A VANISHING THEOREM FOR LOG CANONICAL PAIRS
AFTER DE FERNEX-EIN

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Abstract. We extend the main vanishing theorem in de Fernex-Ein to singular varieties without assuming locally complete intersection.

1. INTRODUCTION

In this note we prove the following Nadel vanishing type theorem,

Theorem 1.1. Given a log canonical pair \((X, \Delta; eZ)\), where \(Z \subset X\) is a pure-dimensional reduced subscheme of codimension \(e\). Suppose that none of the components of \(Z\) is contained in \(\text{Sing}(X) \cup \text{Supp}(\Delta)\), then for any nef line bundles \(A\) and \(M\), such that \(A \otimes \mathcal{O}(-K_X - \Delta)\) is ample and \(M \otimes \mathcal{I}_Z^{\otimes e}\) is globally generated, we have

\[ H^i(X, A \otimes M \otimes \mathcal{I}_Z) = 0 \text{ for } i > 0. \]

In particular, suppose \(Z\) is scheme-theoretically given by

\[ Z = H_1 \cap \cdots \cap H_t \]

for some divisors \(H_i \in |L^{d_i}|\), where \(L\) is a globally generated line bundle such that \(d_1 \geq \cdots \geq d_t\). Then

\[ H^i(X, A \otimes L^{\otimes k} \otimes \mathcal{I}_Z) = 0 \text{ for } i > 0, k \geq e \cdot d_1. \]

This result partially generalizes the main vanishing theorem in de Fernex-Ein, which assumes \(X\) is a locally complete intersection variety with rational singularities.

It has been some efforts generalizing the vanishing theorem in \([6]\). In \([11]\) Niu proves an analogous vanishing theorem for power of ideal sheaves. In \([13]\), using technique of generic linkage, the vanishing theorem in \([6]\) is generalized to pairs \((\mathbb{P}^n, eZ)\) which are log canonical except at finitely many points. In this note, we also prove this type of generalization,

Theorem 1.2. The conclusion of Theorem 1.1 is still true if \((X, \Delta; eZ)\) is log canonical except at finitely many points.

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\[ ^2\text{The lower bound of } k \text{ is bigger than the one in [6].} \]
In [6] the main application is questions related to Castelnuovo-Mumford regularity of singular subvarieties in projective spaces, which generalizes the results in Bertram-Ein-Larzarsfeld [3] and Chardin-Ulrich [5]. (see also Niu [10], [12] for related results).

On the contrary, the main results of this note are in particular suitable for regularity problems of subvarieties in singular varieties. For example, given a curve $Z$ in a Schubert variety $X$. By result of [2], there is some $\Delta$ such that $(X, \Delta)$ is log terminal. and the support of $\Delta$ is in the complement of the biggest Schubert cell $U \subset X$ (a dense smooth open set). So suppose the generic point of $Z$ is contained in $U$, then $(X; \Delta, eZ)$ is log canonical except at finitely many points. Then by Theorem 1.2 we have a bound of regularity in terms of the degrees of generators of $I_Z$.

We make a remark about the the proof in this paper. The assumption of locally complete intersection is crucial to [6] because their strategy is to approximate $I_Z$ by some multiplier ideal sheaf. To achieve this they need to use inversion of adjunction theorem of locally complete intersection varieties.

In this paper, we apply a cohomology tool for log canonical pairs developed by Ambro-Fujino. We consider $\hat{X}$, the normalization of blowing up of $X$ along $Z$. The idea is that when we pull back $A \otimes M \otimes I_Z$ to $\hat{X}$, it becomes a line bundle and has more positivity properties. (Note that $I_Z \cdot \mathcal{O}_{\hat{X}}$ is relative ample.) Using Ambro-Fujino’s theorem (Theorem 2.2) we are able to reduce the vanishing result on $X$ to $\hat{X}$.

The organization of this paper goes as follows. In the second section we recall some notions we need, including the vanishing theorem by Ambro-Fujino. The proofs of the main theorems are in the last section.

