A TWO-PIECE PROPERTY FOR FREE BOUNDARY MINIMAL
HYPERSURFACES IN THE \((n + 1)\)-DIMENSIONAL BALL

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ABSTRACT. We prove that every hyperplane passing through the origin in \(\mathbb{R}^{n+1}\) divides an embedded compact free boundary minimal hypersurface of the euclidean \((n + 1)\)-ball in exactly two connected hypersurfaces. We also show that if a region in the \((n + 1)\)-ball has mean convex boundary and contains a nullhomologous \((n - 1)\)-dimensional equatorial disk, then this region is a closed halfball. Our first result gives evidence to a conjecture by Fraser and Li in any dimension.

1. INTRODUCTION

Inspired by the work of Ros [26] for closed minimal surfaces in \(S^3\), the authors proved in [25] the two-piece property for free boundary minimal surfaces in the unit ball of \(\mathbb{R}^3\). This result gives evidence to a conjecture by Fraser and Li [10] concerning the first Steklov eigenvalue of free boundary minimal surfaces in \(B^3\). Also, Kusner and McGrath [21] used our result of two-piece property in the free boundary context to prove the uniqueness of the critical catenoid among embedded minimal annuli invariant under the antipodal map. This settles a case of another well-known conjecture of [10] on the uniqueness of the critical catenoid.

In the present paper we prove that the two-piece property holds in any dimension. More precisely, we prove the following.

**Theorem A** (The two-piece property). Every hyperplane in \(\mathbb{R}^{n+1}\) passing through the origin divides an embedded compact free boundary minimal hypersurface of the unit \((n + 1)\)-ball \(B^{n+1}\) in exactly two connected components.

We also prove the following result which can be seen as a strong version of the analog of the result by Solomon [29] in the free boundary context.

**Theorem B.** Let \(W \subset B^{n+1}\) be a connected closed region with mean convex boundary such that \(\partial W\) meets \(S^n\) orthogonally along its boundary and \(\partial W\) is smooth. Suppose \(W\) contains a set of the form \(P \cap B^{n+1}\) which is nullhomologous in \(W\) (see Definition 3), where \(P\) is a \((n - 1)\)-dimensional plane in \(\mathbb{R}^{n+1}\) passing through the origin. Then \(W\) is a closed halfball.

Let us remark that exactly as in the case \(n = 2\), Theorem A can be proved by assuming the conjecture by Fraser and Li [10] on the the first Steklov eigenvalue of free boundary minimal hypersurfaces in \(B^{n+1}\); hence, Theorem A gives evidence to this conjecture (see [25, Remark 2]).

The strategy to prove Theorem A and Theorem B is similar to the case \(n = 2\) and uses Geometric Measure Theory to analyze the minimizers of a partially free boundary problem for the area functional. However, in higher dimensions the situation is more delicate since the hypersurfaces obtained as minimizers can have a singular set (see Theorem 1).
Motivated mainly by the celebrated work of Fraser and Schoen [11, 12], the study of free boundary minimal surfaces in $B^3$ saw a rapid development in the last few years, see for instance [22] and the references therein. However, the case of free boundary minimal hypersurfaces in $B^{n+1}$ is not so well-studied. Concerning examples, some free boundary minimal hypersurfaces with symmetry were constructed in [14], and a variational theory has been developed in [23, 31, 32].

Regarding some properties of free boundary minimal hypersurfaces, we can mention that the asymptotic properties of the index of higher-dimensional free boundary minimal catenoids were studied in [28], and in [1] it was proved that the index of a properly embedded free boundary minimal hypersurface in $B^{n+1}, 3 \leq n + 1 \leq 7$, grows linearly with the dimension of its first relative homology group. In [24] the first author proved the index can be controlled from above by a function of the $L^2$ norm of the second fundamental form. Also, compactness results for the space of free boundary minimal hypersurfaces were obtained in [2, 10, 17].

2. Preliminary

2.1. Free boundary minimal hypersurfaces. Let $B^{n+1} \subset \mathbb{R}^{n+1}$ be the unit ball of dimension $n + 1$ with boundary $\partial B^{n+1} = S^n$. Throughout this paper we will denote by $D^n$ the $n$-dimensional equatorial disk which is the intersection of $B^{n+1}$ with a hyperplane passing through the origin. In the following, $\mathcal{H}^s$ denotes the $s$-dimensional Hausdorff measure, where $s > 0$.

Let $\Sigma \subset \mathbb{R}^{n+1}$. Along this section we will use the following notation/assumptions:

- $\Sigma$ is compact and it is contained in $B^{n+1}$.
- $\Sigma$ is an embedded orientable smooth hypersurface with boundary.
- The singular set $S_\Sigma$ is the complement of $\Sigma$ in $\bar{\Sigma}$. We suppose $\mathcal{H}^n(S_\Sigma) = 0$.
- The boundary of $\Sigma$ satisfies $\partial \Sigma = \Gamma_I \cup \Gamma_S$, where $\text{int}(\Gamma_I) \subset \text{int}(B^{n+1})$ and $\Gamma_S \subset S^n$.

We have that, away from the singular set, $\partial \Sigma$ is an embedded smooth submanifold of dimension $n - 1$.

**Definition 1.** Let $\Sigma$ be as above. We say that $\Sigma$ is a minimal hypersurface with free boundary if the mean curvature vector of $\Sigma$ vanishes and $\Sigma$ meets $S^n$ orthogonally along $\partial \Sigma$ (in particular, $\Gamma_I = \emptyset$). We say that $\Sigma$ is a minimal hypersurface with partially free boundary if the mean curvature vector of $\Sigma$ vanishes and its boundary $\Gamma_I \cup \Gamma_S$ satisfies that $\Gamma_I \neq \emptyset$ and $\Sigma$ meets $S^n$ orthogonally along $\Gamma_S$.

