Radiation Damping and Decoherence in Quantum Electrodynamics

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Abstract. The processes of radiation damping and decoherence in Quantum Electrodynamics are studied from an open system’s point of view. Employing functional techniques of field theory, the degrees of freedom of the radiation field are eliminated to obtain the influence phase functional which describes the reduced dynamics of the matter variables. The general theory is applied to the dynamics of a single electron in the radiation field. From a study of the wave packet dynamics a quantitative measure for the degree of decoherence, the decoherence function, is deduced. The latter is shown to describe the emergence of decoherence through the emission of bremsstrahlung caused by the relative motion of interfering wave packets. It is argued that this mechanism is the most fundamental process in Quantum Electrodynamics leading to the destruction of coherence, since it dominates for short times and because it is at work even in the electromagnetic field vacuum at zero temperature. It turns out that decoherence through bremsstrahlung is very small for single electrons but extremely large for superpositions of many-particle states.

1 Introduction

Decoherence may be defined as the (partial) destruction of quantum coherence through the interaction of a quantum mechanical system with its surroundings. In the theoretical analysis decoherence can be studied with the help of simple microscopic models which describe, for example, the interaction of a quantum mechanical system with a collection of an infinite number of harmonic oscillators, representing the environmental degrees of freedom\[\text{[\ref{Breuer1}]}\]. In an open system’s approach to decoherence one derives dynamic equations for the reduced density matrix\[\text{[\ref{Breuer2}]}\] which yields the state of the system of interest as it is obtained from an average over the degrees of freedom of the environment and the resulting loss of information on the entangled state of the combined total system. The strong suppression of coherence can then be explained by showing that the reduced density matrix equation leads to an extremely rapid transitions of a coherent superposition to an incoherent statistical mixture\[\text{[\ref{Breuer3}]}\]. For certain superpositions the associated decoherence time scale is often found to be smaller than the corresponding relaxation or damping time by many orders of magnitude. This is a signature for the fundamental distinction between the notions of decoherence and of dissipation.
A series of interesting experimental investigations of decoherence have been performed as, for example, experiments on Schrödinger cat states of a cavity field mode [6] and on single trapped ions in a controllable environment [7].

If one considers the coherence of charged matter, it is the electromagnetic field which plays the role of the environment. It is the purpose of this paper to study the emergence of decoherence processes in Quantum Electrodynamics (QED) from an open system’s point of view, that is by an elimination of the degrees of freedom of the radiation field. An appropriate technique to achieve this goal is the use of functional methods from field theory. In section 2 we combine these methods with a super-operator approach to derive an exact, relativistic representation for the reduced density matrix of the matter degrees of freedom. This representation involves an influence phase functional that completely describes the influence of the electromagnetic radiation field on the matter dynamics. The influence phase functional may be viewed as a super-operator representation of the Feynman-Vernon influence phase [1] which is usually obtained with the help of path integral techniques.

In section 3 we treat the problem of a single electron in the radiation field within the non-relativistic approximation. Starting from the influence phase functional, we formulate the reduced electron motion in terms of a path integral which involves an effective action functional. The corresponding classical equations of motion are demonstrated to yield the Abraham-Lorentz equation describing the radiation damping of the electron motion. In addition, the influence phase is shown to lead to a decoherence function which provides a measure for the degree of decoherence.

The general theory will be illustrated with the help of two examples, namely a free electron (section 4) and an electron moving in a harmonic potential (section 5). For both cases an analytical expression for the decoherence function is found, which describes how the radiation field affects the electron coherence.

We shall use the obtained expressions to investigate in detail the time-evolution of Gaussian wave packets. We study the influence of the radiation field on the interference pattern which results from the collision of two moving wave packets of a coherent superposition. It turns out that the basic mechanism leading to the decoherence of matter waves is the emission of bremsstrahlung through the moving wave packets. The resultant picture of decoherence is shown to yield expressions for the decoherence time and length scales which differ substantially from the conventional estimates derived from the prominent Caldeira-Leggett master equation. In particular, it will be shown that a superposition of two wave packets with zero velocity does not decohere and, thus, the usual picture of decoherence as a decay of the off-diagonal peaks in the corresponding density matrix does not apply to decoherence through bremsstrahlung.

We investigate in section 6 the possibility of the destruction of coherence of the superposition of many-particle states. It will be argued that, while the
decoherence effect is small for single electrons at non-relativistic speed, it is
drastically amplified for certain superpositions of many-particle states.
Finally, we draw our conclusions in section 7.

2 Reduced Density Matrix of the Matter Degrees of
Freedom

Our aim is to eliminate the variables of the electromagnetic radiation field
to obtain an exact representation for the reduced density matrix \( \rho_m \) of the
matter degrees of freedom. The starting point will be the following formal
equation which relates the density matrix \( \rho_m(t_f) \) of the matter at some final
time \( t_f \) to the density matrix \( \rho(t_i) \) of the combined matter-field system at
some initial time \( t_i \),

\[
\rho_m(t_f) = \text{tr}_f \left\{ T_\leftarrow \exp \left[ \int_{t_i}^{t_f} \text{d}^4x L(x) \right] \rho(t_i) \right\}.
\]

The Liouville super-operator \( L(x) \) is defined as

\[
L(x) \rho \equiv -i[H(x), \rho],
\]

where \( H(x) \) denotes the Hamiltonian density. Space-time coordinates are
written as \( x^\mu = (x^0, \mathbf{x}) = (t, \mathbf{x}) \), where the speed of light \( c \) is set equal
to 1. All fields are taken to be in the interaction picture and \( T_\leftarrow \)
denotes the chronological time-ordering of the interaction picture fields, while \( \text{tr}_f \)
denotes the trace over the variables of the radiation field. Setting \( \hbar = c = 1 \) we
shall use here Heaviside-Lorentz units such that the fine structure constant
is given by

\[
\alpha = \frac{e^2}{4\pi \hbar c} \approx \frac{1}{137}.
\]

To be specific we choose the Coulomb gauge in the following which means
that the Hamiltonian density takes the form

\[
H(x) = H_C(x) + H_{\text{tr}}(x).
\]

Here,

\[
H_{\text{tr}}(x) = j^\mu(x)A_\mu(x)
\]

represents the density of the interaction of the matter current density \( j^\mu(x) \)
with the transversal radiation field,

\[
A^\mu(x) = (0, \mathbf{A}(x)), \quad \nabla \cdot \mathbf{A}(x) = 0,
\]

and

\[
H_C(x) = \frac{1}{2}j^0(x)A^0(x) = \frac{1}{2} \int \text{d}^3y \frac{j^0(x^0, \mathbf{x})j^0(x^0, \mathbf{y})}{4\pi |\mathbf{x} - \mathbf{y}|}
\]

(7)
is the Coulomb energy density such that

$$H_C(x^0) = \frac{1}{2} \int d^3x \int d^3y \frac{j^0(x^0, x) j^0(x^0, y)}{4\pi|x - y|}$$

(8)

is the instantaneous Coulomb energy. Note that we use here the convention that the electron charge $e$ is included in the current density $j^\mu(x)$ of the matter.

Our first step is a decomposition of chronological time-ordering operator $T_\rightarrow$ into a time-ordering operator $T_{j\leftarrow}$ for the matter current and a time-ordering operator $T_{A\leftarrow}$ for the electromagnetic field,

$$T_\rightarrow = T_{j\leftarrow} T_{A\leftarrow}. \quad (9)$$

This enables one to write Eq. (1) as

$$\rho_m(t_f) = T_{j\leftarrow} \left( \text{tr}_f \left\{ T_{A\leftarrow} \exp \left[ \int_{t_i}^{t_f} d^4x \left( L_C(x) + L_{tr}(x) \right) \right] \rho(t_i) \right\} \right), \quad (10)$$

where we have introduced the Liouville super-operators for the densities of the Coulomb field and of the transversal field,

$$L_C(x) \rho \equiv -i[H_C(x), \rho], \quad L_{tr}(x) \rho \equiv -i[j^\mu(x) A^\mu(x), \rho]. \quad (11)$$

The currents $j^\mu$ commute under the time-ordering $T_{j\leftarrow}$. We may therefore treat them formally as commuting $c$-number fields under the time-ordering symbol. Since the super-operator $L_C(x)$ only contains matter variables, the corresponding contribution can be pulled out of the trace. Hence, we have

$$\rho_m(t_f) = T_{j\leftarrow} \left( \exp \left[ \int_{t_i}^{t_f} d^4x L_C(x) \right] \text{tr}_f \left\{ T_{A\leftarrow} \exp \left[ \int_{t_i}^{t_f} d^4x L_{tr}(x) \right] \rho(t_i) \right\} \right). \quad (12)$$

We now proceed by eliminating the time-ordering of the $A$-fields. With the help of the Wick-theorem we get

$$T_{A\leftarrow} \exp \left[ \int_{t_i}^{t_f} d^4x L_{tr}(x) \right] = \exp \left[ \frac{1}{2} \int_{t_i}^{t_f} d^4x \int_{t_i}^{t_f} d^4x' [L_{tr}(x), L_{tr}(x')] \theta(t - t') \right] \exp \left[ \int_{t_i}^{t_f} d^4x L_{tr}(x) \right]. \quad (13)$$

In order to determine the commutator of the Liouville super-operators we invoke the Jacobi identity which yields for an arbitrary test density $\rho$,

$$[L_{tr}(x), L_{tr}(x')] \rho = L_{tr}(x) L_{tr}(x') \rho - L_{tr}(x') L_{tr}(x) \rho
\begin{align*}
&= -[H_{tr}(x), [H_{tr}(x'), \rho]] + [H_{tr}(x'), [H_{tr}(x), \rho]] \\
&= -[[H_{tr}(x), H_{tr}(x')], \rho]. \quad (14)
\end{align*}$$
The commutator of the transversal energy densities may be simplified to read
\[
[H_{tr}(x), H_{tr}(x')] = j^\mu(x) j^\nu(x')[A_\mu(x), A_\nu(x')],
\] (15)
since the contribution involving the commutator of the currents vanishes by virtue of the time-ordering operator \( T^\leftarrow \). Thus, it follows from Eqs. (14) and (15) that the commutator of the Liouville super-operators may be written as
\[
[L_{tr}(x), L_{tr}(x')]\rho = -[A_\mu(x), A_\nu(x')][j^\mu(x) j^\nu(x'), \rho].
\] (16)

It is useful to introduce current super-operators \( J^+_{\mu}(x) \) and \( J^-_{\mu}(x) \) by means of
\[
J^\mu_{\mu}(x) \rho \equiv j^\mu(x) \rho, \quad J^\mu_{\nu}(x) \rho \equiv \rho j^\nu(x).
\] (17)

Thus, \( J^+_{\mu}(x) \) is defined to be the current density acting from the left, while \( J^-_{\mu}(x) \) acts from the right on an arbitrary density. With the help of these definitions we may write the commutator of the Liouville super-operators as
\[
[L_{tr}(x), L_{tr}(x')] = -[A_\mu(x), A_\nu(x')][J^\mu_{\mu}(x) J^\nu_{\nu}(x')
+ [A_\mu(x), A_\nu(x')] J^\mu_{\mu}(x) J^\nu_{\nu}(x')].
\] (18)

Inserting this result into Eq. (13), we can write Eq. (12) as
\[
\rho_m(t_f) = T^\leftarrow \left( \exp \left[ \int_{t_i}^{t_f} \int_{t_i}^{t_f} d^4x L_{tr}(x) \right]
- \frac{1}{2} \int_{t_i}^{t_f} d^4x \int_{t_i}^{t_f} d^4x' \theta(t - t') [A_\mu(x), A_\nu(x')][J^\mu_{\mu}(x) J^\nu_{\nu}(x')
+ [A_\mu(x), A_\nu(x')] J^\mu_{\mu}(x) J^\nu_{\nu}(x')]
\right.
\] (18)

This is an exact formal representation for the reduced density matrix of the matter variables. Note that the time-ordering of the radiation degrees of freedom has been removed and that they enter Eq. (18) only through the functional
\[
W[J^+_{\mu}, J^-_{\mu}] \equiv \text{tr}_f \left\{ \exp \left[ \int_{t_i}^{t_f} d^4x L_{tr}(x) \rho(t_i) \right] \right\},
\] (19)
since the commutator of the \( A \)-fields is a \( c \)-number function.

