REAL K3 SURFACES WITHOUT REAL POINTS,
EQUIVARIANT DETERMINANT OF THE LAPLACIAN,
AND THE BORCHERDS $\Phi$-FUNCTION

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ABSTRACT. We consider an equivariant analogue of a conjecture of Borchers. Let $(Y, \sigma)$ be a real K3 surface without real points. We shall prove that the equivariant determinant of the Laplacian of $(Y, \sigma)$ with respect to a $\sigma$-invariant Ricci-flat Kähler metric is expressed as the norm of the Borchers $\Phi$-function at the “period point”. Here the period of $(Y, \sigma)$ is not the one in algebraic geometry.

1. Introduction

Let $Y$ be an algebraic K3 surface defined over the real number field $\mathbb{R}$. Let $\sigma: Y \to Y$ be the anti-holomorphic involution on $Y$ induced by the complex conjugation. Denote by $\mathbb{Z}_2 = \langle \sigma \rangle$ the group of order 2 of $\mathbb{C}^\infty$ diffeomorphisms of $Y$ generated by $\sigma$. Recall that a point of $Y$ is real if it is fixed by $\sigma$.

By [17], there exists a $\sigma$-invariant Ricci-flat Kähler metric $g$ on $Y$ with Kähler form $\omega_g$. Since $Y$ is defined over $\mathbb{R}$, there exists a nowhere vanishing holomorphic 2-form $\eta_g$ on $Y$ such that

$$\eta_g \wedge \overline{\eta_g} = 2\omega_g^2, \quad \sigma^* \eta_g = \overline{\eta_g}.$$ 

Notice that the choice of $\eta_g$ is unique up to a sign. We identify $\omega_g$ and $\eta_g$ with their cohomology classes.

Let $L_{K3}$ be the K3 lattice, which is an even unimodular lattice with signature $(3, 19)$. Then $H^2(Y, \mathbb{Z})$ equipped with the cup-product is isometric to $L_{K3}$. By [13] or [6], there exists an isometry of lattices $\alpha: H^2(Y, \mathbb{Z}) \cong L_{K3}$ such that the point $[\alpha(\omega_g + \sqrt{-1}\text{Im } \eta_g)] \in \mathbb{P}(L_{K3} \otimes \mathbb{C})$ lies in the period domain for Enriques surfaces.

Let $\Delta_{Y,g}$ be the Laplacian of $(Y, g)$ acting on $C^\infty(Y)$. Following [2] and [11], one can define the equivariant determinant of the Laplacian $\Delta_{Y,g}$ with respect to the anti-holomorphic $\mathbb{Z}_2$-action on $Y$. Notice that $\sigma$ acts on the vector space $C^\infty(Y)$ while it does not act on the vector space of $C^\infty(p, q)$-forms on $Y$ unless $p = q$. Denote by $\det^*_{\mathbb{Z}_2} \Delta_{Y,g}(\sigma)$ the equivariant determinant of the Laplacian $\Delta_{Y,g}$ with respect to $\sigma$. (See Sect. 4.2.)

Recall that Borchers [3] constructed a very interesting automorphic form on the period domain for Enriques surfaces, which is called the Borchers $\Phi$-function and is denoted by $\Phi$. Let $\|\Phi\|$ denote the Petersson norm of $\Phi$. Then $\|\Phi\|^2$ is a $C^\infty$ function on the period domain for Enriques surfaces, which is invariant under the complex conjugation of the period domain. Our result is the following:

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Main Theorem 1.1. There exists an absolute constant $C > 0$ such that for every real $K3$ surface without real points $(Y, \sigma)$ and for every $\sigma$-invariant Ricci-flat Kähler metric $g$ on $Y$ with volume 1,
\[
\det^*_{2\mathbb{Z}} \Delta_{Y,g}(\sigma) = C \|\Phi([\alpha(\omega_g + \sqrt{-1}\text{Im} \eta_g)])\|^4.
\]

Notice that the point $[\alpha(\omega_g + \sqrt{-1}\text{Im} \eta_g)]$ is not the period of the marked $K3$ surface $(Y, \alpha)$, because $\omega_g + \sqrt{-1}\text{Im} \eta_g$ is not a holomorphic 2-form on $Y$. Since $\omega_g$ is the Kähler form of $(Y, g)$, the Main Theorem 1.1 may be regarded as a symplectic analogue of [18, Th. 8.3]. A typical example of a real $K3$ surface without real points is the quartic surface of $\mathbb{P}^3(\mathbb{C})$ defined by the equation $z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$.

To prove the Main Theorem 1.1, we consider an equivariant analogue of the conjecture of Borcherds: Let $X$ be the differentiable manifold underlying a $K3$ surface. In [4, Example 15.1], Borcherds conjectured that the regularized determinant of the Laplacian, regarded as a function on the moduli space of Ricci-flat metrics on $X$ with volume 1, coincides with the automorphic form on the Grassmann $G(L_{K3})$ associated to the elliptic modular form $E_4(\tau)/\Delta(\tau)$; it is worth remarking that the regularized determinant of the Laplacian of a Ricci-flat $K3$ surface can be regarded as an analytic torsion of certain elliptic complex [12].

As an equivariant analogue of the Borcherds conjecture, we shall compare the following two functions on the space of $\sigma$-invariant Ricci-flat metrics on $X$: one is the equivariant determinant of the Laplacian, and the other is the pull-back of the norm of the Borcherds $\Phi$-function via the “period map”. (See Sect. 3.4 for the definition of the period map.) It is a trick of Donaldson [6], [8] that relates these two objects: Let $(I, J, K)$ be a hyper-Kähler structure on $(X, g)$ with $Y = (X, J)$. Then $\sigma$ is holomorphic with respect to another complex structure $I$, while $\sigma$ is anti-holomorphic with respect to the initial complex structure $J$. We shall show that the equivariant determinant of the Laplacain of $(Y, \sigma)$ coincides with the equivariant analytic torsion of $(X, I, \sigma)$. (See Sect. 3.3 and Sect. 4.) After this observation, the Main Theorem 1.1 is a consequence of our result [18, Main Theorem and Th. 8.2].

