Projections of Orbital Measures, Gelfand–Tsetlin Polytopes, and Splines

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Abstract. The unitary group $U(N)$ acts by conjugations on the space $\mathcal{H}(N)$ of $N \times N$ Hermitian matrices, and every orbit of this action carries a unique invariant probability measure called an orbital measure. Consider the projection of the space $\mathcal{H}(N)$ onto the real line assigning to an Hermitian matrix its $(1,1)$-entry. Under this projection, the density of the pushforward of a generic orbital measure is a spline function with $N$ knots. This fact was pointed out by Andrei Okounkov in 1996, and the goal of the paper is to propose a multidimensional generalization. Namely, it turns out that if instead of the $(1,1)$-entry we cut out the upper left matrix corner of arbitrary size $K \times K$, where $K = 2, \ldots, N - 1$, then the pushforward of a generic orbital measure is still computable: its density is given by a $K \times K$ determinant composed from one-dimensional splines. The result can also be reformulated in terms of projections of the Gelfand–Tsetlin polytopes.

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1. Introduction

Orbital measures. Let $\mathcal{H}(N)$ be the space of $N \times N$ Hermitian matrices. For $K = 1, \ldots, N - 1$, we denote by $p_K^N : \mathcal{H}(N) \to \mathcal{H}(K)$ the linear projection consisting in deleting from the matrix $H \in \mathcal{H}(N)$ its last $N - K$ rows and columns. We call $p_K^N(H)$, the image of $H$ under this projection, the $K \times K$ corner of $H$.

The unitary group $U(N)$ acts on $\mathcal{H}(N)$ by conjugations, and because $U(N)$ is compact, each orbit of this action carries a unique invariant probability measure, which we call the orbital measure. Given an orbital measure $\mu$ on $\mathcal{H}(N)$, denote by $p_K^N(\mu)$ its pushforward under projection $p_K^N$. Our goal is to describe $p_K^N(\mu)$.

The orbits in $\mathcal{H}(N)$ (and hence the orbital measures) can be indexed by $N$-tuples of weakly increasing real numbers $X = (x_1 \leq \cdots \leq x_N)$, the matrix

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eigenvalues. Let $\mathcal{X}(N) \subset \mathbb{R}^N$ denote the set of all such $X$’s. Given $X \in \mathcal{X}(N)$, we write $O_X$ and $\mu_X$ for the corresponding orbit and orbital measure, respectively.

Since $p^N_K(\mu_X)$ is a $U(K)$-invariant probability measure on $\mathcal{X}(K)$, it can be uniquely decomposed into a continual convex combination of orbital measures, governed by a probability measure $\nu_{X,K}$ on the parameter space $\mathcal{X}(K)$. That is, $\nu_{X,K}$ is characterized by the property that, for an arbitrary Borel subset $S \subseteq \mathcal{X}(N)$,

$$(p^N_K(\mu_X))(S) = \int_{Y \in \mathcal{X}(K)} \mu_Y(S)\nu_{X,K}(dY).$$

The measure $\nu_{X,K}$ can be called the radial part of measure $p^N_K(\mu_X)$.

**Main result.** Denote by $\mathcal{X}^0(N)$ the interior of $\mathcal{X}(N)$; that is, $\mathcal{X}^0(N)$ consists of $N$-tuples of strictly increasing real numbers. If $X \in \mathcal{X}^0(N)$, then $\nu_{X,K}$ is absolutely continuous with respect to Lebesgue measure on $\mathcal{X}(K) \subset \mathbb{R}^K$, and the main result, Theorem 3.3, gives an explicit formula for the density of $\nu_{X,K}$.

In the case $K = 1$ the target space of the projection is the real line, and the density in question coincides with a $B$-spline, a certain piecewise polynomial function on $\mathbb{R}$ (this fact was observed by Andrei Okounkov). In the general case, it turns out that the density of $\nu_{X,K}$ is expressed through a $K \times K$ determinant composed from some B-splines.

As the reader will see, the proof of Theorem 3.3 is straightforward and elementary. The main reason why I believe the result may be of interest is the very appearance of splines, which are objects of classical and numerical analysis, in a problem of representation-theoretic origin.

