Extending the Best Linear Approximation Framework to the Process Noise Case

Maarten Schoukens, Member, IEEE, Rik Pintelon, Fellow, IEEE,
Tadeusz P. Dobrowiecki, Fellow, IEEE, and Johan Schoukens, Fellow, IEEE

Abstract—The best linear approximation (BLA) framework has already proven to be a valuable tool to analyze nonlinear systems and to start the nonlinear modeling process. The existing BLA framework is limited to systems with additive (colored) noise at the output. Such a noise framework is a simplified representation of reality. Process noise can play an important role in many real-life applications. This paper generalizes the best linear approximation framework to account also for the process noise, both for the open-loop and the closed-loop setting, and shows that the most important properties of the existing BLA framework remain valid. The impact of the process noise contributions on the robust BLA estimation method is also analyzed.

Index Terms—Best linear approximation (BLA), nonlinear systems, process noise, system identification.

I. INTRODUCTION

A linear approximate model of a nonlinear system often offers valuable insight into the linear (but also nonlinear) behavior of that system. The best linear approximation (BLA) framework described in [4], [6], [15], [13], [19], and [20] offers such a well-understood and valuable approximation framework for a wide class of practically important signals and systems (see in detail in Section II). The BLA framework is often used to analyze how nonlinearly a system behaves (see for instance [26] for mechanical systems and [27] for biomechanical systems), to guide the user to select a good nonlinear model structure [22], to obtain linear models in the presence of nonlinearities [15], but also to start the estimation process of a nonlinear model [12], [24], [25]. A recent overview of the BLA framework, and its use in practical applications (e.g., ground vibration testing, combustion engine, robotics, and electronics), is provided in [23].

The BLA framework was initially defined for systems operating in open loop and with additive (colored) noise present at the output only. The extension toward the closed-loop setting has been made in [14] and [16]. The extension of the BLA framework to include process noise is the subject of this paper.

Considering additive (colored) noise at the output only is a simplified representation of reality. This simplification can lead to biased nonlinear model estimates when other noise sources are present, located at other positions inside the system, e.g., process noise passing through a nonlinear subsystem [9]. A more realistic noise framework can be obtained by introducing multiple noise sources, or by placing the noise source at another location in the model structure. Over the last years, quite some effort has been invested in developing nonlinear identification frameworks that offer a tractable handling of the presence of nonlinear process noise sources, e.g., [11], [7], [18]. Nevertheless, the inclusion of process noise sources in the nonlinear model significantly increases the challenges present in the identification process.

It is important to offer the user a theoretical framework with which (s)he can analyze the influence of process noise on the system, and whether or not it is important to include process noise in the nonlinear modeling step. An extended BLA framework could offer this theoretical insight. An initial step toward a generalized BLA analysis is made in [8] where the BLA of a Wiener–Hammerstein system is analyzed in the presence of process noise. The presence of process noise in a nonlinear system can be analyzed and detected, different from the BLA framework, by using nonstationary input signals [31]. However, this approach cannot quantify the level of nonlinearity of the system under test.

This paper introduces first the classical open-loop BLA framework with additive noise at the system output only (see Section II). The extension toward systems with process noise is presented in Section III. It is shown in Section IV that this extension is also valid for systems operating
II. BLA: ADDITIVE NOISE AT THE SYSTEM OUTPUT

A. Signal Class

This paper assumes the input signal \( u(t) \) (see Fig. 1) to belong to the Gaussian signal class, this signal class can be generalized further to the Riemann equivalence class of asymptotically normally distributed excitation signals [15], [20]. This generalization leads to exactly the same theoretical conclusions as for the Gaussian signals, but is omitted here for simplicity (see for instance [15] and the results obtained with the generalized expression for the auto- and cross-correlation for quasi-stationary signals as defined in [10]). The Riemann equivalence class of asymptotically normally distributed excitation signals bridges the Gaussian signal class and the random phase multisine signal class. Therefore, it is also of importance in Sections V and VI of this paper.

Note that the BLA can also be defined for other signal classes. This choice has been made here to follow the framework defined in [15], and to use the same class of systems. However, many of the properties that are derived in this paper are not limited to the chosen input signal class. The impact of the input signal class is studied in [30].

Definition 1: A signal \( u(t) \) belongs to the signal class \( U \) when \( u(t) \) is a Gaussian process with power spectral density \( S_{UU}(j\omega) \), and \( S_{UU}(j\omega) \) is uniformly bounded by \( M_U \) (0 \( \leq \) \( S_{UU}(j\omega) \) \( \leq \) \( M_U \) \( < \) \( \infty \)) for all frequencies \( j\omega \).

B. System Class and Noise Framework

It is assumed that the nonlinear system output can be represented arbitrarily well in the least-squares sense by a fading memory Volterra kernel representation [2], [17]. This system class contains, for instance, systems with a hard saturation nonlinearity, but does not contain bifurcating and chaotic systems. The generality of this system class is discussed in detail by [2].

The Volterra model output consists of the sum of the outputs of the kernels of different degree. The output of a Volterra kernel of degree \( \alpha \) is given in the time domain by

\[
y_\alpha(t) = V_\alpha(u(t)) = \sum_{k_1, \ldots, k_\alpha=0}^{\infty} h_{\alpha}(k_1, \ldots, k_\alpha)u(t-k_1) \cdots u(t-k_\alpha)
\]  

(1)

which results in the following frequency domain representation

\[
Y_\alpha(j\omega_k) = \frac{1}{N^{N/2}} \sum_{k_1, k_2, \ldots, k_{\alpha-1} = -N/2+1}^{N/2-1} H_{\alpha, k_1, k_2, \ldots, k_{\alpha-1}}^\alpha U(j\omega_{k_1})U(j\omega_{k_2}) \cdots U(j\omega_{k_{\alpha-1}})
\]

(2)

where \( L_k = k - k_1 - k_2 - \cdots - k_{\alpha-1} \), \( H_{L_k, k_1, k_2, \ldots, k_{\alpha-1}}^\alpha \) is a symmetrized frequency domain representation of the Volterra kernel \( h_{\alpha} \) of degree \( \alpha \) [15], [17]. \( Y_\alpha(j\omega_k) \) is obtained as the discrete Fourier transform (DFT) of \( y_\alpha(t) \)

\[
y_\alpha(t) = \frac{1}{N^{N/2}} \sum_{k=0}^{N-1} Y_\alpha(j\omega_k) e^{j2\pi tk/N}
\]

(3)

\[
y_\alpha(t) = \frac{1}{N^{N/2}} \sum_{k=0}^{N-1} Y_\alpha(j\omega_k) e^{j2\pi tk/N}
\]

(4)

with \( \omega_k = \frac{2\pi k}{N} \), where \( k \in \{0, 1, \ldots, N - 1\} \).

