Theory of superconducting qubits beyond the lumped element approximation

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In the design and investigation of superconducting qubits and related devices, a lumped element circuit model is the standard theoretical approach. However, many important physical questions lie beyond its scope, e.g. the behavior of circuits with strong Josephson junctions carrying substantial currents and the properties of small superconducting devices. By performing gauge transformations on self-consistent solutions of the Bogoliubov-de Gennes equations, we develop here a formalism that treats Josephson couplings non-perturbatively. We apply the formalism to (a) show that Fermi sea effects can contribute to the effective capacitance of small charge qubits; (b) demonstrate an asymmetry in clockwise and counterclockwise current states in small RF squid qubits; and (c) provide a microscopic wavefunction of superconducting Schrodinger cats suitable for computing the number of entangled electrons.

Over the past quarter century, superconducting qubits [1] have enjoyed dramatic performance improvements and have attracted growing interest and excitement. Several remarkable families of superconducting qubits have been invented and investigated [2]. To describe these designer quantum systems, a quantum mechanical lumped element (LE) circuit theory is employed very broadly and successfully by researchers [3–5]. However, as the degree of control achieved over superconducting qubits becomes ever more exquisite [6–9], a role is emerging for a theoretical description of increased detail and precision. In this paper, we frame such a description which, among its important features, treats the Josephson coupling between islands non-perturbatively.

As shown below, this theory can address physical questions that are beyond the scope of LE theory, such as the behavior of strong Josephson couplings supporting substantial currents or the properties of much smaller superconducting qubits. This description also has the potential to supply answers when ambiguities arise in the application of LE theory (for an example in the case of junctions with Andreev bound states see [10]) and to permit refined calculations of qubit properties such as perturbations of qubit spectroscopy. This paper presents sample calculations on a charge qubit and on an RF squid qubit, showing consistency with LE theory and also revealing uncharted effects. We then leverage the microscopic character of our analysis to study the number of electrons in a superconducting “Schrodinger cat” [11–13].

LE theory describes a superconducting circuit as a system of distinct islands that are coupled weakly by Josephson junctions. The justification for this description typically involves treating a tunneling Hamiltonian at second order in perturbation theory [5, 14, 15]. To go beyond LE theory, and gain new insights, we holistically treat all of the coupled islands of the system as a single superconducting entity.

Assume a microscopic electronic Hamiltonian

\[ H = T + P + W \]  \hspace{1cm} (1)

where the kinetic, potential, and interaction energies are

\[ T = -t \sum_{R} \sum_{a=x,y,z} \left( \frac{t_{r+a,r}}{t} \right) c_{r+a}^\dagger c_{r} \]  \hspace{1cm} (2)

\[ P = \sum_{R} (v(i) - \mu) c_{R}^\dagger c_{R} \]  \hspace{1cm} (3)

\[ W = \frac{1}{2} \sum_{R,R'} W(R,R') c_{R}^\dagger c_{R'}^\dagger c_{R'} c_{R} \]  \hspace{1cm} (4)

Here, \( R \equiv (r, \sigma) \) is a combined position and spin coordinate introduced for notational brevity. The notation is not meant to suggest a tight-binding approximation. Good descriptions of the low-energy eigenstates of \( H \) are obtained by following steps (i)–(iii) below. For concreteness, we focus on a charge qubit comprising a superconducting grain with a junction region (Fig. 1a-b).

![Circuit diagram of charge qubit](image1)

**FIG. 1.** (a) Circuit diagram of charge qubit. (b) Schematic of qubit with Josephson junction in dark region. (c) Magnitude (blue) of superconducting gap \( |\Delta_{r=1,r'=1}| \) depends on component of \( r \). Minimum at Josephson junction is evident. (d) Diagonal phase function \( \langle \theta | H | \theta \rangle \), corresponding to Josephson energy (blue). Energy eigenstates of [8] shown in green, orange, and purple.

**Step (i)** of our approach defines a set of states that will be superposed to describe the low-energy eigenstates of \( H \). The choice of states is motivated by the following conventional rationale. Superconductors have low-energy
excited states that allow them to conduct current. To
develop intuition about the form of such current-carrying
states, imagine boosting each electron in the ground state
by a momentum \(\hbar \delta q\) (henceforth, we set \(\hbar = 1\)). Such a
boost multiplies the superconducting order parameter
\(\Delta_{R, R'} = W(R, R')(\theta_R \theta_{R'})\) \(\text{(5)}\)

by a position-dependent phase \(e^{i \delta q \cdot (r + r')}\). This suggests
that low-energy excitations can be expressed in terms of
phase changes of the superconducting order parameter.

Thus, to describe a charge qubit, we define a set of
states \(|\theta\rangle\), where \(\theta\) denotes the phase drop of the super-
conducting order parameter. The LE approximation also
works with a set of states \(|\theta\rangle\) with dynamics dictated by
\(H_{\text{LE}} = 4E_C n^2 + E_J (1 - \cos \theta)\) \(\text{(6)}\)

where \([\theta, n] = i\) and \(\theta \in (-\pi, \pi]\). In our approach,
however, we eschew the assumption of an abrupt phase
drop and model the junction microscopically. One
consequence is that our \(\theta\) is not restricted to \((-\pi, \pi]\).

To obtain the state \(|\theta = 0\rangle\), we self-consistently solve
the BdG equations \(\text{[14, 16]}\), collecting the positive-energy
\(\pi/\pi\) order parameter \(\Delta_{\text{eff}}\) and leads to results that are more readily compared
with \(\pi/\pi\) sequence is that our

phase of the order parameter, defining
\(\theta, n\) where \(\text{[14]}\)

will drop rapidly across the Josephson junction.
Instead of performing this calculation for all \(\theta\), here we
do so only for \(\theta = \pi\). This both reduces computational
effort and leads to results that are more readily compared
with the LE approximation. The computation yields an
order parameter \(\Delta_{R, R'}\), with phase \(-\pi/2\) on one side of
the system and \(\pi/2\) on the other. We then rescale the
phase of the order parameter, defining
\(\Theta_r = \text{Arg } \Delta_{R, R'}^* (r, r - 1)/\pi\) \(\text{(7)}\)

which takes the value \(-1/2\) on one side of the system and
1/2 on the other. To complete step (i), we define \(\theta\) as
the state obtained by applying a gauge transformation to
the self-consistent BdG solution: \(U_{R, K} \to e^{i \theta \Theta_r/2} U_{R, K}\)
and \(V_{R, K} \to e^{-i \theta \Theta_r/2} V_{R, K}\). This definition ensures
that the phase of the order parameter of \(|\theta\rangle\) indeed changes
by \(\theta\) across the system.

In step (ii) of our approach, each approximate low-
energy eigenstate of \(H\) is written as a superposition of
the states defined in step (i): \(|\psi\rangle = \sum_\theta \psi(\theta)|\theta\rangle\).

Finally, step (iii) entails computing the wavefunction
\(\psi(\theta)\) by solving the Schrodinger equation
\(\sum_\theta \langle \theta | H | \theta' \rangle \psi(\theta') = E \sum_\theta \langle \theta | \theta' \rangle \psi(\theta')\). \(\text{(8)}\)

The overlaps \(\langle \theta | \theta' \rangle\) appear on the right hand side of
this equation because the \(|\theta\rangle\) do not form an orthonormal
basis in general. The Onishi formula \(\text{[18]}\) states
\(|\langle \theta | \theta' \rangle|^2 = \text{det } \mathcal{U}\)

\(\text{(9)}\)
The matrix elements \(\langle \theta | H | \theta' \rangle\) of the Hamiltonian \(\text{[1]}\)
can be computed using \(\text{[18]}\)
\(\frac{\langle \theta' | c_{R, 1} \theta c_{R, \pi} | \theta \rangle}{\langle \theta' | \theta \rangle} = e^{i \theta \Theta_r (V^* \frac{1}{U^T} V^T) R_2 R_1 e^{-i \theta \Theta_r}}\) \(\text{(10)}\)

and a related formula \(\text{[17]}\) for \(\langle \theta' | c_{R, 1} c_{R, \pi} c_{R, 1} \theta \rangle\). To solve \(\text{(6)}\), we transform to an orthonormal basis with
effective Hamiltonian \(\mathcal{H}\) \(\text{[17]}\).

This formalism is suitable for first principles or
phenomenological computation as well as analytical study.
Here we present an example computation, carrying out
steps (i)-(iii) on a model charge qubit formed by a rect-
gle lattice of \(5 \times 5 \times 220\) tight-binding sites with a
mean occupation of 1012 electrons \(\text{[17]}\). An attractive
Hubbard interaction gives rise to superconductivity \(\text{[15]}\).

The magnitude of the order parameter \(|\Delta_{R, R'}^\pi (r, r - 1)|\)
is shown in Fig. \(\text{[4]}\) alongside its rescaled phase \(\text{[1]}\).

The diagonal matrix elements \(\langle \theta | H | \theta \rangle\) are shown in
Fig. \(\text{[1]}\): they correspond to the Josephson term in \(\text{[9]}\)
and originate from terms in \(\text{[3]}\) associated with hopping
across the junction. The off-diagonal elements of \(\langle \theta | H | \theta' \rangle\)
cause transitions from one value of \(|\theta\rangle\) to another; they
 correspond to the capacitive term in \(\text{[9]}. Based on LE
theory, one expects this term to arise from \(\text{[3]}\). However,
in our calculations \(\text{[4]}\) makes only a small contribution
which even deviates from the quadratic \(4E_C n^2\) form; presumably this is because we assume an attractive
Hubbard interaction rather than a Coulomb interaction. Surpris-
ingly, the kinetic energy \(\text{[2]}\) makes the dominant con-
tribution. Physically, the reason is that transferring \(n\)
Cooper pairs across our small charge qubit shrinks the
Fermi sea of one half of the superconducting grain and
grows the Fermi sea of the other. Expanding the total en-
ergy of the Fermi seas in \(n\) gives a quadratic term \(4E_C n^2\).
It may be possible to amplify and exploit such underex-
plored capacitance features in device designs. The eigen-
states in Fig. \(\text{[1]}\) take a form in consonance with the LE
solution of \(\text{[6]}\). Indeed, \(\text{[6]}\) arises from our formalism in
the appropriate abrupt-junction limit \(\text{[17]}\).

We now turn from a charge qubit to a more compli-
cated example: an RF squid qubit threaded by mag-
etic flux (see Fig. \(\text{[2]}\).b). In this case, step (i) involves
In step (ii), we write
\[ |\psi\rangle = \sum_{\theta} \psi(\theta, \theta)|\theta, \theta\rangle + \psi(\theta, \theta)|\theta, \theta\rangle. \tag{12} \]

Step (iii) then requires us to solve
\[
\sum_{\theta'} \left[ \begin{pmatrix} \langle \theta, \theta' | H | \theta \rangle \\ \langle \theta, \theta' | H | \theta \rangle \end{pmatrix} \right] \left[ \begin{pmatrix} \psi(\theta, \theta') \\ \psi(\theta, \theta') \end{pmatrix} \right] = E \sum_{\theta'} \left[ \begin{pmatrix} \langle \theta, \theta' | H | \theta \rangle \\ \langle \theta, \theta' | H | \theta \rangle \end{pmatrix} \right] \left[ \begin{pmatrix} \psi(\theta, \theta') \\ \psi(\theta, \theta') \end{pmatrix} \right]. \tag{13} \]

After transforming to an orthonormal basis, we produce a 2 \times 2 effective Hamiltonian \( H \).

