Persistence of heavy-tailed sample averages occurs by infinitely many jumps

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Abstract: We consider the sample average of a centered random walk in $\mathbb{R}^d$ with regularly varying step size distribution. For the first exit time from a compact convex set $A$ not containing the origin, we show that its tail is of lognormal type. Moreover, we show that the typical way for a large exit time to occur is by having a number of jumps growing logarithmically in the scaling parameter.

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1. Introduction

We consider an exit problem for the sample mean of an $\mathbb{R}^d$-valued random walk with zero mean, where the step size distribution has a distribution which is of multivariate regular variation. Specifically, let $(X_i : i = 1, ..., n)$ be an i.i.d. sequence of random variables in $\mathbb{R}^d$ such that $\mathbb{E}X = 0$ (1.1)

for a generic step $X$. Additionally, we assume that the $\mathbb{R}^d$-valued random vector $X$ has a multivariate regularly varying distribution with index $\alpha$ (writing $X \in \text{RV}(\alpha, \mu)$), that is, there exists an increasing sequence of positive real numbers $(a_n : n \geq 1)$ with $a_n \uparrow \infty$ and a non-null Radon measure $\mu$ on $\mathcal{B}(\mathbb{R}^d \setminus \{0\})$ with

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\( \mu(\mathbb{R}^d \setminus \mathbb{R}^d) = 0 \) such that

\[
\lim_{n \to \infty} n \mathbb{P}\left( a_n^{-1} X \in B \right) = \mu(B) \quad \text{for every } B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).
\]  

(1.2)

The limit measure \( \mu \) necessarily obeys a scaling property, that is, there exists \( \alpha > 0 \) such that

\[
\mu(u \circ B) = u^{-\alpha} \mu(B) \quad \text{for every } u > 0 \text{ and } B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).
\]

(1.3)

With \( (X_i : i = 1, \ldots, n) \), we associate the random walk \( S_k = \sum_{i=1}^{k} X_i \), for all \( k \in \mathbb{N} \). In this paper, we investigate the behavior of the survival probability

\[
P_n := \mathbb{P}\left( k^{-1} S_k \in A \text{ for all } k \in \{1, 2, \ldots, n\} \right)
\]

(1.4)
as \( n \to \infty \), where \( A \) is a compact convex set that does not contain the origin. Thus, (1.3) and the LLN subsequently imply that \( P_n \to 0 \) and our aim is to establish its convergence rate.

Our motivation behind this investigation is two-fold. First of all, \( P_n \) is an example of so-called persistence probability, that is the probability that sample average 'persists' in the set \( A \) for at least \( n \) steps. It can also be interpreted as the survival function \( \mathbb{P}(\tau_A > n) \) of the first time the sample average \( S_k/k \) exits from the set \( A \), or as the probability that the occupation measure of the set \( A \) equals \( n \).

Persistence probabilities and related exit problems have recently received a lot of attention in probability theory and theoretical physics. In many situations of interest, for a stochastic process in discrete or continuous time and some exit time \( \tau_A \), it turns out that the behavior is either polynomial-like, that is \( \lim_{n \to +\infty} \log \mathbb{P}(\tau_A > n)/\log n = -\phi \), or exponential-like, that is \( \lim_{n \to +\infty} \log \mathbb{P}(\tau_A > n)/\log n = -\phi \) for a non-negative parameter \( \phi \) called the persistence exponent (or survival exponent). This exponent usually does not depend on the initial position of the process under consideration. Random walks and Brownian motions have been analysed in \([12, 17, 22, 29]\). For results on Gaussian processes, see \([9, 13]\) and references therein. If the process under consideration is stationary and one-dimensional, and the set \( A \) is a shifted half-line, the law of \( \tau_A \) corresponds to a first passage time. In this case, fluctuation theory may be applied; see e.g. the survey \([3]\) for an overview concerning mainly Lévy processes and (integrated) random walks. Other one-dimensional process have been studied; see for example \([18]\) for autoregressive sequences. Recent work on time-homogeneous Markov chains can be found in \([2]\). For a recent survey on persistence probabilities we refer to \([7]\).

Our investigation distinguishes from the above-mentioned works by focusing on the sample average \( S_k/k, k \geq 1 \), which is a time-inhomogeneous \( \mathbb{R}^d \)-valued Markov chain. As mentioned in \([7]\), the study of sample averages, and more generally, occupation measures is challenging. In the case investigated here, we find out that the asymptotics of \( P_n \) is of lognormal type. That is, there exists a
constant $\phi$ depending on the shape of the set $A$ and $\alpha$ such that

$$\lim_{n \to +\infty} \frac{\log P_n}{(\log n)^2} = -\phi.$$  

Thus, the behavior of $P_n$ is fundamentally different from the two earlier described cases. We manage to identify $\phi$ explicitly. For example, if $d = 1$ and $A = [a, b]$, then the persistence exponent equals

$$\phi = \frac{(\alpha - 1)}{2(\log b - \log a)}.$$  

In the case $d > 1$, we provide a variational characterization of $\phi$.

An explanation of this untypical asymptotics brings us to our second motivation of this paper, which is to obtain a sharper understanding of the nature of heavy-tailed large deviations. In turns out that the problem we consider exhibits a new qualitative phenomenon in the following sense: we prove that the typical way of getting a large exit time is by having a number of jumps which is growing logarithmic in the scaling parameter $n$. Hence persistency in our case is caused by infinitely many large jumps. In other words, the principle of a single big jump used in a significant number of studies (see [16] and references therein) does not hold here.

In addition, heavy-tailed sample-path large deviations theorems such as recently derived in [25] do not apply either. In [25], a sample-path large deviations result for the rescaled random walk $\bar{S}_n(t), t \in [0, 1]$, with $\bar{S}_n(t) = S_{[nt]}/n$ and $S_k = S_k$, has been developed in the case $d = 1$. For a large collection of sets $F$, the results in [25] imply that

$$\log P\left(\bar{S}_n \in F\right) = -(1 + o(1))J_F(\alpha - 1) \log n$$

as $n \to +\infty$ with some rate function $J_F$. This result can be applied to investigate the probability, for fixed $\epsilon > 0$,

$$P_{\epsilon n, n} := P\left(\frac{S_k}{k} \in [a, b] \text{ for all } k \in \{\lceil \epsilon n \rceil, \ldots, n\}\right).$$

If $-\log \epsilon/\log(b/a)$ is not an integer, it can be shown that

$$\lim_{n \to +\infty} \frac{\log P_{\epsilon n, n}}{(-\log \epsilon/\log(b/a))(\alpha - 1) \log n} = 1.$$  

The intuition, which can be made precise using the conditional limit theorems in [25], is that the most likely way for $S_k/k$ to stay in the set $[a, b]$ for $k \in \{\lceil \epsilon n \rceil, \ldots, n\}$ is by having $-\log \epsilon/\log(b/a)$ large jumps. In the case we are interested in, any finite number of jumps will not be sufficient for $S_k/k$ to be persistent. Therefore $P_n$ has different asymptotics. Moreover, note that it is tempting to proceed heuristically, and take $\epsilon = 1/n$ in (1.8). Apart from not being rigorous, the resulting guess of $\phi$ would actually off by a factor $1/2$.  

There exist several approaches that can be used to derive the existence, as well as expressions of persistence exponents. In case of more general processes, the Markovian structure is typically exploited. This allows to relate the persistence exponent to an eigenvalue of an appropriate operator, allowing to marshal analytic methods. This idea is related with identifying so-called quasi-stationary distributions (see [4] for the Brownian motion, [6, 11, 20] for random walks and Lévy processes, [8, 14] for time-homogeneous Markov processes and [1, 15, 21] for continuous-time branching processes and the Fleming-Viot processes).

Our work is based on constructing a typical path for the random walk and show that this path, sometimes also called optimal path, is the most likely way for persistence to occur. For \( d = 1 \) the optimal path is depicted in Figure 1 (where the jumps are coloured by red) and it is constructed in the following way. Fix a positive finite integer \( c_1 \). Suppose that the path stays inside \([ak, bk]\) for all \( k \in \{1, 2, \ldots, c_1\} \) and the path is at \( bc_1 \) at time \( c_1 \). Because of the zero drift assumption, the random walk stays around \( bc_1 \) as long as possible, that is until time \( bc_1/a \). At the time \( bc_1/a \), it makes a 'first big' jump so that it reaches to the maximum height possible and stays there as long as possible. Then it again makes a jump. This strategy can be applied iteratively, and gives a candidate optimal path. This path can be represented by the following function

\[
\sum_{i=1}^{K_n} J_i \delta_{T_i}.
\]

Here, \( \delta_x \) is a Dirac measure putting unit mass at \( x \). \( K_n \) denotes the number of jumps needed till time \( n \), \( J_i \) denotes the size of the \( i \)-th jump and \( T_i \) denotes the time of the \( i \)-th jump. Moreover, \( K_n \sim \log n \), \( T_{i+1} - T_i \sim (b/a)^i \) and \( J_i \sim (b/a)^i \) for all \( i \geq 1 \), where we write \( l(n) \asymp k(n) \) if \( \omega_1 k(n) \leq l(n) \leq \omega_2 k(n) \) for some constants \( \omega_1 \) and \( \omega_2 \). If we agree now that the probability of a jump of size \( J_i \) during \((T_{i-1}, T_i]\) is also of order \((b/a)^{i(1-\alpha)}\), then \( P_n \) is roughly of order \( \prod_{i=1}^{\log n} (b/a)^{i(1-\alpha)} \). This produces the required estimate \( \log P_n \asymp -((\alpha - 1)(\log b/a)^{-1}(\log n)^2)/2 \).

