THE VARCHENKO-GEL’FAND RING OF A CONE

GALEN DORPALEN-BARRY

Abstract. For a hyperplane arrangement in a real vector space, the coefficients of its Poincaré polynomial have many interpretations. An interesting one is provided by the Varchenko-Gel’fand ring, which is the ring of functions from the chambers of the arrangement to the integers with pointwise addition and multiplication. Varchenko and Gel’fand gave a simple presentation for this ring, along with a filtration and associated graded ring whose Hilbert series is the Poincaré polynomial. We generalize these results to cones defined by intersections of halfspaces of some of the hyperplanes and prove a novel result for the Varchenko-Gel’fand ring of an arrangement: when the arrangement is supersolvable the associated graded ring of the arrangement is Koszul.

1. Introduction

Let $\mathcal{A}$ be a central hyperplane arrangement in $V \cong \mathbb{R}^\ell$, i.e. a finite collection of distinct, codimension one subspaces of $V$. A chamber of $\mathcal{A}$ is a connected component of $V \setminus \bigcup_{H \in \mathcal{A}} H$ and we use $\mathcal{C}(\mathcal{A})$ to denote the set of chambers of $\mathcal{A}$. This paper concerns the Varchenko-Gel’fand ring $\mathcal{VG}(\mathcal{A})$ of $\mathcal{A}$, a ring consisting of maps $\mathcal{C}(\mathcal{A}) \to \mathbb{Z}$ with pointwise addition and multiplication. This ring was first introduced by Gel’fand and Varchenko \[29\], who proved that it has a $\mathbb{Z}$-basis of monomials indexed by no broken circuit sets of $\mathcal{A}$ and showed that the degree filtration of $\mathcal{VG}(\mathcal{A})$ yields an associated graded ring with Hilbert series completely determined by $\mathcal{L}(\mathcal{A})$. Since then, Cordovil \[9\], Gel’fand-Rybnikov \[15\], Moseley \[19\], Proudfoot \[22\], and others have studied the Varchenko-Gel’fand ring of an arrangement, and various generalizations.

The main result of this paper, Theorem 1, extends their work to cones, which are intersections of open halfspaces defined by some of the hyperplanes of $\mathcal{A}$. Cones are interesting because they connect central and affine arrangements while generalizing both. They have been studied by various authors including Aguiar-Mahajan \[2\], Brown \[7\], Gente \[16\], Zaslavsky \[31\], and, in Type A, the author together with Kim and Reiner \[11\]. Moreover, cones are conditional oriented matroids, as developed by Bandelt, Chepoi, and Knauer \[4\].

Our extension of Gel’fand and Varchenko’s work utilizes techniques inspired by the theory of Gröbner bases. As a consequence of this approach, we obtain a new result concerning the Varchenko-Gel’fand ring of an arrangement. This result, Theorem 2, is a Varchenko-Gel’fand analogue of a theorem of Peeva for the Orlik-Solomon algebra \[21\].

1.1. Structure of this Paper. In the remainder of this introduction, we state Theorems 1 and 2 (Section 1.2) and then illustrate the main theorem with an example (Section 1.3). Some background on hyperplane arrangements, matroids, the Varchenko-Gel’fand ring, filtrations, and associated graded rings is given in Section 2. We prove Theorem 1 in Section 3 and we prove Theorem 2 in Section 4.

Date: February 13, 2023.

2020 Mathematics Subject Classification. Primary: 52C35; Secondary: 05Axx, 52C40.

Key words and phrases. hyperplane, arrangement, cone, poset, Whitney, Poincaré, Varchenko-Gel’fand ring, Koszul.

The author was supported by NSF grant DMS-1601961.

1 In Russian, Varchenko comes first (alphabetically speaking).
1.2. Statement of Main Results. Now let \( A = \{ H_1, \ldots, H_n \} \) be a central hyperplane arrangement in \( V \cong \mathbb{R}^f \) and \( K \) a cone of \( A \). An intersection of \( A \) is an intersection \( X = \bigcap_i H_i \) of some of the hyperplanes. We use \( \mathcal{L}(A) \) to denote the set of nonempty intersections of \( A \).

We say that \( C \in \mathcal{C}(A) \) is a chamber of the cone \( K \) if it lies inside the open, convex set \( K \) and we say that \( X \in \mathcal{L}(A) \) is an interior intersection of \( K \) is an intersection \( X \in \mathcal{L}(A) \) which cuts through the cone, i.e., for which \( X \cap K \) is nonempty. We denote the set of chambers of the cone \( K \) by \( \mathcal{C}(K) \) and the set of interior intersections by \( \mathcal{L}^{\text{int}}(K) \). We are interested in the poset structure of \( \mathcal{L}^{\text{int}}(K) \) ordered by reverse inclusion and denote its Möbius function by \( \mu \). The poset \( \mathcal{L}^{\text{int}}(K) \) is a ranked poset and the rank of each element \( X \in \mathcal{L}^{\text{int}}(K) \) is its codimension, denoted \( \text{codim}(X) \).

The Varchenko-Gel’fand ring of the cone \( VG(K) \) is the collection of maps \( f: \mathcal{C}(K) \to \mathbb{Z} \) with pointwise addition and multiplication. Our main result takes its cue from a result of Varchenko and Gel’fand, and interprets the Poincaré polynomial of a cone

\[
Poin(K, t) := \sum_{X \in \mathcal{L}^{\text{int}}(K)} |\mu(V, X)| t^{\text{codim}(X)}
\]

as the Hilbert series (defined below) of a certain graded ring, the associated graded ring for a certain filtration on \( VG(K) \). The Poincaré polynomial is interesting in its own right. In [31, Example A], for example, Zaslavsky showed that the Poincaré polynomial evaluated at 1 is precisely the Poincaré polynomial of \( VG(K) \). We will show the same equality for cones and, specializing to the full arrangement, obtain a novel proof of Varchenko and Gel’fand’s original result. We show that when the cone \( K \) is the full arrangement, then \( \mathcal{V}(A) \) is torsion-free and \( \text{Hilb}(\mathcal{V}(A), t) = Poin(A, t) \) [29]. We will show the same equality for cones and, specializing to the full arrangement, obtain a novel proof of Varchenko and Gel’fand’s original result.

Our proof comes from giving a generating set \( \mathcal{G} \) of relations that plays the role of a Gröbner basis presentation for \( VG(K) \) and \( \mathcal{V}(K) \) as quotients of \( \mathbb{Z}[e_1, \ldots, e_n] \); when working over a field instead of \( \mathbb{Z} \), they are Gröbner bases. The relations in \( \mathcal{G} \subseteq \mathbb{Z}[e_1, \ldots, e_n] \) are summarized in Figure \# where we have made the (harmless) assumption that \( K \) is an intersection of \( \mathcal{L} \) positive halfspaces, i.e. \( K = \bigcap_{i \in W} H_i^+ \) where \( W \subseteq [n] \) and \( H_i^+ := \{ x \in \mathbb{R}^f \mid v_i \cdot x > 0 \} \) for some choice of normal vector \( v_i \) to \( H_i \). For a subset \( S \) of \([n]\), we use the notation \( e_S := \prod_{i \in S} e_i \) to describe a squarefree monomial indexed by \( S \) in the variables \( e_1, \ldots, e_n \). Our main theorem will assert that the elements of \( \mathcal{G} \) (given in the second column of Figure \# and \{\text{in}_{\deg}(g) \mid g \in \mathcal{G}\} \) (given in the third column) give presentations for \( VG(K) \) and \( \mathcal{V}(K) \), respectively.

