Branching random walk in random environment with random absorption wall

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Abstract: We consider the branching random walk in random environment with a random absorption wall. When we add this barrier, we discuss some topics related to the survival probability. We assume that the random environment is i.i.d., $S_i$ is a particular i.i.d. random walk depend on the random environment $\mathcal{L}$. Let the random barrier function (the random absorption wall) be $g_i(\mathcal{L}) := a_i^\alpha - S_i$, where $i$ presents the generation. We show that there exists a critical value $a_c > 0$ such that if $a > a_c$, $\alpha = \frac{1}{3}$, the survival probability is positive almost surely and if $a < a_c$, $\alpha = \frac{1}{3}$, the survival probability is zero almost surely. Moreover, if we denote $Z_n$ is the total populations in $n$-th generation in the new system (with barrier), under some conditions, we show $\ln P_{\mathcal{L}}(Z_n > 0)/n^{1/3}$ will converges to a negative constant almost surely if $\alpha \in [0, \frac{1}{3})$.

Keywords: Branching random walk, random environment, barrier.

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1. Introduction

The model named branching random walk on $\mathbb{R}$ with random environment in time (BRWre) has been introduced in [4] and [10]. Let $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n, \ldots)$ be an i.i.d. random sequence of point process law which is also called the environment sequence. More precisely, $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n, \ldots)$ is an i.i.d. sequence of random variables with the values in the space of the distributions on the set of point processes on $\mathbb{R}$. Under a realization $(L_1, L_2, \ldots, L_n, \ldots)$ of $\mathcal{L}$, a time-inhomogeneous branching random walk is driven by the following way. It starts with one individual located at the origin at time 0. This individual dies at time 1 giving birth to children and the children’s position is according to the point process $L_1$. Similarly, at each time $n$ every individual alive at generation $n - 1$ dies and gives birth to children. The position of the children with respect to their parent are given by the point process $L_n$. We denote by $T$ the (random) genealogical tree of the process. For a given
individual $u \in T$ we write $V(u) \in \mathbb{R}$ for the position of $u$ and $|u|$ for the generation at which $u$ is alive. The pair $(T, V)$ is called the branching random walk with i.i.d. random environment $\mathcal{L}$. Thus we can see in the quenched sense, a branching random walk with random environment in time (BRWre) is a branching random walk with a time-inhomogeneous environment sequence $\mathcal{L} = (L_1, L_2, \ldots, L_n, \ldots)$. Conditionally on this environment sequence, we denote $P_{\mathcal{L}}$ for the law of this BRWre $(T, V)$ and $E_{\mathcal{L}}$ for the corresponding expectation. The joint probability of the environment and the branching random walk is written $P$, with the corresponding expectation $E$.

Now we add an absorbing barrier to the BRWre. For a realization of environment $\mathcal{L}$, we write the barrier function $g_{\mathcal{L}}(i)$. At generation $i$, we erase all the individuals whose position is bigger than $g_{\mathcal{L}}(i)$ and its descendants. We denote the new system $(\mathcal{T}, V, g_{\mathcal{L}}(i))$. we call it branching random walk with i.i.d. random environment and random absorbing barrier. This paper is focused on the survival or extinction problem when we add an absorbing barrier and the speed of extinction when we add a barrier which makes the system $(\mathcal{T}, V, g_{\mathcal{L}}(i))$ extinct.

When the environment space is degenerate, in other word, the branching random walk is time-homogeneous, the absorbing barrier problem has been researched by many scholars. Under the boundary case, Biggins et al [2] shows that if we let $g(i) = ai$, then the system $(\mathcal{T}, V, g(i))$ will survival if $a > 0$ and extinct if $a \leq 0$. Jaffuel [5] gives a refinement order of critical barrier function, that is, if we let $g(i) = ai^\frac{1}{3}$, then there exist an $a_c > 0$ such that the system $(\mathcal{T}, V, g(i))$ will survival if $a > a_c$ and extinct if $a \leq a_c$. Furthermore, [1] gives the speed of extinction when we take $g(i) \equiv 0$ and assume that the branching mechanism is $b(b > 2)$ binary tree.

For the model BRWre, Huang and Liu [4] proved that the maximal displacement in the process grows at ballistic speed almost surely, and obtained central limit theorems and large deviations principles for the counting measure of the process. Mallein [?] gives a more precise expression for the asymptotic behaviour of maximal displacement which we will state in detail postponed.

2. Basic assumption and main result

First, we give some notations for the model BRWre. For every $n \in \mathbb{N}$, let

$$\kappa_n(\theta) := \ln E_{\mathcal{L}_n}(\sum_{l \in \mathcal{L}_n} e^{-\theta l}), \quad \theta \in [0, +\infty)$$

be the log-Laplace transform of the point process law $\mathcal{L}_n$. We should notice that for fixed $\theta$ $\kappa_n(\theta)$ is also a random variable defined on environment space. Furthermore, $\{\kappa_n(\theta), n \in \mathbb{N}\}$ is an i.i.d. random sequence since the environment sequence is i.i.d.. Define function $\kappa : [0, +\infty) \to [-\infty, +\infty]$ by $\kappa(\theta) := E(\kappa_n(\theta))$. We assume that the underlying branching process with random environment is supercritical, that is to say,

$$\kappa(0) > 0. \quad (2.1)$$

2
We can see $\kappa(\cdot)$ is a convex function and $C^\infty$ on the interior of the interval $\{ \theta : |\kappa(\theta)| < \infty \}$ interval since $\kappa_n(\cdot)$ is the log-Laplace transform of a measure on $\mathbb{R}$. We assume that the interval $\{ \theta : |\kappa(\theta)| < \infty \}$ is non-empty, then we can find a $\vartheta > 0$ such that

$$\kappa(\vartheta) = \vartheta \kappa'(\vartheta). \quad (2.2)$$

Therefore, $v := \inf_{\vartheta > 0} \frac{\kappa(\vartheta)}{\vartheta} = \kappa'(\vartheta) = E(\kappa'_1(\vartheta))$. Here we also assume that

$$\sigma_Q^2 = \vartheta^2 E(\kappa''(\vartheta)) \in (0, +\infty), \sigma^2 = E((\kappa(\vartheta) - \vartheta \kappa'(\vartheta))^2) \in [0, +\infty). \quad (2.3)$$

We need to add some integrability conditions: There exists a $\alpha_1, \alpha_2 > 0$ such that

$$E(e^{\alpha_1 |\kappa'_1(\vartheta) - \kappa(\vartheta)|}) < +\infty. \quad (2.4)$$

Denote $l_\vartheta := l - \kappa'_1(\vartheta)$, $\mathbb{P}$ a.s. we have

$$\alpha_2 \vartheta E(\sum_{l \in L_1} |l_\vartheta|^3 e^{-\vartheta |l_\vartheta|}) \leq E(\sum_{l \in L_1} l_\vartheta^2 e^{-\vartheta l}) \quad (2.5)$$

There exists $x < y < 0$, and $A \in \mathbb{N}$ such that

$$E\left(\left[ \ln E_\mathcal{L}(1_{\{i \in L_1 \leq A\}} \sum_{l \in L_1} 1_{\{ |l_\vartheta + \kappa'_1(\vartheta) | \in (x, y) |}\}) \right]^8 \right) < +\infty. \quad (2.6)$$

We denote the function $\gamma$ is what we have defined in [6, Theorem 2.1]. That is

$$\lim_{t \to +\infty} -\ln \frac{P^0(\forall s \leq t \exists B_s \in [-\frac{1}{2} + \beta W_s, \frac{1}{2} + \beta W_s]|W)}{t} = \gamma(\beta), \quad \text{a.s.}$$

where $B, W$ are two independent standard Brownian motions. From now on, we let $\beta := \frac{\alpha_1}{\sigma_Q}$ and denote $\sigma_Q$ by $\sigma$ for simplicity.

**Theorem 2.1** Define

$$Z_n(\mathcal{L}) := \mathbb{P}\{ |u| = n : \forall i \leq n, V(u_i) \leq ai^{1/3} - \frac{K_i}{\vartheta} \},$$

$$\mathbb{P}_{L, \text{survive}} = (\exists u \in \mathcal{T}_\infty, \forall i \geq 1, V(u_i) \leq ai^{1/3} - \frac{K_i}{\vartheta}).$$

Under the assumption $\gamma(\beta)$, $\mathbb{P}_{L, \text{survive}} > 0$. Moreover, the function $\vartheta a = \vartheta b + \frac{3\gamma(\beta)\sigma^2}{2b^3\vartheta^3}$ has two solution $b_1, b_2$. For any given $b \in (b_1, b_2)$, for any $\epsilon > 0$, there exist a large enough $M$. we have

$$\mathbb{P}_L(\forall k \leq 1, \frac{\ln \mathbb{P}\{ u \in \mathcal{T}_M, \forall i \leq M, V(u_i) \in [(a - b)i^{1/3} - \frac{K_i}{\vartheta}, ai^{1/3} - \frac{K_i}{\vartheta})\}}{M^k + 1 - M^k} J^{1/3} \geq b - \epsilon) > 0 \text{ a.s.}$$

b). When $a < \frac{3\sqrt{6\gamma(\beta)}\sigma^2}{2b^3\vartheta^3}$, $\mathbb{P}_{L, \text{survive}} = 0$. $\lim_{n \to \infty} \frac{\ln \mathbb{P}_L(Z_n(\mathcal{L}) > 0)}{\sqrt{n}} = c < 0$. a.s.
Theorem 2.2 Define \( g(\cdot) : \mathbb{R} \to \mathbb{R} \) satisfied that \( \lim_{n \to \infty} \frac{g(n)}{\sqrt{n}} = 0 \), \( d := \liminf_{t \to 0} \frac{g(t)}{t^{1/3}} > -\infty \). \( Z_n(L) \) is the surviving population of the generation \( n \) in the system \((T, V, g_L(i))\). That is to say,

\[
Z_n(L) := \# \{|u| = n : \forall i \leq n, V(u_i) \leq g(i) - \frac{K_i}{d}\},
\]

where \( K_i = \sum_{j=1}^i \kappa_j(\theta) \). Under the assumption (2.1)-(2.5). We also assume that there exists an \( \epsilon > 0 \) such that

\[
E\left( \left[ \ln E_L(1_{\{\sum_{l \in L_i} \leq A\}} \sum_{l \in L_i} 1_{\{l \leq x, d \in (x, d - \epsilon)\}}) \right]^8 \right) < +\infty.
\]

Then we have

\[
\lim_{n \to \infty} \frac{\ln P_L(Z_n(L) > 0)}{\sqrt{n}} = -\sqrt{3 \sigma^2} \gamma \left( \frac{2A}{\sigma \gamma} \right), \text{ a.s.}
\]

Remark 2.1 Under our assumption, according the result of [10], we have

\[
\lim_{n \to \infty} \frac{\min_{|u| = n} V(x) + \frac{K_n}{\ln n}}{\ln n} = c, \quad \text{in Probability}
\]

We can see in the random environment, the first order of asymptotic behavior of leftmost position is a random walk, that is way we choose to discuss the barrier function like \( g_L(i) = g(i) - \frac{K_i}{d} \).

3. Some useful lemma

Now we introduce some useful lemmas.

**Bivariate version many-to-one formula in random environment** Many to one formula is essential in studies of extremal behaviour of branching random walks. The random environment version of Many to one formula has been first introduced in [8]. When the environment is degenerate, the bivariate version many-to-one formula can be found in [3]. In this paper we need a bivariate version many-to-one formula in random environment. For every \( n \geq 1 \), we write \( L_n \) for a realisation of the point process with law \( L_n \). Let \((X_i, \xi_i)\) be a random variable taking values in \( \mathbb{R} \times \mathbb{N} \) such that for any measurable non-negative function \( f \),

\[
E_L\left( \left[ \ln E_L(1_{\{\sum_{l \in L_i} \leq A\}} \sum_{l \in L_i} 1_{\{l \leq x, d \in (x, d - \epsilon)\}}) \right]^8 \right) < +\infty.
\]

And we define the probability measure \( \mu_n \) on \( \mathbb{R} \). So in quenched sense, \( \{X_n\}_{n \in \mathbb{N}} \) is a sequence of independent random variables. We set \( S_n = S_0 + \sum_{i=1}^n X_i \). From now on, \( P_L \) stands for the joint law of the BRWre \((T, V)\) and the random variable \((X_i, \xi_i)\), conditionally on the environment \( L \).

