MULTIPLICITY OF SOLUTIONS FOR RESONANT AND NON-RESONANT ASYMPTOTICALLY LINEAR ELLIPTIC PROBLEMS

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Abstract. In this paper we prove the existence of a signed ground state solution (in the mountain pass level) for a class of asymptotically linear elliptic problems, even when the nonlinearity is just continuous in the second variable. A multiplicity result is also proved when \( f \) is odd with respect to the second variable, in which a close relation between \( \lim_{|t| \to \infty} f(x, t)/|t| \) and the number of solutions is established.

1. Introduction

We are interested in studying the existence of ground state and other nontrivial solutions to the following semilinear problem

\[
\begin{aligned}
-\Delta u &= f(x, u) \quad \text{in } \Omega, \\
&\quad u = 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^N \) is a bounded smooth domain, \( N \geq 1 \) and \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function which is assumed to satisfy the following assumptions, where \( \lambda_m \) denotes the \( m \)-th eigenvalue of the laplacian operator with homogeneous Dirichlet boundary condition:

1. \( f_1 \) \( t \mapsto f(x, t)/|t| \) is increasing, for a.e. \( x \in \Omega \);
2. \( f_2 \) there are real numbers \( \alpha_0 \) and \( \alpha_\infty \) such that

\[
0 \leq \alpha_0 := \lim_{t \to 0} f(x, t)/|t| < \lambda_1 \leq \lambda_m < \alpha_\infty := \lim_{|t| \to \infty} f(x, t)/|t| \leq \lambda_{m+1}, \text{ a.e. in } \Omega,
\]

for some \( m \geq 1 \).

Problems satisfying conditions as in \( f_2 \) are known in the literature as asymptotically linear and can be classified as resonant at infinity (if \( \alpha_\infty = \lambda_j \), for some \( j \)) or non-resonant at infinity (if \( \alpha_\infty \neq \lambda_j \), for all \( j \in \mathbb{N} \)). In fact the resonant case is subdivided depending on how small at infinity is the function \( g(x, t) = \alpha_\infty t - f(x, t) \). The worst situation is when, for all \( x \in \Omega \),

\[
\lim_{|t| \to +\infty} g(x, t) = 0 \quad \text{and} \quad \lim_{|t| \to +\infty} \int_0^t g(x, s)ds = b(x) \in \mathbb{R}
\]

and is called the strong resonant case. One of the very first works dealing with this situation is [5] where Bartolo, Benci and Fortunato show the existence and multiplicity of solutions to strong resonant problems in the presence of some symmetry in the autonomous nonlinearity. Their proofs are based on a deformation theorem and pseudo-index theory.

Besides [5], there are so many other works dealing with problem (P). For instance, in the non-resonant case with \( m = 1 \), Amann and Zehnder, in [4], proved that problem (P) has at least one nontrivial solution. Considering yet the non-resonant case with \( m \geq 1 \), Ahmad, in [1], proved the existence of at least two nontrivial solutions. Other interesting results on the same

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issue can be found in [2], [8], [13] and [15]. As far as the resonant case is regarded, the existence of nontrivial solution was studied in [9] by de Figueiredo and Miyagaki and in [12] by Li and Willem. Multiplicity results for problem (P) in the resonant case were also investigated in [13] by Li and Willem, [14] by Liu and Zou, [17] by Su and in [18] by Su and Zhao. It is essential to point out that in all the references above, the nonlinearity $f$ is assumed to be differentiable (or even $C^1$) in the second variable, being this assumption crucial in their arguments.

In this work we prove the existence of a signed ground state solution, in the mountain pass level, for problem (P), when $m = 1$ in $(f_2)_m$, even if the function $f$ is just continuous in the second variable, see Theorem 4.1. Our result includes both, resonant and non-resonant cases. Moreover, assuming that $f$ is odd with respect to the second variable, we prove that if $(f_1) - (f_2)_m$ hold, with $m \geq 2$, then problem (P) admits at least $1 + \sum_{j=2}^{m} d_j$ pairs of nontrivial solutions, where $d_j$ denotes the dimension of $j$-th eigenspace associated to $\lambda_j$, see Theorem 5.5.

In what follows we enumerate the main contributions of this paper: (1) It provides an unified approach to deal concurrently the non-resonant case, the strong and the non-strong resonant cases. (2) Our assumptions on function $f$ are weaker than those that have been most commonly used in the literature to treat this class of problems, mainly because: (a) we are not requiring any differentiability on function $f$, (b) we are not assuming non-quadraticity type conditions as $(F_2)_+$ in [7], (c) even in the strong resonant case, our results encompass nonlinearities which do not satisfy, simultaneously, the hypotheses $(g_1) - (g_2)$ in [5]. (3) The approach used in this work allow us to overcome (see Proposition 3.8) one of the main difficulties in this type of problem, namely, the absence of superquadraticity conditions such as Ambrosetti-Rabinowitz condition (see [3]) or the superquadraticity condition used in [19], which are very important to prove that Palais-Smale sequences are bounded.

Our approach is based on the Nehari method which consists in minimize the energy functional over the so called Nehari manifold, a set which contains all the nontrivial solutions of the problem. This method has been carefully treated in [19] by Szulkin and Weth in the case of superlinear nonlinearities. In fact [19] has been our main influence and inspiration to develop this work and, since we develop a Nehari method in the presence of asymptotically linear nonlinearities, this work can also be seen as a contribution to the results in [19].

In order to make a parallelism with [19], some words about our multiplicity result are in order. Taking a closer look in Theorem 5.5 we can see that (P) has as much solutions as more linear functions with slopes given by the eigenvalues of $(-\Delta, H^1_0(\Omega))$ the function $f(x, \cdot)$ crosses at infinity (see the picture below).

Taking this into account, since in [19] the authors prove that (P) has infinitely many solutions if $f(x, \cdot)$ is odd and superlinear, we can wonder that the reason for such a behavior comes from the fact that in the superlinear setting, $f(x, \cdot)$ crosses at infinity all the linear functions with slopes given by the eigenvalues of $(-\Delta, H^1_0(\Omega))$. In this sense, our result seems to be sharper. This becomes evident since, in the presence of $(f_1)$, assumption $(f_2)_m$ is equivalent to

$$0 \leq \alpha_0 := \lim_{t \to 0} 2F(x, t)/t^2 < \lambda_1 \leq \lambda_m < \alpha_\infty := \lim_{|t| \to \infty} 2F(x, t)/t^2 \leq \lambda_{m+1}, \text{ a.e. in } \Omega.$$  

The paper is organized as follows. In Section 2 we present the variational background and some preliminaries results. In Section 3 we study deeply the Nehari manifold and some of its topological features. In Section 4 we state and prove our main result of existence of ground-state solution. In Section 5 we introduce some classical tools of genus theory and we prove our main result of multiplicity of solutions.
2. Preliminaries

In this section we are going to introduce some variational background and also prove important consequences of hypotheses $(f_1) - (f_2)_1$.

