Coalescence of Two Impurities in a Trapped One-dimensional Bose Gas

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(Dated: June 10, 2022)

We study the ground state of a one-dimensional (1D) trapped Bose gas with two mobile impurity particles. To investigate this set-up, we develop a variational procedure in which the coordinates of the impurity particles are slow-like variables. We validate our method using the exact results obtained for small systems. Then, we discuss energies and pair densities for systems that contain of the order of one hundred atoms. We show that bosonic non-interacting impurities cluster. To explain this clustering, we calculate and discuss induced impurity-impurity potentials in a harmonic trap. Further, we compute the force between static impurities in a ring (a la the Casimir force), and contrast the two effective potentials: the one obtained from the mean-field approximation, and the one due to the one-phonon correction.

We consider a trapped \((2 + N_B)\)-body system, which consists of two impurity particles (mass \(m_I\)) with the coordinates \(\{x_i\}\), and \(N_B\) bosons (mass \(m_B\)) with the coordinates \(\{y_i\}\). The corresponding Hamiltonian is

\[
\mathcal{H} = \sum_{i=1}^{2} H_I(x_i) + \sum_{i=1}^{N_B} H_B(y_i) + g_{II} \sum_{j<k} \delta(x_j - x_k) + g_{IB} \sum_{i=1}^{N_B} \sum_{j=1}^{N_B} \delta(x_i - y_j) + g_{BB} \sum_{j<k} \delta(y_j - y_k),
\]

(1)

where \(H_{I(B)}(x) = -\frac{k^2}{2m_{I(B)}} \frac{\partial^2}{\partial x^2} + V_{I(B)}(x)\) is the one-body Hamiltonian, in which \(V_{I(B)}(x)\) is the external trap for the impurities (bosons). The impurity-impurity, boson-impurity and boson-boson interactions are zero-range with strengths \(g_{II}, g_{IB},\) and \(g_{BB}\), respectively. We focus on the \(g_{II} = 0\) case, which enjoys entirely emergent impurity-impurity correlations. Still, we investigate also the strongly-interacting case \((1/g_{II} = 0)\) as we consider fermionic impurities.

Modern experiments often have \(N_B \approx 100\); such mesoscopic set-ups are our primary interest. We develop a formalism built upon our earlier work on a Bose gas with one impurity \([27]\), adiabatic approximation \([28]\), and the so-called strong coupling approach \([29]\). We treat the Bose gas as one entity whose density profile quickly adapts to the motion of the impurity, i.e., we assume that \(x_1\) and \(x_2\) are slow variables and approximate the wave function

Doped quasi-one-dimensional cold Bose gases \([1, 4]\) is a modern platform for studying 1D polarons – quasiparticles that show up in a medium after renormalizing the impurity parameters \([5, 8]\). The following theoretical model is often used to analyze these systems: A homogeneous and infinite environment with one structureless \([9]\) mobile impurity particle. Its ground state properties have been understood using quantum Monte Carlo \([11, 12]\). Other approaches, such as the renormalization group \([12, 13]\), perturbation theory \([14]\), Bethe ansatz (see \([11, 15, 16]\) and references therein), various mean-field (and beyond) approximations \([12, 13, 17–20]\). Feynman’s variational method \([12, 20]\) give insight into the limiting cases. This model is widely used, however, care is needed when employing it to describe modern 1D cold-atom set-ups, which (often) have: i) more than one impurity, ii) inhomogeneous densities due to the external confinement, iii) a countable number of bosons (usually from a few to a few hundred \([21]\)). It is important to add these features to the model because they influence experimental data. For instance, impurity-impurity scattering is blamed for the observed \([2]\) damping of oscillations at vanishing boson-impurity interactions; a trap affects the dynamics at strong boson-impurity interactions \([12, 23, 24]\). The effect of the confinement can be qualitatively understood from the polaron model within the local density approximation \([12, 23]\). However, to comprehend the role of the impurity-impurity correlations, one must go beyond the polaron model, and investigate systems with at least two impurity particles. Such studies will determine the limits of applicability of the polaron model; moreover, they will guide future research of trapped systems with impurities, and of the corresponding quasiparticles.

We study the ground state of one such model: A weakly-interacting trapped Bose gas with two impurity particles. We argue that for weak boson-impurity interactions the impurities correlate weakly. The situation changes for strong boson-impurity interactions: The bosonic impurities lower their energy by sharing the same distortion of the gas. Unlike the homogeneous case, this impurity-impurity attraction does not require collective excitations and may be observed even in ideal Bose gases. This attraction can be described by the induced impurity-impurity potential that depends on the center-of-mass and relative coordinates.

