WEIGHTED $L^p$ SOBOLEV REGULARITY OF THE BERGMAN PROJECTION ON THE HARTOGS TRIANGLE

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ABSTRACT. We prove a weighted $L^p$ Sobolev estimate of the Bergman projection on the Hartogs triangle, when $p$ is in the interval $(\frac{4}{3}, 4)$, where the weight is some power of the distance to the singularity at the boundary. This method also applies to the weighted Bergman projection on the Hartogs triangle and to the Bergman projection on the $n$-dimensional variants.

1. INTRODUCTION

1.1. Setup and Background. Let $\Omega$ be a domain in $\mathbb{C}^n$. The set of square integrable holomorphic functions on $\Omega$, denoted by $A^2(\Omega)$, forms a closed subspace of the Hilbert space $L^2(\Omega)$. The Bergman projection associated to $\Omega$, is the orthogonal projection

$$\mathcal{B} : L^2(\Omega) \to A^2(\Omega),$$

which has an integral representation

$$(1.1) \quad \mathcal{B}(f)(z) = \int_{\Omega} B(z, \zeta) f(\zeta) \, d(\zeta),$$

for all $f \in L^2(\Omega)$ and $z \in \Omega$. Here the function $B(z, \zeta)$ defined on $\Omega \times \Omega$ is the Bergman kernel.

Different types of regularity of the Bergman projection are of particular interest. When $\Omega$ is bounded, smooth, and strongly pseudoconvex (or weakly pseudoconvex with additional geometric condition on the boundary, e.g. finite type, property (P), and etc.), the regularity of $\mathcal{B}$ in $W^k(\Omega)$ and hence in $C^\infty(\overline{\Omega})$ have been intensively studied through the literature. See, for example, [Str10] and references therein for details.

As well as the regularity in $W^k(\Omega)$, the regularity of $\mathcal{B}$ in $L^p_k(\Omega)$ and the Hölder estimates of $\mathcal{B}$ also have been considerably studied for many years. We mention some important results here. In [PS77], Phong and Stein dealt with bounded smooth strongly pseudoconvex domains by applying the estimates of the Bergman kernel in [Fef74]. In [NRSW89], [MS94] and [CD06], the corresponding authors studied smoothly bounded pseudoconvex domains of finite type under additional assumptions. In [KR14], Khanh and Raich considered smoothly bounded pseudoconvex domains satisfying $f$-property, and hence obtained the regularity for the finite type case and a class of domains of infinite type.

There are also results for irregularity of $\mathcal{B}$ in $L^p_k(\Omega)$ when the underlying domains are smooth, see [BS12], and regularity of $\mathcal{B}$ in $L^p(\Omega)$ when the underlying domains

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are only assumed to be $C^2$ smooth, see [LS12]. From these we see, besides the smoothness of the underlying domain, that we need additional geometric assumption on the boundary. When dealing with non-smooth domains, we cannot expect the $L^p$ regularity holds for all $p \in (1, \infty)$, and the $A^+_p$ class in some sense interprets the geometric condition on the boundary, see [LS04, Zey13, Che14] for details.

1.2. Results. In this article, we consider the $L^p$ Sobolev regularity of $\mathcal{B}$ on the Hartogs triangle $\mathbb{H}$, where the Hartogs triangle is defined as

$$\mathbb{H} = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| < |z_2| < 1\}.$$ 

It is well known that the topological closure of $\mathbb{H}$ does not possess a Stein neighborhood basis, and the solution to the $\overline{\partial}$-equation on $\mathbb{H}$ is not globally regular. Because the boundary at $(0, 0)$ is not even Lipschitz and this singularity may blow things up, we cannot expect to obtain regularity in the ordinary $L^p_k$ spaces, nor for all $p \in (1, \infty)$, see [CS13, Che13, Che14, CZ14].

A natural way to control the boundary behavior of singularity is the use of weights which measure the distance from the points near the boundary to the singularity at the boundary. Since on the Hartogs triangle we have $|z_2| < |z| < \sqrt{2}|z_2|$ where $z = (z_1, z_2) \in \mathbb{H}$, it is reasonable to consider a weight of the form $|z_2|^s$, for some $s \in \mathbb{R}$. Therefore, we consider the following weighted $L^p_k$ spaces.

**Definition 1.1.** On the Hartogs triangle $\mathbb{H}$, for each $k \in \mathbb{Z}^+ \cup \{0\}$, $s \in \mathbb{R}$, and $p \in (1, \infty)$, we define the **weighted Sobolev space** by

$$L^p_k(\mathbb{H}, |z_2|^s) = \{ f \in L^1_{\text{loc}}(\mathbb{H}) \mid \|f\|_{p,k,s} < \infty \},$$

where the norm is defined as

$$\|f\|_{p,k,s} = \left( \int_{\mathbb{H}} \sum_{|\alpha| \leq k} |D^{|\alpha|}_{z_1}(f)(z)|^p |z_2|^s \, dz \right)^{\frac{1}{p}}.$$ 

Here $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is the multi-index running over all $|\alpha| \leq k$, and

$$D^{|\alpha|}_{z_1} = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \partial \overline{z}_1^{\alpha_3} \partial \overline{z}_2^{\alpha_4}}.$$ 

We also denote the **ordinary (unweighted) Sobolev space** by $L^p_k(\mathbb{H})$, with its usual norm

$$\|f\|_{p,k} = \left( \int_{\mathbb{H}} \sum_{|\alpha| \leq k} |D^{|\alpha|}_{z_1}(f)(z)|^p \, dz \right)^{\frac{1}{p}}.$$ 

With the definition above, we can state our main result as follows.

