Secrecy Capacity Region of Binary and Gaussian Multiple Access Channels

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Abstract—A generalized multiple access channel (GMAC) with one confidential message set is studied, where two users (users 1 and 2) attempt to transmit common information to a destination, and user 1 also has confidential information intended for the destination. Moreover, user 1 wishes to keep its confidential information as secret as possible from user 2. A deterministic GMAC is first studied, and the capacity-equivocation region and the secrecy capacity region are obtained. Two main classes of the GMAC are then studied: the binary GMAC and the Gaussian GMAC. For both channels, the capacity-equivocation region and the secrecy capacity region are established.

I. INTRODUCTION

An important security issue in multi-terminal networks is the transmission of confidential information to legitimate destinations while keeping other nodes as ignorant of this information as possible. The secrecy level of a confidential message at a nonlegitimate node (or a wire-tapper) is measured by the equivocation rate, i.e., the entropy rate of the confidential message conditioned on the channel outputs at this node. The secrecy capacity is the maximum rate at which the confidential message can be reliably transmitted to the intended destination with the wire-tapper obtaining no information.

The secrecy capacity was established for a basic wire-tap channel by Wyner in [1], and for a more general model of the broadcast channel with confidential messages by Csiszár and Körner in [2]. The relay channel with confidential messages was studied in [3], where the secrecy rate was given. More recently, a generalized multiple access channel (GMAC) with confidential messages was studied in [4], where each user wishes to transmit a confidential message to a destination, and wishes to keep the other user as ignorant of its confidential message as possible. The secrecy rate region was obtained for the GMAC with two confidential message sets, and the secrecy capacity region was established for the GMAC with one confidential message set. Other work on multiple access channels with confidential messages can be found in [5], [6].

In this paper, we focus on the GMAC with one confidential message set, where the two users have a common message for the destination and only one user (user 1) has a confidential message for the destination. We first study a simple deterministic GMAC, and characterize the capacity-equivocation region and the secrecy capacity region. The focus of this paper is on the two main classes of GMACs: the binary GMAC and the Gaussian GMAC. For both channels, we establish the capacity-equivocation region and the secrecy capacity region explicitly.

In this paper, we use $x^n$ to indicate the vector $(x_1, \ldots, x_n)$, and use $x^n_i$ to indicate the vector $(x_i, \ldots, x_n)$. Throughout the paper, the logarithmic function is to the base 2.

The organization of this paper is as follows. In Section III we introduce the channel model of the GMAC with one confidential message set. In Section III we present the secrecy capacity region of a simple deterministic GMAC with one confidential message set. In Section IV we present the secrecy capacity region for a binary GMAC model with one confidential message set. In Section V we focus on the Gaussian GMAC with one confidential message set, and present our results on the secrecy capacity region.

II. CHANNEL MODEL AND PREVIOUS RESULTS

In this section, we first define the GMAC with one confidential message set, and then review the known results for this model.

Definition 1: A discrete memoryless GMAC consists of two finite channel input alphabets $\mathcal{X}_1$ and $\mathcal{X}_2$, two finite channel output alphabets $\mathcal{Y}$ and $\mathcal{Y}_2$, and a transition probability distribution $p(y, y_2|x_1, x_2)$ (see Fig. 1), where $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$ are channel inputs from users 1 and 2, respectively, and $y \in \mathcal{Y}$ and $y_2 \in \mathcal{Y}_2$ are channel outputs at the destination and user 2, respectively.

Definition 2: The GMAC with one confidential message set is physically degraded if the transition probability distribution satisfies

\[
p(y, y_2|x_1, x_2) = p(y|x_1, x_2)p(y_2|y, x_2),
\]

i.e., $y_2$ is independent of $x_1$ conditioned on $y$ and $x_2$.

Definition 3: A $(2^{nR_0}, 2^{nR_1}, n)$ code consists of the following:

- Two message sets: $\mathcal{W}_0 = \{1, 2, \ldots, 2^{nR_0}\}$ and $\mathcal{W}_1 = \{1, 2, \ldots, 2^{nR_1}\}$. The messages $\mathcal{W}_0$ and $\mathcal{W}_1$ are independent and uniformly distributed over $\mathcal{W}_0$ and $\mathcal{W}_1$, respectively.

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Two (stochastic) encoders, one at user 1: \( \mathcal{W}_0 \times \mathcal{W}_1 \rightarrow \mathcal{X}_1^n \), which maps each message pair \((w_0, w_1) \in \mathcal{W}_0 \times \mathcal{W}_1\) to a codeword \(x_1^n \in \mathcal{X}_1^n\); the other at user 2: \( \mathcal{W}_0 \rightarrow \mathcal{X}_2^n \), which maps each message \(w_0 \in \mathcal{W}_0\) to a codeword \(x_2^n \in \mathcal{X}_2^n\);

- One decoder at the destination: \( \mathcal{Y}^n \rightarrow \mathcal{W}_0 \times \mathcal{W}_1 \), which maps a received sequence \(y^n\) to a message pair \((w_0, w_1) \in \mathcal{W}_0 \times \mathcal{W}_1\).

