ABNORMAL OCCUPATION REVISITED

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1. INTRODUCTION

We shall demonstrate that the Fermi surface of dense neutron matter may experience a rearrangement near the onset of pion condensation, due to strong momentum dependence of the effective interaction induced by spin-isospin fluctuations. In particular, the Fermi surface may take the form of a partially hollow sphere having a spherical hole in its center. Thus, a second (inner) Fermi surface may form as high-momentum single-particle states are filled and low-momentum states are vacated. The influence of this phenomenon on the superfluid transition temperature of the Fermi system is characterized with the help of a separation transformation [1,2] of the BCS gap equation.

The present effort can be viewed as a revival of the search for physical realizations of abnormal occupation in infinite, homogeneous Fermi systems – plausible instances in which the quasiparticle distribution differs from that of an ideal Fermi gas and Fermi liquid theory breaks down. This search (broadened to Bose systems) was in full swing at the time of the Third Workshop on Condensed Matter Theories, held in Buenos Aires in 1979, and it was led by Workshop founders Valdir Aguilera-Navarro, Ruben Barrera, Manuel de Llano, and Angel Plastino, with important contributions from James Vary, John Zabolitzky, and others [3-5].

2. THE BUBBLE REARRANGEMENT

Rearrangement of the characteristic Landau quasiparticle distribution signals a breakdown of Fermi-liquid theory and can lead to profound alteration of the orthodox behaviors we have come to expect based on this pervasive physical picture. For the sake of simplicity, our considerations will be limited to homogeneous systems, for which the Fermi-liquid distribution \( n_F(p) \) coincides with the momentum distribution of an ideal Fermi gas. The actual quasiparticle distribution \( n(p) \) must deviate from \( n_F(p) = \theta(p_F - p) \) if the necessary condition for its stability is violated. At zero temperature, this condition states that the change of the ground-state energy \( E_0 \) re-
mains positive for any admissible variation $\delta n(p)$ away from $n_F(p)$. More specifically, stability of a given quasiparticle distribution implies

$$
\delta E_0 = \int \xi(p, n(p)) \delta n(p) \frac{q^3 d q}{(2\pi)^3} > 0,
$$

where $\xi(p, n(p)) \equiv \varepsilon(p, n(p)) - \mu$ is the quasiparticle energy measured with respect to the chemical potential $\mu$. For $n(p) = n_F(p)$, the condition (1) is violated if $\varepsilon(p)$ rises above $\mu$ at $p < p_F$, or if $\varepsilon(p)$ falls below $\mu$ at $p > p_F$. A rearrangement of quasiparticle occupancies is precipitated when the density $\rho$ attains a critical value $\rho_{cF}$ at which the relation

$$
\xi(p, n_F(p); \rho_{cF}) = 0
$$

exhibits a bifurcation leading to a new root $p = p_0$. This relation usually serves only to determine the Fermi momentum $p_F$.

In homogeneous systems, the simplest type of rearrangement of the momentum distribution $n(p)$ of quasiparticles of given spin and isospin maintains the property that its values are restricted to 0 and 1, but the Fermi sea becomes multiply connected (cf. Refs. [4-7]). In particular, we may suppose that at densities exceeding the critical value $\rho_{cF}$, the normal-state distribution $\theta(p_F - p)$ is altered by the formation of a “bubble,” or particle void, over a range $p_i < p < p_I < p_F$, with the Fermi momentum $p_F$ readjusted to maintain the prescribed density. This distribution is formally represented by $n(p) = \theta(p_i - p) + \theta(p_F - p) \theta(p - p_I)$, where as usual $\theta(x) = 1$ if $x \geq 0$ and vanishes otherwise. One then has three Fermi surfaces: two inner surfaces located at $p_i$ and $p_I$, along with the usual outer surface at $p_F$. However, a more dramatic rearrangement can also occur, resulting in a distribution with partial occupation of quasiparticle states that lacks the distinctive trademark of Fermi-liquid theory, namely a discontinuity of $n(p)$ at the Fermi surface. In this scenario, called fermion condensation, there exists a finite range of momenta over which the quasiparticle energy coincides with the chemical potential, corresponding to the creation of a “fermion condensate” [8-12].

