Good reduction, bad reduction

Notes for a lecture at the conference on
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We give some general properties of good and bad reduction, along with some recent examples (worked out with Dipendra Prasad) of varieties having bad reduction not accounted for by cohomology. We include some consequences of our remarks for varieties over number fields having good reduction everywhere.

1. Good reduction.

Let \( p \) be a prime number and let \( K \) be a finite extension of \( \mathbb{Q}_p \). Denote by \( \mathfrak{o} \) the ring of integers of \( K \) and let \( k \) be the residue field of \( \mathfrak{o} \), the quotient by the unique maximal ideal.

**Definition.** — A smooth proper \( K \)-variety \( X \) is said to have good reduction if it is the generic fibre of a smooth proper \( \mathfrak{o} \)-scheme.

Such an \( \mathfrak{o} \)-scheme is called a smooth model of \( X \). A variety is said to have bad reduction if it does not have good reduction.

As an example, a finite extension \( L \) of \( K \) has good reduction if and only if it is unramified.

There are exactly two conics (i.e. curves of genus 0) over \( K \). The one which has a point, namely \( \mathbb{P}_1 \), has good reduction; the other has bad reduction.

More generally, a twisted form of \( \mathbb{P}_n \) has bad reduction unless it has a \( K \)-point, and is thus isomorphic to \( \mathbb{P}_n \).

A twisted form of an abelian variety has bad reduction if it does not have a \( K \)-point (it may have bad reduction even when it has one).

An elliptic curve has good reduction if and only if the its conductor is (the unit ideal) \( \mathfrak{o} \).

There are some varieties which have (infinitely) many smooth models: A smooth model of \( \mathbb{P}_1 \times \mathbb{P}_1 \) can be found whose special fibre is any of the the Hirzebruch surfaces \( F_0 = \mathbb{P}_1 \times \mathbb{P}_1, F_2, F_4, \ldots \). I don’t know of a variety which has at least two but only finitely many smooth models.

Let \( l \) be a prime number different from \( p \). Let \( \bar{K} \) be an algebraic closure of \( K \). For a \( K \)-variety \( X \), denote by \( \bar{X} = X \times_K \bar{K} \) the change of base field from \( K \) to \( \bar{K} \).
Theorem. — Let $X$ be a (smooth, proper) $K$-variety. If $X$ has good reduction, then the action of $\text{Gal}(\bar{K}|K)$ on $H^i(\bar{X}, \mathbb{Q}_l)$ is unramified.

This means that the action factors through the quotient $\text{Gal}(\bar{K}|K)$, where $K$ is the maximal unramified extension of $K$ in $\bar{K}$: the inertia subgroup $\text{Gal}(\bar{K}|\bar{K})$ acts trivially.

$p$-adic chomology is almost never unramified. It took Fontaine to formulate the correct analogue, and Faltings to prove it in general.

Theorem. — Let $X$ be a (smooth, proper) $K$-variety. If $X$ has good reduction, then the action of $\text{Gal}(\bar{K}|K)$ on $H^i(\bar{X}, \mathbb{Q}_p)$ is crystalline.

This is also a condition on the restriction of the representation to the inertial subgroup. More precisely, Fontaine has constructed a $\mathbb{Q}_p$-algebra $B_{\text{cris}}$ with an action of $\text{Gal}(\bar{K}|K)$ (and some other structures); a finite-dimensional $\mathbb{Q}_p$-representation $V$ of $\text{Gal}(\bar{K}|K)$ is called crystalline if the dimension of $(V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{\text{Gal}(\bar{K}|K)}$ (as a vector space over $B_{\text{cris}}^{\text{Gal}(\bar{K}|K)}$, the maximal unramified extension of $\mathbb{Q}_p$ in $K$) is the same as that of $V$.