We denote by $I : H^1_0(\Omega) \rightarrow \mathbb{R}$ the energy functional associated to problem (P), which is given by

$$I(u) = \frac{1}{2} \|u\|^2 - \int_{\Omega} F(x,u)dx,$$

where $\|u\|^2 = \int_{\Omega} |\nabla u|^2 dx$ and $F(x,t) = \int_{0}^{t} f(x,s)ds$. It is well known that $I \in C^1(H^1_0(\Omega),\mathbb{R})$ and

$$I'(u)\varphi = \int_{\Omega} \nabla u \nabla \varphi dx - \int_{\Omega} f(x,u)\varphi dx.$$

In this way, critical points of $I$ are weak solutions of (P).

The Nehari manifold associated to the functional $I$ is defined by

$$\mathcal{N} = \{u \in H^1_0(\Omega)/\{0\} : I'(u)u = 0\} = \{u \in H^1_0(\Omega)/\{0\} : \|u\|^2 = \int_{\Omega} f(x,u)udx\}.$$

Since $f$ is just a Carathéodory function, we cannot ensure that $\mathcal{N}$ is a smooth manifold.

In the paper, $S$ denotes the unit sphere in $H^1_0(\Omega)$ and

$$\mathcal{A} := \left\{ u \in H^1_0(\Omega) : \|u\|^2 < \alpha_\infty \int_{\Omega} u^2 dx \right\}.$$

Throughout this paper we denote by $e_j$ a normalized (in $H^1_0(\Omega)$ norm) eigenfunction associated to $\lambda_j$.

**Lemma 2.1.** If $f$ satisfies $(f_1) - (f_2)_1$, the following claims hold:

(i) The set $\mathcal{A}$ is open and nonempty;
Proof. (i) Note that, from \((f_2)_1\), if \(u\) is an eigenfunction associated to \(\lambda_1\) then \(u\) belongs to \(A\). Moreover, \(A = \varphi^{-1}(-\infty, 0)\) where \(\varphi : H_0^1(\Omega) \to \mathbb{R}\) is the continuous function defined by \(\varphi(u) = \|u\|^2 - \int_\Omega \alpha_\infty u^2\). The items (ii) and (iii) are straightforward.

(iv) If \(u \in \mathcal{N}\) then, from \((f_1) - (f_2)_1\) and

\[
\|u\|^2 = \int_{[u \neq 0]} \frac{f(x,u)}{u} u^2 dx,
\]

we conclude that

\[
\|u\|^2 < \alpha_\infty \int_\Omega u^2 dx.
\]

(v) It is sufficient to choose a normalized eigenfunction \(e_1\) associated to \(\lambda_1\). Clearly, one has that \(e_1 \in \mathcal{S} \cap A\). \(\square\)

In the sequel we will denote \(\mathcal{S}_A := \mathcal{S} \cap A\). Observe that being \(\mathcal{S}\) a \(C^1\)-submanifold from \(H_0^1(\Omega)\), we conclude from (i) and (v) of Lemma 2.1 that \(\mathcal{S}_A\) is an open set in \(\mathcal{S}\) and, therefore, it is a noncomplete \(C^1\)-submanifold of \(H_0^1(\Omega)\). Moreover, from (ii) and (iii), it is clear that \(\partial \mathcal{S}_A = \{u \in \mathcal{S} : 1 = \alpha_\infty \int_\Omega u^2 dx\}\) and \(\mathcal{S}^c_A = \{u \in \mathcal{S} : 1 \geq \alpha_\infty \int_\Omega u^2 dx\}\), where \(\partial \mathcal{S}_A\) denotes the relative boundary of \(\mathcal{S}_A\) as a topological subspace of \(\mathcal{S}\) and \(\mathcal{S}^c_A\), the set \(\mathcal{S} \setminus \mathcal{S}_A\). Note that, under hypothesis \((f_2)_m\), we have \(e_j \in \mathcal{S}^c_A\) for all \(j \geq 2\). In general, under hypothesis \((f_2)_m\), we have \(e_j \in \mathcal{S}^c_A\), for all \(j \geq m + 1\).

In this paper, we use the symbol \([u \neq 0]\) to denote the set \(\{x \in \Omega : u(x) \neq 0\}\). Moreover, \([|u| \neq 0]\) will denote the Lebesgue’s measure of the set \([u \neq 0]\).

To finish this section we state some consequences of hypothesis \((f_1) - (f_2)_1\) which will be used later on.

Lemma 2.2. Suppose \((f_1)\) holds. Then,

(i) \(t \mapsto (1/2)f(x,t)t - F(x,t)\) is increasing in \((0, \infty)\) and decreasing in \((-\infty, 0)\);

(ii) \(t \mapsto F(x,t)/t^2\) is increasing in \((0, \infty)\) and decreasing in \((-\infty, 0)\);

(iii) \(f(x,t)/t > 2F(x,t)/t^2\) for all \(t \in (0, \infty)\) and \(f(x,t)/t < 2F(x,t)/t^2\) for all \(t \in (-\infty, 0)\);

Proof. (i) Let \(t_1 > t_2 > 0\). Then,

\[
\frac{1}{2} f(x,t_1)t_1 - F(x,t_1) = \frac{1}{2} f(x,t_1)t_1 - F(x,t_2) - \int_{t_2}^{t_1} \frac{f(x,s)}{s} \, ds
\]

\[
> \frac{1}{2} f(x,t_1)t_1 - F(x,t_2) - \frac{f(x,t_1)}{t_1} \int_{t_2}^{t_1} \, ds
\]

\[
= \frac{1}{2} f(x,t_1)t_1 - F(x,t_2) - \frac{f(x,t_1)}{t_1} \left( \frac{t_1^2 - t_2^2}{2} \right)
\]

\[
= \frac{f(x,t_1)}{t_1} \left( \frac{t_1^2 - t_2^2}{2} \right) - F(x,t_2)
\]

\[
> \frac{1}{2} f(x,t_2)t_2 - F(x,t_2),
\]

where it was used \((f_1)\) in the last two inequalities. The other case is analogous. Items (ii) and (iii) follows from (i). \(\square\)
Remark 1. From previous lemma, the following limit is well defined
\[
\beta := \lim_{|t| \to \infty} [(1/2)f(x,t)t - F(x,t)],
\]
which is a positive number or \(\infty\).

3. Topological aspects of the Nehari manifold

The main goal of this section is studying some topological features of the Nehari manifold and the behavior of the energy functional \(I\) on \(\mathcal{N}\).

Proposition 3.1. Suppose that \(f\) verifies \((f_1)-(f_2)_1\) and let \(h_u : [0, \infty) \to \mathbb{R}\) be defined by \(h_u(t) = I(tu)\).

