Poles of degenerate Eisenstein series and Siegel-Weil identities for exceptional split groups

Thesis Submitted in Partial Fulfillment of the Requirements for the Master of Sciences Degree

By: Hezi Halawi

Under the Supervision of: Dr Nadya Gurevich

Beer Sheva, August 2016
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Abstract

Let $G$ be a linear split algebraic group. The degenerate Eisenstein series associated to a maximal parabolic subgroup $E_P(f^0, s, g)$ with the spherical section $f^0$ is studied in the first part of the thesis. In this part, we study the poles of $E_P(f^0, s, g)$ in the region $\text{Re} \, s > 0$. We determine when the leading term in the Laurent expansion of $E_P(f^0, s, g)$ around $s = s_0$ is square integrable. The second part is devoted to finding identities between the leading terms of various Eisenstein series at different points. We present an algorithm to find this data and implement it on SAGE. While both parts can be applied to a general algebraic group, we restrict ourself to the case where $G$ is split exceptional group of type $F_4, E_6, E_7$, and obtain new results.
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Preface

Classical Eisenstein series

Let $G$ be a linear Lie group with a Lie algebra $\mathfrak{g}$. Let $Z(\mathfrak{g})$ be the center of the universal enveloping algebra $U(\mathfrak{g})$. Let $K$ be a maximal compact subgroup of $G$. There is a faithful representation $G \hookrightarrow GL_n$ that allows to define a norm $|||\cdot|||$ on $G$. Finally let $\Gamma$ be a discrete subgroup of $G$.

Let $\mathcal{A}(\Gamma \backslash G)$ be the space of automorphic forms, i.e all functions $f : G \to \mathbb{C}$ that satisfy the following conditions:

1. $f$ is smooth.
2. $f$ is invariant under the action of $\Gamma$ i.e. for every $\gamma \in \Gamma$ $f(\gamma g) = f(g)$.
3. $f$ is $K$–finite, i.e. the space spanned by the translations of $f$ under elements of $K$ is finite dimensional.
4. $f$ is $Z(\mathfrak{g})$ finite.
5. $f$ is of moderate growth i.e. there exists $C > 0$ and $n \in \mathbb{N}$ such that for all $g \in G$ it holds that $||f(g)|| \leq C||g||^n$.

An automorphic form $f$ is called **spherical** if it is in addition right invariant under the action of the maximal compact subgroup $K$. In this case the automorphic function can be consider as function on the space of cosets $\Gamma \backslash G/K$. The classical example is as follows:

Let $G = SL_2(\mathbb{R})$ whose maximal compact subgroup is $K = SO_2(\mathbb{R})$. The space $G/K$ can be identified with the upper half plane $\mathbb{H} = \{x + iy \in \mathbb{C} : y > 0\}$ via $g \to g.i$ where $G$ acts on $\mathbb{H}$ by Mobius transformations. One has $Z(\mathfrak{g}) \simeq \mathbb{C}[\Delta]$ where $\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$. Finally let $\Gamma = SL_2(\mathbb{Z})$. 
For a complex number \( s \) define \( h_s : \mathbb{H} \to \mathbb{C} \) to be

\[
\begin{align*}
  h_s(z) &= \sum_{(c, d) \in \mathbb{Z}^2 \setminus \{(0, 0)\}} \frac{y^{s+\frac{1}{2}}}{|cz + d|^{2s+1}} \\
  z &= x + iy \in \mathbb{H}
\end{align*}
\]

The sum converges absolutely for \( \Re(s) > \frac{1}{2} \). For fix \( z \in \mathbb{H} \) the function \( h_s(z) \) admits a meromorphic continuation to the whole complex plane.

As a function of \( z \in \mathbb{H} \) the series is non-holomorphic but satisfies a differential equation:

\[
\Delta h_s(z) = (s - \frac{1}{2})(s + \frac{1}{2}) h_s(z).
\]

By re-arranging the sum in (1) we get

\[
\begin{align*}
  h_s(z) &= 2\zeta(2s+1) \sum_{(c, d) \in \mathbb{Z}^2 \setminus \{(0, 0)\}} \frac{y^{s+\frac{1}{2}}}{|cz + d|^{2s+1}} = 2\zeta(2s+1) \sum_{\gamma \in B(\mathbb{Z}) \setminus \Gamma} [\Im \gamma z]^{s+\frac{1}{2}}
\end{align*}
\]

where \( B(\mathbb{Z}) \) is the group of upper triangular matrices in \( \Gamma \). Therefore by dropping the \( \zeta \) factor we obtain the first example of an Eisenstein series on \( G \)

\[
E(z, s) = \sum_{\gamma \in B(\mathbb{Z}) \setminus \Gamma} \chi_s(\gamma z), \quad \Re s > \frac{1}{2}
\]

when \( \chi_s(z) = [\Im z]^{s+\frac{1}{2}} \). Notice that \( K \) leaves the point \( i \in \mathbb{H} \) invariant. By abuse of notation we denote by \( \chi_s \) the pullback of \( \chi_s \) to \( G \) as well. Therefore, the Eisenstein series \( E(z, s) \) gives rise to an automorphic function

\[
E(\cdot, s) : \Gamma \backslash SL_2(\mathbb{R})/SO(2, \mathbb{R}) \to \mathbb{C}
\]

defined by

\[
E(g, s) = \sum_{\gamma \in B(\mathbb{Z}) \setminus \Gamma} \chi_s(\gamma g), \quad \Re(s) > \frac{1}{2}.
\]

Let us reinterpret this function from the perspective of representation theory. Recall the Iwasawa decomposition \( G = N \cdot T \cdot K \) where \( N \) is the group of upper triangular unipotent matrices and \( T \) is the group of invertible diagonal matrices of \( G \). The function \( \chi_s : G \to \mathbb{C} \) satisfies:

\[
\chi_s(ntk) = \delta_B(t)^{s+1/2}
\]
and hence can be regarded as a spherical vector in $\text{Ind}_{B(\mathbb{R})}^{\text{SL}_2(\mathbb{R})} \delta_B^s$ (see Definition (1.16) for more details).

More generally we define $E(\cdot, \cdot, s)$ an operator from $\text{Ind}_{B(\mathbb{R})}^{\text{SL}_2(\mathbb{R})} \delta_B^s$ to the space of automorphic forms $\mathcal{A}(\Gamma \backslash \text{SL}_2(\mathbb{R}))$ by

$$E(g, \phi, s) = \sum_{\gamma \in B(\mathbb{Z}) \backslash \Gamma} \phi(\gamma g, s)$$

where $\phi(g, s)$ is any flat section in the induced representation i.e. section $\phi(g, s)$ whose restriction to $K$ it independent on $s$.

**Motivation**

The modern theory of automorphic forms concerns functions on adelic groups. Let $F$ be a number field and $\mathbb{A}$ be its ring of adeles. Let $G$ be a split algebraic group. The group $G(F)$ is a discrete subgroup in $G(\mathbb{A})$. From perspective of representation theory one can think of an Eisenstein series as $G(\mathbb{A})$-equivariant map from induced representations into the space of automorphic forms $\mathcal{A}(G(F) \backslash G(\mathbb{A}))$.

Precisely, let $P = MN$ be a standard parabolic subgroup of a reductive group $G$ and $\sigma$ be an automorphic representation of $M$, i.e. $\sigma$ is realized in the space of automorphic functions on $M$ (see section [?] for more details). Let $X^*(M) = \text{Hom}(M, G_m)$ be the set of algebraic characters of $M$. We denote by $I_P(\lambda \otimes \sigma)$ the induced representation.

The family of operators

$$E(\cdot, \lambda) : I_P(\sigma \otimes \lambda) \to \mathcal{A}(G(F) \backslash G(\mathbb{A})) \quad \lambda \in X^*(M) \otimes \mathbb{C}$$

is defined by

$$E(f, g, \lambda) = \sum_{\gamma \in P(F) \backslash G(F)} f(\gamma g, \lambda)(1)$$

for $\lambda$ in a shifted dominant chamber where the series converges. By a fundamental theorem of Langlands the operator $E$ admits a meromorphic continuation to the space $X^*(M) \otimes \mathbb{C}$.

One of the main goals of the theory of automorphic forms is to decompose $L^2(G(F) \backslash G(\mathbb{A}))$ into irreducible representations and for this, Eisenstein series is an indispensable tool.
The space $L^2(G(F) \setminus G(\mathbb{A}))$ can be written as sum of two orthogonal subspaces

$$L^2(G(F) \setminus G(\mathbb{A})) = L^2_{\text{disc}}(G(F) \setminus G(\mathbb{A})) \oplus L^2_{\text{cont}}(G(F) \setminus G(\mathbb{A}))$$

corresponding respectively to the discrete and continuous parts of the spectrum. Moreover, Langlands showed that $L^2_{\text{disc}}(G(F) \setminus G(\mathbb{A}))$ can be decomposed as

$$L^2_{\text{disc}}(G(F) \setminus G(\mathbb{A})) = L^2_{\text{res}}(G(F) \setminus G(\mathbb{A})) \oplus L^2_{\text{cusp}}(G(F) \setminus G(\mathbb{A}))$$

where $L^2_{\text{res}}$ is spanned by iterated residues of Eisenstein series associated to automorphic cuspidal representations and $L^2_{\text{cusp}}$ is the space of cusp forms. In this thesis we will concentrate of a different type of Eisenstein series, the ones associated to a degenerate principal series $I_P(s) = \text{Ind}^G_P(\delta^s_P)$, where $P$ is a maximal parabolic subgroup. Let us give a partial list of applications of degenerate Eisenstein series.

1. The famous Siegel-Weil formula relates a period of the theta function to a special values of a degenerate Eisenstein series.

2. The degenerate Eisenstein series occur in Rankin-Selberg integral representations of L-functions and hence its poles determine the poles of L-functions. See ([1], [2], [3]).

3. The minimal representation of $G$, when exists, can be realized as a residue of degenerate Eisenstein series. See [4].

In this thesis we study the poles of degenerate Eisenstein series in the right half plane and the relation between their leading terms. Previous studies on this area can be viewed for example in [5],[6],[7]. We obtain new results for split exceptional groups where $G = F_4, E_6, E_7$.

**Strategy to studied poles of Eisenstein series**

By the fundamental theorem of Langlands see[], the poles of the degenerate Eisenstein series $E_P(f, g, s)$ coincide with the poles of its constant term along the the Borel subgroup

$$E^0_P(f, g, s) = \int_{U(F) \setminus U(\mathbb{A})} E_P(f, ug, s)du$$

where $U$ is the unipotent radical of the Borel.
Theorem. The constant term $E_P^0(f, g, s) = \sum_{w \in W_G/W_P} M_w(s)f^0_s$ where $M_w$ is the intertwining operator

The definition and the properties of $M_w(s)$ are summarized in Theorem (1.18). Thus, the union set of poles of various intertwining operators contains the set of poles of Eisenstein series, but by no means is equal to it. The reason is possible cancellations of the poles from various summands. Moreover, even if a pole is not canceled its order can be lower then the order of the pole of each summand as in the case that it is canceled.

Although many examples have been worked out the nature of these cancellations remains a mystery.

The degenerate induced representation associated to the trivial character contains unique spherical section $f^0_s(g)$ normalized such that $f^0_s(1) = 1$. Using Gindikin-Karpelevich formula, Theorem (1.19), for the action of $M_w(s)$ on the spherical function we can evaluate the sum of intertwining operators applied to the vector $f^0_s(g)$ and witness the cancellations.

The approach is computational. We have produced an algorithm that being realized in the SAGE computes the poles and their order of all spherical degenerate Eisenstein series for split groups of small rank.

**Our Results: for the spherical Eisenstein series $E_P(f^0, g, s)$**

We have obtained new results for the split exceptional groups of type $F_4, E_6, E_7$.

Theorem. Let $G = E_7$ and $P = P_4$ then the Eisenstein series $E_P(f^0, s, g)$ admits a pole of order 4 at $s = \frac{1}{8}$.

To the best of our knowledge this is the first example of Eisenstein series with order equal to 4. The results we obtain for $G_2, GL_n, n \leq 8$ and $Sp_n, n < 8$ coincide with the results obtained earlier in [8],[6],[9].

We find out that all the poles for $\text{Re } s > 0$ are real.
Theorem. (Theorem (4.6)) If $F = \mathbb{Q}$ and $s_0 \in \mathbb{R}$ then the order of a pole of $E_P(f^0, g, s)$ at a point $s_0$ is bounded by $d_P(\chi_{s_0})$ where

$$d_P(\chi_{s_0}) = |\alpha > 0, \langle \alpha^\vee, \chi_{s_0} \rangle = 1| - |\alpha > 0, \langle \alpha^\vee, \chi_{s_0} \rangle = 0| - (n-1)$$

and $\chi_{s_0} = \delta_{\frac{s_0}{2}} \otimes \delta_{\frac{1}{P}}$.

Observation. If $F = \mathbb{Q}$ then for all the cases that we studied the order of the spherical Eisenstein series is actually equal to $d_P(\chi_{s_0})$.

The residual representation

Suppose that $E_P(\cdot, g, s)$ admits a pole of order $m$ at $s = s_0$. Consider its Laurent expansion at $s = s_0$

$$E_P(\cdot, s, g) = \sum_{i=-m}^{\infty} \Lambda_{P}^{-i}(\cdot, s_0, g)(s - s_0)^i.$$ 

Proposition. The leading term in the Laurent expansion $\Lambda_{P}^{-m}(\cdot, s_0, g)$, i.e. the first term that it is not zero, defines an intertwining map between $I_P(s_0)$ and the space of automorphic forms.

The image is called residual representation and is denoted by $\Pi$.

The spherical vector in $I_P(s)$ generates a subrepresentation $I_P^0(s)$. If $E_P(f^0, g, s)$ admits a pole of order $m$ at $s = s_0$ then the leading term in the Laurent expansion $\Lambda_{P}^{-m}(f^0, s_0, g)$ defines an intertwining map between $I_P^0(s_0)$ and the space of automorphic forms. The image is called spherical residual representation and is denoted by $\Pi^0$.

It is expected that for any $s$ such that $\text{Re}s > 0$ the highest order of the pole $E_P(\cdot, g, s)$ will be attained by the spherical section. While the analogous statement is known for the intertwining operators $M_w(s)$ the expectation is not proven in general. Indeed the cancellation occurring for the spherical vector might not occur for an arbitrary section.

Square-integrability

To describe the residual representation $\Pi^0$ it is useful to know whether the function $\Lambda_{P}^{-m}(f^0, s_0, g)$ is contained in $L^2(G(F) \backslash G(\mathbb{A}))$. To find this out we use Langlands’
criterion Theorem (2.5).
If this is the case the spherical residual representation is necessary a direct sum of irreducible representations. In particular, \( \Pi^0 \) is isomorphic to the unique spherical irreducible quotient of \( I_P(s_0) \).

In case \( I_P(s_0) \) has unique irreducible quotient and \( \Lambda^m_{P,F}(f^0, s_0, g) \) is square integrable the residual representation is isomorphic to the unique spherical quotient of \( I_P(s_0) \).

All square integrable residual representations coming from degenerate Eisenstein series have cuspidal support \([T, 1]\). In the recent paper [10], the space \( L^2(G(F) \backslash G(\mathbb{A}))^K_{[T, 1]} \), consisting of square integrable functions having cuspidal support \([T, 1]\) and as representation of \( G(\mathbb{A}) \) generated by the spherical section, is studied. It discrete summands correspond to distinguished nilpotent orbits of the Lie algebra \( L^g \) of the dual group. We write explicitly the distinguished orbits corresponding to the residual representations that are square integrable see subsections (3.1.3),(??),(??),(??).

**Identities**

Some automorphic representations can be realized in several ways as leading terms of degenerate Eisenstein series. For example the trivial function can be realized as the residue of \( E_P(f^0, s, g) \) at \( s = \frac{1}{2} \) for any maximal parabolic subgroup \( P \). This is an analogue of the statement that a representation can be a common quotient of several degenerate principal series.

In the second part of the thesis we explore identities involving the leading terms of various Eisenstein series. Generally speaking, we would like to realize a given representation as residues of various Eisenstein series in a hope to gain additional information about it.

For example, the minimal representation of the exceptional group \( E_7 \) has been realized as a residue of a degenerate Eisenstein series associated to a non-Heisenberg parabolic subgroup in [4] and later as a residue at a different point for Eisenstein series associated to a Heisenberg parabolic subgroup. The second realization proved to be useful for certain exceptional theta lift [11].

A necessary condition for the identity between leading terms of Eisenstein series is the induced representations \( I^0_{P,F}(s_0) \), \( I^0_{Q,F}(t_0) \) having a common quotient. These representations are subquotients of \( I_B(\chi_{s_0}) \), \( I_B(\chi_{t_0}) \) respectively where \( \chi_{s_0} = \delta^s_0 \otimes ^s_1 \otimes ^s_{B,F} \) and \( \chi_{t_0} = \delta^t_0 \otimes ^t_1 \otimes ^t_{Q,F} \). This leads to the following definition.
Definition 0.1. The quintuple \((P, s_0, Q, t_0, w)\) where \(P, Q\) are maximal parabolic subgroups, \(s_0, t_0 \in \mathbb{R}\) and \(w \in \mathcal{W}_G\) is called \textbf{admissible data} if

\[
w(\delta_{P}^{s_0 - \frac{1}{2}} \otimes \delta_{B}^{\frac{1}{2}}) = \delta_{Q}^{t_0 - \frac{1}{2}} \otimes \delta_{B}^{\frac{1}{2}}.
\]

If also \(s_0, t_0\) are non negative numbers it is called \textbf{positive admissible data}.

The (positive) admissible data can be found by direct computation as explained in Chapter 6. In this chapter we also list all the positive admissible data for the exceptional groups \(G_2, F_4, E_6, E_7\).

Note the curious chains of the admissible data. We have pairs of the form \((P_1, s_1, P_2, s_2, w_1)\) and \((P_2, s_2, P_3, s_3, w_2)\) leading to the chains of identities. See section 6.5.

Let us state our main theorem regarding the spherical Eisenstein series.

\textbf{Theorem 0.2.} Let \(F = \mathbb{Q}\) and \((P, s_0, Q, t_0, w)\) be an admissible data. Let \(f^0_P, f^0_Q\) be spherical sections in \(I_P(s_0), I_Q(t_0)\) respectively.

There exists a constant \(C \in \mathbb{C}^*\) that depends on the admissible data such that

\[
\Lambda_{-d_P(\chi_{s_0})}^P(f^0_P, s_0, g) = C \Lambda_{-d_Q(\chi_{t_0})}^Q(f^0_Q, t_0, g).
\]

\textbf{Remark 0.2.1.} We assume that \(F = \mathbb{Q}\) since in that case we know that \(\zeta\) does not have any zeros in the real line.

For \(F = \mathbb{Q}\) we observe if \(E_P(f^0, s, g)\) admits a pole of order \(d\) at \(s = s_0\) for \(\text{Re } s_0 > 0\) then \(d = d_P(\chi_{s_0})\). Hence Theorem (0.2) is an identity between the leading terms.

In Chapter 5 we give an explicit formula for the constant in Theorem (0.2). For the groups \(G = F_4, E_6, E_7\) we list all the positive admissible data and determine the constant. Using this we write the identities explicitly.

The identities of this type has been explored before by Dihua Jiang for the symplectic groups in [7].
Thesis structure

Below is the outline of the thesis.

Chapter 1 is introductory. In section 1.1 we set all the notations for an algebraic group $G$ defined over a field $F$, such as root datum and Weyl group.

In section 1.2 we recall the standard definitions and facts from the representation theory of split reductive group $G(F)$, where $F$ is a local field. In particular we define induced representations and intertwining operators between them and recall their properties.

Section 1.3 refers to the space of automorphic forms of $G$. We define a degenerate Eisenstein series, global intertwining operators and recall their properties.

In Chapter 2 we explain our basic algorithm that allows us to compute all the poles and their orders of all degenerate spherical Eisenstein series $E_p(f^0, g, s)$ in the right-half plane for the groups $G = G_2, F_4, E_6, E_7$.

Our algorithm works for any group whose rank is not too big. In particular the results we obtain for $G_2$ and $GL_n, n \leq 8$ agree with the results of [8] and [6].

In Chapter 3 we present the results for the groups of type $G_2, F_4, E_6, E_7$.

In Chapter 4 we prove the preliminary version of our main theorem. Our approach follows Ikeda [12].

Using the $W$ invariance and entireness of the normalized Eisenstein series we prove the identity between two leading terms of degenerate Eisenstein series at an admissible data up to a non-zero constant.

We compute this constant in Chapter 5.

The Chapter 6 is devoted to the search of positive admissible data. We list all of them for the groups $G = F_4, E_6, E_7$. Note the curious chains of length three for the group $E_6$. This phenomenon seems to be new.

The sample of computation for certain parabolic of $G = F_4$ at a fixed point is attached in the appendix A.
Chapter 1

Introduction

1.1 Algebraic groups and their structure

Let $F$ be a field, and let $G$ be an algebraic reductive split group over $F$. We denote by $G(F)$ the group of $F$ points of $G$. For every algebraic group we can define its Lie algebra by the following procedure.

Let $A = F[G]$, we consider the Lie algebra $\text{Der}(A)$ of all $F$– derivations $f : A \rightarrow A$ with the Lie bracket $[f, g] = f \circ g - g \circ f$. On $A$ we can define a left action of $G$ as follows: for every $g \in G$ we denote by $\lambda_g$ the action $\lambda_g(f)(h) = f(g^{-1}h)$ for every $h \in G$.

**Definition 1.1.** A derivation $\delta \in \text{Der}(A)$ is called a left–invariant if it commutes with left translations i.e. for every $g \in G$ it holds $\delta \circ \lambda_g = \lambda_g \circ \delta$.

The left invariant derivations of $A$ form a Lie subalgebra of $\text{Der}(A)$, called the Lie algebra of $G$ and denoted by $\mathfrak{g}$.

We fix a maximal split torus $T$ of $G$. Let $X(T) = \text{Hom}(T, \mathbb{G}_m)$ be the character group of $T$, and let $Y(T) = \text{Hom}(\mathbb{G}_m, T)$ be the cocharacter group. Both $X(T)$ and $Y(T)$ are free abelian groups of rank $\text{dim } T$. Moreover since the $\text{Aut}(\mathbb{G}_m) = \mathbb{Z}$ there is a pairing

$$\langle , \rangle : X(T) \times Y(T) \rightarrow \mathbb{Z}.$$

The maximal torus $T$ acts on the Lie algebra $\mathfrak{g}$ of $G$ via the adjoint action. Since $G$ is reductive - the zero eigenspace of $\mathfrak{g}$ with respect to $T$ is exactly the Lie algebra $\mathfrak{t}$ of
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Therefore we can decompose $g = t \oplus \bigoplus_{\alpha \in \Phi_G} g_\alpha$, where $\Phi_G$ is the set of all $0 \neq \alpha \in X(T)$ such that $g_\alpha = \{x \in g \mid t.x = \alpha(t)x \ \forall t \in T\} \neq 0$. The set $\Phi_G$ is called the set of roots of $G$.

**Definition 1.2.** A subgroup $B$ of $G$ is called a **Borel subgroup** if it is a maximal Zariski closed, connected and solvable.

**Remark 1.2.1.** All Borel subgroups of $G$ are conjugate in $G$.

From now we shall assume that $G$ is a linear group. So we fix $n \in \mathbb{N}$ and an embedding $\iota : G \hookrightarrow GL_n$. We identify $G$ with $\iota(G)$.

**Definition 1.3.** An element $g \in G$ is called an **unipotent** element of $G$, if $g-I_n$ is nilpotent, where $I_n$ is the identity of $GL_n$.

**Definition 1.4.** Let $G$ be a connected algebraic group. The **radical** of $G$, denoted by $R(G)$, is the maximal connected solvable normal subgroup of $G$. The **unipotent radical** $R_u(G)$ of $G$ is the subgroup of unipotent elements of $R(G)$.

We choose a Borel subgroup $B$ such that $T \subset B$, and denote its Lie algebra by $b$. Then there exists a closed connected normal subgroup of $B$ denoted by $U$ such that $B = TU$ is semi–direct product (see [13], section 6). Therefore, since $T$ acts on $B$, it also acts on $b$. Hence by choosing $B$, we also get a choice of positive roots $\Phi^+_G$ defined by $\alpha \in \Phi^+_G$ iff $g_\alpha \subset b$. We define the negative roots to be $\Phi^-_G = \Phi_G \setminus \Phi^+_G$.

**Definition 1.5.** A positive root $\alpha$ is called a **simple root** if it cannot be written as the sum of two positive roots. We denote the set of simple roots of $G$ by $\Delta_G$.

For every root $\alpha \in \Phi_G$ we can associate a homomorphism $x_\alpha : \mathbb{G}_a \to G$ such that for every $c \in \mathbb{G}_a$ and $t \in T$ it holds $tx_\alpha(c)t^{-1} = x_\alpha(\alpha(t)c)$. Moreover, we can define a morphism $\phi_\alpha : SL_2 \to G$ such that

$$
\phi_\alpha\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = x_\alpha(c), \quad \phi_\alpha\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = x_{-\alpha}(c).
$$

Define $\check{\alpha} : \mathbb{G}_m \to T$ by $\check{\alpha}(c) = \phi_\alpha\begin{pmatrix} c & 0 \\ 0 & \frac{1}{c} \end{pmatrix}$. The element $\check{\alpha} \in Y(T)$ and is called the coroot associated to the root $\alpha$. We denote by $\check{\Phi}_G$ the set of all coroots of $G$.

