Superintegrability on \(N\)-dimensional curved spaces: Central potentials, centrifugal terms and monopoles

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Abstract

The \(N\)-dimensional Hamiltonian

\[
\mathcal{H} = \frac{1}{2f(|q|)^2} \left\{ p^2 + \mu^2 q^2 + \sum_{i=1}^{N} \frac{b_i}{q_i^2} \right\} + U(|q|)
\]

is shown to be quasi-maximally superintegrable for any choice of the functions \(f\) and \(U\). This result is proven by making use of the underlying \(sl(2,\mathbb{R})\)-coalgebra symmetry of \(\mathcal{H}\) in order to obtain a set of \((2N-3)\) functionally independent integrals of the motion, that are explicitly given. Such constants of the motion are “universal” since all of them are independent of both \(f\) and \(U\). This Hamiltonian describes the motion of a particle on any \(ND\) spherically symmetric curved space (whose metric is specified by \(f\)) under the action of an arbitrary central potential \(U\), and includes simultaneously a monopole-type contribution together with \(N\) centrifugal terms that break the spherical symmetry. Moreover, we show that two appropriate choices for \(U\) provide the “intrinsic” oscillator and the KC potentials on these curved manifolds. As a byproduct, the MIC–Kepler, the Taub-NUT and the so called multifold Kepler systems are shown to belong to this class of superintegrable Hamiltonians, and new generalizations thereof are obtained. The KC and oscillator potentials on \(N\)-dimensional generalizations of the four Darboux surfaces are discussed as well.

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1 Introduction

An $N$-dimensional (ND) Hamiltonian $\mathcal{H}$ is said to be (Liouville) integrable [1] if it admits the maximum number $(N - 1)$ of functionally independent and Poisson-commuting global first integrals. Similarly, an integrable Hamiltonian $\mathcal{H}$ is called superintegrable if there exists an additional set of $k$ independent constants of motion. It is well-known that $k \leq N - 1$, and in the case where this bound is saturated ($k = N - 1$), the Hamiltonian $\mathcal{H}$ is called maximally superintegrable (hereafter MS). When $k = N - 2$, we shall say that $\mathcal{H}$ is quasi-maximally superintegrable (QMS) [2, 3] and the Hamiltonian $\mathcal{H}$ is only “one integral away” from being MS.

Superintegrable systems have been thoroughly studied because of their significant connections with generalized symmetries [4, 5], isochronous potentials [6, 7], and separability of the associated Hamilton–Jacobi and Schrödinger equations [8, 9, 10]. When one considers a Hamiltonian system in arbitrary dimension $N$ and all the integrals of motion are imposed to be quadratic in the momenta, the list of known $ND$ MS systems becomes strikingly short. To the best of our knowledge, the only known instances are:

- The free motion on the simply connected spaces of constant curvature (see, e.g., [11, 12] for a unified approach in terms of the curvature) which is given by the geodesic flow on these spaces.

- The generalized Kepler–Coulomb (KC) system on the simply connected spaces of constant curvature [12, 13, 14, 15, 16], which is a superposition of the ND KC potential with $N - 1$ “centrifugal” terms.

- The Smorodinsky–Winternitz system on the simply connected spaces of constant curvature [11, 12, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25], that is, the ND harmonic oscillator potential on such spaces together with $N$ centrifugal terms.

- The free motion and the Smorodinsky–Winternitz system on the Darboux space III [26], which is the superposition of an “intrinsic” oscillator and $N$ centrifugal potentials on an ND generalization of the Darboux surface of type III, which is a Riemannian manifold of nonconstant curvature.

Note that the above MS systems on spaces of constant curvature do not only include the flat ND Euclidean case, but also the spherical, hyperbolic, Minkowskian and (anti-)de Sitter spaces (for the relativistic ones see [12, 16]).

On the other hand, if the condition that all the integrals of the motion have to be quadratic in momenta is suppressed, the list of ND MS systems is enlarged with the rational and hyperbolic Calogero–Sutherland–Moser models of type $A_{N-1}$ [27, 28, 29, 30], the nonisotropic oscillator with rational frequencies [31], the nonperiodic Toda lattice [32, 33], the Benenti systems [34, 35] and, very recently, the KC potential on the 3D Euclidean space plus three centrifugal terms [36].
From this viewpoint it is quite natural to wonder whether the number of ND QMS systems is much larger than the number of MS ones (obviously, any MS system is by construction a QMS one). The answer to this question has been recently answered affirmatively in [37] as follows: geodesic flows on any ND spherically symmetric (curved) space define always a QMS Hamiltonian. Moreover, all these systems present the same “universal” and explicit set of \((2N-3)\) integrals in involution with the Hamiltonian that are quadratic in the momenta. The proof of this result is based on the common \(\mathfrak{sl}(2,\mathbb{R})\) Poisson coalgebra symmetry [38, 39, 40] of all these systems, a fact that was also shown for the quadratically MS systems on Riemannian spaces listed above [2, 3, 26, 37, 38].

The aim of this paper is to use the same \(\mathfrak{sl}(2,\mathbb{R})\)-coalgebra symmetry in order to show that such infinite family of QMS free systems in arbitrary dimension can be further enlarged by including the Hamiltonians

\[
\mathcal{H} = \frac{1}{2f(|q|^2)} \left\{ \frac{p^2}{q^2} + \frac{\mu^2}{q^2} + \sum_{i=1}^{N} \frac{b_i}{q_i^2} \right\} + U(|q|), \quad (|q| = \sqrt{\mathbf{q}^2}), \tag{1.1}
\]

where \(f\) and \(U\) are smooth functions and \(\mu, b_1, \ldots, b_N\) are real constants. The proof that this Hamiltonian is indeed QMS is presented in section 2. We stress that, despite that the spherical symmetry is broken by the “centrifugal terms” \(b_i/q_i^2\), the overall \(\mathfrak{sl}(2,\mathbb{R})\)-coalgebra symmetry holds and, as a consequence, the same integrability properties of the geodesic flows are preserved when the potentials are considered. Next sections will be devoted to the study of certain particular cases of (1.1), both from the physical and from the mathematical point of view.

Firstly, note that the term \(\frac{1}{2}p^2/f(|q|^2)\) is just the kinetic energy that defines the geodesic flow on a spherically symmetric space, which is generically of nonconstant curvature (\(f\) playing the role of a “conformal factor” of the flat metric). These are just the aforementioned geodesic flows on ND spherically symmetric spaces [37]. In particular, suitable choices for the function \(f\) allow us to recover the three classical Riemannian spaces of constant curvature [41], ND generalizations [26, 37] of the Darboux surfaces [42, 43] and the Iwai–Katayama spaces [44] (which are themselves a generalization of the Taub–NUT metric [45]). All these QMS geodesic flows are fully described in section 3 by using the \(\mathfrak{sl}(2,\mathbb{R})\)-coalgebra framework, where as a new result the complete integrability and separability of the free Hamiltonian (for any \(f\)) is explicitly shown by using spherical coordinates.