Any change of $n(p)$ from the normal-state distribution $n_F(p)$ must entail an increase of the kinetic energy of the quasiparticle system. Accordingly, the anticipated rearrangement only becomes possible if it is accompanied by a counterbalancing reduction of potential energy, which implies that the effective interaction between quasiparticles has acquired a substantial momentum dependence. The emergence of such a strong momentum dependence is exactly what one expects to occur as the density $\rho$ is increased toward the critical value $\rho_c$ for a second-order phase transition in which a branch of the spectrum $\omega_z(k)$ of collective excitations of the Fermi system collapses at a nonzero value $k_c$ of the wave vector $k$.

To justify this expectation, we follow Dyugaev [13] and consider the behavior of the quasiparticle scattering amplitude $F(p_1, p_2; k) \equiv z^2 \Gamma(p_1, p_2; k, \omega = 0)M^*/M$ near of the phase-transition point. Here $\Gamma(p_1, p_2; k, \omega)$ is the ordinary (in-medium) scattering amplitude, $M^*$ is the effective mass, and $z$ is the renormalization factor specifying the weight of the quasiparticle pole. The amplitude $F$ can be written as the sum $F^r + F^s$ of a regular part $F^r$ and a singular part $F^s$, with the latter taking the universal form

$$
F^s_{\alpha\delta; \beta\gamma}(p_1, p_2; k; \rho \rightarrow \rho_c) = -O_{\alpha\delta} O_{\beta\gamma} D(k) + O_{\alpha\gamma} O_{\beta\delta} D(|p_1 - p_2 + k|)
$$
in terms of the propagator $D(k)$ of the collective excitation. This form has been derived with due attention to the antisymmetry of the two-particle wave function under exchange of the particle coordinates (spatial, spin, isospin). The collective propagator is conveniently parametrized according [13]

$$D^{-1}(k) = \beta^2 + \gamma^2 \left( k^2 / k_c^2 - 1 \right)^2,$$

where the parameter $\beta(\rho)$, with $\beta(\rho_c) = 0$, measures the proximity to the phase-transition point. The vertex $O$ appearing in Eq. (3) determines the structure of the collective-mode operator and is normalized by $\text{Tr}(OO^\dagger) = 1$. Specifically, the choice $O = 1$ is made in treating the rearrangement of the quasiparticle distribution due to collapse of density oscillations [14], while $O = \vec{\sigma}$ is appropriate when studying the rearrangement of $n_F(p)$ triggered by the softening of the spin collective mode [15]. In the present investigation we will be concerned with dense, homogeneous neutron matter in which abnormal occupation is induced by spin-isospin fluctuations; thus the pertinent operator is $O = (\vec{\sigma} \cdot \vec{k}) \tau$.

Cutting through the details, the most important features of the model defined by Eqs. (3) and (4) are that the function $F^s(p_1, p_2, k = 0) \simeq D(p_1 - p_2)$ depends on the difference $p_1 - p_2$ and that in the neighborhood of the soft-mode phase-transition point this dependence becomes very strong.

Eqs. (3) and (4) furnish a suitable basis for efficient evaluation of the single-particle spectrum $\xi(p)$ near the second-order phase transition. We exploit a straightforward connection between $\xi(p)$ and the scattering amplitude $F(p_1, p_2, k = 0)$, thereby circumventing the awkward frequency integration that would be encountered in an RPA approach. This connection is made through the relation

$$\frac{\partial \xi(p)}{\partial p} = \frac{p}{M} + \frac{1}{2} \int F_{\alpha\beta;\alpha\beta}(p_1, \vec{p}_1) \frac{\partial n(p_1)}{\partial \vec{p}_1} \frac{d^3 p_1}{(2\pi)^3} ,$$

which may be derived by means of the Landau-Pitaevskii identities [16-18]. The contribution to Eq. (5) from the singular part (3) of $F$ can be easily integrated over the momentum $p$ to obtain

$$\xi(p) = \frac{p^2}{2M^*_r} + \frac{1}{2} \int D(p - p_1)n(p_1) \frac{d^3 p_1}{(2\pi)^3} .$$

In stating this result, we assume that the contribution to $\xi(p)$ from the regular, nonsingular part of $F$ can be simulated by replacing of the bare mass $M$ appearing in Eq. (5) by an effective mass $M^*_r$. The generally accepted values for this effective mass are in the range 0.7–0.8 for the pertinent densities in the neutron-star interior.

