The Bowen’s topological entropy of the Cartesian product sets

Xiaoyao Zhou \textsuperscript{a}, Ercai Chen\textsuperscript{a,b,*}

\textsuperscript{a} School of Mathematical Sciences and Institute of Mathematics, Nanjing Normal University, Nanjing 210023, Jiangsu, P.R.China

\textsuperscript{b} Center of Nonlinear Science, Nanjing University, Nanjing 210093, Jiangsu, P.R.China

e-mail: zhouxiaoyaodeyouxian@126.com, ecchen@njnu.edu.cn

\textbf{Abstract.} This article is devoted to showing the product theorem for Bowen’s topological entropy.

\textbf{Keywords and phrases:} Bowen’s topological entropy, packing topological entropy, upper capacity topological entropy, product space.

\section{Introduction and Preliminaries}

The purpose of this article is to study the topological entropies of product spaces. The product theorem for topological entropy of the dynamical systems was first investigated by Adler, Konheim and McAndrew \cite{1} and Goodywn \cite{7}. One can see \cite{13} for the product theorem for topological entropy of two compact subsets. Bowen \cite{3} introduced the notion of topological entropy for non-compact sets. A question arises naturally whether the product theorem for Bowen’s topological entropy still holds. The reader is also referred to \cite{2, 5, 8, 9, 10, 11} and references therein for the investigation of dimension of product spaces.

Throughout this article, a topological dynamical system $(X, d, T)$ means a compact metric space $(X, d)$ together with a continuous self-map $T : X \to X$. Let $M(X)$ and $M(X, T)$ denote the sets of all Borel probability measures and $T$-invariant Borel
probability measures, respectively. For \( n \in \mathbb{N} \), the \( n \)-th Bowen metric \( d_n \) on \( X \) is defined by

\[
d_n(x,y) = \max\{d(T^kx, T^ky) : k = 0, 1, \cdots, n - 1\}.
\]

For every \( \epsilon > 0 \), denote by \( B_n(x, \epsilon) \) and \( \overline{B}_n(x, \epsilon) \) the open and closed balls of radius \( \epsilon \) order \( n \) in the metric \( d_n \) around \( x \), i.e.,

\[
B_n(x, \epsilon) = \{ y \in X : d_n(x,y) < \epsilon \} \quad \text{and} \quad \overline{B}_n(x, \epsilon) = \{ y \in X : d_n(x,y) \leq \epsilon \}.
\]

Recently, given \( \mu \in M(X) \), Feng and Huang [6] defined the measure-theoretical lower and upper entropies of \( \mu \) respectively by the idea analogous to Brin and Katok [4] as follows.

**Definition 1.1.** Let \( h_\mu(T) = \int \overline{h}_\mu(T,x)d\mu(x) \) and \( \overline{h}_\mu(T) = \int \overline{h}_\mu(T,x)d\mu(x) \), where

\[
\overline{h}_\mu(T,x) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \mu(B_n(x, \epsilon)),
\]

\[
\overline{h}_\mu(T, x) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \mu(B_n(x, \epsilon)).
\]

Brin and Katok [4] proved that for any \( \mu \in M(X,T) \), \( \overline{h}_\mu(T,x) = \underline{h}_\mu(T,x) \) for \( \mu \)-a.e. \( x \in X \), and \( \int \underline{h}_\mu(T,x)d\mu(x) = h_\mu(T) \). This implies that for any \( \mu \in M(X,T) \), \( \underline{h}_\mu(T) = \overline{h}_\mu(T) = h_\mu(T) \).

A set in a metric space is said to be *analytic* if it is a continuous image of the set \( \mathcal{N} \) of infinite sequences of natural numbers (with its product topology). It is well known that in a Polish space, the analytic subsets are closed under countable unions and intersections, and any Borel set is analytic (see [5]).

## 2 Definitions of Topological Entropies and Main Theorem

In this section, we recall three definitions of topological entropies of subsets in a topological dynamical system: Bowen’s topological entropy, packing topological entropy and upper capacity topological entropy. Since they are analogous to the definitions of dimensions, they are called dimensional entropies.
2.1 Bowen’s topological entropy

Bowen’s topological entropy was first introduced in [3]. Here we use an alternative way to define Bowen’s topological entropy for convenience. See [12] for details.

