Loop symmetry of integrable vertex models at roots of unity

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Abstract

It has been recently discovered in the context of the six vertex or XXZ model in the fundamental representation that new symmetries arise when the anisotropy parameter \((q + q^{-1})/2\) is evaluated at roots of unity \(q^N = 1\). These new symmetries have been linked to an \(U(A^{(1)}_1)\) invariance of the transfer matrix and the corresponding spin-chain Hamiltonian. In this paper these results are generalized for odd primitive roots of unity to all vertex models associated with trigonometric solutions of the Yang-Baxter equation by invoking representation independent methods which only take the algebraic structure of the underlying quantum groups \(U_q(\hat{g})\) into account. Here \(\hat{g}\) is an arbitrary Kac-Moody algebra. Employing the notion of the boost operator it is then found that the Hamiltonian and the transfer matrix of the integrable model are invariant under the action of \(U(\hat{g})\). For the simplest case \(\hat{g} = A^{(1)}_1\) the discussion is also extended to even primitive roots of unity.

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1 Introduction

The six-vertex \[\text{or XXZ model} \] with periodic boundary conditions as defined by the following spin-chain Hamiltonian

\[
\mathcal{H}_{\text{XXZ}}^{s=1/2} = \sum_{j=1}^{L} \left\{ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \frac{q + q^{-1}}{2} \left( \sigma_j^z \sigma_{j+1}^z - 1 \right) \right\}, \quad L + 1 \equiv 1 \tag{1}
\]

has been subject to extensive studies for a long time. Here \(\sigma_j^x, \sigma_j^y, \sigma_j^z\) are the Pauli matrices acting on the \(j^{\text{th}}\) lattice site. Surprisingly the model and its underlying symmetries are still not fully understood. Baxter already noted in 1973 that the model besides its integrable structure for generic anisotropy parameter \(q + q^{-1})/2\) shows additional symmetries when \(q\) becomes an \(N^{\text{th}}\) primitive root of unity (i.e. \(N\) is the smallest integer such that \(q^N = 1\)). Despite numerous articles addressing the energy spectrum and the problem of completeness of the eigenstates for generic \(q\), the symmetry governing the root of unity case has just recently been discovered in \[4, 5\]. The key results obtained by algebraic and numerical methods in the latter articles are the following,

1. As \(q\) approaches a root of unity the transfer matrix of the six-vertex model as well as the associated Hamiltonian exhibit an \(U(A_{1}^{(1)})\) invariance at level zero. (It is for this reason that we have mentioned loop symmetry instead of affine symmetry in the title.) For total spin values being a multiple of \(N' := \begin{cases} \frac{N}{2}, & N \text{ even} \\ N, & N \text{ odd} \end{cases} \tag{2}\)
   
   the generators of this symmetry algebra can be constructed from the quantum group \(U_q(A_{1}^{(1)})\) associated with the \(R\)-matrix of the six-vertex model as \(q^N \to 1\). In addition, the symmetry algebra preserves the momentum, i.e. the \(U(A_{1}^{(1)})\) generators commute with the shift operator.

2. In the framework of the Bethe Ansatz \[6\] the degeneracies manifest themselves in additional string solutions possessing zero energy, which are called exact complete \(N'\)-strings and were first found by Baxter \[3\] (see also the review of Takahashi \[7\]). However, the link between these string solutions and the above symmetry algebra had not been recognized. Moreover, the exact complete \(N'\)-strings lead to a simultaneous vanishing of the numerator and the denominator inside Bethe’s equation, the latter therefore fails to determine the complete set of eigenstates. It has been demonstrated in \[8\] how additional equations can be derived from Bethe’s equation in the limit \(q^N \to 1\) which then allow the determination of the real parts of the exact complete \(N'\)-strings.

*Note that in the articles \[4, 5\] a different convention to parametrize the roots of unity had been chosen. The power \(N\) in the latter articles corresponds to the power \(N'\) in this work.
In this article we demonstrate that the above observations are indeed of a very general nature and not only can be extended for the XXZ model to arbitrary spin $s = 1$ at site $j$. The XXZ model for arbitrary spin has been investigated in [10, 11]. The corresponding Hamiltonian for spin $s > 1$ have not been written down in terms of spin operators but can be defined through the transfer matrix of the associated statistical model. Also here the problem of the degeneracies and the underlying symmetry at roots of unity has not been addressed. Besides the extension to arbitrary spin one can also consider the case of higher rank. For the fundamental representation $V_{\lambda_1} = C^{n+1}$ the R-matrix associated with the $A_n^{(1)} \equiv \widehat{sl}_{n+1}$ vertex models has been found in [12] and the corresponding spin-chain Hamiltonian in [13],

$$\mathcal{H}_{A_n^{(1)}} = \sum_{j=1}^{L} \left\{ \sum_{k \neq l} \left[ E^{kl}_{j} \otimes E^{kl}_{j+1} + E^{kl}_{j} \otimes E^{lk}_{j+1} + iE^{kl}_{j} \otimes iE^{kl}_{j+1} - iE^{kl}_{j} \otimes iE^{lk}_{j+1} \right] - \frac{q + q^{-1}}{2} \left( \sum_{k<l} \left[ E^{kl}_{j} \otimes E^{kl}_{j+1} - E^{lk}_{j} \otimes E^{lk}_{j+1} \right] - \sum_k E^{kk}_{j} \otimes E^{kk}_{j+1} \right) \right\}$$

(4)

where $E^{kl}_{j}$, $1 \leq k, l \leq n+1$ denote the $(n+1) \times (n+1)$ unit matrices, whose entries are all zero except for the entry in the $k$th row and $l$th column which is equal to one. For $n = 1$ one recovers the XXZ Hamiltonian (1). Many R-matrices belonging to algebras different from $\hat{g} = A_n^{(1)}$ have been investigated in e.g. [14].

In this article we demonstrate that the above observations are indeed of a very general nature and not only can be extended for the XXZ model to arbitrary spin but also to the much wider class of integrable vertex models associated with the quantum groups $U_q(\hat{g})$, where $\hat{g}$ is an arbitrary Kac-Moody algebra [15]. The relation between integrable models and the quantum groups $U_q(\hat{g})$ was first established by Drinfel’d [16] and Jimbo [17] who studied trigonometric solutions of the Yang-Baxter equation [18],

$$R_{12}(u)R_{13}(u + v)R_{23}(v) = R_{23}(v)R_{13}(u + v)R_{12}(u).$$

(5)

The Yang-Baxter equation is an operator identity over $V_1 \otimes V_2 \otimes V_3$ with $R_{ab}(u)$ acting on $V_a \otimes V_b$ and $V_3 \cong V$ being some representation space of $U_q(\hat{g})$. An integrable $L' \times L$ vertex
model is now implicitly defined when interpreting the matrix elements of the solution $R(u)$ as Boltzmann weights and taking the partition function and transfer matrix to be

$$Z = \text{Tr}_{V^\otimes L} T'(u), \quad T(u) := \text{Tr}_{V_0} R_{0L}(u) R_{0L-1}(u) \cdots R_{01}(u).$$ (6)

The transfer matrix acts on the tensor product space $V^\otimes L \equiv V_1 \otimes V_2 \cdots \otimes V_L$ and the trace is taken over the boundary values encoded in the auxiliary space $V_0$. As is well known (5) ensures that the transfer matrix when evaluated at different spectral parameters $u$ commutes with itself rendering the model (6) integrable [19, 20]. The corresponding 'spin'-chain Hamiltonian is now generically given by

$$H = i \frac{\partial}{\partial u} \ln T(u) \bigg|_{u=0} + \text{const.}$$ (7)

Up to possible scaling factors depending on different conventions and an additive constant depending on the normalization of the R-matrix this definition specializes for $\hat{g} = A_1^{(1)}$ to the stated examples (1), (3) and (4).

