Infinite-dimensional symmetry for wave equation with additional condition

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Abstract

Symmetries for wave equation with additional conditions are found. Some conditions yield infinite-dimensional symmetry algebra for the nonlinear equation. Ansatzes and solutions corresponding to the new symmetries were constructed.

We discuss conditional symmetries of the Klein-Gordon equation

\[ \Box u = F(x, u) \]

for the real-valued function \( u = u(x_0, x_1, x_2, \ldots, x_n) \), \( x_0 = t \) is the time variable, \( x_0, x_1, x_2, \ldots, x_n \) are space variables, \( n \neq 1 \). \( \Box u \) is the d’Alembert operator

\[ \Box u = -\frac{\partial^2 u}{\partial x_0^2} + \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}. \]

The general equation in the class (1) is not invariant with respect to any operators, with only particular cases having wide symmetry algebras [1].

The maximal invariance algebra of the equation (1) with \( F = F(u) \) (not depending on \( x \)) is the Poincaré algebra \( AP(1, n) \) with the basis operators

\[ p_\mu = ig_{\mu \nu} \frac{\partial}{\partial x_\nu}, \quad J_{\mu \nu} = x_\mu p_\nu - x_\nu p_\mu, \]

where \( \mu, \nu \) take the values 0, 1, 2, \ldots, \( n \); the summation is implied over the repeated indices (if they are small Greek letters) in the following way:

\[ x_\mu x_\nu = x_\nu x_\mu = x_\nu x_\nu = x_0^2 - x_1^2 - x_2^2 - \ldots - x_n^2, \]

\[ g_{\mu \nu} = \text{diag}(1, -1, -1, \ldots, -1). \]

The invariance algebras of the equation (1) will also include dilation operators for \( F = \lambda u^k \) or \( F = \lambda \exp u \) and conformal operators for \( F = \lambda u^{\frac{n+3}{n-1}} \).

The maximal invariance algebra of the equation (1) with \( F = F(x^2, u) \) (\( x^2 = x_\mu x_\nu \)) is a subalgebra of the the Poincaré algebra \( AP(1, 3) \) whose basis operators are Lorentz boosts \( J_{\mu \nu} \).

Symmetry of the linear equation (1) with \( F = 0 \) and \( F = \lambda \exp u \) with \( n = 2 \) is infinite-dimensional.

Similarity solutions for the equation (1) can be found by symmetry reduction with respect to non-equivalent subalgebras of its invariance algebras [1, 2, 3, 4].

Here we present some examples of conditional invariance of the equation (1) - the symmetry with an additional condition being not \( Q \)-conditional symmetry.
The concept of conditional symmetry was derived and discussed in the papers \cite{5 6 7 8 9}, and later it was developed by numerous authors into the theory and a number of algorithms for studying symmetry properties of equations of mathematical physics and for construction of their exact solutions (see e.g.\cite{10}). Here we will work with the following definition of the conditional symmetry:

**Definition 1.** The equation \( \Phi(x, u, u_1, \ldots, u_l) = 0 \), where \( u_k \) is the set of all \( k \)-th order partial derivatives of the function \( u = (u_1, u_2, \ldots, u^m) \), is called conditionally invariant \cite{3} under the operator

\[
Q = \xi^i(x, u)\partial_{x_i} + \eta^r(x, u)\partial_{u^r}
\]

if there is an additional condition

\[
G(x, u, u_1, \ldots, u_l) = 0, \tag{2}
\]

such that the system of two equations \( \Phi = 0, G = 0 \) is invariant under the operator \( Q \).

If (2) has the form \( G = Qu \), then the equation \( \Phi = 0 \) is called \( Q \)-conditionally invariant under the operator \( Q \) \cite{3}.

These definitions of the conditional invariance of some equation are based on what is in reality Lie symmetry (see e.g. the classical texts \cite{11 12 13}) of the same equation with a certain additional condition. Conditional symmetries of wave equation are specifically discussed in \cite{14 15 16}.

The equation (1) with an additional condition

\[
x_\mu u_\mu + \alpha u = 0 \tag{3}
\]

with \( \alpha \neq 0 \) has the maximal symmetry algebra determined by the operators

\[
X = (-1/\alpha) u^\alpha x_\mu \int \Phi u u^\alpha - 1 du + C_{\mu\nu} x_\nu + dx_\mu)p_{x_\mu} + \Phi \partial_u,
\]

with \( \Phi = \Phi(u, u^{\alpha} x_\mu) \) being an arbitrary function of its arguments. \( \partial_u \) designates the operator \( \partial/\partial u \).

With \( \alpha = 0 \) the corresponding algebra is generated by the operator

\[
X = x_0 \phi^\alpha(\frac{x_\alpha}{x_0}, u)p_{x_\mu} + \psi(\frac{x_\alpha}{x_0}, u)\partial_u,
\]

with \( \phi^\alpha, \psi \) being arbitrary functions of their arguments.

