**Abstract**

We propose a natural Higgs sector in $E_6$ grand unified theory (GUT) with anomalous $U(1)_A$ gauge symmetry. In this scenario, the doublet-triplet splitting can be realized, while proton decay via dimension 5 operators is suppressed. Gauge coupling unification is also realized without fine-tuning. The GUT scale obtained in this scenario is generally lower than the usual GUT scale, $2 \times 10^{16}$ GeV, and therefore it should be possible to observe proton decay via dimension 6 operators in near future experiments. The lifetime of a nucleon in this model is roughly estimated as $\tau_p(p \to e^+ \pi^0) \sim 3 \times 10^{33}$ years. It is shown that the Higgs sector is compatible with the matter sector proposed by one of the present authors, which reproduces realistic quark and lepton mass matrices, including a bi-large neutrino mixing angle. Combining the Higgs sector and the matter sector, we can obtain a completely consistent $E_6$ GUT. The input parameters for this model are only eight integer anomalous $U(1)_A$ charges (+3 for singlet Higgs) for the Higgs sector and three (half) integer charges for the matter sector.
1 Introduction

Recently in a series of papers [1, 2, 3] a scenario to construct a realistic GUT has been proposed within the \(SO(10)\) group. In this scenario, anomalous \(U(1)_A\) gauge symmetry [4], whose anomaly is cancelled by the Green-Schwarz mechanism [5], plays a critical role, and it has many interesting features: 1) the interaction is generic, in the sense that all the interactions that are allowed by the symmetry are introduced. Therefore, once we fix the field content with its quantum numbers (integer), all the interactions are determined, except the coefficients of order 1; 2) it naturally solves the so-called doublet-triplet (DT) splitting problem [6], using the Dimopoulos-Wilczek (DW) mechanism [7, 8, 9, 10]; 3) it reproduces realistic structure of quark and lepton mass matrices, including neutrino bi-large mixing [11], using the Froggatt-Nielsen (FN) mechanism [12]; 4) the anomalous \(U(1)_A\) explains the hierarchical structure of the symmetry-breaking scales and the masses of heavy particles; 5) all the fields, except those of the minimal supersymmetric standard model (MSSM), can become heavy; 6) the gauge couplings are unified just below the usual GUT scale \(\Lambda_G \sim 2 \times 10^{16}\) GeV; 7) in spite of the lower GUT scale, proton decay via dimension 6 operators, \(p \rightarrow e^+ \pi^0\), is within experimental limits, leading us to expect that proton decay will soon be observed. 8) the cutoff scale is lower than the Planck scale; 9) the \(\mu\) problem is also solved.

An extension of the above mentioned \(SO(10)\) model to the \(E_6\) gauge group has been carried out in the analysis of fermion masses [13], and it has been found that \(E_6\) is more economical in the sense that we have only to introduce minimal matter fields three \(27\)s for 3-family fermions in contrast to the matter content of three \(16\) plus one \(10\) in the \(SO(10)\) case. Moreover, the charge assignment for realizing bi-large neutrino mixing automatically satisfies the condition for weakening the flavor changing neutral current (FCNC). Specifically, the right-handed down quark and the left-handed lepton of the first and second generations belong to a single multiplet \(27\) as a result of the twisting family structure [14].

In the Higgs sector of \(E_6\) unification, however, the situation is not so simple. The Higgs fields \(\Phi(27)\) and \(\Phi(27)\) are needed to break \(E_6\) into \(SO(10)\), in addition to the Higgs fields \(A(78)\), whose VEV breaks \(SO(10)\) into \(SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}\), and \(\tilde{C}(27)\) and \(C(27)\), whose VEVs break \(SU(2)_R \times U(1)_{B-L}\) into \(U(1)_Y\). The other VEVs, \(\langle \tilde{C} \rangle, \langle C \rangle, \langle \tilde{\Phi} \rangle\) and \(\langle \Phi \rangle\), generally destabilize the DW-type VEV; \(\langle 1_A \rangle = \langle 16_A \rangle = \langle 16_A \rangle = 0, \langle 45_A \rangle = \tau_2 \times \text{diag}(v, v, v, 0, 0),\) which is required to realize DT splitting. Therefore we may have to remove the interaction between \(A\) and \(\tilde{\Phi}\Phi\) as well as that between \(A\) and \(\tilde{C}C\). However, if we forbid interactions between these Higgs fields in the superpotential, then pseudo-Nambu-Goldstone (PNG) fields appear. Since the non-vanishing VEVs \(|\langle \Phi(1,1) \rangle| = |\langle \Phi(1,1) \rangle|\) break \(E_6\) into \(SO(10)\), the Nambu-Goldstone (NG) modes \(16, \overline{16}\) and \(1\) of \(SO(10)\) appear. Also, the VEVs \(|\langle C(16,1) \rangle| = |\langle C(16,1) \rangle|\) break \(E_6\) into another \(SO(10)'\), so \(16, \overline{16}\) and \(1\) of \(SO(10)'\) again appear as NG modes.
Here, the representations of $SO(10)$ and $SU(5)$ are explicitly denoted as the first and second numbers in the parentheses, respectively. In the language of usual $SU(5)$, these NG modes are represented as $(10 + \bar{5} + 1)$, $(\bar{10} + 5 + 1)$ and $1$. The VEV of the adjoint field $\langle 45_A \rangle = \tau_2 \times \text{diag}(v, v, v, 0, 0)$ breaks $E_6$ into $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \times U(1)_Y$, and the NG modes resulting from this breaking are $\underline{16}$ and $\bar{\underline{10}}$ of $SO(10)$ and $(3, 2)_{\frac{1}{2}} + (\bar{3}, 1)_{-\frac{4}{3}} + (3, 2)_{-\frac{5}{6}} + h.c.$ of the standard gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$. Since some of these modes are absorbed by the Higgs mechanism, the remaining PNG modes become $10 + \bar{10} + 2 \times 5 + 2 \times \bar{5} + 4 \times 1$ of $SU(5)$ and $(3, 2)_{\frac{1}{2}} + (\bar{3}, 1)_{-\frac{4}{3}} + h.c.$ of the standard gauge group. If all these PNG modes have only tiny masses around the weak scale, then not only is coupling unification destroyed but also the gauge couplings diverge below the GUT scale. Therefore, we have to give these PNG fields superheavy masses. However, in order to do so, we have to introduce some interactions between Higgs fields, and this requirement is in opposition to that needed to stabilize the DW-type of VEV. This conflict is similar to that existing in the $SO(10)$ case.

This paper aims at obtaining a unified description of the Higgs sector in $E_6$ model, in which the above stated problem and the DT splitting problem are solved. It may seem that $SO(10)$ unified models are promising for this purpose. However, if we proceed to $E_6$, there are more advantages in addition to the natural FCNC suppression. In particular, we have the following:

1. The FN field naturally emerges as the composite operator $\langle \Phi \Phi \rangle$, where $\Phi$ and $\bar{\Phi}$ are needed to break $E_6$ down to $SO(10)$.
2. The usual doublet Higgs field $H$ is already included in the field $\Phi$.
3. In the Higgs sector, the condition for the unification of gauge coupling constants automatically provides “R parity” in terms of anomalous $U(1)_A$ naturally, and therefore we do not have to introduce additional R parity.

Moreover, we can construct a completely consistent and realistic $E_6$ GUT scenario by combining this Higgs sector and the matter sector.

After explaining how the vacuum in the Higgs sector is determined by anomalous $U(1)_A$ charges (§2) and giving a quick review of the $SO(10)$ model (§3), we explain how the above desirable features in the Higgs sector are naturally obtained in the $E_6$ unification (§4) and a completely consistent $E_6$ GUT scenario (§5).

## 2 Vacuum determination

Here we explore some general structures of VEVs that are determined from the superpotential of the Higgs sector. The Higgs sector is the most poorly part, and usually the VEVs of Higgs fields are introduced as input parameters, because
general forms of the potential are usually too arbitrary. However, anomalous 
$U(1)_A$ provides us with very strong constraints on the superpotential $W$ and 
thus dictates the scales of the system in a quite definite way. In our analy-
sis, supersymmetry (SUSY) is essential, because the observation of the following 
vacuum structure is due to the analytic property of SUSY theory.

We study the simplest case, in which all the fields are gauge singlet fields 
$Z_{i}^{\pm}$ ($i = 1, 2, \cdots n_{\pm}$) with charges $z_{i}^{\pm}$ ($z_{i}^{+} > 0$ and $z_{i}^{-} < 0$). Through out this paper we use units in which the cutoff is $\Lambda = 1$, and we denote all the superfields with uppercase letters and their anomalous $U(1)_A$ charges with the corresponding lowercase letters.

We first show that none of the fields with positive anomalous $U(1)_A$ charge 
acquire non-zero VEV if the FN mechanism [12] acts effectively in the vacuum. 
From the $F$-flatness conditions of the superpotential, we get

\[ F_{Z_i} \equiv \frac{\partial W}{\partial Z_i} = 0, \quad D_A = g_A \left( \sum_i z_i |Z_i|^2 + \xi^2 \right) = 0, \quad (2.1) \]

where $\xi^2$ is the coefficient of the Fayet-Iliopoulos (FI) $D$-term. \(^1\) The above equations may seem to be over determined. However, the $F$-flatness conditions are not independent, because of the gauge invariance expressed by

\[ \frac{\partial W}{\partial Z_i} z_i Z_i = 0. \quad (2.2) \]

Therefore, generically a SUSY vacuum with $\langle Z_i \rangle \sim \Lambda$ exists (Vacuum A), because the coefficients of the terms of $F$-flatness conditions are generically of order 1. However, if $n_+ \leq n_-$, we can choose another vacuum (Vacuum B) with $\langle Z_i^+ \rangle = 0$, which automatically satisfy the $F$-flatness conditions $F_{Z_i^-} = 0$, since they contain at least one field with positive charge. Then, the $\langle Z_i^- \rangle$ are determined by the $F$-flatness conditions $F_{Z_i^+} = 0$, with the constraint (2.2), and the $D$-flatness condition $D_A = 0$. Note that if $\xi < 1$, the VEVs of $Z_i^-$ are less than the cutoff scale. This can lead to the FN mechanism.

At this stage, among the fields $Z_i^-$, we define the FN field $\Theta$ as the field whose VEV mainly compensates for the FI parameter $\xi$. If we fix the normalization of $U(1)_A$ charge so that $\theta = -1$, then from $D_A = 0$, the VEV of the FN field $\Theta$ is determined as

\[ \langle \Theta \rangle \equiv \lambda \sim \xi, \quad (2.3) \]

which breaks $U(1)_A$ gauge symmetry. The other VEVs are determined by the $F$-flatness conditions with respect to $Z_i^+$ as $\langle Z_i^- \rangle \sim \lambda^{-z_i^-}$, which is shown below.

\(^1\)In weakly coupled Heterotic string theory, this FI $D$-term can be induced by a stringy loop correction, according to which $\xi^2 = \frac{g_{str}^2 Q_A}{192\pi^2}$. 

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Since $\langle Z_i^+ \rangle = 0$, it is sufficient to examine the terms linear in $Z_i^+$ in the superpotential in order to determine $\langle Z_i^- \rangle$. Therefore, in general, the superpotential to determine the VEVs can be written

$$W = \sum_i^n W_{Z_i^+},$$  \hspace{1cm} (2.4) $$

$$W_{Z_i^+} = \lambda^{z_i^+} Z_i^+ \left(1 + \sum_j^n \lambda^{z_j^-} Z_j^- + \sum_{j,k}^{n-} \lambda^{z_j^- + z_k^-} Z_j^- Z_k^- + \cdots \right)$$

$$= \tilde{Z}_i^+ \left(1 + \sum_j^n \tilde{Z}_j^- + \sum_{j,k}^{n-} \tilde{Z}_j^- \tilde{Z}_k^- + \cdots \right),$$  \hspace{1cm} (2.5) $$

where $\tilde{Z}_i \equiv \lambda^{z_i} Z_i$.\(^2\) The F-flatness conditions of the $Z_i^+$ fields require

$$\lambda^{z_i^+} \left(1 + \sum_j \tilde{Z}_j^- + \cdots \right) = 0,$$  \hspace{1cm} (2.6) $$

which generally lead to solutions $\langle \tilde{Z}_j^- \rangle \sim O(1)$ if these F-flatness conditions determine the VEVs. Thus the F-flatness condition requires

$$\langle Z_j^- \rangle \sim O(\lambda^{-z_j^-}).$$  \hspace{1cm} (2.7) $$

Note that if there is another field $Z_i^-$ which has larger charge than the FN field $\Theta$, then the VEV of $Z_i^-$ becomes larger than $\xi$. This is inconsistent with $D_A = 0$. Therefore it is natural that the field with the largest negative charge becomes the FN field. Note that if $n_+ = n_-$, generically all the VEVs of $Z_i^-$ are fixed, and therefore there is no flat direction in the potential. Hence in this case there is no massless field. Contrasting, if $n_+ < n_-$, generally the $n_+$ equations for the F-flatness and D-flatness conditions do not determine all the VEVs of $n_-$ fields $Z_i^-$. Therefore, there are flat directions in the potential, producing some massless fields. Thus, if we want to realize the case with no massless mode in the Higgs sector, $n_+ = n_-$ must be imposed in the Higgs sector.\(^3\)

\(^2\)The introduction of discrete symmetries or rational number charges disallows some of the interactions in Eq. (2.5). However, the results we obtain do not change unless the situation discussed in the next footnote is realized.

\(^3\)These rough arguments regarding the order of the VEVs and of the number counting are based on the assumption that the Higgs sector has no other structure by which the difference between the number of non-trivial F-flatness conditions and the degree of freedom of non-vanishing VEVs is changed. Such a structure can be realized by imposing a certain symmetry, for example, $Z_2$ parity (or R parity), or by introducing half integer (or rational number) charges. When the number of the negatively charged odd $Z_2$-parity fields is different from that of positively charged odd $Z_2$-parity fields, choosing vanishing VEVs of odd $Z_2$-parity fields changes the difference.
If Vacuum A is selected, the anomalous $U(1)_A$ gauge symmetry is broken at the cutoff scale, and the FN mechanism does not act. Therefore, we cannot surmise the existence of the $U(1)_A$ gauge symmetry from the low energy physics. However, if the Vacuum B is selected, the FN mechanism acts effectively, and the signature of the $U(1)_A$ gauge symmetry can be observed in the low energy physics. Therefore, it is reasonable to assume that Vacuum B is selected in our scenario, in which the $U(1)_A$ gauge symmetry plays an important role in the FN mechanism. The VEVs of the fields $Z_i^+$ vanish. This guarantees that the SUSY zero mechanism acts effectively.

