Abstract

A multiplicativity conjecture for quantum communication channels is formulated, validity of which for the values of parameter $p$ close to 1 is related to the solution of the fundamental problem of additivity of the channel capacity in quantum information theory. The proof of the conjecture is given for the case of natural numbers $p$. 
On the multiplicativity conjecture for quantum channels

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1. Let $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all operators in a finite dimensional unitary space $\mathcal{H}$. We denote the set of states, i.e. positive operators with unit trace (density operators) in $\mathcal{B}(\mathcal{H})$ by $\mathcal{S}(\mathcal{H})$. As is well known, $\mathcal{S}(\mathcal{H})$ is a compact convex subset of $\mathcal{B}(\mathcal{H})$, extreme points of which are pure states, described by one-dimensional projectors in $\mathcal{H}$. The degree of “purity” of arbitrary state $S \in \mathcal{S}(\mathcal{H})$ can be defined with the help of noncommutative $\ell_p$-norms

$$\|S\|_p = \left(\text{Tr}|S|^p\right)^{\frac{1}{p}}, \quad p \geq 1,$$

with the operator norm $\|S\|$ corresponding naturally to the case $p = \infty$. The closer is the value of any norm to the identity, the more “pure” is the state $S$.

A quantum channel $\Phi$ is a completely positive trace preserving linear map of $\mathcal{B}(\mathcal{H})$, i.e. a map admitting the representation

$$\Phi(S) = \sum_k A_k S A_k^*, \quad (1)$$

where $A_k$ are operators satisfying $\sum_k A_k^* A_k = I$ (see e. g. [3] for motivation and background). The channel $\Phi$ maps input state $S$ into output state $\Phi(S)$. In the present paper we consider the multiplicativity problem for the measures of the “highest purity” of outputs of a channel

$$\nu_p(\Phi) = \max_S \|\Phi(S)\|_p, \quad (2)$$

where the maximum is taken with respect to all input density operators $S$. By convexity of the norms, the maximum in the above definition is attained on pure states.
Let $\Phi_1, \ldots, \Phi_n$ be a collection of arbitrary channels in the unitary spaces $H_i; \ i = 1, 2, \ldots, n$. In [1] the following multiplicativity property

$$\nu_p(\Phi_1 \otimes \ldots \otimes \Phi_n) = \nu_p(\Phi_1) \cdot \ldots \cdot \nu_p(\Phi_n)$$

was conjectured. As noticed in [1], validity of this conjecture for values of $p$ close to 1 implies solution of the fundamental problem of additivity of the channel capacity for one important class of quantum channels. For the formulation of the and some partial results see [3], [1], [5]. In the classical case where $B(H)$ is replaced by a commutative algebra of diagonal operators, the states are given by probability distributions, and channels – by transition probabilities, the analog of the formulated additivity/multiplicativity problems has obvious positive solution. The difficulty in the noncommutative case is due to the unusual from a classical viewpoint properties of combined quantum systems described by tensor rather than Cartesian products, and by existence of entangled states in the combined system.

2. We denote by $\ell_p(H)$ the Schatten class of Hermitian operators $A$ in $H$ with the norm $\|A\|_p$.

**Lemma.** The quantity $\nu_p(\Phi)$ is equal to the norm $\|\Phi\|_{1 \rightarrow p}$ of the mapping $\Phi$ acting from the Schatten class $\ell_1(H)$ to $\ell_p(H)$.