**Gelfand–Tsetlin polytopes.** Before explaining a connection with representation theory I want to give a different interpretation of the measure $\nu_{X,K}$.

For $X \in \mathcal{X}(N)$ and $Y \in \mathcal{X}(N - 1)$, write $Y \prec X$ or $X \succ Y$ if the coordinates of $X$ and $Y$ interlace, that is

$$x_1 \leq y_1 \leq x_2 \leq \cdots \leq x_{N-1} \leq y_{N-1} \leq x_N.$$ 

Given $X \in \mathcal{X}(N)$, the corresponding Gelfand–Tsetlin polytope $P_X$ is the compact convex subset in the vector space

$$\mathbb{R}^{N-1} \times \mathbb{R}^{N-2} \times \cdots \times \mathbb{R} = \mathbb{R}^{N(N-1)/2},$$

formed by triangular arrays subject to the interlacement constraints:

$$P_X := \{(Y^{(N-1)}, \ldots, Y^{(1)}) \in \mathbb{R}^{N(N-1)/2} : X \succ Y^{(N-1)} \succ \cdots \succ Y^{(1)}\}.$$

Consider the map assigning to a matrix $H \in O_X$ the array formed by the collections of eigenvalues of its corners $p^N_{N-1}(H), p^N_{N-2}(H), \ldots, p^N_1(H)$. It is well known (see Corollary 3.2 below) that this map projects the orbit $O_X$ onto the polytope $P_X$ and takes $\mu_X$ to the uniform measure on $P_X$ (that is, the normalized Lebesgue measure). Next, given $K = 1, \ldots, N - 1$, consider the natural projection $P_X \to \mathcal{X}(K)$ extracting from the array $(Y^{(N-1)}, \ldots, Y^{(1)})$ its $K$th component $Y^{(K)}$. The measure $\nu_{X,K}$ is nothing else than the pushforward of the uniform measure under the latter projection.
Discrete version of the problem: relative dimension in Gelfand–Tsetlin graph. Let \( \mathcal{GT}_N := \mathcal{A}(N) \cap \mathbb{Z}^N \) be the set of weakly increasing \( N \)-tuples of integers. The elements of \( \mathcal{GT}_N \) are in bijection with the irreducible representations of the group \( U(N) \): with \( X = (x_1, \ldots, x_N) \in \mathcal{GT}_N \) we associate the irreducible representation \( T_X \) with signature (=highest weight) \( \hat{X} := (x_N, \ldots, x_1) \). Here we pass from \( X \) to \( \hat{X} \), because the coordinates of signatures are usually written in the descending order, see Weyl [13].

Let \( X \in \mathcal{GT}_N \) and consider the finite set \( P^\mathcal{G}_X := P_X \cap \mathbb{Z}^{N-1/2} \) consisting of integral points in the polytope \( P_X \). Let us replace the uniform measure on \( P_X \) by the uniform measure on \( P^\mathcal{G}_X \) (that is, the normalized counting measure). Next, given \( K = 1, \ldots, N-1 \), we consider again the same projection \( P_X \to \mathcal{A}(K) \) as before and denote by \( \nu^\mathcal{G}_{X,K} \) the pushforward of the uniform measure on \( P^\mathcal{G}_X \). Evidently, \( \nu^\mathcal{G}_{X,K} \) is a probability measure with finite support.

Elements of \( P^\mathcal{G}_X \) are the Gelfand–Tsetlin schemes (also called Gelfand–Tsetlin patterns) with top row \( X \); they parameterize the elements of Gelfand–Tsetlin basis in \( T_Y \) in the restriction of \( T_X \) to the subgroup \( U(K) \subset U(N) \).

The Gelfand–Tsetlin graph has the vertex set \( \mathcal{GT}_1 \sqcup \mathcal{GT}_2 \sqcup \ldots \) and the edges formed by couples \( Y \prec X \). In the terminology of Borodin–Olshanski [3], \( \nu^\mathcal{G}_{X,K}(Y) \) is the relative dimension of the vertex \( Y \in \mathcal{GT}_K \) with respect to the vertex \( X \in \mathcal{GT}_N \). In [3], we derived a determinantal formula for the relative dimension (see also Petrov [11] for a different proof). That formula can be viewed as a discrete version of the formula of Theorem 3.3.

