SUBADJUNCTION OF LOG CANONICAL DIVISORS
FOR A SUBVARIETY OF CODIMENSION 2

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Abstract. We obtain a formula which relates the log canonical divisor of the ambient space with that of a subvariety of codimension 2 by using Knudsen’s moduli space of pointed stable curves of genus 0.

Introduction

The adjunction formula relates canonical divisors of varieties. Let \( X \) be a smooth variety and \( S \) a smooth divisor. Then we have \((K_X + S)|_S \sim K_S\). But if \( X \) has singularities, then we need to consider the subadjunction as observed by M. Reid ([KMM, 5.1.9]). For example, if \( X \) is a singular conic in \( \mathbb{P}^3 \) and \( S \) one of its generators, then \((K_X + S)|_S \sim Q - \frac{3}{2}H\), while \( K_S \sim -2H \), where \( \sim Q \) stands for the \(\mathbb{Q}\)-linear equivalence and \( H \) is the hyperplane section. So it is more natural to consider the pair \((X, D)\) of the variety and a \(\mathbb{Q}\)-divisor, where \( D = S + \) other components with \( S \) a prime divisor, and compare \( K_X + D \) and \( K_S + D_S \) for some \(\mathbb{Q}\)-divisor \( D_S \) on \( S \).

We are also interested in the relationship between singularities of the pair \((X, D)\) and those of the restricted pair \((S, D_S)\). By using the residue map, we obtain a subadjunction theorem (cf. [K3, Proposition 1.7]): \((K_X + D)|_S \sim Q K_S + D_S\) for a canonically determined effective \(\mathbb{Q}\)-divisor \( D_S \) on \( S \), and if \((X, D)\) is LC (log canonical), then so is \((S, D_S)\).

If there are two smooth divisors \( S_1, S_2 \) on a smooth variety \( X \) with a transversal intersection \( W = S_1 \cap S_2 \), then by using the residue map twice, we obtain \((K_X + S_1 + S_2)|_W \sim K_W\). The purpose of this paper is to extend the subadjunction formula for a codimension 2 subvariety \( W \) even if there are no intermediate divisors \( S_j \), and answer [K3, Question 1.8] in the case of codimension 2 (we use the notation of [KMM] and [K3]):

**Theorem 1.** Let \( X \) be a normal variety with only KLT singularities, \( D \) an effective \(\mathbb{Q}\)-Cartier divisor such that \((X, D)\) is LC, and \( W \) a minimal element of \( CLC(X, D) \) (the set of centers of log canonical singularities). Assume that \( \text{codim} W = 2 \). Then there exist canonically determined effective \(\mathbb{Q}\)-divisors \( M_W \) and \( D_W \) on \( W \) such that \((K_X + D)|_W \sim Q K_W + M_W + D_W\). Moreover, if \( D = D' + D'' \) with \( D' \) (resp. \( D'' \)) the sum for irreducible components which contain (resp. do not contain) \( W \), then \( M_W \) is determined only by the pair \((X, D')\). If \( X \)

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is affine, then there exists an effective $\mathbb{Q}$-divisor $M'_W$ such that $M'_W \sim_{\mathbb{Q}} M_W$ and the pair $(W, M'_W + D_W)$ is KLT.

The divisor $D_W$ is the local contribution and appears by the same reason as in the case of divisorial subadjunction formula, while the divisor $M_W$ is the global contribution which comes from the moduli space of curves. The latter is a new ingredient in the higher codimensional subadjunction. We shall use Knudsen’s moduli space of pointed stable curves of genus 0 in order to define $M'_W$. The naturally determined $\mathbb{Q}$-divisor $M_W$ does not give us the KLT singularities, so we need to introduce $M'_W$ which is not canonically determined. We encountered a similar situation already in [N]. When $W$ an element of $CLC(X, D)$ which is not minimal, we still expect that the pair $(W, M'_W + D_W)$ is LC for some $M'_W$. In order to prove this, one should show that $\mathcal{L}$ in Theorem 4 is semi-ample.