Question. — Is there a $K$-variety $X$ for which the action of $\text{Gal}(\bar{K}|K)$ on $H^i(\bar{X}, \mathbb{Q}_l)$ is unramified, and the action on $H^i(\bar{X}, \mathbb{Q}_p)$ crystalline, and yet which has bad reduction.

We shall provide some examples of such varieties, which have bad reduction but whose “motive” has good reduction.

Twisted abelian varieties

Definition. — An abelian $K$-variety $A$ is said to have abelian reduction if there is an abelian $\sigma$-scheme whose generic fibre is $A$.

The abelian scheme in question is then unique. Clearly, if $A$ has abelian reduction, then it has good reduction. Conversely,

Theorem. — If the variety $A$ has good reduction, then the abelian variety $A$ has abelian reduction.

A direct proof can be found in the book on Néron Models by Bosch, Lütkebohmert Raynaud.

Here is the celebrated $l$-adic criterion (Néron-Ogg-Shafarevich) for good reduction of abelian varieties, proved by Serre and Tate.

Theorem. — Let $A$ be an abelian $K$-variety. If the representation $H^1(X, \mathbb{Q}_l)$ is unramified, then $A$ has good reduction.
So if the variety $A$ has good reduction (disregarding the group law), the $l$-adic cohomology is unramified and hence $A$ has abelian reduction.

The $p$-adic analogue is more recent, and due to Mokrane and Coleman-Iovita.

**Theorem.** — Let $A$ be an abelian $K$-variety. If the representation $H^1(\bar{X}, \mathbb{Q}_p)$ is crystalline, then $A$ has good reduction.

Our observation for abelian varieties amounts to the next two results.

**Proposition.** — Let $T$ be a torsor under an abelian variety $A$. Then the representation of $\text{Gal}(\bar{K}|K)$ on the ($l$-adic or $p$-adic) cohomology of $T$ is the same as the representation on the cohomology of $A$.

In short, the motive of $T$ is the same as that of $A$.

**Proposition.** — Let $T$ be a $K$-variety which is potentially isomorphic to an abelian variety. If $T(K)$ is empty, then $T$ has bad reduction.

Thus taking an $A$ which has good reduction and a torsor $T$ which is not $K$-isomorphic to $A$ (there are many such $T$, by Tate local duality), the variety $T$ has bad reduction but its cohomology is unramified (resp. crystalline).

**Twisted projective spaces**

These are the varieties which become isomorphic to $\mathbb{P}_n$ over a suitable finite (separable) extension of $K$. They are classified by the group $H^2(K, \bar{K}^\times)$, as are similarity classes of simple central $K$-algebras.

**Proposition.** — Let $X$ be a twisted form of $\mathbb{P}_n$ and $A$ the corresponding simple central $K$-algebra. Then the following statements are equivalent:

1) $X$ is $K$-isomorphic to $\mathbb{P}_n$.
2) $X$ has good reduction.
3) $X(K)$ is not empty.
4) $A$ is similar to the matrix algebra.
5) $A$ is the generic fibre of an azumaya $\mathfrak{a}$-algebra.
6) the class of $X$ in $H^2(K, \bar{K}^\times)$ is trivial.

As in the earlier case of abelian varieties, the cohomology of a twisted form is the twist by (the image of) the same 1-cocycle. This gives:
**Proposition.** — The $l$-adic (resp. $p$-adic) cohomology of a twisted form $X$ of $\mathbb{P}_n$ is unramified (resp. crystalline).

Taking $X$ to be different from $\mathbb{P}_n$, we get a variety which has bad reduction but whose cohomology is unramified (resp. crystalline). Recall that $H^2(K, \bar{K}^\times) = \mathbb{Q}/\mathbb{Z}$, so there are many such varieties.

**Rational surfaces**

These are the surfaces which are potentially (i.e. over a suitable finite (separable) extension of $K$) birational to $\mathbb{P}_2$.

Let us confine ourselves to Châtelet surfaces and to $p \neq 2$.