Note that although user 2 can receive channel outputs (see Fig.1), it is only a passive listener in that its encoding function is not affected by the received outputs. However, since its outputs contain the confidential message \(W_1\) sent by user 1, it may extract \(W_1\) from its outputs. We assume that user 1 treats user 2 as a wire-tapper, and wishes to keep it as ignorant of \(W_1\) as possible. The secrecy level of \(W_1\) at user 2 is measured by the following equivocation rate:

\[
\frac{1}{n} H(W_1|Y_2^n, X_2^n, W_0). \tag{2}
\]

The larger the equivocation rate, the higher the level of secrecy. A rate-equivocation triple \((R_0, R_1, R_{1,c})\) is achievable if there exists a sequence of \((2^n, 2^n, n)\) codes with the average error probability \(P_e^n \rightarrow 0\) as \(n\) goes to infinity and with the equivocation rate \(R_{1,c}\) satisfying

\[
R_{1,c} \leq \lim_{n \rightarrow \infty} \frac{1}{n} H(W_1|Y_2^n, X_2^n, W_0). \tag{3}
\]

The rate-equivocation triple \((R_0, R_1, R_{1,c})\) indicates that the rate pair \((R_0, R_1)\) can be achieved at the secrecy level \(R_{1,c}\).

The capacity-equivocation region, denoted by \(\mathcal{C}^d\), is the closure of the set that consists of all achievable rate-equivocation triples \((R_0, R_1, R_{1,c})\).

We are interested in the case where perfect secrecy is achieved, i.e., user 2 does not get any information about the confidential message that user 1 sends to the destination. This happens if \(R_{1,c} = R_1\).

**Definition 4:** The secrecy capacity region \(C_s\) is the region that includes all achievable rate pairs \((R_0, R_1)\) such that \(R_{1,c} = R_1\), i.e.,

\[
C_s = \{(R_0, R_1) : (R_0, R_1, R_{1,c}) \in \mathcal{C}^d\}. \tag{4}
\]

**Definition 5:** For a given rate \(R_0\), the secrecy capacity is the maximum achievable rate \(R_1\) with the confidential message perfectly hidden from user 2, i.e.,

\[
C_s(R_0) = \max_{(R_0, R_1) \in C_s} R_1. \tag{5}
\]

For the GMAC with one confidential message set, inner and outer bounds on the capacity-equivocation region were given in [4]. In particular, the exact secrecy capacity region was established. For the degraded GMAC, the capacity-equivocation region was established, which is given in the following lemma.

**Lemma 1:** (4) For the degraded GMAC with one confidential message set as in Definition 2 the capacity-equivocation region is given by

\[
\mathcal{C}^d = \bigcup_{p(q, x_2)p(x_1|q)}\bigcup_{p(y|x_1, x_2)p(y_2|y, x_2)}\left\{(R_0, R_1, R_{e}) : \right.
\]

\[
\begin{align*}
&R_0 \geq 0, R_1 \geq 0, \\
&R_1 + R_e \leq I(X_1; Y|X_2, Q), \\
&R_0 + R_1 \leq I(X_1; X_2; Y), \\
&0 \leq R_e \leq R_1, \\
&R_e \leq I(X_1; Y|X_2, Q) - I(X_1; Y|X_2, Q), \\
&R_0 + R_e \leq I(X_1; X_2; Y) - I(X_1; Y|X_2, Q) \\
\right\}. \tag{6}
\]

where \(Q\) is bounded in cardinality by \(|Q| \leq |X_1| \cdot |X_2| + 1\).

### III. A Simple Example

In this section, we consider a deterministic discrete memoryless GMAC model with one confidential message set. We obtain the capacity-equivocation region and the secrecy capacity region for this channel.

Consider a binary channel with all channel inputs and outputs having alphabets \(\{0, 1\}\). The MAC from the two users to the destination is a binary multiplier channel, and the channel from user 1 to user 2 is a bias channel. The channel input-output relationship (see Fig. 2) is given by

\[
Y = X_1 \cdot X_2, \quad Y_2 = \begin{cases} 
1, & \text{if } X_1 \leq X_2; \\
0, & \text{if } X_1 > X_2.
\end{cases} \tag{7}
\]

The capacity-equivocation region of the example channel given in (7) is:

\[
\{(R_0, R_1, R_e) : R_0 + R_1 \leq 1, R_e = R_1\}. \tag{8}
\]

The capacity-equivocation region implies that the secrecy capacity region of this channel is:

\[
\{(R_0, R_1) : R_0 + R_1 \leq 1\}. \tag{9}
\]
Note that the region (9) (see Fig. 3) coincides with the capacity region of the binary multiplier MAC given in [7].

To show that perfect secrecy can be achieved for all points in the region (9), we first show that perfect secrecy can be achieved for the two corner points. It is trivial that perfect secrecy can be achieved for the corner point $(R_0 = 0, R_1 = 1)$, i.e., $R_e = 0$ is achievable at this point. For the other corner point $(R_0 = 1, R_1 = 0)$, perfect secrecy is achieved by sending $(x_1 = 0, x_2 = 1)$ for $W_1 = 0$ and $(x_1 = 1, x_2 = 1)$ for $W_1 = 1$. When either of these two codewords is transmitted, user 2 always gets output $Y_2 = 1$, and hence cannot determine whether $W_1 = 0$ or $W_1 = 1$ is sent. Therefore, perfect secrecy is achieved. By time-sharing between these two corner points, perfect secrecy can be achieved for the entire region. Note that since the region (9) is the best possible rate-equivocation region that can be achieved, it is hence the capacity-equivocation region (8).

**Remark 1:** The deterministic GMAC defined in (7) is a nondegraded channel. We hence obtain the capacity-equivocation region for a nondegraded channel.

**IV. THE BINARY GMAC WITH ONE CONFIDENTIAL MESSAGE SET**

In this section, we first follow [8] to introduce notation and useful lemmas for binary channels. We then introduce the binary GMAC model we study and present the capacity-equivocation region for this channel.

