Sufficient conditions for the genericity of feedback stabilisability of switching systems via Lie-algebraic solvability

H. Haimovich† J. H. Braslavsky‡

†CONICET and Laboratorio de Sistemas Dinámicos y Procesamiento de Información, Depto. de Control, Esc. de Ing. Electrónica, FCEIA, Universidad Nacional de Rosario, Riobamba 245bis, 2000 Rosario, Argentina. h.haimovich@gmail.com
‡Australian Commonwealth Scientific and Industrial Research Organisation (CSIRO), Division of Energy Technology, PO Box 330, Newcastle NSW 2300, Australia. Julio.Braslavsky@csiro.au

Abstract—This paper addresses the stabilisation of discrete-time switching linear systems (DTSSs) with control inputs under arbitrary switching, based on the existence of a common quadratic Lyapunov function (CQLF). The authors have begun a line of work dealing with control design based on the Lie-algebraic solvability property. The present paper expands earlier work by deriving sufficient conditions under which the closed-loop system can be caused to satisfy the Lie-algebraic solvability property generically, i.e. for almost every set of system parameters, furthermore admitting straightforward and efficient numerical implementation.

Keywords—Switching systems, Lie algebras, Common eigenvector assignment, Transverse subspaces, Genericity.

1 INTRODUCTION

This paper considers control design for feedback stabilisation of DTSSs under arbitrary switching regimes. Several results for the analysis of stability of switching systems under arbitrary switching exist [10, 9], although most of these deal only with autonomous switching systems, i.e. switching systems without continuous control inputs.

The study of control design methods to achieve closed-loop stability under arbitrary switching for systems with control inputs, in contrast, has been relatively scarce. Amidst the existing work, a computationally appealing approach consists in adapting LMI-based numerical stability tests to the synthesis of feedback controls that will guarantee the existence of a CQLF for the closed-loop switching system (e.g. [11, 12]). Such LMI-based methods for control design, however, provide little information on the structure of the designed closed-loop system, and thus in general offer limited support for analysis.

A structurally-based control design method has been developed by the authors [3, 4, 5] that involves the Lie-algebraic solvability property. A well-known sufficient condition for stability [8] states that an autonomous switching linear system admits a CQLF (and is hence stable under arbitrary switching) if every subsystem is stable and the Lie algebra generated by the subsystem matrices is solvable. Solvability of a matrix Lie algebra is equivalent to the existence of a single similarity transformation that transforms each matrix into upper triangular form. The results of [3, 4, 5] thus “activate” the aforementioned stability analysis result into a control design technique.

A central contribution in [3] is an iterative design algorithm that searches for a set of stabilising feedback matrices that attain the target simultaneously triangularisable closed-loop structure via the application of a common eigenvector assignment (CEA) procedure and state dimension reduction at each iteration. The main theoretical result in [3] establishes that the proposed algorithm will be successful until the state dimension is reduced to 1 if and only if feedback matrices exist so that the corresponding closed-loop subsystem matrices are stable and simultaneously triangularisable, i.e. if and only if feedback matrices exist so that the closed-loop system satisfies the aforementioned Lie-algebraic stability condition. Also in [3], a numerical implementation for the proposed iterative design algorithm and CEA procedure are provided. A key structural condition also is provided which, when satisfied, guarantees a directly computable solution for the CEA procedure. If this structural condition is not satisfied, then the required quantities are sought by means of an optimisation problem.

The aforementioned Lie-algebraic stability condition is (a) restrictive and (b) non-robust, in the sense that (a) it is satisfied for a very limited number of autonomous switching systems and (b) even if it is satisfied for a given system, it is almost surely not satisfied by systems with parameters arbitrarily close to the given one. The work in [4] then provides a robust result by relaxing, for single input systems, the simultaneous triangularisation requirement to approximate (in a specific sense) simultaneous triangularisation. The main theoretical contribution in [4] estab-
lishes that if a system satisfying the aforementioned Lie-algebraic condition exists in a suitably small neighbourhood of the given system data, then the proposed algorithm is guaranteed to find feedback matrices so that the corresponding closed-loop DTSS admits a CQLF even if the Lie-algebraic condition is not met by the given system data.

Even if the aforementioned Lie-algebraic condition is restrictive for autonomous switching systems, the existence of feedback controls causing the corresponding closed-loop system to satisfy such Lie-algebraic condition need not be such a restrictive condition. The restrictiveness of this condition is related to the key structural condition provided in [3]: if such structural condition is satisfied at every iteration of the algorithm, then the problem may be not restrictive at all for systems with the given dimensions. In this regard, the main result in [5] is the identification of the situation that prevents the structural condition from holding at every iteration of the algorithm.

