DE GIORGI-NASH-MOSER AND HÖRMANDER THEORIES: NEW INTERPLAY

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Abstract. We report on recent results and a new line of research at the crossroad of two major theories in the analysis of partial differential equations. The celebrated De Giorgi-Nash-Moser theory shows Hölder estimates and the Harnack inequality for uniformly elliptic or parabolic equations with rough coefficients in divergence form. The theory of hypoellipticity of Hörmander shows, under general “bracket” conditions, the regularity of solutions to partial differential equations combining first and second order derivative operators when ellipticity fails in some directions. We discuss recent extensions of the De Giorgi-Nash-Moser theory to hypoelliptic equations of Kolmogorov (kinetic) type with rough coefficients. These equations combine a first-order skew-symmetric operator with a second-order elliptic operator involving derivatives in only certain variables, and with rough coefficients. We then discuss applications to the Boltzmann and Landau equations in kinetic theory and present a program of research with some open questions.

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1. Introduction

1.1. Kinetic theory. Modern physics goes back to Newton and classical mechanics, and was later expanded into the understanding of electric and magnetic forces were (Ampère, Faraday, Maxwell), large velocities and large scales (Lorentz, Poincaré, Minkowski, Einstein), small-scale particle physics and quantum mechanics (Planck, Einstein, Bohr, Heisenberg, Born, Jordan, Pauli, Fermi, Schrödinger, Dirac, De Broglie, Bose, etc.). However, all these theories are classically devised to study one physical system (planet, ship, motor, battery, electron, spaceship, etc.) or a small number of systems (planets in the Solar system, electrons in a molecule, etc.) In many situations though, one needs to deal with an assembly made up of elements so numerous that their individual tracking is not possible: galaxies made of hundreds of billions of stars, fluids made of more than $10^{20}$ molecules, crowds made of thousands of individuals, etc. Taking such large numbers
into account leads to new effective laws of physics, requiring different models and concepts. This passage from microscopic rules to macroscopic laws is the founding principle of statistical physics. All branches of physics (classical, quantum, relativistic, etc.) can be studied from the point of view of statistical physics, in both stationary and dynamical perspectives. It was first done with the laws of classical mechanics, which resulted in kinetic theory, discovered by Maxwell [104] and Boltzmann [24] in the 19th century after precursory works by D. Bernoulli, Herapath, Waterston, Joule, König, Clausius.

Kinetic theory replaces a huge number of objects, whose physical states are described by a certain phase space, and whose properties are otherwise identical, by a statistical distribution over that phase space. The fundamental role played by the velocity (kinetic) variable inaccessible to observation was counter-intuitive, and accounts for the denomination of kinetic theory. The theory introduces a distinction between three scales: the macroscopic scale of phenomena which are accessible to observation; the microscopic scale of molecules and infinitesimal constituents; and an intermediate scale, loosely defined and often called mesoscopic. This is the scale of phenomena which are not accessible to macroscopic observation but already involve a large number of particles, so that statistical effects are significant.

1.2. Main equations of kinetic theory. Maxwell wrote the first (weak) form of the evolution equation known now as the Boltzmann equation: the unknown is a (non-negative) density function \( f(t, x, v) \), standing for the density of particles at time \( t \) in the phase space \((x, v)\); the equation, in modern writing and assuming the absence of external forces, is

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f).
\]

The left-hand side describes the evolution of \( f \) under the action of transport, with the free streaming operator. The right-hand side describes elastic collisions, with the nonlinear Boltzmann collision operator:

\[
Q(f, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_s, \omega) \left( f(t, x, v') f(t, x, v'_s) - f(t, x, v) f(t, x, v_s) \right) dv_s d\omega.
\]

Note that this operator is localized in \( t \) and \( x \), quadratic, and has the structure of a tensor product with respect to \( f(t, x, \cdot) \). The velocities \( v' \) and \( v'_s \) should be thought of as the velocities of a pair of particles before collision, while \( v \) and \( v_s \) are the velocities after that collision: the formulas are \( v' = v - (v - v_s, \omega) \omega \) and \( v'_s = v_s + (v - v_s, \omega) \omega \). When one computes \( (v, v_s) \) from \( (v', v'_s) \) (or the reverse), conservation laws of the mass, momentum and energy are not enough to yield the result, with only 4 scalar conservation laws for 6 degrees of freedom. The unit vector \( \omega \in \mathbb{S}^2 \) removes this ambiguity: in the case of colliding hard spheres, it can be thought of as the direction of the line joining the two centers of the particles. The kernel \( B(v - v_s, \omega) \) describes the relative frequency of vectors \( \omega \), depending on the relative impact velocity \( v - v_s \); it only depends on the modulus \( |v - v_s| \) and the deflection angle \( \theta \) between \( v - v_s \) and \( v' - v'_s \). Maxwell computed it for hard spheres, where \( B \sim |v - v_s| \sin \theta \), and for inverse power forces, where the kernel factorizes as \( B \sim |v - v_s|^{\gamma} b(\cos \theta) \). Maxwell showed that if the force is repulsive, proportional to \( r^{-\alpha} \) ( \( r \) the inter-particle distance), then \( \gamma = (\alpha - 5)/(\alpha - 1) \) and \( b(\cos \theta) \simeq \theta^{-(1+2\alpha)} \) as \( \theta \to 0 \), where \( 2s = 2/(\alpha - 1) \). In particular, the kernel is usually non-integrable as
a function of the angular variable: this is a general feature of long-range interactions, nowadays sometimes called “non-cutoff property”.

The case $\alpha = 5$, $\gamma = 0$ and $2s = 1/2$ is called Maxwell molecules \cite{104}, the case $\alpha \in (5, +\infty)$, $\gamma > 0$ and $2s \in (0, 1/2)$ is called hard potentials (without cutoff), the case $\alpha \in [3, 5)$, $\gamma \in [-1, 0)$, $2s \in (1/2, 1]$ is called moderately soft potentials (without cutoff), and finally the case $\alpha \in (2, 3)$, $\gamma \in (-3, -1)$, $2s \in (1, 2)$ is called very soft potentials (without cutoff). The limits between hard and soft potentials ($\gamma = 0$) and between moderately and very soft potentials ($\gamma + 2s = 0$) are commonly taken as defining the “hard” / “moderately soft” / “very soft” terminology in any dimension, for kernel of the form $B = |v - v_s|^\gamma b(\cos \theta)$ with $b(\cos \theta) \simeq \theta^{-(1+2s)}$.

