CHROMATIC POLYNOMIAL, COLORED JONES FUNCTION AND Q-BINOMIAL COUNTING

MARTIN LOEBL

ABSTRACT. We define a q-chromatic function on graphs, list some of its properties and provide some formulas in the class of general chordal graphs. Then we relate the q-chromatic function to the colored Jones function of knots. This leads to a curious expression of the colored Jones function of a knot diagram $K$ as a 'defected chromatic operator' applied to a power series whose coefficients are linear combinations of chord diagrams constructed from 'flows' on reduced $K$.

CONTENTS

1. Introduction and Statement of the Results 1
   1.1. q-Bichrome 2
   1.2. Potts partition function 3
   1.3. q-Potts 3
   1.4. Ising partition function 3
   1.5. Van der Waerden Theorem 4
   1.6. Jones polynomial 4
   1.7. Approximating the Jones polynomial 5
   1.8. q-Chromatic function of chordal graphs 5
   1.9. A motivation from the quantum knot theory: colored Jones function 6
   1.10. Chord diagrams, Vassiliev invariants and Kontsevich integral 9
2. Proofs and Comments 9
   2.1. The Principle of Inclusion and Exclusion And The Chromatic Polynomial 9
   2.2. Geometric Summation and Quantum Binomial Formulas 11
   2.3. Proofs 12
   2.4. Proof of theorem 13
   2.5. Categorification of flows: proof of Theorem 14
   References 15

1. Introduction and Statement of the Results

The main purpose of this paper has been a desire to recast the complicated combinatorial construction of 'categorification of flows' of [GL] in more common combinatorial terms. This lead to a definition of the q-chromatic function which I have observed started to live by itself. The final formula may be related to the Kontsevich integral, and perhaps also to the Khovanov’s work on categorification of the Jones polynomial ([KH]). Final push to finish first version of the manuscript came from Stavros Garoufalidis, who informed me about paper [HGR] which establishes a Khovanov homology theory for the chromatic polynomial of graphs.

Let me sketch main steps of the treatment of the colored Jones function $J_n$. We start with the state sum of the $n$-cabling of a knot. Let us denote by $w(s)$ the contribution of state $s$. First we associate to each state $s$ a triple $(f(s), S(s), v(s))$, where $f(s)$ is a non-negative integer flow; you can imagine that the flow lives in...
the reduced knot diagram $K$, even though it turns out to be more convenient to define it on the arc graph of
the knot diagram.

$S(s)$ is a set system on the set of $\sum_e f(s)(e)$ elements where the sum is over all 'jump-up' transitions $e$
of $K$ (which are later associated to 'red edges' of the arc graph). Each such set system will be called simply
'$f(s)$-structure'. The number of $f(s)$-structures for a given flow $f(s)$ is given by a product of binomial
coefficients.

Finally $v(s)$ is a non-negative integer vector of length $\sum_e f(s)(e)$, and for each $i$, $0 \leq v(s)_i \leq (n - 1)$.
We then realise that we can naturally represent $S(s)$ as a chord diagram $ChD(s)$ with $\sum_e f(s)(e)$ chords.
If we denote by $w(n, f, S)$ the sum of $w(s)$ over all states $s$ with $f(s) = f$, $S(s) = S$ and by $G(s)$ the
intersection graph of the chords of $ChD(s)$ then we observe that $w(f, S)$ may be written as

$$Z_{n, f}(t)M_t^{\text{def}}(G(s), n),$$

where $M_t^{\text{def}}$ is a 'defected' q-chromatic function and $Z_{n, f}(t)$ is a Laurent polynomial in $t$ whose precise form
is given in Theorem 10.

This leads to a curious expression of the colored Jones function of a knot diagram $K$ as a power series whose
coefficients are equal to a 'defected chromatic operator' applied to linear combinations of chord diagrams
constructed from 'flows' on reduced $K$.

A graph is a pair $G = (V, E)$ where $V$ is a finite set of vertices and $E$ is a set of unordered pairs of
elements of $V$, called edges. If $e = xy$ is an edge then the vertices $x, y$ are called end-vertices of $e$. A graph
$G' = (V', E')$ is called a subgraph of a graph $G = (V, E)$ if $V' \subset V$ and $E' \subset E$.

1.1. $q$-Bichromate. In this paper we study the following function on graphs:

**Definition 1.1.** Let $G = (V, E)$ be a graph. Let $V = \{1, \ldots, k\}$ and let $V(G, n)$ denote the set of all vectors
$(v_1, \ldots, v_k)$ such that $0 \leq v_i \leq n - 1$ for each $i \leq k$ and $v_i \neq v_j$ whenever $\{i, j\}$ is an edge of $G$. We let

$$M_q(G, n) = \sum_{(v_1, \ldots, v_k) \in V(G, n)} q^{\sum_i v_i}.$$

Note that $M_q(G, z)|_{q=1}$ is the classic chromatic polynomial of $G$. If $G = (V, E)$ is a graph and $A \subset E$ then let $C(A)$ denote the set of the connectivity components of graph $(V, A)$, and if $W \subset C(A)$ then let $|W|$ denote the number of vertices of $W$.

**Theorem 1.**

$$M_q(G, z) = \sum_{A \subset E} (-1)^{|A|} \prod_{W \in C(A)} (z)_q^{|W|}.$$

The following function extensively studied in combinatorics is called bichromate:

$$B(G, a, b) = \sum_{A \subset E} a^{|C(A)|} b^{|A|}.$$

Note that the bichromate is equivalent to the Tutte polynomial (see next section for more details).

For $n > 0$ let $(n)_q = \frac{q^n - 1}{q - 1}$ be a quantum integer. We let $(n)!_q = \prod_{i=1}^n (i)_q$ and for $0 \leq k \leq n$ we define
the quantum binomial coefficients by

$${n \choose k}_q = \frac{(n)!_q}{(k)!_q (n - k)!_q}.$$

The formula of Theorem 1 leads naturally to a definition of $q$-bichromate.

**Definition 1.2.** We let

$$B_q(G, x, y) = \sum_{A \subset E} x^{|A|} \prod_{W \in C(A)} (y)_q^{|W|}.$$
Note that $B_{q=1}(G, x, y) = B(G, x, y)$.

It is well known that the bichromate counts several interesting things in statistical physics. We concentrate on the Potts and Ising partition functions, and on the Jones polynomial, and discuss their q-extensions.

