Exactly solvable quantum impurity model with inverse-square interactions

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We construct an exactly solvable quantum impurity model consists of spin-1/2 conduction fermions and spin-1/2 magnetic moments. The ground-state wave function is a homogeneous Jastrow type polynomial and the parent Hamiltonian has nearest-neighbor hopping terms between the conduction fermions and inverse-square spin-exchange terms between the conduction fermions and the magnetic moments. The spin-spin correlation function between the conduction fermions and the magnetic moments and the low-lying energy levels of the parent Hamiltonian demonstrate that our model captures essential aspects of the spin-1/2 Kondo problem. The low-energy physics of our model is described by a boundary conformal field theory (CFT) and a particular correlator of this CFT reproduces the ground-state wave function. This connection paves the way toward constructing more exactly solvable quantum impurity models using boundary CFT.

Introduction — The study of magnetic impurity in quantum many-body systems has a long history \cite{1, 2}. Its foundation includes two models proposed by Anderson and Kondo that are closely related to each other \cite{3, 4}. The original motivation of Kondo was to understand the mysterious change of resistance in certain metallic samples with dilute magnetic atoms. The divergence found by Kondo signified the inapplicability of perturbative methods and various later attempts trying to save such methods also failed. Based on the idea of renormalization, Wilson developed the numerical renormalization group (NRG) method and brought out a rather complete solution of the Kondo problem \cite{5–7}. It is found that, at energies much lower than the Kondo temperature $T_K$, the model flows to a strong coupling fixed point so the magnetic moment is screened by conduction fermions. The NRG results helped Nozières to formulate a Fermi liquid theory which can describe the low-energy limit of the spin-1/2 Kondo problem \cite{8}. Another important development was the application of boundary conformal field theory (CFT) in the Kondo problem pioneered by Affleck and Ludwig \cite{9, 10}. This approach can reconstruct the Fermi liquid picture for the spin-1/2 Kondo problem, but its applicability further extends to more complicated quantum impurity systems that are not amenable to the Fermi liquid description.

Exactly solvable models are generally very desirable because they can provide valuable physical insights and serve as numerical benchmarks. For the Kondo problem, Andrei and Wiegmann independently constructed a model which can be solved exactly using the Bethe ansatz \cite{11–14}. The model is defined in a semi-infinite continuous system and has fermions with linear dispersion relations. The Kondo physics has also been explored in some other exactly solvable models \cite{15–19}. In this Letter, we propose an exactly solvable quantum impurity model in a one-dimensional open chain with spin-1/2 conduction fermions and one spin-1/2 magnetic moment at each end of the chain. It is different from the model of Andrei and Wiegmann in that it is defined on lattices and ground state exists when the number of sites is even. The most appealing property of our model is that the ground-state wave function is a Jastrow type homogeneous polynomial that can be written as a correlator of the boundary CFT for the model. This is reminiscent of the well-studied connection between edge CFT and bulk wave function in quantum Hall systems \cite{20}. Our work opens up the exciting possibility of constructing a variety of exactly solvable quantum impurity models based on boundary CFT.

The parent Hamiltonian of our model has nearest-neighbor hopping terms between conduction fermions and “inverse-square” spin-spin interactions between the conduction fermions and magnetic moments, i.e., the coupling strengths have trigonometric forms which fall off with distance in an inverse-square manner in the thermodynamic limit. To demonstrate that our model and the usual spin-1/2 Kondo model share the same essential physics, we perform numerical calculations to show that (i) the spin-spin correlation function between the impurity and fermion sites exhibits Kondo screening at short distance and inverse-square powerlaw decay at large distance, in agreement with boundary CFT predictions; (ii) the low-lying energy levels of the parent Hamiltonian resemble that of free fermions, consistent with the Fermi liquid picture of Nozières. It is further suggested that our model may be integrable based on level statistics of the energy spectrum.

Wave function — The system of our interest is an open chain with $L + 2$ sites ($L$ is even) labeled by $j = 0, 1, \cdots, L + 1$ [Fig. 1 (a)]. The chain is placed on the semi-circle with unity radius and projected onto the real line $[-1, 1]$. The angular position of the $j$-th site is $\theta_j = \frac{2\pi}{L + 1}$ and the associated linear position is $u_j = \cos \theta_j$. The motivation for this choice of coordinate will become clear later. The $1 \leq j \leq L$ sites are populated by spin-1/2 conduction fermions described by creation (annihilation) operators $c^\dagger_{j, \sigma}$, $c_{j, \sigma}$ with $\sigma = \uparrow, \downarrow$ being the spin index. The $j = 0$ and $L + 1$ sites are occupied by spin-1/2 magnetic moments described by operators $S_0$ and $S_{L+1}$. The
This can be seen most easily if we rewrite Eq. (1) as

\[ u \text{ system is half-filled. It is worth noting that Eq. (2) is} \]

The coefficient

\[ t \]

FIG. 1. (a) Schematics of the quantum impurity model. (b)-(c) Schematics of the procedure for computing \( t_j \) in the Hamiltonian Eq. (4). (d) \( t_j \) of the \( L = 100 \) system.

magnetic moments can be represented using Abrikosov fermions as \( \mathbf{S}_j = \frac{1}{2} \sum_{\sigma} c_{j, \sigma}^\dagger \sigma \mathbf{f} c_{j, \sigma} \), where \( \mathbf{f} \) are Pauli matrices [2]. The reducance caused by this representation is removed by imposing the single-occupancy constraint

\[ \sum_{\sigma} c_{j, \sigma}^\dagger c_{j, \sigma} = \sum_{\sigma} c_{j+1, \sigma} c_{L+1, \sigma} = 1. \]

The many-body state under consideration is

\[ |\Psi\rangle = \sum_{\{n_j^\uparrow\}, \{n_j^\downarrow\}} \Psi(\{n_j^\uparrow\}, \{n_j^\downarrow\}) (c_{0, \uparrow}^\dagger)^{n_0^\uparrow} \cdots (c_{L, \uparrow}^\dagger)^{n_L^\uparrow} \prod_{\sigma=\uparrow, \downarrow} \prod_{0 \leq j < k \leq L+1} (u_j - u_k)^{n_j^\sigma n_k^\sigma} |0\rangle. \tag{1} \]