**Theorem 1.2.** The Bergman projection $\mathcal{B}$ on the Hartogs triangle $\mathbb{H}$ maps continuously from $L^p_k(\mathbb{H})$ to $L^p_k(\mathbb{H}, |z_2|^pk)$ for $p \in (\frac{4}{3}, 4)$.

That is, for each $k \in \mathbb{Z}^+ \cup \{0\}$ and $p \in (\frac{4}{3}, 4)$, there exits a constant $C_{p,k} > 0$, so that

$$\|\mathcal{B}(f)\|_{p,k, pk} \leq C_{p,k} \|f\|_{p,k},$$

for any $f \in L^p_k(\mathbb{H})$.

**Remark 1.3.** If we let $p = 2$ and replace $k$ by $2k$, then our main result will imply the result in [CS13]. Note that, in our result, there is no loss of smoothness of $\mathcal{B}(f)$. 
1.3. **Organization and Outline.** The idea of the proof of the main result is the following. In section 2, we start with the idea from [CS13] to transfer \( \mathbb{H} \) to the product model \( \mathbb{D} \times \mathbb{D}^* \), as well as to transfer the differential operators \( D^\alpha \) to the ones in new variables. From this, we focus on the integration over the punctured disk \( \mathbb{D}^* \) in section 3. We then use a result in [Str86] to convert \( D^\alpha \) acting on the Bergman kernel in the holomorphic component to the ones acting on the kernel in the anti-holomorphic part. The resulting differential operators can be written as a combination of tangential operators, and therefore, integration by parts applies to the smooth functions. Finally, in section 4, we apply the weighted \( L^p \) estimates in [Che14] to our integral, and the resulting integral is majorized by the weighted \( L^p \) norm of \( D^\alpha(f) \). It will complete the proof if we approximate the weighted \( L^p \) functions by smooth functions and transfer the product model back to \( \mathbb{H} \).

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2. **Transfer to the Product Model**

2.1. **Transfer \( \mathbb{H} \) to \( \mathbb{D} \times \mathbb{D}^* \).** In view of Definition 1.1, we adopt the following notations.

**Definition 2.1.** Let \( \beta = (\beta_1, \beta_2) \) be a multi-index, we use the notations below to denote the differential operators

\[
D^\beta_z = \frac{\partial^{[\beta]}_{\beta_1, \beta_2}}{\partial z_{\beta_1} \partial z_{\beta_2}}
\]

and

\[
D^\beta_{z_j, \zbar_j} = \frac{\partial^{[\beta]}_{\beta_1, \beta_2}}{\partial z_{\beta_1} \partial \zbar_{\beta_2}}
\]

for \( j = 1, 2 \).

From the result [Che13, Theorem 1.1], we see that \( B(f) \in A^p(\mathbb{H}) \) (the set of \( L^p \) functions that is holomorphic), whenever \( p \in (\frac{4}{3}, 4) \) and \( f \in L^p(\mathbb{H}) \). So we can rewrite the weighted \( L^p \) Sobolev norm of \( B(f) \) as

\[
\|B(f)\|_{p,k,pk}^p = \sum_{|\beta| \leq k} \int_{\mathbb{H}} |D^\beta_z(B(f))(z)|^p |z_{2}|^{pk} \, dz,
\]

where \( \beta \) and \( D^\beta_z \) are as in Definition 2.1.

In order to transfer \( \mathbb{H} \) to the product model, we first recall the transformation formula for the Bergman kernels.

**Proposition 2.2.** Let \( \Omega_j \) be a domain in \( \mathbb{C}^n \) and \( B_j \) be its Bergman kernel on \( \Omega_j \times \Omega_j \), \( j = 1, 2 \). Suppose \( \Psi : \Omega_1 \to \Omega_2 \) is a biholomorphism, then for \( (w, \eta) \in \Omega_1 \times \Omega_1 \) we have

\[
\det J_{\mathbb{C}}\Psi(w)B_2(\Psi(w), \Psi(\eta)) \det J_{\mathbb{C}}\Psi(\eta) = B_1(w, \eta).
\]

**Proof.** See, for example, [Kra01, Proposition 1.4.12]. \( \square \)
Now let us consider the biholomorphism
\[ \Phi : \mathbb{H} \to \mathbb{D} \times \mathbb{D}^* \]
with its inverse
\[ \Psi : \mathbb{D} \times \mathbb{D}^* \to \mathbb{H}, \]
where
\[ \Phi(z_1, z_2) = \left( \frac{z_1}{z_2}, z_2 \right) \quad \text{and} \quad \Psi(w_1, w_2) = (w_1 w_2, w_2). \]
A simple computation shows \( \det J_C \Psi(w) = w_2 \), for \( w = (w_1, w_2) \in \mathbb{D} \times \mathbb{D}^* \). Therefore, by the proposition above, we have
\[ B(\Psi(w), \Psi(\eta)) = \frac{1}{w_2 \eta_2} \cdot \frac{1}{(1 - w_1 \eta_1)^2} \cdot \frac{1}{(1 - w_2 \eta_2)^2}, \]
where \( B \) is the Bergman kernel on \( \mathbb{H} \times \mathbb{H} \) as in (1.1) and \( (w, \eta) \in \mathbb{D} \times \mathbb{D}^* \times \mathbb{D} \times \mathbb{D}^* \).

