Connection formulae for new $q$-deformed Laguerre-Gould-Hopper polynomials

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Abstract

We introduce new families of $q$-deformed 2D Laguerre-Gould-Hopper polynomials. For these polynomials we establish connection formulae which extend some known ones.

1 Introduction

The $q$-analysis can be traced back to the earlier works of L. J. Rogers [1]. It has wideranging applications in the analytic number theory and $q$-deformation of well-known functions [2] as well as in the study of solvable models in statistical mechanics [3]. During the 80’s the interest on this analysis increased with quantum groups theory with which models of $q$-deformed oscillators have been developed [4]. The $q$-analogs of boson operators have been defined in [5] where the corresponding wavefunctions were constructed in terms of the continuous $q$-Hermite polynomials of Rogers and other polynomials. Actually, known models of $q$-oscillators are closely related with $q$-orthogonal polynomials.

The connection formulae for orthogonal polynomials are useful in mathematical analysis and also have applications in quantum mechanics, such as finding the relationships between the wavefunctions of some potential functions used for describing physical and chemical properties in atomic and molecular systems [6].

Here we deal with connection formulae for a new class of $q$-deformed Laguerre-Gould-Hopper polynomials which we introduce in an operatorial way by acting on certain newly defined $q$-deformed Gould-Hopper generalized Hermite polynomials [7]. We also consider particular cases of these formulae, which enables us to recover known ones in the literature.

The paper is organized as follows. In section 2, we prepare some needed notations and definitions. We also introduce a family of $q$-deformed Laguerre polynomials and express them by a generating function. In section 3, we define a new class of $q$-deformed Laguerre-Gould-Hopper polynomials for which we give a generating function and we establish some connection formulae. Section 4 is devoted to discuss some particular cases of the obtained results.
2 Notations and definitions

Here, we list notations and special functions we will be using. For the relevant properties and definitions we refer to [8-10]. We also introduce a family of $q$-deformed 2D Laguerre polynomials and we write down their generating function.

1. The $q$-analogues ($q \in \mathbb{C} \setminus \{1\}$) of a natural number $n$, the factorial and semifactorial functions are defined by

\[
[n]_q \equiv \sum_{k=1}^{n} q^{k-1}, \quad [n]_q! = \prod_{k=1}^{n} [k]_q!, \quad [0]_q! \equiv 1,
\]

(2.1)

\[
[mk]_q!! := \prod_{l=1}^{k} [ml]_q, \quad [0]!! := 1.
\]

(2.2)

2. The Euler-Heine-Jackson $q$-difference operator is defined by

\[
D_x^q [f](x) := \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0.
\]

(2.3)

Its inverse $(D_x^q)^{-1}$ is defined in such a way that

\[
(D_x^q)^{-n}(1) = \frac{x^n}{[n]_q!}.
\]

(2.4)

3. The Gauss $q$-binomial coefficient is given by

\[
\begin{aligned}
\left[ \begin{array}{c} n \\ k \end{array} \right]_q & := \frac{[n]_q!}{[n-k]_q! [k]_q!} = \frac{(q;q)_n}{(q;q)_{n-k} (q;q)_k}, \quad k = 0, 1, \ldots, n.
\end{aligned}
\]

(2.5)

where

\[
(a;q)_n := \prod_{k=0}^{n-1} (1-aq^k), \quad (a;q)_0 = 1
\]

(2.6)

denotes the $q$-shifted factorial.

4. The Jackson-Hahn-Cigler (JHC) $q$-addition is the function

\[
(x \oplus_q y)^n := \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q q^{k} x^{n-k} y^k
\]

\[
= x^n \left( -\frac{y}{x} ; q \right)_n ^{n} \equiv P_{n,q}(x,y), \quad n = 0, 1, 2, \ldots
\]

(2.7)

The JHC $q$-substraction is defined by

\[
(x \ominus_q y)^n := P_{n,q}(x,-y), \quad n = 0, 1, \ldots
\]

5. Let $D_\rho = \{ \xi \in \mathbb{C}, |\xi| < \rho \}$ the complex disk of radius $\rho > 0$ and $F(z) := \sum_{n=0}^{\infty} c_n z^n$, $z \in D_\rho$. Define the formal series

\[
F(x \oplus_q y) = \sum_{n=0}^{\infty} c_n(x \oplus_q y)^n.
\]