The additional condition (3) can be presented as \( Du = 0 \), where \( D \) is the dilation operator

\[
D = x_\mu \partial_\mu + i\alpha u \partial_u.
\]

The equation (3) has the general solution

\[
u = x_0^\alpha \phi(\frac{x_\alpha}{x_0}).\tag{7}
\]
where $\phi$ is an arbitrary function. If we use (7) with $\omega_a = \frac{x_a}{x_0}$ as an ansatz for

$$\square u = 0,$$

we get the reduced equation

$$(1 + 2\alpha)\omega_a \phi_{\omega_a} + \omega_a \omega_b \phi_{\omega_a \omega_b} + \alpha(\alpha + 1)\phi - \phi_{\omega_a \omega_a} = 0.$$  \hspace{1cm} (9)

Summation is implied over the repeated indices. The ansatz (7) corresponds to the operator (6) that is a Lie symmetry operator of the equation (8).

We found some particular solutions of the equation (9). If we put $\phi = \phi(\omega)$, $\omega = m_\omega \omega_a$, $m_a$ are parameters with $m_a m_a = 1$, we get an ordinary differential equation

$$(1 + 2\alpha)\omega \phi' + (\omega^2 - 1)\phi'' + \alpha(\alpha + 1)\phi = 0.$$  \hspace{1cm} (10)

Its solution for $\alpha = 0$ is

$$\phi = c_1 \ln |\omega + \sqrt{\omega^2 - 1}| + c_2,$$

for $\alpha = -1$ it is

$$\phi = c_1 \left(\frac{\omega}{2} \sqrt{\omega^2 - 1} - \frac{1}{2} \ln |\omega + \sqrt{\omega^2 - 1}|\right) + c_2.$$

If $\phi = \phi(\omega)$, $\omega = \omega_a \omega_a$, $\alpha = 0$, then a solution of the equation (9) has the form

$$\phi = \int \omega^{-\frac{\alpha}{2}}(\omega - 1)^{-\frac{\alpha}{2} - 1} d\omega.$$

The obtained solution are classical symmetry solutions of the equation. However, by application of the group transformations corresponding to the infinite-dimensional conditional symmetry operators it is possible to multiply these solutions and to obtain new ones that will not be classical symmetry solutions.

The conditional symmetries (4) and (5) can also be considered as a hidden symmetry of the equation (8), that is new symmetries of the reduced equation (9) with $\alpha \neq 0$ or $\alpha = 0$ that is not present for the original equation.

**Definition 2.** An equation is said to have hidden conditional invariance if a reduced equation is conditionally invariant under some additional condition [17].

This definition stems from the definition of the hidden invariance [18].

Further we present an ansatz and solutions for the equation (8) with another additional condition

$$x_\mu x_\nu u_{\mu \nu} + \alpha x_\mu u_\mu = 0.$$  \hspace{1cm} (11)

The condition (11) generates the following ansatz for (11):

$$u = \frac{1}{x_0^{1 - \alpha}} \psi \left(\frac{x_a}{x_0}\right) + \phi \left(\frac{x_a}{x_0}\right) f(x_0),$$  \hspace{1cm} (12)

where $f(x_0) = \ln x_0$ for $\alpha = 1$ or $f(x_0) = 1$ for $\alpha \neq 1$. 

This additional condition gives an ansatz leading to antireduction \[19\]. There will be a system of two reduced equations having the form:

\[
\begin{align*}
2\omega_a \phi_{\omega_a} + \omega_a \omega_b \phi_{\omega_b \omega_a} - \phi_{\omega_a \omega_a} &= 0, \\
2\alpha \omega_a \psi_{\omega_a} + \omega_a \omega_b \psi_{\omega_b \omega_a} - \psi_{\omega_a \omega_a} &= 0
\end{align*}
\]  \[(13)\]

for \(\alpha \neq 1\), and

\[
\begin{align*}
2\omega_a \phi_{\omega_a} + \omega_a \omega_b \phi_{\omega_b \omega_a} - \phi_{\omega_a \omega_a} &= 0, \\
\alpha \omega_a \psi_{\omega_a} + \omega_a \omega_b \psi_{\omega_b \omega_a} - \psi_{\omega_a \omega_a} - \phi - 2\omega_a \phi_{\omega_a} &= 0
\end{align*}
\]  \[(14)\]

for \(\alpha = 1\).

