Abstract. In this paper, we show that the homotopy category of $N$-complexes of projective $R$-modules is triangle equivalent to the homotopy category of projective $\mathbb{T}_{N-1}(R)$-modules where $\mathbb{T}_{N-1}(R)$ is the ring of triangular matrices of order $N-1$ with entries in $R$. We also define the notions of $N$-singularity category and $N$-totally acyclic complexes.

1. Introduction

Given an associative unitary ring $R$, by an $N$-complex $X^*$, we mean a sequence of $R$-modules and $R$-linear maps $\cdots \rightarrow X^{n-1} \rightarrow X^n \rightarrow X^{n+1} \rightarrow \cdots$ such that composition of any $N$ consecutive maps gives the zero map. The notion of $N$-complexes first appeared in the paper [Kap96] by Kapranov. Besides their applications in theoretical physics [CSW07], [Hen08], the homological properties of $N$-complexes have become a subject of study for many authors as in [DV98], [Est07], [Gil12], [GH10], [Tik02]. Iyama and et. al. studied the homotopy category $\mathbb{K}_N(B)$ of $N$-complexes of an additive category $B$ as well as the derived category $\mathbb{D}_N(A)$ of an abelian category $A$. Recall that an abelian category $A$ is an $(Ab4)$-category (resp. $(Ab4)^*$-category) provided that it has any coproduct (resp. product) of objects, and that the coproduct (resp. product) of monomorphisms (resp. epimorphisms) is monic (resp. epic). In the paper, [IKM14], they showed that the well known equivalences between homotopy category of chain complexes and their derived categories also generalize to the case of $N$-complexes. More precisely, if $A$ is an abelian category satisfying the condition $(Ab4)$, then we have triangle equivalence

$$\mathbb{K}_N^3(\text{Prj}-A) \cong \mathbb{D}_N^3(A).$$

where $(1, 1) = (\text{proj}, \text{nothing}), (-, -), ((-, b), b)$ and $\text{Prj}-A$ is the category of projective objects of $A$. As for chain complexes a similar statement is also true for the category $\text{Inj}-A$ of injective objects of $A$ provided that $A$ satisfies the condition $(Ab4)^*$. They also showed that there exists a triangle equivalence

$$\mathbb{D}_N(A) \cong \mathbb{D}(\mathbb{T}_{N-1}(A)).$$

As a consequence of this equivalence they showed that there exists the following triangle equivalences between derived and homotopy categories.
Corollary 1.1. For a ring $R$, we have the following triangle equivalences.

$$D^b_N(\text{Mod}-R) \cong D^b(\text{Mod}-\mathbb{T}_{N-1}(R)),$$

where $i = -, b$.

$$\mathbb{K}_N^i(\text{Prj}-R) \cong \mathbb{K}_N^i(\text{Prj}-\mathbb{T}_{N-1}(R)),$$

where $i = -, b, (-, b)$ and also

$$\mathbb{K}_N^i(\text{prj}-R) \cong \mathbb{K}_N^i(\text{prj}-\mathbb{T}_{N-1}(R)),$$

where $i = -, b, (-, b)$.

In this paper, we show that the homotopy category $\mathbb{K}_N(\text{Prj}-R)$ of $N$-complexes is embedded in the ordinary homotopy category $\mathbb{K}(\text{Prj}-\mathbb{T}_{N-1}(R))$. Having this embedding in hand we are able recover (1.1) by using different techniques than those in [IKM14]. We also show that $\mathbb{K}_N(\text{Prj}-R)$ is equivalent to $\mathbb{K}(\text{Prj}-\mathbb{T}_{N-1}(R))$ whenever $R$ is a left coherent ring.

The explicit construction of such triangle equivalence allows us to prove an $N$-complex version of the following equivalence of triangulated categories given in [Buc87], [Hap91], [BJO14].

$$\mathbb{K}_{\text{tac}}(\text{prj}-R) \rightarrow D^b_{sg}(R)$$

where $\mathbb{K}_{\text{tac}}(\text{prj}-R)$ is the homotopy category of totally acyclic complexes of finitely generated projective $R$-modules and $D^b_{sg}(R)$ is the singularity category.

The paper is organized as follows. In section 2, we recall some generalities on $N$-complexes and provide any background information needed through this paper. Our main result appears in section 3 as Theorem 3.17. In that section, we show that the category $\mathbb{K}_N(\text{Prj}-R)$ embeds as a triangulated subcategory in the category $\mathbb{K}(\text{Prj}-\mathbb{T}_{N-1}(R))$ see proposition 3.9. As an application of this embedding we provide a different proof for the triangle equivalence in (1.1). At the end of this section we show that this embedding is also dense, hence an equivalence.

In section 4 we define an $N$-totally acyclic complex as a complex $X^\bullet$ in $\text{prj}-R$ satisfying the property that for all $P^\bullet \in \mathbb{K}_{N}(\text{prj}-R)$, $\text{Hom}_{\mathbb{K}_{N}(\text{prj}-R)}(P^\bullet, X^\bullet) = \text{Hom}_{\mathbb{K}_{N}(\text{prj}-R)}(X^\bullet, P^\bullet) = 0$. Then we show that the homotopy category $\mathbb{K}_N^{\text{ac}}(\text{prj}-R)$ of $N$-totally acyclic complexes in $\text{prj}-R$ in this sense is triangle equivalent to the homotopy category of ordinary totally acyclic complexes in $\text{prj}-\mathbb{T}_{N-1}(R)$, i.e. $\mathbb{K}_{\text{tac}}(\text{prj}-\mathbb{T}_{N-1}(R))$. We also define a similar notion of singularity category for $N$-complexes $D_N^{\text{sg}}(R)$ and show that it contains $\mathbb{K}_N^{\text{ac}}(\text{prj}-R)$ as a triangulated subcategory. Furthermore, the embedding

$$\mathbb{K}_N^{\text{tac}}(\text{prj}-R) \rightarrow D_N^{\text{sg}}(R)$$

is an equivalence of triangulated categories, when $R$ is a Gorenstein ring.

2. Preliminaries

2.1. The category of $N$-complexes. Throughout, $R$ is an associative ring with identity. $\text{Mod}-R$ denotes the category of all right $R$-modules. We fix a positive integer $N \geq 2$. An $N$-complex $X^\bullet$ is a diagram

$$\cdots \xrightarrow{d_{X^i}^{i-1}} X^i \xrightarrow{d_{X^i}^i} X^{i+1} \xrightarrow{d_{X^{i+1}}^i} \cdots$$

with $X^i \in \text{Mod}-R$ and morphisms $d_{X^i}^i \in \text{Hom}_R(X^i, X^{i+1})$ satisfying $d^N = 0$. That is, composition of any $N$-consecutive maps is 0. A morphism between $N$-complexes is a commutative
We denote by $\mathcal{C}_N(R)$ (resp. $\mathcal{C}_N^{-}(R)$, $\mathcal{C}_N^{+}(R)$, $\mathcal{C}_N^{b}(R)$) the category of unbounded (resp. bounded above, bounded below, bounded) $N$-complexes over $\text{Mod-}R$.

For any object $M$ of $\text{Mod-}R$, $j \in \mathbb{Z}$ and $1 \leq i \leq N$, let

$$D^j_i(M) : \cdots \longrightarrow 0 \longrightarrow X^{j-i+1} \overset{d^{j-i+1}}{\longrightarrow} \cdots X^{j-1} \overset{d^{j-1}}{\longrightarrow} X^j \overset{d^j}{\longrightarrow} 0 \longrightarrow \cdots$$

be an $N$-complex satisfying $X^n = M$ for all $j - i + 1 \leq n \leq j$ and $d^n_X = 1_M$ for all $j - i + 1 \leq n \leq j - 1$.

For $0 \leq r < N$ and $i \in \mathbb{Z}$, we define

$$d^i_X \cdot (r) := d^{i+r-1} \cdots d^i_X.$$

In this notation $d^i_X \cdot (1) = d^i_X$ and $d^i_X \cdot (0) = 1_X$.

**Definition 2.1.** Let $f : X^\bullet \longrightarrow Y^\bullet$ be a morphism in $\mathcal{C}_N(R)$. The mapping cone $C(f)$ of $f$ is defined as follows

$$C(f)^m = Y^m \oplus \bigoplus_{i=m+1}^{m+N-1} X^i, \quad d^m_{C(f)} = \begin{bmatrix} d^m_{Y^\bullet} & f^{m+1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & -d^{m-1}_{X^\bullet(N-1)} & -d^{m+2}_{N-1} & \cdots & -d^{m+N-1}_{X^\bullet} \end{bmatrix}.$$

Let $\mathcal{S}_N(R)$ be the collection of short exact sequences in $\mathcal{C}_N(R)$ of which each term is split short exact in $R$. Then it is easy to see that a category $(\mathcal{C}_N(R), \mathcal{S}_N(R))$ is an exact category such that for every $M \in \text{Mod-}R$ and every $i \in \mathbb{Z}$, $D^{-i+N-1}_N(M)$ is an $\mathcal{S}_N$-projective and $\mathcal{S}_N$-injective object of this category. Hence this category is a Frobenius category, See [IKM14, Proposition 1.5].