We carry out steps (i)-(iii), performing a numerical computation on a model RF squid qubit analogous to the charge qubit computation above [17]. Because of the 2 \times 2 structure of the matrix \( H \), its diagonal gives rise to the 2 potentials \( \langle \theta, \theta | H | \theta, \theta \rangle \) and \( \langle \theta, \theta | H | \theta, \theta \rangle \) depicted in Fig 2d: the curve formed by taking the lower potential at each \( \theta \) produces an RF squid double-well potential as expected. Indeed, under suitable conditions discussed in [17], \( H \) reduces to the LE Hamiltonian
\[
H_{LE} = 4E_C n^2 + E_J (1 - \cos \theta) + E_L (\theta - \phi)^2 / 2. \tag{14} \]

where \( \theta \in \mathbb{R} \) is unbounded, \( [\theta, n] = i \), and \( \phi = 2\pi(2e/h)\Phi \) is fixed by the applied flux \( \Phi \).

The energy eigenstates of \( H \), shown in Fig 2d, accord with those expected within LE theory. However, they exhibit an asymmetry between the two peaks of each wavefunction. This effect occurs because the microscopic electronic Hamiltonian [1] does not possess the symmetry at half a superconducting flux quantum \( h/4e \) that it does at half an electronic flux quantum \( h/2e \): the symmetry between clockwise and counterclockwise superconducting states in the loop emerges only approximately when electrons bind into Cooper pairs. This effect is absent from LE theory; it becomes significant for mesoscopic loop sizes [19] and is captured by our formalism.

The microscopic character of our theory makes it particularly suitable for investigating Schrodinger cat states in superconducting loops. Several remarkable experiments [11,12] have realized these cats, formed by superposing a clockwise supercurrent and a counterclockwise supercurrent. To assess the implications [20] of these experiments, it is of central importance to quantify the size of the cat – the number of entangled particles. We will focus on [11] since it raises the breathtaking possibility that billions of electrons might have been entangled.

Computing the number of entangled electrons, \( \Delta N \), is beyond the scope of LE theory [17]. The state-of-the-art calculation is due to Korsbakken et al. [13]. Given a partition of the many-body Schrodinger cat state into two terms \( \sqrt{1/2} |\theta\rangle + \sqrt{1/2} |\theta\rangle \), they introduced
\[
\Delta N = \frac{1}{2} \sum_{Q} \langle Q | c_{Q}^\dagger c_{Q} | \theta\rangle - \langle Q | c_{Q}^\dagger c_{Q} | \theta\rangle, \tag{15} \]

for all possible partitions of the many-body Schrodinger cat state.
to count how many particles are in different modes in $|\langle \rangle\rangle$ and $|\langle \rangle\rangle$. Here, Q labels a state in whichever basis of single-particle electron states maximizes $\Delta N$, which is the basis in which the matrix $|\langle \rangle\rangle c_Q c_Q^\dagger |\langle \rangle\rangle - |\langle \rangle\rangle c_Q^\dagger c_Q |\langle \rangle\rangle$ is diagonal. Korsbakken et al. evaluated \[15\] by approximating $|\langle \rangle\rangle$ and $|\langle \rangle\rangle$ as Galilean-boosted BCS states rather than using LE theory. They thereby derived the approximation $\Delta N \approx 3 |L_L|/4 ev_F \[17]$. As a check, we evaluate \[15\] using the RF squid qubit eigenstates that we obtained numerically above. For these, their approximation differs from \[15\] by $\sim 30\%$.

Applying their approximation to the experiment \[11\], they concluded \[13\] that $\Delta N \approx 3800 - 5750$ electrons. However, note that the wavefunction realized experimentally in \[11\] takes the form of the green curve in Fig. \[3\] according to LE theory. It differs dramatically from a Schrödinger cat form $\sqrt{\frac{3}{2}} |\langle \rangle\rangle + \sqrt{\frac{1}{2}} |\langle \rangle\rangle$ and particularly so from the pristine wavefunction assumed by Korsbakken et al. (red curve in Fig. \[3\]). A definitive evaluation of the number of entangled electrons in \[11\] would therefore require a tractable alternative to \[15\] that does not assume a Schrödinger cat form $\sqrt{\frac{3}{2}} |\langle \rangle\rangle + \sqrt{\frac{1}{2}} |\langle \rangle\rangle$. Unfortunately, defining such a metric is a vexatious problem \[21\] out of the scope of our study.

In conclusion, we have developed a microscopic theory of superconducting qubits that goes beyond the standard LE approximation. It accords with LE theory both for charge and RF squid qubits and yields several insights inaccessible to LE theory. Numerous applications of this tool are anticipated as experimental progress allows increasingly precise study of superconducting qubits.

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SUPPLEMENTAL MATERIAL

Bogoliubov-de Gennes Equations

This section derives the Bogoliubov-de Gennes (BdG) equations, following pp. 137–145 of [14]. We begin with the definition of the Bogoliubov transformation

\[
c_R = \sum_K U_{R,K} \alpha_K + V_{R,K} \alpha_K^+, \\
c_R^+ = \sum_K U_{K,R}^* \alpha_K^+ + V_{R,K} \alpha_K.
\]  

(S1)

Imposing anticommutation relations, we obtain

\[
\{c_R, c_R^+\} = V_{R,K}^* U_{R,K} + U_{K,R}^* V_{R,K} = 0
\]

which becomes

\[
V^* U^T + UV^\dagger = 0
\]

as a matrix equation. Similarly,

\[
\{c_R^+, c_R^\dagger\} = U_{R,K}^* V_{K,R} + V_{K,R}^* U_{K,R} = \delta_{R,R'}
\]

(S2)

where a Kronecker delta appears on the right hand side. As a matrix equation, this becomes

\[
UU^\dagger + V^* V^T = I.
\]

(S3)

Now, (S1) can be written as a matrix equation

\[
\begin{bmatrix} c_R \\ c_R^\dagger \end{bmatrix} = \begin{bmatrix} U & V^* \\ V^T & U^T \end{bmatrix} \begin{bmatrix} \alpha_R \\ \alpha_R^\dagger \end{bmatrix}.
\]

(S6)

Relations (S3) and (S5) imply

\[
\begin{bmatrix} U & V^* \\ V^T & U^T \end{bmatrix} \begin{bmatrix} U & V^* \\ V^T & U^T \end{bmatrix}^T = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.
\]

(S7)

This equation gives 2 identities; collecting them together with (S3) and (S5) gives a total of 4 identities

\[
UU^\dagger + V^* V^T = I \quad U^\dagger U + V^\dagger V = I \\
V^* U^T + UV^\dagger = 0 \quad V^T U + U^T V = 0.
\]

(S9)

Equation (S8) implies that the inverse of the Bogoliubov transformation is

\[
\alpha_K = \sum_R U_{R,K}^* c_R + V_{R,K}^* c_R^+, \\
\alpha_K^\dagger = \sum_R U_{R,K} c_R^\dagger + V_{R,K} c_R.
\]

(S10)

We use the Bogoliubov transformation to find eigenstates of a mean-field approximation to the Hamiltonian. Starting with

\[
H = \sum_{R,R'} c_R^\dagger h_{R,R'} c_R + \frac{1}{2} W(R',R) c_R^\dagger c_{R'}^\dagger c_R c_{R'},
\]

we make a mean-field approximation

\[
H' = \sum_{R,R'} c_R^\dagger h_{R,R'} c_R + \frac{1}{2} W(R',R) c_R^\dagger c_{R'}^\dagger c_R c_{R'}
\]

(S11)

In each of the three lines in parentheses, we have paired the first operator \(c_R\) in the interaction term \(1/2W(R,R')c_R c_{R'}^\dagger c_R c_{R'}\) one of the remaining 3 operators \((c_R^\dagger, c_R, c_{R'}\) in the first line then \(c_R\) in the second line then \(c_R^\dagger\) in the third line). Within each line, we applied the approximation

\[
ab = [(a - \langle \alpha \rangle) + (\langle \alpha \rangle)] [(b - \langle b \rangle) + \langle b \rangle]
\]

\[
\approx (a - \langle \alpha \rangle)(b - \langle b \rangle) + \langle \alpha \rangle b
\]

\[
= a(b) + (a) b - \langle \alpha \rangle b
\]

taking \(a\) to be the product of \(c_R\) and its pair while taking \(b\) to be the product of the remaining 2 operators. The approximation assumes \((a - \langle \alpha \rangle)(b - \langle b \rangle)\) is the product of 2 small quantities and can be neglected. It is convenient to set

\[
\begin{align*}
H' & = \sum_{R,R'} c_R^\dagger h'_{R,R'} c_R \\
& + \frac{\Delta_{R,R'}}{2} c_R c_R^\dagger + \frac{(\Delta_{R,R'})^*}{2} c_R c_{R'} + \text{const}
\end{align*}
\]

(S13)

where

\[
\Delta_{R,R'} = W(R,R') \langle c_R c_{R'} \rangle
\]

(S14)

and

\[
h'_{R,R'} = h_{R,R'} \\
+ \delta_{R,R'} \sum_{R''} W(R,R'') \langle c_{R''} c_{R''} \rangle - W(R,R') \langle c_R c_{R'} \rangle.
\]

(S15)

To obtain this form of \(h'_{R,R'}\), we have used \(W(R,R') = W(R',R)\). Note that the final term of \(h'_{R,R'}\) is the exchange interaction.

We next demand that the Bogoliubov transformation diagonalize the mean-field Hamiltonian, so that \(H' = E_g + \sum_K \epsilon_K \alpha_K^\dagger \alpha_K\). This implies \([\alpha_K, H'] = \epsilon_K \alpha_K\) and \([\alpha_K, H'] = -\epsilon_K \alpha_K^\dagger\). We compute

\[
[c_R, H'] = \sum_{R'} h'_{R,R'} c_{R'} + \Delta_{R,R'} c_{R'}^\dagger
\]

(S16)
and substitute in the Bogoliubov transformation \( [S1] \). Comparing the coefficients of \( \alpha_K \) and \( \alpha_K^\dagger \) on each side, we obtain the two BdG equations
\[
\epsilon_k U_{R,K} = \sum_{R'} h_{R,R'}^{*} U_{R',K} + \Delta_{R,R'} V_{R',K}, \tag{S17}
\]
\[
\epsilon_k V_{R,K} = \sum_{R'} (-h_{R,R'})^* V_{R',K} + (\Delta_{R,R'})^* U_{R',K}.
\]
Substituting the Bogoliubov transformations in the definitions of \( h_{R,R'} \) and \( \Delta_{R,R'} \) and using the fact that \( \alpha_K \) annihilates the ground state, we obtain
\[
h_{R,R'}' = h_{R,R'} + \delta_{R,R'} \sum_{R''} W(R, R'') \sum_K V_{R'',K} V_{R',K}^* - W(R, R') \sum_K V_{R,K} V_{R',K}^* \tag{S18}
\]
and
\[
\Delta_{R,R'} = W(R, R') \sum_K U_{R,K} V_{R',K}^*. \tag{S19}
\]
The BdG equations are to be solved self-consistently with these expressions for \( h_{R,R'}' \) and \( \Delta_{R,R'} \).

It is sometimes useful to express the BdG many-body ground state in Thouless form \([18]\). To do so, we define \( Z \equiv (V U^{-1})^* \) and write
\[
|0\rangle = \mathcal{N} \exp \left[ \frac{1}{2} \sum_{R,R'} c_{R,R'}^\dagger Z_{R,R'} c_{R,R'}^+ \right] |\text{vac}\rangle. \tag{S20}
\]
Here, \( \mathcal{N} \) is a normalization constant. This is a generalization of the standard BCS ground state wavefunction \( \mathcal{N} \exp \left[ \sum_k (\epsilon_k / u_k) c_{k,1}^{\dagger} c_{k,-1} \right] |\text{vac}\rangle \). It is more general because it does not assume the Cooper pairs form specifically in the momentum basis.

