Turbulent flows are amongst the more well-known problems where the use of standard tools of statistical physics and analysis has met with limited success. One of the factors contributing to this is the intermittent nature of the flow [1, 2] which, in turn, leads to non-Gaussian distributions of observables such as velocity gradients as well as the multiscaling of (suitably-defined) \( q \)-th order moments of the spatial increments of the velocity field: Higher-order moments (and their exponents \( \zeta_q \)) are not trivially (linearly) related to lower-order moments [3–6]. Several experiments and direct numerical simulations (DNSs) of the three-dimensional, incompressible Navier-Stokes equation for fully-developed, statistically homogeneous and isotropic turbulence have now established beyond doubt that not only are distributions of velocity-gradients and fluid acceleration characterised by non-Gaussianity and fat tails [6, 7] but the scaling exponents \( \zeta_q \) are non-linear, convex, monotonically increasing functions of \( q \).

While the inertial range exponents display multiscaling, there is also overwhelming experimental and numerical evidence [8] that the energy dissipation rates show strong temporal and spatial fluctuations [9] characterized by periods of intense bursts and calmness. An important question in this field is to find rigorous estimates for the energy dissipation rate. These results are substantiated through direct numerical simulations for different strengths of rotation rate as well as from simulations of a helical shell model. Our work also shows a surprisingly good agreement between the solutions of the Navier-Stokes equations in a rotating frame with those obtained from low-dimensional dynamical systems. In particular, this agreement extends to the structure of the spatial profile of the energy dissipation rates and the decrease in inertial range intermittency with increasing strengths of rotation.

### I. INTRODUCTION

In this paper, we tackle this question. In particular, we (a) show how with increasing rotation rates not only does the energy dissipation field appear less intermittent with associated changes in its multifractal spectrum (b) establish a relation between the anomalous exponents of equal-time velocity structure functions, measured in the inertial range to the Rényi scaling exponent obtained from partition functions of the dissipation field \( \epsilon(x) \).

Experiments and DNSs are of course primary sources on which theoretical and phenomenological ideas are built. Nevertheless, synthetic models still serve as useful tools to develop insights on the origins of intermittency and the curious nature of energy dissipation [51–58]. A particularly useful example of this is the class of cascade
models known as shell models. Remarkably such models which have very little in common (beyond the formal structure) to the Navier-Stokes equations, are shown, for homogeneous and isotropic turbulence, to mimic the multiscaling of two-point correlation functions with remarkable accuracy and hence have, over the years, proved a remarkable testing ground for theories of various correlation functions which were inaccessible to full-scale simulations or experiments. In this paper, as we will see below, we resort to both full-scale DNSs of the rotating Navier-Stokes equation as well as study its shell model counterpart. The agreement between the two approaches serves as a useful tool to underline the extent and usefulness of modeling rotating turbulence as a low-dimensional dynamical systems model.

II. MODELING FOR ROTATING TURBULENCE

We begin with the incompressible ($\nabla \cdot \mathbf{u} = 0$) Navier-Stokes equation for the velocity field $\mathbf{u}$ of a three-dimensional flow, with density $\rho$ and a kinematic viscosity $\nu$ small enough to generate turbulence, under a solid body rotation $\Omega$:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \nabla)\mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} - 2\Omega \times \mathbf{u} + \mathbf{f}. \quad (1)$$

The pressure $p$ includes the effect of the centrifugal force and the Coriolis force $-2\Omega \times \mathbf{u}$ results from the solid body rotation. Furthermore, an external, large-scale, force $\mathbf{f}$ ensures that the (turbulent) flow remains in a statistically stationary state through the injection of an energy $\epsilon = \langle \mathbf{u} \cdot \mathbf{f} \rangle$. Unlike non-rotating three-dimensional turbulent flows, helicity plays an important role in rotating turbulence; a natural source of helicity is the Coriolis force which results in a helicity injection ($2\mathbf{u} \cdot \nabla (\Omega \cdot \mathbf{u})$); similarly the external drive can also inject helicity at large-scales via helicity $\langle \omega \cdot \mathbf{f} + \mathbf{u} \cdot (\nabla \times \mathbf{f}) \rangle$. (The angular brackets in these definitions imply suitable averaging in space or in time for non-equilibrium stationary states.)

