Network Structure and Naive Sequential Learning

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First version: January 17, 2017
This version: February 25, 2017

Abstract

We study a model of sequential learning with naive agents on a network. The key behavioral assumption is that agents wrongly believe their predecessors act based on only private information, so that correlation between observed actions is ignored. We provide a simple linear formula characterizing agents’ actions in terms of paths in the network and use this formula to determine when society learns correctly in the long-run. Because early agents are disproportionately influential, standard network structures can lead to herding on incorrect beliefs. The probability of mislearning increases when link densities are higher and when networks are more integrated. When actions can only communicate limited information, segregated networks often lead to persistent disagreement between groups.

*We thank Daron Acemoglu, Jetlir Duraj, Ben Enke, Erik Eyster, Drew Fudenberg, Ben Golub, David Laibson, Jonathan Libgober, Pooya Molavi, Xiaosheng Mu, Matthew Rabin, Tomasz Strzalecki, Omer Tamuz, and Muhamet Yildiz for useful comments.

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1 Introduction

In many domains, a sequence of agents must each make a decision in turn, using the behavior of those who came before them to infer those predecessors’ private information. For instance, when consumers choose between rival products, their decision is often informed by the choices of early customers. When doctors decide on a treatment for their patients, they consult best practices established by other clinicians who came before them. And when a new theory or rumor is introduced into a society, individuals are influenced by the discussions of those who have already taken positions on the new idea.

In all of these examples, it seems quite plausible that agents do not process the information contained in their observations in a Bayesian way. A consumer may misinterpret a herd on a product as evidence that everyone in the herd has positive private information about the product’s quality, hence purchase the product despite her own very negative private information about the product. Medieval doctors may have interpreted the widespread use of leeches by their predecessors as evidence of their efficacy, and so kept using this “cure” even when it proved ineffective. Finally, in many online communities a few early opinion makers can make a rumor go viral, due to people not realizing that the vast majority of the viral story’s proponents are just following the herd and possess no private information as to the rumor’s veracity. Moreover, when observation networks are segregated, disagreement between different subgroups can persist even when there are many connections across subgroups. In the product-adoption example, different subcommunities frequently insist upon the superiority of their preferred products. By contrast Acemoglu, Dahleh, Lobel, and Ozdaglar (2011) show that, under mild conditions on network and signal structures, Bayesian agents will eventually agree and be correct.

Indeed, Eyster and Rabin (2014) showed that rational behavior in these social learning settings often requires a counter-intuitive “anti-imitation” behavior, which most subjects fail to perform in lab experiments (Eyster, Rabin, and Weizsacker, 2015). This failure comes from a common inferential naiveté: agents act as if the actions of their predecessors were based only on their private information and not on their observations, thus leading to too much imitation. A recent literature has begun to explore the implications of such naive learning (Eyster and Rabin, 2010; Gagnon-Bartsch and Rabin, 2016) and related inferential mistakes (Gaurino and Jehiel, 2013; Bohren, 2016) in sequential decision-making settings.

The purpose of the present work is to determine the role of the observation network in these settings. We therefore combine two active areas in the literature on sequential learning: first, the investigations of naive inference discussed above and second, studies of Bayesian learning with network observation. For instance, many communities exhibit communication homophily, where individuals are more likely to interact with and listen to predecessors from their own social group than from other social groups. Given that learners are naive, are false
rumors more likely to take hold in a more or less homophilic society? More generally, we are interested in the interaction of network structure with the naive learning behavior: In what sorts of networks will naive agents eventually agree? For what networks will society learn correctly, despite the behavioral bias? What networks features amplify the agents’ inferential error and increase the probability that society converges to a wrong belief?

We are interested in this exercise for at least two reasons. First, as social and technological changes restructure our networks, social learning dynamics change as well. Most notably, the rise of the Internet has dramatically altered the transmission of information about current events, new products, etc. By clarifying how network structures affect learning outcomes, we aim to provide a framework for understanding such changes.

Second, while Eyster and Rabin (2010)’s original model of inferential naïveté considered a setting where every agent observes the actions of all predecessors (i.e. a complete network), we believe agents are more likely to apply this kind of inference heuristic in settings where they either have a limited knowledge of the true network (for instance, they are uncertain as to the order in which their predecessors moved) or the network is known but very complicated. In these settings, the correct Bayesian belief given one’s observations can be very difficult to calculate, so agents are more likely to resort to behavioral heuristics. Indeed, the experiments of Eyster, Rabin, and Weizsacker (2015) and Mueller-Frank and Neri (2015) support this view.

To preview our results, we obtain a simple linear expression of agent actions in terms of private signals and network structure. We exploit this expression to develop a necessary and sufficient condition for a society of naifs to learn correctly: no agent has too much “influence”. This condition echoes the “royal family” example of Bala and Goyal (1998), results on “forceful agents” in Acemoglu, Ozdaglar, and ParandehGheibi (2010) and especially an analogous result in Golub and Jackson (2010). We identify the relevant notion of influence in a sequential learning setting with naive agents where early movers are observed by a great number of successors, which is related to the number of paths terminating at the agent.

We then derive a formula for the probability of asymptotic social mislearning as a function of the network structure. We can apply this formula to a number of specific examples. We find that in an Erdős–Rényi-type model, society is more likely to mislearn when there are more links. We also find that when society is composed of two homophilic subgroups, mislearning is more likely when networks are integrated. A theme is that changes in network structure that would increase the speed of learning in rational and DeGroot models tend to make mislearning more likely in our model, because faster convergence of beliefs creates more redundancy.

Finally, we show that if agents’ actions only coarsely reflect their beliefs and society is segregated, then the two social subgroups can disagree forever. Because of agents’ naïveté
and the limited information conveyed by actions, disagreement can persist even when agents observe the actions of many individuals from the other group. We thus contribute to an active literature on disagreement by demonstrating that network homophily can sustain divergence in beliefs.

This paper complements three recent papers discussing social learning models that include naive sequential learning as special cases. Mueller-Frank and Neri (2015) consider naive learning in a broader class of network observation environments, and discuss when agents agree and when their beliefs are correct. Eyster and Rabin (2014) show that mislearning occurs with positive probability for a general class of social learning rules that can accommodate naive learning with network observation. Bohren (2016) fixes a particular network structure where agents randomly see all or none of their predecessors and allows agents to misperceive the fraction of agents who do have neighbors. We build upon these initial results by exploring the role of the observation network and by distinguishing different outcomes when mislearning is possible.

Our work also relates to a number of papers examining the role of network structure in social learning. There is a strand of literature on characterizing network features that lead to correct asymptotic learning for Bayesian agents who move sequentially, including Acemoglu, Dahleh, Lobel, and Ozdaglar (2011) and Lobel and Sadler (2015). We consider instead a version of the problem where the sequential learners suffer from naive inference. Another strand looks at heuristic learning rules, most often DeGroot learning, and the role that network structure plays in such non-Bayesian learning settings. These papers deal with situations where networked agents repeatedly interact with the same set of neighbors. Our focus is on examining the same question of how network structure interacts with non-Bayesian learning, but in a sequential learning setting where agents move in order and each acts once, such as generations of doctors choosing a medical treatment or consumers deciding whether to purchase a new product. DeMarzo, Vayanos, and Zwiebel (2003) show that consensus opinion in a DeGroot learning setting is a weighted average of agents’ initial opinions, with weights given by an appropriate notion of social influence. They find that consensus is correct if and only if no agent has excess influence relative to the precision of their private signal. Golub and Jackson (2012) show that speed of learning is reduced by the presence of homophily in a society and provide a voting example where time until a correct majority vote is increasing and convex in the extent of homophily. Molavi, Tabbaz-Salehi, and Jadbabaie (2016) axiomatize and compare DeGroot, log-linear learning and other heuristic learning rules.

The insight that certain structures of the observation network can be helpful for rational learners but harmful to behavioral learners has been explored in the setting of word-of-mouth learning. Ellison and Fudenberg (1995) studied this problem for behavioral agents choosing
between two alternatives — each period, a fraction of the population updates their choices by sampling the current-period experiences of $N$ individuals from the society and changing their own action to the alternative with a higher average payoff in the sample. The authors showed that efficient social learning requires an intermediate communication intensity $N$, as society fails to converge to a consensus when $N$ is too large. However, Banerjee and Fudenberg (2004) showed that for rational word-of-mouth learners, a large $N$ ensures efficient social learning.

In our model, each naive agent misperceives her neighbors as only acting on their private information and neglecting any additional information that they might have gained through social learning. This assumption is motivated by recent experimental evidence. In addition to Eyster, Rabin, and Weizsacker (2015), which documents the prevalence of this bias in a sequential learning setting, a similar kind of naiveté was found in settings where agents interact repeatedly with the same set of neighbors, both when the network is common knowledge (Chandrasekhar, Larreguy, and Xandri, 2015) and when agents only have limited information about the network (Mueller-Frank and Neri, 2015). More generally, treating the actions of two predecessors as independent when in fact one learned from the other neglects some of the correlation between their actions. Enke and Zimmermann (2016) show that correlation neglect is prevalent even in simple environments where the observed information sources are known to be correlated but non-strategic.

The rest of the paper is organized as follows: Section 2 introduces the model of naive sequential learning on a network. Section 3 contains our general results: a path-counting interpretation of actions and a necessary and sufficient condition for correct social learning based on a notion of “influence” in the network. Section 4 applies these results to a number of commonly used network structures and obtains comparative statics on the probability of mislearning with respect to network parameters. Section 5 contains results about disagreement in homophilic societies when actions only coarsely reflect beliefs. Section 6 concludes. Appendix A contains omitted proofs. Appendix B presents several simulations to check the robustness of our results to a different interpretation of network weights and to different signal structures.

2 Model

2.1 Social Learning on a Network

There are two possible states of the world, $\omega \in \{0, 1\}$, both equally likely. There is an infinite sequence of agents indexed by $i \in \mathbb{N}$. Agents move in order. On her turn, agent $i$ observes a private signal $s_i \in S$ from signal space $S$, as well as the actions of some previous agents.
Then, $i$ chooses an action $a_i \in [0, 1]$ to maximize the expectation of

$$u_i(a_i, \omega) := -(a_i - \omega)^2$$

given her belief about $\omega$, so her chosen action corresponds to the probability she assigns to the event $\{\omega = 1\}$.

Private signals $(s_i)$ are i.i.d. conditional on the state of the world. To simplify notation, we assume that for every $s \in S$, $0 < \mathbb{P}[\omega = 1|s_i = s] < 1$, so that no signal is fully revealing about the state of the world. In anticipation of our behavioral assumption, we will now assume that the signal structure has unbounded informativeness:

**Assumption 1 (Unbounded Signal Informativeness).** For every $p \in (0, 1)$, there exists some $s \in S$ such that $\mathbb{P}[\omega = 1|s_i = s] = p$.

Our initial results will not require any further assumptions on the signal structure. For our applications in Section 4, we will assume signals are Gaussian and symmetric across the two states:

**Assumption 2 (Gaussian Signals).** $s_i \sim \mathcal{N}(1, \sigma^2)$ when $\omega = 1$, and $s_i \sim \mathcal{N}(-1, \sigma^2)$ when $\omega = 0$.

Write $N_i \subseteq \{1, 2, ..., i - 1\}$ for the subset of $i$’s predecessors whose actions $i$ can observe. Members of $N_i$ are called **neighbors** of $i$. The sets $(N_i)$ define a directed network. The adjacency matrix $M$ of the network is defined by $M_{i,j} = 1$ if $j \in N_i$ and $M_{i,j} = 0$ otherwise. Because of the sequential observation structure, $M$ is lower-triangular with diagonal entries all equal to zero.

If $M_{i,j} = 1$ whenever $j < i$, then $M$ is associated with the complete network (up to the constraint that earlier agents cannot observe the behavior of later agents). In that case, we recover the commonly used social learning model where every agent observes the actions of all predecessors.

### 2.2 Behavioral Assumption

A growing body of recent evidence in psychology and economics demonstrates that agents learning from peers are often not fully rational in their treatment of social structure. When networks are complicated and/or uncertain, determining Bayesian behavior can be very difficult and more naive learning rules are especially prevalent. Motivated by this literature, we assume the following departure from rationality.