To this point, we have examined the VEVs of only singlet fields. We now consider the case in which there are non-trivial representations of the gauge group. The same arguments can be applied if we use a set of independent gauge invariant operators instead of the gauge singlet fields $Z_i$. The gauge invariant operator $O$ with negative charge $\phi$ has non-vanishing VEV $\langle O \rangle \sim \lambda^{-\phi}$ if the $F$-flatness conditions determine the VEV. For example, let us introduce the fundamental representation $C(27)$ and $\bar{C}(\bar{27})$ of $E_6$. The VEV of the gauge singlet operator $CC$ is estimated as $\langle CC \rangle \sim \lambda^{-(c+\bar{c})}$. The essential difference appears in the $D$-flatness condition of $E_6$ gauge theory, which requires

$$|\langle C \rangle| = |\langle \bar{C} \rangle| \sim \lambda^{-(c+\bar{c})/2}. \tag{2.8}$$

Note that these VEVs are also determined by the anomalous $U(1)_A$ charges, but they are different from the naive expectation $\langle C \rangle \sim \lambda^{-c}$. This is because the $D$-flatness condition strongly constrains the VEVs of non-singlet fields. One more important $D$-flatness condition is

$$D_A = g_A \left( \xi^2 + \sum_i \phi_i |\Phi_i|^2 \right) = 0. \tag{2.9}$$

The argument for the singlet fields cannot be applied directly to the case of the non-trivial representation fields $\Phi_i$. For example, if we adopt the fields $\Phi(27)$ and $\bar{\Phi}(\bar{27})$ of $E_6$ with the charges $\phi + \bar{\phi} = -1$, then the above $D$-flatness condition of the anomalous $U(1)_A$ gauge symmetry requires $\xi^2 + \phi |\Phi|^2 + \bar{\phi} |\bar{\Phi}|^2 = 0$. The $D$-flatness condition of the $E_6$ gauge group leads to $|\Phi| = |\bar{\Phi}|$, and therefore we obtain $\xi^2 + (\phi + \bar{\phi}) |\Phi|^2 = 0$, which implies $|\langle \Phi \rangle| = |\langle \bar{\Phi} \rangle| = \xi$. In this case, since $\Phi \bar{\Phi}$ plays the same role as $\Theta$, the unit of the hierarchy becomes $\langle \Phi \bar{\Phi} \rangle = \lambda \sim \xi^2$, which is different from the previous relation (2.3). This implies that even if $\xi$ has a milder hierarchy, the unit of the hierarchy becomes stronger.

Of course we can determine the VEVs of the singlet operators from the same superpotential as (2.4), replacing the singlet fields $Z_i$ by a set of independent

\footnote{Note that if the total charge of an operator is negative, the $U(1)_A$ invariance and analytic property of the superpotential forbids the existence of the operator in the superpotential, since the field $\Theta$ with negative charge cannot compensate for the negative total charge of the operator (SUSY zero mechanism).}
gauge singlet operators. This, however, is not easy. However, the situation can be simplified if all the fields $\Phi_i^+$ (including non-singlets) with positive charges have vanishing VEVs. We can obtain the superpotential to determine the VEVs as

$$W = \sum_{i=1}^{n_+} W_{\Phi_i^+},$$

(2.10)

where $W_X$ denotes the terms linear in the $X$ field. Note that generally fields with positive charges can have non-vanishing VEVs, even if all the gauge singlet operators with positive charges have vanishing VEVs. For example, if we set $\phi = -3$ and $\bar{\phi} = 2$, then the gauge singlet operator $\Phi \Phi$ can have a non-vanishing VEV. This means that $\Phi$ with positive charge $\phi = 2$ has a non-vanishing VEV. In such cases, it is not guaranteed that the $F$-flatness conditions of fields with negative charges are automatically satisfied. Therefore we have to take account of the superpotential $W(\bar{\Phi})$, which includes positively charged fields $\Phi$ with non-vanishing VEVs. We consider such examples below.

In summary, we have the following:

1. Gauge singlet operators with positive total charge have vanishing VEVs, in order for the FN mechanism to act effectively. This guarantees that the SUSY zero mechanism acts effectively.

2. The singlet operator $\Theta$ with the largest negative charge becomes the FN field. When the singlet operator is just a singlet field, the VEV is given by $\langle \Theta \rangle \sim \xi$, which is determined from $D_A = 0$. When the singlet operator is a composite operator, $\Theta \sim \Phi \Phi$, the VEV is given by $\langle \Theta \rangle = \xi^2$.

3. The $F$-flatness conditions of singlet operators with positive charges determine the VEVs of singlet operators $O$ with negative charges $o$ as $\langle O \rangle \sim \lambda^{-o}$, while the $F$-flatness conditions of the singlet operators with negative charges are automatically satisfied. When the operator is a composite operator, $O \sim \bar{C} C$, the $D$-flatness condition requires $|\langle C \rangle| = \langle \bar{C} \rangle \sim \lambda^{-(c+\bar{c})/2}$.

4. If the number of the independent singlet operators (moduli) with positive charges equals that of the fields with negative charges, generically no massless fields appear.

5. The general superpotential to determine the VEVs is expressed as $W = \sum_i W_{O_i^+}$, where $W_{O_i^+}$ is linear in the independent singlet operator $O_i^+$ with positive charges. When all the fields $\Phi^+$ (including non-singlets) with positive charges have vanishing VEVs, the superpotential can be written $W = \sum_{i=1}^{n_+} W_{\Phi_i^+}$. If some of the positively charged fields $\Phi$ have non-vanishing VEVs, the superpotential $W_{NV}$ must be added, which includes only the fields with non-vanishing VEVs.
Table I. Typical values of anomalous $U(1)_A$ charges.

|  | non-vanishing VEV | vanishing VEV |
|---|------------------|---------------|
| 45 | $A(a = -1, -)$ | $A'(a' = 3, -)$ |
| 16 | $C(c = -3, +)$ | $C'(c' = 2, -)$ |
| T6 | $\bar{C}(\bar{c} = 0, +)$ | $\bar{C}'(\bar{c}' = 5, -)$ |
| 10 | $H(h = -3, +)$ | $H'(h' = 4, -)$ |
| 1 | $\Theta(\theta = -1, +), \bar{Z}(z = -2, -), \bar{Z}(\bar{z} = -2, -)$ | $S(s = 3, +)$ |

3 Doublet-triplet splitting in $SO(10)$ GUT

Here we make a quick review of the $SO(10)$ unified scenario proposed by one of the present authors [1, 3].

3.1 Alignment and DT splitting

The content of the Higgs sector in $SO(10) \times U(1)_A$ is listed in Table I. Here the symbols $\pm$ denote the $Z_2$ parity. The adjoint Higgs field $A$, whose VEV $\langle A(45) \rangle_{B-L} = \tau_2 \times \text{diag}(v, v, v, 0, 0)$, breaks $SO(10)$ into $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$. This Dimopoulos-Wilczek form of the VEV plays an important role in solving the DT splitting problem. The spinor Higgs fields $C$ and $\bar{C}$ break $SU(2)_R \times U(1)_{B-L}$ into $U(1)_Y$ by developing $\langle C \rangle = \langle \bar{C} \rangle = \lambda^{-(c+\bar{c})/2}$. The Higgs field $H$ contains the usual $SU(2)_L$ doublet. The gauge singlet operators $A^2$, $\bar{C}C$ and $H^2$ must have negative total anomalous $U(1)_A$ charges to obtain non-vanishing VEVs, as discussed in the previous section. Then, in order to give masses to all the Higgs fields, we have introduced the same number of fields with positive charges, $^5$ which we denote $A'$, $C''$, $\bar{C}'$ and $H'$. This is, in a sense, a minimal set of the Higgs content.

Following the general argument of the previous section, the superpotential required by determination of the VEVs can be written

$$W = W_{H'} + W_{A'} + W_S + W_{C'} + W_{\bar{C}'} + W_{NV}. \quad (3.1)$$

Here $W_X$ denotes the terms linear in the positive charged field $X$, which has vanishing VEV. Note, however, that terms including two fields with vanishing VEVs like $\lambda^{2h'} H' H'$ give contributions to the mass terms but not to the VEVs. All the terms in $W_{NV}$ contain only the fields with non-vanishing VEVs. In the typical charge assignment, it is easily checked that they do not play a significant role in our argument, since they do not include the products of only the neutral

$^5$Strictly speaking, since some of the Higgs fields are absorbed by the Higgs mechanism, in principle, a smaller number of positive fields can give superheavy masses to all the Higgs fields. Here we do not examine the possibilities.
components under the standard gauge group. In the following argument, for simplicity, we ignore the terms that do not include the products of only neutral components under the standard gauge group, like $16^4$, $\overline{16}^4$, $10 \cdot 16^2$, $10 \cdot \overline{16}^2$ and $1 \cdot 10^2$, even if these terms are allowed by the symmetry.$^6$

We now discuss the determination of the VEVs. If $-3a \leq a' < -5a$, the superpotential $W_{A'}$ is in general written

$$W_{A'} = \lambda^{a' + a} \alpha A' A + \lambda^{a' + 3a} (\beta (A' A)_1 (A^2)_1 + \gamma (A' A)_{54} (A^2)_{54}),$$

(3.2)

where the suffixes 1 and 54 indicate the representation of the composite operators under the $SO(10)$ gauge symmetry, and $\alpha$, $\beta$ and $\gamma$ are parameters of order 1. Here we assume $a + a' + c + \bar{c} < 0$ to forbid the term $\bar{C} A' A C$, which destabilizes the DW form of the VEV $\langle A \rangle$. The $D$-flatness condition requires the VEV $\langle A \rangle = \tau_2 \times \text{diag}(x_1, x_2, x_3, x_4, x_5)$, and the $F$-flatness conditions of the $A'$ field requires $x_i (\alpha \lambda^{-2a} + (2 \beta - \frac{\gamma}{\lambda}) (\sum x_i^2) + \gamma x_i^2) = 0$. This allows only two solutions, $x_i^2 = 0$ and $x_i^2 = -\frac{\alpha}{1 - \frac{\gamma}{\lambda} + 2N \beta} \lambda^{-2a}$. Here $N = 0 - 5$ is the number of $x_i \neq 0$ solutions. The DW form is obtained when $N = 3$. Note that the higher terms $A' A^{2L+1}$ $(L > 1)$ are forbidden by the SUSY zero mechanism. If they were allowed, the number of possible VEVs other than the DW form would become larger, and thus it would become less natural to obtain the DW form. This is a critical point of this mechanism, and the anomalous $U(1)_A$ gauge symmetry plays an essential role in forbidding the undesired terms. In this way, the scale of the VEV is automatically determined by the anomalous $U(1)_A$ charge of $A$, as noted in the previous section.

Next, we discuss the $F$-flatness condition of $S$, which determines the scale of the VEV $\langle CC \rangle$. $W_S$ is given by

$$W_S = \lambda^{s+c+\bar{c}} S \left( \langle CC \rangle + \lambda^{-(c+\bar{c})} + \sum_k \lambda^{-(c+\bar{c})+2ka} A^{2k} \right)$$

(3.3)

if $s \geq -(c+\bar{c})$. Then, the $F$-flatness condition of $S$ implies $\langle CC \rangle \sim \lambda^{-(c+\bar{c})}$, and the $D$-flatness condition requires $| \langle C \rangle | = | \langle \bar{C} \rangle | \sim \lambda^{-(c+\bar{c})/2}$. The scale of the VEV is again determined only by the charges of $C$ and $\bar{C}$. If we set $c + \bar{c} = -3$, then we obtain the VEVs of the fields $C$ and $\bar{C}$ as $\lambda^{3/2}$, which differ from the expected values $\lambda^{-c}$ and $\lambda^{-\bar{c}}$ in the case $c \neq \bar{c}$.

Next, we discuss the $F$-flatness of $C'$ and $\bar{C}'$, which causes the alignment of the VEVs $\langle C \rangle$ and $\langle \bar{C} \rangle$ and imparts masses on the PNG fields. This simple mechanism was proposed by Barr and Raby [8]. We can easily assign anomalous $U(1)_A$ charges that allow the following superpotential:

$$W_{C'} = \bar{C}' (\lambda^{d+c+a} A + \lambda^{d+c+z} Z) C',$$

(3.4)

$$W_{\bar{C}'} = C' (\lambda^{d+c+a} A + \lambda^{d+c+z} Z) C.$$  

(3.5)

$^6$It is easy to include these terms in our analysis. They can introduce some constraints on the vacua other than the standard vacuum, but not on the standard vacuum.
The $F$-flatness conditions $F_{C'} = F_{C''} = 0$ give $(\lambda^{a-z} A + Z)C = \bar{C}(\lambda^{a-z} A + \bar{Z}) = 0$. Recall that the VEV of $A$ is proportional to the $B-L$ generator $Q_{B-L}$ (precisely, $\langle A \rangle = \frac{3}{2}vQ_{B-L}$), and that $C$, 16, is decomposed into $(3, 2, 1)_{1/3}$, $(3, 1, 2)_{-1/3}$, $(1, 2, 1)_{-1}$ and $(1, 1, 2)_1$ under $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$. Since $\langle \bar{C}C \rangle \neq 0$, $Z$ is fixed such that $Z \sim -\frac{3}{2}\lambda vQ_{B-L}^0$, where $Q_{B-L}^0$ is the $B-L$ charge of the component of $C$ that has non-vanishing VEV. Once the VEV of $Z$ is determined, no other component fields can have non-vanishing VEVs, because they have different charges $Q_{B-L}$. If the $(1, 1, 2)_1$ field obtains a non-zero VEV (and therefore $\langle Z \rangle \sim -\frac{3}{2}\lambda v$), then the gauge group $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ is broken down to the standard gauge group. Once the direction of the VEV $\langle C \rangle$ is determined, the VEV $\langle \bar{C} \rangle$ must have the same direction, because of the $D$-flatness condition. Therefore, $\langle \bar{Z} \rangle \sim -\frac{3}{2}\lambda v$.

Finally the $F$-flatness condition of $H'$ is examined. $W_{H'}$ is written

$$W_{H'} = \lambda^{h+a+h'}H'AH.$$  \hspace{1cm} (3.6)

$F_{H'}$ leads to a vanishing VEV of the color triplet Higgs, $\langle H_T \rangle = 0$. All VEVs have now been fixed.

There are several terms that must be forbidden for the stability of the DW mechanism. For example, $H^2$, $HZH'$ and $H\bar{Z}H'$ induce a large mass of the doublet Higgs, and the term $\bar{C}A'AC$ would destabilize the DW form of $\langle A \rangle$. We can easily forbid these terms using the SUSY zero mechanism. For example, if we choose $h < 0$, then $H^2$ is forbidden, and if we choose $\bar{c} + c + a + a' < 0$, then $\bar{C}A'AC$ is forbidden. Once these dangerous terms are forbidden by the SUSY zero mechanism, higher-dimensional terms that also become dangerous (for example, $\bar{C}A'A^3C$ and $\bar{C}A'C\bar{C}AC$) are automatically forbidden. This is also an advantageous property of our scenario.

