Beltrami-Klein disk model as viewed for use in impedance trajectory projection

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Abstract: This paper presents a basic consideration of complex impedance projection. We start by recalling the traditional Smith chart. The well-known standing wave ratio is revisited by its natural logarithm, which brings Poincaré metric to geometrical distance on the chart. We then replace the distance by its double to observe the Beltrami-Klein disk model from the aspect of RF engineering. To give fundamental locus examples, we formulate how lumped-constant elements behave on hyperbolic coordinates, and find that they draw their constant-R and -X contours in straight lines or vertical ellipses. For higher frequency applications, we also formulate the behavior of transmission lines to show what trajectories they exhibit on the disk.

Keywords: Smith chart, complex reflectance locus, hyperbolic geometry

Classification: Transmission systems and transmission equipment for communications

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1 Introduction

In radio communication and wireless power transfer systems, conjugate impedance
matching of RF components, antennas, and their feeder lines are indispensable
to avoid unwanted wave reflection and achieve a high transmission efficiency [1-7].
To illustrate complex impedances on a plane, the Smith chart is widely used as a
sophisticated mapping tool. The Smith chart is mathematically based on the
Poincaré disk model of hyperbolic geometry [5-8]. This model provides
a convenient map for the impedance-to-reflectance conversion within a unit circle.
However, at least geometrically speaking, it is not the unique solution for mapping
purposes. In other words, there may exist alternative approaches to chart creation.
With this background, this paper theoretically explores the possibility of the
Beltrami-Klein disk model as a candidate for use as a tool of impedance projection.
By way of complex calculus, we illustrate how lumped-constant elements and
transmission lines draw their trajectories on that disk.

2 Poincaré metric

We begin with a brief review of the Smith chart [1], which is a convenient
engineering tool to display the complex reflectance of RF loads (e.g. antennas) on
a Gaussian plane. Employing the polar coordinates $(\gamma, \theta)$ of the plane, the complex
reflectance is expressed as

$$\Gamma = \gamma e^{i\theta}, \quad \gamma = |\Gamma| = \frac{\rho - 1}{\rho + 1}, \quad \theta = \angle \Gamma$$ (1)

where $j$ stands for the imaginary unit, and $\rho$ denotes the standing wave ratio (SWR)
[5,8]. Introducing the natural logarithm of SWR, namely $D = \ln \rho$, the above
relation can be rewritten as

$$\gamma = \frac{e^D - 1}{e^D + 1} = \frac{e^{D/2} - e^{-D/2}}{e^{D/2} + e^{-D/2}} \therefore \Gamma = e^{i\theta} \tanh \frac{D}{2}.$$ (2)

When we increase $D$ from zero to infinity, the point $\Gamma$ moves from the origin
toward the bound (unit circle) of the Smith chart. On the other hand, when the
phase $\theta$ ranges from 0 to $2\pi$, the point $\Gamma$ revolves counterclockwise around the
origin independently from $D$. Note that the final right-hand side of Eq. (2) is called
a complex reflectance in Poincaré metric, since $D$ is recognized as the geometrical
distance from origin to point $\Gamma$ measured in that metric [5-7].

3 Disk model definition

Some radio broadcasting antennas and wireless power transfer systems transmit
means a resistor connected in series to the load

They start from different points on the unit circle, but commonly end up at point (1, 0). They should be called constant-\(X\) contours, because each of them physically means a resistor connected in series to the load.

Next, we sweep \(X\) with fixed \(R\). Eliminating \(X\) from Eq. (5), we get

\[
50v = X(1-u). \tag{6}
\]

This results in a straight line on disk \(\Omega\). The reactance \(X\) implies the line’s slope.

The above defined \(\Omega\) is called the Beltrami-Klein disk, which was historically the first model of a hyperbolic plane [10]. In contrast to \(\Omega\), the aforementioned \(\Gamma\) was once named the Poincaré disk, but is now widely known to radio and wireless engineers as Smith chart.