3 The Influence Phase Functional of QED

The functional (19) involves an average over the field variables with respect to the initial state \( \rho(t_i) \) of the combined matter-field system. It therefore
contains all correlations in the initial state of the total system. Here, we are interested in the destruction of coherence. Our central goal is thus to investigate how correlations are built up through the interaction between matter and radiation field. We therefore consider now an initial state of low entropy which is given by a product state of the form

\[ \rho(t_i) = \rho_m(t_i) \otimes \rho_f, \]  

(20)

where \( \rho_m(t_i) \) is the density matrix of the matter at the initial time and the density matrix of the radiation field describes an equilibrium state at temperature \( T \),

\[ \rho_f = \frac{1}{Z_f} \exp(-\beta H_f). \]  

(21)

Here, \( H_f \) denotes the Hamiltonian of the free radiation field and the quantity \( Z_f = \text{tr}_f[\exp(-\beta H_f)] \) is the partition function with \( \beta = 1/k_B T \). In the following we shall denote by

\[ \langle O \rangle_f \equiv \text{tr}_f \{ O \rho_f \} \]  

(22)

the average of some quantity \( O \) with respect to the thermal equilibrium state (21).

The influence of the special choice (20) for the initial condition can be eliminated by pushing \( t_i \rightarrow -\infty \) and by switching on the interaction adiabatically. This is the usual procedure used in Quantum Field Theory in order to define asymptotic states and the \( S \)-matrix. The matter and the field variables are then described as \( in \)-fields, obeying free field equations with renormalized mass. These fields generate physical one-particle states from the interacting ground state.

For an arbitrary initial condition \( \rho(t_i) \) the functional \( W[J_+, J_-] \) can be determined, for example, by means of a cumulant expansion. Since the initial state (20) is Gaussian with respect to the field variables and since the Liouville super-operator \( \mathcal{L}_{tr}(x) \) is linear in the radiation field, the cumulant expansion terminates after the second order term. In addition, a linear term does not appear in the expansion because of \( \langle A_\mu(x) \rangle_f = 0 \). Thus we immediately obtain

\[ W[J_+, J_-] = \exp \left[ \frac{1}{2} \int_{t_i}^{t_f} \int_{t_i}^{t_f} d^4x \, d^4x' \langle \mathcal{L}_{tr}(x) \mathcal{L}_{tr}(x') \rangle_f \right] \rho_m(t_i). \]  

(23)

Inserting the definition for the Liouville super-operator \( \mathcal{L}_{tr}(x) \) into the exponent of this expression one finds after some algebra,

\[ \frac{1}{2} \int_{t_i}^{t_f} d^4x \int_{t_i}^{t_f} d^4x' \langle \mathcal{L}_{tr}(x) \mathcal{L}_{tr}(x') \rangle_f \rho_m \]

\[ \equiv -\frac{1}{2} \int_{t_i}^{t_f} d^4x \int_{t_i}^{t_f} d^4x' \text{tr}_f \{ [\mathcal{H}_{tr}(x), [\mathcal{H}_{tr}(x'), \rho_m \otimes \rho_f]] \}. \]
With the help of this notation the density matrix of the matter can now be
expressed as
\begin{equation}
\rho_m(t_f) = T^+_\omega \left( \exp \left[ \int_{t_i}^{t_f} d^4x \mathcal{L}_c(x) \right] \right)
\end{equation}

On using this result together with Eq. (23), Eq. (24) can be cast into the
form,
\begin{equation}
\rho_m(t_f) = T^+_\omega \left( \exp \left[ \int_{t_i}^{t_f} d^4x \mathcal{L}_c(x) \right] \right)
\end{equation}

At this stage it is useful to introduce a new notation for the correlation
functions of the electromagnetic field, namely the Feynman propagator and
its complex conjugated \((T_\omega\) denotes the anti-chronological time-ordering),
\begin{align}
iD_F(x - x')_{\mu\nu} &\equiv \langle T_\omega (A_\mu(x)A_\nu(x')) \rangle_f \\
&= \theta(t - t')[A_\mu(x), A_\nu(x')] - \langle A_\mu(x')A_\nu(x') \rangle_f,
\end{align}

as well as the two-point correlation functions
\begin{align}
D_+(x - x')_{\mu\nu} &\equiv \langle A_\mu(x)A_\nu(x') \rangle_f, \\
D_-(x - x')_{\mu\nu} &\equiv \langle A_\nu(x')A_\mu(x) \rangle_f.
\end{align}

As is easily verified these functions are related through
\begin{align}
-iD_F(x - x')_{\mu\nu} + iD_F(x - x')_{\mu\nu} + D_+(x - x')_{\mu\nu} + D_-(x - x')_{\mu\nu} = 0.
\end{align}

With the help of this notation the density matrix of the matter can now be
written as follows,
\begin{equation}
\rho_m(t_f) = T^+_\omega \left( \exp \left[ \int_{t_i}^{t_f} d^4x \mathcal{L}_c(x) \right] \right)
\end{equation}
One observes that the dynamics of the matter variables is given by a time-ordered exponential function whose exponent is a bilinear function of the current super-operators $J_\pm(x)$. Formally we may write Eq. (28) as

$$\rho_m(t_f) = T^\mu_{\mu'} \exp \{i\Phi[J_+, J_-] \} \rho_m(t_i),$$

where we have introduced an influence phase functional

$$i\Phi[J_+, J_-] = \int_{t_i}^{t_f} d^4x L_C(x) + \frac{1}{2} \int_{t_i}^{t_f} d^4x \int_{t_i}^{t_f} d^4x' \times \left\{ -iD_F(x - x')_{\mu\nu} J_{+}^{\mu}(x) J_{+}^{\nu}(x') + iD_F^*(x - x')_{\mu\nu} J_{-}^{\mu}(x) J_{-}^{\nu}(x') + D_-(x - x')_{\mu\nu} J_{+}^{\mu}(x) J_{+}^{\nu}(x') + D_+(x - x')_{\mu\nu} J_{-}^{\mu}(x) J_{-}^{\nu}(x') \right\}.$$ (30)

It should be remarked that the influence phase $\Phi[J_+, J_-]$ is both a functional of the quantities $J_\pm(x)$ and a super-operator which acts in the space of density matrices of the matter degrees of freedom. There are several alternative methods which could be used to arrive at an expression of the form (30) as, for example, path integral techniques [1] or Schwinger’s closed time-path method [2]. The expression (30) for the influence phase functional has been given in Ref. [12] without the Coulomb term and for the special case of zero temperature. In our derivation we have combined super-operator techniques with methods from field theory, which seems to be the most direct way to obtain a representation of the reduced density matrix.

For the study of decoherence phenomena another equivalent formula for the influence phase functional will be useful. To this end we define the commutator function

$$D(x - x')_{\mu\nu} \equiv i[A_\mu(x), A_\nu(x')]$$

$$= i(D_+(x - x')_{\mu\nu} - D_-(x - x')_{\mu\nu})$$

and the anti-commutator function

$$D_1(x - x')_{\mu\nu} \equiv \langle\{A_\mu(x), A_\nu(x')\}\rangle_f$$

$$= D_+(x - x')_{\mu\nu} + D_-(x - x')_{\mu\nu}.$$ (31) (32)
Of course, the previously introduced correlation functions may be expressed in terms of $D(x - x')_{\mu\nu}$ and $D_1(x - x')_{\mu\nu}$,

$$
D_+(x - x')_{\mu\nu} = \frac{1}{2}D_1(x - x')_{\mu\nu} - \frac{1}{2}D(x - x')_{\mu\nu},
$$

(33)

$$
D_-(x - x')_{\mu\nu} = \frac{1}{2}D_1(x - x')_{\mu\nu} + \frac{1}{2}D(x - x')_{\mu\nu},
$$

(34)

$$
iD_F(x - x')_{\mu\nu} = \frac{1}{2}D_1(x - x')_{\mu\nu} - \frac{1}{2}\text{sign}(t - t')D(x - x')_{\mu\nu},
$$

(35)

$$
-iD_F(x - x')_{\mu\nu} = \frac{1}{2}D_1(x - x')_{\mu\nu} + \frac{1}{2}\text{sign}(t - t')D(x - x')_{\mu\nu}.
$$

(36)

Correspondingly, we define a commutator super-operator $J_c(x)$ and an anti-commutator super-operator $J_a(x)$ by means of

$$
J_{\mu c}(x)\rho \equiv [j_{\mu}(x),\rho], \quad J_{\mu a}(x)\rho \equiv \{j_{\mu}(x),\rho\},
$$

(37)

which are related to the previously introduced super-operators $J_{\pm}(x)$ by

$$
J_{\mu c}(x) = J_{\mu +}(x) - J_{\mu -}(x), \quad J_{\mu a}(x) = J_{\mu +}(x) + J_{\mu -}(x).
$$

(38)

In terms of these quantities the influence phase functional may now be written as

$$
\int_{t_i}^{t_f} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \left\{ \frac{1}{2}D(x - x')_{\mu\nu}J_{\mu c}(x)J_{\nu a}(x') - \frac{1}{2}D_1(x - x')_{\mu\nu}J_{\mu c}(x)J_{\nu c}(x') \right\}.
$$

(39)

This form of the influence phase functional will be particularly useful later on. It represents the influence of the radiation field on the matter dynamics in terms of the two fundamental 2-point correlation functions $D(x - x')$ and $D_1(x - x')$. Note that the double space-time integral in Eq. (39) is already a time-ordered integral since the integration over $t_i$ to $t = x^0$ extends over the time interval from $t_i$ to $t = x^0$.

