Improved Four-state Continuous-variable Quantum Key Distribution with Long Secure Distance

Jian Yang, Bingjie Xu, Xiang Peng† and Hong Guo‡
CREAM Group, State Key Laboratory of Advanced Optical Communication Systems and Networks (Peking University), School of Electronics Engineering and Computer Science, Peking University, Beijing 100871, PR China
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The four-state continuous-variable quantum key distribution (CVQKD) protocol has a long practical secure distance [1], while it has the difficulty of parameter estimation. We propose an improved four-state protocol, where the covariance matrix can be estimated from experimental data without using the linear channel assumption, and thus ensuring its unconditional security in the asymptotical limit. Our new scheme keeps the advantages of high reconciliation efficiency and long secure distance of the four-state protocol, and it can be implemented under current technology.

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I. INTRODUCTION

Quantum key distribution (QKD) is one of the most practical application of quantum information, which allows two remote parties, Alice and Bob, to establish a sequence of secure keys [2]. Continuous-variable quantum key distribution (CVQKD) encodes information into the quadratures $x$ and $p$ of the optical field, and extracts it with homodyne detections, which usually have higher repetition rate than that of single-photon detections. So, CVQKD can potentially generate secure keys with higher speed. Historically, CVQKD protocols are at first based on squeezed states [3, 4]. Later, coherent state protocols with Gaussian modulation were found to be more practical choices [5, 6]. Both protocols have been experimentally demonstrated [7, 8] and have been shown secure against arbitrary collective attacks [9, 10], which are optimal in the asymptotical limit [11].

One remaining problem is that the reconciliation efficiency $\beta$ is quite low for Gaussian modulation, especially when the transmission distance is long. As mentioned in [1], this is the main limiting factor of the secure distance. There are two possible ways to solve this problem. One is to build a good reconciliation code with reasonable efficiency even at low SNR (signal to noise ratio), which has been achieved very recently [12]. The other is to use discrete modulation, such as the four-state protocol, proposed by Leverrier et al. [1]. In this protocol, Alice randomly prepares one of the four coherent states: $|\alpha_m\rangle_B = |\alpha e^{i(2m+1)\pi/4}\rangle_B$ with $m \in \{0, 1, 2, 3\}$ and sends to Bob. Then, Bob randomly measures the $x$ or $p$ quadrature of the signal pulse as his result, the sign of which encodes the bit of the raw key. Since the sign of quadrature has discrete possible values, there exist very good error correction codes when extracting $I(a : b)$, even for extremely low SNR. From this viewpoint, the four-state protocol combines the high reconciliation efficiency of discrete modulation and the security proof of CVQKD together, and improves the secure distance effectively.

However, in this scheme, Alice and Bob can not estimate the covariance matrix from their experimental data without the linear channel assumption (LCA) in practice. In the entanglement-based (E-B) scheme of the four-state protocol, the projection measurement $|\psi_m\rangle\langle\psi_m|, m = 0, 1, 2, 3$ Alice performs only helps to discriminate which coherent state is sent to Bob, but does not measure the quadratures of her mode. So, Alice and Bob are not able to evaluate the covariance matrix $\gamma_{AB}$ from experimental data unless using the LCA, which compromises the security of the protocol. To solve this problem, Leverrier et al. modified their protocols by introducing decoy states [13], such that

$$p\rho_{\text{key}} + (1 - p)\rho_{\text{decoy}} = \rho_G,$$

where $\rho_{\text{key}}$ is the state sent to Bob in the four-state protocol and $\rho_{\text{decoy}}$ is the decoy state. Alice randomly prepares $\rho_{\text{key}}$ and $\rho_{\text{decoy}}$ with probability $p$ and $1 - p$, respectively, so that the mixed state sent to Bob is Gaussian, $\rho_G$. The main difficulty of this method is the decoy state $\rho_{\text{decoy}}$ cannot be accurately prepared.

In this paper, we proposed an improved four-state protocol by modifying its entanglement-based (E-B) scheme, the covariance matrix of which can be directly evaluated from experimental data without using the LCA, and its corresponding prepare and measurement (P&M) scheme is not difficult to implemented under current technology. Using discrete coding, the high reconciliation efficiency and long secure distance can be kept in this protocol.