Kodaira’s canonical bundle formula for an elliptic surface $f : X \to B$

$$K_X = f^*(K_B + \sum_i a_i P_i + \sum_j \frac{m_j - 1}{m_j} Q_j + \sum_k b_k \frac{1}{12} R_k)$$

can be regarded as an adjunction formula, where the $f^{-1}(P_i)$ are the singular fibers of the types $II, III, IV, I^*, II^*, III^*$ or $IV^*$, and the $f^{-1}(Q_j)$ and the $f^{-1}(R_k)$ are of the type $m_j I_{b_k}$. The last sum $\sum_k b_k \frac{1}{12} R_k$ is the global contribution in this case. The main lemma (Theorem 4) is a log version of this kind of formula which is obtained as the integration along fibers (cf. Example 8).

1. Proof of the main theorem

We start with a lemma which gives a standard section of some multiple of log canonical divisors.

**Lemma 2.** Let $D = \sum_{j=1}^k d_j P_j$ be an effective $\mathbb{Q}$-divisor on $C \cong \mathbb{P}^1$ such that $0 < d_j \leq 1$ for all $j$ and $\sum_j d_j = 2$. Let $m$ be a positive integer such that $md_j \in \mathbb{N}$ for all $j$. Then there exist a positive integer $p$ and a canonically defined section $\omega_{D,m} \in \Gamma(C, mp(K_C + D))$ which depend only on $D$ and $m$.

**Proof.** We consider the set

$$P(D, m) = \{ J = (j_1, j'_1, \ldots, j_m, j'_m) \in \mathbb{N}^{2m}; \quad 1 \leq j_i, j'_i \leq k, j_i \neq j'_i \ \text{for all} \ i, \# \{ i | j_i = j \text{ or } j'_i = j \} = md_j \}.$$ 

Let $\omega_{j, j'}$ be a rational 1-form on $C$ such that $\text{div}(\omega_{j, j'}) = P_j + P_{j'}$, $\text{res}_{P_j}(\omega_{j,j'}) = 1$ and $\text{res}_{P_{j'}}(\omega_{j,j'}) = -1$. Then we define

$$\omega_J = \prod_{i=1}^m \omega_{j_i, j'_i} \in \Gamma(C, m(K_C + D))$$

and

$$\omega_{D,m} = \prod_{J \in P(D, m)} \omega_J \in \Gamma(C, mp(K_C + D))$$

where $p = \# P(D, m)$.

$\Box$
Let $\mathcal{M}_{0,n}$ be the moduli space of $n$-pointed stable curves of genus 0, $f_{0,n}: \mathcal{Z}_{0,n} \to \mathcal{M}_{0,n}$ the universal family, and $\mathcal{P}_1, \ldots, \mathcal{P}_n$ the sections of $f_{0,n}$ which correspond to the marked points ([Kn]). $\mathcal{M}_{0,n}$ is smooth. Let $\mathcal{M}_{0,n}^0$ be the open part which parametrizes the smooth curves. $\Delta_{0,n}^{k} = \mathcal{M}_{0,n} \setminus \mathcal{M}_{0,n}^0$ is a normal crossing divisor $\Delta_{0,n} = \sum_S \Delta_S$, where the $S = \{S', S''\}$ run all the decompositions $\{1, \ldots, n\} = S' \cup S''$ such that $\#S', \#S'' \geq 2$. $f_{0,n}^{*} \Delta_S$ has 2 irreducible components $\mathcal{F}_{S'}^{\infty}, \mathcal{F}_{S''}^{\infty}$, to which the $\mathcal{P}_j$ are distributed according to the decomposition $S$.

