Speed–direction description of turbulent flows

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Abstract

In this note we introduce speed and direction variables to describe the motion of incompressible viscous flows. Fluid velocity $u$ is decomposed into $u = ur$, with $u = |u|$ and $r = u/|u|$. We consider a directional split of the Navier–Stokes equations into a coupled system of equations for $u$ and for $r$. Equation for $u$ is particularly simple but solely maintains the energy balance of the system. Under the assumption of a weak correlation between fluctuations in speed and direction in a developed turbulent flow, we further illustrate the application of $u$-$r$ variables to describe mean statistics of a shear turbulence. The standard (full) Reynolds stress tensor does not appear in a resulting equation for the mean flow profile.

1 Introduction

It is most common to describe flow phenomena in terms of fluid velocity, pressure and density, which are natural kinematic and thermodynamic variables. At the same time, the choice of alternative variables may give remarkable new insights. One prominent example is the employment of vorticity and streamfunction to devise a system of equations governing incompressible flow dynamics [3]. In this note, we consider the speed of the flow and its direction as independent Eulerian variables together with (kinematic) pressure to describe the motion of incompressible viscous fluid. For fluid velocity field $u$, transformation to new variables formally takes the form $u = |u|$ and $r = u/|u|$, where for $u = 0$ the direction is ambiguously defined. After a little calculus in section 2 the Navier–Stokes equations are written in terms of $u$, $r$ and $p$. We next split the system by projecting the momentum equation on the flow direction and the orthogonal plane in each point of space–time. The first scalar equation can be interpreted as the equation governing the evolution of $u$. This equation turns out to be particularly simple, but together with the incompressibility condition it encodes an important physics in the form of energy balance. Motivated by these observations, we further employ the new variables to describe turbulent flows. One key observation here is that speed and direction of velocity fluctuations, independent in isotropic turbulence, may still correlate weakly in more practical flows. More precisely, we assume and check using DNS data for the channel turbulence that the second term in (10) can be neglected and so the mean velocity can be written in terms of mean speed and mean direction. Mean speed $\bar{u}$ satisfies an equation, which results from averaging the Navier–Stokes equations projected on the flow direction; see eq. (12). It is interesting to see that the turbulent parts appear in the equation in the form of correlation functions different from the well-known Reynolds stress. The paper discusses and examines all terms in the equation for $\bar{u}$ for the example of a turbulent flow in a channel. For this purpose we make use of the turbulent channel flow data set from the Johns Hopkins turbulence database [1]. In particular, a simple analytical representation of turbulence terms from the equation for $\bar{u}$ leads to an ODE with solution resembling the ‘true’ (recovered from DNS simulations) mean velocity profile with perfect accuracy. A work related to this study is found in [2], where the Navier–Stokes equations are given in angular variables. The author is unaware of any other literature, where the $u$–$r$ variables were used to describe fluid dynamics.

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2 Speed–direction flow variables

Consider a flow of incompressible Newtonian fluid in \( \mathbb{R}^3 \) governed by the Navier–Stokes equations

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu(\Delta \mathbf{u}) + \nabla p &= \mathbf{f} \\
\text{div} \mathbf{u} &= 0
\end{align*}
\]

(1)

with fluid velocity \( \mathbf{u} \), kinematic pressure \( p \), kinematic viscosity coefficient \( \nu > 0 \) and given body forces \( \mathbf{f} \).

We are interested in representing the fluid velocity \( \mathbf{u}(x, t) \) at point \( x \in \mathbb{R}^3 \), \( t \geq 0 \), in terms of its speed \( u(x, t) \in \mathbb{R} \) and direction \( r(x, t) \in S^2 \):

\[
\mathbf{u} = ur, \quad |r| = 1, \quad u \geq 0.
\]

For \( u = 0 \), the choice of \( r \) is not unique. Operators of vector calculus in new variables take the following form:

\[
\nabla \mathbf{u} = u \nabla r + r \otimes (\nabla u), \quad \nabla \times \mathbf{u} = u(\nabla \times r) + (\nabla u) \times r, \quad \text{div} \mathbf{u} = (r \cdot \nabla u) + u \text{div} r.
\]

