GENERAL BILINEAR FORMS

URIYA A. FIRST

Abstract. We introduce the new notion of general bilinear forms (generalizing sesquilinear forms) and prove that for every ring \( R \) (not necessarily commutative, possibly without involution) and every right \( R \)-module \( M \) which is a generator (i.e. \( R_R \) is a summand of \( M^n \) for some \( n \in \mathbb{N} \)), there is a one-to-one correspondence between the anti-automorphisms of \( \text{End}(M) \) and the general regular bilinear forms on \( M \), considered up to similarity. This generalizes a well-known similar correspondence in the case \( R \) is a field. We also demonstrate that there is no such correspondence for arbitrary \( R \)-modules.

We use the generalized correspondence to show that there is a canonical set isomorphism \( \text{Inn}(R) \setminus \text{Aut}^{-}(R) \cong \text{Inn}(M_n(R)) \setminus \text{Aut}^{-}(M_n(R)) \), provided \( R_R \) is the only right \( R \)-module \( N \) satisfying \( N^n \cong R^n \), and also to prove a variant of a theorem of Osborn. Namely, we classify all semisimple rings with involution admitting no non-trivial idempotents that are invariant under the involution.

1. Overview

Unless specified otherwise, all rings are assumed to have a unity and ring homomorphisms are required to preserve it. Subrings are assumed to have the same unity as the ring containing them. Given a ring \( R \), denote its set of invertible elements by \( R^\times \), its center by \( \text{Cent}(R) \) and its inner automorphism group by \( \text{Inn}(R) \). The \( n \times n \) matrices over \( R \) are denoted by \( M_n(R) \) and the category of right (left) \( R \)-modules is denoted by \( \text{Mod-R} \) (\( \text{R-Mod} \)). If a module \( M \) can be considered as a module over several rings, we use \( M_R \) to denote “\( M \), considered as a right \( R \)-module”. Throughout, a semisimple ring means a semisimple artinian ring.

Let \( R \) be an arbitrary (not-necessarily commutative) ring and let \( M \) be a right \( R \)-module satisfying certain mild assumptions (e.g. being a generator). In the main result of this paper, we establish a one-to-one correspondence between the set of anti-automorphisms of the ring \( \text{End}_R(M) \), denoted \( \text{Aut}^{-}(\text{End}_R(M)) \), and the regular bilinear forms on \( M \), considered up to an suitable equivalence relation. Moreover, the correspondence maps involutions to symmetric bilinear forms. The statements just made assumed that there exists a notion of bilinear forms on modules over arbitrary rings. Indeed, to that purpose we introduce general bilinear forms, which generalize sesquilinear forms and other similar notions.

We will use the generalized correspondence to show that there is a canonical set isomorphism

\[
\text{Inn}(R) \setminus \text{Aut}^{-}(R) \cong \text{Inn}(M_n(R)) \setminus \text{Aut}^{-}(M_n(R)),
\]

provided that \( N^n \cong R^n \) implies \( N \cong R_R \) for all \( N \in \text{Mod-R} \) (with \( n \in \mathbb{N} \) fixed). In case \( R \) has an anti-automorphism (e.g. if \( R \) is commutative), this implies that

\[
\text{Inn}(R) \setminus \text{Aut}^{-}(R) = \text{Inn}(M_n(R)) \setminus \text{Aut}^{-}(M_n(R)),
\]

provided that \( N^n \cong R^n \) implies \( N \cong R_R \) for all \( N \in \text{Mod-R} \) (with \( n \in \mathbb{N} \) fixed).

Date: February 6, 2014.

2010 Mathematics Subject Classification. 11E39, 15A63.

Key words and phrases. bilinear form, sesquilinear form, general bilinear form, anti-automorphism, involution.

This research was partially supported by an Israel-US BSF grant #2010/149 and an ERC grant #226135.
\[ \text{Inn}(R) \setminus \text{Aut}(R) \cong \text{Inn}(\text{M}_n(R)) \setminus \text{Aut}(\text{M}_n(R)), \] a statement that can be understood as a Skolem-Noether Theorem; compare with [15, Th. 2.10].

We also use the correspondence to give an easy proof to a variant of a theorem of Osborn ([8, Th. 2]). Osborn’s Theorem determines the structure of rings with involution \((R, \alpha)\) in which 2 is invertible and all \(\alpha\)-invariant elements are invertible or nilpotent. We will determine the structure of all semisimple rings with involution \((R, \alpha)\) such that the only \(\alpha\)-invariant idempotents in \(R\) are 0 and 1. In particular, we will get a new proof of Osborn’s Theorem in the case \(R\) is semilocal.

Further more specialized applications (e.g. [1]) will be published elsewhere.

Our correspondence in fact generalizes a similar well-known correspondence in the case \(R\) is a field:

**Theorem 1.1.** Let \(F\) be a field, let \(\alpha_0\) be an automorphism of \(F\) of order 1 or 2, and let \(V\) be a finite dimensional vector space. Then there exists a one-to-one correspondence between nondegenerate sesquilinear forms \(b : V \times V \rightarrow F\) over \((F, \alpha_0)\), considered up to multiplication by an element of \(F^\times\), and the anti-automorphisms \(\alpha\) of \(\text{End}_F(V)\) whose restriction to \(F = \text{Cent}(\text{End}_F(V))\) is \(\alpha_0\). The correspondence is given by sending \(b : V \times V \rightarrow F\) to the unique anti-automorphism \(\alpha\) satisfying

\[
\begin{align*}
b(wx, y) &= b(x, w^\alpha y) \quad \forall \; x, y \in M, w \in \text{End}_F(V) \\
\end{align*}
\]

and it maps involutions to hermitian forms.

**Proof.** See [4, Ch. 1, Th. 4.2]. □

We will present another proof of Theorem 1.1 which do not use the Skolem-Noether Theorem (as is the case in all proofs we have seen). Note that our correspondence treats all anti-automorphisms of \(\text{End}_F(V)\) and not only those whose order on \(F\) is 1 or 2.

Section 2 defines general bilinear forms and presents some examples. In sections 3 and 4, we construct and discuss the correspondence described above, where in section 5 we prove that our construction does yield a correspondence under mild assumptions. Section 6 examines how the new correspondence interacts with orthogonal sums, and this is used in section 7 to sharpen the results of section 5. In that section, we show the isomorphism in (1) and give a proof of Theorem 1.1. Section 8 proves the variant of Osborn’s Theorem presented above. Finally, section 9 brings various examples of modules over which our correspondence fails.

### 2. General Bilinear Forms

In this section, we define general bilinear forms and study their basic properties. Throughout, \(R\) is a (possibly non-commutative) ring.

**Definition 2.1.** A (right) double \(R\)-module is an additive group \(K\) together with two operations \(\odot_0, \odot_1 : K \times R \rightarrow K\) such that \(K\) is a right \(R\)-module with respect to each of \(\odot_0, \odot_1\) and

\[
(k \odot_0 a) \odot_1 b = (k \odot_1 b) \odot_0 a \quad \forall \; k \in K, \; a, b \in R.
\]

We let \(K'\) denote the \(R\)-module obtained by letting \(R\) act on \(K\) via \(\odot_1\).

The class of (right) double \(R\)-modules is denoted by \(\text{DMod-}R\). For \(K, K' \in \text{DMod-}R\), we define \(\text{Hom}(K, K') = \text{Hom}_R(K_0, K'_0) \cap \text{Hom}_R(K_0, K'_1)\). This makes \(\text{DMod-}R\) into an abelian category. (The category \(\text{DMod-}R\) is isomorphic to \(\text{Mod-}(R \otimes \mathbb{Z} R)\) and also to the category of \((R^{op}, R)\)-bimodules; see Remark 2.2 below for why do we prefer double modules.)
Definition 2.2. Let $K$ be a double $R$-module. An anti-automorphism of $K$ is an additive bijective map $\theta : K \to K$ satisfying

$$\{ k \circ_i a \}_{i=0}^1 = \theta^i \circ_i a \quad \forall a \in R, \ k \in K, \ i \in \{ 0, 1 \}.$$  

If additionally $\theta \circ \theta = \text{id}_K$, then $\theta$ is called an involution.

A (general) bilinear space over $R$ is a triplet $(M, b, K)$ such that $M \in \text{Mod-}R$, $K \in \text{DMod-}R$ and $b : M \times M \to K$ is a biadditive map satisfying

$$b(xr, y) = b(x, y) \circ_0 r \quad \text{and} \quad b(x, yr) = b(x, y) \circ_1 r$$

for all $x, y \in M$ and $r \in R$. In this case, $b$ is called a (general) bilinear form (over $R$). Let $\theta$ be an involution of $K$. The form $b$ is called $\theta$-symmetric if

$$b(x, y) = b(y, x)^\theta \quad \forall x, y \in M.$$  

Example 2.3. Let $(R, \ast)$ be a ring with involution. Recall that a sesquilinear space over $(R, \ast)$ consists of a pair $(M, b)$ such that $M$ is a right $R$-module and $b : M \times M \to R$ is a biadditive map satisfying

$$b(xr, y) = r^* b(x, y) \quad \text{and} \quad b(x, yr) = b(x, y)r$$

for all $x, y \in M$ and $r \in R$. If moreover $b(x, y) = \lambda b(xy, x)^\ast$, then $b$ is called $\lambda$-hermitian (or $(\ast, \lambda)$-hermitian). (See [9] or [3] for an extensive discussion about sesquilinear and hermitian forms.)

We can make any sesquilinear form $b : M \times M \to R$ fit into our definition of general bilinear forms; simply turn $R$ into a double $R$-module by defining $r \circ_0 a = a^r r$ and $r \circ_1 a = ra$ for all $a, r \in R$. Moreover, for every $\lambda \in \text{Cent}(R)$ with $\lambda^2 = 1$, the map $\theta : R \to R$ defined by $r^\theta := \lambda r^\ast$ is an involution of $R$ (as a double $R$-module), and $b$ is $\lambda$-hermitian if and only if $b$ is $\theta$-symmetric.

To avoid any ambiguity, we shall henceforth address sesquilinear forms as classical bilinear forms. General bilinear forms obtained as in Example 2.3 will be called classical as well.

Example 2.4. Let $F$ be a field and let $S : M_{n \times m}(F) \to M_{m \times n}(F)$ be given by

$$A^S = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} A^T \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}.$$  

(Here, $T$ denotes the transpose operation; $S$ reflects $A$ along the diagonal emanating from its top-right corner). It is easy to verify that $(AB)^S = B^S A^S$ for any two matrices over $F$, provided the multiplication is defined.

Let $R$, $K$ be the sets of matrices of the forms

$$\begin{bmatrix} \ast & \ast & \ast \\ \ast & 0 & \ast \\ 0 & \ast & \ast \end{bmatrix}, \quad \begin{bmatrix} \ast & \ast & \ast \\ \ast & 0 & \ast \\ 0 & \ast & \ast \end{bmatrix}$$

with entries in $F$, respectively. Then $R$ is a subring of $M_3(F)$ and $K$ is a double $R$-module w.r.t. the operations

$$k \circ_0 r = r^S k \quad \text{and} \quad k \circ_1 r = kr,$$

where $k \in K$ and $r \in R$. Furthermore, the map $\theta : k \mapsto k^S$ is an involution of $K$. Let $M$ be the set of matrices of the form $[\ast \ast \ast \ast]$ with entries in $F$. Then $M$ is a right $R$-module w.r.t. the standard matrix multiplication. Define $b : M \times M \to K$ by $b(x, y) = x^S y$. Then $(M, b, K)$ is a bilinear space over $R$ and $b$ is $\theta$-symmetric. Nevertheless, $R$ has no anti-automorphism! This can be seen by carefully checking how would an anti-automorphism act on the the standard matrix units in $R$, or by noting that $R$ is an incidence algebra of a partially-ordered set without an anti-automorphism.
We now turn to define nondegenerate and regular\(^4\) bilinear forms. Henceforth, \(K\) is a fixed double \(R\)-module.

Let \(M \in \text{Mod-}R\) and let \(i \in \{0, 1\}\). The \(i\)-\(K\)-dual (or just \(i\)-dual) of \(M\) is defined by

\[
M^{[i]} := \text{Hom}_R(M, K_{1-i}) .
\]

Note \(M^{[i]}\) is naturally a right \(R\)-module w.r.t. the operation \((fr)(m) = (fm) \circ r\) (where \(f \in M^{[i]}, r \in R\) and \(m \in M\)). Moreover, \(M \mapsto M^{[i]}\) is a left-exact contravariant functor from \(\text{Mod-}R\) to itself, which we denote by \([i]\).

Let \(b : M \times M \rightarrow K\) be a (general) bilinear form. The left adjoint and right adjoint of \(b\) are defined as follows:

\[
\text{Ad}^r_b : M \rightarrow M^{[0]}, \quad (\text{Ad}^r_b x)(y) = b(x, y) ,
\]

\[
\text{Ad}^l_b : M \rightarrow M^{[1]}, \quad (\text{Ad}^l_b x)(y) = b(y, x) ,
\]

where \(x, y \in M\). It is straightforward to check that \(\text{Ad}^r_b\) and \(\text{Ad}^l_b\) are right \(R\)-linear. We say that \(b\) is right regular if \(\text{Ad}^r_b\) is injective and right injective or right nondegenerate if \(\text{Ad}^r_b\) is injective. Left regularity and left injectivity are defined in the same manner. Bilinear forms that are not right injective are called right degenerate. This is equivalent to the existence of \(0 \neq x \in M\) such that \(b(M, x) = 0\).

If \(b\) is right regular, then every \(w \in \text{End}_R(M)\) admits a unique element \(w^\alpha \in \text{End}_R(M)\) such that

\[
b(wx, y) = b(x, w^\alpha y) \quad \forall x, y \in M .
\]

Indeed, a straightforward computation shows that \(w^\alpha = (\text{Ad}^r_b)^{-1} \circ w^{[1]} \circ \text{Ad}^r_b\) satisfies these requirements. (Recall that \(w^{[1]} : M^{[1]} \rightarrow M^{[1]}\) is defined by \(w^{[1]}f = f \circ w\).) The map \(w \mapsto w^\alpha\), denoted \(\alpha\), is easily seen to be anti-endomorphism of \(\text{End}_R(M)\) and is thus called the (right) corresponding anti-endomorphism of \(b\). Our usage of the term anti-endomorphism, rather than anti-automorphism, is essential here because \(\alpha\) need not be bijective; see Example 2.6. Nevertheless, in case \(b\) is also left regular, \(\alpha\) is invertible, for the left regularity implies that for all \(w \in W\) there is \(\beta \in W\) such that \(b(x, wy) = b(w^\beta x, y)\), and the map \(\beta\) (called the left corresponding anti-endomorphism of \(b\)) is easily seen to be the inverse of \(\alpha\). Furthermore, if \(b\) is \(\theta\)-symmetric for some involution \(\theta : K \rightarrow K\), then \(\alpha\) is an involution. Indeed,

\[
b(x, wy) = b(wy, x)^\theta = b(y, w^{\alpha x})^\theta = b(w^{\alpha x}, y) = b(x, w^{\alpha \alpha} y)
\]

and this forces \(w = w^{\alpha \alpha}\) (since \(b\) is right regular).