**Assumption 3 (Behavioral Assumption).** Each agent wrongly believes that each predecessor chooses an action to maximize his expected payoff based on only his private signal, and not on his observation of other agents.
Besides this error, agents are Bayesian in all other ways and correctly optimize their expected utility given their mistaken beliefs. Thanks to Unbounded Signal Informativeness (Assumption 1), every action that a predecessor might take can be rationalized using only their private signal.

The behavioral assumption can be equivalently described as misperceiving $N_j = \emptyset$ for all $j < i$. Under this interpretation, agents act as if their neighbors do not observe anyone. A consequence is that actions are independent of knowledge about the network structure or even knowledge about the order of agents.

This assumption was first considered in a sequential learning setting by Eyster and Rabin (2010), who coined the term “best-response trailing naive inference” (BRTNI) to describe this behavior. They find that even in an “information-rich” setting where private signals have full support and actions perfectly reveal beliefs, BRTNI agents on a complete network have a positive probability of converging to playing the wrong action.

Subsequent experimental evidence documented BRTNI behavior in a lab setting. Eyster, Rabin, and Weizsacker (2015) show that BRTNI behavior is especially prevalent when the observation network structure is complicated. They find that under the complete observation network, Bayesian best responses (assuming all previous players are Bayesians) outnumber BRTNI play 14 to 1. However, when players are placed into sets of size 4 and only observe the actions of people in previous sets, BRTNI play outnumbers Bayesian best responses 3 to 2, possibly because it is more difficult to deduce the correct Bayesian response on this more complicated network. Therefore, we view the study of naive learning on more complicated networks as both important and potentially even more realistic than naive learning on the complete network.

Mueller-Frank and Neri (2015) study an equivalent assumption in a more general social learning framework, which they call Quasi-Bayesian learning. In a laboratory experiment, the authors test this assumption in a network setting where agents receive no information about their neighbors’ connections. At least 90% of actions across various specifications are consistent with Quasi-Bayesian learning, indicating the assumption is plausible.

Lemma 1 characterizes the behavior of a naive agent who sees private signal $s_i$ and neighbor actions $(a_j)_{j \in N_i}$. To state the Lemma and our subsequent results, it is convenient to first perform the following change of variables:

Notation 1. \( \tilde{s}_i := \ln \left( \frac{\mathbb{P}[\omega=1|s_i]}{\mathbb{P}[\omega=0|s_i]} \right) \), \( \tilde{\alpha}_i := \ln \left( \frac{\alpha_i}{1-\alpha_i} \right) \)

So, \( \tilde{s}_i \) is the log-likelihood of the event \( \{\omega = 1\} \) given signal $s_i$, while \( \tilde{\alpha}_i \) is the log-likelihood of $\omega = 1$ corresponding to action $a_i$. If $a_i$ is optimal given $i$’s beliefs, then \( \tilde{\alpha}_i \) is the log-likelihood of \( \{\omega = 1\} \) according to $i$’s beliefs. Note the maps from $s_i$ to $\tilde{s}_i$ and from $a_i$ to $\tilde{\alpha}_i$ are bijective, so no information is lost when we relabel variables.
Lemma 1. A naive agent who observes private signal $s_i$ and neighbor actions $(a_j)_{j \in N_i}$ plays $a_i$ such that $\tilde{a}_i = \tilde{s}_i + \sum_{j \in N_i} \tilde{a}_j$.

The key step is to see that, due to naiveté, agent $i$ who observes log action $\tilde{a}_j$ wrongly infers that $j$’s private log signal must satisfy $\tilde{s}_j = \tilde{a}_j$. This means $i$’s belief about the state of the world can be expressed simply in terms of the $\tilde{a}_j$’s that she observes. The action $a_i$ is the product of the relevant likelihoods because a naive agent assumes observed actions are independent, and therefore $\tilde{a}_i$ is the sum of the corresponding log-likelihoods.

The Lemma characterizes agent behavior by expressing actions in terms of observed information. Moreover, the relevant formula is simple and linear. In Theorem 1, we will use Lemma 1 to express actions in terms of only signals and the network structure.

2.3 Weighted Network Edges

In Section 4, we will take comparative statics with respect to the network structure. Because studying continuous changes in the network is more tractable than discrete changes, we extend our network model to allow non-integral weights.

Henceforth our adjacency matrix entries can have real-number weights $M_{i,j}$, which we usually take to lie in the interval $[0, 1]$. Intuitively, this corresponds to the deterministic version of a random network model where link $(i,j)$ is formed with probability $M_{i,j}$. In Appendix B, we check several of the results from Section 4 hold in simulations with random networks.

More formally, we give two interpretations under which a version of Lemma 1 holds, with observed actions given weight proportional to the weight on the corresponding link. The underlying idea is that agents place more trust in or more emphasis on neighbors with whom their connections are stronger.

**Noisy Observation Interpretation:** This interpretation applies when signals are Gaussian as in Assumption 2. Suppose that instead of observing $\tilde{a}_j$, agent $i$ observes $\tilde{a}_j + \epsilon_{i,j}$, where $\epsilon_{i,j} \sim \mathcal{N}(0, \frac{4}{\sigma^2}(\frac{1}{M_{i,j}} - 1))$ is a Gaussian random variable with mean zero and variance decreasing in $M_{i,j}$. Note that $M_{i,j} = 0$ corresponds $i$ observing a signal about $\tilde{a}_j$ with “infinite variance”, i.e. a purely uninformative signal. On the other hand, $M_{i,j} = 1$ means $i$ observes $\tilde{a}_j$ perfectly. We then consider actions in the case where observations are correct (or equivalently, actions of agents who mistakenly perceive observation as noisy).

**Generations Interpretation:** Replace each agent $i$ with continuum of agents $[0, 1]$, interpreted as the $i^{th}$ generation. Each member of generation $i$ has $M_{i,j}$ chance of observing a uniformly random member of generation $j$, where the randomization over observation or no observation is independent across generation $i$ members. Everyone in generation $i$ receives the same private signal $s_i$, and we re-interpret $a_i$ as the average action among generation $i$.
A model where each generation contains a continuum of agents is central to Gagnon-Bartsch and Rabin (2016)’s study of naive social learning. We stipulate that generation $i$ players make i.i.d. observations of generation $j$’s behavior, just as members of the same generation receive i.i.d. private signals in Gagnon-Bartsch and Rabin (2016).

Under either of these interpretations, by arguments analogous to Lemma 1, it is easy to see that:

**Lemma 2.** $\tilde{a}_i = \tilde{s}_i + \sum_{j \in N_i} M_{i,j} \tilde{a}_j$.

### 3 General Results

#### 3.1 Path-Counting Interpretation of Actions

If players were Bayesians, then the vector of their actions $(a_i)$ would be a complicated function of their log signal realizations, $(s_i)$. We now show with naive agents, actions actually have a simple (log-)linear expression in terms of paths in the network. This result requires no further assumptions about the signal structure.

Below we abuse notation by using $M$ to also refer to the $n \times n$ upper-left submatrix of $M$.

**Theorem 1.** For each $n$, the actions of the first $n$ agents are determined by

$$
\begin{pmatrix}
\tilde{a}_1 \\
\vdots \\
\tilde{a}_n
\end{pmatrix} = (I - M)^{-1} \cdot \begin{pmatrix}
\tilde{s}_1 \\
\vdots \\
\tilde{s}_n
\end{pmatrix}.
$$

So, $\tilde{a}_i$ is a linear combination of $(\tilde{s}_j)_{j=1}^n$, with coefficients given by the number of weighted paths from $i$ to $j$ in the network with adjacency matrix $M$.

**Proof.** By Lemma 2, for each $i$ we have $\tilde{a}_i = \tilde{s}_i + \sum_{j \in N_i} M_{i,j} \tilde{a}_j$. In vector notation, we therefore have:

$$
\begin{pmatrix}
\tilde{a}_1 \\
\vdots \\
\tilde{a}_n
\end{pmatrix} = \begin{pmatrix}
\tilde{s}_1 \\
\vdots \\
\tilde{s}_n
\end{pmatrix} + M \cdot \begin{pmatrix}
\tilde{a}_1 \\
\vdots \\
\tilde{a}_n
\end{pmatrix}.
$$

Algebra then yields the desired expression. Note that $(I - M)$ is invertible because $M$ is lower triangular with all diagonal entries equal to zero.

To see the path-counting interpretation, write $(I - M)^{-1} = \sum_{k=0}^{\infty} M^k$. Here, $M^k_{i,j}$ counts the number of weighted paths of length $k$ from $i$ to $j$. 

\qed
The Theorem expresses agent behavior in terms of only the network structure and signal realizations. Actions are a linear combination of signals, where the weights have a simple formula in terms of the adjacency matrix.

From a combinatorial perspective, the formula says that the influence of \( j \)'s signal on \( i \)'s action depends on the number of (weighted) paths from \( i \) to \( j \). Note that weighted paths here means paths counted with multiplicity given by the product of the weights of the edges included in the path.

Notably, this expression does not depend on any functional form assumptions about signals.

**Example 1.** Suppose \( n = 4 \) and every agent observes every predecessor. Agent 1 can only use her own private signals, so \( \tilde{a}_1 = \tilde{s}_1 \). Agent 2 (correctly) believes that agent 1’s decision reflects her private signal, so agent 2's log action is given by 
\[
\tilde{a}_2 = \tilde{a}_1 + \tilde{s}_2 = \tilde{s}_1 + \tilde{s}_2.
\]
From the naive learning rule, agent 3 (wrongly) reasons that all predecessors only had access to their own private signals, so she plays 
\[
\tilde{a}_3 = \tilde{a}_1 + \tilde{a}_2 + \tilde{s}_3 = 2\tilde{s}_1 + \tilde{s}_2 + \tilde{s}_3,
\]
where we substituted expressions for \( \tilde{a}_1 \) and \( \tilde{a}_2 \) just established. By similar arguments, we can get 
\[
\tilde{a}_4 = \tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3 + \tilde{s}_4 = 4\tilde{s}_1 + 2\tilde{s}_2 + \tilde{s}_3 + \tilde{s}_4.
\]

Viewed as a network, the observation structure is the complete network given by:
\[
M = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0
\end{pmatrix}.
\]

According to Theorem 1, for any signal structure, we have
\[
\begin{pmatrix}
\tilde{a}_1 \\
\tilde{a}_2 \\
\tilde{a}_3 \\
\tilde{a}_4
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0
\end{pmatrix}^{-1} \cdot \begin{pmatrix}
\tilde{s}_1 \\
\tilde{s}_2 \\
\tilde{s}_3 \\
\tilde{s}_4
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
2 & 1 & 0 & 0 \\
4 & 2 & 1 & 1
\end{pmatrix} \cdot \begin{pmatrix}
\tilde{s}_1 \\
\tilde{s}_2 \\
\tilde{s}_3 \\
\tilde{s}_4
\end{pmatrix},
\]
where \( I_{4 \times 4} \) is the 4 by 4 identity matrix. This is the same as what we established above.

Next, to illustrate the path-counting interpretation, notice agent 1’s log private signal \( \tilde{s}_1 \) appears with weights 1, 1, 2, and 4 in log actions \( \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4 \). We now verify these these numbers are precisely the number of paths from various agents to agent 1. Agent 1 has one trivial path of length 0 to herself. Agent 2 has one path of length 1 to agent 1. Agent 3 has two paths to agent 1, namely \( 3 \to 2 \to 1 \) and \( 3 \to 1 \). Finally, agent 4 has 4 paths to agent 1, which are \( 4 \to 3 \to 2 \to 1, 4 \to 3 \to 1, 4 \to 2 \to 1, \) and \( 4 \to 1 \).

In general, in a complete network, for \( i > j \) there exist exactly \( 2^{i-j-1} \) paths from \( i \) to \( j \),
so we have \( \tilde{a}_i = \tilde{s}_i + \sum_{j=1}^{i-1} 2^{n-j-1} \tilde{s}_j \).