It is important to note that in the correspondence between quantum groups and integrable vertex models the quantum group $U_q(\hat{g})$ does not define a symmetry of the model and that its generators do not commute with either the Hamiltonian or the transfer matrix. This lack of symmetry comes ultimately from the fact that by construction the transfer matrix (5) and Hamiltonian (7) we are considering are translational invariant in the appropriate space $V^\otimes L$ whereas for generic $q$ the quantum group $U_q(\hat{g})$ does not act in a translational invariant fashion on this space. The situation changes considerably when the deformation parameter $q$ approaches a primitive root of unity, $q^N = 1$ with $N \geq 3$ being odd. Then symmetry generators can be extracted from $U_q(\hat{g})$ in the limiting process $q^N \to 1$ which are translation invariant and which generate the algebra $U(\hat{g})$ at level zero.

The paper is organized as follows. In Section 2 we construct the symmetry algebra $U(\hat{g})$ for odd roots of unity and highest weight representations $\lambda$ obeying $\lambda(h_i) \equiv 0 \mod N$ with $\lambda(h_i)$ being the eigenvalues of the Cartan subalgebra generators $h_i$. In Section 3 we prove the translational invariance of the symmetry generators when they act on the space $V^\otimes L$. In Section 4 we demonstrate for any algebra $\hat{g}$ in a completely generic and representation independent way that the transfer matrix and Hamiltonian associated with $U_q(\hat{g})$ commute with the generators of $U(\hat{g})$ when $q$ is a root of unity. This proof makes use of the boost operator (e.g. [21]) and the quantum group theoretical structure underlying the Yang-Baxter equation as developed by Drinfel’d [16] and Jimbo [17]. Some results of their construction which are relevant to our discussion are reviewed in the appendix. For the simplest case $\hat{g} = A_1^{(1)}$, i.e. the XXZ model, we discuss for arbitrary spin how the results can also be extended to even roots of unity. Finally we conclude in Section 5 with a discussion of our results. By making contact with the representation theory of the symmetry algebra we argue that for all untwisted algebras the degeneracies of the energy eigenstates should be given by powers of the dimension of the fundamental representation $(\text{dim} \lambda)^l$ where $l$ is some integer depending on the multiplet of the energy eigenstate.
2 Constructing $\mathcal{U}(\hat{g})$ from $\mathcal{U}_q(\hat{g})$ at roots of unity

We begin by reviewing the basic definition of $\mathcal{U}_q(\hat{g})$ for arbitrary Kac-Moody algebras $\hat{g}$ in order to introduce our notation (further details can be found in the original references [16, 17] and in numerous monographs e.g. [24, 25]). The quantum universal enveloping algebra $\mathcal{U}_q(\hat{g})$ is the algebra of power series in the Chevalley-Serre generators $\{e_i, f_i, h_i\}_{i=0}^{\text{rank } \hat{g}-1} \cup \{1\}$ subject to the following commutation relations:

(Q1) Let $A$ denote the Cartan matrix associated with the Kac-Moody algebra $\hat{g}$. Then

$$[h_i, h_j] = 0, \quad [h_i, e_j] = A_{ij}e_j, \quad \text{and} \quad [h_i, e_j] = -A_{ij}f_j.$$

(Q2) Considering only highest weight representations in which the $h_i$'s act as multiplication operators, we introduce the exponentiated generators $q^h_i$ with $q_i = q^{\alpha_i^2/2}$ and $\alpha_i$ denoting a simple root of $\hat{g}$. Then one requires

$$[e_i, f_j] = \delta_{ij} q_i^{h_i} q_i^{1-h_i} q_i^{1-q_i^{-1}}.$$

For simplicity we choose throughout this paper the normalization convention $\alpha^2 = 2$ for short roots. That is, for long roots $q_i$ might equal the powers $q, q^2, q^3$ in the deformation parameter.

(Q3) In addition, the generators ought to satisfy the quantum Chevalley-Serre relations

$$\sum_{n=0}^{1-A_{ij}} (-1)^n \left[ \frac{1-A_{ij}}{n} \right]_{q_i} e_i^n e_j e_i^{1-A_{ji}-n} = 0, \quad i \neq j$$

$$\sum_{n=0}^{1-A_{ij}} (-1)^n \left[ \frac{1-A_{ij}}{n} \right]_{q_i} f_i^n f_j f_i^{1-A_{ji}-n} = 0, \quad i \neq j$$

Here $q$-integers have been introduced,

$$\left[ \frac{m}{n} \right]_q := \frac{[m]_q!}{[n]_q! [m-n]_q!}, \quad [n]_q := \prod_{k=1}^{n} [k]_q, \quad \left[ \frac{m}{n} \right]_{q_i} := \frac{[m]_{q_i}!}{[n]_{q_i}! [m-n]_{q_i}!}.$$

We will now focus on the case where the deformation parameter $q$ takes the value of a primitive root of unity, $q^N = 1$. It is known that in this case the elements $e_i^{N'}$ and $f_i^{N''}$ are central elements. We are here interested in representations in which these central elements may be set equal to zero and for these representations it has been shown [22] that the generators

$$e_i^{(N')} := \frac{e_i^{N'}}{[N']_{q_i}!}, \quad f_i^{(N'')} := \frac{f_i^{N''}}{[N'']_{q_i}!}, \quad \text{and} \quad h_i/N'$$
stay well defined in the limit $q^N \to 1$. As it was first observed in [4] for $\hat{g} = A_1^{(1)}$ in the fundamental representation the above set (complemented by unity) generates the non-deformed enveloping algebra $U(\hat{g})$ provided one restricts oneself to highest weight representations $|\lambda\rangle$ satisfying

$$\lambda(h_i) = 0 \mod N. \quad (10)$$

We impose this condition since we will ultimately make use of $q^{-h_i} = 1$ in the calculations (see for example equation (27) in the next section). Note also that this condition will ultimately have to hold for tensor products of highest weight representations, since we are going to consider the action of $U(\hat{g})$ on $V^\otimes L$, see the next section. The proof follows along the lines of [4] and we recall here the key steps in order to keep this article self-contained. First we investigate the commutation relations between the generators $e_i^{(N')}, f_i^{(N')}$ starting from the following relation valid for all $q$ [23],

$$[e_i^{(m)}, f_i^{(n)}] = \sum_{l=0}^{\min(m,n)} f_i^{(n-l)} e_i^{(m-l)} \prod_{r=1}^{l} q_i^{h_i - m_n-r+1} - q_i^{h_i} q_i^{m_n+r-1} q_i^r - q_i^{-r}. \quad (11)$$

