Cluster Partially Observed Graphs via Convex Optimization

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Abstract

This paper considers the problem of clustering a partially observed unweighted graph – i.e. one where for some node pairs we know there is an edge between them, for some others we know there is no edge, and for the remaining we do not know whether or not there is an edge. We want to organize the nodes into disjoint clusters so that there is relatively dense (observed) connectivity within clusters, and sparse across clusters.

We take a novel yet natural approach to this problem, by focusing on finding the clustering that minimizes the number of “disagreements” – i.e. the sum of the number of (observed) missing edges within clusters, and (observed) present edges across clusters. Our algorithm uses convex optimization; its basis is a reduction of disagreement minimization to the problem of recovering an (unknown) low-rank matrix and an (unknown) sparse matrix from their partially observed sum. We evaluate our algorithm on the natural and classical Planted Partition/Stochastic Block Model of clustered random graphs; our main result characterizes when our algorithm succeeds as a function of minimum cluster size, edge noise and observation probability. When there are a constant number of clusters of equal size, our results are optimal up to logarithmic factors.

1 Introduction

This paper is about the following task: given partial observation of an undirected unweighted graph, partition the nodes into disjoint clusters so that there are dense connections within clusters, and sparse connections across clusters. By partial observation, we mean that for some node pairs we know if there is an edge or not, and for the other node pairs we do not know – these pairs are unobserved. This problem arises in several fields across science and engineering. For example, in sponsored search, each cluster is a submarket that represents a specific group of advertisers that do most of their spending on a group of query phrases – see e.g. [Yahoo!-Inc (2009)] for such a project at Yahoo. In VLSI and design automation, it is useful in minimizing signaling between components, layout etc. – see e.g. [Kernighan and Lin (1970)] and references thereof. In social networks, clusters may represent groups of people with similar interest or background; finding clusters enables better recommendations, link prediction, etc. [Mishra et al. (2007)]. In the analysis of document databases, clustering the citation graph is often an essential and informative first step. [Ester et al. (1995)]. In this paper, we will focus not on specific application domains, but rather on the basic graph clustering problem itself.

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Partially observed graphs appear in many applications. For example, in online social networks like Facebook, we observe an edge/no edge between two users when they accept each other as a friend or explicitly decline a friendship suggestion. For the other user pairs, however, we simply have no friendship information between them, which are thus unobserved. More generally, we have partial observations whenever obtaining similarity data is difficult or expensive (e.g. because it requires human participation). In these applications, it is often the case that most pairs are unobserved, which is the regime we are particularly interested in.

As with any clustering problem, this needs a precise mathematical definition of the clustering criterion with potentially a guaranteed performance. There is relatively few existing work with provable performance guarantees for partially observed graphs. Even most existing approaches to clustering fully observed graphs either require an additional input (e.g. the number of clusters \( k \) required for spectral or \( k \)-means clustering approaches), or do not guarantee the performance of the clustering. We review these results in Section 1.1 below, and show that our results extend the known guarantees there for partially observed graphs.

Our Formulation: We focus on a natural formulation, one that does not require any other extraneous input besides the graph itself. It is based on minimizing disagreements, which we now define. Consider any candidate clustering; this will have (a) observed node pairs that are in different clusters, but have an edge between them, and (b) observed node pairs that are in the same cluster, but do not have an edge between them. The total number of node pairs of types (a) and (b) is the number of disagreements between the clustering and the given graph. We focus on the problem of finding the optimal clustering – one that minimizes the number of disagreements. Note that we do not pre-specify the number of clusters. For the special case of fully observed graphs, this formulation is exactly the same as the problem of “Correlation Clustering”, first proposed by Bansal et al. (2002). They showed that exact minimization of the above objective is NP-complete in the worst case – we survey and compare this and other related work in Section 1.1. As we will see, our approach and results are very different.

Our Approach and Results: We aim to achieve the combinatorial disagreement minimization objective using matrix splitting via convex optimization. In particular, as we show in section 2 below, one can represent the adjacency matrix of the given graph as the sum of an unknown low-rank matrix (corresponding to “ideal” clusters) and a sparse matrix (corresponding to disagreements from this “ideal” in the given graph). Our algorithm either returns a clustering, which is guaranteed to be disagreement minimizing, or returns a “failure” – it never returns a sub-optimal clustering. For our main result, we evaluate our algorithm’s performance on the natural and classical Planted Partition/Stochastic Block Model of clustered random graphs with partial observations; the model and results are outlined in Section 2. We prove our theoretical results in Section 3 and provide empirical results in Section 4. Our analysis provides stronger guarantees than are current results on general matrix splitting (Chandrasekaran et al., 2011; Candes et al., 2009; Hsu et al., 2010; Li, 2012; Chen et al., 2012).

1.1 Related Work

Our problem can be interpreted in the general clustering context as one in which the presence of an edge between two points indicates a “similarity”, and the lack of an edge means no similarity. The general field of clustering is of course vast, and a detailed survey of all methods therein is beyond our scope here. We focus instead on the two sets of papers most relevant to the problem here: the work on Correlation
Clustering, and that on the Planted Partition/Stochastic Block-Model.

**Correlation Clustering:** As mentioned, for a completely observed graph, our problem is mathematically precisely the same as Correlation Clustering formulated in Bansal et al. (2002); in particular a “+” in correlation clustering corresponds to an edge in the graph, and a “-” to the lack of an edge. Disagreements are defined in the same way. Thus, this paper can equivalently be considered as an algorithm, and guarantees, for correlation clustering under partial observations. Since correlation clustering is NP-complete in the worst case (Bansal et al., 2002), there has been much work on devising alternative approximation algorithms (Bansal et al., 2002; Becker, 2005; Emmanuel and Fiat, 2003). Approximations using convex optimization, including LP relaxation (Charikar et al., 2003; Demaine and Immorlica, 2003; Demaine et al., 2005) and SDP relaxation (Swamy, 2004; Mathieu and Schudy, 2010), followed by rounding, have also been developed. We emphasize that we use different convex relaxation, and we do not do rounding; rather, our convex program itself yields an optimal clustering.

We note that the result in Mathieu and Schudy (2010) uses a convex formulation with constraints enforcing positive semi-definiteness, triangle inequality and fixed diagonal entries. For the fully observed case, their relaxation is at least as tight as ours, and since they add more constraints, it is possible that there are instances where their convex program works and ours does not. However, this seems hard to prove/disprove. Indeed, in the restricted setting they considered, their theoretical guarantee is identical to ours. Moreover, as we argue in the next section, our guarantees are order-wise optimal in some important cases and thus cannot be improved even with a tighter relaxation. Practically, our method is faster since, to the best of our knowledge, there is no low-complexity algorithm to deal with the positive semi-definite and linear constraints required by Mathieu and Schudy (2010). This means that our method can handle very large graphs while their result is practically restricted to small ones (∼100 nodes). In summary, their approach has much higher computational complexity, and does not provide significant and characterizable gain in performance.

**Planted Partition Model:** The planted partition model, a.k.a. stochastic block-model (Condon and Karp, 2001; Holland et al., 1983), assumes that the graph is generated with in-cluster edge probability \( p \) and inter-cluster edge probability \( q \) \((p > q)\) and fully observed. The goal is to recover the latent cluster structure. A class of this model with \( \tau = \max\{1 - p, q\} \) is often used as benchmark for average case performance for correlation clustering (see, e.g., Mathieu and Schudy (2010)). Our theoretical results are applicable to this model and thus directly comparable with existing work in this area. A detailed comparison is provided in Table 1. For fully observed graphs, our result matches the previous best bounds in both the minimum cluster size and the difference between in-cluster/inter-cluster densities. We would like to point out that nuclear norm minimization has been used to solve the closely related planted clique problem (Alon et al., 1998; Ames and Vavasis, 2011).

**Partially Observed Graphs:** All the previous work listed in Table 1, except Oymak and Hassibi (2011), do not handle partial observations directly. One natural way to proceed is to impute the missing observations with no-edge, or random edges with symmetric probabilities, and then apply any of the results in Table 1. This approach, however, leads to sub-optimal results. Indeed, this is done explicitly in Oymak and Hassibi (2011), and they require the probability of observation \( p_0 \) to satisfy \( p_0 \gtrsim \frac{\sqrt{nK_{\min}}}{n} \), where \( n \) is the number of nodes and \( K_{\min} \) is the minimum cluster size. In contrast, our approach only needs \( p_0 \gtrsim \frac{n}{K_{\min}} \), which is order-wise better. Shamir and Tishby (2011) deals with partial observations directly and shows that \( p_0 \gtrsim \frac{1}{\sqrt{n}} \) suffices for recovering two clusters of size \( \Omega(n) \). Our result applies to much smaller clusters of size \( \tilde{\Omega}(\sqrt{n}) \). In addition, a nice feature of our result is that it gives an explicit tradeoff between the three
relevant parameters: $p_0$, $\tau$, and $K_{\min}$; theoretical result like this is not available in previous works.

