Improved Differentially Private Decentralized Source Separation for fMRI Data

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Abstract

Blind source separation algorithms such as independent component analysis (ICA) are widely used in the analysis of neuroimaging data. In order to leverage larger sample sizes, different data holders/sites may wish to collaboratively learn feature representations. However, such datasets are often privacy-sensitive, precluding centralized analyses that pool the data at a single site. A recently proposed algorithm uses message-passing between sites and a central aggregator to perform a decentralized joint ICA (djICA) without sharing the data. However, this method does not satisfy formal privacy guarantees. We propose a differentially private algorithm for performing ICA in a decentralized data setting. Differential privacy provides a formal and mathematically rigorous privacy guarantee by introducing noise into the messages. Conventional approaches to decentralized differentially private algorithms may require too much noise due to the typically small sample sizes at each site. We leverage a recently proposed correlated noise protocol to remedy the excessive noise problem of the conventional schemes. We investigate the performance of the proposed algorithm on synthetic and real fMRI datasets to show that our algorithm outperforms existing approaches and can sometimes reach the same level of utility as the corresponding non-private algorithm. This indicates that it is possible to have meaningful utility while preserving privacy.
1 Introduction

Sharing data is a major challenge facing researchers in a number of domains. In particular, human health studies often involve a modest number of individuals: privacy concerns often preclude sharing “raw” data with collaborators. Performing a new joint analysis across the individual data points requires access to individuals’ data, so research groups often collaborate by performing meta-analyses which are limited to already-published aggregates or summaries of the data. Ideally, sites would be able to collaborate in new analyses while still protecting subjects’ privacy. An example of such a scenario is in neuroimaging research, in which many individual research groups or sites wish to collaborate [2–5]. Each site may lack a sufficient number of samples to robustly estimate features on their own, but the aggregate number of samples across all sites can yield novel discoveries such as biomarkers for disease. Sending the data samples to a central repository or aggregator can enable efficient feature learning, but privacy concerns and large communication overhead are often prohibitive when sharing “raw” data.

The privacy problem is exacerbated by previous demonstrations of how modern signal processing and machine learning algorithms can potentially reveal information about individuals present in the dataset [6–8]. A mathematically rigorous framework for protection against such information leaks is differential privacy [9]. Under differential privacy, the algorithm outputs are randomized in such a way that the presence or absence of any individual in the dataset does not significantly affect the computation output. In other words, differentially private (DP) algorithms offer a quantifiable plausible deniability to the data owners regarding their participation. The randomization often takes the form of noise introduced somewhere in the computation, resulting in a loss in performance, or utility. Privacy risk is quantified by a parameter or parameters, leading to a privacy-utility tradeoff in DP algorithm design.

In this paper, we consider blind source separation for neuroimaging, in which several individual research groups or sites wish to collaborate. The sample size at each site is moderate at best and is typically not sufficient for an efficient feature learning. However, the aggregate number of samples across all sites is sufficient, as in most statistical learning problems. One option is to send the data samples to a central repository to enable efficient feature learning, but privacy concerns and large communication overhead often preclude sharing “raw” data. This motivates us to develop efficient decentralized privacy-preserving algorithms that provide utility close to centralized case. Unfortunately, conventional approaches to using differential privacy in decentralized settings require introducing too much noise, leading to a poor tradeoff. The primary contribution of this paper is an algorithm for decentralized joint independent component analysis that guarantees differential privacy with an improved privacy-utility tradeoff.

Independent component analysis (ICA) is a matrix factorization algorithm that is one of the most popular blind source separation techniques for neuroimaging studies [10]. It assumes that the observed signals are mixtures of
statistically independent sources and aims to decompose the mixed signals into those sources. ICA has been widely used to estimate intrinsic connectivity networks from brain imaging data (e.g., functional magnetic resonance imaging (fMRI)). Successful application of ICA on fMRI can be attributed to both sparsity and spatial or temporal independence between the underlying sources [10]. We particularly focus on the recently proposed decentralized joint ICA (djICA) algorithm, which can perform temporal ICA of fMRI data [11]. The goal of temporal ICA is to identify temporally independent components that represent activation of different brain regions over time [12]. However, it requires more samples than are typically available from a single study. This is because the ratio of spatial to temporal dimensions often requires the aggregate temporal dimension to be similar to the voxel dimension [11].

A number of modified ICA algorithms exist for joining various data sets together and performing simultaneous decomposition of data from a number of subjects and modalities [13]. For instance, group spatial ICA (GICA) is noteworthy for performing multi-subject analysis of task- and resting-state fMRI data [15–17]. It assumes that the spatial map components are similar across subjects. On the other hand, the joint ICA (jICA) [18] algorithm for multimodal data fusion assumes that the mixing process is similar over a group of subjects. However, group temporal ICA also assumes common spatial maps but pursues statistical independence of timecourses. Consequently, like jICA, the common spatial maps from temporal ICA describe a common mixing process among subjects. While very interesting, temporal ICA of fMRI is typically not investigated because of the small number of time points in each data set, which leads to unreliable estimates [11]. The decentralized jICA approach overcomes that limitation by leveraging datasets from multiple sites.

Our Contributions. In this paper, we propose a new algorithm, capeDJICA, for $(\epsilon, \delta)$-DP decentralized joint ICA. The algorithm significantly improves upon our earlier work [1] by taking advantage of a recently proposed correlation assisted private estimation (CAPE) protocol [19]. Our method adds correlated noise to the output of each site to guarantee privacy locally and a central aggregator combines these noisy outputs to produce an improved estimate. For some parameter regimes, the aggregator can achieve the same level of noise variance as centralized scenario. We employ a randomized gradient descent following the CAPE protocol and show that mitigates the excessive noise problem of conventional decentralized DP gradient descent approaches. To handle the scenarios typically found in practice, we provide two improvements on the original CAPE protocol which may be of independent interest. The first is to extend the CAPE scheme to incorporate asymmetric privacy requirements or sample sizes at the sites. The second is an analysis of capeDJICA using the Rényi Differential Privacy (RDP) [20]. This is necessary because capeDJICA requires multiple rounds of communication. We present an analysis based on the moments accountant [21] for keeping track of the privacy loss at each iteration.

Experimentally, we compare our proposed algorithm with the existing state-of-the-art algorithm and a non-private algorithm. We show that the proposed algorithm outperforms the conventional privacy-preserving algorithm [1] and
can provide utility very close to that of the non-private algorithm \[11\] for some parameter choices. We analyze the variation of utility with different privacy levels, number of samples and some other key parameters. The results show that our algorithm achieves very good utility on both synthetic and real datasets, while providing strong privacy guarantees. By analyzing our proposed algorithm using the moments accountant, we show that we can achieve performance very close to that of non-private algorithm, even for strict privacy requirements.

2 Problem Formulation

**Notation.** We denote vectors, matrices and scalars with bold lower case letters (\(x\)), bold upper case letters (\(X\)) and unbolded letters (\(M\)), respectively. We denote indices with smaller case letters and they typically run from 1 to their upper-case versions (\(m \in \{1, 2, \ldots, M\} \triangleq [M]\)). The \(n\)-th column of the matrix \(X\) is denoted as \(x_n\). We denote the Euclidean (or \(L_2\)) norm of a vector and the spectral norm of a matrix with \(|\cdot|_2\) and the Frobenius norm with \(|\cdot|_F\).

Finally, the density of the standard Normal random variable is given by
\[
\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).
\]

**The ICA Model.** In this paper we consider the generative ICA model as in \[1, 11\]. In the centralized scenario, the independent sources \(S \in \mathbb{R}^{R \times N}\) are composed of \(N\) observations from \(R\) statistically independent components. We have a linear mixing process defined by a mixing matrix \(A \in \mathbb{R}^{D \times R}\), which forms the observed data \(X \in \mathbb{R}^{D \times N}\) as a product \(X = AS\). Many ICA algorithms propose recovering the unmixing matrix \(W = A^{-1}\), assuming \(A\) is invertible \[11\], by trying to maximize independence between rows of the product \(WX\). The maximal information transfer (infomax) \[22\] is a popular heuristic for estimating \(W\) that maximizes an entropy functional related to \(WX\). More specifically, the objective of Infomax ICA can be expressed as
\[
W^* = \arg\max_W h(g(WX)). \tag{1}
\]

Here, \(g(\cdot)\) is the sigmoid function and is given by: \(g(z) = \frac{1}{1+\exp(-z)}\). Additionally, \(h(z)\) is the (differential) entropy of a random vector \(z\) with joint density \(q\).
\[ h(z) = -\int q(z) \log q(z) dz. \] Note that the function \( g(\cdot) \) is applied element-wise for matrix-valued arguments. That is, \( g(Z) \) is a matrix with the same size as \( Z \) and \( [g(Z)]_{ij} = g([Z]_{ij}) \).

**The Decentralized Data Problem.** We consider a decentralized-data model with \( S \) sites (see Figure 1). There is a central node that acts as an aggregator. We assume that all parties are “honest but curious”. That is, all parties follow the protocol honestly but a subset can collude (maybe with an external adversary) to learn other site’s data/function output. Each site \( s \) has a collection of data matrices \( \{X_{s,m} \in \mathbb{R}^{D \times N_t} : m = 1, \ldots, M_s\} \) consisting of a total time course of length \( N_t \) time points over \( D \) voxels for \( M_s \) individuals. We assume the data samples in the local sites are disjoint and come from different individuals. Sites concatenate their local data matrices temporally to form a \( D \times N_s \) data matrix \( X_s \in \mathbb{R}^{D \times N_s} \), where \( N_s = N_t M_s \). Let \( N = \sum_{s=1}^{S} N_s \) be the total number of samples and \( M = \sum_{s=1}^{S} M_s \) be the total number of individuals (across all sites). We assume a global mixing matrix \( A \in \mathbb{R}^{D \times R} \) generates the time courses in \( X_s \) from underlying sources \( S_s \in \mathbb{R}^{R \times N_s} \) at each site. This yields the following model:

\[
X = [A S_1 \ldots A S_S] = [X_1 \ldots X_S] \in \mathbb{R}^{D \times N}.
\] (2)

We want to compute the global unmixing matrix \( W \in \mathbb{R}^{R \times D} \) corresponding to the Moore-Penrose pseudo-inverse of \( A \), denoted \( A^+ \), in the decentralized setting. Because sharing the raw data between sites is often impossible due to privacy concerns and communication cost, we develop methods that guarantee differential privacy \([9]\). More specifically, our goal is to use DP estimates of the local gradients to compute the global unmixing matrix \( W \) such that it closely approximates the true global unmixing matrix while preserving privacy.

