Kasner Metrics and Very Special Geometry

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Abstract

We consider general charged Kasner-like solutions for the theory of five-dimensional supergravity coupled to Abelian vector multiplets in arbitrary space-time signature. These solutions, depending on the choice of coordinates, can be thought of as generalisations of Melvin/Rosen cosmologies, flux-branes and domain walls.
1 Introduction

In recent years an active area of research has been the classification and analysis of supersymmetric gravitational backgrounds in supergravity theories with various space-time dimensions and signatures. The present work deals with non-supersymmetric cosmological and static solutions of the theories of five-dimensional supergravities coupled to vector multiplets in arbitrary space-time signature \([1]\). In these supergravity theories, the dynamics of the bosonic fields can be described in terms of the so called very special geometry \([2]\). Early analysis of supersymmetric solutions, with vanishing fermionic fields, was based on starting with a specific ansatz for the space-time metric and fixing it alongside the bosonic fields by requiring the existence of some Killing spinors (see for example \([3]\)). A systematic classification of solutions admitting Killing spinors for minimal and general five-dimensional supergravity theory was first considered in \([4]\). Later many interesting solutions were found including the novel black ring solutions with an event horizon of topology \(S^1 \times S^2\) \([5]\). Five-dimensional supergravity theories with vector multiplets in all possible space-time signatures were recently considered in \([6,7]\). A rigorous analysis of supersymmetry algebras in all space-time signatures was given in \([8]\). A generalisation of the results of \([4]\) related to time-like solutions to all five-dimensional supergravity was considered in \([9]\).

In addition to the supergravity models based on symmetric scalar manifolds, a large class of supergravities with Lorentzian signature can be obtained as compactifications of eleven-dimensional supergravity \([10]\) on a Calabi-Yau 3-folds \([11]\). Similarly, five-dimensional supergravity theories in various space-time signatures can be obtained via Calabi-Yau 3-folds compactification of the exotic eleven-dimensional supergravity theories constructed by Hull \([12]\).

It is our purpose in the present work to study Kasner-like cosmological and static solutions to five-dimensional supergravity theories. The analysis of classical time-dependent gravitational solutions in string theory and supergravities is important to investigate the role that these theories can play in gravitational physics and cosmology.

The plan of this paper is as follows. In the next section we briefly review the vacuum Kasner metrics in four and five dimensions and their extensions to solutions of Einstein-Maxwell theory. In section three, we present some basic properties of the ungauged five-dimensional \(N = 2\) supergravity theory and the very special geometry underlying its structure as well
as the equations of motion for the metric, gauge and scalar fields of the theory. We present
two classes of solutions with some explicit examples in section four where we also include a
summary of our results.

2 Solutions of Einstein-Maxwell is 4 and 5 dimensions

The original vacuum four-dimensional Euclidean Kasner solution depends only on one vari-

able and is given by [13]

\[ ds^2 = x_1^{2a_1} dx_1^2 + x_1^{2a_2} dx_2^2 + x_1^{2a_3} dx_3^2 + x_1^{2a_4} dx_4^2, \]  

(2.1)

where the constants \( a_i \), known as the Kasner exponents, satisfy two conditions

\[ a_2 + a_3 + a_4 = 1 + a_1, \quad a_2^2 + a_3^2 + a_4^2 = (1 + a_1)^2. \]

The Kasner metric is actually valid for all space-time signatures [14] and with a change of
coordinates can take the form

\[ ds^2 = \epsilon_0 d\tau^2 + (\epsilon_1 \tau^{2a_0} dx^2 + \epsilon_2 \tau^{2b} dy^2 + \epsilon_3 \tau^{2c} dz^2) \]  

(2.2)

where \( \epsilon_i \) take the values \( \pm 1 \) and the constants satisfy the conditions

\[ a + b + c = a^2 + b^2 + c^2 = 1. \]  

(2.3)

Kasner metric is related to the metrics found by Weyl [15], Levi-Civita [16] and Wilson [17]
and was also rediscovered by many authors [18]. For solutions with Lorentzian signature, we
can obtain the vacuum cosmological solutions

\[ ds^2 = -d\tau^2 + \tau^{2a} dx^2 + \tau^{2b} dy^2 + \tau^{2c} dz^2, \]  

(2.4)

as well as the anisotropic vacuum domain wall solutions

\[ ds^2 = -r^{2a} dt^2 + (dr^2 + r^{2b} dy^2 + r^{2c} dz^2). \]  

(2.5)

The vacuum Kasner metric can be promoted to solutions with a non-trivial gauge field [19][20].