We note that there exist other works that consider partial observations, but under rather different settings. For example, Balcan and Gupta (2010), Voevodski et al. (2010) and Krishnamurthy et al. (2012) consider the problem where one samples the rows/columns of the adjacency matrix $A$ rather than its entries. Hunter and Strohmer (2010) consider partial observations in the features, rather than in the similarity graph. Eriksson et al. (2011) show that $\tilde{\Omega}(\sqrt{n})$ actively selected pairwise similarities are sufficient for recovering the hierarchical clustering structure. Their results seem to rely heavily on the hierarchical structure. When disagreements are present, the first split of tree can only recovers clusters of size $\Omega(n)$. Moreover, they require control over the observation process, while we assume random observations.

| Paper               | Cluster size $K$ | Density difference $(1 - 2\tau)$ |
|---------------------|------------------|----------------------------------|
| Boppana (1987)      | $n/2$            | $\Omega(1/\sqrt{n})$            |
| Jerrum and Sorkin (1998) | $n/2$        | $\Omega(1/\sqrt{n^{1/6}-\epsilon})$ |
| Condon and Karp (2001) | $\Omega(n)$     | $\Omega(1/\sqrt{n^{1/2}-\epsilon})$ |
| Carson and Impagliazzo (2001) | $n/2$          | $\tilde{\Omega}(1/\sqrt{n})$    |
| Feige and Kilian (2001) | $n/2$         | $\tilde{\Omega}(1/\sqrt{n})$    |
| McSherry (2001)     | $\tilde{\Omega}(n^{2/3})$ | $\tilde{\Omega}(n^{2/3}/K)$     |
| Bollobás and Scott (2004) | $\tilde{\Omega}(n)$ | $\tilde{\Omega}(1/\sqrt{n})$    |
| Giesen and Mitsche (2005) | $\tilde{\Omega}(\sqrt{n})$ | $\tilde{\Omega}(1/\sqrt{n})$    |
| Shamir and Tsur (2007) | $\tilde{\Omega}(\sqrt{n})$ | $\tilde{\Omega}(1/\sqrt{n})$    |
| Mathieu and Schudy (2010) | $\tilde{\Omega}(\sqrt{n})$ | $\Omega(1)$                  |
| Rohe et al. (2010)  | $\tilde{\Omega}(n^{3/4})$ | $\tilde{\Omega}(n^{3/4}/K)$     |
| Oymak and Hassibi (2011) | $\tilde{\Omega}(\sqrt{n})$ | $\tilde{\Omega}(\sqrt{n}/K)$   |
| Sussman et al. (2012) | $\tilde{\Omega}(n^{3/4})$ | $(\ast)$                        |
| Chaudhuri et al. (2012) | $\tilde{\Omega}(\sqrt{n})$ | $\tilde{\Omega}(\sqrt{n}/K)$   |
| This paper          | $\tilde{\Omega}(\sqrt{n})$ | $\tilde{\Omega}(\sqrt{n}/K)$   |

Table 1: **Comparison with literature.** This table shows the lower-bound requirements on the minimum cluster size $K$ and the density difference $p - q = 1 - 2\tau$ that existing literature needs for exact recovery of the planted partitions, when the graph is fully observed and $\tau \triangleq \max\{1 - p, q\} = \Theta(1)$. Some of the results in the table only guarantee recovering the membership of most, instead of all, nodes. To force a fair comparison, we use the soft-$\Omega$ notation $\tilde{\Omega}(\cdot)$. It hides the logarithmic factors that are necessary for recovering all nodes, which is the goal of this paper. $(\ast)$ The result of Sussman et al. (2012) is stated in terms of parameters related to the factorization of the density matrix, which does not translate directly into the density difference of the standard planted partition model.
2 Main Contributions

2.1 Algorithm

Our algorithm is based on convex optimization, and either (a) outputs a clustering that is guaranteed to be the one that minimizes the number of observed disagreements, or (b) declares “failure”. In particular, it never produces a suboptimal clustering. We now briefly present the main idea, then describe the algorithm, and finally present our main results – analytical characterizations of when the algorithm succeeds.

Setup: We are given a partially observed graph, whose adjacency matrix is $A$ – which has $a_{ij} = 1$ if there is an edge between nodes $i$ and $j$, $a_{ij} = 0$ if there is no edge, and $a_{ij} = ?$ if we do not know. (Here we follow the convention that $a_{ii} = 0$ for all $i$.) We want to find the optimal clustering, i.e. the one that has the minimum number of disagreements in $\Omega_{\text{obs}}$, the set of observed node pairs.

Idea: Consider first the fully observed case, i.e. every $a_{ij} = 0$ or 1. Suppose also that the graph is already ideally clustered – i.e. there is a partition of the nodes such that there are no edges between clusters, and each cluster is a clique. In this case, the matrix $A + I$ is now a low-rank matrix, with the rank equal to the number of clusters. This can be seen by noticing that if we re-ordered the rows and columns so that clusters appear together, the result would be a block-diagonal matrix, with each block being an all-ones sub-matrix – and thus rank one. Of course, this re-ordering does not change the rank of the matrix, and hence $A + I$ is (exactly) low-rank.

Consider now any given graph, still fully observed. In light of the above, we are looking for a decomposition of its $A + I$ into a low-rank part $K^*$ (of block-diagonal all-ones, one block for each cluster) and a remaining $B^*$ (the disagreements) – such that the number of non-zero entries in $B^*$ is as small as possible; i.e. $B^*$ is sparse. Finally, the problem we look at is recovery of the best $K^*$ when we do not observe all entries. The idea is depicted in Figure 1.

Convex Optimization Formulation: We propose to do the matrix splitting using convex optimization, an approach recently taken in Chandrasekaran et al. (2011); Candes et al. (2009); however, we establish stronger results for our special problem. Our approach consists of dropping any additional structural requirements, and just looking for a decomposition of the given $A + I$ as the sum of a sparse matrix $B$ and a low-rank matrix $K$. In particular, let $\Omega_{\text{obs}}$ be the set of observed entries, i.e. the set of elements of $A$ that are known to be 0 or 1; we use the following convex program

$$
\min_{B,K} \quad \lambda \ ||B||_1 + ||K||_* \\
\text{s.t.} \quad \mathcal{P}_{\Omega_{\text{obs}}}(B + K) = \mathcal{P}_{\Omega_{\text{obs}}}(I + A)
$$

Here, for any matrix $M$, the term $\mathcal{P}_{\Omega_{\text{obs}}}(M)$ keeps all elements of $M$ in $\Omega_{\text{obs}}$ unchanged, and sets all other elements to 0; the constraints thus state that the sparse and low-rank matrix should in sum be consistent with the observed entries. $||B||_1 = \sum_{i,j} |b_{ij}|$ is the $\ell_1$ norm of the entries of the matrix, which is well-known to be a convex surrogate for the number of non-zero entries $||B||_0$. The second term $||K||_* = \sum_s \sigma_s(K)$ is the nuclear norm: the sum of singular values of $K$. This has been shown to be the tightest convex surrogate for the rank function (Fazel, 2002). Thus our objective function is a convex surrogate for

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1 In practice, one might be able to use the “failed” output with rounding as an approximate solution. In this paper, we focus on the performance of the exact unrounded algorithm.

2 In particular, it is the $\ell_1$ norm of the singular value vector, while rank is the $\ell_0$ norm of the same.
Figure 1: The adjacency matrix of a graph drawn from the planted partition model before and after proper reordering (i.e. clustering) of the nodes. The figure on the right is indicative of the matrix as a superposition of a sparse matrix and a low-rank one.

the (natural) combinatorial objective $\lambda \|B\|_0 + \text{rank}(K)$. The optimization problem (1) is, in fact, a semi-definite program (SDP) (Chandrasekaran et al., 2011).

