Approximate Sparse Linear Regression

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Abstract

In the Sparse Linear Regression (SLR) problem, given a \( d \times n \) matrix \( M \) and a \( d \)-dimensional vector \( q \), we want to compute a \( k \)-sparse vector \( \tau \) such that the error \( \|M\tau - q\| \) is minimized. In this paper, we present algorithms and conditional lower bounds for several variants of this problem. In particular, we consider (i) the Affine SLR where we add the constraint that \( \sum \tau_i = 1 \) and (ii) the Convex SLR where we further add the constraint that \( \tau \geq 0 \). Furthermore, we consider (i) the batched (offline) setting, where the matrix \( M \) and the vector \( q \) are given as inputs in advance, and (ii) the query (online) setting, where an algorithm preprocesses the matrix \( M \) to quickly answer such queries. All of the aforementioned variants have been well-studied and have many applications in statistics, machine learning and sparse recovery.

We consider the approximate variants of these problems in the "low sparsity regime" where the value of the sparsity bound \( k \) is low. In particular, we show that the online variant of all three problems can be solved with query time \( \tilde{O}(nk^{-1}) \). This provides non-trivial improvements over the naive algorithm that exhaustively searches all \( \binom{n}{k} \) subsets \( B \). We also show that solving the offline variant of all three problems, would require an exponential dependence of the form \( \tilde{\Omega}(n^{k/2}/e^k) \), under a natural complexity-theoretic conjecture. Improving this lower bound for the case of \( k = 4 \) would imply a nontrivial lower bound for the famous Hopcroft’s problem. Moreover, solving the offline variant of affine SLR in \( o(nk^{-1}) \) would imply an upper bound of \( o(n^k) \) for the problem of testing whether a given set of \( n \) points in a \( d \)-dimensional space is degenerate. However, this is conjectured to require \( \Omega(n^d) \) time.

We also present algorithms for some special cases by exploiting the specific structures of the problems. Last but not least, our algorithms involve formulating and solving several interesting subproblems that might find applications in other areas.

1. Introduction

The goal of the Sparse Linear Regression (SLR) problem is to find a sparse linear model explaining a given set of observations. Formally, we are given a matrix \( M \in \mathbb{R}^{d \times n} \), and a vector \( q \in \mathbb{R}^d \), we want to find a vector \( \tau \) that is \( k \)-sparse (has at most \( k \) non-zero entries) and which minimizes \( \|q - M\tau\| \). The problem also has a natural query/online variant where the matrix \( M \) is given in advance (so that it can be preprocessed) and the goal is to quickly find \( \tau \) given \( q \).

Various variants of SLR has been extensively studied, in a wide range of fields including (i) statistics and machine learning [Tib96, Tib11], (ii) compressed sensing [Don06], and (iii) computer vision [WYG+09]. The query/online variant is of particular interest in the application described by Wright et al. [WYG+09], where the matrix \( M \) describes a set of image examples with known labels and \( q \) is a new image that the algorithm wants to label.

If the matrix \( M \) is generated at random or satisfies certain assumptions, it is known that a natural convex relaxation of the problem finds the optimum solution in polynomial time [CRT06, CDS98]. However, in general

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the problem is known to be NP-HARD [Nat95, DMA97], and even hard to approximate up to a polynomial factor [FKT15] (see below for a more detailed discussion). Thus, it is highly likely that any algorithm for this problem that guarantees "low" approximation factor must run in exponential time. A simple upper bound for the offline problem is obtained by enumerating \( \binom{n}{k} \) possible supports of \( \tau \) and then solving an instance of \( d \times k \) least squares problem. This results in \( n^k(d + k)^{O(1)} \) running time, which (to the best of our knowledge) constitutes the fastest known algorithm for this problem. At the same time, one can test whether a given set of \( n \) points in a \( d \)-dimensional space is degenerate by reducing it to \( n \) instances of SLR with sparsity \( d \). The former problem is conjectured to require \( \Omega(n^d) \) time [ES95]. This provides a natural barrier for any running time improvements.

In this paper, we study the complexity of the problem in the case where the sparsity parameter \( k \) is constant. In addition to the formulation above, we also consider two more constrained variants of the problem. First, we consider the **Affine SLR** where the vectors \( \tau \) is required to to satisfy \( \|\tau\|_1 = 1 \), and second, we consider the **Convex SLR** where additionally \( \tau \) should be non-negative. We focus on the approximate version of these problems, where the algorithm is allowed to output a \( k \)-sparse vector \( \tau' \) such that \( \|M\tau' - q\| \) is within a factor of \( 1 + \varepsilon \) of the optimum.

The SLR problem in equivalent to the **Nearest Linear Induced Flat (NIF)** problem defined as follows. Given a set \( P \) of \( n \) points in \( d \) dimensions and a \( d \)-dimensional vector \( q \), the task is to find a \( k \)-dimensional flat spanning a subset \( B \) of \( k \) points in \( P \) and the origin, such that the distance from \( q \) to the flat is minimized. The Affine and Convex variants of SLR respectively correspond to finding the **Nearest Induced Flat (NIF)** and the **Nearest Induced Simplex (NIS)** problems, where the goal is to find the closest \((k - 1)\)-dimensional flat/simplex spanned by a subset of \( k \) points in \( P \) to the query.

### 1.1. Related work

The computational complexity of the approximate sparse linear regression problem has been studied, e.g., in [Nat95, DMA97, FKT15]. In particular, the last paper proved a strong hardness result, showing that the problem is hard even if the algorithm is allowed to output a solution with sparsity \( k' = k2^{\log^{1-\delta} n} \) whose error is within a factor of \( n^\delta m^{1-\alpha} \) from the optimum, for any constants \( \delta, \alpha > 0 \) and \( c > 1 \).

The query/online version of the affine SLR problem can be reduced to the **Nearest k-flat Search Problem**, where the database consists of a set of \( k \)-flats (affine subspaces) of size \( N \) and the goal is to find the closest \( k \)-flat to a given query point \( q \). Let \( P \) be a set of \( n \) points in \( \mathbb{R}^d \) that correspond to the columns of \( M \). The reduction proceeds by creating a database of all \( N = \binom{n}{k} \) possible \( k \)-flats that pass through \( k \) points of \( P \). For the case of \( k = 2 \), it is known that there exists an algorithm that uses \( (dn/\varepsilon)^{O(1)} \) space and \( (1/\varepsilon + d + \log n)^{O(1)} \) query time. This yields an algorithms with space usage \( \frac{n^{12}}{\varepsilon^4} S(n^2, \varepsilon) \) and query time of \( \log^2 nTQ\left(\frac{n^2}{\varepsilon^4}, \varepsilon\right) \) [Mah15]. Similar results can be achieved for the other variants.

The SLR problem has a close relationship with the **Approximate Nearest Neighbor (ANN)** problem. In this problem, we are given a collection of \( N \) points, and the goal is to build a data structure which, given any query point \( q \), reports the data point whose distance to the query is within a \((1 + \varepsilon)\) factor of the distance of the closest point to the query. There are many efficient algorithms known for the latter problem. One of the state of the art results for **ANN** in Euclidean space answers queries in time \( (d \log(N)/\varepsilon^2)^{O(1)} \) using \( (dN)^{O(1/\varepsilon^2)} \) space [KOR00, IM98].

### 1.2. Our results

Our results fall into the following three classes:

- Algorithms to solve the online variant of all three SLR, Affine SLR and Convex SLR problems, for general value of \( k \). This is shown in the following table. The running times are denoted by \( (P(n), Q(n)) \) where \( P(n) \) denotes the space bound (and also the preprocessing time) and \( Q(n) \) denotes the query time. In the table \( S(n, \varepsilon) \) and \( TQ(n, \varepsilon) \) show the space bound (preprocessing time), and the query processing time of the best Approximate Nearest Neighbor (ANN) data-structure.
A conditional lower bound of $\Omega(n^{k/2}/(e^k \log^{(1)} n))$, for the offline variant of all three problems. Our lower bound result presented in Section 5, follows by a reduction from the $k$-sum problem which is conjectured to require $\Omega(n^{[k/2]} / \log^{(1)} n)$ time (see e.g., [PW10], Section 5). This provides further evidence that the problem requires $n^{\Omega(k)}$ time.

Finally for the special case of Convex SLR where $k = 2$, we show the following. First in Section 6 we show that the algorithm of Section 4 can be improved for the case of $\varepsilon \leq 1$, and second in Section 7, we show how to solve the offline variant of the problem in sub-quadratic time.

Our algorithms offer non-trivial tradeoffs that improve the running time (for the offline version) or the space/query time (for the online/query version) over the aforementioned naive algorithm. Moreover, our algorithms involve formulating and solving several interesting subproblems that might find applications in other areas.

### 1.3. Our techniques and sketch of the algorithms

**Nearest Flat.** To solve the NIF problem, we first, fix a set $B$ of $k-1$ points, and search for the closest $(k-1)$-flat among those that contain $B$. Note, that there are at most $n-k+1$ such flats. Each such flat $f$, as well as the query flat $Q_{\text{flat}}$ (containing $B$ and the query $q$), has only one additional degree of freedom, which is represented by a vector $v_H$ ($v_Q$, resp.) in a $d-k+1$ space. The vector $v_H$ that is closest to $v_Q$ corresponds to the flat that is closest to $q$. This can be found approximately using standard ANN data structure, resulting in an algorithm with running time $O(n^{k-1} \cdot T_Q(n, \varepsilon))$. Similarly, by adding the origin to the set $B$, we could solve the NLIF problem in a similar way.

