Universality of the Nonphononic Vibrational Spectrum across Different Classes of Computer Glasses

Richard, D.; González-López, K.; Kapteijns, G.; Pater, R.; Vaknin, T.; Bouchbinder, E.; Lerner, E.

DOI
10.1103/PhysRevLett.125.085502

Publication date
2020

Document Version
Final published version

Published in
Physical Review Letters

Citation for published version (APA):
Richard, D., González-López, K., Kapteijns, G., Pater, R., Vaknin, T., Bouchbinder, E., & Lerner, E. (2020). Universality of the Nonphononic Vibrational Spectrum across Different Classes of Computer Glasses. Physical Review Letters, 125(8), Article 085502. https://doi.org/10.1103/PhysRevLett.125.085502
Universality of the Nonphononic Vibrational Spectrum across Different Classes of Computer Glasses

David Richard,1,* Karina González-López,1,* Geert Kapteijns,1 Robert Pater,1 Talya Vaknin,2 Eran Bouchbinder,2 and Edan Lerner1
1Institute for Theoretical Physics, University of Amsterdam, Science Park 904, 1098 XH Amsterdam, The Netherlands
2Chemical and Biological Physics Department, Weizmann Institute of Science, Rehovot 7610001, Israel

(Received 12 May 2020; revised 9 July 2020; accepted 21 July 2020; published 19 August 2020)

It has been recently established that the low-frequency spectrum of simple computer glass models is populated by soft, quasilocalized nonphononic vibrational modes whose frequencies \( \omega \) follow a gapless, universal distribution \( D(\omega) \sim \omega^4 \). While this universal nonphononic spectrum has been shown to be robust to varying the glass history and spatial dimension, it has so far only been observed in simple computer glasses featuring radially symmetric, pairwise interaction potentials. Consequently, the relevance of the universality of nonphononic spectra seen in simple computer glasses to realistic laboratory glasses remains unclear. Here, we demonstrate the emergence of the universal \( \omega^4 \) nonphononic spectrum in a broad variety of realistic computer glass models, ranging from tetrahedral network glasses with three-body interactions, through molecular glasses and glassy polymers, to bulk metallic glasses. Taken together with previous observations, our results indicate that the low-frequency nonphononic vibrational spectrum of any glassy solid quenched from a melt features the universal \( \omega^4 \) law, independently of the nature of its microscopic interactions.

DOI: 10.1103/PhysRevLett.125.085502

Introduction.—It is common in condensed matter physics that dynamic and thermodynamic phenomena are controlled by low-energy excitations \([1,2]\). For example, in crystalline solids, phonon-phonon interactions control wave attenuation rates and heat transport \([3]\); dislocations (i.e., low-energy topological defects) mediate plastic deformation rates upon external mechanical loading \([4]\); the specific heat grows as the third power of temperature due to the \( \sim \omega^2 \) Debye distribution of phonon frequencies. The same principle is also seen to hold in glassy solids, in which soft two-level systems, and their interactions with phonons, are believed to control thermodynamic and transport properties below 10 K \([5–7]\), and low-energy, quasilocalized excitations—often referred to as shear transformation zones \([8]\)—govern elastoplastic responses \([9]\). Consequently, the complete understanding of the statistical mechanics of soft excitations in solids, and in particular in glasses, is of key importance.

Indeed, much attention has been devoted in the past few decades to understanding the low-frequency spectra of glassy solids \([10–28]\). It is now well accepted that soft, quasilocalized modes dwell at vanishing frequencies \( \omega \to 0 \) in simple computer glasses. These nonphononic excitations were shown to universally feature a disordered core of linear size of about ten particle diameters \([25]\) (see examples in Fig. 1), decorated with algebraically decaying (mostly affine) displacement fields of magnitude \( \sim r^{-(d-1)} \) at distance \( r \) away from the core, in \( d \) spatial dimensions. The frequencies associated with these excitations were shown to follow a universal distribution \( D(\omega) \sim \omega^4 \) \([22]\), independent of spatial dimension \([23]\) or depth of supercooling prior to glass formation \([24,25,27]\). While these numerical observations are supported by various theoretical frameworks