**Lemma 3.1** For any \( k, n \in \mathbb{N} \) and any measurable non-negative function \( f : (\mathbb{R}^n \times \mathbb{N}^n) \to \mathbb{R} \), we have

\[
E_L[ \sum_{|u| = n} f(V(u_1), V(u_2), \ldots, V(u_n))] = E_L[ e^{\beta S_n + K_n} f(S_1, S_2, \ldots, S_n)] \quad P - \text{a.s.}
\]
Due to the time-inhomogeneity, it is useful to consider time-shifts of the environment. For $k \in \mathbb{N}$, we write $\mathbb{P}_L^k$ for the law of the branching random walk with the random environment $(L_{k+j}, j \in \mathbb{N})$. By convention we assume that under By convention we assume that under $\mathbb{P}_L^k$, the corresponding random walk $S$ is $S_n = \sum_{i=1}^n X_{k+i}$. And for any $m, m' \in \mathbb{N}$, the joint law of $(\xi_{k_1}, \xi_{k_2})$, then the shifted version of many to one formula can be written as

$$E_L^k\left[\sum_{|u|=n} f(V(u_1), V(u_2), \ldots, V(u_n))\right] = E_L^k[e^{g_{S_n+K_{n+k}-K_k}} f(S_1, S_2, \ldots, S_n)] \quad \mathbb{P} \text{- a.s.}$$

$$E_L^{k,x}\left[\sum_{|u|=n} f(V(u_1), V(u_2), \ldots, V(u_n))\right] = e^{-\theta x} E_L^{k,x}[e^{g_{S_n+K_{n+k}-K_k}} f(S_1, S_2, \ldots, S_n)] \quad \mathbb{P} \text{- a.s.}$$

$$E_L^{k,x}\left[\sum_{|u|=n} f(V(u_1), V(u_2), \ldots, V(u_n), v(u_1), v(u_2), \ldots, v(u_n))\right] = e^{-\theta x} E_L^{k,x}[e^{g_{S_n+K_{n+k}-K_k}} f(S_1, S_2, \ldots, S_n, \xi_1, \xi_2, \ldots, \xi_n)] \quad \mathbb{P} \text{- a.s.}$$

This can be proved by induction on $n$. Let the distribution of $(X_i, \xi_i)$ be determined by

$$E_L^{i-1,x}\left[\sum_{l \in L_i} f(V(u_1), v(u_1))\right] = e^{-\theta x} E_L^{i-1,x}[e^{g_{S_1+\xi_1}} f(X_1, \xi_1)] \quad \mathbb{P} \text{- a.s.}$$

$$\mathbb{P}_L(X_i \leq x, \xi_i \leq A) = \mathbb{P}_L(\{\sum_{i=k_i}^{i=1} \leq A\} \sum_{l \in L_i} 1{t \leq x} e^{-\theta l - \xi_i(\theta)})$$

We prove it by induction in $n$. For $n = 1$, this is the definition of the distribution of $(X_1, \xi_1)$. Assume the identity proved for $n - 1$. Then, for $n$, we condition on the branching random walk in the first generation; by the branching property, this yields

$$E_L^{k,x}\left[\sum_{|u|=n} f(V(u_1), V(u_2), \ldots, V(u_n), v(u_1), v(u_2), \ldots, v(u_n))\right]$$

$$= E_L^{k,x}\left[\sum_{|z|=1} E_L^{k+1,V(z)}(f(V(z), V(z) + V(u_1), \ldots, V(z) + V(u_{n-1}), v(z), v(u_1), \ldots, v(u_{n-1}))[V(z), v(z)])\right]$$

$$= E_L^{k,x}\left[\sum_{|z|=1} e^{-\theta V(z)} e^{\theta V(z)} e^{\sum_{i=k+1}^{i=k+n} \xi_i(\theta)} f(V(z), v(z) + X_{k+2}, \ldots, v(z) + X_{k+n})[V(z), v(z)])\right]$$

$$= e^{-\theta x} E_L^{k,x}[e^{g_{S_n+K_{n+k}-K_k}} f(S_1, S_2, \ldots, S_n, \xi_1, \xi_2, \ldots, \xi_n)] \quad \mathbb{P} \text{- a.s.}$$
Mogul’skiĭ estimation Mogul’skiĭ estimation is also an essential tool in the barrier problem of Branching random walk. Mogul’skiĭ estimation had first introduced in [11]. Mallein [9] gives the time inhomogeneous version of Mogul’skiĭ estimation. Here we give the random environment version of Mogul’skiĭ estimation (Lemma 3.2), the proof of Lemma 3.2 can be found in [4]. Let \( T_n \) is a random walk with i.i.d. random environment in time and satisfied the following assumption. We use \( \mu = (\mu_1, \mu_2, \cdots, \mu_n, \cdots) \) to present the random environment. Denote

\[
M_n := \mathbb{E}_{\mu}(S_n), \quad U_n := S_n - \mathbb{E}_{\mu}(S_n), \quad \Gamma_n := \mathbb{E}_{\mu}(U_n^2) = \mathbb{E}_{\mu}(S_n^2) - M_n^2.
\]

(H1) \( \mathbb{E}M_1 = 0, \sigma_A^2 := \mathbb{E}(M_1^2) \in [0, +\infty), \sigma_Q^2 := \mathbb{E}(U_1^2) = \mathbb{E}(\Gamma_1) \in (0, +\infty). \)

(H2) There exists \( \lambda_1 > 0, \) such that \( \mathbb{E}(e^{\lambda_1|M_1|}) < +\infty. \)

(H3) There exist \( \lambda_2, \lambda_3 > 0 \) such that \( \mathbb{E}_{\mu}(e^{\lambda_2|U_1|}) \leq \lambda_3 \) almost surely.

(H4) \( \{r_n\}_{n \in \mathbb{N}} \) is a positive sequence such that for any \( \rho > 0, \) \( \lim_{n \to +\infty} \frac{n^\rho}{r_n} = 0. \) \( f(n) \) is an positive integer-valued function such that for any \( \kappa > 0, \) \( \lim_{n \to +\infty} \frac{f(n)}{n^{\kappa}} = 0. \)

Let \( \xi_i \) be a positive random variable whose law is only determined by the \( i \)-th element \( \mu_i \) in a realistic of environment \( \mu. \) Therefore, conditioned on a given environment \( \mu, \{\xi_i\}_{i \in \mathbb{N}} \) is an independent positive random sequence since \( \mu \) is i.i.d.. Moreover, for any measurable function \( \eta, \{\mathbb{E}_{\mu}(\eta(\xi_i))\}_{i \in \mathbb{N}} \) is an i.i.d. random sequence in the environment space.

**Lemma 3.2** Under the assumption \( \mathbb{E}(\xi_1) < +\infty \) and (H1)-(H4), let \( g(s), h(s) \) be two continue functions on \([0,1]\) and \( g(s) < h(s) \) for any \( s \in [0,1], \) \( g(0) = a_0 \leq b_0 < h(0), g(1) \leq a' < b' \leq h(1). \) Recalling the definition of event \( H_n \) in (1.1).Denote \( C^{g_1, g_2} := \int_{g_1}^{g_2} \frac{1}{[h(s) - g(s)]^2} ds, \) then for any \( \alpha \in (0, \frac{1}{2}), \) almost surely there have

\[
\begin{align*}
\limsup_{n \to +\infty} \sup_{x \in \mathbb{R}} \frac{\ln \mathbb{P}_\mu(\forall 0 \leq i \leq n \frac{T_{f(i)+i}}{n^\alpha} \in [g(\frac{i}{n}), h(\frac{i}{n})]|T_f(n) = x)}{n^{1-2\alpha}} & \leq -C_{g,h}^{0,1} \frac{\sigma_A^2}{\sigma_Q^2} \gamma(\frac{\alpha}{\sigma_Q}), \\
\liminf_{n \to +\infty} \inf_{x \in [a_0 n^\alpha, b_0 n^\alpha]} \frac{\ln \mathbb{P}_\mu(\forall 0 \leq i \leq n \frac{T_{f(i)+i}}{T_{f(i)}+i} \in [g(\frac{i}{n}), h(\frac{i}{n})]|T_f(n) = x)}{n^{1-2\alpha}} & \geq -C_{g,h}^{0,1} \frac{\sigma_A^2}{\sigma_Q^2} \gamma(\frac{\alpha}{\sigma_Q}).
\end{align*}
\]

**Corollary 3.1** Under the assumption of Lemma 3.2, \( 0 \leq l < m \leq N, \) almost surely there have

\[
\begin{align*}
\limsup_{k \to +\infty} \sup_{x \in \mathbb{R}} \frac{\ln \mathbb{P}_\mu(\forall k \leq i \leq mk \frac{T_i}{(Nk)^\alpha} \in [g(\frac{i}{Nk}), h(\frac{i}{Nk})]|T_{lk} = x)}{(Nk)^{1-2\alpha}} & \leq -C_{g,h}^{0,1} \frac{\sigma_A^2}{\sigma_Q^2} \gamma(\frac{\alpha}{\sigma_Q}), \\
\end{align*}
\]
Without loss of generality, we assume that

\[ \liminf_{n \to +\infty} \inf_{x \in [a n^{\alpha}, b n^{\alpha}]} \frac{\ln P_\mu(\forall_{l \leq i \leq mk} \frac{T_i}{(Nk)^\alpha} \in [g(\frac{i}{Nk}), h(\frac{i}{Nk})], \frac{T_m}{(Nk)^\alpha} \in [a', b'], \xi_i \leq r_{Nk}|T_l = x)}{(Nk)^{1-2\alpha}} \geq -C_{g,h,\sigma_Q}^{1/2} \gamma(\frac{\sigma_A}{\sigma_Q}). \]

**Proof of Corollary 3.1**

When \( l = 0, m = N \), it has contained in Lemma 3.2. Let \( n := mk - lk, f(n) = \frac{\ln}{m-l} \cdot \frac{S_i}{(mk-lk)^\alpha} \in [(\frac{Nk}{mk-lk})^\alpha g(\frac{i-lk+lk}{mk-lk} \frac{mk-lk}{Nk}), (\frac{Nk}{mk-lk})^\alpha h(\frac{i-lk+lk}{mk-lk} \frac{mk-lk}{Nk})] \)

\[ \limsup_{k \to +\infty} \sup_{x \in \mathbb{R}} \frac{1}{(mk-lk)^{1-2\alpha}} \frac{(mk-lk)^{1-2\alpha}}{(Nk)^{1-2\alpha}} = (\frac{mk-lk}{Nk})^{2\gamma(\beta)^2} \int_0^1 \left[ h(\frac{i-lk+lk}{mk-lk} \frac{mk-lk}{Nk}) - g(\frac{x+lk}{mk-lk} \frac{mk-lk}{Nk}) \right]^{-2} \frac{dx}{(mk-lk)^{1-2\alpha}} \]

\[ = \gamma(\beta)^2 \int_0^{\frac{\pi}{2}} (h(x) - g(x))^{-2} \]

\[ H(x) := h\left(\frac{x+lk}{mk-lk} \frac{mk-lk}{Nk}\right), X := (x+lk) \frac{mk-lk}{Nk} \]

