Approximate Capacity of a Class of Multi-source Gaussian Relay Networks

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Abstract

We study a Gaussian relay network in which multiple source–destination (S–D) pairs communicate through relays without direct links between the sources and the destinations. We observe that the time-varying nature of wireless channels or fading can be used to mitigate the interference. The proposed block Markov amplify-and-forward relaying scheme exploits such channel variations and works for a wide class of channel distributions including Rayleigh fading. We completely characterize the degrees of freedom (DoF) region of the Gaussian relay network. Specifically, the DoF region of the $K$-user $M$-hop Gaussian relay network with $K_m$ nodes in the $m$th layer is the set of all $(d_1, \cdots, d_K)$ such that $d_i \leq 1$ for all $i$ and $\sum_{i=1}^{K} d_i \leq \min\{K_1, \cdots, K_{M+1}\}$, where $d_i$ is the DoF of the $i$th S–D pair and $K = K_1 = K_{M+1}$ is the number of S–D pairs. We further characterize the DoF region of the Gaussian relay network with multi-antenna nodes and general message sets. The resulting DoF regions coincide with the DoF regions assuming perfect cooperation between the relays in each layer.

I. INTRODUCTION

Characterizing the capacity of Gaussian relay networks is one of the fundamental problems in network information theory. However, for Gaussian relay networks, the signal transmitted from a node will be heard by multiple nodes (broadcast) and a node will receive the superposition of the signals transmitted from multiple nodes (interference) and there exist fading and noise, which make the problem complicated. To overcome such difficulties, simplified wireless network models have been developed in [1], [2], [3], [4], [5], [6] that provide intuition towards an approximate capacity characterization of single-source Gaussian relay networks [7].

Unlike the single-source case, the capacity or an approximate capacity characterization of multi-source Gaussian relay networks is very challenging since the transmission of other sessions acts as the inter-user interference. Due to the interference, the extension from the result in [7] is not straightforward. Recently, remarkable progress has been made on multi-source problems in [8], [9], [10], [11], [12] and the references therein. It was proved in [9] that the Han–Kobayashi scheme indeed achieves the capacity of the two-user Gaussian interference channel within one bits/s/Hz. The capacity of the $K$-user Gaussian interference channel has been characterized in [10] as

$$\frac{K}{2} \log(P) + o(\log(P))$$

if channel coefficients are sufficiently independent and drawn from a continuous distribution, where $P$ denotes the signal-to-noise ratio (SNR). To show the degrees of freedom (DoF) or capacity pre-log term of $K/2$, the technique of interference alignment was used, which minimizes the overall interference space by aligning multiple interfering signals from unintended sources at each destination. The concept of interference alignment has also been used to characterize the DoF of the $K$-user multi-antenna Gaussian interference channel [11] and the $X$-network in which each source has independent messages for all destinations [12]. Another alignment technique called ergodic interference alignment has been proposed in [13] showing that, for a broad class of channel distributions, half of the interference-free ergodic capacity is achievable for each user in the $K$-user Gaussian interference channel at any SNR. Exploiting the inseparability of parallel interference channels [14], [15], the ergodic interference alignment

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scheme jointly encodes messages over two specific channel instances to align the interference. A similar concept has been also applied for the finite field case in [13], [16].

Not only can the interference be aligned, but also it can be cancelled or partially cancelled for multi-hop Gaussian relay networks. Assuming amplify-and-forward (AF) relays, each destination may receive multiple copies of an interfering signal from different paths and potentially these copies can cancel each other through a suitable choice of the amplification factors of relays. Reference [17] has shown that partial interference cancellation using AF relays achieves the capacity of two-user two-hop Gaussian networks within a constant bit gap in some scenarios. Also, the interference can be completely removed so that the optimal DoF of $K$ is achievable for $K$-user two-hop Gaussian networks if the number of relays is greater than or equal to $K^2$ [18].

In this paper, we study general multi-source multi-hop Gaussian relay networks. We observe that the time-varying nature of wireless channels or fading can be exploited to cancel the interference. As a simple example, consider a two-user two-hop Gaussian relay network in Fig. 1 in which

$$H_1[t] = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, H_2[t] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

for odd $t$ and

$$H_1[t] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, H_2[t] = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

for even $t$, where $H_m[t]$ is the $m$th hop channel matrix at time $t$. Note that if odd and even time slots are used separately, each S–D pair can only achieve $1/2$ DoF since there is no path between the first S–D pair for even $t$ and the second S–D pair for odd $t$. On the other hand, if the relays amplify and forward their signals with one symbol delay, then the interference can be completely cancelled since $H_2[t+1]H_1[t]$ becomes the identity matrix. Hence every S–D pair can achieve one DoF simultaneously. We generalize this idea to multi-source multi-hop Gaussian relay networks for a wide class of channel distributions including Rayleigh fading. The key ingredient is to set appropriate delays in AF relaying at each layer such that overall channel matrices become diagonal matrices with non-zero diagonal elements, which guarantees interference-free communication. Under this class of channel distributions, we completely characterize the DoF region of multi-source multi-hop Gaussian relay networks. This improves upon our previous result that showed total $K$ DoF is achievable for $K$-user $K$-hop networks with $K$ relays in each layer when $K$ is even and a similar technique has been proposed for linear finite-field multi-hop networks (see the conference papers [18], [19]). We further characterize the DoF region of multi-source multi-hop Gaussian relay networks with multi-antenna nodes and general message sets.

This paper is organized as follows. In Section II we explain the underlying system model and define the DoF region. In Section III we state the main results of this paper, the DoF regions of Gaussian relay networks. In Section IV we propose a block Markov AF relaying scheme and derive its achievable rate region for two-hop and three-hop Gaussian relay networks. From the achievable rate regions, we characterize the DoF region of Gaussian relay networks in Section V. We conclude this paper in Section VI and refer to the Appendix for the proof of the result in Section IV-B.

II. SYSTEM MODEL

In this section, we explain our network model and introduce encoding, relaying, and decoding functions. Based on this model, we define the capacity region and the DoF region. Throughout the paper, we will use $A$, $a$, and $A$...
to denote a matrix, vector, and set, respectively. The notations used in the paper are summarized in Table I.

### A. Gaussian Relay Networks

We study a Gaussian relay network consisting of $M + 1$ layers with $K_m$ nodes in the $m$th layer. We assume that the number of hops $M$ is greater than or equal to two. The nodes in the first layer and the last layer are the sources and the destinations, respectively. Thus $K = K_1 = K_{M+1}$ is the number of S–D pairs. We assume full-duplex relays so that all relays are able to transmit and receive simultaneously, but the results in this paper can be straightforwardly applicable for half-duplex relays by scheduling of hops. For notational simplicity, let us denote the $j$th node in the $m$th layer as node $(i, m)$, where $i \in \{1, \ldots, K_m\}$ and $m \in \{1, \ldots, M + 1\}$.

Consider the $m$th hop transmission in which the nodes in the $m$th layer transmit and the nodes in the $(m+1)$th layer receive. Let $x_{i,m}[t]$ denote the transmit signal of node $(i, m)$ at time $t$ and $y_{j,m}[t]$ denote the received signal of node $(j, m + 1)$ at time $t$. Then the input–output relation of the $m$th hop is given by

$$y_{j,m}[t] = \sum_{i=1}^{K_m} h_{ji,m}[t]x_{i,m}[t] + z_{j,m}[t], \quad (4)$$

where $h_{ji,m}[t]$ is the complex channel from node $(i, m)$ to node $(j, m + 1)$ at time $t$ and $z_{j,m}[t]$ is the additive noise of node $(j, m + 1)$ at time $t$. We assume that $z_{j,m}[t]$’s are independent and identically distributed (i.i.d.) and drawn from $\mathcal{N}(0, 1)$ and each node satisfies the power constraint $P$.

