We propose a new version of generalized probabilistic propositional logic, namely, discrete-
continuous logic (DCL) in which every generalized proposition (GP) is represented as $2 \times 2$ non-
diagonal positive matrix with unit trace. We demonstrate that on the set of propositions of this
kind one can define both the discrete logical operations (connectives) such as negation and strong
logical disjunction and in addition one parameter group of continuous operations (logical rotations).
We prove that an arbitrary classical proposition (which in this logic is represented by the purely
diagonal matrix) can be considered as the result of strong disjunction of two identical GP. This fact
gives one a good reason to presume the DCL as a prime logical substructure underlying to ordinary
propositional logic, which is recorded by our consciousness. We believe that proposed version of
DCL will find many applications both in physics (quantum logic) and also in cognitive sciences
(mental imagery) for better understanding of the peculiar nature of mental brain operations.

PACS numbers: 05.40.-a

I. INTRODUCTION

The Boolean propositional logic is the very structure
that on the one hand accompanies to a certain extent all behavioral acts of most people in everyday life and on
the other hand can be considered as the necessary framework for various scientific theories including the most
subtle ones. Turning to physics we note that the set of all two-valued quantities describing some classical system
(where these quantities admit joint measurement) can be arranged exactly in the form of the Boolean algebra of
corresponding observables. It concerns not only deterministic observables but probabilistic two-valued variables
(that can take two possible values with certain probabilities) as well. In the latter case one must certainly use the
probabilistic generalization of Boolean algebra. However, since the pioneering paper of Birkhoff and von Neumann
[1] it is known that in quantum mechanics the set of noncommuting observables can not be described in simple
terms of Boolean algebra and some generalization of it, that is "quantum logic", is necessary. Although many
ingenious attempts have been made both to formulate the universal version of quantum logic, that, by natural
way, would solve famous quantum paradoxes of the Schrodinger cat type and in addition allows one to consider
consistently the problem of measurement (see e.g. [2] and references therein). Unfortunately, to the best
of my knowledge, similar program to the bitter end was not realized by anyone. Besides, there is another important
reason for the generalization of the Boolean logic (ordinary or probabilistic) arising from famous peculiarities of
human brain activity. As it is well known (see e.g. [3]) the brain consists of two hemispheres which are
roughly of equal size and surface and at first glance are similar to each other. However, unlike from other pair
human organs these two hemispheres do not represent information in an identical manner but perform specific
cognitive and mental functions. It is possible to take for granted that the left hemisphere (LH) produces sequential
processing of information in discrete units. In particular it is responsible for the implementation of various logical
operations and for speech linguistic capacities as well. On the other hand the right hemisphere (RH) produces the
simultaneous and coherent processing of information and it is responsible for such brain functions as for example
image formation, space imagination and for the music perception as well. Summing up one can say: the Boolean logic par excellence is the logic only of the LH. Hence natural question arises: whether there is the generalization of Boolean logic that takes into account the possibility of execution both discrete and continuous logical operations. In present paper we propose the version of DCL that combines both these two types of operations. This logic, in our opinion, could essentially expand the possibilities of currently existing logical devices. It must be emphasized however that in this paper we do not pretend to give the consistent description of real brain activity. Rather our paper should be considered as an attempt to offer the simple phenomenological model which would reflect certain specific peculiarities of brain logic. On the other hand we insist that our approach differs from the most of standard constructions of artificial intellect which as a rule are based on the Boolean logic only. The further part of the paper is organized as follows. In chapter 2 we briefly recall some results of our paper [4] which are necessary for understanding of the present text. The main idea of our approach to logic consists in the possibility to represent all plausible propositions of Boolean probabilistic logic by $2 \times 2$ diagonal positive matrices with unity trace. These representative matrices of propositions can be naturally considered as corresponding density matrices of relevant
two level quantum systems. Moreover, it turns out that all logical operations with plausible propositions (that is logical connectives) can be obtained with the help of positive definite transformations of the type, that we specify exactly. In the chapter 3 which is the basic part of the present paper, we generalize all constructions mentioned above on the case of propositions of more general type (GP) that are now represented by non-diagonal positive 2 × 2 matrices of special form with unit trace. It turns out that in this case apart from few discrete connectives there is also one parameter group of continuous transformations in the space of all GP. For this reason we presume that the "power" of DCL is much stronger compared with the "power" of the ordinary Boolean logic and, in our opinion, bringing it closer to the rules of real brain logic.

