Hypergeometric solutions for variants of the $q$-hypergeometric equation

Taikei Fujii and Takahiko Nobukawa

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Abstract

We introduce a configuration of a $q$-difference equation and characterize the variants of the $q$-hypergeometric equation, which were defined by Hatano-Matsunawa-Sato-Takemura, by configurations. We show integral solutions and series solutions for the variants of the $q$-hypergeometric equation.

1 Introduction

The Gauss hypergeometric equation

$$\left[ x(1-x)\frac{d^2}{dx^2} + (\gamma - (\alpha + \beta + 1)x)\frac{d}{dx} - \alpha\beta \right] f(x) = 0, \tag{1.1} $$

is a standard form of second order Fuchsian differential equations with three singularities on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. The equation (1.1) is characterized by the following Riemann scheme:

$$\begin{bmatrix} 
 x = 0 & x = 1 & x = \infty \\
 0 & 0 & \alpha \\
 1 - \gamma & \gamma - \alpha - \beta & \beta 
\end{bmatrix} \tag{1.2}$$

The equation (1.1) has the integral solution

$$\int_C t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-xt)^{-\beta}dt, \tag{1.3}$$

where $C$ is a suitable path. Also, the equation (1.1) has the series solution

$$2F_1\left( \frac{\alpha, \beta}{\gamma}; x \right) = 1 + \frac{\alpha \cdot \beta}{\gamma - 1} x + \frac{\alpha(\alpha + 1) \cdot \beta(\beta + 1)}{\gamma(\gamma + 1) \cdot 1 \cdot 2} x^2 + \cdots . \tag{1.4}$$

The Riemann scheme (1.2) and the integral solution (1.3), the series solution (1.4) are basic properties of the Gauss equation (1.1).

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*Department of Mathematics, Graduate School of Science, Kobe University, Rokko, Kobe 657-8501, Japan. E-mail: tfujii@math.kobe-u.ac.jp
†Department of Mathematics, Graduate School of Science, Kobe University, Rokko, Kobe 657-8501, Japan. E-mail: tnobukw@math.kobe-u.ac.jp
On the other hand, the Riemann-Papperitz differential equation
\[
\frac{d^2}{dz^2} + \left( \frac{1 - \alpha_1 - \beta_1}{z - t_1} + \frac{1 - \alpha_2 - \beta_2}{z - t_2} + \frac{1 - \alpha_3 - \beta_3}{z - t_3} \right) \frac{d}{dz} + \frac{1}{(z - t_1)(z - t_2)(z - t_3)} g(z) = 0, \quad (1.5)
\]
where \( \alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3 = 1 \), has also similar properties as for the Gauss equation (1.1). The equation (1.5) is characterized by the Riemann scheme
\[
\left\{ \begin{array}{ccc}
  z = t_1 & z = t_2 & z = t_3 \\
  \alpha_1 & \alpha_2 & \alpha_3 \\
  \beta_1 & \beta_2 & \beta_3
\end{array} \right. \quad (1.6)
\]
Thus we put \( x = \frac{z - t_1 t_2 - t_3}{z - t_3 t_2 - t_1} \) and \( h(x) = x^{-\alpha_1}(1 - x)^{-\alpha_2}g(z) \), then the function \( h(x) \) satisfies the Gauss equation (1.1) with \( \alpha = \alpha_1 + \alpha_2 + \alpha_3 \), \( \beta = \alpha_1 + \alpha_2 + \beta_3 \) and \( \gamma = \alpha_1 - \beta_1 + 1 \). Since the Riemann-Papperitz equation (1.5) and the Gauss equation (1.1) are transformed to each other by gauge transformations, the Riemann-Papperitz equation (1.5) has the integral solution and the series solution
\[
(z - t_1)^{\alpha_1}(z - t_2)^{\alpha_2}(z - t_3)^{\alpha_3} \int_C (t - t_1)^{\alpha_2 + \alpha_3 + \beta_1 - 1}(t - t_2)^{\alpha_1 + \alpha_3 + \beta_2 - 1}(t - t_1)^{\alpha_1 + \alpha_2 + \beta_3 - 1}(t - z)^{-\alpha_1 - \alpha_2 - \alpha_3} dt,
\]
(1.7)
\[
\frac{(z - t_1)^{\alpha_1}(z - t_2)^{\alpha_2}}{(z - t_3)^{\alpha_3}} F_2^1 \left( \begin{array}{ccc}
  (1 + \alpha_2 + \alpha_3, 1 + \alpha_2 + \beta_3) \\
  \alpha_1 - \beta_1 + 1 \\
  z - t_1 t_2 - t_3 \\
  z - t_3 t_2 - t_1
\end{array} ; \frac{z - t_1 t_2 - t_3}{z - t_3 t_2 - t_1} \right).
\]
(1.8)
A \( q \)-difference analog of the Gauss hypergeometric equation (1.1) was introduced as follows:
\[
[x(1 - a T_x)(1 - b T_x) - (1 - T_x)(1 - c q^{-1} T_x)] f(x) = 0, \quad (1.9)
\]
where \( T_x \) is the \( q \)-shift operator of \( x \), that is, \( T_x f(x) = f(q x) \). This equation is called the Heine’s \( q \)-hypergeometric equation. In this paper we fix \( q \in \mathbb{C} \) with \( 0 < |q| < 1 \). The equation (1.9) has the integral solution and the series solution
\[
\int_C t^{\alpha - 1} \frac{(a t)^{\infty}}{(c t/a)^{\infty}} \frac{(b t)^{\infty}}{(x t)^{\infty}} dt,
\]
(1.10)
\[
\frac{\varphi_2}{a, b, c, x},
\]
(1.11)
where \( a = q^\alpha \), and
\[
\int_0^t f(t) dt = (1 - q) \sum_{n=0}^{\infty} f(q^n) q^n, \quad \int_{\tau_1}^{\tau_2} f(t) dt = \int_0^{\tau_2} f(t) dt - \int_0^{\tau_1} f(t) dt,
\]
(1.12)
\[
(a)_\infty = \prod_{n=0}^{\infty} (1 - a q^n), \quad (a)_m = \frac{(a)_\infty}{(aq^m)_\infty}, \quad (a_1, \ldots, a_r)_m = (a_1)_m \cdots (a_r)_m,
\]
(1.13)
\[
\varphi_s \left( \begin{array}{c}
  a_1, \ldots, a_r \\
  b_1, \ldots, b_s \\
  x
\end{array} ; x \right) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r)_n}{(b_1, \ldots, b_s, q)_n} (-1)^{(r-s-1)n} q^{(r-s-1)(s)_n},
\]
(1.14)
are called the Jackson integral of \( f(t) \), the \( q \)-Pochhammer symbol and the \( q \)-hypergeometric series, respectively. The integral (1.10) and the series (1.11) are \( q \)-analogues of (1.3) and (1.4).
Here, in the theory of $q$-difference equations, the points $x = 0$ and $x = \infty$ are the fixed points of the $q$-shift operator $T_x$. Thus only the points $x = 0$ and $x = \infty$ can be singularities for $q$-difference equations. The Gauss equation [11] has singularities at $x = 0$ and $x = \infty$, and then it is easy to obtain a $q$-analog of [11]. However, the points $z = 0$ and $z = \infty$ are regular points for the Riemann-Papperitz equation [15]. Thus it is difficult to consider properties for a $q$-analog of [15].

In [9], the variants of the $q$-hypergeometric equation of degree two and degree three were introduced. The variant of the $q$-hypergeometric equation of degree two, i.e. $\text{deg}_x H_2 = 2$, is defined as a special case of the $q$-Heun equation, as follows:

$$ H_2 f(x) = 0, $$

$$ H_2 = \prod_{i=1}^{2} (x - q^{l_i+1/2} t_i) \cdot T_x^{-1} + q^{\alpha_1+\alpha_2} \prod_{i=1}^{2} (x - q^{l_i-1/2} t_i) \cdot T_x - (q^{\alpha_1} + q^{\alpha_2}) x^2 + E x + p(q^{1/2} + q^{-1/2}) t_1 t_2, $$

$$ p = q^{(h_1+h_2+l_1+l_2+\alpha_1+\alpha_2)/2}, \quad E = -p\{ (q^{-h_2} + q^{-l_2}) t_1 + (q^{-h_1} + q^{-l_1}) t_2 \}. $$

The $q$-Heun equation was introduced in [7]. Heun’s differential equation is a second order Fuchsian differential equation with four singularities. When one of four singularities is essentially non-singular, Heun’s differential equation can be transformed to the Gauss hypergeometric equation with some gauge transformation. Here, we say a singularity of a second order differential equation is essentially non-singular when the difference between the characteristic exponents is 1 and the singularity is non-logarithmic. The way to derive the equation $H_2 f(x) = 0$ from the $q$-Heun equation is also to specialize so that the point $x = 0$ is essentially non-singular. The equation $H_2 f(x) = 0$ is a $q$-analog of a second order Fuchsian differential equation with four singularities $\{0, t_1, t_2, \infty\}$, and the point $x = 0$ is essentially non-singular. Thus this equation is essentially a $q$-analog of the Riemann-Papperitz equation with $t_3 = \infty$. Note that by taking the limit $t_2 \to 0$, the variant of the $q$-hypergeometric equation of degree two becomes the Heine’s $q$-hypergeometric equation with some change of parameters. Several solutions of the equation $H_2 f(x) = 0$ were shown in [9] as follows:

$$ x^{-\alpha_1} \Phi^{(1)}(x; a; b_1, b_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{(a)_{n_1+n_2}(b_1)_{n_1}(b_2)_{n_2}}{(c)_{n_1+n_2}(q)_{n_1}(q)_{n_2}} x^{n_1} x_2^{n_2}. $$

The $q$-Appell series $\Phi^{(1)}$ has a Jackson integral representation [2]

$$ \Phi^{(1)} = \frac{(a,c/a)_{\infty}}{(q,c)_{\infty}(1-q)} \int_0^1 t^{a-1} (qt, b_1 t, b_2 x t)_{\infty} d_q t, $$

where $a = q^a$. Therefore the solutions [1.13] and [1.19] are $q$-analogs of the functions [17] and [1.8] with $t_3 = \infty$, respectively. In addition, the $q$-Heun equation was rediscovered in [23] as an eigenvalue problem for the fourth degenerated Ruijsenaars-van Diejen operator [21] of one variable. In [24], the variants of the $q$-Heun equation of degree three and degree four.
were introduced from the viewpoint of degenerations of the Ruijsenaars-van Diejen operator. The variant of the $q$-hypergeometric equation of degree three is also defined by specializing the variant of the $q$-Heun equation of degree three so that $x = 0$ is essentially non-singular, as follows:

$$
\mathcal{H}_3(x) = 0,
$$
(1.22)

$$
\mathcal{H}_3 = \prod_{i=1}^{3}(x - q^{h_i+1/2}t_i) \cdot T_x^{-1} + q^{2a+1} \prod_{i=1}^{3}(x - q^{h_i-1/2}t_i) \cdot T_x
$$

$$
+ q^a \left[-(q + 1)x^3 + q^{1/2} \sum_{i=1}^{3}(q^{h_i} + q^{h_i})t_i x^2
$$

$$
- q^{(h_1+h_2+h_3+l_1+l_2+l_3+1)/2}t_1 t_2 t_3 \sum_{i=1}^{3}((q^{-h_i} + q^{-l_i})/t_i) x
$$

$$
+ q^{(h_1+h_2+h_3+l_1+l_2+l_3)/2}(q + 1)t_1 t_2 t_3 \right].
$$
(1.23)

Note that the points $x = 0$ and $x = \infty$ are essentially non-singular for the variant of the $q$-Heun equation of degree four, and then we cannot obtain some equation by a similar specialization for the variant of the $q$-Heun equation of degree four. The variant of the $q$-hypergeometric equation of degree three is a $q$-analog of a second order Fuchsian differential equation with five singularities $\{0, t_1, t_2, t_3, \infty\}$, and $x = 0$ and $x = \infty$ are essentially non-singular. Therefore the variant of the $q$-hypergeometric equation of degree three is essentially a $q$-analog of the Riemann-Papperitz equation. Note that the equation $\mathcal{H}_3(x) = 0$ becomes $\mathcal{H}_2(x) = 0$ in the limit $t_3 \to \infty$ (see [9] or Remark 2.7). However, the explicit solution for the variant of the $q$-hypergeometric equation of degree three is not known. We have seen only conjectural solutions in [9].

Our aim is to obtain the integral and series solutions for the variants of the $q$-hypergeometric equation, which are $q$-analogs of the integral (1.7) and the series (1.8), respectively. We will introduce a configuration of linear $q$-difference equations in order to characterize $q$-difference equations in Definition 2.3 below. In this paper, we will consider the following equation:

$$
\mathcal{E}_3(x) = 0,
$$
(1.24)

$$
\mathcal{E}_3 = [x^2(B - AT_x)(B - AqT_x) - x^2(e_1(a) - qe_1(b)T_x)(B - AT_x)]
$$

$$
+ x(e_2(a) - qe_2(b)T_x)(1 - T_x) - e_3(a)B^{-1}(1 - q^{-1}T_x)(1 - T_x)]T_x^{-1},
$$
(1.25)

where $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$, $a_1a_2a_3A = q^2b_1b_2b_3B$ and $e_i$ is the elementary symmetric polynomial of degree $i$. The configuration of the equation $\mathcal{E}_3(x) = 0$ is in the same form as the configuration of the equation $\mathcal{H}_3(x) = 0$. Thus by some gauge transformation and changes of parameters, the equation $\mathcal{E}_3(x) = 0$ is identified with the equation $\mathcal{H}_3(x) = 0$ (see Figure 5, Figure 6 and Remark 3.2). Also, in this paper we will consider the following equation $\mathcal{E}_2(x) = 0$ instead of $\mathcal{H}_2(x) = 0$:

$$
\mathcal{E}_2(x) = 0,
$$
(1.26)

$$
\mathcal{E}_2 = [x^2(1 - q^aT_x)(B - AT_x) - x(e_1(a) - qe_1(b)T_x)(1 - T_x) - e_2(a)B^{-1}(1 - q^{-1}T_x)(1 - T_x)]T_x^{-1},
$$
(1.27)

where $a = (a_1, a_2)$, $b = (b_1, b_2)$ and $a_1a_2A = q^{a+1}b_1b_2B$. For the same reason as for the identification of the equation $\mathcal{E}_3(x) = 0$ and $\mathcal{H}_3(x) = 0$, the equation $\mathcal{E}_2(x) = 0$ is identified with $\mathcal{H}_2(x) = 0$ (see Figure 4, Figure 7 and Remark 3.4). We focus on the solution (1.18) for the variant of the $q$-hypergeometric equation of degree two. As mentioned above, the solution
(1.18) has the Jackson integral representation. This integral is a special case of the Jackson integral of Jordan-Pochhammer type defined by

\[
\int_C t^{a-1} \prod_{i=0}^{M} \frac{(a_it)}{b_i t} \, dt.
\]

Note that the integral (1.10) is the case \( M = 1 \) of (1.28). Since a solution for the equation \( \mathcal{H}_2 f(x) = 0 \) is written by the case \( M = 2 \) of the integral (1.28), we expect that there is a solution for the equation \( \mathcal{H}_3 f(x) = 0 \) written by the integral (1.28) with \( M = 3 \). One of the main results is Theorem 4.1, which gives integral solutions for the variant of the \( q \)-hypergeometric equation of degree three.

