IDEAL COTORSION THEORIES IN TRIANGULATED CATEGORIES

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Abstract. We study ideal cotorsion pairs associated to weak proper classes of triangles in extension closed subcategories of triangulated categories. This approach allows us to extend the recent ideal approximations theory developed by Fu, Herzog et al. for exact categories in the above mentioned context, and to provide simplified proofs for the ideal versions of some standard results as Salce’s Lemma, Wakamatsu’s Lemma and Christensen’s Ghost Lemma. In the last part of the paper we apply the theory in order to study connections between projective classes (in particular localization or smashing subcategories) in compactly generated categories and cohomological functors into Grothendieck categories.

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1. Introduction

Approximations of objects by some better understood ones are important tools in the study of various categories. For example they are used to construct resolutions and to do homological algebra: in module theory the existence of injective envelopes, projective precovers and flat covers are often used for defining derived functors, for dealing with invariants as (weak) global dimension etc. The central role in approximation theory for the case of module, or more general abelian or exact, categories is played by the notion of cotorsion pair. On the other hand, in the context of triangulated categories the cotorsion pairs are replaced by $t$-structures, as in [6]. We note that in triangulated categories there are no important differences between torsion and cotorsion theories. The explanation is the fact that the Ext-functor from an abelian category may be computed as a shifted Hom functor in the corresponding derived category. These structures were generalized in [21] to torsion pairs and mutation pairs, and the authors proved that some results which are valid for cotorsion theories in the context of module categories hold also in the context of triangulated categories (e.g. Wakamatsu’s Lemma, [16, Lemma 5.13]).

On the other side, there are situations when the approximations are realized by some ideals which are not object ideals, e.g. the pantom ideal introduced by Ivo Herzog in [17] (in module categories) and the ideal of $P$-null homomorphisms associated to a class $P$ of objects used [11] (in triangulated categories). In [14] and [15] the authors extended, in the context of exact categories, the study of classical cotorsion pairs to ideal cotorsion pairs, and the theory developed in [14] was extended and completed in [13] and [26].

Following these ideas, in the present paper we will study ideal cotorsion pairs in extension closed subcategories of triangulated categories, in order to obtain good information about precovering/preenveloping ideals. Since every exact category can be embedded as an extension closed subcategory of a triangulated category (eventually extending the universe) the theory presented here includes important parts from the theory developed in [14] and [15]. An advantage of this approach is that some proofs are more simple and natural (e.g. in the proof of Salce’s Lemma and Wakamatsu’s lemma, Theorem 51 and Lemma 69, we do not need 3-dimensional diagrams as those used in [13]). On the other side, working in triangulated categories we have only weak (co)kernels, and we have only homotopy pullbacks/pushouts. Therefore, in this context we cannot use the uniqueness parts for corresponding universal properties (these are important ingredients in the case of exact categories, e.g. the reader can compare the proof for Ghost Lemma provided in [15, Theorem 25] with the proof for Theorem 75).

Let $T$ be a triangulated category and $A$ a subcategory of $T$ closed under extensions. In the case of module categories precovering and preenveloping classes are studied in relation with the canonical exact structure on these categories. One of the main question in this context is to establish if some or all precovers (preenvelopes) are deflations (inflations) with respect to this exact structure. In order to approach this problem in our context, let us recall that the usual substitutes for exact structures in triangulated categories are proper classes of triangles, introduced by Beligianis in [7]. For other examples of applications for proper classes we refer to [22] and [23]. Therefore, we will study precovering ideals $I$ such that all $I$-precovers are $E$-deflations, were $E$ is a fixed weak proper class of triangles in $A$. 
(i.e. a class of triangles in \( A \) which is closed with respect to homotopy pullbacks, homotopy pushouts, finite direct sums and contains all splitting triangles).

We will construct an ideal cotorsion theory relative to \( \mathcal{E} \) in the following way: If \( \mathcal{I} \) is an ideal in \( A \) then we can associate to \( \mathcal{I} \) the class \( \mathcal{PB}(\mathcal{I}) \) which consists in all triangles which can be constructed as homotopy pullbacks of triangles from \( \mathcal{E} \) along maps from \( \mathcal{I} \). Therefore, \( \mathcal{PB}(\mathcal{I}) \) is a weak proper subclass of \( \mathcal{E} \). Dually, \( \mathcal{J} \) is an ideal in \( A \) then we can associate to \( \mathcal{J} \) the class \( \mathcal{PO}(\mathcal{J}) \) which consists in all triangles which can be constructed as homotopy pushouts of triangles from \( \mathcal{E} \) along maps from \( \mathcal{J} \). The class \( \mathcal{PO}(\mathcal{J}) \) is also a weak proper subclass of \( \mathcal{E} \). A pair of ideals \( (\mathcal{I}, \mathcal{J}) \) is an ideal cotorsion pair with respect to \( \mathcal{E} \) if \( \mathcal{I} = \mathcal{PO}(\mathcal{J})\)-proj and \( \mathcal{J} = \mathcal{PB}(\mathcal{I})\)-inj. Here, if \( \mathfrak{F} \) is a weak proper class of triangles then we denote by \( \mathfrak{F}\)-proj (respectively \( \mathfrak{F}\)-inj) all maps which are projective (injective) with respect to all triangles from \( \mathfrak{F} \).

We can reverse this process starting with a weak proper subclass \( \mathfrak{F} \) of \( \mathcal{E} \). We associate to \( \mathfrak{F} \) an ideal \( \Phi_{\mathcal{E}}(\mathfrak{F}) \) of those homomorphisms \( \varphi \) with the property that all triangles obtained via homotopy pullbacks of triangles from \( \mathcal{E} \) along \( \varphi \) are in \( \mathfrak{F} \). The elements of \( \Phi_{\mathcal{E}}(\mathfrak{F}) \) are called relative \( \mathfrak{F}\)-phantoms. Dually, we can associate to a weak proper subclass \( \mathfrak{G} \) of \( \mathcal{E} \) the ideal \( \Psi_{\mathcal{E}}(\mathfrak{G}) \) of all maps \( \psi \) such that every triangle obtained via a homotopy pushout of a triangle from \( \mathcal{E} \) along \( \psi \) is a triangle from \( \mathfrak{G} \). In this way we obtain two Galois correspondences between the class of ideals in \( A \) and the class of weak proper subclasses of \( \mathcal{E} \) (Theorem 34).

The ideal cotorsion pairs are studied in Section 3. In the context of cotorsion theories associated to module categories Salce’s Lemma says us that in many cases all precovers/preenvelopes are special (i.e. they can be constructed via some special pushouts/pullbacks). This lemma was extended to ideals associated to exact categories in [14], where it is proved that an ideal \( \mathcal{I} \) is special precovering if and only if it the ideal of phantoms associated to an exact structure which have enough special injective homomorphisms. The results of Section 3 can be summarized in Theorem 61, where it is proved that if we have enough \( \mathcal{E}\)-injective \( \mathcal{E}\)-deflations, then an ideal cotorsion pair \( (\mathcal{I}, \mathcal{J}) \) is complete iff \( \mathcal{I} \) is a precovering ideal or \( \mathcal{J} \) is a preenveloping ideal. These complete ideal cotorsion pairs can be constructed via relative (co)phantoms associated to some weak proper proper subclasses of \( \mathcal{E} \).

In Section 4 we study products and extensions of ideals in order to prove that products of special precover ideals are special precover ideals (Theorem 78). This is an ideal version for Ghost Lemma, [11, Theorem 1.1]. We also include here the ideal version for the above mentioned Wakamatsu’s Lemma (Lemma 69).

As an application of the theory developed here we will prove in the last section of the paper a generalization to projective classes for a result proved by Krause in [20] for smashing subcategories of compactly generated triangulated categories (see Proposition 89). More precisely, let us recall that Krause proved that every smashing subcategory \( \mathcal{B} \) of a compactly generated triangulated category \( \mathcal{T} \) is determined by the ideal \( \mathcal{IB} \) (in the subcategory \( \mathcal{T}_0 \) of all compact objects in \( \mathcal{T} \)) of all homomorphisms between compact objects which factorize through an element of \( \mathcal{B} \). We consider the same ideal \( \mathcal{IB} \) associated to a projective class \( (\mathcal{B}, \mathcal{J}) \), and we prove that if \( H : \mathcal{T} \to \mathcal{A} \) is a cohomological functor into a Grothendieck category \( \mathcal{A} \) such that \( H(\mathcal{IB}) = 0 \) then \( H \) annihilates an ideal of relative phantoms. In the case \( \mathcal{B} \) is smashing this ideal contains of all homomorphisms from \( \mathcal{T} \) which factorizes
through elements from $B$. Moreover, for the case when $H$ is full and $H(I_B) = 0$ we always obtain $H(B) = 0$ (Proposition 94).

For reader’s convenience, the results proved in Section 2 and Section 3 are presented together with their duals since the direct statements and the duals are connected in Theorem 61. The direct statement is denoted by (1) and the dual is denoted by (2). The results proved in Section 4 can be also dualized, but we left to the reader to enunciate these duals.

2. Weak proper classes and ideals

2.1. Weak proper classes of triangles. We refer to [25] for basic properties of triangulated categories. If $T$ is a triangulated category, we denote by $(-)[1]$ the suspension functor associated to $T$. Moreover, $D$ will denote the class of all distinguished triangles in $T$. Since we work only with distinguish triangles, by triangle we mean distinguish triangle. If $d : B \to C \to A \to B[1]$ is a triangle in $T$, we will denote $\gamma$ by $\text{Ph}(d)$. Moreover, if $A$ is a subcategory of $T$ then $A^\perp$ will be the class of all homomorphisms from $A$.

Let $T$ be a triangulated category. If $A$ is a full subcategory (closed with respect to isomorphisms) of $T$, we will say that it is closed under extensions if for every triangle $B \to C \to A \to B[1]$ in $T$ such that $A$ and $B$ are objects in $A$ then $C$ is an object in $A$. We will denote by $D_A$ the class of all distinguished triangles $d : B \to C \to A \to B[1]$ such that $A,B,C \in A$ (and we will say that $d$ is a triangle from $A$). Note that every extension closed subcategory $A$ of $T$ is closed with respect to finite coproducts. We will use the following generalization of the notion of proper class introduced in [7].

**Definition 1.** Let $T$ be a triangulated category, and $A$ a full subcategory of $T$ which is closed under extensions. A class of triangles $\mathcal{E} \subseteq D_A$ is a weak proper class of triangles if

1. $\mathcal{E}$ is closed with respect to isomorphisms of triangles, coproducts and contains all split triangles,
2. $\mathcal{E}$ is closed with respect to base changes (homotopy pullbacks) constructed along homomorphisms from $A$, i.e. if $d : C \to B \to A \to C[1]$ is a triangle in $\mathcal{E}$ and $\alpha : X \to A$ is a homomorphism in $A$ then the top triangle $\varnothing \alpha$ in every homotopy cartesian diagram

$$\varnothing \alpha : \begin{array}{ccc} C & \to & Y & \to & X & \to & C[1] \\ & \parallel & & \parallel & & \alpha \\ & & \downarrow & & \downarrow & & \varnothing \\ & & C & \to & B & \to & A & \to & C[1] \end{array}$$

is in $\mathcal{E}$.
3. $\mathcal{E}$ is closed with respect to cobase changes (homotopy pushouts) constructed along homomorphisms from $A$, i.e. if $d : C \to B \to A \to C[1]$ is a triangle in $\mathcal{E}$ and $\beta : C \to Z$ is a homomorphism in $A$ then the bottom triangle $\beta \varnothing$ in

$$\beta \varnothing : \begin{array}{ccc} C & \to & B & \to & A & \to & C[1] \\ & \parallel & & \parallel & & \varnothing \\ & & Y & \to & X & \to & C[1] \end{array}$$
every homotopy cartesian diagram

\[
\begin{array}{ccc}
C & \rightarrow & B \\
\beta & \downarrow & \beta[1] \\
A & \rightarrow & C[1]
\end{array}
\]

\[
\begin{array}{ccc}
Z & \rightarrow & Y \\
\beta[1] & \downarrow & \beta[1] \\
A & \rightarrow & Z[1]
\end{array}
\]

is in \( \mathcal{E} \).

If \( \mathcal{E} \) is a weak proper class of triangles then a triangle

\[
\sigma : A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\phi} A[1]
\]

which lies in \( \mathcal{E} \) will be called an \( \mathcal{E} \)-triangle. Moreover, we will say that

- \( f \) is an \( \mathcal{E} \)-inflation,
- \( g \) is an \( \mathcal{E} \)-deflation, and
- \( \phi \) is an \( \mathcal{E} \)-phantom.

The class of all \( \mathcal{E} \)-phantoms is denoted by denoted by \( \text{Ph}(\mathcal{E}) \).

Note that a weak proper class \( \mathcal{E} \) as in the definition above depends both on the triangulated category \( \mathcal{T} \) and the full subcategory \( \mathcal{A} \). Therefore, whenever we refer to a weak proper class we assume that it is constructed in an extension closed subcategory \( \mathcal{A} \) of a triangulated category \( \mathcal{T} \).

We present here some standard examples of weak proper classes:

**Example 2.** Let \( \mathcal{A} \) be an extension closed subcategory of a triangulated category. Then \( \mathcal{D}_A \) is a weak proper class of triangles. Moreover, the class \( \mathcal{D}^0_A \) of all splitting triangles from \( \mathcal{D}_A \) is also a weak proper class.

**Example 3.** Let \( \mathcal{A} \) be an abelian category, and we denote by \( \mathbf{D}(\mathcal{A}) \) the derived category associated to \( \mathcal{A} \). Then we can embed \( \mathcal{A} \) in \( \mathbf{D}(\mathcal{A}) \) as a full subcategory closed with respect to extensions by identifying every object \( A \in \mathcal{A} \) with the stalk complex \( A^\ast \) concentrated in 0 such that \( A^0[0] = A \). Then it is obvious that the class \( \mathcal{E} \) of all exact sequences in \( \mathcal{A} \) is a weak proper class. Note that it is possible that the collection of all homomorphisms \( \mathbf{D}(\mathcal{A})(X, Y) \) in \( \mathbf{D}(\mathcal{A}) \) is not a set. We ignore this set theoretic difficulty since in what we do in this paper we can enlarge the universe.

**Example 4.** Recall that a pair of subcategories \( (\mathcal{X}, \mathcal{Y}) \) in \( \mathcal{T} \) is called a torsion theory if \( \mathcal{T}(X, Y) = 0 \) for all \( X \in \mathcal{X} \) and all \( Y \in \mathcal{Y} \) and for every \( A \in \mathcal{T} \) there is a triangle

\[
\sigma : X_A \xrightarrow{i_A} A \xrightarrow{h_A} Y_A \rightarrow X_A[1],
\]

with \( X_A \in \mathcal{X} \) and \( Y_A \in \mathcal{Y} \) (see [21] Definition 2.2). Note that in this definition it is not required any closure of \( \mathcal{X} \) and/or \( \mathcal{Y} \) under shift functors. If \( (\mathcal{X}, \mathcal{Y}) \) is a torsion theory in \( \mathcal{T} \) such that \( \mathcal{X}[1] \subseteq \mathcal{X} \) and \( \mathcal{Y}[-1] \subseteq \mathcal{Y} \) then \((\mathcal{X}, \mathcal{Y}[1])\) is a t-structure in the sense of [6] (see also [1]).

Generalizing the previous example, let \( (\mathcal{X}, \mathcal{Y}) \) be a t-structure in \( \mathcal{T} \). Then the heart of this t-structure is, by definition, \( \mathcal{A} = \mathcal{X} \cap \mathcal{Y} \). By [6], the full subcategory \( \mathcal{A} \) of \( \mathcal{T} \) is abelian. Every short exact sequence \( 0 \rightarrow C \rightarrow B \rightarrow A \rightarrow 0 \) in \( \mathcal{A} \) induces a triangle \( C \rightarrow B \rightarrow A \rightarrow C[1] \) in \( \mathcal{T} \), and the class of all such triangles is a weak proper class in \( \mathcal{A} \).
Example 5. If $\mathcal{A}$ is an exact category in the sense of Quillen then it may be embedded as an extension closed subcategory $\mathcal{A} \subseteq \mathcal{A}'$ of an abelian category. A sequence of composable homomorphisms $C \to B \to A$ from $\mathcal{A}$ is a conflation in $\mathcal{A}$ if and only if it determines a short exact sequence in $\mathcal{A}'$. Hence every conflation in $\mathcal{A}$ induces a triangle $C \to B \to A \to C[1]$ in $D(\mathcal{A}')$. Since the class of all conflations is closed under base and cobase changes, it follows that the class of all triangles constructed as above is a weak proper class in the subcategory $\mathcal{A}$ of $D(\mathcal{A}')$.

Lemma 6. Let $\mathcal{E}$ be a weak proper class of triangles, and let $f : B \to C$ and $g : C \to Y$ be homomorphisms in $\mathcal{A}$.

(1) If $f : B \to C$ is a $D\mathcal{A}$-inflation and $gf$ is an $\mathcal{E}$-inflation then $f$ is an $\mathcal{E}$-inflation. 

(2) If $g$ is a $D\mathcal{A}$-deflation and $gf$ an $\mathcal{E}$-deflation then $g$ is an $\mathcal{E}$-deflation.

Proof. It is enough to prove (1). Consider the triangles

$$B \xrightarrow{f} C \to A \to B[1]$$

and

$$B \xrightarrow{gf} Y \to X \to B[1],$$

and we observe that they are in $D\mathcal{A}$. Then the pair $(1_B, g)$ induces a homomorphism of triangles. The conclusion follows from the fact that $\mathcal{E}$ is closed with respect to base changes. □

2.2. Ideals and phantom ideals. Recall that an ideal in $\mathcal{A}$ is a class of homomorphisms in $\mathcal{A} \to$ which is closed with respect to sums of homomorphisms, and for every chain of composable homomorphisms $A \xrightarrow{f} B \xrightarrow{i} C \xrightarrow{g} D$ in $\mathcal{A}$, if $i \in \mathcal{I}$ then $gif \in \mathcal{I}$.

Definition 7. A class of homomorphisms $\mathcal{E}$ in $\mathcal{T}$ is called a phantom $\mathcal{A}$-ideal if

(i) $\mathcal{E} \subseteq \text{Hom}(\mathcal{A}, \mathcal{A}[1]) = \bigcup_{A,B \in \mathcal{A}} \text{Hom}(A, B[1])$,

(ii) $\mathcal{E}$ is closed with respect to sums of homomorphisms, and

(iii) $\mathcal{A} \to [1]\mathcal{E}\mathcal{A} \subseteq \mathcal{E}$, i.e. for every chain of composable homomorphisms

$$A \xrightarrow{f} B \xrightarrow{i} C \xrightarrow{g} D[1]$$

in $\mathcal{T}$ such that $i \in \mathcal{E}$ and $f, g \in \mathcal{A} \to$ we have $gif \in \mathcal{E}$.

Remark 8. Since we work in additive categories, as in the case of ideals, we can replace the condition (ii) in the definition of phantom $\mathcal{A}$-ideals by

(ii') $\mathcal{E}$ is closed with respect to finite direct sums of homomorphisms.

Remark 9. If $\mathcal{A}[1] = \mathcal{A}$, in particular for $\mathcal{A} = \mathcal{T}$, then a class of homomorphisms $\mathcal{I}$ is a phantom $\mathcal{A}$-ideal if and only if it is an ideal.

Example 10. For a class $\mathcal{X}$ of objects in $\mathcal{A}$ which is closed with respect finite direct sums we put

$$\text{Ideal}(\mathcal{X}) = \{ i \in \mathcal{A} \to \mid i \text{ factors through an object } X \in \mathcal{X} \}. $$

It is not hard to see that Ideal($\mathcal{X}$) is an ideal, and it is called the object ideal associated with $\mathcal{X}$.

Conversely, for every ideal $\mathcal{I}$ in $\mathcal{A}$ we construct the class of objects of $\mathcal{I}$ by:

$$\text{Ob}(\mathcal{I}) = \{ X \in \mathcal{A} \mid 1_X \in \mathcal{I} \}. $$
Obviously we have $\text{Ob}(\text{Ideal}(\mathcal{X})) = \mathcal{X}$ and $\text{Ideal}(\text{Ob}(\mathcal{I})) \subseteq \mathcal{I}$. An ideal is an object ideal if and only if this last inclusion is an equality.

As in [7, Section 2.4] we can apply Baer’s theory techniques to weak proper classes of triangles: Two triangles $\vartheta$ and $\vartheta'$ as in the next commutative diagram are equivalent if there is a homomorphism of triangles of the form:

\[
\begin{array}{ccc}
\vartheta : & C & \rightarrow & B' & \rightarrow & A & \rightarrow & C[1] \\
\vartheta' : & C & \rightarrow & B & \rightarrow & A & \rightarrow & C[1].
\end{array}
\]

In this case we know that $\beta$ has to be an isomorphism, and we defined an equivalence relation on the class of all triangles starting in $C$ and ending in $A$. Using base and cobase changes we can define a sum on the set $\mathcal{E}(A, C)$ of all such triangles modulo this equivalence, and we have an additive bifunctor:

\[
\mathcal{E}(-, -) : A^{\text{op}} \times A \rightarrow \mathbb{A}b
\]

which associate to each pair $(A, C)$ of object from $\mathcal{A}$ the group $\mathcal{E}(A, C)$ of all $\mathcal{E}$-triangles $B \rightarrow C \rightarrow A \rightarrow B[1]$ modulo equivalence of triangles. It is not hard to see that

\[
\text{Ph} : \mathcal{E}(-, -) \rightarrow \text{Ph}(\mathcal{E})(-, -[1])
\]

is an isomorphism of bifunctors.