By the above construction we associate for every algebraic split reductive group a quadruple $(X(T), \Phi_G, Y(T), \check{\Phi}_G)$ with respect to the torus $T$, that is called a **root datum**.
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Let $\Omega = \{\tilde{\omega}_i\}_{i=1}^n$ be elements of $t^*$ such that $\langle \tilde{\omega}_j, \tilde{\alpha}_i \rangle = \delta_{ij}$. These elements are called fundamental weights. Obviously, $X(T) \subset \text{Span}_\mathbb{Z}\{\Omega\}$. Moreover, the quotient is finite.

For every root $\alpha \in \Phi_G$ we define the simple reflection $w_\alpha \in \text{Aut}(X(T)), w_\alpha \in \text{Aut}(Y(T))$ to be

$$w_\alpha(x) = x - \langle x, \tilde{\alpha} \rangle \alpha \quad w_\alpha(y) = y - \langle \alpha, y \rangle \tilde{\alpha}.$$

Let $\text{Norm}_G(T)$ be the normalizer of $T$ in $G$. The Weyl group of $G$ is defined to be $\mathfrak{W}_G =: \text{Norm}_G(T)/T$.

Remark 1.5.1. Since $\mathfrak{W}_G$ acts on $T$ it also acts on $X(T)$.

Proposition 1.5.1. $[13]$ $\mathfrak{W}_G \approx < w_\alpha \mid \alpha \in \Delta_G >$, and this is a finite group.

Parabolic subgroups

Let $G, B, T, \Delta_G$ be as above. For simplicity we denote by $\Delta = \Delta_G$.

Definition 1.6. A subgroup $P$ of $G$ is called a parabolic subgroup (resp. standard) if it contains a Borel subgroup (resp. $B$).

Proposition 1.6.1. $[13]$ Let $I \subset \Delta$, and let $\mathfrak{W}_I$ be the subgroup of $\mathfrak{W}_G$ generated by the subset $S_I = \{w_\alpha \mid \alpha \in I\}$. Denote by $P_I = B\mathfrak{W}_IB$. Then $P_I$ is a standard parabolic subgroup of $G$.

Proposition 1.6.2. $[13]$ The correspondence $I \mapsto P_I$ defines a bijection between subsets of simple roots and standard parabolic subgroups of $G$.

Definition 1.7. A parabolic subgroup $P$ will be called a maximal parabolic subgroup if the only subgroup $H$ of $G$ such that $P \subset H$ is $H = G$.

Remark 1.7.1. According to Proposition (1.6.2) a maximal parabolic subgroup $P$ corresponds to the set of simple roots $\Theta = \Delta \setminus \{\alpha_i\}$ for some $\alpha_i \in \Delta$. In that case we say that $P$ corresponds to $\alpha_i$ and we denote it by $P_i$.

Let $P$ be a parabolic subgroup of $G$ corresponding to $I \subset \Delta$. The set $\Phi_I$, which is the set of all $\alpha \in \Phi$ that are integral linear combinations of elements of $I$, forms a root system, with the Weyl group $\mathfrak{W}_I$. Furthermore, the set of roots of $P$ equals to $\Phi^+ \cup (\Phi^- \cap \Phi_I)$. We denote by $N_I = R_\alpha(P)$.

Proposition 1.7.1. $[13]$ $N_I = \langle x_\alpha(r) \mid \alpha \in \Phi^+ \setminus \Phi_I, \ r \in F \rangle$. 
Let \( T_I = (\cap_{\alpha \in \Phi_I} \ker \alpha)^0 \), and \( M_I = Z_G(T_I) \). The group \( M_I \) normalizes \( N_I \), and it holds that \( P = M_I N_I \) is a semi-direct product. This decomposition is known as the **Levi decomposition** of \( P \). The group \( M_I \) is called the **Levi factor** of \( P_I \).

Let \( P = MN \) be a parabolic subgroup of \( G \) with its Levi decomposition. We define \( \delta_P \in X(T) \otimes \mathbb{Q} \) to be \( \delta_P = \sum_{\alpha \in \Phi_I^+ \setminus \Phi_I^0} \alpha \).

We denote by \( X(M) \) (resp. \( Y(M) \)) the character (resp. cocharacter) group of \( M \).

Let \( P = MN \) be a parabolic subgroup of \( G \). Set

\[
\mathfrak{a}_M = \text{Hom}(X(M), \mathbb{R}),
\]

\[
\text{Re } \mathfrak{a}_M^* = X(M) \otimes_{\mathbb{Z}} \mathbb{R},
\]

\[
\mathfrak{a}_M^* = X(M) \otimes_{\mathbb{Z}} \mathbb{C}.
\]

**Remark 1.7.2.** If \( P = B \) we set \( \mathfrak{a} = \mathfrak{a}_T, \text{ Re } \mathfrak{a}^* = \text{ Re } \mathfrak{a}_T^*, \mathfrak{a}^* = \mathfrak{a}_T^* \).

We denote by \( F = \sum_{\alpha_i \in \Delta_G} \mathbb{C} \alpha_i \) and by \( \text{Re } F = \sum_{\alpha_i \in \Delta_G} \mathbb{R} \alpha_i \). Then it holds \( \mathfrak{a}^* = \mathfrak{z}^* \oplus F \) where \( \mathfrak{z}^* = \{ x \in \mathfrak{a}^* : \langle x, \hat{\alpha} \rangle = 0 \, \forall \alpha \in \Delta_G \} \).

**Remark 1.7.3.** Let \( P \) be a parabolic subgroup of \( G \), corresponds to the set of simple roots \( \Theta = \Delta \setminus \{ \alpha_{i_1}, \ldots, \alpha_{i_l} \} \). Then \( X(M) \simeq \mathbb{Z}^l \). Moreover, the vector space \( \mathfrak{a}_M^* \) is spanned by \( \{ \bar{\omega}_{i_1}, \ldots, \bar{\omega}_{i_l} \} \).

**Definition 1.8.** Two parabolic subgroup of \( G \) are called **opposite** if their intersection is the Levi component of each of them.

**Definition 1.9.** We denote by \( \hat{U} \) unipotent radical of the opposite Borel \( B' \).
1.2 Representation theory of reductive groups over local fields

Let \( G, T, B, \Delta \) be as above.

### 1.2.1 Non–archimedean fields

Let \( F \) be a non–archimedean local field. Let \( O \) be the ring of integers of \( F \), and \( q \) the cardinality of the residue field. By abuse of notation we write \( G \) as the group of \( F \)–points of \( G \). Let \( K = G(O) \), be a maximal open subgroup of \( G \). We fix a Haar measure \( \mu \) on \( G \), such that \( \mu(K) = 1 \).

**Definition 1.10.** A pair \((\pi, V)\) is called a representation of \( G \) if \( V \) is a \( \mathbb{C} \) vector space and \( \pi : G \to \text{GL}(V) \) is a group homomorphism.

**Definition 1.11.** A representation is called irreducible if it does not have a non–trivial invariant subspaces.

**Definition 1.12.** The representation \((\pi, V)\) of \( G \) is smooth if for any \( v \in V \), the stabilizer \( \text{Stab}_G(v) = \{ g \in G \mid \pi(g)(v) = v \} \) of \( v \) in \( G \) is an open subgroup of \( G \).

**Definition 1.13.** A smooth representation \((\pi, V)\) of \( G \) is called admissible if for any compact open subgroup \( C \) of \( G \) the space \( V^C = \{ v \in V \mid \pi(c)v = v \ \forall c \in C \} \) is finite dimensional.

**Definition 1.14.** Let \((\pi, V)\) be a smooth irreducible representation of \( G \). The representation \((\pi, V)\) is called spherical (unramified) if there is a non-zero vector \( v_0 \in V \) which is a \( K \)–fixed vector, i.e. \( \pi(k)v_0 = v_0 \) for all \( k \in K \). A non–zero \( v \in V^K \) is called a spherical vector.

Example: one dimensional representation \( \chi : G(F) \to \mathbb{C}^\times \) is spherical if \( \chi|_{G(O)} = 1 \). Such \( \chi \) is called an unramified character.

**Proposition 1.14.1** ([14],Proposition 4.6.2). Let \((\pi, V)\) be a smooth irreducible representation of \( G \). Then \( \dim V^K \leq 1 \).

Let \( P \) be a parabolic subgroup of \( G \) corresponding to \( I \subset \Delta \). We fix a left Haar measure \( \mu \) on \( P \). Since \( P \) is a locally compact topological group \( \mu \) is unique up to a positive scalar multiple. Since the right translation on \( \mu \) by an element \( p \in P \) is also a left Haar measure \( \mu_p \), there exists a function \( \delta_P : P \to \mathbb{R}_{>0} \) called modular character such that \( \mu_p = \delta_P(p)\mu \).
**Remark 1.14.1.** Let $P = MN$ be a parabolic subgroup with Levi decomposition. Hence it holds that $p = m \oplus n$, where $p, m, n$ is the Lie algebra of $P, M, N$. Then

$$\delta_P = \sum_{\alpha : g_{\alpha} \subset n} \alpha.$$ 

**Definition 1.15.** Let $H$ be a closed subgroup of $G$. Let $(\pi, V)$ be a smooth representation of $H$. We define the **induced representation**, denote by $\text{Ind}^G_H(\pi)$, to be the space of all functions $f : G \to V$ such that:

1. $f(hg) = \pi(h)f(g)$ for all $h \in H, g \in G$.
2. There exists a open compact subgroup $K_1$ of $G$ such that $f(gk) = f(g)$ for all $k \in K_1$.

**Proposition 1.15.1** ([15], Proposition 2.4.5). Let $H_1 \subset H_2$ be closed subgroups of $G$. Let $(\pi, V)$ be a smooth representation of $H_1$. Then

$$\text{Ind}^G_{H_2}(\text{Ind}^{H_2}_{H_1}(\pi)) \simeq \text{Ind}^G_{H_1}(\pi).$$

Induction from parabolic subgroup, which are closed, is an important class of smooth representations.

**Definition 1.16.** Let $P$ be a parabolic subgroup of $G$ with Levi decomposition $P = MN$. Let $(\pi, V)$ be a smooth representation of $M$. We define the **normalized parabolic induction**, denote by $\text{Ind}^G_P(\pi)$, to be the space of all functions $f : G \to V$ such that:

1. $f(mng) = \delta^1_P(m)\pi(m)f(g)$ for all $m \in M, n \in N, g \in G$.
2. There exists a open compact subgroup $K_1$ of $G$ such that $f(gk) = f(g)$ for all $k \in K_1$.

**Remark 1.16.1.** In our thesis we will restrict ourselves only to the case where $\pi$ is an unramified character of $M$.

**Remark 1.16.2.** Let $P_1 \subset P_2$ are parabolic subgroups of $G$, with Levi decomposition $P_i = M_i \times N_i$. Let $\chi$ be representation of $M_1$. Then by proposition (1.15.1) it holds that $\text{Ind}^G_{P_2}(\text{Ind}^{P_2}_{P_1}(\chi)) \simeq \text{Ind}^G_{P_1}(\chi)$.

**Theorem 1.17.** [15] Let $(\pi, V)$ be a spherical representation of $G$. Then $(\pi, V)$ is a constituent of $\text{Ind}^G_T(\chi)$, where $\chi$ is an unramified character of $T$. 
Chapter 1. Introduction

Let $\chi \in X(T) \otimes \mathbb{C}$ be an unramified character of $T$, $w \in \mathcal{W}_G$. We define, formally, the local intertwining operator

$$M_w(\chi) : \text{Ind}_B^G(\chi) \to \text{Ind}_B^G(w\chi)$$

$$(M_w(\chi))(f)(g) = \int_{U \cap wUw^{-1}\backslash U} f(w^{-1}ug)du.$$ 

**Theorem 1.18.** [16] The local intertwining operator $M_w(\chi)$ converges absolutely and uniformly as $g$ varies in a compact set for $\text{Re} \langle \chi, \hat{a} \rangle \gg 0$ for every $\alpha \in \Phi^+_G$. Moreover, it admits an analytic continuation as meromorphic function of $\chi \in X(T) \otimes \mathbb{C}$.

**Proposition 1.18.1.** [12] Let $w = w_1, w_2 \in \mathcal{W}_G$, such that $l(w) = l(w_1) + l(w_2)$ then for every $\chi \in X(T) \otimes \mathbb{C}$ it holds that $M_w(\chi) = M_{w_1}(w_2\chi) \circ M_{w_2}(\chi)$.

**Theorem 1.19.** [12] Let $f^0 \in \text{Ind}_B^G(\chi)$ be the normalized spherical vector. Then for every $w \in \mathcal{W}_G$ it holds that $M_w(\chi)f^0 = C_w(\chi)f^0_w$, where

$$C_w(\chi) = \prod_{\alpha \in \Phi^+_G} \frac{\zeta((\chi, \hat{a}))}{\zeta((\chi, \hat{a}) + 1)},$$

where $\zeta$ is the local $\zeta$ function.

As a corollary we get

**Corollary 1.20.** Let $w = w_1w_2$ be a reduced word. Then for any $\chi \in X(T)$ it holds that

$$C_w(\chi) = C_{w_1}(w_2\chi)C_{w_2}(\chi).$$
Proposition 1.20.1.

1. The induced representation \( \text{Ind}^G_B(\delta_B^{-\frac{1}{2}}) \) has a unique irreducible sub-representation \( \pi \). Moreover, \( \pi \) is the trivial representation.

2. The induced representation \( \text{Ind}^G_B(\delta_B^{\frac{3}{2}}) \) has a unique irreducible quotient \( \pi \). Moreover, \( \pi \) is the trivial representation. In this case the trivial representation is the image of \( M_{w_0} \) where \( w_0 \in W_G \) is the longest element.

Remark 1.20.1. Hence, using Proposition (1.18.1) we deduce that the trivial representation is a quotient of \( IP(\delta_P^{\frac{1}{2}}) \).

Remark 1.20.2. Let \( P = MN \) be a parabolic subgroup of \( G \). Let \( \tilde{\chi} \) be an unramified character of \( M \). Then,

\[
\text{Ind}^G_B(\text{Ind}^P_B(\delta_B^{-\frac{1}{2}})\tilde{\chi}) = \text{Ind}^G_B(\tilde{\chi} \otimes \delta_P^{\frac{3}{2}} \otimes \delta_B^{-\frac{1}{2}}).
\]

Let \( \chi = \tilde{\chi} \otimes \delta_B^{\frac{3}{2}} \otimes \delta_B^{-\frac{1}{2}} \) it holds that \( \text{Ind}^G_B(\chi) \hookrightarrow \text{Ind}^G_B(\tilde{\chi}) \). Thus, it is possible to restrict \( M_w(\chi) \) to \( \text{Ind}^G_B(\tilde{\chi}) \).

Definition 1.21. Let \( P = MN \) be a parabolic subgroup of \( G \). Let \( (\pi, V) \) be an admissible representation of \( G \). Denote by \( V(N) = \text{Span}\{\pi(n)v - v : n \in N, v \in V\} \). Let \( V_N = V/V(N) \) and \( (\pi_N, V_N) \) the corresponding \( M \)-representation. This is called the \textbf{Jacquet module} of \( V \) (with respect to \( P \)), and \( V \mapsto V_N \) is called the \textbf{Jacquet functor} (denoted by \( r_{M,G}(\pi) \)).

Let \( P = MN \) be a parabolic subgroup of \( G \), corresponding to \( I = \Delta_G \setminus J \), where \( J = \{\alpha_i \ldots \alpha_i\} \subset \Delta \). We set

\[
(a_M)_+^* = \{x \in \text{Re } a_M^* : \langle x, \tilde{\alpha}_i \rangle > 0 \ \forall \alpha_i \in J\},
\]

\[
^+a_M^* = \{x \in \text{Re } a_M^* : x = \sum_{\alpha \in J} c_\alpha \alpha \ \alpha \geq 0\}.
\]

Definition 1.22. Let \( \pi \) be an irreducible admissible representation of \( G \). Let

\[
M_{\text{min}} = \{L \text{ standard Levi : } r_{L,G}(\pi) \neq 0 \text{ but } r_{N,G}(\pi) = 0 \text{ for all } N \subset L\},
\]

\[
\text{Exp}(\pi) = \left\{ x \in \text{Re } a^* : x \otimes \rho \leq r_{L,G}(\pi) \text{ for some representation } \rho \text{ of } L \right\}
\]

with unitary central character, \( L \in M_{\text{min}}(\pi) \).
Definition 1.23. [17] Let \((\pi, V)\) be an irreducible representation of \(G\), with a unitary central character. Then

1. \(\pi\) is called square integrable iff \(\text{Exp}(\pi) \subset a_+^*\).

2. \(\pi\) is called tempered iff \(\text{Exp}(\pi) \subset +\overline{a}^*\).

Definition 1.24. Let \(P = MN\) be a parabolic subgroup of \(G\) corresponds to \(\Delta \setminus J\) where \(J \subset \Delta\). Let \(\lambda \in (a_M)^*_+\) and \(\sigma\) be an irreducible tempered representation of \(M\). The triple \((P, \sigma, \lambda)\) will be called Langlands data.

Theorem 1.25. [18] Let \((P, \sigma, \lambda)\) be a Langlands data then \(\text{Ind}^G_P(\sigma \otimes \lambda)\) has a unique irreducible quotient denote by \(J(P, \sigma, \lambda)\). Conversely, if \(\pi\) is an irreducible admissible representation of \(G\), then there exists a unique \((P, \sigma, \lambda)\) such that \(\pi \simeq J(P, \sigma, \lambda)\).
1.3 Automorphic forms

In this section $F$ is a number field and $A$ its ring of adeles. Let $\mathcal{P}$ be the set of places of $F$. For each $\nu \in \mathcal{P}$ we denote by $F_\nu$ the completion of $F$ with respect to $\nu$. Let $G$ be an algebraic reductive split group over $F$. For every $\nu \in \mathcal{P}$, we fix a maximal compact subgroup $K_\nu \subset G(F_\nu)$. Denote by $K = \prod_{\nu \in \mathcal{P}} K_\nu$, the maximal compact subgroup of $G(A)$. We fix a Haar measure $\mu$ on $G(A)$ such that $\mu(K) = 1$.

In our thesis we restrict ourself to very special representations called the automorphic representations. The definition can be read in [19]. The automorphic representations are realized in the space of automorphic functions, in the literature they are known as automorphic forms. The definition of such functions is in [20] (I.2.17). The space of automorphic forms of $G$ is denote by $A(G(F)\backslash G(A))$.

**Definition 1.26.** Let $P(A)$ be a parabolic subgroup of $G(A)$ with Levi decomposition $P(A) = M(A)N(A)$. Let $(\pi, V)$ be an automorphic representation of $M(A)$. We define the normalized parabolic induction, denote by $I_P(\pi) = \text{Ind}_{P(A)}^G(\pi)$, to be the space of all functions $f : G(A) \to V$ such that:

1. $f(mng) = \delta^\frac{1}{2}_P(m)\pi(m)f(g)$ for all $m \in M(A), n \in N(A), g \in G(A)$.

2. $f$ is right $K$–finite.

**Remark 1.26.1.** Let $P$ be a parabolic subgroup of $G$ correspond to $I = \Delta_G \setminus \{\alpha_i, \ldots, \alpha_i\}$. Then by remark (1.7.3), the set $X^\text{un}(M)$ of unramified characters of $M$ is isomorphic $\mathbb{C}^n$ by $(s_1, \ldots, s_n) \leftrightarrow \sum_{i=1}^n s_i \tilde{\omega}_i$.

**Remark 1.26.2.** When $P$ is a maximal parabolic subgroup we define $I_P(s) = \text{Ind}_P^G(\delta_P^s)$.

**Definition 1.27.** Let $B(A) = T(A)U(A)$ be a Borel subgroup of $G(A)$. Let $\chi$ be an automorphic character of $T(A)$. For every $w \in W_G$ we define the global intertwining operator to be:

$$M_w(\chi) : I_B(\chi) \to I_B(w\chi)$$

$$(M_w(\chi)f)(g) = \int_{U(A) \cap wU(A)w^{-1} \setminus U(A)} f(w^{-1}ug)du = \otimes_{\nu}(M_w(\chi_\nu)f_\nu)(g_\nu).$$

**Proposition 1.27.1.** [20] The global intertwining operator converges absolutely and uniformly as $g$ varies in a compact set if $\text{Re} \langle \chi, \tilde{\alpha} \rangle \gg 0$ for every $\alpha \in \Phi_G^+$ such that $w\alpha < 0$. Moreover it admits a meromorphic continuation for all $\chi \in \mathfrak{a}^*$. 
Let $P = MN$ be a parabolic subgroup of $G$. By Iwasawa decomposition $G(\mathbb{A}) = P(\mathbb{A})K$ we conclude that every $f \in IP(\chi s)$ is determined by its restriction to $K$. Moreover, every smooth function $f \in \{ f : K \to C : f(mk) = f(k) \quad \forall m \in M(\mathbb{A}) \cap K \}$ can be extended uniquely to $f_\mathbb{K} \in IP(\chi s) \quad \forall s \in \mathbb{C}^n$. This family $\{ f_\mathbb{K} : s \in \mathbb{C}^n \}$ is called a flat section.

**Definition 1.28.** Let $P$ be a parabolic subgroup of $G$. Given a flat section $f_\mathbb{K} \in IP(\chi s)$ such that $f_\mathbb{K}|_{K(\mathbb{A})} = f$ we define the Eisenstein series on $G(\mathbb{A})$ to be

$$E_P(f, g, s) = \sum_{\gamma \in P(F) \setminus G(F)} f_\mathbb{K}(\gamma g).$$

Whenever this series converges it defines an automorphic form.

Recall the properties of Eisenstein series

**Theorem 1.29.** [21] [Eisenstein series properties]

1. The Eisenstein series is left $G(F)$-invariant when it is defined.

2. There exists an open cone of $a^*_M$ such that the series converges absolutely and uniformly when $g$ varies in a compact set.

3. For any $f$ and $g$ the function $s \mapsto E_P(f, g, s)$ admits a meromorphic continuation to all of $\mathbb{C}^n$.

4. At a point $s_0$ where $E_P(f, g, s_0)$ is holomorphic for all $f$ and $g$, the function $E_P(f, s_0, \cdot)$ is an automorphic form on $G$. Also, the map $f \mapsto E_P(f, s_0, -)$ is a $G(\mathbb{A})$-equivariant map of $IP(\chi s)$ to $A(G(F) \setminus G(\mathbb{A}))$.

5. Let $w \in W_G$. Then

$$E_B(f, g, s) = E_B(M_w(s)f, g, ws).$$

**Definition 1.30.** Let $P$ be a maximal parabolic subgroup of $G$. We say that $s_0 \in \mathbb{C}$ is a pole of order $d$ of the Eisenstein series if $-d = \inf_{f, g} \text{ord}_{s=s_0}\{E_P(f, g, s)\}$.

At the point $s_0$ where $E_P$ has a pole we consider the Laurent expansion of $E_P(f, g, s)$ at $s = s_0$

$$E_P(f, g, s) = \frac{\Lambda^P_d(f, g, s_0)}{(s - s_0)^d} + \frac{\Lambda^P_{d+1}(f, g, s_0)}{(s - s_0)^{d-1}} + \cdots$$

where $d$ is the order of the pole attained at $s_0$ and $\Lambda^P_d(f, g, s_0)$ is the first non-zero term for at least one holomorphic section $f_{s_0} \in IP(s_0)$. 
Proposition 1.30. [21] The function \( f \mapsto \Lambda^P_d(f, g, s_0) \) is a \( G(\mathbb{A}) \)-equivariant map of \( I_P(s_0) \) to \( \mathcal{A}(G(F) \backslash G(\mathbb{A})) \).

Theorem 1.31. [21] The poles of the Eisenstein series are the same as the poles of its constant term along the Borel subgroup.
Chapter 2

Algorithm: Compute pole order

In this chapter we will describe the algorithm to determine the poles of the Eisenstein
series $E_P(f^0, g, s)$ and compute their orders.

The chapter outline is as follows:

The chapter starts with few definitions that introduce the language for describing the
algorithm. The chapter continues with the description of the algorithm itself. Finally
we will prove the correctness of the algorithm.

We fix a maximal parabolic subgroup $P$. We denote by $W(P, G)$ the set of all shortest
representative in $W_G/W_P$. Set $\chi_{P,s} = \delta_P^{1/2} \otimes \delta_B^{-1/2}$ since there is no risk of confusion, we
shall omit the subscript $P$.

Definition 2.1. Let $s_0 \in \mathbb{C}$, $s_0$ is called potential pole of $E_P(f^0, s, g)$ if $\text{Re } s_0 > 0$
and there exists $w \in W(P, G)$ such that $C_w(s)$ (as defined at 1.19) admits a pole at
$s = s_0$.

Definition 2.2. For a potential pole $s_0$ we said that $w, u \in W(P, G)$ are equivalent
if $u.\chi_{s_0} = w.\chi_{s_0}$. We denote it by $u \sim_{s_0} w$. The equivalence class will be denoted by
$[u]_{s_0}$.

Definition 2.3. For a potential pole $s_0$ and equivalence class $[u]_{s_0}$ we define

$$M_u^*(s) = \sum_{w \in [u]_{s_0}} C_w(s)$$

and by $N_u^*(s)$ its numerator.

Definition 2.4. For a potential pole $s_0$, an equivalence class $[u]_{s_0}$ is called square
integrable if $\text{Re}(u.\chi_{s_0})$ can be written as sum of simple roots with negative coefficients.
Chapter 2. Algorithm: Compute pole order

After we put the language we are ready to describe our algorithm.

**Pseudo code**

1. Find out the set of all shortest representative in $\mathbb{W}_G/\mathbb{W}_P$ (denoted by $W(P, G)$).

2. Find out the set of potential poles. (denoted by $\eta(P)$).

3. For each potential pole $s_0 \in \eta(P)$ do:
   
   (a) Divide $W(P, G)$ into equivalence classes.

