Getting the D-brane effective action from BPS configurations

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Abstract: We review a method to find the non-abelian open superstring effective action, thereby settling the issue of the ordering ambiguities. We start from solutions to Yang-Mills which, in D-brane context, define certain BPS configurations. Studying their deformations in the abelian case shows that the Born-Infeld action is the unique deformation which admits solutions of this type. By applying the method to the non-abelian case we calculated the full effective action through $\mathcal{O}(\alpha'^3)$. The presence of derivative terms turns out to be essential. Testing the result by comparing the spectrum in the presence of a constant magnetic background field with the string theory prediction, we obtain perfect agreement.

1 Introduction

A most attractive feature of D-branes is their close connection to gauge theories. Indeed the massless bosonic worldvolume degrees of freedom of a Dp-brane consist of a $U(n)$ gauge field and $9 - p$ $U(n)$-valued scalar fields. In leading order in $\alpha'$ the worldvolume action is precisely the supersymmetric $U(n)$ Yang-Mills theory dimensionally reduced to $p + 1$ dimensions [1]. In the remainder of this paper, we will ignore both the fermions and the scalar fields. When $n = 1$, the abelian case, the full effective action is known for constant fieldstrengths: it is the Born-Infeld action [2]. For the non-abelian case, $n \geq 2$, no such result is known.

A direct calculation requires matching the effective action to $N$-point open superstring amplitudes. This has been done for $N \leq 4$, yielding the full effective action through order $\mathcal{O}(\alpha'^2)$ [3], [4], [5] and derivative terms at higher orders [6]. Pushing the direct calculation to higher orders seems presently infeasible.

In this paper, we review a powerful method which allows for an indirect calculation of the full effective action, including derivative terms, order by order in $\alpha'$ [7]. Stable holomorphic bundles define solutions to Yang-Mills which generalize the standard notion of instantons to arbitrary dimensions. In D-brane context, such solutions correspond to BPS configurations in the weak field limit. Requiring that these solutions, or some deformation thereof, solve the equations of motion of the full effective action allows one to determine both the equations of motion and the deformation of the solution order by order in $\alpha'$. This program was carried out through $\mathcal{O}(\alpha'^3)$ in [8] and tested in [9].
2 BPS states of D-branes

Simple BPS configurations of D-branes arise as follows\[10\]. Start with two coinciding Dp-branes in the (1, 3, ..., 2p−1) directions. Keeping one of them fixed, rotate the other one over angles $\phi_i$ in the (2i-1 2i) plane, for $1 \leq i \leq p$. When

$$\sum_{i=1}^{p} \phi_i = 2\pi n,$$  \hspace{1cm} (1)

holds, one finds that $32/2^p$ supersymmetries are preserved\[11\].

Next, we T-dualize the system in the 2, 4, ..., p directions, ending up with two coinciding D2p-branes with magnetic fields turned on. Indeed, having two D2p-branes extended in the 1, 2, ..., 2p directions with constant magnetic flux $F_{2i-1 2i} = f_i \sigma_3$, $i \in \{1, \ldots, p\}$, and all other components zero, we can choose a gauge in which the potentials are given by,

$$A_{2i-1} = 0, \quad A_{2i} = f_i x^{2i-1} \sigma_3.$$  \hspace{1cm} (2)

T-dualizing back, one ends up with two Dp-branes with transversal coordinates given by

$$X^{2i} = 2\pi \alpha' A_{2i}.$$  \hspace{1cm} (3)

Using eq. (2) in eq. (3), we recognize the original configuration with the two Dp-branes at angles with the angles given by

$$\phi_i = 2 \arctan(2\pi \alpha' f_i).$$  \hspace{1cm} (4)

In terms of the magnetic field, the BPS condition (1) is formulated as,

$$\sum_i 2 \arctan(2\pi \alpha' f_i) = 2\pi n,$$  \hspace{1cm} (5)

or in the limit of weak fields, $\alpha' F \to 0$,

$$\sum_i f_i = 0.$$  \hspace{1cm} (6)

Since a BPS configuration should solve the equations of motion, eq. (6) should provide solutions to the Yang-Mills equations of motion.

Switching to complex coordinates, $z^\alpha = (x^{2\alpha-1} + ix^{2\alpha})/\sqrt{2}$, $\bar{z}^{\bar{\alpha}} = (z^\alpha)^*$, one finds that gauge field configurations satisfying

$$\sum_\alpha F_{\alpha \bar{\alpha}} = 0,$$  \hspace{1cm} (7)

$$F_{\alpha \beta} = 0, \quad F_{\bar{\alpha} \bar{\beta}} = 0.$$  \hspace{1cm} (8)

solve the Yang-Mills equations of motion,

$$D_\alpha F_{\alpha \bar{\beta}} + D_\alpha F_{\alpha \beta} = D_\beta F_{\alpha \bar{\alpha}} = 0,$$  \hspace{1cm} (9)

where we used the Bianchi identities. Furthermore, using the supersymmetry transformation rule, $\delta \psi \propto F_{\mu \nu} \gamma^{\mu \nu \epsilon}$, one discovers that these configurations do preserve $32/2^p$ supersymmetries. The configurations studied in (4) are a special case: the fieldstrengths are constant and in addition, $F_{\alpha \beta} = 0$ for $\alpha \neq \beta$. Eq. (7) reduces then to eq. (6).

