Mirror Symmetry For Zeta Functions

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Abstract

In this paper, we study the relation between the zeta function of a
Calabi-Yau hypersurface and the zeta function of its mirror. Two types
of arithmetic relations are discovered. This motivates us to formulate
two general arithmetic mirror conjectures for the zeta functions of a
mirror pair of Calabi-Yau manifolds.

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1 Introduction

In this section, we describe two mirror relations between the zeta function of a Calabi-Yau hypersurface in a projective space and the zeta function of its mirror manifold. Along the way, we make comments and conjectures about what to expect in the general case.

Let $d$ be a positive integer. Let $X$ and $Y$ be two $d$-dimensional smooth projective Calabi-Yau varieties over $\mathbb{C}$. A necessary condition (the topological mirror test) for $X$ and $Y$ to be a mirror pair is that their Hodge numbers satisfy the Hodge symmetry:

$$h^{i,j}(X) = h^{d-i,j}(Y), \quad 0 \leq i, j \leq d. \quad (1)$$

In particular, their Euler characteristics are related by

$$e(X) = (-1)^d e(Y). \quad (2)$$

In general, there is no known rigorous algebraic geometric definition for a mirror pair, although many examples of mirror pairs are known at least conjecturally. Furthermore, it does not make sense to speak of “the mirror” of $X$ as the mirror variety usually comes in a family. In some cases, the mirror does not exist. This is the case for rigid Calabi-Yau 3-fold $X$, since the rigid condition $h^{2,1}(X) = 0$ would imply that $h^{1,1}(Y) = 0$ which is impossible.

We shall assume that $X$ and $Y$ are a given mirror pair in some sense and are defined over a number field or a finite field. We are interested in how the zeta function of $X$ is related to the zeta function of $Y$. Since there is no algebraic geometric definition for $X$ and $Y$ to be a mirror pair, it is difficult to study the possible symmetry between their zeta functions in full generality. On the other hand, there do have many explicit examples and constructions which at least conjecturally give a mirror pair, most notably in the toric hypersurface setting as constructed by Batyrev \[1\]. Thus, we shall first examine an explicit example and see what kind of relations can be proved for their zeta functions in this case. This would then suggest what to expect in general.

Let $n \geq 2$ be a positive integer. We consider the universal family of Calabi-Yau complex hypersurfaces of degree $n + 1$ in the projective space $\mathbb{P}^n$. Its mirror family is a one parameter family of toric hypersurfaces. To construct the mirror family, we consider the one parameter subfamily $X_{\lambda}$ of complex projective hypersurfaces of degree $n + 1$ in $\mathbb{P}^n$ defined by

$$f(x_1, \cdots, x_{n+1}) = x_1^{n+1} + \cdots + x_{n+1}^{n+1} + \lambda x_1 \cdots x_{n+1} = 0,$$
where $\lambda \in \mathbb{C}$ is the parameter. The variety $X_\lambda$ is a Calabi-Yau manifold when $X_\lambda$ is smooth. Let $\mu_{n+1}$ denote the group of $(n+1)$-th roots of unity. Let

$$G = \{(\zeta_1, \ldots, \zeta_{n+1})|\zeta_i^{n+1} = 1, \zeta_1 \cdots \zeta_{n+1} = 1\}/\mu_{n+1} \cong (\mathbb{Z}/(n+1)\mathbb{Z})^{n-1},$$

where $\mu_{n+1}$ is embedded in $G$ via the diagonal embedding. The finite group $G$ acts on $X_\lambda$ by

$$(\zeta_1, \ldots, \zeta_{n+1})(x_1, \ldots, x_{n+1}) = (\zeta_1 x_1, \ldots, \zeta_{n+1} x_{n+1}).$$

The quotient $X_\lambda/G$ is a projective toric hypersurface $Y_\lambda$ in the toric variety $P_\Delta$, where $P_\Delta$ is the simplex in $\mathbb{R}^n$ with vertices $\{e_1, \ldots, e_n, -(e_1 + \cdots e_n)\}$ and the $e_i$'s are the standard coordinate vectors in $\mathbb{R}^n$. Explicitly, the variety $Y_\lambda$ is the projective closure in $P_\Delta$ of the affine toric hypersurface in $\mathbb{C}^n_{m}$ defined by

$$g(x_1, \ldots, x_n) = x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n} + \lambda = 0.$$  

Assume that $X_\lambda$ is smooth. Then, $Y_\lambda$ is a (singular) mirror of $X_\lambda$. It is an orbifold. If $W_\lambda$ is a smooth crepant resolution of $Y_\lambda$, then the pair $(X_\lambda, W_\lambda)$ is called a mirror pair of Calabi-Yau manifolds. Such a resolution exists for this example but not unique if $n \geq 3$. The number of rational points and the zeta function are independent of the choice of the crepant resolution. We are interested in understanding how the arithmetic of $X_\lambda$ is related to the arithmetic of $W_\lambda$, in particular how the zeta function of $X_\lambda$ is related to the zeta function of $W_\lambda$. Our main concern in this paper is to consider Calabi-Yau manifolds over finite fields, although we shall mention some implications for Calabi-Yau manifolds defined over number fields.

In this example, we see two types of mirror pairs. The first one is the maximally generic mirror pair $\{X_\Lambda, W_\lambda\}$, where $X_\Lambda$ is the universal family of smooth projective Calabi-Yau hypersurfaces of degree $(n+1)$ in $\mathbb{P}^n$ and $W_\lambda$ is the one parameter family of Calabi-Yau manifolds as constructed above. Note that $X_\Lambda$ and $Y_\lambda$ are parametrized by different parameter spaces (of different dimensions). The possible zeta symmetry in this case would then have to be a relation between certain generic property of the zeta function for $X \in X_\Lambda$ and the corresponding generic property of the zeta function for $W \in W_\lambda$.

The second type of mirror pairs is the one parameter family of mirror pairs $\{X_\lambda, W_\lambda\}$ parametrized by the same parameter $\lambda$. This is a stronger type of mirror pair than the first type. For $\lambda \in \mathbb{C}$, we say that $W_\lambda$ is a
strong mirror of $X_\lambda$. For such a strong mirror pair \{$X_\lambda, W_\lambda$\}, we can really ask for the relation between the zeta function of $X_\lambda$ and the zeta function of $W_\lambda$. If $\lambda_1 \neq \lambda_2$, $W_{\lambda_1}$ would not be called a strong mirror for $X_{\lambda_2}$, although they would be an usual weak mirror pair. Apparently, we do not have a definition for a strong mirror pair in general, as there is not even a definition for a generic or weak mirror pair in general.

Let $\mathbb{F}_q$ be a finite field of $q$ elements, where $q = p^r$ and $p$ is a prime. For a scheme $X$ of finite type of dimension $d$ over $\mathbb{F}_q$, let $\#X(\mathbb{F}_q)$ denote the number of $\mathbb{F}_q$-rational points on $X$. Let

\[ Z(X, T) = \exp\left(\sum_{k=1}^{\infty} \frac{T^k}{k} \#X(\mathbb{F}_{q^k})\right) \in 1 + T\mathbb{Z}[T] \]

be the zeta function of $X$. It is well known that $Z(X, T)$ is a rational function in $T$ whose reciprocal zeros and reciprocal poles are Weil $q$-integers. Factor $Z(X, T)$ over the $p$-adic numbers $\mathbb{C}_p$ and write

\[ Z(X, T) = \prod_i (1 - \alpha_i T)^{\pm 1} \]

in reduced form, where the algebraic integers $\alpha_i \in \mathbb{C}_p$. One knows that the slope $\text{ord}_q(\alpha_i)$ is a rational number in the interval $[0, d]$. For two real numbers $s_1 \leq s_2$, we define the slope $[s_1, s_2]$ part of $Z(X, T)$ to be the partial product

\[ Z_{[s_1, s_2]}(X, T) = \prod_{s_1 \leq \text{ord}_q(\alpha_i) \leq s_2} (1 - \alpha_i T)^{\pm 1}. \quad (3) \]

For a half open and half closed interval $[s_1, s_2)$, the slope $[s_1, s_2)$ part $Z_{[s_1, s_2)}(X, T)$ of $Z(X, T)$ is defined in a similar way. These are rational functions with coefficients in $\mathbb{Z}_p$ by the $p$-adic Weierstrass factorization. It is clear that we have the decomposition

\[ Z(X, T) = \prod_{i=0}^{d} Z_{[i, i+1)}(X, T). \]

Our main result of this paper is the following arithmetic mirror theorem.