The main idea works also in dimension \( d > 1 \) by choosing an 'optimal' direction \( \varphi^* \) solving a variational problem (2.2) that is attaining the supremum \( r^* = \sup_{\varphi \in A} U_\varphi/L_\varphi \). Using this, we create a convenient inner set of \( A \) that is big enough to achieve a sharp enough lower bound for \( P_n \). For this inner set, we take a carefully constructed hypercuboid. A key property is then a certain closure property of a class of hypercuboids under a direct sum operation. Another essential feature of our approximation by a sequence of hypercuboids is that we need to continue to allow the fluctuation of the random walk in some directions though the large jumps happen in the optimal direction \( \varphi^* \) only; see Figure 2.

This paper is organized as follows. In Section 2 we present our main results, and several examples and implications. In the sections after that, we provide formal proofs.
2. Main results

In the definition of regular variation on $\mathbb{R}^d$, we have seen that there exists a Radon measure $\mu$ satisfying the scaling property. We first consider $d \geq 2$. The scaling property of $\mu$ implies that $\mu$ can also be written as a product measure on $(0, \infty) \times S^{d-1}$ where $S^{d-1} = \{ x \in \mathbb{R}^d : \|x\| = 1 \}$ and $\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_d^2}$.

We need to introduce the polar coordinate transformation to write down the product measure form of $\mu$. The polar co-ordinate transformation is given by $T : \mathbb{R}^d \setminus \{0\} \mapsto (0, \infty) \times S^{d-1}$ by $T(x) = (\|x\|, x/\|x\|)$. This has inverse transformation $T^{-1} : (0, \infty) \times S^{d-1} \mapsto \mathbb{R}^d \setminus \{0\}$ given by $T^{-1}(r, a) = r \cdot a$ where $r \cdot a$ denotes scalar multiplication of the vector $a$ and a positive real number $r$. The vector $a$ can be interpreted as the direction and $r$ is the distance in the direction $a$.

It is known from the literature (e.g. Theorem 6.1 in [24]) that (1.2) is equivalent to existence of a Radon measure $\varsigma(\cdot)$ on $S^{d-1}$ such that

$$\lim_{n \to \infty} n P\left( \left( a_n^{-1} \|X\|, (\|X\|)^{-1} \cdot X \right) \in C \times D \right) = \nu_\alpha(C) \varsigma(D), \quad (2.1)$$

where $C \in \mathcal{B}((0, \infty))$ and $D \in \mathcal{B}(S^{d-1})$ and $\nu_\alpha(\cdot)$ is a measure on $(0, \infty)$ such that $\nu_\alpha(x, \infty) = x^{-\alpha}$ for any $x > 0$. We shall assume that the spectral (angular) measure $\varsigma$ is absolutely continuous with respect to the Lebesgue measure on the unit sphere. Note that the spectral measure may not satisfy this assumption; for example it can be atomic if we consider the case where the components of the random vector $X$ are independent. Note also that the polar transform is a non-linear transform, that is, the polar transform of a random walk is not a random walk. Thus, the polar transform can not be used directly get an one-dimensional positive random walk and compute the persistence exponent from this simpler object. But this decomposition helps to understand the limit. Intuitively, it is clear that the persistence exponent must be based on the radial part of the set.
under consideration.

We shall write $\Xi(B) = \{\|x\|^{-1} \cdot x : x \in B\}$ for any measurable subset $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$. We consider a compact and convex set $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ which is bounded away from $0$ ($0 \notin \bar{A}$). It is clear that $\Xi(A)$ is also compact. We can then write $A = \{r \cdot \varphi : \varphi \in \Xi(A), r \in [L_{\varphi}, U_{\varphi}]\}$ where $L_{\varphi} := \inf\{r : r \cdot \varphi \in A\}$ and $U_{\varphi} := \sup\{r : r \cdot \varphi \in A\}$. It is clear that $L_{\varphi}$ and $U_{\varphi}$ are continuous functions of $\varphi$ as the boundary of a bounded convex set is continuous and $L_{\varphi} > 0$ for every $\varphi \in \Xi(A)$ as $A$ is bounded away from $0$. So we can conclude that $U_{\varphi}/L_{\varphi}$ is a continuous function of $\varphi$. Define

$$r^* := \sup_{\varphi \in \Xi(A)} U_{\varphi}/L_{\varphi}$$

and it is clear that there exists $\varphi^* \in \Xi(A)$ such that $r^* = U_{\varphi^*}/L_{\varphi^*}$ as $\Xi(A)$ is compact. We assume further that

$$\{y : \|x - y\| < \delta\} \subset A \quad \text{for some } \delta > 0$$

and $x \in A$. This assumption ensures that the set $A$ under consideration is $d$-dimensional. Without loss of generality we can assume that $\varphi^*$ points in the direction of the positive orthant of $\mathbb{R}^d$. If it is not the case, then we can rotate the axes to ensure that it holds. We are now ready to present the main result of this work.

**Theorem 2.1.** Assume that the angular measure $\varsigma$ is absolutely continuous with respect to the Lebesgue measure on the unit sphere and that the set $A$ with non-empty interior is compact and convex such that $0 \notin \bar{A}$. Under the conditions (1.1)-(1.3) and (2.3) we have,

$$\lim_{n \to \infty} \frac{1}{(\log n)^2} \log P\left(k^{-1}S_k \in A \quad \text{for all } k = 1, 2, \ldots, n\right) = -\frac{\alpha - 1}{2(\log r^*)}. \quad (2.4)$$

**Remark 2.2.** The persistence exponent $\phi$ and $r^*$ in particular can be computed by developing an alternative representation for $r^*$. It is not difficult to see that $r^*$ is equal to the largest value of $r$ such that $\text{dist}(A, r \circ A) = 0$ with $\text{dist}(A, B) = \inf\{|x - y| : x \in A, y \in B\}$ and $r \circ A = \{r \cdot x : x \in A\}$. Since any convex set in $\mathbb{R}^d$ is the intersection of a countable number of half-spaces, there exists vectors $a_i$ and constants $b_i$ for $i \geq 1$ such that

$$A = \{x : \langle a_i, x \rangle + b_i \leq 0, i \geq 1\} \quad (2.5)$$

where $\langle a, x \rangle$ denotes the inner product of vectors $x$ and $a$. Defining the convex function

$$H(x) = \max_i \{\langle a_i, x \rangle + b_i\}, \quad (2.6)$$

the problem of maximizing $r$ such that $\text{dist}(A, r \circ A) = 0$ can now equivalently written as the solution of the convex program

$$\max_{r, y} r $$

$$y$$

(2.7)
subject to
\[ H(y) \leq 0, H(r \cdot y) \leq 0. \tag{2.8} \]

In Section 3, we apply a result of [26] to show continuity of \( r \) as function of \( \delta \), if the right-hand-side in the last equations is taken to be equal to \( \delta \). This property is exploited in our proof.

### 2.1. One-dimensional random walk and interval \([a, b]\)

For \( d = 1 \) and the set \( A = [a, b] \) with \( 0 < a < b < \infty \), we consider a collection \((X_i : i \in \mathbb{N})\) of independent copies of the \( \mathbb{R} \)-valued, mean-zero regularly varying random variable \( X \) such that
\[ \Pr(X > x) = x^{-\alpha} L_+(x) \tag{2.9} \]
for \( x > 0 \), such that a tail balance condition
\[ \limsup_{x \to \infty} \frac{\Pr(X < -x)}{\Pr(X > x)} = p_{\pm} \in [0, \infty) \tag{2.10} \]
holds true, where \( L_+ \) is slowly varying functions. This is equivalent to assumption (1.2) in the case \( d = 1 \). With \((X_i : i \in \mathbb{N})\), we consider the associated random walk \((S_k : k \geq 1)\) (without using bold-face).

**Theorem 2.3.** Under the assumptions stated above,
\[ \lim_{n \to \infty} \frac{1}{(\log n)^2} \log \Pr\left( k^{-1} S_k \in [a, b] \text{ for all } k \in \{1, 2, \ldots, n\} \right) = -\frac{(\alpha - 1)}{2(\log b - \log a)} \]
for every \( 0 < a < b < \infty \).

Note that the above theorem is not a straightforward corollary of Theorem 2.1 since the associated angular measure is necessarily atomic in \( d = 1 \). However, we will briefly show later that its proof follows from the same steps as the proof of Theorem 2.1.

Theorem 2.3 (which not surprisingly was proven before Theorem 2.1) can be used to derive an upper bound for the probability in Theorem 2.1 by projecting a \( d \)-dimensional random walk in a certain direction. This leads to a natural upper bound for \( P_n \) in terms of a persistence probability for a one-dimensional random walk. In particular, for any \( d \)-dimensional vector \( c \),
\[ P_n \leq \inf_{c ||c|| = 1} \Pr\left( k^{-1}(c, S_k) \in c \cdot A \text{ for all } k \in \{1, 2, \ldots, n\} \right), \tag{2.11} \]
with \( y \in c \cdot A \) if \( y = \langle c, x \rangle \) for some \( x \in A \). The assumptions on \( A \) and \( c \) imply that \( c \cdot A \) is an interval of the form \([a(c), b(c)]\). A natural question is now whether the bound
\[ \phi \geq \sup_{c ||c|| = 1} \frac{\alpha - 1}{2(\log b(c) - \log a(c))} \tag{2.12} \]
is sharp. This kind of bounding techniques are often applied in light-tailed large
deviations. It can be shown that this bound is sharp if $A$ is a Euclidean ball. However, if $A$ is a rectangle in the positive orthant, then the bound is only sharp if and only if the diagonal connecting the southwest corner and northeast corner of $A$ also passes through the origin. We leave these details as an exercise.

2.2. Nonstandard regular variation

Suppose that $X = (X_1, X_2, \ldots, X_d)$ is a random vector such that $X_i$’s are independent and have regularly varying tails with index of regular variation $\alpha_i$ and slowly varying function $L_i(\cdot)$. This is known by the name of nonstandard regular variation in the theory of regular variation (see [24, Subsect. 6.5.6]). Then exploiting independence of components of $S_k = (S_{k,1}, S_{k,2}, \ldots, S_{k,d})$ we can get the following corollary of Theorem 2.3.