In order to find the stationary solutions, that is, time-independent solutions of \eqref{1.2}, the first step is to identify particular hydrodynamic density functions, which make the collision contribution vanish: these are Gaussian distributions with a scalar co-variance $f(v) = \rho (2\pi T)^{-3/2} e^{-\frac{|v-u|^2}{2T}}$, where the parameters $\rho > 0$, $u \in \mathbb{R}^3$ and $T > 0$ are the local density, mean velocity, and temperature of the fluid. These parameters can be fixed throughout the whole domain (providing in this case an equilibrium distribution), or depend on the position $x$ and time $t$; in both cases collisions will have no effect. As pointed out in Maxwell’s seminar paper, and later proved rigorously at least in some settings \cite{17, 16, 65, 66, 67}, the Boltzmann equation is connected to classical fluid mechanical equations on $\rho, u$ and $T$, and leads to them in certain regimes. This provides a rigorous connection between the mesoscopic (kinetic) level and the macroscopic level. At the other end of the scales, a rigorous derivation of the Boltzmann equation from many-body Newtonian mechanics for short time and short-range interactions was obtained by Lanford \cite{94} for hard spheres; see also \cite{85} for an extension to more general short-range interactions, and the recent works \cite{59, 120} that revisit and complete the initial arguments of Lanford and King. Note however that at the moment the equivalent of Lanford theorem for the Boltzmann equation with long-range interactions is still missing, see \cite{15} for partial progresses.

To summarise the key mathematical points: the Boltzmann equation is an integro-(partial)-differential equation with non-local operator in the kinetic variable $v$. Moreover for long-range interactions with repulsive force $F(r) \sim r^{-\alpha}$, this non-local operator has a singular kernel and shows, as we will see, fractional ellipticity of order $2/(\alpha - 1)$. The Boltzmann equation “contains” the hydrodynamic, and it is a fundamental equation in the sense that it is derived rigorously, at least in some settings, from microscopic first principles. From now on, we consider the position variable in $\mathbb{R}^3$ or in the periodic box $\mathbb{T}^3$.

In the limit case $s \to 1$ (the Coulomb interactions), the Boltzmann collision operator is ill-defined. Landau \cite{93} proposed an alternative operator for these Coulomb interactions that is now called the Landau-Coulomb operator

$$Q(f, f) = \nabla_v \cdot \left( \int_{\mathbb{R}^3} P_{(v-v_s)^\perp} \left( f(t, x, v_s) \nabla_v f(t, x, v) - f(t, x, v) \nabla_v f(t, x, v_s) \right) |v - v_s|^\gamma d\nu(v_s) \right)$$

where $P_{(v-v_s)^\perp}$ is the orthogonal projection along $(v - v_s)^\perp$ and $\gamma = -3$. It writes also

$$Q(f, f) = \nabla_v \cdot \left( A[f] \nabla_v f + B[f] f \right)$$

where

$$A[f] = \frac{1}{|v - v_s|^\gamma} \cdot \left( f(t, x, v_s) \nabla_v f(t, x, v) - f(t, x, v) \nabla_v f(t, x, v_s) \right)$$

$$B[f] = \frac{\gamma}{|v - v_s|^\gamma}$$

and $A[f]$ is the fundamental solution of the Boltzmann equation.
with \[
\begin{align*}
A[f](v) &= \int_{\mathbb{R}^3} \left( I - \frac{w}{|w|} \otimes \frac{w}{|w|} \right) |w|^\gamma + 2 f(t, x, v - w) \, dw, \\
B[f](v) &= -\int_{\mathbb{R}^3} |w|\gamma w f(t, x, v - w) \, dw.
\end{align*}
\]

This operator is a nonlinear drift-diffusion operator with coefficients given by convolution-like averages of the unknown. This is a non-local integro-differential operator, with second-order local ellipticity. The resulting Landau equation (1.1)-(1.3) again “contains” the hydrodynamic. It is also considered fundamental because of its close link to the Boltzmann equation for Coulomb interactions (note however that the equivalent to Lanford theorem for the Landau equation is lacking, even at a formal level, see [23] for partial progresses). Because of the difficulty to handle the very singular kernel of the Landau-Coulomb operator, it is common to introduce artificially a scale of models by letting \( \gamma \) vary in \([-3, 1]\) (or even \([-d, 1]\) in general dimension \(d\)). The terminology hard potentials, Maxwell molecules, soft potentials are used as for the Boltzmann collision operator when \( \gamma > 0, \gamma = 0, \gamma < 0 \) respectively. The terminology moderately soft potentials corresponds here (since \( s = 1 \)) to \( \gamma \in (-2, 0) \).

### 1.3. Open problems and conjectures.

#### 1.3.1. The Cauchy problem.

The first mathematical question when studying the previous fundamental kinetic equations (Boltzmann and Landau equations) is the Cauchy problem, i.e. existence, uniqueness and regularity of solutions. Short-time solutions have been constructed, as well as global solutions close to the trivial stationary solution or with space homogeneity: see [72] and the references therein for some of the most recent results for the Boltzmann equation with short-range interactions, see [12, 10, 8, 11, 70] for the Boltzmann equation with long-range interactions, and see [73] for the Landau equation. However the construction of solutions “in the large” remains a formidable open problem. Since weak “renormalised” solutions have been constructed by DiPerna and Lions [55] that play a similar role to the Leray [95] solutions in fluid mechanics, this open problem can be compared with the millennium problem of the regularity of solutions to 3D incompressible Navier-Stokes equations.

#### 1.3.2. Study of a priori solutions.

Given that the Cauchy problem in the large seems out of reach at the moment, Truesdell and Muncaster [127] remarked almost 40 years ago that: “Much effort has been spent toward proof that place-dependent solutions exist for all time. [...] The main problem is really to discover and specify the circumstances that give rise to solutions which persist forever. Only after having done that can we expect to construct proofs that such solutions exist, are unique, and are regular.” In other words, the \( H \)-theorem and the mathematical understanding of irreversibility are so important in the theory of Maxwell and Boltzmann that it cannot wait for the tremendously difficult issue of global well-posedness to be settled. Cercignani then formulated a precise conjecture along this idea, postulating in [34] a linear relation between the entropy production functional and the relative entropy functional of any a priori given classical solutions. The resolution of this conjecture, for certain interactions, lead to precise new quantitative information on a priori solutions of the Boltzmann and Landau equation (see [51, 48, 111, 72, 83]). And it lead to the related question of the optimal relaxation rates of a priori solutions, with minimal regularity and moments conditions. It is now fairly well
understood for many interactions. The results obtained along this line of research can all 
be summarised into the following general form:

**Conditional relaxation.** Any solution to the Boltzmann (resp. Landau) equation in 
$L^\infty(\mathbb{T}^3, L^1(\mathbb{R}^3, (1+|v|)^k \, dv))$, $k > 2$ (or a closely related functional space as large as possible) converges to the thermodynamic equilibrium with the optimal rate dictated by 
the linearized equation.