FIG. 1. In this work we study five realistic glass-forming models, each representing a different class of disordered solids, as illustrated by the cartoons. Visualizations of quasilocalized modes found in the employed models of (a) an elastic-spheres glass, (b) a network glass, (c) a molecular glass, (d) a polymer glass, and (e) a bulk metallic glass. For visualization purposes, only the largest 1% of components are shown.
Weber network glass model, which employs a three-body term in the potential energy that favors tetrahedral local structures. In some range of its parameters, this model mimics the behavior of amorphous silicon. We refer to this model as SW. (3) A triatomic molecular glass model of soft beads connected by finite extensibility nonlinear elastic nonlinear springs, referred to in the literature as the embedded-atom method (EAM), which gives rise to a spherically symmetric, many-body potential. (4) A binary bulk metallic glass (BMG) alloy composed of Copper (Cu) and Zirconium (Zr) atoms according to the Stillinger-Weber network glass model, which employs a three-body term in the potential energy that favors tetrahedral local structures. (5) A molecular glasses, and bulk metallic glasses (BMGs), which are considerably more realistic representatives of laboratory glasses compared to the simple computer-glass models investigated previously, in order to test whether the universal nonphononic spectrum observed in simple computer glasses remains relevant to laboratory glasses as well. Our main finding is that these realistic glass-forming models also feature the universal $D(\omega) \sim \omega^4$ nonphononic vibrational density of states (VDOS), as seen in simple computer glasses. We thus extend the degree of universality of the $\omega^4$ law, and lend substantial support to the assertion that any glass formed by quenching a melt—and, in particular, laboratory glasses—would feature the gapless $\omega^4$ nonphononic VDOS.

Computer glass models.—We employ five computer glass models, each representing a different class of glassy solids. Here, we briefly review the employed models, keeping a complete description for [30]. Throughout this work we express frequencies in terms of $c_s/a_0$, where $c_s$ is the shear wave speed, and $a_0$ is the typical interparticle distance, both are precisely defined in [30].

The employed models are as follows: (1) An elastic-spheres glass model in which spherical particles interact via the linear-elastic Hertz contact law. At low confining pressures, this model undergoes an unjamming transition [48–50]. We refer to this model as HRZ. (2) The Stillinger-Weber network glass model, which employs a three-body term in the potential energy that favors tetrahedral local structures. In some range of its parameters, this model mimics the behavior of amorphous silicon. We refer to this model as SW. (3) A triatomic molecular glass model inspired by glass-forming models of orthotriphenyl [43,52], referred to in what follows as OTP. (4) A polymer-glass model of soft beads connected by finite extensible nonlinear elastic nonlinear springs [53], referred to in what follows as PG. Monomers between different polymers interact with a Lennard-Jones-like potential. (5) A binary bulk metallic glass (BMG) alloy composed of Copper (Cu) and Zirconium (Zr) atoms according to the embedded-atom method (EAM), which gives rise to a spherically symmetric, many-body potential.

Detailed descriptions about how ensembles of glassy samples were created for each computer glass model are provided in [30]. Briefly described, we generate uncorrelated equilibrium configurations at temperatures much larger than $T_g$, and perform an energy minimization on those configurations to obtain zero-temperature glassy solids.

For each generated glassy sample, we perform a normal mode analysis, which follows from a generalized eigenvalue problem: eigenvectors $\psi$ and eigenfrequencies $\omega$ satisfy the equation

$$\sum_j M_{ij} \psi_j = m_i \omega^2 \psi_i,$$

(1)

where $m_i$ denotes the mass of the $i$th particle, the Hessian matrix reads $M_{ij} = (\partial^2 U/\partial \mathbf{x}_i \partial \mathbf{x}_j)$, where $U$ denotes the potential energy and $\mathbf{x}_j$ is the $d$-dimensional coordinate vector of the $i$th particle, and $\psi_j$ is the $d$-dimensional Cartesian displacement vector of the $j$th particle. Note that no summation over $i$ is implied on the right-hand side of Eq. (1). To obtain the eigenvectors $\psi$ and eigenfrequencies $\omega$, we solve the auxiliary eigenvalue problem

$$\sum_j \frac{M_{ij}}{\sqrt{m_i m_j}} \phi_j = \omega^2 \phi_i,$$

(2)

where $\psi_i = \phi_i/\sqrt{m_i}$. Details about the calculation of $M$ for the SW and BMG systems are provided at length in [30]. The system and ensemble sizes in our simulations were selected such that the lowest-frequency modes appear below the first phononic band, as explained in detail in [22,54].