### Formalism

We consider a trapped \((2 + N_B)\)-body system, which consists of two impurity particles (mass \(m_I\)) with the coordinates \(\{x_i\}\), and \(N_B\) bosons (mass \(m_B\)) with the coordinates \(\{y_i\}\). The corresponding Hamiltonian is

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\( \Psi(x_1, x_2, y_1, \ldots, y_{N_B}) \) by
\[
\Psi \simeq \phi(x_1, x_2) \Phi(y_1, \ldots, y_{N_B} | x_1, x_2),
\]
where \( \Phi \) is the ground state of \( \mathcal{H} \) with \( H_I = 0 \); in other words, \( \Phi \) describes the ground state of the Bose gas with the impurities fixed at \( x_1 \) and \( x_2 \). The corresponding energy is \( N_B \epsilon(x_1, x_2) \). Assuming weak boson-impurity interactions, we write the function \( \Phi \) as the product state: \( \Phi = \prod f(y_i | x_1, x_2) \), where \( f \) solves a variant of the Gross-Pitaevski equation \[30\]. The decomposition \((2)\) might appear similar to the widely used mean-field approximation, in particular, it entangles the \( \chi, \xi \), which is necessary for accurate results for \( g_{1B} \rightarrow \infty \). Furthermore, \( \chi, \xi \) is applicable for bosonic, fermionic, and distinguishable impurities.

Once \( f \) and \( \epsilon \) are calculated, we obtain the equation for \( \phi \) that minimizes the expectation value \( E \) of \( \mathcal{H} \[30\]
\[
\left(-\frac{\hbar^2}{2m_I} \frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m_I} \frac{\partial^2}{\partial x_2^2} + V_{\text{eff}}\right) \phi = E \phi.
\]
We refer to \( E \) as the energy, even though, according to the variational principle, the exact value of the energy is below \( E \). The effective potential \( V_{\text{eff}} \) describes the action of the external trap and the Bose gas on the impurities:
\[
V_{\text{eff}} = \sum_{i=1}^{2} \left(V_I + \frac{\hbar^2 N_B}{2m_I} \left(\left(\frac{\partial f(y | x_1, x_2)}{\partial x_i}\right)^2\right)_y\right) + N_B \epsilon.
\]

Equations \((3)\) and \((4)\) reduce the original \((2 + N_B)\)-body problem to a much simpler two-particle one, in which boson-impurity interactions are hidden in \( V_{\text{eff}} \). The Schrödinger equation \((3)\) can be solved using standard numerical routines; we solve it for the ground state \[38\] utilizing the finite difference approximation.

Note that numerical calculations and analytical analysis \[27\] showed that such the elimination of the bosonic degrees of freedom leads to accurate results for one trapped impurity (see also \[39\] \[41\]). Since the ideas put forward in Ref. \[27\] do not rely on the number of impurities, as long as \( N_B \) is large, it is natural to expect the decomposition \((2)\) to work well also for two trapped impurities. To verify this anticipation, below we compare to the exact ground state of \( \mathcal{H} \), which is computed using the numerical diagonalization method with effective interactions (EEDM) described in detail elsewhere \[39\] \[41\].

**Particles in harmonic traps.** Our first application are systems confined by the harmonic traps \( V_I(x) = m_I \omega^2 x^2/2 \) and \( V_B(y) = m_B \omega^2 y^2/2 \). For convenience, we introduce the length unit \( b = \sqrt{\hbar/(m_B \omega B)} \). For small samples the Hamiltonian \((1)\) can be diagonalized using EEDM. The comparison of the exact results with those of Eq. \((3)\) is shown in Fig. 1. The agreement between the energies is overall satisfactory. The density profiles are reproduced as well. They show that, to minimize the depletion of the condensate at \( g_{1B} \rightarrow \infty \), the impurities are driven out of the bosonic cloud. Similar behavior was observed in the one-impurity case \[12\] \[27\]. The density profile alone does not contain any information regarding the correlations between the impurity particles, thus, below we work with the pair density \( n(x_1, x_2) = \int |\Psi|^2 \, dy_1 \ldots dy_N \simeq |\phi(x_1, x_2)|^2 \), which determines the probability to find one impurity particle at \( x_1 \) if another is placed at \( x_2 \).