**Main Theorem 1.1** (Theorem 3.1). Let \( \tau_1, \tau_2 \in \{ q/a_1, q/a_2, q/a_3, q/(Ax) \} \) and \( \sigma_1, \sigma_2 \in \{ b_1, b_2, b_3, Bx \} \). We suppose \( q^\lambda = B/A \notin q^7 \). Then the integrals

\[
\int_{\tau_1}^{\tau_2} \frac{(Ax, a_1 t, a_2 t, a_3 t)_\infty}{(Bx, b_1 t, b_2 t, b_3 t)_\infty} \, dt,
\]

\[
x^\lambda \int_{\sigma_1}^{\sigma_2} \frac{(q/(Bx), q/b_1, q/s/b_2, q/s/b_3)_\infty}{(q/(Ax), q/a_1, q/a_2, q/a_3)_\infty} \, ds,
\]

satisfy the equation \( \mathcal{E}_3 f(x) = 0 \).

These integrals are \( q \)-anlogs of the integral (1.7). The integral solutions for the variant of the \( q \)-hypergeometric equation of degree two can be obtained by taking some limit to the integrals (1.29), (1.30). For the integral solutions for the variant of the \( q \)-hypergeometric equation of degree two, see Theorem 4.2. Also another main result is Theorem 4.4, which gives series solutions for the variant of the \( q \)-hypergeometric equation of degree three.

**Main Theorem 1.2** (Theorem 4.1). The functions

\[
(Ax q/a_2)_\infty \frac{W_7}{Bx q/a_2}_\infty \left( \frac{a_3 A}{a_2 B}, \frac{q b_1}{a_2 A}, \frac{q b_2}{a_2 A}, \frac{q b_3}{a_2 A}, \frac{A}{a_2 B}, \frac{q Bx}{a_2 B}, \frac{A}{a_1} \right),
\]

\[
(Ax)/(b_1 b_3), \frac{q a_3/(b_2 b_3), q A/(a_2 b_3)}{Bx/(a_2 b_3), q a_3/(a_2 b_3)} \frac{W_7}{(a_2 A_0, a_2 A_0, a_0 A, a_0)} \left( \frac{a_3 A}{a_2 b_3}, \frac{q b_1}{a_2 A}, \frac{q b_2}{a_2 A}, \frac{q b_3}{a_2 A}, \frac{A}{a_2 b_3}, \frac{b_3}{a_1} \right),
\]

\[
1/(q a_1/(Ax), q a_2/(Ax), q a_3/(Ax), q a_1/(b_1 b_2 x), q a_2/(b_2 b_3 x), q a_3/(b_3 b_3)) \frac{W_7}{(a_2 A_0, q b_1/(Ax), q b_2/(Ax), q b_3/(Ax), q a_3/(a_2 b_3))} \left( \frac{a_2 A}{a_2 b_3}, \frac{q b_1}{a_2 A}, \frac{q b_2}{a_2 A}, \frac{q b_3}{a_2 A}, \frac{A}{a_2 b_3}, \frac{b_3}{a_1} \right),
\]

\[
x/(q b_1/(Ax), q b_2/(Ax), q b_3/(Ax), q a_2 b_3/(a_2 a_2/(Ax))) \frac{W_7}{(a_2 A_0, q b_1/(Ax), q b_2/(Ax), q b_3/(Ax), q a_2 b_3/(a_2 a_2/(Ax)))} \left( \frac{a_2 A}{a_2 b_3}, \frac{q b_1}{a_2 A}, \frac{q b_2}{a_2 A}, \frac{q b_3}{a_2 A}, \frac{A}{a_2 b_3}, \frac{b_3}{a_1} \right),
\]

satisfy the equation \( \mathcal{E}_3 f(x) = 0 \).
Here, the function $\text{}_8W_7(a; b, c, d, e, f; x)$ is the very-well-poised $q$-hypergeometric series

$$\text{}_8W_7(a; b, c, d, e, f; x) = \sum_{n=0}^{\infty} \frac{1 - aq^{2n}}{1 - a} \frac{(a, b, c, d, e, f)_n}{(qa/b, qa/c, qa/d, qa/e, qa/f)_n} x^n.$$  \hspace{1cm} (1.37)

Some series in Main Theorem 1.2 are $q$-analogues of the series (1.8). In addition, the equation $H_3 f(x) = 0$ has some symmetries, so we can get many solutions for $H_3 f(x) = 0$ by some transformations acting on solutions in Main Theorem 1.2. For more details, see section 4.

The series solutions for the variant of the $q$-hypergeometric equation of degree two, which are written by $3\varphi_2$, are also obtained by taking $t_3 \to \infty$ (see Theorem 4.2). Moreover, by taking the limit $t_2 \to 0$ to the solutions $3\varphi_2$, we get some series solutions for the Heine’s $q$-hypergeometric equation, which are written by $2\varphi_1, 2\varphi_2, 3\varphi_2, 3\varphi_1$. Many properties of the Heine’s equation (cf. $q$-analogs of Kummer’s 24 solutions [8], some transformation formulas for $2\varphi_1$ [6]) are explained by our results.

The contents of this paper are as follows. In section 2, we introduce a configuration of a $q$-difference equation, and characterize the variants of the $q$-hypergeometric equation by the configurations. In section 3, we show the integral solutions for the variants of the $q$-hypergeometric equation. In section 4, we show the series solutions for the variants of the $q$-hypergeometric equation and the Heine’s $q$-hypergeometric equation. In section 5, we give a summary of this paper and discuss related problems.

### 2 Variants of the $q$-hypergeometric equation

In this section, first we recall some fundamental concepts for linear $q$-difference equations, and define a configuration of a $q$-difference equation. Next, we introduce variants of the $q$-hypergeometric equation defined in [9], and characterize them by configurations. Variants of the $q$-hypergeometric equation are a specialization of the $q$-Heun equation or the variant of it. Thus we introduce and characterize the $q$-Heun equation and the variant.

#### 2.1 Linear $q$-difference equation

In this subsection, we recall some fundamental concepts of linear $q$-difference equations, such as characteristic roots and a non-logarithmic singularity. We then introduce a configuration of a $q$-difference equation. We consider the $q$-difference equation

$$\mathcal{L}f(x) = 0, \quad \mathcal{L} = x^{M'} \sum_{i=0}^{M} \sum_{j=0}^{N} a_{i,j} x^i T_x^j \cdot (T_x)^{N'}.$$ \hspace{1cm} (2.1)

Also, we consider the solution

$$f(x) = x^\lambda \sum_{n=0}^{\infty} c_n x^n, \quad c_0 = 1.$$ \hspace{1cm} (2.2)

The operator $\mathcal{L}$ can be rewritten as

$$\mathcal{L} = x^{M'} (x^M L_M(T_x) + \cdots + x^0 L_0(T_x)),$$ \hspace{1cm} (2.3)

where $L_i$ is some Laurent polynomial. Hence we have

$$\mathcal{L}f(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{M} L_m(q^{\lambda+n}) c_n x^{\lambda+M'+n+m}.$$ \hspace{1cm} (2.4)
The condition of $L f(x) = 0$ is equivalent to the following conditions:

\[
\begin{align*}
c_0 L_0(q^\lambda) &= 0, \\
c_1 L_0(q^{\lambda+1}) + c_0 L_1(q^\lambda) &= 0, \\
&\vdots
\end{align*}
\]

(2.5)

Since $c_0 = 1$, we have $L_0(q^\lambda) = 0$.

**Definition 2.1.** For the equation (2.1), the roots of $L_0(y) = 0$ are called the characteristic roots at $x = 0$. Similarly, the roots of $L_M(y) = 0$ are called the characteristic roots at $x = \infty$.

**Remark 2.1.** If $a$ is a characteristic root at $x = 0$, then we obtain a solution of (2.1) in the form

\[
x^\alpha (1 + O(x)),
\]

(2.6)

where $a = q^\alpha$. Also if $b$ is a characteristic root at $x = \infty$, then we obtain a solution of (2.1) in the form

\[
x^\beta (1 + O(x^{-1})),
\]

(2.7)

where $b = q^\beta$. In the general theory of linear $q$-difference equations, $\alpha$ and $-\beta$ are often called the characteristic exponents.

For general theory of linear $q$-difference equations, if $q^\alpha$ and $q^{\alpha+n}$ are characteristic roots at $x = 0$, where $n \in \mathbb{Z}_{>0}$, then we may need logarithmic terms for some solution. For more details, see [1]. In this case, the equation (2.1) has the solution $x^\alpha \sum_{n=0}^{\infty} c_n x^n$ if this equation satisfies the condition

\[
\sum_{m=0}^{n-1} c_m L_{n-i}(q^{\alpha+i}) = 0.
\]

(2.8)

These can be considered as replacing $x = 0$ and $x = \infty$. In this paper, we will consider the equation that $a$ and $aq$ are the characteristic roots at $x = 0$ or $x = \infty$ (i.e. $n = 1$). In this case, the non-logarithmic condition is simple:

**Lemma 2.1.** For the equation (2.1), we have

1) $x = 0$ is non-logarithmic and $a$, $aq$ are characteristic roots at $x = 0$ if and only if $L_1(a) = 0$.

2) $x = \infty$ is non-logarithmic and $a$, $aq^{-1}$ are characteristic roots at $x = \infty$ if and only if $L_{M-1}(a) = 0$.

**Remark 2.2.** For more general case, see Proposition 3.1 in [20].

As an operator, the $q$-shift operator $T_x$ plays a similar role to the multiplication operator $x$, i.e. $T_x x = qx T_x$. Thus we can consider “characteristic roots at $T_x = 0$ and $T_x = \infty$”. These roots are important for characterizing linear $q$-difference equations.

**Definition 2.2.** The operator $L$ can be rewritten as

\[
L = (P_N(x)T_x^N + \cdots + P_0(x)T_x^0)T_x^N.
\]

(2.9)

The roots of the polynomial $P_0(x)$ (resp. $P_N(x)$) are called the characteristic roots at $T_x = 0$ (resp. $T_x = \infty$).
Definition 2.3. Let \( \{a_j\}_{1 \leq j \leq N}, \{b_j\}_{1 \leq j \leq N}, \{c_i\}_{1 \leq i \leq M}, \{d_i\}_{1 \leq i \leq M} \) be the characteristic roots of the equation (2.1) at \( x = 0, x = \infty, T_x = 0, T_x = \infty \), respectively. Then Figure 1 is called the configuration of (2.1).

![Figure 1: the configuration of the equation \( \mathcal{L} f(x) = 0 \).](image)

Remark 2.3. We rewrite the operator \( \mathcal{L} \) in two form (2.3), (2.9). By equating the coefficients, we find the relation \( a_1 \cdots a_N d_1 \cdots d_M = b_1 \cdots b_N c_1 \cdots c_M \).

Remark 2.4. The name “configuration” comes from the point configuration of algebraic curves. The method to characterize \( q \)-difference equations by the configurations of algebraic curves was discussed in the context of \( q \)-Painlevé equation (cf. \[13\], \[26\]). We use a quantization of this method by the quantum curve \( \mathcal{L} = 0 \) to characterize the linear \( q \)-difference equation \( \mathcal{L} f(x) = 0 \).

2.2 Variants of the \( q \)-hypergeometric equation

In this subsection, we characterize variants of the \( q \)-hypergeometric equation introduced in \[9\] by configurations. As mentioned in section 1, the variants of the \( q \)-hypergeometric equation of degree two and degree three were defined by a special case of the \( q \)-Heun equation and the variant of the \( q \)-Heun equation of degree three, respectively. We characterize not only the variants of the \( q \)-hypergeometric equation but also the \( q \)-Heun equation and the variant of the \( q \)-Heun equation of degree three. The \( q \)-Heun equation was first introduced by \[7\], and rediscovered by the eigenvalue problem for \( A^{(4)} \langle 4 \rangle \) in \[23\]. Here \( A^{(i)} \) is the \( i \)-th degenerated Ruijsenaars-van Diejen operator of one variable. Also the variant of the \( q \)-Heun equation of degree three was defined as the eigenvalue problem for \( A^{(5)} \) in \[24\]. In this paper we do not discuss about the Ruijsenaars-van Diejen operator. For more details, see \[21\], \[25\]. We introduce the \( q \)-Heun equation and the variant of the \( q \)-Heun equation of degree three by the forms of \[24\].

Definition 2.4 (\[24\]). The \( q \)-Heun equation is defined by

\[
(A^{(4)} - E)f(x) = 0,
\]

where

\[
A^{(4)} = x^{-1} \prod_{i=1}^{2} (x - q^{h_i+1/2} t_i) \cdot T_x^{-1} + q^{\alpha_1 + \alpha_2} x^{-1} \prod_{i=1}^{2} (x - q^{h_i-1/2} t_i) \cdot T_x
\]
and the variant of \(q\)-Heun equation of degree three is defined by
\[
(A^{(3)} - E) f(x) = 0,
\]
where
\[
A^{(3)} = - x^{-1} \prod_{i=1}^{3} (x - q^{h_i+1/2} t_i) \cdot T_x^{-1} + x^{-1} \prod_{i=1}^{3} (x - q^{l_i-1/2} t_i) \cdot T_x
- (q^{1/2} + q^{-1/2}) x^2 + \sum_{i=1}^{3} (q^{h_i} + q^{l_i}) t_i x + q^{(l_1+l_2+l_3+h_1+h_2+h_3)/2} (q^{3/2} + q^{-3/2}) t_1 t_2 t_3 x^{-1}.
\]

We can easily find the configurations of the equations (2.10) and (2.12). The configuration of (2.10) is given as follows.

Figure 2: the configuration of the equation \((A^{(4)} - E) f(x) = 0\) (2.10).

Here, \(\lambda_{\pm} = \frac{1}{2}(h_1 + h_2 - l_1 - l_2 - \alpha_1 - \alpha_2 \pm \beta)\). Also, the configuration of (2.12) is given as follows.

