Vibration analysis of rotating porous functionally graded material beams using exact formulation

Mohammadreza Amoozgar® and Len Gelman®

Abstract
In this article, the exact free vibration of porous functionally graded rotating blades is investigated. The nonlinear 3D dynamics of the blade is simulated using the geometrically exact fully intrinsic beam equations, and the corresponding cross-sectional properties of the FG beam are developed. The material properties of the functionally graded material blade are graded through the thickness using a power law distribution. Furthermore, it is assumed that due to the manufacturing process, a level of porosity exists in the material which in turn can affect the material properties of the blade. Two porosity models resembling the even and uneven distributions of porosity are considered. First, the obtained results for a functionally graded material rotating blade are compared with those reported in the literature, and a very good agreement is observed. Furthermore, the effect of various parameters on the vibration of the functionally graded material beam is investigated. It is obtained that the dynamics of the rotating blade is sensitive to the type of the porosity due to manufacturing flaws. Moreover, the numerical results show that the blade length to height ratio, power law index, rotating speed and porosity distribution model affect the dynamics of the beam significantly.

Keywords
rotating functionally graded material blade, fully intrinsic equations, mode veering, porosity, Saint-Venant warping

1. Introduction
Understanding the dynamic behaviour of rotating structures such as helicopter blades, wind turbine blades and gas turbine blades is an important step in design process of such structures. Furthermore, the noise and vibration of rotating blades is one of the main challenges, that the designers face with, which could limit the performance of final design (Dibble et al., 2019; Friedmann and Hodges, 2003; Romani and Casalino, 2019; Tamer et al., 2019; Vouros et al., 2019). It is widely accepted that for high aspect ratio structures, the dynamics of the structure can be modelled, using a beam theory, and, hence, many researchers considered the vibration of rotating beams by considering the effects of rotating speed, blade pre-twist, blade taper, material, etc. (Carnegie, 1959; Dubey et al., 2020; Han et al., 2020; Li et al., 2014; Swaminathan and Rao, 1977). Bossak and Zienkiewicz (1973) studied the effect of centrifugal forces on the free vibration of rotating structures. They showed that the centrifugal force changes the stiffness characteristics of the structure and hence affects the dynamics of beam. The flexural vibration of pre-twisted blade with tapered cross-section was considered by Swaminathan and Rao (1977). They showed that the pre-twist of the blade combined with the rotating speed influences the frequencies of the blade. The dynamic response of rotating tapered blades simulated using Timoshenko beam theory was investigated by Bazoune et al. (2001). A model reduction approach was incorporated, and the accuracy of the reduced-order model was demonstrated by comparing the results with the full-order model. The dynamic stability of pre-twisted rotating beams modelled using the Timoshenko beam theory subjected to lateral excitation was studied by Sabuncu and Evran (2006). It was concluded that the pre-twist of the blade, the coupling and the shear coefficient can influence the dynamics of the beam. There are many other studies concerning the dynamics of rotating structures. Rafiee et al. (2017) presented an extensive literature review on
the vibration of rotating blades. More recently, Dubey et al. (2020) investigated the stability of rotating sandwich blades with the effects of pre-twist, taper and temperature gradient. They highlighted that the angular speed and pre-twist of the blade change the dynamic stability of the beam.

Functionally graded materials (FGMs) are first proposed in early 1980s to be used in high-temperature environments (Koizumi, 1997). The FGMs are normally made from isotropic materials (metals and ceramics) and have remarkable properties such as high strength, low weight, long fatigue life and wear resistance. Therefore, these materials have extensively been used in different applications, and hence, various analyses (static, dynamics...) on the behaviour of structures made of FGMs have been carried out (Cong et al., 2018; Jin et al., 2019; Khaneh Masjedi et al., 2019; Lee and Lee, 2017; Pradhan and Chakraverty, 2013; Sarkar and Ganguli, 2014; Zahedinejad et al., 2020). The dynamic behaviour of FGM structures is an active research area and has attracted the interest of many researchers (Amoozgar et al., 2017; Giunta et al., 2011; Lü et al., 2008; Şimşek and Kocatürk, 2009; Sina et al., 2009; Vo et al., 2014). The free vibration of Euler and Timoshenko beams made of functionally graded materials was considered by Pradhan and Chakraverty (2013). They concluded that different beam theories might result in different frequency values depending on the power law index. Fang et al. (2017) determined the three-dimensional vibration and response of rotating functionally graded beams using Chebyshev polynomial trial functions. It was shown that the rotating speed and power law index affect the frequencies and mode veering of the beam. The free vibration of FGM Euler–Bernoulli beams using an exact transfer matrix expression was analysed by Lee and Lee (2017). The effect of dynamic stiffening on the free vibration of rotating functionally graded beams was considered by Li et al. (2014). They showed that the lag-extension coupling results in mode shift and frequency veering.

The existence of micro-voids and porosities due to the manufacturing processes can affect the mechanical properties of the final product. Therefore, understanding the behaviour of the structure due to possible manufacturing flaws is crucial specially for rotating structures. Ebrahimi and Mokhtari (2015) determined the transverse linear vibration of evenly distributed porous FGM rotating blades using the differential transform method. It was highlighted that the porosity distribution and power law index can affect the dynamic properties of the beam. Shahsavari et al. (2018) investigated the free vibration of FG porous plates resting on elastic foundation using a quasi-3D hyperbolic theory. The effect of evenly and unevenly distributed porosity on the large amplitude vibration of FGM thin plates has been studied by Wang and Zu (2017). They highlighted that the porosity affects the vibration amplitude of the plate. The quasi-3D dynamic behaviour of functionally graded beams using Carrera unified beam formulation was studied by Jin et al. (2019). Zahedinejad et al. (2020) presented a comprehensive literature review of the research studies carried out on the vibration of FGM beams. More recently, Tian et al. (2019) studied the effect of porosity and material grading on the fundamental frequencies of rotating FGM beams and showed that the rotational speed, hub radius and porosity affect the dynamics of the beam significantly.