2.2. Transfer the Differential Operators. We next need to transfer the differential operators \( D^\beta \) to the ones in the new variable \( w \). We need a lemma.

**Lemma 2.3.** Under the above biholomorphism \( \Phi(z) = w \), for each \( \beta \) let \( m = |\beta| \), we have
\[ D^\beta_z = \sum_{a+b \leq m} p_{a,b,\beta}(w_1) \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b}, \]
where \( p_{a,b,\beta}(w_1) \) is a polynomial of degree at most \( m \) in variable \( w_1 \). In addition, if \( |\beta| \leq k \) for some \( k \in \mathbb{Z}^+ \cup \{0\} \), then \( |p_{a,b,\beta}(w_1)| \leq C_k \) on \( \mathbb{D} \) uniformly in \( \beta, a, \) and \( b \), for some constant \( C_k > 0 \) depending only on \( k \).

**Proof.** We prove (2.3) by induction on \( m = |\beta| \). The case \( m = 0 \) is trivial. When \( m = 1 \), a direct computation shows
\[ \frac{\partial}{\partial z_1} = \frac{1}{w_2} \cdot \frac{\partial}{\partial w_1} \]
and
\[ \frac{\partial}{\partial z_2} = -\frac{w_1}{w_2} \cdot \frac{\partial}{\partial w_1} + \frac{\partial}{\partial w_2}. \]
It is obvious that both of \( \frac{\partial}{\partial z_1} \) and \( \frac{\partial}{\partial z_2} \) are of the form in (2.3).

Suppose for all \( \beta \) with \( |\beta| = m \), \( D^\beta_z \)'s are of the form in (2.3), we now check the case \( |\beta'| = m + 1 \). Note that \( D^\beta_z = \frac{\partial}{\partial z_1} \circ D^\beta_z \) or \( D^\beta_z = \frac{\partial}{\partial z_2} \circ D^\beta_z \) for some \( \beta \). By the inductive assumption, we have
\[ \frac{\partial}{\partial z_1} \circ D^\beta_z = \frac{1}{w_2} \cdot \frac{\partial}{\partial w_1} \circ \sum_{a+b \leq m} p_{a,b,\beta}(w_1) \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b}, \]
\[ = \sum_{a+b \leq m} \frac{p'_{a,b,\beta}(w_1)}{w_2^{m+1-b}} \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b} + \frac{p_{a,b,\beta}(w_1)}{w_2^{m+1-b}} \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b}, \]
\[ = \sum_{a+b \leq m+1} p_{a,b,\beta'}(w_1) \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b}, \]
and
\[
\frac{\partial}{\partial z_2} \circ D^\beta_z = \left( -\frac{w_1}{w_2}, \frac{\partial}{\partial w_1} + \frac{\partial}{\partial w_2} \right) \circ \sum_{a+b \leq m} p_{a,b,\beta}(w_1) \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b}
\]

\[
= \sum_{a+b \leq m} \frac{-w_1 p_{a,b,\beta}(w_1)}{w_2^{m+1-b}} \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b} + \frac{-w_1 p_{a,b,\beta}(w_1)}{w_2^{m+1-b}} \cdot \frac{\partial^{a+b+1}}{\partial w_1^a \partial w_2^b}
\]

\[
\left( b - m \right) p_{a,b,\beta}(w_1) \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b} + p_{a,b,\beta}(w_1) \cdot \frac{\partial^{a+b+1}}{\partial w_1^a \partial w_2^b}
\]

\[
= \sum_{a+b \leq m+1} p_{a,b,\beta}(w_1) \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b}.
\]

We see that \( p_{a,b,\beta}(w_1) \) is obvious a polynomial of degree at most \( m + 1 \) and \( D^\beta_z \) has the form in (2.3).

When \( |\beta| \leq k \), all the possible combinations of derivatives in \( D^\beta_z \) are finite. So there are finitely many different coefficients in all of the \( p_{a,b,\beta}(w_1) \)'s. Since \( |w_1| \leq 1 \) on \( \mathbb{D} \) and \( a, b \leq m \leq k \), we obtain \( |p_{a,b,\beta}(w_1)| \leq C_k \) on \( \mathbb{D} \) as desired. \( \square \)