To show that \( |0\rangle \) is the many-body ground state of the BdG equations, we need to demonstrate that it is annihilated by \( \alpha_K \). Using the inverse Bogoliubov transformation \([S10]\), we compute
\[
\left[ \frac{1}{2} \sum_{R,R'} c_{R,R'}^\dagger Z_{R,R'} c_{R,R'}^+ , \alpha_K \right] = \sum_R V_{R,K}^* c_{R,K}^+. \tag{S21}
\]
The argument uses \([S9]\). Note that the right hand side commutes with \( \frac{1}{2} \sum_{R,R'} c_{R,R'}^\dagger Z_{R,R'} c_{R,R'}^+ \). Now, according to a well-known lemma,
\[
e^{X} Y e^{-X} = Y + [X,Y] \tag{S22}
\]
when \([X,[X,Y]] = 0\). (This is proven straightforwardly by differentiating \( e^{sX} Y e^{-sX} \) with respect to \( s \).) Thus, we deduce that
\[
\exp \left[ - \frac{1}{2} \sum_{R,R'} c_{R,R'}^\dagger Z_{R,R'} c_{R,R'}^+ \right] \alpha_K \exp \left[ \frac{1}{2} \sum_{R,R'} c_{R,R'}^\dagger Z_{R,R'} c_{R,R'}^+ \right] = \alpha_K - \sum_R V_{R,K}^* c_{R,K}^+ = \sum_R U_{R,K} c_{R,K}.
\]
It follows that
\[
\alpha_K \exp \left[ \frac{1}{2} \sum_{R,R'} c_{R,R'}^\dagger Z_{R,R'} c_{R,R'}^+ \right] |\text{vac}\rangle = 0. \tag{S24}
\]
We see that \( |0\rangle \) is indeed the many-body ground state of the BdG equations.

The states \( |\theta\rangle \) are defined in the text using the gauge transformation \( U_{R,K} \rightarrow e^{i \theta_\sigma / 2} U_{R,K} \) and \( V_{R,K} \rightarrow e^{-i \theta_\sigma / 2} V_{R,K} \). Given the form \([S20]\) for \( |0\rangle \), this implies
\[
|\theta\rangle = \mathcal{N} \exp \left[ - \frac{1}{2} \sum_{R,R'} e^{i \theta_{\sigma} + \theta_{\sigma'}} / 2 c_{R,R'}^\dagger Z_{R,R'} c_{R,R'}^+ \right] |\text{vac}\rangle. \tag{S25}
\]

**Computational Solution of BdG equations**

Our formalism is compatible with detailed first principles approaches to solving the BdG equations. However, in this paper, for simplicity we model the superconducting qubits using a rectangular lattice with lattice vectors \( a_x, a_y, \) and \( a_z \). The number of tight-binding sites in the \( x, y, \) and \( z \) directions are \( N_x, N_y, \) and \( N_z \) respectively, giving rise to lengths \( L_x = N_x |a_x|, L_y = N_y |a_y|, \) and \( L_z = N_z |a_z| \). Symmetries can facilitate solution of the BdG equations. We employ periodic boundary conditions in the \( x \) and \( y \) directions; as a result of the translational symmetry, our self-consistent quantities \([S18]\) and \([S19]\) are homogeneous in the \( x \) and \( y \) directions.

We choose a spin-independent interaction of the form
\[
W(R, \sigma, \sigma') = w(x - x', y - y', z - z') = \begin{cases} -\frac{\lambda}{|r - r'|} & r = r' \\ \frac{\lambda}{|r - r'|} & r \neq r' \end{cases} \tag{S26}
\]
where the attractive Hubbard interaction at \( r = r' \) gives rise to superconductivity. The BdG equations can be written as
\[
\epsilon_k U_{m_x, m_y, m_z, \sigma, k} = \sum_{\sigma', \sigma'} h'_{m_x, m_y, m_z, \sigma, \sigma', \sigma'} U_{m_x, m_y, m_z, \sigma', \sigma', k} + \Delta_{m_x, m_y, m_z, \sigma, \sigma'} V_{m_x, m_y, m_z, \sigma', \sigma', k},
\]
\[
\epsilon_k V_{m_x, m_y, m_z, \sigma, k} = \sum_{\sigma', \sigma'} (-h'_{m_x, m_y, m_z, \sigma, \sigma', \sigma'})^* V_{m_x, m_y, m_z, \sigma', \sigma', k} + (\Delta_{m_x, m_y, m_z, \sigma, \sigma'} + \sigma')^* U_{m_x, m_y, m_z, \sigma', \sigma', k}, \tag{S27}
\]
where the transverse momenta \( \frac{2 \pi m_x}{L_x} \) and \( \frac{2 \pi m_y}{L_y} \) are good
quantum numbers and

\[ U_{m_x,m_y,z,\sigma,K} = \frac{1}{\sqrt{N_x N_y}} \sum_{x,y} e^{-i \frac{2\pi m_x}{L_x} x - i \frac{2\pi m_y}{L_y} y} U_{x,y,z,\sigma,K} \]

\[ V_{m_x,m_y,z,\sigma,K} = \frac{1}{\sqrt{N_x N_y}} \sum_{x,y} e^{-i \frac{2\pi m_x}{L_x} x - i \frac{2\pi m_y}{L_y} y} V_{x,y,z,\sigma,K} \]

with inverse relation

\[ U_{x,y,z,\sigma,K} = \frac{1}{\sqrt{N_x N_y}} \sum_{m_x,m_y} e^{i \frac{2\pi m_x}{L_x} x + i \frac{2\pi m_y}{L_y} y} U_{m_x,m_y,z,\sigma,K} \]

\[ V_{x,y,z,\sigma,K} = \frac{1}{\sqrt{N_x N_y}} \sum_{m_x,m_y} e^{i \frac{2\pi m_x}{L_x} x + i \frac{2\pi m_y}{L_y} y} V_{m_x,m_y,z,\sigma,K} \]

(S28)

To specify the forms of the operators on the right hand side of \[\text{(S27)}\], we define

\[ \frac{1}{N_x N_y} \sum_{x,y,x',y'} e^{-i \frac{2\pi}{L_x} x - i \frac{2\pi}{L_y} y} e^{-i \frac{2\pi}{L_x} x' - i \frac{2\pi}{L_y} y'} h_{x,y,z,\sigma,x',y',z,\sigma'} = \delta_{x,x'} \delta_{y,y'} h_{m_x,m_y,z,\sigma,z',\sigma'} \]  

(S29)

where

\[ h_{m_x,m_y,z,\sigma,z',\sigma'} = \sum_{x,y} e^{-i \frac{2\pi m_x}{L_x} (x-x') - i \frac{2\pi m_y}{L_y} (y-y')} h_{x,y,z,\sigma,x',y',z,\sigma'} \].  

(S30)

This definition is reasonable because \( h_{x,y,z,\sigma,x',y',z,\sigma'} \) depends on \( x, x', y, \) and \( y' \) only via \( x - x' \) and \( y - y' \). (The use of periodic boundary conditions in the \( x \) and \( y \) directions is essential for this translational invariance.) Similarly, we define

\[ \frac{1}{N_x N_y} \sum_{x,y,x',y'} e^{-i \frac{2\pi}{L_x} x - i \frac{2\pi}{L_y} y} e^{-i \frac{2\pi}{L_x} x' - i \frac{2\pi}{L_y} y'} w(x - x', y - y', z - z') = \delta_{x,x'} \delta_{y,y'} w(m_x,m_y,z,z') \].  

(S31)

Here, we have

\[ w(m_x,m_y,z,z') = \sum_{x,y} e^{-i \frac{2\pi m_x}{L_x} (x-x') - i \frac{2\pi m_y}{L_y} (y-y')} w(x - x', y - y', z - z') \]

\[ = N_x N_y s(m_x,m_y) e^{-2\pi \sqrt{(m_x/L_x)^2 + (m_y/L_y)^2}|z-z'|} \]

\[ - \sum_{n_x,n_y} s(n_x,n_y) \delta_{z,z'} - g \delta_{z,z'} \]  

(S32)

with

\[ s(m_x,m_y) = \frac{\lambda}{L_x L_y} \begin{cases} 2\sqrt{\pi L_x L_y} & m_x = m_y = 0 \\ \frac{1}{\sqrt{(m_x/L_x)^2 + (m_y/L_y)^2}} & \text{otherwise} \end{cases} \]

Then,

\[ h'_{m_x,m_y,z',\sigma',z,\sigma} = h_{m_x,m_y,z',\sigma',z,\sigma} + \delta_{z',z} \delta_{\sigma',\sigma} \sum_{z''} \tilde{w}(0,0,z - z'') \times \]

\[ \frac{1}{N_x N_y} \sum_{\ell_x,\ell_y,K} V_{x,y,z',\sigma',\sigma''} \tilde{w}(\ell_x,\ell_y,z'',z',\sigma',\sigma) \]

\[ - \sum_{\ell_x,\ell_y} \tilde{w}(m_x - \ell_x, m_y - \ell_y, z - z') \times \]

\[ \frac{1}{N_x N_y} \sum_{K} V_{x,y,z',\sigma',\sigma''} \tilde{w}(\ell_x,\ell_y,z'',z',\sigma',\sigma) \]  

(S34)

and

\[ \Delta_{m_x,m_y,z,\sigma,z',\sigma'} = \sum_{\ell_x,\ell_y} \tilde{w}(m_x - \ell_x, m_y - \ell_y, z - z') \times \]

\[ \frac{1}{N_x N_y} \sum_{K} U_{x,y,z',\sigma',\sigma''} \tilde{w}(\ell_x,\ell_y,z'',z',\sigma',\sigma) \]  

(S35)

Alternatively, we also have

\[ \Delta_{m_x,m_y,z,\sigma,z',\sigma'} = - \sum_{\ell_x,\ell_y} \tilde{w}(m_x - \ell_x, m_y - \ell_y, z - z') \times \]

\[ \frac{1}{N_x N_y} \sum_{K} U_{x,y,z',\sigma',\sigma''} \tilde{w}(\ell_x,\ell_y,z'',z',\sigma',\sigma) \]  

(S36)

The total energy of the system is given by

\[ \langle H \rangle = \sum_{\ell_x,\ell_y,z,\sigma,z',\sigma'} \left[ h_{\ell_x,\ell_y,z,\sigma,z',\sigma'} \rho_{\ell_x,\ell_y,z,\sigma,z',\sigma'} + \frac{1}{2 N_x N_y} \sum_{m_x,m_y} \tilde{w}(0,0,z - z') \right] \]

\[ \rho_{\ell_x,\ell_y,z,\sigma,z',\sigma'} = \rho_{\ell_x,\ell_y,z,\sigma,z',\sigma'} \tilde{w}(\ell_x - m_x, \ell_y - m_y, z - z') \]

\[ \rho_{\ell_x,\ell_y,z,\sigma,z',\sigma'} = \rho_{\ell_x,\ell_y,z,\sigma,z',\sigma'} \tilde{w}(\ell_x - m_x, \ell_y - m_y, z - z') \]

\[ K_{\ell_x,\ell_y,z,\sigma,z',\sigma'} = \rho_{\ell_x,\ell_y,z,\sigma,z',\sigma'} \tilde{w}(\ell_x - m_x, \ell_y - m_y, z - z') \]  

(S37)

Here,

\[ \rho_{x,y,z,\sigma,x',y',z',\sigma'} = \langle \hat{c}_{\sigma}^\dagger \hat{c}_{\sigma'} \rangle \]

\[ = \sum_{K} V_{x,y,z',\sigma',\sigma'} \tilde{w}(x,y,z') \]  

(S38)

leading to

\[ \rho_{\ell_x,\ell_y,z,\sigma,z',\sigma'} = \sum_{x,y} e^{-i \frac{2\pi}{L_x} \ell_x (x-x') - i \frac{2\pi}{L_y} \ell_y (y-y')} \rho_{x,y,z,\sigma,x',y',z',\sigma'} \]

\[ = \sum_{K} V_{\ell_x,\ell_y,z,\sigma,z',\sigma'} \tilde{w}(x,y,z') \]  

