What happens to Lattice Fermion near Continuum Limit?

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ABSTRACT: A Ginsparg-Wilson Relation (GWR) is obtained in the presence of chiral symmetry breaking terms. It leads to the PCAC relation as well as an anomaly relation on the lattice. For general fermions, the deviation from the exact GWR is getting small when the block-spin transformations are performed iteratively. Based on a simple geometrical interpretation of the Dirac operator satisfying the GWR, we find some physical properties shared by the lattice fermions near the continuum limit. In two-dimensions, we explicitly construct the GW Dirac operator by using a conformal mapping.

KEYWORDS: Lattice Quantum Field Theory, Renormalization Regularization and Renormalons, Anomalies in Fields and String Theories.
1. Introduction

The chiral symmetry on the lattice has been found by Lüscher [1] based on the Ginsparg-Wilson Relation (GWR) [2]. He has opened a new era in the study of the regularized theory of chiral fermion. His formalism has been applied to chiral gauge theories, numerical calculations of chiral dynamics and supersymmetry on the lattice [3-17].

The purpose of this paper is to investigate the realization of the chiral symmetry near the continuum limit. The related problem was discussed by Hasenfratz based on his idea of the perfect action [18]: he obtained the relation equivalent to the GWR using the block-spin transformation (BST). However, his discussion was restricted to the behaviors of lattice fermions on the renormalized trajectory.

We first consider a BST for chiral non-invariant fermion from a fine-lattice to coarse-lattice, and derive a GWR with symmetry breaking terms. Since the chiral limit can be taken independently of the continuum limit, it is possible to obtain the PCAC relation on the lattice. We also find an anomaly relation between the fermions on the fine-lattice and the coarse-lattice. We next show that the Dirac operator satisfying the conventional GWR can be regarded as a mapping from a torus to a sphere. Based on this identification, it can be understood that a massless state corresponds to the South Pole of the sphere and massive states (doublers) appear at the North Pole. We want to emphasize that massless and massive states can have different charges under a Lüscher’s chiral transformation. That is the reason why the GW fermion escapes from the no-go theorem. We discuss the BSTs of the lattice fermions such as the Wilson fermion and the Domain Wall fermion. It is shown that they must satisfy the GWR as approaching the ultraviolet fixed point (FP). They share the following physical properties near the continuum limit: (i) All doublers have the same mass with order $O(a^{-1})$. This is compatible with the chiral symmetry proposed by Lüscher. (ii) Restoration of the “rotational symmetry”. So, the GW fermion has more symmetries than other lattice fermions.

The present paper is organized as follows. In section 2, we derive the GWR with a breaking term, chiral anomaly term and BST for Dirac operator. Section 3 is devoted to properties of GWR. We analyze block-spin transformed lattice fermion in section 4. In section 5, we explicitly construct, for two dimensional case, the Dirac operator near the continuum limit using a conformal mapping.

2. GWR, Anomaly and BST

In order to investigate the continuum limit which corresponds to a FP, we will use the BST from a microscopic theory to a macroscopic (effective) theory. Let $A_c$ be an action of the high frequency modes $\psi_n$ and $\bar{\psi}_n$ with fine lattice index $n$, and $A$ be the action of the low frequency modes $\Psi_N$ and $\bar{\Psi}_N$ with coarse-lattice index $N$. 


We assume that the action $A_c$ is bi-linear in $\psi_n$ and $\bar{\psi}_n$. The chiral transformation for the fine lattice variables is given by

$$
\delta \psi = -i\gamma_5\psi, \quad \delta \bar{\psi} = \bar{\psi}(-i\gamma_5).
$$

(1)

under which the action transforms as

$$
\delta A_0(\psi, \bar{\psi}) = A_0(\psi + \delta \psi, \bar{\psi} + \delta \bar{\psi}) - A_0(\psi, \bar{\psi}).
$$

(2)

There are two kinds of chiral breaking for which $\delta A_0 \neq 0$: (i) The breaking due to the lattice regularization. For example, the actions for the Wilson fermion and the Domain Wall fermion are not invariant under eq.(1). (ii) Explicit breaking such as the mass term.

We derive a general GWR for the coarse-lattice action $A$ in the presence of the chiral breaking terms. The action is defined as

$$
A \equiv -\ln \int_{\bar{\psi}, \psi} K \exp(-A_0[\psi, \bar{\psi}]) = \bar{\Psi}D\Psi.
$$

(3)

The block spin kernel, $K$, takes of the form

$$
K \sim \exp(-(\bar{\Psi} - \bar{\psi}f^*)\alpha(\Psi - f\psi)),
$$

(4)

where the functions $f(f^*)$ specify the BST and normalized as $f^*_n f_{nm} = \delta_{NM}, \quad f^*_n f_{nm} = \delta_{nm}$. We take $\alpha_{MN} = \alpha \delta_{MN}$ so that $K$ becomes local. The following formula for $K$ is useful:

$$
\frac{\partial}{\partial \Psi} \alpha^{-1}\{\alpha, \gamma_5\} \frac{\partial}{\partial \bar{\Psi}} K = (\bar{\Psi} - \bar{\psi}f^*)\{\alpha, \gamma_5\}(\Psi - f\psi)K.
$$

(5)

We obtain the general GWR,

$$
\bar{\Psi}(\gamma_5 D + D \gamma_5)\Psi = \bar{\Psi}D\alpha^{-1}\{\alpha, \gamma_5\} \alpha^{-1} D\Psi - i <\delta A_0 > |\Psi\psi|,
$$

(6)

where $<\cdots>$ is defined as

$$
<\cdots>= \int_{\bar{\psi}, \psi} \cdots K \exp(-A_0[\psi, \bar{\psi}]) / \int_{\bar{\psi}, \psi} K \exp(-A_0[\psi, \bar{\psi}]),
$$

(7)

and $|\Psi\psi|$ implies the component of $\bar{\Psi}\Psi$. In section 4, we discuss the symmetry breaking term corresponding to the contribution $<\delta A_0 > |\Psi\psi|$ for the Wilson fermions.

