Incoherent matter-wave solitons

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The dynamics of matter-wave solitons in Bose-Einstein condensates (BEC) is considerably affected by the presence of a surrounding thermal cloud and by condensate depletion during its evolution. We analyze these aspects of BEC soliton dynamics, using time-dependent Hartree-Fock-Bogoliubov (TDHFB) theory. The condensate is initially prepared within a harmonic trap at finite temperature, and solitonic behavior is studied by subsequently propagating the TDHFB equations without confinement. Numerical results demonstrate the collapse of the BEC via collisional emission of atom pairs into the thermal cloud, resulting in splitting of the initial density into two solitonic structures with opposite momentum. Each one of these solitary matter waves is a mixture of condensed and noncondensed particles, constituting an analog of optical random-phase solitons.

The physics of quantum-degenerate, interacting Bose gases closely resembles the behavior of light in nonlinear media. The dynamics of Bose-Einstein condensate (BEC) at zero-temperature is well-described by the Gross-Pitaevskii (GP) mean-field theory, given by the nonlinear Schrödinger equation (NLSE) for the condensate order parameter. The same equation describes the evolution of coherent light in nonlinear Kerr media. This analogy has opened the way for the field of nonlinear atom optics1 2 with striking demonstrations of familiar nonlinear optics phenomena such as four wave mixing,3 superradiant Rameigh scattering4, and matter-wave amplification5 6, carried out with matter-waves. One such phenomenon is the formation of matter-wave solitons7 8 9 10 11 12 13 14 15 16. Experimentally, dark solitons1 10 and bright gap solitons1 10 were observed in BECs with repulsive interactions, whereas bright solitons13 14 were demonstrated in systems with attractive interactions. These experimental results are augmented by extensive theoretical work including predictions on bright17 8, and dark11 matter-wave solitons, lattice solitons12, and soliton trains15. The vast majority of previous theoretical efforts on matter-wave solitons have utilized the zero-temperature GP mean-field theory. However, in a realistic system, elementary excitations arising from thermal and/or quantum fluctuations are always present17, and the BEC dynamics may be considerably affected by the motion of excited atoms around it (thermal cloud), and by its dynamical depletion18, giving rise to new nonlinear matter-wave phenomena.

Here we analyze these aspects of BEC soliton dynamics by using the time-dependent Hartree-Fock-Bogoliubov (TDHFB) theory14 20 21. Soliton dynamics is analyzed by first calculating the finite-temperature ground state of the attractively interacting gas within an harmonic trap17. The harmonic confinement is then suddenly turned off, and the partially condensed Bose gas starts to dynamically evolve. Within the TDHFB model, we find a characteristic pattern of evolution of the system whereby pairs of atoms are collisionally excited from the BEC into the thermal cloud causing the initial density to eventually split into two solitonic structures with opposite momentum. Both solitons constitute a mixture of condensed and noncondensed particles. We emphasize that the observed composite waves are a truly novel type of matter-wave solitons, where localization is attained not only in spatial density, but also in spatial correlations. This type of solitons are reminiscent of composite incoherent optical solitons22 23 24, thus highlighting an analogy between incoherent light behavior in nonlinear media and BECs at finite-temperatures.

Starting with a near-unity condensate fraction at very low temperatures, the GP dynamics reproduces, under proper conditions, the well-known zero-temperature BEC solitons, thus demonstrating the condensate’s mechanical stability. However, the evolution of the same initial nearly pure BEC with TDHFB clearly illustrates BEC depletion through pairing, causing these coherent solitons to disintegrate in a characteristic fashion into incoherent solitary matter waves. In all cases, when both the trap and the interparticle interactions are turned off simultaneously, we observe fast matter-wave dispersion.

