Toda lattice hierarchy and Goldstein-Petrich flows for plane curves

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Abstract

A relation between the Goldstein-Petrich hierarchy for plane curves and the Toda lattice hierarchy is investigated. A representation formula for plane curves is given in terms of a special class of \(\tau\)-functions of the Toda lattice hierarchy. A representation formula for discretized plane curves is also discussed.

1 Introduction

Intimate connection between integrable systems and differential geometry of curves and surfaces has been an important topic of intense research \cite{1, 23}. Goldstein and Petrich introduced a hierarchy of commuting flows for plane curves that is related to the modified Korteweg-de Vries (mKdV) hierarchy \cite{6}. The second Goldstein-Petrich flow is defined by the modified Korteweg-de Vries equation,

\[
\frac{\partial \kappa}{\partial t} = \frac{\partial^3 \kappa}{\partial x^3} + \frac{3}{2} \kappa^2 \frac{\partial \kappa}{\partial x},
\]

where \(\kappa = \kappa(x, t)\) denotes the curvature and \(x\) is the arc-length. This result has been extended and investigated from various viewpoints \cite{3, 4, 5, 9, 10, 11, 16, 21, 22}. In \cite{9, 10}, a representation formula for curve motion in terms of the \(\tau\) function with respect to the second Goldstein-Petrich flow has been presented by means of the Hirota bilinear formulation and determinant expression of solutions. The aim of this article is to generalize the results in \cite{9, 10} to the whole hierarchy. We will show how the Goldstein-Petrich hierarchy is embedded in the Toda lattice hierarchy\cite{24, 28}. We remark that the semi-discrete case, discussed in \cite{10}, is not considered in this paper.

An advantage of infinite hierarchical formulation is its relation to integrable discretization. Miwa showed that Hirota’s discrete Toda equation \cite{7} can be obtained by applying a change of coordinate to the KP hierarchy \cite{12, 19, 24}. Using a generalization of Miwa’s approach, we will show that Matsuura’s discretized curve motion \cite{18} can be obtained also from the Toda lattice hierarchy. Another merit of the KP theoretic formulation is Lie algebraic aspect of the hierarchy \cite{12, 20}. We will discuss a relationship between the Goldstein-Petrich hierarchy and a real form of the affine Lie algebra \(\hat{sl}(2, \mathbb{C})\).
2 Goldstein-Petrich flows for Euclidean plane curves

We assume that \( \mathbf{r}(x) = (X(x), Y(x)) \) is a curve in Euclidean plane \( \mathbb{R}^2 \), parameterized by the arc-length \( x \). Define the tangent vector \( \hat{\mathbf{r}} \) and the unit normal \( \hat{\mathbf{n}} \) by

\[
\hat{\mathbf{r}} = x, \quad \hat{\mathbf{n}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \hat{\mathbf{r}}.
\]  

(2.1)

Here the subscript \( x \) indicates differentiation. The Frenet equation for \( \mathbf{r} \) is given by

\[
\hat{\mathbf{r}}_x = \kappa \hat{\mathbf{n}}, \quad \hat{\mathbf{n}}_x = -\kappa \hat{\mathbf{r}},
\]

(2.2)

where \( \kappa \) is the curvature of the curve \( \mathbf{r} \). Goldstein and Petrich [6] considered dynamics of a plane curve described by the equation of the form

\[
\frac{\partial \mathbf{r}}{\partial t_n} = f^{(n)}(\hat{\mathbf{n}}) + g^{(n)}\hat{\mathbf{r}}.
\]

(2.3)

The coefficients \( f^{(n)} = f^{(n)}(x, t) \), \( g^{(n)} = g^{(n)}(x, t) \) \( (t = (t_1, t_2, t_3, \ldots)) \) are differential polynomials in \( \kappa \). We remark that our choice of signature in (2.2) is different from that of [6]. Following the discussion in [6], we choose \( f^{(n)}(x, t), g^{(n)}(x, t) \) as

\[
f^{(1)} = 0, \quad g^{(1)} = 1, \quad f^{(2)} = \kappa_x, \quad g^{(2)} = \kappa^2/2,
\]

\[
g^{(n)} = \kappa f^{(n)}, \quad f^{(n+1)} = \left(f_x^{(n)} + \kappa g^{(n)}\right)_x.
\]

(2.4)

We call as Goldstein-Petrich hierarchy the equations defined by (2.1), (2.2), (2.3) and (2.4).

Applying the condition (2.4) to (2.3), we obtain

\[
\frac{\partial \hat{\mathbf{r}}}{\partial t_n} = \left(f_x^{(n)} + \kappa g^{(n)}\right)\hat{\mathbf{n}}, \quad \frac{\partial \hat{\mathbf{n}}}{\partial t_n} = -\left(f_x^{(n)} + \kappa g^{(n)}\right)\hat{\mathbf{r}}.
\]

(2.5)

The compatibility condition for (2.2) and (2.5) is reduced to

\[
\frac{\partial \kappa}{\partial t_n} = \left(f_x^{(n)} + \kappa g^{(n)}\right)_x = f^{(n+1)}.
\]

(2.6)

The case \( n = 2 \) of (2.6) gives the mKdV equation (1.1). One finds that

\[
f^{(n)} = \Omega f^{(n-1)}, \quad \Omega = \partial_x^2 + \kappa^2 + \kappa \partial_x^{-1} \kappa.
\]

(2.7)

We remark that the operator \( \Omega \) is the recursion operator for the modified KdV hierarchy [2].

We now introduce complex coordinate via a map \( \rho : \mathbb{R}^2 \rightarrow \mathbb{C} \) given by

\[
\rho(X, Y) = X + \sqrt{-1} Y.
\]

(2.8)

and define \( Z, T, N \) as

\[
Z = \rho(\mathbf{r}), \quad T = \rho(\hat{\mathbf{r}}), \quad N = \rho(\hat{\mathbf{n}}) = \sqrt{-1} T.
\]

(2.9)

Since \( |\hat{\mathbf{r}}| = |\hat{\mathbf{n}}| = 1 \), the complex variables \( T \) and \( N \) satisfy \( |T| = |N| = 1 \). The equations (2.1), (2.2), (2.3) are rewritten as

\[
T = Z_x, \quad T_x = \sqrt{-1} \kappa T, \quad \frac{\partial Z}{\partial t_n} = \left(g^{(n)} + \sqrt{-1} f^{(n)}\right) T.
\]