Definition 2: \( S \) is the class of nonlinear systems specified in (1) and (2) such that, when excited by a signal belonging to the signal class \( U \)

\[
\exists C_1 < \infty, \text{s.t.} \sum_{\alpha=1}^{\infty} M_{G^\alpha} M_U^{\alpha} \leq C_1
\]

(5)

with \( M_{G^\alpha} = \max \left| H_{L_k, k_1, k_2, \ldots, k_{\alpha-1}}^\alpha \right| \), where the maximum is taken over the indices \( L_k, k_1, k_2, \ldots, k_{\alpha-1} \).

Definition 2 postulates the existence of a uniformly bounded fading memory Volterra series whose output converges in least-squares sense to the output of the nonlinear system belonging to the system class \( S \) (see [15] for more detail) when the highest degree \( \alpha \) tends to infinity.

The existence of the cross- and autopower spectral densities \( S_{YU} \) and \( S_{UU} \) is guaranteed by Definition 2 combined with Definition 1. They are obtained by

\[
S_{YU}(j\omega) = \lim_{N \to \infty} E\{Y(j\omega)U^*(j\omega)\}
\]

(6)

\[
S_{UU}(j\omega) = \lim_{N \to \infty} E\{U(j\omega)U^*(j\omega)\}
\]

(7)

for \( \omega = 0, 2\pi \frac{1}{N}, \ldots, 2\pi \frac{N-1}{N}, \) and where \( U^* \) denotes the complex conjugate of \( U \). The limit for \( N \) tending to infinity is taken such that the frequency of interest \( \omega_k \) is always part of the frequency grid, e.g., for \( \omega_k = 2\pi \frac{k}{N} \), with \( k \in \{0, 1, \ldots, N - 1\} \), \( N \) being a multiple of \( N_c \). The limit accounts for the leakage terms that are present in the spectra \( Y(j\omega) \) and \( U(j\omega) \). They will be disregarded in the remainder of this paper.

Assumption 1. Output noise framework: An additive, colored zero-mean noise source \( n_y(t) \) with a finite variance \( \sigma_y^2 \) is
present at the output of the system (see Fig. 1)
\[ y(t) = y_0(t) + n_y(t). \]  
This noise \( n_y(t) \) is assumed to be independent of the known input \( u(t) \). \( y(t) \) is the actual measured output signal and a subscript 0 denotes its exact (noise-free) value.

### C. Definition of the BLA

The BLA model of a nonlinear system belonging to system class \( S \) with zero-mean (colored) additive noise at the system output only (see Fig. 1) is a linear time-invariant (LTI) approximation of the behavior of that system. The BLA is defined in [13], [15], [19], and [20] as
\[
G_{bla}(q) = \arg \min_{G(q)} \mathbb{E}_{u,n_y} \left\{ |\tilde{y}(t) - G(q)\tilde{u}(t)|^2 \right\} 
\]
\[
\tilde{u}(t) = u(t) - \mathbb{E}_u \{ u(t) \} 
\]
\[
\tilde{y}(t) = y(t) - \mathbb{E}_{u,n_y} \{ y(t) \} 
\]
where \( \mathbb{E}_{u,n_y} \{ \cdot \} \) denotes the expected value operator taken w.r.t. the random variations due to the input \( u(t) \) and the output noise \( n_y(t) \). Note that this simplified notation for the expected value will be used throughout this paper. This notation can be adopted without confusion since all the random variables over which the expectation is taken are assumed to be independent. Finally, \( G(q) \) belongs to the set of all possible LTI systems. This definition of the BLA is equivalent to the definition of the LTI second-order equivalent model defined in [4], [6], and [11] when the stability and causality restrictions imposed there are omitted.

It is shown that the BLA is given by [4], [6], and [15]
\[
G_{bla}(j\omega) = \frac{S_{YU}(j\omega)}{S_{UU}(j\omega)} 
\]
where \( S_{YU}(j\omega) \) is the cross-power spectrum of \( u(t) \) and \( y(t) \) and \( S_{UU}(j\omega) \) is the autocorrelation of \( u(t) \). Hence, the existence of \( G_{bla}(j\omega) \) is guaranteed if \( S_{UU}(j\omega) \) and \( S_{YU}(j\omega) \) exist, and \( S_{UU}(j\omega) \neq 0 \). The existence of \( S_{YU}(j\omega) \) is guaranteed by the chosen signal (Definition 1) and system class (Definition 2) [15], [17]. The BLA is not defined at the frequencies where \( S_{UU}(j\omega) = 0 \) [4], [15]. A (possibly infinite order, noncausal) transfer function or impulse response representation \( G_{bla}(q) \) can be obtained by fitting \( G_{bla}(j\omega) \) at the excited frequencies.

Three constituents of the BLA framework can be defined (see Fig. 2): the BLA \( G_{bla}(q) \) itself, the stochastic nonlinear contribution \( y_s(t) \) and the noise contribution \( y_n(t) \). The output residuals \( y_i(t) \) represent the total distortion that is present at the output of the system. The total distortion can be split in two contributions based on their nature as is depicted in Fig. 2. The stochastic nonlinear contribution \( y_s(t) \) represents the unmodeled nonlinear contributions, while the noise contribution \( y_n(t) \) is the additive noise that is present at the system output
\[
y_i(t) = \tilde{y}(t) - G_{bla}(q)\tilde{u}(t) = y_s(t) + y_n(t) 
\]
\[
y_s(t) = \tilde{y}(t) - G_{bla}(q)\tilde{u}(t) 
\]
\[
y_n(t) = \tilde{y}(t) - \tilde{y}(t) = n_y(t) 
\]
where \( \tilde{y}(t) = \mathbb{E}_{n_y} \{ \tilde{y}(t) \} \) is the unknown zero-mean noiseless output. The nonlinear distortion \( y_s(t) \) is linearly uncorrelated with the input \( \tilde{u}(t) \) (\( \mathbb{E}_u \{ y_s(t)\tilde{u}(t) \} = 0 \) \( \forall t, \tau \)). The nonlinear distortion \( y_s(t) \) is not independent of the input \( \tilde{u}(t) \); however, [15]. The noise distortion \( y_n(t) \) on the contrary is both uncorrelated with the input and independent of the input \( \tilde{u}(t) \). All three signals \( \tilde{y}_{bla}(t), y_s(t), \) and \( y_n(t) \) are zero-mean.