To end this subsection, we would like to explain how to determine the symmetry and the quantum numbers in the Higgs sector to realize DT splitting. It is essential that dangerous terms be forbidden by the SUSY zero mechanism, and the necessary terms must be allowed by the symmetry. The dangerous terms are

$$H^2, HH', HZH', \bar{C}A'C, \bar{C}A'AC, \bar{C}A'ZC, A'A^4, A'A^5.$$  \hspace{1cm} (3.7)

The terms required to realize DT splitting are

$$A'A, A'A^3, HAH', \bar{C}(A + Z)C, \bar{C}(A + Z)C', S\bar{C}C.$$  \hspace{1cm} (3.8)

Here we denote both $Z$ and $\bar{Z}$ as "$Z$". In order to forbid $HH'$ but not $HAH'$, we introduce $Z_2$ parity.

Of course, the above conditions are necessary but not sufficient. To determine whether a given assignment actually works well, we have to write down the mass matrices of Higgs sector. This is done in the next subsection.
### 3.2 Mass spectrum of the Higgs sector

Under $SO(10) \supset SU(5) \supset SU(3)_C \times SU(2)_L \times U(1)_Y$, the spinor $16$, vector $10$ and the adjoint $45$ are classified in terms of the fields $Q(3,2)\tilde{u}, U(3,1)\tilde{d}, D^c(3,1)\tilde{d}, L(1,2)\tilde{d}, E^c(1,1)_1, N^c(1,1)_0$, $X(3,2)\tilde{d}$ and their conjugate fields, and $G(8,1)_0$ and $W(1,3)_0$ as

\[
\begin{align*}
16 & = \begin{bmatrix} [Q + U^c + E^c] + [D^c + L] + N^c \end{bmatrix}_{10}, \\
10 & = \begin{bmatrix} [D^c + L] + [\bar{D}^c + \bar{L}] \end{bmatrix}_5, \\
45 & = \begin{bmatrix} [G + W + X + \bar{X} + N^c] + [\bar{Q} + \bar{U}^c + \bar{E}^c] + N^c \end{bmatrix}_{24}. \tag{3.9}
\end{align*}
\]

In the following, we study how mass matrices of the above fields are determined by considering an example of the typical charge assignment given in Table I. For the mass terms, we must take account of not only the terms in the previous section but also the terms that contain two fields with vanishing VEVs.

First, we examine the mass spectrum of $5$ and $\bar{5}$ of $SU(5)$. Considering the additional terms $\lambda^{2k} H^c H^c$, $\lambda^{e+\bar{e}} \bar{C}^c C^c$, $\lambda^2 C^c C^c$, $\lambda^{e+e+e} C^c C^c H$, $\lambda^{e+e+e} C^c C^c H$ and $\lambda^{2e+\bar{c}} \bar{C}^2 H^c$, the mass matrices $M_I$ ($I = D^c(H_T), L(H_D)$), whose elements correspond to the mass of the $I$ component of $\bar{5}(10)$ or $5(16)$ and the $\bar{I}$ component of $5(10)$ or $\bar{5}(16)$, are given as

\[
M_I = \begin{pmatrix}
\bar{I} | I \\
10_H(-3) & 16_C(-3) & 10_H(4) & 16_C(2) \\
16_C(0) & 0 & 0 & \lambda^{h+\bar{h}+a} \langle A \rangle \\
10_H'(4) & \lambda^{h+\bar{h}+a} \langle A \rangle & 0 & \lambda^{k}\langle \bar{C} \rangle \\
16_C'(5) & \lambda^{h+\bar{h}+\bar{c}} \langle \bar{C} \rangle & \lambda^{e+e} & \lambda^{h+\bar{e}+\bar{c}} \langle C \rangle \\
\end{pmatrix}, \tag{3.10}
\]

where the vanishing components result from the SUSY zero mechanism, and we indicate typical charges in parentheses.

It is worthwhile examining the general structure of the mass matrices. The first two columns and rows correspond to the fields with non-vanishing VEVs that have smaller charges, and the last two columns and rows correspond to the fields with vanishing VEVs that have larger charges. Therefore, it is useful to divide the matrices into four $2 \times 2$ matrices as

\[
M_I = \begin{pmatrix} 0 & A_I \\ B_I & C_I \end{pmatrix}. \tag{3.11}
\]

It is easily seen that the ranks of $A_L$ and $B_L$ are reduced to 1 when the VEV $\langle A \rangle$ vanishes. This implies that the rank of $M_L$ is reduced, and actually it becomes 3. However, the ranks of $A_{D^c}$ and $B_{D^c}$ remain 2, because the field $A$ becomes

\[
\begin{align*}
16 &= \begin{bmatrix} [Q + U^c + E^c] + [D^c + L] + N^c \end{bmatrix}_{10}, \\
10 &= \begin{bmatrix} [D^c + L] + [\bar{D}^c + \bar{L}] \end{bmatrix}_5, \\
45 &= \begin{bmatrix} [G + W + X + \bar{X} + N^c] + [\bar{Q} + \bar{U}^c + \bar{E}^c] + N^c \end{bmatrix}_{24}. \tag{3.9}
\end{align*}
\]
non-zero on $D^c$. Therefore DT splitting is realized. The mass spectrum of $L$ is easily obtained as $(0, \lambda^{2h}, \lambda^{\ell+c}, \lambda^2)$. The massless modes of the doublet Higgs are approximately given by
\[
5_H, \bar{5}_H + \lambda^{h-c+\frac{1}{2}(e-c)}\bar{5}_C. \quad (3.12)
\]

The elements of the matrices $A_I$ and $B_I$ become generally larger than the elements of the matrices $C_I$ because the total anomalous $U(1)_A$ charge of the corresponding pair of fields in $A_I$ and $B_I$ becomes smaller than that in $C_I$. Therefore, the mass spectrum of $D^c$ is essentially determined by the matrices $A_I$ and $B_I$ as $(\lambda^{h+h'}, \lambda^{h'+h'}, \lambda^{\ell+c}, \lambda^2)$. It is obvious that to realize proton decay, we have to pick up an element of $C_I$. Since such an element is generally smaller than the mass scale of $D^c$, proton decay is suppressed. The effective colored Higgs mass is estimated as $(\lambda^{h+h'})^2/\lambda^{2h'} = \lambda^{2h}$, which is larger than the cutoff scale, because $h < 0$.

Next, we examine the mass matrices for the representations $I = Q, U^c$ and $E^c$, which are contained in the 10 of $SU(5)$, where the additional terms $\lambda^{2a'} A' A'$, $\lambda^{\ell+\ell'} C C'$, $\lambda^{a+a'} C A' C'$ and $\lambda^{\ell+\ell'+a'} C' A' C$ must be taken into account. The mass matrices are written
\[
M_I = \begin{pmatrix}
\bar{I} I & 45_A(-1) & 16_C(-3) & 45_A'(3) & 16_C'(2) \\
45_A(-1) & 0 & 0 & \lambda^{a+a'} C & \lambda^\ell C \\
10_C(0) & 0 & 0 & 0 & \lambda^{\ell+\ell'} F \\
45_A'(3) & \lambda^{a+a'} C & 0 & \lambda^\ell C & \lambda^{\ell+\ell'+a'} C' \\
10_C'(5) & \lambda^{\ell+\ell'+a'} C & \lambda^\ell C & \lambda^{\ell+\ell'+a'} C' & \lambda^\ell C
\end{pmatrix}, \quad (3.13)
\]

where $\alpha_Q = \alpha_{U^c} = 0$ and $\beta_{E^c} = 0$, because there are NG modes in symmetry breaking processes $SO(10) \rightarrow SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{10}$ and $SU(2)_R \times U(1)_{B-L} \rightarrow U(1)_Y$, respectively. Defining $2 \times 2$ matrices as in the $I = L, D^c$ case, it is obvious that the ranks of $A_I$ and $B_I$ are reduced. Thus for each $I$, the 4 × 4 matrices $M_I$ have one vanishing eigenvalue, which corresponds to the NG mode absorbed by the Higgs mechanism. The mass spectrum of the remaining three modes is $(\lambda^{\ell+\ell'}, \lambda^\ell C, \lambda^{2a'})$ for the color-triplet modes $Q$ and $U^c$, and $(\lambda^{a+a'}, \lambda^a C, \lambda^\ell C)$ or $(\lambda^{\ell+\ell'+a'} C')$, $(\lambda^{\ell+\ell'+a'} C'$, $\lambda^{\ell+\ell'+a'} C'$) for the color-singlet modes $E^c$.

The adjoint fields $A$ and $A'$ contain two $G$, two $W$ and two pairs of $X$ and $\bar{X}$, whose mass matrices $M_I (I = G, W, X)$ are given by
\[
M_I = \begin{pmatrix}
\bar{I} I & 45_A(-1) & 45_A'(3) \\
45_A(-1) & 0 & \alpha_I \lambda^{a+a'} \\
45_A'(3) & \alpha_I \lambda^{a+a'} & \lambda^{2a'}
\end{pmatrix}. \quad (3.14)
\]

Two $G$ and two $W$ acquire masses $\lambda^{a+a'}$. Since $\alpha_X = 0$, one pair of $X$ is massless, and this is absorbed by the Higgs mechanism. The other pair has a rather light mass of $\lambda^{2a'}$. 

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3.3 Gauge unification and proton decay

In the minimal SUSY $SU(5)$ GUT, proton stability is not compatible with the success of the gauge coupling unification [15]. Proton stability requires the colored Higgs mass to be larger than $10^{18}$ GeV, which destroys the coupling unification, because it has no other tuning parameter. Of course, if we introduce other superheavy particles with masses smaller than the GUT scale, we may recover the gauge coupling unification by tuning their masses. However, unless we have some mechanism that controls the scale of the masses, generally some fine-tuning is required. In the $SO(10)$ GUT with the DW mechanism, proton stability can be realized in the mass structure

$$M_I = \begin{pmatrix} 0 & \langle A \rangle \\ \langle A \rangle & m \end{pmatrix} (I = L, D_c),$$

(3.15)

if $\langle A \rangle^2/m_{Dc} > 10^{18}$ GeV. However, in order to realize the condition $\langle A \rangle^2/m_{Dc} > 10^{18}$ GeV, the mass scale of the additional doublet Higgs $m$ becomes smaller than the GUT scale $\langle A \rangle$, which generally destroys the success of the gauge coupling unification. Of course, it may be possible to realize gauge coupling unification by tuning the other scale, $\langle C \rangle$, or the mass scales of superheavy particles. Unless we have no mechanism that controls these scales, however, such a situation cannot explain why the gauge couplings meet at a scale $\Lambda_G \sim 2 \times 10^{16}$ GeV in the minimal SUSY standard model (MSSM).

In our scenario, once we determine the anomalous $U(1)_A$ charges, the mass spectra of all superheavy particles and other symmetry breaking scales are determined, and hence we can examine whether or not the running couplings from the low energy scale meet at the unification scale. When the initial values of the gauge coupling constants are replaced by the usual GUT scale, $\Lambda_G \sim 2 \times 10^{16}$ GeV, and the unified gauge coupling, the condition for gauge coupling unification can be converted into a relation between the charges and the ratio of the cutoff scale $\Lambda$ to the usual GUT scale $\Lambda_G \sim 2 \times 10^{16}$ GeV:

$$\frac{\Lambda_G}{\Lambda} \sim \lambda^{-h} \sim \lambda^h.$$  

(3.16)

This leads to

$$\Lambda \sim \Lambda_G,$$

(3.17)

$$h \sim 0.$$  

(3.18)

Here we have used the renormalization group up to the one loop approximation. It is non-trivial that in this relation, all the charges except that of the Higgs doublet

$^7$In Ref. [15], the lower bound of the colored Higgs mass was obtained as $6.5 \times 10^{16}$ GeV. To derive this value, the hadron matrix element parameter $\alpha = 0.003$ and $\tan \beta = 2.5$ were used. If we use the result of a recent lattice calculation, $\alpha \sim 0.015$ [16], and a more reasonable (larger) value of $\tan \beta$, the lower bound easily become larger.
are cancelled out. If we simply had taken $\Lambda = \Lambda_G$ and $h = 0$, the model would exhibit proton decay via dimension 5 operators, because the effective colored Higgs mass becomes $\lambda^2 h \Lambda = \Lambda_G \ll 10^{18}$ GeV. However, we have to use a negative $h$ to forbid the Higgs mass term $H^2$. Therefore, we would like to know how large a negative charge $h$ can be adopted in our scenario. To obtain realistic quark and lepton mass matrices including bi-large neutrino mixing, the maximal value of $h$ is $-3$. In this case, the effective colored Higgs mass becomes $\lambda^2 h \Lambda = \lambda^{-6} \Lambda_G \sim 10^{22}$ GeV, which is much larger than the present experimental limit. Note that even for such a small value of $h$, coupling unification can be realized, using the ambiguities of order 1 coefficients. Since the unified scale becomes $\lambda^{-3} \Lambda$, just below the scale $\Lambda_G$, we believe that proton decay via dimension 6 operators will be observed in the near future. We will return to this point in the next section.

Once we have fixed the anomalous $U(1)_A$ charges, the fact that the gauge couplings meet at the usual GUT scale in the MSSM is non-trivially related to the result that the gauge couplings of the GUT with anomalous $U(1)_A$ gauge symmetry almost meet at the GUT scale $\lambda^{-3} \Lambda$ in our scenario. Therefore, this GUT scenario can explain why the gauge couplings meet at a scale in MSSM with an accuracy up to the one loop approximation.\(^8\)

4 \textit{E}_6 \textit{ unification of the Higgs sector}

In this section, we extend the DT splitting mechanism, discussed in the previous section, to $E_6$ unification. Here we propose the complete Higgs sector with the $E_6$ GUT gauge group.

In order to break the $E_6$ gauge group into the standard gauge group, we introduce the following Higgs content:

1. Higgs fields that break $E_6$ into $SO(10)$: $\Phi(27)$ and $\Phi(\overline{27})$ ($|\langle \Phi(1, 1) \rangle| = |\langle \overline{\Phi}(1, 1) \rangle|$).

2. An adjoint Higgs field that breaks $SO(10)$ into $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$: $A(78)$ ($\langle 45_A \rangle = \tau_2 \times \text{diag}(v, v, v, 0, 0)$).

3. Higgs fields that break $SU(2)_R \times U(1)_{B-L}$ into $U(1)_Y$: $C(27)$ and $\overline{C}(\overline{27})$ ($|\langle C(16, 1) \rangle| = |\langle \overline{C}(\overline{16}, 1) \rangle|$).

Of course, the anomalous $U(1)_A$ charges of the gauge singlet operators, $\Phi \Phi$, $\overline{C}C$ and $A^2$, must be negative.

