Effects of collisions on conservation laws in gyrokinetic field theory

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Effects of collisions on conservation laws for toroidal plasmas are investigated based on the gyrokinetic field theory. Associating the collisional system with a corresponding collisionless system at a given time such that the two systems have the same distribution functions and electromagnetic fields instantaneously, it is shown how the collisionless conservation laws derived from Noether’s theorem are modified by the collision term. Effects of the external source term added into the gyrokinetic equation can be formulated similarly with the collisional effects. Particle, energy, and toroidal momentum balance equations including collisional and turbulent transport fluxes are systematically derived using a novel gyrokinetic collision operator, by which the collisional change rates of energy and canonical toroidal angular momentum per unit volume in the gyrocenter space can be given in the conservative forms. The ensemble-averaged transport equations of particles, energy, and toroidal momentum given in the present work are shown to include classical, neoclassical, and turbulent transport fluxes which agree with those derived from conventional recursive formulations.

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I. INTRODUCTION

Gyrokinetic theories and simulations are powerful means to investigate microinstabilities and turbulent transport processes in magnetically confined plasmas.1–4 Originally, gyrokinetic equations are derived by recursive techniques combined with the WKB or ballooning representation.5–10 On the other hand, modern derivations of the gyrokinetic equations are based on the Lagrangian and/or Hamiltonian formulations11 in which conservation laws for the phase-space volume and the magnetic moment are automatically ensured by Liouville’s theorem and Noether’s theorem, respectively.12 Besides, conservation of the total energy and momentum is naturally obtained in the gyrokinetic field theory, where all governing equations for the distribution functions and the electromagnetic fields are derived from the Lagrangian which describes the whole system consisting of particles and fields.11–13 A subtle point regarding the Lagrangian/Hamiltonian gyrokinetic formulations is that they basically treat collisionless systems so that Noether’s theorem and conservation laws do not hold directly for collisional systems. In this paper, we examine how the collision and external source terms added into the gyrokinetic equations influence the conservation laws derived from Noether’s theorem in the gyrokinetic field theory for collisionless systems.

For a given collisional kinetic system, we can imagine a corresponding collisionless kinetic system such that the two systems have the same distribution functions and electromagnetic fields instantaneously. As an example of two such systems, the Boltzmann-Poisson-Ampère system and the Vlasov-Poisson-Ampère system are considered in Sec. II, where we express the variation of the action integral for the latter collisionless system in terms of the distribution functions and the electromagnetic fields for the former collisional system to show how the conservation laws derived from Noether’s theorem in the collisionless system are modified in the collisional system with external sources of particles, energy, and momentum. There, we confirm the natural result that, when adding no external sources but only the collision term that conserves the energy and momentum, the energy and momentum conservation laws for the Boltzmann-Poisson-Ampère system take the same forms as those for the Vlasov-Poisson-Ampère system. The above-mentioned procedures are repeated in Sec. III to treat the collisional and collisionless gyrokinetic systems. In our previous work,14 using the gyrokinetic Vlasov-Poisson-Ampère system of equations, conservation laws of particles, energy, and toroidal angular momentum are obtained for collisionless toroidal plasmas, in which the slow temporal variation of the background magnetic field is taken into account in order to enable self-consistent treatment of physical processes on transport time scales. Based on these results, the particle, energy, and toroidal angular momentum balance equations for the collisional plasma are derived from the gyrokinetic Boltzmann-Poisson-Ampère system of equations in Secs. IV and V. In Sec. VI, it is shown by taking the ensemble average of these balance equations that the particle, energy, and toroidal angular momentum transport fluxes are given by the sum of the conventional expressions of the classical, neoclassical, and turbulent transport fluxes to the lowest order in the normalized gyroradius parameter. Conclusions are given in Sec. VII and formulas for transformation from particle to gyrocenter coordinates are presented in Appendix A.

Regarding the collision operator for the gyrokinetic equation, several works have been done, which take account of finite-gyroradius effects to modify the Landau collision operator defined in the particle coordinates.15–27
The relation of the collision operator in the gyrocenter coordinates to that in the particle coordinates is explained in Appendix B. The Landau operator for Coulomb collisions conserves particles’ number, kinetic energy, and momentum locally at the particle position although, in the gyrocenter position space, collisions induce transport fluxes of particles, energy, and momentum. Besides, it is emphasized in this work that the collisional change rates of the gyrocenter Hamiltonian (which includes not only the kinetic energy but also the potential energy) and of the canonical momentum (instead of the kinetic momentum) per unit volume in the gyrocenter space should take the conservative (or divergence) forms in order to properly derive the energy and momentum conservation laws for the collisional gyrokinetic system. The approximate collision operator which keeps these conservation properties of the gyrocenter energy and canonical toroidal angular momentum is shown in Appendix C. It is noted that another form of the gyrokinetic collision operator, which satisfies the energy and momentum conservation laws, has recently been presented by Burby et al. Equations (11), (12), and (13) are the governing equations for the Boltzmann-Poisson-Ampère system.

Suppose that $f_a$, $\phi$, and $A$ which satisfy Eqs. (11) are given. Then, for the electromagnetic fields $E = -\nabla \phi - c^{-1} \partial A / \partial t$ and $B = \nabla \times A$ given from $\phi$ and $A$, we consider the distribution function $f_a^V$ which is the solution of the Vlasov equation,

$$ \frac{\partial}{\partial t} f_a^V(x,v,t) + v \cdot \nabla f_a^V(x,v,t) + \frac{e_a}{m_a} \left( E(x,t) + \frac{1}{c} v \times B(x,t) \right) \cdot \nabla_x f_a^V(x,v,t) = 0. \quad (4) $$

We also assume $f_a^V$ to coincide instantaneously with $f_a$ at a given time $t_0$ so that $f_a^V(x,v,t_0) = f_a(x,v,t_0)$. Therefore, equations obtained from Eqs. (2) and (3) with $f_a$ replaced by $f_a^V$ also hold at $t_0$. In other words, $f_a^V$, $\phi$, and $A$ satisfy the Vlasov-Poisson-Ampère system of equations at $t_0$. Note that the Vlasov-Poisson-Ampère system of equations can be derived from the variational principle using the action $I$ defined by Eq. (1) in Ref. 29 where its variation $\delta I$ associated with infinitesimal transformations of independent and dependent variables [see Eqs. (10) and (15) in Ref. 29] are explicitly shown in order to apply Noether’s theorem for obtaining conservation laws of energy and momentum. Now, let us use $f_a^V$, $\phi$, and $A$ to define the action integral $I$ over a small time interval, $t_0 - h/2 \leq t \leq t_0 + h/2$, during which the Vlasov-Poisson-Ampère system of equations are approximately satisfied by $f_a^V$, $\phi$, and $A$ within the errors of order $h$. Then, neglecting the errors of higher order in $h$, the variation $\delta I$ can be written in the same form as in Eq. (15) of Ref. 29,

$$ \delta I = - \int_{t_0 - h/2}^{t_0 + h/2} \! dt \int d^3 x \left[ \frac{\partial}{\partial t} \frac{\partial G}{\partial \phi} (x,t) \right. $$

and any vector field is written as $j = j_L + j_T$, where $j_L = -(4\pi)^{-1} \nabla \int d^3 x (\nabla \cdot j)/|x - x'|$ and $j_T = (4\pi)^{-1} \nabla \times (\nabla \times \int d^3 x j'/|x - x'|)$ represent the longitudinal (or irrotational) and transverse (or solenoidal) parts, respectively. Equations (11), (12), and (13) are the governing equations for the Boltzmann-Poisson-Ampère system.

II. BOLTZMANN-POISSON-AMPÈRE SYSTEM

In this section, conservation laws are investigated for the Boltzmann-Poisson-Ampère system of equations which provide the basis of approximate description by the collisional electromagnetic gyrokinetic system of equations for strongly magnetized plasmas considered in Secs. III–VI. Time evolution of the distribution function $f_a(x,v,t)$ for particle species $a$ is described by the Boltzmann kinetic equation,

$$ \frac{\partial}{\partial t} f_a(x,v,t) + v \cdot \nabla f_a(x,v,t) + \frac{e_a}{m_a} \left( E(x,t) + \frac{1}{c} v \times B(x,t) \right) \cdot \nabla_x f_a(x,v,t) = \mathcal{K}_a(x,v,t), \quad (1) $$

where $\mathcal{K}_a(x,v,t)$ denotes the rate of change in the distribution function $f_a$ due to Coulomb collisions and it may also include other parts representing external particle, momentum, and/or energy sources if any. The electromagnetic fields $E(x,t)$ and $B(x,t)$ are written as $E = -\nabla \phi - c^{-1} \partial A / \partial t$ and $B = \nabla \times A$, where the electrostatic potential $\phi$ and the vector potential $A$ are determined by Poisson’s equation,

$$ \nabla^2 \phi(x,t) = -4\pi \sum_a e_a \int f_a(x,v,t) d^3 v \equiv -4\pi \sigma, \quad (2) $$

and Ampère’s law,

$$ \nabla^2 A(x,t) = -\frac{4\pi}{c} j_T, \quad (3) $$

respectively. Here, the Coulomb (or transverse) gauge condition $\nabla \cdot A = 0$ is used and the current density $j \equiv \sum_a e_a n_a u_a \equiv \sum_a e_a \int f_a(x,v,t) v d^3 v$ (or any vector field) is written as $j = j_L + j_T$, where $j_L = -(4\pi)^{-1} \nabla \int d^3 x (\nabla \cdot j)/|x - x'|$ and $j_T = (4\pi)^{-1} \nabla \times (\nabla \times \int d^3 x j'/|x - x'|)$ represent the longitudinal (or irrotational) and transverse (or solenoidal) parts, respectively. Equations (11), (12), and (13) are the governing equations for the Boltzmann-Poisson-Ampère system.

Suppose that $f_a$, $\phi$, and $A$ which satisfy Eqs. (11) are given. Then, for the electromagnetic fields $E = -\nabla \phi - c^{-1} \partial A / \partial t$ and $B = \nabla \times A$ given from $\phi$ and $A$, we consider the distribution function $f_a^V$ which is the solution of the Vlasov equation,
\[ \Pi_c^V = \sum_a \int d^3 \mathbf{v} f_a^V(x, \mathbf{v}, t) \mathbf{v} \left( m_a \mathbf{v} + \frac{e_a}{c} \mathbf{A} \right) + \frac{1}{8\pi} \left[ (\nabla \phi)^2 - B^2 \right] \mathbf{I} + \frac{1}{4\pi} \left[ -(\nabla \phi)(\nabla \phi) + ((\nabla \mathbf{A}) - (\nabla \mathbf{A})^T) \cdot (\nabla \mathbf{A})^T - \frac{\lambda}{c} (\nabla \mathbf{A})^T \right], \]

respectively, where the superscript \( T \) represents the transpose of the tensor and \( \mathbf{I} \) denotes the unit tensor. The field variable \( \lambda \) which appears in Eq. (7) is introduced in Ref. 29 as the Lagrange undetermined multiplier to derive the Coulomb gauge condition \( \nabla \cdot \mathbf{A} = 0 \) and it is shown from Eq. (8) in Ref. 29 that

\[ \nabla^2 \lambda = 4\pi \nabla \cdot \mathbf{j}. \]  

(8)

Suppose that the variations \( t_E, \delta x_E, \delta \phi, \) and \( \delta \mathbf{A} \) are such that \( \delta I = 0 \) holds for an arbitrary \( x \)-integral domain in Eq. (5). Then, taking the small time interval limit \( \delta t \rightarrow 0 \) in Eq. (6), we find that the conservation law,

\[ \left[ \frac{d}{dt} G_0^V(x, t) \right]_{t = t_0} - \nabla \cdot G^V(x, t_0) = 0, \]  

(9)

should be satisfied. This is the so-called “Noether’s theorem. Recalling that \( f_a^V(x, \mathbf{v}, t_0) = f_a(x, \mathbf{v}, t_0) \) and comparing Eqs. (11) and (12) at \( t = t_0 \), we have

\[ \left[ \frac{d f_a^V(x, \mathbf{v}, t)}{dt} \right]_{t = t_0} = \left[ \frac{d f_a(x, \mathbf{v}, t)}{dt} \right]_{t = t_0} - K_a(x, \mathbf{v}, t_0). \]

We now define \( E_c, \ P_c, \ Q_c, \) and \( \Pi_c \) from \( E_c^V, \ P_c^V, \ Q_c^V, \) and \( \Pi_c^V \), respectively, by replacing \( f_a^V \) with \( f_a \) in Eq. (10). Correspondingly, \( \delta G_0 \) and \( \delta G \) are defined from \( \delta G_0^V \) and \( \delta G^V \) by replacing \( E_c^V, \ P_c^V, \ Q_c^V, \) and \( \Pi_c^V \) with \( E_c, \ P_c, \ Q_c, \) and \( \Pi_c \), respectively, in Eq. (11). These definitions immediately yield \( \delta G^V(x, t_0) = \delta G(x, t_0) \) and

\[ \left[ \frac{d \delta G_0^V(x, t)}{dt} \right]_{t = t_0} = \left[ \frac{d \delta G_0(x, t)}{dt} \right]_{t = t_0} - \delta K_{G0}(x, t_0), \]

(11)

where Eq. (11) is used and \( \delta K_{G0} \) is defined by

\[ \delta K_{G0} = K_{Ec} \delta t_E - K_{Pc} \cdot \delta x_E, \]

\[ K_{Ec} = \sum_a \int d^3 \mathbf{v} K_a \left( \frac{1}{2} m_a |\mathbf{v}|^2 + e_a \phi \right), \]

\[ K_{Pc} = \sum_a \int d^3 \mathbf{v} K_a \left( m_a \mathbf{v} + \frac{e_a}{c} \mathbf{A} \right). \]  

(12)

Substituting Eq. (11) into Eq. (11), we find that the conservation law is modified for the Boltzmann-Poisson-Ampère system as

\[ \frac{d}{dt} \delta G_0(x, t) + \nabla \cdot \delta G(x, t) = \delta K_{G0}, \]  

(13)

where \( t_0 \) is rewritten as \( t \) because \( t_0 \) is an arbitrarily chosen time. Equation (13) shows that \( \delta K_{G0} \) represents effects of \( K_a \) in Eq. (14) on the conservation law. If \( K_a \) is given by the Coulomb collision term only, \( \delta K_{G0} \) defined by Eq. (12) vanishes because the collision term conserves particles’ number, momentum, and energy.

Energy and momentum balance equations can be derived from Eq. (13) using symmetries of the system under infinitesimal time and space translations as shown later. Before deriving them, we first consider the equation for the particle number density \( n_a = \int f_a d^3 \mathbf{v} \) which is obtained by taking the velocity-space integral of Eq. (11) as

\[ \frac{d n_a}{dt} + \nabla \cdot (n_a \mathbf{u}_a) = \int K_a d^3 \mathbf{v}. \]  

(14)

We hereafter assume that \( \sum_a e_a \int K_a d^3 \mathbf{v} = 0 \), which means that the source terms \( K_a \) conserve electric charge even if \( \int K_a d^3 \mathbf{v} \neq 0 \) for each species \( a \). This seems a reasonable assumption in consistency with Eqs. (2) and (3) in which no external source terms are included. Then, multiplying Eq. (14) with the electric charge \( e_a \) and performing the summation over species result in the charge conservation law

\[ \frac{d \sigma}{dt} + \nabla \cdot \mathbf{j} = 0. \]  

(15)

We also find from \( \int e_a \int K_a d^3 \mathbf{v} = 0 \) that, in Eq. (12), the terms including \( \phi \) and \( \mathbf{A} \) vanish and make no contribution to \( K_{Ec} \) and \( K_{Pc} \). As seen from Eqs. (2), (8), and (15), we can put \( \lambda = \partial \phi / \partial t \) which is also used in Ref. 29 to derive energy and momentum conservation laws for the Vlasov-Poisson-Ampère system.