From now on, given a (partially) free boundary minimal hypersurface $\Sigma \subset B^{n+1}$ with boundary $\partial \Sigma = \Gamma_I \cup \Gamma_S$, we will call $\Gamma_I$ its fixed boundary and $\Gamma_S$ its free boundary.

**Definition 2.** Let $\Sigma$ be a partially free boundary minimal hypersurface in $B^{n+1}$. We say that $\Sigma$ is stable if for any function $f \in C^\infty(\Sigma)$ such that $f|_{\Gamma_I} \equiv 0$ and supp$(f)$ is away from the singular set $\bar{\Sigma} \setminus \Sigma$, we have

$$-\int_\Sigma (f \Delta_\Sigma f + |A_\Sigma|^2 f^2) d\mathcal{H}^n + \int_{\Gamma_S} \left( f \frac{\partial f}{\partial \nu} - f^2 \right) d\mathcal{H}^{n-1} \geq 0,$$

or equivalently

$$\int_\Sigma (|\nabla_\Sigma f|^2 - |A_\Sigma|^2 f^2) d\mathcal{H}^n - \int_{\Gamma_S} f^2 d\mathcal{H}^{n-1} \geq 0,$$

where $\nu$ is the outward normal vector field to $\Gamma_S$. 

Observe that if $\Sigma$ is stable then, by an approximation argument, the inequality (2.2) holds for any function $f \in H^1(\Sigma)$ such that $f(p) = 0$ for a.e. $p \in \Gamma_I$ and supp$(f)$ is away from the singular set. In particular (2.2) holds for any Lipschitz function satisfying the boundary condition.

**Lemma 1.** Let $\Sigma$ be a partially free boundary minimal hypersurface in $B^{n+1}$ of finite area and such that the singular set $S_\Sigma = \Sigma \setminus \Sigma$ satisfies $S_\Sigma = S_0 \cup S_1$, where $S_0 \subset \Gamma_I$ and $\mathcal{H}^{n-2}(S_1) = 0$. If $\Gamma_I$ is contained in an $n$-dimensional equatorial disk, then $\Sigma$ is totally geodesic.

**Proof.** Let $\Sigma$ be as in the hypotheses and denote by $D^n$ the equatorial disk that contains $\Gamma_I$. Let $v \in S^n$ be a vector orthogonal to the disk $D^n$ and consider the function $f(x) = \langle x, v \rangle$, $x \in \bar{\Sigma}$. By hypothesis, we know that $f|_{\Gamma_I} \equiv 0$. A standard calculation using that $\Sigma$ is minimal and free boundary yields

$$\Delta_\Sigma f = 0, \quad \frac{\partial f}{\partial \nu} = f.$$

Fix $\epsilon > 0$ and consider a smooth function $\eta_\epsilon : [-1, 1] \to [0, 1]$ so that

- $\eta_\epsilon(s) = 0$ for $|s| < \epsilon$,
- $\eta_\epsilon(s) = 1$ for $|s| > 2\epsilon$,
- $|\eta'_\epsilon| < \frac{C}{\epsilon}$, for some constant $C > 0$.

Define $\phi_{0,\epsilon} : \Sigma \to [0, 1]$ as $\phi_{0,\epsilon}(x) = \eta_\epsilon(f(x))$. In particular, we have $|\nabla_\Sigma \phi_{0,\epsilon}| < C/\epsilon$ in $\Sigma$. Observe that the set $S \subset S_1$ where $\phi_{0,\epsilon}$ is not smooth satisfies $\mathcal{H}^{n-2}(S) = 0$.

Since $S_1 \cap \{|f(x)| \geq \frac{\epsilon}{2}\}$ is compact and $\mathcal{H}^{n-2}(S_1) = 0$, for any $\epsilon' > 0$ there exist balls $B_{r_i}(p_i) \subset \mathbb{R}^{n+1}$, $i = 1, \ldots, m$, such that

$$S_1 \cap \{ |f(x)| \geq \frac{\epsilon}{2} \} \subset \bigcup_{i=1}^{m} B_{r_i}(p_i), \quad \bigcup_{i=1}^{m} r_i^{n-2} \leq \epsilon', \ i = 1, \ldots, m.$$ 

For each $i = 1, \ldots, m$, consider a smooth function $\phi_i : \Sigma \to [0, 1]$ such that

- $\phi_i(s) = 0$ in $B_{r_i}(p_i)$,
- $\phi_i(s) = 1$ in $\mathbb{R}^{n+1} \setminus B_{2r_i}(p_i)$,
- $|\nabla_\Sigma \phi_i| < \frac{2}{r_i}, \ \forall x \in \Sigma$.

Define $\phi_\epsilon, f_\epsilon : \Sigma \to [0, 1]$ by $\phi_\epsilon(x) = \min_{0 \leq i \leq m} \phi_i$, where $\phi_0 = \phi_{0,\epsilon}$, and $f_\epsilon = \phi_\epsilon f$. We have that $f_\epsilon$ is Lipschitz and $f_\epsilon|_{\Gamma_I} \equiv 0$, hence (2.2) holds. Moreover

$$|\nabla_\Sigma f_\epsilon|^2 = \phi_\epsilon^2 |\nabla_\Sigma f|^2 + 2f_\epsilon \langle \nabla_\Sigma f, \nabla_\Sigma \phi_\epsilon \rangle + f_\epsilon^2 |\nabla_\Sigma \phi_\epsilon|^2$$