Suppose \((X, d, T)\) is a topological dynamical system. Given \(Z \subset X, s \geq 0, N \in \mathbb{N}\) and \(\epsilon > 0\), define

\[
M_{N,\epsilon}^s(Z) = \inf \sum_i \exp(-sn_i),
\]

where the infimum ranges over all finite or countable families \(\{B_n(x_i, \epsilon)\}\) such that \(x_i \in X, n_i \geq N\) and \(\bigcup_i B_n(x_i, \epsilon) \supset Z\). Since \(M_{N,\epsilon}^s(Z)\) does not decrease as \(N\) increases and \(\epsilon\) decreases, the following two limits exist:

\[
M_s^\epsilon(Z) = \lim_{N \to \infty} M_{N,\epsilon}^s(Z), \quad M_s(Z) = \lim_{\epsilon \to 0} M_s^\epsilon(Z).
\]

The Bowen’s topological entropy \(h_B(Z)\) is defined as a critical value of the parameter \(s\), where \(M_s^\epsilon(Z)\) jumps from \(\infty\) to 0, i.e.,

\[
M_s^\epsilon(Z) = \begin{cases} 
0, & s > h_B(Z), \\
\infty, & s < h_B(Z).
\end{cases}
\]

2.2 Packing topological entropy

Packing topological entropy was defined by Feng and Huang [6] in a way which resembles the packing dimension. Nowadays, the packing topological entropy is widely believed as important as the Bowen’s topological entropy and an understanding of both the Bowen’s topological entropy and the packing topological entropy of a set provides the basis for a substantially better understanding of the underlying geometry and dynamical behavior of the set.

Given \(Z \subset X, s \geq 0, N \in \mathbb{N}\) and \(\epsilon > 0\), define

\[
P_{N,\epsilon}^s(Z) = \sup \sum_i \exp(-sn_i),
\]

where the supremum runs over all finite or countable pairwise disjoint families \(\{\overline{B}_n(x_i, \epsilon)\}\) such that \(x_i \in Z, n_i \geq N\) for all \(i\). Since \(P_{N,\epsilon}^s(Z)\) does not decrease as \(N, \epsilon\) decrease, the following limit exists:

\[
P_s^\epsilon(Z) = \lim_{N \to \infty} P_{N,\epsilon}^s(Z).
\]
Define

\[ P_s^\epsilon(Z) = \inf \left\{ \sum_{i=1}^{\infty} P_s^\epsilon(Z_i) : \bigcup_{i=1}^{\infty} Z_i \supset Z \right\}. \]

It is obvious that for \( Z \subset \bigcup_{i=1}^{\infty} Z_i \), \( P_s^\epsilon(Z) \leq \sum_{i=1}^{\infty} P_s^\epsilon(Z_i) \). There exists a critical value of the parameter \( s \), denoted by \( h^P(Z, \epsilon) \), where \( P_s^\epsilon(Z) \) jumps from \( \infty \) to 0, i.e.,

\[ P_s^\epsilon(Z) = \begin{cases} 0, & s > h^P(Z, \epsilon), \\ \infty, & s < h^P(Z, \epsilon). \end{cases} \]

Since \( h^P(Z, \epsilon) \) increases when \( \epsilon \) decreases, we call

\[ h^P(Z) := \lim_{\epsilon \to 0} h^P(Z, \epsilon) \]

the packing topological entropy of \( Z \).

Remark that in the definition of \( P_s^\epsilon(Z) \), \( \bigcup_{i=1}^{\infty} Z_i \supset Z \) can be replaced by \( \bigcup_{i=1}^{\infty} Z_i = Z \).

### 2.3 Upper capacity topological entropy

Upper capacity topological entropy is the straightforward generalization of the Adler-Konheim-McAndrew definition of topological entropy to arbitrary subsets.

Given a non-empty subset \( Z \subset X \). For \( \epsilon > 0 \), a set \( E \subset Z \) is called a \( (n, \epsilon) \)-separated set of \( Z \) if \( x, y \in E, x \neq y \) implies that \( d_n(x, y) > \epsilon \); \( E \subset X \) is called \( (n, \epsilon) \)-spanning set of \( Z \), if for any \( x \in Z \), there exists \( y \in E \) with \( d_n(x, y) \leq \epsilon \). Let \( r_n(Z, \epsilon) \) denote the largest cardinality of \( (n, \epsilon) \)-separated sets for \( Z \), and \( \tilde{r}_n(Z, \epsilon) \) the smallest cardinality of \( (n, \epsilon) \)-spanning sets of \( Z \). The upper capacity topological entropy of \( Z \) is given by

\[ h^U(Z) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log r_n(Z, \epsilon) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \tilde{r}_n(Z, \epsilon). \]

Some properties of topological entropies are presented as below.