We now note that the action integral is invariant, namely, \( \delta I = 0 \) under the infinitesimal translations in space and time represented by \( \delta t_E = t_0, \delta x_E = \epsilon \), \( \delta \phi = 0, \) and \( \delta \mathbf{A} = 0 \), where \( \epsilon_0 \) and \( \epsilon \) are constant in time and space. These invariance properties hold because the integrands in the action integral \( I \) depend on \( (x, t) \) only through variational variables [see Eq. (1) in Ref. 29]. Using the time translational symmetry, Eq. (13) reduces to the canonical energy balance equation,

\[ \frac{d \mathcal{E}_c}{dt} + \nabla \cdot \mathbf{Q}_c = \mathcal{K}_{Ec}, \]  

(16)

where the canonical energy density and flux \( (\mathcal{E}_c, \mathcal{Q}_c) \) are given by replacing \( f_a^V \) with \( f_a \) in the definitions of \( (\mathcal{E}_c^V, \mathcal{Q}_c^V) \) in Eq. (17). In the same way as in Ref. 29, we use the kinetic energy density and flux, \( (\mathcal{E}_p, \mathcal{Q}_p) \), defined by

\[ \mathcal{E}_p = \sum_a \int d^3 \mathbf{v} f_a(x, \mathbf{v}, t) \frac{1}{2} m_a |\mathbf{v}|^2, \]

\[ \mathcal{Q}_p = \sum_a \int d^3 \mathbf{v} f_a(x, \mathbf{v}, t) \frac{1}{2} m_a |\mathbf{v}|^2 \mathbf{v}, \]  

(17)
to modify Eq. (19) into more familiar forms. Then, the energy balance equation is finally written as
\[
\frac{\partial}{\partial t} \left( \varepsilon_p + \frac{|E_L|^2 + |B|^2}{8\pi} \right) + \nabla \cdot \left( Q_p + \frac{c}{4\pi} E \times B - \frac{1}{4\pi} \frac{\partial \phi}{\partial t} E_T \right) = \frac{\partial}{\partial t} \left( \varepsilon_p + \frac{|E_L|^2 + 2E_L \cdot E_T + |B|^2}{8\pi} \right) + \nabla \cdot \left( Q_p + \frac{c}{4\pi} E \times B + \frac{1}{4\pi} \frac{\partial E_T}{\partial t} \right) = K_{\varepsilon c} \equiv \sum_a \int d^3v \ K_a \frac{1}{2} m_a |v|^2, \tag{18}
\]
where $E_L \equiv -\nabla \phi$ and $E_T \equiv -c^{-1} \partial A/\partial t$ are the longitudinal and transverse parts of the electric field, respectively.

Next, from the space translational symmetry and Eq. (13), we obtain the canonical momentum balance equation,
\[
\frac{\partial P_c}{\partial t} + \nabla \cdot \Pi_c = K_{P c}, \tag{19}
\]
where the canonical momentum density and tensor $(P_c, \Pi_c)$ are given by replacing $f_a^*$ with $f_a$ in the definitions of $(P_c^c, \Pi_c^c)$ in Eq. (4). Furthermore, in the same way as in Ref. 29, the invariance of $I$ under the infinitesimal rotation is shown to give the equation for the angular momentum, which is used to modify Eq. (19) into the momentum balance equation,
\[
\frac{\partial}{\partial t} (P_p + P_f) + \nabla \cdot (\Pi_p + \Pi_f) = K_{P p} \equiv \sum_a \int d^3v \ K_a m_a v. \tag{20}
\]
Here, the particle parts $(P_p, \Pi_p)$ of the momentum density and the pressure tensor are defined by
\[
P_p = \sum_a \int d^3v \ f_a(x, v, t) m_a v, \quad \Pi_p = \sum_a \int d^3v \ f_a(x, v, t) m_a vv, \tag{21}
\]
and the field parts $(P_f, \Pi_f)$ are given by
\[
P_f = \frac{E_L \times B}{4\pi c}, \quad \Pi_f = \frac{1}{8\pi} (|E_L|^2 + 2E_L \cdot E_T + |B|^2) I - \frac{1}{4\pi} (E_L E_L + E_L E_T + E_T E_L + BB). \tag{22}
\]

Equations (18) and (20) take physically familiar forms of energy and momentum balance equations including external source terms. As mentioned earlier, if $K_{\varepsilon c}$ is given by the Coulomb collision term only, $K_{\varepsilon c}$ and $K_{P c}$ vanish so that the energy and momentum balance equations for the Boltzmann-Poisson-Ampère system take the same forms as those for the Vlasov-Poisson-Ampère system.

### III. Gyrokinetic Boltzmann-Poisson-Ampère System

Let us start from the gyrokinetic Boltzmann equation written as
\[
\left( \frac{\partial}{\partial t} + \frac{dZ_a}{dt} \cdot \frac{\partial}{\partial Z} \right) F_a(Z, t) = \sum_b C_{ab}^Z [F_a, F_b](Z, t) + S_a(Z, t), \tag{23}
\]
where $F_a(Z, t)$ is the gyrocenter distribution function for species $a$, $C_{ab}^Z [F_a, F_b](Z, t)$ represents the rate of change in $F_a(Z, t)$ due to Coulomb collisions between particle species $a$ and $b$, and $S_a(Z, t)$ denotes other parts including external particle, momentum, and/or energy sources if any. The gyrocenter coordinates are written as $Z_a = (X_a, U_a, \mu_a, \xi_a)$, where $X_a$, $U_a$, $\mu_a$, and $\xi_a$ represent the gyrocenter position, parallel velocity, magnetic moment, and gyrophase angle, respectively. Appendix A shows the relation of the gyrocenter coordinates to the particle coordinates in detail. The perturbation expansion parameter in the gyrokinetic theory is denoted by $\delta$ which represents the ratio of the gyroradius $\rho$ to the macroscopic scale length $L$ of the background field. It is shown in Appendix B how the collision operator $C_{ab}^Z [F_a, F_b]$ for the gyrocenter distribution functions $F_a$ and $F_b$ is given from the collision operator $C_{ab}^p [f_a, f_b]$ for the particle distribution functions $f_a$ and $f_b$.

The deviation of each distribution function from the local Maxwellian is regarded as $O(\delta)$, and accordingly the collision term $C_{ab}^p$ is considered to be of $O(\delta^2)$. We assume that the source term $S_a$ is of $O(\delta^2)$ so that its effect appears only in the transport time scale. We also assume that $\sum_a e_a \int dU \int d\mu \int d\xi \ D_a S_a(Z, t) = 0$ in order to prevent the source term from affecting the charge conservation laws [see Eq. (46)]. Here, $D_a$ denotes the Jacobian for the gyrocenter coordinates, $D_a \equiv \det[\theta(x_a, v_a)/\partial (X_a, U_a, \mu_a)]$, where $(x_a, v_a)$ represent the particle coordinates consisting of the particle’s position and velocity vectors.

We treat toroidal systems, for which the equilibrium magnetic field is given in the axisymmetric form as
\[
B_0 = \nabla \times A_0 = I \nabla \zeta + \nabla \chi \times \nabla \chi, \tag{24}
\]
where $I$ and $\chi$ are constant on toroidal flux surfaces labeled by an arbitrary radial coordinate $s$ and $\zeta$ is the toroidal angle. We note that $I$ and $\chi$ represent the covariant toroidal component of the equilibrium field $B_0$ and the poloidal magnetic flux divided by $2\pi$, respectively. The equilibrium field $B_0$ is allowed to be dependent on time. Then, following Ref. 18, the gyrocenter motion equations are written as
\[
\frac{dZ_a}{dt} = \{Z_a, H_a\} + \{Z_a, X_a\} \frac{e_a}{c} \frac{\partial A_a^*}{\partial t}, \tag{25}
\]
where the gyrocenter Hamiltonian, which is independent
of $\xi_a$, is defined by
\begin{equation}
H_a = \frac{1}{2} m_a U_a^2 + \mu_a B_0 + e_a \Psi_a,
\end{equation}
and $A^*_a$ is given by $A^*_a = A_a(X_a, t) + (m_a c / e_a) U_a b(X_a, t)$. Using the nonvanishing components of the Poisson brackets for pairs of the gyrocenter coordinates given by
\begin{equation}
\{X_a, X_b\} = \frac{c}{e_b B^2_{a||}} b \times I, \quad \{X_a, U_b\} = \frac{B^*_{a||}}{m_a B^2_{a||}},
\end{equation}
\begin{equation}
\{\xi_a, \mu_b\} = \frac{e_a}{m_a c},
\end{equation}
the gyrocenter motion equations in Eq. (25) are rewritten as
\begin{equation}
\frac{dX_a}{dt} = \frac{1}{B^2_{a||}} \left[ (U_a + e_a \frac{\partial \Psi_a}{\partial U_a}) B^*_a + e b \times \left( \mu_a \nabla B_0 + \nabla \Psi_a + \frac{1}{c} \frac{\partial A^*_a}{\partial t} \right) \right],
\end{equation}
\begin{equation}
\frac{dU_a}{dt} = \frac{B^*_{a||}}{m_a B^2_{a||}} \left[ \mu_a \nabla B_0 + e_a \left( \nabla \Psi_a + \frac{1}{c} \frac{\partial A^*_a}{\partial t} \right) \right],
\end{equation}
\begin{equation}
\frac{d\mu_a}{dt} = 0,
\end{equation}
and
\begin{equation}
\frac{d\xi_a}{dt} = \Omega_a + \frac{e^2_a}{m_a c} \frac{\partial \Psi_a}{\partial \mu_a}.
\end{equation}
Here, $\Omega_a = \frac{e_a B_0}{(m_a c)}$, $b = B_0 / B_0$, $B^*_{a||} = B^2_a \cdot b$, and $B^*_a = \nabla \times A^*_a$. The field variable $\Psi_a$ is defined by
\begin{equation}
\Psi_a = \langle \psi_a(Z_a, t) \rangle_{\xi_a} + \frac{e_a}{2m_a c^2} \langle |A_1(X_a + \rho_a, t)|^2 \rangle_{\xi_a}
\end{equation}
\begin{equation}
- \frac{e_a}{2B_0} \frac{\partial}{\partial \mu_a} \langle \bar{\psi}_a(Z_a, t) |^2 \rangle_{\xi_a},
\end{equation}
where the field variable $\psi_a$ is defined in terms of the electrostatic potential $\phi$ and the perturbation part of the vector potential $A_1$ as
\begin{equation}
\psi_a(Z_a, t) = \phi(X_a + \rho_a, t) - \frac{1}{c} \bar{v}_0 a(Z_a, t) \cdot A_1(X_a + \rho_a, t).
\end{equation}
The gyroradius vector is given by $\rho_a = b(X_a, t) \times \bar{v}_0 a(Z_a, t) / \Omega_a(X_a, t)$ and the zeroth-order particle velocity $\bar{v}_0 a$ is written in terms of the gyrocenter coordinates as $\bar{v}_0 a(Z_a, t) = U_a b(X_a, t) - [2\mu_a B_0(X_a) / m_a]^{1/2} \sin \xi_a e_1(X_a, t) + \cos \xi_a e_2(X_a, t)$, where the unit vectors $(e_1, e_2, b)$ form a right-handed orthogonal system. The gyroaverage and gyrophase-dependent parts of an arbitrary periodic function $Q(\xi_a)$ of the gyrophase $\xi_a$ are written as
\begin{equation}
\langle Q \rangle_{\xi_a} = \int \frac{d\xi_a}{2\pi} Q(\xi_a) \quad \text{and} \quad \bar{Q} = Q - \langle Q \rangle_{\xi_a},
\end{equation}
respectively. In the gyrocenter motion equations, effects of the time-dependent background magnetic field and those of the fluctuating electromagnetic fields appear through $\partial A^*_a / \partial t$ and $\Psi_a$, respectively. It should be noted that $dZ_a / dt$ on the left-hand side of Eq. (23) is regarded as a function of $(Z, t)$ which is given by the right-hand side of Eq. (23).

We find from Eqs. (A10)–(A14) in Appendix A and Eq. (11) in Appendix B that the gyrophase-dependent part of the right-hand side of Eq. (23) appears from $C_{ab}^g$ and it is of $O(\delta)$. Using $\Omega_a = O(\delta^{-1})$, the gyrophase-dependent part of the left-hand side of Eq. (23) is written as $\Omega_a \partial F_a / \partial \xi_a$ to the lowest order in $\delta$. Then, it is concluded that $F_a = O(\delta^2)$. Taking the gyrophase average of Eq. (23), we obtain
\begin{equation}
\left( \frac{\partial}{\partial t} + dZ_a / dt \frac{\partial}{\partial Z} \right) F_a(Z, t)
= \sum_a \langle C^g_{ab} F_a, F_b \rangle (Z, t) g + S_a(Z, t),
\end{equation}
where $F_a(Z, t)$ and $S_a(Z, t)$ are both regarded as independent of the gyrophase $\xi$ and $\langle \cdots \rangle_\xi$ are omitted from them for simplicity. It is seen from Eq. (11) that effects of $F_a = O(\delta^2)$ on $\langle C^g_{ab} F_a, F_b \rangle \xi$ in the right-hand side of Eq. (34) are estimated as of $O(\delta^3)$. Here and hereafter, we neglect $F_a = O(\delta^2)$ in both sides of the gyrokinetic Boltzmann equation given by Eq. (34). Even so, its moment equations can correctly include the collisional transport fluxes of particles, energy, and toroidal momentum up to the leading order, that is $O(\delta^2)$, as confirmed later. In Appendix C, Eq. (C1) combined with Eqs. (C2), (C10), (C15), and (C16) presents the approximate gyrokinetic collision operator, which has favorable conservation properties and correctly describes collisional transport of energy and toroidal angular momentum.

