SPATIAL DYNAMICS AND OPTIMIZATION METHOD FOR A NETWORK PROPAGATION MODEL IN A SHIFTING ENVIRONMENT

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(Communicated by Shouchuan Hu)

Abstract. In this paper, a reaction-diffusion ISCT rumor propagation model with general incidence rate is proposed in a spatially heterogeneous environment. We first summarize the well-posedness of global solutions. Then the basic reproduction number $R_0$ is introduced for the model which contains the spatial homogeneity as a special case. The threshold-type dynamics are also established in terms of $R_0$, including the global asymptotic stability of the rumor-free steady state and the uniform persistence of all positive solutions. Furthermore, by applying a controller to this model, we investigate the optimal control problem. Employing the operator semigroup theory, we prove the existence, uniqueness and some estimates of the positive strong solution to the controlled system. Subsequently, the existence of the optimal control strategy is established with the aid of minimal sequence techniques and the first order necessary optimality conditions for the optimal control is deduced. Finally, some numerical simulations are performed to validate the main analysis. The results of our study can theoretically promote the regulation of rumor propagation on the Internet.

1. Introduction. Rumor usually refers to the unconfirmed elaboration of issues or annotation of the public interesting things that spreads on a large scale. With the advent of the Internet and information age, the Internet has penetrated into every aspect of our daily life, making rumors spread more rapidly and broadly in online social networks. Rumor, as a significant form of social interaction, sometimes may play a positive role, but most rumors induce panic psychology, reputation damage

2020 Mathematics Subject Classification. Primary: 35K57; Secondary: 92D25.

Key words and phrases. Spatial heterogeneity, reaction-diffusion model, basic reproduction number, stability, uniform persistence.

The first author is supported by National Natural Science Foundation of China (Grant No.12002135), Natural Science Foundation of Jiangsu Province (Grant No.BK20190836), China Postdoctoral Science Foundation (Grant No.2019M661732) and Natural Science Research of Jiangsu Higher Education Institutions of China (Grant No.19KJB110001).

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and potential losses. Therefore, how to reveal, predict or control the propagation of malicious rumors is a valuable research topic with practical significance. To this end, many scholars have become interested in rumor propagation and put forward some strategies for supervising the Internet.

Some pioneering works about rumor propagation by using mathematical models are mostly evolved from the classical epidemiological models [37]. Since the basic rumor propagation model named $DK$ model [6] was introduced by Daley and Kendal in 1964. $DK$ model, together with its variants [15], had been used extensively for quantitative researches of rumor propagation. The early studies opened a new door for the further improvement and enrichment of rumor propagation models. Currently, scholars have made some achievements in the rumor propagation dynamics of complex networks [38]. Several articles, like Ref. [40] studied the dynamic system for rumor propagation in both homogeneous and heterogeneous networks. Ref. [29] introduced a comment mechanism in an $IRS_1CS_2$ information spreading model in online social networks.

In a recent work, Liu and Li [12] divided the whole users in online social networks into four distinct classes: ignoramus (those who have never been exposed to rumors), sharer (those who are capable of spreading rumors), commentator (those who know the rumors and comment), stifler (those who have no response to rumors), which were denoted by $I(t), S(t), C(t), T(t)$ respectively. Without considering scale-free networks, their model can be reduced to the following form in a homogeneous network environment:

$$\begin{align*}
\frac{dI(t)}{dt} &= A - (\beta_1 + \beta_2 + \beta_3) S(t) I(t) - \mu I(t), \\
\frac{dS(t)}{dt} &= \beta_1 S(t) I(t) + \alpha C(t) - (\gamma + \mu) S(t), \\
\frac{dC(t)}{dt} &= \beta_2 S(t) I(t) - (\alpha + \mu) C(t), \\
\frac{dT(t)}{dt} &= \beta_3 S(t) I(t) + \gamma S(t) - \mu T(t),
\end{align*}$$

(1)

where $A$ is the natural entry rate of the new registered users in online social networks. The natural exit rate of each user in online social networks is $\mu$. $\beta_1$, $\beta_2$ and $\beta_3$ represent the sharing rate, the commenting rate and the stifling rate, respectively. The commentators are assumed to have interest in the information and become a sharer at rate $\alpha$. $\gamma$ is the rate at which the sharers translate into the stiflers. All parameters are assumed to be positive constants. The results in Ref. [12] show that system (1) presents the strict threshold dynamics, which in turn implies that negative sharing and comments are not conducive to controlling the rumor propagation.

It has been commonly accepted that the transmission function is an important factor that should be considered in the rumor propagation model. Although some existing researches assume that the incidence rate in most rumor propagation models is bilinear between the ignoramus $I(t)$ and the sharer $S(t)$, actual propagation rates may be not always linear for biological considerations. We write the general transmission function by $\beta_i f(S(t)) I(t)$ ($i = 1, 2, 3$), and function $f(S)$ satisfies the following hypothesis:

$(H_0)$ $f(S)$ is continuously differentiable, and $f(0) = 0$, $f_S(0) > 0$ for all $S \in (0, \infty)$. $S/f(S)$ is monotonically increasing for $S \in (0, +\infty)$.

Recently, various kinds of incidence rates in epidemic and rumor dynamic models have been deeply investigated by many scholars [5]. For example, what we often
Inspired by it, we can improve model (1) as the following general form:

\[
\begin{align*}
\frac{dI(t)}{dt} &= A - (\beta_1 + \beta_2 + \beta_3) f(S(t))I(t) - \mu I(t), \\
\frac{dS(t)}{dt} &= \beta_1 f(S(t))I(t) + \alpha C(t) - (\gamma + \mu) S(t), \\
\frac{dC(t)}{dt} &= \beta_2 f(S(t))I(t) - (\alpha + \mu) C(t), \\
\frac{dI(t)}{dt} &= \beta_3 f(S(t))I(t) + \gamma S(t) - \mu T(t).
\end{align*}
\]  

From the mathematical models themselves, both system (1) and system (2) are derived under the assumption that the environment is “homogeneous mixing”, that is, the location of network users are not taken into account. As is well-known, spatial diffusion and environmental heterogeneity are core factors that affect the spatial spread of infectious diseases and should be reflected in modelling [14, 23]. Reaction-diffusion equations have been shown to be an effective and valuable tool to incorporate the spatial motion of the population into the model by assuming some random movements [39]. In recent years, there are rich mathematical studies on the investigation of the influence of the spatial diffusion and spatial heterogeneity on the dynamics of diseases [13]. Due to the certain similarities between rumor propagation and disease transmission, it is appropriate to consider a user which differs from each other in the network environment when rumor spreads. With these concerns, users are determined not only by the mode of transmission, but also by network resource availability, network interaction, education factors and so on. Therefore, in rumor propagation models, the factor of spatially heterogeneity seems reasonable and needs to be take into consideration.

Denoting by \( I(x, t), S(x, t), C(x, t) \) and \( T(x, t) \), respectively, the density of ignoramuses, sharers, commentators and stiflers in online social networks at position \( x \) and time \( t \). The ISCT rumor propagation model satisfies the transmission rules shown in Fig.1. A natural consideration of a spatially heterogeneous environment inspissus us to the study of the following reaction-diffusion rumor propagation model.

\[
\begin{align*}
\frac{\partial I(x, t)}{\partial t} &= \nabla \cdot d(x) \nabla I + A(x) - \left[ \beta_1(x) + \beta_2(x) + \beta_3(x) \right] f(x, S)I(x, t) - \mu(x)I(x, t), \\
\frac{\partial S(x, t)}{\partial t} &= \nabla \cdot d(x) \nabla S + \beta_1(x) f(x, S)I(x, t) + \alpha(x) C(x, t) - (\gamma(x) + \mu(x)) S(x, t), \\
\frac{\partial C(x, t)}{\partial t} &= \nabla \cdot d(x) \nabla C + \beta_2(x) f(x, S)I(x, t) - \left[ \alpha(x) + \mu(x) \right] C(x, t), \\
\frac{\partial T(x, t)}{\partial t} &= \nabla \cdot d(x) \nabla T + \beta_3(x) f(x, S)I(x, t) + \gamma(x) S(x, t) - \mu(x)T(x, t),
\end{align*}
\]

for \( x \in \Omega, t > 0 \), with the homogeneous Neumann boundary conditions

\[
\frac{\partial I(x, t)}{\partial n} = \frac{\partial S(x, t)}{\partial n} = \frac{\partial C(x, t)}{\partial n} = \frac{\partial T(x, t)}{\partial n} = 0, \quad x \in \partial \Omega, \quad t > 0.
\]

and the initial conditions

\[
\begin{align*}
I(x, 0) &= \phi_1(x) \geq 0, \quad S(x, 0) = \phi_2(x) \geq 0, \\
C(x, 0) &= \phi_3(x) \geq 0, \quad T(x, 0) = \phi_4(x) \geq 0.
\end{align*}
\]
for $x \in \Omega$. This new model is based on some hypotheses. First of all, the within-network environment is spatially heterogeneous, in other words, the natural entry rate $A(x)$, the natural exit rate $\mu(x)$, the sharing rate $\beta_1(x)$, the commenting rate $\beta_2(x)$, the stifling rate $\beta_3(x)$, the recovery rate $\gamma(x)$ and the conversion rate from commentator to sharer $\alpha(x)$ depend on the spatial location $x$. Function $f(x, S)I$ denotes the general incidence rate corresponding to the direct sharing transmission between ignoramuses and sharers. Secondly, for system (3), ignoramuses, sharers, commentators and stiflers can move, which are assumed to follow the Fickian diffusion with the same diffusion coefficient $d(x)$. The fluxes of ignoramuses, sharers, commentators and stiflers are proportional to their density gradient, and flow from high flow information area to low flow information area. According to Ref.[23], we have

$$
\mathbf{J}_I = - \nabla \cdot d(x) \nabla I, \quad \mathbf{J}_S = - \nabla \cdot d(x) \nabla S, \\
\mathbf{J}_C = - \nabla \cdot d(x) \nabla C, \quad \mathbf{J}_T = - \nabla \cdot d(x) \nabla T.
$$

Moreover, let $\Omega \subset \mathbb{R}^n$ be a spatial habitat with smooth boundary $\partial \Omega$. $\frac{\partial}{\partial n}$ represents the normal derivative along $n$ on $\partial \Omega$, where $n$ is the outward normal vector on $\partial \Omega$. The homogeneous Neumann boundary conditions implies there is no user flux across the boundary $\partial \Omega$, that is, all users remain confined to the region $\Omega$ for all time. The symbol $\nabla$ denotes the gradient operator. $\phi_i(x)$ ($i = 1, 2, 3, 4$) are nonnegative continuous functions defined on $\Omega$. Throughout this work, we impose the following hypothesis similar to Ref.[4].

(H1) $d(\cdot), A(\cdot), \mu(\cdot), \beta_1(\cdot), \beta_2(\cdot), \beta_3(\cdot), \alpha(\cdot)$ and $\gamma(\cdot) \in C^2(\bar{\Omega})$ are strictly positive and uniformly bounded on $\bar{\Omega}$.

(H2) $f(\cdot, \cdot) \in C^1(\bar{\Omega} \times \mathbb{R}_+)$, $f(x, 0) = 0$ and $f(x, S) > 0$ for all $x \in \bar{\Omega}$ and $S > 0$. 

