gamma \mu_k E_3^k, \quad A^* E_3^k = c^2 \gamma \mu_k E_2^k - \mu_k^2 E_3^k.
$$

With respect to the orthogonal basis $\{E_j^k\}_{1 \leq j \leq 3}$, $A^*$ has been decoupled into a chain of $3 \times 3$ blocks along the diagonal. If we set $\Sigma_k = (E_1^k, E_2^k, E_3^k)$ and $\xi = (\xi_1, \xi_2, \xi_3)^T$ we can rewrite (19) as

$$
A^* \Sigma_k \xi = \mu_k \Sigma_k R_k \xi,
$$

(20)
where

\[ R_k = \begin{pmatrix} 0 & 1 & 0 \\ -s_k^2 & 0 & c^2 \gamma \\ 0 & -\gamma & -\mu_k \end{pmatrix}. \]  

(21)

Thus if \((\xi, \lambda)\) is an eigenpair of \(R_k\), then \((\Sigma_k \xi, \mu_k \lambda)\) is an eigenpair of \(A^*\).

The characteristic equation of \(R_k\) is

\[ (\lambda + \mu_k)(\lambda^2 + s_k^2) + \lambda \delta = 0. \]  

(22)

The coupling constants \(\gamma\) and \(c\) are generally taken small in comparison to 1, so \(\delta = (\gamma c)^2 = O(1)\). In particular we have \(\delta = O(k^{-2})\) as \(k \to \infty\) which implies that \((\lambda + \mu_k)(\lambda^2 + s_k^2) \to 0\). Asymptotically, the roots of (22) consist of a real root \(\lambda_k\) and a nonreal complex conjugate pair \((\sigma_k, \sigma_k^*\) : \(\lambda_k = -\mu_k, \quad \sigma_k \to \pm is_k \quad \text{as} \quad k \to \infty. \)  

The asymptotic expansions of \(\lambda_k\) and \(\sigma_k\) will be given in the following sections for all \(k \in \mathbb{N}^*\).

Since the eigenfunctions of \(R_k\) are simple then \(R_k\) is diagonalisable, that is

\[ R_k = M_k \Lambda M_k^{-1} \quad \text{where} \quad \Lambda = \text{diag}(\sigma_k, \sigma_k, \lambda_k) \]

and

\[ M_k = \begin{pmatrix} 1 & 1 & \frac{\gamma c^2}{s_k^2 + \lambda_k} \\ \sigma_k & \sigma_k & \lambda_k \gamma c^2 + s_k^2 \\ s_k^2 + \sigma_k^2 & s_k^2 + \gamma c^2 \\ \frac{s_k^2 + \gamma c^2}{\gamma c^2} & 1 \end{pmatrix}. \]  

(24)

The columns of \(M_k\) are the eigenvectors of \(R_k\), corresponding to simple eigenvalues \((\sigma_k, \sigma_k, \lambda_k)\). Hence \(M_k\), is nonsingular for all \(k\). In fact, from (23) we have that \(\frac{s_k^2 + \sigma_k^2}{\gamma c^2} \to 0\) and \(\frac{s_k^2 + \gamma c^2}{\gamma c^2} \to 0\) as \(k \to \infty\), then

\[
\det M_k \to \begin{vmatrix} 1 & 1 & \frac{\gamma c^2}{s_k^2 + \mu_k} \\ is_k & -is_k & -\frac{\mu_k \gamma c^2}{s_k^2 + \mu_k} \\ 0 & 0 & 1 \end{vmatrix} \quad \text{as} \quad k \to \infty
\]

\[
= -2is_k \neq 0,
\]

and

\[
\sup_{k,j,l} |(M_k)_{j,l}| < \infty.
\]

If we denote by \(\Phi_{\sigma_k}, \Phi_{\sigma_k^*}\) and \(\Phi_{\lambda_k}\) \((k \in \mathbb{N}^*)\) the eigenvectors of \(R_k\) corresponding to \(\sigma_k, \sigma_k^*\) and \(\lambda_k\), respectively, then \(\Sigma_k \Phi_{\sigma_k}, \Sigma_k \Phi_{\sigma_k^*}\) and \(\Sigma_k \Phi_{\lambda_k}\) are the corresponding eigenvectors of \(A^*\). Thus we have

\[ R_k \Phi_{\sigma_k} = \sigma_k \Phi_{\sigma_k}, \quad R_k \Phi_{\sigma_k^*} = \sigma_k^* \Phi_{\sigma_k^*}, \quad R_k \Phi_{\lambda_k} = \lambda_k \Phi_{\lambda_k}. \]

Furthermore, from (20) we get

\[ A^* \Sigma_k \Phi_{\sigma_k} = \mu_k \sigma_k \Sigma_k \Phi_{\sigma_k}, \quad A^* \Sigma_k \Phi_{\sigma_k^*} = \mu_k \sigma_k \Sigma_k \Phi_{\sigma_k^*}, \quad A^* \Sigma_k \Phi_{\lambda_k} = \mu_k \lambda_k \Sigma_k \Phi_{\lambda_k}. \]

Thus we have

\[ \begin{pmatrix} \Sigma_k \Phi_{\sigma_k} \\ \Sigma_k \Phi_{\sigma_k^*} \\ \Sigma_k \Phi_{\lambda_k} \end{pmatrix} = M_k^T \begin{pmatrix} E_k^1 \\ E_k^2 \\ E_k^3 \end{pmatrix}. \]  

(25)

Since \(\{E_k^j\}\) is a Riesz basis (being an orthogonal basis) for \(H\) (defined in (12)), we may apply Proposition 3.1, from which it follows that \(\Sigma_k \Phi_{\sigma_k} \cup \Sigma_k \Phi_{\sigma_k^*} \cup \Sigma_k \Phi_{\lambda_k}\) is
a Riesz basis for \( \mathcal{H} \). Thus the eigenfunctions of \( \mathcal{A}^* \) (resp. \( \mathcal{A} \)) form a Riesz basis for \( \mathcal{H} \).