**Definition 2.2.** A morphism $f : X^\bullet \longrightarrow Y^\bullet$ of $N$-complexes is called null-homotopic if there exists $s^i \in \text{Hom}_R(X^i, Y^{i+N-1})$ such that

$$f^i = \sum_{j=0}^{N-1} d^{i-(N-1-j)}_{Y^\bullet(N-1-j)} s^{i+j} d^j_{X^\bullet(j)}.$$

We denote the homotopy category of unbounded $N$-complexes by $\mathbb{K}_N(R)$.

**Definition 2.3.** For $X^\bullet = (X^i, d^i) \in \mathcal{C}_N(R)$, we define a shift functor $\Theta : \mathcal{C}_N(R) \rightarrow \mathcal{C}_N(R)$ by

$$\Theta(X^\bullet)^i = X^{i+1}, \quad \Theta(d)^i = d^{i+1}.$$
We also define suspension functor $\Sigma : \mathbb{K}_N(R) \rightarrow \mathbb{K}_N(R)$ as follows

$$(\Sigma X^*)^m = \bigsqcup_{i=m+1}^{m+N-1} X^i, \quad d_{\Sigma X^*}^m = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & 0 & \\
-d_{(N-1)}^m & \ldots & 0 & 1 \\
\end{bmatrix}$$

$$(\Sigma^{-1} X^*)^m = \bigsqcup_{i=m-1}^{m-N+1} X^i, \quad d_{\Sigma^{-1} X^*}^m = \begin{bmatrix}
-d_{(N-2)}^m & 0 & \cdots & 0 \\
-d_{(2)}^m & 1 & \ddots & \ddots & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & 0 & \\
-e_{(N-1)}^m & \ldots & 0 & 1 \\
\end{bmatrix}$$

It is known that $\mathbb{K}_N(R)$ together with this suspension functor is a triangulated category, see [IKM14, Theorem 1.7].

Let $X^*$,

$$\ldots \xrightarrow{d_{X^*}^{i-1}} X^i \xrightarrow{d_{X^*}^i} X^{i+1} \xrightarrow{d_{X^*}^{i+1}} \ldots$$

be an $N$-complex of $R$-modules. We define

$$Z^i_r(X^*) := \ker d_{X^*}^r, \quad B^i_r(X^*) := \operatorname{im} d_{X^*}^{r-1}$$

$$C^i_r(X^*) := \operatorname{coker} d_{X^*}^r, \quad H^i_r(X^*) := Z^i_r(X^*)/B^i_r(X^*).$$

In each degree we have $N-1$ cycles and clearly $Z^i_N(X^*) = X^*$.

**Remark 2.4.** For any $X^* \in \mathbb{K}_N(R)$ if $H^i_r(X^*) = 0$ for any $i \in \mathbb{Z}$, then we have $H^i_r(X^*) = 0$ for any $i \in \mathbb{Z}$ and $0 < r < N$.

**Definition 2.5.** Let $X^* \in \mathbb{K}_N(R)$. We say $X^*$ is $N$-exact if $H^i_r(X^*) = 0$ for each $i \in \mathbb{Z}$ and all $r = 1, 2, \ldots, N-1$. We denote the full subcategory of $\mathbb{K}_N(R)$ consisting of $N$-exact complexes by $\mathbb{K}^N_{\text{ex}}(R)$.

For a full subcategory $\mathcal{B}$ of $\text{Mod-}R$, we denote by $\mathbb{K}^{\mathcal{B}}_{\mathcal{B}}(\mathcal{B})$ the full subcategory of $\mathbb{K}^{\mathcal{B}}_N(\mathcal{B})$ consisting of $N$-complexes $X^*$ satisfying $H^i_r(X^*) = 0$ for almost all but finitely many $i$ and $r$, where $\mathcal{z} = \text{nothing, -, +}$.

**Definition 2.6.** A morphism $f : X^* \rightarrow Y^*$ is called quasi-isomorphism if the induced morphism $H^i_r(f) : H^i_r(X^*) \rightarrow H^i_r(Y^*)$ is an isomorphism for any $i$ and $1 \leq r \leq N-1$, or equivalently if the mapping cone $C(f)$ belongs to $\mathbb{K}^N_{\text{ex}}(R)$. The derived category $\mathbb{D}_N(R)$ of $N$-complexes is defined as the quotient category $\mathbb{K}_N(R)/\mathbb{K}^N_{\text{ex}}(R)$. 
2.2. Triangular matrix ring. Let $\mathbb{M}_n(R)$ be the set of all $n \times n$ square matrices with coefficients in $R$ for $n \in \mathbb{N}$. $\mathbb{M}_n(R)$ is a ring with respect to the usual matrix addition and multiplication. The identity of $\mathbb{M}_n(R)$ is the matrix $E = \text{diag}(1, \ldots, 1) \in \mathbb{M}_n(R)$ with 1 on the main diagonal and zeros elsewhere. The subset

$$
\mathbb{T}_n(R) = \left[ \begin{array}{ccc}
R & 0 & \cdots & 0 \\
R & R & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
R & R & \cdots & R \\
\end{array} \right]
$$

of $\mathbb{M}_n(R)$ consisting of all triangular matrices $[a_{ij}]$ in $\mathbb{M}_n(R)$ with zeros over the main diagonal is a subring of $\mathbb{M}_n(R)$. It is well known that if $Q$ is the quiver

$$
A_\ell = 1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow n
$$

then $RQ \cong \mathbb{T}_n(R)$, where $RQ$ is a path algebra of quiver $Q$.

Let $Q = (V, E)$ be a quiver. A representation of $Q$ by a ring $R$ is a correspondence which associates an object $M_v$ to each vertex $v$ and a morphism $\varphi_a : M_{(a)} \rightarrow M_{(q(a))}$ to each arrow $a \in E$. Let $\mathcal{X}$ and $\mathcal{Y}$ be two representations by left $R$-modules of the quiver $Q$. A morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a family of homomorphisms $f_v : \mathcal{X}_v \rightarrow \mathcal{Y}_v$ such that $\mathcal{Y}_a \circ f_v = f_w \circ \mathcal{X}_a$ for any arrow $a : v \rightarrow w$. The representations of $Q$ by $R$-modules and $R$-homomorphisms form a category denoted by $\text{Rep}(Q, R)$.

It is known that the category $\text{Rep}(Q, R)$ is equivalent to the category of modules over path algebra $RQ$ whenever $Q$ is finite quiver.

Set $Q = A_\ell$. For $1 \leq i \leq n$, let $e^i : \text{Rep}(Q, R) \rightarrow \text{Mod}-R$ be the evaluation functor defined by $e^i(\mathcal{X}) = \mathcal{X}_i$, for any $\mathcal{X} \in \text{Rep}(Q, R)$. It is proved in [EH99] that $e^i$ has a right adjoint $e^i : \text{Mod}-R \rightarrow \text{Rep}(Q, R)$, where $e^i(M)$ is the following representation

$$
\begin{array}{ccccccc}
M & \rightarrow & M & \rightarrow & \cdots & \rightarrow & M & \rightarrow & \cdots & \rightarrow & 0.
\end{array}
$$

for the $R$-module $M$, where $M$ ends in $i$-th position with identity morphisms beforehand. Moreover, for $1 \leq j \leq n$, it is shown that $e^j$ also admits a left adjoint $e^j$, defined by $e^j(M)$ as follows:

$$
\begin{array}{ccccccc}
0 & \rightarrow & 0 & \rightarrow & \cdots & \rightarrow & M & \rightarrow & \cdots & \rightarrow & M.
\end{array}
$$

for the $R$-module $M$, where $M$ starts in $j$-th position with identity morphisms afterward. It is proved in [EE09] that any projective(resp. injective) representation $\mathcal{X}$ in $\text{Rep}(Q, R)$ is of the form $\bigoplus_{i=1}^n e^i(\mathcal{X}^i)(\mathcal{X}^i)$, where for any $1 \leq i \leq n$, $\mathcal{X}^i$ (resp. $\mathcal{X}^i$), is the cokernel(resp. kernel) of the split monomorphism $\mathcal{X}^{i-1} \rightarrow \mathcal{X}^i$ (resp. epimorphism $\mathcal{X}^i \rightarrow \mathcal{X}^{i+1}$). Hence any projective object in $\text{Mod-} \mathbb{T}_n(R)$ is of the form

$$
P^1 \rightarrow P^1 \oplus P^2 \rightarrow P^1 \oplus P^2 \oplus P^3 \rightarrow \cdots \rightarrow P^1 \oplus P^2 \oplus \cdots \oplus P^n
$$

and an injective object in $\text{Mod-} \mathbb{T}_n(R)$ is of the form

$$
I^1 \oplus I^2 \oplus \cdots \oplus I^n \rightarrow I^1 \oplus I^2 \oplus \cdots \oplus I^{n-1} \rightarrow \cdots \rightarrow I^1 \oplus I^2 \rightarrow I^1
$$

3. SOME TRIANGLE EQUVALENCES BETWEEN HOMOTOPY CATEGORIES

In this section, we show that the homotopy category $\mathbb{K}_N(\text{Prj}(R))$ of $N$-complexes is embedded in the ordinary homotopy category $\mathbb{K}(\text{Prj-} \mathbb{T}_{N-1}(R))$. As a result of this embedding, we show that there exists a triangle equivalence between derived category of $N$-complexes and ordinary derived category of complexes of $\text{Mod-} \mathbb{T}_{N-1}(R)$. At the end of this section we
show that $K_N(\text{Prj}(R)) \cong K(\text{Prj}-T_{N-1}(R))$.