The solution of Eq. (1) is characterized not only by its Reynolds number $Re$ (as would be the case for non-rotating flows) but by a second non-dimensional number, the Rossby number $Ro = \mathbf{u}_{rms}/(2L\Omega)$, which is a measure of the relative importance of the Coriolis and inertial terms in flow; $L$ is the characteristic length of the domain (typically, $2\pi$ in numerical simulations) and $u_{rms}$ is the root-mean-square velocity.

A. Direct Numerical Simulations

We use direct numerical simulations (DNSs), with a standard fully de-aliased pseudo-spectral method, to solve Eq. (1) in a $2\pi$ periodic cubic box, with $N^3 = 512^3$ collocation points, and a fourth-order Runge-Kutta scheme for time-marching. In our simulations, adaptive time-step $\delta t$ consistent with the Courant–Friedrich–Lewy (CFL) condition is employed. We choose $\nu = 10^{-3}$ to obtain a Taylor-scale based Reynolds numbers $Re_\lambda = 137, 44, 50, 56$, and 58 corresponding to the rotation rates $\Omega = 0, 1, 8, 16, \text{and } 32$, respectively. Our choice of forcing

$$f(k) = \frac{\epsilon u(k)}{n_f u(k) \cdot u^*(k)}, \quad (2)$$

allows for a constant-energy injection (and no helicity) at wavenumber $k_f \in [40, 41]$ ($n_f$ is the number of modes at this wavenumber) and drives the system to a statistically stationary, isotropic and homogeneous, turbulent regime. Once we reach such stationary states, we turn on the Coriolis force and choose different Rossby numbers corresponding to $\Omega = 1, 8, 16$ and 32. We then wait sufficiently long for the rotating system to reach a statistically stationary state before measuring the local energy dissipation rates and their statistical properties. All our DNSs were performed by using the open-source code “Tarang” [59, 60] developed at IIT Kanpur.

Direct Numerical Simulations constitute just one part of our strategy to understand small-scale statistics of rotating turbulence. We also perform simulations by using a helical shell model to obtain data at Reynolds numbers much higher than what we can achieve by using DNSs. Shell models thus have the well-known advantage of being useful tools not only for probing and measuring inertial range two-point scaling exponents but also for serving as an important bridge between ideas in real turbulence and low-dimensional dynamical systems.

B. Shell Model

Shell models are essentially low-dimensional dynamical systems which mimic the spectral Navier-Stokes equation without being actually derived from it [1, 15]. The dynamical system is constructed by replacing the three-dimensional Fourier space with a one-dimensional logarithmically-spaced shell-space. We associate with each shell $n$ in this lattice, corresponding to a wavenumber $k_n = k_0 \lambda^n$, a dynamical, complex variable $u_n$ which mimics the velocity increments over a scale $r \sim 1/k_n$ in the Navier-Stokes equation. The actual structure of this shell-space is determined by the constant $k_0$ and $\lambda$. It is this logarithmic construction on a one-dimensional lattice and restricting the non-linear interactions to just the nearest and next-nearest neighbours which allows shell models to achieve extremely high Reynolds numbers—and hence inertial ranges—well beyond those possible through DNSs. Curiously, shell models seem to give very reliable measurements of the anomalous, due to intermittency, scaling exponents of structure functions and the energy spectrum; however not much is known about the dissipation statistics of such models. Thus such models have been used extensively in the past for problems which ranged from studies of static and dynamic multi-scaling in fluid, passive-scalar, binary fluids and magne-
to hydrodynamic turbulence \[63,71\], turbulent flows with polymer-additives and elastic turbulence \[72,74\] as well as the equilibrium solutions of such dynamical systems \[75-77\]. However, the application of such low-dimensional models for rotating turbulence is both sparse and fairly recent \[32,78\].

