Relativistic properties of “marginal” distributions

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Abstract

We study the properties of marginal distributions—projections of the phase space representation of a physical system—under relativistic transforms. We consider the Galileo case as well as the Lorentz transforms exploiting the relativistic oscillator model used for describing the mass spectrum of elementary particles.

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1 Introduction

The concept of phase space arises naturally from the Hamiltonian formulation of classical mechanics, and there have always been considerable efforts to give phase space picture of quantum mechanics too. Much of the thrust of these attempts lies in their ability to exploit classical analogues. Using these techniques, such as \( P \)-representation of Glauber and Sudarshan [4], the Wigner representation [2] and the Husimi representation [3], some quantum systems can be reduced to non-operator systems. However, the essential quantum nature of the problem is present in terms of the interpretation of the (apparently) classical variables.
Moreover, the so called quasi-probability distributions [4] do not have the characteristics of classical probability distributions. Instead, by projecting the quasi-probability in a certain phase subspace, it is possible to obtain a genuine probability. In particular the projection of the Wigner function onto a straight line of the phase space was called ‘marginalization’ procedure, and the obtained distribution ‘marginal’ [2, 5].

Recently, there has been a renewed interest on these marginal probabilities in connection with the tomographic imaging of a quantum state [6]. Along this approach the marginals represent the shadows from which the state (or its phase space representation) is reconstructed [7].

The aim of the present paper is to study the properties of these marginals under relativistic transforms. The case of Galileo transforms results almost trivial, while for the Lorentz one we needed of a model having a covariant phase space picture. To this end we have studied the relativistic oscillator model used for describing the mass spectrum of elementary particles [8].

2 The “marginal” distributions

Referring to the standard definitions given in the literature [2, 3], by ‘marginalization’ one should mean a line integral in the phase space \( \{q, p\} \) of the Wigner function \( W(q, p) \), i.e.

\[
w(x; \theta) = \int dq dp W(q, p) \delta(x - \cos \theta q - \sin \theta p),
\]

where \( \theta \) is the angle orientation of the line. \( w \) becomes a probability distribution for the variable \( x \), depending parametrically on \( \theta \).

One can go beyond this definition [3]; let us consider the phase space transformation as a generic linear combination of position \( q \) and momentum \( p \)

\[
q \rightarrow X = \mu q + \nu p,
\]

\[
p \rightarrow P = \mu' q + \nu' p,
\]

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and consider it as a real symplectic transformation belonging to the group \( Sp(2, R) \), i.e.

\[
ΛσΛ^T = σ, \quad Λ = \begin{pmatrix} μ & ν \\ μ' & ν' \end{pmatrix}, \quad σ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

Then, we can consider the ‘projection’ along the phase subspace characterized by the transformation (2)

\[
w(X; μ, ν) = \int W(q, p)δ(X - μq - νp) \, dq \, dp,
\]

which can be intended as a marginal distribution too. Of course, Eq.(3) represents a particular case of Eq.(4) whenever a mere rotation in the phase space is considered.

The above definition could also be extended to phase spaces of higher dimensions. For example, in the case of two-dimensional system we will have a phase space \( \{ \vec{q} ≡ (q_1, q_2), \vec{p} ≡ (p_1, p_2) \} \), hence we can introduce the transform

\[
(\vec{X}, \vec{P}) = Λ (\vec{q}, \vec{p})^T
\]

where now \( Λ \) is a \( 4 \times 4 \) real symplectic matrix, i.e.

\[
ΛσΛ^T = σ; \quad σ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.
\]

The components \( X_1, X_2, P_1, P_2 \) are related to the homogeneos symplectic group \( Sp(4, R) \).

In particular, \( \vec{X} \) has the following components

\[
X_1 = \vec{μ}q + \vec{ν}p,
\]
\[
X_2 = \vec{μ}'q + \vec{ν}'p,
\]

with \( \vec{μ} = (Λ_{11}, Λ_{12}); \vec{ν} = (Λ_{13}, Λ_{14}); \vec{μ}' = (Λ_{21}, Λ_{22}); \vec{ν}' = (Λ_{23}, Λ_{24}). \)

Therefore, a marginal distribution \( w(\vec{X}; \vec{μ}, \vec{ν}, \vec{μ}', \vec{ν}') \), may be introduced as a probability distribution for the variable \( \vec{X} \), with a dependence upon the parameters characterizing the matrix \( Λ \)

\[
w(\vec{X}; \vec{μ}, \vec{ν}, \vec{μ}', \vec{ν}') = \int d\vec{q} d\vec{p} W(\vec{q}, \vec{p}) \delta(X_1 - \vec{μ}q - \vec{ν}p)
\]
\[ \times \delta(X_2 - \hat{\mu}' \hat{q}' - \hat{\nu}' \hat{p}). \] (10)

3 Marginal distributions and relativity

Now, in order to study the properties of marginal distributions under relativistic transformations we need to know the transformation properties of the wave function of the system.