Let us denote $x_m[t] = [x_{1,m}[t], \ldots, x_{K_m,m}[t]]^T$ and $y_m[t] = [y_{1,m}[t], \ldots, y_{K_{m+1},m}[t]]^T$, which are the $K_m \times 1$ dimensional transmit signal vector and the $K_{m+1} \times 1$ dimensional received signal vector of the $m$th hop, respectively. Then the $m$th hop transmission can be represented as

$$y_m[t] = H_m[t]x_m[t] + z_m[t], \quad (5)$$

where $H_m[t]$ is the $K_{m+1} \times K_m$ dimensional complex channel matrix of the $m$th hop with $[H_m[t]]_{ji} = h_{ji,m}[t]$ and $z_m[t] = [z_{1,m}[t], \ldots, z_{K_{m+1},m}[t]]^T$ is the $K_{m+1} \times 1$ dimensional noise vector of the $m$th hop.

In this paper, we assume time-varying channels such that $h_{ji,m}[t]$’s are i.i.d. drawn from a continuous distribution $p_h(\cdot)$. We consider a class of channel distributions such that $p_h(a) > p_h(b)$ if $|a| < |b|$. The channel state information is assumed to be available at all nodes, i.e., each node knows $H_1[t]$ to $H_M[t]$ at time $t$.

### B. Problem Statement

Based on the network model, we define a set of length $n$ block codes. Let $W_i$ be the message of the $i$th source uniformly distributed over $\{1, \ldots, 2^{nR_i}\}$, where $R_i$ is the rate of the $i$th S–D pair. Then a $(2^{nR_1}, \ldots, 2^{nR_K}; n)$ code consists of the following encoding, relaying, and decoding functions:

- **(Encoding)** For $i \in \{1, \ldots, K\}$, the encoding function of the $i$th source, or node $(i, 1)$, is given by $f_{i,1,t}: \{1, \ldots, 2^{nR_i}\} \rightarrow \mathbb{C}$ such that

$$x_{i,1}[t] = f_{i,1,t}(W_i) \text{ for } t \in \{1, \ldots, n\}.$$
• (Relaying) For \( m \in \{2, \cdots, M\} \) and \( i \in \{1, \cdots, K_m\} \), the relaying function of node \((i, m)\) is given by
\[
x_{i,m}[t] = f_{i,m,t}(y_{i,m-1}[1], \cdots, y_{i,m-1}[t-1]) \quad \text{for} \ t \in \{1, \cdots, n\}.
\]

• (Decoding) For \( i \in \{1, \cdots, K\} \), the decoding function of the \( i^{th} \) destination, or node \((i, M+1)\), is given by
\[
\hat{W}_i = g_i(y_{i,M}[1], \cdots, y_{i,M}[n]).
\]

The probability of error at the \( i^{th} \) destination is given by \( P_{e,i} = \Pr(\hat{W}_i \neq W_i) \). A rate tuple \((R_1, \cdots, R_K)\) is said to be achievable if there exists a sequence of \((2^{nR_1}, \cdots, 2^{nR_K}; n)\) codes with \( P_{e,i} \to 0 \) as \( n \to \infty \) for all \( i \in \{1, \cdots, K\} \). The capacity region \( \mathcal{C} \) is the closure of the set of all achievable rate tuples. In the same manner as for the \( K \)-user interference channel [10], we define the DoF region as
\[
\mathcal{D} = \left\{ (d_1, \cdots, d_K) \in \mathbb{R}_+^K \ \big| \ \forall (w_1, \cdots, w_K) \in \mathbb{R}_+^K, \right. \\
\left. \sum_{i=1}^{K} w_i d_i \leq \limsup_{P \to \infty} \left( \sup_{(R_1, \cdots, R_K) \in \mathcal{C}} \sum_{i=1}^{K} w_i \frac{R_i}{\log P} \right) \right\},
\]
where \( d_i \) is the DoF of the \( i^{th} \) S–D pair.

C. Multi-antenna and General Message Set

We also study a more general case in which each node is equipped with multiple antennas and each source has messages to a subset of destinations. Let \( L_{i,m} \) denote the number of antennas of node \((i, m)\) and \( \mathcal{W}_g \) denote a set of messages, where \( \mathcal{W}_g \subseteq \{W_{i1}, \cdots, W_{Km+K1}\} \). Here, \( W_{ji} \) is the message from the \( i^{th} \) source to the \( j^{th} \) destination and \( K_1 \neq K_{M+1} \) in general. Similar to Section II-B, the achievable rate region \( \mathcal{R}(\mathcal{W}_g) \) and the capacity region \( \mathcal{C}(\mathcal{W}_g) \) can be defined. The DoF region is defined as
\[
\mathcal{D}(\mathcal{W}_g) = \left\{ (d_{ji})_{W_{ji} \in \mathcal{W}_g} \in \mathbb{R}_+^{\text{card}(\mathcal{W}_g)} \right| \forall (w_{ji})_{W_{ji} \in \mathcal{W}_g} \in \mathbb{R}_+^{\text{card}(\mathcal{W}_g)}, \\
\left. \sum_{W_{ji} \in \mathcal{W}_g} w_{ji} d_{ji} \leq \limsup_{P \to \infty} \left( \sup_{(R_1, \cdots, R_K) \in \mathcal{C}(\mathcal{W}_g)} \sum_{W_{ji} \in \mathcal{W}_g} w_{ji} \frac{R_{ji}}{\log P} \right) \right\},
\]
which is a simple extension of (9). Here, \( d_{ji} \) is the DoF from the \( i^{th} \) source to the \( j^{th} \) destination.

III. MAIN RESULTS

Throughout the paper, we characterize the DoF region of the Gaussian relay network. We simply state the main results here and derive them in the remainder of the paper.

**Theorem 1:** The DoF region \( \mathcal{D} \) of the Gaussian relay network is the set of all \((d_1, \cdots, d_K)\) such that
\[
d_i \leq 1 \quad \text{for} \ i \in \{1, \cdots, K\},
\]
\[
\sum_{i=1}^{K} d_i \leq \min_{m \in \{1, \cdots, M+1\}} \{K_m\}.
\]

**Proof:** We refer to Section V-A for the proof.

We notice that the DoF region \( \mathcal{D} \) coincides with the DoF region assuming perfect cooperation between the relays in each layer and there is no penalty due to distributed relays in terms of DoF. This property can be used to characterize the DoF region of more general networks having multi-antenna nodes and general message sets. Fig. 2 plots the DoF region \( \mathcal{D} \) for \( K = 3 \) in which the sum DoF is limited by the minimum number of nodes in each layer.
Theorem 2: The DoF region $\mathcal{D}(W_g)$ of the Gaussian relay network is the set of all $\{d_{ji}\}_{W_{ji} \in W_g}$ such that

$$\sum_{i=1}^{K_1} d_{ji} I_{ji} \leq L_{j,M+1} \text{ for } j \in \{1, \cdots, K_{M+1}\},$$

$$\sum_{j=1}^{K_{M+1}} d_{ji} I_{ji} \leq L_{i,1} \text{ for } i \in \{1, \cdots, K_1\},$$

$$\sum_{i=1}^{K_1} \sum_{j=1}^{K_{M+1}} d_{ji} I_{ji} \leq \min_{m \in \{1, \cdots, M+1\}} \left\{ \sum_{i=1}^{K_m} L_{i,m} \right\},$$

where $I_{ji}$ is the indicator function that is one if $W_{ji} \in W_g$ and zero if $W_{ji} \notin W_g$.