For a physical discussion of these results it may be instructive to compare Eq. (28) with the structure of a Markovian quantum master equation in Lindblad form [3],

$$
\frac{d\rho_m}{dt} = -i[H_m,\rho_m] + \sum_i \left( A_i\rho_mA_i^\dagger - \frac{1}{2}A_i^\dagger A_i\rho_m - \frac{1}{2}\rho_m A_i^\dagger A_i \right),
$$

(40)

where $H_m$ generates the coherent evolution and the $A_i$ denote a set of operators, the Lindblad operators, labeled by some index $i$. One observes that the
terms of the influence phase functional involving the current super-operators
in the combinations \( J_+ J_- \) and \( J_- J_+ \) correspond to the gain terms in the
Lindblad equation having the form \( A_i \rho_m A_i^\dagger \). These terms may be interpreted
as describing the back action on the reduced system of the matter degrees
of freedom induced by “real” processes in which photons are absorbed or
emitted. The presence of these terms leads to a transformation of pure states
into statistical mixtures. Namely, if we disregard the terms containing the
combinations \( J_+ J_- \) and \( J_- J_+ \) the remaining expression takes the form
\[
ρ_m(t_f) ≈ U(t_f, t_i)ρ_m(t_i)U^\dagger(t_f, t_i), \tag{41}
\]
where
\[
U(t_f, t_i) = T_{\Gamma}^f_{\Gamma} \exp \left[ -i \int_{t_i}^{t_f} d^4x \mathcal{H}_C(x) \right. \right. \left. \left. - \frac{i}{2} \int_{t_i}^{t_f} d^4x \int_{t_i}^{t_f} d^4x' D_F(x - x')_{\mu\nu} j_\mu(x) j_\nu(x') \right] \tag{42}
\]
Eq. (41) shows that the contributions involving the Feynman propagators
and the combinations \( J_+ J_- \) and \( J_- J_+ \) of super-operators preserve the purity
of states \[12\]. Recall that all correlations functions have been defined in terms
of the transversal radiation field. We may turn to the covariant form of the
correlation functions if we replace at the same time the current density by
its transversal component \( j_\mu^\text{tr} \). The expression (42) is then seen to contain the
vacuum-to-vacuum amplitude \( A[j] \) of the electromagnetic field in the presence
of a classical, transversal current density \( j_\mu^\text{tr}(x) \) \[13\],
\[
A[j] = \exp \left[ - \frac{i}{2} \int d^4x \int d^4x' D_F(x - x')_{\mu\nu} j_\mu^\text{tr}(x) j_\nu^\text{tr}(x') \right]. \tag{43}
\]
With the help of the decomposition (35) of the Feynman propagator into a
real and an imaginary part we find
\[
A[j] = \exp \left[ i \left( S^{(1)} + i S^{(2)} \right) \right]. \tag{44}
\]
The vacuum-to-vacuum amplitude is thus represented in terms of a complex
action functional with the real part
\[
S^{(1)} = \frac{1}{4} \int d^4x \int d^4x' \text{sign}(t - t') D(x - x')_{\mu\nu} j_\mu^\text{tr}(x) j_\nu^\text{tr}(x'), \tag{45}
\]
and with the imaginary part
\[
S^{(2)} = \frac{1}{4} \int d^4x \int d^4x' D_1(x - x')_{\mu\nu} j_\mu^\text{tr}(x) j_\nu^\text{tr}(x'). \tag{46}
\]
The imaginary part \( S^{(2)} \) yields the probability that no photon is emitted by
the current \( j_\mu^\text{tr} \),
\[
|A[j]|^2 = \exp (-2S^{(2)}). \tag{47}
\]
In covariant form we have

$$D(x - x')_{\mu\nu} = -\frac{1}{2\pi} \text{sign}(t - t') \delta[(x - x')^2] g_{\mu\nu},$$

and, hence,

$$S^{(1)} = -\frac{1}{8\pi} \int d^4x \int d^4x' \delta[(x - x')^2] j^\mu_i(x) j^\mu_i(x').$$

This is the classical Feynman-Wheeler action. It describes the classical motion of a system of charged particles by means of a non-local action which arises after the elimination of the degrees of freedom of the electromagnetic radiation field. In the following sections we will demonstrate that it is just the imaginary part $S^{(2)}$ which leads to the destruction of coherence of the matter degrees of freedom.

4 The Interaction of a Single Electron with the Radiation Field

In this section we shall apply the foregoing general theory to the case of a single electron interacting with the radiation field where we confine ourselves to the non-relativistic approximation. It will be seen that this simple case already contains the basic physical mechanism leading to decoherence.

4.1 Representation of the Electron Density Matrix in the Non-Relativistic Approximation

The starting point will be the representation (29) for the reduced matter density with expression (39) for the influence phase functional $\Phi$. It must be remembered that the correlation functions $D(x - x')_{\mu\nu}$ and $D_1(x - x')_{\mu\nu}$ have been defined in terms of the transversal radiation field using Coulomb gauge and that they thus involve projections onto the transversal component. In fact, we have the replacements,

$$D(x - x')_{\mu\nu} \to D(x - x')_{ij} = -\left(\delta_{ij} - \frac{\partial_i \partial_j}{\Delta}\right) D(x - x')$$

for the commutator functions, and

$$D_1(x - x')_{\mu\nu} \to D_1(x - x')_{ij} = \left(\delta_{ij} - \frac{\partial_i \partial_j}{\Delta}\right) D_1(x - x')$$

for the anti-commutator function, where

$$D(x - x') = -i \int \frac{d^3k}{2(2\pi)^3} \omega \left[ \exp(-ik(x - x')) - \exp(ik(x - x')) \right].$$
and

\[ D_1(x - x') = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega} \left[ \exp(-ik(x - x')) + \exp(ik(x - x')) \right] \coth(\beta\omega/2), \]

with the notation \( \mathbf{k}^n = (\omega, \mathbf{k}) = (|\mathbf{k}|, \mathbf{k}) \) for the components of the wave vector. It should be noted that the commutator function is independent of the temperature, while the anti-commutator function does depend on \( T \) through the factor \( \coth(\beta\omega/2) = 1 + \frac{2N(\omega)}{e^{\beta\omega} - 1} \), where \( N(\omega) \) is the average number of photons in a mode with frequency \( \omega \). Hence, invoking the non-relativistic (dipole) approximation we may replace

\[ D(x - x')_{ij} \rightarrow D(t - t')_{ij} = \delta_{ij}D(t - t') = \delta_{ij} \int_0^\infty \omega J(\omega) \sin(\omega(t - t')), \]

and

\[ D_1(x - x')_{ij} \rightarrow D_1(t - t')_{ij} = \delta_{ij}D_1(t - t') = \delta_{ij} \int_0^\infty \omega J(\omega) \coth(\beta\omega/2) \cos(\omega(t - t')), \]

where we have introduced the spectral density

\[ J(\omega) = \frac{e^2}{3\pi^2 \omega} \Theta(\Omega - \omega), \]

with some ultraviolet cutoff \( \Omega \) (see below). It is important to stress here that the spectral density increases with the first power of the frequency \( \omega \). Had we used dipole coupling \(-e\mathbf{x} \cdot \mathbf{E}\) of the electron coordinate \( \mathbf{x} \) to the electric field strength \( \mathbf{E} \), the corresponding spectral density would be proportional to the third power of the frequency. This means that the coupling to the radiation field in the dipole approximation may be described as a special case of the famous Caldeira-Leggett model \[ \text{(3)} \] and that in the language of the theory of quantum Brownian motion \[ \text{(14)} \] the radiation field constitutes a super-Ohmic environment \[ \text{(15,16)} \]. Note also that we now include the factor \( e^2 \) into the definition of the correlation function. Within the non-relativistic approximation we may thus replace the current density by

\[ j(t, \mathbf{x}) \rightarrow \frac{p(t)}{2m} \delta(\mathbf{x} - \mathbf{x}(t)) + \delta(\mathbf{x} - \mathbf{x}(t)) \frac{p(t)}{2m}, \]

where \( p(t) \) and \( \mathbf{x}(t) \) denote the momentum and position operator of the electron in the interaction picture with respect to the Hamiltonian

\[ H_m = \frac{p^2}{2m} + V(\mathbf{x}) \]

for the electron, \( V(\mathbf{x}) \) being some external potential.
We are thus led to the following non-relativistic approximation of Eq. (29),

$$\rho_m(t_f) = \exp \left[ \int_{t_i}^{t_f} dt \int_{t_i}^{t'} dt' \left\{ \frac{1}{2} D(t-t') \frac{p(t)}{m} \frac{p(t')}{m} - \frac{1}{2} D_1(t-t') \frac{p(t)}{m} \frac{p(t')}{m} \right\} \right] \rho_m(t_i).$$

This equation represents the density matrix (neglecting the spin degree of freedom) for a single electron interacting with the radiation field at temperature $T$. In accordance with the definitions (37) and (38) $p_c$ is a commutator super-operator and $p_a$ an anti-commutator super-operator. In the theory of quantum Brownian motion the function $D(t-t')$ is called the dissipation kernel, whereas $D_1(t-t')$ is referred to as noise kernel.

### 4.2 The Path Integral Representation

The reduced density matrix given in Eq. (55) admits an equivalent path integral representation \cite{14} which may be written as follows,

$$\rho_m(x_f, x'_f, t_f) = \int d^3 x_i \int d^3 x_i' J(x_f, x'_f; x_i, x'_i, t_i) \rho_m(x_i, x'_i, t_i),$$

with the propagator function

$$J(x_f, x'_f; x_i, x'_i, t_i) = \int Dx Dx' \exp \{ i (S_m[x] - S_m[x']) + i\Phi[x, x'] \}.$$  

This is a double path integral which is to be extended over all paths $x(t)$ and $x'(t)$ with the boundary conditions

$$x'(t_i) = x'_i, \quad x'(t_f) = x'_f, \quad x(t_i) = x_i, \quad x(t_f) = x_f.$$  

$S_m[x]$ denotes the action functional for the electron,

$$S_m[x] = \int_{t_i}^{t_f} dt \left( \frac{1}{2} m \ddot{x}^2 - V(x) \right),$$

while the influence phase functional becomes,

$$i\Phi[x, x'] = \int_{t_i}^{t_f} dt \int_{t_i}^{t'} dt' \left\{ \frac{1}{2} D(t-t') (\dot{x}(t) - \dot{x}'(t)) (\dot{x}'(t') + \dot{x}'(t')) - \frac{1}{2} D_1(t-t') (\dot{x}(t) - \dot{x}'(t)) (\dot{x}'(t') - \dot{x}'(t')) \right\}.$$  

We define the new variables

$$q = x - x', \quad r = \frac{1}{2} (x + x'),$$

where $r$ is a Lagrange multiplier to ensure the boundary conditions are satisfied.
and set, for simplicity, the initial time equal to zero, \( t_i = 0 \). We may then write Eq. (56) as

\[
\rho_m(r_f, q_f, t_f) = \int d^3 r_i \int d^3 q_i J(r_f, q_f, t_f; r_i, q_i) \rho_m(r_i, q_i, 0). \tag{62}
\]

The propagator function

\[
J(r_f, q_f, t_f; r_i, q_i) = \int Dr \int Dq \exp\{iA[r, q]\} \tag{63}
\]

is a double path integral over all path \( r(t) \), \( q(t) \) satisfying the boundary conditions,

\[
r(0) = r_i, \quad r(t_f) = r_f, \quad q(0) = q_i, \quad q(t_f) = q_f. \tag{64}
\]