**Nearest Simplex.** The most challenging case is to solve the convex version of the problem. To find the closest $(k-1)$-dimensional simplex, one approach would be to find the closest corresponding flat. This will not work, however, if the projection of the query point onto the closest flat falls outside of the simplex. Because of that, we need to restrict our search to the flats of feasible simplices, i.e., the simplices $S$ such that the projection of the query point onto the corresponding flat falls inside $S$. If we were given the exact distance of the flat of a simplex containing $B$ to the query point, one could determine if the simplex is feasible or not. However, this distance is not known in general. Fortunately, the feasibility property is monotone in the distance: the farther the flat of the simplex is from the query point, the weaker constraints it needs to satisfy. Thus, given a threshold value $r$, our algorithm retrieves the simplices satisfying the restrictions they need to satisfy if they were at a distance $r$ from the query. (This is done by reducing the query to a sequence of queries on canonical sets, where the canonical sets are computed from orthogonal range-searching trees, that are defined in a parameterized space of dimension $k$.) This yields a superset of feasible simplices. The algorithm then finds the closest flat corresponding to the simplices in this superset. Finally, we show that although the reported simplex may not be feasible (the projection of $q$ on to the corresponding flat does not fall inside the simplex), its distance to the query is still approximately at most $r$. Finally, we use random sampling to do binary search for finding the right value of $r$. Our result in this section follows by an intricate combination of low and high dimensional data-structures.

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1In fact, improving this lower bound for the case of $k = 4$ would imply a nontrivial lower bound for the famous Hopcroft’s problem. See Section A for the description.
Lower bound. Our reduction from $k$-sum is randomized, and based on the following idea. First observe that by testing whether the solution to SLR is zero or non-zero we can test whether there is a subset of $k$ numbers and a set of associated $k$ weights such that the \emph{weighted} sum is equal to zero. In order to solve the $k$-sum, however, we need to force the weights to be equal to 1. To this end, we lift the numbers to vectors, by extending each number by a vector selected at random from the standard basis $e_1 \ldots e_k$. We then ensure that in the selected set of $k$ numbers, each element from the basis is represented exactly once and with weight 1. This ensures that the solution to SLR yields a feasible solution to $k$-sum.

Offline Nearest Segment. For the case when the sparsity parameter is $k = 2$, we can further exploit the structure of the problem in the offline/batched variant. For the convex version of the problem, i.e., to find the closest induced segment, consider all segments that have $p_1 \in P$ as one of their endpoints. Note that the closest of them is the one with the minimum angle with the segment $qp_1$. However, we cannot afford building $n$ instances of ANN data-structure for each such $p_1$ as it will not be subquadratic. To circumvent this issue, consider the reflection $p_1'$ of $p_1$ with respect to the query point $q$, and note that the closest segment $p_1p_2$ minimizes the angle with $qp_1'$ as well. Thus we can build only one ANN data-structure around the query point for the set of points $v_p$ where for $p \in P$, $v_p$ is the intersection of a sphere around the query point with the segment $qp$. Note that this is the same data structure for each such $p_1'$. Thus, querying the point $p_1'$ to find the closest segment that has $p_1$ as one of the endpoints, and taking the best over all such segments, yields a subquadratic-time algorithm for this problem.

2. Preliminaries

2.1. Notation

Throughout the paper, we assume $P \subseteq \mathbb{R}^d$ is the set of input points which is of size $n$. Moreover, for two points $a,b \in \mathbb{R}^d$, we denote the segment formed by the two points using $ab$ and the line formed by them using $\overline{ab}$.

**Definition 2.1.** For a set of points $S$, let $f_S = \text{aff}(S) = \left\{ \sum_{i=1}^k \alpha_ip_i \mid k > 0, p_i \in S, \text{ and } \sum_{i=1}^k \alpha_i = 1 \right\}$ be the $(|S| - 1)$-dimensional flat (or $(|S| - 1)$-flat for short) passing through the points in the set $S$ (aka the affine hull of $S$). The $(|S| - 1)$-dimensional simplex ($(k - 1)$-simplex for short) that is formed by the convex-hull of the points of $S$ is denoted by $\triangle_S$. We denote the interior of a simplex $\triangle_S$ by $\text{int}(\triangle_S)$.

In this paper, for simplicity, we assume that the point-sets are non-degenerate, however this assumption is not necessary for the algorithms. Moreover, we use the notation $X \subseteq_i B$ to denote that $X$ is a subset of $B$ of size $i$, and use $0$ to denote the origin.

**Definition 2.2 (distance and nearest-neighbor).** For a point $q \in \mathbb{R}^d$, and a point $p \in \mathbb{R}^d$, we use $d(q, p) = \|q - p\|$ to denote the \emph{distance} between $q$ and $p$. For a close set $X \subseteq \mathbb{R}^d$, we denote by $d(q, X) = \min_{p \in X} \|q - p\|$ the \emph{distance} between $q$ and $X$. The point of $X$ realizing the distance between $q$ and $X$ is the \emph{nearest neighbor} to $q$ in $X$, denoted by $\text{nn}(q, X)$. We sometime refer to $\text{nn}(q, X)$ as the \emph{projection} of $q$ onto $X$.

More generally, given a finite family of such sets $G = \{X_i \subseteq \mathbb{R}^d \mid i = 1, \ldots, m\}$, the \emph{distance} of $q$ from $G$ is $d(q, G) = \min_{X \in G} d(q, X)$. The \emph{nearest-neighbor} $\text{nn}(q, G)$ is defined analogously to the above.

**Assumption 2.3.** Throughout the paper, we assume we have access to a data-structure that can answer $(1 + \varepsilon)$-ANN queries on a set of $n$ points in $\mathbb{R}^d$. We use $S(n, \varepsilon)$ to denote the space requirement of this data structure, and by $T_Q(n, \varepsilon)$ to denote the query time.

2.1.1. Induced stars, bouquets, books, simplices and flats

**Definition 2.4.** Given a point $b$ and a set $P$ of points in $\mathbb{R}^d$, the \emph{star} of $P$, with the base $b$, is the set of segments $\text{star}(b, P) = \{bp \mid p \in P \setminus \{b\}\}$. Similarly, given a set $B$ of points in $\mathbb{R}^d$, with $|B| = k - 1 \leq d$, the \emph{book} of $P$,
with the base $B$, is the set of simplices $\Delta(B, P) = \{ \Delta_{B \cup \{p\}} \mid p \in P \setminus B \}$. Finally, the set of flats induced by these simplices, is the **bouquet** of $P$, denoted by $\text{bqt}(B, P) = \{ f_{B \cup \{p\}} \mid p \in P \setminus B \}$.

If $B$ is a single point, then a book is a star, and a bouquet is a set of lines all passing through a point.

**Definition 2.5.** For a set $P \subseteq \mathbb{R}^d$, let $\mathcal{L}_k(P) = \{ f_{S \cup \{q\}} \mid S \subseteq_k P \}$ be the set of all linear $k$-flats induced by $P$, and $\mathcal{F}_k(P) = \{ f_S \mid S \subseteq_k P \}$ be the set of all $(k-1)$-flats induced by $P$. Similarly, let $\Delta_k(P) = \{ \Delta_S \mid S \subseteq_k P \}$ be the set of all $(k-1)$-simplices induced by $P$.

### 2.2. Problems

In the following, we are given a set $P$ of $n$ points in $\mathbb{R}^d$, a query point $q$ and parameters $k$ and $\varepsilon > 0$. We are interested in the following problems:

I. **NLIF** (Nearest Linear Induced Flat): Compute $\text{nn}(q, \mathcal{L}_k(P))$.

II. **ANLIF** (Approximate Nearest Linear Induced Flat): Compute a $k$-flat $f \in \mathcal{L}_k(P)$, such that $d(q, f) \leq (1 + \varepsilon)d(q, \mathcal{L}_k(P))$.

III. **NIF** (Nearest Induced Flat): Compute $\text{nn}(q, \mathcal{F}_k(P))$.

IV. **ANIF** (Approximate Nearest Induced Flat): Compute a $(k-1)$-flat $f \in \mathcal{F}_k(P)$, such that $d(q, f) \leq (1 + \varepsilon)d(q, \mathcal{F}_k(P))$.

V. **NIS** (Nearest Induced Simplex): Compute $\text{nn}(q, \Delta_k(P))$.

VI. **ANIS** (Approximate Nearest Induced Simplex): Compute a $(k-1)$-simplex $\Delta \in \Delta_k(P)$, such that $d(q, \Delta) \leq (1 + \varepsilon)d(q, \Delta_k(P))$.

Here, the parameter $k$ corresponds to the sparsity of the solution.

**Remark 2.6.** We note that the solutions of NLIF, NIF, and NIS respectively correspond to the solutions of the SLR, Affine SLR and the Convex SLR problems.

### 3. Approximating the Nearest Induced and Nearest Linear Induced Flats

#### 3.1. Approximating the nearest neighbor in a uniform star

**Input & task.** We are given a base point $b$, a set $P$ of $n$ points in $\mathbb{R}^d$, and a parameter $\varepsilon > 0$. We assume that $\|b - p\| = 1$, for all $p \in P$. The task is to build a data-structure that can report quickly, for a query point $q$ that is at distance one from $b$, the closest $(1 + \varepsilon)$-ANN segment to $q$ in $\text{star}(b, P)$.

**Preprocessing.** The algorithm computes the set $V = \{ p - b \mid p \in P \setminus \{b\} \}$, which lies on a unit sphere in $\mathbb{R}^d$.

Next, the algorithm builds a data-structure $\mathcal{D}_V$ for answering $(1 + \varepsilon)$-ANN queries on $V$.

**Answering a query.** For a query point $q$, the algorithm does the following:

(A) Compute $\tau = q - b$.

(B) Compute $(1 + \varepsilon)$-ANN to $\tau$ in $V$, denoted by $u$ using $\mathcal{D}_V$.

(C) Let $y$ be the point in $P$ corresponding to $u$.

(D) Return $\min(d(q, by), 1)$.