We first define the following operation:

$$a * b := a(1 - b) + (1 - a)b$$

for $0 \leq a, b \leq 1$. (10)

We then define the following entropy function

$$h(a) := \begin{cases} -a \log a - (1 - a) \log(1 - a), & \text{if } 0 < a < 1; \\ 0, & \text{if } a = 0 \text{ or } 1. \end{cases}$$

(11)

Note that the function $h(a)$ is one-to-one for $0 \leq a \leq 1/2$. The inverse of the entropy function is limited to $h^{-1}(a) \in [0, 1/2]$.

**Lemma 2:** ([8]) The function $f(u) = h(\rho + h^{-1}(u))$, $0 \leq u \leq 1$ (where $\rho \in (0, 1/2]$ is a fixed parameter) is strictly convex in $u$.

The following useful lemma is a binary version of the entropy power inequality.

**Lemma 3:** ([8]) Consider two binary random vectors $X^n$ and $Y^n$. Let $H(X^n) \geq n\nu$. Let

$$Y_i = X_i \oplus Z_i \quad \text{for } i = 1, \ldots, n$$

(12)

where $Z^n$ is a binary random vector with i.i.d. components and $Z_i$ has distribution $Pr(Z_i = 1) = p_0$ where $0 < p_0 \leq 1/2$.

The vectors $X^n$ and $Y^n$ can be viewed as inputs and outputs of a binary symmetric channel (BSC) with the crossover probability $p_0$. Then,

$$H(Y^n) \geq n\nu(p_0 + h^{-1}(\nu))$$

(13)

with equality if and only if $X^n$ has independent components, and $H(X_i) = v$ for $i = 1, 2, \ldots, n$.

We now consider a discrete memoryless binary GMAC model with all inputs and outputs having the binary alphabet set $\{0, 1\}$. The channel input-output relationship (see Fig. 4) at each time instant satisfies

$$Y_i = X_{1,i} \cdot X_{2,i}, \quad Y_{2,i} = Y_i \oplus Z_{2,i} \quad \text{for } i = 1, \ldots, n$$

(14)

where $Z^n$ is a binary random vector with i.i.d. components and $Z_{2,i}$ has distribution $Pr(Z_{2,i} = 1) = p$ where $0 < p \leq 1/2$. Note that the MAC channel from $(X_1, X_2)$ to $Y$ is a binary multiplier channel. It is clear that this GMAC channel is degraded, and the channel outputs $Y$ and $Y_2$ can be viewed as the input and output of a discrete memoryless BSC with the crossover probability $p$.

We have the following theorem on the capacity-equivocation region.

**Theorem 1:** For the binary GMAC with one confidential message set defined in (14), the capacity-equivocation region is

$$\mathcal{C}_{EB}^B = \bigcup_{0 \leq a \leq 1/2} \left\{ \begin{array}{l} (R_0, R_1, R_e) : \\
R_0 \geq 0, R_1 \geq 0, \\
R_1 \leq h(a), \\
R_0 + R_1 \leq 1, \\
0 \leq R_e \leq R_1, \\
R_e \leq h(a) + h(p) - h(p + a), \\
R_0 + R_e \leq 1 + h(p) - h(p + a) \end{array} \right\}.$$ \hspace{1cm} (15)

The proof of Theorem 1 is given at the end of this section.

**Corollary 1:** The secrecy capacity region of the binary GMAC with one confidential message set defined in (14) is

$$\mathcal{C}_{sE}^B = \bigcup_{0 \leq a \leq 1/2} \left\{ \begin{array}{l} (R_0, R_1) : \\
R_0 \geq 0, R_1 \geq 0, \\
R_1 \leq h(a) + p - h(p + a), \\
R_0 + R_1 \leq 1 + h(p) - h(p + a) \end{array} \right\}.$$ \hspace{1cm} (16)
The secrecy capacity as a function of $R_0$ is given by
\[
C_s^B(R_0) = h(\alpha^*) + h(p) - h(p + \alpha^*)
\] (17)
where $\alpha^*$ is determined by the following equation
\[
R_0 = 1 - h(\alpha^*).
\] (18)

**Remark 2:** The BSC crossover probability parameter $p$ determines how noisy the channel from user 1 to user 2 is compared to the channel from user 1 to the destination. When $p = 0$, user 2 has the same channel from user 1 as the destination, and hence no secrecy can be achieved. As $p$ increases, user 2 has a noisier channel from user 1 than the destination, and hence higher secrecy can be achieved. As $p = \frac{1}{2}$, user 2 is totally confused by confidential messages sent by user 1, and perfect secrecy is achieved.

Fig. 5 plots the secrecy capacity as a function of $R_0$ for four values of $p$. These lines of $C_s^B(R_0)$ also serve as boundaries of the secrecy capacity regions with the vertical axis being viewed as $R_1$. It is clear from Fig. 5 that as $p$ increases, the secrecy capacity region enlarges, because user 2 is further confused about the confidential message sent by user 1.

**Remark 3:** From the achievability proof of Theorem 1 (given at the end of this section), it can be seen that the optimal scheme to achieve the secrecy capacity region uses superposition encoding. To achieve the secrecy capacity corresponding to different values of $R_0$, different values of the superposition parameter $\alpha$ needs to be chosen to generate the codebook. However, if the secrecy constraint is not considered, the capacity region of the binary multiplier MAC can be achieved by a time sharing scheme and superposition encoding is not necessary.