In the present paper, we build upon the results of [5] by providing sufficient conditions for the structural condition to hold at every iteration of the algorithm for almost every set of system parameters with the given dimensions. We thus provide sufficient conditions for the genericity of the property of existence of feedback matrices so that the closed-loop subsystem matrices are stable and generate a solvable Lie algebra.

Notation. The index set \{1,2,\ldots,N\} is denoted \(\mathbb{N}\). The kernel (null space) of a matrix or linear map \(A\) is denoted ker \(A\), its image (range), img \(A\), and its spectral radius, \(\rho(A)\). For \(x \in \mathbb{C}^{n \times m}\), its transpose is denoted \(x'\), its conjugate transpose \(x^*\) and its Moore-Penrose generalised inverse \(x^+\). If \(S, T\) are vector spaces, then \(S \subset T\) means that \(S\) is a subspace of \(T\) and \(d(S)\) denotes the dimension of \(S\).

\section{Problem Formulation}

Consider the DTSS

\[ x_{k+1} = A_{i(k)}x_k + B_{i(k)}u^{i(k)}_k, \]  

where \(x_k \in \mathbb{R}^n\) and \(u^{i(k)}_k \in \mathbb{R}^m\) for all \(k\), \(i(k)\) takes values in \(\mathbb{N}\) for all \(k\), the matrices \(A_i \in \mathbb{R}^{n \times n}\) and \(B_i \in \mathbb{R}^{n \times m}\) are known for all \(i \in \mathbb{N}\), \(B_i\) have full column rank, and \((A_i, B_i)\) is controllable for all \(i \in \mathbb{N}\). We are interested in state-feedback control design of the form

\[ u^{i(k)}_k = K_{i(k)}x_k, \]  

so that the resulting closed-loop system

\[ x_{k+1} = A^{cl}_{i(k)}x_k, \quad \text{where} \quad A^{cl}_i = A_i + B_iK_i, \quad \text{for} \ i \in \mathbb{N}, \]

admit a CQLF and hence be stable under arbitrary switching. Note that at every time instant \(k\), the control law (2) requires knowledge of the “active” subsystem given by \(i(k)\).

As is well-known, ensuring that \(\rho(A^{cl}_i) < 1\) for \(i \in \mathbb{N}\) is necessary but not sufficient to ensure the stability of the DTSS for arbitrary switching. A sufficient condition is given by the following result, which is a minor modification of [11] Theorem 6.18.

\begin{lemma}[Lie-algebraic-solvability stability condition] If \(\rho(A^{cl}_i) < 1\) for \(i \in \mathbb{N}\), and the Lie algebra generated by \(\{A^{cl}_i : i \in \mathbb{N}\}\) is solvable, then \(\mathbb{R}^{n \times n}\) admits a common quadratic Lyapunov function and hence is exponentially stable. \(\square\)
\end{lemma}

In this paper, we specifically consider stabilising state feedback design for the DTSS based on the Lie-algebraic-solvability condition of Lemma 1 and thus focus on the DTSS class defined next.

\begin{definition}[SLASF] A set \(Z = \{(A_i \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n \times m}) : i \in \mathbb{N}\}\) is said to be SLASF (Solvable Lie Algebra with Stability by Feedback) if there exist \(K_i \in \mathbb{R}^{m \times n}\) such that \(A^{cl}_i\) as in (4) generate a solvable Lie algebra and satisfy \(\rho(A^{cl}_i) < 1\). \(\square\)
\end{definition}

In matrix terms, the fact that the Lie algebra generated by the matrices \(A^{cl}_i\) is solvable is equivalent to the existence of an invertible matrix \(T \in \mathbb{C}^{n \times n}\) such that \(T^{-1}A^{cl}_iT\) is upper triangular for \(i \in \mathbb{N}\). That is, each matrix \(A^{cl}_i\) is similar to an upper triangular matrix under a common similarity transformation \(T\). Note that even if the matrices \(A^{cl}_i\) have real entries, those of \(T\) may be complex [2].

\section{Previous Results}

Control design that causes the closed-loop system to be stable by satisfying the conditions of Lemma 1 can be performed iteratively by seeking feedback matrices that assign a common eigenvector with stable corresponding eigenvalues, and reducing the state-space dimension by 1 at every iteration [6,3]. This methodology is given in pseudocode below as Algorithm 1.

Algorithm 1 seeks feedback matrices \(K_1\) so that the closed-loop matrices \(A^{cl}_i\) given by (4) are stable and simultaneously triangularisable.

