Recent results in high temperature
Lattice Gauge Theories

M. Caselle

Dipartimento di Fisica Teorica dell’Università di Torino
Istituto Nazionale di Fisica Nucleare, Sezione di Torino
via P.Giuria 1, I-10125 Torino, Italy

Abstract

We review some analytic results on the deconfinement transition in pure lattice gauge theories. In particular we discuss the relationship between the deconfinement transition in the \((d + 1)\)-dimensional \(SU(2)\) model and the magnetization transition in the \(d\)-dimensional Ising model. This analysis leads to a precise estimate of the deconfinement temperature which agrees well with that obtained with a Montecarlo simulation in the case in which the lattice has only one link in the compactified time direction.

\*

Talk given at the Enrico Fermi International School of Physics on "Selected Topics in Non-Perturbative QCD", June 1995, Varenna, Italy.

\^e-mail: caselle @to.infn.it
1 Introduction

One of the most interesting predictions of QCD is the existence of a deconfinement transition at some critical temperature $T_c$. Finding a precise characterization of this transition is, however, still an open problem. In particular there are two main open questions: the first one is the identification of the order of the transition and, in case it is of second order, of its critical indices. The second one is the precise location of the deconfinement temperature. The natural framework to pose these questions is that of the finite temperature Lattice Gauge Theories (LGT). In this framework, a seminal contribution was given more than ten years ago by B. Svetitsky and L.Yaffe [1] in the case of pure gauge theories. They showed that, if the deconfining transition of a given $(d+1)$ dimensional gauge theory is of second order, then its universality class should coincide with that of the $d$-dimensional spin model, with symmetry group the center of the original gauge group. This result is usually known as the “Svetitsky-Yaffe (SY) conjecture” and has been confirmed in these last ten years by several Montecarlo simulations. It must be noticed however that the SY conjecture gives no information on the location of the deconfinement transition. Trying to answer to this last question will be the main goal of the present contribution. In particular we shall see how far one can go in trying to estimate the critical temperature by using only analytic methods. We shall concentrate only on pure gauge theories with gauge group $SU(2)$, but most of our results can be straightforwardly extended to $SU(N)$ models with $N > 2$.

During these last years the best estimates of the critical temperature have been obtained by means of Montecarlo simulations, which are certainly the most powerful tool to extract quantitative results from LGT. However we think that it is important in itself to have some independent analytic estimate of the location of the critical point, besides the outputs of the computer simulations, to reach a deeper theoretical understanding of the deconfinement transition. The attempts to obtain analytically the critical temperature have a rather long history, starting more than ten years ago [2, 3, 4, 5, 6]. However the strategy has always been essentially the same: first construct an effective action in terms of the Polyakov loops (which, as we shall see below, are the relevant dynamical variables in the physics of deconfinement for pure gauge theories). Second, use a mean field approximation to extract the critical coupling. A common feature of all these attempts was that the effective actions were always constructed neglecting the spacelike part of the action. As a consequence it was impossible to reach a consistent continuum limit for the critical temperature. Moreover, as a consequence of the mean field approximation, the estimates of the critical temperature were in general affected by large systematic errors.

The aim of the present contribution is to show that it is indeed possible to overcome these two difficulties. First we shall construct in the $SU(2)$ case an improved effective action which takes also into account the spacelike part of the original Wilson action. Second we shall avoid the mean field approximation, and shall instead obtain the critical temperature by mapping (following the SY conjecture) the gauge
theory to a suitable Ising-like model, and then using the fact that the critical temperature of the Ising model is known exactly in $d = 2$, and with very high precision in $d = 3$. Let us stress that this is not the only possible way to avoid the mean field approximation. Another interesting possibility is to study the $SU(N)$ models in the $N \rightarrow \infty$ limit where one can use some results recently obtained in the context of random matrix models and two-dimensional exactly solvable gauge theories to discuss the deconfinement transition and fix the critical temperature. We shall not describe here this approach, the interested reader can find in [7] a comprehensive discussion on the subject.

This contribution is organized as follows. Sect.2 will be devoted to a brief introduction to finite temperature LGT and to the Svetitsky-Yaffe conjecture. In sect.3 we shall construct the effective action, which we shall then use in sect.4 to extract the critical deconfinement temperature. Finally sect.5 will be devoted to some concluding remarks.

2 Finite Temperature LGT

2.1 General setting

Let us consider a pure gauge theory with gauge group $SU(N)$, defined on a $d + 1$ dimensional cubic lattice. In order to describe a finite temperature LGT, we have to impose periodic boundary conditions in one direction (which we shall call from now on “time-like” direction), while the boundary conditions in the other $d$ direction (which we shall call “space-like”) can be chosen freely. We take a lattice of $N_t$ ($N_s$) spacings in the time (space) direction, and we work with the pure gauge theory, containing only gauge fields described by the link variables $U_{n;i} \in SU(N)$, where $n \equiv (\vec{x}, t)$ denotes the space-time position of the link and $i$ its direction. It is useful to choose different bare couplings in the time and space directions. Let us call them $\beta_t$ and $\beta_s$ respectively. The Wilson action is then

$$S_W = \sum_n \frac{1}{N} \text{Re} \left\{ \beta_t \sum_i \text{Tr}_f(U_{n;0i}) + \beta_s \sum_{i<j} \text{Tr}_f(U_{n;ij}) \right\} ,$$

where $\text{Tr}_f$ denotes the trace in the fundamental representation and $U_{n;0i}$ ($U_{n;ij}$) are the time-like (space-like) plaquette variables, defined as usual by

$$U_{n;ij} = U_{n;i}U_{n+i;j}U_{n+j;i}U_{n;j}^\dagger .$$

In the following we shall call $S_s$ ($S_t$) the space-like (time-like) part of $S_W$. $\beta_s$ and $\beta_t$ are related to the (bare) gauge coupling $g$ and to the temperature $T$ by the usual relations

$$\frac{4}{g^2} = a^{3-d} \sqrt{\beta_s \beta_t} , \quad T = \frac{1}{N_t a} \sqrt{\frac{\beta_t}{\beta_s}} ,$$

$$a = \sqrt{\frac{\beta_t}{\beta_s}} .$$
where $a$ is the space-like lattice spacing, while $\frac{1}{N_t}$ is the time-like spacing.

In a finite temperature discretization it is possible to define gauge invariant observables which are topologically non-trivial, as a consequence of the periodic boundary conditions in the time directions. The simplest choice is the Polyakov loop, defined in terms of link variables as

$$P(\vec{x}) \equiv \text{Tr}_f P_{\vec{x}} = \text{Tr}_f \prod_{t=1}^{N_t} (U_{\vec{x},t,0}) .$$  \hfill (4)

In the following we shall call $P_{\vec{x}}$, “open Polyakov line”.

As it is well known, the finite temperature theory has a new global symmetry (unrelated to the gauge symmetry), with symmetry group the center $C$ of the gauge group (in our case $Z_2$). The Polyakov loop is a natural order parameter for this symmetry.