**Corollary 3.2**

(H1) Let \( \nu \in (\alpha, 1) \), \( h(t) \) satisfied that \( \liminf_{t \to 0} \frac{h(t)-h(0)}{t^\nu} > -\infty \). There exist a pair of \( x < y < 0 \) such that

\[ \mathbb{E}([-\ln P_\mu(T_1 \in [x,y], \xi_1 \leq A|T_0 = 0])^8) < +\infty. \]

(H2) \( \liminf_{t \to 0} \frac{h(t)-h(0)}{t^\nu} := d > -\infty \). there exists \( x < d \) such that

\[ \mathbb{E}([-\ln P_\mu(T_1 \in [x,d], \xi_1 \leq A|T_0 = 0])^8) < +\infty. \]

If assumption (H1) or (H2) holds, \( \alpha \leq 1/3, a' < b' \). we have

\[ \ln P_\mu(\forall_{f(a) \leq i \leq f(a)+n} \frac{T_i}{(f(a)/n)^\alpha}, \frac{T_m}{(f(a)/n)^\alpha} \in [g(\frac{f(a)-f(a)}{n}), h(\frac{f(a)-f(a)}{n})], \xi_i \leq r_{\frac{Nk}{f(a)+n}}|T_{f(a)}=h(0)n^\alpha) \]

\[ \lim_{n \to +\infty} \frac{\ln P_\mu}{n^{1-2\alpha}} \leq -C_{g,h,\sigma_Q}^{1/2} \gamma(\frac{\sigma_A}{\sigma_Q}). \]

**Proof of Corollary 3.2** Without loss of generality, we assume that H1 holds. By \( \liminf_{l \to 0} \frac{h(t)-h(0)}{t^\nu} > -\infty \), we have \( h(t) - h(0) > -dt^\nu \) when \( t \) is small enough. Choose a \( \delta > 0 \) arbitrarily such that \( \frac{g(0)-h(0)}{y} > \delta \). (Under assumption (H2), we should need \( \frac{g(0)-h(0)}{y} > \delta \).) Let \( N = \lceil \delta n^{\alpha} \rceil \). For the continuity of \( g \), we can see for any small
enough $\epsilon > 0$ such that $g(0) + \epsilon - h(0) \leq y\delta$ then we can find a large enough $n$ such that for any $z \in [0, \frac{N}{n}]$, $g(z) \leq g(0) + \epsilon$. Choose $x \in (g(0) + \epsilon - h(0), y)$ then for any $i \in [f(n), f(n) + N]$ we have

$$
\left[g\left(\frac{i - f(n)}{n}\right) - h(0)\right] n^\alpha \leq x(i - f(n)) \leq y(i - f(n)) \leq \left[h\left(\frac{i - f(n)}{n}\right) - h(0)\right] n^\alpha,
$$

That is because of

$$
y(i - f(n)) \leq d\left(\frac{i - f(n)}{n}\right)^\nu n^\alpha, \quad y(i - f(n))^{1 - \nu} \leq \frac{d}{n^{\nu - \alpha}}.
$$

and

$$
g\left(\frac{i - f(n)}{n}\right) - h(0) \leq g(0) + \epsilon - h(0) \leq x\delta < y\delta.
$$

Let $x < x' < y' < y$, we have

$$
\mathbb{P}_\mu\left(\forall f(n) \leq i \leq f(n) + n\right) \frac{T_i}{n^\alpha} \in \left[g\left(\frac{i - f(n)}{n}\right), h\left(\frac{i - f(n)}{n}\right)\right], \quad \frac{T_{f(n) + n}}{n^\alpha} \in [a', b], \quad \xi_i \leq r_n |T_f(n) = h(0)n^\alpha
$$

$$
\geq \mathbb{P}_\mu\left(\forall f(n) \leq i \leq f(n) + n | T_i \in [x(i - f(n)), y(i - f(n))], \xi_i \leq r_n | T_f(n) = 0\right) \times \inf_{z \in [x'N, y'N]} \mathbb{P}_\mu\left(\forall N \leq i \leq f(n) \leq N^\alpha \frac{T_i}{n^\alpha} \in \left[g\left(\frac{i - f(n)}{n}\right), h\left(\frac{i - f(n)}{n}\right)\right], \quad \xi_i \leq r_n | T_f(n) + N = z\right)
$$

$$
\geq \prod_{m=1}^N \mathbb{P}_\mu(T_f(n) + m \in [x, y], \xi_f(n) + m \leq r_n | T_f(n) + m - 1 = 0) \times \inf_{z \in [x'N, y'N]} \mathbb{P}_\mu\left(\forall N \leq i \leq f(n) \leq N^\alpha \frac{T_i}{n^\alpha} \in \left[g\left(\frac{i - f(n)}{n}\right), h\left(\frac{i - f(n)}{n}\right)\right], \quad \xi_i \leq r_n | T_f(n) + N = z\right)
$$

Let us analyze the last term of that above inequality.

$$
\frac{T_i}{(n - N)^\alpha} \leq \left[\frac{n^\alpha}{(n - N)^\alpha} \left[g\left(\frac{i - f(n)}{n}\right) - h(0)\right], \frac{n^\alpha}{(n - N)^\alpha} \left[h\left(\frac{i - f(n)}{n}\right) - h(0)\right]\right]
$$

$$
\frac{T_i}{(n - N)^\alpha} \leq \left[\frac{n^\alpha}{(n - N)^\alpha} \left[g\left(\frac{i - f(n) - N}{n - N}\right) - h(0)\right], \frac{n^\alpha}{(n - N)^\alpha} \left[h\left(\frac{i - f(n)}{n}\right) - h(0)\right]\right]
$$

$$
\frac{T_i}{(n - N)^\alpha} \leq \left[\frac{n^\alpha}{(n - N)^\alpha} \left[g\left(\frac{i - f(n) - N}{n - N}\right) + \frac{N}{n - N} \frac{n - N}{n}\right] - h(0)\right], \frac{n^\alpha}{(n - N)^\alpha} \left[h\left(\frac{i - f(n)}{n}\right) - h(0)\right]\right]
$$

$$
g\left(\frac{i - f(n) - N}{n - N} + \frac{N}{n - N} \frac{n - N}{n}\right) = g\left((x + \frac{N}{n - N} \frac{n - N}{n}\right) = g\left(x + (1 - x) \frac{N}{n}\right)
$$

$$
L_1^\alpha := \frac{n^\alpha}{(n - N)^\alpha} \left[g\left((x + \frac{N}{n - N} \frac{n - N}{n}\right) - h(0)\right] \xrightarrow{L^1} g(x) - h(0)
$$

$$
L_2^\alpha := \frac{n^\alpha}{(n - N)^\alpha} \left[h\left((x + \frac{N}{n - N} \frac{n - N}{n}\right) - h(0)\right] \xrightarrow{L^1} h(x) - h(0)
$$
By 0-1 law we can see
\[ \delta \]
Proof of Theorem 2.1 (a)

4. Proof

\[ z \]
is a particle in this system such that
\[ T \]
To prove Theorem 2.1 (a), we only need to find a sequence of
\[ v \]
if
\[ \xi, \leq r_n \]
\[ P \]
we complete this proof.

\[ \text{By 0-1 law we can see} \]
\[ \lim \inf_{n \to +\infty} \frac{\ln \inf_{z \in [x,N,y,N]} \mathbb{P}_\mu (\forall N \leq i - f(n) \leq n, \frac{\frac{T_f(n) - T_f(n) - h(0)}{n \epsilon} \in \gamma \left( a^i - h(0), b^i - h(0), \xi, \leq r_n \right)}{n^{1-2\alpha}}} \]
\[ \geq -C_g \frac{\sigma A}{\sigma_Q} \gamma. \]

\[ \mathbb{E}(\ln \mathbb{P}_\mu (T_1 \in [x,y], \xi_1 \leq A, T_0 = 0)) \]
\[ \leq +\infty. \]

By 0-1 law we can see
\[ \lim \inf_{n \to +\infty} \frac{\ln \prod_{m=1}^{N} \mathbb{P}_\mu (T_{f(n)} + m \in [x,y], \xi_{f(n) + m} \leq r_n | T_{f(n)} + m - 1 = 0)}{n^{1-2\alpha}} \]
\[ > -\delta \mathbb{E}(\ln \mathbb{P}_\mu (T_1 \in [x,y], \xi_1 \leq A, T_0 = 0)) \]

Let \( \delta \to 0 \), we complete this proof.

4. Proof

**Proof of Theorem 2.1 (a)** Let \( M \in \mathbb{N} \), define
\[ P_n(\mathcal{L}) := \mathbb{P}(\forall 1 \leq k \leq n, \# \{ u \in T_{M^k}, \forall i \leq M^k, V(u_i) \leq a t^\frac{1}{\alpha} - \frac{K_i}{\vartheta} \} > v_{k-1}). \]
\[ z \]
is a particle in this system such that
\[ V(z) = a M^k t^\frac{1}{\alpha} - \frac{K_i}{\vartheta}, |z| = M^k. \]
Define
\[ Z_k(\mathcal{L}) := \# \{ u \in T_{M^{k+1}} : \forall M^k < i \leq M^{k+1}, V(u_i) \in [(a - b) t^\frac{1}{\alpha} - \frac{K_i}{\vartheta}, a t^\frac{1}{\alpha} - \frac{K_i}{\vartheta}] \}. \]
\[ Y_k(\mathcal{L}) := \# \{ u \in T_{M^k}, \forall i \leq M^k, V(u_i) \leq a t^\frac{1}{\alpha} - \frac{K_i}{\vartheta} \}. \]
It is easy to see
\[ \frac{P_{n+1}(\mathcal{L})}{P_n(\mathcal{L})} := \mathbb{P}(\forall 1 \leq k \leq n + 1, Y_k(\mathcal{L}) > v_{k-1} | \forall 1 \leq k \leq n, Y_k(\mathcal{L}) > v_{k-1}) \]
\[ \geq 1 - \mathbb{P}(Z_n(\mathcal{L}) < v_n)^{v_{n-1}} \]
If we denote \( A_k(\mathcal{L}) := \mathbb{P}(Z_k(\mathcal{L}) > v_k) \), then we have
\[ P_n \geq P_n \prod_{k=n_0}^{n-1} (1 - (1 - A_k)^{v_k}) \geq P_n \prod_{k=n_0}^{n-1} (1 - e^{-A_k v_{k-1}}) \]
To prove Theorem 2.1 (a), we only need to find a sequence of \( v_i \) to satisfied that
\[ \sum_{i=1}^{+\infty} e^{-A_i v_{i-1}} < +\infty. \]
For simplicity, we denote $Z_n(\mathcal{L})$ by $Z_n$ under the probability space $\mathbb{P}_\mathcal{L}$.

Let $v_k := \theta \mathbb{E}_\mathcal{L}(Z_k)$, by Paley-Zygmund inequality, we have

$$A_k := \mathbb{P}_\mathcal{L}(Z_k \geq v_k) \geq (1 - \theta)^2 \frac{\mathbb{E}_\mathcal{L}^2(Z_k)}{\mathbb{E}_\mathcal{L}(Z_k^2)}.$$ 

Define $d_k := M^{k+1} - M^k$.

$$\mathbb{E}_\mathcal{L}(Z_k^2) = \mathbb{E}_\mathcal{L}\left( \sum_{u \neq v \geq r_k} 1_{\{M^{k+1} \leq u \leq M^k, V(u) \in I_1(\mathcal{L}), \gamma(u-1) \leq r_k \}} \right) := \sum_{j=0}^{d_k} B_{k,j}(\mathcal{L})$$

By second moment method, we have $B_{k,j}(\mathcal{L}) \leq \mathbb{E}_\mathcal{L}(Z_k) + (r_k - 1) h_{k,j}(\mathcal{L}) \mathbb{E}_\mathcal{L}(Z_k)$.