\subsection{The Algorithm}

Algorithm 1 begins by setting internal matrices equal to the subsystem matrices of the DTSS to be stabilised \((A^1_1 = A_1, B^1_1 = B_1\) at the Initialisation step). At every iteration \(\ell\) indicates iteration number, see [3], the algorithm executes Procedure CEA [see (5)] on its internal system matrices (the latter matrices are \(A^\ell_i\) and \(B^\ell_i\)). Procedure CEA aims to compute a vector, \(v^\ell_i\), and corresponding feedback matrices, \(F^\ell_i\), so that \(v^\ell_i\) is a feedback-assignable unit eigenvector common to all internal subsystems, with corresponding stable eigenvalues. That is, if Procedure CEA is successful, then \(v^\ell_i\) will satisfy \(|v^\ell_i| = 1\) and \((A^\ell_i + B^\ell_iF^\ell_i)v^\ell_i = \lambda^\ell_i v^\ell_i\) for some scalars \(\lambda^\ell_i\) satisfying \(|\lambda^\ell_i| < 1\), for all \(i \in \mathbb{N}\). Algorithm 1 then computes internal closed-loop matrices \((A^{cl}_{1,\ell,\ell}, \text{in (7)})\), updates internal feedback matrices
Algorithm 1: Iterative triangularisation

**Data:** $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m_i}$ for $i \in \mathbb{N}$

**Output:** $K_i$ for $i \in \mathbb{N}$

**begin**

 Initialization

$A_i^0 \doteq A_i$, $B_i^0 \doteq B_i$, $K_i^0 \doteq 0$, $U_i \doteq 1$, $\ell \leftarrow 0$ ;

repeat

$\ell \leftarrow \ell + 1$, $n_\ell \leftarrow n - \ell + 1$ (5)

$[v^1_\ell, [F^\ell]^N_{i=1}] \leftarrow \text{CEA}(A_i^\ell)^N_{i=1}, [B_i^\ell]^N_{i=1}$ (6)

$A_i^{\ell \text{cl}} \doteq A_i^\ell + B_i^\ell F_i^\ell$, (7)

$K_i^\ell \leftarrow K_i^{\ell - 1} + F_i^\ell \left( \prod_{v=1}^\ell U_v^\ell \right)$ (8)

if $\ell < n$ then

Construct a unitary matrix:

$[v^1_\ell | v^2_\ell | \cdots | v^N_\ell] \in \mathbb{C}^{n_\ell \times n_\ell}$. (9)

Assign $U_{\ell+1} \leftarrow [v^2_\ell | \cdots | v^N_\ell]$.

$A_i^{\ell + 1} \leftarrow U_{\ell+1}^* A_i^\ell U_{\ell+1}$, (11)

$B_i^{\ell + 1} \leftarrow U_{\ell+1}^* B_i^\ell$. (12)

until $\ell = n$;

$K_i \leftarrow K_i^n$ ;

end

$[K_i^n$ in (13), and then reduces the internal state dimension by 1. This reduction occurs at (9)–(12) $[n_\ell$ is the internal state dimension, see (13)]. Note that $v^1_\ell$ is the first column of the unitary matrix (9), and considering (10) then $U_{\ell+1}^* U_{\ell+1} = I$ and $U_{\ell+1}^* v^1_\ell = 0$. Algorithm 1 iterates until the internal state reaches dimension 1. If the given system matrices form a SLASF set (recall Definition 1), the matrices $K_i$ computed by Algorithm 1 will be the required feedback matrices.

If the given system matrices $A_i, B_i, i \in \mathbb{N}$, form a SLASF set, then at every iteration of Algorithm 1 a stable feedback-assignable common eigenvector $v^1_i$ is ensured to exist for the internal system with matrices $A_i^\ell, B_i^\ell$, for $i \in \mathbb{N}$. Conversely, if a feedback-assignable common eigenvector $v^1_i$ exists at every iteration of Algorithm 1 then the given system matrices form a SLASF set. The latter constitutes the main theoretical result that underpins our iterative control design algorithm 1.

3.2 The Procedure

As expressed in the previous paragraph, the existence of a feedback-assignable common eigenvector with corresponding stable eigenvalues is central to our development. This section recalls the structural condition introduced in (3) which, when satisfied, ensures that such a vector exists and allows its computation in a numerically efficient and straightforward way.