(2)

Diffusion and inertia terms are easily computed to be

\[
\Delta \mathbf{u} = u \Delta r + r \Delta u + 2|\nabla r| \nabla u, \quad (u \cdot \nabla \mathbf{u}) = u^2 (r \cdot \nabla) r + u(r \otimes r) \nabla u.
\]

(3)

One finds several useful identities by differentiating equality \( |r|^2 = 1 \) in time and space:

\[
\frac{\partial r}{\partial t} \cdot r = 0, \quad |\nabla r|^2 r = 0, \quad -r \cdot \Delta r = |\nabla r|^2,
\]

(4)

where \( |A|^2 = \text{tr}(A^T A) \). From the second equality we also get

\[
(r \cdot \nabla) r = 2D(r)r = (\nabla \times r) \times r
\]

(5)

where \( D(r) = \frac{1}{2}(\nabla r + |\nabla r|^T) \).

2.0.1 Equations in new variables and the split system

Thanks to the identities (2)–(4) the Navier–Stokes equations (1) in the speed–direction variables take the form

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial r}{\partial t} + u^2 (r \cdot \nabla) r + u(r \otimes r) \nabla u - \nu(\Delta u) r - 2\nu (\nabla u \cdot r) r - \nu u \Delta r + \nabla p &= \mathbf{f} \\
r \cdot \nabla u + u \text{div} r &= 0
\end{align*}
\]

(6)

We now split the momentum equation by projecting it on the velocity direction and orthogonal plane. To this end, we take the scalar product of the momentum equation with \( r \) and note that due to (4) and (5) we have

\[
r \cdot (\nabla u \cdot \nabla) r = r^T |\nabla r| \nabla u = (\nabla u) \cdot (|\nabla r|^T r) = 0
\]

\[
u^2 r \cdot (r \cdot \nabla) r + ur \cdot (r \otimes r) \nabla u = r \cdot ((\nabla \times r) \times r) + ur \cdot \nabla u = \frac{1}{2} r \cdot \nabla u^2.
\]

Employing these identities together with \( |r|^2 = 1 \) and \( \frac{\partial r}{\partial t} \cdot r = 0 \), \( -r \cdot \Delta r = |\nabla r|^2 \) from (3) we obtain the first equation of the split system

\[
\frac{\partial u}{\partial t} - \nu(\Delta u) + \nu|\nabla r|^2 u + r \cdot \nabla (p + \frac{u^2}{2}) = \mathbf{f}_r,
\]

(7)

with \( \mathbf{f}_r = r \cdot \mathbf{f} \). We now project the momentum equation on the orthogonal planes to the fluid velocity directions by multiplying the first equation in (6) by the orthogonal projector \( P = I - r \otimes r \) and use \( Pr = 0 \). For the treatment of the viscous term, we also compute using (4):

\[
P \Delta r = \Delta r - (r \cdot \Delta r) r = \Delta r + |\nabla r|^2 r.
\]
We arrive at the second equation of the split system:

\[
\frac{\partial r}{\partial t} + u^2 (r \cdot \nabla) r - 2\nu (\nabla u \cdot \nabla) r - \nu u (\Delta r + |\nabla r|^2 r) + \mathbf{P} \nabla p = f_p, \tag{8}
\]

with \( f_p = \mathbf{P} f \). Alternative forms of (8) can be derived using (5) and other expressions for the viscous terms.

### 2.0.2 Energy balance

With the help of (2) and (4) we find for the kinetic energy and diffusion densities:

\[
\frac{1}{2}|u|^2 = \frac{1}{2} u^2, \quad |\nabla u|^2 = u^2 |\nabla r|^2 + |\nabla u|^2. \tag{9}
\]

For the energy balance let us assume a flow in a finite volume \( \Omega \) with no-slip boundary condition \( u = 0 \) on the boundary \( \partial \Omega \). Then for any smooth solution to the Navier–Stokes equation the energy equality follows by multiplying (7) by \( u \), integrating over \( \Omega \) and invoking the continuity equation:

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \, dx + \nu \int_{\Omega} \left( |\nabla u|^2 + |\nabla r|^2 u^2 \right) \, dx = \int_{\Omega} f_r u \, dx.
\]

We see that the information about the energy balance is essentially encoded by the equation (7) and the continuity equation. This motivates our focus on (7), when we are interested in we a possible role of the speed–direction decomposition in understanding turbulent flows.