Note however that an interesting remaining open question in this program is to obtain 
a result equivalent to [72, 33] in the case of the Boltzmann equation with long-range 
interactions (with fractional ellipticity in the velocity variable).

### 1.3.3. Regularity conjectures for long-range interactions.

In the case of long-range interactions, the Boltzmann and Landau-Coulomb operators show local ellipticity provided the solution enjoys some pointwise bounds on the hydrodynamic fields $\rho(t,x) := \int_{\mathbb{R}^3} f \, dv$, 
e(t,x) := \int_{\mathbb{R}^3} |v|^2 \, dv$ and the local entropy $h(t,x) := \int_{\mathbb{R}^3} f \ln f \, dv$. Whereas it is clear in the case of the Landau-Coulomb operator, it was understood almost two decades ago in the case of the Boltzmann collision operator [97, 129, 2]. This had lead colleagues working on non-local operators and fully nonlinear elliptic problems like Silvestre and Guillen and co-authors to attempt to use barriers’ techniques reminiscent to the Krylov-Safonov theory [89] in order to obtain pointwise bounds for solutions to these equations. These first attempts, while unsuccessful, later proved crucial in attracting the attention of a larger community on this problem. And these authors rapidly reformulated the initial goal into, again, *conditional conjectures* on the regularity of the form:

**Conditional regularity.** Consider any solution to the Boltzmann equation with long-range interactions (resp. Landau equation) on a time interval $[0,T]$ such that its hydrodynamic fields are bounded:

\begin{equation}
\forall t \in [0,T], \, x \in \mathbb{T}^3, \quad m_0 \leq \rho(t,x) \leq m_1, \quad e(t,x) \leq e_1, \quad h(t,x) \leq h_1
\end{equation}

where $m_0, m_1, e_1, h_1 > 0$. Then the solution is bounded and smooth on $(0,T]$.

Note that this conjecture can be strengthened by removing the assumption that the mass is bounded from below and replacing it by a bound from below on the total mass $\int_{\mathbb{T}^3} \rho(t,x) \, dx \geq M_0 > 0$. Mixing in velocity through collisions combined with transport effects indeed generate lower bounds in many settings, see [110, 58, 28, 27]; moreover it was indeed proved for the Landau equation with moderately soft potentials in [76].

This conjecture is now been partially solved in the case of the Landau equation, when the interaction is “moderately soft” $\gamma \in (-2,0)$. This result has been the joint efforts of several groups [62, 75, 76, 81], and this is the object of the next section. It is currently an ongoing program of research in the case of the Boltzmann equation with hard and moderately soft potentials, and this is the object of the fourth and last section. The conjecture interestingly remains open in the case of very soft potentials for both equations, and making progress in this setting is likely to require new conceptual tools.

### 2. De Giorgi-Nash-Moser meet Hörmander

#### 2.1. The resolution of Hilbert 19-th problem.

The De Giorgi-Nash-Moser theory [41, 42, 112, 108, 109] was born out of the attempts to answer Hilbert’s 19th problem. This problem is about proving the analytic regularity of the minimizers $u$ of an energy
functional $\int_U L(\nabla u) \, dx$, with $u : \mathbb{R}^d \to \mathbb{R}$ and where the Lagrangian $L : \mathbb{R}^d \to \mathbb{R}$ satisfies growth, smoothness and convexity conditions and $U \subset \mathbb{R}^d$ is a compact domain. The Euler-Lagrange equations for the minimizers take the form

$$\nabla \cdot \nabla L(\nabla u) = 0 \quad \text{i.e.} \quad \sum_{i,j=1}^d \frac{\partial_{ij} L(\nabla u)}{b_{ij}} \partial_{ij} u = \sum_{i,j=1}^d b_{ij} \partial_{ij} u = 0.$$ 

For instance the Dirichlet energy $L(p) = |p|^2$ leads to linear Euler-Lagrange equations, whereas the minimal surface energy $L(p) = \sqrt{1 + |p|^2}$ leads to nonlinear Euler-Lagrange equations. With suitable assumptions on $L$ and the domain, the pointwise control of $\nabla u$ was known in the 1950s. However applying the Schauder estimates to get higher regularity requires more information: if $u \in C^{1,\alpha}$ with $\alpha > 0$ then $b_{ij} \in C^\alpha$ and Schauder estimates [122] imply $u \in C^{2,\alpha}$; a bootstrap argument then yields higher regularity, and analyticity follows from this $C^\infty$ regularity [19, 115].

Hence, apart from specific result in two dimensions [107], the missing piece in solving Hilbert 19th problem, in the 1950s, was the proof of the Hölder regularity of $\nabla u$. The equation satisfied by a derivative $f := \partial_k u$ is the divergence form elliptic equation:

$$\sum_{i,j=1}^d \partial_i \left[ \left( \frac{\partial_{ij} L(\nabla u)}{a_{ij}} \right) \partial_j f \right] = \nabla \cdot (A \nabla f) = 0.$$ 

De Giorgi [42] and Nash [112] independently proved this Hölder regularity of $f$ under the sole assumption that the symmetric matrix $A := (a_{ij})$ satisfies the controls $0 < \lambda \leq A \leq \Lambda$, and is measurable (no regularity is assumed). The proof of Nash uses what is now called the “Nash inequality”, an $L \log L$ energy estimate, and refined estimates on the fundamental solution. The proof of De Giorgi uses an iterative argument to gain integrability, and an “isoperimetric argument” to control how oscillations decays when refining the scale of observation. Moser later gave an alternative proof [108, 109] based on one hand on an iterative gain of integrability, formulated differently but similar to that of De Giorgi, and on the other hand on relating Lebesgue norms on $f$ and $1/f$ through energy estimates on the equation satisfied by $g := \ln f$ and the use of a Poincaré inequality; the proof of Moser had an important further contribution in that it also proved, as an intermediate step towards the Hölder regularity, the Harnack inequality for the equation considered, i.e. a universal control on the ratio between local maxima and local minima.