### 4 Lumped-constant elements

Now let us see how the disk \(\Omega\) works for impedance projection. It is well known that impedance and reflectance can be mutually converted by Möbius transform

\[
Z = R + jX = 50 \frac{1+\Gamma}{1-\Gamma}, \quad \Gamma = \frac{Z-50}{Z+50} \tag{4}
\]

where 50 ohm is employed for the reference resistance as an industrial standard.

We decompose a complex variable on the disk defined in Eq. (3) into Cartesian coordinates as \(\Omega = u + jv\). For simplicity, we assume a load consisting of passive elements, i.e., \(R > 0\) or \(u^2 + v^2 < 1\). Applying Eq. (4) back to (3), we get

\[
u = \frac{R^2 + X^2 - 50^2}{R^2 + X^2 + 50^2}, \quad v = \frac{100X}{R^2 + X^2 + 50^2}. \tag{5}\]

This translation enables us to find out what trajectory the impedance makes on the \(u-v\) plane or disk \(\Omega\) when we sweep \(R\) and \(X\).

First, we sweep \(R\) with fixed \(X\). Eliminating \(R\) from Eq. (5), we get

\[
50v = X(1-u). \tag{6}
\]

This results in a straight line on disk \(\Omega\). The reactance \(X\) implies the line’s slope.

Fig. 1 shows the lines projected for \(X = 0, \pm 10, \pm 25, \pm 50, \pm 100, \text{ and } \pm 250\) ohm. They start from different points on the unit circle, but commonly end up at point (1, 0). They should be called constant-\(X\) contours, because each of them physically means a resistor connected in series to the load.
\[(u - 1 + r^2)^2 + r^2v^2 = r^4, \quad r^2 = \frac{50^2}{R^2 + 50^2}. \quad (7)\]

This draws an ellipse of width \(2r^2\), height \(2r\), centered at \((1 - r^2, 0)\), and always passing through the point \((1, 0)\). The axial ratio ranges as \(0 < r < 1\) according to \(R\). Especially for \(R \to 0\), the locus becomes a unit circle, which is exactly the bound of disk \(\Omega\). When \(R \to \infty\), the axial ratio converges to zero and the locus is no longer an ellipse but is reduced to a single point located at \((1, 0)\). Thus, the resistance \(R\) exclusively dominates the ellipse’s aspect and position. Fig. 1 shows the ellipses drawn for \(R = 25, 35, 50, 100, 140,\) and \(200\) ohm. They are called *horocycles* in hyperbolic geometry. However, in the engineering language, we call them *constant-R contours*, because each of them physically implies an inductor or a capacitor connected in series to the load. The arrow in Fig. 1 denotes the direction in which the impedance moves by adding \(L\), \(C\), or \(R\) along a contour.

![Fig. 1 Constant-R and -X contours on a Beltrami-Klein disk. Straight lines: \(X = 0, \pm 10, \pm 25, \pm 50, \pm 100, \pm 250\) ohm; Ellipses: \(R = 25, 35, 50, 100, 140, 200\) ohm.](image)

### 5 Transmission lines

As well as lumped-constant elements, high-frequency systems often employ distributed-constant ones such as transmission lines. A lossless transmission line is fully characterized by its two-port impedance matrix

\[
\begin{bmatrix}
  z_{11} & z_{12} \\
  z_{21} & z_{22}
\end{bmatrix}
= \frac{Z_c}{j\sin\phi}
\begin{bmatrix}
  \cos\phi & 1 \\
  1 & \cos\phi
\end{bmatrix}
\quad (8)
\]

where \(Z_c\) and \(\phi\) denote the line’s characteristic impedance and electrical length or phase delay. Now imagine a transmission line loaded with a resistor \(R_o\) at one port.
Through the matrix’s four components, the line’s opposite port exhibits the impedance

\[ Z = z_{11} - \frac{z_{12} z_{21}}{z_{22} + R_o} . \]  

(9)