(2.8)
6. For \( m = 0, 1, 2, \ldots \), let \( e_q \) and \( E_{qm} \) denote the exponential functions defined by

\[
e_q(x) := \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} \tag{2.9}
\]

and

\[
E_{qm}(x) := \sum_{n=0}^{\infty} \frac{q^{m(n)}(2)}{[n]_{qm}!} x^n. \tag{2.10}
\]

These functions satisfy

\[
e_q(x) E_{qm}(y) = e_q(x \ominus_{q,q} y) \]

where

\[
(a \ominus_{q,q} b)^n = [n]_q! \sum_{k=0}^{n} \frac{(-1)^k q^{m(k)}}{[n-k]_q![k]_{qm}!} a^{n-k} b^k
\]

with \((a \ominus_{q,q} b)^0 := 1\). For \( m = 1 \), we have the rules

\[
e_q(a) E_q(b) = e_q(a \oplus_q b), \quad e_q(a) E_q(-a) = 1. \tag{2.12}
\]

7. For \( m = 0, 1, 2, \ldots \), the function

\[
\varepsilon_{n,q}^m(x) := \sum_{n=0}^{\infty} \frac{(-1)^k q^{m(k)}}{[k]_{qm}![n+k]_{qm}!} x^k, \quad n = 0, 1, 2, \ldots \tag{2.13}
\]

denotes the \( n \)th order of a \((q, m)\)-deformed Bessel-Tricomi function with \( \varepsilon_{0,q}^m(x) := \varepsilon_q^m(x) \).

8. The Gould-Hopper generalized polynomials are defined as ([11], p.58):

\[
g_n^m(x, y) := n! \sum_{k=0}^{[\frac{n}{m}]} \frac{x^{n-mk} y^k}{k!(n-mk)!} \tag{2.14}
\]

where \([a] \) stands for the greatest integer not exceeding \( a \).

9. We define a class of \( q \)-deformed two variables (2D) Laguerre polynomials \((q\text{-2DLP})\) as

\[
m L_n(x, y|q) := E_{qm} \left( (D^m_x)^{-1} (D^m_y)^m \right) (y^n), \quad m, n = 0, 1, 2, \ldots \tag{2.15}
\]

Explicitly,

\[
m L_n(x, y|q) := [n]_q! \sum_{k=0}^{[\frac{n}{m}]} \frac{q^{m(k)} x^k y^{n-mk}}{([k]_{qm}!)^2 [n-mk]_q!}, \quad m, n = 0, 1, 2, \ldots \tag{2.16}
\]

10. For a fixed \( m = 0, 1, 2, \ldots \), a generating function for the \( q \text{-2DLP} \) is given by

\[
\sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} m L_n(x, y|q) = e_q(yt) \varepsilon_q^m(-xt^m) \tag{2.17}
\]

in terms of the Jackson \( q \)-exponential function (2.9) and the \((q, m)\)-deformed Bessel-Tricomi function (2.13). See Appendix A for the proof.
3 $q$-deformed 2D Laguerre-Gould-Hopper polynomials

In this section, we introduce a family of $q$-deformed Laguerre-Gould-Hopper polynomials generalizing both the above $q$-2DLP and the $q$-deformed Gould-Hopper generalized Hermite polynomials which were defined in [7] by

$$G_n^m(x, y|q) = [n]_q! \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k q^{m(n)} x^{n-k} y^k}{[n-mk]_q! [mk]_q!}.$$  \hspace{1cm} (3.1)

We shall establish a connection formulae for these polynomials.