Secondly, section 4 is devoted to the study of the central potential \(U\) and to the analysis of its separability properties which hinges on its Lie–Poisson coalgebra symmetry. In section 5, the ND versions of the potential \(U\) defining the “intrinsic” KC and oscillator potentials on generic spherically symmetric spaces is presented by extending to arbitrary dimension (see [26]) the definitions of such potentials on 3D Riemannian spaces previously introduced in [46, 47, 48, 49, 50]. Moreover, by taking into account that the \(\mu^2/(2f(|q|^2)q^2)\) contribution can be interpreted as a Dirac monopole-type term [44, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61] and by adding \(N\) centrifugal terms to
the KC/oscillator potential $U$, new generalized KC and oscillator systems on arbitrary spherically symmetric spaces are introduced. These general results are explicitly illustrated in section 6 by studying in detail the resulting potentials for some specific spaces. In particular, known superintegrability results for the MIC–Kepler [44, 56, 60, 62, 63, 64] and the Taub-NUT systems [44, 59, 65, 66, 67, 68, 69, 70, 71, 72] are recovered within our framework, and we also derive further results for new Hamiltonians such as the proper KC and oscillator potentials on the $ND$ Darboux spaces introduced in [37]. This way new QMS systems on $ND$ curved spaces are presented and analysed under a unified integrability setting, showing that the integrability properties of several well-known systems rely on a common $\mathfrak{sl}(2, \mathbb{R})$-coalgebra symmetry.

2 The Hamiltonian $\mathcal{H}$ is QMS

The following result holds:

**Theorem 1.** Let $\{q, p\} = \{(q_1, \ldots, q_N), (p_1, \ldots, p_N)\}$ be $N$ pairs of canonical variables. The $ND$ Hamiltonian

$$
\mathcal{H} = \frac{1}{2f(|q|)^2} \left\{ p^2 + \mu^2 \sum_{i=1}^{N} \frac{b_i}{q_i^2} \right\} + \mathcal{U}(|q|), \quad (|q| = \sqrt{q^2}),
$$

where $f$ and $\mathcal{U}$ are smooth functions and $\mu, b_1, \ldots, b_N$ are real constants, is quasi-maximally superintegrable. The $(2N - 3)$ functionally independent and “universal” integrals of the motion for $\mathcal{H}$ are given by

$$
C^{(m)} = \sum_{1 \leq i < j}^{m} \left\{ (q_i p_j - q_j p_i)^2 + \left( b_i \frac{q_i^2}{q_j^2} + b_j \frac{q_j^2}{q_i^2} \right) \right\} + \sum_{i=1}^{m} b_i,
$$

$$
C_{(m)} = \sum_{N-m+1 \leq i < j}^{N} \left\{ (q_i p_j - q_j p_i)^2 + \left( b_i \frac{q_i^2}{q_j^2} + b_j \frac{q_j^2}{q_i^2} \right) \right\} + \sum_{i=N-m+1}^{N} b_i, \quad (2.2)
$$

where $m = 2, \ldots, N$ and $C^{(N)} = C_{(N)}$. Moreover, the sets of $N$ functions $\{\mathcal{H}, C^{(m)}\}$ and $\{\mathcal{H}, C_{(m)}\}$ $(m = 2, \ldots, N)$ are in involution.

**Proof.** Let us define the functions

$$
J_- = q^2, \quad J_3 = q \cdot p, \quad J_+ = p^2 + \sum_{i=1}^{N} \frac{b_i}{q_i^2}, \quad (2.3)
$$

where

$$
q^2 = \sum_{i=1}^{N} q_i^2, \quad p^2 = \sum_{i=1}^{N} p_i^2, \quad q \cdot p = \sum_{i=1}^{N} q_i p_i, \quad b_i \in \mathbb{R}.
$$
It is immediate to check that these three functions close the \( \mathfrak{sl}(2, \mathbb{R}) \) Poisson coalgebra
\[
\{J_3, J_+\} = 2J_+, \quad \{J_3, J_-\} = -2J_-, \quad \{J_-, J_+\} = 4J_3.
\]
(2.4)
Moreover, \( \mathcal{H} \) can be written as the following function of the \( \mathfrak{sl}(2, \mathbb{R}) \)-coalgebra generators:
\[
\mathcal{H} = \frac{J_+ + \mu^2 J_-^{-1}}{2f(\sqrt{J_-})^2} + \mathcal{U}(\sqrt{J_-}).
\]
(2.5)
As a consequence of this coalgebra symmetry, \( \mathcal{H} \) will Poisson-commute with all the integrals \( C^{(m)} \) and \( C_{(m)} \) that respectively come from the left and right \( m \)th-coproducts of the Casimir function for \( \mathfrak{sl}(2, \mathbb{R}) \). This coalgebra symmetry also ensures that the sets of \( N \) functions \( \{\mathcal{H}, C^{(m)}\} \) and \( \{\mathcal{H}, C_{(m)}\} (1 < m \leq N) \) are in involution (see [3, 38, 40] for explicit proofs of all these statements).

In this respect, we recall that any Hamiltonian
\[
\mathcal{H} = \mathcal{H}(J_-, J_+, J_3),
\]
(2.6)
defined as a smooth function on the \( \mathfrak{sl}(2, \mathbb{R}) \) generators is always QMS, and has the same set of “universal” integrals (2.2) that are obtained through the \( m \)D symplectic realization of the Casimir function of \( \mathfrak{sl}(2, \mathbb{R}) \). This underlying coalgebraic structure can be interpreted as a generalization of the spherical symmetry, which is recovered when all \( b_i = 0 \).

Finally, note that the centrifugal terms \( b_i/q_i^2 \) come from the fact that \( \mathfrak{sl}(2, \mathbb{R}) \) allows us to add such a contribution in the corresponding symplectic realization of the \( J_+ \) generator. Therefore, from an algebraic viewpoint these centrifugal contributions are directly related to the kinetic energy term.

## 3 Superintegrability of geodesic flows

The metric of any \( ND \) spherically symmetric space \( \mathcal{M} \) can be written as
\[
ds^2 = f(|q|^2)\,dq^2 = f(r)^2(dr^2 + r^2d\Omega^2),
\]
(3.7)
which is expressed in terms of the following coordinate systems:

- Generic coordinates \( q = (q_1, \ldots, q_N) \) (with \( dq^2 = \sum_{i=1}^N dq_i^2 \)), which in our approach we shall identify with those appearing in the symplectic representation (2.3).
- Spherical coordinates with a radial-type variable \( r = |q| \in \mathbb{R}^+ \) (which does not usually coincide with the geodesic distance) and \( N - 1 \) ordinary angular variables \( \theta_j \in [0, 2\pi) (j = 1, \ldots, N - 1) \) with \( d\Omega^2 \) being the standard metric on the unit \( (N - 1)D \) sphere
  \[
d\Omega^2 = \sum_{j=1}^{N-1} d\theta_j^2 \prod_{k=1}^{j-1} \sin^2 \theta_k.
\]
(3.8)
In both coordinate systems $f(|q|) = f(r)$ is any smooth function, usually interpreted as a conformal factor of the Euclidean metric $ds^2 = dq^2$. We stress that only in the Euclidean case $q$ can be interpreted as Cartesian coordinates. The relations between both sets of coordinates read

$$q_j = r \cos \theta_j \prod_{k=1}^{j-1} \sin \theta_k, \quad 1 \leq j < N, \quad q_N = r \prod_{k=1}^{N-1} \sin \theta_k,$$

where hereafter any product $\prod_l^m$ such that $l > m$ is assumed to be equal to 1. The scalar curvature of (3.7) turns out to be

$$R = -\frac{(N-1)^2 f''(r)}{f(r)^2} + 2 \frac{(N-1) f'(r) - (N-2) f'(r)^2}{f(r)^2},$$

where $f'(r) = df/dr$ and $f''(r) = d^2f/dr^2$. Therefore, we are indeed dealing with $N$D spaces with nonconstant curvature.

