KNOTS, OPERADS AND DOUBLE LOOP SPACES

PAOLO SALVATORE

ABSTRACT. We show that the space of long knots in an euclidean space of dimension larger than three is a double loop space, proving a conjecture by Sinha. We also construct a double loop space structure on framed long knots, and show that the map forgetting the framing is not a double loop map in odd dimension. However there is always such a map in the reverse direction expressing the double loop space of framed long knots as a semidirect product. A similar compatible decomposition holds for the homotopy fiber of the inclusion of long knots into immersions. We also show via string topology that the space of closed knots in a sphere, suitably desuspended, admits an action of the little 2-discs operad in the category of spectra. A fundamental tool is the McClure-Smith cosimplicial machinery, that produces double loop spaces out of topological operads with multiplication.

1. Introduction

The space $Emb_n$ of long knots in $\mathbb{R}^n$ is the space of embeddings $\mathbb{R} \to \mathbb{R}^n$ that agree with a fixed inclusion of a line near infinity. The space $Emb_n$ is equipped with the Whitney topology, and it can be identified up to homotopy with the subspace of based knots in $S^n$ with fixed derivative at the base point. The proof that $Emb_2$ is contractible goes back to Smale. The components of $Emb_3$ correspond to classical knots. The homotopy type of those components has been completely described by Ryan Budney [3]. For $n > 3$ the space $Emb_n$ is connected by Whitney’s theorem. The rational homology of $Emb_n$ for $n > 3$ has been recently computed by Lambrechts, Turchin and Volic [12].

Rescaling and concatenation defines a natural product on the space of long knots that is associative up to higher homotopies. Thus $Emb_n$ is an $A_\infty$-space and in the case $n > 3$, being connected, has the homotopy type of a loop space. The product is homotopy commutative, essentially by passing one knot through the other. This suggested that $Emb_n$ could be (up to weak equivalence) a double loop space. Budney and Sinha proved that two spaces closely related to $Emb_n$ are double loop spaces, for $n > 3$, by different approaches. A framed long knot in $\mathbb{R}^n$ is a long knot in $\mathbb{R}^n$ together with a choice of framing $\mathbb{R} \to SO(n)$, standard near infinity, such that the first vector of the framing gives the unit tangent vector map $\mathbb{R} \to S^{n-1}$ of the knot. Budney shows in [2] that the space $fEmb_n$ of framed long knots in $\mathbb{R}^n$ is a double loop space for $n > 3$. This is achieved by constructing an explicit action of the little 2-cubes operad on a space homotopy equivalent to the group-like space $fEmb_n$. The operad action is also defined for $n = 3$, and makes $fEmb_3$ into a free 2-cubes algebra on the non-connected space of prime long knots. Sinha shows in [18] that the homotopy fiber $Emb'_n$ of the unit tangent vector map $Emb_n \to \Omega S^{n-1}$ is a double loop space, and the map is nullhomotopic. His approach goes via the cosimplicial machinery by McClure and Smith [15] that
produces double loop spaces out of non-symmetric operads in based spaces. Under
this correspondence $\text{Emb}_n'$ is produced by an operad equivalent to the little $n$-discs
operad, the Kontsevich operad. We show that this machinery, applied to an operad
equivalent to the framed little $n$-discs operad, gives a double loop space structure
on framed long knots in $\mathbb{R}^n$, that presumably coincides with the one described by
Budney. We believe that the fact that the framed little discs is a cyclic operad $\mathbb{H}$
together with the McClure-Smith machinery for cyclic objects will lead to a framed
little 2-discs action on framed long knots.

Let us consider the principal fibration

$$\Omega SO(n - 1) \to f\text{Emb}_n \to \text{Emb}_n$$

forgetting the framing. Such fibration is trivial because its classifying map $\text{Emb}_n \to
SO(n - 1)$ is the composite of the (nullhomotopic) unit tangent vector map and the
holonomy $\Omega S^{n-1} \to SO(n - 1)$. Given the splittings

(1) \[ \text{Emb}_n' \simeq \text{Emb}_n \times \Omega^2 S^{n-1} \]

and

(2) \[ f\text{Emb}_n \simeq \text{Emb}_n \times \Omega SO(n - 1) \]

Sinha asked in $[18]$ whether one could restrict the double loop structure to the first
factor. We answer this affirmatively.

**Theorem 1.** The space $\text{Emb}_n$ of long knots in $\mathbb{R}^n$ is a double loop space for $n > 3$.

The double loop space structure is not produced directly from an operad as hoped
in $[18]$, but is deduced by diagram chasing on a diagram of cosimplicial spaces.

The splittings (1) and (2) respect the single loop space structures but not the
double loop space structures, as the projections on the factor $\text{Emb}_n$ are not double
loop maps in general.

**Theorem 2.** The map forgetting the framing $f\text{Emb}_n \to \text{Emb}_n$ and the map from
the homotopy fiber $\text{Emb}_n' \to \text{Emb}_n$ are not double loop maps for $n$ odd.

We prove this by showing that the maps in question do not preserve the Browder
operation, a natural bracket on the homology of double loop spaces. This is based
on computations by Turchin $[19]$.

There are instead double loop maps $\text{Emb}_n \to \text{Emb}_n'$ and $\text{Emb}_n \to f\text{Emb}_n$ that
together with the fiber inclusions $\Omega^2 S^{n-1} \to \text{Emb}_n'$ and $\Omega SO(n - 1) \to f\text{Emb}_n$
produce essentially semidirect product extensions of double loop spaces. We state
this precisely in the following theorem.

**Theorem 3.** There is a commutative diagram of double loop spaces and double
loop maps

\[
\begin{array}{ccc}
\text{Emb}_n & \longrightarrow & \text{Emb}_n' \\
\| & & \| \\
\text{Emb}_n & \longrightarrow & f\text{Emb}_n \\
\| & & \| \\
& \longrightarrow & \Omega^2 S^{n-1} \\
& \longrightarrow & \Omega SO(n - 1)
\end{array}
\]

The rows deloop twice to fibrations with sections, and the vertical maps are in-
duced by the holonomy $\Omega S^{n-1} \to SO(n - 1)$.
Also this theorem develops the approach by Sinha. The double loop spaces and double loop maps are produced by applying the McClure-Smith machinery to suitable operads and operad maps.

At the end of the paper we apply ideas from string topology to show that the shifted homology of the space $\text{Emb}(S^1, S^n)$ of all knots in the $n$-sphere behaves as the homology of a double loop space. More precisely this structure is induced by the action of an operad equivalent to the little 2-cubes at the spectrum level rather than at the space level.

**Theorem 4.** The spectrum $\Sigma^{1-2n} \Sigma^\infty \text{Emb}(S^1, S^n)_+$ is an $E_2$-ring spectrum.

A similar result has been obtained independently by Abbaspour-Chataur-Kallel. The case $n = 3$ is joint work with Kate Gruher [8].