Naively thinking, it appears that we have to introduce at least the same number of superfields with positive charges in order to make them massive. In\(^8\)Actually, if it were the case that the gauge couplings meet at the other scale $\Lambda_O$ in MSSM, then the cutoff scale would be the scale $\Lambda_O$ in our scenario; that is, the GUT scale would be $\lambda^{-a} \Lambda_O$.\)
Table II. Typical values of anomalous $U(1)_A$ charges.

|       | non-vanishing VEV                                         | vanishing VEV            |
|-------|-----------------------------------------------------------|--------------------------|
| 78    | $A(a = -1, -)$                                            | $A'(a' = 4, -)$          |
| 27    | $\Phi(\phi = -3, +) C(c = -6, +)$                        | $C'(c' = 7, -)$          |
| 27    | $\bar{\Phi}(\bar{\phi} = 2, +) \bar{C}($$\bar{c} = -2, +)$ | $C'(c' = 8, -)$          |
| 1     | $Z_2(z_2 = -2, -), Z_5(z_5 = -5, -), \bar{Z}_5(\bar{z}_5 = -5, -)$ |                           |

fact, however, we find this is not the case. This is because some of the Higgs fields with non-vanishing VEVs are absorbed by the Higgs mechanism. Actually, when the $E_6$ gauge group is broken into $SO(10)$ by non-vanishing VEV $|\langle \Phi \rangle| = |\langle \bar{\Phi} \rangle|$, the fields $16_\Phi$ and $\bar{16}_\Phi$ are absorbed by the super-Higgs mechanism. Therefore, if two additional $10$s of $SO(10)$ in the Higgs content with non-vanishing VEVs can be massive, then we can save the superfields with positive charges. At first glance, such a mass term seems to be forbidden by the SUSY zero mechanism. Actually, if all fields with non-vanishing VEVs had negative anomalous $U(1)_A$ charges, their mass term would be forbidden. As discussed in §2, the non-positiveness of the anomalous $U(1)_A$ charges is required only for gauge singlet operators with non-vanishing VEVs, so even fields with positive charges can have non-vanishing VEVs if the total charge of the gauge singlet operators with non-vanishing VEVs is negative. For example, we can set $\phi = -3$ and $\bar{\phi} = 2$, because the total charge of the gauge singlet operator $\bar{\Phi}\Phi$ is negative. Since $\bar{\Phi}$ has positive charge, the term $\bar{\Phi}^3$ is allowed, and it induces a mass of $10_\Phi$ through the non-vanishing VEV $\langle \bar{\Phi} \rangle$. If the term $\Phi^2\bar{C}$ is allowed, masses of the two $10$s, $10_\Phi$ and $10_{\bar{C}}$, are induced, so we can save the superfields with positive charges.

The minimal content of the Higgs sector with $E_6 \times U(1)_A$ gauge symmetry is given in Table II, where the symbols $\pm$ denote the $Z_2$ parity quantum numbers. Here the Higgs field $H$ of the $SO(10)$ model is contained in $\Phi$. This $E_6$ Higgs sector has the same number of superfields with non-trivial representation as the $SO(10)$ Higgs sector, in spite of the fact that the larger group $E_6$ requires additional Higgs fields to break $E_6$ into the $SO(10)$ gauge group. It is interesting that the DT splitting is naturally realized in this minimal Higgs content in a sense.

4.1 DT splitting and alignment

Generally in $E_6$ GUT, the interactions in the superpotential of $27$ and $\bar{27}$ are written in terms of the units $27^3$, $27\bar{27}$ and $\bar{27}^3$. Note that terms like $27^3$ or $\bar{27}^3$ do not contain the product of singlet components of the standard gauge group. Therefore these terms can be ignored when considering the standard vacuum. Of

\footnote{Strictly speaking, a linear combination of $\Phi$, $C$ and $A$ and of $\bar{\Phi}$, $\bar{C}$ and $A$ become massive through the super-Higgs mechanism. The main modes are $16_\Phi$ and $\bar{16}_\Phi$, respectively.}
course, these terms can constrain vacua other than the standard vacuum. This point is discussed below.

The important terms in the superpotential to determine the VEVs are

\[ W = W_A' + W_{C'} + W_{\bar{C}'} + W(\Phi). \] (4.1)

Since we have a positively charged field \( \bar{\Phi} \) which has non-vanishing VEV, we have to take account of the terms \( W(\bar{\Phi}) \), which include the field \( \bar{\Phi} \) but not the fields with vanishing VEVs. Since \( \bar{\Phi} \Phi \) and \( \bar{\Phi} C \) have negative total charges, the superpotential essentially has terms like \( \frac{1}{10} \). Therefore, the superpotential \( W(\bar{\Phi}) \) can constrain vacua other than the standard vacuum.

Let us examine the superpotential \( W(\bar{\Phi}) \) to elucidate the general vacuum structure in the \( E_6 \) model. As discussed in the previous section, the composite operator \( \bar{\Phi} \Phi \) with \( \phi + \bar{\phi} = -1 \) can play the same role as the FN field. In that case, \( \Phi \) and \( \bar{\Phi} \) have non-vanishing VEVs, and the \( E_6 \) \( D \)-flatness condition requires \( \langle \Phi \rangle = \langle \bar{\Phi} \rangle \), up to phases. The VEV of \( \bar{\Phi} \) (and therefore that of \( \Phi \) also, by the \( D \)-flatness condition) can be rotated by the \( E_6 \) gauge transformation into the following form:

\[
\langle \Phi \rangle = \begin{pmatrix}
\bar{u} \\
0 \\
\bar{u}_1 \\
\bar{u}_2 \\
0
\end{pmatrix}
\] = \{SO(10) singlet (real) \} SO(10) \{16 \}
\{the first component of SO(10) 10 (complex) \}
\{the second component of SO(10) 10 (real) \}
\{the third to tenth components of SO(10) 10 \}. (4.2)

For simplicity, we adopt a superpotential of the form

\[ W(\bar{\Phi}) = \bar{\Phi}^3 + \bar{\Phi}^2 \bar{C}. \] (4.3)

Then, the \( F \)-flatness conditions of \( 10_\bar{C} \) and \( 1_\bar{C} \) lead to \( 1_\bar{\Phi} 10_\bar{\Phi} = 0 \) and \( 10_\bar{\Phi}^2 = 0 \). Thus we are allowed to have either the vacuum \( \bar{u} \neq 0, \bar{u}_1 = \bar{u}_2 = 0 \) or the vacuum \( \bar{u} = 0, \bar{u}_1 = i \bar{u}_2 \neq 0 \). This implies that the non-vanishing of the VEV \( \langle 1_\bar{\Phi} \rangle \) requires the vanishing of the VEV \( \langle 10_\bar{\Phi} \rangle \). Therefore, in the first vacuum, the \( E_6 \) gauge group is broken into the \( SO(10) \) gauge group. Moreover, in this vacuum, \( 10_\bar{C} \) has vanishing VEV, because of the \( F \)-flatness conditions of \( 10_\bar{\Phi} \). Interestingly enough, a vacuum alignment occurs naturally. In the following, for simplicity, we often write \( \lambda^n \) in place of the operators \( (\bar{\Phi} \Phi)^n \), though these operators are not always singlets.

The superpotential \( W_{A'} \) is in general written as

\[
W_{A'} = \lambda^{a'+a} A' A + \lambda^{a'+3a} A' A^3 + \lambda^{a'+a+\bar{\phi}+\phi} \bar{\Phi} A' A \Phi \\
+ \lambda^{a'+3a+\bar{\phi}+\phi} \bar{\Phi} A' A^3 \Phi, \] (4.4)

under the condition \(-3a + \bar{\phi} + \phi \leq a' < -5a \). Here we assume \( c + \bar{c}, c + \bar{\phi}, \bar{c} + \phi < -(a' + a) \) to forbid the terms \( \bar{C} A' A C \) (which destabilizes the DW form of the
VEV of $A$, $\bar{C}A'\Phi$ and $\Phi A'AC$ (which may lead to undesired vacua in which $\langle \bar{C} \rangle = \langle C \rangle = 0$).

If $A$ and $(\Phi, \bar{\Phi})$ are separated in the superpotential, PNG fields appear. Since the terms $\Phi A'\Phi$ and $\Phi A'\bar{A}\Phi$ connect $A'$ and $A$ with $\Phi$ and $\bar{\Phi}$, the PNG obtain non-zero masses. Moreover, these terms realize the alignment between the VEVs $\langle \Phi \rangle$ and $\langle A \rangle$. Note that these terms are also important to induce the term $(\Phi A')_54 (\Phi A')_{54}$, which is not included in the term $A'A^3$, because of a cancellation (see Appendix A). In terms of $SO(10)$, which is not broken by the VEV $|\langle \Phi \rangle| = |\langle \bar{\Phi} \rangle|$, the effective superpotential is given by

$$W_{eff}^{A'} = 45_{A'}(1 + 1_A^2 + 45_A^2 + \bar{16}_A 16_A) 45_A + \bar{16}_{A'}(1 + 1_A^2 + 45_A^2 + \bar{16}_A 16_A) 16_A + 16_{A'}(1 + 1_A^2 + 45_A^2 + \bar{16}_A 16_A) \bar{16}_A + 1_A' 1_A(1 + 1_A^2 + 45_A^2 + \bar{16}_A 16_A).$$

The $F$-flatness conditions are written

$$\frac{\partial W}{\partial 45_{A'}} = (1 + 1_A^2 + 45_A^2 + \bar{16}_A 16_A) 45_A,$n \quad (4.6)
$$\frac{\partial W}{\partial \bar{16}_{A'}} = (1 + 1_A^2 + 45_A^2 + \bar{16}_A 16_A) 16_A,$n \quad (4.7)
$$\frac{\partial W}{\partial 16_{A'}} = \bar{16}_A(1 + 1_A^2 + 45_A^2 + \bar{16}_A 16_A),$n \quad (4.8)
$$\frac{\partial W}{\partial 1_{A'}} = 1_A(1 + 1_A^2 + 45_A^2 + \bar{16}_A 1_A).$$

These $F$-flatness conditions and the $D$-flatness conditions of $SO(10)$ determine the VEVs $\langle 16_A \rangle = \langle \bar{16}_A \rangle = 0$. We have two possibilities for the VEV of $1_A$, one vacuum with $\langle 1_A \rangle = 0$ and another vacuum with $\langle 1_A \rangle \neq 0$. In the latter vacuum, the DW mechanism in $E_6$ GUT does not act, because the non-vanishing VEV $\langle 1_A \rangle$ directly gives the bare mass to the doublet Higgs. Therefore, the former vacuum in which $\langle 1_A \rangle = 0$, is desirable to realize DT splitting. Note that if the term $\Phi A'\Phi$ is allowed, the vacuum $\langle 1_A \rangle = 0$ disappears. This destroys the realization of DT splitting. Here this term is forbidden by $Z_2$ parity. As in the $SO(10)$ case, we have several possibilities for the VEV of $45_A$, one of which is the DW-type of the VEV $\langle 45_{A'} \rangle_{B-L} = i\tau_2 \times \text{diag}(v, v, v, 0, 0)$, where $v \sim \lambda^{-a}$. These VEVs break the $SO(10)$ gauge group into $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$.

Next, we discuss the $F$-flatness of $C'$ and $\bar{C}'$, which not only determine the scale of the VEV $\langle \bar{C}C \rangle \sim \lambda^{-(c+e)}$ but also realize the alignment of the VEVs $\langle C \rangle$ and $\langle \bar{C} \rangle$. For simplicity, we assume that $\langle 1_C \rangle = \langle 1_C \rangle = 0$, though there

\footnote{We thank T. Kugo for pointing out this cancellation.}
may be vacua in which these components have non-vanishing VEVs. Then, since \( \langle 10_C \rangle = \langle 10_C' \rangle = 0 \) by the above argument, only the components \( 16_C \) and \( \overline{16}_C \) can have non-vanishing VEVs. The superpotential to determine these VEVs can be written

\[
W_{C'} = \lambda^{\tilde{c}_+ c}_+ \bar{\Phi} (\lambda^{c+ \tilde{c}_+ a} \bar{C} A C + \lambda^{\tilde{z}_5} Z_5 + \lambda^{\tilde{z}_{5}} \bar{Z}_5 + \lambda^{\tilde{z}_2} Z_2 + \lambda^a A) C' \\
+ \lambda^{\tilde{c}_+ c}_+ \bar{C} (\lambda^{\tilde{z}_5} Z_5 + \lambda^{\tilde{z}_5} \bar{Z}_5 + \lambda^{\tilde{z}_2} Z_2 + \lambda^a A) C',
\]

(4.10)

\[
W_{C'} = \lambda^{\tilde{c}_+ c}_+ \bar{C} (\lambda^{\tilde{z}_5} Z_5 + \lambda^{\tilde{z}_5} \bar{Z}_5 + \lambda^{\tilde{z}_2} Z_2 + \lambda^a A) \Phi \\
+ \lambda^{\tilde{c}_+ c}_+ \bar{C} (\lambda^{\tilde{z}_2} Z_2 + \lambda^a A) C,
\]

(4.11)

where we omit the even \( Z_2 \) parity operators with non-vanishing VEVs, like \( A^{2n}, Z_2^2, Z_t A, \) etc., because the VEVs of these operators do not change the power of \( \lambda \). The \( F \)-flatness conditions of \( 1_C' \) and \( 1_C' \) lead to \( \Phi (\lambda^{\tilde{c}_+ c}_+ a \bar{C} A C + \lambda^{\tilde{z}_5} Z_5 + \lambda^{\tilde{z}_5} \bar{Z}_5 + \lambda^{\tilde{z}_2} Z_2 + \lambda^a A) = 0 \) and \( (\lambda^{\tilde{z}_5} Z_5 + \lambda^{\tilde{z}_5} \bar{Z}_5 + \lambda^{\tilde{z}_2} Z_2 + \lambda^a A) \Phi = 0 \), respectively. The vacua are (a) \( \langle \bar{C} C \rangle = 0 \) and (b) \( \langle \bar{C} C \rangle \neq 0 \). The desired vacuum (a) requires the additional \( F \)-flatness conditions of \( 16_C' \) and \( \overline{16}_C' \), which causes the alignment of the VEVs \( \langle A \rangle \) and \( \langle C \rangle \) (\( \langle \bar{C} \rangle \)), as in the \( SO(10) \) cases. Then, the above four \( F \)-flatness conditions with respect to \( 1_C', 1_C', 16_C' \) and \( \overline{16}_C' \) determine the scale of the four VEVs \( \langle \bar{C} C \rangle \sim \lambda^{-(\tilde{c}+\tilde{c})}, \langle Z_i \rangle \sim \lambda^{-z_i} (i = 3, 5) \) and \( \langle \bar{Z}_5 \rangle \sim \lambda^{-z_5} \). The VEVs \( |\langle C \rangle| = |\langle \bar{C} \rangle| \sim \lambda^{-(\tilde{c}+\tilde{c})} \) break \( SU(2)_R \times U(1)_{B-L} \) into \( U(1)_Y \).

Thus all the VEVs are determined by the anomalous \( U(1)_A \) charges.

### 4.2 Mass spectrum of the Higgs sector

Since all the VEVs are fixed, we can derive the mass spectrum of the Higgs sector.

Let the fields be decomposed in terms of the quantum numbers of \( SO(10) \times U(1)_{V'} \) as

\[
27 = 16_1 + 10_{-2} + 1_4, \quad (4.12)
\]

\[
78 = 45_0 + 16_{-3} + \overline{16}_3 + 1_0, \quad (4.13)
\]

which are further decomposed into \( SU(5) \) representations [see Eq. (3.9)].