Let $(C; P_1, \ldots, P_n)$ be a fiber of $f_{0,n}$. $C$ is a tree of rational curves. Let $C_1, \ldots, C_c$ be its irreducible components, and $T(C_i) = \{j; P_j \in C_i\}$. Thus the dual graph $\tau$ of $C$ has $c$ vertices and $n$ tails, and $\{1, \ldots, n\} = \bigsqcup_i T(C_i)$. The locus $\Delta_{\tau} \subset \mathcal{M}_{0,n}$ which parametrizes the $n$-pointed stable curves with the dual graph $\tau$ has codimension $c - 1$. In fact, $\Delta_{\tau} = \bigcap_{k=1}^{c-1} \Delta_{S_k}$, where the $S_k$ run the decompositions which corresponds to $c - 1$ decompositions of $C$ into 2 connected curves $C''^{(k)}$ and $S''^{(k)}$ so that $S_k' = \{j; P_j \in C''^{(k)}\}$ and $S_k'' = \{j; P_j \in S''^{(k)}\}$.

Let $d_j$ ($j = 1, \ldots, n$) be rational numbers such that $0 < d_j \leq 1$ for all $j$ and $\sum_j d_j = 2$. Let $\mathcal{D} = \sum_j d_j P_j$. There is a canonical section $\omega_{\mathcal{D}, m}$ of $\mathcal{O}_{\mathcal{Z}_{0,n}}(mp(K_{\mathcal{Z}_{0,n}/\mathcal{M}_{0,n}} + \mathcal{D}))$ over $f_{0,n}^{-1} \mathcal{O}_{\mathcal{M}_{0,n}}^0$. We set $\alpha(S') = \sum_{j \in S'} d_j$ for a decomposition $S = \{S', S''\}$,

$$\mathcal{F}_S = \begin{cases} (\alpha(S') - 1) \mathcal{F}_{S'} & \text{if } \alpha(S') \geq 1 \\ (1 - \alpha(S')) \mathcal{F}_{S'} & \text{otherwise} \end{cases}$$

and $\mathcal{F} = \sum_S \mathcal{F}_S$.

**Lemma 3.** Let $C$ be any fiber of $f_{0,n}$.

1. $(K_{\mathcal{Z}_{0,n}} + \mathcal{D} - \mathcal{F})|_C \equiv 0$.
2. There exists an irreducible component $C_0$ of $C$ such that $C_0 \not\subset \operatorname{Supp}(\mathcal{F})$.

**Proof.** (1) We shall construct a sequence of closed subschemes $C^{(k)}$ of $C$ with contraction morphisms $C \to C^{(k)}$ for $k = 0, 1, \ldots, c$ inductively as follows. Let $C = C^{(0)}$. Assume that $C^{(k)}$ is already constructed for some $k$. Then pick an irreducible component $C_{i_k}$ of $C^{(k)}$ which intersects the rest of the curve at only 1 point. Let $C^{(k+1)}$ be the curve obtained from $C^{(k)}$ by contracting $C_{i_k}$ to a point, and $C''^{(k)}$ the total inverse image of $C_{i_k}$ by the natural morphism $C \to C^{(k)}$. Then the irreducible components of $C$ other than those of $C''^{(k)}$ make up a connected curve $C''^{(k)}$. Let $S_k$ be the decomposition corresponding to the decomposition $C = C^{(k)} \cup C''^{(k)}$, and $C^{(k)}$ the fiber of $\mathcal{F}_{S_k'}$ over a general point of $\Delta_{S_k}$. Then we have $((K_{\mathcal{Z}_{0,n}} + \mathcal{D} - \mathcal{F}) \cdot C^{(k)}) = ((K_{\mathcal{Z}_{0,n}} + \mathcal{D} - \mathcal{F}) \cdot C_{i_k})$ is the difference of $((K_{\mathcal{Z}_{0,n}} + \mathcal{D} - \mathcal{F}) \cdot C''^{(k)})$ and the sum of some of the $(K_{\mathcal{Z}_{0,n}} + \mathcal{D} - \mathcal{F}) \cdot C''^{(k')}$ for $k' < k$. We obtain our claim.