Finally in chapter 4 of the paper we demonstrate that all concepts of DCL may be properly implemented in relevant quantum systems by the methods of modern quantum engineering. We believe therefore that the manifesto of R. Landauer "Information is physical" can be applied to logic as well as to information.

Now let us begin to the concrete presentation of results of the paper.

II. PRELIMINARY INFORMATION

Let us briefly remind the necessary results of our paper [4] in which the main concepts of probabilistic logic were formulated based on the point of view of quantum theory of open systems (QTOS). In [4] we considered the set of classical plausible propositions (CP), that is, the propositions a truth or falsity of which are known not exactly, but only with certain probability. It is possible to represent plausible propositions with the help of diagonal 2 × 2 matrices with positive elements the sum of which is equal to unity. So, every plausible proposition A can be represented by its matrix ρ(A) = \begin{pmatrix} p_A & 0 \\ 0 & 1 - p_A \end{pmatrix}, where p is the probability for A to be true. In what follows, if there does not lead to confusion, we will sometimes identify plausible propositions with their representative matrices. It turns out that all logical connectives between propositions in this matrix language can be expressed by unified way (that has its roots in QTOS) by some positive definite transformations of representative matrices conserving their diagonal form and traces. Referring the reader for details to [4] we present here only the essence of the approach applied, using some simple examples. So, in order to write down the representative matrix for negation of some proposition A, that is \( \overline{A} \), one must perform the next transformation:

\[
\rho(\overline{A}) = G_N \cdot \rho(A) \cdot G_N^T = \begin{pmatrix} 1 - p_A & 0 \\ 0 & p_A \end{pmatrix},
\]

where \( G_N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) (as usually we denote as \( A^T \) a matrix transposed to \( A \)). In a like manner the representative matrix of arbitrary two-place logical connective can be written in the form: \( \rho_C = G_C [\rho(A) \otimes \rho(B)] G_C^T, \) where \( \rho(A) \otimes \rho(B) \) is the tensor product of matrices corresponding to propositions A and B, and \( G_C \) is a certain 2 × 4 matrix which possesses two defining properties: 1) every element of \( G_C \) is equal 1 or 0 and 2) in each column of \( G_C \) only single element is equal to 1 and all the rest are equal to zero. One can verify directly that these two properties of \( G_C \) exactly ensure the positivity and diagonality of \( \rho_C \) and also the conservation of its trace.

It is clear also that with the help of specified transformations (we call them admissible transformations) one can obtain all logical connectives between any number of propositions as well. Thus any statements relating to probabilistic Boolean propositions can be translated into the language of admissible transformations with the density matrices of relevant two level quantum systems. This language also let one to generalize approach proposed on the case of more extensive class of logical propositions and possible operations with them. Now let us pass to this main topic of the present paper.

III. GENERALIZED PROPOSITIONS AND LOGICAL OPERATIONS WITH THEM

First of all we want to specify in what sense the concept of classical probabilistic propositions can be expanded on more general case. With this end in view let us consider generic non-diagonal 2 × 2 positive matrix with unit trace - \( \rho(A) = \begin{pmatrix} p_A & z_A \\ z_A^* & 1 - p_A \end{pmatrix} \) and try to interpret it as a representative matrix of certain logical proposition. To realize this idea it is necessary to define basic logical connectives on the set of these matrices. Obviously that there is no problem to define negation of proposition A. We can perform this task by the same transformation \( G_N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) as in previous chapter that leads to the required result:

\[
\overline{A} = G_N \cdot A \cdot G_N^T = \begin{pmatrix} 1 - p_A & z_A \\ z_A^* & p_A \end{pmatrix}.
\]  