Having tested the formalism we turn to the 2 + 100 system with \( g_{BB} = 0 \), which is too complex for EEDM. We consider bosonic and fermionic impurities \[22\]; see Fig. 2, where two limits are clearly seen. When the interactions are weak, a simple picture of two non-interacting particles describes well the system; see Figs. \((2)\) \((b)\) and \((c)\). In this case the Bose gas can be treated simply as an external potential \[41\] for the impurities. For \( g_{1B} \rightarrow \infty \) the impurities move to the edge of the trap just as for \( N_B \leq 8 \); see Figs. \((2)\) \((b)\) and \((d)\). The pair density shows that the bosonic impurities cluster and move to the edge as a whole; see Fig. \((2)\) \((c)\). This behavior can be explained by an overall attractive impurity-impurity interaction potential (see below) due to the trap: Impurities in a ho-
Energy $\omega$ 

The “two-impurities energy” print in the energy domain. To unravel it, we calculate two fermionic impurities; see Fig. 2. This interaction, however, is not strong enough to bind two impurities out of the bosonic cloud. These correlations of fermions and we refrain from showing more on this case. Similarly to Fig. 2, the curves in Fig. 3 have distinct behaviors at small and large values $g_{1B}$. However, there are some noticeable changes: The energy difference $E_{diff}(g_{1B} \rightarrow \infty)$ grows with $g_{BB}$, the transition region from the weakly-to strongly-correlated regimes is shifted. The shift is intuitively clear: $g_{1B}$ should be larger than $g_{BB}$ to push the impurities out of the bosonic cloud. These correlations agree qualitatively with the phase separation of two harmonically-trapped Bose gases.

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The clustering demonstrated in Fig. 2 also happens for other values of $N_B$ and $g_{BB}$ [51]. We illustrate this for bosonic impurities in Fig. 3, the correlations of fermions seem to be driven mainly by the Pauli exclusion principle and we refrain from showing more on this case. Similarly to Fig. 2, the curves in Fig. 3 have distinct behaviors at small and large values $g_{1B}$. However, there are some noticeable changes: The energy difference $E_{diff}(g_{1B} \rightarrow \infty)$ grows with $g_{BB}$, the transition region from the weakly-to strongly-correlated regimes is shifted. The shift is intuitively clear: $g_{1B}$ should be larger than $g_{BB}$ to push the impurities out of the bosonic cloud. These correlations agree qualitatively with the phase separation of two harmonically-trapped Bose gases [19, 50].

The clustering may be detected in cold-atom systems. For example, by changing $g_{1B}$ from zero to some large
value: When the system reaches equilibrium, the probability to observe impurities together on one edge of the trap will be larger than in the non-correlated case. The energy \( E_{\text{diff}} \) defines the temperature window, in which this entanglement of impurities is important. For the considered cases, \( E_{\text{diff}} \) is of the order of \( \hbar \omega_B \) (see Fig. 3), which for typical values of \( \omega_B/(2\pi) = (0.1 - 1) \) kHz is \( \sim (4 - 40) mK \times k_B \), here \( k_B \) is the Boltzmann constant.

If \( g_{1B} \) is less than one, this number must become smaller; it is zero, if the impurities fermionize, i.e., \( 1/|g_{1B}| = 0 \).

Effective interaction. The bunching of bosonic impurities follows directly from the form of the effective potential \( V_{\text{eff}} \); see Fig. 4. Weak interactions lead to small modifications to \( V_I(x_1) + V_I(x_2) \). Larger values of \( g_{1B} \) drastically affect \( V_{\text{eff}} \), which now has minima at the edges of the bosonic cloud. The minima at the \( x_1 = x_2 \) line are the deepest leading to the coalescence of impurities. We stress again that \( V_{\text{eff}} \) incorporates the effects of both the trap and the Bose gas. For small trapped systems, it seems impossible to disentangle these forces (see also [37]) and obtain an effective impurity-impurity interaction from \( V_{\text{eff}} \) that depends only on \( |x_1 - x_2| \), which is expected in a homogeneous case, see, e.g., [55, 56].

A thorough study of homogeneous systems is beyond this paper’s scope. To connect our research to previous studies, we discuss \( V_{\text{eff}} \) for two static impurities trapped in a 1D ring-shape trap of length \( L \). This set-up allows us to exclude the trap effects, and to use the existing results on the Casimir-like forces [57, 58] for comparison. We focus on the thermodynamic limit [59], i.e., \( N_B(L) \to \infty \) and \( N_B/L \to \rho \); the length unit now is \( \tilde{b} = 1/\rho \). We calculate the quantity \( V_{\text{eff}}(|x_2 - x_1|) - V_{\text{eff}}(\infty) \), which determines the induced impurity-impurity interaction; see Figs. 4a-e. For comparison, we also plot the Yukawa potential (YP), which is expected [60] in the perturbative regime from the one-phonon exchange (see [30, 57, 63]). The graphs show that for \( g_{1B} \to 0 \) the YP describes well the long-range physics and should be slightly corrected at small values of \( |x_2 - x_1| \). For \( x_2 = x_1 \) the correction can be calculated analytically [30]. For large values of \( g_{1B} \) the YP fails, e.g., it predicts infinite attraction for \( g_{1B} \to \infty \). In this regime our method provides means for calculating emergent impurity-impurity potentials, which are not the YP. These potentials are attractive at large distances within our mean-field approach [30]. Note that tails of induced impurity-impurity interactions can in principle have repulsive regions if one includes quantum fluctuations [60], which are not considered in the present Letter.

