Quantum depletion density of a homogeneous dilute Bose gas
within improved Hartree-Fock approximation

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Abstract

Motivated by the recent experiment [R. Lopes et. al., Phys. Rev. Lett. 119, 190404 (2017)]
with a homogeneous Bose gas, we investigate a homogeneous dilute Bose gas to reproduce the
quantum depletion density. By means of Cornwall-Jackiw-Tomboulis effective action approach
within improved Hartree-Fock approximation, the quantum depletion density is recovered in an
easier manner. Apart from that, the higher-order terms are taken into account for the condensed
fraction.
It is well-known that a number of atoms in a Bose gas will be condensed as the system is cooled to the critical temperature $T_c$ and formed the Bose-Einstein condensate (BEC). The studies on the BEC are rapidly developed after the BEC was created in experiments \cite{1,2}. In a dilute Bose-Einstein condensate, essentially all atoms occupy the same quantum state, and the condensate can be described in terms of a mean-field theory similar to the Hartree-Fock theory for atoms \cite{3}. Theoretically, all of atoms will be in the ground state at zero temperature \cite{4}. In this situation, the ground state is described by a wave function, which is the solution of the Gorss-Pitaevskii (GP) equation \cite{7,8}.

It is reported that we have never reached the absolute zero temperature and even so, some particles are in excited states instead of the ground state \cite{5}. This phenomenon is called the quantum fluctuations. Because of the quantum fluctuations, the particles with nonzero momentum is in excited states and it also called the quantum depletion. The density of quantum depletion was first studied in 1947 by N. N. Bogoliubov \cite{9} up to order 1/2 of the gas parameter by using the second quantization, the main ideal is based on a quantum description, where the particle operators are transformed into quasi-particle operators henceforth yielding an explicit diagonalization of the quantum Hamiltonian. In 1997, using Bogoliubov theory and semiclassical approach, the authors of Ref. \cite{10} investigated the quantum depletion in a Bose gas confined by a harmonic trap. Subsequently, in case of a homogeneous Bose gas, their result deduces exactly to Bogoliubov’s result. Recently, it was reproduced by S. Stringari \cite{11} within the GP theory. However, all of these ways have to use many complicated calculations. The main purpose of the present letter is to give an easier method to recover the density of quantum depletion.

We start with a dilute Bose gas described by the Lagrangian \cite{5},

$$\mathcal{L} = \psi^* \left( -i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2m} \nabla^2 \right) \psi - \mu |\psi|^2 + \frac{g}{2} |\psi|^4 , \quad (1)$$

in which $\hbar$ is the reduced Planck constant, $m$ and $\mu$ are atomic mass and the chemical potential, respectively. The field operator $\psi(\vec{r},t)$ depends on both the coordinate and time. The strength of interaction between the atoms is determined by the coupling constant $g$ which relates to the $s$-wave scattering length $a_s$ in form $g = 4\pi\hbar^2a_s/m$ and $g > 0$ for repulsive interaction.

As usual, we first invoke the Cornwall-Jackiw-Tombouliis (CJT) effective action approach \cite{12} at finite temperature to investigate and then the temperature is set to be zero at the
end of calculations. Let $\psi_0$ be the expectation value of the field operator, in the tree-approximation the GP potential is

$$V_{GP} = -\mu \psi_0^2 + \frac{g}{2} \psi_0^4.$$  \hspace{1cm} (2)