To add to the aforementioned bulk of literature, in this study, the effect of evenly and unevenly distributed porosity and material grading on the nonlinear 3D vibration of rotating functionally graded beams is investigated. The rotating blade is modelled using the exact fully intrinsic beam formulation (Hodges, 2003) which is relatively new and has been used to analyse various beam-like structures (Amoozgar and Shahverdi, 2016, 2019; Mardanpour et al., 2013; Sotoudeh and Hodges, 2013). In none of the studies presented above, the exact 3D vibration of rotating FGM beams with porosity has not been considered. Therefore, in this study, a novel 3D model based on fully intrinsic theory is developed for analysis of porous functionally graded rotating beams. Also, two types of porosity distribution models are considered, and the effects of various geometrical and material properties on the 3D dynamic behaviour of the rotating beam are studied.

2. Problem statement

A rotating blade attached to a rigid hub, as shown in Figure 1(a), is considered, where the length of the blade is denoted by \( L \). The blade is rotating by a constant angular velocity of \( \Omega \). Owing to the aspect ratio of the blade, it is simulated using a beam model. It is assumed that the main load carrying part of the blade is a solid rectangular box spar, which is made of functionally graded material. Also, it is considered that the blade is rotating by a constant angular velocity and the hub radius is very small in comparison with the blade length. The reference coordinate system is fixed at the centre of the hub, in which \( x_1 \) axis lies in the span direction, \( x_2 \) is towards the chord of the blade and \( x_3 \) is upwards in the thickness direction. The material properties of the blade are considered to be graded through the thickness by using a simple power law distribution. Furthermore, to take into account the effect of porosity, two models of porosity distribution are considered. These two models, as shown in Figure 1 (b) and (c), resemble the even and uneven porosity distributions, where the width and height of the rectangular cross-section are denoted by \( h \) and \( h \), respectively. Finally, the effects of material grading, blade geometry and porosity distribution on the frequencies of the rotating blade are studied. This is of high significance to investigate the impacts of
manufacturing flaws on the dynamic behaviour of rotating blades accurately.

3. Governing equations of rotating porous FGM beams

3.1. FGMs with porosities

A functionally graded beam, composed of metal and ceramic materials, is considered here. It is assumed that the porosity is distributed equally in both ceramic and metal phases, and, hence, the general material properties of an FGM beam with evenly distributed porosity can be written as (Cong et al., 2018)

\[
Q(x_3) = Q_c \left[ V_c(x_3) - \frac{\beta}{2} \right] + Q_m \left[ V_m(x_3) - \frac{\beta}{2} \right]
\]

(1)

where \( Q_c \) and \( V_c \) are the effective material properties and volume fraction of the ceramic, \( Q_m \) and \( V_m \) are the effective material properties and volume fraction of the metal and \( \beta (\beta < 1) \) is the porosity volume fraction in both ceramic and metal.

The volume fractions of ceramic \( (V_c) \) and metal \( (V_m) \) are considered to be distributed along the thickness as follows (Cong et al., 2018)

\[
V_c(x_3) = \left( \frac{x_3}{h} + \frac{1}{2} \right)^k
\]

\[
V_m(x_3) = 1 - \left( \frac{x_3}{h} + \frac{1}{2} \right)^k
\]

(2)

where \( k (0 \leq k \leq \infty) \) is the power law index and \( h \) is the beam thickness.

Finally, the Young’s modulus \( (E) \), shear modulus \( (G) \) and density \( (\rho) \) of the FGM beam with even porosity distribution (model I) can be written as (Cong et al., 2018)

Figure 1. (a) Schematic of the functionally graded material rotating blade; (b) even porosity distribution (model I); (c) uneven porosity distribution (model II).
\[
E(x_3) = E_m + (E_c - E_m) \left( \frac{x_3}{h} + \frac{1}{2} \right)^k - \left( 1 - \frac{2|x_3|}{h} \right) (E_c + E_m) \frac{\beta}{2} \\
G(x_3) = G_m + (G_c - G_m) \left( \frac{x_3}{h} + \frac{1}{2} \right)^k - \left( 1 - \frac{2|x_3|}{h} \right) (G_c + G_m) \frac{\beta}{2} \\
\rho(x_3) = \rho_m + (\rho_c - \rho_m) \left( \frac{x_3}{h} + \frac{1}{2} \right)^k - \left( 1 - \frac{2|x_3|}{h} \right) (\rho_c + \rho_m) \frac{\beta}{2}
\]  
(3)

It is noted that Poisson’s ratio is considered to be constant along the thickness. Furthermore, the material properties of the FGM beam with uneven porosity distribution (model II) can be written as (Cong et al., 2018)

\[
E(x_3) = E_m + (E_c - E_m) \left( \frac{x_3}{h} + \frac{1}{2} \right)^k - \left( 1 - \frac{2|x_3|}{h} \right) (E_c + E_m) \frac{\beta}{2} \\
G(x_3) = G_m + (G_c - G_m) \left( \frac{x_3}{h} + \frac{1}{2} \right)^k - \left( 1 - \frac{2|x_3|}{h} \right) (G_c + G_m) \frac{\beta}{2} \\
\rho(x_3) = \rho_m + (\rho_c - \rho_m) \left( \frac{x_3}{h} + \frac{1}{2} \right)^k - \left( 1 - \frac{2|x_3|}{h} \right) (\rho_c + \rho_m) \frac{\beta}{2}
\]  
(4)