In the following, we study how the mass matrices of the above fields are determined by anomalous \( U(1)_A \) charges. Note that for the mass terms, we must take account of not only the terms given in the previous subsection but also the terms that contain two fields with vanishing VEVs (see Appendix C).

Before going into detail, it is worthwhile examining the NG modes that are absorbed by the Higgs mechanism, because in some cases the vanishing eigenvalue in the mass matrices is not obvious. There appear the following NG modes

1. \( 16 + \overline{16} + 1 \) of \( SO(10) \) (namely, \( Q + U^c + E^c + h.c. + N^c \)) in the breaking \( E_6 \rightarrow SO(10) \).
2. \( Q + U^c + X + h.c. \) in the breaking \( SO(10) \rightarrow SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \).

3. \( E^c + h.c. + N^c \) in the breaking \( SU(2)_R \times U(1)_{B-L} \rightarrow U(1)_Y \).

First, we examine the mass matrix of 24 in \( SU(5) \). Considering the additional term \( A^2 \), we write the mass matrices \( M_I \), which correspond to the representations \( I = G, W, X \):

\[
M_I = \frac{24_A}{24_{A'}} \begin{pmatrix} 24_A & 24_{A'} \\ 0 & \frac{\alpha_I \lambda^{a+a}}{\lambda^{2a'}} \end{pmatrix},
\]

where \( \alpha_X = 0 \) and \( \alpha_I \neq 0 \) for \( I = G, W \). One pair of \( X \) is massless. This is absorbed by the Higgs mechanism. The mass spectra are \((0, \lambda^{2a'})\) for \( I = X \) and \((\lambda^{a+a}, \lambda^{a+a})\) for \( I = G, W \).

Next, we examine the mass matrices for the representation \( I = Q, U^c \) and \( E^c \), which are contained in 10 of \( SU(5) \). The mass matrices \( M_I \) are written

\[
\begin{array}{cccccccc}
\text{I} \setminus \bar{I} & \bar{16}_\Phi & \bar{15}_C & \bar{10}_A & 45_A & \bar{16}_C & \bar{10}_A' & 45_{A'} \\
16_\Phi & 0 & 0 & 0 & 0 & \lambda^{c+\phi} & \lambda^{a+\phi} & 0 \\
16_C & 0 & 0 & 0 & 0 & \beta_I \lambda^{c+c} & 0 & 0 \\
16_A & 0 & 0 & 0 & 0 & \lambda^{a+a+\Delta} & \lambda^{a+a} & 0 \\
45_A & 0 & 0 & 0 & 0 & \lambda^{a+a+\Delta} & 0 & \alpha_I \lambda^{a+a} \\
16_{C'} & \lambda^{c+\phi} & \beta_I \lambda^{c+c} & \lambda^{a+c+\Delta} & 0 & \lambda^{a+\phi} & \lambda^{a+\phi} & \lambda^{2a'} \\
16_{A'} & \lambda^{a+\phi} & \lambda^{a+c} & 0 & \lambda^{a+\phi} & \lambda^{2a'} & \lambda^{2a'} \\
45_{A'} & 0 & 0 & 0 & \alpha_I \lambda^{a+a} & \lambda^{a+\phi} & \lambda^{2a'} & \lambda^{2a'}
\end{array}
\]

where we have used the relations \( \lambda^{\phi} \langle \Phi \rangle \sim (\lambda^{\phi} \langle \bar{\Phi} \rangle)^{-1} \sim \lambda^{\Delta\phi} \) and \( \lambda^{C} \sim (\lambda^{C})^{-1} \sim \lambda^{\Delta C} \). Since one pair of \( \bar{16} \) and 16 (whose main modes are \( \bar{16}_\Phi \) and \( 16_\Phi \)) is absorbed by the Higgs mechanism in the process of breaking \( E_6 \) into \( SO(10) \), we simply omit \( 16_\Phi \) and \( \bar{16}_\Phi \) in deriving the mass spectrum. Then, the mass matrices can be written in the form of four 3 \times 3 matrices as

\[
\begin{pmatrix}
0 & A_I \\
B_I & C_I
\end{pmatrix},
\]

as in \( SO(10) \) case. It is obvious that the ranks of \( A_I \) and \( B_I \) reduce to two for \( I = Q, U^c, E^c \) because \((\alpha_I = 0, \beta_I \neq 0)\) for \( I = Q, U^c \) and \((\alpha_I \neq 0, \beta_I = 0)\) for \( I = E^c \), where the vanishing of the parameter values is due to the fact that the NG modes are absorbed by the Higgs mechanism in the breaking \( SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \) (for which the corresponding NG fields are \( Q + U^c + h.c. \)) and in the breaking \( SU(2)_R \times U(1)_{B-L} \) into \( U(1)_Y \) (for which the corresponding NG fields are \( E^c + h.c. \)). The mass spectra become \((0, 0, \lambda^{a+a}, \lambda^{a+a}, \lambda^{c+c}, \lambda^{2a'})\) for \( I = Q, U^c \) and \((0, 0, \lambda^{a+a}, \lambda^{a+a}, \lambda^{a+a}, \lambda^{a+a}, \lambda^{a+a}, \lambda^{c+c})\) for \( I = E^c \).
Finally, we examine the mass matrices of $\mathbf{5}$ and $\mathbf{\bar{5}}$ in $SU(5)$ and show how to realize the DT splitting. Considering the additional terms, we write the mass matrices $M_I$ for the representations $I = D^c(H_T), L(H_D)$ and their conjugates as

$$M_I = \begin{pmatrix} 0 & A_I & 0 \\ B_I & C_I & D_I \\ E_I & F_I & G_I \end{pmatrix},$$

(4.17)

$$A_I = \begin{pmatrix} \bar{16} \setminus I \\ 10_C' \\ 10_{\Phi} \end{pmatrix} \begin{pmatrix} 10_C' & 10_{\bar{C}'} & \mathbf{16}_{C'} & \mathbf{16}_{A'} \\ S_I \lambda^c + \phi + \Delta \phi & \lambda^c + \phi & 0 & 0 \\ 0 & \lambda^c + c & 0 & 0 \\ 0 & 0 & \lambda^c + a + \Delta c & \lambda^c + a \\ 16_C & 16_C' & 16_A' & \end{pmatrix},$$

(4.18)

$$B_I = \begin{pmatrix} \bar{16} \setminus I \\ 10_{C'} \\ 10_C \end{pmatrix} \begin{pmatrix} 10_{C'} & \mathbf{16}_{C} & \mathbf{16}_A \\ S_I A \lambda^c + \phi + \Delta \phi & 0 & 0 \lambda^c + a - \Delta \phi \\ \lambda^c + \phi & \lambda^c + c & \lambda^c + c - \Delta c & \lambda^c + a - \Delta \phi - \Delta c \\ 0 & 0 & \lambda^c + c & \lambda^a + c - \Delta \phi \\ 16_{C'} & 16_A' & \end{pmatrix},$$

(4.19)

$$C_I = \begin{pmatrix} \bar{16} \setminus I \\ 10_{C'} \\ 10_{C'} \end{pmatrix} \begin{pmatrix} 10_{C'} & 10_{C'} & \mathbf{16}_{C'} & \mathbf{16}_{A'} \\ \lambda^{2c + \Delta \phi} & \lambda^{2c + \phi} & \lambda^{2c + \phi - \Delta \phi} & \lambda^{2c + \phi - \rho - \Delta \phi} \\ \lambda^{2c + \phi} & \lambda^{2c + \phi - \rho - \Delta \phi} & \lambda^{2c + \phi - \rho - \Delta \phi} & \lambda^{2c + \phi - \rho - \Delta \phi} \\ \lambda^{2c + \phi} & \lambda^{2c + \phi - \rho - \Delta \phi} & \lambda^{2c + \phi - \rho - \Delta \phi} & \lambda^{2c + \phi - \rho - \Delta \phi} \\ 16_{C'} & 16_{A'} & \end{pmatrix},$$

(4.20)

$$D_I = \begin{pmatrix} \bar{16} \setminus I \\ 10_{C'} \\ 10_{C'} \end{pmatrix} \begin{pmatrix} 10_{C'} & 10_{C'} & \mathbf{16}_{C} & \mathbf{16}_A \\ \lambda^{c + \phi} & \lambda^{c + \phi} & \lambda^{c + \phi - \Delta \phi} & \lambda^{c + \phi - \rho - \Delta \phi} \\ \lambda^{c + \phi - \Delta \phi} & \lambda^{c + \phi - \Delta \phi} & \lambda^{c + \phi - \Delta \phi} & \lambda^{c + \phi - \rho - \Delta \phi} \\ \lambda^{c + \phi} & \lambda^{c + \phi} & \lambda^{c + \phi} & \lambda^{c + \phi} \\ 16_{C'} & 16_A' & \end{pmatrix},$$

(4.21)

$$E_I = \begin{pmatrix} \bar{16} \setminus I \\ 10_{\Phi} \\ 10_C \\ \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \lambda^{2c - \Delta \phi} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^{2c - \Delta \phi} \end{pmatrix},$$

(4.22)

$$F_I = \begin{pmatrix} \bar{16} \setminus I \\ 10_{\Phi} \\ 10_{C} \end{pmatrix} \begin{pmatrix} 10_{C} & 10_{C'} & \mathbf{16}_{C'} & \mathbf{16}_{A'} \\ \lambda^{c + \phi} & \lambda^{c + \phi - \Delta \phi} & \lambda^{c + \phi - \Delta \phi} \lambda^{c + \phi - \rho - \Delta \phi} \\ \lambda^{c + \phi} & \lambda^{c + \phi - \Delta \phi} & \lambda^{c + \phi - \Delta \phi} \lambda^{c + \phi - \rho - \Delta \phi} \\ 0 & \lambda^{c + \phi - \Delta \phi} & \lambda^{c + \phi - \Delta \phi} \lambda^{c + \phi - \rho - \Delta \phi} \\ 10_\Phi & 10_\Phi & \end{pmatrix},$$

(4.23)

$$G_I = \begin{pmatrix} \bar{16} \setminus I \\ 10_{\Phi} \\ 10_C \\ \end{pmatrix} \begin{pmatrix} 0 & \lambda^{2c - \Delta \phi} & \lambda^{2c - \Delta \phi} & \lambda^{2c - \Delta \phi} \\ 0 & \lambda^{2c - \Delta \phi} & \lambda^{2c - \Delta \phi} \lambda^{2c - \Delta \phi} \\ 0 & 0 & 0 & \lambda^{2c - \Delta \phi} \\ 0 & 0 & 0 & \lambda^{2c - \Delta \phi} \end{pmatrix},$$

(4.24)
where \( S_{D^c} \neq 0 \) and \( S_L = 0 \). It is obvious that the rank of \( A_L \) is three, which is smaller than the rank of \( A_{D^c} \). This implies that the rank of \( M_L \) is smaller than the rank of \( M_{D^c} \), and actually the rank of the matrix \( M_I \) is 10 for \( I = D^c \) and 9 for \( I = L \). One pair of massless fields, \( 16 \) and \( \overline{16} \) (whose main modes are \( 16_5 \) and \( \overline{16}_3 \)), is the NG mode, which is absorbed by the Higgs mechanism in the breaking of \( E_6 \) into \( SO(10) \). The other massless mode for \( I = L \) is the so-called doublet Higgs. The massless mode is given by

\[
H_u \sim \bar{L}(10_5) + \lambda^{\phi-e} \bar{L}(10_3),
\]

\[
H_d \sim L(10_5) + \lambda^{\phi-e} L(10_3).
\]

As noted above, \( 16_5 \) and \( \overline{16}_3 \) are absorbed by the Higgs mechanism, and \( 10_5 \) and \( 10_3 \) can become massive through the matrix \( G_I \), whose elements are generally larger than the elements of \( D_I, E_I \) and \( F_I \). Their masses become \((\lambda^{\phi+e-\Delta\phi}, \lambda^{\phi+e-\Delta\phi})\). We simply ignore the matrices \( D_I, E_I, F_I \) and \( G_I \) in the following argument. Since the elements of \( A_I \) and \( B_I \) are generally larger than those of \( C_I \), we can estimate the mass spectrum of the other modes of \( D^c \) from \( A_{D^c} \) and \( B_{D^c} \) as \((\lambda^{e'+\phi+\Delta\phi}, \lambda^{e'+\phi+\Delta\phi}, \lambda^{e'+\phi}, \lambda^{e'+\phi}, \lambda^{e'+\phi}, \lambda^{e'+\phi}, \lambda^{a'+e}, \lambda^{a'+e}, \lambda^{2e'+\Delta\phi})\), and the mass spectrum of the other modes of \( L \) as \((0, \lambda^{\Delta\phi}, \lambda^{\Delta\phi}, \lambda^{\Delta\phi}, \lambda^{\Delta\phi}, \lambda^{\Delta\phi}, \lambda^{\Delta\phi}, \lambda^{\Delta\phi}, \lambda^{\Delta\phi})\). It is obvious that to realize proton decay, we have to pick up an element of \( C_I \). Since such an element is generally smaller than the mass scale of \( D^c \), proton decay is suppressed. The effective colored Higgs mass is estimated as \((\lambda^{e'+\phi+\Delta\phi})^2 / \lambda^{2e'+\Delta\phi} = \lambda^{2\phi+\Delta\phi} \), which is usually larger than the cutoff scale. For example, for the typical charge assignment in Table II, \( 2\phi + \Delta\phi = -17/2 \).