Our Theorem 1 resembles Proposition 2 from Mobius, Phan, and Szeidl (2015), who conducted an experiment on social learning. In a learning environment with binary states of the world and binary signal of either \(-1\) or \(1\), Mobius, Phan, and Szeidl (2015) proposed a learning rule where each agent \(i\) calculates the sum of her private signal and “summary reports” that she receives from others, then communicates this sum as a “summary report” in future interactions. Under this rule, the contribution of \(i\)'s signal to \(j\)'s belief depends on the number of paths between them in the communication network. In lieu of imposing this learning rule, the same sort of path-based dependence arises endogenously if we simply assume agents move sequentially, observe network predecessors’ actions, and infer naively about the source of these actions.

3.2 Condition for Correct Learning

We will now use the representation result of Theorem 1 to study which networks lead to mislearning by naive agents. We first define what it means for society to learn correctly in terms of convergence of actions.

**Definition 1.** Society learns correctly if \((a_n)\) converges to \(\omega\) in probability.

When society learns correctly, agent \(n\) becomes very likely to believe strongly in the true state of the world as \(n\) grows large.

We define below a notion of network influence for the sequential social learning environment, which plays a central role in determining whether society learns correctly for a broad class of signal structures.

**Definition 2.** Let \(b_{i,j} := (I - M)^{-1}_{ij}\) be the number of weighted paths from \(i\) to \(j\).

Without network weights, the entries \(b_{i,j}\) are equal to the number of paths from \(i\) to \(j\). With network weights, each path is counted with multiplicity equal to the product of the weights of the edges included in the path. Because of Theorem 1, these path counts are important to our analysis.

**Definition 3.** For \(n > i\), the influence of \(i\) on \(n\) is \(I(i,n) := b_{n,i}/\sum_{j=1}^{n} b_{n,j}\).

That is to say, influence of \(i\) on \(n\) is the fraction of paths from \(n\) that end at \(i\).

A similar definition of influence appears in Golub and Jackson (2010), who studied naive social learning in a network where agents act simultaneously each period. For them, the influence of an agent \(i\) in an undirected network is the fraction of links in the network that connect to \(i\). Golub and Jackson (2010) showed that, roughly speaking, society learns the
true state of the world with certainty if and only if the influence of every agent converges to 0 as network size grows. We show below that the same condition characterizes correct social learning in our model, but under our definition of influence. Intuitively speaking, a degree-based definition of influence is relevant in learning environments where each agent repeatedly communicates with the same set of neighbors for many periods, whereas a path-based definition of influence matters in learning environments with a sequential move structure where early movers can substantially affect the learning outcome.

We will make the following connectedness assumption on $M$.

**Assumption 4 (Connectedness Assumption).** There is an integer $N$ and constant $C > 0$ such that for all $i > N$, there exists $j < N$ with $b_{i,j} \geq C$.

Intuitively, this says that all sufficiently late agents indirectly observe some early agent. When all links have weight 0 or 1, this assumption holds if and only if there are only finitely many agents who have no neighbors. If such a network fails the connectedness assumption, then clearly the infinitely many agents without neighbors will prevent the society from learning correctly.

**Theorem 2.** Suppose the connectedness assumption holds and $\tilde{s}$ has finite variance. Society learns correctly if and only if $\lim_{n \to \infty} I(i, n) = 0$ for all $i$.

Theorem 2 says that beliefs always converge to the truth if and only if no agent has undue influence in the network. This is a recurring insight in research on social learning on networks, beginning with the “royal family” example and related results in Bala and Goyal (1998). As discussed above, our condition for learning correctly has a similar structure to a condition in Golub and Jackson (2010). The idea that a few influential agents can drive mislearning is also central to Acemoglu, Ozdaglar, and ParandehGheibi (2010). There, influence depends on whether agents are “forceful” and not only on their network position. The main contribution of our Proposition is to identify the relevant measure of influence in a sequential learning setting with naive agents.

The idea behind the proof is that if there were some $i$ and $\epsilon > 0$ such that $I(i, n) > \epsilon$ for infinitely many $n$, then $i$ exerts at least $\epsilon$ influence on all these future players. Since $\tilde{s}_i$ is unbounded, there is a rare but positive probability event where $i$ gets such a strong but wrong private signal so that any future player who puts $\epsilon$ weight on $\tilde{s}_i$ and $(1 - \epsilon)$ weight on other signals would come to believe in the wrong state of the world with high probability. But this would mean infinitely many players have a high probability of believing in the wrong state of the world, so society fails to learn correctly. For the opposite direction, we approximate agent $n$’s action by a Gaussian random variable and then show that the random variable is likely to be large and positive (in state $\omega = 1$) when no agent has much influence.
4 Applications

In this section, we consider several simple network structures for which we can explicitly compute probabilities of correct learning and mislearning using the formulas from Section 3. After computing the probability of learning, we take comparative statics as the link density and degree of homophily vary. The first network structure assigns the same weight to each link. Next, we split agents into two groups and allow different weights on links within groups and between groups. As extensions, we consider variations of these networks where link weights decay exponentially with distance and the network where all agents have the same fixed degree.

For this section, we assume signals are Gaussian and symmetric across the two states as in Assumption 2 in order to get tractable closed-form results.

From simulations using other signal structures in Appendix B, the Gaussian assumption does not seem to substantially change results. The Gaussian signal structure also allows us to draw a parallel between our setting of social learning with sequential movers to a setting of social learning with simultaneous movers, where each agent acts in every period. If we impose the same behavioral assumption and the Gaussian signal assumption in the simultaneous move environment, we find that players act according to the DeGroot heuristic (DeMarzo, Vayanos, and Zwiebel, 2003).

With Gaussian signals, we can give explicit expressions for the distributions of agent actions and conditions for correct learning. To this end, we first show that the associated log-likelihoods also have a Gaussian distribution:

Lemma 3. Under Assumption 2, \( \tilde{s}_i = 2s_i / \sigma^2 \).

We now show the probability that agent \( n \) is correct about the state is related to the ratio of \( \ell_1 \) norm to \( \ell_2 \) norm for the vector of path counts, \( \tilde{b}_n := (b_{n,1}, \ldots, b_{n,n}) \), which has been used as a measure of normalized sparsity in matrix factorization and blind deconvolution (Hoyer, 2004; Krishnan, Tay, and Fergus, 2011).

Lemma 4. When \( \omega = 1 \), the action of agent \( n \) is distributed as \( \tilde{a}_n \sim \mathcal{N} \left( \frac{2}{\sigma^2} \left\| b_n \right\|_1, \frac{4}{\sigma^2} \left\| b_n \right\|_2^2 \right) \).

The action is more likely to be higher when \( \frac{\left\| b_n \right\|_1}{\left\| b_n \right\|_2} \) is large, so the agent is less likely to be correct about the state when the vector of path counts is sparser. Agent \( n \) is likely to be correct when their action is an average of many signals with weights that do not vary too much, and unlikely to be correct when their action puts disproportionally heavy weights on a few signals.

The proof first expresses \( \tilde{a}_n = \sum_{i=1}^n b_{n,i} \tilde{s}_i \) using Theorem 1, then observes \( s_i \) are distributed i.i.d. \( \mathcal{N}(1, \sigma^2) \) conditional on \( \omega = 1 \). This means \( \tilde{s}_i \) are also conditionally i.i.d. Gaussian random variables, due to Lemma 3. As a sum of conditionally i.i.d. Gaussian
random variables, the action $\tilde{a}_n$ is itself Gaussian. The result follows from calculating the mean and variance of this sum.

Our basic technique in this section is to count paths on a given network using an appropriate recurrence relation, and then to apply Lemma 4.

### 4.1 Uniform Weights

The simplest network we consider assigns the same weight $p \in [0, 1]$ to each feasible link. This network is analogous to an Erdos-Renyi random network, which generates each feasible link with a fixed probability $p$. By varying the value of $p$, we can ask how link density affects mislearning.

For the remainder of this section, let the signal variance $\sigma^2 = 1$.

**Proposition 1.** When $p \in (0, 1]$, agents’ actions $a_n$ converge to 0 or 1 a.s. The probability of converging to the incorrect belief is

$$\Phi\left(-\sqrt{\frac{p+2}{p}}\right)$$

where $\Phi$ is the standard Gaussian distribution function. This probability is increasing in $p$.

The proof relies on the recurrence relation $b_{n,i} = (1 + p)b_{n-1,i}$. To see this recurrence holds, let $\Psi_n$ be the set of all paths from $n$ to $i$ and $\Psi_{n-1}$ be the set of all paths from $n-1$ to $i$. For each $\psi \in \Psi_{n-1}$, $\psi = n-1 \rightarrow j_1 \rightarrow j_2 \rightarrow ... \rightarrow i$, we associate two paths $\psi', \psi'' \in \Psi_n$, with $\psi' = n \rightarrow j_1 \rightarrow j_2 \rightarrow ... \rightarrow i$ and $\psi'' = n \rightarrow n - 1 \rightarrow j_1 \rightarrow j_2 \rightarrow ... \rightarrow i$. This association exhaustively enumerates all paths in $\Psi_n$ as we consider all $\psi \in \Psi_{n-1}$. Path $\psi'$ has the same weight as $\psi$ since they have the same length, while path $\psi''$ has $p$ fraction of the weight of $\psi$ since it is longer by one. This shows the weight of all paths in $\Psi_n$ is equal to $1 + p$ times the weight of all paths in $\Psi_{n-1}$, hence $b_{n,i} = (1 + p)b_{n-1,i}$.

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The first statement of the Proposition tells us that agents eventually agree on the state of the world, and these beliefs are arbitrarily strong after some time. These consensus beliefs need not be correct, however. The probability of incorrect beliefs asymptotically is non-zero for all positive $p$, and increases in $p$. When the observational network is more densely connected, society is more likely to be wrong.

When $p$ is low, early agents’ actions convey a large amount of independent information, which facilitates later agents’ learning. For high $p$, early agents’ actions are highly correlated, so later naive agents cannot recover the true state as easily. A related intuition compares agents’ beliefs about network structure to the actual network: as $p$ grows larger, agents’ beliefs about the network structure diverge more and more from the true network.
The case \( p = 1 \) is studied in Eyster and Rabin (2010), which concentrates on a slightly different signal structure. In this case, Eyster and Rabin (2010) show that agents’ beliefs converge to 0 or 1 almost surely and derive a non-zero lower bound on the probability of converging to the incorrect belief. By contrast, our result gives the exact probability of converging to the wrong belief for any \( p \in (0, 1] \), under a Gaussian signal structure.

There is a discontinuity at \( p = 0 \). As \( p \) approaches 0, the probability of society learning correctly approaches 1. But when \( p = 0 \) there is no learning, because all agents choose actions equal to their private signals.

4.2 Two Groups

We next consider a network with two groups and different weights for links within groups and between groups. This is a deterministic version of a simple example of stochastic block models, which are discussed further in Section 5. By varying the link weights, we will consider how homophily changes learning outcomes.

Odd-numbered agents are in one group and even-numbered agents are in a second group. Each within-group link has weight \( p_s \) and each between-groups link has weight \( p_d \), so that \( M_{i,j} = p_s \) if \( i \equiv j \pmod{2} \) and \( M_{i,j} = p_d \) otherwise. We denote the probability of converging to the incorrect belief with weights \( p_s \) and \( p_d \) as \( \alpha(p_s, p_d) \), the mislearning probability.

Proposition 2. When \( p_s \in [0, 1] \) and \( p_d \in (0, 1] \), agents’ actions \( a_n \) converge a.s. to 0 or 1. The derivatives of the mislearning probability \( \alpha(p_s, p_d) \) satisfy

\[
\frac{\partial \alpha}{\partial p_d} > \frac{\partial \alpha}{\partial p_s} > 0,
\]

i.e. the probability is increasing in \( p_s \) and \( p_d \), but increasing \( p_d \) has a larger effect than increasing \( p_s \).