Figure 1. The transmission sketch of the ISCT model.
One of the main focuses for our current work is to investigate the influence of spatiotemporal heterogeneities on the extinction and persistence of the rumors in online social networks. What’s particularly intriguing is the basic reproduction number $R_0$, which will act as the threshold value for persistence and extinction of the rumor. In section 2 of our work, we first summarize the well-posedness of solutions. The basic reproduction number of reaction-diffusion systems which provides a new direction for the later researches [27]. Wang and Zhao have made an in-depth study on the basic reproduction number number of reaction-diffusion systems which provides a new direction for the later researches [27]. Next, the threshold-type dynamics are deeply discussed with respect to $R_0$. More specifically, we demonstrate the local and global asymptotic stabilities of the rumor-free steady state. In order to show the uniform persistence of $I(x,t)$, $S(x,t)$ and $C(x,t)$ in system (3), we adapt the persistence theory of dynamical systems in such a way that it applies to our system. In section 3 of our work, we propose an optimal control problem of system (3) which constitutes the other main focus of our work. It is worth noting that the optimal control including other forms of control of epidemic or rumor propagation models has recently been the area where more and more attention are attracted to [17]. The optimal control problem of a network based on $SIVRS$ infectious disease model with virus variation was analyzed in Ref. [33]. Xiang [31] studied the parameter estimation problem of a $SIS$ reaction-diffusion epidemic model by applying optimal control theories. We suggest readers refer to Ref. [35] as some other related literatures [3, 10, 32] on the optimal control of epidemic and eco-epidemiological models. It is remarked that optimal control has vital practical significance for rumor propagation models. Most of the existing researches on optimal control of rumor propagation either focus on ordinary differential equations or partial differential equations with only consideration of spatially independent system parameters. This constitutes the main difference between our work and the existing researches. The existence of global positive strong solution of controlled system is stated and proved. Furthermore, we deduce the first order necessary optimality conditions for optimal control. In section 4 of our work, we concern with some numerical simulations, such as the extinction and the uniform persistence of rumor propagation, the effects of the diffusion coefficient on the spatiotemporal dynamics of rumor propagation, control strategies and so on. Finally, a further discussion section completes our work.

2. Dynamic analysis of spatially heterogeneous rumor model.

2.1. Preliminaries. In light of the fact that the last equation in system (3) is decoupled from the other three equations, this thus leads us to consider the following subsystem in later work. We consider system:

$$
\begin{align*}
\frac{\partial I(x,t)}{\partial t} &= \nabla \cdot d(x) \nabla I(x,t) + A(x) - \left[\beta_1(x) + \beta_2(x) + \beta_3(x)\right] f(x,S) I(x,t) - \mu(x) I(x,t), \\
\frac{\partial S(x,t)}{\partial t} &= \nabla \cdot d(x) \nabla S(x,t) + \beta_1(x)f(x,S) I(x,t) + \alpha(x) C(x,t) - \left[\gamma(x) + \mu(x)\right] S(x,t), \\
\frac{\partial C(x,t)}{\partial t} &= \nabla \cdot d(x) \nabla C(x,t) + \beta_2(x)f(x,S) I(x,t) - \left[\alpha(x) + \mu(x)\right] C(x,t),
\end{align*}
$$

(7)
for $x \in \Omega, t > 0$, with the homogeneous Neumann boundary conditions
\[
\frac{\partial I(x, t)}{\partial n} = \frac{\partial S(x, t)}{\partial n} = \frac{\partial C(x, t)}{\partial n} = 0, \quad x \in \partial \Omega, \quad t > 0.
\] (8)
and the initial conditions
\[
I(x, 0) = \phi_1(x) \geq 0, S(x, 0) = \phi_2(x) \geq 0, C(x, 0) = \phi_3(x) \geq 0, \quad x \in \Omega.
\] (9)

Throughout this paper, we use the notations similar to Refs.\[14, 23\]. Denote $\mathbb{R}^3_+$ be the positive cone in $\mathbb{R}^3$ by
\[
\mathbb{R}^3_+ = \{\psi = (I, S, C)^T \in \mathbb{R}^3 | I \geq 0, S \geq 0, C \geq 0\}.
\]
Define $C(\bar{\Omega}, \mathbb{R}^3)$ be the Banach space of continuous functions, which is equipped with the supremum norm $\|\cdot\|_{X}$, where
\[
\|\omega\|_{X} := \max \left\{ \sup_{x \in \Omega} |\omega_1(x)|, \sup_{x \in \Omega} |\omega_2(x)|, \sup_{x \in \Omega} |\omega_3(x)| \right\}, \quad \omega = (\omega_1, \omega_2, \omega_3) \in X.
\]

Obviously, the positive cone of $C(\bar{\Omega}, \mathbb{R}^3)$ is $C(\bar{\Omega}, \mathbb{R}^3_+)$. For brevity, set $X := C(\bar{\Omega}, \mathbb{R}^3)$, $X_+ := C(\bar{\Omega}, \mathbb{R}^3_+)$, thus $(X, X_+)$ is an ordered Banach space.

Denote by $T_i(t) : C(\bar{\Omega}, \mathbb{R}) \to C(\bar{\Omega}, \mathbb{R})(i = 1, 2, 3)$ the $C_0$-semigroups in connection with the operators $\nabla \cdot d(\cdot) \nabla - \sigma_i(\cdot)$ under the Neumann boundary conditions, where $\sigma_1(x) = \mu(x)$, $\sigma_2(x) = \gamma(x) + \mu(x)$ and $\sigma_3(x) = \alpha(x) + \mu(x)$ \[21\]. Moreover, the same analysis as in Ref.\[21\] implies that $T_i(t)$ are compact and strongly positive for every $t > 0$.

By appealing to the theory developed in Ref.\[20\], one has
\[
(T_i(t)v)(x) = \int_{\Omega} \Gamma_i(t, x, y)v(y)dy, \quad i = 1, 2, 3, \quad v \in C(\bar{\Omega}, \mathbb{R}), \quad t > 0,
\]
where $\Gamma_i(t, x, y)$ ($i = 1, 2, 3$) are the Green function which associate with $\nabla \cdot d(\cdot) \nabla - \sigma_i(\cdot)$ under the Neumann boundary conditions.

For each initial value function $\phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x))$, we introduce the operators $F = (F_1, F_2, F_3)^T : X_+ \to X$
\[
\begin{cases}
F_1(\phi)(x) = A(x) - \left[\beta_1(x) + \beta_2(x) + \beta_3(x)\right] f(x, \phi_2(x))\phi_1(x), \\
F_2(\phi)(x) = \beta_1(x)f(x, \phi_2(x))\phi_1(x) + \alpha(x)\phi_3(x), \\
F_3(\phi)(x) = \beta_2(x)f(x, \phi_2(x))\phi_1(x).
\end{cases}
\] (10)

Then we can rewrite system (7)-(9) as the integral equations below
\[
u(t, \cdot, \phi) = T(t)\phi + \int_0^t T(t-s)F(u(s, \cdot, \phi))ds, \quad u(t, \cdot, \phi) = \left(I(t, \cdot, \phi), S(t, \cdot, \phi), C(t, \cdot, \phi)\right)^T
\] (11)
where $T(t) = (T_1(t), T_2(t), T_3(t))^T, u(t, \cdot, \phi) = (I(t, \cdot, \phi), S(t, \cdot, \phi), C(t, \cdot, \phi))^T$ is the solution of system (7).

2.2. Global existence and uniqueness of the solutions. In this subsection, we will focus on the existence, nonnegativity as well as the ultimate boundedness of global solutions for system (7). Draw on the methods in Refs.\[9, 28\], one achieves the following theorem.

**Theorem 2.1.** For any initial data function $\phi(x) \in X_+$, system (7)-(9) admits a unique nonnegative and bounded global solution $(I(x, t), S(x, t), C(x, t))$ defined on $\Omega \times [0, \infty)$.

**Proof.** By the similar arguments in Ref.\[35\] and together with Ref.\[1\], we obtain that system (7) admits a unique classical solution $(I(x, t), S(x, t), C(x, t))$ on $\Omega \times [0, \infty)$.
is the globally attractive steady state for the following scalar parabolic equation

where $K(x, t) = I(x, t) + S(x, t) + C(x, t)$, from system (7), we obtain

$$
\frac{\partial K(x, t)}{\partial t} = \nabla \cdot d(x) \nabla K + A(x) - \beta_3(x)f(x, S)I(x, t) - \gamma(x)S(x, t) - \mu(x)K(x, t),
$$

$$
\leq \nabla \cdot d(x) \nabla K + A(x) - \mu(x)K(x, t),
$$

$$
\leq \nabla \cdot d(x) \nabla K + A - \mu K(x, t),
$$

where $A = \max_{x \in \Omega} A(x)$, and $\mu = \min_{x \in \Omega} \mu(x)$.

According to Ref. [13], we see that $\frac{A}{\mu}$ is the globally attractive steady state for the following scalar parabolic equation

$$
\begin{aligned}
&\frac{\partial K(x, t)}{\partial t} = \nabla \cdot d(x) \nabla K + A - \mu K(x, t), & x \in \Omega, t > 0, \\
&\frac{\partial K(x, t)}{\partial n} = 0, & x \in \partial \Omega, t > 0.
\end{aligned}
$$

The standard parabolic comparison theorem [2], together with the nonnegativity of the solution, indicates that $K(x, t)$ is bounded on $[0, \tau_\infty) \times \Omega$. In other words, there exists a positive constant $K_1, K_2, K_3$, such that $0 \leq I(x, t) \leq K_1, 0 \leq S(x, t) \leq K_2$ and $0 \leq C(x, t) \leq K_3$. Hence $\tau_\infty = \infty$. Based on Ref. [16], the solution of system (7) exists globally. This completes the proof.

2.3. The basic reproduction number and some properties. In general, for ordinary differential systems, the next generation matrix method is usually used to calculate $R_0$, but this method is not applicable for reaction-diffusion systems. It has been found that $R_0$ in reaction-diffusion system can be expressed by the spectral radius of the next generation infection operator. In Ref. [26], Wang and Zhao have made an in-depth study on the basic reproduction number of reaction-diffusion systems, and given a standardized procedure for defining the basic reproduction number. In order to accurately judge the prevalence and elimination of rumors in online social networks, we shall adopt the same idea developed in Ref. [26] and follow the framework line of Refs. [24, 7] to obtain the rumor infection basic reproduction number for system (7)-(9), and further discuss its properties.

Step 1. We shall introduce the rumor-free steady state for rumor propagation model (7). If rumor is extinct, let $S(x, t) = 0$ in system (7), then system (7) reduces to

$$
\begin{aligned}
&\frac{\partial I(x, t)}{\partial t} = \nabla \cdot d(x) \nabla I + A(x) - \mu(x)I(x, t), & x \in \Omega, t > 0, \\
&\frac{\partial I(x, t)}{\partial n} = 0, & x \in \partial \Omega, t > 0.
\end{aligned}
$$

By Ref. [13], it then follows that the unique positive steady state $I_0(x)$ of system (14) is globally asymptotically stable. That is to say, system (7) admits a unique rumor-free steady state $E_0(x) = (I_0(x), 0, 0)$. 

It is easily seen that \(S(x,t)\) and \(C(x,t)\) are the rumor-propagation related variables of system (7). Linearizing system (7) around \(E_0(x)\) and we have

\[
\begin{aligned}
\frac{\partial u_S(x,t)}{\partial t} = \nabla \cdot D(x) \nabla u_S + F(x)u_S(x,t) - V(x)u_S(x,t), & \quad x \in \Omega, t > 0, \\
\frac{\partial u_S(x,t)}{\partial n} = 0, & \quad x \in \partial \Omega, t > 0,
\end{aligned}
\]

where \(u_S(x,t) = (S(x,t), C(x,t))^T, D(x) = \text{diag}\{d(x), d(x)\}\), and

\[
F(x) = \begin{pmatrix}
\beta_1(x)I_0(x)f_S(0) & 0 \\
\beta_2(x)I_0(x)f_S(0) & 0
\end{pmatrix}, \quad V(x) = \begin{pmatrix}
\gamma(x) + \mu(x) & -\alpha(x) \\
0 & \alpha(x) + \mu(x)
\end{pmatrix}.
\]

Substituting \(u_S(x,t) = e^{\lambda t}\phi(x)\), \(\phi(x) = (\phi_2(x), \phi_3(x))^T\) into system (15) and dividing each equation by \(e^{\lambda t}\), we acquire the following eigenvalue problem

\[
\begin{aligned}
\lambda \phi(x) = \nabla \cdot D(x) \nabla \phi(x) + F(x)\phi(x) - V(x)\phi(x), & \quad x \in \Omega, t > 0, \\
\frac{\partial \phi(x)}{\partial n} = 0, & \quad x \in \partial \Omega.
\end{aligned}
\]  

(16)

It is easily checked that system (15) is cooperative. According to Refs.\cite{13,27,21}, the eigenvalue problem (16) possesses a principal eigenvalue, denoted by \(\lambda_0 = \lambda_0(d(x), I_0(x))\) which is associated with a strictly positive eigenvector \(\phi_0(x) = (\phi_{02}(x), \phi_{03}(x))^T\), and \(\text{Re}(\lambda) < \lambda_0\) for any other eigenvalue of problem (16).