### 1.2. Potts partition function.

**Definition 1.3.** Let $G = (V, E)$ be a graph, $k \geq 1$ integer and $J_e$ a weight (coupling constant) associated with edge $e \in E$. The Potts model partition function is defined as

$$P^k(G, J_e) = \sum_s e^{E(P^k)(s)},$$

where the sum is over all functions $s$ from $V$ to $\{1, \ldots, k\}$ and

$$E(P^k)(s) = \sum_{\{i,j\} \in E} J_{ij} \delta(s(i), s(j)).$$

Following [VN], we may write

$$P^k(G, J_e) = \sum_s \prod_{\{i,j\} \in E} (1 + v_{ij} \delta(s(i), s(j))) = \sum_{A \subseteq E} k^{|C(A)|} \prod_{\{i,j\} \in A} v_{ij},$$

where $v_{ij} = e^{J_{ij}} - 1$. If all $J_{ij}$ are the same, we get an expression of the Potts partition function in the form of the bichromatic polynomial:

**Theorem 2.**

$$P^k(G, x) = \sum_s \prod_{\{i,j\} \in E} e^{x \delta(s(i), s(j))} = \sum_{A \subseteq E} k^{|C(A)|} (e^x - 1)^{|A|} = B(G, e^x - 1, k).$$

### 1.3. q-Potts.

What happens if we replace $B(G, e^x - 1, k)$ by $B_q(G, e^x - 1, k)$? It turns out that this introduces an additional external field to the Potts model.

**Theorem 3.**

$$\sum_{A \subseteq E} \prod_{W \subseteq C(A)} (k)_q^{|W|} \prod_{\{i,j\} \in A} v_{ij} = \sum_s q^{\sum_{v \in V} s(v)} e^{E(P^k)(s)},$$

where $v_{ij} = e^{J_{ij}} - 1$ as above.

### 1.4. Ising partition function.

The Ising partition function $Z(G)$ of a graph $G$ is equivalent to $P^2(G)$:

$$Z(G, J_e) = \sum_s e^{E(Z)(s)},$$

where the sum is over all functions $s$ from $V$ to $\{1, -1\}$ and

$$E(Z)(s) = \sum_{\{i,j\} \in E} J_{ij} s(i)s(j).$$

We immediately have

$$Z(G, J_e) = e^{-\sum_{\{i,j\} \in E} J_{ij}} P^2(G, 2J_e).$$

Not surprisingly, q-bichromate again adds an external field to the Ising partition function.

**Corollary 1.4.**

$$Z(G, x) = e^{-|E|x} B(G, e^{2x} - 1, 2)$$

and

$$\sum_s q^{\sum_{v \in V} s(v)} e^{E(Z)(s)} = q^{-3|V|x} e^{-|E|x} B_q(G, e^{2x} - 1, 2).$$
Hence the formula follows from the Euler formula for planar graphs. Proof. Note that $\cosh(x) = \frac{e^x + e^{-x}}{2}$, $\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\theta(x) = \frac{\sinh(x)}{\cosh(x)}$.

**Theorem 4.**

\[
\sum_s q^{\sum_{v \in V} s(v)} e^{\sum_{(i,j) \in E} J_{ij} s(i) s(j)} = 
\prod_{(i,j) \in E} \cosh(J_{ij}) \sum_{A \subseteq E} \prod_{(i,j) \in A} \theta(J_{ij}) (q - q^{-1})^{o(A)} (q + q^{-1})^{R - o(A)},
\]

where $o(E)$ denotes the number of vertices of $G$ of an odd degree.

**Theorem 5.** Let $G = (V, E)$ be an oriented knot diagram. The following function $f_K(G)$ is a knot invariant:

\[
f_K(G, A) = (-A)^{-3W(G)} \sum_s (-A^2 - A^{-2})^{S(s) - 1} A^{\sum_{v \in V} \text{sign}(v) e_v(s)},
\]

where $W(G) = \sum_{v \in V} \text{sign}(v)$. Moreover the Jones polynomial equals

\[
J(G, A^{-4}) = f_K(G, A).
\]

Each state $s$ determines a subset of edges $E(s)$ of $M(G)$, which are not cut by the splittings of $s$, and it is easy to see that this gives a natural bijection between the set of states and the subsets of edges of $M(G)$. Moreover for each state $s$

\[
\sum_{v \in V} \text{sign}(v) e_v(s) = - \sum_{e \in E(M(G))} b(e) + 2 \sum_{e \in E(s)} b(e).
\]

**Proposition 1.5.**

\[
S(s) = 2c(E(s)) + |E(s)| - |V(M(G))|,
\]

where $c(E(s))$ denotes the number of connectivity components of $(V(M(G)), E(s))$.

**Proof.** Note that $S(s) = f(E(s)) + c(E(s)) - 1$, where $f(E(s))$ denotes the number of faces of $(V(M(G)), E(s))$. Hence the formula follows from the Euler formula for the planar graphs.
Corollary 1.6.
\[ f_K(G, A) = (-A)^{-3W(G)}(-A^2-A^{-2})^{-|V(M(G))|-1}A^{-\sum_{e \in E(M(G))} b(e)} \sum_s (-A^2-A^{-2})^{2c(E(s))} \prod_{e \in E(s)} (-A^2-A^{-2})A^{2b(e)}. \]

This provides an expression of the Jones polynomial of an arbitrary knot-diagram \( G \) with the same sign \( b \) of each crossing as a bichromate.

Corollary 1.7.
\[ J(G, A^{-1}) = (-A)^{-3V(M(G))}(-A^2-A^{-2})^{-|V(M(G))|-1}A^{-b|E(M(G))|}B(M(G), (-A^2-A^{-2})A^{2b}, (-A^2-A^{-2})^{-2}). \]

Question 1. Is \( B_q(M(G), (-A^2-A^{-2})A^{2b}, (-A^2-A^{-2})^{-2}) \) times an appropriate constant also a knot invariant?

We remark that the Jones polynomial of an alternating link also equals to a specialization of the Tutte polynomial of the median graph of its planar projection, [W] Proposition 5.2.14.

1.7. Approximating the Jones polynomial. A positive link (resp. negative link) is one that has a planar projection with positive (resp. negative) crossings only. Notice that the mirror image of a positive link is negative, and vice versa. As remarked by Stavros Garoufalidis, positive links and positive braids play an important role in symplectic aspects of smooth 4-dimensional topology (such as existence of Lefchetz fibrations). For a discussion of how well-known invariants of links behave when restricted to the class of positive links, see [CM, St].

It is well-known that computing the Jones polynomial is a \#P-complete problem (see [W] Sec.6). On the other hand, one may ask about approximating the Jones polynomial. Partial results on the existence of a Fully polynomial randomized approximation scheme (FPRAS, in short) that approximate values of the Tutte polynomial are known, see [AFW]. Hence, it is still possible that Jones polynomial for alternating and positive links may be well approximable.

