Entangled spins and ghost-spins

Dileep P. Jatkar\textsuperscript{1,2} and K. Narayan\textsuperscript{3}

1. Harish-Chandra Research Institute  
Chhatnag Road, Jhusi, Allahabad 211019, India  
2. Homi Bhabha National Institute  
Training School Complex, Anushakti Nagar, Mumbai 400085, India  
3. Chennai Mathematical Institute,  
SIPCOT IT Park, Siruseri 603103, India.

Abstract

We study patterns of quantum entanglement in systems of spins and ghost-spins regarding them as simple quantum mechanical toy models for theories containing negative norm states. We define a single ghost-spin as in arXiv:1602.06505 [hep-th] as a 2-state spin variable with an indefinite inner product in the state space. We find that whenever the spin sector is disentangled from the ghost-spin sector (both of which could be entangled within themselves), the reduced density matrix obtained by tracing over all the ghost-spins gives rise to positive entanglement entropy for positive norm states, while negative norm states have an entanglement entropy with a negative real part and a constant imaginary part. However when the spins are entangled with the ghost-spins, there are new entanglement patterns in general. For systems where the number of ghost-spins is even, it is possible to find subsectors of the Hilbert space where positive norm states always lead to positive entanglement entropy after tracing over the ghost-spins. With an odd number of ghost-spins however, we find that there always exist positive norm states with negative real part for entanglement entropy after tracing over the ghost-spins.
# Contents

1 Introduction .......................................................... 1

2 Reviewing spins and ghost-spins .................................. 4

3 Tracing over ghost-spins: the reduced density matrix ....... 7

4 Spins disentangled from ghost-spins ............................. 9
   4.1 Two spins and two ghost-spins, disentangled ............... 11

5 One spin entangled with two ghost-spins ....................... 12

6 Spins entangled with one ghost-spin ............................ 17
   6.1 One spin entangled with one ghost-spin ..................... 17
   6.2 Multiple spins entangled with one ghost-spin .......... 20

7 Multi-ghost-spin systems ........................................... 21
   7.1 Three ghost-spins ............................................. 21
   7.2 Multiple ghost-spins ........................................ 22
   7.3 One spin entangled with multiple ghost-spins .......... 23

8 Discussion ............................................................. 24

A Spin & ghost-spin: off-diagonal density matrix ......... 27

B Tracing over a spin and a ghost-spin .......................... 28

# 1 Introduction

The concept of entanglement, in some sense, is at the heart of the interpretation of quantum mechanics. The entanglement entropy is a measure of entanglement of two subsystems of a quantum mechanical system. Initially, systems with a finite number of degrees of freedom were investigated by decomposing them into two disjoint subsets and computing entanglement between these subsets using measures like the von Neumann entropy or the Renyi entropy. Over the last several years, these methods have been extended to computing entanglement entropy in quantum field theories as well. Although the problem of computing entanglement entropy is substantially more complicated in quantum field theories, various techniques have been developed to evaluate it in specific cases. A partial list of references and reviews is [1–10]. In the context of holography [11–14], the Ryu-Takayanagi formulation [15–19] via bulk extremal surfaces has enabled new investigations and perspectives on quantum entanglement in strongly interacting quantum field theories.
In quantum field theories with a gauge symmetry, one naturally encounters degrees of freedom which have negative norm. Although the physical subspace of the Hilbert space has definite norm, in many gauge choices we end up having to deal with degrees of freedom with indefinite norm. In order to ask questions about entanglement in the gauge theories, it would be desirable to have a better understanding of systems which have indefinite norm. Instead of directly attempting this in the gauge theory context, it is easier to look at toy models which are simpler to deal with but at the same time reflect the intricacies of the system with indefinite norm. In this paper, we will consider a system consisting of ordinary spins, which mimic the positive norm part of the Hilbert space, and “ghost-spins” (as defined in [20]), which incorporate the indefinite norm sector. Although in the gauge theory context a certain restricted class of entangled states with mixing of definite and indefinite norm occur, the toy models distilled from them exhibit many interesting possibilities. Restricting to the physical Hilbert space then corresponds to tracing over all the ghost-spins (regarded as invisible) in these toy models and looking at the reduced density matrix of the spin system.

The motivation for defining “ghost-spins” in [20] (where entanglement entropy in certain ghost CFTs was studied) arose from dS/CFT: we will review this in the Discussion section (sec. 7). Here we simply explore patterns of quantum entanglement that emerge in systems containing entangled spins and ghost-spins, regarding them as toy models for subsectors with negative norm states arising in covariant formulations of theories with gauge symmetry, as mentioned earlier. As we will review in sec. 2, while a single spin is defined as a 2-state spin variable with a positive definite inner product $\langle \uparrow | \uparrow \rangle = 1 = \langle \downarrow | \downarrow \rangle$ and $\langle \uparrow | \downarrow \rangle = 0 = \langle \downarrow | \uparrow \rangle$, a single ghost-spin is defined, as in [20], as a 2-state spin variable with the indefinite inner product $\langle \uparrow | \uparrow \rangle = 0 = \langle \downarrow | \downarrow \rangle$ and $\langle \uparrow | \downarrow \rangle = 1 = \langle \downarrow | \uparrow \rangle$, akin to the inner products in the bc-ghost system as is well-known (see e.g. [21]). For multiple variables, the spin Hilbert space has a positive definite metric $g_{ij} = \delta_{ij}$, while the ghost-spin states have a non-positive metric $\gamma_{ij}$, with components $\gamma_{++} = 1$, $\gamma_{--} = -1$, by a basis change $\{| \uparrow \rangle, | \downarrow \rangle \} \rightarrow \{| + \rangle, | - \rangle \}$ which makes negative norm states manifest. Overall these systems of spins and ghost-spins appear to give a broad class of toy models with a lot of flexibility to engineer a variety of quantum mechanical systems containing negative norm states.

In sec. 2, we will briefly review the spin and the ghost-spin system, and patterns of entanglement in the two ghost-spin system studied in [20]. In sec. 3, we define the reduced density matrix $\rho_A^s$ for the remaining spin variables after tracing over all the ghost-spins by requiring that the correlation function $\langle \psi | O_s | \psi \rangle$ (appearing in the expectation value) in any given state $| \psi \rangle$ for any observable $O_s$ of spin variables alone is identical to that calculated using the density matrix of the mixed state of the remaining spins as $\text{tr}_s(O_s \rho_A^s)$. In general, the Hilbert space of spins and ghost-spins contains positive as well as negative norm states. One might ask if the entanglement entropy $S_A$ of $\rho_A^s$ is uniformly positive for all positive
norm states, and uniformly negative for all negative norm states. This can be shown to be identically true (sec. 4), when the spin sector is not entangled with the ghost-spin sector (both of which could be entangled within themselves). In this case the state is a product state comprising spins disentangled from ghost-spins: the sign of the norm of the state enters as an overall sign in $\rho_A$, giving $S_A > 0$ for positive norm states, while for negative norm states, $S_A$ has a negative real part and a constant imaginary part. This is similar to the case of two ghost-spins studied in [20] for the $\rho_A$ obtained after tracing over one ghost-spin.

When the spins are entangled with the ghost-spins, then this straightforward correlation between positive norm states and positivity of the entanglement entropy appears to not be true as we discuss from sec. 5 onwards. The von Neumann entropy contains components of $\rho_A$ which in turn contains linear sub-combinations of the norm of the state. Thus even for positive norm states, some components of $\rho_A$ can be negative in general (while keeping positive the trace of $\rho_A$, which is the norm of the state): this leads to new entanglement patterns in general\(^1\). Requiring that positive norm states give positive entanglement $S_A$ amounts to requiring that the components $(\rho_A)^{IJ}$ are positive ($I, J$ being labels for the remaining spin variables): this is only true for specific subregions of the Hilbert space, \textit{i.e.} only certain families of states. (Correspondingly, negative norm states give negative real part for $S_A$ only for certain families of states.)

More generally, when the spins are entangled with ghost-spins, we can restrict to subfamilies of states which have correlated ghost-spins, \textit{i.e.} the ghost-spin values are the same in each basis state. When the number of ghost-spins is even, this implies that all allowed states are positive norm, \textit{i.e.} negative norm states are excluded. This restricts to half the space of states which are now all positive norm, and the entanglement entropy is manifestly positive. The intuition here is in a sense akin to simulating \textit{e.g.} the $X^\pm + bc$ subsector of the 2-dim sigma model representing the string worldsheet theory: in general negative norm states are cancelled between $X^\pm$ and the $bc$-ghost subsectors in the eventual physical theory. The more general subsectors in the Hilbert space where $\rho_A$ gives positive entanglement entropy for positive norm states can then be interpreted as the component of the state space that is connected to this correlated ghost-spin sector. We demonstrate this explicitly in sec. 5 where we elaborately study the case of a single spin entangled with two ghost-spins.

In general, the family of entangled spin \& ghost-spin systems splits into two sectors. One of them is with an even number of ghost-spins and the other is with an odd number of ghost spins. Unlike the even ghost-spin sector, we find that for systems with odd number of ghost-spins, such a consistent subfamily of correlated ghost-spin states does not exist so it is not possible to uniformly pick a family of entangled states mentioned above such that

\footnote{One can restrict to subcases for which $\rho_A$ is diagonal (for calculational simplicity): this still leaves many parameters and therefore many entanglement patterns.}
positive norm states give positive entanglement entropy. We analyse the case of one spin entangled with one ghost-spin in detail in sec. 6.1 illustrating this. A similar analysis for a system of \( k \) spins entangled with one ghost-spin is discussed in sec. 6.2. In sec. 7, we study systems containing multiple ghost-spins focussing on the case of odd numbers of ghost-spins and show that there always exist positive norm states that lead to entanglement entropy with negative real part.

The indefinite inner products we use for the ghost-spins can be recast, as in the \( \text{bc} \)-ghost system, in terms of a ghost “zero mode” operator insertion. Then the basis states have zero norm and expectation values are nonvanishing only in the presence of the zero mode insertion (which in the \( \text{bc} \)-system was required to cancel background charge). For our present purposes in this paper, we continue to use the indefinite norm language.

2 Reviewing spins and ghost-spins

Here we review the toy model of two “ghost-spins” \([20]\), which abstracts away from the specific technical issues of the ghost CFTs there but mimics some of the key features.

Firstly, for ordinary spin variables with a 2-state Hilbert space consisting of \( \{\uparrow, \downarrow\} \), we take the usual positive definite norms in the Hilbert space

\[
\text{spins} : \quad \langle \uparrow | \uparrow \rangle = \langle \downarrow | \downarrow \rangle = 1, \quad \langle \uparrow | \downarrow \rangle = \langle \downarrow | \uparrow \rangle = 0 . \quad (2.1)
\]

A generic state, its adjoint and positive definite norm are

\[
|\psi\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle, \quad \text{adjoint} \quad \langle \psi| = c_1^* \langle \uparrow | + c_2^* \langle \downarrow | ; \quad \langle \psi|\psi\rangle = |c_1|^2 + |c_2|^2 . \quad (2.2)
\]

Thus we can normalize states as \( \langle \psi|\psi\rangle = 1 \) and pick a representative ray with unit norm (equivalent to calculating expectation values of operators as \( \langle O\rangle = \frac{\langle \psi|O|\psi\rangle}{\langle \psi|\psi\rangle} = \langle \psi|O|\psi\rangle \)). The reduced density matrix obtained by tracing out the second spin is

\[
\rho_A = tr_B |\psi\rangle\langle \psi| = \sum_i \langle i_B|\psi\rangle\langle \psi|i_B \rangle = \langle \uparrow_B|\psi\rangle\langle \psi|\uparrow_B \rangle + \langle \downarrow_B|\psi\rangle\langle \psi|\downarrow_B \rangle . \quad (2.3)
\]

The familiar discussions in 2-spin systems of entanglement entropy via the reduced density matrix are recovered as follows. States of the system such as \( |\psi\rangle = c_1|\uparrow\uparrow\rangle + c_2|\downarrow\downarrow\rangle \) can be normalized as \( \langle \psi|\psi\rangle = 1 = |c_1|^2 + |c_2|^2 \) which is positive definite, and ensure that \( |c_1|, |c_2| \leq 1 \). With these norms, the reduced density matrix (2.3) becomes \( \rho_A = |c_1|^2 |\uparrow\rangle\langle \uparrow| + |c_2|^2 |\downarrow\rangle\langle \downarrow| \). Note that the reduced density matrix is automatically normalized as \( tr\rho_A = 1 \) once the state \( |\psi\rangle \) is normalized. Thus the entanglement entropy given as the von Neumann entropy of \( \rho_A \) is \( S_A = -tr\rho_A \log \rho_A = -\sum_i \rho_A(i) \log \rho_A(i) \) which is positive definite since each eigenvalue
\( \rho_A(i) < 1 \) makes the \(- \log \rho_A(i) > 0 \). For the states above with \( x = |c_1|^2 \), we obtain \( S_A = -x \log x - (1 - x) \log(1 - x) \); this is positive definite since \( 0 < x < 1 \).