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2. Preliminaries

We first recall the notion of singularities of pair. We say $(X, \Delta)$ is a pair if $X$ is normal and $K_X + \Delta$ is $\mathbb{Q}$-Gorenstein. More generally, we consider $(X, \Delta; eZ)$ where $Z$ is any subscheme of $X$ and $e$ is a positive integer. Take a log resolution $f : Y \to X$ such that the support of $\text{Exc}(f) \cup f_*^{-1}\Delta \cup I_Z \cdot \mathcal{O}_Y$ is a union of simple normal crossing divisors.

**Definition 2.1.** Given a resolution $f : Y \to X$ as above, suppose $I_Z \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$, we let $\text{div}(I_Z \cdot \mathcal{O}_Y)$ denote $-F$. In particular, $e \cdot \text{div}(I_Z \cdot \mathcal{O}_Y) = -eF$.

A pair $(X, \Delta; eZ)$ is called log canonical if for any log resolution as above we can write

$$K_Y - f^*(K_X + \Delta) + e \cdot \text{div}(I_Z \cdot \mathcal{O}_Y) = P - N$$
where both $P$ and $N$ are effective, without common components and all coefficients of components of $N$ are less or equal to one.

The following theorem due to Ambro-Fujino is essential to the proof of theorem 1.1. See [1] and [8] Theorem 6.3.

**Theorem 2.2.** Let $(Y, \Delta)$ be a simple normal crossing pair such that $\Delta$ is a boundary $\mathbb{R}$-divisor on $Y$. Let $f: Y \to X$ be a proper morphism between algebraic varieties and let $L$ be a Cartier divisor on $Y$ such that $L - (K_Y + \Delta)$ is $f$-semi-ample.

1. every associated prime of $R^q f_* O_Y(L)$ is the generic point of some stratum of $(Y, \Delta)$.
2. let $\pi: X \to V$ be a projective morphism to an algebraic variety $V$ such that $L - (K_Y + \Delta)$ $\sim_{\mathbb{R}} f^* H$ for some $\pi$-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $H$ on $X$. Then $R^q f_* O_Y(L)$ is $\pi$-acyclic, that is, $R^p \pi_* R^q f_* O_Y(L) = 0$ for every $p > 0$ and $q \geq 0$.

**3. Proofs of main theorems**

Through out this section we consider a pair $(X, \Delta; eZ)$, where $Z \subset X$ is a pure-dimensional reduced subscheme of codimension $e$. We also require that all of the irreducible components of $Z$ are not contained in either $\text{sing}(X)$ or $\text{supp}(\Delta)$. We first prove the following lemma needed later.

**Lemma 3.1.** There is a log resolution $f: Y \to (X, \Delta; eZ)$ such that,

$$K_Y - f^*(K_X + \Delta) + e \cdot \text{div}(I_Z \cdot O_Y) = P - N - E_Z,$$

where both $P$ and $N$ are effective, and $E_Z$ is the sum of all components of the support of $I_Z \cdot O_Y$ mapping to the generic points of $Z$. And the support of $P$ and $N$ does not contain any divisor mapping to any generic points of $Z$. In particular, $f(E_Z) = Z$.

**Proof.** Since no component of $Z$ is contained in $\text{sing}(X)$, we can take factorizing resolution of $Z$ in $X$. Moreover, since no component of $Z$ is contained in $\text{supp}(\Delta)$, by Corollary 3.2 in [7] we can take this resolution to be a log resolution of $(X, \Delta)$. So we have a log resolution $f: Y_1 \to (X, \Delta)$ such that (i) $f_1$ is an isomorphism over the generic points of $Z$. (ii)$I_Z \cdot O_{Y_1} = I_Z \cdot L$, where $L$ is a line bundle. (iii)The strict transform $Z_1 \subset Y_1$ of $Z$ is smooth and $Z_1 \cup \text{exc}(f_1) \cup \text{supp}(f_1^{-1}\Delta)$ is simple normal crossing.