Here is a plan of the paper: in section 2 we recall some background material on operads, cosimplicial spaces and prove theorem 1. In section 3 we study the space of framed knots via cosimplicial techniques and prove theorem 3. In section 4 we recollect some material on the Deligne conjecture and we give a proof of theorem 2. In the last section 5 we develop the string topology of knots proving theorem 4.

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### 2. Cosimplicial spaces and knots

We recall that a topological operad $O$ is a collection of spaces $O(k)$, $k \geq 0$, together with a unit $\iota \in O(1)$ and composition maps

$$\circ_t : O(k) \times O(l) \to O(k + l - 1)$$

for $1 \leq t \leq k$ satisfying appropriate axioms [14]. The operad is symmetric if the symmetric group $\Sigma_k$ acts on $O(k)$ for each $k$, compatibly with the composition maps. We say that a space $A$ is acted on by an operad $O$, or it is an $O$-algebra, if we are given maps $O(n) \times A^n \to A$ satisfying appropriate associativity and unit axioms [14]. The concepts of (symmetric) operads and their algebras can be defined likewise in any (symmetric) monoidal category.

Let $F(\mathbb{R}^n, k)$ be the ordered configuration space of $k$ points in $\mathbb{R}^n$. The direction maps $\theta_{ij} : F(\mathbb{R}^n, k) \to S^{n-1}$ are defined for $i \neq j$ by

$$\theta_{ij}(x_1, \ldots, x_n) = (x_i - x_j) / |x_i - x_j|.$$

Let us write $B_n(k) = (S^{n-1})^{k(k-1)/2}$. We can think of $B_n(k)$ as the space of formal 'directions' between $k$ distinct points in $\mathbb{R}^n$, where the directions are indexed by distinct pairs of integers between 1 and $k$. By convention we set $B_n(1)$ and $B_n(0)$ equal to a point.

**Proposition 5.** [18] The collection $B_n(k)$ forms a symmetric topological operad.

The action of the symmetric group $\Sigma_k$ on $B_n(k)$ permutes both indices. Intuitively the operad composition replaces a point by an infinitesimal configuration and relabels the points. More precisely we must specify the composition rule
\( \circ_t : B_n(k) \times B_n(l) \to B_n(k + l - 1) \) for \( 1 \leq t \leq k \). For elements \( \alpha = (\alpha_{ij})_{1 \leq i < j \leq k} \) and \( \beta = (\beta_{ij})_{1 \leq i < j \leq l} \) the composition is

\[
(\alpha \circ_t \beta)_{ij} = \begin{cases} 
\alpha_{ij} & \text{for } i < j \leq t \\
\beta_{i-t+1,j-t+1} & \text{for } t \leq i < j \leq t + l - 1 \\
\alpha_{i-t+1,j-t+1} & \text{for } t + l \leq i < j \\
\alpha_{i,t} & \text{for } i < t \leq j < t + l \\
\alpha_{t,j} & \text{for } t \leq i < t + l \leq j
\end{cases}
\]

Let \( \theta^k : F(\mathbb{R}^n, k) \to B_n(k) \) be the product of all direction maps \( \theta^k(x) = (\theta_{ij}(x))_{1 \leq i < j \leq n} \). For \( k \geq 2 \) let \( K_n(k) \subset B_n(k) \) be the closure of the image of \( \theta^k \). We set also \( K_n(0) = B_n(0) = \{ * \} \) and \( K_n(1) = B_n(1) = \{ t \} \). The restriction \( \theta^k : F(\mathbb{R}^n, k) \to K_n(k) \) is a \( \Sigma_k \)-equivariant homotopy equivalence.

**Proposition 6.** [18] The collection \( K_n(k) \) forms a suboperad of \( B_n(k) \) that is weakly equivalent to the little \( n \)-discs operad.

The operad \( K_n \) is known as the Kontsevich operad.

We say that a non-symmetric topological operad has a multiplication if there is a choice of base points \( m_k \in O(k) \) for each \( k \) such that the structure maps are based maps. This is the same as a non-symmetric operad in based spaces.

The operads \( B_n \) and \( K_n \) have a multiplication, defined by setting all components \( \theta_{ij} \) of the base points \( m_k \) equal to a fixed direction. We choose the last vector of the canonical basis of \( \mathbb{R}^n \) as fixed direction.

We recall the definition of a cosimplicial space. Let \( \Delta \) be the category with standard ordered sets \( [k] = \{ 0 < \cdots < k \} \) as objects (\( k \in \mathbb{N} \)) and monotone maps as morphisms. A cosimplicial space is a covariant functor from the category \( \Delta \) to the category of topological spaces. For each \( k \) the simplicial set \( \Delta([-k]) \) is also called the simplicial \( k \)-simplex \( \Delta^k \). Its geometric realization is the standard \( k \)-simplex \( \Delta^k \). All simplexes fit together to form a cosimplicial space. In fact if we apply geometric realization to the bisimplicial set (functor from \( \Delta \) to simplicial sets) \( \Delta(\ast, \ast) \) we obtain a cosimplicial space denoted by \( \Delta^\ast \).

The totalization \( \text{Tot}(S^\ast) \) of a cosimplicial space \( S^\ast \) is the space of natural transformations \( \Delta^\ast \to S^\ast \). There is a standard cosimplicial map \( \Delta^\ast \to \Delta^\ast \) where \( \Delta^\ast \) is an appropriate cofibrant resolution. The homotopy totalization \( \overline{\text{Tot}}(S^\ast) \) is the space of natural transformations \( \Delta^\ast \to S^\ast \). This is also the homotopy limit of the functor from \( \Delta \) to spaces defining the cosimplicial space. Precomposition induces a canonical map \( \text{Tot}(S^\ast) \to \overline{\text{Tot}}(S^\ast) \) that is a weak equivalence when \( S^\ast \) is fibrant, in the sense that it satisfies the matching condition [10].

An operad \( (O, p) \) with multiplication defines a cosimplicial space \( O^\ast \) sending \( [k] \) to \( O(k) \). The cofaces \( d^i : O(k) \to O(k + 1) \) is defined by

\[
d^i(x) = x \circ_i m_2 \quad \text{for } 1 \leq i \leq k
\]
\[
d^0(x) = m_2 \circ_1 x
\]
\[
d^{n+1}(x) = m_2 \circ_2 x.
\]

The codegeneracies \( s^i : O(k) \to O(k - 1) \) are defined by \( s^i(x) = x \circ_i m_0 \).

**Theorem 7.** (McClure-Smith) Let \( O \) be an operad with multiplication. Then the totalization \( \text{Tot}(O^\ast) \) (respectively the homotopy totalization \( \overline{\text{Tot}}(O^\ast) \)) admits an
action of an operad \( \mathcal{D}_2 \) (respectively \( \hat{\mathcal{D}}_2 \)) weakly equivalent to the little 2-cubes operad.