Results.—Our key result is displayed in Figs. 2(a)–2(e), where we show the low-frequency regime of the VDOS of all simulated computer glasses. All models feature the universal form $D(\omega) \sim \omega^4$, despite the stark qualitative differences between the microscopic interaction laws that define each model.

A quantitative comparison of the localization properties of quasilocalized modes between our various computer models is made possible by studying those modes’ participation ratio

$$e \equiv \frac{(\sum_i \psi_i \cdot \psi_i)^2}{N\sum_i (\psi_i \cdot \psi_i)^2},$$

(3)

where $\psi_i$ denotes the $d$-dimensional vector of a mode’s Cartesian components pertaining to the $i$th particle. The participation ratio is expected to scale as $1/N$ for localized modes, and should be of order unity for extended modes (e.g., phonons). The product $Ne$ is thus expected to reflect the core size of quasilocalized modes, expressed in terms of the characteristic volume occupied by a single particle.

In Figs. 2(f)–2(j) we show the mean participation ratio $\bar{e}$ of vibrational modes, scaled by system size $N$, binned over and plotted against frequency for all employed computer glasses. The first phonon band frequency $2\pi c_s/L$ is
indicated by the vertical dashed lines, and features $N_{\varepsilon}$ of order of a few thousands, consistent with the system sizes employed. Approaching zero frequency, we see that $N\bar{\varepsilon}$ plateaus at a typical value $N_{\varepsilon}^0$ on the order of a few tens, as marked by the horizontal dashed lines. The estimated values of the plateaus $N_{\varepsilon}^0$ are reported for all investigated computer glass models in Fig. 3(a). Remarkably, the variation of $N_{\varepsilon}^0$ across the different models is very small, of less than a factor of 2 with respect to each other.

Finally, we note that the prefactor $A_g$ of the nonphononic VDOS, namely $D(\omega) = A_g \omega^4$, is an observable with dimensions of an inverse frequency to the fifth power. $A_g$ was discussed at length in [24,25], where it was argued to encompass information both about the number density of soft, quasilocalized modes, and about their characteristic stiffness. In those references it was shown that $A_g$ can be very sensitive to glass history, particularly for glasses that were deeply supercooled prior to their quench to the glass. Here, we compare $A_g \equiv A_g/(a_0/c_s)^5$ across our different computer glasses. The results are displayed in Fig. 3(b); we find that $A_g$ is of order unity in all models, with the exception of the SW network glass model that features $A_g \approx 0.25$.

We note that the quantities $N_{\varepsilon}^0$ and $A_g$ generally depend on glass history [24,25,56]. However, these dependencies are most pronounced for glasses quenched from deeply supercooled liquids, and are generally weak or entirely absent for glasses quenched from high temperature liquid states [24,25,56]. Since in this work we indeed compare glasses quenched from high temperature liquid states (much higher than the computer glass transition temperature), the history dependence of $N_{\varepsilon}^0$ and $A_g$ is expected to be weak or absent, as we demonstrate explicitly in Fig. S2 of [30]. Consequently, our comparison between these observables across different classes of glass-forming models is meaningful. We conclude that the energy landscapes of the computer glasses we investigate here share quantitative similarities that extend beyond the universal scaling of their nonphononic VDOS.

**Summary and outlook.**—In this work we have shown that the low-frequency nonphononic spectra of realistic computer glass models—including network glasses, polymer glasses, and molecular glasses—feature the universal gapless $\omega^4$ law, as seen previously in simple computer glass models [22–27]. We thus expand the degree of universality of the $\omega^4$ law to include several qualitatively different classes of realistic glass-forming models, and reinforce its

---

**FIG. 2.** Low-frequency vibrational modes’ spectra and localization properties, measured in realistic computer glass models. We show $D(\omega)$ vs frequency $\omega$ for a linear-elastic-spheres glass [HRZ, panel (a)], a network glass [SW, panel (b)], a molecular glass [OTP, panel (c)], a polymer glass [PG, panel (d)], and a bulk metallic glass [BMG, panel (e)]. The solid lines indicate $D(\omega) \sim \omega^4$. Panels (f)–(j) show the average participation ratio $\bar{\varepsilon}$ [see definition in Eq. (3)], scaled by the number of particles $N$, binned over and plotted against frequency, for the same models of panels (a)–(e), respectively. The vertical dashed lines mark the first phonon band frequency $2\pi c_s/L$. The horizontal lines represent estimations $N_{\varepsilon}^0$ of the low-frequency plateau, which capture the core size of soft quasilocalized modes, see values reported in Fig. 3(a) and text for further discussions.