In $d > 1$, finite temperature gauge theories admit a deconfinement transition at $T = T_c$, separating the high temperature, deconfined, phase ($T > T_c$) from the low temperature, confining domain ($T < T_c$). The high temperature regime is characterized by the breaking of the global symmetry with respect to the center of the group. In this phase the Polyakov loop has a non-zero expectation value, and it is an element of the center of the gauge group. In the low temperature phase the center symmetry is conserved and the expectation value of the Polyakov loop is zero. The relevant feature of the Polyakov loop is that at the same time it is also the order parameter for the deconfinement transition. In fact its expectation value is related to the free energy of an isolated, static quark as follows:

$$< P > \propto \exp(-F_q) .$$  \hfill (5)

As a consequence, in the low temperature phase it would require an infinite energy to create from the vacuum an isolated quark. Hence in this phase quarks are confined. On the contrary, in the high temperature phase isolated quark can exist: this is the deconfined phase. The critical point in which the center symmetry is broken can thus be interpreted as the deconfinement transition. The corresponding critical temperature $T_c$ will be denoted in the following as the deconfinement temperature.

### 2.2 Svetitsky-Yaffe conjecture

The idea on which the SY conjecture is based is that, if one would be able to integrate out all the gauge degrees of freedom of the original $(d+1)$–dimensional model except those related to the Polyakov loops then the resulting effective theory for the Polyakov loops would be a $d$-dimensional spin system with symmetry group $C$. The deconfinement transition of the original model would become the order–disorder transition of the effective spin system. This effective theory would obviously have very complicated interactions, but Svetitsky and Yaffe were able to argue that all these interactions should be short ranged. As a consequence, if the
transition point of the effective spin system is of second order, near this critical point, where the correlation length becomes infinite, these short ranged interactions can be neglected, and the universality class of the deconfinement transition should coincide with that of the simple spin model with only nearest neighbour interactions and the same global symmetry group. As a consequence all the critical indices describing the two transitions and all the adimensional ratios of scaling quantities should coincide in the limit. In particular the \((d + 1)\)-dimensional \(SU(2)\) gauge theory, which is known to have a second order deconfinement transition, is characterized by the same renormalization group fixed point of the \(d\)-dimensional Ising model.

Unfortunately the SY argument alone cannot help to fix the critical temperature, since the precise mapping between the Ising coupling and that of the original gauge model, requires taking into account exactly those short ranged interaction which we neglected above.

### 2.3 The \(SU(2)\) case: character expansion

In the following we shall concentrate on the \(SU(2)\) model. There are two important features which greatly simplify the analysis in this case. The first one is that according to the SY conjecture the model can be mapped, at the deconfinement point into the spin Ising model, which is exactly solved in \(d = 2\) and very well known in \(d = 3\). The second important feature is that in the \(SU(2)\) case the character expansion (which plays an important role in the construction of the effective action) is very easy to handle.

Let us briefly summarize few results. The character of the group element \(U\) in the \(j^{th}\) representation is:

\[
\chi_j(U) \equiv \text{Tr}_j(U) = \frac{\sin((2j + 1)\theta)}{\sin(\theta)}
\]

where \(\text{Tr}_j\) denotes the trace in the \(j^{th}\) representation and \(\theta\) is defined according to the following parametrization of \(U\) in the fundamental representation:

\[
U = \cos(\theta) \mathbf{1} + i \vec{\sigma} \vec{n} \sin(\theta)
\]

where \(\vec{n}\) is a tridimensional unit vector and \(\sigma_i\) are the three Pauli matrices. Notice, as a side remark, that with this parametrization the Haar measure has the following form:

\[
DU = \sin^2(\theta) \frac{d\theta d^2 \vec{n}}{4\pi^2}
\]

and the Polyakov loop becomes \(P(\vec{x}) = 2 \cos(\theta x)\)

The following orthogonality relations between characters hold:

\[
\int D U \chi_r(U) \chi_s(U) = \delta_{r,s}
\]
\[ \sum_r d_r \chi_r(U V^{-1}) = \delta(U, V) \quad (10) \]

where \(d_r\) denotes the dimensions of the \(r^{th}\) representation: \(d_r = 2r + 1\). In the following we shall use two important properties of the characters:

\[ \int D U \chi_r(U) \chi_s(U^{-1}V) = \delta_{r,s} \frac{\chi_r(V)}{d_r} , \quad (11) \]

\[ \int D U \chi_r(UV_1 U^{-1}V_2) = \frac{1}{d_r} \chi_r(V_1) \chi_r(V_2) . \quad (12) \]

The character expansion of the Wilson action has a particularly simple form:

\[ e^{\frac{\beta}{2} Tr(U)} = \sum_j (2j + 1) \frac{I_{2j+1}(\beta)}{\beta} I_{2j}(U), \quad j = 0, \frac{1}{2}, 1 \cdots (13) \]

where \(I_n(\beta)\) is the \(n^{th}\) modified Bessel function.

In the following we shall often use the normalized version of the character expansion in which the coefficient of the trivial representation is set to 1.

\[ e^{\frac{\beta}{2} Tr(U)} = G(\beta) \sum_j (2j + 1) \frac{I_{2j+1}(\beta)}{I_1(\beta)} I_j(U), \quad j = 0, \frac{1}{2}, 1 \cdots (14) \]

where \(G(\beta) = 2I_1(\beta)/\beta\) is an irrelevant constant that we shall often omit.

### 3 Construction of the Effective Action

In this section our goal is to construct an effective action for the finite temperature LGT in terms of the Polyakov loops only. This implies that we must be able to integrate out exactly all the spacelike variables so that the only remaining degrees of freedom at the end are exactly the Polyakov loops. Notice that in this way the resulting effective action would live in \(d\) dimensions (one dimension less than the starting model). This is exactly along the line of the original Svetitsky-Yaffe program. In trying to follow such a program one must necessarily make some approximation. In order to obtain a good approximation of the original Wilson action, one must identify the physically relevant degrees of freedom, and then try to keep them unchanged when constructing the effective action. Following [1] and [7], we assume that the physics of the deconfinement transition is dominated by the timelike plaquettes, and try to keep as far as possible unchanged this part of the original action. Accordingly we treat the spacelike part of the Wilson action: \(S_s\) as a perturbation of the timelike part \(S_t\) and take care of the contributions coming from \(S_s\) by making a strong coupling expansion in \(\beta_s\). The main difference with respect to the usual approach is that in this case the time-like part of action is treated exactly or, equivalently, that the expansion in \(\beta_t\) is summed up to all orders. The
only remaining expansion parameter is thus $\beta_s$. In particular the zeroth order in $\beta_s$ will contain the timelike plaquettes only. It is not at all obvious that the integration over the spacelike links could be done to all orders in $\beta_t$, but it turns out that it can be done exactly in the framework of the characters expansion order by order in $\beta_s$. In particular we shall discuss the zeroth and first order in $\beta_s$ only, which will be enough to our purposes, but there is in principle no obstruction to go to higher orders. The result for any given order in $\beta_s$ can be expressed as an infinite sum over characters.

