LOCALLY LIPSCHITZ CONTRACTIBILITY AND THE HOMOLOGY OF INTEGRAL CURRENTS

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Abstract. We introduce the notion of weakly locally Lipschitz contractibility for metric spaces. Many fundamental objects in metric geometry, for instance, normed spaces, CAT-spaces and Alexandrov spaces satisfy this condition. We consider the category of all pairs of weakly locally Lipschitz contractible spaces as objects together with all locally Lipschitz maps as morphisms. A main purpose of the present paper is to prove that the singular Lipschitz (and usual singular) homology is naturally isomorphic to the homology of integral currents with compact support on this category.

1. Introduction

The notion of the locally Lipschitz contractibility for metric spaces was introduced by Yamaguchi [Y]. We abbreviate a term “locally Lipschitz contractible” by LLC. Normed spaces and CAT-spaces are typical examples of LLC spaces. Recently, in [MY], they proved that every finite dimensional Alexandrov space is LLC in a strong sense. Here, CAT-spaces and Alexandrov spaces are intrinsic (and complete) metric spaces having a curvature bound locally from above and below, respectively, in a generalized sense. For their definition and basic theory, we refer to the book [BBI]. The locally Lipschitz contractibility is known to have several interesting applications (see [MY]).

In the present paper, we consider and compare three homology theories from certain categories of metric spaces to the category of abelian groups:

- $H_*$ the singular homology,
- $H_*^L$ the singular Lipschitz homology, and
- $H_*^{IC}$ the homology of integral currents with compact support.

For their definitions, we refer to Section 4. Let us denote by $\text{Met}^{\text{LLC}}$ the category of all LLC spaces as objects and all locally Lipschitz maps as morphisms. By $\text{Ab}$, we denote the category of all abelian groups. In [Y], he proved

**Theorem 1.1 (Y Proposition 1.3)**. Two homologies $H_*^L$ and $H_*$ from $\text{Met}^{\text{LLC}}$ to $\text{Ab}$ are naturally isomorphic to each other. Here, the natural
isomorphism is induced by the inclusion from the singular Lipschitz chain complex $S^L_*(X)$ to the usual singular chain complex $S_*(X)$ for each $X \in \text{Met}^{\text{LLC}}$.

Among of the homologies listed above, we will compare mainly $H^L_*$ and $H^{IC}_*$.

Currents in a smooth manifold were introduced and studied by De Rham [DR]. Federer and Fleming [FF] introduced and investigated a restricted classes of currents in a Euclidean space, to solve the Plateau problem in a generalized sense. For a theory of such classical currents, we refer to the book [F]. Currents in metric spaces, often called metric currents, were introduced by Ambrosio and Kirchheim [AK]. There, fundamental and useful theory on currents were established. However, they dealt with only complete metric spaces. It was sufficient for their purpose. On the other hand, we mainly deal with LLC spaces in the present paper. By the definition of the LLC-condition, any open subset of an LLC space is again LLC. Hence, even if $X$ is complete, when we want to consider relative homology $H^{IC}_*(X, A)$ for an open subset $A$ of $X$, we must take care apply the theory in [AK]. Indeed, we will remark that a region of their theory which will be used in this paper can be applied to arbitrary metric spaces (Remark 3.8).

Let us denote by $\text{Met}_2^{\text{LLC}}$ the category of all pairs of LLC spaces together with all locally Lipschitz maps. A main result of the present paper is

**Theorem 1.2.** Three functors $H_*$, $H^L_*$ and $H^{IC}_*$ from $\text{Met}_2^{\text{LLC}}$ to $\text{Ab}$ are naturally isomorphic to each other.

**Corollary 1.3.** Let $X$ be a finite dimensional Alexandrov space. Let $A$ be open or closed convex in $X$. Then,

$$H_*(X, A) \cong H^L_*(X, A) \cong H^{IC}_*(X, A).$$

Riedweg and Schäppi [RS] defined certain notion of “admitting local cone inequalities” and “admitting locally strong Lipschitz contractions” for metric spaces, which are variants of corresponding notion defined by Wenger [W]. They proved that a metric space admitting locally strong Lipschitz contractions admits certain local cone inequalities. And they claimed that for a metric space admitting certain local cone inequalities, considering three homology theories coincide. Normed spaces and CAT-spaces admit locally strong Lipschitz contractions. However, in general, we do not know whether Alexandrov spaces admit locally strong Lipschitz contractions or local cone inequalities.

Further, we will introduce the notion of weakly locally Lipschitz contractibility (Definition 2.2). It will be abbreviated by WLLC. We will check that every metric space which is LLC or which admits locally strongly Lipschitz contractions, is WLLC. And, we will prove
Theorem 1.4. $H_\ast$, $H^L_\ast$ and $H^{IC}_\ast$ are naturally isomorphic to each other on $\text{Met}_{WLLC}^{\text{WLLC}}$. Here, $\text{Met}_{WLLC}^{\text{WLLC}}$ is the fullsubcategory of $\text{Met}_2$ consisting of all WLLC pairs.

Also, we will define the notion of $(H)$-locally triviality for metric spaces (Definition 4.11), where $H : \text{Met} \to \text{Ab}$ is a covariant functor. This condition is also called $H$-locally connectedness or infinitesimally $H$-acyclicity in several literatures. Let $\text{Met}_{LT}^2$ denote the fullsubcategory of $\text{Met}_2$ consisting of all $(X, A) \in \text{Met}_2$ such that both $X$ and $A$ are $(H)$-locally trivial for every $H = \tilde{H}_j$, $\tilde{H}^L_j$ and $H^{IC}_m$ with $j \geq 0$ and $m \geq 1$, where $\tilde{H}_\ast$ (resp. $\tilde{H}^L_\ast$) is the reduced singular (resp. the reduced singular Lipschitz) homology. We prove the following “Poincaré type lemma”.

Theorem 1.5. Every WLLC space is $(H)$-locally trivial for all $H = \tilde{H}_j$, $\tilde{H}^L_j$ and $H^{IC}_m$ for $j \geq 0$ and $m \geq 1$.

Via Theorem 1.5 Theorem 1.4 follows from the following.

Theorem 1.6. Three functors $H_\ast$, $H^L_\ast$ and $H^{IC}_\ast$ are naturally isomorphic to each other on $\text{Met}^{LT}_2$.

We will also check that a metric space admitting a certain local cone inequality in the sense of [RS] is $(H)$-locally trivial for suitable corresponding $H$ in our sense (Proposition 4.15). Therefore, Theorems 1.4 and 1.6 are generalizations of corresponding statements in [RS].

1.1. Strategy. We explain how to prove the coincidence $H^L_\ast$ and $H^{IC}_\ast$ stated in Theorems 1.4 and 1.6

First, we explain a strategy to prove $H^L_\ast \simeq H^{IC}_\ast$ stated in Theorem 1.6. It is purely an algebraic topological argument. We will use the notion of cosheaves. For a general theory of cosheaves and its applications, we refer to the book [B] and the papers [B2] and [DP]. We will review its definition and useful facts in Section 5.

In [DP], De Pauw considered several homologies on several categories of topological spaces or of subspaces in a Euclidean space, and compared them by using cosheaf theory. In particular, he proved that the homology of integral currents with compact support coincides with the Čech homology $\check{H}_\ast$ naturally, on a suitable category consisting of subsets in a Euclidean space ([DP, Theorem 3.14]). Its proof given there was done by using an augmented double complex induced by a corresponding cosheaf with respect to any open covering of a space. And, due to a suitable assumption imposed on spaces and a general theory of homological algebra, taking the inverse limit procedure, a coincidence of the homologies was proved.

To prove a coincidence of the homologies $H^L_\ast$ and $H^{IC}_\ast$ on a certain category of metric spaces, Riedweg and Schäppi [RS] defined a natural transformation $[\cdot]_\ast : H^L_\ast \to H^{IC}_\ast$. They used the slicing of currents,
which was defined in \([AK]\), to decompose a given current into small pieces. And, they compared \(H^L_*\) and \(H^{IC}_*\) directly, by using \([\cdot]_*\), and the slicing.

Our strategy to prove \(H^L_* \cong H^{IC}_*\) as stated in Theorem 1.6 is a hybrid of an argument to compare several homologies by using cosheaves (and using augmented double complexes) developed in \([DP]\) and using the natural transformation \([\cdot]_*\) to compare \(H^L_*\) with \(H^{IC}_*\) directly defined by \([RS]\). In particular, using \([\cdot]_*\), we can avoid to deal with the Čech homology.

Theorem 1.4 is an immediate consequence of a Poincaré type lemma (Theorem 1.5). Therefore, main parts in this paper are to prove Theorems 1.6 and 1.5.

1.2. Organization. In Section 2 we recall the definition of the locally Lipschitz contractibility and introduce the notion of weakly locally Lipschitz contractibility. We also review several notion similar to locally Lipschitz contractibility, and observe relations between them. And, we provide certain categories consisting of metric spaces (or pairs) as objects. In Section 3 we recall the definition of currents in metric spaces, according to \([AK]\). However, differently from \([AK]\), to deal with arbitrary metric spaces, we employ slightly modified definition. We will remark a region of their theory which will be used in this paper can work for arbitrary metric spaces which are need not to be complete. In Section 4 we recall the definition of homologies \(H^L_*\) and \(H^{IC}_*\) which we want to compare, as mentioned in the introduction. We provide a natural transformation \([\cdot]_*\), from \(H^L_*\) to \(H^{IC}_*\), which was defined by \([RS]\). At the end of this section, we prove an important property, which is a counter part of the usual Poincaré’s lemma, for \(H^L_*\) and \(H^{IC}_*\) on WLLC spaces. To compare homologies, we recall the notion of cosheaves and its some useful theory in Section 5. There, we prove that the functor taking the space of integral currents with compact support on each open set in a metric space becomes actually a cosheaf on the space. In Section 6 we prove that \([\cdot]_*\) : \(H^L_* \to H^{IC}_*\) is actually a natural isomorphism on a suitable category. As well as, we prove that the natural transformation \(H^L_* \to H_*\), which is induced by the inclusions between chain complexes, is an isomorphism on a suitable category. In particular, all results mentioned in the introduction are proved.

2. Locally Lipschitz contractibility

In this section, \(X\) and \(Y\) always denote metric spaces. For \(L \geq 0\), a map \(f : X \to Y\) is said to be \(L\)-Lipschitz if it satisfies

\[
d(f(x), f(y)) \leq Ld(x, y)
\]
for all $x, y \in X$. We say that $f$ is Lipschitz if $f$ is $L$-Lipschitz for some $L \geq 0$. The Lipschitz constant of $f$ is the minimum of all $L$ such that $f$ is $L$-Lipschitz, and is denoted by $\text{Lip}(f)$.

A map $f : X \to Y$ is said to be locally Lipschitz if for any $x \in X$, there exists an open set $U$ in $X$ containing $x$ such that the restriction $f|_U$ is Lipschitz.

A map $f : X \to Y$ is called a bi-Lipschitz embedding, or simply, a bi-Lipschitz map if it satisfies
\[ L^{-1}d(x, x') \leq d(f(x), f(x')) \leq Ld(x, x') \]
for all $x, x' \in X$ with some $L \geq 1$. If a bi-Lipschitz map is bijective, then we call it a bi-Lipschitz homeomorphism. A locally bi-Lipschitz homeomorphism is a homeomorphism such that it and its inverse are locally Lipschitz.

### 2.1. Locally Lipschitz contractibility

By $\text{Met}$, we denote the category of all metric spaces as objects together with all locally Lipschitz maps as morphisms.

A homotopy $h : X \times [0, 1] \to Y$ is called a Lipschitz homotopy if it is a Lipschitz map, i.e., there exist constants $C, C' \geq 0$ such that
\[ d(h(x, t), h(x', t')) \leq Cd(x, x') + C'|t - t'| \]
for every $x, x' \in X$ and $t, t' \in [0, 1]$. For a homotopy $h$, we write $h_t = h(\cdot, t)$ for each $t \in [0, 1]$. In this case, $h_0$ and $h_1$ are said to be Lipschitz homotopic to each other.

Let $U$ and $V$ be subsets of $X$ with $U \subset V$. A Lipschitz homotopy $h : U \times [0, 1] \to V$ is called a Lipschitz contraction if there exists a point $x_0 \in V$ such that $h_0$ is the inclusion $U \to V$ and $h_1$ is a constant map valued $x_0$. In this case, we say that $U$ is Lipschitz contractible to $x_0$ in $V$ and precisely say that $h$ is a Lipschitz contraction from $U$ to $x_0$ in $V$.

**Definition 2.1** ([Y], cf. [MY]). Let $X$ be a metric space. We say that $X$ is locally Lipschitz contractible, for short LLC, if for any $x \in X$ and any $r > 0$, there exists $r' \in (0, r]$ such that $U(x, r')$ is Lipschitz contractible to $x$ in $U(x, r)$. Here, $U(x, r)$ denotes the open metric ball centered at $x$ with radius $r$. We denote by $\text{Met}^{\text{LLC}}$ the fullsubcategory of $\text{Met}$ consisting of all LLC spaces as objects.

**Definition 2.2.** We say that a metric space $X$ is weakly locally Lipschitz contractible, for short WLLC, if for any $x \in X$ and any open set $U \subset X$ with $x \in U$, there exists an open set $V \subset X$ with $x \in V \subset U$ such that $V$ is Lipschitz contractible in $U$ to some point of $U$. We denote by $\text{Met}^{\text{WLLC}}$ the fullsubcategory of $\text{Met}$ consisting of all WLLC spaces as objects.

It is obvious that every LLC space is WLLC. Also, the following is trivial:
Proposition 2.3. The LLC-condition (resp. WLLC-condition) is inherited to open subsets and is preserving under locally bi-Lipschitz homeomorphisms. Namely, if $X$ is LLC (resp. WLLC) and $U$ is an open subset of $X$, and if $f : X \to Y$ is a locally bi-Lipschitz homeomorphism, then $U$ and $Y$ are LLC (resp. WLLC).

Remark 2.4. In \cite{MY}, they defined the strongly locally Lipschitz contractibility. We say that a metric space $X$ is strongly locally Lipschitz contractible, for short SLLC, if for any $x \in X$, there exist $r > 0$ and a Lipschitz contraction $h : U(x, r) \times [0, 1] \to U(x, r)$ to $x$ such that for any $r' \in (0, r]$, the image of the restriction of $h$ to $U(x, r') \times [0, 1]$ is $U(x, r')$. It is clear that any SLLC space is LLC. They proved that every finite dimensional Alexandrov space is SLLC.