**Theorem 1.1** Assume that $\lambda \in \mathbb{F}_q$ such that $(X_\lambda, W_\lambda)$ is a strong mirror pair of Calabi-Yau manifolds over $\mathbb{F}_q$. For every positive integer $k$, we have the congruence formula

\[ \#X_\lambda(\mathbb{F}_{q^k}) \equiv \#Y_\lambda(\mathbb{F}_{q^k}) \equiv \#W_\lambda(\mathbb{F}_{q^k}) \pmod{q^k}. \]
Equivalently, the slope \([0, 1)\) part of the zeta function is the same for the mirror varieties \(\{X_\lambda, Y_\lambda, W_\lambda\}\):

\[
Z_{(0,1)}(X_\lambda, T) = Z_{(0,1)}(Y_\lambda, T) = Z_{(0,1)}(W_\lambda, T).
\]

We now discuss a few applications of this theorem. In terms of cohomology theory, this suggests that the semi-simplification of the DeRham-Witt cohomology (in particular, the \(p\)-adic étale cohomology) for \(\{X_\lambda, Y_\lambda, W_\lambda\}\) are all the same. A corollary of the above theorem is that the unit root parts (slope zero parts) of their zeta functions are the same:

\[
Z_{(0,0)}(X_\lambda, T) = Z_{(0,0)}(Y_\lambda, T) = Z_{(0,0)}(W_\lambda, T).
\]

The \(p\)-adic variation of the rational function \(Z_{(0,0)}(X_\lambda, T)\) as \(\lambda\) varies is closely related to the mirror map which we do not discuss here, but see [4] for the case \(n \leq 3\). From arithmetic point of view, the \(p\)-adic variation of the rational function \(Z_{(0,0)}(X_\lambda, T)\) as \(\lambda\) varies is explained by Dwork’s unit root zeta function [5]. We briefly explain the connection here.

Let \(B\) be the parameter variety of \(\lambda\) such that \((X_\lambda, W_\lambda)\) form a strong mirror pair. Let \(\Phi : X_\lambda \to B\) (resp. \(\Psi : W_\lambda \to B\)) be the projection to the base by sending \(X_\lambda\) (resp. \(W_\lambda\)) to \(\lambda\). The pair \((\Phi, \Psi)\) of morphisms to \(B\) is called a \textbf{strong mirror pair of morphisms to} \(B\). Each of its fibres gives a strong mirror pair of Calabi-Yau manifolds. Recall that Dwork’s unit root zeta function attached to the morphism \(\Phi\) is defined to be the formal infinite product

\[
Z_{\text{unit}}(\Phi, T) = \prod_{\lambda \in |B|} Z_{(0,0]}(X_\lambda, T^{\deg(\lambda)}) \in 1 + T\mathbb{Z}_p[[T]],
\]

where \(|B|\) denotes the set of closed points of \(B\) over \(\mathbb{F}_q\). This unit root zeta function is no longer a rational function, but conjectured by Dwork in [5] and proved by the author in [10][11][12] to be a \(p\)-adic meromorphic function in \(T\). The above theorem immediately implies

**Corollary 1.2** Let \((\Phi, \Psi)\) be the above strong mirror pair of morphisms to the base \(B\). Then, their unit root zeta functions are the same:

\[
Z_{\text{unit}}(\Phi, T) = Z_{\text{unit}}(\Psi, T).
\]

If \(\lambda\) is in a number field \(K\), then Theorem 1.1 implies that the Hasse-Weil zeta functions of \(X_\lambda\) and \(Y_\lambda\) differ essentially by the \(L\)-function of a pure motive \(M_n(\lambda)\) of weight \(n - 3\). That is,

\[
\zeta(X_\lambda, s) = \zeta(Y_\lambda, s)L(M_n(\lambda), s - 1).
\]
In the quintic case $n = 4$, the pure weight 1 motive $M_4(\lambda)$ would come from a curve. This curve has been constructed explicitly by Candelas, de la Ossa and Fernando-Rodriquez [3]. The relation between the Hasse-Weil zeta functions of $X_\lambda$ and $W_\lambda$ are similar, differing by a few more factors consisting of Tate twists of the Dedekind zeta function of $K$.

Theorem 1.1 motivates the following more general conjecture.

**Conjecture 1.3 (Congruence mirror conjecture)** Suppose that we are given a strong mirror pair $\{X, Y\}$ of Calabi-Yau manifolds defined over $\mathbb{F}_q$. Then, for every positive integer $k$, we have

$$\#X(\mathbb{F}_{q^k}) \equiv \#Y(\mathbb{F}_{q^k}) \pmod{q^k}.$$ 

Equivalently,

$$Z_{[0,1)}(X, T) = Z_{[0,1)}(Y, T).$$

Equivalently (by functional equation),

$$Z_{(d-1,d]}(X, T) = Z_{(d-1,d]}(Y, T).$$

The condition in the congruence mirror conjecture is vague since one does not know at present an algebraic geometric definition of a strong mirror pair of Calabi-Yau manifolds, although one does know many examples such as the one given above. Thus, a major part of the problem is to make the definition of a strong mirror pair mathematically precise. For an additional evidence of the congruence mirror conjecture, see Theorem 6.2 which can be viewed as a generalization of Theorem 1.1. As indicated before, this conjecture implies that Dwork's unit root zeta functions for the two families forming a strong mirror pair are the same $p$-adic meromorphic functions. This means that under the strong mirror family involution, Dwork's unit root zeta function stays the same.

Just like the zeta function itself, its slope $[0, 1)$ part $Z_{[0,1)}(X_\lambda, T)$ depends heavily on the algebraic parameter $\lambda$, not just on the topological properties of $X_\lambda$. This means that the congruence mirror conjecture is really a continuous type of arithmetic mirror symmetry. This continuous nature requires the use of a strong mirror pair, not just a generic mirror pair.

Assume that $\{X, Y\}$ forms a mirror pair, not necessarily a strong mirror pair. A different type of arithmetic mirror symmetry reflecting the Hodge symmetry, which is discrete and hence generic in nature, is to look for a suitable quantum version $Z_Q(X, T)$ of the zeta function such that

$$Z_Q(X, T) = Z_Q(Y, T)(^{-1})^d,$$
where \( \{X, Y\} \) is a mirror pair of Calabi-Yau manifolds over \( \mathbb{F}_q \) of dimension \( d \). This relation cannot hold for the usual zeta function \( Z(X, T) \) for obvious reasons, even for a strong mirror pair as it contradicts with the congruence mirror conjecture for odd \( d \). No non-trivial candidate for \( Z_Q(X, T) \) has been found. Here we propose a \( p \)-adic quantum version which would have the conjectural properties for most (and hence generic) mirror pairs. We will call our new zeta function to be the slope zeta function as it is based on the slopes of the zeros and poles.

**Definition 1.4** For a scheme \( X \) of finite type over \( \mathbb{F}_q \), write as before

\[
Z(X, T) = \prod_i (1 - \alpha_i T)^{\pm 1}
\]

in reduced form, where \( \alpha_i \in \mathbb{C}_p \). Define the slope zeta function of \( X \) to be the two variable function

\[
S_p(X, u, T) = \prod_i (1 - u^{\text{ord}_q(\alpha_i)} T)^{\pm 1}.
\] (4)

Note that

\[
\alpha_i = q^{\text{ord}_q(\alpha_i)} \beta_i,
\]

where \( \beta_i \) is a \( p \)-adic unit. Thus, the slope zeta function \( S_p(X, u, T) \) is obtained from the \( p \)-adic factorization of \( Z(X, T) \) by dropping the \( p \)-adic unit parts of the roots and replacing \( q \) by the variable \( u \). This is not always a rational function in \( u \) and \( T \). It is rational if all slopes are integers. Note that the definition of the slope zeta function is independent of the choice of the ground field \( \mathbb{F}_q \) where \( X \) is defined. It depends only on \( X \otimes \mathbb{F}_q \) and thus is also a geometric invariant. It would be interesting to see if there is a diophantine interpretation of the slope zeta function.

If \( X \) is a scheme of finite type over \( \mathbb{Z} \), then for each prime number \( p \), the reduction \( X \otimes \mathbb{F}_p \) has the \( p \)-adic slope zeta function \( S_p(X \otimes \mathbb{F}_p, u, T) \). At the first glance, one might think that this gives infinitely many discrete invariants for \( X \) as the set of prime numbers is infinite. However, it can be shown that the set \( \{ S_p(X \otimes \mathbb{F}_p, u, T) \mid p \text{ prime} \} \) contains only finitely many distinct elements. In general, it is a very interesting but difficult problem to determine this set \( \{ S_p(X \otimes \mathbb{F}_p, u, T) \mid p \text{ prime} \} \).

