THE TRANSLATING SOLITON EQUATION

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ABSTRACT. We give an analytic approach to the translating soliton equation with a special emphasis in the study of the Dirichlet problem in convex domains of the plane.

1. HISTORICAL INTRODUCTION AND MOTIVATION

In this paper we consider the equation of mean curvature type

\[ \text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = \frac{1}{\sqrt{1 + |Du|^2}} \]

in a smooth domain \( \Omega \subset \mathbb{R}^2 \), where \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \). We call (1) the translating soliton equation. The geometry behind this equation is the following. Let \((x, y, z)\) be the canonical coordinates in Euclidean space \((\mathbb{R}^3, \langle \cdot, \cdot \rangle)\) and denote \( \Sigma_u = \{(x, y, u(x, y)) : (x, y) \in \Omega \} \) the graph of a function \( u \). The left-hand side of (1) is twice the mean curvature \( H \) of \( \Sigma_u \) at each point \((x, y, u(x, y))\). Here \( H \) is the average of the principal curvatures calculated with respect to the unit normal vector field

\[ N = \frac{1}{\sqrt{1 + |Du|^2}} (-Du, 1). \]

Hence the right-hand side of (1) is the \( z \)-coordinates of \( N \). Consequently, a surface in Euclidean space satisfies locally the translating soliton equation if and only if the mean curvature at each point is the half of the cosine of the angle that makes \( N \) with the vertical direction \( \vec{a} = (0, 0, 1) \).

As far as the author knows, it was S. Bernstein in 1910 the first author that studied equation (1) in a couple of papers [5, 6] in the context of the solvability of the Dirichlet problem of elliptic equations. In [5, p. 240], the translating soliton equation appears numbering as (6) and Bernstein names l’équation des surfaces, dont la courbure en chaque point est proportionnelle (égale) au cosinus de l’angle de la normale en ce point avec l’axe des \( z \). On the other hand, in [6, p. 515] Bernstein considers a family of equations numbered as (2’) in classical notation

\[ (1 + q^2)r - 2pq + (1 + p^2)t = (1 + p^2 + q^2)^{n/2}, \]

where \( n \) is an integer number. In particular, for \( n = 2 \) this equation coincides with (1). The Dirichlet problem consists into find a smooth solution of (2) with boundary data

\[ u = \varphi \quad \text{on} \quad \partial \Omega, \]

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where $\varphi \in C^0(\partial \Omega)$. Bernstein proved that (2)-(3) is solvable for arbitrary analytic functions $\varphi$ when $\Omega$ is an analytic convex domain and $n \leq 2$. In particular, this result holds for the translating soliton equation.

Sixty years later, the second approach to equation (1) is due to J. Serrin. In the eighty-pages article [37], Serrin gave a systematic treatment of the Dirichlet problem for a large class of quasilinear non-uniformly second order elliptic equations. Following the Leray-Schauder fixed point theorem and H"older estimates theory of Ladyzenskaja and Ural'ceva, Serrin establishes the necessary and sufficient conditions for the solvability of the Dirichlet problem for arbitrary boundary data. Possibly, the most known result of [37] is the case of the constant mean curvature equation, that is, when the right-hand side of (1) is replaced by a constant $2H$. In such a case, the Dirichlet problem has a solution for arbitrary smooth boundary data $\varphi$ if and only if the curvature $\kappa$ of $\partial \Omega$ with respect to the inward normal direction satisfies $\kappa \geq 2|H|$. If the solution exists, it is unique. See [38] when the boundary $\partial \Omega$ is not necessarily smooth.

However, the article [37] covers many other types of quasilinear elliptic equations and this is the situation of the translating soliton equation. Exactly in pages 477–478, Serrin considers two families of quasilinear elliptic equations and one of them coincides with (2). The equation (96) of [37] is

$$(1 + q^2)r - 2pq(s + t) = 2H(1 + p^2 + q^2)^{n/2},$$

where now $H$ and $n$ are two real constants: recall that in (2), $n$ is an integer number. Notice that if $n = 3$, the expression (4) is the constant mean curvature equation. As a consequence of the results previously obtained, Serrin proves the following existence result ([37, p. 478]).

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^2$ be a bounded $C^2$-domain. Then (2)-(3) is solvable for arbitrarily given $C^2$ function $\varphi$

1. when $n \leq 2$, if and only if $\kappa \geq 0$, and
2. when $2 < n < 3$, if and only if $\kappa > 0$.

When $n > 3$, the Dirichlet problem is not generally solvable, whatever the domain.

Definitively, for the translating soliton equation, we conclude:

**Corollary 1.2.** Let $\Omega \subset \mathbb{R}^2$ be a bounded $C^2$-domain. Then (1)-(3) is solvable for arbitrarily given $C^2$ function $\varphi$ if and only if the inward curvature satisfies $\kappa \geq 0$.

**Remark 1.3.**

1. The case $n = 3$ in (4), which does not appear in Theorem 1.1, is the constant mean curvature equation, where the solvability occurs if and only if $\kappa \geq 2|H|$.
2. Serrin generalizes the result of Bernstein in [5] changing analyticity by smoothness of $\Omega$.
3. The results of [37] for the equation (3) are established in arbitrary dimension.

Possibly due to the lengthy paper [37], equation (1) seemed to be forgotten in the literature. It is in 80’s when the translating soliton equation appears in two different contexts at the same time.

Firstly in the singularity theory of the mean curvature flow of Huisken and Ilmanen [19, 21]. A **translating soliton** is a surface $\Sigma \subset \mathbb{R}^3$ that is a solution of the mean curvature flow when $\Sigma$ evolves purely by translations along some direction
\( \vec{a} \in \mathbb{R}^3 \setminus \{0\} \). In other words, \( \Sigma \) is a translating soliton if \( \Sigma + t\vec{a}, t \in \mathbb{R} \), satisfies that fixed \( t \), the normal component of the velocity vector \( \vec{a} \) at each point is equal to the mean curvature at that point. For the initial surface \( \Sigma \), this implies that \( 2H = \langle N, \vec{a} \rangle \). After a change of coordinates, if \( \vec{a} = (0, 0, 1) \), then \( 2H = \langle N, \vec{a} \rangle \) coincides locally with (1). Translating solitons appear in the singularity theory of the mean curvature flow. After scaling, near type II-singularity points on the surfaces evolved by mean curvature vector, Huisken, Sinestrari and White demonstrated that the limit flow with initial convex surface is a convex translating soliton ([19, 20, 44]). On the other hand, Ilmanen observed that \( \Sigma \) translates with velocity \( \vec{a} \) if and only if it is stationary for the weighted area
\[
\int_{\Sigma} e^{\langle p, \vec{a} \rangle} dA
\]
In fact, \( 2H = \langle N, \vec{a} \rangle \) is the Euler-Lagrange equation for this functional and thus \( \Sigma \) is a minimal surface with respect to the Riemannian metric \( e^{\langle p, \vec{a} \rangle} \langle , \rangle \).

From the last viewpoint, equation (1) links with the theory of manifolds with density of Gromov ([16]). Exactly, let \( e^{\phi} \) be a positive density function in \( \mathbb{R}^3 \), \( \phi \in C^\infty(\mathbb{R}^3) \), which serves as a weight for the volume and the surface area. Note that this is not equivalent to scaling the metric conformally by \( e^{\phi} \) because the area and the volume change with different scaling factors. For a given compactly supported variation of \( \Sigma_t \) of \( \Sigma \) that fixes the boundary \( \partial \Sigma \) of \( \Sigma \), let \( A_\phi(t) \) and \( V_\phi(t) \) denote the weighted area and the enclosed weighted volume of \( \Sigma_t \), respectively. Then the first variations of \( A_\phi(t) \) and \( V_\phi(t) \) are
\[
A_\phi'(0) = -2 \int_{\Sigma} H_\phi \langle N, \xi \rangle dA_\phi, \quad V_\phi'(0) = \int_{\Sigma} \langle N, \xi \rangle dA_\phi,
\]
where \( \xi \) is the variational vector field of \( \Sigma_t \) and \( H_\phi = H - \langle N, \nabla \phi \rangle / 2 \) is called the weighted mean curvature. If we choose \( \phi(p) = \langle p, \vec{a} \rangle, p \in \mathbb{R}^3 \), then
\[
(5) \quad H_\phi = H - \frac{\langle N, \vec{a} \rangle}{2}.
\]
We say that \( \vec{a} \) is the density vector. Thus we have the next characterizations of a translating soliton.

**Proposition 1.4.** Let \( \Sigma \) be a surface in \( \mathbb{R}^3 \). The following statements are equivalent:

1. \( \Sigma \) satisfies locally (1).
2. \( \Sigma \) translates with velocity \( \vec{a} \) by means of the mean curvature flow.
3. \( \Sigma \) is a critical point of the area \( A_\phi \) for the density \( \phi(p) = \langle p, \vec{a} \rangle \)

In view of both approaches, we point out that similar results of Theorem 1.1 and Corollary 1.2 have been recently treated in the literature. We indicate some of them.

1. Corollary 1.2 appears in [3, Th. 2] assuming \( \Omega \) is contained in a disc of radius 1 and satisfying an enclosing sphere condition. But in a Remark, Bergner asserts that the assumption to be contained in a ball can drop if there exist \( C^0 \) estimates, such as it occurs for (1): see Proposition 3.6 below.
2. Corollary 1.2 appears in [30, Rem. 3]. Initially, it is assumed that \( |\Omega| < 4\pi \) in a general result, but for (1) this hypothesis drops. Using the same proof than in [30], the existence holds for \( n \leq 2 \) in equation (2) under the assumption that \( |\Omega| < 4\pi \).
Theorem 1.1 appears in [22, Lem. 2.2] for $0 < n < 3$ assuming $\kappa > 0$ and $|\Omega| < 4\pi$.

Corollary 1.2 appears in [41, Th. 1.1] assuming that $\text{diam}(\Omega) < 2$.

We finish this section giving two generalizations of the translating soliton equation. First, consider the flow of surfaces by powers of mean curvature according to [35, 36, 40]. If $\alpha > 0$ is a constant, then the surface $z = u(x, y)$ evolves by translations of the $H^{\alpha}$-power of mean curvature flow if

$$\text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = \left( \frac{1}{\sqrt{1 + |Du|^2}} \right)^\alpha.$$  

Notice that this equation coincides with (2) of Bernstein and Serrin with the relation $n = 3 - \alpha$.

