Bayesian Inference via Approximation of Log-likelihood for Priors in Exponential Family

Tohid Ardeshiri, Umut Orguner, and Fredrik Gustafsson

Abstract—In this paper, a Bayesian inference technique based on Taylor series approximation of the logarithm of the likelihood function is presented. The proposed approximation is devised for the case, where the prior distribution belongs to the exponential family of distributions. The logarithm of the likelihood function is linearized with respect to the sufficient statistic of the prior distribution in exponential family such that the posterior obtains the same exponential family form as the prior. Similarities between the proposed method and the extended Kalman filter for nonlinear filtering are illustrated. Furthermore, an extended target measurement update for target models where the target extent is represented by a random matrix having an inverse Wishart distribution is derived. The approximate update covers the important case where the spread of measurement is due to the target extent as well as the measurement noise in the sensor.

Index Terms—Approximate Bayesian inference, exponential family, Bayesian graphical models, extended Kalman filter, extended target tracking, group target tracking, random matrices, inverse Wishart.

I. INTRODUCTION

Determination of the posterior distribution of a latent variable $x$ given the measurements (observed data) $y$ is at the core of Bayesian inference using probabilistic models. These probabilistic models describe the relation between the random latent variables, the deterministic parameters, and the measurements. Such relations are specified by prior distributions of the latent variables $p(x)$, and the likelihood function $p(y|x)$ which give a probabilistic description of the measurements given (some of) the latent variables. Using the probabilistic model and measurements the exact posterior can be expressed in a functional form using the Bayes rule

$$p(x|y) = \frac{p(x)p(y|x)}{\int p(x)p(y|x) \, dx}. \quad (1)$$

The exact posterior distribution can be analytical. A subclass of cases where the posterior is analytical is when the posterior belongs to the same family of distributions as the prior distribution. In such cases, the prior distribution is called a conjugate prior for the likelihood function. A well-known example where an analytical posterior is obtained using conjugate priors is when the latent variable is a priori normal-distributed and the likelihood function given the latent variable as its mean is normal.

The conjugacy is especially useful when measurements are processed sequentially as they appear in the filtering task for stochastic dynamical systems known as hidden Markov models (HMMs) whose probabilistic graphical model is presented in Fig. 1. In such filtering problems, the posterior to the last processed measurement is in the same form as the prior distribution before the measurement update. Thus, the same inference algorithm can be used in a recursive manner.

The exact posterior distribution of a latent variable cannot always be given a compact analytical expression since the integral in the denominator of (1) may not be available in analytical form. Consequently, the number of parameters needed to express the posterior distribution will increase every time the Bayes rule (1) is used for inference. Several methods for approximate inference over probabilistic models are proposed in the literature such as variational Bayes (VB) [1], expectation propagation (EP) [2], integrated nested Laplace approximation (INLA) [4], generalized linear models (GLMs) [5] and, Monte-Carlo (MC) sampling methods [6], [7].

Variational Bayes (VB) and expectation propagation (EP) are two analytical optimization-based solutions for the approximate Bayesian inference [8]. In these two approaches, Kullback-Leibler divergence [9] between the true posterior distribution and an approximate posterior is minimized. INLA is a technique to perform approximate Bayesian inference in latent Gaussian models [10] using the Laplace approximation. GLMs are an extension of ordinary linear regression when errors belong to the exponential family.

Sampling methods such as particle filters and Markov Chain Monte Carlo (MCMC) methods provide a general class of numerical solutions to the approximate Bayesian inference problem. However, in this paper, our focus is on fast analytical approximations which are applicable to large-scale inference problems. The proposed solution is built on properties of the exponential family of distributions and earlier work on extended Kalman filter [11].

In this paper, a Bayesian inference technique based on Taylor series approximation of the logarithm of the likelihood function is presented. The proposed approximation is derived for the case where the prior distribution belongs to the exponential family of distributions. The rest of this paper is organized as follows; In Section I an introduction to exponential family of distribution is provided. In Section III a general algorithm for approximate inference in the exponential family of distributions is suggested. We show how the
II. THE EXPONENTIAL FAMILY

The exponential family of distributions \[^8\] include many common distributions such as Gaussian, beta, gamma and Wishart. For \( x \in X \) the exponential family in its natural form can be represented by

\[ p(x; \eta) = h(x) \exp(\eta \cdot T(x) - A(\eta)), \]

where \( \eta \) is the natural parameter, \( T(x) \) is the sufficient statistic, \( A(\eta) \) is the log-partition function and \( h(x) \) is the base measure. \( \eta \) and \( T(x) \) may be vector-valued. Here \( a \cdot b \) denotes the inner product of \( a \) and \( b \). The log-partition function is defined by the integral

\[ A(\eta) \triangleq \log \int_X h(x) \exp(\eta \cdot T(x)) \, dx. \]

Also, \( \eta \in \Omega = \{ \eta \in \mathbb{R}^m | A(\eta) < +\infty \} \) where \( \Omega \) is the natural parameter space. Moreover, \( \Omega \) is a convex set and \( A(\cdot) \) is a convex function on \( \Omega \).

When a probability density function (PDF) in the exponential family is parametrized by a non-canonical parameter \( \theta \), the PDF in (2) can be written as

\[ p(x; \theta) = h(x) \exp(\eta(\theta) \cdot T(x) - A(\eta(\theta))), \]

where \( \theta \) is the non-canonical parameter vector of the PDF.

Example 1. Consider the normal distribution \( p(x) = \mathcal{N}(x; \mu, \Sigma) \) with mean \( \mu \in \mathbb{R}^d \) and covariance matrix \( \Sigma \). Its PDF can be written in exponential family form given in (2) where

\[ h(x) = (2\pi)^{-\frac{d}{2}}, \]

\[ T(x) = \left[ \begin{array}{c} x \\ \operatorname{vec}(xx^T) \end{array} \right], \]

\[ \eta(\mu, \Sigma) = \left[ \begin{array}{c} \Sigma^{-1}\mu \\ \operatorname{vec}(\Sigma^{-1}) \end{array} \right], \]

\[ A(\eta(\mu, \Sigma)) = \frac{1}{2} \mu^T \Sigma^{-1} \mu + \frac{1}{2} \log |\Sigma|. \]

Remark 1. In the rest of this document, the \( \operatorname{vec}(\cdot) \) operators will be dropped to keep the notation less cluttered. That is, we will use the simplified notation \( T(x) = (x; xx^T) \) instead of (5a) and \( \eta(\mu, \Sigma) = (\Sigma^{-1}\mu, \frac{1}{2} \Sigma^{-1}) \) instead of (5c).

A. Conjugate Priors in Exponential Family

In this section we study the conjugate prior family for a likelihood function belonging to the exponential family. Consider \( m \) independent and identically distributed (IID) measurements \( \mathcal{Y} \triangleq \{ y_j \in \mathbb{R}^d | 1 \leq j \leq m \} \) and the likelihood function \( p(\mathcal{Y}|x, \lambda) \) in exponential family, where \( \lambda \) is a set of hyper-parameters of the likelihood and \( x \) is the latent variable. The likelihood of the \( m \) IID measurements can be written as

\[ p(\mathcal{Y}|x, \lambda) = \left( \prod_{j=1}^m h(y_j) \right) \times \exp \left( \eta(x, \lambda) \cdot \sum_{j=1}^m T(y_j) - mA(\eta(x, \lambda)) \right). \]

We seek a conjugate prior distribution for \( p(x) \). The prior distribution \( p(x) \) is a conjugate prior if the posterior distribution

\[ p(x|\mathcal{Y}) \propto p(\mathcal{Y}|x, \lambda)p(x) \]

is also in the same exponential family as the prior. Let us assume that the prior is in the form

\[ p(x) \propto \exp(\mathcal{F}(x)) \]

for some function \( \mathcal{F}(\cdot) \). Hence,

\[ p(x|\mathcal{Y}) \propto p(\mathcal{Y}|x, \lambda) \exp(\mathcal{F}(x)) \times \exp \left( \eta(x, \lambda) \cdot \sum_{j=1}^m T(y_j) - mA(\eta(x, \lambda)) + \mathcal{F}(x) \right). \]

For the posterior to be in the same exponential family as the prior \( \mathcal{F}(\cdot) \) needs to be in the form

\[ \mathcal{F}(x) = \rho_1 \cdot \eta(x, \lambda) - \rho_0 A(\eta(x, \lambda)) \]

for some \( \rho \triangleq (\rho_0, \rho_1) \), such that

\[ p(x|\mathcal{Y}) \propto \exp \left( \left( \rho_1 + \sum_{j=1}^m T(y_j) \right) \cdot \eta(x, \lambda) - (m + \rho_0)A(\eta(x, \lambda)) \right). \]

Hence, the conjugate prior for the likelihood is parametrized by \( \rho \) and is given by

\[ p(x; \rho) = \frac{1}{Z} \exp \left( \rho_1 \cdot \eta(x, \lambda) - \rho_0 A(\eta(x, \lambda)) \right), \]

where

\[ Z = \int \exp \left( \rho_1 \cdot \eta(x, \lambda) - \rho_0 A(\eta(x, \lambda)) \right) \, dx. \]

In Example 2 the conjugate prior for the normal likelihood is derived using the exponential family form for the normal distribution given in Example 1. In Example 3 a conjugate prior for a more complicated likelihood function is derived.