**Corollary 2.4.** Suppose that the vector $X = (X_1, X_2, \ldots, X_d)$ is such that $X_i$’s are independent and have regularly varying with index of regular variation $\alpha_i$ and each $X_i$ satisfies the assumptions in Theorem 2.3. Then

$$
\lim_{n \to \infty} \frac{1}{(\log n)^2} \log P \left( k^{-1} S_k \in [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_d, b_d] \right. \left. \text{ for all } k = 1, 2, \ldots, n \right) = -\frac{1}{2} \sum_{i=1}^d (\alpha_i - 1)(\log b_i - \log a_i)^{-1}.
$$

(2.13)

Note that this can not be obtained as a corollary of Theorem 2.1 as $X \notin \text{RV}(\alpha, \mu)$ if $\alpha_i$’s are not equal. Even if $\alpha_i = \alpha$ for all $i = 1, 2, \ldots, d$, then it is known in the literature (see Section 6.5.1 in [24]) that the angular measure corresponding to the limit measure $\mu$ is purely atomic and concentrated on the axes which does not fall under the assumptions of Theorem 2.1. Moreover, it is obvious to see that, when all $\alpha$’s are identical, the expression for $\phi$ given in Theorem 2.1 does not coincide with the persistence exponent (2.13).

3. Proof of Theorem 2.1

The proof of Theorem 2.1 will be divided into proving the respective lower and upper bounds.

3.1. Upper bound

We will prove that

$$
\limsup_{n \to \infty} \frac{1}{(\log n)^2} \log P_n \leq -\frac{\alpha - 1}{2 \log r^*}.
$$

(3.1)
Step 1. We divide the set of time slots \{1, 2, \ldots, n\} into appropriate blocks. Fix \(\eta > 0\). Define \(u_1 = C_1\) and

\[ u_{i+1} = [(1 + \eta)^i (r^*)^i C_1] \]  

for \(i \in \mathbb{N}\) and some constant \(C_1\). We define \(\lambda_n\) satisfying \(u_{\lambda_n} = n\). That is,

\[ \lambda_n = [1 + \left(\log(1 + \eta) + \log r^*\right)^{-1} \log n]. \]  

By considering the sets \(B_i = \{u_{i-1} + 1, u_{i-1} + 2, \ldots, u_i\}\) for \(i \geq 2\) and \(B_1 = \{1, 2, \ldots, C_1\}\), we get the following representation of \(P_n\):

\[ P_n = P(k^{-1}S_k \in A \text{ for all } k \in \{1, 2, 3, \ldots, n\}) \]

\[ = \lambda_n \prod_{i=1}^{\lambda_n} P(k^{-1}S_k \in A \text{ for all } k \in B_i) \]

\[ = \lambda_n \prod_{i=1}^{\lambda_n} P(k^{-1}S_k \in A \text{ for all } k \in B_{i+1}| k^{-1}S_k \in A \text{ for all } k \in B_i) \]

\[ \times P(k^{-1}S_k \in A \text{ for all } k \in B_1). \]  

Using the Markov property, we obtain the following upper bound for the product in (3.4)

\[ P(k^{-1}S_k \in A \text{ for all } k \in B_1) \prod_{i=1}^{\lambda_n} P(u_{i+1}^{-1}S_{u_{i+1}} \in A|u_i^{-1}S_{u_i} \in A). \]  

To prove (3.13), we develop a suitable upper bound for the conditional probability in (3.5) for large enough \(i\).

Step 2. In this step we derive an upper bound for

\[ P(u_{i+1}^{-1}S_{u_{i+1}} \in A|u_i^{-1}S_{u_i} \in A) \]

\[ = [P(u_i^{-1}S_{u_i} \in A)]^{-1} P(S_{u_{i+1}} \in u_{i+1} \circ A, S_{u_i} \in u_i \circ A). \]  

In Step 4 below, we shall show that

\[ \text{dist}(u_i \circ A, u_{i+1} \circ A) = C_2 u_i \]  

for some positive constant \(C_2\). This implies \(\{\|S_{u_{i+1}} - S_{u_i}\| \geq C_2 u_i; S_{u_i} \in u_i \circ A\} \supset \{u_{i+1}^{-1}S_{u_{i+1}} \in A; u_i^{-1}S_{u_i} \in A\}\). Using this observation, we obtain the following upper bound for the second probability in (3.6)

\[ P(\|S_{u_{i+1}} - S_{u_i}\| \geq C_2 u_i; S_{u_i} \in u_i \circ A) \]
where the last equality is obtained using the independent increment property of the random walk. Using the upper bound derived in (3.8), we can derive the following upper bound for the ratio in (3.6):

$$P \left( \| S_{u_{i+1}} - u_i \| > C_2 u_i \right) = P \left( \frac{u_i^{-1} S_{u_{i+1}} - u_i}{u_i} \in \{ x \in \mathbb{R}^d : \| x \| > C_2 \} \right).$$ (3.9)

To bound this expression further, we shall use the following estimate, taken from [19, Lemma 2.1]:

$$\frac{P (n^{-1} S_n \in \cdot)}{nP (\| X \| > n)} \xrightarrow{n} \mu (\cdot)$$ (3.10)

on $\mathcal{B} (\mathbb{R}^d \setminus \{0\})$. Note that $\{ x : \| x \| > C_2 \}$ is bounded away from 0 and $\mu$ does not charge any mass at its boundary. We also observe that $u_i^{-1} (u_{i+1} - u_i) > (r_* - 1)$ [note that $r_* > 1$ as $A$ has non-empty interior]. Combining these observations, we obtain the following upper bound for (3.9):

$$P \left( (u_{i+1} - u_i)^{-1} S_{u_{i+1} - u_i} \in \{ x : \| x \| > (r_* - 1) C_2 \} \right)$$

$$\leq [\mu (\{ x : \| x \| > (r_* - 1) C_2 \}) + \epsilon_1] (u_{i+1} - u_i)^{-1 - \alpha} L_{\| \cdot \|} (u_{i+1} - u_i)$$

$$\leq C_3 u_i^{-1 - \alpha} L_{\| \cdot \|} (u_i)$$ (3.11)

for all $i \geq N(\epsilon_1)$ for sufficiently large $N(\epsilon_1)$, where $L_{\| \cdot \|}$ is a slowly varying function related to the regularly varying random variable $\| X \|$ and $C_3 \in (0, \infty)$ is some constant. In addition, we have used the fact that $L_{\| \cdot \|} (u_{i+1} - u_i)/L_{\| \cdot \|} (u_i)$ is bounded above which follows from the facts that $L_{\| \cdot \|}$ is a slowly varying function and $u_i^{-1} (u_{i+1} - u_i) \to (r_* (1 + \eta) - 1)$.

Fix $\epsilon_2 \in (0, \alpha - 1)$. From Potter’s bound (see e.g. [23, Prop. 0.8(ii)]) there exists a large integer $N(\epsilon_2)$ such that $L_{\| \cdot \|} (u_i) \leq u_i^{\epsilon_2}$ for $i \geq N(\epsilon_2)$. Define $N_1 := N(\epsilon_1) \lor N(\epsilon_2)$.

Step 3 Combining what has been achieved in Step 1 and Step 2 we see that

$$P_n \leq P \left( k^{-1} S_k \in A \text{ for all } k \in B_1 \right) \prod_{i=1}^{N_1} P \left( u_i^{-1} S_{u_{i+1}} \in A \mid u_i^{-1} S_u \in A \right)$$

$$C_3^{\alpha n} \prod_{i=N_1+1}^{\alpha n} \frac{1}{1 - \alpha + \epsilon_2} C_3^{\alpha n} \prod_{i=N_1+1}^{\alpha n} \frac{1}{1 - \alpha + \epsilon_2}$$

for some constant $C(N_1)$. Using $\lambda_n = O (\log n)$, a straightforward algebra yields

$$\limsup_{n \to \infty} \frac{1}{(\log n)^2} \log P_n \leq -\frac{\alpha - 1 - \epsilon_2}{2} \left[ \log r^* (1 + \eta) \right]^{-1},$$
which completes the proof of (3.13) after taking $\epsilon_2 \to 0$ and $\eta \to 0$.

**Step 4** We are left to prove only (3.7), that is $\text{dist}(u_i \circ A, u_{i+1} \circ A) = C_2 u_i$.

Note that
\[
\text{dist}(u_i \circ A, u_{i+1} \circ A) = \inf_{x \in A, y \in A} \| u_{i+1} \cdot x - u_i \cdot y \|.
\]
\[
= u_i \inf_{x \in A, y \in A} \| r^*(1 + \eta) \cdot x - y \|. \tag{3.12}
\]

It is sufficient to show that $\inf_{x \in A} \inf_{y \in A} \| r^*(1 + \eta) \cdot x - y \| > 0$. Fix $x \in A$.

Then there are two cases. The first case is that $y$ is in the same direction as $x$ with respect to origin $0$, that is, $y = t \|x\|^{-1} \cdot x$ for some $t > 0$. The optimality of $r^*$ implies that $\| r^*(1 + \eta) \cdot x - y \| > 0$ in this case. The other case is that $y$ is not in the direction of $x$. In this case, it is clear that $y$ can not be written as scalar multiple of $x$ implying $\| r^*(1 + \eta) \cdot x - y \| > 0$. This holds for any $x \in A$ and hence positivity of the expression in (3.12).

### 3.2. Lower bound for $\mathbb{R}^d$

The proof of the lower bound
\[
\liminf_{n \to \infty} \frac{1}{(\log n)^2} \log P_n \geq -\frac{\alpha - 1}{2 \log r^*}. \tag{3.13}
\]
is much more demanding. Using (2.5), and the discussion following that equation, we define $r_\delta$ as the solution of
\[
\max_{r, y} r \tag{3.14}
\]
subject to
\[
H(y) \leq \delta, H(r \cdot y) \leq \delta. \tag{3.15}
\]

We can equivalently write this as the solution of the problem
\[
v(\delta) = -r_\delta = \min_{r, y} -r \tag{3.16}
\]
subject to the constraints $H(y) \leq \delta$ and $H(r \cdot y) \leq \delta$. Since $A$ is compact, $H$ has compact level sets for levels $\delta \leq 0$. Since $H$ is continuous on $A$ and $A$ has non-empty interior, there exists a $\delta < 0$ such that the subset $A^\delta := \{x : H(x) \leq \delta\}$ of $A$ is non-empty, and so we see that $v(\delta) \leq -1 < \infty$ on $\delta$ in a neighborhood of 0. Since $H(r \cdot y)$ is a composition of convex functions, it is jointly convex on $[0, \infty) \times [0, \infty)^d$. Thus, we can apply Corollary 1 of [26] to conclude that $v(\delta)$ is continuous in a neighborhood of 0.

