NON-MARKOVIAN DYNAMICS OF CAVITY LOSSES

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We provide a microscopic derivation for the non-Markovian master equation for an atom-cavity system with cavity losses and show that they can induce population trapping in the atomic excited state, when the environment outside the cavity has a non-flat spectrum. Our results apply to hybrid solid state systems and can turn out to be helpful to find the most appropriate description of leakage in the recent developments of cavity quantum electrodynamics.

Keywords: Cavity quantum electrodynamics; quantum noise; open systems.

1. Introduction

Cavity quantum electrodynamics (CQED) is one of the most important fields of quantum optics, both from a fundamental and an applicative point of view. Indeed, the interaction of an atom and a single mode of the quantized electromagnetic field inside a high-Q cavity can be exploited to study the properties of highly non-classical states of the atom-cavity system and to use their non-classical features for quantum information processing. In this context it is fundamental to take into account the unavoidable coupling between the atom-cavity system and the environment external to it. In general, the coupling with an environment gives rise to dissipation and decoherence phenomena, which can strongly affect the dynamics of the quantum system under study. In previous papers, we derived a master equation for an atom-cavity system which takes into account from the very beginning a coupling between the atom and the cavity described by the Jaynes-Cummings (JC)
mode. The approach followed in Refs. 4 and 5 is different from the one usually reported in the literature to describe cavity losses. Indeed the usual model of cavity losses has been introduced in a phenomenological way, i.e., one microscopically derives the master equation for the cavity only, described as a quantum harmonic oscillator coupled to a bosonic bath, and then assumes that the presence of the atom inside the cavity does not affect the structure of the dissipator in the master equation. Through the microscopic model derived in Refs. 4, 5 it is possible to give a complete justification of the phenomenological model, which turns out to be valid when the spectrum of the environment is flat and its temperature is zero. The situation is quite different if the environment has a non-flat spectral density. This is not an uncommon situation nowadays: indeed there are new-generation CQED experiments which are performed on high-Q cavities created inside photonic bandgap (PBG) materials. For this system we proposed a non-Markovian model of cavity losses which predicts a dynamics completely different from the one predicted by the phenomenological model, giving rise to phenomena such as population trapping due to cavity losses. Scope of this paper is to provide some details of the derivation of the non-Markovian master equation for the JC model and to show that the main predictions of the theory, namely population trapping due to cavity losses, does not depend on the shape of the spectral density one chooses, but only on the values of the asymptotic decay rates corresponding to the transitions of interest. We clarify this point by taking into account two different models of environmental spectral densities and showing that, for the same asymptotic values of the decay rates, the predictions differ only in the details of the very short-time dynamics, while the two densities give the same predictions for longer times. The paper is structured as follows. In Sec. 2 we present the non-Markovian model of cavity losses, in Sec. 3 we present the calculation of the time-dependent decay rate for a Lorentzian spectrum and summarize the dynamics of the system in this case. In Sec. 4 the case of a spectrum with a Lorentzian gap is presented along with some conclusive remarks.

2. The non-Markovian master equation for cavity losses

The system we study consists of a two-level atom interacting with a mode of a cavity coupled to a bosonic environment. Calling $\omega_0$ the Bohr frequency of the atom and $|g\rangle$ and $|e\rangle$ its ground and its excited states respectively, the interaction between the atom and the cavity mode is described, at resonance and in units of $\hbar$, by the JC Hamiltonian

$$H_{JC} = \frac{\omega_0}{2}\sigma_z + \omega_0 a^\dagger a + \Omega (a\sigma_+ + a^\dagger\sigma_-),$$