(S39)
with the minus signs in front of $\ell_x$ and $\ell_y$ in the final expression resulting from the flip in the order of the primed and unprimed variables in (S38). Similarly,

$$K_{x,y,z,\sigma,x',y',z',\sigma'} = \langle c_R | c_R \rangle = \sum_K U_{x',y',z',\sigma',K} V_{x,y,z,\sigma,K}^*$$

and

$$K_{\ell_x,\ell_y,z,\sigma,\ell_x',\ell_y',z',\sigma'} = \sum_{x,y} e^{-i\frac{2\pi}{\nu}\ell_x(x-x')-i\frac{2\pi}{\nu}\ell_y(y-y')} K_{x,y,z,\sigma,x',y',z',\sigma'} = \sum_K U_{-\ell_x,-\ell_y,z',\sigma',K} V_{-\ell_x,-\ell_y,z,\sigma,K}^.$$  \hspace{1cm} (S40)

In the paper, we perform computations on rectangular lattices with $N_x = N_y = 5$ and $N_z = 220$ sites. In the charge qubit case, the tunneling matrix elements in (2) are set to $t_{r+n_{r}} = t$, defining the energy scale of the Hamiltonian. In the RF squid qubit case, a phase is included in $t_{r+n_{r}} = t e^{-i\pi/2(220)}$ since a magnetic field threads the squid loop. In both cases, at a plane in the middle of the lattice of sites, reduced tunneling matrix elements in the $z$ direction to give rise to the Josephson junction. The reduced value is $t_{r+n_{r}} = 0.1t$ in the charge qubit case and $t_{r+n_{r}} = 0.3te^{-i\pi/2(220)}$ in the RF squid qubit case. The potential is set to $v(r) = 0$, and the chemical potential is set to $\mu = -3.42t$ in (3). In (S26), we insert a strong attractive interaction $g = 2.25t$ to ensure a superconducting gap of reasonable magnitude (see Fig. 4 and Fig. 2) despite the nearly 1-dimensional geometry of the lattice. The repulsive interaction was taken small enough that the results were insensitive to its specific value ($\lambda = 5 \times 10^{-2}t$).

**Derivation of Matrix Elements**

As described in the text, we need to compute the matrix elements that appear in the Schrödinger equations (8) and (13). For concreteness, we will focus on the charge qubit case (8); the RF squid case is very similar. The desired matrix elements take the form $\langle \theta'' | O | \theta' \rangle = \langle \theta'' | O (c_{R_1}^\dagger c_{R_2}) | \theta' \rangle$ where the right hand side makes explicit the dependence of $O$ on the operators $c_{R_1}^\dagger$ and $c_{R_2}$. To compute these matrix elements, we follow the derivation in Ref. 18.

The states $|\theta'\rangle$ and $|\theta''\rangle$ are defined in the text by performing gauge transformations from the BdG solutions of (S17):

$$U_{R,K}'' = e^{i\phi_t/2} U_{R,K} \quad V_{R,K}'' = e^{-i\phi_t/2} V_{R,K}$$

and

$$U_{R,K}''' = e^{i\phi_t/2} U_{R,K} \quad V_{R,K}''' = e^{-i\phi_t/2} V_{R,K}.$$  \hspace{1cm} (S42)

These equations imply that $|\theta''\rangle$ is annihilated by the quasiparticle operator $\beta_K$, where

$$c_R = \sum_L U_{R,L}^* |\beta_L\rangle + (V')_{R,L}^* |\beta_L^\dagger\rangle$$

and $|\theta''\rangle$ is annihilated by the quasiparticle operator $\gamma_K$, where

$$c_R = \sum_L U_{R,L}^* |\gamma_L\rangle + (V'')_{R,L}^* |\gamma_L^\dagger\rangle.$$  \hspace{1cm} (S43)

The inverse of this last definition is

$$\gamma_K = \sum_R (U''_{R,K})^* c_R + (V'')_{R,K} c_R^\dagger,$$

$$\gamma_K^\dagger = \sum_R U_{R,K}^* c_R^\dagger + V''_{R,K} c_R,$$  \hspace{1cm} (S45)

combined with (S43), this implies

$$\gamma_K = \sum_L U_{L,K}^* \beta_L + V_{L,K} \beta_L^\dagger,$$

$$\gamma_K^\dagger = \sum_L U_{L,K} \beta_L^\dagger + V_{L,K}^* \beta_L,$$  \hspace{1cm} (S46)

where $U = (U')^\dagger U'' + (V')^\dagger V''$ and $V = (V')^T U'' + (U')^T V''$.

To proceed, we use Thouless’s theorem, which asserts that Bogoliubov states $|\theta''\rangle$ and $|\theta'\rangle$ are related by

$$|\theta''\rangle = |\theta'\rangle e^{\sum_{K,K'} \beta_{K'}^* Z_{K,K'} \beta_{K'}/2} |\theta''\rangle.$$  \hspace{1cm} (S47)

where $Z = (U^{-1})^*$. The theorem assumes that $\langle \theta'' | \theta' \rangle \neq 0$, which we will see implies that $U^{-1}$, and therefore $Z$, exists. To prove Thouless’s theorem, one needs to show that $\gamma_K$ annihilates the expression on the right hand side of (S47). The proof closely parallels the argument leading up to (S24) with $\alpha_K$, $c_R$, and $U_{R,K}$, $V_{R,K}$, and $Z_{R,K}$ replaced by $\gamma_K$, $\beta_L$, $U_{L,K}$, $V_{L,K}$, and $Z_{K,K'}$ respectively. In the first step of the proof, we calculate

$$e^{\sum_{K,K'} \beta_{K'}^* Z_{K,K'}^* \beta_{K'}/2} \beta_L e^{-\sum_{K,K'} \beta_{K'} Z_{K,K'}^* \beta_{K'}/2} = \beta_L$$

and

$$e^{\sum_{K,K'} \beta_{K'}^* Z_{K,K'}^* \beta_{K'}/2} \beta_L e^{-\sum_{K,K'} \beta_{K'} Z_{K,K'}^* \beta_{K'}/2} = \beta_L + \sum_{K} \beta_{K'} Z_{K,K'}^* \beta_{K'}/2, \beta_L$$

$$= \beta_L - \sum_{K} \beta_{K'} Z_{K,K'}^* \beta_{K'}/2.$$  \hspace{1cm} (S48)
by applying lemma (S22). Then, we employ the definition [S45] to obtain the analogues of (S23) and (S24). Note that (S17) reduces to (S20) when we set $U'$ equal to the identity matrix and $V'$ equal to the zero matrix.

Thouless’s theorem allows us to evaluate the overlap $\langle \theta' | \theta'' \rangle$. To do this, we simplify our expression for $|\theta''\rangle$ into a BCS-like form. The theorem of Bloch and Messiah states there are unitary matrices $C$ and $D$ for which $U = DU^T C^T$ and $V = DV^T C^T$ are real and take simple block-diagonal forms. The matrix $U$ has $2 \times 2$ blocks of the form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in which the two states $\mathbf{J}$ and $\mathbf{J}$ are paired with $u_{\mathbf{J}} = u_{\mathbf{J}}$. In the usual BCS case in which pairing occurs in the momentum basis, we would have $J = k, \sigma$ and $J = -k, -\sigma$. The matrix $V$ has $2 \times 2$ blocks of the form $v_{\mathbf{J}} \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$, involving the same two states $\mathbf{J}$ and $\mathbf{J}$. The coefficients satisfy the normalization condition $u_{\mathbf{J}}^2 + v_{\mathbf{J}}^2 = 1$. Defining $\beta_{\mathbf{J}} = \sum_{k,K} \beta_{k,K} D_{\mathbf{J},k}$ in terms of the matrix $D$, we find that $\sum_{k,K} \beta_{k,K}^* \beta_{k,K}^{\dagger} / 2 = \sum_{\mathbf{J}} \beta_{\mathbf{J}}^* \beta_{\mathbf{J}}^{\dagger}$. Thus,

$$|\theta''\rangle = |\theta'\rangle \exp \left( \sum_{\mathbf{J}} \beta_{\mathbf{J}}^* v_{\mathbf{J}} \beta_{\mathbf{J}}^{\dagger} \right) |\theta'\rangle$$

$$= (\langle \theta' | \theta'' \rangle) \Pi_{\mathbf{J}} \left( 1 + \beta_{\mathbf{J}}^* v_{\mathbf{J}} \beta_{\mathbf{J}}^{\dagger} \right) |\theta'\rangle$$

$$= (\langle \theta' | \theta'' \rangle) \Pi_{\mathbf{J}} \left( 1 - \beta_{\mathbf{J}} v_{\mathbf{J}} \right) |\theta'\rangle.$$ 

(S50)

The normalization condition $\langle \theta' | \theta'' \rangle = 1$ then implies $|\langle \theta' | \theta'' \rangle|^2 = \Pi_{\mathbf{J}} v_{\mathbf{J}}^2 = \det U$. The Onishi formula $|\langle \theta' | \theta'' \rangle|^2 = \det U$ leaves the phase of $\langle \theta' | \theta'' \rangle$ undetermined; by choosing the phases of $U^*_{RR,R}$ and $V^*_{RR,R}$ appropriately, we can ensure that $\langle \theta' | \theta'' \rangle$ is real. Then, the correct sign of $\langle \theta' | \theta'' \rangle = \pm \sqrt{\det U}$ is fixed using continuity in $\theta''$ starting with $\langle \theta' | \theta'' \rangle = 1$ at $\theta'' = \theta'$.

Using Thouless’s theorem, we can also calculate matrix elements of the form

$$\langle \theta'' | O | \theta' \rangle / \langle \theta'' | \theta' \rangle = \langle \theta'' | O \sum_{\mathbf{J}} \beta_{\mathbf{J}}^* Z_{k,K} \beta_{k,K}^{\dagger} / 2 | \theta' \rangle / \langle \theta'' | \theta' \rangle =$$

$$\langle \theta'' | \sum_{k,K} \beta_{k,K}^* Z_{k,K} \beta_{k,K}^{\dagger} / 2 O \sum_{\mathbf{J}} \beta_{\mathbf{J}}^* v_{\mathbf{J}} \beta_{\mathbf{J}}^{\dagger} | \theta' \rangle =$$

$$\langle \theta'' | \sum_{k,K} \beta_{k,K}^* Z_{k,K} \beta_{k,K}^{\dagger} / 2 O \sum_{\mathbf{J}} \beta_{\mathbf{J}}^* \beta_{\mathbf{J}}^{\dagger} | \theta' \rangle =$$

$$e^{-\sum_{k,K} \beta_{k,K}^* Z_{k,K} \beta_{k,K}^{\dagger} / 2} \langle \theta'' | O \sum_{\mathbf{J}} \beta_{\mathbf{J}}^* \beta_{\mathbf{J}}^{\dagger} e^{-\sum_{k,K} \beta_{k,K}^* Z_{k,K} \beta_{k,K}^{\dagger} / 2} | \theta' \rangle \rangle.$$

(S51)

where $e^{-\sum_{k,K} \beta_{k,K}^* Z_{k,K} \beta_{k,K}^{\dagger} / 2}$ have been inserted between adjacent operators in $O$ to derive the final line.