For Einstein-Maxwell theory with the action

\[ S = \int d^4x \sqrt{|g|} \left( R - \frac{1}{4} F^2 \right), \]  

(2.6)
one obtains the solutions
\[
\begin{align*}
&d s^2 = \left(1 + \frac{\epsilon_3 Q^2}{16c^2 \tau^{2c}} \right)^2 \left( \epsilon_0 d\tau^2 + \epsilon_1 \tau^{2a_1} dx_1^2 + \epsilon_2 \tau^{2a_2} dx_2^2 \right) + \left(1 + \frac{\epsilon_3 Q^2}{16c^2 \tau^{2c}} \right)^{-2} \epsilon_3 \tau^{2c} dz^2 ,
\end{align*}
\]
with the gauge field-strength two-form given by
\[
F = Q \tau^{2c-1} \left(1 + \frac{\epsilon_3 Q^2}{16c^2 \tau^{2c}} \right) d\tau \wedge dz. \tag{2.7}
\]
The dual magnetic Lorentzian cosmological solutions, with \(\tau\) being a time-like coordinate and \(F = Q dx_1 \wedge dx_2\)\cite{19} are equivalent to the family of anisotropic cosmologies of Rosen \cite{21}.

In five dimensional gravity with arbitrary space-time signature, the Kasner metrics take the form
\[
\begin{align*}
&d s^2 = \epsilon_0 d\tau^2 + \sum_{i=1}^{4} \epsilon_i \tau^{2a_i} dx_i^2 \tag{2.8}
\end{align*}
\]
with the conditions
\[
\sum_{i=1}^{4} a_i = \sum_{i=1}^{4} a_i^2 = 1. \tag{2.9}
\]
For Einstein-Maxwell action in five dimensions
\[
S = \int d^5 x \sqrt{|g|} \left(R - \frac{\epsilon}{4} F^2 \right) \tag{2.10}
\]
with \(\epsilon = \pm 1\), two classes of solutions can be found\cite{20}. One is given by
\[
\begin{align*}
&d s^2 = \left(c_1 + \frac{\epsilon \epsilon_4 Q^2}{12a_4^2 c_1} \tau^{2a_4}\right) \left( \epsilon_0 d\tau^2 + \sum_{i=1}^{3} \epsilon_i \tau^{2a_i} dx_i^2 \right) + \epsilon_4 \left(c_1 + \frac{\epsilon \epsilon_4 Q^2}{12a_4^2 c_1} \tau^{2a_4}\right)^{-2} \tau^{2a_4} dx_4^2 ,
&F = Q \tau^{2a_4-1} \left(c_1 + \frac{\epsilon \epsilon_4 Q^2}{12a_4^2 c_1} \tau^{2a_4}\right)^{-2} d\tau \wedge dx_4 ,
\end{align*}
\]
and the other class of solutions is for \(F = P dx_1 \wedge dx_2\), with constant \(P\) and is given by
\[
\begin{align*}
&d s^2 = e^{2U} \left( \epsilon_0 d\tau^2 + \epsilon_1 \tau^{2a_1} dx_1^2 + \epsilon_2 \tau^{2a_2} dx_2^2 \right) + e^{-U} \left(\epsilon_3 \tau^{2a_3} dx_3^2 + \epsilon_4 \tau^{2a_4} dx_4^2 \right) , \tag{2.11}
\end{align*}
\]
with
\[
e^U = \left(c_1 - \frac{\epsilon \epsilon_0 \epsilon_1 \epsilon_2 P^2}{12 (a_3 + a_4)^2 c_1} \tau^{2(a_3+a_4)} \right) . \tag{2.12}
\]

3 \hspace{0.5cm} 5D \hspace{0.5cm} \text{N = 2 Supergravity and Very Special Geometry}

