SOLITONS DYNAMICS IN THE NONLINEAR SCHRÖDINGER EQUATION OF MULTI-PEAK TYPE

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Abstract. In this paper, we study the Cauchy problem of the nonlinear Schrödinger equation with a nontrivial potential \( V(x) \). Especially, we consider the case where the initial data is close to a superposition of \( k \) solitons with prescribed phase and location, and investigate the evolution of the Schrödinger system. We prove that over a large time interval, all \( k \) solitons will maintain the shape, and the solitons dynamics can be regarded as an approximation of \( k \) particles moving in \( \mathbb{R}^N \) with their accelerations dominated by \( \nabla V \), provided these solitons are not too close to each other.

1. Introduction

We will consider the following Cauchy problem of the nonlinear Schrödinger equation (NSE) in \( \mathbb{R}^1 \times \mathbb{R}^N \):

\[
\begin{cases}
i\varepsilon \partial_t \psi = -\varepsilon^2 \Delta \psi + V(x)\psi - f(\psi), \\
\psi(0, x) = \psi_0(x),
\end{cases}
\] (1.1)

where \( i = \sqrt{-1} \) is the imaginary unit, \( \varepsilon \) denotes the Planck constant, \( \psi : \mathbb{R}^1 \times \mathbb{R}^N \to \mathbb{C} \), \( V(x) \) is a real-valued potential function, \( f(\psi) \) is the nonlinear term mapping the complex Sobolev space \( H^1(\mathbb{R}^N, \mathbb{C}) \) to \( H^{-1}(\mathbb{R}^N, \mathbb{C}) \) satisfying \( f(0) = 0 \) and \( f(e^{i\theta}u) = e^{i\theta}f(u) \). Over last decades, the NSE was investigated by many mathematicians since it arises in the theory of Bose-Einstein condensation, nonlinear optics, theory of water waves and in many other areas.

For simplicity, we will first study a version of (1.1) and (1.2):

\[
\begin{cases}
i\partial_t \psi = -\varepsilon^2 \Delta \psi + V(x)\psi - f(\psi), \\
\psi(0, x) = \psi_0(x),
\end{cases}
\] (1.3)

where we do not scale the time parameter. Then, we can scale on time to revisit the property of solitons to Cauchy problem (1.1) and (1.2). The suitable \( H^1(\mathbb{R}^N, \mathbb{C}) \) norm for this problem is given by

\[
||u||_{H^1}^2 := \int_{\mathbb{R}^N} \frac{1}{\varepsilon^N} (\varepsilon^2 |\nabla u|^2 + |u|^2) dx,
\] (1.4)

with \( |z|^2 = (\text{Re} z)^2 + (\text{Im} z)^2 \) for a complex number \( z \). By classical regularity theory of the NSE, when \( f \) has the following form:

\[
f(\psi) = \lambda |\psi|^{p-1}\psi
\] (1.5)
with $0 < p < 1 + \frac{4}{N}$ and $\lambda > 0$, the nonlinear Schrödinger equation has a global solution $\psi(x, t) \in C(\mathbb{R}^N; H^1(\mathbb{R}^N, \mathbb{C})) \cap C(\mathbb{R}^N; H^{-1}(\mathbb{R}^N, \mathbb{C}))$, provided the initial data $\psi(0, x) = \psi_0(x)$ is in $H^1(\mathbb{R}^N, \mathbb{C})$. Readers can consult [7, 19] for more results on the Cauchy problem of the nonlinear Schrödinger equation. In the next section, we will impose proper constraints on potential $V$ and nonlinearity $f$.

When $V(x) \equiv 0$, if $f$ is given by (1.5) and $\lambda$ is large enough, (1.3) can have a stable solitary wave solution of the form

$$
\psi(x, t) = e^{i(\frac{1}{2}v \cdot x + \frac{1}{5}vt + \gamma)} \eta_{\mu, \varepsilon}(x - a),
$$

where $v \in \mathbb{R}^N$ denotes the velocity of solitary wave, $a = vt + a_0$ with the initial location $a_0 \in \mathbb{R}^N$, $\gamma = \mu t + \frac{1}{4}v^2 t + \gamma_0$ with the initial phase $\gamma_0$, $\eta_{\mu, \varepsilon}$ is a radially symmetric solution of following nonlinear eigenvalue problem

$$
-\varepsilon^2 \Delta u + \mu u - f(u) = 0, \quad x \in \mathbb{R}^N
$$

with $\mu > 0$. In this case, solitary waves travel with a constant velocity $v$ and an oscillatory phase $\mu t - \frac{1}{4}v^2 t + \gamma_0$. From [16] we know that $\eta_{\mu, \varepsilon} > 0$ satisfies

$$
\lim_{\mu, \varepsilon \to \infty} \eta_{\mu, \varepsilon}(x) e^{\alpha \frac{x^2}{\varepsilon^2}} = 0
$$

with some $\alpha > 0$. Thus as $\varepsilon \to 0^+$, the mass of the solitary wave, which is denoted by the $L^2$ norm, will concentrate and behave like a particle.

Solitary waves with this concentrating property are usually called solitons, and the process $\varepsilon \to 0^+$ is known as semiclassical limit of the nonlinear Schrödinger equation. In [4], using a finite dimensional reduction method, Cao et al. construct standing wave solutions with solitons located near nondegenerated and isolated critical points of $V(x)$. For equations with a nonsymmetric nonlinear term $Q(x)|\psi|^{p-1}\psi$, multi-peak solutions are constructed by a similar method in [23, 25]. However, in this case, the location of solitons is determined by $Q(x)$ instead of the potential $V(x)$. For relevant result we refer to [23, 27], in which mathematicians construct different kinds of steady soliton solutions.

Now, we turn to soliton evolution in general case where $V(x) \neq 0$. Inspired by above observation, for the case where initial data is close to a single soliton with given velocity and phase, researchers guess that the soliton will maintain its shape, and move under the action of potential $V(x)$. In [11], for nonlinearity given by (1.5), Bronski and Jerrard proved that if the initial data satisfies foresaid condition (be a small perturbation of $\psi$ given by (1.6)), as $\varepsilon \to 0^+$, the solution $\psi(x, t)$ of (1.1) and (1.2) satisfies

$$
\frac{1}{\varepsilon^N} |\psi((x, t) / \varepsilon)| dx \to ||\eta_{\mu, \varepsilon}||^2_{L^2(\mathbb{R}^N)} \delta_{a(t)}
$$

in the $C^{1s}(\mathbb{R}^N)$, which is the dual to $C^1(\mathbb{R}^N)$. They have shown that $a(t)$, the centre of soliton, satisfies $\frac{1}{2} \ddot{a} = \nabla V(\frac{a}{\varepsilon}) + o(\varepsilon)$. Then Keraani [20, 21] developed their method and gave a more precise estimate for the dynamics. In [8, 10], Benci, Ghimenti and Micheletti also work on this problem. By analysis of total energy and a classical concentration-compactness lemma [23, 24, 25, 26], they obtained similar dynamics of single soliton.
However, in order to ensure the global existance of stable solitons, they gave a blow up to the nonlinearity term so that the energy is dominated by \( f(\psi) \).

Above results were obtained on the basis of orbital stability of NLS solitary waves, which was first proven by Cazenave and Lions for trivial potential \( V(x) \equiv 0 \) in [6]. For \( V(x) \in C^2 \) with a nondegenerate minimum point \( x_0 \), Grillakis et al. [17, 18] proved the orbital stability of standing solitary wave whose center is located near \( x_0 \). To obtain solitons dynamics in an external potential, researchers need a further uniqueness result, which claims a function \( \psi \in H^1(\mathbb{R}^N) \) is close to a ground state solution in \( H^1 \) sense, if and only if its energy is near the minimal energy. Weinstein [28, 29] proved this uniqueness property of ground state solutions in 1 and 3 space dimensions. In [22], Kwong combined with Weinstein’s arguments and proved the same result in all space dimensions.

It is worth mentioning that all relevant studies are on the dynamics of a single soliton with nontrivial potential so far. We may ask following questions: If we put \( k \) solitons in \( \mathbb{R}^N \) and let them enolve under the domination of (1.1) and (1.2), then how will their trajectories be like? And will these solitons influence each other? In this paper, we will exactly focus on this topic and give the dynamics formula of multi-peak evolution under appropriate assumptions. In particular, we will prove the fact that if these \( k \) solitons are not too close to each other, the solitons dynamics are mainly determined by \( \nabla V \), while the interplay among the solitons can be ignored.

To study the solitons dynamics of multi-peak type, traditional method is invalid since we can not use a minimization argument on energy, which is crucial in [8, 9, 10, 11]. We will carry out a new method, which is inspired by Fröhlich et al. [12, 13] investigating the Hartree equations, and [14, 15] on the dynamics of solitary waves. The main idea of this paper is to decompose the solution \( \psi(x,t) \) as \( \Phi(x,t) + \omega(x,t) \), where \( \Phi(x,t) \) is on the multi-soliton manifold \( M_k \) (defined in Section 4 and 5), and \( \omega(x,t) \) is the perturbation term. In order to obtain the dynamic for each single soliton, we will make smooth truncations on \( \psi \) to separate each soliton from others and regard the truncated function as an approximation of solitons. These truncations will cause great difficulty since the conservation of mass and momentum doesn’t hold any longer, and there are gross parts from other \( k - 1 \) solitons in every truncated function. Thanks to the transport formula for mass and momentum in NSE (introduced in Section 3) and the exponential decay rate of ground state solutions (1.8), we can give a precise estimate for each truncated part of \( \psi \) and prove that the interplay among \( k \) solitons is indeed a small term. Compared to [14, 15] mainly concerned with the manifold for a single soliton, our another innovation is to decompose \( \psi \) along the multi-soliton manifold and in its skew orthogonal direction, which is necessary for our purpose to derive the modulation formula from equation of motion. To our knowledge, this is the very first result on the multi-peak evolution of nonlinear Schrödinger equation with nontrivial potential. Sadly, our result does not cover the case where solitons collide with each other. This phenomenon of solitons superposition still needs further research, which is rather difficult.

Our paper is organized as follows: After giving the definition of multi-soliton manifold \( M_k \) and parameters we need in Section 3 and 4, we will prove that the skew orthogonal decomposition is unique for \( \psi \) satisfying a couple of assumptions in Section 5.
we calculate time derivative of each variable to give the modulation formula for solitons dynamics in Section 6. The necessary estimate for time derivative of Lyapunov functional is given in Section 7. In Section 8, we finish our proof by showing that the $H^1$ norm of $\omega(x,t)$ is small for every $t$ such that $\psi$ is in the neighborhood $U_\delta$ of $M_k$.

2. Assumptions and statement of the main result

In this section, we first formulate a couple of assumptions on the potential $V(x)$, the nonlinearity $f$ and the initial data $\psi_0$. These assumptions are important since even a small change of variables can bring great influence to the solitons dynamics.

(A) (Potential) The external potential $V(x) \in C^2(\mathbb{R}^N)$ is bounded and slowly varying, that is,

$$|\partial^\alpha V(x)| \leq C_\alpha \varepsilon^{|\alpha|}_\nu \quad \forall \alpha > 0,$$

where $\varepsilon_\nu$ is a small parameter satisfying $\varepsilon_\nu = O(\varepsilon^h)$ for $h > 1$, which claims that the rate of $\varepsilon_\nu \to 0$ is of polynomial order of $\varepsilon$.

(B) (Nonlinearity) $f(s) = \lambda |s|^{p-1}s$, with $1 < p < 1 + \frac{4}{N}$ and $\lambda > 0$.