**Definition 3.1.** For fixed $m, n, s = 0, 1, 2, \ldots$, a family of $q$-deformed Laguerre-Gould-Hopper polynomials $q$-LGHP is defined by

$$LH_{n}^{(m,s)}(x, y, z|q) := E_q^m ((D_x^q)^{-1}(D_y^q)^m) (G_n^s(x, z|q)).$$  \hspace{1cm} (3.2)

or equivalently

$$LH_{n}^{(m,s)}(x, y, z|q) := E_q^s (z(D_y^q)^s) (m L_n(x, y|q)).$$  \hspace{1cm} (3.3)

Explicitly,

$$LH_{n}^{(m,s)}(x, y, z|q) = [n]_q! \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} q^{s(n)} z^k m L_{n-sk}(x, y|q).$$ \hspace{1cm} (3.4)

**Remark 3.1.** For $z = 0$, the $q$-LGHP reduce to the above $q$-2DLP. That is,

$$LH_{n}^{(m,s)}(x, y, 0|q) = m L_n(x, y|q).$$  \hspace{1cm} (3.5)

**Proposition 3.1.** For each fixed $m, s = 0, 1, 2, \ldots$, the generating function for the $q$-LGHP in (3.2)-(3.5) is given by

$$\sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} LH_{n}^{(m,s)}(x, y, z|q) = c_q(yt) E_q^s (zt^s) E_q^m (-xt^m),$$  \hspace{1cm} (3.6)

in terms of the exponential functions (2.9), (2.10) and the $(q, m)$-deformed Bessel-Tricomi function (2.13).

**Proof.** We start by inserting (3.4) in the l.h.s of (3.6). This gives

$$\sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} LH_{n}^{(m,s)}(x, y, z|q) = \sum_{n=0}^{\infty} t^n \left( \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} q^{s(n)} z^k m L_{n-sk}(x, y|q) \right).$$ \hspace{1cm} (3.7)

The r.h.s of (3.7) also reads successively

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} q^{s(n)} (zt^s)^k m L_{n-sk}(x, y|q) \frac{1}{[k]_q! [n-sk]_q!}.$$ \hspace{1cm} (3.8)

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} q^{s(n)} z^k m L_{n-sk}(x, y|q) t^{n-sk} \frac{1}{[n-sk]_q!}.$$ \hspace{1cm} (3.9)
Now, by applying the following series manipulation
\[\sum_{n=0}^{\infty} \sum_{k=0}^{\left\lfloor \frac{n}{m} \right\rfloor} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k, n - mk),\] (3.10)
where \(m\) is any positive integer ([SM], p.101), the double sum in (3.9) reads
\[\sum_{k=0}^{\infty} \frac{q^s(k)}{[k]_q!} \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} mL_n(x, y|q).\] (3.11)
Recalling (2.11) and (2.17), we arrive at the announced result (3.6). □

**Theorem 3.1.** The \(q\)-LGHP in (3.2) satisfy the connection formula
\[LH_{k+l}^{(m,s)}(x, \xi, \zeta|q) = \sum_{n=r=0}^{k,l} \left[ \begin{array}{c} k \\ n \end{array} \right]_q \left[ \begin{array}{c} l \\ r \end{array} \right]_q q^{r(v-l)} (\xi \ominus_q y)^{n+r} mH_{k+l-n-r}^{(m,s)}(x, y, z|q).\] (3.12)