According to the above argument, the mass spectrum of the superheavy particles are determined only by the anomalous \( U(1)_A \) charges, so we can examine whether coupling unification is realized or not. Before going into discussion of this point (given in the next subsection), we define the reduced mass matrices \( \tilde{M}_I \) by getting rid of the massless modes from the original mass matrices \( M_I \). The ranks of the reduced matrices in our \( E_6 \) model are \( \tilde{r}_X = 1, \tilde{r}_G = \tilde{r}_W = 2, \tilde{r}_Q = \tilde{r}_{U^c} = \tilde{r}_{E^c} = 5, \tilde{r}_L = 9 \) and \( \tilde{r}_{D^c} = 10 \). It is interesting that the determinants of the reduced mass matrices are evaluated mainly as simple sums of the anomalous \( U(1)_A \) charges of massive modes:

\[
\det \tilde{M}_I(I = G, W) = \lambda^{2(a+a')} \]

\[
\det \tilde{M}_X = \lambda^{2a'}. \]

\[
\det \tilde{M}_I(I = Q, U^c) = \lambda^{2a+4a'+e+e+e'+e'}. \]

\[
\det \tilde{M}_{E^c} = \lambda^{4a+4a'+e+e'}. \]

\[
\det \tilde{M}_L = \lambda^{2e+e+e'} + 2(a+a'+\phi+\phi) \]

\[
\det \tilde{M}_{L} = \lambda^{2e+e+e'} + 2(a+a'+\phi+\phi) - \Delta\phi. \]

Note that the last equation for \( \det \tilde{M}_L \) is not determined by a simple sum of the charges of massive modes. The difference is \(-\Delta\phi \). This is because the masses
of the $10$ representation of $SO(10)$ in the multiplets $X_i(27)$ of $E_6$ are derived as $\lambda^{x_i x_j + \Delta \phi}$ from the terms $X_i X_j \Phi$ with VEV $\langle \Phi \rangle \sim \lambda^{-\frac{\Delta \phi}{5}}$, which is different from the naive expectation, $\lambda^{x_i x_j}$. In order to calculate the elements of mass matrices, it is useful to introduce the following ‘effective’ charges:

$$x_i(10(5 + 5), 27) \equiv x_i + \frac{1}{2} \Delta \phi, \quad \bar{x}_i(10(5 + 5), \overline{27}) \equiv \bar{x}_i - \frac{1}{2} \Delta \phi,$$

$$x_i(16(5), 27) \equiv x_i + \Delta c - \frac{1}{2} \Delta \phi, \quad \bar{x}_i(\overline{16}(5), \overline{27}) \equiv \bar{x}_i - \Delta c + \frac{1}{2} \Delta \phi,$$

$$x_i(16(10), 27) \equiv x_i, \quad \bar{x}_i(\overline{16}(10), \overline{27}) \equiv \bar{x}_i,$$

$$a(16(5), 78) \equiv a + \Delta c + \frac{1}{2} \Delta \phi, \quad a(\overline{16}(5), 78) \equiv a - \Delta c - \frac{1}{2} \Delta \phi,$$

$$a(16(10), 78) \equiv a + \Delta \phi, \quad a(\overline{16}(10), 78) \equiv a - \Delta \phi,$$

$$a(45(10), 78) \equiv a + \Delta c, \quad a(45(10), 78) \equiv a - \Delta c,$$

$$a(24, 78) \equiv a.$$

Here, the effective charges $x_i(10(5 + 5), 27)$ are for $10$ of $SO(10)$ from $X_i(27)$ of $E_6$ and $\bar{x}_i(10(5 + 5), \overline{27})$ are for $10$ of $SO(10)$ from $\bar{X}_i(\overline{27})$ of $E_6$, etc.

We thus find that all the elements of the mass matrices can be computed as simple sums of the effective charges of superheavy particles if they are not vanishing, and the determinants of the mass matrices are also determined by simple sums of the effective charges. We will use this result in calculating the running gauge couplings below.

### 4.3 Coupling unification

In this subsection, we apply the general analysis of gauge coupling unification given in Ref. [3] to our scenario.

The pattern of the $E_6$ breaking in our model is as follows. At the scale $\Lambda_\phi \sim \lambda^{-(\phi + \bar{\phi})/2}$, $E_6$ is broken into $SO(10)$. $SO(10)$ is broken into $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ at the scale $\Lambda_A \sim \lambda^{-a}$, which is broken into the standard gauge group at the scale $\Lambda_C \sim \lambda^{-(c+\bar{c})/2}$.

In this paper, we carry out analysis based on the renormalization group equations up to one loop.\footnote{Since we ignore the order 1 coefficients, a higher-order calculation does not improve the accuracy.} The conditions of the gauge coupling unification are given by

$$\alpha_3(\Lambda_A) = \alpha_2(\Lambda_A) = \frac{5}{3} \alpha_Y(\Lambda_A) \equiv \alpha_1(\Lambda_A),$$

where $\alpha^{-1}(\mu > \Lambda_C) \equiv \frac{3}{5} \alpha^{-1}_R(\mu > \Lambda_C) + \frac{2}{5} \alpha^{-1}_{B-L}(\mu > \Lambda_C)$. Here $\alpha_X = g_X^2/4\pi$ and the parameters $g_X(X = 3, 2, R, B - L, Y)$ are the gauge couplings of $SU(3)_C$, $SU(2)_L$, $SU(2)_R$, $U(1)_{B-L}$ and $U(1)_Y$, respectively.
Using the fact that the three gauge couplings of MSSM meet at the scale \( \Lambda_G \sim 2 \times 10^{16} \) GeV, the above conditions for gauge coupling unification can be rewritten

\[
\begin{align*}
  b_1 \ln \left( \frac{\Lambda_A}{\Lambda_G} \right) &+ \sum_I \Delta b_{1I} \ln \left( \frac{\Lambda_{A I}^{\phi}}{\det M_I} \right) - \frac{12}{5} \ln \left( \frac{\Lambda_A}{\Lambda_C} \right) \\
  &= \ b_2 \ln \left( \frac{\Lambda_A}{\Lambda_G} \right) + \sum_I \Delta b_{2I} \ln \left( \frac{\Lambda_{A I}^{\phi}}{\det M_I} \right) \\
  &= \ b_3 \ln \left( \frac{\Lambda_A}{\Lambda_G} \right) + \sum_I \Delta b_{3I} \ln \left( \frac{\Lambda_{A I}^{\phi}}{\det M_I} \right),
\end{align*}
\]

(4.41)

where \((b_1, b_2, b_3) = (33/5, 1, -3)\) are the renormalization group coefficients for MSSM and \(\Delta b_{aI}(a = 1, 2, 3)\) are the corrections to the coefficients from the massive fields \(I = Q + \bar{Q}, U^c + \bar{U}^c, E^c + \bar{E}^c, D^c + \bar{D}^c, L + \bar{L}, G, W \) and \(X + \bar{X}\). The last term in Eq. (4.41) is from the breaking \(SU(2)_R \times U(1)_{B-L} \rightarrow U(1)_Y\) by the VEV \(\langle C \rangle\). Since all the mass matrices and the symmetry breaking scale appearing in the above conditions are determined by the anomalous \(U(1)_A\) charges, these conditions can be translated into a constraint on the effective charge of the doublet Higgs fields,

\[
\phi + \frac{1}{2} \Delta \phi \sim 0,
\]

(4.43)

and into a condition on the cutoff scale,

\[
\Lambda \sim \Lambda_G.
\]

(4.44)

As discussed in Ref. [3], this is a quite general result and independent of the details of the Higgs sector. The essential point is that only the charges of massless modes are important to determine whether coupling unification is realized, and all other effects are cancelled out in the unification conditions, except the charge of the doublet Higgs. Note that the condition (4.43) does not require \(\phi + \frac{1}{2} \Delta \phi = 0\). Actually, even with a typical charge assignment, in which \(\phi + \frac{1}{2} \Delta \phi = -4.25\), the coupling unification is realized, using the ambiguities of the order 1 coefficients (see Fig. 1).

Since the unification scale \(\Lambda_U \sim \lambda \Lambda_G\) is smaller than the usual GUT scale \(\Lambda_G \sim 2 \times 10^{16}\) GeV, proton decay via dimension 6 operators \(p \rightarrow e^+ \pi^0\) may be seen in the near future. If we roughly estimate the lifetime of the proton using the formula in Ref. [17] and a recent result from a lattice calculation for the hadron matrix element parameter \(\alpha\) [16], the lifetime of the proton in our scenario becomes

\[
\tau_p(p \rightarrow e^+ \pi^0) \sim 2.8 \times 10^{33} \left( \frac{\Lambda_U}{5 \times 10^{15} \text{ GeV}} \right)^4 \left( \frac{0.015 \text{ GeV}^3}{\alpha} \right)^2 \text{ years}, \quad (4.45)
\]
because the unification scale is around \(5 \times 10^{15}\) GeV. It is interesting that this value of the lifetime is just around the present experimental lower bound \([18]\)

\[
\tau_{\text{exp}}(p \rightarrow e^+\pi^0) > 2.9 \times 10^{33}\;\text{years}. \tag{4.46}
\]

Of course, since we have ambiguities in the order 1 coefficients and the hadron matrix element parameter \(\alpha\), and because the lifetime of the proton is strongly dependent on the GUT scale and this parameter, this prediction may not be very reliable. However, this rough estimation provide strong motivation for experiments to detect proton decay, because the lifetime of a nucleon via dimension 6 operators must be less than that in the usual SUSY GUT scenario.

We should comment on proton decay via dimension 5 operators. The effective colored Higgs mass is given by \(\lambda^2\phi + \Delta\phi\Lambda\), so the experimental constraint requires \(2\phi + \Delta\phi \leq -3\). With the typical charge assignment in Table II, the effective colored Higgs mass is around \(\sim 10^{22}\) GeV, so that proton decay via dimension 5 operators is suppressed.

### 4.4 How to determine charges

It is worthwhile explaining the method for determining the symmetry and quantum numbers in the Higgs sector to realize DT splitting. There are several terms which must be forbidden in order to realize DT splitting:

1. \(\Phi^3, \Phi^2C, \Phi^2C', \Phi^2C'Z\) induce a large mass of the doublet Higgs.
2. \(\bar{C}A'C, \bar{C}A'AC, \Phi A'\Phi\) would destabilize the DW form of \(\langle A\rangle\).
3. \(\bar{\Phi}A'C, \bar{C}A'\Phi, \bar{\Phi}A'AC, \bar{C}A'\Phi, \bar{C}A'Z\Phi\) lead to the undesired VEV \(\langle 16C\rangle = 0\), unless another singlet field is introduced.
4. \(A'A^n(n \geq 4)\) make it less natural to obtain a DW-type of VEV.

Most of these terms can be easily forbidden by the SUSY zero mechanism. For example, if we choose \(\phi < 0\), then \(\Phi^3\) is forbidden, and if we choose \(\bar{c} + c + a + a' < 0\), then \(\bar{C}A'AC\) is forbidden. Once these terms are forbidden by the SUSY zero mechanism, higher-dimensional terms, which also become dangerous (for example, \(C'\bar{A}'A^3C\) and \(\bar{C}A'CCAC\)), are automatically forbidden, as in \(SO(10)\) cases. Contrastingly, the following terms are necessary:

1. \(A'\Phi, \bar{\Phi}A'\Phi\) to obtain a DW-type VEV \(\langle A\rangle\).
2. \(\Phi^2AC'\) for doublet-triplet splitting.
3. \(\bar{C}'(A + Z)C, \bar{C}(A + Z)C'\) for alignment between the VEVs \(\langle A\rangle\) and \(\langle C\rangle\) and to give superheavy masses to the PNGs.
4. $\Phi A' A \Phi$ for alignment between the VEVs $\langle A \rangle$ and $\langle \Phi \rangle$ and to give the superheavy masses to the PNGs.

5. $\bar{\Phi}^3, \bar{\Phi}^2 C$ to give superheavy masses to two $10$ of $SO(10)$, which save the number of fields with positive charges.

In order to forbid $\Phi^2 C'$ but not $\Phi^2 A'C'$, we have to introduce $Z_2$ parity. The same $Z_2$ parity can forbid $\Phi A' \Phi$, while allowing the term $\Phi A' A \Phi$. We have some ambiguities to assign the $Z_2$ parity, but once the parity is fixed, the above requirements simply become inequalities. In addition to these inequalities, we require that the total charges of the operators $A$, $\Phi \Phi$ and $\bar{C} C$ be negative. If, as discussed in the previous section, we adopt $a = -1$ to realize proton stability, then the inequalities $a' + 3a + \phi + \bar{\phi} \geq 0$ and $a' + 5a < 0$ lead to $\phi + \bar{\phi} = -1$ and $a' = 4$. The relation $\phi + \bar{\phi} = -1$ means that the gauge singlet operator $\Phi \Phi$ can be regarded as the FN field $\Theta$, as discussed in §2. The other inequalities are easily satisfied.

Of course, the above stated conditions are necessary but not sufficient. As in the previous subsection, we have to write down the mass matrices of the Higgs sector to know whether an assignment actually works or not.

### 5 Constraints from the matter sector

For the matter fields, we introduce three $27$, $\Psi_i$ ($i = 1, 2, 3$). As discussed in Refs. [1, 13], we adopt $(\psi_1, \psi_2, \psi_3) = (3 + n, 2 + n, n)$, and the charge of the Higgs $\phi = -2n$ in order to realize a CKM matrix and a large top Yukawa coupling of $O(1)$. As discussed in the previous section, the cutoff scale $\Lambda$ must be around the usual GUT scale $\Lambda_G \sim 2 \times 10^{16}$ GeV. This requires $a \leq -1$. If we have an integer charge for $a$, then we must adopt $a = -1$. As discussed in Ref. [13], the conditions for obtaining realistic quark and lepton mass matrices with bi-large neutrino mixing angles are

$$c - \bar{c} = \phi - \bar{\phi} + 1 = 2n - 9 - l,$$

where there is some ambiguity in the parameter $l$, but as discussed in Ref. [13], for models with $\Lambda \sim \Lambda_G$, we adopt $l = -1$ or $-2$. Since the condition for gauge coupling unification is $\phi + \frac{1}{4}(\phi - \bar{\phi}) = \frac{1}{4}(-6n - 10 - l) \sim 0$, a small value of $n$ is required. From the condition $\phi + \bar{\phi} = -6n + 10 + l \leq -1$, the smallest value of $n$ becomes $3/2$ for $l = -2$. Then, we obtain $\phi = -3$, $\bar{\phi} = 2$ and $c - \bar{c} = -4$. It is non-trivial that the conditions for bi-large neutrino mixing angles and for gauge coupling unification lead to $\phi + \bar{\phi} = -1$, which is required for DT splitting. Since

---

Using the ambiguities of the order 1 coefficients, a rather larger range of values of the parameter $l$ may be allowed, for example, $l = -3, -4$. But in the following discussion, these larger ambiguities do not change the result.
\[ \phi + \bar{\phi} = -1, \] we must adopt \( a' = 4 \) to allow the important term \( \Phi A' A^3 \Phi \) and to forbid the term \( A' A^5 \). Then, in order to forbid the terms \( \bar{\Phi} A' A C, \bar{\Phi} C' A \Phi \) and \( C' A' A C \), we have to take \( c < -5 \) and \( \bar{c} < 0 \). On the other hand, since the term \( \Psi_1 \Psi_3 C \) is required to realize realistic mass matrices of the quark and lepton, as discussed in Ref. [13], \( c \geq -6 \) is needed. Hence we must take \( c = -6 \) and \( \bar{c} = -2 \).

It is interesting that the economical condition for the \( \mu \) problem [2, 3],

\[
-1 \leq 2\phi - (c + \bar{c}) + \frac{1}{2}(\phi - \bar{\phi}) \leq 1, \tag{5.2}
\]

is automatically satisfied and the required terms \( \bar{\Phi}^3 \) and \( \bar{\Phi}^2 \bar{C} \) happen to be allowed. We have some freedom in choosing the charges \( z, c' \) and \( \bar{c}' \). If we take \( z = -2 \), then we must adopt \( c' = 7 \), because the term \( Z C' \Phi^2 \) must be forbidden, while the term \( C' A \Phi^2 \) must be allowed. Also, \( \bar{c}' \geq 8 \) is required to obtain the term \( \bar{C}' (A + Z) C \). In the typical charge assignment, we adopt the minimal value \( c' = 8 \).