(C) (Initial data) Denote

$$\sigma_i(t) := (a_i, v_i, \gamma_i, \mu_i) \in (\mathbb{R}^N, \mathbb{R}^N, \mathbb{R}/[0, 2\pi], \mathbb{R})$$

with $a_i, v_i, \gamma_i, \mu_i$ dependent on $t$ only, and

$$\Psi_{\sigma_i, \varepsilon} := e^{i\frac{1}{\varepsilon}(\frac{x-a_i}{\varepsilon}+\gamma_i)}\eta_{\mu_i, \varepsilon}(x-a_i)$$

with $i = 1, \cdots, k$. The initial data $\psi(0, x) = \psi_0(x)$ satisfies

$$||\psi_0(x) - \sum_{i=1}^k \Psi_{\sigma_i, \varepsilon}(0, x)||_{H^1}^2 \leq c\varepsilon^2_\nu,$$

where $\varepsilon_\nu$ is defined in assumption (A). Moreover, for $\sigma_i(0)$ with $i = 1, \cdots, k$ we impose following constraints: $|a_j(0) - a_i(0)| > 4L$ with $j \neq i$ and $L > 0$ a constant; $|v_i(0)| \leq K$ with $K > 0$ a constant; $\mu_i(0) \subset I$ with $I \subset (\mu_{\min}, \mu_{\max})$ a bounded interval and $0 < \mu_{\min} < \mu_{\max} < +\infty$.

We give a brief explanation to these assumptions. Assumption (A) states that the length scale of potential $V(x)$ is of higher order to the size scale of $\psi$, namely, the potential $V(x)$ varies little over the support of solitons. This guarantees that the velocity change of each soliton is small compared to their initial velocity, so that we can obtain the solitons dynamics over a large time interval.

By assumption (B), the Cauchy problem (1.1) and (1.2) is well-posed in $H^1$ and $H^2$. There exists an energy functional $F : H^1(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}$ corresponding to the nonlinearity $f$, which can be written in an explicit form:

$$F(\psi) = \int_{\mathbb{R}^N} |\psi|^{p+1} dx. \quad (2.1)$$

Obviously, $F(\psi)$ is maintained by a translation $T_a^{tr} : \psi(x) \to \psi(x-a) \ \forall a \in \mathbb{R}^N$, a rotation $T_R : \psi(x) \to \psi(R^{-1}x)$, a gauge transform $T_{\gamma} : \psi(x) \to e^{i\gamma}\psi(x) \ \forall \gamma \in \mathbb{R}/[0, 2\pi]$, a boost
transform $T^b_v : \psi(x) \rightarrow e^{\frac{i}{2}v \cdot x} \psi(x) \quad \forall v \in \mathbb{R}^N$, and a complex conjugation $T^c : \psi(x) \rightarrow \bar{\psi}(x)$. Since $p < 1 + \frac{4}{n}$, the linearized operator $L_{\eta_{\mu, \varepsilon}}$ at $\eta_{\mu, \varepsilon}$ has only one negative eigenvalue, which can be found in the appendix of [14]. In [17, 18], researchers used this fact to prove that solitons to (1.1) and (1.2) generated by $\eta_{\mu, \varepsilon}$ are orbital stable. In Section 4, we will also show that the null space of $L_{\eta_{\mu, \varepsilon}}$ has exact $N + 1$ bases under this assumption.

Assumption (C) claims that the initial data $\psi_0(x)$ is sufficiently close to the superposition of $k$ solitons, and the order of the difference in $H^1$ is $O(\varepsilon v)$. Moreover, $k$ peaks can not get too close to each other at beginning, so that the energy caused by the interaction of two peaks is a small term of exponential decay rate. We also require that the initial velocity has an upper bound $K$, and the parameters $\mu_i > 0$ are in a common range in $\mathbb{R}^+$. Now, we are prepared to state our main result:

**Theorem 2.1.** Assume that assumptions (A) (B) and (C) are satisfied. Then there is a constant $T_0 > 0$ (independent of $\varepsilon$ and $\varepsilon v$ but possibly dependent on constants $L, K$ in assumption (C)), such that for $t \leq \min\{T_0, \frac{L}{K}\}$, the solution to (1.1) and (1.2) has the form

$$\psi(x, t) = \sum_{l=1}^{k} \Psi_{\sigma_l \varepsilon} + w,$$  

(2.2)

where $||w||_{H^1} = O(\varepsilon v)$ and the parameters $a_i, v_i, \gamma_i, \mu_i$ in $\sigma_i$ satisfy the differential equations

$$\frac{1}{2} \ddot{a}_i = -\nabla V(a_i) + O(\frac{\varepsilon^2 v}{\varepsilon}),$$  

(2.3)

$$\dot{a}_i = v_i + O(\varepsilon v),$$  

(2.4)

$$\varepsilon \dot{\gamma}_i = \mu_i - V(a_i) - \frac{1}{4} v_i^2 + O(\varepsilon v),$$  

(2.5)

$$\dot{\mu}_i = O(\frac{\varepsilon^2 v}{\varepsilon}),$$  

(2.6)

with $i = 1, \ldots, k$.

As a consequence of Theorem 2.1, we obtain the integral from of solitons dynamics:

**Corollary 2.2.** Assume that assumptions (A) (B) and (C) are satisfied. Then for $t \leq \min\{T_0, \frac{L}{K}\}$ the same as in Theorem 2.1, the velocity $v_i(t)$, location $a_i(t)$ and phase $\gamma_i(t)$ of soliton $\Psi_{\sigma_i \varepsilon}$ satisfy the integral equations

$$v_i(t) = v_i(0) + 2 \int_0^t \nabla V(a_i(t)) dt + O(\frac{\varepsilon^2 v}{\varepsilon^2}),$$  

(2.7)

$$a_i(t) = a_i(0) + \int_0^t v_i(t) dt + O(\frac{\varepsilon^2 v}{\varepsilon}),$$  

(2.8)

$$\gamma_i(t) = \gamma_i(0) + \frac{1}{\varepsilon} \int_0^t (\mu_i(t) - V(a_i(t)) - \frac{1}{4} v_i(t)^2) dt + O(\frac{\varepsilon^2 v}{\varepsilon^2}),$$  

(2.9)
with $i = 1, \cdots, k$, and it holds
\[
\mu_i(t) = \mu_i(0) + O\left(\frac{\varepsilon^2}{\varepsilon^2}\right) \quad \forall i = 1, \cdots, k. \tag{2.10}
\]

**Remark 2.3.** If these $k$ solitons grow apart from each other at the beginning, namely, the intial velocity and location satisfy additional conditions:
\[
v_i(0) \cdot (a_i(0) - a_j(0)) \geq 0 \quad \forall i, j = 1, \cdots, k, \tag{2.11}
\]
then we can use a bootstrap method to extend the maximum time to $T_0$, which means for this case the solitons dynamics (2.3)-(2.10) can hold for $\forall t \in (0, T]$ with $T \to +\infty$ as $T_0 \to \infty$.

3. Variational structure and conserved quantities of NLS equations

We consider the modified nonlinear Schrödinger equation (1.3) with time parameter not scaled:
\[
\begin{aligned}
&i \partial_t \psi = -\varepsilon^2 \Delta \psi + V(x) \psi - f(\psi), \\
&\psi(0, x) = \psi_0(x),
\end{aligned}
\]
and study its solution in the space $H^1(\mathbb{R}^N, \mathbb{C})$, which can be considered as a direct sum of two real space $H^1(\mathbb{R}^N, \mathbb{R}) \oplus H^1(\mathbb{R}^N, \mathbb{R})$ under the identification $\psi \to (\text{Re} \psi, \text{Im} \psi)$. We equip this complex Sobolev space with the symplectic form
\[
\omega(\psi, \phi) := \text{Im} \int_{\mathbb{R}^N} \frac{1}{\varepsilon^N} \overline{u} \psi \overline{v} dx. \tag{3.1}
\]
The space $H^1(\mathbb{R}^N, \mathbb{C})$ also has a real inner product:
\[
\langle u, v \rangle \varepsilon := \text{Re} \int_{\mathbb{R}^N} \frac{1}{\varepsilon^N} u \overline{v} dx, \tag{3.2}
\]
which induces the $L^2$ norm $||u||_{L^2}^2 = \langle u, u \rangle \varepsilon$. With the identification $\psi \to (\text{Re} \psi, \text{Im} \psi)$, the operator of multiplication by $i^{-1}$ is identified with the operator
\[
J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.3}
\]
Thus we have $\omega(\psi, \phi) = \langle u, J^{-1} v \rangle \varepsilon$.

The Hamiltonian functional related to (1.3) on $H^1(\mathbb{R}^N, \mathbb{C})$ is
\[
\mathcal{H}_{\psi, \varepsilon}(\psi) := \frac{1}{2} \int_{\mathbb{R}^N} \left( \frac{1}{\varepsilon^{N-2}} |\nabla \psi|^2 + \frac{1}{\varepsilon^N} V|\psi|^2 \right) dx - \frac{1}{\varepsilon^N} F(\psi) dx, \tag{3.4}
\]
where $F(\psi)$ is defined in Condition (B) in Section 2. Since the Hamiltonian $\mathcal{H}_{\psi, \varepsilon}(\psi)$ is autonomous and invariant under the gauge transformation $(\mathcal{H}_{\psi, \varepsilon}(e^{i\gamma} \psi) = \mathcal{H}_{\psi, \varepsilon}(\psi))$, under the evolution of (1.3) we have conservation of energy:
\[
\frac{d}{dt} \mathcal{H}_{\psi, \varepsilon} = 0. \tag{3.5}
\]
and conservation of mass
\[ \frac{d}{dt} \left( \frac{1}{2} \int_{\mathbb{R}^N} \frac{1}{\varepsilon} |\psi|^2 dx \right) = 0. \]  
(3.6)

Notice that we do not scale the time parameter in (1.3), so the initial velocity condition for the Cauchy problem (1.3) is
\[ |v_i(0)| \leq K\varepsilon \quad \forall i = 1, \cdots, k. \]  
(3.7)

For the potential term \( \int_{\mathbb{R}^N} \frac{1}{\varepsilon} V |\psi|^2 dx \), we have
\[ \frac{1}{\varepsilon} \frac{d}{dt} \left( \frac{1}{2} \int_{\mathbb{R}^N} \frac{1}{\varepsilon} V |\psi|^2 dx \right) = \langle (\nabla V) i\psi, \varepsilon \nabla \psi \rangle, \]  
(3.8)

which can be found in [14]. And for \( D \subset \mathbb{R}^N \) whose boundary is smooth, the rate of mass exchange on \( \partial D \) is dominated by a flux term, which can be rewritten as an interior divergent form by Green’s theorem:
\[ \frac{1}{\varepsilon} \frac{d}{dt} \left( \frac{1}{2} \frac{1}{\varepsilon} |\psi|^2 \right) = -\nabla \cdot \text{Re}(\varepsilon^{-1} i\psi \nabla \psi). \]  
(3.9)

Similarly, for momentum we have
\[ \frac{1}{\varepsilon} \frac{d}{dt} \left( \text{Re}(\varepsilon^{-1} i\psi \cdot \varepsilon \nabla \psi) \right) = -\frac{1}{\varepsilon^N} \nabla V |\psi|^2 + \frac{1}{\varepsilon^N} \nabla \cdot T, \]  
(3.10)

where \( T \) is the stress tensor with the property
\[ |T| \leq C(\varepsilon^2 |\nabla \psi|^2 + \varepsilon^2 \Delta |\psi|^2 + |\psi|^2 + |\psi|^{p+1}), \]  
(3.11)

see [9]. For more information about conserved quantities of NSE, we refer to [11] and the references therein.

Solutions to the NSE are various with different physical explanations. In particular, we are concerned with \( \eta_{\mu,\varepsilon} \), which is radial decreasing and described as local minimizers of the Hamiltonian \( \mathcal{H}_{\mu=0,\varepsilon}(\psi) \) restricted to the spheres
\[ \{ \psi \in H^1(\mathbb{R}^N, \mathbb{C}) : \frac{1}{2} \int_{\mathbb{R}^N} \frac{1}{\varepsilon} |\psi|^2 dx = m \}, \]  
(3.12)

for \( m > 0 \). Thus \( \eta_{\mu,\varepsilon} \) is a critical point of
\[ \mathcal{E}_{\mu,\varepsilon} := \frac{1}{2} \int_{\mathbb{R}^N} (\frac{1}{\varepsilon^{N-2}} |\nabla \psi|^2 + \frac{1}{\varepsilon^N} \mu |\psi|^2 dx - \frac{1}{\varepsilon^N} F(\psi)) dx \]  
(3.13)

with \( \mu = \mu(m) \) the Lagrange multiplier, known as the ground state solution. The linearized operator corresponding to \( \eta_{\mu,\varepsilon} \) possesses very good properties such as non-degeneracy and coercivity, which we will discuss in following sections.