The potential \( V_{\text{eff}} \) is generated by the mean-field transformation of the Bose gas’ density, hence, we shall refer to \( V_{\text{eff}} \) as the mean-field part of the effective force (motivated by the studies of thin films [64, 65]). In principle, \( V_{\text{eff}} \) can be measured by doing spectroscopy with two tightly trapped atoms [57, 60].

Within our formalism, effective potentials for mobile impurities look similar to the presented in Figs. 4a-e. They have negative net volume [67], hence, they support two-body bound states [69] even for \( g_{1B} \to 0 \).

For \( g_{1B} \to 0 \), though, the binding energy is small [70] and finite temperature will wipe out any emergent correlations. By increasing \( g_{1B} \) the bound state becomes “deeper” and can be observed. One might describe this limit as the dark-bright soliton (see [71]), which, in the present context, corresponds to a multi-polaron bound state that possesses some properties of a quasiparticle.

This work was supported by the Danish Council for Independent Research and the DFF Sapere Aude program. A. G. V. gratefully acknowledges the support of the Humboldt Foundation. We thank Manuel Valiente, Thomas Pohl, Fabio Cinti and Luis Ardila for enlightening discussions about the Casimir-like force. We thank Jens Braun for useful conversations and comments on the manuscript.

Note added: See the recent study in Ref. [72] for details on the contradicting results in the literature described in the footnote in Ref. [69].

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By assumption, the set of functions $\Phi_j$ reproduce very accurately the “one-impurity energy” in the thermodynamic limit [13] giving us another argument in support of the used approach.

Note that the Gross-Pitaevski equation reproduces very well the predictions regarding the form of the tail of the effective interaction for infinite homogeneous systems. Reference [56] uses the phonon Lagrangian with cubic terms to derive results regarding the form of the tail of the effective interaction for infinite homogeneous systems. Reference [56] uses the phonon Lagrangian with cubic terms to predict long-range interactions ($1/r^3$) for static and mobile impurities. This long-range interaction is in contradiction with our results, and the results that rely on the Bogoliobov and renormalization group approaches [57, 58, 59, 61, 62].

To resolve this puzzle one has to resort to ab-initio model-independent methods, e.g., quantum Monte Carlo. Note, however, that this puzzle is for homogeneous infinite systems whose physics differ significantly from that in homogeneous set-ups. For example, the induced interaction in our case is present even for two impurity particles in an ideal Bose gas.

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Technical details

Derivation of Eqs. (3) and (4)

For the system with two impurities fixed at the positions $x_1$ and $x_2$, a complete basis is given by the set of functions $\Phi_j(y_1, \ldots, y_{NB}|x_1, x_2)$ that solve the equation:

$$
\left( \sum_{i=1}^{NB} H_B(y_i) + g_{IB} \sum_{i=1}^{NB} \sum_{j=1}^{NB} \delta(x_i - y_j) + g_{BB} \sum_{i<j} \delta(y_i - y_j) \right) \Phi_j = E_j(x_1, x_2) \Phi_j, \quad (5)
$$

By assumption, the set of functions $\Phi_j$ is orthonormal in the $y$-space. Using this basis we write the wave function $\Psi$ for the system with two mobile impurities

$$
\Psi(x_1, x_2, y_1, \ldots, y_{NB}) = \sum_j \phi_j(x_1, x_2) \Phi_j(y_1, \ldots, y_{NB}|x_1, x_2), \quad (6)
$$

where the functions $\phi_j(x_1, x_2)$ define the decomposition. We normalize the function $\Psi$ to one, and write the corresponding expectation value as

$$
E = \sum_{i,j} \int dx_1 dx_2 \phi_j(x_1, x_2) \left( \sum_{l=1}^{2} \left( H_A(x_l) - P_{j,l}(x_1, x_2) \frac{\hbar^2}{m_l} \frac{\partial}{\partial x_l} + Q_{j,l}(x_1, x_2) \right) + E_j(x_1, x_2) \delta_{ij} \right) \phi_i(x_1, x_2), \quad (7)
$$
Therefore, we use only the lowest-energy solution of Eq. (5), which yields the boson-boson interaction is either vanishing or small, so that the (quasi) Bose-Einstein condensate plays the main role. We leave this investigation for future work.