This note is organized as follows. In Sect. 2, we recall the notion of hyper-Kähler structure on a $K3$ surface. In Sect. 3, we recall the trick of Donaldson. In Sect. 4, we study equivariant determinant of the Laplacian as a function on the space of $\sigma$-invariant Ricci-flat metrics on a $K3$ surface, and we prove the Main Theorem.

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2. $K3$ surfaces and hyper-Kähler structures

2.1. $K3$ surfaces

A compact, connected, smooth complex surface is a $K3$ surface if it is simply connected and has trivial canonical line bundle. Every $K3$ surface is diffeomorphic to a smooth quartic surface in $\mathbb{P}^3(\mathbb{C})$ (cf. [1, Chap. 8 Cor. 8.6]). Throughout this note, $X$ denotes the $C^\infty$ differentiable manifold underlying a $K3$ surface, and $X$ is equipped with the orientation as a complex submanifold of $\mathbb{P}^3(\mathbb{C})$. For a complex structure $I$ on $X$, $X_I$ denotes the $K3$ surface $(X, I)$.

Let $U$ be the lattice of rank 2 associated with the symmetric matrix \[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \], and let $E_8$ be the root lattice of the simple Lie algebra of type $E_8$. We assume that $E_8$ is negative-definite. The even unimodular lattice with signature $(3, 19)$
\[ L_{K3} := U \oplus U \oplus U \oplus E_8 \oplus E_8 \]
is called the K3 lattice. Then $H^2(X, \mathbb{Z})$ equipped with the cup-product $\langle \cdot, \cdot \rangle$, is isometric to $\mathbb{L}_{K3}$ (cf. [1, Chap. 8, Prop. 3.2]).

### 2.2. Hyper-Kähler structures on $X$

In this subsection, we recall Hitchin’s result [9]. Let $E$ be the set of all Ricci-flat metrics on $X$ with volume 1. For every complex structure $I$ on $X$, there exists a Kähler metric on $X_I$ by [1, Chap. 8, Th. 14.5]. For every Kähler class $\kappa$ on $X_I$, there exists by [17] a unique Ricci-flat Kähler form on $X_I$ representing $\kappa$. Hence $E \neq \emptyset$. For $g \in E$, let $dV_g$ denote the volume element of $(X, g)$. Then $\int_X dV_g = 1$ by our assumption.

**Definition 2.1.** A complex structure $I$ on $X$ is compatible with $g \in E$ if $g$ is a Kähler metric on $X_I$, i.e., $I$ is parallel with respect to the Levi-Civita connection of $(X, g)$. For $g \in E$, let $C_g$ denote the set of all complex structures on $X$ compatible with $g$.

Let $g \in E$. By Hitchin [9, Sect. 2, (i) $\iff$ (iii)], we get $C_g \neq \emptyset$. For $I \in C_g$, we define a real closed 2-form $\gamma_I$ on $X$ by

\begin{equation}
\gamma_I(u, v) := g(Iu, v), \quad u, v \in TX.
\end{equation}

Then $\gamma_I$ is a Ricci-flat Kähler form on $X_I$ such that $\gamma_I^2 = 2dV_g$.

**Definition 2.2.** Let $I, J, K \in C_g$. The ordered triplet $(I, J, K)$ is called a hyper-Kähler structure on $(X, g)$ if

\begin{equation}
IJ = -JI = K.
\end{equation}

Let $*: \bigwedge^p T^*X \to \bigwedge^{4-p} T^*X$ be the Hodge star-operator on $(X, g)$. Since $\dim_{\mathbb{R}} X = 4$, we have $*^2 = 1$ on $\bigwedge^2 T^*X$. Recall that a 2-form $f$ on $X$ is self-dual with respect to $g$ if $*g f = f$. Let $\mathcal{H}_+^2(g)$ be the real vector space of self-dual, real harmonic 2-forms on $(X, g)$. Every vector of $\mathcal{H}_+^2(g)$ is parallel with respect to the Levi-Civita connection by [9].

**Theorem 2.3.** Let $I \in C_g$, and let $\eta$ be a nowhere vanishing holomorphic 2-form on $X_I$ such that $\eta \wedge \bar{\eta} = 2\gamma_I^2$. Then there exist complex structures $J, K \in C_g$ satisfying

1. $(I, J, K)$ is a hyper-Kähler structure on $(X, g)$ with $\eta = \gamma_I + \sqrt{-1}\gamma_J$;
2. $\mathcal{H}_+^2(g)$ is a 3-dimensional real vector space spanned by $\{\gamma_I, \gamma_J, \gamma_K\}$;
3. $C_g = \{aI + bJ + cK; (a, b, c) \in \mathbb{R}^3, a^2 + b^2 + c^2 = 1\}$.

**Proof.** See [9, Sect. 2, (i) $\iff$ (iii)] for (1) and (2). Let $I' \in C_g$. Since $\gamma_{I'} \in \mathcal{H}_+^2(g)$ by [9, Sect. 2, (i) $\iff$ (iii)], we can write $\gamma_{I'} = a\gamma_I + b\gamma_J + c\gamma_K$ for some $a, b, c \in \mathbb{R}$. We get $a^2 + b^2 + c^2 = 1$ by the relations $\gamma_I^2 = \gamma_J^2 = 2dV_g, \gamma \wedge \bar{\eta} = 0$, and $\eta \wedge \bar{\eta} = 2\gamma_I^2$. \qed

**Lemma 2.4.** Let $(I, J, K)$ be a hyper-Kähler structure on $(X, g)$. The map from $SO(3)$ to the set of all hyper-Kähler structures on $(X, g)$ defined by

$$A = (a_{ij}) \mapsto (a_{11}I + a_{12}J + a_{13}K, a_{21}I + a_{22}J + a_{23}K, a_{31}I + a_{32}J + a_{33}K)$$

is a bijection.