We now define asymmetry maps. We will only need them briefly in sections 4 and 7 but in general, asymmetries are important tools in studying non-symmetric forms (see [13], [12], [19] for classical applications). Let \(\theta\) be an anti-automorphism of \(K\). A right \(\theta\)-asymmetry (resp. left \(\theta\)-asymmetry) of \(b\) is a map \(\lambda \in \text{End}(M_R)\) such that \(b(x, y)^\theta = b(y, \lambda x)\) (resp. \(b(x, y)^\theta = b(\lambda y, x)\)) for all \(x, y \in M\). A right \(\theta\)-asymmetry need neither exist nor be unique. Nevertheless, the following holds:

**Proposition 2.5.** Let \((M, b, K)\) be a bilinear space over \(R\) and let \(\theta\) be an anti-automorphism of \(K\). Define \(u_{\theta,M} : M^{[0]} \rightarrow M^{[1]}\) by \(u_{\theta,M}(f) = \theta \circ f\). Then:

(i) \(u_{\theta,M}\) is an isomorphism of right \(R\)-modules\(^3\);

(ii) \(\lambda \in \text{End}_R(M)\) is a right \(\theta\)-asymmetry \(\iff\ u_{\theta,M} \circ \text{Ad}^r_b = \text{Ad}^r_b \circ \lambda\).

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1. Some texts use “unimodular” instead of “regular”.
2. The reason that we do not define \(M^{[i]}\) to be \(\text{Hom}_R(M, K_i)\) is because we want \(R^{[i]}\) to be isomorphic to \(K_i\) via \(f \mapsto f(1)\).
3. An anti-endomorphism of a ring is an additive map that preserves the unity and reverses the order of multiplication. It is not required to be bijective (in which case it is called an anti-automorphism).
4. In fact, \(\{u_{\theta,M}\}_{M \in \text{Mod-}R}\) is a natural isomorphism from the functor \([0]\) to the functor \([1]\).
(iii) If $b$ is right regular, then $(\text{Ad}^{\theta}_b)^{-1} \circ u_{\theta,M} \circ \text{Ad}^{\theta}_b$ is a right $\theta$-asymmetry of $b$.
(iv) If $b$ is right injective, then $b$ admits at most one right $\theta$-asymmetry.
(v) The inverse of an invertible right $\theta$-asymmetry of $b$ is a left $\theta^{-1}$-asymmetry of $b$.
(vi) $b$ is $\theta$-symmetric if and only if $\text{id}_M$ is a right $\theta$-asymmetry of $b$.
(vii) Assume $b$ is $\theta$-symmetric. Then $b$ is right regular (injective) if and only if $b$ is left regular (injective).

Proof. (i)–(vi) follow by straightforward computation. (vii) follows from (i), (ii) and (vi).

Example 2.6. Let $R$ be a ring and let $\alpha$ be an anti-endomorphism of $R$. Define $K$ to be the double $R$-module obtained from $R$ by setting

$$k \odot_0 r = r^\alpha k \quad \text{and} \quad k \odot_1 r = kr$$

for all $r, k \in R$. The double $R$-module $K$ is called the standard double $R$-module of $(R, \alpha)$. Define $b : R \times R \to K$ by $b(x, y) = x^\alpha y$. Then $b$ is a bilinear form. As $b(R, x) = 0$ implies $x = 0$ (since $x = b(1, x) = 0$), $b$ is right injective. In addition, it is straightforward to check that for all $f \in R^{[1]} = \text{Hom}_R(R, K_0)$, $\text{Ad}^{\theta}_b(f(1)) = f$, hence $\text{Ad}^{\theta}_b$ is also surjective. Therefore, $b$ is right regular.

Now observe that for all $r, x, y \in R$, $b(rx, y) = (rx)^\alpha y = x^\alpha r^\alpha y = b(x, r^\alpha y)$. Thus, once identifying $\text{End}(R_R)$ with $R$ via $f \leftrightarrow f(1)$, the corresponding anti-endomorphism of $b$ is $\alpha$. It is also straightforward to check that $\ker(\text{Ad}^{\theta}_b) = \ker(\alpha)$ and $\im(\text{Ad}^{\theta}_b) = \im(\alpha)$ (once identifying $R^{[0]} = \text{Hom}_R(R_R, K_1) = \text{End}(R_R)$ with $R$ as before). Thus, $\text{Ad}^{\theta}_b$ is left injective (surjective) if and only if $\alpha$ is. In particular, if $\alpha$ is neither injective nor surjective, then so is $\text{Ad}^{\theta}_b$, despite the fact $\text{Ad}^{\theta}_b$ is bijective. In this case, the corresponding anti-endomorphisms of $b$ is not bijective.

Example 2.7. Let $S, R, M, b, K$ be as in Example 2.6. Then a straightforward (but tedious) computation shows that $b$ is right and left regular. Moreover, $\text{End}_R(M)$ can be identified with the ring of $2 \times 2$ upper-triangular matrices over $F$, acting on $M$ from the left by matrix multiplication. As $b$ is $\theta$-symmetric, it induces an involution on $\text{End}_R(M)$, which is easily seen to be the map $S$. Indeed, for all $x, y \in M$ and $w \in \text{End}_R(M)$, we have

$$b(wx, y) = (wx)^S y = x^S w^S y = b(x, w^S y).$$

Remark 2.8. There is no obvious way to explain general bilinear forms as sesquilinear forms over a hermitian category (see [17] Ch. 7, §2 for definitions). However, it is possible to generalize the notion of hermitian categories to naturally include general bilinear forms. This construction will be published elsewhere.

Remark 2.9. Our double $R$-modules are nothing but $R \otimes_Z R$-modules or $(R^{op}, R)$-bimodules. However, we are led to use this notation for several reasons. First, this notation is shorter and clarifies the computations. (For example, consider the elegant definition of the functors $[0]$ and $[1]$: For every $M \in \text{Mod-} R$ and $i \in \{0, 1\}$, define $M^{[i]} = \text{Hom}_R(M, K_{1-i})$ and make $M^{[i]}$ into a right $R$-module by setting $(fr)m = (fm) \odot_i r$. If we had used the language of $(R^{op}, R)$-bimodules, we would often have to twist the left $R^{op}$-action of $K$ into a right $R$-action, thus causing ambiguity as to which right $R$-module structure is used. (For comparison, here is the definition of $[0]$ and $[1]$ in the language of bimodules: Given $M \in \text{Mod-} R$ and an $(R^{op}, R)$-bimodule $K$, define $M^{[0]} = \text{Hom}_R(M, K_R)$ and $M^{[1]} = \text{Hom}_R^{op}(M, R_{1-R}K)$, where in the definition of $[1]$, we twist $M$ into a left $R^{op}$-module. $M^{[0]}$ admits a standard left $R^{op}$-action, which we consider as a right $R$-action. $M^{[1]}$ has a standard right $R$-action. The shifting from left $R^{op}$-modules to right $R$-modules becomes inconvenient when dealing with objects like $M^{[1][0]}$, which are treated below.) In
addition, it is well known that every left-exact contravariant functor from Mod-$R$ to S-Mod is of the form $\text{Hom}(-, K)$ where $K$ is an $(S, R)$-bimodule. Likewise, it is possible to show that any left-exact contravariant functor from Mod-$R$ to Mod-$S$ is of the form $\text{Hom}(-, K)$ where $K$ is an abelian group admitting a right $R$-module structure and a right $S$-module structure that commute with each other (in a vague sense, applying $\text{Hom}(-, K)$ “kills” the $R$-action and leaves the $S$-action). Finally, our notation allows a natural generalization to multilinear forms: Define a right multi-$R$-module to be an abelian group $(K, +)$ admitting a family of $n$ right $R$-module structures $\circ_1, \ldots, \circ_n : K \times R \to K$ that commute with each other. A multilinear form over $R$ would be a map $b : M \times M \times \cdots \times M \to K$ such that $b$ is additive in all components and $b(\ldots, x_{i-1}, xi, x_{i+1}, \ldots) = b(\ldots, x_{i-1}, x_i, x_{i+1}, \ldots)$ $\circ_1 r$ for all $1 \leq i \leq n, x_1, \ldots, x_n \in M$ and $r \in R$.

3. FROM ANTI-ENDOMORPHISMS TO BILINEAR FORMS

Let $R$ be a ring and let $M$ be a right $R$-module. Set $W = \text{End}_R(M)$ and let $\text{End}^-(W)$ ($\text{Aut}^-(W)$) denote the set of anti-endomorphisms (anti-automorphisms) of $W$. We have seen that any right regular bilinear form $b : M \times M \to K$ induces an anti-endomorphism $\alpha \in \text{End}^-(W)$, which we henceforth denote by $\alpha(b)$. In this section, we construct an “inverse” of the map $b \mapsto \alpha(b)$. That is, for every $\alpha \in \text{End}^-(W)$, we will define a bilinear space $(M, b, K_{\alpha})$ such that $b_{\alpha}(wx, y) = b(x, w^\alpha y)$ $\forall x, y \in M, w \in W$.

This remarkable since, to our best knowledge, over fields, there is no canonical way to construct the classical bilinear form that corresponds to a given anti-automorphism (see Theorem 13). Moreover, the existence of this form is usually shown using “heavy tools” such as the Skolem-Noether Theorem. What allows the unexpected shortcut in the general case is the freedom in choosing the double $R$-module $K_{\alpha}$; we do not have to identify it with a prescribed double $R$-module.

Henceforth, $M$ is a fixed right $R$-module and $W = \text{End}(M_R)$.

We begin by introducing some new notation. Let $\alpha \in \text{End}^-(W)$ and let $A, B$ be two left $W$-modules. Define:

$$A \otimes_{\alpha} B = \{(wa \otimes b - a \otimes w^\alpha b) \mid a \in A, b \in B, w \in W\}.$$ 

For $a \in A$ and $b \in B$, we let $a \otimes_{\alpha} b$ denote the image of $a \otimes b$ in $A \otimes_{\alpha} B$ (the subscript $\alpha$ will be dropped when obvious from the context).

**Remark 3.1.** For any $B \in W$-Mod and $\alpha \in \text{End}^-(W)$, let $B^\alpha$ denote the right $W$-module obtained by twisting $B$ via $\alpha$. Namely, $B^\alpha = B$ as sets, but $B^\alpha$ is equipped with a right action $\circ_{\alpha} : B \times W \to B$ given by $x \circ_{\alpha} w = w^\alpha x$ for all $x \in B$ and $w \in W$. Then the abelian group $A \otimes_{\alpha} B$ can be naturally identified with $B^\alpha \otimes_{\alpha} W$. Therefore, $\circ_{\alpha}$ is a biadditive bifunctor and $W^n \otimes_{\alpha} B \cong B^n$.

Consider $M$ as a left $W$-module and let $\alpha \in \text{End}^-(W)$. Define $K_{\alpha} = M \otimes_{\alpha} M$ and note that $K_{\alpha}$ is a double $R$-module w.r.t. the operations

$$(x \otimes_{\alpha} y) \otimes_0 r = xr \otimes_{\alpha} y \quad \text{and} \quad (x \otimes_{\alpha} y) \otimes_1 r = x \otimes_{\alpha} yr$$

$(x, y \in M, r \in R)$. It is now clear that the map $b_{\alpha} : M \times M \to K_{\alpha}$ defined by $b_{\alpha}(x, y) = x \otimes_{\alpha} y$ is a bilinear form

$$(2) \quad b_{\alpha}(wx, y) = wx \otimes_{\alpha} y = x \otimes_{\alpha} w^\alpha y = b_{\alpha}(x, w^\alpha y)$$

for all $x, y \in M$ and $w \in W$; hence $\alpha(b_{\alpha}) = \alpha$, provided $b_{\alpha}$ is right regular. In fact, the pair $(b_{\alpha}, K_{\alpha})$ is universal w.r.t. satisfying (2) in the sense that if $b : M \times M \to K$ is a bilinear form satisfying $b(wx, y) = b(x, w^\alpha y)$ for all $w \in W$, then there is a
unique double \( R \)-module homomorphism \( f : K_\alpha \rightarrow K \) such that \( b = f \circ b_\alpha \). It is given by \( f(x \otimes \alpha y) = b(x, y) \).

Assume further that \( \alpha \) is an involution. Then \( K_\alpha \) admits an involution \( \theta_\alpha \) given by \( \theta_\alpha(x \otimes \alpha y) = y \otimes \alpha x \), and \( b_\alpha \) is \( \theta_\alpha \)-symmetric. Thus, every involution induces a symmetric form!

**Example 3.2.** Let \( F \) be a field, let \( V \) be a f.d. \( F \)-vector-space and let \( \alpha \) be an anti-automorphism of \( \text{End}_F(V) \) of order 1 or 2 on \( F = \text{Cent}(M_\alpha(F)) \). We will show below (Proposition 7.10) that \( b_\alpha \) is regular and \( K_\alpha \) is isomorphic to the standard double \( F \)-module of \( (F, \alpha|_F) \) (see Example 2.3). In particular, when identifying \( K_\alpha \) with \( F \), \( b_\alpha \) becomes a classical regular bilinear form over \( (F, \alpha|_F) \), and that form (necessarily) corresponds to \( \alpha \) in Theorem 1.1. Moreover, if \( \alpha \) is an orthogonal involution, then \( \theta_\alpha = \text{id}_F \) and \( b_\alpha \) is symmetric, and if \( \alpha \) is a symmetric involution, then \( \theta_\alpha = -\text{id}_F \) and \( b_\alpha \) is alternating.

Recalling Theorem 1.1 we now ask whether the maps \( b \mapsto \alpha(b) \) and \( \alpha \mapsto b_\alpha \) give rise to a one-to-one correspondence between the right regular bilinear forms on \( M \), considered up to a suitable equivalence relation, and the anti-endomorphisms of \( W \). In contrast to Theorem 1.1 the answer is no in general (regardless of the equivalence relation chosen), because \( b_\alpha \) need not be right regular.

**Example 3.3.** Consider the \( \mathbb{Z} \)-module \( M = \mathbb{Z}[\frac{1}{p}] / \mathbb{Z} \). It is well-known that \( \text{End}(M \mathbb{Z}) = \mathbb{Z}_p \), where \( \mathbb{Z}_p \) are the \( p \)-adic integers. (This follows from Matlis’ Duality Theory; see [7] or [5, \S3].) Take \( \alpha = \text{id}_{\mathbb{Z}_p} \in \text{End}^{-}(\mathbb{Z}_p) \), and note that the module \( M \) is \( p \)-divisible. Therefore, for all \( x, y \in M \),

\[
x \otimes \alpha y = p^n p^{-n} x \otimes \alpha y = p^n x \otimes \alpha\alpha(p^n)y = p^n x \otimes \alpha p^n y .
\]

(The “quotient” \( p^n x \) is not uniquely determined, but it does not matter to us.) As \( p^n y = 0 \) for sufficiently large \( n \), it follows that \( x \otimes y = 0 \). This implies \( K_\alpha = 0 \), hence \( b_\alpha = 0 \). Moreover, the universal property of \( b_\alpha \) means that there is no bilinear form \( 0 \neq b' : M \times M \rightarrow K' \) satisfying \( b'(wx, y) = b'(x, w^n y) \) for all \( w \in \mathbb{Z}_p \) and \( x, y \in M \). In particular, \( \alpha \) does not correspond to a right regular form on \( M \).