**Remark 3.1.** Likewise, for \( k \in \mathbb{N}^* \) it can be shown that \( \Sigma_k \Phi_{-\sigma_k}, \Sigma_k \Phi_{-\pi_k}, \) and \( \Sigma_k \Phi_{-\lambda_k} \) (i.e., replace \( \sigma_k \) by \( -\sigma_k \) into \( \Phi_{-\sigma_k} \) and so forth) are eigenvectors of \( \mathcal{A} \) corresponding to the eigenvalues \( \mu_k \sigma_k, \mu_k \pi_k \) and \( \mu_k \lambda_k \), respectively. The union of these eigenvectors is a Riesz basis for \( \mathcal{H} \) which is dual to the Riesz basis defined in (25), i.e.,

\[
(\Sigma_k \Phi_{v_k}, \Sigma_j \Phi_{-n_j})_{\mathcal{H}} = \begin{cases} c_{n_j} & \text{if } v_k = n_j \\ 0 & \text{if } v_k \neq n_j \end{cases}
\]

where \( 0 < c_0 < |c_{n_k}| < C_0 \) (see Young [25]).

4. **The asymptotic expansion of eigenvalues.** In this section we obtain uniform and asymptotic estimates on the location of the spectrum of \( \mathcal{A} \). By virtue of (22) we need only to approximate the roots of the characteristic equation for \( R_k \)

\[
f_k(\lambda) := (\lambda + \mu_k)(\lambda^2 + s_k^2) + \lambda \delta = 0.
\]

In the following we give some properties of the function \( f_k \) on the interval \( I = (-\mu_k - r, -\mu_k + r) \) where \( r \) is a positive real given by

\[
r = \frac{\mu_1}{8(1 + \alpha)}.
\]

Is it clear that the definition of \( r \) always implies the condition

\[
r < \frac{2\mu_k}{3}, \quad \text{for all } k \in \mathbb{N}^*.
\]

In the following we construct a numerical sequence converging to the real solution of (26).

**Proposition 4.1.** Let \( f_k \) denotes the characteristic polynomial of (26). Then \( f_k \) is strictly increasing on \( I = (-\mu_k - r, -\mu_k + r) \) such that

\[
f_k'(\lambda) > \frac{\mu_k^2}{2} + s_k^2 + \delta.
\]

Furthermore there exists a positive constant \( M = 6r \) such that \( f_k' \) is \( M \)-Lipschitz on \( I \).

**Proof.** As \( f_k \in C^\infty(I) \) and \( f_k''(\lambda) = 6\lambda + 2\mu_k \) vanishes only at \( \lambda = -\frac{\mu_k}{3} \not\in I \) (by force of (28)), we have that \( f_k''(\lambda) < 0 \) on \( I \), then \( f_k' \) is strictly decreasing on \( I \) and consequently

\[
f_k'(-\mu_k + r) < f_k'(\lambda).
\]

On the other hand, for all \( k \in \mathbb{N}^* \), we have that

\[
\frac{\mu_k^2}{2} + \frac{3\mu_k^2}{64(1 + \alpha)^2} - \frac{\mu_1 \mu_k}{2(1 + \alpha)} > \frac{\mu_k^2}{2} - \frac{\mu_1 \mu_k}{2(1 + \alpha)} > \frac{\alpha \mu_k^2}{2(1 + \alpha)} > 0,
\]

and consequently

\[
f_k'(-\mu_k + r) = \mu_k^2 + \frac{3\mu_k^2}{64(1 + \alpha)^2} - \frac{\mu_1 \mu_k}{2(1 + \alpha)} + s_k^2 + \delta > \frac{\mu_k^2}{2} + s_k^2 + \delta.
\]

This inequality together with (30) gives (29), which implies that \( f_k \) is strictly increasing on \( I \).

For all \( \lambda, \varsigma \in I \), we have that

\[
|f_k'(\varsigma) - f_k'(\lambda)| = |3(\varsigma^2 - \lambda^2) + 2\mu_k(\varsigma - \lambda)|
\]
Proof. There exists a positive constant $K$ that implies the existence of a positive constant $M = 6r$ such that $f_k'$ is $M$-Lipschitz on $I$.

**Proposition 4.2.** Let $f_k$ denote the characteristic polynomial of (26). The equation (26) has a unique real root in the interval $I = (-\mu_k - r, -\mu_k + r)$.