**Proof.** We have

$$\|\Phi\|_{1 \rightarrow p} = \max_{A \neq 0} \left( \frac{\text{Tr}|\Phi(A)|^p}{\text{Tr}|A|^p} \right)^{1/p},$$

so obviously $\|\Phi\|_{1 \rightarrow p} \geq \nu_p(\Phi)$. Conversely, let $A = A_+ - A_-$ be the decomposition of $A$ into positive and negative parts, then $\text{Tr}|A| = \text{Tr}(A_+ + A_-)$ and $-\Phi(A_+ + A_-) \leq \Phi(A) \leq \Phi(A_+ + A_-)$ by positivity of $\Phi$. From convexity of the function $x^p$, $\text{Tr}|\Phi(A)|^p \leq \text{Tr}\Phi(A_+ + A_-)^p$, indeed, denoting $\{e_j\}$ the basis of eigenvectors of $\Phi(A)$, we have

$$\text{Tr}|\Phi(A)|^p = \sum_j |\langle e_j | \Phi(A) | e_j \rangle|^p \leq \sum_j |\langle e_j | \Phi(|A|) | e_j \rangle|^p$$

$$\leq \sum_j |\langle e_j | \Phi(|A|) | e_j \rangle|^p = \text{Tr}|\Phi(|A|)|^p,$$

and the converse inequality follows. □
Generalizing hypothesis (3), we conjecture that the norms \( \|\Phi\|_{q \rightarrow p} \) have a similar multiplicative property for \( 1 \leq q \leq p \) for completely positive maps \( \Phi_1, \ldots, \Phi_n \). Note that the classical (commutative) counterpart of this conjecture indeed holds for arbitrary (bounded) \( \Phi_1, \ldots, \Phi_n \) ([2], Lemma 2).

In [1] relation (3) was proved for the special case of depolarizing channels \( \Phi_1, \ldots, \Phi_n \) (see the definition in n.3) and \( p = 2, \infty \). Here this property will be established for the depolarizing channels and arbitrary natural number \( p \).

3. Let us consider a collection of unitary spaces \( \mathcal{H}_i; i = 1, 2, \ldots, n \), with \( \dim \mathcal{H}_i = d_i \). Let \( \mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i \), \( d = \prod_{i=1}^{n} d_i \). In what follows we shall use symbols \( I_i \) and \( I = \bigotimes_{i=1}^{n} I_i \) for the identity operators in \( \mathcal{H}_i \) and \( \mathcal{H} \), respectively. For a nonempty subset \( L \subset \{1, 2, \ldots, n\} \) we denote

\[
\mathcal{H}_L = \bigotimes_{i \in L} \mathcal{H}_i, \quad I_L = \bigotimes_{i \in L} I_i, \quad d_L = \prod_{i \in L} d_i = \dim \mathcal{H}_L.
\]

We also let \( d_{\emptyset} = 1 \).

Let \( L_1, \ldots, L_m \) be a collection of nonempty subsets of \( \{1, 2, \ldots, n\} \), and let \( A_1, \ldots, A_m \) be a collection of operators in \( \mathcal{H} \) such that \( A_k = B_k \otimes I_{L_k} \), where \( B_k \) is an operator in \( \mathcal{H}_{L_k} \).

**Lemma.**

\[
|\text{Tr} A_1 \ldots A_m| \leq d_{\bigcap_{k=1}^{m} L_k} ||B_1|| \ldots ||B_m||_1. \tag{4}
\]

**Proof.** By using the singular value decomposition of the operators \( B_k \), we can reduce the problem to the case where \( B_k \) are the rank one operators, \( B_k = |a_k\rangle\langle b_k| \) with unit vectors \( |a_k\rangle, |b_k\rangle \). Moreover, by excluding the common factor \( I_{\bigcap_{k=1}^{m} L_k} \), we can reduce to the case \( \bigcap_{k=1}^{m} L_k = \emptyset \). Then (4) reduces to

\[
|\text{Tr} A_1 \ldots A_m| \leq 1. \tag{5}
\]