Let $I_{k,j}(\mathcal{L}) := [(a - b)(c_k + j) + x - \frac{K_{c_k+j}}{a} + \gamma, a(c_k + j) + x - \frac{K_{c_k+j}}{a} - \frac{K_{c_k+j}}{a}]$, $c_k = M^k$. According to the assumption (2.2), (2.3), (2.4) and (2.5), $\{T_i\}_{i \in \mathbb{N}}$ satisfied the conditions of Lemma 3.2. We can see

$$h_{k,j}(\mathcal{L}) \leq \sup_{x \in I_{k,j}(\mathcal{L})} \mathbb{E}_\mathcal{L}^e \left( \sum_{y \neq d_k+1-j} 1_{\{\forall i \leq d_k+1-j, x + V(y) \in [(a - b)(i+c_k+j) + x - \frac{K_{i+c_k+j}}{a} - \frac{K_{i+c_k+j}}{a}], a(i+c_k+j) + x - \frac{K_{i+c_k+j}}{a} - \frac{K_{i+c_k+j}}{a} \}} \right)$$

$$= \sup_{x \in I_{k,j}(\mathcal{L})} \mathbb{E}_\mathcal{L}^e \left( e^{T_{d_k+1-j}} \sum_{y \neq d_k+1-j} 1_{\{\forall i \leq d_k+1-j, x + T_i \in [(a - b)(i+c_k+j) + x - \frac{K_{i+c_k+j}}{a} - \frac{K_{i+c_k+j}}{a}], a(i+c_k+j) + x - \frac{K_{i+c_k+j}}{a} - \frac{K_{i+c_k+j}}{a} \}} \right)$$

$$\leq \sup_{x \in I_{k,j}(\mathcal{L})} e^{\theta a(c_k+j) + x - \frac{K_{c_k+j}}{a}} \mathbb{E}_\mathcal{L} \left( 1_{\{\forall i \leq d_k+1-j, x + T_i \in [(a - b)(i+c_k+j) + x - \frac{K_{i+c_k+j}}{a} - \frac{K_{i+c_k+j}}{a}], a(i+c_k+j) + x - \frac{K_{i+c_k+j}}{a} - \frac{K_{i+c_k+j}}{a} \}} \right)$$

$$\leq e^{\theta a(c_k+j) + x - \frac{K_{c_k+j}}{a}} \mathbb{E}_\mathcal{L} \left( \sup_{y \in [(a - b)(c_k+j) + x, a(c_k+j) + x]} \mathbb{P}_\mathcal{L} c_k+j, y \sum_{i \leq d_k+1-j, T_i \in [(a - b)(i+c_k+j) + x, a(i+c_k+j) + x]} \right).$$

Where $y := \theta x + K_{c_k+j} \in [(a - b)(c_k+j) + x, a(c_k+j) + x]$. We will divide $d_k = M^{k+1} - M^k = M(M - 1)M^{k-1}$. Denote $K(M) := M^2 - M - 1$, we have

$$\sum_{j=1}^{M^{k+1}} e^{\theta a(c_k+j) + x - \frac{K_{c_k+j}}{a}} \sup_{y \in [(a - b)(c_k+j) + x]} \mathbb{E}_\mathcal{L} c_k+j, y \left( \sum_{i \leq d_k+1-j, T_i \in [(a - b)(i+c_k+j) + x, a(i+c_k+j) + x]} \right)$$

$$\leq K(M) \left( \sum_{l=0}^{M^{k-1} - M^k} e^{\theta a(c_k+lM^{k-1}) + x} \sup_{d_k+1, y \in [(a - b)(c_k+j) + x]} \mathbb{E}_\mathcal{L} c_k+j, y \left( \sum_{i \leq d_k+1-j, T_i \in [(a - b)(i+c_k+j) + x, a(i+c_k+j) + x]} \right) \right)$$

Let $j \in [l, M^{k-1} + 1, lM^{k-1} + M^{k-1}]$, $c_k' = (a - \frac{K}{2})(c_k)$. The next inequalities is
how to overcome the new difficulties which is brought by the random environment.

$$A'_k := \mathbb{P}_L^{c_k,c'_k} \left( \forall i \leq d_k, T_i \in [\theta(a-b)(i+c_k) + \frac{\theta(c_k + (l+1)M^{k-1})}{a_M}, \theta(a(i+c_k)) + \frac{\theta(c_k + (l+1)M^{k-1})}{a_M}] \right)$$

$$\geq \mathbb{P}_L^{c_k,c'_k} \left( \forall i \leq j, T_i \in [\theta(a-b)(i+c_k) + \frac{\theta(c_k + (l+1)M^{k-1})}{a_M}, \theta(a(i+c_k)) + \frac{\theta(c_k + (l+1)M^{k-1})}{a_M}] \right)$$

$$\times \inf_{y \in [\theta(a-b_m)(c_k + j), \theta(a-b_{m+1})(c_k + j)]} \mathbb{P}_L^{c_k+c_j,y} \left( \forall i \leq lM^{k-1}, T_i \in [\theta(a-b)(i+c_k) + \frac{\theta(c_k + (l+1)M^{k-1})}{a_M}, \theta(a(i+c_k)) + \frac{\theta(c_k + (l+1)M^{k-1})}{a_M}] \right)$$

$$\geq \mathbb{P}_L^{c_k,c'_k} \left( \forall i \leq lM^{k-1}, T_i \in [\theta(a-b_m)(c_k + lM^{k-1}) + \frac{\theta(c_k + (l+1)M^{k-1})}{a_M}, \theta(a(i+c_k)) + \frac{\theta(c_k + (l+1)M^{k-1})}{a_M}] \right)$$

$$\times \sup_{y \in [\theta(a-b_m)(c_k + lM^{k-1}) + \frac{\theta(c_k + (l+1)M^{k-1})}{a_M}, \theta(a-b_{m+1})(c_k + lM^{k-1}) + \frac{\theta(c_k + (l+1)M^{k-1})}{a_M}]} \mathbb{P}_L^{c_k+c_j,y} \left( \forall i \leq d_k - j, T_i \in [\theta(a-b)(i+c_k) + \frac{\theta(c_k + (l+1)M^{k-1})}{a_M}, \theta(a(i+c_k)) + \frac{\theta(c_k + (l+1)M^{k-1})}{a_M}] \right)$$

$$:= C_k \times B_k \times \sup_{y \in [\theta(a-b_m)(c_k + j), \theta(a-b_{m+1})(c_k + j)]} \mathbb{P}_L^{c_k+c_j,y} \left( \forall i \leq d_k - j, T_i \in [\theta(a-b)(i+c_k) + \frac{\theta(c_k + (l+1)M^{k-1})}{a_M}, \theta(a(i+c_k)) + \frac{\theta(c_k + (l+1)M^{k-1})}{a_M}] \right)$$

$$\lim_{k \to +\infty} \frac{\ln B_k}{d_k^{1/3}} = -(M^{k-1}/d_k)^{1/3} \gamma(\beta) \sigma^2 \int_0^1 \frac{b\gamma}{a_M} (M + l + x)^{1/3} - 2dx$$

$$= -\left( \frac{1}{M - 1} \right)^{1/3} \frac{1}{3} \frac{\gamma(\beta) \sigma^2 a_M^2}{\sigma^2 b^2} \left[ \sqrt{M + l + 1} - \sqrt{M + l} \right]$$

$$\geq -\left( \frac{1}{M - 1} \right)^{1/3} \frac{3 \gamma(\beta) \sigma^2 a_M^2}{\sigma^2 b^2} \left( \frac{1}{M} \right)^{1/3} \left[ \sqrt{M + 1} - \sqrt{M} \right]$$

$$\geq -\left( \frac{1}{M - 1} \right)^{1/3} \frac{3 \gamma(\beta) \sigma^2 a_M^2}{\sigma^2 b^2} \left[ \sqrt{1 + 1/M} - 1 \right]$$

$$\geq -\left( \frac{1}{M - 1} \right)^{1/3} \frac{3 \gamma(\beta) \sigma^2 a_M^2}{\sigma^2 b^2} \left( \frac{1}{3} \right) M$$
\[
\lim_{k \to +\infty} \frac{\ln C_k}{d_k^{1/3}} = -\left(\frac{lM^{k-1}}{d_k}\right)^{\frac{1}{k}} \gamma(\beta) \sigma^2 \int_0^1 \left| b\vartheta(x + \frac{M}{l}) \right|^2 dx
\]
\[
= -\left(\frac{l}{M - 1}\right)^{\frac{1}{k}} \gamma(\beta) \sigma^2 \left(\frac{3}{\vartheta^2 b^2} \left(\frac{\sqrt{l + M}}{l} - \frac{3 \sqrt{M}}{l}\right)\right)
\]
\[
= -\left(\frac{1}{M - 1}\right)^{\frac{1}{k}} \gamma(\beta) \sigma^2 \left(\frac{3}{\vartheta^2 b^2} \left(\frac{1}{M} + \frac{M}{M(M - 1)} - \frac{1}{M(M - 1)^{3/2}}\right)\right)
\]

Notice that if \( x, y > 0 \), \( \frac{1}{x^2} - \frac{1}{(x+y)^2} \leq \frac{2xy + y^2}{x^2(x+y)^2} \leq \frac{2y}{x^2} + \frac{y}{x^2} \cdot 5 \times \sqrt{2} \leq 9. \)

\[
\lim_{k \to +\infty} \frac{\ln A'_k}{d_k^{1/3}} = -\left(\frac{\gamma(\beta) \sigma^2}{b^2 \vartheta^2}\right) \int_0^1 \left(\frac{1}{x + \frac{1}{M - 1}}\right)^{\frac{1}{k}} + \frac{2b}{a_M} \left(\frac{1}{M - 1} + \frac{l + 1}{M^2 - M}\right)^{\frac{1}{k}} dx
\]
\[
\leq -\frac{\gamma(\beta) \sigma^2}{b^2 \vartheta^2} \left[ \int_0^1 \left(\frac{1}{x + \frac{1}{M - 1}}\right)^{\frac{1}{k}} dx - \frac{5}{2} \int_0^1 \frac{b}{a_M} \left(\frac{1}{M - 1} + \frac{l + 1}{M^2 - M}\right)^{\frac{1}{k}} dx \right]
\]
\[
\leq -\frac{\gamma(\beta) \sigma^2}{b^2 \vartheta^2} \left[ \int_0^1 \left(\frac{1}{x + \frac{1}{M - 1}}\right)^{\frac{1}{k}} dx - \frac{5b \ln M}{2a_M} \left(\frac{1}{M - 1} + \frac{l + 1}{M^2 - M}\right)^{\frac{1}{k}} \right]
\]
\[
\leq -\frac{3\gamma(\beta) \sigma^2}{b^2 \vartheta^2} \left[ \left(\frac{M}{M - 1}\right)^{\frac{1}{k}} - \left(\frac{1}{M - 1}\right)^{\frac{1}{k}} \right] + \frac{9b \gamma(\beta) \sigma^2 \ln M}{a_M b^2 \vartheta^2}
\]

Let \( f(x) = \left(\frac{1}{x + \frac{1}{M - 1}}\right)^{\frac{1}{k}} \), \( a_M = \lfloor M^{2/5} \rfloor \), we have

\[
\lim_{k \to +\infty} \ln \left(\sum_{j=lM^{k-1} + 1}^{(l+1)M^{k-1}} \sup_{y \in [\vartheta(a-b)(c_k + j)^{\frac{1}{k}}, \sqrt{\vartheta(a)(1+c_k+j)^{\frac{1}{k}}}]} \mathbb{P}_{\vartheta c_k + j, y}^{c_k + j, \vartheta} \left(\vartheta a c_k + j, \frac{\vartheta(a-b)(i+c_k+j)^{\frac{1}{k}}}{\sqrt{\vartheta(a)(1+c_k+j)^{\frac{1}{k}}}}\right) \right)
\]
\[
\leq \lim_{k \to +\infty} \frac{\ln M^{k-1} + \ln A'_k - \ln(A_k B_k)}{d_k^{1/3}}
\]
\[
\leq -\frac{3\gamma(\beta) \sigma^2}{b^2 \vartheta^2} \left( f(1) - f(0) \right) + f(0) o(M) + \frac{3\gamma(\beta) \sigma^2}{b^2 \vartheta^2} \left( f\left(\frac{l}{M(M - 1)}\right) - f(0) \right) + f(0) o(M)
\]
\[
= -\frac{3\gamma(\beta) \sigma^2}{b^2 \vartheta^2} \left( f(1) - f\left(\frac{l}{M(M - 1)}\right) \right) + f(0) o(M)
\]