### 3.1 Free Hamiltonians

The metric (3.7) provides the free Lagrangian, which characterizes the geodesic motion of a particle on $\mathcal{M}$ in terms of the velocities $\dot{q}$ or $(\dot{r}, \dot{\theta}_j)$:

$$\mathcal{T} = \frac{1}{2} f(|q|)^2 \dot{q}^2 = \frac{1}{2} f(r)^2 \left( \dot{r}^2 + r^2 \sum_{j=1}^{N-1} \dot{\theta}_j^2 \prod_{k=1}^{j-1} \sin^2 \theta_k \right).$$

Then the canonical momenta $p, p_r, p_{\theta_j}$, conjugate to $q, r, \theta_j$, can be obtained through a Legendre transformation ($p_i = \partial \mathcal{T} / \partial \dot{q}_i$)

$$p = f(|q|)^2 \dot{q}, \quad p_r = f(r)^2 \dot{r}, \quad p_{\theta_j} = f(r)^2 r^2 \dot{\theta}_j \prod_{k=1}^{j-1} \sin^2 \theta_k.$$ (3.12)

From (3.9) and (3.12) the relations between both sets of momenta $p$ and $p_r, p_{\theta_j}$ are found to be

$$p_j = \prod_{k=1}^{j-1} \sin \theta_k \cos \theta_j p_r + \frac{\cos \theta_j}{r} \sum_{l=1}^{j-1} \prod_{k=l+1}^{j-1} \sin \theta_k \cos \theta_l p_{\theta_l} - \frac{\sin \theta_j}{r \prod_{k=1}^{j-1} \sin \theta_k} p_{\theta_j},$$

$$p_N = \prod_{k=1}^{N-1} \sin \theta_k p_r + \frac{1}{r} \sum_{l=1}^{N-1} \prod_{m=l+1}^{N-1} \sin \theta_m \cos \theta_l p_{\theta_l},$$

where $1 \leq j < N$ and from now on any sum $\sum_l^m$ such that $l > m$ is assumed to be zero. Both sets of phase spaces $(q, p)$ and $(r, \theta_j; p_r, p_{\theta_j})$ are canonical coordinates and momenta.
with respect to the usual Poisson bracket

\[
\{F, G\} = \sum_{i=1}^{N} \left( \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i} \frac{\partial F}{\partial p_i} \right) = \frac{\partial F}{\partial \theta} \frac{\partial G}{\partial p_\theta} - \frac{\partial G}{\partial \theta} \frac{\partial F}{\partial p_\theta} + \sum_{j=1}^{N-1} \left( \frac{\partial F}{\partial \theta_j} \frac{\partial G}{\partial p_{\theta_j}} - \frac{\partial G}{\partial \theta_j} \frac{\partial F}{\partial p_{\theta_j}} \right). \tag{3.14}
\]

This implies that the geodesic motion is described through the kinetic energy Hamiltonian

\[
\mathcal{T} = \frac{p^2}{2f(|q|)^2} = \frac{p_r^2 + r^{-2}L^2}{2f(r)^2}, \tag{3.15}
\]

where \(L^2\) is the total “angular momentum”, namely,

\[
L^2 = \sum_{j=1}^{N-1} p_{\theta_j}^2 \prod_{k=1}^{j-1} \frac{1}{\sin^2 \theta_k}. \tag{3.16}
\]

Therefore the kinetic term on \(\mathcal{M}\) arises as an \(\mathfrak{sl}(2, \mathbb{R})\)-coalgebra Hamiltonian (2.6):

\[
\mathcal{T} = \frac{J_+}{2f(\sqrt{J_-})^2}, \tag{3.17}
\]

provided that all \(b_i = 0\). Therefore, this geodesic flow is always QMS for any function \(f\) and its constants of motion are just (2.2).

We remark that in the case of free motion we recover the well-known \(\mathfrak{so}(N)\) Lie–Poisson symmetry of spherically symmetric spaces. Explicitly, the functions \(J_{ij} = q_i p_j - q_j p_i\) with \(i < j\) and \(i, j = 1, \ldots, N\) span an \(\mathfrak{so}(N)\) Lie–Poisson algebra

\[
\{J_{ij}, J_{ik}\} = J_{jk}, \quad \{J_{ij}, J_{jk}\} = -J_{ik}, \quad \{J_{ik}, J_{jk}\} = J_{ij}, \quad i < j < k. \tag{3.18}
\]

Hence the integrals (2.2) can be expressed as sums of such “angular momentum” components

\[
C^{(m)} = \sum_{1 \leq i < j}^{m} J_{ij}^2, \quad C^{(m)} = \sum_{N-m+1 \leq i < j}^{N} J_{ij}^2. \tag{3.19}
\]

Notice that \(C^{(N)} = C_{(N)}\) is the quadratic Casimir of \(\mathfrak{so}(N)\) and, in fact, each \(C^{(m)}\) (or \(C_{(m)}\)) is the quadratic Casimir of certain rotation subalgebra \(\mathfrak{so}(m) \subset \mathfrak{so}(N)\).

By taking into account (3.9) and (3.13) we obtain the generators \(J_{ij}\) written in terms
of the spherical phase space variables; namely,

\[
J_{ij} = \sin \theta_i \cos \theta_j \prod_{k=i}^{j-1} \sin \theta_k p_{\theta_k} - \cos \theta_i \sin \theta_j \prod_{k=i}^{j-1} \sin \theta_k p_{\theta_i} + \cos \theta_i \cos \theta_j \prod_{l=i}^{j-1} \sin \theta_k \cos \theta_l p_{\theta_l}, \quad 1 \leq i < j < N,
\]

\[J_{iN} = \sin \theta_i \prod_{k=i}^{N-1} \sin \theta_k p_{\theta_i} + \cos \theta_i \sum_{l=i}^{N-1} \prod_{m=i}^{N-1} \sin \theta_m \cos \theta_l p_{\theta_l}. \quad (3.20)\]

By using these expression one can readily write down the constants of motion (3.19) in this set of canonical variables. The resulting formulas for \(C(m)\) adopt the very compact form

\[
C(m) = \sum_{j=N-m+1}^{N-1} p_{\theta_j}^2 \prod_{k=N-m+1}^{j-1} \frac{1}{\sin^2 \theta_k}, \quad (3.21)
\]

so that \(C(N) = L^2\) is the total "angular momentum" (3.16). Furthermore, the complete integrability determined by the set of \(N\) functions \(\{T, C(m)\} \ (m = 2, \ldots, N)\) leads to a separable system as a set of \(N\) equations each of them depending only on a pair of canonical quantities. In particular we obtain a common set of \(N - 1\) angular equations for the free Hamiltonian (3.15)

\[
C(2)(\theta_{N-1}, p_{\theta_{N-1}}) = p_{\theta_{N-1}}^2,
\]

\[
C(l)(\theta_{N-l+1}, p_{\theta_{N-l+1}}) = p_{\theta_{N-l+1}}^2 + \frac{C(l-1)}{\sin^2 \theta_{N-l+1}}, \quad l = 3, \ldots, N - 1,
\]

\[
C(N)(\theta_1, p_{\theta_1}) = p_{\theta_1}^2 + \frac{C(N-1)}{\sin^2 \theta_1} \equiv L^2, \quad (3.22)
\]

together with a single radial equation corresponding to the Hamiltonian itself \(T(r, p_r)\).