The gyrokinetic Poisson equation and the gyrokinetic Ampère’s law are written as
\begin{equation}
\nabla^2 \phi(x, t) = -4\pi \sum_a e_a \int d^3Z D_a(Z, t) \delta^3(X + \rho_a - x) \times \left[ F_a(Z, t) + \frac{e_a \bar{\psi}_a}{B_0} \frac{\partial F_a}{\partial \mu} \right],
\end{equation}
and
\begin{equation}
\nabla^2 (A_0 + A_1) = -\frac{4\pi}{c} (j_G)_T,
\end{equation}
respectively, where $(j_G)_T$ is the transverse part of the gyrokinetic current density $j_G$ defined by
\begin{equation}
j_{GT} = \sum_a e_a \int d^3Z D_a(Z) \delta^3(X + \rho_a(Z) - x) \times \left[ F_a(Z, t) \left[ \bar{v}_0 a(Z) - \frac{e_a}{m_a c} A_1(X + \rho_a(Z), t) \right] + \frac{e_a \bar{\psi}_a}{B_0} \frac{\partial F_a}{\partial \mu} \bar{v}_0 a(Z) \right].
\end{equation}
It should also be noted that the Coulomb gauge conditions \( \nabla \cdot \mathbf{A}_0 = 0 \) and \( \nabla \cdot \mathbf{A}_1 = 0 \) for the equilibrium and perturbation parts of the vector potential are used here. The equilibrium vector potential \( \mathbf{A}_0 \) is given by \( \mathbf{A}_0 = -\mathbf{V} \nabla \zeta + \mathbf{\nabla} \times \mathbf{\nabla} \eta \), where \( \eta = \eta(R, Z) \) is the solution of \( \Delta \eta \equiv R^2 \nabla \cdot (R^{-2} \nabla \eta) = I \). In Ref. 18, additional governing equations are derived in order to self-consistently determine \( I \) and \( \chi \) for the time-dependent axisymmetric background field \( \mathbf{B}_0 = \nabla \times \mathbf{A}_0 = \mathbf{I} \nabla \zeta + \nabla \zeta \times \mathbf{\nabla} \chi \). They are given by

\[
I = \oint \frac{d\theta}{2\pi} \left[ \frac{4\pi}{c} \nabla \zeta + \left( B^{(gc)} \right)_\zeta - B_1 \right],
\]

and

\[
\Delta \chi = \frac{4\pi}{c} \left[ \left( j^{(gc)} \right)_T + \nabla \times \mathbf{M} \right] - \nabla \times \mathbf{B}_1 \cdot R^2 \nabla \zeta + \frac{\partial}{\partial \zeta} \Lambda \chi,
\]

where the toroidal-angle average is represented by \( \equiv \int \int \int d\zeta \), the poloidal angle is denoted by \( \theta \), and \( \Lambda \chi = I - \left( 4\pi/c \right) M_\zeta - \left( B^{(gc)} \right)_\zeta - B_1 \). Here, the covariant toroidal component of an arbitrary vector \( \mathbf{V} \) is written as \( \mathbf{V}_\zeta \). The turbulent part of the magnetic field is given by \( \mathbf{B}_1 = \nabla \times \mathbf{A}_1 \) and \( B^{(gc)} \) is defined by \( \nabla \times B^{(gc)} = \left( 4\pi/c \right) j^{(gc)} \). The magnetization \( \mathbf{M} \) can be obtained from the turbulent fields and the distribution functions for all species using Eqs. (41)–(43) in Ref. 18. Thus, Eqs. 53, 51, 47, 59, and 40 constitute the closed system of governing equations which determine \( F_a, \phi, A_1, I, \) and \( \chi \).

For the gyrocenter coordinates which have Poisson brackets given by Eq. (27), the Jacobian is given by \( D_a = B_{a\parallel}/m_a \). It is important to note that the Jacobian \( D_a \) satisfies the gyrocenter phase-space conservation law,

\[
\frac{\partial D_a}{\partial t} + \frac{\partial}{\partial Z} \cdot \left( D_a \frac{d\mathbf{Z}_a}{dt} \right) = 0.
\]

Then, using Eq. (11), the gyrokinetic Boltzmann equation in Eq. (55) can be rewritten as

\[
\frac{\partial}{\partial t} (D_a F_a) + \frac{\partial}{\partial Z} \cdot \left( D_a F_a \frac{d\mathbf{Z}_a}{dt} \right) = D_a K_a,
\]

where \( K_a \) is the gyrophase-independent function given by the right-hand side of Eq. (55),

\[
K_a(Z, t) = \sum_a \langle C_{ab}^g [F_a, F_b]_\xi \rangle_\xi + S_a(Z, t).
\]

We hereafter derive conservation laws for the gyrokinetic Boltzmann-Poisson-Ampère system of equations following the procedures similar to those shown in Sec. II. For that purpose, suppose that \( F_a, \phi, A_1, I, \) and \( \chi \) satisfy Eqs. 55, 50, 47, 59, and 40. Then, we consider the gyrocenter distribution function \( F_a^V \) which obeys the gyrokinetic Vlasov equation,

\[
\left( \frac{\partial}{\partial t} + \frac{d\mathbf{Z}_a}{dt} \cdot \frac{\partial}{\partial \mathbf{Z}} \right) F_a^V = 0,
\]

where \( d\mathbf{Z}_a/dt \) is evaluated by using the above-mentioned fields \((\phi, A_1, I, \chi)\) obtained from the solution of the gyrokinetic Boltzmann-Poisson-Ampère system of equations. Here, it should be noted that, if the distribution functions \( F_a \) and \( F_a^V \) which are given as the solutions of Eqs. 55 and 44, respectively, are initially gyrophase-independent, then they are gyrophase-independent at any time. Besides, \( F_a^V \) is assumed to coincide instantaneously with \( F_a \) at a given time \( t_0 \). Therefore, Eqs. 50, 47, 59, and 40 are all satisfied at that moment even if \( F_a \) is replaced with \( F_a^V \) in these equations. Thus, the gyrokinetic Vlasov-Poisson-Ampère system of equations is instantaneously satisfied by \((F_a^V, \phi, A_1, I, \chi)\) at \( t = t_0 \). In Ref. 18, the action integral \( \mathcal{I} \) is defined to derive all the governing equations for the gyrokinetic Vlasov-Poisson-Ampère system based on the variational principle, and its variation \( \delta \mathcal{I} \) associated with the infinitesimal variable transformations are given to obtain conservation laws from Noether’s theorem. Here, the action integral \( \mathcal{I} \) can be expressed in terms of \((F_a^V, \phi, A_1, I, \chi)\) over a small time interval, \( t_0 - h/2 \leq t \leq t_0 + h/2 \), during which the gyrokinetic Vlasov-Poisson-Ampère system of equations are approximately satisfied by them within the errors of order \( h \). Then, neglecting the errors of higher order in \( h \), we can write the variation \( \delta \mathcal{I} \) in the same form as in Eq. (77) of Ref. 18,

\[
\delta \mathcal{I} = -\int_{t_0-h/2}^{t_0+h/2} dt \int d^3 \mathbf{X} \left[ \frac{\partial}{\partial t} \delta G_a^V(X, t) + \nabla \cdot \delta \mathbf{G}^V(X, t) \right],
\]

with the functions \( \delta G_a^V \) and \( \delta \mathbf{G}^V \) defined by

\[
\delta G_a^V(X, t) = \mathcal{E}_a^V \delta t - \mathbf{P}_a^V \cdot \delta \mathbf{E},
\]

\[
\delta \mathbf{G}^V(X, t) = \mathbf{Q}_a^V \delta t - \mathbf{\Pi}_a^V \cdot \delta \mathbf{E} + \mathbf{S}_a \delta \phi - \mathbf{\Sigma}_a \cdot \delta \mathbf{A}_1 - \mathbf{\Lambda}_a \delta \mathbf{A}_0 + \mathbf{S}_a \delta \chi + \mathbf{T}_a^V,
\]

where \( \mathcal{E}_a^V \) and \( \mathbf{P}_a^V \) are defined by

\[
\mathcal{E}_a^V = \sum_a \int dU \int d\xi \int dZ_a D_a F_a^V H_a + \frac{1}{8\pi} (-\nabla \phi)^2 + | \mathbf{B}_0 + \mathbf{B}_1 |^2),
\]

\[
\mathbf{P}_a^V = \sum_a \int dU \int d\xi \int dZ_a D_a F_a^V \left( m_a U b + \frac{e_a}{c} A_0 \right),
\]

and definitions of other variables \( \mathbf{Q}_a^V, \mathbf{\Pi}_a^V, \mathbf{S}_a, \mathbf{\Sigma}_a, \mathbf{\Lambda}_a \) are shown in Eq. (79) of Ref. 18. The superscript \( V \) in the variables \((\mathcal{E}_a^V, \mathbf{P}_a^V, \cdots)\) implies that they are defined using the distribution function \( F_a^V \) instead of \( F_a \).

As explained in Ref. 18, the integral domain of Eq. (45) is not an arbitrary local one in the \( \mathbf{X} \)-space but it can be
local only in the radial direction in order for Eq. (45) to be valid. Then, if the variations $\delta t_E, \delta x_E, \cdots$ in Eq. (46) are such that $\delta I = 0$ holds for a spatiotemporal integral domain defined by $[t_0 - h/2, t_0 + h/2] \times [s_1, s_2]$ where $[s_1, s_2]$ represents an arbitrary spatial volume region sandwiched between two flux surfaces labeled by $s_1$ and $s_2$, then the conservation law is derived as

$$\left[ \left( \frac{\partial}{\partial t} \delta G_0^V (X, t) + \nabla \cdot \delta G^V (X, t) \right) \right]_{t=t_0}
= \left[ \left( \frac{\partial}{\partial t} \delta G_0^V (X, t) + \frac{1}{V'} \frac{\partial}{\partial s} \left( V' (\delta G^V \cdot \nabla s) \right) \right) \right]_{t=t_0}
= 0. \tag{48}$$

This is Noether’s theorem for the gyrokinetic Vlasov-Poisson-Ampère system. In Eq. (48), $V' = \partial V/\partial s$ represents the derivative of $V(s, t)$ with respect to $s$ and $V(s, t)$ denotes the volume enclosed by the flux surface with the label $s$ at the time $t$.

Using $F^V_a (Z, t_0) = F_a (Z, t_0)$ and comparing Eq. (45) with Eq. (48), we find

$$\left[ \frac{\partial F^V_a (Z, t)}{\partial t} \right]_{t=t_0} = \left[ \frac{\partial F_a (Z, t)}{\partial t} \right]_{t=t_0} - K_a (Z, t_0), \tag{49}$$

where $K$ is defined by Eq. (44). Let us also define $\delta G_0$ and $\delta G$ from $\delta G_0^V$ and $\delta G^V$ by replacing $F^V_a$ with $F_a$. Then, we have

$$G^V (X, t_0) = G (X, t_0) \quad \text{and} \quad \left[ \frac{\partial \delta G_0^V (X, t)}{\partial t} \right]_{t=t_0} = \left[ \frac{\partial \delta G_0 (X, t)}{\partial t} \right]_{t=t_0} - \delta K_{G0} (X, t_0), \tag{50}$$

where

$$\delta K_{G0} = K_{Ec} \delta t_E - K_{Pc} \cdot \delta x_E,$$

$$K_{Ec} = \sum_a e_a \int dU \int d\mu \int d\xi \frac{d}{dt} K_a H_a,$$

$$K_{Pc} = \sum_a e_a \int dU \int d\mu \int d\xi K_a P_a.$$

Here, $p_c^a$ denotes the canonical momentum for species $a$ defined by

$$p_c^a = \frac{e_a}{c} A_a^* = \frac{e_a}{c} A_a + m_a U b.$$

Substituting Eq. (50) into Eq. (45) and rewriting the arbitrarily chosen time $t_0$ as $t$, we obtain the conservation law for the gyrokinetic Boltzmann-Poisson-Ampère system,

$$\left\{ \frac{\partial}{\partial t} \delta G_0 (X, t) + \nabla \cdot \delta G (X, t) \right\}
= \left\{ \frac{\partial}{\partial t} \delta G_0 (X, t) + \frac{1}{V'} \frac{\partial}{\partial s} \left( V' (\delta G \cdot \nabla s) \right) \right\}
= \left\{ \delta K_{G0} \right\}, \tag{53}$$

where $\langle \delta K_{G0} \rangle$ represents effects of the collision and source terms on the conservation law. Under the nonstationary background field $B_0$, flux surfaces may change their shapes and the grid of the flux coordinates moves. Then, Eq. (53) is rewritten as

$$\frac{\partial}{\partial t} \left( V' \langle \delta G_0 \rangle \right) + \frac{\partial}{\partial s} \left( V' \left( \delta G - \delta G_0 \mu_a \cdot \nabla s \right) \right) = V' \langle \delta K_{G0} \rangle,$$  \tag{54}$$

where $\mu_a \cdot \nabla s$ represents the radial velocity of the flux surface labeled by $s$ and $\mu_a$ is defined by $\mu_a = \delta x(s, \theta, \zeta, t)/\partial t$ with the flux coordinates $(s, \theta, \zeta)$ [see Eq. (2.35) in Ref. 31]. In Sec. V, gyrokinetic energy and toroidal angular momentum balance equations are derived from Eq. (53).

IV. EQUATIONS FOR GYROCENTER DENSITIES AND POLARIZATION

In this section, we take the velocity-space integral of the gyrokinetic Boltzmann equation in Eq. (35) to consider the particle transport before treating the energy and toroidal angular momentum transport in Sec. V. We define the gyrocenter density $n^{(gc)}_a$ by

$$n^{(gc)}_a (X, t) = \int dU \int d\mu \int d\xi D_a F_a,$$ \tag{55}$$

and the gyrocenter flux $\Gamma^{(gc)}_a$ by

$$\Gamma^{(gc)}_a = n^{(gc)}_a u^{(gc)}_a = \int dU \int d\mu \int d\xi D_a F_a v^{(gc)}_a,$$ \tag{56}$$

where $u^{(gc)}_a$ represents the gyrocenter fluid velocity and the gyrocenter drift velocity $v^{(gc)}_a = dX_a/dt$ is given by evaluating the right-hand side of Eq. (36) at $(X, U, \mu)$. Then, integrating the gyrokinetic Boltzmann equation, Eq. (43), with respect to the gyrocenter velocity-space coordinates $(U, \mu, \xi)$ and using Eq. (53), we obtain

$$\frac{\partial n^{(gc)}_a}{\partial t} + \nabla \cdot \left( \Gamma^{(gc)}_a + \Gamma^{(C)}_a \right) = \int dU \int d\mu \int d\xi D_a S_a.$$ \tag{57}$$

Using the approximate collision operator given by Eq. (C1) in Appendix C, the particle flux $\Gamma^{(C)}_a$ due to collisions and finite gyroradii is defined by Eq. (C0) with putting $A^0_c (x) = 1$. [If the collision operator given by Eq. (B1) in Appendix B is employed, $\Gamma^{(C)}_a$ is defined by Eq. (B2).]

As shown in Ref. 18, the gyrokinetic Poisson equation in Eq. (36) is rewritten as

$$\sum_a e_a n^{(gc)}_a = \nabla \cdot \left( \frac{E_L}{4\pi} + P^{(pol)} \right),$$ \tag{58}$$

where $E_L = -\nabla \phi$, $\phi = \partial/\partial X$, and $P^{(pol)}$ represents the polarization density defined by

$$P^{(pol)} = \sum_a e_a \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n!} \sum_{i_1, \ldots, i_n} \int dU \int d\mu \int d\xi \frac{\partial^{n-1} (D_a F_a \rho_{ai_1} \cdots \rho_{ai_n})}{\partial X_{i_1} \cdots \partial X_{i_{n-1}}}.$$ \tag{59}$$
Here, $\rho_{ai}$ denotes the $i$th Cartesian component of $\rho_a = b(X, t) \times v_a(z, t) / \Omega_a(X, t)$, and $F_a^* = F_a + (e_a \psi_a/B_0)(\partial F_a/\partial t)$.