Choosing a highest weight such that (10) is satisfied one obtains in the limit $q^N \to 1$,

$$\lim_{q^N \to 1} [e_i^{(N')}, f_i^{(N')} ] = (-1)^{N'-1} \lim_{q^N \to 1} \frac{q_i^{h_i} - q_i^{-h_i}}{q_i^{N'} - q_i^N} = (-1)^{N'-1} q_i^h h_i N^N. \quad (12)$$

Furthermore, one proves easily from (Q1) by induction that

$$[h_i, e_j^{(N')} ] = N' A_{ij} e_j^{(N')} \quad \text{and} \quad [h_i, f_j^{(N')} ] = -N' A_{ij} f_j^{(N')} . \quad (13)$$

It remains to verify the Chevalley-Serre relations for $U(\hat{g})$. For this purpose we employ Lustzig’s higher order Chevalley-Serre relations [22] which are valid for generic $q$. Let $m > -A_{ij} n, n \geq 1$ then the generators $e_i^{(n)} := e_i^n/[n]_{q_i}$ ! satisfy

$$e_i^{(m)} e_j^{(n)} = \sum_{k=0}^{-n_{A_{ij}}} C_{m-k}(q_i) e_i^{(k)} e_j^{(n)} e_i^{(m-k)}, \quad (14)$$

where the coefficient function is given by

$$C_s(q) = \sum_{l=0}^{m+A_{ij} n-1} (-1)^{s+l+1} q^{-s_{l+1-A_{ij} n+m}} \left[ s \atop l \right]_{q} \quad (15)$$

Choosing $m = 1 - A_{ij}, n = 1$ one recovers the usual quantum Chevalley-Serre relations (Q3). Suppose now that the indeterminate $q$ approaches a root of unity $q^N \to 1$. Setting $m = N'(1 - A_{ij})$ and $n = N'$ one verifies for the coefficient function,

$$\lim_{q^N \to 1} C_s(q_i) = \begin{cases} (-1)^{s+1} q_i^{s(N'-1)}, & \text{if } s = 0 \mod N' \\ 0, & \text{else} \end{cases}. \quad (16)$$

\footnote{For the case $\hat{g} = A_1^{(1)}$ considered in [4] this corresponds to condition $S^2 = 0 \mod N'$ with $h_1 = -h_0 = 2S^2$ and $N$ even.}
We now rewrite the higher order Chevalley-Serre equations in terms of powers of the operators $e_i^{(N')}$ by employing the identity

$$e_i^{(N')s} = \frac{[N']_q}{[N's]_{q^s}} e_i^{(N')s} \quad \text{with} \quad \lim_{q^s \to 1} \frac{[N']_q}{[N's]_q} = \frac{N'^{2(s+1)}}{s!}.$$  \hspace{1cm} (17)

Plugging the results (16) and (17) into equation (14) one derives the desired Chevalley-Serre relations of the non-deformed enveloping algebra $U(\hat{g})$ up to certain sign factors,

$$e_i^{(N')(1-A_{ij})} e_j^{(N')} = \sum_{n=0}^{-A_{ij}} (-1)^{N'(1-A_{ij}-n)+1} q_i^{(1-A_{ij}-n)N'(N'-1)} \times q^{-nN'^2(1-A_{ij}-n)} \binom{1-A_{ij}}{n} e_i^{(N')n} e_j^{(N')} e_i^{(N')(1-A_{ij}-n)}.$$ \hspace{1cm} (18)

An analogous equation holds for the generators $f_i^{(N')}$. In order to make contact to the Chevalley-Serre relations of $U(\hat{g})$ one has to discuss carefully the cancellation of the minus signs in the r.h.s. of the above equation. We distinguish the following three cases.

$N$ odd. For odd roots of unity one recovers the correct sign $(-1)^{(1-A_{ij}-n)+1}$ needed for the Chevalley-Serre relations in the r.h.s. of (18). For this case we can therefore conclude that the algebra spanned by the elements (9) can be identified with the non-deformed enveloping algebra $U(\hat{g})$ for all Kac-Moody algebras.

$N'$ even. For even roots of unity one has $q^{N'} = -1$. Provided that $N'$ even and $q_i^{N'} = -1$ for all $i$ one obtains the correct sign factor also in this case. The latter condition requires $\hat{g}$ to be either simply-laced, i.e. $\hat{g} = A_n^{(1)}, D_n^{(1)}, E_6^{(1)}, A_2^{(2)}$, or to be one of the non simply-laced algebras $\hat{g} = G_2^{(1)}, D_4^{(3)}$.

$N'$ odd. In the remaining case of even roots of unity with $N'$ odd one obtains the sign factor $(-1)^{(n+1)(1-A_{ij}-n)+1}$. In general this will not reproduce the correct Chevalley-Serre relations. For the simplest case $\hat{g} = A_1^{(1)}$, however, the signs work out correctly which can be explicitly checked by using $A_{ij} = -2$ for $i \neq j$, compare also [4]. But now one has to pay attention to the sign in (12).

3 Translation invariance

We now establish that the action of the constructed symmetry algebra $U(\hat{g})$ is translation invariant. That is, given some representation space $V$ of $U_q(\hat{g})$ we consider its $L$-fold tensor product $V \otimes^L$ and then show in the limit $q^{N'} \to 1$ that the action of the symmetry algebra on this space commutes with the shift-operator defined as

$$\Pi : V_1 \otimes V_2 \cdots \otimes V_L \to V_2 \otimes \cdots V_L \otimes V_1.$$ \hspace{1cm} (19)
As a preliminary step we first recall the action of $U_q(\hat{g})$ on $V^{\otimes L}$. The latter is determined by the fact that the quantum groups defined through (Q1)-(Q3) are endowed with the structure of an Hopf algebra [16, 17]. This requires in general the notion of a co-unit $\tilde{e}$, an anti-pode $\gamma$ and a coproduct $\Delta$. We will only need the concept of the latter which establishes an algebra homomorphism $U_q(\hat{g}) \rightarrow U_q(\hat{g}) \otimes U_q(\hat{g})$. There are different conventions in the literature how to define the coproduct and we choose the one most convenient for our purposes:

\[
\begin{align*}
\Delta(h_i) &= h_i \otimes 1 + 1 \otimes h_i \\
\Delta(e_i) &= e_i \otimes q_i^{-\frac{h_i}{2}} + q_i^{\frac{h_i}{2}} \otimes e_i \quad \text{and} \quad \Delta(f_i) = f_i \otimes q_i^{-\frac{h_i}{2}} + q_i^{\frac{h_i}{2}} \otimes f_i \\
\Delta(1) &= 1 \otimes 1 
\end{align*}
\]