**Definition 1** \((\epsilon, \delta)-Differential Privacy \([9]\)). An algorithm \( A(D) \) taking values in a set \( T \) provides \((\epsilon, \delta)-differential privacy if \( \Pr[A(D) \in S] \leq \exp(\epsilon) \Pr[A(D') \in S] + \delta \), for all measurable \( S \subseteq T \) and all data sets \( D \) and \( D' \) differing in a single entry (neighboring datasets).

One way to interpret this is that the probability of the output of an algorithm is not changed significantly if the input database is changed by just one entry. Here, \( \epsilon \) and \( \delta \) are privacy parameters, where lower \( \epsilon \) and \( \delta \) ensure more privacy. The parameter \( \delta \) can be interpreted as the probability that the algorithm fails to provide privacy risk \( \epsilon \). Several mechanisms can be employed to ensure that an algorithm satisfies differential privacy. Additive noise mechanisms such as the Gaussian or Laplace mechanisms \([9, 23]\) and random sampling using the exponential mechanism \([24]\) are among the most common ones. For additive noise mechanisms, the standard deviation of the noise is scaled to the sensitivity of the computation.

**Definition 2** \((L_2\text{-sensitivity} \([9]\)). The \( L_2\text{-sensitivity} of a vector-valued function \( f(D) \) is \( \Delta := \max_{D, D'} \| f(D) - f(D') \|_2 \), where \( D \) and \( D' \) are neighboring datasets.
Definition 3 (Gaussian Mechanism [23]). Let \( f : \mathbb{D} \mapsto \mathbb{R}^D \) be an arbitrary \( D \)-dimensional function with \( L_2 \)-sensitivity \( \Delta \). The Gaussian Mechanism with parameter \( \tau \) adds noise scaled to \( \mathcal{N}(0, \tau^2) \) to each of the \( D \) components of the output and satisfies \((\epsilon, \delta)\) differential privacy if \( \tau \geq \frac{\Delta}{\epsilon} \sqrt{2 \log \frac{1.25}{\delta}} \).

Note that, for any given \((\epsilon, \delta)\) pair, we can calculate a noise variance \( \tau^2 \) such that addition of a noise term drawn from \( \mathcal{N}(0, \tau^2) \) guarantees \((\epsilon, \delta)\)-differential privacy. There are infinitely many \((\epsilon, \delta)\) pairs that yield the same \( \tau^2 \). Therefore, we parameterize our methods using \( \tau_2 \) in this paper.

Definition 4 (Rényi Differential Privacy [20]). A randomized mechanism \( \mathcal{A} : \mathbb{D} \mapsto \mathbb{T} \) is \((\alpha, \epsilon_r)\)-Rényi differentially private if, for any adjacent \( D, D' \in \mathbb{D} \), the following holds: \( D_n (\mathcal{A}(D)||\mathcal{A}(D')) \leq \epsilon_r \). Here, \( D_n (P(x)||Q(x)) = \frac{1}{\alpha - 1} \log \mathbb{E}_x \sim Q \left( \frac{P(x)}{Q(x)} \right)^\alpha \) and \( P(x) \) and \( Q(x) \) are probability density functions defined on \( \mathbb{T} \).

In addition to conventional privacy analysis, we present an analysis of our algorithm using RDP. It has been shown that conventional privacy analysis of multi-shot algorithms tend to exaggerate the total privacy loss [20, 21]. RDP offers a much simpler composition rule that is shown to be tight [20].

Conventional Decentralized DP Computations. We use an example [19] to demonstrate the problem with the existing DP decentralized computations. Assume that in our decentralized-data setting (see Figure 3(a)), each site \( s \in [S] \) holds \( N_s \) scalar data samples \( x_{n,s} \in [0, 1] \) as the entries of \( x_s \in \mathbb{R}^{N_s} \). Our goal is to compute and release the DP global mean \( f(\cdot) \) of all the \( N \) samples. We assume \( N_s = \frac{N}{S} \) for simplicity. Without privacy constraints, the sites can send \( a_s = f(x_s) \) to the aggregator. The average computed by aggregator \((\frac{1}{S} \sum_{s=1}^{S} a_s)\) is exactly equal to the average when all the data samples are available in the aggregator node. However, in our honest-but-curious setting, the sites can employ the conventional decentralized DP computation technique to release (send to the aggregator node) a DP estimate of the function \( f(x_s) \) of their local data \( x_s \) as: \( \hat{a}_s = f(x_s) + e_s \), where \( e_s \sim \mathcal{N}(0, \tau^2) \) and \( \tau_s = \frac{1}{\sqrt{S}} \sqrt{2 \log \frac{1.25}{\delta}} = \frac{\sqrt{\frac{2}{\delta}}}{\sqrt{S}} \). The aggregator can then compute the \((\epsilon, \delta)\)-DP approximate average as \( a_{conv} = \frac{1}{S} \sum_{s=1}^{S} \hat{a}_s \). We observe

\[
a_{conv} = \frac{1}{S} \sum_{s=1}^{S} \hat{a}_s = \frac{1}{S} \sum_{s=1}^{S} a_s + \frac{1}{S} \sum_{s=1}^{S} e_s.
\]

The variance of the estimator \( a_{conv} \) is \( S \cdot \frac{\tau_s^2}{S^2} = \frac{\tau_s^2}{S} \). However, if we had all the data samples in the central aggregator (centralized/pooled-data scenario), we could compute the \((\epsilon, \delta)\) DP average as \( a_{pool} = \frac{1}{N} \sum_{n=1}^{N} x_n + e_{pool} \), where \( e_{pool} \sim \mathcal{N}(0, \tau_{pool}^2) \) and \( \tau_{pool} = \frac{1}{N} \sqrt{2 \log \frac{1.25}{\delta}} = \frac{\tau_s}{S} \). We observe the ratio:

\[
\frac{\tau_{pool}^2}{\tau_{conv}^2} = \frac{\tau_s^2}{\tau_{pool}^2} = \frac{1}{S}.
\]

That is, the decentralized DP averaging scheme will always
result in a poorer performance than the pooled data case. The recently proposed CAPE protocol \cite{19} improves the performance of such systems by assuming some reasonable resources.

### 3 Improved Differentially Private djICA

As mentioned before, we focus on the djICA algorithm \cite{11} for performing temporal ICA of fMRI data. Our goal is to compute the privacy-preserving global spatial maps utilizing all samples across all the sites. To that end, we need to perform a multi-round decentralized gradient descent for solving \cite{1}. One option is to employ the conventional DP gradient descent \cite{26,27}: computing the sensitivity \cite{9} of the gradient and then adding noise scaled to the sensitivity. However, this would lead to a significantly noisier estimate of the spatial maps, as outlined in Section 2 (see Section 6 for an empirical validation). For an efficient spatial map estimation that can achieve the same level of noise as the pooled-data scenario, we can employ the recently proposed CAPE scheme. CAPE can benefit a broad class of functions whose sensitivities satisfy some conditions \cite{19}. Examples include the empirical average loss functions used in machine learning. In the following, we first briefly review the CAPE scheme. We then propose an extension to the CAPE scheme so that it can incorporate unequal sample size/privacy requirements at sites. Next, we discuss the decentralized DP Principal Component Analysis (PCA) algorithm \cite{25}, which is a key preprocessing step for our proposed algorithm. Finally, we demonstrate that we can adapt the CAPE scheme to develop a DP djICA algorithm that offers significant improvements over our previous work \cite{1}.

#### 3.1 The Correlation Assisted Private Estimation Scheme

Recall our decentralized data setting and assume that all of the $S$ sites and the central node follow the protocol honestly. However, up to $S_C = \lceil \frac{S}{3} \rceil - 1$ sites can collude with an adversary to learn about some site’s data/function output. The central node is also honest-but-curious (and therefore, can collude with an adversary). An adversary can observe the outputs from each site, as well as the output from the aggregator. Additionally, the adversary can know everything about the colluding sites (including their private data). We denote the number of non-colluding sites with $S_H$ such that $S = S_C + S_H$ (see Figure 1b).
Algorithm 2 Correlation Assisted Private Estimation (CAPE) [19]

Require: Data samples \( \{x_s\} \), local noise variances \( \\{\tau^2_s\} \)

1: for \( s = 1, \ldots, S \) do \\
2: \( \text{Generate } e_s \) according to Algorithm 1 \\
3: \( \text{Generate } g_s \sim \mathcal{N}(0, \tau^2_g) \) with \( \tau^2_g = \frac{\tau^2_s}{S} \) \\
4: Compute and send \( \hat{a}_s \leftarrow f(x_s) + e_s + g_s \) \\
5: end for \\
6: Compute \( a_{\text{cape}} \leftarrow \frac{1}{S} \sum_{s=1}^{S} \hat{a}_s \) \( \triangleright \) at the aggregator \\
7: return \( a_{\text{cape}} \)

CAPE employs a correlated noise addition scheme [19] to: i) ensure \((\epsilon, \delta)\) differential privacy for the algorithm output from each site and ii) achieve the noise level of the pooled data scenario in the final output from the aggregator. Each site \( s \) releases/sends \( \hat{a}_s = f(x_s) + e_s + g_s \), where \( e_s \) and \( g_s \) are two noise terms. The noise \( g_s \sim \mathcal{N}(0, \tau^2_g) \) is generated locally whereas the noise \( e_s \sim \mathcal{N}(0, \tau^2_e) \) is generated jointly with all other sites such that \( \sum_{s=1}^{S} e_s = 0 \). CAPE employs the secure aggregation protocol (SecureAgg) by Bonawitz et al. [28] to generate the zero-sum noise terms \( \{e_s\} \) (see Algorithm 1). The variances of \( e_s \) and \( g_s \) are chosen to ensure that the noise \( e_s + g_s \) is sufficient to provide \((\epsilon, \delta)\)-differential privacy guarantee to \( f(x_s) \). The chosen variance of \( g_s \) also ensures that the output from the aggregator would have the same noise variance as the DP pooled-data scenario. The aggregator computes the following quantity (in Step 6 of Algorithm 2): 

\[
a_{\text{cape}} = \frac{1}{S} \sum_{s=1}^{S} \hat{a}_s = \frac{1}{N} \sum_{n=1}^{N} x_n + \frac{1}{S} \sum_{s=1}^{S} g_s,
\]

where we used \( \sum_{s} e_s = 0 \). The complete procedure is shown in Algorithm 2. The variance of the estimator \( a_{\text{cape}} \) is \( \tau^2_{\text{cape}} = \frac{\tau^2}{S^2} = \tau^2_{\text{pool}} \), which is the exactly the same as if all the data were present at the aggregator (see Lemma 1). The details of the CAPE protocol along with the necessary Theorem, Propositions and Lemma are shown in Appendix B.