Fig. 6 plots the secrecy capacity as a function of $R_0$ (indicated by the solid line) and compares it with the secrecy rate achieved by the time sharing scheme (indicated by the dashed line). The figure demonstrates that the time sharing scheme is strictly suboptimal to provide the secrecy capacity region. As we commented in Remark 3 although the time sharing scheme is optimal to achieve the capacity region of the binary multiplier MAC, it is not optimal to achieve the secrecy capacity region of the binary GMAC, where secrecy is also considered as a performance criterion.

**Proof of Theorem 1**

**Proof of Achievability:**

We apply Lemma 1 to prove that the region [15] is achievable. Let $Q$ and $X'$ be two binary random variables with alphabet $\{0, 1\}$, and assume that $Q$ is independent of $X'$. We choose the following joint distribution:

\[
Pr\{Q = 0\} = \frac{1}{2}; \quad Pr\{X' = 1\} = \alpha, \quad 0 \leq \alpha \leq \frac{1}{2};
\]

\[
Pr\{X_2 = 1\} = 1; \quad X_1 = Q \oplus X'.
\]

We now compute the mutual information terms in (6) given in Lemma 1 based on the preceding joint distribution.

\[
R_1 \leq I(X_1; Y \mid X_2, Q) = H(Y \mid X_2, Q) = Pr\{Q = 0\} H(Y \mid X_2 = 1, Q = 0)
\]

\[
+ Pr\{Q = 1\} H(Y \mid X_2 = 1, Q = 1)
\]

\[
= h(\alpha),
\]

\[
R_0 + R_1 \leq I(X_1, X_2; Y) = H(Y) = 1,
\]

\[
R_0 \leq I(X_1; Y \mid X_2, Q) - I(X_1; Y_2 \mid X_2, Q)
\]

\[
= h(\alpha) - (H(Y_2 \mid X_2, Q) - H(Y_2 \mid X_1, X_2))
\]

\[
= h(\alpha) - [Pr\{Q = 0\} H(Y_2 \mid X_2 = 1, Q = 0)
\]

\[
+ Pr\{Q = 1\} H(Y_2 \mid X_2 = 1, Q = 1)
\]

\[
- Pr\{X_1 = 0\} H(Y_2 \mid X_2 = 1, X_1 = 0)
\]

\[
- Pr\{X_1 = 1\} H(Y_2 \mid X_2 = 1, X_1 = 1)\]

\[
= h(\alpha) - [h(\alpha \ast p) - h(p)]
\]

\[
= h(\alpha) + h(p) - h(\alpha \ast p),
\]
where we have used the deterministic property of the GMAC, $H_{GMAC}$. From (21), we obtain

$$R_0 + R_1 \leq I(X_1; X_2; Y) - I(X_1; Y_2|X_2; Q) = 1 - [h(\alpha * p) - h(p)] = 1 + h(p) - h(\alpha * p).$$

**Proof of the Converse:**

We consider a sequence of $(2^nR_0, 2^nR_1, n)$ codes for the degraded GMAC with one confidential message set with $P^{(n)}_c \rightarrow 0$. Then the probability distribution on $W_0 \times W_1 \times X_1^n \times X_2^n \times Y^n \times Y_2^n$ is given by

$$p(w_0, w_1, x_1^n, x_2^n, y^n, y_2^n) = p(w_0)p(w_1)p(x_1^n|w_0, w_1)p(x_2^n|w_0) \prod_{i=1}^n p(y_i, y_{2,i}|x_{1,i}, x_{2,i}).$$

From [9, Sec. 4.2], we have the following bounds:

$nR_{1,e}$

$$\leq I(X_1^n; Y^n|X_2^n, W_0) - I(X_1^n; Y_2^n|X_2^n, W_0) + n\delta_n$$

where $Q_i := (Y^{i-1}, X_2^n, W_0)$, and $\delta_n \rightarrow 0$ if $P^{(n)}_c \rightarrow 0$.

$$nR_0 + nR_{1,e} \leq I(W_0; Y^n) + nR_{1,e} + n\delta_n$$

$$= \sum_{i=1}^n I(X_1, i; X_2, i|Y_i) - I(X_1, i; Y_2, i|Q_i, X_2, i) + n\delta_n$$

From (19), we obtain

$$nR_0 + nR_{1,e} \leq I(W_0; Y^n) + nR_{1,e} + n\delta_n$$

where we have used the deterministic property of the GMAC, which implies $H(Y^n|X_1^n, X_2^n, W_0) = 0$.

Since $\{Q_i, 1 \leq i \leq n\}$ are binary random variables, $H(Y_i) \leq 1$ for $1 \leq i \leq n$. Hence

$$0 \leq H(Y^n|X_2^n, W_0) \leq \sum_{i=1}^n H(Y_i) \leq n.$$

It is clear that there exists a parameter $\alpha \in [0, 1/2]$ such that

$$H(Y^n|X_2^n, W_0) = nh(\alpha).$$

Substituting the preceding equation into (23), we obtain

$$nR_1 \leq nh(\alpha) + n\delta_n.$$

From (22), we obtain

$$nR_0 + nR_{1,e} \leq I(W_0; Y^n) + nh(\alpha) + n\delta_n \leq H(Y^n) + n\delta_n \leq n + n\delta_n.$$

From (19), we obtain

$$nR_{1,e} \leq I(X_1^n; Y^n|X_2^n, W_0) - I(X_1^n; Y_2^n|X_2^n, W_0) + n\delta_n$$

$$= H(Y^n|X_2^n, W_0) - H(Y_2^n|X_2^n, W_0)$$

$$+ H(Y_2^n|X_1^n, X_2^n, W_0) + n\delta_n$$

In the preceding bound, the first term in (a) follows from Lemma 3, the second term in (a) follows from the fact that $Y^n$ is a deterministic function of $(X_1^n, X_2^n)$, and the third term in (b) follows from the fact that $Y_2^n$ is conditionally independent of everything else given $Y^n$.