The coefficient

\[ \Psi(\{n_j^\uparrow\}, \{n_j^\downarrow\}) = \delta_{n_0=0} \delta_{n_{L+1}=1} \delta_{j \sigma} \sum_j n_j^\uparrow = \sum_j n_j^\downarrow = \frac{L+1}{2} \]

is the wave function in the site-occupation basis, where \( n_j^\sigma = 0, 1 \) is the number of fermion with spin \( \sigma \) at site \( j \), \( n_j^\uparrow = \sum_{\sigma} n_j^\sigma \) is the total number of fermions on site \( j \), the first \( \delta \) imposes the single-occupancy constraints on the magnetic moments, and the second \( \delta \) means that the system is half-filled. It is worth noting that Eq. (2) is a homogeneous polynomial of \( u_j \) analogous to quantum Hall wave functions.

One can also prove that Eq. (2) is a spin-singlet. This can be seen most easily if we rewrite Eq. (1) as a Gutzwiller projected Fermi sea

\[ |\Psi\rangle = P^{G}_{L} P^{G}_{L+1} \prod_{\sigma=\uparrow, \downarrow} \prod_{m=0}^{L+1} \eta_{m, \sigma} |0\rangle, \tag{3} \]

where the Gutzwiller projector \( P^G_L = c_{j+1, \uparrow}^\dagger c_{j, \uparrow} + c_{j+1, \downarrow}^\dagger c_{j, \downarrow} - 2 c_{j+1, \uparrow}^\dagger c_{j, \downarrow} c_{j+1, \downarrow}^\dagger c_{j, \uparrow} \) enforces the single-occupancy constraint on the site \( j \) \((j = 0, L+1)\), and \( \eta_{m, \sigma} = \sum_{j=0}^{L+1} \cos \theta_j c_{j, \sigma} \) are non-orthogonal unnormalized orbitals constructed from linear combinations of conduction and Abrikosov fermions [21]. The participation of the Abrikosov fermions in the Fermi sea indicates hybridization between conduction fermions and magnetic moments.

**Parent Hamiltonian** — We claim that Eq. (1) is the exact ground state of the parent Hamiltonian

\[ H = H_0 + H_K, \tag{4} \]

where

\[ H_0 = - \sum_{j=1}^{L-1} j \sigma \right| \mathbf{S}_{j+1} \rangle \cdot \left( \cot \frac{\theta_j}{2} \mathbf{S}_0 + \tan^2 \frac{\theta_j}{2} \mathbf{S}_{L+1} \right) \tag{5} \]

describes nearest-neighbor hopping of the conduction fermions (with the constraint \( t_j = t_{L-j} \)) and

\[ H_K = \left( \frac{\pi}{2 L+2} \right)^2 \sum_{j=1}^{L} \left( \frac{\theta_j}{2} \right)^2 \mathbf{S}_0^2 + \tan^2 \frac{\theta_j}{2} \mathbf{S}_{L+1} \] stands for the long-range spin-exchange interactions between conduction fermions and magnetic moments. The total number of conduction fermions \( N_f = \sum_{j=1}^{L} \sum_{\sigma} c_{j, \sigma}^\dagger c_{j, \sigma} \), the total spin \( \mathbf{S}^2 = (\sum_{j=0}^{L+1} \mathbf{S}_j)^2 \), and its \( z \)-component \( S_z = \sum_{j=0}^{L+1} S_j^z \) are conserved quantities. The Hamiltonian also has two other symmetries: a mirror symmetry with respect to the vertical axis in Fig. 1 (a) \((c_{j, \sigma} \rightarrow c_{L-j, \sigma}) \) and \((c_{j, \sigma} \rightarrow c_{j+1, \sigma}) \) and a particle-hole symmetry \((c_{j, \sigma} \rightarrow (-1)^j c_{j, \sigma}) \) and \((c_{j, \sigma} \rightarrow (-1)^j c_{j, \sigma}) \).

It seems that the hopping amplitudes do not have any simple analytical forms but they can be computed iteratively. The basic idea is to compute \( \langle \alpha | H | \Psi \rangle \) by acting \( H \) to the left or right for some basis states \( \langle \alpha \rangle \). Let us assume that \( H | \Psi \rangle = E_0 | \Psi \rangle \) with \( E_0 \) being the ground state energy. If \( | \alpha \rangle \) is chosen properly, \( H | \alpha \rangle \) would be a relatively simple linear combination \( \sum_{\alpha} f_{\alpha} | \beta_{\alpha} \rangle \) where the \( f_{\alpha} \)’s depend on the \( t_j \)’s. For certain basis states in which the conduction fermion sites are either doubly occupied or empty, the equations \( \sum_{\alpha} f_{\alpha} | \beta_{\alpha} \rangle = E_0 | \alpha \rangle \) can be solved to express the \( t_j \)’s in terms of \( E_0 \). To be specific, two steps that can give us \( t_{L} \) and \( t_{L-1} \) are illustrated in Fig. 1 (b)-(c). The action of \( H \) on the basis state in Fig. 1 (b) is to move the conduction fermions on the site \( L-1 \). This results in two states with coefficients that depend on \( t_L \), so we get an equation that relates \( t_L \) and \( E_0 \). The basis state in Fig. 1 (c) yields an equation that relates \( t_{L-1} \) and \( t_L \) (already expressed in terms of \( E_0 \), \( t_L = t_{L+1} = t_{L-1} \), and \( E_0 \). The actual numerical value of \( t_j \) can be found using a basis state in which two conduction fermion sites are singly occupied [21]. It turns out that \( t_j \) is almost uniform in the bulk of the lattice (see
fermions and the magnetic moments. The hopping amplitudes given by

\[ t_j \rightarrow \frac{3}{4}, \quad t_1 \rightarrow \frac{3}{7}, \quad \text{and} \quad \left( \frac{\pi}{2(j+\frac{1}{2})} \right)^2 \cot^2 \left( \frac{\pi}{2(j+\frac{1}{2})} \right) \rightarrow j^{-2} \]

(for \( j \ll L \)), so (i) the maximal spin-exchange interaction is of the same order as the band width of the conduction fermions; (ii) the interaction strengths fall off as the inverse-square of distance between the conduction fermions and the magnetic moments.