### 3.2 Unequal Sample Sizes at Sites

It has been shown [19] that CAPE achieves the smallest noise variance at the aggregator (i.e., the ratio \( H(n) = \frac{\tau^2_{\text{cape}}}{\tau^2_{\text{pool}}} \), where \( n \triangleq [N_1, N_2, \ldots, N_S] \), is the smallest) in the symmetric setting (i.e., \( N_s = N/S \)). However, in practice, there would be scenarios where different sites have different privacy requirements and/or sample sizes. Additionally, sites may want the aggregator to use different weights for different sites (possibly according to the quality of the output from a site). A scheme for doing so is shown in [25]. In this work, we propose a generalization of the CAPE scheme that can be applied in asymmetric settings.
Let us assume that site $s$ requires local noise standard deviation $\tau_s$ for ensuring $(\epsilon_s, \delta_s)$-differential privacy for its output, where $\tau_s = \frac{1}{N_s} \epsilon_s \sqrt{2 \log \frac{1.25}{\delta_s}}$, according to the Gaussian mechanism [9]. As before, we intend to parameterize our algorithm using $\{\tau_s\}$ to abstract away $\{(\epsilon_s, \delta_s, N_s)\}$. To initiate the CAPE protocol, each site will generate $\hat{e}_s \sim N(0, \sigma_s^2)$ and $g_s \sim N(0, \tau_{gs}^2)$. The aggregator intends to compute a weighted average of each site’s data/output with weights selected according to some quality measure. For example, if the aggregator knows that a particular site is suffering from more noisy observations than other sites, it can choose to give the output from that site less weight while combining the site results. Let us denote the weights by $\{\mu_s\}$ such that $\sum_{s=1}^{S} \mu_s = 1$ and $\mu_s \geq 0$. First, the aggregator computes $\sum_{s=1}^{S} \mu_s \hat{e}_s$ using the SecureAgg protocol [28] and broadcasts it to all sites. Each site then sets $e_s = \hat{e}_s - \frac{1}{\mu_s} \sum_{i=1}^{S} \mu_i \hat{e}_i$, to achieve $\sum_{s=1}^{S} \mu_s e_s = 0$ and releases $\hat{a}_s = f(x_s) + e_s + g_s$. Now, the aggregator computes

$$a_{\text{cape}} = \sum_{s=1}^{S} \mu_s \hat{a}_s = \sum_{s=1}^{S} \mu_s f(x_s) + \sum_{s=1}^{S} \mu_s g_s,$$

where we used $\sum_{s=1}^{S} \mu_s e_s = 0$. In order to achieve the same utility as the pooled data scenario (i.e. $\tau_{\text{pool}}^2 = \tau_{\text{cape}}^2$), we need

$$\text{Var} \left[ \sum_{s=1}^{S} \mu_s g_s \right] = \tau_{\text{pool}}^2 \implies \sum_{s=1}^{S} \mu_s^2 \tau_{gs}^2 = \tau_{\text{pool}}^2.$$

Additionally, for guaranteeing privacy to the local sites, we need $\tau_{es}^2 + \tau_{gs}^2 \geq \tau_s^2$, where $\tau_{es}^2$ is the variance of $e_s$ and is a function of $\sigma_s^2$. With these constraints, we can formulate a feasibility problem to solve for the unknown noise variances $\{\sigma_s^2, \tau_{gs}^2\}$ as

$$\text{minimize} \quad 0$$

$$\text{subject to} \quad \tau_{es}^2 + \tau_{gs}^2 \geq \tau_s^2; \quad \sum_{s=1}^{S} \mu_s^2 \tau_{gs}^2 = \tau_{\text{pool}}^2,$$

for all $s \in [S]$, where $\{\mu_s\}$, $\tau_{\text{pool}}$ and $\{\tau_s\}$ are known to the aggregator. For this problem, multiple solutions are possible. We present one solution in Appendix A along with the detailed privacy analysis.

### 3.3 Decentralized Differentially Private PCA

Neuroimaging data samples are generally very high dimensional. For a computationally light spatial map estimation process, we require an efficient and privacy-preserving algorithm to reduce the sample dimension. To that end, we employ the recently proposed [25] DP decentralized PCA algorithm (capePCA)
Algorithm 3 Improved Decentralized Differentially Private PCA (capePCA) [25]

Require: Data matrix $X_s \in \mathbb{R}^{D \times N_s}$ with $\|X_{s,n}\|_2 \leq 1$, local noise variances $\{\tau^2_s\}$ for $s \in [S]$; reduced dimension $R$

1: for $s = 1, 2, \ldots, S$ do $\triangleright$ at the local sites
2: Generate $E_s \in \mathbb{R}^{D \times D}$ using Algorithm 1 (element-wise)
3: Compute $C_s \leftarrow \frac{1}{N_s} X_s X_s^\top$
4: Generate $D \times D$ symmetric $G_s$, where $\{[G_s]_{ij} : i \in [D], j \leq i\}$ are drawn i.i.d. $\sim \mathcal{N}(0, \tau^2_g = \frac{1}{S} \tau^2_s)$, $[G_s]_{ij} = [G_s]_{ji}$
5: Compute and send: $\hat{C}_s \leftarrow C_s + E_s + G_s$
6: end for
7: Compute $\hat{C} \leftarrow \frac{1}{S} \sum_{s=1}^S \hat{C}_s$ $\triangleright$ at the aggregator
8: Perform SVD: $\hat{C} = V \Lambda V^\top$
9: Release / send to sites: $V_R$
10: return $V_R$

as a preprocessing step of our proposed capeDJICA algorithm. Recall our honest-but-curious setup: a system of $S$ different sites with disjoint datasets and a central node or aggregator. The $D \times N_s$ data matrix in site $s \in [S]$ is denoted by $X_s$. For simplicity, we assume that the observed samples are mean-centered. We employ capePCA algorithm to approximate the low-rank subspace $V_R$ in the decentralized setting, while guaranteeing differential privacy. We have shown a slightly modified version of the original capePCA algorithm in Algorithm 3 to match the robust CAPE scheme [19] we reviewed in Section 3.1. Note that the DP decentralized PCA scheme proposed in [29] was limited by the larger variance of the additive noise at the local sites due to the smaller sample size. The capePCA alleviates this problem using the CAPE scheme and achieves the same utility as the pooled data scenario in the symmetric setting.

3.4 Proposed Improved Differentially Private djICA

In this section, we propose an algorithm that improves upon our previous decentralized DP djICA algorithm [1] and achieves the same utility as the DP pooled-data scenario in certain regimes. Recall our decentralized data setup: there are $S$ sites and a central aggregator (see Figure 1(b)). Each site $s$ has data from $M_s$ individuals, which are concatenated temporally to form the local data matrix $X_s \in \mathbb{R}^{D \times N_s}$. The global mixing matrix $A \in \mathbb{R}^{D \times R}$ is assumed to generate the time courses in $X_s$ from underlying sources $S_s \in \mathbb{R}^{R \times N_s}$ at each site. That is: $X = [AS_1 \ldots AS_S] \in \mathbb{R}^{D \times N}$. We estimate the DP global unmixing matrix $W \in \mathbb{R}^{R \times D} \approx A^+$ in the decentralized setting with a multi-round gradient descent that employs the CAPE protocol.

As mentioned before, we employ the capePCA algorithm [25] (Algorithm 3) to estimate an efficient and privacy-preserving row-rank subspace and thereby reduce the dimension of the samples before solving the optimization problem of
Algorithm 4 Improved Differentially Private Decentralized Joint ICA (capeDJICA)

**Require:** data \( \{ \mathbf{X}_s^t \in \mathbb{R}^{R \times N_s} : s \in [S]\} \), tolerance level \( t = 10^{-6} \), maximum iterations \( J \), \( \|\Delta \mathbf{W}(0)\|_2^2 = t \), initial learning rate \( \rho = 0.015 / \log(R) \), local noise standard deviations \( \{\tau_{G}^s, \tau_h^n\} \), gradient bounds \( \{B_G, B_h\} \)

1. Initialize \( j = 0 \), \( \mathbf{W} \in \mathbb{R}^{R \times R} \) \( \triangleright \) for example, \( \mathbf{W} = \mathbf{I} \)
2. while \( j < J \), \( \|\Delta \mathbf{W}(j)\|_2^2 \geq t \) do
   3. for all sites \( s = 1, 2, \ldots, S \) do
      4. Generate \( \mathbf{E}_s^G \in \mathbb{R}^{R \times R} \) and \( \mathbf{e}_s^h \in \mathbb{R}^{R} \) using Algorithm 1 (element-wise)
      5. Generate \( \mathbf{K}_s^G \in \mathbb{R}^{R \times R} \) and \( \mathbf{k}_s^h \in \mathbb{R}^{R} \), as described in the text
      6. Compute \( \mathbf{Z}_s(j) = \mathbf{W}(j-1) \mathbf{X}_s + \mathbf{b}(j-1) \mathbf{I} \)
      7. Compute \( \mathbf{Y}_s(j) = 1 - 2g(\mathbf{Z}_s(j)) \)
      8. Compute \( \mathbf{G}_s(j) = \frac{1}{N_s} \sum_{n=1}^{N_s} \mathbf{G}_{s,n}(j) \), where \( \mathbf{G}_{s,n}(j) = (1+\hat{y}_{s,n}) \mathbf{W}(j-1) \cdot \mathbf{Z}_s(j) \)
      9. Compute \( \mathbf{h}_s(j) = \frac{1}{N_s} \sum_{n=1}^{N_s} \mathbf{h}_{s,n}(j) \), where \( \mathbf{h}_{s,n}(j) = \frac{\max(1, \|\mathbf{y}_{s,n}(j)\|_2)}{\max(1, \|\hat{y}_{s,n}(j)\|_2)} \)
     10. Compute \( \mathbf{\hat{G}}_s(j) = \mathbf{G}_s(j) + \mathbf{E}_s^G + \mathbf{K}_s^G \)
     11. Compute \( \mathbf{\hat{h}}_s(j) = \mathbf{h}_s(j) + \mathbf{e}_s^h + \mathbf{k}_s^h \)
     12. Send \( \mathbf{\hat{G}}_s(j) \) and \( \mathbf{\hat{h}}_s(j) \) to the aggregator
     13. end for
   14. Compute \( \Delta \mathbf{W}(j) = \rho \frac{1}{N_s} \sum_{s=1}^{S} \mathbf{\hat{G}}_s(j) \) \( \triangleright \) at the aggregator, update global variables
   15. Compute \( \Delta \mathbf{b}(j) = \rho \frac{1}{N_s} \sum_{s=1}^{S} \mathbf{\hat{h}}_s(j) \)
   16. Compute \( \mathbf{W}(j) = \mathbf{W}(j-1) + \Delta \mathbf{W}(j) \)
   17. Compute \( \mathbf{b}(j) = \mathbf{b}(j-1) + \Delta \mathbf{b}(j) \)
   18. Check upper bound and perform learning rate adjustment (if needed)
   19. Send global \( \mathbf{W}(j) \) and \( \mathbf{b}(j) \) back to each site
   20. \( j \leftarrow j + 1 \)
21. end while
22. return The current \( \mathbf{W}(j) \)