Let $d \in K^\times$ not be a square and $e_1, e_2$ be two distinct elements of $K^\times$.

The ruled surface $X$ given by

$$y^2 - dz^2 = xx'(x - e_1 x')(x - e_2 x')t^2$$

is fibered in conics over $\mathbb{P}_1$ (coordinates $x : x'$), the fibre at each point being a conic in $\mathbb{P}_2$ (coordinates $y : z : t$). The surface is birational over $K(\sqrt{d})$ to $\mathbb{P}_2$.

Without changing the surface $X$, we may assume that $e_1, e_2$ have the same valuation $r$.

**Proposition.** — The surface $X$ has bad reduction if $d$ is not a unit, or if the valuation of $e_1 - e_2$ is $> r$.

The reason is that in these cases the Chow group of 0-cycles of degree 0 turns out to be $\neq 0$, whereas a theorem of Colliot-Thélène says that it should vanish for rational surfaces having good reduction.

**Proposition.** — If $d$ is a unit (and $p \neq 2$), the $l$-adic (resp. $p$-adic) cohomology of $X$ is unramified (resp. crystalline).

This follows from the fact that the cohomology of a rational surface can be computed from its Picard group over $\bar{K}$ (a finitely generated free $\mathbb{Z}$-module with continuous $\text{Gal}(\bar{K}|K)$-action).

Taking $d$ to be a unit and the valuation of $e_1 - e_2$ to be $> r$, we get examples of rational surfaces which have bad reduction but whose “motive” has good reduction.

**Curves of higher genus**

The most natural way of finding examples among curves of genus $\geq 2$ would be to combine Néron-Ogg-Shafarevich with the anabelian $l$-adic criterion of Oda:
THEOREM. — Let $C$ be a curve of genus $\geq 2$. If the outer action of $\text{Gal}(\overline{K}/K)$ on the maximal pro-$l$ quotient of the (étale) fundamental group of $C_{\overline{K}} = C \times_K \overline{K}$ is unramified, then $C$ has good reduction.

**Over number fields**

The classical result of Minkowski about unramified extensions of $\mathbb{Q}$ can be reformulated as:

**Theorem.** — The only point over $\mathbb{Q}$ which has good reduction everywhere is $\text{Spec}(\mathbb{Q})$.

In dimension 1, there is something similar:

**Theorem.** — The only (smooth, proper, absolutely connected) curve over $\mathbb{Q}$ which has good reduction everywhere is $\mathbb{P}_1$.

Our observations suffice to prove this in genus 0 and, combined with the theorem of Tate saying that there is no elliptic curve over $\mathbb{Q}$ having good reduction everywhere, in genus 1. In higher genera, one uses the fact that if a curve has good reduction, then so does its jacobian — an easy consequence of Néron-Ogg-Shafarevich, but there is also a direct proof — and the famous theorem of Fontaine, generalising Tate’s result to higher dimensions:

**Theorem.** — The only abelian variety over $\mathbb{Q}$ which has good reduction everywhere is the point.

These two theorems, about smooth points (resp. curves) over $\mathbb{Z}$, allow N. Fakhruddin to show that the degree-4 (resp. degree-3) embedding $\mathbb{P}_1 \to \mathbb{P}_4$ (resp. $\mathbb{P}_2 \to \mathbb{P}_9$) does not admit a smooth hypersurface section (cf. Poonen’s recent Annals paper).

Our results about twisted forms of projective spaces imply:

**Theorem.** — The only twisted form of $\mathbb{P}_n$ over $\mathbb{Q}$ which has good reduction everywhere is $\mathbb{P}_n$.

Over a given number field, the possible twisted forms of $\mathbb{P}_n$ having good reduction everywhere can easily be listed, basically because there is a local-to-global principle for them (Brauer-Hasse-Noether). For example, over a real quadratic field, there is a unique curve of genus 0, apart from $\mathbb{P}_1$, which has good reduction everywhere.

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