Unfortunately when we try to use the similar method for the definition of arbitrary two place connectives we are faced with insuperable difficulties since the transformations that were admissible for diagonal matrices in non-diagonal case do not conserve the traces of corresponding tensor products. So, we are needed to use another way for the decision of this problem. To this end let us include in our consideration as representative only the matrices of the special form, namely:

\[
\rho(A) = \begin{pmatrix} p_A & i\alpha_A \\ -i\alpha_A & 1 - p_A \end{pmatrix},
\]  

where \( \alpha_A \) is some real number. Now one can define correctly the next two-place connective between two such matrices \( A = \begin{pmatrix} p & i\alpha \\ -i\alpha & 1 - p \end{pmatrix} \) and \( B = \begin{pmatrix} q & i\beta \\ -i\beta & 1 - \beta \end{pmatrix} \)
using the positive definite $2 \times 4$ transformation $G_\Delta = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$. As a result we obtain the proposition $(A\Delta B)$ with the representative matrix:

$$\rho(A\Delta B) = G_\Delta (A \otimes B) G_\Delta^T. \tag{3}$$

$$\rho(A\Delta B) = \begin{pmatrix} p + q - 2pq + 2\alpha \beta & i\alpha (1 - 2q) - i\beta (1 - 2p) \\ -i\alpha (1 - 2q) - i\beta (1 - 2p) & 1 - p - q + 2pq - 2\alpha \beta \end{pmatrix}. \tag{4}$$

Thus one can see that the matrix $\rho(A\Delta B)$ has the required form Eq. (2) and hence belongs to the set of generalized propositions. In the special case when $\alpha = \beta = 0$ propositions $A$, $B$ and $A\Delta B$ become ordinary plausible ones. This means that their representative matrices take diagonal form. Corresponding classical plausible proposition $(A_c, B_c)$ reduces to the strong disjunction of two probabilistic propositions, namely:

$$\rho(A_c \Delta B_c) = \begin{pmatrix} p + q - 2pq & 0 \\ 0 & 1 - p - q + 2pq \end{pmatrix}. \tag{5}$$

(Remind here that in the ordinary nonprobabilistic Boolean logic strong disjunction of two propositions in contrast with usual disjunction is true, in the case when only one from propositions $A, B$ is true and the other is false). Now let us reduce the expression Eq. (6) to more appropriate form. With this end in view we will consider the representative matrices of $A$ and $B$ as density matrices of relevant two level quantum system and write down them in the standard Bloch notation. For example proposition $A$ takes the form: $\rho(A) = \frac{1}{2} \begin{pmatrix} 1 + P_z & P_x - iP_y \\ P_x + iP_y & 1 - P_z \end{pmatrix}$ with corresponding Bloch vector $P$. Comparing this notation with expression Eq. (2) we obtain for components of vector $P$ the relations: $P_z = 0, \alpha = -\frac{p}{2}$ and $p = \frac{1}{2}(1 + P_z)$. We see that the set of GP can be put into one to one correspondence with the set of various mixed states of two level quantum systems polarized in plane $Y - Z$. It is convenient to introduce the complex vector $P = P_z - iP_y$ (we will continue to call it the Bloch vector of the state). Now one can easy verify directly that if proposition $A$ has the Bloch vector $P$, then the negation $\bar{A}$ has the Bloch vector $(-P)$ and proposition $(A\Delta B)$ in this notation can be represented in the form $\rho(A\Delta B) = \frac{1}{2} \begin{pmatrix} 1 + R_z & -iR_y \\ iR_y & 1 - R_z \end{pmatrix}$ with complex vector $R = -PQ$. The components of the Bloch vector $R$ of the proposition $(A\Delta B)$ can be written as: $R_z = P_y Q_y - P_z Q_z$ and $R_y = -(P_y Q_z + P_z Q_y)$. It is worth noting here two simple properties of operation $\Delta$, which can be easily deduced from the expression Eq. (4):

1) $(\bar{A}\Delta B) = (A\Delta \bar{B}) = (A\Delta B)$
2) $(\bar{A}\Delta \bar{B}) = A\Delta B$.