1.8. q-Chromatic function of chordal graphs. Next we study chordal graphs, i.e. graphs such that each cycle of length at least four has a chord. Let \( G = (V, E) \) be a chordal graph. We fix a linear ordering \( x_1, \ldots, x_k \) of the vertices of \( V \) so that for each \( i \), vertex \( x_i \) is a simplicial vertex, i.e. its neighbourhood is complete, in the subgraph induced by vertices \( x_1, \ldots, x_i \); let \( m(i) \) denote the number of vertices in that neighbourhood of \( x_i \). It is well-known that the existence of such an order of vertices characterises the chordal graphs.

A tree is an acyclic connected graph. We consider trees rooted, i.e. a vertex \( r \) is distinguished in each tree. Hence we will denote trees by a triple \( T = (V, E, r) \). For each vertex \( x \neq r \) of \( T \) there is unique path in \( T \) connecting it to \( r \). The neighbour of \( x \) on it is called predecessor of \( x \) and denoted by \( p(x) \). The set of the vertices of the path without \( x \) is denoted by \( P(x) \). We say that a subtree of a rooted tree starts at its unique nearest vertex to the root.

Another well known characterisation says that a graph is chordal if and only if it is an intersection graph of subtrees of a tree. Let \( G = (V, E) \) be a chordal graph and \( x_1, \ldots, x_k \) the specified ordering of its vertices. Let \( T = (W, F) \) be the tree whose subtrees ‘represent’ \( G \), i.e. there are subtrees \( T_v, v \in V \) of \( T \) so that \( T_u \cap T_v \neq \emptyset \) if and only if \( uv \) is an edge of \( G \). We can choose a vertex \( r \in W \) arbitrarily as a root of \( T \) and define sets \( A_w, B_w, w \in W \) as follows: \( A_w = \{v; T_v \text{starts at} w\} \) and \( B_w = \{v; T_v \text{contains but does not start at} w\} \). Then these have the following properties:

1. the \( A_w \)’s are disjoint and \( V = \cup_i A_i \),
2. \( B_w \subset \cup_{w' \in P(w)} A_{w'} \). In particular \( B_r = \emptyset \).
3. if \( i, j, j' \) are vertices of \( T \) such that \( i \in P(j), j' \in P(j), i \in P(j') \) and \( x \in B_j \cap A_i \) then \( x \in B_{j'} \),
4. if \( x \in A_i, y \in A_j \) and \( x < y \) then \( i \notin P(j) \),
5. \( e \in E \) if and only if \( e \subset A_w \cup B_w \) for some \( w \).

This leads to the following definition of a tree structure.
Definition 1.8. Let \( T = (W,F) \) be a tree, \( V = \{x_1, \ldots, x_k\} \) be an ordered set and sets \( A(w), B(w) : w \in W \) satisfy the above properties 1., 2., 3., 4. Moreover let \( |B_w| = b_w \). Then \( (B_w : w \in W) \) is called a \((T, V, (A_w, b_w : w \in W))\) tree structure (tree structure for short). The set of all structures is denoted by \( \Sigma(T, V, (A_w, b_w : w \in W)) \).

What distinguishes tree structures are the sets \( B_w \). \( B_w \) is an arbitrary subset of \( A_p(w) \cup B_p(w) \) of \( b_w \) elements. Hence we get the following observation.

**Proposition 1.9.** The number of tree structures is \( \prod_{r \neq w \in W} (a_p(w) + b_p(w)) b_w \).

**Remark 1.10.** On the other hand, each tree structure on \( T = (W, F, r), V = \{1, \ldots, k\} \) determines a set \( T_v, 1 \leq v \leq k \) of subtrees of \( T \) so that \( T_u \cap T_v \neq \emptyset \) if and only if \( u, v \in A_w \cup B_w \) for some \( w \in W \), by reversing the construction of the tree structure described above.

Definition 1.11. Let \( S \) be a tree structure, \( v \in \{0, \ldots, z-1\}^V \) and \( x \in A_w \) for some \( w \in W \).

- We denote by \( G(S) \) the unique chordal graph with tree structure \( S \) (see the remark above).
- We let \( m(S,x) \) be the number of \( y \in A_w \cup B_w \) such that \( y < x \). Note that \( m(S,x) \) equals \( b_w \) plus the number of elements of \( A_w \) that are smaller than \( x \). Hence \( m(S,x) \) does not depend on \( S \) and we let \( m(S,x) = m(x) \).
- We let \( V(S,z) = \{v \in \{0, \ldots, z-1\}^V : \{x,y\} \subset A_w \cup B_w \) for some \( w \) then \( v_x < v_y \} \).
- We let \( \text{def}(S,v,x) \) equal to the number of \( y \in A_w \cup B_w \) such that \( y < x \) and \( v_y \neq v_x \).

**Theorem 6.**

\[
\sum_{S \in \Sigma(T,V,(A_w,b_w : w \in W))} \prod_{w \in W} \prod_{j=1}^{|A_w|} \frac{(b_w + j)_w - 1}{b_w + j} M_q(G(S),z) = \prod_{r \neq w \in W} (a_p(w) + b_p(w)) \prod_{x \in V} (z - m(x))_q.
\]

1.9. A motivation from the quantum knot theory: colored Jones function. The motivation to the previous discussion comes from a study of the colored Jones function \( J_n \) done jointly with Stavros Garoufalidis in [GL]. Colored Jones function is the quantum group invariant of knots that corresponds to the \((n+1)\)-dimensional irreducible representation of \( \mathfrak{sl}_2 \). In [GL] a new approach to the colored Jones function, based on the Bass-Ihara-Selberg zeta function of a graph, is presented.

Fix a generic planar projection \( K \) of an oriented knot with \( r \) crossings. Let \( c_i \) for \( i = 1, \ldots, r \) denote an ordering of the crossings of \( K \). Then \( K \) consists of \( r \) arcs \( a_i \), which we label so that each arc \( a_i \) ends at the crossing \( i \). We will single out a specific arc of \( K \) which we decorate by \( \ast \). Without loss of generality, we may assume that the crossings of a knot appear in increasing order, when we walk in the direction of the knot, and that the last arc is decorated by \( \ast \).

Given \( K \), we define a weighted directed graph \( G_K \) as follows:

**Definition 1.12.** The arc-graph \( G_K \) has \( r \) vertices \( 1, \ldots, r, r \) blue directed edges \( (v, v+1) \) (\( v \) taken modulo \( r \)) and \( r \) red directed edges \( (u, v) \), where at the crossing \( u \) the arc that crosses over is labeled by \( a_w \).