Figure 3: the configuration of the equation \((A^{(3)} - E) f(x) = 0\) (2.11, 2.12).
Here, \( q^{\mu \pm} = \frac{1}{2}(h_1 + h_2 + h_3 - l_1 - l_2 - l_3 + 3 \pm \beta) \). The double point \( \circ \) in Figure 3 means a non-logarithmic singularity which has characteristic roots \( a, aq \). In this paper, we use the double point in the sense as above. In fact, figures 2 and 3 characterize the equations (2.10) and (2.12), respectively.

**Proposition 2.1.** We consider the \( q \)-difference equations

\[
\mathcal{L}_2 f(x) = \sum_{i=-1}^{1} \sum_{j=-1}^{1} a_{i,j} x^i T_x^j f(x) = 0, \quad (2.14)
\]

\[
\mathcal{L}_3 f(x) = \sum_{i=-1}^{2} \sum_{j=-1}^{1} a_{i,j} x^i T_x^j f(x) = 0. \quad (2.15)
\]

(1) If the configuration of \( \mathcal{L}_2 f(x) = 0 \) is given by Figure 2, then the equation \( \mathcal{L}_2 f(x) = 0 \) coincides with the \( q \)-Heun equation (2.10) up to the eigenvalue \( E \).

(2) If the configuration of \( \mathcal{L}_3 f(x) = 0 \) is given by Figure 3, then the equation \( \mathcal{L}_3 f(x) = 0 \) coincides with the variant of the \( q \)-Heun equation of degree three (2.12) up to the eigenvalue \( E \).

**Proof.** We prove only (2) because we can show (1) in the same way as the proof of (2). We assume that the configuration of \( \mathcal{L}_3 f(x) = 0 \) is given by Figure 3. Since the characteristic roots at \( x = 0 \) are \( q^{\mu +}, q^{\mu -} \), the ratio of \( a_{-1,-1}, a_{-1,0}, a_{-1,1} \) is determined. More precisely, we have \( a_{-1,-1} T_x^{-1} + a_{-1,0} + a_{-1,1} T_x = a_{-1,1} T_x^{-1} (T_x - q^{\mu +})(T_x - q^{\mu -}) \). In the same way, the ratios of \( a_{1,-1}, a_{1,0}, a_{1,1}, a_{0,-1}, a_{0,1}, a_{0,0}, a_{1,-1}, a_{2,-1}, a_{1,-1}, a_{0,1}, a_{1,1}, a_{2,1} \) are determined by the characteristic roots. Using Lemma 2.1 (2), we have \( a_{1,-1} q^{-1/2} + a_{1,0} + a_{1,1} q^{1/2} = 0 \). Since the ratio of \( a_{1,-1}, a_{1,1} \) is already determined, we obtain the ratio of \( a_{1,-1}, a_{1,0} \). Thus we find the ratio of \( \{a_{i,j} \mid (i,j) \neq (0,0)\} \). Therefore \( \mathcal{L}_3 \) coincides with the equation (2.12) up to the eigenvalue \( E \).

**Remark 2.5.** For equations (2.10) and (2.12), the eigenvalue \( E \) is independent of their configurations. It is reasonable to regard \( E \) as an accessory parameter.

Next, we introduce the variants of the \( q \)-difference equation defined by \([9]\). The variant of the \( q \)-hypergeometric equation of degree two (resp. degree three) was defined by specializing the \( q \)-Heun equation (resp. the variant of the \( q \)-Heun equation of degree three) so that the point \( x = 0 \) is essentially non-singular.

**Definition 2.5 (9).** The variant of the \( q \)-hypergeometric equation of degree two is defined by

\[
\mathcal{H}_2 f(x) = 0, \quad (2.16)
\]

where

\[
\mathcal{H}_2 = \prod_{i=1}^{2} \left( x - q^{h_1 + 1/2} t_i \right) \cdot T_x^{-1} + q^{a_1 + a_2} \prod_{i=1}^{2} \left( x - q^{l_1 - 1/2} t_i \right) \cdot T_x - (q^{a_1} + q^{a_2}) x^2 + E x + p(q^{1/2} + q^{-1/2}) t_1 t_2,
\]

\[
p = q^{(h_1 + h_2 + l_1 + l_2 + a_1 + a_2)/2}, \quad E = -p \{ (q^{h_2} + q^{-l_2}) t_1 + (q^{-h_1} + q^{l_1}) t_2 \}. \quad (2.17, 18)
\]

The variant of the \( q \)-hypergeometric equation of degree three is defined by

\[
\mathcal{H}_3 f(x) = 0, \quad (2.19)
\]
where

\[
H_3 = \prod_{i=1}^{3} (x - q^{h_i + 1/2} t_i) \cdot T_x^{-1} + q^{2\alpha+1} \prod_{i=1}^{3} (x - q^{l_i - 1/2} t_i) \cdot T_x
\]

\[
+ q^\alpha \left[-(q+1)x^3 + q^{1/2} \sum_{i=1}^{3} (q^{h_i} + q^{l_i}) t_i x^2 \right. \\
- q^{(h_1 + h_2 + h_3 + l_1 + l_2 + l_3 + 1)/2} \sum_{i=1}^{3} ((q^{-h_i} + q^{-l_i})/t_i) x \\
\left. + q^{(h_1 + h_2 + h_3 + l_1 + l_2 + l_3)/2} (q + 1) t_1 t_2 t_3 \right].
\] (2.20)

We can easily check that Figures 4 and 5 are the configurations of the equations (2.16) and (2.19), respectively. Here, \(\lambda_0 = \frac{1}{2}(h_1 + h_2 - l_1 - l_2 - \alpha_1 - \alpha_2 + 1)\) and \(\nu = \frac{1}{2}(h_1 + h_2 + h_3 - l_1 - l_2 - l_3 + 1)\).

Figure 4: the configuration of the equation \(H_2 f(x) = 0\) (2.16).

Figure 5: the configuration of the equation \(H_3 f(x) = 0\) (2.19).
These configurations characterize the equations (2.16) and (2.19).

**Proposition 2.2.** We consider the q-difference equations

\[ \mathcal{L}_2 f(x) = \sum_{i=0}^{2} \sum_{j=-1}^{1} a_{i,j} x^i T_j^2 f(x) = 0, \quad (2.21) \]

\[ \mathcal{L}_3 f(x) = \sum_{i=0}^{3} \sum_{j=-1}^{1} a_{i,j} x^i T_j^3 f(x) = 0. \quad (2.22) \]

1. If the configuration of \( \mathcal{L}_2 f(x) = 0 \) is given by Figure 4, then the equation \( \mathcal{L}_2 f(x) = 0 \) coincides with the variant of the q-hypergeometric equation of degree two (2.16), that is, \( \mathcal{L}_2 = \alpha \mathcal{H}_2 \). \( \alpha \in \mathbb{C} \).

2. If the configuration of \( \mathcal{L}_3 f(x) = 0 \) is given by Figure 5, then the equation \( \mathcal{L}_3 f(x) = 0 \) coincides with the variant of the q-hypergeometric equation of degree three (2.19), that is, \( \mathcal{L}_3 = \alpha \mathcal{H}_3 \). \( \alpha \in \mathbb{C} \).

**Proof.** We prove (2) only because (1) is proved in the same way. We assume that the configuration of \( \mathcal{L}_3 f(x) = 0 \) is given by Figure 5.\( \alpha \) is determined by the characteristic roots. Also, by the non-logarithmic condition at \( x = 0 \) and \( x = \infty \), we find the ratios of \( a_{0,-1} \) and \( a_{0,0}, a_{1,-1} \) and \( a_{1,0} \), respectively. Thus the ratios of \( \{a_{i,j}\} \) are determined. This completes the proof of (2). \( \square \)

**Remark 2.6.** In [9], the equation \( \mathcal{H}_3 f(x) = 0 \) is a q-analog of the Fuchsian differential equation, which has the following Riemann scheme:

\[ \begin{cases} x = 0 & x = t_1 & x = t_2 & x = t_3 & x = \infty \\ \nu - \alpha & 0 & 0 & 0 & \alpha \\ \nu - \alpha + 1 & l_1 - h_1 & l_2 - h_2 & l_3 - h_3 & \alpha + 1 \end{cases} \quad (2.23) \]

Here, \( x = 0 \) and \( x = \infty \) are essentially non-singular. By some gauge transformation, this equation becomes a Fuchsian differential equation with three singularities \( \{t_1, t_2, t_3\} \). In this sense, the variant of the q-hypergeometric equation of degree three is a q-analog of the Riemann-Papperitz differential equation [13]. Also, the equation \( \mathcal{H}_2 f(x) = 0 \) is a q-analog of the Fuchsian differential equation, which has the following Riemann scheme:

\[ \begin{cases} x = 0 & x = t_1 & x = t_2 & x = \infty \\ \lambda_0 & 0 & 0 & \alpha_1 \\ \lambda_0 + 1 & l_1 - h_1 & l_2 - h_2 & \alpha_2 \end{cases} \quad (2.24) \]

where \( x = 0 \) is essentially non-singular.

**Remark 2.7.** Taking the limit \( t_3 \to \infty \), the equation \( \mathcal{H}_3 f(x) = 0 \) (2.19) becomes the equation \( \mathcal{H}_2 f(x) = 0 \) (2.16) with the parameters \( (\alpha_1, \alpha_2) = (\alpha - \alpha_3 + t_3) \). Also, taking the limit \( t_2 \to 0 \), the equation (2.16) becomes the equation

\[ \mathcal{H}_1 f(x) = 0, \quad (2.25) \]

\[ \mathcal{H}_1 = (x - q^{h_1+1/2} t_1) T_x^{-1} + q^{a_1+\alpha_2} (x - q^{1/2} t_1) T_x - [(q^{a_1+\alpha_2}) x - q^{(b_1+2l_1+b_2+\alpha_1+\alpha_2)/2} (q^{h_2} + q^{-l_2})]. \quad (2.26) \]

By setting

\[ t_1 = 1, \quad h_1 = 1/2, \quad h_2 - l_2 = \alpha_1 + \alpha_2 + l_1 - 3/2, \quad a = q^{a_1}, \quad b = q^{\alpha_2}, \quad c = q^{a_1+\alpha_2+1/2}, \quad (2.27) \]
the Heine’s \( q \)-hypergeometric equation \((1.9)\) is realized. For more details, see [9]. These degenerations can be considered by the configuration as follows:

Here, the right configuration characterize the Heine’s equation. More precisely, the configuration of the Heine’s equation \([x(1-aT_x)(1-bT_x)-(1-T_x)(1-cq^{-1}T_x)]f(x) = 0\) is as follows:

In differential case, these degenerations mean the limit of Fuchsian differential equation with three singularities \( \{t_1, t_2, t_3\} \), that is, \( \{t_1, t_2, t_3\} \rightarrow \{t_1, t_2, \infty\} \rightarrow \{0, t_1, \infty\} \).

Remark 2.8. In Proposition 2.2 we find that the variants of the \( q \)-hypergeometric equation \((2.16), (2.19)\) are rigid by the configuration. This is corresponding to the Riemann-Papperitz equation \((1.5)\) being rigid by the Riemann scheme \((1.6)\).

3 Integral solutions

In this section, we show integral solutions for the variants of the \( q \)-hypergeometric equation. First, we derive a \( q \)-difference equation for the Jackson integral of Jordan-Pochhammer type. Next, we show integral solutions for the variant of the \( q \)-hypergeometric equation of degree three \( H_3f(x) = 0 \) by considering a special case of the Jordan-Pochhammer integral. Integral solutions for the variant of the \( q \)-hypergeometric equation of degree two are obtained in a similar way for some limit of the integral solutions for \( H_3f(x) = 0 \). A linear \( q \)-difference system associated with the Jackson integral of Jordan-Pochhammer type was obtained in [17], [18]. A \( q \)-difference system for the Jackson integral of Selberg type, which contains the system for the Jordan-Pochhammer type, was obtained in [19]. In [11], a \( q \)-difference system for the Jackson integral of Selberg type was also discussed, and above results were summarized. Hence, for more details of the Jackson integral of Jordan-Pochhammer type, see [11, 17, 18, 19]. In this paper, we derive a \( q \)-difference equation for the Jackson integral of Jordan-Pochhammer type by integrating the equation that the integrand satisfies. The configuration of our equation can be calculated more easily than the configuration of an equation which is derived from the above system.
Definition 3.1 ([12]). The Jackson integrals of the function \( f(t) \) are defined as follows:

\[
\int_{0}^{\tau} f(t) \frac{dt}{t} = (1 - q) \sum_{n=0}^{\infty} f(\tau q^n), \tag{3.1}
\]

\[
\int_{0}^{\tau \infty} f(t) \frac{dt}{t} = (1 - q) \sum_{n=-\infty}^{\infty} f(\tau q^n), \tag{3.2}
\]

\[
\int_{\tau_1}^{\tau_2} f(t) \frac{dt}{t} = \int_{\tau_1}^{\tau_2} f(t) \frac{dt}{t} - \int_{0}^{\tau_1} f(t) \frac{dt}{t}. \tag{3.3}
\]

Definition 3.2 ([18]). Let \( \psi \) be the function

\[
\psi(x,t) = t^a (Axt, a_1t, a_2t, \ldots, a_Mt)^{\infty}, \tag{3.4}
\]

The Jackson integral of Jordan-Pochhammer type is defined by

\[
\varphi(x, \tau) = \int_{0}^{\tau \infty} \psi(x, t) \frac{dt}{t}. \tag{3.5}
\]

Lemma 3.1 plays an essential role in deriving the \( q \)-difference equation which the Jackson integral of Jordan-Pochhammer type satisfies.

Lemma 3.1. The integrand (3.4) satisfies the following equation

\[
\sum_{k=0}^{M} (-1)^k x^{M-k}[e_k(a)T_1 T_2^{1} - q^a e_k(b)](B - A T x) \cdots (B - A q^{M-k-1} T x)(1 - q^{-(k-1)} T x) \cdots (1 - T x) \psi = 0. \tag{3.6}
\]

Here, \( e_i \) is the elementary symmetric polynomial of degree \( i \).