Proof: We refer to Section V-B for the proof.

The DoF region $\mathcal{D}(W_g)$ again coincides with DoF region assuming perfect cooperation between the relays in each layer. Fig. 3 plots $\mathcal{D}(W_g)$ when $K_1 = K_{M+1} = 3$, $W_g = \{W_{11}, W_{22}, W_{33}\}$, and $L_{i,1} = L_{i,M+1} = 2$ for all $i \in \{1, 2, 3\}$. Note that this network is the same as that in Fig. 2 except that each node has multiple antennas. Specifically, the number of antennas at each source and destination is two.
IV. Achievability for Two-hop and Three-hop Networks

To prove the main results, we first study the two-hop and three-hop Gaussian relay network with $K$ relays in each layer. We propose a block Markov AF relaying scheme and derive its achievable rate region, which will be used to characterize the DoF region in Section V.

A. Achievable Rate Region for $M = 2$ and $K = K_1 = K_2 = K_3$

In this subsection, we derive an achievable rate region of the Gaussian relay network for $M = 2$ and $K = K_1 = K_2 = K_3$.

1) Channel space partitioning and pairing: As shown in the Introduction, interference-free communication is possible for every S–D pair if the relays amplify and forward their received signals with an appropriate delay $\tau$ such that $H_2[t+\tau]H_1[t]$ becomes a diagonal matrix with non-zero diagonal elements. The relays, however, will have to wait forever in order to pair two channel matrices perfectly since channel coefficients vary according to a continuous distribution. To resolve this problem, we first partition channel spaces into subspaces and then pair subspaces of the first hop to those of the second hop.

Define $Q \triangleq \{H_\delta \mid \text{real}([H_\delta]_{ij}) \leq \delta Q, \text{imag}([H_\delta]_{ij}) \leq \delta Q, i, j \in \{1, \cdots, K\}, H_\delta \in \delta (\mathbb{Z}^{K \times K} + j\mathbb{Z}^{K \times K})\}$, where $\delta > 0$ is the quantization interval and $Q \subset \mathbb{Z}^+$ is related to the number of quantization points, i.e., $\text{card}(Q) = (2\delta + 1)^2 K^2$. We further define $H_f$ as the set of all full-rank matrices in $\mathbb{C}^{K \times K}$. Then we partition the channel space of the first hop into $H_1(H_\delta)$’s such that $H_1(H_\delta) \triangleq \{H \mid \text{real}([H - H_\delta]_{ij}) \leq \delta/2, \text{imag}([H - H_\delta]_{ij}) \leq \delta/2, i, j \in \{1, \cdots, K\}, H \in H_f\}$, where $H_\delta \in Q$. To partition the channel space of the second hop and pair with $H_1(H_\delta)$’s, we define $F : H_f \rightarrow H_f$ such that

$$F(H) \triangleq c(H)H^{-1},$$

where $c(H) > 0$ is set to satisfy $p_{H_3[t]}(F(H)) = p_{H_1[t]}(H)$, i.e.,

$$\prod_{i,j \in \{1, \cdots, K\}} p_h(c(H)[H^{-1}]_{ij}) = \prod_{i,j \in \{1, \cdots, K\}} p_h([H]_{ij}).$$

Then define $H_2(H_\delta) \triangleq \{F(H) \mid H \in H_1(H_\delta)\}$, which is the image of $H_1(H_\delta)$ under $F$.

Note that $c(H)$ can be uniquely determined from $H$ under the considered class of channel distribution. Also $F(H)H$ becomes a diagonal matrix with non-zero diagonal elements and there exists one-to-one correspondence between $H$ and $F(H)$. Along with $p_{H_3[t]}(F(H)) = p_{H_1[t]}(H)$, this property will be used to prove that $\Pr(H_1[t_1] \in H_1(H_\delta)) = \Pr(H_2[t_2] \in H_2(H_\delta))$ in Lemma 7. After the sources transmit their signals at time $t$ satisfying $H_1[t_1] \in H_1(H_\delta)$, the relays will amplify and forward them with the delay $\tau$ such that $H_2[t_1 + \tau] \in H_2(H_\delta)$ (see Fig. 4). For this, we apply block Markov encoding and relaying.

2) Block Markov AF relaying: Let us divide a length $n$ block into $B$ sub-blocks having length $n_B = \frac{n}{B}$ each. We apply block Markov encoding and relaying over two hops so that the number of effective sub-blocks is equal to $B - 1$. Let $T_m^{(b)}(H_\delta)$ be the set of time indices whose channel instances belong to $H_m(H_\delta)$. That is,

$$T_m^{(b)}(H_\delta) \triangleq \{t \mid H_m[t] \in H_m(H_\delta), t \in \{(b - 1)n_B + 1, \cdots, bn_B\}\},$$

where $m \in \{1, 2\}$ and $b \in \{1, \cdots, B\}$. During the $b^{th}$ effective sub-block, the sources and relays transmit using $N(H_\delta) \subset \mathbb{Z}^+$ time indices in $T_1^{(b)}(H_\delta)$ and $T_2^{(b+1)}(H_\delta)$, respectively. The detailed procedure of the $b^{th}$ effective sub-block is as follows:

- (Encoding) For all $H_\delta \in Q$, if $\text{card}(T_1^{(b)}(H_\delta)) < N(H_\delta)$ declare error, otherwise the sources transmit their messages with power $P$ using $N(H_\delta)$ time indices in $T_1^{(b)}(H_\delta)$.
- (Relaying) For all $H_\delta \in Q$, if $\text{card}(T_2^{(b+1)}(H_\delta)) < N(H_\delta)$ declare error, otherwise the relays amplify and forward their received signals that were received during $T_1^{(b)}(H_\delta)$ using $N(H_\delta)$ time indices in $T_2^{(b+1)}(H_\delta)$. Specifically, $x_2[t_2] = \gamma y_1[t_1]$, where $t_1 \in T_1^{(b)}(H_\delta)$ and $t_2 \in T_2^{(b+1)}(H_\delta)$. Here, the amplification factor $\gamma$ is given by

$$\gamma = \left(1 + \max_{i \in \{1, \cdots, K\}} \left\{\frac{\text{card}(H_1[t_1])^2}{P} \right\}\right)^{1/2},$$

(19)
which satisfies the power constraint.

- (Decoding) The destinations decode their messages based on \( N(H_δ) \) received signals for all \( H_δ \in Q \).

Let \( E_{1,i}^{(b)}, E_{2,i}^{(b)}, \) and \( E_{3,i}^{(b)} \) denote the encoding, relaying, and decoding error events of the \( i \)th S–D pair at the \( b \)th effective sub-block, respectively. Notice that \( E_{1,i}^{(b)} \) occurs if \( \text{card}(\mathcal{T}_1^{(b)}(H_δ)) < N(H_δ) \) for any \( H_δ \in Q \) and \( E_{2,i}^{(b)} \) occurs if \( \text{card}(\mathcal{T}_2^{(b+1)}(H_δ)) < N(H_δ) \) for any \( H_δ \in Q \). From the union bound,

\[
P_{e,i} \leq \sum_{b=1}^{B-1} \Pr(E_{1,i}^{(b)}) + \sum_{b=1}^{B-1} \Pr(E_{2,i}^{(b)}) + \sum_{b=1}^{B-1} \Pr(E_{3,i}^{(b)}).
\]

### 3) Achievable rate region

We derive the achievable rate region of the proposed scheme. We first introduce the following lemmas, which will be used to prove the main theorem.

**Lemma 1:** For all \( t_1, t_2, \) and \( H_δ \in Q \), \( \Pr(H_1[t_1] \in \mathcal{H}_1(H_δ)) \) is equal to \( \Pr(H_2[t_2] \in \mathcal{H}_2(H_δ)) \).

