PROPERTY T FOR GENERAL LOCALLY COMPACT QUANTUM GROUPS

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Abstract. In this short article, we obtained some equivalent formulations of property T for a general locally compact quantum group \( G \), in terms of the full quantum group \( C^* \)-algebras \( C_u^0(\hat{G}) \) and the \(*\)-representation of \( C_u^0(\hat{G}) \) associated with the trivial unitary corepresentation (that generalize the corresponding results for locally compact groups). Moreover, if \( G \) is of Kac type, we show that \( G \) has property T if and only if every finite dimensional irreducible \(*\)-representation of \( C_u^0(\hat{G}) \) is an isolated point in the spectrum of \( C_u^0(\hat{G}) \) (this also generalizes the corresponding locally compact group result). In addition, we give a way to construct property T discrete quantum groups using bicrossed products.

1. Introduction

The notion of property T for locally compact groups was first introduced by Kazhdan in the 1960s (see [9]), and this property was proved to be a very useful notion. A locally compact group \( G \) is said to have property T if every unitary representation of \( G \) having almost invariant unit vectors actually has a non-zero invariant vector (see [4, §1.1]). There are several equivalent formulations for property T (see [18] as well as Sections 1.1 and 1.2 of [4]):

(P1) The full group \( C^* \)-algebra \( C^*(G) \) can be decomposed as \( \ker \pi_1 \oplus C \), where \( \pi_1 \) is the \(*\)-representation induced by the trivial one dimensional representation \( 1_G \).

(P2) There exists a minimal projection \( p \in M(C^*(G)) \) such that \( \pi_1(G)(p) = 1 \).

(P3) \( 1_G \) is an isolated point in the topological space \( \hat{G} \) of irreducible unitary representations of \( G \).

(P4) All finite dimensional elements in \( \hat{G} \) are isolated points in \( \hat{G} \).

(P5) There exists a finite dimensional element in \( \hat{G} \) which is an isolated point in \( \hat{G} \).

In [8], P. Fima extended the notion of property T to discrete quantum groups and he showed in [8, Propositions 7 and 8] that discrete quantum groups with property T are of Kac type and finitely generated (in some sense). Recently, D. Kyed and M. Soltan studied property T for discrete quantum groups in [13],

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using the techniques in the theory of matrix quantum groups, and they obtained
the equivalences of property $T$ with the corresponding statements of (P1), (P2)
and (P3) in the discrete quantum groups case. They also proved that in the
case when the discrete quantum group is unimodular (or equivalently, of Kac
type), property $T$ is also equivalent to the corresponding statements of (P4) and
(P5). Furthermore, M. Daws, P. Fima, A. Skalski and S. White extended in
the definition of property $T$ to general locally compact quantum groups and
showed that a locally compact quantum group has both the Haagerup property
and property $T$ if and only if it is compact.

Following the works of Fima, Kyed-Soltan as well as Daws-Fima-Skalski-White,
among others, the present article devotes to the study of property $T$ for locally
compact quantum groups. We will extend the equivalences of property $T$ with
(P1), (P2) and (P3) to locally compact quantum groups, and will verify that they
are equivalent to the corresponding statements of (P4) and (P5) in the case of
locally compact quantum groups of Kac type (which include all locally compact
groups). In fact, by employing $C^*$-algebras technique instead of matrix quantum
groups technique (which do not work in this full generality), our proofs for these
more general results are actually simpler than the ones in [13].

In Section 2, we will recall a basic fact on the spectra of $C^*$-algebras and will
recall some notations and known facts on locally compact quantum groups.

In Section 3, we will give a very short proof for the equivalences of property $T$
with the corresponding statements of (P1), (P2) and (P3) using the materials in
Section 2. On the other hand, in order to show the equivalence of property $T$ with
the corresponding statements of (P4) and (P5), we need to consider contragredient
unitary corepresentation of a unitary corepresentation. We will use the technique
in [15], concerning contragredient corepresentation of Kac algebras, to generalize
Proposition A.1.12] to the quantum case. We then use it to get the desired
equivalence.

At the end of Section 3, we also present a new way to construct property $T$
discrete quantum groups. Up to now, apart from the one given in [8, Example
3.1], the only known examples of property $T$ discrete quantum groups are finite
quantum groups and property $T$ discrete groups, as well as their direct products.
We will show how to construct property $T$ discrete quantum groups of Kac type
using bicrossed products.

2. Notations and preliminary

In this article, we use the convention that the inner product $\langle \cdot, \cdot \rangle$ of a complex
Hilbert space $\mathcal{H}$ is conjugate-linear in the first variable. We denote by $\mathcal{L}(\mathcal{H})$ and
$\mathcal{K}(\mathcal{H})$ the set of bounded linear operators and that of compact operators on $\mathcal{H}$,
respectively. For any $x, y, z \in \mathcal{H}$ and $T \in \mathcal{L}(\mathcal{H})$, we denote by $\omega_{x,y}$ the normal
functional given by

$$\omega_{x,y}(T) := \langle x, Ty \rangle.$$ 

We set $\mathcal{S}_1(\mathcal{H})$ to be the unit sphere of $\mathcal{H}$. 