Let us mention that the De Giorgi-Nash-Moser (DGNM) theory only considers elliptic or parabolic equations in divergence form. An important counterpart result for non-divergence elliptic and parabolic equations was later discovered by Krylov and Safonov [89]. The extension of the DGNM theory to hypoelliptic equations with rough coefficients that we present in this section requires the equation to be in divergent form. It is an open problem whether the Krylov-Safonov theory extends to hypoelliptic non-divergent equations of the form discussed below.

2.2. The theory of hypoellipticity. The DGNM theory has revolutionised the study of nonlinear elliptic and parabolic partial differential equations (PDEs). However it remained limited to PDEs where the diffusion acts in all directions of the phase space.
In kinetic theory, as soon as the solution is non spatially homogeneous, the diffusion or fractional diffusion in velocity is combined to a conservative Hamiltonian dynamic in position and velocity. This structure is called **hypoelliptic**.

The study of regularity properties of such equations can be traced back, at the linear level, to the short note of Kolmogorov [86]. This note considered the combination of free transport with drift-diffusion in velocity: the law satisfies what is now sometimes called the **Kolmogorov equation**, that writes

\[
\partial_t f + v \cdot \partial_x f = \Delta_v f
\]

on \(x, v \in \mathbb{R}^d\) in the simpler case. It is the equation satisfied by the law of a Brownian motion integrated in time. Kolmogorov then wrote the fundamental solution associated with a Dirac distribution \(\delta_{x_0,v_0}\) initial data:

\[
G(t, x, v) = \left(\frac{\sqrt{3}}{2\pi t^2}\right)^d \exp \left[ -\frac{3|x - x_0 - tv_0 - t(v - v_0)/2|^2}{t^3} - \frac{|v - v_0|^2}{4t} \right].
\]

The starting point of Hörmander’s seminal paper [77] is the observation that this fundamental solution shows regularisation in all variables, even though the diffusion acts only in the velocity variable. The regularisation in \((t, x)\) is produced by the interaction between the transport operator \(v \cdot \nabla_x\) and the diffusion in \(v\). Hörmander’s paper then proposes general geometric conditions for this regularisation, called **hypoelliptic**, to hold, based on commutator estimates. In short, given \(X_0, X_1, \ldots, X_n\) a collection of smooth vector fields on \(\mathbb{R}^N\) and the second-order differential operator \(L = -\frac{1}{2} \sum_{i=1}^n X_i^* X_i + X_0\), then the semigroup \(e^{tL}\) is regularising (hypoelliptic) as soon as the Lie algebra generated by \(X_0, \ldots, X_n\) has dimension \(N\) throughout the domain of \(L\).

Let us also mention the connexion with the **Malliavin calculus** in probability, which gives a probabilistic proof to the Hörmander theorem in many settings, see [101] as well as the many subsequent works, for instance [90, 91, 20, 113].

### 2.3. Extending the DGNM theory to hypoelliptic settings.

The main question of interest here is the extension of the DGNM theory to hypoelliptic PDEs of divergent type. Hypoelliptic PDEs of second order \(L = -\frac{1}{2} \sum_{i=1}^n X_i^* X_i + X_0\) naturally split into two classes: the simpler “type I” when \(X_0 = 0\) and the operator is a sum of squares, and the “type II” such as the Kolmogorov equation above, where \(X_0 \neq 0\) and the operator combines a first-order anti-symmetric operator with some partially diffusive second-order operator. Two main research groups had already been working on the question.

The extension of the DGNM theory to hypoelliptic operators of “type I” is relatively straightforward. Regarding the “type II”, Polidoro and collaborators [116] had obtained the Hölder regularity for coefficients with various continuity assumptions, and had obtained the improvement of integrability and pointwise bound for measurable coefficients (see also the isolated result [53] for equations in non-divergence form). Wang and Zhang [133, 134, 137] had extended the proof of Moser for the “type II” equations to obtain Hölder regularity, with technical calculations that did not seem easy to export. Note also that the use of the DGNM theory in kinetic theory had also been advocated almost a decade before in the premonitory lecture notes [131].

We present here the work [62] (see also the two previous related preprints [68, 80]) that (1) provides an elementary and robust proof of the gain of integrability and Hölder regularity in this “type II” hypoelliptic setting, (2) proves the stronger **Harnack inequality** for these equations (i.e. a quantitative version of the strong maximum principle).
Let us consider the following kinetic Fokker-Planck equation
\[ \partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A \nabla_v f) + B \cdot \nabla_v f + s, \quad t \in (0, T), \ (x, v) \in \Omega, \]
where \( \Omega \) is an open set of \( \mathbb{R}^d \), \( f = f(t, x, v) \), \( B \) and \( s \) are bounded measurable coefficients depending on \( (t, x, v) \), and the \( d \times d \) real matrices \( A \) and source term \( s \) are measurable and satisfy
\[ 0 < \lambda I \leq A \leq \Lambda I, \quad |B| \leq \Lambda, \quad s \text{ essentially bounded} \]
for two constants \( \lambda, \Lambda > 0 \). Given \( z_0 = (t_0, x_0, v_0) \in \mathbb{R}^{2d+1} \), we define the “cylinder” \( Q_r(z_0) \) centered at \( z_0 \) of radius \( r \) that respects the invariances of the equation:
\[ Q_r(z_0) := \left\{ (t, x, v) \in \mathbb{R}^{2d+1} : |x - x_0 - (t - t_0)v_0| < r^2, |v - v_0| < r, t \in (t_0 - r^2, t_0) \right\}. \]

The weak solutions to equation (2.1) on \( I \times U_x \times U_v \) with \( U_x \subset \mathbb{R}^d \) open, \( U_v \subset \mathbb{R}^d \) open, \( I = [a, b] \) with \( -\infty < a < b < +\infty \), are defined as functions \( f \in L^\infty_t(L^2(I \times U_x, H^1(U_v))) \cap L^2_t(I \times U_x, H^1(U_v)) \) such that \( \partial_t f + v \cdot \nabla_x f \in L^2_t(I \times U_x, H^1(U_v)) \) and \( f \) satisfies the equation (2.1) in the sense of distributions.