**Proof:** Let us first show that \( F \) is bijective. Since \( F(H) \) is in \( \mathcal{H}_f \), it is enough to prove that \( F \) is injective, i.e., \( F(H^a) \neq F(H^b) \) if \( H^a \neq H^b \) for all \( H^a \in \mathcal{H}_f \) and \( H^b \in \mathcal{H}_f \). Assume that \( F(H^a) = F(H^b) \), equivalently \( H^a = \frac{c(H^b)}{c(H^a)} H^b \). Since \( F(H^a) = F(H^b) \) means \( \prod_{i,j \in \{1,\ldots,K\}} p_h([H^a]_{ij}) = \prod_{i,j \in \{1,\ldots,K\}} p_h([H^b]_{ij}) \) and \( p_h(a) \neq p_h(b) \) if \( |a| \neq |b| \), we get \( H^a = H^b \), which contradicts the condition that \( H^a \neq H^b \). In conclusion, \( F \) is bijective. Then we obtain

\[
\Pr(H_1[t_1] \in \mathcal{H}_1(H_δ)) = \int_{H \in \mathcal{H}_1(H_δ)} p_{H_1[t_1]}(H)dH = \int_{H \in \mathcal{H}_1(H_δ)} p_{H_2[t_2]}(F(H))dH = \int_{H \in \mathcal{H}_2(H_δ)} p_{H_2[t_2]}(H)dH = \Pr(H_2[t_2] \in \mathcal{H}_2(H_δ)),
\]

where the second equality holds since \( p_{H_1[t_1]}(H) = p_{H_2[t_2]}(F(H)) \) and the third equality holds since \( \mathcal{H}_2(H_δ) \) is the image of \( \mathcal{H}_1(H_δ) \) under \( F \) and \( F \) is bijective. In conclusion, Lemma 1 holds.

**Lemma 2 (Csiszár and Körner):** The probability that

\[
\left| \frac{1}{n_B} \text{card}(\mathcal{T}_m^{(b)}(H_δ)) - \Pr(H_δ) \right| \leq \epsilon
\]

for all \( H_δ \in Q \) is greater than \( 1 - \text{card}(Q)/(4n_B^2) \), where \( m \in \{1,2\} \) and \( b \in \{1,\cdots,B\} \).

**Proof:** We refer to Lemma 2.12 in [20] for the proof.

**Lemma 3:** Suppose that \( H \in \mathcal{H}_1(H_δ) \) and \( F(H) + \Delta \in \mathcal{H}_2(H_δ) \), where \( \Delta \) is the quantization error matrix with respect to \( F(H) \). As the quantization interval \( \delta \to 0 \), \( \Delta \) converges to the zero matrix.

**Proof:** From the definition of \( \mathcal{H}_2(H_δ) \), there exists \( H + \Sigma \in \mathcal{H}_1(H_δ) \) that satisfies \( F(H + \Sigma) = F(H) + \Delta \). Since \( ||\Sigma||_2 \leq 2\sqrt{2}\delta \to 0 \) as \( \delta \to 0 \), \( \Sigma \) converges to the zero matrix as \( \delta \to 0 \). Hence, \( \Delta \) also converges to the zero matrix as \( \delta \to 0 \), which completes the proof.

Based on Lemmas 1 to 3, we derive the following theorem.
Theorem 3: For the Gaussian relay network with $M = 2$ and $K = K_1 = K_2 = K_3$,

$$R_i = E \left( \log \left( 1 + \frac{\gamma^2 c^2(H)P}{1 + \gamma^2 c^2(H)||H^{-1}||^2_F} \right) \right)$$  \hspace{1cm} (23)

is achievable for all $i \in \{1, \ldots, K\}$, where $\gamma^2 = P/(1 + \max_{j \in \{1, \ldots, K\}} \{||H_j||^2_F\})$ and $c(H)$ is set to satisfy (17). Here, the distribution of $H$ is given by $\prod_{i,j \in \{1, \ldots, K\}} p_{h_i}(H_{ij})$. 

Proof: Since $\Pr(H_m[t] \in H_m(H_{\delta}))$ does not depend on $t$ or $m$ (Lemma 1), we will use the shorthand notation $\Pr(H_{\delta})$ to denote $\Pr(H_m[t] \in H_m(H_{\delta}))$. From Lemmas 1 and 2 we set $N(H_{\delta}) = n_B(\Pr(H_{\delta}) - \epsilon)$. Then both $\Pr(E_{1,i}^{(b)})$ and $\Pr(E_{2,i}^{(b)})$ are upper bounded by $\frac{(2Q + 1)^{2K^2}}{4n_B^2\epsilon^2}$, where we use $\text{card}(\mathcal{Q}) = (2Q + 1)^{2K^2}$.

Suppose that the sources transmit their signals at $t_1 \in T_1^{(b)}(H_{\delta})$ and the relays amplify and forward them at $t_2 \in T_2^{(b)}(H_{\delta})$. Let us denote $H_1[t_1] = H$ and $H_2[t_2] = F(H) + \Delta$, where $\Delta$ is the quantization error matrix with respect to $F(H)$. From (5),

$$y_2[t_2] = \gamma(c(H)I + \Delta H)x_1[t_1] + \gamma(c(H)H^{-1} + \Delta)z_1[t_1] + z_2[t_2],$$  \hspace{1cm} (24)

where we use $x_2[t_2] = \gamma y_1[t_1]$ and $F(H)H = c(H)I$. Then the received signal-to-interference-and-noise ratio (SINR) at the $i$th destination is given by

$$\text{SINR}_i = \frac{\gamma^2 |c(H) + [\Delta H]_{ii}|^2 P}{\gamma^2 \sum_{j=1,j \neq i}^{K} ||[\Delta H]_{ij}|^2 P + 1 + \gamma^2 ||c(H)H^{-1} + \Delta||^2_F}.$$  \hspace{1cm} (25)

Hence, assuming that each source uses the Gaussian codebook, an achievable rate of the $i$th S–D pair at the $i$th effective sub-block is lower bounded by

$$R_i^{(b)} \geq \sum_{H_{\delta} \in \mathcal{Q}} \min_{H \in H_{\delta}(H_{\delta})} \log(1 + \text{SINR}_i)(\Pr(H_{\delta}) - \epsilon)$$  \hspace{1cm} (26)

with an arbitrarily small probability of decoding error, i.e., $P(E_{3,i}^{(b)}) \rightarrow 0$ as $n_B \rightarrow \infty$. Set $\delta = n_B^{-1/(24K^2)}$, $Q = n_B^{1/(12K^2)}$, and $\epsilon = n_B^{-1/3}$, which are functions of $n_B$. Then

$$\delta = n_B^{-1/(24K^2)} \rightarrow 0$$  \hspace{1cm} (27)

$$\epsilon = n_B^{-1/3} \rightarrow 0$$  \hspace{1cm} (28)

$$\delta Q = n_B^{1/(24K^2)} \rightarrow \infty$$  \hspace{1cm} (29)

$$\frac{(2Q + 1)^{2K^2}}{4n_B^2\epsilon^2} \leq \frac{3^{2K^2}}{4} n_B^{-1/6} \rightarrow 0$$  \hspace{1cm} (30)

as $n_B$ increases. The first condition guarantees an arbitrarily small quantization error, the second condition guarantees an arbitrarily small rate loss due to the randomness of channel realizations, the third condition is needed to use almost all channel instances for transmission, and the fourth condition guarantees arbitrarily small probabilities of encoding and decoding error. From (25) to (30) and the result of Lemma 3, we finally derive that

$$\lim_{n_B \rightarrow \infty} R_i^{(b)} \geq \int_{H \in H_f} \log \left( 1 + \frac{\gamma^2 c^2(H)P}{1 + \gamma^2 c^2(H)||H^{-1}||^2_F} \right) p_{h_m[t]}(H) dH$$  \hspace{1cm} (31)

is achievable with probability approaching one. Since we can make both $n_B$ and $B$ sufficiently large as $n$ increases, $R_i = \frac{B-1}{B} R_i^{(b)} \rightarrow R_i^{(b)}$ with $P_{e,i} \rightarrow 0$ as $n$ increases. From the fact that channel coefficients are i.i.d. drawn from a continuous distribution, the corresponding channel matrix is in $H_f$ with probability one. In conclusion, (23) is achievable, which completes the proof.