[1] Let the output of capePCA to be \( \mathbf{V}_R \in \mathbb{R}^{D \times R} \), which is sent to the sites from the aggregator. Then the reduced dimensional \( (R \times N_s) \) data matrix at site \( s \) is denoted by: \( \mathbf{X}_s^t = \mathbf{V}_R^t \mathbf{X}_s \). These projected samples are the inputs to the proposed capeDJICA algorithm that estimates the unmixing matrix \( \mathbf{W} \). Note that, even though the preprocessing is performed satisfying differential privacy, the maximization to solve [1] itself may leak information about the local data since it relies on iterative message-passing with the aggregator. More specifically, the maximization is performed through a gradient descent [30], where the gradient is to be computed in a decentralized fashion. That is, the algorithm depends on the data samples through the gradients. Our proposed capeDJICA
The algorithm employs the CAPE protocol to perform the privacy-preserving decentralized gradient descent to solve for $W$.

The gradient of the empirical average loss function with respect to $W$ at site $s$ is given \cite{11} by

$$G_s = \frac{1}{N_s} \left( N_s I + (1 - 2Y_s) Z_s^\top \right) W, \quad (3)$$

where $Z_s = WX_s^\top + b1^\top$, $Y_s = g(Z_s)$; $b \in \mathbb{R}^R$ is the bias and $1$ is a vector of ones. If we denote $1 - 2Y_s$ with $\hat{Y}_s$ then we have

$$G_s = \frac{1}{N_s} \left( N_s I + \hat{Y}_s Z_s^\top \right) W = \frac{1}{N_s} \sum_{n=1}^{N_s} (I + \hat{y}_{s,n} z_{s,n}^\top) W.$$

We can consider $(I + \hat{y}_{s,n} z_{s,n}^\top) W$ to be the gradient of the loss function corresponding to a single data sample. Therefore, the gradient of the average loss function at site $s$ is essentially the average of the gradient of the loss function corresponding to each sample. Note that this gradient estimate is needed to be sent to the aggregator from the site. Therefore, we need to approximate this gradient satisfying differential privacy. To that end, let us consider that the gradient due to each sample satisfies

$$\| (I + \hat{y}_{s,n} z_{s,n}^\top) W \|_F \leq B_G, \quad (4)$$

where $B_G$ is some constant. It is easy to see that by changing one data sample (i.e., for a neighboring dataset), the gradient at site $s$ can change by at most $\frac{2B_G}{N_s}$. Therefore, the $L_2$ sensitivity of the function $f(X_s) = G_s$ is

$$\Delta_G^s = \frac{2B_G}{N_s}. \quad (5)$$

In addition to the unmixing matrix $W$, we update a bias term $b$ using a gradient descent \cite{11}. The gradient of the empirical average loss function with respect to the bias at site $s$ is given \cite{11} by

$$h_s = \frac{1}{N_s} \sum_{n=1}^{N_s} \hat{y}_{s,n}. \quad (6)$$

Similar to the case of $G_s$, we can find the $L_2$ sensitivity of the function $f(X_s) = h_s$ as

$$\Delta_h^s = \frac{2B_h}{N_s}, \quad (7)$$

where

$$\|\hat{y}_{s,n}\|_2 \leq B_h. \quad (8)$$
According to the Gaussian mechanism \cite{dp}, computing \((\epsilon, \delta)\) DP approximates of \(G_s\) and \(h_s\) requires additive noise standard deviations \(\tau^G_s\) and \(\tau^h_s\), respectively, satisfy
\[
\tau^G_s = \frac{\Delta^G_s}{\epsilon} \sqrt{2 \log \frac{1.25}{\delta}}, \quad \tau^h_s = \frac{\Delta^h_s}{\epsilon} \sqrt{2 \log \frac{1.25}{\delta}}.
\]  

As mentioned before, we employ the CAPE protocol to combine the gradients from the sites at the aggregator to achieve the same utility level as that of the pooled data scenario. More specifically, each site generates two noise terms: \(E^G_s \in \mathbb{R}^{R \times R}\) and \(e^h_s \in \mathbb{R}^R\), collectively among all sites (element-wise, according to Algorithm 1) at each iteration round. Additionally, each site \(s\) generates the following two noise terms locally at each iteration:

- \(K^G_s \in \mathbb{R}^{R \times R}\) with \([K^G_s]_{ij}\) drawn i.i.d. \(\sim \mathcal{N}(0, \tau^2_{Gk})\) and \(\tau^2_{Gk} = \frac{1}{S} \tau^2_{G}\)
- \(k^h_s \in \mathbb{R}\) with \([k^h_s]_i\) drawn i.i.d. \(\sim \mathcal{N}(0, \tau^2_{hk})\) and \(\tau^2_{hk} = \frac{1}{S} \tau^2_{h}\)

At each iteration round, the sites compute the noisy estimates of the gradients of \(W\) and \(b\):
\[
\hat{G}_s = G_s + E^G_s + K^G_s, \quad \hat{h}_s = h_s + e^h_s + k^h_s.
\]

These two terms are then sent to the aggregator and the aggregator computes: \(\Delta_W = \rho \frac{1}{S} \sum_{s=1}^{S} G_s\) and \(\Delta_b = \rho \frac{1}{S} \sum_{s=1}^{S} h_s\), where \(\rho\) is the learning rate. These gradient estimates are then used to update the variables \(W\) and \(b\). By Lemma 1 the variances of the noise of the two estimates: \(\Delta_W\) and \(\Delta_b\), are exactly the same as the pooled-data scenario in the symmetric setting. The complete algorithm is shown in Algorithm 4. Note that, one does not need to explicitly find the bounds in (4) and (8). Instead, the gradients due to each sample can be clipped to some pre-determined \(B_G\) or \(B_h\) in \(L_2\) norm sense. That is, we can replace \(G_{s,n} = (I + \hat{y}_{s,n} z^T_{s,n}) W\) with
\[
G_{s,n} = \frac{G_{s,n}}{\max \left(1, \frac{\|G_{s,n}\|_F}{B_G} \right)}.
\]

Similarly, we can replace \(h_{s,n} = \hat{y}_{s,n}\) with
\[
h_{s,n} = \frac{h_{s,n}}{\max \left(1, \frac{\|h_{s,n}\|_2}{B_h} \right)}.
\]

We note that this norm clipping has a few consequences \cite{norm_clip}. It destroys the unbiasedness of the gradient estimate. If we choose \(B_G\) and \(B_h\) to be too small, the average clipped gradient may be a poor estimate of the true gradient. Moreover, \(B_G\) and \(B_h\) dictates the additive noise level. In general, clipping prescribes taking a smaller step downhill towards the optimal point \cite{norm_clip} and may slow down the convergence.
4 Privacy Analysis of capeDJICA

In this section, we first present a theorem that provides the privacy guarantee of the capeDJICA algorithm using the conventional composition theorem [23]. However, we note that the capeDJICA algorithm involves a multi-round gradient descent and conventional privacy analysis may exaggerate the privacy loss. For a better characterization of the privacy guarantee, we analyze the capeDJICA algorithm with Rényi Differential Privacy (RDP) [20]. Finally, we compute the overall privacy loss of the capeDJICA algorithm using the moments accountant [21] to keep a better track of the privacy loss at each iteration.

**Theorem 1 (Privacy of capeDJICA Algorithm).** Consider Algorithm 4 in the decentralized data setting of Section 2 with $N_s = N_s^G = \tau_s^G$ and $\tau_s^h = \tau_h$ for all sites $s \in [S]$. Suppose that at most $S_C = \lceil S/3 \rceil - 1$ sites can collude after execution and the required number of iterations is $J^*$. Then Algorithm 4 computes an $(2J^*\epsilon, 2J^*\delta)$-DP approximation to the optimal unmixing matrix $W^*$, where $(\epsilon, \delta)$ satisfy the relation $\delta = 2^{\sigma_z} e^{\frac{\epsilon}{\mu_z} \phi \left(\frac{\epsilon - \mu_z}{\sigma_z}\right)}$ and $(\mu_z, \sigma_z)$ are given by (13) and (14), respectively.

**Proof.** The proof of Theorem 1 follows from using the Gaussian mechanism [9], the decentralized Stochastic Gradient Descent algorithm [21,26,27], the $L_2$ sensitivities of the functions $f(X_s) = G_s$ and $f(X_s) = h_s$ and the privacy of the CAPE scheme [19]. We recall that the data samples in each site are disjoint. By the CAPE scheme (see Theorem 2), each iteration round is $(2\epsilon, 2\delta)$-DP. If the required total number of iterations is $J^*$ then by composition theorem of differential privacy [23], the capeDJICA algorithm satisfies $(2J^*\epsilon, 2J^*\delta)$-differential privacy, where $(\epsilon, \delta)$ satisfy the relation $\delta = 2^{\sigma_z} e^{\frac{\epsilon}{\mu_z} \phi \left(\frac{\epsilon - \mu_z}{\sigma_z}\right)}$. \hfill $\Box$

### 4.1 Privacy Analysis using Rényi Differential Privacy

We now analyze the capeDJICA algorithm with Rényi Differential Privacy [20]. Analyzing the total privacy loss of a multi-shot algorithm, each stage of which is DP, is a challenging task. It has been shown [20,21] that the advanced composition theorem [23] for $(\epsilon, \delta)$-differential privacy can be loose. The main reason is that one can formulate infinitely many $(\epsilon, \delta)$-DP algorithms for a given noise variance $\tau^2$. RDP offers a much simpler composition rule that is shown to be tight [20]. We start our analysis of the capeDJICA algorithm by reviewing some properties of RDP [20].