Remark 2.5. Riedweg and Schäppi \cite{RS} introduced the following notion. A metric space $X$ admits locally strong Lipschitz contractions if for any $x \in X$, there exist $r > 0$ and $\gamma > 0$ such that every subset $S \subset U(x, r)$ admits a strong $\gamma$-Lipschitz contraction in their sense, i.e., there exists a Lipschitz contraction $\phi$ from $S$ to some $y \in X$ in $X$ such that the Lipschitz constant of $\phi$ is estimated as

$$d(\phi(z, t), \phi(z', t')) \leq \gamma \text{diam}(S)|t - t'| + \gamma d(z, z').$$

Since $\gamma$ is depending only on $x$ and the Lipschitz constant of $\phi$ is depending on $\text{diam}(S)$, for $x' \in U(x, r)$, if we choose $r' > 0$ sufficiently small, then we obtain a Lipschitz contraction from $U(x', r')$ to some point of $U(x, r)$ in $U(x, r)$. Therefore, it turns out that any metric space admitting locally strong Lipschitz contractions is WLLC.

For subsets $A \subset X$ and $B \subset Y$, a locally Lipschitz map $f$ from $(X, A)$ to $(Y, B)$ is a locally Lipschitz map $f : X \to Y$ with $f(A) \subset B$. We denote by $\text{Met}_2$ the category of all pairs of metric spaces together with locally Lipschitz maps. A pair $(X, A)$ is said to be WLLC (resp. LLC) if $X$ is WLLC (resp. LLC) and $A$ is WLLC (resp. LLC) with respect to the restriction of the metric of $X$ to $A$. We denote by $\text{Met}_2^{\text{WLLC}}$ (resp. $\text{Met}_2^{\text{LLC}}$) the fullsubcategory of $\text{Met}_2$ consisting of all WLLC (resp. LLC) pairs.

Remark 2.6. Since $(X, A) \in \text{Met}_2^{\text{LLC}}$, then $(X, A) \in \text{Met}_2^{\text{WLLC}}$, Theorem 1.2 and Corollary 1.3 are immediate consequences of Theorem 1.4 via Remark 2.4.

3. Currents

In this section, we denote by $X$ and $Y$ metric spaces. We recall the notion of currents in metric spaces, which is slightly modified from the original one given in \cite{AK}. For its history and fundamental and deep theory, we refer to \cite{AK}. Lemmas 3.4, 3.10 and 3.16 are fundamental and important in our paper. Because the proofs of Lemmas
3.10 and 3.16 could not be found in any literatures, we will give the proof. Lemma 3.4 is maybe well-known, but for the completeness, we will prove it.

3.1. **Basic measure theory.** Let us denote a Borel measure on a metric space $X$. The **support** $\text{spt}(\mu)$ of $\mu$ is defined by

$$\text{spt}(\mu) = \{x \in X \mid \mu(U(x, r)) > 0 \text{ for any } r > 0\}.$$ 

It is closed in $X$. If $\mu$ is finite, i.e., $\mu(X) < \infty$, then its support is separable.

We say that $\mu$ is **concentrated** on a subset $A$ of $X$ if $\mu(X - A) = 0$. In this term, $\text{spt}(\mu)$ is the minimal closed subset of $X$ on which $\mu$ is concentrated. It is easy to check that if $\mu$ is concentrated on a separable set, then $\mu$ is concentrated on its support. If $\mu$ is concentrated on its support, then $\text{spt}(\mu)$ is compact if and only if $\mu$ is concentrated on a compact set.

We say that $\mu$ is **tight** if for any $\varepsilon > 0$, there exists a compact subset $K \subset X$ such that $\mu(X - K) < \varepsilon$. If $\mu$ is finite, then $\mu$ is tight if and only if $\mu$ is concentrated on a $\sigma$-compact set. In this case, $\mu$ is concentrated on its support. Further, the set $\text{Lip}_b(X)$ of all bounded Lipschitz functions is dense in $L^1(X, \mu)$ if $\mu$ is a finite tight Borel measure on $X$.

Let $\mathcal{M}$ be a family of finite Borel measures on $X$. The infimum

$$\bigwedge_{\nu \in \mathcal{M}} \nu(B) := \inf \left\{ \sum_{j=1}^{\infty} \mu_j(B_j) \right\}$$

for all Borel sets $B \subset X$, where the infimum runs over all $\{\mu_j\}_{j=1}^{\infty} \subset \mathcal{M}$ and all Borel partitions $\{B_j\}$ of $B$. Here, a Borel partition $\{B_j\}$ of $B$ is a disjoint countable family consisting of Borel sets satisfying $\bigcup_j B_j = B$. It is easy to check that the infimum of $\mathcal{M}$ is a finite Borel measure. By the definition, $\bigwedge_{\nu \in \mathcal{M}} \nu(B) \leq \nu'(B)$ for any $\nu' \in \mathcal{M}$ and Borel set $B \subset X$. In particular, if some $\nu' \in \mathcal{M}$ is tight, then $\bigwedge_{\nu \in \mathcal{M}} \nu$ is tight.

3.2. **Currents and Normal currents.** From now on, $k$ denotes a nonnegative integer.

Let us denote by $\text{Lip}(X)$ the set of all Lipschitz functions on $X$. Let us denote by $\mathcal{B}^\infty(X)$ the set of all bounded Borel measurable functions on $X$. For any $f \in \mathcal{B}^\infty(X)$, we set the value $|f|_{\infty} = \sup_{x \in X} |f(x)|$. By $\text{Lip}_b(X)$ we denote the set of all bounded Lipschitz functions on $X$, i.e., $\text{Lip}_b(X) = \text{Lip}(X) \cap \mathcal{B}^\infty(X)$. $\text{Lip}(X)$ is a linear space and $\text{Lip}_b(X)$ is its subspace.

We will denote by $1_A$ the indicator function of $A \subset X$, i.e., $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ otherwise.
Definition 3.1 (Domain of currents). We set $\mathcal{D}^0(X) = \text{Lip}_b(X)$. For $k \geq 1$, $\mathcal{D}^k(X) = \text{Lip}_b(X) \times [\text{Lip}(X)]^k$. We will often abbreviate an element $(f, \pi_1, \ldots, \pi_k) \in \mathcal{D}^k(X)$ by $(f, \pi)$.

We consider a condition for multilinear functionals on $\mathcal{D}^k(X)$.

Definition 3.2 (Finite mass axiom). Let $T : \mathcal{D}^k(X) \to \mathbb{R}$ be a multilinear functional. We say that $T$ satisfies the finite mass axiom or has finite mass if there exists a finite tight measure $\mu$ on $X$ such that for any $(f, \pi) \in \mathcal{D}^k(X)$,

\begin{equation}
|T(f, \pi)| \leq \prod_{i=1}^k \text{Lip}(\pi_i) \int_X |f| \, d\mu.
\end{equation}

Here, we regard $\prod \text{Lip}(\pi_i)$ as 1 when $k = 0$.

Definition 3.3 (Mass measure). Let $T : \mathcal{D}^k(X) \to \mathbb{R}$ be a multilinear functional having finite mass. The minimal measure of all finite Borel measures $\mu$ on $X$ satisfying (3.1) is called the mass measure of $T$ and is denoted by $\|T\|$. The support $\text{spt}(T)$ of $T$ is the support of $\|T\|$. The mass $M(T)$ of $T$ is the total measure of $\|T\|$: $M(T) := \|T\|(X)$.

Let us denote by $\mathcal{D}_k(X)$ the set of all multilinear functionals on $\mathcal{D}^k(X)$ having finite mass. We say that $T$ is concentrated on a subset $A \subset X$ if $\|T\|$ is concentrated on it.

Let $T \in \mathcal{D}_k(X)$. By the definition, $\|T\|$ is tight. Hence, $\text{Lip}_b(X)$ is dense in $L^1(X, \|T\|)$. It follows that $T$ is uniquely extended to a multilinear functional on $L^1(X, \|T\|) \times [\text{Lip}(X)]^k$. For a Borel set $B \subset X$, we define the restriction $T|B$ by

$$T|B(f, \pi) = T(1_B f, \pi)$$

for $(f, \pi) \in \mathcal{D}^k(X)$. Then, $T|B \in \mathcal{D}_k(X)$.

By the definition of the mass measure, for any $T, T' \in \mathcal{D}_k(X)$ and any Borel set $B \subset X$,

$$\|T + T'||(B) \leq \|T\||(B) + \|T'\||(B)$$

and

$$\|T|B\| = \|T\|[B].$$

In particular, we obtain

$$\text{spt}(T + T') \subset \text{spt}(T) + \text{spt}(T').$$

It is easy to check that $(\mathcal{D}_k(X), M)$ is a Banach space.

For $k \geq 1$, the boundary of $T \in \mathcal{D}_k(X)$ is given by

$$\partial T(f, \pi) = T(1, f, \pi)$$

for any $(f, \pi) \in \mathcal{D}^{k-1}(X)$, which is a multilinear functional on $\mathcal{D}^{k-1}(X)$. We say that $T \in \mathcal{D}_k(X)$ is normal if $\partial T$ has finite mass. For $T \in \mathcal{D}_0(X)$, we set $\partial T = 0$ and regard every $T \in \mathcal{D}_0(X)$ as normal. The
Lemma 3.4. Let $\phi : X \to Y$ be a Lipschitz map. The push-forward of $T \in \mathcal{D}_k(X)$ is defined by

$$\phi_#T(f, \pi) = T(f \circ \phi, \pi_1 \circ \phi, \ldots, \pi_k \circ \phi)$$

for any $(f, \pi) \in \mathcal{D}^k(Y)$. It is clear that $\phi_#T \in \mathcal{D}_k(Y)$. Indeed,

$$\|\phi_#T\| \leq \text{Lip}(\phi)^k \phi_#\|T\|$$

holds. In particular, $\phi_# : \mathcal{D}_k(X) \to \mathcal{D}_k(Y)$ is a Lipschitz map. If $\|T\|$ is concentrated on $A \subset X$, then $\|\phi_#T\|$ is concentrated on $\phi(A) \subset Y$. In particular, $\|\phi_#T\|$ is tight. If $T$ has compact support, then $\phi_#T$ also has compact support:

(3.2) $\text{spt}(T)$ is compact $\implies$ so is $\text{spt}(\phi_#T)$.

We recall the McShane-Whitney’s Lipschitz extension theorem stated as follows. Let $X$ be a metric space and $A$ a subset of $X$. Let $f : A \to \mathbb{R}$ be an $L$-Lipschitz function. Then, the following functions

$$\begin{align*}
X \ni x &\mapsto \inf \{ f(a) + Ld(x, a) \} \\
X \ni x &\mapsto \sup \{ f(a) - Ld(x, a) \}
\end{align*}$$

are $L$-Lipschitz on $X$ and extensions of $f$. We will often use this theorem in the present paper.

Lemma 3.4. If $\phi : X \to Y$ is a bi-Lipschitz embedding, then $\phi_# : \mathcal{D}_k(X) \to \mathcal{D}_k(Y)$ is injective.

Proof. It suffices to check that

$$\begin{align*}
\text{Lip}(Y) &\ni f \mapsto f \circ \phi \in \text{Lip}(X); \text{ and} \\
\text{Lip}_b(Y) &\ni f \mapsto f \circ \phi \in \text{Lip}_b(X)
\end{align*}$$

are surjective. Let $g : Y \to \mathbb{R}$ be a Lipschitz function. Then, $g \circ \phi^{-1} : \phi(X) \to \mathbb{R}$ is also Lipschitz. By the Lipschitz extension theorem (3.3), there exists $f : Y \to \mathbb{R}$ which is Lipschitz and $f|_{\phi(X)} = g \circ \phi^{-1}$. Then, $f \circ \phi = g$. It turns out that $\text{Lip}(Y) \to \text{Lip}(X)$ is surjective. In addition, we suppose that $g$ is bounded. Let us fix $\infty > C \geq \sup_X |g|$. Let $f$ be taken as above. Then, $h = \max\{\min\{f, C\}, -C\}$ is a bounded Lipschitz function on $Y$ which is an extension of $g \circ \phi^{-1}$. Therefore, the correspondence $\text{Lip}_b(Y) \to \text{Lip}_b(X)$ is also surjective. \hfill $\square$

To define the notion of currents, we provide some conditions for $T : \mathcal{D}^k(X) \to \mathbb{R}$.

Definition 3.5 ([AK] Locality and Continuity). Let $T : \mathcal{D}^k(X) \to \mathbb{R}$ be a multilinear functional. We say that $T$ satisfies the locality if for $f \in \text{Lip}_b(X)$ and $\pi_i \in \text{Lip}(X)$ with $i = 1, \ldots, k$, if for some $i$, $\pi_i$ is constant on $\{f \neq 0\}$, then $T(f, \pi) = 0$. 

set of all normal multilinear functionals on $\mathcal{D}^k(X)$ having finite mass endowed with the norm $N(T) = M(T) + M(\partial T)$ is a Banach space.

Let $Y$ be a metric space and $\phi : X \to Y$ be a Lipschitz map. The push-forward of $T \in \mathcal{D}_k(X)$ is defined by

$$\phi_#T(f, \pi) = T(f \circ \phi, \pi_1 \circ \phi, \ldots, \pi_k \circ \phi)$$

for any $(f, \pi) \in \mathcal{D}^k(Y)$. It is clear that $\phi_#T \in \mathcal{D}_k(Y)$. Indeed,

$$\|\phi_#T\| \leq \text{Lip}(\phi)^k \phi_#\|T\|$$

holds. In particular, $\phi_# : \mathcal{D}_k(X) \to \mathcal{D}_k(Y)$ is a Lipschitz map. If $\|T\|$ is concentrated on $A \subset X$, then $\|\phi_#T\|$ is concentrated on $\phi(A) \subset Y$. In particular, $\|\phi_#T\|$ is tight. If $T$ has compact support, then $\phi_#T$ also has compact support:

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\text{Lip}_b(Y) &\ni f \mapsto f \circ \phi \in \text{Lip}_b(X)
\end{align*}$$

are surjective. Let $g : Y \to \mathbb{R}$ be a Lipschitz function. Then, $g \circ \phi^{-1} : \phi(X) \to \mathbb{R}$ is also Lipschitz. By the Lipschitz extension theorem (3.3), there exists $f : Y \to \mathbb{R}$ which is Lipschitz and $f|_{\phi(X)} = g \circ \phi^{-1}$. Then, $f \circ \phi = g$. It turns out that $\text{Lip}(Y) \to \text{Lip}(X)$ is surjective. In addition, we suppose that $g$ is bounded. Let us fix $\infty > C \geq \sup_X |g|$. Let $f$ be taken as above. Then, $h = \max\{\min\{f, C\}, -C\}$ is a bounded Lipschitz function on $Y$ which is an extension of $g \circ \phi^{-1}$. Therefore, the correspondence $\text{Lip}_b(Y) \to \text{Lip}_b(X)$ is also surjective. \hfill $\square$

To define the notion of currents, we provide some conditions for $T : \mathcal{D}^k(X) \to \mathbb{R}$.