Remarkably enough in the $N_t = 1$ case this series can be summed exactly and the result can be written in a closed form. This is essentially due to the fact that if $N_t = 1$ this same effective action can be obtained in a completely different way, using techniques typical of matrix models (see below), thus allowing a non trivial check of all our strong coupling results. Another interesting feature of the $N_t = 1$ limit is that in this case very precise Montecarlo estimates of $T_c$ exist, with which we can compare our analytic predictions. For instance, in $(3+1)$ dimensions the critical coupling $\beta_c$ at which deconfinement occurs is estimated to be: (for the $SU(2)$ model with $N_t = 1$) $\beta_c = 0.8730(2)$ \textsuperscript{[8]}. Such an impressive precision is due to that fact that in the $N_t = 1$ case (and only in this case) one can simulate the gauge model by using a cluster non-local algorithm (see \textsuperscript{[8]} for the details). This makes the $SU(2)$, $N_t = 1$ model a perfect laboratory to test our techniques, and we shall concentrate on this case in sect.4. We shall briefly comment on the extension of our results to $N_t > 1$ in sect.5.

### 3.1 Expansion in $\beta_s$ of the effective action

The effective action $S_{\text{eff}}$ for the Polyakov loops $P_{\vec{x}} \equiv \prod_{t=1}^{N_t} V_{\vec{x}}$ is obtained integrating over all the spacelike degrees of freedom in the action (1). As explained previously, our approach is to consider the contributions from the spacelike plaquettes up to a certain order in $\beta_s$ only. So, for our purposes, it will be convenient to expand separately the spacelike and the timelike part of the action (1):

$$e^{S_{\text{eff}}} = \int \prod_{\vec{x},t;i} DU_{\vec{x},t;i} \exp S_W$$

$$= \int \prod_{\vec{x},t;i} DU_{\vec{x},t;i} \prod_{\vec{x},t;i} \left( 1 + \sum_{j=\frac{1}{2}}^{\infty} d_j \frac{I_{2j+1}(\beta_t)}{I_1(\beta_t)} \chi_j(U_{\vec{x},t,0i}) \right)$$

$$\times \prod_{\vec{x},t;i,j} \left( 1 + \sum_{l=\frac{1}{2}}^{\infty} d_l \frac{I_{2l+1}(\beta_s)}{I_1(\beta_s)} \chi_l(U_{\vec{x},t,ij}) \right). \quad (15)$$

Specifically, we work out here the effective action up to $O(\beta_s^2)$. This means that in eq. (13) we must look only at the terms containing at most a single space-like plaquette in the adjoint representation, $\chi_1(U_{\vec{x},t,ij})$, or two space-like plaquettes.
in the fundamental, $\chi_{\frac{1}{2}}(U_{\vec{x},t_1;ij})\chi_{\frac{1}{2}}(U_{\vec{y},t_2;kl})$. Due to the orthogonality relations for characters, it's easy to convince oneself that a pair of plaquettes in the fundamental representation do actually contribute to the integral only if they appear in the same spatial position (at two different times $t_1$ and $t_2$); for the same reason a single fundamental plaquette cannot contribute. We are thus lead to the following expression:

$$\exp(S_{\text{eff}}) = \int \prod_{\vec{x},t} DU_{\vec{x},t} \prod_{\vec{x},t} \left( 1 + \sum_{j=\frac{1}{2}}^{\infty} d_j \frac{I_{2j+1}(\beta_t)}{I_1(\beta_t)} \chi_j(U_{\vec{x},t;0}) \right)$$

$$\times \left( 1 + \sum_{\vec{x},i<j}^{N_t} \left[ \sum_{t=1}^{\infty} 3 \frac{I_3(\beta_t)}{I_1(\beta_t)} \chi_1(U_{\vec{x},t;ij}) + \sum_{t_1<t_2}^{4} \left( \frac{I_2(\beta_t)}{I_1(\beta_t)} \right)^2 \chi_{\frac{1}{2}}(U_{\vec{x},t_1;ij}) \chi_{\frac{1}{2}}(U_{\vec{x},t_2;ij}) \right) \right)$$

In the following we shall calculate these integrals for a generic value of $N_t$

### 3.2 Zeroth order approximation

Let us first study the contribution which gives the $O(\beta_0)$ result and corresponds to the “1” in the second factor of eq.(16). In this case the integral only contains the timelike part of the Wilson action:

$$\exp(S_0) = \int \prod_{\vec{x},i} \left[ DU_{\vec{x},i} \left( 1 + \sum_{j=\frac{1}{2}}^{\infty} d_j \frac{I_{2j+1}(\beta_t)}{I_1(\beta_t)} \chi_j(U_{\vec{x},i;V_{\vec{x},i}+i U_{\vec{x},i}^\dagger P_{\vec{x}}}) \right) \right], \quad (17)$$

and it is easy to integrate all the spacelike links. The reason is that each spacelike link only belongs to two timelike plaquettes, hence by making a character expansion, it can be exactly integrated out. Let us do this integration in two steps, for future commodity. First let us integrate (by using eq.(11)) all the spacelike links except the lowermost ones (which, due to the periodic boundary conditions coincides with the uppermost). We obtain:

$$\exp(S_0) = \prod_{\vec{x},i} \left( 1 + \sum_{j=\frac{1}{2}}^{\infty} \left[ \frac{I_{2j+1}(\beta_t)}{I_1(\beta_t)} \right]^{N_t} \chi_j(U_{\vec{x},i;P_{\vec{x},i} U_{\vec{x},i}^\dagger P_{\vec{x}}}) \right). \quad (18)$$

where $P_{\vec{x}}$ is the open Polyakov line (whose trace is the Polyakov loop) in the site $\vec{x}$ and $U_{\vec{x},i}$ are the remaining lowermost spacelike links. Integrating also on $U_{\vec{x},i}$ (this time, by using eq.(11)) we end up with

$$\exp(S_0) = \prod_{\vec{x},i} \left( 1 + \sum_{j=\frac{1}{4}}^{\infty} \left[ \frac{I_{2j+1}(\beta_t)}{I_1(\beta_t)} \right]^{N_t} \chi_j(P_{\vec{x},i}) \chi_j(P_{\vec{x}}) \right). \quad (19)$$
Let us define, for future convenience, the link element of \( \exp(S_0) \) as follows:

\[
C^0_{\vec{x},i} \equiv \sum_{j=0}^{\infty} \left[ \frac{I_{2j+1}(\beta_t)}{I_1(\beta_t)} \right]^N \chi_j(P_{x+i})\chi_j(P_{\vec{x}}^i) .
\] (20)

It is now evident that this basic element, which will be denoted also as \( C^0_{\vec{x},i} = C^0(\theta_{\vec{x}}, \theta_{\vec{x}+i}) \), depends only on \( \theta_{\vec{x}}, \theta_{\vec{x}+i} \), which are the invariant angles for the Polyakov lines \( P_{\vec{x}}, P_{\vec{x}+i} \) in the sites joined by the link. Indeed from now on we will always assume to have gauge-rotated the Polyakov lines to be diagonal:

\[
P_{\vec{x}} = \begin{pmatrix} e^{i\theta_{\vec{x}}} & 0 \\ 0 & e^{-i\theta_{\vec{x}}} \end{pmatrix} .
\] (21)