For a $C^*$-algebra $A$, we use $\text{Rep}(A)$ to denote the collection of unitary equivalence classes of non-degenerate $\ast$-representations of $A$. We consider $\hat{A} \subseteq \text{Rep}(A)$ to be the subset consisting of irreducible representations, equipped with the Fell topology (see [7]). The topological space $\hat{A}$ is known as the spectrum of $A$. Furthermore, all tensor products of $C^*$-algebras in the article, if not specified, are the minimal tensor products.

Let us also recall some well-known facts concerning $\text{Rep}(A)$ and $\hat{A}$. Suppose that $((\mu, H), (\nu, K)) \in \text{Rep}(A)$. We write $\nu \subset \mu$ if there is an isometry $V : K \to H$ such that $\nu(a) = V^* \mu(a) V \ (a \in A)$. Moreover, we write $\nu \prec \mu$ if $\ker \mu \subset \ker \nu$.

The following lemma is well-known. For the equivalence of Statements (1) and (2), one may use [7, Theorems 1.2 and 1.6]. For the equivalence of Statements (1) and (3), one may use the fact that the topology on $\hat{A}$ coincides with the one induced by the hull-kernel topology on $\text{Prim}(A)$. On the other hand, part (b) is a direct consequence of [7, Lemma 1.11].

**Lemma 2.1.** Let $A$ be a $C^*$-algebra and $((\mu, H)) \in \hat{A}$.

(a) The following statements are equivalent.

1. $\mu$ is an isolated point in $\hat{A}$.
2. If $\pi \in \text{Rep}(A)$ satisfying $\mu \prec \pi$, one has $\mu \subset \pi$.
3. $A = \ker \mu \oplus \bigcap_{\nu \in \hat{A} \setminus \{\mu\}} \ker \nu$.

(b) If $\dim H < \infty$, then $\{\mu\}$ is a closed subset of $\hat{A}$.

Next, we recall some materials on locally compact quantum groups. In the following, $(C_0(G), \Delta, \varphi, \psi)$ is a reduced $C^*$-algebraic locally compact quantum group as introduced in [11, Definition 4.1] (for simplicity, we will denote it by $G$). The dual locally compact quantum group of $G$ (as defined in [11, Definition 8.1]) is denoted by $(C_0(G), \hat{\Delta}, \hat{\varphi}, \hat{\psi})$. We use $L^2(G)$ to denote the Hilbert space given by the GNS construction of the left invariant Haar weight $\varphi$ and consider both $C_0(G)$ and $C_0(G)$ as $C^*$-subalgebras of $L(L^2(G))$. There is a unitary $W_G \in M(C_0(G) \otimes C_0(G)) \subseteq L(L^2(G) \otimes L^2(G))$, called the fundamental multiplicative unitary that implements the comultiplication:

$$\Delta(x) = W_G^*(1 \otimes x)W_G \quad (x \in C_0(G)).$$

The von Neumann subalgebra $L^\infty(G)$ generated by $C_0(G)$ in $L(L^2(G))$ is a Hopf von Neumann algebra under a comultiplication $\Delta$ defined by $W_G$ as in the above (see [12] or [17, Section 8.3.4]).
Definition 2.2. For any Hilbert space $\mathcal{H}$, a unitary $U \in M(\mathcal{K}(\mathcal{H}) \otimes C_0(\mathbb{G}))$ is called a unitary corepresentation of $\mathbb{G}$ on $\mathcal{H}$ if
\[
(id \otimes \Delta)(U) = U_{12}U_{13},
\]
where $U_{ij}$ is the usual “leg notation” (see e.g. [1]).

The universal version of $\hat{\mathbb{G}}$ is denoted by $(C_u^0(\hat{\mathbb{G}}), \hat{\Delta}^u)$ (see [10, Section 4 and 5]). As shown in [10, Proposition 5.2], there exists a unitary $V_u^u \in M(C_u^0(\hat{\mathbb{G}}) \otimes C_0(\mathbb{G}))$ that implements a bijection between unitary corepresentations $U$ of $\mathbb{G}$ on $\mathcal{H}$ and non-degenerate $\ast$-representations $\pi_U$ of $C_u^0(\hat{\mathbb{G}})$ on $\mathcal{H}$ through the correspondence
\[
U = (\pi_U \otimes id)(V_u^u).
\]

The identity $1_G$ of $M(C_0(\mathbb{G}))$ is a unitary corepresentation of $\mathbb{G}$ on $\mathcal{H}$ and $\pi_1$ is a character of $C_u^0(\hat{\mathbb{G}})$.

If $W$ is another unitary corepresentation of $\mathbb{G}$ on a Hilbert space $\mathcal{K}$, we denote by $U \odot W$ the unitary corepresentation $U_{13}W_{23}$ on $\mathcal{H} \otimes \mathcal{K}$ and call it the tensor product of $U$ and $W$. In this case,
\[
\pi_U \odot \pi_W = (\pi_U \otimes \pi_W) \circ \hat{\Delta}^u.
\]

Definition 2.3. Let $U \in M(\mathcal{K}(\mathcal{H}) \otimes C_0(\mathbb{G}))$ be a unitary corepresentation.