Next we consider a massive fermion on the fine lattice whose action takes the bilinear form, $A_0 = \bar{\psi}(D_0 + m)\psi$. Then, the general GWR becomes

$$
\bar{\Psi}(\gamma_5 D + D \gamma_5)\Psi = \bar{\Psi}D\alpha^{-1}\{\alpha, \gamma_5\} \alpha^{-1} D\Psi + 2m \bar{\Psi}\alpha(\alpha + m + fD_0 f^*)^{-1}\gamma_5(\alpha + m + fD_0 f^*)^{-1} \alpha \Psi
$$

(8)
Performing the Gaussian integral eq.(3), we find the relation between the Dirac operator $D$ for the coarse-lattice variables and the $D_0$ for fine-lattice variables

$$D^{-1} = \alpha^{-1} + f(D_0 + m)^{-1}f^*.$$ (9)

This leads to the general GWR with the mass breaking term:

$$D\Gamma_5 + \Gamma_5 D = 2m(1 - \frac{D}{\alpha})\Gamma_5,$$ (10)

where $\Gamma_5 \equiv \gamma_5(1 - \frac{D}{\alpha})$. It corresponds to the “PCAC” relation for the coarse-lattice fermions. From this “PCAC” relation on the coarse-lattice, we can consider the chiral symmetry breaking, e.g. the pion mass. In eq.(10), we observe that the effective fermion mass on the coarse lattice should be regarded as $m(1 - \frac{D}{\alpha})$. Note that the l.h.s. of eq.(10) is written entirely with quantities defined for the coarse lattice, while the r.h.s. contains also the microscopic information (microscopic mass, $m$).

The general GWR eq.(10) is obtained from the eq.(8) which is bi-linear in $\Psi$ and $\bar{\Psi}$. For the field-independent relation, we obtain

$$\delta J = -\frac{2}{\alpha} \text{Tr} \gamma_5 \{ D - m(1 - \frac{D}{\alpha}) \}$$ (11)

where $\delta J$ means the shift of path integral measure under the chiral transformation eq.(1) and the l.h.s corresponds to chiral anomaly generated by fine-lattice fermion. This equality implies the anomaly generated by microscopic fields is saturated with coarse-lattice (macroscopic) fields. Since $D$ includes a mass term, we have to eliminate the apparent effect of the mass. From $D - m(1 - \frac{D}{\alpha}) \to D$, the apparent mass dependence of the l.h.s is vanished after redefinition of the Dirac operator $D$.

When $m = 0$, there is a remnant of chiral symmetry which is called Lüscher’s symmetry

$$\delta \Psi = -i\Gamma_5 \Psi, \quad \delta \bar{\Psi} = \bar{\Psi}(-i\Gamma_5).$$ (12)

Eq.(11) implies the relation between microscopic and macroscopic anomalies, since r.h.s of eq.(11) is $\text{Tr} \{2\Gamma_5\}$. It is noted that the transformation of the low frequency modes is given by

$$\delta \Psi = <\delta \psi >, \quad \delta \bar{\Psi} = <\delta \bar{\psi} >.$$ (13)

This gives another interpretation of the Lüscher’s symmetry. Here we note that $<\psi>$ and $<\bar{\psi}>$ do not vanish because of the presence of the external fermions $\Psi$ and $\bar{\Psi}$.

In the proceeding sections, we find the properties of fermions with GWR and carry out BST explicitly for Wilson fermions.
3. GW fermion

In this section we investigate general properties of the free Dirac operator satisfying the GWR. They are useful for discussing in section 4 that Dirac operators are gradually satisfying the GWR by BSTs, and for constructing a Ginsparg-Wilson (GW) Dirac operator in section 5.

3.1. Properties of the GWR solutions

In a momentum space \( \{ p_\mu \} \), \( \mu = 1 \sim d \) \((d = \text{even})\), the parity-even free GW Dirac operator can be written as

\[
D = i\gamma_\mu S_\mu(p) + A(p),
\]

where \( \gamma_\mu \) denote the hermitian gamma matrices, and \( S_\mu(p), A(p) \) are appropriate functions of momentum \( p_\mu \). Using the above expression, the GWR can be expressed as

\[
S_\mu S_\mu + \left( A - \frac{\alpha}{2} \right)^2 = \left( \frac{\alpha}{2} \right)^2.
\]

It represents a d-dimensional spherical surface \( S^d \) with a radius of \(\alpha/2\) in a \((d+1)\)-dimensional space \( \{ S_\mu, A \} \). Since the momentum space \( \{ p_\mu \} \) is equivalent to a d-dimensional torus \( T^d \), we find that the GW Dirac operator is a mapping from \( T^d \) to \( S^d \):

\[
D : T^d \mapsto S^d.
\]

It is noted that eq. (15) has a rotational symmetry in a subspace \( \{ S_\mu \} \). Although this is only a fake symmetry and not a rotational one in the momentum space \( \{ p_\mu \} \), it is expected to become a real rotational symmetry in the continuum limit.

Each point on the \( S^d \) corresponds to each mode of the propagator. For \( A = 0 \), a massless mode appears at the South Pole (SP), and for \( A = \alpha \), massive modes do at the North Pole (NP) in Fig.1. Both Poles are fixed points under the fake rotational symmetry.