(2.10)
3 Toda lattice hierarchy

In this section, we briefly review the theory of Toda lattice hierarchy using the language of difference operators [24, 28] (See also [13, 25, 26]). We denote as $e^{\partial_s}$ the shift operator with respect to $s$: $e^{\partial_s} f(s) = f(s + 1)$. For a difference operator $A(s) = \sum_{-\infty < j < +\infty} a_j(s)e^{i\partial_s}$, we define the non-negative and negative part of $A(s)$ as

$$
(A(s))_{\geq 0} = \sum_{0 \leq j < +\infty} a_j(s)e^{i\partial_s}, \quad (A(s))_{< 0} = \sum_{-\infty < j < 0} a_j(s)e^{i\partial_s}.
$$

Let $L^{(\infty)}(s), L^{(0)}(s)$ be difference operators of the form

$$
L^{(\infty)}(s) = e^{\partial_s} + \sum_{-\infty < j \leq 0} b_j(s)e^{i\partial_s}, \quad L^{(0)}(s) = \sum_{-1 \leq j < +\infty} c_j(s)e^{i\partial_s},
$$

where we assume $c_{-1}(s) \neq 0$ for any $s$. We introduce two sets of infinitely many variables $x = (x_1, x_2, \ldots), y = (y_1, y_2, \ldots)$ and define the weight of the variables as

$$
\text{weight}(x_n) = n, \quad \text{weight}(y_n) = -n \quad (n = 1, 2, \ldots).
$$

Each coefficient of $L^{(\infty)}(s), L^{(0)}(s)$ is a function of $x, y$, i.e. $b_j(s) = b_j(s; x, y), c_j(s) = c_j(s; x, y)$. The Toda lattice hierarchy is defined as the following set of differential equations of Lax-type:

$$
\frac{\partial L^{(\infty)}(s)}{\partial x_n} = [B_n(s), L^{(\infty)}(s)], \quad \frac{\partial L^{(0)}(s)}{\partial x_n} = [B_n(s), L^{(0)}(s)],
$$

$$
B_n(s) = \left(L^{(\infty)}(s)^n\right)_{\geq 0} \quad (n = 1, 2, 3, \ldots),
$$

$$
\frac{\partial L^{(\infty)}(s)}{\partial y_n} = [C_n(s), L^{(\infty)}(s)], \quad \frac{\partial L^{(0)}(s)}{\partial y_n} = [C_n(s), L^{(0)}(s)],
$$

$$
C_n(s) = \left(L^{(0)}(s)^n\right)_{\leq 0} \quad (n = 1, 2, 3, \ldots).
$$

**Proposition 1** [28, Proposition 1.4]. Let $L^{(\infty)}, L^{(0)}$ be difference operators of the form (3.2) and satisfy the differential equations (3.4), (3.5). Then there exist difference operators $\hat{W}^{(\infty)}(s), \hat{W}^{(0)}(s)$ of the form,

$$
\hat{W}^{(\infty)}(s) = 1 + \sum_{j=1}^{\infty} \hat{w}_j^{(\infty)}(s)e^{-j\partial_s},
$$

$$
\hat{W}^{(0)}(s) = \sum_{j=0}^{\infty} \hat{w}_j^{(0)}(s)e^{j\partial_s} \quad (\hat{w}_0^{(0)}(s) \neq 0),
$$

satisfying the following equations:

$$
L^{(\infty)}(s) = \hat{W}^{(\infty)}(s)e^{\partial_s}\hat{W}^{(\infty)}(s)^{-1},
$$

$$
L^{(0)}(s) = \hat{W}^{(0)}(s)e^{-\partial_s}\hat{W}^{(0)}(s)^{-1},
$$

$$
\frac{\partial \hat{W}^{(\infty)}(s)}{\partial x_n} = B_n(s)\hat{W}^{(\infty)}(s) - \hat{W}^{(\infty)}(s)e^{n\partial_s},
$$

$$
\frac{\partial \hat{W}^{(\infty)}(s)}{\partial y_n} = C_n(s)\hat{W}^{(\infty)}(s),
$$

$$
\frac{\partial \hat{W}^{(0)}(s)}{\partial x_n} = B_n(s)\hat{W}^{(0)}(s),
$$

$$
\frac{\partial \hat{W}^{(0)}(s)}{\partial y_n} = C_n(s)\hat{W}^{(0)}(s) - \hat{W}^{(0)}(s)e^{-n\partial_s}.
$$
Proposition 2 ([28], (1.2.18)). The difference operators $\hat{W}^{(s)}(s; x', y') \exp \left[ \sum_{n=1}^{\infty} (x'_n - x_n) e^{\theta x_n} \right] \hat{W}^{(s)}(s; x, y)^{-1}$

$$\hat{W}^{(s)}(s; x', y') \exp \left[ \sum_{n=1}^{\infty} (y'_n - y_n) e^{-\theta y_n} \right] \hat{W}^{(s)}(s; x, y)^{-1}$$

(3.9)

for any $x, x', y, y'$ and any integer $s$.

Define $\hat{W}_j^{(s)}(s; x, y), \hat{W}_j^{(s),*}(s; x, y)$ by expanding $\hat{W}^{(s)}(s; x, y)^{-1}, \hat{W}^{(s)}(s; x, y)^{-1}$ with respect to $e^{\theta x}$:

$$\hat{W}_j^{(s)}(s; x, y)^{-1} = \sum_{j=0}^{\infty} e^{-\theta j} \hat{W}_j^{(s)*}(s + 1; x, y),$$

(3.10)

$$\hat{W}^{(s)}(s; x, y)^{-1} = \sum_{j=0}^{\infty} e^{\theta j} \hat{W}_j^{(s)*}(s + 1; x, y).$$

From (3.6), (3.7) and (3.10), we obtain

$$b_0(s) = \hat{W}_1^{(s)}(s) + \hat{W}_1^{(s)*}(s + 1) = \hat{W}_1^{(s)}(s) - \hat{W}_1^{(s)}(s + 1),$$

$$b_{-n}(s) = \hat{W}_n^{(s)}(s) + \hat{W}_n^{(s)*}(s + 1 - n) + \sum_{j=1}^{n} \hat{W}_j^{(s)}(s) \hat{W}_n^{(s)*}_{n+1-j}(s + 1 - n) \quad (n \geq 1),$$

(3.11)

$$c_n(s) = \sum_{j=0}^{n+1} \hat{W}_j^{(s)}(s) \hat{W}_{n-j+1}^{(s)*}(s + n + 1) \quad (n \geq -1).$$

Theorem 3 ([28], Theorem 1.7). There exists a function $\tau(s) = \tau(s; x, y)$ satisfying

$$\hat{W}_j^{(s)}(s; x, y) = \frac{p_j(-\tilde{\partial}_x) \tau(s; x, y)}{\tau(s; x, y)},$$

$$\hat{W}_j^{(s),*}(s; x, y) = \frac{p_j(-\tilde{\partial}_x) \tau(s + 1; x, y)}{\tau(s; x, y)},$$

$$\hat{W}_j^{(s)*}(s; x, y) = \frac{p_j(\tilde{\partial}_x) \tau(s; x, y)}{\tau(s; x, y)},$$

$$\hat{W}_j^{(s)*}(s; x, y) = \frac{p_j(\tilde{\partial}_x) \tau(s - 1; x, y)}{\tau(s; x, y)}$$