where $a^\dagger$ ($a$) is the creation (annihilation) operator of the mode, $\sigma_- = |g\rangle\langle e|$, $\sigma_+ = |e\rangle\langle g|$, and $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$. The cavity mode interacts with a bosonic reservoir through an interaction Hamiltonian linear in the bosonic operators of the reservoir and of the cavity mode. More precisely we assume that the dynamics of the total system (atom-cavity system and environment) is described by a Hamiltonian $H = H_S + H_E + H_{int}$, where $H_S = H_{JC}$ is relative to the atom-cavity system, $H_E = \sum_k \omega_k b_k^\dagger b_k$ is relative
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The solution of Eq. (1) is the following one:

\[
\rho(t) = \left(1 - \frac{1}{2} e^{-\frac{I_-(t)}{2}} e^{-\frac{I_+(t)}{2}} \right) |E_0\rangle \langle E_0| \left(1 + \frac{1}{2} e^{-\frac{I_-(t)}{2}} e^{\frac{I_+(t)}{2}} \right) + \frac{1}{2} e^{\frac{I_+(t)}{2}} |E_{1,+}\rangle \langle E_{1,+}| + \frac{1}{2} e^{-\frac{I_+(t)}{2}} e^{\frac{I_-(t)}{2}} \left(e^{2\Omega t} |E_{1,-}\rangle \langle E_{1,+}| + \text{h.c.} \right),
\]

where \( I_{\pm}(t) = \int_0^t \gamma(\omega_0 \pm \Omega, t') dt' \). From Eq. (2) one can compute all the populations we are going to show in the following. Below we will study the behavior of the non-Markovian time-dependent rates \( \gamma(\omega_0 \pm \Omega, t) \), which, through the quantities \( I_{\pm}(t) \), lead to non-Markovian behavior and to population trapping.
3. The non-Markovian decay rates for a Lorentzian spectrum

As a first model of environment at zero temperature with non-flat spectrum, we consider the Lorentzian distribution 3:

\[ J(\omega) = \frac{1}{2\pi} \frac{\alpha \lambda^2}{(\omega_1 - \omega)^2 + \lambda^2}, \]

where \( \alpha \) is the system-environment coupling strength, and \( \lambda \) is the width of the distribution, describing also the inverse of the reservoir memory time. The case of Lorentzian spectrum is analytically treatable, while capturing important features of the non-Markovian dynamics we are interested in, i.e., the time-dependence of the decay rates and their different stationary values. We consider the case in which the spectrum is peaked on the frequency of the state \(|E_1, -\rangle\), i.e., \( \omega_1 = \omega_0 - \Omega \), where \( \omega_0 \) is the atomic Bohr frequency and \( \Omega \) is the Rabi splitting due to the JC interaction.

The rate \( \gamma(\omega, t) \) for a generic transition with Bohr frequency \( \omega \) is equal to

\[ \gamma(\omega, t) = 2 \text{Re} \{ \Gamma(\omega, t) \} \]

where \( \Gamma(\omega, t) \) is related to the spectral density \( J(\omega) \) through the relation 9:

\[ \Gamma(\omega, t) = \int_0^t d\tau \int_{-\infty}^{+\infty} d\omega' e^{i(\omega - \omega')\tau} J(\omega'). \]

This relation gives the right Markovian decay rates and Lamb shifts for \( t \to +\infty \).

By performing first the integral with respect to \( \tau \), one obtains:

\[ \int_0^t d\tau e^{i(\omega - \omega')\tau} = \frac{\sin(\omega' - \omega)t}{\omega' - \omega} - \frac{1 - \cos(\omega' - \omega)t}{\omega' - \omega}, \]

which, substituted into Eq. (4), gives:

\[ \Gamma(\omega, t) = \int_{-\infty}^{+\infty} d\omega' J(\omega') \frac{\sin(\omega' - \omega)t}{\omega' - \omega} - i \int_{-\infty}^{+\infty} d\omega' J(\omega') \frac{1 - \cos(\omega' - \omega)t}{\omega' - \omega}. \]

The second term in Eq. (6) gives a time-dependent Lamb shift which in the following will be neglected. The first term, i.e., the real part, is half the time-dependent decay rate \( \gamma(\omega, t) \).