We introduce some notation required to state the aforementioned structural condition. Define $m_i^\ell \doteq \text{rank}(B_i^\ell) = \text{d}(\text{img} B_i^\ell)$, and factor $B_i^\ell = b_i^\ell r_i^\ell$, where $r_i^\ell : \mathbb{R}^{m_i^\ell} \rightarrow \mathbb{R}^{m_i^\ell}$ has full row rank and $b_i^\ell : \mathbb{R}^{m_i^\ell} \rightarrow \mathbb{R}^{n_\ell}$ has full column rank. We adopt the convention that $b_i^\ell$ is an empty matrix if $m_i^\ell = 0$. Note that $\text{img} B_i^\ell = \text{img} b_i^\ell$. Let $\Lambda_i^\ell$ be the vector with components $\lambda_i^\ell, i \in \mathbb{N}$, i.e.

$$\Lambda_i^\ell \doteq [\lambda_1^\ell, \lambda_2^\ell, \ldots, \lambda_N^\ell]^t,$$ (13)

and build the matrix

$$Q_i(\Lambda_i^\ell) \doteq [R_i(\Lambda_i^\ell), -B_i],$$ (14)

$$R_i(\Lambda_i^\ell) \doteq \begin{bmatrix} \lambda_1^\ell I - A_1^\ell \\ \vdots \\ \lambda_N^\ell I - A_N^\ell \end{bmatrix}, B_i \doteq \text{blkdag} [b_1^\ell, \ldots, b_N^\ell],$$

where blkdag denotes block diagonal concatenation.

**Lemma 2** (Structural condition [3, 5]). Let

$$p_\ell \doteq n_\ell + \sum_{i=1}^N m_i^\ell - N n_\ell.$$ (15)

Then,

(a) A vector that can be assigned by feedback as a common eigenvector with corresponding eigenvalues $\lambda_i^\ell$ for $i \in \mathbb{N}$ exists if and only if $\text{d}(\text{ker} Q_i(\Lambda_i^\ell)) > 0$.

(b) If $Q_i(\Lambda_i^\ell) w = 0$ with $w \neq 0$ partitioned as $w \doteq [w', u_1^\ell, \ldots, u_N^\ell]^t$, then $v \neq 0$, and

$$(A_i^\ell + B_i^\ell F_i^\ell) v = \lambda_i^\ell v, \quad \text{for } i \in \mathbb{N},$$ (17)

for every $F_i^\ell$ satisfying $r_i^\ell F_i^\ell v = u_i^\ell$. For each $i \in \mathbb{N}$ one such $F_i^\ell$ always exists and is given by $F_i^\ell = (r_i^\ell)^t u_i^\ell v^t$.

(c) $\text{d}(\text{ker} Q_i(\Lambda_i^\ell)) \geq p_\ell$ for every choice of $\Lambda_i^\ell$ as in (13). Consequently, if $p_\ell > 0$, then a feedback-assignable common eigenvector exists for every choice of corresponding eigenvalues.

**Lemma 2** gives a structural condition, namely $p_\ell > 0$, for a feedback-assignable common eigenvector $v$ to exist for every choice of corresponding eigenvalues $\lambda_i^\ell$. This condition is structural because the quantities involved in the computation of $p_\ell$ are only matrix ranks and dimensions. If the structural condition $p_\ell > 0$ is satisfied, a feedback-assignable common eigenvector $v^1_i$ as required at iteration $\ell$ of Algorithm 1 can be computed by (a) selecting its closed-loop eigenvalues $\lambda_i^\ell$ corresponding to each subsystem, (b) finding a vector $w \neq 0$ with components partitioned as in (13) so that $Q_i(\Lambda_i^\ell) w = 0$, i.e. so that $w \in \text{ker} Q_i(\Lambda_i^\ell)$, (c) taking the first $n_i^\ell$ components of $w$, i.e. the subvector $v$ in (16), and (d) computing $v^1_i = v/\|v\|$. The feedback matrices that assign such eigenvector with corresponding eigenvalues $\lambda_i^\ell$ can be obtained as $F_i^\ell = (r_i^\ell)^t u_i^\ell v^t$. 

□
An implementation of Procedure CEA is thus given below for the case when the structural condition of Lemma 2 is satisfied.

Even if the DTSS matrices $A_i, B_i$ have real entries, those of the matrices $A_i^\ell, B_i^\ell$ internal to Algorithm 1 can be complex at some iteration $\ell$. This is so because the vector $v_i^\ell$ returned by Procedure CEA (a feedback-assignable common eigenvector) can have complex components even if $(a$ feedback-assignable common eigenvector$)$ can have

From (19), then $m_i^\ell - 1$ when $m_i^\ell = n_\ell$, because necessarily in this case $v_i^\ell \in \mathbb{R}^{n_\ell}$ is $\text{img} B_i^\ell$. The following theorem and its corollary follow from (19) and were presented in [14].