More examples of this flavor can be found at section 9. We leave the problem of determining when is \( b_\alpha \) right regular to section 5 and proceed with defining the equivalence relation on the class of right regular bilinear forms on \( M \).

**Definition 3.4.** Call two bilinear forms \( b : M \times M \rightarrow K \) and \( b' : M \times M \rightarrow K' \) similar if there is an isomorphism \( f \in \text{Hom}_{\text{Mod}-R}(K, K') \) such that \( b' = f \circ b \). In this case, \( f \) is called a similarity from \( b \) to \( b' \) and we write \( b \sim b' \).

It is easy to see that two similar regular bilinear forms induce the same anti-endomorphism. In addition, for classical bilinear forms over fields, being similar coincides with being the same up to multiplying by a nonzero scalar, which is the equivalence relation used in Theorem 1.1.

Let us conclude: Denote by \( \text{Bil}_\text{reg}(M) \) the category of regular bilinear forms on \( M \) with similarities as morphisms. We want to have a one-to-one correspondence as follows:

\[
\begin{array}{ccc}
\text{Bil}_\text{reg}(M) & \xrightarrow{\sim} & \text{End}^{-}(W) \\
\cong b \mapsto \alpha(b) & \sim & b_\alpha \in \alpha \\
\end{array}
\]

\[\text{Comment:} \] This is a class rather than a set because the bilinear forms in it can take values in arbitrary double \( R \)-modules.
In order of that to happen, we need to show that:

1. The map $\alpha \mapsto b_{\alpha}$ takes values in $\text{Bil}_{\text{reg}}(M)$. Namely, $b_{\alpha}$ is right regular for all $\alpha \in \text{End}^{-}(W)$.

2. The maps $\alpha \mapsto b_{\alpha}$ and $b \mapsto \alpha(b)$ are mutual inverses. As $\alpha(b_{\alpha}) = \alpha$ when $b_{\alpha}$ is regular, this amounts to showing $b \sim b_{\alpha(b)}$ for all $b \in \text{Bil}_{\text{reg}}(M)$.

(Note that (1) does imply (2); see Example 9.6)

The main result of this paper (Theorem 5.7) asserts that both (1) and (2) hold when $M$ is a \textit{progenerator}, and the analogous weaker claims for (right and left) regular forms and anti-automorphisms of $W$ hold when $M$ is a \textit{generator}. We will also show that (1) holds under other mild assumptions, e.g. when $M$ is finitely generated projective.

\textbf{Remark 3.5.} When (1) holds and (2) fails, it is still possible to obtain a correspondence between forms and anti-endomorphisms by specializing to \textit{generic} forms. A bilinear form $b : M \times M \to K$ is \textit{right generic} if it is right regular and similar to $b_{\alpha}$ for some $\alpha \in \text{End}^{-}(W)$. In this case, we must have $\alpha(b) = \alpha$, implying $b_{\alpha(b)} = b_{\alpha} \sim b$. Letting $\text{Bil}_{\text{gen}}(M)$ denote the category of right generic bilinear forms on $M$ with similarities as morphisms, we easily see that there is a one-to-one correspondence

\[ \text{Bil}_{\text{gen}}(M) \cong \text{End}^{-}(W) \]

provided $b_{\alpha}$ is right regular for all $\alpha \in \text{End}^{-}(W)$. The restriction to $\text{Bil}_{\text{gen}}(M)$ is not so bad because every right regular bilinear form $b$ can be swapped with its \textit{generalization}, defined to be $b_{\alpha(b)}$. (Notice that we are assuming $b_{\alpha}$ is always right regular and hence the generalization of $b$ is right regular. However, it is still open whether the generalization of a right regular form is right regular in general.)

Right generic forms are well-behaved behaved in comparison to right regular forms. This is demonstrated in the following proposition, which fails completely for regular forms (Example 9.6).

\textbf{Proposition 3.6.} Let $b : M \times M \to K$ be a right generic bilinear form. If $\alpha(b)$ is an involution, then $K$ has an involution $\theta$ and $b$ is $\theta$-symmetric. Furthermore, $b$ is left regular.

\textbf{Proof.} We can identify $K$ with $K_{\alpha(b)}$ and $b$ with $b_{\alpha(b)}$. Then $b_{\alpha(b)}$ is $\theta_{\alpha(b)}$-symmetric as explained above. Since $b$ is right regular by assumption and $\theta_{\alpha(b)}$-symmetric, it is also left regular by Proposition 2.3(vii). \hfill $\Box$

\textbf{Remark 3.7.} For a bilinear space $(M, b, K)$, let $\text{im}(b)$ denote the additive group spanned by $\{b(x, y) \mid x, y \in M\}$. (Caution: In general, $\text{im}(b)$ is not the image of $b$ in the usual sense.) It is easy to see that $\text{im}(b)$ is a sub-double-$R$-module of $K$ and that $\text{im}(b_{\alpha}) = K_{\alpha}$ for all $\alpha \in \text{End}^{-}(W)$. However, $\text{im}(b)$ might be strictly smaller than $K$, even when $b$ is regular, in which case $b$ is necessarily not similar to its generalization. This observation suggests that the problem of regular bilinear forms which are not similar to their generalizations might be solved by restricting to bilinear spaces $(M, b, K)$ with $\text{im}(b) = K$. However, this adjustment is not enough in general; the regular bilinear spaces $(M, b, K)$ constructed in Example 9.6 satisfy $\text{im}(b) = K$ and $b \sim b_{\alpha(b)}$.

\textbf{Remark 3.8.} Call two right stable bilinear forms \textit{weakly similar} (denoted $\sim_{w}$) if they have similar generalizations. Then under the assumption that $b_{\alpha}$ is right regular
for all $\alpha \in \text{End}^{-}(W)$, there is a one-to-one correspondence between $\text{Bil}_{\text{reg}}(M)/\sim_{w}$ and $\text{End}^{-}(W)$. However, the author could not find a natural way to make $\text{Bil}_{\text{reg}}(M)$ into a category whose isomorphism classes are the equivalence classes of $\sim_{w}$, i.e. defining weak similarities.

We finish this section by presenting a left analogue of $b_{\alpha}$. Assume $A, B \in W\text{-Mod}$. In the same manner we have defined $A \otimes_{\alpha} B$, we define

$$A_{\alpha} \otimes B = \frac{A \otimes Z \otimes B}{\{a \otimes wb - w^{\alpha}a \otimes b \mid a \in A, b \in B, w \in W\}}.$$  

In addition, we define $\alpha _{K} = M_{\alpha} \otimes M$ and $\alpha _{b} : M \times M \rightarrow _{\alpha} K$ by $b(x, y) = x_{\alpha} \otimes y$. All the results of this paper have left versions obtained by replacing $b_{\alpha}$, $K_{\alpha}$ with $\alpha _{b}$, $\alpha _{K}$ and every right property with its left version.

We also note that if $\alpha$ is bijective, then $A \otimes_{\alpha} B$ is naturally isomorphic to $A_{\alpha^{-1}} \otimes B$ (via $x \otimes_{\alpha} y \leftrightarrow x_{\alpha^{-1}} \otimes y$) and $b_{\alpha}$ is similar to $\alpha^{-1} b$, hence both right and left versions of our results apply.

4. Basic Properties

Let $R$, $M$ and $W$ be as in the previous section and let $\alpha, \beta \in \text{End}^{-}(W)$. In this section, we answer the following questions: Provided $b_{\alpha}$ and $b_{\beta}$ are right regular,

1. when are $K_{\alpha}$ and $K_{\beta}$ isomorphic?
2. when are $b_{\alpha}$ and $b_{\beta}$ weakly isometric (see below)?
3. when does $K_{\alpha}$ have an anti-automorphism? an involution?

The answers are phrased in terms of $W$ and turn out to be independent of $R$ and $M$. They agree with the approach of [4] and other texts that, roughly, isomorphism classes of anti-automorphisms (resp. involutions) are in correspondence with isometry classes of sesquilinear (resp. hermitian) forms considered up to scalar multiplication.

Throughout, $\text{Inn}(W)$ denotes the group of inner automorphisms of $W$ (i.e. those given by conjugation with an invertible element of $W$).

**Proposition 4.1.** Let $\alpha \in \text{End}^{-}(W)$ and $\varphi \in \text{Inn}(W)$. Then $K_{\alpha} \cong K_{\varphi \alpha}$ as double $R$-modules. Conversely, if $\alpha, \beta \in \text{End}^{-}(W)$ are such that $b_{\alpha}$ and $b_{\beta}$ are right regular and $K_{\alpha} \cong K_{\beta}$, then there exists $\varphi \in \text{Inn}(W)$ such that $\beta = \varphi \circ \alpha$.

In particular, if $b_{\alpha}$ is right regular for all $\alpha \in \text{End}^{-}(W)$, then the isomorphism classes of the modules $K_{\alpha}$ correspond to the orbits of the left action of $\text{Inn}(W)$ on $\text{End}^{-}(W)$, i.e. to the set $\text{Inn}(W) / \text{End}^{-}(W)$.

**Proof.** Throughout, $x, y \in M$ and $w \in W$. Let $u \in W^{\times}$ be such that $\varphi(w) = uwu^{-1}$ for all $w \in W$. Define $f : K_{\alpha} \rightarrow K_{\varphi \alpha}$ by $f(x \otimes_{\alpha} y) = x \otimes_{\varphi \alpha} uy$. Then $f$ is well defined since

$$f(wx \otimes_{\alpha} y) = wx \otimes_{\varphi \alpha} uy = x \otimes_{\varphi \alpha} (uw^{\alpha}u^{-1})uy = x \otimes_{\varphi \alpha} uw^{\alpha}y = f(x \otimes w^{\alpha}y)$$

and it is easy to see that $f$ is an isomorphism of double $R$-modules (its inverse is given by $x \otimes_{\varphi \alpha} y \mapsto x \otimes_{\alpha} uy^{-1}y$). Therefore, $K_{\alpha} \cong K_{\varphi \alpha}$.

To prove the second part of the proposition, it is enough to show that if $b, c : M \times M \rightarrow K$ are two right regular bilinear forms with corresponding anti-endomorphisms $\alpha$ and $\beta$, then there exists $\varphi \in \text{Inn}(W)$ s.t. $\beta = \varphi \circ \alpha$. Indeed, define $u = (\text{Ad}_{c})^{-1} \circ \text{Ad}_{c}^{*} \in W^{\times}$. Then for all $x, y \in M$, $c(x, y) = \text{Ad}_{c}^{*}(x) = \text{Ad}_{c}^{*}(uy) = x \otimes uy$. Therefore, for all $w \in W$:

$$c(x, w^{\beta}y) = c(wx, y) = b(wx, uy) = b(x, w^{\alpha}uy) = c(x, u^{-1}w^{\alpha}uy)$$

and it follows that $w^{\beta} = u^{-1}w^{\alpha}u$, as required. \qed
Call two general bilinear spaces \((M, b, K), (M', b', K')\) weakly isometric, denoted \(b \cong_w b'\), if there exist isomorphisms \(\sigma \in \operatorname{End}_R(M, M')\) and \(f \in \operatorname{Hom}_{\operatorname{Mod}_R(K, K')}\) such that \(b'(\sigma x, \sigma y) = f(b(x, y))\). In this case \((\sigma, f)\) is called a weak isometry from \(b\) to \(b'\). For example, two classical bilinear forms over a field \(F\) are weakly isometric if and only if they are isometric after multiplying one of them by a nonzero scalar.

Let \((S, \gamma), (S', \gamma')\) be two rings with anti-endomorphism. Recall that a homomorphism of rings with anti-endomorphism from \((S, \gamma)\) to \((S', \gamma')\) is a ring homomorphism \(\varphi : S \to S'\) such that \(\varphi \circ \gamma = \gamma' \circ \varphi\). In case \(S = S'\), \(\varphi\) is called inner if it is inner as a ring endomorphism of \(S\).

**Proposition 4.2.** Let \(\alpha, \beta \in \operatorname{End}^-(W)\). If there exists an inner isomorphism \(\varphi : (W, \alpha) \to (W, \beta)\), then \(b_\alpha \cong_w b_\beta\). Conversely, if \(b_\alpha\) and \(b_\beta\) are right regular and weakly isomorphic, then there exists an inner isomorphism \(\varphi : (W, \alpha) \to (W, \beta)\).

**Proof.** Let \(\varphi : (W, \alpha) \to (W, \beta)\) be an inner isomorphism and write \(\varphi(w) = uwu^{-1}\) for a suitable \(u \in W\). The equality \(\varphi \circ \alpha = \beta \circ \varphi\) is clearly equivalent to \(w^\alpha = \varphi^{-1}(\varphi(w)^\beta) = (u^\alpha u^{-1})w^\beta (u^\beta u^{-1})\) for all \(w \in W\). Therefore, by the proof of Proposition 4.1, the map \(f : K_\alpha \to K_\beta\) given by \(f(x \otimes_\alpha y) = \otimes_\beta u^\beta uy\) is a well-defined isomorphism of double \(R\)-modules. It is now routine to check that \((u, f)\) is a weak isometry from \(b_\alpha\) to \(b_\beta\).

Now assume we are given a weak isometry \((u, f) : b_\alpha \to b_\beta\). Then for all \(x, y \in M\) and \(w \in W\), we have
\[
 f(b_\alpha(wx, wy)) = b_\beta(ux, uy) = b_\beta(u^{-1}ux, w^\beta u^{-1}uy) = b_\beta(u^{-1}ux, w^{-1}uw^\beta uy) = (u^\alpha u^{-1})w^\beta (u^\beta u^{-1})u.
\]
Since \(f\) is injective and \(b_\alpha\) is right regular, \(w^\alpha = (u^\alpha u^{-1})w^\beta (u^\beta u^{-1})\), and this is equivalent to \(\varphi \circ \alpha = \beta \circ \varphi\) with \(\varphi(w) := uwu^{-1}\). \(\square\)

**Proposition 4.3.** Let \(\alpha \in \operatorname{End}^-(W)\) and assume there exists \(\lambda \in W\) such that \(w^{\alpha\alpha} = \lambda w\) for all \(w \in W\) and \(\lambda^\alpha \lambda \in W^\times\) (e.g. if \(\alpha^2 \in \operatorname{Im}(W)\)). Then the map \(\theta : K_\alpha \to K_\alpha\) given by \((x \otimes_\alpha y)^\theta = y \otimes_\alpha \lambda x\) is a well defined anti-automorphism of \(K_\alpha\) and \(\lambda\) is a right \(\theta\)-asymmetry of \(b_\alpha\). Moreover, if \(\lambda^\alpha \lambda = 1\), then \(\theta\) is an involution. Conversely, if \(b_\alpha\) is right regular and \(K_\alpha\) has an anti-automorphism (or involution) \(\theta\), then there exists \(\lambda \in W\) as above and \(\theta\) is induced from \(\lambda\).

