POLARIZED SUPERSPECIAL SIMPLE ABELIAN SURFACES WITH REAL WEIL NUMBERS

JIANGWEI XUE AND CHIA-FU YU

ABSTRACT. Let \( q \) be an odd power of a prime \( p \in \mathbb{N} \), and \( \text{PPSP}(\sqrt{q}) \) be the finite set of isomorphism classes of principally polarized superspecial abelian surfaces in the simple isogeny class over \( \mathbb{F}_q \) corresponding to the real Weil \( q \)-numbers \( \pm \sqrt{q} \). We produce explicit formulas for \( \text{PPSP}(\sqrt{q}) \) of the following kinds: (i) the class number formula, i.e., the cardinality of \( \text{PPSP}(\sqrt{q}) \); (ii) the type number formula, i.e., the number of endomorphism rings up to isomorphism of the underlying abelian surfaces of \( \text{PPSP}(\sqrt{q}) \). Similar formulas are obtained for other collections of polarized superspecial members of this isogeny class grouped together according to their polarization modules. We observe several surprising identities involving the arithmetic genus of certain Hilbert modular surface on one side and the class number or type number of \((P, P_+)\)-polarized superspecial abelian surfaces in this isogeny class on the other side.

1. Introduction

Let \( p \in \mathbb{N} \) be a prime number, \( q = p^n \) a power of \( p \), and \( \mathbb{F}_q \) the finite field with \( q \) elements. Let \( \overline{\mathbb{Q}} \) be the algebraic closure of \( \mathbb{Q} \) in \( \mathbb{C} \). An algebraic integer \( \pi \in \overline{\mathbb{Q}} \subset \mathbb{C} \) is called a Weil \( q \)-number if \( |\sigma(\pi)| = \sqrt{q} \) for every embedding \( \sigma : \mathbb{Q}(\pi) \hookrightarrow \mathbb{C} \). By the Honda-Tate Theorem \([65, \text{Theorem 1}]\), there is a bijection between the isogeny classes of simple abelian varieties over \( \mathbb{F}_q \) and the \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)-conjugacy classes of Weil \( q \)-numbers. Throughout this paper, we reserve terms such as “isogeny, isomorphism, endomorphism, or polarization” of abelian varieties over \( \mathbb{F}_q \) for those defined over the base field \( \mathbb{F}_q \). Let \( X_{\pi}/\mathbb{F}_q \) be a simple abelian variety in the isogeny class corresponding to \( \pi \). Both the dimension \( g(\pi) := \text{dim}(X_{\pi}) \) and the endomorphism algebra \( \text{End}^0(X_{\pi}) := \text{End}(X_{\pi}) \otimes \mathbb{Q} \) are invariants of the isogeny class and can be determined explicitly from \( \pi \) (ibid.). Recall that \( \text{End}^0(X_{\pi}) \) is a finite-dimensional central division \( \mathbb{Q}(\pi) \)-algebra.

It is well known \([88, \text{4.1}]\) that for each fixed \( g \geq 1 \), there are only finitely many \( g \)-dimensional abelian varieties over \( \mathbb{F}_q \) up to isomorphism. Let \( \text{Isog}(\pi) \) be the finite set of \( \mathbb{F}_q \)-isomorphism classes of simple abelian varieties in the isogeny class corresponding to \( \pi \). Similarly, let \( \text{PPAV}(\pi) \) be the finite set of isomorphism classes of principally polarized abelian varieties \( (X, \lambda)/\mathbb{F}_q \) with \( X/\mathbb{F}_q \) in the simple isogeny class corresponding to \( \pi \). The finiteness of \( \text{PPAV}(\pi) \) is again well known and follows from \([51, \text{Appendix I, Lemma 1}]\) (see also \([44]\)). Therefore, it is natural to ask:

**Question.** What are the cardinalities of \( \text{Isog}(\pi) \) and \( \text{PPAV}(\pi) \)?
In contrast to the algebraically closed base field case \([51, \S 23, \text{Corollary 1}]\), the set \(\PPAV(\pi)\) could be empty. Consequently, an \(\mathbb{F}_q\)-isogeny class \(\mathcal{I}\) of abelian varieties is said to be \textit{principally polarizable} if there exists an abelian variety \(X \in \mathcal{I}\) that admits a principal polarization over \(\mathbb{F}_q\). The question whether a given \(\mathbb{F}_q\)-isogeny class \(\mathcal{I}\) is principally polarizable has been investigated by E. Howe in a series of papers (see \([28]\) for the precise references). Howe, Maisner, Nart and Ritzenthaler \([28]\) Theorem 1] determined all isogeny classes of abelian surfaces that are not principally polarizable. Based on the works of R"{u}ck \([63]\), Maisner-Nart \([46]\), and the aforementioned result, Howe, Maisner, Nart and Ritzenthaler \([28, \text{Theorem 1}]\) determined all isogeny classes of abelian surfaces that are not principally polarizable has been investigated by E. Howe in a series of papers. The question whether a given \(\mathbb{F}_q\)-isogeny class \(\mathcal{I}\) is principally polarizable has been investigated by E. Howe in a series of papers (see \([28]\) for the precise references).

Let \(q = p^n\) be an arbitrary prime power. The Weil \(q\)-numbers \(\pm \sqrt{q}\) are exceptional in several ways. Suppose that \(\pi\) is a Weil \(q\)-number different from \(\pm \sqrt{q}\). Then the number field \(\mathbb{Q}(\pi)\) is always a CM-field (i.e. a totally imaginary quadratic extension of a totally real number field). From \([82, \text{Proposition 2.2}]\),

\[
|\Isog(\pi)| = N_{\pi} \cdot h(\mathbb{Q}(\pi)),
\]

where \(N_{\pi}\) is a positive integer, and \(h(\mathbb{Q}(\pi))\) is the class number of \(\mathbb{Q}(\pi)\). It should be mentioned that \(N_{\pi}\) is highly dependent on \(\pi\) and can be challenging to calculate explicitly in general. See the discussions in \([44, \S 3.2]\) and \([82, \S 2.4]\). The proof of \((1.1)\) relies on a strong approximation argument, which fails for the Weil \(q\)-numbers \(\pm \sqrt{q}\). The distinction is further amplified in the case \(q = p\). If \(\pi\) is a Weil \(p\)-number distinct from \(\pm \sqrt{p}\), then by \([72, \text{Theorem 6.1}]\),

\[
\End^0(X_\pi) = \mathbb{Q}(\pi)
\]

for every abelian variety \(X_\pi\) in the simple \(\mathbb{F}_p\)-isogeny class corresponding to \(\pi\), while \((1.2)\) does not hold for the Weil \(p\)-numbers \(\pm \sqrt{p}\). Consequently, many theories for abelian varieties over \(\mathbb{F}_p\) have to make an exception for the isogeny class corresponding to \(\pm \sqrt{p}\). See \([9, \S 1.3]\) and \([44, \text{Theorem 0.3}]\). Recently, results for counting \(\PPAV(\pi)\) with \(\pi\) being a Weil \(p\)-number distinct from \(\pm \sqrt{p}\) are obtained in \([1]\).

Now suppose that \(\pi = \pm \sqrt{q}\) with \(q = p^n\). There are two cases to consider. First suppose that \(n\) is even. Then \(X_\pi\) is a supersingular elliptic curve with

\[
\End^0(X_\pi) \simeq D_{p,\infty},
\]

where \(D_{p,\infty}\) denotes the unique quaternion \(\mathbb{Q}\)-algebra ramified exactly at \(p\) and \(\infty\). Since each elliptic curve is equipped with the unique canonical principal polarization, we have a canonical bijection \(\PPAV(\pi) \cong \Isog(\pi)\). By \([72, \text{Theorem 4.2}]\), the endomorphism ring \(\End(X_\pi)\) is a maximal order in \(\End^0(X_\pi)\) for every \(X_\pi\), and conversely by \([72, \text{Theorem 3.13}]\), every maximal order in \(D_{p,\infty}\) occurs as an endomorphism ring. A classical result of Deuring \([17]\) and later re-interpreted by Waterhouse \([72, \text{Theorem 4.5}]\) shows that for \(\pi \in \{\pm p^{n/2}\}\) with \(n \in \mathbb{Z}_{>0}\), we have

\[
|\PPAV(\pi)| = |\Isog(\pi)| = h(D_{p,\infty}) = \frac{p - 1}{12} + \frac{1}{4} \left( 1 - \left(\frac{-4}{p}\right) \right) + \frac{1}{3} \left( 1 - \left(\frac{-3}{p}\right) \right).
\]

Here \(h(D_{p,\infty})\) is the class number of \(D_{p,\infty}\), which is first computed by Eichler \([21]\). Igusa \([30]\) also computed \((1.4)\) using another method. In fact, we know more about
PPAV(π) than just its cardinality. Let $t^{pp}(π)$ be the type number of PPAV(π), that is, the number of isomorphism classes of endomorphism rings $\text{End}(X)$ as the isomorphism class $[X, λ]$ ranges over PPAV(π):

$$t^{pp}(π) := \# \{ [\text{End}(X) \mid [X, λ] \in \text{PPAV}(π) \}/\sim \}.$$ (1.5)

By the above discussion, for $π \in \{±p^{n/2}\}$ with $n$ even, $t^{pp}(π)$ coincides with the type number $t(D_{p,∞})$, which was computed by Eichler [21] and Deuring [15] using different methods. If $p = 2, 3$, then $t(D_{p,∞}) = 1$, and

$$2t(D_{p,∞}) = h(D_{p,∞}) + \left[ \frac{1}{2} \left( 1 + \left( \frac{-4}{p} \right) \right) \left( 2 - \left( \frac{2}{p} \right) \right) \right] h(-p) \quad \text{if } p \geq 5.$$ (1.6)

See also [25] (1.10) and (1.11) and (2.5), [29] Remark 3, p. 42 and [85] (1.5) and (1.6).

Next, suppose that $π = ±\sqrt{d} = ±p^{n/2}$ with $n$ odd. It is well known that the isogeny class corresponding to $π$ consists of $F_q$-simple abelian surfaces [65 §1, Examples]. In particular, the two sets PPAV(π) and Isog(π) can no longer be identified with each other. Thanks to [25] Theorem 1, PPAV(π) ≠ ∅. Similar to the previous case as in (1.3), the endomorphism algebra of $X_{π}$ is isomorphic to the unique quaternion $\mathbb{Q}(\sqrt{p})$-algebra $D_{∞,∞}$ ramified exactly at the two infinite places of $\mathbb{Q}(\sqrt{p})$ and splits at all finite places, so we write

$$\text{End}^0(X_{π}) \simeq D_{∞,∞}.$$ (1.7)

However, in the present case, $\text{End}(X_{π})$ is no longer necessarily a maximal order in $\text{End}^0(X_{π})$ by [22] Theorem 6.2, which causes new difficulties. The explicit formula for $|\text{Isog}(\sqrt{p})|$ was previously computed by the present authors together with Tse-Chung Yang in [73, 75], and later extended to a formula for $|\text{Isog}(\sqrt{p})|$ by the present authors in [82] Theorem 4.4. The type number (i.e. the number of endomorphism rings up to isomorphism) of $\text{Isog}(\sqrt{p})$ is calculated in [77]. The primary goal of the current paper is to produce the counterparts of (1.4) and (1.6) for the set PPAV(√p).

For every square-free integer $d ∈ \mathbb{Z}$, let $h(d)$ be the class number of the quadratic field $\mathbb{Q}(\sqrt{d})$. As usual, $(\frac{d}{p})$ denotes the Legendre symbol, and $\zeta_F(s)$ denotes the Dedekind zeta function of the quadratic real field $F := \mathbb{Q}(\sqrt{p})$. See Remark 0.2.3 for methods of computing the special value $\zeta_F(-1)$.

**Theorem 1.1.** (1) $|\text{PPAV}(\sqrt{p})| = 1, 1, 2$ for $p = 2, 3, 5$, respectively.

(2) For $p \geq 13$ and $p = 1 \pmod{4}$,

$$|\text{PPAV}(\sqrt{p})| = \left( 9 - 2 \left( \frac{2}{p} \right) \right) \frac{\zeta_F(-1)}{2} + \frac{3h(-p)}{8} + \left( 3 + \left( \frac{2}{p} \right) \right) \frac{h(-3p)}{6}.$$ (1.8)

(3) For $p \geq 7$ and $p = 3 \pmod{4}$,

$$|\text{PPAV}(\sqrt{p})| = \frac{\zeta_F(-1)}{2} + \left( 11 - 3 \left( \frac{2}{p} \right) \right) \frac{h(-p)}{8} + h(-3p).$$ (1.9)

When $p = 1 \pmod{4}$, the set PPAV(√p) naturally partitions into two subsets $Λ^p_1$ and $Λ^p_1$ according to the endomorphism ring of the underlying abelian surface by Proposition 3.2.5 and we can discuss the class number formulas for $Λ^p_1$ and $Λ^p_1$ individually (see 4.3.1 - 4.3.4). This applies to the type number formulas in Theorem 1.2 below as well.

$^1$The superscript $^{pp}$ stands for “principal polarization”. 

---

**Theorem 1.2.**
The type number of $\text{PPAV}(\sqrt{p})$ is given as follows:

1. $t_{\text{PP}}(\sqrt{p}) = 1, 1, 2$ for $p = 2, 3, 5$, respectively.
2. If $p \equiv 1 \pmod{4}$ and $p \geq 13$, then
   \begin{equation}
   t_{\text{PP}}(\sqrt{p}) = 8\zeta_F(-1) + \frac{h(-p)}{2} + \frac{2h(-3p)}{3}.
   \end{equation}
3. If $p \equiv 3 \pmod{4}$ and $p \geq 7$, then
   \begin{equation}
   t_{\text{PP}}(\sqrt{p}) = \frac{\zeta_F(-1)}{4} + \left(17 - \left(\frac{2}{p}\right)\right) \frac{h(-p)}{16} + \frac{h(-2p)}{8} + \frac{h(-3p)}{12}.
   \end{equation}

Naturally, we would like to generalize Theorems 1.1 and 1.2 to Weil numbers $\pi = \pm\sqrt{q}$, where $q = p^n$ is an arbitrary odd power of $p$. Instead of obtaining a complete formulas for $\text{PPAV}(\sqrt{q})$, we shall restrict ourselves to its subset of superspecial members in this paper. Recall that an abelian variety over a field $k$ of characteristic $p$ is supersingular if it is $\ell$-isogenous to a product of supersingular elliptic curves over an algebraic closure $\bar{k}$ of $k$; it is superspecial [12 §1.7] if it is $\ell$-isomorphic to a product of supersingular elliptic curves over $k$. In the previous papers [72, 76, 81, 85], the authors calculated the number of isomorphism classes of (unpolarized) superspecial abelian surfaces over $\mathbb{F}_q$ for every prime power $q$. By the Manin-Oort Theorem [86 Theorem 2.9], each $X/\mathbb{F}_q$ in the isogeny class corresponding to $\pi = \pm\sqrt{q}$ is supersingular. Moreover, if $q = p$, then each $X$ is superspecial by the proof of [72 Theorem 6.2]. In general, for an odd power $p$ of $q$, let $\text{PPSP}(\sqrt{q})$ be the following subset of $\text{PPAV}(\sqrt{q})$:

$$\text{PPSP}(\sqrt{q}) := \{ [X, \lambda] \in \text{PPAV}(\sqrt{q}) \mid X \text{ is superspecial} \}.$$ 

Similarly, let $\text{Sp}(\sqrt{q}) \subseteq \text{Isog}(\sqrt{q})$ be the subset consisting of the superspecial abelian surfaces. We show in Corollary 3.2.4 that the base change functor $-\otimes_{\mathbb{F}_q, \bar{\mathbb{F}}_q}$ induces canonical identifications

\begin{equation}
\text{Isog}(\sqrt{p}) \sim \text{Sp}(\sqrt{q}) \quad \text{and} \quad \text{PPAV}(\sqrt{p}) \sim \text{PPSP}(\sqrt{q}),
\end{equation}

which preserve the endomorphism rings. In particular, Theorems 1.1 and 1.2 also provide explicit class number and type number formulas for $\text{PPSP}(\sqrt{q})$.

Same as the case of $\text{Isog}(\sqrt{p})$, not every member in $\text{Sp}(\sqrt{q})$ is principally polarizable. Thus, the underlying abelian surfaces of $\text{PPSP}(\sqrt{q})$ form a proper subset of $\text{Sp}(\sqrt{q})$ in general. For the purpose of considering all members of $\text{Sp}(\sqrt{q})$ equipped with a polarization, we make use of polarization modules, which are fundamental invariants appearing in the theory of Hilbert-Blumenthal varieties [16, 61, 83]. We shall recall the notion of $(P, P_\tau)$-polarized abelian varieties in [53] and [66 §X.1] and [24 §2.2.2]. Given a member $X \in \text{Sp}(\sqrt{q})$, we put $R := \mathbb{Q}(\sqrt{p}) \cap \text{End}(X)$, which is an order in $F = \mathbb{Q}(\sqrt{p})$. It is shown in Corollary 5.3 that $X$ is principally polarizable if and only if its polarization module $\mathcal{P}(X)$ represents the trivial element of the narrow class group Pic$(R)$. As a result, our formulas for $|\text{PPAV}(\sqrt{p})|$ may be regarded as special cases of class number formulas for the isomorphic classes of $(P, P_\tau)$-polarized superspecial abelian surfaces in the simple $\mathbb{F}_q$-isogeny class corresponding to $\pi = \sqrt{q}$. Moreover, the same method for computing $|\text{PPAV}(\sqrt{p})|$ applies to these polarized abelian surfaces as well, and the resulting formulas are produced in Theorem 5.9. Surprisingly, these class number formulas coincide with the formulas for the arithmetic genus of certain Hilbert modular surfaces. One example is given as follows.
Example 1.3. Let $O_F$ be the ring of integers of $F = \mathbb{Q}(\sqrt{p})$, and $a \subset F$ be a nonzero fractional $O_F$-ideal. Let $SL(O_F \oplus a)$ be the stabilizer of $O_F \oplus a$ in $SL_2(F)$, and $\Gamma := PSL(O_F \oplus a) = SL(O_F \oplus a)/\{\pm 1\}$. The Hilbert modular surface $Y_\Gamma$ is defined to be the minimal non-singular model of the compactification of $\Gamma \backslash \mathcal{H}^2$ [65, §II.7], where $\mathcal{H} \subset \mathbb{C}$ denotes the Poincare upper half plane as usual. Up to isomorphism, $Y_\Gamma$ depends only on the Gauss genus of $a$ (namely, the coset $Pic_+(O_F)^2[a]_+\in$ Pic $+_+(O_F)$, see (1.20)) and not on the choice of $a$ itself [66, §1.4, p. 12]. Following [64, §III.1, p. 46], the arithmetic genus of $Y_\Gamma$ is defined to be $\sum_{i=0}^2 (-1)^{i}h^i(\mathcal{O}_{Y_\Gamma})$, where $h^i(\mathcal{O}_{Y_\Gamma}) := \dim_{\mathbb{C}} H^i(Y_\Gamma, \mathcal{O}_{Y_\Gamma})$. This is a birational invariant. In particular, we may take $Y_\Gamma$ to be any compact non-singular model of $\Gamma \backslash \mathcal{H}^2$ in the discussion below.

Suppose for the moment that $p \equiv 3 \pmod{4}$, and $a$ belongs to the unique non-principal Gauss genus, i.e. the narrow (strict) ideal class $[a]_+ \in$ Pic $+_+(O_F)$ is not of the form $[b^2]_+$ for any fractional $O_F$-ideal $b$. Comparing Theorem 1.1 with the formula for the arithmetic genus $\chi(Y_\Gamma)$ in [23] Theorems II.5.8–9], we immediately find that

$$\chi(Y_\Gamma) = |PPAV(\sqrt{p})|.$$  

(1.13)

Varying $p$, $a$ and $\Gamma$, we are able to observe several similar identities involving the arithmetic genus of a Hilbert modular surface on one side and the class number of certain kind of $(P, P_+)$-polarized superspecial abelian surfaces on the other side. See Remark 5.10 for the precise identities.

To explain the strategy for computing $|PPAV(\sqrt{p})|$, we draw an analogy between lattices in quadratic spaces and polarized abelian varieties. Let $K$ be a number field, $(V, Q)$ be a finite dimensional nondegenerate quadratic $K$-space, and $G = O(V, Q)$ be the corresponding orthogonal group. Two $O_K$-lattices $L, L'$ in $V$ are isometric if and only if there exists $g \in G(K)$ such that $L' = gL$. Suppose that we want to compute the number $H_{\text{uni}}(V, Q)$ of isometric classes of unimodular (i.e. self-dual) lattices in $V$. One way to do this is to first separate the set of unimodular lattices in $V$ into genera. Following [52, §102], two $O_K$-lattices $L, L'$ in $(V, Q)$ are said to belong to the same genus if for every finite prime $p$ of $K$, the $p$-adic completions $L_p$ and $L'_p$ are isometric. The genus $\mathcal{G}(L)$ is defined to be the set of all $O_K$-lattices $L'$ in $V$ belonging to the same genus as $L$. We write $\Lambda(L) := G(L) \backslash \mathcal{G}(L)$ for the set of isometric classes of lattices within the genus $\mathcal{G}(L)$, and put $h(L) := |\Lambda(L)|$. Let $\hat{K}$ be the ring of finite adeles of $K$. The group $G(\hat{K})$ acts transitively on $\mathcal{G}(L)$, thus inducing a bijection

$$\Lambda(L) \simeq G(K) \backslash G(\hat{K})/U_G(L),$$

(1.14)

where $U_G(L) := \text{Stab}_{G(\hat{K})}(L)$ denotes the stabilizer of $L$ in $G(\hat{K})$. Therefore, $h(L)$ is equal to the class number $h(G, U_G(L)) := |G(K) \backslash G(\hat{K})/U_G(L)|$ of $G = O(V, Q)$ with respect to the open compact subgroup $U_G(L)$ as explained also in [53, Proposition 8.4]. From the local classification of quadratic lattices [52, §§92–93], there are only finitely many genera of unimodular $O_K$-lattices in $(V, Q)$. If we pick a representative $L_i$ from each genus $\mathcal{G}_i$ and write $m$ for the total number of genera of such lattices, then

$$H_{\text{uni}}(V, Q) = \sum_{i=1}^m h(L_i) = \sum_{i=1}^m h(G, U_G(L_i)).$$

(1.15)
For our analogy, the role of a quadratic $O_K$-lattice $(L, Q)$ is played by that of a polarized abelian variety $(X, \lambda)/\mathbb{F}_q$. For each prime $\ell$ (including $\ell = p$), we attach a quasi-polarized $\ell$-divisible group $(X(\ell), \lambda_\ell)$ to $(X, \lambda)$, just like we attach a local lattice $(L_p, Q_p)$ to $(L, Q)$ for each finite prime $p$ of $K$. Equivalently, we can consider the quasi-polarized Tate-module or Dieudonné-module incarnations of the $\ell$-divisible groups, which really are linear objects. The notion of the quadratic space $(V, Q)$ is replaced by that of a $\mathbb{Q}$-isogeny class of polarized abelian varieties.

Given two polarized abelian varieties $(X_1, \lambda_1)$ and $(X_2, \lambda_2)$ over $\mathbb{F}_q$, a $\mathbb{Q}$-isogeny $\varphi : (X_1, \lambda_1) \to (X_2, \lambda_2)$ is an element $\varphi \in \text{Hom}(X_1, X_2) \otimes \mathbb{Q}$ such that $\lambda_1 = \varphi^* \lambda_2 := \varphi^* \circ \lambda_2 \circ \varphi$. Two $\mathbb{Q}$-isogenous polarized abelian varieties $(X_1, \lambda_1)$ and $(X_2, \lambda_2)$ over $\mathbb{F}_q$ are said to belong to the same genus if $(X_1(\ell), \lambda_{1, \ell})$ is isomorphic to $(X_2(\ell), \lambda_{2, \ell})$ for every prime $\ell$. A more in-depth formulation of these concepts will be given in §2. Given a polarized abelian variety $(X, \lambda)/\mathbb{F}_q$, we can attach a linear algebraic $\mathbb{Q}$-group $G^1$ to its $\mathbb{Q}$-isogeny class (see §2), which shall play the role of the orthogonal group $O(V, Q)$. Let $\mathcal{G}(X, \lambda)$ be the genus of $(X, \lambda)$, and $\Lambda(X, \lambda)$ be the isomorphism classes of polarized abelian varieties within $\mathcal{G}(X, \lambda)$. We then have a transitive action of $G^1(\hat{\mathbb{Q}})$ on $\mathcal{G}(X, \lambda)$, which gives rise to a bijection

$$\Lambda(X, \lambda) \simeq G^1(\mathbb{Q}) \backslash G^1(\hat{\mathbb{Q}})/U_{\mathcal{G}^1}(X, \lambda),$$

where $U_{\mathcal{G}^1}(X, \lambda)$ denotes the stabilizer of $(X, \lambda)$ in $G^1(\hat{\mathbb{Q}})$ as usual. The computation of the number of isomorphism classes of polarized abelian varieties within a genus is once again reduced to the computation of the class number of a linear algebraic group with respect to an open compact subgroup.

For the computation of $|\text{PPAV}(\sqrt{p})|$, we put $F := \mathbb{Q}(\sqrt{p})$ as before and write $D$ for the totally definite quaternion $F$-algebra $D_{\infty_1, \infty_2}$ in (1.17). It is straightforward to show that all polarized abelian surfaces $(X, \lambda)/\mathbb{F}_p$ with the Frobenius endomorphism $\pi_X$ satisfying $\pi_X^2 = p$ form a single $\mathbb{Q}$-isogeny class; see Lemma 3.1.3. From §3.1.2, the linear algebraic group $G^1$ attached to this $\mathbb{Q}$-isogeny class is the reduced norm one subgroup of the multiplicative group $D^\times$. After some symplectic local lattice classification that is worked out in §3, we quickly find in Proposition 3.2.20 that $\text{PPAV}(\sqrt{p})$ forms a single genus when $p \not\equiv 1 \pmod{4}$, and it partitions into two different genera $\Lambda_{16}^{pp}$ and $\Lambda_{16}^{pp}$ when $p \equiv 1 \pmod{4}$. For uniformity we also put $\Lambda_{16}^{pp} := PPAV(\sqrt{p})$ when $p \not\equiv 1 \pmod{4}$. Let us pick a representative $[Y_1, \lambda_1] \in \Lambda_{16}^{pp}$ for every prime $p$ and denote $\Omega_1 := \text{End}(Y_1)$. Similarly, if $p \equiv 1 \pmod{4}$, we pick a representative $[Y_{16}, \lambda_{16}] \in \Lambda_{16}^{pp}$ and denote $\Omega_{16} := \text{End}(Y_{16})$. Here the subscripts $r = 1, 16$ are chosen so that $|\Omega : \Omega_r| = r$ for every maximal $O_F$-order $\Omega$ containing $\Omega_r$. Let $\hat{D}$ be the ring of finite adeles of $D$. For any subset $\mathcal{S}$ of $\hat{D}$, we write $\mathcal{S}^\times$ for its subset of elements of reduced norm one, that is, $\mathcal{S}^\times := \{ x \in \mathcal{S} | \text{Nr}(x) = 1 \}$. In particular, $G^1(\mathbb{Q}) = D^1$ and $G^1(\hat{\mathbb{Q}}) = \hat{D}^1$. It turns out that the stabilizer group $U_{\mathcal{G}^1}(Y_r, \lambda_r)$ coincides with $\hat{\Omega}_r^1$, where $\hat{\Omega}_r$ denotes the profinite completion of $\Omega_r$. From (1.16), there is a bijection

$$\Lambda_r^{pp} \simeq D^1 \backslash \hat{D}^1/\hat{\Omega}_r^1, \quad \forall r \in \{1, 16\}.$$  

Denote the cardinality of the double coset space in (1.17) by $h^1(\Omega_r)$. We then have

$$|\text{PPAV}(\sqrt{p})| = \begin{cases} h^1(\Omega_1) & \text{if } p \not\equiv 1 \pmod{4}; \\ h^1(\Omega_1) + h^1(\Omega_{16}) & \text{if } p \equiv 1 \pmod{4}. \end{cases}$$


Several challenges arise when we try to compute the right hand side of \((1.18)\) explicitly. First, we need to produce an order \(\mathcal{O}_r \subset D\) that occurs as \(\text{End}(Y_r)\) for some \([Y_r, \lambda_r] \in A^p\), or at least to describe such an order explicitly enough so that the computation of \(h^1(\mathcal{O}_r)\) is viable. Recall that two \(\mathbb{Z}[^{\sqrt{p}}]\)-orders \(\mathcal{O}, \mathcal{O}'\) in \(D\) are said to \textit{belong to the same genus} if there exists \(x \in \hat{D}^\times\) such that \(\hat{\mathcal{O}} = x\hat{\mathcal{O}}x^{-1}\), or equivalently, if \(\mathcal{O}_\ell \simeq \mathcal{O}'_\ell\) for every prime \(\ell \in \mathbb{N}\). From Tate’s theorem, we can determine explicitly the genera of \(\mathcal{O}_1\) and \(\mathcal{O}_{16}\), that is, the local descriptions of \(\mathcal{O}_r \otimes \mathbb{Z}_\ell\) for all \(r \in \{1, 16\}\) and all prime \(\ell\); see \([11]\) For example, this precisely the reason why we known that \(\mathcal{O}_1\) is always a maximal \(O_F\)-order in \(D\). Conversely from \([72]\) Theorem 3.13, given a maximal \(O_F\)-order \(\mathcal{O} \subset D\), there exists \([X] \in \text{Isog}(\sqrt{p})\) such that \(\text{End}(X) \simeq \mathcal{O}\). However, there is no guarantee in general that we can find such an \(X\) that is principally polarizable. Thus, we need to find a finer classification of orders within a fixed genus of \(D\). To solve this problem we make use of the notion of \textit{spinor genus of orders}. From Definition \([14]\) two \(\mathbb{Z}[^{\sqrt{p}}]\)-orders \(\mathcal{O}, \mathcal{O}'\) in \(D\) are said to \textit{belong to the same spinor genus} if there exists \(x \in D^\times\hat{D}^1\) such that \(\hat{\mathcal{O}} = x\hat{\mathcal{O}}x^{-1}\). Clearly this defines an equivalence relation finer than “being in the same genus”. The key Lemma \([15]\) shows that if \(\mathcal{O}\) and \(\mathcal{O}'\) are two orders in \(D\) belonging to the same spinor genus and \(\mathcal{O}\) occurs as the endomorphism ring of some principally polarizable member of \(\text{Isog}(\sqrt{p})\), then so does \(\mathcal{O}'\). We then determine the spinor genera of endomorphism rings of principally polarizable members of \(\text{Isog}(\sqrt{p})\) in Proposition \([4.6]\) and Lemma \([4.10]\) which is equivalent to finding out the desired \(\mathcal{O}_r\).