In fact, as in the case of proper classes studied in [7], there is an 1-to-1 correspondence between weak proper classes and phantom $\mathcal{A}$-ideals. This is described in the following proposition, whose proof is a simple exercise.

**Proposition 11.** Let $\mathcal{T}$ be a triangulated category, and $\mathcal{A}$ a full subcategory which is closed under extensions. The following are equivalent for class $\mathcal{E} \subseteq \mathcal{D}_\mathcal{A}$ of triangles in $\mathcal{A}$ which is closed with respect to isomorphisms:

(a) $\mathcal{E}$ is a weak proper class of triangles;

(b) $\mathcal{A} \rightarrow [1] \text{Ph}(\mathcal{E}), \mathcal{A} \rightarrow \subseteq \text{Ph}(\mathcal{E})$ is closed with respect (direct) sums of homomorphisms.

Consequently,

(i) If $\mathcal{E}$ is a weak proper class of triangles from $\mathcal{A}$ then $\text{Ph}(\mathcal{E})$ is a phantom $\mathcal{A}$-ideal, and

(ii) for every phantom $\mathcal{A}$-ideal $\mathcal{I}$ the class

\[
\mathcal{D}(\mathcal{I}) = \{ \vartheta \in \mathcal{D} \mid \text{Ph}(\vartheta) \in \mathcal{I} \}
\]

is a weak proper class of triangles.

We already noticed that phantom $\mathcal{A}$-ideals and ideals in $\mathcal{A}$ are different notions, unless $\mathcal{A} = \mathcal{A}[1]$. For avoiding confusions, and having in mind the above correspondence, we prefer to work with weak proper class of triangles instead phantom $\mathcal{A}$-ideals, whenever this is possible. However we kept the notion of phantom ideals because the particular situation in which they coincide to ideals is a motivating example (see [7]).

**Remark 12.** (Base-cobase and cobase-base changes) Let $\mathcal{E}$ be a weak proper class of triangles and let $\vartheta : C \rightarrow B \rightarrow A \overset{\alpha}{\rightarrow} C[1]$ be a triangle in $\mathcal{E}$. If $\alpha : X \rightarrow A$ and $\beta : C \rightarrow Y$ are two homomorphisms, we can construct a triangle $\vartheta \alpha$ as a homotopy
pullback of $d$ along $\alpha$, and then a triangle $\beta(d\alpha)$ as a homotopy pushout of $d\alpha$ along $\beta$. We can also construct a triangle $\beta d$ as a homotopy pushout, and then $(\beta d)\alpha$ as a homotopy pullback. It is easy to see that

$$\text{Ph}(\beta(d\alpha)) = \text{Ph}((\beta d)\alpha) = \beta[1]\varphi\alpha,$$

hence the triangles $\beta(d\alpha)$ and $(\beta d)\alpha$ are equivalent.

A weak proper class $E$ of triangles is saturated provided that it satisfies one of the equivalent conditions in the following:

**Lemma 13.** Let $A$ be a full subcategory closed under extensions of a triangulated category $\mathcal{T}$. The following are equivalent for a weak proper class of triangles $E \subseteq \mathcal{D}_A$:

(a) If $A,C,Y \in A$, $i : C \to Y$ is an $E$-inflation and $\phi : A \to C[1]$, then $i[1]\phi \in \text{Ph}(E)$ implies $\phi \in \text{Ph}(E)$.

(b) If the commutative diagram

$$
\begin{array}{c}
\vspace{1em}
\phi : C \to B \to A \to C[1] \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\text{id} : Y \to Z \to A \to Y[1]
\end{array}
$$

is obtained from the triangle $d \in \mathcal{D}_A$ by a cobase change along an $E$-inflation $i$, such that the bottom triangle is in $E$, then the top triangle $\phi$ lies also in $E$.

(c) If $A,C,X \in A$, $p : X \to A$ is an $E$-deflation and $\phi : A \to C[1]$ then $\phi p \in \text{Ph}(E)$ implies $\phi \in \text{Ph}(E)$.

(d) If the commutative diagram

$$
\begin{array}{c}
\vspace{1em}
\phi p : C \to Y \to X \to C[1] \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\phi : C \to B \to A \to C[1]
\end{array}
$$

is obtained from the bottom triangle $d \in \mathcal{D}_A$ by base change along an $E$-deflation $p$, such that the top triangle $\phi p$ is in $E$, then $\phi \in E$.

**Proof.** The equivalences (a)$\iff$(b) and (c)$\iff$(d) are obvious. Moreover, (a)$\Rightarrow$(c) and (c)$\Rightarrow$(a) are dual to each other, so it is enough to prove (a)$\Rightarrow$(c).

Let $p : X \to A$ be an $E$-deflation and let $\phi : A \to C[1]$ be a map such that $A,C,X \in A$ and $\phi p \in \text{Ph}(E)$. Completing both $p$ and $\phi p$ to triangles we obtain the following commutative diagram:

$$
\begin{array}{c}
\vspace{1em}
B \to X \to A \to C[1] \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
C \to Y \to X \to C[1] \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
C \to Y \to X \to C[1] \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
B[1] \to A \to X \to C[1] \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
B[1] \to A \to X \to C[1] \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
B[1] \to A \to X \to C[1]
\end{array}
$$

By hypothesis, $\psi \in \text{Ph}(E)$, $i$ is an $E$-inflation and $i[1]\phi \in \text{Ph}(E)$. Then (a) implies $\phi \in \text{Ph}(E)$. \qed

**Definition 14.** A proper class of triangles is a weak proper class which is saturated and closed under all suspensions.
Example 15. The proper classes of triangles for the case $A = T$ are studied in [7]. We mention here a basic example: If $\mathcal{H}$ is a class of objects in $T$ such that $\mathcal{H}[1] = \mathcal{H}$ then the class $\mathcal{E}_\mathcal{H}$ of all triangles $A \to B \to C \to A[1]$ such that the sequences of abelian groups
\[ 0 \to T(H, A) \to T(H, B) \to T(H, C) \to 0 \]
are exact for all $H \in \mathcal{H}$ is a saturated weak proper class of triangles. It is easy to see that
\[ \text{Ph}(\mathcal{E}_\mathcal{H}) = \{ f \mid T(H, f) = 0 \text{ for all } H \in \mathcal{H} \}. \]
Since $\mathcal{H}$ is closed under suspensions (i.e. $\mathcal{H}[1] = \mathcal{H}$) then $\text{Ph}(\mathcal{E}_\mathcal{H})$ is also closed under suspensions.

In particular, we mention here the case when $T$ is compactly generated and $\mathcal{H}$ is the class of all compact objects in $T$. Then $\text{Ph}(\mathcal{E}_\mathcal{H})$ is the class of classical phantoms maps (the maps $\phi$ for which $T(H, \phi) = 0$ for all compacts $H \in T$, [20]). Actually this example motivates the name ‘phantom’ chosen for $\text{Ph}(\mathcal{E})$.

Example 16. Let $\mathcal{B}$ be a class of objects in $T$ which is closed with respect to finite direct sums, and let $\mathcal{E}$ be a weak proper class in $A$. Then $\mathcal{B}$ induces a weak proper subclass $\mathcal{E}_\mathcal{B}$ of $\mathcal{E}$ defined by the condition
\[ \text{Ph}(\mathcal{E}_\mathcal{B}) = \{ \varphi \in \text{Ph}(\mathcal{E}) \mid \varphi \text{ factorizes through an object } X \in \mathcal{B} \}. \]

As a particular example we mention that the ideal $I$ used in [20, Theorem A] can be viewed as a phantom $A$-ideal: If $T$ is a compactly generated triangulated category and $T_0$ is the full subcategory of all compact objects in $T$ then every class $\mathcal{B}$ of objects in $T$ which is closed with respect to direct sums induces an ideal $I_\mathcal{B}$ in $T_0$ which consists in all homomorphisms between compact objects which factorize through objects from $\mathcal{B}$.

The next proposition shows that the composition of two inflations (deflations) must be an inflation (respectively a deflation), provided that the weak proper class is saturated. Therefore saturated weak proper classes satisfy all triangulated versions for the axioms of exact categories (see for example [10, Definition 2.1]).

Proposition 17. If $\mathcal{E}$ is a saturated weak proper class of triangles then
(1) The composition of two $\mathcal{E}$-inflations is an $\mathcal{E}$-inflation.
(2) The composition of two $\mathcal{E}$-deflations is an $\mathcal{E}$-deflation.

Proof. Let $f : A \to B$ and $g : B \to C$ two $\mathcal{E}$-inflations. Completing them to triangles and using the octahedral axiom we construct the commutative diagram, whose rows and columns are triangles:
\[
\begin{array}{cc}
C[-1] & C[-1] \\
\downarrow & \downarrow \\
X[-1] & Y[-1] & \downarrow & Z[-1] & \downarrow & X \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
-\psi[-1] & -\psi[-1] & \downarrow & -\phi[-1] & \downarrow & \downarrow \\
X[-1] & A & \downarrow & B & \downarrow & X \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
C & C
\end{array}
\]
We have $f \psi[-1] = \phi[-1] h[-1] \in \text{Ph}(\mathcal{E})[-1]$ and saturation gives us $\psi \in \text{Ph}(\mathcal{E})$ since $f$ is an $\mathcal{E}$-inflation. Therefore $gf$ is an $\mathcal{E}$-inflation too. \qed

2.3. Precovers and preenvelopes. Let $I$ be an ideal in $\mathcal{A}$, and $A$ an object in $\mathcal{A}$. We say that a homomorphism $i : X \to A$ is an $I$-precover for $A$ if $i \in I$ and all homomorphisms $i' : X' \to A$ from $I$ factorize through $i$. Dually, an $I$-preenvelope for an object $B$ in $\mathcal{A}$ is a homomorphism $i : B \to Y$ which lies in $I$ such that every other homomorphism $i' : B \to Y'$ from $I$ factorizes through $i$. The ideal $I$ is a precovers (preenveloping) ideal if every object from $\mathcal{A}$ has an $I$-precover ($I$-preenvelope).

Because the suspension functor is an equivalence we deduce immediately that, for every $n \in \mathbb{Z}$, $i : X \to A$ is an $I$-precover for $A$ if and only if $i[n] : X[n] \to A[n]$ is an $I[n]$-precover for $A[n]$, and a similar statement holds for preenvelopes too.

We extend these notions for phantom $\mathcal{A}$-ideals in the following way: if $\mathcal{E}$ is a phantom $\mathcal{A}$-ideal and $A$ is an object in $\mathcal{A}$, we say that a homomorphism $\phi : X \to A[1]$ is an $\mathcal{E}$-precover for $A[1]$ if $\phi \in \mathcal{E}$ and all homomorphisms $\phi' : X' \to A[1]$ in $\mathcal{E}$ factorize through $\phi$. Dually, an $\mathcal{E}$-preenvelope for an object $B$ in $\mathcal{A}$ is a homomorphism $\phi : B \to Y[1]$ which lies in $\mathcal{E}$ such that every other homomorphism $\phi' : B \to Y'[1]$ from $\mathcal{E}$ factorizes through $\phi$. The phantom $\mathcal{A}$-ideal $\mathcal{E}$ is a precovering (preenveloping) if every object from $\mathcal{A}[1]$ (resp. $\mathcal{A}$) has an $\mathcal{E}$-precover ($\mathcal{E}$-preenvelope).

In the following we will see that precovers (preenvelopes) are strongly connected with some injective (respectively, projective) properties.

Let $\mathcal{E}$ be a weak proper class of triangles in $\mathcal{A}$. We say that a homomorphism $f : X \to A$ from $\mathcal{A}$ is $\mathcal{E}$-projective if $f$ is projective with respect to all triangles in $\mathcal{E}$, i.e. for every triangle $B \to C \to A \xrightarrow{\phi} B[1]$ in $\mathcal{E}$ there is a homomorphism $\mathcal{T} : X \to C$ such that $f = \alpha \mathcal{T}$ ($f$ factorizes through all $\mathcal{E}$-deflations $C \to A$). Dually, $g : C \to Y$ is $\mathcal{E}$-injective if $g$ is injective with respect to all triangles in $\mathcal{E}$, i.e. $f$ factorizes through all $\mathcal{E}$-inflations $B \to C$. We denote by $\mathcal{E}$-proj ($\mathcal{E}$-inj) the class of all $\mathcal{E}$-projective (respectively, $\mathcal{E}$-injective) homomorphisms.

The proof of the following lemma is straightforward.

**Lemma 18.** Let $\mathcal{E}$ be a weak proper class of triangles in $\mathcal{A}$.

(a) A homomorphism $f : X \to A$ from $\mathcal{A}$ is $\mathcal{E}$-projective if and only if $\text{Ph}(\mathcal{E})f = 0$.

(b) A homomorphism $g : B \to Y$ from $\mathcal{A}$ is $\mathcal{E}$-injective if and only if $g[1] \text{Ph}(\mathcal{E}) = 0$.

(c) $\mathcal{E}$-proj and $\mathcal{E}$-inj are ideals in $\mathcal{A}$.

(d) $\mathcal{E}[n]$-proj = $(\mathcal{E}$-proj$)[n]$ and $\mathcal{E}[n]$-inj = $(\mathcal{E}$-inj$)[n]$ for all $n \in \mathbb{Z}$ (where $\mathcal{E}[n]$ is viewed as a weak proper class relative to the full subcategory $\mathcal{A}[n]$).

**Corollary 19.** Let $\mathcal{E}$ be a weak proper class of triangles in in an extension closed subcategory $\mathcal{A}$ of the triangulated category $\mathcal{T}$.

(1) A homomorphism $\alpha : A \to \prod_{i \in I} B_i$ is $\mathcal{E}$-injective if and only if for every $i \in I$ the homomorphism $\pi_i \alpha$ is $\mathcal{E}$-injective ($\pi_i : \prod_{i \in I} B_i \to B_i$ denote the canonical projections).

(2) A homomorphism $\alpha : \bigoplus_{i \in I} B_i \to A$ is $\mathcal{E}$-projective if and only if for every $i \in I$ the homomorphism $\alpha \rho_i$ is $\mathcal{E}$-projective ($\rho_i : B_i \to \bigoplus_{i \in I} B_i$ denote the canonical injections).

*Proof.* This follows from the fact that the family of all canonical projections (injections) associated to a direct product (direct sum) is monomorphic (epimorphic). \qed
The above mentioned connection is presented in the following results:

**Lemma 20.** Let $\mathcal{E}$ be a weak proper class of triangles in in an extension closed subcategory $\mathcal{A}$ of $\mathcal{T}$, and let

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\phi} A[1]$$

be an $\mathcal{E}$-triangle.

1. If $\phi$ is an $\text{Ph}(\mathcal{E})$-precover for $A[1]$ and $d : X \to B$ is a homomorphism such that $gd = 0$ then $d \in \text{E-inj}$. In particular $f$ is an $\mathcal{E}$-inj-preenvelope of $A$. Consequently, the map $\phi$ is an $\text{Ph}(\mathcal{E})$-precover for $A[1]$ if and only if $f$ is $\mathcal{E}$-injective.

2. If $\phi$ is an $\text{Ph}(\mathcal{E})$-preenvelope for $C$ and $d : B \to X$ is a homomorphism such that $df = 0$ then $d \in \mathcal{E}$-proj. In particular $g$ is an $\mathcal{E}$-proj-precover for $C$. Consequently, the map $\phi$ is an $\text{Ph}(\mathcal{E})$-preenvelope for $C$ if and only if $g$ is $\mathcal{E}$-projective.

**Proof.** As usual, it is enough to prove (1).

Let $\psi : Y \to X[1]$ be a homomorphism in $\text{Ph}(\mathcal{E})$. Our initial data consist in the solid part of the following (commutative) diagram:

$$\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
\downarrow{d} & \nearrow{k} & \downarrow{h[1]} \\
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{\phi} & A[1] & \xrightarrow{f[1]} & B[1].
\end{array}$$

Since $gd = 0$ and $f$ is a weak kernel for $g$ we get a factorization $d = fh$ for some homomorphism $h : X \to A$. Now $h[1] \psi : Y \to A[1]$ is in $\text{Ph}(\mathcal{E})$ because $\text{Ph}(\mathcal{E})$ is an phantom $\mathcal{A}$-ideal. Since $\phi$ is an $\text{Ph}(\mathcal{E})$-precover for $A[1]$, we get further a factorization $h[1] \psi = \phi k$, for some $k : Y \to C$. Now $d[1] \psi = f[1] h[1] \psi = f[1] \phi k = 0$. Therefore $d[1] \text{Ph}(\mathcal{E}) = 0$ so $d \in \text{E-inj}$.

Since $gf = 0$ we have $f \in \text{E-inj}$. Moreover, if $f' : A \to B'$ is a homomorphism in $\mathcal{E}$-inj, then $f' \phi [-1] = 0$. Since $f$ is a weak cokernel for $\phi [-1]$, $f'$ has to factor through $f$.

For the last statement, let us observe that if $\phi$ is a precover for $A[1]$ then $f$ is $\mathcal{T}$-injective by what we just proved. Conversely, if $f \in \mathcal{E}$-inj then for every map $\psi : X \to A[1]$ from $\text{Ph}(\mathcal{E})$ we have $f[1] \psi = 0$. But $\phi$ is a weak cokernel for $f[1]$, hence $\psi$ factors through $\phi$. \hfill \square

**Definition 21.** Let $\mathcal{E}$ be a weak proper class in $\mathcal{A}$. We say that there are enough $\mathcal{E}$-injective homomorphisms if for every object $A$ there exists an $\mathcal{E}$-inflation $f : A \to B$ which is $\mathcal{E}$-injective.

Dually, there are enough $\mathcal{E}$-projective homomorphisms if for every object $C$ there exists an $\mathcal{E}$-deflation $g : B \to C$ which is $\mathcal{E}$-projective;

**Theorem 22.** Let $\mathcal{E}$ be a weak proper class of triangles in $\mathcal{A}$.

1. The following are equivalent:
   - (a) there are enough $\mathcal{E}$-injective homomorphisms;
   - (b) $\text{Ph}(\mathcal{E})$ is a precovering phantom $\mathcal{A}$-ideal.

2. The following are equivalent:
   - (a) there are enough $\mathcal{E}$-projective homomorphisms;
(b) \textbf{Ph}(\mathcal{E}) is a preenveloping phantom \mathcal{A}-ideal.}

\textbf{Proof.} (a)⇒(b) Let \(A\) be an object in \(\mathcal{A}\). Using the hypothesis we observe that there exists an \(\mathcal{E}\)-triangle \(A \xrightarrow{f} B \rightarrow C \xrightarrow{\varphi} A[1]\) such that \(f\) is \(\mathcal{E}\)-injective. By (a) and Lemma 20 we conclude that \(\varphi\) is a \(\text{Ph}(\mathcal{E})\)-precover for \(A[1]\).

(b)⇒(a) Suppose that \(\text{Ph}(\mathcal{E})\) is a precovering phantom \(\mathcal{A}\)-ideal. Let \(A\) be an object in \(\mathcal{A}\). If \(\varphi : C \rightarrow A[1]\) is a \(\text{Ph}(\mathcal{E})\)-precover, we consider the \(\mathcal{E}\)-triangle \(A \xrightarrow{f} B \rightarrow C \xrightarrow{\varphi} A[1]\). Using Lemma 20, we conclude that \(f\) is an \(\mathcal{E}\)-injective \(\mathcal{E}\)-inflation. \(\square\)

In the following we will present a method to construct weak proper classes with enough injective/projective homomorphisms. This is an extension of the method presented in \[4, \text{Section 1}\]. We start with an example of a weak proper class which extends Example 15.

\textbf{Example 23.} Let \(I\) be an ideal in \(\mathcal{A}\). Then
\[
\mathcal{E}^I = \{ \varphi \in T(\mathcal{A}, \mathcal{A}[1]) | I[1] \varphi = 0 \}
\]
is a phantom \(\mathcal{A}\)-ideal, hence the class \(\mathcal{E}^I = \mathcal{D}(\mathcal{E}^I)\) is a weak proper class in \(\mathcal{A}\) (Proposition 11). It is easy to see that \(\mathcal{E}^I\) is the class of all triangles \(B \rightarrow C \rightarrow A \rightarrow A[1]\) from \(\mathcal{D}_\mathcal{A}\) with the property that all homomorphisms from \(I\) are injective with respect to these triangles.