(20)

Following [17] we can now use the coproduct $\Delta$ iteratively to generate higher tensor products by setting

\[
\Delta^{(L)} = (\Delta \otimes 1_{L-2})\Delta^{(L-1)} \quad \text{with} \quad \Delta^{(2)} \equiv \Delta. 
\]

(21)

In fact, this defines an algebra homomorphism $\Delta^{(L)} : U_q(\hat{g}) \rightarrow U_q(\hat{g})^{\otimes L}$ and the generators acting on $V^{\otimes L}$ then explicitly read,

\[
\begin{align*}
\Delta^{(L)}(e_i) &\equiv E_i = \sum_{n=1}^{L} E_i(n), \quad E_i(n) := q_i^{\frac{h_i}{2}} \otimes \cdots q_i^{\frac{h_i}{2}} \otimes e_i \otimes q_i^{-\frac{h_i}{2}} \cdots \otimes q_i^{-\frac{h_i}{2}} \\
\Delta^{(L)}(f_i) &\equiv F_i = \sum_{n=1}^{L} F_i(n), \quad F_i(n) := q_i^{\frac{h_i}{2}} \otimes \cdots q_i^{\frac{h_i}{2}} \otimes f_i \otimes q_i^{-\frac{h_i}{2}} \cdots \otimes q_i^{-\frac{h_i}{2}} \\
\Delta^{(L)}(q_i^{h_i}) &\equiv q_i^{H_i} = \prod_{n=1}^{L} q_i^{H_i(n)}, \quad q_i^{H_i(n)} := 1 \otimes \cdots 1 \otimes q_i^{h_i} \otimes 1 \cdots 1. 
\end{align*}
\]

(22)

For completeness we state also the explicit form of the symmetry generators (9) for the $L$-fold tensor product. Starting from the following relation which is easily proved by induction,

\[
\Delta(e_i^n) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q_i^k q_i^{(n-k)\frac{h_i}{2}} \otimes e_i^{-k} q_i^{-k\frac{h_i}{2}} \otimes e_i^{-k} q_i^{-k\frac{h_i}{2}} = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q_i^{-k} q_i^{k\frac{h_i}{2}} \otimes e_i^{k} q_i^{-(n-k)\frac{h_i}{2}} 
\]

(23)

one verifies that the symmetry generators acting on the tensor product space $V^{\otimes L}$ are given by the expression

\[
E_i^{(N)} \equiv \Delta^{(L)}(e_i^{(N)}) = \sum_{0=n_0 \leq n_1 \leq \cdots \leq n_L = N} \otimes q_i^{(N-n_0-n_1)\frac{h_i}{2}}. 
\]

(24)

This formula is immediate to derive by exploiting the fact that the coproduct is an algebra homomorphism $U_q(\hat{g}) \rightarrow U_q(\hat{g}) \otimes U_q(\hat{g})$ and then applying equation (24) to the...
first factor in the tensor product. Note that according to the definition of \( E_i^{(N)} \) we have divided out the factor \([N]_q!\) present in the \( q\)-binomial coefficient in (23). A similar formula holds for \( F_i^{(N)} \).

We are now prepared to generalize the proof of invariance found in [1] for the XXZ model in the fundamental representation. In order to show translation invariance of the symmetry algebra it is obviously sufficient to show that the generators \( E_i^{(N)}, F_i^{(N)}, H_i/N' \) commute with the shift operator. From (22) we see immediately that \([\Pi_i H_i] = 0\) for all \( q\). We now state the proof for \( E_i^{(N)} \), the one for \( F_i^{(N)} \) is completely analogous. For generic \( q \) one finds the following relations (compare [1])

\[
P_i E_i \Pi^{-1} = E_i q_i^{H_i(L)} + E_i(L)(q_i^{-H_i} - 1)q_i^{H_i(L)},
\]

where use has been made of the straightforward identities

\[
P_i E_i(n) \Pi^{-1} = E_i(n - 1)q_i^{H_i(L)} \quad n > 1
\]

\[
q_i^{H_i(n-1)} \Pi^{-1} = q_i^{H_i(n-1)}.
\]

We claim that the transformation property for the \( m \)th power of the generator reads

\[
P_i E_i^m \Pi^{-1} = \sum_{n=0}^{m} E_i^{m-n} E_i(L)^n q_i^{n(m-1)} \left[ \frac{m}{n} \right] q_i^{m H_i(L)} \prod_{l=0}^{n-1} \left( q_i^{-2l-H_i} - 1 \right).
\]

Here it is understood that the product yields one if \( n = 0 \). For the proof we proceed once more by induction. Assume that the above relation holds for \( m \) we calculate

\[
P_i E_i^{m+1} \Pi^{-1} = \sum_{n=0}^{m} E_i^{m-n} E_i(L)^n q_i^{n(m-1)} \left[ \frac{m}{n} \right] q_i^{(m+1) H_i(L)} \prod_{l=1}^{n} \left( q_i^{-2l-H_i} - 1 \right)
\]

\[
+ \sum_{n=0}^{m} E_i^{m-n} E_i(L)^{n+1} q_i^{n(m-1)} \left[ \frac{m}{n} \right] q_i^{2m-H_i} - 1 \right) q_i^{(m+1) H_i(L)} \prod_{l=1}^{n} \left( q_i^{-2l-H_i} - 1 \right)
\]

Employing the commutation relations

\[
E_i(L)^n E_i = q_i^{2n} E_i E_i(L)^n + E_i(L)^{n+1}(1 - q_i^{2n})
\]

and the elementary relation

\[
\left[ m+n \right] = \left[ m \right] q^{-n} + \left[ n \right] q^m
\]

for \( q\)-deformed integers one derives the desired result (26). Now taking the limit \( q^N \to 1 \) one finds by setting \( m = N' \) from (20) that

\[
P_i E_i^{(N')} \Pi^{-1} = E_i^{(N')} q_i^{N'H_i(L)},
\]

since the product always contains a vanishing factor due to the condition (10). We discuss the effect of the factor \( q_i^{N'H_i(L)} \) for the cases of odd and even roots of unity separately.
\textbf{N odd.} For odd roots of unity, \(N = N'\), the factor is always equal to one. Thus, we conclude that the constructed symmetry algebra \(U(\hat{g})\) generated by the elements \(\{E_i^{(N)}, F_i^{(N)}, H_i/N\} \cup \{1\}\) commutes with the shift operator. Recall that for this case the symmetry algebra has been constructed in complete generality, i.e. for all Kac-Moody algebras.