As an example, consider the Dirac operator \( D_{\text{neu}}(p) \) given by Neuberger [20],

\[
D_{\text{neu}}(p) = i\gamma_\mu S^\text{neu}_\mu(p) + A^\text{neu}_\mu(p),
\]

\[
\begin{align*}
S^\text{neu}_\mu(p) &= \sin p_\mu(H^2(p))^{-1/2}, \\
A^\text{neu}_\mu(p) &= 1 + \left( \sum_{\mu=1\sim d} (1 - \cos p_\mu) - M \right)(H^2(p))^{-1/2}, \\
H^2(p) &= \sum_{\nu=1\sim d} \sin^2 p_\nu + \left( \sum_{\nu=1\sim d} (1 - \cos p_\nu) - M \right)^2,
\end{align*}
\]
Figure 1: A massless mode (SP) and massive mode (NP) in \( \{S_1, S_2, A\} \)

where \( 0 < M < 2 \). From a simple calculation, it is found that this Dirac operator satisfies the equation of the d-dimensional sphere \( S^d \), \( S_{\mu} S^{\mu} + (A_{\text{neu}} - 1)^2 = 1 \) with \( \alpha = 2 \) and has the rotational symmetry in a subspace \( \{S_{\mu}^{\text{neu}}\} \). Fig.1 represents this \( S^{d=2} \) parameterized by momentum \( p_\mu \).

The GW Dirac operator defines a smooth mapping from torus \( T^d \) to sphere \( S^d \). However, since they are different in topology, the mapping is not one-to-one and several points on the torus may be mapped to a point on the sphere. For example, we observe in Fig.1 the North Pole is realized for several values of momentum. Fig.2 represents the momentum dependence of \( A(p) \), which shows the massless mode and the degenerate massive modes. From this figure we learn that the North Pole is actually realized three times for this two dimensional example, corresponding to the number of doublers. In section 5, we will observe the same feature in our new GW Dirac operator, different from the Neuberger’s one.

Next, we consider the Lüscher’s chiral transformation of the massless mode \( (S_\mu = 0, A = 0) \) and the massive modes \( (S_\mu = 0, A = \alpha) \). Since the transformation is given by

\[
\begin{align*}
\psi &\rightarrow \left[ 1 + i\theta \gamma_5 \left( 1 - \frac{D}{\alpha} \right) \right] \psi, \\
\bar{\psi} &\rightarrow \bar{\psi} \left[ 1 + i\theta \left( 1 - \frac{D}{\alpha} \right) \gamma_5 \right],
\end{align*}
\]

where \( \theta \) is an infinitesimal parameter, one obtains for the massless mode \( (D = 0) \)
Figure 2: Neuberger’s Dirac operator in \( \{p_1, p_2, A\} \). Three doublers appear in association with three maxima indicated by dots.

\[
\begin{aligned}
\psi_{\text{massless}} &\rightarrow (1 + i\theta \gamma_5) \psi_{\text{massless}}, \\
\bar{\psi}_{\text{massless}} &\rightarrow \bar{\psi}_{\text{massless}} (1 + i\theta \gamma_5),
\end{aligned}
\]

and for the massive modes \( (D = \alpha) \)

\[
\begin{aligned}
\psi_{\text{massive}} &\rightarrow \psi_{\text{massive}}, \\
\bar{\psi}_{\text{massive}} &\rightarrow \bar{\psi}_{\text{massive}}.
\end{aligned}
\]

It should be emphasized that the Lüscher’s chiral transformation allows the mass for the doublers. This fact looks incompatible with the no-go theorem. However, it is not the case, since the Lüscher’s chiral symmetry does not satisfy one of the assumptions for the no-go theorem, the locality of the chiral charge.

In the presence of interactions, the Dirac operator may not take the free fields form \( D = i\gamma_\mu S_\mu (p) + A(p) \), and cannot be interpreted as a d-dimensional sphere \( S^d \). Nevertheless, the GW Dirac operator always can be written as

\[
D = \frac{\alpha}{2} (1 - U),
\]

where \( U \) is a unitary operator. So the eigenvalues \( \lambda \) are given as

\[
\lambda = \frac{\alpha}{2} (1 - e^{i\theta}), \quad (-\pi \leq \theta < \pi).
\]

They distribute on a circle in a \( \{\text{Im}(\lambda), \text{Re}(\lambda)\} \) space [21, 22, 23]. For free field case, it is further possible to consider the Dirac operator in terms of the modes of the propagator.

3.2. The number of free parameters of the GW Dirac operator
Next, we will discuss the number of free parameters of the GW Dirac operator. Let $D_{GW}$ and $D'_{GW}$ be two GW Dirac operators. In order for them to describe the same physics, they should have the same “fermion determinant” and the same “anomaly”. Introducing two hermite matrices, $H = \gamma_5 D_{GW}, H' = \gamma_5 D'_{GW}$, we may express the unitary equivalence conditions as

\[
\begin{align*}
\text{Det } H &= \text{Det } H', \\
\text{Tr } H &= \text{Tr } H', \\
\text{Tr } [H]^n &= \text{Tr } [H']^n ,
\end{align*}
\]

where $2 \leq n \leq N - 1$ and $N$ denotes the total number of color, flavor, spinor indices and lattice points. The equations (20) and (21) are the “fermion determinant” and the “anomaly” conditions. Thus, the number of free parameters of the GW Dirac operator is $N - 2$, which is the number of conditions of eq. (22). This implies that there are many solutions of the GWR. Actually, in section 5 we will construct another GW Dirac operator which differs from $D_{new}$.