Proof. Since $f_k \in C^\infty(I)$ and is strictly increasing on $I$ (by Proposition 4.1), then $f_k$ is one to one on $I$. To prove the proposition it remains just to check that $f_k(-\mu_k - r)f_k(-\mu_k + r) < 0$.

As $f_k(-\mu_k - r) = r[(r + \mu_k)^2 + s_k^2] - r(r + \mu_k)\delta < 0$, then it suffices to prove that $f_k(-\mu_k + r) = (r - \mu_k)[r(r - \mu_k) + \delta] + rs_k^2 > 0$.

From (28), yields $r - \mu_k < -\frac{\gamma}{2} < 0$ and then $r(r - \mu_k) + \delta < \delta - \frac{\gamma^2}{2}$ which implies

$$f_k(-\mu_k + r) > (\mu_k - r)(\frac{r^2}{2} - \delta) + rs_k^2 \text{ (use (28))}$$

$$> \frac{r}{2}(\frac{r^2}{2} - \delta) + rs_k^2 = \frac{r}{2}(\frac{r^2}{2} + c^2(2 - \gamma^2) + 2\alpha s_k^2) > 0$$

since $\gamma$ is taken small in comparison to 1.

**Remark 4.1.** For all $\lambda, \varsigma \in I$ and $t \in [0, 1]$, one can write

$$\varphi(t) = f_k(\lambda + t(\varsigma - \lambda)) - tf_k'(-\lambda)(\varsigma - \lambda).$$

Since $f_k'$ is $M$-Lipschitz on $I$, it is not hard to show that

$$|\varphi'(t)| \leq M|\varsigma - \lambda|^2. \quad (31)$$

Applying the finite-increment theorem to $\varphi$ on $[0, 1]$ and using (31), yields

$$|f_k(\varsigma) - f_k(\lambda) - f_k'(-\lambda)(\varsigma - \lambda)| \leq M|\varsigma - \lambda|^2. \quad (32)$$

**Lemma 4.1.** Let $f_k$ denote the characteristic polynomial given by (26), then the sequence $(\lambda_k^n)_{n \geq 0}$ defined as follows:

$$\lambda_k^0 = -\mu_k,$$

$$\lambda_k^{n+1} = \lambda_k^n - \frac{f_k(\lambda_k^n)}{f_k'(\lambda_k^n)}, \quad (33)$$

has the following properties for all $n \geq 1$,

$$|f_k(\lambda_k^n)| \leq \frac{2\delta^2}{\mu_k(1 + \alpha)}. \quad (34)$$

$$|f_k(\lambda_k^n)| \leq M|\lambda_k^n - \lambda_k^{n-1}|^2. \quad (35)$$

There exists a positive constant $K$ such that

$$|\lambda_k^{n+1} - \lambda_k^n| \leq K|\lambda_k^n - \lambda_k^{n-1}|^2. \quad (36)$$

Proof. To prove (34), we first obtain a first order approximation $\lambda_k^1$ to the real root $\lambda_k$ of $f_k(\lambda)$ by using one step of Newton’s method with the initial guess of $\lambda_k^0 = -\mu_k$,

$$\lambda_k^1 = -\mu_k + \frac{\delta \mu_k}{\mu_k^2 + s_k^2 + \delta}. \quad (37)$$
from which we get
\[(\lambda_k^1)^2 + s_k^2 = (\lambda_k^1 + \mu_k)^2 - 2\mu_k(\lambda_k^1 + \mu_k) - \frac{\delta \lambda_k^1}{\lambda_k^1 + \mu_k}.\]

By this last identity, the expression of \(f_k(\lambda_k^1)\) becomes
\[f_k(\lambda_k^1) = (\lambda_k^1 + \mu_k)((\lambda_k^1 + \mu_k)^2 - 2\mu_k(\lambda_k^1 + \mu_k) - \frac{\delta \lambda_k^1}{\lambda_k^1 + \mu_k}) + \lambda_k^1 \delta = (\lambda_k^1 + \mu_k)^2(\lambda_k^1 - \mu_k).\]

From (37), we get
\[|\lambda_k^1 + \mu_k| \leq \frac{\delta}{\mu_k(1 + \alpha)}, \quad \text{and} \quad |\lambda_k^1 - \mu_k| = \mu_k - \lambda_k^1, \quad (38)\]

which implies
\[|f_k(\lambda_k^1)| \leq \frac{\delta^2}{\mu_k^2(1 + \alpha)^2}(2\mu_k - \frac{\delta \mu_k}{\mu_k + \delta}) \leq \frac{2\delta^2}{\mu_k(1 + \alpha)}.\]

To prove (35), we use Newton’s sequence (33) which can be written as
\[f_k(\lambda_k^n) = f_k(\lambda_k^n) - \left(\frac{f_k(\lambda_k^{n-1}) + f'_k(\lambda_k^{n-1})(\lambda_k^n - \lambda_k^{n-1})}{\lambda_k^n - \lambda_k^{n-1}}\right). \quad (39)\]

Applying (32) to (39) for \(\zeta = \lambda_k^n\) and \(\lambda = \lambda_k^{n-1}\), then (35) follows.