**Proof.** Throughout, \(w \in W, r \in R\) and \(x, y \in M\). The map \(\theta\) is well defined since
\[
(x \otimes_\alpha y)^\theta = \lambda x \otimes_\alpha \lambda y = x \otimes_\alpha \lambda^\alpha \lambda y.
\]
and the map \(x \otimes_\alpha y \mapsto x \otimes_\alpha \lambda^\alpha \lambda y\) has an inverse given by \(x \otimes_\alpha y \mapsto x \otimes_\alpha (\lambda^\alpha \lambda)^{-1} y\). (The latter is well defined since \(\lambda^\alpha \lambda\), and hence \((\lambda^\alpha \lambda)^{-1}\), commutes with \(\operatorname{im}(\alpha)\).

Indeed, \(\lambda^\alpha \lambda^{\alpha\alpha} = \lambda^{\alpha\alpha}(w^{\alpha\alpha})^{\alpha\alpha} = (w^{\alpha\alpha})^{\alpha\alpha} = \lambda^{\alpha\alpha} = \lambda (w^\alpha)^\alpha = w^\alpha \lambda\). That \((k \otimes r)^\theta = k^\theta \otimes 1_r\) for all \(k \in K\) is straightforward and hence \(\theta\) is an anti-automorphism.

In addition, \(\theta\) also implies that \(\theta\) is an involution if \(\lambda^\alpha \lambda = 1\). That \(\lambda\) is a right \(\theta\)-asymmetry of \(b_\alpha\) is routine.

If \(b_\alpha\) is right regular and \(K_\alpha\) has an anti-isomorphism \(\theta\), then by Proposition 2.4, \(b_\alpha\) has a right \(\theta\)-asymmetry \(\lambda \in W\). Now, for all \(w \in W\), we have
\[
b_\alpha(wx, wy) = b_\theta(ux, uy) = b_\theta(ux, w^\alpha uy) = b_\alpha(ux, w^{\alpha\alpha} uy) = b_\alpha(ux, w^{\alpha\alpha} \lambda y)
\]
and since \(b_\alpha\) is right regular, this implies \(\lambda w = w^{\alpha\alpha} \lambda\). In addition, \(b_\alpha(x, y)^\theta = b_\alpha(y, x)^\theta = b_\alpha(x, \lambda y) = b_\alpha(x, \lambda^\alpha \lambda y)\). This means \(\operatorname{Ad}_b^\theta \circ \lambda^\alpha \lambda = \theta \circ u_{\theta^{-1}} \circ \operatorname{Ad}_b^\theta\).
Proof. Let \( \alpha \) be as in section 3. In this section, we present conditions on \( R, M, \) and \( \alpha \) that ensure \( b_\alpha \) is right regular, as well as other supplementary results.

Assume momentarily that \( W \) and \( R \) are arbitrary rings and let \( \text{Mod-}(W,R) \) denote the category of \( (W,R) \)-bimodules. Let \( A \in \text{Mod-W}, B \in \text{Mod-R} \) and \( C \in \text{Mod-}(W,R) \). Then \( \text{Hom}_R(B,C) \) is a right \( W \)-module w.r.t. the action \( (f)w = f(wm) \) (where \( f \in \text{Hom}_R(B,C), w \in W \) and \( m \in M \)) and there is a natural group homomorphism

\[
\Gamma = \Gamma_{A,B,C} : A \otimes_W \text{Hom}_R(B,C) \to \text{Hom}_R(B,A \otimes_W C)
\]

given by \( (\Gamma(a \otimes f))b = a \otimes f(b) \) for all \( f \in \text{Hom}_R(B,C), a \in A \) and \( b \in B \).

Now assume \( M \in \text{Mod-R}, W = \text{End}(M_R) \) and \( \alpha \in \text{End}^{-}(W) \). Then \( M \) can be viewed as a \( (W,R) \)-bimodule. Therefore, we have a map

\[
\Gamma = \Gamma_{M_\alpha,M,M} : M_\alpha \otimes_W \text{Hom}_R(M,M) \to \text{Hom}_R(M,M_\alpha \otimes_W M)
\]

(see Remark 3.4 for the definition of \( M_\alpha \)). In addition, since \( M_\alpha \) has an “unused” right \( R \)-module structure, \( M_\alpha \otimes_W \text{Hom}_R(M,M) \) and \( \text{Hom}_R(M,M_\alpha \otimes_W M) \) can be considered as right \( R \)-modules and \( \Gamma \) becomes an \( R \)-module homomorphism. The following lemma shows that up to certain identifications, \( \Gamma \) is actually \( \text{Ad}_b^\alpha \).

Lemma 5.1. In the previous notation, there is a commutative diagram of right \( R \)-modules

\[
\begin{array}{ccc}
M_\alpha \otimes_W \text{Hom}_R(M,R) & \xrightarrow{\Gamma} & \text{Hom}_R(M,M_\alpha \otimes_W M) \\
\downarrow \psi & & \downarrow \varphi \\
M & \xrightarrow{\text{Ad}_b^\alpha} & M^{[1]}
\end{array}
\]

where \( M^{[1]} = \text{Hom}_R(M, (K_\alpha)_0) \) and \( \psi, \varphi \) are bijective.

Proof. Let \( \psi \) be the identity map \( M_\alpha \to M \) (recall that \( M_\alpha = M \) as sets) composed on the standard isomorphism

\[
M_\alpha \otimes_W \text{Hom}_R(M,R) = M_\alpha \otimes_W W \cong M_\alpha.
\]
Then \( \psi \) is given by \( \psi(m \otimes_W w) = m \circ_a w = w^a m \) and its inverse is \( m \mapsto m \otimes 1 \). The map \( \varphi \) is defined by \( \varphi(f) = \delta \circ f \) where \( \delta \) is the isomorphism \( M^a \otimes_W M \to K_\alpha \) given by \( x \otimes_W y \mapsto y \otimes_\alpha x \). The diagram commutes since
\[
(\text{Ad}_{\alpha}(\psi(x \otimes_W w))) y = (\text{Ad}_{\alpha}(w^a x)) y = b_\alpha(y, w^a x) = y \otimes_\alpha w^a x = wy \otimes_\alpha x
\]
\[
= \delta(x \otimes_W wy) = \delta((\Gamma(x \otimes_W w)) y) = (\varphi(\Gamma(x \otimes_W w))) y
\]
for all \( x, y \in M \) and \( w \in W \).

It is now of interest to find sufficient conditions for \( \Gamma \) to be bijective, or at least injective. This is done in the following lemma.

**Lemma 5.2.** Let \( A \in \text{Mod}-W \), \( B \in \text{R-Mod} \) and \( C \in \text{Mod}-(W,R) \). Then:

(i) If one of the following holds:
   
   (a) \( A \) is finitely generated (abbrev.: f.g.) projective,
   
   (b) \( A \) is projective and \( B \) is f.g.,
   
   (c) \( B \) is f.g. projective,
   
   (d) \( B \) is projective and \( A \) is finitely presented (abbrev.: f.p.).

   Then \( \Gamma \) is bijective.

(ii) If \( A \) is projective, then \( \Gamma \) is injective.

(iii) If \( B \) is projective and \( A \) is f.g., then \( \Gamma \) is surjective.

(iv) If there is an exact sequence \( A_1 \to A_0 \to A \to 0 \) and \( B \) is projective, then:

   (a) \( \Gamma_{A_0,B,C} \) is surjective \( \implies \Gamma_{A,B,C} \) is surjective.

   (b) \( \Gamma_{A_0,B,C} \) is injective and \( \Gamma_{A_1,B,C} \) is surjective \( \implies \Gamma_{A,B,C} \) is bijective.

(v) If there is an exact sequence \( 0 \to A \to A_0 \to A_1 \to 0 \) and \( \text{Hom}_R(B,C) \) is flat (in \( W \text{-Mod} \)), then:

   (a) \( \Gamma_{A_0,B,C} \) is injective \( \implies \Gamma_{A,B,C} \) is injective.

   (b) \( \Gamma_{A_0,B,C} \) is bijective, \( \Gamma_{A_1,B,C} \) is injective and \( WC \) is flat \( \implies \Gamma_{A,B,C} \) is bijective.

(vi) If there is an exact sequence \( B_1 \to B_0 \to B \to 0 \) and \( A \) is flat, then:

   (a) \( \Gamma_{A_0,B,C} \) is injective \( \implies \Gamma_{A,B,C} \) is injective.

   (b) \( \Gamma_{A_0,B,C} \) is injective and \( \Gamma_{A,B,C} \) is bijective \( \implies \Gamma_{A,B,C} \) is bijective.

In particular, this implies that:

(vii) If \( A \) embeds in a free module and \( \text{Hom}_R(B,C) \) is flat, then \( \Gamma \) is injective.

(viii) If \( A \) embeds in a flat module, \( B \) is f.g. and \( \text{Hom}_R(B,C) \) is flat, then \( \Gamma \) is injective.

(ix) If \( A \) is flat and \( B \) is f.p., then \( \Gamma \) is bijective.

**Proof.** We prove (i), (ii) and (iii) together: Since \( \Gamma = \Gamma_{A,B,C} \) is additive in \( A, B, C \) (in the functorial sense), we may replace projective with free and f.g. projective with f.g. free. Assume \( A = \bigoplus_{i \in I} W \), then \( \Gamma \) becomes the standard map \( \bigoplus_{i \in I} \text{Hom}_R(B,C) \to \text{Hom}_R(B,\bigoplus_{i \in I} C) \). This map is clearly injective, and provided \( I \) is finite, it is bijective. In addition, it is also easy to verify it is surjective if \( B \) is f.g. Now assume \( B = \bigoplus_{i \in I} R \). Then \( \Gamma \) becomes the standard map \( \varepsilon : A \otimes \prod_{i \in I} C \to \prod_{i \in I} (A \otimes C) \), which is bijective if \( I \) is finite. In addition, by [5, §4F], \( \varepsilon \) is surjective if \( A \) is f.g. and bijective if \( A \) is finitely presented.

(iv) We have a commutative diagram with exact rows:

\[
\begin{array}{c}
A_1 \otimes \text{Hom}_R(B,C) \longrightarrow A_0 \otimes \text{Hom}_R(B,C) \longrightarrow A \otimes \text{Hom}_R(B,C) \longrightarrow 0 \\
\vert \quad \Gamma_{A_1,B,C} \quad \vert \quad \Gamma_{A_0,B,C} \quad \vert \quad \Gamma_{A,B,C} \\
\text{Hom}_R(B,A_1 \otimes C) \longrightarrow \text{Hom}_R(B,A_0 \otimes C) \longrightarrow \text{Hom}_R(B,A \otimes C) \longrightarrow 0
\end{array}
\]

(The bottom row is exact because \( B \) is projective). Then (a) and (b) now follow from the Four Lemma and the Five Lemma, respectively.
(v) and (vi) are very similar to (iv) and are left to the reader.

(vii) Let \( 0 \to A \to A_0 \to A_1 \) be an exact sequence with \( A_0 \) free. Then \( \Gamma_{A_0,B,C} \) is injective by (ii), hence \( \Gamma_{A_0,B,C} \) is injective by (v), since \( \text{Hom}_R(B,C) \) is flat.

(viii) Let \( 0 \to A \to A_0 \to A_1 \) be an exact sequence with \( A_0 \) flat and let \( B_1 \to B_0 \to B \to 0 \) be a projective resolution with \( B_0 \) finitely generated. Then \( \Gamma_{A_0,B_0,C} \) is injective by (i)-(c), hence \( \Gamma_{A_0,B_0,C} \) is injective by (vi), since \( A_0 \) is flat, so \( \Gamma_{A_0,B_0,C} \) is injective by (v), since \( \text{Hom}_R(B,C) \) is flat.

(ix) Let \( B_1 \to B_0 \to B \to 0 \) be an exact sequence with \( B_1 \) and \( B_0 \) being f.g. projective. Then \( \Gamma_{A,B_1,C} \) and \( \Gamma_{A,B_0,C} \) are injective by (i)-(c), hence \( \Gamma_{A,B,C} \) is injective by (vi), since \( A \) is flat.

\[ \square \]

**Proposition 5.3.** Let \( M \in \text{Mod-R}, W = \text{End}(M_R) \) and \( \alpha \in \text{End}^{-}(W) \). Then:

(i) If \( M \) or \( M^\alpha \) are f.g. projective, then \( \alpha \) is right regular.

(ii) If \( M^\alpha \) is projective and \( M \) is f.g., then \( \alpha \) is right regular.

(iii) If \( M^\alpha \) embeds in a free right \( W \)-module, then \( \alpha \) is right injective.

(iv) If \( M^\alpha \) embeds in flat right \( W \)-module and \( M \) is f.g., then \( \alpha \) is right injective.

**Proof.** By Lemma 5.1 that \( \alpha \) is bijective (injective) is equivalent to \( \Gamma_{M^\alpha,M,M} \) being bijective (injective). Observe that \( \text{Hom}_R(M_R,W,M_R) \cong \text{Hom}_W(M) \) is flat. Parts (i)-(iv) of the corollary now follow from parts (i)-(a), (i)-(b), (vii), and (viii) of Lemma 5.2 respectively.

\[ \square \]

**Remark 5.4.** Let \( \alpha \) be an anti-automorphism of \( W \), then \( M^\alpha \) is (resp. embeds in) a free/projective \( W \)-module if and only if \( M \) is. Since any flat module is a direct limit of f.g. free modules (see [4]) and twisting commutes with direct limits, the previous assertion holds upon replacing “free” with “flat”.

As an immediate corollary, we get:

**Corollary 5.5.** If \( M \) is f.g. projective or f.g. semisimple, then \( \alpha \) is right regular for all \( \alpha \in \text{End}^{-}(W) \). In particular, there exists a one-to-one correspondence between \( \text{Bil}_M(M) \) and \( \text{End}^{-}(W) \) as in [3].

**Proof.** The f.g. projective case follows from Proposition 5.3(i). In case \( M \) is f.g. semisimple, \( W \) is semisimple. Therefore, \( M^\alpha \) is projective, and the corollary follows from Proposition 5.3(ii). \[ \square \]

For the next results, recall that the module \( M \) is a generator if every right \( R \)-module is an epimorphic image of \( \bigoplus_{i \in I} M \) for some \( I \). Equivalently, \( M \) is a generator if \( R \) is summand of \( M^\alpha \) for some \( n \in \mathbb{N} \); see [5] §18B. The module \( M \) is a progenerator if it is a generator, projective and finitely generated.

**Lemma 5.6.** Assume \( M \) is a generator and let \( (M,b,K) \) be a right regular bilinear space with \( \alpha = \alpha(b) \). If \( b_\alpha \) is right regular, then \( b \sim b_\alpha \).