We end this section by giving an estimate to Taylor expansion of \( F(\psi) \) at \( \eta \). If we define
\[ R^{(2)}_{\eta}(w) := \frac{1}{\varepsilon^n} F(\eta + w) - \frac{1}{\varepsilon^n} F(\eta) - \langle F'(\eta), w \rangle_\varepsilon, \]  
(3.14)

\[ R^{(3)}_{\eta}(w) := \frac{1}{\varepsilon^n} F(\eta + w) - \frac{1}{\varepsilon^n} F(\eta) - \langle F'(\eta), w \rangle_\varepsilon - \frac{1}{2} (F''(\eta) w, w)_\varepsilon, \]  
(3.15)

\[ N_{\eta}(w) := F'(\eta + w) - F'(\eta) - F''(\eta) w, \]  
(3.16)
with \( \eta \in H^2(\mathbb{R}^N, \mathbb{C}) \) and \( w \in H^1(\mathbb{R}^N, \mathbb{C}) \), then we have
\[
|R^{(2)}_\eta(w)| \leq C||w||^2_{H^1}, \quad |R^{(3)}_\eta(w)| \leq C||w||^3_{H^1}, \quad ||N_\eta(w)||_{H^{-1}} \leq C||w||^2_{H^1},
\]
where \( C \) is a constant dependent only on \( ||\eta||_{H^1} \) provided \( ||w||_{H^1} \leq 1 \).

4. The Manifold of Solitons

To introduce the manifold of solitons, we first focus on the linearized operator:
\[
\mathcal{L}_{\eta, \varepsilon} := -\varepsilon^2 \Delta + \mu - f'(\eta_{\mu, \varepsilon}).
\]
According to the Appendix C of [14], \( \mathcal{L}_{\eta, \varepsilon} \) has only one negative eigenvalue, and the null space of \( \mathcal{L}_{\eta, \varepsilon} \) is spanned by \( N + 1 \) vectors:
\[
N(\mathcal{L}_{\eta, \varepsilon}) = \text{span}\{i\eta_{\mu, \varepsilon}, \partial_{x_1}\eta_{\mu, \varepsilon}, \ldots, \partial_{x_N}\eta_{\mu, \varepsilon}\}. \tag{4.2}
\]
We will show that the tangent space of soliton manifold are spanned by these \( N + 1 \) zero modes and other \( N + 1 \) symplectically associated zero modes.

Using those transforms defined in Section 2, we can give the combined symmetry transform \( \mathcal{T}_{av\gamma} \):
\[
\psi(x) \to \mathcal{T}_{av\gamma}\psi(x) = \mathcal{T}_a^{\tau_\gamma} \tau_\gamma^b \mathcal{T}_v \psi(x) = e^{\frac{i}{2} \int_{\gamma} \omega_a a - \gamma \psi(x - a). \tag{4.3}
\]
Recall that \( \Psi_{\sigma, \varepsilon} := \mathcal{T}_{av\gamma}\eta_{\mu, \varepsilon}. \) We define the manifold of a single soliton as
\[
M := \{ \Psi_{\sigma, \varepsilon} : a, v, \gamma, \mu \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}/[0, 2\pi) \times I \}, \tag{4.4}
\]
We calculate the derivatives of \( \mathcal{T}_{av\gamma}\eta_{\mu, \varepsilon} \) at origin to give the tangent space to \( M \) at \( \Psi_{\sigma_0, \varepsilon} \) with \( \sigma_0 = (0, 0, 0, \mu) \):
\[
\mathbf{T}_{\Psi_{\sigma_0, \varepsilon}} M = \text{span}\{z_t, z_b, z_g, z_s\}, \tag{4.5}
\]
where
\[
z_t := \nabla_a \mathcal{T}_a^{\tau_\gamma} \eta_{\mu, \varepsilon} \big|_{a=0} = -\nabla \eta_{\mu, \varepsilon}, \tag{4.6}
\]
\[
z_b := 2\nabla_v \mathcal{T}_v \eta_{\mu, \varepsilon} \big|_{v=0} = \frac{i}{\varepsilon} \eta_{\mu, \varepsilon}, \tag{4.7}
\]
\[
z_g := \partial_{\gamma} \mathcal{T}_\gamma \eta_{\mu, \varepsilon} \big|_{\gamma=0} = i\eta_{\mu, \varepsilon}, \tag{4.8}
\]
\[
z_s := \partial_{\mu} \eta_{\mu, \varepsilon}. \tag{4.9}
\]
According to [12], we have
\[
\mathcal{L}_{\eta, \varepsilon} z_t = 0 \quad \text{and} \quad \mathcal{L}_{\eta, \varepsilon} z_g = 0. \tag{4.10}
\]
By taking the derivatives of equation \( \mathcal{E}_{\mu, \varepsilon}(\mathcal{T}_v \eta_{\mu, \varepsilon}) = 0 \) with respect to \( v \) and \( \mu \), we have
\[
\mathcal{L}_{\eta, \varepsilon} z_b = 2iz_t \quad \text{and} \quad \mathcal{L}_{\eta, \varepsilon} z_s = iz_g. \tag{4.11}
\]
Thus \( z_b \) and \( z_s \) are zero modes for the operator \((i\mathcal{L}_{\eta, \varepsilon})^2\):
\[
i\mathcal{L}_{\eta, \varepsilon} (i\mathcal{L}_{\eta, \varepsilon} z_b) = 0 \quad \text{and} \quad i\mathcal{L}_{\eta, \varepsilon} (i\mathcal{L}_{\eta, \varepsilon} z_s) = 0. \tag{4.12}
\]
The tangent space \( \mathbf{T}_{\Psi_{\sigma_0, \varepsilon}} M \) then inherits a symplectic from \((H^1, \omega_\varepsilon)\), which is determined by the operator
\[
\omega_\varepsilon(u, v) := \langle u, \Omega_{\eta, \varepsilon}(v) \rangle_\varepsilon, \quad \forall u, v \in \mathbf{T}_{\Psi_{\sigma_0, \varepsilon}} M \tag{4.13}
\]
where $\Omega_{\eta,\varepsilon}$ is a $(2N+2) \times (2N+2)$ matrix whose component is given by $\Omega^{i,j}_{\eta,\varepsilon} = \langle z_i, J^{-1}z_j \rangle_{\varepsilon}$ with \( \{z_1, \ldots, z_{2d+2}\} := \{z_1, z_b, z_g, \varepsilon\} \). We are going to show this symplectic form is non-degenerate. For this purpose, we denote

\[
m(\mu) := \frac{1}{2} \int_{\mathbb{R}^N} \frac{1}{\varepsilon N} |\eta_{\mu,\varepsilon}|^2 dx.
\]

Lemma 4.1. If $\mu \in I$, then $\Omega_{\eta,\varepsilon}$ is invertible. Consequently $\forall z \in T_{\Psi_{\sigma,\varepsilon}} M$ with $\sigma_0 = (0,0,0,\mu)$, there exists at least one element $z' \in T_{\Psi_{\sigma,\varepsilon}} M$, such that $\omega_{\varepsilon}(z, z') \neq 0$.

Proof. We compute each component $\Omega^{m,n}_{\eta,\varepsilon} = \langle z_m, J^{-1}z_n \rangle$, $1 \leq m, n \leq 2N + 2$, so the matrix $\Omega_{\eta,\varepsilon}$ can be written as

\[
\Omega_{\eta,\varepsilon} = \begin{pmatrix}
0 & -m1 & 0 & 0 \\
-m1 & 0 & 0 & 0 \\
0 & 0 & 0 & m'(\mu) \\
0 & 0 & -m'(\mu) & 0
\end{pmatrix}
\]

with $1$ the $N \times N$ matrix where the elements on diagonal are 1, while others are 0. By assumption (B) in Section 2, $\forall \mu \in I \subset \mathbb{R}^+$ we have

\[
m'(\mu) = \partial_{\mu} \int_{\mathbb{R}^N} \frac{1}{\varepsilon N} \eta^{2}_{\mu,\varepsilon} dx = \partial_{\mu} \left( \mu_{p-1}^2 \int_{\mathbb{R}^N} \frac{1}{\varepsilon N} \eta_{\mu,\varepsilon} (\sqrt{\mu}x)^2 dx \right)
\]

\[
\quad = \partial_{\mu} \left( \mu_{p-1}^2 \int_{\mathbb{R}^N} \frac{1}{\varepsilon N} \eta_{\mu,\varepsilon} (x')^2 dx' \right)
\]

\[
\quad = \partial_{\mu} \left( \mu_{p-1}^2 \right) \int_{\mathbb{R}^N} \frac{1}{N} \eta_{\mu,\varepsilon} (x')^2 dx'
\]

\[
\quad > 0
\]

since $p < 1 + \frac{4}{N}$. Hence the matrix $\Omega_{\eta,\varepsilon}$ is invertible and the corresponding symplectic form is non-degenerate. The rest part of this lemma follows as a consequence. \( \Box \)

When the transform $T_{av,\gamma}$ acts on the soliton $\eta_{\mu,\varepsilon}$, the tangent space $T_{\Psi_{\sigma,\varepsilon}} M$ at $\Psi_{\sigma,\varepsilon}$ is given by

\[
T_{\Psi_{\sigma,\varepsilon}} M := \text{span}\{ \nabla_a \Psi_{\sigma,\varepsilon}, 2\nabla_v \Psi_{\sigma,\varepsilon}, \partial_\mu \Psi_{\sigma,\varepsilon}, \partial_{\mu} \Psi_{\sigma,\varepsilon} \},
\]

where

\[
\nabla_a \Psi_{\sigma,\varepsilon} = -T_{av,\gamma}(\nabla \eta_{\mu,\varepsilon}) - \frac{v}{2\varepsilon} T_{av,\gamma}(i\eta_{\mu,\varepsilon}) = T_{av,\gamma} z_t - \frac{v}{2\varepsilon} T_{av,\gamma} z_g,
\]

\[
2\nabla_v \Psi_{\sigma,\varepsilon} = T_{av,\gamma}(\frac{ix}{\varepsilon} \eta_{\mu,\varepsilon}) = T_{av,\gamma} z_b,
\]

\[
\partial_\gamma \Psi_{\sigma,\varepsilon} = T_{av,\gamma}(i\eta_{\mu,\varepsilon}) = T_{av,\gamma} z_g,
\]

\[
\partial_{\mu} \Psi_{\sigma,\varepsilon} = T_{av,\gamma}(\partial_{\mu} \eta_{\mu,\varepsilon}) = T_{av,\gamma} z_s.
\]

Note that the operator $T_{av,\gamma}$ is canonical, which leaves $\langle \cdot, J^{-1} \cdot \rangle_{\varepsilon}$ unchanged, that is

\[
\langle T_{av,\gamma} u, J^{-1} T_{av,\gamma} v \rangle_{\varepsilon} = \langle u, J^{-1} v \rangle_{\varepsilon}.
\]

So the matrix $\Omega_{\Psi_{\sigma,\varepsilon}}$ induced by (4.18)-(4.21) is related to $\Omega_{\eta,\varepsilon}$ by a similarity transform.
5. SKew ORTHOGONAL DECOMPOSITION

In this section, we will introduce the skew orthogonal decomposition of \( \psi \) along the multi-soliton manifold \( \mathbf{M}_k \) and the skew orthogonal direction, and prove this decomposition is unique when \( \psi \) is sufficiently close to \( \mathbf{M}_k \) (see Arnol’d [1]).

We define the \( \delta \)-neighborhood

\[
U_\delta = \{ \psi \in H^1(\mathbb{R}^N, \mathbb{C}) : \inf_{\sigma \in \Sigma} ||\psi - \sum_{l=1}^{k} \Psi_{\sigma_l, \epsilon}||_{H^1} \leq \delta \}
\]

of the multi-soliton manifold \( \mathbf{M}_k = \{ \sum_{l=1}^{k} \eta_{\sigma_l, \epsilon} : \sigma \in \Sigma \} \), where

\[
\sigma = \{ \sigma_1, \cdots, \sigma_k \}, \quad \Sigma = \Sigma_1 \times \cdots \times \Sigma_k.
\]

and

\[
\Sigma_j := B_L(a_j(0)) \times B_{K\epsilon}(0) \times \mathbb{R}/(0, 2\pi) \times I
\]

for \( j = 1, \cdots, k \) by assumption (C) in Section 2 and (3.7).