Since $Z_2^n$ in (14) is independent of $W_0, X_2^n$ and $Y^n$, we apply Lemma 3 to bound the term $H(Y_2^n|X_2^n, W_0)$.

$$H(Y_2^n|X_2^n, W_0) = nE[h(Y^n|X_2^n, W_0 = w_0)]$$

(a) $$\geq \mathbb{E} \left[ nh(p * h^{-1} \left( \frac{H(Y^n|X_2^n = x_2^n, W_0 = w_0)}{n} \right) \right]$$

(b) $$\geq nh\left( p * h^{-1} \left( \frac{H(Y^n|X_2^n = x_2^n, W_0 = w_0)}{n} \right) \right)$$

(c) $$= nh\left( p * h^{-1} \left( \frac{nh(\alpha)}{n} \right) \right) = nh(p*\alpha)$$

where (a) follows from Lemma 3, (b) follows from Lemma 2 and Jensen’s inequality, and (c) follows from (25).

Substituting (29) into (28), we obtain

$$nR_{1,e} \leq nh(\alpha) + nh(p) - nh(p*\alpha) + n\delta_n.$$
where (a) follows from (19), (b) follows from the chain rule and nonnegativity of mutual information, and (c) follows from the steps in deriving $R_{1,c}$.

In summary, (26), (27), (30) and (31) constitute the converse proof for Theorem 1.

V. GAUSSIAN GMAC WITH ONE CONFIDENTIAL MESSAGE SET

In this section, we study the Gaussian GMAC with one confidential message set, where the channel outputs at the destination and user 2 are corrupted by additive Gaussian noise terms. We assume that the channel is discrete and memoryless, and that the channel input-output relationship at each time instant is given by

\[ Y_i = X_{1,i} + X_{2,i} + Z_i \]
\[ Y_{2,i} = X_{1,i} + X_{2,i} + Z_{2,i} \]

where $Z^n$ and $Z^2$ are independent zero mean Gaussian random vectors with i.i.d. components. We assume that $Z_i$ and $Z_{2,i}$ have variances $N$ and $N_2$, respectively, where $N < N_2$. The channel input sequences $X^n_1$ and $X^n_2$ are subject to the average power constraints $P_1$ and $P_2$, respectively, i.e.,

\[ \frac{1}{n} \sum_{i=1}^{n} X^2_{1,i} \leq P_1, \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^{n} X^2_{2,i} \leq P_2. \]  

(33)

The following theorem states the capacity-equivocation region for the Gaussian GMAC with one confidential message set.

**Theorem 2**: For the Gaussian GMAC with one confidential message set given in (32), the capacity-equivocation region is given by

\[ \mathcal{G}^G = \bigcup_{0 \leq \alpha \leq 1} \left\{ (R_0, R_1, R_c) : \begin{array}{l}
(0, 0, 0), \\
R_0 \geq 0, R_1 \geq 0, \\
R_1 \leq \frac{1}{2} \log \left( 1 + \frac{P_1 N}{N_2^2} \alpha P_1 N_2^2 N \right), \\
R_0 + R_1 \leq \frac{1}{2} \log \left( 1 + \frac{P_1 + P_2 + 2 \sqrt{\alpha P_1 N}}{N} \right), \\
0 \leq R_c \leq R_1, \\
R_c \leq \frac{1}{2} \log \left( 1 + \frac{P_1 N^2}{N_2^2} \right) - \frac{1}{2} \log \left( 1 + \frac{P_1 N}{N_2^2} \right), \\
R_0 + R_c \leq \frac{1}{2} \log \left( 1 + \frac{P_1 + P_2 + 2 \sqrt{\alpha P_1 N}}{N} \right) - \frac{1}{2} \log \left( 1 + \frac{P_1 N}{N_2^2} \right) \\
\end{array} \right\} \]  

(34)

where $\alpha = 1 - \alpha$ indicating the correlation between the inputs from users 1 and 2.

The proof of Theorem 2 is given at the end of this section.

**Corollary 2**: The secrecy capacity region of the Gaussian GMAC with one confidential message set given in (32) is

\[ \mathcal{C}^G_s = \bigcup_{0 \leq \alpha \leq 1} \left\{ (R_0, R_1) : \begin{array}{l}
(0, 0), \\
R_0 \geq 0, R_1 \geq 0, \\
R_1 \leq \frac{1}{2} \log \left( 1 + \frac{P_1 N}{N_2^2} \right) - \frac{1}{2} \log \left( 1 + \frac{P_1 N}{N_2^2} \right), \\
R_0 + R_1 \leq \frac{1}{2} \log \left( 1 + \frac{P_1 + P_2 + 2 \sqrt{\alpha P_1 N}}{N} \right) - \frac{1}{2} \log \left( 1 + \frac{P_1 N}{N_2^2} \right) \\
\end{array} \right\} \]  

(35)