Now comes the real challenge of actually computing \(h^1(\mathcal{O}_r)\). Since \(D\) is totally definite, \(G^1(\mathbb{R})\) is compact, and hence strong approximation fails for \(G^1\). For a long time, the only systematic way to compute \(h^1(\mathcal{O}_r)\) is via the Selberg trace formula, which requires heavy analysis of orbital integrals and can be unwieldy to apply. The situation is quite different if the group \(G^1 = D^1\) is replaced by the multiplicative group \(D^\times\). More generally, let \(\mathcal{F}\) be an arbitrary totally real number field, \(\mathcal{D}\) be a totally definite quaternion \(\mathcal{F}\)-algebra, and \(\mathcal{O}\) be a \(O_\mathcal{F}\)-order in \(\mathcal{D}\). By definition, the class number of the multiplicative group \(\mathcal{D}^\times\) with respect to \(\hat{\mathcal{O}}^\times\) is given by

\[
h(\mathcal{O}) := h(\mathcal{D}^\times, \hat{\mathcal{O}}^\times) = |\mathcal{D}^\times/\hat{\mathcal{D}}^\times|.
\]

The class number \(h(\mathcal{O})\) can be calculated by the Eichler class number formula \([69]\) Corollaire V.2.5, p. 144] [70] Theorem 30.8.6 thanks to the work of Eichler \([22]\), Vigneras \([69]\), Körner \([34]\) and others. A key result underlying this formula is the trace formula \([69]\) Theorem III.5.1] for optimal embeddings from a quadratic \(O_\mathcal{F}\)-order \(B\) into orders in the genus of \(\mathcal{O}\). Eichler’s formula has also been generalized by the current authors together with Tse-Chung Yang to an arbitrary \(\mathbb{Z}\)-order of full rank in \(\mathcal{D}\), and this generalization plays a crucial role for the computation of \(|\text{Isog}(\sqrt{p})|\) in \([73]\). Therefore, it is highly desirable to have a class number formula for \(h^1(\mathcal{O}) = h(\mathcal{D}^1, \hat{\mathcal{O}}^1)\) of a \textit{similar shape} to the Eichler class number formula. The advantage of this approach avoids complicated computations of orbital integrals in the Selberg trace formula. As explained in the previous paragraph, the class number \(h^1(\mathcal{O})\) depends only on the spinor genus of \(\mathcal{O}\). Thus to obtain an Eichler kind class number formula for \(h^1(\mathcal{O})\), it is essential to figure out how the spinor genus information (which boasts a mixture of local and global flavor) would manifest itself in the desired formula. It is at this critical juncture where the spionor selectivity theory comes in.
The selectivity theory is first formulated by Chinburg and Friedman \cite{11} as an refined integral version of the Hasse-Brauer-Noether-Albert Theorem \cite{69} Theorem III.3.8, and has since been further developed by many people. See \cite{70} §31.7.7 for a historical account. We give a brief summary of this theory in §3.1. Except for \cite{5} and \cite{31} Remark, p. 99, most of the literature on selectivity theory focus on quaternion algebras that satisfies the Eichler condition (i.e. the case where strong approximation holds). For the current purpose, the current authors developed the selectivity theory for totally definite quaternion algebras in more depth and obtained in \cite{80} an refinement for the trace formula for optimal embeddings, called the spinor trace formula. In the sequel work \cite{79} based on the Selberg trace formula and spinor trace formula we deduced a formula for $h^1(O)$ of Eichler type for all Eichler orders in totally definite quaternion algebras; see \cite{79} Corollary 4.2 and \cite{6.10}. The proof of this formula has greatly simplified by Voight \cite{79} Appendix A. Using this formula we compute the explicit formula for $h^1(\Omega_1)$.

Note that the general formula for $h^1(O)$ developed in \cite{79} can only be applied to the maximal order $\Omega_1$, which alone is not enough to compute $[PPAV(\sqrt{p})]$ when $p \equiv 1 \pmod 4$. Indeed, in the latter case, $\Omega_{16} \cap F = \mathbb{Z}[\sqrt{p}] \neq O_F$, so $\Omega_{16}$ is not an $O_F$-order. This calls for some ad-hoc methods. Luckily, we are able to show in §7 that $h^1(\Omega_{16}) = h(\Omega_{16})/h(\mathbb{Z}[\sqrt{p}])$, where $h(\Omega_{16})$ has been previously computed in \cite{77}. The establishment of this equality relies on the fact that $h(\mathbb{Q}(\sqrt{p}))$ is odd by \cite{13} Corollary 18.4. This completes a summary of the computation of $[PPAV(\sqrt{p})]$. The type number formulas are obtained along the way using class-type number relations; see Lemma \cite{6.1} for example. A survey of current results of this paper is provided in \cite{78}.

This paper is organized as follows. In §2 we explain our strategy for computing $[PPAV(\pi)]$, which is a simpler version of the Langlands-Kottwitz method. This strategy is divided into three steps as described in §2.10. The first two steps are carried out in detail for $\pi = \pm \sqrt{p}$ in §3 and the third step in §4 except that some class number calculations are postponed to §5.4 and certain symplectic lattice classification is postponed to §8. In §5 we recall the notion of polarization modules, and extend the results of §3.4 to produce both class number and type number formulas for $(P, P_\pi)$-polarized abelian surfaces in the simple $\mathbb{F}_p$-isogeny class corresponding to $\pi = \pm \sqrt{p}$.

Notation. As usual, $\mathbb{Z}_\ell$ denotes the $\ell$-adic completion of $\mathbb{Z}$ at the prime $\ell$. Let $\hat{\mathbb{Z}} = \prod \mathbb{Z}_\ell$ be the pro-finite completion of $\mathbb{Z}$, and $\hat{Q}$ the finite adele ring of $Q$. If $M$ is a finitely generated $\mathbb{Z}$-module or a finite dimensional $Q$-vector space, we put $M_\ell := M \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ and $\hat{M} := M \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. The unique quaternion $Q$-algebra ramified exactly at $p$ and $\infty$ is denoted by $D_{p, \infty}$. Similarly, the unique quaternion $Q(\sqrt{p})$-algebra ramified exactly at the two infinite places is denoted by $D_{\infty_1, \infty_2}$. For simplicity, we often put $D := D_{\infty_1, \infty_2}$. If $B$ is a finite-dimensional $Q$-algebra, we denote by $B^\times$ the algebraic $Q$-group that represents the functor $R \mapsto (B \otimes_Q R)^\times$ for any commutative $Q$-algebra $R$. The group $B^\times$ is called the multiplicative algebraic $Q$-group of $B$. If $K = \prod K_i$ is a finite product of number fields or non-archimedean local fields, then we denote by $O_K$ the maximal order of $K$. For any abelian variety $X$ defined over a finite field $\mathbb{F}_q$, we denote by $\pi_X$ the Frobenius endomorphism of $X$ over $\mathbb{F}_q$. 
2. Method of calculation

Given a Weil $g$-number $\pi$, the study of $\text{Isog}(\pi)$ in terms of ideal classes of certain orders goes back to Waterhouse \[72\]. The general method for counting abelian varieties equipped with a PEL-type structure in an isogeny class is developed by Langlands \[89\] and Kottwitz \[85,59\] for studying the Hasse-Weil zeta functions of Shimura varieties. Employing this method, Lipnowski and Tsimerman \[44\] give asymptotic upper bounds for the number of isomorphism classes of $g$-dimensional PPAVs over $\mathbb{F}_p$ for a fixed $p$ as $g \to \infty$. Nontrivial lower bounds is obtained by Jungin Lee \[40\] in a follow-up work. Recently, results for counting $\text{PPAV}(\pi)$ with $\pi$ being a Weil $p$-number distinct from $\pm \sqrt{p}$ are obtained by Achter et al. \[1\] in terms of a product of local densities and the Tamagawa number of a CM torus. Bergström, Karembeaker and Marseglia \[6\] obtained an algorithmic breakthrough on counting principally polarized abelian varieties with commutative endomorphism algebras over finite fields of small cardinalities. A similar study was previously carried out by Marseglia in \[57\]. In a slight different direction, Oswal and Shankar \[55\] obtained explicit descriptions of almost ordinary abelian varieties over finite fields of odd characteristic that can be applied to counting problems.

For the purpose of this paper, we follow a variant of this method in \[82\], which is previously developed by the second named author in \[54\]. This method treats both the unpolarized case and the principally polarized case uniformly and expresses the cardinalities as sums of class numbers of certain linear algebraic $\mathbb{Q}$-groups. As the method is built upon Tate’s theorem (due to Tate, Zarhin, Faltings and de Jong), its key part is applicable to any finitely generated ground field $k$ (that is, finitely generated over its prime subfield).

Given an abelian variety $X$ over $k$ and a prime number $\ell$ (not necessarily distinct from the characteristic of $k$), we write $X(\ell)$ for the $\ell$-divisible group $\varinjlim X[\ell^n]$ attached to $X$. A $\mathbb{Q}$-isogeny $\varphi : X_1 \to X_2$ between two abelian varieties over $k$ is an element $\varphi \in \text{Hom}(X_1, X_2) \otimes \mathbb{Q}$ such that $N\varphi$ is an isogeny for some $N \in \mathbb{N}$. Similarly, one defines the notion of $\mathbb{Q}_\ell$-isogenies between $\ell$-divisible groups. Clearly, a $\mathbb{Q}$-isogeny $\varphi$ induces a $\mathbb{Q}_\ell$-isogeny $\varphi_\ell : X_1(\ell) \to X_2(\ell)$ for each prime $\ell$, and $\varphi_\ell$ is an isomorphism for almost all $\ell$.

Fix an abelian variety $X_0$ over $k$. Two $\mathbb{Q}$-isogenies $\varphi_1 : X_1 \to X_0$ and $\varphi_2 : X_2 \to X_0$ are said to be equivalent if there exists an isomorphism $\theta : X_1 \to X_2$ such that $\varphi_2 \circ \theta = \varphi_1$. Let $\text{Qisog}(X_0)$ be the set of equivalence classes of $\mathbb{Q}$-isogenies $(X, \varphi)$ to $X_0$. By an abuse of notation, we still write $(X, \varphi)$ for its equivalence class. The endomorphism ring $\text{End}(X)$ is realized as a canonical suborder of $\text{End}^0(X_0)$ via the isomorphism

$$\text{End}^0(X) \cong \text{End}^0(X_0), \quad a \mapsto \varphi a \varphi^{-1}.$$  

The set $\text{Qisog}(X_0)$ contains a distinguished element $(X_0, \text{id}_0)$, where $\text{id}_0$ is the identity map of $X_0$. For any $\mathbb{Q}$-isogeny $\varphi_1 : X_1 \to X_0$, there is a bijection

$$\text{Qisog}(X_0) \to \text{Qisog}(X_1), \quad (X, \varphi) \mapsto (X, \varphi_1^{-1} \varphi).$$

Therefore, we may change the base abelian variety $X_0$ to suit our purpose whenever necessary. Similarly, one defines $\text{Qisog}(X_0(\ell))$ for every prime $\ell$.

Let $G$ be the algebraic $\mathbb{Q}$-group representing the functor

$$R \mapsto G(R) := (\text{End}^0(X_0) \otimes \mathbb{Q})^\times$$
for every commutative \( \mathbb{Q} \)-algebra \( R \). Up to isomorphism, \( G \) depends only on the isogeny class of \( X_0 \). We have \( G(\mathbb{Q}_\ell) = (\text{End}(X_0(\ell)) \otimes \mathbb{Z}_\ell)^\times \) by Tate’s theorem.

Let \( \hat{\mathbb{Q}} := \mathbb{Z} \otimes \mathbb{Q} \) be the ring of finite adeles of \( \mathbb{Q} \). From [82, Lemma 5.2], there is an action of \( G(\hat{\mathbb{Q}}) \) on \( \text{Qisog}(X_0) \) as follows: for any \( (X, \varphi) \in \text{Qisog}(X_0) \) and any \( \alpha = (\alpha_\ell) \in G(\hat{\mathbb{Q}}) \), the member \( (X', \varphi') := \alpha(X, \varphi) \) is uniquely characterized by the following equality in \( \text{Qisog}(X_0(\ell)) \) for every prime \( \ell \):

\[
(2.4) \quad (X'(\ell), \varphi'_\ell) = (X(\ell), \alpha_\ell \varphi_\ell).
\]

In particular, if \( \alpha \in G(\mathbb{Q}) \), then \( \alpha(X, \varphi) = (X, \alpha \varphi) \). From Tate’s theorem and the identification [241], we obtain

\[
(2.5) \quad \text{End}(X') \otimes \hat{\mathbb{Z}} = \alpha(\text{End}(X) \otimes \hat{\mathbb{Z}})\alpha^{-1}.
\]

If \( \text{Qisog}(X_0) \) is equipped with the discrete topology, then the action of \( G(\hat{\mathbb{Q}}) \) on \( \text{Qisog}(X_0) \) is continuous and proper. Indeed, the stabilizer of any \( (X, \varphi) \in \text{Qisog}(X_0) \) coincides with the open compact subgroup \( (\text{End}(X) \otimes \hat{\mathbb{Z}})^\times \subset G(\hat{\mathbb{Q}}) \).

**Definition 2.1.** Let \( H \subseteq G \) be an algebraic subgroup of \( G \) over \( \mathbb{Q} \). Two members \( (X_i, \varphi_i) \in \text{Qisog}(X_0) \) for \( i = 1, 2 \) are said to be in the same \( H \)-genus if there exists \( \alpha \in H(\hat{\mathbb{Q}}) \) such that \( (X_2, \varphi_2) = \alpha(X_1, \varphi_1) \). They are said to be \( H \)-isomorphic if there exists \( \alpha \in H(\mathbb{Q}) \) such that \( (X_2, \varphi_2) = (X_1, \alpha \varphi_1) \).

Thus \( \text{Qisog}(X_0) \) is partitioned into \( H \)-genera, and each \( H \)-genus is further subdivided into \( H \)-isomorphism classes. In particular, it makes sense to talk about two \( H \)-isomorphism classes \( [X_1, \varphi_1]_{i=1,2} \) belonging to the same \( H \)-genus.

**Proposition 2.2.** Let \( \mathcal{G}_H(X, \varphi) \subseteq \text{Qisog}(X_0) \) be the \( H \)-genus containing \( (X, \varphi) \), and \( \Lambda_H(X, \varphi) \) be the set of \( H \)-isomorphism classes within \( \mathcal{G}_H(X, \varphi) \). Let \( U_H(X, \varphi) := \text{Stab}_{H(\hat{\mathbb{Q}})}(X, \varphi) \) be the stabilizer of \( (X, \varphi) \) in \( H(\hat{\mathbb{Q}}) \). Then there is a bijection

\[
\Lambda_H(X, \varphi) \cong H(\mathbb{Q}) \backslash H(\hat{\mathbb{Q}})/U_H(X, \varphi),
\]

sending the \( H \)-isomorphic class \([X, \varphi]\) to the identity class on the right hand side.

The proposition follows directly from definition. From [58 Theorem 8.1], \( \Lambda_H(X, \varphi) \) is a finite set. Proposition 2.2 turns out to be quite versatile. By varying \( H \), it can be used to count abelian varieties with various additional structures. We give two examples below, one for counting unpolarized abelian varieties, and another for counting principally polarized ones. See 3 for counting \((P, P_+)-\)polarized abelian surfaces with real multiplication.

**2.3.** First, let us look at the case \( H = G \). Two members \( (X_i, \varphi_i)_{i=1,2} \in \text{Qisog}(X_0) \) are in the same \( G \)-genus if and only if \( X_1(\ell) \) is isomorphic to \( X_2(\ell) \) for every prime \( \ell \). This matches with the classical notion of genus for unpolarized abelian varieties in an isogeny class, cf. [82 Definition 5.1]. Similarly, \( (X_1, \varphi_1) \) and \( (X_2, \varphi_2) \) are \( G \)-isomorphic if and only if \( X_1 \) and \( X_2 \) are \( k \)-isomorphic abelian varieties. Therefore, Proposition 2.2 recovers [82 Proposition 5.4] in the case \( H = G \).

**2.4.** Next, we look at polarized abelian varieties. Let \( X^t \) be the dual abelian variety of \( X \). A \( \mathbb{Q} \)-isogeny \( \lambda : X \to X^t \) is said to be a \( \mathbb{Q} \)-polarization if \( N \lambda \) is a polarization for some \( N \in \mathbb{N} \). For each prime \( \ell \), the \( \mathbb{Q} \)-polarization \( \lambda \) induces a \( \mathbb{Q}_\ell \)-quasipolarization of \( X(\ell) \) (see [54 §1] and [42 §5.9]). An isomorphism
(resp. $\mathbb{Q}$-isogeny) from a $\mathbb{Q}$-polarized abelian variety $(X_1, \lambda_1)$ to another $(X_2, \lambda_2)$ is an isomorphism (resp. $\mathbb{Q}$-isogeny) $\varphi : X_1 \to X_2$ such that

$$\lambda_1 = \varphi^* \lambda_2 \coloneqq \varphi^t \circ \lambda_2 \circ \varphi.$$  

Fix a $\mathbb{Q}$-polarized abelian variety $(X_0, \lambda_0)$. Two $\mathbb{Q}$-isogenies $\varphi_i : (X_i, \lambda_i) \to (X_0, \lambda_0)$ for $i = 1, 2$ are said to be equivalent if there exists an isomorphism $\theta : (X_1, \lambda_1) \to (X_2, \lambda_2)$ such that $\varphi_1 = \varphi_2 \circ \theta$. We define $\text{Qisog}(X_0, \lambda_0)$ to be the set of equivalence classes of all $\mathbb{Q}$-isogenies $(X, \lambda, \varphi)$ to $(X_0, \lambda_0)$. The forgetful map $(X, \lambda, \varphi) \mapsto (X, \varphi)$ induces a bijection:

$$\mathcal{F}(\lambda_0) : \text{Qisog}(X_0, \lambda_0) \to \text{Qisog}(X_0),$$

whose inverse is given by $(X, \varphi) \mapsto (X, \varphi^* \lambda_0, \varphi)$.

Let $G^1 \subseteq G$ be the algebraic subgroup over $\mathbb{Q}$ that represents the functor

$$R \mapsto G^1(R) := \{ g \in (\text{End}^0(X_0) \otimes \mathbb{Q})^\times \mid g^t \circ \lambda_0 \circ g = \lambda_0 \}$$

for every commutative $\mathbb{Q}$-algebra $R$. In light of the bijection (2.7), two members $(X_i, \varphi_i)_{i=1,2} \in \text{Qisog}(X_0)$ are in the same $G^1$-genus if and only if $(X_1(\ell), \lambda_{1,\ell})$ is isomorphic to $(X_2(\ell), \lambda_{2,\ell})$ for every prime $\ell$. Once again, this recovers the classical notion of “being in the same genus” for $\mathbb{Q}$-polarized abelian varieties in an isogeny class. Similarly, $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ are $G^1$-isomorphic if and only if $(X_1, \lambda_1)$ and $(X_2, \lambda_2)$ are isomorphic $\mathbb{Q}$-polarized abelian varieties over $k$. Therefore, when $H = G^1$, Proposition 2.2 is a special case of [82] Theorem 5.10.

In practice, we are more interested in abelian varieties with integral polarizations than $\mathbb{Q}$-polarizations.

**Lemma 2.5 ([82] Remark 5.7).** Let $\mathcal{G}(X, \lambda, \varphi) \subseteq \text{Qisog}(X_0, \lambda_0)$ be the genus containing $(X, \lambda, \varphi)$. If $\lambda$ is an integral polarization on $X$, then $\lambda'$ is integral for every member $(X', \lambda', \varphi') \in \mathcal{G}(X, \lambda, \varphi)$. If moreover $\lambda$ is principal, then so is $\lambda'$.

2.6. For calculation purpose, it is more convenient to describe genera of $\mathbb{Q}$-polarized abelian varieties in terms of Tate modules and Dieudonné modules rather than $\ell$-divisible groups. Assume that $k$ is the finite field $\mathbb{F}_q$, and $\pi$ is an arbitrary Weil $q$-number. Let

$$F = \mathbb{Q}(\pi), \quad O_F = \text{the ring of integers of } F, \quad A = \mathbb{Z}[\pi] \subseteq O_F.$$

If $F$ is a CM-field, then we write $a \mapsto \bar{a}$ for the complex conjugation map on $F$; otherwise $\pi = \pm \sqrt{q}$ so that $F$ is either $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{p})$, and we put $\bar{a} = a$ for every $a \in F$. Let $X_0/\mathbb{F}_q$ be an abelian variety in the simple $\mathbb{F}_q$-isogeny class corresponding to $\pi$, and $\lambda_0 : X_0 \to X_0^\dagger$ be a $\mathbb{Q}$-polarization. At each prime $\ell \neq p$, the Tate space $V_\ell := T_\ell(X_0) \otimes \mathbb{Q}_\ell$ with its $\text{Gal}(\mathbb{F}_q/\mathbb{F}_q)$-module structure is simply a free $F_\ell$-module of rank $2 \dim(X_0)/[F : \mathbb{Q}]$, and $\lambda_0$ induces a non-degenerate alternating $Q_\ell$-bilinear Weil pairing

$$\psi_\ell : V_\ell \times V_\ell \to \mathbb{Q}_\ell \quad \text{such that} \quad \psi_\ell(ax, y) = \psi_\ell(x, \bar{a}y)$$

for all $a \in F_\ell$ and $x, y \in V_\ell$. A $\mathbb{Q}$-isogeny $\varphi : (X, \lambda) \to (X_0, \lambda_0)$ identifies $T_\ell(X)$ with the $A_\ell$-lattice $\varphi_\ell(T_\ell(X))$ in $(V_\ell, \psi_\ell)$. If $F$ is totally real, then there exists a unique non-degenerate alternating $F_\ell$-bilinear form $\psi_{\ell, F} : V_\ell \times V_\ell \to F_\ell$ such that $\psi_{\ell} = \text{Tr}_{F/\mathbb{Q}} \circ \psi_{\ell, F}$. If $F$ is a CM-field, then after fixing an element $\gamma \in F_\ell^\times$ with $\bar{\gamma} = -\gamma$, there exists a unique non-degenerate hermitian form $\psi_{\ell, F} : V_\ell \times V_\ell \to F_\ell$ such that $\psi_\ell(x, y) = \text{Tr}_{F/\mathbb{Q}}(\gamma \psi_{\ell, F}(x, y))$ for all $x, y \in V_\ell$. 


Similarly, let $M(X_0)$ be the covariant Dieudonné module of $X_0$, which is a free module of rank $2 \dim(X_0)$ over the ring of Witt vectors $W(F_q)$. For simplicity, put
\begin{equation}
Z_q := W(F_q), \quad Q_q := Z_q \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,
\end{equation}
and let $\sigma \in \text{Gal}({\mathbb{Q}_q/\mathbb{Q}_p})$ be the Frobenius automorphism as usual. The Frobenius operator $F$ (resp. Verschiebung operator $V$) acts $(Z_q, \sigma)$-linearly (resp. $(Z_q, \sigma^{-1})$-linearly) on $M(X_0)$, and $\pi$ acts $Z_q$-linearly on $M(X_0)$ via $F^n$. From [42 §5.9], the $\mathbb{Q}$-polarization $\lambda_0$ induces a non-degenerate alternating $\mathbb{Q}_q$-bilinear form $\psi_p$ on the $F$-isocrystal $V_p := M(X_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ satisfying
\begin{equation}
\psi_p(Fx, y) = \psi_p(x, Vy)^\sigma, \quad \forall x, y \in V_p.
\end{equation}
A $\mathbb{Q}$-isogeny $\varphi : (X, \lambda) \to (X_0, \lambda_0)$ again identifies $M(X)$ with the Dieudonné lattice $\varphi_p(M(X))$ in $(V_p, \psi_p)$.

From the factorial equivalence between $\ell$-divisible groups and Dieudonné modules (when $\ell = p$) or Tate modules (when $\ell \neq p$), we see that two members $(X_i, \lambda_i, \varphi_i)_{i=1,2} \in \text{Qisog}(X_0, \lambda_0)$ belong to the same genus if and only if both of the following conditions are satisfied:
\begin{enumerate}
\item at each prime $\ell \neq p$, the two $A_\ell$-lattices $T_\ell(X_i)$ are isometric in $(V_\ell, \psi_\ell)$;
\item at the prime $p$, the two Dieudonné modules $M(X_i)$ are isometric in $(V_p, \psi_p)$.
\end{enumerate}

2.7. Let $\text{Qisog}^{pp}(X_0, \lambda_0)$ be the subset of $\text{Qisog}(X_0, \lambda_0)$ consisting of the principally polarized members. A member $(X, \lambda, \varphi) \in \text{Qisog}(X_0, \lambda_0)$ belongs to $\text{Qisog}^{pp}(X_0, \lambda_0)$ if and only if both of the following conditions are satisfied:
\begin{enumerate}
\item the Dieudonné module $M(X)$ is self-dual in $(V_p, \psi_p)$;
\item the Tate module $T_\ell(X)$ is self-dual in $(V_\ell, \psi_\ell)$ for each $\ell \neq p$.
\end{enumerate}
Let $\mathcal{M}_p$ be the set of isometric classes of self-dual Dieudonné modules in $(V_p, \psi_p)$, and for each prime $\ell \neq p$, let $\mathcal{M}_\ell$ be the set of isometric classes of self-dual $A_\ell$-lattices in $(V_\ell, \psi_\ell)$. Put $\mathcal{M} := \mathcal{M}_p \times \prod_{\ell \neq p} \mathcal{M}_\ell$. There is a canonical map
\begin{equation}
\Phi : \text{Qisog}^{pp}(X_0, \lambda_0) \to \mathcal{M}, \quad (X, \lambda, \varphi) \mapsto ([M(X)], ([T_\ell(X)])_{\ell \neq p}),
\end{equation}
whose fibers are precisely the genera of principally polarized members of $\text{Qisog}(X_0, \lambda_0)$. Let $\mathfrak{d}_A$ be the discriminant of $A$ over $\mathbb{Z}$, and $\mathcal{S}_\ell$ be the following finite set of primes
\begin{equation}
\mathcal{S}_\ell := \{ \ell \mid \ell \text{ divides } \mathfrak{d}_A \} \cup \{ p \}.
\end{equation}
For a prime $\ell \neq p$, we have $\ell \not\in \mathcal{S}_\ell$ if and only if $\ell$ is unramified in $F$ and $A_\ell$ is the maximal order in $F_{\ell}$. We claim that $|\mathcal{M}_\ell| \leq 1$ for every prime $\ell \not\in \mathcal{S}_\ell$. If $\pi = \pm \sqrt{q}$, then $|\mathcal{M}_\ell| = 1$ according to Lemma 8.1; otherwise $F = \mathbb{Q}(\pi)$ is a CM-field, and it follows from [31] §7 that $|\mathcal{M}_\ell| \leq 1$. Therefore, if $\mathcal{M} \neq \emptyset$, then it is bijective to the finite product $\prod_{\ell \in \mathcal{S}_\ell} \mathcal{M}_\ell$.

Lemma 2.8. The map $\Phi$ is surjective if $\mathcal{M} \neq \emptyset$.

Proof. For all but finitely many $\ell \neq p$, the $A_\ell$-lattice $T_\ell(X_0)$ is self-dual in $(V_\ell, \psi_\ell)$. We collect the exceptional $\ell$'s into a finite set $S_0$, which includes $p$ by default. Suppose that $\mathcal{M} \neq \emptyset$. For any member $m \in \mathcal{M}$, let $(M, (T_\ell)_{\ell \neq p})$ be a representative of $m$, and put $S = (S_0 \cup \mathcal{S}_\ell) \setminus \{ p \}$. The finite product of local lattices $(M, (T_\ell)_{\ell \in S}) \subset V_p \times \prod_{\ell \in S} V_\ell$ determines a unique member $(X, \lambda, \varphi) \in \text{Qisog}^{pp}(X_0, \lambda_0)$, which is mapped to $m \in \mathcal{M}$ by $\Phi$. \qed

Corollary 2.9. The genera within $\text{Qisog}^{pp}(X_0, \lambda_0)$ is bijective to the set $\mathcal{M}$.\]
2.10. Thus in principle, to compute \(|PPAV(\pi)|\) for a Weil \(q\)-number \(\pi\), we take the following three steps:

(Step 1) Determine if \(PPAV(\pi) = \emptyset\) or not. If it is nonempty, then separate \(PPAV(\pi)\) into \(\mathbb{Q}\)-isogeny classes. Each such isogeny class determines an algebraic \(\mathbb{Q}\)-group \(G^1\) and a collection of symplectic spaces \((V_\ell, \psi_\ell)\) as in (2.10) and (2.12) indexed by \(\ell \in S_\pi\).

(Step 2) For each \(\mathbb{Q}\)-isogeny class in \(PPAV(\pi)\), separate it further into genera. This amounts to classifying the isometric classes of the following objects:

(a) self-dual Dieudonné modules in \((V_p, \psi_p)\);
(b) self-dual \(A_\ell\)-modules in \((V_\ell, \psi_\ell)\) for every \(\ell \in (S_\pi \setminus \{p\})\).

(Step 3) The cardinality of the genus in \(PPAV(\pi)\) represented by a member \([X, \lambda]\) is equal to the class number

\[
|G^1(\mathbb{Q})\backslash G^1(\hat{\mathbb{Q}})/U_{G^1}(X)|.
\]

Varying \([X, \lambda]\) genus by genus, we obtain \(|PPAV(\pi)|\) by summing up all such class numbers.

However, in this fullest generality, it is nontrivial to even determine if \(PPAV(\pi) = \emptyset\) or not (cf. [27, 28]), let alone separating it into \(\mathbb{Q}\)-isogeny classes. Nevertheless, neither of these problems pose any difficulty when \(\pi = \sqrt{q}\) for an odd power \(q\) of \(p\), as we shall see in \(|3.1.1|\) and Lemma \(|3.1.3|\) in the next section.

3. The classification of \(PPAV(\sqrt{p})\) into genera

We carry out the first two steps of our strategy as described in (2.10) for the Weil \(p\)-number \(\pi = \sqrt{p}\) and classify \(PPAV(\sqrt{p})\) into genera. Certain symplectic lattice classification in Step 2 will be postponed to \(|8|\).

3.1. The first step. For the first step, there is no need to restrict to the prime base field case yet. Assume that \(q = p^n\) is an odd power of \(p\). Every abelian variety \(X/F_q\) in the simple \(F_q\)-isogeny class corresponding to the Weil \(q\)-number \(\pi = \sqrt{q}\) is a supersingular abelian surface. From (1.7), we have \(\text{End}^0(X) \simeq D_{\infty_1, \infty_2}\). The endomorphism ring \(\text{End}(X)\) is a \(\mathbb{Z}[\sqrt{q}]\)-order in \(D_{\infty_1, \infty_2}\) uniquely determined by \(X\) up to an inner automorphism. For simplicity, we put

\[
D := D_{\infty_1, \infty_2}, \quad F := \mathbb{Q}(\sqrt{p}) \quad \text{and} \quad A := \mathbb{Z}[\sqrt{q}].
\]

3.1.1. Let \(E/F_{q^2}\) be an elliptic curve with Frobenius endomorphism \(\pi_E = q\), and \(\lambda_E : E \to E^t\) be the canonical principal polarization of \(E\). Take \(X := \text{Res}_{F_{q^2}/F_q}(E)\), the Weil restriction of \(E\) with respect to \(F_{q^2}/F_q\). Then \(X\) is a superspecial abelian surface with \(\pi_X = q\), and the Weil restriction \(\lambda_X := \text{Res}_{F_{q^2}/F_q}(\lambda_E)\) is a principal polarization on \(X\). Thus, if we write \(PPSP(\sqrt{q})\) for the subset of \(PPAV(\sqrt{q})\) consisting of the members \([X, \lambda]\) with \(X\) being superspecial, then \(PPSP(\sqrt{q}) \neq \emptyset\).