**Numerical results** — The hopping amplitudes given by the procedure outlined above are the only possible values to make Eq. (1) an eigenstate of Eq. (4). However, it is not guaranteed that Eq. (1) would be an eigenstate, let alone being the ground state of Eq. (4) with such hopping amplitudes. While we do not have an analytical proof yet, exact diagonalizations of the Hamiltonian with properly computed \( t_j \) confirm that Eq. (1) is indeed the ground state for the cases with \( L = 2, 4, \cdots, 14 \). The unconventional setting with two impurities and the long-range spin-exchange interactions in the parent Hamiltonian make one wonder whether our model exhibits essential aspects of spin-1/2 Kondo physics. To this end, we have studied the spin-spin correlation function of the ground state and the low-lying energy levels of the parent Hamiltonian.

The spin-spin correlation function \( F_j = \langle \Psi | S_0 \cdot S_j | \Psi \rangle / \langle \Psi | \Psi \rangle \) for the \( L = 40 \) system is shown in Fig. 2 [21]. One can see that \( |F_j| \) decays very rapidly as \( j \) increases so the magnetic moments are screened on an extremely short length scale. The decaying behavior of \( |F_j| \) at large \( j \) is a useful signature of the Kondo physics. To extract such information, we choose \( |F_j| \) at odd \( j > 10 \) (their counterparts on even \( j \) are much smaller) and define the distance between the sites \( 0 \) and \( j \) as \( R_j = \frac{2(L+1)}{\pi} \sin \left( \frac{\theta}{2} \right) \) [22]. The log-log plot of \( |F_j| \) versus \( R_j \) in Fig. 2 (b) has a slope \( -2 \) with a coefficient of determination higher than 99.99%, in perfect agreement with the boundary CFT prediction [23] and other numerical results [24–26].

The low-lying energy levels of the \( L = 14 \) system are presented and analyzed in Fig. 3. In addition to the \( (N_f, S_z) = (14, 0) \) sector which hosts the ground state, three other sectors with \( (13, 1/2), (12, 0), \) and \( (14, 1) \) are also shown. We note that some other sectors are left out because symmetries dictate that they have the same energy spectra as those in Fig. 3 (a): \( (13, -1/2), (15, \pm 1/2) \) are the same as \( (13, 1/2), (16, 0) \) is the same as \( (12, 0) \), and \( (14, -1) \) is the same as \( (14, 1) \) [27]. The lowest energy in the \( (N_f, S_z) \) sector is denoted as \( E(N_f, S_z) \), which helps us to define \( \Delta_0(L) = E(L-1, 1/2) - E(L, 0) \), \( \Delta_1(L) = E(L-2, 0) + E(L, 1)/2 - E(L-1, 1/2) \), and \( S(L) = E(L-2, 0) - E(L, 1) \). The finite-size scaling analysis in Fig. 3 (b)-(d) reveals that (i) \( \Delta_0 \) goes to zero as \( 1/(L+2) \); (ii) both \( \Delta_0 - \Delta_1 \) and \( S \) go to zero as \( 1/(L+2)^2 \). These results signify the emergence of (quasi)-degeneracies in the energy spectrum as \( L \) increases: the ground state is always unique [in \( (L, 0) \)], the first excited state is 4-fold degenerate [one in each of \( (L, \pm 1/2) \)], and the second excited state is 6-fold degenerate [two in \( (L, 0) \) and one in each of \( (L, \pm 2) \), \( (L, \pm 1) \)]. The energy spectrum can be interpreted using the Fermi liquid picture of Nozières as illustrated in Fig. 3 (e): there is a Fermi level at \( \varepsilon_0 = 0 \), the single-particle energy levels are equally spaced at \( \varepsilon_m = \pm \frac{\pi v_F}{L+2} \left( m - \frac{1}{2} \right) \) (\( m = 1, 2, \cdots \)) and \( v_F \) (the Fermi velocity), the ground state is constructed by filling all the single-particle states below the Fermi level, and the excited states are obtained by adding
and/or removing some fermions from the ground state. For an exactly solvable model, it is natural to ask if the model is also integrable. In fact, a large class of models with inverse-square interactions, such as the celebrated Haldane-Shastry model, have been proved to be integrable [28–31]. One diagnostics of integrability is level statistics of the energy spectrum [32, 33]. The eigenvalues \(E_m\) are sorted according to the conserved quantum numbers and arranged in ascending order, so the level spacing \(\delta_m = E_{m+1} - E_m\) and the ratio \(r_m = \min(\delta_m, \delta_{m+1}) / \max(\delta_m, \delta_{m+1})\) can be defined. The probability density \(P(r_m)\) of the ratio would be a Poisson distribution with \(P(r) = 2/(1+r)^2\) if our model is integrable or a Gaussian orthogonal ensemble (GOE) with \(P(r) = [2\pi r^2]/[4(1+r+r^2)^{5/2}]\) otherwise. The results in Fig. 4 (a) and (b) are not conclusive but it seems that a Poisson distribution is more likely [two sectors with \((N_f, S_z) = (9, 1/2)\) are chosen so there is no need to consider the particle-hole symmetry].