3.2. Exact fully intrinsic beam formulation for FGM rotating blades

The dynamics of the rotating FGM blade is modelled by combining the 1D geometrically exact fully intrinsic beam equations (Hodges, 2003) with a 2D cross-sectional analysis. It is noted that in this formulation, no assumption is made for developing the beam equations, except the small strain assumption (the effects of material nonlinearities are ignored). The exact fully intrinsic beam equations for a straight beam without any external force can be written as (Hodges, 2003)

\[
\frac{\partial P_1}{\partial t} + \Omega_2 P_3 - \Omega_3 P_2 = \frac{\partial F_1}{\partial x_1} + K_2 F_3 - K_3 F_2 \\
\frac{\partial P_2}{\partial t} + \Omega_3 P_1 - \Omega_1 P_3 = \frac{\partial F_2}{\partial x_1} + K_3 F_1 - K_1 F_3 \\
\frac{\partial P_3}{\partial t} + \Omega_1 P_2 - \Omega_2 P_1 = \frac{\partial F_3}{\partial x_1} + K_1 F_2 - K_2 F_1
\]

where \( F \) and \( M \) are the internal forces and moments, \( V \) and \( \Omega \) are the linear and angular velocities and \( e_1 \) is a vector in which its arrays are as below (Hodges, 2003)

\[
e_1 = [1 \quad 0 \quad 0]^T
\]
(6)

The generalized linear \( (P) \) and angular \( (H) \) momenta vectors are related to the linear and angular velocities through the cross-sectional mass matrix as follows (Hodges, 2003)

\[
\begin{bmatrix} P \\ H \end{bmatrix} = \begin{bmatrix} \mu \Lambda & -\mu \xi \\ \mu \xi & I \end{bmatrix} \begin{bmatrix} V \\ \Omega \end{bmatrix}
\]
(7)

where \( \mu \) is the mass per unit length of the blade, \( \xi \) is a vector storing the offsets between the beam reference axis and the mass centre, \( \Lambda \) is a unit matrix and \( I \) is the cross-sectional moment of inertia which can be written as (Hodges, 2003)

\[
\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_{13} \\ I_{23} \\ I_3 \end{bmatrix}
\]
(8)

It is noted that in this study, it is assumed that the beam reference axis coincides with the mass centre (\( \xi = 0 \)). Furthermore, the mass per unit length and mass moment of inertia of the FGM beam, which are dependent to the power law index \( (k) \), can be obtained as
\( \mu = \int_A \rho(x_3) dA \)
\( i_2 = \int_A \rho(x_3) x_3^2 dA \)
\( i_3 = \int_A \rho(x_3) x_3^2 dA \)
\( i_{23} = \int_A \rho(x_3) x_2 x_3 dA \)

The internal force and moment vectors can be obtained from the generalized strain measures, using the cross-sectional stiffness matrix as follows (Hodges, 2003)

\[
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
M_1 \\
M_2 \\
M_3
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\
S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\
S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\
S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\
S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\
S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66}
\end{bmatrix}\begin{bmatrix}
\gamma_{11} \\
\gamma_{12} \\
\gamma_{13} \\
\kappa_1 \\
\kappa_2 \\
\kappa_3
\end{bmatrix}
\]

where \( S \) is the cross-sectional stiffness matrix and \( \gamma \) and \( \kappa \) are the generalized strain measures.

It is noted, that in this study, the Euler–Bernoulli beam has been considered and, therefore, the effects of shear

---

**Table 1.** The comparison of the first five natural frequencies of a stationary non-porous functionally graded material beam.

|      | \( m_1 \)  | \( m_2 \)  | \( m_3 \)  | \( m_4 \)  | \( m_5 \)  |
|------|------------|------------|------------|------------|------------|
| Present | 1.9499     | 12.108     | 30.236     | 33.432     | 1.9499     |
| Lee and Lee (2017) | 1.9497 | 12.080 | 30.231 | 33.224 | 1.9497     |
| Present \( k = 0.5 \) | 1.6602 | 10.302 | 27.254 | 28.461 | 1.6602     |
| Lee and Lee (2017) | 1.6601 | 10.284 | 27.265 | 28.303 | 1.6601     |
| Present \( k = 5 \) | 1.3034 | 8.061 | 19.952 | 22.185 | 1.3034     |
| Lee and Lee (2017) | 1.3034 | 8.0526 | 19.946 | 22.099 | 1.3034     |

**Table 2.** The comparison of the natural frequencies of a rotating non-porous functionally graded material beam \( (k = 5, \bar{\Omega} = 10, L/h = 10) \).

| \( L/h = 10 \) | \( m_1 \)  | \( m_2 \)  | \( m_3 \)  | \( m_4 \)  | \( m_5 \)  | \( m_6 \)  | \( m_7 \)  | \( m_8 \)  | \( m_9 \)  |
|----------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| Present | 3.1527 | 10.6512 | 19.0421 | 27.2314 | 28.6510 | 39.4943 | 52.9760 | 54.8006 | 62.1016 |
| Jin et al. (2019) | 3.2699 | 10.7904 | — | 26.1447 | 29.806 | 40.9036 | 53.2677 | 57.9468 | 65.3183 |

**Figure 2.** The effect of cross-sectional warping and rotating speed on the frequencies of the functionally graded material beam for \( k = 1 \) and \( L/h = 10 \).