Specializing to the Lorentzian spectral density given in Eq. (3), the real part of Eq. (6) becomes:

\[ \text{Re} \{ \Gamma(\omega, t) \} = \frac{1}{2\pi} \text{Im} \left\{ e^{-i\omega t} \int_{-\infty}^{+\infty} d\omega' \frac{\alpha \lambda^2 e^{i\omega't}}{[(\omega_1 - \omega')^2 + \lambda^2](\omega' - \omega)} \right\}, \]

whose last form is suitable for an evaluation by means of the method of the residues, by closing the integration path with a circle of ray \( R \) on the upper complex half-plane, where the integral vanishes when \( R \to \infty \). Such an evaluation is straightforward. It is only worth noting that the pole on the real axis, at \( \omega' = \omega \), must be circumventured from below, so that its residue is taken for half its value, with positive sign. The reason of this choice is that, as we will see, it gives a positive stationary
value for the decay rate, while the other choice would give a negative stationary decay rate, which would be unphysical.

By taking twice the quantity in Eq. (7) and evaluating the integral, we finally obtain the following expression for the decay rate $\gamma(\omega, t)$:

$$
\gamma(\omega, t) = \frac{\alpha \lambda^2}{(\omega_1 - \omega)^2 + \lambda^2} \left\{ 1 + \left[ \frac{\omega_1 - \omega}{\lambda} \sin(\omega_1 - \omega)t - \cos(\omega_1 - \omega)t \right] e^{-\lambda t} \right\}.
$$

From Eq. (8) we clearly see the general behavior of the time-dependent rates: all the rates $\gamma(\omega, t)$ are zero at $t = 0$, then they oscillate in time with a time-dependent average value, till they reach stationary values for $t \gg \lambda^{-1}$. These stationary values are proportional to $J(\omega)$, i.e., they are equal to the rates one obtains from a Markovian theory. For these reasons, as anticipated, the quantity $\lambda^{-1}$ can be seen as the memory time of the system-reservoir interaction and non-Markovian effects are expected to occur for times shorter than $\lambda^{-1}$.

In particular, taking the peak of the spectrum in $\omega_1 = \omega_0 - \Omega$ and substituting $\omega = \omega_0 \pm \Omega$, one obtains the decay rates for the two dressed states $|E_{1,\pm}\rangle$, which, along with the solution of the master equation in Eq. (2), allow us to compute the dynamics of the atom cavity system presented in Ref. [9] to which we refer for the plots of the populations considered. The population of the ground state of the atom-cavity system $|0, g\rangle$ increases quadratically for short times, with some oscillations superimposed which are signatures of the oscillations of the decay rates. For larger times, i.e., $t \gg \lambda^{-1}$, it increases in time as the sum of the two exponentials, with rates $\gamma(\omega_0 - \Omega, \infty)$ and $\gamma(\omega_0 + \Omega, \infty)$ respectively. It is easy to see that the smaller the value of $\lambda$ in the spectrum, the smaller the stationary decay rate $\gamma(\omega_0 + \Omega, \infty)$, while $\gamma(\omega_0 - \Omega, \infty) = \alpha$ does not change with $\lambda$. A consequence of this point is the possibility of having situations where, starting with the atom initially excited and the cavity with zero photons, while the state $|E_{1,-}\rangle$ has completely decayed, the population of $|E_{1,+}\rangle$ keeps a constant value for long times, namely the ones in the interval $\gamma(\omega_0 - \Omega, \infty)^{-1} \ll t \ll \gamma(\omega_0 + \Omega, \infty)$. A consequence of this fact is that the population of the atom is trapped in the excited state for an amount close to 25%.