---

**FIG. 3.** Dimensionless characterizers of the nonphononic VDOS, compared across different classes of glassy solids. (a) Low-frequency plateau $N_{\varepsilon}^0$ of the frequency-binned participation ratio $\bar{\varepsilon}$ scaled by $N$, which represents the core size of soft, quasilocalized modes. (b) Dimensionless prefactors $A_g \equiv A_g/(a_0/c_s)^5$ of the universal $D(\omega) = A_g \omega^4$ nonphononic VDOS.
relevance to laboratory glasses. Finally, our results support the description of glasses’ vibrational properties via mesoscopic, coarse-grained approaches that consider interacting oscillators and anharmonicities [11,12], in which the microscopic details play no role in determining the scaling with frequency of the nonphononic VDOS.

Our results underline the timeliness of formulating a first-principles theory that explains the observed universality of nonphononic spectra in glassy solids. Mean-field approaches that are based on a microscopic description [19–21] (rather than a coarse-grained one) predict that the nonphononic VDOS of glassy solids should scale as $\omega^2$, independent of spatial dimension. An important goal for future studies will be to consolidate the predictions of the mesoscopic [11,12] and microscopic [19–21] theoretical approaches.

We wish to acknowledge inspiring discussions with Itamar Procaccia, Corrado Rainone, and Yuri Lubomirsky. D. R. acknowledges support of the Simons Foundation for the “Cracking the Glass Problem Collaboration” Grant No. 348126. K. G.-L. acknowledges the computer resources provided by the Laboratorio Nacional del Sureste de México, CONACYT member of the national laboratories network. E. L. acknowledges support from the NWO (Vidi Grant No. 680-47-554/3259). E. B. acknowledges support from the Minerva Foundation with funding from the German Federal Ministry for Education and Research, the Ben May Center for Chemical Theory and Computation, and the Harold Perlman Family.

Note added.—We note that after the completion of this work, we became aware of the results obtained by Bonfanti et al. [29], which support our conclusions.

These authors contributed equally to this work.