Ignoring the hypermultiplets, the Lagrangian of the bosonic fields of all the five-dimensional supergravity can be given by\cite{14,17}
\[
\mathcal{L}_5 = \sqrt{|g|} \left(R - \frac{\epsilon}{2} G_{IJ} F^I_{\mu \nu} F^{J \mu \nu} - g_{ij} \partial_\mu \phi^i \partial^\mu \phi^j - \frac{1}{24} \epsilon^\mu \nu \sigma \tau C_{IJK} F^I_{\mu \nu} F^J_{\rho \sigma} A^K_\tau \right) . \tag{3.1}
\]
The indices $I, J, K$ take values $0, 1, \ldots, n$, where $n$ is the number of the scalar fields $\phi^i$. The constants $C_{IJK}$ are symmetric on all indices. Note that for supergravity theories originating from Calabi-Yau compactification, $C_{IJK}$ are the intersection numbers. The bosonic sector of the theory can be described in terms of the so called very special geometry \cite{2}. We define the special coordinates $X^I = X^I(\phi)$ satisfying

$$X^I X_I = 1, \quad \mathcal{V} = \frac{1}{6} C_{IJK} X^I X^J X^K = 1,$$

where, $X_I$, the dual coordinate, is defined by

$$X_I = \frac{1}{6} C_{IJK} X^J X^K. \quad (3.3)$$

The gauge coupling metric is derived from the prepotential $\mathcal{V}$ and is given by

$$G_{IJ} = -\frac{1}{2} \frac{\partial}{\partial X^I} \frac{\partial}{\partial X^J} (\ln \mathcal{V})|_{\mathcal{V}=1} = \frac{9}{2} X_I X_J - \frac{1}{2} C_{IJK} X^K. \quad (3.6)$$

Moreover, the scalar metric satisfies

$$g_{ij} = G_{IJ} \partial_i X^I \partial_j X^J|_{\mathcal{V}=1}, \quad (3.4)$$

where we have $\partial_i = \frac{\partial}{\partial \phi^i}$. As such, we have the relation $g_{ij} \partial_i \phi^j \partial^\mu \phi^j = G_{IJ} \partial_i X^I \partial^\mu X^J$. We also have the very special geometry relations

$$G_{IJ} X^J = \frac{3}{2} X_I, \quad dX_I = -\frac{2}{3} G_{IJK} dX^K. \quad (3.5)$$

The Einstein, Maxwell and scalar field equations of motion derived from (3.1) can be written as

$$R_{\mu\nu} - G_{IJ} \partial_\mu X^I \partial_\nu X^J - \epsilon G_{IJ} \left( F_{\mu\alpha} F_\nu^\alpha - \frac{1}{6} g_{\mu\nu} F^2 \right) = 0, \quad (3.6)$$

$$\nabla_\mu (\epsilon G_{IJ} F^{I\mu}) + \frac{1}{16} \epsilon^{\mu\lambda\sigma\tau} C_{IJK} F_{\lambda\sigma} F^K_{\mu\tau} = 0, \quad (3.7)$$

$$\sqrt{|g|} \partial_i G_{IJ} \left( \frac{\epsilon}{2} F_{\mu\nu,i} F^{\mu\nu,j} + \partial_\mu X^I \partial^\mu X^J \right) - 2 \partial_\mu \left( \sqrt{|g|} G_{IJ} \partial^\mu X^J \right) \partial_i X^I = 0. \quad (3.8)$$

\section{4 General Solutions and Examples}

To solve for the equations of motion (3.6)-(3.8), we start with the following metric

$$ds^2 = e^{2U} \left( \epsilon_0 d\tau^2 + \epsilon_1 \tau^{2a_1} dx^2 + \epsilon_2 \tau^{2a_2} dy^2 + \epsilon_3 \tau^{2a_3} dz^2 \right) + \epsilon_4 e^{-4U} \tau^{2a_4} dw^2 \quad (4.1)$$
where the constants $a_i$, satisfying the Kasner conditions (2.9) and $U$ is a function of $\tau$ only. For the gauge fields, we consider solutions with only $F^I_{\tau w}$ non-vanishing and functions of $\tau$, the equations of motion (3.7) then reduce to

$$\partial_\tau \left( \tau e^{2U} G_{IJ} F^{J\tau w} \right) = 0$$

and thus imply the solution

$$G_{IJ} F^{J\tau w} = e^{-2U} \frac{q_I}{\tau}.$$ (4.2)