It should be noted also, that in contrast with the probabilistic Boolean logic, in the case of DCL it is possible, aside from mentioned above discrete logical connectives, to define also the additional one parameter group of continuous logical operations. Indeed, one can make the rotation of the Bloch vector $P$ of every proposition $A$ in the plane $Y - Z$ at an arbitrary angle $\phi$ that results in to the another proposition $A_1$ with corresponding Bloch vector $P_1$ that has the components:

$$P_{1y} = P_y \cos \phi + P_z \sin \phi$$
$$P_{1z} = P_z \cos \phi - P_y \sin \phi \tag{7}$$

The expressions Eq. (4) and Eq. (7) imply that if one rotates the proposition $A$ at an angle $\phi_1$ and the proposition $B$ at an angle $\phi_2$ the proposition $(A\Delta B)$ is rotated at an angle $\phi = (\phi_1 + \phi_2)$. Now let us compare the "logical power" of DCL with the "power" of ordinary probabilistic Boolean logic. At first sight it may seem that in certain respects DCL is more poor logical theory because it possesses only four discrete connectives while the standard Boolean logic has 16 similar connectives. However such conclusion would be hasty. To demonstrate the great opportunities of DCL we are going first of all to prove that every CP (which is represented by purely diagonal matrix) can be obtained from two identical GP by the single discrete operation $\Delta$ (strong disjunction). This fact is in contrast with ordinary Boolean logic where the proposition $(A\Delta A)$ is always false and with probabilistic Boolean logic where the plausible proposition $(A\Delta A)$ has the following form: $(A\Delta A) = \begin{pmatrix} 2p - 2p^2 & 0 \\ 0 & 1 - 2p + 2p^2 \end{pmatrix}$, and hence its plausibility is always less than one half for any proposition $A$. But if we turn to the case of DCL and take as starting the proposition $A = \begin{pmatrix} \frac{1}{2} \alpha & i\beta \\ -i\beta & \frac{1}{2} \alpha \end{pmatrix}$ we, using Eq. (4) result in that strong disjunction $(A\Delta A)$ has
the form: \( (A \Delta A) = \begin{pmatrix} \frac{1}{4} + 2\alpha^2 & 0 \\ 0 & 0 \frac{1}{2} - 2\alpha^2 \end{pmatrix} \) and hence any plausible proposition, whose plausibility is more than \( \frac{1}{2} \), can be obtained in this way by an appropriate choice of \( \alpha \) (recall that parameter \( \alpha \) takes its values in the interval \( -\frac{1}{2} \leq \alpha \leq \frac{1}{2} \)). Note that an arbitrary CP can be represented in the form of strong disjunction in two ways (since the relation \( A \Delta A = \overline{A} \Delta \overline{A} \) is identically holds as we have stated earlier). The proven result gives one the good reason to presume that DCL may precede (as the possible prime substructure) to the ordinary Boolean logic, which is, however, recorded better by our consciousness. It should be noted also that not only classical but arbitrary GP \( A \) can be represented as strong disjunction of two identical propositions, namely: \( A = (A_1 \Delta A_1) \) for some proposition \( A_1 \). The determination of \( A_1 \) can be considered, in figurative sense, as taking the logical squire root from proposition \( A \), since if proposition \( A \) has Bloch vector \( R \) and proposition \( A_1 \) has Bloch vector \( P \) then the relation \( R = -P^2 \) holds as it follows from the foregoing text. One can explicitly write down this relation in components and solve it but we do not dwell on this point. On the other hand the presence of additional one parametric group of continuous operations in the DCL in our opinion greatly expends its opportunities as the tool for various logical and information processing. This fundamental issue certainly deserves of special attention and research but in this paper we restrict ourselves only to the remark that for example, both the concept of thought rotations and the theory of mental imagery which were introduced into cognitive psychology mainly by R. Shepard and S.Kossiin (see e.g. [5], [6]) can be naturally interpreted in the language of similar logical operations. In the remainder of the paper we are going to demonstrate how formal constructions of DCL, described above, can be implemented in relevant quantum systems by the modern quantum engineering tools. Note that our exposition of this problem will be to a large extent sketchy. To get acquainted with the opportunities of this technique at length we recommend the reader turn to the book [7]. So, let us consider several concrete tasks that everyone should be able to perform for simulating main DCL constructions. Obviously first of all one need to create a sufficient reserve of physical realizations of GP. To this end in view it is naturally to use the states of open two- level quantum system whose polarization vector is situated in the \( Y - Z \) plane. A natural way to create such states is to organize the interaction of the open system with its environment suchwise that a component \( P_x \) of initial mixed state rapidly decayed with time. Here we specify only one simple way to achieve this goal. Let us use for the description of evolution of open quantum system the well-known Lindblad master equation that in general case has the following form:

\[
\frac{d\rho}{dt} = -\frac{i}{\hbar} [H, \rho] + \sum_{j=1}^{N} \left[ R_j \rho, R_j^+ \right] + h.c., \tag{8}
\]