From (29), (33) and (35) we get
\[|\lambda_k^{n+1} - \lambda_k^n| \leq \frac{2M}{\mu_k^2 + 2(s_k^2 + \delta)}|\lambda_k^n - \lambda_k^{n-1}|^2.\]

Hence (36) follows for
\[K = \frac{2M}{\mu_k^2 + 2(s_k^2 + \delta)}. \quad (40)\]

\[\square\]

**Lemma 4.2.** The sequence \((\lambda_k^n)_{n \geq 0}\) given by (33) satisfies
\[|\lambda_k^{n+1} - \lambda_k^n| \leq \frac{(K|\lambda_k^1 - \lambda_k^0|)^{2^n}}{K}, \quad \forall n \in \mathbb{N}. \quad (41)\]

and converges to \(\lambda_k\) the unique real root of (26) on \(I\).

**Proof.** We begin by proving the following inequality by recurrence
\[|\lambda_k^{n+1} - \lambda_k^n| \leq K^{1+2+\ldots+2^{n-1}}|\lambda_k^1 - \lambda_k^0|^{2^n}. \quad (42)\]

From (36), we have \(|\lambda_k^2 - \lambda_k^1| \leq K|\lambda_k^1 - \lambda_k^0|^2\) which means that (42) holds for \(n = 1\). We now suppose that (42) holds for \(n\) to prove it for \(n + 1\). From (36), we have
\[|\lambda_k^{n+2} - \lambda_k^{n+1}| \leq K|\lambda_k^{n+1} - \lambda_k^n|^2 \leq K^{1+2+\ldots+2^{n}}|\lambda_k^1 - \lambda_k^0|^{2^{n+1}}.\]

Hence (42) holds for \(n + 1\) and consequently for all \(n \in \mathbb{N}^*\).

By using the geometric sum \(K^{1+2+\ldots+2^{n-1}} = \frac{K^{2^n}}{K}\), we get (41) immediately.
We now use (41) to prove that the sequence (33) converges to \( \lambda_k \). For all \( m \in \mathbb{N}^* \) and \( p \in \mathbb{N}^* \) we have,

\[ |\lambda_k^{m+p} - \lambda_k^m| \leq \sum_{j=1}^{p} |\lambda_k^{m+j} - \lambda_k^{m+j-1}| \]

\[ \leq \frac{1}{K} \sum_{j=1}^{\infty} (K|\lambda_k^1 - \lambda_k^0|)^{2m+j-1} = \frac{1}{K} \sum_{j=2}^{\infty} (K|\lambda_k^1 - \lambda_k^0|)^j \]

\[ \leq \frac{(K|\lambda_k^1 - \lambda_k^0|)^2}{K(1-K|\lambda_k^1 - \lambda_k^0|)}. \]

From (38) and (40), we can easily see that \( K|\lambda_k^1 - \lambda_k^0| = K|\lambda_k^1 + \mu_k| < 1 \), so \( \lim_{n \to +\infty} (K|\lambda_k^1 - \lambda_k^0|)^2 = 0 \). This implies that \( (\lambda_k^n)_{n \in \mathbb{N}} \) is a Cauchy sequence on \( I \) and consequently converges to \( \lambda_k \) the real root of (26).

We now give the asymptotic expansion of \( \lambda_k \).

**Theorem 4.3.** If \( \lambda_k \) denotes the real root of (26) then

\[ \lambda_k = -\mu_k + \frac{\Delta \mu_k}{\mu_k^2 + s_k^2 + \delta} + \Lambda_k, \quad k \in \mathbb{N}^* \] (43)

where

\[ |\Lambda_k| \leq \frac{4\delta^2}{\mu_k^2 (1 + \alpha)}. \] (44)

**Proof.** By steps of Newton’s method with the initial guess of \( \lambda_k^0 = -\mu_k \), we get

\[ \lambda_k^n = \lambda_k^1 - \Lambda_k^n, \]

where

\[ \Lambda_k^n = \sum_{j=1}^{n} \frac{f_k(\lambda_k^j)}{f_k'(\lambda_k^j)} \quad \text{and} \quad \lambda_k^1 = -\mu_k + \frac{\Delta \mu_k}{\mu_k^2 + s_k^2 + \delta}. \]

Since the sequence \( (\lambda_k^n)_{n \geq 0} \) defined by (33) converges to \( \lambda_k \), the sequence \( \Lambda_k^n \) converges to \( \Lambda_k = \lambda_k^1 - \lambda_k \). From the finite-increment theorem, there exists \( c_k \in [\lambda_k^1, \lambda_k] \subseteq I \) such that

\[ f_k(\lambda_k) - f_k(\lambda_k^1) = f_k'(c_k)(\lambda_k - \lambda_k^1). \]

Since \( f_k(\lambda_k) = 0 \) and \( f_k'(c_k) \) is decreasing we get

\[ |\Lambda_k| = |\lambda_k^1 - \lambda_k| = \left| \frac{f_k(\lambda_k^1)}{f_k'(c_k)} \right| \leq \left| \frac{f_k(\lambda_k^1)}{f_k'(\lambda_k)} \right|. \]

By using Eqs. (29) and (34), the estimation (44) follows.

