ON THE EXISTENCE OF CERTAIN WEAK FANO THREEFOLDS OF PICARD NUMBER TWO

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Abstract. We construct smooth weak Fano threefolds of Picard number two with small anti-canonical map obtained by blowing up certain curves on a smooth quadric in $\mathbb{P}^4$ and on smooth del Pezzo threefolds of degrees 4 and 5. In addition, we give the construction of weak Fano threefolds with small anti-canonical map arising as blow-ups of prime Fano threefolds $X_{10}, X_{16}$ and $X_{18}$ along twisted cubics.

1. Introduction

A Fano variety is a smooth projective variety whose anti-canonical class is ample. To study birational maps between Fano varieties it is interesting to relax the positivity condition on the anti-canonical class and require it to be nef and big but not ample. Projective varieties whose anti-canonical class is nef and big but not ample are called weak Fano or almost Fano. In what follows, all varieties are defined over $\mathbb{C}$ and all weak Fano varieties are assumed to be smooth, unless stated otherwise.

Fano threefolds have been classified mainly through the works of Fano, Iskovskikh, Shokurov, Mori and Mukai. As a next step, it is interesting to classify birational maps between Fano threefolds. This question fits into the general framework of Sarkisov program. Within this framework, weak Fano threefolds of Picard number two yield birational maps called Sarkisov links between Fano threefolds of Picard number one. This is one of the motivations to classify weak Fano threefolds of Picard number two.

In [JPR05], [Tak09], [JPR11] and [CM10] one finds the numerical constraints that weak Fano threefolds of Picard number two must satisfy, obtaining a finite list of possibilities, some of which were shown to exist while others were left as numerical possibilities. The recent article [BL] constructs some of the Sarkisov links which were previously not known to exist. The aim of this article is to study further the geometric realizability of the numerical possibilities which were left open in [CM10].
Let $X$ be a weak Fano threefold of Picard number two such that the anti-canonical system $| - K_X|$ is free and gives a small contraction $\psi: X \to X'$. By [Ko], the $K_X$-trivial curves can be flopped. More precisely, there is a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\chi} & X^+ \\
\downarrow{\phi} & & \downarrow{\phi^+} \\
Y & \xrightarrow{\psi} & X' \xrightarrow{\psi^+} Y^+,
\end{array}
\]

where $\chi$ is an isomorphism outside of the exceptional locus of $\psi$. The numerical cases, which we study here, are of E1-E1 type. More precisely, both $\phi$ and $\phi^+$ are assumed to be divisorial contractions of type E1 in the sense of [Mo]. In particular, $Y$ is a smooth Fano variety of Picard number one and $\phi$ is the blow-up of $Y$ along a smooth irreducible curve $C$, whose degree and genus are denoted by $d$ and $g$, respectively. Likewise $\phi^+$ is the blow-up of $Y^+$ along a smooth irreducible curve $C^+$ of degree $d^+$ and genus $g^+$. The exceptional divisors of the blow-ups $\phi$ and $\phi^+$ are denoted by $E$ and $E^+$, respectively.

Let $H$ be the hyperplane class on $Y$. The index of $Y$ is the largest integer $r$ such that $-K_Y = rH$. Abusing notation for the sake of simplicity, the pull-back of $H$ to $X$ is also denoted by $H$. The anti-canonical class of $X$ is $-K_X = rH - E$. We let $l$ and $f$ in $N_1(X)$ denote the classes of the pull-back of a general line on $Y$ and a $\phi$-exceptional curve, respectively. The elements of $N_1(X)$ can be written as $ml - nf$ for some $n, m \in \mathbb{Z}$ and the Mori cone $\text{NE}(X)$ has two extremal rays $f$ and $r$. The extremal ray $r$ has slope

$$\sup\{n/m : ml - nf \text{ can be represented by an effective curve}\}.$$

If $ml - nf$ is represented by a curve $\tilde{\Gamma}$, then $\tilde{\Gamma}$ is the proper transform of a curve $\Gamma \subset Y$ such that $\deg \Gamma = m$ and $C \cdot \Gamma = n$. The intersection pairing is given by the formulas

$$H \cdot l = 1, E \cdot f = -1, H \cdot f = E \cdot l = 0.$$

2. Blow-ups of smooth quadric threefolds

Consider the list of pairs

$$\mathcal{C}(Q) := \{(9, 2), (10, 5), (11, 8), (12, 11), (13, 14), (9, 3), (10, 6), (8, 1), (9, 4), (8, 2), (8, 3), (7, 1)\},$$
which correspond to the numerical possibilities of \((d, g)\) in cases 44-48, 70, 72, 86, 88, 97, 102, 105 in [CM10]. The purpose of this section is to prove that nine of the above twelve numerical possibilities are geometrically realizable.

Throughout this section \(C_{d,g} \subset \mathbb{P}^4\) denotes a smooth non-degenerate curve of degree \(d\) and genus \(g\). For simplicity, we write \(C\) instead of \(C_{d,g}\), when \(d\) and \(g\) are understood from the context. A smooth complete intersection of a quadric and a cubic in \(\mathbb{P}^4\) is denoted by \(S\). The restriction of \(H\) to \(S\) is denoted by \(H_S\).

**Lemma 2.1.** Assume that \((d, g) \in \mathcal{C}(Q)\) and \(C_{d,g} \subset S \subset \mathbb{P}^4\). The divisor \(3H_S - C_{d,g}\) is nef on \(S\), provided

- (A) \(C_{d,g}\) has no \((3m + 1)\)-secant smooth rational curve of degree \(m\) for
  - (1) \(m = 1,\) if \((d, g) \in \{(13, 14)\}\);
  - (2) \(m = 1, 2,\) if \((d, g) \in \{(7, 1), (8, 3), (12, 11)\}\);
  - (3) \(1 \leq m \leq 3,\) if \((d, g) \in \{(8, 2), (9, 4), (10, 6), (11, 8)\}\);
  - (4) \(1 \leq m \leq 4,\) if \((d, g) \in \{(8, 1), (9, 3), (10, 5)\}\);
  - (5) \(1 \leq m \leq 5,\) if \((d, g) = (9, 2)\)

- (B) \(C_{d,g}\) has no \((3m + 1)\)-secant curve of degree 3 or 4 with \(p_a = 1\) if \((d, g) = (10, 5)\) or \((9, 2)\), respectively.

**Proof.** For simplicity we write \(C\) for \(C_{d,g}\). Let \(I_C\) and \(I_S\) be the ideal sheaves of \(C\) and \(S\) in \(\mathbb{P}^4\), respectively. Using the long exact sequence associated to \(0 \to I_C(3) \to \mathcal{O}_{\mathbb{P}^4}(3) \to \mathcal{O}_C(3) \to 0\), we obtain \(h^0(I_C(3)) \geq 9\). Using a similar exact sequence for \(S\), we may check that \(h^0(I_S(3)) = 6\). Therefore, there exist cubics in \(\mathbb{P}^4\) that contain \(C\), but not \(S\).