**Proposition 2.1.**

(i) For \( Z \subset Z' \), \( h^B(Z) \leq h^B(Z') \), \( h^P(Z) \leq h^P(Z') \), \( h^U(Z) \leq h^U(Z') \).

(ii) For \( Z \subset \bigcup_{i=1}^{\infty} Z_i \), \( s \geq 0 \) and \( \epsilon > 0 \), we have
\[ M^\epsilon(Z) \leq \sum_{i=1}^{\infty} M^\epsilon(Z_i), \]
\[ h^B(Z) \leq \sup_{i \geq 1} h^B(Z_i), \]
\[ h^P(Z) \leq \sup_{i \geq 1} h^P(Z_i). \]

(iii) For any \( Z \subset X \), \( h^B(Z) \leq h^P(Z) \leq h^U(Z) \).

(iv) If \( Z \) is \( T \)-invariant and compact, then \( h^B(Z) = h^P(Z) = h^U(Z) \).

(v) For \( Z_1, Z_2 \subset X \), we have \( h^U(Z_1 \times Z_2) \leq h^U(Z_1) + h^U(Z_2) \).

(vi) For \( Z_1, Z_2 \subset X \), we have \( h^U(Z_1 \cup Z_2) = \max \{ h^U(Z_1), h^U(Z_2) \} \).

Proof. (i)-(iv) can be seen in [6]. To see (v), for \( n \in \mathbb{N}, \epsilon > 0 \), suppose \( S_1 \) is an \((n, \epsilon)\)-spanning sets of \( Z_1 \) with minimal cardinality and \( S_2 \) is an \((n, \epsilon)\)-spanning sets of \( Z_2 \) with minimal cardinality, then \( S_1 \times S_2 \) is an \((n, \epsilon)\)-spanning set of \( Z_1 \times Z_2 \). This means \( r_n(Z_1 \times Z_2, \epsilon) \leq r_n(Z_1, \epsilon) r_n(Z_2, \epsilon) \). Furthermore,

\[
h^U(Z_1 \times Z_2) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log r_n(Z_1 \times Z_2, \epsilon)
\leq \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log r_n(Z_1, \epsilon) r_n(Z_2, \epsilon)
\leq \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log r_n(Z_1, \epsilon) + \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log r_n(Z_2, \epsilon)
\leq h^U(Z_1) + h^U(Z_2).
\]

(vi) follows from that for \( n \in \mathbb{N}, \epsilon > 0 \),

\[
\max \{ r_n(Z_1, \epsilon), r_n(Z_2, \epsilon) \} \leq r_n(Z_1 \cup Z_2, \epsilon)
\leq r_n(Z_1, \epsilon) + r_n(Z_2, \epsilon)
\leq 2 \max \{ r_n(Z_1, \epsilon), r_n(Z_2, \epsilon) \}.
\]

The main result of this article is the following theorem.

Theorem 2.1. Suppose \((X, d, T)\) is a topological dynamical system.

- If \( h^B(X) < \infty \) and \( Z_1 \subset X, Z_2 \subset X \) are analytic, then
  \[ h^B(Z_1) + h^B(Z_2) \leq h^B(Z_1 \times Z_2); \]
• If $Z_1 \subset X, Z_2 \subset X$, then
  \[ h_B(Z_1 \times Z_2) \leq h_B(Z_1) + h_B(Z_2); \]
  \[ h_P(Z_1 \times Z_2) \leq h_P(Z_1) + h_P(Z_2); \]
  \[ h_U(Z_1 \times Z_2) \leq h_U(Z_1) + h_U(Z_2). \]

3 Proof of Main Theorem

The main theorem is divided into several theorems, which are proved respectively. The following lemma establishes the variational principles for Bowen and packing topological entropies of arbitrary Borel sets.