Step 2. Our purpose is to define the next infection operator for system (7), which derives from the idea in Ref.\cite{27}. Suppose that \(\phi(x) = (\phi_2(x), \phi_3(x))\) is the density distribution of initial sharers and commentators at the spatial location \(x\). \(\mathcal{T}(t) = (T_1(t), T_2(t))^T\) is obviously the evolution operator of the following problem

\[
\begin{aligned}
\frac{\partial u_S(x,t)}{\partial t} = \nabla \cdot D(x) \nabla u_S - V(x)u_S(x,t), & \quad x \in \Omega, t > 0, \\
\frac{\partial u_S(x,t)}{\partial n} = 0, & \quad x \in \partial \Omega, t > 0.
\end{aligned}
\]  

(17)

Define

\[
L(\phi)(x) := \int_0^{+\infty} F(x)\mathcal{T}(t)\phi(x)dt = F(x) \int_0^{+\infty} \mathcal{T}(t)\phi(x)dt.
\]

Usually, we call \(L\) as the next infection operator which is continuous and positive. Meanwhile, it maps the initial distribution \(\phi\) to the distribution of the total sharers and commentators. By Refs.\cite{27,26}, we define the spectral radius of \(L\) as the basic reproduction number

\[
\mathcal{R}_0 := r(L)
\]

(18)

for system (7).

In the following content, we want to investigate some properties of basic reproduction number. The following lemma is an equivalent characterization of basic reproduction number which can be directly derived from the general results in Ref.\cite{26} and the similar analysis in Ref.\cite{36}.

**Lemma 2.1.** \(\mathcal{R}_0 - 1\) has the same sign as \(\lambda_0\).
Lemma 2.2. The following eigenvalue problem admits a unique principal eigenvalue \( r = r_0 \), and further \( R_0 = r_0 \)

\[
\begin{aligned}
- \nabla \cdot (d(x) \nabla \Phi_1) &= \frac{1}{r} \beta_1(x) I_0(x) f_S(0) \Phi_1 + \alpha(x) \Phi_2 - \left[ \gamma(x) + \mu(x) \right] \Phi_1, \quad x \in \Omega, t > 0, \\
- \nabla \cdot (d(x) \nabla \Phi_2) &= \frac{1}{r} \beta_2(x) I_0(x) f_S(0) \Phi_2 - \left[ \alpha(x) + \mu(x) \right] \Phi_2, \quad x \in \Omega, t > 0, \\
\frac{\partial \Phi_1}{\partial n} &= \frac{\partial \Phi_2}{\partial n} = 0, \quad x \in \partial \Omega, t > 0.
\end{aligned}
\]

(19)

For systems with two rumor infection-related compartments in spatially heterogeneous environments, we are not able to give an explicit expression of the basic reproduction number of the rumor-free equilibrium point is \( E_0 = (I_0, 0, 0) \), where \( I_0 = \frac{\Delta}{\mu} \). Thus, the basic reproduction number is

\[
R_0 = \frac{\left[ \alpha (\beta_1 + \beta_2) + \mu \beta_1 \right] I_0 f_S(0)}{(\alpha + \mu) (\mu + \gamma)}.
\]

Proof. Recall the matrices \( F(x) \) and \( V(x) \) defined earlier, in the case of spatial homogeneity, \( F(x) \) and \( V(x) \) are degenerated to the following constant matrices

\[
F = \begin{pmatrix}
\beta_1 I_0 f_S(0) \\
\beta_2 I_0 f_S(0)
\end{pmatrix}, \quad V = \begin{pmatrix}
\gamma + \mu & -\alpha \\
0 & \alpha + \mu
\end{pmatrix}.
\]

For any \( \varepsilon > 0 \), define

\[
F_\varepsilon(\phi) = F(\phi) + \varepsilon, \quad L_\varepsilon(\phi) = F_\varepsilon(x) \int_0^{+\infty} T(t) \phi dt, \quad \phi \in X.
\]

By virtue of the Krein-Rutman theorem and the similar analysis as in Ref.[28], we further have the following observation

\[
L_\varepsilon(e) = T_\varepsilon(e), \quad \forall e = (e_1, e_2)^T \in \mathbb{R}^2,
\]

where \( T_\varepsilon = F_\varepsilon V^{-1} = (F + \varepsilon E)V^{-1} \). A straightforward calculation shows

\[
T_\varepsilon = \begin{pmatrix}
\beta_1 I_0 f_S(0) + \varepsilon \\
\beta_2 I_0 f_S(0)
\end{pmatrix}, \quad \frac{\gamma + \mu}{\alpha + \mu} = \frac{\alpha [\beta_1 I_0 f_S(0) + \varepsilon]}{(\alpha + \mu)(\gamma + \mu)}.
\]

(20)

Thanks to the uniqueness of the principal eigenvalue of \( L_\varepsilon \), it is necessary that \( r(L_\varepsilon) = r(T_\varepsilon) \) in the present condition. We then obtain the basic reproduction number in the homogeneous environment by letting \( \varepsilon \to 0^+ \) that

\[
R_0 = r(L_\varepsilon) = r(T_\varepsilon) = \frac{[\alpha (\beta_1 + \beta_2) + \mu \beta_1] I_0 f_S(0)}{(\alpha + \mu)(\mu + \gamma)}.
\]

\[\square\]

Remark 2.1. The result of the above theorem asserts that the reaction-diffusion rumor propagation system (7) in a spatially heterogeneous environment shares the same basic reproduction number with its ODE counterpart. In addition, although it is hard to obtain a concrete expression of the basic reproduction number of the system in a spatially heterogeneous environment, we can make an estimate of the value of \( R_0 \) through the method in Ref. [36]. On this basis, we can simulate the complex spatiotemporal dynamics of system (7) in the fourth part of our work.
Theorem 2.3. If $A(x) \equiv A$, $\mu(x) \equiv \mu$, $\alpha(x) \equiv \alpha$ and $\gamma(x) \equiv \gamma$ are all positive constants, denote
\[
\beta_1^m = \min_{x \in \Omega} \beta_1(x), \beta_1^M = \max_{x \in \Omega} \beta_1(x), \n
\beta_2^m = \min_{x \in \Omega} \beta_2(x), \beta_2^M = \max_{x \in \Omega} \beta_2(x).
\]
Then, the basic reproduction number of system (7) satisfies the following inequality
\[
\frac{[\alpha(\beta_1^m + \beta_2^m) + \mu\beta_1^m]A_f(0)}{\mu(\mu + \alpha)(\mu + \gamma)} \leq R_0 \leq \frac{[\alpha(\beta_1^M + \beta_2^M) + \mu\beta_1^M]A_f(0)}{\mu(\mu + \alpha)(\mu + \gamma)}.
\] (21)

Proof. It is not hard to see from Lemma 2.2 and Theorem 2.2 that $R_0$ is monotonically increasing with respect to $\beta_1(x)$ and $\beta_2(x)$, so $R_0$ is minimized when $\beta_1(x)$ and $\beta_2(x)$ are minimized, while $R_0$ is maximized when $\beta_1(x)$ and $\beta_2(x)$ are maximized. This finishes the proof. \qed

2.4. Threshold dynamics. In this subsection, we establish the threshold dynamical behaviours of system (7) with respect to $R_0$. We start with the stability (instability) of the rumor-free steady state $E_0(x)$.

It is easily checked that $(A1)$ – $(A6)$ in Ref.[21] hold, we are able to show some results about the local stability of $E_0(x)$.

Lemma 2.3. For cooperative system (15) with $R_0$ defined before, we have the following conclusions.

(i) If $R_0 < 1$ holds, then the rumor-free steady state $E_0(x)$ is locally asymptotically stable.

(ii) If $R_0 > 1$ holds, then the rumor-free steady state $E_0(x)$ is unstable.

In what follows, we are prepared to demonstrate one of the main results of this subsection, which provides the threshold global dynamics of system (7).

Theorem 2.4. If $R_0 < 1$ holds, then the rumor-free steady state $E_0(x)$ of system (7) is globally asymptotically stable.

Proof. By means of the first equation of system (7), we deduce
\[
\frac{\partial I(x,t)}{\partial t} \leq \nabla \cdot d(x) \nabla I + A(x) - \mu(x)I(x,t), \quad x \in \Omega, \quad t > 0.
\] (22)

Using the parabolic comparison principle, it is easily seen that $\lim_{t \to \infty} \sup_{x \in \Omega} I(x,t) \leq I_0(x)$. Hence, there exists $t_0 > 0$, for fixed $\eta_0 > 0$ such that
\[
I(x,t) \leq I_0(x) + \eta_0, \quad t > t_0.
\] (23)

Note that under assumption $(H_3)$, we obtain
\[
\frac{f(\cdot,S)}{S} \leq \lim_{S \to 0} \frac{f(\cdot,S)}{S} = f_S(\cdot,0).
\] (24)

This, together with (23) as well as the second and third equations of system (7), gives rise to
\[
\begin{align*}
\frac{\partial S(x,t)}{\partial t} & \leq \nabla \cdot d(x) \nabla S + [\beta_1(x)f_S(0)(I_0(x) + \eta_0) - (\gamma(x) + \mu(x))]S(x,t) + \alpha(x)C(x,t), \\
\frac{\partial C(x,t)}{\partial t} & \leq \nabla \cdot d(x) \nabla C + \beta_2(x)f_S(0)(I_0(x) + \eta_0)S(x,t) - \alpha(x) + \mu(x))C(x,t), \\
\frac{\partial S(x,t)}{\partial n} & = \frac{\partial C(x,t)}{\partial n} = 0,
\end{align*}
\] (25)

for $t > t_0$. 
Consider the following eigenvalue problem
\[ \begin{cases} \lambda \phi_2(x) = \nabla \cdot d(x) \nabla \phi_2(x) + [\beta_1(x) f_S(0)(I_0(x) + \eta_0) - (\gamma(x) + \mu(x))] \phi_2(x) + \alpha(x) \phi_3(x), \\ \lambda \phi_3(x) = \nabla \cdot d(x) \nabla \phi_3(x) + \beta_2(x) f_S(0)(I_0(x) + \eta_0) \phi_2(x) - [\alpha(x) + \mu(x)] \phi_3(x), \\ \frac{\partial \phi_2(x)}{\partial n} = \frac{\partial \phi_3(x)}{\partial n} = 0, \end{cases} \tag{26} \]
for \( t > t_0 \), where \( \lambda_0(\eta_0) = \lambda_0(d(x), I_0(x) + \eta_0) \) is the unique principal eigenvalue of system (26) with the eigenvector \( (\phi_{\eta_2}(x), \phi_{\eta_3}(x)) \). Thus, through the comparison principle, the solution \( (S(x, t), C(x, t)) \) of system (25) satisfies
\[ (S(x, t), C(x, t)) \leq M(\phi_{\eta_2}(x), \phi_{\eta_3}(x)) e^{\lambda_0(\eta_0)(t-t_0)}, \quad \forall t \geq t_0, \tag{27} \]
for a fixed large enough constant \( M \).

Due to \( \mathcal{R}_0 < 1 \iff \lambda_0 < 0 \), we have
\[ \lim_{\eta_0 \to 0} \lambda_0(d(x), I_0(x) + \eta_0) = \lambda_0(d(x), I_0(x)) < 0, \]
by the continuity of the principal eigenvalue \( \lambda_0 \).

Let \( t \to +\infty \) in the inequation (27), we obtain that \( \lim_{t \to +\infty, x \in \Omega} (S(x, t), C(x, t)) = (0, 0) \) by the fact of the nonnegativity for \( S(x, t) \) and \( C(x, t) \).

This therefore gives rise to the following limit equation
\[ \begin{cases} \frac{\partial I(x, t)}{\partial t} = \nabla \cdot d(x) \nabla I - \mu(x) I(t, x), & x \in \Omega, \ t > 0, \\ \frac{\partial I(x, t)}{\partial n} = 0, & x \in \partial \Omega, \ t > 0. \tag{28} \end{cases} \]

According to Ref.[13], we further obtain that \( \lim_{t \to +\infty, x \in \Omega} I(x, t) = I_0(x) \). Combining with Lemma 2.3, we can conclude that the rumor-free steady state \( E_0(x) \) is globally asymptotically stable which completes the proof. \( \square \)

In the following, we are in a position to prove the global asymptotic stability of the rumor-free steady state \( E_0(x) \) in terms of Lyapunov function.