According to Migdal and collaborators [19,20] (see also Ref. [21]), a dramatic phase transition can occur when the density $\rho$ of neutron matter in the liquid core of a neutron star reaches a critical density $\rho_{c\pi}$ of some 2–3 times the equilibrium density $\rho_0$ of symmetrical nuclear matter. The spin-isospin collective mode collapses at a finite wave vector $k = k_c \sim p_F$ and a phase transition identified as pion condensation sets in. A prominent feature of the ground state of the system beyond the phase-transition point is the presence of a condensate of spin-isospin density waves. It
should be clear from Eqs. (3) and (4) that spin-isospin fluctuations with \( k \sim k_c \) are strongly amplified in the neighborhood of the transition as a consequence of the divergence of the propagator \( D(k \to k_c, \rho_0) \).

We are thus led to apply Eq. (6) to dense neutron matter close to the onset of neutral pion condensation. Employing the parametrization (4) in Eq. (6), we obtain the working formula

\[
\xi(p) = \frac{p^2}{2M_r^2} + \frac{1}{2} \int \frac{1}{\beta^2 + \gamma^2 ((\mathbf{P} - \mathbf{p_1})^2 - k_c^2)^2 / k_c^4} n(p_1) \frac{d^3p_1}{(2\pi)^3} .
\]  

(7)

Ideally, one would like to extract quantitatively reliable values for the input parameters \( \beta, \gamma, \) and \( k_c \) from a convincing \textit{ab initio} treatment of neutron-star matter. Unfortunately, no such treatment is yet available. Moreover, current predictions of the critical density \( \rho_{c\pi} \) range from 0.2 to 0.5 fm\(^{-3}\) (i.e., some 1–3 times \( \rho_0 \)), depending on what theoretical assumptions are implemented [19-21].

This situation leaves us with no alternative but to carry out calculations for several representative choices the parameters of the microscopic model. Inserting the formula (7) into Eq. (2), we determine the critical density \( \rho_{cF} \) at which the solution of Eq. (2) bifurcates. For \( \rho > \rho_{cF} \) this equation then yields two new momenta \( p_i \) and \( p_1 \) at which \( \xi(p) \) vanishes, and between which \( \xi(p) \) is positive. The bubble region lies between these two momenta.

The necessity for brevity in this presentation precludes the explicit presentation of the numerical results that have been derived for the spectrum \( \xi(p) \) and for the phase diagram of dense neutron matter. The neutron spectrum has been calculated at the critical densities \( \rho_{cF} \) corresponding to three different sets of model parameters: (a) \( \gamma = 1.25m_\pi, k_c = 0.9p_F, \beta^2 = 0.22m_\pi^2 \) (\( \rho_{cF} \simeq 1.19 \rho_0 \)), (b) \( \gamma = 1.25m_\pi, k_c = 0.9p_F, \beta^2 = 0.25m_\pi^2 \) (\( \rho_{cF} \simeq 1.76 \rho_0 \)), and (c) \( \gamma = 1.25m_\pi, k_c = p_F, \beta^2 = 0.13m_\pi^2 \) (\( \rho_{cF} \simeq 1.88 \rho_0 \)), where \( m_\pi \) is the pion mass. The choice \( k_c = 0.9p_F \) for the critical wave number is suggested by earlier numerical investigations [20]. Two different positions were found for the bifurcation point, namely \( p_0 = 0 \) (for parameter sets (a) and (b)) and \( p_0 \simeq 0.12p_F \) (for set (c)). The phase diagram of neutron matter in the variables \( \rho \) (measured in \( \rho_0 \)) and \( \beta^2 \) (measured in \( m_\pi^2 \)) has been constructed at \( k_c = 0.9p_F \) for four different values of \( \gamma \) (1.0, 1.2, 1.4, and 1.6, in units of the pion mass). Plots of the results of these calculations may be found in Refs. [18,22].

Variation of the parameters \( \beta, \gamma, \) and \( k_c \) within sensible bounds can significantly affect the phase diagram and hence the extent, in density, of the phase with rearranged quasiparticle occupation. Even so, our numerical study has revealed four characteristic and generic features of the bubble rearrangement.

(i) The critical density \( \rho_{cF} \) for the rearrangement is less than the critical density \( \rho_{c\pi} \) for pion condensation. Since both phenomena arise from the strong momentum dependence of the amplitude \( F(\mathbf{p_1}, \mathbf{p_2}; \mathbf{k} \to 0) \), rearrangement of the quasiparticle distribution may be regarded as a \textit{precursor} of pion condensation.