(i) For each \(u \in \mathcal{A}\), there is a unique \(t_u > 0\) such that \(h_u'(t) > 0\) in \((0, t_u)\), \(h_u'(t_u) = 0\) and \(h_u'(t) < 0\) in \((t_u, \infty)\). Moreover, \(tu \in \mathcal{N}\) if, and only if, \(t = t_u\);

(ii) for each \(u \in \mathcal{A}^c\), \(h_u'(t) > 0\) for all \(t \in (0, \infty)\).

Proof. (i) First observe that \(h_u(0) = 0\). Moreover, for each \(u \in \mathcal{A}\), we have
\[
(3.1) \quad \frac{h_u(t)}{t^2} = \frac{1}{2} \|u\|^2 - \int_{\{u \neq 0\}} \left[ \frac{F(x,tu)}{(tu)^2} \right] u^2 dx.
\]
Thus, from \((f_1)-(f_2)_1\), L’Hospital rule and Lebesgue’s dominated convergence theorem, it follows that
\[
\lim_{t \to 0} \frac{h_u(t)}{t^2} = \frac{1}{2} \left( \|u\|^2 - \int_{\Omega} \alpha_0 u^2 dx \right) > 0
\]
and
\[
\lim_{t \to \infty} \frac{h_u(t)}{t^2} = \frac{1}{2} \left( \|u\|^2 - \int_{\Omega} \alpha_{\infty} u^2 dx \right) < 0.
\]
Showing that
\[
h_u(t) = \frac{h_u(t)}{t^2} t^2
\]
is positive for \(t\) small and
\[
\lim_{t \to \infty} h_u(t) = \lim_{t \to \infty} \frac{h_u(t)}{t^2} t^2 = -\infty.
\]
Since \(h_u\) is a continuous function the previous arguments implies that there is a global maximum point \(t_u > 0\) of \(h_u\). Now, we are going to show that \(t_u\) is the unique critical point of \(h_u\). In fact, supposing that there exist \(t_1 > t_2 > 0\) such that \(h_u'(t_1) = h_u'(t_2) = 0\), we obtain
\[
0 = \int_{\{u \neq 0\}} \left[ \frac{f(x,t_1 u)}{t_1 u} - \frac{f(x,t_2 u)}{t_2 u} \right] u^2 dx,
\]
and, from \((f_1)\), \(t_1 = t_2\). Thence, the result follows.

(ii) If \(u \in \mathcal{A}^c\), then \(\|u\|^2 \geq \alpha_{\infty} \int_{\Omega} u^2 dx\). Thus, it follows from \((f_1)-(f_2)_1\) that
\[
\frac{h_u(t)}{t} = \|u\|^2 - \int_{\{u \neq 0\}} \frac{f(x,tu)}{tu} u^2 dx \geq \int_{\{u \neq 0\}} \left[ \alpha_{\infty} - \frac{f(x,tu)}{tu} \right] u^2 dx > 0, \quad \forall \ t > 0.
\]
Consequently, \(h_u'(t) = (h_u(t)/t) > 0\) for all \(t \in (0, \infty)\). \(\square\)

Remark 2. It is an immediate consequence of previous proposition that, for each \(u \in \mathcal{A}\) and \(s \in (0, \infty)\), \(t_{su} = t_u/s\). Moreover, it is clear that \(u \in \mathcal{N}\) if, and only if, \(t_u = 1\).

Proposition 3.2. If \(f\) verifies \((f_1)-(f_2)_1\), the following claims hold:
(A1) There is $$\tau > 0$$ independent of $$u$$ such that $$t_u \geq \tau$$ for all $$u \in S_A$$;

(A2) for each compact set $$W \subset S_A$$ there is $$C_W > 0$$ such that $$t_u \leq C_W$$, for all $$u \in W$$;

(A3) The map $$\hat{m} : A \to N$$ given by $$\hat{m}(u) = t_u u$$ is continuous and $$m := \hat{m}|_{S_A}$$ is a homeomorphism between $$A$$ and $$N$$. Moreover, $$m^{-1}(u) = u/\|u\|$$.

Proof. (A1) Suppose there is $$\{u_n\} \subset S_A$$ such that $$t_n := t_{u_n} \to 0$$. In this case, we get $$u \in H^1_0(\Omega)$$ such that $$u_n \rightharpoonup u$$ in $$H^1_0(\Omega)$$. If $$u = 0$$, passing to the limit as $$n \to \infty$$ in

$$\tag{3.2} 1 = \int_{[u_n \neq 0]} \frac{f(x, t_n u_n)}{t_n u_n} u_n^2 dx, \forall \, n \in N,$$

we have a contradiction. Therefore $$u \neq 0$$ and, passing to the limit as $$n \to \infty$$ in (3.2), we get

$$1 = \alpha_0 \int_{\Omega} u^2 dx \leq (\alpha_0/\lambda_1)\|u\|^2 \leq \alpha_0/\lambda_1,$$

a clear contradiction.

(A2) Suppose there is $$\{u_n\} \subset W$$ such that $$t_n := t_{u_n} \to \infty$$. Since $$W$$ is compact, passing to a subsequence, we obtain $$u \in W$$ such that $$u_n \rightharpoonup u$$ in $$H^1_0(\Omega)$$. Whence, passing to the limit as $$n \to \infty$$ in

$$1 = \|u_n\|^2 = \int_{[u_n \neq 0]} \frac{f(x, t_n u_n)}{t_n u_n} u_n^2 dx, \forall \, n \in N$$

and being $$u \neq 0$$, it follows from $$(f_1) - (f_2)$$ and Lebesgue’s dominated convergence theorem that

$$||u||^2 = 1 = \int_{\Omega} \alpha_\infty u^2 dx,$$

showing that $$u \in \partial A$$, leading us to a contradiction since $$u \in W \subset A$$ and $$A$$ is open.

(A3) We first show that $$\hat{m}$$ is continuous. Let $$\{u_n\} \subset A$$ and $$u \in A$$, be such that $$u_n \to u$$ in $$H^1_0(\Omega)$$. From Remark 2 ($$\hat{m}(tw) = \hat{m}(w)$$ for all $$w \in A$$ and $$t > 0$$), we can consider $$\{u_n\} \subset S_A$$. Whence,

$$\tag{3.3} 1 = \|u_n\|^2 = \int_{[u_n \neq 0]} \frac{f(x, t_n u_n)}{t_n u_n} u_n^2 dx,$$

where $$t_n := t_{u_n}$$. From (A1) and (A2), it follows that, passing to a subsequence, $$t_n \to t > 0$$. Thence, passing to the limit as $$n \to \infty$$ in (3.3), we have

$$1 = \|u\|^2 = \int_{[u \neq 0]} \frac{f(x, tu)}{tu} u^2 dx,$$

showing that $$\hat{m}(u_n) = t_n u_n \to tu = \hat{m}(u)$$. The second part of (A3) is straightforward. \(\square\)

Lemma 3.3. The functional $$I$$ is bounded from below on $$N$$, more specifically,

$$I(u) > 0,$$

for all $$u \in N$$.