**Proof.** By the universal property of \( b_\alpha \), there exists a unique double \( R \)-module homomorphism \( f : K_\alpha \to K \) such that \( b = f \circ b_\alpha \). We first claim that \( f \) is onto. Indeed, the image of \( f \) is \( \text{im}(b) \), where \( \text{im}(b) \) is defined as in Remark 5.7. That \( M \) is a generator implies that if \( \text{im}(b) \subseteq K \), then \( \text{Hom}_R(M,\text{im}(b)) \subseteq \text{Hom}_R(M,K) \). Since \( \text{Ad}_b \) takes values in the l.h.s., we get a contradiction to the right regularity of \( b \), so equality must hold. Next, we claim that \( f \) is injective. Indeed, since \( M \) is a generator, \( \text{ker} f = 0 \) if and only if \( \text{Hom}(M,\text{ker} f) = 0 \), so it is enough to show the latter. Let \( \varphi \in \text{Hom}_R(M,\text{ker} f) \subseteq \text{Hom}_R(M,(K_\alpha)_0) \). Since \( b_\alpha \) is right regular, there exists \( x \in M \) such that \( b_\alpha(y,x) = \varphi(y) \) for all \( y \in M \). Applying \( f \) to both sides yields \( b(y,x) = f(\varphi(y)) = 0 \), which implies \( x = 0 \) (since \( b \) is right regular), hence \( \varphi = 0 \). \[ \square \]
Theorem 5.7. Assume \( M \) is a generator (resp. progenerator), then the maps \( b \mapsto \alpha(b) \) and \( \alpha \mapsto b_\alpha \) induce a one-to-one correspondence between the regular (resp. right regular) bilinear forms on \( M \), considered up to similarity, and the elements of \( \text{Aut}^-(W) \) (resp. \( \text{End}^-(W) \)).

Proof. If \( M \) is a progenerator, then \( b_\alpha \) is right regular for all \( \alpha \in \text{End}^-(W) \) by Corollary 5.3. Then Lemma 5.6 then implies that \( b_{\alpha(b)} \sim b \) for every regular bilinear form \( b \), so we are done.

If \( M \) is a generator, then it is well-known that \( W \) is f.g. projective (e.g. see [12, Exer. 4.1.14]). Therefore, for every \( \alpha \in \text{Aut}^-(W) \), \( M^\alpha \) is also f.g. projective. Now, by Proposition 5.3(i), \( b_\alpha \) is right regular, and by symmetry (see the end of section 6), \( b_\alpha \) is also left regular. Again, Lemma 5.6 implies that \( b_{\alpha(b)} \sim b \) for every regular bilinear form \( b \). Finally, notice that every \( \alpha(b) \) is an invertible for every regular bilinear form \( b \) (see section 2) and hence lies in \( \text{Aut}^-(W) \).

We will slightly strengthen Theorem 5.7 in section 6.

Remark 5.8. Theorem 5.7 fails when \( M \) is f.g., faithful and projective because Lemma 5.6 is no longer true (Example 9.6). In addition, if \( M \) is a generator (and not a progenerator), then \( b_\alpha \) need not be right regular when \( \alpha \) is not injective (Examples 9.2 and 9.3).

The following example presents cases in which Theorem 5.7 can be applied to big families of right \( R \)-modules.

Example 5.9. (i) If \( R \) is a Dedekind domain and \( M_R \) is f.g. then \( M \) is a generator when considered as an \( R/\text{ann}(M) \)-module. (This follows from classification of f.g. modules over Dedekind domains; e.g. see [10, Th. 4.14].) Therefore, the conclusions of Theorem 5.7 apply to \( M \).

(ii) A ring \( R \) is called right pseudo-Frobenius (abbrev.: PF) if any faithful right \( R \)-module is a generator. This is equivalent to \( R \) being a right self-injective semilocal ring with essential right socle; see [2, Th. 12.5.2] for other definitions. In this case, Theorem 5.7 applies to all faithful \( R \)-modules. Examples of two-sided PF rings include semisimple rings, artinian rings with a simple socle, Frobenius algebras, and finite group algebras over the previous examples; see [5, Ch. 5].

6. Orthogonal Sums

In this section, we define orthogonal sums and prove a result about how they interact with the map \( \alpha \mapsto b_\alpha \) of section 5. This will be used in the next section.

Let \( K \) be a double \( R \)-module and let \((M, b, K)\) and \((M', b', K)\) be bilinear spaces. The orthogonal sum \((M, b, K) \perp (M', b', K)\) is defined to be \((M \oplus M', b \perp b', K)\) where

\[
(b \perp b')(x \oplus x', y \oplus y') = b(x, y) + b'(x', y') \quad \forall \ x, y \in M, \ x', y' \in M'.
\]

Observe that when viewed as submodules of \( M \oplus M' \), \( M \) and \( M' \) satisfy

\[
(b \perp b')(M, M') = (b \perp b')(M', M) = 0.
\]

Conversely, if \((b'', M'', K)\) is a bilinear space and there are submodules \(M, M' \leq M''\) such that \( M'' = M \oplus M \) and \( b''(M, M') = b''(M', M) = 0\), then \( b'' = b \perp b'\) with \( b = b''|_{M \times M}\) and \( b' = b''|_{M' \times M'}\).

Proposition 6.1. Let \((M_1, b_1, K)\), \((M_2, b_2, K)\) be bilinear spaces. Then \(b_1 \perp b_2\) is right regular (injective) \(\iff\) \(b_1\) and \(b_2\) are right regular (injective).
Proof. Identify $(M_1 \oplus M_2)^{[1]}$ with $M_1^{[1]} \oplus M_2^{[1]}$ in the standard way (i.e. via $f \mapsto (f \mid M_1, f \mid M_2)$). Then it is straightforward to check that $\text{Ad}_{b_1 \cdot b_2} = \text{Ad}_{b_1} \oplus \text{Ad}_{b_2}$. The proposition follows immediately. □

Fix a right $R$-module $M$ and let $W = \text{End}_R(M)$. Then we can identify $\text{End}_R(M^n)$ with $M_n(W)$ for all $n \in \mathbb{N}$ (the elements of $M^n$ are considered as column vectors). For every $\alpha \in \text{End}^{-1}(W)$, let $T_n \alpha$ denote the anti-automorphism of $M_n(W)$ defined by:

$$
\begin{bmatrix}
w_{11} & \ldots & w_{1n} \\
\vdots & \ddots & \vdots \\
w_{n1} & \ldots & w_{nn}
\end{bmatrix}
\mapsto
\begin{bmatrix}
w_{11}^\alpha & \ldots & w_{n1}^\alpha \\
\vdots & \ddots & \vdots \\
w_{1n}^\alpha & \ldots & w_{nn}^\alpha
\end{bmatrix}
$$

(see [1]).

The rest of this section is devoted to showing that $b_{T_n \alpha}$ is always similar to $n \cdot b_n$.

Lemma 6.2. Let $N \in \text{Mod-W}$, $M \in \text{W-Mod}$ and let $e \in W$ be an idempotent satisfying $WeW = W$. Define a group homomorphism $\varphi : Ne \otimes_{e W} e M \to N \otimes W M$ by $\varphi(x \otimes e w, y) = x \otimes W y$. Then $\varphi$ is an isomorphism.

Proof. Write $1_W = \sum_i u_i u_i'$ where $u_1, \ldots, u_n \in W$ and $u_1', \ldots, u_n' \in eW$ and define $\psi : N \otimes W M \to Ne \otimes_{e W} e M$ by $\psi(x \otimes W y) = \sum_i x u_i \otimes_{e W} u_i' y$. Then $\psi$ is well-defined because

$$
\psi(xw \otimes y) = \sum_{i,j} x u_i u_j' w u_i \otimes_{e W} u_j' y = \sum_{i,j} x u_i u_j' w u_i \otimes_{e W} u_j' y = \sum_{i,j} x u_i \otimes_{e W} u_j' y w u_i \otimes_{e W} u_j' y = \psi(x \otimes W y)
$$

and it is straightforward to check that $\psi = \varphi^{-1}$. □

An idempotent $e \in W$ satisfying $WeW = W$ is called full. This condition is equivalent to $eW$ (or $wW$) being a progenerator; see [5, Rm. 18.10]. For example, the standard matrix unit $e_{11}$ is a full idempotent in $M_2(W)$.

Proposition 6.3. For all $\alpha \in \text{End}^{-1}(W)$, we have $b_{T_n \alpha} \sim n \cdot b_n$.

Proof. Let $\{e_{ij}\}$ be the standard matrix units of $U := M_n(W)$, and let $\psi_i : M \to M^n$ be the embedding of $M$ as the $i$-th component of $M^n$. Choose some $1 \leq i \leq n$, and define $f : K_{ii} \to K_{T_n \alpha}$ by $f(x \otimes e \psi_i y) = e_{ij} \psi_i x \otimes T_n \alpha e_{ji} y$ for all $x, y \in M$. It is easy to see that $f$ is well-defined. Furthermore, $f$ is independent of $i$ because

$$
\psi_i x \otimes T_n \alpha \psi_i y = e_{ij} e_{ji} \psi_j x \otimes T_n \alpha \psi_j y = e_{ij} \psi_i x \otimes T_n \alpha e_{ij} \psi_j x \otimes T_n \alpha \psi_j y
$$

This in turn implies that for all $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in M^n$, we have

$$
f((n \cdot b_n)(x, y)) = \sum_{i=1}^n f(b_n(x_i, y_i)) = \sum_{i=1}^n b_{T_n \alpha}(\psi_i x_i, \psi_i y_i) = \sum_{i=1}^n b_{T_n \alpha}(e_{ii} x_i, e_{ii} y_i) = \sum_{i=1}^n b_{T_n \alpha}(x, e_{ii} T_n \alpha e_{ii} y) = b_{T_n \alpha}(x, y).
$$

Therefore, we are done if we show that $f$ is an isomorphism. To see this, identify $M$ with $e_{11} M^n$ and let $e = e_{11}$ and $\alpha' = T_n \alpha |_{e^2 W}$. Recall that $K_{T_n \alpha} = M^n < \otimes_{e W} e M^n$ can be understood as $(M^n)_{T_n \alpha} \otimes_{e W} (M^n)$ (see Remark 3.1), and likewise, we can identify $K_{\alpha} = M \otimes_{e W} e M^n$ with $(e M^n)_{\alpha'} \otimes_{e W} (e M^n) =$
(M^n)^{\alpha} e \otimes_{eUe} e(M^n)$. Now, the map $f$ is just the map $\varphi$ of Lemma [6.2] hence it is an isomorphism because $UeU = U$. \qed

7. The Structure of $K_\alpha$

Let $R, M, W$ be as in section [3] In this section, we use the results of the previous sections to obtain a (relatively) explicit description of $b_\alpha$ and $K_\alpha$ ($\alpha \in \text{End}^{-}(W)$) in case $M$ is a generator. We then use this description for several of applications.

**Definition 7.1.** Let $M$ be a right $R$-module and $W = \text{End}_{R}(M)$. The module $M$ is called faithfully balanced if the standard map $R \rightarrow \text{End}_{W}(M)$ is an isomorphism.

**Example 7.2.** It is well known that any $R$-module which is generator is faithfully balanced; e.g., see [13, Exer. 4.1.14].

Let $M$ be a generator of $\text{Mod}-R$. Then $R_R$ is a summand of $M^n$ for some $n \in \mathbb{N}$. Let $e$ be the projection from $M^n$ onto the summand $R_R$. Then $e$ is an idempotent in $\text{End}_{R}(M^n)$, which we identify with $U := M(e) = M_R(W)$. Observe that $Ue = U M^n = [eUe] \in U$ for any $u \in Ue$. The similarity is just the map $\beta : y \mapsto eUe \alpha x \mapsto y \beta eUe$. The inverse of this isomorphism is given by $\alpha eUe \mapsto (\beta y) eUe$.

**Proposition 7.3.** Keeping the previous notation, let $\alpha \in \text{End}^{-}(W)$ and let $\beta := T_0 e \in \text{End}^{-}(U)$, with $T_0 e$ defined as in section [6]. Make $e\beta Ue$ into a double $R$-module by letting

$$u \circ_0 r = r^\beta u, \quad u \circ_1 r = ur \quad \forall r \in R = eUe, u \in e\beta Ue$$

and define $b : M \times M = e_{11} Ue \times e_{11} Ue \rightarrow e\beta Ue$ by $b(x, y) = x^\beta y$. Then:

(i) $b_\alpha \sim b$. The similarity is given by $x \otimes_\alpha y \mapsto x^\beta y (x, y \in M) = e_{11} Ue$.

(ii) Assume $\alpha$ is an involution. Then once identifying $K_\alpha$ with $Ue$ as $\beta$ in (i), $\theta_\alpha$ is just $\beta | eUe$ (see section [3] for definition of $\theta_\alpha$).

**Proof.** (i) Since $M$ is identified with $e_{11} Ue$, we may identify $\text{End}_{R}(M)$ with $e_{11} Ue_{11}$ and $\alpha$ with $\beta | e_{11} Ue_{11}$. As in the proof of Proposition [6.3], the map $M \otimes_{eUe} M \rightarrow M^n \otimes_{eUe} M^n$ given by $x \otimes_\alpha y \mapsto x \otimes_\alpha y$ is an isomorphism. Identify $K_\beta$ with $(Ue)^{\beta} \otimes_{eUe} Ue$ as in Remark [6.4]. The latter is isomorphic to $(Ue)^{\beta} \otimes_{eUe} Ue$ via $x \otimes_\beta u \mapsto x \otimes_\beta u \mapsto x \otimes u x \mapsto y \otimes u x \mapsto y \otimes u x = x^\beta y$. (This is a general fact; for any $A \in \text{Mod}-U$, $A \otimes_{Ue} Ue \cong A e$).

Part (ii) now follows by composing the isomorphisms $K_\alpha = K_\beta$ and $K_\beta = K_\beta$ as $e\beta Ue$.

Here is the explicit computation:

$$e_{11} Ue \otimes_{eUe} e_{11} Ue \cong Ue \otimes_{eUe} e_{11} Ue \cong (Ue)^{\beta} \otimes_{eUe} Ue \cong e\beta Ue \quad (x \otimes_\alpha y \mapsto x \otimes_\beta y \mapsto y \otimes u x \mapsto y \otimes u x = x^\beta y).$$

(ii) $\beta$ is an involution and identify $K_\alpha$ with $e\beta Ue$. Then for all $x, y \in e_{11} Ue$, $(x \otimes_\beta y)^{\beta_\alpha} = y \otimes_\alpha x$, so under the identification $K_\alpha \cong e\beta Ue$ we get $(x^\beta y)^{\beta_\alpha} = y^\beta x$ and the latter equals $(x^\beta y)^\beta$ since $\beta$ is also an involution. Thus, $\theta_\alpha$ coincides with $\beta$ on $e\beta Ue$.

**Remark 7.4.** In the proposition’s assumptions, it also possible to understand $e\beta Ue$ as $e\beta M^n = \text{im}(e\beta)$. Under this identification, the form $b$ is given by the formula

$$b(x, y) = ([z \mapsto x \cdot e\beta z][\beta])(y) \quad \forall x, y \in M.$$ 

Here, $z \in M^n$ and $M$ has to be identified as one of the summands of $M^n = M \oplus \cdots \oplus M$.