**Figure 3.** The effect of rotating speed on the frequencies of functionally graded material beam for various power law indices and for \( L/h = 10 \).
strains are ignored \((\gamma_{12} = \gamma_{13})\). Therefore, the cross-sectional stiffness matrix for the Euler–Bernoulli beam can be reduced to

\[
\begin{bmatrix}
F_1 \\
M_1 \\
M_2 \\
M_3
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{14} & S_{15} & S_{16} \\
S_{14} & S_{44} & S_{45} & S_{46} \\
S_{15} & S_{45} & S_{55} & S_{56} \\
S_{16} & S_{46} & S_{56} & S_{66}
\end{bmatrix}
\gamma_{11}
\begin{bmatrix}
\kappa_1 \\
\kappa_2 \\
\kappa_3
\end{bmatrix}
\] (11)

The nonzero cross-sectional stiffness values of the above stiffness matrix for the FGM straight prismatic beam with rectangular solid cross-section can be written as

\[
S_{11} = \int E(x_3) dA \\
S_{44} = \int G(x_3)(x_2^2 + x_3^2 + x_2x_3) dA \\
S_{55} = \int E(x_3)x_3^2 dA \\
S_{66} = \int E(x_3)x_2^2 dA \\
S_{14} = \int E(x_3)x_2 dA \\
S_{15} = \int E(x_3)x_3 dA \\
S_{16} = S_{45} = S_{46} = S_{56} = 0
\] (12)

where \((\cdot)\) and \((\cdot)\) are the derivatives with respect to \(x_2\) and \(x_3\) and \(A\) is the cross-sectional area. It is noted that, as shown in equation (12), all these stiffness terms are dependent to the material properties and hence can affect the dynamics of the beam.

Furthermore, \(\psi\) is the Saint-Venant warping, which for a solid rectangular cross-section can be written in the form of an infinite series as follows (Timoshenko and Goodier, 1970)

\[
\psi = -x_2x_3 + \frac{8h^2}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \sinh((2n+1/h)x_2) \cosh((2n+1/2h)\pi b) \times \sin \left( \frac{2n+1}{h} \pi x_3 \right)
\] (13)

This shows that all nonzero values within the stiffness matrix are dependent to the power law index of the FGM and, hence, will affect the dynamics of the beam.

The final dynamic equations are discretized here using a space–time scheme (Hodges, 2003). In this method, all variables \((F, M, V \text{ and } \Omega)\) are defined on the right and left hand sides of each node. The final discretized equations of motion can be written as follows

\[
a_{ij}q_i + b_{ij}q_i + c_{ik}q_iq_k = 0
\] (14)
where \( q \) stores the discretized unknown variables \((F_i, M_i, V_i, \Omega_i, i = 1 \ldots 3)\) at the right and left hand side of each node and \( a, b \) and \( c \) are the matrices containing the linear and nonlinear terms. The natural frequencies of the FGM blade are calculated by obtaining the eigenvalues of the linearized equations. To do this, first, the time derivative variables are removed from the nonlinear equations (equation (14)), and the steady-state condition \((\ddot{q})\) of the below system is obtained using the Newton–Raphson scheme

\[
b_j\ddot{q}_j + c_j\dot{q}_j = 0 \tag{15}
\]

Then, the linearized equations about this steady-state condition can be obtained (as shown in equation (16)), and the eigenvalues of this linearized system are determined

\[
a_j\ddot{q}_j + b_j\dot{q}_j = 0 \tag{16}
\]

4. Numerical results

4.1. Modelling validation

To validate the results, first, the nondimensional natural frequencies of a stationary perfect (non-porous) FGM beam for various power law indices and two length to height ratios are determined and presented in Table 1. The results are compared with those reported by Lee and Lee (2017) for the clamped-free beam, and a very good agreement is observed. It is noted that Lee and Lee (2017) used a linear beam model and the effects of torsional flexibility were ignored.

Next, the three nondimensional frequencies of a square FGM blade are compared with the results provided by Jin et al. (2019) and presented in Table 2.

The nondimensional natural frequencies and rotating speed are considered using the following equations

| Table 3. The effect of beam length to height ratio and power index on natural frequencies of the rotating non-porous functionally graded material beam (\( \Omega = 10, b/h = 1 \)). |
|---|---|---|---|---|---|---|---|---|
| \( L/h = 10 \) | \( k = 0 \) | \( k = 0.5 \) | \( k = 1 \) | \( k = 2 \) | \( k = 3 \) | \( k = 4 \) | \( k = 5 \) | \( k = 10 \) |
| \( \dddot{w} \) | 4.649 | 4.254 | 3.970 | 3.604 | 3.388 | 3.249 | 3.153 | 2.916 | 2.542 |
| \( \dddot{w} \) | 11.129 | 10.919 | 10.807 | 10.711 | 10.677 | 10.661 | 10.651 | 10.619 | 10.465 |
| \( \dddot{w} \) | 30.721 | 26.938 | 24.563 | 21.793 | 20.371 | 19.556 | 19.042 | 17.958 | 15.758 |
| \( \dddot{w} \) | 31.109 | 29.988 | 29.312 | 28.501 | 27.988 | 27.576 | 27.231 | 26.157 | 24.943 |
| \( \dddot{w} \) | 33.192 | 31.075 | 30.063 | 29.216 | 28.887 | 28.737 | 28.651 | 28.394 | 27.265 |