Suppose that \( X \) and \( Y \) form a mirror pair of \( d \)-dimensional Calabi-Yau manifolds over \( \mathbb{F}_q \). For simplicity and for comparison with the Hodge theory, we always assume in this paper that \( X \) and \( Y \) can be lifted to characteristic zero (to the Witt ring of \( \mathbb{F}_q \)). In this good reduction case, the modulo \( p \)
Hodge numbers equal the characteristic zero Hodge numbers. Taking \( u = 1 \)
in the definition of the slope zeta function, we see that the specialization
\[ S_p(X, 1, T) \]
already satisfies the desired relation
\[ S_p(X, 1, T) = (1 - T)^{-e(X)} = (1 - T)^{-(1)^d e(Y)} = S_p(Y, 1, T)^{(-1)^d}. \]

This suggests that there is a chance that the slope zeta function might satisfy
the desired slope mirror symmetry
\[ S_p(X, u, T) = S_p(Y, u, T)^{(-1)^d}. \] (5)

In section 7, we shall show that the slope zeta function satisfies a functional
equation. Furthermore, the expected slope mirror symmetry does hold if
both \( X \) and \( Y \) are ordinary. If either \( X \) or \( Y \) is not ordinary, the expected
slope mirror symmetry is unlikely to hold in general.

If \( d \leq 2 \), the congruence mirror conjecture implies that the slope zeta
function does satisfy the expected slope mirror symmetry for a strong mirror
pair \( \{X, Y\} \), whether \( X \) and \( Y \) are ordinary or not. For \( d \geq 3 \), we believe
that the slope zeta function is still a little bit too strong for the expected
symmetry to hold in general, even if \( \{X, Y\} \) forms a strong mirror pair.
And it should not be too hard to find a counter-example although we have
not done so. However, we believe that the expected slope mirror symmetry
holds for a sufficiently generic pair of 3-dimensional Calabi-Yau manifolds.

Conjecture 1.5 (Slope mirror conjecture) Suppose that we are given
a maximally generic mirror pair \( \{X,Y\} \) of 3-dimensional Calabi-Yau manifolds defined over \( \mathbb{F}_q \). Then, we have the slope mirror symmetry
\[ S_p(X, u, T) = \frac{1}{S_p(Y, u, T)^{(-1)^d}} \] (6)
for generic \( X \) and generic \( Y \).

A main point of this conjecture is that it holds for all prime numbers
\( p \). For arbitrary \( d \geq 4 \), the corresponding slope mirror conjecture might
be false for some prime numbers \( p \), but it should be true for all primes
\( p \equiv 1 \pmod{D} \) for some positive integer \( D \) depending on the mirror family,
if the family comes from the reduction modulo \( p \) of a family defined over a
number field. In the case \( d \leq 3 \), one could take \( D = 1 \) and hence get the
above conjecture.

Again the condition in the slope mirror conjecture is vague as it is not
presently known an algebraic geometric definition of a mirror family, al-
though many examples are known in the toric setting. In a future paper,
using the results in [9][13], we shall prove that the slope mirror conjecture holds in the toric hypersurface case if $d \leq 3$. For example, if $X$ is a generic quintic hypersurface, then $X$ is ordinary by the results in [7][9] for every $p$ and thus one finds

$$S_p(X \otimes \mathbb{F}_p, u, T) = \frac{(1 - T)(1 - uT)^{101}(1 - u^2T)^{101}(1 - u^3T)}{(1 - T)(1 - uT)(1 - u^2T)(1 - u^3T)}.$$

This is independent of $p$. Note that we do not know if the one parameter subfamily $X_\lambda$ is generically ordinary for every $p$. The ordinary property for every $p$ was established only for the universal family of hypersurfaces, not for a one parameter subfamily of hypersurfaces such as $X_\lambda$. If $Y$ denotes the generic mirror of $X$, then by the results in [9][13], $Y$ is ordinary for every $p$ and thus we obtain

$$S_p(Y \otimes \mathbb{F}_p, u, T) = \frac{(1 - T)(1 - uT)(1 - u^2T)(1 - u^3T)}{(1 - T)(1 - uT)^{101}(1 - u^2T)^{101}(1 - u^3T)}.$$

Again, it is independent of $p$. The slope mirror conjecture holds in this example.

For a mirror pair over a number field, we have the following harder conjecture.

**Conjecture 1.6 (Slope mirror conjecture over $\mathbb{Z}$)** Let $\{X, Y\}$ be two schemes of finite type over $\mathbb{Z}$ such that their generic fibres $\{X \otimes \mathbb{Q}, Y \otimes \mathbb{Q}\}$ form a usual (weak) mirror pair of $d$-dimensional Calabi-Yau manifolds defined over $\mathbb{Q}$. Then there are infinitely many prime numbers $p$ (with positive density) such that

$$S_p(X \otimes \mathbb{F}_p, u, T) = S_p(Y \otimes \mathbb{F}_p, u, T)^{(-1)^d}.$$

**Remarks.** If one uses the weight $2 \log_q |\alpha_i|$ instead of the slope $\text{ord}_q \alpha_i$, where $|\cdot|$ denotes the complex absolute value, one can define a two variable weight zeta function in a similar way. It is easy to see that the resulting weight zeta function does not satisfy the desired symmetry as the weight has nothing to do with the Hodge symmetry, while the slopes are related to the Hodge numbers as the Newton polygon (slope polygon) lies above the Hodge polygon.

In practice, one is often given a mirror pair of singular Calabi-Yau orbifolds, where there may not exist a smooth crepant resolution. In such a case, one could define an orbifold zeta function, which would be equal to the zeta function of the smooth crepant resolution whenever such a resolution exists. Similar results and conjectures should carry over to such orbifold zeta functions.
2 A counting formula via Gauss sums

Let $V_1, \ldots, V_m$ be $m$ distinct lattice points in $\mathbb{Z}^n$. For $V_j = (V_{1j}, \ldots, V_{nj})$, write

$$x^{V_j} = x_1^{V_{1j}} \cdots x_n^{V_{nj}}.$$ 

Let $f$ be the Laurent polynomial in $n$ variables written in the form:

$$f(x_1, \ldots, x_n) = \sum_{j=1}^m a_j x^{V_j}, a_j \in \mathbb{F}_q,$$

where not all $a_j$ are zero. Let $M$ be the $n \times m$ matrix

$$M = (V_1, \ldots, V_m),$$

where each $V_j$ is written as a column vector. Let $N_f^*$ denote the number of $\mathbb{F}_q$-rational points on the affine toric hypersurface $f = 0$ in $\mathbb{G}_m^n$. If each $V_j \in \mathbb{Z}^n_{\geq 0}$, we let $N_f$ denote the number of $\mathbb{F}_q$-rational points on the affine hypersurface $f = 0$ in $\mathbb{A}^n$. We first derive a well known formula for both $N_f^*$ and $N_f$ in terms of Gauss sums.

For this purpose, we now recall the definition of Gauss sums. Let $\mathbb{F}_q$ be the finite field of $q$ elements, where $q = p^r$ and $p$ is the characteristic of $\mathbb{F}_q$. Let $\chi$ be the Teichmüller character of the multiplicative group $\mathbb{F}_q^*$. For $a \in \mathbb{F}_q^*$, the value $\chi(a)$ is just the $(q - 1)$-th root of unity in the $p$-adic field $\mathbb{C}_p$ such that $\chi(a)$ modulo $p$ reduces to $a$. Define the $(q - 2)$ Gauss sums over $\mathbb{F}_q$ by

$$G(k) = \sum_{a \in \mathbb{F}_q^*} \chi(a)^{-k} \zeta_p^\text{Tr}(a) \ (1 \leq k \leq q - 2),$$

where $\zeta_p$ is a primitive $p$-th root of unity in $\mathbb{C}_p$ and $\text{Tr}$ denotes the trace map from $\mathbb{F}_q$ to the prime field $\mathbb{F}_p$.

**Lemma 2.1** For all $a \in \mathbb{F}_q$, the Gauss sums satisfy the following interpolation relation

$$\zeta_p^\text{Tr}(a) = \sum_{k=0}^{q-1} G(k) \frac{q-1}{q-1} \chi(a)^k,$$

where

$$G(0) = q - 1, \ G(q - 1) = -q.$$
Proof. By the Vandermonde determinant, there are numbers $C(k)$ ($0 \leq k \leq q - 1$) such that for all $a \in \mathbb{F}_q$, one has
\[ \zeta_p^{\text{Tr}(a)} = \sum_{k=0}^{q-1} \frac{C(k)}{q-1} \chi(a)^k. \]
It suffices to prove that $C(k) = G(k)$ for all $k$. Take $a = 0$, one finds that $C(0)/(q-1) = 1$. This proves that $C(0) = q - 1 = G(0)$. For $1 \leq k \leq q - 2$, one computes that
\[ G(k) = \sum_{a \in \mathbb{F}_q^*} \chi(a)^{-k} \zeta_p^{\text{Tr}(a)} = \frac{C(k)}{q-1}(q-1) = C(k). \]
Finally,
\[ 0 = \sum_{a \in \mathbb{F}_q} \zeta_p^{\text{Tr}(a)} = \frac{C(0)}{q-1}q + \frac{C(q-1)}{q-1}(q-1). \]
This gives $C(q-1) = -q = G(q-1)$. The lemma is proved.

