Particle acceleration in sub-cycle optical cells

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Abstract

A single laser pulse with spot size smaller than half its wavelength ($w_0 < \lambda/2$) can provide a net energy gain to ultra-relativistic particles. In this paper, we discuss the properties of an optical cell consisting of $N$ sub-cycle pulses that propagate in the direction perpendicular to the electron motion. We show that the energy gain produced by the cell is proportional to $N$ and it is sizable even for $\mathcal{O}(1 \text{ TW})$ pulses. The optical cell acts as a defocusing lens with chromatic aberration and can be treated as a linear component in conventional accelerators if the transverse size of the beam is of the order of $\lambda$. 
Modern conventional accelerators transfer only a small amount of energy to particles per unit cell. Large energy gradients are achieved combining the cells in periodic structures and phase locking the particle motion to the RF cavities. On the other hand, phase slippage is difficult to be avoided in laser-based accelerators and acceleration is achieved by a few beam-pulse interactions but, in this case, the energy transfer per pulse can be very large due to the huge electric fields available in lasers. Unlike RF-based accelerators, the interaction time $\tau$ of the particle with the laser is much longer than the period $T$ of the laser fields. Such a long interaction time causes several drawbacks. Plasma-based accelerators tend to be prone to instabilities when $\tau \gg T$. Similarly, laser accelerators in vacuum require special geometrical configurations of the laser pulse to escape the Lawson-Woodward theorem and reach a finite energy gain even if $\tau \gg T$.

The impressive progresses in sub-cycle laser technology achieved in the last decade can change dramatically this landscape. Attosecond optics allows for the focusing of pulses to spot sizes significantly smaller than $\lambda$, and for creation, manipulation and regeneration of isolated sub-cycle pulses. In addition, accelerators based on sub-cycle pulses can be operated in the $\tau < T$ regime: particles that are injected nearly perpendicularly with respect to the pulse propagation axis (see Fig. 1) will experience only a fraction of the oscillating electric field and, if properly phased, they will acquire a net energy gain when they cross the focus of the pulse. A set of $N$ sub-cycle pulses can be employed to transfer a fixed amount of energy to the particles through $N$ beam-pulse interactions. If the overall energy gain is constant and the beam is stable in phase-space, then the $N$-pulse optical lattice acts as a single accelerating unit (optical cell) with a well defined transfer function. Particles crossing an optical cell will experience an energy gain that, in general, will depend on the initial position of the particle; moreover, the particles will be deflected in the transverse plane ($y-z$ in Fig. 1) by the longitudinal electric field $E_z$ and by the ponderomotive force. The transfer function $T$ of the optical cell describes the change in momentum and the deflection angle of the incoming particle as a function of its initial position in phase space. If $T$ is regular - i.e. no instabilities occur during the acceleration process - the optical cell can be embedded in conventional FODO lattices to accelerate particle beams in circular or linear structures. In this paper we will study the accelerating properties of the optical cell and show that, in suitable conditions, this device can provide an energy gain proportional to $N$ and is described by a transfer function similar to the one of a defocusing lens.

The advantage of an optical cell compared to standard RF cells resides mostly in the intensity of the electric field, which can exceed the RF field by more than six orders of magnitude. In addition, the size of the cell in the $x$ direction is $O(N\lambda)$ and, therefore, is suitable for tabletop acceleration. On the other hand, the numerical aperture of an optical cell, i.e. the region in the transverse plane where particles can experience acceleration, is small ($O(\lambda) \sim 1-10 \mu m$) and comparable to the vertical beam size of high energy accelerators or free-electron lasers for optical or IR lasers.

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Figure 1: A N=2 optical cell. The laser pulses propagate along the z axis and the particles are injected along x. Distances are expressed in units of λ: kz (or kx) = 2π corresponds to a shift of λ in the z (x) direction. w₀ is the beam spot (see text).