(a) $\xi \in \mathcal{H}$ is called a $U$-invariant vector if for every $\eta \in L^2(\mathbb{G})$, one has
\[
U(\xi \otimes \eta) = \xi \otimes \eta.
\]

(b) A net $\{\xi_i\}_{i \in I}$ in the unit sphere $S_1(\mathcal{H})$ is called an almost $U$-invariant unit vector if for each $\eta \in L^2(\mathbb{G})$, one has
\[
\|U(\xi_i \otimes \eta) - \xi_i \otimes \eta\| \to 0.
\]

The following proposition can be found in [3, Theorem 5.1] and [5, Proposition 2.7].

Proposition 2.4. Let $U$ be a unitary corepresentation $U$ of $\mathbb{G}$ on $\mathcal{H}$.

(a) An element $\xi \in \mathcal{H}$ is $U$-invariant if and only if for all $x \in C_u^0(\hat{\mathbb{G}})$, one has
\[
\pi_U(x)\xi = \pi_1(x)\xi.
\]

(b) A net $\{\xi_i\}_{i \in I}$ in $S_1(\mathcal{H})$ is an almost $U$-invariant unit vector if and only if for all $x \in C_u^0(\hat{\mathbb{G}})$, one has
\[
\|\pi_U(x)\xi_i - \pi_1(x)\xi_i\| \to 0.
\]

As in the literature, we write $U \subset W$ and $U \prec W$ when $\pi_U \subset \pi_W$ and $\pi_U \prec \pi_W$, respectively (see, e.g., [3, Section 5] or [5, Definition 2.3]). From Proposition 2.4, we can get directly the following corollary.
Corollary 2.5. Let $U$ be a unitary corepresentation of $\mathbb{G}$.
(a) $U$ has a non-zero invariant vector if and only if $1_{\mathbb{G}} \subset U$.
(b) $U$ has almost invariant vectors if and only if $1_{\mathbb{G}} \prec U$.

3. Property $T$ for locally compact quantum groups

Definition 3.1. A locally compact quantum group $\mathbb{G}$ is said to have property $T$ if every unitary corepresentation having an almost invariant unit vector has a non-zero invariant vector.

Let us first generalize the equivalences of property $T$ with the corresponding statements of (P1), (P2) and (P3) to the general case of locally compact quantum groups. Note that our proof here is even simpler than the case of locally compact groups (by using the materials in Section 2).

Proposition 3.2. The following statements are equivalent.

(T1) $\mathbb{G}$ has property $T$

(T2) $\pi_{1_{\mathbb{G}}}$ is an isolated point in $C_{0}(\hat{\mathbb{G}})$.

(T3) $C_{0}(\hat{\mathbb{G}}) = \ker \pi_{1_{\mathbb{G}}} \oplus \mathbb{C}$.

(T4) There is a projection $p_{\mathbb{G}} \in M(C_{0}(\hat{\mathbb{G}}))$ with

$$p_{\mathbb{G}}C_{0}(\hat{\mathbb{G}})p_{\mathbb{G}} = \mathbb{C}p_{\mathbb{G}} \text{ and } \pi_{1_{\mathbb{G}}}(p_{\mathbb{G}}) = 1.$$

Proof: (T1) $\Leftrightarrow$ (T2) This follows from Corollary 2.5 and Lemma 2.1(a).

(T2) $\Rightarrow$ (T3). This follows from Lemma 2.1(a).

(T3) $\Rightarrow$ (T4). One may take $p_{\mathbb{G}} = (0,1) \in \ker \pi_{1_{\mathbb{G}}} \oplus \mathbb{C}$.

(T4) $\Rightarrow$ (T2). Let $x$ be an element in $C_{0}(\hat{\mathbb{G}})$ such that $\pi_{1_{\mathbb{G}}}(x) = 1$. From

$$p_{\mathbb{G}} = p_{\mathbb{G}}xp_{\mathbb{G}},$$

we know that $p_{\mathbb{G}}$ actually belongs to $C_{0}(\hat{\mathbb{G}})$. As $p_{\mathbb{G}}C_{0}(\hat{\mathbb{G}})p_{\mathbb{G}}$ is a hereditary $C^{*}$-subalgebra of $C_{0}(\hat{\mathbb{G}})$, its spectrum can be identified as an open subset of $C_{0}(\hat{\mathbb{G}})$. In fact, this open subset is $\{\pi_{1_{\mathbb{G}}}(x)\}$, by [18, Lemma 1]. On the other hand, by Lemma 2.1(b), $\{\pi_{1_{\mathbb{G}}}(x)\}$ is also a closed subset of $C_{0}(\hat{\mathbb{G}})$.

Example 3.3. Let $G$ be a locally compact group.