Conformal field theory formulation — The successful construction of an exactly solvable model for the spin-1/2 Kondo problem with polynomial wave function motivates us to search for similar scenarios in more complicated quantum impurity models, such as the multichannel and SU(N) Kondo problems. A promising route along this direction manifests itself as soon as we establish a link between the wave function and the associated boundary CFT, which is very similar to the practice of constructing quantum Hall wave functions from edge CFT [20]. The key observation is that Eq. (2) can be rewritten as a CFT correlator

\[
\Psi(\{n^+_j\}, \{n^-_j\}) = \langle A^{n^+_0, n^-_0}(u_0)A^{n^+_1, n^-_1}(u_1) \cdots A^{n^+_L, n^-_L}(u_{L+1}) \rangle
\]

with vertex operators

\[
A^{n^+_j, n^-_j}(u_j) = \begin{cases} \ e^{\sum_\sigma(n^\sigma_j - \frac{1}{2})\phi_\sigma(u_j)} : j = 1, \cdots, L, \\ \delta_{n^+_j, n^-_j} : e^{\sum_\sigma(n^\sigma_j - \frac{1}{2})\phi_\sigma(u_j)} : \text{otherwise} \end{cases}
\]

where \(\phi_\sigma(u_j)\) is a chiral bosonic field in a two-component free boson CFT and \(\cdots : \) denotes normal ordering of operators [21]. This reformulation reveals that the wave function is an infinite matrix product state (iMPS) [34] and turns out to be very illuminating in further analysis. If the vertex operators for the impurity sites were removed from Eq. (7), we would have a wave function that belongs to the iMPS considered in Ref. [35]. It is the ground state of a free fermion with uniform hopping and open boundary condition [36], i.e., its Hamiltonian is Eq. (5) with \(t_j = 1 \forall j\), which provides a lattice realization of a boundary CFT with free boundary condition.

From the boundary CFT perspective, it is known that an impurity site plays the role of boundary condition changing (BCC) operator as it changes the free boundary condition of conduction fermions to the “Kondo boundary condition” [37]. The BCC operator for the spin-1/2 Kondo problem is the Kac-Moody primary of the SU(2)_1 CFT with conformal weight \(h = 1/4\) [37]. In view of the strong evidences for spin-1/2 Kondo physics in our model, one may speculate that the vertex operators on the impurity sites in Eq. (7) also act as BCC operators which change free boundary conditions at both ends of the chain to Kondo boundary conditions. This conjecture can be made persuasive if we switch to the charge (spin) basis \(\phi_{\uparrow}(\sigma) = \frac{1}{\sqrt{2}}(\phi_\uparrow \pm \phi_\downarrow)\) for the chiral bosonic fields. The vertex operators on the impurity sites become : \(e^{\pm i\phi_\uparrow/\sqrt{2}}\) : , which is precisely the BCC operator mentioned above. The identification of BCC operator in Eq. (7) would certainly shed some light on how to design suitable iMPS for more complicated systems because boundary CFT strongly restricts possible forms of the BCC operator. Further investigations along this line is beyond the scope of the present work but is an interesting subject for future study.

Conclusion and discussion — In summary, we have constructed an exactly solvable quantum impurity model whose ground-state wave function is a Jastrow type homogeneous polynomial. The parent Hamiltonian has almost uniform nearest-neighbor hopping terms between conduction fermions and inverse-square spin-exchange terms between conduction fermions and magnetic moments. The spin-spin correlation function and the low-lying energy levels of the parent Hamiltonian demonstrate that our model exhibits essential aspects of the spin-1/2 Kondo problem. The low-energy physics of our model is described by a boundary CFT and a particular correlator of this CFT reproduces the wave function. This work establishes a connection between wave function and boundary CFT in quantum impurity models, which paves the way toward constructing wave functions and exactly solvable models for more complicated systems.

Acknowledgment — We are grateful to Jan von Delft, Biao Huang, Seung-Sup Lee, Frédéric Mila, Germán Sierra, Andreas Weichselbaum, and Guang-Ming Zhang for helpful discussions. This work was supported by the DFG via project A06 of SFB 1143 (HIT), and by the startup grant of HUST (YHW).

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2 A. C. Hewson, *The Kondo Problem to Heavy Fermions* (Cambridge University Press, Cambridge, 1997).
3 P. Coleman, *Introduction to Many-Body Physics* (Cambridge University Press, Cambridge, 2015).
4 J. Kondo, *Prog. Theor. Phys.* 32, 37 (1964).
5 K. G. Wilson, *Rev. Mod. Phys.* 47, 773 (1975).
[6] H. R. Krishna-murthy, J. W. Wilkins, and K. G. Wilson, Phys. Rev. B 21, 1003 (1980).
[7] R. Bulla, T. A. Costi, and T. Pruschke, Rev. Mod. Phys. 80, 395 (2008).
[8] P. Nozières, J. Low Temp. Phys. 17, 31 (1974).
[9] I. Affleck, Nucl. Phys. B 336, 517 (1990).
[10] I. Affleck and A. W. W. Ludwig, Nucl. Phys. B 352, 849 (1991).
[11] N. Andrei, Phys. Rev. Lett. 45, 379 (1980).
[12] P. B. Wiegmann, JETP Lett. 31, 364 (1980).
[13] N. Andrei, K. Furuya, and J. H. Lowenstein, Rev. Mod. Phys. 55, 331 (1983).
[14] A. M. Tsvelik and P. B. Wiegmann, Adv. Phys. 32, 453 (1983).
[15] N. Andrei and H. Johannesson, Phys. Lett. A 100, 108 (1984).
[16] P. Schlottmann, J. Phys.: Condens. Matter 3, 6617 (1991).
[17] E. S. Sorensen, S. Eggert, and I. Affleck, J. Phys. A 26, 6757 (1993).
[18] H. Frahm and A. A. Zvyagin, J. Phys.: Condens. Matter 9, 9939 (1997).
[19] Y. Wang, J. Dai, Z. Hu, and F.-C. Pu, Phys. Rev. Lett. 79, 1901 (1997).
[20] G. Moore and N. Read, Nucl. Phys. B 360, 362 (1991).
[21] See the Appendices for more details on the wave function, the parent Hamiltonian, and the spin-spin correlation function.
[22] This can be achieved if we change the radius to $L + \pi$ in Fig. 1 (a) and use the chord distance between two sites. It ensures that $R_j \approx j$ when $j \ll L$. The linear coordinates $u_j$ are rescaled but the wave function is only modified in a trivial way.
[23] V. Barzykin and I. Affleck, Phys. Rev. B 57, 432 (1998).
[24] T. Hand, J. Kroha, and H. Monien, Phys. Rev. Lett. 97, 136604 (2006).
[25] L. Borda, Phys. Rev. B 75, 041307 (2007).
[26] A. Holzner, I. P. McCulloch, U. Schollwöck, J. von Delft, and F. Heidrich-Meisner, Phys. Rev. B 80, 205114 (2009).
[27] (13, 1/2) and (13, −1/2), (15, 1/2) and (15, −1/2), (14, −1) and (14, 1) are related by the SU(2) spin symmetry. (13, 1/2) and (15, −1/2), (12, 0) and (16, 0) are related by the particle-hole symmetry.
[28] F. D. M. Haldane, Phys. Rev. Lett. 60, 635 (1988).
[29] B. S. Shastry, Phys. Rev. Lett. 60, 639 (1988).
[30] F. D. M. Haldane, Z. N. C. Ha, J. C. Talstra, D. Bernard, and V. Pasquier, Phys. Rev. Lett. 69, 2021 (1992).
[31] D. Bernard, M. Gaudin, F. D. M. Haldane, and V. Pasquier, J. Phys. A 26, 5219 (1993).
[32] D. Poilblanc, T. Ziman, J. Bellissard, F. Mila, and G. Montambaux, Europhys. Lett. 22, 537 (1993).
[33] Y. Y. Atas, E. Bogomolny, O. Giraud, and G. Roux, Phys. Rev. Lett. 110, 084101 (2013).
[34] J. I. Cirac and G. Sierra, Phys. Rev. B 81, 104431 (2010).
[35] H.-H. Tu and G. Sierra, Phys. Rev. B 92, 041119 (2015).
[36] J.-M. Stéphan and F. Pollmann, Phys. Rev. B 95, 035119 (2017).
[37] I. Affleck and A. W. W. Ludwig, J. Phys. A 27, 5375 (1994).