The first statement again says that agents eventually agree on the state and eventually have arbitrarily strong beliefs. The fact that \( \alpha \) is increasing in \( p_s \) and \( p_d \) is another example of higher link density implying more mislearning. The comparison \( \frac{\partial \alpha}{\partial p_d} > \frac{\partial \alpha}{\partial p_s} \) tells us that more integrated networks are more likely to herd on the wrong state of the world. Note that these results hold even when \( p_s < p_d \).

Convergence of beliefs is more subtle with two groups, as we might imagine the two homophilic groups holding different beliefs asymptotically. In Section 5, we will see that this is possible with a coarser action space. By contrast, with continuous actions, agents can convey the strength of their beliefs. As such, eventually one group will develop sufficiently strong beliefs to convince the other.
To see this must occur, observe that the belief of a later agent $n$ depends mostly on the number of paths from that agent to early agents (and those agents’ signal realizations). When $n$ is large, most paths from agent $n$ to an early agent pass between the two groups many times. So the number of paths does not depend substantially on agent $n$’s group. Therefore, agent $n$’s belief does not depend substantially on whether $n$ is in the odd group or the even group.

To interpret the comparative statics, consider changing network weights from $p_s$ and $p_d$ to $p_s - \epsilon$ and $p_d + \epsilon$ for $\epsilon > 0$ sufficiently small. This adds links between groups and removes links within groups, while keeping the degree of later agents approximately constant. As a consequence of the Proposition, this change decreases the probability that society learns correctly.

An important insight from the literature on social learning on networks is that beliefs converge more slowly on more segregated networks. This intuition is formalized for DeGroot learning in Golub and Jackson (2012), which defines a new measure of homophily and shows that measure also describes the speed of learning. In our model, faster convergence of beliefs tends to imply a higher probability of incorrect beliefs. When beliefs converge quickly, agents are putting far too much weight on early movers, while when beliefs converge more slowly agents wait for more independent information. Since agents eventually agree, segregation helps society form strong beliefs more gradually.

### 4.3 Constant Out-Degree

In Subsection 4.1, we considered a network where the number of predecessors observed by an agent grows linearly in the agent’s position in the line. This assumption implies there are late movers who observe arbitrarily many predecessors, which may be unrealistic in some settings. Here we consider an alternate network where all agents have the same number of observations – that is to say, all agents have the same out-degree in the network – and show that a result similar to Proposition 1 still holds.

Suppose there is some positive integer $d$ such that agent $n$ puts weight $d/(n-1)$ on each predecessor. This ensures each agent has an out-degree of $d$ in the observation network $M$. In the analogous random network, this corresponds to each agent having $d$ neighbors on average. (If $d$ is large, then our network has some early agents assigning weight greater than 1 to each of their predecessors. This simplifies proofs, but it is not necessary.)

**Proposition 3.** Society learns correctly if and only if $d = 1$. If $d > 1$, then actions $a_n$ converge to 0 or 1 a.s., and the probability of incorrect beliefs is increasing in $d$. 

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4.4 Decaying Weights

In the example networks we studied so far, the weight placed on the link between agents \(i\) and \(j\) did not depend on the distance between them. In many settings, agents place more weight on recent predecessors than on predecessors who moved long before. For example, recency is often a criterion in deciding which online reviews to display. In other settings, individuals may place less trust in those who are farther away in some sense.

To model these dynamics, we consider networks where network weights decay exponentially in distance. When the rate of decay is not too low, the results of Subsections 4.1 and 4.2 still apply. When the rate of decay is low, agents’ beliefs do not converge. At the threshold between these values, society learns correctly.

We begin with a network where weights depend only on distance. Let \(\delta > 0\) and suppose each link from an agent \(i\) to a predecessor \(j\) has weight \(M_{i,j} = \delta^{i-j}\).

**Proposition 4.** If \(\delta < \frac{1}{2}\), then with probability one the actions \(a_n\) do not converge. If \(\delta \geq \frac{1}{2}\), beliefs converge a.s. to 0 or 1. Society learns correctly when \(\delta = \frac{1}{2}\), while the probability of mislearning is non-zero and increasing in \(\delta\) when \(\delta > \frac{1}{2}\).

The case of \(\delta > \frac{1}{2}\) turns out to have the same recurrence relation as the uniform \(p\) weight network for some suitable \(p\). When \(\delta = \frac{1}{2}\), private signals of all predecessors are given equal weight, so law of large numbers implies correct learning. To see the a.s. non-convergence result for \(\delta < \frac{1}{2}\), we show that there is a uniform bound on the mean and variance of \(\tilde{a}_n\) for all \(n\), hence \(\tilde{a}_n \to \infty\) cannot happen with positive probability. A similar argument shows that \(\tilde{a}_n \to -\infty\) is also a zero-probability event. At the same time, \(\tilde{a}_{n+1} - \tilde{a}_n\) has non-negligible variance, so we also cannot have \(\tilde{a}_n \to c\) for some real number \(c\).

The main takeaway is that our intuitions about link density extend to this setting where weights depend on distance and degree is bounded. There is also a new phenomenon, given by a threshold decay rate that divides networks where convergence does not occur and those where beliefs converge but can be wrong.

At the threshold \(\delta = \frac{1}{2}\), the out-degree of agents approaches one asymptotically. Beliefs converge when agents make more than one direct observation but need not converge with less than one direct observation. Society only learns correctly at the threshold value, which highlights the fragility of learning in our model.

Our results with two groups also extend to networks where weights decay with distance. As in Subsection 4.2, consider a network with odd-numbered agents in one groups and even-numbered agents in another group. The link from a later agent \(i\) to an earlier agent \(j\) has weight \(\delta^{i-j}p_s\) if \(i\) and \(j\) are in the same group (\(i \equiv j \mod 2\)) and weight \(\delta^{i-j}p_d\) if \(i\) and \(j\) are in different groups (\(i \equiv j + 1 \mod 2\)).
Proposition 5. For each \( p_s \) and \( p_d \), there exists a threshold \( \delta_0(p_s, p_d) \) such that (i) with probability one, actions do not converge when \( \delta < \delta_0(p_s, p_d) \); (ii) society learns correctly when \( \delta = \delta_0(p_s, p_d) \); (iii) beliefs converge a.s. to 0 or 1 when \( \delta \geq \delta_0(p_s, p_d) \). In the region where \( \delta > \delta_0(p_s, p_d) \), the probability of mislearning is non-zero and increasing in \( \delta, p_s \) and \( p_d \) but increasing \( p_d \) has a larger effect than increasing \( p_s \).

This Proposition, which extends Propositions 2 and 4, demonstrates the robustness of each of the two previous results.

4.5 Naive and Bayesian Agents

Several recent experiments on learning suggest subjects are divided into a subpopulation of Bayesian agents and another subpopulation of naive agents (Eyster, Rabin, and Weizsacker, 2015; Enke and Zimmermann, 2016). So theoretical models with two agent types would likely be most realistic.

We consider a mix of naive agents and Bayesian agents who (incorrectly) believe that all predecessors are Bayesians. When the network is complete and this is common knowledge, behavior has a simple interpretation: each Bayesian agent is equivalent to a naive agent who sees only their immediate predecessor. So actions are the same on the complete network with a mix of agents as on a modified network with only naive agents. If a positive fraction of agents is naive, there will be a positive probability of incorrect herding.

4.6 Relation to Proposition 4 of Eyster and Rabin (2014)

Our Theorem 2 is related to Proposition 4 of Eyster and Rabin (2014), which provides a set of sufficient conditions under which agents converge to certain beliefs in the incorrect state with positive probability. Their result applies to a more general class of learning environments and learning rules. We view our result as complimentary to theirs in providing a necessary and sufficient characterization of the networks that lead to mislearning in the particular learning environment of naive sequential learning. When specialized to our setting, the “redundancy neglect” condition of Eyster and Rabin (2014)’s Proposition 4 applies if and only if there exist \( c > 0 \) and \( N \in \mathbb{N} \) such that \( \sum_{j=1}^{t-N} M_{t,j} > (1 + c) \) for every \( t \geq N + 1 \). This rules out some network structures such as \( M_{i,j} = \frac{d_{i,j}}{t^{-1}} \) for \( j < i \), which spreads \( d \) weight equally on all predecessors for each agent. But as we have seen in Proposition 3, when \( d > 1 \) we have \( \lim_{n \to \infty} I(1,n) \neq 0 \) and there is positive chance of converging to certain belief in the incorrect state, with this chance increasing in \( d \).
5 Disagreement

In Subsection 4.2, we saw that even on segregated networks agents eventually agree on the state of the world. This agreement relies crucially on the richness of the action space available to agents, which allows agents to communicate the strength of their beliefs. In this section, we modify our model so that the action space is binary. With binary actions, two groups can disagree about the state of the world even when the number of connections across the groups is unbounded.

The contrasting results for binary actions model versus the continuum actions model echo a similar contrast in the rational herding literature, where society herds on the wrong action with positive probability when actions coarsely reflect beliefs (Banerjee, 1992; Bikhchandani, Hirshleifer, and Welch, 1992), but almost surely converges to the correct action when action set is rich enough (Lee, 1993). Interestingly, while the rational herding literature finds that unboundedly informative signal structure prevents herding on the wrong action even when actions coarsely reflect beliefs (Smith and Sørensen, 2000), we will show below that even with Gaussian signals two groups may disagree with positive probability.

Suppose that the state of the world and signal structure are the same as in Section 2, but agents now choose actions $a_i \in \{0, 1\}$. Agents still maximize the expectation of $u_i(a_i, \omega) := -(a_i - \omega)^2$ given their beliefs about $\omega$. This utility function now implies that an agent chooses the action corresponding to the state of world she believes is more likely. With the action space changed, we no longer need unbounded signal informativeness (Assumption 1).

In this section, we consider an actual random network instead of weighted networks (the conclusions are the same either way). Odd- and even-numbered agents are in different groups. Let $p_s$ be the probability of link formation within groups and $p_d$ be the probability of link formation between groups. So agent $i$ observes agent $j$ with probability $p_s$ if $i \equiv j \pmod{2}$, and with probability $p_d$ otherwise. This is a simple directed case of stochastic block networks, which have been used in economics to consider the effect of homophily on speed of learning (Golub and Jackson, 2012) and are widely studied in computer science and related fields (e.g. Airoldi, Blei, Feinberg, and Xing, 2008).

We will state our disagreement result for symmetric binary signals and for Gaussian signals. These functional forms can be relaxed without altering the result, and the key assumptions are stated in the proof. Most importantly, signal structures with fat tails need to be ruled out so that the probability of an agent seeing an extremely informative private signal that overturns the information contained in his neighbor’s actions vanishes to zero fast enough along the sequence.

**Assumption 5** (Binary Signals). $s_i \in \{0, 1\}$, and $s_i = \omega$ with probability $p > \frac{1}{2}$ in either
This says that signals are binary and symmetric. Note that our behavioral assumption does not have a clear interpretation with continuous actions and binary signals, because people will observe actions that cannot arise from only private signals.

**Theorem 3.** Suppose $p_s > p_d$, agents play binary actions, and the environment either has Gaussian signals from Assumption 2 or binary signals from Assumption 5. Then there is a positive probability (with respect to the joint distribution over network structures and signal realizations) that all odd-numbered agents choose action 0 while all even-numbered agents choose action 1.

**Proof.** We will prove a more general statement where disagreement happens if the signal structure satisfies the following three assumptions:

*No Uninformative Signals*: Signals $s$ for which $\Pr[\omega = 1|s] = \Pr[\omega = 0|s]$ have 0 probability. So, we may decompose up to a $\mathbb{P}$-null set $S = S^+ \cup S^-$ where

$$S^+ := \{s \in S : \Pr[\omega = 1|s] > \Pr[\omega = 0|s]\}$$

$$S^- := \{s \in S : \Pr[\omega = 1|s] < \Pr[\omega = 0|s]\}.$$

*Symmetry*:

$$\frac{\Pr[\omega = 1|s \in S^+]}{\Pr[\omega = 0|s \in S^+]} = \frac{\Pr[\omega = 0|s \in S^-]}{\Pr[\omega = 1|s \in S^-]}.$$

*Thin Tails*:

$$\sum_{n=0}^{\infty} \mathbb{P}\left[s \in S : \ln\left(\frac{\Pr[\omega = 1|s]}{\Pr[\omega = 0|s]}\right) > n | \omega = 1\right] < \infty$$

and

$$\sum_{n=0}^{\infty} \mathbb{P}\left[s \in S : \ln\left(\frac{\Pr[\omega = 1|s]}{\Pr[\omega = 0|s]}\right) < -n | \omega = 1\right] < \infty.$$

*No Uninformative Signals* is for convenience and rules out tie-breaking problems. *Thin Tails* says when the state is $\omega = 1$, extremely informative signals in favor of either state are not too common. This assumption is automatically satisfied if the signal structure is boundedly informative, such as in the case of binary signal structure from Assumption 5.