**Theorem 1.** Consider Algorithm 1 at iteration $\ell$ and $p_\ell$ as in (15), with $m_i^\ell = \text{rank}(B_i^\ell)$. Then, $p_{\ell+1} \geq p_\ell - 1$, with equality if and only if

$$v_i^\ell \in \bigcap_{i \in N} \text{img} B_i^\ell.$$  \hspace{1cm} (20)

**Corollary 1.** Let $p_\ell > 0$. Then,

(a) $p_q > 0$ for $q = \ell, \ldots, \ell + p_\ell - 1$.

(b) $p_{\ell+1} > 0$ if $v_i^\ell \notin \text{img} B_k^\ell$ for some $k \in N$.

(c) $p_\ell \neq 0$ if and only if $p_\ell = 1$ and (20).

4 MAIN RESULTS

In this section, we derive conditions to ensure that the structural condition $p_\ell > 0$ will hold for $\ell = 1, \ldots, n$. We will achieve this goal by looking more deeply into the condition (20). In Section 4.2 we recall a property of subspaces that is required for the derivation of our main results in Section 4.3.

4.1 Transversality of Subspaces

We next recall the property of transversality of subspaces (see, e.g., Chapter 0 of [12]).

**Definition 2** (Transverse). Two subspaces $S, T$ of an ambient space $X$ are said to be transverse when the dimension of their intersection is minimal, i.e. when

$$d(S \cap T) = \max \{0, d(S) + d(T) - d(X)\}. \hspace{1cm} (21)$$

Equivalently, $S$ and $T$ are transverse when the dimension of their sum is maximal. We extend this definition to sets of subspaces as follows. Let $S = \{S_1, \ldots, S_N\}$ be a set of subspaces of an ambient space $X$. We say that $S$ is transverse when both the intersection of the subspaces in every subset of $S$ has minimal dimension and the sum of the subspaces in every subset of $S$ has maximal dimension.

The following properties of subspaces can be straightforwardly established.

**Lemma 3.** Let $S = \{S_1, \ldots, S_N\}$ be a set of subspaces of the ambient space $X$, and define

$$p \geq d(X) + \sum_{i \in N} d(S_i) - Nd(X).$$

Then,

(a) $d(S_i \cap S_j) = d(S_i) + d(S_j) - d(S_i + S_j)$.

(b) If $S$ is transverse, then $d\left(\bigcap_{i \in N} S_i\right) = \max \{0, p\}$.

(c) If $S$ is transverse and $p \geq 0$, then $d(S_i + S_j) = d(X)$ for all $i, j \in N$ with $i \neq j$.

(d) Let $J = I \cup \{j\}$, with $J \subset N$ and $\# J = \# I + 1$. Suppose that $p \geq 0$ and that $\{S_i : i \in I\}$ is transverse. Then, $\{S_i : i \in J\}$ is transverse if and only if $\bigcap_{i \in J} S_i + S_j = X$.  

**Procedure CEA (Structural condition satisfied)**

**Input:** $A_i^\ell \in \mathbb{R}^{n_i \times m_i}, B_i^\ell \in \mathbb{R}^{n_i \times m_i}$, for $i \in N$

**Output:** $v_i^\ell, F_i^\ell$ for $i \in N$

Factor $B_i^\ell = b_i^\ell u_i^\ell$ with $b_i^\ell \in \mathbb{R}^{n_i \times m_i}$ and $m_i^\ell = \text{rank}(B_i^\ell)$.

if $p_i = m_\ell + \sum_{i=1}^N m_i^\ell - Nn_\ell > 0$

Select $w_i \in \mathbb{R}$ so that $|w_i| < 1$;

Find $w \neq 0$ such that $Q_i\lambda_i^\ell w = 0$;

Partition $w$ as in (10):

$v_i^\ell = v_i\parallel v_i\parallel$;

$F_i^\ell = (v_i^\ell) u_i, v_i^\ell$, for $i \in N$;

then

$$v_i^\ell \in \bigcap_{i \in N} \text{img} B_i^\ell.$$  \hspace{1cm} (20)
Proof of Lemma 3(d). ($\Rightarrow$) For a set $K \subset \mathbb{N}$, define $p_K = \text{d}(X) + \sum_{i \in K} \text{d}(S_i) - \#K \cdot \text{d}(X)$. Since $p_{\emptyset} = p \geq 0$, then $p_{K} \geq 0$ and $p_{\emptyset} \geq 0$ because $\text{d}(S_i) \leq \text{d}(X)$ for all $i \in \mathbb{N}$. By Lemma 3(b) and since $p_{\emptyset} \geq 0$ and $p_{\emptyset} \geq 0$, then $\text{d}(\bigcap_{i \in I} S_i) = p_I$ and $\text{d}(\bigcap_{i \in J} S_i) = p_J$.