**APPENDIX A: WAVE FUNCTION**

This section provides more details about how to prove that the ground-state wave function

\[ \Psi(\{n^j\}, \{n^j\}) = \delta_{n_0 = n_{L+1}} = \delta_{\sum_j n^j_0 = \sum_j n^j} \prod_{\sigma = \uparrow, \downarrow} \prod_{0 \leq j < k \leq L+1} (u_j - u_k)^{n^j_k n^j_k} \]  

(A1)
can be written as a Gutzwiller projected Fermi sea and a CFT correlator of vertex operators.

**A1: Gutzwiller projected Fermi sea**

This subsection aims to prove Eq. (3) of the main text. To begin with, we can rewrite the ground state as

\[ |\Psi\rangle = \sum_{\{n^j\}, \{\tilde{n}^j\}} \Psi(\{n^j\}, \{\tilde{n}^j\}) (c_{0, \uparrow}^{\dagger})^{n^j_0} \cdots (c_{L+1, \uparrow}^{\dagger})^{n^j_{L+1}} (c_{0, \downarrow}^{\dagger})^{\tilde{n}^j_0} \cdots (c_{L+1, \downarrow}^{\dagger})^{\tilde{n}^j_{L+1}} |0\rangle = P^G_0 P^G_{L+1} |\tilde{\Psi}\rangle. \]  

(A2)

The Gutzwiller projector \( P^G_0 \) enforces the single-occupancy constraint \( \delta_{n_0 = n_{L+1}} = 1 \) on the impurity sites \( j = 0, L + 1 \) and the state \( |\tilde{\Psi}\rangle \) is

\[
\sum_{\{n^j\}, \{\tilde{n}^j\}} \delta_{\sum_j n^j_0 = \sum_j n^j} \prod_{\sigma = \uparrow, \downarrow, 0 \leq j < k \leq L+1} (u_j - u_k)^{n^j_k n^j_k} (c_{0, \uparrow}^{\dagger})^{n^j_0} \cdots (c_{L+1, \uparrow}^{\dagger})^{n^j_{L+1}} (c_{0, \downarrow}^{\dagger})^{\tilde{n}^j_0} \cdots (c_{L+1, \downarrow}^{\dagger})^{\tilde{n}^j_{L+1}} |0\rangle
\]

\[
= \sum_{x_1, \ldots, x_{L+2}} \sum_{y_1, \ldots, y_{L+2}} \prod_{1 \leq j < k \leq L+2} (u_{x_j} - u_{x_k}) \prod_{1 \leq j < k \leq L+2} (u_{y_j} - u_{y_k}) c_{x_1, \uparrow}^{\dagger} \cdots c_{x_{L+2}, \uparrow}^{\dagger} c_{y_1, \downarrow} \cdots c_{y_{L+2}, \downarrow} |0\rangle, \]

(A3)

where \( \{x_1, \ldots, x_{L+2}\} \) and \( \{y_1, \ldots, y_{L+2}\} \) are positions of the spin-up and spin-down fermions, respectively. The Jastrow factor \( \prod_{1 \leq j < k \leq L+2} (u_{x_j} - u_{x_k}) \) can be converted to a Vandermonde determinant

\[
(-1)^{\frac{1}{2}(L+2)(L+3)} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ u_{x_1} & u_{x_2} & \cdots & u_{x_{L+2}} \\ \vdots & \vdots & \ddots & \vdots \\ u_{x_{L+1}} & u_{x_{L+2}} & \cdots & u_{x_{L+2}} \end{pmatrix}, \]

(A4)

so one can see that

\[
|\tilde{\Psi}\rangle \propto \sum_{\{x_j\}, \{y_j\}} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ u_{x_1} & u_{x_2} & \cdots & u_{x_{L+2}} \\ \vdots & \vdots & \ddots & \vdots \\ u_{x_{L+1}} & u_{x_{L+2}} & \cdots & u_{x_{L+2}} \end{pmatrix} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ u_{y_1} & u_{y_2} & \cdots & u_{y_{L+2}} \\ \vdots & \vdots & \ddots & \vdots \\ u_{y_{L+1}} & u_{y_{L+2}} & \cdots & u_{y_{L+2}} \end{pmatrix} c_{x_1, \uparrow}^{\dagger} \cdots c_{x_{L+2}, \uparrow}^{\dagger} c_{y_1, \downarrow} \cdots c_{y_{L+2}, \downarrow} |0\rangle
\]

\[
= \prod_{\sigma = \uparrow, \downarrow} \prod_{m=0}^{L+1} \eta_m |0\rangle, \]

(A5)

with \( \eta_m = \sum_{j=0}^{L+1} u_j^m c_{j, \sigma} \). The substitution of Eq. (A5) into Eq. (A2) completes the proof.