[1] C. Kittel, Introduction to Solid State Physics (Wiley, New York, 2005).
[2] A. M. Tsvelik, Quantum Field Theory in Condensed Matter Physics (Cambridge University Press, Cambridge, England, 2003).
[3] J. B. Ketterson, The Physics of Solids (Oxford University Press, Oxford, 2016).
[4] T. Suzuki, S. Takeuchi, and H. Yoshinaga, Dislocation Dynamics and Plasticity (Springer Science & Business Media, New York, 2013), Vol. 12.
[5] R. C. Zeller and R. O. Pohl, Thermal conductivity and specific heat of noncrystalline solids, Phys. Rev. B 4, 2029 (1971).
[6] P. W. Anderson, B. I. Halperin, and C. M. Varma, Anomalous low-temperature thermal properties of glasses and spin glasses, Philos. Mag. 25, 1 (1972).
[7] W. Phillips, Tunneling states in amorphous solids, J. Low Temp. Phys. 7, 351 (1972).
[8] M. L. Falk and J. S. Langer, Dynamics of viscoplastic deformation in amorphous solids, Phys. Rev. E 57, 7192 (1998).
[9] L. Gartner and E. Lerner, Nonlinear plastic modes in disordered solids, Phys. Rev. E 93, 011001(R) (2016).
[10] U. Buchenau, Y. L. Gurevich, and H. R. Schober, Anharmonic potentials and vibrational localization in glasses, Phys. Rev. B 43, 5039 (1991).
[11] V. L. Gurevich, D. A. Parshin, and H. R. Schober, Anharmonicity, vibrational instability, and the boson peak in glasses, Phys. Rev. B 67, 094203 (2003).
[12] D. A. Parshin, H. R. Schober, and V. L. Gurevich, Vibrational instability, two-level systems, and the boson peak in glasses, Phys. Rev. B 76, 064206 (2007).
[13] F. Leonforte, R. Boissière, A. Tanguy, J. P. Wittmer, and J.-L. Barrat, Continuum limit of amorphous elastic bodies. III. Three-dimensional systems, Phys. Rev. B 72, 224206 (2005).
[14] W. Schirmacher, G. Ruocco, and T. Scopigno, Acoustic Attenuation in Glasses and Its Relation with the Boson Peak, Phys. Rev. Lett. 98, 025501 (2007).
[15] G. Baldi, V. M. Giordano, G. Monaco, and B. Ruta, Sound Attenuation at Terahertz Frequencies and the Boson Peak of Vitreous Silica, Phys. Rev. Lett. 104, 195501 (2010).
[16] M. Wyatt, Scaling of phononic transport with connectivity in amorphous solids, Europhys. Lett. 89, 64001 (2010).
[17] L. Hong, V. N. Novikov, and A. P. Sokolov, Dynamic heterogeneities, boson peak, and activation volume in glass-forming liquids, Phys. Rev. E 83, 061508 (2011).
[18] A. I. Chumakov, G. Monaco, A. Monaco, W. A. Crichton, A. Bosak, R. Rüffer, A. Meyer, F. Kargl, L. Gomez, D. Fioretto, H. Giefer, S. Roitsch, G. Wortmann, M. H. Manghinni, A. Hushur, Q. Williams, J. Balogh, K. Parliński, P. Jochym, and P. Piekarz, Equivalence of the Boson Peak in Glasses to the Transverse Acoustic Van Hove Singularity in Crystals, Phys. Rev. Lett. 106, 225501 (2011).
[19] E. DeGiuli, A. Laversanne-Finot, G. During, E. Lerner, and M. Wyatt, Effects of coordination and pressure on sound attenuation, boson peak and elasticity in amorphous solids, Soft Matter 10, 5628 (2014).
[20] E. DeGiuli, E. Lerner, C. Brito, and M. Wyatt, Force distribution affects vibrational properties in hard-sphere glasses, Proc. Natl. Acad. Sci. U.S.A. 111, 17054 (2014).
[21] S. Franz, G. Parisi, P. Urbani, and F. Zamponi, Universal spectrum of normal modes in low-temperature glasses, Proc. Natl. Acad. Sci. U.S.A. 112, 14539 (2015).
[22] E. Lerner, G. Düring, and E. Bouchbinder, Statistics and Properties of Low-Frequency Vibrational Modes in Structural Glasses, Phys. Rev. Lett. 117, 035501 (2016).
[23] G. Kaperkijn, E. Bouchbinder, and E. Lerner, Universal Nonphononic Density of States in 2D, 3D, and 4D Glasses, Phys. Rev. Lett. 121, 055501 (2018).
[24] E. Lerner and E. Bouchbinder, A characteristic energy scale in glasses, J. Chem. Phys. 148, 214502 (2018).
[25] C. Rainone, E. Bouchbinder, and E. Lerner, Pinching a glass reveals key properties of its soft spots, Proc. Natl. Acad. Sci. U.S.A. 117, 5228 (2020).
[26] H. Mizuno, H. Shiba, and A. Ikeda, Continuum limit of the vibrational properties of amorphous solids, Proc. Natl. Acad. Sci. U.S.A. 114, E9767 (2017).
[27] L. Wang, A. Ninarello, P. Guan, L. Berthier, G. Szamel, and E. Flenner, Low-frequency vibrational modes of stable glasses, Nat. Commun. 10, 26 (2019).

[28] M. Shimada, H. Mizuno, M. Wyart, and A. Ikeda, Spatial structure of quasilocalized vibrations in nearly jammed amorphous solids, Phys. Rev. E 98, 060901(R) (2018).

[29] S. Bonfanti, R. Guerra, C. Mondal, I. Procaccia, and S. M. Shimada, H. Mizuno, M. Wyart, and A. Ikeda, Spatial structure of quasilocalized vibrations in nearly jammed amorphous solids, Phys. Rev. Lett. 125, 085501 (2020).

[30] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.125.085502 for a detailed description of the models and numerical methods employed in our work, and additional supporting data, which includes Refs. [24,25,31–47].

[31] H. R. Hertz, Über die Berührung Fester Elastischer Körper und über Die Härte (Universitätsbibliothek Johann Christian Senckenberg, Frankfurt, Germany, 2006).

[32] F. H. Stillinger and T. A. Weber, Computer simulation of local order in condensed phases of silicon, Phys. Rev. B 31, 5262 (1985).