I first guessed the formula of Theorem 3.3 by degenerating the “discrete” formula of [3]. However, this is not an optimal way of derivation, because the discrete case is much more difficult than the continuous one. I am grateful to Alexei Borodin for the suggestion to study the degeneration of the “discrete” formula. Note that from the comparison of the measures \( \nu_{N,K} \) and \( \nu^\mathcal{G}_{X,K} \) it is seen that the former should be related to the latter by a scaling limit transition.

2. Preliminaries

The fundamental spline with \( n \geq 2 \) knots \( y_1 < \cdots < y_n \) can be characterized as the only function \( a \mapsto M(a; y_1, \ldots, y_n) \) on \( \mathbb{R} \) of class \( C^{n-3} \), vanishing outside the interval \((y_1, y_n)\), equal to a polynomial of degree \( \leq n-2 \) on each interval \((y_i, y_{i+1})\), and normalized by the condition

\[
\int_{-\infty}^{+\infty} M(a; y_1, \ldots, y_n) da = 1.
\]

Here is an explicit expression:

\[
M(a; y_1, \ldots, y_n) := (n-1) \sum_{i: y_i > a} \frac{(y_i - a)^{n-2}}{\prod_{r \neq i} (y_i - y_r)}.
\]
In particular, for $n = 2$

$$M(a; y_1, y_2) = \frac{1_{y_1 \leq a \leq y_2}}{y_2 - y_1}.$$  

**Remark 2.1.** The above definition is taken from Curry–Schoenberg [4]. In the subsequent publications, Schoenberg changed the term to $B$-spline. The latter term became commonly used. However, in the modern literature, it more often refers to the function

$$B(a; y_1, \ldots, y_n) := (y_n - y_1) \sum_{i: y_i > a} \frac{(y_i - a)^{n-2}}{\prod_{r: r \neq i} (y_i - y_r)},$$

(2)

which differs from $M(a; y_1, \ldots, y_n)$ by the numerical factor $(y_n - y_1)/(n - 1)$; see, e.g., de Boor [2] or Phillips [12]. The normalization in (2) has its own advantages, but we will not use it. Note also that $M(a; y_1, \ldots, y_n)$ is a special case of Peano kernel, see Davis [5], Faraut [7].

We need two well-known formulas relating $M(a; y_1, \ldots, y_n)$ to divided differences (see, e.g., [4], [7]).

Recall that the *divided difference* of a function $f(x)$ on points $y_1, \ldots, y_n$ is defined recursively by

$$f[y_1, y_2] = \frac{f(y_2) - f(y_1)}{y_2 - y_1}, \quad f[y_1, y_2, y_3] = \frac{f[y_2, y_3] - f[y_1, y_2]}{y_3 - y_1},$$

and so on; the final step is

$$f[y_1, \ldots, y_n] = \frac{f[y_2, \ldots, y_n] - f[y_1, \ldots, y_{n-1}]}{y_n - y_1}. \quad (3)$$

Next, set

$$x_+^s = \begin{cases} x^s, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

In this notation, the first formula in question is

$$M(a; y_1, \ldots, y_n) = (n - 1)f[y_1, \ldots, y_n], \quad \text{where } f(x) := (x - a)^{n-2}, \quad (4)$$

and the second formula is

$$f[y_1, \ldots, y_n] = \frac{1}{(n - 1)!} \int M(a; y_1, \ldots, y_n)f^{(n-1)}(a)da. \quad (5)$$

In (5), $f$ is assumed being a function on $\mathbb{R}$ with piecewise continuous derivative of order $n - 1$. In particular, (5) is applicable to $f(x) = (x - t)^{n-1}$, which is used in the lemma below.