The most interesting polynomials in \( \mathcal{G} \) are defined by signed circuits. For each hyperplane \( H_i \) of \( A \), fix a normal vector \( v_i \). A pair of disjoint subsets \( D^+, D^- \subseteq [n] \) is a signed dependency if there is a linear relation \( \sum_{i \in D^+} \lambda_i v_i = 0 \) where \( i \in D^+ \) if \( \lambda_i < 0 \) and \( i \in D^- \) if \( \lambda_i > 0 \). To keep track of signed dependencies, we write them as tuples \( D = (D^+, D^-) \) and denote the underlying (unsigned) set by \( \underline{D} = D^+ \cup D^- \). Our presentations for \( VG(K) \) and \( \mathcal{V}(K) \) concern signed circuits \( C = (C^+, C^-) \), which are signed dependencies for which \( \underline{C} \) is minimal under inclusion.
In the main theorem, our presentations of \( VG(K) \) and \( V(K) \) will give a \( \mathbb{Z} \)-basis for both rings in terms of a certain family of monomials indexed by a \( \mathcal{K} \)-no broken circuit sets, which we describe now. One can break a circuit \( C = (C^+, C^-) \), by removing the smallest-indexed element \( i_0 \) of \( C \). The resulting (unsigned) set \( \bigcup \{ i_0 \} \) is called a broken circuit. A subset of \([n] \) not containing any broken circuits is called a no broken circuit set. A no broken circuit set \( N \) is, furthermore, a \( \mathcal{K} \)-no broken circuit set (hereafter \( \mathcal{K} \)-NBC set) if \( X = \bigcap_{i \in N} H_i \) cuts through the cone \( K \), meaning \( X \in L^{\text{int}}(K) \). We denote the set of \( \mathcal{K} \)-NBC sets by \( NBC(K) \).

\( \mathcal{K} \)-NBC sets naturally arise when one studies the Poincaré polynomial of the cone. Since every lower interval \([V, X]\) of \( L^{\text{int}}(K) \) is isomorphic to the corresponding lower interval \([V, X]\) in \( L(A) \), a theorem of Rota \cite{23, Section 7} (c.f. \cite{24, Theorem 1.1}) allows us to compute the Möbius function of the interval \([V, X]\) in \( L^{\text{int}}(K) \) via the \( \mathcal{K} \)-NBC sets, i.e., for all \( X \in L^{\text{int}}(K) \)

\[
\mu(V, X) = (-1)^{\text{codim}(X)} \cdot \# \left\{ N \in NBC(A) \bigg| \bigcap_{i \in N} H_i = X \right\}.
\]

As a result, the Poincaré polynomial of a cone also has an expression in terms of \( \mathcal{K} \)-NBC sets:

\[
Poin(K, t) = \sum_{N \in NBC(K)} t^{\# N}
\]

Taking \( t = 1 \), gives

\[
\#C(K) = \#NBC(K).
\]

**Theorem 1.** Let \( K \) be a cone of an arrangement \( A \) and \( G \) the relations from Figure \[. \] Choose a monomial order \( \prec \) on \( \mathbb{Z}[e_1, \ldots, e_n] \) which refines the ordering by degree. Then \( VG(K), V(K) \) have presentations

\[
VG(K) \cong \mathbb{Z}[e_1, \ldots, e_n]/(G),
\]

\[
V(K) \cong \mathbb{Z}[e_1, \ldots, e_n]/(\text{indeg}(G)),
\]

and free \( \mathbb{Z} \)-modules bases given by the images of the \( \mathcal{K} \)-NBC monomials \( \{ e_N \}_{N \in NBC(K)} \). In particular,

\[
\text{Hilb}(V(K), t) = \sum_{N \in NBC(K)} t^{\# N} = \text{Poin}(K, t).
\]

Recall that \( K = \bigcap_{i \in W} H_i^+ \) for some (possibly-redundant) \( W \subseteq [n] \). The following corollary is an immediate consequence of Theorem \[.

**Corollary.** \( VG(K) \cong \mathbb{Z}[e_1, \ldots, e_n]/I_K \) where \( I_K \) is generated by

- (Idempotent) \( e_i^2 - e_i \) for \( i \in [n] \),
- (Unit) \( e_i - 1 \) for \( i \in W \),
- (Circuit) \( (e_{C^+}) \prod_{j \in C^-} (e_j - 1) - (e_{C^-}) \prod_{i \in C^+} (e_i - 1) \) for signed circuits \( C = (C^+, C^-) \).

Theorem \[ has an interesting consequence when \( A \) is a supersolvable arrangement: we obtain a Varchenko-Gel’fand ring analogue of a result proven by Peeva for the Orlik-Solomon algebra of \( A \). In order to state this result, we work with coefficients in a field \( F \), and consider the ring \( V_{GF}(A) \) of maps from the chambers of \( A \) to a field \( F \), denoting the associated graded ring by \( V_{\bar{F}}(A) \).

**Theorem 2.** If \( A \) is a supersolvable arrangement, then \( V_{\bar{F}}(A) \) is Koszul.

This result uses Björner-Ziegler’s characterization of supersolvable arrangements via broken circuit sets \[, so one might hope that it extends to cones of supersolvable arrangements. Unfortunately, there are cones of supersolvable arrangements for which \( V_{\bar{F}}(K) \) is not Koszul (we provide such an example in Section \[).
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Idempotent & For all $i \in [n]$, & $e_i^2 - e_i$ & $e_i^2$ & $e_i^2$ \\
\hline
Unit & For all $i \in W$, & $e_i - 1$ & $e_i$ & $e_i$ \\
\hline
Cone Circuit & For all circuits $C = (C^+, C^-)$ with $\emptyset \neq W \cap C^+ = W \cap C^-$, & $e_{C^+ \setminus W} \cdot \prod_{j \in C^-} (e_j - 1)$ & $e_{C^+ \setminus W}$ & $e_{C^- \setminus W}$ \\
\hline
Circuit & For signed circuits $C = (C^+, C^-)$ with $\emptyset = W \cap C^-$, & $(e_{C^+} \prod_{j \in C^-} (e_j - 1) - (e_{C^-} \prod_{j \in C^+} (e_j - 1)) + \sum_{i \in C^+} e_{C^+ \setminus \{i\}} - \sum_{j \in C^-} e_{C^- \setminus \{j\}}$ & $e_{C^+ \setminus \{i_0\}}$ & where $i_0 := \min_{\prec}(C)$ \\
\hline
\end{tabular}
\caption{The relations $g$ in $\mathcal{G}$, along with their degree-initial form $\text{in}_{\text{deg}}(g)$ and their initial term $\text{in}_{\prec}(g)$ for any monomial order $\prec$ that satisfies $e_1 \prec \cdots \prec e_n$.}
\end{table}

1.3. An extended example. In this section, we give an extended example illustrating Theorem 1.1. Consider the cone $K$ of a central arrangement in $\mathbb{R}^3$ of which an affine slice is drawn below on the left. We can compute the Poincaré polynomial of the cone from $L^{\text{int}}(K)$ (below, on the right). It is $\text{Poin}(K, t) = 1 + 3t + t^2$.

Figure 2 shows the relations $\mathcal{G}$ for some choice of orientation of $A$ (we omit the the redundant Cone-Circuit relations for which the opposite orientation is already given). Our main theorem says that $\mathcal{V}G(K) \cong \mathbb{Z}[e_1, e_2, e_3, e_4, e_5]/(\mathcal{G})$ and the associated graded ring has presentation $\mathcal{V}(K) \cong \mathbb{Z}[e_1, e_2, e_3, e_4, e_5]/(\text{in}_{\text{deg}} \mathcal{G})$. Furthermore

$$\mathcal{V}(K) \cong \mathbb{Z} \cdot \{1\} \oplus \mathbb{Z} \cdot \{x_1, x_2, x_3\} \oplus \mathbb{Z} \cdot \{x_2x_3\},$$

which means that $\text{Hilb}(\mathcal{V}(K), t) = 1 + 3t + t^2$.