II. THE IMPROVED ENTANGLEMENT-BASED SCHEME OF THE ORIGINAL FOUR-STATE PROTOCOL

In this section, we introduce the improved E-B scheme of the original four-state protocol. In practice, CVQKD protocols are implemented in the P&M scheme, and the secure key rate against collective attacks can be calculated by

$$K_R = \beta I(a : b) - S(b : E),$$

where $K_R$ is the secure key rate using reverse reconciliation, $I(a : b)$ is the classical mutual information between Alice and Bob, $S(b : E)$ is the quantum mutual information between Bob and Eve, and $\beta$ is the reconciliation efficiency.
I(a : b) can be directly estimated from experimental data, while $S(b : E)$ should be estimated using its equivalent E-B scheme. In the E-B scheme of original four-state protocol [1], Alice prepares

$$|\Phi_L\rangle_{AB} = \frac{1}{4} \sum_{m=0}^{3} |\psi_m\rangle_A |\alpha_m\rangle_B, \quad (3)$$

measures mode $A$ with $|\psi_m\rangle_A$, and sends mode $B$ to Bob, where $|\psi_m\rangle_A$ are orthogonal states and $m \in \{0, 1, 2, 3\}$. As mentioned above, the main difficulty of this E-B scheme is parameter estimation, where Alice’s measurement $|\psi_m\rangle(|\psi_m|)$ only helps her to discriminate which state is sent to Bob, but does not provide any information about the quadratures of her mode. Comparatively, in the E-B scheme of Gaussian modulation protocols [14], Alice prepares EPR pairs, and measures her mode with homodyne detection, which not only projects Bob’s mode into coherent states, but also provides the information about the quadratures of Alice’s mode, with which Alice and Bob are able to estimate the covariance matrix $\gamma_{AB}$ from their experimental data.

Our improvement is to substitute $|\psi_m\rangle_A$ with proper states $|\psi'_m\rangle_A$. Obviously, there are at least two conditions that $|\Phi'_L\rangle_{AB} = \sum_{m=0}^{3} C_m |\psi'_m\rangle_A |\alpha_m\rangle_B$ should satisfy, where $C_m$ is the normalization coefficient:

1. Alice’s mode $|\psi'_m\rangle_A$ can be discriminated by homodyne or heterodyne detections, with which Alice are able to measure the quadratures of mode $A$ and the covariance matrix $\gamma_{AB}$ can be estimated from experimental data.

2. The covariance matrix of $|\Phi'_L\rangle$ should be as close as to that of Gaussian state as possible, which ensures the secure bound is tight, since the Gaussian optimality theorem is used when calculating $S(b : E)$.

From this viewpoint, the original E-B model in Eq. (3) satisfies condition 2, since its covariance matrix is close to that of EPR state, especially when the modulation is small. Its main drawback is that Alice does not use homodyne or heterodyne detections, which does not satisfy condition 1.

A natural choice for $|\psi'_m\rangle$ is coherent states, $|\psi'_m\rangle = |\beta_m\rangle = |\beta e^{i(2m+1)\pi/4}\rangle$, where $\beta$ is real and $m = 0, 1, 2, 3$. When $\beta$ is large, states $|\beta_m\rangle$ can be discriminated by heterodyne detection approximately. However, in this case, Alice’s measurement projects Bob’s state $\rho'_b$ into a superposition of coherent states, which is different from the $\rho_R$ of the original four-state protocol, and its equivalent P&M scheme is difficult to implement in real experiment.

### A. The Mixed-state Scheme

To avoid the problems above, we consider that Alice prepares mixed state $\rho_{AB}$ and measures $x$ and $p$ of mode $A$ simultaneously with heterodyne detection, where

$$\rho_{AB} = \frac{1}{4} \sum_{m=0}^{3} |\beta_m\rangle_A (|\beta_m\rangle \otimes |\alpha_m\rangle_B).$$

As illustrated in Fig. 1, Alice projects mode $B$ into a classical mixture of coherent states, which can be implemented in its P&M counterpart. Then, Bob randomly measures $x$ or $p$ of mode $B$ with homodyne detection to extract the information. In this scheme, $\rho_R$ is identical to that of original four-state protocol in Eq. (3) [1]. To calculate $S(b : E)$, we recall that the covariance matrix $\gamma_{AB}$ of $\rho_{AB}$ is defined by

$$(\gamma_{AB})_{ij} = \text{Tr}[\rho_{AB}(|\hat{r}_i - d_i\rangle\langle\hat{r}_j - d_j|)],$$

where the elements of displacement vector $d_i$ and $d_j$ are 0 in this scheme. Without difficult calculation, we find the covariance matrix of $\rho_{AB}$ has the form that