### III. BLA: Process Noise Extension

#### A. Considered System Class and Noise Framework

The considered excitation signal class and the output noise assumptions are unchanged with respect to Section II. The considered system class is extended here to include process noise, and the necessary assumptions on the process noise are formulated.

It is assumed in Section II-B that the underlying nonlinear system is a Volterra system. Here, we extend the standard single-input single-output Volterra kernel representation to a dual-input single-output representation where one of the inputs is excited by the process noise (see Fig. 3). The modeling of the process noise as the second input to the Volterra system describing the measured nonlinear system opens considerable opportunities for a unified treatment of many process noise configurations. The output of a dual-input \( m+n \) th order Volterra kernel with process noise is given by
\[
y_{m,n}(t) = V_{m,n}(u(t), n_x(t)) 
\]
\[
= \sum_{k_1,\ldots,k_m, j_1, \ldots, j_n=0}^{\infty} h_{m,n}(k_1, \ldots, k_m, j_1, \ldots, j_n) 
\]
\[
u(t-k_1) \ldots u(t-k_m) n_x(t-j_1) \ldots n_x(t-j_n). 
\]
\[
A. \text{ Assumption 2. Process noise framework: A colored zero-mean Gaussian noise source } n_x(t) \text{ is present as one of the inputs of the dual-input single-output Volterra representation.}
\]
of the nonlinear system

\[ y_n(t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} V_{m,n}(u(t), n_x(t)). \]  

(17)

The noise \( n_x(t) \) is assumed to be independent of the known input \( u(t) \) and the output noise \( n_y(t) \), its \( n \)th order moments are finite \( \forall n \in \mathbb{N} \).

**Theorem 1:** If \( y_m,n(t) = V_{m,n}(u(t), n_x(t)) \), then \( \bar{y}_{m,n}(t) = \tilde{V}_{m,n}(u(t)) \), where

\[ \bar{y}_{m,n}(t) = E_n_{\text{bla}} \{ y_{m,n}(t) \} \]  

(18)

**Proof:** The signal \( \bar{y}_{m,n}(t) \) is obtained by taking the expectation of (16) w.r.t. the process noise \( n_x \)

\[ \bar{y}_{m,n}(t) = E_n_{\text{bla}} \{ y_{m,n}(t) \} \]

\[ = \sum_{k_1, \ldots, k_m = 0}^{\infty} \sum_{j_1, \ldots, j_n = 0}^{\infty} h_{m,n}(k_1, \ldots, k_m, j_1, \ldots, j_n) \]

\[ u(t-k_1) \ldots u(t-k_m) E_{n_x} \{ n_x(t-j_1) \ldots n_x(t-j_n) \} \]  

(20)

where the expectation \( E_{n_x} \{ n_x(t-j_1) \ldots n_x(t-j_n) \} \) depends on the \( n \)th order moment of \( n_x(t) \) and its decomposition into pairwise autocorrelations [17]. This can be simplified using (20)

\[ \bar{y}_{m,n}(t) = \sum_{k_1, \ldots, k_m = 0}^{\infty} \bar{h}_{m,n}(k_1, \ldots, k_m, j_1, \ldots, j_n) \]

\[ u(t-k_1) \ldots u(t-k_m) \]  

(21)

The sum in (20) is finite since \( h_{m,n}(k_1, \ldots, k_m, j_1, \ldots, j_n) \) is finite and it is assumed in Assumption 2 that the \( n \)th order moments of the process noise \( n_x(t) \) are finite.

This theorem shows that the relation between \( u \) and \( y \) is given by an single input single output Volterra representation after taking the expectation of \( y \) with respect to the random realization of the process noise. These results allow us to formulate the following definition.

**Definition 3:** \( \mathcal{S}_p \) is the class of nonlinear systems specified in (16) such that, when excited by a signal belonging to the signal class \( \mathcal{U} \), the following inequality holds

\[ \exists C_1 < \infty, \text{s.t.} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} M_{G_{m,n}} M_{U}^{m,n} \leq C_1 \]  

(24)

![Fig. 4. BLA of a nonlinear system for a given class of excitation signals](image)

The BLA framework in the presence of process noise is defined in this section. The framework consists of four model components: the BLA \( G_{bla}(q) \) itself, the stochastic nonlinear distortion \( y_s(t) \) due to the randomized input, the process noise contribution \( y_p(t) \) due to the process noise, and the output noise contribution \( y_n(t) \) as depicted in Fig. 4. Note that the process noise contribution \( y_p \) is a new constituent of the extended BLA framework due to the presence of process noise in the system.

Two important design decisions are made in defining the extended BLA framework.

1) The BLA \( G_{bla}(q) \) and the stochastic nonlinear distortion \( y_s \) are defined so that they do not depend on the actual realization of \( n_x \) and \( n_y \).

2) The process noise contribution \( y_p \) is defined so that it does not depend on the actual realization of the output noise \( n_y \).

The BLA and the stochastic distortions \( y_s \) are defined as follows.

**Definition 4:** The BLA in the presence of process noise is defined as

\[ G_{bla}(q) = \arg \min_{G(q)} E_{u,n_x,n_y} \{ |\tilde{y}(t) - G(q)\tilde{u}(t)|^2 \} \]  

(25)

\[ y_s(t) = \tilde{y}(t) - y_{bla}(t) = \tilde{y}(t) - G_{bla}(q)\tilde{u}(t) \]  

(26)

where \( G(q) \) belongs to the set of all possible LTI systems, \( y_{bla}(t) = G_{bla}(q)\tilde{u}(t) \), and \( \tilde{y}(t) \) is defined as

\[ \tilde{y}(t) = E_{u,n_x} \{ y(t) \} \]  

(27)

\( \tilde{u}(t) \) and \( \tilde{y}(t) \) are now defined as

\[ \tilde{u}(t) = u(t) - E_u \{ u(t) \} \]  

(28)

\[ \tilde{y}(t) = y(t) - E_{u,n_x,n_y} \{ y(t) \}. \]  

(29)

**Definition 5:** The process noise contribution \( y_p \) is defined as

\[ y_p(t) = \tilde{y}(t) - \tilde{y}(t) \]  

(30)

\[ = y(t) - G_{bla}(q)\tilde{u}(t) - y_s(t) \]  

(31)
where \( \tilde{y}(t) \) is defined as
\[
\tilde{y}(t) = E_{n_y} \{ \tilde{y}(t) \} = \bar{y}_0(t).
\]

The output noise contribution \( y_n(t) \), the final constituent of the BLA framework, remains to be defined
\[
y_n(t) = \tilde{y}(t) - y(t) = \tilde{y}(t) - \bar{y}_0(t) = \tilde{y}(t) - G_{bla}(q)\tilde{u}(t) - y_s(t) - y_p(t) = n_y(t).
\]

Note that, by definition, the noise contribution \( y_n(t) = n_y(t) \). However, other choices for the definition of \( y_n(t) \) may be made, resulting in different expressions of the noise contribution. This is highlighted in the last paragraph of Section VI.