In order to avoid the injection of any mean helicity, we use a helical shell model which mimics the Navier-Stokes equation which is constructed by decomposing the velocity field in the basis corresponding to the eigenvectors of the curl operator \[79,80\]:

\[
\mathbf{u}(x) = \sum_{k} \mathbf{u}(k) \exp(ik \cdot x)
= \sum_{k} [u^{+}(k)h_{+}^{k} + u^{-}(k)h_{-}^{k}] \exp(ik \cdot x).
\]

(3)

Here \(u^{\pm}\) are the velocity components along the unit eigenvectors \(h_{\pm}\) of the curl operator \(i\mathbf{k} \times \mathbf{h}_{\pm} = \pm k \mathbf{h}_{\pm}\). Such a decomposition is adapted to a shell model framework to yield the following set of ordinary differential equations

\[
\frac{d}{dt}u_{m}^{\pm} = N_{m}^{\pm} - \nu \lambda_{m}^{2} u_{m}^{\pm} + \mathcal{F}_{m}^{\pm} - i\Omega u_{m}^{\mp},
\]

(4)

The non-linear terms \(N_{m}^{\pm}\) are defined as

\[
N_{m}^{\pm} = i \left[ ak_{n+1}u_{n+2}^{\mp}u_{n+1}^{\pm} + bk_{n}u_{n+1}^{\mp}u_{n-1}^{\pm} + ck_{n-1}u_{n-1}^{\mp}u_{n-2}^{\pm} \right]^{*},
\]

(5)

with real coefficients \(a, b\) and \(c\), the superscript * denoting a complex conjugate, and the effective velocity associated with each shell \(u_{n} = \sqrt{|u_{n}^{+}|^{2} + |u_{n}^{-}|^{2}}\). We set the coefficient \(a\) to unity and the coefficients \(b = -\lambda_{m}^{2} + \lambda_{m+1}^{2}\) and \(c = \lambda_{m+1}^{2} + \lambda_{m+1}^{2}\) to ensure the conservation \((\nu = 0)\) of energy \((a + b + c = 0)\) and helicity \((a + b\lambda - c\lambda^{2} = 0)\). The structure of the shell model allows an easy identification of the viscous dissipative term and a forcing term on the \(m^{th}\) shell with \(\mathcal{F}_{m}^{\pm} = c^{\pm}(1 + i)/(u_{m}^{\mp})^{*} ; c^{\pm}\) is the energy input rate to the modes \(u_{m}^{\mp}\) and we choose \(m = 3\). Finally, the last term mimics a Coriolis force and is made explicitly imaginary to ensure that it does not explicitly inject energy into the system.

In our simulations, we use a total of \(N = 32\) shells (with \(k_{0} = 1/16\) and \(\lambda = 1.62\), \(\nu = 10^{-7}\) \((Re \sim 10^{5})\), and for time-marching an exponential fourth-order Runge-Kutta scheme, with a time-step \(\delta t = 10^{-4}\), to factor in the stiffness of these coupled ordinary differential equations. Just like in our DNSs we initialise our velocity field \((u_{n}^{\pm} = \sqrt{k_{n}} \exp(\theta_{n}), for \(n \leq 4\) and \(u_{n}^{\pm} = \sqrt{k_{n}} \exp(-k_{n}^{2} \exp(\theta_{n})) for \(n \geq 5\), where \(\theta \in [0, 2\pi]\) is a random phase) and force the system to a statistically steady state before turning on the Coriolis term.