Let $S_N(R)$ be the collection of short exact sequence in $C_N(R)$ of which each term is split exact then it is shown in [IKM14] that $(C_N(R), S_N(R))$ is a Frobenius category. We need the following definition and lemma from [Hap88].

**Definition 3.1.** Let $(B, S)$ and $(B', S')$ be Frobenius categories. An additive functor $F : B \rightarrow B'$ is called exact if $0 \rightarrow F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \rightarrow 0$ is contained in $S'$ whenever $0 \rightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \rightarrow 0$ is contained in $S$.

If $F$ transforms $S$-injectives into $S'$-injectives then $F$ induces a functor $\widehat{F} : \overline{B} \rightarrow \overline{B'}$. Denote by $T$ (resp. $T'$) the translation functor on stable category $\overline{B}$ (resp. $\overline{B'}$).

**Lemma 3.2.** Let $F$ be an exact functor between Frobenius categories $B$ and $B'$ such that $F$ transforms $S$-injectives into $S'$-injectives. If there exists an invertible natural transformation $\alpha : FT \rightarrow T'F$ then $F$ is an exact functor of triangulated categories.

In order to show that $K_N(\text{Prj}(R))$ embeds in $K(\text{Prj}-T_{N-1}(R))$, we explicitly construct the embedding functor.

**Construction 3.3.** Define the functor $F : C_N(\text{Prj}(R)) \rightarrow C(\text{Prj}-T_{N-1}(R))$ by the following rules.

**On objects:** Let $(P^*, d^*)$ be an object in $C_N(\text{Prj}(R))$. Define $i$-th term of $F(P^*)$ as follows

- For $i = 2r$, let $m = Nr$ and define $F(P^*)^i$ as the following projective representation of $A_{N-1}$:

\[
\begin{array}{cccccccc}
p^m & \rightarrow & p^m \oplus p^{m+1} & \rightarrow & \cdots & \rightarrow & p^m \oplus p^{m+1} \oplus \cdots \oplus p^{m+N-3} & \rightarrow & p^m \oplus \cdots \oplus p^{m+N-2}
\end{array}
\]

- For $i = 2r+1$, let $m = Nr$ and define $F(P^*)^i$ as the following projective representation of $A_{N-1}$:

\[
\begin{array}{cccccccc}
p^{m+N-1} & \rightarrow & p^{m+N-1} \oplus p^{m+N} & \rightarrow & \cdots & \rightarrow & p^{m+N-1} \oplus p^{m+N} \oplus \cdots \oplus p^{m+2N-4} & \rightarrow & p^{m+N-1} \oplus \cdots \oplus p^{m+2N-3}
\end{array}
\]

For the definition of differential of $F(P^*)$, we consider the following two cases:

(i) For $i = 2r$, define $\mu^i : F(P^*)^i \rightarrow F(P^*)^{i+1}$ by $\mu^i = (\mu^i_j)_{1 \leq j \leq N-1}$ where

\[
\mu^i_j = \left[
\begin{array}{cccc}
d_{m-1}^{i-1} & d_{m-2}^{i-1} & \cdots & d_{m-j}^{i-1} \\
0 & d_{m-1}^{i-1} & \cdots & d_{m-j}^{i-1} \\
0 & 0 & \cdots & d_{m-j}^{i-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{array}
\right]
\]

(ii) For $i = 2r + 1$, define $\lambda^i : F(P^*)^i \rightarrow F(P^*)^{i+1}$ by $\lambda^i = (\lambda^i_j)_{1 \leq j \leq N-1}$ where

...
On morphisms: Let \( f^* : Q^* \to P^* \) be a morphism in \( \mathbb{C}_N(\text{Prj-}R) \). We define \( F(f^*) \) as follows:

(i) For \( i = 2r, \) let \( m = Nr \). Define \( \varphi^i : F(Q^*)^i \to F(P^*)^i \) by \( \varphi^i = (\varphi^i_j)_{0 \leq j \leq N-2} \) where

\[
\varphi^i_j = \text{diag}(f^m, ..., f^{m+j})
\]

(ii) For \( i = 2r + 1, \) let \( m = Nr \). Define \( \varphi^i : F(Q^*)^i \to F(P^*)^i \) by \( \varphi^i = (\varphi^i_j)_{0 \leq j \leq N-2} \) where

\[
\varphi^i_j = \text{diag}(f^{m-1+N}, ..., f^{m+N-j}).
\]

It is straightforward to show that this construction defines covariant functor from \( \mathbb{C}_N(\text{Prj-}R) \) to \( \mathbb{C}(\text{Prj-}T_{N-1}(R)) \).

Example 3.4. Let \( N = 3 \). Let \( P^* \)

\[
P^* = \cdots \quad ^{-2} \quad d^0 \quad d^1 \quad d^2 \quad d^3 \quad \cdots
\]

be a 3-complex in \( \mathbb{C}_3(\text{Prj-}R) \). The functor \( F \) maps \( P^* \) to the following complex in \( \mathbb{C}(\text{Prj-}T_2(R)) \)

\[
P^{-1} \quad ^{-1} \quad p^0 \quad d^0 \quad d^1 \quad d^2 \quad d^3 \quad d^4 \quad \cdots
\]

Now consider the morphism \( f^* : (Q^*, e^*) \to (P^*, d^*) \) in \( \mathbb{C}_3(\text{Prj-}R) \) as follows:

\[
\cdots \quad e^{-2} \quad Q^{-1} \quad e^{-1} \quad Q^0 \quad e^{0} \quad Q^1 \quad e^{1} \quad Q^2 \quad e^{2} \quad Q^3 \quad e^{3} \quad \cdots
\]

\[
\cdots \quad d^{-2} \quad P^{-1} \quad d^{-1} \quad P^0 \quad d^{0} \quad P^1 \quad d^{1} \quad P^2 \quad d^{2} \quad P^3 \quad d^{3} \quad \cdots
\]
Lemma 3.5. The functor \( F \), defined above, induces a functor from the category \( \mathcal{K}_N(\text{Prj}-R) \) to the category \( \mathcal{K}(\text{Prj-T}_{N-1}(R)) \) which we denote it again by \( F \).

Proof. We show that if \( f^* : (Q^*, e^*) \to (P^*, d^*) \) is a null homotopic map in \( \mathcal{C}_N(\text{Prj}-R) \) then \( F(f^*) \) is a null homotopic map in \( \mathcal{C}(\text{Prj-T}_{N-1}(R)) \). Since \( f^* \sim 0^* \), by definition there exists \( s^m \in \text{Hom}_R(Q^m, P_{m-N+1}) \) such that

\[
 f^m = \sum_{k=0}^{N-1} d_{m-(N-1-k)}^m \cdot s^m_k \cdot e_{m-k}^k 
\]

We want to construct \( (t^i)_{i \in \mathbb{Z}} \), such that \( (F(f^*))^i = \lambda^i\cdot t^i + t^i+1 \cdot \mu^i \cdot \lambda^j \cdot \nu^i \) when \( i \) is even (resp. odd). We consider the following two cases:

(i) Let \( i = 2r \) and \( m = N \cdot r \). We define \( t^i = (t^i_j)_{1 \leq j \leq N-1} \) where

\[
t^i_j = \begin{bmatrix}
\gamma^N \cdot k = 0 d_{m-(N-1-k)}^m s^m_k e_{m-k}^k \\
\gamma^N \cdot k = 1 d_{m-(N-1-k)}^m s^m_k e_{m-k}^k \\
\gamma^N \cdot k = 2 d_{m-(N-1-k)}^m s^m_k e_{m-k}^k \\
\vdots \\
\gamma^N \cdot N-2 d_{m-(N-1-k)}^m s^m_k e_{m-k}^k \\
\gamma^N \cdot N-1 d_{m-(N-1-k)}^m s^m_k e_{m-k}^k \\
\end{bmatrix}
\]

(ii) Let \( i = 2r + 1 \) and \( m = N \cdot r \). We define:

\[
t^i = (t^i_j)_{1 \leq j \leq N-1} \quad \text{where} \quad t^{i+1} = \begin{bmatrix}
s^{m+N-1} & 0 & \cdots & 0 \\
0 & s^{m+N} & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & s^{m+N+i-2} \\
\end{bmatrix}
\]

\( \square \)

Lemma 3.6. The functor \( F : \mathcal{K}_N(\text{Prj}(R)) \to \mathcal{K}(\text{Prj-T}_{N-1}(R)) \) is a fully faithful functor.