For simplicity, we use the notations

\[
\sigma_j := \{ \sigma_{j,1}, \cdots, \sigma_{j,2N+2} \} := \{ a_j, v_j, \gamma_j, \mu_j \},
\]

\[
\{ z_{j,1}, \cdots, z_{j,2N+2} \} = \{ -\mathcal{T}_{a_j,v_j,\gamma_j}(\nabla \eta_{\mu_j, \epsilon}), \mathcal{T}_{a_j,v_j,\gamma_j}(\frac{ix}{\epsilon}), \mathcal{T}_{a_j,v_j,\gamma_j}(i\eta_{\mu_j, \epsilon}), \mathcal{T}_{a_j,v_j,\gamma_j}(\partial \mu \eta_{\mu_j, \epsilon}) \}
\]

with \( j = 1, \cdots, k \). We have following lemma, which claims that the interplay among \( k \) solitons is a small term.

**Lemma 5.1.** For two bases \( z_{i,m} \) and \( z_{j,n} \) satisfying \( i \neq j \), it holds

\[
|\omega_\epsilon(z_{i,m}, z_{j,n})| = O(e^{-\frac{c}{\epsilon}}),
\]

with \( c \) a positive constant.

**Proof.** According to (1.8), we have

\[
|z_{i,m}| \leq C e^{-\frac{c_1|x-a_i|}{\epsilon}} \quad \text{and} \quad |z_{j,n}| \leq C e^{-\frac{c_2|x-a_j|}{\epsilon}},
\]

where \( a_i \in B_L(a_i(0)) \) and \( a_j \in B_L(a_j(0)) \). By assumption (C) in Section 2, it holds \( |a_i - a_j| \geq L \). If we denote \( x' = (x - a_i)/\epsilon \), we can deduce that

\[
|\omega_\epsilon(z_{i,m}, z_{j,n})| \leq C \int_{\mathbb{R}^N} \frac{1}{\epsilon} e^{-\frac{c_1|x-a_i|}{\epsilon}} e^{-\frac{c_2|x-a_j|}{\epsilon}} dx = C \int_{\mathbb{R}^N} e^{-c_1x'} e^{-c_2|x'-a_j|/\epsilon} dx' \leq C e^{-\frac{c}{\epsilon}}
\]

Hence the proof is complete. \qed
The next proposition is the main result of this section, which shows the skew orthogonal decomposition is unique in $U_\delta$ provided $\delta$ is small enough.

**Proposition 5.2.** If $\delta \ll \inf_{\mu \in I} m'(\mu)$, then there exists a unique $\sigma = \sigma(\psi) \in C^1(U_\delta, \Sigma)$ such that

$$\omega_\varepsilon(\psi - \sum_{l=1}^k \Psi_{\sigma_j, \varepsilon, z} M) = 0, \quad \forall z \in T_{\psi_{\sigma_j, \varepsilon}} M \quad l = 1, \cdots, k \quad (5.9)$$

**Proof.** We denote

$$\partial_{i,n} = \partial_{\sigma_{i,n}} + \frac{v_i}{2\varepsilon} \partial_{\sigma_{i,N+1}} \text{ for } i = 1, \cdots, k, \quad n = 1, \cdots, N \quad (5.10)$$

and

$$\partial_{i,n} = \partial_{\sigma_{i,n}} \text{ for } i = 1, \cdots, k, \quad n = N + 1, \cdots, 2N + 2. \quad (5.11)$$

Then we can apply the implicit function theorem for $G : H^1(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}^{k(2N+2)}$ defined by

$$G_{j,m}(\psi, \sigma) := \langle \psi - \sum_{l=1}^k \Psi_{\sigma_l, \varepsilon, J^{-1}z_{j,m}} \varepsilon, \forall j = 1, \cdots, k \text{ and } \forall m = 1, \cdots, 2N + 2. \quad (5.12)$$

Since both $\eta_{\sigma_l, \varepsilon}$ and $z_{j,m}$ are $C^1$, and $G$ is linear in $\psi$, we deduce $G$ is $C^1$ in $\sigma$. It is obvious that $G_{j,m}(\sum_{l=1}^k \Psi_{\sigma_l, \varepsilon, \sigma_0}) = 0$ for any $\sigma_0 = \{\sigma_1, \cdots, \sigma_k\} \in \Sigma$. We only need to verify that $\partial_{i,n} G_{j,m}(\sum_{l=1}^k \Psi_{\sigma_l, \varepsilon, \sigma})|_{\sigma = \sigma_0}$ is invertible.

By $(5.10)$ $(5.11)$, it holds

$$\partial_{i,n} G_{j,m}(\sum_{l=1}^k \Psi_{\sigma_l, \varepsilon, \sigma})|_{\sigma = \sigma_0} = -\langle z_{i,n}, J^{-1}z_{j,m} \varepsilon \rangle \quad (5.13)$$

Using the canonical property $(4.22)$ of $T_{av\gamma}$, we have

$$\langle z_{i,n}, J^{-1}z_{j,m} \varepsilon \rangle = \Omega_{\eta_{\sigma_l, \varepsilon}}^{m,m}. \quad (5.14)$$

By Lemma 5.1. For $i \neq j$ we have

$$\langle z_{i,n}, J^{-1}z_{j,m} \varepsilon \rangle = O(e^{-\varepsilon}). \quad (5.15)$$

($z_{i,n}$ and $z_{j,m}$ with $i \neq j$ are skew orthogonal in an asymptotic sense.) Thus using Lemma $(4.1)$, we have shown $\partial_{i,n} G_{j,m}(\sum_{l=1}^k \Psi_{\sigma_l, \varepsilon, \sigma})|_{\sigma = \sigma_0}$ is invertible for all $\sigma_0$ when $\varepsilon$ is sufficiently small. The implicit function theorem implies that there exist $\delta \ll \inf_{\mu \in I} m'(\mu)$ and a unique $C^1$ map $\sigma = \sigma(\psi)$ such that $G(\psi, \sigma(\psi)) = 0$ in a neighborhood $V_{\sigma_0}$ of $\sum_{l=1}^k \Psi_{\sigma_0, \varepsilon, \sigma}$.

Since $\Sigma$ is connected and $\sigma_0$ is chosen arbitrarily in $\Sigma$, we can take $V_\sigma \cap V_{\sigma_0} \neq \emptyset$ and expand $V_{\sigma_0}$ to cover the whole neighborhood $U_\delta$ of $M$. This completes the proof of Proposition 5.2. □
We have already known that for given initial data in Section 2, the Cauchy problem (1.3) has a solution in \( C(\mathbb{R}^N; H^1(\mathbb{R}^N, \mathbb{C})) \cap C(\mathbb{R}^N; H^{-1}(\mathbb{R}^N, \mathbb{C})) \). So for \( 0 \leq t \leq T \), if \( \psi \) stay in the neighborhood \( U_\delta \) of \( M_k \), then the \( C^1 \) trajectory \( \sigma(\psi(t)) \) traced out by \( \sigma \) is unique. In the rest part of this paper, we may always assume \( \varepsilon_v \ll \delta \).

6. Equation of motion

To study the dynamics of solitons, we should split \( \psi \) into \( k \) peaks, which is accomplished by making some smooth truncation near each soliton. To simplify notation, let \( T_j = T_{\sigma_j, \psi_j, \varepsilon} \), and define

\[
    u_j := T_j^{-1}(\varphi_j \cdot \psi), \quad j = 1, \ldots, k - 1
\]

and

\[
    u_k := T_k^{-1}(\psi - \sum_{l=1}^{k-1} T_l u_l),
\]

where \( \varphi_j \in C_c^\infty \) are truncating functions satisfying

\[
    \varphi_j(x) = \begin{cases} 
        1 & x \in B_{L}(a_j(0)), \\
        0 & x \in \mathbb{R}^N \setminus B_{2L}(a_j(0))
    \end{cases}
\]

with \( j = 1, \ldots, k - 1 \). According to Proposition 5.2, for \( \psi \in U_\delta \), it has a skew orthogonal decomposition:

\[
    \psi = \sum_{l=1}^{k} T_l \eta_{l, \varepsilon} + w.
\]

Define \( w_j \) as

\[
    w_j := T_j^{-1}(\varphi_j \cdot (\psi - T_j \eta_{j, \varepsilon})) = T_j^{-1}(\varphi_j \cdot (\sum_{l \neq j} T_l \eta_{l, \varepsilon} + w))), \quad j = 1, \ldots, k - 1
\]

and

\[
    w_k := T_k^{-1}(w - \sum_{l=1}^{k-1} T_l w_l).
\]

Since \( |a_i(0) - a_j(0)| > 2L \) and \( |v_j| \leq K \varepsilon \), by the definition of \( H^1 \) norm (1.4), it holds for every fixed time \( t \) that

\[
    \left( \frac{1}{2} - \sup_{l \neq j} |v_l| \right) \sum_{l=1}^{k} ||w||_{H^1} + O(e^{-\frac{\varepsilon}{\varepsilon}}) \leq ||w||_{H^1} \leq (1 + \sup |v_j|) \sum_{l=1}^{k} ||w_l||_{H^1} + O(e^{-\frac{\varepsilon}{\varepsilon}}),
\]

provided that \( \psi \in U_\delta \). From (6.5) and (6.6), we also know that

\[
    T_j u_j(x) - T_j w_j(x) = \begin{cases} 
        \Psi_{\sigma_j, \varepsilon} + O(\sum_{l \neq j} e^{-\frac{|x-a_l|}{\varepsilon}}) & \forall x \in B_{2L}(a_j(0)), \\
        0 & \forall x \in \mathbb{R}^N \setminus B_{2L}(a_j(0)),
    \end{cases}
\]

\( \forall x \in \mathbb{R}^N \setminus B_{2L}(a_j(0)), \)
SOLITONS DYNAMICS IN THE NONLINEAR SCHRÖDINGER EQUATION OF MULTI-PEAK TYPE

for \( j = 1, \ldots, k - 1 \), and

\[
\mathcal{T}_k u_k(x) - \mathcal{T}_k w_k(x) = \begin{cases} 
\Psi_{\sigma_k, \epsilon} + O \left( \sum_{j=1}^{k-1} e^{-\frac{c|x-a_j|}{\epsilon}} \right) & \forall x \in \mathbb{R}^N \setminus \bigcup_{j=1}^{k-1} B_L(a_j(0)), \\
0 & \forall x \in \bigcup_{j=1}^{k-1} B_L(a_j(0)).
\end{cases}
\tag{6.9}
\]

We introduce the generators

\[
\mathcal{K}_{j,m} = -T_j \partial_m, \quad \mathcal{K}_{j,N+m} = \frac{(x - a_j)_m}{\epsilon} i T_j, \quad \mathcal{K}_{j,2N+1} = i T_j, \quad \mathcal{K}_{j,2N+2} = T_j \partial_{\mu_j},
\]

and following coefficients corresponding to the basis \( z_{j,m} \) defined in (5.5)

\[
\beta_{j,m} = \dot{a}_{j,m} \frac{1}{\epsilon} - v_{j,m}, \quad j = 1, \ldots, k, \quad m = 1, \ldots, N,
\tag{6.10}
\]

\[
\beta_{j,N+m} = \frac{1}{2} \dot{v}_{j,m} + \epsilon \partial_{x_m} V(a_j), \quad j = 1, \ldots, k, \quad m = 1, \ldots, N,
\tag{6.11}
\]

\[
\beta_{j,2N+1} = \gamma - \frac{\dot{a}_j}{2\epsilon} \cdot v_j - \mu_j + \frac{|v_j|^2}{4} + V(a_j), \quad j = 1, \ldots, k,
\tag{6.12}
\]

\[
\beta_{j,2N+2} = \mu_j, \quad j = 1, \ldots, k,
\tag{6.13}
\]

where \( a_{j,m} \) and \( v_{j,m} \) are \( m \)-th component of \( a_j \) and \( v_j \). Denote

\[
\beta_j \cdot \mathcal{K}_j := \sum_{m=1}^{2N+2} \beta_{j,m} \cdot \mathcal{K}_{j,m} \quad \text{and} \quad \tilde{\beta}_j \cdot \tilde{\mathcal{K}}_j := \sum_{m=1}^{2N+1} \beta_{j,m} \cdot \mathcal{K}_{j,m}.
\tag{6.14}
\]

We have following theorem, which reparameterize the NSE (1.3) into the form we needed.