The secrecy capacity as a function of $R_0$ is

\[ C_s^G(R_0) = \left\{ \begin{array}{ll}
\frac{1}{2} \log \left( 1 + \frac{P_1 N}{N_2^2} \right) - \frac{1}{2} \log \left( 1 + \frac{P_1 N}{N_2^2} \right), & \text{if } R_0 \leq \frac{1}{2} \log \left( 1 + \frac{P_1 N}{N_2^2} \right) - \frac{1}{2} \log \left( 1 + \frac{P_1 N}{N_2^2} \right), \\
\frac{1}{2} \log \left( 1 + \frac{P_1 N}{N_2^2} \right) - \frac{1}{2} \log \left( 1 + \frac{P_1 N}{N_2^2} \right), & \text{if } R_0 > \frac{1}{2} \log \left( 1 + \frac{P_1 N}{N_2^2} \right) - \frac{1}{2} \log \left( 1 + \frac{P_1 N}{N_2^2} \right) \\
\end{array} \right. \]  

(36)

where $\alpha^*$ is determined by the following equation

\[ R_0 = \frac{1}{2} \log \left( 1 + \frac{P_1 + P_2 + 2 \sqrt{\alpha P_1 N}}{N} \right) - \frac{1}{2} \log \left( 1 + \frac{P_1 N}{N_2^2} \right) \]  

(37)

Fig. 7 plots the secrecy capacity $C^G_s(R_0)$ (solid lines) of Gaussian GMACs with one confidential message set for three user 1-to-user 2 SNR values. The lines of $C^G_s(R_0)$ also serve as boundaries of the secrecy capacity regions if we view the vertical axis as $R_1$. It can be seen that as user 1-to-user 2 SNR decreases, which implies that the noise level at user 2 increases, user 2 gets more confused about the confidential message sent by user 1. Thus the secrecy capacity region enlarges. As this SNR approaches zero, the secrecy capacity region approaches the entire capacity region of the Gaussian MAC, which means that perfect secrecy is achieved for almost all points in the capacity region of the MAC.

**Proof of Theorem 2**

We first note the following useful lemma.

**Lemma 4**: (4)) The capacity-equivocation region of GMACs with one confidential message set depends only on the marginal channel transition probability distributions $p(y|x_1, x_2)$ and $p(y_2|x_1, x_2)$.

To show Theorem 2 we first note that the Gaussian GMAC defined in (32) is not physically degraded according to Definition 2. However, it has the same marginal distributions $p(y|x_1, x_2)$ and $p(y_2|x_1, x_2)$ as the following physically de-
where $Z^n$ is the same as in (32). The random vector $Z^n$ is independent of $Z^n$, and has i.i.d. components with each component having the distribution $N(0, N_2 - N)$. According to Lemma 4, it is sufficient to prove Theorem 2 for the physically degraded Gaussian GMAC defined in (38).

Proof of the Achievability:

The achievability follows by computing the mutual information terms in Lemma 1 with the following joint distribution:

$$Q = \phi, \quad X_2 \sim N(0, P_2)$$

(39)

$$X_i' \sim N(0, \alpha P_i), \text{ and } X_i' \text{ is independent of } X_2$$

$$X_1 = \sqrt{\frac{\alpha P_i}{P_2}} X_2 + X_i'$$

Proof of the Converse:

We apply the bounds (19)-(22), and further derive these bounds for the degraded Gaussian GMAC.

From (21), we obtain

$$n R_1 \leq \sum_{i=1}^n I(X_{1,i}; Y_i X_{2,i}, Q_i) + n \delta_n$$

(40)

$$= \sum_{i=1}^n h(Y_i | X_{2,i}, Q_i) - h(Y_i | X_{1,i}, X_{2,i}, Q_i) + n \delta_n$$

$$= \sum_{i=1}^n h(Y_i | X_{2,i}, Q_i) - h(Z_i | X_{1,i}, X_{2,i}, Q_i) + n \delta_n$$

$$= \sum_{i=1}^n h(Y_i | X_{2,i}, Q_i) - \frac{1}{2} \log 2 \pi e N + n \delta_n$$

For the first term in the preceding inequality, we have

$$\sum_{i=1}^n h(Y_i | X_{2,i}, Q_i)$$

$$= \sum_{i=1}^n h(X_{1,i} + X_{2,i} + Z_i | X_{2,i}, Q_i)$$

$$= \sum_{i=1}^n h(X_{1,i} + Z_i | X_{2,i}, Q_i)$$

$$\leq \sum_{i=1}^n h(X_{1,i} + Z_i) \leq \sum_{i=1}^n \frac{1}{2} \log 2 \pi e (EX^2_{1,i} + N)$$

$$\leq \frac{1}{2} \log 2 \pi e \left( \frac{\sum_{i=1}^n E(X^2_{1,i})}{n} \right)$$

$$\leq \frac{1}{2} \log 2 \pi e (P_1 + N)$$

(41)

On the other hand,

$$\sum_{i=1}^n h(Y_i | X_{2,i}, Q_i)$$

$$\geq \sum_{i=1}^n h(X_{1,i} + X_{2,i} + Z_i | X_{1,i}, X_{2,i}, Q_i)$$

$$= \frac{n}{2} \log 2 \pi e N.$$  

(42)

Combining (41) and (42), we establish that there exists some $\alpha \in [0, 1]$ such that

$$\sum_{i=1}^n h(Y_i | X_{2,i}, Q_i) = \frac{n}{2} \log 2 \pi e (\alpha P_1 + N).$$

(43)

We hence obtain the bound for $R_1$

$$n R_1 \leq \frac{n}{2} \log 2 \pi e (\alpha P_1 + N) - \frac{1}{2} \log 2 \pi e N + n \delta_n$$

$$= \frac{n}{2} \log \left( 1 + \alpha P_1 \right) + n \delta_n.$$  

(44)

For the term $\sum_{i=1}^n h(Y_i | X_{2,i}, Q_i)$, we can also derive the following bound:

$$\sum_{i=1}^n h(Y_i | X_{2,i}, Q_i)$$

(41)

$$= \frac{n}{2} \log 2 \pi e \left( \frac{1}{n} \sum_{i=1}^n E(X^2_{1,i}) \right)$$

$$\leq \frac{n}{2} \log 2 \pi e \left( P_1 - \frac{1}{n} E_{X_{1,i}, Q_i} E^2(X_{1,i} | X_{2,i}, Q_i) + N \right)$$

(45)

where (a) and (b) follows from Jensen’s inequality.