Proposition [7.3] enables us to give another proof for Theorem [5.7] and even strengthen it.
Corollary 7.5. Assume $M \in \text{Mod-R}$ is generator, let $W = \text{End}(M_R)$ and let $\alpha \in \text{End}^-(W)$. If $\alpha$ is injective, then $b_\alpha$ is left injective. If $\alpha$ is bijective, then $b_\alpha$ is (right and left) regular.

Proof. By Proposition 6.3 and Proposition 6.1 we can replace $b_\alpha$ with $n \cdot b_\alpha$, thus assuming $n = 1$, $U = M_1(W) = W$, $e_{11} = 1$ and $\beta = \alpha$ in previous computations. (This step is not really necessary, but it simplifies the arguments to follow.) Define $b$ as in Proposition 7.3. Then it is enough to prove that $b$ is injective/regular. Indeed, $b(x,M) = 0$ implies $x^\alpha \in \text{ann}^f Ue$. Since $U = \text{End}_R(M) = \text{End}_{Ue}(Ue)$, $Ue$ is faithful, so $x^\alpha = 0$. Thus, if $\alpha$ is injective, $x = 0$, hence $b_\alpha$ is left injective.

Now assume $\alpha$ is bijective. We claim that $\text{Ad}_b^\alpha$ is surjective. This is easily seen to be equivalent to showing that any $\gamma \in \text{End}(Ue)$ is induced by left multiplication with an element of $(\text{Hom}_{\mathbb{Z}}(Ue,e^\beta U)) = e^\beta U$. Indeed, viewing $f$ as an endomorphism of $Ue$, we see that $f(x) = ux$ for some unique $u \in U$ (because $U = \text{End}(M_R) = \text{End}(UeR)$). Since $u$ and $e^\beta u$ clearly induce the same endomorphism on $Ue_R$, we must have $u = e^\beta u$, as required. Thus, $\text{Ad}_b^\alpha$ is surjective, hence bijective by the previous paragraph. That $b$ right regular follows by symmetry.

We could neither find nor contradict the existence of a generator $M$ with an injective $\alpha \in \text{End}^-(W)$ such that $b_\alpha$ is not right injective. However, if $\alpha$ is not injective, then it is possible that $b_\alpha$ would be the zero form even when $M$ is a generator; see Example 9.5.

The rest of this section uses Proposition 7.3 to obtain various structural results about $K_\alpha$, provided certain assumptions on $M$ and $R$. In particular, we prove the claims posed in Example 8.2.

Proposition 7.6. Assume $M \in \text{Mod-R}$ is free of rank $n \in \mathbb{N}$, let $W = \text{End}(M_R)$ and let $\alpha \in \text{End}^-(W)$. Then $(K_\alpha)^n \cong R^n$ as right $R$-modules. (Recall that $(K_\alpha)_1$ means "$K_\alpha$, considered as a right $R$-module w.r.t. $\odot_1$".)

Proof. Assume $M = R^n$ and identify $W$ with $M_n(R)$. Let $(e_{ij})$ be the standard matrix units of $W$. Then by Proposition 7.3, $K_\alpha \cong e_{11}W_{\odot_1}$. Consider $K_i := e_{i1}^nW_{\odot_1}$ as a right $R$-module. Then $K_i \cong K_j$ for all $i,j$ (the isomorphism being multiplication on the left by $e_{ij}^n$). Thus, $(K_\alpha)^n \cong K_1 \oplus \cdot \cdot \cdot \oplus K_n = (\sum_i e_{i1}^nW_{\odot_1} = W_{\odot_1} \cong R_R^n)$ as right $R$-modules.

Lemma 7.7. Fix a double $R$-module $K$. For $M \in \text{Mod-R}$, define $\Phi_M : M \to M^{[1]}$ by $(\Phi_M x)f = f(x)$ for all $x \in M$ and $f \in M^{[1]}$ (see section 8 for the definitions of $[0]$ and $[1]$). Then:

(i) $(\Phi_M)_{M \in \text{Mod-R}}$ is a natural transformation from $\text{id}_{\text{Mod-R}}$ to $[0][1]$ (i.e. for all $N,N' \in \text{Mod-R}$ and $f \in \text{Hom}_R(N,N')$, one has $f[1][0] \circ \Phi_N = \Phi_{N'} \circ f$).
(ii) $\Phi$ is additive (i.e. $\Phi_{N \oplus N'} = \Phi_N \oplus \Phi_{N'}$ for all $N,N' \in \text{Mod-R}$).
(iii) $(\text{Ad}_{\Phi_M}^{[0]} \circ \Phi_M = \text{Ad}_{\Phi_M}^{[0]}$ for every general bilinear form $b : M \times M \to K$.
(iv) $R^{[1][0]}$ can be identified with $\text{End}_R(K_1)$. Under that identification, $(\Phi_R x) = k = k \odot_0 r$ for all $r \in R$ and $k \in K$.

Proof. (i)–(iii) are straightforward computation. (Recall that for $f \in \text{Hom}_R(N,N')$, we have $f[1][\varphi] = \varphi \circ f$ for all $\varphi \in N^{[1]}$.) For example, $(\text{Ad}_{\Phi_M}^{[0]} \circ \Phi_M = \text{Ad}_{\Phi_M}^{[0]}$ holds since for all $x,y \in M$, we have $(\text{Ad}_{\Phi_M}^{[0]} x)y = b(x,y) = ((\Phi_M x)(\Phi_M y) = ((\text{Ad}_{\Phi_M}^{[0]} \circ \Phi_M x)(\Phi_M y))$.

(iv) We have $R^{[1]} = \text{Hom}_R(R_R,K_0) \cong K_1$ via $f \mapsto f(1)$ ($f \in R^{[1]}$) and hence $R^{[1][0]} \cong K_1^{[0]} = \text{Hom}_R(K_1,K_1) = \text{End}_R(K_1)$. It is now routine to verify that under that isomorphism, the map $\Phi_R(x)$ is just $[k \mapsto k \odot_0 r] \in \text{End}_R(K_1)$. 

Theorem 7.8. Let \( n \in \mathbb{N}, M \in \text{Mod-}R, W = \text{End}(M_R) \) and \( \alpha \in \text{End}^-(W) \). Assume that \( M^k \cong R^n \) for some \( k \in \mathbb{N} \) and \( N^n \cong R^n \) implies \( N \cong R_R \) for all \( N \in \text{Mod-}R \) (e.g. if \( R \) is semilocal or a principal ideal domain). Then:

(i) There exists \( \gamma = \gamma(\alpha) \in \text{End}^-(R) \) such that \( K_\alpha \) is isomorphic to the standard double \( R \)-module of \((R, \gamma)\) (see Example 2.6). The anti-endomorphism \( \gamma \) is unique up to composition with an inner automorphism of \( R \).

(ii) \( \alpha \in \text{Aut}^-(W) \iff \gamma \in \text{Aut}^-(R) \).

(iii) There exists \( \lambda \in W \) with \( \lambda^\alpha \lambda = 1 \) (resp. \( \lambda^\alpha \lambda \in W^\times \)) and \( u^\alpha \lambda = \lambda w \) for all \( w \in W \iff \) there exists \( \mu \in R \) with \( \mu^\gamma \mu = 1 \) (resp. \( \mu^\gamma \mu \in R^\times \)) and \( r^\gamma \mu = \mu r \) for all \( r \in R \).

(v) The map \( \alpha \mapsto \gamma(\alpha) \) gives rise to injective maps
\[
\text{Inn}(W) \setminus \text{End}^-(W) \to \text{Inn}(R) \setminus \text{End}^-(R), \\
\text{Inn}(W) \setminus \text{Aut}^-(W) \to \text{Inn}(R) \setminus \text{Aut}^-(R).
\]

If \( M \cong R^n \), then these maps are bijective.

Proof. By Proposition 6.3, we may replace \( M \) with \( M^k \) and henceforth assume \( M = R^n \). Throughout, \( S_\gamma \) denotes the standard double \( R \)-module of \((R, \gamma)\). Recall that \( S_\gamma \) is as sets and \( k \circ_0 r = r^\gamma \) for all \( r \in R \) as sets.

(i) By Proposition 7.6, \( (K_\alpha)^1 \cong R^n \), so by assumption, \( (K_\alpha)_1 \cong R_R \). Choose a basis \( \{e_i\} \) to \((K_\alpha)_1\). Then for all \( r \in R \), there is a unique \( r^\gamma \in R \) such that \( k \circ_0 r = k \circ_1 r^\gamma \). The map \( \gamma \) is easily seen to be an anti-automorphism of \( R \), and it is routine to verify that \( S_\gamma \cong K_\alpha \) via \( r \mapsto k \circ_1 r \).

To see that \( \gamma \) is unique up to composition with an inner-automorphism, it enough to show that for all \( \gamma, \gamma' \in \text{End}^-(R) \), \( S_\gamma \cong S_{\gamma'} \iff \) there exists \( u \in R^\times \) such that \( r^\gamma = u^{-1}r^\gamma'u \) for all \( r \in R \). Indeed, since \( (S_\gamma)_1 = (S_{\gamma'})_1 = R_R \), we have \( S_\gamma \cong S_{\gamma'} \iff \) there exists \( u \in R^\times \) such that \( u(r^\gamma k) = r^\gamma(uk) \) for all \( r, k \in R \). However, the same proposition implies \( \lambda^\gamma \mu = \mu \lambda \) for all \( \lambda, \mu \in R \).

(ii) Identify \( K := K_\alpha \) with \( S_\gamma \), let \( b = b_\alpha \), and let \( \Phi_M \) be as in Lemma 7.7. By that lemma, we have \((\text{Ad}_b^\gamma)^{[0]} \circ \Phi_M = \text{Ad}_b^\gamma \). Since \( \text{Ad}_b^\gamma \) is bijective (Theorem 5.7), \( \text{Ad}_b^\gamma \) is bijective if and only if \( \Phi_M \) is bijective, and since \( M = R^n \) and \( \Phi \) is additive, the latter is equivalent to \( R_R \) being bijective. Identifying \( R_R \) with \( \text{End}_R((S_\gamma)_1) = \text{End}_R(R_R) \cong R \) as in Lemma 7.7 (iv), we see that \( \Phi_R \) is just the map \( \gamma \). Therefore, \( \text{Ad}_b^\gamma \) is bijective if and only if \( \gamma \) is.

Now, if \( \gamma \) is bijective, then \( b \) is right and left regular, implying \( \alpha \) is invertible (its inverse is the left corresponding anti-endomorphism of \( b \), which exists since \( b \) is left regular). On the other hand, if \( \alpha \) is invertible, then \( b \) is left regular by Theorem 5.7, implying \( \gamma \in \text{Aut}^-(R) \).

(iii) Define \( b' : R \times R \to S_{\gamma} \) by \( b'(x, y) = x^\gamma y \). By Example 2.6, \( b' \) is right regular with corresponding anti-endomorphism \( \gamma \), so by Theorem 5.7, \( b' \sim b_\gamma \). Now, by Proposition 4.3, the existence of \( \mu \) as above is equivalent to the existence of an involution (resp. anti-automorphism) on \( S_{\gamma} \). However, the same proposition implies that the existence of \( \lambda \) as above is equivalent to the existence of an involution (resp. anti-automorphism) on \( K_\alpha \). Since \( K_\alpha \cong S_{\gamma} \), we are done.

(iv) Let \( \alpha, \alpha' \in \text{End}^-(W) \) and assume \( \text{Inn}(R)\gamma(\alpha) = \text{Inn}(R)\gamma(\alpha') \). By the proof of (i), this means \( K_\alpha \cong K_{\alpha'} \), so by Proposition 4.3, \( \text{Inn}(W)\alpha = \text{Inn}(W)\alpha' \), as required.

To finish, assume \( M \cong R^n \) and let \( \gamma \in \text{End}^-(R) \). Define \( b_\gamma \) as in (iii). Then by Proposition 6.3, \( a \cdot b_\gamma \sim b_{T_\gamma} \). Since \( M = R^n \), we may view \( b_{T_\gamma} \) as a form on \( M \),

---

6 The elements \( \lambda, \mu \) are invertible when \( \alpha, \gamma \) are invertible, respectively; see Remark 4.4.
and hence assume $T_n \gamma \in W$. But now we have $K_{T_n \gamma} \cong K_{\gamma} \cong S_{\gamma}$, so by definition, $\gamma(T_n \gamma) = \gamma$.\hfill $\square$

As a special case of the theorem, we get that for every ring $R$ such that $N^n \cong R^n$ implies $N \cong R_R$ ($N \in \text{Mod-}R$; $n$ fixed), we have a set bijection

$$(6) \quad \text{Inn}(R) \setminus \text{End}^-(R) \cong \text{Inn}(M_n(R)) \setminus \text{End}^-(M_n(R))$$

which maps $\text{Inn}(M_n(R)) \setminus \text{Aut}^-(M_n(R))$ to $\text{Inn}(R) \setminus \text{Aut}^-(R)$. In particular, $R$ has an anti-automorphism (resp. anti-endomorphism) $\iff M_n(R)$ has an anti-automorphism (resp. anti-endomorphism). In case $\text{Aut}^-(R) \neq \phi$ (e.g. if $R$ is commutative), we also get a group isomorphism

$$\text{Out}(R) \cong \text{Out}(M_n(R))$$

where $\text{Out}(R)$ is the outer automorphism group of $R$, namely $\text{Aut}(R)/\text{Inn}(R)$. Indeed, let $W = M_n(R)$ and fix some $\gamma_0 \in \text{Aut}^-(R)$. Observe that $\text{Out}(R) \cong \text{Inn}(R) \setminus \text{Aut}^-(R)$ as sets via $\text{Inn}(R)\varphi \mapsto \text{Inn}(R)(\varphi \circ \gamma_0)$ and likewise, $\text{Out}(W) \cong \text{Inn}(W) \setminus \text{Aut}^-(W)$ via $\text{Inn}(W)\psi \mapsto \text{Inn}(W)(\psi \circ T_n \gamma_0)$. Composing these isomorphisms with the bijection in (6) gives a set bijection $\text{Out}(R) \to \text{Out}(W)$ sending the coset of $\varphi \in \text{Aut}(R)$ to the coset of the automorphism $[(r_{ij})_{i,j} \mapsto (\varphi(r_{ij}))_{i,j}] \in \text{Aut}(W)$, but this bijection is also a group homomorphism.

Remark 7.9. (i) In general, even when $M$ is a progenerator, very little can be said about the structure of $K_n$. For example, the base module $M$ of the regular bilinear form $b : M \times M \to K$ constructed in Example 2.4 is a progenerator. Since $b$ is regular (routine), it is similar to $b_{\alpha}$ for some $\alpha \in \text{End}^{-}(\text{End}_R(M))$ (Theorem 5.7), what allows us to identify $K$ with $K_{b_{\alpha}}$. However, $(K_1)^m$ is not free for all $m \in \mathbb{N}$, in contrast to Proposition 7.8 and Theorem 7.8.\hfill

(ii) It is possible to prove Theorem 7.8 without using the form $b_{\alpha}$ explicitly: Identify $M = R^n$ with $M_n(R)e_{11}$ and define $b : M \times M \to K := e_{11}^n M_n(R)e_{11}$ by $b(xe_{11}, ye_{11}) = e_{11}^n x^w y e_{11}$. Prove directly that $b$ is right regular and $(K_1)^m \cong R^n$, and proceed with $b, K$ in place of $b_{\alpha}, K_{\alpha}$.