4. BST of Lattice Fermions

In this section, we consider the Dirac operators which do not satisfy the GWR, and show that the iterative use of BST makes them satisfy the GWR approximately. The Dirac operators gradually get the properties of the GW fermion:

\[
\begin{align*}
(i) \quad & \text{d-dimensional sphere } S^d \text{ in the } \{S_\mu, A\}\text{space} \\
(ii) \quad & \text{rotational symmetry in the } \{S_\mu\}\text{subspace} \\
(iii) \quad & \text{degenerate doubler masses.}
\end{align*}
\]

For comparison, we also consider a BST of the GW fermion.

4.1. Properties of the Wilson fermion

As a simple example, consider the 2D free Wilson Dirac operator $D_W(p)$ given by

\[
D_W(p) = i\gamma_\mu S^W_\mu (p) + A^W (p),
\]

\[
\begin{align*}
S^W_\mu (p) &= \sin p_\mu, \\
A^W (p) &= \sum_{\mu=1,2} (1 - \cos p_\mu).
\end{align*}
\]
The masses are defined by the values of \( A(p)^W \) for \( S^W_\mu(p) = 0 \). They are given in the case at hand by

\[
A^W(p) = \begin{cases} 
0 & \text{at } p_1 = p_2 = 0: \text{original massless mode} \\
2 & \text{at } p_1 = 0, p_2 = \pi \text{ or } p_1 = \pi, p_2 = 0: \text{doubler} \\
4 & \text{at } p_1 = p_2 = \pi: \text{doubler}.
\end{cases}
\]

The doubler masses split into two values (Fig.3). The Wilson fermion is represented in the \( \{S^W_\mu, A^W\} \) space as Fig.4. It is not a 2-dimensional sphere \( S^2 \), and there is no rotational symmetry in the \( \{S^W_\mu\} \) space. It follows from eq. (24)

\[
A^W = \sum_{\mu=1,2} \left( 1 \pm \sqrt{1 - (S^W_\mu)^2} \right).
\]

In Fig.4 SP, central saddle point (CP), and NP correspond to the mass \( A^W = 0, 2 \) and 4, respectively.

4.2. BST of the Wilson fermion

Next, we investigate that performing the BST eq.(9) of the Wilson fermion \( D^W = i\gamma_\mu S^W_\mu + A^W \), the fermion gets gradually the properties of the GW fermion. By a special choice of \( f_{Nn} \), the BST can be defined as

\[
\left[ -iS^W_\mu(p)\gamma_\mu + A^W(p) \right]_{N} = \frac{1}{2\alpha} + \frac{1}{2} \left[ \frac{-iS^W_\mu(\frac{\pi}{2})\gamma_\mu + A^W(\frac{\pi}{2})}{(S^W_\mu(\frac{\pi}{2}))^2 + (A^W(\frac{\pi}{2}))^2} \right]_{N-1},
\]

where \( 0 < \alpha < \infty \), the \( N \) of \( [\cdots]_N \) denotes the N-th BST. Each BST can be constructed from eq.(4) with a specific choice of the function \( f_{Nn} \). From eq. (27) we can
Figure 4: Wilson fermion in \(\{S_1, S_2, A\}\)

obtain

\[
\begin{align*}
\left[ S^W_\mu(p) \right]_N &= \left[ \frac{X_\mu(\frac{p}{2})}{X^2_\mu(\frac{p}{2}) + Y^2(\frac{p}{2})} \right]_{N-1}, \\
\left[ A^W(p) \right]_N &= \left[ \frac{Y(\frac{p}{2})}{X^2_\mu(\frac{p}{2}) + Y^2(\frac{p}{2})} \right]_{N-1},
\end{align*}
\]

where

\[
\begin{align*}
\left. X_\mu(\frac{p}{2}) \right|_{N-1} &= \frac{1}{2} \left[ \frac{S^W_\mu(\frac{p}{2})}{(S^W_\nu(\frac{p}{2}))^2 + (A^W(\frac{p}{2}))^2} \right]_{N-1}, \\
\left. Y(\frac{p}{2}) \right|_{N-1} &= \frac{1}{2\alpha} + \frac{1}{2} \left[ \frac{A^W(\frac{p}{2})}{(S^W_\nu(\frac{p}{2}))^2 + (A^W(\frac{p}{2}))^2} \right]_{N-1},
\end{align*}
\]

then we can discuss the block-spin transformed Dirac operator in the \(\{S^W_\mu, A^W\}\) space which represented as Fig.5. From this, we can visually understand that the more the Wilson fermion is repeatedly transformed by the BST, the more it accurately satisfies the GW fermion properties. Now verify analytically them. Considering the
Figure 5: BST of massless Wilson fermion. $\alpha$ is set to 2.

GWR,

$$\frac{A(p)}{(S(p))^2 + (A(p))^2} = \frac{1}{\alpha},$$  \hspace{1cm} (30)

and using eqs. (28) and (29), we get

$$\left[ \frac{A^W(p)}{(S^W(p))^2 + (A^W(p))^2} \right]_N = \frac{1}{\alpha} + \left( \frac{1}{2} \right)^N \left\{ \left[ \frac{A^W(\frac{p}{N})}{(S^W(\frac{p}{N}))^2 + (A^W(\frac{p}{N}))^2} \right]_0 - \frac{1}{2\alpha} \right\}$$

$$\rightarrow \frac{1}{\alpha}, \hspace{0.2cm} (N \rightarrow \infty).$$  \hspace{1cm} (31)