The second generalization is by considering critical points of the area $A_\phi$ for a fixed weighted volume. As a consequence of the Lagrange multipliers, $\Sigma$ satisfies that $H_\phi$ is a constant function and thus, in nonparametric form, we have

$$\text{div} \frac{Du}{\sqrt{1 + |Du|^2}} = \frac{1}{\sqrt{1 + |Du|^2}} + \mu,$$

where $\mu$ is a constant. This equation has received a recent interest: [8, 14, 27]. Even more general, we may study the mean curvature flow with a forcing term, so the constant $\mu$ in (6) is replaced by a function $f = f(u, Du)$ ([23, 34]). For example, the mean curvature type equation

$$\text{div} \frac{Du}{\sqrt{1 + |Du|^2}} = H_1(x, u, Du) + H_2(x, u, Du) \frac{1}{\sqrt{1 + |Du|^2}}$$

has been studied in [3, 24, 30].

**Convention.** After a change of coordinates, we will assume that $\vec{a} = (0, 0, 1)$.

This paper is organized as follows. In Section 2 we recall the translating solitons that are invariant by a uniparametric group of translations and of rotations. Section 3 is devoted to the tangency principle and some consequences derived by its applications to control the shape of a compact translating soliton. Sections 4 and 5 solve the Dirichlet problem on bounded convex domains for the translating soliton equation (1) and the constant weighted mean curvature equation (6), respectively. Here the boundary gradient estimates are obtained by means of the classical maximum principle to suitable choices of barrier functions. Finally, in Section 6 we study the Dirichlet problem for (1) in unbounded domains. We will consider two cases, namely, the domain is a strip and the boundary data are two copies of a convex function or the domain is an unbounded convex domain contained in a strip and the boundary data are constant.

2. Examples of translating solitons

Let $\{e_1, e_2, e_3\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ denote the canonical basis of $\mathbb{R}^3$. The Euclidean plane $\mathbb{R}^2$ will identified with plane of equation $z = 0$. We also use the terminology horizontal and vertical to indicate an orthogonal direction to $\vec{a}$ or a parallel direction to $\vec{a}$, respectively.

Notice that any translation of $\mathbb{R}^3$ preserves solutions of the translating soliton equation. The same occurs for a rotation about an axis parallel to $e_3$. Also, equation (1) is preserved by reversing the orientation on the surface.
In this section, we are interested by examples of translating solitons that are invariant by a uniparametric group of motions, more precisely, surfaces invariant along one direction and surfaces of revolution. In both cases, the equation (1) converts into an ODE and one may apply the standard theory.

2.1. Cylindrical surfaces. A cylindrical surface \( \Sigma \) is a surface invariant along a direction \( \vec{v} \in \mathbb{R}^3 \), or in other words, \( \Sigma \) is a ruled surface where all the rulings are parallel to \( \vec{v} \). We ask for those translating solitons of cylindrical type. Notice that there is not an \textit{a priori} relation between the direction \( \vec{v} \) and the density vector \( \vec{a} \).

A parametrization of \( \Sigma \) is \( X(t,s) = \gamma(s) + t\vec{v}, \ t \in \mathbb{R} \) and \( \gamma : I \subset \mathbb{R} \to \mathbb{R}^3 \) is a planar curve orthogonal to \( \vec{v} \). We parametrize \( \gamma = \gamma(s) \) by the arc length \( s \) such that \( \gamma'(s) \times \mathbf{n}(s) = \vec{v} \), being \( \mathbf{n} \) the unit principal normal vector of \( \gamma \). The Gauss map of \( \Sigma \) is \( N(X(s,t)) = \mathbf{n}(s) \) and \( 2H = \kappa(s) \), being \( \kappa \) the inward curvature of \( \gamma \) as a planar curve. Thus \( \Sigma \) is a translating soliton if \( \kappa(s) = \langle \mathbf{n}(s), \vec{a} \rangle \), hence we conclude that there is not a relation between the vectors \( \vec{v} \) and \( \vec{a} \). For example, if \( \vec{v} \) is parallel to \( \vec{a} \), then \( \langle \mathbf{n}(s), \vec{a} \rangle = 0 \) for every \( s \in I \), so \( \gamma \) is a straight line and \( \Sigma \) is a plane parallel to \( \vec{a} \).

In a first step, we investigate the case that \( \vec{v} \) is orthogonal to \( \vec{a} \) which, after a rotation about \( \vec{a} \), we suppose \( \vec{v} = e_1 \). Let us observe that a vertical plane parallel to \( e_1 \), that is, a plane parallel to the \( yz \)-plane, is a translating soliton of cylindrical type. If we write \( \gamma \) as \( z = w(y) \), then (1) converts to

\[
 w'' = 1 + w'^2.
\]

By simple quadratures, the solution of this equation is

\[
 w(y) = -\log(\cos(y + b)) + a, \quad a, b \in \mathbb{R},
\]

and this solution is called the \textit{grim reaper}. Although this holds for graphs \( z = w(y) \), it is true in general: if there is a vertical tangent vector at some point of \( \gamma \), then \( \gamma \) is a vertical line by uniqueness of ODE. This can be also obtained as follows. We parametrize \( \gamma \) by the arc length. Then \( \gamma(s) = (0, y(s), z(s)) \), with \( y'(s) = \cos \psi(s) \), \( z'(s) = \sin \psi(s) \) for some function \( \psi \). Then \( X(t,s) = (t, y(s), z(s)) \) and (1) becomes

\[
 2\psi'(s) = \cos \psi(s).
\]

If \( \gamma \) is not a graph on the \( y \)-axis, there is \( s = s_0 \) such that \( \cos \theta(s_0) = 0 \). By uniqueness, the solution is \( \theta(s) = \pm \pi/2, \gamma(s) = (0, a, \pm s + b), a, b \in \mathbb{R} \), \( \gamma \) is a vertical line and \( \Sigma \) is the vertical plane of equation \( y = a \).

Once obtained the translating solitons of cylindrical type when the vector \( \vec{v} \) is orthogonal to \( \vec{a} \), the rest of cylindrical surfaces are obtained by rotating the about surfaces about a horizontal axis. The resulting surfaces are all translating solitons of cylindrical type (after translations and rotations about a vertical axis). We present these surfaces, which will be called \textit{grim reapers} again (Figure 1).

\textbf{Definition 2.1.} The uniparametric family of grim reapers \( w_\theta = w_\theta(x,y) \) are defined as

\[
 w_\theta(x,y) = -\frac{1}{(\cos \theta)^2} \log(\cos(\cos \theta y)) + (\tan \theta)x + a,
\]

where \( \theta \in (-\pi/2, \pi/2) \), \( a \in \mathbb{R} \).

Here we recall that planes parallel to the \( xz \)-plane are cylindrical translating solitons, which would correspond with the critical values \( \theta = \pm \pi/2 \).

\textbf{Proposition 2.2.} All translating solitons of cylindrical type are planes parallel to the \( xz \)-plane or the grim reapers \( w_\theta \).
Proof. If \( \vec{v} = \vec{a} \), we know that the surface is (7), which coincides, up to a reparametrization, with (8) for the choice \( \theta = 0 \).

Suppose \( \vec{v} \) be a vector which is not orthogonal to \( \vec{a} \). After a rotation with respect to the \( z \)-axis, we assume that \( \vec{v} = \cos \theta \vec{e}_1 + \sin \theta \vec{e}_3 \), \( |\theta| < \pi/2 \). Let \( \vec{e} = -\sin \theta \vec{e}_1 + \cos \theta \vec{e}_3 \). We write the generating curve as a graph \( g = g(s) \) on the \( y \)-axis. Then parametrization of the surface is \( X(t, s) = se_2 + g(s)\vec{e} + t\vec{v} \). A computation shows that (1) writes as \( g'' = \cos \theta (1 + g'^2) \) and its integration gives \( g(s) = -\log(\cos(\cos \theta s + b))/\cos \theta + a, a, b \in \mathbb{R} \). Then

\[
X(t, s) = (-\sin \theta g(s) + t \cos \theta, s, t \sin \theta + \cos \theta g(s)).
\]

Writing \( X(t, s) = (x, y, u(x, y)) \), we deduce easily that \( u \) coincides with the function \( w_\theta \) in (8). □

The maximal domain of \( w_\theta \) is the strip

\[
\Omega^\theta = \left\{ (x, y) \in \mathbb{R}^2 : -\frac{\pi}{2 \cos \theta} < y < \frac{\pi}{2 \cos \theta} \right\}.
\]

In particular, if \( 0 \leq \theta_1 < \theta_2 \), it follows that \( \Omega^{\theta_1} \subset \Omega^{\theta_2} \) and thus the domain \( \Omega^0 \), namely,

\[
\Omega^0 = \left\{ (x, y) \in \mathbb{R}^2 : -\frac{\pi}{2} < y < \frac{\pi}{2} \right\},
\]

is contained in all \( \Omega^\theta \) for any \( \theta \in (-\pi/2, \pi/2) \).

![Figure 1. The grim reapers \( w_\theta \). Left: \( \theta = 0 \); Right: \( \theta = \pi/6 \)](image)

2.2. Rotational surfaces. The second family of translating solitons of our interest are those ones of rotational type. If \( \Sigma \) is a surface of revolution about a rotation axis \( \vec{v} \), we ask about the relation between the vector \( \vec{v} \) and the density vector \( \vec{a} \).

**Proposition 2.3.** Let \( \Sigma \) be a surface of revolution with respect to the vector \( \vec{v} \). If \( \Sigma \) is a translating soliton, then \( \vec{v} \) is parallel to \( \vec{a} \) or \( \Sigma \) is a plane parallel to \( \vec{a} \) where \( \vec{v} \) is orthogonal to \( \vec{a} \).

**Proof.** The value of the mean curvature \( H \) is constant along a parallel of \( \Sigma \). On the other hand, the Gauss map \( N \) makes a constant angle with \( \vec{v} \) along a parallel of the surface. Since \( 2H = \langle N, \vec{a} \rangle \), the function \( \langle N, \vec{a} \rangle \) is constant along every parallel of \( \Sigma \). Hence, we have only two possibilities, namely, \( \vec{v} \) is parallel to \( \vec{a} \) or \( \langle \vec{v}, \vec{a} \rangle = 0 \) with \( \langle N, \vec{a} \rangle = 0 \) on \( \Sigma \). In the latter case, \( \Sigma \) is a plane parallel to \( \vec{a} \). □
After a translation of \( \mathbb{R}^3 \), we will assume that the rotation axis is the \( z \)-axis. If we parametrize \( \Sigma \) as \( z = u(r), r^2 = x^2 + y^2 \), equation (1) becomes

\[
(10) \quad u'' + \frac{u(1 + u^2)}{r} = 1 + u^2.
\]

Therefore, by standard theory of ODE, there are solutions of (10) of initial conditions \( u(r_0) = u_0, \ u'(r_0) = u'_0 \), with \( r_0 > 0 \). The classification of the translating solitons of rotational type was done in [2, 13]: see Figure 2.

**Definition 2.4.** There are two types of rotational translating solitons depending if the surface meets or does not meet the rotation axis:

1. **Bowl solitons.** They are strictly convex entire graphs with a global minimum in the \( z \)-axis and intersect orthogonally the rotation axis. The surfaces are asymptotic to a paraboloid.
2. **Surfaces of winglike shape.** These surfaces do not intersect the rotation axis.