Example 2. Consider the measurement \( y \) with the likelihood \( p(y|x) = \mathcal{N}(y; Cx, R) \). The likelihood can be written in exponential family form as in

\[ p(y|x) = (2\pi)^{-\frac{d}{2}} \exp \left( \eta(Cx, R) \cdot T(y) - A(\eta(Cx, R)) \right) \]

where \( T(y) = (y, yy^T), \eta(Cx, R) = (R^{-1}Cx, -\frac{1}{2} R^{-1}) \) and

\[ A(\eta(Cx, R)) = \frac{1}{2} x^T C^T R^{-1}Cx + \frac{1}{2} \log |R| \]

(16a)

\[ = \frac{1}{2} (C^T R^{-1}C) \cdot xx^T + \frac{1}{2} \log |R|. \]

(16b)
Therefore, the likelihood function can be written in the form
\[
p(y|x) \propto \exp \left( \left( C^T R^{-1} y - \frac{1}{2} C^T R^{-1} C \right) \cdot T(x) \right). \tag{17}
\]
Hence, the conjugate prior for the likelihood function of this example should have \( T(\cdot) \) as its sufficient statistic i.e., the conjugate prior is normal. When the prior distribution is normal i.e., \( p(x) = \mathcal{N}(x; \mu, \Sigma) \) the posterior distribution obeys
\[
p(x|y) \propto \exp \left( \left( C^T R^{-1} y + \Sigma^{-1} \mu - \frac{1}{2} C^T R^{-1} C - \frac{1}{2} \Sigma^{-1} \right) \cdot T(x) \right).
\]

which is the information filter form for the Kalman filter’s measurement update [13].

**Example 3.** Consider the measurement \( y \) with the likelihood
\[
p(y|x) \propto \exp \left( -\frac{1}{24} (y - x)^2 + \cos(y - x) \right). \tag{19}
\]
The likelihood function is a multi-modal likelihood function and is illustrated in Fig. 2 for \( y = 3 \). The likelihood can be written in the form
\[
p(y|x) \propto \exp(\lambda \cdot T(x)) \tag{20}
\]
where \( \lambda = \left[ -\frac{1}{24}, \frac{y}{12}, \cos y \sin y \right]^T \) and \( T(x) = [x^2 \cdot x \cdot \cos x \sin x]^T \). Hence, a conjugate prior for the likelihood function of this example should have \( T(\cdot) \) as its sufficient statistic such as \( p(x) \propto \exp \left( \left[ -\frac{1}{24} 0 0 0 \right] T(x) \right) \). Thus, the posterior for the likelihood function and the prior distribution obeys
\[
p(x|y) \propto \exp \left( \left[ -\frac{1}{24} - \frac{1}{10} \frac{y}{12}, \cos y \sin y \right]^T \cdot T(x) \right).
\]

Although such posteriors can be computed analytically up to a normalization factor, the density may not be available in analytical form. The numerically normalized prior and the posterior are illustrated in Fig. 2.

**Figure 2:** The likelihood function \( 19) \) prior distribution \( 20) \) and the posterior distribution \( 21) \) in Example 3 are plotted.

Similar reasoning as the one presented in this section can be applied to the case where the prior is fixed and the likelihood function is to be selected; The concept of “conjugate likelihoods” will be considered in the sequel.

### Table I: Some continuous exponential family distributions and their sufficient statistic are listed.

| Continuous Exp. Family Distribution | \( T(\cdot) \) |
|--------------------------------------|---------------|
| Exponential distribution             | \( x \)       |
| Normal distribution with known variance \( \sigma^2 \) | \( \frac{x}{\sigma} \) |
| Normal distribution                 | \( (x, xx^T) \) |
| Pareto distribution with known minimum \( \lambda_m \) | \( \log x \) |
| Weibull distribution with known shape \( k \) | \( x^k \) |
| Chi-squared distribution            | \( \log x \) |
| Dirichlet distribution              | \( (\log x_1, \cdots, \log x_N) \) |
| Laplace distribution with known mean \( \mu \) | \( |x - \mu| \) |
| Inverse Gaussian distribution       | \( (\log x, 1/x) \) |
| Scaled inverse Chi-squared distribution | \( \log(1 - x) \) |
| Beta distribution                   | \( \log x, \log(1 - x) \) |
| Lognormal distribution              | \( \log(x, \log(x)^2) \) |
| Gamma distribution                  | \( \log(x, \log x) \) |
| Inverse gamma distribution          | \( \log(x, 1/x) \) |
| Gaussian Gamma distribution         | \( \log(x, |x|, X) \) |
| Wishart distribution                | \( \log(|X|, X^{-1}) \) |

### B. Conjugate Likelihoods in Exponential Family

First, we define the conjugate likelihood functions;

**Definition 1.** A likelihood function \( p(y|x) \) is called conjugate likelihood for prior distribution \( p(x) \) in a family of distributions, if the posterior distribution \( p(x|y) \) belongs to the same family of distributions as the prior distribution.

Now, consider the prior distribution \( p(x; \rho) \) in the exponential family form on the latent variable \( x \in \mathcal{X} \), where \( \rho \) is a set of hyper-parameters for the prior
\[
p(x; \rho) = h(x) \exp \left( \eta(\rho) \cdot T(x) - A(\eta(\rho)) \right). \tag{22}
\]
We will follow the treatment given for characterization of the conjugate priors in Section I-A as well as [12], for the conjugate likelihood functions. Assume now that we have observed \( m \) IID measurements \( \mathcal{Y} \triangleq \{ y_j \in \mathbb{R}^d | 1 \leq j \leq m \} \). Let the likelihood of the measurements be written as
\[
p(y|x) \propto \exp \left( \mathcal{L}(x, y) \right) \tag{23}
\]
for some function \( \mathcal{L}(\cdot) \). We seek a conjugate likelihood function in the exponential family. Hence, the posterior distribution has to be in the exponential family where,
\[
p(x|y) \propto h(x) \exp \left( \eta(\rho) \cdot T(x) - A(\eta(\rho)) + \mathcal{L}(x, y) \right). \tag{24}
\]
For the posterior \( 24 \) to be in the same exponential family, \( \mathcal{L}(\cdot) \) needs to be in the form
\[
\mathcal{L}(x, y) \overset{\pm}{=} \lambda(y) \cdot T(x) \tag{25}
\]
where \( \pm \) means equality up to an additive constant with respect to latent variable \( x \) such that
\[
p(x|y) \propto h(x) \exp \left( (\eta(\rho) + \lambda(y)) \cdot T(x) \right). \tag{26}
\]
Hence, the conjugate likelihood for the prior distribution family \( 22 \) is parametrized by \( \lambda(y) \) and is given by
\[
p(y|x) \propto \exp \left( \lambda(y) \cdot T(x) \right). \tag{27}
\]
The property expressed in \( 27 \) is the necessary condition which should hold for a conjugate likelihood for a given prior distribution family. Another condition which should hold is that the log-partition function of the posterior \( 26 \) should obey
\[
A = \log \int_{\mathcal{X}} h(x) \exp((\eta(\rho) + \lambda(y)) \cdot T(x)) \, dx < \infty \tag{28}
\]
such that the natural parameter of the posterior belongs to the natural parameter space $\Omega$. These properties of the conjugate likelihood are summarized in theorem $\mathbb{I}$

**Theorem 1.** The likelihood function $p(y|x)$ is a conjugate likelihood for the prior distribution in an exponential family $p(x) = h(x) \exp(\eta \cdot T(x) - A(\eta))$ with $x \in \mathcal{X}$ if and only if
1. $p(y|x) \propto \exp(\lambda \cdot T(x))$ for some $\lambda$ and,
2. the likelihood function $p(y|x)$ is integrable with respect to $y$ and,
3. $\log \int_x h(x) \exp((\eta + \lambda) \cdot T(x)) \, dx < \infty$ for all $\eta \in \Omega$.

In Examples $\mathbb{I}$ and $\mathbb{I}$, the conjugate likelihood functions for two prior distributions are derived using the exponential family form.

**Example 4.** Consider the normal distribution $p(x) = \mathcal{N}(x; \mu, \Sigma)$ whose PDF is written in exponential family form in Example $\mathbb{I}$. The conjugate likelihood function has to be in the form
$$p(y|x) \propto \exp \left( \lambda_1 (y - x) + \lambda_2 (y \cdot \text{vec}(xx^T)) \right)$$
(29)
for some permissible $\lambda(\cdot) \triangleq [\lambda_1, \lambda_2]$, which includes $p(y|x) = \mathcal{N}(y; Cx, R)$ for a matrix $C$ with appropriate dimensions and symmetric $R > 0$.

**Example 5.** Consider the prior distribution
$$p(x) \propto \exp \left( \lambda \cdot T(x) \right)$$
(30)
where $T(x) = [x^2 \quad x \quad \cos x \quad \sin x]^T$. The family of conjugate likelihood density functions $p(y|x)$ is parametrized by $\psi \in \mathbb{R}^4$ such that
1. $\log p(y|x) \equiv \psi \cdot T(x)$,
2. $\int_{-\infty}^{\infty} p(y|x) \, dy = 1$, i.e., the likelihood is integrable with respect to $y$,
3. $\int_{-\infty}^{\infty} \exp((\lambda + \psi) \cdot T(x)) \, dx < \infty$, i.e., the posterior is integrable with respect to $x$.

A member of this family is
$$p(y|x) \propto \exp(-\alpha^2 (y - x)^2 + \cos(\beta y - x + \gamma))$$
(31)
for $[\alpha, \beta, \gamma]^T \in \mathbb{R}^3$.

