1. Introduction

Vacuum integrals, i.e. integrals without external momenta (often also called tadpoles or bubbles), constitute an important class of multi-loop Feynman integrals. While the perturbative expansion of quantities like the free energy can be directly expressed in terms of vacuum integrals, they also serve as essential building blocks for many other computations, being the coefficient functions in asymptotic expansions of diagrams with external legs, and encoding the ultraviolet behavior of multi-scale integrals.

A typical perturbative calculation proceeds in four conceptually independent steps. First, all relevant diagrams including their combinatoric factors are generated. For an algorithm that does this for vacuum integrals, see [1]. Second, the Feynman rules of the theory under consideration are inserted, and the color and Lorentz algebra is performed. Since in general individual loop integrals are divergent, a regularization scheme has to be adopted, the most practical one at present being dimensional regularization (DR). Third, linear relations between the regularized integrals are exploited, to systematically reduce all integrals occurring in the computation to a small set of so-called master integrals. In the framework of DR, the most important class of relations can be derived from integration-by-parts (IBP) identities [2]. Fourth, the master integrals have to be evaluated, either, in some fortunate cases, fully analytically, or as an expansion in terms of the regularization parameter, in which case – and only here – the number of dimensions $d$ has to be specified. For results on the 4-loop level, see [3] ($d = 4 - 2\epsilon$) and [4] ($d = 3 - 2\epsilon$).

At higher loop orders, it is inevitable to automate the above setup to a large degree. There exist many approaches to implement automated perturbative calculations, and this is not the place to give a comprehensive review (see e.g. [5]). Instead, it is the third of the above steps that we wish to elaborate on in this contribution.

A computer algebra system that is particularly well suited to cope with the demands of higher order perturbative calculations is FORM [6]. While by no means mandatory to use, we have adopted it to implement our algorithms, and hence we will indicate in a few places which specific FORM commands turned out to be extremely helpful.

2. Notation and general considerations

Consider the generic vacuum topologies of Fig. 1. In this intuitive graphical notation, every line represents a propagator $(p_i^2 + m_i^2)^{-a_i}$, with integer power $a_i > 0$, where the index $i$ labels the different lines with momenta $p_i$, which in turn can be expressed as a linear combination of the loop momenta $k_j$. The vertices do not have any structure, except for assuring momentum conservation. Each diagram can carry a non-trivial numerator structure, which in the general case consists of powers of scalar products of the loop momenta. At $\ell$ loops, there are $\ell(\ell + 1)/2$ different combinations $k_i \cdot k_j$.

Let us distinguish three different representations of our integrals, which naturally appear at various levels of the reduction process: **generic**
Figure 1. The 1+1+3+10 generic vacuum topologies up to four loops. The 0+1+2+6 factorized topologies are not shown here.

Integrals, their standard representations, and the master integrals. The goal of step three is then to formulate the algorithms which transform generic to standard to master integrals:

\[ \int_{k_1 \ldots \ell}^{(d)} \prod_{1 \leq i < j \leq \ell} (k_i \cdot k_j)^{b_{ij}} \prod_{i=1}^{\ell} (p_i^2 + m_i^2)^{a_i} \]

\[ \text{Filter} \int_{k_1 \ldots \ell}^{(d)} \prod_{i=1}^{\ell} (k_i \cdot k_j)^{b_{ij}}_{\text{irred}} = (\{a_i\}, \{b_{ij}, m_i\}) \]

\[ \text{Tables} \sum_j c_j(d) \text{Master}_j^{(d)}(\{m_i\}) \quad (1) \]

Above, the label 'Filter' symbolizes a collection of low-level routines, whose main action is to complete squares in the numerator and cancel against propagators such that only irreducible numerators remain. At this point, it is possible to represent an \( \ell \)-loop vacuum integral by a list of \( \ell (\ell + 1) \) non-negative numbers \((\{a_i\}, \{b_{ij}, m_i\})\), the first half of them collecting the powers of propagators \(a_i\), while the second half contains either the power of an irreducible numerator (if the corresponding \(a_i\) is zero) or the mass of the line. Furthermore, at this step equivalent topologies are re-labeled in a unique way by shifting the loop momenta, i.e. assigning a characteristic pattern of zeroes among the \(a_i\) to each topology of Fig. 1.