To conclude we have that the system output \( \tilde{y}(t) \), as shown in Fig. 4, is given by
\[
\tilde{y}(t) = y_{bla}(t) + y_s(t) \rightarrow E_{n_x, n_y} \{ \tilde{y}(t) \} + y_p(t) \rightarrow E_{n_x} \{ \tilde{y}(t) \} - E_{n_x, n_y} \{ \tilde{y}(t) \} + y_n(t) \rightarrow \tilde{y}(t) - E_{n_y} \{ \tilde{y}(t) \}.
\]

Section V presents how the BLA, and the variances of the signals \( y_p(t) + y_n(t) \) and \( y_s(t) + y_p(t) + y_n(t) \) can be estimated using the robust BLA estimation approach.

C. Properties of the BLA Model Components

This section shows that the stochastic nonlinear distortion \( y_s(t) \) and the process noise contribution \( y_p(t) \) are zero-mean, and linearly uncorrelated with—but not independent of—the input \( \tilde{u}(t) \).

**Theorem 2:** Properties of the stochastic nonlinear distortion and the process noise contribution.

1) The stochastic nonlinear distortion \( y_s(t) \) has zero-mean and is linearly uncorrelated with \( \tilde{u}(t) \)
\[
E_u \{ y_s(t) \} = 0 \quad (39)
\]
\[
E_u \{ y_s(t)\tilde{u}(\tau) \} = 0 \quad \forall \ t, \tau. \quad (40)
\]

2) The process noise contribution \( y_p(t) \) has zero-mean and is linearly uncorrelated with \( \tilde{u}(t) \)
\[
E_{n_x} \{ y_p(t) \} = 0 \quad (41)
\]
\[
E_{n_x} \{ y_p(t)\tilde{u}(\tau) \} = 0 \quad \forall \ t, \tau. \quad (42)
\]

3) The sum of the process noise contribution \( y_p(t) \) and the stochastic nonlinear contribution is uncorrelated with \( \tilde{u}(t) \)
\[
E_{u, n_x} \{ (y_p(t) + y_s(t))\tilde{u}(\tau) \} = 0 \quad \forall \ t, \tau \quad (43)
\]

**Proof:** The expected value of the stochastic nonlinear distortion \( y_s(t) \) with respect to the input signal realization is given by
\[
E_u \{ y_s(t) \} = E_u \{ \tilde{y}(t) \} - E_u \{ G_{bla}(q)\tilde{u}(t) \}.
\]

The second term is equal to zero since \( \tilde{u}(t) \) is zero-mean by construction. The first term is given by
\[
E_u \{ \tilde{y}(t) \} = E_{u, n_x, n_y} \{ \tilde{y}(t) \}
\]
where \( E_u \{ \tilde{y}(t) \} \) is zero by construction. It, hence, follows directly from the definition of \( y_s(t) \) that \( E_u \{ y_s(t) \} = 0 \).

The stochastic nonlinear distortion \( y_s(t) \) is the residual of a linear least squares fit of an LTI model between \( \tilde{y}(t) \) and \( \tilde{u}(t) \) [see (26)]. Hence, \( y_s(t) \) is linearly uncorrelated with \( \tilde{u}(t) \) by construction in the absence of model errors (\( G(q) \) belongs to the set of all possible LTI systems).

The process noise contribution \( y_p(t) \) is given by
\[
y_p(t) = \tilde{y}(t) - y(t) = E_u \{ \tilde{y}(t) \} - E_{n_x, n_y} \{ \tilde{y}(t) \}
\]

The expected value \( E_{n_x} \{ y_p(t) \} \) taken over the process noise realization is, thus, given by
\[
E_{n_x} \{ y_p(t) \} = E_{n_x, n_y} \{ y_p(t) \} - E_{n_x, n_y} \{ \tilde{y}(t) \} = 0. \quad (46)
\]

The input \( \tilde{u} \) does not depend on the process noise \( n_x \) by construction and \( E_{n_x} \{ y_p(t) \} = 0 \) as is shown above. This results in
\[
E_{n_x} \{ y_p(t)\tilde{u}(\tau) \} = E_{n_x} \{ y_p(t) \} \tilde{u}(\tau) = 0. \quad (48)
\]

It follows directly from the proof mentioned above that
\[
E_{u, n_x, n_y} \{ (y_p(t) + y_s(t))\tilde{u}(\tau) \} = 0 \quad \forall \ t, \tau. \quad \blacksquare
\]

Many other properties of the BLA and its constituents can be proven based upon the assumption that the underlying nonlinear system is a Volterra system, and that the input signal belongs to the Riemann equivalence class of asymptotically normally distributed excitation signals [15]. Section III-A showed that if the relationship between \( \tilde{u}, n_x \), and \( y \) is given by a Volterra system, then the relationship between \( \tilde{u} \) and \( y \) is also given by a Volterra system. As a consequence, the theoretical properties of \( y_s \) and \( G_{bla} \) proven for the basic case (see [15] for an overview and a detailed analysis) still hold.

IV. BLA in Feedback

The classical open-loop BLA framework introduced in [19] has been extended to systems operating in closed loop [14], [16]. The generalized BLA applicable to the process noise presented in this paper is complementary with the closed-loop theory and can be similarly extended to closed-loop systems (see Fig. 5 for an overview of the setup). Note, however, that the noise framework has slightly changed compared to [14] and [16] since the pure measurement noise on the output \( y(t) \) is replaced by a noise source in the feedback loop. All the noise sources are again assumed to be zero-mean with a finite variance, to be independent of one another and also to be independent of the reference signal \( r(t) \).