As mentioned before Eq. (21) in Sec. III, $\sum_a e_a \int dU \int d\mu \int d\xi \, D_a S_n(Z, t) = 0$ is assumed. Then, using Eq. (27), we can obtain the charge conservation law,

$$\frac{\partial}{\partial t} \left( \sum_a e_a n_a^{(gc)} \right) + \nabla \cdot (j^{(gc)} + j^C) = 0,$$

where the current density due to the gyrocenter drift and that due to the collisional particle transport are given by $j^{(gc)} = \sum_a e_a \Gamma_a^{(gc)} = \sum_a e_a n_a^{(gc)} u_a^{(gc)}$ and $j^C = \sum_a e_a \Gamma_a^C$, respectively. Note that the magnetization current is solenoidal and, accordingly, it does not contribute to the charge conservation law in Eq. (60). Equation (58) is substituted into Eq. (60) to show

$$j_{L}^{(gc)} + j_{L}^C = -\frac{\partial}{\partial t} \left( \frac{E_L}{4\pi} + P_L^{(pol)} \right),$$

where the subscript $L$ is used to represent the longitudinal part of the vector variable. Then, using Eqs. (58), (60) and (61), we find that the useful formula,

$$\left\langle \frac{\partial}{\partial t} \left( \sum_a e_a n_a^{(gc)} \right) \right\rangle + \left\langle \nabla \cdot \left( A j_{L}^{(gc)} \right) \right\rangle = \left\langle \nabla \cdot \left( A j_{L}^C \right) \right\rangle = \left\langle \nabla \cdot \left( \frac{E_L}{4\pi} + P_L^{(pol)} - A j_{L}^C \right) \right\rangle,$$

holds for any function $A(X, t)$. The relation in Eq. (62) is used in Sec. V.B to derive Eq. (24).

V. GYROKINETIC ENERGY AND TOROIDAL ANGULAR MOMENTUM BALANCE EQUATIONS

In this section, energy and toroidal angular momentum balance equations for gyrokinetic systems including collisional processes are derived by using the results shown in Secs. III–IV.

A. Energy balance equation

The variation $\delta I$ of the action given in Sec. III vanishes under the infinitesimal time translation represented by $\delta t_E = \epsilon$ where $\epsilon$ is an infinitesimally small constant. Here, all other infinitesimal variations $\delta x_E, \delta \phi, \cdots$ are regarded as zero. Then, $\delta G_0$ and $\delta G$ are determined by these conditions for the infinitesimal time translation and they satisfy Eq. (51) which, in the same manner as in Ref. 18, leads to the energy balance equation,

$$\frac{\partial}{\partial t} \left( V' (E) \right) + \frac{\partial}{\partial s} \left( V' (Q^c + Q^R - \mathcal{E} u_s) \cdot \nabla s \right) = V' \langle K_{\mathcal{E}c} \rangle,$$

(63)

Here, the energy density $E$ is defined by

$$E = \mathcal{E} + \nabla \cdot \left( \frac{1}{4\pi} \phi \nabla \phi - \Phi_R \right)$$

$$= \sum_a \int dU \int d\mu \int d\xi \, D_a F_a \left[ H_a v_a^{(gc)} + \left( \frac{m_a}{2} v_a^2 - \frac{e_a}{m_a} \mathcal{A}_1 \right) \right]$$

$$+ \frac{e_a}{2B_0} \frac{\partial}{\partial t} \left( \tilde{\psi}_a \left( 2\tilde{\phi} - \tilde{\psi}_a \right) \right)$$

$$+ \frac{1}{8\pi} \left( |\nabla \phi|^2 + |B_0 + B_1|^2 \right),$$

(64)

and the energy fluxes $Q^c_e$ and $Q^r_R$ are given by

$$Q^c_e = Q_e - \frac{1}{4\pi} \frac{\partial}{\partial t} (\phi \nabla \phi)$$

$$= \sum_a \int dU \int d\mu \int d\xi \, D_a F_a \left[ \left( \frac{m_a U}{B_0} v_a^{(gc)} - N_a \right) \right]$$

$$\times \left( -\frac{1}{4\pi} v_a^{(gc)} \right) + \frac{1}{4\pi} \phi \nabla \phi$$

$$- \frac{1}{4\pi} \frac{\partial (\mathcal{A}_0 + \mathcal{A}_1)}{\partial t} \cdot (B_0 + B_1) + \frac{1}{4\pi} \frac{\partial \mathcal{A}_1}{\partial t}$$

$$+ \frac{\partial \mathcal{A}_0}{\partial t} \right\rangle = \left\langle \nabla \cdot \left( \frac{\partial (\mathcal{A}_0 + \mathcal{A}_1)}{\partial t} \right) \right\rangle$$

$$+ \frac{\partial \mathcal{A}_0}{\partial t} \right\rangle$$

$$= \frac{\partial \mathcal{A}_0}{\partial t} \right\rangle$$

$$= \frac{\partial \mathcal{A}_0}{\partial t} \right\rangle$$

and

$$Q^r_R = Q_R + \frac{\partial \Phi_R}{\partial t},$$

(66)

respectively, where $N_a, Q_R,$ and $\Phi_R$ are defined by Eqs. (43), (85), and (88) in Ref. 18, respectively.

Recalling Eq. (51) and using Eq. (C7) into which we substitute $A_a^0(Z) = H_a(Z)$ and the approximate collision operator given by Eq. (C11) in Appendix C, we rewrite the right-hand side of Eq. (63) as

$$V' \langle K_{\mathcal{E}c} \rangle = -\frac{\partial}{\partial s} \left( V' (Q^c \cdot \nabla s) \right)$$

$$+ V' \sum_a \left\langle \int dU \int d\mu \int d\xi \, D_a S_n H_a \right\rangle,$$

(67)

where the energy flux $Q^c$ due to collisions and finite gyroradii is defined by taking the summation of Eq. (C4) over species $a$ with putting $A_a^0(Z) = \frac{1}{2} m_a v_a^2 + \mu_{0a} B_0 (x_a) + e_a \phi(x_a)$. The derivation of Eq. (67) requires Eq. (C7) which is satisfied by the collision operator in the form of Eq. (C11) with appropriately choosing $\Delta x_a^{(2)}, \Delta v_{||a},$ and $\Delta \mu_{0a}$ as described in Appendix C.
Substituting Eq. (67) into Eq. (68), the energy balance equation is rewritten as

\[
\frac{\partial}{\partial t} (V' \langle \mathcal{E} \rangle) + \frac{\partial}{\partial s} (V' \langle (Q - \mathcal{E} u_s) \cdot \nabla s \rangle) = V' \sum_a \left( \int dU \int d\mu \int d\xi \, D_a S_a H_a \right),
\]  
(68)

where the energy flux \( Q \) is given by

\[
Q = Q_c^* + Q_R^* + Q^C.
\]  
(69)

The right-hand side of Eq. (68) represents the external energy source. It is confirmed later in Sec. VI.B that the ensemble average of \( \langle Q \cdot \nabla s \rangle \) coincides with the well-known expression of the radial energy transport to the lowest order in the \( \delta \)-expansion.

B. Toroidal angular momentum balance equation

The toroidal angular momentum balance equation is derived from the fact that \( \delta L = 0 \) under the infinitesimal toroidal rotation represented by \( \delta \mathbf{X} = \epsilon \mathbf{e}_z (\mathbf{X}) \). Here, \( \epsilon \) is again an infinitesimally small constant, and \( \mathbf{e}_z (\mathbf{X}) \) is defined by \( \mathbf{e}_z (\mathbf{X}) = \partial \mathbf{X} (R, z, \zeta) / \partial \zeta = R^2 \nabla \zeta \) where the right-handed cylindrical spatial coordinates \( (R, z, \zeta) \) are used. We also define \( \mathbf{z} \) by \( \mathbf{z} = R \nabla \zeta \wedge \nabla R \) which represents the unit vector in the \( z \)-direction. Then, if we put in the origin of the position vector \( \mathbf{X} \) at \( (R, z) = (0, 0) \), we have \( \mathbf{e}_z (\mathbf{X}) = \mathbf{X} \wedge \mathbf{z} \). Under the infinitesimal toroidal rotation, the variations of the vector variables are given as \( \delta \mathbf{A}_1 = \epsilon \mathbf{A}_1 \wedge \mathbf{z} \) and \( \delta \mathbf{A}_0 = \epsilon \mathbf{A}_0 \wedge \mathbf{z} \) although the other variations \( \delta t_E, \delta \phi, \ldots \) are all regarded as zero. Then, using these variations of the variables associated with the infinitesimal toroidal rotation, the canonical momentum balance equation is derived from Eq. (68) as

\[
\langle \frac{\partial}{\partial t} (P_c \cdot \mathbf{e}_z) \rangle + \frac{1}{V'} \frac{\partial}{\partial s} \left[ V' \langle \nabla s \cdot (\Pi L \cdot \mathbf{e}_z + (\sum A_1 \wedge A_1 + \sum A_0 \wedge A_0) \wedge \mathbf{z} + P \Pi \rangle \rangle \right] = \langle K_{Pc} \rangle .
\]  
(70)

Here, the density of the canonical toroidal angular momentum is defined by

\[
P_c \cdot \mathbf{e}_z = \sum_a \int dU \int d\mu \int d\xi \, D_a F_a (p_c^a) \zeta ,
\]  
(71)

with the toroidal component of the canonical momentum denoted by

\[
(p^c_\zeta) = \frac{e_a}{c} A^a_\zeta = \frac{e_a}{c} A_0 \zeta + m_a U b_\zeta ,
\]  
(72)

where \( b_\zeta \equiv 1 / b_0 \) represents the covariant toroidal component of \( b \equiv B_0 / B_0 \). Definitions of \( \sum A_1, \sum A_0, \) and \( \Pi \) on the left-hand side of Eq. (70) are given by Eqs. (79) and (100) in Ref. 18. On the right-hand side of Eq. (70), the variation of the canonical toroidal angular momentum due to collisions and external sources is given by

\[
K_{Pc} = K_{Pc} \cdot \mathbf{e}_z = \sum_a \int dU \int d\mu \int d\xi \, D_a K_a (p^a_\zeta) .
\]  
(73)

We follow the same procedures as shown in Sec. V.B of Ref. 18 and use Eqs. (70)–(73) and Eq. (62) with \( A = A_0 \) to write the toroidal angular momentum balance equation as

\[
\langle \frac{\partial}{\partial t} \left[ P_{\Pi \zeta} - \frac{1}{c} \left( P_{L \zeta} + \frac{E_E}{4\pi} \right) \cdot \nabla A_0 \right] \rangle
+ \frac{1}{V'} \frac{\partial}{\partial s} \left[ V' \left\{ \Pi_{\Pi \zeta} + \Pi_{L \zeta} - \frac{1}{4\pi} \langle A_0 (\nabla \times B_1) \cdot \nabla s \rangle \right\} \right]
- \frac{1}{4\pi} \langle E_{\Pi \zeta} E_\Pi + B_{1 \zeta} B_1^* \rangle + \frac{1}{4\pi c} \langle \frac{\partial}{\partial z} A_1^* \rangle
+ \frac{1}{c} \langle \frac{\partial A_0 \zeta}{\partial t} \left( P_{L \zeta} + \frac{E_E}{4\pi} \right) \cdot \nabla s \rangle
= \langle K_{Pc} \rangle + \frac{1}{c} \langle \nabla \cdot (A_0 J^c_\zeta) \rangle ,
\]  
(74)

where

\[
\begin{align*}
P_{\Pi \zeta} &= \sum_a \int dU \int d\mu \int d\xi \, D_a F_a m_a U b_\zeta , \\
\Pi_{\Pi \zeta} &= \sum_a \int dU \int d\mu \int d\xi \, D_a F_a m_a U b_\zeta v_{\Pi \zeta}^{(gc)} \cdot \nabla s , \\
\Pi_{L \zeta} &= P_{L \zeta} \cdot \nabla s ,
\end{align*}
\]  
(75)

Using Eqs. (69) and (72) in Appendix C with putting \( A_0^c = (p_c^a) \zeta \), the right-hand side of Eq. (74) is rewritten as

\[
\langle K_{Pc} \rangle + \frac{1}{c} \langle \nabla \cdot (A_0 J^c_\zeta) \rangle
= - \frac{1}{V'} \frac{\partial}{\partial s} \left[ V' \left\langle \left( J^c_\zeta + \frac{\chi_4 c}{c^2 L} \right) \cdot \nabla s \right\rangle \right]
+ \sum_a \left( \int dU \int d\mu \int d\xi \, D_a S_a m_a U b_\zeta \right) ,
\]  
(76)

where \( J^c_\zeta \) is defined by taking the summation of Eq. (66) over species \( a \) with putting \( A_0^c (x) = (e_a / c) [A_0 (x) + A_1 (x)] + m_a v_\zeta \). In deriving Eq. (76), \( \sum a e_a \int dU \int d\mu \int d\xi \, D_a S_a (Z, t) = 0 \) and Eqs. (C12) are used. The approximate collision operator which satisfies Eq. (C12) is presented in Appendix C.

Substituting Eq. (76) into Eq. (74), the toroidal angu-
lar momentum balance equation is rewritten as
\[
\frac{\partial}{\partial t} \left( V' \left( P_{\parallel \zeta} - \frac{1}{c} \left( P_{L}^{(\text{pol})} + \frac{E_L}{4\pi} \right) \cdot \nabla A_{\zeta} \right) \right) \\
+ \frac{1}{V'} \frac{\partial}{\partial s} \left[ V' \left\{ \Pi_{\parallel \zeta} + \Pi_{\perp \zeta} + (\Pi^C)^s - \frac{1}{4\pi} \times (A_1 \zeta \times B_1) \cdot \nabla s \right\} \\
+ \frac{1}{4\pi c} \left\{ \frac{\partial A}{\partial t} \right\} + \frac{1}{c} \left\{ \frac{\partial A_{\zeta}}{\partial t} \right\} \left( P_{L}^{(\text{pol})} + \frac{E_L}{4\pi} \right) \cdot \nabla s \\
- \left[ \left\{ P_{\parallel \zeta} - \frac{1}{c} \left( P_{L}^{(\text{pol})} + \frac{E_L}{4\pi} \right) \cdot \nabla A_{\zeta} \right\} (u_s \cdot \nabla s) \right] \right] \\
= \sum_a \left\{ \int dU \int d\mu \int d\xi D_a S_a m_a U_b \zeta \right\},
\tag{77}
\]
where the right-hand side represents the external source of the toroidal angular momentum and
\[
(\Pi^C)^s = \left( J_{pc}^C + \frac{\nabla}{c} L_{EL} \right) \cdot \nabla s
\tag{78}
\]
is the radial flux of the toroidal angular momentum due to collisions and finite gyroradii. In Sec. VI.C, we derive the ensemble-averaged toroidal angular momentum balance equation from Eq. (77) in order to confirm that it is consistent with the conventional result up to the second order in \(\delta\).