By the recognition principle [14] if \( \text{Tot}(O^*) \) or \( \overline{\text{Tot}}(O^*) \) is connected then it is weakly equivalent to a double loop space.

Given a simplicial set \( S_* \), considered as simplicial space with discrete values, and a space \( X \), we obtain a cosimplicial space \( \text{map}(S_*, X) \), often denoted \( X^{S_*} \). If \( S_* \) is a simplicial based set and \( X \) is a based space then we obtain similarly a cosimplicial space \( \text{map}_*(X_*, S) \). Let us denote by \( |S| \) the geometric realization of \( S \). The following is standard.

**Proposition 8.** The adjoint maps of the evaluation maps
\[
\text{map}(|S|, X) \times \Delta_k \to \text{map}(S_k, X)
\]
induce a homeomorphism \( \text{map}(|S|, X) \cong \text{Tot}(\text{map}(S_*, X)) \). In the based version we obtain a homeomorphism from the based mapping space
\[
\text{map}_*(|S|, X) \to \text{Tot}(\text{map}_*(S_*, X)).
\]
The canonical maps from these totalizations to the homotopy totalizations are weak equivalences.

Let \( \Delta^k_+ \) be the simplicial \( k \)-simplex, and \( \partial \Delta^k_+ \) its simplicial subset obtained by removing the non-degenerate simplex in dimension \( k \) and its degeneracies. The quotient \( S^k_* := \Delta^k_+ / \partial \Delta^k_+ \) is the simplicial \( k \)-sphere.

**Proposition 9.** [18]

The cosimplicial space \( B^*_n \) is isomorphic to \( \text{Map}_*(S^n_2, S^{n-1}_*). \)

Namely \( B^*_n \) has a factor \( S^{n-1}_j \) for each pair \( 1 \leq i < j \leq k \) and \( \text{map}_*(S^n_2, S^{n-1}_*) \) has a sphere factor for each \( k \)-simplex of \( S^n_2 \), namely for each non-decreasing sequence of length \( k + 1 \) starting with 0 and ending with 2. Then \( i \) corresponds to the position of the last 0 and \( j \) to the position of the last 1. Propositions 8 and 9 imply the following corollary.

**Corollary 10.** The totalization \( \text{Tot}(B^*_n) \) is homeomorphic to \( \Omega^2(S^{n-1}) \).

There is also a cosimplicial space \( K^*_n \times S^{n-1} \), not defined by an operad with multiplication. This is constructed so that \( K^*_n \times S^{n-1} = K_n(k) \times (S^{n-1})^k \). Elements of this space can be thought of as configurations of \( k \) points in \( \mathbb{R}_n \), each labelled by a direction. The composition rule can be defined as follows, via the identification \( S^{n-1} = K_n(2) \). Given \( (x; v_1, \ldots, v_k) \in K_n(k) \times (S^{n-1})^k \), we define for \( 1 \leq i \leq k \)
\[
d^i(x; v_1, \ldots, v_k) = (x \circ_i v_i; v_1, \ldots, v_i, v_{i+1}, \ldots, v_k).
\]
Intuitively these cofaces double a point in the associated direction, at infinitesimal distance. The first and last cofaces add a point labelled by the preferred direction 'before' or 'after' the configuration and are defined by
\[
d^0(x; v_1, \ldots, v_k) = (m_2 \circ_1 x; v_1, \ldots, v_k, m_2)
\]
and
\[
d^{k+1}(x; v_1, \ldots, v_k) = (m_2 \circ_2 x; m_2, v_1, \ldots, v_k).
\]
The codegeneracies forget a point and are defined by
\[
s^i(x; v_1, \ldots, v_k) = (x \circ_i m_0; v_1, \ldots, v_i, v_{i+1}, \ldots, v_k).
\]
The very same rules define a cosimplicial space $B_n^* \times S^{n-1}$ with $B_n^k \times S^{n-1} = (S^{n-1})^{k(k-1)/2} \times (S^{n-1})^k$ so that $K_n^* \times S^{n-1} \subset B_n^* \times S^{n-1}$ is a cosimplicial subspace.

**Theorem 11.** (Sinha) The homotopy totalization of $K_n^* \times S^{n-1}$ is weakly equivalent to $Emb_n$.

The proof of this theorem relies on Goodwillie calculus. From now on we will mean by $Emb_n$ the space of smooth maps from the interval $I$ to the cube $I^n$ sending the extreme points of the interval to centers of opposite faces of the cube, with derivative orthogonal to the faces.

The weak equivalence $Emb_n \to \text{Tot}(K_n^* \times S^{n-1})$ is constructed as follows, by evaluating directions between points of the knot and tangents. Regard an element of the $k$-simplex as a sequence of real numbers $0 \leq x_1 \leq \cdots \leq x_k \leq 1$. There are maps $\beta_k : Emb_n \to \text{map}(\Delta^k, K_n(k) \times (S^{n-1})^k)$ defined by

$$\beta_k(f)(x_1, \ldots, x_k) = \{f^k(f(x_1), \ldots, f(x_k)), f'(x_1)/f'(x_1), \ldots, f'(x_k)/f'(x_k)\}$$

when $x_1 < \cdots < x_k$. If some $x_i = x_j$ for $i < j$ then we must replace the component $\theta_{ij} = f(x_j) - f(x_i)/f'(x_i)$ in the expression above by $f'(x_i)/f'(x_i)$. All maps $\beta_k$ fit together to define a map $\beta : Emb_n \to \text{Tot}(K_n^* \times S^{n-1})$. The composite with the standard map to the homotopy totalization is the desired weak equivalence.

Let us recall some background on homotopy fibers: the homotopy fiber of a based map $f : X \to Y$ is defined by the pullback square

$$
\begin{array}{ccc}
\text{Hofib}(f) & \longrightarrow & X \\
\downarrow & & \downarrow f \\
PY & \longrightarrow & Y
\end{array}
$$

with $PY$ the contractible space of paths in $Y$ sending 0 to the base point, and $ev$ the evaluation at the point 1. If $f$ is a fibration with fiber $F$ then there is a canonical homotopy equivalence $F \to \text{Hofib}(f)$ sending $x = (x, c)$ with $c$ the constant loop at the base point of $Y$. The homotopy fiber is homotopy invariant, namely given a commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow \simeq & & \downarrow \simeq \\
X' & \longrightarrow & Y'
\end{array}
$$

with the vertical arrows weak equivalences, then the induced map $\text{Hofib}(f) \to \text{Hofib}(f')$ is a weak equivalence. This is a special case of the homotopy invariance of homotopy limits (theorem 18.5.3 (2) in [10]).