and

$$\int_{\Sigma} \phi_\epsilon^2 |\nabla_\Sigma f_\epsilon|^2 d\mathcal{H}^n = - \int_{\Sigma} f_\epsilon \phi_\epsilon^2 \Delta_\Sigma f d\mathcal{H}^n - \int_{\Sigma} 2f_\epsilon \phi_\epsilon \langle \nabla_\Sigma \phi_\epsilon, \nabla_\Sigma f \rangle d\mathcal{H}^n + \int_{\partial \Sigma} \phi_\epsilon^2 f \frac{\partial f}{\partial \nu} d\mathcal{H}^{n-1}$$

$$= - \int_{\Sigma} 2f_\epsilon \phi_\epsilon \langle \nabla_\Sigma \phi_\epsilon, \nabla_\Sigma f \rangle d\mathcal{H}^n + \int_{\Gamma_S} \phi_\epsilon^2 f^2 d\mathcal{H}^{n-1},$$

since $\Delta_\Sigma f \equiv 0$, $\frac{\partial f}{\partial \nu} = f$ and $f|_{\Gamma_I} \equiv 0$. Hence, applying it to (2.2), we get

$$\int_{\Sigma} (f^2 |\nabla_\Sigma \phi_\epsilon|^2 - |A_\Sigma|^2 \phi_\epsilon^2 f^2) d\mathcal{H}^n \geq 0. \quad (2.3)$$
On the other hand, along $\Sigma$ we have $f^2 \leq 1$. Since $f$ has support away from the singular set, by the classical monotonicity formula at the interior and at the free boundary, there is $C_{\Sigma, \epsilon} > 0$ such that

$$\mathcal{H}^n(B_{2r_i}(p_i) \cap \Sigma) \leq C_{\Sigma, \epsilon} r_i^n.$$ 

Thus

$$\int_{\Sigma} f^2 |\nabla_{\Sigma} \phi_\epsilon|^2 d\mathcal{H}^n \leq \sum_{i=0}^m \int_{\Sigma} f^2 |\nabla_{\Sigma} \phi_i|^2 d\mathcal{H}^n$$

$$= \int_{\Sigma} f^2 |\nabla_{\Sigma} \phi_0|^2 d\mathcal{H}^n + \sum_{i=1}^m \int_{(B_{2r_i}(p_i) \setminus B_{r_i}(p_i)) \cap \Sigma} f^2 |\nabla_{\Sigma} \phi_i|^2 d\mathcal{H}^n$$

$$\leq 4C \mathcal{H}^n(\Sigma \cap \{|f|^{-1}(\epsilon, 2\epsilon)\}) + \sum_{i=1}^m 4 \mathcal{H}^n(B_{2r_i}(p_i) \cap \Sigma)$$

$$\leq 4C \mathcal{H}^n(\Sigma \cap \{|f|^{-1}(\epsilon, 2\epsilon)\}) + C_{k, \epsilon} \sum_{i=1}^m r_i^{n-2}$$

$$\leq 4C \mathcal{H}^n(\Sigma \cap \{|f|^{-1}(\epsilon, 2\epsilon)\}) + C_{\epsilon} \epsilon'.$$

If we let $\epsilon' \to 0$ first and then $\epsilon \to 0$ we obtain

$$\int_{\Sigma} |A_\Sigma|^2 f^2 d\mathcal{H}^n = 0.$$ 

If $|A_\Sigma| \equiv 0$ then $\Sigma$ is totally geodesic and we are done. If $|A_\Sigma|(x) > 0$ for some $x \in \Sigma$, then we can find a neighborhood $U$ of $x$ in $\Sigma$ such that $|A_\Sigma|$ is strictly positive. This implies $\langle y, v \rangle = 0$ for any $y \in U$. Therefore, $\Sigma$ is entirely contained in the disk $D^n$; in particular, it is totally geodesic. \[\square\]

An equatorial disk $D^n$ divides the ball $B^{n+1}$ into two (open) halfballs. We will denote these two halfballs by $B^+$ and $B^-$, and we have $B^{n+1} \setminus D^n = B^+ \cup B^-$. 

In the next proposition we will summarize some facts about partially free boundary minimal surfaces in $B^{n+1}$ which we will use in the proof of Theorem 3.

**Proposition 1.**

(i) Let $D^n$ be an equatorial disk and let $\Sigma$ be a smooth partially free boundary minimal hypersurface in $B^{n+1}$ contained in one of the closed halfballs determined by $D^n$, say $B^+$, and such that $\partial \Sigma \subset \partial B^+$. If $\Sigma$ is not contained in an equatorial disk, then $\Sigma$ has necessarily nonempty fixed boundary and nonempty free boundary.

(ii) The only smooth (partially) free boundary minimal hypersurface that contains a $(n-1)$-dimensional piece of the free boundary of a $n-$dimensional equatorial disk is (contained in) this equatorial disk itself.

**Proof.** (i) If the free boundary were empty, we could apply the (interior) maximum principle with the family of hyperplanes parallel to the disk $D^n$ and conclude that $\Sigma$ should be contained in the disk $D^n$. On the other hand, if the fixed boundary were empty, then we would have a minimal hypersurface entirely contained in a halfball without fixed boundary; hence, we could apply the (interior or free boundary version of) maximum principle with the family of equatorial disks that are rotations of $D^n$ around a $(n-1)-$dimensional equatorial disk and conclude that $\Sigma$ should be an equatorial disk.
(ii) Let $D^n$ be an equatorial disk and suppose that $\Sigma$ is a (partially) free boundary minimal hypersurface such that $\Sigma \cap D^n$ contains a $(n-1)$-dimensional piece $\Upsilon$ of the free boundary of $D^n$ in $S^n$. Assume, without loss of generality, $D^n \subset \{x_{n+1} = 0\}$.