To proceed, we insert (S48) and (S49) into the defini-

$$e^{-\sum_{k,K} \beta_{k,K}^* Z_{k,K} \beta_{k,K}^{\dagger} / 2} c_{R_{LR}} \langle \theta'' | O \sum_{\mathbf{J}} \beta_{\mathbf{J}}^* \beta_{\mathbf{J}}^{\dagger} | \theta' \rangle =$$

$$\sum_{L} U_{RL}^* \beta_{L} + (V_{RL}^*)^* (\beta_{L}^\dagger - \sum_{\mathbf{J}} \beta_{\mathbf{J}}^* Z_{k,K} \beta_{k,K}^{\dagger})$$

(S52)

and

$$e^{-\sum_{k,K} \beta_{k,K}^* Z_{k,K} \beta_{k,K}^{\dagger} / 2} c_{R_{LR}} \langle \theta'' | O \sum_{\mathbf{J}} \beta_{\mathbf{J}}^* \beta_{\mathbf{J}}^{\dagger} | \theta' \rangle =$$

$$\sum_{L} (U_{RL}^*)^* \beta_{L} - \sum_{\mathbf{J}} \beta_{\mathbf{J}}^* Z_{k,K} \beta_{k,K}^{\dagger} + V_{RL} \beta_{L}.$$ 

(S53)

It follows that

$$\langle \theta'' | c_{R_{LR}}^\dagger c_{R_{L}}^\dagger | \theta' \rangle / \langle \theta'' | \theta' \rangle =$$

$$\sum_{L} (V_{RL}^*)^* (U_{RL}^*)^* Z_{k,K} c_{R_{L}}^\dagger c_{R_{LR}}^\dagger (\theta'' | \theta' \rangle)_{R_{LR}, R_{L}'}$$

$$= (V' (V'')^\dagger - (U'*)^* Z^\dagger (V'')^\dagger)_{R_{LR}, R_{L}'}$$

$$= (V' U^T U' U^T (V'')^\dagger - (U'*)^* Z^\dagger (V'')^\dagger)_{R_{LR}, R_{L}'}$$

$$= (V' ((U')^* U' U' U^T (V'')^\dagger - (U'*)^* Z^\dagger (V'')^\dagger)_{R_{LR}, R_{L}'}$$

(S54)

The third equality uses $Z = -Z^T$, and the final equality is derived using the orthogonality relations $(U'*)^* (V'')^\dagger + V' (U')^\dagger = 0$ and $(V')^* (V'')^\dagger + U' (U')^\dagger = I$. Similarly,

$$\langle \theta'' | c_{R_{LR}}^\dagger c_{R_{L}}^\dagger | \theta' \rangle / \langle \theta'' | \theta' \rangle = -\langle \theta'' | c_{R_{LR}}^\dagger c_{R_{L}}^\dagger | \theta' \rangle / \langle \theta'' | \theta' \rangle =$$

$$= (V' (V'')^\dagger - (U'*)^* Z^\dagger (U'')^\dagger)_{R_{LR}, R_{L}'}$$

(S55)

and

$$\langle \theta'' | c_{R_{LR}}^\dagger c_{R_{L}}^\dagger | \theta' \rangle / \langle \theta'' | \theta' \rangle =$$

$$= (V' (V'')^\dagger - (U'*)^* Z^\dagger (U'')^\dagger)_{R_{LR}, R_{L}'}$$

(S56)
Finally, we have

\[
\langle \theta'' | R_1^c R_2^c R_3^c R_4^c \langle \theta' | (\theta'' | \theta') = (V'(V')^\dagger) R_1 R_2
\]

where \( \langle 0, n \rangle = \int_0^{2\pi} \frac{dt}{2\pi} e^{-in\xi} | 0, \xi \rangle \) is an unnormalized state with \( n \) total pairs (2n total particles) in the system. Clearly, \( \xi \) shows up as the relative phase between states with different numbers of particles. If a superconducting system with a fixed number of pairs \( \bar{n} \) is modeled using the Bogoliubov-de Gennes equations, the state of the system can be described by one of the terms in the sum, i.e. \( |0, \bar{n}\rangle / \sqrt{\bar{n}} |0, 0\rangle \). For mathematical convenience, we often calculate physical properties of the system using the entire superposition \( |0, \xi\rangle \). However, correct physical predictions about the system cannot depend on the value of \( \xi \).

It therefore is jarring that adding a constant to the reseeded order parameter phase \( \bar{\pi} \) does in fact alter the predictions of \( |\theta\rangle \). Indeed, taking \( \Theta_\pi \rightarrow \Theta_\pi + \xi \) multiplies each \( \ell \) in \( (S25) \) by an extra factor of \( e^{i\xi/2} \). Comparing the definition \( (S60) \), we see that \( |\theta\rangle \) has changed to \( |\theta, \xi\rangle \). If we expand

\[
|\theta, \xi\rangle = \sum \frac{e^{in\xi}}{n} |\theta, n\rangle,
\]

the matrix elements in \( |\theta\rangle \) depend on \( \xi \) as

\[
\langle \theta, \xi | H | \theta' \rangle = \sum \frac{e^{in(\theta-\theta')\xi}}{n} |\theta, n\rangle |\theta', n\rangle.
\]

In light of this expansion, one realizes that, for a system with a fixed number of pairs \( \bar{n} \), one would actually like to solve a modified version of \( |\theta\rangle \) with fixed particle number

\[
\sum_\sigma \langle \theta, \bar{n} | H | \theta' \rangle \psi(\theta') = E \sum_\sigma \langle \theta, \bar{n} | \theta', \bar{n} \rangle \psi(\theta').
\]

This equation does not depend on \( \xi \), as required physically.

Nevertheless, the original equation \( (S5) \) can be used provided that we choose a specific value of \( \xi \) satisfying

\[
\langle \theta, \bar{n} | H | \theta' \rangle \psi(\theta') \approx \frac{e^{in(\theta-\theta')\xi}}{\sqrt{\langle \theta, \bar{n} | \theta', \bar{n} \rangle}} |\theta, \bar{n} \rangle |\theta', \bar{n} \rangle
\]

for \( \bar{n} \) equal to the number of pairs in the system. If this condition is satisfied, \( (S5) \) approximately reduces to \( (S62) \). To satisfy \( (S63) \), in this paper we choose \( \xi \) such that \( \bar{n} \) is an antisymmetric function of \( z \), as in Fig. \( \bar{\pi} \). This
antisymmetric on qubit. The rescaled phase function (11) is chosen so it is real for all n:

\[
(e^{i\theta}\xi(\theta, n|\theta', n))^* = e^{-i\theta}\xi(-\theta, n|\theta', n)e^{i\theta}\xi
= e^{i\theta}\xi(\theta, n|\tilde{R}\tilde{R}|\theta', n)e^{-i\theta}\xi
= e^{i\theta}\xi(\theta, n|\theta', n).
\]

In the second equality, we introduced an operator \(\tilde{R}\) that takes \(z\) to \(-z\), and we used the \(z \leftrightarrow -z\) reflection symmetry of the problem about the Josephson junction. The third equality uses the fact that two reflections \(\tilde{R}\tilde{R}\) in succession produce an identity operation. Similarly, our choice of \(\xi\) ensures that \(e^{i\theta\theta'}\xi(\theta, n|H|\theta', n)\) is real for all n:

\[
\left(e^{i\theta\theta'}\xi(\theta, n|H|\theta', n)\right)^*
= e^{i\theta\theta'}\xi(-\theta, n|-\theta', n)e^{-i\theta\theta'}
= e^{i\theta\theta'}\xi(\theta, n|\tilde{R}\tilde{H}\tilde{R}|\theta', n)
= e^{i\theta\theta'}\xi(\theta, n|H|\theta', n).
\]

Now, numerically, we find that the terms in (S61) are strongly peaked as a function of \(n\) about \(n = \bar{n}\). A sample calculation, performed using the charge qubit parameters described in the main text, is shown in Fig. S1 for \(e^{i\theta\theta'}\xi(\theta, n|\theta', n)\) in the case \(\theta = \theta' = 0\). Since they are real and do not have a rapidly varying phase, the terms with \(n\) close to \(\bar{n}\) add constructively and determine the value of the sum. As a result, (S63) is satisfied: we have numerically verified agreement to within \(1\%\) for the overlap \(\langle \theta, \theta|\xi|\theta, \theta\rangle\) and for the one-body operators in the Hamiltonian \(\langle \theta, \theta\rangle\xi|\tilde{R}|\theta', \theta\rangle\xi\) at a few choices of \(\theta, \theta', \tilde{R}\), and \(\tilde{R}'\). (We expect that (S63) would also be satisfied for the two-body operators in the Hamiltonian but did not check this numerically.) Had we chosen \(\xi\) injudiciously, there would have been differences in phase leading to cancellations within the sums (S61). As a result, (S63) would not be well satisfied.

Although this discussion has focused on the case of the charge qubit, analogous remarks apply to the RF squid qubit. The rescaled phase function (11) is chosen so it is antisymmetric on \(z\), as shown in Fig. 1b. This allows us to get physically relevant results by solving (13) without projecting on to states of fixed particle number.

**Definition of Orthonormal Basis**

In the limit of large systems, the states \(|\theta\rangle\) approach orthonormality. This follows from the Onishi formula \(|\langle\theta|\theta\rangle|^2 = \det \mathcal{U}\) in the limit in which the matrix \(\mathcal{U}\) becomes large. To solve (S8) for a finite system, though, it is convenient to construct an orthonormal basis explicitly. Our non-orthonormal basis, defined in the main text, comprises states \(|\theta\rangle\), where \(-\theta_{\text{max}} < \theta \leq \theta_{\text{max}}\). The value of \(\theta_{\text{max}}\) is determined by \(|\theta\rangle = |\theta + 2\pi|\rangle\); based on (S25) a sufficient condition is

\[
e^{2\theta_{\text{max}}}(|\theta\rangle + |\theta + \pi\rangle)/2 = 1 \quad \forall \theta, \theta'.
\]

For instance, if we have a charge qubit in which \(\Theta_{r}/2\) changes from \(-1/4\) to \(1/4\) abruptly across the Josephson junction from one lattice site to the next, then we have \(\theta_{\text{max}} = 2\pi\). If instead \(\Theta_{r}/2\) changes from \(-1/4\) to \(1/4\) linearly over \(\Delta z\) lattice sites, \(\Delta z\) referring to the thickness of the Josephson junction, then \(\theta_{\text{max}} = 2\pi\Delta z\).

We define orthonormal states \(|n\rangle\) as follows. Using expression (S25) for \(|\theta\rangle\) and defining

\[
c_\tilde{R}^\dagger = e^{i\theta\theta}/2c_\tilde{R},
\]

one sees that the overlap matrix elements satisfy \(\langle \theta|\theta\rangle = \langle \theta - \theta'|0\rangle\). Fourier transforming, we define \(o_n\) by

\[
o_n = \frac{\theta_{\text{max}}}{2\theta_{\text{max}}} \int d\theta e^{i\theta n} \frac{1}{\theta_{\text{max}}} \int d\theta' e^{-i\theta' n} \langle \theta - \theta'|0\rangle
= \begin{cases} 1 & n = n' \\ 0 & \text{otherwise} \end{cases}
\]

where

\[
n = 2\pi m/2\theta_{\text{max}}
\]

for \(m\) an integer in the range \(-M/2, \ldots, M/2\) with \(M\) an even integer. It follows that the states

\[
|n\rangle = \frac{1}{\sqrt{o_n}} \frac{1}{\theta_{\text{max}}} \int d\theta e^{-i\theta n} |\theta\rangle
\]

satisfy the orthonormality condition

\[
\langle n|n'\rangle = \begin{cases} 1 & n = n' \\ 0 & \text{otherwise} \end{cases}
\]

Some eigenvalues \(o_n\) approach zero because the basis of \(|\theta\rangle\) states is overcomplete; we exclude the corresponding states \(|n\rangle\) from our basis.

Given the definition of \(|n\rangle\), it is possible to perform a unitary transformation to a coarse-grained \(\bar{\theta}\) basis defined by an inverse Fourier transform over the valid \(|n\rangle\) states:

\[
|\bar{\theta}\rangle = \sum_n e^{i\theta n} |n\rangle/\sqrt{M+1}.
\]

**FIG. S1.** (a) Plot of overlap \(|\langle 0|0,n\rangle|^2\) versus number of particles \(2n\) for charge qubit parameters detailed in main text. Strong peak is evident near \(2n = 1012\) particles. (b) Expanded plot in the range \(2n = 982\) to 1042 particles shows shape of peak.
The number of such states is $M + 1$, and they lie in value between $-\theta_{\text{max}}$ and $\theta_{\text{max}}$, so their allowed values are $\bar{\theta} = 2\theta_{\text{max}}m/(M + 1)$ with $m = -M/2, \ldots, M/2$ integral.