**Theorem 2.5.** Denote
\[ q_1 = \max_{x \in \Omega} \left\{ \frac{\{ \alpha(x) (\beta_1(x) + \beta_2(x)) + \mu(x) \beta_1(x) \} f_S(0) K_1}{[\gamma(x) + \mu(x)] (\alpha(x) + \mu(x))} \right\}. \]
If \( q_1 < 1 \) holds, then the rumor-free steady state \( E_0(x) \) of system (7) is globally asymptotically stable.

**Proof.** Constructing a Lyapunov function in the following form
\[ L_1 = \int_{\Omega} S(x, t) dx + \int_{\Omega} \frac{\alpha(x)}{\alpha(x) + \mu(x)} C(x, t) dx. \tag{29} \]
Taking the time derivative of \( L_1 \) along the positive solutions of system (7), then
\[ \begin{align*}
\left. \frac{dL_1}{dt} \right|_{(7)} &= \int_{\Omega} \left\{ \nabla \cdot d(x) \nabla S + \beta_1(x) f(x, S) I(x, t) + \alpha(x) C(x, t) - [\gamma(x) + \mu(x)] S(x, t) \right\} dx \\
&+ \int_{\Omega} \frac{\alpha(x)}{\alpha(x) + \mu(x)} \left\{ \nabla \cdot d(x) \nabla C + \beta_2(x) f(x, S) I(x, t) - [\alpha(x) + \mu(x)] C(x, t) \right\} dx \\
&= \int_{\Omega} \left\{ \nabla \cdot d(x) \nabla S + \frac{\alpha(x)}{\alpha(x) + \mu(x)} \nabla \cdot d(x) \nabla C + \left[ \frac{\beta_1(x)}{\alpha(x) + \mu(x)} + \frac{\alpha(x) \beta_2(x)}{\alpha(x) + \mu(x)} \right] f(x, S) \right\} dx \\
&= I(x, t) - [\gamma(x) + \mu(x)] S(x, t). \end{align*} \]
Under the homogeneous Neumann boundary conditions (8), the following equations hold

\[
\int_{\Omega} \nabla \cdot d(x) \nabla I(x,t) dx = \int_{\partial \Omega} \frac{\partial I(x,t)}{\partial n} dx = 0,
\]

\[
\int_{\Omega} \nabla \cdot d(x) \nabla S(x,t) dx = \int_{\partial \Omega} \frac{\partial S(x,t)}{\partial n} dx = 0,
\]

\[
\int_{\Omega} \nabla \cdot d(x) \nabla C(x,t) dx = \int_{\partial \Omega} \frac{\partial C(x,t)}{\partial n} dx = 0,
\]

by the Green formula [7]. This fact, together with Theorem 2.1 and the inequation (24), implies that

\[
\frac{dL_1}{dt} \Big|_{(7)} \leq \int_{\Omega} \left\{ \frac{\alpha(x)(\beta_1(x) + \beta_2(x)) + \mu(x)\beta_1(x)}{\alpha(x) + \mu(x)} K_I f_S(0) - \gamma(x) - \mu(x) \right\} S(x,t).
\]

Obviously, \(\frac{dL_1}{dt} \leq 0\) if \(q_1 < 1\) holds, and \(\frac{dL_1}{dt} = 0\) if and only if \(S(x,t) = 0\). By substituting \(S(x,t) = 0\) into system (7), one has \(I(x,t) = I_0(x)\) and \(C(x,t) = 0\). Appealing to the LaSalle’s invariance principle, the rumor-free steady state \(E_0(x)\) is globally asymptotically stable.

**Remark 2.2.** To prove the global stability of the rumor-free steady state \(E_0(x)\), we adopt the method of constructing the Lyapunov function. However, the condition of the theorem is not \(R_0 < 1\) which we have proposed in Theorem 2.4. It is worth noting that in Theorem 2.3 we give an estimate of \(R_0\) when \(A(x) \equiv A\), \(\mu(x) \equiv \mu\), \(\alpha(x) \equiv \alpha\) and \(\gamma(x) \equiv \gamma\) are all positive constants, in which case the condition \(q_1 < 1\) in Theorem 2.5 happens to make the maximum value of \(R_0\) less than 1. In particular, for spatially homogeneous case, we will give a precise description of the global stability of the rumor-free steady state \(E_0(x)\).

**Theorem 2.6.** Denote

\[q_2 = \frac{(\beta_1 + \beta_2 + \beta_3)I_0f_S(0)}{\gamma + \mu}.
\]

If \(q_2 < 1\) holds, then the rumor-free equilibrium point \(E_0 = (I_0, 0, 0)\) of the spatial homogeneity counterpart is globally asymptotically stable.

**Proof.** Constructing the following Lyapunov function

\[L_2 = \int_{\Omega} [I(x,t) - I_0 - I_0\ln\frac{I(x,t)}{I_0}] dx + \int_{\Omega} S(x,t) dx + \int_{\Omega} C(x,t) dx.
\]

Taking the time derivative of \(L_2\) along any positive solutions of system (7), one has

\[
\frac{dL_2}{dt} \Big|_{(7)} = \int_{\Omega} \left(1 - \frac{I_0}{I(x,t)}\right) \left\{d\Delta I + A - (\beta_1 + \beta_2 + \beta_3)f(x,S)I(x,t) - \mu I(x,t)\right\} dx
\]

\[
+ \int_{\Omega} \left\{d\Delta S + \beta_1 f(x,S)I(x,t) + \alpha C(x,t) - (\gamma + \mu)S(x,t)\right\} dx
\]

\[
+ \int_{\Omega} \left\{d\Delta C + \beta_2 f(x,S)I(x,t) - (\alpha + \mu)C(x,t)\right\} dx.
\]
Similar to Theorem 2.5, according to Green's formula and the homogeneous Neumann boundary condition (8), one obtains

\[ \frac{dL_2}{dt} |_{(7)} = \int_{\Omega} \left\{ A - \mu I(x,t) - A \frac{I_0}{I(x,t)} + \mu I_0 + \frac{I_0}{I(x,t)}(\beta_1 + \beta_2 + \beta_3)f(x,S)I(x,t) \right. \\
- \beta_3 f(x,S)I(x,t) - (\gamma + \mu)S(x,t) - \mu C(x,t) \left\} \, dx. \]

Substituting \( A = \mu I_0 \) into the equation above and applying the inequation (24), a straightforward calculation yields

\[ \frac{dL_2}{dt} |_{(7)} = \int_{\Omega} \left\{ \mu I_0 \left( 1 - \frac{I(x,t)}{I_0} \right) \left( 1 - \frac{I_0}{I(x,t)} \right) + (\beta_1 + \beta_2 + \beta_3)I_0 f(x,S) \right. \\
- \beta_3 f(x,S)I(x,t) - (\gamma + \mu)S(x,t) - \mu C(x,t) \left\} \, dx \]

\[ \leq \int_{\Omega} \left\{ - \mu \frac{(I(x,t) - I_0)^2}{I(x,t)} + (\beta_1 + \beta_2 + \beta_3)I_0 f_S(0)S(x,t) \right. \\
- (\gamma + \mu)S(x,t) - \mu C(x,t) \left\} \, dx \]

\[ = \int_{\Omega} -\mu \frac{(I(x,t) - I_0)^2}{I(x,t)} \, dx - (\gamma + \mu) \int_{\Omega} (1 - q_2)S(x,t) \, dx - \int_{\Omega} \mu C(x,t) \, dx. \]

Since \( q_2 < 1 \), it follows that \( \frac{dL_1}{dt} \leq 0 \), and \( \frac{dL_1(t)}{dt} = 0 \) if and only if \( I(x,t) = I_0, S(x,t) = C(x,t) = 0 \). Taking advantage of LaSalle’s invariance principle, then the rumor-free equilibrium point \( E_0 \) is globally asymptotically stable. This completes the proof.

In a bounded region, we are going to show the uniform persistence of all positive solutions in terms of \( R_0 \) which implies that \( R_0 \) is also a threshold standard for rumor persistence in the following part.

**Theorem 2.7.** If \( R_0 > 1 \) holds, then the solution of system (7) with the initial value function \( \phi = (\phi_1, \phi_2, \phi_3) \in X_+ \) and \( \phi_2, \phi_3 \neq 0 \), satisfies

\[ \lim_{t \to \infty} \inf_{u(x,t)} \Omega \geq (\delta, \delta, \delta), \quad (31) \]

where \( u(t,x) = (I(x,t), S(x,t), C(x,t)) \), \( \delta > 0 \) is a constant. Further, system (7) admits at least one rumor-prevailing steady state \( E_*(x) = (I_*(x), S_*(x), C_*(x)) \).

**Proof.** Below we will divide four steps to verify the conditions of Theorem 3 in Ref.[22] one by one by using the methods in Refs.[4, 14, 28].

**Step 1.** With arguments similar as that in Ref.[8], it follows that all solutions \( u(x,t) \) of system (7)-(9) generate a solution semiflow \( \Phi(t) = u(\cdot,t) : X_+ \to X_+ \). It can be see from Theorem 2.1 that all solutions of system (7) are bounded which implies that \( \Phi(t) \) is point dissipative and \( \Phi(t) \) is compact for \( \forall t > 0 \) by the general results in Ref.[30]. According to Theorem 3.4.8 in Ref.[8], we can draw the conclusion that \( \Phi(t) = u(\cdot,t) \) admits a strong global compact attractor in \( X_+ \) which satisfies the condition (C1) in Theorem 3 in Ref.[22].

**Step 2.** Set \( \varepsilon_0 > 0 \) be a sufficiently small constant, if \( R_0 > 1 \), for any solution \( u(x,t) \) of system (7) with \( \phi \in X_+ \) satisfying \( \phi_2 \neq 0 \) and \( \phi_3 \neq 0 \), we want to prove

\[ \lim_{t \to +\infty} \sup_{t \to +\infty} \| u(t,\cdot,\phi) - (I_0(x),0,0) \|_{X_+} \geq \varepsilon_0. \]
In order to prove the conclusion, we argue by contradiction. Assume that there exists some \( \phi \in \mathbb{X}_+ \) with \( \phi_2 \neq 0 \) and \( \phi_3 \neq 0 \) such that

\[
\lim_{t \to +\infty} \sup ||u(t, \cdot, \phi) - (I_0(x), 0, 0)||_{\mathbb{X}_+} < \varepsilon_0.
\]

Thus there exists \( t_1 > 0 \) such that for \( t > t_1 \), we have

\[
\begin{align*}
I_0(x) - \varepsilon_0 &< I(x, t) < I_0(x) + \varepsilon_0, \quad x \in \Omega, \\
0 &< S(x, t) < \varepsilon_0, \quad x \in \Omega, \\
0 &< C(x, t) < \varepsilon_0, \quad x \in \Omega.
\end{align*}
\]  

(32)

By the parabolic maximum principle [18], one has

\[
S(x, t) > 0, \quad C(x, t) > 0, \quad \forall t > 0, \quad x \in \Omega.
\]

(33)

With the help of \((H_3)\), we deduce

\[
f(\cdot, S) / S \geq f(\cdot, \varepsilon_0) / \varepsilon_0 \geq f_S(\cdot, \varepsilon_0).
\]

(34)

This, together with (32), implies that \( S(x, t) \) and \( C(x, t) \) satisfy

\[
\begin{align*}
\frac{\partial S(x, t)}{\partial t} &\geq \nabla \cdot d(x) \nabla S + \left[ \beta_1(x)(I_0(x) - \varepsilon_0)f_S(x, \varepsilon_0) - \gamma(x) - \mu(x) \right] S(x, t) + \alpha(x)C(x, t), \\
\frac{\partial C(x, t)}{\partial t} &\geq \nabla \cdot d(x) \nabla C + \beta_2(x)(I_0(x) - \varepsilon_0)f_S(x, \varepsilon_0)S(x, t) - [\alpha(x) + \mu(x)]C(x, t),
\end{align*}
\]

(35)

for \( t > t_1 \).