**Lemma 3.1.** Suppose $(X, d, T)$ is a topological dynamical system.

(i) If $K \subset X$ is non-empty and compact, then

\[ h_B(K) = \sup \{ h_\mu(T) : \mu \in M(X), \mu(K) = 1 \}. \]

(ii) Assume that $h_B(X) < \infty$. If $Z \subset X$ is analytic, then

\[ h_B(Z) = \sup \{ h_B(K) : K \subset Z \text{ is compact} \}. \]

(iii) If $K \subset X$ is non-empty and compact, then

\[ h_P(K) = \sup \{ h_\mu(T) : \mu \in M(X), \mu(K) = 1 \}. \]

(iv) If $Z \subset X$ is analytic, then

\[ h_P(Z) = \sup \{ h_P(K) : K \subset Z \text{ is compact} \}. \]

It is worth pointing out that the product metric in this article, denoted by $\rho$, on $(X \times X)$ is given by

\[ \rho((x_1, y_1), (x_2, y_2)) = \max \{ d(x_1, x_2), d(y_1, y_2) \} \]

for $x_1, y_1, x_2, y_2 \in X$. Now, we firstly prove the following theorem.
Theorem 3.1. Suppose \((X, d, T)\) is a topological dynamical system.

- If \(h^B(X) < \infty\) and \(Z_1 \subseteq X, Z_2 \subseteq X\) are analytic, then

\[
h^B(Z_1) + h^B(Z_2) \leq h^B(Z_1 \times Z_2).
\]

- If \(Z_1 \subseteq X, Z_2 \subseteq X\), then \(h^B(Z_1 \times Z_2) \leq h^B(Z_1) + h^U(Z_2)\).

Proof. (i) Firstly, we show that \(h^B(Z_1) + h^B(Z_2) \leq h^B(Z_1 \times Z_2)\). It follows from Lemma 3.1 that for any \(\zeta > 0\), there exist \(K_1 \subset Z_1, \mu_1 \in M(X), K_2 \subset Z_2, \mu_2 \in M(X)\) such that

- \(K_1\) and \(K_2\) are compact.
- \(\mu_1(K_1) = 1\) and \(\mu_2(K_2) = 1\).
- \(h^B(Z_1) \leq h_{\mu_1}(T) + \zeta/2\) and \(h^B(Z_2) \leq h_{\mu_2}(T) + \zeta/2\).

Then \(K_1 \times K_2 \subset Z_1 \times Z_2\) is compact, \(\mu_1 \times \mu_2 \in M(X \times X)\), \(\mu_1 \times \mu_2(K_1 \times K_2) = 1\) and

\[
h^B(Z_1 \times Z_2) \geq h_{\mu_1 \times \mu_2}(T \times T)
\]

\[
= \int h_{\mu_1 \times \mu_2}(T \times T, (x, y))d\mu_1 \times \mu_2(x, y)
\]

\[
= \int \lim_{\epsilon \to 0} \lim_{n \to \infty} -\frac{1}{n} \log \mu_1 \times \mu_2 B_n((x, y), \epsilon)d\mu_1 \times \mu_2(x, y)
\]

\[
= \int \lim_{\epsilon \to 0} \lim_{n \to \infty} -\frac{1}{n} \log \mu_1 \times \mu_2 B_n(x, \epsilon) \cap B_n(y, \epsilon)d\mu_1 \times \mu_2(x, y)
\]

\[
\geq \int \lim_{\epsilon \to 0} \lim_{n \to \infty} -\frac{1}{n} \log \mu_1 B_n(x, \epsilon) + \lim_{\epsilon \to 0} \lim_{n \to \infty} -\frac{1}{n} \log \mu_2 B_n(y, \epsilon)d\mu_1 \times \mu_2(x, y)
\]

\[
= h_{\mu_1}(T) + h_{\mu_2}(T)
\]

\[
\geq h^B(Z_1) + h^B(Z_2) - \zeta.
\]

Letting \(\zeta \to 0\), we get the desired result.

Secondly, we prove that \(h^B(Z_1 \times Z_2) \leq h^B(Z_1) + h^U(Z_2)\).