The closed-loop BLA is defined using a two-stage method for linear feedback systems [28], [29], it is based upon the open-loop relations from the exactly known reference signal \( r(t) \) to the system input and the system output (see Fig. 5). Not only the relation from the reference signal \( r(t) \) to the output signal
Fig. 5. Setup for measuring the BLA of a nonlinear system with process noise operating in closed loop. The linear actuator and the feedback dynamics are represented by $G_{\text{act}}(q)$ and $M(q)$, respectively, they are not assumed to be known. $r(t)$ is the known reference signal, $u(t)$ and $y(t)$ are the input and output signal, and $n_y(t)$ and $n_x(t)$ are the output and process noise, respectively.

$y(t)$, but also the relation from $r(t)$ to the input signal $u(t)$ is assumed to belong to the system class $S_p$.

**Definition 6:** The BLA in the presence of process noise in a feedback setting is defined as

$$G_{\text{bla},r}(q) = \arg\min_{G(q)} E_{r,n_x,n_r}\left\{ |\hat{y}(t) - G(q)\hat{r}(t)|^2 \right\}$$

$$G_{\text{bla},y}(q) = \arg\min_{G(q)} E_{r,n_x,n_r}\left\{ |\hat{y}(t) - G(q)\hat{r}(t)|^2 \right\}$$

$$G_{\text{bla}}(q) = G_{\text{bla},y}(q)G_{\text{bla},r}(q)$$

where $G(q)$ belongs to the set of all possible LTI systems, and $\hat{r}(t)$, $\hat{u}(t)$, and $\hat{y}(t)$ are now given by

$$\hat{r}(t) = r(t) - E_r(r(t))$$

$$\hat{u}(t) = u(t) - E_{r,n_x,n_r}\{u(t)\}$$

$$\hat{y}(t) = y(t) - E_{r,n_x,n_y}\{y(t)\}.$$  

This results in the following definition of $y_s(t)$

$$u_s(t) = \hat{u}(t) - G_{\text{bla},r}(q)\hat{r}(t)$$

$$y_s(t) = \hat{y}(t) - G_{\text{bla},y}(q)\hat{r}(t)$$

$$y_{s}(t) = y_s(t) - G_{\text{bla}}(q)u_s(t)$$

with

$$\hat{u}(t) = E_{n_x,n_r}\{\hat{u}(t)\}$$

$$\hat{y}(t) = E_{n_x,n_y}\{\hat{y}(t)\}.$$  

**Definition 7:** The process noise contribution $y_p$ is defined as

$$u_p(t) = \hat{u}(t) - \hat{u}(t)$$

$$y_p(t) = \hat{y}(t) - \hat{y}(t)$$

$$y_{p}(t) = y_{p}(t) - G_{\text{bla}}(q)u_{p}(t)$$

where $\hat{u}(t)$ and $\hat{y}(t)$ are defined as

$$\hat{u}(t) = E_{n_x}\{\hat{u}(t)\}$$

$$\hat{y}(t) = E_{n_y}\{\hat{y}(t)\}.$$  

The output noise contribution $y_{s}$, the final constituent of the BLA framework, remains to be defined

$$u_{n}(t) = \hat{u}(t) - \hat{u}(t)$$

$$y_{n}(t) = \hat{y}(t) - \hat{y}(t)$$

$$y_{n}(t) = y_{n}(t) - G_{\text{bla}}(q)u_{n}(t) = n_{y}(t).$$

It is straightforward that the properties of the BLA constituents proven earlier still hold with one difference: the correlation has now to be evaluated w.r.t. $\hat{r}(t)$ instead of $\hat{u}(t)$.

The current noise setting can further be extended to include measurement noise on both the input and the output signals without complicating the definition of $G_{\text{bla}}$ and $y_{s}(t)$. The definition of the noise contributions does get more involved. This extension and its implications are beyond the scope of this paper.

**V. ESTIMATING THE BLA: THE ROBUST METHOD**

The BLA $G_{\text{bla}}(j\omega)$ can be estimated both parametrically or nonparametrically, an extensive review of the available BLA estimation techniques is provided by [15] and [23]. The presented methods remain valid in the process noise case. Some methods, such as the so-called robust method [15], [21], [23], can also provide an estimate of the noise variance and the total variance. This section first recapitulates the robust BLA estimation method, the behavior of the robust method is analyzed in detail for the process noise case next. The presented method considers the open-loop setting. With some minor modifications, it also holds for the closed-loop case [15], [16].

**A. Robust Method: Algorithm**

The robust BLA estimation approach obtains a frequency-domain estimate of the BLA and its constituents by making use of multiple periods and multiple realizations of a random phase multisine signal [15]. Note, however, that these results are, following the discussion in Section II-B, asymptotically equivalent to random Gaussian noise signals.

The estimated BLA $\hat{G}_{\text{bla}}(j\omega)$ in open loop and with a known input is obtained as follows [15], [21], [23]:

$$\hat{G}^{\text{m,p}}[j\omega] = \frac{Y^{\text{m,p}}[j\omega]}{U^{\text{m,p}}[j\omega]}$$

$$\hat{G}^{\text{m}}[j\omega] = \frac{1}{P} \sum_{p=1}^{P} \hat{G}^{\text{m,p}}[j\omega]$$

$$\hat{G}_{\text{bla}}(j\omega) = \frac{1}{M} \sum_{m=1}^{M} \hat{G}^{\text{m}}[j\omega]$$

where $Y^{\text{m,p}}[j\omega]$ is the DFT of the $p$th period and $m$th realization of the output signal $y(t)$, $U^{\text{m,p}}[j\omega]$ is the $m$th realization of the input signal. Since the $m$th realization of $u(t)$ is noise free, it is equal over all periods. The noise variance $\hat{\sigma}_{\text{bla,n}}^2(j\omega)$ (the
variance on \( \hat{G}_{\text{bla}}(j\omega) \) due to \( y_n(t) \) in the output noise setting) and total variance \( \hat{\sigma}_{\text{bla},t}^2(j\omega) \) (the variance on \( \hat{G}_{\text{bla}}(j\omega) \) due to \( y_n(t) + y_s(t) \) in the output noise setting) estimates are given by

\[
\hat{\sigma}_{\text{bla},n}^2(j\omega) = \frac{1}{M^2} \sum_{m=1}^{M} \sum_{p=1}^{P} \left| \hat{G}_{\text{bla}}[m](j\omega) - \hat{G}_{\text{bla}}^{[m,p]}(j\omega) \right|^2
\]

(59)

\[
\hat{\sigma}_{\text{bla},t}^2(j\omega) = \frac{1}{M(M-1)} \sum_{m=1}^{M} \left| \hat{G}_{\text{bla}}(j\omega) - \hat{G}_{\text{bla}}^{[m]}(j\omega) \right|^2 .
\]

(60)

Observe that \( \hat{\sigma}_{\text{bla},t}^2(j\omega) \) is an estimate of the variance of \( \hat{G}_{\text{bla}}(j\omega) \). This variance is generated by both the noise and the nonlinear behavior of the system.