(2) Let $\alpha(C^{(k)}) = \sum_{j \in C^{(k)}} d_j$, and $k_0$ the smallest integer such that $\alpha(C^{(k)}) \geq 1$. Then $C_0 = C_{i_{k_0}}$ is the one. \qed

**Theorem 4.** There exists a $\mathbb{Q}$-divisor $\mathcal{L}$ on $\mathcal{M}_{0,n}$ which is effective and nef and such that $K_{\mathcal{Z}_{0,n}} + \mathcal{D} - \mathcal{F} \sim_\mathbb{Q} f_{0,n}^* (K_{\mathcal{M}_{0,n}} + \mathcal{L})$.

**Proof.** Since $f_{0,n}^* \mathcal{O}_{\mathcal{Z}_{0,n}} = 0$, we have $K_{\mathcal{Z}_{0,n}} + \mathcal{D} - \mathcal{F} \sim_\mathbb{Q} f_{0,n}^* (K_{\mathcal{M}_{0,n}} + \mathcal{L})$ for some $\mathbb{Q}$-Cartier divisor $\mathcal{L}$. The canonical section $(\omega_{\mathcal{D}, m})$ induces a section of $\mathcal{O}_{\mathcal{Z}_{0,n}}(mp\mathcal{L})$. Therefore, we have $K_{\mathcal{Z}_{0,n}} + \mathcal{D} - \mathcal{F} \sim_\mathbb{Q} f_{0,n}^* (K_{\mathcal{M}_{0,n}} + \mathcal{L})$. \qed
over $\mathcal{M}_{0,n}^0$. We shall prove that this section is extended to the whole $\mathcal{M}_{0,n}$. We have also to show that for any smooth curve $B$ and any morphism $\phi : B \to \mathcal{M}_{0,n}$, we have $\deg \phi^*L \geq 0$. Let $(f : X \to B; P_1, \ldots, P_n)$ be a family of $n$-pointed stable curves corresponding to $\phi$, and $F$ the pull-back of $\mathcal{F}$ on $X$.

First, we assume that $\phi(B) \cap \mathcal{M}_{0,n}^0 \neq \emptyset$. Let $C^1, \ldots, C^t$ be singular fibers of $f$, and $Q_\ell = f(C^\ell)$ for $\ell = 1, \ldots, t$. By Lemma 3, there exists an irreducible component $C^\ell_0$ of $C^\ell$ for each $\ell$ which is not contained in $\text{Supp}(F)$. Let $\mu : X \to Y$ be the birational morphism obtained by contracting all the curves on $C^\ell$ except the $C^\ell_0$. Then the induced morphism $g : Y \to B$ is a $\mathbb{P}^1$-bundle, since it has only reduced and irreducible fibers. Let $\bar{P}_j = \mu_*P_j$ and $\bar{D} = \sum j \bar{P}_j$. The canonical section $\omega_{D,m}$ induces a section of $\mathcal{O}_Y(mp(K_{Y/B} + \bar{D}))$ which vanishes over the points $Q_\ell$, because some of the sections $\bar{P}_j$ meet over these points. Since $g^*\phi^*L = K_{Y/B} + \bar{D}$, we obtain $\deg \phi^*L \geq 0$ in this case. The effectiveness of $L$ is also proved.

Next, we consider the general case. Let $C$ be a general fiber of $f$, and $C_0$ its irreducible component given by the Lemma 3. We apply the previous argument to the irreducible component $X^#$ of $X$ which contains $C_0$. Let $\rho : X \to X^#$ be the natural contraction morphism over $B$. We define $D^# = \rho_*D = \sum_k \bar{d}_k P_k^#$ as follows. The $P_k^#$ are either one of the $P_j$ which are contained in $X^#$ or one of the intersection loci on $X^#$ with other irreducible components of $X$. We set $\bar{d}_k = d_j$ in the former case, and $\bar{d}_k = \sum d_j$ where the sum is taken over those $P_j$ which are mapped to $P_k^#$ by $\rho$ in the latter case. We have $\bar{d}_k \leq 1$ for all $k$, because $C_0 \not\subset \text{Supp}(F)$.