With these definitions the zero-th-order action (19) is simply given by

\[
\exp(S_0) = \prod_{\vec{x},i} C^0_{\vec{x},i} .
\] (22)

However let us stress that the action that was generally used in the previous attempts to obtain mean field estimates of the deconfinement temperature, was actually a simplified version (truncated at the first representation) of eq.(19):

\[
S_p(\beta_t) = \sum_{\vec{x}} \left\{ \beta_t \sum_{i=1}^{d} \cos(\theta_{\vec{x}}) \cos(\theta_{\vec{x}+i}) \right\} .
\] (23)

It is interesting to notice that in the \( N_t = 1 \) case the character expansion contained in eq.(19) can be summed exactly. This can be easily seen by writing the explicit form for the characters in eq.(19):

\[
\exp(S_0) = \prod_{\vec{x},i} \left( 1 + \sum_{j=0}^{\infty} \left[ \frac{I_{2j+1}(\beta_t)}{I_1(\beta_t)} \right] \frac{\sin((2j+1)\theta_{\vec{x}})\sin((2j+1)\theta_{\vec{x}+i})}{\sin(\theta_{\vec{x}})\sin(\theta_{\vec{x}+i})} \right) .
\] (24)

Then using the relation:

\[
2 \sin[(2r+1)\theta_{\vec{x}}] \sin[(2r+1)\theta_{\vec{x}+i}] = \cos[(2r+1)(\theta_{\vec{x}}-\theta_{\vec{x}+i})] - \cos[(2r+1)(\theta_{\vec{x}}+\theta_{\vec{x}+i})]
\] (25)

and the well known expansion:

\[
e^{\beta \cos \theta} = I_0(\beta) + 2 \sum_{k=1}^{\infty} I_k(\beta) \cos(k\theta),
\] (26)

it is easy to obtain:

\[
\exp(S_0) = \prod_{\vec{x},i} \frac{e^{\beta_t \cos(\theta_{\vec{x}}-\theta_{\vec{x}+i})} - e^{\beta_t \cos(\theta_{\vec{x}}+\theta_{\vec{x}+i})}}{4I_1(\beta_t)\sin(\theta_{\vec{x}})\sin(\theta_{\vec{x}+i})} .
\] (27)

\(^1\)The links we are referring to are those of the \( d \)-dimensional spatial lattice, corresponding to one “horizontal” slice in the original \( d + 1 \)-dimensional lattice.
3.3 First order approximation

The $O(\beta^0)$ effective action (22) contains just nearest-neighbour interactions between the Polyakov loops. We shall show below that $O(\beta^2)$ contributions to the effective action are of plaquette type, namely they involve all the four invariant angles of the Polyakov lines which belong to a given plaquette. This interaction is more general than the nearest-neighbour one, but it is still short ranged, in agreement with the hypothesis which is at the basis of the SY conjecture discussed above.

3.3.1 The adjoint representation term.

To calculate the contribution coming from the adjoint representation term, we have to select in eq.(16) the term:

$$3 \frac{I_3(\beta_s)}{I_1(\beta_s)} \int \prod_{\vec{x},t;i} DU_{\vec{x},t;i} \prod_{\vec{x},t;i} \left( 1 + \sum_{j=\frac{1}{2}}^{\infty} d_j \frac{I_{2j+1}(\beta_t)}{I_1(\beta_t)} \chi_j(U_{\vec{x},t,0}) \right) \times \sum_{\vec{x},i<j} \sum_{t=1}^{N_t} \chi_1(U_{\vec{x},t;ij}) . \quad (28)$$

To study the integral (28), we first note that all the spacelike plaquettes in the same spatial position give evidently the same contribution, regardless of the time $t$; therefore the sum over the time positions in (28) simply results in a $N_t$ factor. Secondly, it is convenient to use the following relation for the $SU(2)$ characters:

$$\chi_1 = (\chi_2^2 - 1) . \quad (29)$$

The “$-1$” simply reproduces the zeroth order term, and gives a renormalization of order $\beta^2_s$ to such contribution. The integral along the plaquette can now be decoupled into integrals over a single link matrix, by writing explicitly $[\chi_2(U_{\vec{x},t;ij})]^2$ as a product of elements (in the fundamental representation) of the link matrices. Thus eq. (28) can be rewritten in terms of the following integrals over the unitary spacelike link matrix $U$:

$$B_{\alpha\beta\gamma\delta}(P_{\vec{x}},P_{\vec{x}+i}) = \int DU \left( 1 + \sum_{j=\frac{1}{2}}^{\infty} d_j \left[ \frac{I_{2j+1}(\beta_t)}{I_1(\beta_t)} \right]^{N_t} \chi_j(U P_{\vec{x}+i} U^\dagger P_{\vec{x}}^\dagger) \right) U_{\alpha\beta} U_{\gamma\delta}^\dagger . \quad (30)$$

where $\alpha, \ldots = 1, 2$ are the indices of the $U$ matrix in the fundamental representation. Making use of the invariance of the measure and of the argument of $\chi_j$ in eq. (30) under the transformations

$$U_{\alpha\beta} \rightarrow \omega_{\alpha\alpha} U_{\alpha\beta} , \quad U_{\gamma\delta}^\dagger \rightarrow U_{\gamma\delta}^\dagger (\omega^{-1})_{\delta\delta}$$

and

$$U_{\alpha\beta} \rightarrow U_{\alpha\beta} \omega_{\beta\beta} , \quad U_{\gamma\delta}^\dagger \rightarrow (\omega^{-1})_{\gamma\gamma} U_{\gamma\delta}^\dagger$$

under the transformations
(where the diagonal unitary matrix $\omega$ has the property $\omega^2 = 1$), one can conclude that $B_{\alpha\beta\gamma\delta} = 0$ unless $\beta = \gamma$ and $\alpha = \delta$. As a consequence, the integral (30) depends on the invariant angles of the Polyakov loop only and can be written as follows:

$$B_{\alpha\beta\gamma\delta}(P_{\vec{x}}, P_{\vec{x}+i}) \equiv B_{\alpha\beta\gamma\delta}(\theta_{\vec{x}}, \theta_{\vec{x}+i}) = \delta_{\beta\gamma} \delta_{\delta\alpha} C_{\alpha\beta}(\theta_{\vec{x}}, \theta_{\vec{x}+i})$$