Pick an orthonormal basis \( \{e_{j_s}\} \) in \( \mathcal{H}_s \), and form the factorizable basis \( \{e_J\} \) in the space \( \mathcal{H} \) such that

\[
e_J = e_{j_1} \otimes e_{j_2} \otimes \ldots \otimes e_{j_n},
\]

where \( J = (j_1, j_2, \ldots, j_n) \). Denote \( J_L = (j_s)_{s \in L} \). Then by decomposing the unit vectors \( |a_k\rangle, |b_k\rangle \), we have

\[
|a_k\rangle = \sum_{J_{L_k}} \alpha^k_{J_{L_k}} |e_{J_{L_k}}\rangle, \quad |b_k\rangle = \sum_{J_{L_k}} \beta^k_{J_{L_k}} |e_{J_{L_k}}\rangle,
\]
where $\sum_{J_{L^c_k}} |\alpha_{J_{L^c_k}}|^2 = \sum_{J_{L^c_k}} |\beta_{J_{L^c_k}}|^2 = 1$, so that

$$A_k = \sum_{J_{L^c_k}} \alpha_{J_{L^c_k}} |e_{J_{L^c_k}}\rangle \sum_{J'_{L^c_k}} \beta_{J_{L^c_k}} |e_{J'_{L^c_k}}\rangle \otimes \sum_{J_{L_k}} |e_{J_{L_k}}\rangle \langle e_{J_{L_k}}|.$$  \hspace{1cm} (6)

Now consider the multiindex $J = (J_{L^c_k})_{k=1,...,m}$, the components of which are labeled by pairs $(k, p)$, where $k \in \{1, \ldots, m\}$ and $p \in L^c_k$, and put

$$\alpha_J = \prod_{k=1}^m \alpha_{J_{L^c_k}}, \quad \beta_J = \prod_{k=1}^m \beta_{J_{L^c_k}}.$$  \hspace{1cm} (7)

Then

$$\sum_J |\alpha_J|^2 = \sum_J |\beta_J|^2 = 1.$$  \hspace{1cm} (8)

We are going to show that substituting (6) into the left side of (5), we obtain

$$\text{Tr} A_1 \ldots A_m = \sum_{J_{L^c_k}} |\beta_J| \alpha_{\mathcal{P}_J},$$  \hspace{1cm} (9)

where $\mathcal{P}_J$ is a permutation of the components of the multiindex $J$, and hence by the Cauchy-Schwarz inequality and (7) we have the inequality (8).

The permutation $\mathcal{P}$ arises as follows: let us take $A_k$ of the form (6) in the expression $\text{Tr} A_1 \ldots A_m$, and let $s \in L_k^c$. Then the covectors $\langle e_j |$ are present in the decomposition of $|b_k\rangle$. Let us go right cyclically under the trace starting from $A_k$, passing through the identity operators, and watch when $s$ will first again appear in $L_{k\oplus l}$, where $\oplus$ denotes addition mod $m$. Then the vectors $|e_{j_s}\rangle$ will be present in the decomposition of $|a_{k\oplus l}\rangle$, giving rise to the summation over $j_s$ in (8).

More precisely, let us denote by $\mathcal{A}$ the set of all the pairs $(k, s)$, where $k \in \{1, \ldots, m\}$ and $s \in L_k$. Thus the components of $J$ can be written as $j(k, s)$, where $(k, s) \in \mathcal{A}$. Let $l$ be the minimal mod $m$ positive integer such that $(k \oplus l, s) \in \mathcal{A}$. The mapping $(k, s) \to (k \oplus l, s)$ is a bijection of the set $\mathcal{A}$, therefore it induces a permutation $\mathcal{P}$ of the multiindex $J$, resulting in the formula (8). \hspace{1cm} $\Box$

4. Let us consider a collection of depolarizing channels

$$\Phi_i(S) = (1 - p_i) S + \frac{p_i}{d_i} (\text{Tr} S) I_i, \quad S \in \mathcal{B}(\mathcal{H}_i), \quad 0 < p_i < 1,$$  \hspace{1cm} (10)
in the unitary spaces $H_i$, with parameters $p_i, d_i$; $i = 1, 2, \ldots, n$, and denote $\Phi = \bigotimes_{i=1}^{n} \Phi_i$. It is easy to see that $\Phi_i$ are indeed channels, i.e. can be represented in the form (1). If $S$ is pure state (one-dimensional projection) then operator (9) has the simple eigenvalue $\left(1 - \frac{d_i - 1}{d_i} p_i\right)$ and $d_i - 1$ eigenvalues $\frac{d_i - 1}{d_i} p_i$. Hence

$$\nu_k(\Phi_i) = \left[\left(1 - \left(\frac{d_i - 1}{d_i}\right)^p\right) + (d_i - 1) \left(\frac{p_i}{d_i}\right)^p\right]^{\frac{1}{p}}.$$ 