Moreover,

\[
\frac{\ln e^{\vartheta a c_k + j, \vartheta(a-b)(c_k + lM^{k-1})^{\frac{1}{k}}}}{d_k^{1/3}} = \vartheta a f(1) - \vartheta(a-b) f\left(\frac{l}{M^2 - M}\right)
\]

Let \( r_k = e^{d_k^{1/3}} \) we have
By assumption of (2.6), we can utilize the Corollary 3.2 to get the following limit.

\[
\lim_{k \to +\infty} \frac{(r_k - 1) \sum_{j=0}^{d_k-1} h_{k,j}(L)}{d_k^{1/3}} \\
\leq \max_{t \in \{0, ..., K(M)\}} \left[ \vartheta_a f(1) - \vartheta(a - b)f\left(\frac{l \vartheta}{M^2 - M}\right) - \frac{3 \gamma(\beta)\sigma^2}{b^2 \vartheta^2} \left(f(1) - f\left(\frac{l}{M(M - 1)}\right)\right) + f(0) o(M) \right]
\]

Let us turn to the lower bound of the \( E \rightarrow \lim L_k \) \((\in \mathbb{Z}\), \(0 \leq k \leq K\)) = \( \ln E \infty = \geq e \).

\[
E_L(Z_k) = E_L^{c_k, ac_k} \left( \sum_{u \geq z, |u| = e^{\lambda(k+1)}} 1\{e^{\varsigma} < i \leq e^{(k+1)\lambda}, V(u) \in I_i(L), \gamma(u_{i-1}, k) \leq r_k\} \right)
\]

\[
= E_L^{c_k, ac_k} (e^{T_{d_k} - \vartheta ac_k} - K_{c_k}) 1_{0 < i \leq d_k, \xi_i \leq r_k, T_i \in \theta(a - b)(i + c_k)^{\frac{1}{\gamma} - K_{c_k}, \vartheta a(i + c_k)^{\frac{1}{\gamma} - K_{c_k}}})
\]

\[
= E_L^{c_k, ac_k} (e^{T_{d_k} - \vartheta ac_k} - K_{c_k}) 1_{0 < i \leq d_k, \xi_i \leq r_k, T_i \in \theta(a - b)(i + c_k)^{\frac{1}{\gamma} - \vartheta a(i + c_k)^{\frac{1}{\gamma}}})
\]

\[
\geq e^{(\vartheta a - \varepsilon) c_{k+1} - \vartheta ac_k} P_L^{c_k, ac_k} \left(0 < i \leq d_k, \xi_i \leq r_k, T_i \in \theta(a - b)(i + c_k)^{\frac{1}{\gamma} - \vartheta a(i + c_k)^{\frac{1}{\gamma}}} \right)
\]

By assumption of (2.6), we can utilize the Corollary 3.2 to get the following limit.

\[
\lim_{k \to +\infty} \frac{\ln E_L(Z_k)}{d_k} \geq (a \vartheta - \varepsilon) \left(\frac{M}{M - 1}\right)^{\frac{1}{\gamma}} - a \vartheta \left(\frac{1}{M - 1}\right)^{\frac{1}{\gamma}} - \sigma^2 \gamma(\beta) \int_0^1 \left[ b \vartheta \left(\frac{i}{d_k} + \frac{1}{M - 1}\right) \right]^{-2} dx
\]

\[
= (a \vartheta - \varepsilon) \left(\frac{M}{M - 1}\right)^{\frac{1}{\gamma}} - a \vartheta \left(\frac{1}{M - 1}\right)^{\frac{1}{\gamma}} - \sigma^2 \gamma(\beta) \int_0^1 \left[ b \vartheta (x + \frac{1}{M - 1}) \right]^{-2} dx
\]

\[
\lim_{k \to +\infty} \frac{\ln E_L(Z_k)}{d_k} \geq a \vartheta (f(1) - f(0)) - \frac{3 \gamma(\beta)\sigma^2}{\beta^2 \vartheta^2} (f(1) - f(0))
\]

13
\[
\lim_{k \to +\infty} \frac{\ln A_k v_{k-1}}{d_k} \geq (1 + f(0)^3 \sqrt{\frac{M - 1}{M}})[a \vartheta(f(1) - f(0)) - \frac{3 \gamma(\beta) \sigma^2}{b^2 \vartheta^2}(f(1) - f(0))]
\]
\[
- \max_{t \in \{0, \ldots, K(M)\}} \left[ \vartheta a f(1) - \vartheta (a - b) f\left( \frac{l}{M^2 - M} \right) - \frac{3 \gamma(\beta) \sigma^2}{b^2 \vartheta^2}(f(1) - f(0)) \right] + \frac{l}{M(M - 1)} + f(0)\sigma(M)
\]
\[
\geq -f(0)\left( \frac{1}{M} \right) \frac{1}{2} (a \vartheta - \frac{3 \gamma(\beta) \sigma^2}{b^2 \vartheta^2}) - f(0)\sigma(M) + f\left( \frac{l}{M^2 - M} \right)(a \vartheta - \vartheta a - b) - \frac{3 \gamma(\beta) \sigma^2}{b^2 \vartheta^2}
\]
\[
\geq -f(0)\left( \frac{1}{M} \right) \frac{1}{2} (a \vartheta - \frac{3 \gamma(\beta) \sigma^2}{b^2 \vartheta^2}) - f(0)\sigma(M) + f(0)(\vartheta a - \vartheta b) - \frac{3 \gamma(\beta) \sigma^2}{b^2 \vartheta^2}
\]
\[
\geq f(0)(\vartheta a - \vartheta b) - \frac{3 \gamma(\beta) \sigma^2}{b^2 \vartheta^2}
\]

We can see the when \( b = \frac{3 \sqrt{6 \gamma(\beta) \sigma^2}}{2 \vartheta} \), \( \vartheta a - \vartheta b - \frac{3 \gamma(\beta) \sigma^2}{b^2 \vartheta^2} \) take its minimum value \( \frac{3}{2} \sqrt{6 \gamma(\beta) \sigma^2} \). So if \( a > \frac{3 \sqrt{6 \gamma(\beta) \sigma^2}}{2 \vartheta} \), \( \lim_{k \to +\infty} \frac{\ln A_k v_{k-1}}{d_k} > 0 \). Then we can see (4.1) holds, thus we complete the proof of Theorem 2.1 (a).

**Proof of Theorem 2.1 (b): the upper bound**

Let \( g(i) = ai^\frac{1}{2}, f(\cdot) : [0, 1] \to [0, +\infty) \) is a continue non-negative function. Define

\[
Z_n := \#\{|u| = n, \forall i \leq n, V(u_i) \leq ai^\frac{1}{2} - \frac{K_i}{\vartheta} \}
\]

\[
\mathbb{P}_L(Z_n > 0) = \mathbb{P}_L(\exists u : |u| = n, \forall i \leq n, V(u_i) \leq ai^\frac{1}{2} - \frac{K_i}{\vartheta}) \leq \sum_{j=1}^{n} H_j + H.
\]

Where

\[
H_j := \mathbb{P}_L(\exists |u| = j : \forall i < j, V(u_i) \in [ai^\frac{1}{2} - n^\frac{1}{2} f\left( \frac{i}{n} \right) - \frac{K_i}{\vartheta}, g(i) - \frac{K_i}{\vartheta}], V(u_j) \leq aj^\frac{1}{2} - n^\frac{1}{2} f\left( \frac{j}{n} \right) - \frac{K_j}{\vartheta}).
\]

\[
H := \mathbb{P}_L(\exists |u| = n : \forall i \leq n, V(u_i) \in [ai^\frac{1}{2} - n^\frac{1}{2} f\left( \frac{i}{n} \right) - \frac{K_i}{\vartheta}, g(i) - \frac{K_i}{\vartheta}]).
\]

By Markov inequality and many to one formula, we have

\[
H_j \leq \mathbb{E}_L \left( \sum_{|u| = j} 1_{\{\forall i < j, V(u_i) \in [ai^\frac{1}{2} - n^\frac{1}{2} f\left( \frac{i}{n} \right) - \frac{K_i}{\vartheta}, g(i) - \frac{K_i}{\vartheta}], V(u_j) \leq aj^\frac{1}{2} - n^\frac{1}{2} f\left( \frac{j}{n} \right) - \frac{K_j}{\vartheta} \}} \right)
\]

\[
= \mathbb{E}_L \left( e^{\vartheta g(i) - \vartheta n^\frac{1}{2} f\left( \frac{i}{n} \right)} \mathbb{P}_L(\forall i < j, S_i \in [ai^\frac{1}{2} - n^\frac{1}{2} f\left( \frac{i}{n} \right) - \frac{K_i}{\vartheta}, g(i) - \frac{K_i}{\vartheta}, S_j \leq aj^\frac{1}{2} - n^\frac{1}{2} f\left( \frac{j}{n} \right) - \frac{K_j}{\vartheta} \}) \right)
\]

\[
\leq e^{\vartheta g(i) - \vartheta n^\frac{1}{2} f\left( \frac{i}{n} \right)} \mathbb{P}_L(\forall i < j, T_i \in [ai^\frac{1}{2} - \vartheta n^\frac{1}{2} f\left( \frac{i}{n} \right), g(i) - \frac{K_i}{\vartheta} \})
\]

\[
= e^{\vartheta g(i) - \vartheta n^\frac{1}{2} f\left( \frac{i}{n} \right)} \mathbb{P}_L(\forall i < j, T_i \in [\vartheta ai^\frac{1}{2} - \vartheta n^\frac{1}{2} f\left( \frac{i}{n} \right), \vartheta ai^\frac{1}{2} \}).
\]
\[ H \leq e^{\varrho \langle n \rangle}\mathbb{P}_L \left( \forall i \leq n, S_i \in [a_1^1 - n^1 f(i/n) - K_1^{\varrho}, g(i) - K_1^{\varrho}] \right) . \]

For the monotonicity of \( \mathbb{P}_L(Z_n > 0) \) we only need to consider \( n := Nk, l \in [0, N - 1] \cap \mathbb{N}, f_{l,N} = \inf_{x \in [l/N, (l+1)/N]} f(x) . \)

\[ \sum_{i=kl+1}^{k(l+1)} H_i \leq ke_{\varrho a(l+1)k}^{l+1} - \varrho f_{l,N} \mathbb{P}_L \left( \forall i \leq lk, T_i \in [\varrho a_1^{1/3} - \varrho n^{1/3} f(i/n), \varrho a_1^{1/3}] \right) . \]

By Lemma 3.2, we have

\[ \lim_{k \to +\infty} \frac{\ln \left( \sum_{i=kl+1}^{k(l+1)} H_i \right) }{n^{1/3}} \leq \varrho a(l^{-1}N) \frac{1}{n} - \varrho f_{l,N} - \gamma(\beta) \sigma^2 \int_0^N (\varrho f(x))^{-2} dx \]

\[ \leq \varrho a(l^{-1}N) \frac{1}{n} - \varrho f(l^{-1}N) - \gamma(\beta) \sigma^2 \int_0^N (\varrho f(x))^{-2} dx + 2\epsilon \]

\[ \lim_{k \to +\infty} \frac{\ln H}{n^{1/3}} \leq \varrho a - \gamma(\beta) \sigma^2 \int_0^1 (\varrho f(x))^{-2} dx \]