### 3.2 Some examples of QMS geodesic flows

The \(\mathfrak{sl}(2, \mathbb{R})\)-coalgebra setting describes all spherically symmetric spaces and, consequently, many geodesic motion Hamiltonians that have been studied in the literature by following different procedures can be embedded into this integrability framework. In what follows we shall present three families of examples whose explicit metric and free Hamiltonian are displayed in table 1.

- **The three simply connected Riemannian spaces of constant curvature.** If we take \(f(|q|) = 2/(1 + \kappa q^2)\) we recover the spherical \((\kappa > 0)\), Euclidean \((\kappa = 0)\) and hyperbolic \((\kappa < 0)\) spaces of constant sectional curvature \(\kappa\) (the scalar one is \(N(N-8\))
Table 1: Geodesic flow Hamiltonians with $\mathfrak{sl}(2, \mathbb{R})$-coalgebra symmetry for some particular ND spherically symmetric spaces in the generic coordinates $(q, p)$.

| Space       | Metric $ds^2$ | Hamiltonian $T = T(J_-, J_+)$ |
|-------------|---------------|-------------------------------|
| Euclidean   | $dq^2$        | $\frac{1}{2}J_+$             |
| Spherical   | $\frac{4dq^2}{(1 + q^2)^2}$ | $\frac{1}{2}(1 + J_-^2)J_+$ |
| Hyperbolic  | $\frac{4dq^2}{(1 - q^2)^2}$ | $\frac{1}{2}(1 - J_-^2)J_+$ |
| Darboux I   | $\frac{\ln|q|dq^2}{q^2}$ | $\frac{J_-}{2\ln\sqrt{J_-}}J_+$ |
| Darboux II  | $\frac{1 + \ln^2|q|}{q^2\ln^2|q|}dq^2$ | $\frac{J_-\ln^2\sqrt{J_-}}{2(1 + \ln^2\sqrt{J_-})}J_+$ |
| Darboux IIIa| $\frac{1 + |q|}{q^3}dq^2$ | $\frac{J_2^2J_+}{2(1 + \sqrt{J_-})}J_+$ |
| Darboux IIIb| $(k + q^2) dq^2$ | $\frac{J_+}{2(k + J_-)}$ |
| Darboux IV  | $\frac{a + \cos(\ln|q|)}{q^2\sin^2(\ln|q|)} dq^2$ | $\frac{J_-^2\sin^2(\ln\sqrt{J_-})J_+}{2(a + \cos(\ln\sqrt{J_-}))}$ |
| Taub-NUT    | $\frac{(4m + |q|) dq^2}{|q|}$ | $\frac{\sqrt{J_-}J_+}{2(4m + \sqrt{J_-})}$ |
| $\nu$-fold $a \neq 0$ | $\frac{(a + b|q|^\frac{\nu}{2}) dq^2}{|q|^{2-\frac{2}{\nu}}}$ | $\frac{J_-^{1-\frac{2}{\nu}}J_+}{2(a + bJ_-^{1-\frac{2}{\nu}})}$ |
| $\nu$-fold $a = 0$ | $|q|^{\frac{2}{\nu} - 2}dq^2$ | $\frac{1}{2}J_-^{-\frac{2}{\nu}}J_+$ |

1) $\kappa$. The generic coordinates $q$ are identified with Poincaré coordinates [41] coming from a stereographic projection in $\mathbb{R}^{N+1}$. Note that $r = |q|$ is not a geodesic distance; in fact, the proper geodesic coordinate $\hat{r}$ is related to $r$ by

$$
r = \frac{1}{\sqrt{\kappa}}\tan\left(\sqrt{\kappa} \frac{\hat{r}}{2}\right), \quad ds^2 = d\hat{r}^2 + \frac{\sin^2(\sqrt{\kappa}\hat{r})}{\kappa} d\Omega^2. \tag{3.23}
$$

As it is well-known the geodesic motion on these spaces is MS and their additional quadratic integral of motion in Poincaré coordinates can be found in [2]. In Table 1 we write the resulting expressions for $\kappa = \{\pm 1, 0\}$.

- **ND Darboux spaces.** The four 2D Darboux spaces are surfaces of nonconstant curvature whose geodesic motion is quadratically MS [42, 43]. From the point of view of superintegrability, these surfaces can be seen as the closest analogues of the Riemannian surfaces of constant curvature. An ND spherically symmetric generalization...
for them was recently proposed in [37] by requiring them to be endowed with an $\mathfrak{sl}(2, \mathbb{R})$-coalgebra symmetry. We remark that the Darboux surface of type III admits two (equivalent) generalizations of this kind: the type IIIa given in [37] and the type IIIb constructed in [26]. Only for this Darboux III space the MS property has been proven in arbitrary dimension [26] (and the additional integral is also quadratic), while for the three remaining types the MS problem is still open. Recall also that the parameters $a$ and $k$ appearing in the types IIIb and IV in table 1 are real constants.

**Iwai–Katayama spaces.** These are the ND counterpart of the 3D spaces underlying the so called “multifold Kepler” systems introduced by these authors in [44]. The generic $\nu$-fold Kepler space depends on two real constants, $a$ and $b$, as well on a rational parameter $\nu$ as shown in table 1. For the sake of a further more transparent discussion we have distinguished the cases $a \neq 0$ and $a = 0$, and in the latter case $b$ can always be taken equal to 1. The Iwai–Katayama systems are of physical interest as they are generalizations of the MIC–Kepler and the Taub-NUT ones. As far as the underlying space is concerned the Taub-NUT metric is recovered from the $\nu$-fold Kepler one provided that $\nu = 1$, $a = 4m$ and $b = 1$ (recall that the MIC–Kepler space is just the Euclidean one). We stress that in the 3D case all these Hamiltonians have been shown to be MS, but the additional integral of motion is not, in general, quadratic in the momenta (see [73] for a detailed discussion on the subject).

It is also worth stressing that in 3D the classical Riemannian spaces, the Darboux space of type IIIb and the Iwai–Katayama spaces are just particular cases of the so-called “Bertrand spaces”. The ND version of the whole family of 3D Bertrand spaces has been recently studied in [50, 73] by making use of the same $\mathfrak{sl}(2, \mathbb{R})$-coalgebra symmetry, and their explicit expressions for the abovementioned particular cases are already included in table 1. For the sake of simplicity, we have omitted the generic form of the Bertrand metrics in table 1, but their ND generalization only requires to consider the appropriate conformal factor coming from [50, 73] and to replace the metric $d\Omega^2$ (and its associated 2D angular momentum $L^2$) on the 2D sphere by the metric (3.8) on the $(N - 1)$D sphere (and associated $(N - 1)$D “angular momentum” (3.16)).