**Theorem 6.1.** If \( \psi \in U_\delta \) satisfies (1.3), then it holds

\[
\sum_{i=1}^{k} \tilde{\beta}_i \cdot \tilde{\mathcal{K}}_i u_i + \sum_{i=1}^{k} \mathcal{T}_i u_i = \sum_{i=1}^{k} J \mathcal{T}_i (-\varepsilon^2 \Delta u_i + \mu_i u_i) + \sum_{i=1}^{k} J \mathcal{R}_i \cdot \mathcal{T}_i u_i - J f(\psi).
\tag{6.15}
\]

where

\[
\mathcal{R}_i = V(x) - V(a_i) - \nabla V(a_i) \cdot (x - a_i) = O((\varepsilon^2 x)^2),
\tag{6.16}
\]

and \( J \) is the operator of multiplication by \( i^{-1} \) defined in (3.3).

**Proof.** We can rewrite the first equation of (1.3):

\[
i \partial_t \psi = -\varepsilon^2 \Delta \psi + V(x) \psi - f(\psi),
\]

as

\[
\sum_{i=1}^{k} \partial_t (\mathcal{T}_i u_i) = \sum_{i=1}^{k} J (-\varepsilon^2 \Delta \mathcal{T}_i u_i + V(x) \mathcal{T}_i u_i) - J f(\psi).
\tag{6.17}
\]
Since $T_j v = e^{\frac{j}{\varepsilon} v\cdot \frac{x-a_j}{\varepsilon}} + \gamma_j v(x-a_j)$, for the left side of (6.17), we have

$$
\sum_{l=1}^{k} \partial_t (T_l u_l) = \sum_{l=1}^{k} \left( \frac{1}{2} \dot{v} \cdot \frac{x-a_l}{\varepsilon} - \frac{1}{2} v \cdot \frac{\dot{a}}{\varepsilon} + \dot{\gamma} \right) T_l u_l 
+ \sum_{l=1}^{k} \dot{a}_l \left( -T_l \nabla u_l \right) + \sum_{l=1}^{k} T_l \dot{u}_l, \tag{6.18}
$$

While the right side of (6.17) can be rewritten as

$$
\sum_{l=1}^{k} J(-\varepsilon^2 \Delta T_l u_l + V(x) T_l u_l) - J f(\psi)
= \sum_{l=1}^{k} \left( J T_l (-\varepsilon^2 \Delta u_l) + v_l (-T_l \nabla u_l) - \frac{v^2}{4} (\dot{T_l} u_l) + JV(x)(T_l u_l) \right) - J f(\psi). \tag{6.19}
$$

We add

$$
\sum_{l=1}^{k} \left( \mu_l (-i T_l u_l) - JV(a_l) T_l u_l - J \nabla V(a_l)(x-a_l) T_l u_l \right) \tag{6.20}
$$
to each side of (6.17). By using notations (6.10)-(6.14) to collect parameters into $\tilde{\beta}$, we conclude (6.15).

We want to derive the modulation formula for solitons dynamics. To split the nonlinear term $f(\psi)$, using the definition of $u_j$ and $w_j$, we deduce that

$$
f(\psi) = \sum_{l=1}^{k} T_j f(u_j) + \zeta(w) + O(e^{-\frac{c}{\varepsilon}}), \tag{6.21}
$$

where $\zeta(w) \in H^1_0(\mathbb{R}^N)$ is the boundary term with $\text{supp}\{\zeta(w)\} \subset \bigcup_{l=1}^{k-1} (B_{2L}(a_l(0)) \setminus B_L(a_l(0)))$. Thus for the right side of (6.15), we have

$$
\sum_{l=1}^{k} J T_l (-\varepsilon^2 \Delta u_l + \mu_l u_l) + \sum_{l=1}^{k} J R_l \cdot T_l u_l - J f(\psi)
= \sum_{l=1}^{k} J \mathcal{E}'_{\mu_l,\varepsilon}(u_l) + \sum_{l=1}^{k} J R_l \cdot T_l u_l - J \zeta(w) + O(e^{-\frac{c}{\varepsilon}}).
$$

From (6.8), we obtain

$$
\mathcal{E}'_{\mu_j,\varepsilon}(u_j) = \mathcal{E}'_{\mu_j,\varepsilon}(\eta_{\mu_j,\varepsilon} + w_j) + O(e^{-\frac{c}{\varepsilon}}) = \mathcal{L}_{\eta_{\mu_j,\varepsilon}}(w_j) + N_{\eta_{\mu_j,\varepsilon}}(w_j) + O(e^{-\frac{c}{\varepsilon}}), \tag{6.22}
$$
Since $\eta_{\mu_j,\varepsilon} = \mu_j \partial_{\mu_j} \eta_{\mu_j,\varepsilon}$, we can rewrite (6.13) as
\[
\sum_{l=1}^{k} \left( \beta_l \cdot K\eta_{\mu_l,\varepsilon} + \tilde{\beta}_l \cdot \tilde{K}w_l + T_l \dot{w}_l \right) = \sum_{l=1}^{k} \left( J T_l L_{\eta_{\mu_l,\varepsilon}}(w_l) + J T_l N_{\eta_{\mu_l,\varepsilon}}(w_l) + J R_l \cdot T_l u_l \right)
\]
\[
+ J \zeta(w) + O(e^{-\varepsilon}).
\] (6.23)

If we denote
\[
W(w) := \sum_{l=1}^{k} \left( J T_l L_{\eta_{\mu_l,\varepsilon}}(w_l) + J T_l N_{\eta_{\mu_l,\varepsilon}}(w_l) + J R_l \cdot T_l w_l - \tilde{\beta}_l \cdot \tilde{K}w_l \right) + J \zeta(w),
\] (6.24)

and
\[
Q(\sigma) := \sum_{l=1}^{k} \left( J R_l \cdot T_l \eta_{\mu_l,\varepsilon} - J T_l L_{\eta_{\mu_l,\varepsilon}}(\eta_{\mu_l,\varepsilon}) \right),
\] (6.25)

we obtain
\[
\sum_{l=1}^{k} T_l \dot{w}_l = W(w) + Q(\sigma) + O(e^{-\varepsilon}).
\] (6.26)

We will let $J z_j$ act on (6.23), and hope that the surplus variables can be eliminated by the skew orthogonal condition. For this purpose, we give some useful facts for $z_j \in T_{\Psi_{\sigma_j,\varepsilon}} M$ with $j = 1, \cdots, k$. Using the definition of decomposition in Proposition 7.2, we have $\langle J z_j, J L_{\eta_{\mu_j,\varepsilon}}(w) \rangle_\varepsilon = 0$. By the smooth truncation as we have done in (6.1) and (6.5), it holds
\[
\langle J z_j, J T_j L_{\eta_{\mu_j,\varepsilon}}(w_j) \rangle_\varepsilon = O(e^{-\varepsilon}).
\] (6.27)

Similarly, for $z_i \in T_{\Psi_{\sigma_i,\varepsilon}} M$ and $i \neq j$, we have
\[
\langle J z_i, J T_j L_{\eta_{\mu_j,\varepsilon}}(w_j) \rangle_\varepsilon = O(e^{-\varepsilon}).
\] (6.28)

Then, we consider the time derivatives. For $\bar{z}_j \in T_{\Psi_{\sigma_0,j,\varepsilon}} M$ where $\sigma_{0,j} = (0,0,0,\mu_j)$, and $j = 1, \cdots, k$, it holds
\[
0 = \partial_t \langle J z_j, T_j^{-1} w \rangle_\varepsilon = \langle J z_j, \partial_t (T_j^{-1} w) \rangle_\varepsilon + \mu_j \langle J \partial_{\mu_j} z_j, T_j^{-1} w \rangle_\varepsilon
\]
\[
= \langle J z_j, T_j \partial_t (T_j^{-1} w) \rangle_\varepsilon + \mu_j \langle J \partial_{\mu_j} z_j, w \rangle_\varepsilon,
\] (6.29)

where $z_j = T_j \bar{z}_j \in T_{\Psi_{\sigma_j,\varepsilon}} M$. We can use (6.3). Again to deduce
\[
\langle J z_j, T_j \dot{w}_j \rangle_\varepsilon = -\mu_j \langle J \partial_{\mu_j} z_j, T_j w_j \rangle_\varepsilon + O(e^{-\varepsilon})
\]
\[
= \beta_{j,2N+2} \langle J z_j, K_{j,2N+2} w_j \rangle_\varepsilon + O(e^{-\varepsilon})
\] (6.30)

While for $z_i \in T_{\Psi_{\sigma_i,\varepsilon}} M$ and $i \neq j$ it holds
\[
\langle J z_i, T_j \dot{w}_j \rangle_\varepsilon = O(e^{-\varepsilon}).
\] (6.31)

Now, we let $J z_{i,n}$ act on both sides of (6.23). Since the support of $J \zeta(w)$ is contained in $\cup_{l=1}^{k-1} (B_{2L}(a_l(0)) \setminus B_L(a_l(0)))$, we have $\langle J z_{i,n}, J \zeta(w) \rangle_\varepsilon = O(e^{-\varepsilon})$ for $i = 1, \cdots, k$ and
\( n = 1, \cdots, 2N + 2 \). Recalling that \( K_{j,m}\eta_{\mu_j,\varepsilon} = z_{j,m} \) are exactly the skew orthogonal bases of \( T_{y_{\mu_j,\varepsilon}}M \), we can obtain

\[
\sum_{j=1}^{k} \sum_{m=1}^{2N+2} \Omega_{j,m}^{i,n} \beta_{j,m} = \langle z_{i,n}, \sum_{l=1}^{k} (T_l N_{\eta_{\mu_l,\varepsilon}}(w_l) + R_l \cdot T_l(\eta_{\mu_l,\varepsilon} + w_l)) \rangle \varepsilon
\]

(6.32)

\[
- \sum_{l=1}^{k} \beta_l \langle Jz_{i,n}, K_l w_l \rangle \varepsilon + O(e^{-\frac{2}{\varepsilon}}),
\]

where

\[
\Omega_{j,m}^{i,n} = \langle z_{i,n}, J^{-1}z_{j,m} \rangle \varepsilon = \begin{cases} 
\Omega_{\eta_{\mu_j,\varepsilon}}^{m,n} & i = j, \\
O(e^{-\frac{2}{\varepsilon}}) & i \neq j.
\end{cases}
\]

(6.33)

Thus, using the notations (6.10)-(6.13) and the expression (4.15) of matrix \( \Omega_{\eta_{\mu,\varepsilon}} \), we can rewrite (6.32) into four equations:

\[
\dot{a}_{j,m} = v_{j,m} + (m(\mu_j))^{-1} \langle z_{j,N+m}(T_j N_{\eta_{\mu_j,\varepsilon}}(w_j) + R_j \cdot T_j(w_j)) \rangle \varepsilon - \beta_j \langle Jz_{j,N+m}, K_j w_j \rangle \varepsilon + O(e^{-\frac{2}{\varepsilon}}),
\]

(6.34)

\[
\frac{1}{2} \dot{v}_{j,m} = -\varepsilon \partial x_m V(a_j) - (m(\mu_j))^{-1} \langle z_{j,m}(T_j N_{\eta_{\mu_j,\varepsilon}}(w_j) + R_j \cdot T_j(w_j)) \rangle \varepsilon + \langle z_{j,m}, R_j \cdot T_j(\eta_{\mu_j,\varepsilon}) \rangle \varepsilon \beta_j \langle Jz_{j,m}, K_j w_j \rangle \varepsilon + O(e^{-\frac{2}{\varepsilon}}),
\]

(6.35)

\[
\dot{\gamma}_j = \mu_j + \frac{v_j}{2} \cdot \frac{\dot{a}_j}{\varepsilon} - \frac{|v_j|^2}{4} - V(a_j) - (m'(\mu_j))^{-1} \langle z_{j,2N+2}(T_j N_{\eta_{\mu_j,\varepsilon}}(w_j) + R_j \cdot T_j(w_j)) \rangle \varepsilon + \langle z_{j,2N+2}, R_j \cdot T_j(\eta_{\mu_j,\varepsilon}) \rangle \varepsilon \beta_j \langle Jz_{j,2N+2}, K_j w_j \rangle \varepsilon + O(e^{-\frac{2}{\varepsilon}}),
\]

(6.36)

\[
\dot{\mu}_j = (m'(\mu_j))^{-1} \langle z_{j,2N+1}(T_j N_{\eta_{\mu_j,\varepsilon}}(w_j) + R_j \cdot T_j(w_j)) \rangle \varepsilon - \beta_j \langle Jz_{j,2N+1}, K_j w_j \rangle \varepsilon + O(e^{-\frac{2}{\varepsilon}}),
\]

(6.37)

where \( m(\mu_j), m'(\mu_j) \) are given in (1.11) (1.16), and we use the facts \( \langle J\eta_{\mu_j,\varepsilon}, R_j \eta_{\mu_j,\varepsilon} \rangle \varepsilon = 0, \langle \frac{\partial}{\partial x_m} \eta_{\mu_j,\varepsilon}, R_j \eta_{\mu_j,\varepsilon} \rangle \varepsilon = 0 \) for \( j = 1, \cdots, k \) and \( m = 1, \cdots, N \).