We remark on the above formulation. (a) This formulation does not require specifying the number of clusters; this parameter is effectively learned from the data. The tradeoff parameter $\lambda$ is artificial and can be easily determined: since $K^*$ has trace exactly equal to $n$, we simply choose the smallest $\lambda$ such that the trace of the optimal solution is at least $n$. This can be done by, e.g., bisection, which is described below. (b) It is possible to obtain tighter convex relaxations by adding more constraints, such as the diagonal entries $k_{ii} = 1$, $K$ being positive semi-definite, or even the triangular inequalities $K_{ij} + K_{jk} - K_{ik} \leq 1$. Indeed, this is done by Mathieu and Schudy (2010). We choose not to take this approach for the following reasons. First, and most importantly, even with the extra constraints, Mathieu and Schudy (2010) do not deliver better guarantees (cf. Table 1). In fact, we show in Section 2.3 that our results are near optimal in some important cases, so tighter relaxations do not seem to provide additional benefits; there we elaborate on this point. Second, our formulation can be solved efficiently using Augmented Lagrangian Multiplier methods (Lin et al., 2009), which is due to the fact that the proximal function of the nuclear norm can be easily computed. This is no longer the case with the extra semi-definite and linear constraints, and solving it as a standard SDP is only applicable to small graphs.

Definition: Validity: The convex program (1) is said to produce a valid output if the low-rank matrix part $K$ of the optimum corresponds to a graph of disjoint cliques; i.e. its rows and columns can be re-ordered to yield a block-diagonal matrix with all-one matrices for each block.

Validity of a given $K$ can easily be checked via elementary re-ordering operations\footnote{If we re-order a valid $K$ such that identical rows and columns appear together, it will become block-diagonal.}. Our first simple, but crucial, insight is that whenever the convex program (1) yields a valid solution, it is the disagreement
minimizer.

Theorem 1. For any \( \lambda > 0 \), if the optimum of (1) is valid, then it is the clustering that minimizes the number of observed disagreements.

Algorithm: Our method is given as Algorithm 1. It takes the adjacency matrix of the network \( A \) and outputs either the optimal clustering or declares failure.

### Algorithm 1 Optimal-Cluster(\( A \))

\[
\lambda \leftarrow \frac{1}{32 \sqrt{\bar{p}_0 n}} \\
\text{while not terminated do} \\
\quad \text{Solve (1)} \\
\quad \text{if solution } K \text{ is valid then} \\
\quad \quad \text{Output the clustering w.r.t } K \text{ and EXIT.} \\
\quad \text{else if } \text{trace}(K) > n \text{ then} \\
\quad \quad \lambda \leftarrow \lambda / 2 \\
\quad \text{else if } \text{trace}(K) < n \text{ then} \\
\quad \quad \lambda \leftarrow 2 \lambda \\
\quad \text{end if} \\
\text{end while} \\
\text{Declare Failure.}
\]

To solve the optimization problem (1), we recommend using the fast implementation developed in (Lin et al., 2009), which is tailored for matrix splitting and takes advantage of the sparsity of the observations. Setting the parameter \( \lambda \) is done via binary search, since any valid clustering matrix \( K \) is positive semidefinite and has \( \text{trace}(K) = \|K\|_* = n \). The initial value of \( \lambda \) is not crucial; we use \( \lambda = \frac{1}{32 \sqrt{\bar{p}_0 n}} \) based on our theoretical analysis in the next sub-section, where \( \bar{p}_0 \) is the empirical fraction of observed pairs. Whenever the algorithm results in a valid \( K \), we have found the optimal clustering (by Theorem 1).

2.2 Performance Analysis

For the main analytical contribution of this paper, we provide conditions under which the above algorithm will find the clustering that minimizes the number of disagreements among the observed entries. In particular, we characterize its performance under the standard and classical planted partition/stochastic block model with partial observations, which we now describe.

Planted Partition Model with Partial Observations: Suppose \( n \) nodes are partitioned into \( r \) clusters, each of size at least \( K_{\min} \). Let \( K^* \) be the low-rank matrix corresponding to this clustering (as described above). The adjacency matrix \( A \) of the graph is generated as follows: for each pair of nodes \((i, j)\) in the same cluster, \( a_{ij} = ? \) with probability \( 1 - p_0 \), \( a_{ij} = 1 \) with probability \( p_0 q \), or \( a_{ij} = 0 \) otherwise, independent of all others; similarly, for \((i, j)\) in different clusters, \( a_{ij} = ? \) with probability \( 1 - p_0 \), \( a_{ij} = 1 \) with probability \( p_0 q \), or \( a_{ij} = 0 \) otherwise.

Under the above model, the graph is observed at locations chosen at random with probability \( p_0 \), and in expectation a fraction of \( 1 - p \) (\( q \)) of these observations are in-cluster (across-cluster, respectively) disagreements. Let \( B^* = P_{\Omega_{obs}}(A + I - K^*) \) be the matrix of observed disagreements for the original
clustering. Note that the support of $B^*$ is contained in $\Omega_{obs}$. The following theorem says when our algorithm recovers the original clustering $(K^*, B^*)$ with high probability. Combined with Theorem 1, it also shows that with high probability the original clustering is disagreement minimizing.

**Theorem 2.** Let $\tau = \max\{1 - p, q\}$. For any constant $c > 0$, there exist a constant $C$ such that, with probability at least $1 - cn^{-10}$, the original clustering $(K^*, B^*)$ is the unique optimal solution of (7) with $\lambda = \frac{1}{32\sqrt{np_0}}$ provided that

$$p_0(1 - 2\tau)^2 \geq C\frac{n\log^2 n}{K_{\min}^2}$$

The theorem gives the condition that needs to be satisfied for recovery in terms of the three parameters that define problem: the minimum cluster size $K_{\min}$, the density gap $p - q \geq 1 - 2\tau$, and the observation probability $p_0$. We remark on these parameters.

**Minimum cluster size $K_{\min}$:** Since the left hand side of the condition in Theorem 2 is no more than 1, it imposes a lower-bound $K_{\min} = \tilde{\Omega}(\sqrt{n})$ on the cluster sizes. This means that our method can handle a growing number ($\tilde{\Omega}(\sqrt{n})$) of clusters. The lower-bound is attained when $1 - 2\tau$ and $p_0$ are both $\Theta(1)$, i.e., not decreasing as $n$ grows. Note that all relevant works require a lower-bound at least as strong as ours (cf. Table 1).

**Density gap $1 - 2\tau$:** When $p_0 = \Theta(1)$, our result allows this gap to be vanishingly small, i.e., $\tilde{\Omega}\left(\frac{\sqrt{n}}{K_{\min}}\right)$, with larger $K_{\min}$ allowing smaller gap. As we mentioned before, this matches the best available results (cf. Table 1), including those in (Mathieu and Schudy, 2010) and (Oymak and Hassibi, 2011), which use tighter convex relaxations that are more computationally demanding. We note that applying existing results in the low-rank-plus-sparse literature (Candes et al., 2009; Li, 2012) leads to weaker results, where the gap is required to be $\Theta(1)$.

**Observation probability $p_0$:** When $1 - 2\tau = \Theta(1)$, our result only requires a vanishing fraction of observations, i.e., $p_0$ can be as small as $\tilde{\Theta}\left(\frac{n}{K_{\min}^2}\right)$; larger $K_{\min}$ allows smaller $p_0$. As mentioned in the related work section, this scaling is better than existing results we know of. Moreover, our result provides an explicit trade-off between the observation probability $p_0$ and the density gap $(1 - 2\tau)$, which is new. In particular, $p_0$ can go to zero quadratically faster than $1 - 2\tau$. Consequently, treating missing observations as disagreements would lead to quadratically weaker results. This agrees with the intuition that handling missing entries with known locations is easier than correcting disagreements whose locations are unknown.

Finally, we would like to point out that our algorithm has the capability to handle outliers. Suppose there are some isolated nodes which do not belong to any cluster, and they connect to each node in the clusters and to each other with probability at most $\tau$, with $\tau$ obeying the condition in Theorem 2. Our algorithm will classify all these edges as disagreements - and hence automatically reveal the identity of each outlier. In the output of our algorithm, the low rank part $K$ will have all zeros in the columns and rows corresponding to outliers - all their edges will appear in the disagreement matrix $B$. 

8
2.3 Lower Bounds

In this subsection, we discuss the tightness of Theorem 2. Consider first the case where \( K_{\text{min}} = \Theta(n) \), i.e., there are a constant number of clusters. We establish fundamental lower bounds on the density gap \( 1 - 2\tau \) and the observation probability \( p_0 \) that are required for any algorithm to correctly recover the clusters.

**Theorem 3.** Suppose all clusters have equal size \( K = \Theta(n) \), and \( \tau = \Theta(1) \). Under the planted partition model with partial observations, for any algorithm to correctly identify the clusters with probability at least \( \frac{3}{4} \), we need

\[
p_0(1 - 2\tau)^2 \geq C \frac{1}{n},
\]

where \( C > 0 \) is an absolute constant.