Definition 3.5 ([AK] Locality and Continuity). Let $T : \mathcal{D}^k(X) \to \mathbb{R}$ be a multilinear functional. We say that $T$ satisfies the locality if for $f \in \text{Lip}_b(X)$ and $\pi_i \in \text{Lip}(X)$ with $i = 1, \ldots, k$, if for some $i$, $\pi_i$ is constant on $\{f \neq 0\}$, then $T(f, \pi) = 0$.
We say that $T$ is continuous or satisfies the continuity if for any $f \in \text{Lip}_b(X)$, 
\[
\lim_{j \to \infty} T(f, \pi^j) = T(f, \pi)
\]
whenever $\pi^j \in \text{Lip}(X)$ converges to $\pi$ pointwise on $X$ as $j \to \infty$, with $\sup_{i,j} \text{Lip}(\pi^j) < \infty$, for each $i = 1, \cdots, k$.

**Definition 3.6** (Current and Normal current). An element $T \in \mathcal{D}_k(X)$ is called a $k$-dimensional current in $X$, for short $k$-current, if $T$ satisfies the locality and the continuity. We denote by $\mathcal{M}_k(X)$ the set of all $k$-currents in $X$, and by $\mathcal{N}_k(X)$ the set of all normal $k$-currents in $X$.

The locality and the continuity are preserving under $\mathcal{M}$-convergence and taking the boundary. In particular, $\mathcal{M}_k(X)$ is a Banach subspace of $\mathcal{D}_k(X)$, and $\mathcal{N}_k(X)$ is a Banach subspace of the space of all normal $k$-dimensional multilinear functionals on $\mathcal{D}_k(X)$ having finite mass with respect to the norm $\mathcal{N}$.

If $T \in \mathcal{M}_k(X)$ and $\phi : X \to Y$ is Lipschitz, then $\phi \# T \in \mathcal{M}_k(Y)$. In general, 
\[
\partial \phi \# = \phi \# \partial
\]
holds. So, $\phi \# : \mathcal{N}_k(X) \to \mathcal{N}_k(Y)$ is well-defined. Hence, we obtain

**Proposition 3.7.** $\mathcal{N}_*$ is a covariant functor from the category of all metric spaces together with all Lipschitz maps to $\mathcal{C}(\text{Ab})$. Here, $\mathcal{C}(\text{Ab})$ denotes the category of all chain complexes with all chain maps.

If $T \in \mathcal{M}_k(X)$ and a Borel set $B \subset X$, then $T|B \in \mathcal{M}_k(X)$.

A current $T \in \mathcal{M}_k(X)$ can be regarded as a functional on $L^1(X, \|T\|) \times [\text{Lip}(X)]^k$ as mentioned above. It is known to satisfy the strengthened locality: if $(f, \pi) \in \mathcal{B}^\infty(X) \times [\text{Lip}(X)]^k$ with $\{f \neq 0\} = \bigcup_{i=1}^k B_i$ for some Borel sets $B_i \subset X$ such that $\pi_i$ is constant on $B_i$ for each $i$, then $T(f, \pi) = 0$; and to satisfy the strengthened continuity: for any $f_j \to f$ in $L^1(X, \|T\|)$ as $j \to \infty$, and $\pi^j_i \to \pi_i$ as $j \to \infty$ pointwise on $X$ with $\sup_{i,j} \text{Lip}(\pi^j_i) < \infty$, 
\[
\lim_{j \to \infty} T(f_j, \pi^j) = T(f, \pi).
\]

**Remark 3.8.** There are two different points between our currents and Ambrosio-Kirchheim’s currents. One of them is that they employed the finite mass axiom as the condition about the existence of just a finite Borel measure $\mu$ on $X$ satisfying (3.1). The second one is they only dealt with complete metric spaces.

Instead of the assumption that at least one such a $\mu$ is tight, they assume that
\[
\text{(3.4) any set has the cardinality an Ulam number,}
\]
which condition is known to be consistent with the standard ZFC set theory. From the completeness of spaces, the assumption (3.4) automatically induces the tightness of any finite Borel measure $\mu$ in the
following way. It is known that any finite Borel measure $\mu$ on an arbitrary metric space $E$ (which contains a dense subset) of cardinality an Ulam number is concentrated on its support. In general, $\text{spt}(\mu)$ is separable. Taking a dense sequence $\{x_n\} \subset \text{spt}(\mu)$, for any $\varepsilon > 0$ and $k \in \mathbb{N}$, choosing suitable $h(k, \varepsilon) \in \mathbb{N}$, we set

$$K := \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{h(k, \varepsilon)} U(x_n, 1/k)$$

as constructed in [AK, Lemma 2.9]. By the construction, $\mu(E - K) \leq \varepsilon$ and $K$ is closed and totally bounded. Now, if $E$ is assumed to be complete, then $K$ becomes compact.

In [AK], they mentioned that their theory also works if one only consider currents whose mass measure is tight. On the other hand, it was not mentioned well where the completeness assumption of spaces intrinsically was used. We remark and one can observe that the completeness assumption of spaces was used only to lead the compactness of $K$ in the proof of [AK, Lemma 2.9], within Ambrosio-Kirchheim’s theory of currents related this paper.

Therefore, we employed the finite mass axiom as Definition 3.2. Then, we can deal with arbitrary metric spaces which need not to be complete.

Incidentally, the condition that multilinear functionals have the tight mass measures is closed, i.e., denoting by $\tilde{D}_k(X)$ the set of all multilinear functionals on $D_k(X)$ having finite mass which is not need to be tight, $\tilde{D}_k(X)$ is a Banach space equipped with norm $M(T) = \|T\|(X)$ for $T \in \tilde{D}_k(X)$ and $D_k(X)$ is a closed subspace of $\tilde{D}_k(X)$.

Since the countable infinite cardinality is an Ulam number, if $X$ is complete and separable, then $D_k(X) = \tilde{D}_k(X)$, i.e., there is no difference between our definition and theirs of currents.

Normal currents satisfy the following useful property. It follows from the definition of the normality and the product rule [AK, Theorem 3.5].

**Proposition 3.9** (Equi-continuity of normal currents, [AK, Proposition 5.1]). Let $T \in N_k(X)$. For any $f \in \text{Lip}_b(X)$ and $\pi_i, \pi'_i \in \text{Lip}(X)$ with $\text{Lip}(\pi_i), \text{Lip}(\pi'_i) \leq 1$,

$$|T(f, \pi) - T(f, \pi')| \leq \sum_{i=1}^{k} \left\{ \int_X |f| |\pi_i - \pi'_i| \|\partial T\| + \text{Lip}(f) \int_{\{f \neq 0\}} |\pi_i - \pi'_i| \|\partial T\| \right\}.$$

It is known that if $T$ is a normal $k$-current in $X$, then $\|T\|$ is absolutely continuous in $\mathcal{H}^k_X$ on the Borel $\sigma$-algebra of $X$, i.e., if $B \subset X$ is Borel and $\mathcal{H}^k(B) = 0$, then $\|T\|(B) = 0$ ([AK, Theorem 3.9]).

Let $T \in N_k(X)$ and $\phi \in \text{Lip}(X)$. Due to Localization Lemma 5.3 in [AK], for almost every $r \in \mathbb{R}$, $T[\{\phi > r\}$ is a normal current. Further,
if $T$ and $\partial T$ are concentrated on a $\sigma$-compact set $L$, then $T[\{\phi > r\}$ and $\partial(T[\{\phi > r\})$ are concentrated on $L$ for a.e. $r \in \mathbb{R}$.

By Slicing Theorem 5.6 in [AK], for any normal current $T$, we obtain

$$(3.5) \quad \text{spt}(\partial T) \subset \text{spt}(T).$$

We explain (3.5), however, we do not recall the definition and property of the slicing. For detail, we refer to [AK]. Let us set $\phi = d(\text{spt}(T), \cdot)$. Then, $T = T[\{\phi \leq r\}$ for every $r \geq 0$. Hence, $\partial T = \langle T, \phi, r \rangle - \langle \partial T \rangle[\{\phi \leq r\}$ which is concentrated on $\{\phi \leq r\}$ for a.e. $r > 0$, where $\langle T, \phi, r \rangle$ is the slicing of $T$ by $\phi$ at $r$. Therefore, $\partial T$ is concentrated on $\{\phi = 0\} = \text{spt}(T)$, and hence $\text{spt}(\partial T) \subset \text{spt}(T)$.

Let us define $N^c_k(X)$ by

$$N^c_k(X) = \{T \in N^c_k(X) \mid T \text{ has compact support}\}.$$

**Lemma 3.10.** Let $X$ be a metric space and $A$ a subset of $X$. Let $T \in N^c_k(X)$. We denote by $\iota$ the inclusion $A \rightarrow X$. Then, $T \in \iota_#N^c_k(A)$ if and only if $\text{spt}(T) \subset A$.

**Proof.** Let $T \in N^c_k(X)$. First, we suppose that $T \in \iota_#N^c_k(A)$. Namely, there exists $T' \in N^c_k(A)$ such that $T = \iota_#T'$. Since $T'$ has compact support, by (3.2), we obtain

$$\text{spt}(T) \subset \iota(\text{spt}(T')) \subset A.$$

Let us suppose that $\text{spt}(T) \subset A$. Since $N^c_k$ is covariant, we may assume that $A = \text{spt}(T)$. We define a map $T' : D^k(A) \rightarrow \mathbb{R}$ by

$$(3.6) \quad T'(f, \pi) = T(\hat{f}, \tilde{\pi})$$

for $(f, \pi) \in D^k(A)$. Here, $\tilde{\pi} = (\tilde{\pi}_i)$ and each $\tilde{\pi}_i$ is a Lipschitz extension of $\pi_i$ to $X$ with $\text{Lip}(\tilde{\pi}_i) = \text{Lip}(\pi_i)$. And, $\hat{f}$ is a trivial extension of $f$, i.e., it is defined by

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Since $A = \text{spt}(T)$ is compact, $\hat{f} \in B^\infty(X)$. Thus, $T(\hat{f}, \cdot)$ is actually defined. If $\tilde{\pi}' \in [\text{Lip}(X)]^k$ is another Lipschitz extension of $\pi$, then the strengthened locality of $T$ implies $T(\hat{f}, \tilde{\pi}) = T(\hat{f}, \tilde{\pi}')$. Therefore, the value $T(\hat{f}, \tilde{\pi})$ is well-defined. Since $f \mapsto \hat{f}$ is linear, $T'(f, \pi)$ is linear in $f$. A typical Lipschitz extension procedure $\text{Lip}(A) \ni \tau \mapsto \tilde{\tau} \in \text{Lip}(X)$ is homogeneous. And, it is clear that $\tilde{\tau} + \tilde{\tau}' - \tau + \tau' = 0$ on $A \supset \{\hat{f} \neq 0\}$ for $\tau, \tau' \in \text{Lip}(A)$. By the strengthened locality of $T$, we obtain that $T'$ is multilinear on $D^k(A)$. The locality of $T'$ is implied by the strengthened locality of $T$. We define a Borel measure $\mu$ on $A$ by $\mu(A') = ||T||(A')$ for every Borel set $A' \subset A$. By the tightness of $||T||$,}
μ is also tight. For any \((f, \pi) \in D^k(A)\),
\[
|T'(f, \pi)| = |T(\hat{f}, \hat{\pi})| \leq \prod_{i} \text{Lip}(\pi_i) \int_X |\hat{f}| d|T|
\]
\[
= \prod_{i} \text{Lip}(\pi_i) \int_A |f| d\mu.
\]
Therefore, \(T'\) satisfies the finite mass axiom.

Before checking that \(T'\) is continuous, we observe that (3.7)
\[
T(\hat{f}, \hat{\pi}) = T(\tilde{f}, \tilde{\pi})
\]
for \((f, \pi) \in D^k(A)\). Here, \(\tilde{f}\) is an extension of \(f\) to \(X\) which is bounded Lipschitz with \(\text{Lip}(\tilde{f}) = \text{Lip}(f)\). For instance, taking a constant \(C \geq \sup_A |f|, \max\{\min\{C, \tilde{f}\}, -C\}\) is such \(\tilde{f}\). Then, we have
\[
|T(\hat{f}, \hat{\pi}) - T(\tilde{f}, \tilde{\pi})| \leq \prod_{i} \text{Lip}(\pi_i) \int_X |\hat{f} - \tilde{f}| d|T|
\]
\[
= \prod_{i} \text{Lip}(\pi_i) \int_{\text{spt}(T)} |f - \tilde{f}| d|T| = 0.
\]
Hence, (3.7) holds.

Let us take \(\pi^j_i, \pi_i \in \text{Lip}(A)\) with \(\sup_{i,j} \text{Lip}(\pi^j_i) \leq L < \infty\) such that \(\pi^j_i \rightarrow \pi_i\) pointwise on \(A\) as \(j \rightarrow \infty\) for each \(i\). We may assume that \(L = 1\). The equi-continuity 3.9 of \(T\) implies
\[
|T(\hat{f}, \hat{\pi}^j) - T(\tilde{f}, \tilde{\pi})| \\ \leq \sum_{i=1}^{k} \left\{ \int_X |\hat{f}||\pi^j_i - \pi_i| d|T| + \text{Lip}(\tilde{f}) \int_{\{f \neq 0\}} |\pi^j_i - \pi_i| d|T| \right\}
\]
\[
\leq \sum_{i=1}^{k} \left\{ |f|_{\infty} \int_{\text{spt}(\partial T)} |\pi^j_i - \pi_i| d|T| + \text{Lip}(f) \int_{\text{spt}(T)} |\pi^j_i - \pi_i| d|T| \right\}
\]
\[
\rightarrow 0
\]
as \(j \rightarrow \infty\). Thus, we know that \(T'\) is continuous. \(\square\)

**Example 3.11** (Basic example [AK]). Let \(\theta \in L^1(\mathbb{R}^k)\). We define a current \([\theta] \in M_k(\mathbb{R}^k)\) by
\[
[\theta](f, \pi) = \int_{\mathbb{R}^k} \theta f \, d\pi_1 \wedge \cdots \wedge d\pi_k = \int_{\mathbb{R}^k} \theta f \det(\partial \pi_i/\partial x_j) \, d\mathcal{L}^k(x)
\]
for any \((f, \pi) \in D^k(\mathbb{R}^k)\). Here, \(\mathcal{L}^k\) denotes the \(k\)-dimensional Lebesgue measure. As mentioned in [AK], we explain that \([\theta]\) is actually a \(k\)-current in \(\mathbb{R}^k\). We notice that the derivatives \(\partial \pi_i/\partial x_j\) are defined at almost all points, due to the Rademacher’s theorem. The continuity is implied by the weak star continuity of the determinants in \(W^{1,\infty}(\mathbb{R}^k)\).
One can check that $\|\theta\| = |\theta|\mathcal{L}^k$. In particular, $\|\theta\|$ is concentrated on a $\sigma$-compact set.