Now we can transfer \( \mathbb{H} \) to the product model \( \mathbb{D} \times \mathbb{D}^* \) by the biholomorphism \( \Phi \). Combining (2.2) and (2.3), we see that the right hand side of (2.1) becomes

\[
\sum_{|\beta| \leq k} \int_{\mathbb{D} \times \mathbb{D}^*} \left| \sum_{a+b \leq |\beta|} \int_{\mathbb{D} \times \mathbb{D}^*} K_{a,b,\beta}(w, \eta) f(\Psi(\eta)) |\eta_2|^2 \ d\eta \right|^p |w_2|^p+2 \ dw,
\]

where

\[
K_{a,b,\beta}(w, \eta) = \frac{p_{a,b,\beta}(w_1)}{w_2^{|b|-b}} \cdot \frac{\partial^a}{\partial w_1^a} \left( \frac{1}{(1-w_1 \eta_1)^2} \right) \cdot \frac{\partial^b}{\partial w_2^b} \left( \frac{1}{(1-w_2 \eta_2)^2} \right).
\]

3. Convert the Differential Operators on \( \mathbb{D}^* \)

3.1. Convert to the Anti-holomorphic Part. Since \( \mathbb{D}^* \) is a Reinhardt domain, we can apply a result in [Str86].

**Lemma 3.1.** As in (2.4), for the last factor in \( K_{a,b,\beta}(w, \eta) \), we have

\[
\frac{\partial^b}{\partial w_2^b} \left( \frac{\eta_2}{w_2} \right) \left( \frac{1}{(1-w_2 \eta_2)^2} \right) = \frac{1}{w_2} \frac{\partial^b}{\partial w_2^b} \left( \frac{\eta_2}{w_2} \right) - \frac{1}{w_2^2} \frac{\partial^b}{\partial w_2^b} \left( \frac{1}{(1-w_2 \eta_2)^2} \right).
\]

**Proof.** By [Che14, Lemma 3.1], we see that the kernel in (3.1) is the weighted Bergman kernel associated to \( \mathbb{D}^* \) with the weight \( |\eta|^2 \). So we can argue step by step as [Str86, Lemma 2.1 and Remark 2.3] to complete the proof. \( \square \)

3.2. Integration by Parts. Now we focus on the integration over \( \mathbb{D}^* \) in (2.4). We first define a “tangential” operator.

**Definition 3.2.** Let \( S_w = \frac{\partial}{\partial w} \) be the complex normal differential operator on a neighborhood of \( \partial \mathbb{D} \). We define the **tangential operator** by

\[
T_w = \Im(S_w) = \frac{1}{2i} \left( \frac{\partial}{\partial w} - \frac{w}{\partial \overline{w}} \right).
\]
Remark 3.3. Indeed, $T_w$ is well defined on a neighborhood of $\overline{D}$. Moreover, for any disk $D_{\rho} = \{|w| < \rho\}$ of radius $\rho < 1$ with defining function $r_{\rho}(w) = |w|^{2} - \rho^{2}$, we have
\begin{equation}
T_w(\rho) = 0
\end{equation}
on $\partial D_{\rho}$. That is, $T_w$ is tangential on $\partial D_{\rho}$ for all $\rho < 1$.

In order to make use of integration by parts, we need the following lemma.

**Lemma 3.4.** Let $T_w$ be as above, for $b \in Z^+ \cup \{0\}$, we have
\begin{equation}
T_w^b = \sum_{j=0}^{b} c_j w^j \frac{\partial^j}{\partial w^j} \quad (\text{mod } \frac{\partial}{\partial w}),
\end{equation}
where $c_j$'s are constants, $c_b \neq 0$, and $T_w^b$ denotes the composition of $b$ copies of $T_w$.

**Proof.** We prove (3.3) by induction on $b$. The case $b = 0$ is trivial. When $b = 1$, it is easy to see that
\[
T_w = -\frac{1}{2i} \frac{\partial}{\partial w} \quad (\text{mod } \frac{\partial}{\partial w}).
\]

Suppose (3.3) holds for some $b$, then we see that
\[
T_w^b = \sum_{j=0}^{b} c_j w^j \frac{\partial^j}{\partial w^j} + A \circ \frac{\partial}{\partial w},
\]
for some operator $A$. So for the case $b + 1$, we have
\[
T_w \circ T_w^b = \frac{1}{2i} \left( w \frac{\partial}{\partial w} - \frac{\partial}{\partial w} \right) \circ \left( \sum_{j=0}^{b} c_j w^j \frac{\partial^j}{\partial w^j} + A \circ \frac{\partial}{\partial w} \right)
\]
\[
= \frac{1}{2i} \left( \sum_{j=0}^{b} c_j w^j \frac{\partial^j}{\partial w^j} \frac{\partial}{\partial w} - j c_j w^j \frac{\partial^j}{\partial w^j} - c_j w^{j+1} \frac{\partial^{j+1}}{\partial w^{j+1}} \right) + T_w \circ A \circ \frac{\partial}{\partial w}
\]
\[
= \sum_{j=0}^{b+1} c_j' w^j \frac{\partial^j}{\partial w^j} + A' \circ \frac{\partial}{\partial w},
\]
for some constants $c_j'$'s with $c_b' = -\frac{1}{2i} c_b \neq 0$ and some operator $A'$. Therefore, (3.3) holds for $T_w^{b+1}$.