Dually, if we consider the phantom \(\mathcal{A}\)-ideal
\[
\mathcal{E}_I = \{ \varphi \in T(\mathcal{A}, \mathcal{A}[1]) | \varphi I = 0 \},
\]
we obtain the weak proper class \(\mathcal{E}_I\) of all triangles \(B \rightarrow C \rightarrow A \rightarrow A[1]\) from \(\mathcal{D}_\mathcal{A}\) with the property that all homomorphisms from \(I\) are projective with respect to these triangles.

\textbf{Proposition 24.} Let \(\mathcal{E}\) be a weak proper class in \(\mathcal{A}\).

1. If there are enough \(\mathcal{E}\)-injective homomorphisms then \(\mathcal{E} = \mathcal{E}_{\mathcal{E}^{\text{inj}}}\).
2. If there are enough \(\mathcal{E}\)-projective homomorphisms then \(\mathcal{E} = \mathcal{E}_{\mathcal{E}^{\text{proj}}}\).

\textbf{Proof.} Let \(I\) be the ideal \(\mathcal{E}^{\text{inj}}\). It is enough to prove the inclusion \(\mathcal{E}^I \subseteq \mathcal{E}\).

Let \(\delta : B \rightarrow C \rightarrow A \rightarrow B[1]\) be a triangle in \(\mathcal{E}^I\). If \(\alpha : B \rightarrow E\) is an \(\mathcal{E}\)-injective \(\mathcal{E}\)-inflation then \(\alpha \in I\), so it is injective with respect to \(\delta\). Therefore, we can construct a commutative diagram
\[
\begin{array}{c}
\delta : & B & \longrightarrow & C & \longrightarrow & A & \longrightarrow & B[1] \\
& \| & \downarrow & \| & \downarrow & \| & \downarrow & \| \\
& B & \longrightarrow & E & \longrightarrow & D & \longrightarrow & B[1],
\end{array}
\]
where the horizontal lines are triangles in \(\mathcal{D}_\mathcal{A}\). Since the below triangle is in \(\mathcal{E}\), it follows that the top triangle is also in \(\mathcal{E}\), and the proof is complete. \(\square\)

It is easy to see that if \(\mathcal{F}\) and \(\mathcal{E}\) are weak proper classes such that \(\mathcal{F} \subseteq \mathcal{E}\) then \(\mathcal{E}^{\text{inj}} \subseteq \mathcal{F}^{\text{inj}}\) (and \(\mathcal{E}^{\text{proj}} \subseteq \mathcal{F}^{\text{proj}}\)). We can use the previous proposition to prove a converse for this implication.

\textbf{Corollary 25.} Let \(\mathcal{E}\) and \(\mathcal{F}\) be weak proper classes in \(\mathcal{A}\).

1. Suppose that there are enough \(\mathcal{E}\)-injective homomorphisms. Then \(\mathcal{F} \subseteq \mathcal{E}\) if and only if \(\mathcal{E}^{\text{inj}} \subseteq \mathcal{F}^{\text{inj}}\).
(2) Suppose that there are enough $E$-projective homomorphisms. Then $\mathcal{F} \subseteq \mathcal{E}$ if and only if $\mathcal{E}$-proj $\subseteq \mathcal{F}$-proj.

Proof. Suppose that $\mathcal{I} = \mathcal{E}$-inj $\subseteq \mathcal{F}$-inj $= \mathcal{J}$. Since we have enough $\mathcal{E}$-injective homomorphisms, we can apply the previous proposition to obtain $\mathcal{F} \subseteq \mathcal{E}$-proj $\subseteq \mathcal{E} = \mathcal{E}$.

Proposition 26. Let $\mathcal{F} \subseteq \mathcal{E}$ be weak proper classes in $\mathcal{A}$.

(1) If there are enough $\mathcal{E}$-injective homomorphisms the following are equivalent:
   (a) there are enough $\mathcal{F}$-injective homomorphisms;
   (b) there exists a preenveloping ideal $\mathcal{I}$ in $\mathcal{A}$ such that $\mathcal{E}$-inj $\subseteq \mathcal{I}$, and $\mathcal{F} = \mathcal{E}$-proj.
   In these conditions $\mathcal{F}$-inj $= \mathcal{F}$.

(2) If there are enough $\mathcal{E}$-projective homomorphisms the following are equivalent:
   (a) there are enough $\mathcal{F}$-projective homomorphisms;
   (b) there exists a precovering ideal $\mathcal{I}$ in $\mathcal{A}$ such that $\mathcal{E}$-proj $\subseteq \mathcal{I}$ and $\mathcal{F} = \mathcal{E}$-inj.
   In these conditions $\mathcal{F}$-proj $= \mathcal{F}$.

Proof. (a)$\Rightarrow$(b) Take $\mathcal{I} = \mathcal{F}$-inj. The conclusion follows from Definition 21 and Proposition 24.

(b)$\Rightarrow$(a) We have to prove that $\mathcal{E}$ has enough injective homomorphisms. Let $A$ be in $\mathcal{A}$. We start with an $\mathcal{E}$-triangle $A \rightarrow E \rightarrow X \rightarrow A[1]$ such that $e$ is $\mathcal{E}$-injective. Let $i : A \rightarrow I$ be an $\mathcal{I}$-preenvelope for $A$. Since $e \in \mathcal{I}$, it factorizes through $i$. Then we have a commutative diagram

$$
\begin{array}{ccc}
A & \rightarrow & I \\
\downarrow & & \downarrow \\
A & \rightarrow & Z
\end{array}
\quad
\begin{array}{ccc}
& Z & \rightarrow A[1] \\
& \downarrow & \downarrow \\
& & A[1]
\end{array}
$$

and using the closure of $\mathcal{E}$ with respect to base changes we conclude that $i$ is an $\mathcal{E}$-inflation. Since $i$ is an $\mathcal{I}$-preenvelope, it is easy to see that the triangle $A \rightarrow I \rightarrow Z \rightarrow A[1]$ is in $\mathcal{E}$, hence $i$ is an $\mathcal{E}$-injective $\mathcal{E}$-inflation.

For the last statement, let us observe that for every $A \in \mathcal{A}$ every $\mathcal{I}$-preenvelope $i : A \rightarrow X$ is a $\mathcal{F}$-inflation. Therefore, every $\mathcal{F}$-injective homomorphism $A \rightarrow X$ factorizes through $i$, hence $\mathcal{F}$-inj $\subseteq \mathcal{I}$. The converse inclusion follows from the equality $\mathcal{F} = \mathcal{E}$.

For further reference we mention here the following particular case:

Example 27. If $\mathcal{A} = \mathcal{T}$ and $\mathcal{E} = \mathcal{D}$ is the class of all triangles in $\mathcal{T}$ then $0$ is the ideal of all $\mathcal{E}$-injective ($\mathcal{E}$-projective) homomorphisms. Since in this case all homomorphisms are $\mathcal{E}$-inflations and $\mathcal{E}$-deflations, it follows that we have enough $\mathcal{E}$-injective homomorphisms and $\mathcal{E}$-projective homomorphisms.

If $\mathcal{I}$ is a preenveloping (precovering) ideal, we consider the weak proper class $\mathcal{E}$ (resp. $\mathcal{E}$) of all triangles $\mathcal{D}$ such that all $i \in \mathcal{I}$ are injective (projective) relative to $\mathcal{D}$. By what we just proved we obtain that $\mathcal{E}$ (resp. $\mathcal{E}$) has enough injective (projective) homomorphisms and $\mathcal{E}$-inj $= \mathcal{I}$ (resp. $\mathcal{E}$-proj $= \mathcal{I}$).

3. Relative ideal cotorsion pairs

In this section we extend the ideal cotorsion theory introduced in [14] to triangulated categories. In order to do this we fix a triangulated category $\mathcal{T}$, a full
subcategory $\mathcal{A}$ which is closed under extensions, and a weak proper class of triangles $\mathcal{E}$ in $\mathcal{A}$.

3.1. Two Galois correspondences. We will construct here two Galois correspondences between ideals in $\mathcal{A}$ and weak proper classes in $\mathcal{A}$. In order to do this we will generalize Herzog’s construction of phantoms with respect to pure exact sequences, [17].

Definition 28. Let $\mathcal{F}$ be weak proper class of triangles in $\mathcal{A}$ such that $\mathcal{F} \subseteq \mathcal{E}$. A map $\phi : X \rightarrow A$ from $\mathcal{A}$ is called relative $\mathcal{F}$-phantom (with respect to $\mathcal{E}$), if $h\phi \in \text{Ph}(\mathcal{F})$, whenever $h \in \text{Ph}(\mathcal{E})$. We denote by

$$\Phi_{\mathcal{E}}(\mathcal{F}) = \{ \phi \mid h\phi \in \text{Ph}(\mathcal{F}) \text{ for all } h \in \text{Ph}(\mathcal{E}) \}$$

the class of all relative $\mathcal{F}$-phantom with respect to $\mathcal{E}$.

Dually, a map $\psi : A \rightarrow X$ from $\mathcal{A}$ is called relative $\mathcal{F}$-cophantom (with respect to $\mathcal{E}$), if $\psi h \in \text{Ph}(\mathcal{F})^{-1}$, whenever $h \in \text{Ph}(\mathcal{E})^{-1}$. We denote by

$$\Psi_{\mathcal{E}}(\mathcal{F}) = \{ \psi \mid \psi h \in \text{Ph}(\mathcal{F})^{-1} \text{ for all } h \in \text{Ph}(\mathcal{E})^{-1} \}$$

the class of all relative $\mathcal{F}$-cophantom with respect to $\mathcal{E}$.

The proof of the following lemma is straightforward.

Lemma 29. If $\mathcal{F}$ is a weak proper class of triangles and $\mathcal{F} \subseteq \mathcal{E}$ then $\Phi_{\mathcal{E}}(\mathcal{F})$ and $\Psi_{\mathcal{E}}(\mathcal{F})$ are ideals in $\mathcal{A}$.

Remark 30. a) Informally a map $\phi : X \rightarrow A$ belongs to $\Phi_{\mathcal{E}}(\mathcal{F})$ if and only if for every base change along $\phi$ of a triangle in $\mathcal{E}$,

$$\begin{array}{ccc}
Y & \to & Z \to X \to Y[1] \\
\downarrow & & \downarrow \phi \\
Y & \to & C \to A \to Y[1],
\end{array}$$

the top triangle is in $\mathcal{F}$.

b) Dually, a map $\psi : A \rightarrow X$ belongs to $\Psi_{\mathcal{E}}(\mathcal{F})$ if and only if for every cobase change along $\psi$ of a triangle in $\mathcal{E}$,

$$\begin{array}{ccc}
X & \to & Y \to Z \to X[1] \\
\psi & & \psi[1] \\
A & \to & C \to Z \to A[1],
\end{array}$$

the bottom triangle is in $\mathcal{F}$.

Remark that relative $\mathcal{F}$-phantoms and cophantoms with respect to $\mathcal{E}$ are ideals in $\mathcal{A}$ whereas $\text{Ph}(\mathcal{E})$ is a phantom $\mathcal{A}$-ideal. However, as noticed, if $\mathcal{A} = \mathcal{T}$ there is no difference between phantom $\mathcal{A}$-ideals and ideals in $\mathcal{A}$. Moreover, in this case $\text{Ph}(\mathcal{E})$ may be seen as a particular case of a relative phantom ideal, more precisely, $\text{Ph}(\mathcal{E}) = \Phi_{\mathcal{D}}(\mathcal{E})$, where $\mathcal{D}$ is the (proper) class of all triangles in $\mathcal{T}$.

Example 31. a) Let $\mathcal{A}$ be an exact idemsplit category, and embed it in a triangulated category $\mathcal{T} = \text{D}(\mathcal{A}')$, where $\mathcal{A}'$ is an abelian category containing $\mathcal{A}$ as an extension closed subcategory (see Example [5]). Then the class of all conflations in $\mathcal{A}$ yields to a weak proper class of triangles in $\mathcal{T}$, denoted $\mathcal{E}$. If we consider an exact
substructure of $\mathcal{A}$ then conflations in this substructure are short exact sequences in $\mathcal{A}'$, so they also lead to a weak proper class of triangles in $\mathcal{A}$, denoted by $\mathfrak{F}$. Then $\Phi_\mathcal{E}(\mathfrak{F})$ and $\Psi_\mathcal{E}(\mathfrak{F})$ are exactly the class of phantom respectively cophantom maps considered in [14] and [15].

b) In the case when $\mathfrak{F}$ is the class of all splitting triangles, that is $\mathfrak{F} = \mathfrak{D}_0$, we have $\Phi_\mathcal{E}(\mathfrak{D}_0) = 0$, so $\Phi_\mathcal{E}(\mathfrak{D}_0) = \mathcal{E}$-Proj. Dually $\Psi_\mathcal{E}(\mathfrak{D}_0) = \mathcal{E}$-Inj.

c) If $\mathcal{E} = \mathfrak{D}$ is the class of all triangles in $\mathcal{T}$ then $\Phi_\mathcal{D}(\mathfrak{F}) = \Phi_\mathcal{E}(\mathfrak{F}) = \Psi_\mathcal{D}(\mathfrak{F})$.

In order to going back, from ideals to weak proper classes we consider the following:

**Definition 32.** To every ideal $\mathcal{I}$ in $\mathcal{A}$ we associate the class $\text{PB}(\mathcal{I}) = \text{Ph}(\mathcal{E})\mathcal{I}$ of all homomorphisms which are phantoms for those triangles obtained as homotopy pullbacks of triangles in $\mathcal{E}$ along maps from $\mathcal{I}$. This is a phantom $\mathcal{A}$-ideal (see Lemma 33), called the pullback phantom $\mathcal{A}$-ideal associated to $\mathcal{I}$, and the class of triangles $\mathcal{P}\Phi_\mathcal{E}(\mathcal{I}) = \mathcal{D}(\text{PB}(\mathcal{I}))$ is called the pullback weak proper class associated to $\mathcal{I}$. We write also $\text{PB}_\mathcal{E}(\mathcal{I})$ when we want to emphasize the dependence on $\mathcal{E}$.

Dually, to every ideal $\mathcal{J}$ in $\mathcal{A}$ we will consider the class $\text{PO}(\mathcal{J}) = \mathcal{J}[1] \text{Ph}(\mathcal{E})$, the pushout phantom $\mathcal{A}$-ideal associated to $\mathcal{J}$ and the class of triangles $\mathcal{P}\Psi_\mathcal{E}(\mathcal{J}) = \mathcal{D}(\text{PO}(\mathcal{J}))$, the pushout weak proper class associated to $\mathcal{J}$.

**Lemma 33.** (1) If $\mathcal{I}$ is an ideal then $\text{PB}(\mathcal{I})$ is a phantom $\mathcal{A}$-ideal, hence $\mathcal{P}\Phi_\mathcal{E}(\mathcal{I}) \subseteq \mathcal{E}$ is a weak proper class in $\mathcal{A}$.

(2) If $\mathcal{J}$ is an ideal then $\text{PO}(\mathcal{J})$ is a phantom $\mathcal{A}$-ideal, hence $\mathcal{P}\Psi_\mathcal{E}(\mathcal{J}) \subseteq \mathcal{E}$ is a weak proper class in $\mathcal{A}$.

**Proof.** (1) It is easy to see that $\alpha[1] \text{PB}(\mathcal{I}) \beta \subseteq \text{PB}(\mathcal{I})$ for all $\alpha, \beta \in \mathcal{A}^-$. It is enough to prove that $\text{PB}(\mathcal{I})$ is closed with respect to finite direct sums of homomorphisms. But this is obvious since both $\text{Ph}(\mathcal{E})$ and $\mathcal{I}$ are closed with respect to finite direct sums.

In fact we obtain in this way two Galois correspondences:

**Theorem 34.** Let $\mathcal{T}$ be a triangulated category. We fix a full subcategory $\mathcal{A}$ which is closed under extensions, and a weak proper class of triangles $\mathcal{E}$ in $\mathcal{A}$. The pairs of correspondences

$$\mathcal{P}\Phi_\mathcal{E} : \text{Ideals}(\mathcal{A}) \rightleftarrows \text{WSub}(\mathcal{E}) : \Phi_\mathcal{E},$$

respectively

$$\mathcal{P}\Psi_\mathcal{E} : \text{Ideals}(\mathcal{A}) \rightleftarrows \text{WSub}(\mathcal{E}) : \Psi_\mathcal{E},$$

between the class Ideals of all ideals in $\mathcal{A}$ and the class WSub($\mathcal{E}$) of all weak proper subclasses of $\mathcal{E}$, determine two monotone Galois connections with respect to inclusion.

**Proof.** Let $\mathcal{I}$ be an ideal in $\mathcal{A}$ and let $\mathfrak{F}$ be a weak proper subclass of $\mathcal{E}$. We have to prove that $\mathcal{P}\Phi_\mathcal{E}(\mathcal{I}) \subseteq \mathfrak{F}$ if and only if $\mathcal{I} \subseteq \Phi_\mathcal{E}(\mathfrak{F})$.

The inclusion

$$\mathcal{I} \subseteq \Phi_\mathcal{E}(\mathfrak{F}) = \{ \phi \mid h\phi \in \text{Ph}(\mathfrak{F}) \text{ for all } h \in \text{Ph}(\mathcal{E}) \}$$

is equivalent to $\text{Ph}(\mathcal{E})\mathcal{I} \subseteq \text{Ph}(\mathfrak{F})$. Since $\text{Ph}(\mathcal{E})\mathcal{I} = \text{PB}(\mathcal{I})$, the last inclusion is equivalent to $\mathcal{P}\Phi_\mathcal{E}(\mathcal{I}) \subseteq \mathfrak{F}$.

The proof for the second pair is similar. \qed
Using the standard properties of Galois connections we have the following

**Corollary 35.** If $\mathfrak{H}$ is a weak proper subclass of $\mathfrak{C}$ and $\mathcal{I}$ is an ideal in $\mathfrak{A}$ then:

1. $\mathfrak{P}\mathfrak{B}_{\mathfrak{e}}(\Phi_{\mathfrak{e}}(\mathfrak{H})) \subseteq \mathfrak{H}$ and $\mathcal{I} \subseteq \Phi_{\mathfrak{e}}(\mathfrak{P}\mathfrak{B}_{\mathfrak{e}}(\mathcal{I}));$
2. $\mathfrak{P}\mathfrak{D}_{\mathfrak{e}}(\Psi_{\mathfrak{e}}(\mathfrak{H})) \subseteq \mathfrak{H}$ and $\mathcal{I} \subseteq \Psi_{\mathfrak{e}}(\mathfrak{P}\mathfrak{D}_{\mathfrak{e}}(\mathcal{I})).$

Moreover, for the case when $\mathfrak{H}$ has enough projective homomorphisms we can use the ideal $\mathfrak{H}$-proj to see when an ideal is contained in $\Phi_{\mathfrak{e}}(\mathfrak{H}).$

**Proposition 36.** Let $\mathfrak{H}$ be a weak proper subclass of $\mathfrak{C}$ and $\mathcal{I}$ is an ideal in $\mathfrak{A}.$

1. If there exist enough $\mathfrak{H}$-projective homomorphisms, the following are equivalent:
   - (a) $\mathcal{I} \subseteq \Phi_{\mathfrak{e}}(\mathfrak{H});$
   - (b) $\mathcal{I}(\mathfrak{H}$-proj) $\subseteq \mathfrak{H}$-proj.

2. If there exist enough $\mathfrak{H}$-injective homomorphisms, the following are equivalent:
   - (a) $\mathcal{I} \subseteq \Psi_{\mathfrak{e}}(\mathfrak{H});$
   - (b) $\mathcal{I}(\mathfrak{H}$-inj) $\subseteq \mathfrak{H}$-inj.

**Proof.** (a) $\Rightarrow$ (b) We have $\Phi_{\mathfrak{e}}(\mathfrak{H})(\mathfrak{H}$-proj) $\subseteq \Phi_{\mathfrak{e}}(\mathfrak{H})(\mathfrak{H}$-proj) $= 0,$ hence $\mathcal{I}(\mathfrak{H}$-proj) $\subseteq \mathfrak{H}$-proj.

(b) $\Rightarrow$ (a) We have to prove that $\mathfrak{PB}_{\mathfrak{e}}(\mathcal{I}) \subseteq \mathfrak{H}.$ By Corollary 35, it is enough to prove that $\mathfrak{H}$-proj $\subseteq \mathfrak{PB}_{\mathfrak{e}}(\mathcal{I})$-proj. In order to obtain this, let us observe that

$$\mathfrak{PB}_{\mathfrak{e}}(\mathcal{I})(\mathfrak{H}$-proj) $\subseteq \Phi_{\mathfrak{e}}(\mathfrak{H})(\mathfrak{H}$-proj) $\subseteq \Phi_{\mathfrak{e}}(\mathfrak{H})(\mathfrak{H}$-proj) $= 0,$

and the proof is complete. \qed

The following result shows us that $\mathfrak{C}$-projective $\mathfrak{C}$-deflations (resp. $\mathfrak{C}$-injective $\mathfrak{C}$-inflations) are test maps for relative $\mathfrak{H}$-phantoms (relative $\mathfrak{H}$-cophantoms).