Note that here and hereafter the system is considered without the external field and this no macroscopic part of the condensate moves as a whole so that $\psi_0$ is real and it plays the role of the order parameter. Minimizing this potential with respect to the order parameter one has

$$\psi_0 = \frac{\mu}{g}. \hspace{1cm} (3)$$

The inversion propagator in the tree-approximation has the form

$$D_0^{-1}(k) = \begin{pmatrix} \frac{\hbar^2 k^2}{2m} + 2g\psi_0^2 & -\omega_n \\ \omega_n & \frac{\hbar^2 k^2}{2m} \end{pmatrix}, \hspace{1cm} (4)$$

where $\vec{k}$ is the wave vector in momentum space and $\omega_n$ is the Matsubara frequency. Denote $T$ the temperature, the Matsubara frequency for boson is defined as $\omega_n = 2\pi n/\beta$, $\beta = (k_BT)^{-1}$ with $k_B$ being Boltzmann constant. Requiring the determinant of (4) vanishes one arrives at the dispersion relation

$$E_0(k) = \sqrt{\frac{\hbar^2 k^2}{2m} \left( \frac{\hbar^2 k^2}{2m} + 2g\psi_0^2 \right)}. \hspace{1cm} (5)$$

Undoubtedly, Eq. (5) shows that there is a Goldstone boson.

An important assumption of the tree-approximation is ignoring the quantum fluctuations. To take into account these fluctuations, the field operator need to expand in terms of two real fields $\psi_1, \psi_2$ associated with the fluctuations $\psi \rightarrow \psi_0 + \frac{1}{\sqrt{2}} (\psi_1 + i\psi_2). \hspace{1cm} (6)$

Plugging (6) into (1) one gets the interaction Lagrangian in the Hartree-Fock approximation

$$\mathcal{L}_{int} = \frac{g}{2} \psi_0 \psi_1 (\psi_1^2 + \psi_2^2) + \frac{g}{8} (\psi_1^2 + \psi_2^2)^2. \hspace{1cm} (7)$$

The CJT effective potential can be read-off from (7)

$$V_\beta = -\mu \psi_0^2 + \frac{g}{2} \psi_0^4 + \frac{1}{2} \int \beta \text{ tr} \left[ \ln D^{-1}(k) + D_0^{-1}(k)D(k) - \mathbb{1} \right]$$

$$+ \frac{3g}{8} (P_{11}^2 + P_{22}^2) + \frac{g}{4} P_{11} P_{22}, \hspace{1cm} (8)$$
where the notation

\[ \int_{\beta} f(k) = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^3\vec{k}}{(2\pi)^3} f(\omega_n, \vec{k}). \]

is used. \(D(k)\) is the propagator in this approximation and

\[ P_{11} = \int_{\beta} D_{11}(k), \quad P_{22} = \int_{\beta} D_{22}(k), \quad (9) \]

are the momentum integrals. As pointed out in Ref. [15], the CJT effective potential (8) violates the Goldstone theorem [16]. This problem can be fixed by invoking the method proposed by Ivanov et. al. [17]. According to this method, an extra term

\[ \Delta V = -\frac{g^4}{4}(P_{11}^2 + P_{22}^2) + \frac{g^8}{8}P_{11}P_{22}, \quad (10) \]

need be added into the CJT effective (8) and therefore a new CJT effective potential

\[
V_{\beta} = -\mu \psi_0^2 + \frac{g}{2} \psi_0^4 + \frac{1}{2} \int_{\beta} \text{tr} \left[ \ln D^{-1}(k) + D_0^{-1}(k)D(k) - \mathbf{1} \right] \\
+ \frac{g}{8}(P_{11}^2 + P_{22}^2) + \frac{3g}{8}P_{11}P_{22}. \quad (11)
\]

Minimizing the CJT effective potential (11) with respect to the order parameter one has the gap equation

\[ -\mu + g\psi_0^2 + \Sigma_1 = 0, \quad (12) \]

and, in the same manner, the Schwinger-Dyson (SD) equation is obtained in form

\[ M^2 = -\mu + 3g\psi_0^2 + \Sigma_2, \quad (13) \]

where the self-energies are expressed in terms of the momentum integrals

\[ \Sigma_1 = \frac{3g}{2}P_{11} + \frac{g}{2}P_{22}, \]
\[ \Sigma_2 = \frac{g}{2}P_{11} + \frac{3g}{2}P_{22}, \quad (14) \]