Thus it is found that the Wilson fermion (and also other fermions) satisfies the GWR in FP ($N \rightarrow \infty$). Let us consider the degeneracy of the doubler masses. From eqs. (28) and (29) the masses are given as

$$[A^W]_N = \begin{cases} 
0, \\
\frac{1}{Y(\ell)} \bigg|_{N-1} = \frac{2\alpha [A^W(\frac{\ell}{2})]_{N-1}}{\alpha + [A^W(\frac{\ell}{2})]_{N-1}}, \\
\frac{1}{Y(\ell')} \bigg|_{N-1} = \frac{2\alpha [A^W(\frac{\ell'}{2})]_{N-1}}{\alpha + [A^W(\frac{\ell'}{2})]_{N-1}}, 
\end{cases}$$  \hspace{1cm} (32)
where \( p'_\mu = (0, N\pi), \ p''_\mu = (N\pi, N\pi) \), fold following inequality,

\[
0 < \left[A^W(\frac{p'_\mu}{2}) \right]_{N-1} < \left[A^W(\frac{p''_\mu}{2}) \right]_{N-1},
\]
(33)

and the ratio of the doubler masses \( (\neq 0) \) is

\[
\left[ \frac{A^W(p''_\mu)}{A^W(p'_\mu)} \right]_N = \frac{\left[ A^W(\frac{p''_\mu}{2}) \right]_{N-1} + \alpha \left[ A^W(\frac{p'_\mu}{2}) \right]_{N-1}}{\left[ A^W(\frac{p''_\mu}{2}) \right]_{N-1} + \alpha} \rightarrow 1, \ (N \rightarrow \infty),
\]
(34)

then at the FP the doubler masses \( (\neq 0) \) become degenerate. On the other hand, from eq. (32) we can obtain the mass correction \( \delta^{(N)}m \) by the N-th BST,

\[
[A^W]_N = [A^W]_{N-1} + \delta^{(N)}m,
\]
(35)

\[
\delta^{(N)}m = \left[ A^W(\frac{p''_\mu}{2}) \right]_N - \left[ A^W(p'_\mu) \right]_N.
\]
(36)

This implies that the doubler masses \( (\neq 0) \) approaches to \( \alpha \) by the BST, and the massless mode is a fixed point of the BST. Therefore, as far as \( \alpha \) is finite \( (\alpha < \infty) \), all doubler masses become the same mass \( \alpha \) as \( N \rightarrow \infty \). Thus it is understood that performing the BST on the fermion, the fermion gradually gets the properties of the GW fermion.

4.3. BST of the GW fermion

For comparison, consider the BST of the GW fermion. The Neuberger Dirac operator in eq. (17) satisfies the GWR,

\[
S^\text{neu}_\mu(p)S^\text{neu}_\mu(p) + \left( A^\text{neu}(p) - \frac{\alpha}{2} \right)^2 = \left( \frac{\alpha}{2} \right)^2 \bigg|_{\text{micro}}.
\]
(37)

Thus, the BST is given as

\[
\left[ S^\text{neu}_\mu(p) \right]_{\text{MACRO}} = \frac{2\alpha A^\text{neu}(\frac{p}{2})S^\text{neu}_\mu(\frac{p}{2})}{\left[ S^\text{neu}_\mu(\frac{p}{2}) \right]^2 + 4 \left[ A^\text{neu}(\frac{p}{2}) \right]^2 \bigg|_{\text{micro}}},
\]
(38)

\[
\left[ A^\text{neu}(p) \right]_{\text{MACRO}} = \frac{4\alpha \left[ A^\text{neu}(\frac{p}{2}) \right]^2}{\left[ S^\text{neu}_\mu(\frac{p}{2}) \right]^2 + 4 \left[ A^\text{neu}(\frac{p}{2}) \right]^2 \bigg|_{\text{micro}}},
\]
(39)
and one can show that the $\left[ S_{\mu}^{neu}(p) \right]_{MACRO}$ and the $\left[ A_{\mu}^{neu}(p) \right]_{MACRO}$ satisfy the GWR also,

$$S_{\mu}^{neu}(p)S_{\mu}^{neu}(p) + \left( A_{\mu}^{neu}(p) - \frac{\alpha}{2} \right)^2 = \left( \frac{\alpha}{2} \right)^2_{MACRO}. \tag{40}$$

After all we have obtained the structure of chiral symmetry near the FP. That is, the GW fermion flows into the FP keeping the GWR. The other fermions flow into the FP getting the properties of the GW fermion approximately, and satisfy the GWR at FP at the end. Thus fermions have at least the approximate GWR near FP.

In the above discussion, we have considered only about the massless fermion case which had a bare mass $m_0 = 0$. Now, consider a massive fermion case with a non-zero bare mass ($m_0 \neq 0$). From the fact that all the modes are massive ($\neq 0$) and that the mass correction by the BST is given from eq. (36), the mass will become $\alpha$ at the FP (Fig.6). And the Dirac operator will satisfy the GWR.\footnote{Chandrasekharan introduced a bare mass for the GWR, but his GWR is different from ours\cite{24}.}

### 4.4. The domain wall fermion and massless states

We consider here fermion masses in the domain wall formulation\cite{25, 24, 27}. The domain wall fermion can be interpreted as a multi-flavor Wilson fermion with a negative mass. The number of flavor corresponds to the fifth dimensional size. Its

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig6.png}
\caption{BST of massive Wilson fermion. $\alpha$ is set to 2.}
\end{figure}
rule can be explained in view of the BST or the renormalization group as follows: increasing the flavor number or the fifth dimensional size in the domain wall fermion corresponds to looking at the behavior of the 4-dimensional fermion at larger scale, which would be described by an iteration of the BST’s. In the infinite size limit, it is known that the 4-dimensional effective fermion satisfies the GWR [28]. It just corresponds to the continuum limit. The degeneracy of the doublets mass and the restoration of the “rotational invariance” are essentially the same as those of one flavor Wilson fermion near the continuum limit.