3.1.2. Let \(X/F_q\) be an arbitrary abelian surface with \(\pi_X = q\), and \(\lambda : X \to X^t\) be a \(\mathbb{Q}\)-polarization on \(X\). Since \(D = \text{End}^0(X)\) is totally definite, the Rosati involution on \(\text{End}^0(X)\) induced by \(\lambda\) coincides with the canonical involution \(\alpha \mapsto \bar{\alpha} := \text{Tr}(\alpha) - \alpha\) according to Albert’s classification [51 Theorem 2, §21]. The group \(G^1\) as defined in (2.8) is just the reduced norm one subgroup of \(G = D_X^\times\). Let \(\mathcal{P}(X)\)

\[\footnote{For every prime \(\ell \notin S_\pi\), the space \((V_\ell, \psi_\ell)\) is uniquely determined up to isometry by \(\pi\) itself since it admits a self-dual \(A_\ell\)-lattice.}\]
be the Néron-Severi group of $X$, $\mathcal{P}^0(X) := \mathcal{P}(X) \otimes \mathbb{Q}$, and $\mathcal{P}^0_+(X) \subseteq \mathcal{P}^0(X)$ be the subset consisting of the $\mathbb{Q}$-polarizations of $X$. From [51 §21, Application III], the map $\lambda' \mapsto \lambda^{-1}\lambda'$ with $\lambda'$ ranging in $\mathcal{P}^0(X)$ induces an identification
\begin{equation}
\varrho_\lambda : \mathcal{P}^0(X) \simeq F \quad \text{such that} \quad \varrho_\lambda(\mathcal{P}^0_+(X)) = F^+_x.
\end{equation}
Here $F^+_x$ denotes the subset of totally positive elements of $F^x$.

**Lemma 3.1.3.** For any two $\mathbb{Q}$-polarized abelian surfaces $(X, \lambda)$ and $(X', \lambda')$ over $\mathbb{F}_q$ with $\pi_X^2 = \pi_{X'}^2 = q$, there exists a $\mathbb{Q}$-isogeny $\varphi : X \to X'$ such that $\varphi^*\lambda' = \lambda$.

**Proof.** Take an arbitrary $\mathbb{Q}$-isogeny $\varphi : X \to X'$. By (3.2), $\lambda^{-1}(\varphi^*\lambda')$ lies in $F^+_x$. On the other hand, there exists $\alpha \in D^x$ such that $\alpha\lambda = \lambda^{-1}(\varphi^*\lambda')$ by [69, Theorem III.4.1]. Put $\varphi := \varphi \circ \alpha^{-1}$. Then $\varphi^*\lambda' = \lambda$ as desired. $\square$

**Remark 3.1.4.** Thanks to Lemma 3.1.3, $\text{PPAV}(\sqrt{q})$ forms a single $\mathbb{Q}$-isogeny class. As mentioned in (3.1.2), $G^1$ is just the reduced norm one subgroup of $G = D^\times$. Let $(X_0, \lambda_0)/\mathbb{F}_q$ be a $\mathbb{Q}$-polarized abelian surface with $\pi_{X_0}^2 = q$, and $(V_p, \psi_p)$ (resp. $(V_\ell, \psi_\ell)$) be the quasi-polarized $\mathbb{F}$-isocrystal (resp. Tate space for $\ell \neq p$) attached to $(X_0, \lambda_0)$ as described in (2.6). For each $\ell \neq p$, the Tate space $V_\ell$ is simply a free $F_\ell$-module of rank 2, and $\psi_\ell = \text{Tr}_{F/Q} \circ \psi_{\ell, F}$, where $\psi_{\ell, F}$ is an $F_\ell$-linear symplectic form on $V_\ell$ (hence uniquely determined up to isomorphism). At the prime $p$, the $\mathbb{F}$-isocrystal $V_p$ is a $\mathbb{Q}_q(\sqrt{p})$-space of dimension 2 equipped with a Frobenius operator $F$ such that $F(ax) = a^pFx$ for every $a \in \mathbb{Q}_q$ and $x \in V_p$, and $F^n = \pi_{X_0}$. We shall make no use of this explicit description of the pairing $\psi_p$ for a general $q$, though it can be easily deduced from the prime field case (i.e. $q = p$) below.

### 3.2. The second step.

Now we move on to the classification of $\text{PPAV}(\pi)$ into genera, or equivalently, the classification of self-dual modules in $(V_\ell, \psi_\ell)$ for $\ell \in S_\pi$. From (2.14) and (5.1), $S_\pi = \{2, p\}$ since $\vartheta_A = 4q$. Thus our tasks are two-folds:

(i) to classify the isometric classes of self-dual Dieudonné modules in $(V_p, \psi_p)$;

(ii) to classify the isometric classes of self-dual $A_2$-lattices in $(V_\ell, \psi_\ell)$.

As mentioned previously, we shall restrict ourselves to the classification of the **superspecial** Dieudonné modules and prove the bijections in (1.12). Recall that a Dieudonné module $M \subset V_p$ is superspecial if and only if $FM = VM$.

#### 3.2.1. The superspecial Dieudonné modules.

The structure of Dieudonné modules is very easy to describe when $q = p$ (i.e. $\pi = \sqrt{p}$). In this case, $\sigma \in \text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ is the identity map, and $F, V$ and $\pi$ all act as the same operator on $V_p$. Thus $V_p$ is simply an $F_p$-vector space of dimension 2, and a Dieudonné module in $V_p$ is just a full $A_p$-lattice. Moreover, condition (2.12) becomes identical to (2.10). Therefore, when $\pi = \sqrt{p}$, we treat a Dieudonné module $M(X)$ as if it is a Tate module and put $T_p(X) := M(X)$. Necessarily, $T_p(X)$ is a free $O_{F_p}$-module of rank 2 since $A_p = O_{F_p}$. See also [72, Theorem 6.2]. From Lemma 5.1, we immediately obtain the following lemma.

**Lemma 3.2.1.** Suppose that $\pi = \sqrt{p}$. There is a unique isometric class of self-dual Dieudonné modules (or equivalently here, $O_{F_p}$-lattices) in $(V_p, \psi_p)$.

From a classical result of Oort [53], an abelian variety over a perfect field is superspecial if and only if its Dieudonné module is superspecial. Clearly, every abelian surface $X/F_p$ with $\pi_X^2 = p$ is superspecial.
For a general $q = p^n$ with $n = 2s + 1$, we focus only on the superspecial Dieudonné modules in $(V_p, \psi_p)$ for the sake of simplicity. Given a superspecial Dieudonné module $M \subset V_p$, we define its skeleton as

$$S(M) := \{x \in M \mid Fx = \sqrt{p}x\}.$$  

Define $S(V_p)$ similarly. Clearly, $S(M)$ is a $\mathbb{Z}_p[\sqrt{p}]$-lattice in $S(V_p)$.

**Lemma 3.2.2.** The superspecial Dieudonné module $M \subset V_p$ is canonically isomorphic to $S(M) \otimes_{\mathbb{Z}_p} \mathbb{Z}_q$.

**Proof.** Since $M$ is superspecial, $p^{-1}F^2M = M$. As $p^{-s}F^{2s+1} = \sqrt{p}$, we have

$$F(M) = F(p^{-s}F^{2s}M) = \sqrt{p}M.$$  

Put $F_0 := \sqrt{p}^{-1}F : M \to M$. Then $(M, F_0)$ is a finite free $\mathbb{Z}_q$-module of rank 4 with a $\sigma$-linear automorphism. The pairs $(M, F_0)$ are classified by the first Galois cohomology $H^1(\mathbb{F}_q/\mathbb{F}_p, GL_4(\mathbb{Z}_q))$, which is trivial by “Hilbert’s Theorem 90” [50 Chap. III, Lemma 4.10, p. 124]. It follows that $(M, F_0)$ is isomorphic to the standard $\mathbb{Z}_q$-module $(\mathbb{Z}_4^4, \sigma)$, where $\sigma$ acts coordinate-wisely. Then $S(M) = M^{F_0}$ is of rank 4 as well, and the canonical map $S(M) \otimes_{\mathbb{Z}_p} \mathbb{Z}_q \to M$ is an isomorphism. □

The above lemma allows us to show that every $\mathbb{Q}$-polarized superspecial abelian surface $(Y, \mu)/\mathbb{F}_q$ with $\pi_2^Y = q$ descends to $\mathbb{F}_p$. Let $(X_0, \lambda_0)/\mathbb{F}_p$ be an arbitrary $\mathbb{Q}$-polarized abelian surface with $\pi_2^{X_0} = p$, and put $(Y_0, \mu_0) = (X_0, \lambda_0) \otimes_{\mathbb{F}_q} \mathbb{F}_q$. We have $End^{\mathbb{Z}_p}_{\mathbb{F}_p}(Y_0) = End^{\mathbb{Z}_p}_{\mathbb{F}_p}(X_0) = D$. Thus the group $G(\hat{\mathbb{Q}}) = \hat{D}^\times$ acts on both $\text{Qisog}(X_0)$ and $\text{Qisog}(Y_0)$.

**Lemma 3.2.3.** Let $\text{Qisog}^{\mathbb{Q}}(Y_0)$ be the subset of $\text{Qisog}(Y_0)$ consisting of all the superspecial members. The base change $(X, \varphi) \mapsto (X, \varphi) \otimes_{\mathbb{F}_q} \mathbb{F}_q$ induces a $G(\hat{\mathbb{Q}})$-equivariant bijection $\text{Qisog}(X_0) \to \text{Qisog}^{\mathbb{Q}}(Y_0)$.

**Proof.** From (3.3), the base change map is $G(\hat{\mathbb{Q}})$-equivariant. To prove the bijectivity, it suffices to produce an inverse map. Let $(Y, \phi)$ be an arbitrary member of $\text{Qisog}^{\mathbb{Q}}(Y)$. The $\mathbb{Q}$-isogeny $\phi$ identifies $M(Y)$ with a superspecial Dieudonné module $M$ over $\mathbb{F}_q$ in $V_p(Y_0) = V_p(X_0) \otimes_{\mathbb{Q}_p} \mathbb{Q}_q$. According to Lemma 3.2.2, $S(M)$ is a $\mathbb{Z}_p[\sqrt{p}]$-lattice in $V_p(X_0)$, and $M = S(M) \otimes_{\mathbb{Z}_p} \mathbb{Z}_q$. At each prime $\ell \neq p$, the $\mathbb{Q}$-isogeny $\phi$ identifies $T_\ell(Y)$ with a $\mathbb{Z}_\ell[\sqrt{p}]$-lattice $T_\ell$ in $V_\ell(Y_0) = V_\ell(X_0)$. The inclusion $(S(M), (T_\ell)_{\ell \neq p}) \subset V_p(X_0) \times \prod_{\ell \neq p} V_\ell(X_0)$ corresponds to a unique member $(X, \varphi) \in \text{Qisog}(X_0)$. By our construction, the map $\text{Qisog}^{\mathbb{Q}}(Y_0) \to \text{Qisog}(X_0)$ sending $(Y, \phi)$ to $(X, \varphi)$ is the inverse of the base change map. □

**Corollary 3.2.4.** Let $\text{PolSp}(\sqrt{q})$ be the category of $\mathbb{Q}$-polarized superspecial abelian surfaces $(Y, \mu)/\mathbb{F}_q$ with $\pi_2^Y = q$, whose morphisms are polarized $\mathbb{Q}$-isogenies. Then the base change functor $- \otimes_{\mathbb{F}_q} \mathbb{F}_q : \text{PolSp}(\sqrt{q}) \to \text{PolSp}(\sqrt{q})$ is an equivalence of categories. In particular, it induces a bijection $\text{PPAV}(\sqrt{q}) \simeq \text{PPSP}(\sqrt{q})$.

**Proof.** Let $\varphi : (X_1, \lambda_1) \to (X_0, \lambda_0)$ be a morphism in $\text{PolSp}(\sqrt{q})$. Then

$$\text{Hom}((X_1, \lambda_1), (X_0, \lambda_0)) = D^1\varphi = \text{Hom}((X_1, \lambda_1) \otimes_{\mathbb{F}_q} \mathbb{F}_q, (X_0, \lambda_0) \otimes_{\mathbb{F}_q} \mathbb{F}_q),$$

where $D^1 = G^1(\mathbb{Q})$, the reduced norm one subgroup of $D^\times$. This shows that the base change functor is fully faithful. Fix $(Y_0, \mu_0) := (X_0, \lambda_0) \otimes_{\mathbb{F}_q} \mathbb{F}_q$ and let $(Y, \mu)/\mathbb{F}_q$ be an object of $\text{PolSp}(\sqrt{q})$. From Lemma 3.1.3 there exists a $\mathbb{Q}$-isogeny $\phi : (Y, \mu) \to (Y_0, \mu_0)$. It follows from Lemma 3.2.3 that there exists $(X, \varphi) \in \text{Qisog}(X_0)$ and an
of all members $X$ exclusively on the Weil $p$-number $\pi = \sqrt{p}$. In particular, $A = \mathbb{Z} [\sqrt{p}]$ from now on. From the classification of Dieudonné modules (particularly, Lemma 3.2.1), we see that to classify $\text{PPAV}(\sqrt{p})$ into genera, it is enough to classify the isometric classes of self-dual $A_2$-lattices in $(V_2, \psi_2)$, as $A_\ell$ may be a non-maximal order only when $\ell = 2$.

We first give a brief recount of the genus classification of the unpolarized case in [72, §6] and [73, §6.1]. Recall that $\text{Isog}(\sqrt{p})$ denotes the set of $\mathbb{F}_p$-isomorphism classes of abelian surfaces in the simple isogeny class corresponding to $\pi = \sqrt{p}$.

Let $X_0/\mathbb{F}_p$ be an arbitrary abelian surface with $\pi^2 = p$ and $(X, \varphi)$ be a member of $\text{Qisog}(X_0)$. For every prime $\ell$ (including $\ell = p$), $T_\ell(X)$ is a full $A_\ell$-lattice in $V_\ell \cong \mathbb{F}_p^2$. If either $p \not\equiv 1 \pmod{4}$ or $\ell \not\equiv 2$, then $A_\ell = \mathcal{O}_F$, and hence $T_\ell(X) \cong \mathcal{O}_F^2$. Thus $\text{Isog}(\sqrt{p})$ forms a single genus when $p \not\equiv 1 \pmod{4}$. If $p \equiv 1 \pmod{4}$ and $\ell = 2$, then $[O_{F_2} : A_2] = 2$, and $T_2(X)$ is isomorphic to one of the following three $A_2$-lattices in $(F_2)^2$:

$$O_{F_2}^2, \quad A_2 \oplus O_{F_2}, \quad A_2^2.$$  

Accordingly, $\text{Isog}(\sqrt{p})$ decomposes into three genera when $p \equiv 1 \pmod{4}$:

$$\text{Isog}(\sqrt{p}) = \Lambda_1^{un} \amalg \Lambda_8^{un} \amalg \Lambda_{16}^{un}.  
$$

Here the superscript $\text{un}$ stands for “unpolarized”, and $\Lambda_1^{un}$ is the subset consisting of all members $[X] \in \text{Isog}(\sqrt{p})$ such that $T_2(X) \cong O_{F_2}^2$. The genera $\Lambda_8^{un}$ and $\Lambda_{16}^{un}$ are defined similarly. The subscripts 1, 8, 16 are chosen for $\Lambda_1^{un}$ to indicate that

$$[O : \text{End}(X_p)] = r$$

for every $[X_p] \in \Lambda_1^{un}$ and every maximal $O_F$-order $\mathfrak{D} \subset D$ containing $\text{End}(X_p)$. See [72, Theorem 6.2], [73, Theorem 6.1.2] or [3.1] below. For uniformity, we also put $\Lambda_1^{un} = \text{Isog}(\sqrt{p})$ when $p \not\equiv 1 \pmod{4}$. As a convention, when a result is stated for $\Lambda_r^{un}$ with $r \in \{1, 8, 16\}$, it means that the said result holds for $\Lambda_1^{un}$ for all primes $p$, and also for $\Lambda_8^{un}$ and $\Lambda_{16}^{un}$ when $p \equiv 1 \pmod{4}$.

**Proposition 3.2.5.** Consider the following forgetful map:

$$f : \text{PPAV}(\sqrt{p}) \to \text{Isog}(\sqrt{p}), \quad [X, \lambda] \mapsto [X].$$

Put $\Lambda_1^{pp} := \text{PPAV}(\sqrt{p})$ if $p \not\equiv 1 \pmod{4}$, and $\Lambda_r^{pp} := f^{-1}(\Lambda_r^{un})$ for $r \in \{1, 8, 16\}$ if $p \equiv 1 \pmod{4}$. Then the following holds true:

1. when $p \not\equiv 1 \pmod{4}$, $\Lambda_1^{pp} = \text{PPAV}(\sqrt{p})$ forms a single genus;
2. when $p \equiv 1 \pmod{4}$, $\Lambda_8^{pp} = \emptyset$, and

$$\text{PPAV}(\sqrt{p}) = \Lambda_1^{pp} \amalg \Lambda_{16}^{pp},$$

where each $\Lambda_r^{pp}$ for $r \in \{1, 16\}$ forms a single nonempty genus.

**Proof.** If $p \not\equiv 1 \pmod{4}$, then $A_2 = O_{F_2}$, and there is a unique isometric class of self-dual $O_{F_2}$-lattices in $(V_2, \psi_2)$ by Lemma 8.1. Part (1) of the proposition follows directly, so assume that $p \equiv 1 \pmod{4}$ for the rest of the proof.
We first prove that \( \Lambda_{ss}^{pp} = \emptyset \). It is enough to show that there is no self-dual \( A_2 \)-lattice \( L \) in \( (V_2, \psi_2) \) such that \( L \simeq O_{F_2} \oplus A_2 \). Suppose that such a lattice \( L \) exists. Then \( L = O_{F_2}e_1 + A_2e_2 \) for some \( F_2 \)-basis \( e_1, e_2 \) of \( V_2 \) and \( \psi_2(L, L) \subset \mathbb{Z}_2 \). Denote by 
\[
\psi_{2,F} : V_2 \times V_2 \to F_2
\]
the unique \( F_2 \)-bilinear pairing such that \( \operatorname{Tr}_{F_2/Q_2} \circ \psi_{2,F} = \psi_2 \).
Thus \( G(4.2) \Lambda \)

From Proposition 2.2, there is a canonical bijection

of Proposition 3.2.5. Fix a member \( Y \)

Let \( \Lambda \) be the dual lattice of \( \Lambda \)

According to Proposition 8.4, there is a unique isometric class of self-dual free \( \Lambda \)

We move on to Step 3 of our strategy for computing \( \operatorname{PPAV}(\sqrt{D}) \) as described in 3.2.5, so keep the notation of the previous section. Particularly, \( F = \mathbb{Q}(\sqrt{D}) \) and \( A = \mathbb{Z}[\sqrt{D}] \). Let \( D = D_{1,\infty}^\infty \) be the unique totally definite quaternion \( F \)-algebra that is unramified at all the finite primes of \( F \) as in 3.2.5. Recall that the algebraic group \( G \) defined in 2.2, is just the multiplicative group \( D^\times \), and \( G^1 \) defined in 2.8 is the reduced norm one subgroup of \( D^\times \). Given a subset \( S \subset D \), we write \( S^1 \) for the subset of elements in \( S \) with reduced norm 1, that is, \( S^1 := \{ x \in S \mid \operatorname{Nr}(x) = 1 \} \).

Thus \( G^1(\mathbb{Q}) = D^1 \) and \( G^1(\mathbb{Q}) = D^1 \). Assume that one of the following conditions holds:

- \( p \not\equiv 1 \pmod{4} \) and \( r = 1 \);
- \( p \equiv 1 \pmod{4} \) and \( r \in \{1, 16\} \).

Let \( \Lambda_{pp}^{pp} \) be the genus of principally polarized abelian surfaces as defined in Proposition 3.2.5. Fix a member \( [Y_r, \lambda_r] \in \Lambda_{pp}^{pp} \) and put

\[
\mathcal{O}_r := \operatorname{End}(Y_r).
\]

From Proposition 2.2, there is a canonical bijection

\[
\Lambda_{pp}^{pp} \simeq D^1 \backslash \hat{D}^1 / \hat{D}^1_r
\]

sending \( [Y_r, \lambda_r] \) to the identity class. Thus to compute \( |\Lambda_{pp}^{pp}| \), we need a concrete characterization of the \( A \)-orders in \( D \) that appear as endomorphism rings of the underlying (principally polarizable) abelian surfaces of \( \Lambda_{pp}^{pp} \). The class number calculations will be postponed to Sections 6.7.
4.1. We first recount the description of endomorphism rings in the unpolarized case. Let \([X]\) be a member of \(\Lambda\) and put \(\mathcal{O}_1 \coloneqq \text{End}(X_1)\). Then \(\mathcal{O}_1\) is a maximal order in \(D\) since \(\mathcal{O}_1 = \text{End}_{\Lambda}(T_\ell(X_1)) \simeq \text{Mat}_2(O_{F_\ell})\) for every prime \(\ell\). Fix an isomorphism \(T_2(X_1) \simeq O_{F_2}^2\), which induces an identification \(\mathcal{O}_1 \otimes \mathbb{Z}_2 = \text{Mat}_2(O_{F_2})\).

If \(p \equiv 1 \pmod{4}\), then the inclusion \(A_2 \oplus O_{F_2} \to O_{F_2}^2\) (resp. \(A_2^2 \to O_{F_2}^2\)) gives rise to an isogeny \(X_8 \to X_1\) (resp. \(X_{16} \to X_1\)) with \(\mathcal{O}_r \in \Lambda\) for \(r \in \{8, 16\}\). The endomorphism rings \(\mathcal{O}_r \coloneqq \text{End}(X_r)\) for \(r = 8, 16\) are characterized by

\[
(O_8)_2 \coloneqq O_8 \otimes_{\mathbb{Z}} \mathbb{Z}_2 = \begin{pmatrix} A_2 & 2O_{F_2} \\ O_{F_2} & O_{F_2} \end{pmatrix}, \quad (O_{16})_2 = \text{Mat}_2(A_2),
\]

\((\mathcal{O}_r)_\ell = (\mathcal{O}_1)_\ell \quad \forall \text{ prime } \ell \neq 2, \quad r \in \{8, 16\}.
\]

The order \(\mathcal{O}_r\) has index \(r\) in \(\mathcal{O}_1\).

From Proposition 2.2 there is a natural bijection

\[\Lambda_\text{un} \simeq G(\mathbb{Q}) \backslash G(\mathbb{Q})/\hat{\mathcal{O}}^\times_\ell = D^\times/\hat{D}^\times/\hat{\mathcal{O}}^\times_\ell\]

sending the base member \([X_r]\) to the identity class. This also establishes a bijection between \(\Lambda_\text{un}\) and the set \(\text{Cl}(\mathcal{O}_r)\) of locally principal right ideal \(\mathcal{O}_r\)-classes as discovered by Waterhouse in [72] Theorem 6.2. Indeed, from [69, III.5], \(\text{Cl}(\mathcal{O}_r)\) admits the same adelic double coset description as in [14]. Thus the computation of \(|\text{Isog}(\sqrt{p})|\) is reduced to that of the class numbers of the quaternion orders \(\mathcal{O}_r\).

When \(p \equiv 1 \pmod{4}\), the class number formula for \(h(\mathcal{O}_1) \coloneqq |\text{Cl}(\mathcal{O}_1)|\) is first computed explicitly by Peters [55, p. 363], and also by Kitaoka [33] in light of its relationship to the type number \(h(\mathcal{O}_1)\) as in (1.12) below, which in turn can be interpreted as the proper class number of quaternion positive definite even quadratic lattices of discriminant \(p\) [10, p. 85]. Ponomarev [59, 60] extended Kitaoka’s result to all prime \(p\). Vignéras [67] gave explicit formulas for the class number of any Eichler order of square-free level in a totally definite quaternion algebra over an arbitrary quadratic real field. When \(p \equiv 1 \pmod{4}\), the class number formulas for \(\mathcal{O}_8\) and \(\mathcal{O}_{16}\) are worked out by the present authors together with Tse-Chung Yang in [73].

Remark 4.2. We provide another proof for \(\Lambda_\text{pp} = \emptyset\). Suppose that on the contrary a member \([X, \lambda] \in \Lambda_\text{pp}\) exists. The Rosati involution induced by \(\lambda\) necessarily leaves the endomorphism rings \(\text{End}(X)\) and \(\text{End}(X) \otimes \mathbb{Z}_2\) stable. On the other hand, the Rosati involution coincides with the canonical involution. This already leads to a contradiction since \(\text{End}(X) \otimes \mathbb{Z}_2\), which is conjugate to \(\mathcal{O}_8 \otimes \mathbb{Z}_2\) in (4.3), is not stable under the canonical involution.

4.3. The bijection \(\Lambda_\text{un} \simeq \text{Cl}(\mathcal{O}_r)\) fits into the general framework of the arithmetic of quaternion algebras as follows. Two \(A\)-orders \(\mathcal{O}\) and \(\mathcal{O}'\) in \(D\) are said to belong to the same genus if there exists \(x \in \hat{D}^\times\) such that \(\mathcal{O}' = x\hat{O}x^{-1}\), or equivalently, if \(\mathcal{O}_\ell\) and \(\mathcal{O}'_\ell\) are \(A_\ell\)-isomorphic at every prime \(\ell \in \mathbb{N}\). For example, all maximal orders of \(D\) belong to the same genus. The orders \(\mathcal{O}\) and \(\mathcal{O}'\) are said to be of the same type if they are \(A\)-isomorphic (equivalently, \(D^\times\)-conjugate). Let

\[\text{Tp}(\mathcal{O}) := \{\alpha\mathcal{O}\alpha^{-1} \mid \alpha \in D^\times\}\]

be the type of \(\mathcal{O}\), and \(\text{Tp}(\mathcal{O})\) be the set of types of \(A\)-orders in the genus of \(\mathcal{O}\). It can be described adelically as

\[
(4.5) \quad \text{Tp}(\mathcal{O}) \simeq D^\times/\hat{D}^\times/N(\hat{\mathcal{O}}),
\]
where $N(\hat{O})$ denotes the normalizer of $\hat{O}$ in $\hat{D}$. Given a locally principal right $O$-ideal $I$, the left order of $I$ is defined to be

$$O_l(I) = \{ \alpha \in D \mid \alpha I \subseteq I \}. \tag{4.6}$$

If we write $\hat{I} = x\hat{O}$ for some $x \in \hat{D}$, then $\hat{O}_l(I) = x\hat{O}x^{-1}$, so $O_l(I)$ belongs to the same genus as $O$. There is a natural surjective map

$$\Upsilon : Cl(O) \to Tp(O), \quad \left[ I \right] \mapsto \left[ O_l(I) \right]. \tag{4.7}$$

Now let $O = O_r$ for some $r \in \{1, 8, 16\}$. We have a commutative diagram

$$\begin{array}{ccc}
\Lambda_r^{un} & \xrightarrow{\cong} & Cl(O_r) \\
\downarrow{\cong} & & \downarrow{\cong} \\
D^\times \setminus \hat{D}^{\times}/\hat{O}_r^{\times} & \twoheadrightarrow & D^\times \setminus \hat{D}^{\times}/N(\hat{O}_r)
\end{array} \tag{4.8}$$

where the bottom horizontal map is the canonical projection. From (4.8), the composition of the maps in the top row coincides with the following map

$$\Upsilon^{un} : \Lambda_r^{un} \to Tp(O_r), \quad [X] \mapsto [\text{End}(X)]. \tag{4.9}$$

Thus an $A$-order $O' \subset D$ is isomorphic to the endomorphism ring of some $[X'_r] \in \Lambda_r^{un}$ if and only if it belongs to the same genus as $O_r$. Compare with [72, Theorem 3.13] and Lemma [73] below.

The explicit formula for the type number $t(O_1) := |Tp(O_1)|$ can be traced back to the work of Peters [56], Kitaoka [33], Ponomarev [59, 60] as before. The type number formulas for $O_8$ and $O_{16}$ are computed by the present authors in [77 §4]. In particular, it is shown there that

$$N(O_1) = \hat{F}^{\times}\hat{O}_1^{\times}, \quad N(O_{16}) = \hat{F}^{\times}\hat{O}_{16}^{\times}, \quad \text{while} \quad N(O_8) = \hat{F}^{\times}\hat{O}_4^{\times}. \tag{4.10}$$

Here $O_4 := O_FO_8$, which is the unique suborder of $O_1$ such that

$$O_4 \otimes \mathbb{Z}_2 = \begin{pmatrix} O_{F_2} & 2O_{F_2} \\ O_{F_2} & O_{F_2} \end{pmatrix}, \quad (O_4)t = (O_1)t, \quad \forall t \neq 2. \tag{4.11}$$

According to [77, Proposition 4.1], the class numbers $h(O_r)$ and the type numbers $t(O_r) := |Tp(O_r)|$ are related by

$$h(O_1) = h(O_F)t(O_1), \quad h(O_{16}) = h(A)t(O_{16}), \quad h(O_4) = h(O_F)t(O_8). \tag{4.12}$$

See [7,8] for the formula of $h(A)$.

Now let us move on to the principally polarized case. Recall from (4.1) that $\Omega_r = \text{End}(Y_r)$ denotes the endomorphism ring of a fixed member $[Y_r, \lambda_r] \in \Lambda^{pp}$. For any other member $[Y'_r, \lambda'_r] \in \Lambda^{pp}$, the endomorphism ring $\text{End}(Y'_r)$ belongs to the same genus as $\Omega_r$. However, we would like to emphasize that the converse needs not to be true, that is, a priori not every order in the genus of $\Omega_r$ is necessarily the endomorphism ring of some underlying (principally polarizable) abelian surface of $\Lambda^{pp}$. Thus we need to characterize the image of the map

$$\Upsilon^{pp} : \Lambda^{pp} \to Tp(\Omega_r), \quad [X, \lambda] \mapsto [\text{End}(X)]. \tag{4.13}$$

For this purpose, let us recall the notion of spinor genus of orders from [7 §1].

**Definition 4.4.** Two $A$-orders $O, O' \subset D$ are in the same spinor genus if there exists $x \in D^\times \hat{D}$ such that $\hat{O}' = x\hat{O}_r x^{-1}$. 
Clearly, “being in the same spinor genus” is an equivalence relation that is finer than “being in the same genus” and coarser than “being of the same type”.

**Lemma 4.5.** Keep $\mathcal{O}_r = Y_r$ for some $[Y_r, \lambda_r] \in \Lambda^{pp}_r$ as in (4.1). For any $A$-order $\mathcal{O} \subset D$, there exists $[X, \lambda] \in \Lambda^{pp}$ such that $\text{End}(X) \cong \mathcal{O}$ if and only if $\mathcal{O}$ belongs to the same spinor genus as $\mathcal{O}_r$.

**Proof.** Clearly, $\Upsilon^{pp} : \Lambda^{pp}_r \to Tp(\mathcal{O}_r)$ is the composition of $f : \Lambda^{pp} \to \Lambda^{un}$ with the map $\Upsilon^{un}$ in (4.9) (with $\mathcal{O}_r$ in place of $\mathcal{O}_r$). Combining (4.2) and (4.8), we get a commutative diagram

$$
\Lambda^{pp}_r \xrightarrow{f} \Lambda^{un} \xrightarrow{\Upsilon^{un}} Tp(\mathcal{O}_r)
$$

Both of the bottom horizontal maps are canonical projections. Clearly, $[\mathcal{O}] \in \Upsilon^{pp}(\Lambda^{pp})$ if and only if $\mathcal{O}$ belongs to the same spinor genus as $\mathcal{O}_r$. □

**Proposition 4.6.** (1) If $p \not\equiv 3 \pmod{4}$, then all maximal orders of $D$ form a single spinor genus.

(2) If $p \equiv 1 \pmod{4}$, then all $A$-orders in the genus of $\mathcal{O}_1$ form a single spinor genus.