[33] G. Kapteijns, W. Ji, C. Brito, M. Wyart, and E. Lerner, Fast generation of ultrastable computer glasses by minimization of an augmented potential energy, Phys. Rev. E 99, 012106 (2019).

[34] Y. Q. Cheng, E. Ma, and H. W. Sheng, Atomic Level Structure in Multicomponent Bulk Metallic Glass, Phys. Rev. Lett. 102, 245501 (2009).

[35] Y. Cheng and E. Ma, Atomic-level structure and structure-property relationship in metallic glasses, Prog. Mater. Sci. 56, 379 (2011).

[36] J. F. Lutsko, Generalized expressions for the calculation of elastic constants by computer simulation, J. Appl. Phys. 65, 2991 (1989).

[37] R. B. Lehoucq, D. C. Sorensen, and C. Yang, ARPACK Users’ Guide (Society for Industrial and Applied Mathematics, Philadelphia, 1998), https://epubs.siam.org/doi/abs/10.1137/1.9780898719628.ch3.

[38] D. MacKay, macopt optimizer (2004), http://www.inference .org.uk/mackay/c/macopt.html.

[39] V. V. Vasisht, S. Saw, and S. Sastry, Liquid–liquid critical point in supercooled silicon, Nat. Phys. 7, 549 (2011).

[40] T. Li, D. Donadio, G. Russo, and G. Galli, Homogeneous ice nucleation from supercooled water, Phys. Chem. Chem. Phys. 13, 19807 (2011).

[41] J. Russo, F. Romano, and H. Tanaka, Glass Forming Ability in Systems with Competing Orderings, Phys. Rev. X 8, 021040 (2018).

[42] S. Plimpton, Fast parallel algorithms for short-range molecular dynamics, J. Comput. Phys. 117, 1 (1995).

[43] L. J. Lewis and G. Wahnström, Relaxation of a molecular glass at intermediate times, Solid State Commun. 86, 295 (1993).

[44] H. J. C. Berendsen, J. P. M. Postma, W. F. van Gunsteren, A. DiNola, and J. R. Haak, Molecular dynamics with coupling to an external bath, J. Chem. Phys. 81, 3684 (1984).

[45] K. Kremer and G. S. Grest, Dynamics of entangled linear polymer melts: A molecular dynamics simulation, J. Chem. Phys. 92, 5057 (1990).

[46] J. D. Weeks, D. Chandler, and H. C. Andersen, Role of repulsive forces in determining the equilibrium structure of simple liquids, J. Chem. Phys. 54, 5237 (1971).

[47] Cu-Zr EAM potential (2011), https://sites.google.com/site/ campotentials/Home/CuZr.

[48] C. S. O’Hern, L. E. Silbert, A. J. Liu, and S. R. Nagel, Jamming at zero temperature and zero applied stress: The epitome of disorder, Phys. Rev. E 68, 011306 (2003).

[49] A. J. Liu and S. R. Nagel, The jamming transition and the marginally jammed solid, Annu. Rev. Condens. Matter Phys. 1, 347 (2010).

[50] M. van Hecke, Jamming of soft particles: Geometry, mechanics, scaling and isostaticity, J. Phys. Condens. Matter 22, 033101 (2010).

[51] V. Molinero and E. B. Moore, Water modeled as an intermediate element between carbon and silicon, J. Phys. Chem. B 113, 4008 (2009).

[52] S. Mossa, E. La Nave, H. E. Stanley, C. Donati, F. Sciortino, and P. Tartaglia, Dynamics and configurational entropy in the Lewis-Wahnström model for supercooled ortho-phenyl, Phys. Rev. E 65, 041205 (2002).

[53] F. W. Starr, T. B. Schroder, and S. C. Glotzer, Molecular dynamics simulation of a polymer melt with a nanoscopic particle, Macromolecules 35, 4481 (2002).

[54] E. Lerner, Finite-size effects in the nonphononic density of states in computer glasses, Phys. Rev. E 101, 032120 (2020).

[55] e ∼ 1/N also holds for quasilocalized modes in three or more dimensions, as shown in, e.g., [22].

[56] E. Lerner and E. Bouchbinder, Effect of instantaneous and continuous quenches on the density of vibrational modes in model glasses, Phys. Rev. E 96, 020104(R) (2017).