(no summation over repeated indices). Moreover, it is not difficult to show that $C_{\alpha\beta}$ is a real symmetric matrix. By using these results, we can write the contribution (28) to the effective action only in terms of the invariant angles of the Polyakov line $P_{\vec{x}}$. Eq. (28) becomes:

$$3N_t \frac{I_3(\beta_t)}{I_1(\beta_t)} \left[ \prod_{\vec{x},i} C_{0,i}^0 \right] \left[ \text{Tr} \left[ \hat{C}(\theta_{\vec{x}}, \theta_{\vec{x}+i}) \hat{C}(\theta_{\vec{x}+i}, \theta_{\vec{x}+i+j}) \hat{C}(\theta_{\vec{x}+i+j}, \theta_{\vec{x}+j}) \hat{C}(\theta_{\vec{x}+j}, \theta_{\vec{x}}) \right] - 1 \right]$$

(34)

The subtraction of the term $(-1)$ in (34) is due to the term $(-1)$ in (29), whereas the matrices $\hat{C}(\theta_{\vec{x}}, \theta_{\vec{x}+i})$ are the normalized version of $C$:

$$\hat{C}_{\alpha,\beta}(\theta_{\vec{x}}, \theta_{\vec{x}+i}) = \frac{C_{\alpha,\beta}(\theta_{\vec{x}}, \theta_{\vec{x}+i})}{C_{0,i}^0}$$

(35)

The last step is the explicit evaluation of the matrix elements $C_{\alpha\beta}$. This calculation is described in the Appendix. The final result is:

$$C_{11} = C_{22} = \frac{1}{2} (C_0 + C_1)$$

$$C_{12} = C_{21} = \frac{1}{2} (C_0 - C_1)$$

(36)

with:

$$C_1 = \frac{1}{2 \sin \theta_{\vec{x}} \sin \theta_{\vec{x}+i}} \left\{ \sum_{j=0}^{\infty} \chi_j(\theta_{\vec{x}}) \chi_j(\theta_{\vec{x}+i}) I_{2j+2}(\beta_t) N_t - I_{2j+1}(\beta_t) N_t \right\}$$

$$+ 2 \cos [(2j+1)\theta_{\vec{x}}] \cos [(2j+1)\theta_{\vec{x}+i}] \left( \frac{I_{2j+1}(\beta_t)}{I_1(\beta_t)} \right)^{N_t} + \left[ \frac{I_0(\beta_t)}{I_1(\beta_t)} \right]^{N_t}$$

(37)

$C_0$ was defined in eq.(20), and, as expected, $C_{11} + C_{12} = C_0$.

Eq.(37) could seem a bit complicated, but it is actually very easy to implement it in a mean field analysis or in the SY type mapping described in the next section.

In the $N_t = 1$ case the sum over the representations can be performed exactly, and a closed expression for the $C_{\alpha\beta}$ coefficients can be obtained. This can be done by using the identity:

$$I(\beta)_{n-1} - I(\beta)_{n+1} = 2n I(\beta)_n$$

(38)
are studying becomes a particular instance of the Migdal-Kazakov model. This
interest in the \( N \rightarrow \infty \) limit is that in this case the theory that we shall
be interested in the following we shall omit here the details of this calculation which
can be found in [9].

3.3.2 Pair of fundamental representations.
To calculate the contribution coming from a pair of fundamental representations, we have to select the last term in (16):

\[
C_{12}(N_t = 1) = \frac{4 I_1(\beta_1) \sin(\theta_x) \sin(\theta_{x+i})}{8 \beta_1 I_1(\beta_1) \sin^2(\theta_x) \sin^2(\theta_{x+i})} - \frac{e^{\beta_1 \cos(\theta_x - \theta_{x+i})} - e^{\beta_1 \cos(\theta_x + \theta_{x+i})}}{4 I_1(\beta_1) \sin(\theta_x) \sin(\theta_{x+i})} .
\] (39)

3.4 Independent approach to \( N_t = 1 \)
The interesting feature of the \( N_t = 1 \) limit is that in this case the theory that we are studying becomes a particular instance of the Migdal-Kazakov model. This connection was already noticed in [10] and was the origin of our previous analysis in the \( N \rightarrow \infty \) limit [7]. All the integrals that we have described in the previous sections can be directly evaluated in this case as particular instances of a nontrivial generalization of the so called Itzykson-Zuber integral, evaluated in [11]. We refer the reader to [11] for a comprehensive discussion on this interesting subject and simply report here the results which are useful for our analysis:

zeroth order contribution:

\[
\int dU_{\bar{x};i} \exp \left\{ \frac{\beta_1 t}{2} \text{Tr}_f \left( V(\bar{x}) U_{\bar{x};i} V^{\dagger}(\bar{x} + i) U_{\bar{x};i}^{\dagger} \right) \right\} = e^{\beta_1 t \cos(\theta_x - \theta_{x+i})} - e^{\beta_1 t \cos(\theta_x + \theta_{x+i})} .
\] (41)

The first order contributions can be extracted by the correlators defined in [11] as:

\[
\langle (U_{\bar{x};i})_{\mu,\nu} (U_{\bar{x};i}^{\dagger})^{\rho,\sigma} \rangle = \frac{\int dU_{\bar{x};i} \exp \left\{ \frac{\beta_1 t}{2} \text{Tr}_f \left( V(\bar{x}) U_{\bar{x};i} V^{\dagger}(\bar{x} + i) U_{\bar{x};i}^{\dagger} \right) \right\} (U_{\bar{x};i})_{\mu,\nu} (U_{\bar{x};i}^{\dagger})^{\rho,\sigma} \int dU_{\bar{x};i} \exp \left\{ \frac{\beta_1 t}{2} \text{Tr}_f \left( V(\bar{x}) U_{\bar{x};i} V^{\dagger}(\bar{x} + i) U_{\bar{x};i}^{\dagger} \right) \right\}} \right\} .
\] (42)
It is easy to see that these correlators must be of diagonal form, namely:

\[
\langle (U_{x; i})_{\mu, \nu} (U_{x; i}^\dagger)_{\rho, \sigma} \rangle = \delta_{\mu, \rho} \delta_{\nu, \sigma} \hat{C}_{\mu, \nu}(x; i) \tag{43}
\]

where the \( \hat{C}_{\mu, \nu}(x; i) \) are equivalent, apart from the different normalization, to our \( C_{kl} \) matrix elements. They turn out to be \cite{11}:

\[
\hat{C}_{1,1}(x; i) = \hat{C}_{2,2}(x; i) = \frac{2\beta_t \sin(\theta_x) \sin(\theta_{x+i}) - (1 - e^{-2\beta_t \sin(\theta_x) \sin(\theta_{x+i})})}{(1 - e^{-2\beta_t \sin(\theta_x) \sin(\theta_{x+i})}) (2\beta_t \sin(\theta_x) \sin(\theta_{x+i}))}
\]

\[
\hat{C}_{1,2}(x; i) = \hat{C}_{2,1}(x; i) = \frac{1 - e^{-2\beta_t \sin(\theta_x) \sin(\theta_{x+i})} (1 + 2\beta_t \sin(\theta_x) \sin(\theta_{x+i}))}{(1 - e^{-2\beta_t \sin(\theta_x) \sin(\theta_{x+i})}) (2\beta_t \sin(\theta_x) \sin(\theta_{x+i}))}
\tag{44}
\]

It is now only matter of straightforward algebra (one must also take into account eq.(35) and the normalization constant \( G(\beta) \) defined in \cite{14}) to show that these expressions eq.(41) and (44) are exactly equivalent to our results \cite{27} (39).