   (b) For each equivalence class do:
      
      i. Define $M_w^\#$

      ii. Write Laurent expansion of $N_w^\#$ around $s = s_0$. we denote by $d_0$ the formal order of $N_w^\#$ at $s_0$.

      iii. Find out if the equivalence class is square integrable or not (denote by $\text{square}$).

   (c) $d = \text{Max } d_0$

   (d) $L2 = \bigwedge_{(d_0, \text{square}) | d = d_0} \text{square}$.

   (e) Deduce that the Eisenstein series admits at most a pole of order $d$ at $s = s_0$ and its leading term is square integrable (or not) according to the value of $L2$. 
Correctness

Recall that the poles of the degenerate Eisenstein series coincide with the poles of its constant term along any parabolic. We shall compute the constant term of the Eisenstein series along the Borel subgroup. By its nature, the problem requires a lot of computation (step 3(b)ii) thus we use the computer.

We denote by $W(G, P)$ the set of the shortest representatives in $W_G/W_P$.

**Proposition 2.4.1.** Let $B = TU$ be the Borel subgroup of $G$. Then:

$$
\int_{U(F) \setminus U(A)} E_P(f^0, s, ug) du = \sum_{w \in W(G, P)} M_w(s) f^0_s
$$

**Proof.** The Eisenstein series admits a meromorphic continuation to all $\mathbb{C}$, hence it is enough to prove this proposition only when the Eisenstein series converges absolutely and uniformly. We denote by $U^\gamma = \gamma^{-1} P \cap U$

$$
\int_{U(F) \setminus U(A)} E_P(f^0, s, ug) du = \int_{U(F) \setminus U(A)} \sum_{\gamma \in P(F) \setminus G(F)} f^0_s(\gamma ug) du
$$

$$
= \int_{U(F) \setminus U(A)} \sum_{\gamma \in P(F) \setminus G(F) / B(F)} \sum_{\delta \in \gamma^{-1} P(F) \cap G(F) \setminus B(F)} f^0_s(\gamma \delta ug) du
$$

$$
= \sum_{\gamma \in P(F) \setminus G(F)} \int_{U(F) \setminus U(A)} \sum_{\delta \in U^\gamma(1) \setminus U(F)} f^0_s(\gamma \delta ug) du
$$

$$
= \sum_{\gamma \in P(F) \setminus G(F)} \int_{U(F) \setminus U(A)} \int_{U^\gamma(1) \setminus U(F)} f^0_s(\gamma u_1 u_2 g) du_1 du_2
$$

$$
\#^1 = \sum_{\gamma \in P(F) \setminus G(F)} \int_{U(F) \setminus U(A)} f^0_s(\gamma ug) du
$$

$$
\#^2 = \sum_{w \in W(G, P)} M_w(s) f^0_s(g)
$$

(1.19) $C_w(s) f^0_{w s}(g)$.
The equality (#1) is due to left invariant properties of $f^0$ and taking the measure of $F \setminus A$ to be one. The equality (#2) is the definition of $M_w$. 

From now on, we assume that $w \in W(G, P)$. Therefore the potential poles are the points where $C_w(s)$ has a pole for some $w \in W(G, P)$. Recall that $C_w(s)$ is described by a product of quotients of zeta functions (Theorem 1.19). Let $f^0_s \in I^P(s)$. Since $I^P(s) \subset I^B(\chi_s)$ where $\chi_s = \delta_p^{s+\frac{1}{2}} \otimes \delta_B^{\frac{1}{2}}$. Then

$$C_w(s) = \prod_{\alpha \in \Phi^+_G \text{ w} \in \alpha < 0} \frac{\zeta(\langle \chi_s, \alpha \rangle)}{\zeta(\langle \chi_s, \alpha \rangle + 1)}.$$ 

After performing cancellation we obtained a reduced form

$$C_w(s) = \prod \frac{\zeta(p_i(s))}{\zeta(q_i(s))}$$

where $p_i(s) = \langle \chi_s, \alpha_i \rangle$ and $q_i(s) = \langle \chi_s, \beta_i \rangle + 1$ for some positive roots $\alpha_i, \beta_i$. Hence the pole of each term occurs either when the numerator has a pole (when the term $p_i(s) = 1, 0$) or when denominator is zero.

**Assumption 1.** Let $s \in \mathbb{C}$ such that $\text{Re } s > 0$ then:

For every $w \in W(G, P)$ the denominator of the reduced term of $C_w(s_0)$ is holomorphic and non zero (every $\text{Re } q_i(s) > 1$). In other words the potential poles of $C_w(\chi_s)$ are coming only from its numerator.

**Remark 2.4.1.** This assumption implies that

$$\eta(P) \subset \{ s \in \mathbb{C} : \text{Re } s > 0 \text{ and } \exists \alpha \in N^P \langle \chi_s, \alpha \rangle = 0, 1 \}.$$ 

Moreover, all potential poles are reals.

**Observation.** Let $G$ be a group of rank $\leq 8$. Let $P$ be a maximal parabolic subgroup of $G$. For every $w \in W(G, P)$ the denominator of the reduced term of $C_w(s)$ for $\text{Re } s > 0$ is holomorphic and non zero (every $q_i(s) > 1$). In other words, Assumption 1 holds.

In order to make the algorithm more efficient, we take

$$\eta(P) = \{ s \in \mathbb{C} : \text{Re } s > 0 \text{ and } \exists \alpha \in N^P \langle \chi_s, \alpha \rangle = 0, 1 \}$$

to be the set in step 2. Now we elaborate on step 3. Fix $s_0 \in \eta(P)$. 

1. Reorganize the $\sum_{w \in W(G,P)} M_w(s)f^0$ as follows:

\[
\sum_{w \in W(G,P)} M_w(s)f^0 = \sum_{w \in W(G,P)} \frac{1}{[[w]_{s_0}]} \sum_{u \in [w]_{s_0}} M_u(s)f^0 = \sum_{w \in W(G,P)} \frac{1}{[[w]_{s_0}]} M^\#_w(s)f^0
\]

where $M^\#_w(s)f^0 = \left(\sum_{u \in [w]_{s_0}} M_u(s)\right)f^0$.

**Proposition 2.4.2.** Let $s_0 \in \eta(P)$. The order of the pole of $E_P(f^0, s, g)$ at $s = s_0$ is bounded by $\max_{w \in W(G,P)/\sim_{s_0}} \{d : d = \text{order}_{s=s_0} M^\#_w\}$

**Proof.** If $M^\#_w$ admits a pole of order $d$ it can not be canceled. The reason is

\[
(M^\#_w(s)f^0)(t) = (w\chi_s(t))(M^\#_w(s)f^0)(1) \neq (u\chi_s(t))(M^\#_w(s)f^0)(1) = (M^\#_u(s)f^0)(t)
\]

here $t$ is arbitrary element of the torus and $(u\chi_{s_0}) \neq (w\chi_{s_0})$ for $u \not\sim_{s_0} w$.

**Remark 2.4.2.** This is only an upper bound since it may happened that the leading coefficient is sum of zeta values and we do not know if it is zero or not.

**Remark 2.4.3.** Iwasawa decomposition $g = tuk$, implies that the order of the pole of $(M^\#_u(s)f^0)(g)$ at $s = s_0$ is the same as the order of the pole of $(M^\#_u(s)f^0)(1)$ at $s = s_0$.

After we reorganize the sum, for every equivalence class $u$, we determine the order of the pole of $M^\#_u$ at $s = s_0$. This is done as follows:

For each $w$ we put

\[
m = \max_{u \in W(\chi_s,w)} \{n : M_u \text{ admits a pole of order } n \text{ at } s = s_0\}.
\]

Notice that $M^\#_w(s)$ admits a pole of order at most $m$. Recall that:

\[
(M^\#_w(s)f^0)(1) = \sum_{u \in W(\chi_{s_0},w)} C_u(s) = \sum_{u \in W(\chi_{s_0},w)} \prod_i \zeta(p_{i,u}(s)).
\]

sum all together and do common denominator. By our observation the denominator is holomorphic and non-zero for every $s$ such that $\text{Re } s > 0$ in particular for $s = s_0$. Hence we may ignore it.

Therefore the numerator is of the shape $\sum_i \prod_i \zeta(p_{i}(s))$. For each term in the sum, $\prod_i \zeta(p_{i}(s))$, we expand every factor in the product by the following rule.

If $p_{i}(s_0) > \frac{1}{2}$, we write the Laurent expansion of $\zeta(p_{i}(s_0))$ around $s_0$. If $p_{i}(s_0) \leq \frac{1}{2}$
we use the function equation \( \zeta(s) = \zeta(1 - s) \) and write the Laurent expansion of \( \zeta(1 - p_u(s)) \) around \( s_0 \).

**Remark 2.4.4.** To check possible cancellation it is enough to write the Laurent expansion up to order \( m + 1 \).

**Remark 2.4.5.** Since \( \zeta(s) = \zeta(1 - s) \) for \( s = \frac{1}{2} \) it holds that the \((2n + 1)^{th}\) derivative of \( \zeta(s) \) is zero for every \( n \in \mathbb{N} \). Hence, around \( \frac{1}{2} \)

\[
\zeta(s) = \sum_{j=0}^{\infty} a_{2j}(s - \frac{1}{2})^{2j}, \quad a_{2j} = \frac{\zeta^{(2j)}(\frac{1}{2})}{2j!}
\]

In other words all the odd derivatives of \( \zeta(s) \) at \( s = \frac{1}{2} \) are vanish.

Summing all the terms, we find the order of the pole at \( s_0 \) for this \( N_u^\#(s) \) and also for \( M_u^\#(s) \). The order of the pole at \( s_0 \) is bounded by the maximum of the orders of the pole of the all \( M_u^\#(s) \) at \( s = s_0 \).

This algorithm also gives an answer to the question whenever the residual representation (i.e the leading term \( \Lambda_{P,m}(f^0, s_0, g) \)) is square integrable or not. This is by applying Langlands criterion for \( L^2 \). We recall this criterion

**Theorem 2.5.** [20][Lemma I.4.11] Let \( \phi \) be an automorphic form. Let \( P = M_P N_P \) standard parabolic subgroup of \( G \). Denote be \( \Pi_0(M_P, \phi) \) the cuspidal support of \( \phi \) along \( P \). Then for \( \phi \) to be square integrable, it is necessary and sufficient that for all \( P \) and all \( \pi \in \Pi_0(M_P, \phi) \) the character \( \text{Re} \, \pi \) can be written in the form

\[
\text{Re} \, \pi = \sum_{\alpha \in \Delta_P} x_\alpha \alpha
\]

with coefficients \( x_\alpha \in \mathbb{R}, x_\alpha < 0 \).

**Remark 2.5.1.** Since we are in the degenerate case, the cuspidal support of the leading term is only the Borel subgroup. Hence, this criterion can be written as follows: Assume that \( E_P(f^0, g, s) \) admits a pole of order \( d \) at \( s_0 \). Let

\[
Y = \{ w \in W(G, P) : M_w^\# \text{ contributes a pole of order } d \text{ at } s_0 \}.
\]

Then \( \Lambda_{-d}(f^0, s_0, g) \in L^2(G(F) \backslash G(A)) \) if for every \( w \in Y \) it holds

\[
w \chi_{s_0} = \sum_{\alpha \in \Delta_G} x_\alpha \alpha
\]

with negative coefficients.
Remark 2.5.2. In order to show that the leading term is square integrable, we have to show that all the equivalence classes that may contribute a pole of order $d$ and are non square integrable contribute a pole of order at most $d - 1$.

Observation. If $F = \mathbb{Q}$ (this assumption can be lifted if we assume that several values of zeta function are non zero), and we are in the case that the algorithm found that order $s = s_0 \leq E_p(f^0, s, g) \leq d$ where $d$ is as in step 3c then there exists an equivalence class $[u]_{s_0}$ such that order $s = s_0 M_u^#(s) = d$. Moreover, if the leading term is not square integrable then there exists an equivalence class such that contributes a pole of order exactly $d$ and that is not square integrable.
Chapter 3

Poles of degenerate Eisenstein series

Let $G$ be a split simply connected group of exceptional type $G_2$, $F_4$, $E_6$ or $E_7$. Using the algorithm described in the last chapter we determine the poles of degenerate Eisenstein series $E_P(f^0, g, s)$ for $Re\ s > 0$ associated to various maximal parabolic subgroups, compute their orders, and determine the square integrable ones. For $G = G_2$ our results agree with the results obtained in [8].

In [10] it was shown that there exists a bijection between distinguished unipotent orbits of the dual group $\hat{L}G$ and the spherical residual representations that are square integrable with the cuspidal support $[T, 1]$.

In the case where the residual representation at $s_0$ is square integrable we will determine the distinguished orbit of $\hat{L}G$ related to it. Whenever several residual representations correspond to one orbit there will be identities between the leading terms of the Eisenstein series as will be shown in Chapter 6. If $t(s_0)$ is the representative in the dominant chamber of the Satake conjugacy class of $I_P(s_0)$ then $t(s_0)^2$ is the weighted Dynkin diagram of the distinguished unipotent orbit. For example the principal unipotent orbits corresponds to the trivial representations. The label of the orbits is the same as in [22].
Chapter 3. Results

Chapter structure

In sections 3.1, 3.2 we consider $G = GL_n$ and $G = G_2$. We recall the known results regarding the poles and their orders of degenerate Eisenstein series.

In sections 3.3, 3.4, and 3.5 we compute the order of the poles using the algorithm described in Chapter 2 for the group of type $F_4$, $E_6$, $E_7$. The algorithm has been implemented using the sage packet. The code may be viewed at my homepage. Below we only state the results. A typical output of the program can be seen at Appendix A.

We also determine which leading terms of the spherical Eisenstein series above are square integrable.

For the rest of the chapter let us fix some notations:

Let $P_m$ be maximal parabolic subgroup of $G$ corresponded to $\Delta_G \setminus \{\alpha_m\}$. We denote by $I_{P_m}(s) = Ind_{P_m(K)}^{G(K)}(\delta_{P_m}^s) = \otimes_\nu Ind_{P_m(F_\nu)}^{G(F_\nu)}(\delta_{P_m}^s)$. For the exceptional groups we list our results in the following form:

For each $s_0 \in \mathbb{C}$ such that $\text{Re } s_0 > 0$ and $E_P(f^0, s, g)$ admits a pole at $s = s_0$ we write its order and whenever is square integrable or not. If it is square integrable we also write the orbit that corresponds for it in the dual group.

1https://www.math.bgu.ac.il/~halawi/
3.1 The group $G = GL_n$

Muić and Hanzer in [6] have studied the poles of degenerate Eisenstein series for $G = GL_n$. Let us recall their results.

The Dynkin diagram of $G$ is of type $A_{n-1}$ and we labeled the roots as follows:

![Dynkin diagram](image)

Let $P_m = MN$ be a maximal parabolic subgroup of $G$, with $M \simeq GL_m \times GL_{n-m}$.

Remark 3.0.1. Note that in our notations, the representation $I_{P_m}(s)$ in [6] is denoted $I_{P_m}(\frac{s}{m})$.

3.1.1 Poles of spherical Eisenstein series

For every maximal parabolic subgroup $P_m$ we associate its Eisenstein series $E_{P_m}(f, g, s)$.

**Theorem 3.1 ([6], Theorem 5.1,5.2).** Let $m \leq \frac{n}{2}$, and $Re s \geq 0$. Then

1. For $s \not\in \{\frac{1}{2} - \frac{a}{n} : a \in \mathbb{Z}, \ 0 \leq a \leq m - 1\}$ the Eisenstein series $E_{P_m}(f, g, s)$ is holomorphic and non-zero.

2. For $s_0 \in \{\frac{1}{2} - \frac{a}{n} : a \in \mathbb{Z}, \ 0 \leq a \leq m - 1\}$ the Eisenstein series $E_{P_m}(f, g, s_0)$ admits at most a simple pole, that is attained by the normalized spherical section.

**Remark 3.1.1.** The representation $\Pi_{-1,s_0,P_m}^0$ is not in $L^2_d(G(F)\backslash G(\mathbb{A}))$ except for the case $s_0 = \frac{1}{2}$.

Remark (3.1.1) agrees with the results of [10] since the only distinguished orbit of $GL_n$ is the principal one.
3.2 The group $G = G_2$

We recall the results of Ginzburg and Jiang in [8]. The group $G = G_2$ is an exceptional split group. Its simple roots are labeled as follows:

\[
\begin{array}{c}
1 \\
2
\end{array}
\]

Remark 3.1.2. Note that in our notations, the representation $I_{P_m}(s)$ in [8] is denoted $I_{P_m}(s - \frac{1}{2})$.

3.2.1 Poles of spherical Eisenstein series

Proposition 3.1.1. [8] For $\text{Re } s > 0$ it holds that:

1. The Eisenstein series $E_{P_1}(f^0, g, s)$ is holomorphic except for $s \in \{\frac{1}{10}, \frac{1}{2}\}$. At these points it admits a simple pole.

2. The Eisenstein series $E_{P_2}(f^0, g, s)$ is holomorphic except for $s \in \{\frac{1}{6}, \frac{1}{2}\}$. At $s = \frac{1}{6}$ (resp. $s = \frac{1}{2}$) it admits a pole of order 2 (resp. 1).

3. For the data above, the leading term $\Lambda_d(f, g, s)$ of $E_{P_i}(f, g, s)$ at $s = s_0$ is square integrable.

Remark 3.1.3. In $G_2$ we have 2 distinguished orbits:

1. The principal orbit that corresponds to the trivial representation ($s = \frac{1}{2}$).

2. The subregular orbit, labeled as $G_2(\alpha_1)$, that corresponds to $(P_1, \frac{1}{10}), (P_2, \frac{1}{6})$. 
3.3 The group $G = F_4$

The group $G = F_4$ is an exceptional split group. Its simple roots are labeled as follows:

```
1 ————> 2 ————> 3 ————> 4
```

3.3.1 Poles of spherical Eisenstein series

| $P_1$ | $\frac{1}{8}$ | $\frac{1}{7}$ | $\frac{1}{2}$ |
|-------|---------------|---------------|--------------|
| Pole order | 1 | 1 | 1 |
| $L_2$ | ✓ | ✓ | ✓ |
| Orbit | $F_4(a_2)$ | $F_4(a_1)$ | $F_4$ |

| $P_2$ | $\frac{1}{10}$ | $\frac{1}{5}$ | $\frac{3}{10}$ | $\frac{1}{2}$ |
|-------|---------------|---------------|--------------|--------------|
| Pole order | 3 | 1 | 2 | 1 |
| $L_2$ | ✓ | ✓ | ✓ | ✓ |
| Orbit | $F_4(a_3)$ | $F_4(a_1)$ | $F_4$ |

| $P_3$ | $\frac{1}{14}$ | $\frac{3}{14}$ | $\frac{5}{14}$ | $\frac{1}{2}$ |
|-------|---------------|---------------|--------------|--------------|
| Pole order | 2 | 2 | 1 | 1 |
| $L_2$ | ✓ | ✓ | ✓ | ✓ |
| Orbit | $F_4(a_3)$ | $F_4(a_2)$ | $F_4$ |

| $P_4$ | $\frac{1}{22}$ | $\frac{5}{22}$ | $\frac{1}{2}$ |
|-------|---------------|---------------|--------------|--------------|
| Pole order | 1 | 1 | 1 |
| $L_2$ | ✓ | ✓ | ✓ |
| Orbit | $F_4(a_2)$ | $F_4(a_1)$ | $F_4$ |
3.4 The group $G = E_6$

The group $G = E_6$ is an exceptional split group. Its simple roots are labeled as follows:

$$
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
$$

3.4.1 Poles of spherical Eisenstein series

| $P_1, P_6$ | $\frac{1}{7}$ | $\frac{1}{2}$ |
|------------|-------------|-------------|
| Pole order | 1           | 1           |
| $L_2$      | ✓           | ✓           |
| Orbit      | $E_6(a_1)$  | $E_6$       |

| $P_2$ | $\frac{1}{7}$ | $\frac{5}{7}$ | $\frac{7}{7}$ | $\frac{1}{2}$ |
|--------|-------------|-------------|-------------|-------------|
| Pole order | 1 | 1 | 1 | 1 |
| $L_2$ | ✓ | ✓ | ✓ | ✓ |
| Orbit | $E_6(a_3)$ | $E_6(a_1)$ | $E_6$ |

| $P_3, P_5$ | $\frac{1}{6}$ | $\frac{5}{18}$ | $\frac{7}{18}$ | $\frac{1}{2}$ |
|------------|-------------|-------------|-------------|-------------|
| Pole order | 2 | 1 | 1 | 1 |
| $L_2$ | ✓ | ✗ | ✗ | ✓ |
| Orbit | $E_6(a_3)$ | $E_6$ |

| $P_4$ | $\frac{1}{7}$ | $\frac{3}{7}$ | $\frac{5}{7}$ | $\frac{1}{2}$ |
|--------|-------------|-------------|-------------|-------------|
| Pole order | 2 | 1 | 3 | 2 | 1 |
| $L_2$ | ✗ | ✗ | ✓ | ✓ | ✓ |
| Orbit | $E_6(a_3)$ | $E_6(a_1)$ | $E_6$ |

Remark 3.1.4. For the case $P = P_4$ and $s_0 = \frac{1}{7}$ the result is valid under the assumption that $\zeta(\frac{1}{2}) \neq 0$. 
3.5 The group $G = E_7$

The group $G = E_7$ is an exceptional split group. Its simple roots are labeled as follows:

3.5.1 Poles of spherical Eisenstein series

The following results are valid under the following assumptions:

1. In the case where $P = P_4$ and $s \in \{\frac{1}{16}, \frac{3}{16}\}$ we assume that $\zeta(\frac{1}{2}) \neq 0$.

2. In the case where $P = P_4$ and $s = \frac{1}{12}$ we assume that $\zeta(\frac{2}{3}) \neq 0$. 
## Chapter 3. Results

### Chapter 3: Poles of degenerate Eisenstein series

| \( P_1 \) | \( \frac{1}{77} \) | \( \frac{2}{77} \) | \( \frac{11}{77} \) | \( \frac{1}{2} \) |
|---|---|---|---|---|
| Pole order | 1 | 1 | 1 | 1 |
| \( L_2 \) | ✔ | ✔ | ✔ | ✔ |
| Orbit | \( E_7(a_3) \) | \( E_7(a_2) \) | \( E_7(a_1) \) | \( E_7 \) |

| \( P_2 \) | \( \frac{1}{14} \) | \( \frac{1}{7} \) | \( \frac{3}{14} \) | \( \frac{2}{7} \) | \( \frac{5}{14} \) | \( \frac{1}{2} \) |
|---|---|---|---|---|---|---|
| Pole order | 1 | 1 | 1 | 1 | 1 | 1 |
| \( L_2 \) | ✔ | ✔ | ✔ | ✔ | ✔ | ✔ |
| Orbit | \( E_7(a_3) \) | \( E_7(a_2) \) | \( E_7(a_1) \) | \( E_7 \) |

| \( P_3 \) | \( \frac{1}{33} \) | \( \frac{3}{33} \) | \( \frac{9}{33} \) | \( \frac{7}{33} \) | \( \frac{9}{22} \) | \( \frac{1}{2} \) |
|---|---|---|---|---|---|---|
| Pole order | 2 | 2 | 2 | 1 | 1 | 1 |
| \( L_2 \) | ✔ | ✔ | ✔ | ✔ | ✗ | ✔ |
| Orbit | \( E_7(a_5) \) | \( E_7(a_4) \) | \( E_7(a_3) \) | \( E_7 \) |

| \( P_4 \) | \( \frac{1}{15} \) | \( \frac{1}{7} \) | \( \frac{1}{5} \) | \( \frac{3}{15} \) | \( \frac{1}{5} \) | \( \frac{5}{9} \) | \( \frac{1}{2} \) |
|---|---|---|---|---|---|---|---|
| Pole order | 1 | 1 | 4 | 1 | 3 | 2 | 1 |
| \( L_2 \) | ✗ | ✗ | ✔ | ✗ | ✔ | ✔ | ✔ |
| Orbit | \( E_7(a_5) \) | \( E_7(a_4) \) | \( E_7(a_3) \) | \( E_7(a_1) \) | \( E_7 \) |

| \( P_5 \) | \( \frac{1}{10} \) | \( \frac{3}{20} \) | \( \frac{1}{5} \) | \( \frac{3}{10} \) | \( \frac{22}{2} \) | \( \frac{1}{2} \) |
|---|---|---|---|---|---|---|
| Pole order | 3 | 1 | 2 | 2 | 1 | 1 |
| \( L_2 \) | ✔ | ✗ | ✗ | ✔ | ✔ | ✔ |
| Orbit | \( E_7(a_5) \) | \( E_7(a_4) \) | \( E_7(a_3) \) | \( E_7 \) |

| \( P_6 \) | \( \frac{1}{26} \) | \( \frac{5}{26} \) | \( \frac{7}{26} \) | \( \frac{11}{26} \) | \( \frac{1}{2} \) |
|---|---|---|---|---|---|
| Pole order | 1 | 2 | 1 | 1 | 1 |
| \( L_2 \) | ✔ | ✔ | ✔ | ✗ | ✔ |
| Orbit | \( E_7(a_4) \) | \( E_7(a_3) \) | \( E_7(a_2) \) | \( E_7 \) |

| \( P_7 \) | \( \frac{1}{18} \) | \( \frac{5}{18} \) | \( \frac{1}{2} \) |
|---|---|---|---|
| Pole order | 1 | 1 | 1 |
| \( L_2 \) | ✔ | ✔ | ✔ |
| Orbit | \( E_7(a_2) \) | \( E_7(a_1) \) | \( E_7 \) |
Chapter 4

Normalized Eisenstein series

In this chapter we will see the relations between degenerate Eisenstein series associated to maximal parabolic subgroup and the Eisenstein series associated to the Borel subgroup. This relation stated in [12]. We also introduce the normalized Eisenstein series, that is entire and $W_G$ invariant. Its properties, stated in [12], play a crucial role in the proof of our main theorem. For the convenience of the reader we include the proof of its properties.