**Proposition 37.** (1) Let $K \to P \xrightarrow{p} A \xrightarrow{\psi} K[1]$ be a triangle in $\mathfrak{C}$ such that $p$ is $\mathfrak{C}$-projective. A homomorphism $\varphi : X \to A$ is a relative $\mathfrak{H}$-phantom with respect to $\mathfrak{C}$ (i.e. $\varphi \in \Phi_{\mathfrak{e}}(\mathfrak{H})$) if and only if $\psi\varphi \in \Phi_{\mathfrak{e}}(\mathfrak{H}).$

2. Dually, let $A \xrightarrow{\varphi} E \to L \xrightarrow{\psi} A[1]$ be a triangle in $\mathfrak{C}$ such that $e$ is $\mathfrak{C}$-injective. A homomorphism $\varphi : A \to Y$ is a relative $\mathfrak{H}$-cophantom with respect to $\mathfrak{C}$ if and only if $\varphi[1] \psi \in \Phi_{\mathfrak{e}}(\mathfrak{H}).$

**Proof.** Suppose that $\psi\varphi \in \Phi_{\mathfrak{e}}(\mathfrak{H}).$ We have to show that $\zeta\varphi \in \Phi_{\mathfrak{e}}(\mathfrak{H})$ every homomorphism $\zeta : A \to B[1]$ from $\Phi_{\mathfrak{e}}(\mathfrak{C}).$

Let $\zeta : A \to B[1]$ be a homomorphism from $\Phi_{\mathfrak{e}}(\mathfrak{C}).$ Since $p$ is $\mathfrak{C}$-projective, we have $\zeta p = 0,$ hence there exists a map $g[1] : K[1] \to B[1]$ such that $g[1]\psi = \zeta.$ Moreover, we have $\psi\varphi \in \Phi_{\mathfrak{e}}(\mathfrak{H}),$ and it follows that $\zeta \varphi = g[1]\psi\varphi \in \Phi_{\mathfrak{e}}(\mathfrak{H}),$ hence $\varphi \in \Phi_{\mathfrak{e}}(\mathfrak{H}).$

Conversely, if $\varphi \in \Phi_{\mathfrak{e}}(\mathfrak{H})$ then we apply the definition of $\Phi_{\mathfrak{e}}(\mathfrak{H})$ to obtain that $\psi\varphi \in \Phi_{\mathfrak{e}}(\mathfrak{H}).$ \qed

**3.2. Orthogonality.** We say that a homomorphism $f : X \to A$ from $\mathfrak{A}$ is left orthogonal (with respect to $\mathfrak{C}$) to a homomorphism $g : B \to Y$ from $\mathfrak{A},$ and we denote this by $f \bot g,$ if

$$T(f, g[1])(\Phi_{\mathfrak{e}}(\mathfrak{C})) = 0,$$

i.e. for all homomorphisms $\phi : A \to B[1]$ in $\Phi_{\mathfrak{e}}(\mathfrak{C})$ for which the composition makes sens, we have $g[1]\phi f = 0.$ This means that for every triangle $B \to C \to A \xrightarrow{\psi} B[1]$
from $\mathcal{C}$ the triangle obtained by a base-cobase change

$$
\begin{array}{ccc}
B & \longrightarrow & C \\
\downarrow & & \downarrow \\
B & \longrightarrow & C' \\
\downarrow & & \downarrow \\
Y & \longrightarrow & C'' \\
\end{array}
$$

splits.

**Example 38.** (a) If $\mathcal{A}$ is an abelian category, $\mathcal{T} = \mathcal{D}(\mathcal{A})$ and $\mathcal{C}$ is the class of all short exact sequences in $\mathcal{A}$ then $f \perp g$ if and only if $\text{Ext}^1(f, g) = 0$.

(b) If $\mathcal{A}$ is an exact category, $\mathcal{T} = \mathcal{D}(\mathcal{A}')$, where $\mathcal{A}'$ is an abelian category containing $\mathcal{A}$ as an extension closed category, and $\mathcal{C}$ is the class of triangles coming from conflations in $\mathcal{A}$, then $f \perp g$ if and only if $\text{Ext}(f, g) = 0$.

(c) If $\mathcal{A} = \mathcal{T}$, and $\mathcal{C}$ is a weak proper class of triangles in $\mathcal{T}$ then $f \perp g$ means exactly $\text{Ph}(\mathcal{C})(f, g) = 0$, that is $g[1]|\phi f = 0$ for all $\phi \in \text{Ph}(\mathcal{C})$. In particular, if $\mathcal{C} = \mathcal{D}$ is the (proper) class of all triangles in $\mathcal{T}$ then $f \perp g$ iff $\mathcal{T}(f, g[1]) = 0$, that is $g[1]|\phi f = 0$ for all homomorphisms $\phi$ for which the composition makes sense.

**Remark 39.** Let $\mathcal{A}$ be an abelian category, $\mathcal{T} = \mathcal{D}(\mathcal{A})$ and let $\mathcal{C} = \mathcal{D}$ be the class of all triangles in $\mathcal{T}$. Then a homomorphism $f : A_0 \to A_1$ in $\mathcal{A}$ may be interpreted as a complex, so it gives rise to an object $f$ of $\mathcal{D}(\mathcal{A})$. Clearly if $g : B_0 \to B_1$ is another homomorphism in $\mathcal{A}$, and $g$ is the object in $\mathcal{D}(\mathcal{A})$ by the complex induced by $f$ then we may consider the condition $\mathcal{T}_{\mathcal{D}(\mathcal{A})}(f, g[1]) = 0$, that is there is no other map in $\mathcal{D}(\mathcal{A})$ between the two complexes above than 0 (for example such a condition appears in the definition of a (pre)silting object in [2]). We want to warn that this kind of orthogonality is different from ours.

Indeed, for $\mathcal{A} = \text{Ab}$ the category of all abelian groups, let us consider the homomorphism $f : \mathbb{Z} \to \mathbb{Z}$ which is the multiplication by 2. Since $f$ has projective domain it follows easily that $\phi f = 0$ for all $\phi : \mathbb{Z} \to X[1]$, with $X \in \text{Ab}$. In fact since $\phi$ has also projective domain, even $\phi$ vanishes. Therefore $\mathcal{T}(f, f[1]) = 0$, hence $f \perp f$. On the other side, as object in $\mathcal{D}(\text{Ab})$, $f$ is a bounded complex of projectives, so it is homotopically projective too. It follows that the homomorphisms in $\mathcal{D}(\text{Ab})$ starting in $f$ are exactly homotopy classes of homomorphisms of complexes. But for every two homomorphisms of abelian groups $s_0, s_1$ as in the diagram:

$$
\begin{array}{ccc}
f : & \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \cdots \\
& \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\
& s_0 & \longrightarrow & f & \longrightarrow & s_1 & \longrightarrow & \cdots \\
\end{array}
$$

we have $\text{im}(fs_0 + s_1 f) \subseteq 2\mathbb{Z}$ showing that the homomorphism of complexes which is the identity map in degree 0 and 0 otherwise is not homotopic to 0, so $\mathcal{T}(f, f[1]) \neq 0$.

**Lemma 40.** Let $f : X \to A$ and $g : B \to Y$ be two homomorphisms in $\mathcal{A}$. The following are equivalent:

1. $f \perp g$;
(2) Every homomorphism $\phi : A \to B[1]$ from $\text{Ph}(\mathcal{C})$ induces a triangle homomorphism

\[
\begin{array}{c}
Z \xrightarrow{f} X \xrightarrow{g} A \xrightarrow{\phi} Z[1] \\
\downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
Y \xrightarrow{T} B[1] \xrightarrow{g[1]} Y[1].
\end{array}
\]

Proof. Let $\phi : A \to B[1]$ be a homomorphism in $\text{Ph}(\mathcal{C})$. If we complete $f$ and $g$ to triangles above, respectively below, we obtain a diagram

\[
\begin{array}{c}
Z \xrightarrow{f} X \xrightarrow{\phi} A \xrightarrow{\phi} Z[1] \\
\downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
Y \xrightarrow{T} B[1] \xrightarrow{g[1]} Y[1].
\end{array}
\]

Therefore, $g[1] \phi f = 0$ if and only if there exists a homomorphism $X \to T$ such that the square

\[
\begin{array}{c}
X \xrightarrow{f} A \\
\downarrow \hspace{1cm} \downarrow \\
T \xrightarrow{\phi} B[1]
\end{array}
\]

is commutative. □

Let $\mathcal{M}$ be a class of maps in $\mathcal{A}$. We define

\[
\mathcal{M}^\perp = \{g \in A^\rightarrow \mid m \perp g \text{ for all } m \in \mathcal{M}\},
\]

\[
\perp \mathcal{M} = \{g \in A^\rightarrow \mid g \perp m \text{ for all } m \in \mathcal{M}\}.
\]

The proof of the next lemma is straightforward:

Lemma 41. Let $\mathcal{M}$ be a class of homomorphisms in $\mathcal{A}$. Then

1. $\mathcal{M}^\perp$ and $\perp \mathcal{M}$ are ideals in $\mathcal{A}$.
2. $\mathcal{M}^\perp[n] = (\mathcal{M}[n])^\perp$ and $\perp \mathcal{M}[n] = \perp(\mathcal{M}[n])$ for all $n \in \mathbb{Z}$, where the ideals $(\mathcal{M}[n])^\perp$ and $\perp(\mathcal{M}[n])$ of $\mathcal{A}[n]$ are computed with respect to $\mathcal{E}[n]$.

Definition 42. A pair of ideals $(\mathcal{I}, \mathcal{J})$ from $\mathcal{A}$ is orthogonal if $i \perp j$ for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$, i.e. $\mathcal{J} \subseteq \mathcal{I}^\perp$ and $\mathcal{I} \subseteq \perp \mathcal{J}$.

The following results exhibit connections between orthogonal ideals and some injective/projective properties.

Proposition 43. (1) If $\mathcal{I}$ is an ideal in $\mathcal{A}$ then $\mathcal{I}^\perp = \mathcal{P}\mathcal{B}_\mathcal{E}(\mathcal{I})\text{-inj}$.

(2) If $\mathcal{J}$ is an ideal in $\mathcal{A}$ then $\perp \mathcal{J} = \mathcal{P}\mathcal{O}_\mathcal{E}(\mathcal{J})\text{-proj}$.

Proof. A homomorphism $j : A \to U$ is in $\mathcal{I}^\perp$ if and only if $j[1] \text{Ph}(\mathcal{E})\mathcal{I} = 0$. But $\text{Ph}(\mathcal{E})\mathcal{I} = \text{PB}(\mathcal{I})$, and we apply Lemma 18 to obtain the conclusion. □

Theorem 44. Let $\mathfrak{F}$ be a weak proper subclass of $\mathcal{E}$.

1. (a) The pair $(\Phi_\mathcal{E}(\mathfrak{F}), \mathfrak{F}\text{-inj})$ is orthogonal.

(b) If there are enough $\mathfrak{F}$-injective homomorphisms then $\mathfrak{F}^\perp\text{-inj} = \Phi_\mathcal{E}(\mathfrak{F})$.

2. (a) The pair $(\mathfrak{F}\text{-proj}, \Psi_\mathcal{E}(\mathfrak{F}))$ is orthogonal.
(b) If there are enough $\mathcal{F}$-projective homomorphisms then $\mathcal{F}\text{-proj}^\perp = \Psi E(\mathcal{F})$.

Proof. (a) Let $e : B \to Y$ be an $\mathcal{F}$-injective homomorphism, $f : X \to A \in \Phi E(\mathcal{F})$, and $\varphi : A \to B[1] \in \text{Ph}(\mathcal{E})$. We have $\varphi f \in \text{Ph}(\mathcal{F})$, hence $e[1]\varphi f = 0$ since $e$ is $\mathcal{F}$-injective. Then $f \perp e$.

(b) By (a), it is enough to show that $\perp \mathcal{F}\text{-inj} \subseteq \Phi E(\mathcal{F})$. In order to do this, let us consider $f : X \to A$ a map from $\perp \mathcal{F}\text{-inj}$ and $\varphi : A \to B[1]$ from $\text{Ph}(\mathcal{E})$. Let $e : B \to Y$ be an $\mathcal{F}$-injective $\mathcal{F}$-inflation.

Complete $\varphi$ to a triangle $B \to C \to A \xrightarrow{\varphi} B[1]$ in $\mathcal{E}$, and consider the base-cobase change diagram

$$
\begin{array}{ccc}
B & \longrightarrow & C \\
\downarrow & & \downarrow \\
B & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Z' \\
\end{array}
\begin{array}{ccc}
A & \longrightarrow & B[1] \\
\varphi & & \varphi f \\
\downarrow & & \downarrow \\
0 & \longrightarrow & Y[1] \\
\end{array}
$$

Since $e[1]\varphi f = 0$, it follows that the triangle $Y \to Z' \to X \to Y[1]$ splits. Therefore there exists $z : Z \to Y$ such that $e = zi$. Since $e$ is an $\mathcal{F}$-inflation, by Lemma 6 it follows that $i$ is an $\mathcal{F}$-inflation, hence $\varphi f \in \text{Ph}(\mathcal{F})$. Therefore, $\varphi f \in \text{Ph}(\mathcal{F})$ for all $\varphi \in \text{Ph}(\mathcal{E})$, hence $f \in \Phi E(\mathcal{F})$. $\square$

Corollary 45. (1) If $\mathcal{I}$ is an ideal then

$$
\mathcal{I} \subseteq \Phi E(\text{PPh}(\mathcal{I})) \subseteq \perp (\mathcal{I}^\perp).
$$

(2) If $\mathcal{J}$ is an ideal then

$$
\mathcal{J} \subseteq \Psi E(\text{PO}(\mathcal{J})) \subseteq (\perp \mathcal{J})^\perp.
$$

Proof. The first inclusion is a consequence of Corollary 35.

For the second inclusion, we replace in Theorem 44 the weak proper class $\mathcal{F}$ by $\text{PPh}(\mathcal{I})$, hence we have

$$
\Phi E(\text{PPh}(\mathcal{I})) \subseteq \perp (\text{PPh}(\mathcal{I})\text{-inj}).
$$

By Proposition 43 we have $\text{PPh}(\mathcal{I})\text{-inj} = \mathcal{I}^\perp$, hence $\Phi E(\text{PPh}(\mathcal{I})) \subseteq \perp (\mathcal{I}^\perp)$. $\square$

Corollary 46. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals in $\mathcal{A}$.

(1) If the phantom $\mathcal{A}$-ideal $\text{PB}(\mathcal{I})$ is precovering then $\Phi E(\text{PPh}(\mathcal{I})) = \perp (\mathcal{I}^\perp)$.

(2) If the phantom $\mathcal{A}$-ideal $\text{PO}(\mathcal{J})$ is preenveloping then $\Psi E(\text{PO}(\mathcal{J})) = (\perp \mathcal{J})^\perp$.

Proof. By Theorem 22 we obtain that there are enough $\text{PPh}(\mathcal{I})$-injective homomorphisms. Using Proposition 43 and Theorem 44 we have

$$
\perp (\mathcal{I}^\perp) = \perp (\text{PPh}(\mathcal{I})\text{-inj}) = \Phi E(\text{PPh}(\mathcal{I})),
$$

and the proof is complete. $\square$
3.3. Special precovers and special preenvelopes.

Definition 47. If \( \mathcal{I} \) is an ideal in \( A \), a homomorphism \( i : X \to A \) from \( \mathcal{I} \) is a special \( \mathcal{I} \)-precover (with respect to \( \mathcal{E} \)) if in the corresponding triangle

\[
\begin{align*}
B & \rightarrow X \overset{i}{\rightarrow} A \overset{k}{\rightarrow} B[1] \\
\end{align*}
\]

we have \( k \in (\mathcal{I}^{\perp})[1]\Phi(\mathcal{E}) \), i.e. \( k = j\phi \) for some \( j \in \mathcal{A}^{\rightarrow} \) with \( j\Phi(\mathcal{E})\mathcal{I} = 0 \) and some \( \phi \in \Phi(\mathcal{E}) \). We say that \( \mathcal{I} \) is a special precovering ideal if every object \( A \) in \( \mathcal{T} \) has a special \( \mathcal{I} \)-precover.

Dually, if \( \mathcal{J} \) is an ideal in \( \mathcal{T} \), a homomorphism \( j : B \to Y \) from \( \mathcal{J} \) is a special \( \mathcal{J} \)-preenvelope with respect to \( \mathcal{E} \) if in the corresponding triangle

\[
\begin{align*}
B & \overset{j}{\rightarrow} Y \overset{\ell}{\rightarrow} A \overset{\psi}{\rightarrow} B[1] \\
\end{align*}
\]

we have \( \psi \in \Phi(\mathcal{E})^{\perp}(\mathcal{J}) \), i.e. \( \psi = \phi i \) with \( i \in \mathcal{A}^{\rightarrow} \), \( \phi \in \Phi(\mathcal{E}) \) such that \( \mathcal{J}[1] \Phi(\mathcal{E}) i = 0 \). We say that \( \mathcal{J} \) is a special preenveloping ideal if every object \( A \) in \( \mathcal{A} \) has a special \( \mathcal{J} \)-preenvelope.

Remark 48. A homomorphism \( i : X \to A \) is a special \( \mathcal{I} \)-precover with respect to \( \mathcal{E} \) if there exists a homotopy pushout diagram

\[
\begin{align*}
Y & \longrightarrow Z \longrightarrow A \overset{\phi}{\longrightarrow} Y[1] \\
j & \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad j[1] \\
B & \longrightarrow X \overset{i}{\longrightarrow} A \overset{\psi}{\longrightarrow} B[1] \\
\end{align*}
\]

such that \( j \in \mathcal{I}^{\perp} \) and \( \phi \in \Phi(\mathcal{E}) \).

Dually, a homomorphism \( j : B \to Y \) is a special \( \mathcal{J} \)-preenvelope with respect to \( \mathcal{E} \) if there exists a homotopy pullback diagram

\[
\begin{align*}
B & \overset{j}{\longrightarrow} Y \longrightarrow A \overset{\psi}{\longrightarrow} B[1] \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
B & \longrightarrow Z \longrightarrow X \overset{\phi}{\longrightarrow} B[1] \\
\end{align*}
\]

such that \( i \in \mathcal{J}^{\perp} \) and \( \phi \in \Phi(\mathcal{E}) \).

Observe that in both diagrams (SPC) and (SPE) all horizontal triangles are in \( \mathcal{E} \) (we have automatically \( \psi \in \Phi(\mathcal{E}) \)), hence every special \( \mathcal{I} \)-precover (\( \mathcal{J} \)-preenvelope) is an \( \mathcal{E} \)-deflation (\( \mathcal{E} \)-inflation).

Moreover, we have \( \psi \mathcal{I} = 0 \) in (SPC), respectively \( \mathcal{J}[1] \psi = 0 \) in (SPE). We may see that the terminology of special precover or preenvelope is justified in the sense of the following:

Lemma 49. Let \( \mathcal{I} \) and \( \mathcal{J} \) be ideals.

(1) Every special \( \mathcal{I} \)-precover with respect to \( \mathcal{E} \) is an \( \mathcal{I} \)-precover.

(2) Every special \( \mathcal{J} \)-preenvelope with respect to \( \mathcal{E} \) is a \( \mathcal{J} \)-preenvelope.

Proof. If \( i' : X \to A \) is a map in \( \mathcal{I} \) then in (SPC) we have \( \psi i = j[1]\phi i' = 0 \). Consequently \( i' \) has to factor through \( i \). \qed
Remark 50. If $A = T$ and $E = D$ is the class of all triangles then every precovering ideal is special since every triangle $B \to X \to A \xrightarrow{\psi} B[1]$ can be embedded in a commutative diagram

$$
\begin{array}{cccccc}
A[-1] & \rightarrow & 0 & \rightarrow & A & \rightarrow & A \\
\psi[-1] & \downarrow & & \downarrow & \psi & \downarrow & \\
B & \rightarrow & X & \rightarrow & A & \rightarrow & B[1].
\end{array}
$$

Dually, every preenveloping ideal in $T$ is special with respect to the class $D$ of all triangles in $T$.

The role of special precovers and special preenvelopes is exhibited by the following version of Salce’s Lemma, [16, Lemma 5.20].

Theorem 51. (Salce’s Lemma) Let $I$ and $J$ be ideals in $A$.

1. If there are enough $E$-injective homomorphisms and $I$ is a precovering ideal, then $I^\perp$ is a special preenveloping ideal.
2. If there are enough $E$-projective homomorphisms and $J$ is a preenveloping ideal, then $J^\perp$ is a special precovering ideal.

Proof. It is enough to prove (1).

Consider $A \in A$ and let

$$
(AE) \quad A \xrightarrow{e} E \xrightarrow{\psi} X \xrightarrow{\psi} A[1]
$$

be a triangle such that $e$ is an $E$-inflation which is $E$-injective.

Since $I$ is precovering for $A$ there exists an $I$-precover $i : I \to X$. By cobase change of the triangle $(AE)$ along $i$ we get the commutative diagram

$$
\begin{array}{cccccc}
A & \xrightarrow{a} & Y & \xrightarrow{i} & I & \xrightarrow{\psi} & A[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{e} & E & \xrightarrow{i'} & X & \xrightarrow{\psi} & A[1].
\end{array}
$$

We claim that $a$ is a special $I^\perp$-preenvelope of $A$. In order to prove this, it is enough to show that $a \in I^\perp$ since we have $i \in \perp(I^\perp)$ (see Corollary 43).