### 3 Turbulent variables

Further \( \mathbf{f} \) denotes an ensemble average of quantity \( f \) so that \( f' = f - \mathbf{f} \) is a turbulent fluctuation. For the mean flow velocity it holds

\[
\mathbf{u} = \mathbf{u} \mathbf{r} = \mathbf{u} \mathbf{r} + \mathbf{u} \mathbf{r}'. \tag{10}
\]

If turbulent fluctuations in flow direction and speed are linearly independent (uncorrelated) statistics, then (10) simplifies to

\[
\mathbf{u} = \mathbf{u} \mathbf{r}. \tag{11}
\]

The assumption leading to (11) holds for isotropic turbulence. In general, isotropy is a stronger assump-
Figure 2: Left: Mean velocity profiles $u$ and $u_r$ in viscous units. Right: Temporal auto-correlation in $u$ (matrix norm), $u$ and $r$ (matrix norm) vs. viscous time at $y^+ = 153$. Same faster decay in ACF for $u$ is observed for all $y^+ > 0$ across the channel.

tion then $u'r' = 0$. While for anisotropic turbulence we do not see a reason for $u'$ and $r'$ to be independent, we hypothesize that for many flows with correlated fluctuations in speed and direction the values of $u'r'$ are small relative to the mean flow and \[11\] holds approximately. To back this hypothesis, we evaluate the statistics of interest using the data set from JHU Turbulence Databases \[1\] for the turbulent channel flow with $Re_\tau = 10^3$ (bulk Reynolds number $Re = 4 \times 10^4$, viscous length scale equals $10^3$, friction velocity $u_\tau = 4.9968 \times 10^{-2}$). From the left plot in Figure 1 we see that the relative correlations $u'r'/u\tau$ are of order $10^{-3}$ for all distances from the channel wall, which suggests that \[11\] holds with excellent accuracy. As a result, the mean flow $\overline{u}$ and $\overline{u\tau}$ are in agreement as illustrated in Figure 2 (left), where all three averaged statistics were computed from the database DNS results: The computed $\overline{u\tau}$ and $\overline{u\tau_x}$ virtually coincide. For the reference purpose, the plot shows the linear law and log-law curves. We note that for the shear turbulence (such as the channel turbulence) $u'$ and $r'$ are not necessary uncorrelated, and the right plot in Fig. 1 shows that the correlation is relatively weak but not insignificant between $u'$ and $x, y$ components of $r'$. Nevertheless, it turns out to be reasonable to accept \[11\].

In addition, the right plot in Figure 2 shows normalized temporal autocorrelation functions (ACF) for $u$, $u$ and $r$, where for the vector quantity ACF is defined as the Frobenius norm of the autocorrelation matrix. We see that the flow fluctuations in a point tend to forget their directions faster than the speed. This may be one factor explaining weak correlation between $r'$ and $u'$. The ACF are shown for $y^+ = 153$, but similar picture is observed for other distances.

Motivated by the above observations, we assume \[11\]. In this case, we see that $\overline{u}$ determines the mean flow $\overline{u}$ once the mean direction $\overline{r}$ is known with a reasonable certainty (as in the case of the turbulent channel flow). Therefore, it is interesting to look at the equation for $\overline{u}$. We call $\overline{u}$ the mean flow profile. Taking the ensemble average of \[7\] we arrive at

$$\frac{\partial \overline{u}}{\partial t} - \nu \left( \Delta \overline{u} - |\nabla r|^2 \overline{u} \right) + \overline{r} \cdot \nabla P = \overline{r},$$

with $P = p + \frac{u^2}{2}$, or working out the averages we get

$$\frac{\partial \overline{u}}{\partial t} - \nu \left( \Delta \overline{u} - |\nabla r|^2 \overline{u} - Q \right) + \overline{r} \cdot \nabla \left( \frac{1}{2} \overline{u}^2 + \overline{p} \right) + \overline{P} = \overline{r},$$