In conclusion, We have

\[ \limsup_{n \to +\infty} \frac{\ln \mathbb{P}(Z_n > 0)}{n^{1/3}} \leq \sup_{\alpha \leq 1} \left[ \varrho a_1^{1/3} - \varrho f(\alpha) - \gamma(\beta) \sigma^2 \int_0^\alpha (\varrho f(x))^{-2} dx \right] . \quad (4.2) \]

**Proof of Theorem 2.1 (b): the lower bound**

Define

\[ \Theta := \left\{ u \in \mathcal{U} : \forall 1 \leq i \leq |u|, V(u_i) \in [a_1^{1/3} - \varepsilon |u|^{1/3} - |u|^{1/3} f(i/|u|) - K_1^{\varrho}, a_1^{1/3} - K_1^{\varrho}], \nu(u_{i-1}) \leq r_{|u|} \right\} . \]

\[ Z_n(\Theta) := \sum_{|u| = n} 1_{\{u \in \Theta\}} \text{ and } Z_n(\Theta) = \sum_{|u| = n, u_v \uparrow} 1_{\{u \in \Theta\}} . \]

We have \( \mathbb{P}_L(Z_n > 0) > \mathbb{P}_L(Z_n(\Theta) > 0) \geq \frac{\mathbb{E}_L(Z_n^2(\Theta))}{\mathbb{E}_L(Z_n^2(\Theta))} \). By second moment method,

\[ \mathbb{E}_L(Z_n^2(\Theta)) = \mathbb{E}_L(Z_n(\Theta)) \left[ 1 + (r_n - 1) \sum_{k=1}^n \mathbb{E}_L \left( Z_n(\Theta) \right) \right] . \]

Let \( I_{n,k}(\mathcal{L}) = [a k^{1/3} - \varepsilon n^{1/3} - n^{1/3} f(k/n) - K_1^{\varrho}, a k^{1/3} - K_1^{\varrho}] , \)

\[ \sup_{|u'| = k} \mathbb{E}_L \left( Z_n(\Theta) \right) \leq \sup_{V(u') \in I_{n,k}(\mathcal{L})} \mathbb{E}_L \left( \sum_{|u'| = k} 1_{\{u \in \Theta\}} \right) \]

\[ = \sup_{V(u') \in I_{n,k}(\mathcal{L})} \mathbb{E}_L \left( \sum_{|u'| = k} 1_{\{u \in \Theta\}} \right) \]

\[ = \sup_{x \in I_{n,k}(\mathcal{L})} \mathbb{E}_L^k \left( e^{T_{u-k}} 1_{\{\forall i \leq u-k, V(u_i) \in [\{a(k+i)^{1/3} - \varepsilon n^{1/3} - n^{1/3} f(k+i/n) - K_{i+k}^{\varrho}, a(k+i)^{1/3} - K_{i+k}^{\varrho}] \}} \right) . \]
\[\sup_{|v|=k} \mathbb{E}_L \left( Z_n' (\Theta) \right) \]
\[
\leq e^{\epsilon n^{1/3} - \vartheta k^{1/3} + \vartheta \varepsilon n^{1/3} f (\frac{k}{n})} \sup_{x \in a k^{1/3} - \varepsilon n^{1/3} f (\frac{k}{n})} \mathbb{P}_L \left( \left\{ \forall i \leq n - k, T_i \in \left[ -x + (a k + i)^{1/3} - \vartheta \varepsilon n^{1/3} f (\frac{k+i}{n}) - \frac{k}{\vartheta}, -x + (a k + i)^{1/3} - \vartheta \varepsilon n^{1/3} f (\frac{k+i}{n}) \right] \right\} \right)
\]
\[
\leq e^{\epsilon n^{1/3} - \vartheta k^{1/3} + \varepsilon n^{1/3} f (\frac{k}{n})} \sup_{x \in a k^{1/3} - \varepsilon n^{1/3} f (\frac{k}{n})} \mathbb{P}_L (\forall i \leq n - k, T_i \in \left[ -x + (a k + i)^{1/3} - \vartheta \varepsilon n^{1/3} f (\frac{k+i}{n}) \right] )
\]

Here \( T_n = \vartheta S_n + \sum_{i=k+1}^{k+n} \kappa_i (\vartheta), x := -x - \frac{Kk}{\vartheta} \). Denote

\[ h(n, k) = \vartheta a k^{1/3} - \vartheta \varepsilon n^{1/3} f (\frac{k}{n}) \]

When \( j \in [lk + 1, (l + 1)k] \), \( -h(n, j) \leq -\vartheta a (lk + 1)^{1/3} + \vartheta \varepsilon n^{1/3} f (\frac{l}{N}) + \vartheta \varepsilon n^{1/3} f (\frac{1}{N}) \).

\[ D_n = 1 + \left( r_n - 1 \right) e^{\epsilon n^{1/3} - h(n, j)} \sup_{x \in [h(n, j), a j^{1/3}]} \mathbb{P}_L (\forall i \leq n - j, T_i \in [h(n, j) + i, a (j + i)^{1/3}]) \]

\[ \mathbb{P}_L (Z_n > 0) \geq \frac{\mathbb{E}_L (Z_n (\Theta))}{D_n} \]

Let \( a_N = \lfloor N^{1/3} \rfloor, j \in \lfloor k + 1, k + k \rfloor \), we have

\[ \mathbb{P}_L^{\varepsilon n^{1/3}} \left( \forall i \leq n, T_i \in \left[ \frac{(\varepsilon n^{1/3} + \vartheta \varepsilon n^{1/3} f (\frac{k}{n}))}{\frac{k}{\vartheta}}, \frac{(\varepsilon n^{1/3} + \vartheta \varepsilon n^{1/3} f (\frac{k}{n}))}{\frac{k}{\vartheta}} \right] \right) \]

\[ \mathbb{P}_L^{\varepsilon n^{1/3}} \left( \forall i \leq n, T_i \in \left[ \frac{(\varepsilon n^{1/3} + \vartheta \varepsilon n^{1/3} f (\frac{k}{n}))}{\frac{k}{\vartheta}}, \frac{(\varepsilon n^{1/3} + \vartheta \varepsilon n^{1/3} f (\frac{k}{n}))}{\frac{k}{\vartheta}} \right] \right) \]

\[ \times \inf_{\frac{a j^{1/3} - \frac{m}{a N} (\vartheta n^{1/3} + \vartheta n^{1/3} f (\frac{k}{n}))}{\frac{k}{\vartheta}}} \mathbb{P}_L^{\varepsilon n^{1/3}} \left( \forall i \leq n - j, T_i \in \left[ \frac{\vartheta a (i+j)^{1/3} - \vartheta \varepsilon n^{1/3} f (\frac{i}{n}) - \frac{i}{\vartheta}}{\frac{k}{\vartheta}}, \frac{\vartheta a (i+j)^{1/3} - \vartheta \varepsilon n^{1/3} f (\frac{i}{n}) - \frac{i}{\vartheta}}{\frac{k}{\vartheta}} \right] \right) \]

\[ \geq \mathbb{P}_L^{\varepsilon n^{1/3}} \left( \forall i \leq lk, T_i \in \left[ \frac{(\varepsilon n^{1/3} + \vartheta \varepsilon n^{1/3} f (\frac{k}{n}))}{\frac{k}{\vartheta}}, \frac{(\varepsilon n^{1/3} + \vartheta \varepsilon n^{1/3} f (\frac{k}{n}))}{\frac{k}{\vartheta}} \right] \right) \]

\[ \times \inf_{\frac{a (lk)^{1/3} - \frac{m-0.1}{a N} (\vartheta n^{1/3} + \vartheta n^{1/3} f (\frac{k}{n}))}{\frac{k}{\vartheta}}} \mathbb{P}_L^{\varepsilon n^{1/3}} \left( \forall i \leq k, T_i \in \left[ \frac{\vartheta a (i+lk)^{1/3} - \vartheta \varepsilon n^{1/3} f (\frac{i}{n}) - \frac{i}{\vartheta}}{\frac{k}{\vartheta}}, \frac{\vartheta a (i+lk)^{1/3} - \vartheta \varepsilon n^{1/3} f (\frac{i}{n}) - \frac{i}{\vartheta}}{\frac{k}{\vartheta}} \right] \right) \]

\[ \times \sup_{\frac{a (l+1k)^{1/3} - \frac{m-0.9}{a N} (\vartheta n^{1/3} + \vartheta n^{1/3} f (\frac{l+1}{n}))}{\frac{k}{\vartheta}}} \mathbb{P}_L^{\varepsilon n^{1/3}} \left( \forall i \leq j, T_i \in \left[ \frac{\vartheta a (i+j)^{1/3} - \vartheta \varepsilon n^{1/3} f (\frac{i}{n}) - \frac{i}{\vartheta}}{\frac{k}{\vartheta}}, \frac{\vartheta a (i+j)^{1/3} - \vartheta \varepsilon n^{1/3} f (\frac{i}{n}) - \frac{i}{\vartheta}}{\frac{k}{\vartheta}} \right] \right) \]
We know \( c_N := \max_{l \in \{0, 1, \ldots, N-1\}} \int_{l/N}^{(l+1)/N} f^{-2}(x) \, dx \to 0 \) as \( N \to +\infty \) since \( \int_0^1 f^{-2}(x) \, dx \).

We take \( a_N = c_N^{-1/3} \), then we have

\[
\begin{align*}
\lim_{k \to +\infty} \frac{\ln B_{kN}}{n^{3/2}} &= -(1) \eta \gamma(\beta) \sigma^2 \int_0^{1/N} \left[ \frac{\partial}{\partial x} (e + f(x)) + \frac{2\partial}{a_N} (2e + f(l/N)) \right]^{-2} \, dx \\
&= -(1) \eta \gamma(\beta) \sigma^2 \int_0^{1/N} \left[ \frac{\partial}{\partial x} (e + f(x)) \right]^{-2} \, dx \\
&= -\gamma(\beta) \sigma^2 \int_{1/N}^{1} \left[ \frac{\partial}{\partial x} (\epsilon + f(x)) \right]^{-2} \, dx \\
&= -\gamma(\beta) \sigma^2 \int_{1/N}^{1} \left[ (\epsilon + f(x)) \right]^{-2} \, dx \\
&\geq -\gamma(\beta) \sigma^2 \int_{1/N}^{1} \left[ (\epsilon + f(x)) \right]^{-2} \, dx
\end{align*}
\]

\[
\begin{align*}
\lim_{k \to +\infty} \frac{\ln A_{kN}}{n^{1/3}} &= -\gamma(\beta) \sigma^2 \int_0^{1} \left[ \partial \epsilon + \partial f(x) + \frac{2\partial}{a_N} (2\epsilon + f(l/N)) \right]^{-2} \, dx \\
&\geq -\gamma(\beta) \sigma^2 \int_{1/N}^{1} \left[ (\epsilon + f(x)) \right]^{-2} \, dx \\
&= -\gamma(\beta) \sigma^2 \int_{1/N}^{1} \left[ (\epsilon + f(x)) \right]^{-2} \, dx \\
&\geq -\gamma(\beta) \sigma^2 \int_{1/N}^{1} \left[ (\epsilon + f(x)) \right]^{-2} \, dx
\end{align*}
\]

\[
\begin{align*}
\lim_{k \to +\infty} \frac{\ln D_{kN}}{n^{1/3}} &\leq \partial a - \partial a(l/N)^{1/3} + 2\partial \epsilon + \partial f(l/N) - \gamma(\beta) \sigma^2 \int_{1/N}^{1} (\epsilon + f(x))^{-2} \, dx + o(N) + \gamma(\beta) \sigma^2 \int_{1/N}^{1} [(\epsilon + f(x))^{1/3}]^{-2} \, dx \\
&\leq \partial a - \partial a(l/N)^{1/3} + 2\partial \epsilon + \partial f(l/N) - \gamma(\beta) \sigma^2 \int_{1/N}^{1} (\epsilon + f(x))^{-2} \, dx + o(N)
\end{align*}
\]