In this paper, we consider an optical cell in its simplest configuration (Fig. 1): a set of N pulses reflected back and forth along the z axis and focused to an area < λ² at z = z₀ (z₀ = 0 in Fig. 1). The length of the pulse along z is assumed to be ≫ λ so that the light beam has a stationary focus at z = z₀. The pulse profile at z is circular with radius w(z) = w₀√(1 + (z−z₀)/z_R)², z_R = kw₀²/2 being the Rayleigh length of the pulse and k = 2π/λ the wavenumber of the light at central frequency. The e.m. fields in the proximity of the focus can be derived following the approach of Davis and McDonald [11, 12]. The light beam is modeled employing a vector potential A linearly polarized along x and expressing A as a perturbative expansion of the diffraction angle ϵ = w₀/z_R. Such treatment, which is consistent as long as ϵ < 1 (w₀ > λ/π), provides the electric and magnetic fields in the proximity of the focus as a function of the transformed variables:

$$\xi = x/x₀ , \quad v = y/w₀ , \quad \zeta = z/z_R , \quad r² = x² + y² , \quad \rho = r/w₀$$

and of the plane wave phase η = ωt − kz. At next-to-leading order [13] the fields are:

$$E_x = E₀ w₀/w e^{-\xi²/w} \left\{ S₀ + ϵ² \left[ \xi² S₂ - \rho⁴ S₃/4 \right] + O(ϵ⁴) \right\}$$

$$E_y = E₀ w₀/w e^{-\xi²/w²} \xi v \left\{ ϵ² S₂ + O(ϵ⁴) \right\}$$


\[ E_z = E_0 \frac{w_0}{w} e^{-\frac{x^2}{w^2}} \xi \left\{ \epsilon C_1 + \epsilon^3 \left[ -\frac{C_2}{2} + \rho^2 C_3 - \frac{\rho^4 C_4}{4} \right] + O(\epsilon^5) \right\} \]  

\[ B_x = 0 \]  

\[ B_y = E_0 \frac{w_0}{w} e^{-\frac{x^2}{w^2}} \left\{ \epsilon^2 \left[ \frac{\rho^2 S_2}{2} - \frac{\rho^4 S_3}{4} \right] + O(\epsilon^4) \right\} \]  

\[ B_z = E_0 \frac{w_0}{w} e^{-\frac{x^2}{w^2}} \left\{ \epsilon C_1 + \epsilon^3 \left[ \frac{C_2}{2} + \frac{\rho^2 C_3}{2} - \frac{\rho^4 C_4}{4} \right] + O(\epsilon^4) \right\} \]  

where

\[ S_n = \left( \frac{w_0}{w} \right)^n \sin(\psi + n\psi_G) \quad C_n = \left( \frac{w_0}{w} \right)^n \cos(\psi + n\psi_G) . \]  

The phases entering Eq. 8 are the ones describing standard Gaussian beams [14]: the plane wave phase \( \eta \), the Gouy phase \( \psi_G \equiv \tan^{-1} \zeta \), the initial phase \( \psi_0 \) and the phase advance due to the curvature of the wavefront:

\[ \psi_R = \frac{k r^2}{2(z - z_0) + \frac{r^2}{2}} \]  

The overall phase \( \psi \) is \( \psi = \psi_0 + \eta - \psi_R + \psi_G \). The fields have been computed analytically up to \( \epsilon^{11} \) in Ref. [15]. In the study of the optical cell described below we retain all terms up to \( \epsilon^5 \).

A single pulse \((N = 1)\) optical cell can be operated even in the \( \tau > T \) regime [3, 16] but in this case the particle must be injected at an angle close to the \( z \) direction. For \( \tau > T \) acceleration is mostly driven by \( E_z \) and the ponderomotive force. Employing PW class lasers [17] the energy gain can be very large although the acceleration regime is highly non linear [18, 19]. For \( \tau < T \) [20] the particle of charge \( q \) injected in the proximity of the \( x \) axis experiences only a portion of the \( E_x \) period and net acceleration is caused mostly by the \( qE_x \) linear term. The \( B_y \) and \( E_z \) fields contribute to the deflection angle in the \( x - z \) plane. This can be demonstrated solving numerically the equation of motion of the particle (electron, \( q \equiv -e \)) in the electric fields of Eqs. 2-7:

\[ \frac{d\mathbf{p}}{dt} = -e[\mathbf{E} + e\beta \times \mathbf{B}] , \quad \frac{d\mathbf{E}}{dt} = -ec\beta \cdot \mathbf{E} \]  

Numerical integration of Eq. 10 is eased if the motion is described in units of \( T \) and the fields are expressed as \( \tilde{\mathbf{E}} \equiv (e/mc\omega)\mathbf{E} \) and \( \tilde{\mathbf{B}} \equiv (e/m\omega)\mathbf{B} \). The resulting equation [18] is:

\[ \frac{d\beta}{d\omega t} = \frac{1}{\gamma} \left[ \beta(\beta \cdot \tilde{\mathbf{E}}) - (\tilde{\mathbf{E}} + \beta \times \tilde{\mathbf{B}}) \right] . \]  