Observe that since $\Sigma$ is free boundary we know that $\frac{\partial x_{n+1}}{\partial \eta}|_{\Upsilon} = x_{n+1}|_{\Upsilon} = 0$, where $\eta$ is the conormal vector to $\Upsilon$; and since $\Sigma$ is a minimal hypersurface in $\mathbb{R}^{n+1}$ we have that $x_{n+1}$ is harmonic.

We will show that $x_{n+1}|_{\Sigma} \equiv 0$.

Consider an extension $\hat{\Sigma}$ of $\Sigma$ along $\Upsilon$ such that $\Upsilon \subset \text{int}(\hat{\Sigma})$ and define $\hat{x}_{n+1}$ on $\hat{\Sigma}$ as

\[
\begin{cases}
\hat{x}_{n+1} = x_{n+1} & \text{on } \Sigma \\
\hat{x}_{n+1} = 0 & \text{on } \hat{\Sigma} \setminus \Sigma
\end{cases}
\]

Observe that $\hat{x}_{n+1}|_{\Upsilon} = x_{n+1}|_{\Upsilon} \equiv 0$, $\frac{\partial \hat{x}_{n+1}}{\partial \eta}|_{\Upsilon} = 0$ and $\frac{\partial \hat{x}_{n+1}}{\partial \eta}|_{\Upsilon} = \frac{\partial x_{n+1}}{\partial \eta}|_{\Upsilon} = 0$, where $\hat{\eta}$ is the conormal to $\Upsilon$ pointing towards $\Sigma$ and $\eta$ is the conormal to $\Upsilon$ pointing towards $\hat{\Sigma} \setminus \Sigma$; hence, $\hat{x}_{n+1}$ is $C^1$ in a neighborhood of $\Upsilon$ in $\hat{\Sigma}$.

**Claim 1.** $\hat{x}_{n+1}$ is a weak solution to the Laplacian equation $\Delta u = 0$.

Observe that $\hat{x}_{n+1}$ is a harmonic function on $\hat{\Sigma} \setminus \Upsilon$, so we just need to show the claim in a neighborhood of $\Upsilon$.

Consider a domain $\Omega = \Omega_1 \cup \Omega_2$ where $\partial \overline{\Omega} = \Gamma_1 \cup (\overline{\Omega} \cap \Upsilon)$ with $\Omega_1 \subset \hat{\Sigma} \setminus \Sigma$ and $\Omega_2 \subset \Sigma$ (see Figure 1), and let $\phi : \hat{\Sigma} \to \mathbb{R}$ be a smooth function with compact support contained in $\Omega$.

\[
\text{Figure 1. } \Omega = \Omega_1 \cup \Omega_2.
\]

Integration by parts gives us

\[
\int_{\Omega} \langle \nabla \phi, \nabla \hat{x}_{n+1} \rangle \, d\sigma = - \int_{\Omega} \hat{x}_{n+1} \Delta \phi \, d\sigma + \int_{\partial \Omega} \hat{x}_{n+1} \langle \nabla \phi, \nu \rangle \, dL
\]

since supp($\phi$) $\subset \subset \Omega$, where $\nu$ is the outward conormal to $\partial \Omega$.

Then,

\[
- \int_{\Omega} \hat{x}_{n+1} \Delta \phi \, d\sigma = \int_{\Omega} \langle \nabla \phi, \nabla \hat{x}_{n+1} \rangle \, d\sigma
\]

\[
= \int_{\Omega_1} \langle \nabla \phi, \nabla \hat{x}_{n+1} \rangle \, d\sigma + \int_{\Omega_2} \langle \nabla \phi, \nabla \hat{x}_{n+1} \rangle \, d\sigma.
\]

We have

\[
\int_{\Omega_1} \langle \nabla \phi, \nabla \hat{x}_{n+1} \rangle \, d\sigma = - \int_{\Omega_1} \phi \Delta \hat{x}_{n+1} \, d\sigma + \int_{\partial \Omega_1} \phi \langle \nabla \hat{x}_{n+1}, \nu_1 \rangle \, dL
\]

\[
= \int_{\Upsilon} \phi \langle \nabla \hat{x}_{n+1}, \nu_1 \rangle \, dL
\]

\[
= 0,
\]
where in the first equality we used that $\hat{x}_{n+1}|_{\Omega_1} = 0$ and in the second equality we used the fact that $\frac{\partial \hat{x}_{n+1}}{\partial \nu_1}|_\gamma = 0$, where $\nu_1$ is the outward conormal to $\Upsilon$ with respect to $\Omega_1$.

Analogously, we have
\[
\int_{\Omega_2} \langle \nabla \phi, \nabla \hat{x}_{n+1} \rangle d\sigma = -\int_{\Omega_2} \phi \Delta \hat{x}_{n+1} d\sigma + \int_{\partial \Omega_2} \phi \langle \nabla \hat{x}_{n+1}, \nu_1 \rangle dL = \int_{\Upsilon} \phi \langle \nabla \hat{x}_{n+1}, \nu_2 \rangle dL = 0,
\]
where in the first equality we used that $\hat{x}_{n+1}|_{\Omega_2} = x_{n+1}|_{\Omega_2}$ is harmonic and in the second equality we used the fact that $\frac{\partial \hat{x}_{n+1}}{\partial \nu_2}|_\gamma = 0$, where $\nu_2$ is the outward conormal to $\Upsilon$ with respect to $\Omega_2$.

Therefore, the claim follows and, by the Elliptic theory, $\hat{x}_{n+1}$ has to be a (strong) solution to the Laplacian equation. Moreover, since $\hat{x}_{n+1}$ vanishes on an open set, the unique continuation result implies that $\hat{x}_{n+1} \equiv 0$ on $\Sigma$, that is, $\Sigma$ is (contained in) the equatorial disk $D^n$.