Let $C^1, \ldots, C^t$ be singular fibers of $f^# : X^# \to B$, $Q_\ell = f(C^#\ell)$, and $C^\ell = f^{-1}(Q_\ell)$. If $\rho^\ell : C^\ell \to C^#\ell$ is the natural contraction morphism, then $D^#|_{C^#\ell} = \rho_*^\ell(D|_{C^\ell})$. Since $\bar{d}_k \leq 1$ for all $k$, we have $C^#\ell \not\subset \text{Supp}(F)$, and there exists an irreducible component $C^#\ell_0$ of $C^#\ell$ which is not contained in $\text{Supp}(F)$ as in Lemma 3. Let $\mu : X^# \to Y^#$ be the birational morphism obtained by contracting all the curves on $C^#\ell$ except the $C^#\ell_0$. The induced morphism $g^# : Y^# \to B$ is a $\mathbb{P}^1$-bundle. Let $\bar{P}_k = \mu_*P_k^#$, and $\bar{D} = \sum_k \bar{d}_k P_k$. The canonical section $\omega_{D^m}$ induces a section of $\mathcal{O}_Y(m^#(K_{Y^#B} + \bar{D}))$ which vanishes over the $Q_\ell$. Since $g^#\phi^*L = K_{Y^#B} + \bar{D}$, we obtain $\deg \phi^*L \geq 0$ again.

**Remark 5.** $L$ is not necessarily ample. For example, let $X'$ and $Y''$ be trivial $\mathbb{P}^1$-bundles over a proper smooth curve $B$. Let $P_1, P_2, \Gamma'$ (resp. $P_3, P_4, \Gamma''$) be constant sections of $X'$ (resp. $Y''$), and $\bar{P}_5$ a non-constant section of $Y''$ which intersects the other sections transversally. Let $\mu : X'' \to Y''$ be the blow-up at these intersection points, and denote by $P_3, P_4, P_5, \Gamma''$ the strict transforms of $P_3, P_4, \bar{P}_5, \Gamma''$, respectively. Let $X = X' \cup X''$ be the surface obtained by gluing $\Gamma'$ and $\Gamma''$. Then the natural projection $f : X \to B$ is a non-trivial family of 5-pointed stable curves of genus 0. Let $d_j = \frac{4}{7}, \frac{4}{7}, \frac{2}{7}, \frac{2}{7}$ for $j = 1, 2, 3, 4, 5$, respectively. Then we have $\alpha(S') = \frac{8}{5} > 1$, and $\deg L = 0$ in this case.

But $\mathcal{L}$ might be $\mathbb{Q}$-free, i.e., $|m\mathcal{L}|$ might be free for some positive integer $m$. If this is the case, then we do not need to assume that $X$ is affine at the final part of Theorem 1.

**Theorem 6.** Let $f : X \to B$ be a proper surjective morphism of smooth varieties and $P = \sum P_j$ (resp. $Q = \sum Q_\ell$) a normal crossing divisor on $X$ (resp. $B$) such that $f^{-1}(Q) \subset B$ and $f$ is smooth over $P \setminus Q$ with fibers isomorphic to $\mathbb{P}^1$. Let
$D = \sum_j d_j P_j$ be a $\mathbb{Q}$-divisor on $X$, where $d_j = 0$ is allowed, which satisfies the following conditions:

(i) $D = D^h + D^v$ such that $f : \text{Supp}(D^h) \to B$ is generically finite and etale over $B \setminus Q$, and $f(\text{Supp}(D^v)) \subset Q$. An irreducible component of $D^h$ (resp. $D^v$) is called horizontal (resp. vertical).

(ii) $0 < d_j \leq 1$ if $P_j$ is horizontal and $d_j < 1$ if vertical.

(iii) $K_X + D \sim_Q f^*(K_B + L)$ for some $\mathbb{Q}$-divisor $L$ on $B$.