We define a single “ghost-spin” by a similar 2-state Hilbert space \( \{\uparrow, \downarrow\} \), but with norms

\[ |\uparrow\rangle - |\downarrow\rangle \text{ has norm } -2. \]

This is akin to the normalizations in the \( bc \)-ghost system in \cite{20} (see e.g. \cite{21}, Appendix, vol. 1 where this inner product appears). Now a generic state and its non-positive norm are

\[ |\psi\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle \Rightarrow \langle \psi|\psi\rangle = c_1^*c_2 + c_2^*c_1, \]

where we have taken the adjoint to be \( \langle \psi| = c_1^*\langle \uparrow | + c_2^*\langle \downarrow | \), as in \eqref{eq:inner_product}. Then for instance \( |\uparrow\rangle - |\downarrow\rangle \) has norm \(-2\). It is then convenient to change basis to

\[ |\pm\rangle \equiv \frac{1}{\sqrt{2}} (|\uparrow\rangle \pm |\downarrow\rangle), \quad \langle +|+\rangle = \gamma_{++} = 1, \quad \langle -|\rangle = \gamma_{--} = -1, \quad \langle +|\rangle = \langle -|\rangle = 0. \]

A generic state with nonzero norm can be normalized to norm \( +1 \) or \(-1 \). Then a negative norm state can be written as \( |\psi\rangle = \psi^++|+\rangle + \psi^-|\rangle \) with \( \langle \psi|\psi\rangle = |\psi^++|^2 - |\psi^-|\rangle = -1 \). For every such state (or ray), there is a corresponding state \( |\psi^\perp\rangle \) with norm \( +1 \) orthogonal to \( |\psi\rangle \), i.e. \( \langle \psi^\perp|\psi^\perp\rangle = 1, \quad \langle \psi^\perp|\psi\rangle = 0. \) There are also zero norm states with \( \langle \psi|\psi\rangle = 0, \) i.e. \( |\psi^\perp|^2 = |\psi^-|^2 \), which do not admit any canonical normalization, e.g. \( |\uparrow\rangle, \ |\downarrow\rangle \).

Now considering the two ghost-spin system, basis states are

\[ |s_{AB}\rangle \equiv |\uparrow\uparrow\rangle, \ |\uparrow\downarrow\rangle, \ |\downarrow\uparrow\rangle, \ |\downarrow\downarrow\rangle \equiv |++\rangle, \ |+-\rangle, \ |-+\rangle, \ |--\rangle. \]

The \( |\pm \pm\rangle \) basis is more transparent for our purposes. The inner product or metric on this space of states is not positive definite so the various contractions need to be defined carefully.

We define the states, adjoints and norms as

\[ |\psi\rangle = \sum \psi^{\alpha\beta}|\alpha\beta\rangle, \quad \text{adjoint : } \langle \psi| = \sum \langle \alpha\beta|\psi^{\alpha\beta*}, \]

\[ \langle \psi|\psi\rangle = \langle \kappa|\alpha\rangle\langle \lambda|\beta\rangle \psi^{\alpha\beta}\psi^{\kappa\lambda*} \equiv \gamma_{\alpha\kappa}\gamma_{\beta\lambda}\psi^{\alpha\beta}\psi^{\kappa\lambda*} = \gamma_{\alpha\alpha}\gamma_{\beta\beta}|\psi|^{\alpha\beta}|^2, \]

where repeated indices as usual are summed over: the last expression pertains to the \( |\pm\rangle \) basis where the metric \( \gamma \) is diagonal, with \( \gamma_{++} = 1, \quad \gamma_{--} = -1 \). A generic normalized positive/negative norm state with norm \( \pm 1 \) is

\[ |\psi\rangle = \psi^++|++\rangle + \psi^-|+-\rangle + \psi^-|+-\rangle + \psi^-|--\rangle \Rightarrow \langle \psi|\psi\rangle = |\psi^{++}|^2 + |\psi^-|^2 - |\psi^+|^2 - |\psi^-|^2 = \pm 1. \]

This translates to corresponding conditions on the coefficients \( \psi^{\alpha\beta} \). A simple example of a positive norm state is \( |\psi\rangle = \psi^++|++\rangle + \psi^-|--\rangle \), while \( |\psi\rangle = \psi^-|--\rangle + \psi^+|--\rangle \) has negative norm.
With the density matrix \( \rho = |\psi\rangle\langle\psi| = \sum \psi^{\alpha\beta} \psi^{\kappa\lambda}|\alpha\beta\rangle\langle\kappa\lambda| \), the reduced density matrix obtained by a partial trace over one spin can again be defined via a partial contraction as

\[
\rho_A = tr_B \rho \equiv (\rho_A)^{\alpha\kappa}|\alpha\rangle\langle\kappa| , \quad (\rho_A)^{\alpha\kappa} = \gamma_{\beta\lambda} \psi^{\alpha\beta} \psi^{\kappa\lambda}^* = \gamma_{\beta\lambda} \psi^{\alpha\beta} \psi^{\kappa\lambda}^* ,
\]

(2.10)

\[
\Rightarrow \quad (\rho_A)^{++} = |\psi^{++}|^2 - |\psi^{+-}|^2 , \quad (\rho_A)^{+-} = \psi^{++}\psi^{+-} - \psi^{+-}\psi^{--} , \quad (\rho_A)^{--} = |\psi^{--}|^2 - |\psi^{+-}|^2 ,
\]

(2.11)

Then \( tr\rho_A = \gamma_{\alpha\kappa}(\rho_A)^{\alpha\kappa} = (\rho_A)^{++} - (\rho_A)^{--} \). Thus the reduced density matrix is normalized to have \( tr\rho_A = tr\rho = \pm 1 \) depending on whether the state (2.9) is positive or negative norm. Also, \( \rho_A \) can have some eigenvalues negative.

The entanglement entropy calculated as the von Neumann entropy of \( \rho_A \) is

\[
S_A = -\gamma_{\alpha\beta}(\rho_A \log \rho_A)^{\alpha\beta} = -\gamma_{++}(\rho_A \log \rho_A)^{++} - \gamma_{--}(\rho_A \log \rho_A)^{--}
\]

(2.12)

where the last expression pertains to the \(|\pm\rangle\rangle\) basis with \( \gamma_{\pm\pm} = \pm 1 \). This requires defining \( \log \rho_A \) as an operator: we define this as the usual \( \log \)-expansion

\[
(\log \rho_A)^{\alpha\kappa} = (\log(1 + \rho_A - 1))^{\alpha\kappa} = (\rho_A - 1)^{\alpha\kappa} - \frac{1}{2}(\rho_A - 1)^{\alpha\beta}\gamma_{\beta\lambda}(\rho_A - 1)^{\lambda\kappa} + \ldots
\]

(2.13)

or equivalently via \( (\rho_A)^{\alpha\kappa} = (e^{\log \rho_A})^{\alpha\kappa} = 1^{\alpha\kappa} + (\log \rho_A)^{\alpha\kappa} + \frac{1}{2!}(\log \rho_A)^{\alpha\beta}\gamma_{\beta\lambda}(\log \rho_A)^{\lambda\kappa} + \ldots \)

and the solution thereof. The signs in the contractions in \( \log \rho_A \) are perhaps more easily dealt with if we use the mixed-index reduced density matrix \( (\rho_A)^{\alpha\kappa} \).

To illustrate this, let us for simplicity consider a simple family of states where the reduced density matrix is diagonal, by restricting to \( \psi^{++} = \frac{\psi^{++}}{\psi^{++}} \psi^{--} \). In this case, \( \log \rho_A \) is also diagonal and can be calculated easily. From (2.11) for the state (2.9), this gives

\[
\psi^{++} = \frac{\psi^{++} \psi^{--}^*}{\psi^{++}} \Rightarrow \quad \langle \psi | \psi \rangle = (|\psi^{++}|^2 - |\psi^{+-}|^2) \left( 1 + \frac{|\psi^{--}|^2}{|\psi^{++}|^2} \right) = \pm 1 , \quad \rho_A = \pm \left[ \frac{|\psi^{++}|^2}{|\psi^{++}|^2 + |\psi^{--}|^2} |+\rangle\langle+| - \frac{|\psi^{--}|^2}{|\psi^{++}|^2 + |\psi^{--}|^2} |\rangle\langle-| \right] ,
\]

(2.14)

where the \( \pm \) pertain to positive and negative norm states respectively. The location of the negative eigenvalue is different for positive and negative norm states, leading to different results for the von Neumann entropy. For negative norm states, \( (\rho_A)^{++} < 0 \), \( (\rho_A)^{--} > 0 \). Then the mixed-index reduced density matrix components \( (\rho_A)^{\alpha\kappa}_A = \gamma_{\alpha\beta}(\rho_A)^{\beta\kappa} \) are

\[
(\rho_A)^{+}_A = \pm x \quad , \quad (\rho_A)^{-}_A = \pm (1 - x) \quad , \quad x = \frac{|\psi^{++}|^2}{|\psi^{++}|^2 + |\psi^{--}|^2} , \quad 0 < x < 1 .
\]

(2.15)
Thus \( \text{tr} \rho_A = (\rho_A^+ + (\rho_A)^-) = \pm 1 \) manifestly. Now we obtain \((\log \rho_A)^+ = \log(\pm x)\) and \((\log \rho_A)^- = \log(\pm(1-x))\), the \(\pm\) referring again to positive/negative norm states respectively. The entanglement entropy (2.12) becomes 

\[
S_A = -(\rho_A^+ (\log \rho_A)^+ - (\rho_A^- (\log \rho_A)^-) \quad \text{and so}
\]

\[
\langle \psi | \psi \rangle > 0 : \quad S_A = -x \log x - (1-x) \log(1-x) > 0 , \quad (2.16)
\]

\[
\langle \psi | \psi \rangle < 0 : \quad S_A = x \log(-x) + (1-x) \log(-(1-x)) = x \log x + (1-x) \log(1-x)
\]

\[
= x \log x + (1-x) \log(1-x) + i\pi(2n+1)x + i\pi(2m+1)(1-x) .
\]

For positive norm states, \( S_A \) is manifestly positive since \( x < 1 \), just as in an ordinary 2-spin system. For negative norm states, we note that for the principal branch, \( i.e. n = m \), the imaginary part is independent of \( x \), \( i.e. \) the same for all such negative norm states if we choose the same branch for the logarithms. In our analysis that follows, we will for simplicity consider the principal branch only (with \( n,m = 0 \)), \( i.e. \) we will effectively set \( \log(-1) = i\pi \) henceforth. The real part of entanglement entropy is negative since \( x < 1 \) and the logarithms are negative: apart from the minus sign, it is the same as \( S_A \) for the positive norm states. This real part is minimized when \( x = \frac{1}{2} \) (this value corresponds to maximal entanglement for positive norm states): this “minimal” entanglement is \( S_A = -\log 2 + i\pi \).

The above discussion can also be phrased in terms of the \(|\uparrow\rangle,|\downarrow\rangle\) basis although we have found it convenient to use the \(|\pm\rangle\) basis. It is worth noting that while (2.4) mimics the norms of the \(bc\)-ghost system in [20], there is no obvious analog of the background charge here: in particular tracing over spin\(_A\) instead of spin\(_B\) is equivalent, so that entanglement entropy for the subsystem is the same as that for the complement.

### 3 Tracing over ghost-spins: the reduced density matrix

We consider systems of spins and ghost-spins, possibly entangled. The ghost-spins, representing the negative norm states, are regarded as invisible. The physical system is represented by the spin degrees of freedom and the physical information content thereof is obtained by tracing over the ghost-spins.

Operationally, we start with a state \( |\psi\rangle \) in the full Hilbert space, and the corresponding full density matrix \( \rho = |\psi\rangle\langle\psi| \) and construct a reduced density matrix by tracing over all the ghost-spins, \( i.e. \rho^s = \text{tr}_{gs}(\rho) \). The resulting subsystem is now a mixed state described by the reduced density matrix \( \rho^s \). Since this comprises only physical spin variables, we must require that this be a well-defined physically sensible system. As a minimal requirement in this regard, we expect that any observable \( O_s \) of the spin variables alone has a correlation function in the state \( |\psi\rangle \) that must satisfy

\[
\langle \psi | O_s | \psi \rangle = \text{tr}_s(O_s \rho^s) . \quad (3.1)
\]
Here the left hand side is the correlation function calculated in the full state $|\psi\rangle$ (including the ghost-spins), while the right hand side is calculated in the mixed state $\rho^s$ describing the remaining spins obtained after tracing over the ghost-spins. Since the left hand side contains an implicit trace over the ghost-spins, this gives a definition for the reduced density matrix $\rho^s$. When $O_s$ is the identity operator, (3.1) fixes the normalization of $\rho^s$ as

$$\langle \psi | \psi \rangle = tr_s(\rho^s).$$

(3.2)

In particular for positive norm states normalized as $\langle \psi | \psi \rangle = 1$, we have $tr_s(\rho^s) = 1$. The correlation function above of course appears in the expectation value of the observable as

$$\langle \psi | O_s | \psi \rangle = (\psi^*)^{j,\alpha\beta} O_s^{ij} \psi^{i,\alpha\beta} = (\psi^*)^{j,\alpha\beta} O^s_{ij} \psi^{i,\alpha\beta} = \langle j | O_s | i \rangle \langle i | \rho^s | j \rangle = tr(O_s \rho^s).$$

(3.4)

where $g_{jk} = \langle j | k \rangle$ is the positive definite inner product for the spin states. The above expression has traced over the ghost-spins and shows the resulting reduced density matrix to be

$$(\rho^s)^{ij} = \gamma_{\alpha\alpha} \gamma_{\beta\beta} \psi^{j,\alpha\beta} \psi^{i,\alpha\beta} = \gamma_{\alpha\alpha} \gamma_{\beta\beta} \psi^{j,\alpha\beta} \psi^{i,\alpha\beta},$$

(3.5)

where $\langle \sigma | \alpha \rangle = \gamma_{\alpha\sigma} = \gamma_{\sigma\alpha}$ is the indefinite inner product over the ghost-spin states. In our analysis (and as in [20]), we assume that the ghost-spin states have an inner product given by a real-valued, symmetric metric $\gamma_{\alpha\beta}$. In particular, as reviewed earlier, in the $|\pm\rangle$ basis (2.6), we have $\gamma_{++} = 1$, $\gamma_{--} = -1$. The second expression in (3.5) is specific to this diagonal metric. The resulting reduced density matrix still needs to satisfy positivity properties, if it is to describe a physical spin system: this imposes various conditions on generic states comprising entangled spins and ghost-spins, as we will discuss at length in what follows.