### 4 Determination of \( T_c \)

As we discussed in the introduction, the standard approach to the determination of the critical temperature would be at this point a mean field analysis of the above constructed effective action. However this approach is rather unsatisfactory. For instance if we take the standard mean field approximation of the zeroth order action, truncated at the first representation, eq.(23) (for which the calculation can be performed exactly, see for instance \cite{4}) it is easy to see that the resulting critical coupling in the \( N_t = 1 \) case is \( \beta_c = 2/d \). In (3+1) dimensions we know that \( \beta_c = 0.8730(2) \) \cite{8}, and it is clear that the standard mean field derivation which gives in this case \( \beta_c = 0.666... \) is largely unsatisfactory. It is possible to improve this result keeping the full effective action instead of its truncated version, and improving the mean field approximation. This gives a much better result, which however always remains between 5 and 10 % below the Montecarlo result (see \cite{9} for a discussion of this approach).

In this section we want to discuss a completely different approach, which makes explicit use of the mapping between the \( SU(2) \) model and the \( d \)-dimensional Ising model and turns out to be much more powerful of the mean field approach. Up to our knowledge the only attempt along this line was made by J. Polonyi and K.Szachanyi in \cite{6}, but since they were constrained to keep in the various stages of their analysis only the very first order in \( \beta_t \) (and to neglect \( \beta_s \)) their result was rather unsatisfactory. We shall review their approach below. The novelty of our approach with respect to this previous attempt is twofold. First, we keep all the orders in the \( \beta_t \) expansion of the interaction. Second, and more important, we use the explicit knowledge of the first order in the \( \beta_s \) expansion to map the original gauge model
to the equivalent Ising model, explicitly relating the gauge coupling and the Ising coupling in a new original way, completely different from that of [6]. From the knowledge of the location of the Ising phase transition we can thus reconstruct the exact critical deconfinement temperature.

Let us follow this procedure in the $N_t = 1$ case. The crucial point of the whole approach is the identification of the Ising variable “embedded” in the Polyakov loops. Let us follow as a first example the analysis of ref. [6]. The simplest way to extract these Ising variables is to decompose the Polyakov loops:

$$P(\vec{x}) = 2 \cos(\theta_{\vec{x}}), \quad \theta_{\vec{x}} \in [-\pi, \pi] .$$  \hspace{1cm} (45)

which are $SU(2)$ variables, into the product of a $Z_2$ variable $\sigma(\vec{x})$, which is simply a sign, and a $SO(3)$ variable:

$$\tilde{P}(\vec{x}) = 2 \cos(\theta_{\vec{x}}), \quad \theta_{\vec{x}} \in [-\pi/2, \pi/2] .$$  \hspace{1cm} (46)

Then by integrating over $\tilde{P}$ one ends up with the desired effective action of the Ising type. All these steps can be easily performed if we keep at each stage only the very first order in $\beta_t$ (and neglect $\beta_s$). In this case the first step gives us the effective action $S_p$ of eq.(23). Then, it is possible to see that at the first order in $\beta_t$ the $SO(3)$ variables decouple and can be integrated out exactly. The effective action becomes (neglecting an irrelevant overall constant):

$$S_p(\beta_t) = \sum_{\vec{x}} \left\{ \beta_{Ising} \sum_{i=1}^{d} \sigma(\vec{x}) \sigma(\vec{x} + i) \right\} ,$$  \hspace{1cm} (47)

with $\beta_{Ising} = \frac{16}{9\pi^2} \beta_t$. This is exactly the ordinary Ising action which is known to have, both in two and three dimensions a second order phase transition located at $\beta_{t,Ising}^{(2)} = 0.44068679 \ldots$ and $\beta_{t,Ising}^{(3)} = 0.221652(3)$ in two and three dimensions respectively. From this we obtain the following values for the deconfinement temperature: $\beta_t = 2.447$ and $\beta_t = 1.231$ in $(2+1)$ and $(3+1)$ dimensions respectively. Even if the order of magnitude is essentially correct it is easy to see that these estimates are even worse than the plain mean field analysis on the action eq.(23). Notwithstanding this, let us stress again that this approach is very interesting in itself, because it allows to have a deeper physical insight on the mechanism underlying the deconfinement transition. Let us now improve this analysis by keeping in the various step all the orders in the $\beta_t$ expansion and the first order in $\beta_s$.

First, let us start from eq.(27), which is the exact effective action, taking into account all orders in $\beta_t$ and let us apply the same recipe as above to extract the embedded Ising variables. We obtain after some simple algebra:

$$\exp(S_0) = \prod_{\vec{x}} \left( \prod_{i=1}^{d} \left\{ \frac{e^{\beta_t \sin(\theta_{\vec{x}}) \sin(\theta_{\vec{x}+i})} - e^{-\beta_t \sin(\theta_{\vec{x}}) \sin(\theta_{\vec{x}+i})}}{2\beta_t \sin(\theta_{\vec{x}}) \sin(\theta_{\vec{x}+i})} \right\} \right) \times e^{\beta_t \cos(\theta_{\vec{x}}) \cos(\theta_{\vec{x}+i}) \sigma(\vec{x}) \sigma(\vec{x}+i)} .$$  \hspace{1cm} (48)
Let us define
\[ \alpha(\vec{x}, i) \equiv \cos(\theta_{\vec{x}}) \cos(\theta_{\vec{x}+i}) , \]
\[ \gamma(\vec{x}, i) \equiv \sin(\theta_{\vec{x}}) \sin(\theta_{\vec{x}+i}) . \]

Since at the deconfinement point the SO(3) variables are not critical, we can assume that \( \theta(\vec{x}) \) is described also at the deconfinement point by a constant master field \( \theta_0 \) and that fluctuations in \( \theta \) can be neglected. Let us consistently define:
\[ \alpha_0 \equiv \cos^2(\theta_0) , \quad \gamma_0 \equiv \sin^2(\theta_0) . \]

The problem is thus reduced to the identification of this master field, in terms of which we could write \( \beta_{t,c} = \beta_{c}^{\text{Ising}} / \alpha_0 \).