**Proof.** It is obvious that the map defined as above is injective. Let $(I', J', K')$ be an arbitrary hyper-Kähler structure on $(X, g)$. By Theorem 2.3 (3), there is a real $3 \times 3$ matrix $B = (b_{ij})$ with

$$I' = b_{11}I + b_{12}J + b_{13}K, \quad J' = b_{21}I + b_{22}J + b_{23}K, \quad K' = b_{31}I + b_{32}J + b_{33}K.$$
We get $B \in SO(3)$ by the relations $(I')^2 = (J')^2 = (K')^2 = -1_{TX}$ and $I'J' = -J'I' = K'$. This proves the surjectivity. \hfill \Box

By Lemma 2.4, the element $\gamma_I \wedge \gamma_J \wedge \gamma_K \in \det \mathcal{H}_2^+(g)$ is independent of the choice of a hyper-Kähler structure $(I, J, K)$ on $(X, g)$, and it defines an orientation on $\mathcal{H}_2^+(g)$. In this note, $\mathcal{H}_2^+(g)$ is equipped with this orientation.

Let $A^p(X)$ denote the real vector space of real $C^\infty$ $p$-forms on $X$. For a complex structure $I$ on $X$, $A^p_q(X_I)$ denotes the complex vector space of $C^\infty (p, q)$-forms on $X_I$, and $\Omega^p_X$ denotes the sheaf of holomorphic $p$-forms on $X_I$.

Recall that the $L^2$-inner product on $A^p(X)$ with respect to $g$ is defined by

$$
(f, f')_{L^2} := \int_X f \wedge *_g f' = \int_X \langle f, f' \rangle_x \, dV_g(x), \quad f, f' \in A^p(X).
$$

Equipped with the restriction of $(\cdot, \cdot)_{L^2}$, $\mathcal{H}_2^+(g)$ is a metrized vector space. Then $\{\gamma_I/\sqrt{2}, \gamma_J/\sqrt{2}, \gamma_K/\sqrt{2}\}$ is an oriented orthonormal basis of $\mathcal{H}_2^+(g)$ for every hyper-Kähler structure $(I, J, K)$ on $(X, g)$, because $\gamma = \gamma_I \in A^{1,1}(X_I)$ and $\eta = \gamma_I + \sqrt{-1}\gamma_K \in H^0(X_I, \Omega^2_{X_I})$ satisfy the equations $\gamma \wedge \eta = \eta^2 = 0$.

**Lemma 2.5.** The map from the set of hyper-Kähler structures on $(X, g)$ to the set of oriented orthonormal basis of $\mathcal{H}_2^+(g)$ defined by $(I, J, K) \mapsto \{\gamma_I/\sqrt{2}, \gamma_J/\sqrt{2}, \gamma_K/\sqrt{2}\}$, is a bijection.

**Proof.** The result is an immediate consequence of Lemma 2.4. \hfill \Box

3. **Hyperbolic involutions on $K3$ surfaces and Ricci-flat metrics**

In this section, we recall a trick of Donaldson that relates real $K3$ surfaces and $K3$ surfaces with anti-symplectic holomorphic involution. We follow [6, Chap. 6, Sect. 15] and [8, Sect. 2 pp.21-22].

3.1. **Hyperbolic Involutions**

For a $C^\infty$ involution $\iota$ on $X$, we set

$$
H^2_+(X, \mathbb{Z}) := \{l \in H^2(X, \mathbb{Z}); \iota^*(l) = \pm l \}, \quad r(\iota) := \text{rank}_\mathbb{Z} H^2_+(X, \mathbb{Z}).
$$

By [13, Cor. 1.5.2], $H^2_+(X, \mathbb{Z}) \subset H^2(X, \mathbb{Z})$ is primitive and $2$-elementary.

**Definition 3.1.** A $C^\infty$ involution $\iota: X \to X$ is **hyperbolic** if the following two conditions are satisfied:

1. $H^2_+(X, \mathbb{Z})$ has signature $(1, r(\iota) - 1)$;
2. $\iota$ is holomorphic with respect to a complex structure on $X$.

**Remark 3.2.** The second condition of Definition 3.1 does not seem very natural. We do not know if it is deduced from the first condition. Are there any $C^\infty$ involution on $X$ which is never holomorphic with respect to any complex structure on $X$, such that the invariant lattice $H^2_+(X, \mathbb{Z})$ is hyperbolic?

**Definition 3.3.** For a hyperbolic involution $\iota: X \to X$, set

$$
\mathcal{E}^\iota := \{g \in \mathcal{E}; \iota^*g = g\}.
$$

**Proposition 3.4.** For every hyperbolic involution $\iota: X \to X$, one has $\mathcal{E}^\iota \neq \emptyset$. 

Proof. There exists a complex structure $I$ on $X$ such that $\iota$ is holomorphic with respect to $I$. Since $X_I$ is Kähler, there exists an $\iota$-invariant Kähler class $\kappa$ on $X_I$. Let $\gamma$ be the unique Ricci-flat Kähler form representing $\kappa$. Then $\iota^*\gamma = \gamma$ by the uniqueness of $\gamma$. Let $g$ be the Kähler metric on $X$ whose Kähler form is $\gamma$. Then $g$ is Ricci-flat and $\iota$-invariant. □

Let $\iota : X \to X$ be a hyperbolic involution, and let $g \in \mathcal{E}^i$. Then $\iota$ preserves $H^2_+(g)$. By identifying a real harmonic 2-form on $(X, g)$ with its cohomology class in $H^2(X, \mathbb{R})$, we regard $H^2_+(g)$ as an oriented subspace of $H^2(X, \mathbb{R})$. Since $\ast_g = 1$ on $H^2_+(g)$, the cup-product $(\cdot, \cdot)$ is positive-definite on $H^2_+(g) \subset H^2(X, \mathbb{R})$.