Proof. For any $$u \in S_A$$, we get

$$I(t_u u) = \int_{\Omega} \left[ \frac{1}{2} \frac{f(x, t_u u)}{t_u u} - \frac{F(x, t_u u)}{(t_u u)^2} \right] (t_u u)^2 dx.$$

Now the result follows from Lemma 2.2(iii). \(\square\)
Now we set the maps $\hat{\Psi}: \mathcal{A} \to \mathbb{R}$ and $\Psi: \mathcal{S}_A \to \mathbb{R}$, by

$$\hat{\Psi}(u) = I(\hat{m}(u))$$

These maps will be very important in our arguments mainly because of their properties, which will be presented in the next result. The proof of such a result is a consequence of Proposition 3.2 and the details, in the case where $\mathcal{N}$ is homeomorphic to $\mathcal{S}$, can be found in [19]. Since in this article $\mathcal{N}$ is homeomorphic to a noncompact submanifold of $\mathcal{S}$, for the convenience of the reader, we provide the proof here.

We say that the functional $\Psi$ satisfies the $(PS)_c$ condition in $\mathcal{S}_A$ if for each sequence $\{u_n\} \subset \mathcal{S}_A$, such that $\Psi(u_n) \to c$ and $\Psi'(u_n) \to 0$ in $H^{-1}(\Omega)$, then up to a subsequence there exists $u \in \mathcal{S}_A$ with $u_n \to u$ in $H^1_0(\Omega)$.

**Proposition 3.4.** Suppose that $f$ verifies $(f_1) - (f_2)_1$. Then,

(i) $\hat{\Psi} \in C^1(\mathcal{A}, \mathbb{R})$ and

$$\hat{\Psi}'(u)v = \frac{\|\hat{m}(u)\|}{\|u\|} I'(\hat{m}(u))v,$$

for all $u \in \mathcal{A}$ and all $v \in H^1_0(\Omega)$.

(ii) $\Psi \in C^1(\mathcal{S}_A, \mathbb{R})$ and

$$\Psi'(u)v = \|m(u)\| I'(m(u))v,$$

for all $v \in T_u \mathcal{S}_A$.

(iii) If $\{u_n\}$ is a $(PS)_c$ sequence for $\Phi$ then $\{m(u_n)\}$ is a $(PS)_c$ sequence for $I$. If $\{u_n\} \subset \mathcal{N}$ is a bounded $(PS)_c$ sequence for $I$ then $\{m^{-1}(u_n)\}$ is a $(PS)_c$ sequence for $\Phi$.

(iv) $u$ is a critical point of $\Psi$ if, and only if, $m(u)$ is a nontrivial critical point of $I$.

Moreover, the corresponding critical values coincide and

$$\inf_{\mathcal{S}_A} \Phi = \inf_{\mathcal{N}} I.$$

**Proof.** (i) Consider $u \in \mathcal{A}$ and $v \in H^1_0(\Omega)$. From definitions of $\hat{\Psi}$ and $t_u$ and from the mean value Theorem,

$$\hat{\Psi}(u + sv) - \hat{\Psi}(u) = I(t_{u+sv}(u + sv)) - I(t_u u)$$

$$\leq I(t_{u+sv}(u + sv)) - I(t_{u+sv} u)$$

$$= I'(t_{u+sv}(u + \tau sv)) t_{u+sv} sv,$$

where $|s|$ is small enough and $\tau \in (0, 1)$. On the other hand,

$$\hat{\Psi}(u + sv) - \hat{\Psi}(u) \geq I(t_u(u + sv)) - I(t_u u) = I'(t_u(u + sv)) t_u sv,$$

where $\zeta \in (0, 1)$. Since $u \mapsto t_u$ is a continuous map, it follows from previous inequalities that

$$\lim_{s \to 0} \frac{\hat{\Psi}(u + sv) - \hat{\Psi}(u)}{s} = t_u I'(t_u u) v = \frac{\|\hat{m}(u)\|}{\|u\|} I'(\hat{m}(u))v.$$

Since $I \in C^1(H^1_0(\Omega), \mathbb{R})$, it follows that the Gateaux derivative of $\hat{\Psi}$ is a continuous linear functional in $v$ and a continuous map in $u$. From Proposition 1.3 in [20], $\hat{\Psi} \in C^1(\mathcal{A}, \mathbb{R})$ and

$$\hat{\Psi}'(u)v = \frac{\|\hat{m}(u)\|}{\|u\|} I'(\hat{m}(u))v,$$

for all $u \in \mathcal{A}$ and all $v \in H^1_0(\Omega)$.

(ii) It is a direct consequence of (i).

(iii) Once $H^1_0(\Omega) = T_u \mathcal{S}_A \oplus \mathbb{R} u$ for each $u \in \mathcal{S}_A$, the linear projection $P: H^1_0(\Omega) \to T_u \mathcal{S}_A$, defined by $P(v + tu) = v$, is continuous, namely, there is $C > 0$ such that

$$\|v\| \leq C \|v + tu\|, \forall u \in \mathcal{S}_A, v \in T_u \mathcal{S}_A \text{ and } t \in \mathbb{R}.$$
Thus
\[
\|\Psi'(u)\|_* = \sup_{v \in T_{u_0}S_A \ \|v\| = 1} \Psi'(u)v = \|w\| \sup_{v \in T_{u_0}S_A \ \|v\| = 1} I'(w)v,
\]
where \(w = m(u)\). Since \(w \in \mathcal{N}\), we conclude that
\[
I'(w)u = I'(w)\frac{w}{\|w\|} = 0.
\]
By (3.5), we have
\[
\|\Psi'(u)\|_* \leq \|w\| \|I'(w)\| = \|w\| \sup_{v \in T_{u_0}S_A \ \|v\|= 1 \ \forall t \neq 0} \frac{I'(w)(v + tu)}{\|v + tu\|}.
\]
Hence, from (3.4) and (3.6)
\[
\|\Psi'(u)\|_* \leq \|w\| \|I'(w)\| \leq C\|w\| \sup_{v \in T_{u_0}S_A \ \|v\| \neq 0} \frac{I'(w)(v)}{\|v\|} = C\|\Psi'(u)\|_*,
\]
showing that,
\[
\|\Psi'(u)\|_* \leq \|w\| \|I'(w)\| \leq C\|\Psi'(u)\|_*, \ \forall u \in S_A.
\]
From Proposition 3.2(A1), it follows that there exists \(\tau > 0\) such that \(\|w\| \geq \tau > 0\) for all \(w \in \mathcal{N}\). Therefore, inequality (3.7) together with \(I(w) = \Psi(u)\) conclude the proof of (iii).