At first it is instructive to consider the Galilei transforms

\[ q' = q - vt; \quad p' = p - v; \quad t' = t, \] (11)

where we have considered a particle of unit mass and \( v \) represents the relative velocity between the two reference frames.

Since in this case the wave function transforms, independently of the assumed model, as

\[ \Psi(q, t) \rightarrow \Psi'(q, t) = \exp \left( ivq - i\frac{v^2}{2} t \right) \Psi(q - vt, t), \] (12)

then, the Wigner function correspondingly transforms as

\[ W(q, p, t) \rightarrow W(q', p', t) = W(q - vt, p - v, t). \] (13)

As consequence of (13) we immediatly obtain

\[ w(x, \mu, \nu) \rightarrow w(x, \mu, \nu) = w_0(x - \mu vt - \nu v, \mu, \nu), \] (14)

from which a simple shift in the distribution, in moving from one reference frame to the other, results.

Instead, in the case of Lorentz transformations we have to consider a specific model.

3.1 The relativistic harmonic oscillator

Let us consider the relativistic oscillator model introduced by Feynmann et al. \cite{8} to describe a hadron consisting of two quarks bound together by a harmonic oscillator
potential of unit strength

\[
\begin{aligned}
&\left\{-2 \left[ \left( \frac{\partial}{\partial x^a_\mu} \right)^2 + \left( \frac{\partial}{\partial x^b_\mu} \right)^2 \right] + \left( \frac{1}{16} \right) (x^a_\mu - x^b_\mu)^2 + m^2_0 \right\} \phi(x_a, x_b) = 0 ,
\end{aligned}
\]  

(15)

where \(x_a\) and \(x_b\) are space-time coordinates for the first and second quark respectively (we are using natural units, \(\hbar = c = 1\)). This partial differential equation has many different solutions depending on the choice of variables and boundary conditions. Here we follow the treatment of Ref. [10].

In order to simplify the Eq. (15), let us introduce new coordinate variables

\[
X = (x_a + x_b)/2 ; \quad x = (x_a - x_b)/2 .
\]

(16)

The four-vector \(X\) specifies where the hadron is located in space-time, while the variable \(x\) measures the space-time separation between the quarks. In terms of these variables Eq. (15) can be written as

\[
\left( \frac{\partial^2}{\partial X^2_\mu} - m^2_0 + \frac{1}{2} \left[ \frac{\partial^2}{\partial x^2_\mu} + x^2_\mu \right] \right) \phi(x_a, x_b) = 0 .
\]

(17)

This equation is separable in the \(X\) and \(x\) variables. Thus

\[
\phi(x_a, x_b) = f(X) \psi(x) ,
\]

(18)

and \(f(X)\) and \(\psi(x)\) satisfy the following differential equations respectively

\[
\left( \frac{\partial^2}{\partial X^2_\mu} - m^2_0 - (\lambda + 1) \right) f(X) = 0 ,
\]

(19)

\[
\frac{1}{2} \left( - \frac{\partial^2}{\partial x^2_\mu} + x^2_\mu \right) \psi(x) = (\lambda + 1) \psi(x) .
\]

(20)

Equation (19) is a Klein-Gordon equation, and its solution takes the form

\[
f(X) = \exp[\pm iP\mu X^\mu] ,
\]

(21)

with

\[
-P^2 = -P_\mu P^\mu = M^2 = m^2_0 + (\lambda + 1) ,
\]

(22)
where $M$ and $P$ are the mass and four-momentum of the hadron respectively. The eigenvalue $\lambda$ is determined from the solution of Eq. (20).

As for the four momenta of the quarks $p_a$ and $p_b$, we can combine them into the total four-momentum and momentum-energy separation between the quarks

$$P = p_a + p_b; \quad p = \sqrt{2}(p_a - p_b). \tag{23}$$

$P$ is the hadronic four-momentum conjugate to $X$. The internal momentum-energy separation $p$ is conjugate to $x$ provided that there exist wave functions which can be Fourier transformed.