In this subsection, we concentrate on the massless state in the domain wall fermion related with the the fifth dimensional size.

Let $N_S$ be the fifth dimensional size for the domain wall fermions [23, 26, 27]. Decompose a Dirac spinor $\Psi(x_\mu, x_5)$ as

$$\Psi(x_\mu, x_5) = P_L \Psi_L(x_\mu, x_5) + P_R \Psi_R(x_\mu, x_5),$$

(41)

where $P_{L,R} = (1 \pm \gamma_5)/2$. We impose the following appropriate boundary condition for $x_5$ on $\Psi(x_\mu, x_5)$,

$$\Psi_L(x_\mu, x_5 + N_S) = \Psi_R(x_\mu, x_5),$$

(42)

$$\Psi_R(x_\mu, x_5 + N_S) = \Psi_L(x_\mu, x_5).$$

(43)

The domain wall Dirac operator $D_{DW}(p_\mu, p_5)$ in a momentum space $\{p_\mu, p_5\}$ can be written as

$$D_{DW}(p_\mu, p_5) = i \sum_{\mu=1}^{\sim 4} \gamma_\mu S_{D_W}^{\mu} (p_\mu) + A^{D_W}(p_\mu, p_5),$$

(44)

where

$$\begin{cases}
S_{D_W}^{\mu}(p_\mu) = \sin p_\mu, \\
A^{D_W}(p_\mu, p_5) = -M + (1 - \cos p_5 + i\gamma_5 \sin p_5) + \sum_{\mu=1}^{\sim 4} (1 - \cos p_\mu),
\end{cases}$$

and $0 < M < 2$, $p_5 = \pi n/N_S$, $n = 1 \sim 2N_S$. The lowest mass $m_{\text{eff}}(p_5)$ which depends on $p_5$ is given by

$$m_{\text{eff}}(p_5) = A^{D_W}(p_\mu = 0, p_5) = -M + (1 - \cos p_5 + i\gamma_5 \sin p_5).$$

(45)

For each case of $\gamma_5 = \pm 1$, we denote each mass as

$$m_{\text{eff}}^\pm (p_5) = -M + (1 - \cos p_5 \pm i \sin p_5).$$

(46)

This yields a constraint,

$$-M \leq \text{Re}\{m_{\text{eff}}^\pm (p_5)\} \leq 2 - M.$$  

(47)
In the limit $N_S \to \infty$, each $\text{Re}\{m_{\text{eff}}^\pm(p_5)\}$ has a continuum spectrum with two zeros due to the periodicity of $p_5$. Then the number of total zeros are four. According to the sign of each imaginary part $\text{Im}\{m_{\text{eff}}^\pm\}$, only half of four zeros are allowed by causality. Therefore two allowed zeros are regarded as massless modes corresponding to $\gamma_5 = \pm 1$, and we obtain a massless fermion theory. For a finite $N_S$, each $\text{Re}\{m_{\text{eff}}^\pm(p_5)\}$ has a discrete spectrum. Thus $\text{Re}\{m_{\text{eff}}^\pm(p_5)\}$ does not have zero in general, and we obtain in this case a massive theory.‡ As $N_S$ increases, the minimum value of $|\text{Re}\{m_{\text{eff}}^\pm(p_5)\}|$ decreases, and finally becomes zero which corresponds to a massless mode.

The zero has no mass correction via the BST, that is a fixed point of the BST. Thus we can obtain a massless fermion theory which satisfies the GWR at the FP of the BST.

5. Lattice Fermions near the Continuum Limit

In this section, we show that the Neuberger's Dirac operator is not the unique GW Dirac operator, by constructing another free GW Dirac operator, $D_{GW} = i\gamma_\mu S_\mu + A$, for two dimensional case. As is shown in eq.(16), the construction of $D_{GW}$ is equivalent to find a mapping from a 2D torus $T^2$ to a 2D sphere $S^2$ in the $(S_1, S_2, A)$ space,

$$D_{GW}(p) : T^2 \mapsto S^2. \quad \text{(48)}$$

We obtain this mapping by using a combination of conformal mappings. Before going into the details of the construction, let us explain our idea. We divide the torus equally into four regions (see I below). Take one of the regions and map it to the Southern hemisphere conformally. It is important that this conformal mapping is unique due to the Riemann’s theorem. Each of three other regions is similarly mapped conformally to the Northern hemisphere. Our mapping is constructed by continuously connecting the four conformal mappings.

We will construct such a mapping with the following steps.

I : Divide the momentum space, a complex plane $P = p_1 + ip_2$, into four domains

$$P^{(1)} = \{(p_1, p_2) \mid -\frac{\pi}{2} \leq p_1 \leq \frac{\pi}{2}, -\frac{\pi}{2} \leq p_2 \leq \frac{\pi}{2}\},$$

$$P^{(2)} = \{(p_1, p_2) \mid \frac{\pi}{2} \leq p_1 \leq \frac{3\pi}{2}, -\frac{\pi}{2} \leq p_2 \leq \frac{\pi}{2}\},$$

$$P^{(3)} = \{(p_1, p_2) \mid -\frac{\pi}{2} \leq p_1 \leq \frac{\pi}{2}, \frac{\pi}{2} \leq p_2 \leq \frac{3\pi}{2}\},$$

$$P^{(4)} = \{(p_1, p_2) \mid \frac{\pi}{2} \leq p_1 \leq \frac{3\pi}{2}, \frac{\pi}{2} \leq p_2 \leq \frac{3\pi}{2}\}. \quad \text{(49)}$$

It is arranged in such a way that $(p_1, p_2) = (0, 0)$ in $P^{(1)}$ and $(p_1, p_2) = (\pi, 0), (0, \pi), (\pi, \pi)$ in $P^{(2),(3),(4)}$ correspond to a massless mode and massive modes, respectively.