![Figure 2. Rotational translating solitons. Left: the bowl soliton; Right: surface with winglike-shape](image)

The bowl soliton corresponds with the solution of (10) with initial condition \( u(0) = u'(0) = 0 \) where the existence is not a direct consequence of the standard theory because (10) presents a singularity at \( r = 0 \). On the other hand, the winglike-shape solutions corresponds with solutions of (10) with \( r_0 > 0 \) and \( u'(r_0) = 0 \), whose existence is immediate.

The existence of the bowl soliton was done in [2, Cor. 3.3]. The authors solve (1) in a round disk with Neumann boundary condition \( \partial u/\partial n = \cos \alpha / \sqrt{1 + |Du|^2} \) and, after an argument of continuity varying the parameter \( \alpha \), they obtain the desired rotational solution. In this paper, we give two alternative proofs of the existence of the bowl solitons. One will appear in Remark 3.4 using Corollary 1.2 and an argument by means of the Alexandrov reflection method. We now present the other proof, which follows standard techniques of radial solutions for some equations of mean curvature type ([7, 11]). We write (10) as

\[
(11) \quad \frac{u''(r)}{(1 + u'(r)^2)^{3/2}} + \frac{u'(r)}{r \sqrt{1 + u'(r)^2}} = \frac{1}{\sqrt{1 + u'(r)^2}}.
\]
Multiplying (11) by \( r \), and integration by parts, we wish to establish the existence of a classical solution of

\[
\begin{aligned}
(12) & \quad \begin{cases}
\frac{ru'(r)}{\sqrt{1 + u'(r)^2}} = \frac{r}{\sqrt{1 + u'(r)^2}}, & \text{in } (0, \delta) \\
u(0) = 0, & \text{in } (0, \delta), \\
u'(0) = 0.
\end{cases}
\end{aligned}
\]

Let us observe that equation (12) is singular at \( r = 0 \).

**Proposition 2.5.** The initial value problem (12) has a solution \( u \in C^2([0, R]) \) for some \( R > 0 \) which depends continuously on the initial data.

**Proof.** Define the functions \( g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) and \( f : \mathbb{R} \to \mathbb{R} \) by

\[
g(x, y) = \frac{1}{\sqrt{1 + y^2}}, \quad f(y) = \frac{y}{\sqrt{1 + y^2}}.
\]

It is clear that a function \( u \in C^2([0, \delta]) \), for some \( \delta > 0 \), is a solution of (12) if and only if \( (rf(u'))' = rg(u, u') \) and \( u(0) = 0, \ u'(0) = 0 \).

Fix \( \delta > 0 \) to be determined later, and define the operator \( T \) by

\[
(Tu)(r) = a + \int_0^r f^{-1} \left( \int_0^s \frac{t}{s} g(u') dt \right) ds.
\]

Note that a fixed point of the operator \( T \) is a solution of the initial value problem (12). We claim now that \( T \) is a contraction in the space \( C^1([0, \delta]) \) endowed with the usual norm \( \|u\| = \|u\|_\infty + \|u'\|_\infty \). To see this, the functions \( g \) and \( f^{-1} \) are Lipschitz continuous of constant \( L > 0 \) in \([-\epsilon, \epsilon] \times [-\epsilon, \epsilon] \) and \([-\epsilon, \epsilon] \) respectively, provided \( \epsilon < 1 \). Then for all \( u, v \in B(0, \epsilon) \) and for all \( r \in [0, \delta] \),

\[
\| (Tu)(r) - (Tv)(r) \| \leq \frac{L^2}{4} r^2 \left( \|u - v\|_\infty + \|u' - v'\|_\infty \right)
\]

\[
\| (Tu)'(r) - (Tv)'(r) \| \leq \frac{L^2}{2} r \left( \|u - v\|_\infty + \|u' - v'\|_\infty \right)
\]

By choosing \( \delta > 0 \) small enough, we deduce that \( T \) is a contraction in the closed ball \( B(0, \epsilon) \) in \( C^1([0, \delta]) \). Thus the Schauder Point Fixed theorem proves the existence of one fixed point of \( T \), so the existence of a local solution of the initial value problem (12). This solution lies in \( C^1([0, \delta]) \cap C^2((0, \delta]) \). The \( C^2 \)-regularity up to 0 is verified directly by using the L'Hôpital rule because (11) leads to

\[
u''(0) + \lim_{r \to 0} \frac{u'(r)}{r} = 1,
\]

that is,

\[
\lim_{r \to 0} u''(r) = \frac{1}{2}.
\]

The continuous dependence of local solutions on the initial data is a consequence of the continuous dependence of the fixed points of \( T \). \( \square \)

From the classification of the rotational translating solitons, we observe that there do not exist closed surfaces (compact without boundary). Even more, we prove that there are not closed translating solitons. Usually the proof that appears in the literature of this result uses the touching principle (see Proposition 3.2 below). However, it is easier the following argument that we present, which only utilizes the divergence theorem ([27]).
Proposition 2.6. There do not exist closed translating solitons.

Proof. The proof is by contradiction. Suppose that $\Sigma$ is a closed translating soliton. Since the Laplacian $\Delta$ of the height function $\langle p, \vec{a} \rangle$ is $\Delta \langle p, \vec{a} \rangle = 2H \langle N, \vec{a} \rangle$, and $2H = \langle N, \vec{a} \rangle$, then

$$\Delta \langle p, \vec{a} \rangle = \langle N, \vec{a} \rangle^2.$$  

Integrating in $\Sigma$ and using the divergence theorem, we deduce

$$0 = \int_\Sigma \langle N, \vec{a} \rangle^2 \, d\Sigma,$$

because $\partial \Sigma = \emptyset$. Hence $\langle N, \vec{a} \rangle = 0$ in $\Sigma$. This is a contradiction because on a closed surface, the Gauss map $N$ is surjective on the unit sphere $S^2$. $\square$

3. Properties of the solutions of the translating soliton equation

This section establishes some properties of the solutions $u$ of the translating soliton equation, with a special interest in the control of $|u|$ and $|Du|$ when $\Omega$ is a bounded domain.

It is easily seen that the difference of two solutions of equation (1) satisfies the maximum principle. As a consequence, we give a statement of the comparison principle in our context. First, equation (1) can be expressed as $Q[u] = 0$, where $Q$ is the operator

$$Q[u] = (1 + |Du|^2)\Delta u - u_i u_j u_{ij} - (1 + |Du|^2),$$

being $u_i = \partial u/\partial x_i$, $i = 1, 2$, and we assume the summation convention of repeated indices. The comparison principle asserts ([15, Th. 10.1]):

Proposition 3.1 (Comparison principle). If $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy $Q[u] \geq Q[v]$ in $\Omega$ and $u \leq v$ on $\partial \Omega$, then $u \leq v$ in $\Omega$. If we replace $Q[u] \geq Q[v]$ by $Q[u] > Q[v]$, then $u < v$ in $\Omega$.

As a consequence, we deduce:

Proposition 3.2 (Touching principle). Let $\Sigma_1$ and $\Sigma_2$ be two translating solitons with possibly non-empty boundaries $\partial \Sigma_1$, $\partial \Sigma_2$. If $\Sigma_1$ and $\Sigma_2$ have a common tangent interior point and $\Sigma_1$ lies above $\Sigma_2$ around $p$, then $\Sigma_1$ and $\Sigma_2$ coincide at an open set around $p$. The same statement is also valid if $p$ is a common boundary point and the tangent lines to $\partial \Sigma_i$ coincide at $p$.

The tangency principle allows to control the shape of a given translating soliton by comparing, if possible, with other known surfaces ([26, 28]). For instance, it is easy to deduce that there do not exist closed translating solitons (Proposition 2.6). For this purpose, let $\Sigma$ be a such surface. Take a vertical plane $\Pi$, which is a translating soliton, far from $\Sigma$ so $\Sigma \cap \Pi = \emptyset$ since $\Sigma$ is a compact set. Let us move $\Pi$ towards $\Sigma$ until the first touching point, which occurs necessarily at some interior point because $\partial \Sigma = \emptyset$. Then the tangency principle implies that $\Sigma$ is included in $\Pi$, which is impossible.

In the following proposition, we use the tangency principle for compact translating solitons. By virtue of Proposition 2.6, the boundary of a compact translating soliton is not an empty set. We will see that the boundary of the surface determines, in some sense, the shape of the whole surface that spans. For instance, we characterize the compact translating solitons with circular boundary.
Proposition 3.3. Let $\Sigma$ be a compact translating soliton with boundary $\partial \Sigma$.

(1) If $\partial \Sigma$ is a graph on $\partial \Omega$, $\Omega \subset \mathbb{R}^2$ a bounded domain, then $\Sigma$ is a graph on $\Omega$.

(2) Let $D \subset \mathbb{R}^2$ be the domain bounded by convex hull of the orthogonal projection of $\partial \Sigma$ on $\mathbb{R}^2$. Then $\Sigma$ is contained in the solid cylinder $D \times \mathbb{R}$.

(3) The maximum of the height of $\Sigma$ is attained at some boundary point, that is, $\max_{p \in \Sigma} z(p) = \max_{p \in \partial \Sigma} z(p)$.

As a consequence, if $\partial \Sigma$ is a circle contained in a horizontal plane, then $\Sigma$ is a rotational surface contained in a bowl soliton ([32, 33]).

Proof. (1) Suppose, contrary to our claim, that $\Sigma$ is not a graph on $\Sigma$, in particular, there are two distinct points $p, q \in \text{int}(\Sigma)$ such that their orthogonal projections coincide on $\mathbb{R}^2$. Let $\Sigma^t = \Sigma + t \vec{a}$ be a vertical translation of $\Sigma$ by the vector $t \vec{a}$. Move up $\Sigma$ sufficiently far so $\Sigma^t \cap \Sigma = \emptyset$ for $t$ sufficiently large. Now we come back $\Sigma^t$ by letting $t \downarrow 0$ until the first time $t_1$ such that $\Sigma^{t_1} \cap \Sigma \neq \emptyset$. The existence of the points $p$ and $q$ ensures that $t_1 > 0$ and that this intersection occurs at some common interior point of both surfaces. By the tangency principle, $\Sigma^{t_1} = \Sigma$, a contradiction because their boundaries, namely, $\partial \Sigma^{t_1} = \partial \Sigma + t_1 \vec{a}$ and $\partial \Sigma$, do not coincide because $t_1 \neq 0$.

(2) Let $v \in \mathbb{R}^3$ be a fixed arbitrary horizontal direction. Consider a vertical plane $\Pi$ and orthogonal to $v$. Take $\Pi$ sufficiently far so $\Sigma \cap \Pi = \emptyset$. We move $\Pi$ along the direction $v$ towards $\Sigma$ until the first touching point. By the tangency principle, the intersection must occur at some boundary point of $\Sigma$. By repeating this argument for all horizontal vectors, we conclude the proof.