(3.12)

where $\tilde{\partial}_x = (\partial_{x_1}, \partial_{x_2}/2, \partial_{x_3}/3, \ldots), \tilde{\partial}_y = (\partial_{y_1}, \partial_{y_2}/2, \partial_{y_3}/3, \ldots)$, and the polynomials $p_n(t)$ ($n = 0, 1, 2, \ldots$) are defined by

$$\xi(t, \lambda) = \exp \left[ \sum_{j=1}^{\infty} t_n \lambda^j \right] = \sum_{n=0}^{\infty} p_n(t) \lambda^n, \quad t = (t_1, t_2, \ldots).$$

(3.13)

Furthermore, the $\tau$-function $\tau(s; x, y)$ of the Toda lattice hierarchy is determined uniquely by (3.12) up to a constant multiple factor.

It follows that

$$c_{-1}(s) = \hat{W}_0^{(0)}(s) \hat{W}_0^{(0)*}(s) = \frac{\tau(s + 1) \tau(s - 1)}{\tau(s)^2},$$

$$c_0(s) = \hat{W}_0^{(0)}(s) \hat{W}_1^{(0)*}(s + 1) + \hat{W}_1^{(0)}(s) \hat{W}_0^{(0)*}(s + 1) = \frac{\partial}{\partial y_1} \log \frac{\tau(s)}{\tau(s + 1)}.$$  

(3.14)
Theorem 4 ([28], Theorem 1.11). \(\tau\)-functions of Toda lattice hierarchy satisfy the following equation (bilinear identity):

\[
\oint \tau(s'; x' - [\lambda^{-1}], y')\tau(s; x + [\lambda^{-1}], y)e^{S(x'-x,y',\lambda)}x'^{-s}d\lambda = \oint \tau(s' + 1; x', y' - [\lambda])\tau(s - 1; x, y + [\lambda])e^{S(y'-y,x',\lambda)}x'^{-s}d\lambda,
\]

where \([\lambda] = (\lambda, \lambda^2/2, \lambda^3/3, \ldots)\), and we have used the notation of formal residue,

\[
\oint \left( \sum_n a_n x^n \right) d\lambda = 2\pi \sqrt{-1} a_{-1}.
\]

Conversely, if \(\tau(s; x, y)\) solves the bilinear identity (3.15), then \(\check{W}^{(\infty)}(s; x, y)\) and \(\check{W}^{(0)}(s; x, y)\) defined by (3.6) and (3.12) satisfy (3.8).

4 Time-flows with negative weight with 2-reduction condition

4.1 Reduction to Goldstein-Petrich hierarchy

We now impose the 2-reduction condition [28]

\[
L^{(\infty)}(s)^2 = e^{2d_1}, \quad L^{(0)}(s)^2 = e^{-2d_1},
\]

that implies

\[
W^{(\infty)}(s + 2) = W^{(\infty)}(s), \quad W^{(0)}(s + 2) = W^{(0)}(s),
\]

\[
L^{(\infty)}(s + 2) = L^{(\infty)}(s), \quad L^{(0)}(s + 2) = L^{(0)}(s).
\]

Proposition 5 ([28], Proposition 1.13). Let \(L^{(\infty)}(s; x, y)\), \(L^{(0)}(s; x, y)\) be solutions to the Toda lattice hierarchy (3.4), (3.5), which satisfy the 2-reduction conditions (4.1). Then one finds that

\[
\frac{\partial L^{(\infty)}}{\partial x_{2n}} = \frac{\partial L^{(0)}}{\partial x_{2n}} = \frac{\partial L^{(\infty)}}{\partial y_{2n}} = \frac{\partial L^{(0)}}{\partial y_{2n}} = 0
\]

for \(n = 1, 2, \ldots\).

Proposition 6 ([28], Corollary 1.14). Suppose \(L^{(\infty)}(s; x, y)\), \(L^{(0)}(s; x, y)\) be solutions to the Toda lattice hierarchy (3.4), (3.5), which satisfy the 2-reduction conditions (4.1). Then there exist suitable difference operators \(\check{W}^{(\infty)}(s; x, y)\), \(\check{W}^{(0)}(s; x, y)\) such that the corresponding \(\tau\) functions subject to the following conditions:

\[
\tau(s; x, y) = \tau'(s; x, y)\exp\left(-\sum_{n=1}^{\infty} n x_n y_n\right),
\]

\[
\tau'(s + 2; x, y) = \tau'(s; x, y),
\]

\[
\frac{\partial \tau'(s; x, y)}{\partial x_{2n}} = \frac{\partial \tau'(s; x, y)}{\partial y_{2n}} = 0 \quad (n = 1, 2, \ldots).
\]
We consider the time-evolutions with respect to the variables with negative weight \( y = (y_1, y_2, \ldots) \) under the 2-reduction condition (4.1). In this case, one can write down the difference operators \( C_n(s) (n = 1, 2, \ldots) \) explicitly:

\[
C_{2n}(s) = e^{-2n\partial_s}, \quad C_{2n-1} = \sum_{j=1}^{2n-3} c_j(s)e^{(j-2n)\partial_s}. \tag{4.6}
\]

Applying (4.6) to (3.5) and (3.8), we obtain the following equations \((n = 0, 1, 2, \ldots)\):

\[
\begin{align*}
\frac{\partial w_1(s)}{\partial y_{2n+1}} &= c_{2n-1}(s) \tag{4.7} \\
\frac{\partial c_{2n}(s)}{\partial y_1} &= \frac{\partial c_0(s)}{\partial y_{2n+1}} = c_{-1}(s)c_{2n+1}(s + 1) - c_{-1}(s + 1)c_{2n+1}(s), \tag{4.8} \\
\frac{\partial c_{-1}(s)}{\partial y_{2n+1}} &= \frac{\partial c_{2n-1}(s)}{\partial y_1} = c_{-1}(s)\{c_{2n}(s + 1) - c_{2n}(s)\}, \tag{4.9}
\end{align*}
\]

where we have used the property \( c_j(s + 2) = c_j(s) \).