\[12\]

\[1\]Data was sampled for 50 distances from the wall, which were equally distributed in the log scale for $y_+ \in [0.1, 10^3]$. For each $y_+$, the data (velocity, velocity gradient, and pressure gradient) were collected over 961 points uniformly distributed in the $xz$-plane and for 1000 time instances uniformly distributed in the simulation time interval $[0, 26]$. Ensemble means for each $y_+$ node were computed as averages over the data in $961 \times 1000$ space–time points.

\[2\]For computing AFC we sample data in 30 points randomly distributed in the $xz$-plane and in 4000 time instances uniformly distributed in the simulation time interval $[0, 26]$. 

4
with turbulent correlation functions:

\[ Q = \frac{1}{2} \langle |\nabla r|^2 \rangle u'^2 \text{ and } P = r' \cdot \nabla P' + \frac{1}{2} r' \cdot \nabla |u'|^2. \]

We see that the full Reynolds turbulent stress tensor does not appear in (12) and the action of fluctuations on the mean flow profile comes through the total pressure correlation with \( r' \), variation of \( |u'|^2 \) along mean flow directions, extra viscous terms \( Q \), and \( \frac{1}{2} |\nabla r|^2 \) factor in front of \( P \). Let us take a closer look at these terms for the channel turbulence example, where they all are functions of \( y \).

We start with splitting \( P \) into the parts corresponding to the mean variation of turbulent kinetic energy and pressure along \( r' \) and the variation of \( |u'|^2 \) along \( r' \):

\[ P = P_u + P_p + P_r, \]

where

- \( P_u \) has a meaning of a correlation between fluctuations in the flow direction and fluctuations in the kinetic energy gradient. Far from the wall, the mean kinetic energy has little variation in space so that on average a fluctuation in the flow direction should not cause much of energy flux. Hence we expect \( P_u \) to be small there. In the laminar sublayer \( P_u \) is small for a different reason, namely the turbulent fluctuations are insignificant there. The situation differs in the transition (overlap) region between the viscous sublayer and the outer region. In this region, fluctuations in \( r' \) are significant and they produce the mixing which transfers the energy from the turbulent stream to the viscous layer, where it dissipates. This scenario explains the behaviour of \( P_u \) recovered from the turbulent data\(^4\) and shown in Figure 3 (left), where we see a strong negative correlation reflected by a log-Gaussian type peak around \( y_+ = 8 \).

- In a viscous sublayer, we observe a clear \( P_u \) asymptotic, while understanding the asymptotic of \( P_r \) for \( y_+ \to +\infty \) needs further insights. The same left plot in Figure 3 shows \( P_p \) and \( P_r \) (all quantities are normalized by the mean pressure drop). These functions play minor role in the transition region, but knowing their asymptotics for \( y_+ \to +\infty \) is important for the correct prediction of \( \overline{\tau} \) as we see later.

\[^4\text{Note that } \nabla u, \nabla (u^2) \text{ and } \nabla r \text{ are not available directly in the database and we recover them using the (available) velocity gradient through the equalities:} \]

\[ \nabla u = r'^2 \nabla u, \quad \nabla (u^2) = 2u \nabla u, \quad \text{and} \quad |\nabla r|^2 = \frac{|\nabla \nabla u|^2}{u^2}. \]

The first identity follows from (2) and (4). In turn, (3) and \( \nabla u = r'^2 \nabla u \) implies the expression for \( |\nabla r|^2 \).
To get a rough idea about the $y$-dependence of $|\nabla r|^2$, it can be useful to adopt the view of turbulent flow as a hierarchy of vortices so that the flow on a distance $y$ from the wall is dominated (in an average sense) by vortices of size $O(y)$. The center of a standing 2D vortex is a singular point of $\nabla r$ so that the integral of $|\nabla r|^2$ diverges logarithmically. The situation is more complicated in 3D, but it looks reasonable to suggest that $|\nabla r|^2(y)$ growth proportional to the average number of $O(y)$-vortices filling the layer. Hence we may expect $|\nabla r|^2$ to increase closer to the wall and decrease in the developed turbulent stream.