An upper bound on $M$ is given by $\theta_{\text{max}}N_{\text{pairs}}/\pi$, where $N_{\text{pairs}}$ is the total number of pairs occupying the system. This estimate is obtained by applying the theorem of Bloch and Messiah to (S25). As noted previously, this theorem states there are unitary matrices $U$ and $V$ for which $U = D^I U C^I$ and $V = D^I U C^I$ are real and take simple block-diagonal forms. The matrix $U$ has $2 \times 2$ blocks of the form $u_J \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$, while these blocks take the form $v_J \left[ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right]$ in the case of $V$. Here, the $2 \times 2$ block involving state $J$ involves a paired state that we label $\tilde{J}$. Defining

$$d^J_J = \sum_R c^\dagger_R e^{i\theta_J/2} D^{R,J}_J,$$  

we find

$$\frac{1}{2} \sum_{R,R^\prime} e^{i(\theta_J + \theta_{J^\prime})/2} c^\dagger_R c^\dagger_{R^\prime} Z_{R,R^\prime} e^{i\theta_J/2} = \sum_J d^J_J d^J_{J^\prime}.$$  

Inserting this into (S25) yields

$$|\bar{\theta}| = N \exp \left( \sum_J d^J_J d^J_{J^\prime} \right) \langle \text{vac} \rangle$$  

$$= NL \left( 1 + d^J_J d^J_{J^\prime} \right) \langle \text{vac} \rangle$$  

$$\approx NL \prod_{J \sim \text{Fermi sea}} \left( 1 + d^J_J d^J_{J^\prime} \right) \langle \text{vac} \rangle$$  

$$= NL \sum_{J \sim \text{Fermi sea}} \left( 1 + \sum_J d^J_J d^J_{J^\prime} + \cdots + \prod_{J \sim \text{Fermi sea}} d^J_J d^J_{J^\prime} \right) \langle \text{vac} \rangle$$

(S72)

The notation $J \sim \text{Fermi sea}$ indicates states that are within the Fermi sea or not too far above the Fermi surface. The approximate equality in the third line uses that fact that high-energy states well above the Fermi surface have $v_J \to 0$. The final equality comes from multiplying out the product into a sum of terms. Inserting this expression into (S68) yields

$$|n| \approx \frac{N}{\sqrt{\pi}} \frac{\theta_{\text{max}}}{2\theta_{\text{max}}} \int_{-\theta_{\text{max}}}^{\theta_{\text{max}}} d\theta e^{-n\theta} \left( 1 + \cdots + \prod_{J \sim \text{Fermi sea}} d^J_J d^J_{J^\prime} \right) \langle \text{vac} \rangle$$

$$= \frac{N}{\sqrt{\pi}} \frac{\theta_{\text{max}}}{2\theta_{\text{max}}} \int_{-\theta_{\text{max}}}^{\theta_{\text{max}}} d\theta e^{-n\theta} \left( 1 + \cdots + \prod_{J \sim \text{Fermi sea}} d^J_J d^J_{J^\prime} \right)$$

(S73)

where (S71) has been used on the final line. If $n$ is too large, the right hand side will vanish. To see this, note that

$$\frac{1}{2\theta_{\text{max}}} \int_{-\theta_{\text{max}}}^{\theta_{\text{max}}} d\theta e^{-n\theta} e^{i\theta(\Theta_{1} + \cdots + \Theta_{2T})/2}$$

$$= \begin{cases} 1 & n = (\Theta_{1} + \cdots + \Theta_{2T})/2 \\ 0 & \text{otherwise} \end{cases}$$

(S74)

using (S64). Assuming that $\Theta_T$ is at most $1/2$, the largest possible value of $(\Theta_{1} + \cdots + \Theta_{2T})/2$ is $T/2$; the Kronecker delta then enforces $n = T/2$. Thus, the largest possible value of $n$ that could possibly give a non-vanishing result for (S73) is $n = N_{\text{pairs}}/2$, where $N_{\text{pairs}}$ is the number of occupied pairs appearing in $\prod_{J \sim \text{Fermi sea}} d^J_J d^J_{J^\prime}$.

Similarly, the smallest possible value of $n$ is $-N_{\text{pairs}}/2$. Given (S67), we find that $2\pi(M/2)/2\theta_{\text{max}} \leq N_{\text{shared}}/2$, or $M \leq N_{\text{pairs}}\theta_{\text{max}}/\pi$ as claimed above.

In the case of a charge qubit composed of two weakly coupled superconducting islands with a relatively abrupt phase change, as in Fig. 1c, we can find a better estimate of $M + 1$. The basis size $M + 1$ turns out to scale as the number of pairs $N_{\text{shared}}$ in $|\bar{\theta}|$ that are shared on both islands, which is roughly the number of pairs near the chemical potential of the system. To show this, we write

$$|\bar{\theta}| = \prod_{J \sim \text{Fermi sea}} \left( 1 + d^J_J d^J_{J^\prime} \right) \langle \text{vac} \rangle$$  

$$= \prod_{J \sim \text{Fermi sea}} \left( 1 + d^J_J d^J_{J^\prime} \right) \times$$  

$$\prod_{J \sim \text{Fermi sea}} \left( 1 + d^J_J d^J_{J^\prime} \right) \prod_{J \sim \text{Fermi sea}} \left( 1 + d^J_J d^J_{J^\prime} \right) \langle \text{vac} \rangle$$

$$= N' \prod_{J \sim \text{Fermi sea}} \left( 1 + d^J_J d^J_{J^\prime} \right) \langle \text{vac} \rangle.$$  

(S75)

where $J$ left and $J$ right refer to states that are localized well within the Fermi sea of the left or right island respectively. These states have $v_J/u_J$ very large, permitting the final approximate equality. Let $N_{\text{left}}$ be the number of pairs in the $J$ left product, $N_{\text{right}}$ be the number of pairs in the $J$ right product, and $N_{\text{shared}}$ be the number of pairs in the $J$ shared product. We are considering the case of an abrupt phase change, in which the electrons in the system have $\Theta_R = -1/2$ or $1/2$ depending on whether they inhabit the left island or the right island. Thus, when we calculate (S73), the integrals take the form

$$\frac{1}{2\theta_{\text{max}}} \int_{-\theta_{\text{max}}}^{\theta_{\text{max}}} d\theta e^{-n\theta} e^{i\theta(N_{\text{left}} - N_{\text{right}})/2} e^{i\theta(\Theta_{1} + \cdots + \Theta_{2T})/2}$$

with the largest possible value of $T$ given by the number of pairs $N_{\text{shared}}$ that can wander from the left island to the right island. We see that $n$ ranges from

(S76)
The overlap matrix becomes block diagonal with $2 \times 2$ blocks for composed of 2 orthogonal basis is composed of $|\circ, \theta\rangle$ and $|\circ, \theta\rangle$ approach orthonormality. For smaller systems, we construct an orthonormal basis explicitly as follows. The original, non-orthogonal basis is composed of $|\circ, \theta\rangle$ and $|\circ, \theta\rangle$ with $-\theta_{\max} < \theta \leq \theta_{\max}$ and $\theta_{\max}$ fixed by (S64). For an RF squid qubit, $\theta_{\max}$ will approach $2\pi N_z$, with $N_z$ the number of sites around the circumference of the system. This greatly exceeds the value of $\theta_{\max}$ in the charge qubit case, consistent with the fact that $\theta$ is unbounded in LE theory in the case of an RF squid. The overlap matrix is composed of $2 \times 2$ blocks
\[
\begin{bmatrix}
|\circ, \theta \rangle \langle \circ, \theta'| & |\circ, \theta \rangle \langle \circ, \theta'|
|\circ, \theta' \rangle \langle \circ, \theta | & |\circ, \theta' \rangle \langle \circ, \theta |
\end{bmatrix}
\]
for $-\theta_{\max} < \theta, \theta' \leq \theta_{\max}$. After a Fourier transform, the overlap matrix becomes block diagonal with $2 \times 2$ blocks of the form
\[
\int_{-\theta_{\max}}^{\theta_{\max}} d\theta \int_{-\theta_{\max}}^{\theta_{\max}} d\theta' \frac{e^{i\theta n - i\theta' n'}}{(2\theta_{\max})^2}
\left|
\begin{array}{c}
|\circ, \theta \rangle \langle \circ, \theta'|
|\circ, \theta \rangle \langle \circ, \theta'|
\end{array}
\right|
\left|
\begin{array}{c}
|\circ, \theta' \rangle \langle \circ, \theta | \\
|\circ, \theta' \rangle \langle \circ, \theta |
\end{array}
\right|
\]
where $O_n$ is a $2 \times 2$ matrix and $n, n'$ satisfy (S67). The eigenvalues of $O_n$, labeled $a_{1,n}$ and $a_{2,n}$, have corresponding eigenvectors
\[
O_n \begin{bmatrix}
a_{i,n} \\
b_{i,n}
\end{bmatrix} = a_{i,n} \begin{bmatrix}
a_{i,n} \\
b_{i,n}
\end{bmatrix}
\]
for $i = 1, 2$. We can define orthonormal states in terms of eigenvectors by
\[
|i, n\rangle = \frac{1}{\sqrt{|a_{i,n}|^2 + |b_{i,n}|^2}} e^{i\theta n}(a_{i,n} |\circ, \theta\rangle + b_{i,n} |\circ, \theta\rangle)
\]
for $i = 1, 2$. They satisfy
\[
\langle i, n| i', n'\rangle = \begin{cases} 
1 & i = i' \text{ and } n = n' \\
0 & \text{otherwise}
\end{cases}
\]
(S79)

Because the original, non-orthogonal basis is overcomplete, some eigenvalues $a_{i,n}$ approach zero; the corresponding states $|i, n\rangle$ are omitted from our basis. We denote by $2(M+1)$ the number of valid orthogonal basis states $|i, n\rangle$.

Given this definition of $|i, n\rangle$, it is possible to perform a unitary transformation to a coarse-grained phase basis defined by an inverse Fourier transform over the valid $|i, n\rangle$ states:
\[
|i, \bar{\theta}\rangle = \sum_{n=-M/2}^{M/2} e^{i\theta n}|i, n\rangle/\sqrt{M+1}.
\]
(S80)

Since the number of such states for a given $i$ is $M+1$, the spacing between them is $\Delta \bar{\theta} = 2\theta_{\max}/(M+1)$. As $M+1$ grows with system size, this spacing shrinks to zero.

Applying the theorem of Bloch and Messiah as above to $|\circ, \theta\rangle$ and $|\circ, \theta\rangle$, we can argue that a rough upper bound on $M+1$ is given by the number of occupied pairs in the system. It is important to emphasize that the quantum number $n$ in $|i, n\rangle$ does not admit a simple interpretation in terms of the positions of Cooper pairs in the system.