The divisor \(3H_S - C\) is not nef if and only if there exists an irreducible curve \(\Gamma \subset S\) such that \(\deg \Gamma = m\) and \(C \cdot \Gamma \geq 3m + 1\). In particular, \(\Gamma\) is contained in every cubic and every quadric in \(\mathbb{P}^4\) that contains \(C\). Since there exist cubics containing \(C\) but not \(S\), we must have \(\deg(\Gamma \cup C) \leq 18\). Hence, \(\deg \Gamma \leq 18 - d\).

The Hodge index theorem implies that for any divisor \(D\) on \(S\), \((D \cdot H_S)^2 - D^2 H_S^2 \geq 0\). Applying this inequality to \(D = C + \Gamma\), we obtain \(p_a(C \cup \Gamma) \leq (d + m)^2 / 12 + 1\). Combining this with \(p_a(C \cup \Gamma) \geq p_a(C) \geq p_a(\Gamma) + 3m\) gives

\[
p_a(\Gamma) \leq \frac{(d + m)^2}{12} + 1 - g - 3m.
\]
For each \((d, g) \in \mathcal{C}(Q)\), an elementary calculation, using \(\deg \Gamma \leq 18 - d\) and the above inequality, shows that either \(p_a(\Gamma) = 0\) with possible values of \(\deg \Gamma\) listed in the statement of the lemma, or \(p_a(\Gamma) = 1\) and \(\deg(\Gamma) = 3\) or \(4\). The case \(p_a(\Gamma) = 1, \deg(\Gamma) = 4\) occurs only for \((d, g) = (9, 2)\). The case \(p_a(\Gamma) = 1, \deg(\Gamma) = 3\) occurs for \((d, g) \in \{(8, 1), (9, 2), (9, 3), (10, 5)\}\). For \(d = 8, 9\) the possibility \(p_a(\Gamma) = 1, \deg(\Gamma) = 3\) is ruled out, because \(\Gamma\) would have to be a plane curve and cannot meet \(C\) in \(\geq 10\) points. This proves the lemma. \(\square\)

**Corollary 2.2.** Let \((d, g) \in \mathcal{C}(Q)\) and let \(C_{d,g} \subset S \subset \mathbb{P}^1\) be as above. If \(\text{Pic}(S) = \mathbb{Z}H_S \oplus \mathbb{Z}C_{d,g}\), then the divisor \(3H_S - C_{d,g}\) is free on \(S\).

**Proof.** As before, we write \(C\) for \(C_{d,g}\). First, let us show that \(3H_S - C\) is nef on \(S\). By Lemma 2.1, if \(\Gamma \subset S\) and \(\Gamma \cdot (3H_S - C) < 0\), then \(\deg \Gamma\) and \(p_a(\Gamma)\) must belong to a finite list of possibilities, depending on \((d, g) \in \mathcal{C}(Q)\). Using this observation, we shall check that for each \((d, g) \in \mathcal{C}(Q)\) there is no curve \(\Gamma\) with \(\Gamma \cdot (3H_S - C) < 0\).

There exist \(a, b \in \mathbb{Z}\) such that \(\Gamma = aH_S + bC\). This implies that

\[
\Gamma^2 = 6a^2 + 2dab + b^2(2g-2). \tag{2.1}
\]

Since \(\Gamma \cdot H_S = m = 6a + bd\), then \(b = (m-6a)/d\). For each \((d, g) \in \mathcal{C}(Q)\) and each case in Lemma 2.1 we may check the quadratic (2.1) has no solutions with \(a, b \in \mathbb{Z}\), unless \((d, g) = (13, 14)\). In the case \((13, 14)\), there is one integer solution \((a, b) = (-2,1)\). However, this possibility is ruled out, because \((-2H_S + C) \cdot C = -2d + 2g - 2 = 0\). This shows that \(3H_S - C\) is nef.

Since \(3H_S - C\) is nef, by [S-D] or [Knu, Lem.2.4, p.217], if \(3H_S - C\) is not free, then there exist curves \(E, \Gamma\) and an integer \(k \geq 2\) such that

\[
3H_S - C \sim kE + \Gamma, \quad E^2 = 0, \quad \Gamma^2 = -2, \quad E \cdot \Gamma = 1.
\]

Since \((3H_S - C)^2 = 52 + 2g - 6d\) and \((kE + \Gamma)^2 = 2k - 2\), we see that \(k = 27 + g - 3d\). Also, \(H_S \cdot (kE + \Gamma) = H_S \cdot (3H_S - C) = 18 - d\). Since \(E^2 = 0\), then \(p_a(E) = 1\) and \(H_S \cdot E \geq 3\). This implies that \(H_S \cdot \Gamma \leq 18 - d - 3k\). Consequently, for each \((d, g) \in \mathcal{C}(Q) \setminus \{(9, 2), (10, 5), (11, 8)\}\) we may check that \(H_S \cdot \Gamma \leq 0\), which is a contradiction. When \((d, g)\) is \((9, 2), (10, 5)\) or \((11, 8)\), the above inequality implies that \(H_S \cdot \Gamma\) is at most 3, 2 or 1, respectively. Writing \(\Gamma = aH_S + bC\), the equality \(\Gamma^2 = -2\) becomes

\[
6a^2 + 2dab + b^2(2g-2) = -2 \tag{2.2}
\]
and $H_S \cdot \Gamma = 6a + db$ is 1, 2 or 3. We may check that there are no integer solutions to \((2.2)\) for the given values of \((d,g)\).

\[\Box\]

**Proposition 2.3.** For each \((d,g) \in C(Q)\), there exists a smooth curve \(C\) of degree \(d\) and genus \(g\) on a smooth quadric threefold \(Y\) such that the blow-up of \(Y\) along \(C\) is a smooth weak Fano threefold of Picard number two.

**Proof.** By [Knu, Th.1.1, p.202], for each \((d,g) \in C(Q)\) there exists a smooth curve \(C\) of degree \(d\) and genus \(g\) lying on a smooth K3 surface \(S \subset \mathbb{P}^4\) of degree 6 with the property \(\text{Pic}(S) = \mathbb{Z}H_S \oplus \mathbb{Z}C\).

Let \(I_S\) be the ideal sheaf of \(S\) in \(\mathbb{P}^4\). We have \(h^0(\mathcal{O}_S(2)) = 14\) and \(h^0(\mathcal{O}_{\mathbb{P}^4}(2)) = 15\). Therefore, from the long exact cohomology sequence associated to \(0 \rightarrow I_S(2) \rightarrow \mathcal{O}_{\mathbb{P}^4}(2) \rightarrow \mathcal{O}_S(2) \rightarrow 0\) we obtain \(h^0(I_S(2)) \geq 1\). Since \(\text{deg } S = 6\) this implies that \(S\) is contained in a unique quadric \(Y\), [Bea96, Ex.VIII.14, p.97]. If \(Y\) is singular, then \(Y\) contains a plane, which cuts out a cubic plane curve \(\Gamma\) on \(S\) with \(\Gamma^2 = 0\) and \(\Gamma \cdot H = 3\). We may check that for each \((d,g) \in C(Q)\), the class of \(\Gamma\) cannot be expressed as an integral linear combination of \(H\) and \(C\). Therefore, the quadric \(Y\) must be smooth.