Now consider Gaussian signal structure from Assumption 2. Then *No Uninformative Signals* and *Symmetry* are of course satisfied, with $S^+ = \mathbb{R}_{>0}$ and $S^- = \mathbb{R}_{<0}$. For *Thin Tails*, $\ln\left(\frac{\Pr[\omega = 1|s]}{\Pr[\omega = 0|s]}\right) = 2s/\sigma^2$ under the Gaussian assumption, so we are looking at the sum

$$\sum_{n=0}^{\infty} \mathbb{P}[2N(1, \sigma^2)/\sigma^2 > n].$$

This is convergent because it is known that the Gaussian distribution function tends to 0 faster than geometrically. Similarly we also have $\sum_{n=0}^{\infty} \mathbb{P}[2N(1, \sigma^2)/\sigma^2 < -n] < \infty.$
So it suffices to show the Proposition under assumptions No Uninformative Signals, Symmetry and Thin Tails. We make use of the following concentration inequality from Hoeffding (1963):

**Fact 1.** (Hoeffding) Suppose $X_1, ..., X_n$ are independent random variables on $\mathbb{R}$ such that $a_i \leq X_i \leq b_i$ with probability 1 for each $i$. Write $S_n := \sum_{i=1}^n X_i$. We have $\mathbb{P}[S_n - \mathbb{E}[S_n] \leq -t] \leq \exp(-2t^2/\sum_{i=1}^n (b_i - a_i)^2)$.

Define $\kappa$ to be a naive agent’s log likelihood ratio of state $\omega = 1$ versus state $\omega = 0$ upon observing one neighbor who picks action 1. Then we have:

$$\kappa := \ln \left( \frac{\mathbb{P}[\omega = 1|s \in S^+]}{\mathbb{P}[\omega = 0|s \in S^+]} \right) > 0.$$  

By Symmetry, the log likelihood ratio after observing one neighbor who chooses action 0 is $-\kappa$.

Suppose after $2n$ agents have moved, the actions taken so far involve every odd-numbered agent playing 1, every even-numbered agent playing 0. Write $B_{n,p_s} \sim \text{Binom}(n, p_s)$ for the number of agents that $2n + 1$ observes from his own group, and $B_{n,p_d} \sim \text{Binom}(n, p_d)$ is the number of agents $2n + 1$ observes from the other group. By Hoeffding’s inequality, the probability that $B_{n,p_s} - B_{n,p_d}$ falls below half of its expected value $n(p_s - p_d)$ is:

$$\epsilon_n^{(1)} := \mathbb{P}\left[(B_{n,p_s} - B_{n,p_d}) - n(p_s - p_d) \leq -\frac{1}{2}n(p_s - p_d) \right] \leq \exp(-\frac{1}{2}n^2(p_s - p_d)^2/(2n)).$$

But in the event that $B_{n,p_s} - B_{n,p_d} \geq \frac{1}{2}n(p_s - p_d)$, $2n + 1$ has a log likelihood ratio of at least $\frac{1}{2}n(p_s - p_d)\kappa$ from observations alone. The probability that private signal $s_{2n+1}$ is so strongly in favor of $\omega = 0$ as to make $2n + 1$ play 0 is:

$$\epsilon_n^{(2)} := \mathbb{P}\left[ s \in S : \ln \left( \frac{\mathbb{P}[\omega = 1|s]}{\mathbb{P}[\omega = 0|s]} \right) < \frac{1}{2}n(p_s - p_d)\kappa|\omega = 1 \right]$$

We have $\sum_{n=1}^{\infty} \epsilon_n^{(1)} < \infty$ because $\epsilon_n^{(1)} \sim \exp(-n)$, while $\sum_{n=1}^{\infty} \epsilon_n^{(2)} < \infty$ due to Thin Tails assumptions. This shows that there is positive probability that every odd-numbered agent plays 1.

By Symmetry, an analogous argument establishes that there is also positive probability that every even-numbered agent plays 0.

The two groups can disagree even though agent $n$ observes approximately $np_d/2$ agents from the other group (when $n$ is large) taking opposite actions.

This result adds a new mechanism to the literature on public disagreement. Acemoglu, Como, Fagnani, and Ozdaglar (2013) also study a model generating disagreement via net-
work learning, but their disagreement takes the form of persistent fluctuations in beliefs whereas our model predicts that two subgroups will converge two different beliefs with positive probability. In Acemoglu, Como, Fagnani, and Ozdaglar (2013), disagreement arises from “stubborn” agents who hold different initial opinions and never change their mind. Here, the key forces driving disagreement are the segregated network structure as well as the tendency of naive agents to herd.

The lack of rich actions that perfectly reveal players’ beliefs is central to the disagreement result. This assumption cannot be relaxed even for one of the groups. The next Proposition considers a modification where signals are Gaussian\(^1\) and one group has a rich action set as in the Section 2 model, while the other group has binary actions. No matter how homophilic the two groups are, they cannot disagree in the long-run. Indeed, whenever the group with the rich action set converges, the group with binary actions also converges to playing the action that corresponds to the same state of the world.

**Proposition 6.** Let \(p_s, p_d > 0\) be arbitrary and suppose the environment has Gaussian signals from Assumption 2. Suppose odd-numbered agents choose actions from the interval \([0, 1]\), but even-numbered agents choose actions from the binary set \(\{0, 1\}\). For \(x \in \{0, 1\}\), conditional on the actions of odd-numbered agents converging to \(x\), the actions of even-numbered agents also converge to \(x\) with probability 1.

The intuition is that with a continuum of actions, an agent can express extreme certainty in some state of the world. But binary actions, at least from the perspective of a naive learner, have limited informativeness about predecessors’ beliefs. Although an even-numbered agent tends to observe more predecessors from her own group than from the other group, when the odd-numbered group converges one observation of its continuous action can cancel out an arbitrarily large number of opposing, binary actions from the even-numbered group. In the end, the continuous-action group acts as a leader and the binary-action group follows whatever trend the continuous-action group sets, even in the presence of homophily.

6 Conclusion

In this paper, we have explored the influence of network structures on learning outcomes when agents move sequentially and suffer from inferential naiveté. We provide several testable predictions distinguishing models of sequential learning with naive agents from rational models.

\(^1\)This result also holds under any signal structure satisfying Unbounded Signal Informativeness (Assumption 1) as well as the three technical assumptions of No Uninformative Signals, Symmetry, and Thin Tails defined in the proof of Theorem 3. Assumption 1 is required here, for otherwise our behavioral assumption does not make sense.
Laboratory experiments based on network structures in Section 4 could test whether varying the network parameters leads to the predicted comparative statics in terms of chance of mislearning. Proposition 1, which states that increasing link density increases the chance of mislearning, would be most straightforward to test. The proposed experiments would complement existing tests of inferential naiveté aiming to compare alternate behavioral models directly.

We studied the simplest possible learning social learning environment to focus on the effect of network structure, but several extensions are straightforward. Analogues of our general results hold for finite state spaces with more than two elements, in which we define log-likelihoods for each pair of states. We can also make the order of moves random and unknown, in which case naive behavior with a given turn order is the same as when that order is certain.

Many of our results included network weights, which could be interpreted as link formation probabilities with generations of agents. The simulations in Appendix B suggest that our results extend to actual random networks. The major obstacle to extending proofs was that because our networks are directed and acyclic, the relevant adjacency matrices have no non-zero eigenvalues. As a consequence, most techniques from spectral random graph theory do not apply. Perhaps this obstacle could be avoided via other methods.

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Appendix

A Omitted Proofs

A.1 Proof of Lemma 1

Proof. Due to naiveté, $i$ thinks neighbor $j$ must have received signal $s_j$ such that $\tilde{s}_j = \tilde{a}_j$. The log-likelihood of state $\omega = 1$ conditional on some signal realizations is:

$$
\ln \left( \frac{\mathbb{P}[\omega = 1 | s_i, (s_j)_{j \in N_i}]}{\mathbb{P}[\omega = 0 | s_i, (s_j)_{j \in N_i}]} \right) = \ln \left( \frac{\mathbb{P}[s_i | \omega = 1]}{\mathbb{P}[s_i | \omega = 0]} \cdot \prod_{j \in N_i} \frac{\mathbb{P}[s_j | \omega = 1]}{\mathbb{P}[s_j | \omega = 0]} \right) 
$$

(two states equally likely)

$$
= \ln \left( \frac{\mathbb{P}[s_i | \omega = 1]}{\mathbb{P}[s_i | \omega = 0]} \right) + \sum_{j \in N_i} \ln \left( \frac{\mathbb{P}[s_j | \omega = 1]}{\mathbb{P}[s_j | \omega = 0]} \right) 
$$

(by independence)

$$
= \tilde{s}_i + \sum_{j \in N_i} \tilde{s}_j.
$$

A naive agent $i$ chooses an action with $\tilde{a}_i$ equal to the log-likelihood of state $\omega = 1$ given private signal $s_i$ and neighbors’ signals such that $\tilde{s}_j = \tilde{a}_j$, which is $\tilde{s}_i + \sum_{j \in N_i} \tilde{a}_j$. \hfill \square

A.2 Proof of Lemma 2

Proof. We check the formula for each of the two interpretations. For the noisy observation interpretation, $i$ observes $\tilde{a}_j + \epsilon_{i,j}$ for each neighbor $j$ and believes that $\tilde{s}_j = \tilde{a}_j$. By Assumption 2 and Lemma 3 (which is proven independently), $\tilde{s}_j$ is distributed as $\mathcal{N}(\frac{2}{\sigma^2}, \frac{4}{\sigma^2})$ when $\omega = 1$, so that $i$ believes their observation has distribution $\mathcal{N}(\frac{2}{\sigma^2}, \frac{1}{M_{i,j}} \cdot \frac{4}{\sigma^2})$ when $\omega = 1$ and $\mathcal{N}(\frac{-2}{\sigma^2}, \frac{1}{M_{i,j}} \cdot \frac{4}{\sigma^2})$ when $\omega = -1$. So agent $i$ believes that $\frac{\sigma^2}{2} (\tilde{a}_j + \epsilon_{i,j}) \sim \mathcal{N}(\pm 1, \frac{\sigma^2}{M_{i,j}})$, depending on the state. Applying Lemma 3 again, we find that $i$ then believes the log-likelihood of $\omega = 1$ is $M_{i,j}(\tilde{a}_j + \epsilon_{i,j})$. In the case when observations are correct, this simplifies to $M_{i,j} \tilde{a}_j$. The proof now proceeds as in the proof of Lemma 1.