We denote by $W(G, P)$ the set of the shortest representatives in $W_G/W_P$. Let fix some notations: let $\chi \in X(T)$ be an unramified character of $T$. For every $\alpha \in \Phi$ we denote by $l_+^\alpha(\chi) = \langle \chi, \hat{\alpha} \rangle + 1$ and $F^-_\alpha = \{ \chi \in a_C^* : l_-^\alpha(\chi) = 0 \}$.

**Proposition 4.0.1.** Let $P = P_r$ be maximal parabolic of $G$. Let $\chi_s = \delta_{P_r}^{\frac{1}{2}} \otimes \delta_{B}^{\frac{1}{2}}$. Then the iterated residue of $E_B(\chi, f^0, g)$ along $F_r = F^-_{\alpha_1}, \ldots, F^-_{\alpha_{r-1}}, F^-_{\alpha_{r+1}}, \ldots, F^-_{\alpha_n}$, is equal to $\text{Res}_{F_r} C_{w_p}(\chi) \times E_P(f^0, s, g)$ where $w_p$ is the longest element in $W_P$. Moreover $\text{Res}_{F_r} C_{w_p}(\chi_s) \neq 0$ and is a constant function.

**Proof.** Since $E_B(f^0, \chi_s, g)$ and $E_P(f^0, s, g)$ have the same cuspidal support, it is enough to show that $\text{Res}_{F_r} E_B(f^0, \chi_s, g)$ and $\text{Res}_{F_r} C_{w_p}(\chi) \times E_P(f^0, s, g)$ have the same constant term along the unipotent radical $U$ of the Borel see ([20], Proposition I.3.4).
Chapter 4. Normalized Eisenstein series

Indeed,

\[
E_0^B(f^0, \chi_s, g) = \sum_{w \in \mathcal{W}_G} M_w(\chi_s)f^0_{\chi_s} = \sum_{w \in \mathcal{W}_G/\mathcal{W}_P} \sum_{u \in \mathcal{W}_P} M_{wu}(\chi_s)f^0_{u\chi_s} \tag{1.19}
\]

If \( u \neq w_P \), i.e. \( u \) is not the longest element in \( \mathcal{W}_P \), then there exists a simple root \( \alpha_i \in \Delta_P \) such that \( u\alpha > 0 \). Hence \( C_u(\chi) \) does not contain the factor \( \frac{\zeta((\chi_s, \hat{\alpha}_i))}{\zeta((\chi_s, \hat{\alpha}_i) + 1)} \).

Therefore,

\[
\text{Res}_{\mathcal{F}_r} C_u(\chi) = \lim_{\chi \to \chi_s} \prod_{\alpha \in \Delta_P} ((\chi, \hat{\alpha}) - 1)C_u(\chi) = \begin{cases} 
0 & \text{if } u \neq w_P \\
\text{Res}_{\mathcal{F}_r} C_{w_P}(\chi_s) & \text{if } u = w_P 
\end{cases}
\]

Therefore,

\[
\text{Res}_{\mathcal{F}_r} E_0^B(f^0, \chi_s, g) = \text{Res}_{\mathcal{F}_r} C_{w_P}(\chi_s) \times \sum_{w \in \mathcal{W}_G/\mathcal{W}_P} M_w(w_P\chi_s)f^0_{w_P\chi_s} = \text{Res}_{\mathcal{F}_r} C_{w_P}(\chi_s) \times E_0^B(f^0, \chi, g).
\]

Note that

\[
C_{w_P}(\chi_s) = \prod_{\alpha \in \Phi^+_M} \frac{\zeta((\chi_s, \hat{\alpha}))}{\zeta((\chi_s, \hat{\alpha}) + 1)}.
\]

For given \( \alpha > 0 \) it holds that \( \hat{\alpha} = \sum_{\alpha \in \Delta_G} n^{(\hat{\alpha})}_\alpha \hat{\alpha}_i \) where \( n^{(\hat{\alpha})}_\alpha \in \mathbb{N} \cup \{0\} \). Moreover if \( \alpha \in \Phi^+_M \) it holds that \( n^{(\hat{\alpha})}_\alpha = 0 \). Therefore for every \( \alpha \in \Phi^+_M \setminus \Delta_G \) it holds \( 1 < \langle \chi_s, \hat{\alpha} \rangle \in \mathbb{N} \). Since the zeros of \( \zeta(s) \) lie in \( 0 < \text{Re } s < 1 \), we conclude that the every term corresponds to a non-simple root in that product is a non zero constant. For \( \alpha \in \Phi^+_M \cap \Delta_G \) it holds that \( \langle \chi_s, \hat{\alpha} \rangle = 1 \). Therefore by taking the iterated residue along the \( \mathcal{F}_r \) we get a non zero constant number. \( \square \)

**Definition 4.1.** Let \( f^0 \in Ind^G_B(\chi) \) be the normalized spherical section. We define the normalized Eisenstein series to be

\[
E^#_B(\chi, g) = \prod_{\alpha \in \Phi^+} \zeta(l^+_\alpha(\chi)) \cdot l^+_\alpha(\chi) \cdot l^-_\alpha(\chi) E_B(f^0, \chi, g).
\]
**Theorem 4.2.** The normalized Eisenstein series $E_B^\#$ is entire and $W_G$ invariant, i.e. $E_B^\#(\chi, g) = E_B^\#(w\chi, g)$ for every $w \in W_G$.

**Proof.** We shall prove first the $W_G$ invariance property. Observe that for given $w \in W_G$ it holds that

$$l^+_{\alpha}(w\chi) = l^+_{w^{-1}\alpha}(\chi).$$

Furthermore, since $W_G$ is generated by simple reflections it is enough to show the $W_G$ invariance for simple reflection. Let $w = w_i$ be a simple reflection corresponding to the simple root $\alpha_i$. Note that the following holds:

1. $w^{-1} = w$.
2. $w(\Phi^+ \setminus \{\alpha_i\}) = \Phi^+ \setminus \{\alpha_i\}$.

It implies that:

$$\prod_{\alpha \in \Phi^+ \setminus \{\alpha_i\}} l^+_{\alpha}(\chi)l^-_{\alpha}(\chi) = \prod_{\alpha \in \Phi^+ \setminus \{\alpha_i\}} l^+_{\alpha}(w\chi)l^-_{\alpha}(w\chi)$$ (4.1)

$$\prod_{\alpha \in \Phi^+ \setminus \{\alpha_i\}} \zeta(l^+_{\alpha}(\chi)) = \prod_{\alpha \in \Phi^+ \setminus \{\alpha_i\}} \zeta(l^+_{\alpha}(w\chi)).$$ (4.2)

Therefore, we get:

$$\prod_{\alpha \in \Phi^+ \setminus \{\alpha_i\}} \zeta(l^+_{\alpha}(w\chi))l^+_{\alpha}(w\chi)l^-_{\alpha}(w\chi) = \prod_{\alpha \in \Phi^+ \setminus \{\alpha_i\}} \zeta(l^+_{\alpha}(\chi))l^+_{\alpha}(\chi)l^-_{\alpha}(\chi).$$

So, it remains to show that

$$\zeta(l^+_{\alpha_i}(\chi))l^+_{\alpha_i}(\chi)l^-_{\alpha_i}(\chi)E_B(f^0, \chi, g) = \zeta(l^+_{\alpha_i}(w\chi))l^+_{\alpha_i}(w\chi)l^-_{\alpha_i}(w\chi)E_B(f^0, w\chi, g)$$

Recall that both Eisenstein series and the zeta function admit functional equations:

$$E_B(f^0, \chi, g) = E_B(M_w(\chi)f^0, w\chi, g)$$

$$\zeta(s) = \zeta(1 - s).$$
Using (1.19) it holds:

\[ E_B(f^0, \chi, g) = \frac{\zeta((\chi, \tilde{\alpha}))}{\zeta((\chi, \tilde{\alpha}) + 1)} E_B(f^0, w\chi, g). \]

Observe that \( l_{\alpha_i}^+(w_i\chi) = -l_{\alpha_i}^-(\chi) \). Thus,

\[
\zeta(l_{\alpha_i}^+(w\chi))(l_{\alpha_i}^+(w\chi))(l_{\alpha_i}^-(w\chi)) E_B(f^0, w\chi, g) = \zeta(-l_{\alpha_i}^-(\chi))(l_{\alpha_i}^+(\chi))(l_{\alpha_i}^-(\chi)) E_B(f^0, w\chi, g) = \zeta(l_{\alpha_i}^+(\chi))(l_{\alpha_i}^+(\chi))(l_{\alpha_i}^-(\chi)) E_B(f^0, w\chi, g)
\]

Since \( E_B^\#(\chi, g) \) is a product of two \( w \) invariant functions we conclude that \( E_B^\# \) is \( w \) invariant. Thus \( E_B^\#(\chi, g) \) is \( \mathcal{W}_G \) invariant since it is invariant for every generator of \( \mathcal{W}_G \).

Let’s prove that \( E_B^\#(\chi, g) \) is entire. Recall that the constant term of the degenerate Eisenstein series along the unipotent radical of the Borel subgroup \( E_B^0(f^0, \chi, g) \) and the Eisenstein series itself share the same analytical behavior. Thus, in order to show that \( E_B^\# \) is entire, it will be enough to show that the following function is entire

\[ \Psi(\chi, g) = \prod_{\alpha \in \Phi^+} l_{\alpha}^+(\chi)l_{\alpha}^-(\chi)\zeta(l_{\alpha}^+(\chi))E_B^0(f^0, l, g). \]

Furthermore, by Proposition (2.4.1) the constant term can be written as:

\[ E_B^0(f^0, l, g) = \sum_{w \in \mathcal{W}_G} C_w(\chi)f_{w\chi}^0 = \sum_{w \in \mathcal{W}_G} \prod_{\alpha \in \Phi^+_G} \frac{\zeta((\chi, \tilde{\alpha}))}{\zeta((\chi, \tilde{\alpha}) + 1)} f_{w\chi}^0. \]

Note that \( \Psi(\chi, g) \) can be written as:

\[
\Psi(\chi, g) = \prod_{\alpha \in \Phi^+} l_{\alpha}^+(\chi)l_{\alpha}^-(\chi)\zeta(l_{\alpha}^+(\chi))E_B^0(f^0, l, g)
\]

\[
= \prod_{\alpha \in \Phi^+} l_{\alpha}^+(\chi)l_{\alpha}^-(\chi)\zeta(l_{\alpha}^+(\chi)) \times \sum_{w \in \mathcal{W}_G} \prod_{\alpha \in \Phi^+_G} \frac{\zeta((\chi, \tilde{\alpha}))}{\zeta((\chi, \tilde{\alpha}) + 1)} f_{w\chi}^0
\]

\[
= \prod_{\alpha \in \Phi^+} l_{\alpha}^+(\chi)l_{\alpha}^-(\chi) \times \sum_{w \in \mathcal{W}_G} \prod_{\alpha \in \Phi^+_G} \zeta(l_{\alpha}^+(\chi)) \times \prod_{\alpha \in \Phi^+_G} \zeta((\chi, \tilde{\alpha})) f_{w\chi}^0.
\]
While $L(\chi)$ is entire, $F_w(\chi)$ can have possible poles along the hyperplanes

$$H^\epsilon_\alpha(\chi) = \{ \chi \in \mathfrak{a}_c^* : \langle \chi, \bar{\alpha} \rangle = \epsilon \} \text{ for } \epsilon \in \{-1, 0, 1\}.$$  

We will show that these poles are canceled either by each other or by the zeros of $L(\chi)$. Observe that

$$F_w(\chi) = \left( \prod_{\alpha \in \Phi^+} \zeta(t^{+}_\alpha(\chi)) \times C_w(\chi) \right) \times f^0_w(\chi)$$  

where $C_w(\chi)$ is the factor as in Theorem (1.19).

We recall Hartog theorem.

**Theorem 4.3.** [23, Vol 1, part D, Theorem 4] Suppose that $E$ is an analytic subset of $\mathbb{C}^n$ where $(n \geq 2)$ of complex dimension at most $n - 2$, then every function holomorphic on $\mathbb{C}^n \setminus E$ can be extended holomorphically to $\mathbb{C}^n$.

Let $X = \bigcup_{0 < \alpha \neq \alpha', \epsilon, \epsilon' \in \{-1, 0, 1\}} (H^\epsilon_\alpha \cap H'^{\epsilon'}_{\alpha'})$ and $Y = \bigcup_{0 < \alpha, \epsilon \in \{-1, 0, 1\}} H^\epsilon_\alpha$. Note that $X$ is of codimension 2. Thus, by Hartog theorem, it is enough to show that for every $\chi \in \mathfrak{a}_c^* \setminus X$ the function $\Psi(\chi, g)$ is holomorphic. Moreover, for given $\chi \notin Y$ $\Psi(\chi, g)$ is holomorphic. Hence, it remains to show that the for given $\chi \in Y \setminus X$ the function $\Psi(\chi, g)$ is holomorphic.

For given $\chi \in Y \cap \bigcup_{\alpha > 0} (H^{\pm 1}_\alpha)$ it easy to see that the poles of $F_w(\chi)$ are canceled by the zeros of $L(\chi)$. Hence, it remains to prove the holomorphic property for $\chi_0 \in H^0_\alpha \setminus X$. Recall that function of several complex variables is holomorphic at $s_0 \in \mathbb{C}^n$ if and only if it holomorphic with respect to each variable separately, i.e. it enough to show that $f(s_0 + se_i)$ at $s = 0$, where $e_i$ is the standard basis element of $\mathbb{C}^n$.

Let $\{v_1, \ldots, v_n\}$ be a basis for $\mathbb{C}^n$ such that for every $i$ it holds that $\langle v_i, \bar{\alpha} \rangle = 2$. Therefore it is enough to show that $\Psi(\chi, g)$ is holomoprhic at $\chi = \chi_0$ with respect to this coordinate system.
Along \( v_i \)

Our main goal is to show that \( \Psi(\chi_0 + sv_i, g) \) is holomorphic at \( s = 0 \) for every \( v_i \). It is enough to show that for every \( w \in \mathcal{W}_G/\langle w_\alpha \rangle \) the function

\[
F_{ww_\alpha}(\chi_0 + sv_i) + F_w(\chi_0 + sv_i)
\]

is holomorphic at \( s = 0 \). Without loss of generality \( l(ww_\alpha) = l(w) + 1 \).

\[
F_{ww_\alpha}(\chi) + F_w(\chi) = \left( \prod_{\substack{\alpha \in \Phi^+ \setminus \{\alpha\} \atop \langle\alpha, w\rangle > 0}} \zeta(l^+_\alpha(\chi)) \times \prod_{\substack{\alpha \in \Phi^+ \setminus \{\alpha\} \atop \langle\alpha, w\rangle < 0}} \zeta(\langle\chi, \alpha\rangle) \right) \times \left( \zeta(\langle\chi, \hat{\alpha}\rangle)f^0_{ww_\alpha\chi} + \zeta(\langle\chi, \hat{\alpha}\rangle + 1)f^0_{w\chi} \right).
\]

It is obvious that \( K_1(\chi, w) \) is holomorphic at small neighborhood of \( \chi_0 \) since \( \chi_0 \in H^0_\alpha \setminus X \). Let us show that \( K_2(\chi_0 + sv_i, w) \) is holomorphic at \( s = 0 \). Denote by \( \chi_s = \chi_0 + sv_i \) then for \( s = 0 \) it holds that \( f^0_{ww_\alpha\chi_s} = f^0_{w\chi_s} \) and

\[
\text{Res}_{s=0} K_2(\chi_0 + sv_i, w) = \text{Res}_{s=0} \zeta(2s)f^0_{ww_\alpha\chi_s} + \zeta(2s + 1)f^0_{w\chi_s} = 0.
\]

Hence it holomorphic at \( s = 0 \). To sum up we showed that for every \( \chi_0 \in H^0_\alpha \setminus X \) it holds that \( \Psi(\chi, g) \) is holomorphic for \( \chi = \chi_0 \) with respect to each variable separately. Hence it holomorphic at \( \chi = \chi_0 \). So we are done. \( \square \)

Let us fix some notations: Let \( P \) be a maximal parabolic subgroup of \( G \). Consider \( \chi_s = \delta^s_{P - \frac{1}{2}} \otimes \delta^s_B \).

**Corollary 4.4.** With the notations as above, \( E^\#_B(\chi_s, g) \) can be written as

\[
E^\#_B(\chi_s, g) = A_{wp} \times G_P(\chi_s) \times \prod_{\alpha \in \Delta_p} \zeta(l^+_\alpha(\chi_s))l^+_\alpha(\chi_s) \times E_p(f^0, s, g)
\]

where \( G_P(\chi_s) = \prod_{\alpha \in \Phi^+ \setminus \Delta_p} \zeta(l^+_\alpha(\chi_s))l^+_\alpha(\chi_s)l^-_\alpha(\chi_s) \) and \( A_{wp} = \text{Res}_{\chi} C_{wp}(\chi_s) \).

**Remark 4.4.1.** The product \( \prod_{\alpha \in \Delta_p} \zeta(l^+_\alpha(\chi_s))l^+_\alpha(\chi_s) \) is equal to \( (2\zeta(2))^{n-1} \). since for every \( \alpha \in \Delta_M \) is holds \( \langle\chi_s, \hat{\alpha}\rangle = 1 \).

Let \( d_p(\chi_{s_0}) \) denote the order of the zero of \( G_P(\chi_s) \) at \( s = s_0 \).

**Theorem 4.5.** The Eisenstein series \( E_P(f^0, s, g) \) admits a pole of order at most \( d_p(\chi_{s_0}) \) at \( s = s_0 \).
Proof. Since \( E_B^P(\chi_s, g) \) is entire, the order of the zero of \( G_P(s) \) at \( s = s_0 \) is an upper bound for the order of the pole of \( E_P(f^0, s, g) \) at \( s = s_0 \).

We are mostly interested in the case where \( \text{Re} s > 0 \). Our observation shows that the poles of the degenerate Eisenstein series in the right half plane are real. In that case, we can express the number \( d_P(\chi_{s_0}) \) in geometric terms for \( F = \mathbb{Q} \). Let,

\[
N_{\epsilon}(\chi) = \{ \alpha \in \Phi^+ : \langle \chi, \hat{\alpha} \rangle = \epsilon \} \quad N_{\epsilon_1,...,\epsilon_k}(\chi) = \bigcup_{i=1}^{k} N_{\epsilon_i}(\chi).
\]

**Theorem 4.6.** For \( F = \mathbb{Q} \) and for \( s \in \mathbb{R} \) it holds that

\[
d_P(\chi_s) = |N_1(\chi_{s_0})| - |N_0(\chi_{s_0})| - (n - 1).
\]

**Proof.** Note that \( \Delta_P \subset N_1(\chi_s) \) for every \( s \). Therefore, we can rewrite as

\[
G_P(\chi_s) = \left( \prod_{\alpha \in N_{-1}(\chi_s)} \zeta(l^+_\alpha(\chi_s))l^+_\alpha(\chi_s) \right) \times \prod_{\alpha \in N_{-1}(\chi_s)} l^-\alpha(\chi_s)
\]

\[
\times \left( \prod_{\alpha \in N_0(\chi_s)} l^+_\alpha(\chi_s)l^-\alpha(\chi_s) \right) \times \prod_{\alpha \in N_0(\chi_s)} \zeta(l^+_\alpha(\chi_s))
\]

\[
\times \left( \prod_{\alpha \in N_1(\chi_s) \setminus \Delta_P} \zeta(l^+_\alpha(\chi_s))l^+_\alpha(\chi_s) \right) \times \prod_{\alpha \in N_1(\chi_s) \setminus \Delta_P} l^-\alpha(\chi_s)
\]

\[
\times \left( \prod_{\alpha \in \Phi^+ \setminus N_{\pm 0}(\chi_s)} \zeta(l^+_\alpha(\chi_s))l^+_\alpha(\chi_s)l^-\alpha(\chi_s) \right)
\]

Note that:

\[
\left( \prod_{\alpha \in N_0(\chi_s)} l^+_\alpha(\chi_s)l^-\alpha(\chi_s) \right) = (-1)^{|N_0(\chi_s)|} \quad \prod_{\alpha \in N_{-1}(\chi_s)} l^-\alpha(\chi_s) = (-2)^{|N_{-1}(\chi_s)|}
\]

\[
\left( \prod_{\alpha \in N_1(\chi_s) \setminus \Delta_P} \zeta(l^+_\alpha(\chi_s))l^+_\alpha(\chi_s) \right) = (2\zeta(2))^{|N_1(\chi_s)| - (n - 1)}
\]

Hence,
Chapter 4 Normalized Eisenstein series

\[ G_P(\chi_s) = \left( \prod_{\alpha \in \mathcal{N}_{-1}(\chi_s)} \zeta(l_{\alpha}^+(\chi_s)) \right) \left( \prod_{\alpha \in \mathcal{N}_0(\chi_s)} \zeta(l_{\alpha}^-(\chi_s)) \right) \left( \prod_{\alpha \in \mathcal{N}_1(\chi_s) \setminus \Delta_P} l_{\alpha}^-(\chi_s) \right) \]  \hspace{1cm} (4.4)

\[ \times \left( \prod_{\alpha \in \Phi^+ \setminus \mathcal{N}_{\pm 1,0}(\chi_s)} \zeta(l_{\alpha}^+(\chi_s)) l_{\alpha}^+(\chi_s) l_{\alpha}^-(\chi_s) \right) \]  \hspace{1cm} (4.5)

\[ \times (-1)^{|\mathcal{N}_{0,-1}(\chi_s)|} \times 2^{|\mathcal{N}_{\pm 1}(\chi_s)|-(n-1)} \times \zeta(2)^{|\mathcal{N}_1(\chi_s)|-(n-1)} \]  \hspace{1cm} (4.6)

In the case where \( s \in \mathbb{R} \) it holds that \( l_{\alpha}^+(\chi_s) \in \mathbb{R} \), and hence \( \zeta(l_{\alpha}^+(\chi_s)) \neq 0 \) for \( F = \mathbb{Q} \).

As a result (4.6) is a non zero and holomorphic. The term (4.4) is holomorphic and non zero. Hence the zero order of \( G_P(\chi_s) \) is exactly the zero order of (4.5) which is \( |\mathcal{N}_1(\chi_s)| - |\mathcal{N}_0(\chi_s)| - (n-1) \). To sum up the order of the zero of \( G_P(\chi_s) \) is exactly \( |\mathcal{N}_1(\chi_s)| - |\mathcal{N}_0(\chi_s)| - (n-1) \). \( \square \)

**Corollary 4.7.** For \( F = \mathbb{Q} \), and for \( s_0 \in \mathbb{R} \) there exists a constant \( C \in \mathbb{C}^* \) such that at \( s = s_0 \)

\[ E_B^\#(\chi_{s_0}, g) = C \times A_{d_P(\chi_{s_0})}^P(f^0_{s_0}, s_0, g). \]

**Remark 4.7.1.** Under the assumption that the term (4.6) does not vanish, the assumption that \( F = \mathbb{Q} \) and \( s \in \mathbb{R} \) can be lifted.

Our goal is to find out identities between various leading term. In order to do that, we introduce the notions of (positive) admissible data. As we can see it will help us to find the desire identities.

**Definition 4.8.** The quintuple \((P, s_0, Q, t_0, w)\) where \( P, Q \) are maximal parabolic subgroups, \( s_0, t_0 \in \mathbb{R} \) and \( w \in \mathcal{W}_G \) is called **admissible data** if

\[ w(\delta_P^{s_0-\frac{1}{2}} \otimes \delta_B^{\frac{1}{2}}) = \delta_Q^{t_0-\frac{1}{2}} \otimes \delta_B^{\frac{1}{2}}. \]

If also \( s_0, t_0 \) are non negative numbers it is called **positive admissible data**.
Theorem 4.9. Let $F = \mathbb{Q}$, and let $(P, s_0, Q, t_0, w)$ be an admissible data. Then there exists a constant $C \in \mathbb{C}^*$ such that

$$\Lambda_{-d_P(x_{s_0})}(f^0, s_0, g) = C \times \Lambda_{-d_Q(x_{t_0})}(f^0, t_0, g)$$

where $x_{s_0} = \delta_{P}^{s_0-\frac{1}{2}} \otimes \delta_{B}^{\frac{1}{2}}$ and $x_{t_0} = \delta_{Q}^{t_0-\frac{1}{2}} \otimes \delta_{B}^{\frac{1}{2}}$.