The only way we have to find this master field is to identify also at the level of the \( \mathcal{O}(\beta_2^2) \) effective action the underlying Ising model. However this is clearly a non-trivial task. In fact at a first glance one could think that it is impossible to map our \( \mathcal{O}(\beta_2^2) \) effective action into an equivalent Ising model because the \( \mathcal{O}(\beta_2^2) \) contribution is not of the nearest-neighbour type. For instance, if we separate also in this case the \( Z_2 \) degrees of freedom from the SO(3) ones as we did above, due to the plaquette structure of the action, the \( Z_2 \) degrees of freedom exactly cancel each other, and the underlying Ising model seems to be lost. However there is a completely different way to recognize such an Ising model in the \( \mathcal{O}(\beta_2^2) \) interaction. In fact, a remarkable and non-trivial consequence of the particular form of the \( \mathcal{O}(\beta_2^2) \) interaction is that it can be exactly reorganized as the first term of the strong coupling expansion of an ordinary nearest-neighbour Ising model, thus allowing to complete the identification. In fact, if we keep as above a constant value of \( \theta = \theta_0 \), then the relations \( C_{1,1} = C_{2,2} \) and \( C_{1,2} = C_{2,1} \), and above all the fact that \( C_{1,1} + C_{1,2} = 1 \), allow us to interpret eq.\([34]\) as the strong coupling expansion of a \( d \)-dimensional Ising model with action, say, \( S_{\text{Ising}} = J \sigma(\vec{x})\sigma(\vec{x} + i) \) where the values \( i = 1 \) and \( i = 2 \) of the indices of \( C_{i,j} \) denote the +1 and −1 values of the Ising spin \( \sigma(x) \) and the Ising coupling \( J \) is given by
\[ J = \frac{1}{2} \log \left( \frac{C_{1,1}}{C_{1,2}} \right) = \frac{1}{2} \log \frac{2\beta \gamma_0 - (1 - e^{-2\beta \gamma_0})}{1 - e^{-2\beta \gamma_0} (1 + 2\beta \gamma_0)} . \]  

(49)

This induced Ising model can be considered as the replica at the first order in \( \beta_4 \) of that described above at the zeroth order in \( \beta_4 \). Again we must require the coupling \( J \), to be at its critical value \( J_c \). Solving eq.\([19]\) with respect to \( \gamma_0 \), we find \( \gamma_0 = 1.3957 / \beta_{t,c} \) in \( (2+1) \) dimensions and \( \gamma_0 = 0.67383 / \beta_{t,c} \) in \( (3+1) \) dimensions. Combining these values with the above discussed relation: \( \beta_{t,c} = \beta_{c}^{\text{Ising}} / \alpha_0 \), we finally find:

(2+1) dimensions:
\[ \beta_t = 1.836 , \quad \alpha_0 = 0.2398 , \quad \theta_0 = 0.337\pi \]
(3+1) dimensions:
\[ \beta_t = 0.8954 , \quad \alpha_0 = 0.2475 , \quad \theta_0 = 0.334 \pi . \]

As we anticipated above, these values for \( \beta_c \) are much lower than those of ref. [3], and the one for \( d = 3 \) is in good agreement with the result obtained by Montecarlo simulations.

## 5 Conclusions

The approach outlined in sect.4 can be extended also to \( N_t > 1 \) [3]. In following this extension one must take care of some non-trivial features of the models, like the fact that the critical coupling \( \beta_c \) as a function of \( N_t \) obeys different scaling laws in \( (2+1) \) and \( (3+1) \) dimensions. However the pattern of our approach needs not to be changed. The agreement with the Montecarlo results (when they exist) remains very good. Since our approach is not limited by the magnitude of \( N_t \), we can hope that, as \( N_t \) increases, a sensible continuum limit for \( T_c \) could be taken. To reach this goal we must be able to reconstruct the correct scaling laws in the large \( N_t \) limit. This is certainly possible for the \( (2+1) \) dimensional model (see [9] for details), but it is still an open problem in the \( (3+1) \) dimensional case. In any case, besides the numerical results, we think that the improvements that we have discussed in this contribution both in constructing the effective action and in extracting the critical coupling can help us to have a better and deeper understanding of the physics of the deconfinement transition in lattice gauge theories.

### Acknowledgements

We thank M. Billó, A. D’Adda, F. Gliozzi and S.Panzeri for many helpful discussions.

### Appendix

In this Appendix we shall evaluate the matrix elements \( C_{kl} \).

To this end let us select in the sum over representations contained in \( C_{kl} \) the \( j^{th} \) term:
\[ A_{kl}^j \equiv \int D U |U^{kl}|^2 \chi_j(V(\vec{x})U \ V^\dagger(\vec{x}+i)U^\dagger) \]  

(A.1)

so that \( C_{kl} \) can be written as:
\[ C_{kl} = \sum_{r=0}^{\infty} d_r \left[ \frac{I_{2r+1}(\beta_t)}{T_1(\beta_t)} \right]^{N_t} A_{kl}^r . \]  

(A.2)
Let us use the following relation:

\[ \chi_j(U) = \sum_{k=0}^{\lfloor j \rfloor} \frac{(-1)^k (2j - k)!}{k! (2j - 2k)!} \chi^2_{\frac{j}{2}} - 2k, \quad j = 0, \frac{1}{2}, 1, \cdots \] (A.3)

(where \( \lfloor j \rfloor \) denotes the integer part of \( j \)), which is a direct consequence of the identification of the \( SU(2) \) characters with the Tchebichef polynomials of second kind: \( \chi_j(U) = U_{2j}(\cos(\theta)) \) (where \( \theta \) denotes, as usual, the invariant angle of the matrix \( U \)).

We can rewrite (A.1) as:

\[ A^j_{kl} = \sum_{k=0}^{\lfloor j \rfloor} \frac{(-1)^k (2j - k)!}{k! (2j - 2k)!} \int D U |U^{kl}|^2 \chi^2_{\frac{j}{2}} - 2k (V(\vec{x}) U V^\dagger(\vec{x} + i)U^\dagger). \] (A.4)

Since the \( U \) matrix elements always appear in the form \( |U^{kl}|^2 \) (with the indices in the fundamental representation) it turns out that a very useful parametrization is:

\[ U = a_0 1 + \sum_{i=1}^3 a_i \sigma_i \] (A.5)

where \( \sigma_i \) are the Pauli matrices, the \( a_i \) are real numbers constrained by: \( \sum_{i=0}^3 a_i^2 = 1 \). In this parametrization we have \( |U^{12}|^2 = a_1^2 + a_2^2 \) and \( |U^{11}|^2 = |U^{22}|^2 = a_0^2 + a_3^2 \). Setting \( a_1^2 + a_2^2 = x \) we see that the \( \chi^2_{\frac{j}{2}} \) in eq.(A.4) becomes

\[ \chi^2_{\frac{j}{2}} (V(\vec{x}) U V^\dagger(\vec{x} + i)U^\dagger) = g + hx \] (A.6)

with \( g = 2 \cos(\theta_{\vec{x}} - \theta_{\vec{x}+i}) \) and \( h = -4 \sin \theta_{\vec{x}} \sin \theta_{\vec{x}+i} \). The measure \( DU \) becomes \( dx \), with integration limits 0 and 1, according to the above mentioned constraint on the \( a_i \). The \( A^j_{kl} \) integrals become:

\[ A^j_{11} = A^j_{22} = \sum_{k=0}^{[j]} \frac{(-1)^k (2j - k)!}{k! (2j - 2k)!} \int_0^1 dx (1 - x)(g + hx)^{2j-2k} \] (A.7)

\[ A^j_{12} = \sum_{k=0}^{[j]} \frac{(-1)^k (2j - k)!}{k! (2j - 2k)!} \int_0^1 dx x(g + hx)^{2j-2k} \] (A.8)