On the other hand

\[
\begin{align*}
\mathbb{E}_\mathcal{L}(Z_\nu(\Theta)) &\geq \mathbb{E}_\mathcal{L} \left( \sum_{|u| = n} 1 \{ \forall 1 \leq i \leq n, V(u_i) \in [a_i^{1/3} - \epsilon, a_i^{1/3}] \} \right) \\
&= \mathbb{E}_\mathcal{L} \left( e^{T_n} \sum_{|u| = n} 1 \{ \forall 1 \leq i \leq n, \forall 1 \leq j \leq n, V(u_i, u_j) \in [a_i^{1/3} - \epsilon, a_i^{1/3}] \} \right) \\
&\geq e^{\gamma a_i^{1/3} - \epsilon a_i^{1/3}} \mathbb{P}_\mathcal{L} \left( \forall 1 \leq i \leq n, T_i \in [h(n, i), a_i^{1/3}], \nu(u_{i-1}) \leq r_n \right)
\end{align*}
\]
\[
\lim_{k \to +\infty} \frac{\ln \mathbb{E}_L(Z_n(\Theta)) - \ln D_{kN}}{n^{1/4}} \\
\geq \vartheta a - \vartheta \varepsilon - \vartheta f(1) - \frac{\gamma(\beta)\sigma^2}{\vartheta^2} \int_0^1 (\varepsilon + f(x))^{-2} \, dx \\
- \max_{l \leq N} \left[ \vartheta a - \vartheta a \left( \frac{l}{N} \right)^{1/3} + 2\vartheta \varepsilon + \vartheta f \left( \frac{l}{N} \right) - \frac{\gamma(\beta)\sigma^2}{\vartheta^2} \int_0^1 (\varepsilon + f(x))^{-2} \, dx + o(N) \right]
\]

Combining with (4.2) and (4.3), we complete the proof of Theorem 2.1 (b).

**Proof of Theorem 2.2: the upper bound**

Let \( f(n, k) := -\vartheta \sqrt{n} - k \). We know

\[
\mathbb{P}_L(Z_n > 0) = \mathbb{P}_L(\exists u : |u| = n, V(u_i) \leq g(i) - \frac{K_i}{\vartheta}) \leq \sum_{j=1}^n H_j + H.
\]

Where

\[
H_j := \mathbb{P}_L \left( \exists |u| = j : \forall i < j, V(u_i) \in [f(n, i) - \frac{K_i}{\vartheta}, g(i) - \frac{K_i}{\vartheta}], V(u_j) \leq f(n, j) - \frac{K_j}{\vartheta} \right).
\]

\[
H := \mathbb{P}_L \left( \exists |u| = n : \forall i \leq n, V(u_i) \in [f(n, i) - \frac{K_i}{\vartheta}, g(i) - \frac{K_i}{\vartheta}] \right).
\]

By Markov inequality and many to one formula, we have

\[
H_j \leq \mathbb{E}_L \left( \sum_{|u| = j} 1 \left( \forall i < j, V(u_i) \in [f(n, i) - \frac{K_i}{\vartheta}, g(i) - \frac{K_i}{\vartheta}], V(u_j) \leq f(n, j) - \frac{K_j}{\vartheta} \right) \right)
\]

\[
= \mathbb{E}_L \left( e^{\vartheta f(n, j)} 1 \left( \forall i < j, S_i \in [f(n, i) - \frac{K_i}{\vartheta}, g(i) - \frac{K_i}{\vartheta}], S_j \leq f(n, j) - \frac{K_j}{\vartheta} \right) \right)
\]

\[
\leq e^{\vartheta f(n, j)} \mathbb{P}_L \left( \forall i < j, S_i \in [f(n, i) - \frac{K_i}{\vartheta}, g(i) - \frac{K_i}{\vartheta}] \right).
\]
For the monotonicity of $\mathbb{P}_\mathcal{L}(Z_n > 0)$ we only need to consider $n := Nk, l \in [0, N - 1] \cap \mathbb{N}$.

$$\sum_{i=k+1}^{k(l+1)} H_i \leq ke^{-\vartheta d \sqrt{Nk - k(l+1)}} \mathbb{P}_\mathcal{L}(\forall i \leq lk, T_i \in [\vartheta f(n, i), \vartheta g(i)])$$

By the random version of Mogul’kiĭ estimation, we have

$$\lim_{k \to +\infty} \frac{\ln(\sum_{i=k+1}^{k(l+1)} H_i)}{\frac{1}{n^2}} \leq -\vartheta d \sqrt{1 - \frac{l + 1}{N}} - \gamma(\beta) \sigma^2 \int_0^\frac{1}{N} (\vartheta d \sqrt{1 - x})^{-2} dx$$

$$\leq -\vartheta d \left(1 - \frac{l + 1}{N}\right) \frac{1}{N} - 3\gamma(\beta) \sigma^2 \left[1 - \left(1 - \frac{l}{N}\right)^\frac{1}{2}\right].$$

$$\lim_{k \to +\infty} \frac{\ln H}{\frac{1}{n^2}} \leq -\frac{3\gamma(\beta) \sigma^2}{\vartheta^2 d^2}.$$ 

Let $d := \sqrt{\frac{3\gamma(\beta) \sigma^2}{\vartheta}}$, we have $\limsup_{n \to +\infty} \frac{\ln \mathbb{P}_\mathcal{L}(Z_n > 0)}{n^2} \leq -\frac{3\gamma(\beta) \sigma^2}{\vartheta^2 d^2}$.

**Proof of Theorem 2.2: the lower bound**

Define

$$\Theta := \left\{ u \in \mathcal{T} : \forall 1 \leq i \leq |u|, V(\nu(u_i)) \in [f(|u|, i) - \epsilon |u| \frac{1}{\vartheta} - \frac{K_i}{\vartheta}, g(i) - \frac{K_i}{\vartheta}], \nu(u_{i-1}) \leq \nu(u_i) \right\}.$$

$Z_n(\Theta) := \sum_{|u|=n} 1_{\{u \in \Theta\}}. \mathbb{P}_\mathcal{L}(Z_n > 0) \geq \mathbb{P}_\mathcal{L}(Z_n(\Theta) > 0) \geq \frac{\mathbb{P}_\mathcal{L}(Z_n(\Theta))^2}{\mathbb{E}_\mathcal{L}(Z_n(\Theta))}$.

In this proof we write the so-called second moment method in more detail.

$$\sum_{|u|=n} \sum_{k=0}^{n-1} 1_{\{u \in \Theta\}} Z_{n,k}(\Theta, u_{k+1}) = \sum_{k=0}^{n-1} 1_{\{u \in \Theta\}} Z_{n,k}(\Theta, u_{k+1}) = \sum_{k=0}^{n-1} \sum_{|u|=n} Z_{n,k+1}(\Theta) Z_{n,k}(\Theta, u_{k+1})$$

$$\sum_{|u|=n} 1_{\{u \in \Theta\}} Z_{n,k}(\Theta, u_{k+1}) = \sum_{|u|=n} Z_{n,k+1}(\Theta) Z_{n,k}(\Theta, u_{k+1}).$$

On the other hand

$$\sum_{|u|=n} \sum_{k=0}^{n-1} 1_{\{u \in \Theta\}} Z_{n,k}(\Theta, u_{k+1}) = \sum_{|u|=n} 1_{\{u \in \Theta\}} (Z_n(\Theta) - 1_{\{u \in \Theta\}}) = Z_n^2(\Theta) - Z_n(\Theta).$$

So

$$Z_n^2(\Theta) = Z_n(\Theta) + \sum_{k=0}^{n-1} \sum_{|u|=n} Z_{n,k+1}(\Theta) Z_{n,k}(\Theta, u_{k+1}) = Z_n(\Theta) + \sum_{k=1}^{n} \sum_{|u|=k} Z_{n,k}(\Theta) Z_{n,k}(\Theta, v).$$
\[
\mathbb{E}_L\left( \sum_{|v|=k} Z_n^v(\Theta) Z_n^v(\Theta, v) \right) = \mathbb{E}_L \left( \sum_{|v|=k} \mathbb{E}_L \left( \sum_{v'=br(v)} Z_n^v(\Theta) Z_n^{v'}(\Theta) | \mathcal{F}_k \right) \right) \\
\leq \mathbb{E}_L \left( \sum_{|v|=k} \mathbb{E}_L \left[ \sum_{v'=br(v)} Z_n^v(\Theta) Z_n^{v'}(\Theta) | \mathcal{F}_k \right] \right) \\
\leq \mathbb{E}_L \left( \sum_{|v|=k} \mathbb{E}_L \left[ Z_n^v(\Theta) | \mathcal{F}_k \right] \mathbb{E}_L \left[ Z_n^{v'}(\Theta) | \mathcal{F}_k \right] \right) \\
\leq \mathbb{E}_L \left( \sum_{|v|=k} \mathbb{E}_L \left[ Z_n^v(\Theta) | \mathcal{F}_k \right] \sum_{v'=br(v)} \mathbb{E}_L \left[ Z_n^{v'}(\Theta) | \mathcal{F}_k \right] \right) \\
\leq \mathbb{E}_L \left( (r_n - 1) \sup_{|v'|=k} \mathbb{E}_L \left[ Z_n^{v'}(\Theta) \right] \mathbb{E}_L \left( \sum_{|v|=k} \mathbb{E}_L \left[ Z_n^v(\Theta) | \mathcal{F}_k \right] \right) \right) \\
\leq (r_n - 1) \sup_{|v'|=k} \mathbb{E}_L \left( Z_n^{v'}(\Theta) \right) \mathbb{E}_L \left( \sum_{|v|=k} Z_n^v(\Theta) \right) \\
\leq (r_n - 1) \sup_{|v'|=k} \mathbb{E}_L \left( Z_n^{v'}(\Theta) \right) \mathbb{E}_L \left( \sum_{|v|=k} Z_n^v(\Theta) \right)
\]

\[
\mathbb{E}_L(\mathbb{E}_L(Z_n^2(\Theta))) = \mathbb{E}_L \left( Z_n(\Theta) \right) \left[ 1 + (r_n - 1) \sum_{k=1}^{n} \sup_{|v'|=k} \mathbb{E}_L \left( Z_n^{v'}(\Theta) \right) \right]
\]