We consider a system of N interacting bosons placed in a quasi one-dimensional (Q1D) cigar-shaped harmonic potential \( V_{ext}(x, y, z) = (\omega_x x^2 + \omega_y y^2 + \omega_z z^2)/2 \), where \( \omega_x \gg \omega_x \) denote the transverse and the longitudinal frequencies of the trap, respectively. The interparticle interaction is approximated by the Q1D contact potential \( V(x_1 - x_2) = g_{1D}\delta(x_1 - x_2) \), where \( g_{1D} = -\hbar^2/m a_{1D} \), \( a_{1D} \approx -a_{1D}^2/a_{1D} \) is the effective 1D scattering length25 26 27, \( m \) is the particle mass, \( a_{1D} = \sqrt{\hbar/m\omega_z} \) is the size of the lowest transverse mode, while \( a_{1D} \) is the 3D scattering length. At finite temperatures, the equilibrium state of the system can be described by the HFB theory17:
$$H_{sp} \Phi^s + g_{1D} [n^s_c(x) + 2 \tilde{n}^s(x)] \Phi^s + g_{1D} \tilde{n}^s(x) \Phi^{*,s} = \mu \Phi^s(x),$$

$$\begin{bmatrix} \mathcal{L}^*(x) & \mathcal{M}^*(x) \\ -\mathcal{M}^*(x) & -\mathcal{L}^*(x) \end{bmatrix} \begin{bmatrix} u_j^*(x) \\ v_j^*(x) \end{bmatrix} = E_j \begin{bmatrix} u_j^*(x) \\ v_j^*(x) \end{bmatrix}. $$

Here, $H_{sp} = -\frac{\hbar^2 k^2}{2m} + \frac{1}{2} m \omega_T^2 x^2$ and $\mu$ is the chemical potential. The superscript $s$ denotes the static HFB calculation, e.g. the static order parameter is denoted by $\Phi^s(x)$ and $n^s_c(x)$ is denoted by the static HFB condensate density. The normal density is $\tilde{n}^s(x) = \sum_j |u_j^*(x)|^2 N_j + |v_j^*(x)|^2 (N_j + 1)$, whereas $\tilde{n}^s(x) = -\sum_j u_j^*(x) v_j^*(x)(2N_j + 1)$ is the anomalous density. The population of excited states at temperature $T$ follows the Bose distribution $N_i = \langle e^{E_j/kT} \rangle^{-1}$. In Eq. (2) we denote $\mathcal{L}^s(x) = H_{sp} + 2g_{1D} [n^s_c(x) + \tilde{n}^s(x)] - \mu$ and $\mathcal{M}^s(x) = -g_{1D} [\Phi^{s2}(x) + \tilde{n}^s(x)]$. Two variants of the HFB formalism are the Hartree-Fock (HF) approach, altogether neglecting anomalous terms, and the HFB-Popov approximation wherein the non-condensate anomalous density is explicitly dropped, i.e. $\mathcal{M}^s(x) \approx -g_{1D} \Phi^{s2}(x)$. Unlike the static HFB, the HF and HFB-Popov approximation do not have an unphysical gap in their excitation spectra.

We use the solution of Eqs. (1) and (2) as the initial conditions to study dynamics without confinement within the TDHFB approximation in the modal form.

$$i\hbar \frac{\partial \Phi(x,t)}{\partial t} = H_{sp} \Phi + g_{1D} [n^s_c(x,t) + 2 \tilde{n}(x,t)] \Phi + g_{1D} \tilde{n}(x,t) \Phi^*, (3)$$

$$i\hbar \frac{\partial}{\partial t} \begin{bmatrix} u_j(x,t) \\ v_j(x,t) \end{bmatrix} = \begin{bmatrix} \mathcal{L}(x,t) & \mathcal{M}(x,t) \\ -\mathcal{M}^*(x,t) & -\mathcal{L}^*(x,t) \end{bmatrix} \begin{bmatrix} u_j \\ v_j \end{bmatrix}, (4)$$