The weight factor for the paths \( r(t), q(t) \) is defined in terms of an effective action \( A \) functional,

\[
A[r, q] = \int_0^{t_f} dt \left( m \ddot{r} - V(r + 1/2 q) + V(r - 1/2 q) \right) + \int_0^{t_f} dt \int_0^{t_f} dt' \dot{r}(t')D(t - t')\dot{q}(t)\dot{r}(t') + \frac{i}{4} \int_0^{t_f} dt \int_0^{t_f} dt' D_1(t - t')q(t)\dot{q}(t'). \tag{65}
\]

The first variation of \( A \) is found to be

\[
\delta A = - \int_0^{t_f} dt \left\{ \delta q(t) \left[ m \ddot{r}(t) + \frac{1}{2} \nabla_r (V(r + 1/2 q) + V(r - 1/2 q)) \right. \right.
\]

\[
\left. + \frac{d}{dt} \int_0^t dt' D(t - t')\dot{r}(t') + \frac{i}{2} \frac{d}{dt} \int_0^{t_f} dt' D_1(t - t')\dot{q}(t') \right] \right. \]

\[
+ \delta r(t) \left[ m \ddot{q}(t) + 2 \nabla_q (V(r + 1/2 q) + V(r - 1/2 q)) \right. \right.
\]

\[
\left. + \frac{d}{dt} \int_t^{t_f} dt' D(t' - t)\dot{q}(t') \right] \right\}, \tag{66}
\]

which leads to the classical equations of motion,

\[
m \ddot{r}(t) + \frac{1}{2} \nabla_r (V(r + 1/2 q) + V(r - 1/2 q)) + \frac{d}{dt} \int_0^t dt' D(t - t')\dot{r}(t') = - \frac{i}{2} \frac{d}{dt} \int_0^{t_f} dt' D_1(t - t')\dot{q}(t'), \tag{67}
\]

and

\[
m \ddot{q}(t) + 2 \nabla_q (V(r + 1/2 q) + V(r - 1/2 q)) + \frac{d}{dt} \int_t^{t_f} dt' D(t' - t)\dot{q}(t') = 0. \tag{68}
\]
4.3 The Abraham-Lorentz Equation

The real part of the equation of motion \( \Box \), which is obtained by setting the right-hand side equal to zero, yields the famous Abraham-Lorentz equation for the electron \( \Box \). It describes the radiation damping through the damping kernel \( D(t-t') \) \( \Box \). To see this we write the real part of Eq. (67) as

\[
m \ddot{r}(t) + \frac{d}{dt} \int_0^t dt' D(t-t') \dot{r}(t') = F_{\text{ext}}(t),
\]

where \( F_{\text{ext}}(t) \) denotes an external force derived from the potential \( V \). The damping kernel can be written (see Eqs. (50) and (52))

\[
D(t-t') = \int_0^\Omega d\omega \frac{e^2}{3\pi^2} \omega \sin(\omega(t-t'))
= \frac{e^2}{3\pi^2} \int_0^\Omega d\omega \frac{\sin(\Omega(t-t'))}{t-t'}
= \frac{e^2}{3\pi^2} \int_0^\Omega d\omega \omega f(t-t'),
\]

where we have introduced the function

\[
f(t) \equiv \frac{\sin \Omega t}{t}.
\]

To be specific the UV-cutoff \( \Omega \) is taken to be

\[
h\Omega = mc^2,
\]

which implies that

\[
\Omega = \frac{mc^2}{\hbar} = \frac{c}{\tilde{\lambda}_C},
\]

where

\[
\tilde{\lambda}_C = \frac{\hbar}{mc}
\]

is the Compton wavelength. For an electron we have

\[
\tilde{\lambda}_C \approx 3.8 \times 10^{-13} \text{m} \quad \text{and} \quad \Omega \approx 0.78 \times 10^{21} \text{s}^{-1}.
\]

The term of the equation of motion (69) involving the damping kernel can be written as follows,

\[
\frac{d}{dt} \int_0^t dt' D(t-t') \dot{r}(t') = \frac{e^2}{3\pi^2} \int_0^t dt' \left[ \frac{d}{dt} f(t-t') \right] \dot{r}(t')
= \frac{e^2}{3\pi^2} \int_0^t dt' \left[ -f(t-t') \dot{r}(t') + f(0)\dot{r}(t) - f(t)\dot{r}(0) \right].
\]

For times \( t \) such that \( \Omega t \gg 1 \), i.e. \( t \gg 10^{-21} \text{s} \), we may replace

\[
f(t) \rightarrow \pi \delta(t),
\]
and approximate \( f(t) \approx 0 \), while Eq. (70) yields \( f(0) = \Omega \). Thus we obtain,

\[
\frac{d}{dt} \int_0^t dt' D(t - t') \dot{r}(t') = \frac{e^2}{3\pi^2} \frac{d}{dt} \left[ -\frac{\pi}{2} \dot{r}(t) + \Omega \dot{r}(t) \right],
\]

which finally leads to the equation of motion,

\[
\left( m + \frac{e^2 \Omega}{3\pi^2} \right) \ddot{v}(t) - \frac{e^2}{6\pi} \dot{v}(t) = F_{\text{ext}}(t),
\]

where \( v = \dot{r} \) is the velocity. This is the famous Abraham-Lorentz equation [17]. The term proportional to the third derivative of \( r(t) \) describes the damping of the electron motion through the emitted radiation. This term does not depend on the cutoff frequency, while the cutoff-dependent term yields a renormalization of the electron mass,

\[
m_R = m + \frac{e^2 \Omega}{3\pi^2}.
\]

It is important to note that the electro-magnetic mass \( \Delta m \) diverges linearly with the cutoff. The equation of motion (78) can be obtained heuristically by means of the Larmor formula for the power radiated by an accelerated charge. More rigorously, it has been derived by Abraham and by Lorentz from the conservation law for the field momentum, assuming a spherically symmetric charge distribution and that the momentum is of purely electromagnetic origin [17].

For the cutoff \( \Omega \) chosen above we get

\[
\Delta m = \frac{me^2}{3\pi^2} = \frac{4}{3\pi} \alpha m,
\]

and, hence,

\[
\frac{\Delta m}{m} = \frac{4}{3\pi} \alpha \approx 0.0031.
\]

The decomposition (79) of the mass is, however, unphysical, since the electron is never observed without its self-field and the associated field momentum. In other words, we have to identify the renormalized mass \( m_R \) with the observed physical mass which enables us to write Eq. (78) as

\[
m_R \left[ \dot{v}(t) - \tau_0 \ddot{v}(t) \right] = F_{\text{ext}}(t).
\]

Here, the radiation damping term has been written in terms of a characteristic radiation time scale \( \tau_0 \) given by

\[
\tau_0 = \frac{e^2}{6\pi m_R} = \frac{2}{3} r_e \approx 0.6 \times 10^{-23} \text{s},
\]

where \( r_e \) denotes the classical electron radius,

\[
r_e = \frac{e^2}{4\pi m_R} \approx \alpha \lambda_C \approx 2.8 \times 10^{-15} \text{m}.
\]
It is well-known that Eq. (81), being a classical equation of motion for the electron, leads to the problem of exponentially increasing runaway solutions. Namely, for \( F_{\text{ext}} = 0 \) we have

\[
\dot{v} - \tau_0 \ddot{v} = 0.
\]  

(84)

In addition to the trivial solution of a constant velocity, \( v = \text{const} \), one also finds the solution

\[
\dot{v}(t) = \dot{v}(0) \exp(t/\tau_0),
\]

describing an exponential growth of the acceleration for \( \dot{v}(0) \neq 0 \). In order to exclude these solutions one imposes the boundary condition

\[
\dot{v}(t) \to 0 \quad \text{for} \quad t \to \infty,
\]

if \( F_{\text{ext}} \) also vanishes in this limit. This boundary condition can be implemented by rewriting Eq. (81) as an integro-differential equation

\[
m_R \ddot{r}(t) = \int_0^\infty ds \exp(-s) F_{\text{ext}}(t + \tau_0 s).
\]

(85)

On differentiating Eq. (85) with respect to time, it is easily verified that one is led back to Eq. (84). However, for \( F_{\text{ext}} = 0 \) it follows immediately from Eq. (85) that \( v = \text{const} \), such that runaway solutions are excluded.

On the other hand, Eq. (85) shows that the acceleration depends upon the future value of the force. Hence, the electron reacts to signals lying a time of order \( \tau_0 \) in the future, which is the phenomenon of pre-acceleration. This phenomenon should, however, not be taken too seriously, since the description is only classical. The time scale \( \tau_0 \) corresponds to a length scale \( r_e \) which is smaller than the Compton wavelength \( \lambda_C \) by a factor of \( \alpha \), such that a quantum mechanical treatment of the problem is required.

4.4 Construction of the Decoherence Function

In this subsection we derive the explicit form of the propagator function (63) for the reduced electron density matrix in the case of quadratic potentials,

\[
V(x) = \frac{1}{2} m_{R\omega_0^2} x^2.
\]

(86)

Our aim is to introduce and to determine the decoherence function which provides a quantitative measure for the degree of decoherence. On using

\[
\begin{align*}
V(r + q/2) + V(r - q/2) &= m_{R\omega_0^2} r^2 + m_{R\omega_0^2} q^2/4 \quad (87) \\
-V(r + q/2) + V(r - q/2) &= -m_{R\omega_0^2} r \cdot q,
\end{align*}
\]


the classical equations of motion take the form

\[
m_R \left[ \ddot{r}(t) + \dot{\omega}_0^2 \int_0^\infty ds \exp(-s) r(t + \tau_0 s) \right] = -i \frac{d}{dt} \int_0^{t_f} dt' D_1(t - t') \dot{q}(t') (88)
\]

\[
m_R \left[ \ddot{q}(t) + \dot{\omega}_0^2 \int_0^\infty ds \exp(-s) q(t - \tau_0 s) \right] = 0. \tag{89}
\]

Note, that Eq. (89) is the *backward equation* of the real part of Eq. (88). More precisely, if \( q(t) \) solves Eq. (89), then \( r(t_f - t) \equiv q(t_f - t) \) is a solution of (88) with the right-hand side set equal to zero.

The above equations of motion lead to the following renormalized action functional

\[
A[r, q] = \int_0^{t_f} dt \ m_R \left[ \dot{r}(t) \dot{q}(t) - \dot{\omega}_0^2 q(t) \int_0^\infty ds \exp(-s) r(t + \tau_0 s) \right]
\]

\[
+ \frac{i}{4} \int_0^{t_f} dt \int_0^{t_f} dt' D_1(t - t') \dot{q}(t) \dot{q}(t'). \tag{90}
\]

In the following we shall use this renormalised action functional instead of the action given in Eq. (65). By variation with respect to \( q(t) \) we immediately obtain Eq. (88), whereas the variation with respect to \( r(t) \) yields:

\[
- \int_0^{t_f} dt \ m_R \left[ \dot{q}(t) \delta r(t) + \dot{\omega}_0^2 \int_0^\infty ds \exp(-s) q(t) \delta r(t + \tau_0 s) \right] = 0,
\]

which implies

\[
\int_0^{t_f} dt \dot{q}(t) \delta r(t) + \dot{\omega}_0^2 \int_0^\infty ds \int_0^{t_f} dt \ \exp(-s) q(t) \delta r(t + \tau_0 s)
\]

\[
\int_0^{t_f} dt \dot{q}(t) \delta r(t) + \dot{\omega}_0^2 \int_0^\infty ds \int_{\tau_0 s}^{t_f+\tau_0 s} dt \ \exp(-s) q(t - \tau_0 s) \delta r(t)
\]

\[
= 0. \tag{91}
\]

In the last time integral we may extend the integration over the time interval from 0 to \( t_f \). This is legitimate since \( \tau_0 \) is the radiation time scale: By setting this variation of the action equal to zero we thus neglect times of the order of the pre-acceleration time, which directly leads to the equation of motion (89).