Note that due to the normalization condition on $\Phi$, the coupling $P_{ii;l}$ is zero. In the main text we assume that the boson-boson interaction is either vanishing or small, so that the (quasi) Bose-Einstein condensate plays the main role. Therefore, we use only the lowest-energy solution of Eq. [5], which yields

$$E \approx \int dx_1 dx_2 \phi(x_1, x_2) \left( \sum_{l=1}^{2} (H_A(x_l) + Q_{00;l}(x_1, x_2)) + E_0(x_1, x_2) \right) \phi(x_1, x_2).$$

(10)

Here we have used the notation of the main text $\phi = \phi_0$. The value $E$ is a variational upper bound on the energy. It is minimized if $\phi$ is the ground state of the Hamiltonian

$$h = \sum_{l=1}^{2} [(H_A(x_l) + Q_{00;l}(x_1, x_2)) + E_0(x_1, x_2)].$$

(11)

To find $E_0$ and $Q_{00;l}$ we write $\Phi = \prod f(y_i|x_1, x_2)$, where $f(y_i|x_1, x_2)$ solves the Gross-Pitaevski equation that corresponds to the Schrödinger equation [5] (see the next section). We plot it in Fig. [5]. The function $E_0$ is then simply $N_B \epsilon(x_1, x_2)$, where $\epsilon$ is

$$\epsilon(x_1, x_2) = \int dy f(y|x_1, x_2) \left( H_B(y) + g_B \sum_{i=1}^{2} \delta(x_i - y) + \frac{g_{BB}}{2}(N_{BB} - 1) f(y|x_1, x_2)^2 \right) f(y|x_1, x_2);$$

(12)

the coupling $Q_{00;l}$ has the form

$$Q_{00;l}(x_1, x_2) = N_B \frac{\hbar^2}{2m_l} \int dy \left( \frac{\partial f(y|x_1, x_2)}{\partial x_l} \right)^2.$$

(13)

It is worthwhile noting that the predictive power of our approach can be improved if the couplings $P$ and $Q$ are included. For an ideal Bose gas in a harmonic trap, the function $\Phi$ can be found analytically. For a weakly-interacting Bose gas in a trap, the Bogoliubov approximation should be employed to obtain elementary excitations of the Gross-Pitaevski equation. We leave this investigation for future work.
Two static impurities in a Bose gas

The system with two static impurities within our formalism is described by the Gross-Pitaevski equation

\[- \frac{1}{2} \frac{d^2 f(y)}{dy^2} + g_{BB}(N_{BB} - 1)f(y)^3 + g_{1B}(\delta(x_1 - y) + \delta(x_2 - y))f(y) = \mu f(y), \tag{14}\]

where $\mu$ is the chemical potential, and for convenience, we set $\hbar = m_B = 1$. For simplicity, in what follows we assume that $x_1 = 0$ and $x_2 = r > 0$.

Weakly-interacting case. Let us start by considering a weakly-interacting case, i.e., $g_{1B} \to 0$, in the thermodynamic limit, i.e., $N_{BB} = \rho L \to \infty$. We assume that \( f = (1 + F)/\sqrt{\cal T}, \mu = \mu_0 + \mu_1, \) where $\mu_0 = \frac{g_{BB}(N_{BB} - 1)}{L}$; $F$ and $\mu_0$ solve the following equation

\[- \frac{1}{2} \frac{d^2 F_0}{dy^2} + \mu_0(3F^3 + 3F^2 + F^3) + g_{1B} (\delta(x_1 - y) + \delta(x_2 - y))(1 + F) = \mu_1 + \mu_0 F + \mu_1 F. \tag{15}\]

Since $g_{1B}$ is small we write $F = F_0 + F_1 + \ldots$, such that

\[- \frac{1}{2} \frac{d^2 F_0}{dy^2} + 2\mu_0 F_0 + g_{1B} (\delta(x_1 - y) + \delta(x_2 - y)) = \mu_1, \tag{16}\]

\[- \frac{1}{2} \frac{d^2 F_1}{dy^2} + 2\mu_0 F_1 + 3\mu_0 F_0^2 + g_{1B} (\delta(x_1 - y) + \delta(x_2 - y))F_0 = \mu_1 F_0, \tag{17}\]

\[\ldots. \tag{18}\]