(a) Suppose that $G_1$ and $G_2$ are closed subgroup of $G$ such that the canonical map $\varphi : G_1 \times G_2 \to G$ is a bijective homeomorphism into an open dense subset $\Omega$ of $G$ (and hence $G \setminus \Omega$ has measure zero). We consider $\alpha$ and $\beta$ to be canonical continuous actions of $G_1$ and $G_2$ on $G_1\backslash G$ and $G/G_2$, respectively. Then $G_1$ and $G_2$ is a matched pair of locally compact groups in the sense of [19, Definition 3.6.7]. By considering the trivial cocycles, one obtained from [19, Theorem 3.4.13] the locally compact quantum group $\mathbb{G}$. In fact, the fundamental unitary of $\mathbb{G}$
is the unitary $V$ as given in \cite{2} and one has $C_0(\mathbb{G}) = C_0(G_1 \backslash G) \rtimes \alpha \times r, G_2$ and 
$C_0(\hat{\mathbb{G}}) = C_0(G_1 \backslash G) \rtimes \alpha \times r, G_1$.

Now, suppose that $\Omega = G$, both $G_1$ and $G_2$ are amenable with $G_1$ being non-compact. By \cite[Theorem 6]{16}, $V$ is amenable, or equivalently, $\hat{\mathbb{G}}$ is coamenable. If $\mathbb{G}$ has property $T$, then \cite[Proposition 6.2]{5} implies that $\mathbb{G}$ is compact and hence $C_0(\mathbb{G})$ is unital. This gives the contradiction that $G_1 \cong G/G_2$ is compact.

(b) Suppose that $\hat{\mathbb{G}}$ is the dual group of the locally compact quantum group $\mathbb{G}$ corresponding to $G$. Since $C_u(\mathbb{G}) = C_0(\mathbb{G})$ and $1_{\mathbb{G}}$ corresponding to the evaluation at the identity $e$ of $G$, we know from Proposition 3.2 that $\hat{\mathbb{G}}$ has property $T$ if and only if $G$ is discrete. This part can also be deduced from \cite[Proposition 6.2]{5}.

We recall that $\mathbb{G}$ is said to be of Kac type if $L^\infty(\mathbb{G})$ is a Kac algebra (see e.g. \cite{6}). In this case, the antipode is bounded. We want to extend the equivalences of property $T$ with the corresponding statements of (P4) and (P5) in the Kac type case. Before that we need to generalize \cite[Proposition A.1.12]{14} to this case. Let us set some more notations.

From now on, $\mathbb{G}$ is of Kac type, $U$ and $V$ are unitary corepresentations of $\mathbb{G}$ on $\mathcal{H}$ and $\mathcal{K}$ respectively.

One may regard $U \in \mathcal{L}(\mathcal{H}) \tilde{\otimes} L^\infty(\mathbb{G})$ and $V \in \mathcal{L}(\mathcal{K}) \tilde{\otimes} L^\infty(\mathbb{G})$. As in \cite{14}, we define the contragredient $\tilde{V}$ of $V$ by 
$$\tilde{V} := (\tau \otimes \kappa)(V),$$
where $\tau$ is the canonical anti-isomorphism from $\mathcal{L}(\mathcal{K})$ to $\mathcal{L}(\mathcal{\bar{K}})$ (with $\mathcal{\bar{K}}$ being the conjugate Hilbert space of $\mathcal{K}$) and $\kappa$ is the bounded antipode on $L^\infty(\mathbb{G})$. Then $\tilde{V}$ is a unitary corepresentation of $\mathbb{G}$ on $\tilde{\mathcal{K}}$ (see, e.g., \cite[Corollary A.6(d)]{11} or \cite[Remark 2.2]{15}).

There is a canonical bijective isometry $\Theta$ from $\mathcal{H} \otimes \tilde{\mathcal{K}}$ to the Hilbert space $\mathcal{H}\mathcal{S}(\tilde{\mathcal{K}}, \mathcal{H})$ of Hilbert-Schmidt operators given by 
$$\Theta(x \otimes \tilde{y})(z) := x(y, z),$$
for any $x \in \mathcal{H}$ and $\tilde{y} \in \tilde{\mathcal{K}}$. We set 
$$(U; V)_{\mathcal{H}\mathcal{S}} := (\Theta \otimes \text{id})(U \otimes \tilde{V})(\Theta^* \otimes \text{id}),$$
which is a unitary corepresentation of $\mathbb{G}$ on $\mathcal{H}\mathcal{S}(\tilde{\mathcal{K}}, \mathcal{H})$.

**Lemma 3.4.** $T \in \mathcal{H}\mathcal{S}(\tilde{\mathcal{K}}, \mathcal{H})$ is $(U; V)_{\mathcal{H}\mathcal{S}}$-invariant if and only if 
$$U(T \otimes 1)V^* = T \otimes 1.$$