III. STRUCTURE FUNCTIONS IN ROTATING TURBULENCE

The Coriolis force plays a significant effect on the geometry and hence statistics of turbulence. Dimensionally, rotation sets an (inverse) time-scale in the problem resulting in a characteristic (Zeman) scale \(\Lambda_{T} \sim \sqrt{\nu/\Omega}\) (or wavenumber \(k_{0} \sim \nu^{-1/2}\)) where the rotational and fluid (eddy) turnover time-scales match \[81\]. For finitely small values of the Zeman scale (corresponding to \(Ro \approx 1\)), the two-point statistics, most usefully characterized by the (Fourier space) kinetic energy spectrum \(E(k) = |u(k)|^{2}\), shows a dual-cascade: For wavenumbers \(k < k_{0}\), \(E(k) \sim k^{-2}\) \[32,36,81,82\] and for \(k > k_{0}\), the usual Kolmogorov spectrum \(E(k) \sim k^{-5/3}\). The dual cascade energy spectrum phenomenology is central to theories of rotating turbulence. It is tempting to now ask if low-dimensional dynamical systems, which mimic the formal structure of the Navier-Stokes equations but are neither rigorously derived from them nor sensitive to the geometrical reorganisation of flows under rotation, show any evidence of this new rotation-induced scaling regime. Remarkably, simulations of the shell model—which is devoid of geometry but only respect the formal structure of the Navier-Stokes equation—shows the exact same scaling behaviour (for the shell model, the energy spectrum is

FIG. 1. Log-log plots of the compensated energy spectrum \(k_{n}^{5/3}E(\Lambda_{n})\) versus the wavenumber \(k_{n}\) for different Rossby numbers (see legend) from our simulations of the helical shell model. For \(Ro = \infty\), the plateau (over several decades and shown by the shaded region) confirms the Kolmogorov scaling \(E(k_{0}) \sim k_{0}^{-5/3}\) for non-rotating, homogeneous and isotropic turbulence. As the Rossby number decreases, corresponding to an increased level of rotation, the \(k_{n}^{-5/3}\)-compensated spectrum to the left of the Zeman scale (shown by vertical lines with colors corresponding to the respective \(Ro\) numbers) departs from the plateau with an additional scaling factor which asymptotes to \(k_{n}^{-1/3}\), and hence \(E(k_{n}) \sim k_{n}^{-2}\), as \(Ro \ll 1\) and for wavenumbers lower than the Zeman wavenumber.
$E(k_n) = |u(k_n)|^2 / k_n$ with an associated Zeman scale defined as above.

A convenient way to see the point and extent of departure from the Kolmogorov $k^{-5/3}$ scaling is to look at compensated spectrum $E(k_n) k_n^{5/3}$ plots (versus $k_n$) with different strengths of the Coriolis term. In Fig. 1, we present representative plots of this compensated spectra for a few values of $\Omega$. For clarity, we show by vertical lines, the Zeman wavenumber corresponding to different values of $\Omega$ and shade the inertial range which would have been present in the absence of rotation; for easy comparison we also show results from simulations with $\Omega = 0$. We immediately note the flatness of the compensated spectrum—before falling-off in the deep dissipation range—for all values of $\Omega$ as long as $k_n > k_\Omega$. This is in sharp contrast to the steeper slopes of the spectrum, as evidenced by the departure from the plateau, for finite rotation at scales $k_n < k_\Omega$ for any finite $\Omega$. In the limit of $\Omega \gg 1$, the compensated spectra reaches a slope $E(k_n) \sim k_n^{-1/3}$ (indicated by the black-solid line), for $k_n < k_\Omega$, corresponding to prediction of the secondary scaling regime $E(k) \sim k^{-2}$ for wavenumbers smaller than the Zeman wavenumber.