We also need to use the following classical theorem of Stickelberger.

Lemma 2.2 Let $0 \leq k \leq q - 1$. Write
\[ k = k_0 + k_1 p + \cdots + k_{r-1} p^{r-1} \]
in $p$-adic expansion, where $0 \leq k_i \leq p - 1$. Let $\sigma(k) = k_0 + \cdots + k_{r-1}$ be the sum of the $p$-digits of $k$. Then,
\[ \text{ord}_p G(k) = \frac{\sigma(k)}{p-1}. \]

Now we turn to deriving a counting formula for $N_f$ in terms of Gauss sums. Write $W_j = (1, V_j) \in \mathbb{Z}^{n+1}$. Then,
\[ x_0 f = \sum_{j=1}^{m} a_j x_{W_j} = \sum_{j=1}^{m} a_j x_0 x_{V_{1j}} \cdots x_{V_{nj}}, \]
where $x$ now has $n + 1$ variables $\{x_0, \ldots, x_n\}$. Using the formula
\[ \sum_{t \in \mathbb{F}_q^*} t^k = \begin{cases} 0, & \text{if } (q-1) \nmid k, \\ q-1, & \text{if } (q-1) | k \text{ and } k > 0, \\ q, & \text{if } k = 0, \end{cases} \]
one then calculates that

\[ qN_f = \sum_{x_0, \ldots, x_n \in \mathbb{F}_q} \zeta_p^{\text{Tr}(x_0 f(x))} \]

\[ = \sum_{x_0, \ldots, x_n \in \mathbb{F}_q} \prod_{j=1}^m \zeta_p^{\text{Tr}(a_j x^{W_j})} \]

\[ = \sum_{x_0, \ldots, x_n \in \mathbb{F}_q} \prod_{j=1}^m (q-1)^{q-1} \chi(a_j)^{k_j} \chi(x^{W_j})^{k_j} \]

\[ = \sum_{k_1=0}^{q-1} \cdots \sum_{k_m=0}^{q-1} \left( \sum_{j=1}^m \chi(a_j)^{k_j} \right) \sum_{x_0, \ldots, x_n \in \mathbb{F}_q} \chi(x^{k_1 W_1 + \cdots + k_m W_m}) \]

\[ = \sum_{j=1}^m (q-1)^{s(k)} (q-1)^{q+1-s(k)} \prod_{j=1}^m \chi(a_j)^{k_j} G(k_j), \quad (7) \]

where \( s(k) \) denotes the number of non-zero entries in \( k_1 W_1 + \cdots + k_m W_m \).

Similarly, one calculates that

\[ qN_f^* = \sum_{x_0 \in \mathbb{F}_q, x_1, \ldots, x_n \in \mathbb{F}_q^*} \zeta_p^{\text{Tr}(x_0 f(x))} \]

\[ = (q-1)^n + \sum_{x_0, \ldots, x_n \in \mathbb{F}_q^*} \prod_{j=1}^m \zeta_p^{\text{Tr}(a_j x^{W_j})} \]

\[ = (q-1)^n + \sum_{\sum_{j=1}^m k_j W_j \equiv 0 \pmod{q-1}} \frac{(q-1)^{n+1}}{(q-1)^m} \prod_{j=1}^m \chi(a_j)^{k_j} G(k_j). \quad (8) \]

We shall use these two formulas to study the number of \( \mathbb{F}_q \)-rational points on certain hypersurfaces in next two sections.

### 3 Rational points on Calabi-Yau hypersurfaces

In this section, we apply formula (7) to compute the number of \( \mathbb{F}_q \)-rational points on the projective hypersurface \( X_\lambda \) in \( \mathbb{P}^n \) defined by

\[ f(x_1, \ldots, x_{n+1}) = x_1^{n+1} + \cdots + x_{n+1}^{n+1} + \lambda x_1 \cdots x_{n+1} = 0, \]
where $\lambda$ is an element of $\mathbb{F}_q^*$. We shall handle the easier case $\lambda = 0$ separately. Let $M$ be the $(n + 2) \times (n + 2)$ matrix

\[
M = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
n + 1 & 0 & 0 & \cdots & 0 & 1 \\
0 & n + 1 & 0 & \cdots & 0 & 1 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & n + 1 & 1
\end{pmatrix}
\]

Let $k = (k_1, \cdots, k_{n+2})$ written as a column vector. Let $N_f$ denote the number of $\mathbb{F}_q$-rational points on the affine hypersurface $f = 0$ in $\mathbb{A}^{n+1}$. By formula (7), we deduce that

\[
qN_f = \sum_{Mk \equiv 0 \bmod q-1} (q-1)^{s(k)} q^{n+2-s(k)} (\prod_{j=1}^{n+2} g(k_j)) \chi(\lambda)^{k_{n+2}},
\]

where $s(k)$ denotes the number of non-zero entries in $Mk \in \mathbb{Z}^{n+2}$. The number of $\mathbb{F}_q$-rational points on the projective hypersurface $X_\lambda$ is then given by the formula

\[
\frac{N_f - 1}{q - 1} = -\frac{1}{q - 1} + \sum_{Mk \equiv 0 \bmod q-1} \frac{q^{n+1-s(k)} (\prod_{j=1}^{n+2} g(k_j)) \chi(\lambda)^{k_{n+2}}}{(q-1)^{n+3-s(k)}}.
\]

If $k = (0, \cdots, 0, q - 1)$, then $Mk = (q - 1, \cdots, q - 1)$ and $s(k) = n + 2$. In this case, the corresponding term in the above expression is $-(q - 1)^n$ which is $(-1)^{n-1}$ modulo $q$. If $k = (0, \cdots, 0)$, then $s(k) = 0$ and the corresponding term is $q^{n+1}/(q - 1)$ which is zero modulo $q$.

Thus, we obtain the congruence formula modulo $q$:

\[
\frac{N_f - 1}{q - 1} \equiv 1 + (-1)^{n-1} + \sum^{*}_{Mk \equiv 0 \bmod q-1} \frac{q^{n+1-s(k)} (\prod_{j=1}^{n+2} g(k_j)) \chi(\lambda)^{k_{n+2}}}{(q-1)^{n+3-s(k)}}.
\]

where $\sum^*$ means summing over all those solutions $k = (k_1, \cdots, k_{n+2})$ with $0 \leq k_i \leq q - 1$, $k \neq (0, \cdots, 0)$, and $k \neq (0, \cdots, 0, q - 1)$.

**Lemma 3.1** If $k \neq (0, \cdots, 0)$, then $\prod_{j=1}^{n+2} g(k_j)$ is divisible by $q$.

**Proof.** Let $k$ be a solution of $Mk \equiv 0 \bmod q-1$ such that $k \neq (0, \cdots, 0)$. Then, there are positive integers $\ell_0, \cdots, \ell_{r-1}$ such that

\[
k_1 + \cdots + k_{n+2} = (q - 1)\ell_0,
\]
<p>

<math>
< pk_1 > + \cdots + < pk_{n+2} > = (q - 1)\ell_1,
</math>

\ldots

<math>
< p^{r-1}_1 > + \cdots + < p^{r-1}_{n+2} > = (q - 1)\ell_{r-1},
</math>

where <math>< pk_1 ></math> denotes the unique integer in <math>[0, q - 1]</math> congruent to <math>p k_1</math> modulo <math>(q - 1)</math> and which is 0 (resp. <math>q - 1</math>) if <math>p k_1 = 0</math> (resp., if <math>p k_1</math> is a positive multiple of <math>q - 1</math>). By the Stickelberger theorem, we deduce that

<math>
\text{ord}_p \prod_{j=1}^{n+2} G(k_j) = \frac{\sum_j \sigma(k_j)}{p - 1} = \frac{1}{q - 1} \sum_{i=0}^{r-1} (q - 1)\ell_i = \sum_{i=0}^{r-1} \ell_i.
</math>

Since <math>\ell_i \geq 1</math>, it follows that

<math>
\text{ord}_q \prod_{j=1}^{n+2} G(k_j) = \frac{1}{r} \sum_{i=0}^{r-1} \ell_i \geq 1
</math>

with equality holding if and only if all <math>\ell_i = 1</math>. The lemma is proved.