(iv) It follows from (3.7) that \(\Psi(u) = 0\) if, and only if, \(I'(w) = 0\). Now the result follows from definition of \(\Psi\).

\[\square\]

Proposition 3.5. Suppose \((f_1) - (f_2)_1\) hold. Then,
\[
c_N := \inf_{u \in \mathcal{N}} I(u) = \inf_{u \in A} \max_{t > 0} I(tu) = \inf_{u \in S_A} \max_{t > 0} I(tu) = \inf_{u \in S_A} \Psi(u).
\]

Proof. It follows directly from Propositions 3.1 and 3.2. \(\square\)

Remark 3. It is a consequence of Lemma 3.3 that \(c_N \geq 0\). Moreover, if \(c_N\) is achieved then it is positive.

Proposition 3.6. Suppose \((f_1) - (f_2)_1\) hold. If \(\{u_n\} \subset S_A\) is such that \(\text{dist}(u_n, \partial S_A) \to 0\), then there is \(u \in H^1_0(\Omega)\) such that \(u_n \to u\) in \(H^1_0(\Omega)\), \(u_n \to \infty\) and
\[
\Psi(u_n) \to \beta|\{u \neq 0\}|,
\]
where \(\beta\) was defined in Remark 1;

Proof. Being \(\{u_n\} \subset S_A\) bounded, passing to a subsequence, there exists \(u \in H^1_0(\Omega)\) with \(u_n \to u\) in \(H^1_0(\Omega)\). Since \(\text{dist}(u_n, \partial S_A) \to 0\), we conclude that
\[
\alpha_\infty \int_{\Omega} u_n^2 dx \to 1.
\]
By using compact embedding from \(H^1_0(\Omega)\) in \(L^2(\Omega)\), it follows that
\[
1 = \alpha_\infty \int_{\Omega} u^2 dx.
\]
Thus \(u \neq 0\). Suppose by contradiction that, for some subsequence, \(\{t_{u_n}\}\) is bounded. In this case, passing again to a subsequence, there is \(t_0 > 0\) (see Proposition 3.2(i)) such that
\[
t_{u_n} \to t_0.
\]
It follows from (3.11), \((f_1) - (f_2)_1\) and
\[
1 = \int_{\Omega} \frac{f(x, t_{u_n} u_n)}{t_{u_n} u_n} u_n^2 \, dx, \quad \forall \, n \in \mathbb{N},
\]
that
\[
1 = \int_{\Omega} \frac{f(x, t_0 u)}{t_0 u} u^2 \, dx.
\]
Combining the last equality and \((f_1) - (f_2)_1\), we have
\[
1 < \alpha_\infty \int_{\Omega} u^2 \, dx.
\]
But the previous inequality contradicts (3.10). Showing that \(t_{u_n} \to \infty\).

Finally, from \(t_{u_n} \to \infty\), \((f_1) - (f_2)_1\), L'Hospital rule and (3.10), we get
\[
\int_{\Omega} \frac{F(x, t_{u_n} u_n)}{(t_{u_n} u_n)^2} u_n^2 \, dx \to \frac{1}{2}.
\]

Consequently, from L'Hospital rule
\[
\lim_{n \to \infty} I(t_{u_n} u_n) = \lim_{n \to \infty} \left( \frac{1}{2} - \int_{\Omega} \frac{F(x, t_{u_n} u_n)}{(t_{u_n} u_n)^2} u_n^2 \, dx \right) \rightarrow \frac{1}{2}.
\]

Lemma 3.7. Suppose \((f_1) - (f_2)_1\) hold. Then \(c_N < \beta|\Omega|\). Moreover, if \(\{u_n\} \subset \mathcal{S}_A\) is such that \(\Psi(u_n) \to c\), then \(c \in [c_N, \beta|\Omega|]\).

Proof. Since, for each \(u \in \mathcal{S}_A\), \(m(u) \in \mathcal{N}\), from Lemma 2.2(i) and Remark 1, we obtain
\[
c_N \leq \Psi(u) = I(m(u)) = \int_{\Omega} \left[ \frac{1}{2} f(x, m(u)) m(u) - F(x, m(u)) \right] \, dx < \beta|u \neq 0| \leq \beta|\Omega|.
\]
The second part of the lemma follows by replacing \(u\) by \(u_n\) in the last inequality and by passing to the limit as \(n \to \infty\).

Proposition 3.8. Suppose \((f_1) - (f_2)_1\) hold. Then \(\Psi\) satisfies the \((PS)_c\) condition in \(\mathcal{S}_A\), for \(c \in [c_N, \beta|\Omega|]\).

Proof. By Proposition 3.2(A3) and Proposition 3.4(iii), it is sufficient to show that \(I\) satisfies the \((PS)_c\) condition on \(\mathcal{N}\) for \(c \in [c_N, \beta|\Omega|]\).

For this, let \(\{u_n\} \subset \mathcal{N}\) be a \((PS)_c\) sequence for functional \(I\). We prove that \(\{u_n\}\) is bounded in \(H^1_0(\Omega)\). In fact, suppose by contradiction that, passing to a subsequence, we have \(\|u_n\| \to \infty\). Define \(v_n := u_n/\|u_n\| = m^{-1}(u_n) \in \mathcal{S}_A\). Thus \(\{v_n\}\) is bounded in \(H^1_0(\Omega)\) and
\[
(3.12) \quad \Psi(v_n) \to c.
\]
Consequently, there is \(v \in H^1_0(\Omega)\) such that
\[
(3.13) \quad v_n \rightharpoonup v \text{ in } H^1_0(\Omega).
\]
Suppose by contradiction that \(v = 0\). Since \(\{\Psi(v_n)\}\) is bounded, it follows that there is \(C > 0\) such that
\[
(3.14) \quad C > \Psi(v_n) = I(t_{v_n} v_n) \geq I(t_{v_n}) = \left[ \frac{1}{2} - \int_{|v_n| \neq 0} \frac{F(x, t_{v_n})}{(t_{v_n})^2} v_n^2 \, dx \right] t^2, \quad \forall \, t > 0.
\]
From \((f_1) - (f_2)_1\) and compact embedding, passing to the limit of \(n \to \infty\) in \((3.14)\), we get
\[
C \geq (1/2)t^2, \quad \forall \ t > 0,
\]
a clear contradiction. Thereby, we conclude that \(v \neq 0\).

Since \(\{u_n\} \subset N\) is a \((PS)_c\) sequence for functional \(I\), we get
\[
o_n(1) + \int_{\Omega} \nabla u_n \nabla w dx = \int_{\Omega} f(x, u_n) w dx, \quad \forall \ w \in H_0^1(\Omega).
\]
Dividing last equality by \(\|u_n\|\), we have
\[
o_n(1) + \int_{\Omega} \nabla v_n \nabla w dx = \int_{\Omega} \left[ \frac{f(x, \|u_n\| v_n)}{\|u_n\| v_n} \right] v_n w dx.
\]
Passing to the limit as \(n \to \infty\), from \((f_1) - (f_2)_1\) and Lebesgue’s dominated convergence theorem it follows that
\[
(3.15) \quad \int_{\Omega} \nabla v \nabla w dx = \alpha_{\infty} \int_{\Omega} v w dx, \quad \forall \ w \in H_0^1(\Omega).
\]

Now we are going to consider two cases:

(i) If \(\alpha_{\infty} \neq \lambda_2\), it follows from \((3.15)\) that \(v = 0\). But this is a contradiction. Therefore \(\{u_n\}\) is bounded in \(H_0^1(\Omega)\).