Let \(Z_2 \subseteq X\) and assume \(s > h^U(Z_2)\). For any \(n \in \mathbb{N}\) and \(\epsilon > 0\), let \(R = R_n(Z_2, \epsilon)\) be the largest number so that there is a disjoint family \(\{B_n(x_i, \epsilon)\}_{i=1}^R\) with \(x_i \in Z_2\). Then it is easy to see that for any \(\delta > 0\),

\[
\bigcup_{i=1}^R B_n(x_i, 2\epsilon + \delta) \supseteq Z_2,
\]
which implies that $M_{2\epsilon+\delta}^a(Z_1) \leq R \exp(-ns) \leq 1$. Let $t > h^B(Z_1)$, then $0 = M^t(Z_1) \geq M^t_\delta(Z_1) \geq 0$. Hence, for any $\xi > 0$, there is a finite or countable family $\{B_n(x_i, \epsilon)\}$ such that $x_i \in X, n_i \geq N, \bigcup_i B_n(x_i, \epsilon) \supseteq Z_1$ and $\sum_i \exp(-tn_i) < \xi$. For each $i$, we can cover $Z_2$ with $R_{n_i}(Z_2, \epsilon)$ balls $B_{i,j}$, of order $n_i$ and radius $2\epsilon+\delta, j = 1, 2, \cdots, R_{n_i}(Z_2, \epsilon)$. Then the sets $B_n(x_i, 2\epsilon+\delta) \times B_{i,j}$ together cover $Z_1 \times Z_2$. We have

$$M_{N,2\epsilon+\delta}^a(Z_1 \times Z_2) \leq \sum_i \exp(-n_i(s+t))R_{n_i}(Z_2, \epsilon)$$

$$= \sum_i \exp(-n_i s) \exp(-n_i t)R_{n_i}(Z_2, \epsilon) \leq \xi.$$  

This implies that $h^B(Z_1 \times Z_2) \leq s + t$. Letting $t \to h^B(Z_2), s \to h^U(Z_1)$, this completes the proof.

A question arises naturally whether $h^U$ can be replaced by $h^P$ in Theorem 3.1. For this purpose, we present an equivalent definition of packing topological entropy in the following proposition.

**Proposition 3.1.** $h^P(Z) = \inf \left\{ \sup_i h^U(Z_i) : Z = \bigcup_i Z_i \right\}$.

**Proof.** Given $a > h^P(Z) = \lim_{\epsilon \to 0} h^P(Z, \epsilon) = \lim \inf \left\{ s : P^a_\epsilon(Z) = 0 \right\}$. Since $h^P(Z, \epsilon)$ increases as $\epsilon$ decreases, we have $P^a_\epsilon(Z) = 0$. Furthermore, there exists $\{Z_i\}_i$ such that $\bigcup_i Z_i = Z$ and $\sum_i P^a_\epsilon(Z_i) < 1$. This implies that $P^a_\epsilon(Z_i) < 1$, for such $\{Z_i\}_i$. Since for $N \in \mathbb{N}, \epsilon > 0, s > 0$ and any subset $B \subset X$, we have $r_N(B, 2\epsilon) \exp(-Ns) \leq P^a_{N,\epsilon}(B)$. Then for any $Z_i \in \{Z_i\}_i$,

$$\lim_{\epsilon \to 0} \lim_{N \to \infty} \sup r_N(Z_i, 2\epsilon) \exp(-Na) \leq \lim_{\epsilon \to 0} \lim_{N \to \infty} P^a_{N,\epsilon}(Z_i) = \lim_{\epsilon \to 0} P^a_\epsilon(Z_i) \leq 1.$$  

This implies that $\lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{\log r_N(Z_i, 2\epsilon)}{N} \leq a$, i.e., $h^U(Z_i) \leq a$. Furthermore,

$$\inf \left\{ \sup_i h^U(Z_i) : \bigcup_{i=1}^\infty Z_i = Z \right\} \leq \sup_i h^U(Z_i) \leq a.$$  

