Correlation functions at small quark masses with overlap fermions

L. Giusti\textsuperscript{a}, P. Hernández\textsuperscript{b}, M. Laine\textsuperscript{c}, C. Pena\textsuperscript{d}, P. Weisz\textsuperscript{e}, J. Wennekers\textsuperscript{d} and H. Wittig\textsuperscript{d}

\textsuperscript{a}Centre de Physique Théorique, CNRS Luminy, F-13288 Marseille Cedex 9, France
\textsuperscript{b}Dpto. Física Teórica and IFIC, Edificio Institutos Investigación, E-46071 Valencia, Spain
\textsuperscript{c}Faculty of Physics, University of Bielefeld, D-33501 Bielefeld, Germany
\textsuperscript{d}Deutsches Elektronen-Synchrotron, DESY, Notkestr. 85, D-22603 Hamburg, Germany
\textsuperscript{e}Max-Planck-Institut für Physik, Föhringer Ring 6, D-80805 Munich, Germany

We report on recent work on the determination of low-energy constants describing $\Delta S = 1$ weak transitions, in order to investigate the origins of the $\Delta I = 1/2$ rule. We focus on numerical techniques designed to enhance the statistical signal in three-point correlation functions computed with overlap fermions near the chiral limit.

1. INTRODUCTION

Much recent activity in lattice QCD has focused on the determination of low-energy constants (LECs), which parameterise non-perturbative physics in an effective description at low energies. A promising approach involves computing in the $\epsilon$-regime, the kinematical region of arbitrarily small quark masses $m$ in a finite volume $V$ such that $m\Sigma V \lesssim 1$. Here discretisations that preserve chiral symmetry at non-zero lattice spacing should be used so that small masses and momenta can be reached safely. However, simulations in the $\epsilon$-regime are expensive: owing to spontaneous chiral symmetry breaking the spectrum of the Dirac operator becomes arbitrarily dense near the origin, and thus the operator is ill-conditioned. Furthermore, one observes large statistical fluctuations ("spikes") in correlation functions of local currents \cite{1,2} for small masses. A number of efficient numerical tools have been developed \cite{3,4,2}, in which the low modes of the Dirac operator are treated exactly. In ref. \cite{2} we proposed a method, dubbed “low-mode averaging” (LMA), which was shown to eliminate the spikes in two-point correlation functions for quark masses $m \lesssim 1/\Sigma V$. Here we report on the extension of LMA to three-point correlation functions, in order to study non-leptonic kaon decays.

2. LMA REVISITED

We consider the massive Neuberger operator \cite{5}

$$D_m = (1 - \frac{1}{2} \bar{a}m)D + m, \quad \bar{a} = a/(1 + s),$$ \hspace{2cm} (1)

where $|s| < 1$ is a free parameter and the massless Neuberger operator, $D$, satisfies the Ginsparg-Wilson relation. We are interested in correlation functions of the left-handed flavour current \cite{11,2}

$$[J_\mu]_{\alpha\beta} = (\bar{\psi}_\alpha \gamma_\mu P_- \psi_\beta), \quad P_\pm = \frac{1}{2} (1 \pm \gamma_5),$$ \hspace{2cm} (2)

where $\bar{\psi} \equiv (1 - \bar{a}D/2)\psi$, and $\alpha, \beta$ are flavour indices. Using the left-handed current has the advantage that zero modes of the Dirac operator are projected out in correlation functions. However, modes corresponding to small non-zero eigenvalues can still cause large statistical fluctuations. Let $S(x, y)$ denote the quark propagator from $y$ to $x$. LMA works by separating $n_{\text{low}}$ low-lying modes and using the spectral representation of the propagator. In this way the quark propagator is written as

$$S(x, y) = \sum_{k=1}^{n_{\text{low}}} \frac{e_k(x) \otimes e_k(y)^\dagger}{\alpha_k} + S^h(x, y),$$ \hspace{2cm} (3)