B. Robust Method: Process Noise Analysis

The first step of the robust method takes the average of the output over the periods. Both the output noise contribution \( y_n(t) \) and the process noise contribution \( y_p(t) \) are aperiodic and zero-mean, while the stochastic nonlinear contribution is periodic for the periodic input. Hence, their contribution will be averaged out. In the second step, the average over the input realizations is taken. This step averages out the stochastic nonlinear contribution \( y_s(t) \), but also the remaining contributions of the process noise and output noise \( y_p(t) \) and \( y_n(t) \). More formally, we have that (see Eq. 4)

\[
Y_{[m,p]}(j\omega) = Y_{[m]}(j\omega) + Y_{[p]}(j\omega) + Y_{[n]}(j\omega)
\]

(61)

where \( Y_{[m]}(j\omega) \), \( Y_{[p]}(j\omega) \), and \( Y_{[n]}(j\omega) \) are the DFT of the period \( p \) and realization \( m \) of the signals \( y_s(t) \), \( y_p(t) \), and \( y_n(t) \), respectively. This results in the following expression for \( \hat{G}_{[m,p]}^{[m]}(j\omega) \)

\[
\hat{G}_{[m,p]}^{[m]}(j\omega) = \frac{Y_{[m]}(j\omega)}{U_{[m]}(j\omega)} + \frac{Y_{[p]}(j\omega)}{U_{[m]}(j\omega)} + \frac{Y_{[n]}(j\omega)}{U_{[m]}(j\omega)}
\]

(62)

A closer analysis, analogous to [15], of the expected value of (58), (59), and (60) for the process noise case results in

\[
E\{\hat{G}_{\text{bla}}(j\omega)\} = \hat{G}_{\text{bla}}(j\omega)
\]

(63)

\[
E\{\hat{\sigma}_{\text{bla},n}^2(j\omega)\} = \frac{\sigma_n^2(j\omega) + \sigma_a^2(j\omega)}{M^2 |U(j\omega)|^2}
\]

(64)

\[
E\{\hat{\sigma}_{\text{bla},t}^2(j\omega)\} = \frac{\sigma_n^2(j\omega) + \sigma_a^2(j\omega)}{M |U(j\omega)|^2} + \frac{\sigma_n^2(j\omega) + \sigma_a^2(j\omega)}{M |U(j\omega)|^2}
\]

(65)

where \( \sigma_n^2(j\omega) \), \( \sigma_p^2(j\omega) \), and \( \sigma_a^2(j\omega) \) are the variances of \( Y_{[n]}(j\omega) \), \( Y_{[p]}(j\omega) \), and \( Y_{[a]}(j\omega) \), respectively, and where \( |U(j\omega)|^2 \) is independent of the random phase realization for random phase multisines. The expectations are taken with respect to the input realization, process noise realization and output noise realization.

The robust BLA estimation method is still valid in the process noise case. However, the estimated BLA now depends on the process noise properties (see Section III), and the estimated variance due to noise and the total variance on the BLA have an extra term, which is process noise dependent. The variance due to the process noise contribution \( y_p(t) \) and the output noise contribution \( y_n(t) \) cannot be separated using the robust method: one can only obtain an estimate of \( \sigma_n^2(j\omega) \) on one hand, and of the sum \( \sigma_n^2(j\omega) + \sigma_p^2(j\omega) \) on the other hand.

VI. EXAMPLES

A. Hammerstein Example

1) System: Consider the following Hammerstein system (see Fig. 6), with \( f(x) = x + 0.1x^3 \)

\[
y(t) = S(q) \left[ f(u(t) + n_x(t)) \right] + n_y(t)
\]

(66)

\[
y(t) = S(q) \left[ u(t) + n_x(t) + 0.1u(t)^3 + 0.3u(t)^2 \right] + n_x(t)
\]

(67)

where \( u(t) \), \( n_x(t) \), and \( n_p(t) \) are zero-mean white Gaussian signals with standard deviations \( \sigma_u \), \( \sigma_n \), and \( \sigma_p \), respectively. \( S(q) \) is obtained as a Chebyshev Type 1 digital filter of order 2 with a 20 dB ripple and a cut-off frequency at 0.1 Hz.

\[
S(q) = \frac{0.0049 + 0.0099q^{-1} + 0.0049q^{-2}}{1 - 1.7599q^{-1} + 0.9571q^{-2}}.
\]

(68)

2) Theoretical Analysis: \( \tilde{u}(t) = u(t) \) and \( \tilde{y}(t) = y(t) \) since the input \( u(t) \) has zero-mean and the static nonlinearity is odd. \( \tilde{y}(t) \) and \( \tilde{y}(t) \) are given by

\[
\hat{y}(t) = E_{y_n}\{ y(t) \}
\]

(69)

\[
\hat{y}(t) = E_{n_x}\{ y(t) \}
\]

(70)

Since the input of the static nonlinearity is Gaussian, Bussgang’s Theorem can be applied [3], i.e., the BLA of a static nonlinearity is a static gain depending on the variance of the input \( u(t) \) and the process noise \( n_x(t) \). Based on the results in [5] and [8] we obtain

\[
G_{\text{bla}}(q) = S(q)(1 + 0.3\sigma_n^2 + 0.3\sigma_p^2).
\]

(71)
The BLA constituents $y_{bla}(t)$, $y_s(t)$, $y_p(t)$, $y_n(t)$ are given by

$$\begin{align*}
y_{bla}(t) &= S(q) \left[ (1 + 0.3\sigma_n^2 + 0.3\sigma_x^2)u(t) \right] \\
y_s(t) &= S(q) \left[ 0.1u(t)^3 - 0.3\sigma_n^2 u(t) \right] \\
y_p(t) &= S(q) \left[ n_x(t)^3 + \frac{3}{2}u(t)^2n_x(t) \\
&\quad+ 0.3u(t)(n_x(t)^2 - \sigma_n^2) + 0.1n_x(t)^2 \right] \\
y_n(t) &= n_y(t).
\end{align*}$$

(72)

It can easily be observed that the properties that are derived in Section III-C are valid for this case study. It can also be observed that the BLA does not only depend on the input signal properties, but also on the properties of the disturbing process noise (as it is also the case for the BLA in the feedback framework [16]). It is illustrated in Fig. 7, for the Hammerstein case considered here, that the gain of the BLA depends on the variance of the process noise. The process noise contribution $y_p(t)$ on the other hand does not only depend on the process noise $n_x(t)$, but also on the input signal $u(t)$.