**Proposition 7.** For \( n = 0, 1, 2, \ldots \), the coefficients \( c_n(s; x, y) \) can be represented by \( c_{-1}(s; x, y) \). For example, \( c_0(s; x, y) \) and \( c_1(s; x, y) \) can be written as

\[
c_0(s) = -\frac{1}{2c_{-1}(s)} \frac{\partial c_{-1}(s)}{\partial y_1} = -\frac{1}{2} \frac{\partial}{\partial y_1} \log c_{-1}(s),
\]

\[
c_1(s) = -\frac{c_{-1}(s)}{2} \left\{ c_0(s)^2 + \frac{\partial c_0(s)}{\partial y_1} \right\} = -\frac{c_{-1}(s)}{8} \left[ \left( \frac{\partial}{\partial y_1} \log c_{-1}(s) \right)^2 - 2 \frac{\partial^2}{\partial y_1^2} \log c_{-1}(s) \right]. \tag{4.10}
\]

**Proof.** From (3.2) and (4.1), we have

\[
c_{-1}(s)c_{-1}(s - 1) = 1, \quad c_0(s) + c_0(s - 1) = 0,
\]

\[
c_{-1}(s)c_{k+1}(s - 1) + c_{-1}(s + k + 1)c_{k+1}(s) + \sum_{j=0}^{k} c_j(s)c_{k-j}(s + j) = 0. \tag{4.11}
\]

The desired result can be obtained from (4.8), (4.9) and (4.11). \( \square \)

**Remark:** Under the 2-reduction conditions (4.1), the map

\[
\sum_{n \in \mathbb{Z}} a_n(s)e^{n\partial_s} \mapsto \sum_{n \in \mathbb{Z}} \begin{bmatrix} a_n(0) & 0 \\ 0 & a_n(1) \end{bmatrix} \begin{bmatrix} 0 & 1^a \\ \xi^2 & 0 \end{bmatrix} \]

\[
\tag{4.12}
\]

gives an algebra isomorphism [28]. For example, the operators \( C_1(s), C_3(s) \) are mapped as follows:

\[
C_1(s) \mapsto \begin{bmatrix} c_{-1}(0) & 0 \\ 0 & c_{-1}(1) \end{bmatrix} \begin{bmatrix} 0 & \xi^{-2} \\ \xi^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c_{-1}(0)\xi^{-2} \\ 1/c_{-1}(0) & 0 \end{bmatrix},
\]

\[
C_3(s) \mapsto \begin{bmatrix} c_0(0)\xi^{-2} & c_{-1}(0)\xi^{-4} + c_1(0)\xi^{-2} \\ \xi^{-2}/c_{-1}(0) + c_1(1) & -c_0(0)\xi^{-2} \end{bmatrix}. \tag{4.13}
\]

Applying this isomorphism to the equations (3.4), (3.5), one obtains the Lax equations of \( 2 \times 2 \)-matrix form.
For \( n = 0, 1, 2, \ldots \), define \( F^{(n)}(s) \) and \( G^{(n)}(s) \) as
\[
F^{(n)}(s) = \frac{1}{2} \{ c_{-1}(s + 1)c_{2n-1}(s) - c_{-1}(s)c_{2n-1}(s + 1) \},
\]
\[
G^{(n)}(s) = \frac{1}{2} \{ c_{-1}(s + 1)c_{2n-1}(s) + c_{-1}(s)c_{2n-1}(s + 1) \}.
\]

From (4.7), (4.11) and (4.14), we have
\[
\frac{\partial w_1(s)}{\partial y_{2n+1}} = \frac{F^{(n)}(s) + G^{(n)}(s)}{c_{-1}(s + 1)} = c_{-1}(s) \left\{ F^{(n)}(s) + G^{(n)}(s) \right\}.
\]

It is straightforward to show that
\[
\frac{\partial F^{(n)}(s)}{\partial y_1} = 2c_0(s)G^{(n)}(s) + c_{2n}(s + 1) - c_{2n}(s),
\]
\[
\frac{\partial G^{(n)}(s)}{\partial y_1} = 2c_0(s)F^{(n)}(s).
\]

Next we consider reality condition. Assume \( x_j, y_j \in \mathbb{R} \) (\( j = 1, 2, \ldots \)) and the \( \tau \)-function \( \tau(s; x, y) \) satisfies
\[
\overline{\tau(s; x, y)} = \tau(s + 1; x, y),
\]
where \( \overline{\cdot} \) denotes complex conjugation. Under this condition, the following relations hold:
\[
\hat{w}_j^{(s)}(s) = \hat{w}_j^{(s + 1)}, \quad \hat{w}_j^{(0)}(s) = \hat{w}_j^{(0)}(s + 1),
\]
\[
b_{-n}(s) = b_{-n}(s + 1), \quad c_n(s) = c_n(s + 1),
\]
\[
F^{(n)}(s) = -F^{(n)}(s), \quad G^{(n)}(s) = G^{(n)}(s).
\]

Furthermore, it follows from (3.14) that
\[
c_{-1}(s) c_{-1}(s) = 1, \quad c_0(s) + c_0(s) = 0.
\]

**Theorem 8** (Representation formula in terms of the \( \tau \)-functions). If we set
\[
x = 2y_1, \quad t_n = 2y_{2n-1} \quad (n = 1, 2, \ldots),
\]
\[
Z = \hat{w}_1^{(s)}(s = 0; x, y) = -\frac{\partial}{\partial x_1} \log \tau(0; x, y),
\]
\[
T = \frac{1}{2} c_{-1}(s = 0; x, y) = \frac{\tau(1; x, y)^2}{2\tau(0; x, y)^2},
\]
\[
\kappa = \sqrt{-1} c_0(s = 0; x, y) = \sqrt{-1} \frac{\partial}{\partial y_1} \log \frac{\tau(0; x, y)}{\tau(1; x, y)},
\]
\[
f^{(n)} = -\sqrt{-1} F^{(n-1)}(s = 0), \quad g^{(n)} = G^{(n-1)}(s = 0),
\]
then \( Z, T, \kappa, f^{(n)}, g^{(n)} \) solve the equations (2.4), (2.10).

**Proof.** The first equation of (2.10) follows from (4.7). The second and the third are obtained from (4.9), (4.15). The recurrence relations (2.4) follows from (4.8), (4.14) and (4.16). □
4.2 Discrete mKdV flow on discrete curves

We recall a discrete analogue of the mKdV-flow of plane curve introduced by Matsuura [18]. Let \( \gamma_n^m : \mathbb{Z} \rightarrow \mathbb{C} \) be a map describing the discrete motion of discrete plane curve with segment length \( a_n \):

\[
\begin{align*}
\frac{\gamma_{n+1}^m - \gamma_n^m}{a_n} &= 1, \\
\frac{\gamma_{n+1}^m - \gamma_n^m}{a_n} &= e^{\sqrt{-1} T_n^m} \frac{\gamma_{n+1}^m - \gamma_n^m}{a_{n-1}}, \\
\gamma_{n+1}^m - \gamma_n^m &= e^{\sqrt{-1} W_n^m} (\gamma_{n+1}^m - \gamma_n^m).
\end{align*}
\]  

(4.21)