(3) If $p \equiv 3 \pmod{4}$, then the maximal orders of $D$ separate into two spinor genera. Accordingly the type set $Tp(D)$ of maximal orders in $D$ decomposes into two nonempty subsets

$$
Tp(D) = Tp^+(D) \coprod Tp^{-}(D),
$$

where $Tp^+(D) := \text{img}(\Upsilon^{pp})$, and $Tp^{-}(D) := Tp(D) \smallsetminus Tp^+(D)$.

**Proof.** For any $A$-order $\mathcal{O} \subset D$, we write $[\mathcal{O}]_{\text{sg}}$ for the spinor genus of $\mathcal{O}$, and $\text{SG}(\mathcal{O})$ for the set of spinor genera within the genus of $\mathcal{O}$, regarded as a pointed set with base point $[\mathcal{O}]_{\text{sg}}$; see [80, Definition 2.2]. In other words,

$$
\text{SG}(\mathcal{O}) = \{[\mathcal{O}]_{\text{sg}} \mid \exists x \in \hat{D}^\times \text{ such that } \hat{\mathcal{O}} = x\hat{\mathcal{O}}x^{-1}\}.
$$

From [80] (2.2), there is an adelic description of $\text{SG}(\mathcal{O})$ as follows:

$$
\text{SG}(\mathcal{O}) \simeq (D^\times \hat{D}^1) / \hat{\mathcal{N}}(\hat{\mathcal{O}}) \overset{\text{Nr}}{\cong} F_+^x / \hat{F}_+^x / \text{Nr}(\hat{\mathcal{N}}(\hat{\mathcal{O}})),
$$

where the two double coset spaces are canonically bijective via the reduced norm map. Here we have applied the Hasse-Schilling-Maass theorem [62, Theorem 33.15], [69, Theorem III.4.1] to obtain $\text{Nr}(D^\times) = F_+^x$, the subgroup of totally positive elements of $F^x$.

For simplicity, let us put $R_1 := O_F$, and $R_{16} := A$. Let $\text{Pic}_+(R_r)$ be the narrow class group of $R_r$, which can be described adelically as

$$
\text{Pic}_+(R_r) \simeq \hat{F}_+^x / F_+^x \hat{R}_r^x.
$$

From (4.10), $\hat{\mathcal{N}}(\hat{\mathcal{O}}) = \hat{F}_+^x \hat{R}_r^x$ for $r \in \{1, 16\}$. It follows from (4.13) and (4.17) that

$$
\text{SG}(\mathcal{O}_r) \simeq \hat{F}_+^x / (F_+^x \hat{F}_+^x \hat{R}_r^x),
$$

which is canonically identifiable with the *Gauss genus group* [12, Definition 14.29]

$$
\mathfrak{G}(R_r) := \text{Pic}_+(R_r) / \text{Pic}_+(R_r)^2.
$$
Here \( \text{Pic}_+(R_r)^2 \) denotes the subgroup of \( \text{Pic}_+(R_r) \) consisting of the classes that are perfect squares in \( \text{Pic}_+(R_r) \), and \( \hat{F}^\times \) is defined similarly. First, suppose that \( r = 1 \) so that \( R_1 = O_F \). It is well known [12] Theorem 14.34 that the Gauss genus group \( \mathfrak{G}(O_F) \) has order \( 2t-1 \), where \( t \) is the number of finite ramified primes in \( F/\mathbb{Q} \), so in our case

\[
\begin{cases}
1 & \text{if } p \not\equiv 3 \pmod{4}, \\
2 & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\]

The proves part (1) and (3) of the lemma. Next, suppose that \( p \equiv 1 \pmod{4} \) and \( r = 16 \) so that \( R_{16} = A \). We shall show in Lemma 4.8 that the narrow class number \( h_+(A) := |\text{Pic}_+(A)| \) is odd. It then follows from definition (4.24) that

\[
|\mathfrak{G}(A)| = 1 \quad \text{if } \ p \equiv 1 \pmod{4},
\]

which completes the proof of part (2) of the lemma.

As a result, when \( p \not\equiv 3 \pmod{4} \), for every maximal \( O_F \)-order \( \mathcal{O}_1 \subset D \) there exists some member \( [X_1, \lambda_1] \in \Lambda_{pp}^{gp} \) such that \( \text{End}(X_1) \simeq \mathcal{O}_1 \). A similar statement holds for every \( A \)-order \( \mathcal{O}_{16} \subset D \) satisfying (4.3) and the genus \( \Lambda_{pp}^{gp} \) when \( p \equiv 1 \pmod{4} \). In fact, the member \( [X_r, \lambda_r] \in \Lambda_{pp}^{gp} \) with \( r \in \{1, 16\} \) turns out to be unique by Lemmas (4.7 and (4.8) respectively. Thus we have

**Lemma 4.7.** Suppose that either \( p \equiv 1 \pmod{4} \) and \( r \in \{1, 16\} \) or \( (p, r) = (2, 1) \).

The map \( \Upsilon_{pp} : \Lambda_{pp}^{gp} \to \text{Tp}(\mathcal{O}_r) \) is bijective.

The situation is quite different when \( p \equiv 3 \pmod{4} \). Combining the equations (4.17)-(4.21), we have constructed a map

\[
\Xi : \text{Tp}(D) = \text{Tp}(\mathcal{O}_1) \to \text{SG}(\mathcal{O}_1) \simeq \mathfrak{G}(O_F),
\]

whose neutral (resp. non-neutral) fiber is \( \text{Tp}^+(D) \) (resp. \( \text{Tp}^-(D) \)). A maximal order \( \mathcal{O} \) in \( D \) is said to belong to the principal spinor genus if \( \mathcal{O} \in \text{Tp}^+(D) \), otherwise it is said to belong to the nonprincipal spinor genus. We will produce an explicit maximal order \( \mathcal{O}_0 \) in \( \Lambda_{pp}^{gp} \) such that \( \mathcal{O}_0 \in \text{Tp}^+(D) \). Thus both \( \text{Tp}^+(D) \) and \( \text{Tp}^-(D) \) are characterized purely in terms of quaternion arithmetic.

**4.8.** For the moment let \( p \) be an arbitrary prime. Let \( (X, \lambda_X) = \text{Res}_{p,2}/p_2(E, \lambda_E) \) be as in [3.1.1] where \( E/F_{p,2} \) is an elliptic curve with \( \pi_E = p \). By functoriality, \( \text{End}_{p,2}(E) \otimes \mathbb{Z}[\pi_X] \) acts on \( X \). From [19] Remark 4, this gives rise to an identification

\[
\text{End}_{p,2}^0(X) = \text{End}_{p,2}^0(E) \otimes \mathbb{Q}(\pi_X)
\]

such that

\[
\text{End}_{p,2}(X) \otimes \mathbb{Z}[1/p] = \text{End}_{p,2}^0(E) \otimes \mathbb{Z}[1/p][\pi_X].
\]

In other words, \( \text{End}_{p,2}(X) \otimes \mathbb{Z}[\pi_X] \) differs at most at \( p \). If \( p \equiv 1 \pmod{4} \), then \( \mathbb{Z}[1/p][\pi_X] \otimes \mathbb{Z}[1/p] \mathbb{Z}_2 \simeq A_2 \), and we find that

\[
\text{End}_{p,2}(X) \otimes \mathbb{Z}_2 \simeq \text{End}_{p,2}^0(E) \otimes \mathbb{Z} A_2 \simeq \text{Mat}_2(\mathbb{Z}_2) \otimes \mathbb{Z}_2 A_2 \simeq \text{Mat}_2(A_2).
\]

It follows from (4.3) that \( [X, \lambda_X] \in \Lambda_{pp}^{gp} \) in this case.

When written down explicitly, the identification in (4.24) is just \( D = D_{p, \infty} \otimes F \), where \( D_{p, \infty} \) is the unique quaternion \( \mathbb{Q} \)-algebra ramified precisely at \( p \) and \( \infty \). From [72] Theorem 4.2, \( \otimes : \text{End}_{p,2}(E) \) is a maximal \( \mathbb{Z} \)-order in \( \text{End}_{p,2}^0(E) = D_{p, \infty} \).
and conversely by [72, Theorem 3.13], every maximal $\mathbb{Z}$-order in $D_{p,\infty}$ occurs as the endomorphism ring of some elliptic curve in the isogeny class of $E/\mathbb{F}_{p^2}$. According to [43, Lemma 2.11], there is a unique $A$-order $\mathcal{M}(\wp)$ in $D$ properly containing $\wp \otimes A$ such that $\mathcal{M}(\wp) \otimes \mathbb{Z}_\ell = \wp \otimes A_\ell$ at every prime $\ell \neq p$. Such an $A$-order $\mathcal{M}(\wp)$ is necessarily maximal at $p$, i.e. $\mathcal{M}(\wp) \otimes \mathbb{Z}_p \simeq \text{Mat}_2(O_{F_p})$. In particular, $\mathcal{M}(\wp)$ is a maximal $O_F$-order in $D$ when $p \equiv 1 \pmod{4}$. On the other hand, $\text{End}_{\wp}(X)$ is maximal at $p$ as described in §4.1. Therefore, 
\begin{equation}
\text{End}_{\wp}(X) = \mathcal{M}(\wp).
\end{equation}
We work out an explicit example in the case $p \equiv 3 \pmod{4}$ below.

**Example 4.9.** Suppose that $p \equiv 3 \pmod{4}$. According to [69, Exercise III.5.2], $D_{p,\infty}$ can be presented as \( \left( \frac{-1-p}{q} \right) \), and $\wp_2 := \mathbb{Z}[i, (1+j)/2]$ is a maximal $\mathbb{Z}$-order in $\left( \frac{-1-p}{q} \right)$. Here \{1, i, j, k\} denotes the standard basis of \( \left( \frac{-1-p}{q} \right) \), where we write a subscript $p$ to emphasize that $j^2 = k^2 = -p$. Then $D = D_{p,\infty} \otimes F = \left( \frac{-1-p}{q} \right)$, which has a new standard basis \{1, i, j, k\} by putting $j := j_p/\sqrt{p}$ and $k := k_p/\sqrt{p}$. According to [43, Proposition 5.7], the following is a maximal $O_F$-order in $D$:
\begin{equation}
\mathcal{O}_0 := O_F + O_F i + O_F \sqrt{p} + j + O_F \sqrt{p} i + k \subset \left( \frac{-1, -1}{F} \right).
\end{equation}
Clearly, $\mathcal{O}_0 \supset \mathcal{O}_2 \otimes O_F$, so $\mathcal{O}_0 = \mathcal{M}(\wp_2)$ and $[\mathcal{O}_0] \in \text{Tp}^+(D)$. In fact, $\mathcal{O}_0$ is the unique maximal $O_F$-order in $D$ up to conjugation satisfying $\mathcal{O}_0^\times/O_F^\times \simeq D_4$ (resp. $D_{12}$) if $p \geq 7$ (resp. $p = 3$). Here $D_n$ denotes the dihedral group of order $2n$.

Combining Lemma 4.9 with Example 4.9 we obtain the following result.

**Lemma 4.10.** Suppose that $p \equiv 3 \pmod{4}$. For any maximal $O_F$-order $\mathcal{O} \subset D$, there exists $[X, \lambda] \in \text{PPAV}(\sqrt{p})$ such that $\text{End}(X) \simeq \mathcal{O}$ if and only if $\mathcal{O}$ belongs to the same spinor genus as $\mathcal{O}_0$.

**Proofs of Theorem 1.1 and Theorem 1.2.** For any $A$-order $\mathcal{O}$ in $D$, we put
\begin{equation}
h^1(\mathcal{O}) := |D^1|/|\mathcal{O}^1|,
\end{equation}
which depends only on the spinor genus of $\mathcal{O}$. Let $t(\mathcal{O}) = |\text{Tp}(\mathcal{O})|$ be the type number of $\mathcal{O}$, and $t(\Lambda^{pp})$ be the cardinality of $\Upsilon^{pp}(\Lambda^{pp})$ in (4.13).

From (4.2), Proposition 3.2.5, and Lemmas 4.7 and 4.10 we have
\begin{equation}
h^{pp}(\sqrt{p}) := |\text{PPAV}(\sqrt{p})| = \begin{cases} h^1(\mathcal{O}_1) & \text{if } p = 2; \\
 h^1(\mathcal{O}_1) + h^1(\mathcal{O}_{16}) & \text{if } p \equiv 1 \pmod{4}; \\
 h^1(\mathcal{O}_0) & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\end{equation}
Here $\mathcal{O}_1 \subset D$ is an arbitrary maximal $O_F$-order, $\mathcal{O}_{16} \subset D$ is an arbitrary $A$-order satisfying (4.3), and $\mathcal{O}_0$ is the maximal $O_F$-order in (4.27).

First, suppose that $p = 2$. From Lemma 4.7 and (77 (4.7)), we obtain
\begin{equation}
\text{h}^{pp}(\sqrt{p}) = t^{pp}(\sqrt{2}) = h^1(\mathcal{O}_1) = t(\mathcal{O}_1) = 1.
\end{equation}

\footnote{This order in denoted by $\mathcal{O}_8$ in [43] since $|\mathcal{O}_0^\times/\mathcal{O}_2^\times| = |D_4| = 8$. Unfortunately, we used the same notation $\mathcal{O}_8$ in [25,77] for the $A$-order that is currently denoted as $\mathcal{O}_8$. To make a distinction, throughout this paper the letter $\mathcal{O}$ is reserved for a maximal $O_F$-order, and $\mathcal{O}_0$ is reserved for the maximal order in (4.27).}
Moreover, if \( p \equiv 1 \pmod{4} \). If \( p = 5 \) then it follows from Lemma 1.17 and \([77\), (4.7) and (4.8)]\) that
\[
|\Lambda_{16}^{pp}| = t(\Lambda_{16}^{pp}) = h^1(\mathcal{O}_1) = t(\mathcal{O}_1) = 1,
\]
\[
|\Lambda_{16}^{pp}| = t(\Lambda_{16}^{pp}) = h^1(\mathcal{O}_{16}) = t(\mathcal{O}_{16}) = 1.
\]
Similarly, if \( p \geq 13 \) and \( p \equiv 1 \pmod{4} \), then according to \([77\), (4.10) and (4.12)]\):
\[
|\Lambda_{16}^{pp}| = t(\Lambda_{16}^{pp}) = h^1(\mathcal{O}_1) = t(\mathcal{O}_1) = 1,
\]
\[
|\Lambda_{16}^{pp}| = t(\Lambda_{16}^{pp}) = h^1(\mathcal{O}_{16}) = t(\mathcal{O}_{16}) = 1.
\]
\[
(4.31)\]
\[
(4.32)\]
\[
(4.33)\]
\[
(4.34)\]

The formula for \( h^p(\sqrt{p}) \), which is necessarily identical to that of \( t^p(\sqrt{p}) \) in this case, is obtained by summing up the formulas for \( \Lambda_{16}^{pp} \) and \( \Lambda_{16}^{pp} \).

Lastly, according to Proposition 6.2.1, we have
\[
h^p(\sqrt{3}) = h^1(\mathcal{O}_0) = 1, \quad t^p(\sqrt{3}) = |T^p(D)| = 1 \quad \text{if} \quad p = 3.
\]
Moreover, if \( p \geq 7 \) and \( p \equiv 3 \pmod{4} \), then
\[
h^p(\sqrt{3}) = h^1(\mathcal{O}_0) = \frac{\zeta_F(-1)}{2} + \left(11 - 3 \left(\frac{2}{p}\right)\right) \frac{h(-p)}{8} + \frac{h(-3p)}{6},
\]
\[
t^p(\sqrt{3}) = |T^p(D)| = \frac{\zeta_F(-1)}{4} + \left(17 - 2 \left(\frac{2}{p}\right)\right) \frac{h(-p)}{16} + \frac{h(-2p)}{8} + \frac{h(-3p)}{12}.
\]

The theorems are proved. \( \square \)

5. \((P, P_+)-polarized\ SUPERSPECIAL\ ABELIAN\ SURFACES\)

In this section, we study in details the abelian surfaces in the simple \( \mathbb{F}_p \)-isogeny class corresponding to \( \pi = \sqrt{p} \) equipped with the polarization modules \((16, 61, 66 \text{ \S X.1})\). This allows us to generalize the results of \([33\) to nonprincipally polarized abelian surfaces.

5.1. \((P, P_+)-polarized\ abelian\ varieties\). For the moment let \( F \) be a totally real number field, and \( A \) be a \( \mathbb{Z} \)-order in \( F \). Let \( P \) be a finitely generated torsion free \( A \)-module of rank one, i.e. \( P \otimes_A F \cong F \) so that \( P \) is isomorphic to a fractional \( A \)-ideal in \( F \). We say \( P \) is a proper \( A \)-module if \( \text{End}_A(P) = A \), i.e. \( A = \{a \in F \mid aP \subseteq P\} \). There is a canonical \( \mathbb{R} \)-module isomorphism \( F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{[F:\mathbb{Q}]} \), which we denote by \( P_+ \) the pre-image of the set of totally positive elements \( \mathbb{R}^{[F:\mathbb{Q}]} \). Clearly, \( P_+ \) is closed under \( A_+ \)-linear combinations, where \( A_+ \subset F_+ \) denote the subsets of totally positive elements in \( A \) and \( F \) respectively. Denote by
\[
\text{Pic}_+(A) = \text{the set of isomorphism classes of invertible } A \text{-modules with a notion of positivity.}
\]

Each invertible fractional \( A \)-ideal \( \mathfrak{a} \) has a canonical notion of positivity from the \( \mathbb{R} \)-algebra isomorphism \( F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{[F:\mathbb{Q}]} \). The map \( \mathfrak{a} \mapsto (\mathfrak{a}, \mathfrak{a}_+) \) induces a bijection between the narrow class group of \( A \) and \( \text{Pic}_+(A) \), which equips \( \text{Pic}_+(A) \) with a
canonical abelian group structure. From now on we make no distinction between the narrow class group of $A$ and $\text{Pic}_+(A)$ in the above sense.

Fix a base field $k$. By definition, an $F$-abelian variety over $k$ is a pair $(X, \iota)$, where $X$ is an abelian variety over $k$ and $\iota: F \to \text{End}^0(X)$ is a ring homomorphism. We shall assume that $(X, \iota)$ satisfies the condition $\dim X = [F : \mathbb{Q}]$. For an $F$-abelian variety $\mathcal{X} = (X, \iota)$, we put $A := \iota^{-1}(\text{End}(X)) \subseteq F$ and define

$$\mathcal{P}(\mathcal{X}) := \text{Hom}_A(X, X^t)^{\text{sym}} = \text{the Néron-Severi group of } \mathcal{X} = (X, \iota),$$

(5.1)

$$\mathcal{P}_+(\mathcal{X}) := \text{the subset of } A\text{-linear polarizations on } X \text{ (in } \mathcal{P}(\mathcal{X})).$$

By [61, Propositions 1.12 and 1.18] that $\mathcal{P}(\mathcal{X})$ is a finite torsion-free $A$-module of rank one equipped with notion of positivity $\mathcal{P}_+(\mathcal{X})$. The pair $(\mathcal{P}(\mathcal{X}), \mathcal{P}_+(\mathcal{X}))$ is called the ($F$-linear) polarization module of $\mathcal{X}$. In general, $\mathcal{P}(\mathcal{X})$ need not to be a proper $A$-module nor a projective $A$-module. Nevertheless, any rank one proper module over a Gorenstein order is projective by [32, Characterization B 4.2]. In particular, if $A$ is Bass (see [15, §37] and [41] for the definition and properties of Bass orders), i.e. any order in $F$ containing $A$ is Gorenstein, then $\mathcal{P}(\mathcal{X})$ is a projective $\text{End}_A(\mathcal{P}(\mathcal{X}))$-module.

In general, let $R$ be an order in $F$, and $(P, P_+)$ be an invertible $R$-module with a notion of positivity. A $(P, P_+)$-polarized $F$-abelian variety over $k$ is a triple $(X, \iota, \xi)$, where

- $\mathcal{X} = (X, \iota)$ is an $F$-abelian variety over $k$ with $\dim X = [F : \mathbb{Q}]$, and
- $\xi: (P, P_+) \xrightarrow{\sim} (\mathcal{P}(\mathcal{X}), \mathcal{P}_+(\mathcal{X}))$ is an isomorphism of $R$-modules with notion of positivity.

The isomorphism $\xi$ will be called a $(P, P_+)$-parametrization of $(\mathcal{P}(\mathcal{X}), \mathcal{P}_+(\mathcal{X}))$. Two $(P, P_+)$-polarized abelian varieties $(X_i, \iota_i, \xi_i)/k$ for $i = 1, 2$ are isomorphic if there exists a $k$-isomorphism $\alpha: X_1 \to X_2$ such that $\alpha \iota_1(a) = \iota_2(a) \alpha$ for every $a \in F$ and $\alpha^* \xi_2(b) = \xi_1(b)$ for every $b \in P$.

5.2. $(P, P_+)$-polarized superspecial abelian surfaces. Let $F = \mathbb{Q}(\sqrt{p})$, $A = \mathbb{Z}[\sqrt{p}]$, and $D = D_{\infty_1, \infty_2}$. For any abelian surface $X/\mathbb{F}_p$ with $\pi_X^2 = p$, there is a canonical embedding $\iota: F \to \text{End}^0(X)$ sending $\sqrt{p}$ to $\pi_X$ making $(X, \iota)$ an $F$-abelian variety. For simplicity, we omit $\iota$ from the notation. The polarization module $(\mathcal{P}(X), \mathcal{P}_+(X))$ of $X$ is defined as in (5.1). In the present case, $\mathcal{P}(X)$ coincides with the full Néron-Severi group of $X/\mathbb{F}_p$, and $\mathcal{P}_+(X)$ coincides with the set of polarizations. Since any quadratic $\mathbb{Z}$-order is Bass [41, §2.3],

$$\mathcal{P}(X) := (\mathcal{P}(X), \mathcal{P}_+(X))$$

represents an element in $\text{Pic}_+(R)$ with $R = A$ or $O_F$. The association $X \mapsto \mathcal{P}(X)$ induces a map

$$\mathcal{P}: \text{Isog}(\sqrt{p}) \to \text{Pic}_+(A) \amalg \text{Pic}_+(O_F).$$

When $p \not\equiv 1 \pmod{4}$, the map $\mathcal{P}$ sends $\text{Isog}(\sqrt{p})$ to $\text{Pic}_+(O_F)$. When $p \equiv 1 \pmod{4}$, the set $\text{Isog}(\sqrt{p})$ is the union $\Lambda_1^\text{un} \amalg \Lambda_8^\text{un} \amalg \Lambda_{16}^\text{un}$. We claim that

$$\mathcal{P}(\Lambda_1^\text{un}) \subseteq \text{Pic}_+(O_F), \quad \mathcal{P}(\Lambda_8^\text{un}) \subseteq \text{Pic}_+(O_F), \quad \text{and } \mathcal{P}(\Lambda_{16}^\text{un}) \subseteq \text{Pic}_+(A).$$

Indeed, the first inclusion is obvious, and the middle one follows directly from (3.9), so only the last one needs a proof. Nevertheless, for later applications, we treat the cases $\Lambda_1^\text{un}$ (for all $p$) and $\Lambda_{16}^\text{un}$ (for $p \equiv 1 \pmod{4}$) uniformly. Let $R = O_F$ if $r = 1$ and $R = A$ if $r = 16$. The Tate module $T_\ell(X)$ of any member $[X] \in \Lambda_p^\text{un}$
with \( r \in \{1, 16\} \) is a free \( R_\ell \)-module of rank 2 for every prime \( \ell \) (including \( \ell = p \), see [32, Proposition 3.5]). Let \( \Gamma_{\ell F} = \text{Gal}(\overline{F}/F) \) and \( \mathcal{P}(X)_\ell = \mathcal{P}(X) \otimes \mathbb{Z}_\ell \). We have

\[
\mathcal{P}(X)_\ell = \{ \lambda \in \text{Hom}_{\mathbb{Z}_\ell}(\mathcal{T}_\ell(X), \mathcal{T}_\ell(X^t)) | \lambda^t = -\lambda \}
\]

\[
= \left\{ \psi_\ell \in \text{Alt}_{\mathbb{Z}_\ell}(\mathcal{T}_\ell(X) \otimes \mathcal{T}_\ell(X), \mathbb{Z}_\ell) \left| \psi_\ell(ax, y) = \psi_\ell(x, ay) \forall a \in R_\ell \text{ and } x, y \in T_\ell(X) \right. \right\}
\]

\[
= \text{Hom}_{R_\ell}(\lambda_{R_\ell}^2 T_\ell(X), R_\ell^t)
\]

\[
= (\lambda_{R_\ell}^2 T_\ell(X))^* \otimes_{R_\ell} R_\ell^t,
\]

where \( R_\ell^t \subset F_\ell \) is the dual lattice of \( R_\ell \) with respect to the trace \( Tr_{F_\ell/Q} \). Since \( R_\ell \) is Gorenstein, \( R_\ell^t \) is a free \( R_\ell \)-module of rank one by [32] Proposition 3.5 (which also follows immediately from direct calculations). Therefore, \( \mathcal{P}(X) \) is an invertible \( R \)-module, and our claim is verified.

From now on, whenever we write \( r \in \{1, 8, 16\} \) and \( [P, P_+] \in \text{Pic}_+(R_r) \), we assume that one of the following conditions holds:

\[
(a) \ r = 1 \text{ and } R_1 = O_F,
\]

\[
(b) \ r = 8, \ p \equiv 1 \pmod{4}, \text{ and } R_8 = O_F \text{ or } R_{16} = A.
\]

Observe that for all \( r \in \{1, 8, 16\} \) and \( \mathcal{O}_r \) in [31, 4] we have

\[
\text{Nr}(\mathcal{O}_r^?) = \mathcal{R}_r^?.
\]

Let us define

\[
\Lambda^m_r[P, P_+] := \{ [X] \in \Lambda^m_r | \exists \xi : (P, P_+) \xrightarrow{\sim} (\mathcal{P}(X), \mathcal{P}_+(X)) \}.
\]

In other words, \( \Lambda^m_r[P, P_+] \) is the fiber of the element \( [P, P_+] \in \text{Pic}_+(R_r) \) under the map \( \mathcal{P} : \Lambda^m_r \to \text{Pic}_+(R_r) \). Fix a representative \( (P, P_+) \) for the class \( [P, P_+] \in \text{Pic}_+(R_r) \). Let \( \Lambda^m_r(P, P_+) \) denote the set of isomorphism classes of \( (P, P_+) \)-polarized abelian surfaces \( (X, \xi)/\overline{F}_p \) with \([X] \in \Lambda^m_r[P, P_+] \). There is a surjective forgetful map

\[
f : \Lambda^m_r(P, P_+) \to \Lambda^m_r[P, P_+], \ [X, \xi] \mapsto [X].
\]

Our goal is to give adelic double coset descriptions for both \( \Lambda^m_r(P, P_+) \) and \( \Lambda^m_r[P, P_+] \) and compute their class numbers and type numbers for each \( r \in \{1, 8, 16\} \) and each \( [P, P_+] \in \text{Pic}_+(R_r) \).

Fix an abelian surface \( X_0/\overline{F}_p \) with \( \pi^2_{X_0} = p \). Recall that \( \text{Qisog}(X_0) \) denotes the set of equivalence classes of \( \mathbb{Q} \)-isogenies to \( X_0 \) as in [32]. For each member \( (X, \varphi) \in \text{Qisog}(X_0) \), we realize \( \mathcal{P}(X) \) as an \( A \)-submodule of \( \mathcal{P}^0(X_0) := \mathcal{P}(X_0) \otimes \mathbb{Q} \) by pushing forward along \( \varphi \):

\[
\varphi_* \lambda := (\varphi^t)^{-1} \lambda \varphi^{-1} \in \mathcal{P}^0(X_0), \quad \forall \lambda \in \mathcal{P}(X).
\]

Since push-forwards and pull-backs preserve \( \mathbb{Q} \)-polarizations,

\[
\varphi_* \mathcal{P}_+(X) = \varphi_* \mathcal{P}(X) \cap \mathcal{P}^0(X_0),
\]

\[\text{Here we use the notation } \Lambda^m_r(P, P_+) \text{ instead of } \Lambda^m_r[P, P_+] \text{ since the parametrization } \xi : (P, P_+) \xrightarrow{\sim} (\mathcal{P}(X), \mathcal{P}_+(X)) \text{ depends on } (P, P_+) \text{ itself, not just on its isomorphism class } [P, P_+] \in \text{Pic}_+(R_r). \text{ If } \tau : (P, P_+) \to (P', P'_+) \text{ is an isomorphism of invertible } R_\ell \text{-modules with notion of positivity, then } \tau \text{ induces a bijection } \Lambda^m_r(P, P_+) \to \Lambda^m_r(P', P'_+) \text{ sending each } [X, \xi] \text{ to } [X, \xi \tau^{-1}].\]


where $\mathcal{P}_0^0(X_0)$ denotes the subset of $\mathcal{P}^0(X_0)$ consisting of all $\mathbb{Q}$-polarizations (see §3.1.2). For each order $R \supseteq A$ in $F$, the finite idele group $\hat{F}^\times$ acts transitively on the set of invertible $R$-modules inside $\mathcal{P}^0(X_0)$ by

$$a \cdot P := \mathcal{P}^0(X_0) \cap \prod_{\ell} a_{\ell} P_{\ell}, \quad \forall a = (a_{\ell})_{\ell} \in \hat{F}^\times.$$  

Each such invertible $R$-module is equipped with the canonical notion of positivity induced from $\mathcal{P}_0^0(X_0)$.

Let $G$ (resp. $G^1$) be the algebraic $\mathbb{Q}$-groups $D^\times$ (resp. $D^1$) as before. Recall that there is an action of $\hat{D}^\times = G(\overline{\mathbb{Q}})$ on the set $\text{Qisog}(X_0)$, whose orbit containing the base point $(X_0, \text{id}_0)$ is denoted by $\mathcal{G}(X_0, \text{id}_0)$ (or $\mathcal{G}(X_0)$ for brevity) and called the $G$-classes of underlying abelian varieties for $\mathcal{G}(X_0)$ (see Definition 2.1). The set of isomorphism classes of underlying abelian varieties for $\mathcal{G}(X_0)$ is denoted by $\Lambda^\text{un}(X_0)$. Thus $\Lambda^\text{un}(X_0) = \Lambda^\text{un}_1$ if $p \not\equiv 1 \pmod{4}$, and $\Lambda^\text{un}(X_0) = \Lambda^\text{un}_r$ for some $r \in \{1, 8, 16\}$ if $p \equiv 1 \pmod{4}$.