**Proof:** There are sequences $\{\xi_k\}_{k \in \mathbb{N}}$ and $\{\eta_k\}_{k \in \mathbb{N}}$ in $\mathcal{H}$ and $\tilde{\mathcal{K}}$, respectively, with 
$$\Theta \left( \sum_{k \in \mathbb{N}} \xi_k \otimes \eta_k \right) = T.$$
By [15] Lemma 3.8(a), for any \( \xi \in \mathcal{H} \), \( \eta \in \mathcal{H} \) and \( \alpha, \beta \in L^2(\mathbb{G}) \), one has

\[
\langle \xi \otimes \eta \otimes \beta, (U \overline{\otimes} V)(\Theta^*(T) \otimes \alpha) \rangle = \sum_{k \in \mathbb{N}} \langle U_{1k}(\xi \otimes \eta_k \otimes \beta), V_{2k}^*(\xi_k \otimes \eta \otimes \alpha) \rangle
\]

\[
= \sum_{k \in \mathbb{N}} \langle U^*(\xi \otimes \beta), \xi_k \otimes (\omega_{\eta_k, \eta} \otimes \text{id})(V^* \alpha) \rangle
\]

\[
= \sum_{k \in \mathbb{N}} \langle \xi \otimes \beta, U(\Theta(\xi_k \otimes \eta_k) \otimes 1)V^*(\eta \otimes \alpha) \rangle
\]

\[
= \langle \xi \otimes \beta, U(T \otimes 1)V^*(\eta \otimes \alpha) \rangle,
\]

because \( \Theta(\xi_k \otimes \eta_k)S\eta = \omega_{\eta_k, \eta}(S)\xi_k \) \((S \in \mathcal{L}(\mathfrak{R}))\) and

\[
\langle \xi \otimes \eta \otimes \beta, \Theta^*(T) \otimes \alpha \rangle = \langle \xi \otimes \beta, (T \otimes 1)(\eta \otimes \alpha) \rangle.
\]

Thus, we have

\[
(U \overline{\otimes} V)(\Theta^*(T) \otimes \alpha) = \Theta^*(T) \otimes \alpha \quad (\alpha \in L^2(\mathbb{G}))
\]

if and only if \( U(T \otimes 1)V^* = T \otimes 1 \).

\[\square\]

**Proposition 3.5.** Let \( U \) and \( V \) be as in the above. Then \( 1_G \subset U \overline{\otimes} V \) if and only if there is a finite dimensional unitary corepresentation \( W \) such that \( W \subset U \) and \( W \subset V \).

**Proof:** \( \Rightarrow \). By Corollary 2.5 and Lemma 3.2, there is \( T \in \mathcal{B}(\mathcal{H}(\mathfrak{H})) \setminus \{0\} \) such that \( U(T \otimes 1)V^* = T \otimes 1 \). The proof now proceeds as that of [4] Proposition A.1.12.

More precisely, since \( TT* \in \mathcal{B}(\mathfrak{H})_+ \), there exists \( \lambda \in \sigma(TT*) \setminus \{0\} \) with the corresponding eigenspace \( \mathcal{E}_\lambda \) being finite dimensional. It follows from \( U(TT* \otimes 1)U^* = TT* \otimes 1 \) that

\[
TT^* \pi_U \pi_U = \pi_U TT^*
\]

and \( \epsilon_\lambda \) is \( \pi_U \)-invariant. Moreover, from the equalities

\[
\|T^* \xi\|^2 = \langle \xi, TT^* \xi \rangle = \lambda \|\xi\|^2 \quad (\xi \in \mathcal{E}_\lambda),
\]

we know that \( \lambda^{1/2}T^*|_{\mathcal{E}_\lambda} : \mathcal{E}_\lambda \to T^*(\mathcal{E}_\lambda) \) is a bijective isometry. Furthermore, as

\[
V(T^* \otimes 1) = (T^* \otimes 1)U
\]

and \( \mathcal{E}_\lambda \) is \( \pi_U \)-invariant, we know that \( T^*(\mathcal{E}_\lambda) \) is \( \pi_V \)-invariant and

\[
\pi_U|_{\mathcal{E}_\lambda} \equiv \pi_V|_{T^*|_{\mathcal{E}_\lambda}}
\]

under \( \lambda^{1/2}T^*|_{\mathcal{E}_\lambda} \). Consequently, \( W = (\pi_U|_{\mathcal{E}_\lambda} \otimes \text{id})(V^*_G) \) is the finite dimensional corepresentation that is demanded.

\( \Leftarrow \). Let \( \mathcal{L} \) be a finite dimensional Hilbert space and \( W \in M(\mathcal{B}(\mathcal{L}) \otimes C_0(\mathbb{G})) \). By Lemma 3.1, the identity operator \( 1 \in \mathcal{B}(\mathcal{L}, \mathcal{L}) \) is \((W;W)_{\mathcal{B}(\mathcal{L})}\)-invariant. Thus, Corollary 2.5 gives

\[
1_G \subset W \overline{\otimes} W \subset U \overline{\otimes} V
\]

as required.

\[\square\]

The proof of the following theorem now follows from similar lines of argument as that of [4] Theorem 1.2.5. For completeness, we present the argument here.
Theorem 3.6. Let $G$ be a locally compact quantum group of Kac type. Then the property $T$ of $G$ is also equivalent to the following statements.

(T5) Every finite dimensional irreducible representation of $C^0_0(\hat{G})$ is an isolated point in $C^0_n(\hat{G})$.

(T6) $C^0_0(\hat{G}) \cong B \oplus M_n(\mathbb{C})$ for a $C^*$-algebra $B$ and an $n \in \mathbb{N}$.