In the following, we suppose that

\[ \sqrt{3} < \frac{c}{2(1 + \alpha)}. \] (45)

**Theorem 4.4.** If \( \sigma_k \) denotes the complex root of (26) which belongs to the upper half plane, then

\[ \sigma_k = \frac{-2\Delta \mu_k}{4\mu_k^2 + s_k^2(\delta s_k^2 - 2)^2} + 2is_k \frac{c^2(2 - \gamma^2) + 2(1 + \alpha)\mu_k^2}{4\mu_k^2 + s_k^2(\delta s_k^2 - 2)^2} + \Delta_k, \quad k \in \mathbb{N}^* \] (46)
where $\Delta_k$ is $O(k^{-2})$. Moreover, if (45) holds, then

$$|\Delta_k| \leq \frac{\delta}{(1+\alpha)\mu_k^2}. \quad (47)$$

**Proof.** Let $\sigma_k = is_k + \varepsilon$, it follows that the characteristic equation (26) becomes

$$\varepsilon(-2s_k^2 + 2i\mu_k s_k + \delta) + \varepsilon^2(\mu_k + 3is_k) + \varepsilon^3 + i\delta s_k = 0. \quad (48)$$

Let

$$\varepsilon_1 = \varepsilon_1(k) = \frac{-i\delta s_k}{-2s_k^2 + 2i\mu_k s_k + \delta} \quad \text{and} \quad \varepsilon = \varepsilon_1 + \Delta. \quad (49)$$

Using (49) then Eq. (48) becomes

$$\varepsilon_1^2(\mu_k + 3is_k) + \varepsilon_1^3 + \Delta\left(-2s_k^2 + 2i\mu_k s_k + \delta + 2\varepsilon_1(\mu_k + 3is_k) + 3\varepsilon_1^2\right) + \Delta^2(3is_k + \mu_k + 3\varepsilon_1) + \Delta^3 = 0$$

which can be written as

$$f(\Delta) := A + B\Delta + C\Delta^2 + \Delta^3 = 0,$$

where

$$A = \varepsilon_1^2(\mu_k + 3is_k) + \varepsilon_1^3,$$

$$B = -2s_k^2 + 2i\mu_k s_k + \delta + 2\varepsilon_1(\mu_k + 3is_k) + 3\varepsilon_1^2,$$

$$C = 3is_k + \mu_k + 3\varepsilon_1. \quad (50)$$

Let $g(\Delta) = A + B\Delta$, then $f(\Delta) = g(\Delta) + C\Delta^2 + \Delta^3 = 0$. We now use Rouche’s Theorem to show that $f$ has a root near the root of $g$. So we wish to find a contour $\Gamma$ enclosing $\Delta_0 := -\frac{\delta}{\mu_k}$ such that $|f - g| \leq |g|$ on $\Gamma$.

To obtain an upper bound for $|A|$, we use (49) to get

$$|\varepsilon_1| = \frac{\delta s_k}{\sqrt{(\delta - 2s_k^2)^2 + 4\mu_k^2 s_k^2}} \leq \frac{\delta}{2\mu_k} \quad (51)$$

(we have used $(\delta - 2s_k^2)^2 + 4\mu_k^2 s_k^2 \geq 4\mu_k^2 s_k^2$). Moreover, (51) implies

$$|\varepsilon_1| + \mu_k \leq \frac{\delta}{2\mu_k} + \mu_k \leq 3\mu_k \quad (52)$$

(it is easy to check that $\frac{\delta + 2\mu_k^2}{2\mu_k} - 3\mu_k = \frac{1-4\mu_k^2}{2\mu_k} < 0$ for all $k \in \mathbb{N}^*$). From (50), we infer that

$$|A| \leq |\varepsilon_1|^2(\varepsilon_1 + \mu_k + 3s_k) \quad \text{(use (52))}$$

$$\leq 3|\varepsilon_1|^2(\mu_k + s_k) \quad \text{(use (51))}$$

$$\leq \frac{3\delta^2}{4\mu_k^2}(\mu_k + s_k) \quad \text{(use (45))}$$

$$\leq \frac{3\delta}{16(1+\alpha)^2\mu_k^2}(\mu_k + s_k).$$

By using the following inequalities $a + b < \sqrt{2a^2 + b^2}$ (for $a = \mu_k$ and $b = s_k$) and $c < 1 < \mu_k$ for all $k \in \mathbb{N}^*$, one can get

$$\mu_k + s_k \leq \sqrt{2\mu_k^2 + \alpha \mu_k^2 + c^2}$$

$$\leq \sqrt{2\mu_k^2(2 + \alpha)}$$

$$\leq \frac{\sqrt{2(1 + \alpha)}}{2\mu_k}. \quad (53)$$
Finally, as $|f - g| \leq |\Delta|^2|C + \Delta|$, we obtain from (56) and (57)

$$|f - g| \leq \frac{18\delta}{(1 + \alpha)\mu_k^2} \left[ \sqrt{1 + c\mu_k^2} + \frac{\delta}{\mu_k^2} \right].$$

Since $\sqrt{\delta} < \frac{c}{2}$, $c$ is taken small in comparison to 1 and $\sqrt{1 + c\mu_k^2} > 9$ for all $k \in \mathbb{N}^*$, it follows that

$$\frac{18\delta}{\sqrt{1 + c\mu_k^2}} + \frac{\delta^2}{(1 + \alpha)^2\mu_k^2} < \frac{3c}{2} \left( \frac{3c\mu_k^2}{\sqrt{1 + \alpha\mu_k^2}} + \frac{c^3}{24(1 + \alpha)^2\mu_k^2} \right),$$

$$< \frac{3c}{2}.$$
Then (58) becomes
\[ |f - g| < \frac{3\delta}{2\mu_k(1+\alpha)} < |g| \quad (\Delta \text{ on } \Gamma_k). \]
Thus (47) holds by Rouché’s Theorem. This completes the proof. \qed

We summarize this in the following proposition.