The vertices of \( G_K \) are equipped with a sign, where \( \text{sign}(v) \) is the sign of the corresponding crossing \( v \) of \( K \), and the edges of \( G_K \) are equipped with a weight \( \beta \), where the weight of the blue edge \( (v, v+1) \) is \( t^{-\text{sign}(v)} \), and the weight of the red edge \( (u, v) \) is \( 1 - t^{-\text{sign}(u)} \). Here \( t \) is a variable.

Finally, \( G_K \) denotes the digraph obtained by deleting vertex \( r \) from \( G_K \).

It is clear from the definition that from every vertex of \( G_K \), the blue outdegree is \( 1 \), the red outdegree is \( 1 \), and the blue indegree is \( 1 \). It is also clear that \( G_K \) has a Hamiltonian cycle that consists of all the blue edges. We denote by \( e_i^b \) (\( e_i^r \)) the blue (red) edge leaving vertex \( i \).
Example 1.13. For the figure 8 knot we have:

Its arc-graph $G_K$ with the ordering and signs of its vertices is given by

where the blue edges are the ones with circles on them.

Definition 1.14. A flow $f$ on a digraph $G$ is a function $f : \text{Edges}(G) \rightarrow \mathbb{N}$ of the edges of $G$ that satisfies the (Kirkhoff) conservation law

$$\sum_{e \text{ begins at } v} f(e) = \sum_{e \text{ ends at } v} f(e)$$

at all vertices $v$ of $G$. Let $f(v)$ denote this quantity and let $\mathcal{F}(G)$ denote the set of flows of a digraph $G$.

If $\beta$ is a weight function on the set of edges of $G$ and $f$ is an flow on $G$, then the weight $\beta(f)$ of $f$ is given by $\beta(f) = \prod_e \beta(e)^{f(e)}$, where $\beta(e)$ is the weight of the edge $e$.

In order to express the Jones polynomial as a function of the reduced arc graph $G_K$, we need to add the following two structures, which may be read off from the knot diagram.

- We associate in a standard way a rotation $\text{rot}(e)$ to each edge $e$ of $G_K$; the exact definition is not relevant here; it may be found in [GL].
- We linearly order the set of edges of $G_K$ terminating at vertex $v \in \{1, \ldots, r-1\}$ as follows: if we travel along the arc of $K$ corresponding to vertex $v$, we 'see' one by one the arcs corresponding to the starting vertices of red edges entering $v$: this gives the linear order of the red edges entering $v$. Finally there is at most one blue edge entering $v$, and we make it smaller than all the red edges entering $v$. Let $P(e)$ denote the set of predecessors of an edge $e$ in the corresponding linear order.

With these decorations we define

$$\text{rot}(f) = \sum_{e \in E} f(e) \text{rot}(e), \quad \text{exc}(f) = \sum_v \text{sign}(v) f(e_v^b)(\sum_{e \in P(e_v^r)} f(e)), \quad \delta(f) = \text{exc}(f) - \text{rot}(f).$$

Let $S(G)$ denote the set of all admissible subgraphs $C$ of $G$ such that each component of $C$ is a directed cycle. Note that $S(G)$ may be identified with a finite subset of $\mathcal{F}(G)$ since the characteristic function of $C$ is a flow.

Let $K$ be a knot projection. The writhe of $K$, $\omega(K)$, is the sum of the signs of the crossings of $K$, and $\text{rot}(K)$ is the rotation number of $K$, defined as follows: smoothen all crossings of $K$, and consider the oriented circles that appear; one of them is special, marked by $\star$. The number of circles different from the special one whose orientation agrees with the special one, minus the number of circles whose orientation is opposite to the special one is defined to be $\text{rot}(K)$. We further let $\delta(K, n) = 1/2(n^2 \omega(K) + n \text{rot}(K))$, and $\delta(K) = \delta(K, 1)$.

The following theorem appears in [LW] (see also [GL]).
Theorem 7.

\[ J(K)(t) = t^{\delta(K)} \sum_{c \in S(G_K)} t^{\delta(c)} \beta(c). \]

The colored Jones function equals to the Jones polynomial of a proper 'cabling' of the knot diagram. Using graph theory, this may be described as follows.

Definition 1.15. Fix a red-blue digraph \( G \). Let \( G^{(n)} \) denote the digraph with vertices \( a_k^j \) for \( k = 1, \ldots, r \) and \( j = 1, \ldots, n \). \( G^{(n)} \) contains blue directed edges \( (a_k^l, a_k^{l+1}) \) with weight \( t^{-\epsilon \gamma} \) (where \( \epsilon \in \{-1, +1\} \) is the sign of the crossing \( l \)) for each \( l = 1, \ldots, r \) \((l + 1 \mod r)\) and \( j = 1, \ldots, n \). Moreover, if \( (a_k^l, a_k^{l'}) \) is a red directed edge of \( G \), then \( G^{(n)} \) contains red edges \( (a_k^l, a_k^{l'}) \) for all \( i, j = 1, \ldots, n \) with weight \( t^{(j-1)l}(1-t) \) resp. \( t^{-(n-j)(1-t)} \), if the sign of the crossing is -1 resp. +1.

We will denote the set of admissible even subgraphs of \( G^{(n)} \) by \( S_n(G) \). The following theorem appears in [GL].

Theorem 8. For every knot diagram \( K \) and every \( n \in \mathbb{N} \), we have

\[ J_n(K)(t) = t^{\delta(K,n)} \sum_{c \in S_n(G_K)} t^{\delta(c)} \beta(c). \]

Recall that for an integer \( m \), we denote by

\[ (m)_q = \frac{q^m - 1}{q - 1} \]

the quantum integer \( m \). This defines the quantum factorial and the quantum binomial coefficients by

\[ (m)_q! = (1)_q(2)_q \cdots (m)_q \]

\[ \binom{m}{n}_q = \frac{(m)_q!}{(n)_q!(m-n)_q!} \]

for natural numbers \( m, n \) with \( n \leq m \). We also define

\[ \text{mult}_q(f) = \prod_v f(v)^{-\text{sign}(e_v)} \]

One of the key propositions of [GL] is the following expression of the colored Jones function (as a deformed zeta function of the reduced arc graph).