Combine (3.1) and (3.3), since the kernel in (3.1) is anti-holomorphic in $\eta_2$, the inside integration over $\mathbb{D}^*$ w.r.t. variable $\eta_2$ in (2.4) denoted by $I$ becomes
\[
I = \int_{\mathbb{D}^*} \frac{\partial^b}{\partial w^b} \left( \frac{1}{w^2 \eta_2^2} \cdot \frac{1}{(1 - w^2 \eta_2^2)^2} \right) f(\Psi(\eta)) |\eta_2|^2 \, d\eta_2
\]
\[
= \int_{\mathbb{D}^*} \frac{\eta_2^b}{w^b} \frac{\partial^b}{\partial \eta_2^b} \left( \frac{1}{w^2 \eta_2^2} \cdot \frac{1}{(1 - w^2 \eta_2^2)^2} \right) f(\Psi(\eta)) |\eta_2|^2 \, d\eta_2
\]
\[
= \frac{1}{w^b} \int_{\mathbb{D}^*} \sum_{j=0}^{b} c_j T_{\eta_2}^j \left( \frac{1}{w^2 \eta_2^2} \cdot \frac{1}{(1 - w^2 \eta_2^2)^2} \right) f(\Psi(\eta)) |\eta_2|^2 \, d\eta_2
\]
\[
= \frac{1}{w^b} \sum_{j=0}^{b} c_j \lim_{\epsilon \to 0^+} \int_{\mathbb{D}^*} T_{\eta_2}^j \left( \frac{1}{w^2 \eta_2^2} \cdot \frac{1}{(1 - w^2 \eta_2^2)^2} \right) f(\Psi(\eta)) |\eta_2|^2 \, d\eta_2.
\]
Let us assume in addition for a moment that \( f(\Psi(\eta)) \) belongs to \( C^\infty(\overline{\mathbb{B}} - \{0\}) \) in variable \( \eta_2 \). Then by (3.2) we obtain

\[
I = \frac{1}{w_2^b} \sum_{j=0}^{b} c_j \lim_{\epsilon \to 0^+} \int_{\mathbb{B} - \mathbb{D}} T^{j}_{\eta_2} \left( \frac{1}{w_2^b \eta_2} \cdot \frac{1}{(1 - w_2^b \eta_2)^2} \right) f(\Psi(\eta)) |\eta_2|^2 \, d\eta_2
\]

\[
(3.4) = \frac{1}{w_2^b} \sum_{j=0}^{b} c_j (-1)^j \lim_{\epsilon \to 0^+} \int_{\mathbb{B} - \mathbb{D}} \frac{1}{w_2^b \eta_2} \cdot \frac{1}{(1 - w_2^b \eta_2)^2} T^{j}_{\eta_2} \left( f(\Psi(\eta)) |\eta_2|^2 \right) \, d\eta_2
\]

\[
\sum_{j=0}^{b} (-1)^j c_j \int_{\mathbb{B}} \frac{1}{w_2^b \eta_2} \cdot \frac{1}{(1 - w_2^b \eta_2)^2} T^{j}_{\eta_2} \left( f(\Psi(\eta)) |\eta_2|^2 \right) \, d\eta_2,
\]

where the last line follows from the fact that \( T^{j}_{\eta_2}(|\eta_2|^2) = 0 \).

**Definition 3.5.** We use the following notation

\[
F_j(\eta) = T^{j}_{\eta_2} \left( f(\Psi(\eta)) \right) \cdot \eta_2,
\]

\[
B_{1,a}(g)(w_1) = \int_{\mathbb{B}} \frac{\partial^n}{\partial w_1^n} \left( \frac{1}{w_1 \eta_1} \right) g(\eta_1) \, d\eta_1,
\]

for any \( g \) whenever the integral is well defined, and

\[
B_{2}(h)(w_2) = \int_{\mathbb{B}} \frac{h(\eta_2)}{(1 - w_2^b \eta_2)^2} \, d\eta_2,
\]

for any \( h \) whenever the integral is well defined.

By (3.4) and the notation above (Definition 3.5), we see that (2.4) becomes

\[
(3.5) \sum_{|\beta| \leq k} \int_{\mathbb{D} \times \mathbb{B}^*} \left| \sum_{\alpha + b + |\beta| = |\beta| + 1} \frac{\partial^{a,b}}{\partial w_1^{a} w_2^{b}} \left( w_1 \right) \sum_{j=0}^{b} (-1)^j c_j B_{1,a}(B_{\beta}(\eta)_2)(w) \right|^p |w_2|^{2k+2} \, dw.
\]

### 4. Proof of the Main Theorem

#### 4.1. The \( L^p \) Boundedness

**Proposition 4.1.** The Bergman projection on \( \mathbb{D} \) is bounded from \( L^p_k(\mathbb{B}) \) to itself for \( p \in (1, \infty) \) and all \( k \in \mathbb{Z}^+ \cup \{0\} \).

**Proof.** This is a special case of the classical result. See, for example, [PS77] for details, or [KR14] for treatment of domains with more general boundary. \( \square \)

**Proposition 4.2.** The integral operator \( B_2 \) defined as in Definition 3.5 is bounded from \( L^p(\mathbb{D}^*,|w|^{2-p}) \) to itself for \( p \in \left( \frac{3}{4}, 4 \right) \), where \( L^p(\mathbb{D}^*,|w|^{2-p}) \) is the weighted \( L^p \) space with \( w \in \mathbb{D}^* \).