Let \(\tilde{S} \subset X\) be the birational transform of \(S\) under the blow-up \(\phi: X \rightarrow Y\) along \(C\). We may and shall identify \(\tilde{S}\) with \(S\). Since \(\tilde{S} \in |-K_X| = |3H - E|\), then \(\text{Bs}(|-K_X|) \subset \text{Bs}(|3H - E|_\tilde{S}) = \text{Bs}(|3H_S - C|)\). By Corollary 2.2, the divisor \(3H_S - C\) is free on \(S\), and therefore, \(-K_X\) is also free. In particular, \(-K_X\) is nef.

To see that \(-K_X\) is not ample, it suffices to show that \(C\) has a trisecant line. Indeed, any trisecant \(L\) to \(C\) is necessarily contained in the quadric \(Y\) and the proper transform of \(L\) is \((-K_X)\)-trivial. A formula of Berzolari, see for example [LeB], gives that the number of trisecant lines to a curve of degree \(d\) and genus \(g\) in \(\mathbb{P}^4\) is

\[\theta(d,g) = \binom{d-2}{3} - g(d-4).\]

We may check that for each \((d,g) \in C(Q)\), the number \(\theta(d,g)\) is positive. Therefore, \(C\) has a trisecant line.

Since \(-K_X\) is nef, then to show that \(-K_X\) is big, it suffices to check that \((-K_X)^3 > 0\), [LazI, Thm.2.2.16, p.144]. For this, we use the formula

\[(-K_X)^3 = (-K_Y)^3 + 2K_Y \cdot C - 2 + 2g = 52 - 6d + 2g.\]
see for example [BL, Lem.2.4], and check that for each \((d, g) \in \mathcal{C}(Q)\) we have \((-K_X)^3 > 0\). □

**Theorem 2.4.** The numerical invariants listed in cases 44-46, 48, 70, 86, 88, 97, 105 in [CM10, Table E1-E1] of Sarkisov links are geometrically realizable.

**Proof.** In each of the listed cases the weak Fano threefold \(X\) exists by Proposition 2.3. It suffices to check that \(|-K_X|\) determines a small contraction and \(\phi^+\) is of type E1. The numerical possibilities with \(|-K_X|\) giving a divisorial contraction are classified in [JPR05, Table A.4, p.629]. If the morphism \(\phi^+\) were not of type E1, then the listed numerical possibilities would have appeared either in [JPR11, 7.4, 7.7, p.486] or in the non-E1-E1 tables in [CM10]. □

**Remark 2.5.** The links with numerical invariants listed under 47, 72, 102 in [CM10, Table E1-E1] also appear in the table [JPR05, A4, p.629]. Therefore, to decide if these three numerical links exist in our case, we must check if \(|-K_X|\) determines a small contraction.

3. **Blow-ups of the Intersection of Two Quadrics in \(\mathbb{P}^5\)**

Consider the list

\[
\mathcal{C}(V_4) = \{(7, 0), (8, 2), (9, 4), (10, 6), (11, 8), (7, 1), (8, 3), (7, 2)\}
\]

of pairs \((d, g)\), which correspond to numerical possibilities 30, 34, 37, 39, 41, 64, 67, 84 in [CM10]. In this section \(Y \subset \mathbb{P}^5\) is a smooth intersection of two quadrics and \(S \subset \mathbb{P}^5\) is a smooth complete intersection of three quadrics. The remaining notation and the strategy of proofs are as in Section 2.

**Lemma 3.1.** Assume that \((d, g) \in \mathcal{C}(V_4)\) and \(C_{d,g} \subset S \subset \mathbb{P}^5\). Then the divisor \(2H_S - C_{d,g}\) is nef on \(S\), provided

(A) \(C_{d,g}\) has no \((2m + 1)\)-secant smooth rational curve of degree \(m\) for

1. \(m = 1,\) if \((d, g) = (11, 8)\);
2. \(m = 1, 2,\) if \((d, g) \in \{(7, 2), (8, 3), (10, 6)\}\);
3. \(1 \leq m \leq 3,\) if \((d, g) \in \{(7, 1), (9, 4)\}\);
4. \(1 \leq m \leq 4,\) if \((d, g) = (8, 2)\);
5. \(1 \leq m \leq 5,\) if \((d, g) = (7, 0)\).

(B) \(C_{d,g}\) has no \((2m + 1)\)-secant curve with \(\text{deg} = 3\) and \(p_a = 1\) if \((d, g) = (7, 0)\).
Proof. We proceed as in the proof of Lemma 2.1. For simplicity we write $C$ for $C_{d,g}$, as before. Let $I_C$ and $I_S$ be the ideal sheaves of $C$ and $S$ in $\mathbb{P}^5$, respectively. We may check that $h^0(I_C(2)) \geq 6$ and $h^0(I_S(2)) = 3$. Hence, there exist quadrics in $\mathbb{P}^5$ that contain $C$ but not $S$.

The divisor $2H_S - C$ is not nef if and only if there exists an irreducible curve $\Gamma \subset \mathbb{P}^5$ such that $\deg \Gamma = m$ and $C \cdot \Gamma \geq 2m + 1$. This implies that $\Gamma$ is contained in every quadric that contains $C$ and, in particular, $\Gamma \subset S$. Since there exist quadrics containing $C$ but not $S$, we must have $\deg(\Gamma \cup C) \leq 16$. Hence, $\deg \Gamma \leq 16 - d$.

As in the proof of Lemma 2.1, the Hodge index theorem implies $p_a(C \cup \Gamma) \leq (d + m)^2/16 + 1$. Combining this with $p_a(C \cup \Gamma) \geq p_a(C) + p_a(\Gamma) + 2m$ gives

$$p_a(\Gamma) \leq \frac{(d + m)^2}{16} + 1 - g - 2m.$$  
  
For each $(d, g) \in \mathcal{C}(V_4)$, an elementary calculation, using $\deg \Gamma \leq 16 - d$ and the above inequality, shows that $p_a(\Gamma)$ is either 0 or 1. The case $p_a(\Gamma) = 1$ occurs only for $(d, g) = (7, 0)$ and $m = 3$. In the case $p_a(\Gamma) = 0$, the calculation gives that the possible values of $\deg \Gamma$ are only the ones listed in the statement of the lemma. □

**Corollary 3.2.** Let $(d, g) \in \mathcal{C}(V_4)$ and let $C_{d,g} \subset S \subset \mathbb{P}^5$ be as above. If $\text{Pic}(S) = \mathbb{Z}H_S \oplus \mathbb{Z}C_{d,g}$, then the divisor $2H_S - C_{d,g}$ is free on $S$.