This theorem applies to any algorithm regardless of its computational complexity. It shows that, when \( K_{\text{min}} = \Theta(n) \), the requirement for \( 1 - 2\tau \) and \( p_0 \) in Theorem 2 is optimal up to logarithmic factors, and in particular cannot be significantly improved by using more complicated methods. Moreover, to the best of our knowledge this is the first converse result that characterizes the fundamental tradeoff between \( p_0 \) and \( 1 - 2\tau \).

For the general case \( K_{\text{min}} = O(n) \), only part of the picture is known. Using non-rigorous arguments, Decelle et al. (2011) show that \( 1 - 2\tau \gtrsim \sqrt{\frac{n}{K_{\text{min}}}} \) is necessary when \( \tau = \Theta(1) \) and the graph is fully observed; otherwise recovery is impossible or exponentially hard. According to this lower-bound, our requirement on the density gap \( 1 - 2\tau \) is probably tight (up to log factors) for all \( K_{\text{min}} \). However, a rigorous proof of this claim is still lacking, and seems to be a difficult problem. Similarly, the tightness of our condition on \( p_0 \) and the tradeoff between \( p_0 \) and \( \tau \) is also unclear in this regime.

3 Proofs

3.1 Proof of Theorem 1

In this section, we prove Theorem 1; in particular, that if the optimization problem (1) produces a valid low-rank matrix, i.e. one that corresponds to a clustering of the nodes, then this is the disagreement minimizing clustering. Consider the following non-convex optimization problem

\[
\begin{align*}
\min_{B,K} & \quad \lambda \|B\|_1 + \|K\|_* \\
\text{s.t.} & \quad P_{\Omega_{\text{obs}}}(B + K) = P_{\Omega_{\text{obs}}}(I + A) \\
& \quad K \text{ is valid}
\end{align*}
\]

and let \((B, K)\) be any feasible solution. Since \( K \) represents a valid clustering, it is positive semidefinite and has all ones along its diagonal. Therefore, it obeys \( \|K\|_* = \text{trace}(K) = n \). On the other hand, because both \( K - I \) and \( A \) are adjacency matrices, the entries of \( B = I + A - K \) in \( \Omega_{\text{obs}} \) must be equal to \(-1, 1 \) or \( 0 \) (i.e. it is a disagreement matrix). Clearly any optimal \( B \) must have zero at the entries in \( \Omega_{\text{obs}}^c \). Hence \( \|B\|_1 = \|B\|_0 \) when \( K \) is valid. We thus conclude that the above optimization problem is
equivalent to minimizing $\|\mathbf{B}\|_0$ subject to the constraints in (2). This is exactly the minimization of the number of disagreements on the observed edges. Now notice that (1) is a relaxed version of (2). Therefore, if the optimum of (1) is valid and thus feasible to (2), then it is also optimal to (2), the disagreement minimization problem.

3.2 Proof of Theorem 2

3.2.1 Proof Outline and Preliminaries

We now overview the main steps in the proof of Theorem 2; the following sub-sections provide details. Recall that we would like to show that $K^*$ and $B^*$ corresponding to the true clustering is the unique optimum of our convex program (1). This involves the following steps:

**Step 1:** Show that it suffices to consider an equivalent model for the observation and disagreements. This model is easier to handle, especially when the observation probability and density gap are vanishing, which is the case considered in this paper.

**Step 2:** Write down sub-gradient based first-order sufficient conditions that need to be satisfied for $K^*$, $B^*$ to be the unique optimum of (1). In our case, this involves showing the existence of a matrix $W$ – the dual certificate – that satisfies certain properties. This step is technically involved – requiring us to delve into the intricacies of sub-gradients since our convex function is not smooth – but otherwise standard. Luckily for us, this has been done by (Chandrasekaran et al., 2011; Candes et al., 2009; Li, 2012).

**Step 3:** Using the assumptions made on the true clustering and its disagreements ($K^*$, $B^*$), construct a candidate dual certificate $W$ that meets the requirements – and thus certifies $K^*$, $B^*$ as being the unique optimum.

The crucial third step is where we go beyond the existing literature on matrix splitting (Chandrasekaran et al., 2011; Candes et al., 2009; Li, 2012). These results assume the observation probability and/or density gap is at least a constant, and hence do not apply to our setting. Here we provide a refined analysis, which leads to much more powerful performance guarantees than those that could be obtained via a direct application of existing sparse and low-rank matrix splitting results.

Next, we introduce some notations used in the rest of the proof of the theorem.

**Definitions related to $K^*$:** By symmetry, the SVD of $K^*$ is of the form $U\Sigma U^T$. We define the sub-space $\mathcal{T} = \{UX^T + YU^T : X, Y \in \mathbb{R}^{n \times p}\}$ to be the span of all matrices that share either the same column space or the same row space as $K^*$. For any matrix $M \in \mathbb{R}^{n \times n}$, we can define its orthogonal projection to the space $\mathcal{T}$ as $P_\mathcal{T}(M) = UU^TM + MUU^T - UU^TMUU^T$. We also define the projection onto $\mathcal{T}^\perp$, the complement orthogonal space of $\mathcal{T}$, as $P_{\mathcal{T}^\perp}(M) = M - P_\mathcal{T}(M)$.

**Definitions related to $B^*$ and partial observation:** Let $\Omega^* = \{(i, j) : B^* \neq 0\}$ be the set of matrix entries corresponding to the disagreements. Recall that $\Omega_{\text{obs}}$ is the set of observed entries. For any matrix $N$ and entry set $\Omega_0$, we let $P_{\Omega_0}(N) \in \mathbb{R}^{n \times n}$ be the matrix obtained from $N$ by setting all entries not in the set $\Omega_0$ to zero. We write $\Omega_0 \sim \text{Ber}_0(p)$ if the entry set $\Omega_0$ does not contain the diagonal entries, and each pair $(i, j)$ and $(j, i)$ ($i \neq j$) is contained in $\Omega_0$ with probability $p$, independent all others; $\Omega_0 \sim \text{Ber}_1(p)$ is defined similarly except that $\Omega_0$ contains all the diagonal entries. Our assumption implies
\( \Omega_{\text{obs}} \sim \text{Ber}_1(p_0) \) and \( \Omega^* \sim \text{Ber}_0(\tau) \).

**Norms:** \( \|M\| \) and \( \|M\|_F \) represent the spectral and Frobenius norm of the matrix \( M \) respectively and \( \|M\|_\infty = \max_{i,j} |m_{i,j}| \).

### 3.2.2 Step 1: Equivalent Model for Observation and Disagreements

Notice that the probability of success is completely determined by the distribution of \((\Omega_{\text{obs}}, B^*)\). The first step is to show that it suffices to consider an equivalent model for generating \((\Omega_{\text{obs}}, B^*)\), which results in the same distribution but is easier to handle. This is in the same spirit as (Candes et al., 2009) (Theorem 2.2 and 2.3 therein) and (Li, 2012) (Section 4.1 therein). In particular, we consider the following procedure:

1. Let \( \Gamma \sim \text{Ber}_1(p_0(1 - 2\tau)) \), and \( \Omega \sim \text{Ber}_0(\frac{2p_0\tau}{1-p_0+2p_0\tau}) \). Let \( \Omega_{\text{obs}} = \Gamma \cup \Omega \).

2. Let \( S \) be a symmetric random matrix whose upper-triangular entries are independent and satisfy \( P(s_{i,j} = 1) = P(s_{i,j} = -1) = \frac{1}{2} \).

3. Define \( \Omega' \subseteq \Omega \) as \( \Omega' = \{(i,j) : (i,j) \in \Omega, s_{i,j} = 1 - 2k_{i,j}^* \} \). In other words, \( \Omega' \) is the entries of \( S \) whose signs are consistent with a disagreement matrix.

4. Define \( \Omega^* = \Omega' / \Gamma \), and \( \tilde{\Gamma} = O / \Omega^* \).

5. Let \( B^* = P_{\tilde{\Omega}^*}(S) \).

It is easy to verify that \((\Omega_{\text{obs}}, B^*)\) has the same distribution as the original model. In particular, we have \( P((i,j) \in \Omega_{\text{obs}}) = p_0 \), \( P((i,j) \in \Omega^*, (i,j) \in \Omega_{\text{obs}}) = p_0\tau \) and \( P((i,j) \in \Omega^*, (i,j) \notin \Omega_{\text{obs}}) = 0 \), and observe that \( B^* \) is completely determined by its support \( \Omega^* \).