Let us adduce partial solutions of these reduced equations with \(\phi = \phi(\omega)\), \(\omega = m_a \omega_a\), \(m_a\) are parameters with \(m_a m_a = 1\) for \[(13)\] it is

\[
\begin{align*}
\phi &= c_1 \ln \frac{\omega - 1}{\omega + 1}, \\
\psi &= c_3 \int \frac{d\omega}{(\omega^2 - 1)\alpha}
\end{align*}
\]

for \[(14)\] it is

\[
\begin{align*}
\phi &= c_1 \ln \frac{\omega - 1}{\omega + 1}, \\
\psi &= \frac{1}{\sqrt{\omega^2 - 1}} \left\{ c_2 \ln |\omega + \sqrt{\omega^2 - 1}| - 2 \frac{c_1}{\sqrt{\omega^2 - 1}} + c_1 \int \frac{1}{\sqrt{\omega^2 - 1}} \ln |\omega + \sqrt{\omega^2 - 1}| d\omega \right\}.
\end{align*}
\]

Substituting the found solutions of the reduced equations into the ansatz \[(12)\], we obtain exact solution of the equation \[(8)\].

References

[1] Fushchych W.I. and Serov N.I., The symmetry and some exact solutions of the nonlinear many-dimensional Liouville, d’Alembert and eikonal equations, J. Phys. A, 1983, V.16, 3645–3658.

[2] Tajiri M., Some remarks on similarity and soliton solutions of nonlinear Klein-Gordon equations, J. Phys. Soc. Japan, 1984, V.53, 3759–3764.

[3] Fushchych W.I., Shtelen W.M. and Serov N.I., Symmetry analysis and exact solutions of nonlinear equations of mathematical physics, Kyiv, Naukova Dumka, 1989 (in Russian); Kluwer Publishers, 1993 (in English).

[4] Fushchych W. I., Barannik A. F., On exact solutions of the nonlinear d’Alembert equation in Minkowski space \(R(1,n)\), Dokl. AN Ukr. SSR, Ser.A, 1990, No 6, 31–34.

[5] Olver P.J. and Rosenau P., The construction of special solutions to partial differential equations, Phys. Lett. A, 1986, V.114, 107–112.

[6] Fushchych W.I. and Tayfra I.M., On a reduction and solutions of the nonlinear wave equations with broken symmetry, J. Phys. A, 1987, V.20, L45–L48.
[7] Fushchych W.I. and Zhdanov R.Z., Symmetry and exact solutions of nonlinear spinor equations, *Phys. Reports*, 1989, V.172, 123–174.

[8] Clarkson P. and Kruskal M.D., New similarity solutions of the Boussinesq equation, *J. Math. Phys.*, 1989, V.30, 2201–2213.

[9] Levi D. and Winternitz P., Non-classical symmetry reduction: example of the Boussinesq equation, *J. Phys. A*, 1989, V.22, 2915–2924.

[10] Zhdanov R.Z., Tsyfra I.M. and Popovych R.O., A precise definition of reduction of partial differential equations, *J. Math. Anal. Appl.*, 1999, V.238, N 1, 101–123.

[11] Ovsyannikov L.V., Group analysis of differential equations, New York, Academic Press, 1982.

[12] Olver P., Application of Lie groups to differential equations, New York, Springer Verlag, 1987.

[13] Bluman G.W. and Kumei S., Symmetries and differential equations, New York, Springer Verlag, 1989.

[14] Yehorchenko I. A. and Vorobyova A. I., Conditional invariance and exact solutions of the Klein-Gordon-Fock equation. *Dokl. Akad. Nauk Ukrainy*, 1992, No.3, 19–22.

[15] Fushchych W. I. and Serov M. I., Conditional invariance of the nonlinear equations of d’Alembert, Liouville, Born-Infeld, and Monge-Ampere with respect to the conformal algebra. *Symmetry analysis and solutions of equations of mathematical physics*, 1988, Akad. Nauk Ukrain. SSR, Inst. Mat., Kyiv, 98–102.

[16] Barannyk A.F. and Moskalenko Yu.D., Conditional symmetry and exact solutions of the multidimensional nonlinear d’Alembert equation, *J. Nonlinear Math. Phys.*, 1996, V.3, 336–340.

[17] Yehorchenko I. A., Group classification with respect to hidden symmetry, in Proceedings of Fifth International Conference “Symmetry in Nonlinear Mathematical Physics” (July 23–29, 2003, Kyiv), Editors A.G. Nikitin, V.M. Boyko R.O. Popovych and I.A. Yehorchenko, Kyiv, Institute of Mathematics, 2004, V.50, Part 1, 290–297.

[18] Abraham-Shrauner B., Hidden symmetries, first integrals and reduction of order of nonlinear ordinary differential equations, *J. Nonlin. Math. Phys.*, 2002, V.9, Suppl. 2, 1–9.

[19] Fushchych W. I. and Zhdanov R. Z. Antireduction and exact solutions of nonlinear heat equations, *J. Nonlin. Math. Phys.*, 1994, V.1, 60-64.