Let $\epsilon_L$ be the conditional expectation onto the subalgebra $\mathcal{M}_L$, generated by operators of the form $A_1 \otimes \ldots \otimes A_n$, where $A_i = I_i$ for $i \in L$, and arbitrary otherwise. It is normalized partial trace with respect to $H_L$:

$$\epsilon_L(A) = \text{Tr}_{H_L} A \otimes d_L^{-1} I_L.$$ 

In the following we shall use the expansion

$$\Phi = \sum_{L} \prod_{i=1}^{n} p_i^{\theta_L(i)} (1 - p_i)^{1 - \theta_L(i)} \epsilon_L,$$ 

where $\theta_L(i) = 1$ if $i \in L$ and $\theta_L(i) = 0$ otherwise. Note that $d_L = \prod_{i=1}^{n} d_i^{\theta_L(i)}$.

**Theorem.** For $p \in \mathbb{N}$

$$\nu_k(\Phi) = \prod_{i=1}^{n} \nu_k(\Phi_i) = \prod_{i=1}^{n} \left[\left(1 - \left(\frac{d_i - 1}{d_i}\right)^p\right) + (d_i - 1) \left(\frac{p_i}{d_i}\right)^p\right]^{\frac{1}{p}}.$$ 

**Proof.** Let $S \in S(H)$, then by (10) and by (4)

$$|\text{Tr}_{L_1}(S) \ldots \text{Tr}_{L_p}(S)| \leq \frac{d_{\cap_{i=1}^{n} L_i}}{d_i^{\theta_{L_i}(i)}}.$$ 

The obtained inequality and the expansion (11) imply

$$\text{Tr(} \Phi(S)^{p} = \text{Tr}\left(\sum_{L} \prod_{i=1}^{n} p_i^{\theta_L(i)} (1 - p_i)^{1 - \theta_L(i)} \epsilon_L(S))^p =$$

$$\sum_{L_1, L_2, \ldots, L_p} \prod_{i=1}^{n} p_i^{\sum_{j=1}^{p} \theta_{L_j}(i)} (1 - p_i)^{\sum_{j=1}^{p} (1 - \theta_{L_j}(i))} \text{Tr}_{L_1}(S) \ldots \text{Tr}_{L_p}(S) \leq$$

$$\sum_{L_1, L_2, \ldots, L_p} \prod_{i=1}^{n} \left(\frac{p_i}{d_i}\right)^{\sum_{j=1}^{p} \theta_{L_j}(i)} (1 - p_i)^{\sum_{j=1}^{p} (1 - \theta_{L_j}(i))} \frac{d_i^{\theta_{L_j}(i)}}{d_{L_j}(i)}.$$
The number $\theta_{\bigcap_{j=1}^p L_j}(i)$ is equal to $\min \{ \theta_{L_1}(i), \ldots, \theta_{L_p}(i) \}$ and is equal to 1 if and only if all $\theta_{L_j}(i); j = 1, \ldots, p$, are equal to 1. By using the formula

$$\sum_{L_1, L_2, \ldots, L_p} \prod_{i=1}^n f_i(\theta_{L_1}(i), \ldots, \theta_{L_p}(i)) = \prod_{i=1}^n \sum_{\theta_1, \theta_2, \ldots, \theta_p = 0, 1} f_i(\theta_1, \ldots, \theta_p),$$

we obtain that the last expression is equal to

$$\prod_{i=1}^n \left[ \left( 1 - (d_i - 1)\frac{p_i}{d_i} \right)^p + (d_i - 1) \left( \frac{p_i}{d_i} \right)^p \right]. \quad \square$$

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