Next we take the blow up of $Y_1$ along $Z_1$ and denote it by $f_2: Y_2 \to Y_1$, and let $Z_2$ be the exceptional divisor of $f_2$. In summary, we have the
following diagram.

\[
\begin{array}{ccc}
Z_2 & \to & Y_2 \\
\downarrow & & \downarrow f_2 \\
Z_1 & \to & Y_1 \\
\downarrow & & \downarrow f_1 \\
Z & \to & X
\end{array}
\]

Let \( K_{Y_2} - f_1^*(K_X + \Delta) = \sum a_i E_i \). Since \( f_1 \) is an isomorphism over the generic points of \( Z \), none of the \( E_i \)'s is mapping to generic points of \( Z \). On the other hand, \( f_2 \) is an blowing up of smooth variety along smooth subscheme of codimension \( e \), we have \( K_{Y_2} - f_2^* K_{Y_1} = (e - 1) Z_2 \).

Let \( \mathcal{I}_Z \cdot \mathcal{O}_{Y_2} = \mathcal{O}_{Y_2}( -Z_2 - F) \) where \( Z_2 \) is mapped to generic points of \( Z \) and \( F \) is mapped to non-generic points. Then we have

\[
K_Y - f^*(K_X + \Delta) + e \cdot \text{div}(\mathcal{I}_Z \cdot \mathcal{O}_{Y_2}) = K_{Y_2} - f_2^* K_{Y_1} + f_2^*(K_{Y_1} - f_1^*(K_X + \Delta)) - e \cdot Z_2 - e \cdot F
\]

\[
= (e - 1) \cdot Z_2 + f_2^* (\sum a_i E_i) - e \cdot Z_2 - e \cdot F
\]

\[
:= P = N - Z_2
\]

Since none of the \( E_i \)'s passes through the generic point of \( Z_1 \), the support of \( f_2^*(\sum a_i E_i) \) does not contain any component of \( Z_2 \). In conclusion, the support of \( P \) and \( N \) does not contain any divisor dominating any component of \( Z \). \(\square\)

**Theorem 3.2.** (=Theorem 3.1) Suppose the pair \((X, \Delta; eZ)\) is log canonical. Then for any nef line bundles \( A \) and \( M \), such that \( A \otimes \mathcal{O}(-K_X - \Delta) \) is ample and \( M \otimes \mathcal{I}_Z^{e} \) is globally generated, we have

\[ H^i(X, A \otimes M \otimes \mathcal{I}_Z) = 0 \]

for \( i > 0 \).

**Proof.** Let \( h : \hat{X} \to X \) be the normalization of blowing up of \( X \) along \( Z \), and line bundle \( \mathcal{O}_{\hat{X}}(1) = \mathcal{I}_Z \cdot \mathcal{O}_{\hat{X}} \). Note that \( \mathcal{O}_{\hat{X}}(1) \) is \( h \)-ample and \( h_* \mathcal{O}_{\hat{X}}(1) = \mathcal{I}_Z = \mathcal{I}_Z \) since \( \mathcal{I}_Z \) is a radical ideal.

Take a log resolution \( f : Y \to (X, \Delta; eZ) \) as in lemma 3.1. Since the inverse image \( \mathcal{I}_Z \cdot \mathcal{O}_Y \) is invertible and \( Y \) is smooth, we see that \( f \) factors through \( h \), with morphism \( g : Y \to \hat{X} \). We also note that \( g^* \mathcal{O}_{\hat{X}}(1) = \mathcal{I}_Z \cdot \mathcal{O}_Y \). Then by the log canonical assumption we have

\[
K_Y - f^*(K_X + \Delta) + e \cdot \text{div}(g^* \mathcal{O}_{\hat{X}}(1)) = P - B - E - E_Z,
\]

where \( P \) is effective, \( |B| = 0 \), \( E \) is reduced and \( E_Z \) is the components of the support of \( \mathcal{I}_Z \cdot \mathcal{O}_Y \) mapping to the generic points of \( Z \). Note that the support of \( P \) does not dominate any component of \( Z \) by lemma 3.1.

To apply theorem 2.2, we go as following. First we define \( \Delta_Y = |P| - P + B + E \), then \((Y, \Delta_Y)\) is a simple normal crossing pair. Then we define
by adding $f^*A + f^*M + \Delta_Y$ to both hand sides of equation (3.1), that is,

$$L := f^*A + f^*M + K_Y - f^*(K_X + \Delta) + e \cdot \text{div}(g^*\mathcal{O}_{\hat{X}}(1)) + \Delta_Y = f^*A + f^*M - E_Z + [P]$$

Note that then $L$ is a line bundle.