Using (43), we have

$$\alpha P_1 + N = P_1 - \frac{1}{n} E_{X_{2,i}, Q_i} E^2(X_{1,i} | X_{2,i}, Q_i) + N$$

$$\Rightarrow \frac{1}{n} E_{X_{2,i}, Q_i} E^2(X_{1,i} | X_{2,i}, Q_i) \leq \alpha P_1.$$  

(46)
From (22), we obtain
\[
R_0 + R_1 \leq \sum_{i=1}^{n} I(X_{1,i}, X_{2,i}; Y_i) + n\delta_n
\]
\[
= \sum_{i=1}^{n} h(Y_i) - I(Y_i|X_{1,i}, X_{2,i}) + n\delta_n
\]
\[
= \sum_{i=1}^{n} h(Y_i) - \frac{n}{2} \log 2\pi e N + n\delta_n \quad (47)
\]

For the first term in the preceding inequality, we obtain
\[
\sum_{i=1}^{n} h(Y_i)
\]
\[
= \sum_{i=1}^{n} h(X_i + X_{1,i} + Z_i)
\]
\[
\leq \sum_{i=1}^{n} \frac{1}{2} \log 2\pi e (E(X_{1,i} + X_{2,i})^2 + N)
\]
\[
\leq \frac{n}{2} \log 2\pi e \left( \frac{1}{n} \sum_{i=1}^{n} E(X_{1,i} + X_{2,i})^2 + N \right)
\]
\[
\leq \frac{n}{2} \log 2\pi e \left( \frac{1}{n} \sum_{i=1}^{n} E X_{1,i}^2 + \frac{1}{n} \sum_{i=1}^{n} 2E X_{1,i} X_{2,i} + N \right)
\]
\[
\leq \frac{n}{2} \log 2\pi e \left( P_1 + P_2 + \frac{1}{n} \sum_{i=1}^{n} 2E(X_{1,i} X_{2,i}) + N \right)
\]
\[
\leq \frac{n}{2} \log 2\pi e \left( P_1 + P_2 + \frac{1}{n} \sum_{i=1}^{n} 2E X_{1,i} X_{2,i} + N \right)
\]
\[
\leq \frac{n}{2} \log 2\pi e \left( P_1 + P_2 + \frac{2}{n} \sum_{i=1}^{n} \sqrt{EX_{2,i}^2 \cdot EE^2(X_{1,i} X_{2,i}, Q_i)} + N \right)
\]
\[
\leq \frac{n}{2} \log 2\pi e \left( P_1 + P_2 + 2 \sqrt{\frac{1}{n} \sum_{i=1}^{n} EX_{2,i}^2 \left( \frac{1}{n} \sum_{i=1}^{n} EE^2(X_{1,i} X_{2,i}, Q_i) \right)} + N \right)
\]
\[
\leq \frac{n}{2} \log 2\pi e \left( P_1 + P_2 + 2 \sqrt{\alpha P_1 P_2} + N \right) \quad (48)
\]

Hence,
\[
R_0 + R_1 \leq \frac{n}{2} \log 2\pi e \left( P_1 + P_2 + 2 \sqrt{\alpha P_1 P_2} + N \right)
\]
\[
- \frac{n}{2} \log 2\pi e N + n\delta_n
\]
\[
= \frac{n}{2} \log \left( 1 + \frac{P_1 + P_2 + 2 \sqrt{\alpha P_1 P_2}}{N} \right) + n\delta_n \quad (49)
\]

From (19), we obtain
\[
R_{1,e} \leq \sum_{i=1}^{n} I(X_{1,i}; Y_i|X_{2,i}, Q_i) - I(X_{1,i}; Y_i|X_{2,i}, Q_i) + n\delta_n
\]
\[
= \frac{n}{2} \log \left( 1 + \frac{\alpha P_1}{N} \right) - \sum_{i=1}^{n} h(Y_i|X_{2,i}, Q_i)
\]
\[
+ \frac{n}{2} \log 2\pi e N_2 + n\delta_n \quad (50)
\]

To bound the term \( \sum_{i=1}^{n} h(Y_i|X_{2,i}, Q_i) \) in (50), we first derive the following bound. Since \( Z_i \) is independent of \( Y_i \) given \( X_{2,i} \) and \( Q_i \), by entropy power inequality, we obtain
\[
2^{h(Y_i+Z_i|X_{2,i}=x_{2,i}, Q_i=q_i)} \geq 2^{h(Y_i|X_{2,i}=x_{2,i}, Q_i=q_i)} + 2^{h(Z_i|X_{2,i}=x_{2,i}, Q_i=q_i)}
\]
\[
= 2^{h(Y_i|X_{2,i}=x_{2,i}, Q_i=q_i)} + 2\pi e(N_2 - N)
\]