---

2To see that these expressions are consistent, consider a simple example of spins disentangled from ghost-spins (which we discuss in more detail in sec. 4). The state $|\psi\rangle$ can then be written as a product state $|\psi\rangle = |\psi_s\rangle |\psi_g\rangle$ and its norm is $\langle \psi | \psi \rangle = \langle \psi_s | \psi_s \rangle \langle \psi_g | \psi_g \rangle$. We normalize the norm as $\langle \psi | \psi \rangle = \pm 1$ for positive/negative norm states respectively. The expectation value is $\langle O_s \rangle = \frac{\langle \psi_s | O_s | \psi_s \rangle}{\langle \psi_s | \psi_s \rangle} = \frac{\langle \psi_s | O_s | \psi_s \rangle \langle \psi_s | \psi_s \rangle}{\langle \psi_s | \psi_s \rangle}$. Then $\langle O_s \rho^s \rangle = \langle O_s | \psi \rangle = \langle \psi_s | O_s | \psi_s \rangle \langle \psi_g | \psi_g \rangle = \pm \frac{\langle \psi_s | O_s | \psi_s \rangle}{\langle \psi_s | \psi_s \rangle} = \pm \langle O_s \rangle$. In particular for $O_s$ the identity operator, we have $tr_s(\rho^s) = \pm 1 = tr \rho = \langle \psi | \psi \rangle$ as expected.
For more general spin and ghost-spin systems, the above discussion can be generalized as follows. A generic state is \( |\psi\rangle = \psi^I,\alpha |I\rangle |\alpha\rangle \), where \( I \) is a collective label for states of multiple spin variables, and \( \alpha \) is a collective label for states of multiple ghost-spins. Then

\[
\langle \psi | O_s | \psi \rangle = (\psi^*)^J,\sigma O_s^{KL} \psi^J,\alpha \langle J | K \rangle \langle L | I \rangle \langle \sigma | \alpha \rangle \equiv \text{tr}(O_s \rho^s) \Rightarrow (\rho^s)^{IJ} = \langle \sigma | \alpha \rangle \psi^I,\alpha (\psi^*)^J,\sigma,
\]

where \( \langle J | K \rangle \) symbolises a product of multiple individual inner products \( \prod \langle j | k \rangle \) and \( \langle \sigma | \alpha \rangle \) a product of multiple ghost-spin inner products \( \prod \langle \sigma_k | \alpha_k \rangle \). This defines the reduced density matrix as above, which now has multiple indices \( I, J \).

As we see, the contractions in the reduced density matrix are fixed as given above, and roughly speaking they are correlated with the contraction patterns in the norm of the state. One might ask if there are other contraction schemes that one may cook up formally to trace over the ghost-spins towards defining the reduced density matrix. For instance consider

\[
(\rho^s)^{ik} = \gamma_{\alpha\rho} \gamma_{\beta\sigma} \psi^{i,\alpha\beta} (\psi^*)^{k,\sigma\rho} = \gamma_{\alpha\alpha} \gamma_{\beta\beta} \psi^{i,\alpha\beta} (\psi^*)^{k,\beta\alpha},\]

in the case of one spin and two ghost-spins. Here the complex conjugated state has reversed index list, instead of \( 3.5 \) where the conjugated state has first index contracted with the metric \( \gamma_{\alpha\beta} \). While this appears consistent formally, it does not satisfy the physical conditions \( 3.1 \) and \( 3.2 \), that must hold for the residual physical subsystem of spin variables alone. We will comment on this in specific places in what follows.

To summarise, we have seen how the reduced density matrix \( 3.1 \), \( 3.5 \), \( 3.6 \), arises from tracing over the (invisible) ghost-spins, satisfying the physical requirements \( 3.1 \), \( 3.2 \) expected of the residual physical spin system. In what follows, we will explore the patterns of entanglement that arise from this operation in various categories of spin & ghost-spin systems.

## 4 Spins disentangled from ghost-spins

We will start with a configuration where spins are not entangled with ghost spins. This case is similar in spirit to the longitudinal and time-like degrees of freedom and their ghost counterparts in gauge theories. In the free theory, these sectors are decoupled from the physical sector. As we will see, in the disentangled spin/ghost-spin system we can show in general that the entanglement entropy is positive definite for positive norm states and has negative definite real part for negative norm states.

We define the norm of a state in the Hilbert space by defining the metric on the state space. The spin Hilbert space has a positive definite metric while the ghost-spin states have
a non-positive metric $\gamma_{\alpha\beta}$,
\[
g_{ij} = \delta_{ij}, \quad \gamma_{++} = 1, \quad \gamma_{--} = -1. \tag{4.1}
\]
The $\gamma_{\alpha\beta}$ metric is the same as (2.6), equivalent to the off-diagonal form $\langle \uparrow | \downarrow \rangle = \langle \downarrow | \uparrow \rangle = 1$ in (2.4). This is equivalent to defining the adjoints of the ghost-spin states as
\[
(\langle \uparrow | \rangle)^\dagger = \langle \uparrow | = \langle \downarrow | \downarrow | \rangle = \langle \uparrow | \rangle = \langle \uparrow | c_0, \quad (\langle \downarrow | \rangle)^\dagger = \langle \downarrow | = \langle \uparrow | \rangle = \langle \uparrow | c_0, \tag{4.2}
\]
with $c_0$ a “zero mode” operator, analogous to the ghost zero mode $c_0$ in the $c = -2$ bc-ghost system [20], where nonvanishing correlation functions required an appropriate ghost zero mode insertion to cancel the background charge inherent in these systems. In our discussion throughout this paper, we will however continue to use the non-positive metric for ghost-spin states for simplicity.

Returning to our discussion of entanglement entropy, if the spin sector is not entangled with the ghost-spin sector, then the most general state is of product form
\[
|\psi\rangle = |\psi_s\rangle \ |\psi_{gs}\rangle, \quad \langle\psi|\psi\rangle = \langle\psi_s|\psi_s\rangle \ \langle\psi_{gs}|\psi_{gs}\rangle, \tag{4.3}
\]
\[
\langle\psi_s|\psi_s\rangle = g_{i_1j_1} \cdots g_{i_nj_n} (|\psi_s\rangle^{i_1j_1} \cdots (|\psi_s\rangle^{i_1j_1} \cdots)^{i_nj_n}, \quad \langle\psi_{gs}|\psi_{gs}\rangle = \gamma_{\alpha_1\beta_1} \cdots \gamma_{\alpha_n\beta_n} (|\psi_{gs}\rangle^{\alpha_1\alpha_2} \cdots (|\psi_{gs}\rangle^{\alpha_1\alpha_2} \cdots)^{\beta_1\beta_2} \cdots.
\]
Since $|\psi_s\rangle$ is contracted with $g_{ij}$, this sector is entirely positive norm as expected (with $\langle\psi_s|\psi_s\rangle > 0$): on the other hand, $|\psi_{gs}\rangle$ contracted with $\gamma_{\alpha\beta}$ can give rise to negative norm states if $\langle\psi_{gs}|\psi_{gs}\rangle < 0$.

The reduced density matrix obtained after tracing over all the ghost-spins is
\[
\rho_A^s = tr_{gs} (|\psi_s\rangle \langle\psi_s| \ |\psi_{gs}\rangle \langle\psi_{gs}|),
\]
\[
(\rho_A^s)^{i_1 \cdots k_1} = \gamma_{\alpha_1\beta_1} \cdots \gamma_{\alpha_n\beta_n} (|\psi_{gs}\rangle^{\alpha_1} \cdots (|\psi_{gs}\rangle^{\alpha_1} \cdots)^{\beta_1} \cdots = \langle\psi_{gs}|\psi_{gs}\rangle (|\psi_s\rangle^{i_1} \cdots (|\psi_s\rangle^{i_1} \cdots)^{k_1} \cdots.
\]
We will now normalize positive/negative norm states to have norm $\pm 1$ respectively, i.e.
\[
\langle\psi_{gs}|\psi_{gs}\rangle \geq 0 \quad \Rightarrow \quad \langle\psi|\psi\rangle = \langle\psi_s|\psi_s\rangle \ \langle\psi_{gs}|\psi_{gs}\rangle = \pm 1 \quad [\langle\psi_s|\psi_s\rangle > 0]. \tag{4.5}
\]

\[3\]In the present context also, this implies the existence of a pair of operators satisfying $\{b_0, c_0\} = 1$ with the $\{\uparrow, \downarrow\}$ states forming a representation thereof. We then couple this with the usual positive definite norm on states in the Hilbert space to define expectation values. Nonzero expectation values are obtained only after a $c_0$ insertion: i.e. $\langle \downarrow | \downarrow \rangle = 0 = \langle \uparrow | \uparrow \rangle$, $\langle \downarrow | c_0 | \downarrow \rangle = 1 = \langle \uparrow | c_0 | \uparrow \rangle$. A generic ghost-spin state $|\psi\rangle = c_1 | \uparrow \rangle + c_2 | \downarrow \rangle$ then has adjoint $\langle |\psi\rangle |^\dagger = c^*_1 \langle \downarrow | c_0 + c^*_2 \langle \uparrow | c_0$, recovering the inner product $\langle |\psi\rangle |^\dagger, |\psi\rangle \rangle = \langle \psi|\psi\rangle = c_1 c_2^* + c_2 c_1^*$ identical to (2.5). Thus our analysis can equivalently be phrased using the explicit insertion of this $c_0$ operator in expectation values: in this rephrasing, all expectation values vanish without the insertion and entanglement entropy also vanishes. An analog of the ghost-number operator here would be $N_g \sim c_0 b_0$, which can be used to classify states. Along with this, an analog of the Hamiltonian $L_0$ with appropriate commutation relations would be useful to study dynamics in these systems; our study here is mostly “kinematic”.

10
From (4.4), we then see that $\rho_A^s$ is automatically normalized as

$$( \rho_A^s )^{i_1 \ldots k_1 \ldots} = \pm \frac{1}{\langle \psi_s | \psi_s \rangle} (\psi_s)^{i_1 \ldots} (\psi_s)^{k_1 \ldots} \quad \Rightarrow \quad tr \rho_A^s = \pm 1 \quad (\langle \psi | \psi \rangle \gtrless 0) . \quad (4.6)$$

For positive norm states, $\rho_A^s$ is positive definite with eigenvalues $0 < \lambda_i < 1$ (if the spin sector is entangled) and $\sum_i \lambda_i = 1$: thus we have the usual positive definite entanglement entropy for $\rho_A^s$,

$$S_A = -tr_s \rho_A^s \log \rho_A^s = -\sum_i \lambda_i \log \lambda_i > 0 . \quad (4.7)$$

For negative norm states on the other hand, $\rho_A^s$ is negative definite, with eigenvalues $-\lambda_i$. This gives

$$S_A = -tr_s \rho_A^s \log \rho_A^s = -\sum_i (\lambda_i) \log (-\lambda_i) = \sum_i \lambda_i \log \lambda_i + i\pi , \quad (4.8)$$

with a negative definite real part and constant imaginary part (since $\sum_i \lambda_i = 1$). As mentioned after (2.16), this constant imaginary part here stems from our choice of the branch of the logarithm with $\log(-1) = i\pi$ (for simplicity) and corresponds to an overall minus sign in the reduced density matrix (which is otherwise positive definite, with no relative minus sign amongst the eigenvalues).

Thus when the spin sector is not entangled with the ghost-spin sector (both of which can be entangled within themselves), we see in great generality that positive norm states have positive entanglement entropy while negative norm states have entanglement entropy with a negative definite real part and a constant imaginary part. We recall that the two ghost-spin system exhibited similar behaviour [20].