Example 1: Suppose the Gaussian relay network with $M = 2$, $K = K_1 = K_2 = K_3$ and channel coefficients are i.i.d. drawn from $\mathcal{N}_c(0, 1)$. Then, $c(H)$ is given by $||H||_F/||H^{-1}||_F$. From the cut-set upper bound, one can derive a simple sum rate upper bound. For the cut dividing the sources and the rest of nodes,

$$\sum_{i=1}^{K} R_i \leq E \left( \max_{\sum_{i=1}^{K} ||S_i[t]||_{\cdot,1} \leq K} \log \det (I + PH_1[t]S_{\cdot}[t]H_1^t[t]) \right),$$  \hspace{1cm} (32)
where $\Sigma_x[t] = \frac{1}{T} E(x[t]x^\dagger[t])$ is the normalized covariance matrix. Fig. 5 plots the achievable sum rate in Theorem 3 and its upper bound in (32) for $K = 2, 3, 4$. Numerical results suggest that, at any SNR, sum rate gaps are upper bounded by a constant independent of SNR.

B. Achievable Rate Region for $M = 3$ and $K = K_1 = \cdots = K_4$

In this subsection, we derive an achievable rate region of the Gaussian relay network for $M = 3$ and $K = K_1 = \cdots = K_4$. Since the basic concept and approach are similar to those in Section IV-A, we mainly explain the proposed scheme and its achievable rate region and refer to the Appendix for the detailed proof. Together with the result of the two-hop network, the derived achievable rate region will be used to characterize the DoF region in Section V.

1) Channel space partitioning and pairing: Let $\mathcal{H}_d$ be the set of all full-rank matrices in $\mathbb{C}^{K\times K}$ whose eigenvalues are distinct. By using the eigenvalue decomposition, $\mathbf{H} \in \mathcal{H}_d$ can be represented as $\mathbf{S} \Lambda \mathbf{S}^{-1}$, where $\mathbf{S}$ consists of $K$ eigenvectors and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_K)$ is the diagonal matrix with $K$ ordered eigenvalues $|\lambda_1| > \cdots > |\lambda_K| > 0$. Let $v_i$ be the sign bit of real($\lambda_i$), which is zero if real($\lambda_i$) $\geq 0$ and one if real($\lambda_i$) $< 0$. Define $(w_1, w_2, \ldots, w_K)$ as $(v_K, v_K \oplus v_{K-1}, \ldots, v_K \oplus v_1, v_K)$. Define $F_1(\mathbf{H}) \triangleq \text{diag}((-1)^{w_1}, \lambda_1, \ldots, (-1)^{w_K}, \lambda_K)$, and $F_2(\mathbf{H}) \triangleq \text{diag}((-1)^{w_1}, \lambda_1^2, \ldots, (-1)^{w_K}, \lambda_K^2)$, where $\oplus$ denotes XOR. Now define $F_1 : \mathcal{H}_d \rightarrow \mathcal{H}_d$ and $F_2 : \mathcal{H}_d \rightarrow \mathcal{H}_d$ as

$$F_1(\mathbf{H}) \triangleq c_1(\mathbf{H})\Lambda_1\mathbf{S}^{-1} \text{ and } F_2(\mathbf{H}) \triangleq c_2(\mathbf{H})\Lambda_2\mathbf{S}^{-1},$$

where $c_m(\mathbf{H}) > 0$ is set to satisfy $p_{\mathbf{H}_{m+1}[t]}(F_m(\mathbf{H})) = p_{\mathbf{H}_{m}[t]}(\mathbf{H})$, i.e.,

$$\prod_{i,j \in \{1,\cdots,K\}} p_h(c_m(\mathbf{H})[\mathbf{S} \Lambda_m \mathbf{S}^{-1}]_{ij}) = \prod_{i,j \in \{1,\cdots,K\}} p_h([\mathbf{H}]_{ij}).$$

(34)

We partition the channel space of the first hop in the same manner as was done for $\mathcal{H}_1(\mathbf{H}_d)$ and then partition and pair the channel spaces of the second and third hops based on $F_1$ and $F_2$. Specifically, we define $\mathcal{H}_1(\mathbf{H}_d) \triangleq \mathcal{H}_1(\mathbf{H}_d)$, $\mathcal{H}_2(\mathbf{H}_d) \triangleq \{ F_1(\mathbf{H}) | \mathbf{H} \in \mathcal{H}_1(\mathbf{H}_d) \}$, and $\mathcal{H}_3(\mathbf{H}_d) \triangleq \{ F_2(\mathbf{H}) | \mathbf{H} \in \mathcal{H}_1(\mathbf{H}_d) \}$.

Similar to $F$ in Section IV-A, $c_m(\mathbf{H})$ can be uniquely determined and $F_2(\mathbf{H})F_1(\mathbf{H})$ becomes a diagonal matrix with non-zero diagonal elements. The role of $(w_1, \ldots, w_K)$ is to guarantee the one-to-one correspondence between $\Lambda$ and $\Lambda_m$, which also guarantees the one-to-one correspondence between $\mathbf{H}$ and $F_m(\mathbf{H})$. Without $(w_1, \ldots, w_K)$, $2^K$ different matrices of $\Lambda$ give a same diag $(\lambda_1^2, \ldots, \lambda_K^2)$ (see Fig. 5 for $K = 2$). Along with $p_{\mathbf{H}_{m+1}[t]}(F_m(\mathbf{H})) = p_{\mathbf{H}_{m}[t]}(\mathbf{H})$, this property will be used to prove that the probabilities of paired channel subspaces are the same in Lemma 4.
2) Block Markov AF relaying: We divide a length $n$ block into $B$ sub-blocks having length $n_B = \frac{n}{B}$ and then apply block Markov encoding and relaying. Thus, the number of effective sub-blocks is $B - 2$. Define

$$\mathcal{T}_m^{(b)}(\mathbf{H}_\delta) = \left\{ t | \mathbf{H}_m[t] \in \mathcal{H}_m(\mathbf{H}_\delta), t \in \{(b - 1)n_B + 1, \ldots, bn_B\} \right\},$$

where $m \in \{1, 2, 3\}$ and $b \in \{1, \ldots, B\}$. During the $b^{th}$ effective sub-block, the sources, the relays in the second layer, and the relays in the third layer transmit using $N'(\mathbf{H}_\delta) \in \mathbb{Z}_+$ time indices in $\mathcal{T}_1^{(b)}(\mathbf{H}_\delta)$, $\mathcal{T}_2^{(b+1)}(\mathbf{H}_\delta)$, and $\mathcal{T}_3^{(b+2)}(\mathbf{H}_\delta)$, respectively. The detailed procedure of the $b^{th}$ effective sub-block is as follows:

- (Encoding) For all $\mathbf{H}_\delta \in \mathcal{Q}$, if $\text{card}(\mathcal{T}_1^{(b)}(\mathbf{H}_\delta)) < N'(\mathbf{H}_\delta)$ declare error, otherwise the sources transmit their messages with power $P$ using $N'(\mathbf{H}_\delta)$ time indices in $\mathcal{T}_1^{(b)}(\mathbf{H}_\delta)$.