Proof. Let \( P^* \) and \( Q^* \) be two objects in \( \mathcal{C}_N(\text{Prj}-R) \). We want to show that:

\[
\text{Hom}_{\mathcal{K}_N(\text{Prj}(R))}(Q^*, P^*) \cong \text{Hom}_{\mathcal{K}(\text{Prj-T}_{N-1}(R))}(F(Q^*), F(P^*))
\]
Let \( f^\bullet : (Q^\bullet, e^\bullet) \rightarrow (P^\bullet, d^\bullet) \) be a morphism in \( \mathbb{C}_N(\text{Prj-}R) \) such that \( \mathbf{F}(f^\bullet) = 0 \) in \( \mathbb{K}(\text{Prj-}T_{N-1}(R)) \). We want to show that \( f^\bullet = 0 \) in \( \mathbb{K}(\text{Prj-}R) \). We construct \( s^m : Q^m \rightarrow P^{m-N+1} \) such that

\[
(3.1) \quad f^m = \sum_{k=0}^{N-1} d^{m-(N-1-k)}_{\{N-1-k\}} s^{m+k} e^m_{\{k\}}
\]

for all \( m \in \mathbb{Z} \).

Suppose \( i = Nr \) for some \( r \in \mathbb{Z} \). Consider the following diagram:

\[
\begin{array}{cccccc}
Q^i & \xrightarrow{e^i} & Q^{i+1} & \xrightarrow{e^{i+1}} & \cdots & Q^{i+N-2} & \xrightarrow{e^{i+N-2}} & Q^{i+N-1} \\
\downarrow f^i & & \downarrow f^{i+1} & & \cdots & \downarrow f^{i+N-2} & & \downarrow f^{i+N-1} \\
P^i & \xrightarrow{d^i} & P^{i+1} & \xrightarrow{d^{i+1}} & \cdots & P^{i+N-2} & \xrightarrow{d^{i+N-2}} & P^{i+N-1}
\end{array}
\]

Since \( \mathbf{F}(f^\bullet) \sim 0^\bullet \) there exists

\[
t^n = (t^n_j)_{1 \leq j \leq N-1} \quad \text{where} \quad t^n_j = \begin{bmatrix}
\alpha^n_{11} & \alpha^n_{12} & \cdots & \alpha^n_{1j} \\
0 & \alpha^n_{22} & \cdots & \alpha^n_{2j} \\
& & \ddots & \vdots \\
0 & 0 & \cdots & \alpha^n_{jj}
\end{bmatrix}
\]

such that

\[
(3.2) \quad f^{i+j-1} = d^{i+j-2} \alpha^i_j + \alpha^{i+1}_{jj} e^{i+j-1}_{\{N-1\}} = d^{-N+j}_{\{N-1\}} \alpha^{i-1}_{(j+1)(j+1)} + \alpha^i_{(j+1)(j+1)} e^{i-1}_{\{N-1\}}
\]

for any \( 1 \leq j \leq N-1 \),

\[
(3.3) \quad \alpha^i_{xy} = \sum_{k=x}^{y} d^{-N+x-1+k}_{\{N-1-k\}} \alpha^{i-1}_{(k+1)(k+1)} + \alpha^i_{x(y+1)} e^{i-1}_{\{N-1-k\}}
\]

and

\[
(3.4) \quad \alpha^i_{pq} = d^{-p+1} \alpha^i_{(p-1)q} + \sum_{k=p-1}^{q} \alpha^{i+1}_{1k} e^{-q+k}_{\{N-1-q+k\}}
\]

for any \( 1 \leq x \leq y \leq N-2 \) and \( 2 \leq p \leq q \leq N-1 \). We define homotopy maps \( s^m \), \( i \leq m \leq i + N - 1 \) as

\[
s^m = \begin{cases}
\alpha^{i+1}_{11} & \text{if } m = i + N - 1, \\
\alpha^i_{1(N-1)} & \text{if } m = i + N - 2, \\
\alpha^{i-1}_{(m-i+2)(m-i+2)} + \sum_{k=i}^{N-2-m+i} d^{m-N}_{\{N-1-k\}} \alpha^{i-1}_{(m-i+1)(m-i+1)} e^{i}_{\{k-1\}} & \text{if } i \leq m \leq i + N - 3.
\end{cases}
\]

As \( r \) varies in \( \mathbb{Z} \), the numbers \( i = Nr \) give us a collection of morphisms as above. So we can construct homotopy maps \( (s^m)_{m \in \mathbb{Z}} \). If \( i = Nr \), then it is easy to show that \( f^i, f^{i+1}, \ldots, f^{i+N-1} \) and the homotopy maps \( (s^m)_{i \leq m \leq i+2N-1} \) satisfy relation (3.1).
Now let $\varphi^* \in \text{Hom}_{\mathbb{K}(\text{Prj-T}_{N-1}(R))}(\mathbf{Q}(Q^*),\mathbf{F}(P^*))$. We want to find $f^* \in \text{Hom}_{\mathbb{K}(\text{Prj-R})}(Q^*,P^*)$ such that $\mathbf{F}(f^*) = \varphi^*$. Suppose that $i = Nr$ and $\varphi^i = (\varphi^i_j)_{1 \leq j \leq N - 1}$ where

$$
\varphi^i_j = \begin{bmatrix}
\beta^1_{11} & \beta^1_{12} & \cdots & \beta^1_{1j} \\
0 & \beta^1_{22} & \cdots & \beta^1_{2j} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \beta^1_{jj}
\end{bmatrix}
$$

We again consider the above diagram. Our goal is to construct $(f^m)_{1 \leq m \leq i+N-1}$. By assumption we have $\varphi^i_j(\lambda^i_{Q^*}) = (\lambda^i_{P^*}) \varphi^i_j$ and $\varphi^{i+1}_j(\mu^i_{Q^*}) = (\mu^i_{P^*}) \varphi^{i+1}_j$ for any $1 \leq j \leq N - 1$. Hence we have the following equations:

\begin{align*}
(3.5) & \quad d_i^{i+p-1} \beta^{i-1}_{pq} - \beta^{i-1}_{(p+1)q} = -\beta^i_{pq} + \beta^i_{pq} e^{i+p-2}, \quad 1 \leq p \leq N - 1, \quad p < q \leq N - 1 \\
(3.6) & \quad d_i^{i+p-2} \beta^{i-1}_{pp} = \beta^i_{pp} e^{i+p-2}, \quad 1 \leq p \leq N - 1 \\
\end{align*}

and

\begin{align*}
(3.7) & \quad \sum_{k=2}^{N} d_i^{i+N-k} \beta^i_{(N-k+1)(N-1)} = \sum_{k=1}^{N-1} \beta^i_{1k} e^{i+N-2}.
\end{align*}

We define

$$
f^i = \sum_{k=2}^{N-1} \beta^i_{2k} e^{i_{k-2}} + \beta^i_{1(N-1)} e^{i_{N-2}}.
$$

For $i + 1 \leq j \leq i + N - 2$ define

$$
f^i_j = \sum_{k=2}^{N-1} \beta^j_{2k} e^{j_{k-2}} + \sum_{k=1}^{j+1} d_i^{j+k-2} \beta^j_{k(N-1)} e^{j_{N-j-2}}.
$$

and

$$
f^{i+N-1} = \sum_{k=1}^{N-1} \beta^i_{1k} e^{i+N-1}.
$$

By (3.7) we have

$$
d_i^{i+N-1} f^{i+N-1} = f^{i+N-1} e^{i+N-1}
$$

It is not so hard to see that $\mathbf{F}(f^*) \sim \varphi^*$. \hfill \Box

**Lemma 3.7.** Let $(P^*,d^*) \in \mathbb{K}_{N}^\mathit{ac}(\text{Prj-R})$. The image of $P^*$ under $\mathbf{F}$ is an exact complex in $\mathbb{K}(\text{Prj-T}_{N-1}(R))$. 
Proof. Suppose \( i = 2r \) and \( m = Nr \). The following diagram shows the image of \( P^\bullet \) under \( F \) in degree \( i, i+1, i+2 \) and \( i+3 \).