#### 3.1.1. Analysis

**Lemma 3.1.** Consider a base point $b$, and a set $P$ of $n$ points in $\mathbb{R}^d$ all on $S(b, 1)$, where $S = S(b, 1)$ is the sphere of radius 1 centered at $b$. Given a query point $q \in S$, the above algorithm reports correctly the $(1 + \varepsilon)$-ANN in $\text{star}(b, P)$. The query time is dominated by the time to perform a single $(1 + \varepsilon)$-ANN query.

**Proof:** If $d(q, \text{star}(b, P)) = 1$ the correctness is obvious. Otherwise, $d(q, \text{star}(b, P)) < 1$, and assume for the sake of simplicity of exposition that $b$ is the origin. Let $x$ be the nearest neighbor to $q$ in $P$, and let $x'$ be the nearest point on $bx$ to $q$. Similarly, let $y$ be the point returned by the ANN data-structure for $q$, and let $y'$ be the nearest
point on by to q. Moreover, let \( \alpha = \angle qbx \) and \( \beta = \angle qby \). As \( d(q, \text{star}(b, P)) < 1 \), we can also conclude that \( \alpha \) is smaller than \( \pi/2 \).

We have that \( \|q - x\| \leq \|q - y\| \leq (1 + \varepsilon) \|q - x\| \). Observe that \( \|q - y\| = 2 \sin(\beta/2) \), and \( \|q - x\| = 2 \sin(\alpha/2) \). As such, by the monotonicity of the sine function in the range \([0, \pi/2]\), we conclude that \( \alpha \leq \beta \). This readily implies that \( x \) is the point of \( P \) that minimizes the angle \( \angle qbx \) (i.e., \( \alpha \)), which in turn minimizes the distance to \( q \) on the induced segment. As such, we have \( d(q, \text{star}(b, P)) = \|q - x\| \).

As for the quality of approximation, first suppose that \( \beta \leq \pi/2 \). Then we have \( \|q - y\| = \|q - y\| \cos(\beta/2) \leq 1 + \varepsilon \), since \( \|q - y\| / \|q - x\| \leq 1 + \varepsilon \) and \( \alpha/2 \leq \beta/2 \leq \pi/2 \), which in turn implies that \( \cos(\beta/2) \leq \cos(\alpha/2) \).

Otherwise, we know that \( \beta > \pi/2 \) and thus \( d(q, by) = 1 \). Now if \( 1 \leq (1 + \varepsilon) \|q - x\| \) we are done. Thus, we assume that \( \|q - x\| \leq 1/(1 + \varepsilon) \). Also, we have that \( \|q - x\| \geq \|q - y\| / (1 + \varepsilon) > \sqrt{2}/(1 + \varepsilon) \) as \( \beta > \pi/2 \). This would imply that \( \sqrt{2}/(1 + \varepsilon) < \|q - x\| = \|q - x\| / \cos(\alpha/2) \leq 2/\sqrt{2}(1 + \varepsilon) \), as \( \alpha \leq \pi/2 \), which is a contradiction. Hence, the lemma holds.

\[ \square \]

3.2. Approximating the nearest flat in a bouquet

Definition 3.2. For a set \( X \) and a point \( p \) in \( \mathbb{R}^d \), let \( p' = \text{nn}(p, X) \). We use \( \text{dir}(X, p) \) to denote the unit vector \( (p - p')/\|p - p'\| \), which is the direction of \( p \) in relation to \( X \).

**Input & task.** We are given sets \( B \) and \( P \) of \( k - 1 \) and \( n \) points, respectively, in \( \mathbb{R}^d \), and a parameter \( \varepsilon > 0 \). The task is to build a data-structure that can report quickly, for a query point \( q \), the \((1 + \varepsilon)\)-ANN flat to \( q \) in \( \text{bqt}(B, P) \), see Definition 2.4.

**Preprocessing.** Let \( F = f_B \). The algorithm computes the set \( V = \{ \text{dir}(F, p), -\text{dir}(F, p) \mid p \in P \setminus B \} \), which lies on a \( d - k + 2 \) dimensional unit sphere in \( \mathbb{R}^{d-k+1} \), and then builds a data-structure \( D_V \) for answering (standard) ANN queries on \( V \).

**Answering a query.** For a query point \( q \), the algorithm does the following:

A. Compute \( \tau = \text{dir}(F, q) \).

B. Compute \( \text{ANN} \) to \( \tau \) in \( V \), denoted by \( u \) using the data-structure \( D_V \).

C. Let \( p \) be the point in \( P \) corresponding to \( u \).

D. Return the distance \( d(q, f_B \cup \{p\}) \).

3.2.1. Analysis

Definition 3.3. For two sets \( X, Y \subseteq \mathbb{R}^d \), let \( \text{proj}_X(Y) = \{ \text{nn}(q, X) \mid q \in Y \} \) be the projection of \( Y \) on \( X \).

**Observation 3.4.** Consider two affine subspaces \( F \subseteq G \) with a base point \( b \in F \), and the orthogonal complement affine subspace \( F^\perp = \{ b + \tau \mid \langle \tau, u - v \rangle = 0 \text{ for all } u, v \in F, \tau \in \mathbb{R}^d \} \). For an arbitrary point \( q \in \mathbb{R}^d \), let \( q^\perp = \text{proj}_{F^\perp}(q) \). We have that \( d(q, G) = d(q^\perp, \text{proj}_{F^\perp}(G)) \).

Using the notation of Assumption 2.3 and Definition 2.4, we have the following:

**Lemma 3.5 (ANN flat in a bouquet).** Given sets \( B \) and \( P \) of \( k - 1 \) and \( n \) points, respectively, in \( \mathbb{R}^d \), and a parameter \( \varepsilon > 0 \), one can preprocess them, using a single ANN data-structure, such that given a query point, the algorithm can compute an \((1 + \varepsilon)\)-ANN to the closest \((k - 1)\)-flat in \( \text{bqt}(B, P) \). The algorithm space and preprocessing time is \( O(S(n, \varepsilon)) \), and the query time is \( O(T_Q(n, \varepsilon)) \).
Proof: The construction is described above, and the space and query time bounds follow directly from the algorithm description. As for correctness, pick an arbitrary base point \( b \in B \), and let \( F^\perp \) be the orthogonal complement affine subspace to \( F \) passing through \( b \). Let \( P^\perp = \text{proj}_{F^\perp}(P) \), and observe that \( \text{proj}_{F^\perp}(B) \) is the point \( b \). In particular, the projection of \( B = \text{bqt}(B, P) \) to \( F^\perp \) is the bouquet of lines \( B^\perp = \text{bqt}\{\{b\}, P^\perp\} \). Applying Observation 3.4 to each flat of \( B \), implies that \( d(q, B) = d(q^\perp, B^\perp) \), where \( q^\perp = \text{proj}_{F^\perp}(q) \). Let \( S \) be the sphere of radius \( r = \|q^\perp - b\| \) centered at \( b \). Clearly, the closest line in the bouquet is the closest point to the uniform star formed by clipping the lines of \( B^\perp \) to \( S \). Since all the lines of \( B^\perp \) pass through \( b \), scaling space around \( b \) by some factor \( \alpha \), just scales the distances between \( q \) and \( B^\perp \) by \( x \). As such, scale space so that \( r = 1 \). But then, this is a uniform star with radius 1, and the algorithm of Lemma 3.1 applies (which is exactly what the current algorithm is doing). Thus, it correctly identifies a line in the bouquet that realizes the desired approximation, implying the correctness of the query algorithm.

\[ \square \]

3.3. The Result

In this section, we show a simple algorithm for the \( \text{ANIF} \) and the \( \text{ANLIF} \) problems that works for an arbitrary value of \( k \), whose query time depends on \( n^{k-1} \log^O(1) n \). Here, the input is a set \( P \) of \( n \) points in \( \mathbb{R}^d \), and \( \varepsilon > 0 \) is a prespecified approximation parameter.

Approximating the NIF. As discussed earlier, the goal is to find an approximately closest \((k - 1)\)-dimensional flat that passes through \( k \) points of \( P \), to the query. To this end, we enumerate all possible \( k - 1 \) subsets of points of \( B \subset_{k-1} P \), and build for each such base set \( B \), the data-structure of Lemma 3.5. Given a query, we compute the \( \text{ANN} \) flat in each one of these data-structures, and return the closest one found. We readily get the following.

**Theorem 3.6 (Affine SLR).** Given a set \( P \) of \( n \) points in \( \mathbb{R}^d \), and parameters \( k \) and \( \varepsilon > 0 \), one can preprocess them, such that given a query point, the algorithm can compute an \((1 + \varepsilon)\)-\( \text{ANN} \) to the closest \((k - 1)\)-flat in \( F_k(P) \), see Definition 2.5. The algorithm space and preprocessing time is \( O(n^{k-1}S(n, \varepsilon)) \), and the query time is \( O(n^{k-1}T_Q(n, \varepsilon)) \).

Approximating the NLIF. The goal here is to find an approximately closest \( k \)-dimensional flat that passes through \( k \) points of \( P \) and the origin \( 0 \), to the query. We enumerate all possible \( k - 1 \) subsets of points of \( B' \subset_{k-1} P \), and build for each base set \( B = B' \cup \{0\} \), the data-structure of Lemma 3.5. Given a query, we compute the \( \text{ANN} \) flat in each one of these data-structures, and return the closest one found.

**Theorem 3.7 (SLR).** Given the setup of Theorem 3.6, one can preprocess \( P \), such that given a query point, the algorithm can compute an \((1 + \varepsilon)\)-\( \text{ANN} \) to the closest \( k \)-flat in \( L_k(P) \), see Definition 2.5, with space and preprocessing time of \( O(n^{k-1}S(n, \varepsilon)) \), and the query time of \( O(n^{k-1}T_Q(n, \varepsilon)) \).