Note that the chosen definition of the process noise contribution $y_p$ and the output noise contribution $y_n$ are not unique [see (30) and (34)]. An alternative set of definitions $\hat{y}_p$ and $\hat{y}_n$ could be to assign all the noise terms depending on the input $u$ to the process noise contribution, and assign all the noise terms independent of the input $u$ to the output noise contribution, resulting in the following expressions for $\hat{y}_p$ and $\hat{y}_n$ in this example

$$\begin{align*}
\hat{y}_p(t) &= S(q) \left[ 0.3u(t)^2n_x(t) + 0.3u(t)(n_x(t)^2 - \sigma_n^2) \right] \\
\hat{y}_n(t) &= n_y(t) + S(q) \left[ n_x(t) + 0.1n_x(t)^3 \right].
\end{align*}$$

(73)

However, the original definitions have the merit of being simple extension of the definitions used in the output noise open-loop and closed-loop setting, based on taking the expected value with respect to the output noise $n_y$ and the process noise $n_x$. Note as well that with the chosen definitions the output noise contribution only contains terms due to the output noise, while this is not the case using the alternative definition. For these reasons the authors have chosen to use the definitions that are expressed in (30) and (34).

3) Robust Method Results: This section illustrates how the robust method can be used to estimate the BLA in a process noise setting. The experimentally obtained BLA, total variance and noise variance are compared with the analytically derived total and noise variance. Note that the robust approach does not require any knowledge of the system, while a full knowledge of the system and the (noise) signals distribution is required for the analytical derivation. A total of $M = 10$ realizations is used, each containing two steady-state periods of 4096 points per period. The standard deviation of the input signal and noise signals are $\sigma_u = 1$, $\sigma_n = 0.1$, and $\sigma_{n_y} = 0.003$.

The BLA $\hat{G}_{bla}(j\omega)$ and the variances $\hat{\sigma}^2_{bla,t}(j\omega)$, $\hat{\sigma}^2_{bla,n}(j\omega)$ obtained with the robust BLA estimation method coincide perfectly with their analytical counterparts as can be seen in Fig. 8. Note that the robust approach cannot distinguish between the process noise and the output noise variance, what is shown here is the total variance of the BLA due to both the process noise and the output noise.

Note that the knowledge of the Volterra expansion is not required to obtain an estimate of the BLA, the noise variance and the variance due to the stochastic nonlinear contributions. Such an estimate is obtained based upon experimental data only using the robust method discussed in Section V.

B. Nonlinear Finite Impulse Response (NFIR)

1) System: The following NFIR system is considered

$$\begin{align*}
y(t) &= u(t) + u(t-1)^2 - u(t-2)^2 + u(t-2)^2 n_x(t) \\
&\quad+ u(t-1)^3 n_x(t-1)^2 + n_y(t).
\end{align*}$$

(74)

Again, $u(t)$, $n_x(t)$, and $n_y(t)$ are zero-mean white Gaussian signals with standard deviations $\sigma_u$, $\sigma_n$, and $\sigma_{n_y}$, respectively.
2) Theoretical Analysis: \( \tilde{u}(t) = u(t) \) and \( \tilde{y}(t) = y(t) \) since the input \( u(t) \) has zero-mean and offsets introduced by the even terms cancel out. \( \tilde{y}(t) \) and \( \tilde{y}(t) \) are given by
\[
\tilde{y}(t) = E_{n_t} \{ y(t) \} = u(t) + u(t-1)^2 - u(t-2)^2 + u(t-2)^2 n_x(t) + u(t-1)^3 n_x(t-1)^2
\]
(75)
\[
\tilde{y}(t) = u(t) + u(t-1)^2 - u(t-2)^2 + \sigma_n^2, u(t-1)^3
\]
Since \( u(t) \) is white and Gaussian, Bussgang’s Theorem can be applied [3] for each delayed value of \( u \) separately (i.e., on \( u(t) \), \( u(t-1)^2 + \sigma_n^2 \), \( u(t-2)^2 \), and \( -u(t-2)^2 \)). This results in the following expression:
\[
G_{bla}(q) = 1 + 3\sigma_n^2, \sigma_n^2 q^{-1}.
\]
(76)
The BLA constituents \( y_{bla}(t) \), \( y_s(t) \), \( y_p(t) \), \( y_n(t) \) are given by
\[
y_{bla}(t) = u(t) + 3\sigma_n^2, \sigma_n^2 u(t-1)
\]
(77)
\[
y_s(t) = u(t-1)^2 - u(t-2)^2 + \sigma_n^2, u(t-1)^3
\]
\[
y_p(t) = u(t-2)^2 n_x(t) + u(t-1)^3 (n_x(t-1)^2 - \sigma_n^2)
\]
\[
y_n(t) = n_y(t).
\]
It can again be observed that the properties that are derived in Section III-C are valid for this case study. It can also be observed that the BLA does not only depend on the input signal properties, but also on the properties of the disturbing process noise, which is illustrated in Fig. 9. The process noise contribution \( y_p(t) \) depends on both the process noise \( n_x(t) \) and the input signal \( u(t) \).

3) Robust Method Results: A total of \( M = 100 \) realizations is used, each containing two steady-state periods of 4096 points per period. The standard deviation of the input signal and noise signals are \( \sigma_u = 0.75 \), \( \sigma_n = 0.5 \), and \( \sigma_{n_x} = 0.003 \).

As in the previous example, the BLA \( \hat{G}_{bla}(j\omega) \) and the variances \( \sigma_{bla,s}(j\omega), \sigma_{bla,n}(j\omega) \) obtained with the robust BLA estimation method coincide perfectly with their analytical counterparts as can be seen in Fig. 10. Note that the robust approach cannot distinguish between the process noise and the output noise variance, what is shown here is the total variance of the BLA due to both the process noise and the output noise.