In Table $\mathbb{I}$, the sufficient statistic for some continuous members of the exponential family are given. Some members of the continuous exponential family have conjugate likelihoods which are summarized in Table $\mathbb{I}$.

### III. Measurement Update via Approximation of the Log-likelihood

In this section a method is proposed for the analytical approximation of complex likelihood functions based on linearization of the log-likelihood with respect to sufficient statistic function $T(\cdot)$. We will use the property of conjugate likelihood functions i.e., linearity of log-likelihood function with respect to sufficient statistic of the prior distribution to come up with an approximation of the likelihood function for which there exists analytical posterior distribution.

In the proposed method first, the log-likelihood function is derived in analytical form. Second, the likelihood function is approximated by a dot product of a statistic which depends on the measurement and the sufficient statistic of the prior distribution. For example, consider a likelihood function
$$p(y|x) \propto \exp \left( \mathcal{L}(x, y) \right)$$
(32)
where the log-likelihood function is given by $\mathcal{L}(\cdot)$ and the prior distribution is given by
$$p(x) = h(x) \exp (\eta \cdot T(x) - A(\eta))$$
(33)
where $\eta \in \Omega$. If we approximate $\mathcal{L}(\cdot)$ such that
$$\mathcal{L}(x, y) \approx \tilde{\mathcal{L}}(x, y) \pm \lambda(y) \cdot T(x)$$
(34)
for some $\lambda(\cdot)$ such that for any measurement $y$ belonging to the support of the likelihood function
$$\eta + \lambda(y) \in \Omega$$
(35)
and,
$$\int \exp \left( \tilde{\mathcal{L}}(x, y) \right) \, dy < \infty,$$
(36)
then the approximate likelihood will be conjugate likelihood for the prior distribution with sufficient statistic $T(x)$.

When the prior distribution is normal, the proposed approximation can be obtained using well-known Taylor series approximation at the global maximum of the log-likelihood function. This is due to the fact that $T(x) = (x, xx^T)$ for the normal prior and a second order Taylor series approximation of the log-likelihood approximates the log-likelihood with a function linear in $T(x)$. This mathematical convenience is used in the well-known approximate Bayesian inference technique INLA $\mathbb{I}$. Similarly, when the prior distribution is exponential distribution, a first order Taylor series approximation can be used to approximate the log-likelihood with a function linear in $T(x) = x$.

Taylor series expansion of functions will be used in the proposed approximations in the following. Hence, we establish the notation used in this paper for Taylor series expansion of arbitrary functions before we proceed.

#### A. Taylor series expansion

The Taylor series expansion of a real-valued function $f(x)$ that is infinitely differentiable at a real vector $\hat{x}$ is given by the series
$$f(x) = f(\hat{x}) + \frac{1}{1!} \nabla x f(\hat{x}) \cdot (x - \hat{x}) + \frac{1}{2!} \nabla^2 x f(\hat{x}) \cdot ((x - \hat{x})(x - \hat{x})^T) + \cdots$$
(37)
where $\nabla_x f(\hat{x})$ is the gradient vector which is composed of the partial derivatives of $f(x)$ with respect to elements of $x$, evaluated at $\hat{x}$. Similarly, $\nabla^2_x f(\hat{x})$ is the Hessian matrix of $f(x)$ containing second order partial derivatives and evaluated at $\hat{x}$.

When $f(x)$ is a vector-valued function, i.e., $f(x) \in \mathbb{R}^d$ with $i$th element denoted by $f_i(x)$, first the Taylor series expansion is derived for each element separately. Then, the expansion of $f(x)$ at $\hat{x}$ is constructed by rearranging the terms in a matrix.

$$f(x) = f(\hat{x}) + \frac{1}{1!} (\nabla_x f(\hat{x}))^T (x - \hat{x}) + \frac{1}{2!} \sum_{i=1}^{d} e_i \nabla^2_x f_i(\hat{x}) \cdot ((x - \hat{x})(x - \hat{x})^T) + \cdots$$
(38)
In (38), $\nabla_x f(\hat{x})$ is the Jacobian matrix of $f(x)$ evaluated at $\hat{x}$ and $e_i$ is a unity vector with its $i$th element equal to 1.
Table II: Some prior distributions and their conjugate likelihood functions are listed. The arguments of the PDFs on the left columns are the latent random variables. On the right hand side column the conjugate likelihood functions \( p(y|\cdot) \) are given where \( y \) denotes the measurement. In the middle column the sufficient statistic functions of the prior distributions in the left column are given. The logarithm of the conjugate likelihood functions are linear in sufficient statistic function to their left.

| Prior distribution with sufficient statistic  \( T(\cdot) \) |  \( T(\cdot) \) | Conjugate likelihood function  
|---|---|---|
| \( \mathcal{N}(x; \mu, \Sigma) \) | \( (x, xx^T) \) | \( \mathcal{N}(y; Cx, R) \propto \exp \left( \text{Tr} \left( R^{-1} (y - Cx)(y - Cx)^T \right) \right) \) |
| \( \mathcal{N}(x; \mu, \Sigma) \) | \( (x, xx^T) \) | \( \log -\mathcal{N}(y; x, \sigma^2) \propto \exp \left(-\frac{1}{2\sigma^2} (y - x)^2 \right) \) |
| Gamma(\( \alpha, \beta \)) | \( (\log x, x) \) | Exp(\( y; x \) = \exp(-xy)) |
| Gamma(\( \alpha, \beta \)) | \( (\log x, x) \) | IGamma(\( y; \alpha', x) \propto x^{\alpha'} \exp(-x/y) \) |
| Gamma(\( \alpha, \beta \)) | \( (\log x, x) \) | Gamma(\( y; \alpha', x) \propto x^{\alpha'} \exp(-xy) \) |
| IGamma(\( \alpha; \alpha, \beta \)) | \( (\log x, 1/x) \) | \( \mathcal{N}(y; x, \beta^{-1}) \propto x^\beta \exp(-\frac{1}{\beta} (y - x)^2) \) |
| GaussianGamma(\( x, \tau; \mu, \alpha, \beta \)) | \( (\log x, 1/x) \) | Weibull(\( y; \kappa, x \) \( \propto \frac{1}{x^k} \exp\left(-\frac{x^k}{\kappa} \right) \) |
| \( W(X; n, V) \) | \( (\log |X|, X) \) | \( \mathcal{N}(y; \mu, x^{-1}) \propto |X|^\frac{1}{2} \exp(\text{Tr}(X(x - \mu)(y - \mu)^T)) \) |
| \( TV(X; \nu, \Psi) \) | \( (\log |X|, X^{-1}) \) | \( \mathcal{N}(y; \mu, x^{-1}) \propto |X|^{-\frac{1}{2}} \exp(\text{Tr}(X^{-1}(y - \mu)(y - \mu)^T)) \) |

B. The extended Kalman filter

A well-known problem where a special case of the proposed inference technique is used is the filtering problem for nonlinear state-space models in presence of additive Gaussian noise which will be described here. Consider the discrete-time stochastic nonlinear dynamical system

\[
y_k | x_k \sim \mathcal{N}(y_k; c(x_k), R_k),
\]

\[
x_{k+1} | x_k \sim \mathcal{N}(x_{k+1}; f(x_k), Q_k),
\]

where \( x_k \) and \( y_k \) are the latent variable and the measurement at time index \( k \) and \( c(\cdot) \) and \( f(\cdot) \) are nonlinear functions. A common solution to the inference problem is the extended Kalman filter (EKF) [11].

In the following example, we show that the measurement update for EKF is a special case of the proposed algorithm.

Example 6. Consider the latent variable \( x \) with \( a \) priori PDF \( p(x) = \mathcal{N}(x; \mu, \Sigma) \) and the measurement \( y \) with the likelihood function \( p(y|x) = \mathcal{N}(y; c(x), R) \). The likelihood function can be written in exponential family form as in

\[
p(y|x) = (2\pi)^{-\frac{d}{2}} \exp \left( \eta(\mu, R) \cdot T(y) - A(\eta(\mu, R)) \right)
\]

where \( T(y) = (y, yy^T), \eta(\mu, R) = (R^{-1}c(x), \frac{1}{2} R^{-1}) \) and

\[
A(\eta(\mu, R)) = \frac{1}{2} \text{Tr} \left( R^{-1} c(x) (c(x))^T \right) + \frac{1}{2} \log |R|.
\]

Thus,

\[
p(y|x) \propto \exp \left( R^{-1}c(x) \cdot y - \frac{1}{2} c(x)^T R^{-1} c(x) \right)
\]

and the log-likelihood function can be written as

\[
\mathcal{L}(x) \triangleq y^T R^{-1} c(x) - \frac{1}{2} c(x)^T R^{-1} c(x)
\]

where \( \triangleq \) means equality up to an additive constant with respect to the latent variable \( x \). The second order Taylor series approximation of any function will be linear in the sufficient statistic \( T(x) = (x, xx^T) \). However, such an approximation does not guarantee the integrability of the approximate likelihood and the corresponding approximate posterior due to the dependence of second element of the posterior’s natural parameter on the measurement \( y \).