### 4.1 Two spins and two ghost-spins, disentangled

We will illustrate the above generalities for a simple but illustrative example: consider a system of two spins and two ghost spins. The general state in this case and its norm are

$$|\psi\rangle = \psi^{ij,\alpha\beta}|ij\rangle|\alpha\beta\rangle , \quad \langle \psi | \psi \rangle = g_{ik} g_{jl} \gamma_{\alpha\sigma} \gamma_{\beta\rho} \psi^{ij,\alpha\beta}(\psi^*)^{kl,\sigma\rho} . \quad (4.9)$$

Since there are two $\gamma_{\alpha\beta}$ factors in the contraction, it is clear that terms with a single minus ghost-index $\alpha, \beta$ will acquire a minus sign, giving e.g. $(-|\psi^{+-,-+}|^2)$ while terms like $(|\psi^{++,-+}|^2)$ will contribute with a $+$-sign in the norm.

From sec. 3, tracing over the ghost-spins gives the reduced density matrix ((3.1), (3.5), (3.6))

$$(\rho_A)^{ij,kl} = \gamma_{\alpha\sigma} \gamma_{\beta\rho} \psi^{ij,\alpha\beta}(\psi^*)^{kl,\sigma\rho} = \gamma_{\alpha\alpha} \gamma_{\beta\beta} \psi^{ij,\alpha\beta}(\psi^*)^{kl,\alpha\beta} . \quad (4.10)$$
Taking the spins to be disentangled from the ghost-spins (both of which could be entangled within themselves), the general state is of product form,
\[
|\psi\rangle = |\psi_s\rangle|\psi_{gs}\rangle = (c_{++}|++\rangle + c_{+-}|+-\rangle + c_{-+}|-+\rangle + c_{--}|--\rangle) \times
(\psi^{++}|++\rangle + \psi^{+-}|+-\rangle + \psi^{-+}|-+\rangle + \psi^{--}|--\rangle),
\]
\[
\langle\psi|\psi\rangle = (|c_{++}|^2 + |c_{+-}|^2 + |c_{-+}|^2 + |c_{--}|^2)(|\psi^{++}|^2 - |\psi^{+-}|^2 - |\psi^{-+}|^2 + |\psi^{--}|^2). \tag{4.11}
\]
Thus we have \(\psi^{++}\cdot\cdot\cdot\equiv c_{++}\psi^{++}\) etc. Using (4.10) gives
\[
(r_\varphi)^{ij,kl} = \langle\psi_{gs}|\psi_{gs}\rangle c^i c^ks \longrightarrow (r_\varphi)^{i,k} = g_{ji}(r_\varphi)^{ij,kl}, \tag{4.12}
\]
where we have performed a further trace over one of the spins to obtain the reduced density matrix \((r_\varphi)^{i,k}\) for the remaining spin: this gives
\[
(r_\varphi)^{++} = (|c_{++}|^2 + |c_{+-}|^2)\langle\psi_{gs}|\psi_{gs}\rangle, \quad (r_\varphi)^{+-} = (c^{++}(c^*)^{-+} + c^{+-}(c^*)^{--})\langle\psi_{gs}|\psi_{gs}\rangle, \quad (r_\varphi)^{-+} = (c^{--}(c^*)^{++} + c^{++}(c^*)^{--})\langle\psi_{gs}|\psi_{gs}\rangle, \quad (r_\varphi)^{--} = (|c^{--}|^2 + |c^{++}|^2)\langle\psi_{gs}|\psi_{gs}\rangle. \tag{4.13}
\]
We see that \((r_\varphi)\) inherits the sign from the ghost-spin sector. A simple entangled spin state, its normalization and associated reduced density matrix are
\[
|\psi_s\rangle = c^{++}|++\rangle + c^{--}|--\rangle, \quad \langle\psi_{gs}|\psi_{gs}\rangle = \pm \frac{1}{\langle\psi_s|\psi_s\rangle} = \pm \frac{1}{|c^{++}|^2 + |c^{--}|^2},
\]
\[
(r_\varphi)^{++} = \pm \frac{|c^{++}|^2}{|c^{++}|^2 + |c^{--}|^2} \equiv \pm x, \quad (r_\varphi)^{--} = \pm \frac{|c^{--}|^2}{|c^{++}|^2 + |c^{--}|^2} = \pm (1 - x). \tag{4.14}
\]
We have \(0 < x < 1\). Then the entanglement entropy for this state is
\[
S_\varphi = -(\pm x) \log(\pm x) - (\pm (1 - x)) \log(\pm (1 - x)) \tag{4.15}
\]
which is clearly positive definite for positive norm states (+ sign). For negative norm states, we have \(S_\varphi = x \log x + (1 - x) \log(1 - x) + i\pi\), with a negative real part and a constant imaginary part. This verifies the general structure stated earlier.

In the following sections we will study systems of spins entangled with ghost-spins: this is somewhat more intricate and there are many new entanglement patterns depending on detailed properties of the entangled state.

## 5 One spin entangled with two ghost-spins

We will now consider a single spin entangled with two ghost-spins. This system, as we will see, is quite rich in generating a spectrum of entanglement with complex entanglement entropy with non-constant imaginary part as well as real part correlated with the norm of
the state, *i.e.* there exist families of entangled states with positive entanglement entropy when the norm is positive.

A generic state and its norm are

$$|\psi\rangle = \psi^{i,\alpha\beta} |i\rangle |\alpha\beta\rangle, \quad \langle\psi|\psi\rangle = g_{ij} \gamma_{\alpha\sigma} \gamma_{\beta\rho} \psi^{i,\alpha\beta} (\psi^*)^{j,\sigma\rho} .$$  \hspace{1cm} (5.1)

Since there are two $\gamma_{\alpha\beta}$ factors in the contraction, it is clear that terms with a single minus ghost-index $\alpha, \beta$ will acquire a minus sign, giving *e.g.* $(-|\psi^{+,+}||+\rangle + |\psi^{-,-}||-\rangle)$ while terms like $(|\psi^{+,+}|^2)$ will contribute with a $+\,$-sign in the norm. For instance, a simple entangled state with positive norm is $|\psi\rangle = |\psi^{++,+}||+\rangle + |\psi^{-,-}||-\rangle$ with norm $\langle\psi|\psi\rangle = |\psi^{++,+}|^2 + |\psi^{-,-}|^2$.

Explicitly writing the most general state, we have

$$|\psi\rangle = |\psi^{++,+}|^2 - |\psi^{+,+}||+\rangle - |\psi^{-,-}||-\rangle$$

$$+ |\psi^{-,-}||-\rangle + |\psi^{-,-}||-\rangle + |\psi^{-,-}||-\rangle + |\psi^{-,-}||-\rangle + |\psi^{-,-}||-\rangle$$

with norm

$$\langle\psi|\psi\rangle = |\psi^{++,+}|^2 - |\psi^{+,+}||+\rangle - |\psi^{-,-}||-\rangle$$

$$+ |\psi^{-,-}||-\rangle - |\psi^{-,-}||-\rangle - |\psi^{-,-}||-\rangle - |\psi^{-,-}||-\rangle - |\psi^{-,-}||-\rangle .$$  \hspace{1cm} (5.2)

**Patterns of entanglement:** As discussed in sec. 3, tracing over the ghost-spins gives the reduced density matrix (3.1), (3.5), (3.6),

$$(\rho_A)^{ik} = \gamma_{\alpha\sigma} \gamma_{\beta\rho} \psi^{i,\alpha\beta} (\psi^*)^{k,\sigma\rho} = \gamma_{\alpha\alpha} \gamma_{\beta\beta} \psi^{i,\alpha\beta} (\psi^*)^{k,\alpha\beta} .$$  \hspace{1cm} (5.4)

Explicitly, this reduced density matrix after tracing over both ghost-spins is

$$(\rho_A)^{++} = |\psi^{++,+}|^2 - |\psi^{++,+}|^2 - |\psi^{-,-}|^2$$

$$(\rho_A)^{+-} = |\psi^{++,+}(\psi^*)^{++,+} + |\psi^{-,-}||-\rangle$$

$$(\rho_A)^{-+} = |\psi^{-,-}(\psi^*)^{++,+} + |\psi^{-,-}||-\rangle$$

$$(\rho_A)^{--} = |\psi^{-,-}|^2 - |\psi^{-,-}|^2 - |\psi^{-,-}|^2 .$$  \hspace{1cm} (5.5)

**Physical requirement:** After tracing over the ghost spins, we obtain a reduced density matrix for just ordinary spins alone. On physical grounds, this should be required to be positive definite for positive norm states, since these can equivalently be decomposed into purely physical effective positive norm basis states (even if there were underlying ghost-like states in the full system). Equivalently, since the remaining spins are ordinary spins, they should allow good physical interpretation for positive norm states with positive entanglement. (The negative norm states sector need not allow as clear a physical interpretation.)
Firstly, as in sec.1, if we consider the spins to be disentangled from the ghost-spins (both of which could be entangled within themselves), then the general state (5.2) is of the form

\[ |\psi\rangle = (c^+|+\rangle + c^-|\rangle) \left( \psi^{++}|++\rangle + \psi^{+-}|+-\rangle + \psi^{-+}|-+\rangle + \psi^{--}|--\rangle \right), \]

\[ \langle \psi|\psi\rangle = (|c^+|^2 + |c^-|^2)(|\psi^{++}|^2 - |\psi^{+-}|^2 - |\psi^{-+}|^2 + |\psi^{--}|^2) = \langle \psi_s|\psi_s\rangle \langle \psi_{gs}|\psi_{gs}\rangle. \]

In other words, here \( \psi^{+,++} \equiv c^+\psi^{++} \) etc. This gives

\[ (\rho_A)^{++} = |c^+|^2\langle \psi_{gs}|\psi_{gs}\rangle, \quad (\rho_A)^{+-} = c^+(c^*)\langle \psi_{gs}|\psi_{gs}\rangle, \]

\[ (\rho_A)^{--} = c^-(c^*)\langle \psi_{gs}|\psi_{gs}\rangle, \quad (\rho_A)^{--} = |c^-|^2\langle \psi_{gs}|\psi_{gs}\rangle. \]

We see that \( (\rho_A) \) acquires the sign from the ghost-spin sector. In this case, since there is just a single spin, \( \det(\rho_A) = 0 \) and \( S_A = 0 \) of course. In the two spins disentangled from two ghost-spins, we saw earlier that \( S_A > 0 \) for positive norm states.

Next, we consider the cases where the spin is entangled with the two ghost-spins. We see from (5.4), (5.5), that in general \( \rho_A \) positivity (and thereby \( S_A > 0 \)) for positive norm states is not possible in the entire Hilbert space of states but is possible in subsectors thereof \( (i.e. \) for subfamilies of states), since there are sufficiently many parameters\(^4\). We will analyse various interesting cases in detail below.

First, an interesting subfamily of restricted states is obtained if we require that the ghost-spins are correlated, \( i.e. \) with the ghost-spins being identical in each basis state: then the only allowed states are

\[ |\psi\rangle = \psi^{++,+}|++,+\rangle + \psi^{+-,}|+-,\rangle + \psi^{-+,+}|-,+-\rangle + \psi^{--,}|--,--\rangle. \]

Since there is an even number of minus signs, this entire subfamily of states is manifestly positive norm, from (5.3). In other words, we have excluded all negative norm states and thus we recover positivity of entanglement entropy manifestly (as can be verified from (5.5)).

More generally, there are minus signs in \( \rho_A \). To explore the possibilities for positive norm states giving \( S_A > 0 \), consider the relatively simple but instructive subfamily of states

\[ |\psi\rangle = \psi^{++,+}|++,+\rangle + \psi^{+-,}|+-,\rangle + \psi^{-+,+}|-,+-\rangle + \psi^{--,}|--,--\rangle \]

(whih in general are not product states). These have norm

\[ \langle \psi|\psi\rangle = |\psi^{++,+}|^2 - |\psi^{+,+}|^2 - |\psi^{+-,}|^2 + |\psi^{--,}|^2. \]

\(^4\)For a single spin entangled with a ghost-spin, we will see in the next section that \( \rho_A \) always has a negative eigenvalue: so this sector cannot be salvaged since there are not enough parameters. Interestingly, for the system of two ghost-spins [20] reviewed earlier, this is possible: the signs in \( \rho_A \) are just right!
which are negative if

\[ |\psi^{++,+}|^2 + |\psi^{--,\cdot}|^2 < |\psi^{++,\cdot}|^2 + |\psi^{--\cdot}|^2, \quad (5.10) \]

somewhat similar to the two ghost-spins system reviewed earlier. The states (5.8) have sufficiently many parameters while restricting to a subfamily with \( \rho_A \) diagonal (thus simplifying \( \log \rho_A \) as well) and \( (\rho_A)^{++,\cdot}, (\rho_A)^{--\cdot} > 0 \) for positive norm states. Explicitly, the reduced density matrix (5.4) becomes

\[
(\rho_A)^{++} = |\psi^{++,+}|^2 - |\psi^{++,\cdot}|^2, \quad (\rho_A)^{+-\cdot} = 0,
\]

\[
(\rho_A)^{-+\cdot} = 0, \quad (\rho_A)^{--\cdot} = -|\psi^{--+}|^2 + |\psi^{--\cdot}|^2. \quad (5.11)
\]

Note that \( tr \rho_A = g_{ik} (\rho_A)^{ik} = (\rho_A)^{++,\cdot} + (\rho_A)^{--\cdot} \) and satisfies \( tr \rho_A = tr \rho = \langle \psi | \psi \rangle \). Since the remaining spin has positive definite metric \( g_{ij} \), we have the entanglement entropy

\[
S_A = -g_{ij} (\rho_A \log \rho_A)^{ij} = -(\rho_A \log \rho_A)^{++,\cdot} - (\rho_A \log \rho_A)^{--\cdot}. \quad (5.12)
\]

For states with positive/negative norm, we can normalize as

\[
\langle \psi | \psi \rangle = |\psi^{++,+}|^2 - |\psi^{++,\cdot}|^2 - |\psi^{--+}|^2 + |\psi^{--\cdot}|^2 = \pm 1, \quad (5.13)
\]

so that the entanglement entropy becomes

\[
|\psi^{++,+}|^2 - |\psi^{++,\cdot}|^2 \equiv x, \quad \langle \psi | \psi \rangle = x + (1 - x); \quad (\rho_A)^{++,\cdot} = x, \quad (\rho_A)^{--\cdot} = \pm 1 - x, \quad (5.14)
\]

\[
S_A = -x \log x - (1 - x) \log(1 - x) .
\]

**Positive norm:** Now if \( x > 0 \), then \( (1 - x) > 0 \) also from (5.13), (5.14), implying \( 0 < x < 1 \) and \( \rho_A \) is positive definite. This gives

\[
S_A = -x \log x - (1 - x) \log(1 - x) > 0 . \quad (5.15)
\]

If \( x < 0 \), then \( (1 - x) > 0 \), giving \( (\rho_A)^{++,\cdot} < 0, \ (\rho_A)^{--\cdot} > 0 \), with

\[
S_A = |x| \log |x| - (1 + |x|) \log(1 + |x|) + i\pi |x| , \quad (5.16)
\]

where the real part is negative which shows anti-correlation with the norm. In addition there is an imaginary part which depends linearly on \( x \).