Proof. The equations

\[
x t (B - A T x) \psi = (1 - T x) \psi, \tag{3.7}
\]

\[
\prod_{i=1}^{M} (1 - a_i t) \cdot T_i T_x^{-1} \psi = q^a \prod_{i=1}^{M} (1 - b_i t) \cdot \psi, \tag{3.8}
\]

can be verified by direct calculations. Using the elementary symmetric functions, the equation (3.8) can be rewritten as

\[
\sum_{k=0}^{M} (-t)^k [e_k(a)T_1 T_2^{1} - q^a e_k(b)] \psi = 0. \tag{3.9}
\]

Since \([x t (B - A T x)]^k = (x t)^k (B - A T x)(B - A q T x) \cdots (B - A q^{k-1} T x) \) holds, we have

\[
x^{M}(B - A T x)(B - A q T x) \cdots (B - A q^{M-1} T x)^{k}(T_1 T_x^{-1})^{\varepsilon} \psi
\]

\[
=x^{M-k}(T_1 T_x^{-1})^{\varepsilon}(B - A q^{k-1} T x)(B - A q^{k-1} T x) \cdots (B - A q^{k-1} T x)^{\psi}
\]

\[
=x^{M-k}(T_1 T_x^{-1})^{\varepsilon}(B - A q^{k-1} T x)(B - A q^{k-1} T x) \cdots (B - A q^{k-1} T x)^{\psi}
\]

\[
=x^{M-k}(T_1 T_x^{-1})^{\varepsilon}(B - A q^{k-1} T x)(B - A q^{k-1} T x) \cdots (B - A q^{k-1} T x)^{\psi}, \tag{3.10}
\]

for \( \varepsilon = 0, 1 \). By using the equation (3.7), we get

\[
[(x t)(B - A T x)]^k \psi = (1 - q^{-(k-1)} T x) \cdots (1 - T x) \psi. \tag{3.11}
\]

Therefore, multiplying the equation (3.9) by \( x^{M}(B - A T x)(B - A q T x) \cdots (B - A q^{M-1} T x) \) yields the desired equation (3.6).
If the integral path (the period of integration) is chosen appropriately, then we can derive the \( q \)-difference equation that the Jackson integral of Jordan-Pochhammer type satisfies.

**Proposition 3.1.** Assuming \( T_x^l \tau = q^l \tau \) for \( l \in \mathbb{Z} \), the integral \( (3.3) \) satisfies the \( q \)-difference equation

\[
\sum_{k=0}^{M} (-1)^k x^{M-k} [e_k(a) T_x^{k-1} - q^a e_k(b)] (B - A T_x) \cdots (B - A q^{M-k-1} T_x) (1 - q^{-(k-1)} T_x) \cdots (1 - T_x) \varphi = 0. 
\]

(3.12)

**Proof.** Since \( T_x^l \tau = q^l \tau \), we get

\[
\int_0^{\tau^\infty} \left( T_x^l T_x^j \psi(x, t) \right) \frac{dt}{t} = (1 - q) \sum_{n \in \mathbb{Z}} \psi(T_x^q x, q^{n+j} \tau) \\
= (1 - q) \sum_{n \in \mathbb{Z}} \psi(T_x^q x, q^{n+j-l} T_x \tau) \\
= (1 - q) \sum_{n \in \mathbb{Z}} T_x^q \psi(x, q^n \tau) \\
= (1 - q) \sum_{n \in \mathbb{Z}} T_x^q \psi(x, q^n \tau).
\]

(3.13)

Therefore we have the desired equation \( (3.12) \) by integrating the equation \( (3.6) \).

\( \square \)

**Remark 3.1.** We put \( M = 1 \), then the equation \( (3.12) \) is the Heine’s \( q \)-hypergeometric equation. With the changes of the variable and parameters

\[
z = \frac{q B x}{a_1}, \quad a = q^a, \quad b = \frac{A}{B}, \quad c = q^{a+1} \frac{b_1}{a_1},
\]

(3.14)

we find

\[
x(T_x^{-1} - q^a)(B - AT_x) - (a_1 T_x^{-1} - q^a b_1)(1 - T_x) = T_x^{-1} [z(1 - a T_x)(1 - b T_x) - (1 - c q^{-1} T_x)(1 - T_x)].
\]

(3.15)

Therefore we have integral solutions \( \int_0^{\tau^\infty} t^a \left( \frac{A x t}{B x, b_1 t} \right) \frac{dt}{t} \) for Heine’s equation.

We will reduce the special case of \( (3.12) \) to the variant of the \( q \)-hypergeometric equation of degree three, which yields the integral solutions. We put \( M = 3, \alpha = 1 \) and \( A a_1 a_2 b_3 = q^3 B b_1 b_2 b_3 \). Then the equation \( (3.12) \) can be written as

\[
[x^3(1 - q T_x)(B - AT_x)(B - A q^2 T_x) \\
-x^2(e_1(a) - q e_2(b)) (B - AT_x)(B - A q T_x)(1 - T_x) \\
x(e_2(a) - q e_3(b)) (B - AT_x)(1 - q^{-2} T_x)(1 - T_x) \\
-e_3(b)(1 - T_x)(1 - q^{-2} T_x)(1 - T_x) T_x^{-1} \varphi = 0,
\]

(3.16)

so that the equation becomes

\[
(1 - q^{-2} T_x)(B - A q^{-1} T_x) \phi \varphi = 0,
\]

(3.17)
where $\mathcal{E}_3$ is the $q$-difference operator

\[
\mathcal{E}_3 = x^3(B - AT_x)(B - AqT_x) - x^2(c_1(a) - qe_1(b)T_x)(B - AT_x) \\
+ x(e_2(a) - qe_2(b)T_x)(1 - T_x) - e_3(a)B^{-1}(1 - q^{-1}T_x)(1 - T_x)T_x^{-1}.
\] (3.18)

The configuration of (3.18) is given as follows.

\[
\begin{array}{c|c|c|c|c}
& b_1/A & b_2/A & b_3/A & \hline
1, q & & & & T_x = \infty \\
& & & & B/A, Bq^{-1}/A \\
& 1, q^{-1} & & & \hline
x = 0 & a_1/B & a_2/B & a_3/B & x = \infty \\
& & & & T_x = 0
\end{array}
\]

Figure 6: the configuration of the equation (3.18).

Therefore we identify $\mathcal{E}_3 f(x) = 0$ with the variant of the $q$-hypergeometric equation of degree three $H_3 f(x) = 0$ (2.19). For more details, see the following remark.

**Remark 3.2.** The change of the dependent variable and parameters, which transforms $\mathcal{E}_3 f(x) = 0$ into $H_3 g(x) = 0$, is given as follows:

\[
g(x) = x^{\nu - \alpha} f(x), \quad q^{-\nu} = B/A, \quad q_i^{-1/2}t_i = b_i/A, \quad q_i^{b_i+1/2}t_i = a_i/B.
\] (3.19)

The general solution of $(1 - q^{-2}T_x)(B - Aq^{-1}T_x)f(x) = 0$ is $C_1 x^2 + C_2 x^{\lambda + 1}$, where $C_1, C_2$ are pseudo-constants and $B/A = q^{\lambda}$. If a pseudo-constant $C$ is holomorphic at $x = 0$ or $x = \infty$, then $C$ is a constant. Thus we have

\[
\mathcal{E}_3 \varphi(x, \tau) = C_1 x^2 + C_2 x^{\lambda + 1},
\] (3.20)

where $C_1, C_2$ are constants. Therefore we should calculate these constants $C_1, C_2$. Lemma 3.2 is useful for calculating them.

**Lemma 3.2.** We have

\[
\lim_{z \to 1} (1 - z)^3 \psi_3 \left( \begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2, b_3 \end{array} ; z \right) = \left( \begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2, b_3 \end{array} \right)_{\infty}.
\] (3.21)

Here, $3\psi_3$ is the bilateral $q$-hypergeometric function

\[
3\psi_3 \left( \begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2, b_3 \end{array} ; z \right) = \sum_{n \in \mathbb{Z}} \left( \begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2, b_3 \end{array} \right)_n z^n.
\] (3.22)
Proof. The negative power part of the function $3\psi_3$ is holomorphic at $z = 1$, thus we have

$$\lim_{z \to 1} (1 - z)3\psi_3 \left( \frac{a_1, a_2, a_3}{b_1, b_2, b_3}; z \right) = \lim_{z \to 1} (1 - z) \sum_{n=0}^{\infty} \frac{(a_1, a_2, a_3)_n}{(b_1, b_2, b_3)_n} z^n. \quad (3.23)$$

Using the q-binomial theorem, we obtain

$$\sum_{n=0}^{\infty} \frac{(a_1, a_2, a_3)_n}{(b_1, b_2, b_3)_n} z^n = (a_1, a_2, a_3)_\infty \sum_{n=0}^{\infty} \frac{(b_1 q^n, b_2 q^n, b_3 q^n)_\infty z^n}{(a_1 q^n, a_2 q^n, a_3 q^n)_\infty} \quad (3.24)$$

Therefore we get the desired equation (3.21). \hfill \Box

Lemma 3.3. Suppose $\tau \in \{q/a_1, q/a_2, q/a_3, q/(Ax)\}$ and $B/A \notin q\mathbb{Z}$. Then the Jackson integral of Jordan-Pochhammer type (3.3) satisfies the equation $\mathcal{E}_3\varphi = (1 - q)\varphi(A - B)x^2$.

Proof. For $\tau \in \{q/a_1, q/a_2, q/a_3\}$, the integral \((3.3)\) becomes

$$\varphi(x, \tau) = \int_0^x \frac{(Ax t)_\infty (a_1 t, a_2 t, a_3 t)_\infty q^t}{(Bx t)_\infty (b_1 t, b_2 t, b_3 t)_\infty t} \quad (3.25)$$

Thus we can apply the q-binomial theorem to $\frac{(Ax t q^n)_\infty}{(Bx t q^n)_\infty} \frac{(a_1 t q^n, a_2 t q^n, a_3 t q^n)_\infty}{t^n}$ if $|Bx\tau| < 1$. Using the q-binomial theorem, we have

$$\varphi(x, \tau) = (1 - q) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(A/B)_m (Bx t q^n)_m (a_1 t q^n, a_2 t q^n, a_3 t q^n)_\infty}{(b_1 t q^n, b_2 t q^n, b_3 t q^n)_\infty t^n}. \quad (3.26)$$

Hence $\mathcal{E}_3\varphi(x, \tau)$ can be written as $\mathcal{E}_3\varphi(x, \tau) = \sum_{m=0}^{\infty} \alpha_m x^m$. By (3.20) and the condition $Bq/A \notin q\mathbb{Z}$, we obtain $\mathcal{E}_3\varphi(x, \tau) = a_2 z^2$. By direct calculations, we have

$$\varphi(x, \tau) = \left(1 - q\right) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(A/B)_m (Bx t q^n)_m (a_1 t q^n, a_2 t q^n, a_3 t q^n)_\infty}{(b_1 t q^n, b_2 t q^n, b_3 t q^n)_\infty t^n}. \quad (3.27)$$

The function $3\psi_3 \left( \frac{b_1 \tau, b_2 \tau, b_3 \tau}{a_1 \tau, a_2 \tau, a_3 \tau}; z \right)$ satisfies the q-difference equation

$$[(1 - a_1 \tau q^{-1} T_z)(1 - a_2 \tau q^{-1} T_z)(1 - a_3 \tau q^{-1} T_z) - z(1 - b_1 \tau T_z)(1 - b_2 \tau T_z)(1 - b_3 \tau T_z)]3\psi_3 = 0. \quad (3.28)$$
This equation can be rewritten as
\[
\sum_{k=0}^{3} \left( \frac{-\tau}{q} \right)^{k} e_{k}(a) - \left( -\tau \right)^{k} e_{k}(b) \right) T_{x,3}^{k} \psi_{3} = 0. \tag{3.29}
\]
Therefore we have
\[
\sum_{k=0}^{3} \left( \frac{-\tau}{q} \right)^{k} e_{k}(a) - \left( -\tau \right)^{k} e_{k}(b) \right) \alpha_{3}(\beta_{x,3}, \alpha_{1,2}) = \psi_{3}(\beta_{x,3}, \alpha_{1,2}, \alpha_{3,3} ; z) = \lim_{z \to 1} (1 - z)_{3} \psi_{3} \left( \frac{b_{1,1}, b_{2,2}, b_{3,3}}{a_{1,1}, a_{2,1}, a_{3,3}} ; z \right). \tag{3.30}
\]

Using Lemma 3.2, we get \( \alpha_{2} = (1 - q)q(A - B) \).

Next, we consider the case \( \tau = q/(Ax) \). By changing the variable \( t \rightarrow qt/A \), the integral (3.30) is rewritten as
\[
\varphi(x, \tau) = \frac{q}{Ax} \sum_{n_{1}, n_{2}, n_{3} = 0}^{\infty} \int_{0}^{1} t^{1+n_{1}+n_{2}+n_{3}} \frac{(qt)_{\infty}}{(qBt/A)_{\infty}} \prod_{i=1}^{3} \left( \frac{a_{i}/b_{i}}{q} \right)_{n_{i}} \frac{d_{i}t}{t} x^{-n_{1}-n_{2}-n_{3}}. \tag{3.31}
\]
Applying the \( q \)-binomial theorem to \( \frac{(qt)_{\infty}}{(qBt/A)_{\infty}} \), we get
\[
\varphi(x, \tau) = \frac{q}{Ax} \sum_{n_{1}, n_{2}, n_{3} = 0}^{\infty} \int_{0}^{1} t^{1+n_{1}+n_{2}+n_{3}} \frac{(qt)_{\infty}}{(qBt/A)_{\infty}} \prod_{i=1}^{3} \left( \frac{a_{i}/b_{i}}{q} \right)_{n_{i}} \frac{d_{i}t}{t} x^{-n_{1}-n_{2}-n_{3}}. \tag{3.32}
\]
Thus we find that \( \mathcal{E}_{3} \varphi(x, \tau) = \sum_{m=0}^{\infty} \beta_{m} x^{2-m} \). Similar to the case \( \tau \in \{ q/a_{1}, q/a_{2}, q/a_{3} \} \), we have \( \mathcal{E}_{3} \varphi(x, \tau) = \beta_{0} x^{2} \). From (3.18) and (3.32), we get
\[
\beta_{0} = (B - Aq^{-1})(B - A)q \frac{q}{A} \int_{0}^{1} \frac{(qt)_{\infty}}{(qBt/A)_{\infty}} \frac{d_{t}}{t} = q(1 - q)(A - B). \tag{3.33}
\]
This completes the proof.

We can consider the above discussion when replacing the integrand \( \psi \) with \( \psi C \), where \( C \) is some pseudo-constant. The function
\[
C(t) = t^{\alpha - \beta} \frac{\theta(q^{2}t)}{\theta(q^{2}t)}, \tag{3.34}
\]
is a pseudo-constant, that is, \( T_{t}C(t) = C(t) \). Here, \( \theta(t) = (t, q/t)_{\infty} \). Thus the function
\[
\tilde{C}(x,t) = x^{\lambda} t^{-2} \frac{\theta(Bxt)}{\theta(Axt)} \frac{\theta(b_{1}t)}{\theta(a_{1}t)} \frac{\theta(b_{2}t)}{\theta(a_{2}t)} \frac{\theta(b_{3}t)}{\theta(a_{3}t)} \tag{3.35}
\]
is a pseudo-constant for \( x \) and \( t \). Therefore we find that the integral
\[
\tilde{\varphi}(x, \sigma) = \int_{0}^{\sigma^{-1}} \psi(x,t) \times \tilde{C}(x,t) \frac{d_{t}}{t} = x^{\lambda} \int_{0}^{\sigma^{-1}} \tilde{\psi}(x,s) \frac{d_{s}}{s}, \tag{3.36}
\]
also satisfies the equation (3.19). The following lemma can be proved in the same way as for Lemma 3.3, so we do not prove it here.
Lemma 3.4. Suppose \( \sigma \in \{b_1, b_2, b_3, Bx\} \) and \( B/A \notin \mathbb{Z} \). Then the integral (3.36) satisfies the equation \( E_3 x(\sigma) = (1 - q)^{-1}(B - A)x^{\lambda + 1} \).