4. Approximating the nearest induced simplex

In this section we consider the online variant of the \( \text{ANIS} \) problem. Here, we are given the set \( P \subset \mathbb{R}^d \) of \( n \) points and the parameter \( k \), and the goal is to build a data structure, such that given a query point \( q \), it can find an \((1 + \varepsilon)\)-\( \text{ANN} \) induced \((k - 1)\)-simplex.

4.1. Simplices and distances

4.1.1. Canonical realization

In the following, we fix a sequence \( B = (p_1, \ldots, p_{k-1}) \) of \( k - 1 \) points in \( \mathbb{R}^d \). We are interested in arguing about simplices induced by \( k + 1 \) points – an additional input point \( p_k \), and a query point \( q \). Since the ambient dimension is much higher, it would be useful to have a common canonical space, where we can argue about all entities.
Definition 4.1. For a given set of points \( B \), let \( F = f_B \). Let \( p \notin F \) be a given point in \( \mathbb{R}^d \), and consider the two connected components of \( f_{B \cup \{p\}} \setminus F \), which are \textit{halfflats}. The halfflat containing \( p \) is the \textit{positive halfflat}, and it is denoted by \( f^+(B, p) \).

Fix some arbitrary point \( s^* \in \mathbb{R}^d \setminus F \), and let \( G = f^+(B, s^*) \) be a \textit{canonical} such halfflat. Similarly, for a fixed point \( s^{**} \in \mathbb{R}^d \setminus f_{B \cup \{s^*\}} \), let \( H = f^+(B \cup s^*, s^{**}) \). Conceptually, it would be convenient to consider \( H = \mathbb{R}^{k-2} \times \mathbb{R} \times \mathbb{R}^+ \), where the \( k-2 \) coordinates corresponds to \( F \), and the first \( k-1 \) coordinates corresponds to \( G \) (this can be done by applying a translation and a rotation that maps \( H \) into this desired coordinates system). This is the \textit{canonical parameterization} of \( H \).

Observation 4.2. Given a sequence of distances \( \ell = (\ell_1, \ldots, \ell_{k-1}) \), there might be only one unique point \( p = p_G(\ell) \in G \), such that \( \|p - p_i\| = \ell_i \), for \( i = 1, \ldots, k-1 \). Such a point might not exist at all\(^2\).

Consider a point \( q \in \mathbb{R}^d \setminus F \) (not necessarily the query point), and consider any positive \((k-1)\)-halfflat \( G \) that contains \( B \), and is in distance \( \ell \) from \( q \). Furthermore observe that \( q = d(q, g) < d(q, F) \). Let \( q_g \) be the projection of \( q \) to \( g \). Observe that, by the Pythagorean theorem, we have that \( d_i = \|q_g - p_i\| = \sqrt{\|q - p_i\|^2 - \ell^2} \), for \( i = 1, \ldots, k-1 \). Thus, the above observation implies, that the canonical point \( q_G(\ell) = p_G(d_1, \ldots, d_{k-1}) \) is uniquely defined (which is somewhat counterintuitive – the flat \( g \) and thus the point \( q_g \) are not uniquely defined). Similarly, there is a unique point \( q_H(\ell) \in H \), such that: (i) the projection of \( q_H(\ell) \) to \( G \) is the point \( q_G(\ell) \), (ii) \( \|q_H(\ell) - q_G(\ell)\| = \ell \), and (iii) \( \|q_H(\ell) - p_i\| = \|q - p_i\| \), for \( i = 1, \ldots, d - 1 \).

4.1.2. Orbits

Definition 4.3. Let \( \Phi_B \) be the open set of all points in \( \mathbb{R}^d \), such that their projection into \( F \) lies in the interior of \( \triangle_B = \text{ConvexHull}(B) \). The set \( \Phi_B \) is a \textit{prism}.

Consider a query point \( q \in \Phi_B \), and its projection \( q_B = \text{nn}(q, F) \). Let \( r = r_B(q) = \|q - q_B\| \) be the \textit{radius} of \( q \) in relation to \( B \). Using the above canonical parameterization, we have that \( q_G(0) = (q_B, r) \), and \( q_H(0) = (q_G(0), 0) = (q_B, r, 0) \). More generally, for \( \ell \in [0, r] \), we have

\[
q_G(\ell) = (q_B, \sqrt{r^2 - \ell^2}) \quad \text{and} \quad q_H(\ell) = (q_B, \sqrt{r^2 - \ell^2}, \ell). \tag{4.1}
\]

The curve traced by \( q_H(\ell) \), as \( \ell \) varies from 0 to \( r \), is the \textit{orbit} of \( q \) – it is a quarter circle with radius \( r \).

Lemma 4.4. (i) Let \( \hat{q}(\ell) = (\sqrt{r^2 - \ell^2}, \ell) \), and consider any point \( p = (x, 0) \), where \( x \geq 0 \). Then, the function \( d(\ell) = \|\hat{q}(\ell) - p\| \) is monotonically increasing.

(ii) Let \( q \in \Phi_B \). For any point \( p \) in the halfflat \( G \), the function \( \|q_H(\ell) - p\| \) is monotonically increasing.

\textit{Proof:} (i) \( D(\ell) = (d(\ell))^2 = \left( \sqrt{r^2 - \ell^2 - x} \right)^2 + \ell^2 = r^2 - 2x\sqrt{r^2 - \ell^2} + x^2 \), and clearly this is a monotonically increasing function for \( \ell \in [0, r] \).

(ii) Let \( p_B = \text{nn}(p, F) \), and let \( x \geq 0 \) be a number such that in the canonical representation, we have that \( p = (p_B, x) \). Using Eq. (4.1), we have

\[
D(\ell) = \|q_H(\ell) - p\|^2 = \left\| (q_B, \sqrt{r^2 - \ell^2}, \ell) - (p_B, x, 0) \right\|^2 = \|q_B - p_B\|^2 + (d(\ell))^2,
\]

and the claim readily follows from (i).

\(^2\) \textit{Trilateration} is the process of determining the location of \( p \in G \) given \( \ell \). \textit{Triangulation} is the process of determining the location when one knows the angles (not the distances).
4.1.3. Distance to a simplex via distance to the flat

Definition 4.5. Given a point \( q \), and a distance \( \ell \), let \( \triangle_G(q, \ell) \) be the unique simplex in \( G \), having the points of \( B \) and the point \( q_G(\ell) \) as its vertices. Similarly, let \( \triangle_G(q) = \triangle_G(q,0) \).

Lemma 4.6. Given a query point \( q \in \Phi_B \), and a point \( p_k \in P \setminus B \), for a number \( x > 0 \) we have that
(A) \( q_G(x) \in \triangle_G(p_k) \) and \( d(q_f(B,p_k)) \leq x \implies d(q,\triangle_{B\cup\{p_k\}}) \leq x \).
(B) \( d(q,\triangle_{B\cup\{p_k\}}) \leq x \) and \( q \in \Phi_{B\cup\{p_k\}} \implies q_G(x) \in \triangle_G(p_k) \) and \( d(q,f^+(B,p_k)) \leq x \).

Proof: (A) Let \( q^* = nn(q,f^+(B,p_k)) \). If \( q^* \in F \) then \( q^* \in \text{int}(\triangle_B) \), because \( q \in \Phi_B \). But then, \( d(q,\triangle_{B\cup\{p_k\}}) = \|q - q^*\| \leq x \), as desired.

Observe that the projection of \( q_H(x) \) to \( G \) is the point \( q_G(x) \), and since \( q_G(x) \in \triangle_G(p_k) \), it follows that \( d(q_H(x),\triangle_G(p_k)) = \|q_G(x) - q_H(x)\| \leq x \). As such, let \( \ell = d(q,f^+(B,p_k)) \), and observe that \( d(q,\triangle_{B\cup\{p_k\}}) = d(q_H(\ell),\triangle_G(p_k)) \leq d(q_H(x),\triangle_G(p_k)) = x \) since \( \ell \leq x \) and by Lemma 4.4 (ii).

(B) Since \( q \in \Phi_{B\cup\{p_k\}} \), we have that \( \ell = d(q,f^+(B,p_k)) = d(q,\triangle_{B\cup\{p_k\}}) \leq x \). We also have that \( \ell = d(q,\triangle_{B\cup\{p_k\}}) = d(q_H(\ell),\triangle_G(p_k)) \), which implies that \( q_G(\ell) = nn(q_H(\ell),\triangle_G(p_k)) \in \triangle_G(p_k) \). The Eq. (4.1) and the proof of Lemma 4.4 (ii) implies that \( q_G(y) \), for \( y \in [0,r_B(q)] \), moves continuously and monotonically on a straight segment starting at \( q_G(0) \), and as \( y \) increases, moving towards \( q_G(r) = q_B = nn(q_\ell,\triangle_B) \). Since \( x \geq \ell \), and by the convexity of \( \triangle_G(p_k) \), and the proof of Lemma 4.4 (ii), we have that \( q_G(x) \in q_Bq_G(\ell) \subseteq \triangle_G(p_k) \).

4.2. Approximating the nearest page in a book

Definition 4.7. We are given a set \( P \) of \( n \) points in \( \mathbb{R}^d \), and a sequence \( B = (k-1) \) points. We are interested in the set of simplices having \( B \) and one additional point from \( P \); that is,
\[
\Delta = \Delta(B,P) = \{ \triangle_{B\cup\{p\}} \mid p \in P \setminus B \}.
\]
We refer to the set \( \Delta \) as the book induced by \( (B,P) \), and to a single simplex in this book, as a page.