VI. ENSEMBLE-AVERAGED BALANCE EQUATIONS FOR PARTICLES, ENERGY, AND TOROIDAL ANGULAR MOMENTUM

In this section, the particle, energy, and toroidal angular momentum balance equations derived in Secs. IV and V for ensemble-averaged purposes using their consistency with those obtained by conventional recursive formulations. In the same way as shown in Sec. VI of Ref. 18, we divide an arbitrary physical variable \(Q\) into the average and turbulent parts as
\[
Q = \langle Q \rangle_{\text{ens}} + \hat{Q},
\tag{79}
\]
where \(\langle \cdots \rangle_{\text{ens}}\) represents the ensemble average, and we immediately find \(\langle \hat{Q} \rangle_{\text{ens}} = 0\). We identify the zeroth fields \(A_0\) and \(B_0\) with the ensemble-averaged parts to write \(A_0 = \langle A \rangle_{\text{ens}}\), \(A_1 = A\), \(B_0 = \langle B \rangle_{\text{ens}}\), and \(B_1 = B\). Regarding the electrostatic potential field \(\phi\), it is written as the sum of the average and fluctuation parts, \(\phi(x, t) = \langle \phi(x, t) \rangle_{\text{ens}} + \hat{\phi}(x, t)\). Here, assuming that \(\langle \phi(x, t) \rangle_{\text{ens}} \neq 0\), the background \(E \times B\) flow is retained and its velocity is regarded as \(O(\delta vt)\), where \(\delta\) and \(vt\) represent the drift ordering parameter and the thermal velocity, respectively. Then, using Eq. (53), we have
\[
\hat{\psi}_a = \langle \psi_a \rangle_{\text{ens}} + \hat{\psi}_a,
\tag{80}
\]
where
\[
\langle \psi_a \rangle_{\text{ens}} = \langle \phi \rangle_{\text{ens}}, \quad \hat{\psi}_a = \hat{\phi} - \frac{V_0}{c} \cdot \hat{A}.
\]
We assume that the ensemble average \(\langle Q \rangle_{\text{ens}}\) of any variable \(Q\) considered here has a slow temporal variation subject to the so-called transport ordering, \(\partial \ln \langle Q \rangle_{\text{ens}} / \partial t = O(\delta^2 vt / L)\), and that it has a gradient scale length \(L\) which is on the same order as gradient scale lengths of the equilibrium field and pressure profiles. We also impose the constraint of axisymmetry on \(\langle Q \rangle_{\text{ens}}\) that is written as \(\partial \langle Q \rangle_{\text{ens}} / \partial \zeta = 0\) even though \(Q\) itself is not axisymmetric. On the other hand, the turbulent part \(\hat{Q}\) of \(Q\) is assumed to vary with a characteristic frequency \(\omega = O(vt / L)\) and have gradient scale lengths \(L\) and \(\rho\) in the directions parallel and perpendicular to the equilibrium magnetic field \(B_0\), respectively.

The ensemble-averaged part \(\langle F_a \rangle_{\text{ens}}\) of the distribution function \(F_a\) for species \(a\) consists of the local Maxwellian part and the deviation from it,
\[
\langle F_a \rangle_{\text{ens}} = F_{aM} + \langle F_{a1} \rangle_{\text{ens}}.
\tag{81}
\]

The local Maxwellian distribution function is written as \(F_{aM} = n_{a0} [m_a / (2\pi T_{a0})]^{3/2} \exp[-(m_a U^2 / 2 + \mu B_0) / T_{a0}]\) where the equilibrium density \(n_{a0}\) and temperature \(T_{a0}\) are regarded as uniform on flux surfaces. The first-order ensemble-averaged distribution function \(\langle F_{a1} \rangle_{\text{ens}}\) is determined by the drift kinetic equation, which can be derived by substituting Eq. (81) into the ensemble average of Eq. (29). The derived equation agrees, to \(O(\delta)\), with the well-known linearized drift kinetic equation, on which the neoclassical transport theory is based. The fluctuation part \(\hat{F}_a\) is written as
\[
\hat{F}_a = -F_{aM} \frac{e_a \langle \psi_a \rangle_{\text{ens}}}{T_a} + \hat{h}_a.
\tag{82}
\]
Substituting Eq. (82) into the fluctuation part of the gyrokineastic equation in Eq. (29) yields
\[
\frac{\partial}{\partial t} \langle \hat{h}_a, H_a \rangle \\
= F_{aM} \left[ \frac{e_a}{T_{a0}} \frac{\partial \langle \psi_a \rangle_{\text{ens}}}{\partial t} - \psi^{(gc)}_a \right], \left( \nabla \ln p_{a0} + \frac{e_a}{T_{a0}} \nabla \langle \phi \rangle_{\text{ens}} \right) \\
+ \left( \frac{1}{2} m_a U^2 + \mu B_0 \right) \left( \frac{5}{2} \nabla \ln T_{a0} \right) + C_a^L
\tag{83}
\]
where \(C_a^L\) represents the linear collision term defined by
\[
C_a^L(X) = \sum_b \langle [C_{ab}^{\text{p}}(\hat{h}_a(x - \rho_a), F_{b0})] \rangle_{\text{ens}} \langle [C_{ab}^{\text{p}}(\hat{h}_b(x - \rho_b)) \rangle_{\text{ens}} \rangle_{x = x + \rho_a}.
\tag{84}
\]
Equation (83) is valid to the lowest order in \(\delta\) and agrees with the conventional gyrokineastic equation for the non-diabatic part \(\hat{h}_a\) of the perturbed distribution function derived from using the WKB representation. On the right-hand side of Eq. (83), the turbulent part \(\hat{\psi}_a^{(gc)}\) of the gyrocenter drift velocity \(\psi_a^{(gc)} = dX_a / dt = \{X_a, H_a\}\) is written as
\[
\hat{\psi}_a^{(gc)} = \frac{e_a}{B_0} b \times \nabla \langle \psi_a \rangle_{\text{ens}} (X + \rho_a, t) + O(\delta^2).
\tag{85}
\]
A. Ensemble-averaged particle balance equation

Taking the ensemble average of Eq. \(67\) and subsequently its flux surface average, we obtain

\[
\left\langle \frac{\partial (n_a^{(gc)})_{\text{ens}}}{\partial t} \right\rangle + \frac{1}{V_f} \frac{\partial}{\partial s} \left( V' \left\langle \left( \Gamma_a^{(gc)} + \Gamma_a^C \right) \cdot \nabla s \right\rangle \right) = \left\langle dU \int d\mu \int d\xi D_a S_a \right\rangle,
\]

(86)

where \(n_a^{(gc)}_{\text{ens}} = n_{a0} + \mathcal{O}(\delta)\), and \(\left\langle \cdots \right\rangle\) represents a double average over the flux surface and the ensemble. Here, \(n_{a0}\) is the equilibrium density which is a flux-surface function and characterizes the Maxwellian distribution function \(F_{am}\). On the right-hand side, the source term \(S_a\) is regarded as of \(\mathcal{O}(\delta^2)\) as well as all other terms in Eq. \(86\), and it is assumed to have no turbulent component so that \(S_a = \left\langle S_a \right\rangle_{\text{ens}}\).

It is shown in Ref. 18 that the radial gyrocenter particle flux is given by

\[
(\Gamma_a^{(gc)})^s = \left\langle \left( \Gamma_a^{(gc)} \cdot \nabla s \right) \right\rangle = (\Gamma_a^{NA})^s + (\Gamma_a^C)^s,
\]

(87)

where the nonturbulent part \((\Gamma_a^{NA})^s\) and the turbulence-driven part \((\Gamma_a^C)^s\) are written as

\[
(\Gamma_a^{NA})^s = \left\langle \int dU \int d\mu \int d\xi \; D_a (F_{am})_{\text{ens}} (v_a^{(gc)})_{\text{ens}} \cdot \nabla s \right\rangle
\]

\[
= \left\langle \frac{c}{\epsilon_a B_0} \left[ b \times \langle \nabla \cdot \mathbf{P}_{\text{CGL}} \rangle \right] \cdot \nabla s \right\rangle
+ n_{a0} \left\langle \frac{c}{B_0} \langle E_{\text{ens}} \rangle \times b \right\rangle \cdot \nabla s \right\rangle + \mathcal{O}(\delta^3),
\]

(88)

and

\[
(\Gamma_a^C)^s = \left\langle \int dU \int d\mu \int d\xi \; D_a (\tilde{F}_a v_a^{(gc)})_{\text{ens}} \cdot \nabla s \right\rangle
\]

\[
= - \left\langle \frac{c}{B_0} \int dU \int d\mu \int d\xi \; D_a \tilde{h}_a (\nabla \psi_a \times b) \cdot \nabla s \right\rangle
+ \mathcal{O}(\delta^3),
\]

(89)

respectively. On the right-hand side of Eq. \(88\), \(\mathbf{P}_{\text{CGL}}\) represents the first-order part of the pressure tensor in the Chew-Goldberger-Low (CGL) formalism, defined by \(\mathbf{P}_{\text{CGL}} = \int dU \int d\mu \int d\xi \; D_a (F_{am})_{\text{ens}} \left[ m_a U_a^2 \mathbf{b} b + \mu B_0 (I - \mathbf{b} b) \right]\), and the ensemble-averaged electric field is given by \(\langle E \rangle_{\text{ens}} = -\nabla \langle \phi \rangle_{\text{ens}} - c^{-3} \partial \mathbf{A}_0 / \partial t\). Thus, Eq. \(88\) expresses the neoclassical radial particle flux and the radial \(\mathbf{E} \times \mathbf{B}\) drift which are well-known by the neoclassical transport theory. We also find that Eq. \(89\) agrees with the conventional gyrokinetic theory based on the WKB formalism.

The radial classical particle flux is given by

\[
(\Gamma_a^C)^s = \left\langle \left( \Gamma_a^C \cdot \nabla s \right) \right\rangle = \sum_b \left\langle \frac{m_a c}{\epsilon_a B_0} \int dU \int d\mu \int d\xi \; D_a [(\mathbf{v} \times \mathbf{b}) \cdot \nabla s]
\]

\[
+ \int \frac{C_{ab}^p (\tilde{f}_{a1}, F_{bm}) + C_{ab}^p (F_{am}, \tilde{f}_{bl})} + \mathcal{O}(\delta^3),
\]

\[
= \left\langle \frac{c}{\epsilon_a B_0} (F_{a1} \times \mathbf{b}) \cdot \nabla s \right\rangle + \mathcal{O}(\delta^3),
\]

(90)

where \(F_{a1} \equiv \int d^3v m_a v C_{ab}^p\) is the collisional friction force. It is well-known that the classical transport equation relating \((\Gamma_a^C)^s\) to the gradient forces is immediately derived from Eq. \(89\) because the first-order gyrophase-dependent part of the particle distribution function in Eq. \(89\) is expressed in terms of the gradient of the background Maxwellian distribution function as \(\tilde{f}_{a1} = -\rho_a \cdot \nabla F_{am}\) with the gradient operator \(\nabla\) taken for the fixed energy variable \(\varepsilon = \frac{1}{2} m_a v^2 + \epsilon(\phi)_{\text{ens}}\).

In the same manner as in deriving Eq. \(51\) from Eq. \(53\), the ensemble-averaged particle transport equation can be obtained from Eq. \(89\) as

\[
\frac{\partial}{\partial t} (V' n_{a0}) = \left\langle \frac{\partial}{\partial s} (V' [(\Gamma_a)^s] - n_{a0} \langle \mathbf{u}_a \cdot \nabla s \rangle) \right\rangle
\]

\[
= \left\langle \left( \int dU \int d\mu \int d\xi \; D_a S_a \right) \right\rangle,
\]

(91)

where the total radial particle flux is given by the sum of the classical, neoclassical, and turbulent parts as

\[
(\Gamma_a)^s = (\Gamma_a^{(gc)})^s + (\Gamma_a^C)^s = (\Gamma_a^{NA})^s + (\Gamma_a^C)^s + (\Gamma_a^C)^s.
\]

(92)

As shown above, the well-known expressions of the classical, neoclassical and turbulent particle fluxes are included in \((\Gamma_a^{(gc)})^s\), \((\Gamma_a^{NA})^s\), and \((\Gamma_a^C)^s\), respectively. The latter two fluxes are evaluated by the solutions \((F_{am})_{\text{ens}}\) and \(\tilde{h}_a\) of the first-order drift kinetic and gyrokinetic equations, respectively.

B. Ensemble-averaged energy balance equation

The ensemble average of the energy density defined by Eq. \(64\) is written as

\[
\langle \mathcal{E} \rangle_{\text{ens}} = \frac{3}{2} \sum_a n_{a0} T_{a0} + \frac{B_0^2}{8\pi} + \mathcal{O}(\delta),
\]

(93)

where the energy density of the electric field is neglected as a small quantity of \(\mathcal{O}(\delta^2)\). It is shown in Ref. 18 that the radial components of the first two terms on the
right-hand side of Eq. (63) are double-averaged over the ensemble and the flux surface to give
\[
\sum_a \left\langle \int dU \int d\mu \int d\xi \ D_a \left( H_a V_a^{(gc)} - \mu \frac{\partial A_0}{\partial t} \times b \right) \cdot \nabla s \right\rangle
\]
\[
= \sum_a \left[ \langle q_a^{(gc)} \rangle + \frac{5}{2} T_{a0} \langle \Gamma_a^{(gc)} \rangle \right] + O(\delta^3).
\] (94)

Here, the radial particle flux \( \langle \Gamma_a^{(gc)} \rangle \) is given by Eqs. (97)–(99) and the radial heat flux \( \langle q_a \rangle \) is written as
\[
\langle q_a \rangle = \langle q_a^{NA} \rangle + \langle q_a^\Lambda \rangle
\] (95)
which consists of the classical, neoclassical, and turbulent parts,
\[
\langle q_a^{NA} \rangle = \left\langle \int dU \int d\mu \int d\xi \ D_a \langle F_a \rangle_{ens} \left( V_a^{(gc)} \right)_{ens} \cdot \nabla s \times \left( \frac{1}{2} m_a U^2 + \mu B_0 - \frac{5}{2} T_{a0} \right) \right\rangle + O(\delta^3),
\] (96)
and the turbulence-driven part,
\[
\langle q_a^\Lambda \rangle = - \left\langle \left( \frac{c}{B_0} \int dU \int d\mu \int d\xi \ D_a \hat{\psi}_a (\nabla \hat{\psi}_a \times b) \cdot \nabla s \times \left( \frac{1}{2} m_a U^2 + \mu B_0 - \frac{5}{2} T_{a0} \right) \right\rangle + O(\delta^3).
\] (97)

In Eq. (96), the stress tensor \( \Theta_a^{CGL} \) of the turbulent heat flux in terms of the heat stress tensor \( \Theta_a^{CGL} \) coincides with that of the neoclassical radial heat flux \( [m_a U^2 + \mu B_0 - \frac{5}{2} T_{a0}] [m_a U^2 + \mu B_0 - \frac{5}{2} T_{a0}] \). The expression of Eq. (98) takes the same form as that given by the conventional gyrokinetic theory.12

The radial component of \( \Theta_a^{CGL} \) in Eq. (96) is ensemble-averaged to yield
\[
\left\langle \langle Q^C \cdot \nabla s \rangle \right\rangle = \langle q_a^C \rangle + \frac{5}{2} T_{a0} \langle \Gamma_a^C \rangle + O(\delta^3),
\] (98)
where the radial classical heat flux is
\[
\langle q_a^C \rangle = \sum_b \left[ \frac{m_a c}{e_a B_0} \int dU \int d\mu \int d\xi \ D_a \left( \nabla \cdot \left( \frac{c}{B_0} \right) \cdot \nabla s \right) \right] \times \left( C_{a b}^{\rho} \langle f_{a 1} F_{a 2} \rangle + C_{a b}^{\rho} \langle f_{a 2} F_{a 1} \rangle \right)
\]
\[
= T_{a0} \left\langle \frac{c}{e_a B_0} \left( F_{a 2} \times b \right) \cdot \nabla s \right\rangle.
\] (99)