**Corollary 12.** (Sinha) The homotopy fiber $Emb_n^*$ of the unit tangent vector map $u : Emb_n \to \Omega S^{n-1}$ is weakly equivalent to the homotopy totalization $\text{Tot}(K_n^*)$, and thus is a double loop space for $n > 3$. 

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18: Theorem 11.
Proof: The projection $K_n(k) \times (S^{n-1})^k \to (S^{n-1})^k$ defines a map of cosimplicial spaces $K_n^* \times S^{n-1} \to \text{map}_*(S^1, S^{n-1})$ and there is a commutative square

$$
\begin{array}{ccc}
\text{Emb}_n & \longrightarrow & \widetilde{T}\text{ot}(K_n^* \times S^{n-1}) \\
\downarrow u & & \downarrow \pi \\
\Omega S^{n-1} & \longrightarrow & \widetilde{T}\text{ot}(\text{map}_*(S^1, S^{n-1})).
\end{array}
$$

By theorem 18.5 (2) in [10], the homotopy totalization of a sequence of cosimplicial spaces $X^* \to Y^* \to Z^*$ that are levelwise fibrations is a fibration $\widetilde{T}\text{ot}X^* \to \widetilde{T}\text{ot}Y^* \to \widetilde{T}\text{ot}Z^*$.

Then we have a weak equivalence $\widetilde{T}\text{ot}(K_n^*) \to Hofib(\pi)$ and by homotopy invariance weak equivalences $\text{Emb}_n = Hofib(u) \simeq Hofib(\pi) \simeq \widetilde{T}\text{ot}(K_n^*)$. We conclude by theorem 7.

**Remark:** We may substitute $\Omega S^{n-1}$ in the statement above by the space $Imm(I, I^n)$ of immersions $I \to I^n$ with fixed values and tangent vectors at the boundary, and $u$ by the inclusion $\text{Emb}(I, I^n) \to Imm(I, I^n)$, because the unit tangent vector map induces the Smale homotopy equivalence $Imm(I, I^n) \simeq \Omega S^{n-1}$.

In the next lemma we identify the totalization of $B_n^* \times S^{n-1}$. There are standard simplicial inclusions $d_i^0 : \Delta_i^1 \to \Delta_i^2$ and $d_i^2 : \Delta_i^1 \to \Delta_i^2$ induced by the strictly monotone maps $[1] \to [2]$ avoiding respectively 2 and 0.

**Lemma 13.** The totalization of the levelwise fibration of cosimplicial spaces

$$
B_n^* \to B_n^* \times S^{n-1} \to \text{map}(\Delta_i^1/\partial \Delta_i^1, S^{n-1})
$$

is the fibration

$$
\text{map}_*(\Delta_i^2/\partial \Delta_i^2, S^{n-1}) \to \text{map}_*(\Delta_i^2/(d^0(\Delta_i^1) \cup d^2(\Delta_i^1)), S^{n-1}) \to \text{map}_*(\Delta_i^1/\partial \Delta_i^1, S^{n-1}).
$$

**Proof.** The space $B_n^k \times S^{n-1}$ has a factor $S^{n-1}$ for each pair $1 \leq i < j \leq k$ and a factor $S^{n-1}$ for each $1 \leq l \leq k$. The space

$$
\text{map}_*(\Delta_i^2/\partial \Delta_i^2, S^{n-1}) \cup \Delta_i^1, S^{n-1})
$$

has a factor $S^{n-1}$ for each non-decreasing sequence of length $k + 1$ containing 0, 1, 2 and a factor $S^{n-1}$ for each non-decreasing sequence of length $k + 1$ starting with 0, ending with 2, without 1’s. For these latter sequences $l$ corresponds to the last position containing a 0. For the former sequences we apply the same correspondence as in the proof of proposition 7.

**Proof of theorem 7.** If we map the sequence $K_n^* \to K_n^* \times S^{n-1} \to \text{map}_*(S^1, S^{n-1})$ to the sequence $B_n^* \to B_n^* \times S^{n-1} \to \text{map}_*(S^1, S^{n-1})$ we obtain a commutative diagram of cosimplicial spaces that at level $k$ is
\[
\begin{align*}
\mathcal{K}_n(k) & \longrightarrow (S^{n-1})^{k(k-1)/2} \\
\downarrow & \\
\mathcal{K}_n(k) \times (S^{n-1})^k & \longrightarrow (S^{n-1})^{k(k-1)/2} \times (S^{n-1})^k \\
\downarrow & \\
(S^{n-1})^k & \longrightarrow (S^{n-1})^k.
\end{align*}
\]

The homotopy totalization functor gives a diagram of spaces weakly equivalent to those in the diagram

\[
\begin{align*}
Emb'_n & \longrightarrow \Omega^2 S^{n-1} \\
\downarrow & \\
Emb_n & \longrightarrow P\Omega S^{n-1} \\
\downarrow & \\
\Omega S^{n-1} & \longrightarrow \Omega S^{n-1}.
\end{align*}
\]

Let us analyze the diagram of homotopy totalizations. By naturality the upper row is a map of algebras over the McClure-Smith operad $\overline{D}_2$, and then its homotopy fiber $F$ is also an algebra over $\overline{D}_2$. The homotopy fiber of the second row is weakly equivalent to $Emb_n$ by theorem 11 and because the target is contractible. The homotopy fiber of the third row is contractible. The homotopy fibers of the rows in a diagram whose columns are fibrations form a fibration (18.5.1 in [10]), so that $F \simeq Emb_n$. This space is connected by Whitney’s theorem for $n > 3$, and then by the recognition principle [14] is weakly equivalent to a double loop space. □

3. Framed knots and double loop fibrations

We start by some general considerations on framed knots. By definition $fEmb_n$ is the pullback

\[
\begin{align*}
fEmb_n & \longrightarrow Emb_n \\
\downarrow & \\
\Omega SO(n) & \longrightarrow \Omega S^{n-1}.
\end{align*}
\]

Actually $fEmb_n$ is homeomorphic to the homotopy fiber of the composite

\[
Emb_n \xrightarrow{\mathfrak{h}} \Omega S^{n-1} \xrightarrow{h} SO(n-1)
\]

of the holonomy $h$ and the unit tangent vector map $u$. The homeomorphism is induced by the projection $fEmb_n \rightarrow Emb_n$ and the map $fEmb_n \rightarrow P\Omega SO(n-1)$ considering the difference between the framing induced by the holonomy along the knot and the assigned framing of the framed knot. By naturality of the homotopy fiber construction the holonomy induces a map $Emb'_n \rightarrow fEmb_n$.