**Proof.** This is equivalent to the statement that the weighted Bergman projection associated to \( \mathbb{D}^* \) with the weight \( |w|^2 \) is bounded from \( L^p(\mathbb{D}^*,|w|^2) \) to itself for \( p \in \left( \frac{3}{4}, 4 \right) \), whose proof can be found in [Che14, Proposition 3.5]. \( \square \)
4.2. The Proof under the Additional Assumption. With Proposition 4.1 and Proposition 4.2, we can prove Theorem 1.2 under the additional assumption \( f(\Psi(\eta)) \in C^\infty(\mathbb{D} - \{0\}) \) in variable \( \eta_2 \).

**Proof of Theorem 1.2 under additional assumption.**

By (2.1), (2.4), (3.5) and Lemma 2.3, we obtain

\[
\|B(f)\|_{p,k,pk}^p \leq \sum_{|\beta| \leq k} \sum_{a+b \leq |\beta|} \sum_{j=0}^b C_{p,k} \int_{\mathbb{D} \times \mathbb{D}^*} |B_{1,a}(B_2(F_j))(w)|^p |w_2|^{p k + 2 - p(|\beta| + 1)} \, dw \\
\leq C_{p,k} \sum_{a+b \leq k} \int_{\mathbb{D} \times \mathbb{D}^*} |B_{1,a}(B_2(F_b))(w)|^p |w_2|^{2 - p} \, dw.
\]

By Proposition 4.1 and Definition 3.5, we see that \( B_{1,a} \) is bounded from \( L^p_a(\mathbb{D}) \) to \( L^p(\mathbb{D}) \). Therefore, for \( p \in (1, \infty) \) we have

\[
\|B(f)\|_{p,k,pk}^p \leq C_{p,k} \sum_{|\beta| + b \leq k} \int_{\mathbb{D} \times \mathbb{D}^*} \left( \int_{\mathbb{D}} \sum_{|\beta| \leq a} |D_{w_1,w_1}(B_2(F_b))(w)|^p \, dw_1 \right) |w_2|^{2 - p} \, dw_2 \\
\leq C_{p,k} \sum_{|\beta| + b \leq k} \int_{\mathbb{D} \times \mathbb{D}^*} \left( \int_{\mathbb{D}} |B_2(D_{w_1,w_1}(F_b))(w)|^p |w_2|^{2 - p} \, dw_2 \right) \, dw_1.
\]

Similarly, by Proposition 4.2 and Definition 3.5, for \( p \in (\frac{1}{3}, 4) \) we have

\[
\|B(f)\|_{p,k,pk}^p \leq C_{p,k} \sum_{|\beta| + b \leq k} \int_{\mathbb{D} \times \mathbb{D}^*} \left( \int_{\mathbb{D}} \sum_{|\beta| \leq a} |D_{w_1,w_1}(F_b)(w)|^p \, dw_1 \right) |w_2|^{2 - p} \, dw_2 \\
= C_{p,k} \sum_{|\beta| + b \leq k} \int_{\mathbb{D} \times \mathbb{D}^*} |D_{w_1,w_1}(T_{w_2}(f(\Psi(w))) \cdot w_2|^p |w_2|^{2 - p} \, dw \\
= C_{p,k} \sum_{|\beta| + b \leq k} \int_{\mathbb{D} \times \mathbb{D}^*} |D_{w_1,w_1}T_{w_2}(f(\Psi(w)))|^p |w_2|^2 \, dw \\
\leq C_{p,k} \sum_{|\beta| + b \leq k} \int_{\mathbb{D} \times \mathbb{D}^*} |D^\beta_{w_1,w_1}D_{w_2,w_2}(f(\Psi(w)))|^p |w_2|^2 \, dw,
\]

where the last line follows from \( T_{w_2} = \frac{1}{2i} \left( w_2 \frac{\partial}{\partial w_2} - \overline{w_2} \frac{\partial}{\partial \overline{w_2}} \right) \), \( |w_2| < 1 \) for \( w_2 \in \mathbb{D}^* \), and a similar equation as (3.3).

By the biholomorphism \( \Psi(w) = z \) defined in section 2, we have

\[
\frac{\partial}{\partial w_1} = w_2 \frac{\partial}{\partial z_1} \quad \text{and} \quad \frac{\partial}{\partial \overline{w_1}} = \overline{w_2} \frac{\partial}{\partial \overline{z}_1},
\]

and also

\[
\frac{\partial}{\partial w_2} = w_1 \frac{\partial}{\partial z_2} + \frac{\partial}{\partial \overline{z}_2} \quad \text{and} \quad \frac{\partial}{\partial \overline{w_2}} = \overline{w_1} \frac{\partial}{\partial \overline{z}_2} + \frac{\partial}{\partial \overline{z}_2}.
\]

Again, since \( (w_1, w_2) \in \mathbb{D} \times \mathbb{D}^* \), we have \( |w_1|, |w_2| < 1 \). Therefore, by (4.1) and transferring \( \mathbb{D} \times \mathbb{D}^* \) back to \( \mathbb{H} \), we finally arrive at

\[
\|B(f)\|_{p,k,pk}^p \leq C_{p,k} \sum_{|\alpha| \leq k} \int_{\mathbb{H}} |D^\alpha_{z,w}(f(z))|^p \, dz.
\]
as desired.