Proof. $E_B^\#(\chi, g)$ is a $W_G$ invariant function. In particular,

$$E_B^\#(\chi_{t_0}, g) = E_B^\#(w \chi_{s_0}, g) = E_B^\#(\chi_{s_0}, g).$$

From Corollary (4.7) we deduce that there exists $c_1 \in \mathbb{C}^*$ (resp. $c_2 \in \mathbb{C}^*$) such that

$$E_B^\#(\chi_{s_0}, g) = c_1 \times \Lambda_{-d_P(x_{s_0})}(f^0, s_0, g)$$
$$E_B^\#(\chi_{t_0}, g) = c_2 \times \Lambda_{-d_Q(x_{t_0})}(f^0, t_0, g).$$

Hence

$$\Lambda_{-d_P(x_{s_0})}(f^0, s_0, g) = \frac{c_2}{c_1} \times \Lambda_{-d_Q(x_{t_0})}(f^0, t_0, g).$$

In Chapter 5 we will give an explicit formula for the $C$ in Theorem (4.9). In Chapter 6 we will find all the positive admissible data.
Chapter 5

Refining The Basic Identity

Our main goal for this chapter is to write explicitly the constant appearing in Theorem (4.9). Let $P = P_i$ be a maximal parabolic subgroup of $G$. Let fix some notations:

1. $\delta_{P_i} = b_i \tilde{\omega}_i$

2. $R = \text{Res}_{s=1} \zeta(s)$

3. $\chi_{P,s} = \delta_P^{-\frac{1}{2}} \otimes \delta_B^{\frac{1}{2}}$

4. $\tilde{\alpha} = \sum_{i=1}^n n_i^{(\tilde{\alpha})} \tilde{\alpha}_i$

5. $N_\epsilon(\chi) = \{ \alpha \in \Phi^+ : \langle \chi, \tilde{\alpha} \rangle = \epsilon \}$

6. $N_{\epsilon_1,...,\epsilon_n}(\chi) = \bigcup_{i=1}^n N_{\epsilon_i}(\chi)$

7. $B_\epsilon(\chi) = \{ \alpha \in \Phi : \langle \chi, \tilde{\alpha} \rangle = \epsilon \}$

8. $B_{\epsilon_1,...,\epsilon_n}(\chi) = \bigcup_{i=1}^n B_{\epsilon_i}(\chi)$

9. $h_1(\chi_{P,s}, s_0) = \prod_{\alpha \in N_{-1}(\chi_{P,s})} \zeta(l_+^\alpha(\chi_{P,s})) l_+^\alpha(\chi_{P,s})$

10. $h_2(\chi_{P,s}, s_0) = \prod_{\alpha \in N_0(\chi_{P,s})} \zeta(l_+^\alpha(\chi_{P,s})) \times \prod_{\alpha \in N_1(\chi_{P,s}) \Delta_P} l_+^\alpha(\chi_{P,s})$

11. $h_{3,P}(\chi_{P,s}, s_0) = \prod_{\alpha \in \Phi^+ \setminus N_{\pm 1,0}(\chi_{P,s})} \zeta(l_+^\alpha(\chi_{P,s})) l_+^\alpha(\chi_{P,s}) l_-^\alpha(\chi_{P,s})$

12. $w_{P_i}$ is the longest element in $W_{P_i}$.

13. $\mathcal{F}_{\alpha}^- = \{ \chi \in a_\C^* : l_-^\alpha(\chi) = 0 \}$

14. $\mathcal{F}_r = \bigcap_{i=1}^n \mathcal{F}_{\alpha_i}$

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**Lemma 5.0.1.** Let

\[ \epsilon_{s_0} \times \prod_{\alpha \in \Phi^+ \setminus \mathbb{N}_{s_0}(\chi_{P_i,s_0})} \frac{\zeta((\chi_{P_i,s_0}, \hat{\alpha}))}{\zeta((\chi_{P_i,s_0}, \hat{\alpha}) + 1)} \times \left( \frac{R}{\zeta(2)} \right)^d \times \frac{A_{P_i}}{A_{P_j}} \]

where

\[ A_{P_i} = \text{Res}_{\mathcal{F}_i} C_{w_{P_i}}(\chi_{P_i,s_0}) \]
\[ A_{P_j} = \text{Res}_{\mathcal{F}_j} C_{w_{P_j}}(\chi_{P_j,t_0}) \]

\[ d = |N_1(\chi_{P_j,t_0})| - |N_1(\chi_{P_i,s_0})| \]
\[ \epsilon_{s_0} = b_{\overline{i}}^{[N_1(\chi_{s_0})]-[N_0(\chi_{s_0})]-(n-1)} \times \prod_{\alpha \in \mathbb{N}_0(\chi_{s_0})} n_{i}^{(\overline{\alpha})} \times \prod_{\alpha \in \mathbb{N}_1(\chi_{s_0}) \setminus \Delta_P} n_{i}^{(\overline{\alpha})} \]

In order to prove it we will prove a sequence of Lemmas.

With the notation of Corollary (4.4) it holds that

\[ E_B^\#(\chi_{P,s}, g) = A_{w_P} \times G_P(\chi_{P,s}) \times \prod_{\alpha \in \Delta_P} \zeta(l_\alpha^+(\chi_{P,s}))l_\alpha^+(\chi_{P,s}) \times E_P(f^0, s, g) \]

where \( G_P(\chi_{P,s}) = \prod_{\alpha \in \Phi^+ \setminus \Delta_P} \zeta(l_\alpha^+(\chi_{P,s}))l_\alpha^+(\chi_{P,s})l_\alpha^-(\chi_{P,s}) \) and \( A_{w_P} = \text{Res}_{\mathcal{F}_i} C_{w_P}(\chi_{P_i,s}) \).

Moreover, Theorem (4.6) shows that \( G_P(\chi_s) \) can be written as

\[ G_P(\chi_s) = h_1(\chi_s, s_0) \times h_2(\chi_s, s_0) \times h_3(\chi_s, s_0) \times (\zeta(2)^{[N_1(\chi_s)]-(n-1)} \times (-1)^{[N_0(\chi_s)]} \times 2^{[N_1(\chi_s)]-(n-1)}). \]

**Lemma 5.0.1.** Let \( P = P_i \) be a maximal parabolic subgroup of \( G \). Then

1. \( \lim_{s \to s_0} h_1(\chi_s, s_0) = (-R)^{|N_{-1}(\chi_{s_0})|} \).

2. \( h_2(\chi_s, s_0) \) has a zero of order \( |N_1(\chi_{s_0})| - |N_0(\chi_{s_0})| - (n-1) \) at \( s = s_0 \). Moreover, its leading coefficient is equal to \( \epsilon_{s_0} \times R^{[N_0(\chi_{s_0})]} \) where

\[ \epsilon_{s_0} = b_{\overline{i}}^{[N_1(\chi_{s_0})]-[N_0(\chi_{s_0})]-(n-1)} \times \prod_{\alpha \in \mathbb{N}_0(\chi_{s_0})} n_{i}^{(\overline{\alpha})} \times \prod_{\alpha \in \mathbb{N}_1(\chi_{s_0}) \setminus \Delta_P} n_{i}^{(\overline{\alpha})} \]
Proof. 1. It will be enough to show that if \( \alpha \in N_{-1}(\chi_{s_0}) \) then
\[
\zeta(l^+_\alpha(\chi_{s_0}))l^+_\alpha(\chi_{s_0}) = -R.
\]
But it is just the residue of \( \zeta(s) \) at \( s = 0 \) so we are done.

2. Note that \( h_2(\chi_s, s_0) \) is a meromorphic function (it is a product of such functions) therefore it admits a Laurent expansion around \( s = s_0 \). For every \( \alpha \in \Phi^+ \) it holds that \( l^+_\alpha(\chi_s) \) is a linear function in \( s \) and
\[
l^+_\alpha(\chi_s) = b_i n^{(\alpha)}_i s + m(\alpha)
\]
where \( m(\alpha) \) is a constant depends in \( \alpha \).

In particular, for every \( \alpha \in N_0(\chi_{s_0}) \) the element \( \zeta(l^+_\alpha(\chi_s)) \) can be written as \( \zeta(F(s, \alpha)) \) where
\[
F(s, \alpha) = b_i n^{(\alpha)}_i s + m(\alpha).
\]
By definition \( F(s_0, \alpha) = 1 \). Therefore \( \zeta(F(s, \alpha)) \) admits a simple pole at \( s = s_0 \).

The Laurent expansion of \( \zeta(F(s, \alpha)) \) around \( s_0 \) is
\[
\zeta(F(s, \alpha)) = \sum_{j=-1}^{\infty} a_j (F(s, \alpha) - 1)^j
\]
where \( a_{-1} = R \). Moreover,
\[
F(s, \alpha) - 1 = F(s, \alpha) - F(s_0, \alpha)
\]
\[
= b_i n^{(\alpha)}_i s + m(\alpha) - (b_i n^{(\alpha)}_i s_0 + m(\alpha))
\]
\[
= b_i n^{(\alpha)}_i (s - s_0).
\]

Therefore
\[
\zeta(F(s, \alpha)) = \sum_{j=-1}^{\infty} a_j (b_i n^{(\alpha)}_i)^j (s - s_0)^j.
\]
The leading term around \( s = s_0 \) of
\[
\prod_{\alpha \in N_0(\chi_{s_0})} \zeta(l^+_\alpha(\chi_s))
\]
is the product of the leading term of every \( \zeta(l^+_\alpha(\chi_s)) \) around \( s = s_0 \). Therefore, the product contributes a pole of order \( |N_0(\chi_{s_0})| \) and its leading coefficient equals
to

\[
\prod_{\alpha \in N_0(\chi_{s_0})} \frac{1}{n_\alpha^{(\bar{\alpha})}} \times \left( \frac{R}{b_{ii}} \right)^{[N_0(\chi_{s_0})]},
\]

The product

\[
\prod_{\alpha \in N_1(\chi_{s_0})} \quad \prod_{\alpha \notin \Delta_P} l_\alpha(\chi_s)
\]

has a zero of order \([N_1(\chi_{s_0}) \setminus \Delta_P] = [N_1(\chi_{s_0})] - (n - 1)\). Since every \(\alpha \in \Delta_P\) satisfies \(\langle \chi_s, \bar{\alpha} \rangle = 1\). By a similar argument, its leading coefficient equals to

\[
b_{ii}^{[N_1(\chi_{s_0})] - (n - 1)} \times \prod_{\alpha \in N_1(\chi_{s_0})} n_\alpha^{(\bar{\alpha})}.
\]

Hence, the leading coefficient of \(h_2(\chi_s, s_0)\) around \(s = s_0\) equals to

\[
\epsilon_{s_0} \times R^{[N_0(\chi_{s_0})]} \times R^{[N_0(\chi_{s_0})]}
\]

and it admits a zero of order

\[
N_1(\chi_s) - N_0(\chi_s) - (n - 1).
\]

\[\square\]

**Corollary 5.1.** For the notations as above and for \(F = \mathbb{Q}\) it holds that

\[
E_B^{\#}(\chi_{s_0}, g) = h_3(\chi_{s_0}, s_0)
\]

\[
\times \epsilon_{s_0} \times (\chi_{s_0})^{-1} \times \left[ N_0(\chi_{s_0}) \right] \times 2^{[N_1(\chi_{s_0})]} \times \zeta(2)^{[N_1(\chi_{s_0})]} \times R^{[N_0(\chi_{s_0})]} \times A_{w,P} \times A_{P-d_P(\chi_{s_0})}(f, s_0, h_{3,Q}(\chi_{s_0}, t_0)).
\]

Let \((P, s_0, Q, t_0, w)\) be an admissible data. Our goal is to determine the constant for Theorem (4.9). For this purpose, we shall evaluate the quotient \(\frac{h_{3,Q}(\chi_{s_0}, t_0)}{h_{3,P}(\chi_{s_0}, t_0)}\). We shall need the following lemmas.

**Lemma 5.1.1.** Let \(\chi\) be an unramified character of \(T\). For every \(\epsilon \in \mathbb{C}\) and \(w \in \mathbb{W}_G\) it holds that:

1. \(B_\epsilon(\chi) = -B_{-\epsilon}(\chi)\).
2. \(B_\epsilon(w \chi) = w B_\epsilon(\chi)\).
3. \(|N_{\pm\epsilon}(\chi)| = |N_{\pm\epsilon}(w\chi)|\).

4. \(|N_1(\chi)| - |N_{-1}(w\chi)| = |N_1(w\chi)| - |N_{-1}(\chi)|\).

5. \(|N_0(\chi)| = |N_0(w\chi)|\).

**Proof.** (1) and (2) are immediate from the definition and the \(W_G\) invariance of the paring. (4) and (5) follow from (3). Thus we need to prove only (3).

Note that
\[
B_{\epsilon}(\chi) = \{ \alpha \in \Phi : \langle \chi, \check{\alpha} \rangle = \epsilon \}
\]
\[
= \{ \alpha \in \Phi^+ : \langle \chi, \check{\alpha} \rangle = \epsilon \} \cup \{ \alpha \in \Phi^- : \langle \chi, \check{\alpha} \rangle = -\epsilon \}
\]
\[
= N_\epsilon(\chi) \cup \{-\alpha \in \Phi^+ : \langle \chi, -\check{\alpha} \rangle = -\epsilon \}
\]
\[
= N_\epsilon(\chi) \cup -N_{-\epsilon}(\chi).
\]

Hence,
\[
|B_{\epsilon}(\chi)| = |N_\epsilon(\chi)| + |-N_{-\epsilon}(\chi)| = |N_\epsilon(\chi)| + |N_{-\epsilon}(\chi)|.
\]

By the same argument
\[
|B_{\epsilon}(w\chi)| = |N_\epsilon(w\chi)| + |N_{-\epsilon}(w\chi)|.
\]

From (2) and the injectivity of \(w\) it holds that \(|B_{\epsilon}(w\chi)| = |B_{\epsilon}(\chi)|\). Thus, we are done.

**Lemma 5.1.2.** For every \(\alpha \in \Phi\) let \(F_\alpha : a_\epsilon^* \to \mathbb{C}\) that satisfies \(F_\alpha(w\chi) = F_{w^{-1}\alpha}(\chi)\) for every \(\chi \in a_\epsilon^*, w \in W_G\). Assume that \(F_\alpha(\chi) \neq 0\) for every \(\alpha \in \Phi\) then
\[
\frac{\prod_{\alpha \in \Phi^+ \setminus N_{\pm 1,0}(w\chi)} F_\alpha(w\chi)}{\prod_{\alpha \in \Phi^+ \setminus N_{\pm 1,0}(\chi)} F_\alpha(\chi)} = \prod_{\alpha \in \Phi^+ \setminus N_{\pm 1,0}(\chi), \alpha \neq \alpha} \frac{F_{-\alpha}(\chi)}{F_{\alpha}(\chi)}.
\]
Proof.

\[
\prod_{\alpha \in \Phi^+ \setminus \mathbb{N}_{\pm 1,0}(w\chi)} F_{\alpha}(w\chi) = \prod_{\alpha \in \Phi^+} F_{\alpha}(w\chi) \quad \text{#1} = \prod_{\alpha \notin \mathbb{N}_{\pm 1,0}(w\chi)} F_{w^{-1}\alpha}(\chi).
\]

\[
= \prod_{\alpha \in \Phi^+} F_{w^{-1}\alpha}(\chi) = \prod_{\beta = w^{-1}\alpha} F_{\beta}(\chi)
\]

\[
= \prod_{w\beta \in \Phi^+, \beta \notin \mathbb{N}_{\pm 1,0}(\chi)} F_{\beta}(\chi) \times \prod_{w\beta \in \Phi^-} F_{\beta}(\chi)
\]

\[
= \prod_{w\beta \in \Phi^+, \beta \notin \mathbb{N}_{\pm 1,0}(\chi)} F_{\beta}(\chi)
\]

\[
\text{#2} = \prod_{w\gamma \in \Phi^-} F_{-\gamma}(\chi)
\]

where #1 is due to (5.1.1) item (2) and #2 is due to Lemma (5.1.1) item (1) since

\[
-B_{\pm 1,0}(\chi) = -B_1(\chi) \cup -B_0(\chi) \cup -B_{-1}(\chi)
\]

\[
= B_{-1}(\chi) \cup B_0(\chi) \cup B_1(\chi)
\]

\[
= B_{\pm 1,0}(\chi).
\]

Hence,

\[
\frac{\prod_{\alpha \in \Phi^+ \setminus \mathbb{N}_{\pm 1,0}(w\chi)} F_{\alpha}(w\chi)}{\prod_{\alpha \in \Phi^+ \setminus \mathbb{N}_{\pm 1,0}(\chi)} F_{\alpha}(\chi)} = \prod_{\alpha \in \Phi^+, \alpha \notin \mathbb{N}_{\pm 1,0}(\chi)} \frac{F_{-\alpha}(\chi)}{F_{\alpha}(\chi)}.
\]

\[\square\]
Corollary 5.2. Let \((P, s_0, Q, t_0, w)\) be an admissible data. For \(F = \mathbb{Q}\) it holds that

\[
\frac{h_{3, Q}(\chi_{Q,t}, t_0)}{h_{3, P}(\chi_{P,s}, s_0)} = \prod_{\alpha \in \Phi^+ \setminus N \pm 1, 0(\chi_{P,s_0})} \frac{\zeta((\chi_{P,s_0}, \hat{\alpha}))}{\zeta((\chi_{P,s_0}, \hat{\alpha}) + 1)},
\]

(5.1)

Proof. Notice that \(l^+_\alpha(w\chi) = l^-_{w^{-1}\alpha}(\chi)\). Hence, we can apply Lemma (5.1.2) with the function

\[
F_\alpha(\chi) = l^+_\alpha(\chi)l^-_{\alpha}(\chi)\zeta(l^+_{\alpha}(\chi)).
\]

Since \((P, s_0, Q, t_0, w)\) is an admissible data, it holds that \(w(\chi_{P,s_0}) = \chi_{Q,t_0}\). Hence,

\[
\frac{h_{3, Q}(\chi_{Q,t}, t_0)}{h_{3, P}(\chi_{P,s}, s_0)} = \prod_{\alpha \in \Phi^+ \setminus N \pm 1, 0(\chi_{Q,t_0})} F_\alpha(\chi_{Q,t_0}) = \prod_{\alpha \in \Phi^+ \setminus N \pm 1, 0(\chi_{P,s_0})} \frac{F^-_{-\alpha}(\chi_{P,s_0})}{F_\alpha(\chi_{P,s_0})}.
\]

Notice that

\[
l^+_\alpha(\chi)l^-_{-\alpha}(\chi) = l^+_\alpha(\chi)l^-_{\alpha}(\chi).
\]

Moreover, by functional equation of the \(\zeta\) function it holds that \(\zeta(l^-_{-\alpha}(\chi)) = \zeta((\chi, \hat{\alpha}))\). Hence

\[
\frac{F^-_{-\alpha}(\chi)}{F_\alpha(\chi)} = \frac{\zeta((\chi, \hat{\alpha}))}{\zeta((\chi, \hat{\alpha}) + 1)}.
\]

So the Corollary follows.
Chapter 5. Refining The Basic Identity

Theorem 5.3. Let \((P_i, s_0, P_j, t_0, w)\) be an admissible data. For \(F = \mathbb{Q}\) there exists \(A \in \mathbb{C}\) such that

\[
\lambda_{-d_{P_i}(x_{P_i,-0})}^P(f_{P_i}^0, s_0, g) = A \times \lambda_{-d_{P_j}(x_{P_j,-0})}^P(f_{P_j}^0, t_0, g) \quad \forall g \in G.
\]

Moreover,

\[
A = \frac{\epsilon_{t_0}}{\epsilon_{s_0}} \times \prod_{a \in \Phi^+ \setminus \Delta_{\pm 0}(x_{P_i,-0})} \frac{\zeta(\langle x_{P_i,-0}, \alpha \rangle)}{\zeta(\langle x_{P_i,-0}, \alpha \rangle + 1)} \times \left(\frac{R}{\zeta(2)}\right)^d \times \frac{A_{P_i}}{A_{P_j}}
\]

where

\[
A_{P_i} = \text{Res}_{\mathbb{C}} C_{wP_i}(x_{P_i,-0}) \quad \quad \quad A_{P_j} = \text{Res}_{\mathbb{C}} C_{wP_j}(x_{P_j,-0})
\]

\[
d = |N_{-1}(x_{P_i,-0})| - |N_{-1}(x_{P_i,-0})|
\]

Proof. Using Corollary (5.1) we obtained that:

\[
E_B^\#(x_{P_i,-0}, g) = h_3(x_{P_i,-0}, s_0) \\
\times \epsilon_{s_0} \times (-1)^{|N_0(x_{P_i,-0})|} \times 2^{|N_{\pm 1}(x_{P_i,-0})|} \times \zeta(2)^{|N_1(x_{P_i,-0})|} \\
\times R^{|N_{0,-1}(x_{P_i,-0})|} \times A_{P_i} \times \lambda_{-d_{P_i}(x_{P_i,-0})}^P(f_{P_i}^0, s_0, g).
\]

By a similar argument we get:

\[
E_B^\#(x_{P_j,-0}, g) = h_3(x_{P_j,-0}, t_0) \\
\times \epsilon_{t_0} \times (-1)^{|N_0(x_{P_j,-0})|} \times 2^{|N_{\pm 1}(x_{P_j,-0})|} \times \zeta(2)^{|N_1(x_{P_j,-0})|} \\
\times R^{|N_{0,-1}(x_{P_j,-0})|} \times A_{P_j} \times \lambda_{-d_{P_j}(x_{P_j,-0})}^P(f_{P_j}^0, t_0, g).
\]

Since \(E_B^\#\) is \(W_G\) invariant, both terms are equal. So, we get:

\[
\lambda_{-d_{P_i}(x_{P_i,-0})}^P(f_{P_i}^0, s_0, g) = \frac{A_{P_i}}{A_{P_j}} \times \frac{\epsilon_{t_0}}{\epsilon_{s_0}} \times \frac{h_3(x_{P_j,-0}, t_0)}{h_3(x_{P_i,-0}, s_0)} \\
\times 2^{|N_{\pm 1}(x_{P_j,-0})| - |N_{\pm 1}(x_{P_i,-0})|} \times (-1)^{|N_0(x_{P_j,-0})| - |N_0(x_{P_i,-0})|} \\
\times \zeta(2)^{|N_1(x_{P_j,-0})| - |N_1(x_{P_i,-0})|} \times R^{|N_{0,-1}(x_{P_j,-0})| - |N_{0,-1}(x_{P_i,-0})|} \\
\times \lambda_{-d_{P_j}(x_{P_j,-0})}^P(f_{P_j}^0, t_0, g).
\]

Using Lemma (5.1.1) it holds that:

\[
(-1)^{|N_{\pm 1}(x_{P_j,-0})| - |N_{\pm 1}(x_{P_i,-0})|} = 1, \quad 2^{|N_{\pm 1}(x_{P_j,-0})| - |N_{\pm 1}(x_{P_i,-0})|} = 1, \quad R^{|N_0(x_{P_j,-0})| - |N_0(x_{P_i,-0})|} = 1,
\]
Hence it holds that:

\[
\Lambda_{-d_{P_i}(\chi_{P_i}, s_0)}(f_{P_i}^0, s_0, g) = \frac{A_{P_i}}{A_{P_i}} \times \frac{\varepsilon_{t_0}}{\varepsilon_{s_0}} \times \frac{h_3(\chi_{P_i}, t_0)}{h_3(\chi_{P_i}, s_0)} \times \zeta(2)^{|N_1(\chi_{P_j}, t_0)| - |N_1(\chi_{P_j}, s_0)|} \times R^{|N_1(\chi_{P_j}, t_0)| - |N_1(\chi_{P_j}, s_0)|} \times \Lambda_{-d_{P_j}(\chi_{P_j}, t_0)}(f_{P_j}^0, s_0, g).
\]

Using Lemma (5.1.1) again we obtain

\[
\zeta(2)^{|N_1(\chi_{P_j}, t_0)| - |N_1(\chi_{P_i}, s_0)|} \times R^{|N_1(\chi_{P_j}, t_0)| - |N_1(\chi_{P_i}, s_0)|} = \left( \frac{R}{\zeta(2)} \right)^d.
\]

Corollary (5.2) implies that

\[
\frac{h_3(\chi_{P_i}, t_0)}{h_3(\chi_{P_i}, s_0)} = \prod_{\alpha \in \Phi^+ \setminus N_{1,0}(\chi_{P_i}, s_0), \omega \alpha \in \Phi^-} \frac{\zeta(\langle \chi_{P_i}, s_0, \alpha \rangle)}{\zeta(\langle \chi_{P_i}, s_0, \alpha \rangle + 1)}.
\]

Hence,

\[
\Lambda_{-d_{P_i}(\chi_{P_i}, s_0)}(f_{P_i}^0, s_0, g) = \frac{A_{P_i}}{A_{P_i}} \times \frac{\varepsilon_{t_0}}{\varepsilon_{s_0}} \times \left( \frac{R}{\zeta(2)} \right)^d \times \prod_{\alpha \in \Phi^+ \setminus N_{1,0}(\chi_{P_i}, s_0), \omega \alpha \in \Phi^-} \frac{\zeta(\langle \chi_{P_i}, s_0, \alpha \rangle)}{\zeta(\langle \chi_{P_i}, s_0, \alpha \rangle + 1)} \times \Lambda_{-d_{P_j}(\chi_{P_j}, t_0)}(f_{P_j}^0, s_0, g).
\]

Observe that

\[
d_{P_i}(\chi_{P_i}, s_0) - d = |N_1(\chi_{P_i}, s_0)| - |N_0(\chi_{P_i}, s_0)| - (n - 1) - |N_{-1}(\chi_{P_i}, t_0)| + |N_{-1}(\chi_{P_i}, s_0)|
\]
\[
= |N_1(\chi_{P_i}, s_0)| - |N_0(\chi_{P_i}, s_0)| - (n - 1) + |N_1(\chi_{P_j}, t_0)| - |N_1(\chi_{P_i}, s_0)|
\]
\[
= |N_1(\chi_{P_j}, t_0)| - |N_0(\chi_{P_i}, s_0)| - (n - 1) = d_{P_j}(\chi_{P_j}, t_0).
\]
Chapter 6

Identities

In this chapter we explore the identities between leading terms of various Eisenstein series associated to various maximal parabolic subgroups. The motivation comes from the famous Siegel Weil Formula.