Consider the following comparison system

\[
\begin{align*}
\frac{\partial \xi_2(x, t)}{\partial t} &\geq \nabla \cdot d(x) \nabla \xi_2 + \left[ \beta_1(x)(I_0(x) - \varepsilon_0)f_S(x, \varepsilon_0) - \gamma(x) - \mu(x) \right] \xi_2(x, t) + \alpha(x)\xi_3(x, t), \\
\frac{\partial \xi_3(x, t)}{\partial t} &\geq \nabla \cdot d(x) \nabla \xi_3 + \beta_2(x)(I_0(x) - \varepsilon_0)f_S(x, \varepsilon_0)\xi_2(x, t) - [\alpha(x) + \mu(x)]\xi_3(x, t), \\
\frac{\partial \xi_2(x, t)}{\partial n} &\geq \frac{\partial \xi_3(x, t)}{\partial n} = 0,
\end{align*}
\]

(36)

for \( t > t_1 \).

It is easily checked that \((\xi_2(x, t), \xi_3(x, t)) = (\varphi_{c_2}(x), \varphi_{c_3}(x))e^{\lambda_0(\varepsilon_0)(t-t_1)}\) is a solution to system (36), where \((\varphi_{c_2}(x), \varphi_{c_3}(x))\) is the eigenvector corresponding to the principal eigenvalue \(\lambda_0(\varepsilon_0) = \lambda_0(d(x), I_0(x) + \varepsilon_0)\) of system (36).

By means of the comparison principle, for a given appropriate constant \( m > 0 \), we have

\[
(S(x, t), C(x, t)) \geq m(\varphi_{c_2}(x), \varphi_{c_3}(x))e^{\lambda_0(\varepsilon_0)(t-t_1)}, \quad \forall t \geq t_1.
\]

Similar to the proof in Theorem 2.4, due to \( R_0 > 1 \iff \lambda_0 > 0 \), we can obtain that

\[
\lim_{\varepsilon_0 \to 0} \lambda_0(\varepsilon_0) = \lambda_0(d(x), I_0(x)) > 0.
\]

It then follows that

\[
\lim_{t \to +\infty} S(x, t) = \infty, \quad \lim_{t \to +\infty} C(x, t) = \infty,
\]

which thereby contradicts \( S(x, t) \leq K_2 \) and \( C(x, t) \leq K_3 \). Thus, the hypothesis is not true.

**Step 3.** Before going further, since some necessary definitions are important, we give some notations which will be usually used hereafter. Let

\[
\mathbb{X}_0 = \{ \phi = (\phi_1, \phi_2, \phi_3) \in \mathbb{X}_+ : \phi_2 \neq 0, \phi_3 \neq 0 \}.
\]

It is not hard to see

\[
\partial \mathbb{X}_0 := \mathbb{X}_+ / \mathbb{X}_0 = \{ \phi \in \mathbb{X}_+ : \phi_2 \equiv 0 \text{ or } \phi_3 \equiv 0 \}.
\]
Making use of (33), we would have \( \Phi(t)X_0 \subseteq X_0 \). Consequently, \( X_0 \) is the positively invariant for the solution semiflow \( \Phi(t) \). Define
\[
M_\theta := \{ \phi \in X_+ : \Phi(t) \phi \in \partial X_0, \forall t \geq 0 \},
\]
and \( \omega(\phi) \) is the omega limit set of the forward orbit of \( \Phi(t) \) through \( \phi \in X_+ \).

Let
\[
M_1 := \{(I_0(x),0,0)\},
\]
with the arguments above, then we want to get \( \cup_{\phi \in M_\theta} \omega(\phi) = M_1 \), which is an important condition in Theorem 3 in Ref.[22]. To this end, we shall use similar analysis as in Ref.[14]. On account of some necessary modifications are required, we offer a detailed proof here.

(1) \( M_1 \subset \cup_{\phi \in M_\theta} \omega(\phi) \).

This conclusion comes from the consequence that \( \Phi(t)E_0(x) = E_0(x) \) for all \( t \geq 0 \) by a direct calculation according to the definition of \( \Phi(t) \).

(II) \( \cup_{\phi \in M_\theta} \omega(\phi) \subset M_1 \).

For \( \forall \phi \in M_\theta \), \( \Phi(t) \phi \in \partial X_0 \), thus we can show that for each \( t \geq 0 \) either
\[ S(\cdot, t, \phi) \equiv 0 \]
or
\[ C(\cdot, t, \phi) \equiv 0 \].

Now we shall discuss in the following two cases.

Case 1. ( \( S(\cdot, t, \phi) \equiv 0, \forall t \geq 0 \)). Under this circumstance, by substituting \( S(x, t) = 0 \) into system (7), it is easy to show that
\[
\begin{align*}
\frac{\partial I(x, t)}{\partial t} &= \nabla \cdot d(x) \nabla I + A(x) - \mu(x) I(x, t), \quad x \in \Omega, t > 0, \\
C(x, t) &= 0, \quad x \in \Omega, t > 0, \\
\frac{\partial I(x, t)}{\partial n} &= \frac{\partial C(x, t)}{\partial n} = 0, \quad x \in \partial \Omega, t > 0.
\end{align*}
\]

Thus, we have \( \lim_{t \to +\infty} I(x, t) = I_0(x) \). That is, \( \omega(\phi) = E_0(x), \forall \phi \in M_\theta \).

Case 2. ( \( S(\cdot, t, \phi) \not\equiv 0 \), for some \( t \geq 0 \)). Resort to (33), it is clear that \( S(x, t) > 0 \) for all \( t > t \) and \( x \in \Omega \). Thus we have \( C(t, \phi) \equiv 0 \) for all \( t \geq 0 \). Substituting \( C(t, \phi) \equiv 0 \) into the third equation of system (7), we obtain \( \beta_2(x)f(x)S(x, t)I(x, t) = 0 \). A straightforward calculation shows that
\[
\begin{align*}
\frac{\partial I(x, t)}{\partial t} &= \nabla \cdot d(x) \nabla I + A(x) - \mu(x) I(x, t), \quad x \in \Omega, t > 0, \\
\frac{\partial S(x, t)}{\partial t} &= \nabla \cdot d(x) \nabla S - [\alpha(x) + \mu(x)]S(x, t), \quad x \in \Omega, t > 0, \\
\frac{\partial I(x, t)}{\partial n} &= \frac{\partial S(x, t)}{\partial n} = 0, \quad x \in \partial \Omega, t > 0.
\end{align*}
\]

Now we can conclude that \( \lim_{t \to +\infty, x \in \Omega} I(x, t) = I_0(x) \) and \( \lim_{t \to +\infty, x \in \Omega} S(x, t) = 0 \) by using the theory of asymptotically autonomous semiflows [25]. As a consequence, \( \omega(\phi) = E_0(x), \forall \phi \in M_\theta \).

In summary, \( \cup_{\phi \in M_\theta} \omega(\phi) = M_1 \) holds.

**Step 4.** Similar to Ref.[28], we now define a continuous function \( p : X_+ \to \mathbb{R}^+ \) by
\[
p(\phi) := \min \left\{ \min_{x \in \Omega} \phi_2(x), \min_{x \in \Omega} \phi_3(x) \right\}, \quad \phi = (\phi_1, \phi_2, \phi_3) \in X_+.
\]

It is easily seen that \( p^{-1}(0, +\infty) \subset X_0 \). If \( p(\phi) > 0 \), then \( p(\Phi(t)(\phi)) > 0 \) \( \forall t > 0 \) due to the fact that \( S(x, t) > 0, C(x, t) > 0, \forall t > 0 \) and \( x \in \Omega \). As a consequence, \( p \) is the generalized distance function which is defined in Ref.[22] for the semiflow \( \Phi(t) : X_+ \to X_0 \).
Based on the former fact, we are going to verify the condition (C2) in Theorem 3 in Ref.[22]. Denote \( W^*(M_1) = \{ \phi \in X_+ : \lim_{t \to +\infty} \| u(t, \cdot, \phi) - (I_0(x_0), 0, 0) \|_{X_+} = 0 \} \) as the stable set of \( M_1 \). Obviously, according to the results in Step 2 and Step 3, \( W^*(M_1) \cap W_0 \) is an empty set. Thus, \( W^*(M_1) \cap \Gamma^{-1}(0, +\infty) = 0 \). By Ref.[22], we can find a nonnegative constant \( \delta_1 \) such that

\[
\lim_{t \to +\infty} \inf_{x \in X} S(x, t) \geq \delta_1, \quad \lim_{t \to +\infty} \inf_{x \in X} C(x, t) \geq \delta_1.
\]

The last one we shall cope with is the uniform persistence of \( I(x, t) \). Here we rescale the first equation of system (7) by

\[
\frac{\partial I(x, t)}{\partial t} = \nabla \cdot d(x) \nabla I + A(x) - [\beta_1(x) + \beta_2(x) + \beta_3(x)] f(x, S) I(x, t) - \mu(x) I(x, t),
\]

\[
\geq \nabla \cdot d(x) \nabla I + A - [(\beta_1 + \beta_2 + \beta_3) r + \pi] I(x, t),
\]

where \( A = \min_{x \in X} A(x), \pi = \max_{x \in X} \mu(x), r = \max_{S \in [0, I(x_0)]} f(S), \beta_i = \max_{x \in X} \beta_i(x), i = 1, 2, 3 \). With the help of the comparison principle, we deduce that

\[
\lim_{t \to +\infty} \inf_{x \in X} I(t, x) \geq \frac{A}{(\beta_1 + \beta_2 + \beta_3) r + \pi} \equiv \delta_2.
\]

Thus, we can make sure that \( I(x, t) \) is uniformly persistent.

Choose \( \delta := \min\{\delta_1, \delta_2\} \), such that \( \lim_{t \to +\infty} \inf_{x \in X} u(t, x) \geq (\delta, \delta, \delta) \). As a consequence, the persistence statement (31) is valid. Based on Theorem 1.3.6 in Ref.[34], system (7) has at least one rumor-prevailing steady state \( E_*(x) = (I_*(x), S_*(x), C_*(x)) \).

This completes the proof. \( \square \)

**Remark 2.2.** Here we should mention that although it is shown in Theorem 2.7 that there exists a rumor-prevailing steady state for system (7) when \( R_0 > 1 \), its uniqueness and global stability still remain unsolved. The spatiotemporal dynamic behaviours of the rumor-prevailing steady state will be presented in numerical simulation.

3. Optimal control strategies. In this section, we put forward some control strategies to suppress the diffusion of rumors in online social networks [35, 3, 10, 32]. We consider intervention measures, such as network supervision like forced silence, that is, prohibiting rumor-infected users from speaking online.

Different from the previous parts, we choose a specific incidence function: \( f(x, S) = \frac{S}{1 + aS} \) to study the optimal control problem. It is easily checked that the above nonlinear incidence function satisfies the assumptions \( H_2 \) and \( H_3 \). In addition, for the sake of simplicity, we also let \( d(x) \equiv d, A(x) \equiv A \) in the later work.