2. Background

2.1. Hyperplanes, Cones and Linear Algebra. Let $A = \{H_1, \ldots, H_n\}$ be a central hyperplane arrangement with $n$ distinct hyperplanes $H_i = \{x \in \mathbb{R}^\ell \mid v_i \cdot x = 0\}$ for normal vectors $\{v_i\}_{i \in [n]}$. A chamber of $A$ is an open, connected component of $V \setminus \bigcup_{H \in A} H$ and an intersection $X$ of $A$ is the
|                | $g \in \mathcal{G}$ | $\text{in}_{\text{deg}}(g)$ | $\text{in}_{\prec}(g)$ |
|----------------|----------------------|-----------------------------|-------------------------|
| (Idempotent)   | For all $i \in [5]$  | $e_i^2 - e_i$               | $e_i^2$                 |
| (Unit)         | $e_4 - 1, e_5 - 1$   | $e_4, e_5$                 | $e_4, e_5$             |
| (Circuit)      | (none)               | (none)                     | (none)                 |
| (Cone-Circuit) | $e_1e_2 - e_2, e_1e_2e_3 - e_2e_3, e_1e_3 - e_3$ | $e_1e_2, e_1e_2e_3, e_1e_3, e_1e_3$ | $e_1e_2, e_1e_3, e_1e_3$ |

Figure 2. The elements of $\mathcal{G}$ for some choice of orientation in the extended example in Section 1.3. We omit the the redundant Cone-Circuit relations for which the opposite orientation is already given.

subspace of $V$ defined by intersecting some of the hyperplanes of $\mathcal{A}$. We denote the set of chambers and set of intersections of $\mathcal{A}$ by $\mathcal{C}(\mathcal{A})$ and $\mathcal{L}(\mathcal{A})$, respectively. Note that $\mathcal{L}(\mathcal{A})$ always contains $V$, the intersection of none of the hyperplanes. When ordered by reverse inclusion, $\mathcal{L}(\mathcal{A})$ is ranked by codimension and satisfies these two conditions that define a geometric lattice [26, Definition 3.9]:

- (Upper Semi-Modular) For all $X, Y \in \mathcal{L}(\mathcal{A})$, the codimensions of $X$ and $Y$ satisfy
  \[
  \text{codim}(X) + \text{codim}(Y) \geq \text{codim}(X \cap Y) + \text{codim}(X \cap Y)
  \]
  where $X \cap Y$ is the intersection $X \cap Y$, and $X \cap Y$ denotes the lowest-dimensional subspace $Z \in \mathcal{L}(\mathcal{A})$ containing both $X$ and $Y$.
- (Atomic) Every $X \in \mathcal{L}(\mathcal{A})$ is an intersection of some of the hyperplanes of $\mathcal{A}$.

Our choice of normal vectors $v_1, \ldots, v_n$ induces an orientation of $\mathcal{A}$, which we use to define a pair of halfspaces $H^+$ and $H^-$ for every hyperplane $H \in \mathcal{A}$:

\[
H^+ = \{x \in \mathbb{R} \mid v_i \cdot x > 0\}
\]
\[
H^- = \{x \in \mathbb{R} \mid v_i \cdot x < 0\}.
\]

With this notation, each hyperplane yields a decomposition of $V$ into three sets: the positive halfspace $H^+$, the negative halfspace $H^-$, and the hyperplane itself. To emphasize this relationship, we sometimes use a superscript 0 to denote the hyperplane itself, thus decomposing $V$ into $H^+ \sqcup H^- \sqcup H^0$.

A cone $\mathcal{K}$ of an arrangement $\mathcal{A}$ is an intersection of halfspaces defined by some of the hyperplanes of $\mathcal{A}$. To simplify notation, we assume that $\mathcal{A}$ is oriented so that $\mathcal{K}$ is an intersection of positive halfspaces. We use $W \subseteq [n]$ to denote the set of indices of a (potentially redundant) set of defining hyperplanes of $\mathcal{K}$, chosen among the hyperplanes of $\mathcal{A}$. We call $W$ the set of walls of the cone. Note that there may be several choices of walls which define the same cone $\mathcal{K}$ (when $\mathcal{K}$ is viewed as a subset of the vector space). Our main theorem will show that these different choices of walls for $\mathcal{K}$ produce the same Varchenko-Gelfand ring.
The notions of chambers and intersections naturally extend to cones: a chamber of a cone \( K \) is a chamber of \( A \) contained in the open set \( K \) and an intersection \( X \) of \( K \) is an intersection of \( A \) with \( X \cap K \neq \emptyset \). We denote the set of chambers and intersections of \( K \) by \( \mathcal{C}(K) \) and \( \mathcal{L}^{\text{int}}(K) \), respectively. Although \( \mathcal{L}^{\text{int}}(K) \) is not a geometric lattice when ordered by reverse inclusion, every lower interval, \([V, X]\) for \( X \in \mathcal{L}^{\text{int}}(K) \), is a geometric lattice.

In the introduction, we defined the Poincaré polynomial of a cone and showed that it can be computed from the interior intersection poset by

\[
Poin(K, t) = \sum_{X \in \mathcal{L}^{\text{int}}(K)} |\mu(V, X)| \ t^{rk(X)}.
\]

We noted, furthermore, that a theorem of Zaslavsky relates \( Poin(K, t) \) to the number of chambers in a cone. Combining Zaslavsky’s theorem with the Möbius function definition of the Poincaré polynomial (above) gives

\[
\#\mathcal{C}(K) = \sum_{X \in \mathcal{L}^{\text{int}}(K)} |\mu(V, X)|.
\]

In the introduction, we gave another interpretation of the Poincaré polynomial in terms of the \( K \)-NBC sets, which arise from thinking about the oriented matroid of the arrangement. We will introduce these sets in Section 2.2 and here give an example of the poset-interpretation of \( Poin(K, t) \).

**Example 3.** Consider the arrangement in \( \mathbb{R}^2 \) below on the left with the given orientation. The cone \( K = H_1^+ \) with \( W = \{1\} \) is the shaded region below on the right.

The Hasse diagrams of \( \mathcal{L}(A) \) (left) and \( \mathcal{L}^{\text{int}}(K) \) (right) are below, together with \( \mu(V, X) \) for each \( X \) (circled value).

The associated Poincaré polynomials are \( Poin(A, t) = 1+3t+2t^2 \) and \( Poin(K, t) = 1+2t \), respectively. It is easy to see from the pictures in Example 3 that the arrangement has \( 1+3+2 = 6 \) chambers and that the cone has \( 1+2 = 3 \) chambers.

2.2. **Oriented Matroids.** The collection of normal vectors \( \{v_1, \ldots, v_n\} \) of \( A \) naturally gives rise to an oriented matroid. The theory of (oriented) matroids arising from hyperplane arrangements is well-studied and there are many excellent sources on this topic including [5], [20, Section 2.1], and [26, Lecture 3]. We briefly review some basics but refer the reader to the preceding sources for a more detailed discussion.
Let $E$ be a finite set. A signed set $D = (D^+, D^-)$ of $E$ is a disjoint, ordered pair of subsets $D^+, D^- \subseteq E$. For a signed subset $D$ of $E$ and $e \in E$ and $D$, define

$$D_e := \begin{cases} + & \text{if } e \in D^+ \\ - & \text{if } e \in D^- \\ 0 & \text{else} \end{cases}$$

The collection $D := \{ e \in E : D_e \neq 0 \}$ is called the (unsigned) support set of $E$. Each signed set $D$ has an “opposite” signed set $-D$ with the same (unsigned) support set but $(-D)_e = -(D_e)$ for all $e \in D$. For an arbitrary pair $C, D$ of signed sets, we use their separating set to keep track of the places where their signs are opposite, i.e. if $C, D$ are signed sets of $E$ then the separating set of $C$ and $D$ is

$$S(C, D) := \{ e \in E \mid C_e = -D_e \neq 0 \}.$$ 

It’s easy to see that the separating set of $D$ and $-D$ is $D$. Finally we define the composition product of two signed sets $C, D$ with the same ground set $E$. The composition of $C$ with $D$ is the signed set $C \circ D$ where

$$(C \circ D)_e = \begin{cases} C_e & \text{if } C_e \neq 0 \\ D_e & \text{else} \end{cases} \text{ for } e \in E.$$

**Definition 4.** Let $E$ be a finite set and $\mathcal{D}$ a collection of signed subsets of $E$. The pair $\mathcal{D}$ is the set of vectors of an oriented matroid on $E$ if $\mathcal{D}$ satisfies the vector axioms:

V0. $0 \in \mathcal{D}$
V1. If $D \in \mathcal{D}$, then $-D \in \mathcal{D}$.
V2. If $C, D \in \mathcal{D}$, then $C \circ D \in \mathcal{D}$.
V3. If $C, D \in \mathcal{D}$ and $e \in S(C, D)$ then there exists $F \in \mathcal{D}$ with

$$F_e = 0$$

$$F_f = (C \circ D)_f \text{ for } f \in E \setminus S(C, D).$$

The main proof of this paper concerns the connection between an oriented matroid and its dual. For an oriented matroid $M = (E, \mathcal{D})$, there is a unique oriented matroid $M^* = (E, \mathcal{D}^*)$ with vectors

$$\mathcal{D}^* = \{ (F^+, F^-) \mid D \perp F \text{ for all } D \in \mathcal{D} \}$$

where $F \perp D$ if $F$ and $D$ are orthogonal, i.e. $\{ F_e \cdot D_e \mid e \in E \}$ either equals $\{0\}$ or contains $\{+, -\}$ [17, §6.2.5]. We call $M^*$ the dual matroid to $M$ and call $\mathcal{D}^*$ the covectors of $M$. One can show that the covectors of $M$ also satisfy the vector axioms so that the dual oriented matroid is in fact an oriented matroid.