By Lemma 20 is follows that $\psi$ is a $\text{Ph}(E)$-precover. Then it is not hard to see that $\psi i$ is a $\text{PB}(I)$-precover: if $\psi' i' \in \text{Ph}(E)I$ with $\psi' \in \text{Ph}(E)$ and $i' \in I$ then $\psi' = \psi \beta$ for some homomorphism $\beta$, and $\beta i' = i \gamma$ for some homomorphism $\gamma$, hence $\psi' i' = \psi i \gamma$. Applying again Lemma 20 we obtain that $a$ is $\text{PB}\text{Ph}(E)$-injective. Therefore $a \in I^\perp$ as a consequence of Proposition 43.

For reader’s convenience we present here a direct proof for the fact that the homomorphism $a$ from the above proof is in $I^\perp$.

Proof. In the subcategory $A[-1]$, the homomorphism $i[-1] : I[-1] \to X[-1]$ is an $I[-1]$-precover for $X[-1]$, and we have the solid part of the following commutative
Let $\kappa : Y \to B$ be a map from $\mathcal{I}$ and let $\varphi : B \to A[1]$ be an $\mathcal{E}$-phantom. Since $e$ is $\mathcal{E}$-injective we have $e\varphi[-1] = 0$, hence we can find a map $\zeta : B[-1] \to X[-1]$ such that $\varphi[-1] = -\psi[-1]\zeta$. Since $\mathcal{I}[-1]$ is an ideal in $A[-1]$, we have $\zeta\kappa[-1] \in \mathcal{I}[-1]$, hence $\zeta\kappa[-1]$ factorizes through the $\mathcal{I}[-1]$-precover $i[-1]$. Therefore $\zeta\kappa[-1] = i[-1]\eta$ for some $\eta : Y[-1] \to I[-1]$. Finally
\[
a\varphi[-1]\kappa[-1] = -a\psi[-1]\zeta\kappa[-1] = -a\psi[-1]i[-1]\eta = a\eta = 0,
\]
hence $a[1]\varphi\kappa = 0$, and the proof is complete.

From the proof of Theorem 51 we obtain the following corollary which will be useful in Section 4.

**Corollary 52.** (1) If $\mathcal{I}$ is an ideal and
\[
\begin{align*}
A & \xrightarrow{a} Y \xrightarrow{\eta} I \xrightarrow{i} A[1] \\
A & \xrightarrow{e} E \xrightarrow{\psi} X \xrightarrow{j} A[1]
\end{align*}
\]
is a commutative diagram in $\mathcal{A}$ such that the horizontal lines are triangles in $\mathcal{E}$, the homomorphism $e$ is $\mathcal{E}$-injective and $i$ is an $\mathcal{I}$-precover for $X$, then the homomorphism $a$ is a special $\mathcal{I}^\perp$-preenvelope for $A$.

(2) If $\mathcal{J}$ is an ideal and
\[
\begin{align*}
X & \xrightarrow{p} P \xrightarrow{j} A \xrightarrow{\eta} X[1] \\
J & \xrightarrow{y} Y \xrightarrow{b} A \xrightarrow{j[1]} J[1]
\end{align*}
\]
is a commutative diagram in $\mathcal{A}$ such that the horizontal lines are triangles in $\mathcal{E}$, $p$ is an $\mathcal{E}$-projective map and $j$ is a $\mathcal{J}$-preenvelope for $X$, then $b$ is a special $\perp\mathcal{J}$-precover.

**Corollary 53.** (1) Let us suppose that there are enough $\mathcal{E}$-projective homomorphisms and there are enough $\mathcal{F}$-injective homomorphisms. Then $\Phi_{\mathcal{E}}(\mathcal{F})$ is a special precovering ideal.

(2) Suppose that there are enough $\mathcal{E}$-injective homomorphisms and there are enough $\mathcal{F}$-projective homomorphisms. Then $\Psi_{\mathcal{E}}(\mathcal{F})$ is a special preenveloping ideal.

**Proof.** By Theorem 44 we know that $\Phi_{\mathcal{E}}(\mathcal{F}) = \perp\mathcal{F}$-inj. But $\mathcal{F}$-inj is a preenveloping ideal, hence we can apply Theorem 51 to obtain the conclusion.
Remark 54. We want to point out that Theorem 51 shows us an important differen-
tiation between orthogonal ideals and orthogonal classes of objects (i.e. ob-
ject ideals). Let us suppose that \( A \) has direct products and there are enough \( \mathcal{E} \) -
projective homomorphisms. If we start with an object \( A \in \mathcal{A} \) then the class \( \text{Prod}(A) \) 
of all direct summands in direct products of copies of \( A \) is preenveloping, so the ideal \( \text{Ideal}(\text{Prod}(A)) \) is also preenveloping. Therefore, the ideal \( \perp \text{Ideal}(\text{Prod}(A)) \) is precovering. On the other case, if we look at the category \( \mathcal{A} = \mathcal{A}b \) as in Example 3 (here \( \mathcal{E} \) is the canonical exact structure in \( \mathcal{A}b \)) the class of all abelian groups \( X \) 
such that \( \text{Ext}(X, \text{Ideal}(\text{Prod}(\mathbb{Z}))) = 0 \) is not necessarily precovering, as it is proved 
in [12, Theorem 0.4].

3.4. Ideal cotorsion pairs. An ideal cotorsion-pair (with respect to \( \mathcal{E} \)) is a pair 
of ideals \( (I, J) \) in \( \mathcal{A} \) such that \( J = I \perp \) and \( I = \perp J \). The ideal cotorsion pair 
\( (I, J) \) is complete if \( I \) is a special precovering ideal and \( J \) is a special preenveloping 
ideal.

Theorem 55. (1) If \( \mathcal{I} \) is a special precovering ideal then \( (I, \mathcal{I} \perp) \) is an ideal cotor-
sion pair. Moreover, if there are enough \( \mathcal{E} \)-injective homomorphisms then the ideal 
cotorsion pair \( (I, \mathcal{I} \perp) \) is complete.

(2) Dually, if \( \mathcal{J} \) is a special preenveloping ideal then \( (\perp \mathcal{J}, J) \) is an ideal cotorsion 
pair. Moreover, if there are enough \( \mathcal{E} \)-projective homomorphisms then the ideal 
cotorsion pair \( (\perp \mathcal{J}, J) \) is complete.

Proof. We have to show that \( I = \perp (\mathcal{I} \perp) \). The inclusion \( I \subseteq \perp (\mathcal{I} \perp) \) is obvious.
Let \( i' : X' \to A \) be a homomorphism from \( \perp (\mathcal{I} \perp) \). Since \( \mathcal{I} \) is special precovering 
we can find a triangle \( Y \to X \xrightarrow{i} A \xrightarrow{k} Y[1] \) such that \( i \) is a special \( \mathcal{I} \)-precover 
for \( A \). Then \( k = j[1]\phi \) for some \( j \in \mathcal{I} \perp \) and some \( \phi \in \text{Ph}(\mathcal{E}) \). All these data are 
represented in the solid part of the following commutative diagram:

\[
\begin{array}{c}
T \\
| \downarrow \ \\
Z \\
| \downarrow \ \\
A \\
| \downarrow \phi \\
T[1] \\
\end{array}
\begin{array}{c}
\downarrow \ \\
Y \\
| \downarrow \ \\
X \\
| \downarrow i \\
A \\
| \downarrow k \\
Y[1].
\end{array}
\]

Because \( i' \perp j \) we obtain \( ki' = j[1]\phi i' = 0 \), so \( i' \) factors through the weak kernel 
\( i \) of \( k \), i.e. \( i' = ig \) for some \( g : X' \to X \). Therefore \( i' \in \mathcal{I} \), and the proof for the first 
statement is complete.

The second statement follows from Salce’s lemma. \( \square \)

Corollary 56. (1) If \( \mathcal{I} \) is special precovering then \( \mathcal{I} = \Phi_\mathcal{E}(\mathcal{PB}_\mathcal{E}(\mathcal{I})) = \perp(\mathcal{I} \perp) \).

(2) If \( \mathcal{J} \) is a special preenveloping ideal then \( \mathcal{J} = \Psi_\mathcal{E}(\mathcal{PD}_\mathcal{E}(\mathcal{J})) = (\perp \mathcal{J})^\perp \).

Proof. This follows from Corollary 45 and Theorem 55. \( \square \)

From the proof of Proposition 26 we can deduce that if \( \mathfrak{F} \subseteq \mathcal{E} \) are weak proper 
classes with enough injective homomorphisms then every \( \mathfrak{F} \)-injective \( \mathfrak{F} \)-inflation can 
be obtained as a pullback of an \( \mathcal{E} \)-triangle along a suitable homomorphism from \( \mathcal{A} \).
It is useful to consider some special \( \mathfrak{F} \)-injective \( \mathfrak{F} \)-inflations, defined in the following 
way:
**Definition 57.** An \( \mathfrak{F} \)-injective homomorphism \( e \) is *special with respect to \( \mathcal{E} \)* if it can be embedded in a homotopy pushout diagram

\[
\begin{array}{cccccc}
A & \xrightarrow{e} & C & \xrightarrow{\varphi} & X & \xrightarrow{A[1]} \\
\downarrow & & \downarrow & & \downarrow & \\
\varnothing & \xrightarrow{d} & B & \xrightarrow{\psi} & Y & \xrightarrow{A[1]},
\end{array}
\]

such that \( \varnothing \in \mathcal{E} \) and \( \varphi \in \Phi_{\mathcal{E}}(\mathfrak{F}) \). The notion of *special projective homomorphism* is defined dually.

**Example 58.** From Corollary 56 and Proposition 43 we observe that if \( \mathcal{I} \) is special precovering then every \( \mathcal{I}^\perp \)-special preenvelope is a special \( \mathcal{PB}_{\mathfrak{F}}(\mathcal{I}) \)-injective homomorphism.

**Proposition 59.** Let \( \mathfrak{F} \) be a weak proper subclass of \( \mathcal{E} \). Then

1. Every special \( \mathfrak{F} \)-injective homomorphism is a special \( \mathfrak{F} \)-inj-preenvelope and a special \( \Phi_{\mathcal{E}}(\mathfrak{F}) \)-inj-preenvelope.
2. Every special \( \mathfrak{F} \)-projective homomorphism is a special \( \mathfrak{F} \)-proj-precove and a special \( \Psi_{\mathcal{E}}(\mathfrak{F}) \)-precover.

**Proof.** Using Theorem 44 we observe that \( \Phi_{\mathcal{E}}(\mathfrak{F}) \subseteq \mathcal{F}^\perp \)-inj, hence every special \( \mathfrak{F} \)-injective homomorphism is a special \( \mathfrak{F} \)-inj-preenvelope.

Let \( e \) be a special \( \mathfrak{F} \)-injective homomorphism. By Corollary 55 we have the inclusion \( \mathcal{PB}_{\mathcal{E}}(\Phi_{\mathcal{E}}(\mathfrak{F})) \subseteq \mathfrak{F} \), hence we can apply Proposition 43 to obtain

\[ e \in \mathfrak{F} \text{-inj} \subseteq \mathcal{PB}_{\mathcal{E}}(\Phi_{\mathcal{E}}(\mathfrak{F}))-\text{inj} = \Phi_{\mathcal{E}}(\mathfrak{F})^\perp. \]

Since \( \Phi_{\mathcal{E}}(\mathfrak{F}) \subseteq \mathfrak{F}^\perp \mathcal{F} \)-inj, \( \Phi_{\mathcal{E}}(\mathfrak{F})^\perp \)-inj, we can apply the definition to obtain that \( e \) is a special \( \Phi_{\mathcal{E}}(\mathfrak{F}) \)-inj-preenvelope. \( \square \)

The following result improves Theorem 44.

**Theorem 60.** Let \( \mathfrak{F} \) be a weak proper subclass of \( \mathcal{E} \).

1. If there are enough special \( \mathfrak{F} \)-injective homomorphisms then

\[ (\Phi_{\mathcal{E}}(\mathfrak{F}), \mathfrak{F} \text{-inj}) \]

is a cotorsion pair which is complete if \( \mathcal{E} \) has enough projective homomorphisms.

2. If there are enough special \( \mathfrak{F} \)-projective homomorphisms then

\[ (\mathfrak{F} \text{-proj}, \Psi_{\mathcal{E}}(\mathfrak{F})) \]

is an ideal cotorsion pair which is complete if \( \mathcal{E} \) has enough injective homomorphisms.

**Proof.** Since we have enough special \( \mathfrak{F} \)-injective homomorphisms, it follows that the ideal \( \mathfrak{F} \)-inj is a special preenveloping ideal and there are enough \( \mathfrak{F} \)-injective homomorphisms. Then we can use Theorem 44 to obtain \( \mathfrak{F}^\perp \mathfrak{F} \)-inj = \( \Phi_{\mathcal{E}}(\mathfrak{F}) \). Now the conclusions are consequences of Theorem 55. \( \square \)

We can characterize ideal cotorsion pairs in the case when we have enough \( \mathcal{E} \)-injective \( \mathcal{E} \)-inflations and \( \mathcal{E} \)-projective \( \mathcal{E} \)-deflations.
Theorem 61. Let $\mathcal{E}$ be a weak proper class of triangles such that there are enough $\mathcal{E}$-injective homomorphisms and $\mathcal{E}$-projective homomorphisms, and let $(\mathcal{I}, \mathcal{J})$ be an ideal cotorsion pair.

The following are equivalent:

(a) $\mathcal{I}$ is precovering;
(b) $\mathcal{I}$ is special precovering;
(c) $\mathcal{J}$ is preenveloping;
(d) $\mathcal{J}$ is special preenveloping;
(e) There exists a weak proper subclass $\mathfrak{F} \subseteq \mathcal{E}$ with enough (special) injective homomorphisms such that $\mathcal{I} = \Phi_{\mathcal{E}}(\mathfrak{F})$;
(f) There exists a weak proper subclass $\mathfrak{F} \subseteq \mathcal{E}$ with enough special injective homomorphisms such that $\mathcal{J} = \mathfrak{F}$-$\text{inj}$;
(g) There exists a weak proper subclass $\mathfrak{G} \subseteq \mathcal{E}$ with enough (special) projective homomorphisms such that $\mathcal{J} = \Psi_{\mathcal{E}}(\mathfrak{G})$;
(h) There exists a weak proper subclass $\mathfrak{G} \subseteq \mathcal{E}$ with enough special projective homomorphisms such that $\mathcal{I} = \mathfrak{G}$-$\text{proj}$.

Proof. The equivalences (a)$\iff$(b)$\iff$(c)$\iff$(d) are from Theorem 51.

The implications (e)$\implies$(b) and (h)$\implies$(d) are proved in Theorem 53 (note that here we use only the fact that $\mathfrak{F}$ has enough injective homomorphisms). Finally, (b)$\implies$(e) and (d)$\implies$(h) are in Corollary 56 and Example 58, while the equivalences (e)$\iff$(f) and (h)$\iff$(g) are obtained from Theorem 60. $\square$

Example 62. For the (trivial) case $\mathcal{A} = \mathcal{T}$ and $\mathcal{E} = \mathcal{D}$ we have enough $\mathcal{D}$-injective and $\mathcal{D}$-projective homomorphisms, hence the above results let us to make the following remarks. Since every precovering ideal $\mathcal{I}$ is special we obtain that $(\mathcal{I}, \mathcal{I}^\perp)$ is a complete ideal cotorsion pair, hence $\mathcal{I}^\perp$ is a preenveloping ideal. For this case we obtain the property

$(\ast)$ for every $A \in \mathcal{T}$ there is a triangle

$\mathcal{D}_A : X_A \xrightarrow{i_A} A \xrightarrow{j_A} Y_A \rightarrow X_A[1]$,

with $i_A \in \mathcal{I}$ and $j_A \in \mathcal{I}^\perp$.

Conversely, a pair $(\mathcal{I}, \mathcal{J})$ of ideals in $\mathcal{T}$ is an ideal cotorsion pair (with respect to $\mathcal{D}$) if and only if it has the property $(\ast)$, where $\mathcal{I}^\perp$ is replaced by $\mathcal{J}$.

For every (co)torsion theory $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{T}$, the pair

$(\mathcal{I}, \mathcal{J}) = (\text{Ideal}(\mathcal{X}), \text{Ideal}(\mathcal{Y}))$

is a (complete) ideal cotorsion pair with respect to the proper class $\mathcal{D}$ of all triangles in $\mathcal{T}$. Moreover this ideal cotorsion pair is complete. Clearly, the pair $(\mathcal{X}, \mathcal{Y})$ is a $t$-structure if and only if $\mathcal{I}$ and $\mathcal{J}$ satisfy the additional conditions $\mathcal{I}[1] \subseteq \mathcal{I}$ and $\mathcal{J}[1] \subseteq \mathcal{J}$.

4. Products of ideals and Toda brackets

In this section we continue to fix an extension closed subcategory $\mathcal{A}$ of $\mathcal{T}$ and a weak proper class of triangles $\mathcal{E}$ in $\mathcal{A}$. In addition, $\mathcal{E}$ is supposed to be saturated.
4.1. Toda brackets. In the following we will use the algebraic concept of Toda bracket as it is defined in [28]. This concept let us to generalize the operations $\diamond$ introduced in [27] for (object ideals in) triangulated categories (cf. Proposition 68) and [15] for exact categories (cf. [15, Lemma 6]).

Definition 63. Let

$$\varnothing : Y \xrightarrow{f} Z \xrightarrow{g} X \xrightarrow{\varnothing} Y[1]$$

be a triangle in $\mathcal{T}$. If $i : Y \to U$ and $j : V \to Z$ are two homomorphisms then the Toda bracket $\langle i, j \rangle_{\varnothing}$ is the set of all homomorphisms $\zeta : V \to U$ such that $\zeta = \zeta' \zeta''$, were $\zeta'' : V \to Z$ and $\zeta' : Z \to U$ are homomorphisms which make the diagram

$$\begin{array}{ccc}
V & \xrightarrow{\zeta'} & U \\
\downarrow & & \downarrow \\
X & \xrightarrow{i} & Y[1] \\
\end{array}$$

commutative.

If $\mathcal{I}$ and $\mathcal{J}$ are two classes of homomorphisms then the union of all Toda brackets $\langle i, j \rangle_{\varnothing}$ with $i \in \mathcal{I}, j \in \mathcal{J}$ and $\varnothing \in \mathcal{C}$ is denoted by $\langle \mathcal{I}, \mathcal{J} \rangle_{\mathcal{C}}$, and it is called the Toda bracket of $\mathcal{I}$ and $\mathcal{J}$ induced by $\mathcal{C}$.

Remark 64. Let $i$ and $j$ be two homomorphisms and let $\varnothing$ be a triangle in $\mathcal{T}$. Then $\langle i, j \rangle_{\varnothing} \neq \emptyset$ if and only if $i$ is injective relative to $\varnothing$ and $j$ is projective relative to $\varnothing$.

Remark 65. Let us consider the dual category $\mathcal{T}^*$, and we denote by $\mathcal{I}^*$ and $\mathcal{C}^*$ the ideal, respectively the weak proper class induced by $\mathcal{I}$ and $\mathcal{C}$ in $\mathcal{T}^*$. Then for every two ideals $\mathcal{I}$ and $\mathcal{J}$ in $\mathcal{T}$ we have $\langle \mathcal{I}, \mathcal{J} \rangle_{\mathcal{C}} = (\langle \mathcal{J}, \mathcal{I} \rangle_{\mathcal{C}})^*$. 

Lemma 66. If $\mathcal{I}$ and $\mathcal{J}$ are ideals in $\mathcal{A}$ then $\langle \mathcal{I}, \mathcal{J} \rangle_{\mathcal{C}}$ is also an ideal in $\mathcal{A}$.

Proof. It is straightforward to check that $0 \in \langle 0, 0 \rangle_{\mathcal{C}} \subseteq \langle \mathcal{I}, \mathcal{J} \rangle_{\mathcal{C}}$, and that $\langle \mathcal{I}, \mathcal{J} \rangle_{\mathcal{C}}$ is closed with respect to compositions with arbitrary maps and finite direct sums. \qed

For further references, let us consider the following remark which can be extracted from [15, Lemma 6].

Lemma 67. If $\mathcal{I}$ and $\mathcal{J}$ are ideals and $\xi : V \to U$ is a homomorphism in $\langle \mathcal{I}, \mathcal{J} \rangle_{\mathcal{C}}$ then there exists a commutative diagram

$$\begin{array}{ccc}
Y & \xrightarrow{\alpha} & V \\
\downarrow & & \downarrow^0 \\
U & \xrightarrow{\beta} & Q \\
\end{array} \quad \begin{array}{ccc}
P & \xrightarrow{\xi} & V \\
\downarrow & & \downarrow \\
X & \xrightarrow{\zeta} & U[1] \\
\end{array}$$

such that the horizontal lines are splitting triangles, $i \in \mathcal{I}$ and $j \in \mathcal{J}$, the homomorphisms $\alpha : V \to P$ and $\beta : Q \to U$ are partial inverses for $P \to V$ respectively $U \to Q$ and $\xi = \beta \zeta \alpha$. 