By Lemma 3(b), we have

$$
d(\bigcap_{i \in I} S_i) = \text{d}(\bigcap_{i \in I} S_i) + \text{d}(S_j) - \text{d}(\bigcap_{i \in I} S_i + S_j) = p_J = p_I + \text{d}(S_j) - \text{d}(\bigcap_{i \in I} S_i + S_j). \tag{22}
$$

Necessity is established by substituting the expressions for $p_I$ and $p_J$ into (22) and recalling that $\#J = \#I + 1$.

($\Leftarrow$) Let $K \subset I$. We have $\text{d}(X) = \text{d}(\bigcap_{i \in I} S_i + S_j) \leq \text{d}(\bigcap_{i \in I} K S_i + S_j) \leq \text{d}(X)$. Taking $K = \{k\}$, jointly with the fact that $\{S_i : i \in I\}$ is transverse, establishes that the dimension of the sum of the subspaces in every subset of $\{S_i : i \in J\}$ has maximum dimension. Also, we have

$$
d(\bigcap_{i \in K} S_i \cap S_j) = \text{d}(\bigcap_{i \in K} S_i) + \text{d}(S_j) - \text{d}(\bigcap_{i \in K} S_i + S_j), \tag{23}
$$

which, jointly with the fact that $\{S_i : i \in I\}$ is transverse, establishes that the dimension of the intersection of the subspaces in every subset of $\{S_i : i \in J\}$ has minimum dimension.

As is well known [12], the property of transversality is generic, i.e. it is satisfied for almost every set $S$ composed of a finite number of subspaces of $X$ selected “randomly” among all subspaces of $X$.

4.2 Genericity of the SLASF Property

As previously mentioned, we will look more deeply into the condition (20). We define the following

$$
B_{\ell}^i = \text{img} B_{\ell}^i, \quad B^i = \bigcap_{i \in \mathbb{N}} B_{\ell}^i. \tag{24}
$$

According to (23), then (20) can be rewritten as $v^i \in B^i$. Recall that $v^i$ is a feedback-assignable common eigenvector. The following result is straightforward.

Lemma 4. Let $S_{\ell}^i$ be the set of vectors $v \in B_{\ell}^i$ for which there exist $B_{\ell}^i$ and $\lambda$ with $|\lambda| < 1$ so that

$$
(A_{\ell}^i + B_{\ell}^i F_{\ell}^i) v = \lambda v. \tag{24}
$$

(a) The set $S_{\ell}^i$ is a subspace.

(b) $v \in S_{\ell}^i$ if and only if $v \in B_{\ell}^i$ and $A_{\ell}^i v \in B_{\ell}^i$.

By definition, $S_{\ell}^i$ is the set of feedback-assignable eigenvectors for the subsystem $(A_{\ell}^i, B_{\ell}^i)$ that are contained in $B_{\ell}^i$. Consequently, $v^i \in B_{\ell}^i$ if and only if $v^i \in S_{\ell}^i$. In the sequel, we will employ the following.

$$
\rho_{\ell}^i = \text{d}(S_{\ell}^i), \quad q_{\ell} = n_{\ell} + \sum_{i \in \mathbb{N}} \rho_{\ell}^i - N n_{\ell}, \tag{25}
$$

$$
S^i = \bigcap_{i \in \mathbb{N}} S_{\ell}^i, \quad \rho^i = \text{d}(S^i). \tag{26}
$$

Lemma 5 below relates the dimension of the subspace $S_{\ell}^i$ to the controllability indices of $(A_{\ell}^i, B_{\ell}^i)$.

Lemma 5. The dimension $\text{d}(S_{\ell}^i)$ equals the number of controllability indices equal to 1 of $(A_{\ell}^i, B_{\ell}^i)$.