(ii) The bifurcation point associated with formation of a bubble in the neutron momentum distribution is located at small momenta, \( p_0 < 0.2p_F \), regardless of the applicable value of \( \rho_{c\pi} \).

(iii) The spectrum \( \xi(p) \) exhibits a deep depression for \( p \sim (0.5 - 0.6)p_F \).
The ratios $\rho_{cF}/\rho_{c\pi}$ and $p_0/p_F$ are insensitive to the actual value taken by $\rho_{c\pi}$ within the range of plausible theoretical predictions.

The emergence in neutron matter of one or more new Fermi surfaces positioned at low momentum values would provide a new avenue for rapid direct-Urca neutrino cooling of neutron stars [22]. More broadly, the creation of new Fermi surfaces by the mechanism we have described – as well as the more profound rearrangement involved in fermion condensation – would call for revision of many of the conclusions that have been developed within Fermi liquid theory. Here we shall focus on some elementary properties of pairing in the reconfigured system.

3. PAIRING IN THE PRESENCE OF ABNORMAL OCCUPATION

We assume, for the sake of simplicity, that beyond the instability point there exist only two Fermi surfaces, an outer one corresponding to the usual Fermi momentum $p_F$ and an additional inner one at $p_I$, lying close to the origin in momentum space. Hence we consider the limiting case $p_i = 0$ in our original specification of the “bubble” rearrangement. Also for the sake of simplicity, we restrict the analysis to $^1S_0$ pairing, for which the BCS gap equation has the familiar form

$$\Delta(p) = -\int V(p, p_1)\mathcal{E}^{-1}(p; T)\Delta(p_1)d\tau,$$

where $V$ is the effective particle-particle interaction and we employ notations $d\tau = p^2dp/2\pi^2$ for the volume element and $\mathcal{E}^{-1}(p; T) = [2E(p)]^{-1}\tanh[E(p)/2T]$ for the usual combination of tanh temperature factor and energy denominator $2E(p)$. The appearance of the gap function $\Delta(p)$ in the superfluid quasiparticle energy $E(p) = [\xi^2(p) + \Delta^2(p)]^{1/2}$ renders the gap problem nonlinear. The quantity $\xi(p)$ is to be interpreted as the single-particle spectrum in the system with pairing turned off.

Adopting the strategy for solving gap equations that was introduced in Ref. [1] and elaborated in Refs. [2,3], we write the block $V$, identically, as a separable part plus a remainder that automatically vanishes on the outer Fermi surface. Hence we write

$$V(p, p_1) = V_F\phi_F(p_1)\phi_F(p_2) + W(p, p_1),$$

and take $\phi_F(p) = V(p, p_F)/V_F$, where $V_F = V(p_F, p_F)$. It follows directly that $W(p, p_F) = W(p_F, p) = 0$, as required. In the ordinary case where there is only one Fermi surface, this decomposition allows us to replace the singular nonlinear integral equation (8) by two equivalent equations: (i) a nonsingular quasilinear integral equation for a $T$-independent shape factor $\chi(p) = \Delta(p)/\Delta_F$ and (ii) a nonlinear ‘algebraic’ equation for the $T$-dependent gap value $\Delta_F(T) = \Delta(p_F, T)$. In the present case where there are two Fermi surfaces, we must extend the procedure of Ref. [1] to deal consistently with the inner Fermi surface as well as the outer one. This is accomplished by decomposing the interaction term $W$ appearing in Eq. (9) in the same manner as before, setting

$$W(p_1, p_2) = W_I\phi_I(p_1)\phi_I(p_2) + Y(p_1, p_2)$$
with $\phi_I(p) = W(p, p_I)/W_I$ and $W_I = W(p_I, p_I) \equiv V(p_I, p_I) - V^2(p_F, p_I)/V(p_F, p_F)$, so that $Y(p, p_I) \equiv Y(p_I, p) \equiv Y(p_F, p) \equiv Y(p, p_F) = 0$. The above relations entail the boundary values

$$\phi_F(p_F) = 1, \quad \phi_I(p_I) = 1, \quad \phi_I(p_F) = 0,$$  \hspace{1cm} (11)

while the key quantity $\phi_F(p_I) \propto V(p_I, p_F)$ describes the connection between the quasiparticles of the two Fermi surfaces in the particle-particle channel. If $\phi_F(p_I)$ vanishes, these surfaces are disconnected and the problem is trivialized.