Theorem 9. For oriented knot diagram \( K \) we have:

\[ J_n(K) = t^{\delta(K,n)} \sum_{f \in F(G_K)} \text{mult}_f(f) t^{\delta(f)} \prod_{v \in V_K} t^{-\text{sign}(e_v)f(e_v)} \prod_{e \in \text{red}(e_v) = v} t^{-\text{sign}(e_v)(n-j-S_{e < v} f(e))}. \]

The proof of Theorem 9 is based on a rather complicated combinatorial construction. In this paper we present a curious interpretation of this construction as a deformed q-chromatic operator applied to a power series whose coefficients are linear combinations of chord diagrams.

Definition 1.16. Given reduced knot diagram \( K \) and a flow \( f \) on \( G_K \), we define:

\( J_n(K) = t^{\delta(K,n)} \sum_{f \in F(G_K)} \text{mult}_f(f) t^{\delta(f)} \prod_{v \in V_K} t^{-\text{sign}(e_v)f(e_v)} \prod_{e \in \text{red}(e_v) = v} t^{-\text{sign}(e_v)(n-j-S_{e < v} f(e))}. \)

- A collection \( f_1, \ldots, f_p \) of intervals on \( 1, \ldots, r-1 \) is relevant (for \( K, f \)) if for each \( 1 \leq v \leq r-1 \), the number of intervals starting at \( v \) equals \( \sum f(e): e \text{ red edge entering } v \) and the number of intervals terminating at \( v \) equals \( f(e_v') \). We fix an order on the intervals of the relevant collection starting in the same vertex \( v \), according to \( <_v \).
The Principle of Inclusion and Exclusion

And The Chromatic Polynomial.

A description of the theory of Vassiliev knot invariants and weight systems may be found in a seminal paper \[\text{[BN]}\]. It may be interesting to explore a relationship of the formula of theorem \[\text{10}\] with the Kontsevich integral expression for the colored Vassiliev knot invariants and weight systems.

2. Proofs and Comments

2.1. The Principle of Inclusion and Exclusion And The Chromatic Polynomial. In 1932 Hassler Whitney \[\text{[Wh]}\] deduced a formula for the chromatic polynomial of graphs using the principle of inclusion and exclusion (PIE):
If \( A_1, \ldots, A_n \) are finite sets, and if we let \( \cap (A_i; i \in J) = A_J \) then

\[
| \cup (A_i; i = 1, \ldots, n) | = \sum_{k=1}^{n} (-1)^{k-1} \sum_{J \subseteq \binom{n}{k}} |A_J|.
\]

Let us present one of its folklore proofs, which uses binomial-type counting. We use formula

\[
\prod_{i=1}^{n} (1 + x_i) = \sum_{I \subseteq \{1, \ldots, n\}} \prod_{i \in I} x_i.
\]

Let \( A = \cup_{1 \leq i \leq n} A_i \) and let \( f_i \) denote the characteristic function of \( A_i \) in \( A \). If \( a \in A \) then \( \prod_{i=1}^{n} (1 - f_i(a)) = 0 \), and so by the above formula

\[
\sum_{I \subseteq \{1, \ldots, n\}} (-1)^{|I|} \prod_{i \in I} f_i(a) = 0.
\]

Summing these for each \( a \in A \) we get

\[
0 = \sum_{a \in A} \sum_{I \subseteq \{1, \ldots, n\}} (-1)^{|I|} \prod_{i \in I} f_i(a) = |A| + \sum_{\emptyset \neq I \subseteq \{1, \ldots, n\}} (-1)^{|I|} |\cap_{i \in I} A_i|,
\]

since

\[
\sum_{a \in A} \prod_{i \in I} f_i(a) = |\cap_{i \in I} A_i|.
\]

This is what we wanted to show.

The chromatic polynomial of a graph \( G = (V, E) \), denoted by \( M(G, z) \), equals the number of proper colorings of \( G \) by \( z \) or fewer colors. A proper coloring is assigning one of the colors to each vertex of the graph in such a way that any two vertices which are joined by an edge are of different colors.

Let \( \{v_1, e_1, v_2, e_2, \ldots, v_i, e_i, v_{i+1}, \ldots, e_n, v_{n+1}\} \) be a sequence such that each \( v_j \) is a vertex of a graph \( G \), each \( e_j \) is an edge of \( G \) and \( e_j = v_j v_{j+1} \), and \( v_i \neq v_j \) for \( i < j \) except if \( i = 1 \) and \( j = n + 1 \). If also \( v_1 \neq v_{n+1} \) then \( P \) is called a path of \( G \). If \( v_1 = v_{n+1} \) then \( P \) is called a cycle of \( G \). In both cases the length of \( P \) equals \( n \). When no confusion arises we shall determine paths by listing their edges, namely \( P = (e_1, e_2, \ldots, e_n) \). A graph \( G = (V, E) \) is connected if it has a path between any pair of vertices. If a graph is not connected then its maximum connected subgraphs are called connectivity components. A subgraph of graph \( G \) is spanning if its set of vertices consists of all the vertices of \( G \).

If \( e \in E \) then let \( A_e \) denote the set of the colorings with the property that the end-vertices are of the same color. Then

\[
M(G, z) = z^{|V|} - |\cup_{e \in E} A_e|.
\]

If \( G \) has \((p, s)\) (this is Birkhoff’s symbol) spanning subgraphs of \( s \) edges in \( p \) connectivity components, then by using PIE we get the well-known formula for the chromatic polynomial:

\[
M(G, z) = \sum_{p,s} (p,s)(-1)^{s}z^p.
\]

Let \( G = (V, E) \) be a graph. For \( A \subseteq E \) let \( r(A) = |V| - c(A) \), where \( c(A) \) denotes the number of connectivity components of \( G \). Then we can write

\[
M(z) = z^{c(E)}(-1)^{r(E)} \sum_{A \subseteq E} (-z)^{r(E)-r(A)}(-1)^{|A|-r(A)}.
\]

This leads to Whitney rank generating function \( R(G, u, v) \) defined by

\[
R(G, u, v) = \sum_{A \subseteq E} u^{r(E)-r(A)}v^{|A|-r(A)}.
\]
The Tutte polynomial has been defined by Tutte ([11], [12]) as a minor modification of the Whitney rank generating function.

\[ T(G, x, y) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)}(y - 1)^{|A| - r(A)}. \]

In fact, both Whitney and Tutte polynomials are simply equivalent to a more straightforward but less well-known generalization of the chromatic polynomial, the bichromatic polynomial

\[ B(G, a, b) = \sum_{A \subseteq E} a^{c(A)}b^{|A|}. \]

### 2.2 Geometric Summation and Quantum Binomial Formulas

We all know the geometric summation formula

\[ \sum_{v_1, \ldots, v_k=0}^{n-1} q^{v_1, \ldots, v_k} = \frac{q^n - 1}{q - 1}. \]

The following quantum binomial formula leads to a well-known formula for the summation of the products of distinct powers. We include a proof here in order to keep the paper essentially self-contained.