\textbf{N even.} For even roots of unity \(q^N\) produces in general alternating signs. This can be seen e.g. from the commutation relation (compare (Q1) in Section 2)

\[ E_j(L)(q_i^{N'})H_i(L) = (q_i^{N'})^{-A_{ij}}(q_i^{N'})H_i(L)E_j(L). \] (28)

However, for the special case of the XXZ model \(\hat{g} = A_1^{(1)}\) one has \(|A_{ij}| = 2\) for all \(i, j\) from which we infer that the generators \(E_i^{(N')}, F_i^{(N')}\) of the symmetry algebra \(U(A_1^{(1)})\) either commute or anticommute with the shift operator depending on \(V_L \cong \mathbb{C}^{n+1}\) being either of even or odd highest weight \(n \in \mathbb{N}\), respectively. Here \(n = 2s\) and \(s\) is the spin. This is accordance with the results obtained in [4], where the fundamental representation \(n = 1\) has been considered.

\section{\(U(\hat{g})\) symmetry at roots of unity}

We are now prepared to establish the \(U(\hat{g})\) invariance of the statistical model associated with the affine quantum group \(U_q(\hat{g})\) as defined in (6). The crucial ingredient for this derivation is the R-matrix which provides a solution of the Yang-Baxter equation (5). How every quantum group \(U_q(\hat{g})\) gives rise to such a solution is reviewed in the appendix in order to keep this article self-contained. The proof of invariance then hinges on two observations, namely that the generators of the symmetry algebra commute with the shift operator and the quantum group invariance of the permuted R-matrix (see equation (49) in the appendix.) This allows us to state the symmetry property for the large class of solutions (e.g. [12, 14]) of the Yang-Baxter equation (3) in a completely general fashion.

\subsection{The integrable model}

Suppose now we are given a trigonometric solution \(R(u)\) of (5) associated with \(U_q(\hat{g})\) and which acts on the tensor product \(V \otimes V\) of some representation space \(V\). We choose to normalize the \(R\)-matrix such that

\[ \lim_{u \to 0} R(u) = \pi, \] (29)

where \(\pi\) is the permutation operator. For affine quantum groups we consider this regularity property always holds. The normalization (29) fixes the constant in (7) to be zero. As is well known [13, 21] it follows directly from the Yang-Baxter equation that

\[ [T(u), T(v)] = 0 \] (30)
which implies the integrability of the model. The corresponding infinite set of charges is defined by the following power series expansion of the transfer matrix at vanishing rapidity $u = 0$,

$$\ln T(u) = \sum_{n=0}^{\infty} \frac{u^n}{n!} T^{(n)}$$

with

$$T^{(n)} = \frac{\partial^n}{\partial u^n} \ln T(u) \bigg|_{u=0}.$$  

(31)

From equation (30) we now immediately infer $[T(u), T^{(n)}] = [T^{(n)}, T^{(m)}] = 0$ for all choices of $n, m \in \mathbb{N}$ which manifests the integrability of the model. The zeroth and first order term in the above expansion (31) are of special significance. From the normalization condition (29) of the $R$-matrix one derives for the zeroth term

$$T^{(0)} = \ln T(0) = \ln \Pi^{-1} \equiv -iP ,$$

(32)

where $\Pi$ is the 'shift'-operator introduced in (19) and which generates translations in the horizontal direction of the lattice. This motivates the identification of $T^{(0)}$ as the momentum operator. The first order term $T^{(1)}$ is identical with the (formal) spin-chain Hamiltonian $\widehat{H}$ as defined in the introduction (7),

$$\mathcal{H} = i \frac{\partial}{\partial u} \ln T(u) \bigg|_{u=0} = i \sum_{j=1}^{L} \frac{\partial}{\partial u} R_{jj+1}(u) \bigg|_{u=0} \quad \text{with} \quad L + 1 \equiv 1 .$$

(33)

Here we have defined the operator $R_{jj+1}(u) := \pi_{jj+1} R_{jj+1}(u)$ which acts on the tensor product $V_j \otimes V_{j+1}$ in the chain $V^{\otimes L} = V_1 \otimes \cdots \otimes V_L$. Formula (33) can be derived directly from the following operator identity over $V_0 \otimes V_1 \otimes \cdots \otimes V_L$

$$R_{0L}(u) \cdots R_{01}(u) = \pi_{01} \Pi^{-1} R_{L-1L}(u) \cdots R_{12}(u) R_{01}(u)$$

(34)

The main reason for changing from the original $R$-matrix to $R(u) = \pi R(u)$ comes from the observation that the latter is invariant under the quantum group action (see equation (19) in the appendix). We will now use this fact together with the notion of the boost operator to demonstrate that all conserved charges of the model and especially the transfer matrix are left invariant under the action of $U(\hat{g})$.

### 4.2 The boost operator

The boost operator of the integrable model is implicitly defined by the relation

$$-\frac{\partial}{\partial u} T(u) = [K, T(u)] .$$

(35)

\(^1\text{Notice that this definition of the Hamiltonian is formal in the sense that it is not necessarily always hermitian. For example, as was pointed out for the XXZ model}^{[3]} \text{ hermiticity might restrict for fixed spin the allowed values of the coupling constant } \gamma \text{ incorporated in the deformation parameter } q = e^{i\gamma}.\)
Its explicit expression in terms of R-matrices has been found in e.g. \[21\].

\[
K = \sum_{n \in \mathbb{N}} \sum_{j=1}^{L} (j + nL) \partial_u R_{jj+1}(u) \big|_{u=0}
\]  

(36)

and can be derived by differentiating the Yang-Baxter equation (3) and exploiting translation invariance of the transfer matrix. The name 'boost' operator stems from the observation that $P, H, K$ form a closed algebra which might be interpreted as a lattice version of the Poincaré algebra (see e.g. the article by Thacker \[21\]). From the defining property (35) one infers that under the adjoint action of the boost operator the transfer matrix is shifted in the spectral parameter,

\[
T(u + v) = e^{-vK} T(u) e^{vK} .
\]  

(37)

Therefore, it is sufficient to show that the generators of the constructed algebra $U(\hat{g})$ commute with the shift operator $\Pi = e^{iP}$ and the boost operator $K$ in order to ensure that they also commute with the transfer matrix and the Hamiltonian (as well as all higher charges). Since we have already proven in Section 3 that for roots of unity with $N$ odd the symmetry algebra is translation invariant, i.e. $[X, \Pi] = 0$ for all $X \in U(\hat{g})$, we need only to show the invariance of the boost operator.