We will be concerned with oriented matroids defined by sets of vectors $\{v_1, \ldots, v_n\}$ in $\mathbb{R}^d$, which naturally come equipped with signed dependencies given by linear combinations. Whenever $\sum_{v \in D} \lambda_v v = 0$, one has a signed dependency $D = (D^+, D^-)$ where

$$D_v = \begin{cases} + & \text{if } \lambda_v > 0 \\ - & \text{if } \lambda_v < 0 \\ 0 & \text{else} \end{cases}$$

In this context, one can think of the composition product of as a sum of dependencies where the second dependency is multiplied by a small, positive number.

The covectors of this oriented matroid also have a well-known geometric interpretation via (nonempty) intersections of halfspaces defined by some of the hyperplanes of $\mathcal{A}$, see [2, Section 1.1.3] for example. We will use the fact that if an intersection $\bigcap_i H^e_i$ is nonempty for $\{e_i\}_i$, then there is a covector $F \in \mathcal{D}^*$ such that $F_i = e_i$ for all $i \in E$.
The minimal, nonempty signed dependencies of an oriented matroid are called signed circuits and we denote the set of all signed circuits of \( \mathcal{D} \) by \( \mathcal{C} \), i.e.

\[
\mathcal{C} := \{ C \in \mathcal{D} \mid C \text{ is minimal under inclusion}, C \neq 0 \}.
\]

A theorem of Bland-Las Vergnas and Edmonds-Mandel [5, Theorem 3.7.5] says that every vector \( D \in \mathcal{D} \) is a composition of signed circuits, i.e. there is some \( k \in \mathbb{Z} \) and collection \( C^{(1)}, \ldots, C^{(k)} \in \mathcal{C} \) such that

\[
D = C^{(1)} \circ C^{(2)} \circ \cdots \circ C^{(k)}.
\]

Furthermore, one can select the circuits of this composition so that they conform to \( D \), meaning that for all \( i = 1, \ldots, k \) and \( e \in D \): if \( C^{(k)}_e \) is nonzero then \( C^{(i)}_e = D_e \) [5 Proposition 3.7.2]. We will use a weaker version in the proof of our main theorem: every signed dependence \( D \) can be written as a composition of circuits \( D = C^{(1)} \circ C^{(2)} \circ \cdots \circ C^{(k)} \) and it is easy to see that the first circuit \( C^{(1)} \) always conforms to \( D \).

In order to simplify notation, we will hereafter conflate a vector \( v_i \) with its index \( i \). We take \( E = [n] \), so that \( \mathcal{C} \) is a collection of signed subsets of \([n]\). For \( C \in \mathcal{C} \), we say that \( C - \{i\} \) is a broken circuit if \( i \) is the smallest index (under the usual order on \([n]\) in which \( 1 < 2 < 3 < \cdots < n \)) such that \( i \in C \). We will also consider the no broken circuit sets of \( A \), denoted \( NBC(A) \), which are the subsets of \([n]\) containing no broken circuits. A no broken circuit set \( N \in NBC(A) \) of \( A \) is a \( K \) no broken circuit set or \( K \)-NBC set if

\[
\bigcap_{i \in N} H^0_i \in \mathcal{L}^{\text{int}}(K).
\]

If we take our cone to be the intersection of no halfspaces, i.e. the set of walls is empty, then we recover the full arrangement, and the \( K \)-NBC sets are precisely the usual NBC sets of the arrangement. We will denote the set of \( K \)-NBC sets by \( NBC(K) \). In the introduction, we saw that the \( K \)-NBC sets provide a secondary description for the Poincaré polynomial

\[
Poin(K,t) = \sum_{N \in NBC(K)} t^\#N.
\]

By setting \( t = 1 \), we obtain Equation (1), which says \( \#C(K) = \#NBC(K) \). This equality will be central in our understanding of the Varchenko-Gel’fand ring, which we turn to now.

### 2.3. The Varchenko-Gel’fand Ring

The Varchenko-Gel’fand ring of an arrangement \( A \) is the ring of maps \( f : C(A) \to \mathbb{Z} \) under pointwise addition and multiplication [29]. Similarly, we define the Varchenko-Gel’fand ring of a cone \( K \) to be the ring with underlying set

\[
VG(K) = \{ f : C(K) \to \mathbb{Z} \}
\]

under pointwise addition and multiplication. We can represent elements of \( VG(K) \) as a labelling of the chambers of \( K \) with integers.

**Example 5.** Consider the cone from Example 3. Below are several elements of \( VG(K) \):

\[
x_1 = \begin{array}{c}
H_1 \\
\circ \\
\circ \\
H_2 \\
\circ \\
H_3
\end{array} \quad x_2 = \begin{array}{c}
H_1 \\
\circ \\
\circ \\
H_2 \\
\circ \\
H_3
\end{array} \quad x_3 = \begin{array}{c}
H_1 \\
\circ \\
\circ \\
H_2 \\
\circ \\
H_3
\end{array}
\]

In the preceding example, the elements are suggestively labelled \( x_1, x_2, \) and \( x_3 \) to represent Heaviside functions (defined below) given by some orientation of the hyperplanes \( H_1, H_2, \) and \( H_3 \). The Heaviside function associated to \( H_3 \) is not included, as it would be 1 on every chamber of the cone.
In Varchenko and Gel’fand’s original paper \cite{29}, they observe that $VG(A)$ is generated as a $\mathbb{Z}$-algebra by Heaviside functions

$$x_i(C) = \begin{cases} 1 & \text{if } C \subseteq H_i^+ \\ 0 & \text{else} \end{cases}$$

for each hyperplane $H_i \in \mathcal{L}(A)$. It suffices to check that if $f : \mathcal{C}(A) \to \mathbb{Z}$ is

$$f = \sum_{C \in \mathcal{C}(A)} f(C) \prod_{j \in W_C} x_j \prod_{j \in W_C} (1 - x_j).$$

where $W_C := \{ i \mid H_i \cap C \neq \emptyset \} \subseteq [n]$ is the set of hyperplanes which have a nonempty intersection the closure of $C$. Their proof extends without modification to the cone case, where $x_i(C)$ is 1 when both $C \subseteq H_i^+$ and $C \in \mathcal{C}(K)$, and 0 otherwise. When $W$ is a choice of walls for the cone $K$, this means that $x_i \equiv 1$ for each $i \in W$.

Remark 6. We will usually view the Varchenko-Gel’fand ring of a cone as a ring of functions $\mathcal{C}(K) \to \mathbb{Z}$. However, it is also a quotient of $VG(A)$: one has a surjective restriction map $\mathbf{res} : VG(A) \to VG(K)$ sending $f \mapsto f|_K$ defined by $f|_K(C) := f(C)$ for $C \in \mathcal{C}(K)$.

In the previous section, we introduced oriented matroids and defined a family of sets called the $K$-NBC sets. By the definition of the Varchenko-Gel’fand ring, we have $VG(K) \cong \mathbb{Z}^{\#K}$. Combining this isomorphism with Equation (1) implies

$$VG(K) \cong \mathbb{Z}^{\#K} = \mathbb{Z}^{\#NBC(K)}.$$

This chain of equivalences will be crucial in the proof of the main theorem, which provides an explicit basis for $VG(K)$ in terms of the $K$-NBC monomials $e_N = \prod_{i \in N} e_i$ for $N \in NBC(K)$.