Experimental and numerical simulations also show that for scales $l > l_\Omega$, the equal-time, $q$-th order, longitudinal velocity structure functions $S_q(l) \equiv \langle \delta u \rangle^q$, where $\delta u \equiv u(x+l) - u(x) \cdot \hat{l}$ ($\hat{l}$ is the unit vector along $l$ and $l = |l|$), show a scaling behaviour $S_q(l) \sim l^n$, reminiscent of the standard phenomenology of three-dimensional non-rotating turbulent flows where $S_q(l) \sim l^{3 \delta u}$. The analogous definition for a shell model is $S_q(k_n) \equiv \langle |u_n|^q \rangle \sim k_n^{-\xi_q}$. If we ignore any intermittent, non-Gaussian effects in the probability distribution functions of the velocity increments $\delta u$, we obtain $\xi_q = q/2 \neq \xi_3 = q/3$. Actual measurements, as we also show below, suggests that for any finite Rossby number $\xi_q \neq q/2$ but a non-linear convex function of $q$. As $Ro \rightarrow 0$, the scaling exponents however are much closer to the dimensional prediction.

All of this is consistent with the phenomenology and dimensional predictions which ignore any corrections due to intermittency. Indeed it is well-known that intermittency corrections in the energy spectrum, which is essentially related to the second-order structure function through a Fourier transform, is notoriously hard to detect. Hence we must turn our attention to higher-order structure functions and calculate the scaling exponents $\xi_q$ (for $k_n < k_\Omega$), as defined before, for different values of $\Omega$.

In Fig. 2 we plot the equal-time scaling exponents $\xi_q$ as a function of $q$, from our shell model simulations, for different strengths of the Coriolis force. (For comparison, we also show the exponents $\alpha$ for non-rotation turbulence ($\Omega = 0$) and the associated black solid line indicating the $q/3$ prediction of Kolmogorov.) The black dashed line corresponds to the dimensional prediction $q/2$ and our measurements clearly show $\xi_q \neq q/2$, with the effect becoming more pronounced for $q > 3$. However, as the rotation rate increases, the scaling exponents tend to asymptote to values much more consistent with the dimensional prediction showing a strong depletion of intermittency effects even in our low-dimensional dynamical system. This is completely consistent with earlier studies which showed that strong rotation leads to a depletion of intermittency effects in turbulent flows. What is surprising is that this feature is faithfully reproduced in a shell model which is insensitive to any geometrical effects and the proliferation of columnar vortices in real flows or solutions of the Navier-Stokes equation.

IV. ENERGY DISSIPATION RATE

The three-dimensional Navier-Stokes equation is known to invariant under suitable scaling transformations with a scaling exponent $h$ which allows us to write the (scalar) velocity increments $\delta u$ across a scale $r$ as $\delta u \sim r^h$. Phenomenologically, the scale-dependent mean kinetic energy dissipation rate $c(r) \sim \delta u^2 \sim r^{\alpha-1}$, where $\alpha = 3h$. For homogeneous and isotropic turbulence, Kolmogorov theory, in the absence of intermittency or multifractal statistics, predicts $h = 1/3$ which ensures, in the inertial range, a scale-independent dissipation rate equal to the constant energy flux across these scales. Real
turbulence, though, is multifractal. Thus the dissipation field cannot be characterized by a unique choice of α but rather by its singularity spectrum \( f(\alpha) \) and the mass function of Renyi dimension \( \tau(q) \), both of which we define precisely later.

In a three-dimensional flow such as the one we obtain from our DNSs the local energy dissipation rate

\[
\varepsilon(x) = \frac{\nu}{2} \sum_{i,j} (\partial_i u_j + \partial_j u_i)^2, \tag{6}
\]

is a function of all three spatial directions. Therefore, for our DNSs, a convenient way to carry out a multifractal analysis of such fields is to take several one-dimensional (1D) cuts of \( \varepsilon(x) \) parallel and perpendicular to the direction of rotation \((z-\text{axis})\). These 1D cuts along the axis of rotation yield \( \varepsilon(z) = \varepsilon(x_0, y_0, z) \); similarly, for the plane perpendicular to the axis of rotation, we obtain \( \varepsilon(x) = \varepsilon(x, y_0, z_0) \) and \( \varepsilon(y) = \varepsilon(x_0, y, z_0) \). For reliable statistics, we choose 49 cuts along each direction with different values of \( x_0, y_0 \) and \( z_0 \) lying in the interval \([\pi/4, 3\pi/4]\).