We then obtain
\[
h(Y_i + Z_i|X_{2,i} = x_{2,i}, Q_i = q_i)
\]
\[
\geq \frac{1}{2} \log \left( 2^{2h(Y_i|X_{2,i}=x_{2,i}, Q_i=q_i)} + 2\pi e(N_2 - N) \right)
\]

Taking the expectation on both sides of the preceding equation, we obtain
\[
Eh(Y_i + Z_i|X_{2,i} = x_{2,i}, Q_i = q_i)
\]
\[
\geq \frac{1}{2} \log \left( 2^{2h(Y_i|X_{2,i}=x_{2,i}, Q_i=q_i)} + 2\pi e(N_2 - N) \right)
\]
\[
\geq \frac{1}{2} \log \left( 2^{2Eh(Y_i|X_{2,i}=x_{2,i}, Q_i=q_i)} + 2\pi e(N_2 - N) \right)
\]
\[
= \frac{1}{2} \log \left( 2^{2h(Y_i|X_{2,i}, Q_i)} + 2\pi e(N_2 - N) \right)
\]

where (a) follows from Jensen’s inequality and the fact that \( \log(2^x) \) is a convex function.

Summing over the index \( i \), the preceding inequality becomes
\[
\sum_{i=1}^{n} h(Y_i + Z_i|X_{2,i}, Q_i)
\]
\[
\geq \frac{1}{2} \sum_{i=1}^{n} \log \left( 2^{2h(Y_i|X_{2,i}, Q_i)} + 2\pi e(N_2 - N) \right)
\]
\[
\geq \frac{n}{2} \log \left( 2^{2h(Y_i|X_{2,i}, Q_i)} + 2\pi e(N_2 - N) \right)
\]
\[
\geq \frac{n}{2} \log (2\pi e(\alpha P_1 + N) + 2\pi e(N_2 - N))
\]
\[
= \frac{n}{2} \log (2\pi e(\alpha P_1 + N_2))
\]
where (a) follows from Jensen’s inequality, and (b) follows from (46).

Applying the preceding bound to the term \( \sum_{i=1}^{n} h(Y_{2,i}|X_{2,i}, Q_i) \), we obtain

\[
\sum_{i=1}^{n} h(Y_{2,i}|X_{2,i}, Q_i) = \sum_{i=1}^{n} h(Y_i + Z_i|X_{2,i}, Q_i) \geq \frac{n}{2} \log (2\pi e (\alpha P_1 + N_2)) \tag{51}
\]

Substituting the preceding bound into (50), we obtain

\[
nR_{1,c} \leq \frac{n}{2} \log \left( 1 + \frac{\alpha P_1}{N} \right) - \frac{n}{2} \log (2\pi e (\alpha P_1 + N_2)) + \frac{n}{2} \log 2\pi e N_2 + n\delta_n
\]

\[
\leq \frac{n}{2} \log \left( 1 + \frac{\alpha P_1 + P_2 + 2\sqrt{\alpha P_1 P_2}}{N} \right)
- \sum_{i=1}^{n} h(Y_{2,i}|X_{2,i}, Q_i) + n\delta_n
\]

\[
\leq \frac{n}{2} \log \left( 1 + \frac{P_1 + P_2 + 2\sqrt{P_1 P_2}}{N} \right)
- \frac{n}{2} \log (2\pi e (\alpha P_1 + N_2)) + \frac{n}{2} \log 2\pi e N_2 + n\delta_n
\]

\[
\leq \frac{n}{2} \log \left( 1 + \frac{P_1 + P_2 + 2\sqrt{P_1 P_2}}{N} \right)
- \frac{n}{2} \log (1 + \frac{\alpha P_1}{N_2}) + n\delta_n, \tag{52}
\]

From (20), we obtain

\[
nR_0 + nR_{1,c}
\leq \sum_{i=1}^{n} I(X_{1,i}; X_{2,i}; Y_i) - I(X_{1,i}; Y_{2,i}|X_{2,i}, Q_i) + n\delta_n
\]

\[
\leq \frac{n}{2} \log \left( 1 + \frac{P_1 + P_2 + 2\sqrt{P_1 P_2}}{N} \right)
- \sum_{i=1}^{n} h(Y_{2,i}|X_{2,i}, Q_i) + n\delta_n
\]

\[
\leq \frac{n}{2} \log \left( 1 + \frac{P_1 + P_2 + 2\sqrt{P_1 P_2}}{N} \right)
- \frac{n}{2} \log (2\pi e (\alpha P_1 + N_2)) + \frac{n}{2} \log 2\pi e N_2 + n\delta_n
\]

\[
\leq \frac{n}{2} \log \left( 1 + \frac{P_1 + P_2 + 2\sqrt{P_1 P_2}}{N} \right)
- \frac{n}{2} \log (1 + \frac{P_1}{N_2}) + n\delta_n
\]

which completes the proof.

VI. CONCLUSIONS

We have established the capacity-equivocation region for a binary example GMAC and the Gaussian GMAC with one confidential message set. For the binary GMAC, we have shown that the time-sharing scheme is strictly suboptimal to achieve the secrecy capacity, although it is optimal to achieve the capacity without the secrecy constraint. We have also found that the capacity-equivocation region of GMACs with one confidential message set depends only on the marginal channels \( p(y|x_1, x_2) \) and \( p(y_2|x_1, x_2) \). Based on this observation, we have obtained the capacity-equivocation region for the Gaussian GMAC (not necessarily physically degraded) with one confidential message set.