| Table 4. The effects of beam length to height ratio and power index on natural frequencies of the rotating porous functionally graded material beam, Model I (\( \Omega = 10, b/h = 1 \)). |
|---|---|---|---|---|---|---|---|---|
| \( L/h = 10 \) | \( k = 0 \) | \( k = 0.5 \) | \( k = 1 \) | \( k = 2 \) | \( k = 3 \) | \( k = 4 \) | \( k = 5 \) | \( k = 10 \) |
| \( \dddot{w} \) | 4.649 | 4.761 | 4.254 | 4.301 | 3.604 | 3.391 | 3.153 | 2.605 | 2.916 | 2.209 |
| \( \dddot{w} \) | 11.129 | 11.174 | 10.919 | 10.915 | 10.711 | 10.589 | 3.153 | 2.605 | 2.916 | 2.209 |
| \( \dddot{w} \) | 30.721 | 31.595 | 29.988 | 27.178 | 21.793 | 19.744 | 19.042 | 14.452 | 12.613 |
| \( \dddot{w} \) | 31.109 | 31.129 | 29.988 | 30.301 | 28.501 | 27.929 | 27.231 | 27.007 | 26.157 | 26.011 |
| \( \dddot{w} \) | 33.192 | 33.718 | 31.075 | 31.129 | 29.216 | 29.033 | 28.651 | 27.699 | 28.394 | 27.190 |
\[ \omega_i = \omega_i L^2 \frac{\rho_c A}{E_c I} \]
\[ \Omega = \Omega L^2 \frac{\rho_c A}{E_c I} \]  
(17)

where \( I \) is the area moment of inertia of the cross-section.

It is clear that the results are in very good agreement. It is noted that here the third frequency, which is the first torsion mode, was not reported in Jin et al. (2019).

By considering above two comparisons, it is clear that the developed formulation can accurately predict free vibration of rotating FGM beams. In what follows, the free vibration of FGM rotating beams with and without porosity is considered.

4.2. The effect of rotation on the free vibration of non-porous FGM blades

A square FGM blade \( (b/h = 1) \) with the following material properties is considered

\[ E_c = 380 \text{ GPa}, \quad E_m = 70 \text{ GPa}, \quad \rho_c = 3800 \frac{\text{kg}}{\text{m}^3}, \quad \rho_m = 2700 \frac{\text{kg}}{\text{m}^3}, \quad \nu = 0.3 \]

The blade is rotating with a constant angular speed, and the frequencies are determined. For simplicity, the non-dimensional angular speed and frequency, defined in equation (16), are used.

First, the effect of cross-sectional warping on the natural frequencies of the rotating FGM blade is studied, and the results are shown in Figure 2, ignoring the cross-sectional warping in the formulation results in overpredicting the torsional frequency. Furthermore, it also affects the mode veering and hence changes the frequencies of modes which are coupled with torsion. This is important as mode veering indicates the direction of eigenvectors. This highlights the importance of including the cross-sectional warping in vibration analysis of FGM rotating beams. In all cases from here on, the effect of cross-sectional warping is retained in the formulation.

The effect of the rotating speed on the FGM beam frequencies with a length to height ratio of \( L/b = 10 \) and for various power law indices is investigated first and shown in Figure 3. It is clear that both rotating speed and power law index not only affect the frequency values, but also change the dynamic behaviour of the beam. When the power law index is \( k = 0 \), the veering happens between the fourth and fifth modes, whereas for other cases, the modes contributing in the veering are different. Furthermore, the rotating speed, at which the mode veering happens, is also dependent to the value of \( k \).

Figure 4 shows the mode shapes of the stationary FGM beam for various power law indices \( (k = 0, k = 1, k = 5) \). To this aim, the torsion, flap and lag components of the modes, which have been normalized so that the 2-norm of each mode to be equal to 1, are shown. Figure 4(a) shows the mode shapes of the beam for \( k = 0 \). For this case, the first and third modes are lag, the second and fourth modes are flap and the fifth mode is torsion. This is slightly different when the power law index \( (k) \) is not zero. The mode shapes of the FGM beam for \( k = 1 \) are determined and shown in Figure 4(b). Unlike the previous case, the first and third modes are flap, and the second and fourth modes are lag, and the fifth mode is a combination of torsion and lag.

### Table 5. The effect of beam length to height ratio and power index on the natural frequencies of the rotating porous functionally graded material beam, Model II \( (\Omega = 10, b/h = 1) \).

| \( L/h \) | \( k = 0 \) | \( k = 0.5 \) | \( k = 2 \) | \( k = 5 \) | \( k = 10 \) |
|-----------|----------|----------|----------|----------|----------|
| \( \beta = 0 \) | \( \beta = 0.2 \) | \( \beta = 0 \) | \( \beta = 0.2 \) | \( \beta = 0 \) | \( \beta = 0.2 \) | \( \beta = 0 \) | \( \beta = 0.2 \) |
| \( \omega_1 \) | 4.649    | 4.701    | 4.254    | 4.276    | 3.604    | 3.531    | 3.153    | 2.976    | 2.916    | 2.679    |
| \( \omega_2 \) | 11.129   | 11.169   | 10.919   | 10.943   | 10.711   | 10.696   | 10.651   | 10.620   | 10.619   | 10.588   |
| \( \omega_3 \) | 30.721   | 30.926   | 26.985   | 26.923   | 21.793   | 20.997   | 19.042   | 17.580   | 17.958   | 16.292   |
| \( \omega_4 \) | 31.109   | 31.331   | 29.988   | 30.119   | 28.501   | 28.486   | 27.231   | 27.201   | 26.157   | 26.000   |
| \( \omega_5 \) | 33.192   | 33.590   | 31.075   | 31.297   | 29.216   | 29.173   | 28.651   | 28.471   | 28.394   | 28.240   |

Figure 5. The effect of rotating speed and porosity volume fraction (model I) on the frequencies of the beam with \( L/h = 10, h/b = 1 \) and \( k = 2 \).
Figure 4(c) shows the mode shape of the FGM beam for $k = 5$. The modes for this case are the same as the first case ($k = 0$), except that the fifth mode again is a combination of torsion and lag. This highlights that the power law index of the FGM affects the dynamic behaviour of rotating beams.