Proof. According to the standard construction for the controllability indices of a system (see, e.g. [12]), it follows that the number of controllability indices equal to 1 of $(A_{\ell}^i, B_{\ell}^i)$ is given by $2 m_{\ell} - \text{rank}[\beta_{\ell}^i, A_{\ell}^i \beta_{\ell}^i]$, where $\beta_{\ell}^i$ is any matrix satisfying $\text{img} \beta_{\ell}^i = \text{img} B_{\ell}^i$. Since $S_{\ell}^i \subset B_{\ell}^i$, write $B_{\ell}^i = B_{\ell}^i \oplus S_{\ell}^i$ and let $\alpha = \text{d}(B_{\ell}^i)$. Then, $\rho_{\ell}^i = m_{\ell} - \alpha$. Let $\{b_1, \ldots, b_m\}$ be a basis for $B_{\ell}^i$, $\{b_{\alpha + 1}, \ldots, b_{m_{\ell}}\}$ be a basis for $S_{\ell}^i$, and $\beta_{\ell}^i = \{b_1, \ldots, b_{m_{\ell}}\}$. By Lemma 3(b), $A_{\ell}^i b_k \notin B_{\ell}^i$ for $k = 1, \ldots, \alpha$ and $A_{\ell}^i b_k \in B_{\ell}^i$ for $k = \alpha + 1, \ldots, m_{\ell}$. Therefore, $\text{rank}[\beta_{\ell}^i, A_{\ell}^i \beta_{\ell}^i] \leq m_{\ell} - \alpha$. If $\text{rank}[\beta_{\ell}^i, A_{\ell}^i \beta_{\ell}^i] = m_{\ell} - \alpha$, then $\sum_{j=1}^{\alpha} c_j b_j + \sum_{k=\alpha+1}^{m_{\ell}} d_k A_{\ell}^i b_k = 0$ for some scalars $c_j$ and $d_k$, where not all the $d_k$ are zero. Then, $A_{\ell}^i \sum_{j=1}^{\alpha} c_j b_j + A_{\ell}^i \sum_{k=\alpha+1}^{m_{\ell}} d_k b_k \notin S_{\ell}^i$, a contradiction. Therefore, $\text{rank}[\beta_{\ell}^i, A_{\ell}^i \beta_{\ell}^i] = m_{\ell} - \alpha$. \hfill $\Box$

Our main result is given below as Theorem 2. We will provide comments and explanations after its proof. The proof of Theorem 3 requires an additional result, given as Lemma 6.

Theorem 2. Let $\{S_{\ell}^i : i \in \mathbb{N}\}$ be transverse, and $q_{\ell} > 0$. Then, $\text{d}(S_{\ell}^i) = 0$ for $\ell = 1, \ldots, n$.}

Lemma 6. Consider Algorithm 7 at iteration $\ell$. Suppose that $(A_{\ell}^i, B_{\ell}^i)$ is controllable and $A_{\ell}^{\text{cl}} v^i = \lambda v^i$ with $v^i \neq 0$ and scalar $\lambda^i$. Then, $S_{\ell+1}^i 

\rho_{\ell+1}^i = \begin{cases} 
\rho_{\ell}^i - 1 & \text{if } v^i \in S_{\ell}^i, \\
\rho_{\ell}^i & \text{otherwise.} 
\end{cases} \tag{27}
$$

Proof. Let $\{t_j : j = 1, \ldots, m_{\ell}\}$ be a basis for $B_{\ell}^i$ and let $\kappa_{j,i}$, for $j = 1, \ldots, m_{\ell}$, be the controllability indices of $(A_{\ell}^i, B_{\ell}^i)$. By (27) and the feedback invariance of controllability indices, $\kappa_{j,i}$ also are the controllability indices of the pair $(A_{\ell}^{\text{cl}}, B_{\ell}^i)$. Since $(A_{\ell}^{\text{cl}}, B_{\ell}^i)$ is controllable, then $(A_{\ell}^{\text{cl}}, B_{\ell}^i)$ also is controllable, and $D = \{(A_{\ell}^{\text{cl}})^k t_j : j = 1, \ldots, m_{\ell}, k = 0, \ldots, \kappa_{j,i} - 1\}$ is a basis for $\mathbb{R}^{q_{\ell}}$. Write $v^i$ with respect to the basis $D$: $v^i = \sum_{j,k} c_{j,k}(A_{\ell}^{\text{cl}})^k t_j$, where not all the $c_{j,k}$ are zero. Combining the latter with $A_{\ell}^{\text{cl}} v^i = \lambda^i v^i$ yields

$$
\sum_{j,k} c_{j,k}(A_{\ell}^{\text{cl}})^k t_j = \sum_{j,k} \lambda^i c_{j,k}(A_{\ell}^{\text{cl}})^k t_j. \tag{28}
$$