**Proposition 1 (From RDP to differential privacy [20]).** If $A$ is an $(\alpha, \epsilon_r)$-RDP mechanism, then it also satisfies $\left(\epsilon_r, \frac{\log \frac{\alpha}{\alpha - 1}}{\alpha - 1}, \delta_r\right)$-differential privacy for any $0 < \delta_r < 1$.

**Proposition 2 (Composition of RDP [20]).** Let $A : \mathbb{D} \mapsto T_1$ be $(\alpha, \epsilon_{r1})$-RDP and $B : T_1 \times \mathbb{D} \mapsto T_2$ be $(\alpha, \epsilon_{r2})$-RDP, then the mechanism defined as $(X, Y)$, where $X \sim A(D)$ and $Y \sim B(X, D)$, satisfies $(\alpha, \epsilon_{r1} + \epsilon_{r2})$-RDP.
Proposition 3 (RDP and Gaussian Mechanism [20]). If $A$ has $L_2$ sensitivity 1, then the Gaussian mechanism $G_\sigma(A(D) = A(D) + E$, where $E \sim N(0, \sigma^2)$ satisfies $(\alpha, \frac{\alpha}{2\sigma^2})$-RDP. Additionally, a composition of $J$ Gaussian mechanisms, each with parameter $\sigma$ will have the RDP curve of a Gaussian mechanism with parameter $\frac{\sigma}{\sqrt{J}}$.

The proofs of the Propositions 1, 2 and 3 are provided in [20]. Now, we are in a position to analyze the proposed capeDJICA algorithm with RDP composition. Recall that, at each iteration $j$ of capeDJICA, we compute the noisy estimates of the gradients: $\Delta_W(j)$ and $\Delta_b(j)$. As we employed the CAPE scheme in the symmetric setting, the variances of noise at the aggregator for $\Delta_W(j)$ and $\Delta_b(j)$ are:

\[ \sigma^2_W = \rho^2 \tau^2_{pool} G^2 \] and \[ \sigma^2_b = \rho^2 \tau^2_{pool} h^2. \]

From Proposition 3, we have that the computation of $\Delta_W(j)$ is $(\alpha, \frac{\alpha}{2\sigma^2_W})$-RDP. Similarly, the computation of $\Delta_b(j)$ is $(\alpha, \frac{\alpha}{2\sigma^2_b})$-RDP. By Proposition 2, we have that each iteration step of capeDJICA is $\left(\alpha, \frac{\alpha}{2\sigma^2_{RDP}} \right)$-RDP. If we denote the number of required iterations for reaching convergence in capeDJICA by $J^*$ then, under $J^*$-fold composition of RDP, the overall capeDJICA algorithm is $(\alpha, \frac{\alpha J^*}{2\sigma^2_{RDP}})$-RDP, where

\[ \frac{1}{\sigma^2_{RDP}} = \left( \frac{1}{\sigma^2_W} + \frac{1}{\sigma^2_b} \right). \]

From Proposition 1 we can conclude that the capeDJICA algorithm satisfies $\left(\frac{\alpha J^*}{2\sigma^2_{RDP}} + \frac{\log \frac{1}{\alpha} - 1}{\alpha \delta_r}, \delta_r\right)$-differential privacy for any $0 < \delta_r < 1$. For a given $\delta_r$, we find the optimal $\alpha_{opt}$ as:

\[ \alpha_{opt} = 1 + \sqrt{\frac{2}{J^*} \sigma^2_{RDP} \log \frac{1}{\delta_r}}. \tag{10} \]

Therefore, the capeDJICA algorithm is $\left(\frac{\alpha_{opt} J^*}{2\sigma^2_{RDP}} + \frac{\log \frac{1}{\alpha_{opt}} - 1}{\alpha_{opt} - 1}, \delta_r\right)$-DP for any $0 < \delta_r < 1$.

4.2 Privacy Accounting using Moments Accountant

In this section, we use the moments accountant [21] framework to compute the overall privacy loss of our capeDJICA algorithm. Moments accountant can be used to achieve a much smaller overall $\epsilon$ than the strong composition theorem [23]. As mentioned before, naively employing the additive nature of the privacy loss results in the worst case analysis, i.e., assumes that each iteration step exposes the worst privacy risk and this exaggerates the total privacy loss. However, in practice, the privacy loss is a random variable that depends on the
dataset and is typically well-behaved (concentrated around its expected value). Let us consider the randomized mechanism $A : \mathcal{D} \rightarrow \mathbb{T}$. For a particular outcome $o \in \mathbb{T}$ of the mechanism and neighboring datasets $D, D' \in \mathcal{D}$, the privacy loss random variable is defined \cite{21} as

$$Z = \log \frac{\Pr[A(D) = o]}{\Pr[A(D') = o]} \text{ w.p. } \Pr[A(D) = o]. \quad (11)$$

Note that the basic idea of \cite{21} for accounting for the total privacy loss is to compute the moment generating function (MGF) of $Z$ for each iteration, use composition to get the MGF of the complete algorithm and then use that to compute final privacy parameters (see Theorem 2 of \cite{21}). The stepwise moment for any $t$ at iteration $j$ is defined \cite{21} as

$$\alpha_j(t) = \sup_{D, D'} \log \mathbb{E} [\exp(tZ)]. \quad (12)$$

If total number of iterations is $J^*$ then the overall moment is upper bounded as $\alpha(t) \leq \sum_{j=1}^{J^*} \alpha_j(t)$. Finally, for any given $\epsilon > 0$, the overall mechanism is $(\epsilon, \delta)$ DP for $\delta = \min_t \exp(\alpha(t) - te)$. We now employ the framework to our \texttt{capeDJICA} algorithm and find the best $\epsilon$ for a given $\delta$. For a Gaussian mechanism $G_\sigma A(D) = A(D) + E$, where $E \sim \mathcal{N}(0, \sigma^2)$, the privacy loss random variable defined in (11) can be written as

$$Z = \log \frac{\exp \left( -\frac{1}{2\sigma^2} (o - f_D)^2 \right)}{\exp \left( -\frac{1}{2\sigma^2} (o - f_{D'})^2 \right)} = \frac{1}{2\sigma^2} \left( 2o(f_D - f_{D'}) - (f_D^2 - f_{D'}^2) \right).$$

Now,

$$\mathbb{E}[\exp(tZ)] = \int_{o} \exp \left( \frac{t}{2\sigma^2} \left( 2o(f_D - f_{D'}) - (f_D^2 - f_{D'}^2) \right) \right) \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{1}{2\sigma^2} (o - f_D)^2 \right) do = \frac{\sigma}{\sqrt{f_D - f_{D'}^2}} \exp \left( (f_D - f_{D'})^2 + t(f_D - f_{D'}) \right),$$

where the last equality follows from the moment generating function of generalized Gaussian and some simple algebra. If the $L_2$ sensitivity of the function $A(D)$ is $\Delta$ then

$$\alpha_j(t) = \sup_{D, D'} \log \frac{\sigma}{\sqrt{f_D - f_{D'}^2}} + \frac{f_D - f_{D'}}{2\sigma^2} (t^2\sigma^2 + t(f_D - f_{D'}))$$

$$= \log \frac{\sigma}{\sqrt{\Delta}} + \frac{\Delta}{2\sigma^2} \left( t^2\sigma^2 + t\Delta \right).$$
We can compute the upper bound of the overall moment

\[ \alpha(t) \leq \sum_{j=1}^{J^*} \alpha_j(t) = J^* \left( \log \frac{\sigma}{\sqrt{\Delta}} + \frac{\Delta}{2\sigma^2} \left( t^2 \sigma^2 + t\Delta \right) \right). \]

Now, for any given \( \epsilon > 0 \), we have

\[ \delta = \min_t \exp(\alpha(t) - t\epsilon) = \min_t \exp \left( J^* \left( \log \frac{\sigma}{\sqrt{\Delta}} + \frac{\Delta}{2\sigma^2} \left( t^2 \sigma^2 + t\Delta \right) \right) - t\epsilon \right). \]

We compute the minimizing \( t \) as

\[ t_{opt} = \frac{\epsilon}{J^*\Delta} - \frac{\Delta}{2\sigma^2}. \]

Using this, we find \( \delta_{opt} \)

\[ \delta_{opt} = \exp \left( J^* \log \frac{\sigma}{\sqrt{\Delta}} + \frac{J^*\Delta}{2} \left( \frac{\epsilon}{J^*\Delta} - \frac{\Delta}{2\sigma^2} \right)^2 + \left( \frac{J^*\Delta^2}{2\sigma^2} - \epsilon \right) \left( \frac{\epsilon}{J^*\Delta} - \frac{\Delta}{2\sigma^2} \right) \right). \]

As we are interested in finding the best \( \epsilon \) for a given \( \delta \), we rearrange the above equation to solve for \( \epsilon \) as

\[ \epsilon = \frac{1}{2a} \left( -b \pm \sqrt{b^2 - 4ac} \right), \]

where \( a = \frac{1}{2J^*\Delta}, b = -\frac{\Delta}{2\sigma^2} \), and \( c = \log \delta - J^* \log \frac{\sigma}{\sqrt{\Delta}} + \frac{J^*\Delta^3}{8\sigma^4} \). For our proposed capeDJICA algorithm, we release two noisy gradients: \( \Delta_W(j) \) and \( \Delta_b(j) \), at iteration \( j \) with noise variances \( \sigma_W^2 \) and \( \sigma_b^2 \), respectively. Adjusting for this, we plot the total \( \epsilon \) against the total iterations \( J^* \) for the basic composition and the moments accountant in Figure 2. We observe that the moments accountant method provides a much smaller total \( \epsilon \) than the basic composition (which grows linearly with \( J^* \)).