We next show the Lemma holds under the generations interpretation. By the law of large numbers a fraction $M_{i,j}$ of the agents in generation $i$ observe an agent in generation $j$, and the average observation among this group is equal to $a_j$. The result follows from computing each member of generation $i$’s action as in the proof of Lemma 1 and then averaging. \hfill \square
A.3 Proof of Lemma 3

Proof. The log-likelihood is

\[
\ln \left( \frac{\mathbb{P}[\omega = 1|s_i]}{\mathbb{P}[\omega = 0|s_i]} \right) = \ln \left( \frac{\mathbb{P}[s_i|\omega = 1]}{\mathbb{P}[s_i|\omega = 0]} \right).
\]

\[
= \ln \left( \frac{\exp \left( -\frac{s_i - 1}{2\sigma^2} \right)}{\exp \left( -\frac{s_i + 1}{2\sigma^2} \right)} \right)
\]

\[
= -\frac{(s_i^2 - 2s_i + 1) + (s_i^2 + 2s_i + 1)}{2\sigma^2}
\]

\[
= \frac{2s_i}{\sigma^2}
\]

\[
\square
\]

A.4 Proof of Lemma 4

Proof. By Theorem 1, \( \tilde{a}_n = \sum_{i=1}^{n} b_{n,i} \tilde{s}_i \). This is equal to \( \sum_{i=1}^{n} \frac{2b_{n,i}}{\sigma^2} s_i \) according to Lemma 3. Conditional on \( \omega = 1 \), \( (s_i) \) are i.i.d. \( \mathcal{N}(1, \sigma^2) \) random variables, so

\[
\sum_{i=1}^{n} \frac{2b_{n,i}}{\sigma^2} s_i \sim \mathcal{N} \left( \frac{2}{\sigma^2} \sum_{i=1}^{n} b_{n,i}, \frac{4}{\sigma^2} \sum_{i=1}^{n} b_{n,i}^2 \right) = \mathcal{N} \left( \frac{2}{\sigma^2} \|b_n\|_1, \frac{4}{\sigma^2} \|b_n\|_2^2 \right).
\]

\[
\square
\]

A.5 Proof of Theorem 2

Proof. Without loss of generality, assume \( \omega = 1 \). (The case of \( \omega = 0 \) is exactly analogous and will be omitted.) First suppose that \( \lim_{n \to \infty} I(j,n) \neq 0 \) for some \( j \). Then there exists \( \epsilon > 0 \) such that \( I(j,n) > \epsilon \) for infinitely many \( n \). For each such \( n \), the probability that agent \( n \) chooses an action with \( \tilde{a}_n < 0 \) is equal to the probability that

\[
\sum_{i=1}^{n} I(i,n) \tilde{s}_i
\]

is negative.

Because \( \tilde{s} \) has finite variance, we can find positive constants \( C \) and \( \delta \) independent of \( n \) such that

\[
\sum_{i \neq j} I(i,n) \tilde{s}_i < C
\]

with probability at least \( \delta \) (for example, by applying Markov’s inequality to \( |\tilde{s}_i| \)). Then agent \( n \) will be wrong if \( \tilde{s}_j < -C/\epsilon \), which is a positive probability event since \( \tilde{s} \) is unbounded. So

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the probability that an agent $n$ such that $I(j, n) > \epsilon$ chooses $\tilde{a}_n < 0$ is bounded from below by a positive constant.

For the converse, suppose that $\lim_{n \to \infty} I(i, n) = 0$ for all $i$. By the connectedness assumption, there exists a positive constant $C \leq 1$ and an integer $N$ such that for all $i > N$, there exists $j < N$ such that $b_{ij} \geq C$.

We want to approximate
\[ \sum_{i=1}^{n} I(i, n) \tilde{s}_i \]
by a Gaussian random variable. Define random variables
\[ X_{n,i} = \frac{I(i, n)(\tilde{s}_i - \mathbb{E}[\tilde{s}_i])}{v_n}, \]
where $v_n$ is defined so that $\sum_{i=1}^{n} \mathbb{E}[X_{n,i}^2] = 1$.

We want to apply Lindeberg’s Central Limit Theorem to this triangular array. We must check that for each $\delta > 0$,
\[ \sum_{i=1}^{n} \mathbb{E}[X_{n,i}^2 1_{X_{n,i} \geq \delta}] \to 0 \]
as $n \to \infty$. Because $\sum_{i=1}^{n} \mathbb{E}[X_{n,i}^2] = 1$,
\[ \sum_{i=1}^{n} \mathbb{E}[X_{n,i}^2 1_{X_{n,i} \geq \delta}] \leq \max_i \frac{\mathbb{E}[X_{n,i}^2 1_{X_{n,i} < \delta}]}{\mathbb{E}[X_{n,i}^2]}. \]

By the monotone convergence theorem applied to $X_{n,i}^2 1_{X_{n,i} < \delta}$, the argument $\frac{\mathbb{E}[X_{n,i}^2 1_{X_{n,i} < \delta}]}{\mathbb{E}[X_{n,i}^2]}$ of the right-hand side converges to 0 as $I(i, n) \to 0$.

For each $\epsilon > 0$, we can choose $M_\epsilon$ such that $I(i, n) < \epsilon$ whenever $i < N$ and $n > M_\epsilon$. Then $I(i, n) < \epsilon/C$ for all $i$ and all $n > M_\epsilon$. Letting $\epsilon \to 0$, we see the condition is satisfied.

By Lindeberg’s Central Limit Theorem, $\sum_{i=1}^{n} I(i, n) \tilde{s}_i$ converges to a normal random variable with mean $\|b_n\|_1 \mathbb{E}[\tilde{s}_i]$ and variance $\|b_n\|_2 \text{Var}(\tilde{s}_i)$.

We have
\[ \frac{\|\tilde{b}_n\|_2}{\|\tilde{b}_n\|_1} \leq \max_i \frac{\sqrt{\|b_n\|_1 \cdot \tilde{b}_{n,i}}}{\|\tilde{b}_n\|_1} = \max_i \sqrt{I(i, n)} < \sqrt{\epsilon/C}. \]

Because $\epsilon$ is arbitrary, $\frac{\|\tilde{b}_n\|_1}{\|\tilde{b}_n\|_2}$ converges to infinity. Therefore, the sum $\sum_{i=1}^{n} I(i, n) \tilde{s}_i$ converges to infinity in probability.
A.6 Proof of Proposition 1

Proof. The numbers of paths from various agents to agent $i$ satisfy the recurrence relation

$$b_{n,i} = (1 + p)b_{n-1,i}$$

when $n - i > 1$. By a simple computation, we find that

$$\tilde{a}_n = \sum_{i=1}^{n} p(1 + p)^{n-i-1}\tilde{s}_i.$$ 

The sum above has the same sign as

$$\frac{1}{2}\sum_{i=1}^{n}(1 + p)^{-i}\tilde{s}_i,$$

and this second sum converges to a Gaussian random variable with distribution $\mathcal{N}\left(\frac{1}{p}, \frac{1}{p(p+2)}\right)$ when $\omega = 1$. The probability of this infinite sum being positive is

$$1 - \Phi\left(-\sqrt{\frac{p + 2}{p}}\right).$$

The Proposition follows because for $n$ sufficiently large agent, $n$’s action is close to 1 if the infinite sum is positive and 0 if the infinite sum is negative. 

A.7 Proof of Proposition 2

Proof. Suppose we have two groups, and agents observe predecessors in the same group with weight $p_s$ and predecessors in the other group with weight $p_d$. Then the coefficients $b_{n,i}$ satisfy the recurrence relation

$$b_{n,i} = p_db_{n-1,i} + (1 + p_s)b_{n-2,i}$$

when $n - i > 2$. By a standard algebraic fact,

$$b_{n,i} = c_+\alpha_+^n + c_-\alpha_-^n,$$

where $\alpha_\pm$ are the solutions to the polynomial $x^2 - p_dx - (1 + p_s) = 0$ and $c_+, c_-$ are constants that we can determine from $b_{i+1,i}$ and $b_{i+2,i}$. We compute

$$\alpha_\pm = \frac{p_d \pm \sqrt{4p_s + p_d^2 + 4}}{2},$$
so that $\alpha_+ > |\alpha_-|$. 

The exponential term with base $\alpha_+$ dominates as $n$ grows large, so that beliefs eventually have the same sign as 

$$\sum_{i=0}^{\infty} (\alpha_+)^{-i} \delta_i.$$ 

In particular, all agents agree asymptotically and the probability of mislearning is monotonically increasing in $\alpha^+$. We can take comparative statics:

$$\frac{\partial \alpha_+}{\partial p_d} = \frac{p_d}{2\sqrt{4p_s + p_d^2 + 4}} + \frac{1}{2} \text{ and } \frac{\partial \alpha_+}{\partial p_s} = \frac{1}{\sqrt{4p_s + p_d^2 + 4}}.$$ 

It is easy to see that $\frac{\partial \alpha_+}{\partial p_d} > \frac{\partial \alpha_+}{\partial p_s} > 0$ for all $p_s \geq 0$ and $p_d > 0$. \(\square\)

### A.8 Proof of Proposition 3

**Proof.** We have the recurrence relation $b_{n,i} = \frac{d}{n-1} \cdot b_{n-1,i} + \frac{n^2 - 2}{n-1} \cdot b_{n-1,i}$ when $n-i > 1$. Solving this recurrence relation,

$$b_{n,i} = \frac{d}{n-1} \prod_{j=2}^{n-i-1} \frac{n-j+d}{n-j+1}.$$ 

When $n-i > d$, these coefficients are equal to

$$b_{n,i} = \frac{d}{n-1} \cdot \frac{(n+d-2) \cdot (n+d-3) \cdots (n-1)}{(i+d-1) \cdot (i+d-2) \cdots i}.$$ 

This product approaches $\frac{d^{d-1}}{i^d}$ asymptotically. So for large $n$,

$$\frac{\|\vec{b}_n\|_1}{\|\vec{b}_n\|_2} \approx \frac{\sum_{i=1}^{n} (n^{d-1}/i^d)}{\left(\sum_{i=1}^{n} (n^{2d-2}/i^{2d})\right)^{1/2}} = \frac{\sum_{i=1}^{n} i^{-d}}{\left(\sum_{i=1}^{n} i^{-2d}\right)^{1/2}}.$$ 

If $d = 1$, the numerator diverges and the denominator converges. This argument shows $\lim_{n \to \infty} \frac{\|\vec{b}_n\|_1}{\|\vec{b}_n\|_2} = \infty$. By Lemma 4, this means society learns correctly.

If $d > 1$, then both sums converge and $\lim_{n \to \infty} \frac{\|\vec{b}_n\|_1}{\|\vec{b}_n\|_2} = \frac{\sum_{i=1}^{\infty} i^{-d}}{\left(\sum_{i=1}^{\infty} i^{-2d}\right)^{1/2}} < \infty$. So there is a positive probability of incorrect beliefs asymptotically. From the formula for $b_{n,i}$, when $n-i > d$ we may factor $b_{n,i}$ into a term depending only on $i$ and another term depending only on $n$, $b_{n,i} = \psi(i) \cdot \phi(n)$ with $\psi(i) := (i+d-1)/(i+d-2) \cdots i$ and $\phi(n) := d(n+d-2) \cdots (n-1)$. Because $\sum_{i=1}^{\infty} \psi(i)$ converges, agents agree asymptotically. Because $\phi(n) \to \infty$ as $n \to \infty$, their beliefs tend to 0 or 1 a.s.

We now show the probability of mislearning is increasing in $d$ (for $d$ an integer). Since $\psi(i+1)/\psi(i) = \frac{i}{i+d}$, provided $n-i > d$ we have $b_{n,i}/b_{1,n} = \prod_{j=2}^{d-j} \frac{j}{d-j}$. As $n$ grows large, $\vec{b}_n$
converges to a vector proportional to
\[ \hat{v} := (1, \frac{2}{d+2}, \frac{2}{d+2}, \frac{3}{d+3}, \ldots, \frac{2}{d+2}, \frac{3}{d+3}, \ldots, \frac{i}{d+i}, \ldots). \]

We claim that \( \| \hat{v} \|_1 / \| \hat{v} \|_2 \) is increasing in \( d \). This is a consequence of Lemma 5.

**Lemma 5.** Let \( \vec{a} = (a_1, a_2, \ldots) \) be a vector with non-negative, decreasing entries. For any constant \( 0 \leq c \leq 1 \), let the vector \( \vec{a}' = (a_1, a_2, \ldots, a_{i-1}, ca_i, ca_{i+1}, \ldots) \) have entries equal to \( a_j \) for \( j < i \) and \( ca_j \) for \( j \geq i \). Then
\[ \| \vec{a} \|_1 / \| \vec{a} \|_2 \leq \| \vec{a}' \|_1 / \| \vec{a}' \|_2. \]

**Proof.** It is sufficient to check that \( \| \vec{a} \|_1 / \| \vec{a} \|_2 \) is increasing, or equivalently that
\[ \frac{\partial \| \vec{a}' \|_2}{\partial c} \| \vec{a}' \|_2 \]

is non-negative.