**Proposition 4.3.** Let \( \mathcal{A} \) be the operator defined in (16) and (45) holds. The eigenvalues of \( \mathcal{A} \) consist of a sequence of conjugate pairs \( \{\hat{\sigma}_k, \overline{\sigma}_k\}_{k=1}^\infty \) and a real sequence \( \{\hat{\lambda}_k\}_{k=1}^\infty \) with
\[
\hat{\sigma}_k = -\frac{2\delta \mu_k^2}{4\mu_k^2 + s_k^2(\delta s_k^2 - 2)} + 2\mu_k s_k \frac{c^2(2 - \gamma^2) + 2(1+\alpha)\mu_k^2}{4\mu_k^2 + s_k^2(\delta s_k^2 - 2)^2} + \Delta_k, \quad \forall k \in \mathbb{N}^*
\]
where \( |\Delta_k| \leq \frac{\delta}{(1+\alpha)\mu_k} \), and
\[
\hat{\lambda}_k = -\mu_k^2 + \frac{\delta \mu_k^2}{\mu_k + s_k^2 + \delta} + \hat{\lambda}_k, \quad \forall k \in \mathbb{N}^*
\]
where \( |\hat{\lambda}_k| \leq \frac{4\delta^2}{\mu_k^2(1+\alpha)} \).

The corresponding set of eigenfunctions \( \{\Phi_{\hat{\sigma}_k}, \Phi_{\overline{\sigma}_k}, \Phi_{\hat{\lambda}_k}\} \) is given by
\[
(\Phi_{\hat{\sigma}_k}, \Phi_{\overline{\sigma}_k}, \Phi_{\hat{\lambda}_k}) = \begin{pmatrix}
\cos(\mu_k x) & \cos(\mu_k x) & \frac{s_k^2 + \sigma_k^2}{\gamma c^2} \cos(\mu_k x) \\
\sigma_k \sin(\mu_k x) & \overline{\sigma}_k \sin(\mu_k x) & \hat{\lambda}_k \frac{s_k^2 + \sigma_k^2}{\gamma c^2} \sin(\mu_k x) \\
\frac{s_k^2 + \sigma_k^2}{\gamma c^2} \cos(\mu_k x) & \frac{s_k^2 + \sigma_k^2}{\gamma c^2} \cos(\mu_k x) & \cos(\mu_k x)
\end{pmatrix}.
\]

**Proof.** The eigenvalues of \( \mathcal{A} \) (or \( \mathcal{A}^* \)) given by (59) and (60) are obtained by multiplying \( \sigma_k, \overline{\sigma}_k \) and \( \lambda_k \) by \( \mu_k \).

Since \( \Sigma_k \Phi_{\sigma_k}, \Sigma_k \Phi_{\overline{\sigma}_k} \) and \( \Sigma_k \Phi_{\lambda_k} \) are the corresponding of \( \mathcal{A} \) (or \( \mathcal{A}^* \)), we infer from (24) and (25) that
\[
\begin{pmatrix}
\Sigma_k \Phi_{\sigma_k} \\
\Sigma_k \Phi_{\overline{\sigma}_k} \\
\Sigma_k \Phi_{\lambda_k}
\end{pmatrix} = \begin{pmatrix}
\cos(\mu_k x) & -\sigma_k \sin(\mu_k x) & \frac{s_k^2 + \sigma_k^2}{\gamma c^2} \cos(\mu_k x) \\
\sigma_k \sin(\mu_k x) & \overline{\sigma}_k \sin(\mu_k x) & \hat{\lambda}_k \frac{s_k^2 + \sigma_k^2}{\gamma c^2} \sin(\mu_k x) \\
\frac{s_k^2 + \sigma_k^2}{\gamma c^2} \cos(\mu_k x) & \frac{s_k^2 + \sigma_k^2}{\gamma c^2} \cos(\mu_k x) & \cos(\mu_k x)
\end{pmatrix}.
\]

According to Remark 3.1, the eigenfunctions \( \Phi_{\sigma_k}, \Phi_{\overline{\sigma}_k} \) and \( \Phi_{\lambda_k} \) corresponding to the eigenvalues (59) and (60) are
\[
\begin{pmatrix}
\Phi_{\sigma_k} \\
\Phi_{\overline{\sigma}_k} \\
\Phi_{\lambda_k}
\end{pmatrix} = \begin{pmatrix}
\Sigma_k \Phi_{-\sigma_k} \\
\Sigma_k \Phi_{-\overline{\sigma}_k} \\
\Sigma_k \Phi_{-\lambda_k}
\end{pmatrix} = \begin{pmatrix}
\cos(\mu_k x) & \sigma_k \sin(\mu_k x) & \frac{s_k^2 + \sigma_k^2}{\gamma c^2} \cos(\mu_k x) \\
\cos(\mu_k x) & \overline{\sigma}_k \sin(\mu_k x) & \hat{\lambda}_k \frac{s_k^2 + \sigma_k^2}{\gamma c^2} \sin(\mu_k x) \\
\frac{s_k^2 + \sigma_k^2}{\gamma c^2} \cos(\mu_k x) & \frac{s_k^2 + \sigma_k^2}{\gamma c^2} \cos(\mu_k x) & \cos(\mu_k x)
\end{pmatrix}.
\]