4.3. **Remove the Additional Assumption.** To remove the additional assumption \( f(\Psi(\eta)) \in C^\infty(\overline{\mathbb{D}} - \{0\}) \) in variable \( \eta_2 \), we need the following lemma.

**Lemma 4.3.** The subspace \( C^\infty(\overline{\mathbb{D}} - \{0\}) \cap L_k^p(\mathbb{D}^*, |w|^2) \) is dense in \( L_k^p(\mathbb{D}^*, |w|^2) \) w.r.t. the weighted norm in \( L_k^p(\mathbb{D}^*, |w|^2) \).

**Proof.** The argument is based on [Eva98, §5.3 Theorem 2 and Theorem 3].

Given any \( g \in L_k^p(\mathbb{D}^*, |w|^2) \), fix \( \delta > 0 \). On \( V_0 = \mathbb{D} - \overline{\mathbb{D}_\delta} \), the weighted norm \( L_k^p(V_0, |w|^2) \) is equivalent to the unweighted norm \( L_k^p(V_0) \). Arguing as in the proof of [Eva98, §5.3 Theorem 3], we see that there is a \( g_0 \in C^\infty(\overline{V_0}) \), so that

\[
\|g_0 - g\|_{L_k^p(V_0, |w|^2)} < \frac{\delta}{2j}.
\]

Define \( U_j = \mathbb{D}_\rho - \frac{1}{j} \mathbb{D}_\frac{1}{2} \) for some \( 1 > \rho > \frac{1}{2} \) and for \( j \in \mathbb{Z}^+ \) \((U_1 = \emptyset)\). Let \( V_j = U_{j+3} - U_{j+1} \), then we see \( \bigcup_{j=1}^{\infty} V_j = \mathbb{D}_\rho - \{0\} \). Arguing as in the proof of [Eva98, §5.3 Theorem 2], we can find a smooth partition of unity \( \{\psi_j\}_{j=1}^{\infty} \) subordinate to \( \{V_j\}_{j=1}^{\infty}, \) so that \( \sum_{j=1}^{\infty} \psi_j = 1 \) on \( \mathbb{D}_\rho - \{0\} \). Moreover, for each \( j \), the support of \( \psi_j g \) lies in \( V_j \) \((\text{so } |w| > \frac{1}{j+3})\), and hence \( \psi_j g \in L_k^p(\mathbb{D}_\rho - \{0\}) \). Therefore, we can find smooth function \( g_j \) with support in \( U_{j+4} - U_j \), so that

\[
\|g_j - \psi_j g\|_{L_k^p(\mathbb{D}_\rho - \{0\})} \leq \frac{\delta}{2j},
\]

see [Eva98, §5.3 Theorem 2] for details. Write \( \tilde{g}_0 = \sum_{j=1}^{\infty} g_j \); it is easy to see that \( \tilde{g}_0 \in C^\infty(\mathbb{D}_\rho - \{0\}) \) and

\[
\|\tilde{g}_0 - g\|_{L_k^p(\mathbb{D}_{\rho}, |w|^2)} \leq \|\tilde{g}_0 - g\|_{L_k^p(\mathbb{D}_{\rho}, |w|^2)} \leq \delta,
\]

since \( |w| < 1 \) on \( \mathbb{D}_\rho - \{0\} \).

Let \( V_0' \) be an open set so that \( \partial \mathbb{D} \subset V_0' \) and \( V_0' \cap \overline{\mathbb{D}} = V_0 \), then \( V_0' \cup \mathbb{D}_\rho \) cover \( \overline{\mathbb{D}} \). Take a smooth partition of unity \( \{\tilde{\psi}_1, \tilde{\psi}_2\} \) subordinate to \( \{V_0', \mathbb{D}_\rho\} \), then \( h = \tilde{\psi}_1 g_0 + \tilde{\psi}_2 g_0 \) belongs to \( C^\infty(\overline{\mathbb{D}} - \{0\}) \), and

\[
\|h - g\|_{L_k^p(\mathbb{D}_\rho^*, |w|^2)} \leq C(\|g_0 - g\|_{L_k^p(V_0, |w|^2)} + \|\tilde{g}_0 - g\|_{L_k^p(\mathbb{D}_{\rho}, |w|^2)}) < 2C\delta
\]

as desired. \( \square \)

Now we are ready to remove the extra assumption and prove our main result.