3.1. Optimal control system. The dynamics of the controlled system is expressed, to accommodate control actions, as follows:

\[
\begin{align*}
\frac{\partial I(x, t)}{\partial t} &= d \Delta I(x, t) + A - [\beta_1(x) + \beta_2(x) + \beta_3(x)] \frac{f(x, S) I(x, t)}{1 + a S(x, t)} - \mu(x) I(x, t), \\
\frac{\partial S(x, t)}{\partial t} &= d \Delta S(x, t) + \beta_1(x) I(x, t) S(x, t) + a(x) C(x, t) - [\gamma(x) + \mu(x)] S(x, t) - u(x) T(x, t) S(x, t), \\
\frac{\partial C(x, t)}{\partial t} &= d \Delta C(x, t) + \beta_2(x) I(x, t) S(x, t) + a(x) S(x, t) - [\alpha(x) + \mu(x)] C(x, t), \\
\frac{\partial T(x, t)}{\partial t} &= d \Delta T(x, t) + \beta_3(x) I(x, t) S(x, t) + \gamma(x) S(x, t) - \mu(x) T(x, t) + u(x) T(x, t) S(x, t),
\end{align*}
\]
with \((x, t) \in \Omega_T = \Omega \times (0, T)\), which is equipped with the homogeneous Neumann boundary conditions
\[
\frac{\partial I(x, t)}{\partial n} = \frac{\partial S(x, t)}{\partial n} = \frac{\partial C(x, t)}{\partial n} = \frac{\partial T(x, t)}{\partial n} = 0, \quad (x, t) \in \Sigma_T = \partial \Omega \times (0, T), \quad (42)
\]
and the corresponding initial conditions
\[
\left\{ \begin{array}{l}
I(x, 0) = I^0(x) > 0, 
S(x, 0) = S^0(x) > 0, 
C(x, 0) = C^0(x) > 0, 
T(x, 0) = T^0(x) > 0,
\end{array} \right. \quad (43)
\]
for \(x \in \Omega\). All elements involved in system (41) have the same implications as we defined earlier. Further, \(I^0(x), S^0(x), C^0(x), T^0(x) \in H^2(\Omega)\) and \(\frac{\partial I^0}{\partial n} = \frac{\partial S^0}{\partial n} = \frac{\partial C^0}{\partial n} = \frac{\partial T^0}{\partial n} = 0\) for \(x \in \partial \Omega\) [35].

In the above system we introduce a control \(u(x, t)\), which can be interpreted as measures that are carried out by government such as network supervision. The proportionality factor \(u(x, t)\) is regarded as a control variable, and it is supposed to vary within a finite interval \([0, 1]\). It is easily seen that if \(u = 0\), we obtain the uncontrolled system.

Let \(U\) be the set of admissible controls
\[
U = \{u \in L^2(\Omega_T), 0 \leq u(x, t) \leq 1, (x, t) \in \Omega_T\}. \quad (44)
\]
Our objective is to find a control variable \(u \in U\) such that the density of the ignoramus, sharers and commentators, as well as the cost of implementing control, reach the minimum value at the end of the interval [0, T]. Meanwhile, we want to maximize the density of the stiflers. Thus, we set the following objective function with consideration of the cost control to be minimality, that is:
\[
\inf_{u \in U} J(I, S, C, T, u) = \int_{\Omega_T} \left[ l_1(x, t)I(x, t) + l_2(x, t)S(x, t) + l_3(x, t)C(x, t) - l_4(x, t)T(x, t) + \lambda(x, t)u(x, t) \right] dx dt
\]
\[
+ \int_{\Omega} \left[ \theta_1(x)I(x, T) + \theta_2(x)S(x, T) + \theta_3(x)C(x, T) - \theta_4(x)T(x, T) + \rho(x)u(x, t) \right] dx,
\]
where \(l_i(x, t), \lambda(x, t) \in L^\infty(\Omega_T)\), and \(\theta_i(x), \rho(x) \in L^\infty(\Omega) (i = 1, 2, 3, 4)\) are the weighting functions.

3.2. Existence of the optimal solution. First of all, we rewrite (41) as an abstract Cauchy problem in Hilbert space \(H = (L^2(\Omega))^4\) through Ref.[3]. Let
\[
Z = (I, S, C, T), \quad Z^0 = (I^0, S^0, C^0, T^0).
\]
Denote by \(A : D(A) \subseteq H \rightarrow H\) is a linear operator
\[
AZ = \text{diag}\{d\Delta I, d\Delta S, d\Delta C, d\Delta T\}, \quad \text{for } Z \in D(A),
\]
where the domain of \(A\) is given by
\[
D(A) := \left\{ Z = (I, S, C, T) \in (H^2(\Omega))^4 : \frac{\partial I}{\partial n} = \frac{\partial S}{\partial n} = \frac{\partial C}{\partial n} = \frac{\partial T}{\partial n} = 0, \text{for } x \in \partial \Omega \right\}.
\]
According to the above definition, system (41)-(43) can be rewritten under the form
\[
\left\{ \begin{array}{l}
Z_t = AZ + G(t, Z), \quad t \in [0, T], 
Z(0) = Z^0,
\end{array} \right.
\]
\[
\left\{ \begin{array}{l}
I(x, 0) = I^0(x) > 0, \quad S(x, 0) = S^0(x) > 0, 
C(x, 0) = C^0(x) > 0, \quad T(x, 0) = T^0(x) > 0,
\end{array} \right. \quad (47)
\]
Define \( I, S, C, T \) at the domain

\[
\begin{align*}
D(G) := \{ Z \in H, G(t, Z) \in H, \text{for } t \in [0, T] \}.
\end{align*}
\]

In the following, for a fixed positive integer \( N \), denote

\[
\begin{align*}
D^G_1 &= \{(x, t) : G(t, x) > N \}, \\
D^G_2 &= \{(x, t) : -N \leq G(t, x) \leq N \}, \\
D^G_3 &= \{(x, t) : G(t, x) < -N \}.
\end{align*}
\]

Define \( G^N = (G^N_1, G^N_2, G^N_3) \), where

\[
G^N(x, t) = \begin{cases} 
N, & (x, t) \in D^G_1, \\
G(x, t), & (x, t) \in D^G_2, \\
-N, & (x, t) \in D^G_3.
\end{cases}
\]

It is easily seen that \( G^N \) is Lipschitz continuous in system variables \( I, S, C \) and \( T \), uniformly with respect to \( t \in [0, T] \). Based on the above preparations, by using the standardized method in Refs.\([35, 3, 10]\), such as \( C_0 \)-semigroup theory \([19]\), Green formula \([7]\) and so on, we can obtain the existence, positivity, and boundedness of a strong global solution for system \((41)-(43)\) in the following lemma. Due to the standardization of the method, the proof will not be performed here again.

**Lemma 3.1.** For a given \( u \in \mathcal{U} \), system \((41)-(43)\) has a unique strong solution \((I, S, C, T) \in W^{1,2}(0, T; H)\) such that

\[
\begin{align*}
I(t, x) &\in L^\infty(\Omega_T) \cap L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)), \\
S(t, x) &\in L^\infty(\Omega_T) \cap L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)), \\
C(t, x) &\in L^\infty(\Omega_T) \cap L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)), \\
T(t, x) &\in L^\infty(\Omega_T) \cap L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)).
\end{align*}
\]

More precisely, there exists a positive constant \( D_1 \), irrelevant to \((u(x, t), I(x, t), S(x, t), C(x, t), T(x, t))\) satisfying

\[
\begin{align*}
\left\| \frac{\partial I}{\partial t} \right\|_{L^2(\Omega_T)} + \|I\|_{L^2(0, T; H^2(\Omega))} + \|I\|_{H^1(\Omega)} + \|I\|_{L^\infty(\Omega_T)} &\leq D_1, \\
\left\| \frac{\partial S}{\partial t} \right\|_{L^2(\Omega_T)} + \|S\|_{L^2(0, T; H^2(\Omega))} + \|S\|_{H^1(\Omega)} + \|S\|_{L^\infty(\Omega_T)} &\leq D_1, \\
\left\| \frac{\partial C}{\partial t} \right\|_{L^2(\Omega_T)} + \|C\|_{L^2(0, T; H^2(\Omega))} + \|C\|_{H^1(\Omega)} + \|C\|_{L^\infty(\Omega_T)} &\leq D_1, \\
\left\| \frac{\partial T}{\partial t} \right\|_{L^2(\Omega_T)} + \|T\|_{L^2(0, T; H^2(\Omega))} + \|T\|_{H^1(\Omega)} + \|T\|_{L^\infty(\Omega_T)} &\leq D_1.
\end{align*}
\]

Based on the conclusion in Lemma 3.1, we can discuss the existence of an optimal solution for system \((41)-(43)\) by Refs.\([3, 10]\).
Theorem 3.1. System (41)-(43) exists an optimal solution \((I^*, S^*, C^*, T^*)\) corresponding to the optimal control variable \(u^*\).

Proof. According to Lemma 3.1, obviously \(\inf_{u \in U} \{J(I, S, C, T, u)\}\) is bounded. Denote the optimal objective value as \(J^* = \inf_{u \in U} \{J(I, S, C, T, u)\}\). Then, there exists a sequence \(\{I^n, S^n, C^n, T^n, u^n\}_{n \geq 1} \in W^{1,2}(0, T; H)\) [32] satisfying:

\[
\begin{align*}
\frac{\partial I^n}{\partial t} &= dI^n + A - (\beta_1 + \beta_2 + \beta_3) I^n S^n + \frac{1}{1 + a} \partial S^n - \mu I^n, \\
\frac{\partial S^n}{\partial t} &= dS^n + \frac{1}{1 + a} \partial S^n + \alpha C^n - (\gamma + \mu) S^n - u^n S^n, \\
\frac{\partial C^n}{\partial t} &= dC^n + \frac{1}{1 + a} \partial C^n - (\alpha + \mu) C^n, \\
\frac{\partial T^n}{\partial t} &= dT^n + \gamma S^n - \mu T^n + u^n S^n, \\
\end{align*}
\]

\((x, t) \in \Omega_T\), \(n \geq 1\).

(53)

Inequality (52) gives rise to the estimates that

\[
\begin{align*}
\left\| \frac{\partial I^n}{\partial t} \right\|_{L^2(0, T; H^2(\Omega))} + \| I^n \|_{L^2(0, T; H^2(\Omega))} + \| I^n \|_{H^1(\Omega)} + \| I^n \|_{L^\infty(\Omega_T)} &\leq D_1, \\
\left\| \frac{\partial S^n}{\partial t} \right\|_{L^2(0, T; H^2(\Omega))} + \| S^n \|_{L^2(0, T; H^2(\Omega))} + \| S^n \|_{H^1(\Omega)} + \| S^n \|_{L^\infty(\Omega_T)} &\leq D_1, \\
\left\| \frac{\partial C^n}{\partial t} \right\|_{L^2(0, T; H^2(\Omega))} + \| C^n \|_{L^2(0, T; H^2(\Omega))} + \| C^n \|_{H^1(\Omega)} + \| C^n \|_{L^\infty(\Omega_T)} &\leq D_1, \\
\left\| \frac{\partial T^n}{\partial t} \right\|_{L^2(0, T; H^2(\Omega))} + \| T^n \|_{L^2(0, T; H^2(\Omega))} + \| T^n \|_{H^1(\Omega)} + \| T^n \|_{L^\infty(\Omega_T)} &\leq D_1.
\end{align*}
\]

(54)

It is easily seen that the injection of \(H^1(\Omega)\) into \(L^2(\Omega)\) is compact. Thus one can infer that \(\{I^n(t), S^n(t), C^n(t), T^n(t)\}_{n \geq 1}\) is compact in \((L^2(\Omega))^4\). Using Ascoli-Arzelà theorem [19], selecting further sequence again denoted by: \(\{I^n(t), S^n(t), C^n(t), T^n(t)\}_{n \geq 1}\), we have

\[
\begin{align*}
\lim_{n \to \infty} \sup_{t \in [0, T]} \| I^n(t) - I^*(t) \|_{L^2(\Omega)} &= 0, \\
\lim_{n \to \infty} \sup_{t \in [0, T]} \| S^n(t) - S^*(t) \|_{L^2(\Omega)} &= 0, \\
\lim_{n \to \infty} \sup_{t \in [0, T]} \| C^n(t) - C^*(t) \|_{L^2(\Omega)} &= 0, \\
\lim_{n \to \infty} \sup_{t \in [0, T]} \| T^n(t) - T^*(t) \|_{L^2(\Omega)} &= 0.
\end{align*}
\]

(55)

Based on the method in Refs. [3, 10, 32] and the inequalities (54), one arrives at

\[
\begin{align*}
\frac{\partial I^n}{\partial t} &\to \frac{\partial I^*}{\partial t}, \quad \text{weakly in } L^2(0, T; L^2(\Omega)), \\
\frac{\partial S^n}{\partial t} &\to \frac{\partial S^*}{\partial t}, \quad \text{weakly in } L^2(0, T; L^2(\Omega)), \\
\frac{\partial C^n}{\partial t} &\to \frac{\partial C^*}{\partial t}, \quad \text{weakly in } L^2(0, T; L^2(\Omega)), \\
\frac{\partial T^n}{\partial t} &\to \frac{\partial T^*}{\partial t}, \quad \text{weakly in } L^2(0, T; L^2(\Omega)),
\end{align*}
\]