By taking the difference between \( \varphi(x, \tau_1) \) and \( \varphi(x, \tau_2) \), or between \( \tilde{\varphi}(x, \sigma_1) \) and \( \tilde{\varphi}(x, \sigma_2) \), we obtain integral solutions for the equation \( E_3 f(x) = 0 \).

Theorem 3.1. Let \( \tau_1, \tau_2 \in \{q/a_1, q/a_2, q/a_3, q/(Ax)\} \) and \( \sigma_1, \sigma_2 \in \{b_1, b_2, b_3, Bx\} \). We suppose \( q^\lambda = B/A \notin q^\mathbb{Z} \) and \( Aa_1a_2a_3 = q^2Bb_1b_2b_3 \). Then the integrals

\[
\varphi_3(x, \tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \frac{(Axt, a_1t, a_2t, a_3t)\infty}{(Bxt, b_1t, b_2t, b_3t)\infty} dq_t,
\]

(3.38)

\[
\tilde{\varphi}_3(x, \sigma_1, \sigma_2) = x^\lambda \int_{\sigma_1}^{\sigma_2} \frac{(qs/(Ax), qsa_1, qsa_2, qsa_3)\infty}{(qs/(Ax), qsa_1, qsa_2, qsa_3)\infty} dq_s,
\]

(3.39)

satisfy the equation \( E_3 f(x) = 0 \) \( (3.38) \).

Remark 3.3. By the \( q \)-binomial theorem, these integrals are \( q \)-analogues of the integral (1.7), more precisely, by taking the limit \( q \to 1 \), we have

\[
\int_{\tau_1}^{\tau_2} \frac{(Axt, a_1t, a_2t, a_3t)\infty}{(Bxt, b_1t, b_2t, b_3t)\infty} dq_t \to \int_{\tau_1}^{\tau_2} (1 - t)\nu(1 - tt_1)\nu + t_1 - h_1 - 1(1 - tt_2)\nu + t_2 - h_2 - 1(1 - tt_3)\nu + t_3 - h_3 - 1 dt,
\]

(3.40)

with the change of parameters \( (3.19) \). Here, \( \tau_i \to \tau'_i \).

By considering the case \( M = 2 \) in Proposition \( (3.1) \), we obtain the integral solutions for the variant of the \( q \)-hypergeometric equation of degree two. In the same way as for the case \( M = 3 \), we have the following theorem.

Theorem 3.2. Let \( \tau_1, \tau_2 \in \{0, q/a_1, q/a_2, q/(Ax)\} \) and \( \sigma_1, \sigma_2 \in \{b_1, b_2, Bx, \sigma\infty\} \), where, \( \sigma \) is an arbitrary constant. We suppose \( q^\lambda = B/A \notin q^\mathbb{Z}, q^{\alpha + 1}B/A \notin q^\mathbb{Z} \) and \( Aa_1a_2 = q^{\alpha + 1}Bb_1b_2 \). Then the integrals

\[
\varphi_2(x, \tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} t^{\alpha} \frac{(Axt, a_1t, a_2t)\infty}{(Bxt, b_1t, b_2t)\infty} dq_t,
\]

(3.41)

\[
\tilde{\varphi}_2(x, \sigma_1, \sigma_2) = x^\lambda \int_{\sigma_1}^{\sigma_2} \frac{(qs/(Ax), qsa_1, qsa_2)\infty}{(qs/(Ax), qsa_1, qsa_2)\infty} dq_s,
\]

(3.42)

satisfy the equation \( E_2 f(x) = 0 \), where

\[
E_2 = [x^2(1 - q^\alpha T_x)(B - AT_x) - x(e_1(a) - q^\alpha e_1(b)T_x)(1 - T_x) + e_2(a)B^{-1}(1 - q^{-1}T_x)(1 - T_x)]T_x^{-1}.
\]

(3.43)

Remark 3.4. The configuration of the equation \( E_2 f(x) = 0 \) is given as follows.
Therefore the change of the dependent variable and parameters, which transforms $E_2 f(x) = 0$ into $H_2 g(x) = 0$, is given as follows:

$$g(x) = x^{\lambda_0} f(x), \quad \alpha = \lambda_0 + \alpha_1, \quad B/A = q^{\alpha_2 - \lambda_0}, \quad a_i/B = q^{b_i + 1/2} t_i, \quad b_i/A = q^{b_i - 1/2} t_i. \quad (3.44)$$

In [9], it was shown that the function

$$g_1(x) = x^{-\alpha_1} \Phi^{(1)} \left( \frac{q^{\lambda_0 + \alpha_1}; q^{\lambda_0 + \alpha_1 - b_2 + b_2}; q^{\lambda_0 + \alpha_1 - b_1 + b_1}}{q^{\alpha_1 - \alpha_2 + 1}; q^{1 + 1/2} x; q^{2 + 1/2}} \right), \quad (3.45)$$

satisfies the equation $H_2 g_1(x) = 0$, where $\Phi^{(1)}$ is the $q$-Appell hypergeometric series given by

$$\Phi^{(1)} \left( \frac{a; b_1, b_2}{c}; x_1, x_2 \right) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{(a)_{n_1 + n_2} (b_1)_{n_1} (b_2)_{n_2}}{(c)_{n_1 + n_2} (q)_{n_1} (q)_{n_2}} x_1^{n_1} x_2^{n_2}. \quad (3.46)$$

By using Andrews’s formula [2]

$$\int_0^1 t^{\alpha} (q t, b_1 x t, b_2 x t)^{\infty} \frac{d q t}{t} = (1 - q) \frac{(q, c)^{\infty}}{(q^{a_1}, c/q^{a_1})^{\infty}} \Phi^{(1)} \left( \frac{q^{a_1}; b_1, b_2}{c}; x_1, x_2 \right), \quad (3.47)$$

the solution $g_1(x)$ can be obtained from Theorem 3.2. More precisely, we have

$$g_1(x) = \frac{1}{1 - q} \frac{(q^{\lambda_0 + \alpha_1}; q^{\alpha_2 + 1 - \lambda_0})}{(q, q^{\alpha_1 + 1})^{\infty}} \left( \frac{A}{q} \right)^{\lambda_0 + \alpha_1} x^{\lambda_0} \varphi_2(x, 0, q/(Ax)), \quad (3.48)$$

with the change of parameters (3.44).

**Remark 3.5.** We can also derive Theorem 3.2 by taking a limit in Theorem 3.1. We put $b_3 = q^{\alpha_3 - 1} a_3$ and consider the limit $a_3 \to \infty$. In the same way as for Remark 2.7, the equation $E_3 f(x) = 0$ becomes $E_2 f(x) = 0$ by this limit. Also we find easily that the integrand of the integral $\tilde{\varphi}_3 (3.39)$ becomes the integrand of $\tilde{\varphi}_2 (3.42)$ by the same limit. By multiplying the integrand of $\tilde{\varphi}_2 (3.42)$ by some pseudo-constant, we get the integral (3.41).

We consider the linear independence of the integral solutions. If functions $f(x)$ and $g(x)$ are linear dependent over the field of pseudo-constants $K = \{ C(x) \mid C(qx) = C(x) \}$ for general parameters, then $f(x)$ and $g(x)$ with special parameters are linearly dependent too. Therefore it is enough to consider linear independence for the integrals in some special case.
Proposition 3.2. Suppose \( a_i \neq a_j, \ b_i \neq b_j \) (\( i \neq j \)).

1. Let \( \tau_i \in \{q/a_1, q/a_2, q/a_3, q/(Ax)\} \) for \( i = 1, \ldots, 4 \). If \( \{\tau_1, \tau_2\} \neq \{\tau_3, \tau_4\} \), then the integrals \( \varphi_3(x, \tau_1, \tau_2) \) and \( \varphi_3(x, \tau_3, \tau_4) \), defined by (3.39), are linearly independent over the field of pseudo-constants \( K \).

2. Let \( \sigma_i \in \{b_1, b_2, b_3, Bx\} \) for \( i = 1, \ldots, 4 \). If \( \{\sigma_1, \sigma_2\} \neq \{\sigma_3, \sigma_4\} \), then the integrals \( \tilde{\varphi}_3(x, \sigma_1, \sigma_2) \) and \( \tilde{\varphi}_3(x, \sigma_3, \sigma_4) \), defined by (3.39), are linearly independent over \( K \).

3. Let \( a_0 = Ax \) and \( b_0 = Bx \). If \( \{i, j\} \neq \{k, l\} \) (\( 0 \leq i, j, k, l \leq 3 \)), then the integrals \( \varphi_3(x, q/a_i, q/a_j) \) and \( \tilde{\varphi}_3(x, b_k, b_l) \) are linearly independent over \( K \).

Proof. We consider the special case \( a_i = b_i \) for \( i = 1, 2, 3 \). Because of the condition \( Aa_1a_2a_3 = q^2b_1b_2b_3 \), we have \( A = q^2B \). By a simple calculation, we have

\[
\varphi_3(x, \tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \frac{(q^2Bx)^{\infty}q^t}{(Bx)^{\infty}} dt = \frac{(1-q)(\tau_2-\tau_1)}{(1-Bx\tau_1)(1-Bx\tau_2)}, \tag{3.49}
\]

\[
\tilde{\varphi}_3(x, \sigma_1, \sigma_2) = x^{-2} \int_{\sigma_1}^{\sigma_2} \frac{(qs/(Bx))^{\infty}q^s}{(qs/(q^2Bx))^{\infty}} ds = \frac{(1-q)(qB)^2(\sigma_2-\sigma_1)}{(qBx-\sigma_1)(qBx-\sigma_2)}. \tag{3.50}
\]

We prove only the case (1).

We assume \( C_1(x)\varphi_3(x, \tau_1, \tau_2) + C_2(x)\varphi_3(x, \tau_3, \tau_4) = 0 \), where \( C_1(x), C_2(x) \in K \). By (3.49) and the conditions for \( \tau_i \), the function \( \varphi_3(x, \tau_1, \tau_2)/\varphi_3(x, \tau_3, \tau_4) \) is a non-constant rational function. Since a non-constant rational function is not a pseudo-constant, we get \( C_1 = C_2 = 0 \).

The cases (2) and (3) are proved by the same way as the case (1). More precisely, (2) is proved by considering the ratio of the integrals (3.50), and (3) is proved by considering the ratio of the integrals (3.49) and (3.50).

\[ \square \]

Remark 3.6. In (3) of Proposition 3.2, we consider the case \( (i, j) = (k, l) \). Then the ratio of integrals is a constant. Therefore we cannot determine whether the integrals \( \varphi_3(x, q/a_i, q/a_j) \) and \( \tilde{\varphi}_3(x, b_k, b_l) \) are linearly independent by the above method.

4 Series solutions

In this section, we present solutions for the variant of the \( q \)-hypergeometric solutions of degree three by the very-well-poised-balanced \( q \)-hypergeometric series \( _8\!W_7 \). In section 3 we obtain integral solutions for variants of the \( q \)-hypergeometric equation. The \( _8\!W_7 \) solutions are obtained by transforming the integral. In this section, we use some formulas for the \( q \)-hypergeometric series without proofs. Those formulas are summarized in [6].

Definition 4.1 ([6], (2.1.11)). The very-well-poised \( q \)-hypergeometric series is defined as follows:

\[
_{r+1}W_r(a_1; a_4, \ldots, a_{r+1}; z) = \sum_{n=0}^{\infty} \frac{1-a_1q^{2n}}{1-a_1} \frac{(a_1, a_4, \ldots, a_{r+1})_n}{(q, qa_1/a_4, \ldots, qa_1/a_{r+1})_n} z^n. \tag{4.1}
\]

This series converges for \( |z| < 1 \), and is called balanced if \( z = \frac{(\pm(a_1q)^{1/2})^{r-3}}{a_4a_5\cdots a_{r+1}} \).

Remark 4.1. The very-well-poised \( q \)-hypergeometric series \( _{r+1}W_r \) is rewritten as follows:

\[
_{r+1}W_r(a_1; a_4, \ldots, a_{r+1}; z) = _{r+1}\varphi_r \begin{pmatrix} a_1, qa_1^{1/2} - qa_1^{1/2}, a_4, \ldots, a_{r+1}^{1/2} \cr a_1^{1/2}, -a_1^{1/2}, qa_1/a_4, \ldots, qa_1/a_{r+1} \end{pmatrix} ; z. \tag{4.2}
\]
We introduce some properties of the very-well-poised-balanced $\text{$_8W_7$}$ series without proofs. For proofs and more details, see section 2 of [6].

**Lemma 4.1** ([6], (2.10.19)). If $cd = abcdegh$ and $|ah| < 1$, then we have

\[
\int_a^b \frac{(qt/a, qt/b, ct, dt)}{(et, ft, gt, ht)} \, dq d\t
= b(1 - q) \frac{(q, bq/a, a/b, cd/(eh), cd/(fh), cd/(gh), be, bd)}{(ae, af, ag, be, bf, bg, bh, bcd/h)} \times \text{$_8W_7$} \left( \frac{bce, be, bg, c - d}{ht}; ah \right).
\]

(4.3)

**Lemma 4.2** ([6], (2.10.1)). Let $\mu = qa^2/(bcd)$. We have

\[
\text{$_8W_7$} \left( \frac{a^2q^2}{bcde} \right) = \frac{(aq, ag/(ef), \mu q/e, \mu q/f)}{(aq/e, aq/f, \mu q/(ef))} \times \text{$_8W_7$} \left( \frac{\mu b, \mu c, \mu d}{a}, e, f; \frac{a}{q} \right). \quad (4.4)
\]

if $\max(|aq/(ef)|, |\mu q/(ef)|) < 1$.

By applying Lemma 4.1 to the integral (3.38), we obtain series solutions for the equation (3.18).