and \(M\) is the effective mass. Combining Eqs. (11), (12) and (13), the inversion propagator in this approximation can be derived

\[ D^{-1}(k) = \begin{pmatrix}
\frac{\hbar^2 k^2}{2m} + M^2 - \omega_n \\
\omega_n \\
\omega_n - \frac{\hbar^2 k^2}{2m}
\end{pmatrix}. \quad (15) \]
Eq. (15) confirms that the Goldstone boson is restored in this approximation, this is the reason why it is called the improved Hartree-Fock (IHF) approximation. For all above calculations, note that, for the notational simplicity, the same symbols will be used again from (11) to (15) to denote the corresponding quantities, although their expressions are different from those given in Eq. (8). In this regard, the momentum integrals have the form

\[
P_{11} = \frac{1}{2} \int \frac{d^3 \vec{k}}{(2\pi)^3} \sqrt{\frac{\hbar^2 k^2 / 2m}{\hbar^2 k^2 / 2m + M^2}}, \quad P_{22} = \frac{1}{2} \int \frac{d^3 \vec{k}}{(2\pi)^3} \sqrt{\frac{\hbar^2 k^2 / 2m + M^2}{\hbar^2 k^2 / 2m}},
\]

at zero temperature.

We next investigate the quantum depletion density in the IHF approximation. At first, we note that the pressure is defined as the negative of the CJT effective potential (11) at the minimum, i.e. satisfying the gap and SD equations

\[
\mathcal{P} = -\tilde{V}_\beta \bigg|_{\text{at minimum}}.
\]

The condensate density in the IHF approximation can be derived from the pressure

\[
n = -\frac{\partial \mathcal{P}}{\partial \mu}.
\]

Combining Eqs. (11)-(13), (17) and (18), the condensate density is expressed in terms of the order parameter and the momentum integrals (18),

\[
n = \psi_0^2 + \frac{1}{2}(P_{11} + P_{22}).
\]

The density of condensate depletion is defined as the number of particles in excited states per unit volume [5]. Based on Eq. (19) one easily sees that the condensed fraction for a homogeneous dilute Bose gas is

\[
n_{ex} = \frac{1}{2}(P_{11} + P_{22}).
\]

In order to simplify notations, we use the coherent healing length \( \xi = \hbar^2 / \sqrt{2mgn_0} \) with \( n_0 \) being density in bulk and henceforth several dimensionless quantities are introduced: the reduced order parameter \( \phi_0 = \psi_0 / \sqrt{n_0} \), wave vector \( \kappa = k\xi \) and mass effective \( \mathcal{M} = M/\sqrt{gn_0} \). The momentum integrals (16) reduce

\[
P_{11} = \frac{1}{2\xi^3} \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\kappa}{\sqrt{\kappa^2 + \mathcal{M}^2}}, \quad P_{22} = \frac{1}{2\xi^3} \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\sqrt{\kappa^2 + \mathcal{M}^2}}{\kappa}.
\]
The integration over the dimensionless wave vector in Eq. (21) is ultraviolet divergent. With dimensional regularization, this divergence can be avoided [14],

\[ I_{m,n} = \int \frac{d^d\kappa}{(2\pi)^d} \kappa^{2m-n}(\kappa^2 + \Lambda^2)^{-2n} = \frac{\Omega_d}{(2\pi)^d} \Lambda^{2\varepsilon} M^{d-2m-2n} \frac{\Gamma \left( \frac{d-n}{2} + m \right) \Gamma \left( n - m - \frac{d}{2} \right)}{2\Gamma \left( \frac{d}{2} \right)} \]

(22)

where \( \Gamma(x) \) is the gamma function and \( \Omega_d = 2\pi^{d/2}/\Gamma(d/2) \) and \( \Lambda \) is a renormalization scale.