Using this lemma and the previous congruence formula, we deduce

**Lemma 3.2** Let <math>\lambda \in \mathbb{F}_q^*</math>. We have the congruence formula modulo <math>q</math>:

<math>
\#X_\lambda(\mathbb{F}_q) \equiv 1 + (-1)^{n-1} + \sum_{M \chi(k) \equiv n+2 \pmod{q}} \frac{1}{q(q - 1)} \left( \prod_{j=1}^{n+2} G(k_j) \right) \chi(\lambda)^{k_{n+2}}.
</math>

### 4 Rational points on the mirror hypersurfaces

In this section, we apply formula (8) to compute the number of <math>\mathbb{F}_q^*</math>-rational points on the affine toric hypersurface in <math>\mathbb{G}_m^n</math> defined by the Laurent polynomial equation

<math>
g(x_1, \cdots, x_n) = x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n} + \lambda = 0,
</math>

where <math>\lambda</math> is an element of <math>\mathbb{F}_q^*</math>. Let <math>N</math> be the <math>(n + 1) \times (n + 2)</math> matrix

<math>
N = \begin{pmatrix}
1 & 1 & \cdots & 1 & 1 & 1 \\
1 & 0 & \cdots & 0 & -1 & 0 \\
0 & 1 & \cdots & 0 & -1 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -1 & 0
\end{pmatrix}
</math>
Let \( k = (k_1, \cdots, k_{n+2}) \) written as a column vector. By formula (8), we deduce that
\[
qN_g^* = (q - 1)^n + \sum_{Nk \equiv 0 \pmod{q-1}} \frac{1}{(q-1)} \left( \prod_{j=1}^{n+2} G(k_j) \right) \chi(\lambda)^{k_{n+2}},
\]
where \( k = (k_1, \cdots, k_{n+2}) \) with \( 0 \leq k_i \leq q - 1 \).

The contribution of those trivial terms \( k \) (where each \( k_i \) is either 0 or \( q - 1 \)) is given by
\[
\frac{1}{q-1} \sum_{s=0}^{n+2} (-q)^s(q - 1)^{n+2-s} \binom{n + 2}{s} = \frac{(-1)^n}{q-1}.
\]
Since
\[
(q - 1)^n + \frac{(-1)^n}{q-1} = \frac{(q - 1)^{n+1} + (-1)^n}{q-1} \equiv q(n+1)(-1)^{n-1} \pmod{q^2},
\]
we deduce

**Lemma 4.1** For \( \lambda \in \mathbb{F}_q^* \), we have the following congruence formula modulo \( q \):
\[
N_g^* \equiv (n + 1)(-1)^{n-1} + \sum'_{Nk \equiv 0 \pmod{q-1}} \frac{1}{q(q-1)} \left( \prod_{j=1}^{n+2} G(k_j) \right) \chi(\lambda)^{k_{n+2}},
\]
where \( \sum' \) means summing over all those non-trivial solutions \( k \).

## 5 The mirror congruence formula

**Theorem 5.1** For \( \lambda \in \mathbb{F}_q^* \), we have the congruence formula
\[
\#X_\lambda(\mathbb{F}_q) \equiv N_g^* + 1 - n(-1)^{n-1} \pmod{q}.
\]

**Proof.** If \( k \) is a non-trivial solution of \( Nk \equiv 0 \pmod{q-1} \), then we have
\[
k_1 \equiv k_2 \equiv \cdots \equiv k_n \equiv k_{n+1}(\mod{q-1})
\]
and
\[
k_1 + \cdots + k_{n+1} + k_{n+2} \equiv 0 \pmod{q-1}.
\]
Since $k$ is non-trivial, we must have

$$0 < k_1 = k_2 = \cdots = k_{n+1} < q - 1,$$

$$k_1 + \cdots + k_{n+2} = (n+1)k_1 + k_{n+2} = (n+1)k_2 + k_{n+2} = \cdots \equiv 0 \pmod{q-1}.$$  

This gives all solutions of the equation $Mk \equiv 0 \pmod{q-1}$ with $k_1 = \cdots = k_{n+1}, 0 < k_1 < q - 1$ and $s(k) = n + 2$. The corresponding terms for these $k$’s in $(N_f - 1)/(q - 1)$ and $N_g^*$ are exactly the same.

A solution of $Mk \equiv 0 \pmod{q-1}$ is called admissible if $s(k) = n + 2$ and its first $k + 1$ coordinates $\{k_1, \cdots, k_{n+1}\}$ contain at least two distinct elements. The above results show that we have

$$\frac{N_f - 1}{q - 1} - 1 - (-1)^{n-1} - (N_g^* - (n + 1)(-1)^{n-1}) \equiv \sum_{\text{admissible } k} \frac{1}{q(q-1)} (\prod_{j=1}^{n+2} G(k_j)) \chi(\lambda)^{k_{n+2}} \pmod{q}.$$  

This congruence together with the following lemma completes the proof of the theorem.

**Lemma 5.2** If $k$ is an admissible solution of $Mk \equiv 0 \pmod{q-1}$, then

$$\text{ord}_q \left( \prod_{j=1}^{n+2} G(k_j) \right) \geq 2.$$

**Proof.** If $k$ is an admissible solution, then $< pk >, \cdots, < p^{r-1}k >$ are also admissible solutions. For each $1 \leq i \leq n + 1$, write

$$(n+1)k_i + k_{n+2} = (q - 1)\ell_i,$$

where $\ell_i$ is a positive integer. Adding these equations together, we get

$$(n+1)(k_1 + \cdots + k_{n+1}) + (n+1)k_{n+2} = (q - 1)(\ell_1 + \cdots + \ell_{n+1}).$$

Thus, the integer

$$\frac{k_1 + \cdots + k_{n+2}}{q - 1} = \frac{\ell_1 + \cdots + \ell_{n+1}}{n+1} = \ell \in \mathbb{Z}_{>0}.$$  

It is clear that $\ell = 1$ if and only if each $\ell_i = 1$ which would imply that $k_1 = \cdots = k_{n+1}$ contradicting with the admissibility of $k$. Thus, we must have that $\ell \geq 2$. 

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Similarly, for each $0 \leq i \leq r - 1$, we have

$$< p^i k_1 > + \cdots + < p^i k_{n+2} > = (q - 1)j_i,$$

where $j_i \geq 2$ is a positive integer. We conclude that

$$\text{ord}_q(\prod_{j=1}^{n+2} G(k_j)) = \frac{j_0 + \cdots + j_{r-1}}{r} \geq 2.$$

The lemma is proved.

6 Rational points on the projective mirror

Let $\Delta$ be the convex integral polytope associated with the Laurent polynomial $g$. It is the $n$-dimensional simplex in $\mathbb{R}^n$ with the following vertices:

$$\{e_1, \cdots, e_n, -(e_1 + \cdots + e_n)\},$$

where the $e_i$'s are the standard unit vectors in $\mathbb{R}^n$.

Let $\mathbb{P}_\Delta$ be the projective toric variety associated with the polytope $\Delta$, which contains $\mathbb{G}_m^n$ as an open dense subset. Let $Y_{\lambda}$ be the projective closure in $\mathbb{P}_\Delta$ of the affine toric hypersurface $g = 0$ in $\mathbb{G}_m^n$. The variety $Y_{\lambda}$ is then a projective toric hypersurface in $\mathbb{P}_\Delta$. We are interested in the number of $\mathbb{F}_q$-rational points on $Y_{\lambda}$.

The toric variety $\mathbb{P}_\Delta$ has the following disjoint decomposition:

$$\mathbb{P}_\Delta = \bigcup_{\tau \in \Delta} \mathbb{P}_{\Delta, \tau},$$

where $\tau$ runs over all non-empty faces of $\Delta$ and each $\mathbb{P}_{\Delta, \tau}$ is isomorphic to the torus $\mathbb{G}_m^{\dim \tau}$. Accordingly, the projective toric hypersurface $Y_{\lambda}$ has the corresponding disjoint decomposition

$$Y_{\lambda} = \bigcup_{\tau \in \Delta} Y_{\lambda, \tau}, \quad Y_{\lambda, \tau} = Y_{\lambda} \cap \mathbb{P}_{\Delta, \tau}.$$

For $\tau = \Delta$, the subvariety $Y_{\lambda, \Delta}$ is simply the affine toric hypersurface defined by $g = 0$ in $\mathbb{G}_m^n$. For zero-dimensional $\tau$, $Y_{\lambda, \tau}$ is empty. For a face $\tau$ with $1 \leq \dim \tau \leq n - 1$, one checks that $Y_{\lambda, \tau}$ is isomorphic to the affine toric hypersurface in $\mathbb{G}_m^{\dim \tau}$ defined by

$$1 + x_1 + \cdots + x_{\dim \tau} = 0.$$
For such a $\tau$, the inclusion-exclusion principle shows that
\[
\#Y_{\lambda,\tau}(\mathbb{F}_q) = q^{\dim \tau - 1} - \binom{\dim \tau}{1}q^{\dim \tau - 2} + \ldots + (-1)^{\dim \tau - 1}\binom{\dim \tau}{\dim \tau - 1}.
\]
Thus,
\[
\#Y_{\lambda,\tau}(\mathbb{F}_q) = \frac{1}{q}(q - 1)^{\dim \tau} + (-1)^{\dim \tau + 1}.
\]
This formula holds even for zero-dimensional $\tau$ as both sides would then be zero.