To apply Theorem 2.2 (2), we calculate

$$L - (K_Y + \Delta_Y) = f^*A - f^*(K_X + \Delta) + f^*M + e \cdot \text{div}(g^*\mathcal{O}_{\hat{X}}(1))$$

$$= g^*(h^*A - h^*(K_X + \Delta) + h^*M + e \cdot \mathcal{O}_{\hat{X}}(1))$$

$$:= g^*H$$

because by assumption $A - K_X$ is ample and $M \otimes \mathcal{I}_Z^{\geq e}$ is globally generated.

In particular, $H$ is semiample. Plus the fact that $H.C > 0$ for every curve $C$ on $\hat{X}$ (easy to check), we conclude that $H$ is in fact an ample line bundle on $\hat{X}$. Then by Theorem 2.2 (2), we have $H^i(\hat{X}, g_*L) = 0$ for $i > 0$. Moreover, $H$ is also $h$-ample. So by loc. cit. we have $R^j h_*g_\ast L = 0, \forall j > 0$.

As a result,

$$0 = H^i(\hat{X}, g_*L) = H^i(X, h_*g_*L), \forall i > 0.$$

The theorem follows from this equation and the next claim.

**Claim:** $h_*g_*L = f_*L = A \otimes M \otimes \mathcal{I}_Z$.

**Proof:** Since $L = f^*A + f^*M - E_Z + [P]$, by projection formula it suffices to prove

$$h_*\mathcal{O}_{\hat{X}}(1) = h_*(g_*\mathcal{O}_Y([P] - E_Z)).$$

To this aim, note that

$$\mathcal{I}_Z = h_*\mathcal{O}_{\hat{X}}(1) \subset h_*g_* (\mathcal{O}_Y([P]) + g^*\mathcal{O}_{\hat{X}}(1)) \subset h_*g_* \mathcal{O}_Y([P] - E_Z) \subset \mathcal{O}_X$$

In other words, $h_*g_* \mathcal{O}_Y([P] - E_Z)$ is an ideal sheaf on $X$. By Lemma 3.1, $[P]$ and $E_Z$ do not have common components and $[P]$ is an effective exceptional divisor, so

$$h_*g_* \mathcal{O}_Y([P] - E_Z) = f_* \mathcal{O}_Y([P] - E_Z) = f_* \mathcal{O}_Y(-E_Z).$$

Note that $f_* \mathcal{O}_Y(-E_Z)$ is an ideal sheaf determines a scheme set theoretically equal to $Z$. So we have

$$\mathcal{I}_Z \subset f_* \mathcal{O}_Y(-E_Z) \subset \sqrt{\mathcal{I}_Z}.$$

Hence we have equation (3.3) and the claim. □

With same notations as Theorem 3.2,

**Theorem 3.3.** (=Theorem 1.2) The conclusion of Theorem 3.2 is still true if $(X, \Delta; eZ)$ is log canonical except at finitely many points.
Proof. Apply Lemma 3.1, we have the following equation similar to equation (3.1),

\[ K_Y - f^*(K_X + \Delta) + e \cdot \text{div}(I_Z \cdot O_Y) = P - N - E_Z - F. \]

The only difference is that there is a term \(-F\) denoting the exceptional over the non-log canonical points. We remark that by the assumption \(F\) is non-reduced.

Let \(I^* = f_*O_Y(-E_Z - F)\). Then exactly the same argument as in the proof of Theorem 3.2 shows

\[ H^i(X, A \otimes M \otimes I^*) = 0, \forall i > 0. \]

On the other hand we have,

\[ 0 \to I^* \to I_Z \to Q \to 0, \]

where \(Q\) is a sheaf supported on some close points by assumption. Take the long exact sequence we have

\[ H^i(X, A \otimes M \otimes I_Z) = 0, \forall i > 0. \]

\[ \square \]

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