Transposing this last expression we get the desired result. \qed
Remark 4.2. Asymptotically, the eigenvalues of \( A \) are given by

\[
\hat{\sigma}_k = -\frac{\delta}{2} + i \left( \mu_k s_k + \frac{\delta s_k^2 - \frac{\delta}{2}}{\mu_k s_k} \right) + O(k^{-1}) \quad \text{as} \quad k \to \infty,
\]

\[
\hat{\lambda}_k = -\mu_k^2 + \frac{\delta}{\alpha + 1} + O(k^{-1}) \quad \text{as} \quad k \to \infty.
\]

5. Stability of the system. In this section, we shall discuss the stability of the system (9)-(11) based on the distribution of the spectrum of \( A \).

Proposition 5.1. Let \( A \) be the operator defined in (16) and (45) holds. Then the spectrum-determined growth condition holds true for the \( C_0 \)-semigroup \( e^{At} \), that is,

\[
\omega(A) = s(A) = -\frac{2\delta \mu_k^2}{4\mu_k^2 + s_k^2(\delta s_k^2 - 2)^2},
\]

where

\[
s(A) = \sup\{\Re(\lambda) \mid \lambda \in \sigma(A)\},
\]

is the spectral bound of \( A \) and

\[
\omega(A) = \lim_{t \to \infty} \frac{1}{t} \ln \| e^{At} \|
\]

stands for the growth bound of \( e^{At} \).

Proof. From Lemma 2.1, Theorem 4.3 and Theorem 4.4 it is easy to obtain that the multiplicities of the eigenvalues of \( A \) are uniformly bounded. Therefore, according to Theorem 4.2 of [11], the system (9)-(11) satisfies the spectrum-determined growth condition, i.e; \( s(A) = \omega(A) \).

From equations (26), (43) and (46), one can show after some computations that

\[
\hat{\lambda}_k - \Re\hat{\sigma}_k = \mu_k(\lambda_k - \Re(\sigma_k)) = \mu_k(-\mu_k - 3\Re(\sigma_k)) = -\mu_k^2 \left( 1 - \frac{6\delta}{4\mu_k^2 + s_k^2(\delta s_k^2 - 2)^2} \right) < 0,
\]

so \( s(A) = \Re(\sigma_k) \). The proof is complete.

Theorem 5.1. The solution to system (9)-(11) decays exponentially to zero, that is to say, there exist constants \( M > 1 \) and

\[
\omega = -s(A) = \frac{2\delta \mu_k^2}{4\mu_k^2 + s_k^2(\delta s_k^2 - 2)^2} > 0
\]

such that

\[
\| e^{At} \| \leq M e^{-\omega t},
\]

or equivalently

\[
E(t) \leq ME(0)e^{-\omega t}
\]

where \( E(t) \) is the energy function of system (9)-(11) defined by (14).

Proof. Firstly, we show that this system is asymptotically stable. Since \( A \) is dissipative in \( \mathcal{H} \) and \( 0 \in \rho(A) \), then \( \Re(\lambda) \leq 0, \forall \lambda \in \sigma(A) \). In order to show the asymptotic stability, it is sufficient to show that there is no eigenvalue on the imaginary axis due to the Lyubich and Phong’s Theorem (see [20]). Since

\[
\Re(\hat{\sigma}_k) = -\frac{2\delta \mu_k^2}{4\mu_k^2 + s_k^2(\delta s_k^2 - 2)^2} \neq 0, \forall k \in \mathbb{N}^*,
\]

the Lyubich and Phong’s Theorem asserts that the system (9)-(11) is asymptotically stable.
Secondly, let us show the exponential stability of the system (9)-(11). By the spectrum-determined growth condition claimed by Proposition 5.1 and the asymptotic expressions of the spectrum in Theorems 4.3 and 4.4, $e^{-\lambda t}$ is exponentially stable if and only if $\Re(e^{i\lambda t}) < 0$, $\forall \lambda \in \sigma(A)$.

Note that $\Re(\lambda_k) = -\mu_k^2 \left(1 - \frac{\delta}{\mu_k^2 + 3\delta + 3}\right) < 0$ and $\Re(\sigma_k) = -\frac{2\delta \mu_k^2}{4\mu_k^2 + 3\delta (\delta \mu_k^2 - 2)^2} < 0$, $\forall k \in \mathbb{N}^*$. Since the system (9)-(11) satisfies the spectrum-determined growth condition, by Proposition 5.1, the desired result follows. ☐

6. Conclusion. 1. The asymptotic expansion (2) for the classic thermoelastic system (1) can be obtained from (61) by putting $\alpha = 0$. From (62), the spectrum-determined growth condition in the case $\alpha = 0$ is given by

$$\omega(A)|_{\alpha=0} = s(A)|_{\alpha=0} = \frac{-2\delta \mu_k^2}{4\mu_k^2 + c^2(\delta c^2 - 2)^2}.$$ 

Asymptotically, this condition becomes $\omega(A)|_{\alpha=0} = s(A)|_{\alpha=0} = -\delta/2$. This confirms the result of Henry [14] given by (3).