If we denote \( \mathbf{a}_j = (\frac{a_j}{\varepsilon}, v_j, \gamma_j, \mu_j) \) and \( \mathbf{a} = (\mathbf{a}_1, \cdots, \mathbf{a}_k) \), we can abbreviate (6.34)-(6.37) as

\[
\dot{\mathbf{a}} = X(\mathbf{a}) - X_\beta(\mathbf{a}, w) + O(e^{-\frac{2}{\varepsilon}}),
\]

(6.38)

where

\[
X_{j,m}(\mathbf{a}) = \dot{\mathbf{a}}_{j,m} - \beta_{j,m}
\]

(6.39)
and
\[
X^{i,n}_{\delta}(\sigma, w) = \sum_{j=1}^{k} \sum_{m=1}^{2N+2} (\Omega_{j,m}^i)^{-1} \left( \langle z_{i,n} \rangle \sum_{l=1}^{k} \left( T_l N_{\eta_{\mu,l}}(w_l) + R_l \cdot T_l(\eta_{\mu,l} + w_l) \right) \right) - \sum_{l=1}^{k} \beta_l \langle Jz_{i,n}, K_j w_l \rangle. 
\]

Notice that the last term on the right side of (6.40) is of order \( O(|\beta| k \sum_{l=1}^{k} ||w_l||_{L^2}) \) by the Green’s theorem. On the other hand, it holds \( R_j \leq C \varepsilon_v^2 \) and \( N_{\eta_{\mu,l}}(w_j) \leq C ||w_j||_{H^1}^2 \) for \( j = 1, \ldots, k \). Thus we have
\[
X_{\delta}(\sigma, w) = O(|\beta| \sum_{l=1}^{k} ||w_l||_{L^2} + \varepsilon_v^2 + \sum_{l=1}^{k} ||w_l||_{H^1}^2), \quad (6.41)
\]
provided \( ||w||_{H^1} \leq 1 \). We conclude above calculation as following proposition:

**Proposition 6.2.** \( \sigma \) and perturbation \( w_j \) satisfy
\[
\dot{\sigma} = X(\sigma) - X_{\delta}(\sigma, w) + O(e^{-\frac{c}{\varepsilon}}) \quad (6.42)
\]
and
\[
\sum_{l=1}^{k} T_l \dot{w}_l = W(w) + Q(\sigma) + O(e^{-\frac{c}{\varepsilon}}), \quad (6.43)
\]
where \( W(w) \) and \( Q(\sigma) \) are defined in (6.24) and (6.25). Furthermore, we have the following estimate for the vector field \( X_{\delta} \):
\[
X_{\delta} = O(|\beta| \sum_{l=1}^{k} ||w_l||_{L^2} + \varepsilon_v^2 + \sum_{l=1}^{k} ||w_l||_{H^1}^2), \quad (6.44)
\]
provided \( ||w||_{H^1} \leq 1 \).

In the rest part of this paper, we are going to show \( \sup_{t \in (0, T]} ||w_l||_{H^1} = O(\varepsilon_v) \) for \( T = \frac{T_0}{\varepsilon^2} \) with \( T_0 \) a positive constant, from which \( \sup_{t \in (0, T]} |X_{\delta}| = O(\varepsilon_v^2) \) and the main result follows.

### 7. Time derivative of energy

By the definition of \( u_j \), we observe that \( u_j \) is very close to \( \eta_{\mu_j, \varepsilon} \). In this section, we are going to prove that \( \sum_{l=1}^{k} \partial_t E_{\mu_l, \varepsilon}(u_l) - \sum_{l=1}^{k} \partial_t E_{\mu_l}(\eta_{\mu_l, \varepsilon}) \) is a small term, so that we are able to control \( \sum_{l=1}^{k} ||w_l||_{H^1} \) in a large time interval. We have following lemma, which gives the time derivative of energy functional \( \sum_{l=1}^{k} E_{\mu_l, \varepsilon}(u_l) \):
Lemma 7.1. It holds

\[
\sum_{l=1}^{k} \partial_{t} \mathcal{E}_{\mu, \varepsilon}(u_{l}) = \frac{1}{2} \sum_{l=1}^{k} \dot{\mu}_{l} ||u_{l}||_{L^2}^2 - \sum_{l=1}^{k} \left( \frac{\dot{\varepsilon}_{l}}{2} + \varepsilon \nabla V_{a_{l}} \right) u_{l}, \varepsilon \nabla u_{l} \varepsilon \\
+ C\varepsilon^2 \sum_{l=1}^{k} \cdot ||u_{l}||_{H^1}^2 + O(e^{-\frac{c}{\varepsilon}}),
\]

where \( V_{a}(x) := V(x + a) \).

Proof. According to the definition of \( \mathcal{H}_{v, \varepsilon} \) in (3.4), and the definition of \( \mathcal{E}_{\mu, \varepsilon}(\eta_{\mu, \varepsilon}) \) in (3.5), we have

\[
2\mathcal{H}_{v, \varepsilon}(\psi) = 2\mathcal{H}_{v, \varepsilon}(\sum_{l=1}^{k} \mathcal{T}_{l} u_{l}) \\
= 2 \sum_{l=1}^{k} \mathcal{E}_{\mu, \varepsilon}(u_{l}) - \sum_{l=1}^{k} \left( \frac{|v_{l}|^2}{4} + \mu_{l} \right) ||T_{l} u_{l}||_{L^2}^2 + \sum_{l=1}^{k} v_{l} \cdot \langle \mathcal{T}_{l} u_{l}, \varepsilon \nabla (T_{l} u_{l}) \rangle_{\varepsilon} + \int_{\mathbb{R}^N} V|\psi|^2dx
\]

We calculate the time derivative of each individual term in (7.2). Since \( \partial_{t} \left( |\psi|^2 \right) = 0 \) and \( |v_{j}| \leq K\varepsilon \) for \( j = 1, \cdots, k \), we use condition (C) in Section 2 and (3.9) to deduce that

\[
\sum_{l=1}^{k} \left( \frac{|v_{l}|^2}{4} + \mu_{l} \right) \partial_{t} \left( ||T_{l} u_{l}||_{L^2}^2 \right) \leq \sum_{l=1}^{k} \left( \frac{k\varepsilon^2}{4} + C \right) \partial_{t} \left( ||T_{l} u_{l}||_{L^2}^2 \right) \\
= C\varepsilon \text{Re} \sum_{l=1}^{k-1} \int_{B_{2L}(a_l(0)) \setminus B_{L}(a_l(0))} \frac{i}{\varepsilon^{N-1}} w \cdot \nabla w \cdot \nabla (\varphi_{l}^2)dx + O(e^{-\frac{c}{\varepsilon}}) \\
\leq C\varepsilon \sum_{l=1}^{k-1} \int_{B_{2L}(a_l(0)) \setminus B_{L}(a_l(0))} \frac{1}{\varepsilon^{N-1}} |w|^2 |\nabla^2 (\varphi_{l})|dx + O(e^{-\frac{c}{\varepsilon}}) \\
\leq C\varepsilon^2 \sum_{l=1}^{k} ||u_{l}||_{H^1}^2 + O(e^{-\frac{c}{\varepsilon}}),
\]
Similarly, by (3.10) we have
\[ \partial_t (i T_j u_j, \varepsilon \nabla (T_j u_j)) = \varepsilon (\nabla V) T_j u_j, T_j u_j \varepsilon \]
where the second step is by (3.9) and the smooth truncation we have made in (6.1).

Similarly, by (3.10) we have
\[
\partial_t (i T_k u_k, \varepsilon \nabla (T_k u_k)) = \varepsilon (\nabla V) T_k u_k, T_k u_k \varepsilon
\]
with \( C_2 \) to obtain

\[
\langle \partial_t \psi, \varepsilon \nabla (\psi) \rangle \eta \leq \langle \partial_t \psi, \varepsilon \nabla (\psi) \rangle \eta - \varepsilon (\nabla V) \psi, \varepsilon \nabla \psi \rangle \varepsilon
\]
where we have used the assumption \( |v_i| \leq K \varepsilon \) for \( l = 1, \ldots, k \). By the definition of \( T_j \) and the smooth truncation (6.3), we have

\[
2 \varepsilon (\nabla V) \psi, \varepsilon \nabla \psi \rangle \varepsilon = \sum_{l=1}^{k} \langle \varepsilon (\nabla V) i T_l u_l, \varepsilon \nabla (T_l u_l) \rangle \varepsilon + C \varepsilon \cdot \varepsilon \sum_{l=1}^{k} ||u_i||^2_{H^1} + O(\varepsilon^{-\frac{3}{2}}).
\]

Notice that according to the right side of (6.7), we have

\[
\langle \partial_t \psi, \varepsilon \nabla (\psi) \rangle \eta \leq \langle \partial_t \psi, \varepsilon \nabla (\psi) \rangle \eta - \varepsilon (\nabla V) \psi, \varepsilon \nabla \psi \rangle \varepsilon
\]
Suppose \( ||w||_{H^1} \leq 1 \). Notice that according to the right side of (6.7), we have

\[
||w||_{H^1}^2 \leq C \sum_{l=1}^{k} ||u_l||_{H^1}^2,
\]

with \( C \) a positive constant.

Now using (3.8), (3.10), (7.3), (7.4) and \( \partial_t \mathcal{H}_v, \varepsilon (\psi) = 0 \), we take time derivative of (7.2) to obtain

\[
2 \sum_{l=1}^{k} \partial_t \mathcal{E}_{i \varepsilon l} (u_l) = \sum_{l=1}^{k} \langle \dot{v}_l \cdot \frac{v_l}{2} + \dot{\mu}_l ||T_l u_l||_{L^2}^2 - \sum_{l=1}^{k} \dot{v}_l \cdot \langle i T_l u_l, \varepsilon \nabla (T_l u_l) \rangle \varepsilon - 2 \varepsilon (\nabla V) i \psi, \varepsilon \nabla \psi \rangle \varepsilon
\]

\[
+ \sum_{l=1}^{k} v_l \cdot \langle \varepsilon (\nabla V) T_l u_l, T_l u_l \rangle \varepsilon + C \varepsilon \sum_{l=1}^{k} ||u_l||_{H^1}^2,
\]

provided \( ||w||_{H^1} \leq 1 \). Notice that according to the right side of (6.7), we have
Since $\varepsilon_v = o(\varepsilon)$, we can collect terms of the form $v_j + 2\varepsilon \nabla V_{a_j}$ to give

$$2 \sum_{l=1}^{k} \partial_t \mathcal{E}_{\mu|\varepsilon}(u_l) = \sum_{l=1}^{k} \mu_l||u_l||_{L^2}^2 - \sum_{l=1}^{k} \langle (\dot{v}_l + 2\varepsilon \nabla V_{a_l})i u_l, \varepsilon \nabla u_l \rangle$$

$$+ C \varepsilon^2 \sum_{l=1}^{k} ||w_l||_{H^1}^2 + O(e^{-\frac{\varepsilon}{2}}),$$

which is the desired result. \hfill \Box

By now we can state the main result of this section, which claims the time derivative of energy difference $\sum_{l=1}^{k} \mathcal{E}_{\mu|\varepsilon}(u_l) - \sum_{l=1}^{k} \mathcal{E}_{\mu|\varepsilon}(\eta_{\mu|\varepsilon})$ is a small term depending on $\beta$ and the $H^1$ norm of $w_j$ with $j = 1, \cdots, k$.