**A2: Conformal field theory formulation**

This subsection aims to prove Eq. (7) of the main text. To begin with, we can rewrite the CFT correlator as

\[
\delta_{n_0 = n_{L+1}} = \prod_{\sigma = \uparrow, \downarrow} \left( e^{i(n_{L+1}^{\sigma} - \frac{1}{2})\phi_{\sigma}(u_{L+1})} \right),
\]

(A6)
The chiral correlator of vertex operators can be evaluated as

\[ \langle : e^{i(n_0^+ - \frac{1}{2}) \phi_{\sigma}(u_0)} : \cdots e^{i(n_L^+ - \frac{1}{2}) \phi_{\sigma}(u_{L+1})} : \rangle \]

\[ = \delta \sum_j n_j^+ n_j^- = \frac{L+2}{2} \prod_{0 \leq j < k \leq L+1} (u_j - u_k)(n_j^+ - \frac{1}{2})(n_k^- - \frac{1}{2}) \]

\[ \propto \delta \sum_j n_j^+ n_j^- = \frac{L+2}{2} \prod_{0 \leq j < k \leq L+1} (u_j - u_k)n_j^+ n_k^- \prod_{0 \leq j < k \leq L+1} (u_j - u_k)^{-\frac{1}{2}}(n_j^+ + n_k^-), \tag{A7} \]

where we have dropped an overall factor \( \prod_{0 \leq j < k \leq L+1} (u_j - u_k)^{1/4} \) in the last step. This means that the half-filling constraint is due to the charge neutral condition in the chiral correlator of vertex operators. The last term in Eq. (A7) can be simplified as

\[ \prod_{0 \leq j < k \leq L+1} (u_j - u_k)^{-\frac{1}{2}}(n_j^+ + n_k^-) \]

\[ = \prod_{0 \leq j < k \leq L+1} (u_j - u_k)^{-\frac{1}{2}} n_j^+ \cdot \prod_{0 \leq k < j \leq L+1} (u_k - u_j)^{-\frac{1}{2}} n_j^+ \]

\[ = L \prod_{j=0}^{L+1} (u_j - u_k)^{-\frac{1}{2}} n_j^+ \cdot L \prod_{j=1}^{L+1} (-1)^j j \prod_{k=0}^{j-1} (u_j - u_k)^{-\frac{1}{2}} n_j^+ \]

\[ = (2L + 2)^{-\frac{1}{2}} \sum_{j=0}^{L+1} n_j^+ \cdot 2^{-\frac{1}{2}}(n_0^+ + n_{L+1}^-) \tag{A8} \]

using the identity

\[ (-1)^j j \prod_{k=0,...,j} (u_j - u_k) = \begin{cases} \frac{2L+2}{2j+2} & j = 1, \ldots, L \\ \frac{2L+2}{2} & \text{otherwise} \end{cases} \tag{A9} \]

The substitution of Eq. (A7) and Eq. (A8) into Eq. (A6) gives us

\[ \delta_{n_0=n_{L+1}=1} \delta \sum_j n_j^+ = 2 \frac{L+2}{2} \prod_{\sigma=\uparrow,\downarrow} 0 \leq j \leq L+1 (u_j - u_k)^{-\frac{1}{2}} n_j^+ \cdot \left( \frac{2L + 2}{2L+1} \right)^{-\frac{1}{2}} \sum_{j=0}^{L+1} n_j^+ \cdot 2^{-\frac{1}{2}}(n_0^+ + n_{L+1}^-) \]

\[ \propto \delta_{n_0=n_{L+1}=1} \delta \sum_j n_j^+ = 2 \frac{L+2}{2} \prod_{\sigma=\uparrow,\downarrow} 0 < j \leq L+1 (u_j - u_k)^{-\frac{1}{2}} n_j^+ \cdot \left( \frac{2L + 2}{2L+1} \right)^{-\frac{1}{2}} \sum_{j=0}^{L+1} n_j^+ \cdot 2^{-\frac{1}{2}}(n_0^+ + n_{L+1}^-) \tag{A10} \]

where we have again dropped an overall factor in the last step.

**APPENDIX B: PARENT HAMILTONIAN**

This section provides more detail about how to compute the hopping amplitudes in the parent Hamiltonian. The procedure can be divided into two steps. The first step is to express the \( t_j \)'s in terms of the ground state energy \( E_g \) as outlined in the main text. The second step is to derive a relation between \( E_g \) and the Kondo coupling strengths to obtain the numerical values of the \( t_j \)'s.

For the first step, we select basis states \( |\alpha\rangle \) in which the magnetic moment at the site 0 \((L + 1)\) points up (down) and the fermion sites \(1 \leq j \leq (\frac{L}{2} - 1)\) are doubly occupied. It is convenient to denote them as

\[ |\alpha_1\rangle = |0 \uparrow, 1 \leq j \leq (\frac{L}{2} - 1)\rangle \text{ and } j = \frac{L}{2} \text{ are doubly occupied, others are empty, } (L + 1) \downarrow \]

\[ |\alpha_2\rangle = |0 \uparrow, 1 \leq j \leq (\frac{L}{2} - 1)\rangle \text{ and } j = \frac{L}{2} + 1 \text{ are doubly occupied, others are empty, } (L + 1) \downarrow \]

\[ \cdots \]

\[ |\alpha_{\frac{L}{2}}\rangle = |0 \uparrow, 1 \leq j \leq (\frac{L}{2} - 1)\rangle \text{ and } j = L - 1 \text{ are doubly occupied, others are empty, } (L + 1) \downarrow. \]
The hopping term $H_0$ acts on $|\alpha_k\rangle$ in a simple way because the fermions can only move from the sites $j = \frac{L}{2} - 1$ (for $k > 1$) and $j = \frac{L}{2} + k - 1$ (for all $k$) to their neighbors [see Fig. 1 (b)-(c) of the main text for two examples]. The Kondo term $H_K$ has no effect on $|\alpha_k\rangle$ because the fermion sites are either doubly occupied or empty. One can generate a set of equations by acting $H$ to the left and right in $\langle \alpha_k | H | \Psi \rangle$. The first one is

$$
\langle \alpha_1 | H | \Psi \rangle = \sum_a f_a (\beta_a | \Psi \rangle = E_g (\alpha_1 | \Psi \rangle),
$$