**Theorem 11.**

\[(a - z)(a - qz) \ldots (a - q^{n-1}z) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} q^{i(i-1)/2} a^{n-i}z^i.\]

**Proof.** We proceed by induction on \(n\). It is easy to check the case \(n = 1\). In the induction step assume the statement holds for \(n\) and we want to prove it for \(n + 1\). Let \(y = qz\). We have

\[(a - z)(a - qz) \ldots (a - q^n z) = (a - z)(a - y) \ldots (a - q^{n-1} y) = (a - z) \sum_{i=0}^{n} (-1)^i \binom{n}{i} q^{i(i-1)/2} a^{n-i}y^i =\]

\[\sum_{i=0}^{n} (-1)^i \binom{n}{i} q^{i(i-1)/2} a^{n+1-i}z^i + \sum_{i=0}^{n} (-1)^{i+1} \binom{n}{i} q^{i(i-1)/2} a^{n-i}z^{i+1}q^i =\]

\[\sum_{i=0}^{n} (-1)^i \binom{n}{i} q^{i(i-1)/2} a^{n+1-i}z^i + \sum_{i=1}^{n+1} (-1)^i \binom{n}{i-1} q^{i(i-1)/2} a^{n+1-i}z^i =\]

\[\binom{n}{0} a^{n+1} + (-1)^{n+1} \binom{n}{n} q^{n(n+1)/2} z^{n+1} +\]

\[\sum_{i=1}^{n+1} (-1)^i q^{i(i-1)/2} a^{n+1-i}z^i q^i \binom{n}{i} + q^{n(n+1)/2} z^{n+1} =\]

\[\sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} q^{i(i-1)/2} a^{n+1-i}z^i\]

since it may be observed directly that

\[q^i \binom{n}{i} + q^{n-i} \binom{n}{i-1} = \binom{n+1}{i}.\]

Examining the coefficient of \(z^k\) in the RHS, we get immediately

**Corollary 2.1.**

\[M_q(K_k, n) = k! \binom{n}{k} q^{k(k-1)/2}.\]
2.3. Proofs.

Proof. (of theorem 1)
If $A \subset E$ then let $W(A, z) = \{v \in \{0, \ldots, z - 1\}^{V} \text{ if } \{i, j\} \in A \text{ then } v_i = v_j \}$.
The next considerations connect the PIE with the geometric series formula.

$$M_q(G, z) = \sum_{v \in \{0, \ldots, z - 1\}^V} q^{\sum_i v_i} - \sum_{v \in \cup_{e \in E} J_e} q^{\sum_i v_i},$$

where $J_e, e = \{i, j\} \in E$, denotes the set of all vectors satisfying $v_i = v_j$.

By PIE this equals

$$\sum_{A \subset E} (-1)^{|A|} \sum_{v \in \cup_{e \in A} J_e} t^{\sum_i v_i} = \sum_{A \subset E} (-1)^{|A|} \sum_{v \in W(A, n)} t^{\sum_i v_i} =$$

$$\sum_{A \subset E} (-1)^{|A|} \prod_{W \in C(A)} \sum_{x \in \{0, \ldots, z - 1\}} q^{|W|x} = \sum_{A \subset E} (-1)^{|A|} \prod_{W \in C(A)} (z)_q^{|W|}.$$

Proof. (of theorem 3)
We have

$$P_q^k(G, J_e) = \sum_s q^{\Sigma_{v \in V} s(v)} \prod_{\{i, j\} \in E} (1 + v_{ij} \delta(s(i), s(j))) =$$

$$\sum_s q^{\Sigma_{v \in V} s(v)} \sum_{A \subset E} \prod_{\{i, j\} \in A} v_{ij} \delta(s(i), s(j)) =$$

$$\sum_{A \subset E} \sum_{s \in W(A, k)} q^{\Sigma_{v \in V} s(v)} \prod_{\{i, j\} \in A} v_{ij} =$$

$$\sum_{A \subset E} \prod_{W \in C(A)} (k)_q^{|W|} \prod_{\{i, j\} \in A} v_{ij}.$$

Proof. (of theorem 4)
Using the identity

$$e^{x(s(i)s(j))} = \cosh(x) + s(i)s(j)\sinh(x),$$

we have:

$$\sum_s q^{\Sigma_{v \in V} s(v)} e^{\Sigma_{\{i, j\} \in E} J_{ij}s(i)s(j)} =$$

$$\sum_s q^{\Sigma_{v \in V} s(v)} \prod_{\{i, j\} \in E} [\cosh(J_{ij}) + s(i)s(j)\sinh(J_{ij})] =$$

$$\prod_{\{i, j\} \in E} \cosh(J_{ij}) \sum_s q^{\Sigma_{v \in V} s(v)} \prod_{\{i, j\} \in E} [1 + s(i)s(j)\sinh(J_{ij})] =$$

$$\prod_{\{i, j\} \in E} \cosh(J_{ij}) \sum_s q^{\Sigma_{v \in V} s(v)} \sum_{A \subset E} \prod_{\{i, j\} \in A} s(i)s(j)\sinh(J_{ij}) =$$

$$\prod_{\{i, j\} \in E} \cosh(J_{ij}) \sum_{A \subset E} \prod_{\{i, j\} \in A} \sinh(J_{ij}) U(A),$$

where

$$U(A) = \sum_s q^{\Sigma_{v \in V} s(v)} \prod_{\{i, j\} \in A} s(i)s(j).$$

Theorem now follows from next Lemma [2.2]
**Lemma 2.2.** Let $G = (V, E)$ be a graph. Then
\[
\sum_s q^{\sum_{v \in V} s(v)} \prod_{(i,j) \in E} s(i)s(j) = (q - q^{-1})^{o(E)}(q + q^{-1})^{V-o(E)},
\]
where the first sum is over all functions $s$ from $V$ to $\{-1, 1\}$ and $o(E)$ denotes the number of vertices of $G$ of an odd degree.