Before evaluating these integrals, as a preliminary exercise, let us calculate the simpler integral in which no contribution coming from the spacelike plaquette is present. Let us call it \( A^j_0 \):

\[ A^j_0 = \sum_{k=0}^{[j]} \frac{(-1)^k (2j - k)!}{k! (2j - 2k)!} \int_0^1 dx (g + hx)^{2j-2k} \] (A.9)

If we are able to evaluate the integrals and sum up the series we shall find an alternative way to go from eq.(18) to eq.(19).
The integrals in the sum of eq.(A.9) can be done straightforwardly:
\[ \int_{0}^{1} dx (g + hx)^n = \frac{(g + h)^{n+1} - g^{n+1}}{h(n + 1)} \]  \hspace{1cm} (A.10)
inserting this result in eq.(A.9), and using the explicit expression for \(g\) and \(h\) we have:
\[ A_j^0 = \sum_{k=0}^{[j/2]} (-1)^k (2j - k)! \frac{[2 \cos(\theta_x + \theta_{x+i})]^{n+1} - [2 \cos(\theta_x - \theta_{x+i})]^{n+1}}{k! (2j + 1 - 2k)!} \cdot \frac{4 \sin \theta_x \sin \theta_{x+i}}{n(n+1)} \]  \hspace{1cm} (A.11)
By using the explicit expression of the Tchebichef polynomials of first type:
\[ T_n(\cos(\theta)) = \cos(n\theta) = \frac{\cos(2j+1)(\theta_x - \theta_{x+i}) - \cos(2j+1)(\theta_x + \theta_{x+i})}{2(2j+1)\sin \theta_x \sin \theta_{x+i}} \]  \hspace{1cm} (A.12)
we can rewrite \(A_j^0\) as
\[ A_j^0 = \frac{\cos((2j+1)(\theta_x - \theta_{x+i})) - \cos((2j+1)(\theta_x + \theta_{x+i}))}{2(2j+1)\sin \theta_x \sin \theta_{x+i}} \]  \hspace{1cm} (A.13)
using eq.(25), inserting the result in the sum on the representations, and using the explicit expression for the characters \(\chi_j\) we exactly obtain eq.(19).
Let us now calculate \(A_j^{11}\). Also in this case the integral contained in eq.(A.7) are straightforward:
\[ \int_{0}^{1} dx (1-x)(g + hx)^n = \frac{(g + h)^{n+2} - g^{n+2}}{h^2(n + 1)(n + 2)} - \frac{g^{n+1}}{h(n + 1)} \]  \hspace{1cm} (A.14)
The second term in eq.(A.14) can be treated exactly as we did above for \(A_j^0\). The contribution to \(A_j^{11}\) coming from it is (see eq.(A.13)):
\[ \frac{\cos((2j+1)(\theta_x - \theta_{x+i}))}{2(2j+1)\sin \theta_x \sin \theta_{x+i}} \]  \hspace{1cm} (A.15)
The first term of eq.(A.14), after inserting the expression for \(g\) and \(h\) gives:
\[ \sum_{k=0}^{[j/2]} \frac{(-1)^k (2j - k)!}{k! (2j + 2 - 2k)!} \frac{[2 \cos(\theta_x + \theta_{x+i})]^{n+2} - [2 \cos(\theta_x - \theta_{x+i})]^{n+2}}{16(\sin \theta_x)^2(\sin \theta_{x+i})^2} \]  \hspace{1cm} (A.16)
To sum this series the following trick is needed. Let us divide and multiply for \(2j + 1\) and let us split this factor at the numerator as \((2j + 1 - k) + k\). Then the sum (A.16) is splitted into two sums that, after suitable redefinition of the indices can be reduced to the sum (A.12). Collecting together all the pieces one finds:
The complex function $A_{11}^j$ is defined as:

$$A_{11}^j = \frac{\cos((2j + 2)\theta_x + \theta_{x+i}) - \cos((2j + 2)\theta_x - \theta_{x+i})}{8(2j + 1)(2j + 2)(\sin \theta_x)(\sin \theta_{x+i})^2} - \frac{\cos((2j)\theta_x + \theta_{x+i}) - \cos((2j)\theta_x - \theta_{x+i})}{8(2j + 1)(2j)(\sin \theta_x)(\sin \theta_{x+i})^2} + \frac{\cos((2j + 1)\theta_x - \theta_{x+i})}{2(2j + 1)\sin \theta_x \sin \theta_{x+i}}. \quad (A.17)$$

By using the definitions $(A.7), (A.8)$ and $(A.9)$ one can immediately obtain $A_{12}^j$ as the difference: $A_{12}^j = A_0^j - A_{11}^j$.

Inserting these results in eq. $(A.2)$ one finally obtains the $C_{kl}$ coefficients. Simple trigonometric relations allow to write these coefficients in a compact form:

$$C_{11} = C_{22} = \frac{1}{2}(C_0 + C_1) \quad (A.18)$$

$$C_{12} = C_{21} = \frac{1}{2}(C_0 - C_1) \quad (A.19)$$

with:

$$C_1 = \frac{1}{2 \sin \theta_x \sin \theta_{x+i}} \left\{ \sum_{j=0}^{\infty} \chi_j(\theta_x) \chi_j(\theta_{x+i}) \frac{I_{2j+2}(\beta_t)^{N_t} - I_{2j}(\beta_t)^{N_t}}{(2j+1)I_1(\beta_t)^{N_t}} \right. + 2 \cos[(2j + 1)\theta_x] \cos[(2j + 1)\theta_{x+i}] \left( \frac{I_{2j+1}(\beta_t)}{I_1(\beta_t)} \right)^{N_t} \left. \right\}. \quad (A.20)$$

**References**

[1] B. Svetitsky and L. Yaffe, *Nucl. Phys.* B210 (1982) 423.

[2] M. Ogilvie, Phys. Rev. Lett. 52 (1984) 1369.

[3] J.M. Drouffe, J. Jurkiewicz and A. Krzywicki, Phys. Rev. D29 (1984) 2982.

[4] F. Green and F. Karsch, Nucl. Phys. B238 (1984) 297.

[5] M. Gross and J. F. Wheater, Nucl. Phys. B240 (1984) 253.

[6] J. Polonyi and K. Szlachanyi, Phys. Lett. B110 (1982) 395.

[7] M. Billó, M. Caselle, A. D’Adda, L. Magnea and S. Panzeri, Nucl. Phys. B435 (1995) 172

[8] R. Ben-Av, H. G. Evertz, M. Marcu and S. Solomon, Phys. Rev. D44 (1991) R2953.

[9] M. Billó, M. Caselle, A. D’Adda and S. Panzeri, in preparation
[10] M.Caselle, A.D’Adda and S.Panzeri, Phys. Lett. B302 (1993) 80.

[11] I.I.Kogan et al., Nucl. Phys. B395 (1993) 547.

[12] C.F. Baillie, R.Gupta, K.A.Hawick and G.S.Pawley. Phys. Rev. B45 (1992) 10438.