(A11)

where there are two $|\beta_a\rangle$'s and the $f_a$'s depend on $t_\frac{L}{2}$. It helps us to express $t_\frac{L}{2}$ in terms of $E_g$. The second one is

$$
\langle \alpha_2 | H | \Psi \rangle = \sum_a f_a (\beta_a | \Psi \rangle = E_g (\alpha_2 | \Psi \rangle),
$$

(A12)

where there are six $|\beta_a\rangle$'s and the $f_a$'s depend on $t_{\frac{L}{2}-1}$, $t_{\frac{L}{2}}$, and $t_{\frac{L}{2}+1}$ (= $t_{\frac{L}{2}-1}$). It helps us to express $t_{\frac{L}{2}-1}$ in terms of $E_g$. This procedure can be continued to yield all the hopping amplitudes.

The second step as illustrated in Fig. A1 is to compute the ground state energy $E_g$ using the basis state

$$
|\alpha_g\rangle = |0 \uparrow, 1 \leq j \leq (\frac{L}{2} - 2) \text{ are doubly occupied}, (j = \frac{L}{2} - 1) \uparrow, (j = \frac{L}{2}) \downarrow, \text{ others are empty}, (L + 1) \downarrow \rangle.
$$

(A13)

The hopping term $H_0$ still acts on $|\alpha_g\rangle$ in a simple way. The Kondo term $H_K$ has some effects on $|\alpha_g\rangle$ due to the singly occupied sites. This state leads to the equation

$$
\langle \alpha_g | H | \Psi \rangle = \sum_a f_a (\beta_a | \Psi \rangle = E_g (\alpha_g | \Psi \rangle),
$$

(A14)

where the $f_a$'s depend on $t_{\frac{L}{2}+1}$ (= $t_{\frac{L}{2}-1}$), $t_\frac{L}{2}$ (expressed in terms of $E_g$), and four Kondo coupling strengths (associated with $S_0 \cdot S_{\frac{L}{2}}, S_0 \cdot S_{\frac{L}{2}+1}, S_{L+1} \cdot S_{\frac{L}{2}}, S_{L+1} \cdot S_{\frac{L}{2}+1}$). It gives us the numerical value of $E_g$ so the numerical values of the hopping amplitudes would follow.

The reader might have realized that the coupling strengths in $H_K$ do not have to be fixed by hand but can be derived as well. For this purpose, the ground state energy $E_g$ will serve as the unit of all variables in the Hamiltonian. Let us denote the Kondo term with unknown coefficients as

$$
H_K = \sum_{j=1}^{L} J_{j}^b S_0 \cdot S_j + J_{j}^b S_{L+1} \cdot S_j,
$$

(A15)

where $J_{j}^b = J_{L+1-j}^b$ so it respects the mirror symmetry defined in the main text. If we use the states

$$
|\alpha'_g\rangle = |0 \uparrow, 1 \leq j \leq (\frac{L}{2} - 2) \text{ are doubly occupied}, (j = \frac{L}{2} - 1) \downarrow, (j = \frac{L}{2}) \uparrow, \text{ others are empty}, (L + 1) \downarrow \rangle
$$

(A16)
and $|\alpha\rangle$, we can express $J^x_\nu$ ($= J^x_{\nu+1}$) and $J^y_\nu$ ($= J^y_{\nu+1}$) in terms of $E_\alpha$. The other Kondo coupling strengths can be derived similarly. While this method has been confirmed numerically in some cases, it is rather difficult to prove that $J^y_\nu \sim \cot^2 \frac{\theta}{2}$ in general. The analytical form of $J^y_\nu$ was inspired by explicit calculations in small systems.

**APPENDIX C: SPIN-SPIN CORRELATION FUNCTION**

This section provides more detail about how to compute the spin-spin correlation function of the ground state. To begin with, we compute the overlap of two generic free fermion states. For a set of fermionic operators $c^\dagger_j$ and $c_j$ satisfying the anticommutation relations, we define two sets of operators $\alpha^\dagger_a$ and $\beta^\dagger_b$

$$\alpha_a = \sum_j U_{aj} c_j, \quad \beta_b = \sum_j V_{bj} c_j,$$

where $U_{aj}$ and $V_{bj}$ are two matrices that may not be isometry. It is easy to see that

$$\langle 0 | \alpha_a^\dagger \beta_b^\dagger | 0 \rangle = \sum_{jk} U_{aj} V_{bk} \langle 0 | c_j^\dagger c_k^\dagger | 0 \rangle = \sum_j U_{aj} V_{bj} = \sum_j [UV^\dagger]_{ab} = G_{ab}$$

with $G = UV^\dagger$. The overlap between $\alpha^\dagger_1 \alpha^\dagger_2 \cdots \alpha^\dagger_M | 0 \rangle$ and $\beta^\dagger_1 \beta^\dagger_2 \cdots \beta^\dagger_M | 0 \rangle$ is found to be

$$\langle 0 | \alpha^\dagger_1 \alpha^\dagger_2 \cdots \alpha^\dagger_M | 0 \rangle \beta^\dagger_1 \beta^\dagger_2 \cdots \beta^\dagger_M | 0 \rangle = \langle 0 | \alpha^\dagger_1 \beta^\dagger_1 | 0 \rangle \langle 0 | \alpha^\dagger_2 \beta^\dagger_2 | 0 \rangle \cdots \langle 0 | \alpha^\dagger_M \beta^\dagger_M | 0 \rangle + \text{all possible permutations with signs}
= G_{11} G_{22} \cdots G_{MM} + \text{all possible permutations with signs}
= \det G$$

(A19)

using Eq. (A18) and Wick’s theorem.