Proof. Starting with the diagram (TB) we can construct via a base change and a cobase change the following commutative diagram

\[
\begin{array}{ccc}
V & \downarrow & Y[1] \\
\uparrow & \alpha & \downarrow \\
\vdots & \nabla & \nabla \\
\end{array}
\]

\[
\begin{array}{ccc}
P & \downarrow & V \\
\uparrow & \nabla & \nabla \\
\vdots & \nabla & \nabla \\
\end{array}
\]

\[
\begin{array}{ccc}
Y & \downarrow & P \\
\uparrow & \nabla & \nabla \\
\vdots & \nabla & \nabla \\
\end{array}
\]

The homomorphisms \( \alpha \) and \( \beta \) are constructed via the weak universal property of the homotopy pullback and pushout. Now the conclusion is obvious. \( \square \)

For further applications, let us study Toda brackets associated to object ideals.

**Proposition 68.** Let \( A \) be an extension closed full subcategory of \( T \) and \( E \) a weak proper class of triangles from \( A \). If \( P \) and \( Q \) are two classes of objects in \( A \) closed under finite direct sums, and \( V \) is the class of all objects \( V \) which lie in triangles \( \varphi : Q \to V \to P \to Q[1] \) with \( \varphi \in E \), \( P \in P \) and \( Q \in Q \) then

(a) \( V \) is closed with respect to finite direct sums;
(b) \( \langle \text{Ideal}(Q), \text{Ideal}(P) \rangle_E = \text{Ideal}(V) \).

**Proof.** (a) is a simple exercise.

(b) If \( \zeta : W \to U \) is in \( \langle \text{Ideal}(Q), \text{Ideal}(P) \rangle_E \) then we have a diagram

\[
\begin{array}{ccc}
W & \downarrow & V[1] \\
\uparrow & \pi & \downarrow \\
\vdots & \nabla & \nabla \\
\end{array}
\]

\[
\begin{array}{ccc}
P & \downarrow & A \\
\uparrow & \nabla & \nabla \\
\vdots & \nabla & \nabla \\
\end{array}
\]

\[
\begin{array}{ccc}
Y & \downarrow & P \\
\uparrow & \nabla & \nabla \\
\vdots & \nabla & \nabla \\
\end{array}
\]

such that \( P \in P, Q \in Q, \varphi \in E \) and \( \zeta = \zeta' \zeta'' = \zeta' \alpha \).
Then we construct via a homotopy pushout along \( i \) a triangle \( i \delta j \), hence we have a commutative diagram

\[
\begin{array}{c}
\delta : \quad Y \xrightarrow{f} Z \xrightarrow{g} X \xrightarrow{\alpha} Y[1] \\
\delta j : \quad Y \xrightarrow{f'} A \xrightarrow{\alpha'} P \xrightarrow{j[1]} Y[1] \\
i \delta j : \quad Q \xrightarrow{\rho} V \xrightarrow{\zeta} P \xrightarrow{\zeta' v} Q[1]
\end{array}
\]

Since \( \zeta' \alpha f' = \zeta' f = \rho i \), there exists a homomorphism \( \xi : V \rightarrow U \) such that \( \zeta' \alpha = \xi \alpha' \). Therefore \( \zeta = \zeta' \alpha v = \xi \alpha' v \), and it follows that \( \zeta \) factorizes through \( V \). Then \( \zeta \in \text{Ideal}(V) \).

Conversely, if we have a triangle \( \delta : Q \rightarrow V \rightarrow P \rightarrow Q[1] \) in \( \mathcal{E} \) with \( P \in \mathcal{P} \) and \( Q \in \mathcal{Q} \) then we can construct the commutative diagram

\[
\begin{array}{c}
\delta : \quad Q \xrightarrow{\delta} V \xrightarrow{\alpha'} P \xrightarrow{\alpha} Q[1],
\end{array}
\]

hence \( V \) is an object in the ideal \( \langle \text{Ideal} \mathcal{Q}, \text{Ideal} \mathcal{P} \rangle \mathcal{E} \).

4.2. **Wakamatsu’s Lemma.** We will prove here an ideal version for Wakamatsu’s Lemma which generalizes the corresponding results proved in [15, Lemma 37] for exact categories and in [18, Lemma 2.1] for object ideals in triangulated categories. Let \( \mathcal{I} \) be an ideal in \( \mathcal{A} \). An \( \mathcal{I} \)-precover \( i : Z \rightarrow A \) is an \( \mathcal{I} \)-cover if it is an \( \mathcal{D}_A \)-deflation and for every endomorphism \( \alpha \) of \( Z \) from \( i \alpha = i \) it that follows \( \alpha \) is an isomorphism. We note that there are categories when every precovering (preenveloping) ideal is covering (enveloping), e.g. the category of finitely generated modules over artin algebras, cf. [3, Proposition 1.1] or in the case of \( k \)-linear Hom-finite triangulated categories, [9] Lemma 1.1].

**Lemma 69.** Let \( \mathcal{I} \) be an ideal in \( \mathcal{A} \) which is closed under Toda brackets, that is \( \langle \mathcal{I}, \mathcal{I} \rangle \mathcal{E} \subseteq \mathcal{I} \), and let \( i : Z \rightarrow A \) be an \( \mathcal{I} \)-cover for \( A \). If

\[
K \xrightarrow{\kappa} Z \xrightarrow{i} A \xrightarrow{i'} K[1]
\]

is the corresponding triangle then \( 1_K \in \mathcal{I}^\perp \).

**Proof.** We have to prove that for every \( \varphi : Y \rightarrow K[1] \) from \( \text{Ph}(\mathcal{E}) \) and every \( i' : X \rightarrow Y \) from \( \mathcal{I} \) we have \( \varphi i' = 0 \).
Let \( \varphi \in \text{Ph}(\mathcal{C}) \) and \( i' \in \mathcal{I} \) as before. Using homotopy pullbacks along \( \varphi \) and \( i' \) we obtain the solid part of following commutative diagram

\[
\begin{array}{ccccccc}
Z & \xrightarrow{\upsilon} & U & \xrightarrow{\varphi} & X & \xrightarrow{\eta} & Z[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Z & \xrightarrow{i} & T & \xrightarrow{\alpha} & Y & \xrightarrow{\psi} & Z[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Z & \xrightarrow{\nu} & A & \xrightarrow{\nu} & K[1] & \xrightarrow{\nu} & Z[1].
\end{array}
\]

Since \( \varphi \in \text{Ph}(\mathcal{C}) \) we obtain \( \psi \in \text{Ph}(\mathcal{C}) \), hence the triangle \( Z \to T \to Y \to Z[1] \) is in \( \mathcal{C} \). Moreover, the composition \( U \to X \xrightarrow{i} Y \) is in \( \mathcal{I} \), hence \( \alpha \beta \in \langle \mathcal{I}, \mathcal{I} \rangle_E \subseteq \mathcal{I} \). It follows that \( \alpha \beta \) factorizes through \( i \), hence we can find a homomorphism \( \gamma : U \to Z \) such that \( \alpha \beta = i \gamma \). Then \( i = \alpha \beta \nu = i \gamma \nu \), and it follows that \( \gamma \nu \) is an automorphism of \( Z \). Since \( \gamma \nu \eta [-1] = 0 \) we obtain \( \eta = 0 \). Then the top triangle splits, and it follows that \( \varphi i' \) factorizes through \( \nu \alpha \beta = \nu i \gamma = 0 \). Then \( \varphi i' = 0 \), and the proof is complete. \( \square \)

Now we can apply the previous results to obtain the object version of Wakamatsu’s Lemma. In the case \( \mathcal{C} = \mathcal{D} \) this was proved in [15, Lemma 2.1].

**Corollary 70.** Let \( \mathcal{X} \) be a class of objects in \( \mathcal{T} \). If \( \mathcal{X} \) is closed with respect to \( \mathcal{E} \)-extensions, and

\[
K \to X \xrightarrow{i} A \to K[1]
\]

is an \( \mathcal{E} \)-triangle such that \( i \) is an \( \mathcal{X} \)-cover then \( \text{Hom}(\mathcal{X}, K[1]) \cap \text{Ph}(\mathcal{E}) = 0. \)

**Proof.** Let \( V \) be the class of all objects \( V \) which lie in \( \mathcal{E} \)-conflations \( X \to V \to X' \to X[1] \) with \( X, X' \in \mathcal{X} \). Applying Proposition 68 and the hypothesis we have \( \langle \text{Ideal}(\mathcal{X}), \text{Ideal}(\mathcal{X}) \rangle_E = \text{Ideal}(V) \subseteq \text{Ideal}(\mathcal{X}) \), hence \( \text{Ideal}(\mathcal{X}) \) is closed with respect to Toda brackets. Then the conclusion follows from Lemma 69. \( \square \)

4.3. **Products of ideals.** It is easy to see (as in the proof of Theorem 51) that if \( \mathcal{I} \) and \( \mathcal{J} \) are ideals, \( i : I \to A \) is an \( \mathcal{I} \)-precover for \( A \) and \( j : J \to I \) is a \( \mathcal{J} \)-precover for \( I \) then \( ij \) is an \( \mathcal{I}\mathcal{J} \)-precover for \( A \). Therefore, if \( \mathcal{I} \) and \( \mathcal{J} \) are precovering ideals then \( \mathcal{I}\mathcal{J} \) is also precovering, see [22, Lemma 3.6].

The main aim of this subsection is to prove that if \( \mathcal{I} \) and \( \mathcal{J} \) are special precover ideals (with respect to \( \mathcal{E} \)) then \( \mathcal{I}\mathcal{J} \) is also special precovering, and to compute \( (\mathcal{I}\mathcal{J})^\perp \).

**Lemma 71.** If \( \mathcal{I} \) and \( \mathcal{J} \) are ideals in \( A \) then \( \langle \mathcal{J}^\perp, \mathcal{I}^\perp \rangle_E \subseteq (\mathcal{I}\mathcal{J})^\perp. \)

**Proof.** Let \( \zeta = \zeta' \zeta'' \in \langle \mathcal{J}^\perp, \mathcal{I}^\perp \rangle_E \). In order to prove that \( \zeta' \zeta'' \in (\mathcal{I}\mathcal{J})^\perp \) we consider a chain of composable homomorphisms \( U \xrightarrow{j} T \xrightarrow{i} W \xrightarrow{\phi} V \) such that \( i \in \mathcal{I}, j \in \mathcal{J} \).
and $\phi \in \text{Ph}(\mathfrak{C})$. We have the solid part of the following commutative diagram

\[
\begin{array}{ccc}
U[-1] & \xrightarrow{j[-1]} & T[-1] & \xrightarrow{\iota[-1]} & W[-1] \\
& | & | & | & |
\downarrow & & \downarrow & & \downarrow \\
| & \phi[-1] & & & V \\
& \downarrow & & | & | \\
Y & \xrightarrow{f} & Z & \xrightarrow{g} & X \xrightarrow{\nu} Y[1] \\
& \downarrow & & & \\
U & \xrightarrow{\zeta'} & & & \\
\end{array}
\]

were the row $Y \to Z \to X \to Y[1]$ is a triangle in $\mathfrak{C}$, $\mu \in \mathcal{J}^\perp$ and $\nu \in \mathcal{I}^\perp$.

Then $g\zeta''\phi[-1]i[-1] = \nu \phi[-1]i[-1] = 0$ since $\nu \in \mathcal{I}^\perp$. Therefore $\zeta''\phi[-1]i[-1]$ factors through $f$, i.e. there exists a homomorphism $\phi'[-1] : T[-1] \to Y$ such that $f\phi'[-1] = \zeta''\phi[-1]i[-1]$. We observe that $f[1]j\phi'$ factors through $\phi$ hence $f[1]j\phi' \in \text{Ph}(\mathfrak{C})$. Since $f$ is an $\mathfrak{C}$-inflation, the saturation of $\mathfrak{C}$ implies $\phi' \in \text{Ph}(\mathfrak{C})$.

Finally we have:

\[(\zeta'\zeta'')[1]jij = \zeta'[1]j[1]j = \mu[1]j = 0\]

since $\mu \in \mathcal{J}^\perp$. \hfill \Box

**Corollary 72.** If $\mathcal{I}$ is an idempotent ideal then $\mathcal{I}^\perp$ is closed with respect to Toda brackets.

**Corollary 73.** If $\mathcal{I}$ is an ideal in $\mathcal{A}$ then $\langle \mathcal{I}^\perp, \mathfrak{C}\text{-inj} \rangle_\mathfrak{C} \subseteq \mathcal{I}^\perp$.

**Proof.** Applying Lemma 37 we have

\[\langle \mathcal{I}^\perp, \mathfrak{C}\text{-inj} \rangle_\mathfrak{C} = \langle \mathcal{I}^\perp, (\mathcal{A}^\perp)^\perp \rangle_\mathfrak{C} \subseteq (\mathcal{A}^\perp\mathcal{I})^\perp = \mathcal{I}^\perp,
\]

and the proof is complete. \hfill \Box

**Theorem 74.** Let $\mathcal{I}$ and $\mathcal{J}$ two special precovering ideals in $\mathcal{A}$. Then the product ideal $\mathcal{I}\mathcal{J}$ is also special precovering.

If $A \in \mathcal{A}$, $i : I \to A$ is a special $\mathcal{I}$-precover, and $j : J \to I$ is a special $\mathcal{J}$-precover then $ij : J \to A$ is a special $\mathcal{I}\mathcal{J}$-precover. Moreover, $ij$ can be embedded in a homotopy pushout diagram

\[
\begin{array}{ccc}
Z'' & \xrightarrow{\zeta} & J'' & \xrightarrow{A} & Z[1] \\
\downarrow & & \downarrow & & \downarrow \\
Z & \xrightarrow{ij} & A & \xrightarrow{\zeta} & Z[1] \\
\end{array}
\]

with $\zeta \in \langle \mathcal{J}^\perp, \mathcal{I}^\perp \rangle_\mathfrak{C}$.

**Proof.** Consider the diagrams

\[
\begin{array}{ccc}
X' & \xrightarrow{\xi} & I' & \xrightarrow{A} & X[1] \\
\downarrow & & \downarrow & & \downarrow \xi[1] \\
X & \xrightarrow{I} & A & \xrightarrow{\xi} & X[1] \\
\end{array}
\]

and

\[
\begin{array}{ccc}
X & \xrightarrow{i} & A & \xrightarrow{\xi} & X[1] \\
\downarrow & & \downarrow & & \downarrow \xi[1] \\
X & \xrightarrow{I} & A & \xrightarrow{\xi} & X[1] \\
\end{array}
\]

 respectively.
and

(♯)

\[ Y' \rightarrow J' \rightarrow I \rightarrow Y''[1] \]

\[ \downarrow v \quad \downarrow j \quad \downarrow v[1] \]

\[ Y \rightarrow J \rightarrow I \rightarrow Y[1] \]

with \( \xi \in I^\perp \) and \( v \in J^\perp \), which emphasise the facts that \( i \) and \( j \) are special precovers. By pulling back along \( I' \rightarrow I \) we obtain the commutative diagram

\[ Y' \rightarrow J'' \rightarrow I' \rightarrow Y''[1] \]

\[ \downarrow \downarrow \downarrow \downarrow \]

\[ Y' \rightarrow J' \rightarrow I \rightarrow Y'[1] \]

Using the octahedral axiom, we extend these diagrams to the solid part of the following diagram:

Here all vertical and horizontal lines (from left to right) are triangles in \( \mathcal{C} \) and all squares but the top horizontal square are commutative.

The homomorphism \( \zeta'' \) is constructed as follows: we have the equality

\[ (Z'' \rightarrow X' \xrightarrow{\xi} X \rightarrow I) = (Z'' \rightarrow J'' \rightarrow J' \rightarrow I) \]

and \( Z' \) is a homotopy pullback of the angle \( X \rightarrow I \leftarrow J' \), hence there exists a homomorphism \( \zeta'': Z'' \rightarrow Z' \) making the diagram commutative. The homomorphism
$\zeta' : Z' \to Z$ is obtained in an analogous way by using the equality

$$(Z' \to J' \to J \xrightarrow{\zeta'} I) = (Z' \to X \to I).$$

Finally, we consider a homomorphism $\nu' : Y' \to Y$ such that $(\nu', \zeta', 1_X)$ is a homomorphism of triangles.

We have

$$(Y' \xrightarrow{\nu'} Y \to J) = (Y' \xrightarrow{\nu'} Y \to Z \to J) = (Y' \to Z' \xrightarrow{\zeta'} Z \to J),$$

and the diagram (1) is obtained as a homomopy pushout diagram. Therefore $\nu'$ factorizes through $\nu$. Then $\nu' \in J^\perp$.

We extract from the above diagram the following commutative diagram

\[
\begin{array}{ccc}
Z'' & \xrightarrow{f} & X' \\
\downarrow{\zeta''} & & \downarrow{\xi} \\
Y' & \xrightarrow{g} & X \\
\downarrow{\nu'} & & \downarrow{\zeta'} \\
Y & \xrightarrow{\zeta} & Z
\end{array}
\]

and using Lemma 71 we obtain $\zeta' \zeta'' \in \langle J^\perp, I^\perp \rangle E \subseteq (IJ)^\perp$.

From the commutative diagram

\[
\begin{array}{ccc}
Z'' & \xrightarrow{f} & J'' \\
\downarrow{\zeta' \zeta''} & & \downarrow{=} \\
Z & \xrightarrow{ij} & A \\
\downarrow{=} & & \downarrow{=} \\
I & \xrightarrow{=} & X
\end{array}
\]

we obtain the conclusions stated in theorem.

\[\square\]

**Corollary 75.** If $I$ is a special precovering ideal then the same is true for any ideal in the chain:

$$I = I^1 \supseteq I^2 \supseteq I^3 \supseteq \cdots.$$  

**4.4. Ghost lemma.** In the following we need a result which generalizes Salce’s Lemma (in the case $E$ is saturated).

**Lemma 76.** Let $K$, $L$ be ideals in $A$, and let

(IE) $A \xrightarrow{e} E \to X \to A[1]$

be a triangle in $E$ such that $e \in L$.

Let $i : I \to X$ be a homomorphism which can be embedded in a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & Z \\
\downarrow{h} & & \downarrow{=} \\
W & \xrightarrow{w} & I \\
\downarrow{=} & & \downarrow{=} \\
X & \xrightarrow{=} & W[1]
\end{array}
\]
such that \( g \in \mathcal{K} \) and the rows in this diagram are triangles in \( \mathcal{E} \). If the diagram

\[
\begin{array}{c}
A \rightarrow J \rightarrow I \rightarrow A[1] \\
\alpha \downarrow \downarrow \downarrow i \downarrow \downarrow \\
A \rightarrow E \rightarrow X \rightarrow A[1]
\end{array}
\]

is obtained as a homotopy pullback along \( i \) then

\[ a \in \langle \mathcal{K}, \mathcal{L} \rangle \mathcal{E}. \]

**Proof.** Using a cobase change of the triangle (IE) along \( i \) we complete the diagram (PB) to the commutative diagram

\[
\begin{array}{c}
W \rightarrow W \\
\downarrow \downarrow \downarrow w \downarrow \downarrow \\
A \rightarrow J \rightarrow I \rightarrow A[1] \\
\alpha \downarrow \downarrow \downarrow i \downarrow \downarrow \\
A \rightarrow E \rightarrow X \rightarrow A[1] \\
\phi \downarrow \downarrow \downarrow \phi \downarrow \downarrow \\
W[1] \rightarrow W[1]
\end{array}
\]

Moreover, using this time the homomorphism \( ih \), we can modify the diagram (PO) to obtain the following commutative diagram:

\[
\begin{array}{c}
Y \rightarrow Y \\
\downarrow \downarrow \downarrow y \downarrow \downarrow \\
A \rightarrow C \rightarrow Z \rightarrow A[1] \\
\downarrow \downarrow \downarrow f \downarrow \downarrow \\
A \rightarrow E \rightarrow X \rightarrow A[1] \\
\phi \downarrow \downarrow \downarrow \phi \downarrow \downarrow \\
Y[1] \rightarrow Y[1]
\end{array}
\]

Note that in the above two diagrams all rows and columns are triangles in \( \mathcal{E} \). The horizontal cartesian rectangle from the previous diagram can be obtained as a juxtaposition of two cartesian diagrams

\[
\begin{array}{c}
A \rightarrow C \rightarrow Z \rightarrow A[1] \\
\downarrow \downarrow \downarrow k \downarrow \downarrow \\
A \rightarrow J \rightarrow I \rightarrow A[1] \\
\alpha \downarrow \downarrow \downarrow i \downarrow \downarrow \\
A \rightarrow E \rightarrow X \rightarrow A[1]
\end{array}
\]
and using the octahedral axiom we complete the middle commutative square in the following diagram to a homomorphism of triangles:

\[
\begin{array}{ccc}
Y & \rightarrow & C \\
\downarrow{g'} & & \downarrow{k} \\
W & \rightarrow & J \\
\end{array}
\quad
\begin{array}{ccc}
C & \rightarrow & E \\
\downarrow{f} & & \downarrow{g'[1]} \\
E & \rightarrow & Y[1] \\
\end{array}
\]

Now denote \(\delta = g - g' : Y \rightarrow W\). Since

\[
(Y \xrightarrow{g} W \xrightarrow{w} I) = (Y \rightarrow Z \xrightarrow{h} I) = (Y \rightarrow C \xrightarrow{k} J \rightarrow I) = (Y \xrightarrow{g'} W \rightarrow J \rightarrow I)
\]

we obtain \(w\delta = 0\), hence \(\delta\) factorizes through \(\phi[-1]\). But \(\phi[-1]\) factorizes through \(g\), and it follows that \(g'\) factorizes through \(g\). Therefore \(g' \in K\). Using the commutative diagram

\[
\begin{array}{ccc}
Y & \rightarrow & C \\
\downarrow{g'} & & \downarrow{k} \\
W & \rightarrow & J \\
\end{array}
\quad
\begin{array}{ccc}
A & \rightarrow & E \\
\downarrow{f} & & \downarrow{e} \\
Y & \rightarrow & Y[1] \\
\end{array}
\]

together with \(a = kf\) we obtain \(a \in \mathcal{K} \cup \mathcal{L}\).