The four-dimensional equation (20) is separable in at least thirty-four different coordinate systems [11]. Since we are quite familiar with the three-dimensional harmonic oscillator equation from nonrelativistic quantum mechanics, we are naturally led to consider the separation of the space and time variables, and write the equation (20) as

$$\left(-\nabla^2 + \frac{\partial^2}{\partial t^2} + [x^2 - t^2]\right)\psi(x) = (\lambda + 1)\psi(x). \tag{24}$$

If the hadron moves along the $Z$ direction which is also the $z$ direction, then the hadronic factor $f(X)$ is Lorentz-transformed in the same manner of a scalar field. The Lorentz transformation of the internal coordinates from the laboratory frame to the hadronic rest frame takes the form

\[
\begin{align*}
x' &= x, \quad y' = y, \\
z' &= (z - \beta t)/(1 - \beta^2)^{1/2}, \\
t' &= (t - \beta z)/(1 - \beta^2)^{1/2}, \tag{25}
\end{align*}
\]

where $\beta$ is the velocity of the hadron moving along the $z$ direction. The primed quantities are the coordinate variables in the hadronic rest frame. In terms of the primed variables the oscillator differential equation is

$$\left(-\nabla'^2 + \frac{\partial^2}{\partial t'^2} + [x'^2 - t'^2]\right)\psi(x) = (\lambda + 1)\psi(x). \tag{26}$$
This form is identical to that of Eq. (24), due to the fact that the oscillator differential equation is Lorentz-invariant [12].

Among many possible solutions of the above differential equation, let us consider the form

$$\psi_\beta(x) = \frac{1}{\pi^{1/2}} \left( \frac{1}{2} \right)^{(a+b+n+k)/2} \left( \frac{1}{a!b!n!k!} \right)^{1/2} H_a(x') H_b(y') H_n(z') H_k(t') \times \exp \left[ -\frac{1}{2} (x'^2 + y'^2 + z'^2 + t'^2) \right],$$  (27)

where $a$, $b$, $n$ and $k$ are integers, and $H_a(x')$, $H_b(y')$ ... are the Hermite polynomials. This wave function is normalizable, but the eigenvalue takes the values

$$\lambda = (a + b + n - k).$$  (28)

Thus for a given value of $\lambda$, there are infinitely many possible combinations of $a$, $b$, $n$ and $k$. The most general solution of the oscillator differential equation is infinitely degenerate [13]. The simplest way to avoid this problem (at least to render finite the degeneracy), is to invoke the restriction that there should not be time-like oscillations in the Lorentz frame in which the hadron is at rest, and that the integer $k$ in Eqs. (27) and (28) be zero [13, 14]. In doing so we are led to the question of maintaining the Lorentz covariance with this condition.

When the hadron moves along the $z$ axis, the $k = 0$ condition is equivalent to

$$\left( t' + \frac{\partial}{\partial t'} \right) \psi_\beta(x) = 0. $$  (29)

The most general form of the above condition is

$$p_\mu \left( x^\mu + \frac{\partial}{\partial x^\mu} \right) \psi_\beta(x) = 0. $$  (30)

Thus the $k = 0$ condition is covariant. Once this condition is set, we can write the wave function belonging to this finite set as

$$\psi_\beta(x) = \frac{1}{\pi^{1/2}} \left( \frac{1}{2} \right)^{(a+b+n)/2} \left( \frac{1}{a!b!n!} \right)^{1/2} H_a(x') H_b(y') H_n(z') \times \exp \left[ -\frac{1}{2} (x'^2 + y'^2 + z'^2 + t'^2) \right].$$  (31)
Except for the Gaussian factor in the $t'$ variable, the above expression is the wave function for the three-dimensional isotropic harmonic oscillator.