(ii) If \(\alpha_{\infty} = \lambda_2\), then \((3.15)\) implies that \(v = e_2\), where \(e_2\) is some normalized eigenfunction associated to \(\lambda_2\). From \((3.15)\), it follows also that \(\|v\|^2 = \alpha_{\infty} \int_{\Omega} v^2 dx\), i.e., \(v \in \partial A\). On the other hand,
\[
\alpha_{\infty} \int_{\Omega} v^2 dx = \|v\|^2 \leq \liminf_{n \to \infty} \|v_n\|^2 = 1.
\]
Suppose that
\[
(3.16) \quad \alpha_{\infty} \int_{\Omega} v^2 dx < 1.
\]
In this case, since
\[
(3.17) \quad \|u_n\| = \|t v_n v_n\| = t v_n,
\]
passing to the limit as \(n \to \infty\) in the identity
\[
\Psi(v_n) = \|u_n\|^2 \left\{ 1 - \int_{\Omega} \frac{F(\|u_n\| v_n)}{(\|u_n\| v_n)^2} v_n^2 dx \right\}
\]
and using \((f_1) - (f_2)_1\), L’Hospital rule and \((3.16)\), we conclude that \(\Psi(v_n) \to \infty\), a contradiction with \((3.12)\). Consequently,
\[
(3.18) \quad \|v\|^2 = \alpha_{\infty} \int_{\Omega} v^2 dx = 1,
\]
showing that
\[
(3.19) \quad \|v_n\| \to \|v\|.
\]
By using \((3.13)\) and \((3.19)\), we derive \(v_n \to v\) in \(H_0^1(\Omega)\) with \(v \in \partial S_A\) (see \((3.18)\)). Invoking Proposition \(3.6\), we conclude that
\[
(3.20) \quad \Psi(v_n) \to \beta \|v \neq 0\|.
\]
Since \(v = e_2\), it is a consequence of Theorem 2.2 in [6] that
\[
(3.21) \quad \|v \neq 0\| = |\Omega|.
\]
Thus, from (3.12), (3.20) and (3.21), we obtain $c = \beta |\Omega|$. Since last equality cannot occurs, we conclude that $\{u_n\}$ is bounded.

Being $\{u_n\}$ a bounded sequence, there is $u \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ up to a subsequence.

Since $u_n \rightharpoonup u$, to finish the proof we just have to prove that $\|u_n\| \to \|u\|$. To this end, it is sufficient to note that since $\{u_n\}$ is a $(PS)_c$ sequence, we have
\[
o_n(1) + \int_\Omega \nabla u_n \nabla u dx = \int_\Omega f(x, u_n) u dx.
\]
Passing to the limit as $n \to \infty$ in the previous equality, we get
\[
(3.22) \quad \|u\|^2 = \int_\Omega f(x, u) u dx.
\]
Then (3.22) and Lebesgue’s convergence theorem imply that
\[
\|u_n\|^2 = \int_\Omega f(x, u_n) u_n dx = \int_\Omega f(x, u) u dx + o_n(1) = \|u\|^2 + o_n(1).
\]

\[\square\]

4. Signed ground-state solution

In this section we state and prove our main existence result.

**Theorem 4.1.** Suppose that $f$ satisfies $(f_1)-(f_2)_1$. Then there exists a signed ground-state solution (in the mountain pass level) for problem (P).

**Proof.** Let $\{u_n\} \subset S$ be such that $I(u_n) \to c_S$. Remember that $v_n := u_n/\|u_n\| \in S_A$ (see Proposition 3.2(A$_3$)) and
\[
(4.1) \quad \Psi(v_n) \to c_S.
\]

We will show that $\{v_n\}$ is a $(PS)_{c_S}$ sequence for the functional $\Psi$. To prove this claim, we define the application $\Upsilon : S_A \to \mathbb{R} \cup \{\infty\}$ by
\[
(4.2) \quad \Upsilon(u) = \begin{cases} 
\Psi(u) & \text{if } u \in S_A, \\
\beta\|u \neq 0\| & \text{if } u \in \partial S_A.
\end{cases}
\]

Observe that $\Upsilon$ is well defined and, from Proposition 3.6, $\Upsilon$ is continuous. Since $S_A$ is a complete metric space with metric provided by the norm of $H_0^1(\Omega)$ and $\Upsilon$ is bounded from below (see Lemma 3.3), it follows from Theorem 1.1 in [10] that for each fixed $\varepsilon, \lambda > 0$ and $u \in \Psi^{-1}([c_S, c_S + \epsilon])$, there exists $v \in S_A$ such that
\[
c_S \leq \Upsilon(v) \leq \Psi(u), \quad \|u - v\| \leq \lambda \quad \text{and} \quad \Upsilon(w) > \Upsilon(v) - (\varepsilon/\lambda)\|u - w\|, \quad \forall \, w \neq v.
\]

Up to a subsequence, it follows from (4.1) that we can choose $u = v_n$, $\varepsilon = 1/n^2$ and $\lambda = 1/n$ to get $\hat{v}_n \in S_A$, satisfying
\[
(4.3) \quad \Psi(\hat{v}_n) \to c_S, \quad \|v_n - \hat{v}_n\| \to 0
\]
and
\[
(4.4) \quad \Upsilon(w) > \Upsilon(\hat{v}_n) - (1/n)\|\hat{v}_n - w\|, \quad \forall \, w \neq \hat{v}_n.
\]

Let $\gamma_n : (-\delta_n, \delta_n) \to S_A$ be a differentiable curve, with $\delta_n > 0$ small enough, such that $\gamma_n(0) = \hat{v}_n$ and $\gamma_n(0) = z \in T_{\hat{v}_n}(S_A)$. Choosing $w = \gamma_n(t)$, it follows from (4.4) that
\[
(4.5) \quad - \left[\Psi(\gamma_n(t)) - \Psi(\gamma_n(0))\right] < (1/n)\|\gamma_n(t) - \gamma_n(0)\|.
\]
By the Mean Value Theorem, there exists $c \in \mathbb{R}$ between 0 and $t$, such that
\begin{equation}
\|\gamma_n(t) - \gamma_n(0)\| \leq \|\gamma_n'(c)\|t.
\end{equation}
Thus, multiplying both sides of (4.5) by $1/t$, passing to the limit of $t \to 0$ and using (4.6), we get
\[-\Psi'(\hat{v}_n)z \leq \frac{1}{n} \|z\|,
\]
where $z \in T_{\hat{v}_n}(\mathcal{S}_A)$ is arbitrary. By linearity, we have
\[|\Psi'(\hat{v}_n)z| \leq \frac{1}{n} \|z\|.
\]
Therefore,
\begin{equation}
\|\Psi'(\hat{v}_n)\|_* \to 0,
\end{equation}
as $n \to \infty$, and, by (4.3), we conclude that $\{v_n\}$ is a $(PS)_{c_N}$ sequence for $\Psi$. It follows from Lemma 3.7 and Proposition 3.8 that there exists $v \in \mathcal{S}_A$ such that, passing to a subsequence, $v_n \to v$ in $H^1_0(\Omega)$. Thus $\Psi'(v) = 0$ and $\Psi(v) = c_N$. Defining $u := m(v) \in \mathcal{N}$ and using Proposition 3.4(iv), we conclude that $I'(u) = 0$ and $I(u) = c_N$.