(3) Consider a horizontal plane $P$ above $\Sigma$ and sufficiently far so $\Sigma \cap P = \emptyset$. We move $P$ until the first touching point $p = (x_0, y_0, 0)$ with $\Sigma$ at the height $t_1$. The proof is completed by showing that $p \in \partial \Sigma$. By contradiction, suppose that $p$ is an interior point of $\Sigma$. Consider $P$ as the graph of the function $v(x, y) = t_1$. Similarly, consider $\Sigma$ locally as the graph of a function $u$ around $p$ on some domain $\Omega \subset \mathbb{R}^2$. Then we have $Q|u| = 0$, $Q|v| = -1$, so $Q|v| \leq Q|u|$. In view of $u \leq v$ on $\partial \Omega$ because $\Sigma$ lies below $P$, the comparison principle implies $u < v$ in $\Omega$: a contradiction because $u(x_0, y_0) = v(x_0, y_0)$.

The proof of the last statement is as follows. By items (1) and (3), $\Sigma$ is a graph on the round disc $\Omega$ bounded by $\partial \Sigma$ and the interior of $\Sigma$ lies below the plane $P$ containing $\partial \Sigma$. Then $\Sigma \cup \Omega$ bounds a 3-domain. By using the technique of the Alexandrov reflection by vertical planes ([1]), it is straightforward to see that $\Sigma$ is invariant by any rotation whose axis is the vertical line through the center of $\Omega$. Accordingly, $\Sigma$ a surface of revolution, and since its boundary is a circle, then $\Sigma$ is contained in a bowl soliton.

Remark 3.4. The last statement of the above proposition gives other argument for the existence of the bowl soliton. Indeed, let $\Omega = D_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < r^2\}$ in (1) and take the boundary data $\varphi = 0$ in (3). By Corollary 1.2, the existence and uniqueness of (1)-(3) is assured and Proposition 3.3 asserts that the solution is a radial function. Because the rotation axis meets orthogonally the domain $D_r$, then $\Sigma$ is a surface of revolution intersecting orthogonally the $z$-axis.
Remark 3.5 (Tangency principle). An inspection of the comparison argument in the proof of item (3) in Proposition 3.3 allows to extend the tangency principle as follows. Let $\Sigma_1$ and $\Sigma_2$ be two surfaces with weighted mean curvature $H^1_{\phi}$ and $H^2_{\phi}$, respectively. Suppose that $\Sigma_1$ and $\Sigma_2$ have a common tangent interior point $p$ and the orientations in both surfaces coincide at $p$. If $H^1_{\phi} \leq H^2_{\phi}$ around $p$, and $\Sigma_2$ lies above $\Sigma_1$ around $p$ with respect to $N(p)$, then $\Sigma_1$ and $\Sigma_2$ coincide at an open set around $p$. The same statement holds if $p$ is a common boundary point and the tangent lines to $\partial \Sigma_i$ coincide at $p$.

We derive height and interior gradient estimates for a solution of the translating soliton equation.

Proposition 3.6. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain.

1. The solution of (1)-(3), if exists, is unique.
2. There is a constant $C_1 = C_1(\varphi, \Omega)$ such that if $u$ is a solution of (1)-(3), then
\begin{align*}
    C_1 \leq u \leq \max_{\partial \Omega} \varphi \quad \text{in} \quad \Omega.
\end{align*}
3. If $u$ is a solution of (1)-(3), then
\begin{align*}
    \sup_{\Omega} |Du| = \max_{\partial \Omega} |Du|.
\end{align*}

Proof. (1) The uniqueness of solutions of (1)-(3) is a consequence of the maximum principle.

(2) The inequality in the right-hand side of (15) is immediate from the item (3) of Proposition 3.3.

The lower estimate for $u$ in (15) is obtained by means of bowl soliton as comparison surfaces. Let $R > 0$ be sufficiently large so $\overline{\Omega} \subset D_R$. Let $B$ be a bowl soliton defined by a radial function $b = b(r)$ such that $\partial D_R \subset B$, that is, $b$ is a solution of (1) in $D_R$ with $b = 0$ on $\partial D_R$. Let $B_R$ denote the compact portion of $B$ below the plane $z = 0$. Move vertically down $B_R$ sufficiently far so $\Sigma_u$ lies above $B_R$, that is, if $(x, y, z) \in \Sigma_u$, $(x, y, z') \in B_R$, then $z > z'$. Then move up $B_R$ until the first touching point with $\Sigma_u$. If the first contact occurs at some interior point, then the touching principle implies $\Sigma_u \subset B_R$. The other possibility is that the first contact point occurs when $B_R$ touches a boundary point of $\Sigma_u$. In both cases, we conclude $b(0) \leq u - \min_{\partial \Omega} \varphi$ and consequently, the constant $C_1 = b(0) + \min_{\partial \Omega} \varphi$ satisfies $C_1 \leq u$.

(3) Define the function $v^i = u_i$, $i = 1, 2$, and differentiate (14) with respect to the variable $x_k$, obtaining
\begin{align*}
    ((1 + |Du|^2)\delta_{ij} - u_iu_j)v^k_{i,j} + 2(u_i \Delta u - u_j u_{i,j} - u_i) v^k_i = 0,
\end{align*}
for each $k = 1, 2$. Hence $v^k$ satisfies a linear elliptic equation and by the maximum principle, $|v^k|$ has not a maximum at some interior point. Consequently, the maximum of $|Du|$ on the compact set $\overline{\Omega}$ is attained at some boundary point.

□
4. The Dirichlet problem in bounded convex domains

In this section we prove Corollary 1.2. Recall that the existence result of Serrin is also valid for the general family of equations (4). By completeness of this paper, we do a proof focusing on (1) and following ideas of [37]. We apply the method of continuity which requires the existence of a priori $C^0$ and $C^1$ estimates for a solution in order to provide the necessary compactness properties. These will be derived proving that $u$ admits barriers from above and from below along $\partial \Omega$. The higher order regularities of solutions hold under assuming smoothness hypothesis: [15, Ths. 6.17, 6.19, 13.8].

**Theorem 4.1.** Let $\Omega \subset \mathbb{R}^2$ be a bounded $C^{2,\alpha}$-domain whose inward satisfies $\kappa \geq 0$. If $\varphi \in C^0(\partial \Omega)$, then there is a unique solution of (1)-(3).

In Proposition 3.6, we found height estimates for $u$ and we proved that the interior gradient estimates are obtained once we have gradient estimates of $u$ along $\partial \Omega$. Thus, we now establish these estimates on the boundary.

**Proposition 4.2** (Boundary gradient estimates). Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $C^2$-boundary, $\kappa \geq 0$ and let $\varphi \in C^2(\partial \Omega)$. If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a solution of (1)-(3), then there is a constant $C_2 = C_2(\Omega, C_1, \|\varphi\|_{2,\overline{\Omega}})$ such that

$$\max_{\partial \Omega} |Du| \leq C_2,$$

where $\varphi$ is extended to some tubular neighborhood $\mathcal{N}_\epsilon$ of $\partial \Omega$.

**Proof.** We consider the operator $Q[u]$ defined in (14), which we write now as

$$Q[u] = a_{ij}u_{i;j} - (1 + |Du|^2), \quad a_{ij} = (1 + |Du|^2)\delta_{ij} - u_iu_j. \quad \text{(17)}$$

An upper barrier for $u$ is obtained by considering the solution $v^0$ of the Dirichlet problem for the minimal surface equation in $\Omega$ with the same boundary data $\varphi$: the existence of $v^0$ is assured in [37]). Because $Q[v^0] < 0 = Q[u]$ and $v^0 = u$ on $\partial \Omega$, we conclude $v^0 > u$ in $\Omega$ by the comparison principle.

We now find a lower barrier for $u$. Here we use the distance function in a small tubular neighborhood of $\partial \Omega$ in $\Omega$. Consider on $\overline{\Omega}$ the distance function to $\partial \Omega$, $d(x) = \text{dist}(x, \partial \Omega)$ and let $\epsilon > 0$ be sufficiently small so $\mathcal{N}_\epsilon = \{ x \in \overline{\Omega} : d(x) < \epsilon \}$ is a tubular neighborhood of $\partial \Omega$. We parametrize $\mathcal{N}_\epsilon$ using normal coordinates $x \equiv (t, \pi(x)) \in \mathcal{N}_\epsilon$, where we write $x = \pi(x) + t\nu(\pi(x))$ for some $t \in [0, \epsilon)$, $\pi : \mathcal{N}_\epsilon \to \partial \Omega$ is the orthogonal projection and $\nu$ is the unit normal vector to $\partial \Omega$ pointing to $\Omega$. Among the properties of the function $d$, we know that $d$ is $C^2$, $|Dd|(x) = 1$, and $\Delta d \leq -\kappa(\pi(x))$ for all $x \in \mathcal{N}_\epsilon$.

We extend $\varphi$ on $\mathcal{N}_\epsilon$ by $\varphi(x) = \varphi(\pi(x))$. Define in $\mathcal{N}_\epsilon$ the function

$$w = -h \circ d + \varphi,$$

where

$$h(t) = a \log(1 + bt), \quad a = \frac{c}{\log(1 + b)},$$

where $b > 0$ will be chosen later. Here $c$ is any constant with

$$c > 2 (\|\varphi\|_0 - C_1), \quad \text{(18)}$$

and $C_1$ is the constant of (15). Here and subsequently, $\| \cdot \|$ denotes the norm computed in $\overline{\mathcal{N}_\epsilon}$. It is immediate that $h \in C^\infty([0, \infty))$, $h' > 0$ and $h'' = -h' / a$. 

