GLOBAL WELL-POSEDNESS FOR VOLUME-SURFACE REACTION-DIFFUSION SYSTEMS

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Abstract. We study the global existence of classical solutions to volume-surface reaction-diffusion systems with control of mass. Such systems appear naturally from modeling evolution of concentrations or densities appearing both in a volume domain and its surface, and therefore have attracted considerable attention. Due to the characteristic volume-surface coupling, global existence of solutions to general systems is a challenging issue. In particular, the duality method, which is powerful in dealing with mass conserved systems in domains, is not applicable on its own. In this paper, we introduce a new family of $L^p$-energy functions and combine them with a suitable duality method for volume-surface systems, to ultimately obtain global existence of classical solutions under a general assumption called the intermediate sum condition. For systems that conserve mass, but do not satisfy this condition, global solutions are shown under a quasi-uniform condition, that is, under the assumption that the diffusion coefficients are close to each other. In the case of mass dissipation, we also show that the solution is bounded uniformly in time by studying the system on each time-space cylinder of unit size, and showing that the solution is sup-norm bounded independently of the cylinder. Applications of our results include global existence and boundedness for systems arising from membrane protein clustering or activation of Cdc42 in cell polarization.

Contents

1. Introduction and Main Results 2
   1.1. Problem setting 2
   1.2. State of the art 3
   1.3. Main results and Key ideas 4
2. Construction of $L^p$-energy functions 11
3. Duality method 21
4. Proof of main results 27
   4.1. Theorem 1.2: Global existence 27
   4.2. Theorem 1.2: Uniform-in-time bounds 34
   4.3. Proof of Theorem 1.1 44
   4.4. Proof of Theorem 1.3 45
5. Applications 47
   5.1. Membrane protein clustering 47

2010 Mathematics Subject Classification. 35A01, 35K57, 35K58, 35Q92.
Key words and phrases. Volume-surface reaction-diffusion systems; Global solutions; Intermediate sum condition; $L^p$-energy functions; Duality method.
1. Introduction and Main Results

1.1. Problem setting. Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded domain with smooth boundary $M := \partial \Omega$. Let $m_1 \geq 0$ and $m_2 \geq 0$. We consider the following volume-surface reaction-diffusion system

$$
\begin{align*}
\frac{\partial u_i}{\partial t} &= d_i \Delta u_i + F_i(u), & \text{on } \Omega \times (0, T) & \text{for } i = 1, \ldots, m_1 \\
\frac{\partial u_i}{\partial \eta} &= G_i(u, v), & \text{on } M \times (0, T) & \text{for } i = 1, \ldots, m_1 \\
\frac{\partial v_j}{\partial t} &= \delta_j \Delta_M v_j + H_j(u, v), & \text{on } M \times (0, T) & \text{for } j = 1, \ldots, m_2 \\
u &= u_0 & \text{on } \overline{\Omega} \times \{0\} \\
v &= v_0 & \text{on } M \times \{0\}
\end{align*}
$$

(1.1)

where $\partial / \partial \eta$ denotes the outward normal flux on the boundary $M$, $u = (u_1, u_2, \ldots, u_{m_1})$, $v = (v_1, v_2, \ldots, v_{m_2})$ are vectors of concentrations, $F_i : \mathbb{R}^{m_1} \to \mathbb{R}^{m_1}$ and $G_i, H_j : \mathbb{R}^{m_1 + m_2} \to \mathbb{R}^{m_1 + m_2}$ satisfy assumptions which will be specified later. The operators $\Delta$ and $\Delta_M$ represent the traditional Laplacian in $\Omega$ and Laplace-Beltrami operator on $M$. The initial data $u_0 = (u_{0,i})_{i=1,\ldots,m_1}$ and $v_0 = (v_{0,j})_{j=1,\ldots,m_2}$ are assumed to be smooth, bounded and component-wise non-negative on $\overline{\Omega}$ and $M$, respectively. Also, when $n = 1$, we assume $m_2 = 0$. Throughout this paper, we will assume the following conditions on the domain

(O) $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is bounded domain with smooth boundary $M = \partial \Omega$ such that $\Omega$ lies locally on one side of $M$;

on the diffusion coefficients

(D) $d_i > 0$ and $\delta_j > 0$ for all $i = 1, \ldots, m_1$, $j = 1, \ldots, m_2$;

and on the nonlinearities

(F1) (local Lipschitz continuity) $F_i : \mathbb{R}^{m_1} \to \mathbb{R}$, $G_i : \mathbb{R}^{m_1 + m_2} \to \mathbb{R}$, $H_j : \mathbb{R}^{m_1 + m_2} \to \mathbb{R}$ are locally Lipschitz continuous for $i \in \{1, \ldots, m_1\}$ and $j \in \{1, \ldots, m_2\}$;

(F2) (quasi-positivity)

$$F_i(u), G_i(u, v) \geq 0 \text{ for all } u \in \mathbb{R}^{m_1}, v \in \mathbb{R}^{m_2} \text{ with } u_i = 0, \forall i = 1, \ldots, m_1,$$

and

$$H_j(u, v) \geq 0 \text{ for all } u \in \mathbb{R}^{m_1}, v \in \mathbb{R}^{m_2} \text{ with } v_j = 0, \forall j = 1, \ldots, m_2.$$
(F3) (mass control) there exist \( a_i, b_j > 0 \) and \( L \in \mathbb{R}, K \geq 0 \) such that
\[
\sum_{i=1}^{m_1} a_i F_i(u) \leq L \left( \sum_{i=1}^{m_1} u_i \right) + K, \tag{1.2}
\]
\[
\sum_{i=1}^{m_1} a_i G_i(u, v) + \sum_{j=1}^{m_2} b_j H_j(u, v) \leq L \left( \sum_{i=1}^{m_1} u_i + \sum_{j=1}^{m_2} v_j \right) + K,
\]
for all \( u \in \mathbb{R}_{+}^{m_1} \) and \( v \in \mathbb{R}_{+}^{m_2} \).

(F4) (polynomial growth) there are constants \( K_1, r > 0 \) such that, for all \( i = 1, \ldots, m_1, j = 1, \ldots, m_2 \),
\[
F_i(u), G_i(u, v), H_j(u, v) \leq K_1 \left( \sum_{i=1}^{m_1} u_i^r + \sum_{j=1}^{m_2} v_j^r + 1 \right), \tag{1.3}
\]
for all \( u \in \mathbb{R}_{+}^{m_1} \) and \( v \in \mathbb{R}_{+}^{m_2} \).

With (F1), it can be shown that (1.1) possesses a unique local strong solution on a maximal interval \((0, T_{\text{max}})\), cf. [SM16]. Moreover,
\[
T_{\text{max}} < +\infty \implies \lim \sup_{t \to T_{\text{max}}} \left( \sum_{i=1}^{m_1} \|u_i(t)\|_{L^\infty(\Omega)} + \sum_{j=1}^{m_2} \|v_j(t)\|_{L^\infty(M)} \right) = +\infty. \tag{1.4}
\]

The quasi-positivity assumption (F2) assures that the solution to (1.1) is (component-wise) non-negative (as long as it exists) if the initial data is non-negative. This assumption also has a simple physical interpretation: if a concentration is zero, it cannot be consumed in a reaction. The condition (F3) implies that the total mass of the system is finite for all time, which, together with the non-negativity of solutions, implies that
\[
u_i \in L^\infty(0, T; L^1(\Omega)), \quad v_j \in L^\infty(0, T; L^1(M)) \quad \forall i = 1, \ldots, m_1, j = 1, \ldots, m_2, T > 0.
\]
Comparing to (1.4), these a-priori estimates are far from enough to conclude the global existence of solutions to (1.1). In fact, it was shown in [PS00] that the above assumptions are very likely not enough to prevent blow-up in finite time even for reaction-diffusion systems in domains. The global well-posedness question for (1.1) is, therefore, challenging and generally open.

In this paper, we will show that under the above natural assumptions, and a so-called \textit{intermediate sum condition}, there exists a unique global strong solution to (1.1). Moreover, if the total mass is bounded by a constant independent of time, the solution is sup-norm bounded uniformly for all time.

1.2. \textbf{State of the art.} Volume-surface (or bulk-surface) systems of the form (1.1) have recently attracted a lot of attention. On the one hand, such systems arise naturally from many applications. For example, in asymmetric stem cell division [FRT16, FLT18], in modeling receptor-ligand dynamics in cell biology [GPGG14, AET18], in crystal growth [KD01, YD05], in population modeling [BRR13, BCRR15], in chemistry with fast sorption [SOMB19, AB20], or in fluid mechanics [Mie13, GM13], and so on.
On the other hand, the volume-surface coupling yields new and highly non-trivial challenges in the analysis of such systems. Of interest and importance is the question of global existence of solutions, and most of the existing works rely heavily on special structures of the considered systems. For instance, when the system is linear, or the reactions have at most linear growth, global (weak, strong) solutions were shown in [FRT16, HR18].

For nonlinear coupling on the boundary, [FLT18] shows global bounded solutions for the reversible reaction \(\alpha U \rightleftharpoons \beta V\), where \(U\) and \(V\) are volumic- and surface-concentrations, respectively. For general systems, [Dis20] showed that if the \(L^\infty\)-bound is preserved, then one can get global classical solutions. The works [SM16, SM17] showed global existence, and boundedness of solutions to volume-surface systems assuming a linear upper bound for \(F_i, G_i,\) and the sum of \(G_i + H_j\), i.e. for any \(i \in \{1, \ldots, m_1\}\) and any \(j \in \{1, \ldots, m_2\}\),

\[
F_i(u) \leq L \left( 1 + \sum_{k=1}^{m_1} u_k \right), \quad G_i(u, v) + H_j(u, v) \leq L \left( 1 + \sum_{k=1}^{m_1} u_k + \sum_{l=1}^{m_2} v_l \right),
\]

for some \(L > 0\). Due to these restrictions, the results therein are not applicable in many systems arising from cell biology (see e.g. [SGS+20, BML+20]).

It is also remarked that significant progress has been made lately concerning mass controlled reaction-diffusion systems in domains, i.e. when \(m_2 = 0\) and \(G \equiv 0\). For instance, a major problem concerning the global existence of strong solutions for systems with quadratic nonlinearities in all dimensions has been settled in three recent works [Sou18, CGV19, FMT20]. If the nonlinearities have \(L^1(\Omega \times (0, T))-\)bound a priori, global weak solutions can be shown [Pie10]. In particular, by assuming only an entropy inequality, [Fis15] shows global existence of renormalized solutions without any restriction on the growth of nonlinearities. We emphasize that none of these works seem to readily extend to the case of volume-surface systems.

In this paper, by introducing a new family of \(L^p\)-energy function and combining it with duality methods for volume-surface systems, we show the global existence of solutions to a large class of systems of type (1.1), which include the systems in [SGS+20, BML+20] as special cases.

1.3. Main results and Key ideas. To study the global existence of solutions to (1.1), we assume the so-called intermediate sum condition (see e.g. [MT20]) for the nonlinearities of (1.1), i.e. there exists an \((m_1 + m_2) \times (m_1 + m_2)\) lower triangular matrix \(A\) with non-negative elements everywhere and positive entries on the main diagonal, and constants \(L_2 \geq 0, \mu_M > 0\), such that

\[
A \left[ \frac{F(u)}{\partial_{m_2}} \right] \leq L_2 \left( \sum_{i=1}^{m_2} u_i^{\mu_M} + 1 \right) \hat{1}_{m_1 + m_2},
\]

\[
A \left[ \frac{G(u, v)}{H(u, v)} \right] \leq L_2 \left( \sum_{i=1}^{m_1} u_i^{\mu_M} + \sum_{j=1}^{m_2} v_j^{\mu_M} + 1 \right) \left[ \frac{\hat{1}_{m_1}}{\partial_{m_2}} \right] + L_2 \left( \sum_{i=1}^{m_1} u_i^{\mu_M} + \sum_{j=1}^{m_2} v_j^{\mu_M} + 1 \right) \left[ \frac{\hat{0}_{m_1}}{\partial_{m_2}} \right],
\]

(1.6)
for all \( u \in \mathbb{R}^{m_1}_+ \) and \( v \in \mathbb{R}^{m_2}_+ \), where the exponent \( p_\Omega, p_M \) and \( \mu_M \) will be specified later, \( F(u) = (F_1(u), \ldots, F_{m_1}(u)) \) and \( G(u, v) = (G_1(u, v), \ldots, G_{m_2}(u, v)) \). We denote by \( \bar{1}_m \) (resp. \( \bar{0}_m \)) the vector of all 1s elements (resp. 0s elements) in \( \mathbb{R}^m \). By dividing each row of \( A \) by the diagonal element, we can assume w.l.o.g. that \( a_{ii} = 1 \) for all \( i = 1, \ldots, m_1 + m_2 \), and we will do so for the rest of this paper.

**Remark 1.1.** It’s emphasized that assumptions (1.6) and (1.7) do not require the components of the reaction vector fields \( F, G \) and \( H \) to be at most of order \( p_\Omega, p_M \) or \( \mu_M \). It simply requires a “trade-off” of higher order terms between components of \( F, G \) and \( H \), and additionally requires that \( F_1(u) \) be bounded above by a polynomial in \( u \) of order \( p_\Omega \) and \( G_1(u, v) \) be bounded above by a polynomial in \( u \) and \( v \) of order \( p_M \).

The first main result of this paper is the following theorem.

**Theorem 1.1** ((O)-(D)-(F1)-(F2)-(F3)-(F4)-(1.6)-(1.7)). Assume

\[
1 \leq p_\Omega < 1 + \frac{2}{n}, \quad \text{and} \quad 1 \leq p_M < 1 + \frac{1}{n},
\]

and

\[
1 \leq \mu_M \leq 1 + \frac{4}{n + 1}.
\]

Then for any nonnegative, bounded initial data \( (u_0, v_0) \in (W^{2-2/p}(\Omega))^{m_1} \times (W^{2-2/p}(M))^{m_2} \) for some \( p > n \) satisfying the compatibility condition

\[
d_i \partial_i u_{i,0} = G_i(u_0, v_0) \quad \text{on} \quad M \quad \text{for all} \quad i = 1, \ldots, m_1,
\]

the system (1.1) has a unique global classical solution. Moreover, if \( L < 0 \) or \( L = K = 0 \) in (F3), then

\[
\sup_{i=1,\ldots,m_1} \sup_{j=1,\ldots,m_2} \sup_{t \geq 0} (\|u_i(t)\|_{L^\infty(\Omega)} + \|v_j(t)\|_{L^\infty(M)}) < +\infty.
\]

**Remark 1.2** (Improvements or generalizations).

- The first remark is that the smoothness of initial data as well as the compatibility condition (1.10) are only used to obtain a local classical solution (see e.g. [SM16] or [HR18]). We believe that this local existence can in fact be proved for merely bounded initial data, though, due to the volume-surface coupling, the proof is more delicate comparing to the case of systems in domains (see e.g. [Ali81]). The details are left for the interested reader.

- The intermediate sum conditions involving \( F \) and \( G \) in (1.6) and (1.7) are used to construct \( L^p \)-energy functions, which is essential in the proof of global existence. In fact, one can construct such functions under more general (but technical) assumptions (see Remark 2.1). We choose to present Theorem 1.1 under the intermediate sum conditions (1.6) and (1.7) as they are more constructive, and appear naturally in many applications (see Section 5).
The condition of $\mu_M$ in (1.9) can be slightly improved as
$$\mu_M < 1 + \frac{2(2 + \varepsilon)}{n + 1}$$
for a sufficiently small $\varepsilon > 0$.

The assumption $L < 0$ or $L = K = 0$ is used to obtain uniform-in-time $L^1$-bounds (see Lemma 4.4), i.e.
$$\sup_{t \geq 0} (\|u_i(t)\|_{L^1(\Omega)} + \|v_j(t)\|_{L^1(M)}) < +\infty,$$
which eventually leads to the uniform-in-time $L^\infty$-bound. If this $L^1$-bound can be obtained differently, for instance, by using special structures of a specific model, the assumption $L < 0$ or $L = K = 0$ can be removed.

If $L < 0$ and $K = 0$ in (F3), we can show that the global solution decays exponentially to zero as $t \to \infty$. The details are left for the interested reader.

The sub-critical order of $p_\Omega$ (and $p_M$) in (1.8) is typical for reaction-diffusion systems (in domains) once an $L^1$-bound is known (see e.g. [Ali79]). It is again emphasized that we do not assume the nonlinearities, but only an intermediate sum of them, to have the growth of order $p_\Omega$.

It’s finally remarked that the results of Theorem 1.1 are new even in the case without surface concentrations, i.e. $m_2 = 0$.

Let us recall that for reaction-diffusion systems in domains, i.e. $m_2 = 0$ and homogeneous boundary conditions $G_i \equiv 0, \forall i = 1, \ldots, m_1$, the global existence with control of mass and intermediate sum conditions has been successfully relied on the famous duality method, cf. [Pie10, MT20]. Thanks to this method, the control of mass condition implies an $L^{2+\varepsilon}$-estimate a-priori. This, together with the intermediate sum condition, allows the use of a bootstrap procedure to ultimately obtain $L^\infty$-estimate, which is sufficient for the global existence. It is remarked that the duality is not only crucial to get the initial $L^{2+\varepsilon}$-bound, but also important in the bootstrapping procedure. Such a strategy, unfortunately, fails to apply to volume-surface systems of the form (1.1), see [MS19]. In this paper, our first key idea to overcome this difficulty is to introduce a new family of $L^p$-energy functions. Some preliminary ideas of such $L^p$-energy functions have been carried out also for reaction-diffusion systems, but have been less noticed comparing to the duality method [MX98, Kou01, AK07].

The main aim is to provide a generalization of the usual method of constructing $L^p$-energy functions by (traditionally) multiplying both sides of a parabolic equation for a function $w(x,t)$ by $w(x,t)^{p-1}$. Instead, we consider $L^p$-energy functions consisting of all (mixed) multi-variable polynomials of order $p$ with carefully chosen coefficients. Certainly, the essential difficulty is to choose an $L^p$-energy function

\footnote{Aside from this method’s technicality, another reason for this ignorance might be that the elegant duality method, cf. [PS00], has proved to be very efficient in studying reaction-diffusion systems with control of mass, see also the survey [Pie10]. A recent work [Sha20] studied reaction-diffusion systems with nonlinear boundary conditions to which this $L^p$-energy method is well adapted while the duality method seems not. It is also remarked that the assumption therein do not allow for the use of intermediate sums, and restrict to primary application of the results to two component systems.}
so that it is “compatible” with both diffusion and reaction parts of the system, i.e. its evolution in time should lead to useful a-priori estimates. We show in this paper that the intermediate sum conditions \((1.6)\) and \((1.7)\) allow us to find such an energy function. Yet, this strategy alone is not enough to obtain sufficient estimates for global existence due to the surface concentrations. Our second key idea is, therefore, to combine the constructed \(L^p\)-energy functions with a duality method for volume-surface reaction-diffusion systems. This method has been used in previous work, see e.g. \([SM16, MS19]\), but due to the lack of \(L^p\)-estimates obtained in the current work, it has only applied to systems with restrictive conditions, for instance under assumptions \((1.5)\). We stress, as it will become apparent in our paper, that only the combination of these two ideas makes it possible to obtain global existence of \((1.1)\) under the general assumptions \((1.6)\) and \((1.7)\).

Let us briefly sketch the main ideas of the proof of Theorem 1.1, which can be roughly divided in several steps.

- **Step 1.** Firstly, by \((F3)\) one has
  \[
  \|u_i\|_{L^\infty(0,T;L^1(\Omega))}, \|u_i\|_{L^1(M \times (0,T))}, \|v_j\|_{L^\infty(0,T;L^1(M))} \leq C_T
  \]
  where \(C_T\) is a constant depending on the time horizon \(T > 0\).

- **Step 2.** The intermediate sum condition \((1.6)\) allows us to construct for any \(2 \leq p \in \mathbb{N}\) an \(L^p\)-energy function of the form
  \[
  \mathcal{L}_p[u](t) = \int_\Omega \sum_{|\beta| = p} \left( \frac{p}{\beta} \right) \theta^{\beta} u^\beta(x,t)
  \]
  with the convention \(u^\beta = \prod_{i=1}^{m_1} u_i^{\beta_i}, \theta^{\beta} = \prod_{i=1}^{m_1} \theta_i^{\beta_i}\), and \(\left( \frac{p}{\beta} \right) = \frac{p!}{\beta_1! \beta_2! \cdots \beta_{m_1}!}\), where \(\theta \in (0,\infty)^{m_1}\) are chosen suitably. Since all \(u_i\) are non-negative, \(\mathcal{L}_p[u]^{1/p}\) is an equivalent norm to the usual \(L^p\)-norm of \(u\). Due to the volume-surface coupling, it does not seem possible to show that \(\mathcal{L}_p[u]\) is non-increasing in time, or even bounded. Instead, one obtains for any \(\varepsilon > 0\) a constant \(C_\varepsilon > 0\) such that
  \[
  (\mathcal{L}_p[u])' + C \sum_{i=1}^{m_1} \left( \int_\Omega u_i^{p-1+p\alpha} + \int_M u_i^{p-1+pM} \right) \leq C_\varepsilon + \varepsilon \sum_{j=1}^{m_2} \int_M v_j^{p-1+pM}. \tag{1.11}
  \]
  This estimate is crucial in the analysis of this paper. The “left over” \(L^p\)-integrals of surface concentrations \(v_j\) in \((1.11)\) are treated using a duality method in the next steps.

\[\text{It is remarked that such a function } \mathcal{L}_p[u] \text{ can be constructed under a more general but less constructive condition than } (1.6) \text{ (see Lemma 2.2 and Remark 2.1).}\]
• **Step 3.** We derive an improved duality estimate using the dual problem suited for volume-surface systems

\[
\begin{aligned}
\partial_t \phi + \Delta \phi &= 0, \quad (x, t) \in \Omega \times (0, T), \\
\partial_t \phi + \delta \Delta_M \phi &= -\psi, \quad (x, t) \in M \times (0, T), \\
\phi(x, T) &= 0, \quad x \in \partial \Omega
\end{aligned}
\]

and (1.11) to obtain

\[
\sum_{i=1}^{m_1} \left( \|u_i\|_{L^{2+\varepsilon}(\Omega \times (0,T))}^2 + \|u_i\|_{L^{2+\varepsilon}(M \times (0,T))}^2 \right) + \sum_{j=1}^{m_2} \|v_j\|_{L^{2+\varepsilon}(M \times (0,T))}^2 \leq C_T
\]

for some \(\varepsilon > 0\).

• **Step 4.** By using the estimates in (1.12) and the duality method, we are able to show: for any \(p > 1\), any \(k \in \{1, \ldots, m_2\}\), and any \(\varepsilon > 0\), there exists \(C_{T,\varepsilon} > 0\) such that

\[
\|v_k\|_{L^p(M \times (0,T))} \leq C_{T,\varepsilon} + C_T \sum_{j=1}^{k-1} \|v_j\|_{L^p(M \times (0,T))} + \varepsilon \sum_{j=1}^{m_2} \|v_j\|_{L^p(M \times (0,T))}.
\]

This ultimately leads to the boundedness of \(v_j\) in \(L^p(M \times (0,T))\), and consequently, thanks to (1.11), the boundedness of \(u_i\) in \(L^p(\Omega \times (0,T))\) as well as in \(L^p(M \times (0,T))\). From \(u_i \in L^p(\Omega \times (0,T))\), \(u_i \in L^p(M \times (0,T))\), and \(v_j \in L^p(M \times (0,T))\) for all \(p > 1\), by using the polynomial growth of the nonlinearities in (F4) and regularization of the heat operator with inhomogeneous boundary conditions, we obtain finally \(u_i \in L^\infty(\Omega \times (0,T))\) and \(v_j \in L^\infty(M \times (0,T))\), which concludes the global existence of bounded solutions.