Let $W$ be a symplectic space of dimension $2n$ and $V$ be an orthogonal space of dimension $m$. A. Weil in his seminal paper [24] has evaluated an average of the theta function on the group $\tilde{Sp}(W \otimes V)$ over $O(V)$ as a special value of the degenerate Eisenstein on $Sp(W)$ series associated to the Siegel parabolic subgroup under the assumption that the integral converges. The convergence condition has been lifted in the paper of Kudla and Rallis [25] who considered the regularized average of the theta function above. The regularized integral is naturally related to a residue of a non-Siegel Eisenstein series. However it was essential to find the relation between the average integral the residue of the Siegel Eisenstein series, whose analytic properties can be further related to the standard L-functions by [1]. So Kudla and Rallis have found the identity between the leading terms of the Siegel and non-Siegel Eisenstein series. Later their work has been generalized for all classical groups.

Jiang in [7] has considered various identities for the leading terms of degenerate spherical Eisenstein series on symplectic groups, for various maximal parabolic subgroups.

Finally Ginzburg and Jiang in [8] have considered similar identities for the exceptional group $G_2$.

The kind of identities we expect to see One type of the identities follows from the functional equation satisfied by Eisenstein series that relate the Eisenstein series
associated to maximal parabolic subgroups whose Levi parts are conjugated by elements of the Weyl group.

However, there exists another type of identities that relate between the Eisenstein series associated to maximal parabolic subgroups whose Levi parts are not conjugated by elements of the Weyl group. The simplest example is the trivial representation that can be realized as residue of Eisenstein series associated to any maximal parabolic subgroup \( P \) at \( s = \frac{1}{2} \). This kind of identities is sometimes called Siegel-Weil identities.

Let us briefly describe the context of the chapter. The chapter begins with the proof of the following combinat results

**Theorem 6.1.** For every pair of maximal parabolic subgroup \( P_i, P_j \) there exist \( w \in \mathcal{W}_G \) and \( s_0, t_0 \in \mathbb{R} \) such that \((P_i, s_0, P_j, t_0, w)\) is an admissible data.

During the proof, we give an uniformly formula for those data. As a corollary in the case where \(-1 \in \mathcal{W}_G\) it holds that :

**Corollary.** For every pair of maximal parabolic subgroup \( P_i, P_j \) there exist \( w \in \hat{\mathcal{W}}_G \) and \( s_0, t_0 \in \mathbb{R}\) non negative numbers such that \((P_i, s_0, P_j, t_0, w)\) is a positive admissible data.

The positive admissible data constructed by the above method is called special admissible data if \( s_0, t_0 \) are non negative. However, Theorem (6.1) gives only a partial list of admissible data. Surprisingly, the list of admissible data that can be derived from Theorem (6.1) is almost the complete list of admissible data that exists.

In section 6.2 we describe an algorithm that gives a complete list of admissible data. In sections 6.3, 6.4, 6.5, 6.6 we write explicitly the admissible data and apply Theorem (5.3) for the groups of types \( G_2, F_4, E_6, E_7 \).

### 6.1 Special admissible data

Our goal is to prove Theorem (6.1). We need few notations.

Let \( P_i, P_j \) be maximal parabolic subgroups of \( G \). Denote by \( R = P_i \cap P_j \). Let \( \mathcal{W}_{P_i}, \mathcal{W}_{P_j}, \mathcal{W}_R \) be the Weyl group of \( P_i, P_j, R \). Let \( \hat{w}_{0,i} \) (resp. \( \hat{w}_{0,j} \)) be the shortest representative of the longest Weyl element of \( \hat{\mathcal{W}}_{P_i}/\hat{\mathcal{W}}_R \) (resp. \( \hat{\mathcal{W}}_{P_j}/\hat{\mathcal{W}}_R \)).
Note that since $R$ is the intersection of two maximal parabolic subgroups the set of the unramified characters of $R$ is isomorphic to $\mathbb{C}^2$. Let us define a matrix $B = (b_{ij})$ whose entries satisfy
\[
\delta_{P_i} = b_{ii}\bar{\omega}_i \quad \delta_{P_j} = b_{jj}\bar{\omega}_j \quad \delta_R = b_{ij}\bar{\omega}_i + b_{ji}\bar{\omega}_j \quad \text{if } i < j.
\]

**Lemma 6.1.1.** Let $c_{ij} = \frac{b_{ij}}{b_{ii}} - \frac{1}{2}$ and $c_{ji} = \frac{b_{ij}}{b_{jj}} - \frac{1}{2}$. Then:
\[
\delta_{P_i}^{c_{ij}} \otimes (\delta_{P_j}^{P_R})^{-\frac{1}{2}} = \delta_{P_j}^{c_{ji}} \otimes (\delta_{P_i}^{P_R})^{-\frac{1}{2}}.
\]

**Proof.**
\[
\delta_{P_i}^{c_{ij}} \otimes (\delta_{P_j}^{P_R})^{-\frac{1}{2}} = c_{ij}b_{ii}\bar{\omega}_i - \frac{1}{2}(b_{ij}\bar{\omega}_i + b_{ji}\bar{\omega}_j - b_{ii}\bar{\omega}_i)
= b_{ij}\bar{\omega}_i + \frac{1}{2}b_{ii}\bar{\omega}_i - \frac{1}{2}b_{jj}\bar{\omega}_j + \frac{1}{2}b_{ii}\bar{\omega}_i
= \frac{1}{2}b_{ii}\bar{\omega}_i - \frac{1}{2}b_{jj}\bar{\omega}_j.
\]
On the other hand
\[
\delta_{P_j}^{c_{ij}} \otimes (\delta_{P_i}^{P_R})^{-\frac{1}{2}} = -c_{ji}b_{jj}\bar{\omega}_j - \frac{1}{2}(b_{ij}\bar{\omega}_i + b_{ji}\bar{\omega}_j - b_{jj}\bar{\omega}_j)
= -b_{ij}\bar{\omega}_j + \frac{1}{2}b_{jj}\bar{\omega}_j + \frac{1}{2}b_{ij}\bar{\omega}_i - \frac{1}{2}b_{jj}\bar{\omega}_j
= \frac{1}{2}b_{jj}\bar{\omega}_j - \frac{1}{2}b_{jj}\bar{\omega}_j.
\]
So we are done. □

**Lemma 6.1.2.** Let $\chi_s = \delta_{P}^{\frac{1}{2}} \otimes \delta_{B}^{-\frac{1}{2}}$ and let $w_P \in W_P$ be the longest element. Then $w_P(\chi_s) = \delta_{P}^{\frac{1}{2}} \otimes \delta_{B}^{-\frac{1}{2}}$.

**Proof.** For every $\alpha \in \Phi_G^+ \setminus \Phi_M^+$ it holds that $w_P\alpha \in \Phi_G^+ \setminus \Phi_M^+$ and for every $\beta \in \Phi_M^+$ it holds that $w_P\beta \in \Phi_M^+$. Now rewrite $\delta_B = \delta_P \otimes (\delta_B \otimes \delta_P^{-1})$. Observe that
\[
(\delta_B \otimes \delta_P^{-1}) = \prod_{\alpha \in \Phi_M^+} \alpha.
\]
Chapter 6. Identities

Hence, \( w_p(\delta_B \otimes \delta_P^{-1}) = (\delta_B^{-1} \otimes \delta_P) \). Hence,

\[
w_p(\delta^{s-\frac{1}{2}}_B \otimes \delta^{\frac{1}{2}}_P) = w_p(\delta^{s}_P \otimes \delta^{\frac{1}{2}}_B) \\
= \delta^{s}_P \otimes \delta^{\frac{1}{2}}_P \otimes \delta^{\frac{1}{2}}_B \\
= \delta^{s-\frac{1}{2}}_P \otimes \delta^{\frac{1}{2}}_B.
\]

\[\square\]

**Theorem.** With the notation as above it holds that there exists \( w \in \mathcal{W}_G \) such that

\[
w(\delta^{c_{ij}-\frac{1}{2}}_P \otimes \delta^{\frac{1}{2}}_B) = \delta^{c_{ji}-\frac{1}{2}}_P \otimes \delta^{\frac{1}{2}}_B
\]

i.e \((P, c_{ij}, P, -c_{ji}, w)\) is an admissible data.

**Proof.** By Lemma \((6.1.2)\) we know that

\[
w_{P_i}(\delta^{c_{ij}-\frac{1}{2}}_P \otimes \delta^{\frac{1}{2}}_B) = \delta^{c_{ij}+\frac{1}{2}}_P \otimes \delta^{\frac{1}{2}}_B.
\]

Note that

\[
\delta^{c_{ij}+\frac{1}{2}}_P \otimes \delta^{\frac{1}{2}}_B
\]

\[
= \delta^{c_{ij}-\frac{1}{2}}_P \otimes \delta^{\frac{1}{2}}_R \otimes \delta^{\frac{1}{2}}_R \otimes \delta^{\frac{1}{2}}_B
\]

\[
= \delta^{c_{ij}-\frac{1}{2}}_P \otimes \delta^{\frac{1}{2}}_R \otimes \delta^{\frac{1}{2}}_R \otimes \delta^{\frac{1}{2}}_B
\]

Let \( w_R \) be the longest element in \( \mathcal{W}_R \). Then

\[
w_R \delta_{P_j} = \delta_{P_j}, \quad w_R \delta_R = \delta_R, \quad w_R \delta_B = (\delta_B)^{-1}.
\]

Hence

\[
w_R(\delta^{c_{ji}-\frac{1}{2}}_P \otimes \delta^{\frac{1}{2}}_R \otimes (\delta_B)^{-\frac{1}{2}}) = \delta^{c_{ji}-\frac{1}{2}}_P \otimes \delta^{\frac{1}{2}}_R \otimes (\delta_B^{\frac{1}{2}})
\]

\[
= \delta^{c_{ji}-\frac{1}{2}}_P \otimes \delta^{\frac{1}{2}}_B.
\]

Therefore for \( w = w_R w_{P_i} \) it holds that:

\[
w(\delta^{c_{ij}-\frac{1}{2}}_P \otimes \delta^{\frac{1}{2}}_B) = \delta^{c_{ji}-\frac{1}{2}}_P \otimes \delta^{\frac{1}{2}}_B
\]
Notice that
\[ w = w_R w_{P_i} \]
\[ = w_{P_j} w_{P_j} w_R w_{P_i} \]
\[ = w_{P_j} \tilde{w}_{0,j} w_{P_i} \]
where \( \tilde{w}_{0,j} \) is the shortest representative of the longest element in \( \mathcal{W}_{P_j}/\mathcal{W}_R \).

\[ \square \]

**Corollary 6.2.** Let \( G \) be an algebraic group of type different from \( A_n, D_{2n+1}, E_6 \). Then for \( s_0 = |c_{ij}|, t_0 = |c_{ji}| \) there exists \( w \in \mathcal{W}_G \) such that
\[ w(\delta^{s_0-\frac{1}{2}}_{P_i} \otimes \delta^{\frac{1}{2}}_B) = \delta^{t_0-\frac{1}{2}}_{P_j} \otimes \delta^{\frac{1}{2}}_B. \]

**Proof.** Let \( w_l \in \mathcal{W}_G/\mathcal{W}_{P_j} \) and \( w_r \in \mathcal{W}_G/\mathcal{W}_{P_i} \) be the shortest representative of the longest element. Set
\[ r = \begin{cases} e & \text{if } s_0 = c_{ij} \\ w_r & \text{if } s_0 = -c_{ij} \end{cases} \]
\[ l = \begin{cases} e & \text{if } t_0 = -c_{ji} \\ w_l & \text{if } t_0 = c_{ji} \end{cases} \]
Let \( w \) be as in Proposition (6.1). Then
\[ lwr(\delta^{s_0-\frac{1}{2}}_{P_i} \otimes \delta^{\frac{1}{2}}_B) = \delta^{t_0-\frac{1}{2}}_{P_j} \otimes \delta^{\frac{1}{2}}_B \]
\[ \square \]

**Definition 6.3.** The data \((P_i, s_0, P_j, t_0, w)\) that is constructed by the above method is called **special admissible data** if \( s_0, t_0 \) are non negative.
6.2 Algorithm: Finding Positive Admissible Data

In this section, we introduce a way to compute all the (positive) admissible data. Let us fix some notations.

Let $P, Q$ be maximal parabolic subgroups. Let

$$
\chi_P(s) = \delta_P^{s - \frac{1}{2}} \otimes \delta_B^\frac{1}{2} \quad \chi_Q(t) = \delta_Q^{t - \frac{1}{2}} \otimes \delta_B^\frac{1}{2} \quad (a_T^*)^+ = \{ \lambda \in a_T^* : \langle \lambda, \tilde{\alpha} \rangle \geq 0 \quad \forall \alpha \in \Delta_G \}
$$

It is well known that $|(a_T^*)^+ \cap \{ w\chi_P(s_0) : w \in \tilde{W}_G \}| = 1$. Let

$$
dom_P, dom_Q : \mathbb{R} \to (a_T^*)^+
$$

such that

$$
dom_P(s_0) = (a_T^*)^+ \cap \{ w\chi_P(s_0) : w \in \tilde{W}_G \} \quad dom_Q(s_0) = (a_T^*)^+ \cap \{ w\chi_Q(s_0) : w \in \tilde{W}_G \}
$$

**Remark 6.3.1.** The functions $dom_P, dom_Q$ are piecewise linear.

For every pair of intervals on which $dom_P, dom_Q$ are linear, we solve the system of linear equation $dom_P(s) = dom_Q(t)$. The solution will give rise to admissible data by the following procedure. Let

$$
W_P : \mathbb{R} \to \tilde{W}_G \quad W_Q : \mathbb{R} \to \tilde{W}_G
$$

such that $W_P(s_0)$ and $W_Q(s_0)$ are the shortest Weyl elements such that

$$
W_P(s_0)\chi_P(s_0) = dom_P(s_0) \quad \quad \quad W_Q(s_0)\chi_Q(s_0) = dom_Q(s_0)
$$

**Remark 6.3.2.** The functions $W_P, W_Q$ are piecewise constants.

Suppose $dom_P(s_0) = dom_Q(t_0)$ for some numbers $s_0, t_0$. Then

$$
w(\chi_P(s_0)) = \chi_Q(t_0) \quad \text{for} \quad w = W_Q(t_0)^{-1}W_P(s_0)
$$

Hence $(P, s_0, Q, t_0, w)$ is an admissible data.

**Remark 6.3.3.** Since we are mostly interested in the positive admissible data, we can restrict ourself to the domain $[0, \infty)$. Moreover, for $s > \frac{1}{2}, t > \frac{1}{2}$ we do not have any positive admissible data.
In the following sections we list all the positive admissible data.

**Remark 6.3.4.** As we have mentioned earlier, for every pair of maximal parabolic subgroups \( P_i, P_j \) we have the admissible data \((P_i, \frac{1}{2}, P_j, \frac{1}{2}, e)\) that corresponds to the identity between the trivial representation. Hence we may omit this.

For each section we start by finding all the special data and list the identities. After this we give the complete list of positive admissible data and write the remaining identities.

### 6.3 Group of type \( G_2 \)

**Proposition 6.3.1.** Let \( G = G_2 \) then it holds that

\[
\delta_{P_1} = 5\bar{\omega}_1 \quad \delta_{P_2} = 3\bar{\omega}_2 \quad \delta_B = 2\bar{\omega}_1 + 2\bar{\omega}_2.
\]

**Proposition 6.3.2.** In this case we have only one (special) admissible data \((P_2, \frac{1}{6}, P_1, \frac{1}{10})\)

**Remark 6.3.5.** In this case we have only one identity.

**Proposition 6.3.3.** It holds that:

\[
A_{wP_1} = \frac{R}{\zeta(2)} \quad wP_1 = w_2 \quad A_{wP_2} = \frac{R}{\zeta(2)} \quad wP_2 = w_1
\]

| \( P_i \) | \( s \) | \( P_j \) | \( t \) | \( w \) | \( \frac{dP_i(\chi_s)}{d\tilde{\chi}_s} \) | \( dP_j(\chi_t) \) | \( d \) | \( \epsilon_s \) | \( \epsilon_t \) |
|---|---|---|---|---|---|---|---|---|
| \( P_2 \) | \( \frac{1}{6} \) | \( P_1 \) | \( \frac{1}{10} \) | \( w_1 \) | 1 | 2 | 1 | 54 | 10 |

**Theorem 6.4.** Let \( f^0 \in I_P(s) \) be the normalized spherical section then:

1. \( \Lambda_{-2}(f^0, \frac{1}{6}, g) = \frac{5}{27} \times \frac{R}{\zeta(2)} \times \Lambda_{-1}(f^0, \frac{1}{10}, g) \).
6.4 Group of type $F_4$

Proposition 6.4.1. The matrix $B$ for $G = F_4$ is

$$B = \begin{pmatrix} 8 & 2 & 3 & 5 \\ 4 & 5 & 3 & 4 \\ 5 & 3 & 7 & 6 \\ 6 & 3 & 2 & 11 \end{pmatrix}$$

Proposition 6.4.2. The special admissible data are as follows:

$$\begin{array}{c|cccc}
\left[ s_0, t_0 \right] & P_1 & P_2 & P_3 & P_4 \\
P_1 & \frac{3}{10}, \frac{1}{5} & \frac{1}{10}, \frac{3}{14} & \frac{1}{5}, \frac{1}{14} & \frac{1}{5}, \frac{1}{14} \\
P_2 & \frac{3}{10}, \frac{1}{4} & \frac{1}{10}, \frac{1}{14} & \frac{3}{10}, \frac{5}{22} & \frac{5}{14}, \frac{7}{22} \\
P_3 & \frac{7}{22}, \frac{7}{10} & \frac{5}{22}, \frac{3}{14} & \frac{7}{22}, \frac{5}{14} & \frac{7}{22}, \frac{5}{14} \\
P_4 & \frac{7}{22}, \frac{7}{10} & \frac{5}{22}, \frac{3}{14} & \frac{7}{22}, \frac{5}{14} & \frac{7}{22}, \frac{5}{14} \\
\end{array}$$

Proposition 6.4.3. It holds that:

$$A_{w_{P_1}} = \frac{R^3}{\zeta(4)\zeta(2)\zeta(6)} w_{P_1} = w_1 w_2 w_3 w_4 w_3 w_2 w_2 w_3 w_4 w_3 w_2 w_2$$

$$A_{w_{P_2}} = \frac{R^3}{\zeta(2)^2\zeta(3)} w_{P_2} = w_3 w_1 w_2 w_3 w_2$$

$$A_{w_{P_3}} = \frac{R^3}{\zeta(2)^2\zeta(3)} w_{P_3} = w_4 w_1 w_2 w_3 w_2$$

$$A_{w_{P_4}} = \frac{R^3}{\zeta(6)\zeta(4)\zeta(2)} w_{P_4} = w_3 w_2 w_3 w_1 w_2 w_3 w_2 w_1 w_1 w_2 w_3 w_2$$

$$| P_i | s | P_j | t | w | h_{\chi_1}(\chi_{s}) | d_{P_i}(\chi_{s}) | d_{P_j}(\chi_{t}) | d | \epsilon_p | \epsilon_q |$$

$$\begin{array}{ccccccccccc}
P_1 & 1 & 8 & P_4 & \frac{1}{22} & w_3 w_2 w_1 w_2 & \frac{\zeta(3)}{\zeta(15)} & 1 & 1 & 0 & 8 & 22 \\
P_2 & \frac{3}{10} & P_1 & \frac{1}{4} & w_1 & \zeta(15) & 1 & 2 & 1 & 1 & 50 & 16 \\
P_2 & \frac{1}{10} & P_3 & \frac{1}{14} & w_3 w_4 & \frac{\zeta(2)}{\zeta(13)} & 3 & 2 & 1 & 3000 & 294 \\
P_2 & \frac{3}{10} & P_4 & \frac{5}{22} & w_4 w_3 & \frac{\zeta(2)}{\zeta(13)} & 2 & 1 & 1 & 50 & 11 \\
P_3 & \frac{3}{14} & P_1 & \frac{1}{8} & w_1 w_2 & \frac{\zeta(2)}{\zeta(13)} & 2 & 1 & 1 & 98 & 8 \\
P_3 & \frac{5}{14} & P_4 & \frac{7}{22} & w_4 & \frac{\zeta(2)}{\zeta(13)} & 1 & 1 & 0 & 1 & 7 & 1 \\
\end{array}$$
Chapter 6. Identities- Group of type $F_4$

**Theorem 6.5.** Let $f^0 \in I_{P_1}(s)$ be the normalized spherical section then:

1. $\Lambda_{-1}^{P_1}(f^0, \frac{1}{8}, g) = \frac{11}{4} \times \zeta(3) \zeta(5) \times \Lambda_{-1}^{P_4}(f^0, \frac{1}{22}, g)$.

2. $\Lambda_{-2}^{P_2}(f^0, \frac{3}{10}, g) = \frac{8}{25} \times \zeta(3) \zeta(5) \times \Lambda_{-1}^{P_4}(f^0, \frac{1}{4}, g)$.

3. $\Lambda_{-3}^{P_2}(f^0, \frac{1}{10}, g) = \frac{49}{500} \times \zeta(3) \times \Lambda_{-2}^{P_4}(f^0, \frac{1}{22}, g)$.

4. $\Lambda_{-2}^{P_2}(f^0, \frac{3}{10}, g) = \frac{11}{50} \times \zeta(3) \zeta(5) \times \Lambda_{-1}^{P_4}(f^0, \frac{1}{4}, g)$.

5. $\Lambda_{-2}^{P_2}(f^0, \frac{3}{14}, g) = \frac{1}{10} \times \zeta(3) \zeta(5) \times \Lambda_{-1}^{P_4}(f^0, \frac{1}{4}, g)$.

6. $\Lambda_{-1}^{P_3}(f^0, \frac{5}{14}, g) = \frac{1}{7} \times \zeta(3) \zeta(5) \times \Lambda_{0}^{P_4}(f^0, \frac{7}{22}, g)$.

There exists a positive admissible data which is non special admissible data and cannot be derived from the list above

$$(P_2, \frac{1}{5}, P_1, \frac{1}{16}, w_1 w_2 w_3 w_4).$$

This positive admissible data gives rise to the following identity:

**Theorem 6.6.** Let $f^0 \in I_{P_1}(s)$ be the normalized spherical section then:

1. $\Lambda_{-2}^{P_2}(f^0, \frac{1}{5}, g) = \frac{1}{10} \times \zeta(3) \zeta(5) \times \Lambda_{0}^{P_4}(f^0, \frac{1}{16}, g)$.