where $q_I$ being constants. Substituting the value of the gauge field-strength into the Einstein equations of motion (3.6), we obtain

$$\ddot{U} + 2\dot{U}^2 + (1 - 2a_4) \frac{\dot{U}}{\tau} = -\frac{1}{3} G_{IJ} \dot{X}^I \dot{X}^J,$$ (4.3)

$$3 \left( \ddot{U} + \frac{\dot{U}}{\tau} \right) = \epsilon \epsilon_{IJ} 2a_4 - e^{-4U} G^{IJ} q_I q_J,$$ (4.4)

here the dot symbol indicates differentiation with respect to the coordinate $\tau$. In order to solve these equations, we write the dual coordinate in the form

$$X_I(\tau) = \frac{1}{3} e^{-2U} f_I(\tau),$$ (4.5)

then with the help of the relations of very special geometry, the following equations can be derived

$$\ddot{X}_I = -2\ddot{U} X_I + \frac{1}{3} e^{-2U} \ddot{f}_I,$$
$$\dot{U} = \frac{1}{6} e^{-2U} \dot{f}_I X^I,$$
$$\ddot{U} = \frac{1}{6} e^{-2U} \left( X^I \dddot{f}_I - \frac{1}{2} e^{-2U} G^{IJ} \dot{f}_I \dot{f}_J \right),$$

$$G_{IJ} \dot{X}^I \dot{X}^J = -3\ddot{U} - 6\dot{U}^2 + \frac{1}{2} e^{-2U} X^I \dddot{f}_I,$$ (4.6)

which when substituted in the equations (4.3) and (4.4) give

$$X^I \left[ \dddot{f}_I + (1 - 2a_4) \frac{\dddot{f}_I}{\tau} \right] = 0,$$ (4.7)

$$e^{-2U} X^I \left( \dddot{f}_I + \frac{\dddot{f}_I}{\tau} \right) - e^{-4U} G^{IJ} \left( \frac{1}{2} \dddot{f}_I \dddot{f}_J + 2 \epsilon \epsilon_{IJ} q_I q_J \tau^{2a_4 - 2} \right) = 0.$$ (4.8)
Equation (4.7) admits the solution
\[ f_I = (A_I + B_I \tau^{2a_4}) \] (4.9)
with constant \(A_I\) and \(B_I\). Equation (4.8) then implies the conditions
\[ G^{IJ} (A_J B_I a_4^2 - \epsilon \epsilon_q q_I q_J) = 0. \] (4.10)
The scalar equation of motion (3.8) for our solutions reduces to
\[ \partial_i G^{IJ} \left( \frac{1}{2} j_I j_J - \frac{1}{2} f_J \left( \frac{j_I}{\tau} + \dot{f}_I \right) + 2 \epsilon \epsilon q_I q_J \tau^{2a_4 - 2} \right) = 0 \] (4.11)
which implies the conditions
\[ \partial_i G^{IJ} (A_J B_I a_4^2 - \epsilon \epsilon q_I q_J) = 0. \] (4.12)
Note that (3.2) implies the relation
\[ e^{4U} = \frac{1}{6} G^{IJ} f_I f_J. \] (4.13)
We consider a second class of solutions represented by the metric
\[ ds^2 = e^{2U} \left( \epsilon_0 d\tau^2 + \epsilon_1 \tau^{2a_1} dx^2 + \epsilon_2 \tau^{2a_2} dy^2 \right) + e^{-U} \left( \epsilon_3 \tau^{2a_3} dz^2 + \epsilon_4 \tau^{2a_4} dw^2 \right) \] (4.14)
and the gauge field-strength two-form \( F^I = p^I dx \wedge dy \) with constant \( p^I \). For the metric (4.14), one obtains for the Einstein equations of motion
\[ \ddot{U} + (1 - 2l) \frac{\dot{U}}{\tau} + \dot{U}^2 = -\frac{2}{3} G_{IJ} \dot{X}^I \dot{X}^J, \] (4.15)
\[ \ddot{U} + \frac{\dot{U}}{\tau} = -\frac{2}{3} \epsilon \epsilon_0 \epsilon_1 \epsilon_2 e^{-2U} \tau^{-2s} G_{IJ} p^I p^J \] (4.16)
where we have \( l = a_3 + a_4 \) and \( s = a_1 + a_2 \). To proceed in finding solutions, we set
\[ X^I(\tau) = e^{-U} h^I(\tau), \] (4.17)
then using the relations of very special geometry, we obtain the following relations
\[ \dot{U} = e^{-U} X_I \dot{h}^I, \]
\[ \ddot{U} = -\frac{2}{3} e^{-2U} G_{IJ} \dot{h}^I \dot{h}^J + e^{-U} X_I \ddot{h}^I, \]
\[ G_{IJ} \dot{X}^I \dot{X}^J = -\frac{3}{2} \dot{U}^2 + e^{-2U} G_{IJ} \dot{h}^I \dot{h}^J. \] (4.18)
Using (4.18), equations (4.15) and (4.16) then give