• **Step 5.** To obtain uniform-in-time bounds of the global solution, we use for each \(\tau \in \mathbb{N}\), a smooth cut-off function \(\varphi_\tau : \mathbb{R} \to [0, 1]\) with \(\varphi_\tau \big|_{(-\infty, \tau]} = 0\) and \(\varphi_\tau \big|_{[\tau, \infty)} = 1\) to study (1.1) on the cylinder \(\Omega \times (\tau, \tau + 2)\). By repeating the previous steps on this cylinder, we obtain that \(u_i\) and \(v_j\) are bounded in \(L^\infty(\Omega \times (\tau, \tau + 2))\) and \(L^\infty(M \times (\tau, \tau + 2))\), respectively, uniformly in \(\tau \in \mathbb{N}\). This concludes the uniform-in-time bounds of the global solution, and consequently completes the proof of Theorem 1.1.

It is noticed from the proof of Theorem 1.1 that the \(L^p\)-energy method is used for the volume concentrations, while the duality method is well adapted for the surface concentration. An improved duality method, see e.g. [CDF14], shows that one can deal with higher order nonlinearities in the intermediate sums by assuming quasi-uniform diffusion coefficients, i.e. when the diffusion coefficients are not too different from each other. Following this idea, the second main result of this paper shows global existence of strong solutions to (1.1) with large order \(\mu_M\) of the nonlinearities on the boundary, assuming that the diffusion coefficients \(\delta_j\) are quasi-uniform. To state our second main theorem, we denote by

\[
\delta_{\max} = \max\{\delta_1, \ldots, \delta_{m_2}\}, \quad \delta_{\min} = \min\{\delta_1, \ldots, \delta_{m_2}\}.
\]
**Theorem 1.2** ((O)-(D)-(F1)-(F2)-(F3)-(F4)-(1.6)-(1.7)-(1.8)). Let \( \mu_M > 0 \) be fixed. Assume that there exists a constant \( \Lambda > 0 \) such that
\[
\Lambda \geq \frac{(\mu_M - 1)(n + 1)}{2}, \quad \text{or equivalently} \quad \mu_M \leq 1 + \frac{2\Lambda}{n + 1} \tag{1.14}
\]
and
\[
\frac{\delta_{\max} - \delta_{\min}}{\delta_{\max} + \delta_{\min}} C_{\text{mr,A'}}^{M} < 1, \tag{1.15}
\]
where \( \Lambda' = \Lambda/(\Lambda - 1) \), and \( C_{\text{mr,A'}}^{M} \), which depends only on \( \Lambda \) and \( M \), is the maximal regularity constant in Lemma 3.1. Then for any nonnegative, initial data \((u_0, v_0) \in (W^{2-2/p}(\Omega))^{m_1} \times (W^{2-2/p}(M))^{m_2} \) for some \( p > n \) satisfying the compatibility condition (1.10), the system (1.1) has a unique nonnegative, global strong solution. Moreover, if \( L < 0 \) or \( L = K = 0 \) in (F3), the solution is bounded uniformly in time, i.e.
\[
\sup_{i=1,\ldots,m_1} \sup_{j=1,\ldots,m_2} \sup_{t \geq 0} (\|u_i(t)\|_{L^\infty(\Omega)} + \|v_j(t)\|_{L^\infty(M)}) < +\infty.
\]

**Remark 1.3.**

- We will show in this paper that for any fixed \( \delta_{\max} \) and \( \delta_{\min} \), condition (1.15) is always satisfied with \( \Lambda = 2 \). This implies that Theorem 1.1 is in fact a consequence of Theorem 1.2.
- We see that for fixed \( \mu_M > 0 \), condition (1.15) is fulfilled if we fix \( \Lambda = (\mu_M - 1)(n + 1)/2 \) and require the surface diffusion coefficients \( \delta_j \) to be close enough to each other (relatively to their sums). Since \( C_{\text{mr,q}}^{M} \) is increasing as \( p \searrow 1 \) and \( \lim_{q \searrow 1} C_{\text{mr,q}}^{M} = +\infty \), (1.15) means that \( \delta_j \) are required to get closer to each other as the nonlinearity order \( \mu_M \) in the intermediate sum increases.
- Similarly to Remark 1.2, we believe that the results of Theorem 1.2 still hold true for non-negative, and bounded initial data \((u_0, v_0) \in L^\infty(\Omega)^{m_1} \times L^\infty(M)^{m_2} \).

Thanks to the quasi-uniform condition (1.15), one can use the duality method to obtain some \( L^\Lambda \)-estimates on \( u_i \) and \( v_j \), which should form the starting point of a bootstrap argument. Unfortunately, \( L^\Lambda \)-estimates just fall short in case \( \Lambda \) satisfies (1.14) with an equality. To overcome this difficulty, an important observation is that the condition (1.15) is “open” in the sense that if (1.15) is true for some \( \Lambda \), then it also holds for \( \Lambda + \varepsilon \) with \( \varepsilon > 0 \) small enough\(^3\). This observation allows us to use an improved duality argument to prove that \( u_i \) are bounded in \( L^{\Lambda+\varepsilon}(\Omega \times (0, T)) \) and \( L^{\Lambda+\varepsilon}(M \times (0, T)) \), and \( v_j \) are bounded in \( L^{\Lambda+\varepsilon}(M \times (0, T)) \), for some small \( \varepsilon > 0 \). Starting from these estimates, we can use the duality method as in Step 4 above to get \( L^p \)-estimates for the solution for all \( p \geq 1 \). This is enough to conclude that the solution is global. To show the uniform-in-time bound, we again use the smooth cut-off function \( \varphi_\tau \) and repeat the arguments for global existence on each cylinder \( \Omega \times (\tau, \tau + 2) \), to obtain \( L^\infty \)-bound which are independent of \( \tau \in \mathbb{N} \).

One notices in Theorem 1.2 that by imposing the quasi-uniform condition on diffusion coefficients we are able to improve only the order of the nonlinearities in intermediate

\(^3\)This was observed and utilized in many recent works, see e.g. [CDF14, PSU17].
sums for surface concentrations. The reason is that with the $L^1$-estimates implied from $(F3)$, the upper bound $1 + \frac{2}{n}$ of $p_0$ seems to be the critical exponent to obtain $L^p$-energy estimates for all $p \geq 1$. For a specific system, it might be possible to obtain better a-priori estimates (see Section 5.3), which consequently allows a larger range of $p_0$, $p_M$ and $\mu_M$. More precisely, we have the following conditional result.

**Theorem 1.3** (O)-(D)-(F1)-(F2)-(F4)-(1.6)-(1.7). Assume there exist $a, b \geq 1$ such that, for any $T > 0$, and for all $i = 1, \ldots, m_1$, $j = 1, \ldots, m_2$

$$\|u_i\|_{L^\infty(0,T;L^a(\Omega))} + \|u_i\|_{L^b(M \times (0,T))} + \|v_j\|_{L^b(M \times (0,T))} \leq F(T) \quad (1.16)$$

where $F \in C([0, \infty))$. Assume that

$$1 \leq p_0 < 1 + a \cdot \min \left\{ \frac{2}{n}, \frac{3}{n + 2} \right\}, \quad \text{and} \quad 1 \leq p_M < 1 + \frac{a}{n} \quad (1.17)$$

and

$$1 \leq \mu_M < 1 + \frac{2b}{n + 1}. \quad (1.18)$$

Then for any nonnegative initial data $(u_0, v_0) \in (W^{2-2/p}(\Omega))^{m_1} \times (W^{2-2/p}(M))^{m_2}$ for some $p > n$ satisfying the compatibility condition

$$d_i \partial_q u_{i,0} = G_i(u_0, v_0) \quad \text{on } M \quad \text{for all } i = 1, \ldots, m_1,$$

the system (1.1) has a unique global classical solution. Moreover, if $\sup_{t \geq 0} F(t) < +\infty$, then

$$\sup_{i=1,\ldots,m_1} \sup_{j=1,\ldots,m_2} \sup_{t \geq 0} (\|u_i(t)\|_{L^\infty(\Omega)} + \|v_j(t)\|_{L^\infty(M)}) < +\infty.$$

**Remark 1.4.** It is remarked that Theorem 1.3 does not impose the mass control assumption (F3), since the condition is only used to obtain a-priori estimates of type (1.16), which are now given.

**Organization of the paper.** In the next section, we present the construction of $L^p$-energy functions, and show its relation to the intermediate sum condition (1.6). Section 3 is devoted to an improved duality method for volume-surface systems, where we show that assumption (1.15), in combination with the previously constructed $L^p$-energy functions, gives $L^{A+\varepsilon}$-estimates of the solutions. In Section 4, we start with the proof of Theorem 1.2 in subsections 4.1 and 4.2, where the first subsection shows the global existence while the second one proves the uniform-in-time bound of the solutions. As pointed out in Remark 1.3, we prove Theorem 1.1 in subsection 4.3 by showing that (1.15) is always satisfied for $\Lambda = 2$. The last subsection 4.4 presents the proof of Theorem 1.3. The last section is devoted to applications of our results to some recent models arising from cell biology. It is noted that previous works are unlikely to be applicable to these systems. Finally, we give in the Appendix A two technical lemmas concerning the construction of $L^p$-energy functions.

**Notation.** For the rest of this paper, we will use the following notation:
As some of our intermediate lemmas are of independent interest, we use the convention

**Theorem X. ((A)-(B)-(C))**

to indicate that this theorem assumes only conditions (A), (B), (C) (besides the assumptions stated explicitly therein). It’s also useful to verify which condition is applicable to which lemmas or theorems.

For $0 \leq \tau < T$

$$Q_{\tau,T} := \Omega \times (\tau, T) \quad \text{and} \quad M_{\tau,T} := M \times (\tau, T).$$

When $\tau = 0$, we simply write $Q_T$ and $M_T$.

For $1 \leq p < \infty$,

$$\|f\|_{L^p(Q_{\tau,T})} := \left( \int_\tau^T \int_{\Omega} |f(x,t)|^p \right)^{\frac{1}{p}}$$

and for $p = \infty$,

$$\|f\|_{L^\infty(Q_{\tau,T})} := \text{ess sup}_{Q_{\tau,T}} |f(x,t)|.$$

The spaces $L^p(M_{\tau,T})$ with $1 \leq p \leq \infty$ are defined in the similar way.

For $1 \leq p \leq \infty$,

$$W^{2,1}_p(Q_{\tau,T}) := \{ f \in L^p(Q_{\tau,T}) : \partial_t^r \partial_x^s f \in L^p(Q_{\tau,T}) \forall r, s \in \mathbb{N}, 2r + s \leq 2 \}$$

with the norm

$$\|f\|_{W^{2,1}_p(Q_{\tau,T})} := \sum_{2r+s \leq 2} \|\partial_t^r \partial_x^s f\|_{L^p(Q_{\tau,T})}.$$

2. Construction of $L^p$-energy functions

In this section, we firstly state local existence of (1.1), and provide the blow-up criterion, which were proved in [SM16]. The main part concerns the construction of an $L^p$-like energy function, and its relation to the intermediate sum condition (1.6). Estimates derived from this energy function are crucial in the sequel analysis of this paper.

**Theorem 2.1 ((O)-(D)-(F1)). [SM16, Theorem 3.2]** For any smooth initial data $(u_0, v_0) \in (W^{2-2/p}(\Omega))^{m_1} \times (W^{2-2/p}(M))^{m_2}$ for some $p > n$ satisfying the compatibility condition (1.10), there exists a unique classical solution to (1.1) on a maximal interval $(0, T_{\text{max}})$, i.e. for any $0 < T < T_{\text{max}},$

$$(u, v) \in C([0, T] ; L^p(\Omega)^{m_1} \times L^p(M)^{m_2}) \cap L^\infty(0, T; L^\infty(\Omega)^{m_1} \times L^\infty(M)^{m_2}),$$

for any $p > n,$

$$u \in (C^{2,1}(\overline{\Omega} \times (\tau, T)))^{m_1}, \quad v \in (C^{2,1}(M \times (\tau, T)))^{m_2} \quad \text{for all} \quad 0 < \tau < T,$$

and the equations (1.1) satisfy pointwise.
The following blow-up criterion holds

\[ T_{\text{max}} < +\infty \implies \limsup_{t \searrow T_{\text{max}}} \left( \sum_{i=1}^{m_1} \| u_i(t) \|_{L^\infty(\Omega)} + \sum_{j=1}^{m_2} \| v_j(t) \|_{L^\infty(M)} \right) = +\infty. \] (2.1)

Moreover, if (F2) holds, then \((u(t), v(t))\) is (component-wise) non-negative provided the initial data \((u_0, v_0)\) is non-negative.

Thanks to Theorem 2.1, the global existence of (1.1) follows if we can show that

\[ \sum_{i=1}^{m_1} \| u_i \|_{L^\infty(Q_T)} + \sum_{j=1}^{m_2} \| v_j \|_{L^\infty(M_T)} \leq C_T \] (2.2)

where \(C_T\) depends continuously on \(T > 0\), and \(C_T\) is finite for all \(T > 0\). For simplicity, we consider for the rest of this paper \(0 < T < T_{\text{max}}\), and ultimately prove the estimate (2.2). We first show that, under the mass control condition (F3), the solution is bounded in \(L^\infty(0, T; L^1(\Omega))\) and \(L^\infty(0, T; L^1(M))\).

**Lemma 2.1** ((O)-(D)-(F1)-(F2)-(F3)). There exists a constant \(C_T\) depending on \(T\) such that

\[ \sum_{i=1}^{m_1} \| u_i \|_{L^\infty(0, T; L^1(\Omega))} + \sum_{j=1}^{m_2} \| v_j \|_{L^\infty(0, T; L^1(M))} \leq C_T. \] (2.3)

Moreover, there exists a constant \(C_T\) depending on \(T\) such that

\[ \sum_{i=1}^{m_1} \| u_i \|_{L^1(M_T)} \leq C_T. \] (2.4)

**Proof.** Thanks to (1.2), we have

\[ \partial_t \left( \sum_{i=1}^{m_1} \int_{\Omega} a_i u_i + \sum_{j=1}^{m_2} \int_M b_j v_j \right) \leq L \int_{\Omega} \left( \sum_{i=1}^{m_1} u_i + 1 \right) + L \int_M \left( \sum_{i=1}^{m_1} u_i + \sum_{j=1}^{m_2} v_j + 1 \right). \] (2.5)

Thus, for some constant \(C > 0\),

\[ \sum_{i=1}^{m_1} \int_{\Omega} a_i u_i(t) + \sum_{j=1}^{m_2} \int_M b_j v_j(t) \leq C \int_0^t \left( \sum_{i=1}^{m_1} \int_{\Omega} a_i u_i + \sum_{j=1}^{m_2} \int_M b_j v_j \right) + C \sum_{i=1}^{m_1} \int_0^t \int_M u_i + C. \] (2.6)

We need to deal with the boundary integral of \(u_i\) on the right hand side. Let \(K > 0\) be a constant to be determined later and \(\phi_0\) is a non-negative, smooth function in \(\overline{\Omega}\) satisfying \(\partial_\nu \phi_0 = 1 + K \phi_0\) on \(M\). Suppose \(\phi \in C^{2,1}(\Omega \times [0, t])\) be a nonnegative function such that \(\phi_t + \Delta \phi = 0 \text{ on } \Omega \times (0, t)\), \(\frac{\partial \phi}{\partial \nu} = 1 + K \phi \text{ on } M \times (0, t)\), and \(\phi(\cdot, t) = \phi_0\) on \(\overline{\Omega}\). It follows from the comparison principle that \(0 \leq \phi\). Moreover, if we set \(\theta = -\partial_\nu \phi - \Delta_M \phi\) on \(M \times (0, t)\).
By integration by parts we have for \( i = 1, \ldots, m_1 \)
\[
\int_0^t \int_M a_i d_i u_i (1 + K\phi) = \int_0^t \int_M a_i d_i u_i \frac{\partial \phi}{\partial \eta} \\
\leq \int_0^t \int_M \phi \cdot a_i G_i (u, v) + \int_0^t \int_{\Omega} \phi \cdot a_i F_i (u) + \int_0^t \int_{\Omega} a_i (d_i - 1) \Delta \phi + a_i \int_{\Omega} \phi (\cdot, 0) u_i (\cdot, 0).
\]
(2.7)

Furthermore, \( j = 1, \ldots, m_2 \)
\[
\int_0^t \int_M b_j v_j \theta = \int_0^t \int_M b_j v_j (-\phi_t - \Delta_M \phi) \\
\leq \int_0^t \int_M (\phi \cdot b_j H_j (u, v) + b_j v_j (\delta_j - 1) \Delta_M \phi) + b_j \int_M v_j (\cdot, 0) \phi (\cdot, 0).
\]
(2.8)

Now sum (2.7) from \( i = 1, \ldots, m_1 \), with (2.8) from \( j = 1, \ldots, m_2 \), and apply (1.2), it follows that
\[
\sum_{i=1}^{m_1} \int_0^t \int_M a_i d_i u_i (1 + K\phi) + \sum_{j=1}^{m_2} \int_0^t \int_M b_j v_j \theta \\
\leq L \int_0^t \int_{\Omega} \phi \left( \sum_{i=1}^{m_1} u_i + 1 \right) + L \int_0^t \int_M \phi \left( \sum_{i=1}^{m_1} u_i + \sum_{j=1}^{m_2} v_j + 1 \right) \\
+ \sum_{i=1}^{m_1} \int_0^t \int_{\Omega} a_i u_i (d_i - 1) \Delta \phi + \sum_{i=1}^{m_1} a_i \int_{\Omega} \phi (\cdot, 0) u_i,0 \\
+ \sum_{j=1}^{m_2} \int_0^t \int_M b_j v_j (\delta_j - 1) \Delta_M \phi + \sum_{j=1}^{m_2} \int_M \phi (\cdot, 0) v_j,0 \\
\leq L \sum_{i=1}^{m_1} \int_0^t \int_M \phi u_i + C \sum_{i=1}^{m_1} \int_0^t \int_{\Omega} a_i u_i + C \sum_{j=1}^{m_2} \int_0^t \int_M b_j v_j + C(1 + t)
\]

where the last step uses \( \phi \in C^{2,1} (\Omega \times [0, t]) \). By choosing \( K \) large enough such that \( K d_i a_i \geq L \) for all \( i = 1, \ldots, m_1 \), and using \( v_j \geq 0 \) and \( \theta \in L^\infty (M_T) \), we obtain
\[
\sum_{i=1}^{m_1} \int_0^t \int_M a_i d_i u_i \leq C \sum_{i=1}^{m_1} \int_0^t \int_{\Omega} a_i u_i + C \sum_{j=1}^{m_2} \int_0^t \int_M b_j v_j + C(1 + t).
\]
(2.9)

Inserting this into (2.6) yields
\[
\sum_{i=1}^{m_1} \int_{\Omega} a_i u_i (t) + \sum_{j=1}^{m_2} \int_M b_j v_j (t) \leq C \int_0^t \left( \sum_{i=1}^{m_1} \int_{\Omega} a_i u_i + \sum_{j=1}^{m_2} \int_M b_j v_j \right) + C(1 + t).
\]

A direct application of Gronwall’s inequality gives the estimates (2.3). The bound (2.4) follows directly from (2.9).\qed
Lemma 2.2 ((1.6)). There exist componentwise increasing functions \( g_j : \mathbb{R}^{m_1 - j} \to \mathbb{R} \) for \( j = 1, \ldots, m_1 - 1 \), such that if \( \ell \in (0, \infty)^{m_1} \) with \( \ell_j > g_j(\ell_{j+1}, \ldots, \ell_{m_1}) \) for \( j = 1, \ldots, m_1 - 1 \) then there exists \( L_\ell > 0 \) such that

\[
\sum_{i=1}^{m_1} \ell_i F_i(u) \leq L_\ell \left( \sum_{i=1}^{m_1} u_{i0}^p + 1 \right) \quad \text{for all } u \in \mathbb{R}_{+}^{m_1},
\]

and

\[
\sum_{i=1}^{m_1} \ell_i G_i(u, v) \leq L_\ell \left( \sum_{i=1}^{m_1} u_{i0}^p + \sum_{j=1}^{m_2} v_{j0}^p + 1 \right) \quad \text{for all } u \in \mathbb{R}_{+}^{m_1}, v \in \mathbb{R}_{+}^{m_2}.
\]

Proof. Without loss of generality we assume that and \( a_{i,j} > 0 \) for \( i > j \) with \( i, j \in \{1, \ldots, m_1\} \). We construct two sequences of functions \( g_j : \mathbb{R}^{m_1 - j} \to \mathbb{R}, \ j = m_1 - 1, m_1 - 2, \ldots, 1 \) and \( \alpha : \mathbb{R}^{m_1 - j + 1} \to \mathbb{R}, \ j = m_1, \ldots, 1 \) inductively as follows:

- \( g_{m_1-1}(x_{m_1}) := a_{m_1,m_1-1}x_{m_1} \). We also define the function \( \alpha_{m_1}(x_{m_1}) := x_{m_1} \). Note that \( \alpha_{m_1} := \alpha_{m_1}(\ell_{m_1}) = \ell_{m_1} > 0 \).
- For \( i = m_1 - 2, m_1 - 3, \ldots, 2, 1 \), we constructed the function \( g_i \) using established functions \( g_j \) and \( \alpha_j \) for \( j \geq i + 1 \). More precisely, we define

\[
g_i(x_{i+1}, x_{i+2}, \ldots, x_{m_1}) := \sum_{j=i+1}^{m_1} a_{j,i} \alpha_j(x_{i}, x_{i+1}, \ldots, x_{m_1}),
\]

and

\[
\alpha_i(x_{i}, \ldots, x_{m_1}) := x_{i} - g_i(x_{i+1}, x_{i+2}, \ldots, x_{m_1}).
\]

Due to the assumptions of \((\ell_i)_{i=1,\ldots,m_1}\),

\[
\hat{\alpha}_i := \alpha_i(\ell_{i}, \ell_{i+1}, \ldots, \ell_{m_1}) = \ell_{i} - g_i(\ell_{i+1}, \ldots, \ell_{m_1}) > 0.
\]

Therefore, we have

\[
\sum_{i=1}^{m_1} \ell_i F_i(u) = \sum_{i=1}^{m_1} \left[ \hat{\alpha}_i + g_i(\ell_{i+1}, \ldots, \ell_{m_1}) \right] F_i(u)
\]

\[
= \sum_{i=1}^{m_1} \left[ \hat{\alpha}_i + \sum_{j=i+1}^{m_1} \hat{\alpha}_j a_{j,i} \right] F_i(u)
\]

\[
= \sum_{i=1}^{m_1} \sum_{j=i}^{m_1} \hat{\alpha}_j a_{j,i} F_i(u) \quad \text{(since } a_{i,i} = 1 \text{)}
\]

\[
= \sum_{i=1}^{m_1} \hat{\alpha}_i \sum_{j=1}^{i} a_{i,j} F_j(u)
\]
which implies directly (2.10). The inequality (2.11) can be verified in the same way, so we omit it here.

**Remark 2.1.** The above lemma shows that the intermediate sum conditions involving $F$ and $G$ in (1.6) and (1.7) imply the existence of functions $g_j$, $j = 1, \ldots, m_1 - 1$. It will be shown later on that once we have these functions $g_j$, an $L^p$-energy can be constructed. This means that the conclusion of Theorem 1.1 still holds true if we assume the existence of $g_j$, $j = 1, \ldots, m_1 - 1$, instead of the intermediate sum conditions (involving $F$ and $G$) (1.6) and (1.7). We choose to present Theorem 1.1 under (1.6) and (1.7) as they are more constructive, and they appear naturally in many applications (see Section 5).

It is important to remark that the existence of functions $g_j$ is more general than the intermediate sum conditions (1.6)–(1.7). For example, the nonlinearities

\[
F_1(u_1, u_2) = u_1 u_2^3 - u_1^4 \\
F_2(u_1, u_2) = u_1^4 - u_1 u_2^4,
\]

clearly do not satisfy (1.6) for high dimensions $n \geq 3$. However, if we define $g_1(x) = x$, then for any $\ell = (\ell_1, \ell_2)$ with $\ell_1 > g_1(\ell_2) = \ell_2$ we have

\[
\ell_1 F_1(u_1, u_2) + \ell_2 F_2(u_1, u_2) = u_1 [\ell_1 u_2^3 - \ell_2 u_2^4 + (\ell_2 - \ell_1) u_1^3] \\
\leq u_1 [\ell_1 u_2^3 - \ell_2 u_2^4] \leq L_\ell (1 + u_1)
\]

for all $u_1, u_2 \geq 0$, and a constant $L_\ell$ depending only on $\ell$.