### 4.3 Invariance of the boost operator

According to equation (39) in the appendix one easily verifies that the operators $R_{jj+1}(u)$ commute with the generators (22) of the quantum group $U_q(\hat{g})^\otimes L$ in the evaluation representation at generic $q$ for $j < L$,

\[
[\Delta_u^{(L)}(x), R_{jj+1}(u)] = 0 , \quad x \in U_q(\hat{g}) , \ j < L
\]  

(38)

Recall that when taking the root of unity limit a universal R-matrix may not always exist, since then the elements $E_i^N, F_i^N, q_i^{\pm H_i}$ are central and the quasitriangular structure imposes constraints on their spectral values, see e.g. the article by E. Date et al. \[14\]. However, we are only interested in representations where $E_i^N, F_i^N \equiv 0$. Then these restrictions do not exist and a universal R-matrix can be defined. Therefore, property (35) remains true when the limit $q^N \to 1$ is taken and, consequently, all we need to show is that $R_{L1}(u)$ commutes with $X \in U(\hat{g})$. But since we have established translation invariance of the symmetry algebra in the root of unity limit, one immediately verifies that

\[
[X, R_{L1}(u)] = \Pi^{-1} [X, \Pi R_{L1}(u) \Pi^{-1}] \Pi = \Pi^{-1} [X, R_{L-1L}(u)] \Pi = 0 , \quad X \in U(\hat{g})
\]  

(39)

From (35) and the fact that $R(0) = 1$ we conclude that the boost operator commutes with all elements of the algebra $U(\hat{g})$ in the evaluation representation with $e^u \to 1$. Notice that by the same arguments it also follows directly that the Hamiltonian (33) is invariant. We thus conclude that the integrable model (6) associated with the quantum group $U_q(\hat{g})$ exhibits an $U(\hat{g})$ invariance as $q^N \to 1$ with $N$ being odd.
4.4 Even roots of unity and $\hat{g} = A_1^{(1)}$

In Section 2 we have seen that the construction of the algebra $U(\hat{g})$ for the XXZ model $\hat{g} = A_1^{(1)}$ also holds for even roots of unity. As has become apparent in Section 3 the difference to case of odd roots occurs when the behaviour of the symmetry algebra under translation is investigated. Depending on the choice of the spin $s = n/2$ the constructed algebra generators $E_i^{(N')}, F_i^{(N')}, i = 0, 1$ commute or anticommute with the shift operator if we choose the representation spaces $V_n$ in the $L$-fold tensor product to be of highest weight $n$ even or odd. It is evident from equation (37) and (39) that in the latter case the symmetry algebra still commutes with the boost operator, but that the generators $E_i^{(N')}, F_i^{(N')}$ now anticommute with the transfer matrix. We therefore conclude that

$$XT(u) = (-1)^{2s}T(u)X, \quad X = E_i^{(N')}, F_i^{(N')}$$

which generalizes the results obtained in [4] to arbitrary spin $s \in \frac{1}{2}\mathbb{N}$. In contrast, the Hamiltonian (33) obviously commutes with all elements of the symmetry algebra for even and odd roots independent of the chosen spin value.

5 Conclusions

In this article we have shown that the loop symmetry of the Hamiltonian and the transfer matrix first observed in the context of the XXZ and the six vertex model for spin $s = 1/2$ at roots of unity [4] is of a general nature. For odd roots of unity we demonstrated that it is present for generic integrable vertex models associated with trigonometric solutions of the Yang-Baxter equation with underlying quantum group $U_q(\hat{g})$, $\hat{g}$ being any Kac-Moody algebra. The invariance has been shown to be a direct consequence of both the quasi-triangular structure of the quantum group $U_q(\hat{g})$ and the translation invariance of the symmetry algebra $U(\hat{g})$. While for generic algebras we had to restrict ourselves for the construction to roots of unity $q^N = 1$ with $N$ odd the loop symmetry could be extended also to $N$ even for the XXZ model thereby generalizing the results obtained for the fundamental representation [4] to arbitrary spin. The restriction on the highest weight representations ([4]) might only be of technical nature since the numerical investigations performed in the context of the XXZ model [4, 5] point out that the loop symmetry is present in general. We expect, however, the construction of the symmetry algebra in the other cases to be more involved.

We therefore conclude that the spectrum of the Hamiltonian [4] organizes in multiplets of finite dimensional representations of $U(\hat{g})$. That is, given an eigenstate of the Hamiltonian or transfer matrix it belongs to some highest weight representation $\lambda = (\lambda_1, ..., \lambda_r)$ with $r = \text{rank } g$ being the rank and $\lambda_i = \lambda(H_i)$ the eigenvalues of the Cartan generators $H_i$ acting on $V^\otimes L$. (We assume that $\lambda$ satisfies the condition $\lambda_i \mod N = 0$.) The irreducible finite dimensional representations of untwisted algebras have been shown to be isomorphic to tensor products of evaluation representations

$$V_{\lambda_1}(a_1) \otimes V_{\lambda_2}(a_2) \cdots \otimes V_{\lambda_l}(a_l),$$

(41)
whose evaluation parameters \( a_k, k = 1, \ldots, l \) are determined by the roots of rank \( g \) Drinfeld’s Polynomials (see \[26\], pp 11). Here \( g \) is the finite dimensional algebra whose affinization is \( \hat{g} \) and \( l \) is some integer depending on the multiplet \( \lambda \) of the energy eigenstate. Let \( n_{i,k} \) be the multiplicity of the root \( a_k \) in the \( i \)th Polynomial then the weights appearing in the above product are given by \( \Lambda_k = \sum_{i=1}^{\text{rank} g} n_{i,k} \omega_i \) with \( \omega_i \) denoting the fundamental weights of \( g \). Thus, the dimension of the representation, i.e. the degeneracy of the energy eigenstate is given by

\[
\prod_{k=1}^{l} \dim \Lambda_k \quad \text{with} \quad \dim \Lambda = \prod_{\alpha > 0} \frac{\langle \Lambda + \varrho, \alpha \rangle}{\langle \varrho, \alpha \rangle}.
\]

(42)

Here the dimension is calculated from the Weyl formula with the product running over the positive roots of \( g \) and \( \varrho \) denotes the Weyl vector. The numerical work in the context of the XXZ model \[4, 5\] indicates that in the above tensor product only the fundamental representation appears, explaining the degeneracy factors \( 2^l \).