(where \( H \) is some hermitian operator and operators \( R_j, \quad R_j^+ \) are a set, generally speaking, nonhermitian operators). Taking jointly these operators describe both internal dynamics of the system in question and its interaction with environment. In the case of two- level quantum system, that we are only interested in, it is convenient to use the Bloch representation for it density matrix, namely:

\[
\rho = \frac{1}{\pi} P \overline{P}, \quad \text{where} \quad \overline{P} = (\sigma_i, R_j^+, R_j) \quad (1 \leq i, j \leq 2) \quad \text{are standard Pauli matrices}.
\]

Taking into account that any \( 2 \times 2 \) hermitian matrix can be decomposed in Pauli matrices one can write down all operators entering in Eq. \( (8) \) in the following form: \( H = 2 \hbar \overline{P} \overline{P}, \quad R_j = \overline{A}_j \overline{P} + i \overline{B}_j \overline{P} \). The set of vectors: \( \hbar, \overline{A}_j, \overline{B}_j \) are completely characterizes the dynamics of two- level open system within the Lindblad equation approach. Based on Eq. \( (8) \) after elementary algebra one can obtain the required evolution equation for the polarization vector \( \overline{P} \) that reads as:

\[
\frac{d\overline{P}}{dt} = \left( \hbar \times \overline{P} \right) + \sum_{j=1}^{N} 2 \left( \overline{A}_j \times \overline{B}_j \right) - \overline{A}_j \times \left( \overline{P} \times \overline{A}_j \right) - \overline{B}_j \times \left( \overline{P} \times \overline{B}_j \right). \tag{9}
\]

Let us consider the simplest situation when \( N = 1 \) and \( \hbar = 0 \) and choose as \( R \) the hermitian operator: \( R = \overline{A} \overline{P} \overline{P} \overline{A} \overline{P} \)

In this case the Eq. \( (9) \) for the polarization vector \( P \) takes the simple form:

\[
\frac{d\overline{P}}{dt} = -\left( \overline{A} \times \left( \overline{P} \times \overline{A} \right) \right). \tag{10}
\]

It is clear that evolution of the system according to Eq. \( (10) \) physically corresponds to the continuous measurement of the observable \( O \equiv \overline{A} \overline{P} \overline{P} \overline{A} \overline{P} \) in an arbitrary two-level system. Therefore if one will choose the observable \( O \) in the form: \( O_i = a_i \sigma_y + b_i \sigma_z \) (where \( a_i \) and \( b_i \) some numerical coefficients) the terminal state of given open system will simulate a certain proposition from DCL (since its polarization vector will be situated in the \( Y - Z \) plane). Varying the constants \( a_i \) and \( b_i \) by appropriate way, one
is able to create initial reserve of relevant mixed states which are the physical realizations of required generalized propositions. The next necessary stage towards the realization of main constructions of the DCL is a simulation of various logical operations with these states, that are considered now as already present. Among these operations there are certain unitary operations such as negation and all logical rotations. They will not be considered in this paper, since it is well known that any unitary operator can be realized by the relevant quantum circuit consisting of some universal gates (see e.g. [5]). Therefore we will discuss here only the method of implementation relating to the single operation, namely, the strong disjunction which, clearly, can not be reduced to any unitary operator. Remind that, as it was shown above, the strong disjunction is determined by the following not quadratic 2 × 4 matrix: 

$$G_\Delta = \begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
\end{pmatrix}$$

acts on the tensor product \(A \otimes B\) of two propositions \(A = \begin{pmatrix} p & i\alpha \\
- i\alpha & 1 - p \end{pmatrix}\) and \(B = \begin{pmatrix} q & i\beta \\
- i\beta & 1 - q \end{pmatrix}\) which in explicit form has the form:

$$A \otimes B = \begin{pmatrix}
 pq & i\beta p & i\alpha q & -\alpha i \beta \\
- i\beta p & p (1 - q) & -\alpha i \beta & i\alpha (1 - q) \\
- i\alpha q & - \alpha i \beta & q (1 - p) & -\beta (1 - p) (1 - q) \\
- \alpha i \beta & i\alpha (1 - q) & - \beta (1 - p) (1 - q) & \end{pmatrix},$$