3.3. **Integral currents.** Let $\mathcal{H}_X^k = \mathcal{H}_X^k(X)$ denote the $k$-dimensional Hausdorff measure on a metric space $X$. A subset $S$ of $X$ is called a *countably $\mathcal{H}_X^k$-rectifiable* set if there exist countably many Lipschitz maps $\phi_j$ from Borel subsets $B_j$ of $\mathbb{R}^k$ to $X$ and

$$\mathcal{H}_X^k \left( S - \bigcup_{j=1}^{\infty} \phi_j(B_j) \right) = 0.$$  

Kirchheim [K] proved that all $B_j$ and $\phi_j$ can be chosen so that each $B_j$ is compact, each $\phi_j$ is bi-Lipschitz on $B_j$ and the family $\{\phi_j(B_j)\}$ is disjoint. This fact is important for the theory of rectifiable currents in [AK].

**Definition 3.12** ((Integer) Rectifiable current [AK]). Let $k \geq 1$. A current $T \in M_k(X)$ is said to be *rectifiable* if $T$ is concentrated on a countably $\mathcal{H}_X^k$-rectifiable set and $\|T\|$ is absolutely continuous in $\mathcal{H}_X^k$ on the Borel $\sigma$-algebra of $X$.

Further, $T$ is said to be *integer rectifiable* if it is rectifiable and for any open subset $O \subset X$ and Lipschitz map $\phi : X \to \mathbb{R}^k$, there exists $\theta \in L^1(\mathbb{R}^k, \mathbb{Z})$ such that $\phi_#(T|O) = [\theta]$.

We say that a 0-dimensional current $T \in M_0(X)$ is *rectifiable* if there exist countably many points $x_j \in X$ and $\theta_j \in \mathbb{R}$ such that $T$ is represented by

$$T(f) = \sum_{j=1}^{\infty} \theta_j f(x_j)$$

for each $f \in \mathcal{B}_X^\infty$ ($\mathcal{B}$). We write such a $T$ as

$$T = \sum_j \theta_j [x_j].$$

(3.8)

We say that $T \in M_0(X)$ is *integer rectifiable* if it is represented as (3.8) so that $\theta_j$ can be chosen to be integers.

**Remark 3.13.** As mentioned in [RS], every integer rectifiable 0-current can be represented as (3.8) by finitely many points $x_j$ and integers $\theta_j$. Indeed, let us take an integer rectifiable 0-current $T$ as (3.8) for some formally countably many $x_j \in X$ and $\theta_j \in \mathbb{Z}$. Let us consider a function

$$f = \sum_{\theta_j \geq 1} 1_{\{x_j\}}$$

which is bounded Borel on $X$. Since the value

$$T(f) = \sum_{\theta_j \geq 1} \theta_j$$

for each $f \in \mathcal{B}_X^\infty$.
is bounded, the number of all \( j \) with \( \theta_j \geq 1 \) is finite. As well as, the number of all \( j \) with \( \theta_j \leq -1 \) is finite. Therefore, the index set \( \{ j \} \) can be chosen to be finite.

**Definition 3.14** (Integral current [AK]). We say that a current \( T \in \mathcal{M}_k(X) \) is *integral* if it is integer rectifiable and normal. The set of all \( k \)-dimensional integral currents in \( X \) is denoted by \( \mathcal{I}_k(X) \). Also, we denote by \( \mathcal{I}_k^c(X) \) the set of all \( k \)-dimensional integral currents in \( X \) with compact support.

**Lemma 3.15.** Let \( \phi : X \to Y \) be a Lipschitz map, \( \psi \in \text{Lip}(X) \) and let \( T \in \mathcal{N}_k(X) \) be a normal current. If \( T \) is rectifiable (resp. integer rectifiable), then so are \( \phi_{\#}T \) and \( T\{\psi > r\} \) for a.e. \( r \in \mathbb{R} \).

**Proof.** Recall that if \( T \in \mathcal{M}_k(X) \) is concentrated on some subset \( A \subset X \), then \( \phi_{\#}T \) is concentrated on \( \phi(A) \). If \( A \) is countably \( \mathcal{H}^k \)-rectifiable in \( X \), then so is \( \phi(A) \) in \( Y \). Hence, if \( T \) is concentrated on a countably \( \mathcal{H}^k \)-rectifiable set, then so is \( \phi_{\#}T \).

Let us suppose that \( T \) is normal. Since any normal current is absolutely continuous in \( \mathcal{H}^k \) and the normality is preserving under the push-forward, \( \phi_{\#}T \) is absolutely continuous in \( \mathcal{H}^k \).

In addition, we suppose that \( T \) is integer rectifiable. Let us take an open subset \( O \subset Y \) and Lipschitz map \( f \in \text{Lip}(Y, \mathbb{R}^k) \). By the definition of integer rectifiability, there exists \( \theta \in L^1(\mathbb{R}^k, \mathbb{Z}) \) such that

\[
[\theta] = (f \circ \phi)_{\#}(T|_{\phi^{-1}(O)}) = f_{\#}(\phi_{\#}T|_O)
\]

Thus, \( \phi_{\#}T \) is integer rectifiable.

Let \( \psi : X \to \mathbb{R} \) be a Lipschitz function. Due to Localization Lemma 5.3 in [AK], \( T\{\psi > r\} \) is normal for a.e. \( r \in \mathbb{R} \). Then, \( T\{\psi > r\} \) is absolutely continuous in \( \mathcal{H}^k \). Let \( S \) be a countably \( \mathcal{H}^k \)-rectifiable set in \( X \) on which \( T \) is concentrated. Then,

\[
\|T\{\psi > r\}\|(X - S) = \|T\|(\{\psi > r\} - S) = 0.
\]

Let us take an open subset \( O \subset X \) and \( f \in \text{Lip}(X, \mathbb{R}^k) \). Since

\[
(T\{\psi > r\}|_O = T\{(\psi > r) \cap O),
\]

there exists \( \theta \in L^1(\mathbb{R}^k, \mathbb{Z}) \) such that

\[
f_{\#}\left(T\{\psi > r\}\right) = [\theta].
\]

Thus, \( T\{\psi > r\} \) is integer rectifiable (resp. rectifiable) if so is \( T \). \( \Box \)

**Lemma 3.16.** Let \( X \) be a metric space and \( A \) a subset. Let \( T \in \mathcal{N}_k^c(X) \). We denote by \( \iota \) the inclusion \( A \to X \). Then, \( T \in \mathcal{I}_k^c(X) \) and \( \text{spt}(T) \subset A \) if and only if \( T \in \iota_{\#}\mathcal{I}_k^c(A) \).

**Proof.** Let \( (X, A) \in \text{Met}_2 \) and \( T \in \mathcal{N}_k^c(X) \). By Lemma 3.15 if \( T \in \iota_{\#}\mathcal{I}_k^c(A) \), then \( \text{spt}(T) \subset A \) and \( T \in \mathcal{I}_k^c(X) \).
Let us suppose that \( T \in \mathfrak{I}^k_X \) and \( \text{spt}(T) \subset A \). We may assume that \( \text{spt}(T) = A \). Let \( T' \in \mathfrak{N}^k_A \) be defined as (3.10) in the proof of Lemma 3.10. Then, \( \iota_\# T' = T \) and \( \mu = \|T\| \|A\) satisfies the condition of the finite mass axiom for \( T' \) on \( \mathcal{D}^k(A) \). We check that \( T' \) is integer rectifiable in \( A \). Since \( T' \) is a normal current in \( A \), \( T' \) is absolutely continuous with respect to \( \mathcal{H}^k_A \). Let \( S \) be countably \( \mathcal{H}^k_A \)-rectifiable set in \( X \) on which \( T \) is concentrated. We may assume that \( S = \bigcup_{j=1}^\infty f_j(K_j) \) for some compact sets \( K_j \subset \mathbb{R}^k \) and Lipschitz maps \( f_j : K_j \rightarrow X \). Since \( \|T\|(X - A) = 0 \), we obtain \( \mu(A - S) = \|T\|(A - S) = 0 \). By the compactness of \( A \), \( K_j' = f_j^{-1}(f_j(K_j) \cap A) \) is a Borel set in \( \mathbb{R}^k \) for each \( j \). Hence, \( \mu \) is concentrated on the countably \( \mathcal{H}^k_A \)-rectifiable set \( A \cap S = \bigcup_j f_j(K_j') \) in \( A \). Therefore, \( \|T'\| \) is also concentrated on \( A \cap S \). We prove that \( T' \) is integer rectifiable. Let us take an open subset \( O \) of \( A \) and a Lipschitz map \( f : A \rightarrow \mathbb{R}^k \). Then, there exist an open subset \( U \) of \( X \) and a Lipschitz map \( g : X \rightarrow \mathbb{R}^k \) such that \( O = U \cap A \) and \( g|_A = f \). Since \( T \) is integer rectifiable, there exists \( \theta \in L^1(\mathbb{R}^k, \mathbb{Z}) \) such that \( g\#(T|U) = [\theta] \). We see that \( T|U = (\iota_\# T')|U = \iota_\#(T'|O) \). Indeed, for any \((f, \pi) \in \mathcal{D}^k(X)\),
\[
(\iota_\# T')(U(f, \pi)) = T'(1_\Omega f \circ \iota, (\pi_i \circ \iota)) = \iota_\#(T'|O)(f, \pi).
\]
Therefore, we obtain
\[
[\theta] = g\#(T|U) = g\#(T'|O) = f\#(T'|O).
\]
Hence, \( T' \) is integer rectifiable. It completes the proof. \( \square \)

4. Homologies which will be compared

In this section, \( X \) and \( Y \) always denote metric spaces. We will recall the definitions of the singular Lipschitz (relative) homology and the (relative) homology of integral currents with compact supports. To compare these theories, we provide a natural transformation between them, according to [RS].

4.1. Singular Lipschitz homology. For a metric space \( X \), a singular Lipschitz simplex, chain, complex and homology of \( X \), are defined by a similar manner to corresponding usual singular ones of \( X \), replaced continuous by Lipschitz. For the completeness, we recall their definition.

Let \( \triangle^k \) denote a regular \( k \)-dimensional simplex. We consider \( \triangle^k \) as a subset of the Euclidean space \( \mathbb{R}^k \) and equip on it a metric defined by the restriction of the Euclidean metric. As usual way, choosing some geometrically independent vectors \( e_0, e_1, \ldots, e_k \in \mathbb{R}^k \), we write
\[
\triangle^k = [e_0, \cdots, e_k].
\]

A singular Lipschitz \( k \)-simplex of \( X \) is just a Lipschitz map from \( \triangle^k \) to \( X \). We denote by \( S^L_k(X) \) the subgroup of the usual group \( S_k(X) \) of singular \( k \)-chains of \( X \) with integer coefficients, generated by singular
Lipschitz \( k \)-simplexes of \( X \). An element of \( S^L_k(X) \) is called a singular Lipschitz \( k \)-chain of \( X \). Let \( b \) denote the boundary operator of the singular complex \( S_*(X) \) defined by

\[
b \sigma = \sum_{i=0}^k (-1)^i \sigma|e_0,...,\hat{e_i},...,e_k|
\]

for each \( k \)-simplex \( \sigma : [e_0,\cdots,e_k] \to X \). Then, \( S^L_*(X) \) becomes a subcomplex of \( S_*(X) \) with respect to \( b \), which is called the singular Lipschitz complex of \( X \). Its homology is called the singular Lipschitz homology of \( X \) and denoted by \( H^L_* \).

A locally Lipschitz map \( \phi : X \to Y \) induces the chain map \( \phi^# : S^L_*(X) \to S^L_*(Y) \), called the push-forward of \( \phi \), which is defined by a linear extension of the map \( \phi^# \sigma = \sigma \circ \phi \) for each singular Lipschitz simplex \( \sigma \) in \( X \).

Let \( A \) be a subset of \( X \). The inclusion \( \iota : A \to X \) is an isometric embedding. Then, the image \( \iota^#S^L_*(A) \) of \( S^L_*(A) \) under the push-forward \( \iota^# \) becomes a subcomplex of \( S^L_*(X) \). Since \( \iota^# \) is injective, we will not distinguish \( \iota^#S^L_*(A) \) and \( S^L_*(A) \) and regard \( S^L_*(A) \) as a subcomplex of \( S^L_*(X) \). The quotient complex

\[
S^L_*(X,A) = S^L_*(A)/S^L_*(A)
\]

is called the singular Lipschitz complex of the pair \((X,A)\). Its homology is denoted by \( H^L_*(X,A) \) which is called the singular Lipschitz relative homology of \((X,A)\).

We immediately obtain

**Proposition 4.1.** \( H^L_* \) is a covariant functor from \( \text{Met}_2 \) to \( \text{Ab} \).

Like the usual singular homology theory, we can also consider the reduced singular Lipschitz homology \( \tilde{H}^L_* \) for each \( X \in \text{Met} \), which is the homology of an augmented chain complex

\[
\cdots \to S^L_k(X) \to S^L_{k-1}(X) \to \cdots \to S^L_0(X) \xrightarrow{\varepsilon'} \mathbb{Z} \to 0,
\]

where \( \varepsilon' \) is given by \( \varepsilon'(\sum_{x \in X} a_x x) = \sum_{x \in X} a_x \). The map \( \varepsilon' \) is called an augmentation. Note that \( \varepsilon' \) is surjective. Then, \( \tilde{H}^L_* \) becomes a covariant functor from \( \text{hMet} \) to \( \text{Ab} \). It is clear that \( H^L_k(X) = \tilde{H}^L_k(X) \) for every \( k \geq 1 \).
4.2. Homology of integral currents. We already defined integral currents in a metric space $X$ in Section 3. The whole set of them was denoted by $I_*(X)$.

**Theorem 4.2** (Boundary Rectifiability Theorem [AK Theorem 8.6]). $I_*(X)$ is a subcomplex of $N_*(X)$. Namely, if $T \in I_k(X)$, then $\partial T \in I_{k-1}(X)$.

We recall that $I^c_*(X)$ is the whole set of integral currents in $X$ with compact supports.

**Proposition 4.3.** $I_*$ is a covariant functor from the category of all metric spaces as objects together with Lipschitz maps as morphisms to $\mathcal{C}(\text{Ab})$, and $I^c_*$ is a covariant functor from $\mathcal{M}$ to $\mathcal{C}(\text{Ab})$.