**Theorem 4.1.** If $a_1a_2a_3A = q^2b_1b_2b_3B$, then the functions

\[
\frac{\text{$_{Ax}/(a_2)$}}{(Bx/ax)^a_2} \text{$_{W_7}$} \left( \frac{a_3A, qBx}{Bx/a_2}, b_2, \frac{qBx}{Bx/a_2}, a_3, A, qBx \right),
\]

(4.5)

\[
\frac{\text{$_{Ax}/(a_2)$}}{(Bx/ax)^a_2} \text{$_{W_7}$} \left( \frac{a_2A, qBx}{Bx/a_2}, b_2, \frac{qBx}{Bx/a_2}, a_2, Bx/a_1 \right),
\]

(4.6)

\[
\frac{1}{x (\text{$_{Ax}/(a_2)$})} \times \text{$_{W_7}$} \left( \frac{a_2A, qBx}{Bx/a_2}, b_2, \frac{qBx}{Bx/a_2}, a_2, Bx/a_1 \right).
\]

(4.7)

satisfy the equation $E_3f(x) = 0$ (3.18).

**Remark 4.2.** The series (4.5) and (4.6), (4.7) and (4.8), (4.9) and (4.10) correspond to the integrals $\varphi_3(x, q/a_1, q/a_2)$, $\varphi_3(x, q/a_1, q/(Ax))$, $\varphi_3(x, q/(Ax), q/a_1)$, respectively. The right-hand-side of the equation (4.3) in Lemma 4.1 is symmetric for $e$, $f$, $g$ and also for $c$, $d$. Thus the series representations for the integrals changes depending essentially only on the choices of $a$, $b$, $h$. In this case, we have two or four representations for each integral.

The solutions that correspond to the integral $\varphi_3$ are obtained by acting with $s_2$, defined in (4.18) below, to the solutions in Theorem 4.1.
Remark 4.3. The very-well-poised-balanced $q$-hypergeometric series $sW_7$ is called the Askey-Wilson function $[15]$. Well known linear $q$-difference equations satisfied by the very-well-poised-balanced $sW_7$ are the ones considered by Ismail and Rahman $[10]$. We put $\phi = sW_7 \left( a; b, c, d, e, f; \frac{(aq)^2}{bcdef} \right)$, and $T_1 = T_0 T_s^{-1}$, $T_2 = T_0^a T_c T_d T_s T_f$. In $[10]$, linear relations for $T_1 \phi$, $T_1^{-1} \phi$, and for $T_2 \phi$, $T_2^{-1} \phi$ were obtained. These equations are as follows:

\[
\left[ (b - c) \left( 1 - \frac{aq}{bc} \right) \left( 1 - \frac{a^2 q^2}{bcdef} \right) + L + M \right] \phi = \left( \frac{1 - c}{1 - b} \right) (1 - T_1^{-1} \phi + \frac{(1 - b)(1 - a/b)}{(1 - c)(1 - aq/c)} T_1 \phi, \right.
\]

\[
\left. \frac{(1 - a)(1 - b)}{(1 - a/b)(1 - aq/b)} \phi + (1 - a/b)(1 - aq/b)bcdef \right)
\]

\[
= \left( \frac{(1 - a)(1 - d)(1 - a/e)(1 - a/f)}{(1 - a/b)(1 - aq/b)} \phi \right)
\]

\[
+ \frac{a^2 (1 - c)(1 - d)(1 - e)(1 - f)q}{(1 - a/b)(1 - aq/b)bcdef} + (1 - b) \left( 1 - \frac{a^2 q}{bcdef} \right) \right) \phi
\]

\[
= \frac{(1 - a)(1 - d)(1 - e)(1 - f)(aq)_2 (1 - aq/(bd))(1 - aq/(be))(1 - aq/(bf))(1 - aq/(cf))}{(a/b)(aq/b)_2 (1 - aq/d)(1 - aq/e)(1 - aq/f)} \phi \]

\[
T_2 \phi,
\]

(4.11)

where

\[
L = \frac{c(1 - c/q)(1 - aq/(cd))(1 - aq/(ce))(1 - aq/(cf))}{b - c/q},
\]

\[
M = \frac{b(1 - b/q)(1 - aq/(bd))(1 - aq/(be))(1 - aq/(bf))}{c - b/q}.
\]

(4.12)

(4.13)

(4.14)

On the other hand, we find that the function $sW_7$ satisfies the equation $H_3 f(x) = 0$ under some gauge transformation. Our result gives linear relations for $T_1 \phi$, $T_1^{-1} \phi$, and for $T_2 \phi$, $T_2^{-1} \phi$, and $T_3 \phi$, $T_3^{-1} \phi$. Here, $T_3 = T_0$, $T_4 = T_0 T_c T_s$, and $T_5 = T_0^a T_c T_s T_d T_s T_f$. Therefore we find that the equation $H_3 f(x) = 0$ can be regarded as a contiguity relation for the Askey-Wilson function $sW_7$. In $[9]$, it was mentioned that some solution for the equation $H_2 f(x) = 0$ is related to the big $q$-Jacobi polynomial. The big $q$-Jacobi polynomial can be obtained from the Askey-Wilson polynomial by taking some limit. In addition, the little $q$-Jacobi polynomial, which is related to the Heine’s $q$-hypergeometric equation, can be obtained from the big $q$-Jacobi polynomial. Degenerations of orthogonal polynomials are summarized in the Askey scheme $[10]$. We find that the degeneration $H_3 \rightarrow H_2 \rightarrow H_1$ is related to (a part of) the Askey scheme.

Remark 4.4. Taking the limit $q \to 1$, we have

\[
\frac{(q^a X)_\infty}{(q^b X)_\infty} \to (1 - X)^{3 - \alpha}, \quad (q^a X)_n \to (1 - X)^n, \quad \frac{(q^a)_n}{(1 - q)^n} \to \alpha(\alpha + 1) \cdots (\alpha + n - 1).
\]

(4.15)

Thus, by taking $q \to 1$ with the change of parameters $3.19$, some of the solutions in Theorem 4.1 become the Gauss hypergeometric function:

\[
x^{\nu - \alpha}(x - t_1)^{\mu_1}(x - t_2)^{\mu_2}(x - t_3)^{\mu_3} F_1 \left( \begin{array}{c} \nu_1, \nu_2, \nu_3 \\ \nu_4 \end{array} \right) \frac{x - t_4, t_k - t_j}{x - t_4, t_k - t_i}.
\]

(4.16)

Here, $\mu_i, \nu_j$ are suitable parameters determined by $\nu, \alpha, l_i, h_i$. In this sense, some of the solutions in Theorem 4.1 are $q$-analogos of series solutions of Riemann-Papperitz equation $[13]$.  

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The equation $\mathcal{E}_3 f(x) = 0$ has some symmetries. We recall the configuration of $\mathcal{E}_3 f(x) = 0$, that is

$$
\begin{array}{ccc}
& b_1/A & b_2/A & b_3/A \\
T_x = \infty & \bullet & \bullet & \bullet \\
\end{array}
$$

$$
\begin{array}{ccc}
1, q & & \odot B/A, Bq^{-1}/A \\
\end{array}
$$

$$
\begin{array}{ccc}
& a_1/B & a_2/B & a_3/B \\
T_x = 0 & \bullet & \bullet & \bullet \\
x = 0 & x = \infty \\
\end{array}
$$

By the gauge transformation $g(x) = \frac{(qBx/a_1)_{\infty}}{(Ax/b_1)_{\infty}} f(x)$, this configuration is transformed to the following configuration:

$$
\begin{array}{ccc}
& a_1/(qB) & b_2/A & b_3/A \\
T_x = \infty & \bullet & \bullet & \bullet \\
\end{array}
$$

$$
\begin{array}{ccc}
1, q & & \odot a_1/(q b_1), a_1/(q^2 b_1) \\
\end{array}
$$

$$
\begin{array}{ccc}
& q b_1/A & a_2/B & a_3/B \\
T_x = 0 & \bullet & \bullet & \bullet \\
x = 0 & x = \infty \\
\end{array}
$$

Thus, if a function $f(a_1, a_2, a_3, b_1, b_2, b_3, A, B, x)$ is a solution of $\mathcal{E}_3 f(x) = 0$, then the function

$$
\frac{(Ax/b_1)_{\infty}}{(qBx/a_1)_{\infty}} f \left( a_1, a_2, a_1 A, a_2 A, a_3 A, b_2, b_3, A, \frac{a_1}{q b_1} A, x \right)
$$

satisfies the same equation. Also by the transformation $\{g(x) = x^{-\lambda} f(x), \ z = x^{-1}\}$, then we get the configuration
Thus the function $x^3 f\left( \frac{AB}{b_1}, \frac{AB}{b_2}, \frac{AB}{b_3}, a_1, a_2, a_3; A, B, \frac{1}{x} \right)$ satisfies the same equation. In addition, the equation $E_3 f(x) = 0$ is symmetric for $a_1, a_2, a_3$, and for $b_1, b_2, b_3$. Therefore the function $f(a_1, a_2, a_3, b_1, b_2, b_3, A, B, x)$ also satisfies $E_3 f(x) = 0$, where $(i, j, k)$ and $(i', j', k')$ are permutations of $(1, 2, 3)$. By acting by group $G_3 = \{s_i \mid 1 \leq i \leq 6\}$ to the solutions in Theorem 4.1 we obtain many solutions. Here,

\begin{align*}
\begin{array}{l}
s_1: (a_1, a_2, a_3, b_1, b_2, b_3, A, B, x) \mapsto \left( a_1, a_2, a_2, a_2, a_1, a_2, b_2, b_3, A, \frac{a_1}{q b_1} A, x \right), \\
\quad \tag{4.17}
\end{array}
\end{align*}

\begin{align*}
\begin{array}{l}
s_2: (a_1, a_2, a_3, b_1, b_2, b_3, A, B, x) \mapsto \left( \frac{AB}{b_1}, \frac{AB}{b_2}, \frac{AB}{b_3}, \frac{AB}{a_2}, \frac{AB}{a_3}, A, \frac{1}{x} \right), \\
\quad \tag{4.18}
\end{array}
\end{align*}

\begin{align*}
\begin{array}{l}
s_3: a_1 \leftrightarrow a_2, s_4: a_2 \leftrightarrow a_3, s_5: b_1 \leftrightarrow b_2, s_6: b_2 \leftrightarrow b_3. \\
\quad \tag{4.19}
\end{array}
\end{align*}

The relations among the $\{s_i\}$ are as follows:

\begin{align*}
\begin{array}{l}
s_i^2 = \text{id} \quad (i = 1, \ldots, 6), \\
(s_1 s_2)^2 = (s_1 s_4)^2 = (s_1 s_6)^2 = (s_1 s_3)^3 = (s_1 s_5)^3 = \text{id}, \\
(s_3 s_4)^2 = (s_3 s_6)^2 = (s_4 s_6)^2 = (s_4 s_5)^3 = (s_5 s_6)^3 = \text{id}, \\
(s_2 s_3)^4 = (s_2 s_4)^4 = (s_2 s_5)^4 = (s_2 s_6)^4 = \text{id}, \\
\quad \tag{4.20}
\end{array}
\end{align*}

\begin{align*}
\begin{array}{l}
s_2 s_3 s_5 s_2 s_4 s_2 s_6 = \text{id}. \\
\quad \tag{4.21}
\end{array}
\end{align*}

We find $G_3 = \{s_1, s_3, s_4, s_5, s_6\} \ltimes (s_2) \simeq \mathfrak{S}_6 \ltimes \mathfrak{S}_2$.

For the same reason as in Remark 3.5 we can obtain series solutions for $E_2 f(x) = 0$ by taking the limit $a_3 \to \infty$ with $b_3 = q^{a-1} a_3$. Thus we consider the limit of series solutions for $E_3 f(x) = 0$. The limit of the very-well-poised-balanced $q$-hypergeometric series $\sum W_{7}$ is expressed by the $q$-hypergeometric series $\sum \varphi_2$ as follows:

\begin{align*}
\begin{array}{l}
\lim_{t \to \infty} \sum_{W_{7}} \left( a; bl, cl, d, e, f; \frac{(alq)^2}{(bl)(cl)def} \right) = 3\varphi_2 \left( d, e, f; \frac{aq}{bcdef} \right), \\
\quad \tag{4.25}
\end{array}
\end{align*}

\begin{align*}
\begin{array}{l}
\lim_{t \to 0} \sum_{W_{7}} \left( a; bl, cl, d, e, f; \frac{(alq)^2}{(bl)(cl)def} \right) = 3\varphi_2 \left( d, e, f; \frac{aq}{bcdef} \right). \\
\quad \tag{4.26}
\end{array}
\end{align*}

Using the transformation [4.4], we have

\begin{align*}
\begin{array}{l}
\lim_{b \to \infty} \sum_{W_{7}} \left( a; b, c, d, e, f; \frac{(aq)^2}{bcdef} \right) = \lim_{b \to \infty} \left( \sum_{q} \frac{(aq, aq/(ef), \mu q/e, \mu q/f)}{\infty} \sum_{W_{7}} \left( \mu; \frac{\mu b}{a}, \frac{\mu c}{a}, \frac{\mu d}{a}, e, f; \frac{aq}{ef} \right) \right).
\quad \tag{4.24}
\end{array}
\end{align*}
and $b$ as the limits of the series (4.5). Thus we only show the results. We have some series solutions 

If Theorem 4.2.

\[
\lim_{b \to 0} \left( \frac{\mu a}{\mu b}, \frac{\mu a}{\mu c}, \frac{\mu d}{\mu e}, f, \frac{aq}{ef} \right) = \lim_{b \to 0} \left( \frac{a q/a}{b c d e f} \right)
\]

\[
= (aq/e, aq/f)_{\infty} 3 \frac{\varphi_2}{(aq/e, aq/f)_{\infty}} \frac{aq/(cd), e, f}{aq/e, aq/d, \varphi}, \frac{aq}{ef}
\]

\[
= \lim_{t \to \infty} \left( \frac{aq/ef, (aq)^2/(bcde), (aq)^2/(bcdf)}{\infty} \right)_{\infty} W_7 \left( \frac{\mu b}{a}, \frac{\mu c}{a}, \frac{\mu d}{a}, e, f, \frac{aq}{ef} \right)
\]

\[
= (aq/e, aq/f)_{\infty} 3 \frac{\varphi_2}{(aq/e, aq/f)_{\infty}} \frac{aq/(cd), e, f}{aq/e, aq/d, \varphi}, \frac{aq}{ef}
\]

\[
= \lim_{t \to \infty} \left( \frac{aq/ef, (aq)^2/(bcde), (aq)^2/(bcdf)}{\infty} \right)_{\infty} W_7 \left( \frac{\mu b}{a}, \frac{\mu c}{a}, \frac{\mu d}{a}, e, f, \frac{aq}{ef} \right)
\]

\[
= (aq/e, aq/f)_{\infty} 3 \frac{\varphi_2}{(aq/e, aq/f)_{\infty}} \frac{aq/(cd), e, f}{aq/e, aq/d, \varphi}, \frac{aq}{ef}
\]

This equation is obtained by 

\[
\lim_{a_1 \to \infty} \frac{(A q a / a_2)_{\infty}}{(B q a / a_2)_{\infty}} W_7 \left( \frac{a_3 a B}{a_2 B}, \frac{q b_1}{a_2 a_1}, \frac{q^2 a^3}{a_2 a_1}, \frac{a_3}{A}, \frac{A}{B}, \frac{A}{a_1} \right)
\]

\[
= (A q a / a_2)_{\infty} 3 \varphi_2 \left( \frac{q a^2}{A a_1, A a_1/(B a_1)}, \frac{q b_1}{a_2 a_1}, \frac{a_1}{a_2} \right)
\]

Here we use the relation $a_1 a_2 A = q^{a_1 + b_1} b_2 B$. This relation is obtained by $a_1 a_2 A = q^{b_1} b_2 b_3 B$ and $b_3 = q^{a_1 - a_3}$. The limits of the other series in Theorem 4.1 can be obtained in the same way as the limits of the series (4.1). Thus we only show the results. We have some series solutions for the equation $E_2 f(x) = 0$, which is expressed by $3 \varphi_2$.