To shorten the notation, let us abbreviate $Y := (y_1 < \cdots < y_n)$. 
Lemma 2.2. Fix \( n = 2, 3, \ldots \) and an \( n \)-tuple \( Y = (y_1 < \cdots < y_n) \in \mathcal{X}^0(N) \). For an arbitrary \( b \in \mathbb{R} \) set

\[
f_b(x) := (x - b)^{n-1}, \quad x \in \mathbb{R}.
\]

One has

\[
\begin{align*}
\int_{-\infty}^c M(a; Y) da &= 1 - f_c[Y], \quad c \in \mathbb{R}, \quad (6) \\
\int_{b}^c M(a; Y) da &= f_b[Y] - f_c[Y], \quad b < c, \quad (7) \\
\int_{b}^{+\infty} M(a; Y) da &= f_b[Y], \quad b \in \mathbb{R}. \quad (8)
\end{align*}
\]

Proof. To check (7), we apply (5) to \( f(x) = f_b(x) - f_c(x) \), which is justified, see the comment just after (5). Then in the left-hand side of (5) we get \( f_b[Y] - f_c[Y] \). Next, observe that the \((n-1)\)th derivative of \( f_t(x) \) equals \((n-1)!!\) for \( a \geq b \), so that

\[
f^{(n-1)}(a) = (n-1)!!(1_{a \geq b} - 1_{a \geq c}) = (n-1)!!1_{b \leq a < c}.
\]

Therefore, in the right-hand side we get \( \int_{b}^{c} M(a; Y) da \), which proves (7).

Now (8) follows from (7) by setting \( c = +\infty \), and (6) follows from (8), because the total integral of the \( M(a; Y) \) equals 1.

3. Projections of orbital measures

We keep to the notation of Sections 1 and 2.

Given \( X \in \mathcal{X}(N) \), the pushforward of the orbital measure \( \mu_X \) under the map

\[
O_X \ni H \mapsto \text{the spectrum of } p^N_{N-1}(H)
\]
can be viewed as a probability measure on \( \mathcal{X}(N-1) \) depending on \( X \) as a parameter; let us denote it by \( \Lambda^N_{N-1}(X, \cdot) \) or \( \Lambda^N_{N-1}(X, dY) \). We regard \( \Lambda(X, dY) \) as a Markov kernel.

By classical Rayleigh’s theorem, the eigenvalues of a matrix \( H \in \mathcal{X}(N) \) and its corner \( p^N_{N-1}(H) \) interlace. Therefore, the measure \( \Lambda^N_{N-1}(X, \cdot) \) is concentrated on the subset

\[
\{ Y \in \mathcal{X}(N-1) : Y \prec X \} \subset R^{N-1}. \quad (9)
\]

Proposition 3.1. Assume \( X = (x_1, \ldots, x_N) \in \mathcal{X}^0(N) \). Then the measure \( \Lambda^N_{N-1}(X, \cdot) \) is absolutely continuous with respect to Lebesgue measure on the set \( \{ Y \in \mathcal{X}(N-1) : Y \prec X \} \), and the density of \( \Lambda^N_{N-1}(X, \cdot) \), denoted by \( \Lambda^N_{N-1}(X, Y) \), is given by

\[
\Lambda^N_{N-1}(X, Y) = (N-1)! \frac{V(Y)}{V(X)} 1_{Y \prec X}, \quad (10)
\]

where we use the notation

\[
V(X) = \prod_{j>i} (x_j - x_i)
\]
and the symbol $1_{Y \prec X}$ equals 1 or 0 depending on whether $Y \prec X$ or not.

**Proof.** To the best of my knowledge, a published proof first appeared in Baryshnikov [1, Proposition 4.2]. However, the argument given in [1] was known earlier: it is hidden in the first computation of the spherical functions of the groups $SL(N, \mathbb{C})$ due to Gelfand and Naimark, see [8, §9]. Note also that a more general result can be found in Neretin [9].

Here is a different proof. Consider the Laplace transform of the orbital measure $\mu_X$:

$$\hat{\mu}_X(Z) := \int e^{\text{Tr}(ZH)} \mu_X(dH),$$

where $Z$ is a complex $N \times N$ matrix. The integral in the right-hand side is often called the *Harish-Chandra–Itzykson–Zuber integral*. Its value is given by a well-known formula (see, e.g., Olshanski–Vershik [10, Corollary 5.2]):

$$\hat{\mu}_X(Z) = c_N \prod_{j>i} \frac{\det[e^{z_i z_j}]_{i,j=1}^{N}}{(z_j - z_i)(x_j - x_i)},$$

where $z_1, \ldots, z_N$ are the eigenvalues of $Z$ and

$$c_N = (N - 1)! (N - 2)! \ldots 0!$$

(note that the right-hand side of (12) does not depend on the enumeration of the eigenvalues of $Z$).