We will give next an operadic interpretation of framed knots. We recall [17] that a topological group $G$ acts on a topological operad $O$ if each $O(n)$ is a $G$-space and the operadic composition maps are $G$-equivariant. In other words $O$ is an operad
in the category of $G$-spaces. In such case one can define the semidirect product

\[ O \rtimes G \] with \(n\)-ary space \(O(n) \rtimes G^n\) and composition

\[ (p; g_1, \ldots, g_n) \circ (q; h_1, \ldots, h_m) = (p; g_1(q); g_1, \ldots, g_i h_1, \ldots, g_i h_m, \ldots, g_n). \]

For example, the (trivial) action of a group $G$ on the commutative operad $\text{Com}$ defines a semidirect product $\mathcal{G} := \text{Com} \rtimes G$ such that $\mathcal{G}(n) = G^n$. The framed little $n$-discs operad is isomorphic to the semidirect product $fD_n = D_n \rtimes SO(n)$, where $SO(n)$ rotates the picture of the little discs.

The natural action of $SO(n)$ on $S^{n-1}$ defines a $SO(n)$-action on the operad $B_n$, given that $B_n(k) = (S^{n-1})^{k(k-1)/2}$. This action restricts to an action on the operad $\mathcal{K}_n$. The arguments giving the weak equivalence between $\mathcal{K}_n$ and the little $n$-discs operad $D_n$ extend to show that the semidirect product operad $f\mathcal{K}_n = \mathcal{K}_n \rtimes SO(n)$ is weakly equivalent to the framed little $n$-discs operad. Namely in [16] we constructed a diagram of weak equivalences of operads $D_n \leftarrow WD_n \to F_n$, where $F_n$ is the Fulton-MacPherson operad. These arrows and the projection $F_n \to \mathcal{K}_n$, that is also a weak equivalence [18], are $SO(n)$-equivariant.

**Proposition 14.** The homotopy totalization of the cosimplicial space $f\mathcal{K}_n^*$, for \(n > 3\), is weakly equivalent to the space $f\text{Emb}_n$ of framed long knots in $\mathbb{R}^n$.

**Proof.** The sequence of cosimplicial spaces

\[ \text{map}_\bullet(S^1, SO(n-1)) \to f\mathcal{K}_n^* \to \mathcal{K}_n^* \rtimes S^{n-1} \]

is levelwise the fibration $SO(n-1)^k \to SO(n)^k \times \mathcal{K}_n(k) \to (S^{n-1})^k \times \mathcal{K}_n(k)$. There is a commutative diagram

\[ \begin{array}{ccc}
\Omega SO(n-1) & \longrightarrow & f\text{Emb}_n \\
\downarrow & & \downarrow f\beta \\
\widetilde{\text{Tot}}(\text{map}_\bullet(S^1, SO(n-1))) & \longrightarrow & \widetilde{\text{Tot}}(f\mathcal{K}_n^*) \\
\downarrow \widetilde{f\beta} & & \downarrow \widetilde{f\beta} \\
\widetilde{\text{Tot}}(f\mathcal{K}_n^*) & \longrightarrow & \widetilde{\text{Tot}}(f\mathcal{K}_n^* \rtimes S^{n-1})
\end{array} \]

where the rows are fibrations. The middle arrow $\widetilde{f\beta}$ is the composite of a map $f\beta$ and the canonical map $\widetilde{\text{Tot}}(f\mathcal{K}_n^*) \to \widetilde{\text{Tot}}(f\mathcal{K}_n^*)$, where $f\beta$ is adjoint to a collection of maps $f\text{Emb}_n \rtimes \Delta_k \to \mathcal{K}_n(k) \times SO(n)^k$ that evaluate directions between points of the framed knot as before and in addition evaluate the framings at those points.

The left and right vertical maps are weak equivalences, and hence the middle vertical map $\widetilde{f\beta}$ is a weak equivalence.

\[ \square \]

Now $f\mathcal{K}_n$ is an operad with multiplication, so that $\widetilde{\text{Tot}}(f\mathcal{K}_n)$ has an action of the McClure-Smith operad $D_2$ by theorem [7]. The space $f\text{Emb}_n \simeq \text{Emb}_n \rtimes \Omega SO(n-1)$ is grouplike for $n > 3$, in the sense that its components form a group, namely $\mathbb{Z}_2$.

By the recognition principle [14] we readily obtain:

**Corollary 15.** The space of framed long knots in $\mathbb{R}^n$ is weakly equivalent to a double loop space for $n > 3$.

This recovers the result by Budney [2].

We characterize next the semidirect product operad $B_n \rtimes SO(n)$, that we will also call $fB_n$. We observe that there is an operad inclusion $i_n : SO(n-1) \to fB_n$ that we define next. Let us identify $SO(n-1)$ to the subgroup of $SO(n)$ fixing the
preferred direction \( m_2 \in S^{n-1} = B_n(2) \). We recall that \( m_k \in B_n(k) \) is the base point. Then \( i_n \) sends \((g_1, \ldots, g_k) \in SO(n-1)^k\) to \((m_k, g_1, \ldots, g_k) \in B_n(k) \times SO(n)^k\). We visualize the image as a configuration of points on a line parallel to the preferred direction, with the assigned framings. Clearly \( i_n \) factors through the operad \( fK_n \). We remark that \( i_n \) does not extend to a section \( SO(n) \to fB_n \) of the projection \( fB_n \to SO(n) \).

**Proposition 16.** The map \( i_n : SO(n-1) \to fB_n \) induces on the (homotopy) totalizations of the associated cosimplicial spaces a homotopy equivalence that is a double loop map, so that

\[
\Omega SO(n-1) \simeq Tot(fB^*_n).
\]

**Proof.** We have a pullback diagram of cosimplicial spaces

\[
\begin{array}{ccc}
fB^*_n & \to & B^*_n \times S^{n-1} \\
\downarrow & & \downarrow \\
m_{\bullet}(S^1, SO(n)) & \to & m_{\bullet}(S^1, S^{n-1}).
\end{array}
\]

On totalizations we obtain the pullback diagram

\[
\begin{array}{ccc}
Tot(fB^*_n) & \to & POS^{n-1} \\
\downarrow & & \downarrow \\
\Omega SO(n) & \to & \Omega S^{n-1}.
\end{array}
\]

The inclusion \( SO(n-1) \to fB_n \) induces on totalizations the standard homotopy equivalence from \( \Omega SO(n-1) \) to \( \Omega SO(n) \to Tot(fB^*_n) \), the homotopy fiber of the looped projection \( \Omega SO(n) \to \Omega S^{n-1} \). We can replace totalizations by homotopy totalizations in the proposition since all cosimplicial spaces involved are fibrant.

\[\square\]

**Proof of theorem 3** We have a diagram of operads

\[
\begin{array}{ccc}
K_n & \to & B_n \\
\downarrow & & \downarrow \\
SO(n-1) & \to & fK_n \to fB_n \\
\downarrow & & \downarrow \\
SO(n) & \to & fB_n
\end{array}
\]