In the general case where $V(p_I, p_F) \neq 0$, substitution of Eqs. (9) and (10) into the BCS gap equation (8) gives

$$\Delta(p) = -V_F \phi_F(p) \int \phi_F(p_1) \mathcal{E}^{-1}(p_1; T) \Delta(p_1) d\tau_1 - W_I \phi_I(p) \int \phi_I(p_1) \mathcal{E}^{-1}(p_1; T) \Delta(p_1) d\tau_1$$

$$- \int Y(p, p_1) \mathcal{E}^{-1}(p_1; T) \Delta(p_1) d\tau_1. \hspace{1cm} (12)$$

This equation is conveniently rewritten as

$$\Delta(p) = B_F \chi_F(p) + B_I \chi_I(p), \hspace{1cm} (13)$$

with

$$B_F = -V_F \int \phi_F(p) \mathcal{E}^{-1}(p; T) \Delta(p) d\tau,$$

$$B_I = -W_I \int \phi_I(p) \mathcal{E}^{-1}(p; T) \Delta(p) d\tau, \hspace{1cm} (14)$$

and

$$\chi_F(p) = \phi_F(p) - \int Y(p, p_1) \mathcal{E}^{-1}(p_1; T) \chi_F(p_1) d\tau_1,$$

$$\chi_I(p) = \phi_I(p) - \int Y(p, p_1) \mathcal{E}^{-1}(p_1; T) \chi_I(p_1) d\tau_1. \hspace{1cm} (15)$$

Referring to the relations (11), we observe that

$$\chi_I(p_I) = \chi_F(p_F) = 1, \quad \chi_I(p_F) = 0, \quad \chi_F(p_I) = \phi_F(p_I) = V(p_I, p_F)/V(p_F, p_F), \hspace{1cm} (16)$$

because the block $Y$ is zero when either of its arguments lies on a Fermi surface. By this same property, it is permissible, inside the quantity $\mathcal{E}^{-1}$ appearing in the integral equations (15), to replace the superfluid quasiparticle energy $E(p_1)$ by $|\xi(p_1)|$ and the temperature factor $\tanh(E(p_1)/2T)$ by unity. Because the energy gaps involved are generally quite tiny compared to the Fermi energy, these replacements are valid to a superb approximation. We are left with the linear integral equations

$$\chi_F(p) = \phi_F(p) - \int Y(p, p_1) \frac{1}{2|\xi(p_1)|} \chi_F(p_1) d\tau_1,$$

$$\chi_I(p) = \phi_I(p) - \int Y(p, p_1) \frac{1}{2|\xi(p_1)|} \chi_I(p_1) d\tau_1, \hspace{1cm} (17)$$
for the two shape functions needed to construct the gap function $\Delta(p)$ using Eq. (13). Since there remains no trace of the temperature $T$ in Eqs. (17), we are free to regard the solutions $\chi_I(p)$ and $\chi_F(p)$ as $T$-independent quantities.

Appealing to the properties (16), Eq. (13) yields

$$\Delta_F \equiv \Delta(p_F) = B_F, \quad \Delta_I \equiv \Delta(p_I) = B_I + B_F \phi_F(p_I). \quad (18)$$

Inserting the decomposition (13) into Eqs. (14), we arrive at a system of two equations

$$B_F = -V_F L_{FF} B_F - V_F L_{FI} B_I, \quad B_I = -W_I L_{IF} B_F - W_I L_{II} B_I, \quad (19)$$

for determination of the amplitudes $B_F$ and $B_I$ entering Eq. (13), where

$$L_{FF} = \int \phi_F(p) \mathcal{E}^{-1}(p; T) \chi_F(p) d\tau, \quad L_{II} = \int \phi_I(p) \mathcal{E}^{-1}(p; T) \chi_I(p) d\tau,$$

$$L_{IF} \equiv L_{FI} = \int \phi_I(p) \mathcal{E}^{-1}(p; T) \chi_F(p) d\tau \equiv \int \phi_F(p) \mathcal{E}^{-1}(p; T) \chi_I(p) d\tau. \quad (20)$$