\[
\begin{array}{cccccccc}
p^m & \rightarrow & p^m \oplus p^{m+1} & \rightarrow & \cdots & \rightarrow & p^m \oplus \cdots \oplus p^{m+N-2} & \\
p^{m+N-1} & \rightarrow & p^{m+N-1} \oplus p^{m+N} & \rightarrow & \cdots & \rightarrow & p^{m+N-1} \oplus \cdots \oplus p^{m+2N-3} & \\
p^{m+N} & \rightarrow & p^{m+N} \oplus p^{m+N+1} & \rightarrow & \cdots & \rightarrow & p^{m+N} \oplus \cdots \oplus p^{m+2N-2} & \\
p^{m+2N-1} & \rightarrow & p^{m+2N-1} \oplus p^{m+2N} & \rightarrow & \cdots & \rightarrow & p^{m+2N-1} \oplus \cdots \oplus p^{m+3N-3} & \\
\end{array}
\]

We want to show that \( \text{Im}(\mu_j^i) = \text{Ker}(\lambda_j^{i+1}) \) for any \( 1 \leq j \leq N-1 \). Clearly \( \text{Im}(\mu_j^i) \subseteq \text{Ker}(\lambda_j^{i+1}) \). Let \( (x_1, x_2, \ldots, x_j) \in \text{Ker}(\lambda_j^{i+1}) \). It is easy to show that there exists \( y_t \in p^{m+t-1} \) for all \( 1 \leq t \leq j \) such that

\[
x_p = \sum_{k=1}^{j-p+1} d_{\{N-k\}}^{i+p-1+(k-1)}(y_{p+k-1}),
\]

for all \( 1 \leq p \leq t \). Hence \( (x_1, x_2, \ldots, x_j) = \mu_j^i(y_1, y_2, \ldots, y_j) \).

Likewise suppose that \( i = 2r+1 \) and \( m = Nr \). We show that \( \text{Im}(\lambda_j^{i+1}) = \text{Ker}(\mu_j^{i+2}) \) for any \( 1 \leq j \leq N-1 \). Clearly \( \text{Im}(\lambda_j^{i+1}) \subseteq \text{Ker}(\mu_j^{i+2}) \). Let \( (x_1, x_2, \ldots, x_j) \in \text{Ker}(\mu_j^{i+2}) \). It is easy to show that there exists \( y_t \in p^{m+N-2+t} \) for all \( 1 \leq t \leq j \) such that

\[
x_j = d^{m+2N-3}(y_j),
\]

and

\[
x_q = -y_{q+1} + d^{m+N+q-2}(y_q)
\]

for any \( 1 \leq q \leq t \) and \( q \neq j \). Hence \( (x_1, x_2, \ldots, x_j) = \lambda_j^{i+1}(y_1, y_2, \ldots, y_j) \).

\[\square\]

Lemma 3.8. The functor \( F : \mathbb{K}_N(\text{Prj}-(R)) \rightarrow \mathbb{K}(\text{Prj-T}_N-1(R)) \) is an exact triangulated functor.

Proof. Clearly \( F \) is an exact functor. Since \( F \) preserves direct sum it is enough to show that \( F \) transforms \( D^N(P) \) to a projective object of \( \mathbb{C}(\text{Prj-T}_N-1(R)) \). By lemma 3.7 \( F(D^N(P)) \) is a bounded exact complex in \( \mathbb{C}(\text{Prj-T}_N-1(R)) \). Since \( D^N(P) \) is an \( N \)-exact complex. Hence \( F(D^N(P)) \) is a projective object in \( \mathbb{C}(\text{Prj-T}_N-1(R)) \). Now we show that

\[
F(\Sigma P^\bullet) \cong F(P^\bullet)[1]
\]
Let $P^\bullet$ be a complex in $\mathbb{K}N(\text{Prj}(R))$. For $i \equiv 0 \pmod{2}$ let $m = \frac{N}{2}$. For $1 \leq j \leq N - 1$ and $1 \leq k \leq j$ let $\alpha^i_{j,k} : (\Sigma P^\bullet)^{m+k-1} \to \oplus_{l=m+N-1}^{m+N-2+j} P^l$ be a morphism given as

$$\alpha^i_{j,k} = \begin{pmatrix} \alpha^i_{j,1} & \alpha^i_{j,2} & \cdots & \alpha^i_{j,j} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha^i_{k,1} & \alpha^i_{k,2} & \cdots & \alpha^i_{k,k} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha^i_{j,1} & \alpha^i_{j,2} & \cdots & \alpha^i_{j,j} \end{pmatrix}$$

These morphisms define a map $\alpha^i_j : F(\Sigma P^\bullet)^j \to (F(P^\bullet)[1])^j$ as

$$\alpha^i_j = \begin{pmatrix} \alpha^i_{j,1} & \alpha^i_{j,2} & \cdots & \alpha^i_{j,j} \end{pmatrix}.$$  

For $i \equiv 1 \pmod{2}$ let $m = \frac{(i+1)N}{2}$. For $1 \leq j \leq N - 1$ and $1 \leq k \leq j$ let $\alpha^i_{j,k} : (\Sigma P^\bullet)^{m+k-1} \to \oplus_{l=m}^{m+j-1} P^l$ be a morphism given as

$$\alpha^i_{j,k} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ k-1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ k & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ j & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Likewise these morphisms give a map $\alpha^i_j : F(\Sigma P^\bullet)^j \to (F(P^\bullet)[1])^j$ defined by

$$\alpha^i_j = \begin{pmatrix} \alpha^i_{j,1} & \alpha^i_{j,2} & \cdots & \alpha^i_{j,j} \end{pmatrix}.$$  

In the other direction for $i \equiv 0 \pmod{2}$, define $\beta^i_j : (F(P^\bullet)[1])^j \to F(\Sigma P^\bullet)^j$ as

$$\beta^i_j = \begin{pmatrix} \beta^i_{j,1} \\ \beta^i_{j,2} \\ \vdots \\ \beta^i_{j,j} \end{pmatrix}$$

where,

$$\beta^i_{j,k} = \begin{pmatrix} 1 & \cdots & k-1 & k & k+1 & \cdots & j \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \cdots & 0 & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$
For \( i \equiv 1( \mod 2) \) define \( \beta_j^i : (F(P^*)[1])_j^i \rightarrow F(\Sigma P^*)_j^i \) as

\[
\beta_j^i = \begin{bmatrix}
\beta_j^i,1 \\
\beta_j^i,2 \\
\vdots \\
\beta_j^i,j
\end{bmatrix}
\]

where,

\[
\beta_j^i,_{j,k} = \begin{bmatrix} 0_{j-k+1 \times k-1} & I_{j-k+1} \\
0_{k-1 \times k-1} & 0_{k-1 \times j-k+1} \end{bmatrix},
\]

in which, \( I_{j-k+1} \) is the identity matrix of order \( j - k + 1 \), and the other three entries are zero matrices of given size.

It is not so hard to see that the composition \( \alpha^i \circ \beta^i : (F(P^*)[1])^i \rightarrow (F(P^*)[1])^i \) is the identity morphism. One can show that \( \beta \circ \alpha - 1 \) is null-homotopic where the homotopy maps are defined as follows: For \( i \equiv 0( \mod 2) \), \( 1 \leq j \leq N - 1 \) and \( 1 \leq k \leq j \) let

\[
\psi_k^i = \begin{bmatrix} 0_{k \times N - k - 1} & 0_{k \times k} \\
-I_{N-k-1} & 0_{N-k-1 \times k} \end{bmatrix}
\]

and define

\[
s_j^i = \begin{bmatrix}
\psi_1^i & \psi_2^i & \cdots & \psi_j^i \\
0 & \psi_1^i & \ddots & \psi_{j-1}^i \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \psi_1^i
\end{bmatrix}
\]

For \( i \equiv 1( \mod 2) \) and \( 1 \leq j \leq N - 1 \) define \( s_j^i = 0 \). It is quite tedious to show that the morphisms \( \alpha : F(\Sigma P^*) \rightarrow F(P^*[1]) \) and \( \beta : F(P^*)[1] \rightarrow F(\Sigma P^*) \) are natural in \( P^* \).

Putting it all together, we have

**Proposition 3.9.** The functor \( F : K_N(Prj-R) \rightarrow K(Prj-T_{N-1}(R)) \) is a fully faithful triangle functor.