Since the above oscillator wave functions are separable in the Cartesian coordinate system, and since the transverse coordinate variables are not affected by the boost along the $z$ direction, we can omit the factors depending on the $x$ and $y$ variables when studying their Lorentz transformation properties. Hence, the solutions satisfying the subsidiary condition (30) take the simple form

$$\psi_n^\beta(z, t) = \left( \frac{1}{\pi^{2n^2}} \right)^{1/2} H_n(z') \exp \left[ -\frac{1}{2} (z'^2 + t'^2) \right],$$

(32)

with $\lambda = n$. This normalizable wave function, without excitations along the $t'$ axis, describes the internal space-time structure of the hadron moving along the $z$ direction with the velocity parameter $\beta$. If $\beta = 0$, then the wave function becomes

$$\psi_n^0(x, t) = \left( \frac{1}{\pi^{2n^2}} \right)^{1/2} H_n(z) \exp \left[ -\frac{1}{2} (z^2 + t^2) \right].$$

(33)

Thus

$$\psi_n^\beta(z, t) = \psi_n^0(z', t').$$

(34)

We have therefore obtained the Lorentz-boosted wave function by making a passive coordinate transformation on the $z$ and $t$ coordinate variables.

### 3.2 Covariant phase space

It is possible to construct a covariant phase space for the relativistic harmonic oscillator by following Ref. [15]. Let us consider at first the Gaussian factor of the wave function (33), which practically corresponds to the ground state,

$$\psi_0^0(z, t) = \left( \frac{1}{\pi} \right)^{1/2} \exp \left( -(z^2 + t^2)/2 \right),$$

(35)

and introduce the light cone coordinates

$$u = (z + t)/\sqrt{2}, \quad v = (z - t)/\sqrt{2}.$$
The latter transform as
\[ u' = \left(\frac{1 + \beta}{1 - \beta}\right)^{1/2} u, \quad v' = \left(\frac{1 - \beta}{1 + \beta}\right)^{1/2} v. \tag{37} \]

It is easy to see that the product \( uv \) is Lorentz invariant. By using such coordinates, the wave function \( \psi_0 \) can be rewritten as
\[ \psi_0^0(z, t) = \psi_0^0(u, v) = \left(\frac{1}{\pi}\right)^{1/2} \exp\left(-\left(u^2 + v^2\right)/2\right), \tag{38} \]
and, if the system is boosted, it becomes
\[ \psi_0^\beta(z, t) = \left(\frac{1}{\pi}\right)^{1/2} \exp\left\{-\left(\frac{1}{2}\right)\left(\frac{1 - \beta^2}{1 + \beta^2} u^2 + \frac{1 + \beta^2}{1 - \beta^2} v^2\right)\right\}, \tag{39} \]
Practically, it undergoes a continuous deformation as \( \beta \) increases.

Analogously to the Eq. \( \text{(38)} \), we may define for the momentum and energy the variables
\[ p_u = \left(\frac{p_z - p_0}{\sqrt{2}}\right), \quad p_v = \left(\frac{p_z + p_0}{\sqrt{2}}\right), \tag{40} \]
and the momentum-energy wave function will be given by
\[ \phi_\beta^0(p_u, p_v) = \left(\frac{1}{2\pi}\right) \int \psi_\beta^0(z, t)e^{-i(p_z - p_0)z}dzdt, \tag{41} \]
\[ = \left(\frac{1}{\pi}\right)^{1/2} \exp\left\{-\left(\frac{1}{2}\right)\left(\frac{1 + \beta^2}{1 - \beta^2} p_u^2 + \frac{1 - \beta^2}{1 + \beta^2} p_v^2\right)\right\}. \tag{42} \]

Hence, we deal a four dimensional phase space \( \{u, v, p_u, p_v\} \) where the Wigner function can be defined in a canonical way
\[ W_\beta^0(u, p_u; v, p_v) = \left(\frac{1}{\pi}\right) \int \left(\psi_\beta^0(u + x, v + y)\right)^* \psi_\beta^0(u - x, v - y) \times \exp\left[2i(p_u x + p_v y]\right] dxdy. \tag{43} \]

After the evaluation of the integral we obtain
\[ W_\beta^0(u, p_u; v, p_v) = \left(\frac{1}{\pi}\right)^2 \exp\left\{-\left(\frac{1}{2}\right)\left(\frac{1 - \beta^2}{1 + \beta^2} p_u^2 + \frac{1 + \beta^2}{1 - \beta^2} p_v^2\right)\right\} \times \exp\left\{-\left(\frac{1}{2}\right)\left(\frac{1 + \beta^2}{1 - \beta^2} p_u^2 + \frac{1 - \beta^2}{1 + \beta^2} p_v^2\right)\right\}, \tag{44} \]
which is manifestly covariant.
3.3 The properties of marginal probabilities