Without loss of generality, we can now assume that $\phi^* \in \Xi^0(A)$ for an interior $\Xi^0(A)$ of $\Xi(A)$, that is $\{y \in \mathbb{S}^{d-1} : \|y - \phi^*\| < \epsilon\} \subset A$ for some $\epsilon > 0$. Indeed, if $\phi^* \in \partial \Xi(A) = \Xi(A) \setminus \Xi^0(A)$, then one can consider set $A^\delta$ instead
and then take $\delta \downarrow 0$ at the last step of the proof.

**Step 1.** Define $C_\epsilon(\varphi^*) = \{ y \in \mathbb{S}^{d-1} : \| y - \varphi^* \| \leq \epsilon \}$ for some $\epsilon > 0$. From the above assumption $\varphi^* \in \Xi^0(A)$, we can fix an $\epsilon > 0$ satisfying $C_\epsilon(\varphi^*) \subset \Xi(A)$. This implies that the solid cone $C = \{ r \cdot y : y \in C_\epsilon(\varphi^*), r > 0 \}$ has non-empty intersection with $A$. We shall say a hypercuboid is aligned in the direction $\varphi^*$ if the hypercuboid is specified by the orthogonal set of unit vectors $(\epsilon_i : 1 \leq i \leq d)$ with $\epsilon_1 = \varphi^*$. We define $[\cdot](\epsilon)$ to be the largest hypercuboid inscribed in $C \cap A$. It is clear that as $\epsilon \to 0$, $C_\epsilon$ converges to the straight line $\{ r \cdot \varphi^* : r > 0 \}$. Hence it is clear that $C_\epsilon \cap A$ converges to $\{ r \cdot \varphi^* : r \in [L_{\varphi^*}, U_{\varphi^*}] \}$. These observations can be used to obtain that $[\cdot](\epsilon)$ converges to $\{ r \cdot \varphi^* : r \in [L_{\varphi^*}, U_{\varphi^*}] \}$ using the notion of convergence of sets (see [27, Definition 4.1]).

To specify a $d$-dimensional hypercuboid $[\cdot](\epsilon)$, define

$$[\cdot](\epsilon) = \left\{ x : (x, \epsilon_i) \in [\beta_i^{(i)}(\epsilon), \beta_u^{(i)}(\epsilon)] \text{ for all } i = 1, 2, \ldots, d \right\}. \quad (3.17)$$

Note that $[\cdot](\epsilon) \subset A$. Moreover, we have chosen $([\cdot](\epsilon) : \epsilon > 0)$ in such a way that

$$\beta_i^{(i)}(\epsilon) \uparrow 0 \quad \text{and} \quad \beta_u^{(i)}(\epsilon) \downarrow 0 \quad \text{as} \quad \epsilon \to 0 \quad \text{for all} \quad i = 2, \ldots, d \quad (3.18)$$

and

$$\beta_l^{(1)}(\epsilon) \downarrow L_{\varphi^*} \quad \text{and} \quad \beta_u^{(1)}(\epsilon) \uparrow U_{\varphi^*} \quad \text{as} \quad \epsilon \to 0. \quad (3.19)$$

We have

$$P\left( k^{-1}S_k \in A \text{ for all } k = 1, 2, 3, \ldots, n \right) \geq P\left( k^{-1}S_k \in [\cdot](\epsilon) \text{ for all } k = 1, 2, 3, \ldots, n \right). \quad (3.20)$$

**Step 2.** As in Step 1 of the upper bound, we divide $\{1, 2, \ldots, n\}$ into smaller pieces. Define

$$r^{(\epsilon)} = \beta_u^{(1)}(\epsilon)/\beta_l^{(1)}(\epsilon) \quad (3.21)$$

and note that $r^{(\epsilon)} \to r_{\varphi^*}$ as $\epsilon \to 0$. Let

$$m_1 = C_1, \quad m_i = \lceil (r^{(\epsilon)})^{i-1}C_1 \rceil \quad \text{for} \quad i \geq 2, \quad (3.22)$$

where $C_1$ is some positive constant, and for $i \geq 1$,

$$D_1 = \{1, 2, \ldots, m_1\} \quad \text{and} \quad D_{i+1} = \{m_i + 1, m_i + 2, \ldots, m_{i+1}\}. \quad (3.23)$$

Then we get following expression for the probability in $(3.20)$,

$$P\left( k^{-1}S_k \in [\cdot](\epsilon) \text{ for all } k = 1, 2, \ldots, n \right) \geq P\left( k^{-1}S_k \in [\cdot](\epsilon) \text{ for all } k \in D_1 \right) \prod_{i=1}^{n} P\left( k^{-1}S_k \in [\cdot](\epsilon) \text{ for all } k \in D_{i+1} | m_i^{-1}S_{m_i} \in [\cdot](\epsilon) \right) \quad (3.24)$$
Fig 2. Approximation by constructing a narrow hypercuboid: The blue line denotes the optimal direction. We have constructed the largest possible hypercuboid $\blacklozenge(\epsilon)$ inscribed in the intersection of $A$ and the cone.

for

$$\kappa_n = [1 + (\log nC_{1}^{-1})]/\log r^{(\epsilon)}].$$

(3.25)

We will derive a lower bound for each term in the product of (3.24) by taking

$$P \left( k^{-1}S_k \in \blacklozenge(\epsilon) \text{ for all } k \in D_{i+1} \right) m_i^{-1}S_{m_i} \in \blacklozenge(\epsilon) \geq T_1 \times T_2,$$

(3.26)

where

$$T_1 = P \left( k^{-1}S_k \in \blacklozenge(\epsilon) \text{ for all } k \in D_{i+1} \right) S_{m_i} \in \square_i,$$

(3.27)

$$T_2 = P \left( S_{m_i} \in \square_i \text{ for all } m_i \in \square_i \right) m_i^{-1}S_{m_i} \in \blacklozenge(\epsilon),$$

(3.28)

for the set

$$\square_i = \left\{ x : \langle x, e_j \rangle \in [\eta_{i,1}^{(j)}m_i + \eta_{i,2}^{(j)}m_i^{1/\alpha_0 + \delta}, \eta_{i,1}^{(j)}m_i + \eta_{i,2}^{(j)}m_i^{1/\alpha_0 + \delta}] \text{ for all } j = 1, 2, \ldots, d \right\}.$$  

(3.29)

is fixed and $\alpha_0 = \alpha \wedge 2$. The full specification of all constants that define $\square_i$ is given in Lemma 3.1. We consider the following partition of the set $D_{i+1} = D_{i+1}^{(1)} \cup D_{i+1}^{(2)} \cup D_{i+1}^{(3)}$ where

$$D_{i+1}^{(1)} = \left\{ m_i, m_i + 1, m_i + 2, \ldots, \lfloor f_1m_{i+1} \rfloor \right\}$$

$$D_{i+1}^{(2)} = \left\{ \lfloor f_1m_i \rfloor + 1, \lfloor f_1m_i \rfloor + 2, \ldots, \lfloor f_2m_i \rfloor \right\}$$

$$D_{i+1}^{(3)} = \left\{ \lfloor f_2m_i \rfloor + 1, \lfloor f_2m_i \rfloor + 2, \ldots, m_{i+1} \right\}.$$
with
\[ f_1 = 1 + (r^\varepsilon - 1)/3 \quad \text{and} \quad f_2 = 1 + 2(r^\varepsilon - 1)/3. \] (3.30)

We also define the following sets:
\[ \square_{i,1} = \left\{ \mathbf{x} : (\mathbf{x}, \mathbf{e}_j) \in [\eta^{(j)}_{i,1} m_i + (\eta^{(j)}_{i,2} - 1) m_i^{1/\alpha_0 + \delta}, \eta^{(j)}_{u,1} m_i + (\eta^{(j)}_{u,2} + 1) m_i^{1/\alpha_0 + \delta}] \right\} \]
for all \( j = 1, 2, \ldots, d \). \[ \square_{i,2} = \left\{ \mathbf{x} : (\mathbf{x}, \mathbf{e}_j) \in [\eta^{(j)}_{i,1} m_i + (\eta^{(j)}_{i,2} - 2) m_i^{1/\alpha_0 + \delta}, \eta^{(j)}_{u,1} m_i + \eta^{(j)}_{u,2} m_i^{1/\alpha_0 + \delta}] \right\} \]
for all \( j = 1, 2, \ldots, d \). \[ \square_{i,3} = \left\{ \mathbf{x} : (\mathbf{x}, \mathbf{e}_j) \in [\eta^{(j)}_{i,1} m_i + (\eta^{(j)}_{i,2} - 1) m_i^{1/\alpha_0 + \delta}, \eta^{(j)}_{u,1} m_i + \eta^{(j)}_{u,2} m_i^{1/\alpha_0 + \delta}] \right\} \]
for all \( j = 1, 2, \ldots, d \). \[ \square_{i,4} = \left\{ \mathbf{x} : (\mathbf{x}, \mathbf{e}_j) \in [\eta^{(j)}_{i,1} m_i + (\eta^{(j)}_{i,2} - 1) m_i^{1/\alpha_0 + \delta}, \eta^{(j)}_{u,1} m_i + (\eta^{(j)}_{u,2} + 1) m_i^{1/\alpha_0 + \delta}] \right\} \]
for all \( j = 1, 2, \ldots, d \),

where the detailed specifications of all constants appearing in \( \square_{i,3} \) are given in the following Lemma, whose proof is given in the Appendix.