2.2. **Integer rectifiable varifolds.** A set $M \subset \mathbb{R}^{n+1}$ is called countably $k$-rectifiable if $M$ is $\mathcal{H}^k$-measurable and if
\[
M \subset \bigcup_{j=0}^{\infty} M_j,
\]
where $\mathcal{H}^k(M_0) = 0$ and for $j \geq 1$, $M_j$ is an $k$-dimensional $C^1$-submanifold of $\mathbb{R}^{n+1}$. Such $M$ possesses $\mathcal{H}^k$-a.e. an approximate tangent space $T_xM$.

Let $G(n+1, k)$ be the Grassmannian of $k$-hyperplanes in $\mathbb{R}^{n+1}$. An integer multiplicity rectifiable $k$-varifold $\mathcal{V} = v(M, \theta)$ is a Radon measure on $U \times G(n+1, k)$, defined by
\[
\mathcal{V}(f) = \int_M f(x, T_xM) \theta(x) d\mathcal{H}^k, \ f \in C_0^\infty(U \times G(n+1, k)),
\]
where $M \subset U$ is countably $k$-rectifiable and $\theta > 0$ is a locally $\mathcal{H}^k$-integrable integer valued function. Also, we say $\mathcal{V} = v(M, \theta)$ is stationary if
\[
\int_M (\text{div}_M \zeta) \theta d\mathcal{H}^k = 0, \tag{2.4}
\]
for any $C^1$-vector field $\zeta$ of compact support.

Then we have the following result, see [20].

**Lemma 2.** Let $\Sigma \subset \mathbb{R}^{n+1}$ be an embedded $C^1$-hypersurface such that $\mathcal{H}^{n-1}(\Sigma \setminus \Sigma) \cap U = 0$, for every open set $U \subset \mathbb{R}^{n+1}$ with compact closure. Let $\theta > 0$ be a integer valued function which is locally constant. Then, the following conditions are equivalent:

1. $\mathcal{V} = v(\Sigma, \theta)$ is stationary.
2. $\vec{H}_\Sigma = 0$, and there is $C_\Sigma > 0$ such that for any ball $B_r(p) \subset \mathbb{R}^{n+1}$ we have
\[
\mathcal{H}^{n}(B_r(p) \cap \Sigma) \leq C_\Sigma r^n.
\]

2.3. **Minimizing Currents with Partially Free Boundary.** In this section we will use the following notation.

- $U \subset \mathbb{R}^{n+1}$ is an open set;
- $\mathcal{D}^k(U) = \{C^\infty \omega; \ spt \ \omega \subset U\}$;
- $\mathcal{D}_k(U)$ denotes the dual of $\mathcal{D}^k(U)$, and its elements are called $k$-currents with support in $U$;
The mass of $T \in \mathcal{D}_k(U)$ in $W$ is defined by
\[ M_W(T) := \sup \{ T(\omega); \omega \in \mathcal{D}^k(U), \text{spt} \omega \subset W, |\omega| \leq 1 \} \leq +\infty; \]

The boundary of $T \in \mathcal{D}_k(U)$ is the $(k-1)$-current $\partial T \in \mathcal{D}_{k-1}(U)$ given by
\[ \partial T(\omega) := T(d\omega), \]
where $d$ denotes the exterior derivative operator.

Consider a compact domain $W \subset \mathbb{R}^{n+1}$ such that $\partial W = S \cup M$, where $S$ is a compact $C^2$-hypersurface (not necessarily connected) with boundary, $M$ is a smooth compact mean convex hypersurface with boundary, which intersects $S$ orthogonally along $\partial S$, and such that $\text{int}(S) \cap \text{int}(M) = \emptyset$.

Let $\Omega \subset W$ be a compact hypersurface with boundary $\Gamma = \partial \Omega$. We assume that $\Gamma \cap \text{int}(W)$ is an embedded $C^2$-submanifold of dimension $n-1$ away from a singular set $S_0$ such that $\mathcal{H}^{n-1}(S_0) = 0$.

Define the class $\mathcal{C}$ of admissible currents by
\[ \mathcal{C} = \{ T \in \mathcal{D}_n(\mathbb{R}^{n+1}); T \text{ is integer multiplicity rectifiable}, \text{spt} T \subset W \text{ and is compact}, \text{and spt}(\Gamma - \partial T) \subset S \}, \]
where $[\Gamma]$ is the current associated to $\Gamma$ with multiplicity one. We want to minimize area in $\mathcal{C}$, that is, we are looking for $T \in \mathcal{C}$ such that
\[ M(T) = \inf \{ M(\tilde{T}); \tilde{T} \in \mathcal{C} \}. \tag{2.5} \]

Observe that $\mathcal{C} \neq \emptyset$ since $[\Omega] \in \mathcal{C}$. Hence, it follows from [8, 5.1.6(1)], that the variational problem (2.5) has a solution (see also [15]). If $T \in \mathcal{C}$ is a solution we have
\[ M(T) \leq \mathcal{M}(T + X), \tag{2.6} \]
\[ \text{spt} T \subset W, \tag{2.7} \]
\[ \mu T(S) = 0, \tag{2.8} \]
for any integer multiplicity current $X \in \mathcal{D}_n(\mathbb{R}^{n+1})$ with compact support such that $\text{spt} X \subset W$ and $\text{spt} \partial X \subset S$.

In order to apply the known regularity theory for $T$ we need the following result, whose proof is the same as that of the case $n = 2$ (see Section 3 in [25]).