We compute
\[ \frac{\partial \| \vec{a}' \|_2}{\partial c} \| \vec{a}' \|_2 = \frac{\partial}{\partial c} \left( \sum_{j=1}^{i-1} a_j + \sum_{j=i}^{\infty} ca_j \right)^2. \]

Differentiating gives a fraction with positive denominator and numerator
\[ 2 \sum_{j=i}^{\infty} a_j (\sum_{j=1}^{i-1} a_j + c^2 \sum_{j=i}^{\infty} a_j^2) (\sum_{j=1}^{i-1} a_j + c \sum_{j=i}^{\infty} a_j) - 2c \sum_{j=1}^{i-1} a_j^2 (\sum_{j=1}^{i-1} a_j + c \sum_{j=i}^{\infty} a_j)^2. \]

This expression has the same sign as
\[ \sum_{j=1}^{\infty} a_j \sum_{j=1}^{i-1} a_j^2 + c^2 \sum_{j=i}^{\infty} a_j^2 - c \sum_{j=1}^{\infty} a_j^2 \sum_{j=1}^{i-1} a_j - c \sum_{j=1}^{\infty} a_j \sum_{j=i}^{\infty} a_j^2 = \sum_{j=1}^{i-1} a_j^2 \sum_{j=1}^{\infty} a_j - c \sum_{j=1}^{\infty} a_j \sum_{j=i}^{\infty} a_j^2 \]
\[ = \sum_{j=1}^{i-1} \sum_{k=i}^{\infty} a_j a_k (a_j - ca_k) \geq 0 \]

because \( a_j \geq a_k \) for all \( j < k \) and \( c \leq 1 \).

**A.9 Proof of Proposition 4**

**Proof.** The coefficients \( b_{i,n} \) satisfy the recurrence relation \( b_{n,i} = 2\delta b_{n-1,i} \) whenever \( n - i > 1 \).
Recall that the recurrence relation in the uniform $p$ weight network is $b_{n,i} = (1+p)b_{n-1,i}$.

When $\delta > \frac{1}{2}$, the coefficients $b_{n,i}$ for the $\delta$ decay network is the same as the coefficients in the uniform $p$ weight network with $p = 2\delta - 1 > 0$. By Proposition 1, there is positive probability of converging to full certainty in the wrong state of the world. Furthermore, as $\delta \mapsto 2\delta - 1$ is increasing, again by Proposition 1 we see the probability of converging to the wrong belief is increasing in $\delta$ for $\delta > \frac{1}{2}$.

When $\delta = \frac{1}{2}$, all predecessors’ signals are given equal weight, so by the law of large numbers actions converge to 1 a.s. conditional on $\omega = 1$.

The final case is $\delta < \frac{1}{2}$. It is sufficient to show that the sequence $\tilde{a}_n$ does not converge. From the recurrence relation for the coefficients $b_{n,i}$, it is easy to check that $\tilde{a}_n = 2\delta \tilde{a}_{n-1} + \tilde{s}_n - \delta \tilde{s}_{n-1}$ for each $n$.

Evidently $b_{i+1,i} = \delta$, so from recursion $b_{n,i} = (2\delta)^n - 1 - i \delta$ for $i \leq n - 1$, $b_{n,n} = 1$. So, $\tilde{a}_n = \tilde{s}_n + \delta \sum_{j=0}^{n-2}(2\delta)^j \cdot \tilde{s}_{n-1-j}$, meaning $\tilde{a}_n \sim \mathcal{N}(1 + \frac{\delta}{1-2\delta}, 1 + \frac{\delta^2}{(1-2\delta)^2})$, with a limiting distribution of $\mathcal{N}(1 + \frac{\delta}{1-2\delta}, 1 + \frac{\delta^2}{4})$.

Let the probability space be $\mathbb{R}^\infty$, the space of all sequences of private signal realizations, with typical member $s = (s_i)_{i \in \mathbb{N}}$.

Assume by way of contradiction that $\tilde{a}_n$ diverges to $\infty$ with positive probability $2\epsilon > 0$.

We can find a large enough $K$ so that $\mathcal{N}(1 + \frac{\delta}{1-2\delta}, 1 + \frac{\delta^2}{4}) > K$ has probability less than $\epsilon$. Observing the mean and variance of $\tilde{a}_n$ are both increasing in $n$, this says $\mathbb{P}[\tilde{a}_n > K] \leq \mathbb{P}[\mathcal{N}(1 + \frac{\delta}{1-2\delta}, 1 + \frac{\delta^2}{4}) > K] < \epsilon$ for every $n$. Writing $E_n \subseteq \mathbb{R}^\infty$ for those signal realizations $s$ where $\tilde{a}_n(s) > K$, we get $\mathbb{P}[E_n] < \epsilon$ for every $n$.

But since we assumed $\tilde{a}_n$ diverges to $\infty$ with probability $2\epsilon$, this means there is an event $F_* \subseteq \mathbb{R}^\infty$, where $\mathbb{P}[F_*] = 2\epsilon$ and $\lim_{n \to \infty} \tilde{a}_n(s) = \infty$ for every $s \in F_*$. There is a function $N(s)$ so that $\tilde{a}_n(s) > K$ for all $s \in F_*$, $n \geq N(s)$. Let $F_n := \{s \in F_* : N(s) \leq n\}$. We have $F_n \uparrow F_*$ so there is some $N$ where $\mathbb{P}[F_N] \geq \frac{1}{2} \mathbb{P}[F_*] = \epsilon$. We claim that $F_N \subseteq E_N$. This is because at every $s \in F_N$, $n \geq N(s)$ so $\tilde{a}_N(s) > K$. And yet this would say $\mathbb{P}[E_N] \geq \epsilon$, which contradicts $\mathbb{P}[E_n] < \epsilon$ for every $n$. Therefore, the event where $\tilde{a}_n$ diverges to $\infty$ must have 0 probability.

A similar argument establishes $\tilde{a}_n$ diverges to $-\infty$ with 0 probability.

The only remaining case is $\tilde{a}_n$ converging to some real number. By way of contradiction, assume this event has probability $2\epsilon > 0$.

By algebra, we expand the difference in consecutive log actions $\tilde{a}_{n+1} - \tilde{a}_n$ as the sum of some independent Gaussians,

$$\tilde{a}_{n+1} - \tilde{a}_n = \tilde{s}_{n+1} - (1 - \delta)\tilde{s}_n - (1 - 2\delta)\delta \cdot \sum_{j=0}^{n-2} [(2\delta)^j] \cdot \tilde{s}_{n-1-j}$$

The limiting distribution of RHS is $\mathcal{N}(0, \tilde{\sigma}^2)$ for some $\tilde{\sigma} > 0$. We can pick $\delta > 0$ small.
enough so $\mathcal{N}(0, \delta^2) \in [-h/2, h/2]$ has probability less than $\epsilon$. So there exists $N_1$ so that for each $n \geq N_1$, $\mathbb{P}[|\tilde{a}_{n+1} - \tilde{a}_n| < h] < \epsilon$. Write $G_n$ for the event where $|\tilde{a}_{n+1} - \tilde{a}_n| < h$.

To derive a contradiction, write $J_* := \{ s \in \mathbb{R}^\infty : \lim_{n \to \infty} \tilde{a}_n(s) = c \text{ for some } c \in \mathbb{R} \}$. We have functions $N_2 : J_* \to \mathbb{N}$ and $c : J_* \to \mathbb{R}$ so that whenever $s \in J_*$ and $n \geq N_2(s)$, $|\tilde{a}_n(s) - c(s)| < \frac{1}{2}h$. Let $J_n := \{ s \in J_* : N_2(s) \leq n \}$. We have $J_n \uparrow J_*$ so there is some $N_2$ where $\mathbb{P}[J_{N_2}] \geq \frac{1}{2}\mathbb{P}[J_*] = \epsilon$. It is without loss to pick $N_2 \geq N_1$. We claim that $J_{N_2} \subseteq G_{N_2}$. This is because for each $s \in J_{N_2}$, $N_2(s) \leq N_2$ so $|\tilde{a}_{N_2(s)}(s) - c(s)| < \frac{1}{2}h$ and also $|\tilde{a}_{N_2(s)}(s) - c(s)| < \frac{1}{2}h$. By triangle inequality, $|\tilde{a}_{N_2(s)}(s) - \tilde{a}_{N_2(s)+1}(s)| < h$. But this would show $\mathbb{P}[G_{N_2}] \geq \epsilon$, which is a contradiction since $N_2 \geq N_1$.

\[ \square \]

### A.10 Proof of Proposition 5

**Proof.** On this network, the coefficients $b_{n,1}$ satisfy the recurrence relation

$$b_{n,1} = \delta p_d b_{n-1,1} + \delta^2(1 + p_s)b_{n-2,1}$$

whenever $n - i > 2$. As before,

$$b_{n,1} = c_+ \alpha_+^n + c_- \alpha_-^n,$$

where $\alpha_\pm$ are now the solutions to the polynomial $x^2 - \delta p_d x - \delta^2(1 + p_s) = 0$ and $c_+, c_-$ are constants that we can determine from $b_1,1$ and $b_{2,1}$. We compute

$$\alpha_\pm = \frac{\delta \cdot p_d \pm \sqrt{4p_s + p_d^2 + 4}}{2^d},$$

so that $\alpha_+ > |\alpha_-|$.

The convergence of beliefs now depends on how $\delta$ compares to $\delta_0(p_s, p_d) = \frac{2}{p_d + \sqrt{4p_s + p_d^2 + 4}}$. The results about convergence in the first statement follow as in the proof of Proposition 4. More precisely, the proof of Proposition 4 depends on the network structure only through two conditions: (A) for every $\epsilon > 0$, there exist $K > 0$ so that $\mathbb{P}[^\tilde{a}_n > K] < \epsilon$ for every $n$; (B) for every $\epsilon > 0$, there exists an $h > 0$ so that $\mathbb{P}[|\tilde{a}_{n+1} - \tilde{a}_n| < h] < \epsilon$ for every $n$. We now verify these two conditions in the context of Proposition 5.

To verify (A), we have $\tilde{a}_n = \sum_{i=1}^n b_{n,i} \cdot \tilde{s}_i = \sum_{j=1}^n b_{j,1} \cdot \tilde{s}_{n-j+1}$, where we used $b_{n,i} = b_{n-i,1,1}$. Let $\mu_n := \sum_{j=1}^n b_{j,1}$ and $\sigma_n^2 := \sum_{j=1}^n b_{j,1}^2$ and we have $\tilde{a}_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$. But whenever $\delta < \delta_0(p_s, p_d)$, we have $|\alpha_-| < |\alpha_+| < 1$, so:

$$\mu_n = \sum_{j=1}^n b_{j,1} \leq \sum_{j=1}^n |b_{j,1}| \leq \sum_{j=1}^\infty (|c_+| + |c_-|) \cdot \alpha_+^n =: \mu_\infty < \infty$$
\[ \sigma_n^2 = \sum_{j=1}^n b_{j,1}^2 \leq \sum_{j=1}^n b_{j,1}^2 \leq \sum_{j=1}^\infty (|c_+| + |c_-|)^2 \cdot \alpha_{2n}^2 =: \sigma_\infty^2 < \infty \]

This implies for every \( K > \mu_\infty \), \( \mathbb{P}[\mathcal{N}(\mu_\infty, \sigma_\infty^2) > K] > \mathbb{P}[\mathcal{N}(\mu_n, \sigma_n^2) > K] = \mathbb{P}[\tilde{a}_n > K] \). Clearly we may choose \( K > \mu_\infty > 0 \) large enough so that \( \mathbb{P}[\mathcal{N}(\mu_\infty, \sigma_\infty^2) > K] < \epsilon \).