Here, \( F_{a 2} \equiv \int d^3 \psi (m_a v^2 / 2T_a - 5/2 m_s v \psi^2 / 2) \) is the collisional heat friction. The expression of the classical heat flux \( \langle q_a^C \rangle \) agrees with the conventional one12 and it immediately gives the classical heat transport equation relating \( \langle q_a^C \rangle \) to the gradient forces in the same way as mentioned after Eq. (98) for the classical particle flux \( \langle \Gamma_a^C \rangle \). Now, Eq. (68) is rewritten as
\[
\frac{\partial}{\partial s} \left( V' \left[ \frac{3}{2} \sum_a n_{a0} T_{a0} + \frac{B_0^2}{8\pi} \right] \right)
\]
\[
+ \frac{\partial}{\partial t} \left( V' \left[ \sum_a \langle q_a \rangle + \frac{5}{2} T_{a0} \langle \Gamma_a \rangle \right] \right)
\]
\[
- \left( \frac{3}{2} \sum_a n_{a0} T_{a0} + \frac{B_0^2}{8\pi} \right) \left( \nabla s \right)
\]
\[
= V' \sum_a \left\langle \left( \int dU \int d\mu \int d\xi \ D_a S_a \left( \frac{1}{2} m_a U^2 + \mu B_0 \right) \right) + O(\delta^3) \right\rangle,
\] (100)
where the total radial heat flux is given by the sum of the classical, neoclassical, and turbulent parts as
\[
\langle q_a \rangle = \langle q_a^{(gc)} \rangle + \langle q_a^\Lambda \rangle
\] (101)
and \( S^{(Poynting)} \equiv \langle c/4\pi \rangle (E)_{ens} \times B_0 \) represents the non-turbulent part of the Poynting vector. Using the relation \( \langle \partial (B_0^2 / 8\pi) / \partial t \rangle = -(V')^{-1} \partial (V' \langle S^{(Poynting)} \cdot \nabla s \rangle) / \partial s - \langle J_0 \cdot (E)_{ens} \rangle \) shown in Ref. 18, we also obtain
\[
\frac{\partial}{\partial t} \left( V' \left[ \frac{3}{2} \sum_a n_{a0} T_{a0} + \frac{B_0^2}{8\pi} \right] \right)
\]
\[
- \left( \frac{3}{2} \sum_a n_{a0} T_{a0} \langle \nabla \cdot \nabla s \rangle \right)
\]
\[
= V' \langle J_0 \cdot (E)_{ens} \rangle + V' \sum_a \left\langle \left( \int dU \int d\mu \int d\xi \ D_a S_a \left( \frac{1}{2} m_a U^2 + \mu B_0 \right) \right) \right\rangle + O(\delta^3).
\] (102)

Equations (101) and (102) take the well-known forms of the energy balance equations except that the terms associated with the electric field energy and the kinetic energies due to the fluid velocities are neglected here as small quantities of higher order in \( \delta \).
C. Ensemble-averaged toroidal angular momentum balance equation

The ensemble-averaged toroidal angular momentum balance equation is written as

$$\frac{\partial}{\partial t}\left( V' \sum_a n_{a0}m_a (u_{a\parallel} + u_E) + \frac{S^{\text{Poynting}}}{c^2} \right) \cdot e_\zeta + \frac{\partial}{\partial s}\left( V' \sum_a \{ (\Pi_a^{NA})^s + (\Pi_a^A)^s + (\Pi_a^C)^s \} \right) - \left\{ \sum_a n_{a0}m_a (u_{a\parallel} + u_E) + \frac{S^{\text{Poynting}}}{c^2} \right\} \cdot e_\zeta (u_s \cdot \nabla s) \right)$$

$$- \frac{1}{4\pi} \left( \nabla_s \cdot \left[ b_g \cdot \nabla_s \right] \right) + \mathcal{O}(\delta^3), \quad (103)$$

where $u_{a\parallel}$ represents the nonturbulent part of the parallel fluid velocity for particle species $a$ defined by $n_{a0}u_{a\parallel} = \int dU \int d\mu \int d\xi D_a (F_{a1})_{\text{ens}} u_{a\parallel}$ and $u_E = c(E_{\text{ens}} \times B_0$ is the nonturbulent part of the $E \times B$ drift velocity. Equation (103) is derived from Eq. (77) following the same procedures as shown in Ref. 18 except that the additional transport flux $\Pi_a^C$ defined in Eq. (107) and the external momentum source are newly included in the present case.

On the left-hand side of Eq. (103), the terms including $(u_{a\parallel} + u_E)$ and $S^{\text{Poynting}}$ are of $\mathcal{O}(\delta^3)$ although they are written down to explicitly show the inertia-term part. The nonturbulent and turbulence-driven parts of the radial flux of the toroidal angular momentum are defined by

$$(\Pi_a^{NA})^s = \left\langle \int dU \int d\mu \int d\xi \left\{ D_a (F_{a1})_{\text{ens}} \times m_a U_b \zeta (v_{a\parallel}^{gc})_{\text{ens}} \cdot \nabla s \right\} \right\rangle, \quad (104)$$

and

$$(\Pi_a^A)^s = \left\langle \left\langle \int dU \int d\mu \int d\xi \left\{ D_a \hat{h}_{a\parallel} \times m_a (U_b + v_{a\parallel}) \cdot e_\zeta (v_{a\parallel}^{gc}) \cdot \nabla s \right\} \right\rangle \right\rangle, \quad (105)$$

respectively. It is shown in Appendix D that Eq. (78) is ensemble-averaged to give

$$\left\langle \left\langle (\Pi_a^C)^s \right\rangle \right\rangle = \sum_a \Pi_a^C + \mathcal{O}(\delta^3), \quad (106)$$

where the radial transport flux of the toroidal angular momentum for species $a$ due to the collision term and finite gyroradius is defined by

$$\Pi_a^C = - \sum_b \left\{ \frac{m_ac}{2e_aB_0} \frac{\partial \chi}{\partial s} \int dU \int d\mu \int d\xi D_{ab}^\mu \times \left[ C^{p\parallel}_a ((F_{a1})_{\text{ens}}, F_{bM}) + C^{p\parallel}_a ((F_{a1}), (F_{b1})_{\text{ens}}) \right] \right\}, \quad (107)$$

The expressions for the toroidal momentum fluxes shown in Eqs. (104)–(107) agree with those given by conventional recursive formulations in Refs. 33–35. [Since the so-called high-flow ordering is used in Refs. 33 and 34, the expressions for the toroidal momentum fluxes in it reduce to those in the present work in the low-flow-speed limit.] As argued in Refs. 18 and 35, when there exists the up-down symmetry of the background magnetic field, all toroidal momentum fluxes vanish to $\mathcal{O}(\delta^3)$ and the nontrivial toroidal momentum balance equation is of $\mathcal{O}(\delta^3)$. In this case, gyrokinetic systems equations of higher-order accuracy in $\delta$ are required for the correct derivation of this $\mathcal{O}(\delta^3)$ toroidal momentum balance equation to determine the profile of the radial electric field although we should note, at the same time, that the radial electric field is not necessary to determine the particle and energy transport fluxes to the lowest order in $\delta$.\[\text{35}\]

VII. CONCLUSIONS

In this paper, particle, energy, and toroidal momentum balance equations including collisional and turbulent transport fluxes are systematically derived from the gyrokinetic Boltzmann-Poisson-Ampère system of equations. Considering an imaginary collisionless system, for which the distribution functions and electromagnetic fields coincide instantaneously with those for the considered collisional system, and expressing the variation of the action integral for the collisionless system in terms of the solution to the governing equations for the collisional system clarify effects of the collision and external source terms on the collisionless conservation laws derived from Noether’s theorem. The gyrokinetic collision operator is newly presented, by which the collisional changes in the velocity-space integrals of the gyrocenter Hamiltonian and the canonical toroidal angular momentum can be written in the conservative (or divergence) forms. It is confirmed that, to the lowest order in the normalized gyroradius, the ensemble-averaged fluxes in the derived particle, energy, and toroidal angular momentum balance equations can be written by the sum of conventional expressions of classical, neoclassical, and turbulent transport fluxes. The extension of the present work to the case of the high-flow ordering remains as a future task.

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Appendix A: COORDINATE TRANSFORMATION

We consider the transformation of the phase-space coordinates in this Appendix, where the subscript representing the particle species is omitted as far as it is unnecessary. In terms of the position \( \mathbf{x} \) and the velocity \( \mathbf{v} \) of a given particle, we define the parallel velocity \( v_{\parallel} = \mathbf{v} \cdot \mathbf{b} \), the perpendicular velocity \( v_{\perp} = \mathbf{v} - v_{\parallel} \mathbf{b} \), and the zeroth-order magnetic moment,

\[
\mu_0 = \frac{mv_{\perp}^2}{2B_0(x,t)} \tag{A1}
\]

where the equilibrium field at position \( x \) and time \( t \) is denoted by \( B_0(x,t) = B_0(\mathbf{x},t) \). We also define the zeroth-order gyrophase by \( \xi_0 = \tan^{-1}(\mathbf{v} \cdot \mathbf{e}_1)/(\mathbf{v} \cdot \mathbf{e}_2) \) where \((\mathbf{e}_1, \mathbf{e}_2, \mathbf{b})\) are unit vectors which form a right-handed orthogonal system at \( x, t \). Then, the gyrocenter coordinates \( \mathbf{Z} = (X, U, \mu, \xi) \) are represented in terms of the particle coordinates \( \mathbf{z} = (x, v_{\parallel}, \mu_0, \xi_0) \) as

\[
(X, U, \mu, \xi) = (x, v_{\parallel}, \mu_0, \xi_0) + (\Delta x, \Delta v_{\parallel}, \Delta \mu_0, \Delta \xi_0), \tag{A2}
\]

where

\[
\Delta x = -\rho + O(\delta^2)
\]

\[
\Delta v_{\parallel} = -v_{\parallel} \mathbf{b} \cdot \nabla \mathbf{b} \cdot \rho + \frac{1}{4} (3\mathbf{p} \cdot \nabla \mathbf{b} \cdot v_{\perp} \mathbf{v}_{\perp} - v_{\perp} \cdot \nabla \mathbf{b} \cdot \rho)
\]

\[
+ \frac{e}{mc} A_{\parallel} + O(\delta^2)
\]

\[
\Delta \mu_0 = \frac{m}{B_0} \left[ v_{\parallel} \mathbf{b} \cdot \nabla \mathbf{b} \cdot \rho + \frac{v_{\parallel}^2}{4\Omega B_0} \mathbf{b} \cdot \nabla \mathbf{b} \cdot \mathbf{v}_{\perp} \right]
\]

\[
+ \frac{e}{c B_0} \mathbf{v}_{\perp} \cdot \mathbf{A}_{\perp} + \frac{e}{B_0} \psi
\]

\[
+ O(\delta^2)
\]

\[
\Delta \xi_0 = \frac{1}{2B_0} \left( v_{\perp} \mathbf{v}_{\perp} \cdot \nabla \mathbf{b} \cdot \rho \right)
\]

\[
+ \frac{v_{\parallel}^2}{4\Omega B_0} \mathbf{v}_{\perp} \cdot \nabla \mathbf{b} \cdot \mathbf{v}_{\perp} + \frac{v_{\parallel}^2}{2\Omega B_0} (\mathbf{b} \cdot \nabla \mathbf{b} \cdot \mathbf{v}_{\perp})
\]

\[
+ \frac{e}{c B_0} \mathbf{v}_{\perp} \cdot \mathbf{A}_{\perp} + \frac{e}{B_0} \psi - \frac{1}{B_0} \left( \int \frac{\partial \psi}{\partial \mu_0} d\xi_0 \right) + O(\delta^2). \tag{A3}
\]

The formulas for \( \Delta v_{\parallel}, \Delta \mu_0, \) and \( \Delta \xi_0 \) in Eq. (A3) are obtained by combining the guiding center and gyrocenter coordinate transformations. Here, effects of the background electric field and turbulent electromagnetic fields are included through \( \psi \) [see Eq. (A3)] and \( A_1 \). When the background electric field and turbulent electromagnetic fields vanish, \( \Delta v_{\parallel}, \Delta \mu_0, \) and \( \Delta \xi_0 \) in Eq. (A3) agree with the results in Ref. 37.

Denoting the coordinate transformation by \( T \), Eq. (A2) is rewritten as

\[
\mathbf{Z} = T(\mathbf{z}) = \mathbf{z} + \Delta \mathbf{z}. \tag{A4}
\]

An arbitrary scalar field \( \mathcal{A} \) on the phase space can be expressed in terms of either the gyrocenter coordinates \( \mathbf{Z} = (X, U, \mu, \xi) \) or the particle coordinates \( \mathbf{z} = (x, v_{\parallel}, \mu_0, \xi_0) \) as

\[
\mathcal{A}^g(\mathbf{Z}) = \mathcal{A}^p(\mathbf{z}). \tag{A5}
\]

Using Eqs. (A4), (A5), and the Taylor series expansion, we obtain

\[
\mathcal{A}^p(\mathbf{z}) = (T^* \mathcal{A}^g)(\mathbf{z}) = \mathcal{A}^g(\mathbf{Z}) + \Delta \mathbf{z} \cdot \nabla \mathcal{A}^g(\mathbf{Z})
\]

\[
+ \frac{1}{n!} \sum_{i_1, \ldots, i_n} \Delta z^{i_1} \cdots \Delta z^{i_n} \frac{\partial^n \mathcal{A}^g(\mathbf{Z})}{\partial z^{i_1} \cdots \partial z^{i_n}}. \tag{A6}
\]

where \( T^* \mathcal{A}^g \) denotes the pullback transformation of \( \mathcal{A}^g \) by \( T \). Using the inverse transformation \( T^{-1} \), we also have \( \mathcal{A}^p(\mathbf{Z}) = (T^{-1} \mathcal{A}^p)(\mathbf{Z}) = \mathcal{A}^p(T^{-1}(\mathbf{Z})) \).

The Jacobians \( D^p \) and \( D^g \) for the two coordinate systems \( \mathbf{z} \) and \( \mathbf{Z} \) are related to each other by

\[
D^p(\mathbf{z}) = \text{det} \left[ \frac{\partial (\mathbf{Z})}{\partial (\mathbf{z})} \right] D^g(\mathbf{Z}), \tag{A7}
\]

where \( \partial (\mathbf{Z})/\partial (\mathbf{z}) \) denotes the Jacobian matrix. Then, we use the following formula,

\[
\delta^g(\mathbf{z} + \Delta \mathbf{z} - \mathbf{Z}) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1, \ldots, i_n} \Delta z^{i_1} \cdots \Delta z^{i_n} \frac{\partial^n \delta^g(\mathbf{z} - \mathbf{Z})}{\partial z^{i_1} \cdots \partial z^{i_n}}, \tag{A8}
\]

and partial integrals to derive the relation between the expressions of the scalar density \( \delta \mathcal{A} \) in the gyrocenter and particle coordinate systems as

\[
D^g(\mathbf{Z}) \delta^g(\mathbf{Z})
\]

\[
= \int d^6\mathbf{Z}' \delta^g(\mathbf{Z}' - \mathbf{Z}) D^g(\mathbf{Z}) \delta^g(\mathbf{Z}')
\]

\[
= \int d^6\mathbf{z} \delta^g(\mathbf{z} + \Delta \mathbf{z} - \mathbf{Z}) D^p(\mathbf{z}) \delta^p(\mathbf{z})
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{i_1, \ldots, i_n} \left[ \frac{\partial^n [\Delta z^{i_1} \cdots \Delta z^{i_n} D^p(\mathbf{z}) \delta^p(\mathbf{z})]}{\partial z^{i_1} \cdots \partial z^{i_n}} \right]_{z=\mathbf{Z}} \tag{A9}
\]

where the replacement of \( \mathbf{z} \) with \( \mathbf{Z} \) is represented by \( [\cdots]_{\mathbf{z}=\mathbf{Z}} \equiv \int d^6\mathbf{z} \delta^g(\mathbf{z} - \mathbf{Z}) \cdots \).