Since the action functional is quadratic the propagator function can be determined exactly by evaluating the action along the classical solution and by taking into account Gaussian fluctuations around the classical path. We therefore assume that \( r(t) \) and \( q(t) \) are solutions of the classical equations of motion (88) and (89) with boundary conditions (64). The effective action along these solutions may be written as

\[
A_{cl}[r, q] = m_R \left[ \dot{r}_f q_f - \dot{r}_i q_i \right]
\]
To prove this statement we first deduce from Eq. (95) that
\[ -\int_0^{t_f} \, dt \, m_R \ddot{q}(t) \left[ \dddot{r}(t) + \omega_0^2 \int_0^\infty ds \exp(-s) r(t + \tau_0 s) \right] \]
\[ + \frac{i}{4} \int_0^{t_f} \, dt \int_0^{t_f} \, dt' D_1(t-t') \dot{q}(t) \dot{q}(t'), \]  
(92)
or, equivalently,
\[ A_{cl}[r, q] = m_R [\ddot{r} + \dddot{r} q - \dddot{r}_i q_i] + \frac{i}{2} \int_0^{t_f} \, dt \, q(t) \frac{d}{dt} \int_0^{t_f} \, dt' D_1(t-t') \dot{q}(t) \dot{q}(t'), \]
\[ + \frac{i}{4} \int_0^{t_f} \, dt \int_0^{t_f} \, dt' D_1(t-t') \dot{q}(t) \dot{q}(t'). \]  
(93)

Eq. (93) shows that the solution \( r(t) \) is, in general, complex due to the coupling to \( q(t) \) via the noise kernel \( D_1(t-t') \). Consider the decomposition of \( r(t) \) into real and imaginary part,
\[ r(t) = r^{(1)}(t) + i r^{(2)}(t), \]  
(94)
where \( r^{(1)} \) is a solution of the real part of Eq. (93), while \( r^{(2)} \) solves its imaginary part,
\[ m_R \left[ \dddot{r}^{(2)}(t) + \omega_0^2 \int_0^\infty ds \exp(-s) r^{(2)}(t + \tau_0 s) \right] = -\frac{i}{2} \frac{d}{dt} \int_0^{t_f} \, dt' D_1(t-t') \dot{q}(t'). \]  
(95)

We now demonstrate that, in order to determine the action along the classical paths, it suffices to find the homogeneous solution \( r^{(1)} \) and to insert it in the action functional (14). In other words we have
\[ A_{cl}[r^{(1)}, q] = A_{cl}[r, q], \]  
(96)
where
\[ A_{cl}[r^{(1)}, q] = m_R [\ddot{r} + \dddot{r}_i q_i] \]
\[ + \frac{i}{4} \int_0^{t_f} \, dt \int_0^{t_f} \, dt' D_1(t-t') \dot{q}(t) \dot{q}(t'). \]  
(97)

To proof this statement we first deduce from Eq. (98) that
\[ \frac{i}{2} \int_0^{t_f} \, dt \, q(t) \frac{d}{dt} \int_0^{t_f} \, dt' D_1(t-t') \dot{q}(t') \]
\[ = -im_R \int_0^{t_f} \, dt \, q(t) \left[ \dddot{r}^{(2)}(t) + \omega_0^2 \int_0^\infty ds \exp(-s) r^{(2)}(t + \tau_0 s) \right] \]
\[ = -im_R [\dddot{r}_i^{(2)} q_i - \dddot{r}_i^{(1)} q_i] \]
\[ -im_R \int_0^{t_f} \, dt \left[ r^{(2)}(t) \dddot{q}(t) + \omega_0^2 \int_0^\infty ds \exp(-s) r^{(2)}(t + \tau_0 s) q(t) \right]. \]  
(98)
The term within the square brackets is seen to vanish if one employs Eq. (89) and the same arguments that were used to derive the equation of motion from the variation of the action functional. Furthermore, we made use of \( r^{(2)}(0) = r^{(2)}(t_f) = 0 \) which means that the real part \( r^{(1)}(t) \) of the solution satisfies the given boundary conditions. Hence we find

\[
\frac{i}{2} \int _0 ^{t_f} dt \, q(t) \frac{d}{dt} \int _0 ^{t_f} dt' D_1(t - t') \dot{q}(t') = -im \Re [r_f^{(2)} q_f - \dot{r}_i^{(2)} q_i], \tag{99}
\]

from which we finally obtain with the help of (93),

\[
A_{\text{cl}}[r, q] = m \Re [\dot{r}_f q_f - \dot{r}_i q_i] - im \Re [\dot{r}_f^{(2)} q_f - \dot{r}_i^{(2)} q_i] + \frac{i}{4} \int _0 ^{t_f} dt \int _0 ^{t_f} dt' D_1(t - t') \dot{q}(t) \dot{q}(t')
\]

\[
= A_{\text{cl}}[r^{(1)}, q]. \tag{100}
\]

This completes the proof of the above statement.

Summarizing, the procedure to determine the propagator function for the electron can now be given as follows. One first solves the equations of motion

\[
\ddot{r}(t) + \omega_0 ^2 \int _0 ^{\infty} ds \exp(-s) r(t + \tau_0 s) = 0, \tag{101}
\]

\[
\ddot{q}(t) + \omega_0 ^2 \int _0 ^{\infty} ds \exp(-s) q(t - \tau_0 s) = 0, \tag{102}
\]

together with the boundary conditions (64). With the help of these solutions one then evaluates the classical action,

\[
A_{\text{cl}}[r, q] = m \Re [\dot{r}_f q_f - \dot{r}_i q_i] + \frac{i}{4} \int _0 ^{t_f} dt \int _0 ^{t_f} dt' D_1(t - t') \dot{q}(t) \dot{q}(t'), \tag{103}
\]

which immediately yields the propagator function

\[
J(r_f, q_f, t_f; r_i, q_i) = N \exp \{i A_{\text{cl}}[r, q]\}
\]

\[
= N \exp \{im \Re [\dot{r}_f q_f - \dot{r}_i q_i] + \Gamma(q_f, q_i, t_f)\} . \tag{104}
\]

Here, \( N \) is a normalization factor which is determined from the normalization condition

\[
\int d^3r_f J(r_f, q_f = 0, t_f; r_i, q_i) = \delta(q_i). \tag{105}
\]

The function \( \Gamma(q_f, q_i, t_f) \) introduced in Eq. (104) will be referred to as the decoherence function. It is given in terms of the noise kernel \( D_1(t - t') \) as

\[
\Gamma(q_f, q_i, t_f) = -\frac{i}{4} \int _0 ^{t_f} dt \int _0 ^{t_f} dt' D_1(t - t') \dot{q}(t) \dot{q}(t'). \tag{106}
\]
Explicitly we find with the help of Eq. (51),
\[ \Gamma = -\frac{1}{4} \int_0^{t_f} dt \int_0^{t_f} dt' \int_0^{\infty} d\omega J(\omega) \coth(\beta \omega/2) \cos(\omega(t-t') \dot{q}(t) \dot{q}(t')). \] (107)

The double time-integral can be written as
\[ \text{Re} \int_0^{t_f} dt \int_0^{t_f} dt' \exp[i \omega(t-t')] \dot{q}(t) \dot{q}(t') = \left| \int_0^{t_f} dt \exp(i \omega t) \dot{q}(t) \right|^2. \] (108)

Hence, the decoherence function takes the form
\[ \Gamma(q_f, q_i, t_f) = -\frac{1}{4} \int_0^{\infty} d\omega J(\omega) \coth(\beta \omega/2) |Q(\omega)|^2, \] (109)
where we have introduced
\[ Q(\omega) \equiv \int_0^{t_f} dt \exp(i \omega t) \dot{q}(t). \] (110)

It can be seen from the above expressions that \( \Gamma \) is a non-positive function. The decoherence function will be demonstrated below to describe the reduction of electron coherence through the influence of the radiation field.

5 Decoherence Through the Emission of Bremsstrahlung

As an example we shall investigate in this section the most simple case, namely that of a free electron coupled to the radiation field. This case is of particular interest since it allows an exact analytical determination of the decoherence function and already yields a clear physical picture for the decoherence mechanism. Having determined the decoherence function, we proceed with an investigation of its influence on the propagation of electronic wave packets.

5.1 Determination of the Decoherence Function

We set \( \omega_0 = 0 \) to describe the free electron. The equations of motion (101) and (102) with the boundary conditions (64) can easily be solved to yield
\[ r(t) = r_i + \frac{r_f - r_i}{t_f} t, \quad q(t) = q_i + \frac{q_f - q_i}{t_f} t. \] (111)

Making use of Eq. (104) and determining the normalization factor from Eq. (105) we thus get the propagator function,
\[ J(r_f, q_f, t_f; r_i, q_i) = \left( \frac{m_R}{2\pi t_f} \right)^3 \exp \left\{ i \frac{m_R}{t_f} (r_f - r_i)(q_f - q_i) + \Gamma(q_f, q_i, t_f) \right\}. \] (112)
As must have been expected $J$ is invariant under space translations since it depends only on the difference $r_f - r_i$. Furthermore, one easily recognizes that the contribution
\[ G(r_f - r_i, q_f - q_i, t_f) \equiv \left( \frac{m_R}{2 \pi t_f} \right)^3 \exp \left\{ i \frac{m_R}{t_f} (r_f - r_i)(q_f - q_i) \right\} \] (113)
is simply the propagator function for the density matrix of a free electron with mass $m_R$ for a vanishing coupling to the radiation field. We can thus write the electron density matrix as follows,
\[ \rho_m(r_f, q_f, t_f) = \int d^3 r_i \int d^3 q_i G(r_f - r_i, q_f - q_i, t_f) \times \exp \{ \Gamma(q_f, q_i, t_f) \} \rho_m(r_i, q_i, 0), \] (114)
which exhibits that the decoherence function $\Gamma$ describes the influence of the radiation field on the electron motion.