The compatibility condition for (4.21) implies the existence of the function \( \theta_n^m \) defined by

\[
W_n^m = \frac{\theta_{n+1}^m - \theta_n^m}{2}, \quad K_n^m = \frac{\theta_{n+1}^m - \theta_n^m}{2}.
\]  

(4.22)

Then the isoperimetric condition (the first equation in (4.21)) implies that \( \theta_n^m \) satisfies the discrete potential mKdV equation [8]

\[
\tan \left( \frac{\theta_{n+1}^m - \theta_n^m}{2} \right) = \frac{b_m + a_n}{b_m - a_n} \tan \left( \frac{\theta_{n+1}^m - \theta_n^m + 1}{2} \right).
\]  

(4.23)

In what follows, we will show that the equations (4.21) can be obtained from the Toda lattice hierarchy. We introduce discrete variables \( m, n \in \mathbb{Z} \) and assume \( y_k \) depends on \( m, n \) as

\[
y_k(m, n) = -\sum_{n'} \frac{d_{n'}^k}{k} - \sum_{m'} \frac{b_{m'}^k}{k} \quad (k = 1, 2, 3, \ldots),
\]  

(4.24)

which is a non-autonomous version of Miwa transformation [30]. We remark that if \( a_n = a \) and \( b_m = b \) for any \( n, m \) then (4.24) is reduced to original Miwa transformation [19]:

\[
y_k(n, m) = -\frac{na_k^k}{k} - \frac{mb_k^k}{k} \quad (k = 1, 2, 3, \ldots).
\]  

(4.25)

To consider the dependence on \( m, n \), we use the following abbreviation:

\[
\hat{W}^{(0)}(s; m, n) = \hat{W}^{(0)}(s; x, y = \hat{y}(m, n)),
\]  

(4.26)

Proposition 9. \( \hat{W}^{(0)}(s; m, n) \) and \( \hat{W}^{(0)}(s; m, n) \) satisfy

\[
\begin{align*}
\hat{W}^{(0)}(s; m, n + 1) &= \left\{ 1 - a_n \hat{u}(s; m, n)e^{-\theta_s} \right\} \hat{W}^{(0)}(s; m, n), \\
\hat{W}^{(0)}(s; m, n + 1) &= \left\{ 1 - a_n \hat{u}(s; m, n)e^{-\theta_s} \right\} \hat{W}^{(0)}(s; m, n), \\
\hat{W}^{(0)}(s; m + 1, n) &= \left\{ 1 - b_m \hat{v}(s; m, n)e^{-\theta_s} \right\} \hat{W}^{(0)}(s; m, n), \\
\hat{W}^{(0)}(s; m + 1, n) &= \left\{ 1 - b_m \hat{v}(s; m, n)e^{-\theta_s} \right\} \hat{W}^{(0)}(s; m, n),
\end{align*}
\]  

(4.27)

where

\[
\begin{align*}
\hat{u}(s; m, n) &= \frac{\hat{w}_0^0(s; m, n + 1)}{\hat{w}_0^0(s - 1; m, n)} = \frac{\tau(s - 1; m, n)\tau(s + 1; m, n + 1)}{\tau(s; m, n)\tau(s; m, n + 1)}, \\
\hat{v}(s; m, n) &= \frac{\hat{w}_0^0(s; m + 1, n)}{\hat{w}_0^0(s - 1; m, n)} = \frac{\tau(s - 1; m, n)\tau(s + 1; m + 1, n)}{\tau(s; m, n)\tau(s; m + 1, n)}.
\end{align*}
\]  

(4.28)
Proof: Setting \( x'_k = x_k, y'_k = \tilde{y}(m, n + 1), y_k = \tilde{y}(m, n) \) \((k = 1, 2, \ldots)\) in (3.9), we have
\[
\hat{W}^{(\infty)}(s; m, n + 1)\hat{W}^{(\infty)}(s; m, n)^{-1} = \hat{W}^{(0)}(s; m, n + 1)\left(1 - a_ne^{-\partial_s}\right)\hat{W}^{(0)}(s; m, n)^{-1},
\]
where we have used the formula \( \exp(-\sum_{n=0}^{\infty} a^n/n) = 1 - z \). Since the left-hand side of (4.29) is of non-positive order with respect to \( e^{\partial_s} \), it follows that it is of the form
\[
(4.29) = \tilde{c}_0(s; m, n) + \tilde{c}_1(s; m, n)e^{-\partial_s}.
\]
Inserting \( \hat{W}^{(\infty)} \) and \( \hat{W}^{(0)} \) of (3.6) to (4.29) with (4.30), we obtain the first and the second equation of (4.27). The third and the fourth can be obtained in the same fashion. \( \square \)

Remark: Tsujimoto [27] proposed and investigated the equations (4.27) as a discrete analogue of (3.8). In our approach, the results in [27] can be obtained directly from (3.9) with the Miwa transformation.

Hereafter in this section, we impose the 2-reduction condition \( \tau(s + 2; m, n) = \tau(s; m, n) \). From the first and the third equations of (4.27), we obtain
\[
\begin{align*}
\hat{w}_1^{(\infty)}(s; m, n + 1) &= \hat{w}_1^{(\infty)}(s; m, n) - a_n\tilde{u}(s; m, n), \\
\hat{w}_1^{(\infty)}(s; m, n + 1) &= \hat{w}_1^{(\infty)}(s; m, n) - b_m\tilde{v}(s; m, n).
\end{align*}
\]
It follows that
\[
\begin{align*}
\frac{\hat{w}_1^{(\infty)}(s; m, n + 1) - \hat{w}_1^{(\infty)}(s; m, n)}{a_n} &= \mathcal{K}(s; m, n)\frac{\hat{w}_1^{(\infty)}(s; m, n) - \hat{w}_1^{(\infty)}(s; m, n - 1)}{a_{n-1}}, \\
\frac{\hat{w}_1^{(\infty)}(s; m, n + 1) - \hat{w}_1^{(\infty)}(s; m, n)}{b_m} &= \mathcal{W}(s; m, n)\frac{\hat{w}_1^{(\infty)}(s; m, n + 1) - \hat{w}_1^{(\infty)}(s; m, n)}{a_n},
\end{align*}
\]
with
\[
\begin{align*}
\mathcal{K}(s; m, n) &= \frac{\tilde{u}(s; m, n)}{\tilde{u}(s; m, n - 1)} = \frac{\tau(s + 1; m, n + 1)\tau(s; m, n - 1)}{\tau(s; m, n + 1)\tau(s + 1; m, n - 1)}, \\
\mathcal{W}(s; m, n) &= \frac{\tilde{v}(s; m, n)}{\tilde{u}(s; m, n)} = \frac{\tau(s + 1; m + 1, n)\tau(s; m, n + 1)}{\tau(s; m + 1, n + 1)\tau(s + 1; m, n + 1)}.
\end{align*}
\]
If we introduce \( \Theta(s; m, n) \) as
\[
\Theta(s; m, n) = \frac{\tau(s + 1; m, n + 1)\tau(s; m, n - 1)}{\tau(s; m, n + 1)\tau(s + 1; m, n - 1)},
\]
then \( \mathcal{K}(s; m, n) \) and \( \mathcal{W}(s; m, n) \) are written as
\[
\begin{align*}
\mathcal{K}(s; m, n) &= \frac{\Theta(s; m, n + 1)}{\Theta(s; m, n - 1)}, \\
\mathcal{W}(s; m, n) &= \frac{\Theta(s; m + 1, n)}{\Theta(s; m, n + 1)}.
\end{align*}
\]
We furthermore impose the reality condition (4.17). Under the condition, \( \Theta(s; m, n) \) satisfies \( |\Theta(s; m, n)| = 1 \) and one can set
\[
e^{\sqrt{-1}\Theta_n} = \Theta(s = 0; m, n) = \tau(1; m, n)/\tau(0; m, n).
\]
Theorem 10 (Representation formula for discrete curves in terms of the $\tau$-functions). If we set