The charges of the matter sector \( \Psi_i(27) \) \((i = 1, 2, 3)\) become half integers as \((\psi_1, \psi_2, \psi_3) = (9/2, 7/2, 3/2)\) in this case. It is interesting that we do not have to introduce R-parity, because half integer anomalous \( U(1)_A \) charges can play the same role.

From the above consideration, all the charges are fixed except the singlets. Therefore we can calculate the running flows of the gauge couplings (see Fig. 1). Here we use the ambiguities of the coefficients expressed by \( 0.5 \leq y \leq 2 \). It is shown that the three gauge couplings actually meet around \( \lambda^{-a} \Lambda_G \sim 5 \times 10^{15} \) GeV. Note that the unified gauge coupling at the cutoff scale is still finite, although it is so large that the perturbative calculation is not reliable. Since the value of the unified gauge coupling is strongly dependent on the actual charge assignment and the value \( \lambda \) (which is weakly dependent on the coefficients of order 1), we expect that it is large but finite at the cutoff scale.

### 6 Discussion and summary

We emphasize that the effective \( SO(10) \) theory that is obtained with the non-vanishing VEVs \( \langle \Phi \rangle = \langle \bar{\Phi} \rangle \) from \( E_6 \) GUT is generally different from the \( SO(10) \) GUT with anomalous \( U(1)_A \) gauge symmetry. This is because the VEV \( \langle \Phi \rangle \sim \lambda^{-\frac{1}{2}(\phi + \bar{\phi})} \) is generally different from the naively expected value, \( \langle \Phi \rangle \sim \lambda^{-\phi} \). For example, the mass of \( 10_F \) in \( SO(10) \) is obtained from the term \( \lambda^{2f + \phi} F(27)^2 \Phi(27) \) by developing the VEV \( \langle \Phi \rangle \sim \lambda^{-\frac{1}{2}(\phi + \bar{\phi})} \) as \( m_F \sim \lambda^{2f + \frac{1}{2}(\phi - \bar{\phi})} \), which is different from the naively expected value, \( m_F \sim \lambda^{2f} \). This is the same effect as that

\[ \text{In order to solve the } \mu \text{ problem economically, we need the term } \bar{C}' \Phi Z(\Phi^3) \text{ or } \bar{C}' \Phi(\bar{\Phi}^3) \text{ when we adopt even } Z_2 \text{ parity for } C'. \text{ This leads to } \bar{c}' \geq 14 \text{ or } \bar{c}' \geq 12. \text{ However, in these cases, the gauge couplings seem to diverge below the GUT scale. Therefore, the additional gauge singlet field } S \text{ with positive charge } s \geq 3 \phi \text{ may be required.} \]
Figure 1: Here we adopt $\lambda = 0.25$, $\alpha_1^{-1}(M_Z) = 59.47$, $\alpha_2^{-1}(M_Z) = 29.81$, $\alpha_3^{-1}(M_Z) = 8.40$, and the SUSY breaking scale $m_{SB} \sim 1$ TeV. We also use the anomalous $U(1)_A$ charges $a' = 4$, $a = -1$, $(\psi_1, \psi_2, \psi_3) = (9/2, 7/2, 3/2)$, $\phi = -3$, $\tilde{\phi} = 2$, $c = -6$, $\tilde{c} = -2$, $z = -2$, $c' = 7$ and $\tilde{c}' = 8$. Using the ambiguities of coefficients expressed by $0.5 \leq y \leq 2$, the three gauge couplings meet at around $\lambda^{-a} \Lambda_G \sim 5 \times 10^{15}$ GeV.

discussed in Ref. [1] with regard to the right-handed neutrino mass in $SO(10)$ GUT. The right-handed neutrino mass has been estimated as $m_{\nu_R} \sim \lambda^{2f+\frac{1}{2}(c-\tilde{c})}$, which is different from the naively expected value, $\lambda^{2f}$. Therefore, it is obvious that the $SO(10)$ or $E_6$ GUT with anomalous $U(1)_A$ gauge symmetry is essentially different from the MSSM with anomalous $U(1)_A$ gauge symmetry. Note that the difference is caused by $\Delta \phi$ and $\Delta c$.

In this paper, we have proposed a realistic Higgs sector in $E_6$ grand unified theory, which can realize DT splitting and proton stability. In this scenario, the anomalous $U(1)_A$ gauge symmetry plays a critical role. Moreover, using the matter sector in $E_6$ GUT given in Ref. [13], we have proposed a completely consistent $E_6$ GUT scenario. Since we have introduced all the interactions that are allowed by the symmetry, the model can be defined by the symmetry and the quantum numbers of the fields. In our scenario, after deciding the field content (Higgs and matter), the model can be defined with only 8 integer charges (+3 charges for singlet fields) in the Higgs sector and 3 (half-) integer charges in
the matter sector. It is quite intriguing that by choosing only 11 (half-) integer
carried charges (+3 charges for singlet fields), not only are DT splitting with proton
stability and gauge coupling unification realized but also the realistic structure of
quark and lepton mass matrices, including bi-large neutrino mixing, are obtained.
Moreover, the FCNC process is automatically suppressed. In other words, the
charge assignment, which is almost determined by realizing realistic quark and
lepton mass matrices and gauge coupling unification, also solves the DT splitting
with proton stability. For example, the relation \( \bar{\phi} + \phi = -1 \) is required to obtain
realistic quark and lepton mass matrices and gauge coupling unification, and it
is also independently needed to realize DT splitting. Moreover, since the half-
integer charges of the matter sector play the same role as \( R \)-parity, we do not
have to introduce \( R \)-parity.

Of course if it is allowed for the half-integer charges to be assigned to the Higgs
sector, there are other possibilities. For example, we can adopt another charge
assignment, in which the half integer charges play the same role as \( Z_2 \) parity.
If we use \( a = -1/2, a' = 5/2, \phi = -3, \bar{\phi} = 2, c = -5, \bar{c} = -1, c' = 13/2, \bar{c}' = 13/2, z_i = -i/2(i = 3, 7, 11), \psi_1 = 9/2, \psi_2 = 7/2, \psi_3 = 3/2 \) with odd \( R \)-parity in
the matter sector \( \Psi_i \), we can obtain the completely consistent \( E_6 \) GUT again.
Since the absolute values of the charges of this model are smaller than those of
the previous model, the unified gauge coupling at the unified scale \( \lambda^{-a} \) becomes
smaller than that of the previous model. Therefore, the unified gauge coupling
at the cutoff scale must be finite in this model. Then, since the unification scale
\( \lambda^{-a} \) is larger than that of previous model, the model predicts a longer lifetime of
the nucleon, which is roughly estimated as

\[
\tau_p(p \to e^+\pi^0) \sim 4.5 \times 10^{34} \left( \frac{\Lambda_U}{10^{16} \text{GeV}} \right)^4 \left( \frac{0.015 \text{ GeV}^3}{\alpha} \right)^2 \text{ years. (6.1)}
\]

Though this predicted value is significantly longer than the present experimental
lower bound, we hope that the next generation of experiments can reach this
value.\(^{14}\)

Though the requirement on \( E_6 \) GUT is so severe that the possible charge
assignments are fairly restricted, there are several possibilities for this assignment.
However, since our scenario requires only several integer charges as the input
parameters to provide realistic results both in the matter sector and in the Higgs
sector, we believe that this scenario indeed describes our world.

\(^{14}\)Unfortunately, since the term \( A'A^5 \) is allowed by the symmetry in this charge assignment,
it is less natural to obtain a DW-type VEV in this model than in the previous model. However,
the number of VEVs is still finite.
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A Factorization

As mentioned in §4, the naive extension of DT splitting in $SO(10)$ GUT into $E_6$ GUT does not work. In the $SO(10)$ DT splitting, the interaction $(A'A)_{54}(A^2)_{54}$ plays an essential role. In $E_6$ GUT, however, the term $A'A^3$ does not include the interaction $(45A'45A)_{54}(45^2A)_{54}$. Therefore the superpotential

$$W_A' = \lambda_a A'_a A + \lambda_a^3 (A' A)_{11} (A^2)_{11} + \gamma (A' A)_{650} (A^2)_{650}$$  \hspace{1cm} (A.1)

does not realize the DW VEV naturally. Here, we show that the term $A'A^3$ of $E_6$ actually does not include the interaction $(45A'45A)_{54}(45^2A)_{54}$ of $SO(10)$.

The VEV of $SO(10)$ adjoint Higgs can be represented in the form

$$\langle A \rangle = \tau_2 \times \text{diag}(x_1, x_2, x_3, x_4, x_5),$$  \hspace{1cm} (A.2)

because of the $SO(10)$ rotation and D-flatness condition (see Appendix B). In this gauge,

$$(A'A)_{54} = 2 \sum_i x'_i x_i,$$  \hspace{1cm} (A.3)

$$(A'A)_{54}(A^2)_{54} = 2 \sum_i x'_i x_i^3 - 2 \left( \sum_i x'_i x_i \right) \left( \sum_j x'_j x_j \right).$$  \hspace{1cm} (A.4)

In the same manner, the VEV of $E_6$ adjoint Higgs can be represented in the form

$$\langle A \rangle = y, \langle 16_A \rangle = \langle 16 \rangle = 0, \langle 45_A \rangle = \tau_2 \times \text{diag}(x_1, x_2, x_3, x_4, x_5).$$

In this gauge, the VEV $\langle A \rangle$ can be represented as $27 \times 27$ matrix as

$$\langle A \rangle = \begin{pmatrix} \frac{2}{\sqrt{3}} y & 0 & \theta^{MN} T_{16}^{MN} + \frac{1}{2 \sqrt{3}} y \mathbf{1}_{16} & 0 \\ 0 & \theta^{MN} T_{10}^{MN} - \frac{1}{\sqrt{3}} y \mathbf{1}_{10} \\ 0 & 0 & \theta^{MN} T_{16}^{MN} + \frac{1}{2 \sqrt{3}} y \mathbf{1}_{16} & 0 \\ 0 & \theta^{MN} T_{10}^{MN} - \frac{1}{\sqrt{3}} y \mathbf{1}_{10} \end{pmatrix}.$$  \hspace{1cm} (A.5)

Here, $T_{ij}^{MN}$ is the $i \times i$ matrix representation of $SO(10)$ generators and the summation of the indices $M$ and $N$ is understood from 1 to 10 with $M > N$. Also, $\mathbf{1}_i$ is the $i \times i$ unit matrix. Explicitly, we have

$$(T_{10}^{MN})_{KL} = -i (\delta^M_K \delta^N_L - \delta^M_L \delta^N_K),$$  \hspace{1cm} (A.6)

$$(T_{16}^{MN})_{\alpha \beta} = \frac{1}{2} (\sigma^{MN})_{\alpha \beta}$$

$$= \frac{1}{4i} ([\gamma^M, \gamma^N] P_R)_{\alpha \beta},$$  \hspace{1cm} (A.7)

$$\theta^{MN} = \begin{cases} x_n & M + 1 = N = 2n, \ (n = 1, \cdots, 5) \\ 0 & \text{otherwise} \end{cases}.$$
where the $\gamma^M$ are $SO(10)$ $\gamma$-matrices and $P_R$ is the right-handed projector, which can be written

$$
\begin{align*}
\gamma^1 &= \tau_1 \otimes 1 \otimes 1 \otimes 1 \otimes 1, \quad (A.8) \\
\gamma^2 &= \tau_3 \otimes 1 \otimes 1 \otimes 1 \otimes 1, \quad (A.9) \\
\gamma^3 &= \tau_2 \otimes \tau_1 \otimes 1 \otimes 1 \otimes 1, \quad (A.10) \\
\gamma^4 &= \tau_2 \otimes \tau_3 \otimes 1 \otimes 1 \otimes 1, \quad (A.11) \\
\gamma^5 &= \tau_2 \otimes \tau_2 \otimes \tau_1 \otimes 1 \otimes 1, \quad (A.12) \\
\gamma^6 &= \tau_2 \otimes \tau_2 \otimes \tau_3 \otimes 1 \otimes 1, \quad (A.13) \\
\gamma^7 &= \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_1 \otimes 1, \quad (A.14) \\
\gamma^8 &= \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_3 \otimes 1, \quad (A.15) \\
\gamma^9 &= \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_1, \quad (A.16) \\
\gamma^{10} &= \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2, \quad (A.17) \\
\gamma^{11} &= i\gamma^1 \gamma^2 \gamma^3 \gamma^4 \gamma^5 \gamma^6 \gamma^7 \gamma^8 \gamma^9 \gamma^{10} \\
&= \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2, \quad (A.18) \\
P_R &= \frac{1 + \gamma^{11}}{2}. \quad (A.19)
\end{align*}
$$

In this basis, we have

$$
\begin{align*}
\theta^M N T_{16}^{M N} &= -\frac{1}{2} (x_1 \tau_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \\
&\quad + x_3 1 \otimes \tau_2 \otimes 1 \otimes 1 \otimes 1 \\
&\quad + x_3 1 \otimes 1 \otimes \tau_2 \otimes 1 \otimes 1 \\
&\quad + x_4 1 \otimes 1 \otimes 1 \otimes \tau_2 \otimes 1 \\
&\quad + x_5 1 \otimes 1 \otimes 1 \otimes 1 \otimes \tau_2) P_R, \\
&\equiv B, \quad (A.20) \\
\theta^M N T_{10}^{M N} &= \tau_2 \otimes \text{diag}(x_1, x_2, x_3, x_4, x_5). \\
&\equiv C. \quad (A.21)
\end{align*}
$$

Before beginning the calculation, we should determine what coupling can occur in the term $A' A^3$ of $E_6$. Because $78 \times 78 = 1_s + 78_a + 650_s + 2430_s + 2925_a$, $A' A^3 \ni (A'A)_1(A^2)_1, (A'A)_{650}(A^2)_{650}, (A'A)_{2430}(A^2)_{2430}$. On the other hand, because of the completeness,

$$
(A_1 A_2)_{2430} (A_3 A_4)_{2430} = \sum_{I=1,78,650,2430,2925} \lambda_I (A_1 A_4)_I (A_3 A_2)_I. \quad (A.22)
$$

Therefore,

$$
(A'A)_{2430} (A^2)_{2430} = \sum_{I=1,650,2430} \lambda_I (A'A)_I (A^2)_I, \quad (A.23)
$$
which implies that the above three couplings are not independent, and it is sufficient to examine the first two. They are essentially described as \( \text{tr} A' A \text{tr} A^2 \) and \( \text{tr} A' A^3 \) in matrix language. If the desirable coupling existed, it would apparently be included only in \( (A'A)_{650}(A^2)_{650} \) and \( \text{tr} A' A^3 \). Thus we can conclude that it does not exist if \( \text{tr} A' A^3 \) does not include \( \sum_i x'_i x_i^3 \).