Having a four dimensional (covariant) phase space, the marginal distributions can be defined analogously to the case of Eq. (10). Practically, a marginal distribution will be a projection on the plane \{U, V\} determined by the equations

\[ U = \mu_1 u + \mu_2 p_v + \nu_1 v + \nu_2 p_u, \]
\[ V = \zeta_1 u + \zeta_2 p_v + \eta_1 v + \eta_2 p_u, \]

where we have \( \mu_i, \nu_i, \zeta_i, \eta_i \in \mathbb{R}, \ (i = 1, 2). \)

Without lost of generality we do not specify the constraints on these parameters, since they will be related to the space-time asimmetry in the commutation relations, which is an hard problem to face in making the relativistic quantum mechanics, and goes beyond the scope of the present paper.

Then, we define

\[
w_{\beta}(U, V; \sigma) = \int du dv dp_u dp_v W_{\beta}(u, p_u; v, p_v) \times \delta (U - \mu_1 u - \mu_2 p_v - \nu_1 v - \nu_2 p_u) \times \delta (V - \zeta_1 u - \zeta_2 p_v - \eta_1 v - \eta_2 p_u),
\]

where \( \sigma = \{\mu_i, \nu_i, \zeta_i, \eta_i\}, \ (i = 1, 2). \) As limiting cases we have \( \mu_1 = \eta_1 = 1, \) and all the other parameters equal to zero, then

\[
w_{\beta}(U, V; \sigma) = \left| \psi_{\beta}^0(u, v) \right|^2;
\]

or otherwise, for \( \nu_2 = \zeta_2 = 1, \) and all the other parameters zero, then

\[
w_{\beta}(U, V; \sigma) = \left| \phi_{\beta}^0(p_u, p_v) \right|^2.
\]

It is clear from the Eq. (47) that the marginal distribution, with such definition, is not covariant, but we may rescale the variables, due to the Wigner function covariance,
to have

\[ w_\beta^0(U, V; \sigma) = \int du' dv' dp'_u dp'_v W_0^0(u', p'_u; v', p'_v) \times \delta(U - \bar{\mu}_1 u' - \bar{\tau}_1 v' - \bar{\tau}_2 p'_u - \bar{\mu}_2 p'_v) \times \delta(V - \bar{\zeta}_1 u' - \bar{\eta}_1 v' - \bar{\eta}_2 p'_u - \bar{\zeta}_2 p'_v) \]  

where we take into account the invariance of the measure and we set

\[ \bar{\mu}_i = \left[ \frac{1}{1 + \beta} \right]^{1/2} \mu_i, \quad \bar{\nu}_i = \left[ \frac{1}{1 - \beta} \right]^{1/2} \nu_i, \quad \bar{\zeta}_i = \left[ \frac{1}{1 + \beta} \right]^{1/2} \zeta_i, \quad \bar{\eta}_i = \left[ \frac{1}{1 - \beta} \right]^{1/2} \eta_i, \quad (i = 1, 2) \]  

From Eq. (50) it immediately follows

\[ w_\beta^0(U, V; \sigma) = w_0^0(U, V; \sigma_\beta), \]  

where \( \sigma_\beta \) indicates the parameters (51).

Eq. (52) defines the transformation properties of the marginal distributions; practically we get that different marginals correspond to the same measurement, but performed in different frames. This means that the boosts connecting several frames could be useful to vary the parameters characterizing the marginal distribution. This interpretation is in agreement with that given in Ref. [16] for the two-mode ‘symplectic tomography’.

The above results can be extended to the excited states of the relativistic oscillator as well; the only additive factor one has to consider is the Hermite polynomial multiplying the Gaussian of the ground state.

Of course, the result of Galilei transforms cannot be obtained as a limiting case of the Lorentz transforms.

4 Conclusions

In conclusion we have studied the properties of the marginal distributions under relativistic transformations. Since they contain all the information about the quantum
state of a system, other probabilities related to different observables, could be derived from them, as well as their properties.

Finally, the discussed properties do not concern only fundamental questions, but could become interesting in quantum optics where “optical mesons” enter in the reality [17], and in particle physics where, hopefully, the quantum state tomography concept could be applied. Since by repeated measurements one can build up the marginal probabilities, one is lead to ask the following question: which observables should be measured to this goal for example in high energy processes? This subject will be addressed in future works.

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