The variation of nondimensional frequencies of the non-porous rotating FGM beam is determined for various power law indices and presented in Table 3 for three values of length to height ratios. In all cases, the rotating speed of the beam is equal to $\Omega = 10$, and the length to height ratio of the beam is $L/h = 10$, $L/h = 30$ and $L/h = 80$, respectively. The frequencies decrease when the power law index increases; but the rate of variation is not monotonic. At first, there is a sharp reduction in the frequency, and then the frequency gradually decreases until a point, at which the frequency remains almost constant. Furthermore, for all values of $k$, when the length to height ratio increases from $L/h = 10$ to $L/h = 30$, the frequencies increase, but the rate of change is not constant for all modes. On the other hand, when the length to height ratio increases from $L/h = 30$ to $L/h = 80$, the frequencies do not change too much.

4.3. The free vibration of evenly distributed porous rotating FGM beam (Model I)

Here, it is assumed that a uniformly distributed porosity (model I) available in the material and the effect of porosity volume fraction ($\beta$) on the natural frequencies of the rotating FG beam is obtained. First, the effect of the power law index on natural frequencies of the FGM beam with and without porosity is determined and presented in Table 4. In this case, the length to height ratio of the FG beam is $L/h = 10$, and the beam is rotating with a constant speed of $\Omega = 10$. The results show that the porosity volume fraction affects the dynamics of the FG beam, and depending on the value of the power law index, it could increase or decrease the frequency values. Therefore, it is necessary to consider the effect of manufacturing flaws in the design of such structures.
Furthermore, for power law indices less than 0.5 ($k < 0.5$), the porosity increases all five frequencies of the beam, whereas for $k > 0.5$, the porosity decreases the frequencies. Next, the effect of rotating speed on the frequencies of the beam with and without porosity is studied. Figure 5 shows the variation of first five frequencies of the FG beam for $k = 2$ and $L/h = 10$. It is noted that as the porosity directly affects the material properties (equations (3) and (4)), it could also affect the beam frequencies. As shown in Figure 5, the first and fifth modes are affected less than the other modes by the porosity. The third mode is affected the most by the porosity among all modes, and the frequency increase is dependent to the value of rotating speed. Furthermore, the porosity volume fraction shifts the rotating speed at which the mode veering happens.

### 4.4. The free vibration of unevenly distributed porous rotating FGM beam (Model II)

As it was mentioned, in this article, two porosity distribution models are considered. In this section, the effect of unevenly distributed porosity (model II) on the free vibration of the FGM rotating beam is studied. Table 5 presents the variation of the frequencies of the beam with and without uneven porosity for $L/h = 10$ and $\beta = 0.2$. Similar to the first porosity model (model I), when the uneven porosity is available in the material, depending on the power law index value, the frequencies might increase or decrease.

Furthermore, Figure 6 shows the effect of rotating speed and type of porosity distribution on the frequencies of the FGM beam with $k = 2$. The type of porosity distribution not only affects the values of frequencies, but also it shifts the location at which the mode veering happens. Furthermore, for this case ($k = 2$), the model with uneven porosity distribution results in higher frequencies than the even porosity model. To elaborate this more, the variation of the first three frequencies of the beam for different porosity volume fractions is studied next.

Figure 7 compares the variation of the first three frequencies of the rotating FGM beam for two types of porosity distributions and two values of power law indices. In this case, the beam is rotating with a constant speed of $V = 5$, and the length to height ratio is $L/h = 10$. When the power law index is $k = 2$, the uneven porosity model predicts higher frequencies than the even porosity model. Furthermore, the rate of change is higher in the even model than the uneven model. Moreover, for the FG beam with the power law index of $k = 0.5$, the type of porosity model affects the modes differently. In this case, the uneven porosity model predicts the first-mode frequencies lower than the even model. However, the uneven porosity distribution

| $L/h = 10$ | $k = 0$ | $k = 1$ | $k = 2$ | $k = 3$ | $k = 5$ |
|-----------|--------|--------|--------|--------|--------|
|           | Model I | Model II | Model I | Model II | Model I | Model II | Model I | Model II | Model I | Model II |
| $\omega_1$ | 4.761   | 4.701   | 3.936   | 3.960   | 3.023   | 3.268   | 2.605   | 2.976   |
| $\omega_2$ | 11.174  | 11.169  | 10.754  | 10.814  | 10.517  | 10.652  | 10.454  | 10.620  |
| $\omega_3$ | 31.595  | 30.926  | 24.017  | 24.234  | 17.107  | 19.247  | 14.452  | 17.580  |
| $\omega_4$ | 31.616  | 31.331  | 29.223  | 29.382  | 27.416  | 27.968  | 27.007  | 27.201  |
| $\omega_5$ | 33.718  | 33.590  | 30.000  | 30.150  | 28.446  | 28.761  | 27.699  | 28.471  |