**Proof.** By replacing \(t\) by \(u \oplus_q t\) in (3.6) we get
\[e_q(y (u \oplus_q t)) E_{q^r} (z (u \oplus_q t)^r) \mathcal{E}_q^m(-x (u \oplus_q t)^m) = \sum_{n=0}^{\infty} \frac{(u \oplus_q t)^n}{[n]_q!} LH_{n}^{(m,s)}(x, y, z|q).\] (3.13)
We now apply to the r.h.s of (3.13) the identity
\[\sum_{j=0}^{\infty} F(j) \frac{(x \oplus_q y)^j}{[j]_q!} = \sum_{j,s=0}^{\infty} F(j + s) q^{(\frac{j}{2})} (\frac{x^j y^s}{[j]_q! [s]_q!}).\] (3.14)
satisfied by the JHC \(q\)-addition. So that (3.13) becomes
\[\mathcal{E}_q^m(-x (u \oplus_q t)^m) = \frac{1}{e_q(y (u \oplus_q t)) E_{q^r} (z (u \oplus_q t)^r)} \sum_{k,l=0}^{\infty} \frac{u^{k+l}}{[k]_q! [l]_q!} q^{\left(\frac{j}{2}\right)} LH_{k+l}^{(m,s)}(x, y, z|q).\] (3.15)
Note that the r.h.s of (3.15) is independent of variables \(y\) and \(z\) so that we can write for any two variables \(\xi, \zeta\) the following equality
\[\sum_{k,l=0}^{\infty} \frac{u^{k+l}}{[k]_q! [l]_q!} q^{\left(\frac{j}{2}\right)} LH_{k+l}^{(m,s)}(x, \xi, \zeta|q) = \lambda_{q,s}^{u,t} (\xi, \zeta; y, z) \sum_{k,l=0}^{\infty} \frac{u^{k+l}}{[k]_q! [l]_q!} q^{\left(\frac{j}{2}\right)} LH_{k+l}^{(m,s)}(x, y, z|q).\] (3.16)
where
\[\lambda_{q,s}^{u,t} (\xi, \zeta; y, z) := \frac{e_q(\xi (u \oplus_q t)) E_{q}^r (\zeta (u \oplus_q t)^r)}{e_q(y (u \oplus_q t)) E_{q^r} (z (u \oplus_q t)^r)}.\] (3.17)
By using the rules in (2.12), on can check that the quantity (3.17) also reads
\[\lambda_{q,s}^{u,t} (\xi, \zeta; y, z) = e_q((\xi \ominus_q y) (u \oplus_q t)) E_{q^r} ((\zeta \ominus_q z) (u \oplus_q t)^r).\] (3.19)
On another hand, the r.h.s of (3.19) coincides with the generating function
\[\sum_{n,r=0}^{\infty} \frac{u^n t^r}{[n]_q! [r]_q!} q^{\left(\frac{j}{2}\right)} G_{n+r}^{(m,s)}(\xi \ominus_q y, \zeta \ominus_q z|q)\] (3.20)
involving the $q$-deformed Gould-Hopper generalized Hermite polynomials. Summarizing the above calculations in (3.16) – (3.20), we arrive at the sum

$$
\sum_{n,r=0}^{\infty} \frac{u^{nl}q^n}{[n]_q![l]_q!} q^{(\xi)} \mathcal{G}_{n+r}^{(s)}(\xi \ominus q y, \zeta \ominus q z|q) \sum_{k,l=0}^{\infty} \frac{u^{kl}t^l}{[k]_q![l]_q!} q^{(\zeta)} L_{H_{k+l}}^{(m,s)}(x, y, z|q)
= \sum_{k,l=0}^{\infty} \frac{u^{kl}t^l}{[k]_q![l]_q!} q^{(\zeta)} L_{H_{k+l}}^{(m,s)}(x, \xi, \zeta|q). \tag{3.21}
$$

Next, applying the series manipulation ([12], p.100):

$$
\sum_{p=0}^{\infty} \sum_{s=0}^{\infty} A(p, s) = \sum_{p=0}^{\infty} \sum_{s=0}^{\infty} A(s, p - s). \tag{3.22}
$$

to the l.h.s of (3.21), we obtain that

$$
\sum_{k,l=0}^{\infty} \sum_{n,r=0}^{\infty} \frac{u^{kl}t^lq^{(\xi)+(\zeta)}}{[k]_q![l]_q!} \left[ k \atop n \right] \left[ l \atop r \right] q^{(\xi)} \mathcal{G}_{n+r}^{(s)}(\xi \ominus q y, \zeta \ominus q z|q)L_{H_{k+l}}^{(m,s)}(x, y, z|q)
= \sum_{k,l=0}^{\infty} \frac{u^{kl}t^l}{[k]_q![l]_q!} q^{(\zeta)} L_{H_{k+l}}^{(m,s)}(x, \xi, \zeta|q). \tag{3.23}
$$

By equating terms with $u^{kl}t^l/[k]_q![l]_q!$ and using the simple combinatorial fact

$$
\left( \begin{array}{c} r \\ 2 \end{array} \right) + \left( \begin{array}{c} l - r \\ 2 \end{array} \right) - \left( \begin{array}{c} l \\ 2 \end{array} \right) = r (r - l), \tag{3.24}
$$

we arrive at the following result

$$
L_{H_{k+l}}^{(m,s)}(x, \xi, \zeta|q) = \sum_{n,r=0}^{\infty} q^{(r-l)} \left[ k \atop n \right] \left[ l \atop r \right] q^{(\xi)} \mathcal{G}_{n+r}^{(s)}(\xi \ominus q y, \zeta \ominus q z|q)L_{H_{k+l-n-r}}^{(m,s)}(x, y, z|q). \tag{3.25}
$$