From (28), it follows that $c_{j,k} \neq 0$ for at least one pair of indices $(j, k)$ such that $k = \kappa_{j,i} - 1$, or otherwise the vectors in $D$ would be linearly dependent, a contradiction. Let $k = \max_{j}(\kappa_{j,i} - 1)$, and let $i$ be such that $c_{i,k} \neq 0$. From
Proof of Theorem 2. Let $T = \{S_i^{\ell+1} : i \in I\}$, with $I \subset \mathbb{N}$ and $\#I = \alpha$, and consider $R = T \cup \{S_i^{\ell+1}\}$ so that $\#R = \alpha + 1$. By Lemma 5 and properties of maps and subspaces, we have

$$\bigcup_{i \in I} S_i^{\ell+1} + S_j^{\ell+1} \supset \bigcup_{i \in I} U_i^{\ell+1} S_i^{\ell+1} + U_j^{\ell+1} S_j^{\ell+1} \supset U_i^{\ell+1} \bigcup_{i \in I} S_i^{\ell+1} + S_j^{\ell+1}. \quad (31)$$

By (30) and since $I \subset \mathbb{N}$, then $\{S_i^{\ell+1} : i \in I\}$ is transverse. By Lemma 5, then $d(\bigcap_{i \in I} S_i^{\ell+1} + S_j^{\ell+1}) = n_\ell$. Combining the latter equality with (31), then $R$ is transverse. We have thus established that our induction hypothesis is valid for $\alpha + 1$ and we conclude that $\{S_i^{\ell+1} : i \in \mathbb{N}\}$ is transverse. By Lemma 1, then $\rho^{\ell+1} = \max(0, q_{\ell+1})$. From (25) and (27), it follows that the minimum value for $q_{\ell+1}$ is $q_\ell - 1$, and this happens only if $\rho^{\ell+1} = \rho_\ell - 1$ for all $i \in \mathbb{N}$. However, if $q_\ell = 0$, then $\rho_\ell^{\ell+1} \geq \rho_\ell^{\ell}$ for at least one $i \in \mathbb{N}$ because, since $\rho_\ell = d(S_i^{\ell+1}) = q_\ell$, then $v_i \notin S_i^{\ell+1}$. Consequently $q_{\ell+1} \geq 0$ and hence we have established (30) for $\ell = 1, \ldots, \alpha$.

Theorem 2 gives a condition, namely $\{S_i^{\ell} : i \in \mathbb{N}\}$ transverse and $q_\ell \geq 0$, under which the structural condition of Lemma 1 is satisfied at every iteration of Algorithm 1. The quantity $q_\ell$ depends on the dimensions of the subspaces $S_i^{\ell}$ for $i \in \mathbb{N}$ which, by Lemma 5 equals the number of controllability indices equal to one of $(A_i^{\ell+1}, B_j^{\ell+1})$. A consequence of Corollary 5.4 of [12] is that the latter number is generically (i.e. for almost every matrices $A_i$ and $B_i$) nonzero when $m_i > n/2$ and if the latter holds, generically equal to $m_i - (n \mod m_i)$. If $m_i > n/2$, arbitrary choices for the entries of $A_i$ and $B_i$ yield arbitrary $S_i^{\ell}$, although generically of dimension $m_i - (n \mod m_i)$. Therefore, if the system dimensions $m$ and $n$ for $i \in \mathbb{N}$ are such that $q_\ell \geq 0$, then $\{S_i^{\ell} : i \in \mathbb{N}\}$ will be transverse generically in the space of parameters of the matrices $A_i$ and $B_i$, for $i \in \mathbb{N}$. For example, a DTSS with two subsystems ($N = 2$), order $n = 6$, subsystem 1 having 4 inputs ($m_1 = 4$) and subsystem 2 having 5 inputs ($m_2 = 5$) will generically satisfy the hypotheses of Theorem 2 since $\rho_1 = m_1 - (n \mod m_1) = 2$, $\rho_2 = 4$ and hence $q_1 = 6 + (2+4) - 2 - 6 = 0$. From the preceding analysis, it follows that the hypotheses of Theorem 2 can hold only for DTSSs where each subsystem has “a lot of” inputs ($m_i > n/2$, $q_i \geq 0$).

5 CONCLUSIONS

We have addressed feedback stabilisation of discrete-time switched linear systems with control inputs. The control strategy employed is to seek feedback matrices so that the closed-loop subsystem matrices are stable and generate a solvable Lie algebra. The problem of feedback stabilisation by means of the latter strategy...
is known to not always have a solution. In this context, we have derived conditions under which this problem is ensured to have a solution for most possible sets of parameters. These conditions hold only for DTSSs of specific system dimensions, where each subsystem has a considerable number of inputs, as compared with the system dimension.

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