The advantage of the above model is that \( \Gamma \) and \( \Omega \) are independent of each other, and \( S \) has random signed entries. This facilitates the construction of the dual certificate, especially in the regime of vanishing \( \rho \) and \( (\frac{1}{2} - \tau) \) considered in this paper. We use this equivalent model in the rest of the proof.

### 3.2.3 Step 2: Sufficient Conditions for Optimality of \((K^*, B^*)\)

We state the first-order conditions that guarantee \( K^* \) and \( B^* \) to be the unique optimum of (1) with high probability. Here and henceforth, by *with high probability* we mean with probability at least \( 1 - cn^{-10} \) for some constant \( c > 0 \). The following lemma follows from Theorem 4.4 in Li (2012) and the discussion thereafter.

**Lemma 1 (Probabilistic Sufficient Optimality).** Suppose \( \frac{1}{(1-2\tau)p_0}P_{\mathcal{T}}P_{\mathcal{T}^\top}P_{\mathcal{T}} - P_{\mathcal{T}} \| \leq \frac{1}{2} \). Then \( K^* \) and \( B^* \) are unique solutions to (1) with high probability provided that there exists \( W \in \mathbb{R}^{n \times n} \) such that

1. \( \|P_{\mathcal{T}}(W + \lambda P_{\mathcal{G}}S - UU^\top)\|_F \leq \frac{\lambda}{\tau^2} \)
2. \( \|P_{\mathcal{T}^\perp}(W + \lambda P_{\mathcal{G}}S)\| \leq \frac{1}{4} \)
3. $\mathcal{P}_T(W) = 0$

4. $\|\mathcal{P}_T(W)\|_\infty \leq \frac{\lambda}{4}$

Lemma 3 in the appendix guarantees that the condition $\|\frac{1}{1-2\tau} p_0 \mathcal{P}_T \mathcal{P}_U \mathcal{P}_T - \mathcal{P}_T\| \leq \frac{1}{2}$ is satisfied with high probability under the assumption of Theorem 2. Thus it remains to show the existence of the dual certificate $W$.

### 3.2.4 Step 3: Dual Certificate construction

We use a variant of the so-called Golfing Scheme (Candes et al., 2009; Gross, 2009) to construct $W$. Our application of Golfing Scheme, as well as its analysis, is different from (Candes et al., 2009) and leads to stronger guarantees. In particular, we go beyond existing results by allowing the fraction of observed entries and the density gap to be vanishing.

By definition, $\Gamma$ obeys $\Gamma \sim \text{Ber}(p_0(1 - 2\tau))$. Observe that $\Gamma$ may be considered to be generated by $\Gamma = \bigcup_{1 \leq k \leq k_0} \Gamma_k$, where the sets $\Gamma_k \sim \text{Ber}(q)$ are independent; here the parameter $q$ obeys $p_0(1 - 2\tau) = 1 - (1 - q)^{k_0}$, and $k_0$ is chosen to be $\lceil 5\log n \rceil$. This implies $q \geq p_0(1 - 2\tau)/k_0 \geq C_0 n^{\log n} K_{\min}$ for some constant $C_0$, with the last inequality following from the assumption of Theorem 2. For any random entry set $\Omega_0 \sim \text{Ber}(p)$, define the operator $R_{\Omega_0}$ by

$$
R_{\Omega_0}(M) = \sum_{i=1}^{n} m_{i,i} e_i e_i^T + p^{-1} \sum_{1 \leq i < j \leq n} \delta_{ij} m_{i,j} (e_i e_i^T + e_j e_j^T),
$$

where $\delta_{ij} = 1$ if $(i, j) \in \Omega_0$ and 0 otherwise, and $e_i$ is the $i$-th standard basis – i.e., the $n \times 1$ column vector with 1 in its $i$-th entry and 0 elsewhere.

We now define our dual certificate. Let $W := W_{k_0}$, where $W_{k_0}$ is defined recursively by setting $W_0 = 0$ and for all $k = 1, 2, \ldots, k_0$,

$$
W_k = W_{k-1} + R_{\Gamma_k} \mathcal{P}_T (UU^T - \lambda \mathcal{P}_T (\mathcal{P}_U(S)) - W_{k-1}).
$$

Clearly the equality condition in Lemma 1 is satisfied. It remains to show that $W$ also satisfies the inequality conditions with high probability. The proof makes use of the auxiliary lemmas given in the appendix. For convenience of notation, we define the quantity $\Delta_k = UU^T - \lambda \mathcal{P}_T (\mathcal{P}_U(S)) - \mathcal{P}_T(W_k)$, and write

$$
\prod_{i=1}^{k} (\mathcal{P}_T - \mathcal{P}_T R_{\Gamma_i} \mathcal{P}_T) = (\mathcal{P}_T - \mathcal{P}_T R_{\Gamma_k} \mathcal{P}_T) \cdots (\mathcal{P}_T - \mathcal{P}_T R_{\Gamma_1} \mathcal{P}_T),
$$

where the order of multiplication is important. Observe that by construction of $W$, we have

$$
\Delta_k = \prod_{i=1}^{k} (\mathcal{P}_T - \mathcal{P}_T R_{\Gamma_i} \mathcal{P}_T) (UU^T - \lambda \mathcal{P}_T (\mathcal{P}_U(S))), \forall k = 1, \ldots, k_0, \quad (3)
$$

$$
W_{k_0} = \sum_{k=1}^{k_0} R_{\Gamma_k} \Delta_{k-1}. \quad (4)
$$

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Inequality 1: Bounding $\| \mathcal{P}_T (W + \lambda \mathcal{P}_\Omega S - UU^T) \|_F$.

We have the following geometric convergence thanks to (3).

$$
\| \mathcal{P}_T (W + \lambda \mathcal{P}_\Omega S - UU^T) \|_F = \| \Delta_{k_0} \|_F \\
\leq \left( \prod_{k=1}^{k_0} \| \mathcal{P}_T - \mathcal{P}_T \mathcal{R}_{\Gamma_k} \mathcal{P}_T \| \right) \| UU^T - \lambda \mathcal{P}_T \mathcal{P}_\Omega (S) \|_F
\leq e^{-k_0} (\| UU^T \|_F + \| \mathcal{P}_T \mathcal{P}_\Omega (S) \|_F) \leq n^{-5} (n + \lambda \cdot n)
\leq (1 + \lambda)n^{-4}.
$$

Here, (a) uses Lemma 3 with $\epsilon_1 = e^{-1}$, (b) uses our choices of $\lambda$ and $k_0$ and the fact that $\| \mathcal{P}_T \mathcal{P}_\Omega (S) \|_F \leq \| \mathcal{P}_\Omega (S) \|_F \leq n$, and (c) uses $\lambda \geq \frac{1}{32\sqrt{n}}$.

Inequality 4: Bounding $\| \mathcal{P}_\Gamma (W) \|_\infty$.

We have

$$
\| \mathcal{P}_\Gamma (W) \|_\infty = \| \mathcal{P}_\Gamma (W_{k_0}) \|_\infty \\
\leq \sum_{k=1}^{k_0} \| \mathcal{R}_{\Gamma_k} \Delta_{i-1} \|_\infty \leq q^{-1} \sum_{k=1}^{k_0} \| \Delta_{k-1} \|_\infty
$$

Here, (a) uses (4). We proceed as follows

$$
\sum_{k=1}^{k_0} \| \Delta_{k-1} \|_\infty \\
\leq \sum_{k=1}^{k_0} \prod_{i=1}^{k-1} \| \mathcal{P}_T - \mathcal{P}_T \mathcal{R}_{\Gamma_i} \mathcal{P}_T \| (UU^T - \lambda \mathcal{P}_T \mathcal{P}_\Omega (S)) \|_\infty
\leq \sum_{k=1}^{k_0} \left( \frac{1}{2} \right)^k \| UU^T - \lambda \mathcal{P}_T \mathcal{P}_\Omega (S) \|_\infty
\leq \frac{1}{K_{\text{min}}} + \lambda \sqrt{\frac{p_0 n \log n}{K_{\text{min}}^2}},
$$

where (b) uses (3), (c) uses Lemma 5 and (d) uses Lemma 6. It follows that

$$
\| \mathcal{P}_\Gamma (W) \|_\infty \\
\leq \frac{1}{q} \left( \frac{1}{K_{\text{min}}} + \frac{n \log n}{K_{\text{min}}^2} \lambda \right)
\leq \frac{k_0}{p_0 (1 - 2\tau)} \left( \frac{1}{K_{\text{min}}} + \lambda \sqrt{\frac{p_0 n \log n}{K_{\text{min}}^2}} \right)
\leq \frac{1}{4} \lambda,
$$
where the last inequality holds under the assumption of Theorem 2.