The following interpolation inequality might be of independent interest.

**Lemma 2.3 ((O)).** Assume that $w: \Omega \to [0, \infty)$ with $\|w\|_{L^p(\Omega)} \leq K$. Then for any $p \geq 2a$, $\epsilon > 0$, and any $r_\Omega, r_M$ satisfying

\[
1 \leq r_\Omega < 1 + \frac{2a}{n} \quad \text{and} \quad 1 \leq r_M < 1 + \frac{a}{n},
\]

there exists a constant $C_{p, \epsilon, K}$ depending on $p, \epsilon$ and $K$, but not on $w$, such that

\[
\int_\Omega w^{p-1+r_\Omega} + \int_M w^{p-1+r_M} \leq \epsilon \left( \int_\Omega w^{p-2} |\nabla w|^2 + \int_\Omega w^p \right) + C_{p, \epsilon, K}. \tag{2.12}
\]

**Proof.** We will estimate the domain term on the left hand side of (2.12), while the boundary term will follow as a consequence. First, by using Sobolev’s embedding we have

\[
\int_\Omega w^{p-2} |\nabla w|^2 + \int_\Omega w^p = \frac{4}{p^2} \int_\Omega |\nabla (w^{p/2})|^2 + \int_\Omega (w^{p/2})^2 \geq \frac{4}{p^2} \|w^{p/2}\|_{H^1(\Omega)}^2. \tag{2.13}
\]

Define $r_\Omega = 1 + \eta$ and $\beta = \frac{2a}{p}$. Then we have $\eta \in [0, 2a/n)$ and $\beta \in [0, \frac{4a}{np})$, and

\[
\int_\Omega w^{p-1+r_\Omega} = \int_\Omega w^{p+\eta} = \int_\Omega |w^{p/2}|^{2+\beta} = \|y\|_{L^{2+\beta}(\Omega)}^{2+\beta},
\]
where $y := w^{p/2}$. Thanks to the Gagliardo-Nirenberg inequality, we have
$$\|y\|_{L^{2+\beta}(\Omega)}^{2+\beta} \leq C_{GN}\|y\|_{H^1(\Omega)}^{\alpha(2+\beta)}\|y\|_{L^1(\Omega)}^{(1-\alpha)(2+\beta)}$$
where $\alpha \in (0, 1)$ satisfies
$$\frac{1}{2 + \beta} = \left(\frac{1}{2} - \frac{1}{n}\right)\alpha + \frac{1 - \alpha}{1}.$$ It follows that
$$\alpha(2 + \beta) = \frac{2n(\beta + 1)}{n + 2} \quad \text{and} \quad (1 - \alpha)(2 + \beta) = \frac{4 + 2\beta - \beta n}{n + 2}.$$ Therefore,
$$\|y\|_{L^{2+\beta}(\Omega)}^{2+\beta} \leq C_{GN}\|y\|_{H^1(\Omega)}^{\frac{2n(\beta + 1)}{n + 2}}\|y\|_{L^1(\Omega)}^{\frac{4 + 2\beta - \beta n}{n + 2}}.$$ Since $\beta < 4a/(np)$ and $p \geq 2a$, it holds $\frac{2n(\beta + 1)}{n + 2} < 2$. We can then use Young’s inequality to estimate
$$\|y\|_{L^{2+\beta}(\Omega)}^{2+\beta} \leq \varepsilon\|y\|_{H^1(\Omega)}^2 + C_{\varepsilon}\|y\|_{L^1(\Omega)}^{\frac{4 + 2\beta - \beta n}{2(2-n\beta)}}. \quad (2.14)$$ Changing the variable $y = w^{p/2}$ we have
$$\|y\|_{L^{4+2\beta-\beta n}/2(2-n\beta)} = \|w\|_{L^{\frac{p(4+2\beta-\beta n)}{2(2-n\beta)}}}^{\frac{p(4+2\beta-\beta n)}{2(2-n\beta)}}. \quad (2.15)$$ If $p = 2a$, then this term is bounded by a constant depending on $K$, since $\|w\|_{L^a(\Omega)} \leq K$. If $p > 2a$, we use interpolation inequality to have
$$\|w\|_{L^{\frac{p}{p+\eta}}(\Omega)} \leq \|w\|_{L^{p+\eta}(\Omega)}^{\frac{\theta}{p}}\|w\|_{L^p(\Omega)}^{1-\theta} \leq K^{1-\theta}\|w\|_{L^{p+\eta}(\Omega)}^\theta \quad (2.16)$$ where $\theta \in (0, 1)$ satisfies
$$\frac{2}{p} = \frac{\theta}{p+\eta} + \frac{1-\theta}{a}.$$ Note that
$$\frac{\theta p(4+2\beta-\beta n)}{2(2-n\beta)} = \frac{(p+\eta)(p-2a)(4+2\beta-\beta n)}{2(2-n\beta)(p+\eta-a)} < p + \eta \quad (2.17)$$ due to $\beta = 2\eta/p$ and $\eta < 2a/n$. From (2.15)–(2.17) and Young’s inequality, it follows that
$$C_\varepsilon\|y\|_{L^1(\Omega)}^{\frac{4+2\beta-\beta n}{2(2-n\beta)}} \leq C_\varepsilon C_K\|w\|_{L^{p+\eta}(\Omega)}^{\frac{p(4+2\beta-\beta n)}{2(2-n\beta)(p+\eta-a)}} \leq \frac{1}{2}\|w\|_{L^{p+\eta}(\Omega)}^{p+\eta} + C_{p,\varepsilon,K}.$$ Inserting this into (2.14), we get
$$\|w\|_{L^{p+\eta}(\Omega)}^{p+\eta} \leq 2\varepsilon\|w^{p/2}\|_{H^1(\Omega)}^2 + C_{p,\varepsilon,K}.$$ Combining this with (2.13) leads to the desired estimate for the domain term, i.e.
$$\int_{\Omega} w^{p-1+p\alpha} \leq \varepsilon \left(\int_{\Omega} w^{p-2} |\nabla w|^2 + \int_{\Omega} w^p\right) + C_{p,\varepsilon,K}. \quad (2.18)$$ To treat the boundary term $\int_M w^{p-1+p\alpha}$, we first define $p_M = 1 + \xi$ for $\xi \in [0, a/n)$. We
then use the following interpolation trace inequality, see e.g. [Gri11, Proof of Theorem 1.5.1.10, page 41],

\[
\int_M w^{p-1+pM} = \int_M w^{p+\xi} \leq C \int_\Omega w^{p+\xi-1} |\nabla w| + C \int_\Omega w^{p+\xi} \\
\leq \frac{\varepsilon}{2} \int_\Omega w^{p-2} |\nabla w|^2 + C \varepsilon \int_\Omega w^{p+2\xi} + C \int_\Omega w^{p+\xi} \\
\leq \frac{\varepsilon}{2} \int_\Omega w^{p-2} |\nabla w|^2 + C \varepsilon \int_\Omega w^{p+2\xi} + C.
\]

Since \( \xi \in [0,a/n] \), \( 2\xi \in [0,2a/n] \). Therefore, we can use (2.18) to show

\[
C \varepsilon \int_\Omega w^{p+2\xi} \leq \frac{\varepsilon}{2} \left( \int_\Omega w^{p-2} |\nabla w|^2 + \int_\Omega w^p \right) + C_{p,\varepsilon,K}.
\]

Therefore, we obtain the estimate for the boundary term in (2.12), and thus finish the proof of Lemma 2.3. \( \square \)

To construct our \( L^p \)-energy function, we write \( \mathbb{Z}_+^k \) for the set of all \( k \)-tuples of non negative integers. Addition and scalar multiplication by non negative integers of elements in \( \mathbb{Z}_+^k \) is understood in the usual manner. If \( \beta = (\beta_1, \ldots, \beta_k) \in \mathbb{Z}_+^k \) and \( p \in \mathbb{N} \), then we define \( \beta^p = ((\beta_1)^p, \ldots, (\beta_k)^p) \). Also, if \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}_+^k \), then we define \( |\alpha| = \sum_{i=1}^k \alpha_i \). Finally, if \( z = (z_1, \ldots, z_k) \in \mathbb{R}_+^k \) and \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}_+^k \), then we define \( z^\alpha = z_1^{\alpha_1} \cdot \ldots \cdot z_k^{\alpha_k} \), where we interpret \( 0^0 \) to be 1. For \( 2 \leq p \in \mathbb{N} \), we build our \( L^p \)-energy function of the form

\[
\mathcal{L}_p[u](t) = \int_\Omega \mathcal{H}_p[u](t),
\]

where

\[
\mathcal{H}_p[u](t) = \sum_{\beta \in \mathbb{Z}_+^{m_1}, |\beta|=p} \left( \frac{p}{\beta} \right) \theta^{\beta^2} u^{\beta}(t),
\]

and the positive constants \( \theta = (\theta_1, \ldots, \theta_{m_1}) \) are to be chosen. For convenience, hereafter we drop the subscript \( \beta \in \mathbb{Z}_+^{m_1} \) in the sum as it should be clear.

The main result of this section is the following lemma.

**Lemma 2.4** ((O)-(D)-(F1)-(F2)-(F3)-(1.6)-(1.7)-(1.8)). For any positive integer \( p \geq 2 \) and any constant \( \varepsilon > 0 \), there exists \( K_{p,\varepsilon} > 0 \) such that

\[
\sum_{i=1}^{m_1} \left( \|u_1\|_{L^{p-1+P}}^{p-1+P} + \|u_1\|_{L^{p-1+P}M}^{p-1+P} \right) \leq K_{p,\varepsilon}(1 + T) + \varepsilon \sum_{j=1}^{m_2} \|v_j\|_{L^{p-1+P}M}^{p-1+P} \tag{2.21}
\]

for a possibly different constant \( K_{p,\varepsilon} \).

Consequently, for any \( 1 < p < \infty \) and any \( \varepsilon > 0 \), there exists a constant \( K_{p,\varepsilon} > 0 \) such that

\[
\sum_{i=1}^{m_1} \left( \|u_i\|_{L^p} + \|u_i\|_{L^p M} \right) \leq K_{p,\varepsilon}(1 + T) + \varepsilon \sum_{j=1}^{m_2} \|v_j\|_{L^p M}. \tag{2.22}
\]
Proof. First, we choose \( \theta = (\theta_1, \ldots, \theta_{m_1}) \) with \( \theta_i \geq 1 \) for all \( i = 1, \ldots, m_1 \) such that

(\theta 1) The matrix \( \mathcal{M} = (\mathcal{M}_{i,j}) \) is positive definite, where

\[
\mathcal{M}_{i,j} = \begin{cases} 
    d_i \theta_i^2, & \text{if } i = j \\
    \frac{d_i + d_j}{2}, & \text{if } i \neq j 
\end{cases} \tag{2.23}
\]

(\theta 2) \( \theta_j > \max_{i_1, \ldots, i_{m_1-1} \in \{1, \ldots, 2p-1\}} g_j (\theta_{j+1}, \ldots, \theta_{m_1-j}^{i_{m_1-j}}) \) for all \( j = 1, \ldots, m_1 - 1 \), where the functions \( g_j \) are given by Lemma 2.2.

Such a choice of \( \theta \) is possible since the off-diagonal elements of \( \mathcal{M} \) are fixed, therefore we choose successively \( \theta_{m_1} > 0, \theta_{m_1-1}, \ldots, \theta_1 \) such that (\theta 2) is fulfilled, and \( \theta_j \) is large enough such that \( \mathcal{M} \) is diagonally dominant, which implies its positive definiteness. With this chosen \( \theta \), we define for \( 2 \leq p \in \mathbb{Z} \) our \( L^p \)-energy function \( \mathcal{L}_p[u] \) as in (2.19). Then, thanks to Lemma A.1

\[
(\mathcal{L}_p[u])' (t) = \int_{\Omega} \sum_{|\beta|=p-1} \left( \begin{array}{c} p \\
\beta 
\end{array} \right) \theta^{2\beta} u^\beta \sum_{i=1}^{m_1} \frac{\partial}{\partial t} u_i 
\]

\[
= \int_{\Omega} \sum_{|\beta|=p-1} \left( \begin{array}{c} p \\
\beta 
\end{array} \right) \theta^{2\beta} u^\beta \sum_{i=1}^{m_1} \theta_i^{2\beta_i+1} (d_i \Delta u_i + F_i(u)) . \tag{2.24}
\]

Integration by parts gives

\[
\int_{\Omega} \sum_{|\beta|=p-1} \left( \begin{array}{c} p \\
\beta 
\end{array} \right) \theta^{2\beta} u^\beta \sum_{i=1}^{m_1} d_i \Delta u_i =: (I) + (II),
\]

where

\[
(I) = \int_{M} \sum_{|\beta|=p-1} \left( \begin{array}{c} p \\
\beta 
\end{array} \right) \theta^{2\beta} u^\beta \sum_{i=1}^{m_1} \theta_i^{2\beta_i+1} G_i(u, v),
\]

and, thanks to Lemma A.2,

\[
(II) = - \int_{\Omega} \sum_{|\beta|=p-2} \left( \begin{array}{c} p \\
\beta 
\end{array} \right) \theta^{2\beta} u^\beta \sum_{i=1}^{m_1} \sum_{i,j=1}^{m_1} a_{i,j} \frac{\partial}{\partial x_i} u_i \frac{\partial}{\partial x_j} u_j \tag{2.25}
\]

with

\[
a_{i,j} = \begin{cases} 
    d_j \theta_i^{2\beta_i+1} \theta_j^{2\beta_j+1}, & \text{if } i \neq j, \\
    d_i \theta_i^{4\beta_i+4}, & \text{if } i = j. 
\end{cases} \tag{2.26}
\]

Note that

\[
\sum_{i,j=1}^{m_1} a_{i,j} \frac{\partial}{\partial x_i} u_i \frac{\partial}{\partial x_j} u_j = \sum_{i,j=1}^{m_1} b_{i,j} \frac{\partial}{\partial x_i} u_i \frac{\partial}{\partial x_j} u_j \tag{2.27}
\]

where

\[
b_{i,j} = \begin{cases} 
    d_j + d_i \theta_i^{2\beta_i+1} \theta_j^{2\beta_j+1}, & \text{if } i \neq j, \\
    d_i \theta_i^{4\beta_i+4}, & \text{if } i = j. 
\end{cases}
\]
Furthermore, if we define \( \mathcal{B} = (b_{i,j}) \), and the \( m_1 \times m_1 \) diagonal matrix

\[
\mathcal{C} = \text{diag} \left( \theta_i^{-2\beta_i-1} \right)
\]

then

\[
\mathcal{C}^{-1} \mathcal{M} \mathcal{C}^{-1} = \mathcal{B}
\]

where \( \mathcal{M} \) is defined in (2.23). Consequently, from the choice of \( \theta \), the matrix \( \mathcal{B} \) is positive definite. Therefore, there exists \( \lambda > 0 \) such that

\[
\sum_{i,j=1}^{m_1} b_{i,j} \frac{\partial}{\partial x_i} u_i \frac{\partial}{\partial x_j} u_j \geq \lambda \sum_{i=1}^{m_1} \left| \frac{\partial}{\partial x_i} u_i \right|^2.
\]

Inserting this into (2.27) and (2.25) leads to, for some \( c_p > 0 \),

\[
(II) \leq -\lambda \int_{\Omega} \sum_{|\beta|=p-2} \left( \begin{array}{c} p \\ \beta \end{array} \right) \theta^{\beta_1} u^{\beta} \sum_{l=1}^{n} \sum_{i=1}^{m_1} \left| \frac{\partial}{\partial x_l} u_i \right|^2 \leq -c_p \int_{\Omega} u^{-2} \left| \nabla u \right|^2. \tag{2.28}
\]

Also, \( \theta_i \leq \theta_i^{2\beta_i+1} \leq \theta_i^{2p-1} \) for all \( i = 1, ..., m_1 \). Therefore, from Lemma 2.2, (1.6)–(1.7) and the choice of \( \theta \), there is a value \( L_\theta > 0 \) such that

\[
\sum_{i=1}^{m_1} \theta_i^{2\beta_i+1} G_i(u, v) \leq L_\theta \left( \sum_{i=1}^{m_1} u_i^{p^M} + \sum_{j=1}^{m_2} v_j^{p^M} + 1 \right)
\]

and

\[
\sum_{i=1}^{m_1} \theta_i^{2\beta_i+1} F_i(u) \leq L_\theta \left( \sum_{i=1}^{m_1} u_i^{p^M} + 1 \right).
\]

As a result, there exist \( c_p, K_{p,\theta} > 0 \) so that

\[
\sum_{|\beta|=p-1} \left( \begin{array}{c} p \\ \beta \end{array} \right) \theta^{\beta_1} u^{\beta} \sum_{i=1}^{m_1} \theta_i^{2\beta_i+1} G_i(u, v) \leq K_{p,\theta} \sum_{j=1}^{m_2} u_j^{p-1} \left( \sum_{i=1}^{m_1} u_i^{p^M} + \sum_{j=1}^{m_2} v_j^{p^M} + 1 \right), \tag{2.29}
\]

and

\[
\sum_{|\beta|=p-1} \left( \begin{array}{c} p \\ \beta \end{array} \right) \theta^{\beta_1} u^{\beta} \sum_{i=1}^{m_1} \theta_i^{2\beta_i+1} F_i(u) \leq K_{p,\theta} \left( \sum_{i=1}^{m_1} u_i^{p-1+p^M} + 1 \right). \tag{2.30}
\]
By applying (2.28), (2.29) and (2.30) into (2.24), there exists \( K_{p,\theta} > 0 \) such that

\[
(L_p[u])'(t) + c_p \sum_{i=1}^{m_1} \int_{\Omega} u_i^{p-2} |\nabla u_i|^2 \\
\leq K_{p,\theta} \left[ 1 + \int_{\Omega} \sum_{j=1}^{m_1} u_j^{p-1+p\alpha} + \sum_{i=1}^{m_1} \int_{M} u_i^{p-1+pM} + \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \int_{M} u_i^{p-1} v_j^{pM} \right] \tag{2.31}
\]

\[
\leq K_{p,\theta,\varepsilon} \left[ 1 + \sum_{i=1}^{m_1} \int_{\Omega} u_i^{p-1+p\alpha} + \sum_{i=1}^{m_1} \int_{M} u_i^{p-1+pM} \right] + \varepsilon \sum_{j=1}^{m_2} \int_{M} v_j^{p-1+pM},
\]

where we used Young’s inequality at the last step. Adding

\[
\sum_{i=1}^{m_1} \left( \frac{c_p}{2} \int_{\Omega} u_i^p + \int_{\Omega} u_i^{p-1+p\alpha} + \int_{M} u_i^{p-1+pM} \right)
\]

to both sides gives

\[
(L_p[u])'(t) + c_p \sum_{i=1}^{m_1} \left( \int_{\Omega} u_i^{p-2} |\nabla u_i|^2 + \int_{\Omega} u_i^p \right) + \frac{1}{2} \sum_{i=1}^{m_1} \left( \int_{\Omega} u_i^{p-1+p\alpha} + \int_{M} u_i^{p-1+pM} \right)
\]

\[
\leq \sum_{i=1}^{m_1} \left( \frac{c_p}{2} \int_{\Omega} u_i^p + \int_{\Omega} u_i^{p-1+p\alpha} + \int_{M} u_i^{p-1+pM} \right) + K_{p,\theta,\varepsilon} + \varepsilon \sum_{j=1}^{m_2} \int_{M} v_j^{p-1+pM}
\]

\[
\leq K_{p,\theta,\varepsilon} + \varepsilon \sum_{i=1}^{m_1} \left( \int_{\Omega} u_i^{p-2} |\nabla u_i|^2 + \int_{\Omega} u_i^p \right) + \varepsilon \sum_{j=1}^{m_2} \int_{M} v_j^{p-1+pM} \tag{2.32}
\]

where we used Lemma 2.3 at the last step. Integrating the resultant on \((0, T)\) finishes the proof of (2.21).

To prove (2.22), we first show that for each \( p \in \mathbb{N}, \varepsilon > 0 \), there exists \( K_{p,\varepsilon} \) such that

\[
\sum_{i=1}^{m_1} \left\| u_i \right\|_{L^{p-1+pM}(Q_T)}^{p-1+pM} \leq K_{p,\varepsilon,T} + \varepsilon \sum_{j=1}^{m_2} \left\| v_j \right\|_{L^{p-1+pM}(M_T)}^{p-1+pM}. \tag{2.33}
\]

Indeed, from the \( L^1 \)-bound in Lemma 2.3 and interpolation inequality we have, for \( \gamma \in (0, 1) \) with \( \frac{1}{p-1+pM} = \gamma + \frac{1-\gamma}{p-1+p\alpha} \),

\[
\sum_{i=1}^{m_1} \left\| u_i \right\|_{L^{p-1+pM}(Q_T)}^{p-1+pM} \leq \sum_{i=1}^{m_1} \left\| u_i \right\|_{L^1(Q_T)}^{\gamma(p-1+pM)} \sum_{i=1}^{m_1} \left\| u_i \right\|_{L^{p-1+p\alpha}(Q_T)}^{(1-\gamma)(p-1+pM)}
\]

\[
\leq K_T \sum_{i=1}^{m_1} \left\| u_i \right\|_{L^{p-1+p\alpha}(Q_T)}^{(1-\gamma)(p-1+pM)}
\]

\[
\leq K_T \left( 1 + \sum_{i=1}^{m_1} \left\| u_i \right\|_{L^{p-1+p\alpha}(Q_T)}^{p-1+p\alpha} \right)
\]
\[ \leq K_{p, \varepsilon, T} + \varepsilon \sum_{j=1}^{m_2} \| v_j \|_{L^p(M^\varepsilon, M, \tau)}^{p-1+pm}. \]

The estimate (2.22) now follows from (2.21), (2.33), and interpolation.

3. Duality method

We first collect some useful results which will be used in the following sections. The next lemma is about the $L^p$-maximal regularity for heat equation in a smooth manifold without boundary.

**Lemma 3.1** (O). Let $1 < p < \infty$, $0 \leq \tau < T < \infty$. There exists a constant $C_{\text{mr}, p}^M$ (Mr stands for “maximal regularity”), which depends only on $p$, $M$, if the dimension $n$, such that, for any $F \in L^p(M, T)$, and $U$ is the solution to
\[
\begin{cases}
\partial_t U - \Delta_M U = F, & (x, t) \in M, \\
U(x, 0) = 0, & x \in M,
\end{cases}
\]
we have the following estimate
\[ \| \Delta_M U \|_{L^p(M, T)} \leq C_{\text{mr}, p}^M \| F \|_{L^p(M, T)}. \]

**Proof.** The proof of this lemma can be found in [Lam87, Theorem 1]4. We emphasize that fact that the constant $C_{\text{mr}, p}^M$ is independent of $T$ and $\tau$. \qed

The following crucial lemma provides the regularity of the duality problem suited for volume-surface systems.