Putting these calculations together, we deduce that
\[
\#Y_{\lambda}(\mathbb{F}_q) = N_g^* - \frac{(q - 1)^n + (-1)^{n+1}}{q} + \sum_{\tau \in \Delta} \frac{1}{q}((q - 1)^{\dim \tau} + (-1)^{\dim \tau + 1}),
\]
where $\tau$ runs over all non-empty faces of $\Delta$ including $\Delta$ itself. Since $\Delta$ is a simplex, one computes that
\[
\sum_{\tau \in \Delta} ((q - 1)^{\dim \tau} + (-1)^{\dim \tau + 1}) = \frac{q^{n+1} - 1}{q - 1} + (-1) = \frac{q^n - 1}{q - 1}.
\]
This implies that
\[
\#Y_{\lambda}(\mathbb{F}_q) = N_g^* - \frac{(q - 1)^n + (-1)^{n+1}}{q} + \frac{q^n - 1}{q - 1}.
\] (11)
This equality holds for all $\lambda \in \mathbb{F}_q$, including the case $\lambda = 0$. Reducing modulo $q$, we get
\[
\#Y_{\lambda}(\mathbb{F}_q) \equiv N_g^* + 1 - n(-1)^{n-1} \pmod{q}.
\] (12)
This and Theorem 5.1 prove the case $\lambda \neq 0$ of the following theorem.

**Theorem 6.1** For every finite field $\mathbb{F}_q$ with $\lambda \in \mathbb{F}_q$, we have the congruence formula
\[
\#X_{\lambda}(\mathbb{F}_q) \equiv \#Y_{\lambda}(\mathbb{F}_q) \pmod{q}.
\]
If furthermore, $\lambda \in \mathbb{F}_q$ such that $g$ is $\Delta$-regular and $W_{\lambda}$ is a mirror manifold of $X_{\lambda}$, then
\[
\#Y_{\lambda}(\mathbb{F}_q) \equiv \#W_{\lambda}(\mathbb{F}_q) \pmod{q}.
\]
**Proof.** For the first part, it remains to check the case \( \lambda = 0 \). The proof is similar and in fact somewhat simpler than the case \( \lambda \neq 0 \). We give an outline. Since \( \lambda = 0 \), we can take \( k_{n+2} = 0 \) in the calculations of \( N_f \) and \( N_g^* \). One finds then

\[
\# X_0(\mathbb{F}_q) \equiv 1 + \sum^* \frac{1}{q(q-1)} \left( \prod_{j=1}^{n+2} G(k_j) \right),
\]

where \( \sum^* \) means summing over all those solutions \( k = (k_1, \ldots, k_{n+1}, 0) \) with \( 0 \leq k_i \leq q-1 \) and \( k \neq (0, \ldots, 0) \).

Similarly, one computes that

\[
N_g^* \equiv n(-1)^{n-1} + \sum' \frac{1}{q(q-1)} \left( \prod_{j=1}^{n+2} G(k_j) \right),
\]

where \( \sum' \) means summing over all those non-trivial solutions \( k \) with \( k_{n+2} = 0 \). By (12), we deduce

\[
\# Y_0(\mathbb{F}_q) \equiv 1 + \sum' \frac{1}{q(q-1)} \left( \prod_{j=1}^{n+2} G(k_j) \right).
\]

As before, one checks that

\[
\sum^* \frac{n+2}{q(q-1)} \left( \prod_{j=1}^{n+2} G(k_j) \right) \equiv \sum' \frac{n+2}{q(q-1)} \left( \prod_{j=1}^{n+2} G(k_j) \right) (\mod q^2).
\]

The first part of the theorem follows.

To prove the second part of the theorem, let \( \Delta^* \) be the dual polytope of \( \Delta \). One checks that \( \Delta^* \) is the simplex in \( \mathbb{R}^n \) with the vertices

\[(n+1)e_i - \sum_{j=1}^n e_j \quad (i = 1, \ldots, n), \quad -\sum_{j=1}^n e_j.\]

This is the \((n+1)\)-multiple of a basic (regular) simplex in \( \mathbb{R}^n \). In particular, the codimension 1 faces of \( \Delta^* \) are \((n+1)\)-multiples of a basic simplex in \( \mathbb{R}^{n-1} \). By the parallel hyperplane decomposition in [6], one deduces that the codimension 1 faces of \( \Delta^* \) have a triangulation into basic simplices. Fix such a triangulation which produces a smooth crepant resolution \( \phi : W_\lambda \to Y_\lambda \).

One checks [2] that for each point \( y \in Y_\lambda(\mathbb{F}_q) \), the fibre \( \phi^{-1}(\lambda) \) is stratified...
by affine spaces over $\mathbb{F}_q$. Since the fibres are connected, it follows that the number of $\mathbb{F}_q$-rational points on $\phi^{-1}(\lambda)$ is congruent to 1 modulo $q$. Thus, modulo $q$, we have the congruence

$$\#W_\lambda(\mathbb{F}_q) \equiv \sum_{y \in Y_\lambda(\mathbb{F}_q)} \phi^{-1}(\lambda)(\mathbb{F}_q) \equiv \sum_{y \in Y_\lambda(\mathbb{F}_q)} 1 = \#Y_\lambda(\mathbb{F}_q).$$

The proof is complete.

In terms of zeta functions, the above theorem says that the slope $[0, 1)$ part of the zeta function for $X_\lambda$ equals the slope $[0, 1)$ part of the zeta function for $Y_\lambda$.

The above elementary calculations can be used to treat some other examples of toric hypersurfaces and complete intersections. In a forthcoming joint work with Lei Fu, we can prove the following generalization.

**Theorem 6.2** Let $X$ be a smooth connected Calabi-Yau variety defined over the ring $W(\mathbb{F}_q)$ of Witt vectors of $\mathbb{F}_q$. Let $G$ be a finite group acting on $X$. Assume that $G$ fixes the non-zero global section of the canonical bundle of $X$. Then, for each positive integer $k$, we have the congruence formula

$$\#(X \otimes \mathbb{F}_q)(\mathbb{F}_q^k) \equiv \#(X/G \otimes \mathbb{F}_q)(\mathbb{F}_q^k)(\text{mod } q^k).$$

Strictly speaking, this is not a complete generalization of Theorem 6.1 yet, since Theorem 6.1 includes singular cases as well.

## 7 Applications to zeta functions

In this section, we compare the two zeta functions $Z(X_\lambda, T)$ and $Z(Y_\lambda, T)$, where $\{X_\lambda, Y_\lambda\}$ is our strong mirror pair.

First, we recall what is known about $Z(X_\lambda, T)$. Let $\lambda \in \mathbb{F}_q$ such that $X_\lambda$ is smooth projective. By the Weil conjectures, the zeta function of $X_\lambda$ over $\mathbb{F}_q$ has the following form

$$Z(X_\lambda, T) = \frac{P(\lambda, T)(-1)^n}{(1 - T)(1 - qT) \cdots (1 - q^{n-1}T)},$$

where $P(\lambda, T) \in 1 + T\mathbb{Z}[T]$ is a polynomial of degree $n(n - (-1)^n)/(n + 1)$, pure of weight $n - 1$. By the results in [7][9], the universal family of hypersurfaces of degree $n + 1$ is generically ordinary for every $p$ (Mazur’s conjecture). However, we do not know if the one parameter family $X_\lambda$ of hypersurfaces is generically ordinary for every $p$. Thus, we raise
Question 7.1 Is the one parameter family $X_\lambda$ of degree $n+1$ hypersurfaces in $\mathbb{P}^n$ generically ordinary for every prime number $p$ not dividing $(n+1)$?

The answer is yes if $p \equiv 1 \pmod{n+1}$ since the fibre for $\lambda = 0$ is already ordinary if $p \equiv 1 \pmod{n+1}$. It is also true if $n \leq 3$. The first unknown case is when $n = 4$, the quintic case.

Next, we recall what is known about $Z(Y_\lambda, T)$. Let $\lambda \in \mathbb{F}_q$ such that $g$ is $\Delta$-regular. This is equivalent to assuming that $\lambda^n \neq (n+1)^{n+1}$. Then, the zeta function of the affine toric hypersurface $g = 0$ over $\mathbb{F}_q$ in $\mathbb{G}_m^n$ has the following form (see [14])

$$Z(g, T) = Q(\lambda, T)(-1)^n \prod_{i=0}^{n-1} (1 - q^i T)^{(-1)^{n-i} \binom{n}{n+i}},$$

where $Q(\lambda, T) \in 1 + T \mathbb{Z}[T]$ is a polynomial of degree $n$, pure of weight $n-1$.