**Theorem 1** (Hölder continuity \[62\]). Let \( f \) be a weak solution of (2.1) in \( Q_0 := Q_{t_0}(z_0) \) and let \( Q_1 := Q_{r_1}(z_0) \) with \( r_1 < r_0 \). Then \( f \) is \( \alpha \)-Hölder continuous with respect to \((x, v, t)\) in \( Q_1 \) and
\[ \|f\|_{C^{\alpha}(Q_1)} \leq C \left( \|f\|_{L^2(Q_0)} + \|s\|_{L^\infty(Q_0)} \right) \]
for some \( \alpha \in (0, 1) \) and \( C > 0 \) only depending on \( d, \lambda, \Lambda, r_0, r_1 \) (plus \( z_0 \) for \( C \)).

In order to prove such a result, we first prove that \( L^2 \) sub-solutions are locally bounded; we refer to such a result as an \( L^2 - L^\infty \) estimate. We then prove that solutions are Hölder continuous by proving a lemma which is a hypoelliptic counterpart of De Giorgi’s “isoperimetric lemma”.

We moreover prove the Harnack inequality:

**Theorem 2** (Harnack inequality \[62\]). If \( f \) is a non-negative weak solution of (2.1) in \( Q_1(0, 0, 0) \), then
\[ \sup_{Q^-} f \leq C \left( \inf_{Q^+} f + \|s\|_{L^\infty(Q_1(0, 0, 0))} \right) \]
where \( Q^+ := Q_R(0, 0, 0) \) and \( Q^- := Q_R(0, 0, -\Delta) \) and \( C > 1 \) and \( R, \Delta \in (0, 1) \) are small (in particular \( Q^2 \subset Q_1(0, 0, 0) \) and they are disjoint), and universal, i.e. only depend on dimension and ellipticity constants.

Note that using the transformation \( T_{z_0}(t, x, v) = (t_0 + t, x_0 + x + tv_0, v_0 + v) \), we get a Harnack inequality for cylinders centered at an arbitrary point \( z_0 = (t_0, x_0, v_0) \).

Our proof combines the key ideas of De Giorgi and Moser and the velocity averaging method, which is a special type of smoothing effect for solutions of the free transport equation \( (\partial_t + v \cdot \nabla_x) f = S \) observed for the first time in \[11 \[64\] \) independently, later improved and generalized in \[63 \[54\]. This smoothing effect concerns averages of \( f \) in the velocity variable \( v \), i.e. expressions of the form \( \int_{\mathbb{R}^d} f(t, x, v) \phi(v) \, dv \) with, say, \( \phi \in C_c^\infty \). Of course, no smoothing on \( f \) itself can be observed, since the transport operator is hyperbolic and propagates the singularities. However, when \( S \) is of the form
\[ S = \nabla_v \cdot (A(t, x, v) \nabla_v f) + s, \]

...
where \( s \) is a given source term in \( L^2 \), the smoothing effect of velocity averaging can be combined with the \( H^1 \) regularity in the \( v \) variable implied by the energy inequality in order to obtain some regularity in all directions. A first observation of this type (at the level of a compactness argument) can be found in [95]; Bouchut [25] had then obtained quantitative Sobolev regularity estimates.

Our proof of the \( L^2 - L^\infty \) gain of integrability follows the so-called “De Giorgi-Moser iteration”, see [62] where it is presented in both the equivalent formulations of De Giorgi and of Moser. We emphasize that, in both approaches, the main ingredient is a local gain of integrability of non-negative sub-solutions. This latter is obtained by combining a comparison principle and a fractional Sobolev regularity estimate following from (1) the velocity averaging method discussed above and (2) energy estimates. We then prove the Hölder continuity through a De Giorgi type argument on the decrease of oscillation for solutions. We also derive the Harnack inequality by combining the decrease of oscillation with a result about how positive lower bounds on non-negative solutions deteriorate with time. It is worth mentioning here that our “hypoelliptic isoperimetric argument” is proved non-constructively, by a contradiction method, whereas the original isoperimetric argument of De Giorgi is obtained by a quantitative direct argument. It is an interesting open problem to obtain such quantitative estimates in the hypoelliptic case.

3. Conditional regularity of the Landau equation

3.1. Previous works and a conjecture. The infinite smoothing of solutions to the Landau equation has been investigated so far in two different settings. On the one hand, it has been investigated for weak spatially homogeneous solutions (non-negative in \( L^1 \) and with finite energy), see [52] and the subsequent follow-up papers [4, 78, 7, 5, 106, 14, 45, 128, 49], and see also the related entropy dissipation estimates in [50, 46], and see the analytic regularisation of weak spatially homogeneous solutions for Maxwellian or hard potentials in [55]. Furthermore, Silvestre [124] derives an \( L^\infty \) bound (gain of integrability) for spatially homogeneous solutions in the case of moderately soft potentials without relying on energy methods. Let us also mention works studying modified Landau equations [87, 69] and the work [71] that shows, using barrier arguments, that any weak radial solution to the Landau-Coulomb equation that belongs to \( L^{3/2} \) is automatically bounded and \( C^2 \). On the other hand, fewer investigations of the regularity of spatially heterogeneous solutions have been done, focusing on the regularisation of classical solutions [36, 98].

The general question of conditional regularity hence suggests the following question in the context of the Landau equation:

**Conjecture 1.** Any solutions to the Landau equation (1.1)-(1.3) (with Coulomb interaction \( \gamma = -3 \)) on \([0, T]\) satisfying (1.4) is bounded and smooth on \([0, T]\).

An important progress has been made by solving a weaker version of this conjecture when the exponent \( \gamma \in (-2, 0) \), which corresponds to *moderately soft potentials*, i.e. \( \gamma + 2s > 0 \) since here \( s = 1 \). We describe in this section the different steps and combined efforts of different groups.

3.2. DGNM theory and local Hölder regularity. The first step is the work [62] already mentioned. A corollary of Theorem [1] is the following:
Theorem 3 (Local Hölder regularity for the LE [62]). Given any $\gamma \in [-3, 1]$, there are universal constants $C > 0$, $\alpha \in (0, 1)$ such that any essentially bounded weak solution $f$ of \((1.1)-(1.3)\) in $(-1, 0) \times B_1 \times B_1$ satisfying \((1.4)\) is $\alpha$-Hölder continuous with respect to $(t, x, v) \in (-1/2, 0] \times B_{1/2} \times B_{1/2}$ and

$$
\|f\|_{C^\alpha(-1/2,0) \times B_{1/2} \times B_{1/2}} \leq C \left( \|f\|_{L^2((-1,0) \times B_1 \times B_1)} + \|f\|_{L^\infty((-1,0) \times B_1 \times B_1)} \right).
$$

Note that this theorem includes the physical case of Coulomb interactions $\gamma = -3$. The adjective “universal” for the constants refers to their independence from the solution.