To verify (B),

\[ \tilde{a}_{n+1} - \tilde{a}_n = \sum_{i=1}^n d_{n+1,i} \tilde{s}_i + \tilde{s}_{n+1} \]

where \( d_{n+1,i} \) is the difference in weight, \( d_{n+1,i} = b_{n+1,i} - b_{n,i} \).

Here \( \left\{ \sum_{i=1}^n d_{n+1,i} \tilde{s}_i \right\} \) is some Gaussian random variable and \( \tilde{s}_{n+1} \) is independent of it. Even if the first piece \( \sum_{i=1}^n d_{n+1,i} \tilde{s}_i \) were always 0, clearly we may pick \( h \) small enough so \( \mathbb{P}[|\tilde{s}_{n+1}| < h] < \epsilon \).

The proofs of the results in the case \( \delta > \delta_0(p_s, p_d) \) are essentially the same as the proof of Proposition 2.

\[ \square \]

A.11 Proof of Proposition 6

\textbf{Proof.} As a matter of preliminary notation, let \( -\kappa \) be a naive agent’s log likelihood ratio of state \( \omega = 1 \) versus state \( \omega = 0 \) upon observing an even-numbered predecessor play the action 0. Then we have:

\[ -\kappa := \ln \left( \frac{\mathbb{P}[\omega = 1|s < 0]}{\mathbb{P}[\omega = 0|s < 0]} \right) < 0. \]

Write \( O \) for the event where the odd-numbered agents converge to playing 1. Write \( E \) for the event where the even-numbered agents converge to playing 1. We will prove that \( \mathbb{P}[E|O] = 1. \)

(The other case of odd-numbered agents converging to playing 0 is symmetric and omitted.)

Below, we will partition \( O \) into countably many subevents, \( O_{M,N} \) for \( M, N \in \{0, 1, \ldots, \} \). We will construct for each \( (M, N) \) a \( \mathbb{P} \)-null event \( H_{M,N} \) such that \( O_{M,N} \cap H_{M,N}^c \subseteq E \). This implies \( \mathbb{P}[O_{M,N} \cap E] = \mathbb{P}[O_{M,N}] \) for every \( (M, N) \), so by countable additivity \( \mathbb{P}[O \cap E] = \mathbb{P}[O] \).

To begin this partitioning, pick \( \epsilon > 0 \) small enough such that

\[ -2\kappa p_s + \frac{1}{12} p_d \tilde{a}^{1-\epsilon} > 0, \tag{1} \]

where \( \tilde{a}^{1-\epsilon} \) is the log odds associated with the action \( 1 - \epsilon \).

Now, let \( O_{M,N} \subseteq O \) be the event where:

1. \( (a_{2n+1}) \) converge to 1;
2. \( \left| \sum_{(n,\tilde{a}_{2n+1}<0)} \tilde{a}_{2n+1} \right| = -M; \)

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3. \( N = \min\{n : \inf_{j \geq n} a_{2j+1} \geq 1 - \epsilon\} \).

That is, the (floor of) sum of negative log-actions in \( \tilde{a}_{2n+1} \) equals to \(-M\), while all actions are above \( 1 - \epsilon \) starting from player \( 2N + 1 \). To see that \((O_{M,N})\) partition \( O \), observe that every convergent sequence \( a_{2n+1} \rightarrow 1 \) satisfies (2) and (3) for some \( M \) and \( N \), because only finitely many terms can have \( a_{2n+1} < 0 \) and also we must have \( a_{2n+1} \geq 1 - \epsilon \) for all large enough \( n \).

We now proceed to construct the \( \mathbb{P}\)-null event \( H_{M,N} \). We will do this through a sequence of triplets of events \( (F_1^n, F_2^n, F_3^n) \) with \( \sum_{n=1}^{\infty} \mathbb{P}[F_1^n \cup F_2^n \cup F_3^n] < \infty \) and put

\[
H_{M,N} := \{(F_1^n \cup F_2^n \cup F_3^n) \text{ infinitely often},
\]

so that \( \mathbb{P}[H_{M,N}] = 0 \) by the Borel–Cantelli lemma.

We start by defining \( F_1^n \). Let \( B_{n,p_s} \) be the number of even-numbered predecessors that agent \( 2n \) observes, so \( B_{n,p_s} \sim \text{Binom}(n-1, p_s) \). Define \( F_1^n \) as the event \( B_{n,p_s} > 2np_s \). Straightforward application of Hoeffding’s inequality as in the proof of Theorem 3 shows that \( \mathbb{P}[F_1^n] \leq \exp(-2np_s^2) \). It is clear that \( \sum_n \mathbb{P}[F_1^n] < \infty \).

Then, we define \( F_2^n \). Let \( B_{n,d}^{(N)} \) be the number of odd-numbered predecessors in positions \( 2N + 1 \) or later that agent \( 2n \) observes, so \( B_{n,d}^{(N)} \sim \text{Binom}(n(N) \wedge 0, p_d) \). Define \( F_2^n \) as the event \( B_{n,d}^{(N)} < \frac{1}{2} np_d \). For \( n > 2N \), we have \( \mathbb{E}[B_{n,d}^{(N)}] = (n-N)p_d \geq \frac{1}{2} np_d \), so applying Hoeffding’s inequality gives

\[
\mathbb{P}[F_2^n] \leq \exp(-2 \cdot (\frac{1}{4} np_d)^2/(n-N)) \leq \exp(-2 \cdot (\frac{1}{4} np_d)^2/n) = \exp(-\frac{1}{8} np_d^2).
\]

So when \( n > 2N \), \( \mathbb{P}[F_2^n] \) decreases to 0 exponentially, meaning \( \sum_n \mathbb{P}[F_2^n] < \infty \).

Next, we define \( F_3^n \). Intuitively, this is the event that agent \( 2n \) gets a “strong enough” private signal to overturn her observations. Specifically, \( F_3^n \) is the event

\[
\tilde{s}_{2n} + \frac{1}{12} np_d \cdot \tilde{a}^{1-\epsilon} \leq 0,
\]

where \( \tilde{a}^{1-\epsilon} \) was defined in Equation (1). Since \( \ln \left( \frac{\mathbb{P}[\omega = 1|\sigma]}{\mathbb{P}[\omega = 0|\sigma]} \right) = 2s/\sigma^2 \) under the Gaussian assumption, \( \mathbb{P}[F_3^n] = \mathbb{P}[2N(1, \sigma^2)/\sigma^2 < -\frac{1}{12} np_d \cdot \tilde{a}^{1-\epsilon}] \). But the Gaussian distribution function tends to 0 faster than geometrically, so \( \sum_n \mathbb{P}[F_3^n] < \infty \) as well.

Finally, we need to argue that the even-numbered agents always converge to playing 1 on the event \( O_{M,N} \cap H_{M,N}^c \). On the event \( H_{M,N}^c \), there exists \( G_1 \in \mathbb{N} \) so that \( n \geq G_1 \) implies player \( 2n \) gets no more than \( 2np_s \) observations from her own group, no fewer than \( \frac{1}{4} np_d \) observations from the odd-numbered predecessors past position \( 2N + 1 \), and gets a private signal \( s_{2n} \) with \( \tilde{s}_{2n} + \frac{1}{12} np_d \cdot \tilde{a}^{1-\epsilon} > 0 \). Find also \( G_2 \in \mathbb{N} \) so that \( \frac{1}{12} G_2 p_d \tilde{a}^{1-\epsilon} > M \). We show that \( a_{2n} = 1 \) for every \( n \geq \max(G_1, G_2) \).
Agent $2n$ observes three kinds of predecessors: even-numbered predecessors, odd-numbered predecessors earlier than position $2N + 1$, and odd-numbered predecessors later than position $2N + 1$. The no more than $2np_s$ observations of the first kind contribute in the worst case a log odds of $-\kappa \cdot (2np_s)$. Observations of the second kind, on the event $O_{M,N}$, contribute a worst-case log odds of $-M$. Observations of the third kind, which number at least $\frac{1}{4} np_d$ on the event $H^c_{M,N}$, contribute at least a log odds of $\frac{1}{3} np_d \cdot \tilde{a}^{1-\epsilon}$ on the event $O_{M,N}$. So $2n$’s log odds is at least:

$$n \cdot (-2\kappa p_s + \frac{1}{12} p_d \tilde{a}^{1-\epsilon}) + (-M + \frac{1}{12} np_d \tilde{a}^{1-\epsilon}) + (\hat{s}_2n + \frac{1}{12} np_d \tilde{a}^{1-\epsilon})$$

The first summand is positive by choice of $\epsilon$ in Equation (1). The second summand is positive since $n \geq G_2$. And the third summand is positive since $n \geq G_1$.

B Simulations Using Random Networks

In this paper, we have focused on a learning model with a deterministic network, where non-integral weights on network edges $M_{i,j} \in [0,1]$ can be assigned either a noise interpretation or a generation interpretation, as discussed in Subsection 2.3. However, it is also natural to consider a random network in which edge $(i,j)$ materializes with probability $M_{i,j}$, where realizations of the network edges and the private signals are mutually independent. To verify the robustness of our conclusions in Section 4, we performed simulations where $M_{i,j}$ are assigned random network interpretations.

We first conducted simulations using Gaussian private signals on Erdos-Renyi random networks. We choose a signal variance of $\sigma^2 = 2$. For each linking probability $p \in \{0.1, 0.2, ..., 1.0\}$, we simulated 100,000 joint realizations of the Erdos-Renyi random network and private signals for a society of 150 agents. We report the probabilities of mislearning in Figure 1, where we find increasing link probability increases the likelihood of mislearning, in agreement with Proposition 1.
| link probability | chance of mislearning |
|------------------|----------------------|
| 0.1              | 0.03043              |
| 0.2              | 0.06982              |
| 0.3              | 0.09923              |
| 0.4              | 0.12346              |
| 0.5              | 0.13796              |
| 0.6              | 0.15270              |
| 0.7              | 0.16580              |
| 0.8              | 0.17776              |
| 0.9              | 0.18595              |
| 1.0              | 0.19349              |

Figure 1: Simulation results for the probability of mislearning on Erdos-Renyi networks with Gaussian signals of variance $\sigma^2 = 2$. We simulated 150 agents in each society and performed 100,000 simulations at each link probability.

To further test the robustness of our results to a different signal structure, we consider a case where the space of signals is $S = [0, 1]$ and the density functions over signals are $f_0(s) = 2 - 2s$ and $f_1(s) = 2s$ in states $\omega = 0$ and $\omega = 1$ respectively. This is the triangular distribution that leads to the inference $P[\omega = 1|s] = s$ and was previously used in Eyster and Rabin (2010) to illustrate the probability of mislearning for naive agents on a complete network. In Figure 2, we find that the conclusion of Proposition 1 continues to hold on Erdos-Renyi random networks with triangular signals: higher link probability leads to higher chance of mislearning. Notice the probability of mislearning at a linking probability of 1 is 11.26%, which is similar to the simulation result of 11.15% of mislearning at the 10th naive player in Table 1 of Eyster and Rabin (2010).
In Figures 3 and 4, we report our simulation results for random networks under the two-groups model, with Gaussian and triangular signals respectively. The conclusions of Proposition 2 appear generally robust to this shift to random networks and new signal structure. For both signal structures, increasing within-group link probability and between-group link probability increases mislearning. Also, for each signal structure and at each \((p_s, p_d) \in \{0.1, ..., 0.9\} \times \{0.1, ..., 0.9\}\), we checked whether mislearning is more prevalent at \((p_s + 0.1, p_d)\) or \((p_s, p_d + 0.1)\). Under each signal structure, we found that out of the 81 cells increasing \(p_d\) led to more mislearning than increasing \(p_s\) in 79 cases, in agreement with Proposition 2. The results for the two remaining cells may be attributed to noise in the simulation outcomes.
Figure 3: Simulation results for the probability of mislearning on two-groups random networks with Gaussian signals of variance $\sigma^2 = 2$. We simulated 150 agents in each society and performed 100,000 simulations at each link probability.
Figure 4: Simulation results for the probability of mislearning on two-groups random networks with triangular signals. We simulated 150 agents in each society and performed 100,000 simulations at each link probability.