(iii) Let $\alpha \in \text{End}^{-}(W)$ with $W$ as in Theorem 7.8. If there is $\varphi \in \text{Inn}(W)$ such that $\varphi \circ \alpha$ is an involution, then there exists $\lambda \in W$ with $\lambda^2 \lambda = 1$ and $w^w \lambda = \lambda w$ for all $w \in W$ (write $\varphi(w) = wu^{-1}$ and take $\lambda = u^{-1} w^w$). The converse is false in general, but it is true when $W$ is semisimple (see [11] Pr. 2.1, for instance). Together with Theorem 7.8, this can be used to show that when $R$ is simple artinian, all involutions on $W = \text{End}_R(M)$ are induced by regular $\mu$-hermitian forms over $R$ ($\mu \in \text{Cent}(R)$). The details are left to the reader.

We finish this section by specializing to the case where $R$ is a field.

Proposition 7.10. Let $F$ be a field and let $V$ be a f.d. right $F$-vector-space. Let $W = \text{End}_F(V)$ and identify $F$ with $\text{Cent}(W)$. For all $\alpha \in \text{End}^{-}(W)$ we have:

(i) $\alpha(\text{Cent}(W)) \subseteq \text{Cent}(W)$, hence $\alpha$ can be understood as an (anti-)endomorphism of $F$. In addition, $\alpha|_F$ is bijective precisely when $\alpha$ is.

(ii) $K_\alpha$ is isomorphic to the standard double $F$-module of $(F, \alpha|_F)$.

(iii) There is a one-to-one correspondence between right regular bilinear forms on $V$, considered up to similarity, and anti-endomorphisms of $W$. More precisely, the form $b_\alpha$ is right regular and left injective. It is left regular if and only if $\alpha$ is bijective.

(iv) If $(\alpha|_F)^2 = \text{id}_F$, then $b_\alpha$ is similar to a classical regular bilinear form (i.e. a sesquilinear form) over $(F, \alpha|_F)$. That bilinear form is unique up to multiplying by a scalar (in $F^\times$). Conversely, if $\beta$ is an involution of $F$ and
(v) If $\alpha$ is an involution, then one of the following is true:

(1) $\alpha|_F = \text{id}_F$ and $b_\alpha$ is similar to a symmetric bilinear form over $F$.
(2) $\alpha|_F = \text{id}_F$ and $b_\alpha$ is similar to an alternating bilinear form over $F$.
(3) $\alpha|_F \neq \text{id}_F$ and $b_\alpha$ is similar to a $1$-hermitian form over $(F, \alpha|_F)$.

If moreover $\text{char } F \neq 2$, then precisely one of the above is true. (By definition, the cases (1),(2),(3) correspond to the cases where $\alpha$ is orthogonal, symplectic or unitary, respectively.)

Proof. (i) Since $b_\alpha$ is simple, $\ker \alpha = 0$. Fix some matrix units $\{c^\alpha_{ij}\}$ in $W$. Then $\{c^\alpha_{ij}\}$ are also matrix units for $W$ and it is easy to see that every element that commutes with $\{c^\alpha_{ij}\}$ lies in $\text{Cent}(W)$. As $\alpha(\text{Cent}(W)) \subseteq \text{Cent}(W)$, we get that $\alpha(\text{Cent}(W)) \subseteq \text{Cent}(W)$. If $\alpha$ is bijective, then by the same argument, $\alpha^{-1}(\text{Cent}(W)) \subseteq \text{Cent}(W)$, which implies $(\alpha|_F)^{-1} = \alpha^{-1}|_F$. On the other hand, if $\alpha|_F$ is bijective, then $\alpha(W)$ is a $F$-subalgebra of $W$ (since $\alpha(F) = F$) of dimension $(\dim V)^2$ (since $\ker \alpha = 0$). Thus, $\alpha$ is surjective, and hence bijective.

(ii) By Theorem 7.8, $K_\alpha$ is isomorphic to the standard double $F$-module of $(F, \gamma)$ for some $\gamma \in \text{End}^{-}(F)$. (In fact, $\gamma$ is unique since $\text{Inn}(F) = \{\text{id}_F\}$.) For all $a \in F = \text{Cent}(W)$ and $x, y \in V$, we have $b_\alpha(x, y) \circ_0 a = b_\alpha(ax, y) = b_\alpha(x, a^\alpha y) = b_\alpha(x, y) \circ_1 a^\alpha$. This forces $\gamma = \alpha|_F$, as required.

(iii) Observe that any $\alpha \in \text{End}^{-}(W)$ is injective because $W$ is simple. Everything now follows from Theorem 5.7 and Corollary 7.6.

(iv) Assume $(\alpha|_F)^2 = \text{id}_F$. Then $\alpha|_F$ is bijective, hence $\alpha$ is bijective (by (i)). Thus, $b_\alpha$ is regular (by (iii)) and $K_\alpha$ is the standard double $F$-module of $(F, \alpha|_F)$, which, according to Example 2.3, means that $b_\alpha$ is similar to a classical bilinear form $b : V \times V \rightarrow F$. That $b$ is unique to multiplying by an element of $F^\times$ easily follows from the fact that $b$ is unique up to similarity. Conversely, if $b_\alpha$ is similar to a regular hermitian form $b : V \times V \rightarrow F$ over $(F, \beta)$, then for all $a \in F$ and $x, y \in V$, $b(ax, y) = a^\beta b(x, y) = b(x, ya^\beta)$. Thus, $\alpha|_F = \beta$, as required.

(v) That $\alpha$ is an involution implies $(\alpha|_F)^2 = \text{id}_F$, hence by (iv), $b_\alpha$ is similar to a classical bilinear form $b : V \times V \rightarrow F$ over $(F, \alpha|_F)$. Identify $K_\alpha$ with $F$ via the similarity $b_\alpha \sim b$ and consider $\theta := \theta_\alpha$ (see section 2) as an involution of the standard double $F$-module of $(F, \alpha|_F)$ (rather than $K_\alpha$). Let $\lambda = 1^\theta$. Then for all $k \in F$, $k^\theta = (1 \circ_0 k)^\theta = k \circ_0 \lambda k \lambda^{-1}$ and $1 = 1^\theta = \lambda^\theta = \lambda\lambda^{-1}$. As $b$ is $\theta$-symmetric, this means that $b$ is $\lambda$-hermitian. Now, if $\alpha|_F = \text{id}_F$, then $\lambda^2 = 1$, hence $\lambda = 1$ or $\lambda = -1$ which implies (1) or (2), respectively. If $\alpha|_F \neq \text{id}_F$, then by Hilbert’s Theorem 90, there is $u \in F$ with $u^\alpha u^{-1} = \lambda$. Define $b'(x, y) = ub(x, y)$. Then $b'$ is similar to $b$ and it is routine to check that $b'$ is a 1-hermitian form over $(F, \alpha|_F)$, i.e. (3) holds.

Remark 7.11. The results of the Proposition 7.10 are not typical; in general, even when $M$ is free and $R$ has an involution, there is no “nice” characterization of the anti-endomorphisms of $W$ that correspond to classical bilinear forms.

8. On a Result of Osborn

In this section, we use the results of sections 5 and 6 to prove a variant of a theorem of Osborn. Throughout, $\text{Jac}(R)$ is the Jacobson radical of the ring $R$.

Theorem 8.1 (Osborn). Let $(W, \alpha)$ be a ring with involution such that $2 \in W^\times$ and every element $w \in W$ with $w^\alpha = w$ is either a unit or nilpotent. Let $\alpha'$ denote the induced involution on $W/\text{Jac}(W)$. Then $\text{Jac}(W) \cap \{w \in W : w^\alpha = w\}$ consists of nilpotent elements and one of the following holds:

(i) $W/\text{Jac}(W)$ is a division ring.
(ii) $W/Jac(W) \cong D \times D^{op}$ for some division ring $D$ and under that isomorphism $\alpha'$ exchanges $D$ and $D^{op}$.

(iii) $W/Jac(W) \cong M_2(F)$ for some field $F$ and under that isomorphism $\alpha'$ is a symplectic involution (i.e. it is induced by a classical alternating bilinear form over $F$; see [4, Ch. 1]).

Proof. See [8, §4].

Osborn’s result has several generalizations (see papers related to [8]) and his proof is based on Jordan algebras. We will prove Osborn’s Theorem in the case $W$ is semisimple, but under milder assumptions. Our proof actually implies Osborn’s Theorem in the case $\alpha$-invariant.

Proof. Let $(W, \alpha)$ be a ring with involution such that $W$ is semisimple and the only $\alpha$-invariant idempotents in $W$ are 0 and 1. Then one of the following holds:

1. $W$ is a division ring.
2. $W \cong D \times D^{op}$ for some division ring $D$ and under that isomorphism $\alpha$ exchanges $D$ and $D^{op}$.
3. $W \cong M_2(F)$ for some field $F$ and under that isomorphism $\alpha$ is a symplectic involution.

Proof. We may assume $W$ is not the zero ring. Let $\{e_1, \ldots, e_n\}$ be the primitive idempotents of $\text{Cent}(W)$. Then $\alpha$ permutes $e_1, \ldots, e_n$. Assume $n > 1$. Then we have $e_i \neq e_i^\alpha$ for all $i$. This implies $e_1 + e_1^\alpha$ is a nonzero $\alpha$-invariant idempotent, hence $e_1 + e_1^\alpha = 1$. Thus, $n = 2$ and $e_1^\alpha = e_2$. Write $W_1 = e_1W$. Then $W \cong W_1 \times W_2$ and $\alpha$ exchanges $W_1$ and $W_2$. If $0 \neq e \in W_1$ is an idempotent, then $e^\alpha \in W_2$, hence $e + e^\alpha$ is a non-zero $\alpha$-invariant idempotent, hence $e + e^\alpha = 1$ and $e = 1W_1$. This means $W_1$ is a simple artinian ring with no non-trivial idempotents, hence it is a division ring. As $W_2 \cong W_1^{op}$ via $\alpha$, (ii) holds.

Now assume $n = 1$. Then $W$ is simple artinian and we can write $W = \text{End}_D(V)$ for some division ring $D$ and a f.d. right $D$-vector-space $V$. Let $b = b_\alpha$, $K = K_\alpha$ and $\theta = \theta_\alpha$. Then $b : V \times V \rightarrow K$ is a regular $\theta$-symmetric bilinear form by Theorem 5.7. Moreover, by Proposition 7.4 $\dim_D K_1 = 1$.

We claim that if $V = U_1 \oplus U_2$ with $b(U_1, U_2) = b(U_2, U_1) = 0$, then $U_1 = 0$ or $U_2 = 0$. Indeed, let $e \in \text{End}_D(V)$ be the projection onto $U_1$ with kernel $U_2$. It is straightforward to check that $b(ex, y) = b(ex, ey) = b(x, ey)$, hence $e^\alpha = e$. Therefore, $e = 1$ or $e = 0$, so $U_1 = V$ or $U_1 = 0$.

Assume there is $x \in V$ such that $b(x, x) \neq 0$ and define

$$L = x^\perp = \{y \in V \mid b(x, y) = 0\}.$$ 

We claim that $V = L \oplus xD$. Clearly $x \notin L$, hence $xD \cap L = 0$. On the other hand, for all $v \in V$, there exists $d \in D$ such that $b(x, x) \circ_1 d = b(x, v)$ (because $\dim_D K_1 = 1$), hence $b(x, v - xd) = b(x, v) - b(x, x) \circ_1 d = 0$ and this implies $v = xd + (v - xd) \in xD + L$. Now, since $b$ is $\theta$-symmetric $b(L, xD) = b(xD, L)^\theta = 0$, so by the previous paragraph, $L = 0$. But this means $\dim_D V = 1$, so $W \cong D$ and (i) holds.

We may now assume that $b(x, x) = 0$ for all $x \in V$. Then $0 = b(x + y, x + y) = b(x, y) + b(y, x) = b(x, y) + b(x, y)^\theta$ for all $x, y \in V$, hence $\theta = -\text{id}_K$. Furthermore, for all $x, y \in V$ and $a \in D$ we have $b(x, y) \circ_0 a = b(xa, y) = -b(y, xa) = -b(y, x) \circ_1 a = b(x, y) \circ_1 a$, hence $\circ_0 = \circ_1$. This implies that for any $0 \neq k \in K$ and $a, b \in D$, $w^\alpha = w$. Some texts use $\alpha$-symmetric instead of $\alpha$-invariant.
we have $k \odot (ab) = (k \odot a) \odot b = (k \odot b) \odot a = (k \odot b) \odot (ba)$, hence $ab = ba$. Therefore, $D$ is a field and $K$ is isomorphic as a double $D$-module to the standard double module of $(D, \text{id}_D)$. As $b(x, x) = 0$ for all $x \in V$, $b$ is a classical alternating bilinear form. We are thus finished if we prove that $\dim_P V = 2$ (as this would imply $W \cong \text{M}_2(D)$, as in (iii)). However, this follows from the well-known fact that every regular alternating form over a field is the orthogonal sum of 2-dimensional alternating forms (and $b$ cannot be the orthogonal sum of two non-trivial forms, as argued above).

Theorem 8.2 is false for rings $W$ with $\text{Jac}(W) = 0$ (and hence it does not imply Osborn’s Theorem for general rings). For example, for any simple domain $S$ which is not a division ring (e.g. a Weyl algebra), the ring $W = S \times S^{\text{op}}$ has an involution and satisfies $\text{Jac}(W) = 0$, but it fails to satisfy any of the conditions (i)–(iii) of Theorem 8.2.

9. Counterexamples

This last section presents counterexamples. We begin with demonstrating that $b_\alpha$ can be degenerate even when the base ring is a finite dimensional algebra over a field.

Example 9.1. Let $F$ be a field and let $R$ be the commutative subring of $\text{M}_3(F)$ consisting of matrices of the form:

$$
\begin{bmatrix}
a & b & a \\
b & c & a \\
a & a & a 
\end{bmatrix}.
$$

Let $x = e_{21}$ and $y = e_{31}$ (where $\{e_{ij}\}$ are the standard matrix units of $\text{M}_3(F)$). Then $\{1, x, y\}$ is an $F$-basis of $R$. Consider the elements of $M = F^3$ as row vectors and let $\{e_1, e_2, e_3\}$ be the standard $F$-basis of $M$. Then $M$ is naturally a right $R$-module (the action of $R$ being matrix multiplication on the right) and a straightforward computation shows that $\text{End}(M_R) \cong R$, i.e. all $R$-linear maps $f : M \to M$ are of the form $m \mapsto mr$ for some $r \in R$. Let $\alpha = \text{id}_R \in \text{Aut}^{-}(R)$. Then $M \otimes_R M$ is just $M \otimes M$. We now have:

$$
\begin{align*}
b_\alpha(e_1, e_1) &= e_1 \otimes e_1 = e_2 x \otimes e_1 = e_2 x \otimes e_1 = 0, \\
b_\alpha(e_2, e_1) &= e_2 \otimes e_1 = e_2 \otimes e_3 y = e_2 y \otimes e_3 = 0, \\
b_\alpha(e_3, e_1) &= e_3 \otimes e_1 = e_3 \otimes e_2 x = e_3 x \otimes e_2 = 0.
\end{align*}
$$

Therefore, $b_\alpha(M, e_1) = 0$, hence $b_\alpha$ is not right injective. In particular, the correspondence $[4]$ of Remark 8.3 fails for $M$.