**Proof.** First note that if $E$ is a cycle and $s$ arbitrary then $\prod_{(i,j) \in E} s(i)s(j) = 1$. Hence, we can delete from $G$ any cycle without changing the LHS
\[
\sum_s q^{\sum_{v \in V} s(v)} \prod_{(i,j) \in E} s(i)s(j).
\]
This reduces the proof to the case that $E$ is acyclic. If $E$ is a path, then it follows from the observation above that for $s$ arbitrary, $\prod_{(i,j) \in E} s(i)s(j) = 1$ if and only if $s$ is constant on the end-vertices of $E$. Hence, we can delete from $E$ any maximal path and replace it by the edge between its end-vertices, without changing the LHS. Hence it suffices to prove the proposition for the case that each component of $G$ contains at most one edge. This is however simply true.

\[\square\]

2.4. **Proof of theorem** Let $G$ be a chordal graph and $S$ its tree structure. Then
\[
\sum_{v = (v_1, \ldots, v_k) \in V(G, z)} q^{\sum_i v_i - \text{def}(S, v, i)} = \prod_{i=1}^k (z - m(i))_q.
\]

**Proof.** The basis for the calculation is the following Claim.

**Claim.** Fix numbers $v_1, \ldots, v_{k-1}$ between 0 and $z-1$ so that no edge of $G$ receives two equal numbers. Then

- \[\sum_{v_k : v = (v_1, \ldots, v_k) \in V(G, z)} t^{v_k - \text{def}(v, k)} = A - B + C,\]
- \[A = \sum_{v_k : v \in V(G, z)} t^{v_k}, \quad B = \sum_{i=1}^{m(k)} t^{z - i}, \quad \text{and} \quad C = \sum_{(i, k) \in E(G)} t^{v_i}.\]

**Proof of Claim.** Note that the second part simply follows from the first one. Let $v'_1 < \cdots < v'_{m(k)}$ be a reordering of $\{v_i : \{i, k\} \in E(G)\}$. We may write $v'_i = z - i_1, \ldots, v'_{m(k)} = z - i_{m(k)}, 1 \leq i_{m(k)} < \cdots < i_1$. The LHS becomes
\[
\left( \sum_{v_k : v \in V(G, z)} t^{v_k} \right) - t^{z-i_1+1} - \ldots - t^{z-i_2-1} - t^{z-i_2+1} - \ldots - t^{z-i_{m(k)}-1} - t^{z-i_{m(k)}+1} - \ldots - t^{z-1} + t^{z-i_1} + \ldots +
\]
\[
t^{z-i_2-2} + t^{z-i_2-1} + \ldots + t^{z-m(k)-1}].
\]
This equals to the RHS of the equality we wanted to show. The Proposition simply follows from the Claim.

\[\square\]

The proof of Proposition yields the following

**Proposition 2.4.** Let $S$ be a structure. Then
\[
\sum_{v \in V(S, z)} \prod_{x \in V} q^{v_x - \text{def}(S, v, x)} = \prod_{x \in V} (z - m(x))_q.
\]

13
Hence
\[ \sum_{v \in V(S,z)} \prod_{x \in V} q^{v_x - \text{def}(S,v,x)} \]
is invariant for \( S \in \Sigma(T,V,(A_w,b_w : w \in W)) \). Note that the same is not true for the non-defected version: path of three edges and star of three edges, with their tree being the path, provide a contraexample.

**Proof.** (of Theorem 4)

\[
\sum_{S \in \Sigma(T,V,(A_w,b_w : w \in W))} \sum_{v \in V(S,z)} \prod_{x \in V} q^{v_x - \text{def}(S,v,x)} = \sum_{S = (B_w : w \in W)} \sum_{v \in V(S,z)} \prod_{x \in V} q^{v_x - \text{def}(S,v,x)} =
\]

\[
\sum_{v_x : x \in A_r, B_w : p(w) = r} \prod_{x \in A_r} q^{v_x - \text{def}(S,v_x)} \sum_{(B_w : p(w) \neq r)} \sum_{v_x \notin A_r, v \in V(S,z)} \prod_{x \in V} q^{v_x - \text{def}(S,v,x)} =
\]

\[
\sum_{v(r)' : B_w : p(w) = r} (|A_r|)!q^{-1} \prod_{|A_r|} q^{v(r)'_x} \sum_{(B_w : p(w) \neq r)} \sum_{v_x \notin A_r, v \in V(S,z)} \prod_{x \in V} q^{v_x - \text{def}(S,v,x)},
\]

where the second sum is over all vectors \( v(r)' = (v(r)'_x : x \in A_r) \) so that \( v(r)'_x > v(r)_y \) for \( x < y \). In the above equality we used

\[
\sum_{\pi} q^{\text{number}(i,j); i < j, \pi(i) < \pi(j)} = (n)_q!
\]

This further equals

\[
\sum_{B_w : p(w) = r} (|A_r|)!q^{-1} \sum_{v(r)' : A_r} \prod_{|A_r|} q^{v(r)'_x} \sum_{(B_w : p(w) \neq r)} \sum_{v_x \notin A_r, v \in V(S,z)} \prod_{x \in V} q^{v_x - \text{def}(S,v,x)} =
\]

\[
\sum_{B_w : p(w) = r} \frac{|A_r|!}{|A_r|!} q^{-1} \sum_{v_x : x \in A_r} \prod_{|A_r|} q^{v_x} \sum_{(B_w : p(w) \neq r)} \sum_{v_x \notin A_r, v \in V(S,z)} \prod_{x \in V} q^{v_x - \text{def}(S,v,x)} =
\]

\[
\sum_{S \in \Sigma(T,V,(A_w,b_w : w \in W))} \prod_{j = 1}^{n_A} \frac{(b_w + j)q^{-1}}{b_w + j} \prod_{v \in V(S,z)} q^{v_x}.
\]

Theorem now follows from Proposition 2.1. \( \square \)

### 2.5. Categorification of flows: proof of Theorem 10

Recall Theorem 8. Each \( c \in S_n(G_K) \) projects to a flow on \( G_K \). An analysis of the contribution of each flow is obtained via categorification of the flows and their multiplicities in \( \Sigma_1 \). Next we briefly describe this.

Let \( f \) be a flow on \( G_K \). Let \( F \) (resp. \( F_r \)) denote the multiset that contains each edge (resp. red edge) \( e \) of \( G_K \) with multiplicity \( f(e) \).

Let \( F_r^e \) denote the set of all red edges of \( F \) which leave a vertex with + sign. Let \( f_r^+ = |F_r^e| \). Analogously we define \( F_r^- \), . . . .

If \( e \) is an edge of \( G_K \) then we let \( F(e) \subset F \) be the set of all copies of \( e \) in \( F \), and fix an arbitrary total order on each \( F(e) \). We also denote by \( t(e) \) the terminal vertex of \( e \).