We proceed with an explicit calculation of the decoherence function. It follows from Eqs. (110) and (111) that
\[ Q(\omega) = \int_0^{t_f} dt \exp(i \omega t) \frac{q_f - q_i}{t_f} = \frac{\exp(i \omega t_f) - 1}{i \omega} w, \] (115)
where
\[ w \equiv \frac{1}{t_f} (q_f - q_i). \] (116)
Therefore, the decoherence function is found to be
\[ \Gamma = \frac{e^2 \omega^2}{6 \pi^2} \int_0^\Omega d\omega \frac{1 - \cos \omega t_f}{\omega} \coth(\beta \omega/2), \] (117)
where we have used expression (52) for the spectral density $J(\omega)$. The decoherence function may be decomposed into a vacuum contribution $\Gamma_{\text{vac}}$ and a thermal contribution $\Gamma_{\text{th}},$
\[ \Gamma = \Gamma_{\text{vac}} + \Gamma_{\text{th}}, \] (118)
where
\[ \Gamma_{\text{vac}} = -\frac{e^2 \omega^2}{6 \pi^2} \int_0^\Omega d\omega \frac{1 - \cos \omega t_f}{\omega} \] (119)
and
\[ \Gamma_{\text{th}} = -\frac{e^2 \omega^2}{6 \pi^2} \int_0^\Omega d\omega \frac{1 - \cos \omega t_f}{\omega} [\coth(\beta \omega/2) - 1]. \] (120)

The frequency integral appearing in the vacuum contribution can be evaluated in the following way. Substituting $x = \omega t_f$ we get
\[ \int_0^\Omega d\omega \frac{1 - \cos \omega t_f}{\omega} = \int_0^{\Omega t_f} dx \frac{1 - \cos x}{x} = \ln \Omega t_f + C + O \left( \frac{1}{\Omega t_f} \right), \] (121)
where \( C \approx 0.577 \) is Euler’s constant \([18]\). For \( \Omega t_f \gg 1 \) we obtain asymptotically
\[
\Gamma_{\text{vac}} \approx -\frac{e^2 w^2}{6\pi^2} \ln \Omega t_f = -\frac{e^2}{6\pi^2} \frac{\Omega t_f (q_f - q_i)^2}{t_f^2}.
\] (122)

To determine the thermal contribution \( \Gamma_{\text{th}} \) we first write Eq. (120) as follows,
\[
\Gamma_{\text{th}} = -\frac{e^2 w^2}{6\pi^2} \int_0^{t_f} \int \frac{d\omega}{\beta \omega} \left[ \coth \left( \frac{\beta \omega}{2} \right) - 1 \right] \sin \omega t \equiv -\frac{e^2 w^2}{6\pi^2} I. \] (123)

Introducing the integration variable \( x = \beta \omega \) we can cast the double integral \( I \) into the form
\[
I = \frac{1}{\beta} \int_0^{t_f} \int_0^{\beta \Omega} dx \frac{\coth \left( \frac{x}{2} \right) - 1}{\sin \left( \frac{tx}{\beta} \right)}.
\]

Here, we have \( \beta \Omega = \hbar \Omega / k_B T \) and, using the cutoff \( \hbar \Omega = mc^2 \), we get
\[
\beta \Omega = \frac{mc^2}{k_B T}.
\]

For temperatures \( T \) obeying
\[
k_B T \ll mc^2 \] (124)

the upper limit of the \( x \)-integral may be shifted from \( \beta \Omega \) to \( \infty \). Condition (124) states that
\[
\frac{h^2}{mk_B T} \gg \frac{h^2}{m^2 c^2},
\]
which means that the thermal wavelength \( \lambda_{\text{th}} = \hbar / \sqrt{2mk_B T} \) is much larger than the Compton wavelength,
\[
\lambda_{\text{th}} \gg \lambda_C. \] (125)

Thermal and Compton wavelength are of equal size at a temperature of about \( 10^9 \) Kelvin. Condition (124) therefore means that \( T \ll 10^9 \) K. Under this condition we now obtain
\[
I \approx \frac{1}{\beta} \int_0^{t_f} \int_0^{\infty} dx \left[ \coth \left( \frac{x}{2} \right) - 1 \right] \sin \left( \frac{tx}{\beta} \right)
= \frac{1}{\beta} \int_0^{t_f} dt \left[ \pi \coth \left( \frac{\pi t}{\beta} \right) - \frac{\beta}{t} \right]
= \ln \left( \frac{\sinh \left( \frac{\pi t_f}{\beta} \right)}{\pi t_f / \beta} \right), \] (126)
where we have employed the formula
\[ \int_0^\infty dx \left[ \coth(x/2) - 1 \right] \sin \tau x = \pi \coth(\pi \tau) - \frac{1}{\tau}. \] (127)

The quantity
\[ \tau_B \equiv \frac{\beta}{\pi} = \frac{\hbar}{\pi k_B T} \approx 2.4 \cdot 10^{-12} \text{s/K} \] (128)
represents the correlation time of the thermal radiation field. Putting these results together we get the following expression for the thermal contribution to the decoherence function,
\[ \Gamma_{\text{th}} \approx -\frac{e^2}{6\pi^2} \ln \left( \frac{\sinh(t_f/\tau_B)}{t_f/\tau_B} \right) \frac{(q_f - q_i)^2}{t_f^2}. \] (129)

Adding this expression to the vacuum contribution (122) and introducing \( \alpha = e^2/4\pi \hbar c \) and further factors of \( c \), we can finally write the expression for the decoherence function as
\[ \Gamma(q_f, q_i, t_f) \approx -\frac{2\alpha}{3\pi} \left[ \ln \Omega t_f + \ln \left( \frac{\sinh(t_f/\tau_B)}{t_f/\tau_B} \right) \right] \frac{(q_f - q_i)^2}{(ct_f)^2}. \] (130)

Alternatively, we may write
\[ \Gamma(q_f, q_i, t_f) = -\frac{(q_f - q_i)^2}{2L(t_f)^2}, \] (131)

where the quantity \( L(t_f) \) defined by
\[ L(t_f)^2 \equiv \frac{3\pi}{4\alpha} \left[ \ln \Omega t_f + \ln \left( \frac{\sinh(t_f/\tau_B)}{t_f/\tau_B} \right) \right]^{-1} : (ct_f)^2 \] (132)
may be interpreted as a time-dependent coherence length.

The vacuum contribution \( \Gamma_{\text{vac}} \) to the decoherence function (130) apparently diverges with the logarithm of the cutoff \( \Omega \). This is, however, an artificial divergence which can be seen as follows. The decoherence function is defined in terms of the Fourier transform \( \mathcal{Q}(\omega) \) of \( \dot{q}(t) \), see Eqs. (109) and (110). Evaluating \( \mathcal{Q}(\omega) \) as in Eq. (115) we assume that the velocity is zero prior to the initial time \( t = 0 \), that it suddenly jumps to the value given by Eq. (116), and that it again jumps to zero at time \( t_f \). This implies a force having the shape of two \( \delta \)-function pulses around \( t = 0 \) and \( t = t_f \). Such a force acts over two infinitely small time intervals and leads to sharp edges in the classical path. More realistically one has to consider a finite time scale \( \tau_p \) for the action of the force which must be still large compared to the radiation time scale \( \tau_0 \). We may interpret the time scale \( \tau_p \) as a preparation time since it represents the time required to prepare the initial state of a moving electron.
A natural, physical cutoff frequency of the order \( \Omega \sim 1/\tau_p \) is thus introduced by the preparation time scale \( \tau_p \) and we may set

\[
\Omega t_f = \frac{t_f}{\tau_p}
\]  

(133)

in the following. It should be noted that the weak logarithmic dependence on \( \Omega \) shows that the precise value of the preparation time scale \( \tau_p \) is rather irrelevant. The important point is that the preparation time introduces a new time scale which removes the dependence on the cutoff. The vacuum decoherence function can thus be written,

\[
\Gamma_{\text{vac}} \approx -\frac{2\alpha}{3\pi} \ln \left( \frac{t_f}{\tau_p} \right) \frac{(q_f - q_i)^2}{(ct_f)^2},
\]  

(134)

showing that it vanishes for large times essentially as \( t_f^{-2} \).

The thermal contribution \( \Gamma_{\text{th}} \) is determined by the thermal correlation time \( \tau_B \). For \( T \to 0 \) we have \( \tau_B \to \infty \), and this contribution vanishes. For large times \( t_f \gg \tau_B \) the thermal decoherence function may be approximated by

\[
\Gamma_{\text{th}} \approx -\frac{2\alpha}{3\pi} \frac{t_f}{\tau_B} \frac{(q_f - q_i)^2}{(ct_f)^2},
\]  

(135)

which shows that \( \Gamma_{\text{th}} \) vanishes as \( t_f^{-1} \). Thus, for short times the vacuum contribution dominates, whereas the thermal contribution is dominant for large times. Both contributions \( \Gamma_{\text{vac}} \) and \( \Gamma_{\text{th}} \) are plotted separately in Fig. 1 which clearly shows the crossover between the two regions of time.

Eq. (132) implies that the vacuum coherence length is roughly of the order

\[
L(t_f)_{\text{vac}} \sim c \cdot t_f.
\]  

(136)

To see this let us assume a typical preparation time scale of the order \( \tau_p \sim 10^{-21} \text{s} \). If we take \( t_f \) to be of the order of 1s we find that \( \ln(t_f/\tau_p) \sim 48 \). In the rather extreme case \( t_f \sim 10^{17} \), which is of the order of the age of the universe, we get \( \ln(t_f/\tau_p) \sim 87 \). On using \( 3\pi/4\alpha \approx 322 \) and Eq. (132) for \( T = 0 \) one is led to the estimate (136).

5.2 Wave Packet Propagation

Having obtained an expression for the decoherence function \( \Gamma \) we now proceed with a detailed discussion of its physical significance. For this purpose it will be helpful to investigate first how \( \Gamma \) affects the time-evolution of an electronic wave packet. We consider the initial wave function at time \( t = 0 \),

\[
\psi_0(x) = \left( \frac{1}{2\pi \sigma_0^2} \right)^{3/4} \exp \left[ -\frac{(x - a)^2}{4\sigma_0^2} - ik_0(x - a) \right],
\]  

(137)
Fig. 1. The vacuum contribution $\Gamma_{\text{vac}}$ and the thermal contribution $\Gamma_{\text{th}}$ of the decoherence function $\Gamma$ (Eq. (130)). For a fixed value $|q_f - q_i| = 0.1 \cdot c \tau_B$, the two contributions are plotted against the time $t_f$ which is measured in units of the thermal correlation time $\tau_B$. The temperature was chosen to be $T = 1$K. One observes the decrease of both contributions for increasing time, demonstrating the vanishing of decoherence effects for long times. The thermal contribution $\Gamma_{\text{th}}$ vanishes as $t_f^{-1}$, while the vacuum contribution $\Gamma_{\text{vac}}$ decays essentially as $t_f^{-2}$, leading to a crossover between two regimes dominated by the vacuum and by the thermal contribution, respectively.

describing a Gaussian wave packet centered at $x = a$ with width $\sigma_0$. With the help of Eqs. (113), (114) and (131) we get the position space probability density at the final time $t_f$,

\begin{equation}
\rho_m(r_f, t_f) = \rho_m(r_f, q_f = 0, t_f) = \rho_m(r_f, 0, t_f) = \int d^3r_i \int d^3q_i \left( \frac{m R}{2 \pi t_f} \right)^3 \exp \left[ -\frac{i m R}{t_f} (r_f - r_i) - \frac{q_i^2}{2L(t_f)^2} \right] \times \psi_0(r_i + \frac{1}{2} q_i) \psi_0^*(r_i - \frac{1}{2} q_i).
\end{equation}

The Gaussian integrals may easily be evaluated with the result,

\begin{equation}
\rho_m(r_f, t_f) = \left( \frac{1}{2 \pi \sigma(t_f)} \right)^{3/2} \exp \left[ -\frac{(r_f - b)^2}{2 \sigma(t_f)^2} \right],
\end{equation}
where
\[ b \equiv a - \frac{k_0 t_f}{m_R} \quad (140) \]
and
\[ \sigma(t_f)^2 \equiv \sigma_0^2 + \frac{t_f^2}{4m_R^2 \sigma_0^2} + \frac{t_f^2}{m_R^2} \quad (141) \]
This shows that the wave packet propagates very much like that of a free Schrödinger particle with physical mass \( m_R \). The centre \( b \) of the probability density moves with velocity \(-k_0/m_R\), while its spreading, given by Eq. (141), is similar to the spreading \( \sigma(t_f)^2_{\text{free}} \) which is obtained from the free Schrödinger equation,
\[ \sigma(t_f)^2_{\text{free}} = \sigma_0^2 + \frac{t_f^2}{4m_R^2 \sigma_0^2} \quad (142) \]
If we write
\[ \sigma(t_f)^2 = \sigma_0^2 + \frac{t_f^2}{4m_R^2 \sigma_0^2} \left( 1 + \frac{4\sigma_0^2}{L(t_f)^2} \right) \quad (143) \]
we observe that the decoherence function affects the probability density only through the width \( \sigma(t_f) \) and leads to an increase of the spreading. In view of the estimate (136) the correction term in Eq. (143) is, however, small for times satisfying
\[ L(t_f) \sim c \cdot t_f \gg \sigma_0. \quad (144) \]
This means that the influence of the radiation field can safely be neglected for times which are large compared to the time it takes a light signal to travel the width of the wave packet.