\[ X_I \left[ \ddot{h}^I + (1 - 2l) \frac{\dot{h}^I}{\tau} \right] = 0, \]  \hspace{1cm} (4.19)

\[ G_{IJ} \left[ \dot{h}^I \dot{h}^J - h^J \left( \ddot{h}^I + \frac{\dot{h}^I}{\tau} \right) - \epsilon \epsilon_0 \epsilon_1 \epsilon_2 \tau^{-2} p^I p^J \right] = 0, \]  \hspace{1cm} (4.20)

which can be solved by

\[ h^I = C^I + D^I \tau^{2l}, \]  \hspace{1cm} (4.21)

with the conditions

\[ G_{IJ} \left( 4l^2 C^J D^I + \epsilon \epsilon_0 \epsilon_1 \epsilon_2 p^I p^J \right) = 0. \]  \hspace{1cm} (4.22)

The scalar equation of motion also implies the conditions

\[ \partial_i G_{IJ} \left( 4l^2 C^J D^I + \epsilon \epsilon_0 \epsilon_1 \epsilon_2 p^I p^J \right) = 0. \]  \hspace{1cm} (4.23)

Moreover, the condition (3.2) for our solutions implies

\[ e^{3U} = \frac{1}{6} C_{IJK} h^I h^J h^K. \]  \hspace{1cm} (4.24)

In what follows, some explicit solutions are constructed. We start with the supergravity models where the scalar manifold is a symmetric space. For such theories, the prepotential \( V \) defined in (3.2) is related to the norm forms of degree three Euclidean Jordan algebras. For non-simple Jordan algebras, the corresponding symmetric scalar manifolds \( M \) are given by

\[ M = SO(1,1) \times \frac{SO(n-1,1)}{SO(n-1)} \]  \hspace{1cm} (4.25)

and the prepotential \( V \) factorizes into a linear times a quadratic form in \( n-1 \) scalars and takes the form

\[ V = \frac{1}{2} X^1 \left( \eta_{ab} X^a X^b \right), \quad a, b = 2, \ldots, n. \]  \hspace{1cm} (4.26)

For these models, the following relations hold

\[ X^I = \frac{9}{2} C^{IJK} X_J X_K, \quad G^{IJ} = 2X^I X^J - 6C^{IJK} X_K, \]  \hspace{1cm} (4.27)

where \( C^{IJK} \) is defined by

\[ C^{IJK} = \delta^{II'} \delta^{JJ'} \delta^{KK'} C_{I'J'K'}. \]  \hspace{1cm} (4.28)

Explicitly, the components of \( G_{IJ} \) and its inverse are given by
\[ G_{11} = \frac{9}{2} (X^1)^2, \quad G_{1a} = 0, \quad G_{ab} = \frac{9}{2} X_a X_b - \frac{1}{2} \eta_{ab} X^1, \]
\[ G^{11} = 2 \left( X^1 \right)^2, \quad G^{1a} = 0, \quad G^{ab} = 2 X^a X^b - 6 \eta^{ab} X^1. \]

(4.29)