**Lemma 3.2** (O). Assume that $0 < \tau < T$, $1 < q < \infty$ and $\psi \in L^q(M, T)$. Let $\phi$ be the solution to
\[
\begin{cases}
\partial_t \phi + \Delta \phi = 0, & \text{on } Q_{\tau, T}, \\
\partial_t \phi + \delta \Delta_M \phi = -\psi, & \text{on } M, \\
\phi(x, T) = 0 & \text{on } \Omega.
\end{cases}
\]
Then, we have the following estimate
\[ \| \Delta_M \phi \|_{L^q(M, T)} \leq \frac{C_{\text{mr}, q}^M}{\delta} \| \psi \|_{L^q(M, T)}, \]
where $C_{\text{mr}, q}^M$ is the maximal regularity constant in Lemma 3.1. Moreover, with $\xi = \frac{q}{n+1}$ we have
\[ \| \phi \|_{W^{2,1}_{q,\xi}(Q_{\tau, T})} + \| \phi \|_{W^{2,1}_q(M, T)} + \| \phi(\tau) \|_{L^{q+\xi}(\Omega)} + \| \phi(\tau) \|_{L^q(M)} \leq C_{T-\tau} \| \psi \|_{L^q(M, T)}, \]
and
\[ \| \partial_\eta \phi \|_{L^{q+\xi}(M, T)} \leq C_{T-\tau} \| \psi \|_{L^q(M, T)}. \]
Consequently,
\[ \|\phi\|_{L^q(Q_{\tau,T})} + \|\phi\|_{L^q(M_{\tau,T})} \leq C_{T-\tau}\|\psi\|_{L^q(M_{\tau,T})} \tag{3.6} \]
where
\[ q^+ = \begin{cases} \frac{(n+2)q}{n+2-2q} & \text{if } q < \frac{n+1}{2}, \\ < +\infty \text{ arbitrary} & \text{if } q = \frac{n+1}{2}, \\ +\infty & \text{if } q > \frac{n+1}{2}. \end{cases} \]

Moreover, if \( \psi \geq 0 \) a.e. in \( M_{\tau,T} \), then \( \phi \geq 0 \).

Remark 3.1.

- At the first glance, the dual problem (3.2) looks like a backward parabolic equation. However, since the “initial data” is considered at \( t = T \), by the change of variable \( s = T - t \), (3.2) transforms into the usual forward parabolic equation.
- To solve the dual problem, we first solve the boundary equation \( \partial_t \phi + \delta \Delta_M \phi = -\psi \) and define \( \Psi := \phi|_{M_{\tau,T}} \). Then, we consider the heat operator with inhomogeneous Dirichlet boundary condition: \( \partial_t \phi + \Delta \phi = 0 \) in \( Q_{\tau,T} \) and \( \phi|_{M_{\tau,T}} = \Psi \). See more details in [SM16].

Proof. The bound
\[ \|\phi\|_{W^{q,1}_q(M_{\tau,T})} \leq C_{T-\tau}\|\psi\|_{L^q(M_{\tau,T})} \]
can be found in [SM16]. In particular,
\[ \|\partial_t \phi\|_{L^q(M_{\tau,T})} \leq C_{T-\tau}\|\psi\|_{L^q(M_{\tau,T})}. \]

Therefore, by Hölder’s inequality
\[ \|\phi(\tau)\|_{L^q(M)}^q = \int_M \left( \int_\tau^T \partial_t \phi \right)^q \leq (T - \tau)^{q-1} \int_M \int_\tau^T |\partial_t \phi|^q \leq C_{T-\tau}\|\psi\|_{L^q(M_{\tau,T})}^q \]
which implies
\[ \|\phi(\tau)\|_{L^q(M)} \leq C_{T-\tau}\|\psi\|_{L^q(M_{\tau,T})}. \]

To show the improved bound in \( Q_{\tau,T} \) and for the normal derivative, we use the following embedding, cf. [LSU88, Lemma 3.3],
\[ W^{2,1}_q(M_{\tau,T}) \hookrightarrow L^r(\tau, T; W^{2-\frac{1}{r},r}_q(M)) \cap W^{1-\frac{1}{r},r}(0, T; L^r(M)) \]
for all
\[ r \leq \frac{(n+2)q}{n+1} = q + \frac{q}{n+1}. \]

Therefore, by choosing \( \xi = \frac{r}{n+1} \) and defining \( r = q + \xi \), we have \( \Psi := \phi|_{M_{\tau,T}} \in L^r(\tau, T; W^{2-\frac{1}{r},r}_q(M)) \cap W^{1-\frac{1}{r},r}(0, T; L^r(M)) \). Now we can apply the maximal regularity for equation with inhomogeneous Dirichlet boundary condition, see e.g. [Prü02],
\[
\begin{align*}
\partial_t \phi + \Delta \phi &= 0, & \text{on } Q_{\tau,T}, \\
\phi &= \Psi, & \text{on } M_{\tau,T}, \\
\phi(x, T) &= 0, & \text{on } \overline{\Omega},
\end{align*}
\]
to obtain
\[ \|\phi\|_{W^{2,1,1}_{q+\xi}(Q_{\tau, T})} \leq C_{T-\tau} \|\Psi\|_{L^r(\tau,T; W^{2-\frac{1}{q},1}(M))} \cap W^{1,1=r}(0,T; L^r(M))} \leq C_{T-\tau} \|\phi\|_{W^{2,1}_{q+\xi}(M_{\tau, T})} \leq C_{T-\tau} \|\psi\|_{L^q(M \times (\tau, T))}. \]

The estimate of the normal derivative (3.5) follows from the bound of \( \phi \) in \( W^{2,1,1}_{q+\xi}(Q_{\tau, T}) \) and Lemma [SM16, Lemma 6.9].

The estimate (3.6) then follows from (3.4) and the embedding theory ([LSU88, Lemma 3.3]).

It remains to show (3.3). We define scaled functions \( \hat{\phi}(x, t) = \phi(x, t/\delta) \) and \( \hat{\psi}(x, t) = \psi(x, t/\delta) \). From the equation of \( \phi \), it follows that
\[
\begin{cases}
\partial_t \hat{\phi} - \Delta_M \hat{\phi} = \frac{1}{\delta} \hat{\psi}, & (x, t) \in M_{\delta\tau, \delta T}, \\
\hat{\phi}(x, 0) = 0, & x \in M.
\end{cases}
\]

From Lemma 3.1,
\[ \|\Delta_M \hat{\phi}\|_{L^q(M_{\delta\tau, \delta T})} \leq \frac{C_{mr, A}}{\delta} \|\hat{\psi}\|_{L^q(M_{\delta\tau, \delta T})}. \]

By switching back to the original variables we obtain easily
\[ \delta \max - \delta \min \|\Delta_M \phi\|_{L^q(M_{\tau, T})} = \int_{\delta\tau}^{\delta T} \int_M |\Delta_M \hat{\phi}|^q \leq \left( \frac{C_{mr, A}}{\delta} \right)^q \int_{\delta\tau}^{\delta T} \int_M |\hat{\psi}|^q = \left( \frac{C_{mr, A}}{\delta^{q-1}} \|\hat{\psi}\|_{L^q(M_{\tau, T})}^q \right), \]

which yields the desired inequality (3.3).

We show that if (1.15) is true for some \( \Lambda \), then it’s also true for \( \Lambda + \kappa \) for small enough \( \kappa > 0 \) in the following sense.

**Lemma 3.3.** Assume that (1.15) holds for some \( \Lambda \). Then there exists \( \kappa_0 > 0 \) such that
\[ \frac{\delta \max - \delta \min}{\delta \max + \delta \min} C_{mr,(\Lambda + \kappa)^{\prime}} < 1 \quad \text{for all} \quad 0 < \kappa < \kappa_0 \]
where
\[ (\Lambda + \kappa)^{\prime} = \frac{\Lambda + \kappa}{\Lambda + \kappa - 1} \]
is the Hölder conjugate exponent of \( \Lambda + \kappa \).

**Proof.** It’s sufficient to show that
\[ (C_{mr, A})^{-} := \liminf_{\eta \to 0^+} C_{mr, A - \eta} \leq C_{mr, A}. \]

Let \( \Lambda_{\eta} \) satisfy
\[ \frac{1}{\Lambda_{\eta}} = \frac{1}{2} \left[ \frac{1}{A'} + \frac{1}{A' - \eta} \right] \quad \text{or equivalently} \quad \Lambda_{\eta} = \Lambda' - \frac{A' \eta}{2A' - \eta}. \]

By the Riesz-Thorin interpolation theorem (cf. [Lun18, Chapter 2]),
\[ C_{mr, A_{\eta}} \leq C_{mr, A}^{1/2} C_{mr, A - \eta}^{1/2}. \]
By letting $\eta \to 0$, 
\[
(C^M_{\text{mr}, \Lambda'})^{-} \leq C^{1/2}_{\text{mr}, \Lambda'} \left[ (C^M_{\text{mr}, \Lambda'})^{-} \right]^{1/2}
\]
which yields the desired claim (3.8) and thus finishes the proof of Lemma 3.3. □

**Proposition 3.1** ((O)-(D)-(F1)-(F2)-(F3)-(1.6)-(1.7)-(1.8)-(1.15)). There exist constants $C_T$ and $\gamma > 0$ such that
\[
\sum_{i=1}^{m_1} \left( \| u_i \|_{L^{\Lambda+\gamma}(Q_T)} + \| u_i \|_{L^{\Lambda+\gamma}(M_T)} \right) + \sum_{j=1}^{m_2} \| v_j \|_{L^{\Lambda+\gamma}(M_T)} \leq C_T.
\]

**Proof.** Let $\varkappa_0$ be given in Lemma 3.3, and choose $\varkappa \in (0, \varkappa_0)$ small enough such that
\[
\Lambda + \varkappa \frac{n+1}{\Lambda-1} > \frac{\varkappa}{\Lambda-1}.
\]
Note that this is equivalent to
\[
\Lambda' = \frac{\Lambda}{\Lambda-1} < (\Lambda + \varkappa)' + \frac{(\Lambda + \varkappa)'}{n+1}
\]
where $(\Lambda + \varkappa)'$ is the Hölder conjugate exponent of $\Lambda + \varkappa$. Since we only need (3.9) for $\varkappa$ sufficiently small and positive, note that (3.9) is true when $\varkappa = 0$, and therefore it is true for small positive $\varkappa$.

Let $0 \leq \psi \in L^{(\Lambda+\varkappa)'}(M_T)$ with $\| \psi \|_{L^{(\Lambda+\varkappa)'}(M_T)} = 1$, and let $\phi$ be the solution to the dual problem (3.2) with
\[
\delta = \frac{\delta_{\text{max}} + \delta_{\text{min}}}{2},
\]
where $\delta_{\text{max}}$ and $\delta_{\text{min}}$ are defined in (1.13). Thanks to (3.4) in Lemma 3.2 and (3.9), we have
\[
\| \phi \|_{W^{2,1}_{\Lambda'}(Q_T)} + \| \partial_t \phi \|_{L^{\Lambda'}(M_T)} + \| \phi \|_{W^{2,1}_{(\Lambda+\varkappa)'}(M_T)} + \| \phi(0) \|_{L^{\Lambda'}(\Omega)} + \| \phi(0) \|_{L^{(\Lambda+\varkappa)'}(M)} \leq C_{\Lambda', T}.
\]

In particular, thanks to Lemma 3.2,
\[
\| \Delta_M \phi \|_{L^{(\Lambda+\varkappa)'}(M_T)} \leq \frac{C^M_{\text{mr}, (\Lambda+\varkappa)'}}{\delta} = \frac{2C^M_{\text{mr}, (\Lambda+\varkappa)'}}{\delta_{\text{max}} + \delta_{\text{min}}}.
\]

By integration by parts, we have
\[
0 = -\sum_{i=1}^{m_1} \int_{Q_T} a_i u_i (\partial_t \phi + \Delta \phi)
\]
\[
= \sum_{i=1}^{m_1} \int_{Q_T} a_i u_i \phi(0) - \sum_{i=1}^{m_1} \int_{M_T} a_i d_i u_i \partial_t \phi + \sum_{i=1}^{m_1} \int_{M_T} a_i \phi d_i \partial_t u_i
\]
\[
+ \sum_{i=1}^{m_1} \int_{Q_T} a_i \phi (\partial_t u_i - d_i \Delta u_i) + \sum_{i=1}^{m_1} \int_{Q_T} (d_i - 1) a_i u_i \Delta \phi
\]
On the other hand, we have

\[
\leq C \sum_{i=1}^{m_1} \left\| u_{i,0} \right\|_{L^\Lambda(\Omega)} \left\| \phi(0) \right\|_{L^{\Lambda'}(\Omega)} + C \sum_{i=1}^{m_1} \left\| u_i \right\|_{L^\Lambda(M_T)} \left\| \partial_t \phi \right\|_{L^{\Lambda'}(M_T)}
\]
\[
+ \iint_{M_T} \varphi \left( \sum_{i=1}^{m_1} a_i G_i(u) \right) + \iint_{Q_T} \varphi \left( \sum_{i=1}^{m_1} a_i F_i(u) \right) + C \sum_{i=1}^{m_1} \left\| u_i \right\|_{L^\Lambda(Q_T)} \left\| \Delta \phi \right\|_{L^{\Lambda'}(Q_T)}
\]
\[
\leq C_T \sum_{i=1}^{m_1} \left( \left\| u_{i,0} \right\|_{L^\Lambda(\Omega)} + \left\| u_i \right\|_{L^\Lambda(Q_T)} + \left\| u_i \right\|_{L^\Lambda(M_T)} \right)
\]
\[
+ L \iint_{Q_T} \varphi \left( \sum_{i=1}^{m_1} u_i + 1 \right) + \iint_{M_T} \varphi \left( \sum_{i=1}^{m_1} a_i G_i(u, v) \right)
\]
\[
\leq C_T \sum_{i=1}^{m_1} \left( 1 + \left\| u_{i,0} \right\|_{L^\Lambda(\Omega)} + \left\| u_i \right\|_{L^\Lambda(Q_T)} + \left\| u_i \right\|_{L^\Lambda(M_T)} \right) + \iint_{M_T} \varphi \left( \sum_{i=1}^{m_1} a_i G_i(u, v) \right).
\]

(3.11)

On the other hand, we have

\[
\iint_{M_T} \left( \sum_{j=1}^{m_2} b_j v_j \right) \psi
\]
\[
= - \sum_{j=1}^{m_2} \iint_{M_T} b_j v_j (\partial_t \phi + \delta \Delta_M \phi)
\]
\[
= \sum_{j=1}^{m_2} \int_{M} v_{j,0} \phi(0) + \sum_{j=1}^{m_2} \iint_{M_T} b_j \phi(\partial_t v_j - \delta_j \Delta_M v_j) + \sum_{j=1}^{m_2} \iint_{M_T} b_j (\delta_j - \delta) v_j \Delta_M \phi
\]
\[
\leq \sum_{j=1}^{m_2} \left\| v_{j,0} \right\|_{L^{\Lambda+\kappa}(M)} \left\| \phi(0) \right\|_{L^{(\Lambda+\kappa)'(M)}} + \sum_{j=1}^{m_2} \iint_{M_T} \phi b_j H_j(u, v)
\]
\[
+ \frac{\delta_{\max} - \delta_{\min}}{2} \left\| \sum_{j=1}^{m_2} b_j v_j \right\|_{L^{\Lambda+\kappa}(M_T)} \left\| \Delta_M \phi \right\|_{L^{(\Lambda+\kappa)'(M_T)}}
\]
\[
\leq C_{(\Lambda+\kappa), T} \sum_{j=1}^{m_2} \left\| v_{j,0} \right\|_{L^{\Lambda+\kappa}(M)} + \iint_{M_T} \phi \left( \sum_{j=1}^{m_2} b_j H_j(u, v) \right)
\]
\[
+ \frac{\delta_{\max} - \delta_{\min}}{\delta_{\max} + \delta_{\min}} C_{mr,(\Lambda+\kappa)'} \left\| \sum_{j=1}^{m_2} b_j v_j \right\|_{L^{\Lambda+\kappa}(M_T)}.
\]

(3.12)
By summing (3.11) and (3.12) and using the second inequality (F3), we have

\[
\begin{align*}
\int_M \left( \sum_{j=1}^{m_2} b_j v_j \right) \psi \\
&\leq C_T \sum_{i=1}^{m_1} \left( 1 + \|u_{i,0}\|_{L^\Lambda(\Omega)} + \|u_i\|_{L^\Lambda(\Omega')} + \|u_i\|_{L^\Lambda(M_T)} \right) + C_T \sum_{j=1}^{m_2} \|v_{j,0}\|_{L^{\Lambda+\kappa}(M_T)} \\
&+ \int_M \phi \left( \sum_{i=1}^{m_1} u_i + \sum_{j=1}^{m_2} v_j + 1 \right) + \frac{\delta_{\max} - \delta_{\min}}{\delta_{\max} + \delta_{\min}} C_M \sum_{j=1}^{m_2} \|b_j v_j\|_{L^{\Lambda+\kappa}(M_T)}.
\end{align*}
\]  

(3.13)

We show now that by choosing \( \kappa \) small enough, we have \( \|\phi\|_{L^{\Lambda}(M_T)} \leq C_T \). Indeed, from the boundary bound in (3.10) we can use the embedding theorem to have

\[
\|\phi\|_{L^{h(\Lambda)}(M_T)} \leq C_T \|\phi\|_{W^{2,1}_{(\Lambda+\kappa)'(M_T)}} \leq C_T
\]

where \( h(\kappa) = \frac{\left( n+1 \right) (\Lambda + \kappa)'}{n+1-2(\Lambda + \kappa)'} \) when \( (\Lambda + \kappa)'< \frac{n+1}{2} \), and \( h(\kappa) \) can be chosen arbitrarily large if \( (\Lambda + \kappa)'> \frac{n+1}{2} \). Since \( h(\kappa) \) is strictly increasing in \( \kappa \) and \( h(0) > \Lambda' \), we can choose \( \kappa \in (0, \kappa_0) \) small enough such that

\[
\Lambda' < h(\kappa) = \frac{\left( n+1 \right) (\Lambda + \kappa)'}{n+1-2(\Lambda + \kappa)'}
\]

(3.14)

and consequently,

\[
\|\phi\|_{L^{\Lambda'}(M_T)} \leq C_T \|\phi\|_{L^{h(\Lambda)}(M_T)} \leq C_T
\]

Thus, we can estimate

\[
\begin{align*}
\int_M \phi \left( \sum_{i=1}^{m_1} u_i + \sum_{j=1}^{m_2} v_j + 1 \right) \\
&\leq \left( C_T + \sum_{i=1}^{m_1} \|u_i\|_{L^\Lambda(M_T)} \right) \|\phi\|_{L^{\Lambda'}(M_T)} + \frac{1}{\min_{j=1,\ldots,m_2} b_j} \|\phi\|_{L^{\Lambda'}(M_T)} \sum_{j=1}^{m_2} \|b_j v_j\|_{L^{\Lambda}(M_T)} \\
&\leq C_T \left( 1 + \sum_{i=1}^{m_1} \|u_i\|_{L^\Lambda(M_T)} + \sum_{j=1}^{m_2} \|b_j v_j\|_{L^{\Lambda}(M_T)} \right).
\end{align*}
\]  

(3.15)

Thanks to the interpolation inequality and the \( L^1 \)-bound in Lemma 2.1 we have

\[
\| \sum_{j=1}^{m_2} b_j v_j \|_{L^\Lambda(M_T)} \leq \| \sum_{j=1}^{m_2} b_j v_j \|_{L^1(M_T)}^\beta \| \sum_{j=1}^{m_2} b_j v_j \|_{L^{\Lambda+\kappa}(M_T)}^{1-\beta} \leq C_{T,\beta} \| \sum_{j=1}^{m_2} b_j v_j \|_{L^{\Lambda+\kappa}(M_T)}^{1-\beta}
\]  

(3.16)
where $\beta \in (0, 1)$ satisfies $\frac{1}{\Lambda} = \frac{1}{\alpha} + \frac{1-\beta}{\Lambda+\varepsilon}$. Inserting (3.15) and (3.16) into (3.13), and applying (2.21) we have for any $\varepsilon > 0$

$$
\int_{M_T} \left( \sum_{j=1}^{m_2} b_j v_j \right) \psi \leq C_{T,\varepsilon} \sum_{j=1}^{m_1} \left( 1 + \|u_{i,0}\|_{L^\Lambda(\Omega)} \right) + C_T \sum_{j=1}^{m_2} \|v_{j,0}\|_{L^{\Lambda+\varepsilon}(M_T)} + \varepsilon \left\| \sum_{j=1}^{m_2} b_j v_j \right\|_{L^{\Lambda+\varepsilon}(M_T)}
$$

$$
+ \frac{\delta_{\max} - \delta_{\min}}{\delta_{\max} + \delta_{\min}} C'_M \left( \sum_{j=1}^{m_2} b_j v_j \right)_{L^{\Lambda+\varepsilon}(M)} + C_{T,\beta} \left\| \sum_{j=1}^{m_2} b_j v_j \right\|_{L^{\Lambda+\varepsilon}(M)}^{1-\beta}.
$$

By Young’s inequality we have

$$
C_{T,\beta} \left\| \sum_{j=1}^{m_2} b_j v_j \right\|_{L^{\Lambda+\varepsilon}(M)}^{1-\beta} \leq C_{T,\varepsilon,\beta} + \varepsilon \left\| \sum_{j=1}^{m_2} b_j v_j \right\|_{L^{\Lambda+\varepsilon}(M_T)},
$$

and consequently,

$$
\int_{M_T} \left( \sum_{j=1}^{m_2} b_j v_j \right) \psi \leq C_{T,\varepsilon,\beta} + \left( 2\varepsilon + \frac{\delta_{\max} - \delta_{\min}}{\delta_{\max} + \delta_{\min}} C'_M \right) \left\| \sum_{j=1}^{m_2} b_j v_j \right\|_{L^{\Lambda+\varepsilon}(M_T)}.
$$

Since $0 \leq \psi \in L^{(\Lambda+\varepsilon)'}(M_T)$ with $\|\psi\|_{L^{(\Lambda+\varepsilon)'}(M_T)} = 1$ arbitrary, we obtain by duality

$$
\left\| \sum_{j=1}^{m_2} b_j v_j \right\|_{L^{\Lambda+\varepsilon}(M_T)} \leq C_{T,\varepsilon,\beta} + \left( 2\varepsilon + \frac{\delta_{\max} - \delta_{\min}}{\delta_{\max} + \delta_{\min}} C'_M \right) \left\| \sum_{j=1}^{m_2} b_j v_j \right\|_{L^{\Lambda+\varepsilon}(M_T)}.
$$

Now thanks to Lemma 3.3 we can choose $\varepsilon$ small enough such that, for all $\kappa \in (0, \kappa_0)$ verifying (3.14),

$$
2\varepsilon + \frac{\delta_{\max} - \delta_{\min}}{\delta_{\max} + \delta_{\min}} C'_M < 1,
$$

and finally obtain

$$
\left\| \sum_{j=1}^{m_2} b_j v_j \right\|_{L^{\Lambda+\varepsilon}(M_T)} \leq \left( 1 - 2\varepsilon - \frac{\delta_{\max} - \delta_{\min}}{\delta_{\max} + \delta_{\min}} C'_M \right)^{-1} C_{T,\varepsilon,\beta}.
$$

Combining this with (2.21) we can finish the proof of Proposition 3.1. \hfill \Box

4. Proof of main results

4.1. Theorem 1.2: Global existence.

Lemma 4.1. Let $\{y_j\}_{j=1,\ldots,m_2}$ be a sequence of non-negative numbers. Assume that there is a constant $K > 0$ such that, for any $\varepsilon > 0$, there exists $C_{\varepsilon} > 0$ independent of $\{y_k\}$ such that if $k \in \{1, \ldots, m_2\}$ then

$$
y_k \leq C_{\varepsilon} + K \sum_{j=1}^{k-1} y_j + \varepsilon \sum_{j=1}^{m_2} y_j,
$$
where if \( k = 1 \), the sum \( \sum_{j=1}^{k-1} y_j \) is neglected. Then there exists a constant \( C \) independent of the sequence \( \{y_j\} \) such that

\[
\sum_{j=1}^{m_2} y_j \leq C. \tag{4.1}
\]

Proof. We prove by induction that for each \( k \in \{1, \ldots, m_2\} \), there exists a constant \( B_k, R_k > 0 \) independent of \( \varepsilon \) such that

\[
y_k \leq B_k + R_k \varepsilon \sum_{j=1}^{m_2} y_j. \tag{4.2}
\]

With \( k = 1 \), (4.2) follows for \( B_1 = C \varepsilon \) and \( R_1 = 1 \). Assume that this is true for all \( j = 1, \ldots, k-1 \). Then

\[
y_k \leq C \varepsilon + K \sum_{j=1}^{k-1} y_j + \varepsilon \sum_{j=1}^{m_2} y_j \leq C \varepsilon + K \sum_{j=1}^{k-1} \left( B_j + R_j \varepsilon \sum_{j=1}^{m_2} y_j \right) + \varepsilon \sum_{j=1}^{m_2} y_j \\
\leq \left[ C \varepsilon + K \sum_{j=1}^{k-1} B_j \right] + \left[ K \sum_{j=1}^{k-1} R_j + 1 \right] \varepsilon \sum_{j=1}^{m_2} y_j
\]

which proves (4.2). Now summing (4.2) for \( k = 1, \ldots, m_2 \) gives

\[
\sum_{k=1}^{m_2} y_k \leq \sum_{k=1}^{m_2} B_k + \left( \sum_{k=1}^{m_2} R_k \right) \varepsilon \sum_{j=1}^{m_2} y_j,
\]

and consequently, by choosing \( \varepsilon = \frac{1}{2} \left( \sum_{k=1}^{m_2} R_k \right)^{-1} \), we obtain the desired bound (4.1). \( \square \)

Lemma 4.2 ((O)-(D)-(F1)-(F2)-(F3)-(1.6)-(1.7)-(1.8)-(1.14)-(1.15)). For any \( p > \Lambda + \kappa \), any \( \varepsilon > 0 \) and any \( k \in \{1, \ldots, m_2\} \), there exists a constant \( C_{T,\varepsilon} \) depending on \( T \) and \( \varepsilon \) such that

\[
\|v_k\|_{L^p(M_T)} \leq C_{T,\varepsilon} + C_T \sum_{j=1}^{k-1} \|v_j\|_{L^p(M_T)} + \varepsilon \sum_{j=1}^{m_2} \|v_j\|_{L^p(M_T)}. \tag{4.3}
\]

Naturally, when \( k = 1 \), the first sum on the right-hand side is neglected.