The claim of the proposition is equivalent to the following equality: Assume that the entries in the last row and column of $Z$ equal 0, so that $Z$ has the form

$$Z = \begin{bmatrix} \tilde{Z} & 0 \\ 0 & 0 \end{bmatrix},$$

where $\tilde{Z}$ is a complex matrix of size $(N - 1) \times (N - 1)$; then

$$\hat{\mu}_X(Z) = \frac{(N - 1)!}{V(X)} \int_{Y \prec X} V(Y) \hat{\mu}_Y(\tilde{Z}) dY.$$  

(14)

To prove (14), consider the matrix $T := [e^{z_i z_j}]$ in the right-hand side of (12). Since $Z$ has the form (13), at least one of the eigenvalues $z_1, \ldots, z_N$ equals 0. It is convenient to slightly change the enumeration and denote the eigenvalues as $z_0 = 0, z_1, \ldots, z_{N-1}$. In accordance to this we will assume that the row number $i$ of $T$ ranges over $\{0, \ldots, N - 1\}$ while the column index $j$ ranges over $\{1, \ldots, N\}$. Since $z_0 = 0$, the 0th row of $T$ is $(1, \ldots, 1)$. Let us subtract the $(N-1)$th column from the $N$th one, then subtract the $(N-2)$th column from the $(N-1)$th one, etc. This gives $\det T = \det \tilde{T}$, where $\tilde{T}$ stands for the matrix of order $N-1$ with the entries

$$\tilde{T}_{i,j} = e^{z_i z_j+1} - e^{z_i z_j} = z_i \int_{z_i}^{z_j} e^{z_i y} dy_j, \quad i, j = 1, \ldots, N - 1.$$
It follows
\[ \det \tilde{T} = z_1 \ldots z_N \int_{Y < X} dY \det [e^{z_i y_j}]_{i,j=1}^{N-1}, \]
so that
\[ \hat{\mu}_X(Z) = c_N \frac{z_1 \ldots z_N \int_{Y < X} dY \det [e^{z_i y_j}]_{i,j=1}^{N-1}}{\prod_{N-1 \geq j > i \geq 0} (z_j - z_i) \cdot V(X)}. \]

Next, because \( z_0 = 0 \), the product over \( j > i \) in the denominator equals
\[ z_1 \ldots z_N \prod_{N-1 \geq j > i \geq 1} (z_j - z_i), \]
so that the product \( z_1 \ldots z_N \) is cancelled out. Taking into account the fact that \( \hat{\mu}_Y(\tilde{Z}) \) is given by the determinantal formula similar to (12) and using the obvious relation \( c_N = (N - 1)! c_{N-1} \) we finally get the desired equality (14). 

From Proposition 3.1 it is easy to deduce the following corollary (see also [1, Proposition 4.7] and Defosseux [6]).

**Corollary 3.2.** Fix \( X \in \mathcal{X}^0(N) \) and let \( H \) range over \( O_X \). The map assigning to \( H \) the collection of the eigenvalues of the corners \( p^N_K(H) \), where \( K = N - 1, N - 2, \ldots, 1 \), projects \( O_X \) onto the Gelfand–Tsetlin polytope \( P_X \) and takes the measure \( \mu_X \) to the Lebesgue measure multiplied by the constant
\[ \frac{(N-1)!(N-2)! \ldots 0!}{V(X)}. \]
In particular, the volume of \( P_X \) in the natural coordinates is equal to the inverse of the above quantity.

Recall that \( \nu_{X,K} \) stands for the radial part of the \( K \times K \) corner of the random matrix \( H \in O_X \), driven by the orbital measure \( \mu_X \) (see Section 1), and \( M(a; y_1, \ldots, y_n) \) denotes the fundamental spline with \( n \) knots \( y_1, \ldots, y_n \) (see [1] and (1)).