This behaviour of \( S_A \) can be interpreted as follows. Choosing \( x > 0 \) means we assign higher probability to getting the \(+ + \) ghost-spin state than the \(+ - \) state (and likewise the \(- - \) versus \(- + \)). Since the \(+ + \) and \(- - \) are positive norm states (in the sense of (5.7) with correlated ghost-spins), \( x > 0 \) corresponds to the component of the Hilbert
space continuously connected to the correlated ghost-spin sector (which contains only $|++\rangle$ and $|--\rangle$ ghost-spin basis states). Equivalently starting with the correlated ghost-spin sector of the Hilbert space, small deformations of the state vector by turning on small $|+-\rangle$ components (or $|--\rangle$) are still positive norm only if $x > 0$. For $x < 0$, this feature does not exist simply because the corresponding state is not continuously connected to the positive norm correlated ghost-spin sector.

**Negative norm:** Now $\rho_A$ is negative definite if $x < 0$ and $(-1 - x) < 0$, so that $0 < |x| < 1$, giving

$$S_A = |x| \log |x| + (1 - |x|) \log(1 - |x|) + i\pi ,$$

(5.17)

with a negative real part since the logs are negative, and a constant imaginary part. With $x < -1$, we have $(\rho_A)^{++} < 0$ and $(\rho_A)^{--} = -1 + |x| > 0$, which gives

$$S_A = |x| \log |x| - (|x| - 1) \log(|x| - 1) + i\pi |x| ,$$

(5.18)

where the real part is positive but the imaginary part is not constant anymore. Likewise when we have $(\rho_A)^{++} > 0$, $(\rho_A)^{--} < 0$, which corresponds to $x > 0$ and $-1 - x < 0$, we get

$$S_A = -x \log x + (1 + x) \log(1 + x) + i\pi (1 + x) ,$$

(5.19)

where the real part is again positive. Finally the choice $(\rho_A)^{++} > 0$, $(\rho_A)^{--} > 0$, gives $x > 0$, $-1 - x > 0$, which is clearly not possible.

To investigate other possibilities, let us consider restricting to a diagonal $\rho_A$, this corresponds to setting $(\rho_A)^{++}, (\rho_A)^{--} = 0$. This gives rise to the reduced density matrix corresponding to yet another subfamily of entangled states. Setting to zero the off-diagonal terms in the reduced density matrix (5.5) gives

$$(\psi^*)^{-+ -} = \frac{1}{\psi^{++}} (\psi^{++})(\psi^*)^{-+} - \psi^{+- -} (\psi^*)^{-+} + \psi^{++ -} (\psi^*)^{++}$$

(5.20)

and a complex conjugated condition. Using these conditions the norm of the state as well as the diagonal components of the reduced density matrix can be written as

$$\langle \psi | \psi \rangle = |\psi^{++}|^2 - |\psi^{+-}|^2 - |\psi^{+-}|^2 + |\psi^{--}|^2 + |\psi^{++}|^2 - |\psi^{+-}|^2 + |\psi^{++}|^2 - |\psi^{+-}|^2$$

$$-\frac{1}{|\psi^{++}|^2} ((\psi^{++})(\psi^*)^{-+} - \psi^{+- -} (\psi^*)^{-+} + \psi^{++ -} (\psi^*)^{++})^2$$

(5.21)

and

$$(\rho_A)^{++} = |\psi^{++}|^2 - |\psi^{+-}|^2 - |\psi^{+-}|^2 + |\psi^{++}|^2,$$

(5.22)

$$(\rho_A)^{--} = |\psi^{--+}|^2 - |\psi^{-+-}|^2 - |\psi^{-+-}|^2 + |\psi^{--+}|^2 .$$
This form of $\rho_A$ is again quite flexible, i.e. there are subregions in the Hilbert space where positive norm states give $S_A > 0$, which is along the lines of (5.14). The resulting analysis therefore follows the same pattern. Although the generic state does not give positive $S_A$ for positive norm states unless we impose further restrictions, if we allow all such basis states, then we can always find subfamilies of states where $S_A > 0$ for positive norm states.

6 Spins entangled with one ghost-spin

In this section we will consider entangled systems containing one ghost-spin. We will start with one spin entangled with one ghost-spin and later generalize it to multiple spins entangled with one ghost-spin.

6.1 One spin entangled with one ghost-spin

Here we will demonstrate that whenever we have one ghost-spin entangled with an ordinary spin, it is not possible to find an entangled state which has positive entanglement entropy for positive norm states and negative entanglement entropy for negative norm states after tracing over the ghost-spin. Although we will use one spin and one ghost-spin system, it is easy to see that the conclusion is independent of the number of spins in the system (after tracing out the ghost-spin) as we will see in the next section. The reason is that the outcome completely depends on the ghost-spin system.

A point to note here is that the entanglement entropy, no matter whether we have positive norm states or negative norm states, is necessarily a complex quantity with non-constant imaginary part. This is to be contrasted with the disentangled system where we have complex entanglement entropy for negative norm states but the imaginary part was constant. We also find that the real part of the entanglement entropy is anti-correlated with the norm, i.e. positive norm states have negative definite real part of the entropy and vice versa.

A generic state for one spin and one ghost-spin system is

$$|\psi\rangle = \psi^{i,\alpha}|i\rangle|\alpha\rangle$$

where $i = \pm$ refers to the spin index while $\alpha = \pm$ refers to the ghost-spin index. Then the norm is

$$\langle\psi|\psi\rangle = g_{ij}\gamma_{\alpha\beta}\psi^{i,\alpha}\psi^{j,\beta} = \sum_{i,\alpha}\gamma_{\alpha\alpha}\psi^{i,\alpha}\psi^{*i,\alpha}$$

$$= |\psi^{+,+}|^2 - |\psi^{+,-}|^2 + |\psi^{-,+}|^2 - |\psi^{-,-}|^2 = \pm |1|,$$

where the normalization $\pm 1$ refers to positive/negative norm states respectively. This is negative norm if $|\psi^{+,+}|^2 - |\psi^{+,-}|^2 + |\psi^{-,+}|^2 - |\psi^{-,-}|^2 < 0$. 

17
From sec. 3 (see (3.1), (3.5), (3.6)), the reduced density matrix obtained by tracing out
the ghost-spin in the present case is

\[(\rho_A)^{ik} = \gamma_{\alpha\beta}\psi^{i,\alpha}(\psi^*)^{k,\beta} = \gamma_{\alpha\alpha}\psi^{i,\alpha}(\psi^*)^{k,\alpha}\]  

(6.3)

\[
\Rightarrow (\rho_A)^{++} = |\psi^{+,+}|^2 - |\psi^{+-}|^2, \quad (\rho_A)^{+-} = \psi^{+,+}(\psi^*)^{-+,+} - \psi^{+,+}(\psi^*)^{-+,+},
\]

(6.4)

\[
(\rho_A)^{-+} = \psi^{-+}(\psi^*)^{+,+} - \psi^{-+}(\psi^*)^{+,+}, \quad (\rho_A)^{--} = |\psi^{-+}|^2 - |\psi^{-+}|^2.
\]

This is identical to the case of the two ghost-spin system, except that \(\rho_A\) is now contracted
with the positive definite spin metric. The mixed-index reduced density matrix is obtained
by raising an index with the (positive definite) spin metric, giving

\[(\rho_A)^{i} = (\rho_A)_{++}, \quad (\rho_A)^{--} = (\rho_A)_{--}.\]

Focussing on the subfamily with \((\rho_A)^{+-} = 0\) such that
the reduced density matrix is diagonal, we have (see Appendix A for non-diagonal \(\rho_A\))

\[
(\psi^*)^{-,+} = \frac{\psi^{-,+}(\psi^*)^{-,-}}{\psi^{+,+}}, \quad \frac{1}{|\psi^{+,+}|^2}(|\psi^{+,+}|^2 - |\psi^{-+}|^2)(|\psi^{+,+}|^2 - |\psi^{-+}|^2) = \pm 1,
\]

(6.5)

\[
(\rho_A)^{++} = |\psi^{+,+}|^2 - |\psi^{+-}|^2, \quad (\rho_A)^{--} = -\frac{|\psi^{-+}|^2}{|\psi^{+,+}|^2}(|\psi^{+,+}|^2 - |\psi^{-,+}|^2).
\]

We see that \((\rho_A)^{--}\) necessarily has sign opposite to \((\rho_A)^{++}\), so all states, including positive
norm states, necessarily have a negative eigenvalue. Simplifying gives

\[
(\rho_A)^{++} = \pm \frac{|\psi^{+,+}|^2}{|\psi^{+,+}|^2 - |\psi^{-+}|^2} \equiv \mp x, \quad (\rho_A)^{--} = \mp \frac{|\psi^{-+}|^2}{|\psi^{+,+}|^2 - |\psi^{-,+}|^2} = \pm (1 - x).
\]

(6.6)

Due to the relative sign between the two terms in the denominator we end up with \(|x| > 1\).
Then tracing over the remaining spin using its positive definite metric gives the entanglement
entropy

\[
S_A = -g_{ij}(\rho_A \log \rho_A)^{ij} = -(\rho_A \log \rho_A)^{++} - (\rho_A \log \rho_A)^{--}
\]

\[
= -(\mp x) \log(\pm x) - (\pm (1 - x)) \log(\pm (1 - x)).
\]

(6.7)

Since \(1 - x\) is necessarily negative for positive norm states, we have an imaginary component
in the entanglement entropy,

\[
S_A = -x \log x + (x - 1) \log(x - 1) + i\pi(x - 1).
\]

(6.8)

Note that the real part of the entropy is negative and the imaginary part is \(x\)-dependent.
In other words, \(S_A\) is not positive for positive norm states. It turns out that this is a
generic feature of this system. For example we could try to consider restricted cases, \(i.e.\)
we can fix some of the parameters to see if special entangled states can exhibit positivity of
The real part of identical to the one so far. In particular for the example above, with conditions: let us therefore consider the analog of (density matrix yields something useful, although this is not expected to satisfy the physical γ
glement entropy is obtained by contracting with the metric ρ
+ giving
S
A
= −|ψ
+−|2 log (|ψ
+−|2) + |ψ
+−|2 log (|ψ
+−|2) + |ψ
−−|2(iπ) .

We see that this now depends on whether the state is positive or negative norm, which are normalized respectively as ±1. For positive norm states, we take |ψ
−−|2 = |ψ
+−|2 − 1, so
S
A
= −x log x + (x − 1) log(x − 1) + (x − 1)(iπ) , x = |ψ
+−|2 , 1 ≤ x < ∞ . (6.10)
The real part of S
A
is always negative definite, although this is a positive norm state: also there is an imaginary part. For negative norm states, we take |ψ
−−|2 = |ψ
+−|2 + 1, giving
S
A
= −x log x + (x + 1) log(x + 1) + (x + 1)(iπ) , x = |ψ
+−|2 , 0 ≤ x < ∞ . (6.11)
In this case, the real part of S
A
is positive definite, although this is a negative norm state.

Tracing over the spin first gives (ρ
A
)αβ = g
i
jψ
i
jα(ψ*)β which has no negative signs since g
i
j is positive definite. But the mixed-index reduced density matrix is (ρ
A
)β = γαδ(ρ
A
)δβ, identical to the one so far. In particular for the example above, with ψ
+− = 0, we have
(ρ
A
)++ = |ψ
+−|2, (ρ
A
)−− = |ψ
−−|2, while (ρ
A
)++ = |ψ
+−|2, (ρ
A
)−− = −|ψ
−−|2. The entanglement entropy is obtained by contracting with the metric γαβ of the remaining ghost-spin, giving S
A
= −γαβ(ρ
A
log ρ
A
)αβ = −(ρ
A
)++(log ρ
A
)−− − (ρ
A
)−−(log ρ
A
)−− as before.