### 2.4. Some Commutative Algebra

This section reviews some commutative algebra material on polynomial rings over $\mathbb{Z}$ and their quotients. For more details, see \cite{13,12}.

#### 2.4.1. Monomial orders

A polynomial in $\mathbb{Z}[e_1, \ldots, e_n]$ is a sum

$$f = \sum_{a = (a_1, \ldots, a_n)} c_a e_1^{a_1} \cdots e_n^{a_n}$$

where $c_a \in \mathbb{Z}$. When $c_a \neq 0$, one calls $c_a e_1^{a_1} \cdots e_n^{a_n}$ a term of $f$, and $e_1^{a_1} \cdots e_n^{a_n}$ a monomial of $f$. Define $\deg(f) := \max \{ \sum_i a_i : c_a \neq 0 \}$ and then the degree-initial form of $f$ is

$$\text{in}_{\deg}(f) := \sum_{\sum_i a_i = \deg(f)} c_a e_1^{a_1} \cdots e_n^{a_n}.$$

A monomial ordering is a total (linear) order $\prec$ well-ordering on the set of all monomials $m$ in $\mathbb{Z}[e_1, \ldots, e_n]$ which respects multiplication in the sense that $m \prec m'$ implies $m \cdot m'' \prec m' \cdot m''$ for all monomials $m, m', m'' \in \mathbb{Z}[e_1, \ldots, e_n]$. Define the $\prec$-leading monomial $\text{in}_\prec(f)$ to be the $\prec$-highest monomial of $f$. Say that $\prec$ is a degree order if it is compatible with $\text{in}_{\deg}$ in the sense that $\text{in}_\prec(f) = \text{in}_\prec(\text{in}_{\deg}(f))$ for all $f$; see Sturmfels \cite{27} Chapter 1 for more on these notions. Given a collection $\mathcal{G} = \{ g_i \}_{i \in I}$ of polynomials, say that a monomial $m$ is in $\prec(\mathcal{G})$-standard if it is divisible by none of $\{ \text{in}_\prec(g_i) \}_{i \in I}$. 
2.4.2. Filtrations and Associated Graded Rings. Let $R$ be a commutative ring with unit. An (ascending) filtration of $R$ is a sequence $F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots$ of nested $\mathbb{Z}$-submodules of $R$ with the property that if $f \in F_i$ and $g \in F_d$, then $f \cdot g \in F_{i+d}$. In this paper, we consider the degree filtration $\{F_d\}_{d \geq 0}$ for quotient rings $R = \mathbb{Z}[e_1, \ldots, e_n]/I$, where $I$ is an ideal of $\mathbb{Z}[e_1, \ldots, e_n]$: define $F_d$ to be the image within $R$ of the polynomials in $\mathbb{Z}[e_1, \ldots, e_n]$ having degree at most $d$. Define the associated graded ring

$$\text{gr}(R) := \text{gr}_{\mathcal{F}}(R) := \bigoplus_{d \geq 0} F_d/F_{d-1}$$

where we define $F_{-1} := 0$.

Recall that the rank of a $\mathbb{Z}$-module $M$ is $\text{rank}_\mathbb{Z}(M) := \dim_\mathbb{Q} \left( \mathbb{Q} \otimes \mathbb{Z} M \right)$, see [12, Section 11.6]. In the setting of a degree filtration, each $F_d$ is a finitely generated $\mathbb{Z}$-module, allowing us to define the Hilbert series of the associated graded ring:

$$\text{Hilb}(\text{gr}(R), t) := \sum_{d \geq 0} \text{rank}_\mathbb{Z}(F_d/F_{d-1}) \ t^d.$$ 

For example, we will wish to consider the associated graded ring of the Varchenko-Gel’fand ring with its degree filtration $V(K) := \text{gr}(V(G(K)))$, with Hilbert series $\text{Hilb}(V(K), t)$.

The proof of Theorem 1 in Section 3 uses a certain general lemma, which we state and prove now. Experts may recognize this lemma as a standard fact from Gröbner basis theory when the polynomial rings are defined over a field, but the modification here relates to polynomial rings over $\mathbb{Z}$; see Remark 8 below.

**Lemma 7.** Let $\preceq$ be a monomial order and assume one has a $\mathbb{Z}$-algebra surjection $S := \mathbb{Z}[e_1, \ldots, e_n] \xrightarrow{\psi} R$ in which $R$ is a free $\mathbb{Z}$-module of rank $r$, and $G = \{g_i\}_{i \in I} \subseteq S$ has these properties:

(i) $G \subset \ker \varphi$.

(ii) Each $g_i$ is $\preceq$-monic, meaning $\text{in}_{\preceq}(g_i)$ has coefficient $\pm 1$ in $g_i$.

(iii) The set of in$_{\preceq} G$-standard monomials $N = \{m_1, \ldots, m_t\}$ has cardinality $t \leq r$.

Then one has these implications:

(a) $\ker(\varphi) = (G)$, so that $\varphi$ induces a $\mathbb{Z}$-algebra isomorphism $S/(G) \cong R$.

(b) The cardinality $\# N = t = r$, and $R$ has $\varphi(N) = \{\varphi(m_1), \ldots, \varphi(m_r)\}$ as a $\mathbb{Z}$-basis.

If $\preceq$ is also a degree ordering, then one has two further implications:

(c) The map $S \xrightarrow{\psi} \text{gr}(R)$ sending $e_i \mapsto \bar{e}_i$ in $F_1/F_0$ is surjective, with $\ker(\psi) = (\text{in}_{\deg}(G))$, so that it induces a $\mathbb{Z}$-algebra isomorphism

$$S/(\text{in}_{\deg}(G)) \xrightarrow{\psi} \text{gr}(R)$$

(d) For $d \geq 0$, each $F_d/F_{d-1}$ is a free $\mathbb{Z}$-module on the basis $\{m_i \in N : \deg(m_i) = d\}$, so that

$$\text{Hilb}(\text{gr}(R), t) = \sum_{i=1}^{r} t^{\deg(m_i)}.$$ 

**Proof.** Note since each $g_i$ in $G$ is $\preceq$-monic, the usual multivariate division algorithm with respect to $G$ using the order $\prec$ (see Cox, Little and O’Shea [10, §2.3, Theorem 3]) shows that every $f$ in $S$ lies in $\mathbb{Z}m_1 + \cdots + \mathbb{Z}m_t + (G)$). Therefore, if one defines a $\mathbb{Z}$-module map $\mathbb{Z}^t \to S$ that sends the $i^{th}$ standard basis element of $\mathbb{Z}^t$ to the in$_{\prec}(G)$-standard monomial $m_i$, then the composite $\mathbb{Z}^t \to S \to S/(G)$ is surjective. This composite is the map $\alpha$ in this sequence of $\mathbb{Z}$-module surjections/isomorphisms

$$\mathbb{Z}^t \xrightarrow{\alpha} S/(G) \xrightarrow{\beta} S/\ker(\varphi) \cong R \cong \mathbb{Z}^r,$$

where $\beta$ comes from our assumption (i) above. It is well-known (see Chapter 2, Exercise 12, for example) that if $M$ is a $\mathbb{Z}$-module with $\text{rank}_\mathbb{Z}(M) = r \geq t$, then any surjection $\mathbb{Z}^t \to M$ must in fact
be an isomorphism, with $t = r$. It follows that the composite of all maps in (2) is an isomorphism. Thus $\beta$ and $\beta \circ \alpha$ are isomorphisms, proving assertions (a) and (b), respectively.