Putting $\zeta = z$ in the last equation we establish the result in(3.12). □

**Theorem 3.2.** The following summation formula for the product of $q$-LGHP

$$
L_{H_n}^{(m,s)}(x, \xi, \zeta|q)L_{H_r}^{(m,s)}(X, \Omega, U|q) = \sum_{k,p=0}^{n,r} \left[ k \atop n \right] \left[ l \atop r \right] q^{(\xi)} \mathcal{G}_p^{(s)}(\Omega \ominus q Y, U \ominus q Z|q)L_{H_{n-k}^{(m,s)}}(x, y, z|q)L_{H_{r-p}^{(m,s)}}(X, Y, Z|q)\tag{3.26}
$$

holds true.

**Proof.** From the generating function (3.6), we have

$$
\sum_{n,r=0}^{\infty} L_{H_n}^{(m,s)}(x, y, z|q)L_{H_r}^{(m,s)}(X, Y, Z|q) \frac{t^m T^r}{[m]_q![r]_q} = \sum_{n,r=0}^{\infty} L_{H_n}^{(m,s)}(x, y, z|q)L_{H_r}^{(m,s)}(X, Y, Z|q) \frac{t^m T^r}{[n]_q![r]_q}.
$$

Replacing in (3.27) $y$ by $\xi$, $z$ by $\zeta$, $Y$ by $\Omega$ and $Z$ by $U$, we get
By replacing

\[
\mathcal{E}^m_q(-x t^m) \mathcal{E}^m_q(-X T^m) = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \mathcal{L}H^{(m,s)}_n(x, \xi, \zeta|q) \mathcal{L}H^{(m,s)}_r(X, \Omega, U|q) \frac{t^n T^r}{[n]_q! [r]_q!}.
\]

in the l.h.s. of (3.28) and using (2.12) one gets, after expanding the exponentials in series, the following

\[
\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \mathcal{L}H^{(m,s)}_n(x, \xi, \zeta|q) \mathcal{L}H^{(m,s)}_r(X, \Omega, U|q) \frac{t^n T^r}{[n]_q! [r]_q!} = e_q(t) \mathcal{E}_q^m((\zeta \ominus q)(x t^m)) e_q((\Omega \ominus q)(Y T)) e_q((U \ominus q)(Z T^m)) = \sum_{n,k=0}^{\infty} \mathcal{G}_q^{l}(\xi \ominus q y, \zeta \ominus q z|q) \mathcal{L}H^{(m,s)}_n(x, y, z|q) \frac{t^{n+k}}{[n]_q! [k]_q!}.
\]

Finally, by replacing \( n \) by \( n - k \) and \( r \) by \( r - p \) in the r.h.s. of (3.29), the proof is completed. \( \square \)

4 Particular cases

1. Putting \( z = 0 \) in Theorem 3.1 leads to a connection formula for the \( q \)-2DLP as:

\[
mL_{k+l}(x, y|q) = \sum_{n,r=0}^{k,l} \binom{k}{n}_q \binom{l}{r}_q q^{r(l-r)} (\xi \ominus q y)^{n+r} mL_{k+l-n-r}(x, y|q). \quad (4.1)
\]

In particular,

\[
mL_k(x, y|q) = \sum_{n=0}^{k} \binom{k}{n}_q (\xi \ominus q y)^n mL_{k-n}(x, y|q) \quad (4.2)
\]

and

\[
mL_l(x, y|q) = \sum_{r=0}^{l} \binom{l}{r}_q q^{r(l-r)} (\xi \ominus q y)^r mL_{l-r}(x, y|q) \quad (4.3)
\]

are obtained by setting \( l = 0 \) and \( k = 0 \) in (4.1) respectively.