**Proof.** As before, we write $C$ for $C_{d,g}$. First, let us show that $2H_S - C$ is nef on $S$. By Lemma 3.1 if $\Gamma \subset S$ and $\Gamma \cdot (2H_S - C) < 0$, then $\deg \Gamma$ and $p_a(\Gamma)$ must belong to a finite list of possibilities, depending on $(d, g) \in \mathcal{C}(V_4)$. Using the assumption $\text{Pic}(S) = \mathbb{Z}H_S \oplus \mathbb{Z}C_{d,g}$, we shall eliminate these possibilities.

There exist $a, b \in \mathbb{Z}$ such that $\Gamma = aH_S + bC$. This implies that

$$\Gamma^2 = 8a^2 + 2dab + b^2(2g - 2). \quad (3.1)$$

Since $\Gamma \cdot H_S = m = 8a + bd$, then $b = (m - 8a)/d$. For each $(d, g) \in \mathcal{C}(V_4)$ and each case in Lemma 3.1 we may check the quadratic (2.2) has no solutions with $a, b \in \mathbb{Z}$, unless $(d, g) = (10, 6)$ and $m = 2$. In the case $(10, 6)$, there is one integer solution $(a, b) = (-1, 1)$. However, this possibility is ruled out, because $(-H_S + C) \cdot C = -d + 2g - 2 = 0$. This shows that $3H_S - C$ is nef.
Since $2H_S - C$ is nef, by [S-D] or [Knu, Lem.2.4, p.217], if $2H_S - C$ is not free, then there exist curves $E, \Gamma$ and an integer $k \geq 2$ such that

$$2H_S - C \sim kE + \Gamma, \quad E^2 = 0, \quad \Gamma^2 = -2, \quad E \cdot \Gamma = 1.$$ 

Since $(2H_S - C)^2 = 30 - 4d + 2g$ and $(kE + \Gamma)^2 = 2k - 2$, we see that $k = 16 - 2d + g$. Also, $H_S \cdot (kE + \Gamma) = H_S \cdot (2H_S - C) = 16 - d$. Since $E^2 = 0$, then $p_a(E) = 1$ and $H_S \cdot E \geq 3$. This implies that $H_S \cdot \Gamma \leq 16 - d - 3k$. As a consequence, for each $(d, g) \in C(V_4) \setminus \{(7, 0), (8, 2), (9, 4)\}$ we may check that $H_S \cdot \Gamma$ is at most 3, 2 or 1, respectively. Writing $\Gamma = aH_S + bC$, the equality $\Gamma^2 = -2$ becomes

$$8a^2 + 2dab + b^2(2g - 2) = -2 \quad (3.2)$$

and $H_S \cdot \Gamma = 8a + db$ is 1, 2 or 3. We may check that there are no integer solutions to (3.2) for the given values of $(d, g)$. □

**Proposition 3.3.** For each $(d, g) \in C(V_4)$, there exists a smooth curve $C$ of degree $d$ and genus $g$ on a smooth intersection $Y$ of two quadrics in $\mathbb{P}^5$ such that the blow-up of $Y$ along $C$ is a smooth weak Fano threefold of Picard number two.

**Proof.** By [Knu, Th.1.1, p.202], for each $(d, g) \in C(V_4)$ there exists a smooth curve $C$ of degree $d$ and genus $g$ lying on a smooth complete intersection K3 surface $S \subset \mathbb{P}^5$ of degree 8 with the property $\text{Pic}(S) = \mathbb{Z}H_S \oplus \mathbb{Z}C$.

The linear system $|2H_S - S|$ of quadrics containing $S$ has dimension two and its base locus is $S$. Since $S$ is non-singular, no quadric in $|2H_S - S|$ can be singular at a point of $S$. By Bertini’s theorem, a general member of $|2H_S - S|$ is also smooth outside of $S$. Therefore, a general member $Q \in |2H_S - S|$ is a smooth quadric. The restricted linear system $|2H_S - S|_Q$ is a pencil on $Q$ with base locus $S$. Again by Bertini’s theorem, for a general quadric $Q' \in |2H_S - S|$ the intersection $Q \cap Q'$ is smooth outside of $S$. Since $S$ is a complete intersection of quadrics, then $Q \cap Q'$ must be smooth along $S$ as well. This shows that $S$ is contained in a smooth intersection of two quadrics in $\mathbb{P}^5$, which is the promised threefold $Y$.

Let $X \to Y$ the blow-up of $C$ and let $\tilde{S}$ be the birational transform of $S$. By Corollary 3.2 the divisor $2H_S - C$ is free on $S$. Since $\tilde{S} \in |-K_X| = |2H - E|$, then, as in the proof of
we conclude that $-K_X$ is also free and, in particular, nef. Since $-K_X$ is nef and $(-K_X)^3 > 0$, then $-K_X$ is big. By classification \[IP99\] 12.3, p.217, $-K_X$ is not ample. □

**Theorem 3.4.** The numerical invariants listed in cases 30, 34, 37, 41, 64, 84 in \[CM10\] Table E1-E1] of Sarkisov links are geometrically realizable.

*Proof.* The proof is the same as that of Theorem 2.4. □

**Remark 3.5.** The links with numerical invariants listed under 39 and 67 in \[CM10\] Table E1-E1] also appear in the table \[JPR05\] A4, p.629]. Therefore, to decide if these two numerical links exist in our case, we must check if $| - K_X |$ determines a small contraction.

### 4. Blow-ups of $V_5$

In this section $Y \subset \mathbb{P}^9$ is a smooth section of the Plücker-embedded Grassmannian $G(1, 4) \subset \mathbb{P}^9$ of lines in $\mathbb{P}^4$ by a linear subspace of codimension three. The remaining notation is as before. The open cases are 31, 35, 38, 40, 42, 43, 65, 68, 69, 81, 83, 94, 96, 101 with $(d, g)$ in

$$
C(V_5) := \{(9, 0), (10, 2), (11, 4), (12, 6), (13, 8), (14, 10), (9, 1), (10, 3), (12, 7), (9, 2), (8, 0), (9, 3), (8, 1), (7, 0)\}.
$$

In this section we shall show that all of the above numerical cases are geometrically realizable, except the case with $(d, g) = (14, 10)$, which is not realizable.

In the sequel, unless otherwise stated, we fix $(d, g) \in C(V_5) \setminus \{(12, 7), (13, 8), (14, 10)\}$. Let $S \subset \mathbb{P}^6$ be a smooth K3 surface of degree 10 and genus 6 with the following properties. The surface $S$ is cut out by quadrics and $\text{Pic}(S) = \mathbb{Z}T \oplus \mathbb{Z}C$, where $T$ is a general hyperplane section of $S$ (a smooth canonical curve of genus 6) and $C$ is a smooth curve of degree $d$ and genus $g$.