Appendix B: COLLISION OPERATOR INGYROCENTER COORDINATES

We can regard the collision term as a scalar field \( C \) on the phase space. When using the particle coordinates, we represent the collision term for collisions between species \( a \) and \( b \) by \( C_{ab}^p \). A well-established collision operator \( C_{ab}^p(f_a, f_b) \) for the particle distribution functions \( f_a \) and \( f_b \) is known as the Landau operator [see, for example,
Eq. (3.22) in Ref. 32. Then, the collision term $C_{ab}$ represented in the gyrocenter coordinates is related to $C_{ab}$ by
\begin{equation}
C_{ab}^{\theta}(F_a, F_b) = T_a^{-1} C_{ab}^\varphi (T_a^-F_a, T_b^{-1}F_b),
\end{equation}
(\text{B1})
where the distribution function for species $a$ ($b$) in the particle coordinates is written as the pullback $f_a = T_a^-F_a$ ($f_b = T_b^{-1}F_b$) of that in the gyrocenter coordinates $F_a$ ($F_b$) by the coordinate transformation $T_a$ ($T_b$) described in Appendix A, and $T_a^{-1}$ transforms the collision term as a function of the particle coordinates into that of the gyrocenter coordinates.

In order to see collisional effects on conservation laws, it is convenient to represent the collision term in the gyrocenter coordinate using the transformation formula for the scalar density $D_a C_{ab}$ rather than that for the scalar $C_{ab}$ shown in Eq. \text{[B1]}. Using Eq. \text{[A3]}, we can derive
\begin{equation}
D_a^\theta (Z_a) C_{ab}^\theta (F_a, F_b) (Z_a) A_b^\varphi (Z_a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{i_1, \ldots, i_n} \left[ \frac{\partial^n}{\partial z_a^{i_1} \cdots \partial z_a^{i_n}} D_a^\theta (Z_a) C_{ab}^\theta (f_a, f_b) (Z_a) A_b^\varphi (Z_a) \right] \bigg|_{z_a = Z_a},
\end{equation}
(\text{B2})
where $A_b$ is an arbitrary scalar field depending on particle species and $f_a = T_a^-F_a$ is rewritten by using Eq. \text{[A3]} as
\begin{equation}
f_a (Z_a) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1, \ldots, i_n} \Delta z_a^{i_1} \cdots \Delta z_a^{i_n} \frac{\partial^n F_a (Z_a)}{\partial z_a^{i_1} \cdots \partial z_a^{i_n}}.
\end{equation}
(\text{B3})
Then, the gyrocenter representation of the collision operator $C_{ab}$ acting on $F_a$ and $F_b$ is obtained by Eq. \text{B2} with putting $A^\theta = A^\varphi = 1$ and using Eq. \text{B3} to express $f_a$ and $f_b$ in terms of $F_a$ and $F_b$, respectively. Integrating Eq. \text{B2} with respect to $(U, \mu, \xi)$ and taking the summation over species $b$ yield
\begin{align}
\int dU \int d\mu \int d\xi \int d^3v \ C_a^\varphi (Z_a) A_b^\varphi (Z_a) &= \left[ \int d^3v \ C_a^\varphi (Z_a) A_b^\varphi (Z_a) \right]_{Z_a = Z} - \nabla \cdot J_a^C,
\end{align}
(\text{B4})
where $C_a^\varphi = \sum_b C_{ab}^\varphi$ and $\nabla = \partial / \partial X$ are used and $\int d^3v = \int d\mu \int d\xi \int d^3v_a D_a^\varphi (Z_a)$ denotes the velocity-space integral using the particle coordinates. Here, the transport flux $J_a^C$ of the quantity $A_a$ due to collisions and finite gyroradii of particles is defined by
\begin{align}
J_a^C &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \sum_{i_1, \ldots, i_n} \frac{\partial^n}{\partial X_{i_1} \cdots \partial X_{i_n}} \left[ \int d^3v \ \Delta X_a \Delta x_{i_1}^a \cdots \Delta x_{i_n}^a C_a^\varphi (Z_a) A_b^\varphi (Z_a) \right]_{x = X} \\
&= \left[ \int d^3v \ \Delta x_a C_a^\varphi (Z_a) A_b^\varphi (Z_a) \right]_{x = X} + \cdots.
\end{align}
(\text{B5})
The integral of an arbitrary scalar field $A_a$ over the whole phase space is written in either the gyrocenter or particle coordinate system as
\begin{align}
\int d^6Z \ D_a^\theta (Z_a) C_{ab}^\theta (Z_a) A_b^\varphi (Z_a) &= \int d^6z \ D_a^\theta (z) C_{ab}^\theta (z) A_b^\varphi (z).
\end{align}
(\text{B6})
For the case of $A_a = 1$, Eqs. \text{B4} and \text{B5} reduce to
\begin{align}
\int dU \int d\mu \int d\xi \int d^3v \ D_a^\theta (Z_a) C_{ab}^\theta (Z_a) &= -\nabla \cdot \Gamma_a^C (X),
\end{align}
(\text{B7})
and
\begin{align}
\Gamma_a^C (X) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \sum_{i_1, \ldots, i_n} \frac{\partial^n}{\partial X_{i_1} \cdots \partial X_{i_n}} \left[ \int d^3v \ \Delta X_a \Delta x_{i_1}^a \cdots \Delta x_{i_n}^a C_a^\varphi (Z_a) \right]_{x = X} \\
&= \left[ \int d^3v \ \Delta x_a C_a^\varphi (Z_a) \right]_{x = X} + \cdots,
\end{align}
(\text{B8})
respectively, where $\int d^3v C_a^\varphi (z) = 0$ is used. Here, $\Gamma_a^C$ is regarded as the classical particle flux which occurs due to collisions and finite gyroradii. In fact, using $\Delta X_a \approx -\rho_a$, we see that the primary term of $\Gamma_a^C$ shown in the last line of Eq. \text{B5} is identical to the conventional definition of the classical particle flux $\Gamma_a^{\text{cl}} \equiv (c/e_a B_0) F_{a1} \times b$, where $F_{a1} = \int d^3v \rho_a v C_a^\varphi$ is the collisional friction force. Thus, we have $\Gamma_a^C = \Gamma_a^{\text{cl}} + [1 + O(\delta)]$.

Let us take the kinetic energy of the particle as $A_a$ and put $A_b^\varphi = \frac{1}{2} m_a v_{a1}^2 = \frac{1}{2} m_a v_{a1}^2 + \mu_{a0} B_0 (X_a)$. Then, it is written in terms of the gyrocenter coordinates as
\begin{align}
A_b^\varphi &= \frac{1}{2} m_a v_{a1}^2 \\
&= \frac{1}{2} m_a U_a^2 + \mu_a B_0 (X_a) + \mu_a \rho_a \cdot \nabla B_0 (X_a) \\
&\quad - m_a U_a (\Delta \mu_{a0})_{\mu_a} = z_{a0} - (\Delta \mu_{a0})_{\mu_a} = z_a - B_0 (X_a) + \cdots \\
&= \frac{1}{2} m_a v_{a1}^2 + \mu_a B_0 (X_a) + \mu_a \rho_a (\psi_a (Z_a))_{\mu_a} - \mu_a (T_a^{-1} \phi) (Z_a) + O(\delta^2),
\end{align}
(\text{B9})
where the inverse $T_a^{-1}$ of the transformation $T_a$ given by Eq. \text{[A3]} is used. In this case, taking the summation
of Eq. (B11) over species \( a \) and using the conservation property \( \sum_a \int d^3v \, C_{a}^{p} \frac{m_a v^2}{2} = 0 \), we have

\[
\sum_a \int dU \int d\mu \int d\xi \, D_a^{\mu}(Z) C_{a}^{p}(Z) T_a^{-1+} \left( \frac{1}{2} m_a v^2 \right) = -\nabla \cdot Q^C, \tag{B10}
\]

where \( Q^C \) represents the transport flux of the total kinetic energy due to collisions and finite gyroradii defined by

\[
Q^C(X) = \sum_a \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \sum_{i_1, \ldots, i_n} \left( \frac{1}{2} m_a v^2 \right) 
\times \left[ \int d^3v \, \Delta x_a \Delta x_{a,i_1} \cdots \Delta x_{a,i_n} C_{a}^{p}(z) \right]_{x=X} + \cdots. \tag{B11}
\]

To the lowest order in \( \delta \), the collisional energy flux \( Q^C \) is approximately written as \( Q^C \approx \sum_a (q_{a}^{cl} + \frac{1}{2} T_a \Gamma_{a}^{cl}) \). Here, the classical heat flux for species \( a \) is defined by \( q_{a}^{cl} \equiv (eT_a/c_a B_0) \mathbf{F}_{a2} \times \mathbf{b} \), where \( \mathbf{F}_{a2} \equiv \int d^3v (m_a v^2/2T_a - 5/2) m_a v C_{a}^{p} \) is the collisional heat friction. We note from Eq. (B9) that the expression of the kinetic energy in the gyrocenter coordinates should be generally given by

\[
\int d^3v \, \Delta x_a \Delta x_{a,i_1} \cdots \Delta x_{a,i_n} C_{a}^{p}(z) \left( \frac{1}{2} m_a v^2 \right) \] for the lowest order terms in Eq. (B9) and evaluate the gyrocenter velocity-space integral of the collisional rate of change of the gyrocenter coordinates.

The above-mentioned subtle relation between expressions of the collisional energy conservation properties in the particle and gyrocenter coordinate systems is also found when considering the collisional momentum conservation. It should be recalled that the perturbative expansions in \( \delta \) are truncated up to finite orders in deriving gyrokinetic equations as shown in Sec. III although the conservative form of equations for the energy and the toroidal angular momentum are obtained even from these approximate equations for the collisionless case since they are constructed based on the variational principle. Thus, from the viewpoint of practical applications, it is desirable for the approximate collision operator in the gyrocenter coordinates to keep the conservation properties. More rigorously speaking, we want the gyrokinetic collisional velocity-space integrals \( \sum_a \int dU \int d\mu \int d\xi \, D_a^{\mu}(Z) C_{a}^{p}(Z) H_a(Z) \) and \( \sum_a \int dU \int d\mu \int d\xi \, D_a^{\mu}(Z) C_{a}^{p}(Z) \) to take the divergence forms and include no local source or sink terms where \( H_a(Z) \) and \( (p_{a}^{cl})_{i_1}^{\mu}(Z) \equiv (e_a/c) A_{a,i_1}^{cl}(Z) \) are the gyrocenter Hamiltonian and the canonical toroidal angular momentum defined by Eqs. (B9) and (B12), respectively. Here, it should be noted that only kinetic parts of energy and toroidal momentum but also contributions from scalar and vector potentials are included in \( H_a(Z) \) and \( (e_a/c) A_{a,i_1}^{cl}(Z) \). In Appendix C, we find how to construct the approximate gyrokinetic collision operator, by which the two integrals mentioned above are written in the divergence forms.

We now consider the entropy per unit volume defined in terms of the gyrocenter distribution functions as \( S^g \equiv -\sum_a \int dU \int d\mu \int d\xi D_a^{\mu}(Z) \log F_a(Z) \), in which the rate of change is given by \( \frac{dS^g}{dt} = -\sum_a \int dU \int d\mu \int d\xi \, D_a^{\mu}(Z) \left[ \log F_a(Z) + 1 \right] (dF_a/dt) \). Then, the rate of change in \( S^g \) due to collisions is obtained by putting \( A_{a}^{g} = -\int dF_a(z) [\log F_a(Z) + 1] \) in Eq. (B11) and taking the summation over species \( a \) as

\[
-\sum_a \int dU \int d\mu \int d\xi \, D_a^{\mu}(Z) C_{a}^{p}(z) \log F_a(z) + 1]
= -\sum_a \left[ \int d^3v \, C_{a}^{p}(z) \log f_a(z) \right]_{x=X} - \nabla \cdot J_S^C, \tag{B12}
\]

where \( \int d^3v \, C_{a}^{p}(z) = 0 \) is used although we should recall that \( \int dU \int d\mu \int d\xi D_a^{\mu}(Z) C_{a}^{p}(Z) \) does not vanish generally as seen from Eq. (B11). It is well-known that, when Landau’s collision operator is used for \( C_{a}^{p} \), the collisional entropy production rate given by the first term on the right-hand side of Eq. (B12) is nonnegative. This is Boltzmann’s H-theorem which proves the second law of thermodynamics. The collisional transport flux \( J_S^C \) of the entropy in Eq. (B12) is defined by

\[
J_S^C(X) = \sum_a \int dU \int d\mu \int d\xi \, D_a^{\mu}(Z) C_{a}^{p}(z) \left[ \log f_a(z) + 1 \right]_{x=X} + \cdots. \tag{B13}
\]

It is shown that, to the lowest order in \( \delta \), the collisional entropy transport flux is written as \( J_S^C = \sum_a (S_0 u_{a}^{cl} + q_{a}^{cl}/T_a) \) where the lowest-order entropy density \( S_0 \) for species \( a \) is given in terms of the local Maxwellian distribution function \( f_{aM} \) as \( S_0 \equiv -\int dU \int d\mu \int d\xi \, f_{aM} \log f_{aM} \), and \( u_{a}^{cl} \) is defined by \( u_{a}^{cl} = \Gamma_{a}^{cl}/n_a \). Here, we note again that the infinite series expansion in \( \delta \) as given by Eq. (B12) is used in deriving Eq. (B13). When the expansion is truncated to finite order; the collisional entropy production term is represented by \( -\sum_a \int d^3v \, C_{a}^{p} \log f_a + \text{residual error terms of higher order in} \, \delta \), and thus the H-theorem is only approximately satisfied.
Appendix C: Collision Operator Relevant for Gyrokinetic Conservation Laws

In this Appendix, we consider an approximate gyrokinetic collision operator instead of the one given by Eq. (131) [or Eq. (122)] with \( A_\parallel = A_\perp = 1 \) in order to get the gyrokinetic collisional velocity-space integrals of energy and canonical toroidal angular momentum to take desirable conservative (or divergence) forms. The approximate collision operator is written in the gyrocenter coordinates as

\[
D_\parallel^0(Z_a)(C_{ab}^g[F_a,F_b](Z_a))\xi_a = \left( \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \sum_{i_1,\ldots,i_n} \frac{\partial^n}{\partial x_{i_1} \cdots \partial x_{i_n}} \left[ \delta_x^{(2)} D_a^p(z_a) C_{ab}^g(f_a,f_b)(z_a) \right] - \frac{\partial}{\partial v_{\parallel a}} \left[ \delta v_{\parallel a}^{(2)} D_a^p(z_a) C_{ab}^g(f_a,f_b)(z_a) \right] - \frac{\partial}{\partial \mu_{0 a}} \left[ \delta \mu_{0 a}^{(2)} D_a^p(z_a) C_{ab}^g(f_a,f_b)(z_a) \right] \right]_{z_a = Z_a} \right)_{\xi_a},
\]

(C1)

which \( \Delta v_{\parallel a} \) and \( \Delta \mu_{0 a} \) are written as

\[
\Delta x_a = -\rho_a + \Delta x_a^{(2)}, \quad \Delta v_{\parallel a} = \Delta v_{\parallel a}^{(1)} + \Delta v_{\parallel a}^{(2)}, \quad \Delta \mu_{0 a} = \Delta \mu_{0 a}^{(1)} + \Delta \mu_{0 a}^{(2)},
\]

(C2)

and \( f_a \) is given from \( F_a \) by \( f_a(z_a) = F_a(x_a + \Delta x_a, v_{\parallel a} + \Delta v_{\parallel a}, \mu_{0 a} + \Delta \mu_{0 a}) \). Here, \( \Delta x_a^{(1)}, \Delta v_{\parallel a}^{(1)} \), and \( \Delta \mu_{0 a}^{(1)} \) are the \( O(\delta) \) parts of \( \Delta v_{\parallel a} \) and \( \Delta \mu_{0 a} \) given in Eq. (A3). In this Appendix, we do not derive expressions for the \( O(\delta^2) \) parts \( \Delta x_a^{(2)}, \Delta v_{\parallel a}^{(2)}, \) and \( \Delta \mu_{0 a}^{(2)} \) by the Lie perturbation expansion method which is used to define the gyrocenter coordinates with the well-conserved magnetic moment because it would unnecessarily give higher-order accuracy to the coordinate transformation than the accuracy of the gyrocenter motion equations themselves shown in Eqs. (22)–(31). Instead, we determine these \( O(\delta^2) \) terms from the conditions that the collisional change rates of energy and canonical toroidal angular momentum per unit volume in the gyrocenter space can be given in the conservative forms as shown below. Thus, the \( O(\delta^2) \) terms are introduced not for accuracy of higher order in \( \delta \) but for satisfying the conservation property of the collision operator.