**Definition 3.10.** We call an \( N \)-complex \( P^* \), \( K \)-projective, if \( P^* \in \mathbb{C}_N(Prj-R) \) and

\[
\text{Hom}_{K_{N(R)}}(P^*, Y^*) = 0
\]

for all \( Y^* \in K^\mathbb{Z}_N(R) \). We denote by \( K_{N(Prj-R)}^\mathbb{Z} \) the category of all \( K \)-projective \( N \)-complexes. Dually we define the triangulated full subcategory \( K_N^{Inj}(Inj-R) \) consisting of complexes \( I^* \) of injectives such that \( \text{Hom}_{K_{N(R)}}(X^*, I^*) = 0 \) for all \( X^* \in K^\mathbb{Z}_N(R) \)

**Theorem 3.11.** We have the following triangle equivalences:

\[
K_{N(Prj-R)}^\mathbb{Z} \cong \mathbb{D}_N(R) \quad \text{and} \quad K_N^{Inj}(Inj-R) \cong \mathbb{D}_N(R)
\]

**Proof.** See [IKM14, Theorem 2.22.].

**Definition 3.12.** For an additive category \( A \) with arbitrary coproducts, an object \( C \) is called compact in \( A \) if the canonical morphism \( \prod_i \text{Hom}_A(C, X_i) \rightarrow \text{Hom}_A(C, \prod_i X_i) \) is an isomorphism for any coproduct \( \prod_i X_i \) in \( A \). We denote by \( A^c \) the subcategory of \( A \) consisting of all compact objects.
Definition 3.13. Let $T$ be a triangulated category. A non-empty subcategory $S$ of $T$ is said to be thick if it is a triangulated subcategory of $T$ that is closed under retracts. If, in addition, $S$ is closed under all coproducts allowed in $T$, then it is localizing; if it is closed under all products in $T$ it is colocalizing.

The following remark gives us a better understanding of the objects in a thick subcategory, see [Kra06].

Remark 3.14. Let $S$ be a class of objects of a triangulated category $T$. Then
- $	ext{Thick}(S) = \bigcup_{n \in \mathbb{N}} \langle S \rangle_n$, where
  - $\langle S \rangle_1$ is the full subcategory of $T$ containing $S$ and closed under all the objects $S$ such that there is a distinguished triangle $Y \to X \to Z \to \cdots$ in $T$ with $Y \in \langle S \rangle_1$, and $Z \in \langle S \rangle_i$ such that $i, j < n$ and $S$ is a direct summand of shifting of $X$.

The following theorem has been proved by Iyama and et. al. in [IKM14]. As a result of proposition 3.9, we present another proof for this theorem.

Theorem 3.15. For a ring $R$, we have the following triangle equivalence:
\[ \mathbb{D}^b(N)(R) \cong \mathbb{D}(\mathbb{T}_{N-1}(R)) \]

Proof. By theorem 3.11 $\mathbb{K}_{N}^{Prj}(\text{Prj}-R) \cong \mathbb{D}(N)(R)$ and $\mathbb{K}_{Prj}^{T}(\text{Prj}-T_{N-1}(R)) \cong \mathbb{D}(\mathbb{T}_{N-1}(R))$, and we have the following diagram:

\[
\begin{array}{ccc}
\mathbb{K}_{N}(\text{Prj}-R) & \xrightarrow{F} & \mathbb{K}(\text{Prj}-\mathbb{T}_{N-1}(R)) \\
\downarrow & & \downarrow \\
\mathbb{K}_{Prj}^{Prj}(\text{Prj}-R) & \xrightarrow{F} & \mathbb{K}_{Prj}^{T}(\text{Prj}-\mathbb{T}_{N-1}(R))
\end{array}
\]

In addition $\mathbb{D}(\mathbb{T}_{N-1}(R))^c \cong K^{b}(\text{prj}-\mathbb{T}_{N-1}(R))$. For $1 \leq i \leq N - 1$ let $\mathcal{R}_i$ be the following projective representation of $A_{N-1}:
\[0 \to 0 \to \cdots \to R \to R \to \cdots \]
where $R$ start in $i$-th position with identity morphisms afterward.

We can show that $\mathbb{K}^{b}(\text{prj}-\mathbb{T}_{N-1}(R)) = \text{Thick}((\mathcal{R}_1^*, ..., \mathcal{R}_{N-1}^*))$ whenever $\mathcal{R}_i^*$ is a complex $\cdots \to 0 \to \mathcal{R}_i^* \to 0 \to 0 \to \cdots$ concentrated in degree $0$. Now we show that each $\mathcal{R}_i^*$ belong to $\text{Im} F$. Let $R^*$ be a complex $\cdots \to 0 \to \mathcal{R}_i^* \to 0 \to 0 \to \cdots$ concentrated in degree $0$. $F(\Sigma^{\mathcal{R}_1^*}) = \mathcal{R}_{N-1}^*$ and $F(\Theta^{-1} R^*) = \mathcal{R}_1^*$, hence $\mathcal{R}_{N-1}^*, \mathcal{R}_1^* \in \text{Im} F$. On the other hand $F(\Theta^{-3} R^*) = \cdots \to 0 \to \mathcal{R}_{N-1} \to \mathcal{R}_{N-2} \to 0 \to \cdots$ and therefore there exists a short exact sequence $0^* \to \mathcal{R}_{N-2}^* \to F(\Theta^{-3} R^*) \to \mathcal{R}_{N-1}^*[1] \to 0^*$ with degree-wise split exact sequences. So there exists a triangle $\mathcal{R}_{N-2}^* \to F(\Theta^{-3} R^*) \to \mathcal{R}_{N-1}^*[1] \to \cdots$ in $\mathbb{K}(\text{prj}-\mathbb{T}_{N-1}(R))$, hence $\mathcal{R}_{N-2}^* \in \text{Im} F$, since $F(\Theta^{-3} R^*), \mathcal{R}_{N-1}^*[1] \in \text{Im} F$. Similarly by induction we can say that $\mathcal{R}_i^* \in \text{Im} F$ for $2 \leq i \leq N - 3$. Hence $\text{Thick}((\mathcal{R}_1^*, ..., \mathcal{R}_{N-1}^*)) \subseteq \text{Im} F \subseteq \mathbb{D}(\mathbb{T}_{N-1}(R))$. But $\text{Im} F$ is closed under coproduct and contains compact objects, therefore the restriction of functor $F$ to $\mathbb{D}(R)$ is dense hence $\mathbb{D}(N)(R) \cong \mathbb{D}(\mathbb{T}_{N-1}(R))$. 

According to the above theorem, we have a triangle equivalence
\[ \mathbb{D}^b(N)(R) \cong \mathbb{D}^-(\mathbb{T}_{N-1}(R)) \]
It is easy to check that $\mathbf{Q}$ as follows:

Now we define a functor $\mathbf{P}$ to modules with split epimorphism maps.

Let $\mathbf{Q}$ be the quiver of type $A_n$. Any projective representation $\mathbf{P}$ of $Q$ is of the form $\mathbf{P} = \oplus_{i=1}^n e_i^\lambda(P^i)$, where $P^i$ is the cokernel of split monomorphism $P_{i-1} \to P_i$. For any projective representation $\mathbf{P} = \oplus_{i=1}^n e_i^\lambda(P^i)$ of $Q$, set $\tilde{\mathbf{P}} = \oplus_{i=1}^n e_i^\lambda(P^i)$. Clearly $\tilde{\mathbf{P}}$ is an object in the category $\text{Prj}^{op}(A_n)$, where $\text{Prj}^{op}(A_n)$ is the category of all representations by projective modules with split epimorphism maps.

Now we define a functor $\hat{\imath} : \text{Prj}^{op}(A_n) \to \text{Prj}^{op}(A_n)$ such that any $\mathbf{P} \in \text{Prj}^{op}(A_n)$ is mapped under $\hat{\imath}$ to $\tilde{\mathbf{P}}$, as defined above, and for any morphism $\varphi = (\varphi_i)_{1 \leq i \leq n}$ in $\text{Hom}(e_i^\lambda(P^i), e_j^\lambda(P^j))$ define $\hat{\varphi}$ as follows:

$$
\hat{\varphi} = \begin{cases} 
0 & i < j \\
\langle \varphi_i \rangle & i \geq j 
\end{cases}
$$

It is easy to check that $\hat{\imath}$ is in fact an equivalence of categories. We also see that for a finite quiver $Q$, $\hat{\imath}$ is an equivalence of categories, see [AEHS11]. The functor $\hat{\imath}$ can be naturally extended to a functor $\mathbb{K}(\text{Prj}^{op}(A_n)) \to \mathbb{K}(\text{Prj}^{op}(A_n))$ which we denote again by $\hat{\imath}$. So for any $X^\bullet \in \mathbb{K}(\text{Prj}^{op}(A_n))$, let $\hat{X}^\bullet$ be the complex with $\hat{X}_i$ as its $i$-th term and $\hat{d}_i$ as its $i$-th differential. The functor $\hat{\imath}$ also is an equivalence of homotopy categories.

we also need the following description of compact objects in $\mathbb{K}(\text{Prj}^{op}(A_n))$. Neeman [Nee08] showed that an object $X^\bullet$ of $\mathbb{K}(\text{Prj}^{op}(A_n))$ is compact if and only if it is isomorphic, in $\mathbb{K}(\text{Prj}^{op}(A_n))$, to a complex $Y^\bullet$ satisfying

(i) $Y$ is a complex of finitely generated projective modules.

(ii) $Y^i = 0$ if $i \ll 0$.

(iii) $H^i(Y^\bullet) = 0$ if $i \ll 0$, where $Y^\bullet = \text{Hom}(Y^\bullet, R)$. 