**Theorem 3.3.** Fix \( X = (x_1, \ldots, x_N) \in \mathcal{X}^0(N) \). For any \( K = 1, \ldots, N-1 \), the measure \( \nu_{X,K} \) on \( \mathcal{X}(K) \) is absolutely continuous with respect to Lebesgue measure and has the density
\[ M(a_1, \ldots, a_K; x_1, \ldots, x_N) := c_{N,K} \frac{V(A) \det [M(a_j; x_i, \ldots, x_{N-K+i})]_{i,j=1}^{K}}{\prod_{(j,i): j-i \geq N-K+1} (x_j - x_i)}, \]  
where
\[ c_{N,K} = \prod_{i=1}^{K-1} \binom{N - K + i}{i}. \]
Note that for \( K = 1 \) the right-hand side reduces to the fundamental spline with knots \( x_1, \ldots, x_N \). Thus, in the case \( K = 1 \) the theorem says that the density of the measure \( \nu_{N,1} \) on \( \mathbb{R} \) coincides with the spline \( M(a; x_1, \ldots, x_N) \). This simple but important claim is due to Andrei Okounkov, see [10, Proposition 8.2].

**Proof.** We argue by induction on \( K \), starting with \( K = N - 1 \) and ending at \( K = 1 \).

**Step 1.** Examine the case \( K = N - 1 \), which is the base of induction. We have \( \nu_{X,N-1}(dA) = \Lambda_{N-1}^N(X,dA) \). By proposition 3.1, the measure \( \Lambda_{N-1}^N(X,\cdot) \) on \( \mathcal{K}(N-1) \) is absolutely continuous with respect to Lebesgue measure and has density \( \Lambda_{N-1}(X,A) \) given by [10]. Thus, we have to check that \( \Lambda_{N-1}(X,A) \) coincides with the quantity \( M(a_1, \ldots, a_{N-1}; x_1, \ldots, x_N) \) given by the right-hand side of (15), where we have to take \( K = N - 1 \). That is, the desired equality has the form

\[
(N-1)! \frac{V(A)}{V(X)} 1_{A\prec X} = c_{N,N-1} \frac{V(A)}{M(a_1; x_1, x_1+1))} \frac{\det M(a_j; x_i, x_{i+1}))_{i,j=1}^{N-1}}{\prod_{(j,i): j-i \geq 2} (x_j - x_i)}.
\]

Since \( c_{N,N-1} = (N-1)! \), the desired equality reduces to

\[
\det M(a_j; x_i, x_{i+1))_{i,j=1}^{N-1} = \frac{1_{A\prec X}}{(x_2 - x_1)(x_3 - x_2) \cdots (x_N - x_{N-1})}.
\]

Observe that the \((i,j)\)-entry in the determinant is the quantity

\[
M(a_j; x_i, x_{i+1}) = \frac{1_{x_i \leq a_j \leq x_{i+1}}}{x_{i+1} - x_i},
\]

which vanishes unless \( a_j \in [x_i, x_{i+1}] \). Since \( a_1 \leq \cdots \leq a_{N-1} \), the determinant vanishes unless \( A \prec X \). Furthermore, if \( A \prec X \), then the matrix under the sign of determinant is diagonal, so the determinant equals the product of the diagonal entries, which equals

\[
\frac{1}{(x_2 - x_1)(x_3 - x_2) \cdots (x_N - x_{N-1})},
\]

as required.

**Step 2.** Given \( K = 1, \ldots, N - 1 \), we consider the superposition of Markov kernels

\[
\Lambda_K^N := \Lambda_{N-1}^N \Lambda_{N-2}^{N-1} \ldots \Lambda_K^{K+1}.
\]

In more detail, the result is a Markov kernel on \( \mathcal{K}(N) \times \mathcal{K}(K) \) given by

\[
\Lambda_K^N(X, dA) = \int \Lambda_K^N(X, dY^{(N-1)}) \Lambda_{N-2}^{N-1}(Y^{(N-1)}, dY^{(N-2)}) \ldots \Lambda_K^{K+1}(Y^{(K+1)}, dA),
\]

where the integral is taken over variables \( Y^{(N-1)}, \ldots, Y^{(K+1)} \). Obviously, \( \Lambda_K^N(X, dA) = \nu_{X,K}(dA) \), which entails the recurrence relation

\[
\nu_{X,K-1} = \nu_{X,K} \Lambda_K^{K-1}, \quad K = N - 1, N - 2, \ldots, 2,
\]

(16)
where, by definition, $\nu_{X,K}^{K-1}$ is the measure on $\mathcal{X}(K-1)$ given by
\begin{equation}
(\nu_{X,K}^{K-1})(dB) = \int_{A \in \mathcal{X}(K)} \nu_{X,K}(dA)\Lambda_{K-1}^{K}(A,dB).
\end{equation}

**Step 3.** Assume now that the claim of the theorem holds for some $K \geq 2$ and deduce from this that it also holds for $K-1$. To do this we employ (16) and (17).