To show that $u$ does not change sign, observe that if $u^\pm \neq 0$, then $u^\pm \in \mathcal{N}$ (to verify it just calculate $I'(u)u^\pm$). Then
\begin{equation}
c_N = I(u) = I(u^+) + I(u^-) \geq 2c_N,
\end{equation}
which is a clear contradiction. Then it follows that either $u^+ = 0$ or $u^- = 0$ and then $u$ is a signed solution.

\section{5. Multiplicity of solutions}

The main goal of this section is to prove that the problem (P) has as many pairs of solutions as we want, provided that the continuous function $f$ is odd with respect to the second variable and $\alpha_\infty$ is large enough. For this, we are assuming throughout this section that $f$ is odd with respect to the second variable and
\begin{equation}
(f_2)_m
\end{equation}
\[0 \leq \alpha_0 := \lim_{t \to 0} f(x,t)/|t| < \lambda_1 < \lambda_m < \alpha_\infty := \lim_{|t| \to \infty} f(x,t)/|t|, \text{ a.e. } \Omega,
\]
for some $m \geq 2$. Our results will be proven through Krasnoselski’s genus theory. Thus, we start defining some preliminaries notations:
\[\gamma_j := \{B \in \mathcal{E} : B \subset \mathcal{S}_A \text{ and } \gamma(B) \geq j\},
\]
where
\[\mathcal{E} = \{B \subset H^1_0(\Omega) \setminus \{0\} : B \text{ is closed and } B = -B\}
\]
and $\gamma : \mathcal{E} \to \mathbb{Z} \cup \{\infty\}$ is the Krasnoselski’s genus function, which is defined by
\begin{equation}
\gamma(B) = \begin{cases} 
n := \min\{m \in \mathbb{N} : \text{ there is an odd map } \varphi \in C(B, \mathbb{R}^m \setminus \{0\})\}, \\
\infty, \text{ if there is no map } \varphi \in C(B, \mathbb{R}^m \setminus \{0\})\}, \\
0, \text{ if } B = \emptyset.
\end{cases}
\end{equation}
It is important to note that, since $\mathcal{S}_A = -\mathcal{S}_A$, $\gamma_j$ is well defined.

In the sequel we will state some standard properties of the genus which will be used in this work. More information about this subject can be found, for instance, in [3] or [11].

\begin{lemma}
Let $B$ and $C$ be sets in $\mathcal{E}$.
\end{lemma}
(i) If \( x \neq 0 \), then \( \gamma(\{x\} \cup \{-x\}) = 1 \);
(ii) If there exists an odd map \( \varphi \in C(B, C) \), then \( \gamma(B) \leq \gamma(C) \). In particular, if \( B \subset C \) then \( \gamma(B) \leq \gamma(C) \).
(iii) If there exists an odd homeomorphism \( \varphi : B \rightarrow C \), then \( \gamma(B) = \gamma(C) \). In particular, if \( B \) is homeomorphic to the unit sphere in \( \mathbb{R}^n \), then \( \gamma(B) = n \).
(iv) If \( B \) is a compact set, then there exists a neighborhood \( K \in \mathcal{E} \) of \( B \) such that \( \gamma(B) = \gamma(K) \).
(v) If \( \gamma(C) < \infty \), then \( \overline{\gamma(B \setminus C)} \geq \gamma(B) - \gamma(C) \).
(vi) If \( \gamma(A) \geq 2 \), then \( A \) has infinitely many points.

From now on, we denote by \( s_m \) the sum of the dimensions of all eigenspaces \( V_j \) associated to eigenvalues \( \lambda_j \), where \( 1 \leq j \leq m \).

**Lemma 5.2.** Suppose \( (f_2)_m \) holds.

(i) \( \gamma_{s_m} \neq \emptyset \);
(ii) \( \gamma_1 \supset \gamma_2 \supset \ldots \supset \gamma_{s_m} \);
(iii) If \( \varphi \in C(S_A, S_A) \) and is odd, then \( \varphi(\gamma_j) \subset \gamma_j \), for all \( 1 \leq j \leq s_m \);
(iv) If \( B \in \gamma_j \) and \( C \in \mathcal{E} \) with \( \gamma(C) \leq s < j \leq s_m \), then \( \overline{B \setminus C} \in \gamma_{j-s} \).

Proof. (i) Let \( S_{s_m} \) be the \( (s_m \)-dimensional) unit sphere of \( V_1 \oplus V_2 \oplus \ldots \oplus V_m \). From \( (f_2)_m \), it is clear that \( S_{s_m} \subset S_A \). Moreover, from Lemma 5.1(iii), we have \( \gamma(S_{s_m}) = s_m \). Showing that \( S_{s_m} \in \gamma_{s_m} \). (ii) It is immediate. (iii) Follows directly from Lemma 5.1(ii). (iv) It is a consequence of Lemma 5.1(v).

Now, for each \( 1 \leq j \leq s_m \), we define the following minimax levels

\[
c_j = \inf_{\substack{B \in \gamma_j \\, \sup \Psi(u)}} \sup_{u \in B} \Psi(u)
\]

**Lemma 5.3.** Suppose \( (f_1) - (f_2)_m \) hold. Then,

\[
0 < c_{N'} = c_1 \leq c_2 \leq \ldots \leq c_{s_m} < \beta|\Omega|.
\]

Proof. The first inequality follows from Lemma 3.3 (see also Remark 3). The equality \( c_{N'} = c_1 \) can be easily derivated from Lemma 5.1(i) and from definition of \( c_1 \). On the other hand, the monotonicity of the levels \( c_j \) is a consequence of Lemma 5.2(ii). Finally, choosing \( B = S_{s_m} \), there exists \( w \in B \) such that

\[
c_{s_m} \leq \sup_{u \in S_{s_m}} \Psi(u) = \Psi(w) = \int_{\Omega} \left[ \frac{1}{2} f(m(w))m(w) - F(m(w)) \right] \, dx < \beta|\Omega|.
\]

the result follows.

Next proposition is crucial to ensure the multiplicity of solutions.