Proof: (T2) ⇒ (T5). Let $\mu \in C^n_0(\hat{G})$ and $\pi \in \text{Rep}(C^n_0(\hat{G}))$ such that $\mu$ is finite dimensional and $\mu \prec \pi$. If we set

$$U := (\pi \otimes \text{id})(V^n_G) \quad \text{and} \quad V := (\mu \otimes \text{id})(V^n_G),$$

then $V \prec U$. Therefore, by (2.2) and Proposition 3.5 one has

$$1_G \subset V \otimes \bar{V} \prec U \bar{V}.$$  

Hence, $1_G \subset U \bar{V}$ by Lemma 2.1(a) and Proposition 3.5. This gives a unitary corepresentation $W$ with $W \subset U$ and $W \subset V$ (again, by Proposition 3.5) and the irreducibility implies $V = W \subset U$ (or equivalently, $\mu \subset \pi$). Now, Lemma 2.1(a) gives the required conclusion.

(T5) ⇒ (T6). This follows from Lemma 2.1(a).

(T6) ⇒ (T2). Let $\mu$ be the irreducible $*$-representation of $C^0_0(\hat{G})$ corresponding to the summand $M_n(\mathbb{C})$ and denote

$$U := (\mu \otimes \text{id})(V^n_G).$$

As $\mu$ is finite dimensional, one has

$$(\mu \otimes \hat{\mu}) \circ \hat{\Delta}^u = \mu_0 \oplus \cdots \oplus \mu_n$$

for some $\mu_0, \ldots, \mu_n \in C^n_0(\hat{G})$. By Proposition 3.5, we may assume that $\mu_0 = \pi_{1_{\mathcal{I}}}$. Moreover, Lemma 2.1(b) tells us that all such $\{\mu_k\}$ are closed subset of $\mathcal{C}^0_0(\hat{G})$. Suppose on the contrary that $\{\pi_{1_{\mathcal{I}}}\}$ is not open in $\mathcal{C}^0_0(\hat{G})$. Then there is a net $\{\sigma_i\}_{i \in \mathcal{I}} \subset \mathcal{C}^0_0(\hat{G}) \setminus \{\mu_0, \ldots, \mu_n\}$ that converges to $\pi_{1_{\mathcal{I}}}$. Thus, $\pi_{1_{\mathcal{I}}} \prec \bigoplus_{i \in \mathcal{I}} \sigma_i$, which implies

$$\mu \prec \bigoplus_{i \in \mathcal{I}} (\sigma_i \otimes \mu) \circ \hat{\Delta}^u.$$  

By Lemma 2.1(a), one has $\mu \subset \bigoplus_{i \in \mathcal{I}} (\sigma_i \otimes \mu) \circ \hat{\Delta}^u$, and hence,

$$\mu \subset (\sigma_{i_0} \otimes \mu) \circ \hat{\Delta}^u,$$

for some $i_0 \in \mathcal{I}$ (as $\mu$ is irreducible). If we put

$$V_0 := (\sigma_{i_0} \otimes \text{id})(V^n_G),$$

then by Proposition 3.5, we know that

$$1_G \subset U \bar{V} \subset V_0 \bar{V} \subset V_0 \bar{V} \circledast U \bar{V} = \bigoplus_{k=0}^n V_0 \bar{V} \circledast (\mu_k \otimes \text{id})(V^n_G).$$

Now, Proposition 3.5 and the irreducibility of $\mu_k$ $(k = 0, \ldots, n)$ again tells us that there is $k_0 \in \{0, \ldots, n\}$ with

$$(\mu_{k_0} \otimes \text{id})(V^n_G) \subset V_0.$$
However, this will produce the contradiction that $\sigma_{i_0} = \mu_{k_0}$.

The following corollary is a direct consequence of Proposition 3.2 and Theorem 3.6.

**Corollary 3.7.** Let $\mathbb{H}$ be another locally compact quantum group such that there is a surjective $^*$-homomorphism $\Phi : C^*_0(G) \to C^*_0(\mathbb{H})$. Suppose that $G$ has property $T$. If either $\pi_1 = \pi_1 \circ \Phi$ or $G$ is of Kac type, then $\mathbb{H}$ has property $T$.

Let us end this paper with a “non-trivial” construction of discrete quantum groups with property $T$. Suppose that $G_1$ is a property $T$ discrete group acting non-trivially on a finite group $G_2$ by group automorphisms and take $G$ to be the semi-direct product $G_2 \rtimes G_1$. For example, if $[G_1, G_1] \neq G_1$, then there exist a non-trivial action of the finite abelian group $G_1/[G_1, G_1]$ on some finite group $G_2$. If $\beta$ is the action of $G_2$ on $G_1$ as in Example 3.3(a), then $\beta$ is trivial. Thus, the resulting locally compact quantum group $G$ in Example 3.3(a) is of Kac type (see [19 Corollary 3.6.17]).

Furthermore, the following theorem tells us that $G$ has property $T$. Observe that an essential part of the proof of this theorem is to verify that $C^*_0(G)$ is a quotient C*-algebra of the full crossed product $C(G_2) \rtimes_{\alpha} G_1$. This could be a known fact, but since we do not find it in the literature, we give an argument here for the sake of completeness.