From (A.4), we find

\[
\text{tr} A'A = \frac{4}{3} y'y + \text{tr}_{16} \left[ B'B + \frac{1}{2\sqrt{3}} B'y + \frac{1}{2\sqrt{3}} y'B + \frac{1}{12} y'y \right] \\
+ \text{tr}_{10} \left[ C'C - \frac{1}{\sqrt{3}} C'y - \frac{1}{\sqrt{3}} y'C + \frac{1}{3} y'y \right] \\
= \left( \frac{4}{3} + \frac{16}{12} + \frac{10}{3} \right) y'y + \left( 16 \frac{1}{22} + 2 \right) \sum_i x'_i x_i \\
= 6 \left( y'y + \sum_i x'_i x_i \right). 
\]  

Similarly,

\[
\text{tr} A' A^3 = \frac{16}{9} y'y^3 + \text{tr}_{16} \left[ B'B^3 + 3 \frac{1}{12} \left( B'B y^2 + y'y B^2 \right) + \frac{1}{144} y'y^3 \right] \\
+ \text{tr}_{10} \left[ C'C^3 + 3 \frac{1}{3} \left( C'C y^2 + y'y C^2 \right) + \frac{1}{9} y'y^3 \right] \\
= \frac{16}{9} y'y^3 + 16 \left[ \frac{1}{24} \left( \sum x'_i x_i \sum x_i^2 - 2 \sum x'_i x_i^3 \right) \\
+ \frac{3}{12} \left( \sum x'_i x_i y^2 + y'y \sum x'_i x_i \right) + \frac{1}{144} y'y^3 \right] \\
+ \left[ 2 \sum x'_i x_i^3 + \frac{3}{3} \left( 2 \sum x'_i x_i y^2 + y'y 2 \sum x'_i x_i \right) + \frac{10}{9} y'y^3 \right] \\
= 3 \left( y'y + \sum_i x'_i x_i y^2 \right) \left( y^2 + \sum_i x_i^2 \right) \\
= \frac{1}{12} \text{tr} A'A \text{tr} A^2. 
\]  

It is thus seen that desirable coupling does not exist because of the group theoretical cancellation between the contributions from the \( \text{tr}_{16} \) part and the \( \text{tr}_{10} \) part.

There are several solutions, and the simplest one is to use the term \( \Phi A'A^3 \Phi \). At first glance, it seems to have no effect, because \( \Phi \Phi \) is written as

\[
\Phi \Phi = \begin{pmatrix}
\langle \Phi \Phi \rangle & 0 & 0 \\
0 & 0_{16} & 0 \\
0 & 0 & 0_{10}
\end{pmatrix}. 
\]  

(A.26)
However this form is a special combination of \((\Phi \Phi)_1\), \((\Phi \Phi)_{78}\) and \((\Phi \Phi)_{650}\). In fact, we have

\[
\begin{pmatrix} \langle \Phi \Phi \rangle & 0 & 0 \\ 0 & 0_{16} & 0 \\ 0 & 0 & 0_{10} \end{pmatrix} = \frac{\langle \Phi \Phi \rangle}{54} \left[ 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_{16} & 0 \\ 0 & 0 & 1_{10} \end{pmatrix} + 3 \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1_{16} & 0 \\ 0 & 0 & -2 \times 1_{10} \end{pmatrix} \\
+ \begin{pmatrix} 40 & 0 & 0 \\ 0 & -5 \times 1_{16} & 0 \\ 0 & 0 & 4 \times 1_{10} \end{pmatrix} \right],
\]  

(A.27)

where the three matrices on the rhs are proportional to the \(SO(10)\) singlets of \(1\), \(78\) and \(650\), respectively. Since the interactions for each representation have independent couplings, generically the cancellation does not happen without fine-tuning.

There are several other solutions for this problem. The essential ingredient is the interaction between \(A' A^3\) and some other operator, whose VEV breaks \(E_6\), because the cancellation is due to a feature of the \(E_6\) group. We now present some of these solutions.

- Allowing the higher-dimensional term \(A' A^5\). Since \(\langle A^2 \rangle\) breaks \(E_6\), the cancellation can be avoided, which can be shown by a straightforward calculation. Since the number of solutions of the \(F\)-flatness conditions increases, it becomes less natural to obtain a DW VEV. But the number of vacua is still finite.

- Introducing additional adjoint Higgs fields \(B'\) and \(B\), and giving \(B\) the VEV pointing to a \(SO(10)\)-singlet. Then \(B\) plays the same role as the above \((\Phi \Phi)_{78}\). Examining the superpotential

\[
W = B' B + \bar{\Phi} B' \Phi,
\]  

(A.28)

the desired VEV \(\langle 1_B \rangle \neq 0\) and \(\langle 45_B \rangle = 0\) is easily obtained.

## B The VEV of an adjoint Higgs

In this appendix, we show that there is a gauge in which the VEV of an adjoint Higgs points in the direction of the Cartan subalgebra (CSA).

Suppose that one Higgs \(A\), which belongs to the adjoint representation of the group \(G\) (of dimension \(d\) and rank \(r\)), obtains a non-vanishing VEV. Then the D-flatness condition is as follows:

\[
0 = (A^b)^* (T^a_G)_{bc} A^c \\
= -i (A^b)^* f^{abc} A^c
\]
For infinitesimal rotations, the transformation law of \( A \) becomes
\[
0 = \left( A^b \right)^* f^{abc} A^c (T^a)
\]
\[
= \left( A^b \right)^* \left[ T^b, T^c \right] A^c
\]
\[
= \left[ A^i, A \right]. \quad \text{(B.1)}
\]

Next, we expand the VEV in the basis \( \{ H^a, E_\alpha, E_{-\alpha} \} \). Here the \( H^a \) \((a = 1, \cdots, r)\) are contained in the CSA, the values \( \alpha \) are \( \frac{d-r}{2} \) positive roots, and \( E^+_\alpha = E_{-\alpha} \).

The following commutation relations hold:
\[
\begin{align*}
[H^a, H^b] &= 0, \quad \text{(B.2)} \\
[H^a, E_{\pm \alpha}] &= \pm \alpha^a E_{\mp \alpha}, \quad \text{(B.3)} \\
[E_\alpha, E_{-\alpha}] &= \alpha^a H^a, \quad \text{(B.4)} \\
[E_\alpha, E_{\pm \beta}] &= N_{\alpha, \pm \beta} E_{\alpha \pm \beta}, \quad \text{(B.5)}
\end{align*}
\]

where the values \( N_{\alpha, \pm \beta} \) are constants that depend on \( \alpha \) and \( \beta \), which are nonzero only if \( \alpha \pm \beta \) \ is also a root. In this basis, the VEV is written as
\[
A = A^a H^a + \sum_{\text{positive root } \alpha} \left\{ A^\alpha_+ E_\alpha + A^\alpha_- E_{-\alpha} \right\}, \quad \text{(B.6)}
\]
\[
A^i = (A^a)^* H^a + \sum_{\alpha} \left\{ (A^\alpha_+)^* E_\alpha + (A^\alpha_-)^* E_{-\alpha} \right\}. \quad \text{(B.7)}
\]

Then, extracting the part proportional to the CSA from the lhs of (B.1), it becomes
\[
\left[ A^i, A \right] = \sum_{\alpha} \left( |A^\alpha_+|^2 - |A^\alpha_-|^2 \right) \alpha^a H_a + \cdots
\]
\[
= 0. \quad \text{(B.8)}
\]

Therefore, if all the values \( A^\alpha_+ \) are zero, then \( \sum_{\alpha} |A^\alpha_+|^2 \alpha^a \) \ is zero, and therefore all the values \( A^\alpha_- \) \ are also zero.\(^{15}\)

Now, we show that all the \( A^\alpha_- \) can be rotated away through the gauge rotation. For infinitesimal rotations, the transformation law of \( A \) is
\[
-i \delta A = \left[ \theta^a H^a + \sum_{\alpha} \left\{ \theta^\alpha_+ E_\alpha + \theta^\alpha_- E_{-\alpha} \right\}, A^a H^a + \sum_{\alpha} \left\{ A^\alpha_+ E_\alpha + A^\alpha_- E_{-\alpha} \right\} \right], \quad \text{(B.9)}
\]

where \( \theta \) is the parameter of the rotation composed of \( d \) real numbers corresponding to \( d \) generators. Then, extracting the part proportional to \( E_\alpha \), we have
\[
-i \delta A^\alpha = \left( A^\alpha \theta^\alpha_- - A^\alpha \theta^\alpha_+ \right) \alpha^a + \sum_{\beta, \gamma = -\alpha} \left( A^\alpha_\beta \theta^\beta_- \theta^\gamma_+ \right) N_{\beta, -\gamma} - \sum_{\beta, \gamma = \alpha} A^\alpha_\beta \theta^\beta_- N_{\beta, \gamma} \cdot \quad \text{(B.10)}
\]

\(^{15}\) Though even positive roots may have negative components, such roots must have a positive component at smaller values of \( a \) by the definition of a positive root. Therefore, examining the conditions from that of a smaller value of \( a \) to that of larger one successively, \( \sum_{\alpha} |A^\alpha_+|^2 \alpha^a = 0 \) is easily confirmed.
It is seen that there are $d - 1$ gauge degrees of freedom, except for the one corresponding to the maximum root, compared with the $\frac{d+r}{2}$ complex VEVs corresponding to $d - r$ real numbers. Therefore, all the $A^c_\alpha$ can be rotated away, and in this gauge, the D-flatness condition forces all the $A^c_\alpha$ to be zero.

Now, we have shown that the VEV of an adjoint Higgs can be expressed as pointing to the CSA. In other words, the VEV of an adjoint Higgs is gauge equivalent to that pointing to the CSA in the supersymmetric limit.

C Operators that induce mass matrices

In this appendix, we give the operators that induce the mass matrices of super-heavy particles in $E_6$ GUT.

First, we examine the operator matrix $O_{24}$ of $24$ in $SU(5)$, which induces the mass matrices $M_I$ ($I = X, G, W$),

$$O_{24} = \begin{pmatrix} 0 & A^\prime A \\ A^\prime A & A^\prime A^2 \end{pmatrix},$$

where the numbers in the parentheses denote typical charges.

Next, we examine the operator matrix $O_{10}$ of $10$ in $SU(5)$, which induces the mass matrices $M_I$ ($I = Q, U^c, E^c$),

$$O_{10} = \begin{pmatrix} 0 & A^\prime A \\ A^\prime A & A^\prime A^2 \\ A^\prime A^2 & 0 \end{pmatrix},$$

where we have given only one example, even if there are several corresponding operators.

Finally, we examine the operator matrix $O_5$ of $5$ and $\bar{5}$ in $SU(5)$, which induces the mass matrices $M_I$ ($I = L, D^c$),

$$O_5 = \begin{pmatrix} 0 & A_5 & 0 \\ B_5 & C_5 & D_5 \\ E_5 & F_5 & G_5 \end{pmatrix},$$
\[
A_5 = \begin{pmatrix}
\bar{1}/I & 10_{C'}(7) & 10_{C'}(8) & \bar{10}_{C'}(8) & \bar{10}_{A'}(4) \\
10_{\Phi}(-3) & C'\Phi^2 & C'(A + Z)\Phi & 0 & 0 \\
10_{C'}(-6) & 0 & C'(A + Z)C & 0 & 0 \\
16_{C'}(-6) & 0 & 0 & \bar{C}'(A + Z)C & 0 \\
16_{A'}(-1) & 0 & \bar{C}'AC & \bar{C}'A\Phi & A'A \\
\end{pmatrix}, \quad (C.4)
\]

\[
B_5 = \begin{pmatrix}
\bar{1}/I & 10_{\Phi}(-3) & 10_{C'}(-6) & \bar{16}_{C'}(-2) & \bar{16}_{A'}(-1) \\
10_{C'}(7) & 0 & 0 & \bar{C}AC' & \bar{C}AC' \\
10_{C'}(8) & \bar{C}'(A + Z)\Phi & \bar{C}'(A + Z)C & \bar{C}'(A + Z)C^2 & \bar{C}'A\Phi\bar{C} \\
16_{C'}(7) & 0 & 0 & C'(A + Z)C' & \Phi AC' \\
16_{A'}(4) & 0 & 0 & 0 & A'A \\
\end{pmatrix}, \quad (C.5)
\]

\[
C_5 = \begin{pmatrix}
\bar{1}/I & 10_{\Phi}(2) & 10_{C'}(-2) & \bar{10}_{\Phi}(2) \\
10_{C'}(7) & C'\Phi^2 & C'C' & C'C'C' & \Phi CC'\Phi \\
10_{C'}(8) & C'\Phi^2 & C'\Phi & C'\Phi & \Phi CC'\Phi \\
16_{C'}(7) & \bar{C}'AC'C' & \bar{C}'AC'C & \bar{C}'AC'C & \Phi (A + Z)C' \\
16_{A'}(4) & C'\Phi AC' & C'\Phi AC' & C'\Phi AC' & C'\Phi AC' \\
\end{pmatrix}, \quad (C.6)
\]

\[
D_5 = \begin{pmatrix}
\bar{1}/I & 10_{\Phi}(2) & 10_{C'}(-2) & \bar{10}_{\Phi}(2) \\
10_{C'}(7) & \bar{C}'(A + Z)C' & C'(A + Z)C' & \Phi CC'\Phi \\
10_{C'}(8) & C'\Phi^2 & C'\Phi & C'\Phi & \Phi CC'\Phi \\
16_{C'}(7) & \bar{C}'AC'C' & \bar{C}'AC'C & \bar{C}'AC'C & \Phi (A + Z)C' \\
16_{A'}(4) & C'\Phi AC' & C'\Phi AC' & C'\Phi AC' & C'\Phi AC' \\
\end{pmatrix}, \quad (C.7)
\]

\[
E_5 = \begin{pmatrix}
\bar{1}/I & 10_{\Phi}(-3) & 10_{C'}(-6) & \bar{10}_{A'}(-1) \\
10_{\Phi}(2) & 0 & 0 & 0 & \bar{\Phi}^2 A'\bar{C} \\
16_{\Phi}(-3) & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad (C.8)
\]

\[
F_5 = \begin{pmatrix}
\bar{1}/I & 10_{C'}(7) & 10_{C'}(8) & \bar{10}_{C'}(8) & \bar{10}_{A'}(4) \\
10_{\Phi}(2) & \Phi (A + Z)C' & C'\Phi^2 & C'\Phi & \bar{\Phi} A'\bar{C} \\
16_{C'}(-3) & 0 & C'\Phi AC' & C'(A + Z)\Phi & \Phi A'\Phi \\
\end{pmatrix}, \quad (C.9)
\]

\[
G_5 = \begin{pmatrix}
\bar{1}/I & 10_{\Phi}(2) & 10_{C'}(-2) & \bar{10}_{\Phi}(2) \\
10_{\Phi}(2) & \bar{\Phi}^3 & \bar{\Phi}^2 \bar{C} & \bar{\Phi}^2 \bar{C} \\
16_{\Phi}(-3) & 0 & 0 & 0 \\
\end{pmatrix}. \quad (C.10)
\]

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