**Proof.** By Theorem 4.2 and Lemma 3.15, we know that $I_*$ is a covariant functor from the category of metric spaces together with Lipschitz maps to $\mathcal{C}(\text{Ab})$. Let $X$ be a metric space and $T \in I_k(X)$. By (3.5), $\text{spt}(\partial T)$ is compact, and hence $\partial T \in I^c_{k-1}(X)$. Let $\phi : X \to Y$ be a locally Lipschitz map. The push-forward $\phi_#T$ has the meaning as follows. Let $\iota : \text{spt}(T) \to X$ be the inclusion. By Lemma 3.16 there exists a unique $T' \in I_k^c(\text{spt}(T))$ such that $\iota_#T' = T$. Then, $\phi_#T$ is defined by $(\phi \circ \iota)_#T'$, which has the meaning, because $\phi \circ \iota : \text{spt}(T) \to Y$ is Lipschitz. Hence, we have $\phi_#T \in I_k(Y)$. In general, $(\phi \circ \iota)_#T'$ is concentrated on $\phi \circ \iota : \text{spt}(T)$ = $\phi(\text{spt}(T))$. Since $\text{spt}(T)$ is compact, we obtain $\text{spt}(\phi_#T) \subset \phi(\text{spt}(T))$. Therefore, $\phi_#T$ has compact support. $\square$

For a subset $A$ of $X$, the inclusion $\iota : A \to X$ induces a subcomplex $\iota_#I^c_*(A)$ of $I^c_*(X)$. Since $\iota_#$ is injective, we regard $I^c_*(A)$ itself as a subcomplex of $I^c_*(X)$. We set

$$I^c_*(X,A) = I^c_*(X)/I^c_*(A).$$

This chain complex is called the complex of integral currents in $(X,A)$ with compact supports. Its homology is called the relative homology of integral currents in $(X,A)$ with compact supports and is denoted by

$$H^c_*(X,A).$$

For $T \in M_k(X)$ and $t \in [0,1]$, we define $T \times [t] \in M_k(X \times [0,1])$ by

$$T \times [t] = i_t#T,$$

where $i_t : X \to X \times [0,1]$ is an isometric embedding $x \mapsto (x,t)$.

**Definition 4.4** ([W Definition 3.1], cf. [AK Definition 10.1]). For a metric space $X$ and a current $T \in M_k(X)$, we define a multilinear functional $T \times [0,1] \in \mathcal{D}^{k+1}(X \times [0,1])$ by

$$(T \times [0,1])(f, \pi_1, \ldots, \pi_{k+1}) :=
\sum_{i=1}^{k+1} (-1)^i \int_0^1 T(f_t \frac{\partial \pi_{i,t}}{\partial t}, \pi_{1,t}, \ldots, \pi_{i,t}, \ldots, \pi_{k+1,t}) dt$$
for \((f, \pi_1, \ldots, \pi_{k+1}) \in D^{k+1}(X \times [0, 1])\).

In general, because of the product \(T \times [0, 1]\) is defined by using
the derivatives \(\partial \pi_i / \partial t\), it is not expected that the continuity holds.
However, if \(T\) is a normal current, then the following holds.

**Theorem 4.5** ([W] Theorem 3.2 cf. [AK] Propositions 10.2 and 10.4]). For \(T \in N^k(X)\) with bounded support, \(T \times [0, 1] \in N_{k+1}(X \times [0, 1])\) with
the boundary \(\partial (T \times [0, 1]) = T \times [1] - T \times [0] - \partial T \times [0, 1]\). Moreover,
if \(T \in I^k(X)\), then \(T \times [0, 1] \in I^k_{k+1}(X \times [0, 1])\).

In particular, Theorem 4.5 implies that the correspondence
\(\cdot \times [0, 1] : I^k(X) \to I^k_{k+1}(X \times [0, 1])\) is a chain homotopy between
\(i_0#\) and \(i_1#\), where \(i_t : X \to X \times [0, 1]\) is an isometric embedding \(x \mapsto (x, t)\) for \(t \in \{0, 1\}\). It follows
Proposition 4.6. \(HIC^*\) is a covariant functor from \(\text{Met}_2\) to \(\text{Ab}\).

4.3. A natural transformation. We provide a natural transformation from \(SL^k\) to \(I^k\) on \(\text{Met}_2\). It was defined by [RS].

**Definition 4.7** ([RS]). Let \(X\) be a metric space and \(k \geq 0\). A map
\([\cdot] : [\cdot]_X : SL^k(X) \to I^k(X)\) is defined by the linear extension of the map
\([\sigma] = \sigma#[1_{\Delta^k}]\)
for each singular Lipschitz \(k\)-simplex \(\sigma : \Delta^k \to X\). Explicitly, for a singular Lipschitz \(k\)-chain \(c = \sum a_\sigma \sigma \in SL^k(X)\), \([c]\) is given by
\([c](f, \pi) = \sum a_\sigma \int_{\Delta^k} f \circ \sigma(x) \cdot \det(\nabla(\pi \circ \sigma)(x))\ d\mathcal{L}^k(x)\)
for all \((f, \pi) \in D^k(X)\).

For \(c \in SL^k(X)\), \([c]\) is actually an integral current with compact support, because \([1_{\Delta^k}]\) is an integral current in \(\Delta^k\). And, \([\cdot] : SL^k(X) \to I^k(X)\) is a chain map. Namely, for any singular Lipschitz \(k\)-simplex \(\sigma\) in \(X\), we have
\(\partial [\sigma] = [b\sigma]\).

It follows from the following Stokes’s theorem:
\(\int_{\Delta^k} df_1 \wedge df_2 \wedge \cdots \wedge df_k = \int_{\partial \Delta^k} f_1 df_2 \wedge \cdots \wedge df_k\)
holds, for Lipschitz functions \(f_1, \cdots, f_k\) on \(\Delta^k\). This is shown by a standard mollifier smoothing argument.

We note that
\(\text{spt}([c]) \subset \text{im}(c)\)
for any singular Lipschitz chain \(c\) in \(X\). In general, it is not an equality.
For instance, let \(\sigma : \Delta^k \to X\) be a singular \(k\)-simplex with \(k \geq 1\). If
Assume $\sigma$ is a constant map, then $\|\sigma\|(X) = 0$. In particular, $[\sigma] = 0$ and $\text{spt}([\sigma]) = \emptyset$ which is a proper subset of $\text{im}(\sigma)$.

**Proposition 4.8.** [ ] is a natural transformation from $S^L_*$ to $I^c_*$.

*Proof.* Let $X$ and $Y$ be metric spaces and $\phi : X \to Y$ a locally Lipschitz map. For each singular Lipschitz $k$-simplex $\sigma : \Delta^k \to X$, we have

$$\phi_# [\sigma] = \phi_# [1_{\Delta^k}] = [\phi_# \sigma].$$

Therefore, [] is natural. It complete the proof.

Further, one can define a natural transformation [ ] from $S^L_* (\cdot, \cdot)$ to $I^c_* (\cdot, \cdot)$ on the category $\text{Met}_2$ in a similar manner to defining the above [ ]. It induces a natural transformation

$$[] : H^L_* \to H^IC_*$$

on the category $\text{Met}_2$. We recall that a main purpose of the present paper is to prove that the natural transformation (4.1) is a natural isomorphism on a suitable category containing $\text{Met}_2^{\text{WLLC}}$.

For 0-th chains, we remark

**Lemma 4.9.** $[] : S^L_0 (X) \to I^c_0 (X)$ is isomorphic for every $X \in \text{Met}$.

*Proof.* Due to Remark 3.13 we note that $I^c_0 (X) = I_0 (X)$. Then, we already know that [ ] : $S^L_0 (X) \to I^c_0 (X)$ is surjective. Let us take singular (Lipschitz) 0-chain $c = \sum_{x \in X} a_x x \in S^L_0 (X)$ with $c \neq 0$. Let us choose $x \in X$ with $a_x \neq 0$. Then, we have

$$[c](1_{\{x\}}) = a_x \neq 0.$$

It follows that [] is injective on $S^L_0 (X)$.

**4.4. Poincaré type lemma.** In this subsection, we prove an important property for homologies $\widetilde{H}^L_*$ and $H^IC_*$ on WLLC spaces.

**Lemma 4.10.** Let $H$ denote one of the homologies $H^IC_*$ and $\widetilde{H}^L_j$ for $k \geq 1$ and $j \geq 0$. Let $V$ be a metric space and a subset $U \subset V$. If $U$ is Lipschitz contractible in $V$, then the inclusion $\iota : U \to V$ induces a trivial map $\iota_* : H(U) \to H(V)$.

*Proof.* We first consider the case that $H = \widetilde{H}^L_j$. Let $\varphi$ be a Lipschitz contraction from $U$ to some point $x_0 \in V$ in $V$. Namely, $\varphi : U \times [0, 1] \to V$ is a Lipschitz map such that $\varphi_0$ is the inclusion $\iota : U \to V$ and $\varphi_1$ is the constant map with value $x_0$. By Theorem 4.5, for $k \geq 1$, the map

$$\varphi_# (\cdot \times [0, 1]) : I^c_k (U) \to I^c_{k+1} (V)$$

satisfies that for every $T \in I^c_k (U)$,

$$\partial \varphi_# (T \times [0, 1]) = -\iota_# T - \varphi_# (\partial T \times [0, 1]).$$

Therefore, if $T \in I^c_k (U)$ satisfies $\partial T = 0$, then $\varphi_# (T \times [0, 1]) \in I^c_k (V)$ has the boundary

$$\partial \varphi_# (T \times [0, 1]) = -\iota_# T.$$
Hence, the induced map \( \iota_* : H^IC_k(U) \to H^IC_k(V) \) is trivial.

We note that the standard chain homotopy operator defined on \( S_* (X) \) preserves Lipschitz-ness of singular chains. By using it, one can prove Lemma 4.10 for \( H = \hat{H}^L_j \). It completes the proof. \( \square \)

**Definition 4.11.** Let \( C \) be one of \( \text{Ab} \) and \( \text{C(\text{Ab})} \). For a covariant functor \( H : \text{Met} \to C \), a metric space \( X \) is said to be \((H)\)-\textit{locally trivial} if for any \( x \in X \) and any open set \( V \) in \( X \) with \( x \in V \), there exists an open set \( U \) with \( x \in U \subset V \) such that the map \( H(\iota) : H(U) \to H(V) \) induced by the inclusion \( \iota : U \to V \) is trivial.

Let us denote by \( \text{Met}^{(H)\text{-LT}} \) the fullsubcategory of \( \text{Met} \) consisting of all \((H)\)-locally trivial metric spaces.

In Section 5 we will recall the notion of (pre)cosheaves. By using this terminology, \( X \in \text{Met} \) is \((H)\)-locally trivial if and only if regarding \( H \) as a precosheaf on \( X \), \( H \) is locally zero.

Obviously, every \( X \in \text{Met}^{\text{WLLC}\text{-LT}} \) is \((\hat{H}_j)\)-locally trivial for every \( j \geq 0 \), where \( \hat{H}_j \) is the usual reduced singular homology. In several literatures about algebraic topology, the \((\hat{H}_j)\)-locally triviality is also called the locally connectedness.

**Proposition 4.12.** The \((H)\)-locally triviality is inherited to open subsets and is stable under locally bi-Lipschitz homeomorphisms.

**Proof.** Let \( X \in \text{Met}^{(H)\text{-LT}} \) and \( X' \) an open subset of \( X \). Let us take \( x \in X' \) and an open neighborhood \( U \) of \( x \) in \( X' \). Then, there exists \( r' > 0 \) such that \( U_X(x,r') \subset U \). Here, \( U_Y(y,s) \) denotes the open ball centered at \( y \) with radius \( s \) in a metric space \( Y \). Since \( X \) is \((H)\)-locally trivial, there exists \( r \in (0,r'] \) such that the inclusion \( \iota : U_X(x,r) = U_X(x,r') \to U_X(x,r') \) implies a trivial map \( H(\iota) = 0 \). Since inclusions \( \iota' : U_X(x,r') \to U \) and \( \iota'' : U_Y(y,r) \to U \) satisfy \( \iota'' = \iota' \circ \iota \), the induced map is trivial:

\[
H(\iota'') = H(\iota') \circ H(\iota) = 0.
\]

Therefore, \( X' \) is \((H)\)-locally trivial.

Let \( X \in \text{Met}^{(H)\text{-LT}} \), \( Y \in \text{Met} \). Let \( f : X \to Y \) be a locally bi-Lipschitz homeomorphism. We may assume that \( f \) is a bi-Lipschitz homeomorphism. Let us take \( y \in Y \) and an open neighborhood \( V \) of \( y \) in \( Y \). Set \( x = f^{-1}(y) \in X \), \( L = \max\{\text{Lip}(f), \text{Lip}(f^{-1})\} \), then we obtain \( r > 0 \) such that \( U_X(x,r) \subset f^{-1}(V) \) and \( H(\iota) = 0 \) for the inclusion \( \iota : U_X(x,r) \to f^{-1}(V) \). Then, \( U_Y(y,L^{-1}r) \subset V \) and the inclusion \( \iota' : U_Y(y,L^{-1}r) \to U \) is given by \( \iota' = f^{-1} \circ \iota \circ f \), and hence \( H(\iota') = 0 \). \( \square \)

By Lemma 4.10 we immediately obtain

**Corollary 4.13 (Theorem 1.5).** If \( X \in \text{Met}^{\text{WLLC}} \), then \( X \in \text{Met}^{(H)\text{-LT}} \) for \( H = \hat{H}_j, \hat{H}^L_j \) and \( H^IC_k \) with \( j \geq 0 \) and \( k \geq 1 \).
We define a condition for metric spaces by using unified terminology, which is corresponding to a condition introduced by Riedweg and Schäppi at [RS] Definition 2.5.

**Definition 4.14.** Let us consider a covariant functor $\mathcal{C} : \text{Met} \to \mathcal{C}(\text{Ab})$ satisfying the following:

- For $X \in \text{Met}$ and any open set $U \subset X$, the map $\iota_\# : \mathcal{C}(U) \to \mathcal{C}(X)$ induced by the inclusion $\iota : U \to X$ is injective.
- For $j \geq 0$, $X \in \text{Met}$ and $T \in \mathcal{C}_j(X)$, there exist a unique compact set $K(T) \subset X$ such that for every open set $U \subset X$ with $K(T) \subset U$, there exists $T' \in \mathcal{C}_j(U)$ with $\iota_\#(T') = T$, where $\iota : U \to X$ is the inclusion.
- For $j \geq 0$, $X \in \text{Met}$ and $S \in \mathcal{C}_{j+1}(X)$, $K(bS) \subset K(S)$, where $b$ is the boundary operator of the chain complex $\mathcal{C}(X)$.
- For each $X \in \text{Met}$, an augmentation $\varepsilon = \varepsilon_X : \mathcal{C}_0(X) \to A(X)$ of the chain complex $\mathcal{C}(X)$ is given, i.e., $\varepsilon b_1 = 0$, for some $A(X) \in \text{Ab}$.

Let $j \in \mathbb{Z}_{\geq 0}$. A metric space $X$ admits a local cone inequality for $\mathcal{C}_j$ if for any $x \in X$, there exist $r > 0$ and a continuous non-decreasing function $F : [0, \infty) \to [0, \infty)$ with $F(0) = 0$ such that for every $T \in \mathcal{C}_j(X)$ with $K(T) \subset U(x, r)$ such that $\varepsilon T = 0$ when $j \geq 1$ and that $\varepsilon T = 0$ when $j = 0$, there exists $S \in \mathcal{C}_{j+1}(X)$ satisfying $bS = T$ and $\text{diam} K(S) \leq F(\text{diam} K(T))$.