**Inequality 2: Bounding** $\|P_{T^\perp}(W + \lambda P_{\Omega}(S))\|$.

Observe that by triangle inequality, we have

$$\|P_{T^\perp}(W + \lambda P_{\Omega}(S))\| \leq \lambda \|P_{T^\perp}(P_{\Omega}(S))\| + \|P_{T^\perp}(W_{k_0})\|.$$  

For the first term, a standard argument about the norm of a matrix with i.i.d. entries (e.g., see Vershynin (2007)) gives

$$\lambda \|P_{T^\perp}(P_{\Omega}(S))\| \leq \lambda \|P_{\Omega}(S)\| \leq \frac{1}{32 \sqrt{p_0 n}} \cdot 4 \sqrt{\frac{2 p_0 \tau n}{1 - p_0 + 2 p_0 \tau}} \leq \frac{1}{8}$$

It remains to show that the second term is bounded by $\frac{1}{8}$. To this end, we observe that

$$\|P_{T^\perp}(W_{k_0})\| \leq \sum_{k=1}^{k_0} \|P_{T^\perp}(R_{\Gamma_k} \Delta_{k-1} - I)\| \leq \sum_{k=1}^{k_0} \|(R_{\Gamma_k} - I) \Delta_{k-1}\| \leq C \sqrt{\frac{n \log n}{q}} \sum_{k=1}^{k_0} \|\Delta_{k-1}\|_\infty \leq C \sqrt{\frac{k_0 n \log n}{p_0 (1 - 2 \tau)}} \left( \frac{1}{K_{\min}} + \lambda \sqrt{\frac{p_0 n \log n}{K_{\min}^2}} \right) \leq \frac{1}{8},$$

where (a) uses (4) and the fact that $\Delta_k \in T$, and (b) uses Lemma 4.

This completes the proof of Theorem 2.

### 4 Experimental Results

We explore the performance of our algorithm as a function of various graph parameters ($n, K, p_0, \tau$) via simulation. We see that the performance matches well with the theory.

In the experiment, each test case is constructed by generating a graph with $n$ nodes divided into clusters of equal size $K_{\min}$, and then placing a disagreement on each pair of node with probability $\tau$, independent of all others. Each node pair is observed with probability $p_0$. The optimization problem (1) is solved using the fast algorithm in (Lin et al., 2009) with $\lambda$ set via binary search described in Algorithm 1. We check if the algorithm successfully outputs a solution that equals to the underlying true clusters. In the first set of experiments, we fix $\tau = 0.2$ and $K_{\min} = n/4$ and vary $(p_0, n)$. For each $(p_0, n)$, we repeat the experiment for 5 times and plot the probability of success in the left pane of Fig. 2.
Figure 2: Simulation results verifying the performance of our algorithm as a function of the observation probability \( p_0 \) and the graph size \( n \). The left pane shows the probability of successful recovery under different \( p_0 \) and \( n \) with fixed \( \tau = 0.2 \) and \( K_{\min} = n/4 \); each point is an average over 5 trials. After proper rescaling of the x-axis, the curves align as shown in the right pane, indicating a good match with the theory.

One observes that our algorithm has better performance with larger \( p_0 \) and \( n \), and the success probability exhibits a phase transition. Theorem 2 predicts that, with \( \tau \) fixed and \( K_{\min} = n/4 \), the transition occurs at \( p_0 \propto \frac{n \log^2 n}{K_{\min}^2} \propto \frac{\log^2 n}{n} \); in particular, if we plot the success probability versus the control parameter \( \frac{p_0 n}{\log n} \), all curves should align with each other. Indeed, this is precisely what we see if we use \( \frac{p_0 n}{\log n} \) as the control parameter (right pane of Fig. 2). This shows that Theorem 2 gives the correct scaling between \( p_0 \) and \( n \) up to an extra log factor.

In a similar fashion, we run another three sets of experiments with the following settings: (1) \( n = 1000 \) and \( \tau = 0.2 \) with varying \( (p_0, K_{\min}) \); (2) \( K_{\min} = n/4 \) and \( p_0 = 0.2 \) with varying \( (\tau, n) \); (3) \( n = 1000 \) and \( p_0 = 0.6 \) with varying \( (\tau, K_{\min}) \). The results are shown in Fig. 3, 4, and 5; note that each x-axis represents a control parameter chosen according to the scaling predicted by Theorem 2. Again we observe that all the curves roughly align, indicating a good match with the theory. In particular, by comparing Fig. 2 and 4 (or Fig. 3 and 5) one verifies the quadratic tradeoff between observations and disagreements (\( p_0 \) vs. \( 1 - 2\tau \)) as predicted by Theorem 2.
Finally, we compare the performance of our method with spectral clustering, a popular method for graph clustering. For spectral clustering, we first impute the missing entries of the adjacency matrix with either zeros or random 1/0’s. We then compute the largest $k$ principal components of the adjacency matrix, and run $k$-means clustering on the principal components (Von Luxburg, 2007); here we set $k$ equal to the correct number of clusters. The adjacency matrix is generated in a similar fashion as before using the following parameters: $n = 2000$, $K_{\text{min}} = 200$, and $\tau = 0.1$. We vary the observation probability $p_0$ and plot the success probability (Fig. 6). It can be observed that our method outperforms spectral clustering with both imputation schemes; in particular, it requires fewer observations.

5 Conclusion

We proposed a convex optimization formulation, based on a reduction to decomposing low-rank and sparse matrices, to address the problem of clustering partially observed graphs. We showed that under a wide range of parameters of the classical planted partition model, our method guarantees to find the optimal (disagreement-minimizing) clustering. In particular, our method succeeds under higher levels of noise,
Figure 6: Comparison of our method and spectral clustering for partially observed graphs under different observation probability $p_0$. For spectral clustering, two imputation schemes are considered: Spectral (Zero), where the missing entries are imputed with zeros, and Spectral (Rand), where they are imputed with 0/1 random variables with symmetric probabilities. The result shows that our method recovers the underlying clusters with fewer observations.

and/or missing observations than existing methods in this setting. The effectiveness of the proposed method and the scaling of the theoretical results are validated by simulation studies.

This work is motivated by graph clustering applications where obtaining similarity data is expensive and it is desirable to use as few observations as possible. As such, potential directions for future work include considering different sampling schemes such as active sampling, as well as dealing with sparse graphs with very few connections.

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Appendices

A Auxiliary Lemmas

In this section, we provide several auxiliary lemmas required in the proof of Theorem 2. We will make use of the non-commutative Bernstein inequality. We use a version given in (Tropp, 2010).

Lemma 2. [Tropp (2010)] Consider a finite sequence \( \{M_i\} \) of independent, random \( n_1 \times n_2 \) matrices that satisfy the assumption \( E M_i = 0 \) and \( \|M_i\| \leq D \) almost surely. Let

\[
\sigma^2 = \max \left\{ \left\| \sum_i E [M_i M_i^\top] \right\|, \left\| \sum_i E [M_i^\top M_i] \right\| \right\}.
\]

Then for all \( t > 0 \) we have

\[
P \left[ \left\| \sum M_i \right\| \geq t \right] \leq (n_1 + n_2) \exp \left( -\frac{t^2}{2\sigma^2 + 2Dt/3} \right).
\]

\[
\leq \begin{cases}  
(n_1 + n_2) \exp \left( -\frac{3t^2}{8\sigma^2} \right), & \text{for } t \leq \frac{\sigma^2}{D}; \\
(n_1 + n_2) \exp \left( -\frac{3t}{8\sigma} \right), & \text{for } t \geq \frac{\sigma^2}{D}.
\end{cases} \tag{7}
\]

Remark 1. When \( n_1 = n_2 = 1 \), this becomes the standard two-sided Bernstein inequality.

We will also make use of the following estimate, which follows from the structure of \( U \).

\[
\left\| P_T(e_i e_j^\top) \right\|_F^2 = \|UU^T e_i\|^2 + \|UU^T e_j\|^2 - \|UU^T e_i\|^2 \|UU^T e_j\|^2 \leq \frac{2n}{K_{\min}^2}, \quad \forall 1 \leq i, j \leq n
\]
The first auxiliary lemma is similar to Theorem 4.1 in Candes et al. (2009), but applies to the symmetric matrix case. Our proof is different from Candes et al. (2009).

**Lemma 3.** Suppose $\Omega_0$ is a set of entries obeying $\Omega_0 \sim \text{Ber}_1(p)$. Consider the operator $P_T - P_T R_{\Omega_0} P_T$. For some constant $C_0 > 0$, we have

$$\|P_T - P_T R_{\Omega_0} P_T\| < \epsilon_1$$

with high probability provided that $p \geq C_0 \frac{n \log n}{\epsilon_1 K_{\min}}$ and $\epsilon_1 \leq 1$.