3.3. Maximum principles and pointwise bounds. This line of research originates in the work of Silvestre both on the spatially homogeneous Boltzmann (SHBE) and Landau (SHLE) equations [123, 125]. These papers build upon the ideas of “nonlinear maximum principles” introduced in [40] in the case of the Boltzmann collision operator, and upon the so-called “Aleksandrov-Bakelman-Pucci Maximum Principle” in the case of the Landau collision operator, see for instance [31, 30].

The main result of [125] is:

Theorem 4 (Pointwise bound for the SHLE). Let $\gamma \in [-2, 0]$ (moderately soft potentials) and let $f$ be a classical non-negative spatially homogeneous solution to the Landau equation \((1.1)-(1.3)\) on $[0, T] \times \mathbb{R}^d$ for some $T > 0$, and satisfying the assumptions \((1.4)\). Then $f \lesssim 1 + t^{-3/2}$ with constant depending only on the bounds \((1.4)\).

As noted by the author, this estimate implies quite straightforwardly existence, uniqueness and infinite regularity for the spatially homogeneous solution. For the difficult case of very soft potentials $\gamma \in [-3, 0)$, this paper includes a weaker result where the $L^\infty$ bound depends on a certain weighted Lebesgue norm; unfortunately it is not yet known exactly how to control such norm along time. This conceptual barrier, when crossing the “very soft potentials threshold”, is reminiscent of the state of the art for the Cauchy theory in Lebesgue and Sobolev spaces by energy estimates, for both the spatially homogeneous Boltzmann with long-range interactions [47] and Landau equation [6, 136].

The pointwise bounds estimates were then extended to the spatially inhomogeneous case in [32]. The main result in this paper is:

Theorem 5 (Pointwise bound for the LE). Let $\gamma \in (-2, 0]$ (moderately soft potentials without the limit case) and let $f$ be a bounded non-negative weak solution to the Landau equation \((1.1)-(1.3)\) on $[0, T] \times \mathbb{R}^{2d}$ for some $T > 0$, satisfying the assumptions \((1.4)\). Then $f \lesssim (1 + t^{-3/2})(1 + |v|)^{-1}$ with constant depending only on the bounds \((1.4)\) (and not on the $L^\infty$ norm of the solution). Moreover if $f_{\text{in}}(x, v) \leq C_0 e^{-\alpha |v|^2}$, for some $C_0 > 0$ and a sufficiently small $\alpha > 0$ (depending on $\gamma$ and \((1.4)\)), then $f(t, x, v) \leq C_1 e^{-\alpha |v|^2}$ with $C_1 > 0$ depending only on $C_0$, $\gamma$ and the bounds \((1.4)\).

The proof relies on using locally the Harnack inequality in Theorem 2 adapted to the Landau equation and on devising a clever change of variable to track how this local estimate behaves at large velocities. The Gaussian bound is then obtained by combining existing maximum principle arguments at large velocities (using that well-constructed Gaussians provide supersolutions at large $v$) in the spirit of [60], and the previous pointwise bound for not-so-large velocities. Finally the authors remarked that the Hölder regularity estimate of Theorem 3 can be made global using the Gaussian decay bound.
3.4. Schauder estimates and higher regularity. Once the $L^\infty$ norm and the Hölder regularity is under control, the next step is to obtain higher-order regularity. The classical tool is the so-called Schauder estimates \[122\]. The purpose of such estimates in general is to show that the solution to an elliptic or parabolic equation whose coefficients are Hölder continuous gains two derivatives with respect to the data (source term, initial data). The gain of the two derivatives is obtained in Hölder spaces: $C^\delta \to C^{2+\delta}$.

Two works have been obtained independently along this line of research. The first one \[75\] focuses on the use of combination of Hölder estimates, maximum principles and Schauder estimates to obtain conditional infinite regularity for solutions to the Landau equation with moderately soft potentials $\gamma \in (-2,0)$. The second one \[81\] focuses on the use of these ingredients to “break the super-criticality” of the nonlinearity for a toy model of the Landau equation. Both these works develop, in different technical ways, Schauder estimates for this hypoelliptic equation. The main result in \[75\] is:

**Theorem 6** (Conditional regularity for LE). Let $\gamma \in (-2,0)$ (moderately soft potentials without the limit case) and let $f$ be a bounded non-negative weak solution to the Landau equation \[1.1\] on $[0,T] \times \mathbb{R}^d$ for some $T > 0$, satisfying the assumptions \[1.4\] and \[f_{in}(x,v) \leq C_0 e^{-\alpha |v|^2}\] for some $C_0 > 0$ and a sufficiently small $\alpha > 0$ (depending on $\gamma$ and \[1.4\]). Then $f$ is smooth and its derivatives have some (possibly weaker) Gaussian decay.

Note that: (1) the regularity and decay bounds are uniform in time, as long as the bounds \[1.4\] remain uniformly bounded in time, (2) further conditional regularity are given in the paper for very soft potentials $\gamma \in [-3,-2]$ but they require higher $L^\infty_t L^1_v (1+|v|)^q$ moments and the constants depend on time when $\gamma \in [-3,-5/2]$ in dimension 3, (3) a useful complementary result is provided by \[76\] where a local existence is proved in weighted locally uniform Sobolev spaces and the lower bound on the mass is relaxed by using the regularity to find a ball where the solution is uniformly positive: the combination of the two papers provide a conditional existence, uniqueness and regularity result for moderately soft potentials, conditionally to upper bounds on the local mass, energy and entropy.

The work \[81\] considers the toy model:

\[\partial f + v \cdot \nabla_x f = \rho[f] \nabla_v (\nabla_v f + vf) , \quad \rho[f] := \int_{\mathbb{R}^d} f \, dv,\]

in $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$, $d \geq 1$. This model preserves the form of the steady state, the ellipticity in $v$, the non-locality, the bilinearity and the mass conservation of the LE. It however greatly simplifies the underlying hydrodynamic and the maximum principle structure. Here $H^k(\mathbb{T}^d \times \mathbb{R}^d)$ denotes the standard $L^2$-based Sobolev space. The main result states (note that solutions are constructed and not conditional here):

**Theorem 7.** For all non-negative initial data $f_{in}$ such that $f_{in}/\sqrt{\mu} \in H^k(\mathbb{T}^d \times \mathbb{R}^d)$ with $k > d/2$ and satisfying $C_1 \mu \leq f_{in} \leq C_2 \mu$ for some $C_1, C_2 > 0$, there exists a unique global-in-time solution $f$ to \[3.1\] with initial data $f_{in}$ satisfying for all time $t > 0$: $f(t)/\sqrt{\mu} \in H^k(\mathbb{T}^d \times \mathbb{R}^d)$ and $C_1 \mu \leq f \leq C_2 \mu$ and $f(t,\cdot,\cdot) \in C^\infty$.