$$\begin{pmatrix} V_A & C_{AB} \\ C_{AB} & V_B \end{pmatrix}.$$

where $V_A$ and $V_B$ are the variances of mode $A$ and $B$, and $C_{AB}$ are their correlations. After channel transmission, the covariance matrix is changed to

$$\begin{pmatrix} V_A \eta & \sqrt{\eta} C_{AB} \\ \sqrt{\eta} C_{AB} & V_B \chi \end{pmatrix},$$

where $\eta$ and $\chi = (1-\eta)/\eta + \epsilon$ are the channel parameters. Both parameters can be estimated from experimental data. Here, for the ease of theoretical research, we suppose the channel is linear, and $\eta$ and $\epsilon$ are the transmittance and excess noise, respectively. It should be emphasized that this assumption is just for simplifying the simulation, but not necessary in this scheme [15].

The classical mutual information $I(a : b)$ can be calculated by $1 - H(e)$, where $e$ is the bit error rate and $H(e)$ is the Shannon entropy. The calculation of $S(b : E)$ is a little more complex. To maximize Eve’s information, Eve is supposed to purify the whole system $\rho_{AB}$ and the quantum mutual information $S(b : E)$ is calculated by

$$S(b : E) = S(E) - S(E|b) = S(AB) - S(A|b),$$

where $S(AB)$ and $S(A|b)$ can be derived from $\gamma_{AB}$, using the Gaussian optimality theorem [16, 17].

It is not surprising that we can not acquire positive secure key rate $K_R$ with this E-B scheme, and there are two reasons. First, when calculating $S(b : E)$, we suppose Eve is able to
purify the whole system to maximize the information leaked to her. So, this scheme just overestimates Eve’s information, since $\rho_{AB}$ is initially in a mixed state, and it is not difficult to find that $S(b : E) > 0$, even if the transmittance $\eta$ is 1. Second, the secure key rate in E-B scheme is related to how much pure entangled pairs can be extracted from $\rho_{AB}$, while in this E-B scheme, $\rho_{AB}$ is separable and contains little entanglement. Though Alice and Bob are classically correlated, they can not distill secret information from experimental data. Nevertheless, this attempt is very enlightening for our improved E-B scheme in the following.

B. The Improved Entanglement-based Scheme

In this subsection, we proposed our improved E-B scheme, which is illustrated in Fig. 2. Instead of mixed state $\rho_{AB}$, Alice prepares four-mode pure state $|\Psi_i\rangle_{FGAB}$, where $I$ denotes the improved E-B scheme and its subsystem $AB$ is identical to the mixed state $\rho_{AB}$ in Eq. (4), where

$$T_{FG}|\Psi_i\rangle_{FGAB} \langle \Psi_i| = \rho_{AB}. \tag{10}$$

The reason why we introduce two ancilla modes $FG$ is to guarantee that the pure state $|\Psi_i\rangle_{FGAB}$ does exist. In this scheme, modes $F$ and $G$ are used as neutral parties, the information of which is controlled neither by Eve nor by Alice and Bob. Alice measures $x$ and $p$ of mode $A$ simultaneously with heterodyne detection, and then sends mode $B$ to Bob. It is not difficult to verify that $\rho_B$ in this case is identical to that of the original four-state protocol. The classical mutual information $I(a : b)$ can be directly calculated by $1 - H(e)$, while Eve’s knowledge about Bob’s data $S(b : E)$ depends on the covariance matrix of $|\Psi_i\rangle_{FGAB}$. Certainly, we can derive the exact expression of $|\Psi_i\rangle_{FGAB}$ and calculate its covariance matrix, while using our previous technique [15], we find that this work is not necessary.

To compute $S(b : E)$, we consider a pure state

$$|\Psi_L\rangle_{FGAB} = |0\rangle_F|0\rangle_G|\Psi_L\rangle_{AB}, \tag{9}$$

where $|0\rangle$ is the vacuum state and $|\Psi_L\rangle_{AB}$ is Leverrier’s E-B model in Eq. (3). It is not difficult to verify the covariance matrix of $|\Psi_L\rangle_{FGAB}$ can be written as

$$\gamma'_{FGAB} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & V_{BZ} & Z \sigma_z \\ 0 & 0 & Z \sigma_z & V_{BZ} \end{pmatrix}. \tag{11}$$