**Proof.** For each $(i,j)$, define the indicator random variable $\delta_{ij} = 1_{\{(i,j) \in \Omega_0\}}$. We observe that for any matrix $M \in T$

$$(P_T R_{\Omega_0} P_T - P_T) M = \sum_{1 \leq i < j \leq n} (p^{-1} \delta_{ij} - 1) \left( P_T (e_i e_i^\top), \ M \right) P_T (e_j e_j^\top + e_j e_i^\top)$$

$$\triangleq \sum_{1 \leq i < j \leq n} S_{ij}(M).$$

Here $S_{ij} : \mathbb{R}^{n \times n} \mapsto \mathbb{R}^{n \times n}$ is a linear self-adjoint operator with $\mathbb{E}[S_{ij}] = 0$. We also have the bounds

$$\|S_{ij}\| \leq p^{-1} \left\| P_T (e_i e_i^\top) \right\| \left\| P_T (e_j e_j^\top + e_j e_i^\top) \right\| F$$

$$\leq p^{-1} \cdot 2 \left\| P_T (e_i e_j^\top) \right\| F \leq \frac{4n}{K_{\min}^2 p},$$

and

$$\left\| \mathbb{E} \left[ \sum_{1 \leq i < j \leq n} S_{ij}^2(M) \right] \right\| F$$

$$= \left\| \sum_{1 \leq i < j \leq n} \mathbb{E} \left[ (p^{-1} \delta_{ij}^{(k)} - 1)^2 \right] \left( P_T (e_i e_j^\top), \ M \right) \left( P_T (e_j e_j^\top + e_j e_i^\top), e_i e_j^\top \right) P_T (e_j e_j^\top + e_j e_i^\top) \right\| F$$

$$= (p^{-1} - 1) \left\| \sum_{1 \leq i < j \leq n} 2 \left\| P_T (e_i e_j^\top) \right\| F \ m_{i,j} P_T (e_j e_j^\top + e_j e_i^\top) \right\| F$$

$$\leq (p^{-1} - 1) \left\| \sum_{1 \leq i < j \leq n} 2 \left\| P_T (e_i e_j^\top) \right\| F \ m_{i,j} (e_i e_j^\top + e_j e_i^\top) \right\| F$$

$$\leq (p^{-1} - 1) \frac{4n}{K_{\min}^2} \left\| \sum_{1 \leq i < j \leq n} m_{i,j} (e_i e_j^\top + e_j e_i^\top) \right\| F$$

$$= (p^{-1} - 1) \frac{4n}{K_{\min}^2} \|M\|_F,$$
which means \[ \left\| \mathbb{E} \left[ \sum_{1 \leq i < j \leq n} S_{ij}^2 \right] \right\| \leq \frac{4n}{K_{\min^p}}; \] here we use the fact that \( \mathcal{P}_T(e_i e_j^\top) = (\mathcal{P}_T(e_j e_i^\top))^\top \) and \( M \) is symmetric. An application of the Bernstein inequality (first inequality of (7)) then yields
\[ P \left[ \left\| \sum_{1 \leq i < j \leq n} S_{ij} \right\| \geq \epsilon_1 \right] \leq 2n^{2-2\beta} \]

provided \( p \geq \frac{64\beta n \log n}{3K_{\min^p}^2} \) and \( \epsilon_1 < 1 \).

The next lemma is similar to Theorem 6.3 in (Candes and Recht, 2009) but applies to the symmetric matrix case. The proof is again different.

**Lemma 4.** Suppose \( \Omega_0 \) is a set of entries obeying \( \Omega_0 \sim \text{Ber}_1(p) \), and \( M \) is a fixed \( n \times n \) symmetric matrix. Then for some constant \( C_0 > 0 \), we have
\[ \left\| (I - R_{\Omega_0}) M \right\| < \sqrt{C_0 \frac{n \log n}{p} \| M \|_\infty}, \]

with high probability provided that \( p \geq C_0 \frac{\log n}{n} \).

**Proof.** Define \( \delta_{ij} \) as before. Notice that
\[ R_{\Omega_0}(M) - M = \sum_{i < j} (q^{-1} \delta_{ij} - 1)m_{i,j} \left( e_i e_j^\top + e_j e_i^\top \right) \triangleq \sum_{i < j} S_{ij}. \]

Here the symmetric matrix \( S_{ij} \in \mathbb{R}^{n \times n} \) satisfies \( \mathbb{E} [S_{ij}] = 0 \), \( \| S_{ij} \| \leq 2p^{-1} \| M \|_\infty \) and
\[ \left\| \mathbb{E} \left[ \sum_{i < j} S_{ij}^2 \right] \right\| = (p^{-1} - 1) \sum_{i < j} \left\| m_{i,j}^2 \left( e_i e_j^\top + e_j e_i^\top \right) \right\| \]
\[ \leq (p^{-1} - 1) \left\| \text{diag} \left( \sum_j m_{1,j}^2, \ldots, \sum_j m_{n,j}^2 \right) \right\| \]
\[ \leq (p^{-1} - 1) n \| M \|_\infty^2 \leq 2p^{-1} n \| M \|_\infty^2. \]

When \( p \geq \frac{16\beta \log n}{3n} \), we apply the Bernstein inequality (first inequality of (7)) and obtain
\[ P \left[ \left\| \sum_{i < j} S_{ij} \right\| \geq \sqrt{\frac{16\beta n \log n}{3p} \| M \|_\infty} \right] \leq 2n \exp \left( -\frac{3 \cdot 16\beta n \log n}{8 \cdot \frac{2n}{p} \| M \|_\infty^2} \right) \]
\[ \leq 2n^{1-\beta}. \]

The conclusion follows by choosing a sufficiently large \( \beta \).

The third lemma is similar to Lemma 3.1 in (Candes et al., 2009), but extended to the symmetric matrix case.
**Lemma 5.** Suppose $\Omega_0$ is a set of entries obeying $\Omega_0 \sim \text{Ber}(p)$, and $M \in T$ is a fixed symmetric $n \times n$ matrix. Then for some constant $C_0 > 0$, we have

$$\| (P_T - P_T R_{\Omega_0} P_T) M \|_\infty < \epsilon_3 \| M \|_\infty,$$

with high probability, provided that $p \geq C_0 \frac{n \log n}{\epsilon_3 K_{\min}^2}$ and $\epsilon_3 \leq 1$.

**Proof.** Define $\delta_{ij}$ as before. Fix an entry index $(a, b)$. Notice that

$$(P_T R_{\Omega_0} P_T M - P_T M)_{a, b} = \sum_{i<j} \left< \left( (p-1)\delta_{ij}^{(k)} - 1 \right) m_{i,j} P_T \left( e_i e_j^T + e_j e_i^T \right), e_a e_b^T \right>$$

where $E[\xi_{ij}] = 0$. We have the bounds

$$|\xi_{ij}| \leq 2p^{-1} \left\| P_T (e_i e_j^T) \right\|_F \left\| P_T (e_a e_b^T) \right\|_F |m_{i,j}|$$

$$\leq \frac{4n}{K_{\min}^2} \| M \|_\infty$$

and

$$E\left[ \sum_{i<j} \xi_{ij}^2 \right] = \sum_{i<j} E \left[ (p^{-1} \delta_{ij}^{(k)} - 1)^2 \right] m_{i,j}^2 \left\langle P_T \left( e_i e_j^T + e_j e_i^T \right), e_a e_b^T \right\rangle^2$$

$$\leq (p^{-1} - 1) \| M \|_\infty^2 \sum_{i<j} \left\langle e_i e_j^T + e_j e_i^T, P_T (e_a e_b^T) \right\rangle^2$$

$$\leq 2 (p^{-1} - 1) \| M \|_\infty^2 \left\| P_T (e_a e_b^T) \right\|_F^2$$

$$\leq 2 (p^{-1} - 1) \frac{2n}{K_{\min}^2} \| M \|_\infty^2$$

$$\leq \frac{4n}{K_{\min}^2} \| M \|_\infty^2.$$

When $p \geq \frac{64 \beta n \log n}{3K_{\min}^2 \epsilon_3^2}$ and $\epsilon_3 \leq 1$, we apply the standard Bernstein inequality (first inequality of (7)) and obtain

$$\mathbb{P} \left[ \left\| (P_T R_{\Omega_0} P_T M - P_T M)_{a, b} \right\| \geq \epsilon_3 \| M \|_\infty \right] \leq 2 \exp \left( - \frac{3 \epsilon_3^2 \| M \|_\infty^2}{8 \frac{4n}{K_{\min}^2} \| M \|_\infty^2} \right) \leq 2n^{-2\beta}.$$ 

Union bound then yields

$$\mathbb{P} \left[ \| P_T R_{\Omega_0} P_T M - P_T M \|_\infty \geq \epsilon_3 \| M \|_\infty \right] \leq 2n^{2-2\beta}.$$

The last lemma bounds the matrix infinity norm of $P_T P_\Omega(S)$. 

\[ \square \]
Lemma 6. Suppose \( \Omega \sim \text{Ber}_0\left(\frac{2p_0\tau}{1-p_0+2p_0\tau}\right) \) and \( \mathbf{S} \in \mathbb{R}^{n \times n} \) has i.i.d. symmetric \( \pm 1 \) entries. Under the assumption of Theorem 2 for some constant \( C_0 \), we have

\[
\|P_T P_\Omega(\mathbf{S})\|_\infty \leq C_0 \sqrt{\frac{p_0 n \log n}{K_{\min}^2}}
\]

with high probability.