For example, $S_*, S_*^L$ and $I_\varepsilon$ satisfy conditions imposed to $\mathcal{C}$ with respect to $K(\cdot) = \text{im}(\cdot)$ and $K(\cdot) = \text{spt}(\cdot)$, respectively, for any augmentations.

**Proposition 4.15.** Let $\mathcal{C} : \text{Met} \to \mathcal{C}(\text{Ab})$ be a covariant functor satisfying the property written in Definition 4.14. Let $j \geq 0$. If a metric space $X$ admits a local cone inequality for $\mathcal{C}_j$, then $X$ is $(H_j(\mathcal{C}))$-locally trivial. Here, $\tilde{H}_\varepsilon(\mathcal{C})$ is the reduced homology of $\mathcal{C}$ with respect to the augmentation $\varepsilon$ given in Definition 4.14.

**Proof.** Let $X$ admit a local cone inequality for $\mathcal{C}_j$. Then, for $x \in X$, there exists $r > 0$ and $F : [0, \infty) \to [0, \infty)$ satisfying the condition written in Definition 4.14. For any $s \in (0, r)$, we choose $s' \in (0, s)$ with $s' + F(2s') < s$.

Let us take any $T \in \mathcal{C}_j(X)$ with $K(T) \subset U(x, s')$ such that $\varepsilon T = 0$ when $j \geq 1$ and that $\varepsilon T = 0$ when $j = 0$. Since $s' < s < r$, there exists $S \in \mathcal{C}_{j+1}(X)$ such that $bS = T$ and

$$\text{diam} K(S) \leq F(\text{diam} K(T)).$$

For any $y \in K(T) \subset K(S)$ and $z \in K(S)$, we have

$$d(x, z) \leq d(x, y) + d(y, z) < s' + F(2s') < s.$$
Hence, $K(S) \subset U(x, s)$. Therefore, the morphism
\[
\tilde{H}_j(\mathcal{C}(U(x, s'))) \to \tilde{H}_j(\mathcal{C}(U(x, s)))
\]
is trivial. It completes the proof. \qed

5. Cosheaves

In this section, we recall the notion of (pre)cosheaves on a topological space, together with its fundamental theory. This is useful to compare two different homologies. For detail, we refer to the book [B] and the papers [B2] and [DP].

We will define certain precosheaves of currents on a metric space, and prove that they are actually cosheaves.

5.1. Cosheaves. Let $X$ be a topological space. Let us denote by $\mathcal{O}(X)$ the set of all open subsets of $X$. We regard $\mathcal{O}(X)$ as a category as follows: objects are all $U \in \mathcal{O}(X)$ and there exists a unique arrow $U \to V$ if and only if $U \subset V$ and there is no arrow from $U$ to $V$ otherwise.

Let us denote by $\text{Ab}$ the category of all abelian groups and by $\mathcal{C}(\text{Ab})$ the category of all chain complexes. Let $C$ be one of $\text{Ab}$ and $\mathcal{C}(\text{Ab})$. We call a covariant functor $A : \mathcal{O}(X) \to C$ a precosheaf on $X$ (to $C$). We only consider a precosheaf $A$ satisfying $A(\emptyset) = 0$. When $U \subset V$, the morphism $A(U \to V)$ is denoted by $i_U : A(U) \to A(V)$.

A precosheaf is called a cosheaf if for every family $U = \{U_\alpha\}_{\alpha \in I}$ of open subsets in $X$ with $U = \bigcup_\alpha U_\alpha$, the sequence
\[
\bigoplus_{\alpha_0, \alpha_1 \in I} A(U_{\alpha_0} \cap U_{\alpha_1}) \xrightarrow{f} \bigoplus_{\alpha \in I} A(U_\alpha) \xrightarrow{\varepsilon} A(U) \to 0
\]
is exact, where $\varepsilon = \sum_\alpha i_U, U_\alpha$ and $f = \sum_{\alpha_0, \alpha_1} i_{U_{\alpha_1}, U_{\alpha_0} \cap U_{\alpha_1}} - i_{U_{\alpha_0}, U_{\alpha_0} \cap U_{\alpha_1}}$. In general, $\varepsilon \circ f = 0$.

There is another characterization of cosheaves:

**Proposition 5.1** ([B, Chapter VI, Proposition 1.4]). Let $\mathcal{A}$ be a precosheaf on $X$. Then, $\mathcal{A}$ is a cosheaf if and only if it satisfies the following two conditions:

- For any open sets $U$ and $V$ in $X$, the sequence
  \[
  \mathcal{A}(U \cap V) \xrightarrow{f} \mathcal{A}(U) \oplus \mathcal{A}(V) \xrightarrow{\varepsilon} \mathcal{A}(U \cup V) \to 0
  \]
is exact.

- If a family $\{U_\alpha\}$ of open sets in $X$ is directed upwards by inclusion, then the map
  \[
  \lim_\leftarrow \mathcal{A}(U_\alpha) \to \mathcal{A}(U)
  \]
  induced by $\{i_U, U_\alpha\}$, is isomorphic, where $U = \bigcup_\alpha U_\alpha$.

We say that a precosheaf $\mathcal{A}$ on $X$ is flabby if for any $U \in \mathcal{O}(X)$, the morphism $\mathcal{A}(U) \to \mathcal{A}(X)$ is a monomorphism.
5.2. **Singular (Lipschitz) cosheaves.** We provide elementary examples of (pre)cosheaves.

**Example 5.2 ([B], Singular cosheaf \( \mathcal{G}_s \)).** Let \( X \) be a topological space and \( k \geq 0 \). We recall that \( S_k(X) \) is the usual singular chain \( k \)-th complex of \( X \). Then, the correspondence \( \mathcal{O}(X) \ni U \mapsto S_k(U) \in \text{Ab} \) is a flabby precosheaf on \( X \). In general, \( S_k \) is not cosheaf.

Let us consider a sequence of the barycentric subdivisions

\[
S_k(X) \xrightarrow{S_d} S_k(X) \xrightarrow{S_d} \cdots \xrightarrow{S_d} S_k(X) \xrightarrow{S_d} \cdots
\]

Its direct limit is denoted by \( \mathcal{G}_k(X) \). Namely, \( \mathcal{G}_k(X) \) is the quotient group of \( S_k(X) \) identifying \( c \) and \( c' \in S_k(X) \) whenever \( S_d^m(c) = S_d^{m'}(c') \) for some \( m, m' \geq 0 \). Then, \( \mathcal{O}(X) \ni U \mapsto \mathcal{G}_k(U) \in \text{Ab} \) becomes a flabby cosheaf. Since the subdivision \( S_d \) and the identity \( \text{Id} \) are chain homotopic to each other, there is a natural isomorphism

\[
H_*(X) \cong H_*(\mathcal{G}_s(X)).
\]

We remark that for any subset \( A \subseteq X \), \( \mathcal{G}(A) \to \mathcal{G}(X) \) is injective. Obviously, \( S_0(X) = \mathcal{G}_0(X) \).

**Example 5.3 (Singular Lipschitz cosheaf \( \mathcal{G}^L_s \)).** Let us suppose that \( X \) is a metric space. By a similar manner to Example 5.2 we can define an abelian group

\[
\mathcal{G}^L_s(X) := \lim_{\to} (S^L_k(X) \xrightarrow{S_d} S^L_k(X) \xrightarrow{S_d} \cdots).
\]

Then, \( \mathcal{O}(X) \ni U \mapsto \mathcal{G}^L_s(U) \in \text{Ab} \) is a flabby cosheaf on \( X \) and naturally

\[
H^L_*(X) \cong H_*(\mathcal{G}^L_s(X)).
\]

Let \( X \) be a metric space. Since \( [S_d^m c] = [c] \) for every \( m \geq 0 \) and \( c \in S^L_k(X) \),

\[
(5.2) \quad [\cdot] : \mathcal{G}^L_s(X) \to \mathcal{I}^L_s(X)
\]

can be defined. We will use this to compare with \( H^L_* \) and \( H^{1C}_* \).

5.3. **Cosheaves of currents.** Let us suppose that \( X \) is a metric space. Let us regard \( \mathcal{N}^c_s \) and \( \mathcal{I}^c_s \) as precosheaves on \( X \) to \( \mathcal{C}(\text{Ab}) \). By Lemma 3.4, they are flabby. We prove

**Proposition 5.4.** \( \mathcal{N}^c_s \) and \( \mathcal{I}^c_s \) are flabby cosheaves on \( X \) to \( \mathcal{C}(\text{Ab}) \).

**Proof.** We only check two conditions in Proposition 5.1. We first consider \( \mathcal{N}^c_s \). Let us prove that if a family \( \{U_\alpha\} \) of open sets in \( X \) is directed upwards by inclusion, then the homomorphism

\[
(5.3) \quad \lim_{\to} i_{U,U_\alpha} : \lim_{\to} \mathcal{N}^c_s(U_\alpha) \to \mathcal{N}^c_s(U)
\]

is isomorphic, where \( U = \bigcup_\alpha U_\alpha \). Since taking the direct limit is an exact functor and \( \mathcal{N}^c_s \) is flabby, (5.3) is injective. Let us take \( T \in \mathcal{N}^c_s(U) \). Since \( \text{spt}(T) \) is compact and \( \{U_\alpha\} \) is directed by inclusion, there exists \( \alpha \) with \( \text{spt}(T) \subseteq U_\alpha \). Hence, (5.3) is surjective.
Let $U, V \in \mathcal{O}(X)$. We prove that
\begin{equation}
0 \to \mathbf{N}_k^c(U \cap V) \xleftarrow{f} \mathbf{N}_k^c(U) \oplus \mathbf{N}_k^c(V) \xrightarrow{\varepsilon} \mathbf{N}_k^c(U \cup V) \to 0
\end{equation}
is exact. Here, $f(T) = (-T, T)$ and $\varepsilon(T, T') = T + T'$. It is trivial that $f$ is injective. To check that $\varepsilon$ is surjective, let us take $T \in \mathbf{N}_k^c(U \cup V)$.
We consider the distance function $d$ from $X - V$. Since $\text{spt}(T)$ is compact, there exists $r > 0$ such that
\[ \text{spt}(T) \cap \{d \leq r\} \subset U. \]
Retaking $r$ if necessary, by Localization Lemma 5.3 in [AK], we may assume that $T[\{d \leq r\}]$ is normal. By the choice of $r$, we have $S := T[\{d \leq r\}] \in \mathbf{N}_k^c(U)$ and $S' := T[\{d > r\}] \in \mathbf{N}_k^c(V)$. Note that we now use Lemma 3.10 intrinsically. Hence, $\varepsilon$ is surjective. Next, we prove that $\text{im} \ f \supset \ker \varepsilon$. To do this, we let us take $(S, S') \in \mathbf{N}_k^c(U) \oplus \mathbf{N}_k^c(V)$ satisfying $S + S' = 0$. We see that $\text{spt}(S) \subset V$. If it fails, then there exists $x \in \text{spt}(S) - V$ such that for every $r > 0$, $\|S\|(U(x, r)) > 0$. Since
\[ \|S\|(U(x, r)) \leq \|S + S'\|(U(x, r)) + \|S'\|(U(x, r)), \]
we have $\|S'\|(U(x, r)) > 0$. However, $x \notin \text{spt}(S')$. It is a contradiction. Hence, $\text{spt}(S) \subset U \cap V$. As well as, we obtain $\text{spt}(S') \subset U \cap V$. It turns out that (5.4) is exact. Due to Proposition 5.1, $\mathbf{N}_k^c$ is a cosheaf on $X$.

One can also prove that $\mathcal{I}_k^c$ is a cosheaf on $X$ as well as $\mathbf{N}_k^c$. It just suffices to remember that if $T$ is integral, then so is $T[\{d \leq r\}]$ for a.e. $r \in \mathbb{R}$. \hfill \Box

**Remark 5.5.** De Pauw proved a similar statement to Proposition 5.4 for currents in subsets in a Euclidean space ([DP]).

### 5.4. Čech chain complex

Let $X$ be a topological space. We denote by $\text{Cov}(X)$ the class of all open coverings of $X$. Namely, an element $\mathcal{U} \in \text{Cov}(X)$ is a map $\mathcal{U} : I \ni \alpha \mapsto U_\alpha \in \mathcal{O}(X)$ with an index set $I$ such that $\bigcup_{\alpha \in I} U_\alpha = X$. We also regard $\mathcal{U}$ as a family $\{U_\alpha\}_{\alpha \in I}$ of open sets. For a while, we fix one open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of $X$. The *nerve* $N(\mathcal{U})$ of $\mathcal{U}$ is an abstract simplicial complex defined as follows. For indexes $\alpha_0, \alpha_1, \ldots, \alpha_k \in I$, we set $U_{\alpha_0 \alpha_1 \cdots \alpha_k} := U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$. The zero-simplexes are all elements $\alpha \in I$ with $U_\alpha \neq \emptyset$, the one-simplexes are all pair $(\alpha_0, \alpha_1) \in I \times I$ with $U_{\alpha_0 \alpha_1} \neq \emptyset$, the two-simplexes are triple $(\alpha_0, \alpha_1, \alpha_2)$ of indexes with $U_{\alpha_0 \alpha_1 \alpha_2} \neq \emptyset$ and so on.

Let $\mathcal{A}$ be a precosheaf on $X$. For $k \geq 0$, we set an abelian group
\[ \check{C}_k(\mathcal{U}; \mathcal{A}) := \bigoplus_{(\alpha_0, \ldots, \alpha_k) : k\text{-simplex in } N(\mathcal{U})} \mathcal{A}(U_{\alpha_0 \cdots \alpha_k}) \]
and call it the *Čech $k$-chains of $\mathcal{U}$ coefficients in $\mathcal{A}$*. For $k \geq 1$, we define a map
\[ \Phi_k = \Phi_k^{\mathcal{U}, \mathcal{A}} : \check{C}_k(\mathcal{U}; \mathcal{A}) \to \check{C}_{k-1}(\mathcal{U}; \mathcal{A}) \]
by

\[
\Phi_k^{U, A} = \sum_{(\alpha_0, \alpha_1, \ldots, \alpha_k)} \sum_{j=0}^k (-1)^j i_{U\alpha_0 \ldots \hat{\alpha}_j \ldots \alpha_k, U\alpha_0 \ldots \alpha_k}.
\]

The map \( \Phi_k^{U, A} \) is none other than \( f \) in (5.1). We set \( \Phi_0 = 0 \). The sequence of maps \( \{ \Phi_k^{U, A} \}_{k \geq 0} \)

\[
\cdots \rightarrow \check{C}_k(U; A) \rightarrow \cdots \rightarrow \check{C}_1(U; A) \rightarrow \check{C}_0(U; A) \rightarrow 0
\]

becomes a chain complex. Its homology

\[
\check{H}_k(U; A) = \ker \Phi_k / \text{im} \Phi_{k+1}
\]

for \( k \geq 0 \), is called the Čech homology of \( U \) coefficients in \( A \).