**Proposition 7.2.** It holds

$$\sum_{l=1}^{k} \partial_t \mathcal{E}_{\mu|\varepsilon}(u_l) - \sum_{l=1}^{k} \partial_t \mathcal{E}_{\mu|\varepsilon}(\eta_{\mu|\varepsilon}) = O(|\beta| \sum_{l=1}^{k} ||w_l||_{H^1}^2 + \varepsilon^2 \sum_{l=1}^{k} ||w_l||_{H^1}^2 + \varepsilon^2 \sum_{l=1}^{k} ||w_l||_{H^1} + e^{-\frac{\varepsilon}{2}}).$$

(7.6)

**Proof.** Since $\eta_{\mu|\varepsilon}$ is a critical point of $\mathcal{E}_{\mu|\varepsilon}$, we have

$$\partial_t \mathcal{E}(\eta_{\mu|\varepsilon}) = \frac{1}{2} \dot{\mu}_j ||\eta_{\mu|\varepsilon}||_{L^2}^2.$$  (7.7)

From Lemma [7.4] we can divide $\sum_{l=1}^{k} \partial_t \mathcal{E}_{\mu|\varepsilon}(u_l) - \sum_{l=1}^{k} \partial_t \mathcal{E}_{\mu|\varepsilon}(\eta_{\mu|\varepsilon})$ into three parts:

$$I_1 = \frac{1}{2} \sum_{l=1}^{k} \dot{\mu}_l (||u_l||_{L^2}^2 - ||\eta_{\mu|\varepsilon}||_{L^2}^2),$$

(7.8)

$$I_2 = - \sum_{l=1}^{k} \langle (\frac{\dot{v}_l}{2} + \varepsilon \nabla V_{a_l})i u_l, \varepsilon \nabla u_l \rangle,$$

(7.9)

$$I_3 = C \varepsilon^2 \sum_{l=1}^{k} ||w_l||_{H^1}^2 + O(e^{-\frac{\varepsilon}{2}}).$$

(7.10)

Using the decomposition $\psi = \sum_{l=1}^{k} T_{\eta_{\mu|\varepsilon}} + w$ to find $0 = \langle T_\beta \eta_{\mu|\varepsilon}, w \rangle = \langle \eta_{\mu|\varepsilon}, T_{\beta}^{-1} w \rangle$, and by the smooth truncation [6.1], we have

$$\langle \eta_{\mu|\varepsilon}, w_j \rangle = O(e^{-\frac{\varepsilon}{2}}).$$

(7.11)

Recall $\dot{\mu}_j = \beta_{j,2N+2}$. Hence for the first part $I_1$, it holds

$$I_1 = \frac{1}{2} \sum_{l=1}^{k} \dot{\mu}_l ||w_l||_{L^2}^2 + O(e^{-\frac{\varepsilon}{2}}) = O(|\beta| \sum_{l=1}^{k} ||w_l||_{H^1}^2 + e^{-\frac{\varepsilon}{2}}).$$

(7.12)
Similar to (7.11), we can prove
\[
\langle i \eta_{\mu_j, \epsilon} \partial \nabla w_j \rangle_{\epsilon} = O(e^{-\frac{\epsilon}{\mu_j}})
\] (7.13)
and
\[
\langle i \epsilon \partial \nabla w_j, \eta_{\mu_j, \epsilon} \rangle_{\epsilon} = O(e^{-\frac{\epsilon}{\mu_j}}).
\] (7.14)
Notice that \( \nabla V_a - \nabla V(a) = O(\epsilon^2 |x|) \) and \( |x| \eta_{\mu_j, \epsilon}, \epsilon |x| \nabla \eta_{\mu_j, \epsilon} \in L^2 \) for \( j = 1, \ldots, k \). Combining (7.13) and (7.14) with \( \epsilon \partial \nabla V_{\alpha_j} = \beta_{j,N+1} \), we can estimate the second part \( I_2 \) as follows:
\[
I_2 = - \sum_{l=1}^{k} \langle (\partial_l ^{2} + \epsilon \nabla V_{a_l}) i w_l, \nabla \eta_{\mu_j, \epsilon} \rangle_{\epsilon} - \sum_{l=1}^{k} \langle \epsilon (\nabla V_{a_l}) i \eta_{\mu_j, \epsilon} \rangle_{\epsilon} - \sum_{l=1}^{k} \langle (\epsilon (\nabla V_{a_l} - \nabla V(a_l))) i w_l, \nabla \eta_{\mu_j, \epsilon} \rangle_{\epsilon}
\]
\[
= - \sum_{l=1}^{k} \langle (\partial_l ^{2} + \epsilon \nabla V(a_l)) i w_l, \nabla \eta_{\mu_j, \epsilon} \rangle_{\epsilon} - \sum_{l=1}^{k} \langle (\epsilon (V_{a_l} - \nabla V(a_l))) i w_l, \nabla \eta_{\mu_j, \epsilon} \rangle_{\epsilon}
\]
\[
= O(\| \beta \| \sum_{l=1}^{k} \| w_l \|_{H^1}^2 + \epsilon \cdot \epsilon \sum_{l=1}^{k} \| w_l \|_{H^1}^2 + \epsilon \cdot \epsilon \sum_{l=1}^{k} \| w_l \|_{H^1}^2 + e^{-\frac{\epsilon}{\mu_j}})
\] (7.15)

From condition (A) in Section 2, we have \( \epsilon_v = o(\epsilon) \). Thus we complete our proof by adding up (7.10) (7.12) and (7.15).

8. Proof of the main result

The next lemma is a well-known result about coercivity of \( \mathcal{L}_{\eta_{\mu, \epsilon}} \), whose proof can be found in [14] and [29]. It relies on the fact that \( \mathcal{L}_{\eta_{\mu, \epsilon}} \) has only one negative eigenvalue and the structure of its null space.

Lemma 8.1. For \( w \in H^1(\mathbb{R}^N, \mathbb{C}) \) satisfying \( \omega_{\epsilon}(w, z) = 0, \forall z \in \mathbf{T}_{\Psi_{\sigma_0, m}} M \) with \( \sigma_0 = (0, 0, 0, \mu) \) and \( \mu \in I \), it holds
\[
\langle \mathcal{L}_{\eta_{\mu, \epsilon}} w, w \rangle_{\epsilon} \geq \rho \| w \|_{H^1}^2
\] (8.1)
where the linearized operator \( \mathcal{L}_{\eta_{\mu, \epsilon}} \) is defined in (4.1), and \( \rho > 0 \) is a constant.

Unfortunately, Lemma 8.1 cannot be applied directly for this multi-peak case. Thus we introduce following corollary:

Corollary 8.2. For \( u_j \) and \( w_j \) defined in (6.1) (6.2) (6.3) (6.6), it holds
\[
\mathcal{E}_{\mu_j, \epsilon}(u_j) - \mathcal{E}_{\mu_j, \epsilon}(\eta_{\mu, \epsilon}) \geq \frac{\rho_j}{2} \| w_j \|_{H^1}^2 - C \| w_j \|_{H^1}^3 + O(e^{-\frac{\epsilon}{\mu_j}}) \quad \text{for } j = 1, \ldots, k,
\] (8.2)
where $E_{\mu,\varepsilon}$ is defined in (3.12), and $\rho_j, c$ are two positive constants.

**Proof.** We will use Lemma (8.1). Define $\tilde{w}_j = u_j - \eta_{\mu_j, \varepsilon}$, then by (6.8) and (6.9) we have

$$||\tilde{w}_j - w_j||_{H^1} = O(e^{-\frac{\varepsilon}{2}}) \quad \text{for} \quad j = 1, \cdots, k.$$  

(8.3)

Moreover, since $\omega_{\varepsilon}(w, T_j z) = \omega_{\varepsilon}(T_j^{-1} w, z) = 0$, $\forall z \in T_{\Psi_{\sigma_0, j, \varepsilon}} M$ with $\sigma_{0, j} = (0, 0, \mu_j)$, it holds

$$\omega_{\varepsilon}(\tilde{w}_j, z) = O(e^{-\frac{\varepsilon}{2}}) \quad \forall z \in T_{\Psi_{\sigma_0, j, \varepsilon}} M.$$  

(8.4)

Thus we can write $\tilde{w}_j$ as

$$\tilde{w}_j = \hat{w}_j + \sum_{m=1}^{2N+2} r_{j,m} z_{j,m},$$

(8.5)

where $\hat{w} \in H^1(\mathbb{R}^N, \mathbb{C})$ is the function such that

$$\omega_{\varepsilon}(\hat{w}_j, z) = 0, \quad \forall z \in T_{\Psi_{\sigma_0, j, \varepsilon}} M,$$

and $|r_{j,m}| = O(e^{-\frac{\varepsilon}{2}})$ for $m = 1, \cdots, 2N + 2$. By (8.1) (8.3) (8.5), and using the fact that $\eta_{\mu_j, \varepsilon}$ is a radial ground state solution, we have

$$2E_{\mu_j, \varepsilon}(u_j) - 2E_{\mu_j, \varepsilon}(\eta_{\mu_j, \varepsilon}) = \langle L_{\eta_{\mu_j, \varepsilon}} \tilde{w}_j, \tilde{w}_j \rangle_{\varepsilon} - C||\tilde{w}_j||_{H^1}^3$$

$$= \langle L_{\eta_{\mu_j, \varepsilon}} \hat{w}_j, \hat{w}_j \rangle_{\varepsilon} - C||\hat{w}_j||_{H^1}^3 + O(e^{-\frac{\varepsilon}{2}})$$

$$\geq \rho||\hat{w}_j||_{H^1}^2 - C||w_j||_{H^1}^3 + O(e^{-\frac{\varepsilon}{2}})$$

$$\geq \rho||w_j||_{H^1}^2 - C||w_j||_{H^1}^3 + O(e^{-\frac{\varepsilon}{2}}).$$

(8.6)

Hence we have proved (8.2). \qed

To study the change of $H^1$ norm for the perturbation term $w$, we define the norm

$$|||w|||_t = \sup_{s \in (0,t]} ||w(s)||_{H^1}$$

(8.7)

We are going to prove $|||w|||_t \leq c_{\varepsilon, c}$ for $t = \min\left\{ \frac{T_0}{\varepsilon^2}, \frac{L}{K} \right\}$ with $T_0$ some positive constant dependent on $\rho_0 := \min\{\rho_1, \cdots, \rho_k\}$.

**Proposition 8.3.** There are positive constants $T_0, c, c'$ independent of $\varepsilon$ and $\varepsilon_v$ such that for $0 < t \leq \min\{ \frac{T_0}{\varepsilon^2}, \frac{L}{K\varepsilon} \}$, it holds

$$||w||_{H^1} \leq c_{\varepsilon, c}$$

(8.8)

and

$$|\beta| \leq c'_{{\varepsilon, c}^2},$$

(8.9)

where $\beta$ is defined in (6.10)-(6.13).