Letting $a \to h^P(Z)$, we have

$$\inf \left\{ \sup_i h^U(Z_i) : \bigcup_{i=1}^\infty Z_i = Z \right\} \leq h^P(Z).$$
To prove the opposite inequality, let \( 0 < t < s < h^P(Z) \). Given \( \{Z_i\}_{i=1}^\infty \) such that \( \bigcup_{i=1}^\infty Z_i = Z \), it is enough to show that \( h^U(Z_i) \geq t \) for some \( i \). Since \( h^P(Z) = \lim_{\epsilon \to 0} h^P(Z, \epsilon) \), there exists \( \epsilon^* > 0 \) such that for any \( 0 < \epsilon < \epsilon^* \), \( h^P(Z, \epsilon) > s \). Then
\[
\inf \left\{ \sum_{i=1}^\infty P_s^\epsilon(Z_i) : \bigcup_{i=1}^\infty Z_i = Z \right\} = P_s^\epsilon(Z) > 0.
\]
Furthermore, there exist \( \alpha \) and \( i \), such that \( P_s^\epsilon(Z_i) > \alpha > 0 \). Since \( P_{N,\epsilon}^s(Z_i) \) decreases as \( N \) increases, we have \( P_{N,\epsilon}^s(Z_i) > \alpha \). There exist \( N_1 \) and a finite or countable pairwise disjoint family \( \{B_{n_j}(x_j, \epsilon)\}_{j=1}^\infty \) such that \( x_j \in Z_i, n_j \geq N_1 \) for all \( j \) and \( \sum_{j=1}^\infty \exp(-sn_j) > \alpha \). For each \( k \in \mathbb{N} \), let \( m_k \) be the number of \( j \) such that \( n_j = k \). Then we have \( \sum_{k=N_1}^\infty m_k \exp(-ks) > \alpha \). This yields for some \( N \geq N_1, m_N \geq \exp(Nt)(1 - \exp(t - s))\alpha \), since otherwise
\[
\sum_{j=1}^\infty \exp(-sn_j) = \sum_{k=N_1}^\infty m_k \exp(-ks) < \sum_{k=0}^\infty \exp(k(t-s))(1 - \exp(t - s))\alpha = \alpha.
\]
Furthermore,
\[
h^U(Z_i) = \lim_{\epsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \log r_N(Z_i, \epsilon) \geq \limsup_{N \to \infty} \frac{1}{N} \log m_N \geq t.
\]
This completes the proof.

**Theorem 3.2.** Suppose \((X, d, T)\) is a topological dynamical system. If \( Z_1 \subset X, Z_2 \subset X \), then
\[
h^B(Z_1 \times Z_2) \leq h^B(Z_1) + h^P(Z_2).
\]

**Proof.** Given \( \{Z_i^j\}_{i=1}^\infty \) such that \( \bigcup_{i=1}^\infty Z_i^j = Z_2 \).
\[
h^B(Z_1 \times Z_2) = h^B(Z_1 \times \bigcup_{i=1}^\infty Z_i^j) = h^B(\bigcup_{i=1}^\infty (Z_1 \times Z_i^j)) \leq \sup_{i \geq 1} h^B(Z_1 \times Z_i^j) \leq h^B(Z_1) + \sup_{i \geq 1} h^U(Z_i).
\]
This together with Proposition 3.1 implies that \( h^B(Z_1 \times Z_2) \leq h^B(Z_1) + h^P(Z_2) \).
Theorem 3.3. Suppose \((X, d, T)\) is a topological dynamical system. If \(Z_1 \subset X, Z_2 \subset X\), then

\[ h^P(Z_1 \times Z_2) \leq h^P(Z_1) + h^P(Z_2). \]

Proof. This follows from Proposition 3.1 and (v) of Proposition 2.1.

Corollary 3.1. For any \(A \subset X\), and \(\epsilon > 0\) there exists an increasing sequence \(A_1 \subset A_2 \subset \cdots \subset A\) such that \(A = \bigcup_{i=1}^{\infty} A_i\) and \(h^U(A_i) \leq h^P(A) + \epsilon\).

Proof. By Proposition 3.1 for \(\epsilon\), there exists \(A_1\) such that \(h^U(A_1) < h^P(A) + \epsilon\). Then for \(\epsilon/2\), there exists \(A_2^1\) such that \(h^U(A_2^1) < h^P(A) + \epsilon/2\). Let \(A_2 = A_1 \cup A_2^1\). Then \(h^U(A_2) = \max\{h^U(A_1), h^U(A_2^1)\} < h^P(A) + \epsilon\). We can construct a sequence \(\{A_i\}_{i=1}^{\infty}\) like this. Such \(\{A_i\}_{i=1}^{\infty}\) is desired.

It is worth pointing out that the above results hold for two different topological dynamical systems, i.e., suppose \((X_1, d_1, T_1), (X_2, d_2, T_2)\) are two topological dynamical systems and the product metric \(\rho\) on \((X_1 \times X_2, T_1 \times T_2)\) is given by \(\rho((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}\) for any \(x_1, y_1 \in X_1, x_2, y_2 \in X_2\).