The existence of such surfaces was established in \[Knu\]. Although both $S$ and $C$ depend on $(d, g)$, we omit $d, g$ from the notation for simplicity.

We shall use Gushel’s method (see \[Gu82\] and \[Gu92\]) to show that $S$ can be embedded into a smooth del Pezzo threefold $V_5$. This method relies on Maruyama’s construction of regular vector bundles, see \[Ma\]. First, we shall construct a globally generated rank two vector bundle $M$ on $S$ such that $h^0(M) = 5$. Let $\mathbb{P}^1_S := \mathbb{P}^1 \times S$ be the trivial $\mathbb{P}^1$-bundle over $S$ and let $D$ be a $g_4^1$ on $T$. Since $S$ is cut out by quadrics, so is $T$. Therefore, by Enriques-Babbage theorem $T$ is non-trigonal. In particular, this implies that $|D|$ is free. Fix a point $p \in \mathbb{P}^1$ and let $Y$ be a
general member of the linear system $|\{p\} \times T + \mathbb{P}^1 \times D|$ on $\mathbb{P}^1_T$. The situation is summarized in the following diagram

$$
\begin{array}{c}
Y \\ \downarrow \scriptstyle{T} \\
\mathbb{P}^1_T \\ \downarrow
\end{array}
\begin{array}{c}
\mathbb{P}^1 \\
\downarrow \\
S
\end{array}
$$

where the square is Cartesian and $\pi$ is the natural projection. Let $I_Y$ be the ideal sheaf of $Y$ in $\mathbb{P}^1_S$ and let $G := \{p\} \times S + \mathbb{P}^1 \times T$. Define $M = \pi_* (I_Y \otimes \mathcal{O}_{\mathbb{P}^1_S}(G))$. By [Ma, Principle 2.6, p.112] or [Gu82, Lem.1.4], $M$ is a vector bundle of rank two on $S$.

Let us show that $M$ is globally generated. Let $W = H^0(D)$. As in [Gu82, 1.5.2], there is a short exact sequence

$$
0 \to W^* \otimes \mathcal{O}_S \to M \to \mathcal{O}_T(K_T - D) \to 0 \tag{4.1}
$$

(note that $T^2$ in Gushel’s notation is a hyperplane section of $T$, which in our case is a canonical divisor $K_T$). Using the facts $h^1(W^* \otimes \mathcal{O}_S) = h^1(\mathcal{O}_S) = 0$, dim $W^* = 2$, and $h^0(K_T - D) = 3$, the long exact sequence associated to (4.1) gives $h^0(M) = 5$. By [Gu92, Prop.1.6], the vector bundle $M$ is globally generated if and only if $|K_T - D|$ is free. The linear system $|K_T - D|$ is a $g_6^5$. Since $T$ is non-hyperelliptic and is cut out by quadrics then by Enriques-Babbage theorem $T$ is non-trigonal and not a plane quintic. This implies that $|K_T - D|$ is free and completes the proof that $M$ is globally generated. Therefore, the evaluation homomorphism $H^0(M) \otimes \mathcal{O}_S \to M$ determines a morphism $\alpha : S \to \mathbb{G}(1,4)$. By [Ma, Cor.2.19.1, p.121], $c_1(M) = T$ and $c_2(M) = D$. Since $\alpha^* \mathcal{O}_{\mathbb{G}(1,4)}(1) = c_1(M)$, the morphism $\alpha$ is given by a subsystem of $|T|$. In particular, $\alpha$ is a finite morphism.

In what follows we shall use the notation of [KL] for Schubert calculus. In this notation, given a flag $A_0 \subseteq A_1 \subseteq \mathbb{P}^4$ with $a_i = \dim A_i$, the symbols $\Omega(a_0,a_1) = \Omega(A_0,A_1)$ denote the Schubert variety of lines $L$ in $\mathbb{P}^4$ such that $\dim (L \cap A_i) \geq i$. The class of the Schubert variety in the cohomology ring $H^*(\mathbb{G}(1,4),\mathbb{Z})$ is denoted by the same symbol as the variety itself.

**Lemma 4.1.** Every element of $|T|$ is irreducible and reduced.

**Proof.** Recall the ongoing assumption that $(d, g) \in \mathcal{C}(V_5) \setminus \{(12, 7), (13, 8), (14, 10)\}$. Suppose $T \sim D_1 + D_2$ with $D_1, D_2$ non-trivial and write $D_1 \sim aT + bC$ for some $a, b \in \mathbb{Z}$. Since $T$ is very ample,

$$
0 < D_1 \cdot T < T^2 = 10. \tag{4.2}
$$
If $C$ is not a component of $D_1$ and $D_2$, then $D_i \cdot C \geq 0$, and therefore,
\[ 0 \leq D_1 \cdot C \leq T \cdot C. \quad (4.3) \]

If $C$ is a component of either $D_1$ or $D_2$, without loss of generality, we shall assume that $D_1 = C$ and $D_2 = T - C$.

Assume $(d, g) = (7, 0)$, then the inequalities (4.2) and (4.3) become $0 < 10a + 7b < 10$ and $0 \leq 7a - 2b \leq 7$, respectively. Since either $a > 0$ or $1 - a > 0$, we may check that the two inequalities may not hold simultaneously. Therefore, it remains to consider the case when $D_1 = C$ and $D_2 = T - C$. In this case we may check that $\deg D_2 = 3$ and $D_2^2 = -6$. This implies that $D_2$ must have a line as a component. However, using $\text{Pic}(S) = \mathbb{Z}T \oplus \mathbb{Z}C$, we may check that $S$ does not contain lines. The remaining cases can be handled analogously and we omit the details. \qed

**Remark 4.2.** By [Laz86], $T$ is Brill-Noether general.

**Lemma 4.3.** The composition of $\alpha : S \to \mathbb{G}(1, 4)$ with the Plücker embedding $\mathbb{G}(1, 4) \hookrightarrow \mathbb{P}^9$ has degree one.

**Proof.** By the basis theorem [KL, p.1071], the class of $\alpha(S)$ in $H^*(\mathbb{G}(1, 4), \mathbb{Z})$ can be expressed as $[\alpha(S)] = a\Omega(0, 3) + b\Omega(1, 2)$ for some $a, b \in \mathbb{Z}$. Therefore, we may compute
\[ 4 = \alpha_* \left( c_2(M) \right) = \alpha_* \alpha^* \Omega(2, 3) = \deg(\alpha)(\Omega(2, 3) \cdot [\alpha(S)]) = \deg(\alpha)b. \]

Furthermore, since $c_1(M)^2 = T^2 = 10$, the degree of $\alpha$ must divide 10. Hence, we have only two possibilities: (1) $\deg(\alpha) = 1, a = 6, b = 4$; (2) $\deg(\alpha) = 2, a = 3, b = 2$.