A common solution to the log-likelihood linearization problem has been the first order Taylor series approximation of \( c(x) \) about \( \hat{x} = \mu \) (instead of second order Taylor series approximation of \( L(x) \)), i.e.,

\[
c(x) \approx c(\mu) + c'(\mu)(x - \mu)
\]

(44)

where

\[
c'(\mu) \triangleq \nabla_x c(x)|_{x=\mu}.
\]

With this approximation the approximate log-likelihood becomes

\[
\mathcal{L}(x) \approx (c'(\mu)R^{-1}(y^T - c(\mu) - c'(\mu)\mu)) \cdot x
\]

\[
- \frac{1}{2} c'(\mu)R^{-1}c'(\mu)^T \cdot xx^T.
\]

(46)

Hence, a conjugate likelihood function for prior distribution is obtained by approximating the log-likelihood function by a function that is linear in sufficient statistic of the prior distribution. Also note that the posterior distribution obeys

\[
p(x|y) \propto \exp (\phi \cdot T(x))
\]

(47)

where

\[
\phi = \left( c'(\mu)R^{-1}(y^T - c(\mu) - c'(\mu)\mu) + \Sigma^{-1} \right)
\]

\[
- \frac{1}{2} c'(\mu)R^{-1}c'(\mu)^T - \frac{1}{2} \Sigma^{-1}
\]

(48)

which is the extended information filter form for the extended Kalman filter [14]. Advantages of this solution over Taylor series approximation of the log-likelihood is that the natural parameter of the posterior will be in natural parameter space by construction i.e., \( \frac{1}{2} c'(\mu)R^{-1}c'(\mu)^T + \frac{1}{2} \Sigma^{-1} \succeq 0 \).

C. A general linearization guideline

The approximation of a general scalar function \( L(\cdot) \) with another function \( \hat{L}(\cdot) \) linear in a general vector valued function \( t(\cdot) \) i.e., \( \hat{L}(\cdot) \triangleq \lambda \cdot t(\cdot) \) can be expressed as an optimization problem where various cost functions are suggested and minimized. Study of such procedures is considered outside the scope of this paper and a subject of future research. However,
a few tricks are introduced here which can be used for the purpose of linearization.

**Lemma 1.** Let $L(x)$ be an arbitrary scalar-valued continuous and differentiable function and $t(x)$ be a vector-valued invertible function with independent continuous elements such that $x = t^{-1}(z)$ and $L(t^{-1}(z))$ is differentiable with respect to $z$. Then, $L(x)$ can be linearized with respect to $t(x)$ about $t(\hat{x})$ such that

$$L(x) \approx L(\hat{x}) + \Phi \cdot (t(x) - t(\hat{x}))$$

and $\Phi = \nabla_x L(t^{-1}(z))|_{z = t(\hat{x})}$.

**Proof.** Let $z = t(x)$ and $Q(z) \triangleq L(t^{-1}(z))$. Then, $x = t^{-1}(z)$ and $Q(z) = L(x)$ for $z = t(x)$. Using the first-order Taylor series approximation of the scalar function $Q(z)$ about $\hat{z} \triangleq t(\hat{x})$ we obtain

$$Q(z) \approx Q(\hat{z}) + \nabla_z Q(z)|_{z = \hat{z}} \cdot (z - \hat{z}).$$

Hence, the proof follows.

**Remark 2.** The variable $z$ and the function $t(x)$ can be scalar-valued, vector-valued or even matrix-valued. Further, $\nabla_z Q(z)$ can be computed using the chain rule and the fact that $Q(z) = L(t^{-1}(z))$.

The elements of sufficient statistic of most continuous members of the exponential family are dependent, such as $x$ and $xx^T$ for normal distribution. Furthermore, some of them are not invertible such as $\log |X|$ for Wishart distribution. However, the linearization method given in Lemma 1 can be applied to individual summand terms of the log-likelihood function and individual elements of the sufficient statistic. Due to the freedom in choosing the element of the sufficient statistic to linearize the log-likelihood with respect to, there is no unique solution for the linearization problem. In Example 7 we will use Lemma 1 and linearize a log-likelihood function in various ways and describe their properties.

**Example 7.** Consider the scalar latent variable $x$ with a priori PDF $p(x) = IGamma(x; \alpha, \beta)$ and the measurement $y$ with the likelihood function $p(y|x) = N(y; 0, x + x^T)$. Since the exact posterior density is not analytically tractable, an approximation is needed.

The prior distribution can be written in the exponential family form where the natural parameter is $\eta = (-\alpha - 1, -\beta)$ and the sufficient statistic is $T(x) = (\log x, x^{-1})$. The logarithm of the prior distribution is given by

$$\log p(x) \propto - (\alpha + 1) \log x - \beta x^{-1}. \quad (51)$$

In order to compute the posterior using the proposed linearization approach, first the log-likelihood function is computed

$$-2L(x, y) \propto \log(x + \sigma^2) + \frac{y^2}{x + \sigma^2}. \quad (52)$$

The linearization of the log-likelihood with respect to the sufficient statistic of the prior distribution can be carried out in several ways, four of which are listed in the following:

**Solution 1:** We can linearize the right-hand-side of (52) with respect to $\log x$ using Lemma 1 by letting $z \triangleq \log x$ and subsequent linearization with respect to $z$ about $\hat{z} \triangleq \log \hat{x}$ which gives

$$-2L(x, y) \approx \left(\frac{\hat{x}}{x + \sigma^2} + \frac{-y^2}{(x + \sigma^2)2}\right) \log x, \quad (53)$$

where $\approx$ means approximations up to an additive constant with respect to all sufficient statistics (i.e., the elements of $T(x)$).

The approximate likelihood can be found as

$$\hat{p}_1(y|x) \propto x^{-\frac{\hat{x}}{2(\hat{x} + \sigma^2)}(\hat{x} + \sigma^2 - y^2)}$$

which is not integrable with respect to $y$ since $\hat{x} > 0$ and $\hat{p}_1(y|x)$ goes to infinity as $y$ goes to infinity.

**Solution 2:** We can linearize the right-hand-side of (52) with respect to $x^{-1}$ by letting $z \triangleq x^{-1}$ and subsequent posterior obtains the form

$$\hat{p}(x|y) \propto \exp\left(-\frac{\hat{x}^2}{2(\hat{x} + \sigma^2)^2} + \frac{y^2\hat{x}^2}{(\hat{x} + \sigma^2)^2} \right)x^{-1}. \quad (55)$$

The approximate likelihood can be found as

$$\hat{p}_2(y|x) \propto \exp\left(-\frac{\hat{x}^2}{2(\hat{x} + \sigma^2)^2} \right) (y^2 - \hat{x} - \sigma^2), \quad (56)$$

which is integrable with respect to $y$. However, the approximate posterior obtains the form

$$\hat{p}(x|y) \propto \exp\left(- (\alpha + 1) \log x - \left(\frac{\hat{x}^2(y^2 - \hat{x} - \sigma^2)}{2(\hat{x} + \sigma^2)^2} + \beta\right)x^{-1}\right), \quad (57)$$

which is not integrable with respect to $x$ for a small $y$ such that $\frac{\hat{x}^2(y^2 - \hat{x} - \sigma^2)}{2(\hat{x} + \sigma^2)^2} + \beta < 0$.

**Solution 3:** We can linearize the first term on the right-hand-side of (52) with respect to $\log x$ and the second term with respect to $x^{-1}$ about $\log \hat{x}$ and $\hat{x}^{-1}$, respectively, using Lemma 1 which gives

$$-2L(x, y) \approx \frac{\hat{x}}{x + \sigma^2} \log x + \frac{y^2\hat{x}^2}{(x + \sigma^2)^2} x^{-1}. \quad (58)$$

The approximate likelihood can be found as

$$\hat{p}_3(y|x) \propto x^{-\frac{\hat{x}}{2(\hat{x} + \sigma^2)} \exp\left(-\frac{y^2\hat{x}^2}{2(\hat{x} + \sigma^2)^2} \right)}, \quad (59)$$

which is integrable with respect to $y$. The approximate posterior remains integrable with respect to $x$ for $\hat{x} > 0$.

**Solution 4:** We can first expand the log-likelihood as in

$$-2L(x, y) \propto \log x + \log \left(1 + \frac{\sigma^2}{x}\right) + \frac{y^2}{x + \sigma^2} \quad (60)$$

and then linearize the last two terms on the right hand side of (60) with respect to $z \triangleq x^{-1}$ about the nominal point $\hat{z} = \hat{x}^{-1}$ using Lemma 1 which gives

$$-2L(x, y) \approx \log x + \left(\frac{\sigma^2\hat{x}}{\hat{x} + \sigma^2} + \frac{y^2\hat{x}^2}{(\hat{x} + \sigma^2)^2}\right) x^{-1}. \quad (61)$$

The approximate likelihood can be found as

$$\hat{p}_4(y|x) \propto x^{-\frac{1}{2}} \exp\left(-\frac{1}{2\hat{x}} \left(\frac{\sigma^2\hat{x}}{\hat{x} + \sigma^2} + \frac{y^2\hat{x}^2}{(\hat{x} + \sigma^2)^2}\right)\right) \quad (62)$$

which is integrable with respect to $y$. The posterior remains integrable with respect to $x$ for $\hat{x} > 0$.

Integrability of the approximate posterior is not guaranteed for the first two solutions, while the last two solutions will result in an inverse gamma approximate posterior distribution.
The parameters of the posterior depends on the linearization point \( \hat{x} \) among other factors. A candidate point for linearization is the prior mean \( \bar{x} = \frac{x_0}{P} \).

As pointed out in Theorem [1] and illustrated in Example [7] linearity of the log-likelihood with respect to sufficient statistic of the prior is a necessary condition for conjugacy. Hence, even after succeeding in linearization of the log-likelihood with respect to \( T(x) \), the integrability of the posterior distribution should be examined. Consequently, the linearization should be done skillfully such that the approximation error is minimized and the approximate posterior remains integrable.