Proof. It’s enough to show (4.3) for \( p \) large enough. Let \( 0 \leq \psi \in L^{p'}(M_T) \) with \( \|\psi\|_{L^{p'}(M_T)} = 1 \), and \( \phi \) be the solution to (3.2) with \( \delta = \delta_k \). Recall that from the assumption (1.7) we have for any \( k = 1, \ldots, m_2 \)

\[
\sum_{i=1}^{m_1} a_{(k+m_1)i} G_i(u, v) + \sum_{j=1}^{k} a_{(k+m_1)(j+m_1)} H_j(u, v) \leq L_2 \left[ \sum_{i=1}^{m_1} u_{i}^{\mu_M} + \sum_{j=1}^{m_2} v_{j}^{\mu_M} + 1 \right]
\]
which implies, recalling \( a_{kk} = 1 \) for all \( k = 1, \ldots, m_1 + m_2 \),

\[
H_k(u, v) \leq - \sum_{i=1}^{m_1} a_{(k+m_1)i} G_i(u, v) - \sum_{j=1}^{k-1} a_{(k+m_1)(j+m_1)} H_j(u, v)
+ L_2 \left[ \sum_{i=1}^{m_1} u_i^\mu + \sum_{j=1}^{m_2} v_j^\mu + 1 \right].
\] (4.4)

By integration by parts we have

\[
\begin{align*}
\int_M v_k \psi &= - \int_M v_k (\partial_t \phi + \delta_k \Delta_M \phi) \\
&= \int_M v_k \phi(0) + \int_M \phi (\partial_t v_k - \delta_k \Delta_M v_k) \\
&= \int_M v_k \phi(0) + \int_M \phi H_k(u, v) \\
&=: (A) + (B).
\end{align*}
\] (4.5)

From Lemma 3.2 we have

\[
|(A)| \leq C \| v_k.0 \|_{L^p(M)} \| \phi(0) \|_{L^{p'}(M)} \leq C \| v_k.0 \|_{L^p(M)}.
\] (4.6)

To estimate \((B)\) we use \( \phi \geq 0 \) and (4.4) to have

\[
(B) \leq - \sum_{i=1}^{m_1} \int_{M_T} a_{(k+m_1)i} G_i(u, v) \phi - \sum_{j=1}^{k-1} \int_{M_T} a_{(k+m_1)(j+m_1)} H_j(u, v) \phi
+ L_2 \int_{M_T} \phi \left[ \sum_{i=1}^{m_1} u_i^\mu + \sum_{j=1}^{m_2} v_j^\mu + 1 \right]
=: (B1) + (B2) + (B3). 
\] (4.7)

**Estimate of \((B1)\).** From the equation (1.1) we have

\[
(B1) = - \sum_{i=1}^{m_1} \int_{M_T} a_{(k+m_1)i} G_i(u, v) \phi
\]

\[
= - \sum_{i=1}^{m_1} \int_{M_T} a_{(k+m_1)i} (d_i \partial_\eta u_i) \phi
\]

\[
= - \sum_{i=1}^{m_1} a_{(k+m_1)i} \left[ \int_{Q_T} d_i \Delta u_i \phi + \int_{M_T} d_i u_i \partial_\eta \phi - \int_{Q_T} d_i u_i \Delta \phi \right]
\]

\[
= - \sum_{i=1}^{m_1} a_{(k+m_1)i} \left[ \int_{Q_T} [\partial_\eta u_i - F_i(u)] \phi + \int_{M_T} d_i u_i \partial_\eta \phi - \int_{Q_T} d_i u_i \Delta \phi \right]
\]
$$= \sum_{i=1}^{m_1} a_{(k+m_1)i} \int_{\Omega} u_{i,0} \phi(0) + \iint_{Q_T} \phi \left[ \sum_{i=1}^{m_1} a_{(k+m_1)i} F_i(u) \right]$$

$$+ \sum_{i=1}^{m_1} a_{(k+m_1)i} d_i \iint_{M_T} u_i \partial_y \phi + \sum_{i=1}^{m_1} a_{(k+m_1)i} \iint_{Q_T} u_i \left[ \partial_t \phi + d_i \Delta \phi \right]$$

$$=: (B11) + (B12) + (B13) + (B14).$$

By using Lemma 3.2 we have thanks to H"older’s inequality

$$|(B11)| \leq C \sum_{i=1}^{m_1} \|u_{i,0}\|_{L^p(\Omega)} \|\phi(0)\|_{L^{p'}(\Omega)} \leq C_T \sum_{i=1}^{m_1} \|u_{i,0}\|_{L^p(\Omega)}. \quad (4.8)$$

From (1.6) we have

$$\sum_{i=1}^{m_1} a_{(k+m_1)i} F_i(u) \leq L_2 \left[ \sum_{i=1}^{m_1} u_i^{p_{\Omega}} + 1 \right].$$

Therefore, by H"older’s inequality,

$$|(B12)| \leq L_2 \iint_{Q_T} \phi \left[ \sum_{i=1}^{m_1} u_i^{p_{\Omega}} + 1 \right] \leq L_2 \sum_{i=1}^{m_1} \iint_{Q_T} \phi u_i^{p_{\Omega}} + C_T \|\phi\|_{L^{p'}(Q_T)}. \quad (4.9)$$

From Lemma 3.2 we have

$$\|\phi\|_{L^{(p')\dagger}(Q_T)} \leq C_T \quad (4.10)$$

with $(p')\dagger$ defined similarly to $q^\dagger$ in (3.7). For any $\beta \in (0, 1)$, it follows from H"older’s inequality that

$$\iint_{Q_T} \phi u_i^{p_{\Omega}} \leq \left( \iint_{Q_T} \phi^{(p')\dagger} \right)^{\frac{1}{(p')\dagger}} \left( \iint_{Q_T} u_i^{(p_{\Omega} - \beta)s} \right)^{\frac{1}{s}} \left( \iint_{Q_T} u_i^p \right)^{\frac{\beta}{p}} \quad (4.11)$$

where

$$\frac{1}{(p')\dagger} + \frac{1}{s} + \frac{\beta}{p} = 1. \quad (4.12)$$

This implies

$$s = \frac{(n + 2)p}{(n + 2)(p - \beta) + 2p - (n + 1)(p - 1)}.$$

Therefore, from (1.8) and the fact that $\Lambda \geq 2$, we can always choose $\beta \in (0, 1)$ and $p$ large enough such that

$$(p_{\Omega} - \beta)s < \Lambda + \varkappa,$$

and consequently

$$\|u_i\|_{L^{(p_{\Omega} - \beta)s}(Q_T)} \leq C_T \|u_i\|_{L^{\Lambda + \varkappa}(Q_T)} \leq C_T.$$

Thus it follows from (4.11) and (4.10) that

$$\iint_{Q_T} \phi u_i^{p_{\Omega}} \leq \|\phi\|_{L^{(p')\dagger}(Q_T)} \|u_i\|_{L^{(p_{\Omega} - \beta)s}(Q_T)} \|u_i\|_{L^p(Q_T)} \leq C_T \|u_i\|_{L^p(Q_T)}. \quad (4.13)$$
Therefore, from (4.9) we get the estimate for (B12),

$$|(B12)| \leq C_T + \sum_{i=1}^{m_1} \|u_i\|_{L^p(Q_T)}^\beta \leq C_T + \sum_{i=1}^{m_1} \|u_i\|_{L^p(Q_T)}^\beta, \quad (4.14)$$

since $\beta \in (0, 1)$. Due to Hölder’s inequality, Lemmas 2.4 and 3.2 we have

$$|(B13)| \leq C \sum_{i=1}^{m_1} \|u_i\|_{L^p(M_T)} \|\partial_t \phi\|_{L^p(M_T)} \leq C_T \sum_{i=1}^{m_1} \|u_i\|_{L^p(M_T)} \quad (4.15)$$

Finally, since $\partial_t \phi + \Delta \phi = 0$ in $Q_T$ we estimate (B14) as

$$|(B14)| \leq \sum_{i=1}^{m_1} a_{(k+m_1)i}|d_i - 1| \int_{Q_T} |u_i| \Delta \phi \leq C \sum_{i=1}^{m_1} \|u_i\|_{L^p(Q_T)} \|\Delta \phi\|_{L^p(Q_T)} \quad (4.16)$$

$$\leq C \sum_{i=1}^{m_1} \|u_i\|_{L^p(Q_T)}.$$

From (4.8), (4.14), (4.15), and (4.16) we obtain the estimate for (B1) as

$$|(B1)| \leq C_T + \sum_{i=1}^{m_1} \left( \|u_i\|_{L^p(Q_T)} + \|u_i\|_{L^p(M_T)} \right). \quad (4.17)$$

**Estimate of (B2).** Since $\partial_t v_j - \delta_j \Delta_M v_j = H_j(u, v)$, we have

$$|(B2)| = \left| \sum_{j=1}^{k-1} a_{(k+m_1)(j+m_1)} \int_{M_T} (\partial_t v_j - \delta_j \Delta_M v_j) \phi \right|$$

$$= \left| \sum_{j=1}^{k-1} a_{(k+m_1)(j+m_1)} \left[ \int_M v_j(\phi(0) - \int_{M_T} v_j(\partial_t \phi + \delta_j \Delta_M \phi) \right] \right|$$

$$\leq C \sum_{j=1}^{k-1} \left[ \|v_j\|_{L^p(M)} \|\phi(0)\|_{L^p(M)} + \|v_j\|_{L^p(M_T)} \left( |\delta_j - \delta_k| \|\Delta_M \phi\|_{L^p(M_T)} + \|\psi\|_{L^p(M_T)} \right) \right] \quad (4.18)$$

$$\leq C \sum_{j=1}^{k-1} \left( \|v_j\|_{L^p(M)} + \|v_j\|_{L^p(M_T)} \right)$$

where we used Lemma 3.2 at the last step.

**Estimate of (B3).** We split (B3) into three parts, as

$$(B3) = L_2 \sum_{i=1}^{m_1} \int_{M_T} \phi \mu_i^M + L_2 \sum_{j=1}^{m_2} \int_{M_T} \phi \mu_j^M + L_2 \int_{M_T} \phi =: (B31) + (B32) + (B33).$$
The term \((B33)\) can be estimated directly as
\[
|(B33)| \leq C_T \|\phi\|_{L^{p'}(M_T)} \leq C_T. \tag{4.19}
\]
For any \(\alpha \in (0, 1)\) we use Hölder’s inequality to estimate
\[
\iint_{M_T} \phi u_i^{\mu_M} \leq \left(\iint_{M_T} \phi (p')^\alpha \right)^{\frac{1}{p'}} \left(\iint_{M_T} u_i^{\mu_{M-\alpha}}\right)^{\frac{1}{p}} \left(\iint_{M_T} u_i^\alpha\right)^{\frac{1}{p}}
\]
where \((p')^\alpha\) is defined similarly as \(q^\alpha\) in \((3.7)\), and
\[
\frac{1}{(p')^\alpha} + \frac{1}{r} + \frac{\alpha}{p} = 1. \tag{4.21}
\]
It follows that
\[
r = \frac{(n + 1)p}{(n + 1)(1 - \alpha) + 2p}. \tag{4.22}
\]
We now choose \(\alpha \in (0, 1)\) such that
\[
1 - \alpha < \frac{2\kappa}{(n + 1)\left[1 - \frac{\Lambda + \kappa}{p}\right]}. \tag{4.23}
\]
Combining this with \((1.14)\) gives
\[
\mu_M - \alpha < \frac{(\Lambda + \kappa)[(n + 1)(1 - \alpha) + 2p]}{(n + 1)p},
\]
and consequently
\[
(\mu_M - \alpha)r < \Lambda + \kappa.
\]
Therefore, it follows from \((4.20)\) that
\[
\iint_{M_T} \phi u_i^{\mu_M} \leq \|\phi\|_{L^{(p')^\alpha}(M_T)} \left\|u_i\right\|_{L^{(\mu_{M-\alpha})^\alpha}(M_T)} \left\|u_i\right\|_{L^p(M_T)} \leq C_T \left\|u_i\right\|_{L^p(M_T)}^{\alpha}
\]
thanks to \((3.6)\) and \(\left\|u_i\right\|_{L^{(\mu_{M-\alpha})^\alpha}(M_T)} \leq C_T \|u_i\|_{L^{\Lambda + \kappa}(M_T)} \leq C_T\). Therefore, we have
\[
|(B31)| \leq C_T \sum_{i=1}^{m_1} \|u_i\|_{L^p(M_T)} \leq C_T + \sum_{i=1}^{m_1} \|u_i\|_{L^p(M_T)}. \tag{4.24}
\]
In the same way we can estimate
\[
|(B32)| \leq C_T \sum_{j=1}^{m_2} \|v_j\|_{L^p(M_T)}^{\alpha}. \tag{4.25}
\]
From \((4.24)\), \((4.19)\), and \((4.25)\) we obtain
\[
|(B3)| \leq C_T + \sum_{i=1}^{m_1} \|u_i\|_{L^p(M_T)} + \sum_{j=1}^{m_2} \|v_j\|_{L^p(M_T)}^{\alpha}, \tag{4.26}
\]
with \(\alpha \in (0, 1)\) satisfying \((4.23)\).
From the estimates of (B1), (B2), (B3) in (4.17), (4.18), (4.26), we get 
\[ |B| \leq C_T + C_T \left( \sum_{j=1}^{k-1} \|v_j\|_{L^p(M_T)} + \sum_{j=1}^{m_2} \|v_j\|_{L^p(Q_T)} + \sum_{i=1}^{m_1} \left( \|u_i\|_{L^p(Q_T)} + \|u_i\|_{L^p(M_T)} \right) \right). \]

By using estimate (2.22) and Young’s inequality, we find 
\[ |B| \leq C_T + C_T \left( \sum_{j=1}^{k-1} \|v_j\|_{L^p(M_T)} + \varepsilon \sum_{j=1}^{m_2} \|v_j\|_{L^p(M_T)} \right). \]
Together with (4.6), we get finally from (4.5),
\[ \int_{M_T} v_k \psi \leq C_T + C_T \left( \sum_{j=1}^{k-1} \|v_j\|_{L^p(M_T)} + \varepsilon \sum_{j=1}^{m_2} \|v_j\|_{L^p(M_T)} \right), \]
which yields the desired estimate (4.3) thanks to duality. □

By combining Lemmas 2.4, 4.1, and 4.2 we get

**Lemma 4.3** ((O)-(D)-(F1)-(F2)-(F3)-(1.6)-(1.7)-(1.8)-(1.14)-(1.15)). For any \( 2 \leq p \in \mathbb{Z} \), there exists a constant \( C_{T,p} \) such that 
\[ \sum_{i=1}^{m_1} \left( \|u_i\|_{L^p(Q_T)} + \|u_i\|_{L^p(M_T)} \right) + \sum_{j=1}^{m_2} \|v_j\|_{L^p(M_T)} \leq C_{T,p}. \]

**Proof.** From Lemmas 4.2 and 4.1 we obtain
\[ \sum_{j=1}^{m_2} \|v_j\|_{L^p(M_T)} \leq C_{T,p}. \]
This and Lemma 2.4 imply the desired estimate. □

As a consequence, we obtain the global existence of (1.1) in Theorem 1.2 part (i).

**Theorem 4.1** ((O)-(D)-(F1)-(F2)-(F3)-(F4)-(1.6)-(1.7)-(1.8)-(1.14)-(1.15)). Let \( p > n \). For any non-negative initial data \( (u_0, v_0) \in (W^{2-2/p}(\Omega))^{m_1} \times (W^{2-2/p}(M))^{m_2} \) satisfying the compatibility condition (1.10), the system (1.1) has a unique global strong solution in all dimensions.

**Proof.** From Lemma 4.3 and (F4), the nonlinearities \( F_i(u) \) are bounded in \( L^p(Q_T) \), and \( G_i(u, v) \) are bounded in \( L^p(M_T) \), for all \( p \geq 1 \). It follows that
\[
\begin{cases}
\partial_t u_i - d_i \Delta u_i = F_i(u) \in L^p(Q_T), \\
d_i \partial_n u_i = G_i(u, v) \in L^p(M_T),
\end{cases}
\]
for any \( p \geq 1 \). By the regularizing effect of linear parabolic equations with inhomogeneous boundary conditions, cf. [Nit14, Proposition 3.1], it follows that \( \|u_i\|_{L^\infty(Q_T)} \) is bounded. Similarly, \( \|v_j\|_{L^\infty(M_T)} \) is bounded. This implies that the solution is bounded in \( L^\infty \), which implies the desired global existence. □
4.2. **Theorem 1.2:** Uniform-in-time bounds. To complete the proof of Theorem 1.1, it remains to show the uniform-in-time bound of the solution. To this end, we study the system (1.1) on each cylinder $Q_{\tau,\tau+1} = \Omega \times (\tau, \tau + 1)$, $\tau \in \mathbb{N}$.

For the rest of this section, all constants are independent of $\tau$ unless otherwise stated. We also consider an increasing, smooth function $\varphi \in C^\infty(\mathbb{R}; [0, 1])$ such that $\varphi(s) = 0$ for $s \in (-\infty, 0]$, $\varphi(s) = 1$ for $s \geq 1$, and its shifted version $\varphi(\cdot) = \varphi(\cdot - \tau)$. By multiplying the system (1.1) by $\varphi_\tau$ we have the truncated system

$$
\begin{cases}
\partial_t(\varphi_\tau u_i) = d_i \Delta(\varphi_\tau u_i) + \varphi'_\tau u_i + \varphi_\tau F_i(u), & (x, t) \in Q_{\tau,\tau+2}, \ i = 1, \ldots, m_1, \\
\partial_t \partial_\eta(\varphi_\tau u_i) = \varphi_\tau G_i(u, v), & (x, t) \in M_{\tau,\tau+2}, \ i = 1, \ldots, m_1, \\
\partial_t(\varphi_\tau v_j) = \delta_j \Delta_M(\varphi_\tau v_j) + \varphi'_\tau v_j + \varphi_\tau H_j(u, v), & (x, t) \in M_{\tau,\tau+2}, \ j = 1, \ldots, m_2,
\end{cases}
$$

(4.27)

with zero initial data

$$
\begin{align*}
(\varphi_\tau u_i)(x, \tau) &= 0, \quad x \in \Omega, \ i = 1, \ldots, m_1, \\
(\varphi_\tau v_j)(x, \tau) &= 0, \quad x \in M, \ j = 1, \ldots, m_2.
\end{align*}
$$

(4.28)

**Lemma 4.4 ((O)-(D)-(F1)-(F2)-(F3)).** If $L < 0$ or $L = K = 0$ in (F3), then

$$
\sup_{t \geq 0} \sum_{i=1}^{m_1} \|u_i(t)\|_{L^1(\Omega)} + \sup_{t \geq 0} \sum_{j=1}^{m_2} \|v_j(t)\|_{L^1(M)} + \sup_{\tau \in \mathbb{N}} \sum_{i=1}^{m_1} \|u_i\|_{L^1(M_{\tau,\tau+1})} \leq C.
$$

**Proof.** From (F3) it follows that

$$
\frac{d}{dt} \left( \sum_{i=1}^{m_1} \int_{\Omega} a_i u_i + \sum_{j=1}^{m_2} \int_{M} b_j v_j \right) \leq L \left( \sum_{i=1}^{m_1} \int_{\Omega} u_i + \sum_{j=1}^{m_2} \int_{M} v_j \right) + K.
$$

If $L = K = 0$, it yields directly by integrating on $(0, t)$ that

$$
\sup_{t \geq 0} \sum_{i=1}^{m_1} \|u_i(t)\|_{L^1(\Omega)} + \sup_{t \geq 0} \sum_{j=1}^{m_2} \|v_j(t)\|_{L^1(M)} \leq C.
$$

(4.29)

If $L < 0$, there exists $\sigma > 0$ such that

$$
\frac{d}{dt} \left( \sum_{i=1}^{m_1} \int_{\Omega} a_i u_i + \sum_{j=1}^{m_2} \int_{M} b_j v_j \right) \leq -\sigma \left( \sum_{i=1}^{m_1} \int_{\Omega} a_i u_i + \sum_{j=1}^{m_2} \int_{M} b_j v_j \right) + K,
$$

which also implies (4.29) thanks to Gronwall’s lemma. Let $\phi \in C^{2,1}(\bar{\Omega} \times [\tau, \tau + 2])$ be a nonnegative function such that $\phi_t + \Delta \phi = 0$ on $Q_{\tau,\tau+2}$, $\partial_\eta \phi = 1$ on $M_{\tau,\tau+2}$ and $\phi(\cdot, \tau + 2) = 0$.
in $\hat{\Omega}$. Define $\theta = -\partial_t \phi - \Delta_M \phi$ on $M \times (\tau, \tau + 2)$. By integration by parts,

\[
\begin{align*}
\iint_{M_{\tau,\tau+2}} a_i d_i (\varphi \partial_t u_i) &= \iint_{M_{\tau,\tau+2}} a_i d_i (\varphi \partial_t u_i) \partial_n \phi \\
&= \iint_{M_{\tau,\tau+2}} \varphi \partial_t \phi \cdot a_i G_i (u, v) + a_i \iint_{Q_{\tau,\tau+2}} \phi \varphi' u_i \\
&\quad + \iint_{Q_{\tau,\tau+2}} \varphi \partial_t \phi \cdot a_i F_i (u) + \iint_{Q_{\tau,\tau+2}} a_i (\varphi \partial_t u_i) (d_i - 1) \Delta \phi,
\end{align*}
\]

(4.30)

and

\[
\begin{align*}
\iint_{M_{\tau,\tau+2}} b_j (\varphi \partial_t v_j) \theta &= \iint_{M_{\tau,\tau+2}} [\varphi \partial t \phi \cdot b_j H_j (u, v) + \varphi' \partial t v_j \phi + b_j v_j (\delta_j - 1) \Delta_M \phi].
\end{align*}
\]

(4.31)