where $Z$ is the correlation between Alice and Bob’s quadratures [1]. As shown in [19], since $|\Psi_i\rangle_{AB}$ and $|\Psi_L\rangle_{AB}$ are different purifications of $\rho_B$, there exist a unitary transformation $U_{FGA}$ on mode $F$, $G$, and $A$, that

$$|\Psi_i\rangle_{FGAB} = U_{FGA}|\Psi_L\rangle_{FGAB}, \tag{12}$$

which does not change the mutual information $S(b : E)$, since $U_{FGA}$ is commuted with $U_{BE}$, where $U_{BE}$ denotes Eve’s operation on mode $B$ and $E$. So, we can safely calculate $S(b : E)$ by substituting $|\Psi_i\rangle_{FGAB}$ with $|\Psi_L\rangle_{FGAB}$, the elements of which are known. This result can also be understood physically. In reverse reconciliation, Both Alice and Eve performs error correction according to Bob’s data. Whenever $|\Psi_i\rangle$ or $|\Psi_L\rangle$ is used, the mode $B$ sent to Bob is in the same state $\rho_B$. Since Eve is not able to discriminate which E-B source is used, she has to perform the same strategy to eavesdrop the information, and the leaked information $S(b : E)$ should be same.

Since $|\Psi_L\rangle_{FGAB}$ is a pure state, we have

$$S(E : b) = S(E) - S(E|b) \tag{12}$$

where van Neumann entropies $S(AB)$ and $S(A|b)$ can be calculated with the symplectic eigenvalues of covariance matrices $\gamma_{AB}$ and $\gamma_{\nu}$. [18]

The performance of our improved E-B scheme is illustrated in Fig. 3, where we use $\alpha = 0.5$ and $\beta = 20$. For small $\alpha$, the CM of $|\Psi_L\rangle$ is close to that of EPR state, which ensures a high secure key rate. For large $\beta$, coherent states $|\beta_m\rangle$ are approximately orthogonal to each other, which are easier to be discriminated by heterodyne detection. The variance of excess noise is set to be 0.002, 0.004, 0.006, 0.008 and 0.01, respectively. The secure distance is a little shorter than that of original four-state scheme, and this is mainly because coherent states $|\beta_m\rangle$ can not be discriminated deterministically.

The reason why $\rho_{AB}$’s purification $|\Psi_i\rangle_{FGAB}$ can be used to generate secure keys is based on two sides.
First, $|\Psi_I\rangle_{FGAB}$ is a pure state, and Eve is not benefit from her purification at the very beginning. It is not difficult to verify that $S(b : E) = 0$, when $\eta = 1$. Second, though $\rho_{AB}$ contains little entanglement, the whole system $FGAB$ is generally an entangled-state, which can be used to extract secure keys. From this viewpoint, our improved E-B scheme just combines $\rho_{AB}$’s advantages in parameter estimation and $|\Phi_L\rangle$’s advantages in computing $S(b : E)$ together, which ensures a long secure distance.

III. THE PREPARE AND MEASUREMENT SCHEME

Though the E-B scheme is convenient for theoretical research, it is difficult to implement directly. In this section, we will present its equivalent P&M scheme. As mentioned above, in the E-B scheme, Alice measures quadratures $x$ and $p$ of mode $A$ simultaneously. To do this, Alice should use a 50 : 50 beamsplitter to separate mode $A$ into two parts, $A_1$ and $A_2$, and the whole state is changed to

$$\rho_{A_1A_2B} = \frac{1}{4} \sum_{m=0}^{3} \frac{\beta_m}{\sqrt{2}} a_1(\beta_m |\beta_m\rangle \langle |\beta_m\rangle)_{A_2} \frac{\beta_m}{\sqrt{2}} \langle |\sigma_m\rangle_B |\sigma_m\rangle_m. \quad (13)$$

Then, Alice measures $x$ of mode $A_1$, measures $p$ of mode $A_2$, and projects Bob’s state to

$$\rho_B|_{x_m, p_A} = \frac{1}{4} \sum_{m=0}^{3} C_m^{(x_m,p_A)} |x_m\rangle_1 |\sigma_m\rangle_m. \quad (14)$$