(11)

The equations of the first group have the following form:

$$\frac{d\rho_{11}}{dt} = - \frac{\rho_{11}}{\tau_1}, \quad \frac{d\rho_{33}}{dt} = - \frac{\rho_{33}}{\tau_3},$$

and the equations of the second group that read as:

$$\frac{d\rho_{12}}{dt} = - \frac{\rho_{12}}{\tau_{12}}, \quad \frac{d\rho_{13}}{dt} = - \frac{\rho_{13}}{\tau_{13}},$$

$$\frac{d\rho_{14}}{dt} = - \frac{\rho_{14}}{\tau_{14}}, \quad \frac{d\rho_{23}}{dt} = - \frac{\rho_{23}}{\tau_{23}},$$

$$\frac{d\rho_{24}}{dt} = - \frac{\rho_{24}}{\tau_{24}}, \quad \frac{d\rho_{34}}{dt} = - \frac{\rho_{34}}{\tau_{34}},$$

(13)

(14)

(in equations Eq. 13 and Eq. 14 we use the notation \(\tau_i, \tau_{ik}\) for the relaxation times of corresponding states and transitions ). We do not write down here the similar equations for the remaining matrix elements \(\rho_{ik}\) since we assume that their evolution is determined uniquely by the principle of detailed balance. The elementary analysis of Eq. 12, and Eq. 13 shows that when \(t\) tends to infinity (that is for the stationary density matrix) diagonal matrix elements \(\rho_{11}, \rho_{33}\) and also non-diagonal matrix elements: \(\rho_{12}, \rho_{13}, \rho_{14}, \rho_{23}, \rho_{34}\) vanish. Note that the system Eq. 12, Eq. 13 have two integrals of motion namely \(I_1 = \rho_{22} + \rho_{33} + \rho_{23} - \rho_{14}\) and \(I_2 = \rho_{24} + \rho_{34} - \rho_{12} - \rho_{13}\). Taking into account that in initial state \(I_1 = p + q - 2pq + 2\alpha \beta\) and \(I_2 = i\alpha (1 - 2q) + i\beta (1 - 2p)\) one can conclude that
for terminal state \( \rho_{22} = I_1 = p + q - 2pq + 2\alpha\beta \) and \( \rho_{24} = i\alpha (1 - 2q) + i\beta (1 - 2p) \). QED. Thus we prove that it is possible to organize the interaction of open composite quantum system with its environment suchwise that its initial state \( A \otimes B \) eventually transforms in required state namely \( \rho (A \Delta B) \) in one of its subsystems. It should be noted although we describe the relevant evolution of the system directly in the language of differential equations the same result can be reached also by the appropriate Lindblad equation since the open quantum system of interest is undoubtedly the quantum Markov system.

In conclusion we want emphasize once more that the main subject of our study in present paper is, using the semiotic language, only the syntax of the DCL. The other equally important issues relating to semantics of this theory, that is an interpretation of all its basic constructions remained out of our scope. Also we are not touched the possible concrete applications of the DCL in physics although, for example, the study of the quantum logic problems from this point of view suggests itself. All mentioned issues we hope to study and discuss in detail in our further publications.

[1] Birkhoff, G., von Neumann, J.: Ann. Math.37, 823 (1936).
[2] Maria- Luisa Dalla Chiara, R. Giuntini. "Quantum Logics" in D. Gabbay and F. Guenthner (eds.), Handbook of Philosophical Logic, Vol.6, Dordrecht, Kluwer, 2002, 129-228
[3] S. P. Springer, Left Brain, Right Brain, W. H. Freeman, New York, (1989).
[4] E.D. Vol, Int J Theor Phys (2013), 52, 514-523
[5] R.N. Shepard, American Psychologist, 33, 125, (1978)
[6] S.M. Kosslyn, Science, 240, 1621, (1988)
[7] Zagoskin, A.M.: Quantum Engineering, Theory and Design of Quantum Coherent Structures, Cambridge University Press, Cambridge (2011)
[8] Nielsen, M.A., Chuang, I.L. Quantum Computation and Quantum Information. Cambridge University Press, Cambridge (2001)