where $S^h$ is the propagator in the orthogonal complement of the subspace spanned by the $n_{\text{low}}$
lowest modes. The vector \( e_k \) is given by
\[
e_k = P_\sigma u_k + P_{-\sigma} D_\sigma u_k,
\]
where \(-\sigma\) denotes the chirality of the sector containing zero modes (if any), and \( u_k \) is an approximate eigenmode of \( D_m^\dagger D_m \):
\[
P_\sigma D_m^\dagger D_m P_{-\sigma} u_k = \alpha_k u_k + r_k, \quad (u_k, r_k) = 0.
\]
After inserting the rhs. of eq. (3) into the expression for the correlator \( C(t) \equiv \sum_x \langle J_0(x) J_0(0) \rangle \), one picks up three contributions:
\[
C(t) = C^{ll}(t) + C^{lh}(t) + C^{hh}(t).
\]
As explained in [2], the statistical signal for the correlator is enhanced by exploiting translational invariance in \( C^{ll}(t) \) and \( C^{lh}(t) \), so that these contributions are sampled over many different source points. Applications of LMA to the calculation of \( F_\pi \) in the \( \epsilon \)-regime can be found in [2].

3. \( K \to \pi\pi \) WITH ACTIVE CHARM

In a recent paper [6] we proposed a strategy to investigate the origins of the \( \Delta I = 1/2 \) rule in \( K \to \pi\pi \) decays, i.e. the observed enhancement of the transition amplitude for a two-pion final state with isospin \( 0, |A_0|/|A_2| = 22.1 \).

The goal of our programme is to quantify separately the contributions to the \( \Delta I = 1/2 \) rule from physics at the charm quark mass scale and from “intrinsic” QCD effects due to soft-gluon exchange. A crucial part of our strategy is to keep an active charm quark, so that the theory has an approximate SU(4)_L \times SU(4)_R chiral symmetry. The \( \Delta S = 1 \) effective weak Hamiltonian after the Operator Product Expansion contains the four-quark operators \( Q^\pm_1 \) and \( Q^-_1 \) which are given by
\[
Q^\pm_1 = (\bar{\psi}_\gamma \mu P_\gamma \bar{u})(\bar{\psi}_\gamma \mu P_\gamma \bar{d}) \pm (\bar{\psi}_\gamma \mu P_\gamma \bar{d})(\bar{\psi}_\gamma \mu P_\gamma \bar{u}) \quad (u \to c).
\]
\( Q^\pm_1 \) and \( Q^-_1 \) are singlets under SU(4)_R and transform under irreducible representations of SU(4)_L of dimensions 84 and 20, respectively.

In Chiral Perturbation Theory (ChPT), the ratio of amplitudes \( |A_0|/|A_2| \) is, at leading order, related to a ratio of LECs via
\[
\frac{|A_0|}{|A_2|} = \frac{1}{\sqrt{2}} \left( \frac{1}{2} + \frac{3}{2} \frac{g_1^-}{g_1^+} \right).
\]
Here, the LECs \( g_1^\pm \) multiply the counterparts of the four-quark operators \( Q^\pm_1 \) in the low-energy theory. They can be determined by studying suitable correlation functions of \( Q^\pm_1 \) in QCD and matching them to the corresponding expressions in ChPT. The fact that in the \( \epsilon \)-regime this matching can be performed at NLO without the presence of additional interaction terms with unknown coefficients [6], makes this an attractive setting for carrying out our programme.

The intrinsic QCD contributions to the enhancement can be isolated by determining \( g_1^-/g_1^+ \) in the theory with \( m_u = m_d = m_s = m_c \). Then, by studying the ratio of LECs as \( m_c \) departs from the mass-degenerate limit, the specific contribution of the charm quark to the \( \Delta I = 1/2 \) rule can be investigated in detail.

The complicated renormalisation and mixing patterns of four-fermion operators usually encountered in lattice formulations can be avoided through the use of Ginsparg-Wilson fermions. Indeed, the use of the modified fields \( \bar{\psi} = (1 - \bar{\psi} D/2)\psi \) in eq. (7) guarantees that the renormalisation and mixing of \( Q^\pm_1 \) are like in the continuum theory [6]. In particular, no mixings with lower-dimensional operators can occur. For full details we refer the reader to ref. [6]. An investigation of the effects of the charm quark based on ChPT has been published in [6].