**Proposition 3.5.** The orientation on $H^2_+(g)$ is preserved by $\iota$.

**Proof.** Since $\iota$ is a diffeomorphism of $X$, the result follows from [7, Prop. 6.2]. □

**Proposition 3.6.** (1) There exists a hyper-Kähler structure $(I, J, K)$ on $(X, g)$ with

\[
\iota_+ I = I_{\iota_+}, \quad \iota_+ J = -J_{\iota_+}, \quad \iota_+ K = -K_{\iota_+}.
\]

(2) If $(I', J', K')$ is another hyper-Kähler structure satisfying (3.1), then there exists $\psi \in \mathbb{R}$ satisfying one of the following two equations:

\[
(I', J', K') = \begin{cases} 
(I, \cos \psi J - \sin \psi K, \sin \psi J + \cos \psi K), \\
(-I, \cos \psi J + \sin \psi K, \sin \psi J - \cos \psi K).
\end{cases}
\]

**Proof.** Set $\Pi(g)_\pm := \{ \gamma \in H^2_+(g) : \iota^*\gamma = \pm \gamma \}$. Since the cup-product is positive definite on $H^2_+(g)$, the hyperbolicity of $\iota$ implies that $\dim \Pi(g)_\pm \leq 1$. Since $\det \iota^*|H^2_+(g) = 1$ by Proposition 3.5, we get $\dim \Pi(g)_+ = 1$ and $\dim \Pi(g)_- = 2$.

Since $\iota$ is an involution preserving the $L^2$-inner product $(\cdot, \cdot)_{L^2}$, $\iota^*$ is symmetric with respect to $(\cdot, \cdot)_{L^2}$. Hence there exists an oriented orthonormal basis $\{\gamma_1, \gamma_2, \gamma_3\} \subset H^2_+(g)$ with

\[
\iota^*\gamma_1 = \gamma_1, \quad \iota^*\gamma_2 = -\gamma_2, \quad \iota^*\gamma_3 = -\gamma_3.
\]

By Lemma 2.5, there exists a hyper-Kähler structure $(I, J, K)$ on $(X, g)$ satisfying $\gamma_1 = \gamma_1/\sqrt{2}$, $\gamma_2 = \gamma_2/\sqrt{2}$, $\gamma_3 = \gamma_3/\sqrt{2}$. These equations, together with (2.1), (3.3) and $\iota^*g = g$, yields (3.1). This proves (1).

Since $\dim \Pi(g)_+ = 1$, there exists $l \in \mathbb{R}$ such that $\gamma_{I'} = l\gamma_I$. This, together with $\gamma_1^2 = \gamma_2^2 = 2dv_g$, implies that $I' = \pm I$. Since $\{\omega_J/\sqrt{2}, \omega_K/\sqrt{2}\}$ are orthonormal bases of $\Pi(g)_-$, there exists $\psi \in \mathbb{R}$ with

\[
(J', K') = (\cos \psi J \mp \sin \psi K, \sin \psi J \pm \cos \psi K).
\]

Since $J'K' = I$ when $I' = I$ and since $J'K' = -I$ when $I' = -I$, we get (3.2). □

**Definition 3.7.** A hyper-Kähler structure $(I, J, K)$ on $(X, g)$ is compatible with $\iota$ if Eq. (3.1) holds.

3.2. 2-elementary K3 surfaces. Let $Y$ be a K3 surface, and let $\theta : Y \to Y$ be a holomorphic involution. Then $\theta$ is anti-symplectic if

\[
\theta^*\eta = -\eta, \quad \forall \eta \in H^0(Y, \Omega_Y^2).
\]

**Definition 3.8.** A K3 surface equipped with an anti-symplectic holomorphic involution is called a 2-elementary K3 surface.
 Proposition 3.9. Let $(Y, \theta)$ be a 2-elementary K3 surface equipped with a \(\theta\)-invariant Ricci-flat Kähler metric \(g\). Let \(I\) be the complex structure on \(X\) such that \(Y = X_I\), let \(\gamma\) be a holomorphic 2-form on \(Y\) such that \(\eta \wedge \bar{\eta} = 2\gamma^2\), and let \(J, K \in \mathbb{C}_g\) be the complex structures such that \(\gamma_J = \text{Re}(\eta)\) and \(\gamma_K = \text{Im}(\eta)\). Then
(1) \(\theta\) is a hyperbolic involution and \(g \in E^\theta\);
(2) the hyper-Kähler structure \((I, J, K)\) on \((X, g)\) is compatible with \(\theta\).

Proof. By (3.4) and the \(\theta\)-invariance of \(\gamma_J\), we get (3.1). The hyperbolicity of \(\theta\) follows from e.g. [6], [13], [18, Lemma 1.3 (1)]. \(\square\)

We refer to [6], [10], [15] for more details about 2-elementary K3 surfaces.

3.3. Real K3 surfaces

After [6], [10], [15, Sect. 2 and Sect. 3], we make the following:

Definition 3.10. A K3 surface equipped with an anti-holomorphic involution is called a real K3 surface. A point of a real K3 surface is real if it is fixed by the anti-holomorphic involution.

Example 3.11. Let \(Y\) be an algebraic K3 surface defined over \(\mathbb{R}\). Then there exists a projective embedding \(j : Y \hookrightarrow \mathbb{P}^N(\mathbb{C})\) defined over \(\mathbb{R}\). The complex conjugation \(\mathbb{P}^N(\mathbb{C}) \ni (z_1 : \cdots : z_N) \rightarrow (\bar{z}_1 : \cdots : \bar{z}_N) \in \mathbb{P}^N(\mathbb{C})\) acts on \(Y\) as an anti-holomorphic involution. Let \(\sigma : Y \rightarrow Y\) be the involution induced by the complex conjugation on \(\mathbb{P}^N(\mathbb{C})\). Then the pair \((Y, \sigma)\) is a real K3 surface. We refer to [6], [10], [13], [15, Sect. 2] for more details about this example.

Let \((Y, \sigma)\) be a real K3 surface. Let \(g\) be a Kähler metric on \(Y\) with Kähler form \(\gamma\). Then \(\sigma^*g\) is a Kähler metric with Kähler form \(-\sigma^*\gamma\). Indeed, if \(Y = X_I\), we get
(3.5) \((\sigma^*g)(J(u), v) = g(\sigma_*J(u), \sigma_*(v)) = -g(J\sigma_*(u), \sigma_*(v)) = -(\sigma^*\gamma)(u, v)\)
for all \(u, v \in TX\). Hence \(Y\) admits a \(\sigma\)-invariant Kähler metric e.g. \(g + \sigma^*g\). By (3.5), the Kähler form and the Kähler class of a \(\sigma\)-invariant Kähler metric are anti-invariant with respect to the \(\sigma\)-action. In particular, there exists a Kähler class \(\kappa\) on \(Y\) with \(\sigma^*\kappa = -\kappa\).