In the remainder, we will specialize on two different general classes of vacuum diagrams. First, we will consider all lines to have the same mass, \(m_i = m\). This class of integrals is useful when computing infrared-safe quantities like renormalization coefficients, in which case the infrared sector of individual diagrams can be regulated by introducing masses into massless propagators. Second, we will allow for all \(m_i\) to be either zero or \(m\), with the restriction that the number of massive lines at each vertex be even. This includes theories like QED and gauge+Higgs models, whence we call this class 'QED-like'.

The label 'Tables' in Eq. (1) symbolizes a lookup in a database, which contains the necessary relations in a tabulated form. These tables are the main ingredient of the reduction step, and their organization and generation, which systematically exploits IBP identities, will be described in more detail below.

The intermediate step of applying the 'Filter' algorithms not only serves the purpose of allowing for a fairly compact representation of the integral, but can also be used to keep the number of entries in the database, the memory requirements, and the CPU time needed for their derivation, in manageable bounds. To this end, we found it advantageous to add further routines to the 'Filter' package:

- Early detection of zeroes: massless (sub-) tadpoles are zero in DR, as are integrals whose integrand does not depend on one of the loop momenta.
- Symmetrization of the integrand: use the full symmetry group of the corresponding topology to order the list, and hence enable early cancellations in big expressions.
- Decouple scalar products involving the loop momentum of a factorized one-loop tadpole:
  \[ \int_{k_1, \ldots, k_n} \frac{d^n k}{(k^2 + m^2)^{\nu+1}} \]
  vanishes for odd \(n\) and is proportional to a totally symmetric combination of metric tensors \(g_{\mu_1\mu_2 \ldots g_{\mu_{n-1}\mu_n}}\). The FORM function \texttt{dd} is perfectly suited for this symmetrization. This eliminates the need to derive relations for 8 (out of 9, the 9th being the two-loop \(\times\) two-loop case) of the factorized topologies, since after decoupling the numerator, factorization into scalar vacuum integrals of the type of Eq. (1) is complete.
- Reduce powers of factorized one-loop tadpoles to one:
  \[ \int \frac{d^d k}{(k^2 + m^2)^{\nu+1}} = -\frac{d - 2a}{2am^2} \int \frac{d^d k}{(k^2 + m^2)^\nu} \]
- Employ the 'triangle relation' [2]: for integrals involving massless lines, this helps
to reduce the number of different topologies that have to be treated in the database considerably.

Another potentially useful routine, which we have however not implemented, would be to use $T$-operators [7] in order to trade all numerator structure for higher dimensions of the integral measure, hence also immediately decoupling the factorized (two-loop $\times$ two-loop)-topology.

One more practical note: To not miss cancellations, it is important to have a unique representation for coefficients. Partial fractioning of terms like $\frac{1}{d-a}$ and $\frac{1}{d-a_1} - \frac{1}{d-a_2}$ helps here, ensuring the coefficients $c_i(d)$ to be a sum of powers of simple poles $\frac{1}{d-a}$ and powers of $d$.

In principle however, all these further relations are redundant since they would be automatically covered by the IBP identities. As mentioned above, their sole purpose is to optimize the derivation of relations among the integrals, to be discussed next.

3. Reduction

Integration by parts relies on the fact that an integral over a total derivative of any of the loop momenta vanishes in dimensional regularization. For the case of vacuum integrals, which we are interested in here, the IBP identities read

$$0 = \int \frac{d^d k_j}{(2\pi)^d} \partial_{p_\mu} q_\mu \left( \{a_i\}, \{b_{ij}, m_i\} \right),$$

where $p, q \in \{k_1, \ldots, k_\ell\}$ cover all $\ell^2$ different $\ell$-loop identities, and we have made use of the standard representation introduced above.

There are two possible general strategies implementing the IBP identities to find relations useful for reducing the integrals from their standard representation to master integrals.

The first strategy is to derive general relations, valid for symbolic list-entries. These general symbolic relations can then be applied repeatedly to any integral of the specified class, no matter how large the powers are, to achieve the reduction. In practice however, it turns out that it is quite an art to shuffle IBP identities for integrals with symbolic indices such as to obtain useful reduction relations. In absence of a generic algorithmic formulation, it involves extensive handwork, and typically there are many special cases to be considered when pre-factors vanish at special parameter values. At lower loop orders, there are complete solutions, see e.g. [9] for two-loop twopoint functions with general masses, or [10] for three-loop vacuum integrals with one mass.