For the first class of solutions in (4.1), the non-vanishing gauge fields and scalars are given by

\[
F^{1\tau w} = e^{-2U} G^{11} \frac{q_1}{\tau},
\]
\[
F^{a\tau w} = e^{-2U} G^{ab} \frac{q_b}{\tau},
\]
\[
X_1 = \frac{1}{3} e^{-2U} (A_1 + B_1 \tau^{2\alpha_1}),
\]
\[
X_a = \frac{1}{3} e^{-2U} (A_a + B_a \tau^{2\alpha_1}),
\]
\[
X^1 = \frac{1}{2} \eta^{ab} e^{-4U} (A_a + B_a \tau^{2\alpha_1}) (A_b + B_b \tau^{2\alpha_1}),
\]
\[
X^a = \eta^{ab} e^{-4U} (A_1 + B_1 \tau^{2\alpha_1}) (A_b + B_b \tau^{2\alpha_1}). \]

(4.30)

Note that for these models we have

\[
X^1 X_1 = \frac{1}{3}, \quad X^a X_a = \frac{2}{3},
\]

thus we obtain for our solution

\[
e^{6U} = \frac{1}{2} \eta^{ab} (A_a + B_a \tau^{2\alpha_1}) (A_b + B_b \tau^{2\alpha_1}) (A_1 + B_1 \tau^{2\alpha_1}). \]

(4.31)

The conditions (4.10) and (4.12) are satisfied by allowing for two independent charges \( q_1 \) and \( q_a = Q \), and \( A_a = B_a = A \), with

\[
(q_1)^2 = \epsilon \epsilon_4 A_1 a_1^2, \quad Q^2 = \epsilon \epsilon_4 A^2 a_4^2.
\]

(4.32)

Similarly one can construct solutions with two independent charges for the second class of metrics given in (4.14).

Next we consider the \( U(1)^3 \) supergravity with the prepotential \( V = X^1 X^2 X^3 = 1 \) and \( G_{IJ} = \frac{1}{2(X^I)^2} \delta_{IJ} \). We first consider the first class of solutions given in (4.1). Using (4.5), we obtain

\[
X_1 = \frac{1}{3} X^2 X^3 = \frac{1}{3} e^{-2U} H_1,
\]

(4.33)
\[
X_2 = \frac{1}{3} X^1 X^3 = \frac{1}{3} e^{-2U} H_2,
\]

(4.34)
\[
X_3 = \frac{1}{3} X^1 X^2 = \frac{1}{3} e^{-2U} H_3.
\]

(4.35)
where \( H_I = (a_I + b_I \tau^{2a_4}) \). The special coordinates and the metric are given by

\[
X^I = \frac{(H_1 H_2 H_3)^{1/3}}{H_I},
\]

(4.36)

and

\[
ds^2 = (H_1 H_2 H_3)^{1/3} \left( \epsilon_0 \tau^2 + \epsilon_1 \tau^{2a_1} dx^2 + \epsilon_2 \tau^{2a_2} dy^2 + \epsilon_3 \tau^{2a_3} dz^2 \right) + \epsilon_4 (H_1 H_2 H_3)^{-2/3} \tau^{2a_4} dw^2.
\]

(4.37)

Using (4.2), we obtain for the gauge fields (with no summation over the index \( I \))

\[
F^{I\tau w} = \frac{2q_I}{\tau H_I^2} (H_1 H_2 H_3)^{1/3}
\]

(4.38)

with the three independent charges given by \( q_I^2 = \epsilon \epsilon_4 a_I b_I a_1^2 \).

Using the general class of solutions given in (4.14), we obtain for the \( U(1)^3 \) supergravity

\[
ds^2 = (H_1 H_2 H_3)^{2/3} \left( \epsilon_0 \tau^2 + \epsilon_1 \tau^{2a_1} dx^2 + \epsilon_2 \tau^{2a_2} dy^2 \right) + (H_1 H_2 H_3)^{-1/3} \left( \epsilon_3 \tau^{2a_3} dz^2 + \epsilon_4 \tau^{2a_4} dw^2 \right)
\]

and

\[
F^I = q^I dx \wedge dy, \quad X^I = \frac{H^I}{(H_1 H_2 H_3)^{1/3}}
\]

(4.39)

where

\[
H^I = A^I + B^I \tau^{2l}
\]

(4.40)

and the three independent charges are given by

\[
(q^I)^2 = -4 \epsilon \epsilon_0 \epsilon_1 \epsilon_2 l^2 A^I B^I.
\]

(4.41)