Now assume further that $\prec$ is a degree ordering. Since $\deg(\indeg(g_i)) = \deg(g_i)$, replacing $G$ with $\indeg(G)$, we conclude as in the above proof that the composite map
\[
Z^r \to S \to S/(\indeg(G))
\]
is surjective. The fact that $S \overset{\psi}{\to} R$ is surjective implies that $S \overset{\psi}{\to} \text{gr}(R)$ is also surjective. Furthermore, the definitions of $\indeg(-)$ and $\text{gr}(R)$, together with $G \subset \ker(S \overset{\psi}{\to} R)$, imply
\[
\indeg(G) \subset \ker(S \overset{\psi}{\to} \text{gr}(R)).
\]
Hence we again have a sequence of surjections and isomorphisms:
\[
Z^r \overset{\gamma}{\to} S/(\indeg(G)) \overset{\delta}{\to} S/\ker(\psi) \cong \text{gr}(R).
\]
Note that $\text{rank}_Z(\text{gr}(R)) = \text{rank}_Z R = r$, since $\text{rank}_Z(-)$ is additive along short exact sequences and direct sums. Thus we can again conclude that the composite of the surjections in (3) is an isomorphism. Hence $\delta$ is an isomorphism, proving (c). Then (d) follows from $\delta \circ \gamma$ being an isomorphism, upon noting that a monomial $m$ in $S$ has $\psi(m)$ lying in $F_d/F_{d-1}$ where $d = \deg(m)$.

**Remark 8.** Replacing $\mathbb{Z}$ by a field $\mathbb{F}$, and replacing $\text{rank}_Z(-)$ with $\text{dim}_F(-)$, the proof of Lemma 7 shows $G$ and $\indeg(G)$ give Gröbner bases for the ideals presenting the rings $R$ and $\text{gr}(R)$.

### 3. Relations Among the Heaviside Functions and the $\indeg(G)$-Standard Monomials

Given a cone $K$ in an arrangement $A$, let $G$ be the elements shown in the second column of the table in Figure 1. In this section, we give two propositions regarding $G$, which together prove Theorem 1. First we show that the polynomials $G$ lie in the kernel of the map $\varphi : \mathbb{Z}[e_1, \ldots, e_n] \to VG(K)$ which sends the variable $e_i$ to the Heaviside function $x_i$ for each $i \in [n]$. After that, we fix a monomial order $\prec$ on $\mathbb{Z}[e_1, \ldots, e_n]$ whose restriction to the variables is $e_1 \prec \cdots \prec e_n$. We will show that the $K$-$\text{NBC}$ monomials are exactly the $\indeg(G)$-standard monomials. In fact, since Equation (11) implies $VG(K) \cong \mathbb{Z}^\#C(K) = \mathbb{Z}^\#N_{\text{NBC}}(K)$, using Lemma 7 it suffices to show that the $\indeg(G)$-standard monomials are a subset of the $K$-$\text{NBC}$ monomials.

**Proposition 9.** Every polynomial in $G$ lies in $\ker \varphi$.

**Proof.** This holds for idempotent relations $e_i^2 - e_i$ since Heaviside functions $x_i$ have $x_i(C) \in \{0, 1\}$. It holds for Unit relations $e_i - 1$ with $i \in W$, since then $H^+_i \supseteq K$, so $x_i(C) \equiv 1$ for all $C$ in $C(K)$.

To understand the Circuit and Cone Circuit relations, note that the existence of a signed circuit $C = (C^+, C^-)$ implies that these two intersections are empty, and hence contain no chambers:
\[
\bigcap_{i \in C^+} H^+_i \cap \bigcap_{j \in C^-} H^-_j = \emptyset = \bigcap_{i \in C^-} H^+_i \cap \bigcap_{j \in C^+} H^-_j
\]
Consequently, if one writes the Circuit relation as the difference $f_+ - f_-$ of these two products
\[
f_+ := e_{C^+} \cdot \prod_{j \in C^-} (e_j - 1) \quad \text{and} \quad f_- := e_{C^-} \prod_{j \in C^+} (e_j - 1),
\]
one finds that both $f_+, f_-$ lie in $\ker \varphi$, and hence so does the Circuit relation $f_+ - f_-$. For a Cone Circuit relation, assume without loss of generality that the signed circuit $C = (C^+, C^-)$ has $\emptyset \neq W \cap C^+ = W \cap C$. Then since $\ker \varphi$ contains the product $f_+$ defined in (4) along with the Unit relations $e_i - 1$ for $i \in W \cap C^+$, it also contains the Cone Circuit relation $e_{C^+ \setminus W} \cdot \prod_{j \in C^-} (e_j - 1)$.
Remark 10. Note that if the signed circuit $C = (C^+, C^-)$ has $W \cap C^+ \neq \emptyset$, the element $f_-$ is divisible by Unit relations $e_i - 1$ for $i \in W \cap C^+$, and hence is superfluous in generating $\ker \varphi$. Similarly, if $W \cap C^- \neq \emptyset$, then $f_+$ is a redundant generator. Combining these: if both $W \cap C^- \neq \emptyset$ and $W \cap C^+ \neq \emptyset$, then both $f_+, f_-$ are redundant and so is the corresponding Circuit relation.

Also note, we $H_i \in A$ is neither one of the chosen set of defining hyperplanes nor the union $W \cup \{i\}$ contains a signed circuit $C = (C^+, C^-)$. Furthermore, one can show that $i \in C$ and that the Cone-Circuit relation does not vanish, i.e. one of $e_i - 1$ or $e_i$ is in $G$.

**Proposition 11.** The in$_<(G)$-standard monomials are (a subset of the) $\mathcal{K}$-NBC monomials.

*Proof.* Let $m \in \mathbb{Z}[e_1, \ldots, e_n]$ be any in$_<(G)$-standard monomial. We show that $m$ is a $\mathcal{K}$-NBC monomial in several reduction steps.

**Reduction 1.** Since $e_i^2 \in$ in$_<(G)$ for $1 \leq i \leq n$, we may assume that $m = e_N$ for some $N \subseteq \{1, \ldots, n\}$.

**Reduction 2.** Since $e_i \in$ in$_<(G)$ for $i \in W$, we may assume that $m = e_N$ with $W \cap N = \emptyset$.

**Reduction 3.** We can assume that $m = e_N$ where $N$ contains no broken circuits, i.e. $N \in NBC(A)$. To see this, suppose $N$ contains a signed circuit $C$ with $i_0 = \min(C)$ such that the corresponding broken circuit $C \backslash \{i_0\}$ is contained in $N$.

Since $W \cap N = \emptyset$ (from Reduction 2) and $C \backslash \{i_0\} \subseteq N$, either $i_0 \in W$ or $W \cap C$ is empty. We obtain a contradiction in both cases. First, if $W \cap C = \{i_0\}$, then

$$N \supseteq C \backslash \{i_0\} = C \backslash W,$$

forcing $e_N$ to be divisible by $e_{C \backslash W}$, and contradicting that $e_N$ is in$_<(G)$-standard. On the other hand, if $\# W \cap C = 0$, then $e_N$ is divisible by $e_{C \backslash \{i_0\}}$ which contradicts the assumption that $e_N$ is in$_<(G)$-standard.

**Reduction 4.** Assuming that $m = e_N$ where $N$ is in $NBC(A)$, we will show that it also lies in $NBC(\mathcal{K})$, that is, $X := \cap_{j \in N} H_j^0$ has $\mathcal{K} \cap X \neq \emptyset$. For the sake of contradiction, assume

$$\mathcal{K} \cap X = \bigcap_{i \in W} H_i^+ \cap \bigcap_{j \in N} H_j^0 = \emptyset.$$ 

It is easy to see that there is a choice of signs $\varepsilon \in \{+,-\}^N$ for which

$$\bigcap_{i \in W} H_i^+ \cap \bigcap_{j \in N} H_j^{\varepsilon_j} = \emptyset$$

(see Observation 12 below, for details). Translating Equation (5) into the language of oriented matroids, we have that there is no covector $F = (F^+, F^-)$ of the matroid on $E = W \cup N$ with

$$F_j = \begin{cases} + & \text{if } j \in W \\ \varepsilon_j & \text{if } j \in N. \end{cases}$$

From Observation 13 (below) or, equivalently, Gordan’s Theorem [18], the fact that no such $F$ exists, means that there does exist a (nonzero) signed dependence $D = (D^+, D^-) \in \mathcal{D}$ with $D \subseteq W \cup N$ and $(W \cap D) \subseteq D^+$.