4 Superintegrable potentials on curved spaces

At this point, the Hamiltonian (1.1) can be thought of as the kinetic energy (3.15) plus a $\mu^2$-term and a function $U(|q|)$, and by considering a symplectic realization with non-vanishing $b_i$’s. This system can also be expressed in terms of the spherical phase space variables (3.9) and (3.13) by taking into account that the generators $\{J_\pm, J_3\}$ turn out to
be
\[ J_- = r^2, \quad J_3 = rp_r, \]
\[ J_+ = p_r^2 + r^{-2}L^2 + \sum_{j=1}^{N-1} \frac{b_j^2}{r^2 \cos^2 \theta_j \prod_{k=1}^{j-1} \sin^2 \theta_k} + \frac{b_N}{r^2 \prod_{k=1}^{N-1} \sin^2 \theta_k}. \]  \hfill (4.1)

In this way we find that the Hamiltonian (1.1) can be rewritten as
\[ \mathcal{H} = \frac{J_+ + \mu^2 J_-}{2f(\sqrt{J_-})^2} + U(\sqrt{J_-}) = \frac{p_r^2 + r^{-2}L^2}{2f(r)^2} + \frac{\mu^2}{2r^2 f(r)^2} + U(r) \]
\[ + \frac{1}{2f(r)^2} \sum_{j=1}^{N-1} \frac{b_j^2}{r^2 \cos^2 \theta_j \prod_{k=1}^{j-1} \sin^2 \theta_k} + \frac{b_N}{2f(r)^2 \prod_{k=1}^{N-1} \sin^2 \theta_k}. \]  \hfill (4.2)

Again, we stress that the symplectic realization of the generator \( J_+ \) is the essential tool to incorporate in a natural way up to \( N \) centrifugal potentials associated to the \( b_i \)'s.

Therefore this Hamiltonian is QMS as it is endowed with the \((2N - 3)\) integrals of motion given by (2.2). However, the existence of an additional integral providing the MS property for \( \mathcal{H} \) is, in general, not guaranteed. Nevertheless, as it already happened with the free motion, \( \mathcal{H} \) is separable in the latter form (4.2). In this case the integrals of motion \( C_{(m)} \) (2.2), with arbitrary centrifugal terms, are given by
\[ C_{(m)} = \sum_{j=N-m+1}^{N-1} \left( p_{\theta_j}^2 + \frac{b_j}{\cos^2 \theta_j} \right) \prod_{k=N-m+1}^{j-1} \frac{1}{\sin^2 \theta_k} + \frac{b_N}{\prod_{l=1}^{N-m+1} \sin^2 \theta_l}. \]  \hfill (4.3)

Consequently, we obtain again a common set of \( N - 1 \) angular equations
\[ C_{(2)}(\theta_{N-1}, p_{\theta_{N-1}}) = p_{\theta_{N-1}}^2 + \frac{b_{N-1}}{\cos^2 \theta_{N-1}} + \frac{b_N}{\sin^2 \theta_{N-1}}, \]
\[ C_{(l)}(\theta_{N-l+1}, p_{\theta_{N-l+1}}) = p_{\theta_{N-l+1}}^2 + \frac{C_{(l-1)}}{\sin^2 \theta_{N-l+1}} + \frac{b_{N-l+1}}{\cos^2 \theta_{N-l+1}}, \quad l = 3, \ldots, N - 1, \]
\[ C_{(N)}(\theta_1, p_{\theta_1}) = p_{\theta_1}^2 + \frac{C_{(N-1)}}{\sin^2 \theta_1} + \frac{b_1}{\cos^2 \theta_1}, \]
\[ = L^2 + \sum_{j=1}^{N-1} \frac{b_j^2}{\cos^2 \theta_j \prod_{k=1}^{j-1} \sin^2 \theta_k} + \frac{b_N}{\prod_{l=1}^{N-1} \sin^2 \theta_l}. \]  \hfill (4.4)

(to be compared with (3.22)) plus one radial equation which depends on the specific Hamiltonian under consideration
\[ \mathcal{H}(r, p_r) = \frac{p_r^2 + r^{-2}C_{(N)} + r^{-2}\mu^2}{2f(r)^2} + U(r). \]  \hfill (4.5)

Thus when any \( b_i \neq 0 \) the spherical symmetry is broken and the “total angular momentum” \( L^2 \) (3.16) is no longer a constant of motion, its role being now played by \( C_{(N)} \) (4.4)
(only when all $b_i = 0$, $C_{(N)}$ reduces to $L^2$). In the case with $b_i \neq 0$, the generator $J_+$ (4.1) simply reads

$$J_+ = p_r^2 + r^{-2}C_{(N)},$$

(4.6)

and the Poisson brackets among the $\mathfrak{sl}(2, \mathbb{R})$ generators (4.1) and $L^2$ turn out to be

$$\{J_-, L^2\} = \{J_3, L^2\} = 0,$$

$$\{J_+, L^2\} = \frac{4}{r^2} \sum_{j=1}^{N-1} \sin^2 \theta_j \tan \theta_j p_{\theta_j} \left( b_j \tan^4 \theta_j - \sum_{l=j+1}^{N-1} \frac{b_l}{\cos^2 \theta_l \prod_{k=j+1}^{l-1} \sin^2 \theta_k} - \frac{b_N}{\prod_{s=j+1}^{N-1} \sin^2 \theta_s} \right).$$

(4.7)

Summing up, once a single nonzero parameter $b_i$ is allowed, the spherical symmetry is broken, and the chain of rotation subalgebras $\mathfrak{so}(m) \subset \mathfrak{so}(N)$ do not provide symmetries for the Hamiltonian, as it was the case of the Casimirs (3.19) for the free motion. Therefore, the $\mathfrak{sl}(2, \mathbb{R})$ Lie–Poisson coalgebra symmetry can be understood as the appropriate generalization of the usual rotational symmetry which allows to deal with the general case above.

We also remark, as pointed out in the introduction, that the $\mu^2$-potential in (4.5) (or (4.2)) can be interpreted as a Dirac monopole-type term [44, 56, 57, 60]. Hence since $C_{(N)}$ is an integral of the motion, the appearance of this monopole-type potential can be seen, from a dynamical viewpoint, as coming from a radial centrifugal term ruled by $C_{(N)}$. In particular, when there are no centrifugal potentials ($b_i = 0$), the $\mu^2$-term has already been interpreted, for some concrete systems, as a proper Dirac monopole [44, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61] that actually comes from the “total angular momentum” through $C_{(N)} \equiv L^2 \rightarrow L^2 + \mu^2$.

5 KC and oscillator potentials on curved spaces

Among all the possible choices of the central function $U(r)$, two of them should correspond to the appropriate ND definition of the intrinsic KC and oscillator potentials on $\mathcal{M}$. As we shall see in the sequel, a solution to this problem can be found through a suitable generalization of the 3D construction for such potentials.

Let us consider the Laplace–Beltrami operator on the 3D spherically symmetric manifold $\mathcal{M}^3$ with metric (3.7) in the coordinates $q$:

$$\Delta_{\mathcal{M}^3} = \sum_{i,j=1}^{3} \frac{1}{\sqrt{g}} \frac{\partial}{\partial q_i} \sqrt{g^{ij}} \frac{\partial}{\partial q_j},$$

(5.8)

where $g^{ij}$ is the inverse of the metric tensor $g_{ij}$ and $g$ is the determinant of $g_{ij}$. The radial symmetric Green function $U(|q|) = U(r)$ on $\mathcal{M}^3$ (up to multiplicative and additive
constants) is defined as the positive nonconstant solution to the equation

$$\Delta_{\mathcal{M}} U(r) = \frac{1}{r^2 f(r)^3} \frac{d}{dr} \left( r^2 f(r) \frac{dU(r)}{dr} \right) = 0 \quad \text{on} \quad \mathcal{M}^3 \setminus \{0\}, \quad (5.9)$$

so that

$$U(r) = \int^r \frac{dr'}{r'^2 f(r')}. \quad (5.10)$$

Now, the essential point in the ND case [26] is the fact that we can keep the very same definitions given in [46, 47, 48, 49, 50] for the KC and oscillator potentials on a 3D spherically symmetric space. In particular, the intrinsic KC potential on the ND space $\mathcal{M}$ will be defined by

$$U_{\text{KC}}(r) := \alpha U(r), \quad (5.11)$$

while the intrinsic oscillator potential is defined to be proportional to the inverse square of the KC potential

$$U_{\text{O}}(r) := \frac{\beta}{U^2(r)}, \quad (5.12)$$

where $\alpha$ and $\beta$ are real constants.