**Proposition 5.4.** Suppose that \( f \) satisfies \( (f_1) - (f_2)_m \). If \( c_j = \ldots = c_{j+p} \equiv c \), \( j + p \leq s_m \), then \( \gamma(K_c) \geq p + 1 \), where \( K_c := \{ v \in S_A : \Psi(v) = c \} \) and \( \Psi'(v) = 0 \).

Proof. Suppose that \( \gamma(K_c) \leq p \). It follows from Proposition 3.8 and Lemma 5.3 that \( K_c \) is a compact set. Thus, by Lemma 5.1(iv), there is a compact neighborhood \( K \) (in \( H_0^1(\Omega) \)) of \( K_c \) such that \( \gamma(K) \leq p \). Defining \( M := K \cap S_A \), we derive from Lemma 5.1(ii) that \( \gamma(M) \leq p \). Despite the noncompleteness of \( S_A \) we can yet using Theorem 3.11 in [16] (see also Remark 3.12 in [16]) to ensure the existence of an odd homeomorphisms family \( \eta(., t) \) of \( S_A \) such that, for each \( u \in S_A \), the map

\[
t \mapsto \Psi(\eta(u, t))
\]

is non-increasing.
Observe that, although $S_A$ is non-complete, from Proposition 3.6, (5.3) and $\Psi(u) < \beta|\Omega|$ for all $u \in S_A$, the maps $t \mapsto \eta(u, t)$ are well defined in $t \in [0, \infty)$. Consequently, it makes sense the third claim of Theorem 3.11 in [16], namely,

\begin{equation}
\eta(\Psi_{c+\varepsilon}|M, 1) \subset \Psi_{c-\varepsilon}.
\end{equation}

Let us choose $B \in \gamma_{j+p}$ such that $\sup_B \Psi \leq c + \varepsilon$. From Lemma 5.2(iv), $\overline{B \setminus M} \in \gamma_j$. It follows again from Lemma 5.2(iii) that $\eta(\overline{B \setminus M}, 1) \in \gamma_j$. Therefore, from (5.4) and the definition of $c$, we have

\begin{equation*}
c \leq \sup_{\eta(\overline{B \setminus M}, 1)} \Psi \leq c - \varepsilon,
\end{equation*}

that is a contradiction. Then $\gamma(K_c) \geq p + 1$. \hfill \Box

We are now ready to prove the following multiplicity result:

**Theorem 5.5.** Suppose that $f(x, \cdot)$ is odd a.e. $x \in \Omega$ and satisfies $(f_1) - (f_2)_m$. Then, problem (P) has at least $s_m$ pairs of nontrivial solutions, where $s_m = 1 + \sum_{j=2}^{m} d_j$ is the sum of the dimensions $d_j$ of the first $m$ eigenspaces $V_j$ associated to $(-\Delta, H^1_0(\Omega))$.

**Proof.** First of all, note that the levels $0 < c_j < \infty$ are critical levels of $\Psi$. In fact, suppose by contradiction that $c_j$ is regular for some $j$. Invoking Theorem 3.11 in [16], with $\beta = c_j$, $\varepsilon = 1$, $N = \emptyset$, there are $\varepsilon > 0$ and a family of odd homeomorphisms $\eta(\cdot, t)$ satisfying the properties of referred theorem. Choosing $B \in \gamma_j$ such that $\sup_B \Psi < c_j + \varepsilon$ and arguing as in the proof of Proposition 5.4 we get a contradiction.

Finally, if the levels $c_j$, $1 \leq j \leq s_m$, are different from each other, it follows from Proposition 3.4(iv) that the result is proved. On the other hand, if $c_j = c_{j+1}$ for some $1 \leq j \leq s_m$, it follows from Proposition 5.4 that $\gamma(K_c) \geq 2$. Combining last inequality with Lemma 5.1(vi) and Proposition 3.4(iv), we conclude that (P) has infinitely many pairs of nontrivial solutions. \hfill \Box

### References

[1] S. Ahmad, Multiple non-trivial solutions of resonant and non-resonant asymptotically linear problems, *Proc. Amer. Math. Soc.*, 96(1987),401-409.

[2] A. Ambrosetti, Differential equations with multiple solutions and nonlinear functional analysis, *Equadiff*. 1982; LN in Math.

[3] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Functional Analysis*, 14 (1973) 349-381.

[4] H. Amann and E. Zehnder, Nontrivial solutions for a class of non resonant problems and applications to nonlinear differential equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, (4)7(1980)539-603.

[5] P. Bartolo, V. Benti and D. Fortunato, Abstract critical point theorems and applications to some nonlinear problems with strong” resonance at infinity, *Nonlinear Anal.*, (9)7(1983)981-1012.

[6] S. Cheng, Eigenfunctions and nodal sets, *Comment. Math. Helvetici*, 51(1976)43-55.

[7] D. G. Costa and C. A. Magalhães, Variational elliptic problems which are nonquadratic at infinity, *Nonlin. Analysis*, 23 (1994) 1401-1412.

[8] E. N. Dancer, Degenerate critical points, homotopy indices and Morse inequalities, *J. Reine Angew. Math.*, 350(1984)1-22.

[9] D. G. de Figueiredo and O. H. Miyagaki, Semilinear elliptic equations with the primitive of the nonlinearity away from the spectrum, *Nonlinear Anal.*, 17, (1991)1201-1219.

[10] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.*, 47 (1974)324-353.

[11] M. A. Krasnoselskii, Topological methods in the theory of nonlinear integral equations, *MacMillan*, New York, 1964.

[12] S. J. Li and M. Willem, Applications of local linking to critical point theory, *J. Math. Anal. Appl.*, 189(1995)6-32.

[13] S. Li and M. Willem, Multiple solutions for asymptotically linear boundary value problems in which the nonlinearity crosses at least one eigenvalue, *NoDEA*, 5(1998)479-490.
[14] J. Q. Liu and W. Zou, Multiple Solutions for Resonant Elliptic Equations via Local Linking Theory and Morse Theory, *Journal of Differential Equations*, **170**(2001)68 -95. 2
[15] M. Struwe, Infinitely many critical points for functionals which are not even and applications to nonlinear BVP, *Manuscripta Math.*, **32**(1982)753-770. 2
[16] M. Struwe, Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems. *Springer*, 2008. 13, 14
[17] J. Su, Multiplicity results for asymptotically linear elliptic problems at resonance, *J. Math. Anal. Appl.*, **278**(2003)397-408. 2
[18] J. Su and L. Zhao, An elliptic resonance problem with multiple solutions, *J. Math. Anal. Appl.*, **319**(2006)604-616. 2
[19] A. Szulkin and T. Weth, The method of Nehari manifold, *Handbook of Nonconvex Analysis and Applications*. D.Y. Gao and D. Montreanu eds., International Press, Boston, 2010, 597-632. 2, 7
[20] M. Willem, Minimax Theorems. *Birkhauser*, 1996. 7

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