In the rest of this paper we will focus on exemplifying the application of the linearization technique in a specific Bayesian inference problem in Section [IV] and the related numerical simulations in Section [V]. In this specific example, two of the sufficient statistics is not invertible as it is required in Lemma [1]. Hence, a remedy is suggested and used in the example.

IV. EXTENDED TARGET TRACKING

A. The problem formulation

In the Bayesian extended target tracking (ETT) framework [15] the state of an extended target consists of a kinematic state and a symmetric positive definite matrix representing the extent state. Suppose that the extended target’s kinematic state and the target’s extent state are denoted by \( x_k \) and \( X_k \), respectively. Further, we assume that the measurements \( Y_k \) \( \triangleq \{y^i_k \in \mathbb{R}^{d \times m_k}\}_{j=1}^{m_k} \) generated by the extended target are independent and identically distributed (conditioned on \( x_k \) and \( X_k \)) as \( y^i_k \sim N(y^i_k \mid Hx_k, sX_k + R) \), see Fig. 3. The following measurement likelihood which will be used in this paper was first introduced in [16]

\[
p(y_k | x_k, X_k) = \prod_{j=1}^{m_k} N(y^j_k \mid Hx_k, sX_k + R),
\]

where

- \( m_k \) is the number of measurements at time \( k \).
- \( s \) is a real positive scalar constant.
- \( R \) is the measurement noise covariance.

We assume the following prior form on the kinematic state and target extent state.

\[
p(x_k, X_k | y_{0:k-1}) = \mathcal{N}(x_k | x_{k|k-1}, P_{k|k-1}) \mathcal{IW}(X_k | \nu_{k|k-1}, V_{k|k-1})
\]

where \( x_{k|k-1} \in \mathbb{R}^n, X_{k|k-1} \in \mathbb{R}^{d \times d} \) and the double time index “\( a|b \)” is read “at time \( a \) using measurements up to and including time \( b \)”.

The inverse Wishart distribution \( \mathcal{IW}(X; \nu, V) \) we use in this work is given in the following form.

\[
\mathcal{IW}(X; \nu, V) \triangleq \left\{ X \mid \frac{1}{2} \left( \nu - d - 1 \right) \exp \left( -\frac{1}{2} VX^{-1} \right) \right\} \frac{1}{2^{\frac{1}{2} (\nu - d - 1)} \Gamma_d \left[ \frac{1}{2} (\nu - d - 1) \right]} |X|^\frac{\nu - d}{2}
\]

where \( X \) is a symmetric positive definite matrix of dimension \( d \times d \), \( \nu > 2d \) is the scalar degrees of freedom and \( V \) is a symmetric positive definite matrix of dimension \( d \times d \) and is called the scale matrix. This form of the inverse Wishart distribution is the same one used in the well-known reference [17].

No analytical solution for the posterior exists. The reason for the lack of an analytical solution for the posterior corresponding to the likelihood (63) and the prior (64) is both the covariance addition in the Gaussian distributions on the right hand side of (63) and the intertwined nature of the kinematic state and extent state in the likelihood. In order to get rid of the covariance addition, latent variables are defined in [18] and the problem of intertwined kinematic and extent states are solved using variational approximation. In [16] the authors design an unbiased estimator. Both of these measurement updates can be used in a recursive estimation framework which requires the posterior to be in the same form as the prior as in

\[
p(x_k, X_k | y_{0:k}) \approx \mathcal{N}(x_k | x_{k|k}, P_{k|k}) \mathcal{IW}(X_k | \nu_{k|k}, V_{k|k}).
\]

B. Solution proposed by Feldmann et al. [16]

In [16] the authors cleverly design a measurement update based on unbiasedness properties for the measurement model (63) to calculate \( x_{k|k}, P_{k|k}, \nu_{k|k} \) and \( V_{k|k} \) in the posterior (66).

The kinematic state \( x_k \) is updated as follows:

\[
x_{k|k} = x_{k|k-1} + K_k(\bar{y}_k - Hx_{k|k-1})
\]

\[
P_{k|k} = P_{k|k-1} - K_k S_k K_k^T
\]

where

\[
K_k = P_{k|k-1} H^T \left( H P_{k|k-1} H^T + \frac{1}{m_k}(sX_{k|k-1} + R) \right)^{-1}
\]

\[
S_k = H P_{k|k-1} H^T + \frac{1}{m_k}(sX_{k|k-1} + R)
\]

\[
X_{k|k-1} = \frac{V_{k|k-1}}{m_k - 2d - 2}
\]

and

\[
\bar{y}_k \triangleq \frac{1}{m_k} \sum_{j=1}^{m_k} y^j_k
\]

is the mean of the measurements at time \( k \).

The extent state updates given in [16] is in the following form

\[
\nu_{k|k} = \nu_{k|k-1} + m_k
\]

\[
V_{k|k} = V_{k|k-1} + M_k^FFK
\]

where \( M_k^FFK \) is a given positive definite update matrix and the superscript -FFK which is used to ease presentation consists of the authors initials in [16]. Before describing the construction of the matrix \( M_k^FFK \) of [16] let us define the measurement spread \( Y_k \) from the predicted measurement \( Hx_{k|k-1} \) as follows.

\[
Y_k \triangleq \frac{1}{m_k} \sum_{j=1}^{m_k} (y^j_k - Hx_{k|k-1})(y^j_k - Hx_{k|k-1})^T
\]
Feldmann et al. first write the measurement spread $Y_k$ as
\[ Y_k = Y_k^1 + Y_k^2. \] (76)
The summands $Y_k^1$ and $Y_k^2$ are defined as
\[ Y_k^1 \triangleq (\tilde{y}_k - H \hat{x}_{k|k-1})(\tilde{y}_k - H \hat{x}_{k|k-1})^T, \] (77)
\[ Y_k^2 \triangleq \frac{1}{m_k} \sum_{j=1}^{m_k} (y_k^j - \tilde{y}_k)(y_k^j - \tilde{y}_k)^T. \] (78)

When we take the conditional expected values of $Y_k^1$ and $Y_k^2$, we obtain
\[ Y_k^1 \triangleq E[Y_k^1 | X_k = X_{k|k-1}] = HP_{k|k-1}H^T + sX_{k|k-1} + R, \] (79)
\[ Y_k^2 \triangleq E[Y_k^2 | X_k = X_{k|k-1}] = \frac{m_k - 1}{m_k}(sX_{k|k-1} + R). \] (80)

The matrix $M^F_{k}$ is then proposed to be
\[ M^F_{k} = X_{k|k-1}^{1/2} (Y_k^1)^{-1/2} Y_k^1 (Y_k^1)^{-1/2} X_{k|k-1}^{1/2} + (m_k - 1)X_{k|k-1}^{1/2}(Y_k^2)^{-1/2} Y_k^2 (Y_k^2)^{-1/2} X_{k|k-1}^{1/2}. \] (81)
The conditional expected value of $M^F_{k}$ is
\[ E[M^F_{k} | X_k = X_{k|k-1}] = m_kX_{k|k-1}, \] (82)
which results in an unbiased estimator.

C. ETT via log-likelihood linearization

A new measurement update is obtained by performing linearization of the logarithm of the likelihood function (63) with respect to sufficient statistic given in (83). The sufficient statistics of the prior distribution (64) are given as follows.
\[ T(x_k, X_k) = (x_k, x_kx_k^T, X_k^{-1}, \log |X_k|) \] (83)

As seen above the second and the fourth elements of $T(\cdot)$ are not invertible. In the following lemma, using log-likelihood linearization (LLL) via a first order Taylor series expansion, we approximate the likelihood function (63) to obtain a conjugate likelihood (with respect to the prior (64)) with sufficient statistics given in (83).

Lemma 2. The likelihood $\prod_{j=1}^{m} \mathcal{N}(y_j^j; Hx, sX + R)$ can be approximated up to a multiplicative constant by a first order Taylor series approximation around the nominal points $\hat{X}$ (for variable $X$) and $\hat{x}$ (for variable $x$) as
\[ \prod_{j=1}^{m} \mathcal{N}(y_j^j; Hx, s\hat{X} + R) \approx \left( \prod_{j=1}^{m} \mathcal{N}(y_j^j; Hx, s\hat{X} + R) \right) \mathcal{N}(X; m, M) \] (84)
where
\[ M = m\hat{X} + ms\hat{X}(s\hat{X} + R)^{-1} \times \left[ Y_k - (s\hat{X} + R) \right] (s\hat{X} + R)^{-1} \hat{X}. \] (85)
\[ ^{\text{i.e., constant with respect to the variables } X \text{ and } x.} \]

Proof. Proof is given in Appendix A for the sake of clarity. \qed

Lemma 2 states that the likelihood (63) can be approximately factorized into two independent likelihood terms corresponding to kinematic and extent states. This type of factorization gives independent measurement updates for the kinematic state and the extent state. For this purpose, one can set $\hat{X} = \frac{\nu_{k|k-1} - \nu_{k|k-1}}{\nu_{k|k-1}}$ and $\hat{x} = x_{k|k-1}$ in order to find the factors of Lemma 2. Using the conjugate likelihood factors in (83), the posterior density $p(x_k, \hat{X}_k | Y_{0:k})$ is given in the form of (66) with the update parameters given below.
\[ x_{k|k} = P_{k|k} \]
\[ \times \left( P_{k|k-1}^{-1} + m_kH^T(sX_{k|k-1} + R)^{-1} \right) \hat{y}_k \] (86)
\[ P_{k|k} = \left( P_{k|k-1}^{-1} + m_kH^T(sX_{k|k-1} + R)^{-1} \right)^{-1}, \] (87)
where
\[ X_{k|k-1} = \frac{V_{k|k-1}}{\nu_{k|k-1} - 2d - 2}, \] (88)
\[ \hat{y}_k = \frac{1}{m_k} \sum_{j=1}^{m_k} y_k^j. \] (89)

Note that the kinematic updates are the same as the kinematic updates proposed in (16).