The first and second derivatives of \( w \) are \( w_i = -h'd_i + \varphi_i \) and \( w_{ij} = -h''d_id_j - h'd_{ij} + \varphi_{ij} \). The computation of \( Q[w] \) leads to

\[
Q[w] = -h''a_{ij}d_id_j - h'a_{ij}d_{ij} + a_{ij}\varphi_{ij} - (1 + |Dw|^2).
\]

From \(|Dd| = 1\), it follows that \( \langle D(Dd)\xi, Dd(x) \rangle = 0 \) for all \( \xi \in \mathbb{R}^2 \). If \( \{v_1, v_2 \} \) is the canonical basis of \( \mathbb{R}^2 \), by taking \( \xi = v_i \), we find \( d_{ij}d_j = 0 \). Thus

\[
w_iw_jd_{ij} = (h'd_i + \varphi_i)(-h'd_j + \varphi_j)d_{ij} = (h'^2d_i - 2h'\varphi_i)d_jd_{ij} + \varphi_i\varphi_jd_{ij} \]

\[
= \varphi_i\varphi_jd_{ij} \geq |D\varphi|^2\Delta d,
\]

where the last inequality is due to \( D^2d \) is negative. Using this inequality and the definition of \( a_{ij} \) in (17), we derive

\[
a_{ij}d_{ij} = (1 + |Dw|^2)\Delta d - w_iw_jd_{ij} \leq (1 + |Dw|^2) - |D\varphi|^2\Delta d.
\]

Notice that

\[
|Dw|^2 = h'^2 + |D\varphi|^2 - 2h'(Dd, D\varphi).
\]

Then

\[
1 + |Dw|^2 - |D\varphi|^2 = 1 + h'^2 - 2h'(D\varphi, Dd) \geq 1 + h'^2 - 2h'|D\varphi|
\]

\[
= 1 + \frac{c^2b^2}{\log(1 + b)(1 + bt)} - \frac{2cb}{\log(1 + b)(1 + bt)}|D\varphi| > 0
\]

if \( b \) is sufficiently large, with \( b \) a constant depending on \( \partial\Omega, c \) and \( |D\varphi| \). Since \( \Delta d \leq 0 \) because \( D^2d \) is negative, we deduce from (20) that \( a_{ij}d_{ij} \leq 0 \).

The ellipticity of \( A = (a_{ij}) \) can be written as \( |\xi|^2 \leq a_{ij}\xi_i\xi_j \leq (1 + |Dw|^2)|\xi|^2 \) for all \( \xi \in \mathbb{R}^2 \). Taking \( \xi = Dd \), then \( 1 = |Dd|^2 \leq a_{ij}d_id_j \leq (1 + |Dw|^2) \). Since \( h'' < 0 \), we have

\[
h''(a_{ij}d_id_j) \leq h''.
\]

On the other hand, if \( \cdot \) is the usual scalar product in the set of the square matrix,

\[
|A|^2 = A \cdot A = 1 + (1 + |Dw|^2)^2 \leq 2(1 + |Dw|^2)^2,
\]

hence

\[
a_{ij}\varphi_{ij} = A \cdot D^2\varphi \geq -|A||D^2\varphi| \geq -\sqrt{2}|D^2\varphi|(1 + |Dw|^2).
\]

By combining this inequality with \( \Delta d \leq 0 \), and inserting (20) and (22) in (19), we deduce

\[
Q[w] \geq -h'' - h'(1 + |Dw|^2 - |D\varphi|^2)\Delta d - (1 + \sqrt{2}|D^2\varphi|(1 + |Dw|^2))
\]

\[
\geq -h'' - (1 + \sqrt{2}|D^2\varphi|(1 + |Dw|^2)) \geq -h'' - \beta(1 + |Dw|^2),
\]

where \( \beta = 1 + \sqrt{2}|D^2\varphi|_0 \). Take \( b \) sufficiently large if necessary, to ensure that \( 1/a - \beta > 0 \), so \( \beta \) depends on \( \|D^2\varphi\|_0, C_1 \) and \( \|\varphi\|_0 \). Using that \( h'' = -h''/a \) and (21), we obtain

\[
Q[w] \geq \frac{h'^2}{a} - \beta(1 + |Dw|^2) = \left( \frac{1}{a} - \beta \right) h'^2 + 2h'(Dd, D\varphi) - \beta(1 + |D\varphi|^2_0)
\]

\[
\geq \left( \frac{1}{a} - \beta \right) h'^2 - 2h'(D\varphi)\|D\varphi\|_0 - \beta(1 + |D\varphi|^2_0)
\]

\[
= \left( \frac{1}{a} - \beta \right) \frac{a^2b^2}{(1 + bt)^2} - 2\beta \frac{ab}{1 + bt}\|D\varphi\|_0 - \beta(1 + |D\varphi|^2_0).
\]
We write the last term as a function on $\mathcal{N}_t$, namely, $g(x) = g(t, \pi(x))$. At $t = 0$,

$$
g(0) = \left(\frac{1}{a} - \beta\right) \frac{cb^2 \log(1 + b)^2}{\log(1 + b)} - 2\beta \frac{c}{b} \frac{\|D\phi\|_0 - \beta(1 + \|D\phi\|_0^2)}{\log(1 + b)}$$

$$= \frac{cb}{\log(1 + b)} \left(\frac{1}{a} - \beta\right) \frac{\log(1 + b)}{\log(1 + b)} \frac{c}{b} \frac{\|D\phi\|_0 - \beta(1 + \|D\phi\|_0^2)}{\log(1 + b)}$$

Therefore, if $b$ is sufficiently large, $g(0) > 0$. Since $\partial \Omega$ is compact, by an argument of continuity, $b$ can be chosen sufficiently large to ensure that $g(t) > 0$ in $\mathcal{N}_t$. For this choice of $b$, we find $Q[w] > 0$.

In order to assure that $w$ is a local lower barrier in $\mathcal{N}_t$, we have to see that

$$w \leq u \text{ in } \partial \mathcal{N}_t.$$  

In $\partial \mathcal{N}_t \cap \partial \Omega$, the distance function is $d = 0$, so $w = \varphi = u$. On the other hand, let us further require $b$ large enough so $\log(1 + bc)/\log(1 + b) \geq 1/2$. Then in $\partial \mathcal{N}_t \setminus \partial \Omega$, we find from (18) that

$$w = -c \frac{\log(1 + bc)}{\log(1 + b)} + \varphi \leq -\frac{\|\varphi\|_0 - C_1 \log(1 + b)}{2} + \varphi$$

$$\leq C_1 - \|\varphi\|_0 + \varphi \leq C_1 \leq u$$

in $\partial \mathcal{N}_t \setminus \partial \Omega$. Definitively, (24) holds in $\partial \mathcal{N}_t \setminus \partial \Omega$. Because $Q[w] > 0 = Q[u]$, we conclude $w \leq w$ in $\mathcal{N}_t$ by the comparison principle.

Consequently, we have proved the existence of lower and upper barriers for $u$ in $\mathcal{N}_t$, namely, $w \leq u \leq v^0$.

Hence

$$\max_{\partial \Omega} |Du| \leq C_2 := \max\{\|Dw\|_{0; \partial \Omega}, \|Dv^0\|_{0; \partial \Omega}\}$$

and both values $\|Dw\|_{0; \partial \Omega}, \|Dv^0\|_{0; \partial \Omega}$ depend only on $\Omega$, $C_1$ and $\varphi$. This completes the proof of proposition. $\square$

**Remark 4.3.** (1) It is possible, instead the function $v^0$, to use $w = h \circ d + \varphi$ for an upper barrier of $u$.

(2) The use of the auxiliary function $b(d) = a \log(1 + bd)$ for obtaining boundary gradient estimates is standard in the theory of elliptic equations (see [15, Ch. 14] as a general reference). It should also be mentioned that Bernstein was the first author whose employed this function to construct barriers for solutions in elliptic equations in two variables, assuming analytic hypothesis: [4, pp. 265-6].

**Proof of Theorem 4.1.** In a first step, we demonstrate the theorem when $\varphi \in C^{2, \alpha}(\Omega)$. We establish the solvability of the Dirichlet problem (1)-(3) by applying a slightly modification of the method of continuity ([15, Sec. 17.2]). Define the family of Dirichlet problems parametrized by $t \in [0, 1]$ by

$$\left\{ \begin{array}{lcl} Q_t[u] &=& 0 \text{ in } \Omega \\ u &=& \varphi \text{ on } \partial \Omega, \end{array} \right.$$  

where

$$Q_t[u] = (1 + |Du|^2) \Delta u - u_i u_j u_{i,j} - t(1 + |Du|^2).$$

As usual, let

$$\mathcal{A} = \{t \in [0, 1] : \exists u_t \in C^{2, \alpha}(\Omega), Q_t[u_t] = 0, u_t|_{\partial \Omega} = \varphi\}.$$
The theorem is established if $1 \in \mathcal{A}$. For this purpose, we prove that $\mathcal{A}$ is a non-empty open and closed subset of $[0, 1]$.

1. The set $\mathcal{A}$ is not empty. Let us observe that $0 \in \mathcal{A}$ because the minimal solution $v^0$ defined in Proposition 4.2 corresponds with $t = 0$.

2. The set $\mathcal{A}$ is open in $[0, 1]$. Given $t_0 \in \mathcal{A}$, we need to prove that there exists $\epsilon > 0$ such that $(t_0 - \epsilon, t_0 + \epsilon) \cap [0, 1] \subset \mathcal{A}$. Define the map $T(t, u) = Q_t[u]$ for $t \in \mathbb{R}$ and $u \in C^{2,\alpha}(\Omega)$. Then $t_0 \in \mathcal{A}$ if and only if $T(t_0, u_{t_0}) = 0$. If we show that the derivative of $Q_t$ with respect to $u$, say $(DQ_t)_u$, at the point $u_{t_0}$ is an isomorphism, it follows from the Implicit Function Theorem the existence of an open set $V \subset C^{2,\alpha}(\Omega)$, with $u_{t_0} \in V$ and a $C^1$ function $\psi: (t_0 - \epsilon, t_0 + \epsilon) \to V$ for some $\epsilon > 0$, such that $\psi(t_0) = u_{t_0} > 0$ and $T(t, \psi(t)) = 0$ for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$: this guarantees that $\mathcal{A}$ is an open set of $[0, 1]$.

To show that $(DQ_t)_u$ is one-to-one is equivalent that say that for any $f \in C^\alpha(\Omega)$, there is a unique solution $v \in C^{2,\alpha}(\Omega)$ of the linear equation $Lv := (DQ_t)_u(v) = f$ in $\Omega$ and $v = \varphi$ on $\partial \Omega$. The computation of $L$ was done in Proposition 3.6, namely,

$$Lv = (DQ_t)_u v = a_{ij}(Du)v_{ij} + B_i(Du, D^2u)v_i,$$

where $a_{ij}$ is as in (14) and $B_i = 2(u_i\Delta u - u_{ij}u_{ij} - tu_i)$. The existence and uniqueness is assured by standard theory ([15, Th. 6.14]).