We truncate this set of equations at $F_2$ and beyond, which will provide the energy in the $g_{1B}$ order. The truncated system can be easily solved. The function $F_0$ is

\[F_0(y) = \frac{\mu_1}{\mu_0} + \alpha_1 e^{-2\sqrt{\mu_0}y} + \beta_1 e^{2\sqrt{\mu_0}y}, \quad 0 < y < r, \tag{19}\]

\[F_0(y) = \frac{\mu_1}{\mu_0} + \alpha e^{-2\sqrt{\mu_0}y} + \beta e^{2\sqrt{\mu_0}y}, \quad r < y < L. \tag{20}\]

The coefficients $\alpha_1, \beta_1, \alpha$ and $\beta$ are obtained from

\[\lim_{\epsilon \to 0^+} F_0(\epsilon) = \lim_{\epsilon \to 0^+} F_0(L - \epsilon), \tag{21}\]

\[\lim_{\epsilon \to 0^+} F_0(r - \epsilon) = \lim_{\epsilon \to 0^+} F_0(r + \epsilon), \tag{22}\]

\[\lim_{\epsilon \to 0^+} \frac{dF_0}{dy}(\epsilon) - \lim_{\epsilon \to 0^+} \frac{dF_0}{dy}(L - \epsilon) = 2g_{1B}, \tag{23}\]

\[\lim_{\epsilon \to 0^+} \frac{dF_0}{dy}(r + \epsilon) - \lim_{\epsilon \to 0^+} \frac{dF_0}{dy}(r - \epsilon) = 2g_{1B}. \tag{24}\]

The function $F_1(y)$ reads

\[F_1(y) = -\frac{1}{8} \frac{\mu_1^2}{\mu_0^2} - 3\alpha_1 \beta_1 + (c_1 + \frac{\mu_1 \alpha_1}{\sqrt{\mu_0}}) e^{-2\sqrt{\mu_0}y} + (d_1 + \frac{\mu_1 \beta_1}{\sqrt{\mu_0}}) e^{2\sqrt{\mu_0}y} + \frac{a_1^2}{2} e^{-4\sqrt{\mu_0}y} + \frac{b_1^2}{2} e^{4\sqrt{\mu_0}y}, \quad 0 < y < r, \tag{25}\]

\[F_1(y) = -\frac{1}{8} \frac{\mu_1^2}{\mu_0^2} - 3\alpha \beta + (c + \frac{\mu_1 \alpha}{\sqrt{\mu_0}}) e^{-2\sqrt{\mu_0}y} + (d + \frac{\mu_1 \beta}{\sqrt{\mu_0}}) e^{2\sqrt{\mu_0}y} + \frac{a^2}{2} e^{-4\sqrt{\mu_0}y} + \frac{b^2}{2} e^{4\sqrt{\mu_0}y}, \quad r < y < L, \tag{26}\]

where $c_1, d_1, c$ and $d$ are determined from the boundary conditions

\[\lim_{\epsilon \to 0^+} F_1(\epsilon) = \lim_{\epsilon \to 0^+} F_1(L - \epsilon), \tag{27}\]

\[\lim_{\epsilon \to 0^+} F_1(r - \epsilon) = \lim_{\epsilon \to 0^+} F_1(r + \epsilon), \tag{28}\]

\[\lim_{\epsilon \to 0^+} \frac{dF_1}{dy}(\epsilon) - \lim_{\epsilon \to 0^+} \frac{dF_1}{dy}(L - \epsilon) = 2g_{1B} F_0(0), \tag{29}\]

\[\lim_{\epsilon \to 0^+} \frac{dF_1}{dy}(r + \epsilon) - \lim_{\epsilon \to 0^+} \frac{dF_1}{dy}(r - \epsilon) = 2g_{1B} F_0(r). \tag{30}\]
The parameter $\mu_1$, which defines the chemical potential, is determined from the normalization condition, i.e., $\int (1 + 2F_0 + F_0^2 + 2F_1) dy = L$:

$$\mu_1 = \frac{2g_{IB}}{L} - \frac{g_{IB}^2}{2L\sqrt{\mu_0}} + \frac{g_{IB}^2}{2L\sqrt{\mu_0}} e^{-2\sqrt{\mu_0}} (-1 + 2\sqrt{\mu_0}).$$

The corresponding “two-impurities energy” is

$$\mu_1 N_{BB} - \frac{\mu_0}{2L} N_{BB} \int_0^L (2F_0 + 5F_0^2 + 2F_1) dy = \frac{2g_{IB} N_{BB}}{L} - \frac{g_{IB}^2 N_{BB}}{L\sqrt{\mu_0}} - \frac{g_{IB}^2 e^{-2\sqrt{\mu_0}} N_{BB}}{L\sqrt{\mu_0}}.$$  

The first two terms define the energy of two non-correlated impurities up to the $g_{IB}^2$ order. The last term is the Yukawa potential, which defines the effective impurity-impurity interaction, cf. Ref. [57]. Note that this potential is accurate only if $g_{IB} \to 0$ defines the smallest energy scale of the problem.