In Figs. 3(a) and (b) we show representative plots of one such cut for the reconstructed \( \varepsilon(x) \) and \( \varepsilon(z) \), respectively, normalised by the global mean, at a single instant of time for the non-rotating case and one with Rossby number \( Ro = 0.001 \) which illustrates that highly intermittent nature of the dissipation field persists even for such 1D cuts.

For shell models, given its lack of spatial structure, obtaining the analogue of such a one-dimensional dissipation field amenable to a multifractal analysis is less obvious. Let us nevertheless assume that the shell model describes the flow in a spatial domain of size \( L \) and that the energy dissipation rate can be defined at any spatial position \( x \in [0, L] \). Thus, keeping in mind the Richardson picture of energy cascade, it is natural to assume that beginning with the largest eddy of size \( L \), an energy cascade is set up in the system such that each eddy in a given generation \( m \) of the cascade breaks up into 2 to provide the eddies of the next generation. This suggests a hierarchical transfer of energy, scale-by-scale, such that at each scale \( m \in \{0, 1, 2, \cdots, K\} \), the number of eddies is \( 2^m \) and the typical size of each eddy is \( l_m = L/2^m \sim 1/k_m \) (since the wave-numbers in a shell model correspond to the inverse of the spatial scales). The smallest scales, set by \( K \), correspond to the viscosity-dominated Kolmogorov scale of the flow. Let us now focus on the \( i \)-th eddy (of size \( l_m = L/2^m \sim 1/k_m \)) out of the \( 2^m \) eddies of generation \( m \). Denoting the energy of this eddy by \( E_i^{(m)} \), the total energy at scale \( l_m \) must correspond to the kinetic-energy of the \( m \)-th shell in our shell model:

\[
|u_m|^2 = 2 \sum_{i=1}^{2^m} E_i^{(m)} \quad \text{with an associated energy density} \quad E_i^{(m)}/l_m \quad \text{in the \( i \)-th eddy.}
\]

Thus we adapt the multifractal cascade ideas of Meneveau and Sreenivasan [89] and construct it for our shell model. Furthermore, we choose different fractions \( p \in (0.5, 0.9) \) of the energy distribution amongst the daughter eddies and find that our results are qualitatively insensitive to the particular choice of \( p \); in this paper we report results for \( p = 0.7 \).

With these definitions, following Lepreti et al. [88], the kinetic energy density \( \varepsilon(x) \) at a spatial location \( x \in [0, L] \) is given by the contributions from eddies of all scales which have their imprints on a specific point \( x: \varepsilon(x) = \sum_{m=0}^{K} E_{s_m(x)}/l_m \) where \( s_m(x) := [(x - 1)/l_m] \) and thence the one-dimensional energy dissipation rate

\[
\varepsilon(x) = 2\nu \sum_{m=0}^{K} k_m^2 E_{s_m(x)}/l_m \tag{7}
\]

in the shell model. We refer to the reader to Ref. [88] for a detailed description on how \( \varepsilon(x) \) is evaluated in the shell model at any given instant of time from the knowledge of the energy content of the eddies in the previous time step. In our calculations, we choose the largest length scale \( L \) as the one associated with the forcing shell, i.e., \( n = 3 \), and \( K = 23 \) to obtain a grid resolution \( L/2^{20} \). Finally, to obtain reliable statistics, we extract the spatial distribution \( \varepsilon(x) \) from 100 different, statistically-independent velocity configurations in the steady state.