We now report on our attempts to extract the ratio \( g_1^-/g_1^+ \) in the SU(4)-symmetric theory. In this case only “figure-8”-graphs must be considered, since “eye”-graphs cancel exactly for \( m_c = m_u \). We define the following three-point functions of \( Q^\pm_1 \) and the left-handed flavour current:
\[
C^+_1(x_0, y_0) = \sum_{x,y} \langle [J_0(x)]_u | Q^+(0) | [J_0(y)]_s \rangle.
\]

The relation between \( g_1^-/g_1^+ \) and the ratio of correlators is given by
\[
\frac{g_1^-}{g_1^+} H(x_0, y_0) = \frac{k_1^-}{k_1^+} \frac{C^-_1(x_0, y_0)}{C^+_1(x_0, y_0)} \big| \text{ren},
\]
where the factor \( H(x_0, y_0) \) has been computed in ChPT in the \( \epsilon \)-regime at NLO [8]. The factors \( k_1^\pm \) are Wilson coefficients, and it is assumed that \( C^-_1/C^+_1 \) is renormalised in some scheme. From
Figure 1. The fitted value of the ratio \( C^-_1 / C^+_1 \) plotted versus \( m\Sigma V \) for \( n_{\text{low}} = 5 \). We have set \( \Sigma = (250 \text{ MeV})^3 \). eq. 10 one sees that the ratio of correlators is directly proportional to the ratio of LECs.

Our main results were obtained from 638 configurations on a lattice of size \( 12^3 \cdot 30 \) at \( \beta = 5.8485 \), using the 5 lowest modes in the implementation of LMA for figure-8 graphs. In Fig. 1 we plot the ratio \( C^-_1 / C^+_1 \) as a function of the quark mass in units of \( \Sigma V \). It is seen that \( C^-_1 / C^+_1 \) increases as the quark mass is tuned towards smaller values, which, at first sight, might be interpreted as an enhancement in the ratio of LECs from QCD effects alone. However, the quark masses used in the simulations are too heavy to lie inside the \( \epsilon \)-regime, so that the computed NLO matching factor \( H(x_0, y_0) \) cannot be applied to yield \( g^-_1 / g^+_1 \). For smaller masses the statistical signal for the ratio of three-point functions was lost even with LMA.

In order to demonstrate the feasibility of our strategy one has to find ways of obtaining a strong signal in the \( \epsilon \)-regime. We have thus investigated the quality of the signal for three-point functions using LMA with a larger number of low modes [8]. Preliminary results on a lattice of size \( 8^3 \cdot 20 \) at \( \beta = 5.8 \) are shown in Fig. 2 where we have plotted the bootstrap distributions for the ratio of three-point functions \( C^-_1 / C^+_1 \). The plot shows that using \( n_{\text{low}} = 8 \) hardly improves the signal at all, while for \( n_{\text{low}} = 20 \) the distribution is much narrower, so that the statistical signal is indeed enhanced. We conclude that LMA leads to an improvement also for three-point functions, but only if \( n_{\text{low}} \) is increased relative to the previously studied case of two-point correlators. Obviously, using larger values of \( n_{\text{low}} \) leads to a computational overhead, which, however, is outweighed by far through the observed enhancement of the signal. We are currently running on larger spatial volumes in order to corroborate these findings.

Acknowledgements: Simulations were performed on PC clusters at DESY Hamburg, LRZ Munich and University of Valencia. L.G. was supported by the EU under contract HPRN-CT-2002-00311 (EURIDICE). P.H. was supported by the CICYT (FPA2002-00162) and by the Generalitat Valenciana (CTIDIA/2002/5).

REFERENCES
1. W. Bietenholz et al., JHEP 0402 (2004) 023
2. L. Giusti et al., JHEP 0404 (2004) 013
3. L. Giusti, C. Hoelbling, M. Lüscher & H. Wittig, Comput. Phys. Commun. 153 (2003) 31
4. T. DeGrand and S. Schaefer, Comput. Phys. Commun. 159 (2004) 185
5. H. Neuberger, Phys. Lett. B417 (1998) 141; ibid. B427 (1998) 353
6. L. Giusti, P. Hernández, M. Laine, P. Weisz and H. Wittig, hep-lat/0407007
7. P. Hernández and M. Laine, hep-ph/0407086
8. L. Giusti, P. Hernández, C. Pena, P. Weisz, J. Wennekers & H. Wittig, in preparation.