Proof. By triangle inequality, we have

\[
\|P_T P_\Omega(\mathbf{S})\|_\infty \leq \|UU^T P_\Omega(\mathbf{S})\|_\infty + \|P_\Omega(\mathbf{S})UU^T\|_\infty + \|UU^T P_\Omega(\mathbf{S})UU^T\|_\infty,
\]

so it suffices to show that each of these three terms are bounded by \( C \sqrt{\frac{p_0 n \log n}{K_{\min}^2}} \) w.h.p. for some constant \( C \). Under our assumption on \( \Omega \) and \( \mathbf{S} \), each pair of symmetric entries of \( P_\Omega(\mathbf{S}) \) equals \( \pm 1 \) with probability \( p \triangleq \frac{p_0 \tau}{1-p_0+2p_0\tau} \) and 0 otherwise; notice that \( p \leq \frac{p_0}{2} \) since \( \tau \leq \frac{1}{2} \). Let \( (s^{(i)})^T \) be the \( i \)th row of \( \mathbf{U}U^T \). From the structure of \( \mathbf{U} \), we know

\[
|s^{(i)}_j| \leq \frac{1}{K_{\min}}, \quad \forall i, j.
\]

and

\[
\sum_{j=1}^n (s^{(i)}_j)^2 \leq \frac{1}{K_{\min}}, \quad \forall i.
\]

We now bound \( \|UU^T P_\Omega(\mathbf{S})\|_\infty \). For simplicity, we focus on the \((1,1)\) entry of \(UU^T P_\Omega(\mathbf{S})\) and denote this random variable as \( X \). Observe that \( X = \sum_is^{(1)}_i(P_\Omega(\mathbf{S}))_{i,1} \), and

\[
\mathbb{E}
\left[
\left|
\left|
\left(P_\Omega(\mathbf{S})\right)_{i,1}^{(1)}
\right|
\right|
\right]
= 0
\]

\[
\left|
\left|
\left(P_\Omega(\mathbf{S})\right)_{i,1}^{(1)}
\right|
\right|
\leq \mathbb{E}
\left[
\left|
\left|
\left(P_\Omega(\mathbf{S})\right)_{i,1}^{(1)}
\right|
\right|
\right] \leq \frac{1}{K_{\min}}, \quad \text{a.s.}
\]

\[
\text{Var}(X) = \sum_{i:(i,1) \in \Omega} (s^{(1)}_i)^2 |2p| \leq \frac{p_0}{K_{\min}}.
\]

Standard Bernstein inequality thus gives

\[
\mathbb{P}\left[
|X| > C \sqrt{\frac{p_0 n \log n}{K_{\min}^2}}\right] \leq 2 \exp\left[- \left( C^2 \frac{p_0 n \log n}{K_{\min}^2} \right) / \left( 2 \frac{p_0}{K_{\min}} + 2C \sqrt{\frac{p_0 n \log n}{K_{\min}^2}} \right) \right].
\]

Under the assumption of Theorem 2, the right hand side is bounded by \( 2n^{-12} \). It follows from a union bound that \( \|UU^T P_\Omega(\mathbf{S})\|_\infty \leq C \sqrt{\frac{p_0 n \log n}{K_{\min}^2}} \) w.h.p. Clearly, the same bound applies to \( \|P_\Omega(\mathbf{S})UU^T\|_\infty \).

Finally, let \( K \) be the size of the cluster that node \( j \) is in; observe that due to the structure of \( \mathbf{U}U^T \), we have

\[
(UU^T P_\Omega(\mathbf{S})UU^T)_{i,j} = \sum_i (UU^T P_\Omega(\mathbf{S}))_{i,i} (UU^T)_{i,j} \leq \frac{1}{K} \cdot K \cdot \|UU^T P_\Omega(\mathbf{S})\|_\infty,
\]

which implies \( \|UU^T P_\Omega(\mathbf{S})UU^T\|_\infty \leq \|UU^T P_\Omega(\mathbf{S})\|_\infty \). This completes the proof of the lemma. \( \square \)
B Proof of Theorem 3

Proof. We use a standard information theoretical argument, which improves upon the proof of Theorem 5 in (Chaudhuri et al., 2012). For simplicity we assume \( n/K \) is an integer. Let \( \mathcal{F} \) be the set of all possible partition of \( n \) nodes into \( n/K \) clusters of equal size \( K \). Using Stirling’s approximation, we have

\[
M \triangleq |\mathcal{F}| = \frac{1}{(n/K)!} \left( \frac{n}{K} \right) \left( \frac{n-K}{K} \right) \cdots \left( \frac{n}{3K} \right)^{n(1-K)} \geq c_1^n
\]

when \( K = \Theta(n) \).

Suppose the clustering \( Y \) is chosen uniformly at random from \( \mathcal{F} \), and the graph \( A \) is generated from \( Y \) according to the planted partition model with partial observations, where we use \( a_{ij} = ? \) for unobserved pairs. We use \( P_{A|Y} \) to denote the distribution of \( A \) given \( Y \). Let \( \hat{Y} \) be any measurable function of the observation \( A \). A standard application of Fano’s inequality (Yang and Barron, 1999) gives

\[
\sup_{Y \in \mathcal{F}} \mathbb{P} \left[ \hat{Y} \neq Y | Y \right] \geq 1 - \frac{\left( \frac{M(M-1)}{2} \right)^{-1} \sum_{Y(1) \neq Y(2) \in \mathcal{F}} D \left( P_{A|Y(1)} \| P_{A|Y(2)} \right) + \log 2}{\log M}, \tag{8}
\]

where \( D(\|) \) is the KL-divergence. We now upper bound this divergence. Given \( Y \), the \( a_{ij} \)'s are independent of each other, so we have

\[
D \left( P_{A|Y(1)} \| P_{A|Y(2)} \right) = \sum_{(i,j)} D \left( P_{a_{ij}|Y(1)} \| P_{a_{ij}|Y(2)} \right).
\]

For each pair \((i,j)\), the KL-divergence is zero if \( y_{ij}^{(1)} = y_{ij}^{(2)} \), and otherwise

\[
D \left( P_{a_{ij}|Y(1)} \| P_{a_{ij}|Y(2)} \right) \leq p_0(1-\tau) \log \frac{p_0(1-\tau)}{p_0\tau} + p_0\tau \log \frac{p_0\tau}{p_0(1-\tau)} + (1-p_0) \log \frac{1-p_0}{1-p_0} \\
= p_0(1-2\tau) \log \frac{1-\tau}{\tau} \\
\leq p_0(1-2\tau) \left( \frac{1-\tau}{\tau} - 1 \right) \\
\leq c_2 p_0(1-2\tau)^2,
\]

where \( c_2 > 0 \) is an absolute constant and the last inequality holds when \( \tau = \Theta(1) \). Let \( N \) be the number of pairs \((i,j)\) such that \( y_{ij}^{(1)} \neq y_{ij}^{(2)} \). When \( K = \Theta(n) \), we have

\[
N \leq |\{(i,j) : y_{ij}^{(1)} = 1\} \cup \{(i,j) : y_{ij}^{(2)} = 1\}| \leq n^2.
\]

It follows that \( D \left( P_{A|Y(1)} \| P_{A|Y(2)} \right) \leq N \cdot c_2 p_0(1-2\tau)^2 \leq c_2 n^2 p_0(1-2\tau)^2 \).

Combining pieces, for the left hand side of (8) to be less than \( 1/4 \), we need

\[
p_0(1-2\tau)^2 \geq C \frac{1}{n}.
\]

\qed

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