As a first application, we improve Corollary 73.

**Corollary 77.** Suppose that there are enough \(\mathcal{E}\)-injective homomorphisms. If \(I\) is a special precovering ideal then \(I^\perp = \langle I^\perp, \mathcal{E}\text{-inj} \rangle_{\mathcal{E}}\).

**Proof.** Using Corollary 62 we can construct for every object \(A\) in \(\mathcal{A}\) a special \(I^\perp\)-preenvelope via a pullback diagram

\[
\begin{array}{ccc}
A & \rightarrow & K \\
\downarrow{a} & & \downarrow{i} \\
A[1] & \rightarrow & I \\
\end{array}
\quad
\begin{array}{ccc}
A & \rightarrow & E \\
\downarrow{a} & & \downarrow{i} \\
A[1] & \rightarrow & X \\
\end{array}
\]

such that \(e\) is injective and \(i\) is a special precover for \(X\). By Lemma 48 it follows that \(a \in \langle I^\perp, \mathcal{E}\text{-inj} \rangle_{\mathcal{E}}\), hence \(I^\perp \subseteq \langle I^\perp, \mathcal{E}\text{-inj} \rangle_{\mathcal{E}}\). Using Corollary 73 we obtain \(I^\perp = \langle I^\perp, \mathcal{E}\text{-inj} \rangle_{\mathcal{E}}\).

**Theorem 78.** Suppose that there are enough \(\mathcal{E}\)-injective homomorphisms. If \(I\) and \(J\) are special precovering ideals in \(\mathcal{A}\) then \((IJ)^\perp = \langle J^\perp, I^\perp \rangle_{\mathcal{E}}\).

**Proof.** By Lemma 71 we only have to prove the inclusion \((IJ)^\perp \subseteq \langle J^\perp, I^\perp \rangle_{\mathcal{E}}\). Since \(I_J\) is a special precovering ideal, it follows that the ideal \((IJ)^\perp\) is special preenveloping. Therefore, it is enough to prove that for every object \(A\) in \(\mathcal{A}\) there exists a special \((IJ)^\perp\)-preenvelope which belongs to \(\langle J^\perp, I^\perp \rangle_{\mathcal{E}}\).
Let $A$ be an object in $\mathcal{A}$. As in the proof of Corollary 77, we use Corollary 52 to construct for every object $A$ in $\mathcal{A}$ a special $(\mathcal{I}, \mathcal{J})^\perp$-preenvelope $a : A \to K$ via a pullback diagram

$$
\begin{array}{ccc}
A & \xrightarrow{a} & K \\
\downarrow & & \downarrow \\
A & \xrightarrow{e} & E
\end{array}
\begin{array}{ccc}
J & \xrightarrow{j} & A[1] \\
\downarrow & & \downarrow \\
X & \xrightarrow{i} & A[1]
\end{array}
$$

such that $i : I \to X$ is a special $\mathcal{I}$-precover for $X$ and $j : J \to I$ is a special $\mathcal{J}$-precover for $I$. If we consider the homotopy pullback of the triangle

$$A \to E \to X \to A[1]$$

along $i$, we can assume that the above commutative diagram is constructed using two homotopy pullbacks as in the following commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{a} & K \\
\downarrow & & \downarrow \\
A & \xrightarrow{b} & L \\
\downarrow & & \downarrow \\
A & \xrightarrow{c} & E
\end{array}
\begin{array}{ccc}
J & \xrightarrow{j} & A[1] \\
\downarrow & & \downarrow \\
I & \xrightarrow{\phi} & W[1]
\end{array}
$$

where both horizontal rectangles are cartesian.

By Corollary 52 we have $b \in I^\perp$. Moreover, since $j$ is a special precover, it can be embedded in a commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{g} & Z \\
\downarrow & & \downarrow \\
W & \xrightarrow{w} & J
\end{array}
\begin{array}{ccc}
X & \xrightarrow{h} & I \\
\downarrow & & \downarrow \\
Y[1] & \xrightarrow{g[1]} & W[1]
\end{array}
$$

such that $g \in J^\perp$. Then we can apply Lemma 76 for the top rectangle which lies in diagram (PB') to obtain $a \in \langle J^\perp, I^\perp \rangle_E$. Since $a$ is an $(\mathcal{I}, \mathcal{J})^\perp$-preenvelope we obtain $(\mathcal{I}, \mathcal{J})^\perp \subseteq \langle \mathcal{I}^\perp, \mathcal{J}^\perp \rangle_E$. □

We have a converse for Corollary 72.

**Corollary 79.** Suppose that there are enough $\mathcal{E}$-injective homomorphisms. A special precovering ideal $\mathcal{I}$ in $\mathcal{A}$ is idempotent (i.e. $\mathcal{I}^2 = \mathcal{I}$) if and only if $\mathcal{I}^\perp$ is closed with respect to Toda brackets.

**Proof.** If $\mathcal{I}$ is idempotent then $\langle \mathcal{I}^\perp, \mathcal{I}^\perp \rangle_E = (\mathcal{I}\mathcal{I})^\perp = \mathcal{I}^\perp$.

Conversely, from $\mathcal{I}^\perp \subseteq (\mathcal{I}^2)^\perp = (\mathcal{I}^\perp, \mathcal{I}^\perp)_E \subseteq \mathcal{I}^\perp$ it follows that $\mathcal{I}^\perp = (\mathcal{I}^2)^\perp$. By Theorem 55 using the fact that both ideal $\mathcal{I}$ and $\mathcal{I}^2$ are special precovering, we have $\mathcal{I}^2 = (\mathcal{I}^2)^\perp = ^\perp(I^\perp) = \mathcal{I}$. □

As in the study of ideal cotorsion pairs in exact categories, we can state the following version of (co-)Ghost Lemma.

**Corollary 80.** Suppose that there are enough $\mathcal{E}$-injective and $\mathcal{E}$-projective homomorphisms. Then
(1) the class of special precovering ideals is closed with respect to products and Toda brackets;
(2) the class of special preenveloping ideals is closed with respect to products and Toda brackets;
(3) If \((\mathcal{I}, \mathcal{J})\) and \((\mathcal{K}, \mathcal{L})\) are two complete ideal cotorsion pairs then
(a) \((\mathcal{I}\mathcal{K})^\perp = \langle \mathcal{L}, \mathcal{J} \rangle_\mathcal{E}\) and \((\langle \mathcal{I}, \mathcal{K} \rangle)^\perp \mathcal{E} = \langle \mathcal{L}, \mathcal{J} \rangle\);
(b) \((\langle \mathcal{I}, \mathcal{K} \rangle)^\perp \mathcal{E} = \mathcal{L}\mathcal{J}\) and \(\mathcal{I}\mathcal{K} = \langle \mathcal{L}, \mathcal{J} \rangle_\mathcal{E}\)\)

Remark 81. From the above result it follows that if we have enough \(\mathcal{E}\)-injective and \(\mathcal{E}\)-projective homomorphisms the Toda bracket operation is associative on the class of special precovering (resp. preenveloping) ideals. In particular, we obtain that the Toda bracket operation computed with respect to the class \(\mathcal{D}\) of all triangles from \(\mathcal{T}\) is associative for precovering (resp. preenveloping) ideals. This is used in \[5,4\]. In the case of ideals in exact categories the associativity is proved in the general setting in \[11, Proposition 8\]. This is also valid for object ideals in triangulated category (as a consequence of \[8, Lemma 1.3.10\]). We are not able to prove decide if this property is valid for arbitrary ideals and proper classes.

5. Applications

5.1. Projective classes. We recall from \[11, Definition 2.5 and Proposition 2.6\] that a projective class in \(\mathcal{T}\) is a pair \((\mathcal{P}, \mathcal{J})\) where \(\mathcal{P}\) is a class of objects and \(\mathcal{J}\) is an ideal in \(\mathcal{T}\) such that

\[
\mathcal{P} = \{ P \in \mathcal{T} \mid \mathcal{T}(P, \phi) = 0 \text{ for all } \phi \in \mathcal{J} \},
\]

and every \(X \in \mathcal{T}\) lies in a triangle \(P \rightarrow X \xrightarrow{\phi} Y \rightarrow P[1]\), with \(P \in \mathcal{P}\) and \(\phi \in \mathcal{J}\). As in \[11, Section 3\], we consider the case when \(\mathcal{P}\) and \(\mathcal{J}\) are suspension closed.

Proposition 82. If \((\mathcal{P}, \mathcal{J})\) is a projective class such that \(\mathcal{P}\) and \(\mathcal{J}\) are suspension closed and \(\mathcal{I} = \text{Ideal}(\mathcal{P})\), then \((\mathcal{I}, \mathcal{J})\) is a complete cotorsion theory with respect to \(\mathcal{D}\).

Conversely, if \(\mathcal{Q}\) is a class of objects closed with respect to direct sums such that \((\text{Ideal}(\mathcal{Q}), \mathcal{J})\) is a (special) cotorsion pair with respect to \(\mathcal{D}\) then \((\text{add}(\mathcal{Q}), \mathcal{J})\) is a projective class.

Proof. The first statement follows from Example [62].

For the second statement, let \(\mathcal{Q}\) be a class closed with respect to finite direct sums. For every object \(A\) we fix a special \(\mathcal{J}\)-preenvelope \(j : A \rightarrow Y\). By Corollary [62] via the commutative diagram (constructed as in Example [50])

\[
\begin{array}{ccc}
A[-1] & \xrightarrow{j[-1]} & 0 \xrightarrow{} A \xrightarrow{} A \\
\downarrow & & \downarrow j \\
Y[-1] & \xrightarrow{i} & L \xrightarrow{\iota} A \xrightarrow{j} Y,
\end{array}
\]

we obtain that the cocone \(i : L \rightarrow A\) of \(j\) is a special \(\text{Ideal}(\mathcal{Q})\)-precover. Since \(i\) factorizes through an object from \(\mathcal{Q}\), there exists \(Q \in \mathcal{Q}\) and a commutative
Since \( \varphi \in J \), we apply the remarks stated in Example 10 to conclude the proof. \( \square \)

In particular we obtain Christensen’s Ghost Lemma.

**Example 83.** Let \((P, J)\) and \((Q, L)\) be two projective classes in \(T\). From Proposition 82 we know that \((\text{Ideal}(P), J)\) and \((\text{Ideal}(Q), L)\) are complete ideal cotorsion pairs with respect to the proper class \(D\) of all triangles in \(T\). Then \((\langle \text{Ideal}(Q), \text{Ideal}(P) \rangle_D, JL)\) is a complete ideal cotorsion pair. By Proposition 68 we know that 
\[
\langle \text{Ideal}(Q), \text{Ideal}(P) \rangle_D = \text{Ideal}(V),
\]
where \(V\) is the class of all objects \(V\) which lie in triangles \(Q \to V \to P \to P[1]\) with \(P \in P\) and \(Q \in Q\). Applying again Proposition 82, it follows that \((\text{Ob}(\text{Ideal}(V)), JL)\) is a projective class, and it is easy to see that \(X \in \text{Ob}(\text{Ideal}(V))\) if and only if \(X\) is a direct summand of an object from \(V\), hence
\[
(\text{add}(V), JL)
\]
is a projective class.

**Remark 84.** Dually, we can consider injective classes \((I, Q)\), and the duals of above results are also valid. For the case when \(T\) is a \(k\)-category \((k\) is a field) and the homomorphisms groups \(T(A, B)\) are finitely dimensional for all objects \(A\) and \(B\) in \(T\) then it is easy to see that for every object \(A\) in \(T\) the class \(\text{add}(A)\) is precovering and preenveloping. Therefore it induces a projective class \((\text{add}(A), I)\) and an injective class \((I, \text{add}(A))\). Here the homomorphisms from \(J\) (resp. \(I\)) are called \(A\)-ghosts (co-ghosts). A direct application of (co-)Ghost Lemma 80 and Proposition 82 lead us to the Ghost/Co-ghost Lemma and Converse proved in [5, Lemma 2.17].

Moreover, we have the following dual of Christensen’s Ghost Lemma:

**Corollary 85.** Let \((P, J)\) and \((Q, K)\) be two projective classes in \(T\), and denote by \(\mathcal{T}(P, Q)\) the ideal of all homomorphisms which factorize through a homomorphism \(P \to Q\) with \(P \in P\) and \(Q \in Q\). Then the pair
\[
(\mathcal{T}(P, Q), \langle J, K \rangle_D)
\]
is an ideal cotorsion pair with respect to the class of all triangles in \(T\).

**Proof.** This follows from Corollary 80 since \(\text{Ideal}(Q) \text{Ideal}(P) = \mathcal{T}(P, Q)\). \( \square \)

### 5.2. Krause’s telescope theorem for projective classes

We will apply the previous results to extend [21, Proposition 4.6] to projective classes in compactly generated triangulated categories.

Let \(T\) be a compactly generated category and denote by \(T_0\) a representative set of compact objects in \(T\). Then this induces a projective class \((\mathcal{P}, \mathcal{P}h)\), where \(\mathcal{P} = \text{Add}(T_0)\) is the class of pure-projective objects in \(T\) and \(\mathcal{P}h\) is the class of (classical) phantoms in \(T\). If \((B, J)\) is another projective class, we observe that the ideal \(\mathcal{T}(B, P)\) of all homomorphisms which factorize through homomorphisms \(B \to P\) with \(B \in B\) and \(P \in P\) is a precovering ideal. Then, as in Example 27 we
consider the weak proper class \( \mathcal{C} = \mathcal{C}_{\mathcal{T}(B, P)} \) of all triangles such that all elements in \( \mathcal{T}(B, P) \) are projective with respect to these triangles. We also consider the weak proper class \( \mathfrak{F} = \mathfrak{F}_P \) of all pure triangles in \( \mathcal{T} \), i.e. \( \mathbf{Ph}(\mathfrak{F}) = \mathcal{P}h \). It is easy to see that \( \mathfrak{F} \subseteq \mathcal{C} \), so we can consider the class \( \Phi_\mathcal{C}(\mathfrak{F}) \) of all relative \( \mathfrak{F} \)-phantoms associated to \( \mathcal{C} \). These relative phantoms can be characterized in the following way:

**Lemma 86.** The following are equivalent for a homomorphism \( \varphi : A \to B \):

1. \( \varphi \in \Phi_\mathcal{C}(\mathfrak{F}) \);
2. for every compact object \( C \) and every homomorphism \( \alpha : C \to A \) we have \( \varphi \alpha \in \mathcal{T}(B, P) \).

**Proof.** (1)\(\Rightarrow\)(2) From Proposition 36 it follows that for every compact object \( C \) and every homomorphism \( \alpha : C \to A \) the homomorphism \( \varphi \alpha \) is \( \mathcal{C} \)-projective. Using Example 27 we obtain that \( \varphi \alpha \in \mathcal{T}(B, P) \).

(2)\(\Rightarrow\)(1) In order to apply Proposition 36 we have to prove that for every pure-projective object \( P \) and every homomorphism \( \alpha : P \to A \) the homomorphism \( \varphi \alpha \) is \( \mathcal{C} \)-projective. Since \( P \) is a direct summand of a direct sum of compact objects, we can assume w.l.o.g. that \( P = \oplus_{i \in I} C_i \) is a direct sum of compact objects. Then for every \( i \in I \) we have \( \varphi \alpha u_i \) is \( \mathcal{C} \)-projective (\( u_i \) denotes the canonical map \( C_i \to \oplus_{i \in I} C_i \)) and we apply Lemma 19 to obtain the conclusion. \( \square \)

**Lemma 87.** Let \( \mathcal{T} \) be an additive category. If \( u : C \to A, v : A \to D_1 \oplus D_2 \) and \( \alpha : C \to D_1 \) are homomorphisms in \( \mathcal{T} \) such that \( \nu u = \rho \alpha \) (i.e. \( \nu u \) factorizes through \( \rho \)), where \( \rho : D_1 \to D_1 \oplus D_2 \) is the canonical homomorphism, then \( \alpha = \nu \pi \nu u \), where \( \pi : D_1 \oplus D_2 \to D_1 \) is the canonical projection (i.e. \( \alpha \) factorizes through \( A \)).

**Proof.** From \( \rho \alpha = \nu u \) we obtain \( \rho \pi \nu u = \rho \pi \rho \alpha = \rho \alpha \), hence \( \nu \pi \nu u = \alpha \) since \( \rho \) is split mono. \( \square \)

**Lemma 88.** Let \( \mathcal{T} \) be compactly generated triangulated category and let \( \mathcal{B} \) be a class of objects in \( \mathcal{T} \).

We denote by \( \mathcal{T}_\mathcal{B} \) the set of all homomorphisms between compact objects which factorize through an object \( B \in \mathcal{B} \), and by

\[
\mathcal{T}_\mathcal{B} = \text{Ideal}(\text{Add}(\mathcal{T}_0))(\mathcal{B} \text{Ideal}(\text{Add}(\mathcal{T}_0)))
\]

the ideal generated by class of all homomorphisms between pure-projective objects which factorize through an object \( B \in \mathcal{B} \).

If \( \mathcal{C} \) is a category with direct sums, and \( F : \mathcal{T} \to \mathcal{C} \) is a functor which commutes with direct sums, the following are equivalent:

(a) \( F(\mathcal{T}_\mathcal{B}) = 0 \);
(b) \( F(\mathcal{T}_B) = 0 \).

**Proof.** (a)\(\Rightarrow\)(b) It is enough to prove that if we consider two arbitrary families \( (C_\lambda)_{\lambda \in \Lambda} \) and \( (D_\kappa)_{\kappa \in \Lambda} \) then for every homomorphism

\[
\alpha : \oplus_{\lambda \in \Lambda} C_\lambda \to A \to \oplus_{\kappa \in \Lambda} D_\kappa
\]

with \( A \in \mathcal{B} \) we have \( F(\alpha) = 0 \). If \( \alpha : \oplus_{\lambda \in \Lambda} C_\lambda \to A \to \oplus_{\kappa \in \Lambda} D_\kappa \) then we observe that

\[
F(\alpha) : \oplus_{\lambda \in \Lambda} F(C_\lambda) \to F(A) \to F(\oplus_{\kappa \in \Lambda} D_\kappa).
\]

Since \( F \) commutes with respect to direct sums, \( F(\oplus_{\lambda \in \Lambda} C_\lambda) \) is the direct sum of the family \( (F(C_\lambda))_{\lambda \in \Lambda} \), and the canonical homomorphisms associated to this direct
sum are \( F(u_\lambda), \lambda \in \Lambda \), where \( u_\lambda \) are the canonical homomorphisms associated to the direct sum \( \oplus_{\lambda \in \Lambda} C_\lambda \). Hence \( F(\alpha) \) can be identified to a family \((F(\alpha_\lambda))_{\lambda \in \Lambda}\) of homomorphisms \( \alpha_\lambda : C_\lambda \to A \to \oplus_{\kappa \in K} D_\kappa \). Since every \( C_\lambda \) is compact, using Lemma 87 we observe that every homomorphism \( \alpha_\lambda \) can be viewed as a homomorphism \( \alpha'_\lambda : C_\lambda \to A \to \oplus_{\kappa \in \kappa_\lambda} D_\kappa \), where \( \kappa_\lambda \) are finite subsets of \( K \) for all \( \lambda \in \Lambda \).

\[ \text{Since } F(\alpha'_\lambda) = 0 \text{ for all } \lambda, \text{ it follows that } F(\alpha) = 0. \square \]

We recall that a covariant functor \( H : \mathcal{T} \to \mathcal{A} \), where \( \mathcal{A} \) is abelian, is called **cohomological** if it sends triangles to exact sequences.