Assume that there exist a finite surjective morphism $\pi_0 : \tilde{B} \to B$ from a smooth variety $\tilde{B}$ with a normal crossing divisor $\tilde{Q} = \pi_0^{-1}(Q) = \sum_m \tilde{Q}_m$, a morphism $\phi_0 : \tilde{B} \to M_{0,n}$, and a common desingularization $\tilde{X}$ of $X \times_B \tilde{B}$ and $Z_{0,n} \times M_{0,n} \tilde{B}$ over $\tilde{B}$ as in the following commutative diagram:

$$
\begin{array}{ccc}
X & \xleftarrow{\pi} & \tilde{X} \\
\downarrow f & & \downarrow \phi \\
B & \xleftarrow{\pi_0} & \tilde{B} \\
\end{array}
\quad \quad \begin{array}{c}
z_{0,n} \\
f_{0,n} \\
\end{array}
$$

such that $\tilde{P} = \pi^{-1}(P) = \sum_k \tilde{P}_k$ is a normal crossing divisor, and the horizontal components of $\pi^*D$ and $\phi^*\tilde{D}$ coincide. Let

$$
\tilde{d}_j = \frac{d_j + w_{\ell j} - 1}{w_{\ell j}} \text{ if } f(P_j) = Q_\ell \\
\delta_\ell = \max\{ \tilde{d}_j : f(P_j) = Q_\ell \} \\
\Delta = \sum_\ell \delta_\ell Q_\ell \\
L = M + \Delta.
$$

Then $\delta_\ell < 1$ for all $\ell$, and $M$ is effective and nef. Moreover, if $1 - w_{\ell j} \leq d_j$ for some $j$, then $0 \leq \delta_\ell$.

**Proof.** The only non-trivial statement is that $M$ is effective and nef. We define a $\mathbb{Q}$-divisor $\tilde{D}$ on $\tilde{X}$ by $\pi^*(K_{\tilde{X}/\tilde{B}} + D - f^*\Delta) = K_{\tilde{X}/\tilde{B}} + \tilde{D}$. We write $\tilde{D} = \sum_k \tilde{d}_k \tilde{P}_k$.

Let us take any $\tilde{P}_k$ such that $\tilde{f}(\tilde{P}_k) = \tilde{Q}_m$. Let $Q_\ell = \pi_0(\tilde{Q}_m)$ and $P_j = \pi(\tilde{P}_k)$. Let $e_m$ (resp. $e_k$) be the ramification index of $\pi_0$ (resp. $\pi$) along $\tilde{Q}_m$ (resp. $\tilde{P}_k$). Since $e_m = e_k w_{\ell j}$, we have

$$
\tilde{d}_k = e_k (d_j - \delta_\ell w_{\ell j}) - (e_k - 1) + (e_m - 1) = e_k (\tilde{d}_j - \delta_\ell).
$$

Therefore, the vertical component of $-\tilde{D}$ over the generic point $\tilde{\eta}_m$ of $\tilde{Q}_m$ is effective and its support does not contain the whole fiber $\tilde{f}^{-1}(\eta_\ell)$. So we have $K_{\tilde{X}/\tilde{B}} + \tilde{D} \sim_Q \phi^*(K_{z_{0,n}/M_{0,n}} + D - F)$, since both hand sides are pull-backs of some $\mathbb{Q}$-divisors on $\tilde{B}$ anyway. Hence $\pi^* f^* (L - \Delta) \sim_Q \phi^* f_{0,n}^* L$, and $M$ is effective and nef. $\Box$

**Remark 7.** $\Delta$ is independent of the birational model of $X$ in the following sense. If we blow-up $X$ along the intersection of $P_{j'}$ and $P_{j''}$ over $Q_\ell$ and $P_j$ is the exceptional divisor, then we have $w_j = w_{j'} + w_{j''}$ and $d_j = d_{j'} + d_{j''} - 1$. So $\delta_\ell$ does not change.
Example 8. Let $g : Y \to B$ be a $\mathbb{P}^1$-bundle over a smooth curve with 4 sections $\bar{P}_j$ ($j = 1, \ldots, 4$). Assume that $\bar{P}_1$ and $\bar{P}_2$ intersect at $P \in Y$ with multiplicity $n$ and that there are no other intersections among the $\bar{P}_j$. Let $\mu : X \to Y$ be the composition of $n$ blow-ups over $P$ which makes the strict transforms $P_j = h^{-1}_s \bar{P}_j$ disjoint. Let $f = g \circ \mu$ and $Q = g(P)$. Then we have

$$K_X + \frac{1}{2} \sum_j P_j = \mu^*(K_Y + \frac{1}{2} \sum_j \bar{P}_j) = f^*(K_B + \frac{n}{6}Q).$$