The operad inclusion \( fK_n \to fB_n \) gives on homotopy totalizations, by naturality of the McClure-Smith construction, a map of \( \tilde{D}_2 \)-algebras \( \tilde{Tot}(fK_n) \to \tilde{Tot}(fB_n) \), that by naturality of the recognition principle is a double loop map. Its homotopy fiber \( F \) is weakly equivalent to \( E\text{mb}_n \) as double loop space, by comparison with the homotopy fiber of \( \tilde{Tot}(K_n) \to \tilde{Tot}(B_n) \) and by the arguments in the proof of theorem. The double loop map \( \tilde{Tot}(fK_n) \to \tilde{Tot}(fB_n) \) has a double loop section because the operad inclusion \( SO(n-1) \to fK_n \to fB_n \) induces a weak equivalence.
that is a double loop map on homotopy totalizations (proposition 10). This gives the fiber sequence of double loop maps with section

\[ Emb_n \to fEmb_n \xleftarrow{\sim} \Omega SO(n-1). \]

Now there is a commutative diagram

\[
\begin{array}{c}
\Omega SO(n-1) \xrightarrow{j} fEmb_n \xrightarrow{\sim} \Omega SO(n) \\
\downarrow \cong \hspace{1cm} \downarrow \cong \\
\widetilde{Tot}(SO(n-1)^*) \xrightarrow{\sim} \widetilde{Tot}(fK_n^*) \xrightarrow{\sim} \widetilde{Tot}(SO(n)^*)
\end{array}
\]

and the inclusion \( j : \Omega SO(n-1) \subset fEmb_n \) represents the subspace of all framings of the trivial knot. We conclude the proof by taking homotopy fibers over \( \widetilde{Tot}(SO(n)^*) \).

Namely the homotopy fiber \( K' \) of \( \widetilde{Tot}(fK_n^*) \) (resp. \( B' \) of \( \widetilde{Tot}(fB_n^*) \)) is canonically weakly equivalent to \( \Omega^2 S^{n-1} \). Let \( \Omega' \) be the homotopy fiber of \( \widetilde{Tot}(SO(n-1)^*) \) to \( \widetilde{Tot}(SO(n)^*) \), canonically weakly equivalent to \( \Omega^2 S^{n-1} \) as double loop space. Then the double loop map \( K' \to B' \) has a double loop section because the composite \( \Omega' \to K' \to B' \) is a weak equivalence and a double loop map. This gives the fiber sequence of double loop maps with section

\[ Emb_n \to Emb'_n \xleftarrow{\sim} \Omega^2 S^{n-1}. \quad \square \]

4. AN OBSTRUCTION TO DOUBLE LOOP MAPS

In this section we will prove theorem 2 by showing that the projection \( fEmb_n \to Emb_n \) from framed knots to knots and the map \( p : Emb'_n \to Emb_n \) from section 2 do not preserve the Browder operation in rational homology for \( n \) odd. We need to review some notions on homology operations of double loop spaces.

**Definition 17.** An \( n \)-algebra is an algebra over the homology operad of the little \( n \)-discs operad.

In particular a 2-algebra is called a Gerstenhaber algebra. A (graded) \( n \)-algebra \( A \) for \( n > 1 \) is described by assigning a product and a bracket

\[
\begin{align*}
- \ast & : A_i \otimes A_j \to A_{i+j} \\
\{\cdot, \cdot\} & : A_i \otimes A_j \to A_{i+j+n-1}
\end{align*}
\]

that satisfy essentially the axioms of a Poisson algebra, except for signs. We refer to [17] for a full definition. The action of the little \( n \)-discs operad on an \( n \)-fold loop space gives a natural \( n \)-algebra structure on its homology, such that the product is the Pontrjagin product and the bracket is called the Browder operation. In particular the homologies of the double loop spaces \( Emb_n, Emb'_n \) and \( fEmb_n \) have a natural structure of Gerstenhaber algebras.

Originally Gerstenhaber introduced the algebraic structure bearing his name while studying the Hochschild complex of associative algebras. More generally Gerstenhaber and Voronov introduced this structure on the Hochschild homology of an operad with multiplication in vector spaces. Let \( O \) be an operad in vector spaces together with a multiplication, i.e. an operad map \( Ass \to O \) from the
associative operad. The image of the multiplication in $Ass$ is an element $m \in O(2)$. The operad composition maps define a bracket

$$\llbracket \, \cdot \, , \, \cdot \, \rrbracket : O(k) \otimes O(l) \to O(k + l - 1)$$

by

$$[x, y] = \sum_{i=1}^{k} \pm x \circ_i y - \sum_{i=1}^{l} \pm y \circ_i x$$

for appropriate signs [19]. The multiplication defines a star product

$$\ast : O(k) \otimes O(l) \to O(k + l)$$

by

$$x \ast y = m(x, y) := (m \circ_2 y) \circ_1 x.$$

**Definition 18.** The Hochschild complex of $O$ is the chain complex $(\bigoplus s^{-k}O(k), \partial)$, where $s^{-k}$ is degree desuspension, and the differential is $\partial(x) = [m, x]$. The Hochschild homology $HH(O)$ of $O$ is the homology of such complex.

**Proposition 19.** [9] The bracket and the star product induce a Gerstenhaber algebra structure on the Hochschild homology of an operad with multiplication in vector spaces.

Since the operad describing Gerstenhaber algebras is the homology of the little 2-discs operad $D_2$, Deligne asked his famous question, now known as the Deligne conjecture, whether the homological action could be induced by an action of (singular) chains of the little discs $C_*(D_2)$ on the Hochschild complex. Many authors proved that indeed there was a natural action of a suitable operad quasi-isomorphic to $C_*(D_2)$ on the Hochschild complex.

If we work instead with operads with multiplications in chain complexes then the Deligne conjecture holds for the normalized Hochschild complex. In this context we say that an operad $O$ in chain complexes has a unital multiplication if we have a morphism of operads $Ass_* \to O$, where $Ass_*$ is the operad describing unital associative algebras. This latter operad is also isomorphic as non-symmetric operad to the homology $H_*(D_1)$ of the little 1-discs. The image of the generator in $Ass_*(0)$ defining the unit is an element $u \in O(0)$.

**Definition 20.** The normalized Hochschild complex of a chain operad with (unital) multiplication is the subcomplex of the (full) Hochschild complex consisting of those elements $x \in O(k)$, $k \in \mathbb{N}$ such that $x \circ_i u = 0$ for all $1 \leq i \leq k$.

**Proposition 21.** (McClure-Smith) [15] The normalized Hochschild complex of a chain operad $O$ with unital multiplication has an action of an operad quasi-isomorphic to the singular chain operad of the little discs $C_*(D_2)$.