Recall from [5] Theorem 3.7.5] that every signed dependence $D$ is a composition of circuits and that at least one of these circuits must conform to $D$. Let $C$ be such a circuit conforming to $D$. Then $C$ has $C \subseteq W \cup N$ and $(W \cap C) \subseteq C^+$. Since $N$ is an NBC set, the corresponding collection of vectors $\{v_i\}_{i \in N}$ is independent and we can assume that $W \cap C$ is nonempty. Thus its Cone-Circuit relation has initial form $e_{C \backslash W}$ dividing $e_N$, contradicting $e_N$ being in$_<(G)$-standard. \[\square\]
Combining the preceding proposition with Lemma [7] gives a proof of Theorem [1]. For completeness, we now state two observations about oriented matroids, which were used in the preceding proof. The first concerns the geometric interpretation of covectors as faces of hyperplane arrangements and the second observation connects the non-existence of a covector to the existence of a vector.

**Observation 12.** If \( K \cap X = \bigcap_{i \in W} H_i^+ \cap \bigcap_{j \in N} H_j^\varepsilon_j = \emptyset \) then there is some choice of signs \((\varepsilon_j)_{j \in N}\) in \(\{+, -\}^N\) such that

\[
K \cap X = \bigcap_{i \in W} H_i^+ \cap \bigcap_{j \in N} H_j^{\varepsilon_j} = \emptyset.
\]

In particular, the signed set \( F^\varepsilon = (F^\varepsilon)^+, (F^\varepsilon)^-\) with

\[
F^\varepsilon_i = \begin{cases} + & \text{if } i \in W \\ \varepsilon_i & \text{if } i \in N. \end{cases}
\]

is not a covector of the oriented matroid on ground set \( E = W \cup N \).

Another way to phrase \( X \cap K = \emptyset \) is: there are no covectors \( F \) having \( F_i = + \) for all \( i \in W \) and \( F_j = 0 \) for all \( j \in N \). With that in mind, the observation holds because if no such choice of signs \((\varepsilon_j)_{j \in N}\) existed, one would obtain a family of covectors \( \{F^\varepsilon\}_{\varepsilon \in \{+, -\}^N} \) to which one could repeatedly apply the elimination axiom V3 and reach such a covector \( F \) having \( F_j = 0 \) for \( j \in N \).

**Observation 13 ( [32] Section 6.3).** From the definition of an oriented matroid dual, we know that every \( F \in \mathcal{D}^* \) is orthogonal to every vector \( D \in \mathcal{D} \). In particular if there is a signed set \( F \) on \([n]\) that is not in \( \mathcal{D}^* \), then there is some \( D \in \mathcal{D} \) such that \( \{F_e \cdot D_e \mid e \in E\} \) contains exactly one of + or −.

**Remark 14.** The crux of the preceding proof only uses statements about vectors and covectors valid for oriented matroids. Hence our results remain valid in that setting, as in the generalization by Gel’fand and Rybnikov [15] of the work by Gel’fand and Varchenko [29] to oriented matroids.

## 4. Proof of Theorem [2]

In this section we prove Theorem [2], asserting that when one works over a field \( \mathbb{F} \), the associated graded ring \( V_G(\mathcal{A}) \) is Koszul whenever \( \mathcal{A} \) is a supersolvable arrangement. Some of the most well-studied hyperplane arrangements are supersolvable, and supersolvable arrangements are interesting, for example, because their Poincaré polynomial \( \text{Poin}(\mathcal{A}, t) \) factors into linear factors [4] [26] Corollary 4.9. Koszulity, on the other hand, is also interesting from an algebraic perspective. Koszul algebras come equipped with a natural Koszul dual quadratic algebra \( A^! \), and the relationship between \( A, A^! \) has implications for the coefficients of the Hilbert series of \( A \).

Let \( \mathbb{F} \) be a field. Recall that \( V G_{\mathbb{F}}(\mathcal{K}) \) is the collection of maps \( \{f : C(\mathcal{K}) \to \mathbb{F}\} \) with pointwise addition and multiplication. Theorem [1] extends without modification to \( V G_{\mathbb{F}}(\mathcal{K}) \), and in fact some of the proofs are easier since \( G \) forms a Gröbner basis (see Remark [3]).

Before beginning the proof of Theorem [2], we remind the reader of some standard results relating to supersolvable lattices and Koszul algebras. For a more detailed reference on Koszul algebras, we point the reader toward [13]. Let \( R \) be a *commutative standard graded* \( \mathbb{F} \)-*algebra*, i.e. \( R \cong \mathbb{F}[e_1, \ldots, e_n]/I \) where \( I \) is a homogeneous ideal and each \( e_i \) has degree exactly 1. Suppose

\[
F_* : \cdots \xrightarrow{\varphi_3} R^{\beta_3} \xrightarrow{\varphi_2} R^{\beta_1} \xrightarrow{\varphi_1} R \longrightarrow \mathbb{F}
\]

is a *minimal free resolution* of the \( R \)-module \( \mathbb{F} = R/R_+ \) where \( R_+ \) is the maximal homogeneous ideal, consisting of all elements of positive degree. For details on free resolutions, see [12]. We say \( R \) is *Koszul* if the nonzero entries of each \( \varphi_i \) matrix are homogeneous of degree 1. Say that an ideal

\footnote{This does not hold for cones of supersolvable arrangements, see [11] Remark 5.6.}
is monomial if it is generated by monomials. A monomial ideal is $G$-quadratic if it has a Gröbner basis of monomials of degree two. It is well-known (see [8], [12] Chapter 15, [13] Section 4, for example) that:

**Proposition 15.** If $I$ a homogeneous ideal in $\mathbb{F}[e_1, \ldots, e_n]$ is generated by a Gröbner basis $G$ consisting of quadratic elements for some monomial order $\prec$, then $R = \mathbb{F}[e_1, \ldots, e_n]/I$ is Koszul.

We are now prepared to define a supersolvable arrangement.

**Definition 16** (Definition 1.1, [25] Definition 4.13]). A lattice $L$ is supersolvable if there exists a maximal chain $\Delta$ satisfying: for every chain $K$ of $L$, the sublattice generated by $\Delta$ and $K$ is distributive. An arrangement $\mathcal{A}$ is supersolvable if $L(\mathcal{A})$ is supersolvable.

The following result and its proof are analogous to a result of Peeva [21, Theorem 4.3].

**Theorem 2** If $\mathcal{A}$ is a supersolvable arrangement, then $V_{\mathbb{F}}(\mathcal{A})$ is Koszul.

**Proof.** A theorem of Björner and Ziegler [6] Theorem 2.8] tells us that when $\mathcal{A}$ is a supersolvable arrangement, one can choose a linear ordering of the hyperplanes $H_1, \ldots, H_n$ such that every broken circuit $C \setminus \{i_0\}$ contains some broken circuit of size two. Choose $\prec$ a degree monomial order on $\mathbb{Z}[e_1, \ldots, e_n]$ which restricts to the same linear order on the variables $e_1 \prec \cdots \prec e_n$.

We wish to use the presentation $V_{\mathbb{F}}(\mathcal{A}) = \mathbb{F}[e_1, \ldots, e_n]/I$ where $I = (\text{in}_{\deg}(G))$ that comes from Theorem [1]. From Remark [8], the generators $\text{in}_{\deg}(G)$ form a Gröbner basis for $I \subset \mathbb{F}[e_1, \ldots, e_n]$. Because $\mathcal{A}$ is a full arrangement, not a cone, $\text{in}_{\deg}(G)$ will contain only $\text{in}_{\deg}(g)$ for idempotent and Circuit relations $g$. The idempotent relations correspond to generators $\text{in}_{\deg}(g) = e_i^2$ that are all quadratic.

Each Circuit relation corresponds to a generator $\text{in}_{\deg}(g)$ which may not be quadratic: its degree is the size of the broken circuit $C \setminus \{i_0\}$, with $\text{in}_{\deg}(g) = e_{C \setminus \{i_0\}}$. Since each is a squarefree monomial, it suffices to consider the monomials who indexing set is minimal under inclusion. The Björner-Ziegler result [6] Theorem 2.8] implies that the minimal (under inclusion) broken circuits all have cardinality 2. From Proposition [15] it follows that $V_{\mathbb{F}}(\mathcal{A})$ is Koszul.