**Proposition 5.1.** Let $(X, \varphi) \in \text{Qisog}(X_0)$ be a member in the same $G$-genus of $(X_0, \text{id}_0)$ so that there exists $g \in \hat{D}^\times$ such that $(X, \varphi) = g(X_0, \text{id}_0)$. Then $\varphi_* \mathcal{P}(X) = \text{Nr}(g)^{-1} \mathcal{P}(X_0)$. In particular, there is a commutative diagram

$$\begin{array}{ccc}
D^\times \backslash \hat{D}^\times / \hat{\mathcal{O}}_0^\times & \rightarrow & \Lambda^\text{un}(X_0) \\
\downarrow \text{Nr}^{-1} & & \downarrow \mathcal{E} \\
F^\times \backslash \hat{F}^\times / \text{Nr}(\hat{\mathcal{O}}_0^\times) & \rightarrow & \text{Pic}_+(R)
\end{array}$$

(5.10)

where $\mathcal{O}_0 := \text{End}(X_0)$ and $R := \text{End}_A(\mathcal{P}(X_0))$. Moreover, the lower horizontal bijection is induced from sending each $a \in \hat{F}^\times$ to the $R$-module $a \cdot \mathcal{P}(X_0)$ with the induced notion of positivity from $\mathcal{P}_0^0(X_0)$.

**Proof.** First, suppose that $[X_0] \in \Lambda^\text{un}_r$ with $r \in \{1, 16\}$. Correspondingly, $R = O_F$ if $r = 1$ and $R = A$ if $r = 16$. Write $(X, \varphi) = g(X_0, \text{id}_0)$ for $g = (g_{\ell})_{\ell} \in \hat{D}^\times$. From Definition 2.1, $\varphi_* T_\ell(X) = T_\ell(X_0)$ for every prime $\ell$. Plugging in (5.3), we obtain

$$\varphi_* \mathcal{P}(X)_{\ell} = (\wedge^2 \mathcal{P}(X))^* \otimes_{R_{\ell}} R_{\ell}^\vee = (\wedge^2 \mathcal{P}(X_0))^* \otimes_{R_{\ell}} R_{\ell}^\vee = \text{Nr}(g_{\ell})^{-1} (\wedge^2 T_\ell(X_0))^* \otimes_{R_{\ell}} R_{\ell}^\vee = \text{Nr}(g_{\ell})^{-1} \mathcal{P}(X_0)_{\ell}.$$  

(5.11)

This proves that $\varphi_* \mathcal{P}(X) = \text{Nr}(g)^{-1} \mathcal{P}(X_0)$, and the commutative diagram follows easily from (5.5) and (5.11).

Next we treat the case $[X_0] \in \Lambda^\text{un}_8$, so $R = O_F$ in this case. For each $(X, \varphi) \in \mathcal{G}(X_0)$, let $\eta : X \to \overline{X}$ be the minimal isogeny constructed in Remark 3.2.7. There is a $\hat{D}^\times$-equivariant map:

$$\mathcal{G}(X_0) \to \mathcal{G}(\overline{X}_0), \quad (X, \varphi) \to (\overline{X}, \overline{\varphi} := \eta \varphi \eta^{-1}).$$

(5.12)

\[\text{It should be emphasized that this bijection is not the canonical group isomorphism as in (4.15). Rather it is the composition of this group isomorphism together with the translation by the narrow class of } \mathcal{P}(X_0).\]
It follows that the surjective map $\Lambda_{\text{un}}^m \to \Lambda_1^m$ sending each $[X] \mapsto [	ilde{X}]$ fits into a commutative diagram

$$
\begin{array}{ccc}
D^\times \backslash \hat{D}^\times / \hat{\mathcal{O}}_0^\times & \xleftarrow{\sim} & \Lambda_{\text{un}}^m \\
\downarrow & & \downarrow \\
D^\times \backslash \hat{D}^\times / \mathcal{O}_0^\times & \xleftarrow{\sim} & \Lambda_1^m
\end{array}
$$

(5.13)

where $\mathcal{O}_0 := \text{End}(\tilde{X}_0)$, and the left vertical map is the canonical projection. From (3.9), the map $\mathcal{P} : \Lambda_{\text{un}}^m \to \text{Pic}(O_F)$ factors through $\Lambda_{\text{un}}^m \to \Lambda^m$. Thus the proposition for the case $r = 8$ is reduced to the $r = 1$ case. □

Corollary 5.2. (1) If $p \not\equiv 1 \pmod{4}$, then $\mathcal{P}([\text{Isog}(\sqrt{p})]) = \text{Pic}_+(O_F)$.

(2) If $p \equiv 1 \pmod{4}$, then

$$
\mathcal{P}(\Lambda_{\text{un}}^m) = \begin{cases} 
\text{Pic}_+(O_F) & \text{for } r = 1, 8; \\
\text{Pic}_+(A) & \text{for } r = 16.
\end{cases}
$$

(5.14)

Proof. This follows directly from the commutative diagram (5.10) since the left vertical map $\text{Pic}^r$ is surjective. □

Corollary 5.3. A member $[X] \in \text{Isog}(\sqrt{p})$ is principal polarizable if and only if $[X] \in \Lambda_1^m[O_F, O_{F,+}] \Pi \Lambda_{16}^m[A, A_+]$.

Moreover, there are canonical bijections:

$$
\Lambda_{16}^m(O_F, O_{F,+}) \cong \Lambda_{16}^P, \quad \Lambda_{16}^m(A, A_+) \cong \Lambda_{16}^P.
$$

(5.15)

Proof. First, suppose that $X$ is equipped with a principal polarization $\lambda : X \to X^t$. Necessarily, $[X] \in \Lambda_1^m \cup \Lambda_{16}^m$ by Proposition 3.2.5. The map

$$
\mathcal{P}(X) \mapsto R := F \cap \text{End}(X), \quad \lambda' \mapsto \lambda^{-1}\lambda'
$$

establishes an isomorphism of $R$-modules that identifies $\mathcal{P}_+(X)$ with $R_{++}$.

Conversely, suppose that $[X] \in \Lambda_1^m \cup \Lambda_{16}^m$ and $(\mathcal{P}(X), \mathcal{P}_+(X))$ represents the trivial class in $\text{Pic}_+(R)$. According to Proposition 3.2.5, there exists a principally polarizable $X_0$ in the same genus of $X$. Let $\varphi : X \to X_0$ be a $\mathbb{Q}$-isogeny and pick $g \in \hat{D}^\times$ such that $(X, \varphi) = g(X_0, \text{id}_0)$ in $\text{Isog}(X_0)$. From Proposition 5.1, $(\mathcal{P}(X), \mathcal{P}_+(X))$ is isomorphic to $\text{Nr}(g)^{-1}(R, R_+)$, so our assumption implies that $\text{Nr}(g) \in F^+_0 = \text{Nr}(\hat{\mathcal{O}}_0^\times)$. Pick $\alpha \in D^\times$ and $u \in \hat{\mathcal{O}}_0^\times$ such that $\text{Nr}(\alpha g u) = 1$. Let $g' = \alpha g u$ and $\varphi' = \alpha \varphi$. Then $g' \in \hat{D}^1$ and $(X, \varphi') = g'(X_0, \text{id}_0)$. Now it follows from Lemma 2.5 that the pull-back of any principal polarization on $X_0$ along $\varphi'$ is a principal polarization on $X$.

Lastly, let $[X, \xi]$ be a member of $\Lambda_{r}^m(R, R_+)$ with $r = 1, 16$ and $R = O_F$ or $A$ accordingly. Then there exists $u \in R_+$ such that $\xi(u)$ is a principal polarization. Comparing the degrees on both sides of $\xi(u) = u\xi(1)$, one immediately sees that $u \in R_+^\times$ and $\xi(1)$ is a principal polarization as well. It follows the map $[X, \xi] \mapsto [X, \xi(1)]$ establishes a bijection $\Lambda_{r}^m(R, R_+) \cong \Lambda_{r}^P$. □

Remark 5.4. Suppose that $p = 2$. Then $\text{Pic}_+(\mathbb{Z}[\sqrt{2}])$ is trivial, and from Theorem 1.2, there is a unique abelian surface $X/F_2$ up to isomorphism satisfying $\pi_2^2 = 2$. It admits a unique principal polarization up to isomorphism by (4.30). Moreover, $\text{End}(X)$ represents the unique type of maximal orders in the quaternion
\(\mathbb{Q}(\sqrt{2})\)-algebra \(D_{\infty_1, \infty_2}\). Therefore, for the rest of this section we assume that \(p\) is an odd prime (unless specified otherwise).

**Proposition 5.5.** Let \(p, r, and \(R_r\) be as in \([5.4]\), and let \((P, P_+)\) be an invertible \(R_r\)-module with a notion of positivity. Fix a member \([X_r, \xi_r] \in \Lambda^\odot_r(P, P_+)\) and put \(\mathcal{O}_r := \text{End}_{\mathbb{F}_p}(X_r)\). Let \(\mathcal{T}_r[P, P_+] \subseteq \text{Tp}(\mathcal{O}_r)\) be the image of the map

\[
(5.16) \quad \mathcal{Y}_r : \Lambda^\odot_r[P, P_+] \to \text{Tp}(\mathcal{O}_r) \quad [X] \mapsto \text{ord}(X).
\]

There is a commutative diagram

\[
\begin{array}{ccc}
\Lambda^\odot_r(P, P_+) & \xrightarrow{f} & \Lambda^\odot_r[P, P_+] \\
\cong & \cong & \cong \\
D^1 \tilde{D}^1 / \tilde{O}^1 & \cong & D^\times \backslash D^\times \tilde{D}^1 \tilde{O}^\times / \tilde{O}^\times \\
\end{array}
\]

where \(f\) is the forgetful map in \([5.7]\), and the bottom two horizontal maps are canonical projections.

**Proof.** From \((5.10)\), \(\Lambda^\odot_r[P, P_+]\) can be identified with the neutral fiber of the map

\[
D^\times \tilde{D}^1 / \tilde{O}^\times \xrightarrow{\text{Nr}^{-1}} \mathcal{F}_+^\times \tilde{F}_+^\times / \text{Nr}(\tilde{O}^\times),
\]

which shows that the middle vertical map is a bijection. The commutativity of right square follows from that of \((5.8)\), and bijectivity of the right vertical map follows from the definition of \(\mathcal{T}_r[P, P_+]\).

To get the left vertical bijection, let \(\mathcal{G}_G^\odot_r(X_r)\) be the subset of the genus \(\mathcal{G}_G(X_r)\) consisting of members \((X, \varphi : X \to X_r)\) such that \(\varphi_* P(X) = P(X_r)\). Given \((X, \varphi) \in \mathcal{G}_G^\odot_r(X_r)\), write \((X, \varphi) = g(X_r, \text{id}_r)\) for some \(g \in \tilde{D}^\times\), where \(\text{id}_r\) denotes the identity map of \(X_r\). Then \(\text{Nr}(g) \in \tilde{R}_r^\times\) by Proposition 5.4, so there exists \(u \in \tilde{O}^\times\) such that \(\text{Nr}(g) = \text{Nr}(u)\) by \((5.8)\). Put \(g' = gu^{-1}\). Then \(\tilde{D}^1\) and \((X, \varphi) = g'(X_r, \text{id}_r)\). It follows that \(\tilde{D}^1\) acts transitively on \(\mathcal{G}_G^\odot_r(X_r)\), that is,

\[
\mathcal{G}_G^\odot_r(X_r) = \mathcal{G}_G^\times(X_r) \simeq \tilde{D}^1 / \tilde{O}^\times.
\]

Each member \((X, \varphi) \in \mathcal{G}_G^\odot_r(X_r)\) gives rise to a \((P, P_+)\)-polarized abelian surface \((X, \xi)\) by putting \(\xi = \varphi^* \xi_r\), i.e. \(\xi(b) = \varphi^* \xi_r(b)\) for every \(b \in P\). If \((X, \varphi)\) and \((X', \varphi')\) in \(\mathcal{G}_G^\odot_r(X_r)\) give rise to isomorphic pairs \((X, \xi)\) and \((X', \xi')\), then there exists an isomorphism \(\alpha : X \to X'\) such that \(\alpha^* \xi' = \xi\) for every \(b \in P\). Let \(\beta = \varphi' \alpha^* \varphi^{-1} \in D^\times\). Then \(\beta(X, \varphi) = (X', \varphi')\) and \(\beta^* \lambda_r = \lambda_r\) for every \(\lambda_r \in \mathcal{P}(X_r)\). This implies that \(\beta \in D^1\) since the Rosati involution induced by every polarization \(\lambda_r \in \mathcal{P}(X_r)\) coincides with the canonical involution on \(D = \text{End}^0(X_r)\). Conversely, if \(\beta(X, \varphi) = (X', \varphi')\) for some \(\beta \in D^1\), then it is straightforward to show that \((X, \xi)\) and \((X', \xi')\) are isomorphic. This proves that there is an injective map

\[
(5.18) \quad D^1 \tilde{D}^1 / \tilde{O}^\times \hookrightarrow \Lambda^\odot_r(P, P_+)
\]

sending the neutral class to \([X_r, \xi_r]\).

Let \((X, \xi)\) be a \((P, P_+)\)-polarized abelian surface with \([X] \in \Lambda^\odot_r\). Fix \(b_r \in P_+\) and put \(\lambda_r := \xi_r(b_r) \in \mathcal{P}_r(X)\) and \(\lambda := \xi(b_r) \in \mathcal{P}_r(X)\). By Lemma 3.1.3 there exists a quasi-isogeny \(\varphi : X \to X_r\) such that \(\varphi^* \lambda_r = \lambda\). As any element \(b \in P\) is of the form \(ab_r\) for some \(a \in F\), we get

\[
(5.19) \quad \varphi^* \xi_r(b) = \varphi^* \xi_r(ab_r) = a \varphi^* \xi_r(b_r) = a \varphi^* \xi_r(b_r) = a \xi(b_r) = \xi(b).
\]
It follows that \((X, \varphi) \in \mathcal{O} \) is bijective as well.

The left square commutes because both double coset descriptions of \(\Lambda^\mu(P, P_+)\) and \(\Lambda^\mu[P, P_+]\) are induced from the same \(\hat{D}^\times\)-action on \(\mathcal{O}(X, r) \subseteq \text{Qisog}(X, r)\).

Let \(O_r\) be as in Proposition 5.5. It is clear from (5.17) that

\[
(5.20) \quad T_r[P, P_+] = \{[O] \mid O \text{ is in the same spinor genus as } O_r\} = T\text{pg}(O),
\]

which generalizes Lemma 4.5. When \(p \equiv 1 \pmod{4}\) and \(r \in \{1, 16\}\), we have seen in (4.21) and (4.22) that \(T\text{pg}(O_r)\) in fact forms a single spinor genus (i.e. \(T\text{pg}(O_r) = T\text{pg}(O_r)\)), so \(T_r[P, P_+] = T\text{pg}(O_r)\) for \(r \in \{1, 16\}\). A similar result holds when \(r = 8\) according to the following corollary.

**Corollary 5.6.** Suppose that \(p \equiv 1 \pmod{4}\). Then the map

\[
(5.21) \quad T^{\mu}_{\text{pg}} : \Lambda^\mu(P, P_+) \to T\text{pg}(O_r)
\]

is bijective if \(r \in \{1, 16\}\), and it is surjective if \(r = 8\). In particular, \(f : \Lambda^\mu(P, P_+) \to \Lambda^\mu[P, P_+] \) is bijective for \(r \in \{1, 16\}\), and \(T_r[P, P_+] = T\text{pg}(O_r)\) for all \(r \in \{1, 8, 16\}\).

**Proof.** In light of Corollary 5.5, the present corollary for the cases \(r \in \{1, 16\}\) is just a generalization of Lemma 4.7 and follow from Lemma 7.1 or Lemma 7.3 as before.

Suppose that \(r = 8\). From (4.10), \(N(\hat{O}_8) = \hat{F}^\times \hat{O}_4^\times\), where \(O_4\) is the Eichler order given in (4.11). Interpreted adelically, the map \(T^{\mu}_{\text{pg}} : \Lambda^\mu(P, P_+) \to T\text{pg}(O_8)\) is the composition of the following canonical projections:

\[
D^\times \backslash \hat{D}^\times / \hat{O}_4^\times \to D^\times \backslash \hat{D}^\times / \hat{F}^\times \hat{O}_4^\times.
\]

The first map is obviously surjective, and the second map is bijective by Lemma 7.1. Thus the composition is surjective.

If \(p \equiv 3 \pmod{4}\), then we have constructed a map \(\Xi : T\text{p}(D) \to \mathcal{O}(O_F)\) in (4.23) from the type set \(T\text{p}(D)\) of maximal orders in \(D\) to the Gauss genus group \(\mathcal{O}(O_F) / \text{Pic}_+ (O_F)^2\), whose fibers are precisely the distinct spinor genera of maximal orders in \(D\). Since \(|\mathcal{O}(O_F)| = 2\) when \(p \equiv 3 \pmod{4}\), \(T\text{p}(D)\) accordingly decomposes into a disjoint union of two spinor genera \(T\text{p}(D) = T\text{p}(D) \cup T\text{p}(D)\), where the principal spinor genus \(T\text{p}(D)\) (i.e. the neutral fiber) consists of the types of maximal orders that occur as endomorphism rings of principally polarizable members of \(\text{Qisog}(\sqrt{\mathcal{O}})\). From (5.20), \(T_1[P, P_+]\) coincides with either \(T\text{p}(D)\) or \(T\text{p}(D)\).

**Lemma 5.7.** Suppose that \(p \equiv 3 \pmod{4}\). The map \(T^{\mu} : \Lambda^\mu(P, P_+) \to T_1[P, P_+]\) is bijective for every \([P, P_+] \in \text{Pic}_+(O_F)\). Moreover,

\[
(5.22) \quad T_1[P, P_+] = \begin{cases} 
T\text{p}(D) & \text{if } [P, P_+] \in \text{Pic}_+(O_F)^2; \\
T\text{p}(D) & \text{otherwise.}
\end{cases}
\]

**Proof.** In light of the surjectivity of \(\Lambda^\mu \to T\text{p}(D)\) in (4.9) and the adellic description of the map \(T^{\mu} : \Lambda^\mu(P, P_+) \to T_1[P, P_+]\) in (5.17), to show that \(T^{\mu}\) is bijective, it is equivalent to prove that the canonical projection map

\[
D^\times \backslash D^\times \hat{D}^\times / \hat{O}_4^\times \to D^\times \backslash D^\times \hat{D}^\times \hat{N}(\hat{O}) / \hat{N}(\hat{O})
\]

is bijective for every maximal \(O_F\)-order \(\hat{O} \subset D\). The proof will be worked out in Lemma 6.1.
To prove (5.22), let us fix a principally polarizable abelian surface $Y_1/F_p$ with $\pi_1 = p$ and put $\Omega_1 := \text{End}(Y_1)$ as in (4.11). Then $[\Omega_1] \in \tilde{T}p^+ (D)$ by definition, and $[\mathcal{O}(Y_1)] \simeq (O_F, O_{F,+})$ by Corollary 5.3. Combining (4.8) and (5.10), we obtain a diagram

\[
\begin{array}{ccc}
\Lambda^\text{un} & \hookrightarrow & D \times \tilde{D}^\times / \tilde{\Omega}_1^\times \\
\downarrow & & \downarrow \text{Nr}^{-1} \\
\text{Pic}_+ (O_F) & \hookrightarrow & F^\times / \tilde{F}^\times \tilde{\Omega}_1^\times \\
\end{array}
\]

Here the commutativity of the right square follows from the construction of $\Xi$ in (4.17), (4.23), and the fact that $|\mathcal{O}(O_F)| = 2$. The middle square is obviously commutative (Recall that ). The choice of $Y_1/F_p$ guarantees that the left square is commutative as well. The composition of the maps in the top row is none other than $\Upsilon^\text{un} : \Lambda^\text{un}_1 \to \tilde{T}p (D)$, and the composition for the bottom row is just the canonical projection $\text{Pic}_+ (O_F) \to \mathcal{O}(O_F)$.

By definition, $\Lambda^\text{un}_1 [P, P_+]$ is the fiber of the map $\mathcal{P}$ over $[P, P_+] \in \text{Pic}_+ (O_F)$, and $\mathcal{T}_1 [P, P_+]$ is the image of $\Lambda^\text{un}_1 [P, P_+]$ under the map $\Upsilon^\text{un}$. From the commutativity of (5.23), we have $\mathcal{T}_1 [P, P_+] \subseteq \tilde{T}p^+ (D)$ if $[P, P_+] \in \text{Pic}_+ (O_F)^2$, and $\mathcal{T}_1 [P, P_+] \subseteq \tilde{T}p^{-} (D)$ otherwise. The equalities now follow from (5.20).

**Remark 5.8.** Let $\mathcal{O}$ be an arbitrary $\Lambda$-order in $D$. Following [7, §1], we say two locally principal right $\mathcal{O}$-ideals $I$ and $I'$ are in the same spinor class if there exists $x \in D^\times \tilde{D}^1$ such that $\tilde{P} = x \tilde{I}$. We denote by $\text{Cl}_{\text{sc}} (\mathcal{O})$ the set of locally principal right $\mathcal{O}$-ideal classes within the spinor class of $\mathcal{O}$ itself (regarded as a principal right $\mathcal{O}$-ideal), and put $h_{\text{sc}} (\mathcal{O}) := |\text{Cl}_{\text{sc}} (\mathcal{O})|$. The set $\text{Cl}_{\text{sc}} (\mathcal{O})$ can be described adelically as

\[
\text{Cl}_{\text{sc}} (\mathcal{O}) \simeq D^\times / (D^\times \tilde{D}^1 \tilde{\Omega}^\times) / \tilde{\Omega}^\times.
\]

Let $\text{SCI} (\mathcal{O})$ be the set of spinor classes of locally principal right $\mathcal{O}$-ideals. Similar to (4.17), $\text{SCI} (\mathcal{O})$ admits an adellic description as follows:

\[
\text{SCI} (\mathcal{O}) \simeq (D^\times \tilde{D}^1) / \tilde{\Omega}^\times / \tilde{\Omega}^\times \overset{\text{Nr}}{\to} F^\times / \tilde{F}^\times / \text{Nr} (\tilde{\Omega}^\times).
\]

Let $[X_r, \xi_r] \in \Lambda^\text{un}_1 [P, P_+]$ and $\mathcal{O}_r = \text{End}(X_r)$ be as in Proposition 5.5. Combining (5.24) with (5.26), we see that the reduced norm map induces an isomorphism

\[
\text{SCI} (\mathcal{O}_r) \simeq \text{Pic}_+ (R_r)
\]

sending the spinor class of $\mathcal{O}_r$ to the identity of $\text{Pic}_+ (R_r)$. On the other hand, if we combine (5.24) with (5.17), then we get a bijection

\[
\Lambda^\text{un}_1 [P, P_+] \simeq \text{Cl}_{\text{sc}} (\mathcal{O}_r)
\]

sending the prior fixed class $[X_r] \in \Lambda^\text{un}_1 [P, P_+]$ to the principal right $\mathcal{O}_r$-ideal class $[\mathcal{O}_r]$. This is a refinement of the bijection $\Lambda^\text{un}_1 \simeq \text{Cl} (\mathcal{O}_r)$ in (4.8). See also [8, (2.11)]. Similarly, under the identifications (5.20) and (5.27), the map $\Upsilon^\text{un} : \Lambda^\text{un}_1 [P, P_+] \to \mathcal{T}_1 [P, P_+]$ can be identified with

\[
\text{Cl}_{\text{sc}} (\mathcal{O}_r) \to \text{Tp}_{\text{sg}} (\mathcal{O}_r), \quad [I] \mapsto [\mathcal{O}(I)],
\]

which refines the map $\Upsilon : \text{Cl} (\mathcal{O}_r) \to \text{Tp} (\mathcal{O}_r)$ in (4.7).
To state the main theorem of class number and type number formulas for our abelian surfaces equipped with polarization modules, we need some extra notation. If \( p \equiv 1 \pmod{4} \), then \( A = \mathbb{Z}[\sqrt{P}] \) is a suborder of index 2 in \( O_F \). We define
\[
\varpi := [O_F^\times : A^\times].
\]
According to \( [73, \S \delta] \), \( \varpi \in \{1, 3\} \), and \( \varpi = 1 \) if \( p \equiv 1 \pmod{8} \). Let \( \delta_{3, \varpi} \) be the Kronecker \( \delta \)-symbol, which takes value 1 if \( \varpi = 3 \), and 0 otherwise. If \( p \equiv 3 \pmod{4} \), then thanks to Lemma 5.7, our formulas are separated into two cases according to whether \( [P, P_+] \in \text{Pic}_+(O_F)^2 \) or not.

**Theorem 5.9.** Let \( p, r \) and \( R_r \) with \( r \in \{1, 8, 16\} \) be as in (5.4). For each \( [P, P_+] \in \text{Pic}_+(R_r) \), we list the formulas for following quantities
\[
h_{r+}^{pm}[P, P_+] := |\Lambda_{r+}^{pm}(P, P_+)|, \quad h_{r+}^{un}[P, P_+] := |\Lambda_{r+}^{un}(P, P_+)|, \quad t_r[P, P_+] := |T_r[P, P_+]|.
\]

1. If \( p = 2, 3, 5 \), then
\[
h_1^{pm}[P, P_+] = h_1^{un}[P, P_+] = t_1[P, P_+] = 1, \quad \forall [P, P_+] \in \text{Pic}_+(O_F).
\]

2. Suppose that \( p \equiv 1 \pmod{4} \). Then
\[
h_1^{pm}[P, P_+] = h_1^{un}[P, P_+] = t_1[P, P_+]
\]
\[
= \frac{\zeta_F(-1)}{2} + \frac{h(-p)}{8} + \frac{h(-3p)}{6} \quad \text{for } p > 5;
\]
\[
h_8^{pm}[P, P_+] = \frac{3}{2} \left(4 - \left(\frac{2}{p}\right)\right) \zeta_F(-1) + \left(2 - \left(\frac{2}{p}\right)\right) \frac{h(-p)}{8},
\]
\[
h_8^{un}[P, P_+] = \frac{3}{2\varpi} \left(4 - \left(\frac{2}{p}\right)\right) \zeta_F(-1) + \left(2 - \left(\frac{2}{p}\right)\right) \frac{h(-p)}{8\varpi} + \frac{\delta_{3, \varpi}}{\varpi} h(-3p),
\]
\[
t_8^{un}[P, P_+] = \frac{1}{2} \left(7 + 2 \left(\frac{2}{p}\right)\right) \zeta_F(-1) + \frac{h(-p)}{8} + \left(1 - \left(\frac{2}{p}\right)\right) \frac{h(-3p)}{6},
\]
\[
h_{16}^{pm}[P, P_+] = h_{16}^{un}[P, P_+] = t_{16}[P, P_+]
\]
\[
= 4 - \left(\frac{2}{p}\right) \zeta_F(-1) + \frac{h(-p)}{4} + \left(2 + \left(\frac{2}{p}\right)\right) \frac{h(-3p)}{6}.
\]

3. Suppose that \( p \equiv 3 \pmod{4} \) and \( p \geq 7 \).

3a. If \( [P, P_+] \in \text{Pic}_+(O_F)^2 \), then
\[
h_1^{pm}[P, P_+] = \frac{\zeta_F(-1)}{2} + \left(11 - 3 \left(\frac{2}{p}\right)\right) \frac{h(-p)}{8} + \frac{h(-3p)}{6},
\]
\[
h_1^{un}[P, P_+] = t_1[P, P_+]
\]
\[
= \frac{\zeta_F(-1)}{4} + \left(17 - \left(\frac{2}{p}\right)\right) \frac{h(-p)}{16} + \frac{h(-2p)}{8} + \frac{h(-3p)}{12}.
\]

3b. If \( [P, P_+] \notin \text{Pic}_+(O_F)^2 \), then
\[
h_1^{pm}[P, P_+] = \frac{\zeta_F(-1)}{2} + 3 \left(1 - \left(\frac{2}{p}\right)\right) \frac{h(-p)}{8} + \frac{h(-3p)}{6},
\]
\[
h_1^{un}[P, P_+] = t_1[P, P_+]
\]
\[
= \frac{\zeta_F(-1)}{4} + 9 \left(1 - \left(\frac{2}{p}\right)\right) \frac{h(-p)}{16} + \frac{h(-2p)}{8} + \frac{h(-3p)}{12}.
\]
Proof. Let \([X_r, \zeta_r]\) be an arbitrary member of \(\Lambda^\text{pm}(P, P_+),\) and put \(O_r := \text{End}_r(X_r)\). It follows from (5.17), Remarks 5.8 and (5.20) that
\[
(5.40) \quad h^\text{pm}_r[P, P_+] = h^1(O_r), \quad h^\text{un}_r[P, P_+] = h_{sc}(O_r), \quad t_r[P, P_+] = |Tp_{sg}(O_r)|.
\]

The following inequalities always hold true:
\[
(5.41) \quad h^\text{pm}_r[P, P_+] \geq h^\text{un}_r[P, P_+] \geq t_r[P, P_+].
\]

Moreover, if \(p \equiv 1 \pmod{4}\) and \(r \in \{1, 16\}\), all three quantities are equal by Corollary 5.6 if \(p \equiv 3 \pmod{4}\) (hence necessarily \(r = 1\)), then \(h^\text{un}_r[P, P_+] = t_1[P, P_+]\) according to Lemma 5.7.

Already, this explains why the formula in (5.31) (resp. (5.35)) is identical to the one in (4.33) (resp. (4.34)). A priori, (4.34) is stated for primes \(p \equiv 1 \pmod{4}\) and \(p \geq 13\). Nevertheless, a direct calculation in the case \(p = 5\) (cf. (4.32)) shows that it holds for all primes \(p \equiv 1 \pmod{4}\).

Next, suppose that \(p \equiv 1 \pmod{4}\) and \(r = 8\). The formula for \(h^\text{pm}_r[P, P_+] = h^1(O_8)\) is calculated in Proposition 7.3. According to Corollary 5.6, \(T_8[P, P_+] = \text{Tp}(O_8)\). We find that \(t_5[P, P_+] = t(O_8)\), which is calculated in \(\text{[77]}\) (4.11). Similar to the \(r = 16\) case, the formula for \(t(O_8)\) in (5.34) holds for all primes \(p \equiv 1 \pmod{4}\) (cf. (77) (4.8)). To calculate \(h^\text{un}_r[P, P_+]\), we claim that its value is independent of the choice of \([P, P_+] \in \text{Pic}_+(O_F)\). Indeed, the class number \(h_{sc}(O_8)\) depends only on the spinor genus of \(O_8\), while the entire genus of \(O_8\) forms a single spinor genus by combining (5.20) and Corollary 5.6. This verifies the claim and shows that
\[
(5.42) \quad h^\text{un}_r[P, P_+] = \frac{|\Lambda^\text{un}_r|}{|\text{Pic}_+(O_F)|} = \frac{h(O_8)}{h(F)} = \frac{h(O_8)}{h(F)}.
\]

The class number formula for \(O_8\) has already been calculated in \(\text{[75]}\) (6.8) and (6.10). One then use a result of Herglotz (see \(\text{[78]}\) §2.10) to factor \(h(F)\) out from \(h(O_8)\) to obtain (5.33).

Now suppose that \(p \equiv 3 \pmod{4}\). It is shown in Lemma 5.7 that \(O_1\) belongs to the principal spinor genus of maximal \(O_F\)-orders in \(D\) if and only if \([P, P_+] \in \text{Pic}_+(O_F)^2\). In particular, \(t_1[P, P_+] = |\text{Tp}^+(D)|\) if \([P, P_+] \in \text{Pic}_+(O_F)^2\), and \(t_1[P, P_+] = |\text{Tp}^-(D)|\) otherwise. The class number formulas for \(h^1(O_1)\) (for both the principal and nonprincipal cases) and type number formulas for \(|\text{Tp}^+(D)|\) and \(|\text{Tp}^-(D)|\) are produced in Proposition 6.2.2. This takes care of (5.30) for \(p = 3\) and the formulas (5.36) (5.39) for \(p \equiv 3 \pmod{4}\) and \(p > 7\).