It is helpful to recast the system (19) in an equivalent form

$$[1 + V_F L_{FF}(T) - V_F \phi_F(p_I) L_{FI}(T)] \Delta_F + V_F L_{FI}(T) \Delta_I,$$

$$[W_I L_{FI}(T) - (1 + W_I L_{II}(T)) \phi_F(p_I)] \Delta_F + [1 + W_I L_{II}(T)] \Delta_I = 0. \quad (21)$$

For a solution to exist, the determinant $D(T)$ of (19) or (21) must equal zero for any $T$. Together with either of the two equations (19) [or either of (21)], the dispersion relation $D(T) = 0$ forms a closed system that permits one to determine all characteristics of the superfluid system feeding upon the two Fermi surfaces located at $p_I$ and $p_F$.

We begin to explore the implications of the formalism we have developed by examining the effect of the additional (inner) Fermi surface on the superfluid transition temperature $T_c$. Observe that Eqs. (19) [or (21)] become decoupled if $L_{IF} = 0$. Assume, as a first case (Case I), that both of the interaction parameters $V_F$ and $W_I$ are negative, so that Cooper pairing could exist at both Fermi surfaces when they are disconnected. The pairing effect is naturally stronger at the “main” or outer surface due to a greater density of states. From the two solutions of the problem as stated, we therefore select $\Delta(p) = \Delta_F \chi_F(p)$ with $\Delta_I = \phi_F(p_I) \Delta_F$, implying that the individual critical temperatures $T_c^F$ and $T_c^I$ satisfy $T_c^F > T_c^I$. It should be noted that in spite of this inequality, the magnitude of the ratio $\Delta_I/\Delta_F \sim \phi_F(p_I)$ is not necessarily less than unity (see below).

Working in the vicinity of the transition temperature $T_c$, standard arguments and manipulations in the spirit of BCS theory reveal the behaviors

$$L_{FF}(T \to T_c) \to N_F(0) \{ (1 + g^2_{IF})(L + \alpha_1\tau) - \alpha_2 \left[ D_F^2 + g^2_{IF} D_I^2 \right] \},$$

$$L_{II}(T \to T_c) \to N_I(0) \left[ L + \alpha_1\tau - \alpha_2 D_I^2 \right],$$

$$L_{IF}(T \to T_c) \to N_I(0) \phi_F(p_I) \left[ L + \alpha_1\tau - \alpha_2 D_I^2 \right], \quad (22)$$
in terms of the three dimensionless parameters \( \tau = (T_c - T) / T_c \), \( D_F = \Delta_F / T_c \), and \( D_I = \Delta_I / T_c \). In Eqs. (22), \( N_F(0) \) and \( N_I(0) \) are respectively the densities of states at the outer and inner Fermi surfaces, \( g_F^2 = \phi_F^2(p_I)N_I(0) / N_F(0) \) is an effective coupling constant, and \( L = \ln(\varepsilon_F^0 / \pi T_c) + C \) measures the transition temperature, where \( \varepsilon_F^0 \) is the free Fermi energy and \( C \) is Euler’s constant (0.577). Certain constants entering the derivation of the limiting behaviors of \( L_{II}(T) \), \( L_{IF}(T) \), and \( L_{FF}(T) \) have no material role in our arguments; they merely effect a renormalization of the critical temperatures \( T^F_c \) and \( T^I_c \) and may be thus be suppressed in forming Eqs. (22). The relevant temperature dependences are determined entirely by the ratio \( \alpha_1 / \alpha_2 = 8\pi^2 / 7\zeta(3) \).

After substituting the results (22) into Eqs. (21), we arrive at

\[
[1 + V_F N_F(0)(L + \alpha_1 \tau - \alpha_2 D_F^2)]D_F + V_F N_I(0)\phi_F(p_I)(L + \alpha_1 \tau - \alpha_2 D_I^2)D_I = 0 , \\
-\phi_F(p_I)D_F + [1 + W_I N_I(0)](L + \alpha_1 \tau - \alpha_2 D_I^2)D_I = 0 .
\]

(23)

Setting \( T = T_c \) and evaluating the determinant \( D(T_c) \) of this system, we obtain a closed formula for the new critical temperature \( T_c \) in terms of the individual critical temperatures \( T^F_c \) and \( T^I_c \) for the uncoupled system:

\[
(L - l_I)(L - l_F) - g_{II}^2 l_F l_I L = 0 .
\]

(24)

The constants \( l_F \) and \( l_I \) entering this relation are defined by \( l_F \equiv \ln(\varepsilon_F^0 / \pi T^F_c) + C = -1 / V_F N_F(0) \) and \( l_I \equiv \ln(\varepsilon_I^0 / \pi T^I_c) + C = -1 / W_I N_I(0) \). The inequality \( T^F_c > T^I_c \) clearly implies \( l_I > l_F \).