The task at hand, is to preprocess \( \Delta \) for ANN queries, as long as the nearest point lies in the interior of one of these simplices. To this end, we consider the canonical representation of this set of simplices
\[
\triangle_G = \{ \triangle_G(p) \mid p \in P \setminus B \}.
\] (4.2)

Idea. The algorithm follows Lemma 4.6 (A). Given a query point, using standard range-searching techniques, we extract a small number of canonical sets of the points, that in the parametric space their simplex contains the parameterized query point – this is described in Section 4.2.1. For each of these canonical sets, we use the data-structure of Lemma 3.5 to quickly query each one of these canonical sets for their nearest positive flat, see Remark 4.12 below, this would give us the desired ANN.

4.2.1. Reporting all simplices containing a point

Definition 4.8. Let \( B = (p_1,\ldots,p_{k-1}) \) be a sequence of \( k-1 \) points in \( \mathbb{R}^d \). For a point \( p \in \mathbb{R}^d \), consider the \( (k-1) \)-simplex \( \triangle_{B\cup\{p\}} \), which is a full dimensional simplex in the flat \( f_{B\cup\{p\}} \) (see Definition 2.1). The base angles of \( p \) (with respect to \( B \)), is the \( (k-1) \)-tuple \( \alpha_B(p) = (\alpha_1(p),\ldots,\alpha_{k-1}(p)) \), where \( \alpha_i(p) \) is the dihedral angle between the facet \( \triangle_{B\cup\{p\}\setminus\{p_i\}} \) and the base facet \( \triangle_B \). See Figure 4.1.

Observation 4.9 (Inclusion and base angles). Let \( B \) be a set of \( k-1 \) points in \( \mathbb{R}^{k-1} \) all with their \( (k-1) \)th coordinate being zero, and let \( p \) be an additional point with its \( (k-1) \)th coordinate being a positive number. Then, for a point \( q \in \mathbb{R}^{k-1} \), we have that \( q \in \triangle_{B\cup\{p\}} \iff \alpha_B(q) \leq \alpha_B(p) \) (i.e., \( \forall i : \alpha_i(q) \leq \alpha_i(p) \)).
Lemma 4.10. Given a set $n$ of $(k-1)$-simplices $\Delta_G$ in $\mathbb{R}^{k-1}$, that all share common $k-1$ vertices, then one can build a data-structure of size $O(n \log^{k-1} n)$, such that given a query point $q \in \mathbb{R}^{k-1}$, one can compute $O(\log^{k-1} n)$ disjoint canonical sets, such that the union of these sets, is the set of all simplices in $\Delta_G$ that contain $q$. The query time is $O(\log^{k-1} n)$.

Proof: By a rigid transformation of the space, we can assume that $F$ is the hyperplane $x_{k-1} = 0$, and furthermore, all the vertices of $\Delta_G$ have $x_{k-1} \geq 0$ (we can handle the simplices $x_{k-1} \leq 0$ in a similar separate data-structure). Let $B$ be the vertices of $\Delta_G$ not lying on $F$. We generate the corresponding set of base angles $P_\Delta = \{ \alpha_B(p) \mid p \in P \}$. Preprocess this set for orthogonal range searching, say, using range-trees [BCK08].

Given a query point $q$, the desired simplices correspond to all points in $p \in P$, such that $\alpha_B(p) \geq \alpha_B(q)$, which is an unbounded box query in the range tree of $P_\Delta$, with the aforementioned performance.

Remark 4.11. The data-structure of Lemma 4.10 can be used to report all simplices that contain a specific point $p$, and do not contain another point $p'$, which is vertically above $p$. This corresponds to a box query instead of quadrant query in the orthogonal data-structure. The query time and number of canonical sets remain the same.

4.2.2. Data-structure and correctness

Remark 4.12. For a set of points $P$ and a base set $B$, consider the set of positive halfspaces (the **positive bouquet** $bqt^+(B, P) = \{ f^+(B, p) \mid p \in P \setminus B \}$). We can preprocess such a set for ANN queries readily, by using the data-structure of Lemma 3.5, the modification being that every positive flat corresponds to one vector, instead of two, in the data-structure.

Preprocessing. The algorithm computes the set of canonical simplices $\Delta_G$, see Eq. (4.2). Next, the algorithm builds the data-structure of Lemma 4.10 for this set of simplices. For each canonical set $V$ in this data-structure, for the corresponding set of original points, we build the data-structure of Remark 4.12 to answer ANN queries on the positive bouquet $bqt^+(B, V)$. (Observe that the total size of these canonical sets is $O(n \log^{k-1} n)$.)

Answering a query. Given a query point $q \in \Phi_B$, the algorithm computes its projection $q_B = \text{nn}(q, F)$, where $F = f_B$. Let $r = \|q - q_B\|$ be the radius of $q$. The desired ANN distance is somewhere in the interval $[0, r]$, and the algorithm would maintain an interval $[\alpha, \beta]$ where this distance lies, and use binary search to keep pruning away on this interval, till reaching the desired approximation.

For every point $p \in P$, there is a critical value $\gamma(p)$, such that for $x \geq \gamma(p)$, the parameterized point $q_G(x)$ is inside the simplex $\Delta_G(p)$, and is outside if $x < \gamma(p)$, see Definition 4.5. By Remark 4.11, we can compute a polylogarithmic number of canonical sets, such that the union of these sets, are (exactly) all the points with critical values in the range $[\alpha, \beta]$. As long as the number of critical values is at least one, we randomly pick one of these values (by sampling from the canonical sets – one can assume each canonical set is stored in an array), and let $\gamma$ be this value. We have to decide if the desired ANN is smaller or larger than $\gamma$. To this end, we compute a representation, by polylogarithmic number of canonical sets, of all the points of $P$ such that their simplex contain the parameterized point $q_G(\gamma)$, using Lemma 4.10. For each such canonical set, we compute the approximate closest positive halfflat by Remark 4.12. Let $\tau$ be the minimum distance of such a
halfflat computed. If this distance is smaller than γ, then the desired ANN is smaller than γ, and the algorithm continues the search in the interval [α, γ), otherwise, the algorithm continues the search in the interval [γ, β).

After logarithmic number of steps, in expectation, we have an interval [α′, β′), that contains no critical value in it, and the desired ANN distance lies in this interval. We compute the ANN positive flats for all the points that their parameterized simplex contains q∗(β′), and we return this as the desired ANN distance.

**Correctness.** For the sake of simplicity, we first assume the ANN data-structure returns the exact nearest-neighbor Lemma 4.6 (A) readily implies that whatever simplex is being returned, its distance from the query point is as claimed by the data-structure. The other direction is more interesting – consider the unknown point pk, such that the (desired) nearest point to the query point lies in the interior of the simplex ΔB∪{pk}. Lemma 4.6 (B) implies that the distance to this simplex is always going to be inside the active interval, as the algorithm searches (if not, then the algorithm had found even closer simplex, which is a contradiction).

To adapt the proof to the approximate case, suppose that the data-structure returns a (1 + ε) approximate nearest-neighbor. Let P ⊆ P be the set of all points p such that the simplex ΔG(p) contains qG(γ), and let A be the set of simplices ΔB∪{p} corresponding to the points in P, and F be the set of half-flats f+(B, p) corresponding to the points in P.

Suppose that ℓ* is the minimum distance of the half-flats in F to the query, and let τ be the distance of the half-flat reported by the ANN data structure to the query. Thus, we have τ ≤ ℓ*(1 + ε). Note that, if τ < γ, we know that the optimal distance ℓ* is also less than γ and recursing on the interval [α, γ) would work as it satisfies the precondition. However, if τ ≥ γ, we know that either ℓ* ≥ γ as well, in which case the recursion would work for the same reason, or ℓ* < γ ≤ τ ≤ ℓ*(1 + ε).

In the latter case, let p be the reported point corresponding to τ. We know that the distance of the query to the half-flat f+(B, p) is τ which is at most ℓ*(1 + ε). Now, if the simplex ΔG(p) contains the point qG(τ), as well, then the distance of the query to the simplex ΔB∪{p} is equal to its distance to the half-flat f+(B, p) which is τ ≤ ℓ*(1 + ε). Therefore, we can assume that qG(τ) is outside of the simplex ΔG(p). However, in this case, the distance of the query to the simplex ΔB∪{p}, using Eq. (4.1) is at most

\[
d(q, ΔB∪{p}) = d(qH(τ), ΔG(p)) \leq d(qH(τ), qG(γ)) = \sqrt{τ^2 + d(qG(τ), qG(γ))^2} \\
\leq \sqrt{τ^2 + τ^2 - γ^2} \leq ℓ* \sqrt{2(1 + ε)^2 - 1} < ℓ*(1 + 2ε)
\]

This means that the distance of the query to the simplex ΔB∪{p} is itself approximately the closest. Thus, it is enough to change the algorithm so that at each iteration, it checks the distance of simplex it finds to the query and reports the best one found in all iterations.

**Query time.** The algorithm performs \(O(\log^{k-1} n)\) ANN queries in each iteration of the search, and the search takes \(O(\log n)\) iterations in expectation (and also with high-probability). As such, the query time is \(O(T_Q(n, ε) \log^k n)\).

**Lemma 4.13 (Approximate nearest induced page).** Given a set \(P\) of \(n\) points in \(\mathbb{R}^d\), and a set \(B\) of \(k - 1\) points, and a parameter \(ε > 0\), one can preprocess them, such that given a query point, the algorithm can compute an \((1 + ε)\)-ANN to the closest page in \(Δ(B, P)\), see Definition 4.7. The algorithm space and preprocessing time is \(O(S(n, ε) \log^k n)\), and the query time is \(O(T_Q(n, ε) \log^k n)\).