Given the result so far, it is worth asking if any other contraction scheme for the reduced density matrix yields something useful, although this is not expected to satisfy the physical conditions: let us therefore consider the analog of (3.7) in sec. 3. In the present case, tracing over the ghost-spin would correspond to

(ρ
A
)ik = γαβψ
i
jα(ψ*)βk = γαβψ
j
kα(ψ*)β . (6.12)

See also the discussion around eq.(3.7) for a similar scheme for the case of one spin and two ghost-spins. The components of the reduced density matrix are

(ρ
A
)++ = |ψ
+−|2 − ψ
+−(ψ*)−− , (ρ
A
)−− = ψ
+−(ψ*)−− − ψ
+−(ψ*)−− ,
(ρ
A
)−− = ψ
−−(ψ*)−− − ψ
−−(ψ*)−− . (6.13)

Requiring the hermiticity condition on ρ
A
then implies ψ
+− = ψ
−−. Substituting this condition back into the form of the reduced density matrix gives

(ρ
A
)++ = |ψ
+−|2 − |ψ
+−|2 , (ρ
A
)−− = ψ
+−(ψ*)−− − ψ
+−(ψ*)−− ,
(ρ
A
)−− = |ψ
−−|2 − |ψ
−−|2 . (6.14)
This is diagonal if $|\psi^{+,+}|^2 = |\psi^{-,-}|^2$, however $\rho_A$ is again not positive definite. As a result, this different contraction rule also does not help improve the situation (although this was simply a technical exercise, not accounting for the physical conditions). It is just as well that the conundrum of the anti-correlation of the sign of the entanglement entropy with the norm is unaffected by this alternate contraction scheme. This is because a single ghost-spin system is analogous to a gauge theory with a single set of indefinite norm states. We know that we need a second set of indefinite norm fields, namely the ghost fields, to effectively impose the restriction to the physical subspace.

6.2 Multiple spins entangled with one ghost-spin

We will now show that the results obtained for the system studied in the previous subsection 6.1 can be easily extended to an arbitrary number of spins entangled with one ghost-spin. This establishes that the anti-correlation between the norm of the state and sign of the real part of the entanglement entropy is an effect entirely due to tracing over the single ghost-spin degree of freedom.

Let us consider a system with $k$ spins entangled with a single ghost-spin. As usual we will denote the ghost spin by $\pm$; however, to denote the $k$-tuple of ordinary spins, we will use indices $I, J, \cdots$. They run over $2^k$ possible configurations of $k$ spins. A state with spin configuration $I$ and ghost spin $+$ is denoted as $|I, +\rangle$. Using this notation, we can write the reduced density matrix after tracing over the ghost-spin degrees of freedom as

$$
(\rho_A)^{I,J} = \psi^{I,+} (\psi^{J,+})^* - (\psi^{I,-})^* \psi^{J,-} - (\rho_A)^{I,I} |\psi^{I,+}|^2 - |\psi^{I,-}|^2.
$$

(6.15)

To illustrate the point, let us restrict to the diagonal form of the reduced density matrix. We will see in a moment that there is no loss of generality in doing this. Setting the off diagonal components of the density matrix to zero gives the condition

$$
\psi^{I,+} = \frac{(\psi^{I,-})^* \psi^{J,-}}{(\psi^{J,+})^*}.
$$

(6.16)

We can now relate the diagonal components of $\rho_A$ using the condition (6.16), obtaining

$$
(\rho_A)^{I,J} = -(\rho_A)^{I,I} \frac{|\psi^{I,+}|^2}{|\psi^{I,-}|^2}.
$$

(6.17)

This relation is the multi-spin generalization of (6.6) and is valid for any pair $(I, J)$. Therefore, further analysis of this system has similarities with that in section 6.1. In particular we see that as long as we have entangled states with a single ghost-spin and we trace over it then, in general, we always end up obtaining negative eigenvalues in the reduced density.
matrix $\rho_A$. Thus positive norm states do not lead to positive entanglement entropy. This result is independent of the number of ordinary spins in the entangled state.

We will now comment on the choice of the reduced density matrix. Suppose we are considering a conventional spin system. Since the metric on this space is positive definite, the entropy of entangled states is always positive definite, no matter what choice of basis we select to denote these spin states. Let us now consider this spin system entangled with a ghost-spin. Since our aim is to trace over the ghost-spin and write down the reduced density matrix for the conventional spins, the choice of basis in the conventional spin space does not affect the form of the reduced density matrix.

## 7 Multi-ghost-spin systems

In this section we will consider multiple entangled ghost-spin systems. As a warm up, we will first look at the three entangled ghost-spins system and then generalize it to multiple entangled ghost-spins. We find that in a certain class of entangled states it is easy to distinguish entangled states of an even number of ghost-spins from those involving an odd number of ghost-spins. When the number of ghost-spins is even, then after tracing over all the ghost-spins except one, we get manifestly positive definite entanglement entropy. However, when the number of ghost-spins is odd, then following the same procedure of tracing over all ghost-spins except one gives a negative definite real part of the entanglement entropy for positive norm states. We will also consider entangling this multi-ghost-spin system with one spin. After tracing over all ghost-spins, the resulting entanglement entropy exhibits the same odd vs. even distinction as the pure multi-ghost-spin system.

### 7.1 Three ghost-spins

We will begin with a system of three entangled ghost-spins. A generic state and its norm are

$$|\psi\rangle = \psi^{\alpha\beta\gamma}|\alpha\beta\gamma\rangle, \quad \langle\psi|\psi\rangle = \gamma_{\alpha\delta\gamma}\gamma_{\beta\sigma\gamma}\psi^{\alpha\beta\gamma}(\psi^*)^{\delta\sigma\rho}.$$  

(7.1)

Explicitly writing the most general state, we have

$$|\psi\rangle = \psi^{++}|++\rangle + \psi^{+-}|+-\rangle + \psi^{-+}|+\rangle + \psi^{--}| \rangle + \psi^{++}|++\rangle + \psi^{+-}|+-\rangle + \psi^{-+}|+\rangle + \psi^{--}| \rangle$$

(7.2)

with norm

$$\langle\psi|\psi\rangle = |\psi^{++}|^2 - |\psi^{+-}|^2 - |\psi^{-+}|^2 + |\psi^{--}|^2$$

$$- |\psi^{+-}|^2 + |\psi^{-+}|^2 + |\psi^{--}|^2 - |\psi^{--}|^2.$$  

(7.3)
The reduced density matrix $(\rho_A)^{\alpha\delta} = \gamma_{\beta\gamma} \gamma_{\rho} \psi^{\alpha\beta\gamma}(\psi^*)^{\delta\rho} = \gamma_{\beta\lambda} \gamma_{\rho} \psi^{\alpha\beta\lambda}(\psi^*)^{\delta\rho}$ for the last ghost-spin after tracing over two ghost-spins is

$$(\rho_A)^{++} = |\psi^{++}|^2 - |\psi^{+-}|^2 - |\psi^{+\gamma}|^2 + |\psi^{-}\gamma|^2,$$

$$(\rho_A)^{+-} = \psi^{++}(\psi^*)^{+-} - \psi^{+-}(\psi^*)^{++} - \psi^{+\gamma}(\psi^*)^{-\gamma} + \psi^{-\gamma}(\psi^*)^{+\gamma},$$

$$(\rho_A)^{-+} = \psi^{++}(\psi^*)^{-+} - \psi^{+-}(\psi^*)^{++} - \psi^{-\gamma}(\psi^*)^{+\gamma} + \psi^{+\gamma}(\psi^*)^{-\gamma},$$

$$(\rho_A)^{--} = |\psi^{--}|^2 - |\psi^{+-}|^2 - |\psi^{+\gamma}|^2 + |\psi^{-\gamma}|^2.$$

Consider the relatively simple but instructive subfamily of states

$$|\psi\rangle = |\psi^{++}| + |++\rangle + |+-\rangle + |+\rangle + |\psi^{--}| + |\psi^{--}|$$

(7.5)

(which in general are not product states) with normalized norm

$$\langle \psi | \psi \rangle = |\psi^{++}|^2 - |\psi^{++}|^2 - |\psi^{+-}|^2 - |\psi^{--}|^2 = \pm 1.$$  

(7.6)

The reduced density matrix becomes

$$(\rho_A)^{++} = |\psi^{++}|^2 - |\psi^{++}|^2,$$

$$(\rho_A)^{+-} = 0,$$

$$(\rho_A)^{-+} = 0,$$

$$(\rho_A)^{--} = -|\psi^{--}|^2 + |\psi^{--}|^2,$$

(7.7)

and the mixed-index $(\rho_A)^{\alpha\beta}_\gamma = \gamma_{\alpha\beta} (\rho_A)_\gamma$ is $(\rho_A)^{+}_+ = (\rho_A)^{++}, (\rho_A)^{-}_- = -(\rho_A)^{--}$, so that $tr \rho_A = (\rho_A)^{+}_+ + (\rho_A)^{-}_- = \langle \psi | \psi \rangle$. Also $\log(\rho_A)^{+}_+ = \log((\rho_A)^{++})$ etc. So the entanglement entropy becomes $S_A = -(\rho_A)^{+}_+(\log \rho_A)^{+}_+ - (\rho_A)^{-}_-(\log \rho_A)^{-}_-$. We then have (similar to (5.14) in the case of one spin and two ghost-spins)

$$|\psi^{++}|^2 - |\psi^{+-}|^2 \equiv x,$$

$$\langle \psi | \psi \rangle = x + (\pm 1 - x);$$

$$\rho_A^{+}_+ = x, \quad \rho_A^{-}_- = \pm 1 - x,$$

$$S_A = -x \log x - (\pm 1 - x) \log(\pm 1 - x).$$

(7.8)

As in the discussion following (5.14), we can see that positive norm states with $x > 0$ have $S_A > 0$ but when $x < 0$ we obtain $Re(S_A) < 0$ and $Im(S_A) \neq 0$. However unlike in that case, there is no correlated ghost-spin subsector here so no reason to restrict to $x > 0$ states.

### 7.2 Multiple ghost-spins

Consider a system of $n$ ghost-spins and the entangled state (and its norm)

$$|\psi\rangle = \psi^{++} \cdots | + + \cdots \rangle + \psi^{--} \cdots | - - \cdots \rangle,$$  

$$\langle \psi | \psi \rangle = |\psi^{++} \cdots |^2 + (-1)^n |\psi^{--} \cdots |^2.$$  

(7.9)
This state is a linear combination of a state with all + and another with all −. The mixed-index reduced density matrix for a subsystem comprising a single ghost-spin after tracing over the remaining $n-1$ ghost-spins, and associated entanglement entropy are

$$
(r_A)^{++}_+ = (r_A)^{++} = |\psi^{++...}|^2 , \quad (r_A)^{- -} = -(r_A)^{- -} = (-1)^n|\psi^{- -...}|^2 , \\
S_A = -(r_A)^{++}_+ (\log r_A)^{++}_+ - (r_A)^{- -}_- (\log r_A)^{- -}_- .
$$

(7.10)

For $n$ even, the state is clearly positive norm and so has manifestly positive $r_A$ and thus positive entanglement entropy. For $n$ odd however, we have $|\psi^{++...}|^2 - |\psi^{- -...}|^2 = \pm 1$ for normalized positive/negative norm states, giving

$$
S_A = -|\psi^{++...}|^2 \log (|\psi^{++...}|^2) + |\psi^{- -...}|^2 \log (|\psi^{- -...}|^2) + |\psi^{- -...} (i\pi) ,
$$

(7.11)

very similar to the case of one spin entangled with one ghost-spin (see e.g. eqs.(6.9), (6.10), and (6.11)). In particular, for positive norm states, $S_A$ has a negative definite real part (and an imaginary part) while for negative norm states $Re(S_A) > 0$. The states (7.9) always exist and have this counter-intuitive structure for odd numbers of ghost-spins.

In the above, we could implicitly regard the $n$ ghost-spins as a “ghost-spin-chain” in a 1-dimensional space with the ghost-spins located at lattice sites. Then the single ghost-spin comprises a subchain whose entanglement with the rest of the chain has the above structure.

### 7.3 One spin entangled with multiple ghost-spins

We will generalize the case studied in the last subsection by coupling it to one ordinary spin. As we will see, the outcome is identical in the sense that there is a clear distinction between states with an odd number of ghost-spins and those with an even number of ghost-spins. The reduced density matrix after tracing over all ghost-spins is positive definite for even number of ghost-spins which contains positive norm states. In the case of odd numbers of ghost-spins, we exhibit simple entangled states which always exhibit anticorrelation between their norm and the sign of the real part of the entanglement entropy.