![Fig. 2.](image)

**Fig. 2.** Sketch of the interference experiment used to determine the decoherence factor. Two Gaussian wave packets with initial separation \( 2a \) approach each other with opposite velocities of equal magnitude \( v = k_0/m_R \).

Let us now study the evolution of a superposition of two Gaussian wave packets separated by a distance \( 2a \). This case has been studied already by
Barone and Caldeira [15] who find, however, a different result. We assume that the packets have equal widths \( \sigma_0 \) and that they are centered initially at \( x = \pm a = \pm (a, 0, 0) \). The packets are supposed to approach each other with the speed \( v = k_0/m_R > 0 \) (see Fig. 2). For simplicity the motion is assumed to occur along the \( x \)-axis. Thus we have the initial state

\[
\psi_0(x) = A_1 \left( \frac{1}{2\pi \sigma_0^2} \right)^{3/4} \exp \left[ -\frac{(x - a)^2}{4\sigma_0^2} - i k_0(x - a) \right] + A_2 \left( \frac{1}{2\pi \sigma_0^2} \right)^{3/4} \exp \left[ -\frac{(x + a)^2}{4\sigma_0^2} + i k_0(x + a) \right],
\]

(145)

where \( k_0 = (k_0, 0, 0) \) and \( A_1, A_2 \) are complex amplitudes. Our aim is to determine the interference pattern that arises in the moment of collision of the two packets at \( x = 0 \). Using again Eqs. (113), (114) and (131) and doing the Gaussian integrals we find

\[
\rho_m(r_f, t_f) = \left( \frac{1}{2\pi \sigma(t_f)^2} \right)^{3/2} \exp \left[ -\frac{r_f^2}{2\sigma(t_f)^2} \right] \times \left( |A_1|^2 + |A_2|^2 + 2\text{Re}A_1A_2^\ast \exp(\varphi(r_f)) \right).
\]

(146)

We recognize a Gaussian envelope centered at \( r_f = 0 \) with width \( \sigma(t_f) \), an incoherent sum \( |A_1|^2 + |A_2|^2 \), and an interference term proportional to \( A_1A_2^\ast \). The interference term involves a complex phase given by

\[
\varphi(r_f) = -2i k_0 r_f (1 - \varepsilon) - \frac{2a^2}{L(t_f)^2}(1 - \varepsilon).
\]

(147)

The term \(-2i k_0 r_f \) describes the usual interference pattern as it occurs for a free Schrödinger particle, while the contribution \( 2i k_0 r_f \varepsilon \) leads to a modification of the period of the pattern. The final time \( t_f = am_R/k_0 \) is the collision time and

\[
v = \frac{a}{t_f} = \frac{k_0}{m_R}
\]

(148)

is the speed of the wave packets. The factor \( \varepsilon \) is given by

\[
\varepsilon \equiv \frac{t_f^2}{m_R^2 L(t_f)^2 \sigma(t_f)^2} = \left( 1 + \frac{L(t_f)^2}{4\sigma_0^2} + \frac{m_R^2 \sigma_0^2 t_f^2}{L(t_f)^2} \right)^{-1}.
\]

(149)

Obviously we always have \( 0 < \varepsilon < 1 \). Furthermore, for the situation considered in Eq. (144) we have \( \varepsilon \ll 1 \). Thus, we get

\[
\varphi(r_f) = -2i k_0 r_f - \frac{2a^2}{L(t_f)^2}.
\]

(150)
The last expression clearly reveals that the real part of the phase $\varphi(r_f)$ describes decoherence, namely a reduction of the interference contrast described by the factor

$$D = \exp \left[ -\frac{(2\alpha)^2}{2L(t_f)^2} \right] = \exp \left[ -\frac{\text{distance}^2}{2(\text{coherence length})^2} \right],$$

which multiplies the interference term. As was to be expected from the general formula for $\Gamma$, the decoherence factor $D$ is determined by the ratio of the distance of the two wave packets to the coherence length.

Alternatively, we can write the decoherence factor in terms of the velocity of the wave packets. In the vacuum case we then get

$$D_{\text{vac}} = \exp \left[ -\frac{8\alpha^3}{3\pi} \ln \left( \frac{t_f}{\tau_p} \right) \left( \frac{v}{c} \right)^2 \right].$$

This clearly demonstrates that it is the motion of the wave packets which is responsible for the reduction of the interference contrast: If one sets into relative motion the two components of the superposition in order to check locally their capability to interfere, a decoherence effect is caused by the creation of a radiation field. As can be seen from Eq. (117) the spectrum of the radiation field emitted through the moving charge is proportional to $1/\omega$ which is a typical signature for the emission of bremsstrahlung. Thus we observe that the physical origin for the loss of coherence described by the decoherence function is the creation of bremsstrahlung.

It is important to recognize that the frequency integral of Eq. (117) converges for $\omega \to 0$, see Eq. (121). The decoherence function $\Gamma$ is thus infrared convergent which is obviously due to the fact that we consider here a process on a finite time scale $t_f$. This means that we have a natural infrared cutoff of the order of $\Omega_{\text{min}} \sim 1/t_f$, in addition to the natural ultraviolet cutoff $\Omega \sim 1/\tau_p$ introduced earlier. The important conclusion is that the decoherence function is therefore infrared as well as ultraviolet convergent.

It might be instructive, finally, to compare our results with the corresponding expressions which are derived from the famous Caldeira-Leggett master equation in the high-temperature limit (see, e.g. [3]). From the latter one finds the following expression for the coherence length

$$L(t_f)^2_{\text{CL}} = \frac{\hbar^2}{2\gamma t_f},$$

where $\gamma$ is the relaxation rate. This is to be compared with the expressions (132) for the coherence length. For large temperatures we have the following dominant time and temperature dependence,

$$L(t_f)^2 \sim \frac{t_f}{T} \quad \text{and} \quad L(t_f)^2_{\text{CL}} \sim \frac{1}{T t_f}.$$

$$L(t_f)^2 \sim \frac{t_f}{T} \quad \text{and} \quad L(t_f)^2_{\text{CL}} \sim \frac{1}{T t_f}. \quad (154)$$
Hence, while both expressions for the coherence length are proportional to the inverse temperature, the time dependence is completely different. Namely, for $t_f \to \infty$ we have

$$L(t_f)^2 \to \infty \quad \text{and} \quad L(t_f)^2_{\text{CL}} \to 0,$$

and, therefore, complete coherence in the case of bremsstrahlung and total destruction of coherence in the Caldeira-Leggett case.

6 The Harmonically Bound Electron in the Radiation Field

As a further illustration let us investigate briefly the case of an electron in the radiation field moving in a harmonic external potential. Another approach to this problem may be found in [19], where the authors arrive, however, at the conclusion that there is no decoherence effect in the vacuum case.

We take $\omega_0 > 0$ and solve the equation of motion (101) with the help of the ansatz

$$r(t) = r_0 \exp(zt),$$

where, for simplicity, we consider the motion to be one-dimensional. Substituting this ansatz into (101) one is led to a cubic equation for $z$,

$$z^2 - \tau_0 z^3 + \omega_0^2 = 0.$$  \hspace{1cm} (157)

For vanishing coupling to the radiation field ($\tau_0 = 0$) the solutions are located at $z_{\pm} = \pm i\omega_0$, describing the free motion of a harmonic oscillator with frequency $\omega_0$.

For $\tau_0 > 0$ the cubic equation has three roots, one is real and the other two are complex conjugated to each other. The real root corresponds to the runaway solution and must be discarded. Let us assume that the period of the oscillator is large compared to the radiation time,

$$\tau_0 \ll \frac{1}{\omega_0}.$$  \hspace{1cm} (158)

Because of $\tau_0 \sim 10^{-24}$ s this assumption is well satisfied even in the regime of optical frequencies. We may thus determine the complex roots to lowest order in $\omega_0\tau_0$,

$$z_{\pm} = \pm i\omega_0 - \frac{1}{2} \tau_0 \omega_0^2.$$  \hspace{1cm} (159)

The purely imaginary roots $\pm i\omega_0$ of the undisturbed harmonic oscillator are thus shifted into the negative half plane under the influence of the radiation field. The negative real part describes the radiative damping. In fact, we see that $r(t)$ decays as $\exp(-\gamma t/2)$, where

$$\gamma = \tau_0 \omega_0^2 = \frac{2}{3} \frac{\hbar \omega_0^2}{m_{R}c^2},$$  \hspace{1cm} (160)
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is the damping constant for radiation damping \[17\]. In the following we consider times \(t_f\) of the order of magnitude of one period \(\omega_0 t_f \sim 1\). Because of \(\gamma t_f = (\omega_0 \tau_0)(\omega_0 t_f)\) we then have \(\gamma t_f \sim \tau_0 \omega_0 \ll 1\). In this case the damping can be neglected and we may use the free solution in order to determine the decoherence function.