The extent state is updated as given in
\[ \nu_{k|k} = \nu_{k|k-1} + m_k \] (90)
\[ V_{k|k} = V_{k|k-1} + M^L_{k}, \] (91)
where
\[ M^L_{k} = m_kX_{k|k-1} + m_k sx_{k|k-1}(sX_{k|k-1} + R)^{-1} \times \left[ Y_k - (sX_{k|k-1} + R) \right] (sX_{k|k-1} + R)^{-1} X_{k|k-1}. \] (92)

This form of the update suggests that the matrix $Y_k$ serves as a pseudo-measurement for the quantity $sX_k + R$. If $Y_k$ is larger than the current predicted estimate of $sX_k + R$ (i.e., $sX_{k|k-1} + R$) then a positive matrix quantity is added to the statistics $V_{k|k-1}$ and vice versa.

We here note that
\[ \mathbb{E}[Y_k | X_k = X_{k|k-1}] = HP_{k|k-1}H^T + sX_{k|k-1} + R \] (93)
where the expectation is taken over all the measurements at time $k$ and the estimate $x_{k|k-1}$. Hence we conclude that expected value of the second term on the right hand side of (85) is always positive which makes the current update biased. Another drawback of the update is that the uncertainty of the predicted estimate $x_{k|k-1}$ does not affect the update. In order to solve both problems we modify the quantity $M^L_{k}$ as follows.
\[ M^L_{k} \triangleq m_kX_{k|k-1} + m_k sx_{k|k-1}(HP_{k|k-1}H^T + sX_{k|k-1} + R)^{-1} \times \left[ Y_k - (HP_{k|k-1}H^T + sX_{k|k-1} + R) \right] \times (HP_{k|k-1}H^T + sX_{k|k-1} + R)^{-1} X_{k|k-1}. \] (94)

Note that in the current update the difference
\[ Y_k - (HP_{k|k-1}H^T + sX_{k|k-1} + R) \] (95)
has zero conditional mean which makes the update unbiased and the terms $(HP_{k|k-1}H^T + sX_{k|k-1} + R)^{-1}$ decrease
with increasing uncertainty in the predicted estimate \(x_{k|k-1}\) which makes the update term smaller. In the following we will call the proposed unbiased measurement update based on linearization of the likelihood in the log domain as ULL for ease of presentation.

V. Numerical simulation

In this section the newly proposed measurement update ULL, is compared with the update based on variational Bayes [18] and update proposed by Feldmann et al. in [16] which will be referred to as VB and FFK, respectively. In section V-A we will use a simulation scenario which is partly based on the simulation scenario described in [18, Section IV] which in turn is based on [19, Section 6.1]. In section V-B an ETT scenario based on the numerical simulation described in [16, VI-B] will be presented.

A. Monte-Carlo simulations

A planar extended target in the two dimensional space whose true kinematic state \(x_k^0\) and the extent state \(X_k^0\) are

\[
x_k^0 = [0 \ m, 0 \ m, 100 \ m/s, 100 \ m/s]^T \tag{96a}
\]

\[
X_k^0 = E_k \text{Diag}([300^2 \ m^2, 200^2 \ m^2])E_k^T \tag{96b}
\]
is considered. Here, \(E_k \triangleq [e_1, e_2]\) is a \(2 \times 2\) matrix whose columns \(e_1\) and \(e_2\) are the normalized eigenvectors of \(X_k^0\) which are \(e_1 \triangleq \frac{1}{\sqrt{2}}[1, 1]^T\) and \(e_2 \triangleq \frac{1}{\sqrt{2}}[1, -1]^T\). For the extended target with these true parameters, we conduct a Monte Carlo (MC) simulations to quantify the differences between the measurement updates FFK, ULL and VB. For each MC run we assume that the initial predicted target density for all three methods is the same and has the following structure:

\[
p(x_k, X_k|y_{0:k-1}) = \mathcal{N}(x_k, X_k; P_{k|k-1}, 0, \nu_{k|k-1}^j, |V_{k|k-1}^j|) \tag{97}
\]

where superscript \(j\) indicates the \(j\)th MC run. The quantities \(x_{j|k-1}^j, P_{k|k-1}\) for the kinematic state are selected as

\[
x_{j|k-1}^j \sim \mathcal{N}(x_k^0, P_{k|k-1}^j) \tag{98a}
\]

\[
P_{k|k-1} = \text{Diag}([50^2, 50^2, 10^2, 10^2]) \tag{98b}
\]

where the scalar \(\alpha_k\) is the scaling parameter for the covariance of the density in \(98a\) to adjust the distribution of \(x_{j|k-1}^j\) in the MC simulation study. The quantities \(\nu_{k|k-1}^j\) and \(|V_{k|k-1}^j|\) are selected as

\[
\nu_{k|k-1}^j = \max(7, \nu_{k|k-1}^j \sim \text{Poisson}(100)) \tag{99a}
\]

\[
\frac{V_{k|k-1}^j}{\nu_{k|k-1}^j - 2d - 2} \sim \mathcal{W}(\nu_{k|k-1}^j, X_k^0) \tag{99b}
\]

where the functions \(\text{Poisson}(\lambda)\) and \(\mathcal{W}(\nu; n, \Psi)\) represent the Poisson density with expected value \(\lambda\) and Wishart density with degrees of freedom \(n\) and scale matrix \(\Psi\). The scalar \(\delta_k\) controls the variance of the Wishart density in this study. For the \(j\)th MC run, \(m_{j|k}^j\) measurements are generated according to

\[
m_{j|k}^j = \max(2, m_{j|k}^j \sim \text{Poisson}(10)) \tag{100a}
\]

\[
y_{j|k}^j \sim \mathcal{N}(\cdot; H_{x_k^0}, s_{X_k^0} + R) \tag{100b}
\]

where \(s = 0.25, R = 100^2 I_2, m^2\) and \(H = [I_2, 0_{2 \times 2}]\). Matrices \(I_2\) and \(0_{2 \times 2}\) are the identity matrix and zero matrix of size \(2 \times 2\), respectively. The performance of the measurement updates are compared for various levels of accuracy of the initial predicted target density in view of the optimal Bayesian solution. To this end, the optimal solution is numerically computed via importance sampling using the predicted density as the proposal density. Total of \(10^5\) samples are generated from the predicted density and the normalized importance weights are calculated from the likelihood function given in (63). Using the importance weights, the posterior expected values of the kinematic state \(x_k\) and extent state \(X_k\) are calculated. The resulting expected values are referred to as \(x_{k|k}^{\text{opt}}\) and \(X_{k|k}^{\text{opt}}\) respectively.

To change the level of accuracy for the predicted target density, the parameters \(\alpha_k\) are selected on a linear grid of 40 values between 1 and 50 while values of \(\delta_k\) were selected on a logarithmic grid of 40 values between 2 and 1000. For each pair of \(\alpha_k\) and \(\delta_k\) out of the 40 pairs, \(N_{MC} = 1000\) runs have been made by selecting the parameters \(x_{j|k-1}^j, P_{k|k-1}, \nu_{k|k-1}^j, V_{k|k-1}^j\) and \(\nu_{k|k-1}^j\) as above and the estimates \(x_{k|k}^{U,j}\) and \(X_{k|k}^{U,j}\) for \(U \in \{\text{FFK}, \text{VB}, \text{ULL}\}\) are calculated.

Using the results of the MC runs, the average error metrics \(E_x^U\) and \(E_X^U\) for \(U \in \{\text{FFK}, \text{VB}, \text{ULL}\}\) for the kinematic state and the extent state, respectively, are calculated as follows:

\[
E_x^U = \left(\frac{1}{dN_{MC}} \sum_{j=1}^{N_{MC}} \left\|H(x_{k|k}^{U,j} - x_{k|k}^{\text{opt},j})\right\|_2^2\right)^{\frac{1}{2}} \tag{101}
\]

\[
E_X^U = \left(\frac{1}{d^2N_{MC}} \sum_{j=1}^{N_{MC}} \text{tr}(X_{k|k}^{U,j} - X_{k|k}^{\text{opt},j})^2\right)^{\frac{1}{4}} \tag{102}
\]

\(^1\)In each MC run 20 iterations were performed for the VB solution, although the VB solution would usually converge within the first 5 iterates.
where $X_{k|k-1}$ is the respective graphs coincide.

Table III: Comparison of measurement updates for ETT with respect to cycle time and estimation errors. The data are obtained from the single ETT simulation scenario presented here. Since the VB update diverged several times, the errors exceeding $24m$ are set to $24m$ to make the quantitative comparison meaningful.

| Update  | $E_x[m]$ Mean ± St.Dev. | $E_x[m]$ Mean ± St.Dev. | Cycle times [s] Mean± St.Dev. |
|---------|-------------------------|-------------------------|-------------------------------|
| FFK     | 15.4581 ± 0.9943        | 19.4354 ± 0.6894        | 0.1627 ± 0.0000              |
| VB      | 16.3447 ± 1.1913        | 19.8340 ± 0.7764        | 3.2547 ± 0.0002              |
| ULL     | 15.5204 ± 0.9685        | 19.2356 ± 0.6680        | 1.2933 ± 0.0000              |

The errors are illustrated in Fig. 4 and Fig. 5. Since FFK and ULL use identical kinematic state through all the simulations, the true initial state and its true extension are as shown in Fig. 6.