Note that the initial regularity exponent $k$ could be relaxed with more work. A key step of the proof is the Schauder estimate. It gives the following additional information...
on this solution: the hypoelliptic Hölder norm $\mathcal{H}^\alpha$ (defined below) of $f / \sqrt{\mu}$ is uniformly bounded in terms of the $L^2$ norm of $f_{in} / \sqrt{\mu}$ for times away from 0. This norm is defined on a given open connected set $Q$ by
\[
\|g\|_{\mathcal{H}^\alpha(Q)} := \sup_Q |g| + \sup_Q \left(|(\partial_t + v \cdot \nabla_x)g| + \sup_Q \left|D^2_v g\right| + \left|[(\partial_t + v \cdot \nabla_x)g]_{C^{0,\alpha}(Q)} + [D^2_v g]_{C^{0,\alpha}(Q)}\right|\right)
\]
where $[\cdot]_{C^{0,\alpha}(Q)}$ is a Hölder anisotropic semi-norm, i.e. the smallest $C > 0$ such that
\[
\forall z_0 \in Q, \quad r > 0 \quad \text{s.t.} \quad Q_r(z_0) \subset Q, \quad \|g - g(z_0)\|_{L^\infty(Q_r(z_0))} \leq Cr^\alpha
\]
where
\[
Q_r(z_0) := \left\{z : \frac{1}{r}(z_0^{-1} \circ z) \in Q_1\right\}
\]
and $rz := (r^2t, r^3x, rv)$ and $z_1 \circ z_2 := (t_1 + t_2, x_1 + x_2 + t_2 v_1, v_1 + v_2)$.

The specific contribution of this work is the study of the Cauchy problem: the maximum principle provides Gaussian upper and lower bounds on the solution, and we then provide energy estimates and a blow-up criterion à la Beale-Kato-Majda [18]. We then use the extensions of the DGNM and Schauder theories to control the $L^\infty_x(H^1_v)$ type norm that governs the blow-up. We prove Hölder regularity through the method of [32]. We then develop Schauder estimates following the method of [88] (see also [116, 102, 53, 26, 100, 121, 75]). New difficulties arise compared with the parabolic case treated in [88] in relation with the hypoelliptic structure and we develop hypoelliptic commutator estimates directly at the level of trajectories to solve them. We also borrow some ideas from hypocoercivity [132] in the proof of the so-called gradient estimate.

Note that it would be interesting to give a proof of Schauder estimates for such hypoelliptic equations that is entirely based on scaling arguments in the spirit [126] (see also the use of such scaling arguments in [74], in the elliptic-parabolic case). This might indeed prove useful for generalising such estimates to the integral Boltzmann collision operator, see the next section.

4. Conditional regularity of the Boltzmann equation

4.1. Previous works and a conjecture. Short time existence of solutions to (1.1)-(1.2) was obtained in [9] for sufficiently regular initial data $f_0$. Global existence was obtained in [47] for moderately soft potentials in the spatially homogeneous case. In the next subsections, we present the progresses made so far in the case of moderately soft potentials: the estimate in $L^\infty$ for $t > 0$ was obtained in [123], the local Hölder regularity in [83], and finally the polynomial pointwise decay estimates in [82]. The bootstrap mechanism to obtain higher regularity through Schauder estimates remains however unsolved at now.

Let us briefly review the existing results about regularisation. The very first mathematical observation that long-range interactions are associated with fractional ellipticity in the kinetic variable goes back to Desvillettes [44] in the mid 1990s. In [9], the authors prove that if the solution $f$ has five derivatives in $L^2$, with respect to all variables $t, x$ and $v$, weighted by $(1 + |v|)^q$ for arbitrarily large powers $q$, and in addition the mass density is bounded below, then the solution $f$ is $C^\infty$. It is not known however
whether these hypotheses are implied by (1.4). Note also the previous partial results
in the spatially homogeneous case and with less assumptions on
the initial data, and the work in the spatially inhomogeneous case but with much
stronger a priori assumptions.

Note that, drawing inspiration from the case of the Landau equation, in order for the
iterative gain of regularity in to work, it is necessary to start with a solution
that decays, as \(|v| \to \infty\), faster than any algebraic power rate \(|v|^{-\beta}\). We expect the same
general principle to apply to the Boltzmann equation, even if the appropriate Schauder
type estimates for kinetic integro-differential equations to carry out an iterative gain in
regularity are not yet available.

The question of conditional regularity suggests the following conjecture in the context
of the Boltzmann equation with long-range interactions:

**Conjecture 2.** Any solutions to the Boltzmann equation (1.1)-(1.2) with long-range
interactions (\(\gamma \in (-3,1)\), \(s \in (0,1)\), \(\gamma + 2s \in (-1,1)\)) on \([0,T]\) satisfying (1.4) is bounded
and smooth on \((0,T]\).

The rest of this section is devoted to describing the partial progresses made in the case
of, again, moderately soft potentials \(\gamma + 2s > 0\).

4.2. **Maximum principle and pointwise \(L^\infty\) bound.** This first breakthrough is due
to Silvestre [123]. This article draws inspiration from his own previous works on non-local
operators and from the “nonlinear maximum principle” of Constantin and Vicol [10]. It
is based on a maximum principle argument for a barrier supersolution that is constant
in \(x,v\) and blowing-up as \(t \to 0^+\); it uses the decomposition of the collision operator
and “cancellation lemma” going back to [3], the identification of a cone of direction for
\((v' - v)\) is order to obtain lower bounds on the \(f\)-dependent kernel of the elliptic part of the
operator, and finally some Chebycheff inequality and nonlinear lower bound on the
collision integral. The main result is:

**Theorem 8 (Pointwise bound for the BE [123]).** Let \(\gamma \in [-2,1]\), \(s \in (0,1)\) with \(\gamma + 2s > 0\)
(moderately soft potentials). Let \(f\) be a classical non-negative solution to the Boltzmann
equation (1.1)-(1.3) on \([0,T] \times \mathbb{T}^d \times \mathbb{R}^d\) for some \(T > 0\), satisfying the assumptions (1.4).
Then \(f \leq C(1 + t^{-\beta})\) with \(C > 0\) and \(\beta > 0\) depending only on \(\gamma\), \(s\) and the bounds (1.4).