The next example demonstrates that $b_\alpha$ can be injective even when it is not regular.

Example 9.2. Let $F$ be a field, let $R = F[s, t]$ and let $M = \langle s, t \rangle := sR + tR$. It is routine to verify that $\text{End}_R(M) \cong R$ (in the sense of the previous example). Let $\alpha = \text{id}_R \in \text{Aut}^{-}(\text{End}_R(M))$. We claim that $b_\alpha$ is injective, but not regular.

Indeed, we can identify $K_\alpha$ with $M \otimes_R M$ as above. Using this isomorphism, it is not hard (but tedious) to verify that the set

$$
\{s \otimes a, s \otimes t, t \otimes s, t \otimes t, s \otimes a, t^k | i, j, k \in \mathbb{N}, j + k > 2\}
$$

forms an $F$-basis to $K_\alpha$. Let $\varphi \in M^{[1]} = \text{Hom}_R(M, (K_\alpha)_0)$ be defined by $\varphi(x) = t \otimes a x$ (in fact $\varphi = \text{Ad}_{a}^{t}$). Then for all $x \in M$, we have $(\text{Ad}_{a}^{t} x)s = s \otimes a x \neq t \otimes s$, $s = \varphi(s)$, hence $\text{Ad}_{a}^{t} x \neq \varphi$. It follows that $\text{Ad}_{a}^{t}$ is not onto, hence $b_\alpha$ is not right regular.
Now consider the classical form \( b : M \times M \to R \) over \((R, \text{id}_R)\) given by \( b(x, y) = xy \). It is easy to check that \( b \) is nondegenerate and \( b(rx, y) = b(x, ry) \) for all \( r \in R \), hence there is a double \( R\)-module homomorphism \( f : K_\alpha \to K \) such that \( f \circ b_0 = b \). If there is \( x \in M \) such that \( b_0(M, x) = 0 \), then \( b(M, x) = f(b_0(M, x)) = f(0) = 0 \), implying \( x = 0 \). Thus, \( b_0 \) is right injective. That \( b_0 \) is left injective and not left regular follows by symmetry.

**Remark 9.3.** Assume \( b_0 \) is right injective but not right regular (e.g. as in Example 9.2). Then the correspondence in (4) fails. However, it is still possible to recover \( \alpha \) from \( b_0 \) in this case, because for all \( w \in W \), there exists a unique \( w^\alpha \in W \) such that \( b_0(wx, y) = b_0(x, w^\alpha y) \) for all \( x, y \in M \). This suggests the following definition: Call a bilinear form \( b : M \times M \to K \) right stable if for all \( w \in W := \text{End}_R(M) \), there exists a unique \( w^\alpha \in W \) such that \( b(wx, y) = b(x, w^\alpha y) \) for all \( x, y \in M \). The map \( w \mapsto w^\alpha \) is then a well-defined anti-endomorphism of \( W \). We can now ask whether there is a correspondence between stable forms on \( M \), up to some similarity, and elements of \( \text{Aut}^{-}(W) \). It turns out that the answer is "yes" in many cases, e.g. in the setting of Example 9.2. Moreover, it is possible that \( b_0 \) would be degenerate and stable. The details of these results will be published elsewhere.

The next two examples demonstrate what might happen when \( M \) is a generator, but \( \alpha \in \text{End}^{-}(\text{End}(M)) \) is not bijective. They imply that there need not be a one-to-one correspondence between the anti-endomorphisms of \( \text{End}(M) \) and the right regular forms on \( M \), despite the fact the the anti-auto-endomorphisms of \( \text{End}(M) \) correspond to regular forms in this case (Theorem 5.7). In addition, they imply that the injectivity of \( \alpha \) in Corollary 7.3 is essential.

**Example 9.4.** Let \( N \) be any nonzero torsion \( \mathbb{Z} \)-module and let \( M = \mathbb{Z} \oplus N \in \text{Mod}-\mathbb{Z} \). We consider the elements of \( M \) as column vectors. Then

\[
W := \text{End}_\mathbb{Z}(M) = \begin{bmatrix}
\text{End}(\mathbb{Z}_\mathbb{Z}) & \text{Hom}(N, \mathbb{Z}) \\
\text{Hom}(\mathbb{Z}, N) & \text{End}(N)\end{bmatrix} = \begin{bmatrix}
\mathbb{Z} & 0 \\
N & \text{End}(N)\end{bmatrix}.
\]

Note that \( M \) is a generator and \( e := [1 0] \in W \) is a projection from \( M \) onto \( \mathbb{Z} \).

Thus, we can identify \( M \) with

\[
We = \begin{bmatrix}
\mathbb{Z} & 0 \\
N & 0\end{bmatrix}.
\]

Define \( \alpha \in \text{End}^{-}(W) \) by \([a \ b \ c \ d]^\alpha = [a \ b \ c \ d] \pi\) where \( \pi \) is the image of \( a \in \mathbb{Z} \) in \( \text{End}(N) \). Then by Proposition 7.3, \( b_\alpha \) is similar to \( b : M \times M \to \text{End}_R(M) \) defined by \( b([x \ y \ 0 \ 1]) = [x \ 0 \ 0 \ 0] + [x \ 0 \ 0 \ 0] = [x \ 0 \ 0 \ 0] \). It is now easy to see that \( b \) is right injective but not left injective. In addition, \( b \) is not right regular. To see this, let \( 0 \neq f \in \text{End}(N) \) and note that the homomorphism \([x \ 0 \ 0 \ 0] \mapsto [0 \ 0 \ 0] \in \text{Hom}_\mathbb{Z}(M, (K_\alpha)\mathbb{Z}) \) does not lie in \( \text{im}(\text{Ad}_b) \).

**Example 9.5.** View \( N := \mathbb{Z}[\frac{1}{p}] / \mathbb{Z} \) as a \( \mathbb{Z}_p \)-module as in Example 8.3. Then \( \text{End}(N) = \mathbb{Z}_p \). Define \( M = \mathbb{Z}_p \oplus N \in \text{Mod-}\mathbb{Z}_p \) and consider the elements of \( M \) as column vectors. Then

\[
W := \text{End}_{\mathbb{Z}_p}(M) = \begin{bmatrix}
\text{End}(\mathbb{Z}_p) & \text{Hom}(N, \mathbb{Z}_p) \\
\text{Hom}(\mathbb{Z}_p, N) & \text{End}(N)\end{bmatrix} = \begin{bmatrix}
\mathbb{Z}_p & 0 \\
N & \text{End}(N)\end{bmatrix}.
\]

Let \( e = [1 0] \in W \). As in the previous example, we can identify \( M \) with \( We \). Define \( \alpha \in \text{End}^{-}(W) \) by \([x \ y \ z]^\alpha = [x \ y \ z]^\alpha \). Then by Proposition 7.3, \( K_\alpha \) is isomorphic to \( e^{\alpha}We = 0We = 0 \), so \( b_\alpha \) is the zero form!

Our last example demonstrates that a regular bilinear form \( b \) need not be similar to its generization, \( b_\alpha(b) \). The construction also shows several other interesting phenomena, which are pointed out at the end.
Example 9.6. Let $1 < n \in \mathbb{N}$ and let $F$ be a field. Denote by $R$ the ring of upper-triangular matrices over $F$. For $0 \leq i \leq n$, let $M_i$ denote the right $R$-module consisting of row vectors of the form

$$(0, \ldots, 0, \ast, \ldots, \ast) \in F^n$$

with $R$ acting by matrix multiplication on the right. It is not hard to verify that $\dim_F \text{Hom}_R(M_n, M_i) = 1$ for all $0 \leq i < n$. In addition, $\text{End}_R(M_n/M_i) = F$ for all $0 \leq i < n$. The latter implies the following fact, to be used later, which follows from the Krull-Schmidt Theorem (see [14, Th. 2.9.17]): Assume that $a_0 + \cdots + a_n = b_0 + \cdots + b_n$ and $\bigoplus_{i=0}^n (M_n/M_i)^{a_i} \cong \bigoplus_{i=0}^n (M_n/M_i)^{b_i}$, then $a_i = b_i$ for all $0 \leq i \leq n$. (The assumption $a_0 + \cdots + a_n = b_0 + \cdots + b_n$ is needed because $M_n/M_i$ is the zero module.)

Let $T$ be the transpose involution on $K := M_n(F)$. We make $K$ into a double $R$-module by defining

$$A \circ_0 B = B^T A \quad \text{and} \quad A \circ_1 B = AB$$

for all $A \in K$ and $B \in R$. Then $b : M_n \times M_n \to K$ defined by $b(x, y) = x^T y$ is a bilinear form. For $0 \leq u, v \leq n$, let $K_{u,v}$ denote the matrices $A = (A_{ij}) \in K$ for which $A_{ij} = 0$ if $i \leq u$ or $j \leq v$. For example, when $n = 3$, $K_{1,2}$ and $K_{2,0}$ consist of matrices of the forms

$$
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
, 
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\ast & \ast & \ast
\end{bmatrix}
$$

respectively. Then $K_{u,v}$ is a sub-double-$R$-module of $K$, hence $K/K_{u,v}$ is a double $R$-module, and $b_{u,v} : M_n \times M_n \to K/K_{u,v}$ defined by $b(x, y) = x^T y + K_{u,v}$ is a bilinear form.

Recall that we use $K_i$ to denote the $R$-module obtained from $K$ by letting $R$ act via $\otimes_i$. Then $(K/K_{u,v})_0 \cong M_n^u \oplus (M_n/M_{n-u})^{n-u}$ (the summands are the columns of $K/K_{u,v} = M_n(F)/K_{u,v}$) and $(K/K_{u,v})_1 \cong M_n^u \oplus (M_n/M_{n-u})^{n-u}$ (the summands are the rows of $K/K_{u,v} = M_n(F)/K_{u,v}$). Therefore, by the fact recorded above, for $0 \leq u, u', v, v' < n$ and $i \in \{0, 1\}$, we have

$$(K/K_{u,v})_i \cong (K/K_{u',v',i}) \quad \iff \quad (u, v) = (u', v')$$

and

$$(K/K_{u,v})_{1-i} \cong (K/K_{u',v',1-i}) \quad \iff \quad (u, v) = (v', u').$$

We now claim that $b_{u,v}$ is right regular when $u > 0$. Indeed, it is easy to check that $b_{u,v}$ is right injective in this case. In addition, we have

$$\dim_F M^{[1]} = \dim_F \text{Hom}_R(M_n, (K/K_{u,v})_0) = v \dim_F \text{Hom}_R(M_n, M_n) + (n - v) \dim_F \text{Hom}_R(M_n, M_{n-u}) = n.$$

Therefore, dimension considerations imply $A d_{b_{u,v}}^v$ is bijective, i.e., $b_{u,v}$ is right regular. Similarly, $b_{u,v}$ is left regular when $v > 0$.

Now, observe that $\text{End}_R(M_n) \cong F$ and $b_{u,v}(ax, y) = a b_{u,v}(x, y)$ for all $x, y \in M_n$ and $a \in F$. Therefore, provided $b_{u,v}$ is right regular, its corresponding anti-endomorphism is $\text{id}_F$. It follows that the forms $\{b_{u,v} \mid 0 < u\}$ are right regular and have the same corresponding anti-automorphism (which is in fact an involution), and hence the same generalization, which is regular since $M_n$ is f.g. projective (Corollary 5.5). In addition, the double $R$-modules $K/K_{u,v}$ for $0 < u, v < n$ are pairwise non-isomorphic, hence the forms $\{b_{u,v} \mid 0 < u, v < n\}$ are pairwise non-similar. This means that the forms $\{b_{u,v} \mid 0 < u, v < n\}$ are regular, pairwise non-similar,
but nevertheless have the same generization, which is regular. In particular, necessarily all but possibly one of them is not similar to its generization. Let us show that this is in fact true for all of these forms.

We claim that common generization of \( \{ b_{u,v} \mid 0 < u \} \) is similar to \( b \). Indeed, let \( \alpha = \text{id}_F \in \text{End}^-(F) \). Then \( \dim_F K_n = \dim_F M_n \otimes \alpha M_n = \dim_F M_n \otimes \alpha M_n = n^2 \).

The universality of \( b \) implies that there is a double \( R \)-module homomorphism \( f : K_n \to K \), that must be onto since \( \text{im}(b) = K \). (Recall that \( \text{im}(b) \) was defined to be the additive group spanned by \( \{ b(x,y) \mid x, y \in M_n \} \).) As \( f \) is clearly \( F \)-linear and \( \dim_F K = n^2 \), dimension considerations imply that \( f \) is an isomorphism. Thus, \( b \) is the generization of all the forms \( \{ b_{u,v} \mid 0 < u \} \), and since \( \dim_F K > \dim_F K/K_{u,v} \) whenever \( u, v < n \), we see that \( b_{u,v} \sim b \) for all \( 0 < u, v < n \), as required.

Note that since \( M_n \) is f.g. projective, there is a one-to-one correspondence between \( \text{End}^-(\text{End}_R(M_n)) \) and the right generic forms on \( M_n \), up to similarity (Remark 6.5 and Corollary 6.6). However, we have just shown that there is no correspondence between \( \text{End}^-(\text{End}_R(M_n)) \) the right regular forms on \( M_n \) (since the maps \( \alpha \mapsto b_{\alpha} \) and \( b \mapsto \alpha(b) \) of section 6.3 are not inverse to each other).

Finally, let \( 0 < u, v < n \) be distinct. Then \( b_{u,v} \) is regular, \( \alpha(b_{u,v}) \) is an involution, but \( \text{im}(b) = K/K_{u,v} \) does not have an anti-automorphism since \( (K/K_{u,v}) \not\cong (K/K_{u,v})^1 \). Furthermore, \( b_{u,0} \) is right regular, but left degenerate. This shows that Proposition 6.7 fails for non-generic forms.

We could neither find nor contradict the existence of the following:

- An anti-automorphism \( \alpha \) such that \( b_{\alpha} \) is right regular but not left regular. (In this case \( \alpha^2 \) cannot be inner by Corollary 6.3)
- A generator \( M \) and an injective \( \alpha \in \text{End}^-(\text{End}_R(M)) \) such that \( b_{\alpha} \) is not right injective.

ACKNOWLEDGMENTS

I deeply thank to my supervisor, Uzi Vishne, for guiding me through the research, to Lance Small for a very beneficial conversation, and to Jean-Pierre Tignol for his comments on earlier versions.

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EINSTEIN INSTITUTE OF MATHEMATICS, HEBREW UNIVERSITY OF JERUSALEM

E-mail address: uriya.first@gmail.com