**Definition 2.5.** Fix a flow \( f \) on \( G_K \). A flow configuration of \( f \) is a sequence \( C = (C_1, \ldots, C_{r-2}) \) so that \( C_1 \) is a subset of \( \{ e \in F_r ; e \text{ terminates in vertex 1} \} \) of \( f(e^1) \) elements and for each \( 2 \leq i < r - 1 \), \( C_i \) is a subset of \( C_{i-1} \cup \{ e \in F_r ; e \text{ terminates in vertex } i \} \) of \( f(e^i) \) elements.

Let us denote by \( C(f) \) the set of all flow configurations of \( f \).

**Definition 2.6.** Let \( f \) be a flow on \( G_K \), \( n > 0 \) a natural number, \( C \in C(f) \), and \( v \in \{ 0, \ldots, n-1 \}^F_r \). We say that a pair \( P = (C,v) \), is admissible if, for every two edges \( e, e' \in F_r \) such that \( v_e = v_{e'} \) and \( e \) ends in vertex \( i \) and \( e' \) ends in vertex \( j \) and \( j \geq i \), there exists an \( l, i \leq l < j \) such that \( e \notin C_l \). We denote the set of admissible flow configurations by \( AC(f) \).
Definition 2.7. Let \( e' \in F(e) \). We define set \( P(f, e') \) as follows: if \( e_1, e_1' \in F(e) \) then \( e_1' \in P(f, e) \) if \( e_1 \in P(e) \) in \( G_K \) or \( e = e_1 \) and \( e_1' < e' \) in our fixed total order of \( F(e) \).

Definition 2.8. Let \( e \in F_r \). We define:

- \( \text{def}_1(C, v, e) = \{ \{ e' \in P(f, e) : v_{e'} < v_e \} \} \),
- \( \text{def}_2(C, v, e) = \{ \{ e' \in C_d(e) : v_{e'} < v_e \} \} \), where \( d(e) \) is the biggest index such that \( d(e) \geq t(e) \) and \( e \not\in C_d(e) \).

Definition 2.9. If \( v \in \{0, \ldots, n-1\}^F \) then we define \( f_r^-(v) = \sum_{e \in F_r^-} v_e \) and we define \( f_r^+(v) \) analogously.

The following theorem appears in [GL].

Theorem 12.

\[
J_n(K)(t) = t^{5(n,n)} \sum_{f \in F(G_K)} \prod_{e \in F_r^+} t^{-n-1|P(f, e)|} \prod_{t: \text{sign}(t) = +} \delta(f, t) \delta(e, t) \sum_{(C, v) \in AC(f, n)} \prod_{e \in F_r} v_e - \text{def}_1(C, v, e) - \text{def}_2(C, v, e).
\]

Finally we observe the relation of the flow structures and relevant collections of intervals. Hence Theorem [12] implies Theorem [10].

From flow structures to relevant collections of intervals. The formula of Theorem 12 may be interpreted in terms of chordal graphs. The basic observation is that each flow structure is a \((T, V, (A_w, b_w : w \in W)) \)– tree structure where \( T \) is a path with vertices \( 1, \ldots, r-1 \) rooted at \( 1, B_{w+1} = C_w \) and \( A_w = \{ e \in F_r^+ : e \) terminates in vertex \( w \} \). Now we recall that each chordal graph is the intersection graph of subtrees of a tree, and this representation may be obtained from its tree structure (see subsection [CN]). However, if a tree structure of a graph is indexed by a path, then it is the intersection graph of subpaths (intervals) of the path. This directly leads to the relevant collection of intervals, and to the proof of Theorem [10].

References

[K] C. Kassel, Quantum Groups, Springer-Verlag 1995.

[Wh] H. Whitney, A logical expansion in mathematics, Bull. Amer. Math. Soc. 38, 572-579, 1932.

[T1] W.T. Tutte, A ring in graph theory, Proc. Camb. Phil. Soc. 43, 26-40, 1947.

[GL] S. Garoufalidis, M. Loeb, A probabilistic view of the Jones polynomial, 2004 manuscript.

[T2] W.T. Tutte, A contribution to the theory of chromatic polynomials, Canad. J. Math. 6, 80-91, 1954.

[VN] O.A. Vasilyev, S.K. Nechaev, Thermodynamics and Topology of Disordered Systems: Statistics of the random Knot Diagrams on Finite Lattices, J. Experimental and Theoretical Physics 92, 1119-1136, 2001.

[BN] D. Bar-Natan, On the Vassiliev knot invariants, Topology 34, 423-475, 1995.

[K1] L.H. Kauffman, State models and the Jones polynomial, Topology 26, 395, 1987.

[AFW] N. Alan, A. Frieze and D. Welsh, Polynomial time randomized approximation schemes for Tutte-Gröthendieck invariants: the dense case, Random Structures Algorithms 6 (1995) 459-478.

[CP] Y. Chari and A. Pressley, A guide to quantum groups, Cambridge University Press, 1994.

[CM] P.R. Cromwell and H.R. Morton, Positivity of knot polynomials on positive links, J. Knot Theory Ramif. 1 (1992), 203-206.

[HGR] L. Helme-Guizon and Y. Rong, A Categorification for the Chromatic Polynomial. math.CO/0412264v1.

[J1] V.F.R. Jones, Hecke algebra representation of braid groups and link polynomials, Annals Math. 126 (1987) p. 335–388.

[Kh] M. Khovanov, A Categorification of the Jones Polynomial, Duke Math. J. 101 (2000) 359-426.

[J2] On knot invariants related to some statistical mechanical models, Pacific J. Math. 137 (1989) 311-334.

[St] A. Stoimenow, Positive knots, closed braids and the Jones polynomial, preprint math.GT/9805078.

[W] D.J.A. Welsh, Complexity: knots, colourings and counting, London Math. Soc. Lecture Note Series 186 Cambridge Univ. Press, 1993.

[B] H. Bass, The Ihara-Selberg zeta function of a tree lattice, Intern. J. Math. 3 (1992) 717-797.

[FZ] D. Foata and D. Zeilberger, A combinatorial proof of Bass’s evaluation of the Ihara-Selberg zeta function for graphs, Transactions Amer. Math. Soc. 351 (1999) 2257-2274.

[LW] and Z. Wang, Random Walk on Knot Diagrams, Colored Jones Polynomial and Ihara-Selberg Zeta Function, preprint 1998 math.GT/9812039.
[FRT] L. Fadeev, N. Reshetikhin, L. Takhtadjian, *Quantization of Lie groups and Lie algebras*, Leningrad Math Journal 1(1990), 193-225.

Dept. of Applied Mathematics and, Institute of Theoretical Computer Science (ITI), Charles University, Malostranske n. 25, 118 00 Praha 1, Czech Republic.

E-mail address: loebl@kam.mff.cuni.cz