Consider a system consisting of a spin and $n$ ghost-spins. We will look at a simple entangled state which is a straightforward generalization of the $n$ ghost-spin state in (7.9). We denote the state with the first index representing the spin and the rest representing the ghost-spins,

$$
|\psi\rangle_{(1,n)} = \psi^{++...} (++, ++...) + \psi^{- -...} (-,-,-...) ,
$$

(7.12)

$$
_{(1,n)}\langle \psi|\psi\rangle_{(1,n)} = |\psi^{++...}|^2 + (-1)^n|\psi^{- -...}|^2 .
$$

The reduced density matrix in the ordinary spin sector with mixed indices obtained after tracing over all $n$ ghost-spins has the form

$$
(r_A)^{++}_+ = (r_A)^{++} = |\psi^{++...}|^2 , \quad (r_A)^{- -}_- = (r_A)^{- -} = (-1)^n|\psi^{- -...}|^2 .
$$

(7.13)
The entanglement entropy for the single spin subsystem is

\[ S_A = - (\rho_A^+ \log \rho_A^+ - \rho_A^- \log \rho_A^-) . \]  

(7.14)

It is obvious from the expressions for the components of the reduced density matrix (7.13) that for \( n \) even the state has positive norm and the entanglement entropy is positive definite. However, when \( n \) is odd then we are back to the situation where positive norm does not lead to positive entanglement entropy. This is identical to the situation encountered in the system with a single spin entangled with a single ghost-spin. This conclusion also applies if there are multiple spins instead of one.

We therefore conclude that the multi-ghost-spin systems fall into two categories. Whereas the even number of ghost-spins case gives rise to positive norm states and positive entanglement entropy, the odd number of ghost-spins case always contain states such as (7.9), (7.12), which exhibit the unphysical anticorrelation between the norm and the sign of the real part of the entanglement entropy\(^5\) that we first encountered in the case of the single ghost-spin system. This fits well with our interpretation that an even number of indefinite norm states is needed to get a sensible reduction to the definite norm subsector of the theory. In this sense, the odd number of ghost-spins systems are analogous to partial gauge fixed or gauge unfixed systems.

\section*{8 Discussion}

We have studied patterns of quantum entanglement in spin & ghost-spin systems. When the spins and ghost-spins are disentangled (both sectors possibly entangled within themselves), the reduced density matrix obtained by tracing out the ghost-spins leads to positive entanglement entropy for positive norm states. Negative norm states give rise to entanglement entropy with a negative real part and a constant imaginary part. However, for entangled spins and ghost-spins, the entanglement patterns are richer. For even numbers of ghost-spins, there are always subsectors of the Hilbert space where positive norm states give positive entanglement entropy. For odd numbers of ghost-spins, we have seen the existence of positive norm states which always have negative real part for entanglement entropy. These toy models in a sense contain only entanglement information: we have not utilized any description of time evolution and dynamics. It would be interesting to further explore dynamical models which lead to the toy models here. It would also be interesting to explore inter-relations of these models with recent studies of entanglement in gauge theories \( e.g. \) [22–26]. In this regard, it is interesting to note [27] who point out the necessity of ghost fields to account for entanglement in gauge theories: see also related discussions more recently in \( e.g. \) [28,29].

\(^5\)although there are also states with positive norm and positive entanglement, \( e.g. \) (7.5), (7.6), (7.7), (7.8).
A related obvious question in the present context has to do with how precisely the physical positive norm subspace arises in the full theory containing the negative norm sectors. It would appear that such a truncation must dovetail with a better understanding of the partial trace in the reduced density matrix over the extended Hilbert space and the gauge fixed theory including the ghost sector arising from gauge fixing. In general understanding how the physical subspace arises is unclear within the present work, possibly admitting a clear answer in the context of toy models with dynamics e.g. [30] and ongoing investigations on the role of a BRST symmetry and associated cohomology. It would appear that the latter will provide a truncation to a physical subspace which is entirely positive norm, thereby leading to positive definite reduced density matrices and positive entanglement (which can be expected to satisfy known universal properties such as strong subadditivity). In general this may not be as simple as truncating to e.g. correlated ghost-spin subsectors (although that subsector is indeed entirely positive norm in simple toy models). We hope to clarify some of these issues in future work.

The motivation for defining “ghost-spins” in [20] arose from dS/CFT, which, although not directly relevant to the present context, is useful to review briefly. Certain generalizations of gauge/gravity duality to de Sitter space or dS/CFT [31–33] conjecture that de Sitter space is dual to a hypothetical Euclidean non-unitary CFT that lives on the future boundary I\(^+\). The late-time wavefunction of the universe \(\Psi_{dS}\) with appropriate boundary conditions is equated with the dual CFT partition function \(Z_{CFT}\) [33], which is a useful way to organize de Sitter perturbations (independent of the actual existence of the CFT). The dual CFT\(_d\) energy-momentum tensor correlator \(\langle TT\rangle\) in a semiclassical approximation \(\Psi \sim e^{iS}\) reveals central charge coefficients \(C_d \sim i^{1-d} \frac{\kappa_{d-2}}{G_{d+1}}\) in \(dS_{d+1}\), real and negative in \(dS_4\), and pure imaginary in \(dS_3, dS_5\) etc (effectively analytic continuations from AdS/CFT). \(dS_4/CFT_3\) is thus reminiscent of ghost-like non-unitary theories. In [34], a higher spin \(dS_4\) duality was conjectured involving a 3-dim CFT of anti-commuting \(Sp(N)\) (ghost) scalars.

Certain attempts at generalizing the Ryu-Takayanagi formulation [15–19] to dS/CFT were carried out in [35, 36]: while appropriate real surfaces were found to have vanishing area, the areas of certain complex codim-2 extremal surfaces (involving an imaginary bulk time parametrization) were found to have structural resemblance with entanglement entropy of dual Euclidean CFTs. These end up being equivalent to analytic continuation from the Ryu-Takayanagi expressions in AdS/CFT. In \(dS_4\) the areas are real and negative. Towards gaining some insight into whether such a negative entanglement entropy can at all arise in a field theoretic calculation, certain 2-dim ghost conformal field theories with negative central charge were studied as toy models for the replica calculation in [20]. Specifically certain \(c = -2\) ghost-CFTs were focussed upon, where (i) the \(SL(2)\) vacuum coincides with the ghost ground state and (ii) correlation functions are calculated in the presence
of appropriate ghost zero mode insertions which cancel the background charge inherent in these systems. The replica formulation via twist operator 2-point correlation functions then gives the entanglement entropy for a single interval of size $l$ as the usual $\frac{c}{3} \log l$ behaviour (with $\epsilon$ the ultraviolet cutoff): this is negative and has various odd properties as discussed there. Also studied in [20] was a toy model of two ghost-spins with a view to exploring a simple quantum mechanical system with negative norm states, with the ghost-spin defined as we have reviewed in sec. 2. The reduced density matrix obtained by tracing over one ghost-spin then reveals that positive norm states give positive von Neumann entropy while negative norm states give entanglement entropy with a negative real part and a constant imaginary part. Overall these are perhaps best regarded as formal generalizations of the ideas and techniques of the usual notions of entanglement entropy in more familiar quantum systems. While a deeper understanding, if any, of this dual entanglement entropy (although consistent with negative central charge) as a probe of $dS/CFT$ remains open, our interest in the present paper has been to study the resulting object in toy quantum mechanical systems of entangled spins and ghost-spins towards exploring patterns of quantum entanglement in systems containing negative norm states that are expected to arise in systems with a gauge symmetry as mentioned earlier.

Finally, in light of the present analysis where we have seen that generically negative norm states give a complex-valued entanglement entropy, we recall that the replica calculation for the $c = -2$ 2-dim ghost CFTs in [20] recovered only the negative real part, with no imaginary part. It is interesting to note that an extra phase $(-1)^n$ in the reduced density matrix $\rho^n_A \rightarrow (-1)^n \rho^n_A$ in the replica theory gives a contribution $S_A \rightarrow S_A - \partial_n \log (-1)^n = S_A \pm i\pi$ in the $n \rightarrow 1$ limit. We hope to understand this and related issues better.

Acknowledgements: It is a pleasure to thank Anshuman Maharana and especially Ashoke Sen for several useful discussions. KN thanks the hospitality of the String Group, HRI, Allahabad, where this work began, and the Organizers of the Simons Workshop in Mathematics and Physics, 2016, Simons Center, Stony Brook, USA for hospitality while this work was in progress. The work of KN is partially supported by a grant to CMI from the Infosys Foundation and of DPJ by the DAE project 12-R&D-HRI-5.02-0303.

although note that none of these models gives a pure imaginary entanglement entropy, as might arise in the case of $dS_3/CFT_2$ where the central charge is pure imaginary.
A Spin & ghost-spin: off-diagonal density matrix

We will consider a system of one spin and one ghost-spin here. A general state in this set up contains 4 parameters. In sec. 6.1, we tried to impose a constraint on these parameters so that the resulting density matrix was diagonal.

Here we will not demand the diagonal form of the density matrix to start with. However, it can always be diagonalised by a change of basis. We can then try to find conditions under which the reduced density matrix is positive. Recall the general form of \( \rho_A \) in this system is

\[
\begin{bmatrix}
|\psi^{++}|^2 - |\psi^{+-}|^2 & \psi^{++}(\psi^*)^{+-} - \psi^{+-}(\psi^*)^{++} \\
\psi^{+-}(\psi^*)^{++} - \psi^{++}(\psi^*)^{+-} & |\psi^{+-}|^2 - |\psi^{--}|^2
\end{bmatrix}
\]

We can diagonalise this by solving the quadratic equation

\[
0 = \lambda^2 - (|\psi^{++}|^2 - |\psi^{+-}|^2 + |\psi^{--}|^2 - |\psi^{--}|^2)\lambda
+ (|\psi^{++}|^2 - |\psi^{+-}|^2)(|\psi^{--}|^2 - |\psi^{--}|^2)
- (\psi^{++}(\psi^*)^{+-} - \psi^{+-}(\psi^*)^{++})(\psi^{++}(\psi^*)^{++} - \psi^{+-}(\psi^*)^{+-})
\]

We can use the fact that \( \text{Tr}\rho = \pm 1 \) to write

\[
0 = \lambda^2 + \lambda + (|\psi^{++}|^2 - |\psi^{+-}|^2)(|\psi^{--}|^2 - |\psi^{--}|^2)
- (\psi^{++}(\psi^*)^{+-} - \psi^{+-}(\psi^*)^{++})(\psi^{++}(\psi^*)^{++} - \psi^{+-}(\psi^*)^{+-})
\]

The solution to this equation is

\[
\lambda = \pm \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + 4(\psi^{++}\psi^{--} - \psi^{+-}\psi^{-+})(\psi^*)^{++}(\psi^*)^{--} - (\psi^*)^{+-}(\psi^*)^{+-})}
\]

Clearly we need to impose a variety of constraints to ensure that the eigenvalues are not complex and to ensure that they are positive definite. First of all, if the trace of the density matrix is \( \pm 1 \) then we get both positive (negative) eigenvalues if

\[
-\frac{1}{4} \leq (\psi^{++}\psi^{--} - \psi^{+-}\psi^{-+})(\psi^*)^{++}(\psi^*)^{--} - (\psi^*)^{+-}(\psi^*)^{+-} < 0
\]

This condition is not satisfied because the quantity we are looking at is the modulus square of \( (\psi^{++}\psi^{--} - \psi^{+-}\psi^{-+}) \) which is positive semi-definite and as a result we have one positive and one negative eigenvalue.

Thus we see that if we do not impose any conditions on the parameters of the entangled state we seem to get one positive and one negative eigenvalue of the density matrix.
B  Tracing over a spin and a ghost-spin

In sec. 6.1 and sec. 6.2, we had analysed tracing over the single ghost-spin with the result that positive norm states are not correlated with positive entanglement. One could ask if any alternate mechanism of tracing over a subsector could make sense in this system. To explore that let us now consider two spins entangled with one ghost-spin. As we saw above this is similar to the case of one spin and one ghost-spin. The minus signs that arise in the $(\rho^A)^i_k$ components come from the trace over the single ghost-spin since the spin indices contract with $\delta_{ij}$. Therefore it appears that the fate of this system is more in the hands of the ghost-spin sector than the spin sector.

However suppose we view this system as a single spin entangled with an entangled system of one spin and one ghost-spin, and trace over the latter, i.e. trace over the entire entangled one spin–one ghost-spin system. We can then ask if the reduced density matrix $\rho_A$ for the single spin looks physically reasonable, or not. In particular, does $\rho_A$ possess positivity?