Also, there is a canonical augmentation

\[
\varepsilon = \varepsilon^{U, A} : \check{C}_0(U; A) \rightarrow A(X)
\]

defined by \( \varepsilon^{U, A} = \sum_{\alpha} i_{X, U\alpha} \), which is none other than \( \varepsilon \) in (5.1). Recall that if \( A \) is a cosheaf, then \( \varepsilon \) is surjective and \( \ker \varepsilon / \text{im} \Phi_0^{A, U} = 0 \). In addition, if \( A \) is flabby, then the following holds.

**Proposition 5.6** ([13, Chapter VI, Corollary 4.5]). Let \( X \) be a topological space and \( A \) be a flabby cosheaf. Then, \( \check{H}_k(U; A) = 0 \) for any \( k \geq 1 \) and any \( U \in \text{Cov}(X) \).

For two open coverings \( U \) and \( V \) of \( X \), we say that \( V \) is a refinement of \( U \) if there exists a map \( \lambda : V \rightarrow U \) such that for every \( V \in V \), \( V \subset \lambda(V) \). We denote by \( V \succeq U \) this situation and call such a map \( \lambda \) a refinement projection. A refinement projection \( \lambda : V \rightarrow U \) implies a chain map \( \lambda_# : \check{C}_k(V; A) \rightarrow \check{C}_k(U; A) \). We remark that

\[
\varepsilon^{U, A} \circ \lambda_# = \varepsilon^{V, A}
\]

holds. We will use this formula in the proofs of main results.

Incidentally, for a refinement \( V \succeq U \), a morphism \( \lambda_* : \check{H}_*(V; A) \rightarrow \check{H}_*(U; A) \) is induced and is determined independently on the choice of refinement projection \( \lambda : V \rightarrow U \). The inverse limit

\[
\check{H}_*(X; A) := \varprojlim \check{H}_*(U; A)
\]

with respect to \( \succeq \), is called the Čech homology of \( X \) coefficients in \( A \).

5.5. **Locally triviality.** A precosheaf \( H \) on \( X \) is said to be locally zero or locally trivial if for every \( x \in X \) and every \( U \in O(X) \) with \( x \in U \), there exists \( U' \in O(X) \) such that \( x \in U' \subset U \) and \( H(U') \rightarrow H(U) \) is trivial.

A topological space \( X \) is paracompact if for any open covering of \( X \), there exists a locally finite refinement of it. We remark that every metric space is paracompact.
Proposition 5.7 ([B2 Theorem 4.4], [DP] Proposition 2.8). Let $X$ be a paracompact space. Let $H$ be a locally zero precosheaf on $X$. Then, for any open covering $\mathcal{U} \in \text{Cov}(X)$, there exists its refinement $\mathcal{V} \in \text{Cov}(X)$ with a refinement projection $\lambda : \mathcal{V} \to \mathcal{U}$ such that if $V_0, \ldots, V_p \in \mathcal{V}$ with $V_0 \cap \cdots \cap V_p \neq \emptyset$, then

$$H(V_0 \cap \cdots \cap V_p) \to H(\lambda(V_0) \cap \cdots \cap \lambda(V_p))$$

is trivial.

6. Comparing the homologies

In this section, we prove that the natural transformation $[\_]_*$ is a natural isomorphism from $H_*^L$ to $H_*^C$ on a suitable category containing $\text{Met}^\text{WLLC}$ (or $\text{Met}_2^\text{WLLC}$). At the end of this section, we also prove the inclusion $\iota : S_*^L \to S_*$ induces the natural isomorphism between the functors $H_*$ and $H_*^L$ on $\text{Met}_2^\text{WLLC}$.

6.1. Double complex. In this subsection, we recall some basic fact (Proposition 6.1) to compare homologies by using augmented double complexes. For precisely, we refer to [DP] or some book about homological algebra. If one well know about Proposition 6.1, then one can skip this subsection.

Let us consider the following diagram

$$
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \\
\Psi & \Psi & \Psi & \partial & \\
\cdots & \Phi & A_{2,2} & \Phi & A_{1,2} & \Phi & A_{0,2} & \varepsilon & M_2 & \rightarrow & 0 \\
\Psi & \Psi & \Psi & \partial & \\
\cdots & \Phi & A_{2,1} & \Phi & A_{1,1} & \Phi & A_{0,1} & \varepsilon & M_1 & \rightarrow & 0 \\
\Psi & \Psi & \Psi & \partial & \\
\cdots & \Phi & A_{2,0} & \Phi & A_{1,0} & \Phi & A_{0,0} & \varepsilon & M_0 & \rightarrow & 0 \\
\varepsilon' & \varepsilon' & \varepsilon' & \varepsilon' & \\
\cdots & M'_2 & M'_1 & M'_0 & \\
0 & 0 & 0 & 0 & \\
\end{array}
$$

of morphisms among abelian groups $A_{\*\*}$, $M_*$ and $M'_*$. Suppose that all rows and columns are chain complexes and that this diagram commutes. In this subsection, we does not use an information about $M'_*$ and $\varepsilon'$. So, we may consider as $M'_* = 0$, for a while. Such data $A_{\*\*} = (A_{\*\*}, \Phi, \Psi)$ is called a double complex and $(A_{\*\*}, M_*, \varepsilon) =$
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\((A_{*,*}, \Phi, \Psi), (M_*, \partial, \varepsilon)\) is called an augmented double complex (by an augmentation \(\varepsilon\)). In other words, the sequence of chain maps

\[
\cdots \xrightarrow{\Phi} A_{p,*} \xrightarrow{\Phi} A_{p-1,*} \xrightarrow{\Phi} \cdots \xrightarrow{\Phi} A_{0,*} \xrightarrow{\varepsilon} M_* \xrightarrow{} 0
\]
is a chain complex of chain complexes.

The total complex of \(A_{*,*}\) is a chain complex \(A_*\) defined by

\[
A_m = \bigoplus_{p+q=m} A_{p,q}
\]

with the boundary map \(\Delta = \sum_{p+q=m} \Phi + (-1)^p \Psi_p : A_m \to A_{m-1}\). Its homology is denoted by \(H_*(A_*) := H_*(A_*)\).

The morphisms \(\varepsilon : A_{0,*} \to M_*\) induces a morphism

\[
\varepsilon_* : H_*(A_{*,*}) \to H_*(M_*)
\]

**Proposition 6.1.** Let \((A_{*,*}, M_*, \varepsilon)\) be an augmented double complex as above. Let \(m \geq 0\). If the sequences

\[
A_{m-q,q} \to A_{m,q} \to \cdots \to A_{1,q} \to A_{0,q} \to M_q \to 0
\]
in \([6.1]\) are exact for \(q = 0, 1, \ldots, m\), then \(\varepsilon_* : H_*(A_{*,*}) \to H_*(M_*)\) is surjective. Further, \(\varepsilon_* : H_{m-1}(A_{*,*}) \to H_{m-1}(M_*)\) is isomorphic, whenever \(m \geq 1\).

**Sketch of the proof.** The complete proof is written in several literatures, for instance, it exists in [DP]. We just prove that \(\varepsilon_*\) is surjective. Let \(c \in M_m\) with \(\partial c = 0\). Since \(\varepsilon : A_{0,m} \to M_m\) is surjective, there is an element \(c_{0,m} \in A_{0,m}\) such that \(\varepsilon c_{0,m} = c\). By the assumption on \(c\), we have \(\partial \varepsilon c_{0,m} = \varepsilon \partial c_{0,m} = 0\). Since \(A_{1,m-1} \xrightarrow{\Phi} A_{0,m-1} \xrightarrow{\varepsilon} M_{m-1}\) is exact, there exists \(c_{1,m-1} \in A_{1,m-1}\) such that \(\Phi c_{1,m-1} = \varepsilon c_{0,m}\).

By using the assumption and repeating such an argument, we obtain \(c_{k,m-k} \in A_{k,m-k}\) such that

\[
\Phi c_{k,m-k} = \Psi c_{k-1,m-k+1}
\]
for \(1 \leq k \leq m\). So, we find out \((c_{k,m-k})_k \in \bigoplus_{k=0}^m A_{k,m-k} = A_m\) with \(\Delta((c_{k,m-k})) = 0\). Therefore, \(\varepsilon_* : H_*(A_{*,*}) \to H_*(M_*)\) is surjective. \(\square\)

6.2. **Fixing the notation.** To compare \(H_*^{L_\nu}\) and \(H_*^{IC}\), we fix the notation. Let \(X \in \text{Met}\) and let \(\nu = \{V_\sigma\} \in \text{Cov}(X)\). Let \(\Phi\) denote a map \(\Phi^\nu : \mathcal{C}^L_\nu \to \mathcal{C}^IC\) defined as \([5.5]\), which is the boundary map of the \(\check{\text{C}}ech\) chain complexes (of \(\nu\) coefficients in \(\mathcal{C}^L_\nu\) or in \(\mathcal{C}^{IC}\)). Let \(\varepsilon\) denote \(\varepsilon^\nu : \mathcal{C}^L_\nu \to \check{\mathcal{C}}^L_\nu\) defined as \([5.6]\), which is the canonical augmentation of the \(\check{\text{C}}ech\) chain complexes.

Let \(q \geq 1\). The morphisms

\[
b : \mathcal{C}^L_q(\nu) \to \mathcal{C}^L_{q-1}(\nu) \quad \text{and} \quad \partial : \mathcal{I}^\nu_q(\nu) \to \mathcal{I}^\nu_{q-1}(\nu)
\]
for \(\sigma \in \mathcal{N}(\nu)\) imply the morphisms

\[
b : \check{\mathcal{C}}_p(\nu; \mathcal{C}^L_q) \to \check{\mathcal{C}}_p(\nu; \mathcal{C}^L_{q-1}) \quad \text{and} \quad \partial : \check{\mathcal{C}}_p(\nu; \mathcal{I}^\nu_q) \to \check{\mathcal{C}}_p(\nu; \mathcal{I}^\nu_{q-1})
\]
for $p \geq 0$. Thus, we regard $\hat{C}_s(\mathcal{V}; \mathcal{G}_s^L)$ and $\hat{C}_s(\mathcal{V}; \mathcal{I}_s^*)$ as double complexes. Further, let $\mathcal{U} \in \text{Cov}(X)$ with a refinement projection $\lambda : \mathcal{V} \to \mathcal{U}$. Then,

$$\lambda \circ b = b \circ \lambda \quad \text{and} \quad \lambda \circ \partial = \partial \circ \lambda$$

hold.

Let $p \geq 0$. Let us set $C_p(N(\mathcal{V})) = \bigoplus_{\sigma = (\alpha_0, \cdots, \alpha_p)} \mathbb{Z} \langle \sigma \rangle$ which is the $p$-th chain complex of the nerve $N(\mathcal{V})$ as abstract simplicial complex. We will not use the boundary map of this chain complex. For each $p$-chain $\sigma = (\alpha_0, \cdots, \alpha_p) \in N(\mathcal{V})$, a canonical map $\varepsilon' : \mathcal{G}_0^L(V_\sigma) \to \mathbb{Z} \langle \sigma \rangle$ is defined by

$$\varepsilon' \left( \sum_j a_j x_j \right) = \left( \sum_j a_j \right) \langle \sigma \rangle$$

where $a_j \in \mathbb{Z}$ and $x_j \in V_\sigma$. This induces an augmentation

$$\varepsilon' : \hat{C}_p(\mathcal{V}; \mathcal{G}_0^L) \to C_p(N(\mathcal{V}))$$

of the chain complex $(\hat{C}_p(\mathcal{V}; \mathcal{G}_0^L), b)$. As well as, we can define an augmentation

$$\varepsilon' : \hat{C}_p(\mathcal{V}; \mathcal{I}_0^L) \to C_p(N(\mathcal{V}))$$

of the chain complex $(\hat{C}_p(\mathcal{V}; \mathcal{I}_0^L), \partial)$, via an identification $[\cdot] : \mathcal{G}_0^L(V_\sigma) \xrightarrow{\cong} \mathcal{I}_0(V_\sigma)$ for each $\sigma \in N(\mathcal{U})$. We does not distinguish the two symbols $\varepsilon'$.

Let $\mathcal{V} = \{V_\alpha\}$, $\mathcal{U} = \{U_\beta\} \in \text{Cov}(X)$ with a refinement projection $\lambda : \mathcal{V} \to \mathcal{U}$. Then, a map $\lambda \circ : C_p(N(\mathcal{V})) \to C_p(N(\mathcal{U}))$ is induced by

$$\lambda \circ \sigma = (\lambda(\alpha_0), \cdots, \lambda(\alpha_p))$$

for each $p$-simplex $\sigma = (\alpha_0, \cdots, \alpha_p) \in N(\mathcal{V})$. Further,

$$\lambda \circ \varepsilon' = \varepsilon' \lambda \circ$$

holds. Indeed, the diagram

$$\begin{array}{ccc}
\mathcal{G}_0^L(V_\sigma) & \xrightarrow{\lambda \circ} & \mathcal{G}_0^L(U_{\lambda \circ \sigma}) \\
\varepsilon' \downarrow & & \varepsilon' \downarrow \\
\mathbb{Z} \langle \sigma \rangle & \xrightarrow{\lambda \circ} & \mathbb{Z} \langle \lambda \circ \sigma \rangle
\end{array}$$

commutes for every $\sigma \in N(\mathcal{V})$. 

6.3. Injectivity. We prove

**Theorem 6.2.** Let \( m \geq 0 \). If \( X \) is an \((H)\)-locally trivial space for \( H = \tilde{H}^j \) with \( 0 \leq j \leq m \), then \([\cdot]_s : H^L_m(X) \to H^L_m(C)(X)\) is injective.

**Proof.** Let us choose \( \mathcal{U}_0, \cdots, \mathcal{U}_{m+1} \in \text{Cov}(X) \) such that for each \( 0 \leq p \leq m, \mathcal{U}_p \) is a refinement of \( \mathcal{U}_{p+1} \) together with a refinement projection \( \lambda_p : \mathcal{U}_p \to \mathcal{U}_{p+1} \) satisfying the following: for finitely many elements \( \mathcal{U}_{a_0}, \cdots, \mathcal{U}_{a_k} \in \mathcal{U}_p \), the morphism

\[
\tilde{H}_p^L(\mathcal{U}_{a_0} \cap \cdots \cap \mathcal{U}_{a_k}) \to \tilde{H}_p^L(\lambda_p(\mathcal{U}_{a_0}) \cap \cdots \cap \lambda_p(\mathcal{U}_{a_k}))
\]

is trivial.