5 Performance Analysis of capeDJICA

5.1 Performance Gain with Correlated Noise

The existing DP djICA algorithm [1] achieved \( J^*\epsilon \)-differential privacy (where \( J^* \) is the total number of iterations required for convergence) by adding a noise term to the local estimate of the source (i.e., \( Z_s(j) \)). Although the algorithm
Figure 2: Variation of total $\epsilon$ with number of iterations $J^*$: $\sigma^2_W = \sigma^2 = 0.001$ and $\delta = 1/N$. Moments accountant method provides a much smaller overall $\epsilon$ than the basic composition.

offered a “pure” DP djICA procedure, there are a few shortcomings. The cost of achieving pure differential-privacy (i.e., employing the Laplace mechanism [9]) was that the noise variance was dependent on $R$ and the norm of the most-recent $W$. Moreover, the neighboring dataset condition was met by restricting the $L_2$-norm of the samples to satisfy $\|x_n\|_2 \leq \frac{1}{2\sqrt{D}}$, which can be too limiting for datasets with large ambient dimensions. Last but not the least, the DP PCA preprocessing step was less fault tolerant than the one employed in this paper. By employing the CAPE protocol in the preprocessing stage and also in the optimization process, we expect to gain a significant performance boost. We validate the performance gain in the Experimental Results (Section 6).

5.2 Convergence of capeDJICA Algorithm

We note that the gradient estimate at the aggregator (Step 14 in Algorithm 4) essentially contains the noise $\frac{1}{S} \sum_{s=1}^{S} K_s^G$, which is zero mean. Therefore, the convergence of Algorithm 4 is guaranteed [31]. Since the total additive noise variance is smaller than the conventional case by a factor of $S$, the convergence rate is faster than the conventional case.

5.3 Communication Cost

We analyze the total communication cost associated with the proposed capeDJICA algorithm. At each iteration round, we need to generate two zero-sum noise terms, which entails $O(S + R^2)$ communication complexity of the sites and $O(S^2 + SR^2)$ communication complexity of the aggregator [28]. Each site computes the noisy gradient and sends one $R \times R$ matrix and one $R$ dimensional
vector to the aggregator. And finally, the aggregator sends the $R \times R$ updated weight matrix and $R$ dimensional bias estimate to the sites. If the number of iterations required to achieve convergence is denoted by $J^*$ then the total communication cost is proportional to $4J^*(SR^2 + SR)$ or $O(S + R^2)$ for the sites and $O(S^2 + SR^2)$ for the central node. This is expected as we are estimating an $R \times R$ matrix in a decentralized setting.

6 Experimental Results

In this section, we empirically show the effectiveness of the proposed capeDJICA algorithm. We note the intricate relationship between $\epsilon$ and $\delta$ (see Theorem 1) due to the correlated noise scheme and the challenge of characterizing the overall privacy loss in our multi-round capeDJICA algorithm. We designed the experiments to better demonstrate the trade-off between performance and several parameters: $\epsilon$, $\delta$ and $M$. We show the simulation results to compare the performance of our capeDJICA algorithm with the existing DP djICA algorithm [1].
Figure 5: Variation of $q^{NGI}$ and overall $\epsilon$ with privacy parameter $\delta$: (a) synthetic and (b) real fMRI data. Fixed parameters: $S=4$, $\epsilon_i=0.5$. capeDJICA achieves very close utility to the non-private djICA with small overall $\epsilon$.

(DP – djICA), the non-private djICA algorithm [11] and a DP ICA algorithm operating on only local data (local DP – ICA). We modified the base non-private djICA algorithm to incorporate the gradient bounds $B_G$ and $B_h$. Although we are proposing an algorithm for decentralized setting, we included the performance indices for the local setting to show the effect of smaller sample sizes on the performance. We note that the DP – djICA algorithm [1] offers $\epsilon$-differential privacy as opposed to $(\epsilon, \delta)$-differential privacy offered by capeDJICA. For both synthetic and real datasets, we consider the symmetric setting (i.e., $N_s = N_S$, $\tau^*_G = \tau_G$ and $\tau^*_h = \tau_h$) and show the average performance over 10 runs of the algorithms.

**Synthetic Data.** We generated the synthetic data from the same model as [11]. The source signals $S$ were simulated using the generalized autoregressive (AR) conditional heteroscedastic (GARCH) model [32,33]. We used $M = 1024$ simulated subjects in our experiments. For each subject, we generated $R = 20$ time courses with 250 time points. The data samples are equally divided into $S = 4$ sites. For each subject, the fMRI images are $30 \times 30$ dimensional. We employ the capePCA algorithm [25] as a preprocessing stage to reduce the sample dimension from $D = 900$ to $R = 20$. The capeDJICA is carried out upon the $R$-dimensional samples.

**Real Data.** We use the same data and preprocessing as Baker et al. [11]: the data were collected using a 3-T Siemens Trio scanner with a 12-channel radio frequency coil, according to the protocol in Allen et al. [15]. In the dataset, the resting-state scan durations range from 2 min 8 sec to 10 min 2 sec, with an average of 5 min 16 sec [11]. We used a total of $M = 1548$ subjects from the dataset and estimated $R = 50$ independent components using the algorithms under consideration. Preprocessing included rigid body alignment for head motion, slice-timing correction, spatial normalization to MNI space, regression of 6 motion parameters and their derivatives in addition to any trends (up to cubic or quintic), and spatial smoothing using a $10mm^3$ full-width at half-maximum Gaussian kernel. We also projected the data onto a 50-dimensional PCA subspace estimated using pooled non-private PCA. As we do not have the ground truth for the real data, we computed a pseudo ground truth [11] by performing a pooled non-private analysis on the data and estimating the unmixing matrix. The performance of capeDJICA, djICA, DP – djICA and local DP – ICA algorithms are evaluated against this pseudo ground truth.

**Performance Index.** We set $\tau^*_G = \frac{\Delta^*_G}{\epsilon_i} \sqrt{2 \log \frac{1.25}{\epsilon_i^2}}$ and $\tau^*_h = \frac{\Delta^*_h}{\epsilon_i} \sqrt{2 \log \frac{1.25}{\epsilon_i^2}}$.
for our experiments, where $\epsilon_i$ is the privacy parameter per iteration, $\Delta G_s$ and $\Delta h_s$ are the $L_2$ sensitivities of $G$ and $h$, respectively. To evaluate the performance of the algorithms, we consider the quality of the estimated unmixing matrix $W$. More specifically, we utilize the normalized gain index $q_{NGI}$ that quantizes the quality of $W$. The normalized gain index $q_{NGI}$ varies from 0 to 1, with 0 indicating that the unmixing matrix is an identity matrix.

Note that, in addition to a small $q_{NGI}$, we want to attain a strict privacy guarantee, i.e. small overall $(\epsilon, \delta)$. Recall from Section 4.2 that the overall $\epsilon$ is a function of the number of iterations, the overall $\delta$ and $\{\tau G_s, \tau h_s\}$. For all of our experimental analyses, we plotted the overall $\epsilon$ (with solid lines on the right $y$-axis) along with $q_{NGI}$ (with dashed lines on the left $y$-axis) as a means for visualizing how the privacy-utility trade-off varies with different parameters.

**Performance Variation with $\epsilon$.** First, we explore how the privacy-utility tradeoff between $q_{NGI}$ and the overall “privacy risk” $\epsilon$ varies with $\epsilon_i$. In Figs. 3(a) - (b), we show the variation of $q_{NGI}$ and overall $\epsilon$ for different algorithms.
with $\epsilon_i$ on synthetic data. We kept the number of sites $S = 4$ and the target $\delta = 10^{-5}$ fixed. As mentioned before, we compare the performance of capeDJICA with those of the djICA, the DP – djICA and local DP – ICA. We show the performance indices for two different $M$ values, namely $M = 256$ and $M = 1024$. We observe from the figures that the proposed capeDJICA outperforms the existing DP – djICA by a large margin. This is expected as DP – djICA suffers from too much noise (see Section 5.1 for the explanation). capeDJICA also guarantees the smallest overall $\epsilon$ among the privacy-preserving methods. capeDJICA can reach the utility level of the non-private djICA for some parameter choices and naturally outperforms local DP – ICA as estimation of the sources is much accurate when more samples are available. For the same privacy loss (i.e., for a fixed $\epsilon$), one can achieve better performance by increasing the number of subjects. In Figs. 3(c) - (d), we show the variation of $q_{NGI}$ and overall $\epsilon$ for different algorithms with $\epsilon_i$ on real data. We show the performance indices for $M = 256$ and $M = 1024$. We observe that, similar to the synthetic data, the proposed capeDJICA outperforms the existing DP – djICA by a large margin. The proposed capeDJICA can reach the utility level of the non-private djICA even for small overall $\epsilon$ values and outperforms local DP – ICA. Again we observe that, for a fixed $\epsilon$, we can achieve better performance by increasing the number of subjects. For both synthetic and real data, we note that assigning a higher $\epsilon_i$ may provide a good $q_{NGI}$ but does not guarantee a small overall $\epsilon$. The user needs to choose the $\epsilon_i$ based on the “privacy budget” and the required performance.

**Performance Variation with $M$.** Next, in Figure 4(a) - (b), we show the variation of $q_{NGI}$ and the overall $\epsilon$ with the total number of subjects $M$ for two different $\epsilon_i$ values on synthetic data. We kept the number of sites $S = 4$ and target $\delta = 10^{-5}$ fixed. We observe similar trends in performance as in the case of varying $\epsilon_i$. The capeDJICA algorithm outperforms the DP – djICA and the local DP – ICA: with respect to both $q_{NGI}$ and the overall $\epsilon$. For the $q_{NGI}$, the capeDJICA performs very closely to the non-private djICA. The performance gain over DP – djICA is particularly noteworthy. For a fixed number of subjects, increasing $\epsilon$ results in a slightly better utility, albeit at the cost of greater privacy loss. In Figure 4(c) - (d), we show the variation of $q_{NGI}$ and overall $\epsilon$ with the total number of subjects $M$ for $\epsilon_i = 0.1$ and $\epsilon_i = 1.0$ on real data. We observe a very similar trend as the synthetic data simulations: the capeDJICA algorithm comfortably outperforms the DP – djICA and the local DP – ICA and achieves utility close to the non-private djICA even for moderate $M$ values, while guaranteeing the smallest overall $\epsilon$.

**Performance Variation with $\delta$.** Recall that, the proposed capeDJICA algorithm guarantees $(\epsilon, \delta)$ differential privacy, where $(\epsilon, \delta)$ satisfy the relation

$$\delta = 2 \frac{\sigma_x}{\epsilon} \phi \left( \frac{\epsilon - \mu_x}{\sigma_x} \right).$$

In Figure 5, we show the variation of $q_{NGI}$ with overall $\delta$ on synthetic and real data. Recall that $\delta$ is essentially the probability of failure of a DP algorithm. Therefore, we want $\delta$ to be small. However, a smaller $\delta$ also results in a larger noise variance, which affects the utility. From the figure, we can observe how the performance of the proposed capeDJICA algorithm varies
with $\delta$. We demonstrate the performance indices keeping $\epsilon_i = 0.5$ and $S = 4$ fixed. We set the number of colluding sites to be $S_C = \lceil \frac{S}{3} \rceil - 1$. The proposed algorithm achieves very close utility to the non-private djICA for both synthetic and real data. For both cases, the overall $\epsilon$ is also very small. However, we can opt for even smaller $\delta$ values at the cost of performance.