In these formulas, \(-e\) and \( m \) are the charge and electron mass in SI units, \( \beta \) is the electron velocity normalized to \( c \) and \( \gamma = (1 - \beta^2)^{-1/2} \). Fig. 2 shows the net energy gain \( \Delta E = (\gamma - \gamma_0)mc^2 \) for a 100 MeV electron \( (\gamma_0 = 195.7) \) injected along the \( x \) axis as a function of \( w_0 \). The initial phase \( \psi_0 \) is chosen in order to have the electron at the
center of the focus when $|E_x(\omega t)|$ reaches its maximum. The laser pulse considered corresponds to $P_0=1$ TW and 10 TW power Gaussian beams focused to a beam spot $w_0$. In Fig. 2 $w_0$ ranges from $w_0 = \lambda/\pi$ to $\lambda/2$. The power crossing the disk of area $\pi w_0^2$ at $z_0 = 0$ is $P_0(1 - e^{-2}) \simeq 0.865 P_0$. The laser intensity $I_0$ at $z = z_0$ and $r^2 = 0$ is proportional to the overall beam power: $P_0 = \pi w_0^2 I_0/2$. The electric field is

$$E_0 = \frac{1}{w_0} \sqrt{\frac{4P_0}{\pi\epsilon_0 c}}.$$  

For $w_0 = \lambda/3$ and $P_0=1$ TW, $I_0$ is $5.7 \times 10^{20}$ W/cm$^2$ and $E_0 = 6.6 \times 10^{13}$ V/m.

As expected, a net energy gain due to the electric field $E_x$ is visible as long as the particle experiences just a fraction of the oscillation period. In particular, no energy gain is observed when $w_0 > \lambda/2$ because the interaction time for an ultra-relativistic particle $\tau \simeq 2 w_0/c$ is larger than an optical cycle and hence, the electron experiences both an accelerating and a decelerating electric field with nearly equal strength. For $w_0 < \lambda/2$ the energy gain $\Delta E$ grows linearly with $E_0$ and is proportional to $\sqrt{P_0}$. This scaling is in agreement with results obtained under paraxial conditions [21] and in non-paraxial treatments [22]. It is worth mentioning that an increase of the laser wavelength at fixed $P_0$ does not increase the energy gain per pulse for ultra-relativistic particles: since $E_0 \sim w_0^{-1} \sim 1/\lambda$ and the interaction time $\tau \sim 2 w_0/c$, the momentum change experienced during the pulse-electron interaction is $-e \int_0^\tau E_x dt \simeq -e \tau E_0$ and it is thus independent of $\lambda$. On the other hand, a longer $\lambda$, as in CO$_2$ lasers [23], increases the numerical aperture of the cell and eases substantially the practical implementation of this acceleration scheme. The electron motion in the $z$-direction is determined by the interplay between $B_y$ and $E_z$. In the proximity of the focus the former boosts the electron toward the direction of motion of the laser pulse, while for $z \to z_0$ and $\tau \to 0$, $E_z$ is $\simeq 0$ and the effect of the longitudinal field $E_z$ is marginal. Far from the focus, $B_y$ decreases steeply and the contribution of $E_z$ becomes sizable. Fig. 3 shows the net energy gain $\Delta E$ for a 100 MeV electron injected along $x$ at $z_0 = 0$ and crossing three pulses ($P_0 = 1$ TW, $w_0 = \lambda/3$) located at a distance $3\lambda$ in $x$ ($\Delta kx = 6\pi$). The accelerating field $E_x$ experienced along its trajectory is also shown.