**Proposition 2.** If $T$ is a solution of (2.5), then either $\text{spt} T \setminus \Gamma \subset W \setminus M$ or $\text{spt} T \subset M$.

For any given $n$-dimensional compact set $K \subset W$ we call corners the set of points of $\partial K$ which also belong to $S$. We then have the following regularity result.

**Theorem 1.** Let $T$ be a solution of (2.5). Then there is a set $S_1 \subset \text{spt} T$ such that, away from $S_0 \cup S_1 \cup \Gamma$, $T$ is supported in a oriented embedded minimal $C^2$-hypersurface, which meets $S$ orthogonally along $\text{spt}(\Gamma - \partial T)$. Moreover
\[ S_1 = \emptyset, \quad \text{if } n \leq 6, \]
\[ S_1 \text{ is discrete}, \quad \text{if } n = 7, \]
\[ \mathcal{H}^{n-7+\delta}(S_1) = 0, \quad \forall \delta > 0, \quad \text{if } n > 7. \tag{2.9} \]

**Proof.** Let us write
\[ \text{spt} T \setminus (S_0) = \mathcal{R} \cup S_1, \]
where the union is disjoint and $\mathcal{R}$ consists of the points $x \in \text{spt } T$ such that there is a neighborhood $U \subset \mathbb{R}^{n+1}$ of $x$ where $T \mathcal{L} U$ is given by $m$-times ($m \in \mathbb{N}$) integration over an embedded $C^2$-hypersurface with boundary. To complete the proof we will prove the following:

- **Regularity at the interior**: $(\text{spt } T \setminus \text{spt } \partial T) \cap \mathcal{R} \neq \emptyset$ and $(\text{spt } T \setminus \text{spt } \partial T) \cap \mathcal{S}_1$ satisfies (2.9);
- **Regularity at the free-boundary**: $\text{spt}(\mathcal{L} - \partial T) \cap \mathcal{R} \neq \emptyset$ and $\text{spt}(\mathcal{L} - \partial T) \cap \mathcal{S}_1$ satisfies (2.9);

The interior regularity is a classical result, see [8, Section 5.3]. Since by Proposition 2 the free part of the boundary is contained in $\mathcal{S} \setminus \partial \mathcal{S}$, we can use the result by Gruter [16] to conclude the regularity at the free boundary (away from the corners). □

### 3. The Two-Piece Property and Other Results

**Definition 3.** Let $W$ be a region in $B^{n+1}$ and let $\Upsilon \subset W$ be a $(n-1)$-dimensional equatorial disk (that is, the intersection of $B^{n+1}$ with an $(n-1)$-dimensional plane passing through the origin). We say that $\Upsilon$ is nullhomologous in $W$ if there exists a compact hypersurface $M \subset W$ such that $\partial M = \Upsilon \cup \Gamma$, where $\Gamma$ is a $(n-1)$-dimensional compact set contained in $S^n$ (see Figure 2).

**Figure 2.** In this region $W$, any $(n-1)$-dimensional equatorial disk $\Upsilon \subset W$ is nullhomologous.

The boundary of the region $W$ can be written as $U \cup V$, where $\text{int}(U) \subset \text{int}(B^{n+1})$ and $V \subset S^n$. In the next theorem we will denote by $\partial W$ the closure of the component $U$, that is, $\partial W = \overline{U}$.

**Theorem 2.** Let $W \subset B^{n+1}$ be a connected closed region with (not necessarily strictly) mean convex boundary such that $\partial W$ meets $S^n$ orthogonally along its boundary and $\partial W$ is smooth. If $W$ contains a $(n-1)$-dimensional equatorial disk $\Upsilon$, and $\Upsilon$ is nullhomologous in $W$, then $W$ is a closed $(n+1)$-dimensional halfball.

**Proof.** Up to a rotation of $\Upsilon$ around the origin, we can assume that $\Upsilon \cap \partial W$ is nonempty. Since $\Upsilon$ is nullhomologous in $W$, there exists a compact hypersurface $M$ contained in $W$ such that $\partial M = \Upsilon \cup \Gamma$, where $\Gamma$ is a $(n-1)$-dimensional compact set contained in $S^n$. We consider the class of admissible currents

$$\mathcal{C} = \{ T \in \mathcal{D}_o(\mathbb{R}^{n+1}); T \text{ is integer multiplicity rectifiable, } \text{spt } T \subset W \text{ and is compact, and } \text{spt}(\mathcal{L} - \partial T) \subset S^n \cap W \},$$

where $\mathcal{L} - \partial M$ is the current associated to $\partial M$ with multiplicity one, and we minimize area (mass) in $\mathcal{C}$. Then, by the results presented in Section 2.3, we get a compact embedded
(orientable) partially free boundary minimal hypersurface $\Sigma \subset W$ which minimizes area among compact hypersurfaces in $W$ with boundary on the class $\Gamma = \Upsilon \cup \tilde{\Gamma}$; in particular, its fixed boundary is exactly $\Upsilon$. Moreover, by Proposition 2 in Section 2.3, either $\Sigma \subset \partial W$ or $\Sigma \cap \partial W \subset \Upsilon$.

Now the same arguments we used in the proof of Theorem 1 in [25] can be applied. In fact:

**Claim 2.** $\Sigma$ is stable.

In the case $\partial W \cap \Sigma \subset \Gamma$, $\Sigma$ is automatically stable in the sense of Definition 2, since it minimizes area for all local deformations.