Lastly, if \(p \in \{2, 5\}\), then \(\text{Pic}_+(O_F)\) consists a single member, namely \([O_F, O_{F,+}]\). In light of Corollary 5.5 (5.50) is just a restatement of (4.30) (resp. (4.31)) in the case \(p = 2\) (resp. \(p = 5\)).

Remark 5.10. We would like to exhibit some identities relating the following three kinds of objects:

- the arithmetic genera of some Hilbert modular surfaces;
- the class numbers \(h^\text{pm}_r[P, P_+]\), \(h^\text{un}_r[P, P_+]\) and type numbers \(t_1[P, P_+]\);
- the proper class numbers of quaternary positive definite even quadratic lattices within certain genera.

Let \(\text{GL}_2^+(F)\) be the subgroup of \(\text{GL}_2(F)\) consisting of the elements with totally positive determinants, and \(\text{PGL}_2^+(F)\) be its canonical image in \(\text{PGL}_2(F)\). For each nonzero fractional ideal \(a\) of \(O_F\), we define three arithmetic subgroups
\[
\Gamma(O_F \oplus a) \subseteq \Gamma'(O_F \oplus a) \subseteq \Gamma_m(O_F \oplus a)
\]
of \( \text{PGL}_2^+ (F) \) as follows. We write \( \text{SL}(O_F \oplus \mathfrak{a}) \) for the stabilizer of \( O_F \oplus \mathfrak{a} \) in \( \text{SL}_2 (F) \), and \( \Gamma(O_F \oplus \mathfrak{a}) \) for its image in \( \text{PGL}_2^+ (F) \). Let \( K_a \subset \text{GL}_2 (\hat{F}) \) be the stabilizer of the \( \hat{O}_F \)-lattice \( \hat{O}_F \oplus \hat{\mathfrak{a}} \), and let \( \hat{\Gamma}(O_F \oplus \mathfrak{a}) \subset \text{PGL}_2^+ (F) \) be the image of the subgroup \( K_a \cap \text{GL}_2^+ (F) \). We refer the definition of the Hurwitz-Maass extension \( \Gamma_m (O_F \oplus \mathfrak{a}) \) of \( \Gamma(\hat{O}_F \oplus \hat{\mathfrak{a}}) \) to [66 §1.4]. It is shown that \( \Gamma_m (O_F \oplus \mathfrak{a}) \) is the maximal discrete subgroup of \( \text{PGL}_2^+ (\mathbb{R}) [F^\mathfrak{a}] \) that contains \( \hat{\Gamma}(O_F \oplus \mathfrak{a}) \); see [66, p. 13]. One can show that this is the same as the image of the subgroup \( \text{GL}_2^+ (F) \cap N(K_a) \) in \( \text{PGL}_2^+ (F) \), where \( N(K_a) \subset \text{GL}_2 (\hat{F}) \) is the normalizer of \( K_a \).

Up to conjugation in \( \text{PGL}_2^+ (F) \), these three groups depend only on the Gauss genus \( \gamma := \text{Pic}_+ (O_F)^2 [\mathfrak{a}]_+ \in \text{Pic}_+ (O_F) / \text{Pic}_+ (O_F)^2 \) represented by \( \mathfrak{a} \).

Let \( \mathcal{H} \) be the upper half plane of \( \mathbb{C} \) as usual. For \( \Gamma = \Gamma(O_F \oplus \mathfrak{a}) \) or \( \Gamma_m (O_F \oplus \mathfrak{a}) \), the Hilbert modular surface \( Y_\mathcal{H} \) is defined to be the minimal non-singular model of the compactification of \( \Gamma \backslash \mathcal{H}^2 \) [66 §§II.7]. The arithmetic genus \( \chi (Y_\mathcal{H}) \) is an important global invariant of \( Y_\mathcal{H} \) (see [66 §§VI]) and \( \Gamma_m (O_F \oplus \mathfrak{a}) \) depend only on the Gauss genus \( \gamma = \text{Pic}_+ (O_F)^2 [\mathfrak{a}]_+ \), we denote them as \( Y(\mathfrak{d}_F, \gamma) \) and \( Y_m (\mathfrak{d}_F, \gamma) \) respectively, where \( \mathfrak{d}_F \) is the discriminant of \( \mathbb{Q}(\sqrt{p}) \) as in Remark [66, §5.4]. Similarly, let us put

\[
(5.43) \quad h_1^{un} (\gamma) := h_1^{un} [\mathfrak{a}, \mathfrak{a}_+] , \quad h_1^{un} (\gamma) := h_1^{un} [\mathfrak{a}, \mathfrak{a}_+] , \quad t_1 (\gamma) = t_1 [\mathfrak{a}, \mathfrak{a}_+] ,
\]

where \( \mathfrak{a} \) is equipped with the canonical notion of positivity \( \mathfrak{a}_+ \) (see §3.1). Lastly, according to [10], the genera of quaternary positive definite even quadratic lattices of discriminant \( \mathfrak{d}_F \) can be labeled by the Gauss genus group \( \text{Pic}_+ (O_F) / \text{Pic}_+ (O_F)^2 \).

Let \( H^+ (\mathfrak{d}_F, \gamma) \) denote the proper class number of such lattices in the genus labeled by \( \gamma \) (cf. [10 §3.2]).

The formulas for \( \chi (Y(\mathfrak{d}_F, \gamma)) \) can be found in [23 Theorems II.5.8-9], and those for \( \chi (Y_m (\mathfrak{d}_F, \gamma)) \) and \( H^+ (\mathfrak{d}_F, \gamma) \) can be found in [10 §1]. Comparing these formulas with the ones in Theorem 5.4, we immediately obtain several interesting identities (5.44)–(5.47) described below.

First suppose that \( p \not\equiv 3 \pmod{4} \). Then there is a unique Gauss genus \( \gamma \), and hence a unique genus of quaternary positive definite even quadratic lattices of discriminant \( \mathfrak{d}_F \). The two groups \( \Gamma_m (O_F \oplus \mathfrak{a}) \) and \( \Gamma(O_F \oplus \mathfrak{a}) \) coincide by [66 §1.4, p. 13], which implies that \( Y(\mathfrak{d}_F, \gamma) = Y_m (\mathfrak{d}_F, \gamma) \). In this case we have

\[
(5.44) \quad \chi (Y(\mathfrak{d}_F, \gamma)) = \chi (Y_m (\mathfrak{d}_F, \gamma)) = h_1^{un} (\gamma) = h_1^{un} (\gamma) = t_1 (\gamma) = H^+ (\mathfrak{d}_F, \gamma).
\]

The coincidence of the arithmetic genus \( \chi (Y(\mathfrak{d}_F, \gamma)) \) with \( H^+ (\mathfrak{d}_F, \gamma) \) has been observed by many authors [10, 37, 65].

Now suppose that \( p \equiv 3 \pmod{4} \). There are two Gauss genera, so we denote the principal Gauss genus by \( \gamma^+ \), and the nonprincipal one by \( \gamma^- \). In this case

\[
(5.45) \quad \chi (Y(4p, \gamma^+)) = h_1^{un} (\gamma^-) , \quad \chi (Y(4p, \gamma^-)) = h_1^{un} (\gamma^+) .
\]

Moreover,

\[
(5.46) \quad \chi (Y_m (4p, \gamma^+)) = h_1^{un} (\gamma^-) = t_1 (\gamma^-) = H^+ (4p, \gamma^+),
\]

\[
(5.47) \quad \chi (Y_m (4p, \gamma^-)) = h_1^{un} (\gamma^+) = t_1 (\gamma^+) = H^+ (4p, \gamma^-). \]

Once again, the coincidence between the arithmetic genus \( \chi (Y_m (4p, \gamma)) \) and the proper class number \( H^+ (4p, \gamma) \) has been observed by Chan and Peters [10].
6. The class and type number formulas for maximal orders

Throughout this section, \( p \in \mathbb{N} \) denotes a prime number, \( F = \mathbb{Q}(\sqrt{p}) \), and \( D = D_{x_1, \infty} \) denotes the totally definite quaternion \( F \)-algebra that splits at all finite primes of \( F \). All maximal \( OF \)-orders in \( D \) form a single genus. Let \( \text{Tp}(D) \) be the type set of maximal orders in \( D \), and \( \mathcal{O} \subset D \) be an arbitrary maximal order. The goal of this section is to compute the following class numbers

\[
\begin{align*}
(6.1) \quad & h^1(\mathcal{O}) := |D^1 \setminus \mathcal{D}^1 / \mathcal{D}^1|, \\
& h_{\text{sc}}(\mathcal{O}) := |\text{Cl}_{\text{sc}}(\mathcal{O})| = |\mathcal{D}^1 \setminus (\mathcal{D}^1 \mathcal{N}(\mathcal{O})) / \mathcal{N}(\mathcal{O})|,
\end{align*}
\]

where \( \mathcal{D}^1 \) denotes the reduced norm one subgroup of \( \mathcal{D} \), and \( \mathcal{D}^1, \mathcal{O}^1 \) are defined similarly. Moreover, \( \text{Cl}_{\text{sc}}(\mathcal{O}) \) denotes the set of locally principal right \( \mathcal{O} \)-ideal classes within the spinor class of principal ideal \( \mathcal{O} \) itself as defined inside Remark 5.8. We first show that in our case the class number \( h_{\text{sc}}(\mathcal{O}) \) coincides with the type number of maximal orders within the spinor genus of \( \mathcal{O} \).

**Lemma 6.1.** Let \( \text{Tp}_{\text{sg}}(\mathcal{O}) \) be the subset of \( \text{Tp}(D) \) consisting of types of maximal orders in the spinor genus of \( \mathcal{O} \) as in (6.20). Then the following map is bijective:

\[
(6.2) \quad \mathcal{Y} : \text{Cl}_{\text{sc}}(\mathcal{O}) \to \text{Tp}_{\text{sg}}(\mathcal{O}), \quad [I] \mapsto [\mathcal{O}_I(I)].
\]

**Proof.** Adelically, this map is given by the canonical projection

\[
(6.3) \quad \mathcal{N}(\mathcal{O}) \to \mathcal{N}(\mathcal{O}) / \mathcal{N}(\mathcal{O}),
\]

where \( \mathcal{N}(\mathcal{O}) \) denotes the normalizer of \( \mathcal{O} \) in \( \mathcal{D} \). Since \( D \) splits at all the finite primes of \( F \) and \( \mathcal{O} \) is maximal, we have \( \mathcal{N}(\mathcal{O}) = \mathcal{F}^\times \mathcal{O}^\times \) as in (4.10). Thus to show that the map in (6.3) is injective, it is enough to show that the following \( D^\times \)-equivariant surjective canonical projection map

\[
(6.4) \quad (\mathcal{D}^1 \mathcal{N}(\mathcal{O})) / (\mathcal{D}^1 \mathcal{N}(\mathcal{O})) \to (\mathcal{D}^1 \mathcal{N}(\mathcal{O})) / \mathcal{N}(\mathcal{O})
\]

is injective as well. Since \( \mathcal{D}^1 \) is normal in \( \mathcal{D} \) with an abelian quotient \( \mathcal{D} \mathcal{D}^1 \mathcal{N}(\mathcal{O}) \approx \mathcal{F}^\times \mathcal{D}^1 \approx \mathcal{F}^\times \mathcal{D}^1 \mathcal{N}(\mathcal{O}) \) both \( \mathcal{D}^1 \mathcal{N}(\mathcal{O}) \) and \( \mathcal{D}^1 \mathcal{N}(\mathcal{O}) / \mathcal{D}^1 \mathcal{N}(\mathcal{O}) \) are subgroups of \( \mathcal{D}^\times \).

According to [11], \( h(F) = h(\mathbb{Q}(\sqrt{p}))) \) is odd for every prime \( p \), so \( F^\times_+ \cap \mathcal{F}^\times_2 \mathcal{O}_F^\times = F^\times_2 \mathcal{O}_F^\times \), where \( F^\times_+ \) denotes the subgroup of totally positive elements of \( F^\times \), and \( \mathcal{O}_F^\times \) denotes the group of totally positive units. Indeed, given \( \alpha \in F^\times_+ \cap \mathcal{F}^\times_2 \mathcal{O}_F^\times \), the principal ideal \( \alpha \mathcal{O}_F \) is a perfect square, that is, \( \alpha \mathcal{O}_F = \beta^2 \) for some fractional \( \mathcal{O}_F \)-ideal \( \beta \subseteq F \). Now it follows from the class number parity statement above that \( \beta \) is principal as well. Write \( \beta = \beta \mathcal{O}_F \) for some \( \beta \in F^\times \). Then \( \alpha = \beta^2 u \) for some \( u \in \mathcal{O}_F^\times_+ \).

Thus if \( x \in \mathcal{D}^1 \mathcal{D}^1 \mathcal{N}(\mathcal{O}) \cap \mathcal{D}^\times \mathcal{O}^\times \), then \( \text{Nr}(x) \in (\mathcal{F}^\times_2 \mathcal{O}_F^\times \cap \mathcal{F}^\times_2 \mathcal{O}_F^\times) = F^\times_2 \mathcal{O}_F^\times \), and hence there exists \( a \in F^\times \) such that \( \text{Nr}(ax) \in \mathcal{O}_F^\times \). On the other hand, any element \( y \in \mathcal{F}^\times \mathcal{O}^\times \) with \( \text{Nr}(y) \in \mathcal{O}_F^\times \) necessarily lies in \( \mathcal{O}_F^\times \). It follows that

\[
\mathcal{D}^1 \mathcal{N}(\mathcal{O}) \cap \mathcal{D}^\times \mathcal{O}^\times = F^\times \mathcal{O}_F^\times,
\]

which in turn implies the injectivity of (6.4). The lemma is proved.

As it turns out, if \( p \not\equiv 3 \pmod{4} \) then the computations of both \( h^1(\mathcal{O}) \) and \( h_{\text{sc}}(\mathcal{O}) \) are relatively easy because we have even more coincidences of class numbers and type numbers. Recall that \( \zeta_F(s) \) denotes the Dedekind zeta function of the quadratic real field \( F = \mathbb{Q}(\sqrt{p}) \), and \( h(d) \) denotes the class number of the quadratic field \( \mathbb{Q}(\sqrt{d}) \) for each square-free integer \( d \in \mathbb{Z} \).
Superspecial Abelian Surfaces 35

Proposition 6.2. If \( p \not\equiv 3 \pmod{4} \), then

\[
(6.5) \quad h^1(\mathcal{O}) = h_{sc}(\mathcal{O}) = t(\mathcal{O}) = h(\mathcal{O})/h(F),
\]

where \( t(\mathcal{O}) := |\text{Tp}(D)| \) denotes the type number of \( \mathcal{O} \), and \( h(\mathcal{O}) := |\text{Cl}(\mathcal{O})| \) denotes the class number of \( \mathcal{O} \). In particular,

\[
(6.6) \quad h^1(\mathcal{O}) = h_{sc}(\mathcal{O}) = 1 \quad \text{if} \quad p \in \{2, 5\}, \quad \text{and}
\]

\[
(6.7) \quad h^1(\mathcal{O}) = h_{sc}(\mathcal{O}) = \frac{\zeta_F(-1)}{2} + \frac{h(-p)}{8} + \frac{h(-3p)}{6} \quad \text{if} \quad p \equiv 1(\text{mod} 4) \quad \text{and} \quad p > 5.
\]

Proof. From Lemma 6.1 \( h_{sc}(\mathcal{O}) = |\text{Tp}_{sg}(\mathcal{O})| = t(\mathcal{O}) \), where the last equality follows from the fact that all maximal orders in \( D \) belong to the same spinor genus in this case by Proposition 1.6. The equality \( t(\mathcal{O}) = h(\mathcal{O})/h(F) \) has been proved in [77] Proposition 4.1, and we shall prove the equality \( h^1(\mathcal{O}) = t(\mathcal{O}) \) in Lemma 6.1. The formulas in (6.6) and (6.7) are reproduced from the type number formulas in [77] (4.5) and (4.8). In light of the class-type number relationship in (6.5), these formulas are not new. See §4.1 for a brief historical note. \( \Box \)

The computations for \( h^1(\mathcal{O}) \) and \( h_{sc}(\mathcal{O}) \) when \( p \equiv 3 \pmod{4} \) are made difficult by the existence of two spinor genera of maximal orders in \( D \) according to Proposition 1.6. A systematic study of these class number formulas were carried out by the current authors in [79]. For the moment let \( F \) be a totally real number field, \( D \) be a totally definite quaternion \( F \)-algebra, and \( \mathcal{O} \subset D \) be an Eichler order. We are able to derive in [79] Corollary 4.2 a class number formula for \( h^1(\mathcal{O}) := |D^{1}\backslash D^{1}/\hat{\mathcal{O}}^{1}| \) using the Selberg trace formula. The proof of this formula is greatly simplified by Voight [79] Appendix A], who gives an alternative proof by replacing calculations with the Selberg trace formula with a direct, conceptual argument. A formula for \( h_{sc}(\mathcal{O}) := |D^{\times}\backslash (D^{\times}D^{1}\hat{\mathcal{O}}^{1}\backslash \hat{\mathcal{O}}^{\times})| \) is also obtained in [79] Theorem 3.2 by refining the original proof of the classical Eichler class number formula [69] Corollaire V.2.5, p. 144 [70] Theorem 30.8.6]. However, different from Eichler’s formula, which depends purely on the local datum of \( \mathcal{O} \) (namely, the genus of \( \mathcal{O} \)), the formulas for both \( h^1(\mathcal{O}) \) and \( h_{sc}(\mathcal{O}) \) depend more subtly on the spinor genus of \( \mathcal{O} \), which has a mixture of local and global flavor; see Definition 4.4. In particular, our formulas require certain new invariants arising from the theory of optimal spinor selectivity, which we recall briefly below.

6.1. Optimal spinor selectivity. The selectivity theory is first formulated by Chinburg and Friedman [11] as an refined integral version of the Hasse-Brauer-Noether-Albert Theorem [69] Theorem III.3.8], and has since been further developed by many people. See [70] §31.7.7] for a brief historical account. We shall follow closely the exposition in [80], which is most suitable for the current purpose.

Let \( F, D \) be as above, and \( \mathfrak{n} \subseteq O_F \) be an integral ideal coprime to the discriminant of \( D \). All Eichler orders of level \( \mathfrak{n} \) in \( D \) form a single genus \( \mathcal{G}_n \). The spinor genus of \( \mathcal{O} \) is denoted by \( [\mathcal{O}]_{sg} \). Let \( \text{SG}(\mathcal{G}_n) \) be the set of spinor genera within \( \mathcal{G}_n \). In other words, \( \text{SG}(\mathcal{G}_n) = \{ [\mathcal{O}]_{sg} \mid \mathcal{O} \in \mathcal{G}_n \} \).

Since \( D \) is totally definite, a quadratic field extension \( K/F \) embeds into \( D \) only if \( K/F \) is a CM-extension. An \( O_F \)-order \( B \) (of full rank) in a CM-extension of \( F \) will be called a CM \( O_F \)-order. Given a CM \( O_F \)-order \( B \) with fractional field \( K \) and an Eichler order \( \mathcal{O} \in \mathcal{G}_n \), we write \( \text{Emb}(B, \mathcal{O}) \) for the set of optimal embeddings of
\( B \) into \( \mathcal{O} \), that is
\[
(6.8) \quad \text{Emb}(B, \mathcal{O}) := \{ \varphi \in \text{Hom}_F(\mathcal{K}, \mathcal{D}) \mid \varphi(\mathcal{K}) \cap \mathcal{O} = \varphi(B) \}.
\]
Similarly, at each finite prime \( p \) of \( \mathcal{F} \), we write \( \text{Emb}(B_p, \mathcal{O}_p) \) for the set of optimal embeddings of \( B_p \) into \( \mathcal{O}_p \). Clearly, if \( \text{Emb}(B, \mathcal{O}) \neq \emptyset \), then
\[
(6.9) \quad \text{Emb}(B_p, \mathcal{O}_p) \neq \emptyset \quad \text{for every finite prime } p \text{ of } \mathcal{F}.
\]
Conversely, if condition (6.9) holds\(^6\), then there exists an order \( \mathcal{O}_0 \in \mathcal{G}_n \) such that \( \text{Emb}(B, \mathcal{O}_0) \neq \emptyset \) by [70, Corollary 30.4.18]. However, in the totally definite case, it does not make sense to expect \( \text{Emb}(B, \mathcal{O}') \neq \emptyset \) for every order \( \mathcal{O}' \in \mathcal{G}_n \); see [80, Example 2.1]. Instead, one asks whether condition (6.9) implies the existence of an order \( \mathcal{O}' \) in every spinor genus of \( \mathcal{G}_n \) such that \( \text{Emb}(B, \mathcal{O}') \neq \emptyset \). It turns out that while the answer is positive for most CM \( \mathcal{O}_F \)-orders and Eichler orders, there exist some particular cases where the answer is negative. The selectivity theory is developed to give a concrete criterion for when the answer is negative.

Let us first define the \textit{optimal spinor selectivity symbol} as follows
\[
(6.10) \quad \Delta(B, \mathcal{O}) = \begin{cases} 1 & \text{if } \exists \mathcal{O}' \in [\mathcal{O}]_{\mathcal{G}} \text{ such that } \text{Emb}(B, \mathcal{O}') \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases}
\]

\textbf{Definition 6.1.1.} Suppose that condition (6.9) holds. We say \( B \) is \textit{optimally spinor selective} (selective for short) for the genus \( \mathcal{G}_n \) if \( \Delta(B, \mathcal{O}) = 0 \) for some (but not all) \( [\mathcal{O}]_{\mathcal{G}} \in \text{SG}(\mathcal{G}_n) \). If \( B \) is selective for \( \mathcal{G}_n \), then a spinor genus \( [\mathcal{O}]_{\mathcal{G}} \) with \( \Delta(B, \mathcal{O}) = 1 \) is said to be \textit{selected} by \( B \).

To state the main theorem of optimal spinor selectivity, we need some more notation. For each finite prime \( p \) of \( \mathcal{O}_F \), we write \( \nu_p : \mathcal{F}^\times \rightarrow \mathbb{Z} \) for the normalized discrete valuation of \( \mathcal{F} \) attached to \( p \). Let \( \mathcal{O}(\mathcal{O}) \) be the \textit{reduced discriminant} of \( \mathcal{O} \in \mathcal{G}_n \), which is the product of \( n \) with all the finite ramified primes of \( \mathcal{D}/\mathcal{F} \). Given a CM \( \mathcal{O}_F \)-order \( B \) with fractional field \( \mathcal{K} \), we write \( \mathcal{I}(B) \) for the \textit{conductor} of \( B \). In other words, \( \mathcal{I}(B) \) is the unique integral ideal of \( \mathcal{O}_F \) such that \( B = \mathcal{O}_F + \mathcal{I}(B)\mathcal{K} \). If \( B' \) is another \( \mathcal{O}_F \)-order in \( \mathcal{K} \), then we put \( \mathcal{I}(B'/B) := \mathcal{I}(B')^{-1}\mathcal{I}(B) \) and call it the \textit{relative conductor} of \( B \) with respect to \( B' \), which is now a fractional ideal of \( \mathcal{O}_F \).

The following theorem is a combination of [80, Theorem 2.15, Lemma 2.17, and Theorem 3.2].

\textbf{Theorem 6.1.2.} Let \( \mathcal{G}_n \) be the genus of Eichler orders of level \( n \) in the totally definite quaternion \( \mathcal{F} \)-algebra \( \mathcal{D} \), and \( B \) be a CM \( \mathcal{O}_F \)-order with fractional field \( \mathcal{K} \). Suppose that condition (6.9) holds for \( B \) and \( \mathcal{G}_n \). Then
\begin{enumerate}
  \item \( B \) is optimally spinor selective for the genus \( \mathcal{G}_n \) if and only if both of the following conditions hold:
    \begin{enumerate}
      \item both \( \mathcal{K} \) and \( \mathcal{D} \) are unramified at every finite prime \( p \) of \( \mathcal{F} \);
      \item \( \mathcal{K}/\mathcal{F} \) splits at every finite prime \( p \) of \( \mathcal{F} \) satisfying \( \nu_p(\mathcal{O}(\mathcal{O})) \equiv 1 \pmod{2} \).
    \end{enumerate}
  \item If \( B \) is optimally spinor selective for \( \mathcal{G}_n \), then \( \Delta(B, \mathcal{O}) = 1 \) for exactly half of the spinor genera \( [\mathcal{O}]_{\mathcal{G}} \in \text{SG}(\mathcal{G}_n) \).
\end{enumerate}

\(^6\)A concrete criterion for the existence of local optimal embeddings into Eichler orders is given by Guo and Qin in [26, Lemma 2.2], which in turn is based on a theorem of Brzezinski [5, Theorem 1.8]. In our case, condition (6.9) holds automatically for every maximal \( \mathcal{O}_F \)-order \( \mathcal{O} \subseteq \mathcal{D}_{\infty_1, \infty_2} \) and every \( \mathcal{O}_F \)-order \( B \) in a CM-extension of \( \mathcal{F} = \mathbb{Q}(\sqrt{\mathcal{P}}) \) by [69, Theoreme II.3.2].
(3) Suppose that \( \mathcal{K} \) and \( \mathcal{D} \) satisfy both of the conditions in part (1) of the theorem so that \( B \) is optimally spinor selective for \( \mathcal{G}_n \). Let \( B' \) be another \( O_F \)-order in \( \mathcal{K} \) satisfying condition (3.1). Then for every \( \mathcal{O} \in \mathcal{G}_n \), we have

\[
\Delta(B, \mathcal{O}) = (\{B'/B), \mathcal{K}/F\} + \Delta(B', \mathcal{O}),
\]

where \( \{B'/B), \mathcal{K}/F\} \in \text{Gal}(\mathcal{K}/F) \) is the Artin symbol [38, §X.1], and the summation on the right hand side of (6.11) is taken inside \( \mathbb{Z}/2\mathbb{Z} \) with the canonical identification \( \text{Gal}(\mathcal{K}/F) = \mathbb{Z}/2\mathbb{Z} \).

In fact, if \( B \) is selective for the genus \( \mathcal{G}_n \), then starting from a known selectivity symbol \( \Delta(B, \mathcal{O}) \) for some \( \mathcal{O} \in \mathcal{G}_n \), we can express any other \( \Delta(B, \mathcal{O}') \) with \( \mathcal{O}' \in \mathcal{G}_n \) in terms of \( \Delta(B, \mathcal{O}) \) and the “relative position” of \( \mathcal{O}' \) in \( \mathcal{G}_n \). See [80, Theorem 2.15] or [70, §31.1.9] for more details. In general, for an arbitrary CM \( O_F \)-order \( B \) with fractional field \( \mathcal{K} \), we put

\[
s(B, \mathcal{O}) := \begin{cases} 
1 & \text{if } \mathcal{K}, \mathcal{D} \text{ satisfy conditions (a) and (b) in Theorem 6.1.2} \\
0 & \text{otherwise.}
\end{cases}
\]

By definition, \( s(B, \mathcal{O}) \) depends only on \( \mathcal{K}, \mathcal{D} \) and \( n \).

Theorem 6.1.2 was first obtained by Maclachlan [45] for Eichler orders of square-free levels in indefinite quaternion algebras. Independently, Arenas et al. [4] and Voight [70, Chapter 31] removed the square-free condition and obtained the first two parts of the theorem for Eichler orders of arbitrary levels. The generalization of (6.11) to Eichler orders of arbitrary levels are due to the current authors [80]. This concludes our brief account of the optimal spinor selectivity theory.

6.2. The class number formulas for \( h^1(\mathcal{O}) \) and \( h_\infty(\mathcal{O}) \) when \( p \equiv 3 \pmod{4} \). We return to the original setup where \( F = \mathbb{Q}(\sqrt{p}) \) and \( \mathcal{O} \) is a maximal \( O_F \)-order in the quaternion \( F \)-algebra \( D = D_{\infty_1, \infty_2} \). Assume that \( p \equiv 3 \pmod{4} \) for the rest of this section unless specified otherwise.

Let \( O_{F, +}^\times := F_+ \cap O_F^\times \) be the group of totally positive units of \( F \), and \( \varepsilon > 1 \) be the fundamental unit of \( O_F^\times \). By [2] §11.5 or [13, Corollary 18.4bis], \( \varepsilon \) is totally positive (i.e. \( N_{F/\mathbb{Q}}(\varepsilon) = 1 \)) since \( p \equiv 3 \pmod{4} \). Hence \( O_{F, +}^\times = \langle \varepsilon \rangle \) while \( O_F^{\times 2} = \langle \varepsilon^2 \rangle \), from which it follows that

\[
[O_{F, +}^\times : O_F^{\times 2}] = 2.
\]

According to [13, Lemma 11.6], the narrow class number \( h_+(O_F) \) is related to the (wide) class number \( h(F) \) by \( h_+(F) = h(F)[O_{F, +}^\times : O_F^{\times 2}] \), and hence

\[
h_+(F) = 2h(F).
\]

From [13, Corollary 18.4], \( h(F) \) is odd.

Given a CM \( O_F \)-order \( B \), we write \( \mu(B) \) for the subgroup of roots of unity of \( B^\times \). Let \( h(B) \) be the class number of \( B \), and put

\[
w(B) := [B^\times : O_F^\times].
\]

According to [57, Remarks, p. 92] (cf. [33] §3.1 and [75] §3.3), there exist only finitely many CM \( O_F \)-orders \( B \) with \( w(B) > 1 \), so we collect them into a finite set \( \mathcal{B} \). The CM \( O_F \)-orders \( B \) with \( |\mu(B)| > 2 \) form a subset of \( \mathcal{B} \) which will be denoted by \( \mathcal{B}^1 \).
Applying \[79\] Corollary 4.2 and (3.13)] to the maximal order \( \mathcal{O} \subseteq D \), we obtain
\[
(6.16) \quad h^1(\mathcal{O}) = \frac{1}{2} \zeta_F(-1) + \frac{1}{4h(F)} \sum_{B \in \mathcal{B}} 2^{s(B, \mathcal{O})} \Delta(B, \mathcal{O})(|\mu(B)| - 2)h(B)/w(B).
\]
Similarly, combining (6.13), (6.14) with \[79\] Theorem 3.2 and (3.8)\], we obtain
\[
(6.17) \quad h_{sc}(\mathcal{O}) = \frac{\zeta_F(-1)}{4} + \frac{1}{4h(F)} \sum_{B \in \mathcal{B}} 2^{s(B, \mathcal{O})} \Delta(B, \mathcal{O}) \left(1 - \frac{1}{w(B)}\right)h(B).
\]

Note that the summations in (6.16) and (6.17) range over \( \mathcal{B} \) and \( \mathcal{B}^1 \) respectively. It remains to classify these CM \( O_F \)-orders and work out the respective invariants. According to \[13\] (4.4) and §7.2 or \[73\] §2.8\], if \( p \geq 7 \) and \( B \in \mathcal{B} \), then the fractional field \( K : \text{Frac}(B) \) coincides with \( F(\sqrt{-1}), F(\sqrt{p}) \) or \( F(\sqrt{-3}) \), where \( F(\sqrt{-2}) = F(\sqrt{-\varepsilon}) \) by \[73\] Proposition 2.6. Moreover, \( K \in \{ F(\sqrt{-1}), F(\sqrt{-3}) \} \) if \( B \in \mathcal{B}^1 \). The prime \( p = 3 \) will be treated separately. For simplicity, we put
\[
(6.18) \quad K_m := F(\sqrt{-m}) \quad \text{for} \quad m \in \{1, 2, 3\}.
\]

We make no use of 2-adic or 3-adic completions for the rest of this section, so the notation should cause no confusion.