**Theorem 3.8.** If $G, G_1$ and $G_2$ are as in the above, then the discrete quantum group $G$ as in Example 3.3(a) has property $T$.

**Proof:** Let $\alpha$ and $\beta$ be the actions as in Example 3.3(a). We denote by $\Delta, \Delta_1$ and $\Delta_2$ the coproducts on $C_0(G)$, $C_0(G_1)$ and $C^*_0(G_2)$ respectively. By abuse of notations, we use

$$
\alpha : C_0(G_2) \to C_0(G_1 \times G_2) \quad \text{and} \quad \beta : C_0(G_1) \to C_0(G_1 \times G_2)
$$

denote the maps induced by the actions $\alpha$ and $\beta$ as in [19 p.275], respectively. In particular,

$$
\alpha(y)(g,s) = y(\alpha_g(s)) \quad (y \in C_0(G_2); g \in G_1; s \in G_2).
$$

The triviality of $\beta$ implies that $\beta(\phi) = \phi \otimes 1$ ($\phi \in C_0(G_1)$). Suppose that $W^{(1)} \in M(C_0(G_1) \otimes C^*_0(G_1))$ and $W^{(2)} \in M(C^*_0(G_2) \otimes C_0(G_2))$ are the fundamental unitary corresponding to $G_1$ and the dual of $G_2$, respectively. As in [19 Definition 3.4.2], the fundamental unitary of $G$ is given by

$$
W_G = W^{(1)}_{13} ((1 \otimes (\text{id} \otimes \alpha))(W^{(2)})) \in \mathcal{L}(L^2(G_1 \times G_2 \times G_1 \times G_2)).
$$

Consider $1_{G_2}$ to be the trivial representation of $G_2$ and $\eta_0 : C_0(G_1) \to \mathbb{C}$ to be the evaluation at the identity $e$ of $G_1$. Since $G_2$ is finite and $\beta$ is trivial, the
\(C^\ast\)-algebra \(C_0(\mathbb{G})\) coincides with \(C_0(G_1) \otimes C^\ast(G_2)\). For any \(z \in C_0(\mathbb{G})\), we have

\[
(\eta_e \otimes \text{id} \otimes \eta_e \otimes \text{id})\Delta(z) = (\eta_e \otimes \text{id} \otimes \eta_e \otimes \text{id})(W^2_G(1 \otimes z)W_G) = (\text{id} \otimes (\eta_e \otimes \text{id})\alpha)(\hat{W}'^2) \ast (1 \otimes (\eta_e \otimes \text{id})(z))(\text{id} \otimes (\eta_e \otimes \text{id})\alpha)(\hat{W}'^2) = \hat{\Delta}_2((\eta_e \otimes \text{id})(z)),
\]

as well as

\[
(\text{id} \otimes \pi_{1G_2} \otimes \text{id} \otimes \pi_{1G_2})\Delta(z) = (\text{id} \otimes \pi_{1G_2} \otimes \text{id} \otimes \pi_{1G_2})(W^2_G(1 \otimes z)W_G) = (W^{(1)} \otimes 1)^\ast (1 \otimes z)(W^{(1)} \otimes 1),
\]

which implies that

\[
(\text{id} \otimes \pi_{1G_2} \otimes \text{id} \otimes \pi_{1G_2})\Delta(z) = \Delta_1((\text{id} \otimes \pi_{1G_2})(z)).
\]

On the other hand, for any \(g \in G_1\) and \(r \in G_2\), we put \(\delta_g, \lambda_g, \delta_r\) and \(\lambda_r\) to be the corresponding elements in \(C_0(G_1), C^\ast(G_1), C_0(G_2)\) and \(C^\ast(G_2)\), respectively. Note that \(W^{(1)} = \sum_{g \in G_1} \delta_g \otimes \lambda_g\) and \(W^{(2)} = \sum_{r \in G_2} \lambda_r \otimes \delta_{r^{-1}}\). Hence,

\[
(W^{(1)}_{13} \ast (1 \otimes 1 \otimes \delta_g \otimes \lambda_r)W^{(1)}_{13}) = \sum_{k \in G_1} \delta_k \otimes 1 \otimes \delta_{k^{-1}g} \otimes \lambda_r.
\]

Moreover, Relation 3.1 implies that \(\alpha(\delta_r) = \sum_{h \in G_1} \delta_{h^{-1}} \otimes \delta_{\alpha_h(r)}\), and so

\[
(\text{id} \otimes \alpha)(\hat{W}'^2) = \sum_{h \in G_1} \sum_{t \in G_2} \lambda_{t^{-1}} \otimes \delta_{h^{-1}} \otimes \delta_{\alpha_h(t)}.
\]