$$
\gamma^m_n = \hat{\omega}^{(\infty)}(s = 0; m, n) = -\frac{\partial}{\partial x_1} \log \tau(0; m, n),
$$

$$
\theta^m_n = \frac{1}{\sqrt{-1}} \log \Theta(s = 0; m, n) = \frac{1}{\sqrt{-1}} \log \frac{\tau(1; m, n)}{\tau(0; m, n)},
$$

then $\gamma^m_n$ and $\theta^m_n$ solve the equations (4.21) and (4.22).

Proof. From (4.31) and (4.28), it follows that

$$
\left| \frac{\hat{\omega}^{(\infty)}(s; m, n + 1) - \hat{\omega}^{(\infty)}(s; m, n)}{a_n} \right| = \left| \frac{\tau(s - 1; m, n)\tau(s + 1; m, n + 1)}{\tau(s; m, n)\tau(s; m, n + 1)} \right| = 1
$$

under the condition (4.17). This is equivalent to the first equation of (4.21). The remaining equations follow directly from (4.32), (4.35) and (4.36). \qed

5 Fermionic construction of $\tau$-functions

In [25, 26], Takebe described $\tau$-functions for the Toda hierarchy as expectation values of fermionic operators (See also [24]). We firstly recall the definition of charged free fermions [12, 20].

Let $\mathcal{A}$ be an associative unital $\mathbb{C}$-algebra generated by $\psi_i, \psi_i^* (i \in \mathbb{Z})$ satisfying the relations

$$
\psi_i \psi_j^* + \psi_j \psi_i = \delta_{ij}, \quad \psi_i \psi_j + \psi_j \psi_i = \psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0.
$$

We consider a class of infinite matrices $A = [a_{ij}]_{i,j \in \mathbb{Z}}$ that satisfies the following condition:

there exists $N > 0$ such that $a_{ij} = 0$ for all $i, j$ with $|i - j| > N$. (5.2)

Define the Lie algebra $\mathfrak{gl}(\infty)$ as [12]

$$
\mathfrak{gl}(\infty) = \left\{ \sum_{i,j \in \mathbb{Z}} a_{ij} : \psi_i \psi_j^* : \right\} A = [a_{ij}]_{i,j \in \mathbb{Z}} \text{ satisfies (5.2)} \oplus \mathbb{C}
$$

where $: \cdot :$ indicates the normal ordering

$$
: \psi_i \psi_j^* : = \begin{cases} \psi_i \psi_j^* & \text{if } i \neq j \text{ or } i = j \geq 0, \\ -\psi_j^* \psi_i & \text{if } i = j < 0. \end{cases}
$$

We also define the group $\mathbf{G}$ corresponds to $\mathfrak{gl}(\infty)$ to be

$$
\mathbf{G} = \left\{ e^{X_1} e^{X_2} \cdots e^{X_i} \mid X_i \in \mathfrak{gl}(\infty) \right\}.
$$

Consider a left $\mathcal{A}$-module with a cyclic vector $|\text{vac}\rangle$ satisfying

$$
\psi_j |\text{vac}\rangle = 0 \quad (j < 0), \quad \psi_j^* |\text{vac}\rangle = 0 \quad (k \geq 0).
$$

The $\mathcal{A}$-module $\mathcal{A}|\text{vac}\rangle$ is called the fermion Fock space $\mathcal{F}$, which we denote $\mathcal{F}$. We also consider a right $\mathcal{A}$-module (the dual Fock space $\mathcal{F}^*$) with a cyclic vector $\langle \text{vac}|$ satisfying

$$
\langle \text{vac}| \psi_j = 0 \quad (j \geq 0), \quad \langle \text{vac}| \psi_j^* = 0 \quad (k < 0).
$$

10
We further define the generalized vacuum vectors $|s\rangle$, $\langle s|$ ($s \in \mathbb{Z}$) as

$$
|s\rangle = \begin{cases} 
\psi_s^* \cdots \psi_1^* |\text{vac}\rangle & \text{for } s < 0, \\
|\text{vac}\rangle & \text{for } s = 0, \\
\psi_{s-1} \cdots \psi_0 |\text{vac}\rangle & \text{for } s > 0,
\end{cases}
$$

(5.8)

$$
\langle s| = \begin{cases} 
\langle \text{vac}| \psi_{s-1} \cdots \psi_1 & \text{for } s < 0, \\
\langle \text{vac}| & \text{for } s = 0, \\
\langle \text{vac}| \psi_s^* \cdots \psi_{s-1}^* & \text{for } s > 0.
\end{cases}
$$

There exists a unique linear map (the vacuum expectation value) $\mathcal{F}^* \otimes \mathcal{A} \mathcal{F} \rightarrow \mathbb{C}$ such that $\langle \text{vac}| \otimes |\text{vac}\rangle \mapsto 1$. For $a \in \mathcal{A}$ we denote by $\langle \text{vac}|a|\text{vac}\rangle$ the vacuum expectation value of the vector $\langle \text{vac}|a|\text{vac}\rangle = \langle \text{vac}|a \otimes |\text{vac}\rangle$ in $\mathcal{F}^* \otimes \mathcal{A} \mathcal{F}$.

**Theorem 11** ([25] §2, [26] §2). For $s \in \mathbb{Z}$ and $g \in \mathcal{G}$, define $\tau_g(s; x, y)$ as

$$
\tau_g(s; x, y) = \langle s| e^{H(x)} g e^{-H(y)} |s\rangle,
$$

(5.9)

where

$$
H(x) = \sum_{n=1}^{\infty} x_n \sum_{j \in \mathbb{Z}} \psi_j \psi_j^*, \quad H(y) = \sum_{n=1}^{\infty} y_n \sum_{j \in \mathbb{Z}} \psi_{j+n} \psi_j^*.
$$

(5.10)

Then $\tau_g(s; x, y)$ satisfies the bilinear identity (3.15).