Analogous to the work for the spin \( s = 1/2 \) XXZ model one can now investigate the relation of the generally established symmetry to the Bethe Ansatz. As mentioned already in the introduction it was found \[4, 5\] that the degeneracies are related to ambiguities inside Bethe’s equation which fails to determine the complete set of string solutions at roots of unity. The missing solutions were shown to be complete exact \( N \)-strings

\[
v_k^{(N)} = \alpha + ik2\gamma, \quad 1 \leq k \leq N, \quad q = e^{i\gamma}, \quad \frac{\gamma}{\pi} = \frac{m}{N} \in \mathbb{Q}
\]

which have momentum \( P = 0, \pi \) and zero energy and give thus rise to degeneracies in the energy spectrum. These observations can also be generalized. For example we observe that Bethe’s equation for the higher spin XXZ model \[10\]

\[
\left( \frac{\sinh \frac{1}{2}(v_j + i2s\gamma)}{\sinh \frac{1}{2}(v_j - i2s\gamma)} \right)^{L} = \prod_{k \neq j} \frac{\sinh \frac{1}{2}(v_j - v_k + i2\gamma)}{\sinh \frac{1}{2}(v_j - v_k - i2\gamma)}
\]

incorporates the spin dependence only on the l.h.s. while the ambiguous factors 0/0 due to the complete exact \( N \)-strings occur on the r.h.s. of the above equation. Hence, the statements made in the context of the spin \( s = 1/2 \) case can be generalized in a straightforward way. In particular, the derivation of the equations determining the real parts \( \alpha \) of the complete exact \( N \)-strings in the limit \( q^N \rightarrow 1 \) follows exactly along the lines of \[5\]. In view of these considerations one might now anticipate that the regular solutions to the Bethe’s equation give the highest weight states w.r.t. to the symmetry algebra and that the action of the latter corresponds to adding complete exact \( N \)-strings to this solution. However, to understand this relation fully in terms of representation theory of the symmetry algebra and how to extract from the Bethe Ansatz the evaluation parameters \( a_k \) are both open problems at the moment. They require a deeper and more profound understanding of the quantum group theoretical structure of the Bethe Ansatz.
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A The Yang-Baxter equation

In order to keep this paper self-contained we briefly review the connection between quantum groups and trigonometric solutions to the Yang-Baxter equation \([16, 17]\).

First one introduces for arbitrary spectral parameter \(u\) the following automorphism \(D_u\) on \(U_q(\hat{g})\) which equals the identity on all generators except for

\[
D_u(e_0) = e^u e_0 \quad \text{and} \quad D_u(f_0) = e^{-u} f_0 .
\]

Loosely, speaking it can be thought of as conjugating by \((e^u)^d\), where \(d\) is the homogeneous degree operator of the affine Lie algebra \(\hat{g}\). Analogously, one might also consider the principal grading \(\hat{g} = h^\vee d + \varrho\) where \(h^\vee\) is the dual Coxeter number and \(\varrho\) the Weyl vector, see e.g. \([27]\). Suppose we are given a finite-dimensional representation \(\rho: U_q(\hat{g}) \to \text{End}(V)\), such that \(\rho\) viewed as representation of \(U_q(g)\) has finite length and all irreducible subrepresentations are highest weight \([25]\). Here \(g\) denotes the finite dimensional simple Lie algebra whose affinization is \(\hat{g}\). Then an evaluation representation \(V(u)\) is defined through the following composition of maps,

\[
\rho_{V(u)} = \rho \circ D_u .
\]

Originally the finite-dimensional representations \(\rho\) have been explicitly constructed via evaluation homomorphisms \(p: U_q(\hat{g}) \to U_q(g)\) by Jimbo \([17]\) for the series \(\hat{g} = A_n^{(1)}\). Since they have been implicitly used for the construction in \([4]\) we review them at the end of this section. However, these evaluation homomorphisms do not exist in general \([24, 25]\) whence the above definition of evaluation representations is the more generic one.

Secondly we must specify how solutions of the Yang-Baxter equation come about in the setting of evaluation representations. The R-matrix naturally arises in the context of the coproduct structure \([20]\) which is part of the definition of \(U_q(\hat{g})\) as Hopf algebra \([16, 17]\). As already mentioned in the main text the latter is needed to build up tensor products of representations giving the state space \(V^{\otimes L}\) of the statistical model. As we infer from the definition of the coproduct \([20]\) these tensor products carry an ‘orientation’ since the quantum group \(U_q(\hat{g})\) viewed as an Hopf algebra is in general non-cocommutative. In formulas, this means that the action of the ‘opposite’ coproduct

\[
\Delta^{\text{op}} \equiv \pi \circ \Delta ,
\]

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where $\pi$ denotes the permutation operator, does not coincide with the action of $\Delta$. Thus, the products $V \otimes W$ and $W \otimes V$ of two representations $V, W$ are distinct. However, as is well known the quantum groups belong to the class of quasi-triangular Hopf algebras where the two different coproduct structures can be related by conjugation via some invertible element, the 'universal $R$-matrix' $R \in U_q(\hat{g}) \otimes U_q(\hat{g})$ (see e.g. [24] for further details),

$$\Delta^{\text{op}}(x) = R \Delta(x) R^{-1}$$

This universal $R$-matrix as well as the coproduct acquire a spectral parameter dependence when we specialize to evaluation representations $V(u), W(v)$. We emphasize that both $V$ and $W$ are considered to be finite dimensional, whence we work in a level zero representation of $U_q(\hat{g})$. The spectral parameter dependent $R$-matrix and coproduct are then defined via

$$R_{V,W}(u-v) := (\rho_{V(u)} \otimes \rho_{W(v)}) R \quad \text{and} \quad \Delta_{V(u),W(v)} = (\rho_{V(u)} \otimes \rho_{W(v)}) \Delta$$

(46)

The intertwining property for the coproduct $\Delta$ and its counterpart $\Delta^{\text{op}}$ then reads

$$R_{V,W}(u-v) \Delta_{V(u),W(v)}(x) = \Delta^{\text{op}}_{V(u),W(v)}(x) R_{V,W}(u-v), \quad x \in U_q(\hat{g}).$$

(47)

As indicated the $R$-matrix depends only on the difference $u-v$ (see e.g. [17]), whence we might set $v = 0$ in the above relations. In addition, we choose in what follows $V = W$ and drop the dependence on the representation $V$ in order to unburden the notation. Besides the intertwining property (17) the $R$-matrix is subject to several other requirements. For example, in order that the quasi-triangular structure is compatible with coassociativity, $(\Delta \otimes 1) \Delta = (1 \otimes \Delta) \Delta$, the $R$-matrix is subject to the following equations sometimes referred to as 'fusion laws' (see e.g. [24],

\begin{align*}
(\Delta_u \otimes 1)(R(v)) &= R_{13}(u+v)R_{23}(v) \\
(1 \otimes \Delta_u)(R(v)) &= R_{13}(v-u)R_{12}(v),
\end{align*}

(48)

The Yang-Baxter equation (5) is now an immediate consequence of the first identity and the intertwining property. This establishes the link between affine quantum groups and integrable models as found by Drinfel’d and Jimbo.