Let us consider again the case of a superposition of two Gaussian wave packets in the harmonic potential. The packets are initially separated by a distance \(2a\) and approach each other with opposite velocities of equal magnitude such that they collide after a quarter of a period, \(t_f = \pi/2\omega_0\). The corresponding free solution \(q(t)\) is therefore given by

\[
q(t) = q_i \cos \omega_0 t + q_f \sin \omega_0 t.
\] (161)

To describe the situation we have in mind we take \(q_i = 2a\) (initial separation of the wave packets) and \(q_f = 0\) (to get the probability density). Hence, we have

\[
\dot{q}(t) = -2a \omega_0 \sin \omega_0 t,
\] (162)

and we evaluate the Fourier transform,

\[
Q(\omega) = \int_0^{t_f} dt \exp(i\omega t) \dot{q}(t)
= a \omega_0 \left[ \frac{\exp(i[\omega + \omega_0]t_f) - 1}{\omega + \omega_0} - \frac{\exp(i[\omega - \omega_0]t_f) - 1}{\omega - \omega_0} \right].
\]

This yields the decoherence function

\[
\Gamma \equiv \Gamma(q_i = 0, q_i, t_f) = e^{2(a\omega_0)^2} \int_0^{\Omega} d\omega \omega \left[ \frac{1 - \cos(\omega + \omega_0)t_f}{(\omega + \omega_0)^2} + \frac{1 - \cos(\omega - \omega_0)t_f}{(\omega - \omega_0)^2} \right] \coth \left( \frac{\beta \omega}{2} \right).\] (163)

We discuss the case of zero temperature. The frequency integral in Eq. (163) then approaches asymptotically the value \(2 \ln \Omega t_f\) which leads to the following expression for the decoherence factor,

\[
D_{\text{vac}} = \exp \Gamma_{\text{vac}} = \exp \left[ -\frac{8a}{3\pi} \ln \left( \frac{t_f}{\tau_p} \right) \langle (v/c)^2 \rangle \right].\] (164)

The interesting point to note here is that this equation is the same as Eq. (152) for the free electron, with the only difference that the square \((v/c)^2\) of the velocity, which was constant in the previous case, must now be replaced with its time averaged value \(\langle (v/c)^2 \rangle\).

7 Destruction of Coherence of Many-Particle States

For a single electron the vacuum decoherence factor (152) turns out to be very close to 1, as can be illustrated by means of the following numerical example.
We take $\tau_p$ to be of the order of $10^{-21}$s and $t_f$ of the order of 1s. Using a velocity $v$ which is already as large as 1/10 of the speed of light, one finds that $\Gamma_{\text{vac}} \sim 10^{-2}$, corresponding to a reduction of the interference contrast of about 1%. This demonstrates that the electromagnetic field vacuum is quite ineffective in destroying the coherence of single electrons.

For a superposition of many-particle states the above picture can lead, however, to a dramatic increase of the decoherence effect. Consider the superposition

$$|\psi\rangle = |\psi_1\rangle + |\psi_2\rangle$$

(165)
of two well-localized, spatially separated $N$-particle states $|\psi_1\rangle$ and $|\psi_2\rangle$. We have seen that decoherence results from the imaginary part of the influence phase functional $\Phi[J_c, J_a]$, that is from the last term on the right-hand side of Eq. (39) involving the anti-commutator function $D_1(x-x')_{\mu\nu}$ of the electromagnetic field. Thus, it is the functional

$$\Gamma[J_c] = -\frac{1}{4} \int_{t_i}^{t_f} d^4x \int_{t_i}^{t_f} d^4x' \, D_1(x-x')_{\mu\nu} J_{c\mu}(x) J_{c\nu}(x'),$$

(166)

which is responsible for decoherence. This shows that the decoherence function for $N$-electron states scales with the square $N^2$ of the particle number. Thus we conclude that for the case of the superposition (165) the decoherence function must be multiplied by a factor of $N^2$, that is the decoherence factor for $N$-particle states takes the form,

$$D_{\text{vac}}^N \sim \exp \left[ -\frac{8\alpha}{3\pi} \ln \left( \frac{t_f}{\tau_p} \right) \left( \frac{v}{c} \right)^2 N^2 \right].$$

(167)

This scaling with the particle number obviously leads to a dramatic increase of decoherence for the superposition of $N$-particle states. To give an example we take $N = 6 \cdot 10^{23}$, corresponding to 1 mol, and ask for the maximal velocity $v$ leading to a 1% suppression of interference. With the help of (167) we find that $v \sim 10^{-16}$ m/s. This means that, in order to perform an interference experiment with 1 mol electrons with only 1% decoherence, a velocity of at most $10^{-16}$ m/s may be used. For a distance of 1m this implies, for example, that the experiment would take $3 \times 10^8$ years!

8 Conclusions

In this paper the equations governing a basic decoherence mechanism occurring in QED have been developed, namely the suppression of coherence through the emission of bremsstrahlung. The latter is created whenever two spatially separated wave packets of a coherent superposition are moved to one place, which is indispensable if one intends to check locally their capability to interfere. We have seen that the decoherence effect through the electromagnetic radiation field is extremely small for single, non-relativistic electrons.
The decoherence mechanism is thus very ineffective on the Compton length scale. An important conclusion is that decoherence does not lead to a localization of the particle on arbitrarily small length scales and that no problems with associated UV-divergences arise here.

The decoherence mechanism through bremsstrahlung exhibits a highly non-Markovian character. As a result the usual picture of decoherence as a decay of the off-diagonals in the reduced density matrix does not apply. In fact, consider a superposition of two wave packets with zero velocity. The expression \( L(t_f)_{\text{vac}} \sim c \cdot t_f \) for the vacuum coherence length \( L(t_f)_{\text{vac}} \) show that decoherence effects are negligible for times \( t_f \) which are large in comparison to the time it takes light to travel the distance between the wave packets. The off-diagonal terms of the reduced density matrix for the electron do therefore not decay at all, which shows the profound difference between the decoherence mechanism through bremsstrahlung and other decoherence mechanisms (see, e.g. [20]).

A result of particular interest from a fundamental point of view is that coherence can already be destroyed by the presence of the electromagnetic field vacuum if superpositions of many-particle states are considered. An important conclusion which can be drawn from this picture of decoherence in QED refers to various alternative approaches to decoherence and the closely related measurement problem of quantum mechanics: In recent years several attempts have been made to modify the Schrödinger equation by the addition of stochastic terms with the aim to explain the non-existence of macroscopic superpositions through some kind of macrorealism. Namely, the random terms in the Schrödinger equation lead to a spontaneous destruction of superpositions in such a way that macroscopic objects are practically always in definite localized states. Such approaches obviously require the introduction of previously unknown physical constants. In the stochastic theory of Ghirardi, Pearle and Rimini [21], for example, a single particle microscopic jump rate of about \( 10^{-16}\text{s}^{-1} \) has to be introduced such that decoherence is extremely weak for single particles but acts sufficiently strong for many particle assemblies. It is interesting to observe that the decoherence effect caused by the presence of the quantum field vacuum yields a similar time scale in a completely natural way without the introduction of new physical parameters. Thus, QED indeed provides a consistent picture of decoherence and it seems unnecessary to propose new ad hoc theories for this purpose.

It must be emphasized that the above picture of decoherence in QED has been derived from the well-established basic postulates of quantum mechanics and quantum field theory. It therefore does not, of course, constitute a logical disprove of alternative approaches. However, it does represent an example for a basic decoherence mechanism in a microscopic quantum field theory. In particular, it provides a unified explanation of decoherence which does not suffer from problems with renormalization (as they occur, e.g. in
alternative theories \cite{22}) and which does not exclude a priori the existence of macroscopic quantum coherence. Only under certain well-defined conditions regarding time scales, relative velocities and the structure of the state vector, it is true that decoherence becomes important. Thus, decoherence is traced back to a dynamical effect and not to a modification of the basic principles of quantum mechanics.

In this paper we have discussed in detail only the non-relativistic approximation of the reduced electron dynamics. For a treatment of the full relativistic theory, including a Lorentz invariant characterization of the decoherence induced by the vacuum field, one can start from the formal development given in section 2. An investigation along these lines could also be of great interest for the study of measurement processes in the relativistic domain \cite{23}.

References

1. Feynman R. P., Vernon F. L. (1963): The Theory of a General Quantum System Interacting with a Linear Dissipative System. Ann. Phys. (N.Y.) 24, 118-173.
2. Caldeira A. O., Leggett A. J. (1983): Quantum Tunneling in a Dissipative System. Ann. Phys. (N.Y.) 149, 374-456; (1984) 153, 445(E).
3. Gardiner C. W., Zoller P. (2000): Quantum Noise. 2nd. ed., Springer-Verlag, Berlin.
4. Zurek W. H. (1991): Decoherence and the transition from quantum to classical. Phys. Today 44, 36-44.
5. Giulini D. et al. (1996): Decoherence and the Appearance of a Classical World in Quantum Theory. Springer-Verlag, Berlin.
6. Brune M. et al. (1996): Observing the progressive decoherence of the "meter" in a quantum measurement. Phys. Rev. Lett. 77, 4887-4890.
7. Myatt C. J. et al. (2000): Decoherence of quantum superpositions through coupling to engineered reservoirs. Nature 403, 269-273.
8. Weinberg S. (1996): The Quantum Theory of Fields, Volume I, Foundations. Cambridge University Press, Cambridge.
9. Jauch J. M., Rohrlich F. (1980): The Theory of Photons and Electrons. Springer-Verlag, New York.
10. Cohen–Tannoudji C., Dupont–Roc J., Grynberg G. (1998): Atom-Photon Interactions. John Wiley, New York.
11. Chou K.-c., Su Z.-b., Hao B.-l., Yu, L. (1985): Equilibrium and Nonequilibrium Formalisms Made Unified. Phys. Rep. 118, 1-131.
12. Diosi L. (1990): Landau’s Density Matrix in Quantum Electrodynamics. Found. Phys. 20, 63-70.
13. Feynman R. P., Hibbs A. R. (1965): Quantum Mechanics and Path Integrals. McGraw-Hill, New York.
14. Grabert H., Schramm P., Ingold G.-L. (1988): Quantum Brownian Motion: The Functional Integral Approach. Phys. Rep. 168, 115-207.
15. Barone P. M. V. B., Caldeira A. O. (1991): Quantum mechanics of radiation damping. Phys. Rev. A43, 57-63.
16. Anglin J. R., Paz J. P., Zurek W. H. (1997): Deconstructing Decoherence. Phys. Rev. A55, 4041-4053.
17. Jackson J. D. (1999): Classical Electrodynamics. Third Edition, John Wiley, New York.
18. Gradshteyn I. S., Ryzhik I. M. (1980): Table of Integral, Series, and Products. Academic Press, New York.
19. Dürr D., Spohn H. (2000): Decoherence Through Coupling to the Radiation Field. In: Blanchard Ph., Giuliani D., Joos E., Kiefer C., Stamatescu I.-O. (Eds.) Decoherence: Theoretical, Experimental, and Conceptual Problems. Springer-Verlag, Berlin, 77-86.
20. Joos E., Zeh H. D. (1985): The Emergence of Classical Properties Through Interaction with the Environment. Z. Phys. B59, 223-243.
21. Ghirardi G. C., Pearle P., Rimini A. (1990) Markov processes in Hilbert space and continuous spontaneous localization of systems of identical particles. Phys. Rev. A42, 78-89.
22. Ghirardi G. C., Grassi R., Pearle P. (1990): Relativistic Dynamical Reduction Models: General Framework and Examples. Found. Phys. 20, 1271-1316.
23. Breuer H. P., Petruccione F. (1999): Stochastic Unravelings of Relativistic Quantum Measurements. In: Breuer H. P., Petruccione F. (Eds.) (1999): Open Systems and Measurement in Relativistic Quantum Theory. Springer-Verlag, Berlin, 81-116.