Therefore we get the follows:

\[
\begin{array}{c}
\text{I}_{P_2}(\frac{3}{10}) & \xrightarrow{\text{I}_{P_1}(\frac{1}{14})} & \text{I}_{P_3}(\frac{5}{17}) \\
& & \\
\text{I}_{P_2}(\frac{1}{10}) & \xrightarrow{\text{I}_{P_1}(\frac{1}{14})} & \text{I}_{P_3}(\frac{5}{17}) \\
& & \\
\text{I}_{P_2}(\frac{1}{9}) & \xrightarrow{\text{I}_{P_1}(\frac{1}{14})} & \text{I}_{P_3}(\frac{5}{17})
\end{array}
\]
6.5 Group of type $E_6$

**Proposition 6.6.1.** The matrix $B$ for $G = E_6$ is

$$B = \begin{pmatrix}
12 & 6 & 2 & 3 & 5 & 8 \\
8 & 11 & 5 & 2 & 5 & 8 \\
8 & 6 & 9 & 3 & 5 & 7 \\
6 & 6 & 5 & 7 & 5 & 6 \\
7 & 6 & 5 & 3 & 9 & 8 \\
8 & 6 & 5 & 3 & 2 & 12
\end{pmatrix}$$

**Proposition 6.6.2.** The special admissible data are as follows:

| $[s_0, t_0]$ | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ | $P_6$ |
|--------------|-------|-------|-------|-------|-------|-------|
| $P_1$        |       |       |       |       |       |       |
| $P_2$        | $[\frac{5}{22}, 0]$ |       |       |       |       |       |
| $P_3$        | $\frac{7}{14}, \frac{1}{7}$ | $\frac{5}{14}, \frac{1}{7}$ |       |       |       |       |
| $P_4$        | $\frac{5}{14}, \frac{1}{14}$ | $\frac{5}{14}, \frac{7}{22}$ | $\frac{3}{14}, \frac{1}{6}$ |       |       |       |
| $P_5$        | $\frac{5}{18}, \frac{1}{12}$ | $\frac{1}{6}, \frac{1}{22}$ |       | $\frac{3}{14}, \frac{1}{6}$ | $\frac{5}{14}, \frac{7}{12}$ |       |
| $P_6$        |       |       |       |       |       | $\frac{7}{18}, \frac{1}{3}$ |

**Proposition 6.6.3.** It holds that:

\[
A_{w_{P_1}} = \frac{R^5}{\zeta(2)\zeta(4)\zeta(5)\zeta(6)\zeta(8)} \quad w_{P_1} = w_6w_5w_3w_2w_4w_5w_6w_5w_4w_3w_2w_4w_3w_2w_4w_3w_2w_2
\]

\[
A_{w_{P_2}} = \frac{R^5}{\zeta(2)\zeta(3)\zeta(4)\zeta(5)\zeta(6)} \quad w_{P_2} = w_1w_3w_4w_5w_6w_1w_3w_4w_5w_6w_1w_3w_4w_1w_3w_1
\]

\[
A_{w_{P_3}} = \frac{R^5}{\zeta(2)^2\zeta(3)\zeta(4)\zeta(5)} \quad w_{P_3} = w_2w_4w_5w_6w_2w_4w_5w_2w_4w_2w_1
\]

\[
A_{w_{P_4}} = \frac{R^5}{\zeta(2)^3\zeta(3)^2} \quad w_{P_4} = w_5w_6w_3w_1w_3w_2w_1
\]

\[
A_{w_{P_5}} = \frac{R^5}{\zeta(2)^2\zeta(3)\zeta(4)\zeta(5)} \quad w_{P_5} = w_6w_3w_4w_1w_3w_2w_4w_1w_3w_2w_1
\]

\[
A_{w_{P_6}} = \frac{R^5}{\zeta(2)\zeta(4)\zeta(5)\zeta(6)\zeta(8)} \quad w_{P_6} = w_2w_4w_5w_3w_4w_1w_3w_2w_4w_5w_3w_4w_1w_3w_2w_4w_1w_3w_2w_1
\]
Theorem 6.7. Let $f^0 \in I_{P_i}(s)$ be the normalized spherical section then:

1. $\Lambda_{P_2}^0(f^0, \frac{5}{22}, g) = \frac{1}{11} \times \frac{R(3)}{\zeta(6)(8)} \times \Lambda_{P_1}^0(f^0, 0, g)$.
2. $\Lambda_{P_2}^0(f^0, \frac{5}{22}, g) = \frac{1}{11} \times \frac{R(3)}{\zeta(6)(8)} \times \Lambda_{P_0}^0(f^0, 0, g)$.
3. $\Lambda_{P_2}^0(f^0, \frac{7}{18}, g) = \frac{1}{3} \times \frac{R(2)(3)}{\zeta(6)(8)} \times \Lambda_{P_1}^0(f^0, \frac{1}{3}, g)$.
4. $\Lambda_{P_2}^0(f^0, \frac{7}{22}, g) = \frac{1}{81} \times \frac{R(5)(2)}{\zeta(6)(8)} \times \Lambda_{P_2}^0(f^0, \frac{1}{22}, g)$.
5. $\Lambda_{P_2}^0(f^0, \frac{7}{18}, g) = \frac{1}{3} \times \frac{R(5)(2)(3)}{\zeta(6)(8)} \times \Lambda_{P_1}^0(f^0, \frac{1}{12}, g)$.
6. $\Lambda_{P_3}^0(f^0, \frac{5}{14}, g) = \frac{12}{49} \times \frac{R(2)(5)(3)}{\zeta(4)(5)(6)(8)} \times \Lambda_{P_1}^0(f^0, \frac{1}{4}, g)$.
7. $\Lambda_{P_3}^0(f^0, \frac{3}{14}, g) = \frac{81}{343} \times \frac{R(2)(5)}{\zeta(4)(5)} \times \Lambda_{P_2}^0(f^0, \frac{1}{6}, g)$.
8. $\Lambda_{P_3}^0(f^0, \frac{3}{14}, g) = \frac{81}{343} \times \frac{R(2)(5)}{\zeta(4)(5)} \times \Lambda_{P_2}^0(f^0, \frac{1}{6}, g)$.
9. $\Lambda_{P_3}^0(f^0, \frac{7}{18}, g) = \frac{12}{49} \times \frac{R(2)(5)(3)}{\zeta(4)(5)(6)(8)} \times \Lambda_{P_1}^0(f^0, \frac{1}{4}, g)$.
10. $\Lambda_{P_3}^0(f^0, \frac{7}{18}, g) = \frac{12}{49} \times \frac{R(2)(5)(3)}{\zeta(4)(5)(6)(8)} \times \Lambda_{P_1}^0(f^0, \frac{1}{4}, g)$.
11. $\Lambda_{P_3}^0(f^0, \frac{7}{18}, g) = \frac{12}{49} \times \frac{R(2)(5)(3)}{\zeta(4)(5)(6)(8)} \times \Lambda_{P_1}^0(f^0, \frac{1}{4}, g)$.
12. $\Lambda_{P_3}^0(f^0, \frac{7}{18}, g) = \frac{12}{49} \times \frac{R(2)(5)(3)}{\zeta(4)(5)(6)(8)} \times \Lambda_{P_1}^0(f^0, \frac{1}{4}, g)$.
13. $\Lambda_{P_3}^0(f^0, \frac{7}{18}, g) = \frac{12}{49} \times \frac{R(2)(5)(3)}{\zeta(4)(5)(6)(8)} \times \Lambda_{P_1}^0(f^0, \frac{1}{4}, g)$.
However in that case we are able to find another positive admissible data which are not special and cannot be derived from the list above.

\[
(P_4, \frac{1}{7}, P_3, 0, w_3w_4w_1w_3w_2w_4w_5w_6)
\]
\[
(P_4, \frac{1}{7}, P_3, 0, w_3w_5w_4w_3w_2w_3w_1)
\]

The following Theorem together with Theorem (6.7) give the complete list of the identities corresponds to positive admissible data.

**Theorem 6.8.** Let \( f^0 \in I_{P_1}(s) \) be the normalized spherical section then:

1. \( \Lambda_{-1} P_1(f^0, \frac{1}{4}, g) = \frac{11}{12} \times \frac{\zeta(8)}{\zeta(2)} \times \Lambda_{-1} P_2(f^0, \frac{7}{22}, g) \).  
2. \( \Lambda_0 P_3(f^0, 0, g) = 1 \times \Lambda_{0} P_6(f^0, 0, g) \).  
3. \( \Lambda_{-1} P_1(f^0, \frac{1}{4}, g) = 1 \times \Lambda_{-1} P_6(f^0, \frac{1}{4}, g) \).  
4. \( \Lambda_{-3} P_4(f^0, \frac{3}{14}, g) = \frac{11}{343} \times \frac{R^2 \zeta(2)^2}{\zeta(4) \zeta(5) \zeta(6)} \times \Lambda_{-1} P_2(f^0, \frac{1}{22}, g) \).  
5. \( \Lambda_{-1} P_2(f^0, \frac{7}{22}, g) = \frac{12}{11} \times \frac{\zeta(2)}{\zeta(6)} \times \Lambda_{-1} P_6(f^0, \frac{1}{4}, g) \).  
6. \( \Lambda_{-3} P_4(f^0, \frac{1}{7}, g) = \frac{1}{11} \times \frac{R \zeta(2) \zeta(4) \zeta(5) \zeta(7)}{\zeta(4) \zeta(5) \zeta(7) \zeta(2)} \times \Lambda_{0} P_3(f^0, 0, g) \).  
7. \( \Lambda_0 P_3(f^0, 0, g) = 1 \times \Lambda_{0} P_6(f^0, 0, g) \).  
8. \( \Lambda_{-2} P_4(f^0, \frac{1}{7}, g) = 1 \times \Lambda_{-2} P_6(f^0, \frac{1}{7}, g) \).  
9. \( \Lambda_{-1} P_4(f^0, \frac{1}{7}, g) = \frac{1}{11} \times \frac{R \zeta(2) \zeta(4) \zeta(5) \zeta(7)}{\zeta(4) \zeta(5) \zeta(7) \zeta(2)} \times \Lambda_{0} P_6(f^0, 0, g) \).
| $P_i$ | $s$ | $P_j$ | $t$ | $w$ | $\frac{h_3(\chi_s)}{h_3(\chi_t)}$ | $d_{P_i}(\chi_s)$ | $d_{P_j}(\chi_t)$ | $d$ | $\epsilon_p$ | $\epsilon_q$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $P_1$ | $\frac{1}{2}$ | $P_2$ | $\frac{7}{22}$ | $w_3w_2w_1$ | $\frac{\zeta(3)}{\zeta(2)}$ | 1 | 1 | 0 | 12 | 11 |
| $P_1$ | 0 | $P_5$ | 0 | $w_5w_6w_4w_5w_3w_4w_1w_3$ | 1 | 0 | 0 | 0 | 1 | 1 |
| $P_1$ | $\frac{1}{2}$ | $P_6$ | $\frac{1}{4}$ | $w_6w_5w_3w_1$ | 1 | 1 | 1 | 0 | 12 | 12 |
| $P_4$ | $\frac{3}{14}$ | $P_2$ | $\frac{7}{22}$ | $w_2w_4w_5w_6w_3w_1$ | $\frac{\zeta(2)^2}{\zeta(1)^2\zeta(5)}$ | 3 | 1 | 2 | 686 | 22 |
| $P_2$ | $\frac{7}{22}$ | $P_6$ | $\frac{1}{4}$ | $w_6w_5w_2$ | $\frac{\zeta(2)}{\zeta(3)}$ | 1 | 1 | 0 | 11 | 12 |
| $P_4$ | $\frac{1}{7}$ | $P_3$ | 0 | $w_3w_4w_1w_5w_6w_3w_4w_5w_6$ | $\frac{\zeta(4)^2\zeta(2)}{\zeta(1)^2\zeta(3)^2\zeta(5)}$ | 1 | 0 | 1 | 14 | 1 |
| $P_3$ | 0 | $P_5$ | 0 | $w_4w_5w_6w_2w_4w_5w_3w_4w_1w_3w_2w_4$ | 1 | 0 | 0 | 0 | 1 | 1 |
| $P_3$ | $\frac{1}{6}$ | $P_3$ | $\frac{1}{6}$ | $w_5w_6w_1w_3$ | 1 | 2 | 2 | 0 | 162 | 162 |
| $P_4$ | $\frac{1}{7}$ | $P_5$ | 0 | $w_5w_6w_4w_5w_2w_4w_3w_1$ | $\frac{\zeta(4)^2\zeta(2)}{\zeta(1)^2\zeta(3)^2\zeta(5)}$ | 1 | 0 | 1 | 14 | 1 |
Therefore we get the follows:

\[
\begin{array}{c}
I_P(0) \\
I_P\left(\frac{5}{77}\right) \\
I_P\left(\frac{7}{18}\right) \\
I_P\left(\frac{10}{18}\right) \\
I_P\left(\frac{15}{18}\right) \\
I_P\left(\frac{5}{12}\right) \\
I_P\left(\frac{1}{22}\right) \\
I_P\left(\frac{5}{7}\right)
\end{array}
\]
6.6 Group of type $E_7$

Proposition 6.8.1. The matrix $B$ for $G = E_7$ is

$$B = \begin{pmatrix}
17 & 7 & 2 & 3 & 5 & 8 & 12 \\
10 & 14 & 6 & 2 & 5 & 8 & 11 \\
10 & 7 & 11 & 3 & 5 & 7 & 9 \\
7 & 7 & 6 & 8 & 5 & 6 & 7 \\
8 & 7 & 6 & 4 & 10 & 8 & 9 \\
9 & 7 & 6 & 4 & 3 & 13 & 12 \\
10 & 7 & 6 & 4 & 3 & 2 & 18
\end{pmatrix}$$

Proposition 6.8.2. The special admissible data are as follows:

| $[8_0, t_0]$ | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ | $P_6$ | $P_7$ |
|--------------|-------|-------|-------|-------|-------|-------|-------|
| $P_1$        | $\begin{pmatrix} 3 & 3 \\ 34 & 34 \end{pmatrix}$ | $\begin{pmatrix} 13 & 9 \\ 34 & 22 \end{pmatrix}$ | $\begin{pmatrix} 11 & 3 \\ 34 & 8 \end{pmatrix}$ | $\begin{pmatrix} 7 & 3 \\ 34 & 10 \end{pmatrix}$ | $\begin{pmatrix} 1 & 5 \\ 34 & 26 \end{pmatrix}$ | $\begin{pmatrix} 7 & 1 \\ 34 & 18 \end{pmatrix}$ |
| $P_2$        | $\begin{pmatrix} 3 & 3 \\ 34 & 34 \end{pmatrix}$ | $\begin{pmatrix} 9 & 13 \\ 22 & 22 \end{pmatrix}$ | $\begin{pmatrix} 3 & 1 \\ 22 & 14 \end{pmatrix}$ | $\begin{pmatrix} 5 & 1 \\ 22 & 14 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 \\ 22 & 14 \end{pmatrix}$ | $\begin{pmatrix} 3 & 1 \\ 22 & 14 \end{pmatrix}$ | $\begin{pmatrix} 7 & 1 \\ 34 & 18 \end{pmatrix}$ |
| $P_3$        | $\begin{pmatrix} 3 & 5 \\ 13 & 34 \end{pmatrix}$ | $\begin{pmatrix} 3 & 5 \\ 22 & 34 \end{pmatrix}$ | $\begin{pmatrix} 1 & 5 \\ 22 & 34 \end{pmatrix}$ | $\begin{pmatrix} 1 & 5 \\ 22 & 34 \end{pmatrix}$ | $\begin{pmatrix} 1 & 5 \\ 22 & 34 \end{pmatrix}$ | $\begin{pmatrix} 3 & 5 \\ 22 & 34 \end{pmatrix}$ | $\begin{pmatrix} 7 & 1 \\ 34 & 18 \end{pmatrix}$ |
| $P_4$        | $\begin{pmatrix} 3 & 5 \\ 11 & 34 \end{pmatrix}$ | $\begin{pmatrix} 3 & 5 \\ 11 & 22 \end{pmatrix}$ | $\begin{pmatrix} 1 & 5 \\ 11 & 22 \end{pmatrix}$ | $\begin{pmatrix} 1 & 5 \\ 11 & 22 \end{pmatrix}$ | $\begin{pmatrix} 1 & 5 \\ 11 & 22 \end{pmatrix}$ | $\begin{pmatrix} 3 & 5 \\ 11 & 22 \end{pmatrix}$ | $\begin{pmatrix} 7 & 1 \\ 11 & 26 \end{pmatrix}$ |
| $P_5$        | $\begin{pmatrix} 3 & 5 \\ 11 & 34 \end{pmatrix}$ | $\begin{pmatrix} 3 & 5 \\ 11 & 22 \end{pmatrix}$ | $\begin{pmatrix} 1 & 5 \\ 11 & 22 \end{pmatrix}$ | $\begin{pmatrix} 1 & 5 \\ 11 & 22 \end{pmatrix}$ | $\begin{pmatrix} 1 & 5 \\ 11 & 22 \end{pmatrix}$ | $\begin{pmatrix} 3 & 5 \\ 11 & 22 \end{pmatrix}$ | $\begin{pmatrix} 7 & 1 \\ 11 & 26 \end{pmatrix}$ |
| $P_6$        | $\begin{pmatrix} 1 & 5 \\ 34 & 26 \end{pmatrix}$ | $\begin{pmatrix} 1 & 5 \\ 34 & 26 \end{pmatrix}$ | $\begin{pmatrix} 1 & 5 \\ 34 & 26 \end{pmatrix}$ | $\begin{pmatrix} 1 & 5 \\ 34 & 26 \end{pmatrix}$ | $\begin{pmatrix} 1 & 5 \\ 34 & 26 \end{pmatrix}$ | $\begin{pmatrix} 1 & 5 \\ 34 & 26 \end{pmatrix}$ | $\begin{pmatrix} 11 & 7 \\ 26 & 18 \end{pmatrix}$ |
| $P_7$        | $\begin{pmatrix} 1 & 5 \\ 34 & 26 \end{pmatrix}$ | $\begin{pmatrix} 1 & 5 \\ 34 & 26 \end{pmatrix}$ | $\begin{pmatrix} 1 & 5 \\ 34 & 26 \end{pmatrix}$ | $\begin{pmatrix} 1 & 5 \\ 34 & 26 \end{pmatrix}$ | $\begin{pmatrix} 1 & 5 \\ 34 & 26 \end{pmatrix}$ | $\begin{pmatrix} 1 & 5 \\ 34 & 26 \end{pmatrix}$ | $\begin{pmatrix} 11 & 7 \\ 26 & 18 \end{pmatrix}$ |

Proposition 6.8.3. It holds that:

$$A_{w_{P_1}} = \frac{R^6}{\zeta(2)\zeta(4)\zeta(6)\zeta(8)\zeta(10)} \quad \text{with} \quad w_{P_1} = w_7w_6w_5w_4w_3w_2w_4w_5w_6w_7w_6w_5w_4w_3$$

$$A_{w_{P_2}} = \frac{R^6}{\zeta(2)\zeta(3)\zeta(4)\zeta(5)\zeta(6)\zeta(7)} \quad \text{with} \quad w_{P_2} = w_1w_3w_4w_5w_6w_7w_4w_3w_4w_5w_6w_1w_3w_4w_5w_1w_3w_4w_1$$

$$A_{w_{P_3}} = \frac{R^6}{\zeta(2)^2\zeta(3)\zeta(4)\zeta(5)\zeta(6)} \quad \text{with} \quad w_{P_3} = w_2w_4w_5w_6w_7w_2w_4w_5w_6w_2w_4w_5w_2w_4w_2w_1$$

$$A_{w_{P_4}} = \frac{R^6}{\zeta(2)^3\zeta(3)^2\zeta(4)} \quad \text{with} \quad w_{P_4} = w_5w_6w_7w_5w_6w_5w_1w_3w_2w_1$$

$$A_{w_{P_5}} = \frac{R^6}{\zeta(2)^2\zeta(3)^2\zeta(4)\zeta(5)} \quad \text{with} \quad w_{P_5} = w_6w_7w_5w_4w_1w_3w_4w_1w_3w_2w_1$$
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Chapter 6 Identities

$$A_{w_P} = \frac{R^6}{\zeta(2)\zeta(5)\zeta(6)\zeta(8)}$$

$$w_{P_6} = w_7w_2w_4w_5w_1w_3w_2w_4w_5w_3w_4w_1w_3w_2w_4w_1w_3$$

$$A_{w_{P_7}} = \frac{R^6}{\zeta(2)\zeta(5)\zeta(6)\zeta(8)\zeta(12)\zeta(9)}$$

$$w_{P_7} = w_1w_3w_4w_5w_6w_2w_4w_5w_3w_4w_1w_3w_4w_5w_6w_4w_5w_3$$

$$w_4w_1w_3w_2w_4w_5w_3w_4w_1w_3w_2w_4w_1w_3$$

| $P_i$ | $s$ | $P_j$ | $t$ | $w$ | $h_i(x_i)$ | $d_{P_i}(x_i) | d_{P_j}(x_t)$ | $d$ | $\epsilon_p$ | $\epsilon_q$ |
|-------|-----|-------|-----|-----|-------------|----------------|----------------|-----|-------------|-------------|
| $P_1$ | $\frac{7}{54}$ | $P_7$ | $\frac{1}{12}$ | $w_7w_6w_5w_4w_1w_3w_2w_4$ | $\frac{\zeta(5)\zeta(12)}{\zeta(8)}$ | 1 | 1 | 0 | 17 | 18 |
| $P_2$ | $\frac{3}{14}$ | $P_1$ | $\frac{3}{34}$ | $w_1w_3w_4w_5w_6w_7$ | $\frac{\zeta(2)\zeta(17)}{\zeta(16)}$ | 1 | 0 | 1 | 14 | 1 |
| $P_2$ | $\frac{1}{14}$ | $P_6$ | $\frac{1}{26}$ | $w_5w_6w_7w_3w_4w_5w_6w_2w_4w_5$ | $\frac{\zeta(2)\zeta(5)}{\zeta(7)}$ | 1 | 1 | 0 | 14 | 26 |
| $P_2$ | $\frac{5}{7}$ | $P_2$ | $\frac{5}{7}$ | $w_7w_6w_5w_4w_3w_1$ | $\frac{\zeta(2)\zeta(17)}{\zeta(16)}$ | 1 | 0 | 1 | 14 | 1 |
| $P_3$ | $\frac{9}{22}$ | $P_1$ | $\frac{13}{34}$ | $w_1$ | $\frac{\zeta(2)\zeta(6)}{\zeta(4)}$ | 1 | 1 | 0 | 11 | 1 |
| $P_3$ | $\frac{3}{22}$ | $P_2$ | $\frac{1}{14}$ | $w_2w_3w_5w_6w_7$ | $\frac{\zeta(2)\zeta(17)}{\zeta(16)}$ | 2 | 1 | 1 | 242 | 14 |
| $P_3$ | $\frac{3}{22}$ | $P_6$ | $\frac{1}{26}$ | $w_6w_7w_5w_6w_4w_5w_3w_2w_4$ | $\frac{\zeta(2)\zeta(17)}{\zeta(16)}$ | 2 | 1 | 1 | 242 | 26 |
| $P_3$ | $\frac{7}{22}$ | $P_7$ | $\frac{1}{6}$ | $w_7w_6w_5w_4w_2$ | $\frac{\zeta(2)\zeta(17)}{\zeta(16)}$ | 1 | 0 | 1 | 11 | 1 |
| $P_4$ | $\frac{3}{8}$ | $P_1$ | $\frac{11}{34}$ | $w_1w_3$ | $\frac{\zeta(2)\zeta(3)}{\zeta(4)}$ | 2 | 1 | 1 | 64 | 17 |
| $P_4$ | $\frac{3}{8}$ | $P_2$ | $\frac{5}{14}$ | $w_2$ | $\frac{\zeta(2)\zeta(3)}{\zeta(4)}$ | 2 | 1 | 1 | 64 | 14 |
| $P_4$ | $\frac{1}{4}$ | $P_3$ | $\frac{3}{22}$ | $w_3w_1$ | $\frac{\zeta(2)\zeta(3)}{\zeta(4)}$ | 3 | 2 | 1 | 1024 | 242 |
| $P_4$ | $\frac{1}{4}$ | $P_5$ | $\frac{1}{10}$ | $w_5w_5w_7$ | $\frac{\zeta(2)\zeta(17)}{\zeta(16)}$ | 4 | 3 | 1 | 49152 | 6000 |
| $P_4$ | $\frac{1}{4}$ | $P_6$ | $\frac{5}{26}$ | $w_6w_7w_5w_6$ | $\frac{\zeta(2)\zeta(17)}{\zeta(16)}$ | 3 | 2 | 1 | 1024 | 338 |
| $P_4$ | $\frac{3}{8}$ | $P_7$ | $\frac{5}{18}$ | $w_7w_5w_5$ | $\frac{\zeta(2)\zeta(17)}{\zeta(16)}$ | 2 | 1 | 1 | 64 | 18 |
| $P_5$ | $\frac{3}{10}$ | $P_1$ | $\frac{7}{34}$ | $w_1w_3w_4w_2$ | $\frac{\zeta(2)\zeta(17)}{\zeta(16)}$ | 2 | 1 | 1 | 100 | 17 |
| $P_5$ | $\frac{1}{4}$ | $P_2$ | $\frac{1}{7}$ | $w_2w_4w_5w_3w_1$ | $\frac{\zeta(2)\zeta(17)}{\zeta(16)}$ | 2 | 1 | 1 | 200 | 28 |
| $P_5$ | $\frac{1}{10}$ | $P_3$ | $\frac{1}{22}$ | $w_3w_4w_1w_3w_2w_4$ | $\frac{\zeta(2)\zeta(17)}{\zeta(16)}$ | 3 | 2 | 1 | 6000 | 726 |
| $P_5$ | $\frac{3}{10}$ | $P_4$ | $\frac{7}{26}$ | $w_6w_7$ | $\frac{\zeta(2)\zeta(3)}{\zeta(4)}$ | 2 | 1 | 1 | 100 | 13 |
| $P_5$ | $\frac{3}{7}$ | $P_7$ | $\frac{1}{7}$ | $w_7w_6$ | $\frac{\zeta(2)\zeta(3)}{\zeta(4)}$ | 1 | 0 | 1 | 10 | 1 |
| $P_6$ | $\frac{5}{22}$ | $P_1$ | $\frac{1}{17}$ | $w_1w_3w_4w_5w_2w_4w_3w_1$ | $\frac{\zeta(2)\zeta(17)}{\zeta(16)}$ | 2 | 1 | 1 | 338 | 34 |
| $P_6$ | $\frac{11}{26}$ | $P_7$ | $\frac{7}{18}$ | $w_7$ | $\frac{\zeta(2)\zeta(3)}{\zeta(4)}$ | 1 | 1 | 0 | 13 | 1 |

Theorem 6.9. Let $f^0 \in I_{P_i}(s)$ be the normalized spherical section then:

1. $\Lambda_{P_1}(f^0, \frac{7}{54}, g) = \frac{18}{17} \times \frac{\zeta(2)\zeta(6)\zeta(10)}{\zeta(8)\zeta(12)\zeta(9)} \times \Lambda_{P_1}(f^0, \frac{1}{18}, g)$.
2. $\Lambda_{P_2}(f^0, \frac{3}{14}, g) = \frac{14}{17} \times \frac{R\zeta(3)\zeta(5)}{\zeta(6)\zeta(8)\zeta(10)} \times \Lambda_{P_1}(f^0, \frac{3}{34}, g)$.
3. $\Lambda_{P_2}(f^0, \frac{5}{22}, g) = \frac{13}{17} \times \frac{\zeta(3)\zeta(7)}{\zeta(5)\zeta(8)} \times \Lambda_{P_1}(f^0, \frac{1}{21}, g)$.
4. $\Lambda_{-1}(f^0, \frac{2}{7}, g) = \frac{1}{11} \times \frac{RC(3, \zeta(4))}{\zeta(8) \zeta(12) \zeta(9)} \times \Lambda_{0}^{P_{2}}(f^0, \frac{1}{9}, g)$.

5. $\Lambda_{-1}(f^0, \frac{9}{22}, g) = \frac{1}{11} \times \frac{RC(3, \zeta(5))}{\zeta(6) \zeta(8) \zeta(10)} \times \Lambda_{0}^{P_{1}}(f^0, \frac{13}{34}, g)$.

6. $\Lambda_{-2}(f^0, \frac{3}{22}, g) = \frac{7}{121} \times \frac{RC(2)}{\zeta(6) \zeta(7)} \times \Lambda_{-1}(f^0, \frac{1}{11}, g)$.