- If \(h^B(X_1) < \infty\) and \(h^B(X_2) < \infty\), and \(Z_1 \subset X_1, Z_2 \subset X_2\) are analytic, then

\[ h^B(Z_1) + h^B(Z_2) \leq h^B(Z_1 \times Z_2); \]

- If \(Z_1 \subset X_1, Z_2 \subset X_2\), then

\[
\begin{align*}
h^B(Z_1 \times Z_2) & \leq h^B(Z_1) + h^P(Z_2); \\
h^P(Z_1 \times Z_2) & \leq h^P(Z_1) + h^P(Z_2); \\
h^U(Z_1 \times Z_2) & \leq h^U(Z_1) + h^U(Z_2).
\end{align*}
\]

This together with (iv) of Proposition 2.1 leads to the following corollary.

Corollary 3.2. Suppose \((X_1, d_1, T_1)\) and \((X_2, d_2, T_2)\) are two topological dynamical systems. If \(Z_1 \subset X_1, Z_2 \subset X_2\) are analytic and \(Z_2\) is \(T_2\)-invariant and compact (or \(Z_1\) is \(T_1\)-invariant and compact), then \(h^B(Z_1 \times Z_2) = h^B(Z_1) + h^B(Z_2)\).

At last, we give an example as follows.

Example: We take a topological dynamical system \((X, T)\) and \(D = \{\frac{1}{n}\}_{n \in \mathbb{N}} \cup \{0\}\) and let \(Z = X \times D\). Define \(R: Z \to Z\) satisfying \(R(x, \frac{1}{n+1}) = (x, \frac{1}{n})\), \(n \in \mathbb{N}\); \(R(x, 1) = (Tx, 1)\) and \(R(x, 0) = (x, 0)\) for \(x \in X\). Then \((Z, R)\) is a topological dynamical system.
If we identity \((x, 1)\) with \(x\) for each \(x \in X\), then \(X\) can be viewed as a closed subset of \(Z\) and \(R|_X = T\). Since \(h^P(D) \leq \max\left\{ \sup_{n \in \mathbb{N}} h^U(\{\frac{1}{n}\}), h^U(\{0\}) \right\} = 0\), we have \(h^B(Z) = h^B(X)\).

**Acknowledgements.** The research was supported by the National Natural Science Foundation of China (Grant No. 11271191) and National Basic Research Program of China (Grant No. 2013CB834100) and the Foundation for Innovative Program of Jiangsu Province (Grant No. CXZZ12 0380).

**References**

[1] R. Adler, A. Konheim and M. McAndrew. Topological entropy. *Trans. Amer. Math. Soc.* **114** (1965), 309-319.

[2] A. Besicovitch and P. Moran. The measure of product and cylinder sets. *J. London Math. Soc.* **20** (1945), 110-120.

[3] R. Bowen. Topological entropy for non-compact sets. *Trans. Amer. Math. Soc.* **184** (1973), 125-136.

[4] M. Brin and A. Katok. On local entropy. in: Geometric Dynamics, Rio de Janeiro, 1981, in: *Lecture Notes in Math. 1007*, Springer, Berlin, (1983), 30-38.

[5] H. Federer. Geometric Measure Theory, Springer-Verlag, New York, 1969.

[6] D. Feng and W. Huang. Variational principles for topological entropies of subsets. *J. Funct. Anal.* **263** (2012), 2228-2254.

[7] L. Goodwyn. The product theorem for topological entropy. *Trans. Amer. Math. Soc.* **158**(2) (1971), 445-452.

[8] J. Howroyd. On Hausdorff and packing dimension of product spaces. *Math. Proc. Camb. Phil. Soc.* **119** (1996), 715-727.

[9] J. Kelly. A method for constructing measures appropriate for the study of Cartesian products. *Proc. London Math. Soc.* **26** (1973), 521-546.

[10] P. Mattila. Geometry of sets and measures in Euclidean spaces. Cambridge University Press, 1995.

[11] J. Marstrand. The dimension of the Cartesian product sets. *Math. Proc. Camb. Phil. Soc.* **50** (1954), 198-202.
[12] Y. Pesin. Dimension theory in dynamical systems. *Contemporary Views and Applications* University of Chicago Press, Chicago, IL, 1997.

[13] P. Walters. An introduction to ergodic theory. Springer-Verlag. New York-Berlin. 1982.