2. The following formulæ for the \( q \)-LGHP :

\[
\mathcal{L}H^{(m,s)}_n(x, \xi, \zeta|q) = \sum_{k=0}^{n} \binom{n}{k}_q \mathcal{G}_k^{l}(\xi \ominus q y, \zeta \ominus q z|q) \mathcal{L}H^{(m,s)}_{n-k}(x, y, z|q) \quad (4.4)
\]
or

\[ LH^{(m,s)}_n(x, \xi, \zeta|q) = \sum_{k=0}^{n} \binom{n}{k}_q q^{k(n-k)} G^s_n(\xi \ominus_q y, \xi \ominus_q z|q) \cdot LH^{(m,s)}_{n-k}(x, y, z|q) \]  \hspace{1cm} (4.5) \]

and

\[ LH^{(m,s)}_n(x, \xi \ominus_q y, \zeta|q) = \sum_{k=0}^{n} \binom{n}{k}_q \xi^k LH^{(m,s)}_{n-k}(x, y, z|q) \]  \hspace{1cm} (4.6) \]

or

\[ LH^{(m,s)}_n(x, \xi \ominus_q y, \zeta|q) = \sum_{k=0}^{n} \binom{n}{k}_q q^{k(n-k)} \xi^k LH^{(m,s)}_{n-k}(x, y, z|q) \]  \hspace{1cm} (4.7) \]

are valid.

3. The following formula for the product of two q-LGHP:

\[ LH^{(m,s)}_n(x, \xi, \zeta \oplus_q z|q) \cdot LH^{(m,s)}_r(x, \Omega, U \oplus_q Z|q) = \]

\[ \sum_{k,p=0}^{n,r} \binom{n}{k}_q \binom{r}{p}_q (\xi \oplus_q y)^k (\Omega \oplus_q Y)^p \cdot LH^{(m,s)}_{n-k}(x, y, z|q) \cdot LH^{(m,s)}_{r-p}(X, Y, Z|q) \]  \hspace{1cm} (4.8) \]

holds true. The proof is immediate by replacing \( \zeta \) by \( \zeta \oplus_q z \) and \( U \) by \( U \oplus_q Z \) in Eq.(3.26).

4. Taking \( l = 0 \), in Eq.(3.25) and replacing \( \xi \) by \( \xi \oplus_q y, \zeta \) by \( \zeta \oplus_q z \) in the resultant equation, we get

\[ LH^{(m,s)}_k(x, \xi \oplus_q y, \zeta \oplus_q z|q) = \sum_{n=0}^{k} \binom{k}{n}_q G^s_n(\xi, \zeta|q) \cdot LH^{(m,s)}_{k-n}(x, y, z|q) \]  \hspace{1cm} (4.9) \]

which by taking \( \zeta = 0 \), yields

\[ LH^{(m,s)}_k(x, \xi \oplus_q y, z|q) = \sum_{n=0}^{k} \binom{k}{n}_q \xi^n LH^{(m,s)}_{k-n}(x, y, z|q). \]  \hspace{1cm} (4.10) \]

Similarly, taking \( k = 0 \), in Eq.(3.25) and replacing \( \xi \) by \( \xi \oplus_q y, \zeta \) by \( \zeta \oplus_q z \) in the resultant equation, we get

\[ LH^{(m,s)}_l(x, \xi \oplus_q y, \zeta \oplus_q z|q) = \sum_{r=0}^{l} \binom{l}{r}_q q^{r(n-l)} G^s_r(\xi, \zeta|q) \cdot LH^{(m,s)}_{l-r}(x, y, z|q), \]  \hspace{1cm} (4.11) \]

which by taking \( \zeta = 0 \), yields

\[ LH^{(m,s)}_l(x, \xi \oplus_q y, z|q) = \sum_{r=0}^{l} \binom{l}{r}_q q^{r(n-l)} \xi^r LH^{(m,s)}_{l-r}(x, y, z|q). \]  \hspace{1cm} (4.12) \]