(56)
\[
\begin{align*}
&\Delta I^n \to \Delta S^*, \quad \text{weakly in } L^2(0, T; L^2(\Omega)), \\
&\Delta S^n \to \Delta I^*, \quad \text{weakly in } L^2(0, T; L^2(\Omega)), \\
&\Delta C^n \to \Delta C^*, \quad \text{weakly in } L^2(0, T; L^2(\Omega)), \\
&\Delta T^n \to \Delta R^*, \quad \text{weakly in } L^2(0, T; L^2(\Omega)), \\
\end{align*}
\]

and
\[
\begin{align*}
&I^n \to I^*, \quad \text{weak star in } L^\infty(0, T; H^1(\Omega)), \\
&S^n \to S^*, \quad \text{weak star in } L^\infty(0, T; H^1(\Omega)), \\
&C^n \to C^*, \quad \text{weak star in } L^\infty(0, T; H^1(\Omega)), \\
&T^n \to T^*, \quad \text{weak star in } L^\infty(0, T; H^1(\Omega)).
\end{align*}
\]

A direct calculation yields
\[
\begin{align*}
&I^n S^n - I^* S^* = \frac{S^n (I^n - I^*) + I^* (S^n - S^*) + a S^* S^n (I^n - I^*)}{(1 + a S^n) (1 + a S^*)}, \\
&w^n S^n - u^n S^* = u^n (S^n - S^*) + S^* (u^n - u^*).
\end{align*}
\]

Taking advantages of convergences \(I^n \to I^*, S^n \to S^*\) in \(L^2(\Omega)\) as well as the boundedness of \(I^n, S^n\) in \(L^\infty(\Omega)\), we have \(I^n S^n \to I^* S^*\) in \(L^2(\Omega_T)\). Similar to Ref.[3], denote a further sequence \(\{u^n\}_{n \geq 1}\) by itself, such that
\[
u^n \to u^*, \text{ weakly in } L^2(\Omega_T).
\]

Combined with (60) and the fact that \(U\) is a convex and closed set in \(L^2(\Omega)\), one has \(u^* \in U\). Consequently, by the second equation in Eq.(59), we can easily show that \(w^n S^n \to u^* S^*\) in \(L^2(\Omega_T)\).

Thus, by considering \(n \to \infty\) in system (53), we deduce that \((I^*, S^*, C^*, T^*)\) is an optimal solution of system (41)-(43) corresponding to the optimal control variable \(u^*\) and \(J^* = \lim_{n \to \infty} J(I^n, S^n, C^n, T^n, u^n)\). This completes the proof.

3.3. Necessary optimality conditions. In this subsection, we are going to show the necessary optimality conditions to system (41)-(43). Let \((I^*, S^*, C^*, T^*)\) be an optimal solution of system (41)-(43) corresponding to the optimal control variable \(u^*\), and \((I^*, S^*, C^*, T^*)\) be a unique strong positive solution of system (41)-(43) corresponding to the control variable \(u^* = u^* + \epsilon \tilde{u} \in U\) [32], where \(\epsilon > 0\) and \(\tilde{u} \in L^2(\Omega_T)\). Further, define
\[
(z_{11}^\epsilon, z_{12}^\epsilon, z_{13}^\epsilon, z_{14}^\epsilon) = \left(\frac{I^* - I^*}{\epsilon}, \frac{S^* - S^*}{\epsilon}, \frac{C^* - C^*}{\epsilon}, \frac{T^* - T^*}{\epsilon}\right).
\]

Subtracting system (41)-(43) corresponding to \(u^*\) from the system corresponding to \(u^*\), and dividing both sides by \(\epsilon\), one achieves that:
\[
\begin{align*}
\frac{\partial z_1^\epsilon}{\partial \theta} = & \Delta z_1^\epsilon - \left[\frac{\mu + (\beta_1 + \beta_2 + \beta_3) S^*}{1 + a S^*}\right] z_1^\epsilon - \left[\frac{(\beta_1 + \beta_2 + \beta_3) I^*}{(1 + a S^*) (1 + a S^*)}\right] z_2^\epsilon, \\
\frac{\partial z_2^\epsilon}{\partial \theta} = & \Delta z_2^\epsilon + \beta_1 S^* z_1^\epsilon + \left[\frac{\beta_1 I^*}{1 + a S^* (1 + a S^*)}\right] z_2^\epsilon - \left[(\mu + \gamma + u^*)\right] z_3^\epsilon + \alpha z_3^\epsilon - \tilde{u} S^*, \\
\frac{\partial z_3^\epsilon}{\partial \theta} = & \Delta z_3^\epsilon + \frac{\beta_2 S^*}{1 + a S^*} z_1^\epsilon + \left[\frac{\beta_2 I^*}{1 + a S^* (1 + a S^*)}\right] z_2^\epsilon - (\alpha + \mu) z_3^\epsilon, \\
\frac{\partial z_4^\epsilon}{\partial \theta} = & \Delta z_4^\epsilon + \frac{\beta_3 S^*}{1 + a S^*} z_1^\epsilon + \left[\frac{\beta_3 I^*}{1 + a S^* (1 + a S^*)} + \gamma + u^*\right] z_3^\epsilon - \mu z_3^\epsilon + \tilde{u} S^*,
\end{align*}
\]
Theorem 3.2. gives the necessary optimality conditions for the optimal control problem. To prove the uniqueness of the positive strong solution to the adjoint system (66), results hold.

According to the proof process in Theorem 3.1, as $\epsilon \to 0$, the convergences that $I^\epsilon \to I^*, S^\epsilon \to S^*, C^\epsilon \to C^*, T^\epsilon \to T^*$ and $z_i^\epsilon \to z_i$ hold in $L^2(\Omega_T)$ [10]. Meanwhile, $z = (z_1, z_2, z_3, z_4)$ is the solution of the following system

\[
\begin{align*}
\frac{\partial z_1}{\partial t} &= d\Delta z_1 - \left[ \mu + (\beta_1 + \beta_2 + \beta_3) S^* \frac{1}{1 + aS^*} \right] z_1 - \frac{\beta_1 + \beta_2 + \beta_3}{(1 + aS^*)^2} z_2, \\
\frac{\partial z_2}{\partial t} &= d\Delta z_2 + \frac{\beta_1 S^*}{1 + aS^*} z_1 + \left[ \frac{\beta_1 I^*}{(1 + aS^*)^2} - (\mu + \gamma + u^*) \right] z_2 + \alpha z_3 - \tilde{u} S^*, \\
\frac{\partial z_3}{\partial t} &= d\Delta z_3 + \frac{\beta_2 S^*}{1 + aS^*} z_1 + \left[ \frac{\beta_2 I^*}{(1 + aS^*)^2} - (\alpha + \mu) z_3, \right. \\
\frac{\partial z_4}{\partial t} &= d\Delta z_4 + \frac{\beta_3 S^*}{1 + aS^*} z_1 + \left[ \frac{\beta_3 I^*}{(1 + aS^*)^2} + \gamma + u^* \right] z_2 - \mu z_4 + \tilde{u} S^*,
\end{align*}
\]

for $(x, t) \in \Omega_T$ and with the initial and boundary conditions

\[
\begin{align*}
&z_1(x, 0) = z_2(x, 0) = z_3(x, 0) = z_4(x, 0) = 0, \quad x \in \Omega, \\
&\frac{\partial z_1}{\partial n} = \frac{\partial z_2}{\partial n} = \frac{\partial z_3}{\partial n} = \frac{\partial z_4}{\partial n} = 0, \quad (x, t) \in \Sigma_T.
\end{align*}
\]

Using the standard method, next we want to establish the necessary optimality conditions for the optimal control, which prompts us to introduce the following adjoint system corresponding to system (41)-(43)

\[
\begin{align*}
\frac{\partial p_1}{\partial t} &= -d\Delta p_1 + \left[ \mu + (\beta_1 + \beta_2 + \beta_3) S^* \frac{1}{1 + aS^*} \right] p_1 - \frac{\beta_1 S^*}{1 + aS^*} p_2 - \frac{\beta_2 S^*}{1 + aS^*} p_3 - \frac{\beta_3 S^*}{1 + aS^*} p_4 + l_1, \\
\frac{\partial p_2}{\partial t} &= -d\Delta p_2 + \left[ \beta_1 S^* \frac{1}{1 + aS^*} \right] p_1 - \left[ \frac{\beta_1 I^*}{(1 + aS^*)^2} - (\mu + \gamma + u^*) \right] p_2 - \frac{\beta_2 I^*}{(1 + aS^*)^2} p_3 - \frac{\beta_3 I^*}{(1 + aS^*)^2} p_4 + l_2, \\
\frac{\partial p_3}{\partial t} &= -d\Delta p_3 + \alpha p_2 + (\alpha + \mu) p_3 + l_3, \\
\frac{\partial p_4}{\partial t} &= -d\Delta p_4 + \mu p_4 + l_4,
\end{align*}
\]

with the initial and boundary conditions

\[
\begin{align*}
&p_1(x, T) = -\theta_1, p_2(x, T) = -\theta_2, p_3(x, T) = -\theta_3, p_4(x, T) = \theta_4, \quad x \in \Omega, \\
&\frac{\partial p_1}{\partial n} = \frac{\partial p_2}{\partial n} = \frac{\partial p_3}{\partial n} = \frac{\partial p_4}{\partial n} = 0, \quad (x, t) \in \Sigma_T.
\end{align*}
\]

where $p = (p_1, p_2, p_3, p_4)$ is the adjoint variable. Employing the same methods of proving Theorem 3.1 to system (66), one can easily declare the existence and uniqueness of the positive strong solution to the adjoint system (66).

In the following, we are ready to show the main result of this subsection, which gives the necessary optimality conditions for the optimal control problem.

**Theorem 3.2.** Let $p = (p_1, p_2, p_3, p_4)$ be given in the adjoint system (66), if $(I^*, S^*, C^*, T^*, u^*)$ is an optimal solution for system (41)-(43), then the following results hold.
(i) If $\rho(x) \neq 0$ for $x \in \Omega$, then
\[
\int_{\Omega_T} [\lambda + S^*(p_2 - p_4)] (u_0 - u^*)(x, t)dxdt \geq - \int_{\Omega_T} \rho(x)(u_0 - u^*)(x, T)dx.
\] 
(ii) If $\rho(x) \equiv 0$ for $x \in \Omega$, then
\[
u^* = \begin{cases}
1, & \text{in } \{(x, t) \in \Omega : [\lambda + S^*(p_2 - p_4)](x, t) < 0\}, \\
0, & \text{in } \{(x, t) \in \Omega : [\lambda + S^*(p_2 - p_4)](x, t) \geq 0\},
\end{cases}
\]
that is to say, the optimal control $u^*$ is of a Bang-Bang form.