At the end of this section we show that the functor $\mathbf{F}$ is dense, hence there exist an triangle equivalence between $\mathbb{K}(\text{Prj}^{op}(A_n))$ and $\mathbb{K}(\text{Prj}^{op}(A_n))$. Before we give the proof we need to introduce another functor.

Moreover $\mathbf{F}$ induces some triangle equivalences between subcategories of $\mathbb{K}_N(\text{Prj}(R))$ and subcategories of $\mathbb{K}^{-}(\text{Prj}^{op}(R))$, see [IKM14, Corollary 4.15]. We summarize all of these equivalences in the following diagram. Note that the existence of the first row follows from proposition 3.9.

At the end of this section we show that the functor $\mathbf{F}$ induces an equivalence of categories, see [IKM14, Corollaries 4.11, 2.17]. We summarize all of these equivalences in the following diagram. Note that the existence of the first row follows from proposition 3.9.
He also showed that when the compact objects generate the category $K(\text{Prj-}R)$.

**Proposition 3.16.** If $R$ is a left coherent ring, then the category $K(\text{Prj-}R)$ is compactly generated.

This idea enables us to prove our main theorem:

**Theorem 3.17.** For a left coherent ring $R$, we have triangle equivalence

$$K_N(\text{Prj-}R) \cong K(\text{Prj-}T_{N-1}(R)).$$

**Proof.** In view of proposition 3.9, the functor $F$ is full and faithful. Now we show that $F$ is dense. Let $\mathcal{T}$ be a triangle subcategory of $K_N(\text{Prj-}R)$ such that for all $T \in \mathcal{T}$ we have the following conditions

1. $T \in K_N^{+}(\text{prj-}R)$;
2. There exists an integer $n \in \mathbb{Z}$ such that for every $i < n$ and $1 \leq r \leq N-1$, $H_i^T \cdot \mathbf{T}^* = 0$

where $\mathbf{T}^*$ denotes the induced complex $\text{Hom}_{\mathbf{p} \cdot \mathbf{T}}(\mathbf{T}, R)$.

Clearly the duality $\text{Hom}(-, R) : \text{prj-}R \to \text{prj-}R^{op}$ induces an equivalence $\mathcal{T}^{op} \cong K_N^{-}(\text{prj-}R^{op})$ of triangulated categories. On the other hand, by restricting the functor $F$ to $\mathcal{T}$, we have

$$K_N(\text{Prj-}R) \xrightarrow{F} K(\text{Prj-}T_{N-1}(R)) \xrightarrow{\mathcal{T}} K(\text{prj-}T_{N-1}(R))^c.$$

We want to show that $F|_\mathcal{T}$ is an equivalence. It is easy to check that we have the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{F|_\mathcal{T}} & K(\text{prj-}T_{N-1}(R))^c \\
\cong & & \cong \\
K_N^{-}(\text{prj-}R^{op}) & \xrightarrow{\text{Hom}(-, R) \cdot \mathbf{T}^*} & K_N^{-}(\text{prj-}T_{N-1}(R^{op}))
\end{array}
$$

where $F^{op}$ is the functor $F$ in construction 3.3 whenever we define it on $C_N(\text{prj-}R^{op})$ and $\cdot \mathbf{T}^*$ is the quasi-inverse of the functor $\cdot \mathbf{T}$. Therefore $F|_\mathcal{T} = \gamma \circ \text{Hom}(-, R) \circ F^{op} \circ \text{Hom}(-, R)$, hence it is an equivalence. It shows that $K(\text{prj-}T_{N-1}(R))^c \subseteq \text{Im}F$. On the other hand $\text{Im}F$ is closed under coproduct and contains compact objects, therefore $\text{Im}F = K(\text{Prj-}T_{N-1}(R))$. □

**Corollary 3.18.** If $R$ is a left coherent ring, then the category $K_N(\text{Prj-}R)$ is compactly generated.

In a dual manner, in view of construction 3.3 and lemmas 3.5, 3.6 and 3.8 we can embed the category $K_N(\text{Inj-}R)$ into the category $K(\text{Inj-}T_{N-1}(R))$. Since the compact objects of $K(\text{Inj-}T_{N-1}(R))$ are different from $K(\text{Prj-}T_{N-1}(R))$, the proof of theorem 3.17 dose not work. However, when $R$ is an artin algebra the embedding is dense.

**Proposition 3.19.** Let $\Lambda$ be an artin algebra. We have a triangle equivalence

$$K_N(\text{Inj-}\Lambda) \cong K(\text{Inj-}T_{N-1}(\Lambda)).$$
Proof. Let $D$ denote the duality between right and left $\Lambda$-modules. The adjoint pair of functors $- \otimes_A D(\Lambda)$ and $\text{Hom}_A(D(\Lambda), -)$ induces an equivalence between $\text{Prj}-\Lambda$ and $\text{Inj}-\Lambda$, which restricts to an equivalence between $\text{prj}-\Lambda$ and $\text{inj}-\Lambda$. Therefore the adjoint pair induces an equivalence between $\mathcal{K}_N(\text{Prj}-\Lambda)$ and $\mathcal{K}_N(\text{Inj}-\Lambda)$. So if we denote the embedding $\mathcal{K}_N(\text{Inj}-R) \rightarrow \mathcal{K}(\text{Inj}-\mathcal{T}_{N-1}(\Lambda))$ by $G$, then we have the following diagram:

$$
\begin{array}{ccc}
\mathcal{K}_N(\text{Prj}-\Lambda) & \xrightarrow{\cong} & \mathcal{K}_N(\text{Inj}-\Lambda) \\
\cong & & \cong \\
\mathcal{K}(\text{Prj}-\mathcal{T}_{N-1}(\Lambda)) & \xrightarrow{G} & \mathcal{K}(\text{Inj}-\mathcal{T}_{N-1}(\Lambda))
\end{array}
$$

Hence $G$ is an equivalence of categories. \qed

As a final remark we will discuss about $N$-dualizing complex.

Remark 3.20. Let $R$ be a commutative noetherian ring with a dualizing complex $D$ and $Q$ be a finite quiver. In [AEHS11] they showed that Grothendieck duality is extendable to path algebra $RQ$. Therefore when $Q = A_{N-1}$, we have

$$
\mathcal{D}^b(\text{mod-}\mathcal{T}_{N-1}(R))^{\text{op}} \cong \mathcal{D}^b(\text{mod-}\mathcal{T}_{N-1}(R)).
$$

On the other hand by [IKM14] we have

$$
\mathcal{D}^b(\text{mod-R}) \cong \mathcal{D}^b(\text{mod-}\mathcal{T}_{N-1}(R)).
$$

Hence

$$
\mathcal{D}^b(\text{mod-R})^{\text{op}} \cong \mathcal{D}^b(\text{mod-R}).
$$

So in case $R$ has a dualizing complex, there exists a Grothendieck duality for derived category of $N$-complexes and the question is "What could be the definition of an $N$-dualizing complex?".

4. $N$-Totally Acyclic Complexes:

Let $\mathcal{A}$ be an additive category. We say that a complex $X^\bullet$ in $\mathcal{A}$ is acyclic if the complex $\text{Hom}_\mathcal{A}(A, X^\bullet)$ of abelian groups is acyclic for all $A \in \mathcal{A}$. If in addition $\text{Hom}_\mathcal{A}(X^\bullet, A)$ is acyclic for all $A \in \mathcal{A}$, then $X^\bullet$ is called totally acyclic. Let $\mathcal{C}_{\text{tac}}(\mathcal{A})$ denote the full subcategory of $\mathcal{C}(\mathcal{A})$ consisting of totally acyclic complexes. Note that these definitions are up to isomorphism in $\mathcal{K}(\mathcal{A})$. The full triangulated subcategory of $\mathcal{K}(\mathcal{A})$ consisting of totally acyclic complexes, will be denoted by $\mathcal{K}_{\text{tac}}(\mathcal{A})$. For instance if $\mathcal{A} = \text{Prj}-R$ is the class of projective objects in $\text{Mod-}R$ then the object $X^\bullet$ of $\mathcal{K}_{\text{tac}}(\text{Prj}-R)$ is an exact complex such that $\text{Hom}_R(X^\bullet, P)$ is acyclic for all $P \in \text{Prj}-R$. The objects of $\mathcal{K}_{\text{tac}}(\text{Prj}-R)$ will be called totally acyclic complexes of projectives.

Remark 4.1. It is easy to see that $X^\bullet \in \mathcal{K}_{\text{tac}}(\text{Prj}-R)$ if and only if $\text{Hom}_{\mathcal{K}(\text{Prj}-R)}(P^\bullet, X^\bullet) = 0$ and $\text{Hom}_{\mathcal{K}(\text{Prj}-R)}(X^\bullet, P^\bullet) = 0$ for all $P^\bullet \in \mathcal{K}_b(\text{Prj}-R)$.