Note that the paper also includes further results in the case of very soft potentials but
conditionally to additional estimates of the form \(L^\infty_{t,x} L^p_v(1 + |v|^q)\) for some \(p > 1\), \(q > 0\);
it is not known at present how to deduce the latter estimates from the hydrodynamic
bounds (1.4).

4.3. **Weak Harnack inequality and local Hölder regularity.** The second break-
through is the paper [33]. In comparison to the Landau equation, the Boltzmann equation
has a more complicated integral structure, that shares similarity with “fully nonlinear”
fractional elliptic operators. The main result proved is:

**Theorem 9 (Local Hölder regularity for the BE [33]).** Given any \(\gamma \in (-3,1)\) and
\(s \in (0,1)\) with \(\gamma + 2s > 0\), there are universal constants \(C > 0\), \(\alpha \in (0,1)\) such that
any essentially bounded non-negative weak solution \(f\) to (1.1)-(1.2) in \((-1,0) \times B_1 \times \mathbb{R}^3\)
satisfying (1.4) is \(\alpha\)-Hölder continuous with respect to \((t,x,v) \in (-1/2,0] \times B_{1/2} \times B_{1/2},\)
where \(C,\alpha\) only depend on the \(L^\infty\) bound of \(f\) and the bounds (1.4).
The proof goes in two steps. The first step is a local $L^2 \to L^\infty$ gain of integrability, following the approach of De Giorgi and Moser as reformulated in a kinetic context in \[114\] and \[62\]. It requires further technical work to formulate the De Giorgi iteration for such integro-differential equations with degenerate kernels (see also the related works \[84, 57, 29\]). The regularity mechanism at the core of the averaging velocity method is used; it is however presented differently than in most papers on this topic, by relying on explicit calculations on the fundamental solution of the fractional Kolmogorov equation.

In the second step of the proof, the authors establish a weak Harnack inequality, i.e. the control from above of local $L^\epsilon_{t,x,v}$ averages with $\epsilon > 0$ small by a local infimum multiplied by a universal constant. This inequality is sufficient to deduce the Hölder regularity. Two different strategies are used depending on whether $s \in (0, 1/2)$ or $s \in [1/2, 1)$. In the first case, they construct a barrier function to propagate lower bounds as in the method by Krylov and Safonov for equations in nondivergence form. In the second case, they use a variant of the isometric argument of De Giorgi proved by compactness as in \[62\]. Again the regularity of velocity averages plays a crucial role but is exploited by direct calculation on the fundamental solution of the fractional Kolmogorov equation.

4.4. Maximum principle and decay at large velocities. Finally in the paper \[82\], the nonlinear maximum principle argument of \[123\] is refined to obtain “pointwise counterpart” of velocity moments. The main result established in this paper is:

**Theorem 10** (Decay at large velocities for the BE \[82\]). Given any $\gamma \in (-3, 1]$ and $s \in (0, 1)$ with $\gamma + 2s \geq 0$, there are universal constants $C > 0$, $\alpha \in (0, 1)$ such that for any classical non-negative solution $f$ to (1.1)-(1.2) in $[0, T] \times T^3 \times \mathbb{R}^3$ satisfying (1.4), it holds for any $q > 0$: (i) if $f_{\infty}^\epsilon \lesssim (1 + |v|)^{-q}$ then $f(t, x, v) \leq C(1 + |v|)^{-q}$ for all $t > 0$, (ii) assuming furthermore that $\gamma > 0$, then $f(t, x, v) \leq C'(1 + t^{-\beta})(1 + |v|)^{-q}$ for all $t > 0$. All the constants depend on $\gamma$, $s$, $q$ and the bounds (1.4).

The study of large velocity decay in weighted $L^1$ spaces, known as the study of moments, is an old and important question in kinetic equations. The study of moments was initiated, for spatially homogeneous solutions, in \[79\] for Maxwellian potentials ($\gamma = 0$). In the case of hard potentials ($\gamma > 0$), Povzner identities \[119, 56, 135, 21\] play an important role. For instance, Elmroth \[56\] used them to prove that if moments are initially bounded, then they remain bounded for all times. Desvillettes \[43\] then proved that only one moment of order $q > 2$ is necessary for the same conclusion to hold true. It is shown in \[135, 105\] that even the condition on one moment of order $q > 2$ can be dispensed with, in both (homogeneous) cutoff and non-cutoff case. These moment estimates were used by Bobylev \[21\] in order to derive (integral) Gaussian tail estimates. In the case of soft potentials, Desvillettes \[43\] proved for $\gamma \in (-1, 0)$ that initially bounded moments grow at most linearly with time and it is explained in \[130\] that the method applies to $\gamma \in [-2, 0)$. The case of measure-valued solutions is considered in \[99\].

However the extension of these integral moments estimates to spatially inhomogeneous solutions is a hard and unclear question at the moment. The only result available is \[72\] Lemma 5.9 & 5.11] which proves the propagation and appearance of certain exponential moments for the spatially inhomogeneous Boltzmann equation for hard spheres (or hard potentials with cutoff), however in a space of the form $W^{3,1}_x L^1(1 + |v|^q)$. Another line of research opened by \[60\] consists in establishing exponential Gaussian pointwise decay by
maximum principle arguments (see also [22, 13, 61]). But these works assumes exponential integral moments whose propagation in time is not known, therefore it is not clear how to use them in this context.

We finally recall that the last part of the research program, the Schauder estimates, is missing for the Boltzmann equation with moderately soft potentials, and is an interesting open question for future research.

Acknowledgement. This review article greatly benefited from the many discussions of the author with Cyril Imbert and Luis Silvestre, in particular concerning the works on the conditional regularity initiated by Luis Silvestre that form a key part of this review, and concerning their important joint work on the weak Harnack inequality for the Boltzmann equation. Moreover the author gratefully acknowledges the many crucial discussions with Alexis Vasseur about his insights on the De Giorgi method, and with François Golse about averaging lemma and homogeneisation. The author acknowledges partial funding by the ERC grants MATKIT 2011-2016 and MAFRAN 2017-2022.

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