The transverse motion of the particle is described in Fig. 4. The continuous and short-dashed lines indicate the position and $\beta_z$ of the particle along $kz$ ($kz = 2\pi$ corresponds to $z = \lambda$) as a function of $\omega t$. The long-dashed and dotted lines show the values of the fields $\tilde{B}_y$ and $\tilde{E}_z$ (divided by 30 to ease the reading of the plot) experienced along the trajectory. For an ultra-relativistic particle (100 MeV electron in Fig. 4) starting at $kx_0 = -2\pi$, the focus is reached at $\omega t = 2\pi$. Far from the focus, both $E_z$ and $B_y$ are positive and deflect the particle toward negative values of $z$. Near the focus, $B_y$ is large and negative and accelerate the electron along positive $z$. The net result is that the two effects partially cancel when the electron crosses the laser beam, and the overall deflection is much smaller than one could envisage from the $\beta \times B$ force only. This cancellation effect has been noted in the 90’s and it is known to severely limits the energy transfer along $z$ in the $\tau > T$ regime [3, 24]. If $\tau < T$, however, the
Figure 2: Net energy gain $\Delta E$ in MeV versus $w_0$ for a 100 MeV electron injected into the focus of a 1 TW (black continuous line) and 10 TW (dashed line) Gaussian beam ($\lambda = 1 \mu m$). The smallest value $w_0 = \lambda/\pi$ corresponds to the limit of the perturbative expansion of the fields ($\epsilon = 1$). The largest value corresponds to $w_0 = \lambda/2$.

Figure 3: Black line: net energy gain $\Delta E$ in MeV versus $\omega t$ for a 100 MeV electron injected into the focus of three 1 TW Gaussian beams ($\lambda = 1 \mu m, w_0 = \lambda/3$). The accelerating field $\tilde{E}_x$ experienced by the particle at the time $\omega t$ is also shown (dashed line).
acceleration is due to $E_z$ and the cancellation above helps to stabilize the particles in the transverse plane.

In an optical cell of $N$ pulses, the final energy reached by the particle and its trajectory in the transverse plane depends on the initial position of the particle itself. If particles are ultra-relativistic, they will always arrive in phase with the next pulse, except for the slippage in $kx$ due to the fact that $\beta_z \neq 0$ and the trajectory is no more rectilinear. The small distance among the pulses and the cancellation effect of $B_y$ and $E_z$ in the transverse plane make the phase slippage quite small and the transfer function $T$ of the cell very regular. This is demonstrated for $P_0 = 1$ TW, $w_0 = \lambda/3$ pulses and 100 MeV electrons in Figs. 5 and 6. Fig. 5 shows the energy gain as a function of $N$ for particles injected at different $\Delta \equiv kz(t = 0) - kz_0$ (in the plot, $\Delta$ ranges between $-\pi/3$ and $\pi/3$). Note that the maximum gain is observed for electrons that are injected with a slight offset with respect to $z_0$ ($\Delta \simeq \pi/12$). This is due to the residual bending in the transverse plane experienced by particle injected directly at the focus ($\Delta = 0$, see Fig. 4). It is also visible in Fig. 6 where the $kz$ position of the particle after $N$ electron-pulse interactions is shown. The energy spread at $N = 10$ is 13% and the maximum transverse distance is comparable to the size of the pulse in the $z$ direction ($\mathcal{O}(\lambda)$).

Fig. 6 shows that the cell operates as a defocusing magnetic lens. It is, thus, inconvenient to build optical cells with $N \gg 10$ before proceeding to particle refocusing because peripheral particles will not experience large values of $E_z$ at the end of the
cell. The overall energy gain of an optical cell is independent of the initial energy of
the electron as far as the particle is ultra-relativistic but, as for standard lenses, the
cell causes chromatic aberration. For a 1 GeV initial energy electron the $N_0 = 10$ cell
accelerates particles up to $\Delta E \simeq 80$ MeV but, in this case, the spread in the transverse
plane is $\sim 10$ times smaller because the magnetic rigidity is proportional to the particle
momentum. At larger momenta and ultra-relativistic particles ($E \simeq pc$), it is thus
appropriate to employ cells with $N \simeq N_0 \cdot p$(MeV)/100.

In conclusion, even the simplest configuration of the optical cell can be operated
with ultra-relativistic particles in order to increase the energy to an amount propor-
tional to $N$. The energy gain is sizable if the spot size is smaller than $\lambda/2$ and scales
as $P_0^{1/2}$. The optical cell can be operated with $N \simeq 10$ for 100 MeV electrons and
the optimum number of pulses increases linearly with the initial momentum of the
particles. The cell acts as a defocusing lens with a chromatic aberration that depends
on $p$ and exhibits a numerical aperture of the order of $2\lambda/3$.

**Acknowledgments**

The author gratefully acknowledges comments and suggestions from S.V. Bulanov,
K.T. McDonald and F. Pegoraro.
Figure 6: $k_z$ position of the particles after crossing $N$ laser pulses for 100 MeV electrons injected along $x$ at different positions in the $z$ axis.

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