First of all, (16) and (17) imply that $\nu_{X,K}^{K-1}$ is absolutely continuous with respect to Lebesgue measure on $\mathcal{X}(K-1)$ and has the density
\begin{equation}
(\nu_{X,K}^{K-1})(B) = \int_{A \in \mathcal{X}(K)} \nu_{X,K}(dA)\Lambda_{K-1}^{K}(A,B), \quad B \in \mathcal{X}(K-1).
\end{equation}

Let us compute the integral explicitly. By the induction assumption, $\nu_{X,K}$ is absolutely continuous and has density (15). Therefore, integral (18) can be written in the form
\begin{equation}
\int_{A \in \mathcal{X}(K)} M(a_1, \ldots, a_K; x_1, \ldots, x_N)\Lambda_{K-1}^{K}(A,B)da_1 \ldots da_K.
\end{equation}

Write $B = (b_1, \ldots, b_K)$. Substituting the explicit expression for $\Lambda_{K-1}^{K}(A,B)$ given by Proposition 3.1 we rewrite this as
\begin{equation}
(K-1)!V(B) \int_{A} \frac{M(a_1, \ldots, a_K; x_1, \ldots, x_N)}{V(A)}da_1 \ldots da_K,
\end{equation}
where the integration domain is
\begin{equation}
-\infty < a_1 \leq b_1, \ldots, b_i \leq a_{i+1} \leq b_{i+1}, \ldots, b_{K-1} \leq a_K < +\infty.
\end{equation}

Next, plug in into (19) the explicit expression for $M(a_1, \ldots, a_K; x_1, \ldots, x_N)$ given by (15). Then the factor $V(A)$ is cancelled out and we get
\begin{equation}
\frac{c_{N,K}(K-1)!V(B)}{\prod_{(j,i): j-i \geq N-K+1}(x_j-x_i)} \int_{A} \det [M(a_j; x_i, \ldots, x_{N-K+i})]_{i,j=1}^{K}da_1 \ldots da_K
\end{equation}
with the same integration domain (20).

Put aside the pre-integral factor in (21) and examine the integral itself. It can be written as a $K \times K$ determinant,
\begin{equation}
\det[F(i,j)]_{i,j=1}^{K},
\end{equation}
where
\begin{equation}
F(i,j) := \int_{b_{j-1}}^{b_j} M(a; Y_i)da
\end{equation}
and
\begin{equation}
Y_i := (x_i, \ldots, x_{N-K+i})
\end{equation}
with the understanding that $b_0 = -\infty$ and $b_K = +\infty$. 

We are going to prove that
\[ \det[F(i, j)]_{i,j=1}^K = (N - K + 1)^{K-1} \prod_{i=1}^{K-1} (x_{N-K+i+1} - x_i) \times \det[M(b_j; x_i, \ldots, x_{N-K+i+1})]_{i,j=1}^{K-1}. \] (22)

This will justify the induction step, because
\[ c_{N,K} = c_{N,K-1} \cdot \frac{(N - K + 1)^{K-1}}{(K - 1)!} \]

and
\[ \prod_{i=1}^{K-1} (x_{N-K+i+1} - x_i) \prod_{(j,i): j-i \geq N-K+1} (x_j - x_i). \]

**Step 4.** It remains to prove (22). We evaluate the quantities \( F(i, j) \) using Lemma 2.2, where we substitute \( n = N - K + 1 \) and \( Y = Y_i \). Then we get that the matrix entries \( F(i, j) \) are given by the following formulas:

- The entries of the first column have the form \( F(i, 1) = 1 - f_{b_i}[Y_i] \) by (6).
- The entries of the \( j \)th column, \( 2 \leq j \leq K - 1 \), have the form \( F(i, j) = f_{b_{j-1}}[Y_i] - f_{b_j}[Y_i] \) by (7).
- The entries of the last column have the form \( F(i, K) = f_{b_K}[Y_i] \) by (8).