Clearly, the intrinsic KC and oscillator potentials do preserve the $\mathfrak{sl}(2, \mathbb{R})$-coalgebra symmetry since they are defined through a function of $|q| = r = \sqrt{J_-}$. As a straightforward consequence, such a symmetry allows us to give the superposition of either the intrinsic KC (5.11) or oscillator (5.12) potential with $N$ centrifugal terms and also with a monopole-type term. In terms of the symplectic realization (2.3) and by considering arbitrary parameters $b_i$, the curved KC system read

$$\mathcal{H}_{\text{KC}} := \frac{J_+ + \mu^2 J_-^{-1}}{2f(\sqrt{J_-})^2} + U_{\text{KC}}(\sqrt{J_-}) = \frac{p^2 + \mu^2 q^{-2} + \sum_{i=1}^{N} b_i q_i^{-2}}{2f(|q|)^2} + \alpha U(|q|) \quad (5.13)$$

and the intrinsic oscillator on a curved space is defined as

$$\mathcal{H}_{\text{O}} := \frac{J_+ + \mu^2 J_-^{-1}}{2f(\sqrt{J_-})^2} + U_{\text{O}}(\sqrt{J_-}) = \frac{p^2 + \mu^2 q^{-2} + \sum_{i=1}^{N} b_i q_i^{-2}}{2f(|q|)^2} + \frac{\beta}{U(|q|)^2}. \quad (5.14)$$

Both systems can be immediately written in terms of the spherical phase space variables by means of (4.1).

### 6 Some physically relevant examples

The results of the previous section are explicitly illustrated by writing in table 2 the QMS Hamiltonians (5.13) and (5.14) associated to all the particular spaces given in table 1. For each ND curved space we display the KC, oscillator, monopole-type and centrifugal potentials (with parameters $\alpha$, $\beta$, $\mu^2$ and $b_i$, respectively). A specific QMS Hamiltonian

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\( \mathcal{H} \) thus comes from the superposition of some (or all) of the corresponding terms. Notice that, in some cases, the multiplicative constants that arise when computing the Green function (5.10) have been absorbed within \( \alpha \) and \( \beta \).

In what follows we explicitly analyse these systems for the three families of spaces discussed in section 3.2, since they are indeed the most relevant ones from the physical point of view.

Table 2: Intrinsic KC, oscillator, monopole-type and centrifugal potentials with \( \mathfrak{sl}(2, \mathbb{R}) \)-coalgebra symmetry in the generic coordinates \( \mathbf{q} \) corresponding to the free Hamiltonians on the particular \( ND \) spherically symmetric spaces given in table 1.

| Space         | KC             | Oscillator | Monopole        | Centrifugal terms |
|---------------|----------------|------------|-----------------|-------------------|
| Euclidean     | \( \frac{\alpha}{|\mathbf{q}|} \) | \( \beta |\mathbf{q}|^2 \) | \( \frac{\mu^2}{2|\mathbf{q}|^2} \) | \( \frac{1}{2} \sum_{i=1}^{N} \frac{b_i}{q_i^2} \) |
| Spherical     | \( \frac{\alpha(|\mathbf{q}|^2 - 1)}{|\mathbf{q}|} \) | \( \frac{\beta |\mathbf{q}|}{(|\mathbf{q}|^2 - 1)^2} \) | \( \frac{\mu^2(|\mathbf{q}|^2 + 1)^2}{2|\mathbf{q}|^2} \) | \( \frac{1}{2} \sum_{i=1}^{N} \frac{b_i}{q_i^2} \) |
| Hyperbolic    | \( \frac{\alpha(|\mathbf{q}|^2 + 1)}{|\mathbf{q}|} \) | \( \frac{\beta |\mathbf{q}|}{(|\mathbf{q}|^2 + 1)^2} \) | \( \frac{\mu^2(|\mathbf{q}|^2 - 1)^2}{2|\mathbf{q}|^2} \) | \( \frac{1}{2} \sum_{i=1}^{N} \frac{b_i}{q_i^2} \) |

| Darboux I     | \( \alpha \ln |\mathbf{q}| \) | \( \frac{\beta}{\ln |\mathbf{q}|} \) | \( \frac{\mu^2}{2 \ln |\mathbf{q}|} \) | \( \frac{q^2}{2 \ln |\mathbf{q}|} + \frac{1}{2} \sum_{i=1}^{N} \frac{b_i}{q_i^2} \) |
| Darboux II    | \( \alpha \sqrt{1 + \ln^2 |\mathbf{q}|} \) | \( \frac{\beta}{1 + \ln^2 |\mathbf{q}|} \) | \( \frac{\mu^2 \ln^2 |\mathbf{q}|}{2(1 + \ln^2 |\mathbf{q}|)} \) | \( \frac{q^2 \ln^2 |\mathbf{q}|}{2(1 + \ln^2 |\mathbf{q}|)} + \frac{1}{2} \sum_{i=1}^{N} \frac{b_i}{q_i^2} \) |
| Darboux IIIa  | \( \alpha \sqrt{1 + |\mathbf{q}|} \) | \( \frac{\beta}{1 + |\mathbf{q}|} \) | \( \frac{\mu^2 |\mathbf{q}|^2}{2(1 + |\mathbf{q}|)} \) | \( \frac{q^4}{2(1 + |\mathbf{q}|)} + \sum_{i=1}^{N} \frac{b_i}{q_i^2} \) |
| Darboux IIIb  | \( \alpha \sqrt{k + |\mathbf{q}|^2} \) | \( \frac{\beta |\mathbf{q}|^2}{k + |\mathbf{q}|^2} \) | \( \frac{\mu^2}{2k^2(k + |\mathbf{q}|^2)} \) | \( \frac{1}{2k^2} \sum_{i=1}^{N} \frac{b_i}{q_i^2} \) |
| Darboux IV    | \( \alpha \sqrt{a + \cos(\ln |\mathbf{q}|)} \) | \( \frac{\beta}{a + \cos(\ln |\mathbf{q}|)} \) | \( \frac{\mu^2 \sin^2(\ln |\mathbf{q}|)}{2(a + \cos(\ln |\mathbf{q}|))} \) | \( \frac{q^2 \sin^2(\ln |\mathbf{q}|)}{2(a + \cos(\ln |\mathbf{q}|))} + \sum_{i=1}^{N} \frac{b_i}{q_i^2} \) |
| Taub-NUT      | \( \alpha \sqrt{4m|\mathbf{q}|^{-1} + 1} \) | \( \frac{\beta |\mathbf{q}|}{4m + |\mathbf{q}|} \) | \( \frac{\mu^2}{2|\mathbf{q}|(4m + |\mathbf{q}|)} \) | \( \frac{|\mathbf{q}|}{2(4m + |\mathbf{q}|)} + \sum_{i=1}^{N} \frac{b_i}{q_i^2} \) |
| \( \nu \)-fold \( a \neq 0 \) | \( \alpha \sqrt{a|\mathbf{q}|^{-b} + b} \) | \( \frac{\beta}{a|\mathbf{q}|^{-b} + b} \) | \( \frac{\mu^2}{2|\mathbf{q}|^2(a + b|\mathbf{q}|^b)} \) | \( \frac{|\mathbf{q}|^2}{2(a + b|\mathbf{q}|^b)} + \sum_{i=1}^{N} \frac{b_i}{q_i^2} \) |
| \( \nu \)-fold \( a = 0 \) | \( \frac{\alpha}{|\mathbf{q}|^b} \) | \( \frac{\beta |\mathbf{q}|^2}{|\mathbf{q}|^2} \) | \( \frac{\mu^2}{2|\mathbf{q}|^2} \) | \( \frac{1}{2} |\mathbf{q}|^{2-\frac{b}{2}} + \sum_{i=1}^{N} \frac{b_i}{q_i^2} \) |
6.1 The classical Riemannian spaces of constant curvature