**Analytical approach.** The Casimir-like force discussed in the paper can be obtained analytically within our model not only for $g_{IB} \to 0$, because the Gross-Pitaevski equation (14) is solvable on a ring of length $L$ (cf. Refs. [13] [31] [32]). The corresponding solution is

$$f(y) = \sqrt{\frac{4K(p_1)^2 p_1}{g_{BB} \delta_1^2 r^2 (N_{BB} - 1)^2}} \text{sn} \left( 2K(p_1) \left[ y - \frac{x}{\delta_1 r} + 1 \right] \frac{1}{2}, p_1 \right), \quad 0 \leq y \leq r,$$

$$f(y) = \sqrt{\frac{4K(p_2)^2 p_2}{g_{BB} \delta_2^2 (L - r)^2 (N_{BB} - 1)^2}} \text{sn} \left( 2K(p_2) \left[ y - \frac{L + x}{\delta_2 (L - r)} + 1 \right] \frac{1}{2}, p_2 \right), \quad r < y < L,$$

where $K(p)$ is the complete elliptic integral of the first kind, and $\text{sn}(x|p)$ is the Jacobi elliptic function [33]. The parameters $p_1, p_2, \delta_1$ and $\delta_2$ are determined from normalization and the boundary conditions, i.e.,

$$\int_0^L f(y)^2 dy = 1, \quad \lim_{\epsilon \to 0^+} f(r - \epsilon) = \lim_{\epsilon \to 0^+} f(r + \epsilon),$$

$$2g_{IB} f(r) = \lim_{\epsilon \to 0^+} \frac{df}{dy} \bigg|_{r+\epsilon} - \lim_{\epsilon \to 0^+} \frac{df}{dy} \bigg|_{r-\epsilon},$$

here $0^+$ means that we deal with one-sided limits. In addition, we have the condition from the chemical potential

$$\mu = \frac{2K(p_1)^2 (1 + p_1)}{\delta_1^2 r^2} = \frac{2K(p_2)^2 (1 + p_2)}{\delta_2^2 (L - r)^2}.$$  

Note that these quantities are available in the literature for a Bose gas with one impurity particle [13], which can be used to solve the problem with two impurities if $r = 0$ and $r \to \infty$. This provides us with the energy release $V_{eff}(0) - V_{eff}(\infty)$ for two static impurities. Indeed, the “one-impurity” energy is [13]

$$\varepsilon(g_{IB} = 0) = \frac{\rho^2}{3} \sqrt{\frac{g_{BB}}{\rho}} \left[ 4 + (-4 + \text{sech}^2(d)) \tanh(d) \right], \quad \text{where} \quad d = \frac{1}{2} \text{asinh} \left( \frac{2\rho}{g_{IB} \sqrt{g_{BB}}} \right),$$

here $\text{sech}(x)$, $\tanh(x)$ and $\text{asinh}(x)$ are standard hyperbolic functions. The energy release is then

$$V_{eff}(0) - V_{eff}(\infty) = \frac{\rho^2}{3} \sqrt{\frac{g_{BB}}{\rho}} \left[ -4 + (-4 + \text{sech}^2(D)) \tanh(D) - (-8 + 2\text{sech}^2(D)) \tanh(d) \right],$$

where $D = \frac{1}{2} \text{asinh} \left( \frac{\rho}{g_{IB} \sqrt{g_{BB}}} \right)$. The function in Eq. (40) is plotted in Fig. 6, which shows that the energy release is always negative, thus, two static impurities minimize the energy when they cluster; in the limit $g_{IB} \to \infty$ the value of $V_{eff}(0) - V_{eff}(\infty)$ is determined by the activation energy of one dark soliton, which is $-4\rho^{3/2} \sqrt{g_{BB}/3}$ (see below).