### 4.3. Result: Nearest Induced Simplex

The idea is to handle recursively the query for \(≤ (k - 2)\)-simplices induced by the given point set. As such, the remaining task is to handle the \((k - 1)\)-simplices. To this end, we generate the \(\binom{n}{k-1}\) choices for \(B \subseteq P\), and for each one of them we build the data-structure of Lemma 4.13, and query each one of them, returning the closet one found.
Remark 4.14. Note that for a set of $k$ points $A \subset_k P$, if the projection of the query onto the simplex $\Delta_A$ falls inside the simplex, i.e. $q \in \Phi_A$, then there exists a subset of $k-1$ points $B \subset_{k-1} A$ such that the projection of the query onto the simplex $\Delta_B$ falls inside the simplex, i.e., $q \in \Phi_B$. Therefore, either the recursive component of the algorithm finds an ANN, or there exists a set $B$ for which the corresponding data structure reports the correct ANN.

We thus get the following result.

**Theorem 4.15 (Convex SLR).** Given a set $P$ of $n$ points in $\mathbb{R}^d$, and parameters $k$ and $\varepsilon > 0$, one can preprocess them, such that given a query point, the algorithm can compute an $(1+\varepsilon)$-ANN to the closest $(k-1)$-simplex in $\Delta_k(P)$, see Definition 2.5. The algorithm space and preprocessing time is $O(n^{k-1}S(n, \varepsilon) \log^k n)$, and the query time is $O(n^{k-1}T_Q(n, \varepsilon) \log^k n)$.

5. Lower bound

In this section, we reduce the $k$-sum problem to offline variant of all of our problems, ANLIF, ANIF and ANIS problems, providing an evidence that the time needed to solve these problems is $\Omega(n^{k/2}/e^k)$. In the $k$-sum problem, we are given $n$ integer numbers $a_1, \ldots, a_n$. The goal is to determine if there exist $k$ numbers among them $a_{i_1}, \ldots, a_{i_k}$ such that their sum equals zero. The problem is conjectured to require $\Omega(n^{k/2}/\log^2 n)$ time, see [PW10], Section 5.

We reduce this problem as follows. Let $P = \{v_1, \ldots, v_n\}$ be a set of $n$ vectors of dimension $k + 1$. More precisely, each $v_i \in \mathbb{R}^{k+1}$ has its first coordinate equal to $a_i$ and all the other coordinates are 0 except for one coordinate chosen uniformly at random from $\{2, \ldots, k + 1\}$, whose value we set to 1. The query is also a vector of dimension $(k + 1)$ and is of the form $q = [0, 1/k, 1/k, \ldots, 1/k]^T$. We query the point $q$ and let $v_{i_1}, \ldots, v_{i_k}$ be the points corresponding to the approximate closest flat/simplex reported by the algorithm. We then check if $\sum_{j=1}^k a_{i_j} = 0$, and if so, we report $\{a_{i_1}, \ldots, a_{i_k}\}$. Otherwise, we report that no such $k$ numbers exist. Next, we prove the correctness via the following two lemmas.

**Lemma 5.1.** If there is no solution to the $k$-sum problem, that is if there is no set $\{i_1, \ldots, i_k\}$ such that $\sum_{j=1}^k a_{i_j} = 0$, then the distance of the query to the closest flat/simplex is non-zero.

**Proof:** Suppose that the distance of the query to the closest flat/simplex is zero. Thus, there exist $k$ vectors $v_{i_1}, \ldots, v_{i_k}$ and the coefficients $c_1, \ldots, c_k$, such that $c_1v_{i_1} + \cdots + c_kv_{i_k} = q$. Let $t_1, \ldots, t_k$ be the coordinates ($t_i$ has value from 2 to $(k + 1)$) such that $v_{i_j}$ has nonzero value in its $t_j$th coordinate. Note that $q$ has exactly $k$ nonzero coordinates, and each $v_{i_j}$ has exactly one non-zero coordinate from 2 to $(k + 1)$th coordinates, and there are $k$ such vectors. Thus $t_1, \ldots, t_k$ should be a permutation from 2 to $k + 1$. Therefore all $c_j$’s should be equal to 1/k. Hence, we also have that $a_{i_1} + \cdots + a_{i_k} = 0$ which is a contradiction. Therefore the lemma holds.

**Lemma 5.2.** If there exist $a_{i_1}, \ldots, a_{i_k}$ such that $\sum_{j=1}^k a_{i_j} = 0$, then with probability $e^{-k}$ the solution of the NIF/NIS/ANLIF would be a flat/simplex which contains the query $q$.

**Proof:** We consider the sum of $\frac{1}{k}v_{i_1} + \cdots + \frac{1}{k}v_{i_k}$ and show that it equals $q$ with probability $e^{-k}$. Let $t_j$ be the position (from 2 to $(k + 1)$) of the coordinate with value 1 in vector $v_{i_j}$. Then if $t_1, \ldots, t_k$ is a permutation from 2 to $k + 1$, we have that $\sum_{j=1}^k \frac{1}{k} \cdot v_{i_j} = q$ and thus the solution of the NIF/NIS/ANLIF would contain $q$ (notice that the coefficients are positive and they sum to 1, so they satisfy the required constraints). The probability that $t_1, \ldots, t_k$ is a permutation is $\frac{k!}{k^k} \approx \frac{\sqrt{2\pi k} (e/k)^k}{k^k} \geq e^{-k}$.

Therefore we repeat this process $e^k$ times, and if any of the reported flats/simplices contained the query point, then we report the corresponding solution $\{a_{i_1}, \ldots, a_{i_k}\}$. Otherwise, we report that no such $a_{i_1}, \ldots, a_{i_k}$ exists. This algorithm would report the answer correctly with constant probability by the above lemmas. Moreover as the algorithm needs to detect the case when the optimal distance is zero or not, this lower bound works for any approximation of the problem. Thus we get the following theorem.
6. Approximating the nearest induced segment

In this section, we consider the online / query variant of the ANIS problem for the case of \( k = 2 \). That is, given a set of \( n \) points \( P \in \mathbb{R}^d \), the goal is to preprocess this set, such that given a query point \( q \), we can find the approximate closest segment formed by any of the two points in \( P \) to \( q \).

6.1. Approximating the nearest neighbor in a star

Here, we show how to handle the non-uniform star case – namely, we have a set of \( n \) segments (of arbitrary length), all sharing an endpoint, and given a query point, we would like to compute quickly the \( \text{ANN} \) on the star to this query.

6.1.1. Preliminaries

Lemma 6.1. Let \( P = \{p_1, \ldots, p_n\} \) be an ordered set of \( n \) points in \( \mathbb{R}^d \). Given a data-structure that can answer \((1 + \varepsilon)\)-\( \text{ANN} \) queries for \( n \) points in \( \mathbb{R}^d \), with space \( S(n, \varepsilon) \) and query time \( T_Q(n, \varepsilon) \), then one can build a data-structure, such that for any integer \( i \) and a query \( q \), one can answer \((1 + \varepsilon)\)-\( \text{ANN} \) queries on the prefix set \( P_i = \{p_1, \ldots, p_i\} \). The query time and space requirement becomes \( O(S(n, \varepsilon) \log n) \) and \( O(T_Q(n, \varepsilon) \log n) \), respectively.

Proof: This is a standard technique used for example in building range trees [BCKO08]. Build a balanced binary tree over \( P \), with the leaves of the tree ordered in the same way as in \( P \). Build for any internal node in this tree, the \( \text{ANN} \) data-structure for the canonical set of points stored in this subtree. Now, an \((i, q)\)-\( \text{ANN} \) query, can be decomposed into \( \text{ANN} \) queries over \( O(\log n) \) such canonical sets. For each canonical set, the algorithm uses the \( \text{ANN} \) data-structure built for it. The algorithm returns the best \( \text{ANN} \) found.

Given a set of points \( P \), and a base point \( c \), its star is the collection of segments \( \text{star}(c, P) = \bigcup_{y \in P} cy \) as defined in Definition 2.4. The set of points in such a star at distance \( r \) from \( c \), is the set of points

\[
\ominus(c, r, P) = \left\{ v \mid y \in P, \|c - y\| \geq r, \text{ and } v = cy \cap S(c, r) \right\},
\]

where \( S(c, r) \) denotes the sphere of radius \( r \) centered at \( c \).

Lemma 6.2. Let \( c \) be a point, and let \( P \) be a set of \( n \) points in \( \mathbb{R}^d \). Given a data-structure that can answer \((1 + \varepsilon)\)-\( \text{ANN} \) queries for \( n \) points in \( \mathbb{R}^d \), with space \( S(n, \varepsilon) \) and query time \( T_Q(n, \varepsilon) \), then one can build a data-structure, such that given a query point \( q \) and radius \( r \), one can answer \((1 + \varepsilon)\)-\( \text{ANN} \) queries on the set of points \( \ominus(c, r, P) \) (see Eq. (6.1)). The query time and space bounds are \( O(S(n, \varepsilon) \log n) \) and \( O(T_Q(n, \varepsilon) \log n) \), respectively.

Proof: Let \( P = \{p_1, \ldots, p_n\} \) be the ordering of \( P \) such that the points are in decreasing distance from \( c \). Let \( V = \{v_1, \ldots, v_n\} \), be the ordering of the direction vectors of the points of \( P \), where \( v_i = (p_i - c)/\|p_i - c\| \). Build the data-structure of Lemma 6.1 on the (ordered) point set \( V \).

Given a query point \( q \), using a balanced binary search tree, find the maximal \( i \), such that \( \ell = \|c - p_i\| \geq r \). Compute the affine transformation \( M(q) = (q - c)/r \), and compute the \((1 + \varepsilon)\)-\( \text{ANN} \) to \( M(q) \) in \( V_i \), and let \( v_j \) be this point. The algorithm returns \( p_j \) is the desired \( \text{ANN} \).