The second strategy, nowadays constituting the mainstream of higher-loop computations, is a more brute-force approach, which however has the huge advantage of being perfectly suited to be completely automated. The main idea is to write down IBP identities for specific values of the indices. Introducing a lexicographic ordering among the integrals [8], it is then possible to solve every single one of the IBP identities for the 'most difficult' integral occurring. By starting from simple topologies (low number of lines), one systematically generates relations which express 'difficult' integrals in terms of 'simpler' in the sense of the ordering) ones. Solving an adequate set of fixed-index IBP relations, it is possible to express every integral of interest in terms of a few simple ones, ultimately the master integrals.

The reason why the second strategy is sufficient for most computations is that in practice, one does not meet the most general integrals, but only a subset, typically characterized by an upper cutoff on the sum of indices. Indeed, dealing with a concrete model like QCD, knowledge of the vertex and propagator structure allows to constrain the set of possible indices $\{\{a_i\}, \{b_{ij}, m_i\}\}$, hence rendering the search-space to be covered with IBP identities finite.

Building up the relations proceeds as follows:
Pick a list of indices \( \{a_i\}, \{b_{ij}, m_i\} \) that is 'simple', typically meaning a low number of loops, a low number of different lines, a low number of extra powers on the propagators, a low number of powers on the irreducible numerators. In FORM, these lists are most naturally represented by sparse tables.

- Generate the first of the IBP identities.
- Call 'Filter' to transform the resulting sum of integrals to the standard representation.
- Label the 'most difficult' integral, according to the lexicographic ordering. A global operation like that became possible with the introduction of '$\xi$-variables' in FORM v3.
- Invert its coefficient and multiply it into the equation. We do so only if we can factorize the coefficient into terms which are linear in \( d \), to preserve the generic structure of coefficients. While (at present) there is no factorization algorithm in FORM, we implemented one by 'guessing' zeroes, utilizing the fact that since the coefficients are generated by IBP, most of them have factors \( (n d \pm a) \) where \( n \) is not bigger than the number of loops, and \( a \) is an integer of moderate size. To check that no relations are missed when factorization fails, it is useful to keep track of those cases and check in the end.
- Bring the most difficult integral (having coefficient 1) to the left-hand-side, taking the generated equation as a definition. In FORM, this is done by the \texttt{fill} statement.
- Take the next IBP identity, repeat the above steps. Increase list-indices. Repeat...
- Write the relations found to disk in intervals. Large intervals ensure a high degree of re-substitution (of relations for integrals that are found later but that appeared on the right-hand-sides earlier), but are risky when the program execution crashes.

Solving the IBP relations one by one like described above seems to be simpler than solving large systems of linear equations at once. In the end, it might be advantageous to re-substitute relations into memory and re-writing them to disk.

In the end, one has to check whether the set of generated identities is sufficiently large to achieve a reduction of all integrals occurring in the physics problem at hand. While a first educated guess on the maximum powers needed can be obtained by scanning the terms to be calculated after application of the 'Filter' package, it might be necessary to enlarge the set of relations in further runs. To this end, the \texttt{tablebase} statement of FORM, implemented in version 3.1, allows for a good control over large amounts of data in the form of tables and table elements.

The resulting master integrals are depicted in Fig. 2 for the 'QED-like' case, and in Figs. 3,1 for the fully massive case.

4. Master integrals

Once the reduction algorithm 'stops', are we guaranteed to arrive at the desired minimal set of master integrals? If we had followed the path of deriving generic reduction relations, valid for symbolic indices, the answer would be yes. For the implementation in terms of specific indices, one can however not be absolutely sure not to miss a relation which would only be detected when increasing the upper cutoff on indices of the integrand. For most practical purposes it might already be sufficient to work with an incomplete, but small, basis.

In the case of gauge theories, it is also amusing to watch the gauge-parameter dependence as an indicator of how 'close' one is to the minimal set, since in a full reduction gauge-parameter dependent terms cancel at an algebraic level, in \( d \) dimensions, before evaluating the master integrals.

The basis of master integrals is of course not unique, but depends on the actual choice of the lexicographic ordering. While we label an integral with unit numerator as 'simpler' than one with in-
5. Discussion

We did not comment on the problem of so-called spurious poles here. Spurious poles are singular pre-factors, which can occur in the reduction relations. They are difficult to avoid in general, if one is not willing to specify the dimension yet in the reduction process. However, in our four-loop computation of the QCD free energy, we treated them after the reduction was performed successfully, by changing basis with the help of the tables.

In principle, the package at hand can be used for other calculations requiring a four-loop reduction of massive vacuum bubbles. One such application would be the re-evaluation of the QCD beta function.

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