Suppose $\Sigma \subset \partial W$. For any $f \in C_c^\infty(\Sigma)$ with $f|_{\Upsilon} \equiv 0$, consider $Q(f, f)$ defined by

$$Q(f, f) = \frac{\int_{\Sigma} (|\nabla_{\Sigma} f|^2 - |A_{\Sigma}|^2 f^2) \, d\mathcal{H}^n - \int_{\Gamma} f^2 \, d\mathcal{H}^{n-1}}{\int_{\Sigma} f^2 \, d\mathcal{H}^n},$$

and let $f_1$ be a first eigenfunction, i.e., $Q(f_1, f_1) = \inf f Q(f, f)$.

Observe that although differently from the classical stability quotient (we have an extra term that depends on the boundary of $\Sigma$) we can still guarantee the existence of a first eigenfunction. In fact, since for any $\delta > 0$ there exists $C_\delta > 0$ such that $||f||_{L^2(\partial\Sigma)} \leq \delta ||\nabla f||_{L^2(\Sigma)} + C_\delta ||f||_{L^2(\Sigma)}$, for any $f \in W^{1,2}(\Sigma)$, we can use this inequality to prove that the infimum is finite. Once this is established the classical arguments to show the existence of a first eigenfunction work.

Since $|\nabla |f_1|| = |\nabla f_1|$ a.e., we have $Q(f_1, f_1) = Q(|f_1|, |f_1|)$, that is, $|f_1|$ is also a first eigenfunction. Since $|f_1| \geq 0$, the maximum principle implies that $|f_1| > 0$ in $\Sigma \setminus \partial\Sigma$, in particular, $f_1$ does not change sign in $\Sigma \setminus \partial\Sigma$. Then we can assume that $f_1 > 0$ in $\Sigma \setminus \partial\Sigma$ and, by continuity, we get $f_1 \geq 0$ in $\Gamma$. Therefore, we can use $f_1$ as a test function to our variational problem: Let $\zeta$ be a smooth vector field such that $\zeta(x) \in T_x S^n$, for all $x \in S^n$, $\zeta(x) \in (T_x \Sigma)^\perp$, for all $x \in \Sigma$, and $\zeta$ points towards $W$ along $\Sigma$. Let $\Phi$ be the flow of $\zeta$. For $\varepsilon$ small enough the hypersurfaces $\Sigma_t = \{\Phi(x, tf_1) : x \in \Sigma, 0 < t < \varepsilon\}$ are contained in $W$. Since $\Sigma$ has least area among the hypersurfaces $\Sigma_t$, we know that

$$0 \leq \left. \frac{d^2 |\Sigma_t|}{dt^2} \right|_{t=0} = \int_{\Sigma} (|\nabla_{\Sigma} f_1|^2 - |A_{\Sigma}|^2 f_1^2) \, d\mathcal{H}^n - \int_{\Gamma} f_1^2 \, d\mathcal{H}^{n-1},$$

which implies that $Q(f_1, f_1) \geq 0$. Since $f_1$ is a first eigenfunction, we get that $Q(f, f) \geq 0$ for any $f \in C_c^\infty(\Sigma)$ with $f|_{\Upsilon} \equiv 0$. Therefore, we have stability for $\Sigma$.

Then, since $\Upsilon$ is contained in an equatorial disk $D^n$, Lemma 1 implies that $\Sigma$ is necessarily a half $n$-dimensional equatorial disk. If $\Sigma \subset \partial W$, then we already conclude that $W$ has to be a $(n + 1)$-dimensional halfball.

Suppose $\Sigma \cap \partial W \subset \Upsilon$. Rotate $\Sigma$ around $\Upsilon$ until the last time it remains in $W$ (this last time exists once $\Sigma \cap \partial W$ is nonempty), and let us still denote this rotated hypersurface by $\Sigma$. In particular, there exists a point $p$ where $\Sigma$ and $\partial W$ are tangent. We will conclude that $W$ is necessarily a $(n + 1)$-dimensional halfball.

In fact, if $p \in \text{int}(\Upsilon)$, we can write $\partial W$ locally as a graph over $\Sigma$ around $p$ and apply the classical Hopf Lemma; if $p \in \partial\Upsilon$, we can use the Serrin’s Maximum Principle at a corner (see Appendix A in [25] for the details); and if $p \in \Sigma \setminus \Upsilon$ we can apply (the interior or the free boundary version of) the maximum principle. In any case, we get that $W$ is a $(n + 1)$-dimensional halfball. \qed

Now we prove the two-piece property for free boundary minimal hypersurfaces in $B^{n+1}$. 
**Theorem 3.** Let $M$ be a compact embedded smooth free boundary minimal hypersurface in $B^{n+1}$. Then for any equatorial disk $D^n$, $M \cap B^+$ and $M \cap B^-$ are connected.

**Proof.** If $M$ is an equatorial disk, then the result is trivial. So let us assume this is not the case.

Suppose that, for some equatorial disk $D^n$, $M \cap B^+$ is a disjoint union of two nonempty open hypersurfaces $M_1$ and $M_2$, $M_1$ being connected. Notice that by Proposition 1(i) both $\overline{M_1}$ and (all components of) $\overline{M_2}$ have nonempty fixed boundary and nonempty free boundary. Let us denote by $\Gamma_I = \partial \overline{M_1} \cap D^n$ the fixed boundary of $\overline{M_1}$ which might be disconnected. If $M_1$ and $D^n$ are transverse, then $\Gamma_I$ is an embedded smooth submanifold of dimension $n-1$. If $M_1$ and $D^n$ are tangent, then the local description of nodal sets of elliptic PDE’s (see for instance [19]) imply that int($\Gamma_I$) is an embedded smooth submanifold of dimension $n-1$, away from a singular set $S_0$ such that $\mathcal{H}^{n-1}(S_0) = 0$.