where $\mathcal{L}(x,t) = H_{sp} + 2g_{1D} [n^s_c(x,t) + \tilde{n}(x,t)]$, $\mathcal{M}(x,t) = -g_{1D} [\Phi^{s2}(x,t) + \tilde{n}(x,t)]$, $\tilde{n}(x,t) = \sum_j |u_j^2(x,t) + |v_j^2(x,t)|^2 (N_j + 1)$ is the normal density, and $\tilde{n}(x,t) = -\sum_j u_j^* v_j^*(x,t)(2N_j + 1)$ is the anomalous density; at $t = 0$, all dynamical quantities are identical to their static counterparts, e.g., $\Phi(x,0) = \Phi^s(x)$ etc. The TDHFB model is usually given in the form of coupled equations for the condensate order parameter and the single particle density matrix (e.g. see Ref. 21). A tedious but straightforward calculation shows its full equivalence to the modal form (4). As expected, when the time-dependence of the BEC and quasiparticle functions is $\Phi(x,t) = \Phi^s(x) \exp(-i\mu t/\hbar)$, $\tilde{n}(x,t) = \tilde{n}^s_c(x) \exp(-iE_j + \mu t/\hbar)$, and $\tilde{n}^{s,*}(x) \exp(-iE_j - \mu t/\hbar)$, the equations of motion (3) and (4) reduce to the time-independent HFB equations (1) and (2).

In what follows we present numerical results based on the described formalism, demonstrating the effect of thermal particles and condensate depletion on matter-wave soliton dynamics. The parameters of the calculation are chosen to resemble the experimental parameters of Ref. 15. We consider $N = 2.2 \times 10^4$ $^7$Li atoms in a harmonic trap with $\omega_\perp = 4907$ Hz ($\omega_\perp = \sqrt{\hbar/m\omega_\perp} \approx 1.35 \mu$m). The 3D scattering length $a_{3D} = 3.1 \times 10^{-11}$ m corresponds to a nonlinear parameter of $N|a_{3D}| \approx 0.68 \mu$m, and is tunable by the Feshbach resonance technique 12. We present two calculations for different $T$ and $\omega_\perp$ to show the influence of excitations on the soliton dynamics.

First we consider the gas at higher temperature, where the condensate fraction is approximately 25%. The trapping frequency is $\omega_\perp = 141$ Hz ($\omega_\perp = \sqrt{\hbar/m\omega_\perp} \approx 8.0 \mu$m), while $k_BT/\hbar\omega_\perp = 60$. Fig. (a) illustrates the total density $n^s_c(x) + n^s_c(x)$ of the stationary HFB-Popov calculation. The total density profile is double humped,
which is a clear signature of the significant population of the first excited state (approximately $\approx 10\%$ of atoms), which has a dipole-like shape shown in Fig. 1(b). We have numerically checked the stability of the solution with respect to small perturbations; the stability is underpinned by the use of parameters resembling experiment [13]. When the trap is turned off, the system is suddenly taken out of equilibrium, and consequently starts to evolve; we simulate the dynamics with the full TDHFB model. In the spirit of Ref. [13], we compare the $x$-unconfined dynamics of the system in the presence of interparticle interactions [Fig. 1(c)] to its time evolution when both the confinement in $x$ and the interactions are turned off [Fig. 1(d)]. In the absence of interactions, we clearly observe fast matter-wave dispersion. In contrast, when interactions are present, the two humps begin to separate, because the trapping potential which provided a balance to the kinetic energy term is no longer present [Fig. 1(c)]. During evolution, the condensate is slowly depleted [Fig. 1(f)]. Consequently, at one point during the evolution the two humps split, each forming a solitonic structure with opposite momentum. Each of these solitonic structures contains both Bose-condensed atoms, and a significant portion of the non-condensed particles, i.e., they are partially coherent matter-wave solitons. The uniqueness of such random-phase structures is elucidated by their complex degree of coherence, $\mu(x, x', t) = \rho(x, x', t)/\sqrt{\rho(x, x')\rho(x', x, t)}$, where $\rho(x, x', t) = \Phi(x', t)\Phi^*(x, t) + \langle \hat{\Psi}(x', t)\hat{\Psi}(x, t) \rangle$, plotted at different times in Fig. 1(c). We observe that spatial correlation is localized in space at $t = 0$. Correlations change as the two humps split. While retaining spatial localization, the phases at the two separated peaks are well-correlated, and out of phase, resembling the behavior of out-of-phase adjacent solitons from Ref. [14]. We emphasize that for zero-temperature GPE solitons, the pair correlation function factorizes as $\rho(x, x') = \Phi^*(x)\Phi(x')$, which yields $\mu(x, x') = 1$, corresponding to coherent matter-waves. Our incoherent matter-wave solitons are thus rather special in that they correspond to localization of entropy and spatial correlation, as well as to localization of density.