The plot of $|\nabla r|^2$ reconstructed from the DNS data in Figure 4 (left) confirms this hypothesis and shows that $|\nabla r|^2 = O(y^{-1})$ can be a reasonable (though not perfect) approximation.

The derivative of the mean-flow kinetic energy along the mean direction $\bar{r}$ plays also an important role in (12). Figure 4 reveals that although the deviation of $\bar{r}$ from the $x$-direction is less than 1%, the $y$-component of $\bar{r}$ demonstrates quite distinctive behaviour, first growing in the viscous layer close to $r(y) \approx y = 10^{-3}y_+$ and in the transition region changing this growth to a slow decay, with the log function turning out to be a very good fit. Finally, $\nu Q$ is small due to the scaling with viscosity coefficient and can be neglected in the equation for $\bar{r}$.

For the example of turbulent channel flow we assume the statistically stationary turbulence with $\bar{u}$ and coefficients in (12) independent of $x$ and $z$. The equation reduces to

$$-\nu \left( \frac{d^2\bar{r}}{dy^2} - |\nabla r|^2 \bar{r} \right) + r_y \bar{u} \frac{d\bar{r}}{dy} + \mathcal{P} = -r_x \frac{d\bar{r}}{dx}, \quad (13)$$

Based on the above discussion we use the following

$$\text{Figure 5: Re-scaled by } u_*^{-1} \text{ solution to (13) with data-reconstructed coefficients vs computed } \bar{u}.$$
analytical representations for the statistics appearing in (13):

\[
\begin{align*}
\mathbf{r} & \approx r_g(y_+) = 3 \times 10^{-3} y_+^{-1}, \quad Q = 0, \\
F_y & \approx r_y(y_+) = \begin{cases} 10^{-3} y_+ & \text{for } y_+ \leq 8, \\
10^{-3}(11.1 - 1.6 \ln y_+) & \text{for } y_+ > 8, 
\end{cases} \\
\mathcal{P} + r_x \frac{d\mathcal{P}}{dx} & \approx r_p(y_+) = \begin{cases} c_0 & \text{for } y_+ \leq 3, \\
c_1 \exp \left( - \frac{(\ln y_+ - \ln (y_0))^2}{2\sigma^2} \right) & \text{for } 3 \leq y_+ \leq 40, \\
c_2 y_+^4 & \text{for } y_+ > 40. 
\end{cases}
\end{align*}
\]

The solution of (13) was not sensitive to \(c_0\) and we set \(c_0 = 0\), while the correct position and amplitude of the log-Gaussian and the proper decay of \(\mathcal{P}\) in the interior were found to be both important for (13) to correctly reproduce the 'true' mean profile. We set \(c_1 = -0.25\), \(\bar{y}_0 = 7.9\), \(\sigma = 0.66\) and \(c_2 = -2.75\), \(\alpha = 1.33\). The solution to (13) with the coefficients defined in (14) and boundary conditions \(\bar{\pi}(0) = 0\), \(d_x \bar{u}(1) = 0\) is shown in Figure 5. We see that it predicts the correct mean profile with a perfect accuracy.

3.1 Energy balance

We comment on the energy balance for the averaged turbulent flow in the speed–direction variables. Similar to the mean flow velocity, mean profile can be related to the kinetic energy \(\mathcal{E}\):

\[
2\mathcal{E} = \bar{u}^2 = \overline{\mathbf{u}^2 + |\mathbf{u}|^2}. \tag{15}
\]