Applying (22) for (21) with \( d = 3 \) one attains

\[ P_{11} = \frac{M^3}{6\pi^2\xi^3}, \quad P_{22} = -\frac{M^3}{12\pi^2\xi^3}. \]

(23)

To proceed further we note that a dilute Bose gas is constrained by a condition for the gas parameter \( n_s = n_0 a^3_s \ll 1 \) [14]. Inserting (23) into (12), the gap equation is rewritten in dimensionless form

\[ -1 + \phi_0^2 + \frac{10\sqrt{2}n_s^{1/2}}{3\sqrt{\pi}} M^3 = 0. \]

(24)

Similarly, the SD equation (13) becomes

\[ M^2 = -1 + 3\phi_0^2 - \frac{2\sqrt{2}n_s^{1/2}}{3\sqrt{\pi}} M^3. \]

(25)

To derive Eqs. (24) and (25), the chemical potential is set to be \( \mu = gn_0 \). This means that our system is considered in the grand canonical ensemble. Solving Eqs. (24) and (25) one easily finds the dimensionless effective mass

\[ M = \sqrt{2} - \frac{32\sqrt{2}}{3\sqrt{\pi}} n_s^{1/2}. \]

(26)

Finally, substituting (26) into (20) one finds the condensed fraction

\[ \frac{n_{ex}}{n_0} = \gamma n_s^{1/2} - \frac{256}{3\pi} n_s + \frac{8192 n_s^{3/2}}{9\pi^{3/2}} + \mathcal{O}(n_s^2), \]

(27)

in which \( \gamma = \frac{8}{3\sqrt{\pi}} \approx 1.50 \). Note that here, resemble [9], \( n_0 \) is assumed to be the density of condensate, which is expectation value of the field operator (3) in the tree-approximation.

The evolution of the condensed fraction versus the gas parameter is depicted in Fig. 11, in which the red line corresponds to the first term in right hand size of Eq. (27) is kept, the green line associates with all terms in right hand size of Eq. (27), the blue line is the numerical solution for the condensed fraction, whereas the magenta dots are experimental data of Ref. [19].

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FIG. 1. (Color online) The condensed fraction \( \frac{n_{ex}}{n_0} \) of a homogeneous Bose gas as a function of the gas parameter \( n_s = n_0 a_s^3 \). The red and green lines correspond to the Bogoliubov’s result and \( (27) \). The blue line is the numerical solution for the condensed fraction. The magenta dots are experimental data \[19\].

Recently, Lopes et. al. have set up an experiment \[19\] to consider the quantum depletion density of a dilute homogeneous Bose gas \(^{39}\)K as a function of the gas parameter, in which the interaction strength was controlled by a magnetic Feshbach resonance \[20\]. With the particle density \( n_0 = 3.5 \times 10^{11} \text{ cm}^{-3} \) in the lowest hyperfine state \( |F = 1, m_F = 1\rangle \). Within a 15% statistical error and 20% systematic effects, the authors found \( \gamma = 1.5(2) \). This work confirmed the Bogoliubov’s result and, of course, verified our result. However, the higher-order terms are kept in our result, this makes our result coincide more with the experimental data in the extremely small gas parameter region.

In conclusion, in this letter we have recovered the Bogoliubov’s result for the quantum depletion density in the homogeneous Bose gas, in which, instead of many complicated calculations, the CJT effective action approach has been employed within IHF approximation. In this way, the quantum depletion density has been found within easier calculations and the higher-order terms of the gas parameter are kept. Moreover, the CJT effective action approach can be employed to reproduce the relation for the ground state energy of a Bose gas, which was attained in Refs. \[21, 22\], therefore compared with the relevant experimental result in \[23\]. In addition, the thermal fluctuations can also be similarly investigated.