**Derivation of Lumped Element Equations**

In the coarse-grained $\bar{\theta}$ basis, the Schrödinger equation (S3) becomes an $(M+1) \times (M+1)$ matrix equation
\[
\sum_{\theta'} \langle \theta| H| \theta'\rangle \psi(\theta') = E \psi(\theta')
\]
(S81)

where
\[
\langle \theta| H| \theta'\rangle = \frac{1}{M+1} \left( \frac{1}{(2\theta_{\max})^2} \sum_{n=-M/2}^{M/2} e^{-i\theta n} \theta_{\max} \right)
\int_{-\theta_{\max}}^{\theta_{\max}} d\theta e^{-i\theta n'} \langle \theta| H| \theta'\rangle.
\]
(S82)

As the size of the system, and the basis size $M+1$, grows, the overlap matrix elements $\langle \theta| \theta'\rangle$ decay rapidly for $\theta \neq \theta'$ as a consequence of (S67). The original $|\theta\rangle$ basis tends toward orthonormality, and one expects the orthonormal basis $|\theta\rangle$ that we constructed to approach $|\theta\rangle|\theta=\bar{\theta}$, the original basis state with $\theta$ evaluated at $\bar{\theta}$. Thus, in this limit we approximate $\langle \theta| H| \theta'\rangle \approx \langle \theta| H| \theta'\rangle|\theta=\bar{\theta}, \theta'=\bar{\theta}$. Recall the decomposition (1), $\langle \theta| H| \theta'\rangle = \langle \theta| T + P + W| \theta'\rangle$. Employing (S65), we can show that the potential and interaction energies satisfy $\langle \theta| P + W| \theta'\rangle = \langle \theta-\theta'| P + W| 0\rangle$. Therefore, along the diagonal of the Hamiltonian matrix in (S81), $\langle \theta| P + W| \theta\rangle = \langle 0| P + W| 0\rangle$ contributes an overall constant that simply shifts $E$ in (S81). Only $\langle \theta| T| \theta\rangle$ depends upon $\theta$; this dependence is physically important and should not be neglected. However, in the off-diagonal elements, we will approximate $\langle \theta| T| \theta'\rangle$ as $\langle \theta-\theta'| T| 0\rangle$, which is reasonable if the error incurred thereby is small:
\[ |\theta - \theta'| T|0\rangle - |\theta T|0\rangle \ll |\theta - \theta'| T + P + W|0\rangle. \] Summarizing, we have

\[
\langle \bar{\theta} | H(\bar{\theta}) \approx \delta_{\theta,\bar{\theta}} \langle 0 (T|\theta) - (0|T|0) \rangle_{\theta=\bar{\theta}, \theta'=\bar{\theta}} + \langle \theta - \theta'| T + P + W|0\rangle_{\theta=\bar{\theta}, \theta'=\bar{\theta}} \tag{S83}\]

Clearly, the first line is diagonal in the phase basis. The second line, since it depends only on \( \theta - \theta' \), becomes diagonal in the basis of \( |n\rangle \) states \( \langle n | \). We can develop the analysis further in the case of a charge qubit composed of two weakly coupled superconducting islands with a relatively abrupt phase change, as in Fig. 15. The matrix element of the tunneling Hamiltonian is

\[
\langle \theta | T | \theta' \rangle = - (\theta - \theta') \sum_{R} \sum_{n=\pm, \pm} t_{R+n,R} c^{\dagger}_{R+n} c_{R} |0\rangle \tag{S84}\]

where we have used (S65). It is useful to decompose \( T = T_{\text{near}} + T_{\text{far}} \), where \( T_{\text{near}} \) contains the small fraction of terms in which \( R \) is near the junction and \( T_{\text{far}} \) contains all other tunneling terms. As a result of the form of Fig. 15, far from the junction, \( \Theta_{R+n} \approx \Theta_{R} \), so that \( e^{i\theta (\Theta_{R}-\Theta_{R+n})/2} \approx 1 \). The expression (S84) then implies \( \langle \theta | T_{\text{near}} | \theta' \rangle \approx \langle \theta - \theta' | T_{\text{far}} | 0 \rangle \). So, neglecting the contribution of \( T_{\text{near}} \), we conclude that \( T_{\text{far}} = T + W \) is diagonal in the basis \( \langle n | \) \( \rangle \) \( \langle n | \). A reasonable approximation \( \langle n | T_{\text{near}} | n' \rangle = E_{\text{seas}} n^2 \delta_{n,n'} \langle n | W | n' \rangle = 4E_{C} n^2 \delta_{n,n'} \) and \( \langle n | P | n' \rangle = -\bar{\mu} n \delta_{n,n'} \). Here, \( \bar{\mu} \) denotes the electrochemical potential difference between the islands. The capacitive energy, \( 4E_{C} = (2e)^{3}/2C \) in terms of an appropriate capacitance \( C \), is familiar from LE theory. The unfamiliar coefficient \( E_{\text{seas}} \) is determined by the total energy of the two Fermi seas residing in the theory. For simplicity, assume that each of the islands has volume \( \bar{L}^{3} \). Let the total number of electrons in the system be \( 2N_{\text{tot}} \), with \( N_{\text{tot}} + 2n \) residing on one island and \( N_{\text{tot}} - 2n \) residing on the other island. Then total energy of the two Fermi seas is \( \frac{3}{2} \frac{k^{2}}{2mL} (3\pi^{2})^{3/2} ((N_{\text{tot}} + 2n)5/3 + (N_{\text{tot}} - 2n)5/3) \). Taylor expanding this expression to second order, we deduce that \( E_{\text{seas}} = \frac{2}{2mL} (3\pi^{2})^{3/2} n^{-1/3} = \frac{8}{3} E_{F} n^{-1/3} \), where \( E_{F} \) is the Fermi energy of each island when \( n = 0 \). For physical devices, \( E_{\text{seas}} \) is much smaller than \( 4E_{C} \). However, in simulations such as the one performed in the text, where the Coulomb interaction is replaced with a short-range Hubbard interaction, \( E_{\text{seas}} \) becomes the important energy.

The contribution of \( T_{\text{near}} \) to the diagonal matrix elements takes the form \( \langle \theta | T_{\text{near}} | \theta \rangle \approx -E_{J} \cos \theta /2 \). Here, we have introduced the parameter \( E_{J} \) and have used the fact that \( \Theta_{R} - \Theta_{R+n} \approx 1 \) when \( R \) and \( R+n \) stand on opposite sides of the junction, implying \( e^{i\theta (\Theta_{R}-\Theta_{R+n})/2} \approx e^{i\theta /2} \) in (S84). Assembling these results, we make a continuum approximation to our matrix equation (S81) for \( M+1 \) large, obtaining the lumped element equation

\[
4E_{C} \left( -i \frac{d}{d\theta} - n_{0} \right)^{2} \psi(\theta) + E_{J} \left( 1 - \cos \frac{\theta}{2} \right) \psi(\theta) = E \psi(\theta) \tag{S85}\]

where \( n_{0} = \bar{\mu}/8E_{C} \). This should be compared with the standard lumped element expression \( [3] \). There is a factor of 2 difference in the potential stemming from the fact that \(-2\pi < \theta < 2\pi \) in our theory. Setting \( \theta = \theta /2, \ E_{C} = E_{C}/4 \) and \( n_{0} = 2n_{0} \), we have

\[
4E_{C} \left( -i \frac{d}{d\theta} - n_{0} \right)^{2} \psi(\theta) + E_{J} \left( 1 - \cos \theta \right) \psi(\theta) = E \psi(\theta) \tag{S86}\]

with \(-\pi < \theta < \pi \). This agrees with \[4\].

We can make a similar argument in the case of an RF squid qubit. The transformation to the orthogonal basis changes \( \langle \theta, \bar{\theta} | \) to the matrix equation

\[
\sum_{\theta'} \langle \bar{\theta}, \theta | H | i', \theta' \rangle = \frac{1}{M+1} \left( \frac{1}{(2\theta_{\text{max}})^{2}} \right) \sum_{n=-M/2}^{M/2} \frac{e^{-i\theta_{n} n}}{\theta_{n}} \left( \frac{2\theta_{\text{max}}}{2\theta_{\text{max}}^{2}} \right) \sum_{n'=0}^{2\theta_{\text{max}}} \int d\theta' e^{-i\theta_{n'} n'} \left[ a_{i,n'} \langle \bar{\theta}, \theta | H | i', \theta' \rangle \right] \langle \bar{\theta}, \theta | H | i', \theta' \rangle \left[ b_{i,n} \right] \tag{S87}\]

For large systems, the overlap matrix (S76) tends to that of an orthonormal set of states, so we can approximate \( |1, \bar{\theta} \rangle = |\langle \bar{\theta}, \theta | \rangle \rangle_{\theta=\bar{\theta}} \) and \( |2, \bar{\theta} \rangle = |\langle \bar{\theta}, \theta | \rangle \rangle_{\theta=\bar{\theta}} \). Then, our matrix equation becomes

\[
\sum_{\theta'} \left[ \langle \bar{\theta}, \bar{\theta} | H | i', \theta' \rangle \langle \bar{\theta}, \theta | H | i', \theta' \rangle \right]_{\theta=\bar{\theta}, \theta'=\bar{\theta}} \psi(\bar{\theta}, \theta) = E \psi(\bar{\theta}, \theta) \tag{S88}\]

The potential and interaction matrix elements satisfy

\[
\left[ \langle \bar{\theta}, \theta | P + W | \bar{\theta}, \theta' \rangle \langle \bar{\theta}, \theta | P + W | \bar{\theta}, \theta' \rangle \right] = \left[ \langle \bar{\theta}, \theta - \theta' | P + W | \bar{\theta}, \theta' \rangle \langle \bar{\theta}, \theta - \theta' | P + W | \bar{\theta}, \theta' \rangle \right] \tag{S89}\]

which can be verified using a substitution like (S69).
For $\theta \neq \theta'$, we adopt the approximation
\[
\begin{bmatrix}
\langle \psi, \theta | T | \psi, \theta' \rangle \\
\langle \psi, \theta | T | \psi, \theta' \rangle & \langle \psi, \theta | T | \psi, \theta' \rangle \\
\langle \psi, \theta | T | \psi, \theta' \rangle & \langle \psi, \theta | T | \psi, \theta' \rangle
\end{bmatrix}
\approx
\begin{bmatrix}
\langle \psi, \theta - \theta' | T | \psi, 0 \rangle \\
\langle \psi, \theta - \theta' | T | \psi, 0 \rangle & \langle \psi, \theta - \theta' | T | \psi, 0 \rangle
\end{bmatrix}
\]
for the tunneling matrix elements. This relies on the approximation $e^{i\theta' (\Theta_r - \Theta_{r+\delta})/2} \approx 1$, which is roughly true since $\Theta_r$ changes gradually along the long circumference of the RF squid qubit (see Fig. 2). When $\theta = \theta'$, on the diagonal of the Hamiltonian matrix, we retain the explicit $\theta$ dependence seen in the double-well potential in Fig. 2. This leads to

\[
\begin{bmatrix}
\langle 1, \theta | T | 1, \theta' \rangle \\
\langle 2, \theta | T | 2, \theta' \rangle
\end{bmatrix}
\approx
\delta_{\theta, \theta'}
\begin{bmatrix}
\langle \psi, \theta | T | \psi, \theta \rangle - \langle \psi, 0 | T | \psi, 0 \rangle \\
\langle \psi, \theta | T | \psi, \theta \rangle - \langle \psi, 0 | T | \psi, 0 \rangle
\end{bmatrix}
\]
\[
+ \begin{bmatrix}
\langle \psi, \theta - \theta' | T + P + W | \psi, 0 \rangle \\
\langle \psi, \theta - \theta' | T + P + W | \psi, 0 \rangle \\
\langle \psi, \theta - \theta' | T + P + W | \psi, 0 \rangle \\
\langle \psi, \theta - \theta' | T + P + W | \psi, 0 \rangle
\end{bmatrix}
\]

(S89)

In the first term on the right hand side, we approximate
\[
\langle \psi, \theta | T | \psi, \theta \rangle - \langle \psi, 0 | T | \psi, 0 \rangle = -E_J \cos \frac{1}{2} (\theta - \theta^2) + \frac{1}{2} E_L (\theta - \theta^2)^2
\]
\[
\langle \psi, \theta | T | \psi, \theta \rangle - \langle \psi, 0 | T | \psi, 0 \rangle = -E_J \cos \frac{1}{2} (\theta - \theta^2) + \frac{1}{2} E_L (\theta - \theta^2)^2.
\]
In each equation, the contribution proportional to $E_L$ comes from tunneling terms in distant from the Josephson junction, while the contribution proportional to $E_J$ comes from tunneling through the Josephson junction.