By definition, the quaternion \( F \)-algebra \( D = D_{1,1} \) is unramified at all the finite primes of \( F \), so part of condition (a) in Theorem 6.1.2 is already satisfied. Moreover, condition (b) is vacuous in this case since the reduced discriminant \( \delta(\mathcal{O}) = O_F \) as \( \mathcal{O} \) is maximal. On the other hand, recall from \[12\] \S15G (cf. \[14\] \S6) that the strict genus field \( \Sigma/F \) is defined to be the maximal unramified\(^7\) elementary abelian 2-extension of \( F \). The Artin reciprocity map induces an isomorphism between the Gauss genus group \( \mathfrak{G}(O_F) = \text{Pic}_+(O_F)/\text{Pic}_+(O_F)^2 \) and the Galois group of \( \Sigma/F \):

\[
(6.19) \quad \mathfrak{G}(O_F) \cong \text{Gal}(\Sigma/F), \quad \text{Pic}_+(O_F)^2[\mathfrak{a}]_+ \mapsto (\mathfrak{a}, \Sigma/F).
\]

From (4.21), \( |\mathfrak{G}(O_F)| = 2 \) since \( p \equiv 3 \pmod{4} \). One checks directly that \( F(\sqrt{-1})/F \) is unramified for \( p \equiv 3 \pmod{4} \), so it coincides with \( \Sigma \) and is the unique unramified quadratic extension of \( F \). According to Theorem 6.1.2 an CM \( O_F \)-order \( B \in \mathcal{B} \) is selective for the genus of maximal orders of \( D \) if and only if its fractional field is \( K_1 = F(\sqrt{-1}) \), so we have
\[
(6.20) \quad s(B, \mathcal{O}) = \begin{cases} 1 & \text{if } \text{Frac}(B) = F(\sqrt{-1}), \\ 0 & \text{otherwise}. \end{cases}
\]

To apply part (3) of Theorem 6.1.2 we need a more concrete description of the Artin reciprocity map in (6.19) for \( \Sigma = K_1 \). For every narrow ideal class in \( \text{Pic}_+(O_F) \), there is an integral ideal \( \mathfrak{a} \subseteq O_F \) coprime to \( p \) representing this class. If we identify \( \text{Gal}(K_1/F) \) with the multiplicative group \( \{ \pm 1 \} \), then the canonical map \( \text{Pic}_+(O_F) \rightarrow \mathfrak{G}(O_F) \approx \text{Gal}(K_1/F) \) is identified with the unique nontrivial genus character \[12\] \S14G, p. 150]
\[
(6.21) \quad \chi : \text{Pic}_+(O_F) \rightarrow \{ \pm 1 \},
\]
which can be expressed in terms of the Kronecker symbol \[12\] \S10D as follows:
\[
(6.22) \quad \chi([\mathfrak{a}]_+) = \left( -\frac{p}{N(\mathfrak{a})} \right), \quad \forall [\mathfrak{a}]_+ \in \text{Pic}_+(O_F) \quad \text{with } \gcd(\mathfrak{a}, p) = 1.
\]

---

\(^7\)Here by “unramified” we mean “unramified at all the finite primes of \( F \)”.
For example, if \( q \) is the unique dyadic prime of \( F \), then

\[
(6.23) \quad (q, K_1/F) = \chi([q]_+) = \left( \frac{-p}{2} \right) = \begin{cases} 1 & \text{if } p \equiv 7 \pmod{8}, \\ -1 & \text{if } p \equiv 3 \pmod{8}. \end{cases}
\]

This can also be re-interpreted as follows. By [43, Lemma 3] or [89, Lemma 3.2(1)], there exists \( \vartheta \in F^\times \) such that \( \varepsilon = 2\vartheta^2 \), so \( q = 2\vartheta O_F \). In particular, \([q]_+\) is a 2-torsion in \( \text{Pic}_+ (O_F) \), and it has order 2 if and only if \( N_{F/Q}(\vartheta) < 0 \). From (6.24), we have \( N_{F/Q}(\vartheta) = \frac{1}{2} \left( \frac{q}{F} \right) \) (cf. the proof of [43, Lemma 6.2.6]).

From Proposition 6.2.1, there are two spinor genera of maximal \( O_F \)-orders in \( D \) in this case, which leads to a decomposition of the type set as follows:

\[
(6.24) \quad Tp(D) = Tp^+(D) \sqcup Tp^-(D).
\]

A maximal order \( \mathcal{O} \) in \( D \) is said to belong to the principal spinor genus if \([\mathcal{O}] \in Tp^+(D)\), otherwise it is said to belong to the nonprincipal spinor genus. Following Example 4.9, if we present \( D \) as \( (1/2)^{-1} \) and write \( \{1, i, j, k\} \) for its standard basis, then \( Tp^+(D) = Tp_{s\!c}(\mathcal{O}_0) \), where

\[
(6.25) \quad \mathcal{O}_0 = O_F + O_F i + O_F \sqrt{p + j}/2 + O_F \sqrt{p + k}/2 \subset \left( -1, -1 \right).
\]

**Proposition 6.2.1.** Suppose that \( p \equiv 3 \pmod{4} \). Let \( \mathcal{O} \) (resp. \( \mathcal{O}' \)) be a maximal \( O_F \)-order in \( D \) belonging to the principal (resp. nonprincipal) spinor genus. If \( p = 3 \), then

\[
(6.26) \quad h^1(\mathcal{O}) = h_{s\!c}(\mathcal{O}) = |Tp^+(D)| = 1 = h^1(\mathcal{O}') = h_{s\!c}(\mathcal{O}') = |Tp^-(D)|.
\]

If \( p \geq 7 \), then

\[
\begin{align*}
& h^1(\mathcal{O}) = \frac{\zeta_F(-1)}{2} + \left( 11 - 3 \left( \frac{2}{p} \right) \right) \frac{h(-p)}{8} + \frac{h(-3p)}{6}, \\
& h^1(\mathcal{O}') = \frac{\zeta_F(-1)}{2} + \left( 3 - 3 \left( \frac{2}{p} \right) \right) \frac{h(-p)}{8} + \frac{h(-3p)}{6}, \\
& |Tp^+(D)| = h_{s\!c}(\mathcal{O}) = \frac{\zeta_F(-1)}{4} + \left( 17 - 9 \left( \frac{2}{p} \right) \right) \frac{h(-p)}{16} + \frac{h(-2p)}{8} + \frac{h(-3p)}{12}, \\
& |Tp^-(D)| = h_{s\!c}(\mathcal{O}') = \frac{\zeta_F(-1)}{4} + \left( 9 - 9 \left( \frac{2}{p} \right) \right) \frac{h(-p)}{16} + \frac{h(-2p)}{8} + \frac{h(-3p)}{12}.
\end{align*}
\]

**Proof.** We focus on the values of \( \Delta(B, \mathcal{O}) \) and \( \Delta(B, \mathcal{O}') \) for \( B \subseteq K_1 \), since the rest of the data required by (6.10) and (6.17) are routine calculations (see [43, §7] and [73, §3]). From part (2) of Theorem 6.1.2, exactly one of the two spinor genera of maximal orders is selected by \( B \), so we have

\[
(6.27) \quad \Delta(B, \mathcal{O}) + \Delta(B, \mathcal{O}') = 1 \quad \text{if} \quad B \subseteq K_1.
\]

We first treat the case \( p \geq 7 \). Define

\[
(6.28) \quad B_{1,2} := O_F + qO_{K_1} = Z + Z\sqrt{p} + Z\sqrt{-1} + Z(1 + \sqrt{-1})(1 + \sqrt{p})/2,
\]

where \( q = 2\vartheta O_F \) is the unique dyadic prime of \( F \). According to [43, §7], we have

\[
(6.29) \quad \mathcal{B} = \{ O_F[\sqrt{-1}], \quad B_{1,2}, \quad O_{K_1}, \quad O_{K_2}, \quad O_{K_3} \} \quad \text{if} \quad p \geq 7.
\]
Here we have used the fact that \( OF[\sqrt{-7}] \) coincides with the maximal order \( O_{K_2} \) in \( K_2 = F(\sqrt{-2}) \) by [43 Proposition 2.6] (see also [43 Lemma 7.2.5]). Moreover, 
\[
\mathcal{B} = \mathcal{B} \setminus \{ O_{K_2} \} \quad \text{if} \quad p \geq 7.
\]

It is clear from (6.29) that \( \mathcal{O}_0 \cap F(i) = O_F[i] \simeq O_F[\sqrt{-1}] \), and \( \mathcal{O}_0 \cap F(j) = O_F(j) \simeq O_{K_1} \). Since \( \mathcal{O}_0 \) belongs to the principal spinor genus, we have
\[
\Delta(O_F[\sqrt{-1}], \mathcal{O}) = 1, \quad \Delta(O_{K_1}, \mathcal{O}) = 1, \quad \text{and hence}
\]
\[
\Delta(O_F[\sqrt{-1}], \mathcal{O}') = 0, \quad \Delta(O_{K_1}, \mathcal{O}') = 0 \quad \text{by (6.27)}.
\]

Applying (6.11) with \( B = B_{1,2} \) and \( B' = O_{K_1} \), we obtain from (6.23) that
\[
\Delta(B_{1,2}, \mathcal{O}) = \frac{1}{2} \left( 1 + \left( \frac{2}{p} \right) \right), \quad \Delta(B_{1,2}, \mathcal{O}') = \frac{1}{2} \left( 1 - \left( \frac{2}{p} \right) \right).
\]

The data required by (6.16) and (6.17) are gathered in the following table (see [43] §8.2):

| \( B \) | \( |\mu(B)| \) | \( w(B) \) | \( h(B)/h(F) \) | \( s(B, \mathcal{O}) \) | \( \Delta(B, \mathcal{O}) \) | \( \Delta(B, \mathcal{O}') \) |
|---|---|---|---|---|---|---|
| \( O_F[\sqrt{-1}] \) | 4 | 2 | \( 2 - \left( \frac{2}{p} \right) \) \( h(-p) \) | 1 | 1 | 0 |
| \( B_{1,2} \) | 4 | 4 | \( 2 - \left( \frac{2}{p} \right) \) \( h(-p) \) | 1 | \( \frac{1}{2} (1 + \left( \frac{2}{p} \right)) \) | \( \frac{1}{2} (1 - \left( \frac{2}{p} \right)) \) |
| \( O_{K_1} \) | 4 | 4 | \( h(-p) \) | 1 | 1 | 0 |
| \( O_{K_2} \) | 6 | 3 | \( h(-3p)/2 \) | 0 | 1 | 1 |
| \( O_{K_2} \) | 2 | 2 | \( h(-2p) \) | 0 | 1 | 1 |

Here the data for \( O_{K_2} \) is separated from the rest to emphasize that \( O_{K_2} \) is the unique order in \( \mathcal{B} \setminus \mathcal{B}^1 \). The class number formulas for \( p \geq 7 \) in the proposition now follow by a straightforward calculation.

Next, suppose that \( p = 3 \). According to [43] Theorem 1.6, we have \( |\text{Tp}(D)| = 2 \), and hence every spinor genus of maximal orders contains precisely one type. Now it follows from Lemma 6.11 that
\[
h_{\text{sc}}(\mathcal{O}) = |\text{Tp}^+(D)| = 1 = |\text{Tp}^-(D)| = h_{\text{sc}}(\mathcal{O}').
\]

To compute \( h^1(\mathcal{O}) \) and \( h^1(\mathcal{O}') \), note that \( K_1 = K_3 = \mathbb{Q}(\sqrt{3}, \sqrt{-7}) \) in this case. Let \( B_{1,3} := \mathbb{Z}[\sqrt{3}, (1 + \sqrt{-3})/2] \) as in [74] §6.2.6], which has conductor \( \sqrt{3}O_F \) in \( O_{K_1} \). Since \( \sqrt{3}O_F \) represents the unique nontrivial element of the order 2 group \( \text{Pic}_+(O_F) \) here, we combine (6.11) and (6.21) to obtain
\[
\Delta(B_{1,3}, \mathcal{O}) = 0, \quad \Delta(B_{1,3}, \mathcal{O}') = 1.
\]

The relevant data required by (6.16) is now given by the following table

| \( B \) | \( |\mu(B)| \) | \( w(B) \) | \( h(B)/h(F) \) | \( s(B, \mathcal{O}) \) | \( \Delta(B, \mathcal{O}) \) | \( \Delta(B, \mathcal{O}') \) |
|---|---|---|---|---|---|---|
| \( O_F[\sqrt{-1}] \) | 4 | 2 | 1 | 1 | 1 | 0 |
| \( B_{1,2} \) | 4 | 4 | 1 | 1 | 0 | 1 |
| \( B_{1,3} \) | 6 | 3 | 1 | 1 | 0 | 1 |
| \( O_{K_1} \) | 12 | 12 | 1 | 1 | 1 | 0 |
Recall that \( \zeta_F(-1) = 1/6 \) by \([75, \text{§6.2.6}]\) again, we get \( h^1(\mathcal{O}) = h^1(\mathcal{O}') = 1 \) when \( p = 3 \).

**Remark 6.2.2.** Summing up the formulas for \( |T_p^+(D)| \) and \( |T_p^-(D)| \), we recover the type number formula of \( |T_p(D)| \) for \( p \equiv 3 \pmod{4} \) and \( p \geq 7 \), which was previously computed in \([77, \text{(4.7)}]\):

\[
|T_p(D)| \approx \frac{\zeta_F(-1)}{2} + \left( 13 - 5 \left( \frac{2}{p} \right) \right) \frac{h(-p)}{8} + \frac{h(-2p)}{4} + \frac{h(-3p)}{6}.
\]

Similar to the \( p \equiv 1 \pmod{4} \) case discussed in \([\text{§4.1}]\) the type numbers \( |T_p^+(D)| \) and \( |T_p^-(D)| \) can be interpreted as proper class numbers of even definite quaternionic quadratic forms of discriminant \( 4p \) within a fixed genus. In this setting such formulas have been previously obtained by Ponomarev \([60]\) and by Chan and Peters \([10]\) using another method.

**Remark 6.2.3.** Let \( p \in \mathbb{N} \) be an arbitrary prime and \( \mathfrak{d}_F \) be the discriminant of \( F \), i.e. \( \mathfrak{d}_F = p \) if \( p \equiv 1 \pmod{4} \), and \( \mathfrak{d}_F = 4p \) otherwise. The special value \( \zeta_F(-1) \) can be calculated by Siegel’s formula (see \([87, \text{Table 2, p. 70}]\), \([66, \text{Theorem I.6.5}]\)):

\[
(6.34) \quad \zeta_F(-1) = \frac{1}{60} \sum_{b^2 + 4ac = \mathfrak{d}_F, a,c > 0} a,
\]

where \( b \in \mathbb{Z} \) and \( a,c \in \mathbb{N}_{>0} \). According to \([82, \text{Remark 3.4}]\), we can also express \( \zeta_F(-1) \) in terms of the second generalized Bernoulli number (see \([3, \text{§4.2}]\), \([71, \text{Exercise 4.2(a)}]\)):

\[
(6.35) \quad \zeta_F(-1) = \frac{B_2 \chi_F}{2^3 \cdot 3} = \frac{1}{24\mathfrak{d}_F} \sum_{a=1}^{\mathfrak{d}_F} \chi_F(a)a^2,
\]

where \( \chi_F \) is the quadratic character associated to \( F/\mathbb{Q} \), that is, \( \chi_F \) is the unique even (i.e. \( \chi_F(-1) = 1 \)) quadratic character of conductor \( \mathfrak{d}_F \).

### 7. Class Number Calculations When \( p \not\equiv 3 \pmod{4} \)

Throughout this section, we assume that \( p \in \mathbb{N} \) is a prime number that is not congruent to 3 modulo 4. Let \( F = \mathbb{Q}(\sqrt{p}) \), and \( D = D_{\infty_1, \infty_2} \) be the totally definite quaternion \( F \)-algebra that splits at all finite primes of \( F \). If \( p \equiv 1 \pmod{4} \), then \( A := \mathbb{Z}[\sqrt{p}] \) is a suborder of \( \mathcal{O}_F \) of index 2. The main goal of this section is to compute the class numbers

\[
h^1(\mathcal{O}_r) := |D^1/\mathcal{D}^1|/|\mathcal{O}_F^1| \quad \text{for} \quad r \in \{8, 16\} \text{ and } p \equiv 1 \pmod{4},
\]

where \( \mathcal{O}_8 \) and \( \mathcal{O}_{16} \) are the \( A \)-orders in \( D \) defined in \([4.3]\). In particular, \( \mathcal{O}_r \cap F = A \) for both \( r \in \{8, 16\} \). As far as we know, there are no systematic ways to compute \( h^1(\mathcal{O}) \) when the center of the quaternion order \( \mathcal{O} \) is non-maximal (other than the Selberg trace formula). Thus we adopt some ad-hoc methods.

Let \( \varepsilon > 1 \) be the fundamental unit of \( \mathcal{O}_F^* \). By \([73, \text{Lemma 2.4}]\), \( N_{F/Q}(\varepsilon) = -1 \) if \( p \not\equiv 3 \pmod{4} \), which implies that in this case

\[
(7.1) \quad \text{Pic}_+(\mathcal{O}_F) \cong \text{Pic}(\mathcal{O}_F) \quad \text{and} \quad \mathcal{O}_{F,+} = \mathcal{O}_F^2 = \langle \varepsilon^2 \rangle.
\]
Lemma 7.1. Suppose that $p \not\equiv 3 \pmod{4}$. Then for every Eichler $O_F$-order $O$ in $D$, the following canonical projection is a bijection
\begin{equation}
D^1 \backslash \hat{D}^1 \rightarrow D^x \backslash \hat{D}^x / \hat{F}^x \hat{O}^x.
\end{equation}
Moreover, $h^1(O) = h(O)/h(F)$.

Proof. According to [13 Corollary 18.5], the narrow class number $h_+(O_F)$ is odd, and $O_{F,+}^x = O_F^x$ by (7.1). It follows that
\begin{equation}
\hat{F}^x = F_+ \hat{F}^x \hat{O}^x, \quad \text{and} \quad F_+ \cap \hat{F}^x \hat{O}^x = F_x^x O_{F,+}^x = F^x.
\end{equation}
From [69 Theorem 4.1], $\text{Nr}(D^x) = F^x$, so $\text{Nr}(D^x \hat{F}^x \hat{O}^x) = F_+^x \hat{F}^x \hat{O}^x = \hat{F}^x$. Thus for any $x \in \hat{D}^x$, there exists $\alpha \in D^x$ and $y \in \hat{F}^x \hat{O}^x$ such that $\text{Nr}(\alpha x y) = 1$, which shows that the map in (7.2) is surjective.

If $D^1 x_i \hat{O}^1$ with $x_i \in \hat{D}^1$ for $i = 1, 2$ are mapped to the same element, then there exists $\alpha \in D^x$ and $y \in \hat{F}^x \hat{O}^x$ such that $x_1 = \alpha x_2 y$. Taking the reduced norm on both sides, we get $\text{Nr}(\alpha) = \text{Nr}(y^{-1}) = F_+^x \cap (\hat{F}^x \hat{O}^x)$. It follows from (7.3) that there exists $a \in F^x$ such that $\beta := a \alpha^{-1} \in D^1$. On the other hand,
\begin{equation}
(\hat{F}^x \hat{O}^x) \cap \hat{D}^1 = \hat{O}^1.
\end{equation}
If we put $u := ay$, then $u \in \hat{O}^1$ and $x_1 = \beta x_2 u$. This shows that the map in (7.2) is injective.

Lastly, we prove that $h^1(O) = h(O)/h(F)$. The ideal class group $\text{Cl}(O_F)$ acts naturally on the class set $\text{Cl}(O)$ of locally principal right $O$-ideal classes by
\begin{equation}
\text{Cl}(O_F) \times \text{Cl}(O) \rightarrow \text{Cl}(O), \quad ([a], [I]) \mapsto [aI].
\end{equation}
Let $\text{Cl}(O)$ be the set of orbits of this action, which can be described adelically as
\begin{equation}
\overline{\text{Cl}}(O) \simeq D^x \backslash \hat{D}^x / \hat{F}^x \hat{O}^x.
\end{equation}
Since $h(F)$ is odd, the action is free by [77 Corollary 2.5], and hence $h^1(O) = |\text{Cl}(O)| = h(O)/h(F)$. \hfill $\square$

From now on we assume that $p \equiv 1 \pmod{4}$ for the rest of this section. We first show that the narrow class number of $A = \mathbb{Z}[\sqrt{p}]$ is odd.

Lemma 7.2. Put $\varpi := [O_F^x : A^x]$. Then
\begin{equation}
\text{Pic}_+(A) \cong \text{Pic}(A), \quad A^x_+ = A^x \quad \text{and} \quad h_+(A) = h(A) = \left(2 - \frac{2}{p}\right) \frac{h(O_F)}{\varpi}.
\end{equation}
In particular, $h_+(A)$ is odd.

Proof. According to [73 §4.2], $\varpi \in \{1, 3\}$, and $\varpi = 1$ if $p \equiv 1 \pmod{8}$. Thus, $\varepsilon^3 \in A^x$. As $N_{F/Q}(\varepsilon^3) = (-1)^3 = -1$, we find that $\text{Pic}_+(A) \cong \text{Pic}(A)$. The equality $A^x_+ = A^x$ can be verified case by case according to $\varpi = 1$ or $3$ (See [72 §4.3]). To compute the class number $h(A)$, we apply directly Dedekind’s formula [69 p. 95] for the class number of quadratic orders. Lastly, according to [13 Corollary 18.4], $h(O_F)$ is odd for every prime $p$. It is clear from (7.8) that $h_+(A)$ is odd as well. \hfill $\square$

Let us recall the definition of the $A$-order $O_{16}$. Fix a maximal $O_F$-order $O_1$ in $D$ and also an identification $O_1 \otimes \mathbb{Z}_2 = \text{Mat}_2(O_{F_2})$. Then $O_{16}$ is defined to be the unique suborder of $O_1$ of index 16 such that $O_{16} \otimes \mathbb{Z}_2 = \text{Mat}_2(A_2)$. 


Lemma 7.3. The following canonical projection is a bijection
\[ D^1 / \hat{D}^1 / \hat{O}_{16} \rightarrow D^x / \hat{D}^x / \hat{F}^x \hat{O}_{16}^x. \]
In particular, \( h^1(\mathcal{O}_{16}) \) coincides with the type number \( t(\mathcal{O}_{16}) \).

Proof. Once we prove that the equalities \( \text{(7.9)} \) and \( \text{(7.5)} \) hold again when \( O_F \) and \( \mathcal{O} \) are replaced by \( A \) and \( \mathcal{O}_{16} \) respectively, the same proof as that of Lemma 7.4 applies to the bijectivity of \( \text{(7.9)} \).

From Lemma 7.2, \( h_+(A) \) is odd, and \( A_A^x = A^x \), so the equalities in \( \text{(7.3)} \) still hold if \( O_F \) is replaced by \( A \). If \( \text{Nr}(au) = 1 \) for some \( a \in \hat{F}^x \) and \( u \in \hat{O}_{16}^x \), then \( a^2 = \text{Nr}(a u^{-1}) \in \text{Nr}(\hat{O}_{16}^x) = \hat{A}^x \), which implies that \( a \in \hat{O}_{16}^x \). Since \( [\hat{O}_{16}^x : \hat{A}^x] = [\hat{O}_{F_2} : A_A^x] = 2 - \left( \frac{q}{4} \right) \) (which is odd), we get \( a \in \hat{A}^x \subseteq \hat{O}_{16}^x \), and hence \( au \in \hat{O}_{16}^x \). This shows that \( (\hat{F}^x \hat{O}_{16}^x) \cap \hat{D}^1 = \hat{O}_{16}^x \). The rest of the proof of \( \text{(7.9)} \) runs the same as that of Lemma 7.4.

Lastly, recall that \( N(\mathcal{O}_{16}) = \hat{F}^x \hat{O}_{16}^x \) as in \( \text{(4.10)} \). It follows that the cardinality of the right hand side of \( \text{(7.9)} \) is precisely the type number \( t(\mathcal{O}_{16}) \), which is calculated in \( \text{(7.7)} \) \( (4.12) \) and reproduced in \( \text{(4.34)} \) \( (\text{resp. (4.32)}) \) for \( p \geq 13 \) \( (\text{resp. } p = 5) \).

Keep the maximal order \( \mathcal{O}_1 \subset D \) fixed as before. We write \( \mathcal{O}_8 \) and \( \mathcal{O}_4 \) for the unique \( A \)-suborders of \( \mathcal{O}_1 \) of index 8 and 4 respectively such that
\[ \text{(7.10)} \quad \mathcal{O}_8 \otimes \mathbb{Z}_2 = \left( \begin{array}{ll} A_2 & 2OF_2 \\
OF_2 & O_2 \end{array} \right), \quad \mathcal{O}_4 \otimes \mathbb{Z}_2 = \left( \begin{array}{ll} OF_2 & 2OF_2 \nOF_2 & O_2 \end{array} \right). \]
By definition, \( \mathcal{O}_4 \) is an Eichler order of level \( 2OF_2 \). Clearly, \( A/2OF_2 ) \cong A_2/2OF_2 \cong \mathbb{F}_2 \).

Proposition 7.4. Let \( \zeta_F(s) \) be the Dedekind zeta function of \( F = \mathbb{Q}(\sqrt{-p}) \), and \( h(-p) \) be the class number \( h(-p) \). Then
\[ \text{(7.11)} \quad h^1(\mathcal{O}_8) = \frac{3}{2} \left(\frac{p}{f}\right) \zeta_F(-1) + \left(2 - \left(\frac{p}{f}\right)\right) h(-p)/8. \]
To compute \( h^1(\mathcal{O}_8) \), we first study its relation with \( h^1(\mathcal{O}_4) \).

Lemma 7.5. We have
\[ \text{(7.12)} \quad h^1(\mathcal{O}_8) = \begin{cases} h^1(\mathcal{O}_4) & \text{if } p \equiv 1 \pmod{8}, \\ 3h^1(\mathcal{O}_4) - 2S & \text{if } p \equiv 5 \pmod{8}, \end{cases} \]
where \( S \) denotes the following set
\[ \{ [x] = D^1 x \hat{O}_4^1 \in D^1 / \hat{O}_4^1 \mid \exists \zeta \in D^1 \cap x \hat{O}_4^1 x^{-1} \text{ such that } \text{ord}(\zeta) = 3\}. \]

Proof. Let \( \theta : D^1 / \hat{O}_8^1 \rightarrow D^1 / \hat{O}_4^1 \) be the canonical projection map. If \( p \equiv 1 \pmod{8} \), then \( OF_2 = Z_2 \times Z_2 \), and hence \( A^x_A = O_F^x \). It follows that \( \hat{O}_8^1 = \hat{O}_4^1 \) in this case, so \( \theta \) is a bijection.

Assume that \( p \equiv 5 \pmod{8} \) for the rest of the proof. For each \( [x] = D^1 x \hat{O}_4^1 \), \( \theta^{-1}( [x] ) = D^1 ( \langle x^{-1} D^1 x \hat{O}_4^1 \rangle ) / \hat{O}_8^1 \).

Multiplying \( D^1 x \hat{O}_4^1 \) from the left by \( x^{-1} \) induces bijections
\[ \text{(7.13)} \quad D^1 / (D^1 x \hat{O}_4^1) / \hat{O}_8^1 \simeq \langle x^{-1} D^1 x \rangle / \langle x^{-1} D^1 x \hat{O}_4^1 \rangle / \hat{O}_8^1 \simeq \langle x^{-1} D^1 x \hat{O}_4^1 \rangle / \hat{O}_8^1. \]
From \( \text{(7.10)} \), \( (\mathcal{O}_8 \otimes \mathbb{Z}_2)^1 \) is normal in \( (\mathcal{O}_4 \otimes \mathbb{Z}_2)^1 \), and
\[ \text{(7.14)} \quad (\mathcal{O}_4 \otimes \mathbb{Z}_2)^1 / (\mathcal{O}_8 \otimes \mathbb{Z}_2)^1 \simeq F_4^x, \quad \left( \begin{array}{cc} a & b \\
c & d \end{array} \right) \mapsto (a \mod{2OF_2}). \]
Clearly, \( \hat{O}_4^1/\hat{O}_8^1 \simeq (O_4 \otimes \mathbb{Z}_2)^1/((O_8 \otimes \mathbb{Z}_2)^1 \), and the action of \( (x^{-1}D^1x \cap \hat{O}_4^1) \) on \( \hat{O}_4^1/\hat{O}_8^1 \) factors through the quotient

\[
(x^{-1}D^1x \cap \hat{O}_4^1) \rightarrow \hat{O}_4^1/\hat{O}_8^1.
\]

Hence \( |\theta^{-1}([x])| \in \{1,3\} \), and it takes value 1 if and only if the above homomorphism is surjective. If \( \zeta' \) is an element of \( (x^{-1}D^1x \cap \hat{O}_4^1) \) with \( \text{ord}(\zeta') = 3 \) | \( \text{ord}(\zeta) \), then it follows from \[7.3\] §2.8 that \( F(\zeta') \simeq F(\sqrt{-3}) \) and \( \zeta'^2 + \zeta' + 1 = 0 \). This automatically implies that \( \zeta' \not\in \hat{O}_8^1 \) since \( \text{Tr}(\zeta) \equiv 0 \) (mod 2\( \hat{O}_F \)) for every \( \zeta \in \hat{O}_4^1 \). Therefore, \( |\theta^{-1}([x])| = 1 \) if and only if \( [x] \in S \). In other words, \( \theta \) is a 3 : 1 cover “ramified” above the set \( S \), and the lemma follows.

**Proof of Proposition 7.4.** First, suppose that \( p \equiv 1 \) (mod 8). Combining the results of Lemma 7.5 and Lemma 7.1, we see that \( h^1(O_8) = h(O_4)/h(F) \), which also coincides with \( t(O_8) \) by \[4.12\]. The formulas for \( h(O_4) \) and \( t(O_8) \) are computed in \[7.7\] Lemma 4.2 and \[7.7\] (4.11)] respectively, so we get

\[
(7.15) \quad h^1(O_8) = \frac{9}{2} \zeta_F(-1) + \frac{h(-p)}{8} \quad \text{if} \quad p \equiv 1 \pmod{8}.
\]

Next, suppose that \( p \equiv 5 \) (mod 8). Similar to the previous case, we combine Lemma 7.1 with \[7.7\] Lemma 4.2] to obtain

\[
(7.16) \quad h^1(O_4) = \frac{5}{2} \zeta_F(-1) + h(-p)/8 + h(-3p)/3,
\]

where \( h(-3p) \) denotes the class number of \( \mathbb{Q}(\sqrt{-3p}) \).