Lemma 3.12. (1) There exists \(\eta \in H^0(Y, \Omega^2_Y) \setminus \{0\}\) with
(3.6) \(\sigma^*\eta = \bar{\eta}\).
(2) Let \(\kappa\) be a Kähler class on \(Y\) with \(\sigma^*\kappa = -\kappa\), and let \(\gamma\) be the Ricci-flat Kähler form representing \(\kappa\). Then
(3.7) \(\sigma^*\gamma = -\gamma\).
(3) There exists a \(\sigma\)-invariant Ricci-flat Kähler metric on \(Y\).

Proof. (1) Let \(\xi\) be a nowhere vanishing holomorphic 2-form on \(Y\). Since \(\sigma\) is anti-holomorphic, \(\sigma^*\xi\) is a holomorphic 2-form on \(Y\). Then either \(\xi + \sigma^*\xi\) or \((\xi - \sigma^*\xi)/\sqrt{-1}\) is a nowhere vanishing holomorphic 2-form on \(Y\) satisfying (3.6).
(2) Let \(g\) be the Riemannian metric on \(Y\) whose Kähler form is \(\gamma\). By (3.5), \(-\sigma^*\gamma\) is the Kähler form of \(\sigma^*g\) representing \(\kappa\). By the Ricci-flatness of \(\gamma\), there exists a real non-zero constant \(C\) with \(C\gamma^2 = \eta \wedge \bar{\eta}\). This, together with (3.6), yields that
\[ C(-\sigma^*\gamma)^2 = \sigma^*\eta \wedge \sigma^*\bar{\eta} = \bar{\eta} \wedge \eta = \eta \wedge \bar{\eta}. \]
This implies the Ricci-flatness of $-\sigma^*\gamma$. By the uniqueness of the Ricci-flat Kähler form in the Kähler class $\kappa$, we get (3.7).

(3) By (2), there exists a Ricci-flat Kähler metric $g$ on $Y$ whose Kähler form satisfies (3.7). Since $\sigma$ is anti-holomorphic, we get $\sigma^*g = g$ by (3.7).

**Definition 3.13.** A holomorphic 2-form $\eta$ on a real K3 surface $(Y, \sigma)$ is defined over $\mathbb{R}$ if Eq. (3.6) holds.

**Proposition 3.14.** Let $(Y, \sigma)$ be a real K3 surface equipped with a $\sigma$-invariant Ricci-flat Kähler metric $g$. Let $J$ be the complex structure on $X$ with $Y = X_J$, let $\eta$ be a holomorphic 2-form on $Y$ defined over $\mathbb{R}$ with $\eta \wedge \bar{\eta} = 2\gamma_J^2$, and let $I, K \in \mathcal{C}_g$ be the complex structures with $\gamma_I = -\text{Re} \eta$ and $\gamma_K = \text{Im} \eta$. Then

1. $\sigma$ is a hyperbolic involution and $g \in \mathcal{E}^\sigma$;
2. the hyper-Kähler structure $(I, J, K)$ is compatible with $(g, \sigma)$.

**Proof.** By (3.6) and (3.7), we get

$$(3.8) \quad \sigma^*\gamma_I = \gamma_I, \quad \sigma^*\gamma_J = -\gamma_J, \quad \sigma^*\gamma_K = -\gamma_K,$$

which, together with $\sigma^*g = g$, implies (3.1). Hence it suffices to verify the hyperbolicity of $\sigma$. Consider the K3 surface $X_I$. By (3.1) and (3.8), $\sigma: X_I \to X_I$ is an anti-symplectic holomorphic involution. Hence $\sigma$ is hyperbolic. \qed

**Proposition 3.15.** Let $\iota: X \to X$ be a hyperbolic involution, and let $g \in \mathcal{E}^\iota$. Let $(I, J, K)$ be a hyper-Kähler structure on $(X, g)$ compatible with $\iota$. Then

1. $(X_I, \iota)$ is a 2-elementary K3 surface, and $\gamma_I + \sqrt{-1}\gamma_K$ is a holomorphic 2-form on $X_I$;
2. $(X_J, \iota)$ is a real K3 surface, and $\gamma_I + \sqrt{-1}\gamma_K$ is a holomorphic 2-form on $X_I$ defined over $\mathbb{R}$.

**Proof.** The result follows from (3.1) and Propositions 3.9 and 3.14. \qed

3.4. **The period map for Ricci-flat metrics compatible with involution**

Let $M \subset \mathbb{L}_{K3}$ be a sublattice.

**Definition 3.16.** A hyperbolic involution $\iota: X \to X$ is of type $M$ if there exists an isometry of lattices $\alpha: H^2(X, \mathbb{Z}) \cong \mathbb{L}_{K3}$ such that $M = \alpha(H^2(X, \mathbb{Z}))$. An isometry $\alpha: H^2(X, \mathbb{Z}) \cong \mathbb{L}_{K3}$ with this property is called a marking of type $M$.

Let $\iota$ be a hyperbolic involution of type $M$, and let $\alpha: H^2(X, \mathbb{Z}) \cong \mathbb{L}_{K3}$ be a marking of type $M$. Then $M \subset \mathbb{L}_{K3}$ is a primitive, 2-elementary, hyperbolic sublattice by [13, Cor 1.5.2]. The orthogonal complement of $M$ in $\mathbb{L}_{K3}$ is denoted by $M^\perp$. Then $M^\perp = \alpha(H^2(X, \mathbb{Z}))$. We set $r(M) := \text{rank}_{\mathbb{Z}} M$ and

$$\Omega_M := \{[\eta] \in \mathbb{P}(M^\perp \otimes \mathbb{C}); \langle \eta, \eta \rangle = 0, \langle \eta, \bar{\eta} \rangle > 0\}.$$

Since $M^\perp$ has signature $(2, 20 - r(M))$, $\Omega_M$ consists of two connected components, each of which is isomorphic to a symmetric bounded domains of type IV of dimension $20 - r(M)$ (cf. [1, p.282, Lemma 20.1]). Then $\Omega_M$ is the period domain for 2-elementary K3 surfaces of type $M$ by [18, Sect. 1.4]. Notice that the two connected components of $\Omega_M$ are exchanged by the complex conjugation on $\mathbb{P}(M^\perp \otimes \mathbb{C})$.