The coefficient $C_m^{(x_m,p_A)}$ is calculated by

$$C_m^{(x_m,p_A)} = \text{Tr}(M_{A_1}(p_A)M_{A_1}(x_A)\rho_{A_2A_1}(m)M_{A_1}(x_A)M_{A_1}^\dagger(p_A)), \quad (15)$$

where operators $M_{A_1}(x_A) = |x_A\rangle_1 a_1(x_A), M_{A_1}(p_A) = |\rho_{A_1}\rangle_{A_1} a_1(p_A)$, and $\rho_{A_1A_2}(m) = [\frac{\beta_m}{\sqrt{2}}] a_1(\frac{\beta_m}{\sqrt{2}} \otimes \frac{\beta_m}{\sqrt{2}} a_{12}(\frac{\beta_m}{\sqrt{2}} a_{12}).$ This is a classical mixture of coherent states $|\langle x_m|_1 \sigma_m\rangle_m\rangle$, where the probability $C_m^{(x_m,p_A)}$ is a Gaussian function of Alice’s measurement result $(x_A, p_A)$. The calculation of $C_m^{(x_m,p_A)}$ is straight with the methods in [20], while we omit the detail here and focus on its experimental realization. In this section, we propose two possible P&M schemes to implement this protocol. One is the true random number generator (TRNG) based scheme and the other is the beamsplitter based scheme.

A. TRNG based Scheme

In TRNG-based scheme, each time Alice uses TRNG1 to generate random pairs $(x_A, p_A)$ with probability

$$\Pr(x_A) = \text{Tr}(M_{A_1}(x_A)\rho_{A_1} M_{A_1}^\dagger(x_A))$$

and

$$\Pr(p_A) = \text{Tr}(M_{A_1}(p_A)\rho_{A_1} M_{A_1}^\dagger(p_A)),$$

respectively, where density operators $\rho_{A_1} = \text{tr}_{A_2}(\rho_{A_1A_2B})$ and $\rho_{A_1} = \text{tr}_{A_2}(\rho_{A_1A_2B})$. To prepare $\rho_B|_{x_m, p_A}$, Alice randomly prepares a coherent state $|\sigma_m\rangle_m$ from $|\rho_{A_1}\rangle_{A_1}$, $m = 0, 1, 2, 3$ with probability $C_m^{(x_m,p_A)}$ and sends it to Bob. As illustrated in Fig. 4, the TRNG-based scheme can be realized within current technology, while it is still a little complicated, since each time two random numbers are generated and the probability $C_m^{(x_m,p_A)}$ depends on the random pair $(x_A, p_A)$.

B. Beamsplitter based Scheme

To simplify the experimental implementation, we propose a beamsplitter-based scheme. Noticing that $\rho_{AB} = \frac{1}{4} \sum_{m=0}^{3} [\beta_m]_A \langle \beta_m | \otimes |\sigma_m\rangle_m |\sigma_m\rangle_m$, we find it can be directly implemented with a beamsplitter. As illustrated in Fig. 5, Alice prepares a coherent state $|y\rangle$, and modulates it with a phase modulator, driven by TRNG2. Then, the modulated coherent state is separated by a beamsplitter, the output states of which are $|\beta_m\rangle_A$ and $|\sigma_m\rangle_B$, respectively. Then Alice measures the $x$ and $p$ of mode $A$ simultaneously, and sends mode $B$ to Bob. In this scheme, TRNG2 generates only 4 possible values $m = 0, 1, 2, 3$, which are easier to implement than TRNG1 in Fig. 4.

IV. DISCUSSION AND CONCLUSION

To sum up, we propose an improved long-distance CVQKD protocol by modifying the E-B scheme. We find that the mixed state $\rho_{AB}$ is helpful to establish a classical correlation.
between Alice and Bob, and then purifies $\rho_{AB}$ with two ancilla mode $F$ and $G$ as the improved E-B model, $|\Psi_I\rangle_{FGAB}$. The parameter estimation can be performed in this scheme without using the LNA. Further, based on [15], we find the mutual information $S(b:E)$ of our improved scheme is identical to that of Leverrier’s one, so we can derive a security bound for reverse reconciliation, without deriving the exact expression of the ancilla states $F$ and $G$. From this viewpoint, our improved E-B scheme combines the high reconciliation efficiency of discrete coding and the facility of parameter estimation together, and hence ensures a long secure distance with unconditional security. Also, we present two potential equivalent P&M schemes to implement the improved protocol experimentally.

There are also several remaining problems to study. First, the E-B model $|\Psi_I\rangle_{FGAB}$ can be further optimized to make its density matrix closer to that of a EPR, which may further improve the secure key rate. Second, in four-state protocol, the optimal value of $\alpha$ is less than 1, which is still not easy to detect by homodyne detections in the experiment. At last, its unconditional security against coherent attack need to be reconsidered when the finite size effect is taken into account.

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