**B. Single extended target tracking scenario**

The performance of measurement update is evaluated in a single extended target tracking scenario where an extended target follows a trajectory in a two-dimensional space illustrated in Fig. 6. In this simulation scenario, an elliptical extended target with diameters 340 and 80 m, and it moves with a constant velocity of about $50$ km/h. The extended target’s true initial kinematic state $x_0^j$ and true initial state $X_0^j$ are as follows:

$$x_0^j = [0m, 0m, 9.8m/s, -9.8m/s]^T$$

$$X_0^j = E_k \text{Diag}([170^2 m^2, 400^2 m^2])E_k^T$$

where $E_k$ is the $2 \times 2$ matrix whose columns are the normalized eigenvectors of $X_0^j$ which are $e_1 \equiv \frac{1}{\sqrt{2}}[-1, 1]^T$ and $e_2 \equiv \frac{1}{\sqrt{2}}[1, 1]^T$, i.e., $E_k \equiv [e_1, e_2]$.

We conduct MC study to quantify the difference between different measurement updates in a single extended target tracking scenario. The measurement set $Y_k^j = \{y_k^{j,i}\}^{m_k^j}_{i=1}$ for the $j$th MC run is generated according to (100) where $s = 0.25$, $R = 20^2 I_2 m^2$ and $H = [I_2, 0_{2 \times 2}]$ and $1 \leq k \leq 181$.

In the filters, the kinematic state vector consists of the position and the velocity $x_k = [p_x, p_y, \dot{p}_x, \dot{p}_y]^T$, which evolves according to the discrete-time constant velocity model in two dimensional Cartesian coordinates (20) where $p(x_{k+1}|x_k) = \mathcal{N}(x_{k+1}; A_kx_k, Q_k)$ and

$$A_k = \begin{bmatrix} I_2 & \tau I_2 \\ 0_2 & I_2 \end{bmatrix}, \quad Q_k = \sigma_v^2 \begin{bmatrix} \frac{\tau^2}{2} I_2 & \frac{\tau^2}{2} I_2 \\ \frac{\tau^2}{2} I_2 & \frac{\tau^2}{2} I_2 \end{bmatrix},$$

$$\tau = 10s, \quad \sigma_v = 0.1m/s^2.$$  

In the filters the initial prior density for the kinematic state and extent state are

$$p(x_1, X_1) = \mathcal{N}(x_0; x_0^1, P_{1|0}) \mathcal{N}(X_0; \nu_{1|0}^j, V_{1|0}^j)$$

where

$$x_{1|0}^j \sim \mathcal{N}(x_0; x_0^0, \frac{P_0}{\alpha_0})$$

$$P_{1|0} = \text{Diag}(50^2, 50^2, 10^2, 10^2)$$

$$\nu_{1|0}^j = \max(7, \nu^j \sim \text{Poisson}(10))$$

$$V_{1|0}^j = \nu_{1|0}^j - 2d - 2 \sim \mathcal{N}(0; \frac{X_0^0}{\delta_0})$$
The state is assumed to be slowly varying. Hence, exponential forgetting strategy \cite{15, 21, 22} can be used to account for possible changes in the parameters in time. In \cite{23} it is shown that using the exponential forgetting factor will produce maximum entropy distribution in the time update for the processes which are slowly varying with unknown dynamics but bounded by a Kullback-Leibler divergence (KLD) constraint. Forgetting factor is applied to the parameters of the inverse Wishart distribution as follows:

\begin{align}
\nu_k^{j+1} &= \exp(-\tau/\tau_0)\nu_k^j, \\
V_k^{j+1} &= \nu_k^{j+1} - 2d - 2V_k^j.
\end{align}

The exponential decay time constant was selected \( \tau_0 = 15 \).

2) Evaluation of filters: 50,000 MC runs are performed where the parameters \( x_{1|0}^j, P_{1|0}^j, \nu_0^j, V_0^j \), and \( Y_k^j \) for \( 1 \leq k \leq 181 \) are selected as above. Using the results of the MC runs, the average error metrics \( E_x^{U,j} \) and \( E_x^{U,j} \) for \( U \in \{FFK, VB, ULL\} \) and \( j \)th MC run for the target's kinematic state and the extent state, respectively, are calculated as follows:

\begin{align}
E_x^{U,j} &\triangleq \left( \frac{1}{dK} \sum_{k=1}^K \left| H(x_{k|k}^M - x_{k}^j) \right|^2 \right)^{1/2}, \\
E_x^{U,j} &\triangleq \left( \frac{1}{dK} \sum_{k=1}^K \text{tr}(X_{k|k}^M - X_{k}^j)^2 \right)^{1/2}.
\end{align}

VI. CONCLUSION

A Bayesian inference technique based on Taylor series approximation of the logarithm of the likelihood function is presented. The proposed approximation is devised for the case where the prior distribution belongs to the exponential family of distribution and is continuous. The logarithm of the likelihood function is linearized with respect to the sufficient statistic of the prior distribution in exponential family such that the posterior obtains the same exponential family form as the prior. Taylor series approximation is used for linearization with respect to the sufficient statistic. However, Taylor series approximation is a local approximation which is used for approximation of the posterior over the entire support. In spite of this inherent weakness, the proposed algorithm performs well in the numerical simulations that are presented here for extended target tracking. Furthermore, there are numerous successful applications of extended Kalman filter, which is a special case of the proposed algorithm, in theory and practice.

When a Bayesian posterior estimate is needed with limited computational budget which prohibits the use of Monte-Carlo methods or even variational Bayes approximation as in real-time filtering problems, the proposed algorithm offers a good alternative.

The comparison of possible choices for the linearization point and linearization methods with respect to the sufficient statistic are among the future research problems.

VII. ACKNOWLEDGMENTS

The authors would like to thank Henri Nurminen for proof reading and providing comments on an earlier version of the manuscript.

REFERENCES

\cite{1} M. Jordan, Learning in Graphical Models. MIT Press, 1999.
\cite{2} M. I. Jordan, Z. Ghahramani, T. S. Jaakkola, and L. K. Saul, “An introduction to variational methods for graphical models,” Mach. Learn., vol. 37, no. 2, pp. 183–233, Nov. 1999. [Online]. Available: http://dx.doi.org/10.1023/A:1007665509718.
\cite{3} T. Minka, “Expectation propagation for approximate Bayesian inference,” in Proceedings of the Seventeenth Conference Annual Conference on Uncertainty in Artificial Intelligence (UAI-01). San Francisco, CA: Morgan Kaufmann, 2001, pp. 362–369.
\cite{4} H. Rue, S. Martino, and N. Chopin, “Approximate Bayesian inference for latent Gaussian models by using integrated nested Laplace approximations,” Journal of the Royal Statistical Society: Series B (Statistical Methodology), vol. 71, no. 2, pp. 319–392, 2009.
\cite{5} J. A. Nelder and R. W. M. Wedderburn, “Generalized linear models,” Journal of the Royal Statistical Society. Series A (General), vol. 135, no. 3, pp. 370–384, 1972.
\cite{6} W. Hastings, “Monte Carlo sampling methods using Markov chains and their applications,” Biometrika, vol. 57, pp. 97–109, 1970.
\cite{7} S. Geman and D. Geman, “Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images,” IEEE Transactions on Pattern Analysis and Machine Intelligence, vol. 6, no. 6, pp. 721–741, 1984.
\cite{8} M. Wainwright and M. Jordan, Graphical Models, Exponential Families, and Variational Inference, ser. Foundations and trends in machine learning. Now Publishers, 2008.
\cite{9} T. M. Cover and J. Thomas, Elements of Information Theory. John Wiley and Sons, 2006.
\cite{10} W. Hennevegol, L. Fahrmeir, and G. Tutz, Multivariate Statistical Modelling Based on Generalized Linear Models, ser. Springer Series in Statistics. Springer New York, 2001.
\cite{11} G. Smith, S. Schmidt, L. McGee, U. S. N. Aeronautics, and S. Administration, Application of Statistical Filter Theory to the Optimal Estimation of Position and Velocity on Board a Circular Vehicle, ser. NASA technical report. National Aeronautics and Space Administration, 1962.
\cite{12} M. I. Jordan, “Lecture notes on conjugacy and exponential family,” 2014.
\cite{13} R. E. Kalman, “A new approach to linear filtering and prediction problems,” Transactions of the ASME–Journal of Basic Engineering, vol. 82, no. Series D, pp. 35–45, 1960.
\cite{14} S. Thrun, W. Burgard, and D. Fox, Probabilistic Robotics, ser. Intelligent robotics and autonomous agents. MIT Press, 2005.
Writing (113) in terms of \( Z \) and \( N = [N^1, \ldots, N^m] \), we obtain

\[
-2 \sum_{j=1}^{m} \log \mathcal{N}(y^j; Hx, sX + R) \pm m \log |Z^{-1}|
+ m \log \left| sI + Z^{1/2} R Z^{1/2} \right| + \sum_{j=1}^{m} \text{tr} \left[ N^j (sZ^{-1} + R)^{-1} \right]
\]

(117)