**Lemma 3.1.** Let the constants appearing in \( \square_i \) be given by
\[ \eta^{(1)}_{i,1} = \beta^{(1)}_i (r^\varepsilon + 1)/3, \quad \eta^{(2)}_{i,1} = 3, \quad \eta^{(1)}_{u,1} = \beta^{(1)}_u (5r^\varepsilon + 1)/6, \quad \eta^{(1)}_{u,2} = 1, \]
\[ \eta^{(j)}_{i,1} = \beta^{(j)}_i (r^\varepsilon)/3, \quad \eta^{(j)}_{u,1} = \beta^{(j)}_u (r^\varepsilon)/3, \quad \eta^{(j)}_{u,2} = 2, \quad \eta^{(j)}_{u,2} = 1 \] (3.31)
for all \( j = 2, 3, \ldots, d \). In addition, let the constants appearing in \( \square_{i,3} \) be given by
\[ \eta^{(1)}_{i,1} = \beta^{(1)}_i (e^\varepsilon + 1)/3, \quad \eta^{(1)}_{i,2} = 2, \quad \eta^{(1)}_{u,1} = \beta^{(1)}_u (e^\varepsilon + 2)/3, \quad \eta^{(1)}_{u,2} = 0, \]
\[ \eta^{(j)}_{i,1} = 2\beta^{(j)}_i (e^\varepsilon)/3, \quad \eta^{(j)}_{u,1} = 2\beta^{(j)}_u (e^\varepsilon)/3, \quad \eta^{(j)}_{u,2} = 2, \quad \eta^{(j)}_{u,2} = 0 \] (3.32)
for all \( j = 2, 3, \ldots, d \). Then, we have for large enough \( i \),

1. \( \square_i \subset \left\{ m_i \circ \mathbb{1}(\varepsilon) \right\} \cap \left\{ (f_2 m_i) \circ \mathbb{1}(\varepsilon) \right\}; \)
2. \( \square_{i,1} \subset \left\{ (f_1 m_i) \circ \mathbb{1}(\varepsilon) \right\}; \)
3. \( \square_{i,2} \subset \left\{ (f_2 m_i) \circ \mathbb{1}(\varepsilon) \right\} \cap \left\{ (f_1 m_i) \circ \mathbb{1}(\varepsilon) \right\}; \)
4. \( \square_{i,3} \subset \left\{ (f_2 m_i) \circ \mathbb{1}(\varepsilon) \right\} \cap \left\{ m_{i+1} \circ \mathbb{1}(\varepsilon) \right\}; \)
5. \( \square_{i,4} \subset \left\{ m_{i+1} \circ \mathbb{1}(\varepsilon) \right\}. \)

To analyze the event \( \{k^{-1} S_k \in \mathbb{1}(\varepsilon) \} \) for all \( k \in D_{i+1} \) appearing in (3.24), we conclude from Lemma 3.1 that
\[ \left\{ S_k \in \square_{i,1} \text{ for all } k \in D_{i+1}^{(1)} \right\} \subset \left\{ k^{-1} S_k \in \mathbb{1}(\varepsilon) \text{ for all } k \in D_{i+1}^{(1)} \right\}; \]
\[ \{ S_k \in □_{i,2} \text{ for all } k \in D^{(2)}_{i+1} \} \setminus \{ |f_2 m_i| \} : S_{|f_2 m_i|} \in □_{i,3} \}\]
\[ \subset \{ k^{-1}S_k \in A \text{ for all } k \in D^{(2)}_{i+1} \}; \]
\[ \{ S_k \in □_{i,4} \text{ for all } k \in D^{(3)}_{i+1} \} \subset \{ k^{-1}S_k \in □(\epsilon) \text{ for all } k \in D^{(3)}_{i+1} \}. \]

The motivation for considering these inclusions is as follows. Note that the segment of the random walk \( (S_k : m_i \leq k \leq m_{i+1}) \) starts from \( □_i \subset m_i \circ □(\epsilon) \) and reaches \( m_{i+1} \circ □(\epsilon) \) and hence \( ∥S_{m_{i+1}} - S_m∥ = O(m_i) \) as \( m_{i+1} - m_i = O(m_i) \) for large enough \( i \). This phenomenon can be explained by the principle of a single big jump for the shifted random walk \( (S_k - S_{m_i} : m_i + 1 \leq k \leq m_{i+1}) \). So the shifted random walk \( (S_k - S_{m_i} : m_i + 1 \leq k \leq m_{i+1}) \) must have exactly one jump of order \( O(m_i) \) and the total contribution of the other \( (m_{i+1} - m_i - 1) \) jumps will be of order \( o(m_i) \). To make this precise, we split the segment into three parts and among them \( (S_k : k \in D^{(2)}_{i+1}) \) contains the necessary large jump (see \( T_4 \) in (3.33)) and the contributions from other parts are negligible (see \( T_3 \) and \( T_5 \) in (3.33)). This will be proved in the next step.

Step 3. In this part of the proof we estimate \( T_1 = P(k^{-1}S_k \in □(\epsilon) \text{ for all } k \in D_{i+1} \mid S_{m_i} \in □_i) \) in (3.27). Combining the above inclusions, we get that
\[ T_1 \geq T_3 \times T_4 \times T_5, \] (3.33)

where
\[ T_3 = P(S_k \in □_{i,1} \text{ for all } k \in D^{(1)}_{i+1} \mid S_{m_i} \in □_i); \]
\[ T_4 = P(S_k \in □_{i,2} \text{ for all } k \in D^{(2)}_{i+1} \setminus \{ |f_2 m_i| \}; S_{|f_2 m_i|} \in □_{i,3} \mid S_{|f_1 m_i|} \in □_{i,1}); \]
\[ T_5 = P(S_k \in □_{i,3} \text{ for all } k \in D^{(3)}_{i+1} \mid S_{|f_2 m_i|} \in □_{i,3}). \]

TERM \( T_3 \). We shall deal with each of the term separately. We start analysis with \( T_3 \). Note that
\[ P(S_k \in □_{i,1} \text{ for all } k \in D^{(1)}_{i+1} \mid S_{m_i} \in □_i) \]
\[ = \left[ P(S_{m_i} \in □_i) \right]^{-1} P(S_k \in □_{i,1} \text{ for all } k \in D^{(1)}_{i+1} \mid S_{m_i} \in □_i) \]
\[ \geq \left[ P(S_{m_i} \in □_i) \right]^{-1} P(S_k - S_{m_i}, e_j) \in [-m_i^{1/\alpha_0 + \delta}, m_i^{1/\alpha_0 + \delta}] \text{ for all } \]
\[ j = 1, 2, \ldots , d \text{ and } k \in D^{(1)}_{i+1} \mid S_{m_i} \in □_i) \]
\[ = P(\bigcap_{j=1}^{d} \left\{ \min_{1 \leq k \leq |f_j m_i| - m_i} \langle S_k, e_j \rangle > -m_i^{1/\alpha_0 + \delta} \quad \text{ and } \quad \max_{1 \leq k \leq |f_j m_i| - m_i} \langle S_k, e_j \rangle < m_i^{1/\alpha_0 + \delta} \right\}) \]
\[ \geq 1 - d \sum_{j=1}^{d} P(\min_{1 \leq k \leq |f_j m_i| - m_i} \langle S_k, e_j \rangle < -m_i^{1/\alpha_0 + \delta} \text{ or } \max_{1 \leq k \leq |f_j m_i| - m_i} \langle S_k, e_j \rangle > m_i^{1/\alpha_0 + \delta}) \]
Lemma 3.2. Let $\mathbf{X} \in \text{RV}(\alpha, \mu)$ and $\mu = \nu_\alpha \otimes \varsigma$ on $(0, \infty) \times \mathbb{S}^{d-1}$ with $\varsigma$ being absolutely continuous with respect to the Lebesgue measure. Then for any direction vector $\mathbf{u} \in \mathbb{S}^{d-1}$, we have $\langle \mathbf{u}, \mathbf{X} \rangle \in \text{RV}(\alpha, \vartheta_\alpha)$ where $\vartheta_\alpha$ is a Radon measure on $\mathbb{R}\setminus\{0\}$ with

$$
\vartheta_\alpha(dx) = \alpha \mu(\{y : \langle \mathbf{u}, \mathbf{y} \rangle > 1\}) x^{-\alpha-1} dx \mathbb{1}(x > 0)
+ \alpha \mu(\{y : \langle \mathbf{u}, \mathbf{y} \rangle < -1\}) (-x)^{-\alpha-1} \mathbb{1}(x < 0).
$$

(3.35)

Fix $\delta_{t,j}$ for $t = 1, 2, \ldots, d$ and $j = 1, 2$ such that $\delta_{t,j} \in (0, 1/2d)$. Then we get that

$$
P \left( \max_{1 \leq k \leq |f_i m_i| - m_i} \langle \mathbf{S}_k, \mathbf{e}_j \rangle > m_i^{1/\alpha_0 + \delta} \right) < \delta_{j,2}
$$

(3.36)

for sufficiently large $i$.