By summing (4.30) in $i = 1, \ldots, m_1$, summing (4.31) in $j = 1, \ldots, m_2$, and adding the resultants we can apply (F3) (recalling $L = 0$) to get

\[
\begin{align*}
\sum_{i=1}^{m_1} \iint_{M_{\tau,\tau+2}} a_i d_i (\varphi \partial_t u_i) + \sum_{j=1}^{m_2} \iint_{M_{\tau,\tau+2}} b_j (\varphi \partial_t v_j) \theta \\
\leq \sum_{i=1}^{m_1} \left[ \iint_{Q_{\tau,\tau+2}} \phi \varphi' u_i + \iint_{Q_{\tau,\tau+2}} a_i (\varphi \partial_t u_i) (d_i - 1) \Delta \phi \right] + K \left( \iint_{Q_{\tau,\tau+2}} \varphi \partial t \phi + \iint_{M_{\tau,\tau+2}} \varphi \partial t \phi \right) \\
\quad + \sum_{j=1}^{m_2} \left[ \iint_{M_{\tau,\tau+2}} \phi \varphi' u_i + \iint_{M_{\tau,\tau+2}} b_j (\varphi \partial t v_j) (\delta_j - 1) \Delta_M \phi \right].
\end{align*}
\]

Thanks to the fact that $\theta \in L^\infty (M_{\tau,\tau+2})$, $\sup_{t \geq 0} \| u_i (t) \|_{L^1 (\Omega)} \leq C$ and $\sup_{t \geq 0} \| v_j (t) \|_{L^1 (M)} \leq C$, $\phi \in C^{2,1} (\hat{\Omega} \times [\tau, \tau + 2])$, we conclude that

\[
\sum_{i=1}^{m_1} \iint_{M_{\tau,\tau+2}} \varphi \partial_t u_i \leq C \quad \text{for all} \quad \tau \in \mathbb{N}.
\]

Since $\varphi \geq 0$ and $\varphi |_{[\tau, \tau + 2]} = 1$, we get finally

\[
\sup_{\tau \in \mathbb{N}} \sum_{i=1}^{m_1} \| u_i \|_{L^1 (M_{\tau,\tau+1})} \leq C.
\]

\[\Box\]

Lemma 4.5 ((O)-(D)-(F1)-(F2)-(F3)-(1.6)-(1.7)-(1.8)). If $L < 0$ or $L = K = 0$ in (F3), then for any positive integer $p \geq 2$, and any $\varepsilon > 0$, there exists $K_{p,\varepsilon} > 0$ such that

\[
\begin{align*}
\sum_{i=1}^{m_1} \left( \iint_{Q_{\tau,\tau+2}} (\varphi u_i)^{p-1+pM} + \iint_{M_{\tau,\tau+2}} (\varphi u_i)^{p-1+pM} \right) \\
\leq K_{p,\varepsilon} + \varepsilon \left[ \sum_{i=1}^{m_1} \left( \iint_{Q_{\tau,\tau+2}} u_i^{p-1+pM} + \iint_{M_{\tau,\tau+2}} u_i^{p-1+pM} \right) + \sum_{j=1}^{m_2} \iint_{M_{\tau,\tau+2}} v_j^{p-1+pM} \right].
\end{align*}
\]

(4.32)
As a consequence, for any $1 < p < \infty$ and any $\varepsilon > 0$, there exists $K_{p,\varepsilon} > 0$ such that

$$
\sum_{i=1}^{m_1} \left( \| \varphi \tau u_i \|_{L^p(Q_{\tau,\tau+2})} + \| \varphi \tau u_i \|_{L^p(M_{\tau,\tau+2})} \right) 
\leq K_{p,\varepsilon} + \varepsilon \sum_{i=1}^{m_1} \left( \| u_i \|_{L^p(Q_{\tau,\tau+2})} + \| u_i \|_{L^p(M_{\tau,\tau+2})} \right) + \sum_{j=1}^{m_2} \| v_j \|_{L^p(M_{\tau,\tau+2})}.
$$

(4.33)

**Proof.** Recall the Lyapunov-like function $L_p[u]$ in (2.19), with $\theta$ is chosen in $(\theta 1)$ and $(\theta 2)$. Thanks to (2.32),

$$(L_p[u])' (t) + C \sum_{i=1}^{m_1} \left( \int_{\Omega} \varphi \tau u_i^{p-1+p\eta} + \int_{M} \varphi \tau u_i^{p-1+p\eta} \right) \leq K_{p,\theta} \left[ 1 + \varepsilon \sum_{j=1}^{m_2} \int_{M} v_j^{p-1+pM} \right].$$

Therefore, we have

$$(\varphi \tau L_p[u])' + C \sum_{i=1}^{m_1} \left( \int_{\Omega} \varphi \tau u_i^{p-1+p\eta} + \int_{M} \varphi \tau u_i^{p-1+p\eta} \right) \leq \varphi '_\tau \sigma_p (t) + K_{p,\theta,\varepsilon} \varphi \tau + \varepsilon \sum_{j=1}^{m_2} \int_{M} \varphi \tau v_j^{p-1+pM}.$$  

(4.34)

Since $0 \leq \varphi \tau \leq 1$,

$$\varphi \tau u_i^{p-1+p\eta} \geq (\varphi \tau u_i)^{p-1+p\eta}, \quad \varphi \tau u_i^{p-1+pM} \geq (\varphi \tau u_i)^{p-1+pM}.$$  

From (2.19) and $0 \leq \varphi '_\tau \leq C$, it follows that

$$|\varphi '_\tau L_p[u] (t)| \leq C \sum_{i=1}^{m_1} \int_{\Omega} u_i^{p} \leq C \varepsilon + \varepsilon \sum_{i=1}^{m_1} \int_{\Omega} u_i^{p-1+pM}.$$  

Putting all these into (4.34) and integrating the resultant on $(\tau, \tau + 2)$, noticing that $\varphi \tau (\tau) = 0$, we obtain

$$
\int_{Q_{\tau,\tau+2}} (\varphi \tau u_i)^{p-1+p\eta} + \int_{M_{\tau,\tau+2}} (\varphi \tau u_i)^{p-1+pM} 
\leq K_{p,\theta,\varepsilon} + \varepsilon \sum_{i=1}^{m_1} \left( \int_{Q_{\tau,\tau+2}} u_i^{p-1+p\eta} + \int_{M_{\tau,\tau+2}} u_i^{p-1+pM} \right) + \sum_{j=1}^{m_2} \int_{M_{\tau,\tau+2}} v_j^{p-1+pM},
$$

which is the desired estimate (4.32). From this, (4.33) can be obtained similarly to the last step of Lemma 2.4.

We need the following elementary result whose proof is straightforward.

**Lemma 4.6.** Let $\{y_n\}_{n \geq 0}$ be a nonnegative sequence and $\mathcal{N} = \{n \in \mathbb{N} : y_{n-1} \leq y_n\}$. If there exists $K > 0$ (independent of $n$) such that

$$y_n \leq K \quad \text{for all} \quad n \in \mathcal{N},$$

then $y_n$ is bounded.
then
\[ y_n \leq \max\{y_0, K\} \quad \text{for all} \quad n \in \mathbb{N}. \]

**Lemma 4.7** (\((O)-(D)-(F1)-(F2)-(F3)-(1.6)-(1.7)-(1.8)-(1.15)\)). There exist constants \( C > 0 \) and \( \gamma > 0 \) such that for all \( \tau \in \mathbb{N} \),
\[
\sum_{i=1}^{m_1} \left( \|u_i\|_{L^{\Lambda+\gamma}(Q_{\tau+2})} + \|u_i\|_{L^{\Lambda+\gamma}(M_{\tau+2})} \right) + \sum_{j=1}^{m_2} \|v_j\|_{L^{\Lambda+\gamma}(M_{\tau+2})} \leq C. \tag{4.35}
\]

**Proof.** As in Lemma 3.1, we choose \( \epsilon > 0 \) small enough such that (3.9) holds. Let \( 0 \leq \psi \in L^{(\Lambda+\epsilon)'}(M_{\tau+2}) \) with \( \|\psi\|_{L^{(\Lambda+\epsilon)'}(M_{\tau+2})} = 1 \), and let \( \phi \) be the solution to (3.2) with \( T = \tau + 2 \) and
\[
\delta = \frac{\delta_{\max} + \delta_{\min}}{2}
\]
with \( \delta_{\max} \) and \( \delta_{\min} \) are in (1.13). From Propositions 3.2 and (3.9), we have
\[
\|\phi\|_{W^{2,1}_\Lambda(Q_{\tau+2})} + \|\partial_t \phi\|_{L^{\Lambda'}(M_{\tau+2})} + \|\phi\|_{W^{2,1}_{(\Lambda+\epsilon)'}(M_{\tau+2})} \leq C. \tag{4.36}
\]
In particular,
\[
\|\Delta_M \phi\|_{L^{(\Lambda+\epsilon)'}(M_{\tau+2})} \leq \frac{2C_{\text{mr},(\Lambda+\epsilon)'}}{\delta_{\max} + \delta_{\min}}. \tag{4.37}
\]
By integration by parts (see the proof of Lemma 3.1) we have
\[
0 = -\sum_{i=1}^{m_1} \iint_{Q_{\tau+2}} a_i(\varphi_t u_i)(\partial_t \phi + \Delta \phi)
\]
\[
= \sum_{i=1}^{m_1} \iint_{Q_{\tau+2}} a_i \phi(\varphi'_t u_i + \varphi_t F_i(u)) - \sum_{i=1}^{m_1} \iint_{M_{\tau+2}} d_ia_i \varphi_t u_i \partial_\eta \phi
\]
\[
+ \sum_{i=1}^{m_1} \iint_{M_{\tau+2}} \phi \varphi_t a_i G_i(u, v) + \sum_{j=1}^{m_2} \iint_{Q_{\tau+2}} a_i(d_i - 1) \varphi_t u_i \Delta \phi
\]
and
\[
\iint_{M_{\tau+2}} \left( \sum_{j=1}^{m_2} b_j \varphi_t v_j \right) \psi = \sum_{j=1}^{m_2} \iint_{M_{\tau+2}} b_j \phi \varphi'_t v_j + \sum_{j=1}^{m_2} \iint_{M_{\tau+2}} \phi \varphi_t b_j H_j(u, v)
\]
\[
+ \sum_{j=1}^{m_2} \iint_{M_{\tau+2}} b_j(\delta_j - \delta)(\varphi_t v_j) \Delta_M \phi. \tag{4.39}
\]
Sum (4.38) and (4.39), and note that either \( L < 0 \) or \( L = K = 0 \) in (1.6) and (1.7). We provide the argument in the case when \( L = K = 0 \), and leave the similar case when \( L < 0 \)
for the reader. We calculate

\[
\iint_{M_{\tau, \tau+2}} \left( \sum_{j=1}^{m_2} b_j \varphi_{\tau} v_j \right) \psi \leq \sum_{i=1}^{m_1} \iint_{Q_{\tau, \tau+2}} \phi a_i \varphi_{\tau} u_i + \sum_{i=1}^{m_1} \iint_{Q_{\tau, \tau+2}} a_i (d_i - 1) \varphi_{\tau} u_i \Delta \phi - \sum_{i=1}^{m_1} \iint_{M_{\tau, \tau+2}} a_i d_i \varphi_{\tau} u_i \partial_{\eta} \phi + \sum_{j=1}^{m_2} \iint_{M_{\tau, \tau+2}} b_j \varphi_{\tau} v_j \tag{4.40}
\]

\[
+ \sum_{j=1}^{m_2} \iint_{M_{\tau, \tau+2}} b_j (\delta_j - \delta) (\varphi_{\tau} v_j) \Delta_M \phi
\]

\[=: (I) + (II) + (III) + (IV) + (V).\]

We estimate five terms on the right hand side of (4.40) as follows. We use (4.36) to estimate, for \( \gamma \in (0, 1) \) such that \( \frac{1}{\lambda} = \frac{1}{1} + \frac{\gamma}{\lambda+\kappa} \) and any \( \varepsilon > 0 \),

\[
|I| \leq C \sum_{i=1}^{m_1} \| \phi \|_{L^\lambda(Q_{\tau, \tau+2})} \| u_i \|_{L^\lambda(Q_{\tau, \tau+2})} \leq C \sum_{i=1}^{m_1} \| u_i \|_{L^1(Q_{\tau, \tau+2})} \| u_i \|_{L^{\lambda+\kappa}(Q_{\tau, \tau+2})} \tag{4.41}
\]

\[
\leq C \varepsilon \sum_{i=1}^{m_1} \| u_i \|_{L^{\lambda+\kappa}(Q_{\tau, \tau+2})}.
\]

Similarly,

\[
|II| \leq C \varepsilon \sum_{i=1}^{m_1} \| u_i \|_{L^\lambda(Q_{\tau, \tau+2})} \| \Delta \phi \|_{L^\lambda(Q_{\tau, \tau+2})} \leq C \varepsilon \sum_{i=1}^{m_1} \| u_i \|_{L^{\lambda+\kappa}(Q_{\tau, \tau+2})}, \tag{4.42}
\]

\[
|III| \leq C \varepsilon \sum_{i=1}^{m_1} \| u_i \|_{L^\lambda(M_{\tau, \tau+2})} \| \partial_{\eta} \phi \|_{L^\lambda(M_{\tau, \tau+2})} \leq C \varepsilon \sum_{i=1}^{m_1} \| u_i \|_{L^{\lambda+\kappa}(M_{\tau, \tau+2})}, \tag{4.43}
\]

\[
|IV| \leq C \varepsilon \sum_{j=1}^{m_2} \| \phi \|_{L^\lambda(M_{\tau, \tau+2})} \| v_j \|_{L^\lambda(M_{\tau, \tau+2})} \leq C \varepsilon \sum_{j=1}^{m_2} \| v_j \|_{L^{\lambda+\kappa}(M_{\tau, \tau+2})}. \tag{4.44}
\]

For (V) we estimate using (4.37)

\[
|V| \leq \frac{\delta_{\max} - \delta_{\min}}{\delta_{\max} + \delta_{\min}} \iint_{M_{\tau, \tau+2}} \sum_{j=1}^{m_2} b_j \varphi_{\tau} v_j \left| \Delta_M \phi \right| \leq \frac{\delta_{\max} - \delta_{\min}}{\delta_{\max} + \delta_{\min}} \left\| \sum_{j=1}^{m_2} b_j \varphi_{\tau} v_j \right\|_{L^{\lambda+\kappa}(M_{\tau, \tau+2})} \| \Delta_M \|_{L^{\lambda+\kappa}(M_{\tau, \tau+2})} \tag{4.45}
\]

\[
\leq \frac{\delta_{\max} - \delta_{\min}}{\delta_{\max} + \delta_{\min}} C_{m_2, \lambda+\kappa} \left( \sum_{j=1}^{m_2} b_j \varphi_{\tau} v_j \right) \left\| \Delta_M \|_{L^{\lambda+\kappa}(M_{\tau, \tau+2})} \right. .
\]
Using (4.41)–(4.45) into (4.40), it follows from duality, (1.15) and Lemma 3.3, that
\[
\left\| \sum_{j=1}^{m_2} b_j \varphi_j v_j \right\|_{L^{\Lambda + \kappa}(M_{\tau}, \tau + 2)} 
\leq C \varepsilon + \varepsilon \left[ \sum_{i=1}^{m_1} \left( \| u_i \|_{L^{\Lambda + \kappa}(Q_{\tau}, \tau + 2)} + \| u_i \|_{L^{\Lambda + \kappa}(M_{\tau}, \tau + 2)} \right) + \sum_{j=1}^{m_2} \| v_j \|_{L^{\Lambda + \kappa}(M_{\tau}, \tau + 2)} \right].
\tag{4.46}
\]

Combining (4.46) with with (4.33) in Lemma 4.5 (choosing \( p = \Lambda + \kappa \)) we have
\[
\sum_{i=1}^{m_1} \left( \| \varphi_{\tau} u_i \|_{L^{\Lambda + \kappa}(Q_{\tau}, \tau + 2)} + \| \varphi_{\tau} u_i \|_{L^{\Lambda + \kappa}(M_{\tau}, \tau + 2)} \right) + \left\| \varphi_{\tau} \sum_{j=1}^{m_2} b_j v_j \right\|_{L^{\Lambda + \kappa}(M_{\tau}, \tau + 2)} 
\leq C \varepsilon + \varepsilon \left[ \sum_{i=1}^{m_1} \left( \| u_i \|_{L^{\Lambda + \kappa}(Q_{\tau}, \tau + 2)} + \| u_i \|_{L^{\Lambda + \kappa}(M_{\tau}, \tau + 2)} \right) + \sum_{j=1}^{m_2} \| v_j \|_{L^{\Lambda + \kappa}(M_{\tau}, \tau + 2)} \right].
\tag{4.47}
\]

Recall that \( \varphi_{\tau} \geq 0 \) and \( \varphi_{\tau}[\tau, \tau+1] \equiv 1 \), it follows from (4.47) that
\[
\sum_{i=1}^{m_1} \left( \| u_i \|_{L^{\Lambda + \kappa}(Q_{\tau}, \tau + 2)} + \| u_i \|_{L^{\Lambda + \kappa}(M_{\tau}, \tau + 2)} \right) + \sum_{j=1}^{m_2} \| b_j v_j \|_{L^{\Lambda + \kappa}(M_{\tau}, \tau + 2)} 
\leq C \varepsilon + C \varepsilon \left[ \sum_{i=1}^{m_1} \left( \| u_i \|_{L^{\Lambda + \kappa}(Q_{\tau}, \tau + 2)} + \| u_i \|_{L^{\Lambda + \kappa}(M_{\tau}, \tau + 2)} \right) + \sum_{j=1}^{m_2} \| b_j v_j \|_{L^{\Lambda + \kappa}(M_{\tau}, \tau + 2)} \right].
\tag{4.48}
\]

For \( \tau \in \mathbb{N} \), we define
\[
y_{\tau} := \sum_{i=1}^{m_1} \left( \| u_i \|_{L^{\Lambda + \kappa}(Q_{\tau}, \tau + 2)} + \| u_i \|_{L^{\Lambda + \kappa}(M_{\tau}, \tau + 2)} \right) + \sum_{j=1}^{m_2} \| b_j v_j \|_{L^{\Lambda + \kappa}(M_{\tau}, \tau + 2)}.
\]

Inequality (4.48) implies
\[
y_{\tau} \leq C + C \varepsilon (y_{\tau} + y_{\tau+1}).
\tag{4.49}
\]

Define \( \mathcal{N} = \{ \tau \in \mathbb{N} : y_{\tau} \leq y_{\tau+1} \} \). Then for any \( \tau \in \mathcal{N} \), by choosing \( \varepsilon \) sufficiently small, we obtain from (4.49)
\[
y_{\tau} \leq C,
\]
where \( C \) is independent of \( \tau \). From Lemma 4.1, we have
\[
y_{\tau} \leq C \quad \text{for all} \quad \tau \in \mathbb{N},
\]
which proves the desired estimate (4.35). \( \square \)

**Lemma 4.8** \(((O)-(D)-(F1)-(F2)-(F3)-(1.6)-(1.7)-(1.8)-(1.14)-(1.15))\). Assume that \( L < 0 \) or \( L = K = 0 \) in (F3). For any \( \tau \in \mathbb{N} \), \( 2 \leq p \), any \( k \in \{1, \ldots, m_2\} \), and any \( \varepsilon > 0 \), there
exists a constant \( C_\varepsilon > 0 \) such that

\[
\| \varphi \tau v_k \|_{L^p(M_{r, \tau+2})} \leq C_\varepsilon + C_\varepsilon \sum_{j=1}^{k-1} \| \varphi \tau v_j \|_{L^p(M_{r, \tau+2})} + \varepsilon \sum_{i=1}^{m_1} (\| u_i \|_{L^p(Q_{r, \tau+2})} + \| u_i \|_{L^p(M_{r, \tau+2})}) + \varepsilon \sum_{j=1}^{m_2} \| v_j \|_{L^p(M_{r, \tau+2})}.
\]

Consequently, for any \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that

\[
\| \varphi \tau v_k \|_{L^p(M_{r, \tau+2})} \leq C_\varepsilon + \varepsilon \sum_{i=1}^{m_1} (\| u_i \|_{L^p(Q_{r, \tau+2})} + \| u_i \|_{L^p(M_{r, \tau+2})}) + \varepsilon \sum_{j=1}^{m_2} \| v_j \|_{L^p(M_{r, \tau+2})}
\]

(4.50)

for all \( k = 1, \ldots, m_2 \).

**Proof.** Let \( 0 \leq \psi \in L^{p'}(M_{r, \tau+2}) \) with \( \| \psi \|_{L^{p'}(M_{r, \tau+2})} = 1 \), and \( \phi \) be the solution to (3.2) with \( T = \tau + 2 \). Thanks to the calculations of Lemma 4.2, we can write

\[
\int_{M_{r, \tau+2}} (\varphi \tau v_k) \psi
\]

\[
\leq \int_{M_{r, \tau+2}} \phi \varphi' \tau v_k + \sum_{i=1}^{m_1} \int_{Q_{r, \tau+2}} a_{k+m_1}(\varphi \tau u_i) + \sum_{i=1}^{m_1} \int_{Q_{r, \tau+2}} a_{k+m_1} d_i(\varphi \tau u_i) \partial_0 \phi
\]

\[
- \sum_{j=1}^{k-1} \int_{M_{r, \tau+2}} a_{k+m_1+m+j}(\varphi \tau \partial_0 v_j - \delta_j \Delta_M v_j) + L_2 \int_{M_{r, \tau+2}} \phi \varphi' \left( \sum_{i=1}^{m_1} u_i^{\mu M} + \sum_{j=1}^{m_2} v_j^{\mu M} + 1 \right)
\]

(4.52)

\[
=: (I) + (II) + (III) + (IV) + (V) + (VI) + (VII).
\]

We estimate the terms on the right-hand side of (4.52) separately.

- **Estimate (I).** From Lemma 3.2, there exists \( q > p' \) such that

\[
\| \phi \|_{L^q(M_{r, \tau+2})} \leq C \| \psi \|_{L^{p'}(M_{r, \tau+2})} = C.
\]

Therefore, by Young’s inequality and \( q' = \frac{q}{q-1} \) is the Hölder conjugate exponent of \( q \), we have

\[
| (I) | \leq C \| \phi \|_{L^q(M_{r, \tau+2})} \| v_k \|_{L^{p'}(M_{r, \tau+2})} \leq C \| v_k \|_{L^{p'}(M_{r, \tau+2})}.
\]

Since \( q > p' = \frac{p}{p-1} \), we have \( q' < p \). Therefore, it follows from Hölder’s inequality that

\[
\| v_k \|_{L^{p'}(M_{r, \tau+2})} \leq \| v_k \|_{L^q(M_{r, \tau+2})} \| v_k \|_{L^{p'}(M_{r, \tau+2})} \leq C \| v_k \|_{L^{q'}(M_{r, \tau+2})} \leq C_\varepsilon + \varepsilon \| v_k \|_{L^p(M_{r, \tau+2})},
\]
with $\theta_0 \in (0,1)$ satisfying $\frac{1}{q'} = \frac{\theta_0}{1} + \frac{1-\theta_0}{p}$. Therefore,

$$|(I)| \leq C_\varepsilon + \varepsilon \|v_k\|_{L^p(M_{r'',r'+2})}.$$  \hfill (4.53)

- **Estimate (II).** Similarly to the estimate of (I), we can use Lemma 3.2 to estimate

$$|(II)| \leq C \sum_{i=1}^{m_1} \|u_i\|_{L^p(Q_{r',r'+2})}^{1-\theta_1} \leq C_\varepsilon + \varepsilon \sum_{i=1}^{m_1} \|u_i\|_{L^p(Q_{r',r'+2})}$$  \hfill (4.54)

for some $\theta_1 \in (0,1)$.