If we make a first order Taylor series expansion for the second and the third terms on the right hand side of (117) with respect to the variables \( Z \) and \( \mathcal{N} \) around \( \hat{Z} \) and \( \hat{\mathcal{N}} \), we obtain

\[
-2 \sum_{j=1}^{m} \log \mathcal{N}(y^j; Hx, sX + R) \approx
+m \log |Z^{-1}| + m \text{tr} \left( (sR^{-1} + \hat{X})^{-1} Z \right)
+ \sum_{j=1}^{m} \text{tr} \left[ N^j (s\hat{Z}^{-1} + R)^{-1} \right]
+ s \sum_{j=1}^{m} \text{tr} \left[ (s\hat{Z}^{-1} + R)^{-1} \hat{N}^j (s\hat{Z}^{-1} + R)^{-1} \hat{Z}^{-1} Z \right]
\]

(118)

where \( \approx \) represents an approximation up to an additive constant. The first order Taylor series approximations for the scalar valued functions of matrix variables of interest are given in Appendix B. We now substitute back the relations (115) and (116) into (118) to obtain the approximation

\[
-2 \sum_{j=1}^{m} \log \mathcal{N}(y^j; Hx, sX + R)
\approx m \log |X| + m \text{tr} \left( (sR^{-1} + \hat{X})^{-1} X^{-1} \right)
+ \sum_{j=1}^{m} \text{tr} \left[ (y^j - Hx)(y^j - Hx)^T (s\hat{X} + R)^{-1} \right]
+ s \sum_{j=1}^{m} \text{tr} \left[ (s\hat{X} + R)^{-1} (y^j - H\hat{x})(y^j - H\hat{x})^T \right]
\times (s\hat{X} + R)^{-1} \hat{X} X^{-1}
\]

(119)

= m \log |X| + m \text{tr} \left( (sR^{-1} + \hat{X})^{-1} X^{-1} \right)
+ \sum_{j=1}^{m} (y^j - Hx)^T (s\hat{X} + R)^{-1} (y^j - Hx)
+ s \text{tr} \left[ \hat{X} (s\hat{X} + R)^{-1} \sum_{j=1}^{m} (y^j - H\hat{x})(y^j - H\hat{x})^T \right]
\times (s\hat{X} + R)^{-1} \hat{X} X^{-1}
\]

(120)

Rearranging the terms in (120), dividing both sides by \( -2 \) and then taking the exponential of both sides, we can see that

\[
\prod_{j=1}^{m} \mathcal{N}(y^j; Hx, sX + R)
\approx \left( \prod_{j=1}^{m} \mathcal{N}(y^j; Hx, s\hat{X} + R) \right) 
\mathcal{W} (X; m, M)
\]

(121)
Another form for the inverse Wishart parameter \( M \) can be written using the matrix inversion lemma as follows.

\[
M = m \bar{X} + ms \bar{X}(s \bar{X} + R)^{-1} 
\times \left( \frac{1}{m} \sum_{j=1}^{m} (y^j - H \bar{x})(y^j - H \bar{x})^T \right) - (s \bar{X} + R) 
\times (s \bar{X} + R)^{-1} \bar{X} 
\]

(123)

Proof is complete.

B. First Order Taylor Series Approximations for Some Scalar Valued Functions of Matrix Variables

In this section, some first order Taylor series approximations of some scalar valued functions of matrix arguments will be studied. The two functions we consider are given as

- Function 1

\[
f_1(Z) \triangleq \log \left| sI + Z^{1/2} R Z^{1/2} \right| \quad (124)
\]

- Function 2

\[
f_2(Z) \triangleq \text{tr} \left( N(sZ^{-1} + R)^{-1} \right) \quad (125)
\]

Both functions can be approximated with a first order Taylor series approximation around a nominal point \( \hat{Z} \) as follows.

\[
f_k(Z) \approx f_k(\hat{Z}) + \text{tr} \left( F_k^T (Z - \hat{Z}) \right) \quad (126)
\]

for \( k = 1, 2 \) where \( F_k \) is defined as

\[
[F_k]_{ij} \triangleq \frac{\partial f_k}{\partial z_{ij}} \bigg|_{Z=\hat{Z}} \quad (127)
\]

for \( 1 \leq i, j \leq d \) where the notation \( [\cdot]_{ij} \) denotes the element corresponding to \( i \)th row and \( j \)th column of the argument matrix and \( z_{ij} \triangleq [Z]_{ij} \). Hence, in order to construct the required Taylor series approximation, we are only required to calculate the matrices \( F_1 \) and \( F_2 \).

1) Calculation of \( F_1 \):

We first write \( f_1(\cdot) \) as

\[
f_1(Z) \triangleq \log \left| sI + Z^{1/2} R Z^{1/2} \right| = \log |Z| + \log |sZ^{-1} + R| \quad (128)
\]

Now, we can calculate the related derivative using two well-known matrix derivatives

\[
\frac{\partial \ln |U|}{\partial x} = \text{tr} \left( U^{-1} \frac{\partial U}{\partial x} \right) \quad (130)
\]

\[
\frac{\partial U^{-1}}{\partial x} = - U^{-1} \frac{\partial U}{\partial x} U^{-1} \quad (131)
\]

\[
\frac{\partial f_1}{\partial z_{ij}} = \text{tr} \left( Z^{-1} \frac{\partial Z}{\partial z_{ij}} \right) + \text{tr} \left( (sZ^{-1} + R)^{-1} \frac{\partial(sZ^{-1} + R)}{\partial z_{ij}} \right) \quad (132)
\]

\[
= \text{tr} \left( Z^{-1} E_{ij} \right) + s \text{tr} \left( (sZ^{-1} + R)^{-1} \frac{\partial Z}{\partial z_{ij}} \right) \quad (133)
\]

\[
= \left[ Z^{-1} \right]_{ji} - s \text{tr} \left( (sZ^{-1} + R)^{-1} \frac{\partial Z}{\partial z_{ij}} Z^{-1} \right) \quad (134)
\]

\[
= \left[ Z^{-1} \right]_{ji} - s \text{tr} \left( Z^{-1} (sZ^{-1} + R)^{-1} Z^{-1} E_{ij} \right) \quad (135)
\]

\[
= \left[ Z^{-1} \right]_{ji} - s \text{tr} \left( [Z^{-1} (sZ^{-1} + R)^{-1} Z^{-1}]_{ji} \right) \quad (136)
\]

\[
= \left[ (Z + sR^{-1})^{-1} \right]_{ji} \quad (137)
\]

where \( E_{ij} \) is a \( d \times d \) matrix filled with zeros except the \( ij \)th element which is unity. The last expression (137) is equivalent to

\[
F_1 = (Z + sR^{-1})^{-T} \quad (139)
\]

which gives

\[
\log \left| sI + Z^{1/2} R Z^{1/2} \right| \approx \log \left| sI + R \hat{Z} \right| + \text{tr} \left( (\hat{Z} + sR^{-1})^{-1} (Z - \hat{Z}) \right) \quad (140)
\]

\[2) \text{Calculation of } F_2:\]

\[
\frac{\partial f_2}{\partial z_{ij}} = \text{tr} \left( N \frac{\partial (sZ^{-1} + R)^{-1}}{\partial z_{ij}} \right) \quad (141)
\]

\[
= \text{tr} \left( N \frac{\partial(sZ^{-1} + R)}{\partial z_{ij}} \right) \quad (142)
\]

\[
= - s \text{tr} \left( N (sZ^{-1} + R)^{-1} \frac{\partial(sZ^{-1} + R)}{\partial z_{ij}} \right) \quad (143)
\]

\[
= - s \text{tr} \left( (sZ^{-1} + R)^{-1} \frac{\partial Z}{\partial z_{ij}} (sZ^{-1} + R)^{-1} \right) \quad (144)
\]

\[
= s \text{tr} \left( (sZ^{-1} + R)^{-1} \frac{\partial Z}{\partial z_{ij}} \right) \quad (145)
\]

\[
= s \text{tr} \left( N (sZ^{-1} + R)^{-1} Z^{-1} E_{ij} (sZ^{-1} + R)^{-1} \right) \quad (146)
\]

\[
= s \text{tr} \left( (sZ^{-1} + R)^{-1} N (sZ^{-1} + R)^{-1} Z^{-1} E_{ij} \right) \quad (147)
\]

\[
= s \left[ Z^{-1} (sZ^{-1} + R)^{-1} N (sZ^{-1} + R)^{-1} Z^{-1} \right]_{ji} \quad (148)
\]

The last expression (148) is equivalent to

\[
F_2 = s \left[ Z^{-1} (sZ^{-1} + R)^{-1} N (sZ^{-1} + R)^{-1} Z^{-1} \right]^T \quad (149)
\]

which gives

\[
\text{tr} \left( N (s\hat{Z}^{-1} + R)^{-1} \right) \approx \text{tr} \left( N (s\hat{Z}^{-1} + R)^{-1} \right) \quad (150)
\]

\[+ s \text{tr} \left( \hat{Z}^{-1} (s\hat{Z}^{-1} + R)^{-1} N (s\hat{Z}^{-1} + R)^{-1} \hat{Z}^{-1} (Z - \hat{Z}) \right) \]

\[+ s \text{tr} \left( (s\hat{Z}^{-1} + R)^{-1} N (s\hat{Z}^{-1} + R)^{-1} \hat{Z}^{-1} (Z - \hat{Z}) \right) \]

\[= s \text{tr} \left( (s\hat{Z}^{-1} + R)^{-1} N (s\hat{Z}^{-1} + R)^{-1} \hat{Z}^{-1} (Z - \hat{Z}) \right) \]