Solutions satisfying eqs. (7) and (8) define a stable holomorphic vector bundle. For $p = 2$ ($d = 4$) they are equivalent to the standard instanton equations.

\[2\]Note that it is possible to preserve more supersymmetries with more stringent BPS conditions.
3 Abelian Born-Infeld

The next step is to look for deformations of the Yang-Mills action which still allow for this type of solutions. We work in the slowly varying field limit, so that we do not have to consider derivative terms. We expect condition (8) not to get any corrections because of its geometric origin. Condition (7) on the other hand will get corrections.

As an illustration we will consider the calculation through order $\alpha'^2$. The most general lagrangian through this order reads

$$\frac{1}{4}tr F^2 + \lambda_{(0,1)}(tr F^4) + \lambda_{(2)}(tr F^2)^2 + O(\alpha'^4),$$

where $\lambda_{(0,1)} + 4\lambda_{(2)} = 0$. The last line vanishes by virtue of eq. (7), the last line in eq. (13) will contribute to the equations of motion at $O(\alpha'^4)$. This will eventually lead to conditions between coefficients at different orders in $\alpha'$.

For general terms in the lagrangian:

$$\lambda_{(p_1,\ldots,p_n)}(tr F^2)^{p_1}\cdots(tr F^{2n})^{p_n},$$

we find by continuing the same procedure:

$$\lambda_{(p_1,\ldots,p_n)} = (-1)^{p_1+\cdots+p_n+1} \frac{1}{p_1!\cdots p_n!} \frac{1}{1^p_1\cdots n^p_n}$$

which corresponds to the expansion of the abelian Born-Infeld action $-\sqrt{\det (\delta^\mu_\nu + F^\mu_\nu)}$. The deformed stability condition is

$$F_{a\bar{a}} + \frac{1}{3}F_{a\bar{a}}^3 + \frac{1}{5}F_{a\bar{a}}^5 + \cdots = 0,$$

or using $F_{a\bar{a}} = iF_{x^{2\alpha-1}x^{2\alpha}}$ exactly condition (5) for $n = 0$.

We therefore conclude that the abelian Born-Infeld action is the unique deformation of Yang-Mills (without derivative terms) that admits generalized stable holomorphic vector bundles as a solution.

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3 Order $\alpha'$ is zero in the abelian case because of antisymmetry.
4 From now on we put $2\pi\alpha' = 1$. 
4 From the abelian to the non-abelian case

The same method works essentially for the non-abelian case too, but there are several complications:

- Because of identities of the form $[D_\mu, D_\nu]F_{\rho\sigma} = [F_{\mu\nu}, F_{\rho\sigma}]$, there is no unambiguous notion of slowly varying field strengths as in the abelian case. In fact, our method clearly shows that from order $\alpha'^4$ on, derivative terms are unavoidable.

- Once derivative terms are included, field redefinition ambiguities have to be dealt with! This fact should also be taken into account when comparing to results in the literature.

- In the non-abelian case, there is a huge amount of possible terms (derivative terms, permutations of the $F$s), that are connected by a complex web of identities (partial integration identities, Bianchi identities and $[D, D]F = [F, F]$-identities, field redefinitions). So we wrote a computer program to keep track of all these.

A (very rough) flowchart of the calculations would look as in figure 1.

In this way we found up to order $\alpha'^3$ and modulo field redefinitions:

$$
\mathcal{L} = \frac{1}{g^2} \text{Tr} \left( -\frac{1}{4} F_{\mu_1}^{\mu_2} F_{\mu_2}^{\mu_1} - \frac{1}{24} F_{\mu_1}^{\mu_2} F_{\mu_2}^{\mu_3} F_{\mu_3}^{\mu_4} F_{\mu_4}^{\mu_1} - \frac{1}{12} F_{\mu_1}^{\mu_2} F_{\mu_2}^{\mu_3} F_{\mu_3}^{\mu_4} F_{\mu_4}^{\mu_1} + \frac{1}{48} F_{\mu_1}^{\mu_2} F_{\mu_2}^{\mu_1} F_{\mu_3}^{\mu_4} F_{\mu_4}^{\mu_3} + \frac{1}{96} F_{\mu_1}^{\mu_2} F_{\mu_2}^{\mu_3} F_{\mu_3}^{\mu_4} F_{\mu_4}^{\mu_1} F_{\mu_4}^{\mu_3} - \Lambda \left( F_{\mu_1}^{\mu_2} F_{\mu_2}^{\mu_3} F_{\mu_3}^{\mu_4} F_{\mu_4}^{\mu_1} F_{\mu_4}^{\mu_3} + F_{\mu_1}^{\mu_2} F_{\mu_1}^{\mu_3} F_{\mu_2}^{\mu_4} F_{\mu_3}^{\mu_5} F_{\mu_4}^{\mu_1} F_{\mu_5}^{\mu_3} F_{\mu_5}^{\mu_4} F_{\mu_4}^{\mu_3} - \frac{1}{2} F_{\mu_1}^{\mu_2} F_{\mu_2}^{\mu_3} F_{\mu_3}^{\mu_5} F_{\mu_4}^{\mu_1} F_{\mu_4}^{\mu_5} + F_{\mu_1}^{\mu_2} (D^{\mu_1} F_{\mu_2}^{\mu_3} (D_{\mu_3} F_{\mu_2}^{\mu_4} (D_{\mu_5} F_{\mu_2}^{\mu_5} F_{\mu_4}^{\mu_5})) F_{\mu_4}^{\mu_5} - \frac{1}{2} (D^{\mu_1} F_{\mu_2}^{\mu_3} (D_{\mu_3} F_{\mu_2}^{\mu_4} F_{\mu_5}^{\mu_2} F_{\mu_4}^{\mu_5} - \frac{1}{2} (D^{\mu_1} F_{\mu_2}^{\mu_3} (D_{\mu_3} F_{\mu_2}^{\mu_4} F_{\mu_5}^{\mu_2} F_{\mu_4}^{\mu_5} - \frac{1}{8} (D^{\mu_1} F_{\mu_2}^{\mu_3} (D_{\mu_3} F_{\mu_2}^{\mu_4} F_{\mu_5}^{\mu_2} (D_{\mu_5} F_{\mu_3}^{\mu_4} (D_{\mu_5} F_{\mu_2}^{\mu_5} F_{\mu_4}^{\mu_5})))) \right) \right),
\right)
$$

(17)
where $\text{Tr}$ is the group trace. $\Lambda$ is an arbitrary constant which can be fixed by comparing to string scattering calculations, see for instance [\ref{6}]:

$$\Lambda = -\frac{2\zeta(3)}{\pi^3}. \tag{18}$$

String theory tells us that the order $\alpha'^m$ correction to the effective action is proportional to $\zeta(m)/\pi^m$. Since Euler, we know that for $m$ even this is rational while for $m$ odd it is not. Our method clearly only yields rational numbers. So it is most fortunate that we obtained a free parameter at order $\alpha'^3$. In fact we expect our method to fully fix the action at even orders but leaving free parameters at odd orders.

5 Checks on the result

- Internal check: before finding eq. (17) our program had to solve a set of 156 homogeneous equations in 63 unknowns. The fact that we found a solution is encouraging!

- Fluctuation spectrum: string theory predicts the following spectrum for strings stretching between the two $Dp$-branes at angles from section 2:

$$M^2 = \left( \sum_j (2n_j + 1)\phi_j \right) \pm 2\phi_i, \tag{19}$$

with $\phi_i = 2\arctan f_i$.

Fluctuations around Yang-Mills (order $\alpha'^0$) on the 2$p$-torus lead to the following spectrum:

$$M^2 = \left( \sum_j 2(2n_j + 1)f_j \right) \pm 4f_i, \tag{20}$$

and the higher orders in the $f_j$ must come from higher order terms in $\alpha'$. We see immediately that the odd orders should not contribute.

Careful calculations [\ref{9}] reveal that indeed our order $\alpha'^3$ does not contribute to the spectrum.

6 Conclusions and future research

Because of these checks, we are fairly confident that the result is indeed the correct non-abelian open superstring effective action through $\mathcal{O}(\alpha'^3)$.

Furthermore our program managed to calculate the lagrangian through $\mathcal{O}(\alpha'^4)$. Unfortunately, at this order not only the calculation but also the result is very complicated. Due to the web of identities, and especially the field redefinitions, we lack a sort of “canonical” form. This is probably not the way to get all order results in $\alpha'$.

Therefore, we plan to make an expansion in the degree of non-abelianity, where the zeroth order would be the symmetrized trace Born-Infeld action. Using our method, we try to calculate corrections — at the first non trivial order in this degree and at all orders in $\alpha'$.

Finally, let us make a remark about the D3-brane effective action versus the $d = 4, N = 4$ effective super Yang-Mills action. In the abelian case, the $F^8$ term in the one-loop $N = 4$ super Yang-Mills effective action is different in structure, [\ref{1}], from the $F^8$
term in the Born-infeld action \cite{12}. In the non-abelian case, this discrepancy shows up at lower order. Indeed, in \cite{13}, the terms of the same dimensions as the ones discussed in this paper ($F^5$ and $D^2 F^4$), in the one-loop effective action of $N = 4$ supersymmetric Yang-Mills in four dimensions were calculated. Not only are these terms different from the ones calculated in this paper \cite{8}, they do not pass the test in \cite{9} as well.

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