One may wonder how strongly this effect could depend on the particular model of spectrum we have chosen. In fact, the possibility of having a certain amount of population trapping in the system depends on the stationary decay rates only, so that the effect of population trapping involves only the values of the spectrum at the Bohr frequencies of the transitions of interest, and not the shape of the spectrum over all the real axis. To clarify this point, in the next section we show the behavior of the same populations when the spectrum of the environment has a shape different from the one considered in this section.
4. Evolution of the system in the case of a structured spectrum

As a second model for a non-flat spectrum, we take a simple model of structured reservoir, consisting in a Lorentzian background with a Lorentzian gap [3]:

\[ J(\omega) = \frac{1}{2\pi} \frac{\alpha_1 \lambda_1^2}{(\omega_1 - \omega)^2 + \lambda_1^2} - \frac{1}{2\pi} \frac{\alpha_2 \lambda_2^2}{(\omega_2 - \omega)^2 + \lambda_2^2}, \]  

(9)

where the gap is given by the inverted Lorentzian peak at frequency \( \omega = \omega_2 \).

From the calculation of the previous section, it is straightforward to show that each Lorentzian peak gives a similar contribution, but with opposite sign, so that the time-dependent decay rate of a transition of Bohr frequency \( \omega \) is given by:

\[ \gamma(\omega, t) = \alpha_1 \lambda_1^2 \left\{ 1 + \left[ \frac{\omega_1 - \omega}{\lambda_1} \sin(\omega_1 - \omega) t - \cos(\omega_1 - \omega) t \right] e^{-\lambda_1 t} \right\} - \alpha_2 \lambda_2^2 \left\{ 1 + \left[ \frac{\omega_2 - \omega}{\lambda_2} \sin(\omega_2 - \omega) t - \cos(\omega_2 - \omega) t \right] e^{-\lambda_2 t} \right\}. \]  

(10)

In the following we will take \( \omega_1 = \omega_2 = \omega_0 + \Omega \), and \( \alpha_1 > \alpha_2 \) and \( \lambda_1 > \lambda_2 \); in this way we assume that the Bohr frequency corresponding to the transition \( |E_{1,-}\rangle \rightarrow |E_0\rangle \) corresponds to the minimum of the spectral density, in analogy with what done in the case of a single Lorentzian peak. The choice of \( \alpha_1 = 0.1 * 2\Omega \), \( \alpha_2 = 0.099 * 2\Omega \), \( \lambda_1 = 100 * 2\Omega \) and \( \lambda_2 = 0.1 * 2\Omega \) gives a ratio 1/100 between the stationary values of the two decay rates of interest \( \gamma(\omega_0 \pm \Omega, \infty) \), a situation which is close to the ideal case of perfect population trapping.

As we see from Fig. 1-(a), the short-time behavior of the rates is rather different from the case of the single Lorentzian peak, indeed in the case of the gap the decay rate of \( |E_{1,+}\rangle \) initially increases in the same way the rate of \( |E_{1,-}\rangle \) does, and, after reaching its maximum value close to \( 0.1 * 2\Omega \), it decays exponentially to its stationary value \( 0.01 * 2\Omega \). Anyway this difference in the behavior of the rates does not lead to observable effects in the dynamics of the atomic populations. Indeed, comparing the population of the atomic ground state in Fig. 1-(b) with the corresponding...
population in Ref. [9] we see that the behavior predicted by both the single-peak Lorentzian model and the Lorentzian gap model is exactly the same: for a long time a population trapping occurs in the excited atomic state, for an amount of about 25%, no matter which of the two models one chooses.

Summarizing, in the situation we have analyzed, the essential point to properly choose the model is to check the appropriate ratio between the stationary values of the decay rates. In this sense, the most important property is the value of the spectrum at the Bohr frequencies of the transitions of interest. On the other hand, the particular shape of the spectrum one chooses affects only the details of the short-time dynamics, especially the short-time behavior of the decay rates, but the main aspects of the dynamics of the atom-cavity system are not affected by its choice.

We feel that these conclusions are quite general and that they apply to a wide variety of situations involving lossy cavities with non-flat spectra, as in the most advanced quantum electrodynamic systems[8].

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