3. The set $\mathcal{A}$ is closed in $[0, 1]$. Let $\{t_k\} \subset \mathcal{A}$ with $t_k \to t \in [0, 1]$. For each $k \in \mathbb{N}$, there is $u_k \in C^{2,\alpha}(\Omega)$ such that $Q_{t_k}[u_k] = 0$ in $\Omega$ and $u_k = \varphi$ in $\partial \Omega$. Define the set

$$\mathcal{S} = \{u \in C^{2,\alpha}(\Omega) : \exists \hat{t} \in [0, 1] \text{ such that } Q_{\hat{t}}[u] = 0 \text{ in } \Omega, u|_{\partial \Omega} = \varphi\}.$$

Then $\{u_k\} \subset \mathcal{S}$. If we see that the set $\mathcal{S}$ is bounded in $C^{1,\beta}(\Omega)$ for some $\beta \in [0, \alpha]$, and since $a_{ij} = a_{ij}(Du)$ in (14), the Schauder theory proves that $\mathcal{S}$ is bounded in $C^{2,\beta}(\Omega)$, in particular, $\mathcal{S}$ is precompact in $C^{2}(\Omega)$ (Th. 6.6 and Lem. 6.36 in [15]). Hence there is a subsequence $\{u_{k_l}\} \subset \{u_k\}$ converging to some $u \in C^2(\Omega)$ in $C^0(\Omega)$. Since $T: [0, 1] \times C^{2}(\Omega) \to C^0(\Omega)$ is continuous, we obtain $Q_{t_k}[u] = T(t_k, u_{t_k}) = 0$ in $\Omega$. Moreover, $u|_{\partial \Omega} = \lim_{k \to \infty} u_{k_l}|_{\partial \Omega} = \varphi$ on $\partial \Omega$, so $u \in C^{2,\alpha}(\Omega)$ and consequently, $t \in \mathcal{A}$.

Definitively, $\mathcal{A}$ is closed in $[0, 1]$ provided we find a constant $M$ independent on $t \in \mathcal{A}$, such that

$$\|u_t\|_{C^0(\Omega)} = \sup_{\Omega} |u_t| + \sup_{\Omega} |Du_t| \leq M.$$

Let $t_1 < t_2, t_i \in [0, 1], i = 1, 2$. Then $Q_{t_i}[u_{t_i}] = 0$ and

$$Q_{t_1}[u_{t_2}] = (t_2 - t_1)(1 + |Du_{t_2}|^2) > 0 = Q_{t_1}[u_{t_1}].$$

Since $u_{t_1} = u_{t_2}$ on $\partial \Omega$, the comparison principle yields $u_{t_2} < u_{t_1}$ in $\Omega$. This proves that the solutions $u_{t_i}$ are ordered in decreasing sense according the parameter $t$. It turns out that $u_1 \leq u_t < v^0$ for all $t$, where $u_1$ is the solution of (1)-(3). According to (28), we have $C_1 \leq u_t \leq \sup_{\Omega} u_0 \leq \max_{\partial \Omega} \varphi$ and we conclude

$$\|u_t\|_{0,\Omega} \leq C_3, \quad C_3 = \max\{|C_1|, \|\varphi\|_{0,\partial \Omega}\}.$$
In order to find the desired gradient estimates for the solution \( u_t \), by Proposition 3.6, we have to find estimates of \( |Du_t| \) along \( \partial \Omega \). On the other hand, the same computations given in Proposition 4.2 conclude that \( \sup_{\partial \Omega} |Du_t| \) is bounded by a constant depending on \( \Omega \), \( \varphi \) and \( \|u_t\|_{0,L^1} \). However, and by using (25), the value \( \|u_t\|_{0,L^1} \) is bounded by \( C_3 \), which depends only on \( \varphi \) and \( \Omega \).

Until here, we have proved the part of existence in Theorem 1.1. The uniqueness is a consequence of Proposition 3.6 and this completes the proof of theorem if \( \varphi \in C^2(\partial \Omega) \).

Finally we suppose \( \varphi \in C^0(\partial \Omega) \). Let \( \{ \varphi_k^+ \} \subset C^{2,\alpha}(\partial \Omega) \) be a monotonic sequence of functions converging from above and from below to \( \varphi \) in the \( C^0 \) norm. By virtue of the first part of this proof, there are solutions \( u_k^+ \) and \( u_k^- \subset C^{2,\alpha}(\Omega) \) of the translating soliton equation (1) such that \( u_k^+ \mid_{\partial \Omega} = \varphi_k^+ \) and \( u_k^- \mid_{\partial \Omega} = \varphi_k^- \). By the comparison principle, we find

\[
u_k^- \leq \ldots \leq u_k^- \leq u_k^- + 1 \leq \ldots \leq u_k^- + 1 \leq \ldots \leq u_k^+ \leq \ldots \leq u_k^+ \leq \ldots \leq u_k^+ \text{ for every } k,
\]

hence the sequences \( \{ u_k^\pm \} \) are uniformly bounded in the \( C^0 \) norm. By the proof of Theorem 4.1, the sequences \( \{ u_k^\pm \} \) have a priori \( C^1 \) estimates depending only on \( \Omega \) and \( \varphi \). Using classical Schauder theory again ([15, Th. 6.6]), the sequence \( \{ u_k^\pm \} \) contains a subsequence \( \{ u_k \} \subset C^{2,\alpha}(\Omega) \) converging uniformly on the \( C^2 \) norm on compacts subsets of \( \Omega \) to a solution \( u \subset C^2(\Omega) \) of (1). Since \( \{ u_k^\pm \} = \{ \varphi_k^\pm \} \) and \( \{ \varphi_k^\pm \} \) converge to \( \varphi \), we conclude that \( u \) extends continuously to \( \Omega \) and \( u \mid_{\partial \Omega} = \varphi \).

5. The Dirichlet problem for the constant weighted mean curvature

In this section we solve the Dirichlet problem for the case that \( H \phi \) is constant in (5):

\[
\begin{align*}
\text{div} \frac{Du}{\sqrt{1+|Du|^2}} &= \frac{1}{\sqrt{1+|Du|^2}} + \mu \quad \text{in } \Omega \\
u &= \varphi \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \mu \) is a constant and \( u \subset C^2(\Omega) \cap C^0(\bar{\Omega}) \). The motivation of this problem is twofold. First, equation (26) is the analogous to the constant mean curvature equation in Euclidean space in the context of manifolds with density, whereas (1) corresponds with the minimal surface equation. Second, the solvability of the constant mean curvature equation holds for any \( \varphi \) if \( \kappa \geq 2|H| \geq 0 \) ([37]) and the next result for (26)-(27) establishes a similar result.

**Theorem 5.1.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded \( C^{2,\alpha} \)-domain with inward curvature \( \kappa \). If \( \kappa \geq \mu \geq 0 \) and \( \varphi \subset C^0(\partial \Omega) \), then there is a unique solution of (26)-(27).

Notice that the solvability of this Dirichlet problem was proved in [24] in a context more general where the domain may be not bounded. As a difference, the present proof uses a comparison argument with rotational surfaces in order to obtain the \( C^0 \) estimates. The uniqueness is again a consequence of the maximum principle for the equation (26). After Theorem 4.1, we assume that \( \mu > 0 \). The proof now differs only in minor details from that of the preceding section, which are left to the reader. We point out that the assumption \( \mu > 0 \) will be use strongly.
Lemma 5.2. If $\Omega \subset \mathbb{R}^2$ is a bounded domain with $\operatorname{diam}(\Omega) < 1/\mu$, then there is a constant $C_1 = C_1(\varphi, \Omega)$ such that if $u$ is a solution of (26)-(27), then

\begin{equation}
C_1 \leq u \leq \max_{\partial \Omega} \varphi \quad \text{in } \Omega.
\end{equation}

Proof. Using $\mu > 0$, the right-hand side of (26) is non-negative, the maximum principle implies $\sup_{\Omega} u = \max_{\partial \Omega} u = \max_{\partial \Omega} \varphi$, proving the inequality in the right-hand side of (28). The lower estimate for $u$ in (28) is obtained by using radial solutions of (26). It was proved in [26] that if $\mu > 0$, any radial solution intersecting the rotational axis (and necessarily perpendicularly) converges to a right circular cylinder of radius $1/\mu$. More exactly, let $D_r \subset \mathbb{R}^2$ be a disc centered at the origin of radius $r$. If $\mu > 1/2$, there is a radial solution of (26) on $D_r$ for some $r_0 > 1/\mu$ and if $0 < \mu \leq 1/2$, there is a radial solution of (26) on $D_r$ for any $r < 1/\mu$.

Since $\operatorname{diam}(\Omega) < 1/\mu$, let $r > 0$ such that $\operatorname{diam}(\Omega) < r < 1/\mu$ and denote by $v$ the radial solution of (26) on $D_r$ with $v = 0$ on $\partial D_r$. After a horizontal translation if necessary, we suppose $\Omega \subset D_r$. Now the argument works the same as in Proposition 3.6 with the graph $\Sigma_v$, where now $C_1 = v(0) + \min_{\partial \Omega} \varphi$. \hfill \Box

Lemma 5.3 (Interior gradient estimates). If $u$ is a solution of (26)-(27), then

$$\sup_{\Omega} |Du| = \max_{\partial \Omega} |Du|.$$  

Proof. Now the corresponding equation (16) for (26) is

\begin{equation}
(1 + |\nabla v|^2) \delta_{ij} - v_i v_j ) z^{k}_{i,j} + 2 \left( v_i \Delta v - v_i v_j u_i - v_j - \frac{3}{2} \mu (1 + |\nabla v|^2) \right) z^{k}_{i} = 0,
\end{equation}

and the arguments are similar. \hfill \Box

Lemma 5.4 (Boundary gradient estimates). Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $C^2$-boundary, $\kappa \geq \mu > 0$ and let $\varphi \in C^2(\partial \Omega)$. If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a solution of (26)-(27), then there is a constant $C_2 = C_2(\Omega, C_1, \|\varphi\|_2)$ such that

$$\max_{\partial \Omega} |Du| \leq C_2.$$  

Proof. We consider the operator

\begin{equation}
Q[u] = a_{ij} u_{ij} - (1 + |Du|^2) - \mu (1 + |Du|^2)^{3/2}.
\end{equation}

The minimal solution $v^0$ is an upper barrier for $u$. For the lower barrier for $u$, we use again the function $w = -h \circ d + \varphi$. Now

\begin{equation}
Q[w] = -h'' a_{ij} d_i d_j - h' a_{ij} d_{ij} + a_{ij} \varphi_{ij} - (1 + |Dw|^2) - \mu (1 + |Dw|^2)^{3/2}.
\end{equation}

Taking into account $(1 + |Dw|^2)^{3/2} \leq 1 + |Dw|$ and $|Dw|^2 \leq h'^{2} + |D\varphi|^2 + 2h' |D\varphi| \leq (h' + |D\varphi|)^2$, we deduce from (31)

\begin{align*}
Q[w] &\geq -h'' - h' (1 + |Dw|^2) - |D\varphi|^2 \Delta d - (1 + \sqrt{2} |D\varphi|)(1 + |Dw|^2) - \mu (1 + |Dw|^2)^{3/2} \\
&\geq -h'' - h' (1 + |Dw|^2) - |D\varphi|^2 \Delta d - (1 + \sqrt{2} |D\varphi|)(1 + |Dw|^2) - \mu (1 + |Dw|^2)(1 + |Dw|) \\
&\geq -h'' - h' (1 + |Dw|^2) - |D\varphi|^2 \Delta d + \mu - h' |D\varphi|^2 \\
&\geq (\mu (1 + |D\varphi|) + 1 + \sqrt{2} |D\varphi|)(1 + |Dw|^2).
\end{align*}
Let $\beta = \mu(1 + ||D\varphi||_0) + 1 + \sqrt{2}||D^2\varphi||_0$. Since $\Delta d + \mu \leq -\kappa + \mu \leq 0$, it follows

$$Q[u] \geq \frac{h^2}{a} - \beta(1 + |Dw|^2) - \mu h'|D\varphi|^2$$

$$\geq \left(\frac{1}{a} - \beta\right) h^2 - h'(2\beta||D\varphi||_0 + \mu||D\varphi||_0^2) - \beta(1 + ||D\varphi||_0^2).$$

The rest of the proof runs as in Proposition 4.2. \qed

With the help of the preceding three lemmas we can now prove Theorem 5.1.