In Fig. 3(c), we show a representative plot of \( \varepsilon(x) \) obtained from our shell model data as described above.
clear though that given the much higher Reynolds number in the non-rotating flow and one with $Ro = 0.006$. Fig. 3(c) suggests that the behaviour of a temporal trace of the energy dissipation rate, at any given instant in time, obtained from the low-dimensional model is fairly consistent with what is seen from in panels (a) and (b) in the same figure but obtained from DNSs. It is clear though that given the much higher Reynolds number that our shell model achieves, the intensity of the intermittent peaks in the dissipation rate in Fig. 3(c) are much stronger than what is seen in Figs. 3(a) and (b). Furthermore, when we compare the cuts of these dissipation rates for the rotating and non-rotating cases, we do see a suggestion—evidenced by the relatively calmer traces of $\varepsilon$—that intermittency is suppressed (along with the degree of multifractality) as soon as we have a small enough Rossby number. However, this visual evidence is
hardly compelling and in order to substantiate our claim, we must resort to a more quantitative characterisation of this phenomenon through a full multifractal analysis.

V. MULTIFRACTAL ANALYSIS

We set the stage for this analysis by defining, through the 1D cuts of the dissipation field \( \varepsilon(x) \), a scale-dependent energy dissipation \( \varepsilon_r \equiv \int_{x \in I_r} \varepsilon(x)dx \) integrated in an interval \( I_r \) of size \( r \). If we choose the interval all the way up to the integral scale \( L \), this gives a reference scale-dependent dissipation \( \varepsilon_L \) which allows us to define the Rényi scaling exponent \( \tau_q \) via

\[
\langle \varepsilon^q \rangle \sim \varepsilon_L^q \left( \frac{r}{L} \right)^{\tau_q}
\]  

(8)

for \( L \gg r \gg \eta \). By using this exponent, we can define the generalised dimension \( D_q = \tau_q/(q-1) \) and the multifractal singularity spectrum through a Legendre transform of \( \tau_q \) as \( f(\alpha) = \min_{q}(q\alpha - \tau_q) \) where \( \alpha = d\tau_q/dq \). As is traditional in this field, we characterize intermittency through the exponent \( \mu = -(d^2\tau_q/dq^2)_{q=0} \) \((\mu \approx 0.26\) in non-rotating case\) and the width of the singularity spectrum \( \Delta\alpha = \alpha_{\text{max}} - \alpha_{\text{min}} \) where \( \alpha_{\text{min}} \) and \( \alpha_{\text{max}} \) are the strongest and weakest singularities respectively.

Fig. 4 summarises our results from the multifractal analysis of the dissipation rates obtained from DNSs. In panels (a) and (d) of Fig. 4 we show plots of the generalised dimensions as a function of \( q \) in the planes perpendicular and parallel to the axis of rotation, respectively, for different degrees of rotations. We note that these results, in the no-rotation limit, are consistent with the results obtained earlier for homogeneous and isotropic turbulence. With decreasing Rossby numbers, these curves progressively flatten as a first indicator of decreasing intermittency. This is confirmed in panels (b), for the plane perpendicular and in (c), for planes parallel to the axis of rotation, where we show the singularity spectrum \( f(\alpha) \) and the widths \( \Delta\alpha \) as insets. With increasing rotation, the spectrum narrows as quantified by \( \Delta\alpha \) which shows a monotonic decrease with \( \text{Ro} \). Finally, we calculate directly the exponents \( \tau_q \) (as a function of \( q \)), again for planes perpendicular (panel (e)) and parallel (panel (f)) to the rotation axis. Our results show a clear decrease in the curvature of \( \tau_q \), with decreasing Rossby numbers, which is quantified more carefully by the measurement of the intermittency exponents \( \mu \), as shown in the insets of panels (c) and (f).

Some of the understanding of the properties of a rotating turbulent flow stems from the geometrical reorganisation of three-dimensional flows. Shell models by definition should be insensitive to such effects. Before we engage in an interpretation of the results summarised in Fig. 4 it is useful to ask if low-dimensional models replicate these features. In Fig. 6 we show results obtained from our shell model studies for (a) the generalised dimension \( D_q \), (b) the singularity spectrum \( f(\alpha) \) (with the widths \( \Delta\alpha \) shown in the inset), and (c) the exponents \( \tau_q \) vs \( q \), with the intermittency exponent \( \mu \) vs \( \text{Ro} \) in the inset. Remarkably, a comparison of Figs. 4 and 5, and consequently intermittent behaviour, seems to weaken with increased degrees of rotation. This is most evident, e.g., in plots of \( \tau_q \) vs \( q \) (Figs. 4(c) and (f) as well as 5(c)) which show a progressively linear trend as \( \text{Ro} \ll 1 \) along with a decrease in the values of the intermittency exponent \( \mu \) (insets in the same figures).