**Proposition 89.** Let \( \mathcal{T} \) be a compactly generated triangulated categories, and let \((\mathcal{B}, \mathcal{J})\) be a projective class in \( \mathcal{T} \). The following are equivalent for a Grothendieck category \( \mathcal{A} \) and a cohomological functor \( H : \mathcal{T} \to \mathcal{A} \) which commutes with direct sums:

(a) \( H(\Phi_\mathcal{E}(\mathfrak{F})) = 0; \)

(b) \( H(\mathcal{I}_B) = 0. \)

**Proof.** (a)\(\Rightarrow\)(b) is obvious since \( \mathcal{I}_B \subseteq \mathcal{E-proj} \subseteq \Phi_\mathcal{E}(\mathfrak{F}). \)

(b)\(\Rightarrow\)(a) Let \( \psi : X \to A \) be a homomorphism from \( \Phi_\mathcal{E}(\mathfrak{F}). \)

It is easy to see that \( \mathcal{I}_B = \text{Ideal}(\mathcal{P}) \text{Ideal}(\mathcal{B}) \text{Ideal}(\mathcal{P}) = \overline{\mathcal{T}(B, \mathcal{P}) \text{Ideal}(\mathcal{P})} \)

is a precovering ideal, so the \( \mathcal{I}_B \)-orthogonal ideal with respect the class \( \mathcal{D} \) of all triangles is

\[ \mathcal{I}_B^\perp = \langle \mathcal{P} \mathcal{h}, \overline{\mathcal{T}(B, \mathcal{P}) \text{Ideal}(\mathcal{P})} \rangle_{\mathcal{D}}, \]

and it is a (special) preenveloping ideal.

Therefore every object \( A \) from \( \mathcal{T} \) has a special \( \mathcal{I}_B^\perp \)-preenvelope \( \gamma_A : A \to A^* \)

which can be obtained as a composition \( A \xrightarrow{\mu} Y \xrightarrow{\nu} A^* \) of two homomorphisms which lie in the solid part of the commutative diagram

\[
\begin{array}{c}
\text{(ENV)}
\end{array}
\]
where \( \alpha \in \mathcal{T}(\mathcal{B}, \mathcal{P})^\perp \), \( \varphi \in \mathcal{P} h \), and the horizontal line is a triangle in \( \mathcal{T} \). Moreover, since \( \gamma_A \) is a special \( \mathcal{T}_B^\perp \)-preenvelope, we have a commutative diagram

\[
\begin{array}{ccccccc}
I[-1] & \longrightarrow & A & \xrightarrow{\gamma_A} & A^* & \longrightarrow & I & \longrightarrow & A[1] \\
| & | & | & \Downarrow & | & | & | & | & | \\
X[-1] & \longrightarrow & A & \xrightarrow{e} & E & \longrightarrow & X & \longrightarrow & A[1]
\end{array}
\]

with \( i \in \mathcal{T}_B \). By Lemma 88 since \( H(\mathcal{I}_B) = 0 \), we obtain \( H(\mathcal{I}_B) = 0 \), hence \( H(\gamma_A) \) is a monomorphism. So, in order to obtain \( H(\psi) = 0 \) it is enough to prove that \( H(\gamma_A \psi) = 0 \).

Let \( C \) be a compact object and \( \xi : C \rightarrow X \) a homomorphism. By Lemma 88 we obtain that \( \psi \xi \in \mathcal{T}(\mathcal{B}, \mathcal{P}) \), hence \( \gamma A \xi \) factorizes through \( f \), hence \( \mu \gamma A \xi \) factorizes through \( \varphi \). But \( \varphi \) is a phantom and \( C \) is compact, and this implies \( \gamma A \psi \xi = 0 \). Then \( \mu \gamma A \psi \xi \in \mathcal{P} h \), hence \( \gamma_A \psi \xi \) is a phantom. By [20, Corollary 2.5] we obtain \( H(\gamma_A \psi) = 0 \), and the proof is complete.

We apply this result for the particular case when \( B \) is a the kernel of a localizing functor. For further reference, let us remark that in this case \( B \) is a localizing subcategory, i.e. it is a full triangulated subcategory of \( \mathcal{T} \) which is closed under direct sums. Let \( \mathcal{T} \) and \( \mathcal{C} \) be compactly generated triangulated categories, and let \( F : \mathcal{T} \rightarrow \mathcal{C} \) be a functor. We recall that \( F \) is a localizing functor if it has a right adjoint \( G : \mathcal{C} \rightarrow \mathcal{T} \) such that the induced natural transformation \( FG \rightarrow 1_\mathcal{C} \) is an isomorphism. Note that every localizing functor commutes with respect to direct sums.

Let \( F : \mathcal{T} \rightarrow \mathcal{C} \) a localization functor. If \( G : \mathcal{C} \rightarrow \mathcal{T} \) is its right adjoint and \( \eta : 1_\mathcal{T} \rightarrow GF \) is the induced natural transformation then for every \( X \in \mathcal{T} \) we can fix a triangle

\[ X' \xrightarrow{\nu_X} X \xrightarrow{\eta_X} GF(X) \rightarrow X'[1]. \]

Applying \( F \) we obtain that \( F(\nu_X) \) is an isomorphism, and it follows that \( F(X') = 0 \), hence \( X' \in B \).

Let \( \mathcal{B} = \{ X \in \mathcal{T} \mid F(X) = 0 \} \) be the kernel of \( F \). For every \( B \in \mathcal{B} \) we have \( \mathcal{T}(\mathcal{B}, GF(X)) \cong \mathcal{T}(F(B), F(X)) = 0 \), hence \( \eta_X \in B \perp \). Since \( \mathcal{B} \) is closed with respect to direct summands, we can apply [11, Lemma 3.2] to conclude that \( (\mathcal{B}, B \perp) \) is a projective class (the orthogonal class \( B^{\perp} \) is computed with respect to the class \( \mathfrak{D} \) of all triangles). In fact the ideal \( B^{\perp} \) is in this case an object ideal.

For every \( B \in \mathcal{B} \) the abelian group homomorphism \( \text{Hom}(B, \nu_X) \) is an isomorphism, hence \( \nu_X \) is an \( \text{Ideal}(\mathcal{B}) \)-precover and \( \eta_X \) is a \( B \perp \)-preenvelope for all \( X \in \mathcal{T} \).

We have the following corollary.

**Corollary 90.** Let \( F : \mathcal{T} \rightarrow \mathcal{C} \) be a localizing functor between the compactly generated triangulated categories \( \mathcal{T} \) and \( \mathcal{C} \). If \( \mathcal{B} = \text{Ker}(F) \) and we keep the notations used in this subsection then \( \text{Ob}(\Phi_{\mathcal{E}}(\mathfrak{F})) \subseteq \mathcal{B} \).

**Proof.** As in [20] we consider the Grothendieck category \( \text{Mod-}\mathcal{C}_0 \) of all contravariant functors \( \mathcal{C}_0 \rightarrow Ab \), and the functor \( h_\mathcal{C} : \mathcal{C} \rightarrow \text{Mod-}\mathcal{C}_0 \) defined by \( h_\mathcal{C}(X) = \mathcal{C}(\cdot, X)_{|\mathcal{C}_0} \). Then \( h_\mathcal{C} F : \mathcal{T} \rightarrow Ab \) is a cohomological functor such that \( \text{Ker}(h_\mathcal{C} F) = \mathcal{B} \). Then \( h_\mathcal{C} F(\mathcal{I}_B) = 0 \), and it follows that \( \text{Ob}(\Phi_{\mathcal{E}}(\mathfrak{F})) \subseteq \text{Ker}(h_\mathcal{C} F) = \mathcal{B} \).
5.3. **Smashing subcategories.** Let $F$ be a localizing functor and $G$ its right adjoint. If $G$ also commutes with respect to direct sums then $F$ is *smashing*. A subcategory $B$ of $\mathcal{T}$ is a *smashing subcategory* if and only if there exists a smashing functor $F$ such that $B = \text{Ker}(F)$. Note that a subcategory $B$ of $\mathcal{T}$ is a smashing if and only if $B$ is a localizing subcategory of $\mathcal{T}$ such that every homomorphism $C \to B$ with $C \in \mathcal{T}_0$ and $B \in B$ can be factorized as

$$(C \to B) = (C \to B' \to C' \to B)$$

with $B' \in B$ and $C' \in \mathcal{T}_0$, [20, Theorem 4.2]. Therefore, using Lemma [59] it is easy to see that a localizing functor $F$ is smashing if and only if $\text{Ob}(\Phi_{E_0}(F)) = B$, so Proposition [59] is a generalization for [20, Proposition 4.6].

We will say that a smashing subcategory $B$ of a compactly generated triangulated category $\mathcal{T}$ satisfies the telescope conjecture if for every compactly generated triangulated category $\mathcal{C}$ and every exact functor $H : \mathcal{T} \to \mathcal{C}$ which preserves direct sums and annihilates the subcategory $B_0$ of all compact objects $C \in B$ we obtain $H(B) = 0$. Note that this is equivalent to the fact that $B$ is the smallest smashing subcategory which contains $B_0$. In the following we will present a characterization for this properties using relative phantom ideals. In order to do this, let us consider the weak proper class $E_0$ induced by $B_0$ as in Example 15, i.e.

$$\text{Ph}(E_0) = \{ \varphi \mid T(B_0, \varphi) = 0 \}.$$  

Note that $E_0$ has enough projective homomorphisms and a homomorphism is $E_0$-projective if and only if it factorizes through an object from $\text{Add}(B_0)$. Moreover, we have

**Lemma 91.** If $\mathcal{E}$ and $\mathcal{E}_0$ are defined as before, we have $\mathfrak{E} \subseteq \mathcal{E} \subseteq \mathcal{E}_0$, hence $\Phi_{\mathcal{E}_0}(\mathfrak{E}) \subseteq \Phi_{\mathcal{E}}(\mathfrak{E})$.

**Proof.** Let us denote by $\mathcal{P}_0$ the class $\text{Add}(B_0)$. Hence $\mathcal{P}_0 \subseteq B$ and $\mathcal{P}_0 \subseteq \mathcal{P}$, and we have

$$\mathcal{T}(B, \mathcal{P}) \supseteq \mathcal{T}(\mathcal{P}_0, \mathcal{P}_0) = \text{Ideal}(\mathcal{P}_0).$$

Now the conclusion is obvious. \hfill \Box

Now we will prove the promised characterization.

**Proposition 92.** Let $B$ be a smashing subcategory of $\mathcal{T}$. If $\mathcal{E}_0$ is defined as before, the following are true:

(a) $\text{Ob}(\Phi_{\mathcal{E}_0}(\mathfrak{E}))$ is a smashing subcategory of $\mathcal{T}$;

(b) $B$ satisfies the telescope conjecture if and only if $\text{Ob}(\Phi_{\mathcal{E}_0}(\mathfrak{E})) = B$.

**Proof.** (a) In order to prove that $\text{Ob}(\Phi_{\mathcal{E}_0}(\mathfrak{E}))$ is a triangulated subcategory, we consider a triangle $Y \xrightarrow{\beta} X \xrightarrow{\alpha} Z \to Y[1]$ in $\mathcal{T}$ such that $Y, Z \in \text{Ob}(\Phi_{\mathcal{E}_0}(\mathfrak{E}))$. Let $C$ be a compact object, and $\gamma : C \to X$ a homomorphism. We have to prove that $\gamma$ is projective with respect to $\mathcal{E}_0$, i.e. $\gamma$ factorizes through an object from $B_0$.

Since $Z \in \text{Ob}(\Phi_{\mathcal{E}_0}(\mathfrak{E}))$ and $C$ is compact we know that $\beta \gamma$ factorizes through an object $B \in B_0$. Therefore we have a commutative diagram

$$
\begin{array}{ccccccccc}
C' & \xrightarrow{\zeta} & C & \xrightarrow{\delta} & B & \xrightarrow{\gamma} & C'[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & Z & \to Y[1]
\end{array}
$$
such that the horizontal lines are triangles in $\mathcal{T}$. Since $C'$ is compact and $Y \in \text{Ob}(\Phi_{\mathcal{E}_0}(\mathcal{F}))$ the homomorphism $\delta$ factorizes through an object $B' \in \mathcal{B}_0$, i.e. $\delta = \mu\nu$ with $\nu : C' \to B'$ and $\mu : B' \to Y$. Using a cobase change we can complete the above commutative diagram to the following commutative diagram

\[
\begin{array}{ccc}
B' & \longrightarrow & D \\
\mu & \downarrow & \downarrow \\
C' & \longrightarrow & C \\
\delta & \downarrow & \downarrow \\
Y & \alpha & X \\
\end{array}
\]

where the dotted arrow exists since $\gamma\zeta = \alpha\delta = \alpha\mu\nu$. It follows that $\gamma$ factorizes through $D$. Since $\mathcal{B}$ is a triangulated subcategory, we obtain that $D \in \mathcal{B}_0$, hence $X \in \text{Ob}(\Phi_{\mathcal{E}_0}(\mathcal{F}))$.

It is easy to see that $\text{Ob}(\Phi_{\mathcal{E}_0}(\mathcal{F}))$ is closed under direct sums, hence $\text{Ob}(\Phi_{\mathcal{E}_0}(\mathcal{F}))$ is a localizing subcategory in $\mathcal{T}$.

Moreover if $X \in \text{Ob}(\Phi_{\mathcal{E}_0}(\mathcal{F}))$ and $C$ is a compact object in $\mathcal{T}$ then every homomorphism $C \to X$ factorizes through an object $B \in \mathcal{B}_0$, and we can write

\[
(C \to X) = (C \to B \xrightarrow{=} B \to X).
\]

Since $\mathcal{B}_0 \subseteq \text{Ob}(\Phi_{\mathcal{E}_0}(\mathcal{F}))$ we can apply [20, Theorem 4.2] to obtain that $\text{Ob}(\Phi_{\mathcal{E}_0}(\mathcal{F}))$ is a smashing subcategory.

(b) If $\mathcal{B}$ satisfies the telescope conjecture then we have $\mathcal{B} = \text{Ob}(\Phi_{\mathcal{E}_0}(\mathcal{F}))$ since $\text{Ob}(\Phi_{\mathcal{E}_0}(\mathcal{F})) \subseteq \mathcal{B}$ is smashing.

Conversely, let $H : \mathcal{T} \to \mathcal{C}$ be an exact functor between compactly generated triangulated categories such that it preserves direct sums and $H(\mathcal{B}_0) = 0$. We observe that all homomorphisms from $I_B$ factorize through objects from $\mathcal{B}_0$, hence $H(\mathcal{B}_0) = 0$. We apply Proposition [20] to obtain $h\mathcal{C}H(\mathcal{B}) = 0$, hence $H(\mathcal{B}) = 0$. □

As a corollary we obtain the following characterization, proved by H. Krause in [19, Theorem 13.4].

Corollary 93. Let $\mathcal{B}$ be a smashing subcategory of a compactly generated subcategory $\mathcal{T}$. The following are equivalent:

(a) $\mathcal{B}$ satisfies the telescope conjecture;
(b) for every compact object $C$ and for every object $B \in \mathcal{B}$ every homomorphism $C \to B$ factorizes through an object from $\mathcal{B} \cap \mathcal{T}_0$;
(c) $\mathcal{B}$ is a compactly generated as a triangulated category.

Proof. (a)$\iff$(b) follows from Proposition [22]

(b)$\implies$(c) For every non-zero object $B \in \mathcal{B}$ we can find a non-zero homomorphism $C \to B$ with $C$ a compact object in $\mathcal{T}$. Applying (b) this homomorphism factorizes through a (non-zero) homomorphism $C_0 \to B$ with $C_0 \in \mathcal{B} \cap \mathcal{T}_0$.

(c)$\implies$(a) Let $F$ be a localizing functor which induces $\mathcal{B}$, and let $G$ be its right adjoint. As before, we denote by $\eta : 1_{\mathcal{T}} \to GF$ is the induced natural transformation.

We first observe that if $B$ is a compact from $\mathcal{B}$ and $\alpha : B \to X = \bigoplus_{i \in I} X_i$ is a homomorphism in $\mathcal{T}$ then $\eta_X \alpha = 0$. If we embed every object $Y$ in the canonical
we observe that \( X' = \bigoplus_{i \in I} X'_i \in \mathcal{B}, \nu_X = \bigoplus_{i \in I} \nu_{X_i} \), and \( \alpha \) factorizes through \( \nu_X \).

Since every homomorphism \( B \to \bigoplus_{i \in I} X'_i \) factorizes through a finite subset of \( I \), it is easy to see that \( \alpha \) has the same property. Therefore every compact from \( \mathcal{B} \) is compact in \( \mathcal{T} \).

Since every object from \( \mathcal{B} \) is a homotopy colimit of pure-projective objects (cf. the proof of [24, Theorem 3.1 and Lemma 3.2]) and the homotopy colimits are computed in the same way in \( \mathcal{B} \) as in \( \mathcal{T} \) (as cones of Milnor’s triangles), we can apply [24, Lemma 2.8] to obtain the conclusion. \( \square \)

5.4. **Full functors.** Let \((\mathcal{B}, \mathcal{J})\) be a projective class. If we keep the notations used in this section, we can apply Corollary 80 (Ghost Lemma) and Remark 81 to compute the right \( \mathcal{I}_B \)-orthogonal ideal (with respect to the class \( \mathcal{D} \) of all triangles)

\[
\mathcal{I}_B = \langle \mathcal{P}h, \mathcal{J}, \mathcal{P}h \rangle_{\mathcal{D}}.
\]

Therefore for every object \( A \) from \( \mathcal{T} \) the \( \mathcal{I}_B \)-preenvelope \( \gamma_A : A \to A^* \) can be obtained as a composition \( A \overset{\mu}{\to} V \overset{\nu}{\to} A^* \) of two homomorphisms which lie in a commutative diagram

\[
(\text{ENV}')
\]

\[
\begin{array}{c}
A \\
\downarrow \varphi \\
X \quad Y \quad Z \\
\downarrow \beta \\
U \\
\downarrow \psi \\
A^*
\end{array}
\]

such that \( \varphi, \psi \in \mathcal{P}h \) and \( \beta \in \text{Ideal}(\mathcal{B})^\perp \).

We will use this diagram to prove that in the case of full functors the hypothesis \( F(\mathcal{I}_B) = 0 \) always implies \( F(\mathcal{B}) = 0 \).

**Proposition 94.** Let \( \mathcal{T} \) be a compactly generated triangulated category and \( B \) an object in \( \mathcal{T} \). We denote by \( \mathcal{I}_B \) the set of all homomorphisms between compact objects which factorize through \( B \). If \( A \) is a Grothendieck category and \( F : \mathcal{T} \to A \) a full cohomological functor which commutes with direct sums, the following are equivalent:

(a) \( F(B[n]) = 0 \) for all \( n \in \mathbb{Z} \);

(b) \( F(\mathcal{I}_B[n]) = 0 \) for all \( n \in \mathbb{Z} \).

**Proof.** We will work with the projective class \((\text{Add}(B), \mathcal{J})\), and we consider the ideal \( \mathcal{I} = \mathcal{I}_{\text{Add}(B)} \).

Then for an object \( A \) the \( \mathcal{I}^\perp \)-preenvelope \( \gamma_A : A \to A^* \) can be embedded in a commutative diagram \((\text{ENV}')\) such that \( \varphi, \psi \in \mathcal{P}h \) and \( \beta \in \text{Ideal}(\text{Add}(B))^\perp \).
Using [20, Corollary 2.5] we obtain that \( F(\varphi) = 0 \) and \( F(\psi) = 0 \). Then applying \( F \) to the above diagram we obtain the solid part of the following commutative diagram:

\[
\begin{array}{ccccccccc}
F(A) & \rightarrow & F(X) & \rightarrow & F(Y) & \rightarrow & F(Z) & \rightarrow & F(X[1]) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & F(U) & \rightarrow & F(V) & \rightarrow & F(W) & \rightarrow & F(U[1]) \\
\downarrow & & 0 & & \downarrow & & \downarrow & & \\
F(A^*) & & & & & & & & \\
\end{array}
\]

Since the horizontal lines are exact sequences then we can complete the diagram with the homomorphism \( F(A) \rightarrow F(X) \). But \( F \) is full, hence we can find a homomorphism \( A \rightarrow X \) such that \( F(A \rightarrow X) = F(A) \rightarrow F(X) \).

If \( A \in \text{Add}(B) \) then we have

\[ F(A) \rightarrow F(X) \rightarrow F(W) = F(A \rightarrow X \delta W) = 0 \]

since \( \delta \in \text{Ideal}(\text{Add}(B)) \), hence we can find the homomorphism \( F(A) \rightarrow F(U) \).

It follows that \( F(\gamma A) = 0 \). Since \( \gamma A \) is a special preenveloping, as in the proof of Proposition [9] we have a commutative diagram

\[
\begin{array}{ccccccccc}
I[-1] & \rightarrow & A & \rightarrow & A^* & \rightarrow & I & \rightarrow & A[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
X[-1] & \rightarrow & A & \rightarrow & E & \rightarrow & X & \rightarrow & A[1] \\
\end{array}
\]

with \( i \in I \). Applying \( F \) we obtain the exact sequence

\[ F(I[-1]) \rightarrow F(A) \rightarrow F(A^*), \]

hence \( F(A) = 0 \). \qed

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