‡It is possible, however, to make $\text{Re}\{m_{\text{eff}}^\pm(p_5)\} = 0$ for a certain value $p_5$ by making a fine-tuning for $M$. Thus we may construct a fine-tuned massless fermion theory for finite $N_S$. 

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II : Regarding the \((S_1, S_2)\) space as a complex plane with \(S = S_1 + iS_2\) and \(\bar{S} = S_1 - iS_2\), the GWR is written as
\[
SS + \left( A - \frac{\alpha}{2} \right)^2 = \left( \frac{\alpha}{2} \right)^2.
\]
(50)

This gives
\[
A_{\pm} = \frac{\alpha}{2} \pm \sqrt{\left( \frac{\alpha}{2} \right)^2 - SS},
\]
(51)
where \(A_{-}(A_{+})\) corresponds to the Southern (Northern) Hemisphere. It contains a massless (massive) mode as the South (North) Pole of \(S^2\). We construct conformal mappings \(S^{(1),(2),(3),(4)}\) from \(P^{(1),(2),(3),(4)}\) to disks with a radius of \(\alpha/2\),
\[
S^{(i)}(P^{(i)}) = \left\{ (S_1, S_2) \mid SS = S_1S_1 + S_2S_2 \leq \left( \frac{\alpha}{2} \right)^2 \right\}, \quad i = 1 \sim 4.
\]
(52)

III : From eqs. (51) and (52) we can get \(A_{-}\) and \(A_{+}\) corresponding to \(P^{(1)}\) and \(P^{(2),(3),(4)}\), respectively, as explicit function of momenta \(p_{\mu}\).

In this way we can construct a solution of the GWR, \(D_{GW} = i\gamma_{\mu}S_{\mu} + A\).

First let us construct the disk \(S^{(1)}\) as the conformal mapping, \(P^{(1)} \mapsto V^{(1)} \mapsto S^{(1)}\), where \(V^{(1)}\) denotes upper half plane mapped from \(P^{(1)}\). The conformal mapping from \(P^{(1)}\) to \(V^{(1)}\) is defined as
\[
V^{(1)} = \text{sn} \left( \frac{P^{(1)} + iK'}{2}, k \right),
\]
(53)
where \(\text{sn}\) is an elliptic function with modulus \(k\), and \(K'\) is the complete elliptic integral of the first kind with complementary modulus \(k' = \sqrt{1 - k'^2}\). The elliptic function \(\text{sn}(P, k)\) is a periodic function of \(4K, 2iK'\). In our case \(K = \frac{\pi}{2}, K' = \pi\), and \(k = (\sqrt{2} - 1)^2\). Also considering the conformal mapping from \(V^{(1)}\) to \(S^{(1)}\), we obtain the mapping \(P^{(1)} \mapsto S^{(1)}\),
\[
S^{(1)} = \frac{V^{(1)} - i(\sqrt{2} + 1)}{V^{(1)} + i(\sqrt{2} + 1)} \cdot \frac{i\alpha}{2}
= \frac{\text{sn} \left( \frac{P^{(1)} + \frac{i}{2}K'}{2}, k \right) - i(\sqrt{2} + 1)}{\text{sn} \left( \frac{P^{(1)} + \frac{i}{2}K'}{2}, k \right) + i(\sqrt{2} + 1)} \cdot \frac{i\alpha}{2}.
\]
(54)

*In order for the four points, \((p_1, p_2) = (0, 0), (\pi, 0), (0, \pi), (\pi, \pi)\) in \(P^{(1),(2),(3),(4)}\), to behave as poles, we require \(S^{(1),(2),(3),(4)} = 0\) at these points. This is indeed the case thanks to the Riemann’s mapping theorem which guarantees that \(S_{\mu} \sim ip_{\mu}\) for \(|p_{\mu}| \ll 1\).
Similarly, we construct the mapping, \( P^{(2),(3),(4)} \mapsto S^{(2),(3),(4)} \), paying attention to continuity at boundaries. The mapping, \( P^{(4)} \mapsto V^{(4)} \) is given by

\[
V^{(4)} = \text{sn} \left( -\left( \frac{P^{(4)}}{2} - 2K - iK' \right) + \frac{i}{2}K', k \right) \\
= \text{sn} \left( -\left( P^{(4)} - iK' \right) + \frac{i}{2}K', k \right) \\
= -\text{sn} \left( \frac{P^{(4)}}{2} + \frac{i}{2}K' \right) + 2iK', k \right) \\
= \text{sn} \left( P^{(4)} + \frac{i}{2}K', k \right),
\]

(55)

where we used the properties \( \text{sn}(P + 2K, k) = -\text{sn}(P, k) \), \( \text{sn}(P + 2iK', k) = \text{sn}(P, k) \), \( \text{sn}(-P, k) = -\text{sn}(P, k) \). Thus we obtain the mapping, \( P^{(4)} \mapsto S^{(4)} \)

\[
S^{(4)} = \frac{\text{sn} \left( \frac{P^{(4)}}{2} + \frac{i}{2}K', k \right) - i(\sqrt{2} + 1)}{\text{sn} \left( \frac{P^{(4)}}{2} + \frac{i}{2}K', k \right) + i(\sqrt{2} + 1)} \cdot \frac{i\alpha}{2}.
\]