For any simply connected Riemannian space with constant sectional curvature $\kappa$, the metric function $f$ in Poincaré coordinates [2] reads

$$f(|q|) = \frac{2}{1 + \kappa q^2},$$

and the Green function (5.10) is easily computed:

$$U(|q|) = \frac{\kappa q^2 - 1}{|q|}.$$ 

As a consequence, the KC and oscillator potentials are the ones shown in table 2 for $\kappa = \{\pm 1, 0\}$. Such expressions can be rewritten in a more usual form by introducing the geodesic radial coordinate $\hat{r}$ (3.23) which gives

$$U_{\text{KC}}(\hat{r}) = -\alpha \sqrt{\kappa} \tan(\sqrt{\kappa} \hat{r}), \quad U_{\text{O}}(\hat{r}) = \beta \tan^2(\sqrt{\kappa} \hat{r}) \kappa.$$ 

We recall that when all the $N$ parameters $b_i$ are different from zero, the generalized KC Hamiltonian is QMS but not quadratically MS. Moreover, only when at least one of the $b_i$'s vanishes an additional quadratic integral of motion arises [2] (which is a component of the Laplace–Runge–Lenz $N$-vector). Nevertheless, it has been recently proven in [36] that when the KC potential is constructed on the 3D Euclidean space it is possible to consider the three centrifugal potentials obtaining a MS Hamiltonian with an additional $quartic$ integral. In contrast, the so-called Smorodinsky–Winternitz system (i.e., the oscillator plus $N$ centrifugal terms) is always MS for any value of the $b_i$'s.

We stress that the MIC–Kepler Hamiltonian [44, 56, 60, 62, 63, 64] is also recovered when the KC and the monopole potential are considered in the flat Euclidean space:

$$\mathcal{H}_{\text{MIC–Kepler}} = \frac{1}{2}(J_+ - \frac{\alpha}{\sqrt{J_-}}) \frac{\mu^2}{2J_-} = \frac{1}{2}P^2 - \frac{\alpha}{|q|} + \frac{\mu^2}{2q^2}. \quad (6.15)$$

Therefore, as a byproduct of our procedure, the curved MIC–Kepler analogue on the $N$D spherical and hyperbolic spaces can be constructed from table 2. Namely,

$$\mathcal{H}_{\text{MIC–Kepler}} = \frac{1}{2}(1 + J_-)^2 J_+ + \frac{\alpha(J_+ - 1)}{\sqrt{J_-}} + \frac{\mu^2(1 + J_-)^2}{2J_-}$$

$$= \frac{1}{2}(1 - q^2)^2 p^2 + \frac{\alpha(q^2 - 1)}{|q|} + \frac{\mu^2(1 + q^2)^2}{2q^2}, \quad (6.16)$$

$$\mathcal{H}_{\text{MIC–Kepler}} = \frac{1}{2}(1 - J_-)^2 J_+ - \frac{\alpha(J_+ + 1)}{\sqrt{J_-}} + \frac{\mu^2(1 - J_-)^2}{2J_-}$$

$$= \frac{1}{2}(1 - q^2)^2 p^2 - \frac{\alpha(q^2 + 1)}{|q|} + \frac{\mu^2(1 - q^2)^2}{2q^2}. \quad (6.17)$$
It can be easily checked that in this way we have exactly recovered (in Poincaré coordinates) the curved MIC–Kepler systems studied in [74, 75, 76]. Obviously, all these Hamiltonians can be generalized by adding the $b_i$-centrifugal terms given in table 2, and in that case the QMS property is, by construction, fully preserved.

### 6.2 Darboux spaces

A thorough discussion of the MS potentials on the four types (I, II, IIIa and IV) of 2D Darboux surfaces was given in [43]. Hence it is natural to compare with them the 2D versions of the ND potentials given in table 2. To carry out this analysis, we firstly recall that the Darboux metrics given in [43] depend on two coordinates $(u, v)$, and their associated metrics and free Hamiltonians read

\[ ds^2 = F(u)^2 (du^2 + dv^2), \quad H = \frac{p_u^2 + p_v^2}{F(u)^2}, \]  

\[ (6.18) \]

where $(p_u, p_v)$ are the conjugate momenta and the function $F(u)^2$ is given by

- **Type I**: $F(u)^2 = u$
- **Type II**: $F(u)^2 = 1 + u^{-2}$
- **Type IIIa**: $F(u)^2 = e^{-2u} + e^{-u}$
- **Type IV**: $F(u)^2 = \frac{a + \cos u}{\sin^2 u}$  

\[ (6.19) \]

Secondly, we also recall that the ND spaces given in table 2 are just an ND spherically symmetric generalization of these four spaces that was constructed in [37] through the maps

\[ u \rightarrow \ln r = \ln |q|, \quad dv^2 \rightarrow d\Omega^2. \]  

\[ (6.20) \]

Now, if we consider our ND KC and oscillator potentials given in table 2 for these four spaces and we perform the substitution $u \equiv \ln |q|$, we immediately see that, in any dimension, we recover expressions for the potentials that depend only on the variable $u$. Now, if we go back to the classification given in [43], the unique MS potentials in 2D which are functions of $u$ alone are

- **Type I**: $\frac{1}{u} \rightarrow \frac{1}{\ln |q|}$
- **Type II**: $\frac{1}{1 + u^2} \rightarrow \frac{1}{1 + \ln^2 |q|}$
- **Type IIIa**: $\frac{1}{1 + e^u} \rightarrow \frac{1}{1 + |q|}$
- **Type IV**: $\frac{1}{a + \cos u} \rightarrow \frac{1}{a + \cos(\ln |q|)}$.  

\[ (6.21) \]

Surprisingly, we find that these four potentials are just the intrinsic oscillators of the four 2D Darboux spaces, and there is no KC potential from table 2 appearing in the classification given in [43]. This fact suggests that in the case of 2D spaces of non-constant curvature, the intrinsic oscillator potential seems to be more fundamental than the KC one from the integrability viewpoint, since the former would be a MS system whilst the later would be only a QMS one.
A remark is in order: in the classification [43], the MS potentials associated to the Darboux spaces II and IIIa contain also other terms depending on \( u \), namely \( \frac{1}{1 + u^2} \) and \( \frac{1}{1 + e^u} \), respectively, which in fact can be obtained by adding a constant \( \gamma \) to the above oscillator potentials with constant \( \beta 
\)

\[
\text{Type II: } \frac{\beta}{1 + u^2} + \gamma = \frac{\beta + \gamma}{1 + u^2} + \frac{\gamma}{1 + u^2}; \tag{6.22}
\]

\[
\text{Type IIIa: } \frac{\beta}{1 + e^u} + \gamma = \frac{\beta + \gamma}{1 + e^u} + \frac{\gamma}{1 + e^u}.
\]

Finally, recall that in arbitrary dimension \( N \), only the oscillator potential with \( N \) centrifugal terms for the Darboux space IIIb has been proven to be MS [26]. Therefore the MS property remains as an open problem for all the remaining types of \( ND \) intrinsic oscillator potentials.