One might ask if Theorem 2 has a cone analogue. Sadly there are cones $\mathcal{K}$ of supersolvable arrangements whose $V_{\mathbb{F}}(\mathcal{K})$ are not Koszul as we demonstrate by example. A particularly well-studied family of supersolvable arrangements are the Type $A$ reflection arrangements or braid arrangements; see [26] Cor. 4.10, Example 4.11(c)]. The braid arrangement $A_n - 1$, consists of the $\binom{n}{2}$ hyperplanes

$$H_{ij} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_i - x_j = 0\}$$

for each pair $\{i, j\}$.

We wish to exhibit a cone $\mathcal{K}$ inside a braid arrangement for which $V(\mathcal{K})$ is not Koszul. One way to prove something is not Koszul uses the following.

**Theorem 17** (Section 4]). Let $A$ be a Koszul algebra. Then there is another algebra $A^t$, the quadratic dual of $A$, whose Hilbert series is $\text{Hilb}(A^t, t) = 1/\text{Hilb}(A, -t)$. In particular, if $A$ is Koszul, then $1/\text{Hilb}(A, -t)$ has positive coefficients when considered as a power series in $\mathbb{Z}[t]$.

**Example 18.** The cone of $A_5$ given by

$$\mathcal{K} = \{x \in \mathbb{R}^6 \mid x_1 \leq x_2, x_3 \leq x_4, x_5 \leq x_6\}$$

does not yield a Koszul $V_{\mathbb{F}}(\mathcal{K})$. The Hilbert series of $V_{\mathbb{F}}(\mathcal{K})$ is

$$\text{Hilb}(V_{\mathbb{F}}(\mathcal{K}), t) = 1 + 12t + 43t^3 + 30t^3 + 4t^4.$$
The first few terms of \( \frac{1}{\text{Hilb}(\mathcal{V}_F(K), -t)} \) are

\[
\frac{1}{1 - 12t + 43t^3 - 30t^3 + 4t^4} = 1 + 12t + 101t^2 + 725t^3 + 4725t^4 + 28464t^5 + 159769t^6 \\
+ 832122t^7 + 3950417t^8 + 16302972t^9 + 50092317t^{10} \\
+ 15264030t^{11} - 1497513779t^{12} + \cdots
\]

The coefficient of \( t^{12} \) is negative, meaning that \( \frac{1}{\text{Hilb}(\mathcal{V}_F(K), -t)} \) is not the Hilbert series of a ring.

This counterexample begs the following question:

**Question 19.** Is there some simple, combinatorial condition on \( L(\mathcal{A}) \) for a cone \( K \), in the spirit of supersolvability for the full lattice \( L(\mathcal{A}) \) of the arrangement, that implies Koszulity of \( \mathcal{V}_F(K) \)?

Even in the arrangement case, the connection between supersolvable arrangements and Koszul \( \mathcal{V}_F(A) \) remains opaque. Fröberg tells us that the converse to Proposition 15 is false in general [14, Note on p.39], but one might ask: if \( \mathcal{V}_F(A) \) is Koszul, is \( A \) supersolvable? The analogous question is famously open for Orlik-Solomon algebras [21, Example 4.5], [30, Example 6.22].

**Acknowledgements**

The author thanks Victor Reiner for motivating discussions, Franco Saliola for enlightening conversations about SAGE [28], and the anonymous reviewer for many helpful comments. She also thanks Sarah Brauner, Darij Grinberg, and Trevor Karn for pointing out typos/giving feedback on drafts of this paper.

**References**

[1] William W. Adams and Philippe Loustaunau. *An introduction to Gröbner bases*, volume 3 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 1994.

[2] Marcelo Aguiar and Swapneel Mahajan. *Topics in hyperplane arrangements*, volume 226 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2017.

[3] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Series in Mathematics. Westview Press, Boulder, CO, economy edition, 2016.

[4] Hans-Jürgen Bandelt, Victor Chepoi, and Kolja Knauer. COMs: complexes of oriented matroids. *J. Combin. Theory Ser. A*, 156:195–237, 2018.

[5] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler. *Oriented matroids*, volume 46 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, second edition, 1999.

[6] Anders Björner and Günter M. Ziegler. Broken circuit complexes: factorizations and generalizations. *J. Combin. Theory Ser. B*, 51(1):96–126, 1991.

[7] Kenneth S. Brown. Semigroups, rings, and Markov chains. *J. Theoret. Probab.*, 13(3):871–938, 2000.

[8] Aldo Conca. Gröbner bases for spaces of quadrics of low codimension. *Adv. in Appl. Math.*, 24(2):111–124, 2000.

[9] R. Cordovil. A commutative algebra for oriented matroids. volume 27, pages 73–84. 2000. Geometric combinatorics (San Francisco, CA/Davis, CA, 2000).

[10] David A. Cox, John Little, and Donal O’Shea. *Ideals, varieties, and algorithms*. Undergraduate Texts in Mathematics. Springer, Cham, fourth edition, 2015. An introduction to computational algebraic geometry and commutative algebra.

[11] Galen Dorpalen-Barry, Jang Soo Kim, and Victor Reiner. Whitney numbers for poset cones. *Order*, 2021.

[12] David Eisenbud. *Commutative algebra*, volume 150 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[13] R. Fröberg. Koszul algebras. In *Advances in commutative ring theory (Fez, 1997)*, volume 205 of Lecture Notes in Pure and Appl. Math., pages 337–350. Dekker, New York, 1999.

[14] Ralph Fröberg. Determination of a class of Poincaré series. *Math. Scand.*, 37(1):29–39, 1975.

[15] I. M. Gel’fand and G. L. Rybnikov. Algebraic and topological invariants of oriented matroids. *Dokl. Akad. Nauk SSSR*, 307(4):791–795, 1989.

[16] Regina Gente. *The Varchenko Matrix for Cones*. PhD thesis, Universität Marburg, 2013.
Jacob E. Goodman, Joseph O’Rourke, and Csaba D. Tóth, editors. *Handbook of discrete and computational geometry*. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, 2018. Third edition of [MR1730156].

P. Gordan. Ueber die Auflösung linearer Gleichungen mit reellen Coefficienten. *Math. Ann.*, 6(1):23–28, 1873.

Daniel Moseley. Equivariant cohomology and the Varchenko-Gelfand filtration. *J. Algebra*, 472:95–114, 2017.

Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

Irena Peeva. Hyperplane arrangements and linear strands in resolutions. *Trans. Amer. Math. Soc.*, 355(2):609–618, 2003.

Nicholas Proudfoot. The equivariant Orlik-Solomon algebra. *J. Algebra*, 305(2):1186–1196, 2006.

Gian-Carlo Rota. On the foundations of combinatorial theory. I. Theory of Möbius functions. *Z. Wahrscheinlichkeitsrechnung und Verw. Gebiete*, 2:340–368 (1964), 1964.

Bruce E. Sagan. A generalization of Rota’s NBC theorem. *Adv. Math.*, 111(2):195–207, 1995.

R. P. Stanley. Supersolvable lattices. *Algebra Universalis*, 2:197–217, 1972.

Richard P. Stanley. An introduction to hyperplane arrangements. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 389–496. Amer. Math. Soc., Providence, RI, 2007.

Bernd Sturmfels. *Gröbner bases and convex polytopes*, volume 8 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1996.

The Sage Developers. *SageMath, the Sage Mathematics Software System (Version x.y.z)*, 2021. [https://www.sagemath.org](https://www.sagemath.org).

A. N. Varchenko and I. M. Gel’fand. Heaviside functions of a configuration of hyperplanes. *Funktsional. Anal. i Prilozhen.*, 21(4):1–18, 96, 1987.

S. Yuzvinskii. Orlik-Solomon algebras in algebra and topology. *Uspekhi Mat. Nauk*, 56(2(338)):87–166, 2001.

Thomas Zaslavsky. A combinatorial analysis of topological dissections. *Advances in Math.*, 25(3):267–285, 1977.

Günter M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

Fakultät für Mathematik, Ruhr-Universität Bochum, Germany

Email address: galen.dorpalen-barry@rub.de