**Lemma 3.17.** Let $\iota: X \to X$ be a hyperbolic involution of type $M$, and let $\alpha$ be a marking of type $M$. Let $g \in \mathcal{E}^\iota$, and let $(I, J, K)$ be a hyper-Kähler structure on
(X, g) compatible with \( \iota \). Then the pair of conjugate points \([\alpha(\gamma_J \pm \sqrt{-1}\gamma_K)] \in \Omega_M\)

is independent of the choice of \((I, J, K)\) compatible with \( \iota \).

**Proof.** By Proposition 3.15 (1), \([\alpha(\gamma_J + \sqrt{-1}\gamma_K)]\) is the period of a marked 2-

elementary K3 surface of type \( M \). Hence \([\alpha(\gamma_J + \sqrt{-1}\gamma_K)] \in \Omega_M\) by [18, Sect. 1.4]. Since the complex conjugation preserves \( \Omega_M \), we get \([\alpha(\gamma_J \pm \sqrt{-1}\gamma_K)] \in \Omega_M\).

Let \((I', J', K')\) be an arbitrary hyper-Kähler structure on \((X, g)\) compatible with \( \iota \). By Proposition 3.6 (2), there exists \( \psi \in \mathbb{R} \) such that

\[
\gamma_J + \sqrt{-1}\gamma_{K'} = e^{\sqrt{-1}\psi}(\gamma_J \pm \sqrt{-1}\gamma_K).
\]

Hence \([\alpha(\gamma_J \pm \sqrt{-1}\gamma_K)] = [\alpha(\gamma_J \pm \sqrt{-1}\gamma_{K'})] \in \Omega_M\). \( \square \)

**Definition 3.18.** With the same notation as in Lemma 3.17, the pair of conjugate

points \([\alpha(\gamma_J \pm \sqrt{-1}\gamma_K)] \in \Omega_M\) is called the *period* of \((g, \alpha)\) and is denoted by

\[\varpi_M(g, \alpha) := [\alpha(\gamma_J \pm \sqrt{-1}\gamma_K)].\]

4. An invariant of Ricci-flat metric compatible with involution

Throughout this section, we fix the following notation. Let \( \iota : X \to X \) be a

hyperbolic involution of type \( M \), and let \( \alpha : H^2(X, \mathbb{Z}) \cong L_{K3} \) be a marking of type \( M \). Let \( \mathbb{Z}_2 = \langle \iota \rangle \) be the group of diffeomorphisms of \( X \) generated by \( \iota \). Let \( g \in E^\iota \).

4.1. **Equivariant determinant of the Laplacian**

Let \( d^* : A^1(X) \to C^\infty(X) \) be the formal adjoint of the exterior derivative \( d : C^\infty(X) \to A^1(X) \) with respect to the \( L^2 \)-inner product induced by \( g \). The Laplacian of \((X, g)\)

is defined as \( \Delta_g = \frac{1}{2} d^* d \). We define

\[ C^\infty_{\pm}(X) := \{ f \in C^\infty(X); \iota^* f = \pm f \}. \]

Since \( \iota \) preserves \( g \), \( \Delta_g \) commutes with the \( \iota \)-action on \( C^\infty(X) \). Hence \( \Delta_g \) preserves the subspaces \( C^\infty_{\pm}(X) \). We set

\[ \Delta_{g, \pm} := \Delta_g |_{C^\infty_{\pm}(X)}. \]

Define the spectral zeta function of \( \Delta_{g, \pm} \) as

\[ \zeta_{g, \pm}(s) := \text{Tr} \left\{ \Delta_g \pm \iota \Delta_g \right\}^{-s} = \text{Tr} \left[ \frac{1 \pm \iota}{2} \circ (\Delta_g \pm \iota \Delta_g)^{-s} \right], \quad \text{Re} \ s \gg 0. \]

Then \( \zeta_{g, \pm}(s) \) converges absolutely for \( \text{Re} \ s \gg 0 \), it extends meromorphically to the complex plane \( \mathbb{C} \), and it is holomorphic at \( s = 0 \).

**Definition 4.1.** (1) The equivariant determinant of \( \Delta_g \) with respect to \( \mathbb{Z}_2 = \langle \iota \rangle \)

is defined by

\[ \det^*_\mathbb{Z}_2 \Delta_g(\iota) := \exp[-\zeta_{g, +}'(0) + \zeta_{g, -}'(0)]. \]

(2) For a real K3 surface \((Y, \sigma)\) and a \( \sigma \)-invariant Ricci-flat Kähler metric \( g \), set

\[ \det^*_\mathbb{Z}_2 \Delta_{Y, g}(\sigma) := \det^*_\mathbb{Z}_2 \Delta_g(\sigma). \]
4.2. Equivariant determinant of the Laplacian and equivariant analytic torsion. Let \((I, J, K)\) be a hyper-Kähler structure on \((X, g)\) compatible with \(\iota\). By Proposition 3.15, \(\iota\) is an anti-symplectic holomorphic involution on \(X_I\).