A generic state for two spins and one ghost-spin system and its norm are

$$|\psi\rangle = \psi^{ij,\alpha}|ij\rangle|\alpha\rangle, \quad \langle\psi|\psi\rangle = g_{ij} g_{kl} \gamma_{\alpha\beta} \psi^{ij,\alpha}(\psi^*)^{kl,\beta}. \quad (B.1)$$

Since we have only one ghost-spin, the norm has just one $\gamma_{ik}$ factor. For instance, the state

$$|\psi\rangle = \psi^{++,+}|++\rangle + \psi^{+,+-}|+-\rangle + \psi^{--,+}|--\rangle + \psi^{--,-}|--\rangle$$

which in general is not a product state has the norm

$$\langle\psi|\psi\rangle = |\psi^{++,+}|^2 - |\psi^{++,+}|^2 + |\psi^{--,-}|^2 - |\psi^{--,-}|^2 = \pm 1. \quad (B.3)$$

The single spin reduced density matrix obtained by tracing over the ghost-spin and one spin is

$$(\rho_A)^{ik} = g_{ij} \gamma_{\alpha\beta} \psi^{ij,\alpha}(\psi^*)^{kl,\beta} = g_{ij} \gamma_{\alpha\alpha} \psi^{ij,\alpha}(\psi^*)^{kj,\alpha}. \quad (B.4)$$

For the choice of entangled state $(B.2)$ we end up getting the diagonal form of the reduced density matrix

$$(\rho_A)^{++,+} = |\psi^{++,+}|^2 - |\psi^{++,+}|^2, \quad (\rho_A)^{++,+} = 0, \quad (\rho_A)^{--,+} = 0, \quad (\rho_A)^{--,--} = |\psi^{--,-}|^2 - |\psi^{--,-}|^2. \quad (B.5)$$

As a consequence of the diagonal form of $\rho_A$, $\log \rho_A$ also has a simple form. Note that $tr\rho_A = g_{ik}(\rho_A)^{ik} = (\rho_A)^{++,+} + (\rho_A)^{--,--}$ and satisfies $tr\rho_A = tr\rho = \langle\psi|\psi\rangle$. Since the remaining spin has positive definite metric $g_{ij}$, we have the entanglement entropy

$$S_A = -g_{ij} (\rho_A \log \rho_A)^{ij} = -(\rho_A \log \rho_A)^{++,+} - (\rho_A \log \rho_A)^{--,--}. \quad (B.6)$$
Then the entanglement entropy is

\[
|\psi^{++}|^2 - |\psi^{+-}|^2 \equiv x, \quad \langle \psi|\psi \rangle = x + (\pm 1 - x);
\]

\[
(r_A)^{++,} = x, \quad (r_A)^{-+} = \pm 1 - x, \tag{B.7}
\]

\[
S_A = -x \log x - (\pm 1 - x) \log(\pm 1 - x).
\]

Curiously this structure is similar to the case of one spin entangled with two ghost-spins. It would be interesting to relate this system to a more physical situation to gain insight into this pattern of entanglement.

To see whether this conclusion survives when we make a different choice of entangled state, let us consider the most general state

\[
|\psi\rangle = |\psi^{++}|^+|^+\rangle + |\psi^{+-}|^+|^-angle + |\psi^{+-}|^-|^+\rangle + |\psi^{-+}|^-|^-angle + |\psi^{--}|^+|^+\rangle + |\psi^{--}|^-|^-angle \tag{B.8}
\]

The norm of this state is

\[
\langle \psi|\psi \rangle = |\psi^{++}|^2 - |\psi^{+-}|^2 + |\psi^{-+}|^2 - |\psi^{--}|^2
\]

\[
+ |\psi^{-+}|^2 - |\psi^{+\ast,+}|^2 + |\psi^{+\ast,\ast,-}|^2 - |\psi^{+\ast,-}|^2
\]

\[
\tag{B.9}
\]

The reduced density matrix after tracing over a spin and a ghost spin is

\[
(r_A)^{++,} = |\psi^{++}|^2 - |\psi^{+-}|^2 + |\psi^{-+}|^2 - |\psi^{--}|^2,
\]

\[
(r_A)^{+-} = \psi^{++}(\psi^{+\ast})^{--} - \psi^{+-}(\psi^{+\ast})^{+\ast,-} + \psi^{-+}(\psi^{+\ast})^{+-,-} - \psi^{--}(\psi^{+\ast})^{--,-},
\]

\[
(r_A)^{-+} = \psi^{-\ast,+}(\psi^{+\ast})^{++} - \psi^{-\ast,-}(\psi^{+\ast})^{+-,-} + \psi^{-\ast,-}(\psi^{+\ast})^{+\ast,-} - \psi^{-\ast,-}(\psi^{+\ast})^{--,-},
\]

\[
(r_A)^{-\ast} = |\psi^{-\ast,+}|^2 - |\psi^{-\ast,-}|^2 + |\psi^{++}|^2 - |\psi^{--}|^2. \tag{B.10}
\]

If we set the off diagonal components of $r_A$ to zero then

\[
\psi^{++,} = \frac{1}{(\psi^{+\ast})^{++}} (\psi^{++}(\psi^{+\ast})^{++} - \psi^{+-}(\psi^{+\ast})^{+\ast,-} + \psi^{-+}(\psi^{+\ast})^{+-,-} - \psi^{--}(\psi^{+\ast})^{--,-})
\]

\[
\psi^{--,-} = \frac{1}{(\psi^{+\ast})^{+\ast,-}} (\psi^{-\ast,+}(\psi^{+\ast})^{++} - \psi^{-\ast,-}(\psi^{+\ast})^{+-,-} + \psi^{-\ast,-}(\psi^{+\ast})^{+\ast,-} - \psi^{-\ast,-}(\psi^{+\ast})^{--,-}). \tag{B.11}
\]
The diagonal components can be rewritten using eq. (B.11),

\[
\begin{align*}
(r_A)^++ &= \frac{1}{|\psi^{++}|^2} |\psi^{--}|^2(|\psi^{++}|^2 - |\psi^{--}|^2) + |\psi^{+-}|^2(|\psi^{+-}|^2 + |\psi^{++}|^2) \\
&\quad + |\psi^{+-}|^2(|\psi^{--}|^2 - |\psi^{+-}|^2) + \left\{ \psi^{+-}\psi^{-+} - \psi^{-+}\psi^{+-} + c.c. \right\}, \\
(r_A)^-- &= \frac{1}{|\psi^{--}|^2} |\psi^{++}|^2(|\psi^{--}|^2 - |\psi^{++}|^2) + |\psi^{+-}|^2(|\psi^{+-}|^2 + |\psi^{--}|^2) \\
&\quad + |\psi^{+-}|^2(|\psi^{++}|^2 - |\psi^{+-}|^2) + \left\{ \psi^{+-}\psi^{-+} - \psi^{-+}\psi^{+-} + c.c. \right\}.
\end{align*}
\]  

(B.12)

The form of the diagonal components is rich enough to allow various possibilities, which clearly include cases where we get positive entanglement entropy. For example, if we demand that each term in \((r_A)^{++}\) is positive definite then we need to impose three conditions. Two of which are

\[
|\psi^{+-}|^2 \geq |\psi^{++}|^2, \quad |\psi^{--}|^2 \geq |\psi^{--}|^2,
\]

and the third condition puts imposes the positivity condition on the curly bracket term in the expression for \((r_A)^{++}\) in eq. (B.12). Similarly demanding that \((r_A)^{--}\) is negative definite gives following conditions

\[
|\psi^{++}|^2 \geq |\psi^{--}|^2, \quad |\psi^{+-}|^2 \geq |\psi^{+-}|^2,
\]

and a positivity constraint on the curly bracket term in the expression for \((r_A)^{--}\) in eq. (B.12).

References

[1] L. Bombelli, R. K. Koul, J. Lee and R. D. Sorkin, “A Quantum Source of Entropy for Black Holes,” Phys. Rev. D34 (1986) 373.

[2] M. Srednicki, “Entropy and area,” Phys. Rev. Lett. 71 (1993) 666 [arXiv:hep-th/9303048].

[3] C. Holzhey, F. Larsen and F. Wilczek, “Geometric and renormalized entropy in conformal field theory,” Nucl. Phys. B 424 (1994) 443 [arXiv:hep-th/9403108].

[4] G. Vidal, J. I. Latorre, E. Rico and A. Kitaev, “Entanglement in quantum critical phenomena,” Phys. Rev. Lett. 90 (2003) 227902 [arXiv:quant-ph/0211074].
[5] J. I. Latorre, E. Rico and G. Vidal, “Ground state entanglement in quantum spin chains,” Quant. Inf. Comput. 4, 48 (2004) [arXiv:quant-ph/0304098].

[6] P. Calabrese and J. L. Cardy, “Entanglement entropy and quantum field theory,” J. Stat. Mech. 0406, P06002 (2004) [arXiv:hep-th/0405152].

[7] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, “Quantum entanglement,” Rev. Mod. Phys. 81 (2009) 865 [arXiv:quant-ph/0702225].

[8] J. Eisert, M. Cramer and M. B. Plenio, “Area laws for the entanglement entropy - a review,” Rev. Mod. Phys. 82 (2010) 277 [arXiv:0808.3773[quant-ph]].

[9] P. Calabrese and J. Cardy, “Entanglement entropy and conformal field theory,” J. Phys. A 42, 504005 (2009) doi:10.1088/1751-8113/42/50/504005 [arXiv:0905.4013[cond-mat.stat-mech]].

[10] H. Casini and M. Huerta, “Entanglement entropy in free quantum field theory,” J. Phys. A 42, 504007 (2009) doi:10.1088/1751-8113/42/50/504007 [arXiv:0905.2562[hep-th]].

[11] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2 (1998) 231 [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].

[12] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B428 (1998) 105 [arXiv:hep-th/9802109].

[13] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2 (1998) 253 [arXiv:hep-th/9802150].

[14] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. 323 (2000) 183 [arXiv:hep-th/9905111].

[15] S. Ryu and T. Takayanagi, “Holographic derivation of entanglement entropy from AdS/CFT,” Phys. Rev. Lett. 96 (2006) 181602 [arXiv:hep-th/0603001].

[16] S. Ryu and T. Takayanagi, “Aspects of Holographic Entanglement Entropy,” JHEP 08 (2006) 045 [arXiv:hep-th/0605073].

[17] V. E. Hubeny, M. Rangamani and T. Takayanagi, “A Covariant holographic entanglement entropy proposal,” JHEP 07 (2007) 062 [arXiv:0705.0016[hep-th]].

[18] T. Nishioka, S. Ryu and T. Takayanagi, “Holographic Entanglement Entropy: An Overview,” J. Phys. A 42, 504008 (2009) doi:10.1088/1751-8113/42/50/504008 [arXiv:0905.0932[hep-th]].

[19] T. Takayanagi, “Entanglement Entropy from a Holographic Viewpoint,” Class. Quant. Grav. 29 (2012) 153001 [arXiv:1204.2450[gr-qc]].

[20] K. Narayan, “On dS4 extremal surfaces and entanglement entropy in some ghost CFTs,” Phys. Rev. D94 (2016) 046001 [arXiv:1602.06505[hep-th]].

[21] J. Polchinski, String Theory, Vol. 1,2. Cambridge University Press (1998).
[22] H. Casini, M. Huerta and J. A. Rosabal, “Remarks on entanglement entropy for gauge fields,” Phys. Rev. D89 (2014) 085012 [arXiv:1312.1183[hep-th]].

[23] S. Ghosh, R. M. Soni and S. P. Trivedi, “On The Entanglement Entropy For Gauge Theories,” JHEP 09 (2015) 069 doi:10.1007/JHEP09(2015)069 [arXiv:1501.02593[hep-th]].

[24] R. M. Soni and S. P. Trivedi, “Aspects of Entanglement Entropy for Gauge Theories,” JHEP 01 (2016) 136 doi:10.1007/JHEP01(2016)136 [arXiv:1510.07455[hep-th]].

[25] H. Casini and M. Huerta, “Entanglement entropy of a Maxwell field on the sphere,” Phys. Rev. D93 (2016) 105031 [arXiv:1512.06182[hep-th]].

[26] R. M. Soni and S. P. Trivedi, “Entanglement Entropy in (3+1)-d Free U(1) Gauge Theory,” [arXiv:1608.00353[hep-th]].

[27] D. N. Kabat, “Black hole entropy and entropy of entanglement,” Nucl. Phys. B 453, 281 (1995) doi:10.1016/0550-3213(95)00443-V [arXiv:hep-th/9503016].

[28] W. Donnelly and A. C. Wall, “Geometric entropy and edge modes of the electromagnetic field,” Phys. Rev. D 94, no. 10, 104053 (2016) doi:10.1103/PhysRevD.94.104053 [arXiv:1506.05792[hep-th]].

[29] D. Harlow, “Wormholes, Emergent Gauge Fields, and the Weak Gravity Conjecture,” JHEP 1601, 122 (2016) doi:10.1007/JHEP01(2016)122 [arXiv:1510.07911[hep-th]].

[30] D. P. Jatkar and K. Narayan, “Ghost-spin chains, entanglement and bc-ghost CFTs,” [arXiv:1706.06828[hep-th]].

[31] A. Strominger, “The dS / CFT correspondence,” JHEP 10 (2001) 034 [arXiv:hep-th/0106113].

[32] E. Witten, “Quantum gravity in de Sitter space,” [arXiv:hep-th/0106109].

[33] J. M. Maldacena, “Non-Gaussian features of primordial fluctuations in single field inflationary models,” JHEP 05 (2003) 013 [arXiv:astro-ph/0210603].

[34] D. Anninos, T. Hartman and A. Strominger, “Higher Spin Realization of the dS/CFT Correspondence,” [arXiv:1108.5735[hep-th]].

[35] K. Narayan, “de Sitter extremal surfaces,” Phys. Rev. D91 (2015) 126011 [arXiv:1501.03019[hep-th]].

[36] K. Narayan, “de Sitter space and extremal surfaces for spheres,” Phys. Lett. B753 (2016) 308 doi:10.1016/j.physletb.2015.12.019 [arXiv:1504.07430[hep-th]].