The product of the trivial factors in $Z(g, T)$ is simply the zeta function of this sequence

$$\frac{(q^k - 1)^n + (-1)^{n+1} q^k}{q^k}, \quad k = 1, 2, \ldots$$

From this and (11), one deduces that the zeta function of the projective toric hypersurface $Y_\lambda$ has the form

$$Z(Y_\lambda, T) = \frac{Q(\lambda, T)(-1)^n}{(1 - T)(1 - qT) \cdots (1 - q^{n-1} T)}. \quad (14)$$

By the results in [9][13], this one parameter family $Y_\lambda$ of toric hypersurfaces is generically ordinary for every $n$ and every prime number $p$.

Now, we are ready to compare the two zeta functions $Z(X_\lambda, T)$ and $Z(Y_\lambda, T)$. Let now $\lambda \in \mathbb{F}_q$ such that $X_\lambda$ is smooth and $g$ is $\Delta$-regular. The above description shows that

$$\frac{Z(X_\lambda, T)}{Z(Y_\lambda, T)} = \left( \frac{P(\lambda, T)}{Q(\lambda, T)} \right)(-1)^n.$$

To understand this quotient of zeta functions, it suffices to understand the quotient $P(\lambda, T)/Q(\lambda, T)$.

Lemma 7.2 The polynomial $Q(\lambda, T)$ divides $P(\lambda, T)$.

Proof. We consider the finite Galois covering $X_\lambda \rightarrow Y_\lambda$ with Galois group $G$, where $G = (\mathbb{Z}/(n+1)\mathbb{Z})^{n-1}$ is an abelian group. For an $\ell$-adic
representation $\rho : G \to \text{GL}(V_\rho)$, let $L(X_\lambda, \rho, T)$ denote the corresponding $L$-function of $\rho$ associated to this Galois covering. Then, we have the standard factorization

$$Z(X_\lambda, T) = \prod_\rho L(X_\lambda, \rho, T),$$

where $\rho$ runs over all irreducible (necessarily one-dimensional) $\ell$-adic representations of $G$. If $\rho = 1$ is the trivial representation, then

$$L(X_\lambda, 1, T) = Z(Y_\lambda, T).$$

For a prime number $\ell \neq p$, the $\ell$-adic trace formula for $Z(X_\lambda, T)$ is

$$Z(X_\lambda, T) = \prod_{i=0}^{2(n-1)} \det(I - TFrob_q|H^i(X_\lambda \otimes \overline{F}_q, \mathbb{Q}_\ell)^G)^{(-1)^{i-1}},$$

where $\text{Frob}_q$ denotes the geometric Frobenius element over $\mathbb{F}_q$. Since $X_\lambda$ is a smooth projective hypersurface of dimension $n-1$, one has the more precise form of the zeta function:

$$Z(X_\lambda, T) = \frac{\det(I - TFrob_q|H^{n-1}(X_\lambda \otimes \overline{F}_q, \mathbb{Q}_\ell)\otimes V_\rho)^G)^{(-1)^n}}{(1 - T)(1 - qT)\cdots(1 - q^{n-1}T)}. \quad (15)$$

Similarly, the $\ell$-adic trace formula for the $L$-function is

$$L(X_\lambda, \rho, T) = \prod_{i=0}^{2(n-1)} \det(I - T(Frob_q \otimes 1)|(H^i(X_\lambda \otimes \overline{F}_q, \mathbb{Q}_\ell) \otimes V_\rho)^G)^{(-1)^{i-1}}.$$

For odd $i \neq n - 1$,

$$H^i(X_\lambda \otimes \overline{F}_q, \mathbb{Q}_\ell) = 0, \quad (H^i(X_\lambda \otimes \overline{F}_q, \mathbb{Q}_\ell) \otimes V_\rho)^G = 0.$$

For even $i = 2k \neq n - 1$ with $0 \leq k \leq n - 1$,

$$H^{2k}(X_\lambda \otimes \overline{F}_q, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-k), \quad (H^{2k}(X_\lambda \otimes \overline{F}_q, \mathbb{Q}_\ell) \otimes V_\rho)^G = 0$$

for non-trivial irreducible $\rho$. This proves that for irreducible $\rho \neq 1$, we have

$$L(X_\lambda, \rho, T) = \det(I - T(Frob_q \otimes 1)|(H^{n-1}(X_\lambda \otimes \overline{F}_q, \mathbb{Q}_\ell) \otimes V_\rho)^G)^{(-1)^n}.$$

Similarly, taking $\rho = 1$, one finds that

$$Z(Y_\lambda, T) = \frac{\det(I - TFrob_q|H^{n-1}(X_\lambda \otimes \overline{F}_q, \mathbb{Q}_\ell))^G)^{(-1)^n}}{(1 - T)(1 - qT)\cdots(1 - q^{n-1}T)}. \quad (16)$$
Comparing (13)-(16), we conclude that

\[ P(\lambda, T) = \det(I - T\text{Frob}_q|H^{n-1}(X_\lambda \otimes \overline{F}_q, \mathbb{Q}_\ell)), \]

\[ Q(\lambda, T) = \det(I - T\text{Frob}_q|(H^{n-1}(X_\lambda \otimes \overline{F}_q, \mathbb{Q}_\ell))^G). \]

Furthermore, the quotient

\[ \frac{P(\lambda, T)}{Q(\lambda, T)} = \prod_{\rho \neq 1} \det(I - T(\text{Frob}_q \otimes 1)|(H^{n-1}(X_\lambda \otimes \overline{F}_q, \mathbb{Q}_\ell) \otimes V_{\rho})^G) \]

is a polynomial with integer coefficients of degree \( \frac{n(n^n - (-1)^n)}{n+1} - n \), pure of weight \( n - 1 \). The lemma is proved.

This lemma together with Theorem 6.1 gives the following result.

**Theorem 7.3** There is a polynomial \( R_n(\lambda, T) \in 1 + T\mathbb{Z}[T] \) which is pure of weight \( n - 3 \) and of degree \( \frac{n(n^n - (-1)^n)}{n+1} - n \), such that

\[ \frac{P(\lambda, T)}{Q(\lambda, T)} = R_n(\lambda, qT). \]

The polynomial \( R_n(\lambda, T) \) measures how far the zeta function of \( Y_\lambda \) differs from the zeta function of \( X_\lambda \). Being of integral pure weight \( n - 3 \), the polynomial \( R_n(\lambda, T) \) should come from the zeta function of a variety (or motive \( M_n(\lambda) \)) of dimension \( n - 3 \). It would be interesting to find this variety or motive \( M_n(\lambda) \) parametrized by \( \lambda \). In this direction, the following is known.

If \( n = 2 \), then \( n - 3 < 0 \), \( M_2(\lambda) \) is empty and we have \( R_2(\lambda, T) = 1 \). If \( n = 3 \), then \( n - 3 = 0 \) and

\[ R_3(\lambda, T) = \prod_{i=1}^{18} (1 - \alpha_i(\lambda)T) \]

is a polynomial of degree 18 with \( \alpha_i(\lambda) \) being roots of unity. In fact, Dwork \[4\] proved that all \( \alpha_i(\lambda) = \pm 1 \) in this case. Thus, \( R_3(\lambda, T) \) comes from the the zeta function of a zero-dimensional variety \( M_3(\lambda) \) parameterized by \( \lambda \). What is this zero-dimensional variety \( M_3(\lambda) \)? For every \( p \) and generic \( \lambda \), the slope zeta function has the form \( S_p(Y_\lambda, u, T) = 1 \) and

\[ S_p(X_\lambda, u, T) = \frac{1}{(1 - T)^2(1 - uT)^20(1 - u^2T)^2}. \]
Note that $Y_\lambda$ is singular and not a smooth mirror of $X_\lambda$ yet. Thus, it is not surprising that the two slope zeta functions $S_p(X_\lambda, u, T)$ and $S_p(Y_\lambda, u, T)$ do not satisfy the expected slope mirror symmetry.