| $L/h = 30$ | $k = 0$ | $k = 1$ | $k = 2$ | $k = 3$ | $k = 5$ |
|-----------|--------|--------|--------|--------|--------|
|           | Model I | Model II | Model I | Model II | Model I | Model II | Model I | Model II | Model I | Model II |
| $\omega_1$ | 5.102   | 5.048   | 4.449   | 4.452   | 3.850   | 3.947   | 3.560   | 3.717   |
| $\omega_2$ | 11.238  | 11.236  | 10.822  | 10.886  | 10.607  | 10.742  | 10.570  | 10.721  |
| $\omega_3$ | 32.528  | 32.266  | 29.531  | 29.546  | 27.117  | 27.512  | 26.046  | 26.691  |
| $\omega_4$ | 34.054  | 34.033  | 29.895  | 30.497  | 27.993  | 29.157  | 27.693  | 28.976  |
| $\omega_5$ | 75.552  | 74.664  | 60.992  | 63.285  | 53.470  | 58.082  | 52.128  | 55.269  |

| $L/h = 80$ | $k = 0$ | $k = 1$ | $k = 2$ | $k = 3$ | $k = 5$ |
|-----------|--------|--------|--------|--------|--------|
|           | Model I | Model II | Model I | Model II | Model I | Model II | Model I | Model II | Model I | Model II |
| $\omega_1$ | 5.140   | 5.087   | 4.509   | 4.509   | 3.949   | 4.029   | 3.677   | 3.809   |
| $\omega_2$ | 11.245  | 11.243  | 10.829  | 10.893  | 10.616  | 10.751  | 10.580  | 10.731  |
| $\omega_3$ | 32.630  | 32.368  | 29.693  | 29.695  | 27.480  | 27.773  | 26.551  | 26.996  |
| $\omega_4$ | 34.134  | 34.117  | 29.963  | 30.572  | 28.055  | 29.234  | 27.758  | 29.057  |
| $\omega_5$ | 75.938  | 75.048  | 61.340  | 63.658  | 53.841  | 58.523  | 52.632  | 55.999  |
model results in an increase in the frequencies of the second and third modes in comparison to the even model. So, this shows that both the type of porosity distribution and the power law index affect the beam frequencies significantly.

Finally, the effect of length to height ratio and porosity model on the frequencies of the beam are presented in Table 6. This highlights that by increasing the length to height ratio of the beam, the nondimensional frequencies increase for both porosity models, but the rate of increase is not monotonic. It must be noted that the modes show different behaviour for \( k = 0 \) in comparison to other power law indices for all length to height ratios. For this power law index \( (k = 0) \), the even porosity model results in higher frequency values than the uneven model for all length to height ratios. However, for all other values of \( k \), shown in this table, the uneven model predicts higher frequencies than the even porosity distribution. Therefore, to accurately calculate the dynamics of rotating FGM beams, it is important to know which type of porosity distribution is available in the material.

Conclusion

The free vibration of a porous rotating functionally graded blade is studied. In this study, the application of geometrically exact fully intrinsic beam equations is extended to study the free vibration of rotating FGM beams. To this aim, the cross-sectional stiffness and inertial matrices of the FGM beam are presented, and the Saint-Venant warping has been incorporated to determine the torsional rigidity of the rectangular cross section. Two types of porosity models representing the even and uneven porosity distributions are considered to take into account the possible porosities, that might appear during the manufacturing processes. The proposed exact formulation is used to study the effect of material grading on the dynamics of the rotating beam. Furthermore, the effect of porosity distribution model on the free vibration of the blade is investigated. It has been observed that as given in the following:

1. The proposed exact beam formulation can accurately predict the dynamics of rotating FGM beams.
2. The material gradation affects both the frequency values and the location at which the modes veer away from each other.
3. The blade dynamics is sensitive to the material porosities due to manufacturing flaws.
4. For power law indices lower than \( k \leq 0.5 \), both even and uneven porosity distributions result in an increase in the natural frequencies of the rotating beam.
5. For the range of power law indices of higher than \( k \leq 0.5 \), all frequencies decrease due to the porosity.
6. Various modes behave differently depending on the type of porosity distribution and power law index.
7. Finally, ignoring the cross-sectional warping in the formulation results in overpredicting the torsional frequency.

Declaration of Conflicting Interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The author(s) received no financial support for the research, authorship, and/or publication of this article.

ORCID iDs

Mohammadreza Amoozgar @ https://orcid.org/0000-0003-1670-9762
Len Gelman @ https://orcid.org/0000-0001-5464-6227

References

Amoozgar MR and Shahverdi H (2016) Analysis of nonlinear fully intrinsic equations of geometrically exact beams using generalized differential quadrature method. Acta Mechanica 227: 1265–1277.
Amoozgar MR and Shahverdi H (2019) Aeroelastic stability analysis of curved composite blades in hover using fully intrinsic equations. International Journal of Aeronautical and Space Sciences 20: 653–663.
Amoozgar MR, Shahverdi H and Ovesy HR (2017) Nonlinear response of functionally graded panels with stiffeners in supersonic flow. Journal of Aeroelasticity and Structural Dynamics 5: 1–16.
Bazoune A, Khulief YA, Stephen NG, et al. (2001) Dynamic response of spinning tapered Timoshenko beams using modal reduction. Finite Elements in Analysis and Design 37: 199–219.
Bossak MAJ and Zienkiewicz OC (1973) Free vibration of initially stressed solids, with particular reference to centrifugal-force effects in rotating machinery. Journal of Strain Analysis 8: 245–252.
Carnegie W (1959) Vibrations of rotating cantilever blading: theoretical approaches to the frequency problem based on energy methods. Journal of Mechanical Engineering Science 1: 235–240.
Cong PH, Chien TM, Khoa ND, et al. (2018) Nonlinear thermomechanical buckling and post-buckling response of porous FGM plates using Reddy’s HSDT. Aerospace Science and Technology 77: 419–428.
Dibble R, Ondra V and Titurus B (2019) Resonance avoidance for variable speed rotor blades using an applied compressive load. Aerospace Science and Technology 88: 222–232.
Dubey A, Nayak CR, Nayak DK, et al. (2020) Stability of a tapered, pretwisted, and rotating sandwich beam under temperature gradient. Journal of Aerospace Engineering 33: 04020059.
Ebrahimi F and Mokhtari M (2015) Transverse vibration analysis of rotating porous beam with functionally graded microstructure using the differential transform method. Journal of the Brazilian Society of Mechanical Sciences and Engineering 37: 1435–1444.
Fang J, Zhou D and Dong Y (2017) Three-dimensional vibration of functionally graded beams. *Journal of Vibration and Control* 24: 3292–3306.