We conclude by recalling that from the defining relations (15), (16), and the property (17) it is straightforward to check that for $V = W$ the composition $\mathcal{R}(u) := \pi R(u)$ is quantum group invariant

$$[\mathcal{R}(u), \Delta_u(x)] = 0, \quad x \in U_q(\hat{g}).$$

(49)

Here $\pi$ is the permutation operator introduced earlier. It is this invariance property of the intertwiner $\mathcal{R}(u)$ under $U_q(\hat{g})$ which we exploit to prove the invariance of the transfer matrix and its associated higher charges under the algebra $U(\hat{g})$ at roots of unity. We also note that if $(U_q(\hat{g}), \Delta_u, R(u))$ forms a quasitriangular Hopf algebra, so do the combinations $(U_q(\hat{g}), \Delta^{\text{op}}_u, R(u)^{-1})$ and $(U_q(\hat{g}), \Delta^{\text{op}}_u, R^{\text{op}}(u))$ where $R^{\text{op}}(u) = \pi R(u) \pi$, see e.g. [24]. Hence, one has the additional relations $[\pi R(u)^{-1}, \Delta^{\text{op}}_u(x)] = [\pi R^{\text{op}}(u), \Delta^{\text{op}}_u(x)] = 0$. 

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A.1 The evaluation representation for $U_q(A_1^{(1)})$

As a concrete example for the abstract considerations outlined above we now explicitly state the evaluation representation for the quantum group $U_q(A_1^{(1)})$. We will start from the evaluation representation $p_z : U_q(A_1^{(1)}) \to U_q(A_1)$ found by Jimbo [17] and then construct in a second step an evaluation representation for the $L$-fold tensor product $p_z^{(L)} : U_q(A_1^{(1)})^\otimes L \to U_q(A_1)^\otimes L$ which is the physical case of interest since the transfer matrix lives as an operator on the quantum state space $V^{\otimes L}$. Therefore, we need to construct a representation of our symmetry algebra which acts on the same space. Note that this is not simply accomplished by taking the $L$-fold tensor product of $p_z$, since the evaluation homomorphism does not constitute an Hopf algebra homomorphism, i.e. the coproduct structure is not preserved [17].

The quantum group $U_q(A_1^{(1)})$ consists of all power series in terms of the generators $\{e_i, f_i, h_i\}_{i=0,1} \cup \{1\}$. The commutation and Chevalley-Serre relations can be deduced from the Cartan matrix $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ according to the definitions (Q1)-(Q3). The homomorphism found in [17] reads

\[
\begin{align*}
& e_0 \to p_u(e_0) = e^u f \\
& f_0 \to p_u(f_0) = e^{-u} e \\
& h_0 \to p_u(h_0) = -h \\
& e_1 \to p_u(e_1) = e \\
& f_1 \to p_u(f_1) = f \\
& h_1 \to p_u(h_1) = h
\end{align*}
\]

It is a straightforward exercise to verify that the above identification of the generators not only preserve the commutation relations of $U_q(A_1^{(1)})$ but also the Chevalley-Serre relations. For simplicity we set in the following $u = 0$, since this is the only case which will be relevant later on. The generalization to non-zero spectral parameter is straightforward.

To construct now the generators of the $L$-fold tensor product $U_q(A_1^{(1)})^\otimes L$ we apply the coproduct $\Delta$ iteratively as defined in [21] and obtain the generators [22] for $i = 0, 1$. Remembering that we have $h_0 = -h_1, e_0 = f_1$ and $f_0 = e_1$ in the evaluation representation we might rewrite the generators [22] in terms of the generators of $U_q(A_1)$ by exploiting both of the non-affine coproduct structures $\Delta'$ and $\Delta^{\text{cop}} = \pi \circ \Delta'$. The latter are obtained by restricting the affine coproduct $\Delta$ to the subalgebra $\{e_1 = e, f_1 = f, h_1 = h\}$. The following identities then hold

\[
\begin{align*}
E_0 & \equiv \Delta^{\text{cop}(L)}(f), \\
F_0 & \equiv \Delta^{\text{cop}(L)}(e), \\
q^{H_0} & \equiv \Delta^{(L)}(q^{-h}) \\
E_1 & \equiv \Delta^{(L)}(e), \\
F_1 & \equiv \Delta^{(L)}(f), \\
q^{H_1} & \equiv \Delta^{(L)}(q^h)
\end{align*}
\]

Here we have made use of the fact that the opposite coproduct $\Delta^{\text{cop}} = \pi \circ \Delta'$ of $U_q(A_1)$ is easily seen to be obtained by formally replacing $q \to q^{-1}$ (compare [20]). This establishes an homomorphism $U_q(A_1^{(1)})^\otimes L \to U_q(A_1)^\otimes L$, since $\Delta^{(L)}, \Delta^{\text{cop}(L)}, \Delta^{(L)}$ are all algebra homomorphisms. In particular, the affine quantum Chevalley-Serre relations of $U_q(A_1^{(1)})^\otimes L$ are also satisfied in $U_q(A_1)^\otimes L$ under this representation. This matches the construction of $U_q(A_1^{(1)})^\otimes L$ obtained in [18] for the fundamental representation $V \cong \mathbb{C}^2$. 

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A.2 The evaluation representation for $U_q(A_n^{(1)})$

Following Jimbo [17] we recall for completeness how the evaluation representations for the higher rank algebras are constructed. The homomorphism $p : U_q(A_n^{(1)}) \rightarrow U_q(A_n)$ is constructed by defining iteratively elements $e_{ij} \in U_q(A_n)$, $i \neq j$: Setting $e_{ii+1} := e_i$ and $e_{i+1i} := f_i$ assume that these elements had been constructed for $|i - j| < k$, then the elements with $|i - j| = k$ are given by

$$e_{ij} = \begin{cases} 
    e_{ii+1}e_{i+1j} - qe_{i+1j}e_{ii+1}, & i < j \\
    e_{ii-1}e_{i-1j} - q^{-1}e_{i-1j}e_{ii-1}, & i > j.
\end{cases} \quad (52)$$

It has been shown [17] that the so defined elements satisfy the relation

$$e_{ij} = e_{il}e_{lj} - q^{\pm 1}e_{lj}e_{il}, \quad i \leq l \leq j \quad (53)$$

from which one may verify that the mapping

$$p(e_0) = e_{n1}, \quad p(f_0) = e_{1n} \quad \text{and} \quad p(q^{h_0}) = \prod_{j=1}^{n-1} q^{-h_j} \quad (54)$$

fixes an algebra homomorphism $U_q(A_n^{(1)}) \rightarrow U_q(A_n)$ when identifying the remaining generators in the natural way, $p(e_i) = e_i$, $p(f_i) = f_i$, $p(q^{h_i}) = q^{h_i}$ with $i > 0$. One might then apply the affine coproduct (20) in this evaluation representation in order to obtain expressions for the generators in the physical state space analogous to (22) and (24). Note that the interplay between the affine coproduct $\Delta$ and the non-affine coproducts $\Delta', \Delta^{\text{op}}$ is special to the $A_1 \equiv sl_2$ case and does not apply for $n > 1$.

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