If $n = 4$, then $n - 3 = 1$ and

$$R_4(\lambda, T) = \prod_{i=1}^{200}(1 - \alpha_i(\lambda)T)$$

is a polynomial of degree 200 with $\alpha_i(\lambda) = \sqrt{q}$. Thus, $M_4(\lambda)$ should come from some curve parameterized by $\lambda$. This curves has been constructed explicitly in a recent paper by Candelas, de la Ossa and Fernando-Rodriquez [3]. For every $p$ and generic $\lambda$, we know that $S_p(Y_\lambda, u, T) = 1$, but as indicated at the beginning of this section, we do not know if the slope zeta function of $X_\lambda$ for a generic $\lambda$ has the form

$$S_p(X_\lambda, u, T) = (1 - T)(1 - uT)^{101}(1 - u^2T)^{101}(1 - u^3T)/(1 - T)(1 - uT)(1 - u^2T)(1 - u^3T).$$

For general $n$ and $\lambda \in K$ for some field $K$, in terms of $\ell$-adic Galois representations, the pure motive $M_n(\lambda)$ is simply given by

$$M_n(\lambda) = (\bigoplus_{\rho \not\equiv 1}(H^{n-1}(X_\lambda \otimes \bar{K}, \mathbb{Q}_\ell) \otimes V_\rho)^G) \otimes \mathbb{Q}_\ell(-1),$$

where $\mathbb{Q}_\ell(-1)$ denotes the Tate twist. If $\lambda$ is in a number field $K$, this implies that the Hasse-Weil zeta functions of $X_\lambda$ and $Y_\lambda$ are related by

$$\zeta(X_\lambda, s) = \zeta(Y_\lambda, s)L(M_n(\lambda), s - 1).$$

### 8 Slope zeta functions

The slope zeta function satisfies a functional equation. This follows from the usual functional equation which in turn is a consequence of the Poincare duality for $\ell$-adic cohomology.

**Proposition 8.1** Let $X$ be a connected smooth projective variety of dimension $d$ over $\mathbb{F}_q$. Then the slope zeta function $S_p(X, u, T)$ satisfies the following functional equation

$$S_p(X, u, -1/u^dT) = S_p(X, u, T)(-u^{d/2}T)^{e(X)},$$

where $e(X)$ denotes the the $\ell$-adic Euler characteristic of $X$. 

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**Proof.** Let $P_i(T)$ denote the characteristic polynomial of the geometric Frobenius acting on the $i$-th $\ell$-adic cohomology of $X \otimes \overline{F}_q$. Then,

$$Z(X, T) = \prod_{i=0}^{2d} P_i(T)^{(-1)^{i+1}}.$$ 

Let $s_{ij}$ ($j = 1, \cdots, b_i$) denote the slopes of the polynomial $P_i(T)$, where $b_i$ is the degree of $P_i(T)$ which is the $i$-th Betti number. Write

$$Q_i(T) = \prod_{j=1}^{b_i}(1 - u^{s_{ij}}T).$$

Then, by the definition of the slope zeta function, we have

$$S_p(X, u, T) = \prod_{i=0}^{2d} Q_i(T)^{(-1)^{i+1}}.$$ 

For each $0 \leq i \leq 2d$, the slopes of $P_i(T)$ satisfies the determinant relation

$$\sum_{j=1}^{b_i} s_{ij} = \frac{i}{2} b_i.$$ 

Using this, one computes that

$$Q_i\left(\frac{1}{T}\right) = (-1/T)^{b_i} u^{ib_i/2} \prod_{j=1}^{b_i}(1 - u^{-s_{ij}}T).$$

Replacing $T$ by $u^{dT}$, we get

$$Q_i\left(\frac{1}{u^{dT}}\right) = (-1/u^{dT})^{b_i} u^{ib_i/2} \prod_{j=1}^{b_i}(1 - u^{d-s_{ij}}T).$$

The functional equation for the usual zeta function $Z(X, T)$ implies that $d - s_{ij}$ ($j = 1, \cdots, b_i$) are exactly the slopes for $P_{2d-i}(T)$. Thus,

$$Q_i\left(\frac{1}{u^{dT}}\right) = (-1/u^{dT})^{b_i} u^{ib_i/2} Q_{2d-i}(T).$$

We deduce that

$$S_p(X, u, \frac{1}{u^{dT}}) = \prod_{i=0}^{2d} (Q_{2d-i}(T) (-1/u^{dT})^{b_i} u^{ib_i/2} (-1)^{i+1}).$$
Since $b_i = b_{2d-i}$, it is clear that

$$
\sum_{i=0}^{2d} (-1)^i \frac{i}{2} b_i = \frac{d}{2} e(X).
$$

We conclude that

$$
S_p(X, u, \frac{1}{u^d T}) = S_p(X, u, T) (-T)^{e(X)} u^\frac{d}{2} e(X).
$$

The proposition is proved.

From now on, we assume that $X$ is a smooth projective scheme over $W(F_q)$. Assume that the reduction $X \otimes \mathbb{F}_q$ is ordinary, i.e., the $p$-adic Newton polygon coincides with the Hodge polygon [8]. This means that the slopes of $P_i(T)$ are exactly $j$ ($0 \leq j \leq i$) with multiplicity $h^{j,i}(X)$. In this case, one gets the explicit formula

$$
S_p(X \otimes \mathbb{F}_q, u, T) = \prod_{j=0}^{d} (1 - u^j T)^{e_j(X)},
$$

where

$$
e_j(X) = (-1)^j \sum_{i=0}^{d} (-1)^{i-1} h^{j,i}(X).
$$

If $X$ and $Y$ form a mirror pair over the Witt ring $W(F_q)$, the Hodge symmetry $h^{j,i}(X) = h^{j,d-i}(Y)$ implies for each $j$,

$$
e_j(X) = (-1)^j \sum_{i=0}^{d} (-1)^{i-1} h^{j,d-i}(Y) = (-1)^j e_j(Y).
$$

We obtain the following result.

**Proposition 8.2** Let $X$ and $Y$ be a mirror pair of $d$-dimensional smooth projective Calabi-Yau schemes over $W(F_q)$. Assume that both $X \otimes \mathbb{F}_q$ and $Y \otimes \mathbb{F}_q$ are ordinary. Then, we have the following symmetry for the slope zeta function:

$$
S_p(X \otimes \mathbb{F}_q, u, T) = S_p(Y \otimes \mathbb{F}_q, u, T)^{(-1)^d}.
$$

The converse of this proposition may not be always true. The slope mirror conjecture follows from the following slightly stronger
Conjecture 8.3 (Generically ordinary conjecture) Let $d \leq 3$. Suppose that $\{X, Y\}$ form a maximally generic mirror pair of $d$-dimensional smooth projective Calabi-Yau schemes over $W(\mathbb{F}_q)$. Then, both $X \otimes \mathbb{F}_q$ and $Y \otimes \mathbb{F}_q$ are generically ordinary.

For $d \leq 3$, this conjecture can be proved in the toric hypersurface case using the results in [9][13]. For $d \geq 4$, we expect that the same conjecture holds if $p \equiv 1 \pmod{D}$ for some positive integer $D$. This is again provable in the toric hypersurface case using the results in [9]. But we do not know if we can always take $D = 1$, even in the toric hypersurface case if $d \geq 4$.

References

[1] V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, J. Algebraic Geom., 3(1994), no. 3, 493-535.

[2] V. Batyrev and D.I. Dais, Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry, Topology, 35(1996), no. 4, 901-929.

[3] P. Candelas, X. de la Ossa and F. Rodriguez-Villegas, Calabi-Yau manifolds over finite fields, II, preprint, 2004. [arXiv:hep-th/0402133].

[4] B. Dwork, $p$-Adic cycles, Publ. Math., IHES, 39(1969), 327-415.

[5] B. Dwork, Normalized period matrices II, Ann. Math., 98(1973), 1-57.

[6] G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat, Toroidal Embeddings, Lecture Notes in Math., 339, Springer-Verlag, 1973.

[7] L. Illusie, Ordinarité des intersections complètes générales, in Grothendieck Festschrift, Volume II (1990), 375-405.

[8] B. Mazur, Frobenius and the Hodge filtration, Bull. Amer. Math. Soc., 78(1972), 653-667.

[9] D. Wan, Newton polygons of zeta functions and L-functions, Ann. Math., 137(1993), 247-293.

[10] D. Wan, Dwork’s conjecture on unit root zeta functions, Ann. Math., 150(1999), 867-927.
[11] D. Wan, Higher rank case of Dwork’s conjecture, J. Amer. Math. Soc., 13(2000), 807-852.

[12] D. Wan, Rank one case of Dwork’s conjecture, J. Amer. Math. Soc., 13(2000), 853-908.

[13] D. Wan, Variation of $p$-adic Newton polygons for $L$-functions of exponential sums, Asian J. Math., Vol. 8, 3(2004), 427-474.

[14] D. Wan, Zeta functions of toric Calabi-Yau hypersurfaces, Course notes given at the Arizona Winter School 2004, 22 pages. [http://swc.math.arizona.edu/oldaws/04GenInfo.html](http://swc.math.arizona.edu/oldaws/04GenInfo.html).