- **Estimate (III).** We use the condition (1.6) to find

$$(III) \leq L_2 \iint_{Q_{r',r'+2}} \phi \varphi \left[ \sum_{i=1}^{m_1} u_i^{p_1} + 1 \right].$$

Looking at the estimate of (B12) in Lemma 4.2, and using $|\varphi'| \leq C$, we have

$$|(III)| \leq C \sum_{i=1}^{m_1} \iint_{Q_{r',r'+2}} \phi u_i^{p_0} + C \|\phi\|_{L^p(Q_{r',r'+2})} \leq C \sum_{i=1}^{m_1} \iint_{Q_{r',r'+2}} \phi u_i^{p_0} + C.$$  \hfill (4.55)

Similarly to (4.11)–(4.13),

$$\iint_{Q_{r',r'+2}} \phi u_i^{p_0} \leq \|\phi\|_{L^p(Q_{r',r'+2})} \|u_i\|_{L^p(Q_{r',r'+2})}^{p_0 - \beta} \leq C \|u_i\|_{L^p(Q_{r',r'+2})}^{\beta} \leq C_\varepsilon + \varepsilon \|u_i\|_{L^p(Q_{r',r'+2})}$$

where $\beta$ and $s$ are in (4.12). Therefore,

$$|(III)| \leq C_\varepsilon + \varepsilon \sum_{i=1}^{m_1} \|u_i\|_{L^p(Q_{r',r'+2})}.$$  \hfill (4.55)

- **Estimate (IV).** Thanks to Lemma 3.2 we have for some $s > p'$,

$$\|\Delta \phi\|_{L^s(Q_{r',r'+2})} \leq C \|\psi\|_{L^{p'}(M_{r',r'+2})} \leq C.$$  \hfill (4.56)

Therefore, for $s' = \frac{s}{s - 1}$,

$$|(IV)| \leq C \|\Delta \phi\|_{L^s(Q_{r',r'+2})} \sum_{i=1}^{m_1} \|u_i\|_{L^{s'}(Q_{r',r'+2})} \leq C \sum_{i=1}^{m_1} \|u_i\|_{L^{s'}(Q_{r',r'+2})}.$$  \hfill (4.56)

Since $s > p'$, $s' < p = \frac{p'}{p' - 1}$. Therefore, by interpolation inequality

$$\|u_i\|_{L^{s'}(Q_{r',r'+2})} \leq \|u_i\|_{L^{p'}(Q_{r',r'+2})}^{\theta_3} \|u_i\|_{L^p(Q_{r',r'+2})}^{1-\theta_3} \leq C \|u_i\|_{L^p(Q_{r',r'+2})}^{1-\theta_3}$$

with $\theta_3 \in (0,1)$ satisfying $\frac{1}{s'} = \frac{\theta_3}{1} + \frac{1-\theta_3}{p}$. From that we obtain

$$|(IV)| \leq C \sum_{i=1}^{m_1} \|u_i\|_{L^{p-1+p\theta_3}(Q_{r',r'+2})}^{1-\theta_3} \leq C_\varepsilon + \varepsilon \sum_{i=1}^{m_1} \|u_i\|_{L^p(Q_{r',r'+2})}.$$
• Estimate (V). From (3.5) in Lemma 3.2 we have for $\xi = \frac{p'}{n+1}$

$$\|\partial_t \phi\|_{L^{p' + \xi}(M_{r,r+2})} \leq C \|\psi\|_{L^{p'}(M_{r,r+2})} \leq C.$$  

Therefore, with $s = \frac{p' + \xi}{p' + 1} = \frac{p + \xi(p - 1)}{p + 1} < p$, we can estimate

$$|\langle V \rangle| \leq C \sum_{i=1}^{m_1} \| u_i \|_{L^{p' + \xi}(M_{r,r+2})} \| \partial_t \phi \|_{L^{p' + \xi}(M_{r,r+2})}$$

$$\leq C \sum_{i=1}^{m_1} \| u_i \|_{L^{p' + \xi}(M_{r,r+2})}^{\theta_1} \| u_i \|_{L^{p}(M_{r,r+2})}^{1 - \theta_1} \left( \text{with } \frac{1}{p' + \xi} = \frac{\theta_1}{1} + \frac{1 - \theta_1}{p} \right)$$

$$\leq C \varepsilon + \varepsilon \sum_{i=1}^{m_1} \| u_i \|_{L^{p}(M_{r,r+2})}.$$  

• Estimate (VI). By integration by parts,

$$\langle VI \rangle = \sum_{j=1}^{k-1} \int_{M_{r,r+2}} \left[ a_{(k+m_1)(k+j)} v_j \phi \varphi' \tau + v_j \varphi'_\tau \left( (\delta_j - \delta_k) \Delta_M \phi + \psi \right) \right].$$

Similar to the estimate of (I) above

$$\left| \sum_{j=1}^{k-1} \int_{M_{r,r+2}} a_{(k+m_1)(k+j)} v_j \phi \varphi'_\tau \right| \leq C \sum_{j=1}^{k-1} \| v_j \|_{L^{p}(M_{r,r+2})}^{1 - \theta_5} \leq C \varepsilon + \varepsilon \sum_{j=1}^{m_2} \| v_j \|_{L^{p}(M_{r,r+2})}$$

for some $\theta_5 \in (0, 1)$. By Hölder’s inequality and

$$\| \Delta_M \phi \|_{L^{p'}(M_{r,r+2})} \leq C \| \psi \|_{L^{p'}(M_{r,r+2})} \leq C,$$

we get

$$\left| \sum_{j=1}^{k-1} \int_{M_{r,r+2}} v_j \varphi'_\tau \left( (\delta_j - \delta_k) \Delta_M \phi + \psi \right) \right|$$

$$\leq C \sum_{j=1}^{k-1} \| \varphi'_\tau v_j \|_{L^{p}(M_{r,r+2})} \left( \| \Delta_M \phi \|_{L^{p'}(M_{r,r+2})} + \| \psi \|_{L^{p'}(M_{r,r+2})} \right)$$

$$\leq C \sum_{j=1}^{k-1} \| \varphi'_\tau v_j \|_{L^{p}(M_{r,r+2})}.$$
Therefore, we have
\[ |(VI)| \leq C \varepsilon + \varepsilon \sum_{j=1}^{m_2} \|v_j\|_{L^p(M_{r,\tau+2})} + C \sum_{j=1}^{k-1} \|\varphi \tau v_j\|_{L^p(M_{r,\tau+2})}. \quad (4.58) \]

**Estimate (VII).** We use similar estimates to that of (B3) in (4.19)–(4.26). More precisely, with \( \alpha \) and \( r \) are in (4.21)–(4.22) we have
\[
\|u_i\|_{L^{\mu M-\alpha}(M_{r,\tau+2})} \leq C \|u_i\|_{L^{\Lambda+\kappa}(M_{r,\tau+2})} \leq C.
\]

From Lemma 3.2,
\[
\|\phi\|_{L^{(\nu')\ast}(M_{r,\tau+2})} \leq C \|\psi\|_{L^{\nu'}(M_{r,\tau+2})} \leq C.
\]

Therefore
\[
\sum_{i=1}^{m_1} \int_{M_{r,\tau+2}} \phi u_i^M \leq C \sum_{i=1}^{m_1} \|\phi\|_{L^{(\nu')\ast}(M_{r,\tau+2})} \|u_i\|_{L^{\mu M-\alpha}(M_{r,\tau+2})} \|u_i\|_{L^{\nu}(M_{r,\tau+2})} \leq C \sum_{i=1}^{m_1} \|u_i\|_{L^{\nu}(M_{r,\tau+2})}
\]
\[
\leq C \varepsilon + \varepsilon \sum_{i=1}^{m_1} \|u_i\|_{L^{\nu}(M_{r,\tau+2})}.
\]

Similarly,
\[
\sum_{i=1}^{m_1} \int_{M_{r,\tau+2}} \phi v_j^M \leq C \sum_{j=1}^{m_2} \|v_j\|_{L^{\nu}(M_{r,\tau+2})} \leq C \varepsilon + \varepsilon \sum_{j=1}^{m_2} \|v_j\|_{L^{\nu}(M_{r,\tau+2})}.
\]

Finally,
\[
L_2 \int_{M_{r,\tau+2}} \phi \varphi \tau \leq C \|\phi\|_{L^{\nu'}(M_{r,\tau+2})} \leq C.
\]

Therefore,
\[
|(VII)| \leq C \varepsilon + \varepsilon \sum_{i=1}^{m_1} \|u_i\|_{L^{p}(M_{r,\tau+2})} + \varepsilon \sum_{j=1}^{m_2} \|v_j\|_{L^{p}(M_{r,\tau+2})}. \quad (4.59)
\]

Applying all the estimates of (I) to (VII) in (4.53), (4.54), (4.55), (4.56), (4.57), (4.58), (4.59) into (4.52) we obtain
\[
\int_{M_{r,\tau+2}} (\varphi \tau v_k) \psi \leq C \varepsilon + \varepsilon \sum_{j=1}^{k-1} \|\varphi \tau v_j\|_{L^p(M_{r,\tau+2})}
\]
\[
+ \varepsilon \sum_{i=1}^{m_1} \left( \|u_i\|_{L^p(Q_{r,\tau+2})} + \sum_{i=1}^{m_1} \|u_i\|_{L^p(M_{r,\tau+2})} \right) + \varepsilon \sum_{j=1}^{m_2} \|v_j\|_{L^p(M_{r,\tau+2})}.
\]

From this we get the estimate (4.50) due to duality. Finally (4.51) follows from (4.50) by induction. \( \square \)
We are now ready to show the uniform-in-time bound in Theorem 4.1.

**Theorem 4.2** ((O)-(D)-(F1)-(F2)-(F3)-(F4)-(1.6)-(1.7)-(1.8)-(1.14)-(1.15)). Assume that \( L < 0 \) or \( L = K = 0 \) in (F3). The global solution to (1.1) is bounded uniformly in time, i.e.

\[
\sup_{i=1,\ldots,m_1} \sup_{j=1,\ldots,m_2} \sup_{t \geq 0} \left[ \|u_i(t)\|_{L^\infty(\Omega)} + \|v_j(t)\|_{L^\infty(M)} \right] < +\infty.
\]

**Proof.** We claim that, for any \( 2 \leq p \), there exists a constant \( C_p > 0 \) such that

\[
\sum_{i=1}^{m_1} \left( \|u_i\|_{L^p(Q_{\tau,\tau+1})} + \|u_i\|_{L^p(M_{\tau,\tau+1})} \right) + \sum_{j=1}^{m_2} \|v_j\|_{L^p(M_{\tau,\tau+1})} \leq C_p \quad \text{for all} \quad \tau \in \mathbb{N}. \tag{4.60}
\]

Indeed, from (4.33) in Lemma 4.5 and (4.51) in Lemma 4.8, we get for any \( \varepsilon > 0 \) a constant \( C_\varepsilon > 0 \) such that

\[
\sum_{i=1}^{m_1} \left( \|\varphi_\tau u_i\|_{L^p(Q_{\tau,\tau+2})} + \|\varphi_\tau u_i\|_{L^p(M_{\tau,\tau+2})} \right) + \sum_{j=1}^{m_2} \|\varphi_\tau v_j\|_{L^p(M_{\tau,\tau+2})} \leq C_\varepsilon + \varepsilon \left[ \sum_{i=1}^{m_1} \left( \|u_i\|_{L^p(Q_{\tau,\tau+2})} + \|u_i\|_{L^p(M_{\tau,\tau+2})} \right) + \sum_{j=1}^{m_2} \|v_j\|_{L^p(M_{\tau,\tau+2})} \right].
\]

Using the same arguments as at the end of the proof of Lemma 4.7, we obtain (4.60).

Now we can use (4.60) in the truncated system (4.27), with \( p \) is large enough, to conclude that there exists \( C_\infty > 0 \) independent of \( \tau \in \mathbb{N} \) such that

\[
\sum_{i=1}^{m_1} \|u_i\|_{L^\infty(Q_{\tau,\tau+1})} + \sum_{j=1}^{m_2} \|v_j\|_{L^\infty(M_{\tau,\tau+1})} \leq C_\infty \quad \text{for all} \quad \tau \in \mathbb{N},
\]

which finishes the proof of Theorem 4.2. \( \square \)

### 4.3. Proof of Theorem 1.1

**Proof of Theorem 1.1.** Thanks to Theorem 1.2, it’s sufficient to show that (1.15) always holds for \( \Lambda = 2 \). Indeed, in this case \( \Lambda' = 2 \). We only need to show that

\[
C_{mr,2}^M \leq 1. \tag{4.61}
\]

To do that, we multiply the equation (3.1) by \(-\Delta_M U\) in \( L^2(\Omega)\), to get

\[
\frac{1}{2} \frac{d}{dt} \|\nabla_M U\|_{L^2(M)}^2 + \|\Delta_M U\|_{L^2(\Omega)}^2 = - \int_M F \Delta_M U \leq \frac{1}{2} \|F\|_{L^2(M)}^2 + \frac{1}{2} \|\Delta_M U\|_{L^2(M)}^2;
\]

implying

\[
\frac{d}{dt} \|\nabla_M U\|_{L^2(M)}^2 + \|\Delta_M U\|_{L^2(M)}^2 \leq \|F\|_{L^2(M)}^2.
\]

Integrating this on \((0,T)\) and using \( U(\cdot,0) = 0 \) we obtain

\[
\|\nabla_M U(\cdot,T)\|_{L^2(M)}^2 + \|\Delta_M U\|_{L^2(M_T)}^2 \leq \|F\|_{L^2(M_T)}^2.
\]
which implies
\[ \| \Delta_M U \|_{L^2(M_T)} \leq \| F \|_{L^2(M_T)}, \]
and hence, the desired estimate (4.61).

\[ \square \]

4.4. Proof of Theorem 1.3.

Proof of Theorem 1.3. First, thanks to Lemmas 2.3 and 2.4, we have, similarly to (2.21) and (2.22), for any \( p > 1 \) and \( \varepsilon > 0 \) a constant \( K_{p,\varepsilon} > 0 \) such that
\[ \sum_{i=1}^{m_1} (\| u_i \|_{L^p(Q_T)} + \| u_i \|_{L^p(M_T)}) \leq K_{p,\varepsilon} (1 + T) + \varepsilon \sum_{j=1}^{m_2} \| v_j \|_{L^p(M_T)}. \]

Following Lemma 4.2, we will show that for any \( \varepsilon > 0 \), and any \( k \in \{1, \ldots, m_2\} \), there exists \( C_{T,\varepsilon} > 0 \) such that
\[ \| v_k \|_{L^p(M_T)} \leq C_{T,\varepsilon} + C_T \sum_{j=1}^{k-1} \| v_j \|_{L^p(M_T)} + \varepsilon \sum_{j=1}^{m_2} \| v_j \|_{L^p(M_T)}. \quad (4.62) \]

We consider the duality equation (3.2) with \( 0 \leq \psi \in L^{p'}(M_T) \) satisfying \( \| \psi \|_{L^{p'}(M_T)} = 1 \). The same integration by parts arguments in Lemma 4.2 gives
\[ \iint_{M_T} v_k \psi \leq \int_M v_{k,0} \phi(0) + \sum_{i=1}^{m_1} a_{(k+m_1)i} \int_\Omega u_{i,0} \phi(0) + L_2 \iint_{Q_T} \phi \left[ \sum_{i=1}^{m_1} u_i^{p_i} + 1 \right] \]
\[ + \sum_{i=1}^{m_1} a_{(k+m_1)i} d_i \int_{M_T} u_i \partial_\nu \phi + \sum_{i=1}^{m_1} a_{(k+m_1)i} \iint_{Q_T} u_i [\partial_t \phi + d_i \Delta \phi] \]
\[ + \sum_{j=1}^{k-1} \sum_{l=1}^{m_2} a_{(k+m_1)(k+j)} \int_{M_T} v_{j,0} \phi(0) - \sum_{j=1}^{k-1} \iint_{M_T} v_j [\partial_t \phi + \delta_j \Delta_M \phi] \]
\[ + L_2 \iint_{M_T} \phi \left[ \sum_{i=1}^{m_1} u_i^{\mu_iM} + \sum_{j=1}^{m_2} u_j^{\mu_jM} + 1 \right]. \quad (4.63) \]

All the terms on the right hand side of (4.63) can be estimated similarly to Lemma 4.2 except for two sums
\[ (A) = \sum_{i=1}^{m_1} \iint_{Q_T} \phi u_i^{p_i} \quad \text{and} \quad (B) = \sum_{i=1}^{m_1} \iint_{M_T} \phi u_i^{\mu_iM} + \sum_{j=1}^{m_2} \iint_{M_T} \phi u_j^{\mu_jM}. \]

To estimate (A) we first use Lemma 3.2 to have
\[ \| \phi \|_{L^{p'_1}(Q_T)} \leq C_T \| \psi \|_{L^{p'}(Q_T)} = C_T. \]

By Hölder’s inequality
\[ \int_0^T \int_\Omega \phi u_i^{p_i-\alpha} u_i^\alpha \leq \| \phi \|_{L^{p'_1}(Q_T)} \| u_i \|_{L^{p_i-\alpha}(Q_T)} \| u_i \|_{L^{p}(Q_T)} \leq C_T \| u_i \|_{L^{p_i-\alpha}(Q_T)} \| u_i \|_{L^{p}(Q_T)}. \quad (4.64) \]
with 
\begin{equation}
\frac{1}{(p')^s} + \frac{1}{s} + \frac{\alpha}{p} = 1.
\end{equation}

From this
\begin{equation}
\frac{1}{s} = 1 - \frac{\alpha}{p} - \frac{1}{(p')^s} = 1 - \frac{\alpha}{p} - \frac{n + 1 - 2p'}{(n + 2)p'} = \frac{n(1 - \alpha) + 3p + 1 - 2\alpha}{(n + 2)p}
\end{equation}
or equivalently
\begin{equation}
s = \frac{(n + 2)p}{n(1 - \alpha) + 3p + 1 - 2\alpha}.
\end{equation}

Note that as \( p \to \infty \), \( s \to \frac{n+2}{3} \). Since
\begin{equation}
p_\Omega < 1 + \frac{3a}{n + 2}
\end{equation}
we can choose \( \alpha \in (0, 1) \) and \( p \) large enough such that
\begin{equation}
(p_\Omega - \alpha)s \leq \frac{3a}{n + 2} \frac{n + 2}{3} = a.
\end{equation}

We then use this and the assumption \( \|u_i\|_{L^\infty(0,T;L^p(\Omega))} \leq \mathcal{F}(T) \) in (4.64) to get
\begin{equation}
\int_{Q_T} \phi u_i^{p_\Omega} \leq C_T \mathcal{F}(T)^{p_\Omega - a} \|u_i\|^{p_\Omega}_{L^p(Q_T)} \leq C_{T,\varepsilon} + \varepsilon \|u_i\|_{L^p(Q_T)},
\end{equation}
which gives the estimate
\begin{equation}
(A) \leq C_{T,\varepsilon} + \varepsilon \sum_{i=1}^{m_1} \|u_i\|_{L^p(Q_T)}.
\end{equation}

The estimate of (B) can be done in the same way as for (B3) in the proof of Lemma 4.2, where the integrability \( b \) plays the role of \( \Lambda + \kappa \) therein, so we omit it here. Ultimately, we have for any \( \varepsilon > 0 \) a constant \( C_{T,\varepsilon} > 0 \) such that
\begin{equation}
(B) \leq C_{T,\varepsilon} + \sum_{i=1}^{m_1} \|u_i\|_{L^p(M_T)} + \sum_{j=1}^{m_2} \|v_j\|_{L^p(M_T)}.
\end{equation}

With the estimates of (A) and (B), one can proceed from (4.63) the same way in Lemma 4.2 to finally obtain (4.62). Lemma 4.1 then gives
\begin{equation}
\sum_{j=1}^{m_2} \|v_j\|_{L^p(M_T)} \leq C_T
\end{equation}
and consequently
\begin{equation}
\sum_{i=1}^{m_1} (\|u_i\|_{L^p(Q_T)} + \|v_j\|_{L^p(M_T)}) \leq C_T
\end{equation}
for all \( p \geq 1 \). This is enough to conclude the global existence of solutions to (1.1). The uniform-in-time bound is obtained by using the truncated function \( \varphi_T \) and working on each
cylinder \( Q_{\tau,\tau+2} \) for \( \tau \in \mathbb{N} \). We omit the details since they are very similar to that of Section 4.2.

\[ \square \]

5. Applications

In this section, we show the application of our theorems to some models arising from cell biology. It’s worth emphasizing that, all known results for volume-surface systems seem not applicable to these systems.

5.1. Membrane protein clustering. In a recent work of Lucas M. Stolerman et al [SGS+20], the authors focus on stability analysis of a bulk-surface reaction-diffusion model of membrane protein clustering. In their model, \( U \) represents a volume component which diffuses in the cytoplasm. It binds with the membrane and forms a surface monomer \( A_1 \) via a reaction flux (see below). Then subsequent oligomerization at the membrane is given by \( A_{j-1} + A_1 \rightleftharpoons A_j \) for \( j = 2, 3, \ldots, N \). If we model the cytoplasm as a smooth bounded region \( \Omega \subset \mathbb{R}^n \), and the membrane (boundary of \( \Omega \)) by \( M \), then the model in [SGS+20] has the form

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d \Delta u, & & \text{on } \Omega \times (0, T) \\
\frac{\partial u}{\partial n} &= F(u, v_1, \ldots, v_N), & & \text{on } M \times (0, T) \\
\frac{\partial v_i}{\partial t} &= \delta_i \Delta_M v_i + G_i(u, v_1, \ldots, v_N), & & \text{on } M \times (0, T) \\
\vdots & & & \\
\frac{\partial v_N}{\partial t} &= \delta_N \Delta_M v_N + G_N(u, v_1, \ldots, v_N), & & \text{on } M \times (0, T) \\
u &= u_0, & & \text{on } \Omega \times 0 \\
v &= v_0, & & \text{on } M \times 0
\end{align*}
\]

(5.1)

Here, \( d \) and \( \delta_i \) are positive diffusion coefficients, for \( i = 1, \ldots, N \), \( u \) represents the cytoplasm concentration density of \( U \), and \( v_1, \ldots, v_N \) represent the membrane concentration densities of \( A_1, \ldots, A_N \), \( u_0 \) and the vector \( v_0 = (v_{0i}) \) represent bounded nonnegative initial data, and \( k_0, k_b, k_d \) and \( k_2, \ldots, k_N \) are positive constants. Also,

\[
F(u, v_1, \ldots, v_N) = -(K_0 + k_b v_N)u + k_d v_1
\]

\[
G_1(u, v_1, \ldots, v_N) = (K_0 + k_b v_N)u - k_d v_1 - 2k_m v_1^2 + 2k_2 v_2 - k_g v_1 \sum_{i=1}^{N-1} v_i + \sum_{j=3}^N k_j v_j
\]

\[
G_2(u, v_1, \ldots, v_N) = k_m v_1^2 - k_g v_1 v_2 - k_2 v_2 + k_3 v_3
\]

\[
G_j(u, v_1, \ldots, v_N) = k_g v_1 v_{j-1} - k_g v_1 v_j - k_j v_j + k_{j+1} v_{j+1}, \text{ for } j = 3, \ldots, N - 1
\]

\[
G_N(u, v_1, \ldots, v_N) = k_g v_N v_{N-1} - k_N v_N
\]

Note that the nonlinearities \( F \) and \( G_j \), \( j = 1, \ldots, N \), are locally Lipschitz continuous,

\[
F(0, v_1, \ldots, v_N) \geq 0 \text{ for all } (v_1, \ldots, v_N) \in \mathbb{R}_+^N
\]

(5.2)

and

\[
G_i(u, v_1, \ldots, v_N) \geq 0 \text{ for all } u, v_1, \ldots, v_N \geq 0 \text{ with } v_i = 0 \text{ for } i = 1, \ldots, N
\]

(5.3)
and they are polynomial. That means the assumptions (F1), (F2), (F4) are fulfilled. It’s also simple to check that

$$F(u, v_1, ..., v_N) + \sum_{j=1}^{N} jG_j(u, v_1, ..., v_N) = 0, \text{ for all } u, v_1, ..., v_N \geq 0,$$  \hspace{1cm} (5.4)

and thus (F3) is satisfied with $L = K = 0$. It remains to check the intermediate sum conditions (1.6) and (1.7), i.e. there is $(N+1) \times (N+1)$ lower triangular matrix $A = (a_{i,j})$ with positive entries on the diagonal, and nonnegative entries below the diagonal, and constants $K_1, K_2 \geq 0$ so that

$$A \begin{bmatrix} F(u, v_1, ..., v_N) \\ G_1(u, v_1, ..., v_N) \\ \vdots \\ G_N(u, v_1, ..., v_N) \end{bmatrix} \leq K_1 \bar{I} \left( u + \sum_{j=1}^{n} v_j + K_2 \right), \text{ for all } u, v_1, ..., v_N \geq 0.$$  \hspace{1cm} (5.5)

Indeed, this is done by choosing $K_1 = 2 \max_{i=2, ..., m} k_i$, $K_2 = 0$, and

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 1 \end{bmatrix}.$$  

Since all the assumptions in Theorem 1.1 are satisfied, we have the following

**Theorem 5.1.** For any non-negative initial data $(u_0, v_0) \in W^{2-2/p}(\Omega) \times (W^{2-2/p}(M))^N$ for $p > n$ satisfying the compatibility condition

$$d\partial_\eta u_0 = F(u_0, v_{1,0}, \ldots, v_{N,0}) \quad \text{on} \quad M,$$

there exists a unique global classical solution to (5.1) which is uniformly bounded in time, i.e.

$$\sup_{t \geq 0} \left( \|u(t)\|_{L^\infty(\Omega)} + \sum_{j=1}^{N} \|v_j(t)\|_{L^\infty(M)} \right) < +\infty.$$  

5.2. **Activation of Cdc42 in cell polarization.** This second example is from the recent paper [BML+20] where the authors derive a mathematical model for the activation of
Cdc42 in cell polarization. The system reads as\(^5\)
\[
\begin{align*}
\partial_t G &= D_G \Delta G, & (x, t) \in Q_T, \\
D_G \partial_t G &= Q(A, I, G), & (x, t) \in M_T, \\
\partial_t A &= D_A \Delta_M A + F(A, I), & (x, t) \in M_T, \\
\partial_t I &= D_I \Delta_M I + H(A, I, G), & (x, t) \in M_T, \\
G(x, 0) &= G_0(x), & x \in \Omega, \\
A(x, 0) &= A_0(x), & I(x, 0) = I_0(x), & x \in M,
\end{align*}
\]
\tag{5.6}
and
\[
Q(A, I, G) = -k_1 G (k_{\text{max}} - (A + I))_+ + k_- I, \\
F(A, I) = k_2 I - k_- A + k_3 A^2 I
\]
with positive constant rates \(k_1, k_-, k_2, k_- A, k_{\text{max}}\). Here \(G, A, I\) are concentrations of the GTP-, GDP-, and GDI-bound form respectively. The interested reader is referred to \[\text{[BML+20]}\] for more details. It’s easy to check that all the assumptions (F1), (F2), (F3) (with \(L = K = 0\)) and (F4) are fulfilled. It’s also clear that
\[
\begin{bmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
Q \\
H \\
F
\end{bmatrix}
\leq
\begin{bmatrix}
k_- I \\
k_- A \\
0
\end{bmatrix},
\]
which means that the intermediate sum condition (1.7) is satisfied (condition (1.6) is trivially fulfilled since we have no nonlinearities for the equation of \(G\)). Therefore Theorem 1.1 applies and we have the following result.