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REFERENCES

[1] S. N. Bose, *Planck’s Law and Light Quantum Hypothesis*, Z. Physik **26**, 178 (1924).
[2] A. Einstein, Sitz. Ber. Preussischen Akad. Wiss. Phys. Math. Kl. **261**, 3 (1924).
[3] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman and E. A. Cornell, *Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor*, Science **269**, 198 (1995).
[4] K. B. Davis, M. -O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, *Bose-Einstein Condensation in a Gas of Sodium Atoms*, Phys. Rev. Lett. **75**, 3969 (1995).
[5] C. J. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases*, Cambridge: Cambridge University Press, 2008.
[6] L. P. Pitaevskii and S. Stringari, *Bose-Einstein Condensation*, Oxford: Oxford University Press, 2003.
[7] E. P. Gross, *Structure of a Quantized Vortex in Boron Systems*, Nuovo Cimento, vol. 20, p. 454, 1961.
[8] L. P. Pitaevskii, *Vortex lines in an imperfect Bose gas*, Sov. Phys. JETP, vol. **13**, p. 451, 1961.
[9] N. N. Bogolyubov, *On the theory of superfluidity*, J. Phys. (USSR), vol. **11**, p. 23, 1947.
[10] F. Dalfovo, S. Giorgini, M. Guilleumas, L. Pitaevskii, and S. Stringari, *Collective and single-particle excitations of a trapped Bose gas*, Phys. Rev. A **56**, .
[11] S. Stringari, *Quantum Fluctuations and Gross-Pitaevskii Theory*, J. Expr. Theor. Phys., vol. **127**, p. 844, 2018.
[12] J. M. Cornwall, R. Jackiw, E. Tomboulis, *Effective action composite operators*, Phys. Rev. D **10**, 2428 (1974).
[13] S. Floerchinger and C. Wetterich, *Superfluid Bose gas in two dimensions*, Phys. Rev. A, vol. **79**, p. 013601, 2009.
[14] J. O. Andersen, *Theory of the weakly interacting Bose gas*, Rev. Mod. Phys., vol. **76**, p. 599, 2004.
[15] N. V. Thu and P. T. Song, *Casimir effect in a weakly interacting Bose gas confined by a parallel plate geometry in improved Hartree-Fock*, Physica A, vol. 540, p. 123018, 2020.

[16] J. Goldstone, *Field theories with ”Superconductor” solutions*, Nuovo Cimento 19, 154-164 (1961).

[17] Y. B. Ivanov, F. Riek and J. Knoll, *Gapless Hartree-Fock resummation scheme for the O(N) model*, Phys. Rev. D, vol. 71, p. 105016, 2005.

[18] T. H. Phat, L. V. Hoa, N. T. Anh, and N. V. Long, *Bose-Einstein condensation in binary mixture of Bose gases*, Ann. Phys. 324, 2074 (2009).

[19] R. Lopes, C. Eigen, N. Navon, D. Clement, R. P. Smith and Z. Hadzibabic, *Quantum depletion of a homogeneous Bose-Einstein condensate*, Phys. Rev. Lett., vol. 119, p. 190404, 2017.

[20] S. Inouye, M. R. Andrews, J. Stenger, H. -J. Miesner, D. M. Stamper-Kurn, W. Ketterle, *Observation of Feshbach resonances in a Bose-Einstein condensate*, Nature (London) 392, 151 (1998).

[21] T. D. Lee, K. Huang, and C. N. Yang, *Eigenvalues and eigenfunctions of a Bose system of hard spheres and its low-temperature properties*, Phys. Rev. 106, 1135 (1957).

[22] T. D. Lee and C. N. Yang, *Many-body problem in quantum statistical mechanics V. Degenerate phase in Bose-Einstein condensation*, Phys. Rev. 117, 897 (1960).

[23] N. Navon, S. Piatecki, K. Gunter, B. Rem, T. C. Nguyen, *Dynamics and Thermodynamics of the Low-Temperature Strongly Interacting Bose Gas*, Phys. Rev. Lett. 107, 135301 (2011).