To see why this procedure is correct, observe that \( \ominus(c, r, P_i) = \ominus(c, r, P) \), where \( P_i = \{p_1, \ldots, p_i\} \). Furthermore, \( V_i = \{v_1, \ldots, v_i\} = M(\ominus(c, r, P_i)) \). In particular, since \( M \) is only translation and scaling, it preserves order between distances of pairs of points. In particular, if \( y \) is the \( \text{ANN} \) to \( q \) in \( \ominus(c, r, P) \), then \( M(y) \) is an \( \text{ANN} \) to \( M(q) \) in \( M(\ominus(c, r, P)) = V_i \), implying the correctness of the above.
6.1.2. The query algorithm

The algorithm in detail.

We are given a set of $n$ points $P$ and a center point $c \in P$, and we are interested in answering $(1 + \varepsilon)$-ANN queries on $\star = \text{star}(c, P)$.

**Preprocessing.** We sort the points of $P$ to be in decreasing distance from $c$, and let $P = \{p_1, \ldots, p_n\}$ be the points in this sorted order. We build the data-structure $D$ of Lemma 6.2 for the points of $P$ to answer $(1 + \varepsilon/4)$-ANN queries.

**Answering a query.** Given a query point $q$, let $r = \|q - c\|$. As a first step, we perform $(1 + \varepsilon/4)$-ANN query on $P$. Next, let $r_i = i(\varepsilon^2/16)r$, for $i = 1, \ldots, N = 32/\varepsilon^2$. For each $i$, find the $(1 + \varepsilon/4)$-ANN in the point set $\odot(c, r_i, P)$, using the data-structure $D$. Return the nearest point to $q$ found as the desired ANN.

6.1.3. Correctness

Observe that $d(q, \star) \leq r$, since $c \in P \subseteq \star = \text{star}(c, P)$.

**Lemma 6.3.** If $d(q, \star) \geq r\sqrt{\varepsilon}/2$ then the above algorithm returns a point $p \in \star$, such that $\|q - p\| \leq (1 + \varepsilon)d(q, \star)$, for any $0 < \varepsilon \leq 1$.

*Proof:* Let $\ell = d(q, \star)$. Consider the point set $U = \bigcup_{i=1}^{N} \odot(c, r_i, P)$. The algorithm effectively performs $(1 + \varepsilon/4)$-ANN query over this point set. So, let $q' = \text{nnd}(q, \star)$, and let $y \in P$ be the point, such that $q' \in cy$.

By construction, there is a point $u \in U \cap cy$, such that $\|u - q'\| \leq (\varepsilon^2/16)r$. Namely, we have $d(q, U) \leq d(q, \star) + (\varepsilon^2/16)r$. As such, the $(1 + \varepsilon/4)$-ANN point returned for $q$ in $U$, is in distance at most

$$(1 + \varepsilon/4)\|q - u\| \leq (1 + \varepsilon/4)(\|q - q'\| + (\varepsilon^2/16)r) \leq (1 + \varepsilon/4)\|q - q'\| + (\varepsilon^2/8)r \leq (1 + \varepsilon/2)d(q, \star),$$

since $d(q, cy) = d(q, \star)$.

The above lemma implies (conceptually) the hard case is when the ANN distance is small (i.e., $\leq O(\sqrt{\varepsilon} r)$). The intuition is that in this case the (regular) ANN query on $P$ and $\odot(c, r, P)$ would “capture” this distance, and returns us the correct ANN. The following somewhat tedious lemma testifies to this effect.

**Lemma 6.4.** Let $c, q$ be two given points, where $r = \|q - c\|$, and let $0 < \varepsilon \leq 1$ be a parameter. Consider a set of $n$ points $Z = \{p_1, \ldots, p_n\} \subseteq b(c, r)$, where $b(c, r)$ denotes the ball centered at $c$ of radius $r$. Let $\star = \text{star}(c, Z)$, and assume that $d(q, \star) \leq r\sqrt{\varepsilon}/2$. Then, we have that $d(q, Z) \leq (1 + \varepsilon/4)d(q, \star)$.

*Proof:* Observe that by definition $d(q, \star) \leq d(q, Z)$. Let $q' = \text{nnd}(q, \star)$, and consider the clipped cone $C$ of all points $y \in Z$ in distance at most $r$ from $c$, such that $\angle ycq \leq \phi = \sqrt{\varepsilon}/2$.

Assume that $q'$ is in $C$. Then, there must be a point $y \in Z \cap C$ such that $q' \in cy$. Observe that $d(q, \star) = \|q - q'\| = d(q, cy)$. If $d(q, cy) = \|q - y\| \geq d(q, Z)$, which implies that $d(q, Z) = d(q, \star)$, and we are done.

As such, consider the case that $d(q, cy) < \|q - y\|$, which is depicted in Figure 6.1. Clearly, $\beta = \angle ycq$ is minimized when $\|c - y\| = r$, and $\angle ycq = \phi$. But then, this angle is equal to $\pi/2 - \phi/2$, which implies that $\beta \geq \pi/2 - \phi/2$.

Now, $\|q - q'\| = \|q - y\| \sin \beta$, and $\|q - y\| \leq 2r \sin(\phi/2)$. As such, $\Delta = \|q - q'\| = \|q - y\| \sin \beta - \|q - y\| \leq \|q - y\| \left| 1 - \sin \beta \right|$. In particular, we have $\left| 1 - \sin \beta \right| \leq 1 - \sin(\pi/2 - \phi/2) = 1 - \cos(\phi/2) \leq 1 - (1 - (\phi/2)^2/2) = \phi^2/8 = \varepsilon/32$, since $\cos(x) \geq 1 - x^2/2$, for $x \leq \pi/4$. We conclude that

$$\|q - y\| - \|q - q'\| \leq (\varepsilon/32)\|q - y\|,$$
which implies that $(1 - \varepsilon/32)\|q - y\| \leq \|q - q'\|$. But then, we have \(d(q, Z) \leq \|q - y\| \leq \frac{1}{1-\varepsilon/32}\|q - q'\| \leq (1 + \varepsilon/16)d(q, \star)\), implying the claim.

The remaining case is that \(q'\) is in \(b(c, r) \setminus C\). This implies that \(d(q, \star) \geq d(q, b(c, r) \setminus C)\), which is at least \(r\sin \phi \geq \sqrt{\varepsilon}/4\), since \(\sin x \geq x/2\), for \(x \leq 1/2\). But this is a contradiction to the assumption that \(d(q, \star) \leq r\sqrt{\varepsilon}/2\).

\begin{lemma}
The query algorithm of Section 6.1.2 returns a \((1 + \varepsilon)\)-ANN to \(d(q, \star)\) for \(0 < \varepsilon \leq 1\).
\end{lemma}

\begin{proof}
If \(d(q, \star) \geq r\sqrt{\varepsilon}/2\) then the claim follows from Lemma 6.3. Otherwise, consider the point \(p \in P\), such that \(nn(q, \star)\) lies on \(cp\). If \(\|c - p\| \leq r\), then by Lemma 6.4, the \((1 + \varepsilon/4)\)-ANN query on \(b(c, r) \cap P \subseteq P\) would return the desired ANN. (Formally, the result is worse by a factor of \((1 + \varepsilon/4)\), which is clearly smaller than the desired threshold of \((1 + \varepsilon)\).) Note, that the algorithm performs ANN query on \(P\), and this resolves this case.

As such, we remain with the long case, that is \(\|c - p\| > r\). But then, \(q' = nn(q, \star)\) must lie inside \(b(c, r)\), where \(\star = \text{star}(c, P)\). As such, in this case \(nn(q, \star) = nn(q, \text{star}(c, \cap(c, r, P)))\). As such, again by Lemma 6.4, the \((1 + \varepsilon/4)\)-ANN to \(\cap(c, r, P)\) is the desired approximation. Note, that the algorithm performs explicitly an ANN query on this point-set, implying the claim.
\end{proof}

\textbf{Running time analysis and result.} The algorithm built the data-structure of Lemma 6.1 once, and we performed \(O(1/\varepsilon^2)\) ANN queries on it. This results in \(O((\log n)/\varepsilon^2)\) queries on the original ANN data-structure. We thus conclude the following:

\begin{lemma}[ANN in a star]
Let \(P = \{p_1, \ldots, p_n\}\) be a set of \(n\) points in \(\mathbb{R}^d\) and let \(0 < \varepsilon \leq 1\) be a parameter. Given a data-structure that can answer \((1 + \varepsilon)\)-ANN queries for \(n\) points in \(\mathbb{R}^d\), using \(S(n, \varepsilon)\) space and \(T_Q(n, \varepsilon)\) query time, then one can build a data-structure, such that for any query point \(q\), it returns the \((1 + \varepsilon)\)-ANN to \(q\) in \(\star = \text{star}(c, P)\). The space needed is \(O(S(n, \varepsilon)\log n)\), and the query time is \(O(T_Q(n, \varepsilon)\varepsilon^{-2} \log n)\).
\end{lemma}

\subsection{The result – approximate nearest induced segment}

By building the data-structure of Lemma 6.6 around each point of \(P\), and querying each of these data-structures, we get the following result.

\begin{theorem}
Let \(P = \{p_1, \ldots, p_n\}\) be a set of \(n\) points in \(\mathbb{R}^d\) and let \(0 < \varepsilon \leq 1\) be a parameter. Given a data-structure that can answer \((1 + \varepsilon)\)-ANN queries for \(n\) points in \(\mathbb{R}^d\), using \(S(n, \varepsilon)\) space and \(T_Q(n, \varepsilon)\) query time, then one can build a data-structure, such that for any query point \(q\), it returns a segment induced by two points of \(P\), which is \((1 + \varepsilon)\)-ANN to the closest such segment. The space needed is \(O(S(n, \varepsilon)n\log n)\), and the query time is \(O(T_Q(n, \varepsilon)n\varepsilon^{-2} \log n)\).
\end{theorem}
7. Offline nearest induced segment problem

In this section, we consider the offline variant of the NIS problem, for the case of \( k = 2 \). We present an algorithm which achieves constant factor approximation and has sub-quadratic running time.