**Reconstructed Spatial Maps.** Finally, we intend to demonstrate how the estimated spatial maps actually look like, as interpretability is one of the most important concerns for fMRI applications. In Figure 6, we show the true spatial map and the one estimated from the non-private djICA algorithm. In Figure 7, we show the estimated spatial maps resulting from the proposed capeDJICA algorithm along with the overall $\epsilon$ for a variety of combinations of $\epsilon_i$ and $M$. We observe that when sufficiently large number of subjects are available, the estimated spatial maps closely resemble the true one, even for strict privacy guarantee (small overall $\epsilon$). For smaller number of samples, we may need to compensate by allowing larger $\epsilon$ values to achieve good utility. In general, we observe that capeDJICA can achieve very good approximate to the true spatial map, almost indistinguishable from the non-private spatial map. This emphasizes the effectiveness of the proposed capeDJICA in the sense that very meaningful utility can be achieved even with strict privacy guarantee.

## 7 Conclusion

In this paper, we proposed a new and improved algorithm for DP decentralized joint independent component analysis. The proposed algorithm offers significant improvement upon our previous work and achieves the same level of additive noise variance as the pooled data scenario in certain regimes. Therefore, we attained the same utility as the DP pooled data scenario in a decentralized setting. This is achieved due to the employment of the correlated noise protocol, assuming the availability of some reasonable resources. We proposed an extension of the CAPE scheme that can incorporate asymmetric network/privacy settings, which are more prevalent in medical research scenarios, such as ours. We analyzed our capeDJICA algorithm using Rényi differential privacy and provided a better account of the privacy loss per iteration using the moments accountant method. We empirically compared the performance of the proposed algorithm with those of existing, pooled and local algorithms on synthetic and real datasets. We varied privacy parameters and relevant dataset parameters to show that the proposed algorithm outperformed the existing and local algorithms comfortably and matched the performance of the non-private algorithm for some parameter choices. In general, the proposed algorithm offered very good utility even for strong privacy guarantees – indicating achievability of meaningful privacy even without losing much utility.
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Appendix

A Extension of CAPE: Unequal Sample Sizes at Sites

Solution to the optimization problem of Section 3.2. We observe that the variance \( \tau_{es}^2 \) of the zero-mean random variable \( e_s = \hat{e}_s - \frac{1}{\mu_s S} \sum_{i=1}^{S} \mu_i \hat{e}_i \) can be computed as

\[
\tau_{es}^2 = \text{Var}\left[ \hat{e}_s - \frac{1}{\mu_s S} \sum_{i=1}^{S} \mu_i \hat{e}_i \right]
= \mathbb{E}\left[ \hat{e}_s^2 + \frac{1}{\mu_s^2 S^2} \left( \sum_{i=1}^{S} \mu_i \hat{e}_i \right)^2 - \frac{2}{\mu_s S} \hat{e}_s \sum_{i=1}^{S} \mu_i \hat{e}_i \right]
= \sigma_s^2 + \frac{1}{\mu_s^2 S^2} \sum_{i=1}^{S} \mu_i^2 \sigma_i^2 - \frac{2}{\mu_s S} \cdot \mu_s \sigma_s^2
= \left( 1 - \frac{2}{S} \right) \sigma_s^2 + \frac{1}{\mu_s^2 S^2} \sum_{i=1}^{S} \mu_i^2 \sigma_i^2.
\]

Note that we need \( \sum_{s=1}^{S} \mu_s^2 \tau_{gs}^2 = \tau_{pool}^2 \). One solution is to set \( \tau_{gs}^2 = \frac{1}{\mu_s^2 S} \tau_{pool}^2 \).

Using the constraint and the expression for \( \tau_{gs}^2 \), we have

\[
\tau_{es}^2 + \tau_{gs}^2 = \tau_s^2
\]

\[
\Rightarrow \left( 1 - \frac{2}{S} \right) \sigma_s^2 + \frac{1}{\mu_s^2 S^2} \sum_{i=1}^{S} \mu_i^2 \sigma_i^2 + \frac{1}{\mu_s^2 S} \tau_{pool}^2 = \tau_s^2
\]

\[
\Rightarrow \left( 1 - \frac{1}{S} \right)^2 \sigma_s^2 + \frac{1}{\mu_s^2 S^2} \sum_{i \neq s} \mu_i^2 \sigma_i^2 = \tau_s^2 - \frac{1}{\mu_s^2 S} \tau_{pool}^2.
\]
We can write this expression for all $s \in [S]$ in matrix form and solve for $[\sigma_1^2 \sigma_2^2 \ldots \sigma_S^2]^\top$ as

$$
\begin{bmatrix}
(1 - \frac{1}{S})^2 & \frac{\mu_1^2}{\mu_1^2 + \mu_2^2 S^2} & \cdots & \frac{\mu_S^2}{\mu_S^2 S^2} \\
\frac{\mu_1^2}{\mu_1^2 S^2} & (1 - \frac{1}{S})^2 & \cdots & \frac{\mu_S^2}{\mu_S^2 S^2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\mu_1^2}{\mu_1^2 S^2} & \frac{\mu_2^2}{\mu_2^2 S^2} & \cdots & (1 - \frac{1}{S})^2
\end{bmatrix}^{-1} \begin{bmatrix}
\tau_1^2 - \frac{\tau_{pool}}{\mu_1^2 S^2} \\
\tau_2^2 - \frac{\tau_{pool}}{\mu_2^2 S^2} \\
\vdots \\
\tau_S^2 - \frac{\tau_{pool}}{\mu_S^2 S^2}
\end{bmatrix}
$$

Privacy Analysis in Asymmetric Setting. We present an analysis of privacy for the aforementioned scheme in asymmetric setting. Recall that the adversary can observe $\hat{a} = [\hat{a}_1, \ldots, \hat{a}_S] \in \mathbb{R}^{S_H}$ and $\hat{e} = \sum_{s \in S_H} \hat{e}_s$. In other words, the vector $y = [\hat{a}^\top, \hat{e}] \in \mathbb{R}^{S_H+1}$ is what the adversary can observe to make inference about the non-colluding sites. To prove differential privacy guarantee, we must show that $\log \frac{g(y|a)}{g(y|a')} \leq \epsilon$ holds with probability (over the randomness of the mechanism) at least $1 - \delta$. Here, $a = [f(x_1), \ldots, f(x_{S_H})]^\top$ and $g(y|a)$ and $g(y|a')$ are the probability density functions of $y$ under $a$ and $a'$, respectively. The vectors $a$ and $a'$ differ in only one coordinate (neighboring). Without loss of generality, we assume that $a$ and $a'$ differ in the first coordinate. We note that the maximum difference is $1/N_S$ as the sensitivity of the function $f(x) = \frac{1}{N_S}$. Recall that we release $\hat{a}_s = f(x_s) + e_s + g_s$ from each site. We observe

$$
\mathbb{E}(\hat{a}_s) = f(x_s), \quad \text{Var}(\hat{a}_s) = \tau_s^2, \quad \forall s \in [S]
$$

$$
\mathbb{E}(\hat{a}_s \hat{a}_{s_2}) = f(x_s) f(x_{s_2}) - \frac{\mu_{s_1} \sigma_{s_1}^2}{\mu_{s_1} S} + \frac{\mu_{s_2} \sigma_{s_2}^2}{\mu_{s_2} S} + \frac{1}{\mu_{s_1} \mu_{s_2} S^2} \sum_{i=1}^{S} \mu_i^2 \sigma_i^2, \quad \forall s_1 \neq s_2 \in [S].
$$

Without loss of generality, we can assume $[23]$ that $a = 0$ and $a' = a - v$, where $v = \left[ \frac{1}{N_S}, 0, \ldots, 0 \right]^\top$. That is, the random variable $\hat{a}$ is $\mathcal{N}(0, \Sigma_{a})$, where

$$
\Sigma_a = \begin{bmatrix}
\tau_1^2 & \Psi(1, 2) & \cdots & \Psi(1, S) \\
\Psi(2, 1) & \tau_2^2 & \cdots & \Psi(2, S) \\
\vdots & \vdots & \ddots & \vdots \\
\Psi(S, 1) & \Psi(S, 2) & \cdots & \tau_S^2
\end{bmatrix},
$$

and $\Psi(i, j) = -\frac{1}{S} \left( \frac{\mu_i \sigma_i^2}{\mu_i} + \frac{\mu_j \sigma_j^2}{\mu_j} \right) + \sum_{s=1}^{S_H} \frac{\mu_s^2 \sigma_s^2}{\mu_s^2 S^2}$. Additionally, the random variable $\hat{e}$ is $\mathcal{N}(0, \tau_0^2)$, where $\tau_0^2 = \sum_{s=1}^{S_H} \sigma_s^2$. Therefore, $g(y|a)$ is the density of $\mathcal{N}(0, \Sigma)$,
where
\[
\Sigma = \begin{bmatrix} \Sigma_{\mathbf{a}} & \Sigma_{\mathbf{a}^T \mathbf{e}} \\ \Sigma^{T}_{\mathbf{a}^T \mathbf{e}} & \tau^2_{\mathbf{e}} \end{bmatrix} \in \mathbb{R}^{(S_H+1) \times (S_H+1)}.
\]

With some simple algebra, we can find the expression for \(\Sigma_{\mathbf{a}^T \mathbf{e}}\):
\[
\Sigma_{\mathbf{a}^T \mathbf{e}} = \begin{bmatrix} 
\sigma^2_1 - \frac{1}{\mu_{zS}} \sum_{i=1}^{S_H} \mu_i^2 \sigma_i^2 \\
\sigma^2_2 - \frac{1}{\mu_{zS}} \sum_{i=1}^{S_H} \mu_i^2 \sigma_i^2 \\
\vdots \\
\sigma^2_{S_H} - \frac{1}{\mu_{zS}} \sum_{i=1}^{S_H} \mu_i^2 \sigma_i^2 
\end{bmatrix}.
\]

The rest of the proof proceeds as the proof of Theorem 2. Note that, due to the complex nature of the expression of \(\Sigma\), we do not have a closed form solution for \(\mu_z\) and \(\sigma_z\). However, we can numerically evaluate their values and consequently, the resulting \(\delta\) guarantee.
B Details of the CAPE Scheme

In this section, we review the recently proposed CAPE scheme. We assume that all of the $S$ sites and the central node follow the protocol honestly. However, up to $S_C = \left\lceil \frac{S}{3} \right\rceil - 1$ sites can collude with an adversary to learn about another site's data/function output. The central node is also honest-but-curious (and therefore, can collude with an adversary). An adversary can observe the outputs from each site, as well as the output from the aggregator. Additionally, the adversary can know everything about the colluding sites (including their private data). We denote the number of non-colluding sites with $S_H$ such that $S = S_C + S_H$. Without loss of generality, we designate the non-colluding sites with numbers $1, \ldots, S_H$ (see Figure 1(b)).