As as special case, we can obtain from (4.37) a Lorentzian solution for which all Kasner exponents vanish except for \( a_4 \). This takes the form

\[
ds^2 = (H_1 H_2 H_3)^{1/3} \left( -d\tau^2 + dx^2 + dy^2 + dz^2 \right) + (H_1 H_2 H_3)^{-2/3} \tau^2 dw^2,
\]

(4.42)

where the gauge fields and scalars are as in (4.36) and (4.38) and we choose

\[
H_I = (1 + \epsilon q_I^2 \tau^2).
\]

(4.43)

This solution can be thought of as a Melvin cosmology with non-trivial scalar fields and three independent electric charges. Another special solution can be obtained by analytic
continuation or by simply setting $\epsilon_0 = -\epsilon_1 = \epsilon_2 = \epsilon_3 = a_4 = 1$ in (4.37). This can be given (after relabelling of the coordinates) by

$$ds^2 = (H_1 H_2 H_3)^{1/3} \left(-dt^2 + dr^2 + dy^2 + dz^2\right) + (H_1 H_2 H_3)^{-2/3} r^2 dw^2$$  \hspace{1cm} (4.44)

with

$$H_I = \left(1 + \epsilon q^2 r^2\right), \hspace{1cm} X^I = \frac{(H_1 H_2 H_3)^{1/3}}{H_I}, \hspace{1cm} F^{Irw} = \frac{2q_I}{r H_I^2} (H_1 H_2 H_3)^{1/3}.$$  \hspace{1cm} (4.45)

This solution can be thought of an anisotropic Melvin domain wall with non-trivial gauge and scalar fields. Solutions with cyclic $w$ correspond to five dimensional generalisations of Melvin fluxtubes [22]. One can also obtain from (4.39), magnetically charged cosmological solutions represented by

$$ds^2 = (H_1 H_2 H_3)^{2/3} \left(-d\tau^2 + dx^2 + dy^2\right) + (H_1 H_2 H_3)^{-1/3} (dz^2 + \tau^2 dw^2)$$

$$H^I = 1 + \epsilon \left(q^I\right)^2 4^{-1/2},$$  \hspace{1cm} (4.46)

with bosonic fields as given in (4.40). Static solutions can also be obtained from (4.46) by analytic continuation.

It is known that any solution of the five-dimensional $U(1)^3$ supergravity can be uplifted to a solution of eleven-dimensional supergravity and a solution of type IIB supergravity. The eleven-dimensional solutions take the form [5]

$$ds^2_{11} = ds^2_5 + X^1 (dz_1^2 + dz_2^2) + X^2 (dz_3^2 + dz_4^2) + X^3 (dz_5^2 + dz_6^2)$$  \hspace{1cm} (4.47)

where $ds^2_5$ represents the five-dimensional metric and the coordinates $z^i, i = 1, \ldots, 6$, parametrize $T^6$. The type IIB solution is given by

$$ds^2_{10} = \left(X^3\right)^{1/2} ds^2_5 + \left(X^3\right)^{-3/2} (dz + A^3)^2 + X^1 \left(X^3\right)^{1/2} (dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2)$$  \hspace{1cm} (4.48)

with the dilaton $\phi$ and the Ramond-Ramond 3-form field strength $F_{(3)}$ given by

$$e^{2\phi} = \frac{X^1}{X^2}, \hspace{1cm} F_{(3)} = \left(X^1\right)^{-2} *_{5} F_1 + F_2 \wedge (dz + A^3),$$  \hspace{1cm} (4.49)

with $*_{5}$ being the Hodge dual with respect to $ds^2_5$. 

10
In conclusion, time-dependent and static gravitational solutions for the theory of five-dimensional supergravity with non-trivial scalar and gauge fields in various space-time dimensions have been considered. As in the original metric constructed by Kasner, our solutions depend only on one parameter which we have denoted by \( \tau \). Two classes of solutions were found and some explicit solutions with two and three charges were constructed. Depending on the choice of coordinates, the resulting solutions can be thought of as generalizations of the electric Melvin or magnetic Rosen cosmologies, flux-branes and domain walls. Our results can be generalised to supergravity with various space-time dimensions. Another interesting direction for further investigation is to consider solutions of gauged supergravity theories which have a cosmological constant or a scalar potential. Those solutions will have potential applications to (A)dS/CFT correspondence. We hope to report on this in our future work.

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