It is crucial that the normalized Hochschild complex of a chain operad with unital multiplication $O$ can be seen also as (co)normalization of a cosimplicial chain complex $O^*$ defined from the operad $O$ in a manner completely analogous as in the topological category (section 2). We recall that the (co)normalization of a cosimplicial chain complex $O^*$ is the chain complex of cosimplicial maps $\Delta^* \otimes \mathbb{Z} \to O^*$, with differential induced by the cosimplicial chain complex $\Delta^* \otimes \mathbb{Z}$. This construction is the algebraic analog of the totalization of a cosimplicial space. Thus Theorem 7 can be seen as a topological analog of the Deligne conjecture. We make this analogy precise in the following statement.
Proposition 22. Let $O$ be a topological operad with multiplication. The Hochschild homology of the operad $C_*(O)$ of singular chains on $O$ is isomorphic to the homology of $\text{Tot}(O^*)$. The bracket and the star product under the isomorphism $HH(C_*(O)) \cong H_*(\text{Tot}(O^*))$ correspond respectively to the Browder operation and the Pontrjagin product.

The Gerstenhaber algebra structure interacts well with a spectral sequence computing the homology of $\text{Tot}(O^*)$, the Bousfield spectral sequence.

Proposition 23. Given a cosimplicial space $K^*$, there is a second quadrant spectral sequence computing the homology of $\text{Tot}K^*$. Its $E^1$-term is $E^1_{p,q} = H_q(K^p)$, with the differential $\sum_{i=0}^{p+1}(-1)^i d_i^* : H_q(K^p) \to H_q(K^{p+1})$.

The filtration giving the spectral sequence is the decreasing filtration by cosimplicial degree in the normalization of $C_*(K^*)$.

Proposition 24. Let $O$ be a topological operad with multiplication. Then the Bousfield spectral sequence for $H_*(\text{Tot}O^*)$ is a spectral sequence of Gerstenhaber algebras with bracket

$$[-,-] : E^r_{-p,q} \otimes E^r_{-r',q'} \to E^r_{-p-r'+1,q+q'}$$

and product

$$- \cdot - : E^r_{-p,q} \otimes E^r_{-r',q'} \to E^r_{-p-r',q+q'}.$$ 

The $E_2$-term is the Hochschild homology of the homology operad $H_*(O)$ as a Gerstenhaber algebra.

Proof. The star product sums filtration indices on elements in $C_*(O)$. The bracket $[x,y]$ sits in the $(m+n-1)$-th filtration term if $x$ sits in the $m$-th term and $y$ in the $n$-th term.

The Bousfield spectral sequence does not always converge, but it does for $K^* = K^*_n$ or $K^* = K^*_n \times S^{n-1}$, as observed by Sinha [13]. Arone, Lambrechts and Volic have recently announced a proof that in these two cases (for $n > 3$) the spectral sequence collapses at the $E^2$-term over the rational numbers [12]. A key ingredient in their proof is a result by Kontsevich showing the formality of the little $n$-discs operad [11], in the sense that the chain operad $C_*(D_n, \mathbb{Q})$ is quasi-isomorphic to its homology $H_*(D_n, \mathbb{Q})$. The same idea can be used to show that for $K = K_n$ there are no extension issues, in the sense that the $E^2$-term is isomorphic to $H_*(\text{Emb}'_n, \mathbb{Q}) \cong H_*(\text{Tot}(K_n))$ as a Gerstenhaber algebra. We will not need these collapse results here because in low degree the spectral sequence must collapse and there are no extension issues.

The $E^2$ term is the Hochschild homology of the little $n$-discs operad homology $H_*(D_n)$, and has been extensively studied by Turchin [19].

As we have seen the operad $H_*(D_n)$ is generated by a product $x_1 \cdot x_2 \in H_0(D_n(2))$ and a bracket $\{x_1, x_2\} \in H_{n-1}(D_n(2))$. We use different symbols to avoid confusion with the product and the bracket in the Hochschild complex.

Proof of theorem. If $p : \text{Emb}'_n \to \text{Emb}_n$ is homotopic to a double loop map then it should induce on homology a homomorphism of Gerstenhaber algebras. We
will show that this is not the case because the kernel of \( p_* \) is not an ideal with respect to the bracket.

We are considering the case \( n \) odd and \( n > 3 \) over rational coefficients. The lowest dimensional class in the \( E^2 \)-term for \( \text{Emb}' \) is the element \( \alpha = \{x_1, x_2\} \in E_{-2, n-1} \). There is no class that can kill it, so this element survives and represents the generator of \( H_{n-3}(\text{Emb}'_n) \cong \mathbb{Q} \), coming from the factor \( \Omega^2 S^{n-1} \) with respect to the splitting \( \text{Emb}'_n \cong \text{Emb}_n \times \Omega^2 S^{n-1} \). For similar reasons \( H_{2n-6}(\text{Emb}'_n) \cong \mathbb{Q}^2 \) is generated by the surviving elements \( \beta = \{x_1, x_3\} \cdot \{x_2, x_4\} \) and \( \alpha^2 = \alpha \ast \alpha = \{x_1, x_2\} \cdot \{x_3, x_4\} \).

The cosimplicial inclusion \( p^*: \mathcal{K}_n^* \to \mathcal{K}_n^* \times S^{n-1} \) induces a morphism of spectral sequences, and on homotopy totalizations gives a map that we can identify to \( p: \text{Emb}'_n \to \text{Emb}_n \).

The lowest dimensional class in the \( E^2 \)-term for \( \text{Emb}_n \cong \tilde{Tot}(\mathcal{K}_n^* \times S^{n-1}) \) is the image \( E^2(p)(\beta) \). This class survives to a class \( p_*(\beta) \) generating \( H_{2n-6}(\text{Emb}_n) \cong \mathbb{Q} \). The computation by Turchin given in formula 2.9.21 of [19] indicates that the \( E^2 \)-term for \( \text{Emb}_n \) in dimension \( 3n - 8 \) has two generators, \( [\alpha, \beta] \) and \( [\alpha, \alpha^2] = 2\alpha[\alpha, \alpha] \), that survive, so that \( H_{3n-8}(\text{Emb}_n) \cong \mathbb{Q}^2 \). The \( E^2 \)-term for \( \text{Emb}_n \) in the same dimension has one generator, \( E^2(p)[\alpha, \beta] \), that survives so that \( H_{3n-8}(\text{Emb}_n) \cong \mathbb{Q} \) is generated by \( p_*(\alpha) = \beta \neq 0 \). But by dimensional reason \( p_*(\alpha) = 0 \), so the bracket is not preserved by \( p_* \).

Thus \( p \) is not a double loop map. Actually this shows more: there is no double loop space splitting \( \text{Emb}'_n \cong \text{Emb}_n \times \Omega^2 S^{n-1} \).