Friedmann PP and Hodges DH (2003) Rotary wing aeroelasticity-a historical perspective. *Journal of Aircraft* 40: 1019–1046.

Giunta G, Crisafulli D, Belouettar S, et al. (2011) Hierarchical theories for the free vibration analysis of functionally graded beams. *Composite Structures* 94: 68–74.

Han H, Liu L and Cao D (2020) Dynamic modeling for rotating composite Timoshenko beam and analysis on its bending-torsion coupled vibration. *Applied Mathematical Modelling* 78: 773–791.

Hodges DH (2003) Geometrically exact, intrinsic theory for dynamics of curved and twisted anisotropic beams. *AIAA Journal* 41: 1131–1137.

Jin G, Chen Y, Li S, et al. (2019) Quasi-3D dynamic analysis of rotating FGM beams using a modified Fourier spectral approach. *International Journal of Mechanical Sciences* 163: 105087.

Khane Masjedi P, Maheri A and Weaver PM (2019) Large deflection of functionally graded porous beams based on a geometrically exact theory with a fully intrinsic formulation. *Applied Mathematical Modelling* 76: 938–957.

Koizumi M (1997) FGM activities in Japan. *Composites Part B: Engineering* 28: 1–4.

Lee JW and Lee JY (2017) Free vibration analysis of functionally graded Bernoulli-Euler beams using an exact transfer matrix expression. *International Journal of Mechanical Sciences* 122: 1–17.

Li L, Zhang DG and Zhu WD (2014) Free vibration analysis of a rotating hub-functionally graded material beam system with the dynamic stiffening effect. *Journal of Sound and Vibration* 333: 1526–1541.

Lü CF, Chen WQ, Xu RQ, et al. (2008) Semi-analytical elasticity solutions for bi-directional functionally graded beams. *International Journal of Solids and Structures* 45: 258–275.

Mardanpour P, Hodges DH, Neuhart R, et al. (2013) Engine placement effect on nonlinear trim and stability of flying wing aircraft. *Journal of Aircraft* 50: 1716–1725.

Pradhan KK and Chakraverty S (2013) Free vibration of Euler and Timoshenko functionally graded beams by Rayleigh-Ritz method. *Composites Part B: Engineering* 51: 175–184.

Rafiee M, Nitzsche F and Labrosse M (2017) Dynamics, vibration and control of rotating composite beams and blades: a critical review. *Thin-Walled Structures* 119: 795–819.

Romani G and Casalino D (2019) Rotorcraft blade-vortex interaction noise prediction using the Lattice-Boltzmann method. *Aerospace Science and Technology* 88: 147–157.

Sabuncu M and Evran K (2006) The dynamic stability of a rotating pre-twisted asymmetric cross-section Timoshenko beam subjected to lateral parametric excitation. *International Journal of Mechanical Sciences* 48: 878–888.

Shahsavari D, Shahsavari M, Li L, et al. (2018) A novel quasi-3D hyperbolic theory for free vibration of FG plates with porosities resting on Winkler/Pasternak/Kerr foundation. *Aerospace Science and Technology* 72: 134–149.

Șimşek M and Kocatürk T (2009) Free and forced vibration of a functionally graded beam subjected to a concentrated moving harmonic load. *Composite Structures* 90: 465–473.

Sina SA, Navazi HM and Haddadpour H (2009) An analytical method for free vibration analysis of functionally graded beams. *Materials & Design* 30: 741–747.

Sotoudeh Z and Hodges DH (2013) Structural dynamics analysis of rotating blades using fully intrinsic equations, Part II: dual-load-path configurations. *Journal of the American Helicopter Society* 58: 1–9.

Swaminathan M and Rao JS (1977) Vibrations of rotating, pre-twisted and tapered blades. *Mechanism and Machine Theory* 12: 331–337.

Tamer A, Muscarello V, Masarati P, et al. (2019) Evaluation of vibration reduction devices for helicopter ride quality improvement. *Aerospace Science and Technology* 95: 105456.

Tian J, Zhang Z and Hua H (2019) Free vibration analysis of rotating functionally graded double-tapered beam including porosities. *International Journal of Mechanical Sciences* 150: 526–538.

Timoshenko SP and Goodier JN (1970) *Theory of Elasticity*. Maidenhead, UK: McGraw-Hill, Chap. 11.

Vo TP, Thai H-T, Nguyen T-K, et al. (2014) Static and vibration analysis of functionally graded beams using refined shear deformation theory. *Mechanica* 49: 155–168.

Vouros S, Goulos I and Pachidis V (2019) Integrated methodology for the prediction of helicopter rotor noise at mission level. *Aerospace Science and Technology* 89: 136–149.

Wang YQ and Zu JW (2017) Large-amplitude vibration of sigmoid functionally graded thin plates with porosities. *Thin-Walled Structures* 119: 911–924.

Zahedinejad P, Zhang C, Zhang H, et al. (2020) A comprehensive review on vibration analysis of functionally graded beams. *International Journal of Structural Stability and Dynamics* 20: 2030002.