Decomposing \( Z \) into resistance and reactance \( R + jX \), and substituting Eq. (8) into each of them, we obtain

\[ R = \frac{Z_c^2 R_o}{Z_c^2 \cos^2 \phi + R_o^2 \sin^2 \phi}, \quad X = \frac{Z_c(Z_c^2 - R_o^2) \cos \phi \sin \phi}{Z_c^2 \cos^2 \phi + R_o^2 \sin^2 \phi}. \]  

(10)

Eliminating \( \phi \) from this set of equations, we reach

\[ R_cX^2 = (R - R_o)(Z_c^2 - RR_o). \]  

(11)

By way of Eq. (5), we rewrite this equation as \( u-v \) quadratic form

\[ Au^2 + 2Bu + Cv^2 + D = 0 \]  

(12)

with the four coefficients representing

\[ A = (R_o^2 + 50^2)(Z_c^4 + 50^2 R_o^2), \quad B = R_o^2(50^4 - Z_c^4) \]  
\[ C = 50^2(Z_c^2 + R_o^2)^2, \quad D = (R_o^2 - 50^2)(Z_c^4 - 50^2 R_o^2). \]  

(13)

This is the general equation of transmission-line trajectories observed on the \( u-v \) plane or disk \( \Omega \) in 50-ohm reference. Looking at the coefficient formulas, we find that \( A > 0 \) and \( C > 0 \). Therefore, the above quadratic equation draws a circle or an ellipse as the trajectory. Since the equation involves neither first-order term of \( v \) nor \( u-v \) cross term, the trajectory is always centered somewhere on the horizon, and two elliptic axes are never slant but exactly horizontal and vertical. Focusing on \( A \) and \( C \) again in detail, we also notice that their subtraction makes

\[ A - C = R_o^2(Z_c^2 - 50^2)^2 \quad \therefore \quad A \geq C . \]  

(14)

If and only if \( Z_c = 50 \) ohm, we have \( A = C \), which implies a pure circle. Otherwise, namely if \( A > C \), it results in a vertical-long ellipse on the disk.

Let us experience what locus the equation actually displays by giving three instances for typical sets of \( R_o \) and \( Z_c \). First, when \( R_o = 50 \) and \( Z_c = 25 \), the general equation reduces to

\[ 34u^2 + 30u + 25v^2 = 0 . \]  

(15)

This draws a vertical ellipse passing through the origin \((0, 0)\). Next, when \( R_o = 25 \) and \( Z_c = 50 \), it reduces to

\[ 25u^2 + 25v^2 = 9 . \]  

(16)

This draws a circle of radius 3/5 centered at the origin. The last example is for \( R_o = 100 \) and \( Z_c = 200 \), resulting in

\[ 325u^2 - 510u + 100v^2 + 189 = 0 . \]  

(17)

This draws a vertical ellipse again, but is remote from the origin. These trajectories
are illustrated on the disk in Fig. 2. We find them all stay in symmetry to the horizon. The axial ratio changes with the line impedance. Ellipses located away from the origin are slenderer than ones near the origin.

![Diagram of calculated trajectory](image)

**Fig. 2** Calculated trajectory starting from resistive load $R_o$ and revolving clockwise along a circle or an ellipse.

(a) $R_o = 50$, $Z_e = 25$; (b) $R_o = 25$, $Z_e = 50$; (c) $R_o = 100$, $Z_e = 200$.

### 6 Conclusion

We have explored theoretically the Beltrami-Klein disk from the viewpoint of impedance projection. In comparison to the Smith chart, the mapping scale is quadruple in area near the origin. The disk is highly elegant, with the constant-$X$ contours drawing straight lines that commonly converge at the point of infinity. In contrast, it is less elegant that the constant-$R$ contours make ellipses rather than circles. We therefore conclude that the disk works in a similar manner to the Smith chart on the whole topology, but exhibits a clear distinction in the local mapping scale and slope angle. Although centuries have passed since the disk was discovered, it is still worth exploiting for RF system design and development. We hope this discussion will stimulate study and open up new geometrical vistas in future radio and wireless engineering.

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