The decomposition (15) is, in general, different from \(2\mathcal{E} = |\mathbf{u}|^2 = |\mathbf{u}^2 + |\mathbf{u}|^2|\), where the evolution of the mean flow energy \(\frac{1}{2} \bar{u}^2\) is driven by viscous tensions and Reynold’s stresses. Turbulent stresses does not appear in equation for \(\pi^2\), which we get by multiplying (12) with \(\bar{\pi}\):

\[
\frac{1}{2} \frac{\partial|\pi|^2}{\partial t} - \nu (\Delta \pi) \bar{\pi} + \nu \left( |\nabla \bar{r}|^2 |\pi|^2 + Q\pi \right) + \bar{\mathbf{u}} \cdot \nabla \mathcal{P} - \bar{u} \cdot r' \cdot \nabla \mathcal{P} + \bar{\pi} \cdot \nabla \mathcal{P} = \bar{\Gamma} \bar{\pi}, \quad \mathcal{P} = \frac{1}{2} \bar{u}^2 + \bar{p},
\]

where we used (10). Again under the assumption about a weak correlation between speed and direction in fluctuations the term \(\bar{u} \cdot r' \cdot \nabla \mathcal{P}\) can be omitted and the above identity simplifies to

\[
\frac{1}{2} \frac{\partial|\pi|^2}{\partial t} - \nu (\Delta \pi) \bar{\pi} + \nu \left( |\nabla \bar{r}|^2 |\pi|^2 + Q\pi \right) + \bar{\mathbf{u}} \cdot \nabla \mathcal{P} + \bar{\pi} \cdot \nabla \mathcal{P} = \bar{\Gamma} \bar{\pi}, \tag{16}
\]

Denote by \(\frac{\delta}{\delta t} = \frac{d}{dt} + (\bar{\mathbf{u}} \cdot \nabla)\cdot\) the material derivative along mean flow trajectories. One can regroup terms as

\[
\frac{1}{2} \frac{\partial|\pi|^2}{\partial t} + \bar{\mathbf{u}} \cdot \nabla \mathcal{P} + \bar{\pi} \mathcal{P} = \frac{1}{2} \frac{d\pi^2}{dt} + \bar{\mathbf{u}} \cdot \nabla \bar{p} + \bar{\mathbf{u}} \cdot (\mathcal{P}_u + P_p), \quad \text{with } \bar{p} = \bar{\pi} + \frac{1}{2} |\mathbf{u}'|^2.
\]

Now let \(\mathcal{V}\) be a material volume evolving with the mean flow field \(\bar{\mathbf{u}}\). Integrating the energy equation (16) over \(\mathcal{V}\), using \(\text{div} \bar{\mathbf{u}} = 0\) and the above relation we obtain:

\[
\frac{1}{2} \frac{d}{dt} \int_\mathcal{V} |\pi|^2 = - \nu \int_\mathcal{V} \left( |\nabla \pi|^2 + |\nabla \mathbf{r}|^2 |\pi|^2 + Q\pi \right) + \int_\partial\mathcal{V} \nu (\mathbf{n} \cdot \nabla \pi) + (\mathbf{n} \cdot \mathbf{n}) \bar{p} - \int_\mathcal{V} \bar{\mathbf{u}} \cdot (\mathcal{P}_u + P_p) + \int_\partial\mathcal{V} \bar{\Gamma} \bar{\pi}.
\]

We see that the effect of turbulent fluctuations on the mean flow energy balance is present through the work of correlation functions \(\mathcal{P}_u\) and \(P_p\) and the boundary flux of \(|\mathbf{u}'|^2\).

4 Conclusions

The speed–direction variables have a potential to become a useful alternative approach to describe the motion of fluids, and in particular of turbulent flows. A mostly data-driven approach was taken here to understand turbulent (correlation) functions arising in the mean profile equation. More study is required to model them for more general flows. The paper does not discuss the complementing equation (8). A suitable way to use it in modelling and analysis has to be found.
Acknowledgements

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References

[1] Kalin Kanov, Randal Burns, Cristian Lalescu, and Gregory Eyink. The Johns Hopkins turbulence databases: an open simulation laboratory for turbulence research. *Computing in Science & Engineering*, 17(5):10–17, 2015.

[2] A.A. Kovalishin and V.I. Lebedev. Transformation of the Navier-Stokes equations to angular variables. In *Doklady Mathematics*, volume 77, pages 223–225. SP MAIK Nauka/Interperiodica, 2008.

[3] Andrew J. Majda and Andrea L. Bertozzi. *Vorticity and Incompressible Flow*, volume 27. Cambridge University Press, 2002.