To summarise, our multifractal analysis suggests that the lack of self-similarity in the statistics of the dissipation rate as measured through the generalised dimension \( D_q \), the singularity spectrum \( f(\alpha) \) or the exponent \( \tau_q \), seems to weaken with increased degrees of rotation. This is most evident, e.g., in plots of \( \tau_q \) vs \( q \) (Figs. 4(c) and (f)) which show a progressively linear trend as \( \text{Ro} \ll 1 \) along with a decrease in the values of the intermittency exponent \( \mu \) (insets in the same figures).

All of this brings us to the central question that we address in this work. For rotating turbulence (especially in the limit \( \text{Ro} \ll 1 \)), is there a way to bridge the statistics of the dissipation rates, characterised by generalised dimension \( D_q \) with the (anomalous) inertial range scaling exponents \( \xi_q \)? It is useful to recall that for non-rotating flow, such a relationship \( \xi_q = q/3+(q/3-1)(D_q/3-1) \) exists. (We have verified this from the data obtained from our simulations of the non-rotating shell model.)

In the same spirit, we make the following ansatz when \( \text{Ro} \neq \infty \):

\[
\xi_q = hq + (hq - 1)(D_{hq} - 1)
\]

(9)

which can be re-arranged in the more useful form (for what is to follow):

\[
\frac{\xi_q - 1}{D_{hq}} + 1 = hq.
\]

(10)

(For \( \text{Ro} = \infty \) or \( \Omega = 0 \), the scaling exponent \( h = 1/3 \).)

Combining Eq. (10) with data obtained from our numerical simulations for scales larger than the Zeman
scale, we obtain (see Fig. 6) the scaling exponent $h$ as a function of $Ro$. Remarkably, and as we would expect both from phenomenology as well as from Fig. 1 as $Ro \to 0$, the exponent $h \to 1/2$. This leads us to conjecture, in the limit $Ro \to 0$, the following relation bridging the dissipation and inertial range statistics:

$$\xi_q = q/2 + (q/2 - 1)(D_{q/2} - 1); \quad Ro \to 0. \quad (11)$$

Furthermore, this relationship shows how the emergence of an approximate self-similarity ($\Delta \alpha \to 0$ and a $q$-independent generalised dimension) as $Ro \to 0$, leads to a progressively simple scaling (and not multi-scaling) of the exponents of the equal-time structure functions as shown in Fig. 2.

VI. CONCLUSION

Eqn. (11), thus captures what, in our opinion, is the central result in this study by bridging, in the limit of strongly rotating ($Ro \to 0$) turbulence, the statistics of the energy dissipation rate characterised by the generalised dimension $D_q$ with the scaling exponents $\xi_q$ of the moments of velocity increments over scales $r$ in the inertial range. Furthermore our analysis provides a more complete description of the depletion of intermittency in flows affected strongly by a Coriolis force. As has been stressed in our paper, we also find remarkable agreement in the multifractal analysis of the data obtained from our direct numerical simulations to those obtained from the low-dimensional shell model (albeit with a higher Reynolds number) that, by construction, is insensitive to the spatial reorganisation of the flow due to rotation. Indeed, our work shows that following Lepreti et al. [88] (see also Meneveau et al. [90]), it is possible to extract a spatial profile of the dissipation rates in such shell models that can be used for future studies of the small-scale statistics of different forms of turbulence for whom such a low-dimensional dynamical systems representation exists.

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