The second line on the right hand side of (S89) depends only on $\theta - \theta'$. It therefore simplifies in the basis (S78). As the system size grows and the overlap matrix (S76) tends to that of an orthonormal set of states, the matrix $O_n$ becomes nearly diagonal: its eigenvectors tend to $a_{i,n} = 1, b_{i,n} = 0$ and $a_{i,n} = 0, b_{i,n} = 1$. Considering the expression (S78), it becomes reasonable to adopt the notation $| \psi, n \rangle = | 1, n \rangle$ and $| \psi, n \rangle = | 2, n \rangle$. In this basis, the second line on the right hand side of (S89) is block diagonal with 2 \times 2 blocks
\[
\begin{bmatrix}
E_{\psi, n | \psi, n} E_{\psi, n+1 | \psi, n} \\
E_{\psi, n+1 | \psi, n} E_{\psi, n+1 | \psi, n+1}
\end{bmatrix}.
\]
(S90)

Now, the matrix elements in the second line of (S89) can be assumed real since one can introduce phases if necessary into the state definitions. (It is true that conditions such as $| \psi, n_{\max} \rangle = | 1, n_{\max} \rangle$ could prevent the introduction of such phases consistently for all $\theta - \theta'$ in (S89). However, since $n_{\max}$ is so large for an RF squid qubit, states such as $| \psi, n_{\max} \rangle$ have extremely high energy and play no role in the accessible energy eigenstates of the system. Therefore, the matrix elements in

the second line of (S89) can be written as a real part plus a correction that vanishes for energetically accessible states.) Because the second line of (S89) should decay with $| \theta - \theta' |$, we make a tight-binding approximation, retaining neighbors with $| \theta - \theta' |\leq \theta_{\theta', \theta} = \delta \bar{\theta}$. (Recall that $\Delta \bar{\theta} = 2 \theta_{\max}/(M + 1)$ as argued after (S80).) Then, each of the 4 functions in (S90) equals a constant plus a term proportional to $\cos n \Delta \bar{\theta}$. As system size increases and $\Delta \bar{\theta}$ shrinks, we can truncate the cosine at second order to obtain
\[
\begin{bmatrix}
E_{\psi, \psi} \\
E_{\psi, \psi}
\end{bmatrix} + \begin{bmatrix}
E_{\psi, \psi} \\
E_{\psi, \psi}
\end{bmatrix} n^2.
\]
(S91)

This result may be reminiscent of the energy proportional to $\psi$ obtained in the case of a charge qubit with abrupt junction, where $| n \rangle$ is a state with $2n$ extra Cooper pairs on one side of the Josephson junction. (See discussion below (S84).) However, it has been derived quite differently here; in particular, we avoided any claim that the states $| i, n \rangle$ have a simple physical interpretation in terms of the positions of Cooper pairs.

Assembling our results, we find the continuum limit of our matrix equation (S88)
\[
- \begin{bmatrix}
E_{\psi, \psi} \\
E_{\psi, \psi}
\end{bmatrix} \frac{d^2}{d\bar{\theta}^2} \begin{bmatrix}
\langle \psi, \bar{\theta} | \\
\psi, \bar{\theta}
\end{bmatrix}
+ \begin{bmatrix}
E_{\psi, \psi} \\
E_{\psi, \psi}
\end{bmatrix} \frac{d^2}{d\bar{\theta}^2} \begin{bmatrix}
\langle \psi, \bar{\theta} | \\
\psi, \bar{\theta}
\end{bmatrix}
\]
\[
= E \begin{bmatrix}
\langle \psi, \bar{\theta} | \\
\psi, \bar{\theta}
\end{bmatrix}.
\]
(S92)

An approximate one-component equation can be ob-
tained by adopting the ansatz
\[
\begin{bmatrix}
\psi(\bar{\theta}, \bar{\theta}) \\
\psi(\bar{\theta}, \bar{\theta})
\end{bmatrix}
\approx \begin{cases}
\begin{bmatrix}
\psi(\bar{\theta}) \\
0
\end{bmatrix} & \bar{\theta} \leq \bar{\theta}_m \\
\begin{bmatrix}
0 \\
\psi(\bar{\theta})
\end{bmatrix} & \bar{\theta} > \bar{\theta}_m
\end{cases} = \begin{bmatrix}
1 - f(\bar{\theta}) \\
f(\bar{\theta})
\end{bmatrix} \psi(\bar{\theta}).
\]
(S93)

Here \(f(\bar{\theta})\) is a step function that increases from 0 to 1 when \(\bar{\theta}\) transitions through the local maximum \(\bar{\theta}_m\) defined by \(E_\theta - E_f \cos \frac{1}{2}(\bar{\theta}_m - \bar{\theta}_c^2) + \frac{1}{2}E_L(\bar{\theta}_m - \bar{\theta}_c^2)^2 = E_\theta - E_f \cos \frac{1}{2}(\bar{\theta}_m - \bar{\theta}_c^2) + \frac{1}{2}E_L(\bar{\theta}_m - \bar{\theta}_c^2)^2\). When the RF squid qubit is threaded by a half superconducting flux quantum, the symmetric double-well potential depicted in Fig. 2d has a local maximum at \(\bar{\theta}_m = 0\). Inserting the ansatz, we are left with
\[
-4 E_C(\bar{\theta}) \frac{d^2}{d\theta^2} \psi(\bar{\theta}) + E_I(\bar{\theta}) \psi(\bar{\theta}) = E \psi(\bar{\theta}) \quad (S94)
\]
with
\[
4 E_C(\bar{\theta}) = \begin{cases}
E_{\psi \psi} & \bar{\theta} \leq \bar{\theta}_m \\
E_{\psi \psi} & \bar{\theta} > \bar{\theta}_m
\end{cases}
\]
and
\[
E_I(\bar{\theta}) = \begin{cases}
E_{\psi \psi} - E_f \cos \frac{1}{2}(\bar{\theta} - \bar{\theta}_c^2) + \frac{1}{2}E_L(\bar{\theta} - \bar{\theta}_c^2)^2 & \bar{\theta} \leq \bar{\theta}_m \\
E_{\psi \psi} - E_f \cos \frac{1}{2}(\bar{\theta} - \bar{\theta}_c^2) + \frac{1}{2}E_L(\bar{\theta} - \bar{\theta}_c^2)^2 & \bar{\theta} > \bar{\theta}_m
\end{cases}.
\]

In this derivation, we neglect the off-diagonal values \(E_{\psi \psi}\) and \(E_{\psi \psi}\), assuming that they are small compared to \(E_I(\bar{\theta})\). We also neglect terms proportional to \(\frac{d^2 f(\bar{\theta})}{d\theta^2} \frac{d^2 \psi(\bar{\theta})}{d\theta^2}\) or \(\frac{d^2 f(\bar{\theta})}{d\theta^2} \frac{d^2 \psi(\bar{\theta})}{d\theta^2}\). This is justified if \(\psi(\bar{\theta})\) and \(\frac{d^2 \psi(\bar{\theta})}{d\theta^2}\) are small at the local maximum \(\bar{\theta}_m\), the only point at which \(\frac{d^2 f(\bar{\theta})}{d\theta^2} \frac{d^2 \psi(\bar{\theta})}{d\theta^2}\) and \(\frac{d^2 f(\bar{\theta})}{d\theta^2} \frac{d^2 \psi(\bar{\theta})}{d\theta^2}\) do not vanish. For instance, this approximation seems particularly appropriate for low-energy eigenstates of a double-well potential that nearly vanish inside the potential barrier (see Fig. 2a).

### Number of Entangled Electrons

Computing the number of entangled electrons in an RF squid qubit is beyond the scope of LE theory. One might attempt an answer within LE theory (see, e.g., Ref. [S1]) by working in the basis \(|n\rangle\) and regarding \(n\) as the number of Cooper pairs on the capacitors shunting the Josephson junction of the qubit. For example, one might associate the uncertainty \(\Delta n\) with the number of Cooper pairs participating in the supercurrent by flowing on and off the junction capacitance. However, this association leads to unphysical conclusions. For the parameters of the RF squid qubit experiment [11], for example, one computes \(\Delta n \sim 50\) pairs in the entangled state. Traveling with speed \(v\) around a ring of circumference \(L_z\), they should produce a current \(I\) satisfying \(\Delta n \sim IL_z/2ev\). Inserting \(L_z \sim 500\mu m\) and \(I \sim \mu A\), we find agreement only if \(v\) approaches the speed of light \(c\). But physically, the maximum plausible speed is the Fermi velocity \(v_F \sim 0.01c\).

Our theory provides a microscopic many-body quantum state of the RF squid qubit, which is not provided by LE theory. We use this quantum state in the main text to evaluate (14). In this section, we give a short derivation of an approximate expression [13] for (15) that assumes that the many-body state is a superposition of displaced Fermi seas counterpropagating in a ring of circumference \(L_z\). A displaced Fermi sea is a sphere of momentum eigenstates that is centered at a non-zero momentum. Thus, the basis that diagonalizes the expression \(|\psi\rangle \langle \psi|\) for \(|\psi\rangle \langle \psi|^{\dagger}\) appearing in (15) is given by momentum and spin: \(Q = (q, \sigma)\).

If the displaced Fermi seas carry current \(\pm I/2\) and are centered at momentum \(\pm q\), then the total number of electrons below the Fermi surface of each sea is \(N \sim IL_z/2e(q/m)\), where \(m\) is the electron mass. However, not all \(N\) electrons are entangled when we superpose displaced Fermi seas. The core electronic states are occupied in both displaced Fermi seas; these core electronic states do not participate in the entanglement [13] and the sum (15) is deliberately defined so that they do not contribute. In Fig. 2a, which depicts the two displaced Fermi seas graphically, these core electrons occupy the white region. Only the 4 colored slivers in Fig. 2a make contributions to (15). And, recalling the factor of \(1/2\) in (15), the quantity \(\Delta N\) equals the number of electronic states in 1 blue sliver plus 1 purple sliver. The slivers are approximately congruent, and we evaluate the volume of a blue sliver with the assistance of Fig. 2a. If \(p_F\) is the Fermi momentum, the volume of the blue sliver in momentum space is
\[
\int_{\text{Sliver}} d^3 p = \left[ 2\pi \int_0^{p_F} \int_0^{\sqrt{p_F^2 - (q/2)^2}} dp_r p_r \int_0^{\sqrt{p_F^2 - (q/2)^2}} dp_z + 2\pi p_F \int_{\sqrt{p_F^2 - (q/2)^2}}^{p_F} dp_r \int_0^{\sqrt{p_F^2 - p_r^2}} dp_z \right]
\]
\[
= \pi p_F^2 q - \frac{\pi}{24} q^3 \approx \pi p_F^2 q.
\]
We have performed the integral using cylindrical coordinates. In the final line, we assume \(p_F \gg q\). Intuitively, the light blue region of Fig. 2a, rotated around the \(z\) axis, approximately has the volume of cylinder of base \(\pi p_F^2\) and height \(q\). The number of electronic states in 1 blue sliver plus 1 purple sliver in Fig. 2a is \(\Delta N \approx N(2\pi p_F^2 q)/(4\pi p_F^2/3) = N q/2p_F\), using the fact that the Fermi sea has volume \(4\pi p_F^3/3\) in momentum space. Substituting in the expression for \(N\) above, we find \(\Delta N \sim 3IL_z/4e(p_F/m) = 3IL_z/4ev_F\).
FIG. S2. (a) Cross-section of three displaced Fermi seas, one shifted up by $q$, one shifted down by $-q$, and one undisplaced. If the blue slivers are exchanged and the purple approximate slivers are exchanged, the displaced Fermi seas are mapped into one other. Thus, the number of entangled electrons is the number of electronic states occupying 1 blue sliver and 1 purple sliver. (b) Diagram used to evaluate volume of sliver in momentum space. Upper circle, centered at $q$, and middle circle, centered at 0, from (a) are depicted. Sliver is decomposed into several regions (first term in $S95$ proceeds over light blue part and second term in $S95$ over the dark blue part).