Let \( I_{n,k}(\mathcal{L}) = [f(n,k) - \varepsilon n^{\frac{1}{2}} - \frac{K_k}{\vartheta}, g(k) - \frac{K_k}{\vartheta}] \), then we have

\[
\sup_{|v'|=k} \mathbb{E}_L \left( Z_n^{v'}(\Theta) \right) \leq \sup_{V(v') \in I_{n,k}(\mathcal{L})} \mathbb{E}_L \left( \sum_{|u|=n,u,v'=v'} 1 \{u \in \Theta\} \right) \\
\leq \sup_{V(v') \in I_{n,k}(\mathcal{L})} \mathbb{E}_L \left( \sum_{|u|=n,u,v'=v'} 1 \{v_i \leq n-k, v(u_{k+i}) \in [f(n,i+k) - \frac{K_{i+k}}{\vartheta}, g(i+k) - \frac{K_{i+k}}{\vartheta}]\} \right) \\
= \sup_{v \in I_{n,k}(\mathcal{L})} \mathbb{E}_L^k \left( \sum_{|u|=n-k} 1 \{v_i \leq n-k, v(y_i) \in [-x+f(n,i+k) - \frac{K_{i+k}}{\vartheta}, -x+g(i+k) - \frac{K_{i+k}}{\vartheta}]\} \right) \\
= \sup_{v \in I_{n,k}(\mathcal{L})} \mathbb{E}_L^k \left( e^{T_{n-k} - k} 1 \{v_i \leq n-k, T_i \in [-x+f(n,i+k) - \frac{K_{i+k}}{\vartheta}, -x+g(i+k) - \frac{K_{i+k}}{\vartheta}]\} \right)
\]

\[
\sup_{|v'|=k} \mathbb{E}_L \left( Z_n^{v'}(\Theta) \right) \leq e^\vartheta f(n,k) \sup_{x \in I_{n,k}(\mathcal{L})} \mathbb{P}_L^k \left( \forall i \leq n-k, \frac{T_i}{\vartheta} + x + \frac{K_k}{\vartheta} \in [f(n,i+k), g(i+k)] \right) \\
\leq e^\vartheta f(n,k) \sup_{x \in [\vartheta f(n,k), \vartheta g(k)]} \mathbb{P}_L^{k,x} \left( \forall i \leq n-k, T_i \in [\vartheta f(n,i+k), \vartheta g(i+k)] \right) \\
\leq e^{-\vartheta f(n,k)} \sup_{x \in [\vartheta f(n,k), 0]} \mathbb{P}_L^{k,x} \left( \forall i \leq n-k, T_i \in [\vartheta f(n,i+k), 0] \right)
\]
Here $T_n = \partial S_n + \sum_{i=k+1}^{k+n} \kappa_i(\vartheta). -x - \frac{K}{\sqrt{n}} \in [a k^\alpha, \varepsilon n^{\frac{1}{2}} + d \sqrt{n} - k]$

\[
\mathbb{P}_\mathcal{L}(Z_n > 0) \geq \frac{\mathbb{E}_\mathcal{L}(Z_n(\Theta))}{1 + (r_n - 1) \sum_{j=1}^{n} e^{-\vartheta f(n,j)} \sup_{x \in [\vartheta f(n,j), 0]} \mathbb{P}_\mathcal{L}^{i,x}(\forall i \leq n - j, T_i \in [\vartheta f(n, i + j), 0])}
\]

Let $n = Nk$.

\[
\sum_{j=1}^{n} e^{-\vartheta f(n,j)} \sup_{x \in [\vartheta f(n,j), 0]} \mathbb{P}_\mathcal{L}^{i,x}(\forall i \leq n - j, T_i \in [\vartheta f(n, i + j), 0])
\]

\[
\leq \sum_{l=0}^{N-1} e^{\varepsilon l \vartheta f(n,j) + \varepsilon d \sqrt{kn - kl}} \sum_{j=kl+1}^{k(l+1)} \sup_{x \in [\vartheta f(n,j), 0]} \mathbb{P}_\mathcal{L}^{i,x}(\forall i \leq k(N - l - 1), T_i \in [\vartheta f(n, i + j), 0])
\]

\[
\leq \sum_{l=0}^{N-1} e^{\varepsilon l \vartheta f(n,j) + \varepsilon d \sqrt{kn - kl}} \sum_{j=kl+1}^{k(l+1)} \sup_{x \in [\vartheta f(n,j), 0]} \mathbb{P}_\mathcal{L}^{i,x}(\forall i \leq n - j, T_i \in [\vartheta f(n, i + j), 0])
\]

\[
\leq \sum_{l=0}^{N-1} e^{\varepsilon l \vartheta f(n,j) + \varepsilon d \sqrt{kn - kl}} \sum_{j=kl+1}^{k(l+1)} \sum_{m=2N}^{k(l+1)+1} \mathbb{P}_\mathcal{L}^{i,x}(\forall i \leq n - j, T_i \in [\vartheta f(n, i + j), 0])
\]

\[
\sum_{j=kl+1}^{k(l+1)} \sum_{m=2N}^{k(l+1)+1} \mathbb{P}_\mathcal{L}^{i,x}(\forall i \leq n - j, T_i \in [\vartheta f(n, i + j), 0])
\]

\[
\leq k \sum_{m=2N}^{k(l+1)+1} \frac{\mathbb{P}_\mathcal{L}^{i,x}(\forall i \leq n, T_i \in [f(n,i) + \frac{f(n,ik)}{a_N}, -\frac{f(n,ik)}{a_N}])}{B_m \times C_m}
\]

\[
\lim_{n \to \infty} \frac{\ln \mathbb{P}_\mathcal{L}^{i,x}(\forall i \leq n, T_i \in [f(n,i) + \frac{f(n,ik)}{a_N}, -\frac{f(n,ik)}{a_N}])}{n^{\frac{1}{3}}}
\]

\[
= -\gamma(\beta)\sigma^2 \int_0^1 (\varepsilon \vartheta + d \vartheta (1 - x)^{\frac{1}{3}} + \frac{2d\vartheta}{a_N} (1 - \frac{l}{N})^{\frac{1}{3}})^{-2}dx
\]

\[
\lim_{n \to \infty} \frac{\ln B_m}{n^{\frac{1}{3}}} = -\gamma(\beta)\sigma^2 \int_0^1 (\varepsilon \vartheta + d \vartheta (1 - x)^{\frac{1}{3}} + \frac{2d\vartheta}{a_N} (1 - \frac{l}{N})^{\frac{1}{3}})^{-2}dx
\]

\[
\lim_{n \to \infty} \frac{\ln C_m}{n^{\frac{1}{3}}} = -\gamma(\beta)\sigma^2 \int_0^1 (\varepsilon \vartheta + d \vartheta (1 - x)^{\frac{1}{3}} + \frac{2d\vartheta}{a_N} (1 - \frac{l}{N})^{\frac{1}{3}})^{-2}dx
\]
Let \( a_N = \lfloor N^{\frac{j}{n}} \rfloor \), \( j \in [k, l + k, k] \).

\[
\begin{align*}
\mathbb{P}_{\mathcal{L}}^{-\varepsilon_n \frac{1}{n}} \left( \forall i \leq n, T_i \in [f(n, i) + \frac{f(n, lk)}{a_N}, -\frac{f(n, lk)}{a_N}] \right) \\
\geq \mathbb{P}_{\mathcal{L}}^{-\varepsilon_n \frac{1}{n}} \left( \forall i \leq j, T_i \in [f(n, i) + \frac{f(n, lk)}{a_N}, -\frac{f(n, lk)}{a_N}], T_j \in \left[ \frac{mf(n, j)}{a_N}, \frac{(m - 1)f(n, j)}{a_N} \right] \right) \\
\quad \times \inf_{x \in \left[ \frac{mf(n, j)}{a_N}, \frac{(m - 1)f(n, j)}{a_N} \right]} \mathbb{P}_{\mathcal{L}}^{j, x} \left( \forall i \leq j - lk, T_i \in \left[ \frac{mf(n, lk + i)}{a_N}, \frac{(m - 1)f(n, lk + i)}{a_N} \right] \right) \\
\geq \mathbb{P}_{\mathcal{L}}^{-\varepsilon_n \frac{1}{n}} \left( \forall i \leq j, T_i \in [f(n, i) + \frac{f(n, lk)}{a_N}, -\frac{f(n, lk)}{a_N}], T_j \in \left[ \frac{mf(n, j)}{a_N}, \frac{(m - 1)f(n, j)}{a_N} \right] \right) \\
\quad \times \inf_{x \in \left[ \frac{mf(n, j)}{a_N}, \frac{(m - 1)f(n, j)}{a_N} \right]} \mathbb{P}_{\mathcal{L}}^{j, x} \left( \forall i \leq j - lk, T_i \in \left[ \frac{mf(n, lk + i)}{a_N}, \frac{(m - 1)f(n, lk + i)}{a_N} \right] \right) \\
\quad \times \sup_{x \in \left[ \frac{mf(n, j)}{a_N}, \frac{(m - 1)f(n, j)}{a_N} \right]} \mathbb{P}_{\mathcal{L}}^{i, x} \left( \forall i \leq n - k, T_i \in [f(n, i + k, 0)] \right) \\
\geq C_{k, m} \times B_{k, m} \times \sup_{x \in \left[ \frac{mf(n, j)}{a_N}, \frac{(m - 1)f(n, j)}{a_N} \right]} \mathbb{P}_{\mathcal{L}}^{j, x} \left( \forall i \leq n - k, T_i \in [f(n, i + k, 0)] \right) \\
\end{align*}
\]

\[
\begin{align*}
\lim_{k \to \infty} \left( \mathbb{P}_{\mathcal{L}}^{-\varepsilon_n \frac{1}{n}} \left( \forall i \leq n, T_i \in [f(n, i) + \frac{f(n, lk)}{a_N}, -\frac{f(n, lk)}{a_N}] \right) \right) \\
\leq \lim_{k \to \infty} \left( \mathbb{P}_{\mathcal{L}}^{-\varepsilon_n \frac{1}{n}} \left( \forall i \leq n, T_i \in [f(n, i) + \frac{f(n, lk)}{a_N}, -\frac{f(n, lk)}{a_N}] \right) \right) \\
\leq \varepsilon \theta + \vartheta \sqrt{1 - \frac{1}{N}} + \gamma(\beta) \sigma^2 \int_{0}^{1} \left( \varepsilon \theta + \vartheta \sqrt{1 - x} \right)^{\frac{3}{2}} + \frac{2 \vartheta \theta}{a_N} \left( 1 - \frac{l}{N} \right)^{\frac{3}{2}} - 2 dx \\
+ \gamma(\beta) \sigma^2 \int_{0}^{1} \vartheta \sqrt{1 - x} \left( \frac{d\theta(1 - x)}{a_N} \right)^{\frac{3}{2}} + \frac{2 \vartheta \theta}{a_N} \left( 1 - \frac{l}{N} \right)^{\frac{3}{2}} - 2 dx
\end{align*}
\]
On the other hand, we have
\[ \mathbb{E}_L(Z_n(\Theta)) = \mathbb{E}_L\left( \sum_{|u|=n, 1 \leq i \leq n} 1 \{ \forall 1 \leq i \leq n, V(u_i) \in [f(n,i) - \varepsilon n^{1/3} g(i)/n^{1/3}, \nu(u_{i-1}) \leq r_n] \} \right) = \mathbb{E}_L\left( e^{T_n} 1 \{ \forall 1 \leq i \leq n, T_i \in [\vartheta f(n,i) - \vartheta \varepsilon n^{1/3} g(i), \nu(u_{i-1}) \leq r_n] \} \right) \geq e^{-\varepsilon n^{1/3}} \mathbb{P}_L( \forall 1 \leq i \leq n, T_i \in [\vartheta f(n,i) - \vartheta \varepsilon n^{1/3} g(i), \nu(u_{i-1}) \leq r_n] ) \]
\[ \sup_{\alpha = \frac{\vartheta}{\vartheta} \in [0,1]} \frac{\ln \mathbb{E}_L(Z_n(\Theta))}{n^{1/3}} \geq -\varepsilon - \gamma(\beta) \sigma^2 \int_0^1 (\varepsilon \vartheta + d\vartheta (1-x)^{1/3})^{-2} dx \geq \frac{-3\gamma(\beta) \sigma^2}{d^2 \vartheta^2} \]
\[ \sup_{\alpha = \frac{\vartheta}{\vartheta} \in [0,1]} \vartheta d \sqrt{1 - \alpha} - \gamma(\beta) \sigma^2 \int_0^1 (d\vartheta (1-x)^{1/3})^{-2} dx = \sup_{\alpha = \frac{\vartheta}{\vartheta} \in [0,1]} (\vartheta d - 3\gamma(\beta) \sigma^2 \int_0^1 \sqrt{1 - \alpha} \vartheta d \sqrt{1 - \alpha}) \]
When \( \vartheta d = \frac{3\gamma(\beta) \sigma^2}{d^2 \vartheta^2} \), we have \( \liminf_{n \to \infty} \frac{\mathbb{P}_L(Z_n > 0)}{n^{1/3}} \geq -\sqrt{3\gamma(\beta) \sigma^2} \). This is the end of the proof of Theorem 2.2.

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