In Eq. (C1), the expansions in \( \Delta x_a^{(2)}, \Delta v_{\parallel a}, \Delta \mu_{0 a} \) are truncated to the first order while the infinite series expansion in \( \Delta x_a^{(1)} = -\rho_a \) is retained because fluctuations’ wavelengths in the directions perpendicular to the equilibrium magnetic field can be of order of the gyroradius \( \rho_a \). In the WKB (or ballooning) representation, the above-mentioned infinite series expansion can be treated using the Bessel functions of the gyroradius normalized by the perpendicular wavelength. We should also note that the gyrophase average \( \langle \cdots \rangle_{\xi_a} \) is taken so that the gyrokinetic equation with the collision term is solved only for the gyrophase-averaged part of the gyrocenter distribution function. For an arbitrary function \( A_a^g(Z_a) \) which is independent of the gyrophase \( \xi_a \), we obtain the following formula,

\[
D_a^g(Z_a)(C_{ab}^g[F_a,F_b](Z_a))\xi_a A_a^g(Z_a) = \left[ D_a^g(z_a) C_{ab}^g(f_a,f_b)(z_a) A_a^g(z_a) + \sum_{i_1,\ldots,i_n} \frac{\partial^n}{\partial x_{i_1} \cdots \partial x_{i_n}} \left[ \delta_x^{(2)} D_a^p(z_a) C_{ab}^g(f_a,f_b)(z_a) A_a^g(z_a) \right] \right]_{z_a = Z_a} \xi_a
\]

(C3)

where \( A_a^g(z_a) \) and \( A_a^{p g}(z_a) \) are defined by

\[
A_a^g(z_a) = A_a^g(Z_a), \quad A_a^{p g}(z_a) = A_a^{p g}(Z_a) - \rho_a (v_{\parallel a}, \mu_{0 a}).
\]

(C4)

We should note that the function \( A_a^g(Z_a) \) defined from \( A_a^g(Z_a) \) in Eq. (C3) does not exactly coincide with that given in Eq. (A6) in Appendix A by the second- and higher-order terms in the series expansion with respect to \( \Delta x_a \). Integrating Eq. (C3) over the gyrocenter velocity space, we immediately obtain

\[
\int dU \int d\mu \int d\xi \int d^3X D_a^g(Z_a) C_{ab}^g(Z_a) A_a^g(Z) = \int d^3v C_a^g(z) A_a^g(z) - \nabla \cdot J_{A_a}^g \]

(C5)

where the transport flux \( J_{A_a}^g \) due to collisions and finite gyroradii is defined by

\[
J_{A_a}^g(X) = \int d^3v \left[ -\rho_a + \Delta x_a^{(2)} C_a^g(z) A_a^g(z) \right]_{X = X} + \sum_{n=1}^{\infty} \frac{-1}{(n+1)!} \sum_{i_1,\ldots,i_n} \frac{\partial^n}{\partial X_{i_1} \cdots \partial X_{i_n}} \int d^3v \rho_a \rho_a^i \cdots \rho_a^{i_n} C_a^g(z) A_a^g(z)_{X = X}.
\]

(C6)
To the lowest order in $\delta$, Eqs. (C8), (C9), and (C10) derived from the approximate collision operator in Eq. (C11) agree with Eqs. (B2), (B3), and (B5) given in Appendix B, respectively. The particle flux $\Gamma_a^C$ due to collisions and finite gyroradii is given from Eq. (C6) with putting $A_p^f(x) = 1$ in the same way as in Eqs. (B5).

Now, let us take $A_p^f(Z) = H_a(Z)$ in Eq. (C5). Here, $H_a(Z)$ denotes the gyrocenter Hamiltonian defined by Eq. (27). It is desirable that the gyrocenter velocity-space integral $\sum_a \int dU \int du \int d\xi \ D_a^f(Z) C_a^f(Z) H_a(Z)$ takes the conservative form, which implies that the integral is expressed by the divergence term only and

$$\sum_a \int d^3v \ C_a^f(Z) H_a^p(z) = 0 \quad \text{(C7)}$$

holds. Here, using Eqs. (26) and (44), $H_a^p(z)$ is given by

$$H_a^p(z) = \frac{1}{2} m_a v_a^2 + \mu B_0(x_a) + c_a$$(C8)

where

$$\Delta H_a(z_a) \equiv \left( \Delta x_a^{(2)} \frac{\partial}{\partial \Delta x_a} + \Delta v^{(2)}_a \frac{\partial}{\partial \Delta v^{(2)}_a} + \Delta \mu_{0a} \frac{\partial}{\partial \mu_{0a}} \right) H_a(z_a) + \mu \left[ B_0(x_a - \rho_a) - B_0(x_a) \right] + c_a \left[ \delta_x(x_a - \rho_a, \eta_a, \mu_{0a}) - \phi(x_a) \right]. \quad \text{(C9)}$$

It is easily seen that Eq. (A7) is satisfied if $\Delta H_a(z_a) = 0$. Then, substituting Eq. (22) into Eq. (9) and using $\Delta H_a(z_a) = 0$, we have

$$\Delta x_a^{(2)} \cdot \nabla (p_{\xi_a}^f)^p(z_a) = -\frac{e_a}{c_b \mu_0} \left[ \frac{3}{4} \left( 3 \rho_a \cdot \nabla b \cdot \nabla \cdot \rho_a \right) - \frac{1}{2} \frac{m_a c^2}{e_a} A_1 \right] \left( \frac{\partial}{\partial v^{\parallel}_a} - \frac{m_a v_a^\parallel}{B_0} \frac{\partial}{\partial \mu_{0a}} \right) \Psi. \quad \text{(C10)}$$

We find that the right-hand side of Eq. (C10) is of $O(\delta^2)$. Then, as remarked after Eq. (C14), we can choose $\Delta x_a^{(2)}$, $\Delta v^{(2)}_a$, and $\Delta \mu_{0a}$ which satisfy Eq. (C10) and are of $O(\delta^2)$ so as to be consistent with Eq. (A3).

When we use $\Delta x^{(2)} = 0$, $\Delta v^{(2)} = \Delta \delta^{(1)}$, and $\Delta \mu_{0a} = \Delta \delta^{(2)}$ for Eq. (C11) by putting $\Delta v^{(2)} = \Delta \delta^{(2)} = 0$, we have $\Delta H_a(z_a) = O(\delta^2)$ and $\sum_a \int d^3v \ C_a^f(z) H_a^p(z) = O(\delta^3)$ because $C_a^f(z) = O(\delta)$ holds for the distribution function, the zeroth order of which is given by the local Maxwellian. Therefore, even for this case

where $\sum_a \int dU \int du \int d\xi \ D_a^f(Z) C_a^f(Z) H_a(Z)$ is not completely given in the conservative form, the residual term $\sum_a \int d^3v C_a^f(z) H_a^p(z) = O(\delta^3)$ is smaller by a factor of $\delta$ than other transport terms of $O(\delta^2)$ in the lowest-order energy balance equation given by Eq. (A10) in Sec. VI.B.

We next put $A_p^f(Z) = (p_{\xi_a}^f)^p(Z)$ in Eq. (C3). Here, $(p_{\xi_a}^f)^p(Z)$ denotes the canonical toroidal angular momentum defined by

$$\left( p_{\xi_a}^f(Z) \right)^p = \frac{e_a}{c_b A_0} \left( A_0(X) + A_1(X) \right) + m_a v_a + (p_{\xi_a}^p(Z), Z) \quad \text{(C12)}$$

Here, using Eqs. (C4) and (C11), $(p_{\xi_a}^p(Z))$ is given by

$$\left( p_{\xi_a}^p(Z) \right)^p = \frac{e_a}{c_b A_0(X) + A_1(X)} + m_a v_a + (p_{\xi_a}^p(Z), Z) \quad \text{(C13)}$$

where

$$\Delta (p_{\xi_a}^p(Z), Z) = \left( \Delta x_a^{(2)} \frac{\partial}{\partial x_a} + \Delta v^{(2)}_a \frac{\partial}{\partial v^{(2)}_a} \right) (p_{\xi_a}^p(Z), v^{\parallel}_a) \quad \text{(C14)}$$

Again, we easily see that Eq. (C12) is satisfied if $\Delta (p_{\xi_a}^p(Z), Z) = 0$. The $O(\delta^2)$ variables, $\Delta x_a^{(2)}$ and $v^{\parallel}_a$, which meet the condition that $\Delta (p_{\xi_a}^p(Z), Z) = 0$, are given by

$$\Delta x_a^{(2)} \cdot \nabla (p_{\xi_a}^f)^p(z_a) = -\frac{e_a}{c_b A_0(X) + A_1(X)} \left( \frac{3}{4} \left( 3 \rho_a \cdot \nabla b \cdot \nabla \cdot \rho_a \right) - \frac{1}{2} \frac{m_a c^2}{e_a} A_1 \right) \left( \frac{\partial}{\partial v^{\parallel}_a} - \frac{m_a v_a^\parallel}{B_0} \frac{\partial}{\partial \mu_{0a}} \right) \Psi. \quad \text{(C15)}$$

and

$$\Delta v^{\parallel}_a \frac{\partial}{\partial v^{\parallel}_a} (p_{\xi_a}^f)^p(z_a) = m_a b_c(x_a) \Delta v^{\parallel}_a \quad \text{(C16)}$$

where $\nabla \equiv \partial/\partial x_a$ and $W_c = -(\nabla \chi \cdot \nabla \chi)/(RB_0) + \frac{1}{2} b_c \cdot (\nabla \times b)$. As a solution to Eq. (C15), we can assume $\Delta x_a^{(2)}$ to be given in the form $\Delta x_a^{(2)} \cdot \nabla \chi$. We should note that $\Delta x_a^{(2)} \cdot \nabla (p_{\xi_a}^f)^p(Z) = O(\delta^3)$ and $\Delta v^{\parallel}_a \frac{\partial}{\partial v^{\parallel}_a} (p_{\xi_a}^f)^p(Z) = m_a b_c = O(\delta)$ while the right-hand sides of Eqs. (C15) and (C16) are of $O(\delta)$ and $O(\delta^2)$, respectively. Therefore, Eqs. (C15) and (C16) give $\Delta x_a^{(2)}$ and $\Delta v^{\parallel}_a$, which
are both of $O(\delta^2)$, consistently with Eq. (A3). Then, these $\Delta x^a_\alpha (2)$ and $\Delta v^a_\alpha (2)$ are substituted into Eq. (C10) to determine $\Delta \mu^a_\alpha (2)$ of $O(\delta^2)$ as well.

Thus, the collision operator, which has the desired conservation properties as well as the accuracy required for correct description of collisional transport of the energy and the toroidal angular momentum, is given by Eq. (C7), in which $\Delta x^a_\alpha (2)$, $\Delta v^a_\alpha (2)$, and $\Delta \mu^a_\alpha (2)$ are defined by Eqs. (C2), (C10), (C14), and (C16). Using this collision operator, putting $A^p(z) = \frac{1}{2} m_a v_\parallel^2 + \mu_0 B_0(x) + e_a \phi(x)$ and $A^p_a(z) = (e_a/c)[A_0(x) + A_1(x)] + m_a v_\parallel$ in Eq. (C3) and taking their summation over species $a$ define the transport fluxes $Q^C_a$ and $J^C_a$ of the energy and the canonical toroidal angular momentum, respectively, which appear in the energy and toroidal angular momentum balance equations in Secs. V.A and B [see Eqs. (19), (20), (21), and (22)]. In the definition of $Q^C_a$ mentioned above, the contribution of the potential energy part $e_a \phi$ is written as $\phi \sum_a e_a f_a$ which is smaller than the contribution of the kinetic energy part by a factor of $\delta$ because the classical particle fluxes represented by the lowest-order part of $\Gamma^C_a$ are intrinsically ambipolar. Therefore, the energy flux $Q^C_a$ defined here agrees with Eq. (B11) to the lowest order in $\delta$. Regarding the entropy production rate [corresponding to the first term on the left-hand side of Eq. (B12)] is only approximately shown by using the present model collision operator in Eq. (C3) with $A^p_a(z) = -[\log F_a(z) + 1]$ because $A^p_a(z) = -[\log f_a(z) + 1]$ is not rigorously derived from Eq. (C4) without the infinite series expansion in $\Delta z_a$ as in Eq. (A6).

Appendix D: Derivation of Eqs. (106) and (107)

In this Appendix, it is shown how to derive Eqs. (106) and (107) by using the collision operator given in Appendix C. On the right-hand side of Eq. (78) where the radial flux $(\Pi^C)^{s}_\alpha$ of the toroidal angular momentum due to collisions and finite gyroradii is defined, the two types of fluxes $j^C_a = \sum_a e_a \Gamma^C_a$ and $j^C_{\phi \alpha} = \sum_a j^C_{\phi \alpha}^a$ are evaluated by taking the summation of Eq. (C6) over species $a$ with putting $A^p_a(z) = e_a$ and $A^p_a(z) = (p^C_a)^\parallel_a(z)$, respectively. Here, $(p^C_a)^\parallel_a(z) = (e_a/c)[A_0(x) + A_1(x)] + m_a v_\parallel$ is used for the collision operator which conserves the toroidal angular momentum as explained in Appendix C. Consequently, the ensemble average of $(\Pi^C)^{s}_\alpha$ is expressed explicitely up to $O(\delta^2)$ as

$$
\langle (\Pi^C)^{s}_\alpha \rangle_{\text{ens}} = \left\langle \left( J^C_{\phi \alpha} \mathbf{c} \cdot \nabla s \right)_{\text{ens}} \right\rangle = \sum_a \int d^3 \mathbf{v}_s \left( C^p_a(z) \right)_{\text{ens}} \frac{e_a}{2 \pi c} \mathbf{P}_a : \nabla \chi \nabla s - m_a \mathbf{P}_a \cdot \mathbf{v}_s + e_a \nabla \phi + O(\delta^3)
$$

[D1]

from which Eqs. (106) and (107) are immediately obtained. It is noted that the $O(\delta^3)$ part of $\langle (\Pi^C)^{s}_\alpha \rangle_{\text{ens}}$ has no contribution from the gyrophase-dependent part of the distribution function, the lowest-order part of which is given by $f_{\text{in}} = -\mathbf{P}_a : \nabla F_{aM}$ with the gradient operator $\nabla$ taken for the fixed energy variable $\varepsilon = \frac{1}{2} m_a v^2 + e(\phi)_{\text{ens}}$.

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