2. The method can be illustrated by an example of beam equations with Dirichlet boundary conditions in $(0, 1) \times \mathbb{R}^+$:

$$\begin{align*}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^4 u}{\partial x^4} - \nu \frac{\partial^2 \theta}{\partial x^2}, \\
\frac{\partial \theta}{\partial t} &= \frac{\partial^2 \theta}{\partial x^2} + \nu \frac{\partial^2 u}{\partial x^2}, \\
u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) &= 0, \\
\theta(0,t) = \theta(1,t) &= 0, \quad \forall t \in \mathbb{R}^+, \\
u(x,0) = u_0(x), \quad u_t(x,0) &= u_1(x), \\
\theta(x,0) &= \theta_0(x), \quad x \in (0,1). 
\end{align*}$$ (64)

In this example, we take into account the rotational inertia of the beam filaments (i.e., $\gamma > 0$). In some models, the later is neglected (i.e., $\gamma = 0$). The positive constant $\nu$ is a measure of the mechanical-thermal coupling. By applying the same approach developed in this paper to (64), we obtain the following result.

Theorem 6.1. For $0 < \nu \leq \frac{1}{2\sqrt{2}}$, the spectrum of the linear operator $A$ associated to the system (64) is given by

$$\hat{\sigma}_k = \frac{-2\nu^2 \mu_k^2}{4c_k^2 + (\nu^2 - 2)^2} + i\mu_k^2 c_k^{-1} \frac{4c_k^2 + 2(2 - \nu^2)}{4c_k^2 + (\nu^2 - 2)^2} + \Delta_k, \quad \forall k \in \mathbb{N}^*$$

where $|\Delta_k| \leq \frac{\nu^2 \mu_k^2}{c_k^2}$ and

$$\hat{\lambda}_k = -\mu_k^2 + \frac{\nu^2 \mu_k^2}{c_k^2 + \nu^2 + 1} + \Lambda_k, \quad \forall k \in \mathbb{N}^*$$

where $|\Lambda_k| \leq \frac{4\nu^2 \mu_k^2}{c_k^2}$, $c_k = \sqrt{1 + \gamma \mu_k^2}$ and $\mu_k = k\pi$.

The spectrum-determined growth condition holds true for $e^{-\lambda t}$, that is,

$$s(A) = \omega(A) = -\frac{2\nu^2 \mu_k^2}{(\nu^2 - 2)^2 + 4c_k^2},$$ (65)

and the solution to system (64) decays exponentially to zero.
Asymptotically, the above eigenvalues are given by
\[ \hat{\sigma}_k = -\frac{\nu}{2\gamma} - i\frac{\mu_k}{\sqrt{\gamma}} + O(k^{-1}) \quad \text{as} \quad k \to \infty, \]
\[ \hat{\lambda}_k = -\mu_k^2 + \frac{\alpha^2}{\gamma} + O(k^{-2}) \quad \text{as} \quad k \to \infty, \]
which is exactly the same asymptotic expansion given in [13].

Asymptotically, the condition (65) becomes
\[ s(A) = \omega(A) = -\frac{\nu^2}{2\gamma}. \]
This confirms the well-known result that the associated semigroup loses its exponential stability when \( \gamma = 0 \) (see [6] for more details).

3. For the system (9)-(11), we also observe that the higher order gradients of displacement \( (\alpha \neq 0) \) has the effect of slowing the decay rate. However, if we set \( \omega_0 = \omega|_{\alpha=0} \), from (63) we obtain
\[ \omega_0 = \frac{2\delta \mu_k^2}{c^2 s_k^2 + c^2(\delta c^{-2} - 2)^2} \]
and we find that
\[ \omega - \omega_0 = \frac{2\delta \mu_k^2}{c^2 s_k^2} \left( \frac{(4c^2 s_k^2 - \delta^2)(c^2 - s_k^2)}{(4\mu_k^2 + (\delta c^{-1} - 2s_k^2))(4\mu_k^2 + (\delta c^{-1} - 2c^2))} \right). \]
By \( s_k^2 = c^2 + \alpha \mu_k^2 \), we can see that \( c^2 - s_k^2 < 0 \). On the other hand, from (45) we have that \( \delta < 2cs_k \), which implies that \( \omega < \omega_0 \). This is due to the dissipative nature of the motion equation. In fact, the more we take into account higher order gradients of displacement, the more the motion equation becomes dissipative. As a consequence, the decay of solutions slows down.

4. The results presented in this paper should prove useful in the study of control problems and \( L^p-L^q \) decay estimations for nonsimple materials which is a topic that has not yet been addressed in the literature. This approach can also be extended to other linear systems, such as elastic, Euler-Bernoulli and other equations coupled with different diffusion equations.

**Acknowledgments.** The authors would like to thank the referees for their critical review and valuable comments which thoroughly improved the paper.

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Received October 2014; 1st revision May 2015; 2nd revision July 2015.

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