Indeed, using Lemma 3.2, note that $\langle \mathbf{S}_k, \mathbf{e}_j \rangle = \sum_{t=1}^k Y_t$ is a mean 0 random walk with steps $Y_t = \langle \mathbf{X}_t, \mathbf{e}_j \rangle \in \text{RV}(\alpha, \vartheta_\alpha)$. For $\alpha \in (1, 2]$ we will apply the generalized Kolmogorov inequality given in [28]:

$$
P \left( \max_{1 \leq k \leq m} \sum_{t=1}^k Y_t \geq x \right) \leq C_4 m x^{-2} \mathbb{E} \left[ X^2 \mathbb{1}(|X| < x) \right],
$$

(3.37)

where $C_4$ is some constant. In this case, as [28] noted, $\mathbb{E} \left[ X^2 \mathbb{1}(|X| < x) \right]$ is regularly varying with index 2 $- \alpha$ (or slowly varying if $\alpha = 2$). For $\alpha > 2$ we can apply the classical Kolmogorov inequality. In both cases we can bound

$$
P \left( \max_{1 \leq k \leq |f_i m_i| - m_i} \langle \mathbf{S}_k, \mathbf{e}_j \rangle > m_i^{1/\alpha_0 + \delta} \right) \leq C_5 m_i^{-\alpha_0 \delta + \eta},$$

where $\eta$ appears due to Potter’s bound applied to the slowly varying part of $\mathbb{E} \left[ X^2 \mathbb{1}(|X| < x) \right]$ and $C_5$ is some constant. For $\eta > 0$ sufficiently small this upper bound gives (3.36) as $m_i \to \infty$ with $i \to \infty$.

Similarly, we can prove that

$$
P \left( \min_{1 \leq k \leq |f_i m_i| - m_i} \langle \mathbf{S}_k, \mathbf{e}_j \rangle < -m_i^{1/\alpha_0 + \delta} \right) < \delta_{j,2}$$

(3.34)
for large enough $i$. Hence we get that

$$T_3 \geq \left( 1 - \sum_{j=1}^{d} \sum_{t=1}^{2} \delta_{i,t} \right)$$

(3.38)

for large enough $i$.

**TERM $T_4$.** Define

$$\mathcal{E} = \left\{ x : \langle x, e_j \rangle \in \left[ (\tilde{\eta}(j)_{u,1} - \eta_{u,1})m_i + (\tilde{\eta}(j)_{u,2} - \eta_{u,2}) + 2 \right] m_i^{1/\alpha_0 + \delta}, \right.$$

$$\left. (\tilde{\eta}(j)_{u,1} - \eta_{u,1})m_i + (\tilde{\eta}(j)_{u,2} - \eta_{u,2})m_i^{1/\alpha_0 + \delta} \right\} \text{ for all } j = 1, 2, \ldots, d \right\}.$$  (3.39)

Note that

$$T_4 = P \left( S_k \in \square_{i,2} \text{ for all } k \in D_{i+1}^{(2)} \setminus \{ [f_2m_i] \}; S_{[f_2m_i]} \in \square_{i,3}, S_{[f_1m_i]} \in \square_{i,1} \right)$$

$$\geq \left[ P \left( S_{[f_1m_i]} \in \square_{i,1} \right) \right]^{-1} \left[ P \left( \left( S_k - S_{[f_1m_i]}, e_j \right) \in [-m_i^{1/\alpha_0 + \delta}, \right. \right.$$

$$\left. \left( (\tilde{\eta}(j)_{u,1} - \eta_{u,1})m_i + (\tilde{\eta}(j)_{u,2} - \eta_{u,2})m_i^{1/\alpha_0 + \delta} \right\} \text{ for all } j = 1, 2, \ldots, d \right)$$

$$\text{ and } k \in D_{i+1}^{(2)} \setminus \{ [f_2m_i] \}; S_{[f_2m_i]} - S_{[f_1m_i]} \in \mathcal{E}; S_{[f_1m_i]} \in \square_{i,1} \right)$$

$$= P \left( \langle S_k - S_{[f_1m_i]}, e_j \rangle \in [-m_i^{1/\alpha_0 + \delta}, \eta(\tilde{\eta}(j)_{u,1} - \eta_{u,1})m_i + (\tilde{\eta}(j)_{u,2} - \eta_{u,2})m_i^{1/\alpha_0 + \delta} \right)$$

$$\text{ for all } j = 1, 2, \ldots, d \text{ and } k \in D_{i+1}^{(2)} \setminus \{ [f_2m_i] \}; S_{[f_2m_i]} - S_{[f_1m_i]} \in \mathcal{E}. \right)$$

(3.40)

The event inside the probability in (3.40) will be written as disjoint union of the following events

$$E_t = \left\{ X_t \in \mathcal{E} : \max_{1 \leq k \leq t-1} \langle S_k, e_j \rangle < m_i^{1/\alpha_0 + \delta} \text{ and } \min_{1 \leq k \leq t-1} \langle S_k, e_j \rangle > m_i^{1/\alpha_0 + \delta} \right\}$$

$$\text{ for all } j = 1, 2, \ldots, d; \text{ and } t \geq 1 \text{ with } t+1 \leq \left[ f_2m_i \right] - \left[ f_1m_i \right] \text{ and }$$

$$\min_{t+1 \leq k \leq \left[ f_2m_i \right] - \left[ f_1m_i \right]} \langle S_k - X_t, e_j \rangle > -m_i^{1/\alpha_0 + \delta} \right\}$$

(3.41)

for all $t = 1, 2, \ldots, \left[ f_2m_i \right] - \left[ f_1m_i \right]$. Using the fact that $(E_t : 1 \leq t \leq \left[ f_2m_i \right] - \left[ f_1m_i \right])$ is a collection of disjoint and exchangeable events we can conclude that

$$T_4 \geq \left| D_{i+1}^{(2)} \right| P \left( X_1 \in \mathcal{E}, \max_{2 \leq k \leq \left[ f_2m_i \right] - \left[ f_1m_i \right]} \langle S_k - X_1, e_j \rangle < m_i^{1/\alpha_0 + \delta} \right.$$

$$\min_{2 \leq k \leq \left[ f_2m_i \right] - \left[ f_1m_i \right]} \langle S_k - X_1, e_j \rangle > -m_i^{1/\alpha_0 + \delta} \text{ for all } j = 1, 2, \ldots, d \right)$$

$$\left| D_{i+1}^{(2)} \right| P \left( X_1 \in \mathcal{E}, \max_{1 \leq k \leq \left[ f_2m_i \right] - \left[ f_1m_i \right]} \langle S_k, e_j \rangle < m_i^{1/\alpha_0 + \delta} \right.$$
Combining the lower bounds obtained in (3.42) and (3.48) for all \( j = 1, 2, \ldots, d \)

\[
\min_{1 \leq k \leq |f_2 m_i| - |f_1 m_i| - 1} \langle S_k, e_j \rangle > -m_i^{1/\alpha_0 + \delta}
\]

\[
:= T_6 \times T_7. 
\]

(3.42)

\( T_7 \) is similar to \( T_3 \) and so can be dealt in similar way to get

\[
T_7 \geq \left( 1 - \sum_{j=1}^{d} \sum_{t=1}^{2} \delta_{j,t} \right)
\]

(3.43)

for large enough \( i \). To estimate \( T_6 \), note that

\[
T_6 = |D_{i+1}^{(2)}| P( X_1 \in \mathbb{B}_i )
\]

\[
= (|f_2 m_i| - |f_1 m_i|) P( X_1 \in \mathbb{B}_i )
\]

\[
= (f_2 - f_1) m_i P( X_1, e_j ) \in [ (\tilde{\eta}_{l,1}^{(j)} - \eta_{l,1}^{(j)}) m_i + (\tilde{\eta}_{l,2}^{(j)} - \eta_{l,2}^{(j)} + 2) m_i^{1/\alpha_0 + \delta},
\]

\[
(\tilde{\eta}_{l,1}^{(j)} - \eta_{l,1}^{(j)} + 2) m_i + (\tilde{\eta}_{l,2}^{(j)} - \eta_{l,2}^{(j)} - 2) m_i^{1/\alpha_0 + \delta} ] \text{ for all } j = 1, 2, \ldots, d
\]

\[
\geq (f_2 - f_1) m_i P( m_i^{-1} X_1 \in \mathbb{B}_i ),
\]

(3.44)

where

\[
\mathbb{B}_i = \left\{ x : \langle x, e_j \rangle \in [ \tilde{\eta}_{l,1}^{(j)} - \eta_{l,1}^{(j)} - \eta_{l,1}^{(j)} ] \text{ for all } j = 1, 2, \ldots, d \right\}.
\]

(3.45)

We know from the definition of regular variation that

\[
\lim_{i \to \infty} \frac{P( m_i^{-1} X_1 \in \mathbb{B}_i )}{P( \| X_1 \| > m_i )} = c \mu( \mathbb{B}_i )
\]

(3.46)

for some constant \( c > 0 \); see e.g. [24, Thm. 6.1, p. 173]. It is straightforward to check that \( \mu( \mathbb{B}_i ) \in (0, \infty) \) as \( \mathbb{B}_i \) contains a ball (non-trivial angular and radial part) which is bounded away from \( 0 \). Hence

\[
P( m_i^{-1} X_1 \in \mathbb{B}_i ) \geq (c \mu( \mathbb{B}_i ) - \delta_1) P( \| X_1 \| \geq m_i )
\]

(3.47)

for large enough \( i \) and \( \delta_1 \in (0, c \mu( \mathbb{B}_i )) \). It is also known that \( P( \| X_1 \| > m_i ) = m_i^{-1} L_{\| \cdot \|}(m_i) \) for some slowly varying function \( L_{\| \cdot \|} \). Thus, we get the following lower bound for \( T_6 \),

\[
(f_2 - f_1) m_i^{1-\alpha} L_{\| \cdot \|}(m_i) \left( c \mu( \mathbb{B}_i ) - \delta_1 \right).
\]

(3.48)

Combining the lower bounds obtained in (3.48) and (3.43), we obtain

\[
T_4 \geq (f_2 - f_1) \left( 1 - \sum_{j=1}^{d} \sum_{t=1}^{2} \delta_{j,t} \right) \left( c \mu( \mathbb{B}_i ) - \delta_1 \right) m_i^{1-\alpha} L_{\| \cdot \|}(m_i).
\]

(3.49)
TERM T5. Observe that
\[ T_b = P \left( S_k \in \square_{i,4} \text{ for all } k \in D_{i+1}^{(3)} | S_{[f_2 m_i]} \in \square_{i,3} \right) \]
\[ = \left[ P \left( S_{[f_2 m_i]} \in \square_{i,3} \right) \right]^{-1} P \left( S_k \in \square_{i,4} \text{ for all } k \in D_{i+1}^{(3)}, S_{[f_2 m_i]} \in \square_{i,3} \right) \]
\[ \geq P \left( \langle S_k - [f_2 m_i], e_j \rangle \in [-m_i^{1/\alpha_0 + \delta}, m_i^{1/\alpha_0 + \delta}] \text{ for all } k \in D_{i+1}^{(3)} \text{ and } j = 1, 2, \ldots, d \right) \]
\[ \geq \left( 1 - \sum_{j=1}^{d} \sum_{t=1}^{2} \delta_{j,t} \right) \] (3.50)
using the same arguments used to get lower bound for \( T_4 \). Summing up, from (3.33), (3.38), (3.49) and (3.50) we get that
\[ T_1 \geq (f_2 - f_1) \left( 1 - \sum_{j=1}^{d} \sum_{t=1}^{2} \delta_{j,t} \right)^3 \left( c\mu(\square_1) - \tilde{\delta}_1 \right) m_i^{-\alpha} L_{\|\cdot\|}(m_i) \] (3.51)
for sufficiently large \( i \).