Denote by $W$ and $W'$ the closures of the two components of $B^{n+1} \setminus M$. They are compact domains with mean convex boundary. Hence, we can minimize area for the following partially free boundary problem (see Section 2.3):

We consider the class of admissible currents

$$
\mathcal{C} = \{ T \in D_n(\mathbb{R}^{n+1}) ; \ T \text{ is integer multiplicity rectifiable, } \text{spt}\ T \subset W \text{ and is compact, and } \text{spt}(\| \partial \overline{M_1} \| - \partial T) \subset S^2 \cap W \},
$$

where $\| \partial \overline{M_1} \|$ is the current associated to $\partial \overline{M_1}$ with multiplicity one, and we minimize area (mass) in $\mathcal{C}$. Then, by the results presented in Section 2.3, we get a compact embedded (orientable) partially free boundary minimal hypersurface $\Sigma \subset W$ which minimizes area among compact hypersurfaces in $W$ with the same fixed boundary as $\overline{M_1}$, which is contained in $D^n$. Moreover, by Proposition 2 in Section 2.3, either $\Sigma \subset \partial W$ or $\Sigma \cap \partial W \subset \partial \Sigma$.

Arguing as in Claim 2 of Theorem 2, we can prove the stability of $\Sigma$. Also, observe that by Theorem 1 the singular set $S_1$ of $\Sigma \setminus D$ is empty or satisfies $\mathcal{H}^{n-1+\delta}(S_1) = 0$, $\forall \delta > 0$, in particular $\mathcal{H}^{n-2}(S_1) = 0$. So, we can apply Lemma 1 and conclude that each component of $\Sigma$ is a piece of an equatorial disk.

The case $\Sigma \subset \partial W$ can not happen because this would imply that $M$ is a disk, and we are assuming it is not. Therefore, only the second case can happen, that is, any component of $\Sigma$ meets $\partial W$ only at points of $\partial \Sigma$. Observe that each component of $\Sigma$ that is not bounded by a $(n-1)$-dimensional equatorial disk is necessarily contained in $D^n$. If some component of $\Sigma$ were bounded by a $(n-1)$-dimensional equatorial disk, then we could apply Theorem 2 and would conclude that $M$ is a $n$-dimensional equatorial disk, which is not the case. Then $\Sigma$ is entirely contained in $D^n$ and, since $\Sigma \cap \partial W \subset \partial \Sigma$, $M \subset \partial W$ and $M \cap D^n$ does not contain any $(n-1)$-dimensional piece of $\partial D^n$ (Proposition 1(ii)), we have $\Sigma \cap M = \Gamma_I$.

Doing the same procedure as in the last paragraph for $W'$, we can construct another compact hypersurface $\Sigma'$ of $D^n$ with fixed boundary $\partial \Sigma' = \Gamma_I$ and such that $\Sigma' \subset W'$ and $\Sigma' \cap M = \Gamma_I$. Notice that $\Sigma \cup \Sigma'$ is a hypersurface without fixed boundary of $D^n$, therefore $\Sigma \cup \Sigma' = D^n$. In fact, let us denote by $T$ and $T'$ the minimizing currents associated to $\Sigma$ and $\Sigma'$ respectively, that is, spt $T = \Sigma$ and spt $T' = \Sigma'$. First observe that spt $\partial(T - T') \subset \partial D^n$ and $\partial\partial(T - T') = 0$; hence, by the Constancy Theorem, we know that $\partial(T - T') = k \partial D^n$, for some integer $k$. Now, since spt$(T - T' - kD^n) \subset D^n$ and $\partial(T - T' - kD^n) = 0$, the Constancy Theorem implies that $T - T' = kD^n$; but since $\Gamma_I$ has multiplicity one, this also holds for $T$ and $T'$ and therefore $k = 1$ necessarily. Hence, $\Sigma \cup \Sigma' = \text{spt}(T - T') = D^n$. 

In particular, $M \cap D^n = \Gamma_I$, which implies that $M_2 = M \cap B^+ \setminus M_1$ has fixed boundary contained in $\Gamma_I$. For $n = 2$, since $M$ is embedded and $\Gamma_I$ has singularities of $n$-prong type (if any), we know that the fixed boundaries of $M_1$ and $M_2$ are disjoint, in particular, the fixed boundary of $\overline{M_2}$ is necessarily empty and this yields a contradiction by Proposition 1(i). It remains to analyse the case when $n \geq 3$.

Let us assume, without loss of generality, that $D^n = B \cap \{x_{n+1} = 0\}$; hence, we have $M \cap D^n = \Gamma_I = \{q \in M; x_{n+1}(q) = 0\}$ which is the nodal set of the Steklov eigenfunction $x_{n+1}: M \to \mathbb{R}$.

Observe that if $q \in \Gamma_I$ and $\nabla_M x_{n+1}(q) \neq 0$ then, since $M$ is embedded, we know that in a neighborhood of $q$ we have $M \cap D^n = \overline{M_1} \cap D^n$; in particular, $q$ can not be contained in $\partial \overline{M_2}$.

Now let us analyse the singular set $\mathcal{S} = \{x_{n+1} = 0\} \cap \{\nabla_M x_{n+1} = 0\} \subset \Gamma_I$. By Theorem 1.7 in [19], the Hausdorff dimension of $\mathcal{S}$ is less than or equal to $n - 2$; in particular, $\mathcal{H}^{n-1}(\mathcal{S}) = 0$ and therefore by Lemma 2 $\overline{M_2}$ is stationary. By [30] we can conclude that either $\overline{M_2} \cap D^n = 0$ or $D^n \subset \overline{M_2}$ (which we already know is not possible). Therefore, $\overline{M_2}$ has empty fixed boundary which is a contradiction by Proposition 1(i).

Therefore, the theorem is proved. \hfill \Box

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