**Theorem 4.2.** If $a_1 a_2 A = q^{a_1 + b_1} b_2 B$, then the functions

\[
3 \varphi_2 \left( \frac{q a^2, A/B, A_1/(B_1), A_1/(B_2)}{a_1/(B_1)}, \frac{q B x}{a_2} \right)
\]

\[
= \lim_{a_1 \to \infty} \frac{(B q x / a_2)_{\infty}}{(A q x / a_2)_{\infty}} 3 \varphi_2 \left( \frac{q b_1}{a_2 a_1}, \frac{A x / b_1}{a_2}, \frac{b_1}{b_2} \right)
\]

\[
= \lim_{a_1 \to \infty} \frac{(B q x / a_2)_{\infty}}{(A q x / a_2)_{\infty}} 3 \varphi_2 \left( \frac{q b_1}{a_2 a_1}, \frac{A x / b_1}{a_2}, \frac{b_1}{b_2} \right)
\]

\[
= \lim_{a_1 \to \infty} \frac{(A x / b_2)_{\infty}}{(b_1 B x / (a_1 a_2))_{\infty}} 3 \varphi_2 \left( \frac{q a_1}{a_1 a_2 / (b_1 B x), a_1 a_2 / (b_1 B x)}{a_2/(b_1 B x)}, \frac{q b_1}{A} \right)
\]

\[
= \lim_{a_1 \to \infty} \frac{(A x / b_2)_{\infty}}{(b_1 B x / (a_1 a_2))_{\infty}} 3 \varphi_2 \left( \frac{q a_1}{a_1 a_2 / (b_1 B x), a_1 a_2 / (b_1 B x)}{a_2/(b_1 B x)}, \frac{q b_1}{A} \right)
\]

\[
= 3 \varphi_2 \left( \frac{q a_1}{a_1 A/(b_1 B), a_2 A/(b_1 B)} \right)
\]

satisfy the equation $E_2 f(x) = 0$. 

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Remark 4.5. With the change (3.44), these solutions are \(q\)-analogs of the series (1.8) with \(t_3 = \infty\), i.e. \(q\)-analogs of the following series:

\[
(x - t_1)^{\mu_1} (x - t_2)^{\mu_2} {}_2F_1 \left( \frac{\nu_1, \nu_2}{\nu_3}; \frac{t_1 - t_2}{x - t_2} \right), \tag{4.38}
\]

\[
(x - t_1)^{\mu_1} (x - t_2)^{\mu_2} {}_2F_1 \left( \frac{\nu_1, \nu_2}{\nu_3}; \frac{x - t_2}{t_1 - t_2} \right), \tag{4.39}
\]

\[
(x - t_1)^{\mu_1} (x - t_2)^{\mu_2} {}_2F_1 \left( \frac{\nu_1, \nu_2}{\nu_3}; \frac{x - t_1}{x - t_2} \right). \tag{4.40}
\]

As is mentioned in section 1, it was shown in [9] that the function

\[
g_2(x) = x^{\lambda_0} \phi_2 \left( \frac{x}{(q^{l_1-1/2} t_1), q^{\lambda_0+\alpha_1}, q^{\lambda_0+\alpha_2}} ; q \right), \tag{4.41}
\]

satisfies the equation \(\mathcal{H}_2 g_2(x) = 0\). The solution \(g_2(x)\) corresponds to the series (4.37) with the change (3.44).

Remark 4.6. Some of the limits of the function \(8W_7\) are not \(q\)-analogs of \(2F_1\), but \(q\)-analogs of the integral (1.7). For example, we get a solution for \(E_2 f(x) = 0\):

\[
\frac{(qAx/a_2)_\infty}{(qBx/a_2)_\infty} q^2 \left( \frac{a_2}{q} \right) \int_0^{q/2} t^{\alpha-1} \left( \frac{Axt, a_1 t_1, a_2 t_2}{Bxt, b_1 t_1, b_2 t_2} \right)_\infty d_q t. \tag{4.42}
\]

by taking the limit for \(8W_7\). This solution is rewritten by the Jackson integral as follows:

\[
\frac{(qb_1/a_2, qb_2/a_2)_\infty}{(q, q a_1/a_2)_\infty (1 - q)} \left( \frac{a_2}{q} \right)^\alpha \int_0^{q/2} t^{\alpha-1} \left( \frac{Axt, a_1 t_1, a_2 t_2}{Bxt, b_1 t_1, b_2 t_2} \right)_\infty d_q t. \tag{4.43}
\]

Note that this integral was discussed in section 3 more precisely it corresponds to \(\varphi_2(x, 0, q/a_2)\).

Remark 4.7. We recall the configuration of \(E_2 f(x) = 0\):

\[
\begin{array}{ccc}
1, q & b_1/A & b_2/A \\
& & T_x = \infty \\
& 1/a & B/A \\
a_1/B & & a_2/B \\
& & T_x = 0 \\
x = 0 & & x = \infty
\end{array}
\]

Here, \(a = q^\alpha\). By the gauge transformation \(g(x) = \frac{(qBx/a_1)_\infty}{(Ax/b_1)_\infty} f(x)\), we get the configuration
Thus, if a function \( f(a, a_1, a_2, b_1, b_2, A, B, x) \) is a solution of \( E_2 f(x) = 0 \), then the function

\[
\frac{a_1}{(qB)} \quad \frac{a_1}{(qb_1)} \\
\frac{b_2}{A} \quad T_x = \infty
\]

\[
\frac{a_1}{A/(qab_1B)} \\
\frac{1}{q} \quad T_x = 0
\]

\[
f((aqb_1B/a_1A, a_1, a_2, a_1A/qb_1B, a_1A/qb_1, b_2, A, a_1/qb_1, A, x) \)
\]

satisfies the same equation too. Also, the following transformations preserve the configuration of \( E_2 f(x) = 0 \):

\[
s_1: (a, a_1, a_2, b_1, b_2, A, B) \rightarrow \left( \frac{A}{B}, a_1, a_2, \frac{a_1A}{qab_1B}, \frac{a_1A}{qab_1}, b_1, b_2, A, \frac{A}{a} \right),
\]

\[
(4.44)
\]

\[
s_2: a_1 \leftrightarrow a_2, \quad s_3: b_1 \leftrightarrow b_2.
\]

(4.45)

Therefore we obtain many solutions for \( E_2 f(x) = 0 \) by considering the symmetries of the group 

\[
G_2 = \langle s_1, s_2, s_3, s_4 \rangle,
\]

where

\[
s_4: (a, a_1, a_2, b_1, b_2, A, B, x) \rightarrow \left( \frac{aqb_1B}{a_1A}, a_1, a_2, \frac{a_1A}{qab_1B}, \frac{a_1A}{qab_1}, b_1, b_2, A, \frac{a_1}{qb_1} A \right).
\]

(4.46)

The relations among \( \{s_i\} \) are as follows:

\[
s_i^2 = \text{id} \quad (1 \leq i \leq 4),
\]

\[
(s_1s_2)^2 = \text{id} \quad (2 \leq i \leq 4),
\]

\[
(s_2s_3)^2 = (s_3s_4)^3 = (s_4s_1)^3 = \text{id}.
\]

(4.47)

(4.48)

(4.49)

Then we find 

\[
G_2 \simeq \langle s_1 \rangle \times \langle s_2, s_3, s_4 \rangle \simeq \mathfrak{S}_2 \times \mathfrak{S}_4.
\]

From Remark 2.7, series solutions of the Heine’s \( q \)-difference equation are obtained by taking some limit for \( 3 \varphi_2 \) solutions in Theorem 4.2. We put \( b_2 = da_2 \) and take \( a_2 \to 0 \), then the equation \( E_2 f(x) = 0 \) becomes

\[
x[(1 - q^\alpha T_x)(B - AT_x) - (a_1 - q^\alpha b_1 T_x)(1 - T_x)]T_x^{-1} f(x) = 0.
\]

Here, 

\[
a_1 A = q^{\alpha + 1} b_1 dB.
\]

Changing the variables and parameters

\[
z = \frac{qBx}{a_1}, \quad a = q^\alpha, \quad b = \frac{A}{B}, \quad c = \frac{q^{\alpha + 1} b_1}{a_1},
\]

then we have

\[
[z(1 - aT_x)(1 - bT_z) - (1 - cq^{-1} T_z)(1 - T_z)]f(z) = 0.
\]

(4.51)

Therefore by taking the limit for \( 3 \varphi_2 \) in Theorem 4.2 we obtain series solutions of the Heine’s equation. We can easily calculate the limits of those series, thus we only show the results. The following series satisfy the Heine’s \( q \)-difference equation (4.51),

\[
2 \varphi_1 \left( \frac{a, b}{c} ; z \right),
\]

(4.52)
\[
\begin{align*}
\phi_2(c/a, c/b; abz/c) &= (abz/c)_{\infty} 2\phi_1\left( c/a, c/b; abz/c \right), \\
&= (z)_{\infty} 2\phi_1\left( c/a, c/b; abz/c \right), \\
&= z^{1-\gamma} 2\phi_1\left( aq/c, bq/c; \frac{q}{a} \right), \\
&= z^{1-\gamma} \frac{(abz/c)_{\infty}}{(z)_{\infty}} 2\phi_1\left( q/a, q/b; abz/c \right), \\
&= z^{-\alpha} 2\phi_1\left( a, aq/c; \frac{cq}{abz} \right), \\
&= z^{-\alpha} \frac{(q/z)_{\infty}}{(cq/(abz))_{\infty}} 2\phi_1\left( q/b, c/b; \frac{q}{z} \right), \\
&= z^{-\beta} 2\phi_1\left( b, bq/c; \frac{cq}{abz} \right), \\
&= z^{-\beta} \frac{(q/z)_{\infty}}{(cq/(abz))_{\infty}} 2\phi_1\left( q/a, c/a; \frac{q}{z} \right), \\
&= 3\phi_2\left( a, b, abz/c; abq/c, 0 : q \right), \\
&= \frac{(abz/c)_{\infty}}{(z)_{\infty}} 3\phi_2\left( c/a, c/b, z; cq/(ab), 0 : q \right), \\
&= z^{1-\gamma} 3\phi_2\left( aq/c, bq/c, abz/c; abq/c, 0 ; q \right), \\
&= z^{1-\gamma} \frac{(abz/c)_{\infty}}{(z)_{\infty}} 3\phi_2\left( q/a, q/b, z; cq/(ab), 0 ; q \right), \\
&= z^{-\alpha} 3\phi_2\left( a, aq/c, q/z; abq/c, 0 ; q \right), \\
&= z^{-\alpha} \frac{(q/z)_{\infty}}{(cq/(abz))_{\infty}} 3\phi_2\left( q/b, c/b, cq/(abz), cq/(ab), 0 ; q \right), \\
&= z^{-\beta} 3\phi_2\left( a, aq/c, q/z; abq/c, 0 ; q \right), \\
&= z^{-\beta} \frac{(q/z)_{\infty}}{(cq/(abz))_{\infty}} 3\phi_2\left( q/a, c/b, q/z; cq/(ab), 0 ; q \right), \\
&= \frac{(az)_{\infty}}{(z)_{\infty}} 2\phi_2\left( c/b, a; bz \right), \\
&= \frac{(bz)_{\infty}}{(z)_{\infty}} 2\phi_2\left( b, c/a; bz \right), \\
&= \frac{(az)_{\infty}}{(z)_{\infty}} 2\phi_2\left( aq/c, q/b; q^2/c, aqz/c; bz/c \right), \\
&= \frac{(bz)_{\infty}}{(z)_{\infty}} 2\phi_2\left( bq/c, q/a; q^2/c, bqz/c; aqz/c \right), \\
&= \frac{(abz/c)_{\infty}}{(bz/c)_{\infty}} 2\phi_2\left( c/b, a; \frac{q^2}{bz} \right), \\
&= \frac{(abz/c)_{\infty}}{(bz/c)_{\infty}} 2\phi_2\left( bq/a, cq/(az); \frac{q^2}{az} \right), \\
\end{align*}
\]

\[
z^{1-\frac{1}{2}} \frac{(abz/c)_{\infty}}{(bz/c)_{\infty}} {\phi}_2 \left( \frac{aq/c, q/b}{(aq/b, q^2/(bz))} ; \frac{cq}{bz} \right),
\]
\[
z^{1-\frac{1}{2}} \frac{(abz/c)_{\infty}}{(bz/c)_{\infty}} {\phi}_2 \left( \frac{bq/c, q/a}{(bq/a, q^2/(az))} ; \frac{cq}{az} \right),
\]
\[
\frac{(az)_{\infty}}{(z)_{\infty}} \frac{3\phi_2}{(z)_{\infty}} \left( \frac{c/b, a, 0}{aq/b, az} ; q \right),
\]
\[
\frac{(bz)_{\infty}}{(z)_{\infty}} \frac{3\phi_2}{(z)_{\infty}} \left( \frac{c/a, b, 0}{bq/a, bz} ; q \right)
\]
\[
z^{1-\frac{1}{2}} \frac{(aqz/c)_{\infty}}{(z)_{\infty}} {\phi}_2 \left( \frac{q/b, aq/c, 0}{bq/a, aqz/c} ; q \right),
\]
\[
z^{1-\frac{1}{2}} \frac{(bzq/c)_{\infty}}{(z)_{\infty}} {\phi}_2 \left( \frac{q/a, bq/c, 0}{aq/b, bqz/c} ; q \right),
\]
\[
z^{-\alpha} \frac{(cq/(bz))_{\infty}}{(cq/(abz))_{\infty}} \frac{3\phi_2}{(z)_{\infty}} \left( \frac{c/b, a, 0}{c, cq/(b)} ; q \right),
\]
\[
z^{-\alpha} \frac{(q^2/(bz))_{\infty}}{(cq/(abz))_{\infty}} \frac{3\phi_2}{(z)_{\infty}} \left( \frac{q/b, aq/c, 0}{q^2/c, q^2/(bz)} ; q \right),
\]
\[
z^{-\beta} \frac{(cq/(az))_{\infty}}{(cq/(abz))_{\infty}} \frac{3\phi_2}{(z)_{\infty}} \left( \frac{c/a, b, 0}{c, cq/(az)} ; q \right),
\]
\[
z^{-\beta} \frac{(q^2/(az))_{\infty}}{(cq/(abz))_{\infty}} \frac{3\phi_2}{(z)_{\infty}} \left( \frac{q/a, bq/c, 0}{q^2/c, q^2/(az)} ; q \right)
\]

Here, \(q^\alpha = a, q^\beta = b, q^\gamma = c\).