While the temperature $k_BT$ in the previous example is higher than the transverse level spacing $\hbar\omega_{\perp}$ whereas a ‘true’ 1D geometry calls for $k_BT < \hbar\omega_{\perp}$ [27], the use of a Q1D formalism is still justified because the first $\omega_x/\omega_{\perp} \sim 35$ states are essentially 1D (they are in the lowest state of the transverse Hamiltonian). Furthermore, only condensed atoms, and atoms from lower excited states determine the outcome of the motion. Therefore, a proper inclusion of the transverse dimension in the calculation would lead to some rescaling of the parameters, but would not influence the dynamics observed in our quasi-1D calculation. Moreover, the simulations as well as the experiment of [13], are all in the weak interaction regime $\hbar\omega_{\perp}/a_x \sim 10^8 \gg 1$ [28], thus justifying the use of a mean-field approach.

Next, we consider a gas prepared at near-zero temperature, where the condensate fraction is 99%. The trapping frequency is $\omega_{\perp} = 439$ Hz ($a_x = \sqrt{\hbar/m\omega_x} \approx 4.51$ μm), while $k_BT/\hbar\omega_{\perp} = 5$. As previously, the trap is turned off and the dynamics is calculated using both the GP equation (Fig. 2a) and the TDHFB formalism (Fig. 2b). The mechanical stability of the evolving BEC is demonstrated by propagating the time-dependent GP equation, showing small oscillations of the condensate width, rather than a mechanical collapse to a point [Fig. 2(a)]. This motion corresponds to a coherent matter wave soliton. However, as clearly evident from propagation of the TDHFB equations [Fig. 2(b)], allowing BEC depletion [Fig. 2(c)], the BEC collapses via a pairing instability [51, 52] whereby pairs of atoms are collisionally pulled out of the BEC into the thermal cloud, thus gaining a mean-field energy which goes into their relative motion. Such pairing collapse with little or no mechanical shrinking is indeed observed in 3D collapse experiments [32, 34]. Our results show that in 1D it results in two separate solitonic structures of opposite momentum, containing both a condensed part and a significant thermal population. We note that similar structures were observed in stochastic simulations of molecular BEC dissociation in 1D geometry [27], indicating that incoherent matter-wave solitons can also be produced in this system. Figure 2(d) shows that the correlations of each solitonic structure are localized, and the two structures are partially correlated.

Before closing, we note that within the time-dependent Hartree-Fock (TDHF) approximation [13], where con-
densate depletion is not accounted for during the dynamics, we find solutions with double-humped density profile that begin to split as in TDHFB model. However, as the condensate is not depleted within TDHF, the two peaks may be pulled back and merge to almost recover the initial density profile. Such motion is also characteristic of composite incoherent solitons in optics.

In conclusion, we have used the time-dependent Hartree-Fock-Bogoliubov theory to analyze the influence of the thermal cloud and condensate depletion onto the dynamics of BEC solitons. We find that condensate depletion induced by pairing, and the presence of a thermal cloud cause the particle density to split into two solitonic structures, each being a mixture of condensed and noncondensed particles. The predicted incoherent matter-wave structures represent novel correlation solitons which resemble localized second-sound entropy waves. Such random-phase matter-wave solitons correspond to incoherent solitons in nonlinear optics [22, 23, 24], which points at the analogy between partially condensed Bose gases and nonlinear partially coherent optical waves.

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