Assume that $\deg(\alpha) = 2$. Let $\langle \alpha(S) \rangle$ denote the linear span of $\alpha(S)$ in the Plücker embedding. Since $h^0(\mathcal{O}_S(T)) = 7$ and $|T|$ is very ample, then $\deg(\alpha) = 2$ implies $3 \leq \dim(\alpha(S)) \leq 5$. We shall eliminate the possibilities $\dim(\alpha(S)) = 3, 4, 5$ and conclude that $\alpha$ has degree one.

Suppose $\dim(\alpha(S)) = 3$. We may check that $\langle \alpha(S) \rangle$ is the intersection of all quadrics containing $\alpha(S)$ and since $\mathbb{G}(1, 4) \subset \mathbb{P}^9$ is also an intersection of quadrics then $\langle \alpha(S) \rangle \subset \mathbb{G}(1, 4)$. Therefore, $\langle \alpha(S) \rangle$ is a maximal linear subvariety of $\mathbb{G}(1, 4)$ and its class in $H^*(\mathbb{G}(1, 4), \mathbb{Z})$ must be $\Omega(0, 4)$. Since $\deg(\alpha) = 2$, the surface $\alpha(S)$ has degree 5 and is the intersection of $\langle \alpha(S) \rangle$ with a quintic hypersurface in $\mathbb{P}^9$. Therefore,
\[ [\alpha(S)] = 5\Omega(2, 4) \cdot \Omega(0, 4) = 5\Omega(0, 3), \]
which is impossible because \([\alpha(S)] = 3\Omega(0, 3) + 2\Omega(1, 2)\).

Suppose \(\dim<\alpha(S)> = 4\). Let \(Y\) be the intersection of all quadrics containing \(\alpha(S) \subset \mathbb{P}^9\). Then \(Y \subset <\alpha(S)> \cap \mathbb{G}(1, 4)\). Since \(\mathbb{G}(1, 4)\) does not contain any linear spaces of dimension 4, \(<\alpha(S)>\) is not in \(\mathbb{G}(1, 4)\). Hence, \(Y \neq <\alpha(S)>\). Also, since \(\alpha(S)\) has degree 5 and is in \(\mathbb{P}^4\), \(Y \neq \alpha(S)\). This shows that \(Y\) is a quadric hypersurface in \(\mathbb{P}^4\). Since the divisor \(\alpha(S) \subset Y\) has degree 5, \(<\alpha(S)>\) is not in \(\mathbb{G}(1, 4)\). Hence, \(Y \neq <\alpha(S)>\). Also, since \(\alpha(S)\) has degree 5 and is in \(\mathbb{P}^4\), \(Y \neq \alpha(S)\). This shows that \(Y\) is a quadric hypersurface in \(\mathbb{P}^4\). Since the divisor \(\alpha(S) \subset Y\) has degree 5, \(Y\) must be singular. Let \(\alpha(S) \dasharrow Q\) be the map induced by the projection \(Y \dasharrow Q\) from a singular point on \(Y\). The variety \(Q \subset \mathbb{P}^3\) is either a smooth quadric or a quadric cone. In any case, a general hyperplane section of \(Q\) is reducible and its pull-back to \(S\) gives a reducible hyperplane section of \(S\). This contradicts Lemma \([4.1]\).

Suppose \(\dim<\alpha(S)> = 5\). In this case the morphism \(\alpha: S \rightarrow \mathbb{G}(1, 4) \subset \mathbb{P}^9\) can be factored into the closed embedding \(S \hookrightarrow \mathbb{P}^6\) given by \([T]\) followed by the projection \(\pi: \mathbb{P}^6 \dasharrow \mathbb{P}^5 \subset \mathbb{P}^9\) from some point \(P \in \mathbb{P}^6\). The inverse image \(\Lambda := \pi^{-1}(\alpha(S))\) is a cone whose vertex we shall denote by \(P\). Since \(S \subset \mathbb{P}^6\) is the intersection of quadrics, there is a quadric \(Q \subset \mathbb{P}^6\) such that \(S = \Lambda \cap Q\) (a priori \(S \subset \Lambda \cap Q\), but \(\deg S = \deg(\Lambda \cap Q) = 10\) implies the equality).

Since \(S\) is smooth, \(\Lambda\) must be smooth along \(S\). This implies that \(\alpha(S)\) is smooth. Indeed, if \(\alpha(S)\) were singular, \(\Lambda\) would contain a line of singularities, which necessarily meets \(Q\), giving a singular point of \(\Lambda\) which lies on \(S\). Therefore, \(\alpha(S) \subset \mathbb{P}^5\) is a smooth non-degenerate surface of degree 5, hence must be a del Pezzo surface. The Picard numbers of \(S\) and \(\alpha(S)\) are 2 and 5, respectively. This is impossible because \(S \rightarrow \alpha(S)\) is finite of degree 2.

**Lemma 4.4.** The surface \(\alpha(S)\) is not contained in any Schubert variety of type \(\Omega(2, 4)\) in \(\mathbb{G}(1, 4)\).

**Proof.** Suppose that \(\alpha(S)\) is contained in the Schubert variety \(\Omega(P_2, P_4)\), consisting of lines that meet a fixed 2-plane \(P_2 \subset P_4 = \mathbb{P}^4\). Consider the diagram

\[
\begin{array}{ccc}
\mathbb{P}(M) & \xrightarrow{\pi} & \mathbb{P}^5 \\
p \downarrow & & \downarrow \\
S & \xrightarrow{p} & P_4,
\end{array}
\]

where \(p\) is the natural projection and \(\pi\) is the morphism given by the complete linear system \(|\mathcal{O}_{\mathbb{P}(M)}(1)|\). Let \(B\) be the image of \(\pi\).
Let \( S' \) be the blow-up of \( S \) along four general points that belong to a \( g^1_4 \) on \( T \subset S \). We may check that \( \pi(S') \) is a hyperplane section of \( B \). Since \( \alpha(S) \subset \Omega(P_2, P_4) \), the image of every fiber of \( p \) under \( \pi \) intersects \( P_2 \). There are two cases to consider.

First, if \( P_2 \subset B \) then \( L := P_2 \cap \pi(S') \) is a line. If \( \pi(S') \) is general then \( \pi^{-1}(L) \) is not a union of fibers of \( p \) (otherwise, \( B \subset P_2 \), which is impossible, because \( c_1(O_{\mathbb{P}(M)}(1))^3 = 6 \) and \( \dim B = 3 \)). Thus, we may and shall assume that \( \pi^{-1}(L) \) is not a union of fibers of \( p \). A general hyperplane through \( L \) cuts \( \pi(S') \) in a reducible curve, whose proper transform on \( S \) is a reducible hyperplane section of \( S \). This contradicts Lemma 4.1.