- (Relaying) For all $m \in \{1, 2\}$ and $\mathbf{H}_\delta \in \mathcal{Q}$, if $\text{card}(\mathcal{T}_m^{(b)}(\mathbf{H}_\delta)) < N'(\mathbf{H}_\delta)$ declare error, otherwise the relays amplify and forward their received signals that were received during $\mathcal{T}_m^{(b+1)}(\mathbf{H}_\delta)$ using $N'(\mathbf{H}_\delta)$ time indices in $\mathcal{T}_m^{(b+1)}(\mathbf{H}_\delta)$. Specifically, $x_{m+1}[t_{m+1}] = y_{m+1}[t_{m+1}]$, where $t_1 \in \mathcal{T}_1^{(b)}(\mathbf{H}_\delta)$, $t_2 \in \mathcal{T}_2^{(b+1)}(\mathbf{H}_\delta)$, and $t_3 \in \mathcal{T}_3^{(b+2)}(\mathbf{H}_\delta)$. Here, the power amplification factors are given by

$$\gamma_1 = \left( \frac{P}{1 + \max_{i \in \{1, \ldots, K\}} \left\{ \| \mathbf{H}_1[t_1]_i \|_F^2 \} P \right} \right)^{1/2}$$

and

$$\gamma_2 = \left( \frac{P}{1 + \gamma_1^2 \max_{i \in \{1, \ldots, K\}} \left\{ \| \mathbf{H}_2[t_2]_i \|_F^2 + \| \mathbf{H}_2[t_2]_i \mathbf{H}_1[t_1]_i \|_F^2 \} P \right} \right)^{1/2},$$

which satisfy the power constraint.

- (Decoding) The destinations decode their messages based on $N'(\mathbf{H}_\delta)$ received signals for all $\mathbf{H}_\delta \in \mathcal{Q}$.

3) Achievable rate region: We derive the achievable rate region of the proposed scheme. Because the overall proof is similar to that of Theorem 3, we explain the main result here and refer to the Appendix for the detailed proof.
Sum rate gaps are upper bounded by a constant independent of SNR. The sum rate gaps are larger than those of the two-hop network, numerical results still suggest that, at any SNR, the sum rate is achievable for all $i \in \{1, \ldots, K\}$, where $\gamma_1^2$ and $\gamma_2^2$ are given by $P/(1 + \max_{i \in \{1, \ldots, K\}} \{\|H\|_F^2\}) P$ and $P/(1 + \gamma_1^2 \max_{i \in \{1, \ldots, K\}} \{\|F_1(H)\|_i^2\}) P$, respectively. The definition of $F_m(H)$ is given by (33) and $c_m(H)$ is set to satisfy (32). Here, the distribution of $H$ is given by $\prod_{i,j \in \{1, \ldots, K\}} p_h([H]_{ij})$.

**Proof:** We refer to the Appendix for the proof.

**Example 2:** Suppose the Gaussian relay network with $M = 3$ and $K = K_1 = \cdots = K_4$. If channel coefficients are i.i.d. drawn from $\mathcal{CN}(0, 1)$, then $c_1(H)$ and $c_2(H)$ are given by $\|H\|_F/\|S\Lambda_1 S^{-1}\|_F$ and $\|H\|_F/\|S\Lambda_2 S^{-1}\|_F$, respectively. Fig. 7 plots the achievable sum rate in Theorem 4 and its upper bound in (32) for $K = 2, 3, 4$. Although the sum rate gaps are larger than those of the two-hop network, numerical results still suggest that, at any SNR, sum rate gaps are upper bounded by a constant independent of SNR.

**V. DoF REGION**

In this section, we characterize the DoF region $\mathcal{D}$ of the Gaussian relay network based on the results in Section IV. We further characterize the DoF region $\mathcal{D}(W_g)$ assuming multi-antenna nodes and general message sets.

**A. DoF Region of Gaussian Relay Networks**

In this subsection, we prove Theorem 1. The converse can be shown from a simple cut set upper bound. From the cut dividing the $i$th source and the rest of nodes, $R_i$ is upper bounded by $1 \times K_2$ single-input single-output (SIMO) capacity, which gives $d_i \leq 1$, where $i \in \{1, \ldots, K\}$. From the cut dividing the nodes in the $m$th layer and the nodes in the $(m+1)$th layer, $\sum_{i=1}^{K} R_i$ is upper bounded by $K_m \times K_{m+1}$ multiple-input multiple-output (MIMO) capacity, which gives $\sum_{i=1}^{K} d_i \leq \min\{K_m, K_{m+1}\}$, where $m \in \{1, \ldots, M\}$. Hence we obtain (11) and (12).

Let us now show the achievability. We first prove that every S–D pair achieves one DoF if $K = K_1 = \cdots = K_{M+1}$. Consider the case where $M = 2$. We recall the rate expression (23) in Theorem 3 that is lower bounded by

$$R_i \geq E \left( \log \left( 1 + \frac{\gamma_1^2 c_1^2(H) P}{1 + c_2^2(H)\|H\|_F^2} \right) \right).$$

**Fig. 7. Sum rate comparison for i.i.d. Rayleigh fading with $M = 3$ and $K = K_1 = \cdots = K_4$.**
Assuming $P \geq 1$, we derive $1/(1 + \|H\|^2_F) \leq \gamma^2 \leq K/\|H\|^2_F$, showing that the bounds are independent of $P$. Because the absolute values of channel coefficients are non-zero and finite with probability one, the related quantities such as $\gamma^2(H)$, $\|H\|^2_F$, and $\|H^{-1}\|^2_F$ are also non-zero and finite with probability one. Hence $\limsup_{P \rightarrow \infty} R_i / \log P = 1$. Since the above bounds hold for any $i$, every S–D pair achieves one DoF. Consider the case where $M = 3$. We recall the rate expression (38) in Theorem 4 that is lower bounded by

$$R_i \geq E \left( \log \left( 1 \right. + \frac{\gamma^2_2 \gamma^2_3 (H) c^2_i(H) P}{1 + \gamma^2_2 \|F_2(H)\|^2_F + \gamma^2_2 \gamma^2_3 (H) c^2_i(H) P} \right) \right). \quad (40)$$

Assuming $P \geq 1$, we derive $1/(1 + \|H\|^2_F) \leq \gamma^2 \leq K/\|H\|^2_F$ and $1/(1 + \gamma^2 (\|F_1(H)\|^2_F + \|F_1(H)\|^2_F)) \leq \gamma^2 \leq K/(\gamma^2_2 \|F_1(H)\|^2_F)$. Note that the bounds are again independent of $P$. Hence $\limsup_{P \rightarrow \infty} R_i / \log P = 1$. For $M \geq 4$, one can immediately obtain that every S–D pair achieves one DoF from the results of $M = 2$ and $M = 3$. If $M = 5$, for example, we can apply the result of $M = 2$ over the first two hops and the result of $M = 3$ over the rest three hops.

Let us now consider the case where $K_m \neq K$ in general. In this case, $D$ has corner points $(d^*_1, \ldots, d^*_K)$ such that $\sum_{i=1}^K d^*_i = \min_m \{K_m\}$ and $d^*_i \in \{0, 1\}$ for all $i \in \{1, \ldots, K\}$. To achieve $(d^*_1, \ldots, d^*_K)$, $\min_m \{K_m\}$ S–D pairs with $d^*_i = 1$ participate in communication and $\min_m \{K_m\}$ relays in each layer participate in relaying. Hence one DoF is achievable for each of the corresponding $\min_m \{K_m\}$ S–D pairs, where we use the result for the case $K = K_1 = \cdots = K_{M+1}$. Note that any point in the dominant face can be achieved by time sharing between corner points. In conclusion, Theorem 1 holds.