**Theorem 5.2.** For any non-negative initial data \((G_0, A_0, I_0) \in W^{2-2/p}(\Omega) \times (W^{2-2/p}(M))^2\) for \(p > n\) satisfying the compatibility condition
\[
D_G \partial_\eta G_0 = Q(A_0, I_0, G_0) \quad \text{on} \quad M,
\]
there exists a unique global classical solution to (5.6) which is bounded uniformly in time,
\[
\sup_{t \geq 0} \left( \|G(t)\|_{L^\infty(\Omega)} + \|A(t)\|_{L^\infty(M)} + \|I(t)\|_{L^\infty(M)} \right) < +\infty.
\]

\(^5\)The model considered herein is slightly different from that of \[\text{[BML+20]}\], where \(Q(A, I, G) = k_1 G (k_{\text{max}} - (A + I))\), since we take into account the saturation. It’s noted that their choice of non-linearity might lead to negative concentrations, for instance, when \(I_0 \equiv 0, \beta \geq A_0(x) > k_{\text{max}} + 1\) for \(x \in \Omega\), and \(G_0(x) \geq \alpha > 0\) for a large enough constant \(\alpha\).
5.3. **A system with better a-priori estimates.** In this section we consider a system where better a-priori estimates can be derived using the system’s special structures, which in turn allows us to obtain global existence for higher order nonlinearities. More precisely, the system reads

\[
\begin{align*}
\partial_t u_1 - d_1 \Delta u_1 &= f_1(u) = u_1^3 - u_2^3, & x &\in \Omega, \\
\partial_t u_2 - d_2 \Delta u_2 &= f_2(u) = -u_1^3 + u_2^3, & x &\in \Omega, \\
d_1 \partial_t u_1 &= g_1(u,v) = u_1 - u_2 - 2u_1^3 + u_2^2 - v_2^2, & x &\in M, \\
d_2 \partial_t u_2 &= g_2(u,v) = -u_1 + u_2 - u_2^3 - v_1^2, & x &\in M, \\
\partial_t v_1 - \delta_1 \Delta_M v_1 &= h_1(u,v) = 2u_1^3 + u_1 u_2^2 - v_6^2, & x &\in M, \\
\partial_t v_2 - \delta_2 \Delta_M v_2 &= h_2(u,v) = 2u_2^3 + u_1^3 - v_6, & x &\in M, \\
u(x,0) &= u_0(x), & x &\in \Omega, \\
v(x,0) &= v_0(x), w(x,0) = w_0(x), & x &\in M.
\end{align*}
\]  

(5.7)

It’s obvious that

\[
f_1(u) + f_2(u) \leq 0.
\]

By using the Young’s inequality

\[
u_1 u_2^2 \leq \frac{u_1^3}{3} + \frac{2u_2^3}{3}
\]

we can check that

\[
2g_1(u,v) + 2g_2(u,v) + h_1(u,v) + h_2(u,v) \leq -u_1^3 + u_1 u_2^2 - 2u_3^2 \leq -\frac{2}{3}u_1^3 - \frac{4}{3}u_2^3 \leq 0.
\]

Therefore, assumption (F3) is satisfied. It’s easy to check that by choosing the matrix

\[
A = \begin{pmatrix} 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \end{pmatrix}
\]

the assumptions (1.6) and (1.7) are satisfied with

\[
p_\Omega = 3, \quad p_M = 2, \quad \text{and} \quad \mu_M = 3.
\]

Therefore, the results of Theorem 1.1 are not applicable to obtain global existence to (5.7). We show here that by utilizing the special structure of (5.7), one can use Theorem 1.3 to still get global solutions. Indeed, direct computations give

\[
\begin{align*}
\partial_t \left( \int_{\Omega} (u_1^4 + u_2^4) + \int_M (v_1 + v_2) \right) &+ 12d_1 \|u_1 \nabla u_1\|^2_{L^2(\Omega)} + 12d_2 \|u_2 \nabla u_2\|^2_{L^2(\Omega)} \\
&= 4 \int_{\Omega} (f_1(u)u_1^3 + f_2(u)u_2^3) + 4 \int_M (g_1(u,v)u_1^3 + g_2(u,v)u_2^3) + \int_M (h_1(u,v) + h_2(u,v)).
\end{align*}
\]

(5.9)

We have

\[
f_1(u)u_1^3 + f_2(u)u_2^3 = -(u_1^3 - u_2^3)^2 \leq 0,
\]
and
\[
4 \left( g_1(u, v)u_1^4 + g_2(u, v)u_2^4 \right) + h_1(u, v) + h_2(u, v) \\
\leq 4u_1^4 - 8u_1^6 + 4u_1^3u_2^2 + 4u_1^2 - 4u_1^6 + 2u_1^3 + u_1u_2^2 - v_2^6 + 2u_2^3 + u_1^3 - v_1^6 \\
\leq -C(u_1^6 + u_2^6 + v_1^6 + v_2^6) + C
\]
where we used (5.8) and
\[
4u_1^4 + 4u_1^3u_2^2 + 4u_1^2 + 2u_1^3 + u_1u_2^2 + 2u_2^3 + u_1^3 \leq \varepsilon (u_1^6 + u_2^6) + C\varepsilon
\]
for any \( \varepsilon > 0 \). Thus we get from (5.9) that
\[
\partial_t \left( \int_{\Omega} (u_1^4 + u_2^4) + \int_{M} (v_1 + v_2) \right) + C \int_{M} (u_1^6 + u_2^6 + v_1^6 + v_2^6) \leq C.
\]
By integrating in time we get
\[
\sup_{i=1,2} \|u_i\|_{L^\infty(0,T;L^4(\Omega))} + \sup_{i=1,2} \left( \|u_i\|_{L^6(M)}^{6\varepsilon} + \|v_i\|_{L^6(M)}^{6\varepsilon} \right) \leq CT.
\]
Thus, Theorem 1.3 is applicable with \( a = 4 \) and \( b = 6 \), which allows us to get global existence of (5.8) in the physical dimension \( n = 3 \). The uniform-in-time bound of solutions remains unclear.

**Theorem 5.3.** Let \( n \leq 3 \). For any non-negative initial data \((u_0, v_0) \in W^{2-2/p}(\Omega)^2 \times W^{2-2/p}(M)^2\) satisfying the compatibility condition
\[
d_1\partial_t u_{1,0} = g_1(u_0, v_0), \quad d_2\partial_t u_{2,0} = g_2(u_0, v_0), \quad \text{on} \ M,
\]
there exists a unique global classical solution to (5.8).

**APPENDIX A. TECHNICAL LEMMAS**

In this appendix, we prove two technical lemmas that are used in Lemma 2.4, more precisely the time derivative of the function \( \mathcal{H}_p[u] \) and the integration by parts in (2.25)–(2.26).

**Lemma A.1.** Suppose \( m_1 \in \mathbb{N}, \theta = (\theta_1, \ldots, \theta_{m_1}) \), where \( \theta_1, \ldots, \theta_{m_1} \) are positive real numbers, \( \beta \in \mathbb{Z}_{+}^{m_1} \), and \( \mathcal{H}_p[u] \) is defined in (2.20). Then
\[
\frac{\partial}{\partial t} \mathcal{H}_0[u](t) = 0, \quad \frac{\partial}{\partial t} \mathcal{H}_1[u](t) = \sum_{j=1}^{m_1} \theta_j \frac{\partial}{\partial t} u_j(t),
\]
and for \( p \in \mathbb{N} \) such that \( p \geq 2 \),
\[
\frac{\partial}{\partial t} \mathcal{H}_p[u](t) = \sum_{|\beta|=p-1} \binom{p}{\beta} \theta^{p-1} u(t) \sum_{j=1}^{m_1} \theta_j^{p-1} \frac{\partial}{\partial t} u_j(t).
\]

**Proof.** The results for \( \mathcal{H}_0[u](t) \) and \( \mathcal{H}_1[u](t) \) are trivial. The same is true for the case when \( m_1 = 1 \). Suppose \( p \geq 2 \) and \( m_1 \geq 2 \). We proceed by induction on the value \( m_1 \), and assume \( k \in \mathbb{N} \) such that the result is true for \( m_1 = k \). Suppose \( m_1 = k + 1 \) and denote
\[
\tilde{\beta} = (\beta_2, \ldots, \beta_{m_1}) \quad \text{and} \quad \tilde{u} = (u_2, \ldots, u_{m_1}).
\]
Then we can rewrite $\mathcal{H}_p[u]$ as

$$
\mathcal{H}_p[u] = \sum_{\beta_1=0}^{p} \frac{1}{\beta_1!} \theta_1^{\beta_1} u_1^{\beta_1} \sum_{|\beta|=p-\beta_1} \left( \frac{p}{\beta} \right) \theta^{\beta_1} \tilde{u}^\beta .
$$

(A.1)

Consequently,

$$
\frac{\partial}{\partial t} \mathcal{H}_p[u] = \sum_{\beta_1=1}^{p} \frac{1}{\beta_1!} \theta_1^{\beta_1} u_1^{\beta_1-1} \frac{\partial}{\partial t} u_1 \sum_{|\beta|=p-\beta_1} \left( \frac{p}{\beta} \right) \theta^{\beta_1} \tilde{u}^\beta
$$

$$
+ \sum_{\beta_1=0}^{p} \frac{1}{\beta_1!} \theta_1^{\beta_1} u_1^{\beta_1} \frac{\partial}{\partial t} \left( \sum_{|\beta|=p-\beta_1} \left( \frac{p}{\beta} \right) \theta^{\beta_1} \tilde{u}^\beta \right)
$$

$$
= \sum_{\beta_1=1}^{p} \frac{1}{\beta_1!} \theta_1^{\beta_1} u_1^{\beta_1-1} \frac{\partial}{\partial t} u_1 \sum_{|\beta|=p-\beta_1} \left( \frac{p}{\beta} \right) \theta^{\beta_1} \tilde{u}^\beta
$$

$$
+ \sum_{\beta_1=0}^{p-1} \frac{1}{\beta_1!} \theta_1^{\beta_1} u_1^{\beta_1} \frac{p!}{(p-\beta_1)!} \sum_{|\beta|=p-\beta_1-1} \left( \frac{p}{\beta} \right) \theta^{\beta_1} \tilde{u}^\beta \sum_{j=1}^{m_1-1} \tilde{\theta}_j^{2\beta_j+1} \frac{\partial}{\partial t} \tilde{u}_j.
$$

(A.2)

Now, from our induction hypothesis,

$$
\frac{\partial}{\partial t} \mathcal{H}_{p-\beta_1}[\tilde{u}] = \sum_{|\beta|=p-\beta_1-1} \left( \frac{p-\beta_1}{\beta} \right) \tilde{\theta}^{\beta_1} \tilde{u}^\beta \sum_{j=1}^{m_1-1} \tilde{\theta}_j^{2\beta_j+1} \frac{\partial}{\partial t} \tilde{u}_j.
$$

(A.3)

Therefore, substituting (A.3) into (A.2), and noting that $\tilde{u}_j = u_{j+1}$ and $\tilde{\theta}_j = \theta_{j+1}$, gives

$$
\frac{\partial}{\partial t} \mathcal{H}_p[u] = \sum_{\beta_1=1}^{p} \frac{1}{\beta_1!} \theta_1^{\beta_1} u_1^{\beta_1-1} \frac{\partial}{\partial t} u_1 \sum_{|\beta|=p-\beta_1} \left( \frac{p}{\beta} \right) \theta^{\beta_1} \tilde{u}^\beta
$$

$$
+ \sum_{\beta_1=0}^{p-1} \frac{1}{\beta_1!} \theta_1^{\beta_1} u_1^{\beta_1} \frac{p!}{(p-\beta_1)!} \sum_{|\beta|=p-\beta_1-1} \left( \frac{p}{\beta} \right) \theta^{\beta_1} \tilde{u}^\beta \sum_{j=1}^{m_1-1} \tilde{\theta}_j^{2\beta_j+1} \frac{\partial}{\partial t} \tilde{u}_j
$$

$$
= \sum_{\beta_1=0}^{p-1} \frac{1}{\beta_1!} \theta_1^{\beta_1} u_1^{\beta_1} \frac{p!}{(p-\beta_1)!} \sum_{|\beta|=p-\beta_1-1} \left( \frac{p}{\beta} \right) \theta^{\beta_1} \tilde{u}^\beta \sum_{j=1}^{m_1-1} \tilde{\theta}_j^{2\beta_j+1} \frac{\partial}{\partial t} \tilde{u}_j
$$

$$
= \sum_{|\beta|=p-1} \left( \frac{p}{\beta} \right) \theta^{\beta} u^{\beta_1+1} \frac{\partial}{\partial t} u_1 + \sum_{|\beta|=p-1} \left( \frac{p}{\beta} \right) \theta^{\beta} u^{\beta} \sum_{j=2}^{m_1-1} \tilde{\theta}_j^{2\beta_j+1} \frac{\partial}{\partial t} \tilde{u}_j.
$$
Therefore, the result follows from induction.

Lemma A.2. Suppose \( m_1 \in \mathbb{N} \), \( \theta = (\theta_1, \ldots, \theta_{m_1}) \), where \( \theta_1, \ldots, \theta_{m_1} \) are positive real numbers. If \( p \in \mathbb{N} \) such that \( p \geq 2 \), then

\[
\sum_{|\beta|=p-1} \binom{p}{\beta} \theta^{\beta} \sum_{i=1}^{m_1} \theta_i^{2\beta_i+1} d_i \nabla u_i \cdot \nabla u^\beta = \sum_{|\beta|=p-2} \binom{p}{\beta} \theta^\beta \sum_{l=1}^{n} \sum_{i,j=1}^{m_1} a_{i,j} \frac{\partial u_i}{\partial x_l} \frac{\partial u_j}{\partial x_l},
\]

where \( (a_{i,j}) \) is the \( m_1 \times m_1 \) symmetric matrix with entries

\[
a_{i,j} = \begin{cases} \frac{d_i+d_j}{d_i \theta_i^{2\beta_i+4}}, & \text{if } i \neq j, \\ \frac{1}{\theta_i^{2\beta_i+4}}, & \text{if } i = j. \end{cases}
\]  

Proof. The result is easily verified when \( m_1 = 1 \), regardless of the choice of \( p \), and for \( p = 2 \), regardless of the choice of \( m_1 \). Suppose \( p \geq 2 \) and \( m_1 \geq 2 \). We proceed by induction on the value \( m_1 \), and assume \( k \in \mathbb{N} \) such that the result is true for \( m_1 = k \). Suppose \( m_1 = k + 1 \) and (as in the proof of Lemma A.1) denote

\[
\tilde{\beta} = (\beta_2, \ldots, \beta_{m_1}) \quad \text{and} \quad \tilde{u} = (u_2, \ldots, u_{m_1}).
\]

Then

\[
\sum_{|\beta|=p-1} \binom{p}{\beta} \theta^{\beta} \sum_{i=1}^{m_1} \theta_i^{2\beta_i+1} d_i \nabla u_i \cdot \nabla u^\beta = \sum_{|\beta|=p-2} \binom{p}{\beta} \theta^\beta \sum_{l=1}^{n} \sum_{i,j=1}^{m_1} a_{i,j} \frac{\partial u_i}{\partial x_l} \frac{\partial u_j}{\partial x_l},
\]

\[
+ \sum_{i=1}^{m_1} \theta_i^{2\beta_i+1} d_{i+1} \nabla \tilde{u}_i \cdot \nabla \left( u^{\beta_i+1} \tilde{u} \right). \tag{A.5}
\]

Note that for \( 1 \leq \beta_1 \leq p - 2 \) and \( |\tilde{\beta}| = p - \beta_1 - 1 \)

\[
\nabla \left( u^{\beta_1} \tilde{u} \right) = \beta_1 u^{\beta_1-1} \tilde{u} \nabla u_1 + \sum_{j=1, \beta_j \neq 0}^{m_1-1} \beta_j u^{\beta_1-1} \tilde{u} \nabla u_j. \tag{A.6}
\]

Therefore, from (A.5) and (A.6), we have

\[
\sum_{|\beta|=p-1} \binom{p}{\beta} \theta^{\beta} \sum_{i=1}^{m_1} \theta_i^{2\beta_i+1} d_i \nabla u_i \cdot \nabla u^\beta = I + \Pi, \tag{A.7}
\]

where

\[
I = \sum_{\beta_1=0}^{p-1} \sum_{|\beta|=p-\beta_1-1} \binom{p}{\beta} \theta_1^{\beta_1} \theta^{\beta_2} \left[ \theta_1^{2\beta_1+1} d_1 \nabla u_1 \cdot \nabla u^\beta \right].
\]
where the last step follows from the induction hypothesis. Now let’s investigate I. We begin
above, $e_j$ denotes row $j$ of the $(m_1 - 1) \times (m_1 - 1)$ identity matrix. We start with the
analysis of II. We can rewrite

\begin{equation}
II = \sum_{\beta_1 = 0}^{p-2} \sum_{|\beta| = p - \beta_1 - 1} \left( \frac{p}{\beta} \right) \theta_1^{\beta_2} \tilde{\theta}^{\beta_2} \sum_{i,j=1,\beta_j \neq 0}^{m_1 - 1} \tilde{\theta}_i^{2\beta_1 + 1} d_{i+1} \nabla \tilde{u}_i \cdot \beta_j u_1^{\beta_1 - 1} \tilde{u}^{\beta_i} \nabla u_1 \right) \tag{A.8}
\end{equation}

and

\begin{equation}
II = \sum_{\beta_1 = 0}^{p-2} \sum_{|\beta| = p - \beta_1 - 1} \left( \frac{p}{\beta} \right) \theta_1^{\beta_2} \tilde{\theta}^{\beta_2} \sum_{i,j=1,\beta_j \neq 0}^{m_1 - 1} \tilde{\theta}_i^{2\beta_1 + 1} d_{i+1} \nabla \tilde{u}_i \cdot \beta_j u_1^{\beta_1 - 1} \tilde{u}^{\beta_i} \nabla u_j. \tag{A.9}
\end{equation}

Above, $e_j$ denotes row $j$ of the $(m_1 - 1) \times (m_1 - 1)$ identity matrix. We start with
the analysis of II. We can rewrite

\begin{equation}
II = \sum_{\beta_1 = 0}^{p-2} \frac{1}{\beta_1!} \theta_1^{\beta_2} u_1^{\beta_1} \sum_{|\beta| = p - \beta_1 - 1} \left( \frac{p}{\beta} \right) \tilde{\theta}^{\beta_2} \sum_{i,j=1,\beta_j \neq 0}^{m_1 - 1} \tilde{\theta}_i^{2\beta_1 + 1} d_{i+1} \nabla \tilde{u}_i \cdot \beta_j u_1^{\beta_1 - 1} \tilde{u}^{\beta_i} \nabla u_j \tag{A.10}
\end{equation}

where the last step follows from the induction hypothesis. Now let’s investigate I. We begin
by expanding the $\nabla u^\beta$ term to find

\begin{equation}
I = \sum_{\beta_1 = 0}^{p-1} \sum_{|\beta| = p - \beta_1 - 1} \left( \frac{p}{\beta} \right) \theta_1^{\beta_2} \tilde{\theta}^{\beta_2} \left[ \tilde{\theta}_1^{2\beta_1 + 1} d_1 \nabla u_1 \cdot \left( \beta_1 u_1^{\beta_1 - 1} \tilde{u}^{\beta} \nabla u_1 \right) \right.
\end{equation}

\begin{equation}
+ \sum_{i=1}^{m_1 - 1} \tilde{\theta}_i^{2\beta_1 + 1} \beta_i u_1^{\beta_1 - 1} \tilde{u}^{\beta_i} \nabla u_i \right) + \sum_{i=1,\beta_i \neq 0}^{m_1 - 1} \tilde{\theta}_i^{2\beta_i + 1} d_{i+1} \nabla \tilde{u}_i \cdot \beta_i u_1^{\beta_i - 1} \tilde{u}^{\beta_i} \nabla u_1 \right]
\end{equation}

\begin{equation}
= I_{1,1} + \sum_{i=2}^{m_1} I_{i,1}, \tag{A.11}
\end{equation}

where

\begin{equation}
I_{1,1} = \sum_{\beta_1 = 1}^{p-1} \sum_{|\beta| = p - \beta_1 - 1} \left( \frac{p}{\beta} \right) \theta_1^{\beta_2} \tilde{\theta}^{\beta_2} \sum_{i,j=1}^{m_1} a_{i,j} \frac{\partial u_i}{\partial x_l} \frac{\partial u_j}{\partial x_l}
\end{equation}
The result follows by combining (A.7), (A.8), (A.10), (A.11), (A.12) and (A.13).

\[ \text{Acknowledgement.} \] This work is partially supported by NAWI Graz, and the International Research Training Group IGDK 1754 “Optimization and Numerical Analysis for Partial Differential Equations with Nonsmooth Structures”, funded by the German Research Council (DFG) project number 188264188/GRK1754 and the Austrian Science Fund (FWF) under grant number W 1244-N18.

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