Let \(\square_{g,i}^0\) be the \(\bar{\partial}\)-Laplacian acting on \((0, q)\)-forms on the Kähler manifold \((X_I, \gamma_I)\). By the definition of \(\Delta_{g,i}\) and the Kähler identities, one has \(\Delta_{g,i} = \square_{g,i}^0\). We set

\[
\zeta^{0,q}(g, I, \iota)(s) := \text{Tr} \left[ \iota^* (\square_{g,i}^0)_{|\ker \square_{g,i}^0})^{-s} \right], \quad \text{Re } s > 0.
\]

Then

\[
\zeta^{0,1}(g, I, \iota)(s) = \zeta^{0,0}(g, I, \iota)(s) + \zeta^{0,2}(g, I, \iota)(s),
\]

\[
\zeta^{0,0}(g, I, \iota)(s) = \zeta^+_g(s) - \zeta^-_g(s).
\]

After [2] and [11], we make the following:

**Definition 4.2.** The equivariant analytic torsion of \((X_I, \gamma_I, \iota)\) is defined by

\[
\tau_{\mathbb{Z}_2}(g, I, \iota) := \exp \left[ \zeta^{0,1}(g, I, \iota)'(0) - 2\zeta^{0,2}(g, I, \iota)'(0) \right].
\]

**Lemma 4.3.** The following identity holds

\[
\tau_{\mathbb{Z}_2}(g, I, \iota) = (\det^*_{\mathbb{Z}_2} \Delta_{g,i}(\iota))^{-2}.
\]

**Proof.** Let \(K_{X_I}\) be the canonical line bundle of \(X_I\), and set \(\eta_I = \gamma_I + \sqrt{-1}\gamma_K \in H^0(X_I, K_{X_I})\). Since \(\gamma_J\) and \(\gamma_K\) are parallel with respect to the Levi-Civita connection of \((X, g)\), so is \(\eta_I\). The isomorphism of complex line bundles \(\otimes \eta_I : O_{X_I} \cong K_{X_I}\) induces an isometry with respect to the \(L^2\)-inner products:

\[
\otimes \eta_I/\sqrt{2} : C^\infty(X) \ni f \rightarrow f \cdot \eta_I/\sqrt{2} \in A^{0,2}(X_I).
\]

Let \(E_g(\lambda)\) (resp. \(E_{g,i}^{0,2}(\lambda)\)) be the eigenspace of \(\Delta_{g,i}\) (resp. \(\square_{g,i}^0\)) with respect to the eigenvalue \(\lambda \in \mathbb{R}\). Then \(\iota\) preserves \(E_g(\lambda)\) and \(E_{g,i}^{0,2}(\lambda)\). Let \(E_g(\lambda)_{\pm}\) and \(E_{g,i}^{0,2}(\lambda)_{\pm}\) be the \(\pm 1\)-eigenspaces of the \(\iota\)-actions on \(E_g(\lambda)\) and \(E_{g,i}^{0,2}(\lambda)\), respectively. Since \(\iota^* \eta_I = -\eta_I\) and

\[
\square_{g,i}^0(f \cdot \eta_I) = (\Delta_{g,i}f) \cdot \eta_I, \quad f \in C^\infty(X),
\]

we get the isomorphism \(\otimes \eta_I/\sqrt{2} : E_g(\lambda)_{\pm} \cong E_{g,i}^{0,2}(\lambda)_{\mp}\) for all \(\lambda \in \mathbb{R}\), which yields that

\[
\zeta^{0,2}(g, I, \iota)(s) = -\zeta^+_g(s) + \zeta^-_g(s), \quad s \in \mathbb{C}.
\]

By (4.1), (4.2) and (4.3), we get

\[
\log \tau_{\mathbb{Z}_2}(g, I, \iota) = \zeta^{0,1}(g, I, \iota)'(0) - 2\zeta^{0,2}(g, I, \iota)'(0)
\]

\[
= \zeta^{0,0}(g, I, \iota)'(0) - \zeta^{0,2}(g, I, \iota)'(0)
\]

\[
= 2 \left[ \frac{d}{ds} \right]_{s=0} (\zeta^+_g(s) - \zeta^-_g(s)) = -2 \log \det^*_{\mathbb{Z}_2} \Delta_{g,i}(\iota).
\]

This completes the proof of Lemma 4.3.
4.3. A function $\tau_\iota$ on $E^\iota$

Let $X^\iota$ be the set of fixed points of $\iota$:

$$X^\iota := \{ x \in X; \iota(x) = x \}.$$ 

By [13, Th. 3.10.6] or [14, Th. 4.2.2], $X^\iota$ is either the empty set or the disjoint union of finitely many compact, connected, orientable two-dimensional manifolds. Moreover, $r(\iota) = 10$ when $X^\iota = \emptyset$.

When $X^\iota \neq \emptyset$, the Riemannian metric $g|_{X^\iota}$ induces a complex structure on $X^\iota$ such that $g|_{X^\iota}$ is Kähler. Equipped with this complex structure, $X^\iota$ is a complex submanifold of $X_I$, since $\iota$ is holomorphic with respect to $I$. Let

$$X^\iota = \coprod_i C_i$$

be the decomposition into the connected components. Let $\Delta_{(C_i,g|_{C_i})} := \frac{1}{2}d^*d$ be the Laplacian of the Riemannian manifold $(C_i,g|_{C_i})$, and let

$$\zeta_{(C_i,g|_{C_i})}(s) := \text{Tr} \left[ \Delta_{(C_i,g|_{C_i})}(\ker \Delta_{(C_i,g|_{C_i})})^s \right]$$

be the spectral zeta function of $\Delta_{(C_i,g|_{C_i})}$. The regularized determinant of $\Delta_{(C_i,g|_{C_i})}$ is defined as

$$\text{det}^s \Delta_{(C_i,g|_{C_i})} := \exp \left( -\zeta_{(C_i,g|_{C_i})}'(0) \right).$$

Similarly, let $\tau(C_i,I,\gamma_I|_{C_i})$ be the analytic torsion of the trivial Hermitian line bundle on the Kähler manifold $(C_i,I,\gamma_I|_{C_i})$ (cf. [16]). For all $i$, one has

$$\tau(C_i,I,\gamma_I|_{C_i}) = (\text{det}^s \Delta_{(C_i,g|_{C_i})})^{-1}. \quad (4.5)$$

We define a function $\tau_\iota$ on $E^\iota$ and a function $\tau_M$ on the moduli space of 2-elementary $K3$ surfaces of type $M$ (cf. [18, Def. 5.1]) as follows:

**Definition 4.4.** Let $(I,J,K)$ be a hyper-Kähler structure on $(X,g)$ compatible with $\iota$. When $X^\iota \neq \emptyset$, set

$$\tau_\iota(g) := \left( \text{det}^s \Delta_g(\iota) \right)^{-2} \prod_i \text{Vol}(C_i,g|_{C_i}) (\text{det}^s \Delta_{(C_i,g|_{C_i})})^{-1},$$

$$\tau_M(X_I,\iota) := \tau_2^+ (X_I,\gamma_I)(\iota) \prod_i \text{Vol}(C_i,\gamma_I|_{C_i}) \tau(C_i,I,\gamma_I|_{C_i}).$$

When $X^\iota = \emptyset$, set

$$\tau_\iota(g) := \left( \text{det}^s \Delta_g(\iota) \right)^{-2}, \quad \tau_M(X_I,\iota) := \tau_2^+ (X_I,\gamma_I)(\iota).$$

Notice that $(X,g)$ has volume 1 for $g \in E^\iota$. By [18, Th. 5.7], $\tau_M(X_I,\iota)$ is independent of the choice of an $\iota$-invariant Ricci-flat Kähler metric on $X_I$.