7. $\Lambda_{-2}(f^0, \frac{3}{22}, g) = \frac{13}{121} \times \frac{RC(2, \zeta(3))}{\zeta(5) \zeta(6) \zeta(8) \zeta(9)} \times \Lambda_{-1}^{P_{1}}(f^0, \frac{5}{22}, g)$.

8. $\Lambda_{-2}(f^0, \frac{7}{22}, g) = \frac{1}{11} \times \frac{RC(2, \zeta(3), \zeta(4))}{\zeta(6) \zeta(8) \zeta(12) \zeta(9)} \times \Lambda_{0}^{P_{2}}(f^0, \frac{1}{6}, g)$.

9. $\Lambda_{-2}(f^0, \frac{3}{8}, g) = \frac{17}{64} \times \frac{RC(2)^{2}}{\zeta(6) \zeta(8) \zeta(10)} \times \Lambda_{-1}^{P_{1}}(f^0, \frac{11}{34}, g)$.

10. $\Lambda_{-2}(f^0, \frac{3}{8}, g) = \frac{7}{32} \times \frac{RC(2)}{\zeta(5) \zeta(6) \zeta(7)} \times \Lambda_{-1}^{P_{2}}(f^0, \frac{5}{14}, g)$.

11. $\Lambda_{-3}(f^0, \frac{1}{4}, g) = \frac{121}{512} \times \frac{RC(2)}{\zeta(5) \zeta(6)} \times \Lambda_{-2}^{P_{3}}(f^0, \frac{5}{22}, g)$.

12. $\Lambda_{-4}(f^0, \frac{1}{8}, g) = \frac{125}{1024} \times \frac{RC(2)}{\zeta(4) \zeta(5)} \times \Lambda_{-3}(f^0, \frac{1}{10}, g)$.

13. $\Lambda_{-3}(f^0, \frac{1}{4}, g) = \frac{169}{512} \times \frac{RC(2)^{2}}{\zeta(5) \zeta(6) \zeta(8) \zeta(9)} \times \Lambda_{-2}^{P_{2}}(f^0, \frac{5}{26}, g)$.

14. $\Lambda_{-2}(f^0, \frac{3}{8}, g) = \frac{9}{32} \times \frac{RC(2)^{2}}{\zeta(5) \zeta(6) \zeta(8) \zeta(9)} \times \Lambda_{-1}^{P_{2}}(f^0, \frac{5}{18}, g)$.

15. $\Lambda_{-2}(f^0, \frac{3}{10}, g) = \frac{17}{100} \times \frac{RC(2)^{2}}{\zeta(5) \zeta(6) \zeta(8) \zeta(9)} \times \Lambda_{-1}(f^0, \frac{7}{34}, g)$.

16. $\Lambda_{-2}(f^0, \frac{1}{5}, g) = \frac{7}{56} \times \frac{RC(2)}{\zeta(5) \zeta(6) \zeta(7)} \times \Lambda_{-1}^{P_{1}}(f^0, \frac{1}{7}, g)$.

17. $\Lambda_{-3}(f^0, \frac{1}{10}, g) = \frac{121}{1000} \times \frac{RC(2)}{\zeta(4) \zeta(5) \zeta(6)} \times \Lambda_{-2}^{P_{3}}(f^0, \frac{1}{22}, g)$.

18. $\Lambda_{-2}(f^0, \frac{3}{10}, g) = \frac{13}{1000} \times \frac{RC(2)}{\zeta(6) \zeta(8) \zeta(10)} \times \Lambda_{-1}^{P_{2}}(f^0, \frac{7}{26}, g)$.

19. $\Lambda_{-2}(f^0, \frac{2}{3}, g) = \frac{1}{10} \times \frac{RC(2)}{\zeta(6) \zeta(8) \zeta(10) \zeta(9)} \times \Lambda_{-1}^{P_{3}}(f^0, \frac{1}{3}, g)$.

20. $\Lambda_{-2}(f^0, \frac{5}{26}, g) = \frac{17}{169} \times \frac{RC(2)}{\zeta(6) \zeta(8) \zeta(9)} \times \Lambda_{-1}^{P_{1}}(f^0, \frac{1}{34}, g)$.

21. $\Lambda_{-2}(f^0, \frac{11}{26}, g) = \frac{1}{13} \times \frac{RC(4)}{\zeta(12) \zeta(9)} \times \Lambda_{0}^{P_{2}}(f^0, \frac{7}{18}, g)$.
| $P_i$ | $s$ | $P_j$ | $t$ | $w$ | $\frac{h_s(x_t)}{h_s(x_3)}$ | $d_{P_i}(x_s)$ | $d_{P_i}(x_t)$ | $d$ | $\epsilon_p$ | $\epsilon_q$ |
|------|----|------|----|-----|-----------------|----------------|----------------|---|-------------|-------------|
| $P_3$ | $\frac{5}{22}$ | $P_1$ | $\frac{1}{34}$ | $w_1w_3w_4w_5w_6w_7w_2w_4w_5w_6$ | $\frac{\mathfrak{C}(2)^2}{\mathfrak{C}(5)^3(8)}$ | 2 | 1 | 1 | 242 | 34 |
| $P_1$ | $\frac{11}{34}$ | $P_2$ | $\frac{5}{14}$ | $w_3w_2w_1$ | $\frac{\mathfrak{C}(3)}{\mathfrak{C}(2)}$ | 1 | 1 | 0 | 17 | 14 |
| $P_4$ | $\frac{1}{4}$ | $P_1$ | $\frac{1}{34}$ | $w_1w_3w_4w_5w_6w_7w_2w_4w_5w_6w_3w_1$ | $\frac{\mathfrak{C}(2)^2}{\mathfrak{C}(3)^3(5)(8)}$ | 3 | 1 | 2 | 1024 | 34 |
| $P_4$ | $\frac{1}{34}$ | $P_5$ | $\frac{1}{25}$ | $w_6w_7w_2w_4w_3w_1$ | $\frac{\mathfrak{C}(5)}{\mathfrak{C}(3)}$ | 1 | 1 | 0 | 17 | 13 |
| $P_1$ | $\frac{11}{34}$ | $P_2$ | $\frac{5}{13}$ | $w_7w_6w_5w_3w_1$ | $\frac{\mathfrak{C}(3)}{\mathfrak{C}(4)}$ | 1 | 1 | 0 | 17 | 18 |
| $P_5$ | $\frac{3}{20}$ | $P_2$ | $\frac{1}{38}$ | $w_2w_4w_5w_3w_4w_1w_3w_2w_4w_5w_6w_7$ | $\frac{\mathfrak{C}(2)^2}{\mathfrak{C}(5)^3(8)}$ | 1 | 0 | 1 | 20 | 1 |
| $P_2$ | $\frac{5}{14}$ | $P_7$ | $\frac{5}{18}$ | $w_7w_6w_5w_2$ | $\frac{\mathfrak{C}(2)}{\mathfrak{C}(4)}$ | 1 | 1 | 0 | 14 | 18 |
| $P_4$ | $\frac{1}{8}$ | $P_3$ | $\frac{1}{27}$ | $w_3w_4w_1w_3w_2w_4w_5w_6w_7$ | $\frac{\mathfrak{C}(2)^2}{\mathfrak{C}(4)^3(8)}$ | 4 | 2 | 2 | 49152 | 726 |
| $P_3$ | $\frac{5}{32}$ | $P_6$ | $\frac{5}{16}$ | $w_6w_7w_5w_6w_1w_3$ | $\frac{\mathfrak{C}(2)}{\mathfrak{C}(4)}$ | 2 | 2 | 0 | 242 | 338 |
| $P_4$ | $\frac{1}{12}$ | $P_5$ | $\frac{1}{39}$ | $w_5w_6w_7w_4w_5w_6w_3w_4w_5w_6w_3w_4w_2$ | $\frac{\mathfrak{C}(5)^3(4)^3}{\mathfrak{C}(5)^3(8)^3}$ | 1 | 0 | 1 | 24 | 1 |
| $P_4$ | $\frac{3}{16}$ | $P_6$ | $\frac{1}{13}$ | $w_6w_7w_5w_6w_4w_5w_3w_2w_4w_3w_1$ | $\frac{\mathfrak{C}(2)^2}{\mathfrak{C}(1)^4(8)}$ | 1 | 0 | 1 | 16 | 1 |
| $P_5$ | $\frac{4}{10}$ | $P_7$ | $\frac{1}{15}$ | $w_7w_6w_5w_4w_3w_2w_4w_1w_3$ | $\frac{\mathfrak{C}(2)^2}{\mathfrak{C}(4)^3(8)}$ | 2 | 1 | 1 | 100 | 18 |
| $P_6$ | $\frac{11}{26}$ | $P_7$ | $\frac{1}{18}$ | $w_7$ | $\frac{\mathfrak{C}(2)}{\mathfrak{C}(4)^3(8)}$ | 1 | 1 | 0 | 13 | 1 |
The following Theorem together with Theorem (6.9) give the complete list of the identities corresponds to positive admissible data.

**Theorem 6.10.** Let \( f^0 \in I_{P_r}(s) \) be the normalized spherical section then:

1. \( \Lambda_{-2}^{P_2}(f^0, \frac{5}{22}, g) = 17 \frac{121}{17} \times \frac{R(2\zeta(3))}{\zeta(5\zeta(7))} \times \Lambda_{-1}^{P_1}(f^0, \frac{1}{34}, g). \)

2. \( \Lambda_{-1}^{P_1}(f^0, \frac{11}{34}, g) = 14 \frac{17}{17} \times \frac{\zeta(10\zeta(10))}{\zeta(2\zeta(5\zeta(7))} \times \Lambda_{-1}^{P_1}(f^0, \frac{5}{14}, g). \)

3. \( \Lambda_{-3}^{P_3}(f^0, \frac{1}{4}, g) = 17 \frac{172}{342} \times \frac{R(2\zeta(3))^{3\zeta(3)}}{\zeta(2\zeta(5\zeta(7))} \times \Lambda_{-1}^{P_1}(f^0, \frac{1}{34}, g). \)

4. \( \Lambda_{-1}^{P_1}(f^0, \frac{7}{34}, g) = 13 \frac{17}{17} \times \frac{\zeta(10\zeta(10))}{\zeta(2\zeta(5\zeta(7))} \times \Lambda_{-1}^{P_1}(f^0, \frac{7}{26}, g). \)

5. \( \Lambda_{-1}^{P_1}(f^0, \frac{11}{34}, g) = 18 \frac{17}{17} \times \frac{\zeta(10\zeta(10))}{\zeta(2\zeta(5\zeta(7))} \times \Lambda_{-1}^{P_1}(f^0, \frac{5}{12}, g). \)

6. \( \Lambda_{-1}^{P_1}(f^0, \frac{3}{20}, g) = 1 \frac{5}{20} \times \frac{R(2\zeta(4))^{3\zeta(3)}}{\zeta(2\zeta(5\zeta(7))} \times \Lambda_{-1}^{P_1}(f^0, \frac{1}{28}, g). \)

7. \( \Lambda_{-1}^{P_1}(f^0, \frac{5}{14}, g) = 9 \frac{7}{7} \times \frac{\zeta(2\zeta(3)(7))}{\zeta(8\zeta(12\zeta(9))} \times \Lambda_{-1}^{P_1}(f^0, \frac{5}{18}, g). \)

8. \( \Lambda_{-1}^{P_1}(f^0, \frac{1}{3}, g) = 121 \frac{121}{121} \times \frac{R(2\zeta(3))^{3\zeta(3)}}{\zeta(2\zeta(5\zeta(7))} \times \Lambda_{-1}^{P_1}(f^0, \frac{1}{22}, g). \)

9. \( \Lambda_{-1}^{P_1}(f^0, \frac{5}{22}, g) = 169 \frac{169}{121} \times \frac{\zeta(3\zeta(3))}{\zeta(4\zeta(8))} \times \Lambda_{-1}^{P_1}(f^0, \frac{5}{26}, g). \)

10. \( \Lambda_{-1}^{P_1}(f^0, \frac{1}{12}, g) = 1 \frac{24}{24} \times \frac{R(2\zeta(3))^{3\zeta(3)}}{\zeta(2\zeta(5\zeta(7))} \times \Lambda_{-1}^{P_1}(f^0, \frac{1}{30}, g). \)

11. \( \Lambda_{-1}^{P_1}(f^0, \frac{3}{16}, g) = 121 \frac{16}{16} \times \frac{R(2\zeta(3))^{3\zeta(3)}}{\zeta(2\zeta(5\zeta(7))} \times \Lambda_{-1}^{P_1}(f^0, \frac{1}{13}, g). \)

12. \( \Lambda_{-1}^{P_1}(f^0, \frac{1}{8}, g) = 1 \frac{9}{9} \times \frac{R(2\zeta(3))^{3\zeta(3)}}{\zeta(2\zeta(5\zeta(7))} \times \Lambda_{-1}^{P_1}(f^0, \frac{1}{18}, g). \)

13. \( \Lambda_{-1}^{P_1}(f^0, \frac{11}{26}, g) = 1 \frac{13}{13} \times \frac{R(2\zeta(3))^{3\zeta(3)}}{\zeta(2\zeta(5\zeta(7))} \times \Lambda_{-1}^{P_1}(f^0, \frac{7}{18}, g). \)

**Remark 6.10.1.** Observe that the following positive admissible data that are not special.

\[
(P_5, \frac{3}{10}, P_7, \frac{1}{18}, w_7w_ww_5w_4w_3w_2w_4w_1w_3)
\]

\[
(P_5, \frac{3}{20}, P_2, \frac{1}{28}, w_2w_4w_5w_3w_4w_1w_3w_2w_4w_5w_6w_7)
\]

\[
(P_4, \frac{3}{16}, P_6, \frac{1}{13}, w_6w_7w_5w_6w_4w_5w_2w_4w_3w_1)
\]

\[
(P_4, \frac{1}{12}, P_5, \frac{1}{30}, w_5w_6w_7w_4w_5w_6w_4w_5w_1w_3w_4w_2)
\]

Therefore we get the following list:
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\[
\begin{align*}
I_{P_5}(\frac{4}{17}) & \quad I_{P_1}(\frac{3}{17}) \\
I_{P_1}(\frac{1}{12}) & \quad I_{P_2}(\frac{1}{17}) \\
I_{P_3}(\frac{3}{27}) & \quad I_{P_1}(\frac{1}{27}) \\
I_{P_3}(\frac{7}{27}) & \quad I_{P_2}(\frac{1}{8}) \\
I_{P_5}(\frac{1}{28}) & \quad I_{P_1}(\frac{3}{26}) \\
I_{P_1}(\frac{1}{12}) & \quad I_{P_2}(\frac{1}{36}) \\
I_{P_8}(\frac{4}{110}) & \quad I_{P_1}(\frac{7}{17}) \\
I_{P_1}(\frac{1}{12}) & \quad I_{P_2}(\frac{1}{17}) \\
I_{P_8}(\frac{3}{10}) & \quad I_{P_1}(\frac{7}{18}) \\
I_{P_4}(\frac{1}{13}) & \quad I_{P_3}(\frac{1}{26})
\end{align*}
\]
Appendix A

Appendix 1

**Definition A.1.** For every $a \in \mathbb{C} \setminus \{0,1\}$ we write the Taylor series of $\zeta(s)$ around $s = a$ as follows

$$
\zeta(s) = \sum_{j=0}^{\infty} \zeta(a)_j (s-a)^j.
$$

For $a = 1$ we write

$$
\zeta(s) = \sum_{j=-1}^{\infty} c_j (s-1)^j.
$$

**A.1 Standard output**

**Theorem A.2.** Let $G = F_4$, and let $P = P_1$. Then $E_P(f^0, g, s)$ admits a pole of order 1 at $s = \frac{1}{4}$. Moreover, the leading term $\Lambda_{-1}(f^0, g, \frac{1}{4})$ is in $L^2$.

**Proof.** As we can see from Table the exponent $[-5, -9, -12, -5]$ contributes a pole of order 1 and it cannot be canceled. Hence it remains to show that the exponent $[-5, -9, -13, -7]$ does not contribute a pole of order 2.

For the exp. $[-5, -9, -13, -7]$ we do the follows:

First of all, we take only the operators that contribute an order great or equal to 1. In that case, all the elements in the exponent contribute a pole of order 2. Let:

$$
y_1 = \zeta(16s - 3) \quad y_2 = \zeta(8s - 2) \quad y_3 = \zeta(8s) \quad y_4 = \zeta(16s + 4)
$$

$$
y_5 = \zeta(8s + 1) \quad y_6 = \zeta(8s + 4) \quad y_7 = \zeta(8s - 1)
$$
Therefore after doing a common denominator we get the following expression

\[
\frac{y_1 y_2 y_3 + y_1 y_5 y_7}{y_4 y_5 y_6}.
\]

Since the denominator is holomorphic and non zero we may ignore it. For every \( y_i \) we write its Laurent expansion around \( \frac{1}{4} \):

\[
y_1 = \frac{\frac{1}{16} c_{-1}}{(s - \frac{1}{4})} + c_0 + 16c_1(s - \frac{1}{4}) + 256c_2(s - \frac{1}{4})^2 + 4096c_3(s - \frac{1}{4})^3 + \ldots
\]

\[
y_2 = \frac{-\frac{1}{8} c_{-1}}{(s - \frac{1}{4})} + c_0 - 8c_1(s - \frac{1}{4}) + 64c_2(s - \frac{1}{4})^2 - 512c_3(s - \frac{1}{4})^3 + \ldots
\]

\[
y_3 = \zeta(2)_0 + 8\zeta(2)_1(s - \frac{1}{4}) + 64\zeta(2)_2(s - \frac{1}{4})^2 + 512\zeta(2)_3(s - \frac{1}{4})^3 + \ldots
\]

\[
y_4 = \zeta(8)_0 + 16\zeta(8)_1(s - \frac{1}{4}) + 256\zeta(8)_2(s - \frac{1}{4})^2 + 4096\zeta(8)_3(s - \frac{1}{4})^3 + \ldots
\]

\[
y_5 = \zeta(3)_0 + 8\zeta(3)_1(s - \frac{1}{4}) + 64\zeta(3)_2(s - \frac{1}{4})^2 + 512\zeta(3)_3(s - \frac{1}{4})^3 + \ldots
\]

\[
y_6 = \zeta(6)_0 + 8\zeta(6)_1(s - \frac{1}{4}) + 64\zeta(6)_2(s - \frac{1}{4})^2 + 512\zeta(6)_3(s - \frac{1}{4})^3 + \ldots
\]

\[
y_7 = \frac{\frac{1}{8} c_{-1}}{(s - \frac{1}{4})} + c_0 + 8c_1(s - \frac{1}{4}) + 64c_2(s - \frac{1}{4})^2 + 512c_3(s - \frac{1}{4})^3 + \ldots
\]

For the summand: \( y_1 y_2 y_3 \) we get:

\[
\frac{-\frac{1}{128} c^2_{-1} \zeta(2)_0}{(s - \frac{1}{4})^2} + \frac{-\frac{1}{16} c_{-1} c_0 \zeta(2)_0 - \frac{1}{16} c^2_{-1} \zeta(2)_1}{(s - \frac{1}{4})} + O(1)
\]

For the summand: \( y_1 y_5 y_7 \) we get:

\[
\frac{\frac{1}{128} c^2_{-1} \zeta(2)_0}{(s - \frac{1}{4})^2} + \frac{\frac{3}{16} c_{-1} c_0 \zeta(2)_0 + \frac{1}{16} c^2_{-1} \zeta(2)_1}{(s - \frac{1}{4})} + O(1)
\]

In conclusion the final sum is:

\[
\frac{\frac{1}{8} c_{-1} c_0 \zeta(2)_0}{(s - \frac{1}{4})} + O(1).
\]

Hence, this exponent contributes a pole of at most order 1. The leading term is in \( L^2 \) by Langlands’ criterion.
| pole | order | operator                                      | factor                                      | exp       |
|------|-------|----------------------------------------------|---------------------------------------------|-----------|
| 1/4  | 2     | $w_2 w_3 w_4 w_2 w_3 w_1 w_2 w_3 w_4 w_1 w_2 w_3 w_2 w_1$ | $\zeta(16s-1)\zeta(8s-2)\zeta(8s)$ | $[-5, -9, -13, -7]$ |
| 1/4  | 2     | $w_3 w_4 w_2 w_3 w_1 w_2 w_3 w_4 w_1 w_2 w_3 w_2 w_1$ | $\zeta(16s+4)\zeta(8s+1)\zeta(8s+4)$ | $[-5, -9, -13, -7]$ |
| 1/4  | 1     | $w_2 w_3 w_1 w_2 w_3 w_4 w_1 w_2 w_3 w_2 w_1$ | $\zeta(8s)(8s+1)\zeta(8s+4)$ | $[-5, -9, -12, -5]$ |
| 1/4  | 1     | $w_1 w_2 w_3 w_1 w_2 w_3 w_4 w_1 w_2 w_3 w_2 w_1$ | $\zeta(16s+4)\zeta(8s)(8s)$ | $[-4, -9, -13, -7]$ |
| 1/4  | 1     | $w_4 w_2 w_3 w_1 w_2 w_3 w_4 w_1 w_2 w_3 w_2 w_1$ | $\zeta(8s)(8s+1)(8s-2)$ | $[-5, -9, -12, -7]$ |
| 1/4  | 0     | 1                                            | 1                                           | $[4, 3, 3, 1]$ |
| 1/4  | 0     | $w_1$                                        | $\frac{\zeta(8s+3)}{\zeta(8s+4)}$       | $[-1, 3, 3, 1]$ |
| 1/4  | 0     | $w_2 w_1$                                    | $\zeta(8s+2)\zeta(8s+4)$                 | $[-1, -1, 3, 1]$ |
| 1/4  | 0     | $w_3 w_2 w_1$                                | $\frac{\zeta(16s+3)\zeta(8s+2)}{\zeta(16s+4)\zeta(8s+4)}$ | $[-1, -1, -4, 1]$ |
| 1/4  | 0     | $w_2 w_3 w_2 w_1$                            | $\frac{\zeta(16s+3)\zeta(8s+1)}{\zeta(16s+4)\zeta(8s+4)}$ | $[-1, -4, -4, 1]$ |
| 1/4  | 0     | $w_1 w_2 w_3 w_2 w_1$                        | $\frac{\zeta(16s+3)\zeta(8s)}{\zeta(16s+4)\zeta(8s+4)}$ | $[-3, -4, -4, 1]$ |
| 1/4  | 0     | $w_4 w_3 w_2 w_1$                            | $\frac{\zeta(8s+4)\zeta(16s+2)}{\zeta(16s+4)\zeta(8s+4)}$ | $[-1, -1, -4, -5]$ |
| 1/4  | 0     | $w_1 w_2 w_3 w_2 w_1$                        | $\frac{\zeta(8s+1)\zeta(16s+2)}{\zeta(16s+4)\zeta(8s+4)}$ | $[-1, -4, -4, -5]$ |
| 1/4  | 0     | $w_3 w_4 w_2 w_3 w_2 w_1$                    | $\zeta(8s)(8s+1)\zeta(16s+2)$ | $[-1, -4, -9, -5]$ |
| 1/4  | 0     | $w_4 w_3 w_4 w_2 w_3 w_2 w_1$                | $\zeta(8s)(8s+1)\zeta(16s+4)$ | $[-1, -4, -9, -5]$ |
| 1/4  | 0     | $w_2 w_3 w_3 w_2 w_1$                        | $\zeta(8s)(8s)(8s)$ | $[-1, -6, -9, -5]$ |
| 1/4  | 0     | $w_4 w_1 w_2 w_3 w_2 w_1$                    | $\zeta(8s)(8s+1)\zeta(16s+2)$ | $[-3, -4, -4, -5]$ |
| 1/4  | 0     | $w_3 w_4 w_1 w_2 w_3 w_2 w_1$                | $\zeta(8s)(8s+1)\zeta(16s+4)$ | $[-3, -4, -9, -5]$ |
| 1/4  | 0     | $w_2 w_3 w_4 w_2 w_3 w_2 w_1$                | $\zeta(16s)(8s)$ | $[-3, -8, -9, -5]$ |
| 1/4  | 0     | $w_3 w_2 w_3 w_4 w_1 w_2 w_3 w_2 w_1$        | $\zeta(16s)(8s+1)$ | $[-3, -8, -12, -5]$ |
| 1/4  | 0     | $w_1 w_2 w_3 w_4 w_2 w_3 w_2 w_1$            | $\zeta(16s)(8s)$ | $[-5, -6, -9, -5]$ |
| $\frac{1}{4}$ | 0 | $w_1 w_2 w_3 w_4 w_1 w_2 w_3 w_2 w_1$ | $\zeta(16s)[\zeta(8s)]^2$ | $\zeta(16s+4)[\zeta(8s+1)]\zeta(8s+4)$ | $[-5, -8, -9, -5]$ |
| $\frac{1}{4}$ | 0 | $w_3 w_1 w_2 w_3 w_4 w_1 w_2 w_3 w_2 w_1$ | $\zeta(8s)[\zeta(16s-1)]$ | $\zeta(16s+4)[\zeta(8s+1)]\zeta(8s+4)$ | $[-5, -8, -12, -5]$ |
| $\frac{1}{4}$ | 0 | $w_4 w_3 w_2 w_3 w_4 w_1 w_2 w_3 w_2 w_1$ | $\zeta(8s)[\zeta(16s-2)]$ | $\zeta(16s+4)[\zeta(8s+4)]$ | $[-3, -8, -12, -7]$ |
| $\frac{1}{4}$ | 0 | $w_4 w_3 w_2 w_3 w_4 w_1 w_2 w_3 w_2 w_1$ | $\zeta(8s)[\zeta(16s-2)]$ | $\zeta(16s+4)[\zeta(8s+1)]\zeta(8s+4)$ | $[-5, -8, -12, -7]$ |
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