(56)

As for \( V^{(2),(3)} \), it is convenient to take the complex conjugate \( \overline{P} \). The mappings, \( \overline{P}^{(2),(3)} \mapsto V^{(2),(3)} \mapsto S^{(2),(3)} \) are given by

\[
V^{(2)} = \text{sn} \left( -\left( \frac{P^{(2)}}{2} - 2K \right) + \frac{i}{2}K', k \right) \\
= -\text{sn} \left( \frac{P^{(2)}}{2} + \frac{i}{2}K', k \right) \\
= \text{sn} \left( P^{(2)} + \frac{i}{2}K', k \right) \\
= \text{sn} \left( P^{(2)} + \frac{i}{2}K', k \right),
\]

(57)

and

\[
V^{(3)} = \text{sn} \left( \frac{P^{(3)}}{2} - iK' \right) + \frac{i}{2}K', k \right) \\
= \text{sn} \left( \frac{P^{(3)}}{2} + \frac{i}{2}K' \right) + 2iK', k \right) \\
= \text{sn} \left( P^{(3)} + \frac{i}{2}K', k \right).
\]

(58)

We find that

\[
S^{(2),(3)} = \frac{\text{sn} \left( P^{(2),(3)} + \frac{i}{2}K', k \right) - i(\sqrt{2} + 1)}{\text{sn} \left( P^{(2),(3)} + \frac{i}{2}K', k \right) + i(\sqrt{2} + 1)} \cdot \frac{i\alpha}{2}.
\]

(59)

Using eqs. (51),(54),(56),(59), and \( P = P_1 + iP_2, S = S_1 + iS_2 \), we obtain a solution of the 2D free GW Dirac operator, \( D_{GW} = i\gamma_\mu S_\mu + A \). For the momentum domains
\( P^{(1),(4)} \),

\[
S_1 = 2 \text{Re} \left( \text{sn} \left( P + \frac{i}{2} K', k \right) \right) \frac{\alpha f_+}{2}, \quad \text{(60)}
\]

\[
S_2 = \left\{ (\sqrt{2} - 1) \text{sn} \left( P + \frac{i}{2} K', k \right) \text{sn} \left( P + \frac{i}{2} K', k \right) + (\sqrt{2} + 1) \right\} \frac{\alpha f_+}{2}, \quad \text{(61)}
\]

\[
A = \frac{\alpha}{2} \left\{ 1 + \sqrt{4 \text{Im} \left( \text{sn} \left( P + \frac{i}{2} K', k \right) \right) f_+} \right\}, \quad \text{(62)}
\]

where \(-\) and \(+\) in eq. (62) correspond to \( P^{(1)} \) and \( P^{(4)} \), respectively, and \( f_{\pm}^{-1} \) are defined as

\[
(f_{\pm})^{-1} = (\sqrt{2} - 1) \text{sn} \left( P + \frac{i}{2} K', k \right) \text{sn} \left( P + \frac{i}{2} K', k \right) + (\sqrt{2} + 1) \pm 2 \text{Im} \left( \text{sn} \left( P + \frac{i}{2} K', k \right) \right). \quad \text{(63)}
\]

For the momentum domains \( P^{(2),(3)} \),

\[
S_1 = 2 \text{Re} \left( \text{sn} \left( P + \frac{i}{2} K', k \right) \right) \frac{\alpha f_-}{2}, \quad \text{(64)}
\]

\[
S_2 = \left\{ (\sqrt{2} - 1) \text{sn} \left( P + \frac{i}{2} K', k \right) \text{sn} \left( P + \frac{i}{2} K', k \right) - (\sqrt{2} + 1) \right\} \frac{\alpha f_-}{2}, \quad \text{(65)}
\]

\[
A = \frac{\alpha}{2} \left\{ 1 + \sqrt{4 \text{Im} \left( \text{sn} \left( P + \frac{i}{2} K', k \right) \right) f_-} \right\}. \quad \text{(66)}
\]

In Fig.7, we show this solution in the \( \{S_1, S_2, A\} \) space. Although singular points appear on the equator,† the Dirac operator \( D_{GW} \) is a smooth function of \( p_\mu \).

6. Summary and Discussions

We have investigated behaviors of lattice Dirac operators near the fixed point (the continuum limit). For the purpose of our approach, we used a block-spin transformation method. The analysis lead us to the chiral anomaly and the GWR with chiral breaking terms. The breaking measures the distance between the fixed point action and the general lattice fermions. It was found that features of the operators are (i) masses of doublers are completely degenerate (ii) “rotational” symmetry. When we start from a Wilson fermion with a chiral breaking, it acquires these two properties after a few block-spin steps.

†In Fig.7, the white stripe around the equator is generated by a bug of the graphical software we use. After analytical calculations, it can be shown that upper hemisphere and lower hemisphere are continuously connected with each other.
In the case of multi-flavor Wilson fermions (the domain wall fermion), we found massless modes as a fifth dimension size becomes infinite. Even if we adopt more general lattice fermion, we could find massless modes because the fifth dimension size essentially controls the macroscopic fermion mass as the BST does.

In the 2-dimensional case, we constructed a Dirac operator, in addition to Neu- berger’s one, which satisfies the GWR using elliptic functions. This example teaches us the unique existence of the fixed point action (chiral symmetric lattice fermion), if it is possible to exchange between a massless mode and one of doublers because of uniqueness for conformal mapping by the Riemann’s theorem.

We are grateful to Y. Igarashi and K. Itoh for reading our manuscript carefully and invaluable comments. H.S. is supported in part by the Grants-in-Aid for Scientific Research No. 12640259 from Japan Society for the Promotion of Science.

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