5. Taking \( z = 0 = Z \) in Eq.(3.26) and Eq.(4.8), respectively, we get the following formulae for the q-2DLP:

\[ mL_{k+l}(x, \xi|q) = \sum_{n,r=0}^{k,l} \binom{k}{n}_q \binom{l}{r}_q q^{r(n-l)}(\xi \oplus_q y)^{n+r} mL_{k+l-n-r}(x, y|q) \]  \hspace{1cm} (4.13) \]

and

\[ mL_n(x, \xi|q) \cdot mL_{r}(X, \Omega|q) = \]

\[ \sum_{k,p=0}^{n,r} \binom{n}{k}_q \binom{r}{p}_q (\xi \oplus_q y)^k (\Omega \oplus_q Y)^p mL_{n-k}(x, y|q) \cdot mL_{r-p}(X, Y|q). \]  \hspace{1cm} (4.14) \]
6. Taking \( x = 0 \) and \( z \to \frac{-z}{[8]_q} \) in Eq.(3.25), we get the following formulae for the \( q \)-deformed Gould-Hopper generalized Hermite polynomials:

\[
G_{k+l}^s(\xi,\zeta|q) = \sum_{n,r=0}^{k,l} \left[ \begin{array}{c} k \\ n \end{array} \right]_{q} \left[ \begin{array}{c} l \\ r \end{array} \right]_{q} q^{r(r-1)} G_{n+r}^s(\xi \ominus_q y, \zeta \ominus_q z|q) G_{k+l-n-r}^s(y, z|q). \tag{4.15}
\]

7. Taking \( s = 2 \) and \( x = 0 \) in (3.25), we get the following connection formulae for the \( q \)-deformed Hermite polynomials \( H_n(x, y|q) \) \cite{7}:

\[
H_{k+l}(\xi, z|q) = \sum_{n,r=0}^{k,l} \left[ \begin{array}{c} k \\ n \end{array} \right]_{q} \left[ \begin{array}{c} l \\ r \end{array} \right]_{q} q^{r(r-1)} (\xi \ominus_q y)^{n+r} H_{k+l-n-r}(y, z|q), \tag{4.16}
\]

\[
H_n(\xi, z|q) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_{q} (\xi \ominus_q y)^k H_{n-k}(y, z|q), \tag{4.17}
\]

\[
H_n(\xi \oplus_q y, z|q) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_{q} q^{k(k-1)} (\xi \ominus_q y)^k H_{n-k}(y, z|q), \tag{4.18}
\]

\[
H_n(\xi \oplus_q y, z|q) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_{q} q^{k(k-1)} \xi^{n-k} H_k(y, z|q), \tag{4.19}
\]

Again, taking \( s = 2, x = 0 = X, z = \zeta \) and \( U = Z \), in equation (3.26), we get that

\[
H_n(\xi, z|q) \ H_r(\Omega, Z|q) = \sum_{k,p=0}^{n,r} \left[ \begin{array}{c} n \\ k \end{array} \right]_{q} \left[ \begin{array}{c} r \\ p \end{array} \right]_{q} (\xi \ominus_q y)^k (\Omega \ominus_q Y)^p H_{n-k}(y, z|q) H_{r-p}(Y, Z|q). \tag{4.21}
\]

8. When \( q \to 1 \) in (3.25) and (3.26), we obtain the following summation formulae for the Laguerre-Gould-Hopper polynomials \( L H_{n}^{(m,s)}(x, y, z) \) \cite{7}:

\[
L H_{k+l}^{(m,s)}(x, \xi, \zeta) = \sum_{n,r=0}^{k,l} \left[ \begin{array}{c} k \\ n \end{array} \right] L H_{n+r}^{(m,s)}(\xi - y, \zeta - z) \ L H_{k+l-n-r}^{(m,s)}(x, y, z), \tag{4.22}
\]

and

\[
L H_{n}^{(m,s)}(x, \xi, \zeta) \ L H_{r}^{(m,s)}(X, \Omega, U) = \sum_{k,p=0}^{n,r} \left[ \begin{array}{c} n \\ k \end{array} \right] \left[ \begin{array}{c} r \\ p \end{array} \right] g_p^s(\Omega - Y, U - Z) \ L H_{n-k}^{(m,s)}(x, y, z) \ L H_{r-p}^{(m,s)}(X, Y, Z). \tag{4.23}
\]