Let us take \( c \in \mathcal{G}_m^L(X) \) with \( bc = 0 \). Since \( \mathcal{G}_m^L \) is a cosheaf on \( X \), by Proposition 6.1 we obtain \( c_{k,m-k} \in \tilde{C}_k(\mathcal{U}; \mathcal{G}_m^{L-k}) \) for \( 0 \leq k \leq m \) such that

\[
\varepsilon c_{0,m} = c \quad \text{and} \quad \Phi c_{k,m-k} = bc_{k-1,m-k+1}
\]

for \( 1 \leq k \leq m \).

Let us assume that there exists \( T \in \mathcal{I}_{m+1}^L(X) \) with \( \partial T = [c] \). It suffices to show that there is \( c' \in \mathcal{G}_m^{L+1}(X) \) such that \( bc' = c \). Since \( \varepsilon : \tilde{C}_0(\mathcal{U}; \mathcal{G}_m^{L+1}) \to \mathcal{I}_{m+1}^L(X) \) is surjective, there exists \( T_{0,m+1} \in \tilde{C}_0(\mathcal{U}; \mathcal{G}_m^{L+1}) \) such that \( \varepsilon(T_{0,m+1}) = T \). We obtain \( \varepsilon(\partial T_{0,m+1} - [c_{0,m}]) = 0 \). Since \( \tilde{C}_1(\mathcal{U}; \mathcal{G}_m^L) \xrightarrow{\Phi} \tilde{C}_0(\mathcal{U}; \mathcal{G}_m^{L+1}) \xrightarrow{\varepsilon} \mathcal{I}_m^L(X) \) is exact, there exists \( T_{1,m} \in \tilde{C}_1(\mathcal{U}; \mathcal{G}_m^L) \) such that

\[
\Phi(T_{1,m}) = \partial T_{0,m+1} - [c_{0,m}].
\]

Then, we obtain \( \Phi(\partial T_{1,m} + [c_{1,m-1}]) = -\partial[c_{0,m}] + [bc_{0,m}] = 0 \). The exactness of the sequence \( \tilde{C}_2(\mathcal{U}; \mathcal{G}_m^{L+1}) \xrightarrow{\Phi} \tilde{C}_1(\mathcal{U}; \mathcal{G}_m^{L+1}) \xrightarrow{\varepsilon} \tilde{C}_0(\mathcal{U}; \mathcal{G}_m^{L+1}) \) implies the existence of \( T_{2,m-1} \in \tilde{C}_2(\mathcal{U}; \mathcal{G}_m^{L+1}) \) such that

\[
\Phi(T_{2,m-1}) = \partial T_{1,m} + [c_{1,m-1}].
\]

By repeating this argument, we obtain \( T_{k,m+1-k} \in \tilde{C}_k(\mathcal{U}; \mathcal{G}_{m+1-k}^L) \) for \( 0 \leq k \leq m+1 \) such that

\[
(6.2) \quad \Phi(T_{k,m+1-k}) = \partial T_{k-1,m+2-k} + (-1)^k[c_{k-1,m+1-k}]
\]
holds for \( 1 \leq k \leq m+1 \).

We consider \( T_{m+1,0} \in \tilde{C}_{m+1}(\mathcal{U}; \mathcal{I}_0^L) \). Since \( \tilde{C}_{m+1}(\mathcal{U}; \mathcal{I}_0^L) \) is isomorphic to \( \tilde{C}_{m+1}(\mathcal{U}; \mathcal{G}_0^L) \) naturally, there exists a unique element \( c_{m+1,0} \in \tilde{C}_{m+1}(\mathcal{U}; \mathcal{G}_0^L) \) such that \( [c_{m+1,0}] = T_{m+1,0} \). We calculate \( \varepsilon' \Phi(c_{m+1,0}) \) as follows.

\[
\varepsilon' \Phi(c_{m+1,0}) = \varepsilon' \Phi([c_{m+1,0}]) = (-1)^{m+1} \varepsilon' c_{m,0} = (-1)^{m+1} \varepsilon' c_{m,0}.
\]

By the choice of \( \mathcal{U} = \mathcal{U}_0 \) and \( \lambda_0 : \mathcal{U}_0 \to \mathcal{U}_1 \), there exists \( c_{m,1} \in \tilde{C}_m(\mathcal{U}_1; \mathcal{G}_1^L) \) such that

\[
b(c_{m,1}) = \lambda_0 \# \Phi(c_{m+1,0}) + (-1)^m c_{m,0}.
\]

Then, we have

\[
b \Phi(c_{m,1}) = (-1)^m \lambda_0 \# c_{m,0} = (-1)^m \lambda_0 \# bc_{m-1,1}.
\]
By the choice of $\lambda_1$, there exists $c_{m-1,2} \in \tilde{C}_{m-1}(\mathcal{U}_2; \mathfrak{S}_2^L)$ such that
\[
 b(c_{m-1,2}) = \lambda_1 \# \left( \Phi(c_{m,1}) + (-1)^{m-1} \lambda_0 \# c_{m-1,1} \right).
\]
By repeating such an argument, we obtain $c_{m-k,k+1} \in \tilde{C}_{m-k}(\mathcal{U}_{k+1}; \mathfrak{I}_{k+1}^L)$ for $1 \leq k \leq m$ such that
\[
 b(c_{m-k,k+1}) = \lambda_k \# \left( \Phi(c_{m-k+1,k}) + (-1)^{m-k} \lambda_{k-1} \# c_{m-k,k} \right)
\]
holds for $1 \leq k \leq m$. Here, $\lambda_k = \lambda_{k-1} \circ \cdots \circ \lambda_1 \circ \lambda_0$.
Let us consider $\varepsilon(c_{0,m+1}) \in \mathfrak{S}_L^{m+1}(X)$. It satisfies
\[
 b\varepsilon(c_{0,m+1}) = \varepsilon \lambda_m \# c_{0,m} = \varepsilon c_{0,m} = c.
\]
In particular, we know that $c$ is the boundary. Therefore, $[\cdot]_*$ is injective. □

**Corollary 6.3.** If $X$ is an $(\tilde{H}_0^L)$-locally trivial space, then $[\cdot]_* : H_0^L(X) \to H_0^{IC}(X)$ is an isomorphic.

**Proof.** In general, $[\cdot] : \mathfrak{S}_0^L(X) \to \mathfrak{I}_0^L(X)$ is an isomorphic. Hence, $[\cdot]_* : H_0^L(X) \to H_0^{IC}(X)$ is surjective. By Theorem 6.2, $[\cdot]_*$ is injective. It completes the proof. □

### 6.4. Surjectivity.

We prove

**Theorem 6.4.** Let $m \geq 1$. If $X$ is an $(H)$-locally trivial space for $H = \tilde{H}_0^L$ and $H_0^{IC}$ with $0 \leq j \leq m-1$ and $1 \leq k \leq m$, then $[\cdot]_* : H^L_m(X) \to H^{IC}_m(X)$ is surjective.

**Proof.** Let us choose a sequence of open coverings
\[
 \mathcal{V}_0 \geq \mathcal{U}_1 \geq \mathcal{V}_1 \geq \mathcal{U}_2 \geq \cdots \geq \mathcal{V}_{m-1} \geq \mathcal{U}_m
\]
together with refinement projections
\[
 \lambda_{k-1} : \mathcal{V}_{k-1} \to \mathcal{U}_k \text{ and } \mu_k : \mathcal{U}_k \to \mathcal{V}_k
\]
for $1 \leq k \leq m$ such that for any finitely many $V_0, \ldots, V_p \in \mathcal{V}_{k-1}$ and $U_1, \ldots, U_p \in \mathcal{U}_k$, the morphisms
\[
 (\lambda_{k-1})_* : \tilde{H}_k^{IC}(V_0 \cap \cdots \cap V_p) \to \tilde{H}_k^{IC}(\lambda_{k-1}(V_0) \cap \cdots \cap \lambda_{k-1}(V_p)),
\]
\[
 (\mu_k)_* : \tilde{H}_k^L(U_0 \cap \cdots \cap U_p) \to \tilde{H}_k^{IC}(\mu_k(U_0) \cap \cdots \cap \mu_k(U_p))
\]
are trivial for all $p \geq 0$ and $1 \leq k \leq m$.
Let us take $T \in \mathfrak{I}_m^L(X)$ with $\partial T = 0$. Since $\mathfrak{I}_m^L$ is a cosheaf, by Proposition 6.1, there exists $T_{k,m-k} \in \tilde{C}_k(V_0; \mathfrak{I}_{m-k})$ satisfying
\[
 \Phi(T_{k,m-k}) = \partial T_{k-1,m-k+1} \text{ and } \varepsilon T_{0,m} = T
\]
for every $1 \leq k \leq m$.
Since $[\cdot] : \tilde{C}_m(V_0; \mathfrak{S}_0^L) \to \tilde{C}_m(V_0; \mathfrak{I}_0^L)$ is isomorphic, there exists a unique $c_{m,0} \in \tilde{C}_m(V_0; \mathfrak{I}_0^L)$ with $[c_{m,0}] = T_{m,0}$. Then, we have
\[
 \varepsilon' \Phi c_{m,0} = \varepsilon' \Phi T_{m,0} = \varepsilon' \partial T_{m-1,1} = 0.
\]
Hence, there exists $c_{m-1,1} \in \tilde{C}_{m-1}(\mathcal{U}; \mathcal{G}_1^L)$ such that
\[ bc_{m-1,1} = \lambda_0 \Phi c_{m,0}. \]
We have
\[ b\Phi c_{m-1,1} = \Phi bc_{m-1,1} = \lambda_0 \Phi^2 c_{m,0} = 0, \]
\[ \partial(c_{m-1,1}) = \lambda_0 \Phi T_{m,0} = \partial \lambda_0 \Phi T_{m-1,1}. \]
By the choice of $\mu_1 : \mathcal{U}_1 \to \mathcal{V}_1$, there exists $S_{m-1,2} \in \tilde{C}_{m-1}(\mathcal{V}_1; \mathcal{I}_2^c)$ such that
\[ \partial S_{m-2} = \mu_1 \Phi c_{m-1,1}. \]
By repeating this argument, we obtain
\[ S_{m-2} = \lambda_1 \Phi c_{m-1,1}. \]

On the other hand, by the choice of $\lambda_1 : \mathcal{V}_1 \to \mathcal{U}_2$, there exists $c_{m-2,2} \in \tilde{C}_{m-2}(\mathcal{U}_2; \mathcal{G}_2^L)$ such that
\[ bc_{m-2,2} = \lambda_1 \mu_1 \Phi c_{m-1,1}. \]

6.5. Comparing relative homologies. Let $(X, A) \in \text{Met}_2$ and $f : (X, A) \to (Y, B)$ a locally Lipschitz map. There is the following commutative diagram
\[
\begin{array}{cccccc}
0 & \longrightarrow & S^L_+(A) & \longrightarrow & S^L_+(X) & \longrightarrow & S^L_+(X, A) & \longrightarrow & 0 \\
\downarrow & & | & & | & & | & & | \\
0 & \longrightarrow & \mathcal{I}^c_+(A) & \longrightarrow & \mathcal{I}^c_+(X) & \longrightarrow & \mathcal{I}^c_+(X, A) & \longrightarrow & 0
\end{array}
\]
of chain maps among chain complexes, where the two rows are exact. According to the snake lemma, we obtain the following commutative diagram
\[
\begin{array}{cccccc}
\cdots & \longrightarrow & H^L_m(A) & \longrightarrow & H^L_m(X) & \longrightarrow & H^L_m(X, A) & \overset{\partial}{\longrightarrow} & H^L_{m-1}(A) & \longrightarrow & \cdots \\
| & & | & & | & & | & & | & & | \\
\cdots & \longrightarrow & H^IC_m(A) & \longrightarrow & H^IC_m(X) & \longrightarrow & H^IC_m(X, A) & \overset{\delta}{\longrightarrow} & H^IC_{m-1}(A) & \longrightarrow & \cdots
\end{array}
\]
such that the long two rows are exact, where $\delta$ is a connection morphism.
Now, we prepare some category. Let us denote by \( \text{Met}^{(m)} \) the full-subcategory of \( \text{Met}_2 \) consisting of all pairs \((X, A)\) of metric spaces such that \( X \) is \((H)\)-locally trivial for \( H = \check{H}^L_j \) and \( H^IC_k \) with \( 0 \leq j \leq m \) and \( 1 \leq k \leq m \) and \( A \) is \((H')\)-locally trivial for \( H' = \check{H}^L_p \) and \( H'^IC_q \) with \( 0 \leq p \leq m - 1 \) and \( 1 \leq q \leq m \). By the five lemma, we obtain

**Corollary 6.5.** On the category \( \text{Met}^{(m)}_2 \), the natural map \([·]_* : H^L_m \to H^IC_m \) is a natural isomorphism.

Immediately, we obtain

**Corollary 6.6.** On the category \( \text{Met}^{WLLC}_2 \), the natural map \([·]_* : H^L_m \to H^IC_m \) is a natural isomorphism for every \( m \geq 0 \).

**6.6. Comparing \( H_* \) and \( H^L_* \).** We prove that \( H_* \) and \( H^L_* \) are naturally isomorphic to each other on (a category containing) \( \text{Met}^{WLLC}_2 \). A natural transformation from \( S^L_* \) to \( S_* \) consists of the inclusions \( \iota_X : \check{S}^L_0(X) \to S_0(X) \) of chain complexes for each \( X \in \text{Met} \). We denote by \( \iota_* : H^L_* \to H_* \) a natural transformation induced by \( \{\iota_X\}_X \).

By a similar way to prove a coincidence of \( H^L_* \) and \( H'^IC_* \) on a suitable category, one can prove the following statement.

Let \( \text{Met}^{[m]}_2 \) denote the full-subcategory of \( \text{Met}_2 \) consisting of all pairs \((X, A)\) such that \( X \) is \((H)\)-locally trivial for \( H = \check{H}^L_j \) and \( H_k \) with \( 0 \leq j \leq m \) and \( 1 \leq k \leq m \) and \( A \) is \((H')\)-locally trivial for \( H' = \check{H}^L_p \) and \( H_q \) with \( 0 \leq p \leq m - 1 \) and \( 1 \leq q \leq m \).

**Theorem 6.7.** On the category \( \text{Met}^{[m]}_2 \), the natural transformation \( \iota_* : H^L_m \to H_m \) is an isomorphism.

**Proof.** The proof is done by using the proof of Corollary 6.6 replaced \( H'^IC_* \) and \([·]_* \) by \( H_* \) and \( \iota_* \). It suffices to notice that
\[
S^L_0(X) = \check{S}^L_0(X) \xrightarrow{\iota_X} S_0(X) = S_0(X)
\]
is an isomorphism for every \( X \in \text{Met} \). \( \Box \)

Since \( \text{Met}^{WLLC}_2 \) is contained in \( \text{Met}^{[m]}_2 \), we obtain

**Corollary 6.8.** On the category \( \text{Met}^{WLLC}_2 \), the natural transformation \( \iota_* : H^L_m \to H_m \) is an isomorphism.

By Corollaries 6.8 and 6.6, we obtain the conclusion of Theorem 1.6.

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