Let $\rho : Z \to X$ be the double cover whose ramification divisor is $\sum_j P_j$. Then $K_Z = \rho^*(K_X + \frac{1}{2} \sum_j P_j)$ and the induced morphism $e : Z \to B$ gives an elliptic surface with a degenerate fiber of type $I_{2\nu}$ over $Q$. By the canonical bundle formula of Kodaira, we have $K_Z = e^*(K_B + \frac{2\nu}{12}Q)$.

Proof of Theorem 1. Let $\mu : Y \to X$ be an embedded resolution of the pair $(X, D)$. We write

$$K_Y + E + F = \mu^*(K_X + D)$$

where $E$ is the place of canonical singularities corresponding to $W$. We may assume that there is a resolution of singularities $\nu : V \to W$ which factors $\mu : E \to W$ and such that $f : E \to V$ and $F|_E$ satisfy the conditions of Theorem 6 in places of $f : X \to B$ and $D$, respectively, since

$$\mu^*(K_X + D)|_E = (K_Y + E + F)|_E = K_E + F|_E.$$ 

So there exist $\mathbb{Q}$-divisors $M$ and $\Delta$ on $V$ such that $K_E + F|_E = f^*(K_V + M + \Delta)$ where $M$ is effective and nef and the coefficients of $\Delta$ are less than 1. We put $M_W = f_* M$ and $D_W = f_* \Delta$.

Now we use the notation of Theorem 6. By using Kawamata-Viehweg vanishing theorem, we conclude that the natural injective homomorphism $\mathcal{O}_W \to \mu_* \mathcal{O}_E(\nu - F)$ is surjective ([K3, Theorem 1.6]). Therefore, if $\nu_* Q_\ell \neq 0$, then there exists a $j$ such that $-d_j \leq w_{ij} - 1$. Hence $D_W$ is effective.

Assume that $X$ is affine. Since $M$ is nef, we may assume that $M - \epsilon \sum_\ell q_\ell Q_\ell$ is ample for $0 < \epsilon \ll 1$ and for some rational numbers $q_\ell$ such that $q_\ell > 0$ (resp. $= 0$) if $\nu_* Q_\ell = 0$ (resp. $\neq 0$). We take an effective $\mathbb{Q}$-divisor $M' \sim_\mathbb{Q} M - \epsilon \sum_\ell q_\ell Q_\ell$ which has smooth support and transversal to the $Q_\ell$, and let $M'_W = \nu_* M'$. Then we have

$$\nu^*(K_W + M'_W + D_W) = K_V + M' + \Delta + \epsilon \sum_\ell q_\ell Q_\ell$$

since the pull-back of the right hand side by $f$ is equal to a pull-back by $\mu$. If $\epsilon$ is chosen small enough, then the coefficients of the $Q_\ell$ on the right hand side are less than 1, and $(W, M'_W + D_W)$ is KLT.

Remark 9. One could use the moduli space of log surfaces [A] in order to extend our results to the case of codimension 3. The argument of this paper is similar to the one used in the proof of the additivity of the logarithmic Kodaira dimension of algebraic fiber spaces of open curves ([K1]). The first result toward the additivity of the (non-log) Kodaira dimension used some properties of the moduli space of stable curves ([V]), but these were replaced later by some positivity results from the Hodge theory in order to generalize to the case of higher dimensional fibers ([K2]). The most important statement of Theorem 6 is that $M$ is nef, and this might be generalized by the Hodge theoretic method to the case of higher dimensional fibers, hence the higher codimensional subadjunction.
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