We design the noise generation procedure such that: i) we can ensure $(\epsilon, \delta)$ differential privacy of the algorithm output from each site and ii) achieve the noise level of the pooled data scenario in the final output from the aggregator. We achieve that by employing a correlated noise addition scheme. Considering the same decentralized averaging problem as Section 2, we intend to release (and send to the aggregator) $\hat{a}_s = f(x_s) + e_s + g_s$ from each site $s$, where $e_s$ and $g_s$ are two noise terms. The variances of $e_s$ and $g_s$ are chosen to ensure that the noise $e_s + g_s$ is sufficient to guarantee $(\epsilon, \delta)$-differential privacy to $f(x_s)$. Here, each site generates the noise $g_s \sim \mathcal{N}(0, \tau^2_g)$ locally and the noise $e_s \sim \mathcal{N}(0, \tau^2_e)$ jointly with all other sites such that $\sum_{s=1}^{S} e_s = 0$. We employ the recently proposed secure aggregation protocol (SecureAgg) by Bonawitz et al. to generate $e_s$ that ensures $\sum_{s=1}^{S} e_s = 0$. The SecureAgg protocol utilizes Shamir’s $t$-out-of-$n$ secret sharing and is communication-efficient.

Now, to initiate the CAPE protocol, each site $s \in [S]$ generates a noise term $\hat{e}_s \sim \mathcal{N}(0, \tau^2_e)$ independently. The aggregator computes $\sum_{s=1}^{S} \hat{e}_s$ according to the SecureAgg protocol and broadcasts it to all the sites. Each site then sets

$$e_s = \hat{e}_s - \frac{1}{S} \sum_{s'=1}^{S} \hat{e}_{s'}$$

to achieve $\sum_{s=1}^{S} e_s = 0$. We show the complete noise generation procedure in Algorithm 1. Note that, in the original form, the SecureAgg protocol is intended for computing sum of $D$-dimensional vectors in a finite field $\mathbb{Z}_D^D$. For our purposes, we need to perform the summation of Gaussian random variables over $\mathbb{R}$ or $\mathbb{R}^D$. To accomplish this, each site can employ a mapping map : $\mathbb{R} \mapsto \mathbb{Z}_\lambda$ that performs a stochastic quantization for large-enough $\lambda$. The aggregator can compute the sum in the finite field according to SecureAgg and then invoke a reverse mapping remap : $\mathbb{Z}_\lambda \mapsto \mathbb{R}$ before broadcasting $\sum_{s=1}^{S} \hat{e}_s$ to the sites. Algorithm 1 can be readily extended to generate array-valued zero-sum noise terms.

For our approach of generating $\hat{e}_s$, the variance of $e_s$ is given by $\tau^2_e = (1 - \frac{1}{3}) \tau^2_s$. Additionally, we choose $\tau^2_g = \frac{\tau^2_s}{3}$. Each site then generates the noise $g_s \sim \mathcal{N}(0, \tau^2_g)$ independently and sends $\hat{a}_s = f(x_s) + e_s + g_s$ to the aggregator.
Note that neither of the terms $e_s$ and $g_s$ has large enough variance to provide $(\epsilon, \delta)$-differential privacy guarantee to $f(x_s)$. Instead, we chose the variances of $e_s$ and $g_s$ to ensure that the $e_s + g_s$ is sufficient to ensure a DP guarantee to $f(x_s)$ at site $s$. The chosen variance of $g_s$ also ensures that the output from the aggregator would have the same noise variance as the DP pooled-data scenario.

To see this, observe that we compute the following quantity at the aggregator (in Step 6 of Algorithm 2):

$$a_{\text{cape}} = \frac{1}{S} \sum_{s=1}^{S} f(x_s) + \frac{1}{S} \sum_{s=1}^{S} e_s + \frac{1}{S} \sum_{s=1}^{S} g_s$$

$$= \frac{1}{N} \sum_{n=1}^{N} x_n + \frac{1}{S} \sum_{s=1}^{S} g_s,$$

where we used $\sum_s e_s = 0$. The variance of the estimator $a_{\text{cape}}$ is $\tau_{\text{cape}}^2 = \tau_{\text{pool}}^2$, which is the exactly the same as if all the data were present at the aggregator. This claim is formalized in Lemma 1. We show the complete algorithm in Algorithm 2. The privacy of Algorithm 2 is given by Theorem 2.

**Theorem 2 (Privacy of CAPE Algorithm (Algorithm 2)).** Consider Algorithm 2 in the decentralized data setting of Section 2 with $N_s = \frac{N}{S}$ and $\tau^2 = \tau^2$ for all sites $s \in [S]$. Suppose that at most $S_C = \left\lceil \frac{S}{3} \right\rceil - 1$ sites can collude after execution. Then Algorithm 2 guarantees $(\epsilon, \delta)$-differential privacy for each site, where $(\epsilon, \delta)$ satisfy the relation $\delta = 2 \frac{\sigma_z}{\epsilon - \mu_z} \phi \left( \frac{\epsilon - \mu_z}{\sigma_z} \right)$ and $(\mu_z, \sigma_z)$ are given by

$$\mu_z = \frac{S^3}{2\tau^2 N^2 (1 + S)} \left( \frac{S - S_C + 2}{S - S_C} + \frac{9}{S - S_C} S_C^2 \right),$$

$$\sigma_z^2 = \frac{S^3}{\tau^2 N^2 (1 + S)} \left( \frac{S - S_C + 2}{S - S_C} + \frac{9}{S - S_C} S_C^2 \right).$$

**Proof.** The proof is given in [19].

**Remark 1.** Theorem 2 is stated for the symmetric setting: $N_s = \frac{N}{S}$ and $\tau^2 = \tau^2 \forall s \in [S]$. As with many algorithms using the approximate differential privacy, the guarantee holds for a range of $(\epsilon, \delta)$ pairs subject to a tradeoff constraint between $\epsilon$ and $\delta$, as in the simple case (see Definition 3).

**Remark 2 (Utility Analysis).** The goal is to ensure $(\epsilon, \delta)$-differential privacy for each site and achieve $\tau_{\text{cape}}^2 = \tau_{\text{pool}}^2$ at the aggregator (see Lemma 1). The CAPE protocol guarantees $(\epsilon, \delta)$-differential privacy with $\delta = 2 \frac{\sigma_z}{\epsilon - \mu_z} \phi \left( \frac{\epsilon - \mu_z}{\sigma_z} \right)$.

Inttiaz et al. empirically showed [19] that this $\delta$ guarantee is much better than the $\delta$ guarantee in the conventional decentralized DP scheme. In particular, they compared $\delta$ with $\delta_{\text{conv}}$, where $\delta_{\text{conv}}$ is the smallest $\delta$ guarantee we can afford in the conventional decentralized DP scheme to achieve the same noise variance.

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as the pooled-data scenario for a given $\epsilon$. The authors observed that $\delta$ is always smaller than $\delta_{\text{conv}}$. That is, for achieving the same noise level at the aggregator output (and therefore the same utility) as the pooled data scenario, CAPE ensures a much better privacy guarantee than the conventional approach.

**Lemma 1.** Consider the symmetric setting: $N_s = \frac{N}{S}$ and $\tau^2_s = \tau^2$ for all sites $s \in [S]$. Let the variances of the noise terms $e_s$ and $g_s$ (Step 4 of Algorithm 2) be $\tau^2_e = \left(1 - \frac{1}{S}\right)\tau^2$ and $\tau^2_g = \frac{\tau^2}{S}$, respectively. If we denote the variance of the additive noise (for preserving privacy) in the pooled data scenario by $\tau^2_{\text{pool}}$ and the variance of the estimator $a_{\text{cape}}$ (Step 6 of Algorithm 2) by $\tau^2_{\text{cape}}$ then Algorithm 2 achieves the same expected error as the pooled-data scenario (i.e., $\tau^2_{\text{pool}} = \tau^2_{\text{cape}}$).

**Proof.** The proof is given in [19].

**Remark 3.** Note that the CAPE algorithm achieves the same noise variance as the pooled-data scenario (i.e., $\tau^2_{\text{cape}} = \tau^2_{\text{pool}}$) for the symmetric setting: $N_s = \frac{N}{S}$ and $\tau^2_s = \tau^2$ for all sites $s \in [S]$. In general, the ratio $H(n) = \frac{\tau^2_{\text{cape}}}{\tau^2_{\text{pool}}}$, where $n \triangleq [N_1, N_2, \ldots, N_S]$, is a function of the sample sizes in the sites. We observe: $H(n) = \frac{N^2}{S} \sum_{s=1}^{S} \frac{1}{N_s^2}$. As $H(n)$ is a Schur-convex function, it can be shown using majorization theory [37] that $1 \leq H(n) \leq \frac{N^2}{S} \left(\frac{N-1}{(N-1+S)^2} + S - 1\right)$, where the minimum is achieved for the symmetric setting. That is, CAPE achieves the smallest noise variance at the aggregator in the symmetric setting.

**Proposition 4.** (Performance gain) If the local noise variances are $\{\tau^2_s\}$ for $s \in [S]$ then the CAPE algorithm achieves a gain of $G = \frac{\tau^2_{\text{conv}}}{\tau^2_{\text{cape}}} = S$, where $\tau^2_{\text{conv}}$ and $\tau^2_{\text{cape}}$ are the noise variances of the final estimate at the aggregator in the conventional decentralized DP scheme and the CAPE scheme, respectively.

**Proof.** The proof is given in [19].

**Remark 4** (Site Dropouts). Even in the case of site drop-out, the CAPE scheme achieves $\sum_s e_s = 0$, as long as the number of active sites is above some threshold (see Bonawitz et al. [28] for details). Therefore, the performance gain of CAPE remains the same irrespective of the number of dropped-out sites.