Let $R$ be a Noetherian ring. For the rest of this section we are only considering the category $\text{prj}-R$, i.e. the category of finitely generated projective left $R$-modules. Given an integer $n$ and a complex

$$
\cdots \xrightarrow{d^{n-3}} X^{n-2} \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} X^{n+2} \cdots
$$

and a complex $X^\bullet$ in $\mathcal{C}(\mathcal{A})$.

\begin{itemize}
\item $X^\bullet$ is acyclic if the complex $\text{Hom}_\mathcal{A}(A, X^\bullet)$ is acyclic for all $A \in \mathcal{A}$.
\item $X^\bullet$ is called totally acyclic if $\text{Hom}_\mathcal{A}(X^\bullet, A)$ is acyclic for all $A \in \mathcal{A}$.
\end{itemize}
in \text{Mod-}R, \text{we denote its brutal truncation at degree n by}

\[ \beta_{\leq n}(X^\bullet) : \cdots \xrightarrow{d^m} X^{n-2} \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^0} 0 \xrightarrow{d^0} \cdots \]

If \( X^\bullet \in \mathbb{K}_\text{tac}(\text{prj-}R) \), then \( \beta_{\leq n}(X^\bullet) \in \mathbb{K}^{-b}(\text{prj-}R) \). The brutal truncation at degree zero induces a map from the category \( \mathbb{K}_\text{tac}(\text{prj-}R) \) to the category \( \mathbb{K}^{-b}(\text{prj-}R) \). However, this map is not a functor. Now consider the singularity category

\[ \mathbb{D}^b_{\text{sg}}(R) = \mathbb{K}^{-b}(\text{prj-}R)/\mathbb{K}^b(\text{prj-}R). \]

The brutal truncation induces a triangle functor

\[ \beta_{\text{proj}} : \mathbb{K}_\text{tac}(\text{prj-}R) \longrightarrow \mathbb{D}^b_{\text{sg}}(R). \]

This functor is always full and faithful. If \( R \) is either an Artin ring or commutative Noetherian local ring, then the functor \( \beta_{\text{proj}} \) is dense if and only if \( R \) is Gorenstein, see \cite{Buc87, Hap91} and \cite{BJO14}. Motivated by the discussion above, in this section we want to introduce \( N \)-totally acyclic complexes and \( N \)-singularity category. We show that the restriction of functor \( \mathbb{F} \) to this category is an equivalence. As a result of this equivalence, we show that there exists equivalences between \( N \)-singularity category of \( \text{Mod-}R \) and usual singularity category of \( \text{Mod-}T_{N-1}(R) \).

It is easy to see that an \( N \)-complex \( X^\bullet \in \mathbb{K}_N(\text{prj-}R) \) is \( N \)-acyclic if and only if

\[ \text{Hom}_{\mathbb{K}_N(\text{prj-}R)}(P^\bullet, X^\bullet) = 0 \]

for all \( P^\bullet \in \mathbb{K}_N^b(\text{prj-}R) \).

**Definition 4.2.** An \( N \)-complex \( X^\bullet \in \mathbb{K}_N(\text{prj-}R) \) is called \( N \)-totally acyclic if and only if

\[ \text{Hom}_{\mathbb{K}_N(\text{prj-}R)}(P^\bullet, X^\bullet) = 0 \quad \text{and} \quad \text{Hom}_{\mathbb{K}_N(\text{prj-}R)}(X^\bullet, P^\bullet) = 0 \]

for all \( P^\bullet \in \mathbb{K}_N^b(\text{prj-}R) \). We denote by \( \mathbb{K}_N^\text{ac}(\text{prj-}R) \) the category of all \( N \)-totally acyclic complexes in \( \text{prj-}R \).

Clearly, if \( X^\bullet \in \mathbb{K}_N^\text{ac}(\text{prj-}R) \) then \( X^\bullet \) is an \( N \)-acyclic complex.

**Proposition 4.3.** For a left coherent ring \( R \), we have the following triangle equivalences.

(i) \( \mathbb{K}_N(\text{Prj-}R) \cong \mathbb{K}_\text{ac}(\text{prj-}T_{N-1}(R)). \)

(ii) \( \mathbb{K}_N^\text{ac}(\text{prj-}R) \cong \mathbb{K}_\text{tac}(\text{prj-}T_{N-1}(R)). \)

**Proof.** (i) By theorem 3.17, the functor \( \mathbb{F} \) induced an equivalence

\[
\begin{array}{ccc}
\mathbb{K}_N(\text{Prj-}R) & \xrightarrow{\mathbb{F}} & \mathbb{K}(\text{Prj-}T_{N-1}(R)) \\
\downarrow & & \downarrow \\
\mathbb{K}_N^\text{ac}(\text{prj-}R) & \cong & \mathbb{K}_\text{ac}(\text{prj-}T_{N-1}(R))
\end{array}
\]

Let \( P^\bullet \) be an object of \( \mathbb{K}_N^\text{ac}(\text{prj-}R) \). Hence \( \text{Hom}_{\mathbb{K}_N(\text{prj-}R)}(Q^\bullet, P^\bullet) = 0 \) for all \( Q^\bullet \in \mathbb{K}_N^\text{ac}(\text{prj-}R) \). Now let \( \mathcal{P}^\bullet \in \mathbb{K}_N^b(\text{prj-}T_{N-1}(R)) \). There exists an object \( Q^\bullet \in \mathbb{K}_N^b(\text{prj-}R) \) such that \( \mathbb{F}(Q^\bullet) = \mathcal{P}^\bullet \). We have

\[ \text{Hom}_{\mathbb{K}_N(\text{prj-}T_{N-1}(R))}(\mathcal{P}^\bullet, \mathbb{F}(P^\bullet)) = \text{Hom}_{\mathbb{K}_N(\text{prj-}T_{N-1}(R))}(\mathbb{F}(Q^\bullet), \mathbb{F}(P^\bullet)) \cong \text{Hom}_{\mathbb{K}_N(\text{prj-}R)}(Q^\bullet, P^\bullet) = 0. \]

It shows that \( \mathbb{F}(P^\bullet) \in \mathbb{K}_\text{ac}(\text{prj-}T_{N-1}(R)) \). Hence the functor \( \mathbb{F} \) sends any object of the subcategory \( \mathbb{K}_N^\text{ac}(\text{prj-}R) \) of \( \mathbb{K}_N(\text{prj-}R) \) to an object of the subcategory \( \mathbb{K}_\text{ac}(\text{prj-}T_{N-1}(R)) \) of \( \mathbb{K}(\text{Prj-}T_{N-1}(R)) \).

Let \( \mathcal{P}^\bullet \in \mathbb{K}_\text{ac}(\text{prj-}T_{N-1}(R)) \). In a similar way there exists \( P^\bullet \in \mathbb{K}_N^\text{ac}(\text{prj-}R) \) such that \( \mathbb{F}(P^\bullet) = \mathcal{P}^\bullet \).
$\mathcal{P}^\ast$. Hence $F|_{K_N^{-b}(\text{prj}-R)}$ is dense.
(ii) It is similar to (i) by use of remark 4.1. □

We define the $N$-singularity category $\mathbb{D}_N^{\text{sg}}(R)$ of $R$ as a Verdier quotient

$$K^{-b}_N(\text{prj}-R)/K^b_N(\text{prj}-R).$$

**Remark 4.4.** The brutal truncation at degree zero induces a triangle functor

$$K_{\text{trac}}^{\text{ac}}(\text{prj}-R) \to \mathbb{D}_N^{\text{sg}}(R),$$

and the following diagram shows that this functor is always full and faithful. Moreover it is a triangle equivalence of categories when $R$ is a Gorenstein ring.

$$\begin{array}{ccc}
K_{\text{trac}}^{\text{ac}}(\text{prj}-T_{N-1}(R)) & \overset{\beta_{\text{prac}}}{\longrightarrow} & \mathbb{D}_N^{\text{sg}}(T_{N-1}(R)) = K^{-b}_N(\text{prj}-T_{N-1}(R))/K^b_N(\text{prj}-R)
\end{array}$$

The functor $\tilde{F}$ is induced from equivalences in diagram after theorem 3.15.

Remark 4.4 provides us with another interpretation of quotient category

$$K^{\infty, b}(\text{prj}-R)/K^b(\text{prj}-R)$$

where $K^{\infty, b}(\text{prj}-R)$ is the homotopy category of unbounded complexes with bounded homologies. Iyama and et al. showed that there is a triangle equivalence between the above quotient category and $\mathbb{D}_N^{\text{sg}}(T_2(R))$, whenever $R$ is a Gorenstein ring, see [IKM11]. Hence by remark 4.4 we have the following equivalence of categories

$$K^{\infty, b}(\text{prj}-R)/K^b(\text{prj}-R) \cong \mathbb{D}_3^{\text{sg}}(R).$$

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