**Proof of Theorem 5.1.** For the method of continuity, let

$$Q_t[u] = (1 + |Du|^2)\Delta u - u_iu_ju_{ij} - (1 + |Du|^2) - t\mu(1 + |Du|^2)^{3/2},$$

and

$$\mathcal{A} = \{t \in [0,1] : \exists u_t \in C^{2,\alpha}(\overline{\Omega}), Q_t[u_t] = 0, u_t|_{\partial\Omega} = \varphi\}.$$

The set $\mathcal{A}$ is not empty because the solution of Theorem 4.1 corresponds with the value $t = 0$. For the openness of $\mathcal{A}$, the computation of $L$ leads to

$$Lv = (DQ_t)uv = a_{ij}(Du)v_{ij} + B_i(Du, D^2u)v,$$

where $B_i = 2(u_i\Delta u - u_ju_{ij} - u_i - 3t(1 + |Du|^2)/2)$. Then the proof works again.

Finally, we show that the set $\mathcal{A}$ is closed in $[0,1]$. For the height and gradient estimates for $u_t$, we use lemmas 5.2, 5.3 and 5.4. The arguments are similar once we prove that the solutions $u_t$ are ordered in decreasing sense. If $t_1 < t_2$, then $Q_{t_1}[u_{t_1}] = 0$ and

$$Q_{t_1}[u_{t_2}] = (t_2 - t_1)\mu(1 + |Du_{t_2}|^2)^{3/2} > 0 = Q_{t_1}[u_{t_1}].$$

Since $u_{t_1} = u_{t_2}$ on $\partial\Omega$, the comparison principle yields $u_{t_2} < u_{t_1}$ in $\Omega$. \qed

6. The Dirichlet Problem in Unbounded Domains

We study in this section the Dirichlet problem (1)-(3) in unbounded convex domains contained in a strip. We have two cases depending if the domain is or is not a strip.

The first result assumes that $\Omega$ is a strip. In such a case, the motivation comes from the grim reapers that appeared in (8). For each $\theta$, the surface $\Sigma_{w_\theta}$ is a graph defined in the (maximal) strip $\Omega^\theta$, with $w_\theta(x, y) \to +\infty$ as $|y| \to \pi/(2\cos \theta)$. If we narrow the strip to $|y| < b$, with $0 < b < \pi/(2\cos \theta)$, then the value of $w_\theta$ on $|y| = b$ is the linear function $x \mapsto \varphi(x, \pm b) = w_\theta(x, b)$ and $\partial\Sigma_{w_\theta}$ is formed by two parallel straight lines.

Our purpose is to consider the Dirichlet problem when $\Omega$ is a strip and $\varphi$ is formed by two copies of a convex function. Let $\Omega_m = \{(x, y) \in \mathbb{R}^2 : -m < y < m\}$, $m > 0$, be the strip of width $2m$. For each smooth convex function $f$ defined in $\mathbb{R}$, we extend $f$ to a function $\varphi_f$ on $\partial\Omega_m$ by $\varphi_f(x, \pm m) = f(x)$. The result of existence is established by our next theorem ([29]).

**Theorem 6.1.** If $m < \pi/2$, then for each convex function $f$, there is a solution of (1)-(3) for boundary values $\varphi_f$ on $\partial\Omega_m$.

The proof uses the classical Perron method of sub and supersolutions: see [12, pp. 306-312], [15, Sec. 6.3]. We consider the operator $Q$ defined in (14), where we know that $Q[u] = 0$ if and only if $u$ is a solution of the translating soliton equation.
The existence result of Theorem 4.1 holds in disks, so we can proceed to apply the Perron process when the domain is a strip.

First we need a subsolution of (1)-(3). In the following result, \( f \) is not necessarily a convex function (\([9]\)).

**Proposition 6.2.** Let \( \Omega_m \subset \mathbb{R}^2 \) be a strip. If \( f \) is a continuous function defined in \( \mathbb{R} \), then there is a solution \( \bar{v}^0 \) of the Dirichlet problem

\[
\begin{align*}
\text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) &= 0 \quad \text{in } \Omega_m \\
u = \varphi_f &\quad \text{on } \partial \Omega_m
\end{align*}
\]

with the property \( f(x) < \bar{v}^0(x, y) \) for all \((x, y) \in \Omega_m\).

Let \( u \in C^0(\overline{\Omega_m}) \) be a continuous function and let \( D \) be a closed round disk in \( \Omega_m \). We denote by \( \bar{u} \in C^2(D) \) the unique solution of the Dirichlet problem

\[
\begin{align*}
Q[\bar{u}] &= 0 \quad \text{in } D \\
\bar{u} &= u \quad \text{on } \partial D,
\end{align*}
\]

whose existence and uniqueness is assured by Theorem 4.1. We extend \( \bar{u} \) to \( \Omega_m \) by continuity as

\[ M_D[u] = \begin{cases} 
\bar{u} & \text{in } D \\
u & \text{in } \Omega_m \setminus D.
\end{cases} \]

The function \( u \) is said to be a supersolution in \( \Omega_m \) if \( M_D[u] \leq u \) for every closed round disk \( D \) in \( \Omega_m \). For example, for any domain \( \Omega \subset \mathbb{R}^2 \), the function \( u = 0 \) in \( \Omega \) is a supersolution in \( \Omega \). Indeed, if \( D \subset \Omega \) is a closed round disk, then \( \bar{u} < 0 \) since \( Q[0] = -1 < 0 = Q[\bar{u}] \) and the comparison principle applies. Thus \( M_D[u] \leq 0 \).

On the other hand, for each \( p \in \Omega \), there is a supersolution \( u \) with \( u(p) < 0 \). To this end, consider \( D \subset \Omega \) a closed round disk centered at the origin of \( \mathbb{R}^2 \). Let \( b = b(r) \) be the bowl soliton with \( b|_{\partial D} = 0 \). Then the function \( u \) defined as \( u = b \) in \( D \) and \( u = 0 \) in \( \Omega \setminus D \) is a supersolution.

**Definition 6.3.** A function \( u \in C^0(\overline{\Omega_m}) \) is called a superfunction relative to \( f \) if \( u \) is a supersolution in \( \Omega_m \) and \( f \leq u \) on \( \partial \Omega_m \). Denote by \( S_f \) the class of all superfunctions relative to \( f \),

\[ S_f = \{ u \in C^0(\overline{\Omega_m}) : M_D[u] \leq u \text{ for every closed disk } D \subset \Omega_m, f \leq u \text{ on } \partial \Omega_m \}. \]

**Lemma 6.4.** The set \( S_f \) is not empty.

**Proof.** We claim that \( \bar{v}^0 \in S_f \), where \( \bar{v}^0 \) is the minimal solution given in Proposition 6.2. Let \( D \subset \Omega_m \) be a closed round disk. Since \( \bar{v}^0 \) is a minimal surface, \( Q[\bar{v}^0] = -1 + |Dv^0|^2 \leq 0 \) and because \( \bar{v}^0 = \bar{v}^0 \) on \( \partial D \), the comparison principle implies \( M_D[\bar{v}^0] = \bar{v}^0 \leq \bar{v}^0 \) in \( D \). On the other hand, \( \bar{v}^0 = f \) on \( \partial \Omega_m \), proving definitively that \( \bar{v}^0 \in S_f \). \( \square \)

We now give some properties about superfunctions whose proofs are straightforward: in the case of the constant mean curvature equation, we refer [25]; in the context of translating solitons, see [22, Lems. 4.2–4.4].

**Lemma 6.5.**

1. If \( \{u_1, \ldots, u_n\} \subset S_f \), then \( \min\{u_1, \ldots, u_n\} \in S_f \).
2. The operator \( M_D \) is increasing in \( S_f \).
3. If \( u \in S_f \) and \( D \) is a closed round disk in \( \Omega_m \), then \( M_D[u] \in S_f \).
Consider the family of grim reaper \( w_\theta \) of (8). Since \( w_\theta \) is defined in the strip \( \Omega^\theta \) and, by assumption, \( m < \pi/2 \), then \( \Omega_m \subset \Omega^0 \subset \Omega^\theta \) for any \( \theta \). Thus it makes sense to restrict \( w_\theta \) to the strip \( \Omega_m \) and we keep the same notation for its restriction in \( \Omega_m \). Consequently \( w_\theta \) is a linear function on \( \partial \Omega_m \) and \( \partial \Sigma_{w_\theta} \) consists of two parallel lines.

Consider the subfamily of grim reapers
\[
\mathcal{G} = \{ w_\theta : w_\theta \leq f \text{ on } \partial \Omega_m, \theta \in (-\pi/2, \pi/2) \}.
\]
Notice that the set \( \mathcal{G} \) is not empty because \( f \) is convex. Furthermore, the minimal surface \( v^0 \) with \( v^0 = f \) on \( \partial \Omega_m \) satisfies \( Q[v^0] < 0 = Q[w_\theta] = 0 \) for all \( w_\theta \in \mathcal{G} \). Hence, the comparison principle asserts that \( w_\theta < v^0 \) in \( \Omega_m \) for all \( \theta \in (-\pi/2, \pi/2) \). This implies that \( v^0 \) plays the role of a subsolution for (1)-(3).

We now construct a solution of equation (1) between the grim reapers of \( \mathcal{G} \) and the minimal surface \( v^0 \). Let
\[
\mathcal{S}_f^* = \{ u \in \mathcal{S}_f : w_\theta \leq u \leq v^0, \text{ for every } w_\theta \in \mathcal{G} \}.
\]
We point out that \( \mathcal{S}_f^* \) is not empty because \( v^0 \in \mathcal{S}_f^* \). By using the maximum principle, it is not difficult to see that set \( \mathcal{S}_f^* \) is stable for the operator \( M_D \), that is, if \( u \in \mathcal{S}_f^* \), then \( M_D[u] \in \mathcal{S}_f^* \). The key point is the next proposition.

**Proposition 6.6** (Perron process). The function \( v : \Omega_m \to \mathbb{R} \) given by
\[
v(x, y) = \inf\{ u(x, y) : u \in \mathcal{S}_f^* \}
\]
is a solution of (1) with \( v = \varphi_f \) on \( \partial \Omega_m \).