Second, \( P_2 \cap B \) is a plane curve \( \Gamma \). If the images of all the fibers of \( p \) pass through a single point, then \( \alpha(S) \) is contained in a Schubert variety of type \( \Omega(0, 4) \), which is a \( \mathbb{P}^3 \) in the Plücker embedding. By an argument as in the third paragraph of the proof of Lemma 4.3 this is impossible because \([\alpha(S)] = 6\Omega(0, 3) + 4\Omega(1, 2)\). Thus, the images of the fibers of \( p \) do not pass through a single point. Therefore, there is an irreducible component of \( \Gamma \), which we also denote by \( \Gamma \), such that for every \( x \in \Gamma \), \( p(\pi^{-1}(x)) \) is a curve in \( S \). A general hyperplane section of \( B \) can be written as \( \pi(S') \), where \( S' \) is as above. Let \( x \in \Gamma \cap \pi(S') \) and take a general hyperplane section \( H \) of \( \pi(S') \) passing through \( x \) (here general means that \( H \) does not contain any of the images of the four exceptional divisors of the blow-up \( S' \to S \)). The proper transform of \( H \) in \( S \) is a reducible hyperplane section of \( S \), which contradicts Lemma 4.1.

**Proposition 4.5.** The composition of \( \alpha: S \to \mathbb{G}(1, 4) \) with the Plücker embedding \( \mathbb{G}(1, 4) \hookrightarrow \mathbb{P}^9 \) is given by the complete linear system \(|T|\). In particular, the linear span \( \langle \alpha(S) \rangle \) in \( \mathbb{P}^9 \) is isomorphic to \( \mathbb{P}^6 \).

**Proof.** By Lemma 4.3 the morphism \( S \to \mathbb{P}^9 \) has degree one. Therefore, it suffices to show that \( \langle \alpha(S) \rangle \) has dimension 6. Suppose \( \dim \langle \alpha(S) \rangle \leq 5 \), then \( \alpha(S) \) is contained in a 3-dimensional family of hyperplanes in \( \mathbb{P}^9 \). The subvariety of \( (\mathbb{P}^9)^* \) parametrizing hyperplanes \( H \subset \mathbb{P}^9 \) such that \( H \cap \mathbb{G}(1, 4) \) is of type \( \Omega(2, 4) \) has dimension 6. By Bézout’s theorem, this implies that \( \alpha(S) \) is contained in a Schubert variety of type \( \Omega(2, 4) \), which contradicts Lemma 4.4.

To simplify the notation in the sequel, we shall denote the image of \( S \) in \( \mathbb{P}^9 \) by \( S \) as well.

**Corollary 4.6.** The complete intersection of the linear span \( \langle S \rangle \) with \( \mathbb{G}(1, 4) \) in \( \mathbb{P}^9 \) is a smooth del Pezzo threefold \( V_5 \).
Proof. By Proposition 4.5, \( \langle S \rangle \simeq \mathbb{P}^6 \). First, let us show that if \( G(1,4) \cap \langle S \rangle \) is singular, then there is a hyperplane \( H \subset \mathbb{P}^9 \) containing \( \langle S \rangle \) such that \( G(1,4) \cap H \) is also singular. Let 
\[ V = \langle S \rangle \cap G(1,4) \]
and suppose \( p \) is a singular point of \( V \). Since \( \dim V = 3 \), the tangent space \( T_p V \) has dimension \( \geq 4 \). Assume that \( \dim T_p V = 4 \). Let \( v_1, v_2 \in T_p G(1,4) \) be such that the linear span \( \langle T_p V, v_1, v_2 \rangle \) is all of \( T_p G(1,4) \). Let \( l_1, l_2, l_3 \) be the linear forms on \( \mathbb{P}^9 \), whose zero locus is \( \langle S \rangle \). The linear forms \( l_i \) vanish on \( T_p V \). Furthermore, we may find scalars \( a, b, c \) such that the linear form 
\[ l := al_1 + bl_2 + cl_3 \]
vanishes on \( v_1 \) and \( v_2 \), hence on all of \( T_p G(1,4) \). The zero locus \( H := Z(l) \) is the desired hyperplane. When \( \dim T_p V > 4 \), the construction of \( H \) is similar.

Suppose \( H \) is a hyperplane containing \( \langle S \rangle \) such that \( G(1,4) \cap H \) is singular. We shall show that \( G(1,4) \cap H \) is a Schubert variety of type \( \Omega(2,4) \). The hyperplane section \( H \) is given by a skew-symmetric \( 5 \times 5 \) matrix \( A \). We may check that the singular locus of \( G(1,4) \cap H \) is the Grassmannian \( G(1, \mathbb{P}(\ker A)) \). Since \( G(1,4) \cap H \) is singular by assumption, \( \dim \ker A = 3 \). We may check that \( G(1,4) \cap H = \Omega(\mathbb{P}(\ker A), \mathbb{P}^4) \).

By Lemma 4.4, the image of \( S \) is not contained in any hyperplane section of type \( \Omega(2,4) \). Therefore, the intersection \( \langle S \rangle \cap G(1,4) \) is smooth and the lemma is proved.

\( \square \)

Theorem 4.7. For each \((d,g) \in C(V_5) \setminus \{(14,10)\} \), the associated weak Fano threefold with small anti-canonical map exists. The numerical case with \((d,g) = (14,10)\) is not realizable.

Proof. Let us first show that the case with \((d,g) = (14,10)\) is not realizable. Suppose to the contrary that there is a weak Fano threefold \( X \) obtained by blowing up a smooth curve \( C \) of degree 14 and genus 10. A general member \( \tilde{S} \in | - K_X | \) is a smooth surface, whose image in \( V_5 \) is a smooth K3 surface \( S \) containing \( C \). Let \( H_S \) be a hyperplane section of \( S \). By [JPR05], Prop.2.5, p.451], \( | - K_X | \) is free, and therefore, the linear system \( | 2H_S - C | \) is also free on \( S \) (by the same reasoning as in the proof of Proposition 2.3). A general member \( C' \in | 2H_S - C | \) is a smooth curve of degree 6 and genus 2. The curve \( C' \) lies on \( V_5 \) and the linear span of \( C' \) in \( \mathbb{P}^6 \) has dimension at most 4. Thus, \( C' \) is contained in two distinct hyperplanes \( H_1 \) and \( H_2 \) in \( \mathbb{P}^6 \). Since \( V_5 \) is linearly normal and \( \rho(V_5) = 1 \), the intersection \( V_5 \cap H_1 \cap H_2 \) is a curve of degree 5, which contains \( C' \). However, \( \deg C' = 6 \) and we have reached a contradiction. This shows that the case with \((d,g) = (14,10)\) is not realizable.