Step 4. We next turn to
\[ T_2 = \frac{P \left( S_{m_i} \in \square_i \right)}{P \left( m_i^{-1} S_{m_i} \in \square(\epsilon) \right)} \]
given in (3.28). To analyze the asymptotic behavior of the numerator and denominator of \( T_2 \), we will use again (3.10). From this fact, we get that
\[ \lim_{i \to \infty} \left( m_i P(\|X\| > m_i) \right)^{-1} P \left( m_i^{-1} S_{m_i} \in \square(\epsilon) \right) = c\mu(\square(\epsilon)) \] (3.52)
and hence
\[ P \left( m_i^{-1} S_{m_i} \in \square(\epsilon) \right) \leq m_i P(\|X\| > m_i) \left( c\mu(\square(\epsilon)) + \tilde{\delta}_2 \right) \] (3.53)
for large enough \( i \) where \( \tilde{\delta}_2 \) is small enough constant. A similar reasoning used to show that \( \mu(\square_1) \in (0, \infty) \) can be used to see that \( \mu(\square(\epsilon)) \in (0, \infty) \). Moreover,
\[ P \left( S_{m_i} \in \square_i \right) \]
\[ = P \left( \langle S_{m_i}, e_j \rangle \in [\eta_{i,1}^{(j)} m_i + \eta_{i,2}^{(j)} m_i^{1/\alpha_0 + \delta}, \eta_{i,1}^{(j)} m_i + \eta_{i,2}^{(j)} m_i^{1/\alpha_0 + \delta}] \text{ for all } j = 1, 2, \ldots, d \right) \]
\[ \geq P \left( m_i^{-1} \langle S_{m_i}, e_j \rangle \in [\eta_{i,1}^{(j)} + \xi_j, \eta_{i,1}^{(j)}] \text{ for all } j = 1, 2, \ldots, d \right) \]
\[ \geq m_i P \left( \|X_i\| > m_i \right) \left( c\mu(\square_2) - \tilde{\delta}_3 \right) \] (3.54)
where \( (\xi : j = 1, 2, \ldots, d) \) and \( \tilde{\delta}_3 \in (0, \mu(\square_2)) \) are small enough positive numbers such that \( \mu(\square_2) \in (0, \infty) \) for
\[ \square_2 = \left\{ x : \langle x, e_j \rangle \in [\eta_{i,1}^{(j)} + \xi_j, \eta_{i,1}^{(j)}] \text{ for all } j = 1, 2, \ldots, d \right\} \] (3.55)
It is straightforward to check that \( \mu(\mathbb{M}_2) \in (0, \infty) \). Then we get

\[
T_2 \geq \left( c\mu(\mathbb{M}_2) - \delta_3 \right) \left( c\mu(\mathbb{M}(\epsilon)) + \bar{\delta}_2 \right)^{-1}.
\] (3.56)

**Step 5.** We now put all estimates together to arrive at our lower bound (3.13). Combining the lower bounds obtained in (3.56), (3.38), (3.49) and (3.50) yields

\[
P\left( k^{-1}S_k \in \mathbb{M}(\epsilon) \text{ for all } k \in D_{i+1} | S_{m_i} \in \mathbb{M}_i \right)
\geq \left( 1 - \sum_{j=1}^d \sum_{l=1}^{2d} \delta_{j,l} \right)^{3} (f_2 - f_1)(c\mu(\mathbb{M}_1) - \delta_1)
\geq \left( c\mu(\mathbb{M}(\epsilon)) + \bar{\delta}_2 \right)^{-1} (c\mu(\mathbb{M}_2) - \bar{\delta}_3) m_i^{1-\alpha} L_\| \| (m_i)
:= C_6 m_i^{1-\alpha} L_\| \| (m_i)
\] (3.57)

for some constant \( C_6 \) and for large enough \( i \). Note that a similar expression for the upper bound is obtained in (3.11) with a different constant. Now we can use the same arguments that were used in getting the upper bound to conclude that

\[
\liminf_{n \to \infty} \frac{1}{(\log n)^2} \log P\left( k^{-1}S_k \in A \text{ for all } k = 1, 2, \ldots, n \right) \geq -\frac{1}{2} \alpha (\alpha - 1)(\log r(\epsilon))^{-1}.
\]

Letting \( \epsilon \to 0 \) completes the proof of (3.13). \( \square \)

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**4. Appendix**

**Proof of Lemma 3.1.** 1. We start with proving that \( \square_i \subset \{ m_i \circ \mathbb{M}(\epsilon) \} \). To do this it suffices to prove that \( \beta^{(j)}_1(\epsilon) \leq \eta^{(j)}_{r,2} m_i^{1/\alpha + \delta - 1} < \eta^{(j)}_{u,1} + \eta^{(j)}_{u,2} m_i^{1/\alpha + \delta - 1} < \beta^{(j)}_u(\epsilon) \) for \( j = 1, 2, \ldots, d \). Note that we can ignore the terms \( \eta^{(j)}_{r,2} \) and \( \eta^{(j)}_{u,2} \) as \( m_i^{1/\alpha + \delta - 1} \) can be made arbitrarily small choosing \( i \) large enough. Thus it is enough to show that \( \eta^{(j)}_{r,1} \leq \eta^{(j)}_{u,1} \leq \beta^{(j)}_u(\epsilon) \) for each \( j = 1, 2, \ldots, d \). It follows from (3.31) that the above inequalities hold for \( j = 1, 2, 3, \ldots, d \).

Further, we have \( (2r^{(\epsilon)}+1)\beta^{(1)}_r(\epsilon) = \eta^{(1)}_{r,1} \) and \( \eta^{(1)}_{r,1} < \beta^{(1)}_u(\epsilon) < (2r^{(\epsilon)} + 1)\beta^{(1)}_u(\epsilon)/3 \). These inequalities hold for other \( j \)'s from the definition in (3.31) as well. This observation gives the inclusion \( \square_i \subset \{ (m_2m_i) \circ \mathbb{M}(\epsilon) \} \).

2. The proof of the inclusion \( \square_{i+1} \subset \{ (m_2m_i) \circ \mathbb{M}(\epsilon) \} \cap \{ (m_2m_i) \circ \mathbb{M}(\epsilon) \} \) is similar as only the coefficients of \( m_i^{1/\alpha + \delta} \) changes which can be ignored.
3. Ignoring the coefficients of $m_i^{1/\alpha_0 + \delta}$, we need to check that $f_1\beta_l^{(j)}(\epsilon) < \tilde{\eta}_l^{(j)} < \tilde{\eta}_u^{(j)} < f_1\beta_u^{(j)}(\epsilon)$ for all $j \geq 1$ to show $\square_{i, 2} \subset \{(f_1m_i) \circ \square(\epsilon)\}$. This follows from the definitions given in (3.31) and (3.32) for $f_1 = 2$. For $f_1 = 1$, we can observe that $f_1\beta_u^{(j)}(\epsilon) < \tilde{\eta}_u^{(1)} < \beta_u^{(1)}(\epsilon)(r^{(e)} + 2)/3 = \tilde{\eta}_u^{(1)} = f_1\beta_u^{(1)}(\epsilon)$. which completes the proof of this inclusions.

We shall show next that $\square_{i, 2} \subset \{(f_2m_i) \circ \square(\epsilon)\}$. So it is enough to show that $f_2\beta_l^{(j)}(\epsilon) \leq \tilde{\eta}_l^{(j)} < f_2\beta_u^{(j)}(\epsilon)$ for all $j = 1, 2, \ldots, d$. Note that $f_2\beta_l^{(j)}(\epsilon) = (2^{r^{(e)}} + 1)\beta_l^{(j)}(\epsilon)/3 < \beta_l^{(j)}(\epsilon)/3 = \tilde{\eta}_l^{(j)} = \tilde{\eta}_u^{(j)} = 2\beta_u^{(j)}(\epsilon)/3 < (2^{r^{(e)}} + 1)\beta_u^{(j)}(\epsilon)/3 = f_2\beta_u^{(j)}(\epsilon)$ for all $j = 2, 3, \ldots, d$. Also note that $f_2\beta_l^{(1)}(\epsilon) = (2^{r^{(e)}} + 1)\beta_l^{(1)}(\epsilon)/3 = \tilde{\eta}_l^{(1)} = \tilde{\eta}_u^{(1)} = \beta_u^{(1)}(\epsilon)(r^{(e)} + 2)/3 = \tilde{\eta}_u^{(1)} = 2\beta_u^{(1)}(\epsilon)/3 = f_2\beta_u^{(1)}(\epsilon)$ as $r^{(e)} > 1$.