Note that the analytical results for the total variance and the noise variance shown in Fig. 10 are obtained as
\[
\sigma_{bla,s}^2(j\omega_k) = \frac{\sum_{i=1}^{M} |Y_s[i](j\omega_k) + Y_p[i](j\omega_k) + Y_n[i](j\omega_k)|^2}{\sum_{i=1}^{M} |U[i](j\omega_k)|^2} \]
\[
\sigma_{bla,n}^2(j\omega_k) = \frac{\sum_{i=1}^{M} |Y_p[i](j\omega_k) + Y_n[i](j\omega_k)|^2}{\sum_{i=1}^{M} |U[i](j\omega_k)|^2}
\]
(78)
where \( U[i](j\omega), Y_s[i](j\omega), Y_p[i](j\omega), \) and \( Y_n[i](j\omega) \) are obtained as the DFT of the signals \( u[i](t), y_s[i](t), y_p[i](t), \) and \( y_n[i](t) \) specified in (77). The superscript \( [i] \) denotes that the signal is computed using the i-th realization of the input and noise signals.

VII. CONCLUSION

The BLA framework is extended to the process noise case, both for the open-loop and the closed-loop setting. The process noise acts as a second input of a Volterra system, resulting in a very general process noise framework. It is proven that the stochastic nonlinear contributions and the process noise contribution are zero-mean and uncorrelated with the input. It is also illustrated that the BLA can depend of the properties on the process noise, and that both the process noise contribution and the stochastic nonlinear distortion are uncorrelated but not independent of the input excitation. The BLA, together with the total and the noise variance can be obtained using the robust estimation method in the case of process noise.
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Maarten Schoukens (SM’12–M’15) received the master’s degree in electrical engineering and the Ph.D. degree in engineering from the Vrije Universiteit Brussel (VUB), Brussels, Belgium, in 2010 and 2015, respectively.

He is currently an Assistant Professor with the Control Systems Group, Department of Electrical Engineering, Eindhoven University of Technology (TU/e), Eindhoven, The Netherlands. From 2015 to 2017, he has been a Postdoctoral Researcher with the ELEC Department, VUB. In October 2017, he joined the Control Systems Research Group, TU/e where he became an Assistant Professor in 2018. His main research interests include the measurement and data-driven modeling of linear parameter-varying and nonlinear dynamical systems using system identification, and machine learning techniques.

Dr. Schoukens was the recipient of an FWO Ph.D. Fellowship in 2011, and a Marie Skłodowska-Curie Individual Fellowship in 2018.

Rik Pintelon (F’98) was born on December 4, 1959, in Gent, Belgium. He received the master’s degree in electrical engineering, the doctorate (Ph.D.) degree in engineering, and the qualification to teach at university level (geagregerde voor het hoger onderwijs), all from the Vrije Universiteit Brussel (VUB), Brussels, Belgium, in 1982, 1988, and 1994, respectively, and the Doctor of Science (D.Sc.) degree from the University of Warwick, Coventry, U.K., in 2014, for publications with the collective title “Frequency Domain System Identification: A Mature Modeling Tool.”

From 1982 to 1984 and 1986 to 2000, he was a Researcher with the Belgian National Fund for Scientific Research (FWO-Vlaanderen), Electrical Engineering (ELEC) Department, VUB. From 1984 to 1986, he did his military service overseas in Tunisia with the Institut National Agronomique de Tunis. From 1991 to 2000, he was a part-time Lecturer with the Department ELEC, VUB, and since 2000, he has been a full-time Professor in electrical engineering with the same department. From 2009 to 2018, he was a Visiting Professor with the Department of Computer Sciences, Katholieke Universiteit Leuven, and from 2013 to 2018, he was a Honorary Professor with the School of Engineering of the University of Warwick. His main research interests include system identification, signal processing, and measurement techniques.

Dr. Pintelon is the coauthor of four books on System Identification and the coauthor of more than 200 articles in refereed international journals. He was the recipient of the 2012 IEEE Joseph F. Keithley Award in Instrumentation and Measurement (IEEE Technical Field Award).
Tadeusz P. Dobrowiecki (F’07) was born in Warsaw, Poland, in 1952. He received the M.Sc. degree in electrical engineering in 1975 from the Technical University of Budapest, Budapest, Hungary, the Ph.D. (candidate of sciences) and the D.Sc. (Doctor of Sciences) degrees in informatics from the Hungarian Academy of Sciences, Budapest, Hungary, in 1981 and 2017, respectively.

After spending one year as a Professional System Engineer, he joined the staff of the Department of Measurement and Information Systems, Budapest University of Technology and Economics, where he is working since then and is currently a Full Professor. He is engaged in teaching artificial intelligence, signal processing, and system identification. His research interests concern advanced signal processing algorithms, technical applications of artificial intelligence, and nonlinear system identification problems.

Johan Schoukens (F’97) received the master’s degree in electrical engineering, the Ph.D. degree in engineering sciences, and the Geagregreerde voor het Hoger Onderwijs degree from the Vrije Universiteit Brussel (VUB), Brussels, Belgium, the Doctor Honoris Causa degree from the Budapest University of Technology and Economics, Budapest, Hungary, in 1980, 1985, 1991, and 2011, respectively, and the Doctor of Science degree from The University of Warwick, Coventry, U.K., in 2014.

From 1981 to 2000, he was a Researcher with the Belgian National Fund for Scientific Research (FWO-Vlaanderen) and was a full-time Professor in electrical engineering with the VUB till 2018. Since 2018, he is an Emeritus Professor with the INDI Department, VUB, and member of the Department of Electrical Engineering, Eindhoven University of Technology. From 2009 to 2016, he was a Visiting Professor with the Department of Computer Sciences, Katholieke Universiteit Leuven. Since 2013, he has been an Honorary Professor with the University of Warwick. His main research interests include system identification, signal processing, and measurement techniques.

Dr. Schoukens was the recipient of the 2002 Andrew R. Chi Best Paper Award of the IEEE TRANSACTIONS ON INSTRUMENTATION AND MEASUREMENT, the 2002 Society Distinguished Service Award from the IEEE Instrumentation and Measurement Society, and the 2007 Belgian Francqui Chair at the Université Libre de Bruxelles (Belgium). Since 2010, he is a member of the Royal Flemish Academy of Belgium for Sciences and the Arts.