Consequently,

\[
\Delta(\delta_g \otimes \lambda_r) = \sum_{s,t \in G_1} \sum_{f,h,k \in G_1} \delta_k \otimes \lambda_{t^{-1}} \otimes \delta_h \delta_{k^{-1}g} \delta_f \otimes \delta_{\alpha_{h^{-1}}(t)} \lambda_r \delta_{\alpha_{f^{-1}}(s)}
\]

\[
= \sum_{s,t \in G_1} \sum_{k \in G_1} \delta_k \otimes \lambda_{t^{-1}} \otimes \delta_{k^{-1}g} \otimes \delta_{\alpha_g^{-1}(s)} \delta_{\alpha_{g^{-1}}(s)} \lambda_r
\]

\[
= \sum_{t \in G_2} \sum_{k \in G_1} \delta_k \otimes \lambda_{k^{-1}g}(r) \otimes \delta_{k^{-1}g} \otimes \delta_{\alpha_g^{-1}(s)} \lambda_r
\]

\[
= \sum_{h \in G_1} \delta_{h^{-1}} \otimes \lambda_{\alpha_g(r)} \otimes \delta_{hg} \otimes \lambda_r.
\]

Now, suppose that \(U \in M(\mathcal{X}(\mathbb{G}) \otimes C_0(\mathbb{G}))\) is a unitary corepresentation of \(C_0(\mathbb{G})\). Then

\[
U = \sum_{r \in G_2} \sum_{g \in G_1} U_{g,r} \otimes \delta_g \otimes \lambda_r
\]
for some $U_{g,r} \in \mathcal{L}(\mathfrak{g})$ (here we use $\sum_{g \in G_1} U_{g,r} \otimes \delta_g$ to denote the map $g \mapsto U_{g,r}$).

Relations (2.1) and (3.4) produce
\[ \sum_{k,l} \sum_{s,t} U_{k,s} U_{l,t} \otimes \delta_k \otimes \lambda_\delta \otimes \lambda_\ell = \sum_{g,h \in G_1} \sum_{r \in G_2} U_{g,r} \otimes \delta_{h^{-1}} \otimes \lambda_{\alpha_{g}(r)} \otimes \delta_{h} \otimes \lambda_r = \sum_{k,l} \sum_{s,t} U_{k,l,t} \otimes \delta_k \otimes \lambda_{\alpha_{s}(t)} \otimes \delta_{t} \otimes \lambda_t. \]

This tells us that
\[ U_{k,s} U_{l,t} = \begin{cases} U_{k,l,t} & \text{if } s = \alpha_l(t) \\ 0 & \text{otherwise.} \end{cases} \quad (3.5) \]

Now, we set $U^{(1)} := (\text{id} \otimes \text{id} \otimes \pi_{1G_2})(U)$ and $U^{(2)} := (\text{id} \otimes \eta \otimes \text{id})(U)$. By Relations (3.2) and (3.3), we know that $U^{(1)}$ and $U^{(2)}$ are unitary corepresentations of $C_0(G_1)$ and $C^*(G_2)$, respectively. They induces, respectively, a unitary representation $\mu_U : G_1 \rightarrow \mathcal{L}(\mathfrak{g})$ and an $^*$-representation $\Psi_U : C(G_2) \rightarrow \mathcal{L}(\mathfrak{g})$. Clearly, for any $h \in G_1$ and $t \in G_2$, one has
\[ \mu_U(h) = \sum_{r \in G_2} U_{h,r} \quad \text{and} \quad \Psi_U(\delta_t) = U_{e,t}. \]

It is now easy to verify, using Relation (3.5), that
\[ \Psi_U(\delta_{\alpha_s(t)}) \mu_U(h) = U_{h,t} = \mu_U(h) \Psi_U(\delta_t) \]
and $(\Psi_U, \mu_U)$ is covariant for the action $\alpha$ and hence induces a $^*$-representation $\Psi_U \times \mu_U : C(G_2) \rtimes_\alpha G_1 \rightarrow \mathcal{L}(\mathfrak{g})$.

On the other hand, as $L^\infty(\mathfrak{g}) = L^\infty(G_1) = G_2 = L^\infty(G_1) \otimes L(G_2)$ (see [19] Definition 3.4.2) or [21 Proposition 1.1]), its predual $L_1(\mathfrak{g})$ is generated by \{ $\hat{\lambda}_g \otimes \delta_s : g \in G_1; s \in G_2$ \}, where $\hat{\lambda}_g \in L_1(G_1)$ and $\delta_s \in A(G_2)$ are the images of $g$ and $s$, respectively. Thus,
\[ \pi_U(\lambda_g \otimes \delta_s) = U_{g,s} = \mu_U(g) \Psi_U(\delta_s) = (\Psi_U \times \mu_U)(\hat{\lambda}_g \hat{\delta}_s), \]
where $\hat{\delta}_s$ and $\hat{\lambda}_g$ are the images of $\delta_s$ and $\lambda_g$, respectively, in the full crossed product $C(G_2) \rtimes_\alpha G_1$.

The above shows that $C_0^0(\mathfrak{g})$ is a quotient $C^*$-algebra of $C(G_2) \rtimes_\alpha G_1$. Since $C(G_2)$ has strong property $T$ (see Example 5.1 and Lemma 4.2 of [14]), we know, through Theorem 4.6 and Lemma 4.1 of [13], that the unital $C^*$-algebra $C_0^0(\mathfrak{g})$ has strong property $T$, and hence has property $T$. Now, [13] Theorem 5.2 concludes that $\mathfrak{g}$ has property $T$.

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