We have \( \det F = \det G \), where the \( K \times K \) matrix \( G \) is defined by
\[ G(i, j) := F(i, j) + \cdots + F(i, K). \]

The entries of the matrix \( G \) are
\[ G(i, 1) = 1, \quad G(i, j) = f_{b_{j-1}}[Y_i], \quad 2 \leq j \leq K. \]

Next, we get \( \det G = \det H \) with the \((K - 1) \times (K - 1)\) matrix \( H \) defined by
\[ H(i, j) := F(i + 1, j + 1) - F(i, j), \quad 1 \leq i, j \leq K - 1. \]

Observe now that
\[ H(i, j) = f_{b_j}[Y_{i+1}] - f_{b_i}[Y_i], \]
which can be rewritten as
\[ H(i, j) = (x_{N-K+i+1} - x_i) f_{b_j}[x_{i+1}, \ldots, x_{N-K+i+1}] - f_{b_i}[x_i, \ldots, x_{N-K+i+1}] \]
\[ = (x_{N-K+i+1} - x_i) f_{b_j}[x_{i+1}, \ldots, x_{N-K+i+1}] \] by (3)
\[ = \frac{1}{N - K + 1} (x_{N-K+i+1} - x_i) M(b_j; x_i, \ldots, x_{N-K+i+1}) \] by (4).
\[ \] (23) (24) (25)

This shows that the determinant \( \det H = \det[H(i, j)]_{i,j=1}^{K-1} \) equals the right-hand side of (22). Since \( \det H = \det F \), this completes the proof.

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References

[1] Baryshnikov, Yu, *GUEs and queues*, Prob. Theory Rel. Fields **119** (2001), 256–274.

[2] Boor, C. de, “A practical guide to splines,” Springer, 1978.

[3] Borodin, A., and G. Olshanski, *The boundary of the Gelfand–Tsetlin graph: A new approach*, Adv. Math. **230** (2012), 1738–1779.

[4] Curry, H. B., and I. J. Schoenberg, *On Polya frequency functions IV: The fundamental spline functions and their limits*, J. Analyse Math. **17** (1966), 71–107.

[5] Davis, P. J., “Interpolation and approximation,” Dover, 1975.

[6] Defosseux, M., *Orbit measures, random matrix theory and interlaced determinantal processes*, Ann. Inst. Henri Poincaré (B) Prob. Stat. **46** (2010), 209–249.

[7] Faraut, J., *Noyau de Peano et intégrales orbitales*, Glob. J. Pure Appl. Math. **1** (2005), 306–320.

[8] Gelfand, I. M., and M. A. Naimark, *Unitary representations of classical groups*, Proc. Steklov Math. Institute **36** (1950) (Russian); German translation: I. M. Gelfand, M. A. Neumark, „Unitäre Darstellungen der klassischen Gruppen“, Mathematische Lehrbücher und Monographien. II. Abt. Band **6**. Berlin: Akademie–Verlag (1957).

[9] Neretin, Yu. A., *Rayleigh triangles and nonmatrix interpolation of matrix beta integrals*, Sbornik: Mathematics **194** (2003), 515–540.

[10] Olshanski, G., and A. Vershik, *Ergodic unitarily invariant measures on the space of infinite Hermitian matrices*, in: “Contemporary Mathematical Physics. F. A. Berezin’s memorial volume”, American Mathematical Society Translations, Series 2, Vol. **175** (Advances in the Mathematical Sciences—31), R. L. Dobrushin, R. A. Minlos, M. A. Shubin, A. M. Vershik, eds., Amer. Math. Soc., Providence, RI, 1996, 137–175.

[11] Petrov, L., *The boundary of the Gelfand-Tsetlin graph: New proof of Borodin-Olshanski’s formula, and its q-analogue*, Moscow Math. J., to appear. arXiv:1208.3443.

[12] Phillips, G. M., “Approximation and interpolation by polynomials,” CMS Books in Math. bf14, Springer, 2003.
[13] Weyl, H., “The classical groups. Their invariants and representations,” Princeton Univ. Press, 1939; 1997 (fifth edition).