Next, taking \( \zeta = z \) and \( \zeta = z \), \( U = Z \) in (4.22) and (4.23) respectively, one obtains

\[
L H_{k+l}^{(m,s)}(x, \xi, z) = \sum_{n,r=0}^{k,l} \left[ \begin{array}{c} k \\ n \end{array} \right] \left[ \begin{array}{c} l \\ r \end{array} \right] (\xi - y)^{n+r} \ L H_{k+l-n-r}^{(m,s)}(x, y, z) \tag{4.24}
\]
and

\[ L^m_n(x, \xi, y) L^s_r(x, \Omega, z) = \sum_{k,p=0}^{n,r} \binom{n}{k} \binom{r}{p} (\xi - y)^k (\Omega - Y)^p L^m_{n-k} (x, y, z) L^s_{r-p} (x, Y, Z). \]  

(4.25)

9. Taking \( s = 2, x = 0 \) and \( q \to 1 \), the \( q-L\)GHP are reduced to the higher-order Hermite polynomials. That is,

\[ L^m_n(x, y, z|q) \equiv g^m_n(y, z) \]  

(4.26)

and summation formulae (3.25) and (3.26) reduce to the ones in [7]. That is,

\[ g^m_k(l, y) = \sum_{n,r=0}^{l,k} \binom{n}{k} \binom{l}{r} (\xi - x)^n (\Omega - Y)^p g^m_{l-n-r}(x, y), \]  

(4.27)

\[ g^m_k(x + l, y) = \sum_{n,k=0}^{n,k} \binom{n}{k} (\xi - x)^k g^m_k(x, y), \]  

(4.28)

\[ g^m_k(\xi + x, y) = \sum_{n,k=0}^{n,k} \binom{n}{k} (\xi - x)^n g^m_k(x, y) \]  

(4.29)

and

\[ g^m_k(\xi, l) g^m_r(\Omega, U) = \sum_{n,r=0}^{n,r} \binom{n}{k} \binom{r}{p} g^m_k(\xi - y, \zeta - z) \times g^m_p(\Omega - Y, U - Z) g^m_{n-k}(y, z) g^m_{r-p}(Y, Z), \]  

(4.30)

which, by taking \( \zeta = z \) and \( U = Z \), yields

\[ g^m_k(\xi, z) g^m_r(\Omega, Z) = \sum_{n,r=0}^{n,r} \binom{n}{k} \binom{r}{p} (\xi - y)^k (\Omega - Y)^p g^m_{n-k}(y, z) g^m_{r-p}(Y, Z). \]  

(4.31)

Appendix A: The proof of (2.17)

To obtain a closed form for the generating function

\[ G_q(t; x, y; m) := \sum_{n=0}^{\infty} \binom{\frac{n}{m}}{q} t^n L_n(x, y|q) \]  

(A.1)

we start by replacing in the r.h.s of (A.1) the \( q\)-2DLP \( mL_n(x, y|q) \) by its explicit expression (2.16) as follows

\[ G_q(t; x, y; m) = \sum_{n=0}^{\infty} t^n \left( \left[ \frac{n}{m} \right] \sum_{k=0}^{n-mk} \binom{n}{m} q^m(k) x^k y^{n-mk} \right). \]  

(A.2)

The right hand side of (A.2) also reads

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{\left[ \frac{n}{q} \right]} q^m(k) (xt)^k (yt)^{n-mk} \]  

\[ \binom{k}{q} t^n \left[ \frac{n}{m} \right] \]  

(A.3)
which can be put in form
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/m \rfloor} q^{m(k)}_q (x^m t^k) \frac{(yt)^{n-mk}}{([k]_q^m)!} \left[\frac{n}{m}\right]_q!
\tag{A.4}
\]
Using the same manipulation in (3.10). This enable us to write (A.4) as
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} q^{m(k)}_q (x^m t^k) \frac{(yt)^n}{([k]_q^m)!} \left[\frac{n}{m}\right]_q!
\tag{A.5}
\]
which coincides with
\[
e_q(yt)\varepsilon^m_q(-xt^m) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} q^{m(k)}_q (x^m t^k) \frac{(yt)^n}{([k]_q^m)!} \left[\frac{n}{m}\right]_q!
\tag{A.6}
\]
This completes the proof. □

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