4. We need to show that $\square_{i, 3} \subset \{(f_2m_i) \circ \square(\epsilon)\}$. It is enough to show that $f_2\beta_l^{(j)}(\epsilon) \leq \tilde{\eta}_l^{(j)} < \tilde{\eta}_u^{(j)} < f_2\beta_u^{(j)}(\epsilon)$ for all $j = 1, 2, \ldots, d$. The above inequalities follow immediately from definition (3.32) for $f_2 = 2$. For $f_2 = 1$, we can show $f_2\beta_l^{(j)}(\epsilon) \leq \tilde{\eta}_l^{(j)} < \tilde{\eta}_u^{(j)} < f_2\beta_u^{(j)}(\epsilon)$ as $r^{(e)} > 1$ and $\beta_l^{(j)}(\epsilon) = \tilde{\eta}_l^{(j)} = \tilde{\eta}_u^{(j)} = \beta_u^{(1)}(\epsilon)(r^{(e)} + 2)/3 = \tilde{\eta}_u^{(1)} = 2\beta_u^{(1)}(\epsilon)/3 = f_2\beta_u^{(1)}(\epsilon)$. So we are done.

5. Note that $\square_{i, 4}$ differs from $\square_{i, 3}$ only in the coefficients of $m_i^{1/\alpha_0 + \delta}$ which can be ignored. So $\square_{i, 4} \subset \{m_{i+1} \circ \square(\epsilon)\}$ follows from the fact $\square_{i, 3} \subset \{m_{i+1} \circ \square(\epsilon)\}$.

Proof of Lemma 3.2. To prove this lemma, we need to find $(b_n : n \geq 1)$ such that

$$\lim_{n \to \infty} nP\left(b_n^{-1}(X, u) \in B\right) = \vartheta_\alpha(B) \in (0, \infty) \quad (4.1)$$

for any $B \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ such that $\vartheta_\alpha(\partial B) = 0$. It is enough to show convergence in (4.1) for the collection of sets $\{(-\infty, -t_1) \cup (t_2, \infty) : t_1 > 0, t_2 > 0\}$ as these collection of intervals is a $\pi$-system (see [24, Lem. 6.1]). We consider the case $B = (t, \infty)$ for $t > 0$. The set $(-\infty, t)$ with $t < 0$ can be handled similarly. If we consider $b_n = a_n$, we get

$$\lim_{n \to \infty} nP\left(a_n^{-1}(X, u) > t\right) = \lim_{n \to \infty} nP\left(a_n^{-1}X \in \{x : (u, x) > t\}\right) = t^{-\alpha} \mu\left(\{x : (u, x) > 1\}\right) \quad (4.2)$$

as $\{x : (u, x) > 1\}$ is bounded away from 0 and it can be proved that $\mu$ does not put any mass at the boundary of this set. Thus the limit exists and satisfies
the scaling homogeneity property. To complete the proof it suffices to show that 
\[ \mu\left(\{x : \langle u, x \rangle > 1\}\right) > 0. \] We show this by using polar decomposition, invoking our assumption on the angular measure. Note that

\[
\mu\left(\{x : \langle u, x \rangle > 1\}\right) \\
\quad = \nu_\alpha \otimes \varsigma\left(\{(r, y) \in (0, \infty) \times S^{d-1} : r\langle u, y \rangle > 1\}\right) \\
\quad = \int_{\{y \in S^{d-1} : \langle u, y \rangle > 0\}} \varsigma(dy) \int_{r>(\langle u, y \rangle)^{-1}} \nu_\alpha(dr) \\
\quad = \int_{\{y \in S^{d-1} : \langle u, y \rangle > 0\}} (\langle u, y \rangle)^{\alpha} d_\nu \left(\frac{dy}{d\text{Leb}}\right)\text{Leb}(dy). \tag{4.3}
\]

It is enough now to prove that \(\text{Leb}\left(\{y : \langle u, y \rangle > 0\}\right) > 0.\) Note that if \(x \in \{y \in S^{d-1} : \langle u, y \rangle > 0\},\) then \(-x \in \{y \in S^{d-1} : \langle u, y \rangle < 0\}.\) This implies that \(\text{Leb}\left(\{y \in S^{d-1} : \langle u, y \rangle > 0\}\right) = \text{Leb}\left(\{y \in S^{d-1} : \langle u, y \rangle \neq 0\}\right)/2.\) Finally, we note that \(\text{Leb}\left(\{y : \langle u, y \rangle = 0\}\right) = \text{Leb}(S^{d-1}) - \text{Leb}\left(\{y : \langle u, y \rangle = 0\}\right)\) is strictly positive, since \(\{y : \langle u, y \rangle = 0\}\) contains only \(2(d-1)\) elements. Hence \(\text{Leb}\left(\{y : \langle u, y \rangle = 0\}\right) = 0.\)

4.1. Proof of Theorem 2.3

The proof is similar to the proof given in Section 3. So we shall provide a brief sketch of the proof below to indicate the similarity and the obvious differences between these two cases.

For the upper bound, we shall follow the steps given in Subsection 3.1. We follow Step 1 with \(r^* = b/a\) in the definition of \(u_i\) in (3.2). Then (3.6) in Step 2 becomes

\[
P\left(S_{u_{i+1}} \in [au_{i+1}, bu_{i+1}] \big| S_{u_i} \in [au_i, bu_i]\right) \leq P\left(S_{u_{i+1} - u_i} > b\eta u_i\right) \tag{4.4}
\]

using the independent increment property of the random walk. We can again use [19, Lemma 2.1] with \(d = 1\) to obtain the upper bound in (3.11). Then Step 3 produces the desired upper bound.

We shall follow the steps in Subsection 3.2 to derive a lower bound. The main difference is that we do not need any approximation by hypercuboids and so skip Step 1. So we can start directly with Step 2 and define \(m_i\) with \(r^{(i)} = r = b/a.\) Next, we replace

\[ \square_i = \left[am_i(2r+1)/3 + 3m_i^{1/\alpha_0 + \delta}, a(5r+1)m_i/6 + m_i^{1/\alpha_0 + \delta}\right] \]

with \(\alpha_0 = \alpha \wedge 2.\) This produces \(T_1\) and \(T_2.\) To produce the other terms we need to specify

\[ \square_{i, 1} = \left[am_i(2r+1)/3 + 2m_i^{1/\alpha_0 + \delta}, a(5r+1)m_i/6 + 2m_i^{1/\alpha_0 + \delta}\right]. \]
\[ \square_{i,2} = [am_i(2r + 1)/3 + m_i^{1/\alpha_0 + \delta}, b(r + 2)m_i/3]; \]
\[ \square_{i,3} = [bm_i + 2m_i^{1/\alpha_0 + \delta}, b(r + 2)m_i/3]; \]
\[ \square_{i,4} = [bm_i + m_i^{1/\alpha_0 + \delta}, b(r + 2)m_i/3 + m_i^{1/\alpha_0 + \delta}]. \]

Note that this sets are similar to the sets \( \square_{i,j} \) used in the \( d \)-dimensional case projected in the optimal direction \( e_1 \) and \( r^{(c)} \) replaced by \( r = b/a \) for \( j = 1, 2, 3, 4 \). So we are done with Step 3.

We shall discuss briefly the asymptotics for \( T_3 \) and \( T_4 \) and the other terms can be dealt with similarly. Note that \( T_3 \) equals

\[ P(S_k \in \square_{i,1} \text{ for all } k \in D_{i+1}^{(1)} | S_{m_i} \in \square_i) \geq P(S_{k-m_i} \in [-m_i^{1/\alpha_0 + \delta}, m_i^{1/\alpha_0 + \delta}] \text{ for all } k \in D_{i+1}^{(1)}) \tag{4.5} \]

using the independent increment property of the random walk. It is clear that the probability in (4.5) can be given following the lower bound

\[ 1 - P \left( \min_{1 \leq k \leq [(r+2)m_i/3]-m_i} S_k \leq -m_i^{1/\alpha_0 + \delta} \right) \]
\[ - P \left( \max_{1 \leq k \leq [(r+2)m_i/3]-m_i} S_k \geq m_i^{1/\alpha_0 + \delta} \right) \]

These two probabilities can be made arbitrarily small using (3.37). The reasoning is very similar to that used to make the probabilities in (3.34) arbitrarily small. So we have similar lower bound derived in (3.38).

**TERM T_4.** Recall that \( T_4 \) equals

\[ P(S_k \in \square_{i,2} \text{ for all } k \in D_{i+1}^{(2)} \setminus \{[f_2m_i]\}; S_{[f_2m_i]} \in \square_{i,3}|S_{[f_1m_i]} \in \square_{i,1}). \tag{4.6} \]

The event inside can be written in terms of \( (S_{k-[f_1m_i]} : k \in D_{i+1}^{(2)}) \) using independent increment property again. Next, we can write down this event in terms of pairwise disjoint and exchangeable events \( (E_j : j \in D_{i+1}^{(2)}) \) where

\[ E_j = \left\{ X_{j-[f_1m_i/3]} \in [(b-a)m_i/3 + 3m_i^{1/\alpha_0 + \delta}, (2br-a)m_i/6 - 2m_i^{1/\alpha_0 + \delta}]; \right. \]
\[ S_{k-[f_1m_i]} \in [-m_i^{1/\alpha_0 + \delta}, m_i^{1/\alpha_0 + \delta}] \text{ for all } k \in \{[f_1m_i] + 1, [f_1m_i] + 2, \]
\[ \ldots, [f_1m_i] + j - 1\}; S_{k-[f_1m_i]} - X_{j-[f_1m_i]} \in [-m_i^{1/\alpha_0 + \delta}, \]
\[ m_i^{1/\alpha_0 + \delta}] \text{ for all } k \in \{[f_1m_i] + j, [f_1m_i] + j + 1, \ldots, [f_2m_i]\} \right\}. \]

After that, we can the large deviation estimate in (3.10) with \( d = 1 \) and (3.37) again to derive the following lower bound

\[ T_4 \geq C_3m_i^{1-\alpha}L_+((b-a)m_i/3), \]

where \( L_+ \) is the slowly varying function appearing in the right-tail of \( X \). The reasoning is very similar to the \( d \)-dimensional case. The other terms can be dealt in a similar way producing the desired lower bound.
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