Now \( p \) factors through \( f\text{Emb}_n \) via a double loop map \( p': \text{Emb}'_n \to f\text{Emb}_n \), that is induced by the operad inclusion \( \mathcal{K}_n \to f\mathcal{K}_n \). This map \( p' \) can be identified to the map \( \text{Emb}_n \times \Omega^2 S^{n-1} \to \text{Emb}_n \times \Omega SO(n-1) \) induced by looping the holonomy \( \Omega S^{n-1} \to SO(n-1) \). It is well known that \( p'_*(\alpha) \) is non-trivial so by the same reason the projection \( f\text{Emb}_n \to \text{Emb}_n \) is not a double loop map. \( \square \)

We remark that the obstruction argument does not work rationally for \( n \) even because in that case there is a Gerstenhaber structure on the \( E^2 \)-term for \( \text{Emb}_n \) such that \( E^2(p) \) is a map of Gerstenhaber algebras. Namely additively this \( E^2 \)-term is identified to the Hochschild homology of the Batalin-Vilkovisky operad \( BV_2 \). [19] This operad in vector spaces is the semidirect product of the little \( n \)-discs homology \( H_*(D_n) \) and the exterior algebra on a generator in dimension \( (n-1) \) [17]. Then \( E^2(p) \) is naturally the map of Gerstenhaber algebras induced in Hochschild homology by the operad inclusion \( H_*(D_n) \to BV_2 \).

However only for \( n = 2 \) the operad \( BV_2 \) is the homology of a topological operad, the framed little 2-discs operad \( fD_2 \). It might be possible that torsion operations like Dyer-Lashof operations still give obstructions to a double loop structure on the projection \( f\text{Emb}_n \to \text{Emb}_n \) for \( n \) even.

5. String topology of knots

We will show that the suspension spectrum of the space of knots in a sphere, suitably desuspended, is an \( E_2 \)-ring spectrum, proving theorem 4.

We proved this for \( n = 3 \) in our joint paper with Kate Gruher [8]. The original proof was based on the work by Budney, and on a generalized approach to string topology, expanding on fundamental ideas by Chas-Sullivan [6] and Cohen-Jones [7]. Now, knowing that \( \text{Emb}_n \) is a double loop space, we can produce a proof for
n > 3. We recall some terminology and we refer to [2] for details. We recall that an $E_2$-operad is a topological operad weakly equivalent to the little 2-discs operad. Similarly an $E_2$-operad spectrum is an operad in the category of (symmetric) spectra weakly equivalent to the suspension spectrum of the little 2-discs operad. For us an $E_2$-ring spectrum will be an algebra over an $E_2$-operad spectrum in the weak sense, meaning that the associativity and unit axioms hold up to homotopy. Given a manifold $M$ with tangent bundle $TM$ we denote by $-TM$ the opposite virtual bundle.

**Lemma 25. (Gruher-S.)**

Let $X$ be an algebra over an $E_2$-operad $O$, $G$ a compact Lie group and $H \subset G$ a closed subgroup. Suppose that $H$ acts on $X$ and the structure maps are $H$-equivariant. Let $p: G \times_H X \to G/H$ be the projection. Then the Thom spectrum of the virtual bundle $p^*(-T(G/H))$ over $G \times_H X$ is an $E_2$-ring spectrum.

Let $Emb(S^1, S^n)$ be the space of smooth embeddings $S^1 \to S^n$.

**Proof of thm 25**

It is convenient to use the model for the space of long knots $Emb_n$ given by embeddings of the interval into a cylinder $I \to D_{n-1} \times I$, with $D_{n-1}$ the unit $(n-1)$-disc, sending 0 and 1 to $(0,0)$ and $(0,1)$ respectively with tangents directed along the positive direction of the long axis, namely the last coordinate axis. There is a natural action by $SO(n-1)$ on $Emb_n$ rotating long knots around the long axis.

We have seen in section 2 that $Emb_n$ is weakly equivalent to the homotopy fiber $F$ of $\text{Tot}K_n \to \text{Tot}B_n^*$, by a sequence of weak equivalences

$$F \to F' \to \text{Tot}(K_n^* \times S^{n-1}) \leftrightarrow Emb_n,$$

where $F'$ is the homotopy fiber of $\text{Tot}(K_n^* \times S^{n-1}) \to \text{Tot}(B_n^* \times S^{n-1})$. Actually all maps in the sequence are $SO(n-1)$-equivariant maps between $SO(n-1)$-spaces. Namely the action of $SO(n-1) \subset SO(n)$ on $S^{n-1}$ makes $B_n^*$ and $B_n^* \times S^{n-1}$ into cosimplicial $SO(n-1)$-spaces, such that respectively $K_n^*$ and $K_n^* \times S^{n-1}$ are $SO(n-1)$-invariant cosimplicial subspaces. Thus the induced maps on homotopy totalizations are $SO(n-1)$-equivariant. Moreover it is easy to see that the evaluation $Emb_n \to \text{Tot}(K_n^* \times S^{n-1})$ is $SO(n-1)$-equivariant. Thus $SO(n+1) \times SO(n-1) Emb_n$ is weakly equivalent to $SO(n+1) \times SO(n-1) F$. As observed by Budney and Cohen [2] there is a homotopy equivalence $Emb(S^1, S^n) \simeq SO(n+1) \times SO(n-1) Emb_n$. We obtain then a weak equivalence $Emb(S^1, S^n) \simeq SO(n+1) \times SO(n-1) F$.

The $SO(n-1)$-action makes $K_n$ and $B_n$ into operads in the category of based $SO(n-1)$-spaces. Thus the homotopy totalizations of $K_n^*$ and $B_n^*$ are algebras over the operad $\mathcal{D}_2$ in the category of $SO(n-1)$-spaces, where a trivial $SO(n-1)$-action is assumed on $\mathcal{D}_2$. The inclusion $\text{Tot}K_n^* \to \text{Tot}B_n^*$ respects this structure, so that the homotopy fiber $F$ is also an algebra over $\mathcal{D}_2$ in $SO(n-1)$ spaces. By lemma 25 with $G = SO(n+1)$, $H = SO(n-1)$ and $O = \mathcal{D}_2$, $SO(n+1) \times SO(n-1) F^{-T(SO(n+1)/SO(n-1))}$ is an $E_2$-ring spectrum. But $SO(n+1)/SO(n-1)$ is (stably) parallelizable and has dimension $2n - 1$, so that

$$(SO(n+1) \times SO(n-1) F)^{-T(SO(n+1)/SO(n-1))} \simeq \Sigma^{1-2n} \Sigma^\infty Emb(S^1, S^n)_+$$

is an $E_2$-ring spectrum.

The following corollary has been proved independently by Abbaspour-Chataur-Kallel, who describe also a BV-algebra structure.
Corollary 26. The homology $H_{*+2n−1}(\text{Emb}(S^1, S^n))$ has a natural structure of Gerstenhaber algebra.

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Dipartimento di matematica, Università di Roma “Tor Vergata”, Roma, Italy
E-mail address: salvator@mat.uniroma2.it