### 6.3 Iwai–Katayama spaces

We have also written in table 2 the resulting potentials corresponding to the Iwai–Katayama spaces described in section 3.2 by considering either \( a \neq 0 \) or \( a = 0 \) (with \( b = 1 \)). We remark that, although the \( \nu \)-fold Kepler metric given in table 1 depends continuously on \( a \), a glance at the potentials (displayed in table 2) reveals that it is convenient to perform a separate analysis of the cases \( a \neq 0 \) and \( a = 0 \).

Firstly, let us recall that the \( ND \) version of the 3D multifold Kepler Hamiltonian introduced in [44] is given by

\[
H_{\nu\text{-fold–Kepler}} = \frac{|\mathbf{q}|^{2 - \frac{1}{\nu}}}{2(a + b|\mathbf{q}|^{\frac{1}{\nu}})} \left( \mathbf{p}^2 + \mu^2 |\mathbf{q}|^{-2} + \frac{\mu^2}{2|\mathbf{q}|^{\frac{1}{\nu}}(a + b|\mathbf{q}|^{\frac{1}{\nu}})} + \mu^2 d |\mathbf{q}|^{2 - 2} \right), \tag{6.23}\]

where \( \nu \) is a rational parameter and \( a, b, c \) and \( d \) are real constants. By expanding this expression we find that

\[
H_{\nu\text{-fold–Kepler}} = \frac{|\mathbf{q}|^{2 - \frac{1}{\nu}}\mathbf{p}^2}{2(a + b|\mathbf{q}|^{\frac{1}{\nu}})} + \frac{\mu^2}{2(2(a + b|\mathbf{q}|^{\frac{1}{\nu}})} + \frac{\mu^2}{2|\mathbf{q}|^{\frac{1}{\nu}}(a + b|\mathbf{q}|^{\frac{1}{\nu}})} + \frac{\mu^2 c}{2(a + b|\mathbf{q}|^{\frac{1}{\nu}})}. \tag{6.24}\]

Hence, table 2 allows us to provide a more clear interpretation of the four terms forming this Hamiltonian, which indeed \textit{does} depend on the value of the constant \( a \). Namely:

- If \( a \neq 0 \), the first term in the Hamiltonian (6.24) is the kinetic term written in table 1, the second is an intrinsic \textit{oscillator} with \( \beta = \mu^2 d/2 \), the third is the Dirac monopole and the fourth comes out by adding a constant \( \gamma \) to the corresponding oscillator potential (this trick is the same as the one used in (6.22)):

\[
\frac{\beta}{a|\mathbf{q}|^{\frac{1}{\nu}} + b} + \gamma = \frac{\beta + b\gamma}{a|\mathbf{q}|^{\frac{1}{\nu}} + b} + \frac{a\gamma}{a + b|\mathbf{q}|^{\frac{1}{\nu}}}, \tag{6.25}\]

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so that $\gamma = \mu^2 c/(2a)$. Consequently, from this approach we can state that the multifold Kepler systems with $a \neq 0$ are, in fact, multifold oscillator systems. This interpretation was already given in [50] for the 3D case.

- If $a = 0$ (and $b = 1$), the multifold Kepler Hamiltonian (6.24) reduces to

$$H_{\text{fold-Kepler; } a=0} = \frac{1}{2}|q|^2 \frac{p^2}{2} + \frac{\mu^2 c}{2|q|} + \frac{\mu^2}{2|q|^2} + \frac{\mu^2 d}{2}.$$  \hspace{2cm} (6.26)

Therefore in this case the first term is the kinetic energy given in table 1, the second is an intrinsic KC potential with $\alpha = -\mu^2 c/2$, the third is the monopole and the fourth is an additive constant. Thus only in the case $a = 0$ the Hamiltonian (6.24) is a proper multifold Kepler system.

From the viewpoint adopted in this paper, it is apparent that the case $a \neq 0$ corresponds to the $\nu$-fold generalization of the Taub-NUT system [44, 59, 65, 66, 67, 68, 69, 70, 71, 72], while $a = 0$ is the $\nu$-fold version of the MIC–Kepler model [44, 56, 60, 62, 63, 64]. Explicitly:

- The Taub-NUT system arises as the particular case of (6.24) with $\nu = 1$, $a = 4m$, $b = 1$, $c = 1/(2m)$ and $d = 1/(4m)^2$ (see [44]) which yields

$$H_{\text{Taub-NUT}} = \frac{|q|^2 p^2}{2(4m + |q|)} + \frac{\mu^2 |q|/(4m)^2}{2(4m + |q|)} + \frac{\mu^2}{2|q|(4m + |q|)} + \frac{\mu^2 / (4m)}{4m + |q|} \left(1 + \frac{4m}{|q|}\right),$$  \hspace{2cm} (6.27)

so that the same interpretation as for the multifold Kepler system with $a \neq 0$ holds. In the first line of (6.27), the first term is the kinetic energy, the second is an intrinsic oscillator with $\beta = \mu^2/(2(4m)^2)$, the third is the monopole and the fourth corresponds to adding a constant $\gamma = \mu^2/(4m)^2$ to the oscillator potential, leading to $\beta \rightarrow \beta + \gamma$.

- The (flat) MIC–Kepler system is recovered from (6.24) or (6.26) when $\nu = 1$, $a = d = 0$, $b = 1$ and $c = -2\alpha/\mu^2$, giving rise to the expression (6.15).

Note also that in 3D, the generic Hamiltonian (6.23) (for any value of $a$) has been recently shown to be MS [73]. This Hamiltonian has an additional integral of motion (coming from a generalized Laplace–Runge–Lenz vector) which, in general, is not quadratic in the momenta.

Let us end by stressing that all the ND Hamiltonians that we have presented are QMS, but some of them could be MS, as we have already commented. As a matter of fact, one of the models that we have described (namely, the Darboux III oscillator
potential with monopole and centrifugal terms) is, to the best of our knowledge, the only known quadratically MS system living in an \( N \)D space of nonconstant curvature; the additional integral can be consulted in [26]. The search for an additional integral for the rest of the systems in the Darboux spaces presented here remains a challenging open problem. In the cases where such a constant of the motion exists, it cannot be derived from the \( \mathfrak{sl}(2,\mathbb{R}) \)-coalgebra symmetry, and it reflects the exceptional superintegrability properties of the Riemannian manifold determined by \( f(r) \).

On the other hand, it is worth remarking that the results presented throughout the paper can be extended as well to curved Lorentzian spaces through an analytic continuation method analogous to the one applied in [37] for some free Hamiltonians.

Finally, we would like to emphasize that the underlying \( \mathfrak{sl}(2,\mathbb{R}) \)-coalgebra symmetry can also be implemented in the Quantum Mechanical analogues of the systems here presented. In particular, the quantum counterparts of the integrals of the motion (2.2) can be readily obtained after dealing with the ordering problems which arise in the quantization of (1.1) due to the term \( p^2/f(|q|)^2 \). We shall report on these and other related issues elsewhere.

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