**Impenetrable impurity.** In general it is a difficult task to find the form of the potential. Here we show the derivations for impenetrable impurities, i.e., $g_{IB} \to \infty$, assuming that $r$ is much larger than all relevant [physical] length scales. Note that in the thermodynamic limit $p_1 = 1 - m_1$, $0 < m_i \ll 1$ for $i = 1, 2$ and from Eq. (38) we obtain

$$m_1 \simeq 16e^{-r_1 \sqrt{\pi}}, \quad m_2 \simeq 16e^{-(L-r)\delta_2 \sqrt{\pi}}.$$  


calculate the “two-impurities energy”

\[ \text{Eq. (47) describes the impurity-impurity correlations. To calculate the effective potential between impurities we} \]

\[ \text{The comparison of this result with the chemical potential of the} \]

\[ \text{Bose gas [34]. Therefore, we see that at} \]

\[ \text{FIG. 6. The energy release defined in Eq. (40) as a function of} \]

\[ \text{For impenetrable impurities,} \]

\[ \text{expressions in Eq. (45) to obtain the equation for the chemical potential} \]

\[ \text{where} \]

\[ \text{The parameters} \]

\[ \text{through the normalization condition} \]

\[ \text{The parameters} \]

\[ \text{The parameters} \]

\[ \text{The expressions in Eq. (45) to obtain the equation for the chemical potential} \]

\[ \text{from which we obtain} \]

\[ \text{The comparison of this result with the chemical potential of the} \]

\[ \text{To calculate the effective potential between impurities we calculate the “two-impurities energy”} \]

\[ \text{The quantity} \]

\[ \text{is the energy of a dark soliton, which is also the energy of a static impenetrable impurity in a} \]

\[ \text{BoSh} \]

\[ \text{Therefore, we see that at} \]

\[ \text{repulsive} \]

\[ \text{potential.} \]
The highlighted discussion from the original submission contains an error that was clarified in the Erratum to the Letter. Below, we present the discussion from the Erratum.

According to the Supplemental Material (SM), the chemical potential for large systems \((L \to \infty)\) can be written as

\[
\mu \simeq \frac{4K(p_2)^2}{(L-r)^2},
\]

(49)

where \(r\) is the distance between impurities. Using this expression, we derive:

\[
\frac{\mu}{\mu_0} = 1 + \frac{1}{\sqrt{\mu_0}L} \left[ 2 + \frac{r}{\sqrt{\mu_0}} \left( \frac{4K(p_1)(K(p_1) - E(p_1))}{\sqrt{\mu_0}r} \right) \right],
\]

(50)

where \(\mu_0\) is the chemical potential without impurities.

To calculate \(p_1\), we notice that \(r = \sqrt{2(1+p_1)/\mu K(p_1)}\). Therefore, for \(r \to \infty\), we must have \(p_1 \to 1\), which leads to

\[
r\sqrt{\mu} \simeq \ln \left( \frac{16}{1-p_1} \right) \to p_1 \simeq 1 - 16e^{-\sqrt{\mu}r}.
\]

(51)

If we use this expansion directly in Eq. (50), then we obtain the chemical potential presented in the SM

\[
\frac{\mu}{\mu_0} = 1 + \frac{1}{\sqrt{\mu_0}L} \left[ 4 + 16e^{-\sqrt{\mu_0}r} \left( 1 - \frac{1}{\sqrt{\mu_0}r} \right) \right].
\]

(52)

However, it turns out that this result is not correct as higher-order terms in the expansion \([51]\) modify it.

The simplest way to derive the correct expression for the chemical potential in the limit \(r \to \infty\) is to re-write Eq. (50) as

\[
\frac{\mu}{\mu_0} = 1 + \frac{1}{\sqrt{\mu_0}L} \left[ 2 + \sqrt{2(1+p_1)K(p_1)} - \frac{4(K(p_1) - E(p_1))}{\sqrt{2(1+p_1)}} \right],
\]

(53)

where \(r\) has been excluded using \(r \simeq \sqrt{2(1+p_1)/\mu_0 K(p_1)}\). Equation (53) depends on a single variable \((p_1)\), allowing us to avoid any self-consistency issues when using the expansion \([51]\). We derive

\[
\frac{\mu}{\mu_0} \simeq 1 + \frac{1}{\sqrt{\mu_0}L} \left[ 4 + 16\sqrt{\mu_0}re^{-2\sqrt{\mu_0}r} - 24e^{-2\sqrt{\mu_0}r} \right].
\]

(54)

The procedure outlined above allows us to compute the tail of the induced impurity-impurity interaction, which is \(-16\sqrt{\mu_0}\rho e^{-2\sqrt{\mu_0}r}\) (\(\rho\) is the density of the Bose gas without the impurities). Therefore, the impurity-impurity potential is attractive for particles placed far from each other.