We introduce an automorphism $\iota_l$ of $\mathcal{A}$ by

$$
\iota_l(\psi_i) = \psi_{i-l}, \quad \iota_l(\psi_i^*) = \psi_{i-l}^*,
$$

(5.11)

which satisfies

$$
\langle s'|a|s\rangle = \langle s' - l\iota_l(a)|s - l\rangle
$$

(5.12)

for any $s, s', l$ and any $a \in \mathcal{A}$.

**Proposition 12.** If $g \in \mathcal{G}$ satisfies

$$
\iota_l(g) = \overline{g},
$$

(5.13)

then the $\tau$-function corresponds to $g$ gives a solution of the Goldstein-Petrich hierarchy.

**Proof.** From (5.12) and (5.13), it is clear that (4.17) holds. \hfill \Box

To construct soliton-type solutions, we choose $g$ as

$$
g_N(c_j, \{p_j\}, \{q_j\}) = \prod_{j=1}^{N} e^{c_j \psi(p_j) \psi^*(q_j)},
$$

$$
\psi(p) = \sum_{j \in \mathbb{Z}} \psi_j p^j, \quad \psi^*(q) = \sum_{j \in \mathbb{Z}} \psi_j^* q^{-j}.
$$

(5.14)

We remark that the vacuum expectation value of $e^{c \psi(p) \psi^*(q)}$ makes sense even when $X = c \psi(p) \psi^*(q)$ does not satisfy the condition (5.2):

$$
\langle s| e^{c \psi(p) \psi^*(q)} |s\rangle = \langle s| \{1 + c \psi(p) \psi^*(q)\} |s\rangle = 1 + \left(\frac{p}{q}\right)^s \frac{cq}{p - q}.
$$

(5.15)

We consider the following two types of conditions for the parameters in (5.14):

11
A. (Soliton solutions)

\[ c_j \in \sqrt{-1}\mathbb{R}, \quad p_j \in \mathbb{R}, \quad q_j = -p_j \quad (j = 1, 2, \ldots, N), \]  \hspace{1cm} (5.16)

B. (Breather solutions)

\[ N = 2M, \quad c_{2k-1} = -c_{2k}, \quad p_{2k-1} = p_{2k} \quad (k = 1, 2, \ldots, M), \]
\[ q_j = -p_j \quad (j = 1, 2, \ldots, N). \]  \hspace{1cm} (5.17)

An straightforward calculation shows that \( g_N(c_j, \{p_j\}, \{q_j\}) \) satisfies (5.13) under each of the conditions (5.16), (5.17). The \( \tau \)-functions under these conditions provide the solutions given in [9, 10].

We now consider Lie algebraic meaning of the condition (5.13). We recall the facts about a fermionic representation of the affine Lie algebra \( \widehat{\mathfrak{sl}}(2, \mathbb{C}) \). The affine Lie algebra \( \widehat{\mathfrak{sl}}(2, \mathbb{C}) \) is generated by the Chevalley generators \( \{e_0, e_1, f_0, f_1, h_0, h_1\} \) that satisfy

\[
\begin{align*}
[h_i, h_j] &= 0, \quad [e_i, f_j] = \delta_{ij} h_i \text{ for all } i, j, \\
h_i, e_j &= \begin{cases} 2e_j & \text{if } i = j, \\ -2e_j & \text{if } i \neq j, \end{cases} \quad [h_i, f_j] = \begin{cases} -2e_j & \text{if } i = j, \\ 2e_j & \text{if } i \neq j, \end{cases} \\
[e_i, [e_i, e_j]] = [f_i, [f_i, f_j]] &= 0 \text{ if } i \neq j.
\end{align*}
\]  \hspace{1cm} (5.18)

Define a linear map \( \pi : \widehat{\mathfrak{sl}}(2, \mathbb{C}) \to \mathfrak{gl}(\infty) \) as

\[
\begin{align*}
\pi(e_j) &= \sum_{n \equiv j \mod 2} \psi_{n-1}\psi_n^*, \quad \pi(f_j) = \sum_{n \equiv j \mod 2} \psi_n\psi_{n-1}^*, \\
\pi(h_j) &= \sum_{n \equiv j \mod 2} (\psi_{n-1}\psi_{n-1}^* - \psi_n\psi_n^*) + \delta_{j0} \quad (j = 0, 1).
\end{align*}
\]  \hspace{1cm} (5.19)

**Theorem 13** ([12, 20]). \((\pi, \mathcal{F})\) is a representation of \( \widehat{\mathfrak{sl}}(2, \mathbb{C}) \).

Note that \( \iota_1 \) works as an involutive automorphism:

\[ \iota_1(e_0) = e_1, \quad \iota_1(f_0) = f_1, \quad \iota_1(e_1) = e_0, \quad \iota_1(f_1) = f_0, \]  \hspace{1cm} (5.20)

which defines a real form of \( \widehat{\mathfrak{sl}}(2, \mathbb{C}) \). Kobayashi [15] classified automorphisms of prime order of the affine Lie algebra \( \widehat{\mathfrak{sl}}(n, \mathbb{C}) \). The involutive automorphism \( \iota_1 \) under consideration is labeled as \( (1\alpha') \)-type ([15], Theorem 3). We remark that the same real form of \( \widehat{\mathfrak{sl}}(2, \mathbb{C}) \) appeared also in construction of solutions of a derivative nonlinear Schrödinger equation [14].

**Appendix: Time-flows with positive weight**

So far, we have used the time-evolutions with respect to the variables with negative weight \( y = (y_1, y_2, \ldots) \) to derive the Goldstein-Petrich hierarchy. In this appendix, we use \( x = (x_1, x_2, \ldots) \) and show that the mKdV hierarchy can be obtained under the 2-reduction condition (4.1). Applying the condition (4.1), one can show that

\[ B_{2n-1}(s) = e^{(2n-1)\partial_s} + \sum_{-2(n-1)\leq j \leq 0} b_j(s) e^{(2n-2+j)\partial_s}, \]  \hspace{1cm} (A.1)

\[ B_{2n}(s) = e^{2n\partial_s} \quad (n = 1, 2, \ldots). \]
From (3.2) and (4.1), we obtain
\[ b_0(s + 1) + b_0(s) = 0, \]
\[ b_{-k-1}(s + 1) + b_{-k-1}(s) + \sum_{j=0}^{L} b_j(s)b_{j-k}(s-j) = 0 \quad (k = 0, 1, 2, \ldots). \]  