**B. Multi-antenna and General Message Set**

In this subsection, we prove Theorem 2. The converse can be shown from the cut-set upper bound. From the cut dividing the $j$th destination and the rest of nodes, $\sum_{i=1}^{K_j} R_{ji} I_{ji}$ is upper bounded by $(\sum_{i=1}^{K_i} L_{i,M}) \times L_{j,M+1}$ MIMO capacity, which gives (13). From the cut dividing the $i$th source and the rest of nodes, $\sum_{j=1}^{K_j+1} R_{ji} I_{ji}$ is upper bounded by $L_{i,j} \times (\sum_{j=1}^{K_j} L_{j,2})$ MIMO capacity, which gives (14). Lastly, from the cut dividing the nodes in the $m$th layer and the nodes in the $(m+1)$th layer, $\sum_{i=1}^{K_i} \sum_{j=1}^{K_j} R_{ji} I_{ji}$ is upper bounded by $(\sum_{i=1}^{K_i} L_{i,m}) \times (\sum_{i=1}^{K_{m+1}} L_{i,m+1})$ MIMO capacity, which gives (15).

Consider the achievability. The greedy allocation of $\{d_{ji}\}_{W_{ji} \in W_g}$ based on a specific order that satisfies (13) to (15) is one of the corner points in $D(W_g)$, and $\{d^*_{ji}\}_{W_{ji} \in W_g}$ be the result of the greedy allocation according to a specific order. For $W_{ji} \in W_g$, we can choose $d^*_{ji}$ antennas at the $i$th source and $d^*_{ji}$ antennas at the $j$th destination and pair them as $d^*_{ji}$ virtual S–D pairs. As a result, we can establish total $\sum_{i=1}^{K_i} \sum_{j=1}^{K_{j+1}} d^*_{ji}$ virtual S–D pairs because $\{d^*_{ji}\}_{W_{ji} \in W_g}$ satisfies (13) and (14). We can also choose total $\sum_{i=1}^{K_i} \sum_{j=1}^{K_{j+1}} d^*_{ji} I_{ji}$ virtual relays in each layer because $\{d^*_{ji}\}_{W_{ji} \in W_g}$ satisfies (15). The resulting network consists of $\sum_{i=1}^{K_i} \sum_{j=1}^{K_{j+1}} d^*_{ji} I_{ji}$ virtual S–D pairs with $\sum_{i=1}^{K_i} \sum_{j=1}^{K_{j+1}} d^*_{ji} I_{ji}$ virtual relays in each layer. Therefore, from the result of Theorem 1 all virtual S–D pairs can achieve one DoF, equivalently $\{d^*_{ji}\}_{W_{ji} \in W_g}$ is achievable. Note that any point in the dominant face can be achieved by time sharing between corner points, which completes the proof.

**VI. CONCLUSION**

In this paper, we study the $K$-user $M$-hop Gaussian relay network consisting of $K_m$ nodes in the $m$th layer. The proposed block Markov AF relaying exploits channel fluctuation to cancel the inter-user interference at each destination and works for a wide class of channel distributions including i.i.d. Rayleigh fading. Under this class of channel distributions, we completely characterize the DoF region of the Gaussian relay network and further characterize the DoF region of more general networks with multi-antenna nodes and general message sets. The resulting DoF regions coincide with the DoF regions assuming perfect cooperation between relays in each layer and there is no penalty due to distributed relaying in terms of DoF.

**APPENDIX**

**ACHIEVABLE RATE REGION FOR $M = 3$ AND $K = K_1 = \cdots = K_4$**

In this appendix, we derive the achievable rate region in Theorem 4. The overall steps are similar to those in Section IV-A. We first introduce a useful property, which will be used to prove the following lemmas. Let
$H^a = S^a \Lambda^a (S^a)^{-1}$ and $H^b = S^b \Lambda^b (S^b)^{-1}$, where $H^a \in \mathcal{H}_d$ and $H^b \in \mathcal{H}_d$. Then $H^a = H^b$ if and only if $\Lambda^a = \Lambda^b$ and there exists a diagonal matrix $\Theta$ satisfying $S^a = S^b \Theta$. Similar to Lemmas 1 and 3, we derive the following two lemmas.

**Lemma 4:** For all $t_1, t_2, t_3$, and $H_d \in \mathcal{Q}$,

$$\Pr(H_1[t_1] \in \mathcal{H}_d(H_3)) = \Pr(H_2[t_2] \in \mathcal{H}_d(H_3)) = \Pr(H_3[t_3] \in \mathcal{H}_d(H_3)).$$  

(41)

**Proof:** The overall proof is similar to that of Lemma 1. To show that $F_m$ is bijective, we need to prove $F_m(H^a) \neq F_m(H^b)$ if $H^a \neq H^b$ for all $H^a \in \mathcal{H}_d$ and $H^b \in \mathcal{H}_d$. Assume that $F_m(H^a) = F_m(H^b)$. Then $S^a \Lambda^a (S^a)^{-1} = c(H^a) S^b \Lambda^b (S^b)^{-1}$, where $S^a$ and $\Lambda^a$ ($S^b$ and $\Lambda^b$) are the eigenvector matrix and the eigenvalue matrix of $H^a$ ($H^b$), respectively. Hence $F_m(H^a) = F_m(H^b)$ if and only if $\Lambda^a = \frac{c(H^b)}{c(H^a)} \Lambda^b$ and there exists a diagonal matrix $\Theta$ satisfying $S^a = S^b \Theta$. Since there is a one-to-one correspondence between $\Lambda$ and $\Delta_m$, we get $\Lambda^a = c \Lambda^b$, where $c > 0$ is a constant. As a result, we obtain

$$H^a = S^a \Lambda^a (S^a)^{-1} = c(S^b \Theta \Lambda^b (S^b \Theta)^{-1}) = cH^b.$$  

(42)

Since $F_m(H^a) = F_m(H^b)$ means $\prod i,j \in \{1, \ldots, K\} p_h[[H^a]_{ij}] = \prod i,j \in \{1, \ldots, K\} p_h[[H^b]_{ij}]$ and $p_h(a) \neq p_h(b)$ if $|a| \neq |b|$, we get $H^a = H^b$, which contradicts the condition that $H^a \neq H^b$. In conclusion, $F$ is bijective. Then similar to (27), we obtain $\Pr(H_1[t_1] \in \mathcal{H}_d(H_3)) = \Pr(H_2[t_2] \in \mathcal{H}_d(H_3))$, which completes the proof. ■

**Lemma 5:** Suppose that $H \in \mathcal{H}_d(H_3)$ and $F(H) + \Delta_{m-1} \in \mathcal{H}_d(H_3)$, where $m \in \{2, 3\}$. As $\delta \to 0$, $\Delta_1$ and $\Delta_2$ converge to the zero matrix.