**Proof.** Since $(I^*, S^*, C^*, T^*, U^*)$ is an optimal solution, then it follows that
\[J(I^*, S^*, C^*, T^*, u^*) \leq J(I^*, S^*, C^*, T^*, u^*), \quad \forall \epsilon > 0.\] 
A straightforward calculation yields
\[
\int_{\Omega_T} [l_1(I^* - I^*) + l_2(S^* - S^*) + l_3(C^* - C^*) - l_4(T^* - T^*) + \lambda \epsilon \bar{u}] (x, t)dxdt
\]
\[
+ \int_{\Omega} [\theta_1(I^* - I^*) + \theta_2(S^* - S^*) + \theta_3(C^* - C^*) - \theta_4(T^* - T^*) + \rho \epsilon \bar{u}] (x, T)dt \geq 0.
\]
Combining (61) and taking the limit as $\epsilon \to 0$, it arrives at
\[
\int_{\Omega_T} [l_1 z_1 + l_2 z_2 + l_3 z_3 - l_4 z_4 + \lambda \bar{u}] (x, t)dxdt
\]
\[
+ \int_{\Omega} [\theta_1 z_1 + \theta_2 z_2 + \theta_3 z_3 - \theta_4 z_4 + \rho \bar{u}] (x, T)dt \geq 0.
\]
Multiplying the equations from (66) and the equations from (64) by $z_1, z_2, z_3, z_4$ and $p_1, p_2, p_3, p_4$, respectively, we obtain
\[
\sum_{i=1}^{4} \left( p_i \frac{\partial z_i}{\partial t} + z_i \frac{\partial p_i}{\partial t} \right) = \sum_{i=1}^{4} d (p_i \Delta z_i - z_i \Delta p_i) + l_1 z_1 + l_2 z_2 + l_3 z_3 - l_4 z_4 + \bar{u}(S^* p_4 - S^* p_2).
\]
Observe that
\[
\sum_{i=1}^{4} \left( p_i \frac{\partial z_i}{\partial t} + z_i \frac{\partial p_i}{\partial t} \right) = \frac{\partial}{\partial t} (p_1 z_1 + p_2 z_2 + p_3 z_3 + p_4 z_4).
\]
Reviewing the initial and boundary conditions, and substituting (74) into (73), we have
\[
\int_{\Omega_T} (p_1 z_1 + p_2 z_2 + p_3 z_3 + p_4 z_4)(x, T)dx
\]
\[
= \int_{\Omega_T} (l_1 z_1 + l_2 z_2 + l_3 z_3 - l_4 z_4)(x, t)dxdt
\]
\[
+ \int_{\Omega_T} \bar{u} S^* (p_4 - p_2) dxdt,
\]
by integrating over $\Omega_T$. Employing (67), and we obtain
\[
\int_{\Omega} (\theta_1 z_1 + \theta_2 z_2 + \theta_3 z_3 - \theta_4 z_4)(x, T)dx
\]
\[
+ \int_{\Omega_T} (l_1 z_1 + l_2 z_2 + l_3 z_3 - l_4 z_4)(x, t)dxdt
\]
\[
= \int_{\Omega_T} \bar{u} S^* (p_4 - p_2) dxdt.
\]
Combining (72), we have
\[ \int_{\Omega_T} \tilde{u} S^* (p_2 - p_4) dx dt \geq - \int_{\Omega_T} (\lambda \tilde{u})(x,t) dx dt - \int_{\Omega} (\rho \tilde{u})(x,T) dx. \] (75)

According to the arbitrariness of \( \tilde{u} \in L^2(\Omega_T) \), let \( \tilde{u} = u_0 - u^* \), \( u_0 \in U \), one obtains
\[ \int_{\Omega_T} [\lambda + S^* (p_2 - p_4)] (u_0 - u^*)(x,t) dx dt \geq - \int_{\Omega} \rho(x)(u_0 - u^*)(x,T) dx. \] (76)

In particular, if \( \rho(x) \equiv 0 \), inequality (76) can be simplified to
\[ \int_{\Omega_T} [\lambda + S^* (p_2 - p_4)] (u_0 - u^*)(x,t) dx dt \geq 0, \] (77)
under the optimality condition (69) for the optimal control function \( u^* \). This completes the proof.

4. Numerical simulation. In this section, we will use Matlab to show some numerical examples which illustrates the applications of the results given in the theoretical analysis before. In spatial heterogeneous case, choose \( \Omega = [0, \pi] \). For system (7)-(9), we demonstrate the complex spatiotemporal dynamics of rumor propagation, including the global asymptotic stability of rumor-free steady state \( E_0(x) \), the uniformly persistence of positive solutions, and the effects of diffusion coefficients on the dynamics of rumor propagation. Further, the control dynamics of the model (41)-(43) is numerically simulated when the system parameters are constant.

4.1. The extinction of rumor propagation. According to Theorem 2.3, here we only consider the sharing rate \( \beta_1(x) \) and the commenting rate \( \beta_2(x) \) are spatially related parameters. Thus selecting
\[
\begin{cases}
  d(x) = 0.001, A(x) = 0.45, \mu(x) = 0.45, \\
  \alpha(x) = 0.1, \gamma(x) = 0.4, \beta_3(x) = 0.2, \\
  \beta_1(x) = 0.4(1 + c_1 \cos x), c_1 = 0.1, \\
  \beta_2(x) = 0.2(1 + c_2 \cos x), c_2 = 0.2, \\
  f(S) = S/(1 + aS^2), a = 0.2,
\end{cases}
\]
where \( 0 \leq c_1, c_2 \leq 1 \) are constants which reflect the heterogeneity degree of \( \beta_1(x) \) and \( \beta_2(x) \) respectively [4]. By Theorem 2.3, we have
\[
\mathcal{R}_0 = \frac{[\alpha(\beta_1^M + \beta_2^M) + \mu \beta_1^M]A_{fs}(0)}{\mu(\mu + \alpha)(\mu + \gamma)} = \frac{0.1 \times [0.4 \times (1 + 0.1) + 0.2 \times (1 + 0.2)] + 0.45 \times 0.4 \times (1 + 0.1) \times 0.45}{0.45 \times (0.45 + 0.1) \times (0.45 + 0.4)}
\]
\[ = 0.5690 < 1. \]

Applying Theorem 2.4, it is easily checked that the rumor-free steady state \( E_0(x) \) is globally asymptotically stable. From Fig.2, the solutions of system (7)-(9) converge to \( E_0(x) \) which implies that rumor is extinct.
Figure 2. The rumor-free steady state $E_0(x)$ is globally asymptotically stable.

4.2. The uniform persistence of rumor propagation. For some parameters fixed as follows

\[
\begin{aligned}
    d(x) &= 0.001, A(x) = 0.3, \mu(x) = 0.3, \\
    \alpha(x) &= 0.5, \gamma(x) = 0.1, \beta_3(x) = 0.2, \\
    \beta_1(x) &= 0.6(1 + c_1 \cos x), c_1 = 0.1, \\
    \beta_2(x) &= 0.5(1 + c_2 \cos x), c_2 = 0.1, \\
    f(S) &= S/(1 + aS^2), a = 0.2.
\end{aligned}
\]

Similarly
\[
\mathcal{R}_0 = \frac{[\alpha (\beta_1^m + \beta_2^m) + \mu \beta_1^m] Af_S(0)}{\mu (\mu + \alpha)(\mu + \gamma)}
= \frac{\{0.5 \times [0.6 \times (1 - 0.1) + 0.5 \times (1 - 0.1)] + 0.3 \times 0.6 \times (1 - 0.1)\} \times 0.3}{0.3 \times (0.3 + 0.5)(0.3 + 0.1)}
= 1.9750 > 1.
\]

which satisfies the condition in Theorem 2.7. Thus the positive solutions of system (7)-(9) are uniformly persistent in Fig.3. This environment suggests that rumors continue to circulate.

Figure 3. The positive solutions of system (7)-(9) are uniformly persistent.
4.3. The effects of the diffusion coefficient \( d(x) \) on the spatiotemporal dynamics of rumor propagation. We take the system parameters below

\[
\begin{align*}
\beta_1(x) &= 0.6(1 + c_1 \cos x), c_1 = 0.1, \\
\beta_2(x) &= 0.5(1 + c_2 \cos x), c_2 = 0.2, \\
\beta_3(x) &= 0.2(1 + c_3 \cos x), c_3 = 0.1, \\
A(x) &= 0.3 + 0.02 \cos x, \\
\alpha(x) &= 0.5 + 0.03 \cos x, \\
\mu(x) &= 0.3 + 0.01 \cos x, \\
\gamma(x) &= 0.1 + 0.02 \cos x, \\
f(S) &= S/(1 + aS^2), a = 0.3,
\end{align*}
\]

(78)

To observe the complex spatiotemporal dynamics of system (7)-(9) by changing the diffusion coefficient \( d(x) = 0.0001, 0.1, 1, 5 \). As shown in Fig.4, the density distribution of ignoramuses, sharers and commentators are greatly influenced by the diffusion coefficient \( d(x) \). The larger the diffusion coefficient is, the smaller the variation region of the density distribution is. This phenomenon tells us that it is necessary to consider the influence of spatial mobility of network users on the propagation process in the study of network rumor propagation.

![Figure 4. Projection diagram in the tx-plane.](image-url)
4.4. Optimal control strategies. The parameters used in system (41) are selected as $A = 0.5$, $\mu = 0.15$, $\beta_1 = 0.2$, $\beta_2 = 0.1$, $\beta_3 = 0.5$, $\alpha = 0.8$, $\gamma = 0.3$, $\alpha = 0.15$ and $d = 0.005$. The initial functions are $I_0(x) = 0.12 + 0.32(1.2x^2 + \sin 15x)$, $S_0(x) = 0.21x^2 \cos(0.9 + 0.3x)$, $C_0(x) = 0.29 + 0.45(1 + \cos 10x)$, $T_0(x) = 0.32(1 - x)^2 \sin(1.5 + 0.4x)$. In addition, the parameters used in the cost functional are $\theta_1 = 1$, $\theta_2 = 0.5$, $\theta_3 = 1.2$, $\theta_4 = 0.8$, $l_1 = 0.6$, $l_2 = 1$, $l_3 = 0.8$, $l_4 = 1.2$, $\lambda = 1$, $\rho = 0$, and the control domain is $\Omega_T = [0,1] \times [0,1]$.

For the state system (41)-(43) and the adjoint system (66)-(67), we use mathematical software to get the images of the solution surface of state equations, the solution surface of the adjoint equations and the solution surface of optimal control during the interactive process (see Fig.5-6). We further see from Fig.6 that the optimal control $u(x,t)$ is of a Bang-Bang form. It is consistent with the results in Theorem 3.3. Fig.7 and Fig.8 have shown the control effect of the optimal control $u^*$ at the end of the control. The optimal controller $u^*$ effectively reduces the density of the sharer $S(x,t)$ and increases the density of the stifler $T(x,t)$. This control method achieves the ultimate control goal, which reduces the risk of rumor propagation in online social networks.
Figure 5. The solution surface to state equations and adjoint equations on $\Omega_T$.

Figure 6. The optimal control $u(x,t)$ on $\Omega_T$. 
5. Conclusions and discussions. This paper focuses on threshold dynamics of a reaction-diffusion ISCT rumor propagation model in a spatial heterogeneous environment. Furthermore, we apply the optimal control theory to our spatiotemporal rumor propagation model. The combination of diffusion and the spatial heterogeneity of the environment is an important feature that we want to reveal in this paper. We first summarize the well-posedness of global solutions. The basic reproduction number $R_0$ is calculated for the model which contains the spatial homogeneity as a special case. Next, with the aid of the comparison principle and Lyapunov function, we obtain the global asymptotic stability of the rumor-free steady state $E_0(x)$. The uniform persistence of all positive solutions are also established by using the persistence theory of dynamical systems. In addition, we have studied an optimal control problem for the ISCT model. Similarly, we demonstrate the well-posedness of the controlled system and some estimates of the solution are given. Then the existence of the optimal control problem have been proved. After that, the first condition of optimality has been deduced. It’s worth mentioning that the optimal control of system (41) is of the Bang-Bang form if $\rho(x, T) = 0$. That is to say, the cost of interventions in terminal time $T$ is not taken into consideration. Numerical
simulations verify the correctness of the theories. Some results of the optimal control problem in this paper contribute certain value to designing the optimal control of various models of rumor propagation.

Undoubtedly, our work is not perfect which inspires us to continue our research. We mention that there are quite a few spaces to improve and generalize our rumor propagation model. On the one hand, our model is based on the assumption that all network users enjoy the same diffusion rate $d(x)$, which plays a key role in proving the uniform boundedness of solutions. In other words, if we consider different diffusion rate, then we can’t easily get the corresponding result. Actually, due to the spatial heterogeneity of the environment, individuals may spread at different rates, so it is of practical significance to consider such a model with different diffusion rate. On the other hand, in recent years, age structure has been widely taken into account in epidemic models. It would be interesting to incorporate age structure into the current model to expound the spatial-temporal propagation of rumor. We will leave these problems for further study.

Acknowledgments. This research is supported by the National Natural Science Foundation of China (Grant No.12002135), the Natural Science Foundation of Jiangsu Province (Grant No.BK20190836), the Natural Science Research of Jiangsu Higher Education Institutions of China (Grant No.19KJB110001) and the China Postdoctoral Science Foundation (Grant No.2019M661732).

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Received April 2020; 1st revision August 2020; 2nd revision August 2020.

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