**Input & task.** We are given a set of \( n \) points \( P = \{p_1, \cdots, p_n\} \) along with the query point \( q \) and a parameter \( \varepsilon > 1 \). The task is to find the \((1 + \varepsilon)\)-approximate closest segment \( p_ip_j \) to the query point \( q \), where \( p_i, p_j \in P \).

**Algorithm.** Let \( r = \min_i \|p_i - q\| \) be the distance of the query to the closest point in \( P \) which can be found in \( O(n) \) time. Let \( * = \text{star}(q, P) \), and let \( V = \otimes(q, r, P) = \{v_i \mid v_i = qp_i \cap S(q, r)\} \), as defined previously, be the intersection of the star \( * \) with the sphere of radius \( r \) around the query. Next, the algorithm builds a data-structure \( D_V \) for answering \((1 + \varepsilon)\)-ANN queries on \( V \).

Next, the algorithm queries this data structure \( n \) times. For each point \( p_i \in P \), let \( p'_i = q - r \frac{p_i - q}{\|p_i - q\|} \), which is the reflected and scaled projection of \( p_i \) with respect to \( q \). More precisely, \( p'_i \) is one of the two intersections of the line \( qp_i \) with the sphere \( S(q, r) \) that is on the other side of the query than \( p_i \) (see Figure 7.1). The algorithm queries the data structure \( D_V \) for the point \( p'_i \), and let \( v_j \in V \) be the approximate nearest neighbor of \( p'_i \). The algorithm then computes the distance of the query \( q \) with the segment \( p_ip_j \) to see if it improves the closest segment found so far.

**Lemma 7.1.** The space usage of the algorithm is \( O(S(n, \varepsilon)) \) and running time of the algorithm is of the form \( O(nT_Q(n, \varepsilon) + S(n, \varepsilon)) \). If we use the result of [AI06] for the ANN in the Euclidean space, the running time of the algorithm would be \( n^{1+O(\frac{1}{(1+\varepsilon)^2})} \), i.e., sub-quadratic in \( n \).

### 7.1. Correctness

**Lemma 7.2.** The algorithm reports a segment whose distance to the query is within a factor of \( O(\varepsilon) \) of the distance of the closest segment to the query, assuming \( \varepsilon \geq 1 \).

**Proof:** Let \( s^* = p_1p_2 \) be the closest segment to the query point \( q \) and suppose that it is formed by the two points \( p_1 \) and \( p_2 \) and assume that \( p_1 \) is closer to \( q \) than \( p_2 \). Also, let \( p'_1 \) be the scaled reflection of the point \( p_1 \) as defined in the algorithm.

Note that we can assume that \( d(q, s^*) \leq r/(4\varepsilon) \), otherwise any segment formed by the closest point to the query is itself an \( O(\varepsilon) \) factor approximation to the optimal segment, as its distance to query is at most \( r \). Thus, the segment \( s^* \) intersects with the sphere \( S(q, r) \), and thus the closest point on the segment formed by \( p_1 \) and \( p_2 \) lies in the interior of the segment.

![Figure 7.1: An illustration of the points.](image-url)
Let \( v_2 \) be the intersection of the sphere with the segment \( qp_2 \) as defined in the algorithm. Now consider the iteration in which we are querying the point \( p_1' \) and suppose that \( v_2 \) has not been reported as the ANN of \( p_1' \). Instead, suppose that some other point such as \( v_3 \) has been reported as the ANN of \( p_1' \) (see Figure 7.1). Thus, we have that \( \|p_1' - v_3\| \leq \|p_1' - v_2\| (1 + \varepsilon) \). We will prove that the distance of \( q \) to the line \( \ell_3 = p_1p_3 \) is an \( O(\varepsilon) \) factor approximation of the optimal distance of \( q \) to \( \ell_2 = p_1p_2 \). It will also shortly be clear that the closest point on the line \( \ell_3 \) will be inside the segment \( p_1p_3 \), which completes the proof.

Let \( \ell_2' \) and \( \ell_3' \) be two lines that pass through \( p_1 \) such that \( \ell_2' \) is parallel to \( \overline{qp_2} \) and \( \ell_3' \) is parallel to \( \overline{qv_3} \).

**Claim 7.3.** We have that \( d(q, \ell_3) \leq d(q, \ell_3') \)

**Proof:** Let \( x \) be the intersection of the segment \( p_1q \) with the sphere \( S(q, r) \) (reflection of \( p_1' \)). Then clearly, \( d(q, \overline{qv_2}) \leq d(q, \ell_2) \leq r/(4\varepsilon) \). Therefore, we have \( \|p_1' - v_2\| \leq r/(2\varepsilon) \). Furthermore, by the fact that \( \varepsilon \geq 1 \), we have \( \|p_1' - v_3\| \leq (1 + \varepsilon)r/(2\varepsilon) \leq r \). Therefore, the angle \( \angle p_1'q_3 \) is at most \( \pi/3 \). Note that this shows that the closest point on the line \( \ell_3 \) lies on the segment formed by \( p_1p_3 \) as promised earlier.

Next, consider the triangle \( \triangle qp_1p_3 \) and note that the angle of \( \ell_3 \) with \( \overline{p_1p_1'} \), i.e., \( \angle q_1p_3 \leq \angle p_1q_3 \). Since \( \ell_3' \) is parallel to \( \overline{qv_3} \), we get that \( d(q, \ell_3) \leq d(q, \ell_3') \).

**Claim 7.4.** \( d(q, \ell_2') \leq 2d(q, \ell_2) \)

**Proof:** First consider the triangle \( \triangle q_1p_2p_1 \). Since by assumption, we have \( \|q - p_1\| \leq \|q - p_2\| \), then we have that \( \angle q_1p_2p_1 \geq \angle q_2p_1p_1 \). Therefore, since the angle between \( \overline{qp_1} \) and the line \( \ell_2' \) is equal to \( \angle p_1q_3 \), it is bounded by at most \( \angle q_1p_2 + \angle q_2p_1 \leq 2\angle q_1p_2p_1 \). Hence, the claim follows.

**Claim 7.5.** \( d(q, \ell_3) \leq 2(1 + \varepsilon)d(q, \ell_2') \)

**Proof:** First note that we have the following equalities.

\[
\frac{d(q, \ell_3)}{d(q, \ell_2')} = \frac{\|q - p_1\| \sin(\overline{qp_1}, \ell_3')}{\|q - p_1\| \sin(\overline{qp_1}, \ell_2')} = \frac{\|q - p_1\| \sin(\angle p_1q_3)}{\|q - p_1\| \sin(\angle p_1q_2)} = \frac{d(p_1, \overline{qp_3})}{d(p_1, \overline{qp_2})}
\]

where \( \sin(l_1, l_2) \) used in the second term above, denotes the angle between the two lines \( l_1 \) and \( l_2 \). Also, we have that

\[
d(p_1, \overline{qp_3}) \leq \|p_1' - v_3\| \leq (1 + \varepsilon)\|p_1' - v_2\| \leq (1 + \varepsilon)2d(p_1', \overline{qp_2})
\]

where the last inequality holds by a similar argument as in the proof of Claim 7.3 showing that the angle \( \angle p_1'q_2 \) is also less than \( \pi/3 \).

The above three claims prove the lemma.

\[
d(q, \ell_3) \leq d(q, \ell_3') \leq 2(1 + \varepsilon)d(q, \ell_2') \leq 4(1 + \varepsilon)d(q, \ell_2)
\]

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[BCKO08] M. de Berg, O. Cheong, M. van Kreveld, and M. H. Overmars. *Computational Geometry: Algorithms and Applications*. Springer-Verlag, Santa Clara, CA, USA, 3rd edition, 2008.
A. Connection to Hopcroft’s problem

In the Hopcroft’s problem, we are given two sets \( U \) and \( V \), each consisting of \( N \) vectors in \( \mathbb{R}^d \) and the goal is to check whether there exists \( u \in U \) and \( v \in V \) such that \( u \) and \( v \) are orthogonal.

Given an instance of the Affine SLR with \( n \) points and \( k = 4 \), we proceed as follows. Suppose that the input is a set of \( n \) points in \( \mathbb{R}^d \) and the query is the origin \( \mathbf{0} \). Moreover, suppose that \( d = 4 \). So the goal is to decide whether there exist four points \( a, b, c, d \) such that the three dimensional flat that passes through them also passes through the origin. This is equivalent to checking whether the determinant of the matrix which is formed by concatenating these four points as its columns, is zero or not. We can pre-process pairs of points to solve it fast.

Take all \( \binom{n}{2} \) pairs of points \( a = (a_1, a_2, a_3, a_4) \), \( b = (b_1, b_2, b_3, b_4) \) and preprocess them by constructing a vector \( u \) in 24 dimensional space such that \( u = (a_1b_2, -a_1b_2, -a_1b_3, a_1b_3, \cdots, -a_4b_3, a_4b_3) \) and let \( U \) be
the set of such vectors $u$. Also, for each pair of points $c = (c_1, c_2, c_3, c_4)$, $d = (d_1, d_2, d_3, d_4)$, we construct $v = (c_3d_4, c_4d_3, c_2d_4, c_4d_2, \ldots, c_1d_2, c_2d_1)$ and let $V$ be the set of $\binom{n}{2}$ such vectors $v$. It is easy to check that the determinant is zero, if and only if the inner product of $u$ and $v$ is zero.

Thus, we have two collections $U$ and $V$ of $\binom{n}{2}$ vectors in $\mathbb{R}^{24}$ and would like to check whether there exists two points, one from each collection, that are orthogonal. So any lower bound better than $\omega(n^{k/2}) = \omega(n^2)$ would imply a super linear lower bound $\omega(N)$ for the Hopcroft’s problem.