Next, let us show that the cases with \((d,g) = (12,7)\) and \((13,8)\) are geometrically realizable. By [JPR05], A4, No.10 there is a weak Fano threefold obtained by blowing up a smooth curve
C' ⊂ V_5 of degree 8 and genus 3. By the argument as in the previous paragraph, the curve C' lies on a smooth K3 surface S ⊂ V_5 and the linear system |2H_S−C'| is free. A general member C ∈ |2H_S−C'| is a smooth curve of degree 12 and genus 7. The linear system |2H_S−C| has no base points outside of its fixed components by [S-D, Cor.3.2, p.611]. Also, since C' ∈ |2H_S−C| and C' is not rational, it follows from [S-D, 2.7, p.610] that |2H_S−C| is free. This implies that the blow-up of V_5 along C is a weak Fano threefold, whose anti-canonical map must be small, because this numerical case does not appear on the tables in [JPR05]. The case (d, g) = (13, 8) can be handled in a similar way by starting with the weak Fano threefold appearing as No.7 on Table E1-E5 in [CM10].

In the remaining cases (d, g) ∈ C(V_5)\{(12, 7), (13, 8), (14, 10)}. By Corollary 4.6, there is a smooth del Pezzo threefold V_5 containing Knutsen’s K3 surface S such that C ⊂ S. Using the method introduced in Section 2, we may check that the blow-up of C on V_5 gives a weak Fano threefold with small anti-canonical map. □

5. Cases 3, 94, 100

In the cases 3 and 100 the variety Y is X_{10} and X_{18}, respectively. In the case 94, the variety Y^+ is X_{16}. Note that case 94 is realizable by Section 4 but here we shall give an alternative construction. The existence of twisted cubics on the anti-canonical models of X_{10}, X_{16}, and X_{18} follows from [IP99, Lemma 4.6.1]. As a consequence of [IP99, Lemma 4.6.2], blowing up any twisted cubic on X_{16} and X_{18}, and a sufficiently general twisted cubic on X_{10}, gives rise to weak Fano threefolds with free anti-canonical system. The proof of Lemma 4.6.2 was only sketched [IP99]. For completeness of presentation we shall include the full argument.

In the cases when Y is X_{10} or X_{18}, to prove that |−K_X| is free, it suffices to check that the linear span ⟨C⟩ ∼ P^3 of the twisted cubic C ⊂ Y in the anti-canonical embedding of Y satisfies ⟨C⟩ ∩ Y = C. Indeed, this ensures that Bs(|H−C|) = C, which implies that |−K_X| is free. Since the anti-canonical model of Y is cut out by quadrics [IP99, Cor.4.1.13, p.70], the intersection ⟨C⟩ ∩ Y is a curve cut out by quadrics. If ⟨C⟩ ∩ Y ≠ C, we must have ⟨C⟩ ∩ Y = C ∪ Z, where Z is a line which is either a bisecant or a tangent of C. If such a line Z existed, then C ⋅ (−K_Y−2Z) < 0 and C ⊂ Bs(|−K_Y−2Z|). However, since the genus of Y is ≥ 7, the base locus of |−K_Y−2Z| is precisely the set of lines meeting Z by [Isk89, Lem.1, p.268], which is a contradiction. This shows ⟨C⟩ ∩ Y = C, as required.
The case $Y = X_{10}$ needs a different argument. It follows from [Isk89, Lem.1, p.268] that $h^0(-K_Y - 2Z) = 1$ and since $\rho(Y) = 1$, the unique surface $F \in |-K_Y - 2Z|$ is irreducible. If a general twisted cubic $C \subset Y$ has a bisecant or a tangent line $Z \subset Y$, then for some line $Z$ the surface $F$ has a two-dimensional family of twisted cubics. The following lemma shows that this is impossible.

**Lemma 5.1 (Prokhorov).** The surface $F$ may not contain a two-parameter family of rational cubics.

**Proof.** Assume that $F$ contains a two-parameter family of rational cubics. Let $C$ be a rational cubic on $F$ and let $H$ be a hyperplane section of $F$. Take $\nu : F' \to F$ to be the normalization, $C'$ and $H'$ be the inverse images of $C$ and $H$. Let $\rho : \tilde{F} \to F'$ be the minimal resolution, i.e., $\tilde{F}$ is non-singular and $K_{\tilde{F}}$ is $\rho$-nef. Let $\tilde{C}$ and $\tilde{H}$ be the proper transforms of $C'$ and $H'$. Note that $\tilde{H}^2 = H^2 = 10$. The proof proceeds in several steps.

If $\rho(\tilde{F}) \leq 2$, then $\tilde{F}$ is isomorphic to $\mathbb{P}^2$ or one of the Hirzebruch surfaces $\mathbb{F}_n$ for $n \geq 0$. Since $\mathbb{P}^2$ has no divisors of square 10, $\tilde{F}$ is not isomorphic to $\mathbb{P}^2$. If $\tilde{F} \simeq \mathbb{F}_n$ for some $n \geq 0$, a calculation shows that at least one of the numerical constraints: $\tilde{H}$ is nef, $H^2 = 10$, $\tilde{C} \cdot \tilde{H} = 3$, and $-2 = (K_{\tilde{F}} + \tilde{C}) \cdot \tilde{C}$ is not satisfied. This implies that $\tilde{F}$ is not isomorphic to $\mathbb{F}_n$ for any $n \geq 0$. Therefore, by classification of surfaces, we conclude $\rho(\tilde{F}) \geq 3$.

The divisor $K_{\tilde{F}} + \tilde{H}$ must be nef. Otherwise, there is a $(-1)$-curve $E \subset \tilde{F}$ with $0 > (K_{\tilde{F}} + \tilde{H}) \cdot E = -1 + \tilde{H} \cdot E$, which implies $\tilde{H} \cdot E = 0$. Hence, $E$ must be $\rho$-exceptional and $K_{\tilde{F}} \cdot E = -1$ contradicts the fact that $K_{\tilde{F}}$ is $\rho$-nef.

Since $0 \leq (K_{\tilde{F}} + \tilde{H}) \cdot \tilde{C} = 1 - \tilde{C}^2$, then we must have $\tilde{C}^2 = 1$ and $(K_{\tilde{F}} + \tilde{H}) \cdot \tilde{C} = 0$. By the base-point free theorem for surfaces, $K_{\tilde{F}} + \tilde{H}$ is semi-ample. Since $\tilde{F}$ contains a two-parameter family of $(K_{\tilde{F}} + \tilde{H})$-trivial curves, the divisor $K_{\tilde{F}} + \tilde{H}$ is numerically trivial. This shows that $-K_{\tilde{F}}$ is nef and big, and therefore, $\tilde{F}$ is a smooth weak Fano surface. By Noether’s formula $K_{\tilde{F}}^2 \leq 9$. On the other hand, $K_{\tilde{F}}^2 = \tilde{H}^2 = 10$, which is a contradiction. \qed

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