**Proof:** Since the overall proof is the same as that in Lemma 3, we refer to the proof of Lemma 3. ■

We are now ready to prove Theorem 4 by using the previous lemmas. The overall procedure is similar to that of Theorem 3. Since $\Pr(H_m[t] \in \mathcal{H}_d(H_3))$ does not depend on $t$ or $m$ (Lemma 4), we will again use the shorthand notation $\Pr(H_d)$ to denote $\Pr(H_m[t] \in \mathcal{H}_d(H_3))$. We set $N'(H) = n_B(\Pr(H_d) - \epsilon)$. Then the probability of encoding and relaying error of the $b^{th}$ effective sub-block is upper bounded by $\frac{3(2Q+1)^2}{4n_B^2}$. Suppose that the sources transmit their signals at $t_1 \in \mathcal{T}_1^b(H_d)$ and the relays amplify and forward them at $t_2 \in \mathcal{T}_2^{b+1}(H_d)$ for the second hop and $t_3 \in \mathcal{T}_3^{b+2}(H_d)$ for the third hop. Let us denote $H_1[t_1] = H$, $H_2[t_2] = F_1(H) + \Delta_1$, $H_3[t_3] = F_2(H) + \Delta_2$, where $\Delta_1$ and $\Delta_2$ are the quantization error matrices with respect to $F_1(H)$ and $F_2(H)$, respectively. Then

$$y_3[t_3] = \gamma_2 \gamma_1 c_2(H)c_1(H) \mathbf{x}_1[t_1] + \gamma_2 \gamma_1 (F_2(H) + \Delta_2)(F_1(H) + \Delta_1) \mathbf{z}_1[t_1] + \gamma_2 (F_2(H) + \Delta_2) \mathbf{z}_2[t_2] + \mathbf{z}_3[t_3],$$  

(43)

where

$$\Delta_{tot} = F_2(H) \Delta_1 H + \Delta_2 F_1(H) H + \Delta_2 \Delta_1 H.$$  

(44)

Hence the received SINR at the $i^{th}$ destination is given by

$$\text{SINR}_i = \frac{\gamma_2 \gamma_1^2 c_2(H)c_1(H)}{\gamma_2^2 \gamma_1^2 \sum_{j=1,j\neq i}^K ||\Delta_{tot}||_j^2 P + \sigma_{tot,i}^2},$$  

(45)

where $\sigma_{tot,i}^2 = 1 + \gamma_2^2 \|F_2(H) + \Delta_2\|_F^2 + \gamma_2^2 \gamma_1^2 \|F_2(H) + \Delta_2\|_F^2 + \gamma_2 (F_2(H) + \Delta_2)(F_1(H) + \Delta_1)\|_F^2$. Hence,

$$R_i^{(b)} \geq \min_{H \in \mathcal{Q}} \log(1 + \text{SINR}_i/(\Pr(H_d) - \epsilon))$$  

(46)

is achievable with an arbitrary small probability of decoding error. Setting $\delta = n_B^{-1/(24K^2)}$, $Q = n_B^{1/(12K^2)}$, and $\epsilon = n_B^{-1/3}$ satisfies the conditions in (27) to (30). Hence,

$$\lim_{n_B \to \infty} R_i^{(b)} \geq \int_{H \in \mathcal{H}_d} \log \left(1 + \frac{\gamma_2 \gamma_1^2 c_2(H)c_1(H) P}{1 + \gamma_2 \|F_2(H)\|_F^2 + \gamma_2 \gamma_1 \|F_2(H)F_1(H)\|_F^2} \right) \Pr(H_m[t](H)) dH$$  

(47)

is achievable with probability approaching one, where we use the result of Lemma 5. Since we can make both $n_B$ and $B$ sufficiently large as $n$ increases, $R_i = \frac{B-2}{B} R_i^{(b)} \to R_i^{(b)}$ with $P_{e,i} \to 0$ as $n$ increases. In conclusion, (38) is achievable, which completes the proof.
REFERENCES

[1] M. R. Aref, “Information Flow in Relay Networks,” Ph.D. dissertation, Stanford Univ., 1980.

[2] S. Ray, M. Méard, and J. Abounadi, “Random coding in noise-free multiple access networks over finite fields,” in Proc. IEEE GLOBECOM, San Francisco, CA, Dec. 2003.

[3] S. Bhadra, P. Gupta, and S. Shakkottai, “On network coding for interference networks,” in Proc. IEEE Int. Symp. Information Theory (ISIT), Seattle, WA, Jul. 2006.

[4] N. Ratnakar and G. Kramer, “The multicast capacity of deterministic relay networks with no interference,” IEEE Trans. Inf. Theory, vol. 52, pp. 2425–2432, Jun. 2006.

[5] B. Smith and S. Vishwanath, “Unicast transmission over multiple access erasure networks: capacity and duality,” in Proc. IEEE Information Theory Workshop, Lake Tahoe, CA, Sep. 2007.

[6] A. S. Avestimehr, S. N. Diggavi, and D. Tse, “Wireless network information flow,” in Proc. 45th Annu. Allerton Conf. Communication, Control, and Computing, Monticello, IL, Sep. 2007.

[7] ———, “Approximate capacity of Gaussian relay networks,” in Proc. IEEE Int. Symp. Information Theory (ISIT), Toronto, Canada, Jul. 2008.

[8] G. Bresler, A. Parekh, and D. Tse, “The approximate capacity of the many-to-one and one-to-many Gaussian interference channels,” in Proc. 45th Annu. Allerton Conf. Communication, Control, and Computing, Monticello, IL, Sep. 2007.

[9] R. H. Etkin, D. Tse, and H. Wang, “Gaussian interference channel capacity to within one bit,” IEEE Trans. Inf. Theory, vol. 54, pp. 5534–5562, Dec. 2008.

[10] V. R. Cadambe and S. A. Jafar, “Interference alignment and degrees of freedom of the K-user interference channel,” IEEE Trans. Inf. Theory, vol. 54, pp. 3425–3441, Aug. 2008.

[11] T. Gou and S. A. Jafar, “Degrees of freedom of the K user M × N MIMO interference channel,” in arXiv:cs.IT/0809.0099, Sep. 2008.

[12] V. R. Cadambe and S. A. Jafar, “Interference alignment and the degrees of freedom of wireless X networks,” IEEE Trans. Inf. Theory, vol. 55, pp. 3893–3908, Sep. 2009.

[13] B. Nazer, M. Gastpar, S. A. Jafar, and S. Vishwanath, “Ergodic interference alignment,” in Proc. IEEE Int. Symp. Information Theory (ISIT), Seoul, South Korea, Jun./Jul. 2009.

[14] L. Sankar, X. Shang, E. Erkip, and H. V. Poor, “Ergodic two-user interference channels: is separability optimal?” in Proc. 46th Annu. Allerton Conf. Communication, Control, and Computing, Monticello, IL, Sep. 2008.

[15] V. R. Cadambe and S. A. Jafar, “Parallel Gaussian interference channels are not always separable,” IEEE Trans. Inf. Theory, vol. 55, pp. 3983–3990, Sep. 2009.

[16] S.-W. Jeon and S.-Y. Chung, “Capacity of a class of multi-source relay networks,” in Information Theory and Applications Workshop, University of California San Diego, La Jolla , CA, Feb. 2009.

[17] S. Mohajer, S. N. Diggavi, and D. Tse, “Approximate capacity of a class of Gaussian relay-interference networks,” in Proc. IEEE Int. Symp. Information Theory (ISIT), Seoul, South Korea, Jun./Jul. 2009.

[18] S.-W. Jeon, S.-Y. Chung, and S. A. Jafar, “Degrees of freedom of multi-source relay networks,” in Proc. 47th Annu. Allerton Conf. Communication, Control, and Computing, Monticello, IL, Sep. 2009.

[19] S.-W. Jeon and S.-Y. Chung, “Sum capacity of multi-source linear finite-field relay networks with fading,” in Proc. IEEE Int. Symp. Information Theory (ISIT), Seoul, South Korea, Jun./Jul. 2009 (full version available at arXiv:cs.IT/0907.2510).

[20] I. Csizsár and J. Körner, Information Theory: Coding Theorems for Discrete Memoryless Systems. New York: Academic Press, 1981.