The situation at small coupling, \( g_{II}^2 \ll 1 \), is especially transparent. In this case the quantity \( L \) (which measures \( T_c \)) is not much different from \( l_F \) (which measures \( T^F_c \)), allowing us to replace \( L \) by \( l_F \) in the last term of the determinantal condition (24). The solution of Eq. (24) is then

\[
L_\pm = \frac{l_I + l_F}{2} \pm \left[ \frac{(l_I - l_F)^2}{4} + g_{II}^2 l_F l_I \right]^{1/2} .
\]

(25)

This result reminds us of the familiar textbook solution of the two-level problem. In that problem, the two energy levels repel each other when the off-diagonal interaction is switched on. The lower level moves downward and the upper level moves upward. In the current problem, the greater logarithm (in this case, \( L_+ \)) increases while the smaller logarithm (\( L_- \)) decreases. In particular,

\[
L_- - l_F \simeq -g_{II}^2 l_I l_F / l_I - l_F .
\]

(26)

In the case being considered, \( l_I \) and \( l_F \) are both positive, with \( l_I > l_F \). We may therefore conclude that the emergence of the second Fermi surface increases the critical temperature \( T_c \) relative to \( T^F_c \).

The picture changes nontrivially when we consider the more interesting case (Case II) in which pairing is absent at the new Fermi surface when the two surfaces are disconnected, but is still present at the original surface. When the coupling is
turned on, a pairing gap is found to exist on the new Fermi surface as well as the old one, a feature that is directly evident from either of the equations (21). The result (25) continues to apply (again assuming small coupling). However, in contrast to Case I, the value of \( l_I \) becomes negative while \( l_F \) remains positive. Consequently, the single acceptable value of \( L \) derived from Eq. (25) increases relative to \( l_F \), implying a decrease of \( T_c \) with respect to \( T_c^F \). This behavior should not be surprising: the value of the pairing gap depends on the shape of the single-particle spectrum, and if the spectrum becomes flatter in a region where the interaction is repulsive, there must be a suppression of the gap value and a concomitant suppression of \( T_c \). We must stress that the situation is now quite different from that of perturbation theory, where the gap increases independently of the sign of the perturbating interaction. The distinctive behavior we have described is indicative of a failure of perturbation theory in Case II. We should also point out the close resemblance between the predicted behavior and the proximity effect observed in junctions between a superconductor and a normal metal: the superconductor tends to induce superconditivity on the normal side of the junction, while suppressing its strength on the superconducting side.

Here we have confined our attention to the effect of the bubble rearrangement on the superfluid transition temperature \( T_c \), considering the possible scenarios for pairing when there are two concentric Fermi surfaces. Results are also available [18] for the modifications produced in the specific-heat discontinuity at \( T_c \) and for the relation between \( T_c \) and the energy gap \( \Delta_F \) at \( T = 0 \).

4. CONCLUSIONS

We have studied the rearrangement of single-particle degrees of freedom that precedes the onset of pion condensation, and we have found that this rearrangement may express itself in the emergence of a bubble in the quasiparticle momentum distribution, i.e., the formation of a new Fermi sea with a spherical hole in the middle. This is in fact one of the scenarios considered some two decades ago by de Llano, Plastino, and their collaborators. The formalism we have developed and the results we have obtained can be applied more widely in the theory of strongly correlated Fermi systems. In this spirit, it will be of special interest to re-examine the case of superfluid \(^3\)He, which offers a realistic example of a Fermi liquid existing near an antiferromagnetic phase transition. In our view, rearrangements of the single-particle degrees of freedom arising from momentum-dependent effective interactions have a generic character. Such rearrangements include not only bubble configurations, but also the phenomenon of fermion condensation [8-12]. We call attention especially to Ref. [11], where the competition between bubble rearrangement and fermion condensation is studied under variation of the temperature.

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