It remains to compute the cardinality of the set \( S \) defined in Lemma 7.5. Put

\[
\tilde{S} := \{ [I] \in \text{Cl}(O_4) \mid \exists \zeta \in \text{Cl}(I)^\times \text{ such that } \text{ord}(\zeta) = 3 \}.
\]

Here \( O_l(I) \) denotes the left order of \( I \) as in \[4.6\]. If we write \( I = D \cap x\hat{O} \) for some \( x \in \hat{D}^\times \), then \( O_l(I) = D \cap x\hat{D} \). Therefore, \( [I] \) belongs to \( \tilde{S} \) if and only if the image of \( [I] \) belongs to \( S \) under the following composition of maps

\[
\text{Cl}(O_4) \rightarrow \text{Cl}(O_4) \simeq D^\times/\hat{D}^\times / \hat{F}^\times \hat{O}_4^\times \simeq D^1/\hat{D}^1/\hat{O}_4^1.
\]

Since the action of \( \text{Cl}(O_F) \) on \( \text{Cl}(O_4) \) is free and \( O_l(aI) = O_l(I) \) for every \( [a] \in \text{Cl}(O_F) \), we see that

\[
(7.17) \quad |\tilde{S}| = h(F)/|S|.
\]

For simplicity, put \( K = F(\sqrt{-3}) \). For each \( \zeta \in \text{Cl}(I)^\times \text{ with } \text{ord}(\zeta) = 3 \), we have \( F(\zeta) \simeq K \) and \( F(\zeta) \cap O_l(I) \simeq O_K \) by \[4.3\] Proposition 3.1] (cf. \[4.3\] Table 7.2]. Thus each such \( \zeta \in \text{Cl}(I)^\times \) gives rise to an optimal embedding \( O_K \rightarrow O_l(I) \) and vice versa. For each \( O_F \)-order \( \mathcal{O} \) in \( D \), let \( m(O_K,\mathcal{O},\mathcal{O}^\times) \) be the number of \( \mathcal{O}^\times \)-conjugacy classes of optimal embeddings from \( O_K \) into \( \mathcal{O} \) (cf. \[4.3\] Section 3.2]). We claim that

\[
(7.18) \quad m(O_K,\mathcal{O}_l(I),\mathcal{O}_l(I)^\times) = \begin{cases} 2 & \text{if } [I] \in \tilde{S}, \\ 0 & \text{otherwise}. \end{cases}
\]

The equality clearly holds if \( [I] \not\in \tilde{S} \). Suppose that \( [I] \in \tilde{S} \). If \( p > 5 \), then \( O_l(\mathcal{O}^\times)^\times/O_F^\times \) is a finite group isomorphic to \( \mathbb{C}_3, \mathbb{D}_3 \) or \( \mathbb{A}_4 \) according to \[4.3\] Tables 4.1 and 4.2], where \( \mathbb{C}_3 \) denotes the cyclic group of order 3, \( \mathbb{D}_3 \) the dihedral group of order 6, and \( \mathbb{A}_4 \) the alternating group on 4-letters. However if \( O_l(\mathcal{O}^\times)^\times/O_F^\times \simeq \mathbb{D}_3 \), then by \[4.3\] Proposition 4.3.5], \( O_l(I) \) necessarily contains an \( O_F \)-suborder
isomorphic to the order \( \mathcal{O}_3 \) in [43 Table 4.3]. This leads to a contradiction since \( \mathcal{O}_3 \) has reduced discriminant \( 3QF \) while \( \mathcal{O}_3(I) \) is an Eichler order of level \( 2QF \).
If \( p = 5 \), then \( h(O_1) = 1 \) by [77 Lemma 4.2] (which implies that \( \mathcal{O}_1(I) = \mathcal{O}_4 \)), and \( \text{Mass}(O_4) = 1/[O_4^\times : O_F^\times] = 1/12 \) by a direct calculation using the mass formula [69 Corollaire IV.2.3]. We find that \( \mathcal{O}_4^\times / O_F^\times \simeq A_4 \) when \( p = 5 \) from the classification in [43 §8.1]. In conclusion, if \( |I| \in \mathcal{S} \), then \( \mathcal{O}_0(I)^\times / O_F^\times \) is isomorphic to either \( C_3 \) or \( A_4 \), so our claim follows directly from [43 Proposition 3.4].

Lastly, we apply the trace formula [69 Theorem III.5.11] to \( O_K \) and \( O_4 \) to get

\[
\sum_{[I] \in \text{Cl}(O_4)} m(O_K, \mathcal{O}_I(I), \mathcal{O}_I(I)^\times) = h(O_K) \left( 1 + \left( \frac{O_K}{2O_F} \right) \right) = 2h(K).
\]

(7.19)

Here \( (\frac{O_K}{p}) \) is the Eichler symbol [69 p. 94] for a prime ideal \( p \subseteq O_F \), and it takes value 1 at \( p = 2O_F \), which splits in \( K \). Combining (7.17), (7.18) and (7.19), we get

\[
[S] = \frac{2h(K)}{2h(F)} = \frac{h(-3p)}{2}.
\]

(7.20)

Plugging (7.16) and (7.20) into the equality \( h^1(O_S) = 3h^1(O_4) - 2[S] \) in Lemma 7.5, we obtain the following class number formula

\[
h^1(O_S) = \frac{15}{2} \zeta_F(-1) + \frac{3}{8} h(-p) \quad \text{if} \ p \equiv 5 \pmod{8}.
\]

(7.21)

Proposition 7.4 follows by combining (7.15) with (7.21) and interpolating. \( \square \)

8. Classification of self-dual local lattices

We consider some variants of lattices in symplectic spaces over local fields that are used in this paper. As usual, \( \delta_{ij} \) denotes Kronecker’s symbol, namely \( \delta_{ij} = 1 \) or \( 0 \) according to whether \( i = j \) or not.

**Lemma 8.1.** Let \( K_0 \) be a field, and \( O_{K_0} \subseteq K_0 \) be a Dedekind domain with quotient field \( K_0 \). Let \( K = \prod_i K_i \) be a commutative separable \( K_0 \)-algebra, where each \( K_i \) is a finite separable field extension of \( K_0 \), and let \( O_K \) be its maximal \( O_{K_0} \)-order. Let \( (V, \psi) \) be a non-degenerate alternating \( K_0 \)-valued \( K \)-module such that \( \psi(ax, y) = \psi(x, ay) \) for all \( a \in K \) and all \( x, y \in V \). Then

1. There exists a self-dual \( O_{K_0} \)-valued \( O_K \)-lattice.
2. Any two self-dual \( O_{K_0} \)-valued \( O_K \)-lattices \( L \) and \( L' \) are isometric.

**Proof.** Let \( L \) and \( L' \) be \( O_K \)-lattices in \( V \). Decompose \( L = \oplus L_i \) and \( L' = \oplus L'_i \) into \( O_K \)-modules with respect to the decomposition \( O_K = \prod_i O_{K_i} \) into the product of Dedekind domains. Clearly this decomposition is orthogonal with respect to the pairing \( \psi \). The \( O_K \)-lattice \( L \) is self-dual if and only if each component \( L_i \) is self-dual. Moreover, \( (L, \psi) \simeq (L', \psi) \) if and only if \( (L_i, \psi) \simeq (L'_i, \psi) \) for all \( i \). Thus, without loss of generality, we may assume that \( K \) is a finite separable field extension of \( K_0 \).

Let \( \psi_K : V \times V \to K \) be the unique \( K \)-bilinear form such that \( \text{Tr}_{K/K_0} \circ \psi_K = \psi \). The form \( \psi_K \) remains non-degenerate and alternating. Denote the inverse different of \( K/K_0 \) by \( D_{K/K_0}^{-1} \). The dual lattice of \( L \) with respect to \( \psi \) is defined as

\[
L' := \{ x \in V \mid \psi(x, L) \subseteq O_{K_0} \}.
\]
Since $\psi(x, L) \subseteq O_{K_0}$ if and only if $\psi_K(x, L) \subseteq D_{K/K_0}^{-1}$, we have
\[
L^\vee = \{ x \in V \mid \psi_K(x, L) \subseteq D_{K/K_0}^{-1} \}. 
\]

(8.2) According to [64, Proposition 1.3], there exist a $K$-basis $\{x_1, \cdots, x_n, y_1, \cdots, y_n\}$ of $V$ and fractional $O_K$-ideals $a_1, \cdots, a_n$ such that
\[
\psi_K(x_i, x_j) = \psi_K(y_i, y_j) = 0, \quad \psi_K(x_i, y_j) = \delta_{ij}, \quad \text{for all } 1 \leq i, j \leq n,
\]
\[
(8.3) \quad L = \sum_{i=1}^{n} O_K x_i + \sum_{j=1}^{n} a_j y_j, \quad a_1 \supseteq a_2 \supseteq \cdots \supseteq a_n.
\]

The fractional ideals $a_1 \supseteq \cdots \supseteq a_n$ are called the invariant factors of $L$ with respect to $\psi_K$. A direct calculation shows that
\[
(8.4) \quad L^\vee = D_{K/K_0}^{-1} \left( \sum_{i=1}^{n} a_i^{-1} x_i + \sum_{j=1}^{n} O_K y_j \right).
\]

It follows that $L$ is self-dual if and only if $a_i = D_{K/K_0}^{-1}$ for all $1 \leq i \leq n$. Therefore, a self-dual $O_K$-lattice in $(V, \psi)$ exists and is unique up to isometry. □

**Remark 8.2.** The first invariant factor $a_1$ is called the norm of $L$ with respect to $\psi_K$ since it coincides with the $O_K$-ideal generated by $\psi_K(x, y)$ for all $x, y \in L$. Following [64, §1.4], an $O_K$-lattice $M$ in $(V, \psi_K)$ is said to be maximal if $M$ is a maximal one among the $O_K$-lattices in $(V, \psi_K)$ with the same norm. We have shown in the proof of Lemma 8.1 that an $O_K$-lattice $L$ in $(V, \psi)$ is self-dual if and only if it is a maximal lattice with norm $D_{K/K_0}^{-1}$ in $(V, \psi_K)$.

Let $a$ be an invertible $O_K$-ideal in $K$. An $O_K$-lattice $L$ is said to be $a$-modular in $(V, \psi)$ if $a(L^\vee) = L$.

**Corollary 8.3.** Keep the notation and assumption of Lemma 8.1. For any invertible ideal $a$ in $K$, there exists a unique $a$-modular $O_K$-lattices in $(V, \psi)$ up to isometry.

**Proof.** Once again, we reduce to the case $K$ being a finite separable field extension of $K_0$. From the proof of Lemma 8.1, an $O_K$-lattice $L$ is $a$-modular if and only if all its invariant factors $a_i = a D_{K/K_0}^{-1}$. The corollary follows easily. □

We would like a similar result as Lemma 8.1 with a weaker condition than $O_K$ being a product of Dedekind domains. The best that we can manage at the moment is for Gorenstein orders over complete discrete valuation rings. Notation as in Lemma 8.1 an $O_{K_0}$-order $R$ in $K$ is called Gorenstein if every short exact sequence of $R$-lattices
\[
0 \rightarrow R \rightarrow M \rightarrow N \rightarrow 0
\]
splits. It is known that $R$ is Gorenstein if and only if its dual $R^\vee := \text{Hom}_{O_{K_0}}(R, O_{K_0})$ is an invertible $R$-module [20, Prop. 6.1, p. 1363]; see also [32, Proposition 3.5].

**Lemma 8.4.** Let $K_0$ be a complete discrete valuation field, and $O_{K_0}$ be its valuation ring. Let $K$, $(V, \psi)$ be as in Lemma 8.1 and let $R$ be a $O_{K_0}$-order in $K$. Assume that $V$ is free over $K$ and $R$ is Gorenstein. Then

1. There exists a self-dual $O_{K_0}$-valued free $R$-lattice in $(V, \psi)$.
2. Any two self-dual $O_{K_0}$-valued free $R$-lattices $L$ and $L'$ in $(V, \psi)$ are isometric.
where Noetherian rings, any finitely generated stably free be reduced to those of self-dual lattices in \((V, \psi)\)

Lemma 8.5. Let \(R\) be a Noetherian commutative ring such that any finitely generated stably free \(R\)-module is free, and let \(L\) be a free \(R\)-module of rank \(2n\). Then for any perfect alternating pairing \(\psi : L \times L \to R\), there is a Lagrangian basis with respect to \(\psi\), that is, a basis \(\{x_1, \ldots, x_n, y_1, \ldots, y_n\}\) such that
\[
\psi(x_i, y_j) = \delta_{ij} \quad \text{and} \quad \psi(x_i, x_j) = \psi(y_i, y_j) = 0, \quad \forall i, j.
\]

In particular, there is only one perfect alternating pairing up to isomorphism.

Proof. We prove by induction on \(n\). Let \(\{e_1, \ldots, e_n\}\) be an \(R\)-basis of \(L\). Since \(\psi\) is a perfect pairing, the map \(x \mapsto \psi(x, \cdot)\) establishes an isomorphism of \(R\)-modules \(L \to \text{Hom}_R(L, R)\). In particular, there exists \(y_1 \in L\) such that \(\psi(e_1, y_1) = 1\). Put \(x_1 = e_1\). We note that the \(R\)-submodule \(\langle x_1, y_1 \rangle_R \subseteq L\) spanned by \(x_1\) and \(y_1\) is free of rank 2. Indeed, if \(ry_1 \in Rx_1\) for some \(r \in R\), then \(r = \psi(x_1, ry_1) = 0\). Let \(L' = \{x \in L \mid \psi(x, x_1) = \psi(x, y_1) = 0\}\). For every \(z \in L\), put \(a = \psi(z, y_1)\), \(b = \psi(z, x_1)\), and \(z' = z - az_1 + by_1\). Then \(z' \in L'\), and hence \(L = \langle x_1, y_1 \rangle_R \oplus L'\). Clearly, \(L'\) is finitely generated and stably free. Thus it is free of rank \(2n - 2\) by our assumption on \(R\). The restriction of \(\psi\) to \(L'\) is necessarily perfect. Now the lemma follows by induction.

Remark 8.6. The condition for \(R\) in Lemma 8.5 is satisfied if \(R\) is one of the following: a local ring, a Dedekind domain, a commutative Bass order \([15, \text{Section 37}][41]\), a polynomial ring over a field (Serre’s conjecture, now the Quillen-Suslin theorem), or any finite product of them. In particular, if in Lemma 8.4 we assume that \(O_K\) is discrete valuation ring (without the completeness condition), and suppose further that \(R\) is a Bass \(O_K\)-order, then the results still hold. Indeed, in this case \(R\) is a semi-local ring, so the invertible \(R\)-module \(D_R^{−1}O_K\) is again free, and we can still apply Lemma 8.5 thanks to the preceding Remark.

Example 8.7. We provide an example of a non-projective self-dual lattice \(L\) over a Bass order \(R\) in a symplectic space. Let \(d \in \mathbb{Z}\) be a square-free integer coprime to 6, \(K = \mathbb{Q}(\sqrt{d})\) and \(R = \mathbb{Z}[\sqrt{d}]\). Let \((V, \psi_K)\) be a symplectic space over \(K\) of dimension 4 with a canonical basis \(\{x_1, y_1, x_2, y_2\}\) satisfying (8.3). We put
\[L = R_1x_1 + b_1y_1 + R_2x_2 + b_2y_2,\]
where \(R_1 = \mathbb{Z}[2\sqrt{d}]\), \(R_2 = \mathbb{Z}[3\sqrt{d}]\) and \(b_1 = (R : R_1) = \{x \in K \mid xR_1 \subseteq R\}\). More explicitly, \(b_1 = 3R_1\) and \(b_2 = 2R_2\), and both of them are integral ideals of \(R\). From
2.6(ii), \( (R : b_i) = R_i \) for each \( i \), which is also immediate from calculations. This implies that
\[
L = \{ x \in V \mid \psi_K(x, L) \subseteq R \} \simeq \text{Hom}_R(L, R).
\]

The fractional \( R \)-ideal \( \langle \psi_K(x, y) \mid x, y \in L \rangle_R \) coincides with \( R \) itself, so \( L \) is also a maximal \( R \)-lattice with norm \( R \). Clearly, \( L \) is not projective over \( R \), otherwise both \( R_i \) are projective \( R \)-modules, which is absurd.

Acknowledgments

The authors would like to express their gratitude to Tomoyoshi Ibukiyama, and Chao Zhang for stimulating discussions. They thank the organizers of the 2019 Nagoya supersingular conference held September 30–October 4 of 2019 for their kind invitation. They also thank Sushi Harashita for his kind invitation to the RIMS supersingular conference held October 13–15, 2020, where the present work was presented. Xue is partially supported by the National Natural Science Foundation of China grant No. 12271410 and No. 12331002. Yu is partially supported by the grants NSTC 109-2115-M-001-002-MY3 and 112-2115-M-001-009. Part of the manuscript was prepared during the first author’s 2024 visit to Institute of Mathematics, Academia Sinica. He thanks the institute for the warm hospitality and great working conditions.

References

[1] Jeffrey D. Achter, S. Ali Altug, Luis Garcia, and Julia Gordon. Counting abelian varieties over finite fields via Frobenius densities. Algebra Number Theory, 17(7):1239–1280, 2023.

[2] Saan Alaca and Kenneth S. Williams. Introductory algebraic number theory. Cambridge University Press, Cambridge, 2004.

[3] Tsuneo Arakawa, Tomoyoshi Ibukiyama, and Masanobu Kaneko. Bernoulli numbers and zeta functions. Springer Monographs in Mathematics. Springer, Tokyo, 2014. With an appendix by Don Zagier.

[4] Manuel Arenas, Luis Arenas-Carmona, and Jaime Contreras. On optimal embeddings and trees. J. Number Theory, 193:91–117, 2018.

[5] Luis Arenas-Carmona. Applications of spinor class fields: embeddings of orders and quaternionic lattices. Ann. Inst. Fourier (Grenoble), 53(7):2021–2038, 2003.

[6] Jonas Bergström, Valentin Karemaker, and Stefano Marseglia. Polarizations of abelian varieties over finite fields via canonical liftings. Int. Math. Res. Not. IMRN, (4):3194–3248, 2023.

[7] Juliusz Bracinski. Spinor class groups of orders. J. Algebra, 84(2):468–481, 1983.

[8] Juliusz Bracinski. On embedding numbers into quaternion orders. Comment. Math. Helv., 66(2):302–318, 1991.

[9] Tommaso Giorgio Centeleghe and Jakob Stix. Categories of abelian varieties over finite fields, I: Abelian varieties over \( F_q \). Algebra Number Theory, 9(1):225–265, 2015.

[10] Wai Kiu Chan and Meinhard Peters. Quaternary quadratic forms and Hilbert modular surfaces. In Algebraic and arithmetic theory of quadratic forms, volume 344 of Contemp. Math., pages 85–97. Amer. Math. Soc., Providence, RI, 2004.

[11] Ted Chinburg and Eduardo Friedman. An embedding theorem for quaternion algebras. J. London Math. Soc. (2), 60(1):33–44, 1999.

[12] Harvey Cohn. A classical invitation to algebraic numbers and class fields. Springer-Verlag, New York-Heidelberg, 1978. With two appendices by Olga Taussky: “Artin’s 1932 Göttingen lectures on class field theory” and “Connections between algebraic number theory and integral matrices”, Universitext.

[13] Pierre E. Conner and Jürgen Hurrelbrink. Class number parity, volume 8 of Series in Pure Mathematics. World Scientific Publishing Co., Singapore, 1988.
[14] David A. Cox. *Primes of the form $x^2 + ny^2$*. Pure and Applied Mathematics (Hoboken). John Wiley & Sons, Inc., Hoboken, NJ, second edition, 2013. Fermat, class field theory, and complex multiplication.

[15] Charles W. Curtis and Irving Reiner. *Methods of representation theory. Vol. I*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1990. With applications to finite groups and orders, Reprint of the 1981 original, A Wiley-Interscience Publication.

[16] Pierre Deligne and Georgios Pappas. *Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant*. Compositio Math., 90(1):59–79, 1994.

[17] Max Deuring. *Die Typen der Multiplikatorenringe elliptischer Funktionenkörper*. Abh. Math. Sem. Hanssischen Univ., 14:197–272, 1941.

[18] Max Deuring. *Die Anzahl der Typen von Maximalordnungen einer definiter Quaternionen-Algebra mit primer Grundzahl*. Jber. Deutsch. Math. Verein., 54:24–41, 1950.

[19] Claus Diem and Niko Naumann. *On the structure of Weil restrictions of abelian varieties*. J. Ramanujan Math. Soc., 18(2):153–174, 2003.

[20] Ju. A. Drozd, V. V. Kiričenko, and A. V. Roiter. *Hereditary and Bass orders*. Izv. Akad. Nauk SSSR Ser. Mat., 31:1415–1436, 1967.

[21] Martin Eichler. *Über die Idealklassenzahl total definiter Quaternionenalgebren*. Math. Z., 43(1):102–109, 1938.

[22] Martin Eichler. *Zur Zahlentheorie der Quaternionen-Algebren*. J. Reine Angew. Math., 195:127–151 (1956), 1955.

[23] Eberhard Freitag. *Hilbert modular forms*. Springer-Verlag, Berlin, 1990.

[24] Eyal Z. Goren. *Lectures on Hilbert modular varieties and modular forms*, volume 14 of CRM Monograph Series. American Mathematical Society, Providence, RI, 2002. With the assistance of Marc-Hubert Nicole.

[25] Benedict H. Gross. *Heights and the special values of $L$-series*. In *Number theory (Montreal, Que., 1985)*, volume 7 of CMS Conf. Proc., pages 115–187. Amer. Math. Soc., Providence, RI, 1987.

[26] Xuejun Guo and Hourong Qin. *An embedding theorem for Eichler orders*. J. Number Theory, 107(2):207–214, 2004.

[27] Everett W. Howe. *Kernels of polarizations of abelian varieties over finite fields*. J. Algebraic Geom., 5(3):583–608, 1996.

[28] Everett W. Howe, Daniel Maisner, Enric Nart, and Christophe Ritzenthaler. *Principally polarizable isogeny classes of abelian surfaces over finite fields*. Math. Res. Lett., 15(1):121–127, 2008.

[29] Tomoyoshi Ibukiyama and Toshiyuki Katsura. *On the field of definition of superspecial polarized abelian varieties and type numbers*. Compositio Math., 91(1):37–46, 1994.

[30] Jun-ichi Igusa. *Class number of a definite quaternion with prime discriminant*. Proc. Nat. Acad. Sci. U.S.A., 44:312–314, 1958.

[31] Ronald Jacobowitz. *Hermitian forms over local fields*. Amer. J. Math., 84:441–465, 1962.

[32] Christian U. Jensen and Anders Thorup. *Gorenstein orders*. J. Pure Appl. Algebra, 219(3):551–562, 2015.

[33] Yoshiyuki Kitaoka. *Quaternary even positive definite quadratic forms of prime discriminant*. Nagoya Math. J., 52:147–161, 1973.

[34] Otto Körner. *Traces of Eichler-Brandt matrices and type numbers of quaternion orders*. Proc. Indian Acad. Sci. Math. Sci., 97(1-3):189–199 (1988), 1987.

[35] Robert E. Kottwitz. *Shimura varieties and $ł$-adic representations*. In *Automorphic forms, Shimura varieties, and $L$-functions, Vol. I (Ann Arbor, MI, 1988)*, volume 10 of Perspect. Math., pages 161–209. Academic Press, Boston, MA, 1990.

[36] Robert E. Kottwitz. *Points on some Shimura varieties over finite fields*. J. Amer. Math. Soc., 5(2):373–444, 1992.

[37] Jürg Kramer. *On the linear independence of certain theta-series*. Math. Ann., 281(2):219–228, 1988.

[38] Serge Lang. *Algebraic number theory*, volume 110 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1994.

[39] R. P. Langlands. *Modular forms and $ł$-adic representations*. In *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 361–500. Lecture Notes in Math., Vol. 349, 1973.
[40] Jungin Lee. On the lower bound of the number of abelian varieties over $\mathbb{F}_p$. *Int. Math. Res. Not. IMRN*, (6):4290–4317, 2022.

[41] Lawrence S. Levy and Roger Wiegand. Dedekind-like behavior of rings with 2-generated ideals. *J. Pure Appl. Algebra*, 37(1):41–58, 1985.

[42] Ke-Zheng Li and Frans Oort. *Moduli of supersingular abelian varieties*, volume 1680 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1998.

[43] Qin Li, Jiangwei Xue, and Chia-Fu Yu. Unit groups of maximal orders in totally definite quaternion algebras over real quadratic fields. *Trans. Amer. Math. Soc.*, 374(8):5349–5403, 2021.

[44] Michael Lipnowski and Jacob Tsimerman. How large is $\mathcal{A}_g(\mathbb{F}_q)$? *Duke Math. J.*, 167(18):3403–3453, 2018.

[45] C. Maclachlan. Optimal embeddings in quaternion algebras. *J. Number Theory*, 128(10):2852–2860, 2008.

[46] Daniel Maisner and Enric Nart. Abelian surfaces over finite fields as Jacobians. *Experiment. Math.*, 11(3):321–337, 2002. With an appendix by Everett W. Howe.

[47] Stefano Marseglia. Computing square-free polarized abelian varieties over finite fields. *Math. Comp.*, 90(328):953–971, 2021.

[48] Thomas M. McCall, Charles J. Parry, and Ramona Ranalli. Imaginary bicyclic biquadratic fields with cyclic 2-class group. *J. Number Theory*, 53(1):88–99, 1995.

[49] J. S. Milne. *Abelian varieties*. In *Arithmetic geometry (Storrs, Conn., 1984)*, pages 103–150. Springer, New York, 1986.

[50] Frans Oort. *Which abelian surfaces are products of elliptic curves?* *Math. Ann.*, 214:35–47, 1975.

[51] Abhishek Oswal and Ananth N. Shankar. Almost ordinary abelian varieties over finite fields. *J. Number Theory*, 264:76–102, 1973.

[52] M. Peters. *Ternäre und quaternäre quadratische Formen und Quaternionenalgebren*. *Acta Arith.*, 15:329–365, 1968/1969.

[53] Arnold K. Pizer. Type numbers of Eichler orders. *J. Reine Angew. Math.*, 264:76–102, 1973.

[54] Vladimir Platonov and Andrei Rapinchuk. *Algebraic groups and number theory*, volume 139 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.

[55] Paul Ponomarev. Arithmetic of quaternary quadratic forms. *Acta Arith.*, 39(1):95–104, 1981.

[56] M. Rapoport. Compactifications de l'espace de modules de Hilbert-Blumenthal. *Compositio Math.*, 36(3):255–335, 1978.

[57] I. Reiner. *Maximal orders*, volume 28 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, Oxford, 2003. Corrected reprint of the 1975 original, With a foreword by M. J. Taylor.

[58] Hans-Georg Rück. Abelian surfaces and Jacobian varieties over finite fields. *Compositio Math.*, 76(3):351–366, 1990.

[59] Goro Shimura. Arithmetic of alternating forms and quaternion hermitian forms. *J. Math. Soc. Japan*, 15:33–65, 1963.

[60] John Tate. Classes d’isogénie des variétés abéliennes sur un corps fini (d’après T. Honda). In *Séminaire Bourbaki. Vol. 1968/69: Exposés 347–363*, volume 175 of *Lecture Notes in Math.*, pages Exp. No. 352, 95–110. Springer, Berlin, 1971.
Gerard van der Geer. Hilbert modular surfaces, volume 16 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1988.

Marie-France Vignéras. Nombre de classes d’un ordre d’Eichler et valeur au point $-1$ de la fonction zêta d’un corps quadratique réel. Enseignement Math. (2), 21(1):69–105, 1975.

Marie-France Vignéras. Invariants numériques des groupes de Hilbert. Math. Ann., 224(3):189–215, 1976.

Marie-France Vignéras. Arithmétique des algèbres de quaternions, volume 800 of Lecture Notes in Mathematics. Springer, Berlin, 1980.

John Voight. Quaternion algebras, volume 288 of Graduate Texts in Mathematics. Springer, Cham, 2021 ©2021. stable post-publication version (v.1.0.5, January 10, 2024).

Lawrence C. Washington. Introduction to cyclotomic fields, volume 83 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1997.

William C. Waterhouse. Abelian varieties over finite fields. Ann. Sci. École Norm. Sup. (4), 2:521–560, 1969.

Jiangwei Xue, Tse-Chung Yang, and Chia-Fu Yu. Numerical invariants of totally imaginary quadratic $\mathbb{Z}[\sqrt{p}]$-orders. Taiwanese J. Math., 20(4):723–741, 2016.

Jiangwei Xue, Tse-Chung Yang, and Chia-Fu Yu. On superspecial abelian surfaces over finite fields. Doc. Math., 21:1607–1643, 2016.

Jiangwei Xue, Tse-Chung Yang, and Chia-Fu Yu. Supersingular abelian surfaces and Eichler’s class number formula. Asian J. Math., 23(4):651–689, 2019.

Jiangwei Xue, Tse-Chung Yang, and Chia-Fu Yu. On superspecial abelian surfaces over finite fields II. J. Math. Soc. Japan, 72(1):303–331, 2020.

Jiangwei Xue and Chia-Fu Yu. On superspecial abelian surfaces and type numbers of totally definite quaternion algebras. Indiana Univ. Math. J., 70(2):781–808, 2021.

Jiangwei Xue and Chia-Fu Yu. Polarized superspecial simple abelian surfaces with real Weil numbers: a survey. In Theory and Applications of Supersingular Curves and Supersingular Abelian Varieties, volume B90 of RIMS Kôkyûroku Bessatsu, pages 39–57. Res. Inst. Math. Sci. (RIMS), Kyoto, 2022.

Jiangwei Xue and Chia-Fu Yu. Trace formulas for the norm one group of totally definite quaternion algebras, with an appendix by John Voight. arXiv e-prints, Aug 2023, arXiv:1909.11858v4 to appear in the Indiana Univ. Math. J.

Jiangwei Xue and Chia-Fu Yu. Optimal spinor selectivity for quaternion orders. J. Number Theory, 255:166–187, 2024.

Jiangwei Xue, Chia-Fu Yu, and Yuqiang Zheng. On superspecial abelian surfaces over finite fields III. Res. Number Theory, 8(1):Paper No. 9, 2022.

Jiangwei Xue and Chia-Fu Yu. On Counting Certain Abelian Varieties Over Finite Fields. Acta Math. Sin. (Engl. Ser.), 37(1):205–228, 2021.

Chia-Fu Yu. On reduction of Hilbert-Blumenthal varieties. Ann. Inst. Fourier (Grenoble), 53(7):2105–2154, 2003.

Chia-Fu Yu. Simple mass formulas on Shimura varieties of PEL-type. Forum Math., 22(3):565–582, 2010.

Chia-Fu Yu. Superspecial abelian varieties over finite prime fields. J. Pure Appl. Algebra, 216(6):1418–1427, 2012.

Chia-Fu Yu. Endomorphism algebras of QM abelian surfaces. J. Pure Appl. Algebra, 217(5):907–914, 2013.

Don Zagier. On the values at negative integers of the zeta-function of a real quadratic field. Enseignement Math. (2), 22(1-2):55–95, 1976.

Ju. G. Zarhin. Endomorphisms of abelian varieties and points of finite order in characteristic $p$. Mat. Zametki, 21(6):737–744, 1977.

Zhe Zhang and Qin Yue. Fundamental units of real quadratic fields of odd class number. J. Number Theory, 137:122–129, 2014.
(Xue) 
Hubei Key Laboratory of Computational Science (Wuhan University), Wuhan, Hubei, 430072, P.R. China.
Email address: xue_j@whu.edu.cn

(Yu) 
Institute of Mathematics, Academia Sinica, Astronomy-Mathematics Building, No. 1, Sec. 4, Roosevelt Road, Taipei 10617, TAIWAN.

(Yu) National Center for Theoretical Sciences, Astronomy-Mathematics Building, No. 1, Sec. 4, Roosevelt Road, Taipei 10617, TAIWAN.
Email address: chiafu@math.sinica.edu.tw