**Proof.** The proof consists of two parts.

Claim 1. The function \( v \) is a solution of equation (1).

The proof is standard and here we follow [15]. Let \( p \in \Omega_m \) be an arbitrary fixed point of \( \Omega_m \). Consider a sequence \( \{ u_n \} \subset \mathcal{S}_f^* \) such that \( u_n(p) \to v(p) \) when \( n \to \infty \). Let \( D \) be a closed round disk centered at \( p \) and contained in \( \Omega_m \). For each \( n \), define on \( \Omega_m \) the function
\[
v_n(q) = \min\{ u_1(q), \ldots, u_n(q) \}, \quad q \in \overline{\Omega_m}.
\]
Then \( v_n \in \mathcal{S}_f^* \) by Lemma 6.5. Since \( M_D[v_n] \in \mathcal{S}_f^* \), we deduce \( M_D[v_n](p) \to v(p) \) as \( n \to \infty \). Set \( V_n = M_D[v_n] \). Then \( \{ V_n \} \) is a decreasing sequence bounded from below by \( w_\theta \) for all \( w_\theta \in \mathcal{G} \) and satisfying (1) in the disk \( D \). It turns out that the functions \( V_n \) are uniformly bounded on compact sets \( K \) of \( D \). In each compact set \( K \), the norms of the gradients \( |DV_n| \) are bounded by a constant depending only on \( K \) and using Hölder estimates of Ladyzhenskaya and Ural’tseva, there exist uniform \( C^{1,\beta} \) estimates for the sequence \( \{ V_n \} \) on \( K \) ([17]). By compactness, there is a subsequence of \( V_n \), that we denote \( V_n \) again, such that \( \{ V_n \} \) converges on \( K \) to a \( C^2 \) function \( V \) in the \( C^2 \) topology and by continuity, \( V \) satisfies (1). Moreover, by construction, at the fixed point \( p \) we have \( V(p) = v(p) \).

It remains to prove that \( V = v \) in \( \text{int}(D) \). For \( q \in \text{int}(D) \), the same argument as above gives the existence of \( \{ \tilde{u}_n \} \subset \mathcal{S}_f^* \) with \( \tilde{u}_n(q) \to v(q) \). Let \( \tilde{v}_n = \min\{ V_n, \tilde{u}_n \} \) and \( \tilde{V}_n = M_D[\tilde{v}_n] \). Again \( \tilde{V}_n \) converges on \( D \) in the \( C^2 \) topology to a \( C^2 \) function \( \tilde{V} \) satisfying (1) and \( \tilde{V}(q) = v(q) \). By construction, \( \tilde{V}_n \leq \tilde{v}_n \leq V_n \), hence \( \tilde{V} \leq V \).

In view that \( v \leq \tilde{V} \), we infer \( \tilde{V}(p) = v(p) = V(p) \). Thus \( V \) and \( \tilde{V} \) coincide at an interior point of \( D \), namely, the point \( p \), and both functions \( V \) and \( \tilde{V} \) satisfy the translating soliton equation. Because \( \tilde{V} \leq V \), the touching principle implies \( V = \tilde{V} \). 

in int(\(D\)). In particular, \(V(q) = \hat{V}(q) = v(q)\). This shows that \(V = v\) in int(\(D\)) and the claim is proved.

In order to finish the proof of Theorem 6.1, we prove that the function \(v\) takes the value \(\varphi_f\) on \(\partial \Omega_m\) and consequently, \(v\) is continuous up to \(\partial \Omega_m\) proving that \(v \in C^2(\Omega_m) \cap C^0(\overline{\Omega_m})\). In contrast to with the proof of Theorem 4.1, here we will find local barriers for each boundary point \(p \in \partial \Omega_m\).

Claim 2. The function \(v\) is continuous up to \(\partial \Omega_m\) with \(v = \varphi_f\) on \(\partial \Omega_m\).

The graph of \(\varphi_f\) consists of two copies of \(f\),
\[
\Gamma_{\varphi_f} = \Gamma_1 \cup \Gamma_2 = \{(x, m, f(x)) : x \in \mathbb{R}\} \cup \{(x, -m, f(x)) : x \in \mathbb{R}\}.
\]
Let \(p = (x_0, m) \in \partial \Omega_m\) be a boundary point (similar argument if \(p = (x_0, -m)\)). Because of the convexity of \(f\) in the plane of equation \(y = m\) the tangent line \(L_p\) to the planar curve \(\Gamma_1\) leaves \(\Gamma_1\) above \(L_p\). We choose the number \(\theta\) such that the grim reaper \(w_\theta\) takes the values \(L_p\) on \(\partial \Omega_m\): exactly, \(\theta\) is chosen so that \(\theta\) is the slope of \(L_p\). Recall that all rulings of this grim reaper are parallel to \(L_p\). Let \(w_\theta^p = w_\theta\) denote this grim reaper in order to indicate its dependence on the point \(p\).

Taking into account the symmetry of \(\varphi_f\) and the convexity of \(f\), we have \(w_\theta^p(p) = f(x_0)\) and \(w_\theta^p < f \text{ in } \Gamma_{\varphi_f} \setminus \{(x_0, m, f(x_0)), (x_0, -m, f(x_0))\}\), or in other words, \(\partial \Sigma_{w_\theta^p}\) lies strictly below \(\partial \Sigma_v\), except at the points \((x_0, m, f(x_0))\) and \((x_0, -m, f(x_0))\), where both graphs coincide.

Therefore the function \(w_\theta^p\) and the minimal surface \(v^0\) form a modulus of continuity in a neighborhood of \(p\), namely, \(w_\theta^p \leq v \leq v^0\). Because \(w_\theta^p(p) = v^0(p) = f(p)\), we infer that \(v(p) = f(p)\) and this completes the proof of Theorem 6.1. \(\square\)

We finish this section with the second type of domains, that is, when \(\Omega\) is an unbounded convex domain contained in a strip. Under this situation, we will suppose \(\varphi = 0\) on \(\partial \Omega\).

**Theorem 6.7.** Let \(\Omega\) be an unbounded convex domain contained in a strip of width strictly less than \(\pi\). Then there is a solution of the translating soliton equation (1) in \(\Omega\) with \(\varphi = 0\) on \(\partial \Omega\).

**Proof.** If \(\Omega\) is a strip, then the result was established in Theorem 6.1. In fact, if \(\Omega = \Omega_m, m < \pi/2\), the solution is \(w(x, y) = -\log(\cos(y)) + \log(\cos(m))\).

Suppose now that \(\Omega\) is not a strip. After a change of coordinates, we assume that the narrowest strip containing \(\Omega\) is \(\Omega_m\). Since \(\Omega\) is an unbounded domain contained in a strip, then \(\partial \Omega\) has two branches asymptotic to the boundary set \(\partial \Omega_m\) and the \(x\)-coordinate function is bounded in \(\partial \Omega\) from above or from below.

We follow the same reasoning as in Theorem 6.1, and we only point out the differences. The subsolution is the function \(v^0 = 0\), which is a solution of the minimal surface equation. We consider the family of operators \(M_D\) and
\[
\mathcal{S} = \{u \in C^0(\overline{\Omega}) : M_D[u] \leq u \text{ for every closed round disk } D \subset \Omega, 0 \leq u \text{ on } \partial \Omega\}.
\]
Let the grim reaper \(w(x, y) = -\log(\cos(y))\) whose domain is the strip \(\Omega^0\) of width \(\pi\) and define \(\omega(x, y) = -\log(\cos(y)) + \log(\cos(m))\). Note that \(\omega = 0\) on \(\partial \Omega_m\) and \(\omega < 0\) on \(\partial \Omega\) because \(\Omega \subset \Omega_m\). We construct a solution of equation (1) between the grim reaper \(\omega\) and the minimal surface \(v^0\). Let
\[
\mathcal{S}^* = \{u \in \mathcal{S} : \omega \leq u \leq 0\}.
\]
Note that $S^*$ is not empty because $0 \in S^*$: indeed, $Q(0) = -1 < 0 = Q(\omega)$ and $\omega < v_0$ in $\partial \Omega$, hence $\omega < 0$ in $\Omega$ by the comparison principle. Again, the function
\[
v(x, y) = \inf\{u(x, y) : u \in S^*\} = \inf\{M_D[u](x, y) : u \in S^*\}
\]
is a solution of (1) and it remains to prove that the function $v$ is continuous up to $\partial \Omega$ with $v = 0$ on $\partial \Omega$. Here the barrier construction in the proof of Theorem 6.1 can be adapted to provide boundary modulus of continuity estimates.

Let $p = (x_0, y_0) \in \partial \Omega$ be a boundary point of $\Omega$. We rotate $\Omega$ with respect to the $z$-axis and translate along a horizontal direction if necessary, in such way that the tangent line $L$ to $\Omega$ at $p$ is one of the boundaries of $\Omega_m$ and a neighborhood $U_p$ of $p$ in $\Omega$ is contained in $\Omega_m$: this is possible by the convexity of $\Omega$. There is no loss of generality in assuming that $L = \{(x, m, 0) : x \in \mathbb{R}\}$. We take now the restriction of $\omega$, $\omega^* = \omega|_{\Sigma^*}$, in the half-strip $\Omega^*_m = \{(x, y) \in \mathbb{R}^2 : 0 < y \leq m\}$. Let $\Sigma_{\omega^*}$ be the graph of $\omega^*$. Notice that $\partial \Sigma_{\omega^*}$ is formed by two parallel lines, one is $L$ and the other one is $L' = \{(x, 0, \omega(0)) : x \in \mathbb{R}\}$.

Let $n(p) = (0, 1, 0)$ be the unit outward normal vector to $\partial \Omega$ at $p$. Let us move horizontally $\Sigma^*$ in the direction $n(p)$ until $\Sigma^*$ does not intersect $\Sigma^*$. Then we come back in the direction $-n(p)$ until the first touching point $q$ between $\Sigma^*$ and $\Sigma^*$. Since $\omega < v < 0$ in $\Omega$, it is not possible that $q \in L'$. By the tangency principle, $q \in L$ and by the convexity of $\Omega$, the point $q$ coincides with $p$. Accordingly, we have proved that in the interior of the neighborhood $U$, we have $\omega < v < 0$. Since $\omega(p) = v(p) = 0$, the functions $\omega$ and $0$ are a modulus of continuity in $U$ of $p$, hence $v(p) = 0$. This completes the proof of Theorem 6.7.

We point out that the domain $\Omega$ is not necessarily strictly convex. Thus in the last part of the above proof, the intersection between $\Sigma_{\omega^*}$ and $\Sigma^*$ at the first touching point, may occur along a segment of $L$. In any case, we can take that a first contact point is the very point $p$.

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