Applying (A.1) to (3.4), we obtain
\[ \frac{\partial b_0(s)}{\partial x_{2n-1}} = b_{-2n+1}(s + 1) - b_{-2n+1}(s). \]  

Define \( L_1(x, y), L_2(x, y) \) by
\[ L_1(x, y) = \frac{1}{2} \left\{ L^{(0)}(s = 0; x, y) - L^{(0)}(s = 1; x, y) \right\}, \]
\[ L_2(x, y) = \frac{1}{2} \left\{ L^{(0)}(s = 0; x, y) + L^{(0)}(s = 1; x, y) \right\}, \]
which have the following form:
\[ L_1(x, y) = \sum_{n=0}^{\infty} q_n(x, y)e^{-n\delta_s}, \quad L_2(x, y) = e^{\delta_s} + \sum_{n=1}^{\infty} r_n(x, y)e^{-n\delta_s}, \]
\[ q_n(x, y) = \frac{b_{-n}(s = 0, x, y) - b_{-n}(s = 1, x, y)}{2} \quad (n = 0, 1, 2, \ldots), \]
\[ r_n(x, y) = \frac{b_{-n}(s = 0, x, y) + b_{-n}(s = 1, x, y)}{2} \quad (n = 1, 2, 3, \ldots). \]

We remark that \( q_n \) and \( r_n \) are eigenfunctions of \( e^{\delta_s} \):
\[ e^{\delta_s}q_n = -q_n, \quad e^{\delta_s}r_n = r_n. \]  
Applying the notation (A.5) to (A.3), we have
\[ \frac{\partial q_0}{\partial x_{2n-1}} = -2q_{2n-1} \]  
Since \( B_1(0), B_1(1) \) are of the form
\[ B_1(0) = e^{\delta_s} + q_0, \quad B_1(1) = e^{\delta_s} - q_0, \]  
it follows that
\[ \frac{\partial L_1}{\partial x_1} = -2L_1e^{\delta_s} + [q_0, L_2], \quad \frac{\partial L_2}{\partial x_1} = [q_0, L_1], \]
and hence
\[ \frac{\partial q_{2n-1}}{\partial x_1} = -2q_{2n} + 2q_0r_{2n-1}, \quad \frac{\partial q_{2n}}{\partial x_1} = -2q_{2n+1}, \]
\[ \frac{\partial r_{2n-1}}{\partial x_1} = 2q_0q_{2n-1}, \quad \frac{\partial r_{2n}}{\partial x_1} = 0. \]  
From (A.7) and (A.10), we have
\[ \frac{\partial q_0}{\partial x_{2n+1}} = \left( \frac{1}{4} \frac{\partial^2}{\partial x_1^2} - q_0^2 - \frac{\partial q_0}{\partial x_1} \right) \frac{\partial q_0}{\partial x_{2n-1}}. \]
Especially for the case $n = 1$,

$$\frac{\partial q_0}{\partial x_3} = \frac{1}{4} \frac{\partial^3 q_0}{\partial x_1^3} - \frac{3}{2} \frac{\partial q_0}{\partial x_1}.$$  \hfill (A.12)

After suitable scaling, the linear operator appeared in the right-hand side of (A.11) yields the recursion operator $\Omega$ in (2.7), and the equation (A.12) yields the mKdV equation (1.1).

We remark that another derivation of the recursion operator $\Omega$ in terms of bilinear differential equations of Hirota-type was given in [29]. Here we briefly summarize the approach in [29]. We use the Hirota differential operators $D_x, D_y, \ldots$, defined by

$$D^n_x D^n_y f(x,y) \cdot g(x,y) = (\partial_x - \partial_x')^n (\partial_y - \partial_y')^n f(x,y)g(x',y') \big|_{x'=x, y'=y}.$$ \hfill (A.13)

Setting $s'=0$, $s=1$ $y'_n = y_n$, $x'_n = x_n + a_n$ ($n = 1, 2, \ldots$), the bilinear identity (3.15) is reduced to

$$\int \tau(0; x'-[\lambda^{-1}], y) \tau(1; x + [\lambda^{-1}], y) e^{\delta(x'-x, y) \lambda^{-1} d\lambda} = \tau(1; x', y) \tau(0; x, y),$$ \hfill (A.14)

or, using the Hirota operators $\tilde{D} = (D_1, D_2/2, D_3/3, \ldots), D_j = D_{x_j}$ ($j = 1, 2, \ldots$), we can write

$$\sum_{j=0}^\infty p_j(-2a)p_j(\tilde{D}) \left( \sum_{k=1}^\infty a_k D_k \right) \tau(0) \cdot \tau(1) = \exp \left( \sum_{k=1}^\infty a_k D_k \right) \tau(0) \cdot \tau(0),$$ \hfill (A.15)

for any $a = (a_1, a_2, \ldots)$ (cf. [17]). Expanding (A.15) with respect to the variables $a = (a_1, a_2, \ldots)$, we obtain

$$\left( p_m(\tilde{D}) - D_m \right) \tau(1) \cdot \tau(0) = 0$$ \hfill (A.16)

from the coefficient of $a_m$, and

$$\left( -2p_{m+k}(\tilde{D}) + p_m(\tilde{D}) D_k + p_k(\tilde{D}) D_m \right) \tau(1) \cdot \tau(0) = 0$$ \hfill (A.17)

from the coefficient of $a_m a_k$. Using (A.16) to eliminate the first term in (A.17), we have

$$\left( -2D_{m+k} + p_m(\tilde{D}) D_k + p_k(\tilde{D}) D_m \right) \tau(1) \cdot \tau(0) = 0.$$ \hfill (A.18)

Hereafter we impose the 2-reduction condition $\partial_{x_2}, \tau = 0$ ($n = 1, 2, \ldots$). Setting $k = 2$, the bilinear equations (A.16), (A.18) yield

$$D^2_1 \tau(1) \cdot \tau(0) = 0, \quad (-4D_{m+2} + D^2_1 D_m) \tau(1) \cdot \tau(0) = 0.$$ \hfill (A.19)

If we set

$$\psi = \log \left( \frac{\tau(1)}{\tau(0)} \right), \quad \phi = \log \left( \frac{\tau(0)}{\tau(1)} \right),$$ \hfill (A.20)

it follows that

$$(\partial_1 \psi)^2 + \partial^2_1 \phi = 0, \quad -4\partial_{m+2} \psi + \partial^2_1 \partial_m \psi + 2(\partial_1 \psi)(\partial_1 \partial_m \phi) = 0,$$ \hfill (A.21)

from (A.19), where $\partial_n = \partial/\partial x_n$. Setting

$$q_0 = \partial_1 \psi = \partial_1 \left( \log \frac{\tau(1)}{\tau(0)} \right),$$ \hfill (A.22)

we have the recursion relation (A.11).
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