The ground state can be written as

$$|\Psi\rangle = \sum_{j=1,\{x_1<\}}^{\frac{4}{3}-1} \sum_{k=1,\{y_1<\}}^{\frac{4}{3}+1} \Psi_{\uparrow \downarrow} (\{x_j\}; \{y_k\}) \sum_{j=1}^{\frac{4}{3}-1} c_{0,\uparrow}^\dagger \prod_{j=1}^{\frac{4}{3}-1} c_{x_j,\uparrow}^\dagger \prod_{k=1}^{\frac{4}{3}+1} c_{y_k,\downarrow}^\dagger |0\rangle$$

$$+ \sum_{j=1,\{x_1<\}}^{\frac{4}{3}} \sum_{k=1,\{y_1<\}}^{\frac{4}{3}} \Psi_{\uparrow \downarrow} (\{x_j\}; \{y_k\}) \sum_{j=1}^{\frac{4}{3}} c_{0,\uparrow}^\dagger \prod_{j=1}^{\frac{4}{3}} c_{x_j,\uparrow}^\dagger \prod_{k=1}^{\frac{4}{3}+1} c_{y_k,\downarrow}^\dagger |0\rangle$$

$$+ \sum_{j=1,\{x_1<\}}^{\frac{4}{3}} \sum_{k=1,\{y_1<\}}^{\frac{4}{3}} \Psi_{\downarrow \uparrow} (\{x_j\}; \{y_k\}) \sum_{j=1}^{\frac{4}{3}} c_{1,\downarrow}^\dagger \prod_{j=1}^{\frac{4}{3}} c_{x_j,\uparrow}^\dagger \prod_{k=1}^{\frac{4}{3}} c_{y_k,\downarrow}^\dagger |0\rangle$$

(A20)

where $\{x_1<\}$ means that the arguments are ordered as $x_1 < x_2 < \cdots < x_{\frac{4}{3}-1}$. The next step is to decompose the coefficients $\Psi_{\sigma\tau}(\{x_j\}; \{y_k\})$ in certain ways and combine them with the creation operators such that

$$(-1)^{(L^2-2L+4)/4} |\Psi\rangle = 2 c_{0,\uparrow}^\dagger \prod_{a=1}^{\frac{4}{3}-1} \alpha_{a,\uparrow}^\dagger \prod_{b=1}^{\frac{4}{3}+1} \beta_{b,\downarrow}^\dagger |0\rangle - c_{0,\uparrow}^\dagger \prod_{a=1}^{\frac{4}{3}} \mu_{a,\uparrow}^\dagger \prod_{b=1}^{\frac{4}{3}} \nu_{b,\uparrow}^\dagger c_{L+1,\downarrow}^\dagger |0\rangle$$

$$- c_{1,\downarrow}^\dagger \prod_{a=1}^{\frac{4}{3}} c_{x,a,\uparrow}^\dagger c_{0,\uparrow}^\dagger \prod_{b=1}^{\frac{4}{3}} \mu_{b,\uparrow}^\dagger |0\rangle + 2 c_{1,\downarrow}^\dagger \prod_{a=1}^{\frac{4}{3}} \beta_{a,\uparrow}^\dagger \prod_{b=1}^{\frac{4}{3}} c_{y,b,\uparrow}^\dagger c_{L+1,\downarrow}^\dagger |0\rangle,$$

(A21)
where

\[
\alpha_{a,\sigma}^\dagger = \sum_{j=1}^{L} (\sin \theta_j)^2 (\cos \theta_j)^{a-1} c_{j,\sigma}^\dagger \quad a = 1, 2, \cdots, \frac{L}{2} - 1
\]

\[
\beta_{a,\sigma}^\dagger = \sum_{j=1}^{L} (\cos \theta_j)^{a-1} c_{j,\sigma}^\dagger \quad a = 1, 2, \cdots, \frac{L}{2} + 1
\]

\[
\mu_{a,\sigma}^\dagger = \sum_{j=1}^{L} 2(\sin \frac{\theta_j}{2})^2 (\cos \theta_j)^{a-1} c_{j,\sigma}^\dagger \quad a = 1, 2, \cdots, \frac{L}{2}
\]

\[
\nu_{a,\sigma}^\dagger = \sum_{j=1}^{L} 2(\cos \frac{\theta_j}{2})^2 (\cos \theta_j)^{a-1} c_{j,\sigma}^\dagger \quad a = 1, 2, \cdots, \frac{L}{2}.
\] (A22)

The numerator of the spin-spin correlation function can be simplified using SU(2) spin symmetry as

\[
\langle \Psi | S_0 \cdot S_j | \Psi \rangle = \frac{3}{2} \langle \Psi | S_0^+ S_j^- | \Psi \rangle.
\] (A23)

It can be proved using Eq. (A21) that

\[
\langle \Psi | S_0^+ S_j^- | \Psi \rangle = 2 \langle 0 | \prod_{b=\frac{L}{2}}^{1/2} \nu_{b,\downarrow} \prod_{a=\frac{L}{2}}^{1} \mu_{a,\uparrow} \prod_{b=1}^{\frac{L}{2}+1} \beta_{b,\downarrow} \prod_{a=\frac{L}{2}-1}^{1} \alpha_{a,\uparrow} c_{j,\uparrow} \prod_{a=\frac{L}{2}}^{1} \nu_{a,\uparrow} c_{j,\downarrow} \prod_{b=1}^{\frac{L}{2}} \mu_{b,\downarrow} | 0 \rangle
\]

\[
+ 2 \langle 0 | \prod_{b=\frac{L}{2}+1}^{L} \beta_{b,\downarrow} \prod_{a=\frac{L}{2}-1}^{1} \alpha_{a,\uparrow} c_{j,\uparrow} \prod_{a=\frac{L}{2}}^{1} \nu_{a,\uparrow} c_{j,\downarrow} \prod_{b=1}^{\frac{L}{2}} \mu_{b,\downarrow} | 0 \rangle,
\] (A24)

where \( \prod_{a=\frac{L}{2}}^{1} \mu_{a,\uparrow} \) means \( \mu_{\frac{L}{2}+1,\uparrow} \cdots \mu_{2,\uparrow} \mu_{1,\uparrow} \). The unnormalized overlap \( \langle \Psi | \Psi \rangle \) can also be expressed using the \( \alpha, \beta, \mu, \nu \) operators. The numerical values can be evaluated using Eq. (A19).