**Lemma 4.5.** If the hyper-Kähler structure $(I,J,K)$ on $(X,g)$ is compatible with $\iota$, then

$$\tau_\iota(g) = \tau_M(X_I,\iota). \quad (4.6)$$

In particular, one has

$$\tau_M(X_I,\iota) = \tau_M(X_{-I},\iota). \quad (4.7)$$

**Proof.** The first result follows from Lemma 4.3 and (4.5). If $(I,J,K)$ is compatible with $\iota$, so is $(-I,J,-K)$. Hence the second result follows from the first one. \(\square\)
In the next theorem, we shall use the notion of automorphic forms on $\Omega_M$, for which we refer to [18, Sect. 3]. For an automorphic form $\Psi$ on $\Omega_M$, its norm $\|\Psi\|$ is a function on $\Omega_M$ defined in [18, Def. 3.16]. If $X^\ell = \emptyset$ or if every connected component of $X^\ell$ is diffeomorphic to a 2-sphere, then $\Psi$ is an automorphic form in the classical sense and $\|\Psi\|$ coincides with the Petersson norm of $\Psi$.

**Theorem 4.6.** There exist $\nu(M) \in \mathbb{N}$ and an automorphic form $\Phi_M$ on $\Omega_M$ of weight $((r(M) - 6)\nu(M), 4\nu(M))$ for some cofinite subgroup of $O(M^\perp)$ satisfying

1. $\|\Phi_M([\eta])\| = \|\Phi_M([\overline{\eta}])\|$ for all $[\eta] \in \Omega_M$;
2. For all $g \in \mathcal{E}$,

\begin{equation}
\tau_\nu(g) = \|\Phi_M(\varpi_M(g, \alpha))\|^{-\frac{1}{\nu(M)}} .
\end{equation}

**Proof.** Let $\Phi_M$ be the automorphic form as in [18, Th. 5.2]. Let $(I, J, K)$ be a hyper-Kähler structure on $(X, g)$ compatible with $\iota$. Let $(X_{-I, \iota})$ be a 2-elementary $K3$ surface of type $M$. Then so is $(X_{-I, \iota})$. Since an anti-holomorphic 2-form on $X_I$ is a holomorphic 2-form on $X_{-I}$, the Griffiths period of $(X_{-I, \iota})$ in the sense of [18, (1.11)] is the complex conjugate of the Griffiths period of $(X_I, \iota)$. This, together with [18, Th. 5.2] and (4.7), implies the first assertion. Since $\varpi_M(g, \alpha) = \alpha(\gamma_I \pm \sqrt{-1}\gamma_K)$ and since $\gamma_I + \sqrt{-1}\gamma_K \in H^0(X_I, \Omega_{X_I}^2)$, the second assertion follows from [18, Th. 5.2] and (4.6). \qed

We assume that $\iota$ has no fixed points. By Proposition 3.15 (1), $\iota$ is a holomorphic involution on $X_I$ without fixed points, so that the quotient $X_I/\iota$ is an Enriques surface by [1, Chap. 8, Lemma 15.1]. By [1, Chap. 8, Lemma 19.1], there exists an isometry $\alpha: H^2(X, \mathbb{Z}) \cong \mathbb{L}_K$ such that

\[ \alpha \ i^* \ a^{-1}(a, b, c, x, y) = (b, a, -c, y, x), \quad a, b, c \in \mathbb{U}, \quad x, y \in \mathbb{E}_8. \]

Set $\mathcal{L} := \alpha(H^2_+(X, \mathbb{Z}))$. Then $\iota$ is of type $\mathcal{L}$. We refer to [1, Chap. 8, Sects. 15-21] for more details about Enriques surfaces.

Let $\Phi$ be the **Borcherds $\Phi$-function**, which is an automorphic form of weight 4 on the period domain for Enriques surfaces by [3]. By [18, Th. 8.2], there exists a constant $C_{\mathcal{L}} \neq 0$ such that

\begin{equation}
\Phi_{\mathcal{L}} = C_{\mathcal{L}} \Phi.
\end{equation}

Since $\iota$ has no fixed points, we may choose $\nu(\mathcal{L}) = 1$ in Theorem 4.6 by the definition of $\nu(M)$ in [18, pp. 79].

**Corollary 4.7.** Let $(Y, \sigma)$ be a real $K3$ surface without real points. Let $g$ be a $\sigma$-invariant Ricci-flat Kähler metric on $Y$ with volume 1. Let $\omega_g$ be the Kähler form of $g$, and let $\eta_g$ be a holomorphic 2-form on $Y$ defined over $\mathbb{R}$ such that $\eta_g \wedge \overline{\eta}_g = 2\omega_g^2$. Let $\alpha$ be a marking of type $\mathcal{L}$. Under the identifications of $\omega_g$ and $\eta_g$ with their cohomology classes, the following identity holds:

\[ \det^\sharp \omega-g(\sigma) = C_{\mathcal{L}}^2 \|\Phi([-\alpha(\gamma_g + \sqrt{-1}\Im \eta_g)] \|^{\frac{1}{2}}. \]

**Proof.** By Proposition 3.14 and Definition 3.18, we get $\varpi_{\mathcal{L}}(g, \alpha) = \|\alpha(\gamma_g + \sqrt{-1}\Im \eta_g)]\|$. Substituting this equality and (4.9) into (4.8), we get the result. \qed
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