**Proof of Theorem 1.2.**

For any \( f \in L_k^p(\mathbb{D}) \), we have \( f(\Psi(w)) \in L_k^p(\mathbb{D}^*, |w_2|^2) \) in variable \( w_2 \). Then by Lemma 4.3, we can find a sequence \( \{h_j(w)\} \subset C^\infty(\overline{\mathbb{D}} - \{0\}) \) tending to \( f(\Psi(w)) \) in variable \( w_2 \) w.r.t. the norm in \( L_k^p(\mathbb{D}^*, |w_2|^2) \). We have already seen that (4.1) holds for each \( h_j(w) \) replacing \( f(\Psi(w)) \). Indeed, if we focus on the integration over \( \mathbb{D}^* \), by comparing with (2.4), we see that (4.1) is just the following: for each \( b = 0, 1, \ldots, k \)

\[
(4.2) \quad \int_{\mathbb{D}^*} \left| \frac{\partial^b}{\partial w^2} (B_3(h_j)) \right|^p |w_2|^2 \, dw_2 \leq C_{p,k} \|h_j\|_{L_k^p(\mathbb{D}^*, |w_2|^2)},
\]

where \( B_3 \) is the weighted Bergman projection associated to \( \mathbb{D}^* \) with the weight \(|w_2|^2\).
Now letting \( j \to \infty \), in view of the boundedness of \( B_3 \), see [Che14, Proposition 3.5], we see that \( w_2 \frac{\partial f}{\partial w_2}(B_3(h_j)) \) indeed tends to \( w_2 \frac{\partial f}{\partial w_2}(B_3(f(\Psi))) \) in \( L^p(\mathbb{D}^*, |w_2|^2) \) for each \( h = 0, 1, \ldots, k \). Therefore, (4.2) is valid for general \( f(\Psi(w)) \in L^p_k(\mathbb{D}^*, |w_2|^2) \), which completes the proof for any general \( f \in L^p_k(\mathbb{D}) \).

5. Remarks

Remark 5.1. The method here applies to the \( n \)-dimensional variants of the Hartogs triangle. To be precise, for \( j = 1, \ldots, l \), let \( \Omega_j \) be a bounded smooth domain in \( \mathbb{C}^{m_j} \) with a biholomorphic mapping \( \phi_j : \Omega_j \to \mathbb{B}^{m_j} \) between \( \Omega_j \) and the unit ball \( \mathbb{B}^{m_j} \) in \( \mathbb{C}^{m_j} \). We use the notation \( \tilde{z}_j \) to denote the \( j \)th \( m_j \)-tuple in \( z \in \mathbb{C}^{m_1+\cdots+m_l} \), that is \( z = (\tilde{z}_1, \ldots, \tilde{z}_l) \). Let \( n = m_1 + \cdots + m_l + n' \), \( n - n' \geq 1 \), and \( n' \geq 1 \), we define the \( n \)-dimensional Hartogs triangle by

\[
\mathbb{H}^n_{\phi_j} = \left\{ (z, z') \in \mathbb{C}^{m_1+\cdots+m_l+n'} : \max_{1 \leq j \leq l} |\phi_j(\tilde{z}_j)| < |z'_1| < |z'_2| < \cdots < |z'_{n'}| < 1 \right\}.
\]

Then we have the following generalization of Theorem 1.2.

Theorem 5.2. The Bergman projection \( B \) on \( \mathbb{H}^n_{\phi_j} \) is bounded from \( L^p_k(\mathbb{H}^n_{\phi_j}) \) to \( L^p_k(\mathbb{H}^n_{\phi_j}, |z|^p) \) for \( p \in (\frac{2}{n+1}, \frac{2}{n+2}) \).

The idea of the proof remains the same. However, the weight \( |z'_1| \) is no longer comparable to \( (z, z') \), the distance from points near the boundary to the singularity at the boundary.

Remark 5.3. It is not difficult to see that the same method also applies to the weighted Bergman projection associated to the Hartogs triangle with the usual weight. We have the following weighted version of Theorem 1.2.

Theorem 5.4. Let \( B' \) be the weighted Bergman projection associated to the Hartogs triangle with the weight \( |z|^s' \), where \( z = (z_1, z_2) \in \mathbb{H}^n_{\phi_j} \) and \( s' \geq 0 \) with the unique expression \( s' = s + 2l \) for \( l \in \mathbb{Z} \) and \( s \in (0, 2) \). Then \( B' \) is bounded from \( L^p_k(\mathbb{H}^n_{\phi_j}, |z|^s) \) to \( L^p_k(\mathbb{H}^n_{\phi_j}, |z'|^{p+k}) \) for \( p \in \left( \frac{s+2l+4}{s+4}, \frac{s+2l+4}{l+2} \right) \).

We refer the readers to [Che14] for a full consideration of the weighted Bergman projection \( B' \) associated to the Hartogs triangle \( \mathbb{H}^n_{\phi_j} \) with the weight \( |z|^s' \).

Concluding Remark. We have proved a weighted \( L^p \) Sobolev estimate of the Bergman projection on the Hartogs triangle. The estimate seems quite sharp in sense that we cannot derive boundedness of the Bergman projection when \( k = 0 \) and \( p \) equals the endpoints of the given interval, see [Che13]. In the \( n \)-dimensional case, as we mentioned above, \( |z'_1| \) is not comparable to \( (z, z') \). So other methods are required, if we want to consider the \( n \)-dimensional variant \( \mathbb{H}^n_{\phi_j} \) with the weight \( |(z, z')|^{p+k} \) —not \( |z'_1|^{p+k} \). It would be also interesting to consider other types of variants of the Hartogs triangle, and study the \( L^p \) Sobolev regularity of the corresponding Bergman projection.

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