**Remark 4.8.** For the Gauss hypergeometric differential equation (1.1), there are solutions which are written by power series of \(x\), \(1/x\), \(1 - x\), \(1/(1 - x)\), \((x - 1)/x\), \(x/(x - 1)\). These solutions are called Kummer’s 24 solutions. The solutions (4.52), ..., (4.83) are \(q\)-analogs of Kummer’s 24 solutions. In \([8]\), \(q\)-analogs of Kummer’s 24 solutions were obtained. More precisely, the \(3\phi_2\) solution (4.60) and the \(2\phi_2\) solution (4.72) were obtained. Note that the solution (4.76) was not discussed in \([8]\).

The dimension of the space of solutions for the Heine’s equation, which are regular at \(z = 0\), is 1, and then we have

\[
2\varphi_1 \left( \frac{a b}{c} ; z \right) = \frac{(abz/c)_{\infty}}{(z)_{\infty}} \frac{(abz/c)_{\infty}}{2\varphi_1} \left( \frac{c/a, b/c}{c} ; \frac{abz}{c} \right)
\]

Also if \(a = q^{-n}\ (n \in \mathbb{Z}_{\geq 0})\), the solutions (4.60), (4.72), (4.76) are regular at \(z = 0\). Moreover if \(a = q^{-n}\), then the function

\[
z^{\alpha} \frac{\theta(abz/c)}{\theta(bz/c)} \cdot z^{-\alpha} \frac{(cq/(bz))_{\infty}}{(cq/(abz))_{\infty}} \frac{3\phi_2}{(z)_{\infty}} \left( \frac{c/b, a, 0}{c, cq/(b)} ; q \right) = (q^{-n}bz/c)_{n} \cdot 3\phi_2 \left( \frac{c/b, q^{-n}, 0}{c, cq/(b)} ; q \right),
\]

is regular at \(z = 0\), too. Therefore we get

\[
2\varphi_1 \left( \frac{q^{-n}b}{c} ; z \right) = (q^{-n}z)_{n} \cdot 2\varphi_2 \left( \frac{q^{-n}, c/b}{c, q^{-n}z} ; bz \right)
\]

\[
= \left( \frac{c/b}{c} \right)_{n} 3\phi_2 \left( \frac{q^{-n}, b, q^{-n}bz/c}{q^{1-n}b/c, 0} ; q \right) = \left( \frac{b}{c} \right)_{n} (q^{-n}z)_{n} \cdot 3\phi_2 \left( \frac{q^{-n}, c/b, 0}{q^{1-n}b, q^{-n}z} ; q \right)
\]

(4.85)
\[ (q^{-n}bz/c)_{n}3\varphi_{2}\left( \begin{array}{c} \frac{c}{b}, q^{-n}, 0 \\ c, cq/(bz) \end{array} ; q \right). \] (4.86)

We used the \( q \)-Vandermonde formula
\[ 2\varphi_{1}\left( \frac{a, q^{-n}}{c} ; q \right) = \frac{c/a}{(c)_{n}} a^{n} \quad (n \in \mathbb{Z}_{\geq 0}), \] (4.87)

to lead the formulas (4.86). The \( q \)-Vandermonde formula can be obtained by the Heine’s transformation formula (4.92). Heine’s transformation formula also can be found from the viewpoint of solutions for the Heine’s equation (see Remark 4.9 below). Note that a part of the equation (4.86) holds with the general parameter \( a \):
\[ 2\varphi_{1}\left( \frac{a, b}{c} ; z \right) = \frac{(az)_{\infty}}{(z)_{\infty}} 2\varphi_{2}\left( \frac{a, c/b}{c, az} ; bz \right), \] (4.88)

Some of the above formulas are summarized in [6].

**Remark 4.9.** We find that the functions
\[ 3\varphi_{1}\left( \frac{a, b, q/z}{abq/c} ; \frac{z}{c} \right), \] (4.89)
\[ \frac{(bz)_{\infty}}{(z)_{\infty}} 2\varphi_{1}\left( \frac{c/a, z}{bz} ; a \right), \] (4.90)
also satisfy the Heine’s equation in the same way.

1. The series \( 3\varphi_{1} \) is a divergent series. Thus this solution is formal if the series does not terminate. If \( a = q^{-n} \) \( (n \in \mathbb{Z}_{\geq 0}) \), we have
\[ 2\varphi_{1}\left( \frac{q^{-n}, b}{c} ; z \right) = \frac{(c/b)_{n}}{(c)_{n}} b^{n} 3\varphi_{1}\left( \frac{q^{-n}, b, q/z}{q^{1-n}b/c} ; \frac{z}{c} \right). \] (4.91)

2. The series (4.90) is a \( q \)-analog of the integral solution of the Gauss’s equation (1.3). This series is regular at \( z = 0 \), and then we get
\[ 2\varphi_{1}\left( \frac{a, b}{c} ; z \right) = C \frac{(bz)_{\infty}}{(z)_{\infty}} 2\varphi_{1}\left( \frac{c/a, z}{bz} ; a \right), \] (4.92)

and by using the \( q \)-binomial theorem, we have \( C = \frac{(a)_{\infty}}{(c)_{\infty}} \). The transformation (4.92) is well known as Heine’s transformation formula.

**Remark 4.10.** We consider the configuration of the Heine’s equation \([z(1 - aT_{z})(1 - bT_{z}) - (1 - T_{z})(1 - cq^{-1}T_{z})]f(z) = 0 \) below:
By the gauge transformation \( f(z) = z^{1-\gamma} g(z) \), this configuration is transformed as follows:

Thus if a function \( f(a, b, c, z) \) satisfies the Heine’s equation, then the function \( z^{1-\gamma} f \left( \frac{bq}{c}, \frac{aq}{c}, \frac{q^2}{c}, z \right) \) satisfies the Heine’s equation, too. Also, by the transformations \( \{ g(z) = f(z) \frac{(abz/c)_\infty}{(z)_\infty}, z' = \frac{abz}{c} \} \), \( \{ g(z) = z^{-\alpha} f(z), \ z' = \frac{cq}{abz} \}, \{ a \leftrightarrow b \} \), we obtain these configurations, respectively:
Therefore the functions 
\[
\binom{abz/c}{z}_\infty \left( \frac{c}{b}, \frac{c}{a}, \frac{abz}{c} \right), \quad z^{-\alpha} f \left( a, \frac{aq}{c}, \frac{aq}{b}, \frac{cq}{abz} \right), \quad f(b, a, c, z)
\]
satisfy the Heine’s equation, too. We obtain the above solutions by taking some limit for \(3\varphi_2\) here. On the other hand, we can obtain the solutions by these symmetries too. We put

\[
H_1 f(x) = 0 \quad \Rightarrow \quad H_2 f(x) = 0 \quad \Rightarrow \quad H_3 f(x) = 0
\]

and then we have \(s_1^2 = \text{id}\) for \(1 \leq i \leq 4\), \(s_is_j = s_js_i\) for \(1 \leq i \neq j \leq 3\), \((s_is_4)^4 = \text{id}\) for \(1 \leq i \leq 3\), \((s_is_js_4)^2 = \text{id}\). We define \(G_1 = \langle s_i \mid 1 \leq i \leq 4 \rangle\), and then we have \(G_1 = \langle s_1s_4 \rangle \times \langle s_1, s_2, s_3 \rangle \simeq (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})\). The order of \(G_1\) is 32. This would be related to the 32 solutions \((4.52)\), ..., \((4.83)\).

## 5 Summary and discussion

In this paper, we obtained two main results for the variant of the \(q\)-hypergeometric equation of degree three \(\mathcal{H}_3 f(x) = 0\). Also, by taking some limits, we obtained many results for the variant of the \(q\)-hypergeometric equation of degree two \(\mathcal{H}_2 f(x) = 0\), and Heine’s \(q\)-hypergeometric equation \(\mathcal{H}_1 f(x) = 0\). A summary of these results is as follows.

Main results are Theorem 3.1 and Theorem 4.1 which give the integral solutions and the series solutions, respectively. These integrals and series are \(q\)-analogs of the integral \((1.7)\) and the series \((1.8)\), respectively. Because the series solutions are expressed by the very-well-poised-balanced \(q\)-hypergeometric series \(\mathcal{W}_7\), we found that the variant of the \(q\)-hypergeometric equation of degree three is regarded as a \(q\)-difference equation of the Askey-Wilson function with respect to some parameter. By taking some limits, we obtain integral and series solutions for the variant of the \(q\)-hypergeometric equation of degree two and the Heine’s \(q\)-hypergeometric equation. The following table is a summary of these limits.

| configurations | \(\mathcal{H}_3 f(x) = 0\) | \(\mathcal{H}_2 f(x) = 0\) | \(\mathcal{H}_1 f(x) = 0\) |
|----------------|--------------------------|--------------------------|--------------------------|
|                | \(\bullet\)              | \(\bullet\)              | \(\bullet\)              |
|                | \(\bullet\)              | \(\bullet\)              | \(\bullet\)              |
|                | \(\bullet\)              | \(\bullet\)              | \(\bullet\)              |
|                | \(\bullet\)              | \(\bullet\)              | \(\bullet\)              |

We also found that \(q\)-analogs of Kummer’s 24 solutions, which were obtained in \([8]\), are rediscovered by taking the limit of the solutions \(\mathcal{W}_7\). Moreover, we found that some formulas for \(2\varphi_1\) can be considered from the viewpoint of linear \(q\)-difference equations.

There are many problems related to our results. We mention three of them here.

1. In section 3, we obtained integral solutions for variants of the \(q\)-hypergeometric equation. In Theorem 3.1 we found 12 solutions for the equation \(\mathcal{E}_3 f(x) = 0\). Since the rank of the
In Proposition 2.2, we showed that variants of the
The variants of the
$q$-hypergeometric solutions of the
equation for the $q$-Heun equation are rigid by
Theorem 2.7, we showed that variants of the
$q$-Heun equation are rigid by

$$1 -1 1 0 0 0 
1 0 0 -1 1 0 
0 1 0 -1 0 1 
0 0 1 0 -1 1$$
$$\left( \begin{array}{c}
\varphi_3(x, q/a_1, q/a_2) \\
\varphi_3(x, q/a_1, q/a_3) \\
\varphi_3(x, q/a_2, q/a_3) \\
\varphi_3(x, q/a_1, q/(Ax)) \\
\varphi_3(x, q/a_2, q/(Ax)) \\
\varphi_3(x, q/a_3, q/(Ax)) 
\end{array} \right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$ (5.1)

The rank of this $4 \times 6$ matrix is 3. In the same way we find 3 linear relations for $\tilde{\varphi}_3(x, \sigma_1, \sigma_2)$. There are still 4 other linear relations for the integrals. We hope these relations are discovered. From Remark 3.6, we expect that the integrals $\varphi_3(x, q/a_1, q/a_j)$, $\tilde{\varphi}_3(x, b_j, b_j)$ are linearly dependent over the field of pseudo-constant $K$ in general parameters. In the general theory of linear $q$-difference equations (cf. [3, 4]), the connection problem for linear $q$-difference equations is to find the connection matrix $C(x)$ which satisfies $\mathbf{y}_{\infty}(x) = C(x)\mathbf{y}_a(x)$. Here $\mathbf{y}_a$ is a fundamental solution of the $q$-difference equation at $x = a$. However, the points $x = 0$ and $x = \infty$ are essentially non-singular for the variant of the $q$-difference equation of degree three. We are sure that the linear relations for the integral solutions should be considered instead of the connection problem between solutions at $x = 0$ and $x = \infty$. According to Remark 2.7, the variant of the $q$-hypergeometric equation of degree three becomes Heine’s $q$-hypergeometric equation by some limits. Taking the same limits, we expect that the linear relations of integrals become connection coefficients for solutions of Heine’s equation.

(2) The variants of the $q$-hypergeometric equation is a special case of the $q$-Heun equation or the variant of the $q$-Heun equation of degree three. Therefore our integrals and series can be considered as the special solutions of the $q$-Heun equation or its variant of degree three. The $q$-Heun equation and their variants are expressed by the eigenvalue problem for the degenerated Ruijsenaars-van Diejen operator $A$ of one variable. It is expected that special solutions of the equation $A^{(i)} f(x) = E f(x)$ are obtained by some hypergeometric function. Here, $A^{(i)}$ is the $i$-th degeneration of $A$. In [23], it was found that the degenerations of the Ruijsenaars-van Diejen operator correspond to the $q$-Painlevé equations. More precisely, the eigenvalue problem for the degenerations of the Ruijsenaars-van Diejen operator is obtained by a specialization of the linear $q$-difference equation of the Lax pair for the $q$-Painlevé equation. Special solutions for the $q$-Painlevé equation of type $E_7^{(1)}$ and $E_6^{(1)}$ are also written by using the functions $sW_7$ and $3\varphi_2$, respectively. In addition, a special solution for the $q$-Painlevé equation of type $E_8^{(1)}$ is written by $10W_9$. For more details about $q$-hypergeometric solutions of the $q$-Painlevé equations, see [13]. Therefore, we expect that the eigenvalue problem of the degenerated Ruijsenaars-van Diejen operator has a special solution which is written by $10W_9$.

(3) In Proposition 2.2 we showed that variants of the $q$-hypergeometric equation are rigid by the configuration. In the differential case, a Fuchsian differential equation has an integral solution if the equation is irreducible and rigid. In [14], it was shown that any irreducible rigid Fuchsian differential equation is reduced to a rank 1 equation by a finite iteration of addition and middle convolution. This is called Katz’s theory. Also in [5], addition and middle convolution are interpreted as the operations of residue matrices of the Fuchsian equation. A $q$-analog of middle convolution was introduced in [22]. However, a $q$-analog of Katz’s theory is not obtained yet. We hope that a $q$-analog of Katz’s theory is established,
and many $q$-hypergeometric integrals are explained by this theory. It seems that our results (Theorem 3.1 and 3.2) are examples of a $q$-analog of Katz’s theory.

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