**Proof.** By condition (C) in Section 2, we have $||w(0)||_{H^1} \leq c_{\varepsilon, c}$, which implies

$$\sum_{j=1}^{k} ||w_j(0)||_{H^1} \leq c_{\varepsilon, c}$$

(8.10)
from (6.7). So for radial ground state solutions \( \eta_{\mu_j, \varepsilon} \) where \( j = 1, \cdots, k \), we have
\[
2 \sum_{l=1}^{k} (\mathcal{E}_{\mu_l, \varepsilon}(u_l(0)) - \mathcal{E}_{\mu_l, \varepsilon}(\eta_{\mu_l, \varepsilon}(0))) = 2 \sum_{l=1}^{k} (|\varepsilon \nabla w_l(0)|^2_{L^2} + \mu_l |w_j(0)|^2_{L^2} - R^{(2)}_{\eta_{\mu_l, \varepsilon}}(w_l(0))).
\]

(8.11)

According to (3.16), it holds
\[
\sum_{l=1}^{k} |\varepsilon| |w_l(0)|^2_{L^2} = \min_{\mu_l} \left\{ \varepsilon |w_l(0)|^2_{L^2} \right\}.
\]

(8.12)

So we have
\[
2 \sum_{l=1}^{k} (\mathcal{E}_{\mu_l, \varepsilon}(u_l(0)) - \mathcal{E}_{\mu_l, \varepsilon}(\eta_{\mu_l, \varepsilon}(0))) \leq c \varepsilon^2_v.
\]

(8.13)

On the other hand, from Corollary 8.2 we know that
\[
2 \sum_{l=1}^{k} (\mathcal{E}_{\mu_l, \varepsilon}(u_l(0)) - \mathcal{E}_{\mu_l, \varepsilon}(\eta_{\mu_l, \varepsilon}(0))) \geq \rho_0 \sum_{l=1}^{k} |w_l|_{H^1}^2 + C \sum_{l=1}^{k} |w_l|_{H^3}^3 + O(e^{-\frac{\beta}{2}}),
\]

(8.14)

where \( \rho_0 = \min \{\rho_1, \cdots, \rho_k\} \). Then we can use Proposition 7.2 to get the integral energy from 0 to \( t \), and deduce that for \( s \in (0, t] \) it holds:
\[
\rho_0 \sum_{l=1}^{k} |w_l(s)|_{H^1}^2 - C \sum_{l=1}^{k} |w_l(s)|_{H^3}^3 + O(e^{-\frac{\beta}{2}})
\]
\[
\leq 2 \sum_{l=1}^{k} (\mathcal{E}_{\mu_l, \varepsilon}(u_l(0)) - \mathcal{E}_{\mu_l, \varepsilon}(\eta_{\mu_l, \varepsilon}(0))) + 2t \partial_t \left( \sum_{l=1}^{k} \mathcal{E}_{\mu_l, \varepsilon}(u_l) - \sum_{l=1}^{k} \mathcal{E}_{\mu_l}(\eta_{\mu_l, \varepsilon}) \right)
\]
\[
\leq c \varepsilon^2_v + c'' t (|\beta| \cdot ||w||_{L^2}^2 + \varepsilon^2 ||w||_{L^2}^2 + \varepsilon \cdot \varepsilon_v \cdot ||w||_{L^2} + O(e^{-\frac{\beta}{2}})),
\]

(8.15)

where we have used (6.7) and (8.13) in the last inequality. Thus by (6.7) and Cauchy-Schwarz inequality, we have
\[
\sum_{l=1}^{k} |w_l(s)|_{H^1}^2 \geq \frac{1}{k} (\sum_{l=1}^{k} |w_l(s)|_{H^1}^2)^2 \geq \frac{1}{4k} |w(s)|_{H^1}^2
\]
and
\[
\sum_{l=1}^{k} |w_l(s)|_{H^3}^3 \leq (\sum_{l=1}^{k} |w_l(s)|_{H^3}^3)^{\frac{1}{3}} \leq C |w(s)|_{H^3}^3.
\]

For \( t \) satisfying \( e'' t (|\beta| + \varepsilon^2) = \rho_0/8k \), we take the sup\( \sup_{s \in (0, t]} \) of \( |w(s)|_{H^1} \) to obtain
\[
\frac{\rho_0}{8k} ||w||_{L^2}^2 \leq c \varepsilon^2_v + O(1) \cdot \varepsilon_v ||w||_{L^2} + C ||w||_{L^2}^3 + O(e^{-\frac{\beta}{2}}),
\]

(8.16)

which implies that for \( t = \rho_0(8ke''(\varepsilon^2_v + \varepsilon^2))^{-1} \), it holds
\[
||w||_{L^2}^2 \leq c \varepsilon^2_v
\]

(8.17)
provided \( ||w||_{H^1} \leq \rho_0/16kC \) and \( \varepsilon \) is sufficiently small. Since \( \psi \in U_\delta \), it is always true that \( ||w||_{t} \leq \delta \) and we can let \( \delta \leq \rho_0/16kC \). From (6.42) we have \( \beta_{j,m} = X_{\delta}^{j,m} + O(e^{-\frac{\pi}{4}}) \). Combining this and (8.17), we can deduce that

\[
|\beta| \leq C\varepsilon^2
\]

for \( t \leq \rho_0(8k\varepsilon''(\varepsilon_v^2 + \varepsilon^2))^{-1} \). According to condition (A) in Section 2 we have \( \varepsilon_v = o(\varepsilon) \), while \( \psi \in U_\delta \) requires that \( t \leq \frac{L}{K\varepsilon} \). Thus (8.8) and (8.9) hold over the time interval

\[
0 < t \leq \min\{\frac{T_0}{\varepsilon^2}, \frac{L}{K\varepsilon}\},
\]

with \( T_0 = \rho_0/16k\varepsilon'' < \infty \) a positive constant, which completes the proof of this proposition.

\[\Box\]

**Proof of Theorem 2.1:** From (6.34)-(6.37) and Proposition 8.3 for \( t \in (0, \min\{\frac{T_0}{\varepsilon^2}, \frac{L}{K\varepsilon}\}) \) and \( i = 1, \cdots, k \) we have

\[
\frac{1}{2}\dot{v}_i = -\varepsilon \nabla V(a_i) + O(\varepsilon_v^2),
\]

\[
\dot{a}_i = v_i + O(\varepsilon_v^2),
\]

\[
\dot{\gamma}_i = \mu_i - V(a_i) - \frac{1}{4}v_i^2 + O(\varepsilon_v^2),
\]

\[
\dot{\mu}_i = O(\varepsilon_v^2)
\]

Note that all the estimates we have done are about the solutions to Equation (1.3). However, we want to derive the corresponding result for Cauchy problem (1.1) and (1.2). For this purpose, we can let \( t' \) be the time parameter in (1.1) and (1.2) and deduce

\[
t' = \varepsilon t \quad \text{and} \quad \varepsilon \partial_{t'} = \partial_t,
\]

where \( t \) is the time parameter in (1.3), so that we can rewrite (8.20)-(8.23) to get the asymptotic estimates for Cauchy problem (1.1) and (1.2) over the time interval \( (0, \min\{\frac{T_0}{\varepsilon^2}, \frac{L}{K\varepsilon}\}) \]. In conclusion, we have

\[
||w||_{H^1} = O(\varepsilon_v)
\]

and

\[
\frac{1}{2}\dot{v}_i = -\nabla V(a_i) + O(\frac{\varepsilon_v^2}{\varepsilon}),
\]

\[
\dot{a}_i = v_i + O(\varepsilon_v^2),
\]

\[
\varepsilon \dot{\gamma}_i = \mu_i - V(a_i) - \frac{1}{4}v_i^2 + O(\varepsilon_v^2),
\]

\[
\varepsilon \dot{\mu}_i = O(\varepsilon_v^2),
\]

with \( i = 1, \cdots, k \), and the proof of Theorem 2.1 is complete.  \[\Box\]
references

[1] V.I. Arnol'd, Mathematical methods of classical mechanics, *Number 60 in Graduate Texts in Mathematics*, Springer-Verlag, New York, Second edition, 1989.

[2] D. Cao, E.S. Noussair and S. Yan, Existence and uniqueness results on single-peaked solutions of a semilinear problem, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 15 (1998), no. 1, 73–111.

[3] D. Cao, E.S. Noussair and S. Yan, Solutions with multiple peaks for nonlinear elliptic equations, *Proc. Roy. Soc. Edinburgh*, 129A (1999), no. 2, 235–264.

[4] D. Cao and E.S. Noussair, Multi-bump standing waves with a critical frequency for nonlinear Schrödinger equations, *J. Differential Equation*, 203 (2004), no. 2, 292–312.

[5] D. Cao, E.S. Noussair and S. Yan, Multiscale-bump standing waves with a critical frequency for nonlinear Schrödinger equations, *Trans. Amer. Math. Soc.*, 360 (2008), no. 7, 3813–3837.

[6] T. Cazenave and P.L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, *Comm. Math. Phys.*, 85 (1982), 549–561.

[7] T. Cazenave, Semilinear Schrödinger equations, *Courant Lecture Notes in Mathematics, Vol. 10.*, New York University Courant Institute of Mathematical Sciences, New York, 2003.

[8] V. Benci, M. Ghimenti and A.M. Micheletti, Nonlinear Schrödinger equation: soliton dynamics, *J. Differential Equations*, 249 (2010), 3312–3341.

[9] V. Benci, M. Ghimenti and A.M. Micheletti, On the dynamics of solitons in the nonlinear Schrödinger equation, *Arch. Rational Mech. Anal.*, 205 (2012), 467–492.

[10] V. Benci and D. Fortunato, A minimization method and applications to the study of solitons, *Nonlinear Anal.*, 75 (2012), 4398–4421.

[11] J.C. Bronski and R.L. Jerrard, Soliton dynamics in a potential, *Math. Res. Lett.*, 7 (2000), no. 2–3, 329–342.

[12] J. Fröhlich, T.P. Tsai and H.T. Yau, On a classical limit of quantum theory and the non-linear Hartree equation, *Geom. Funct. Anal.*, Special Volume, Part I, 2000, 57–78.

[13] J. Fröhlich, T.P. Tsai and H.T. Yau, On the point-particle (Newtonian) limit of the non-linear Hartree equation, *Comm. Math. Phys.*, 225 (2002), no. 2, 223–274.

[14] J. Fröhlich, S. Gustafson, B.L.G. Jonsson and I.M. Sigal, Solitary wave dynamics in an external potential, *Comm. Math. Phys.*, 250 (2004), no. 3, 613–642.

[15] J. Fröhlich, S. Gustafson, B.L.G. Jonsson and I.M. Sigal, Long time motion of NLS solitary waves in a confining potential. *Ann. Henri Poincaré*, 7 (2006), no. 4, 621–620.

[16] B. Gidas, W.M. Ni and L. Nirenberg, symmetry and related properties via the maximum principle, *Comm. Math. Phys.*, 68 (1979), no. 3, 209–243.

[17] M. Grillakis, H. Shatah and W. Strauss, Stability theory of solitary waves in the presence of symmetry. I. *J. Funct. Anal.*, 74 (1987), no. 1, 160–197.

[18] M. Grillakis, H. Shatah and W. Strauss, Stability theory of solitary waves in the presence of symmetry. II. *J. Funct. Anal.*, 94 (1990), no. 2, 308–348.

[19] T. Kato, Nonlinear Schrödinger equations, *Lecture Notes in Phys, Vol. 345.*, Springer, Berlin, 1989, 218–263.

[20] S. Keraani, Semiclassical limit of a class of Schroedinger equations with potential, *Comm. Partial Differential Equations*, 27 (2002), 693–704.

[21] S. Keraani, Semiclassical limit of a class of Schroedinger equations with potential II, *Asymptot. Anal.*, 47 (2006), 171–186.

[22] M.K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$, *Arch. Rational Mech. Anal.*, 105 (1989), 243–266.

[23] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. I, *Ann. Inst. H. Poincaré Anal. Non Lineairé*, 1 (1984), no. 2, 109–145.

[24] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. II, *Ann. Inst. H. Poincaré Anal. Non Lineairé*, 1 (1984), no. 4, 223–283.
[25] M. Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities, *Math. Z.*, 187 (1984), no. 4, 511–517.
[26] M. Willem, Minimax theorems, *Progress in Nonlinear Differential Equations and their Applications*, 24. Birkhäuser Boston, Inc., Boston, MA, 1996.
[27] J. Wei and S. Yan, Infinitely many positive solutions for nonlinear Schrödinger equations in $\mathbb{R}^N$, *Calc. Var. Partial Differential Equations*, 37 (2010), no. 3–4, 423–439.
[28] M. Weinstein, Modulational stability of ground states of nonlinear Schrödinger equations, *SIAM J. Math. Anal.*, 16 (1985), 472–491.
[29] M. Weinstein, Lyapunov stability of ground states of nonlinear dispersive evolution equations, *Comm. Pure Appl. Math.*, 39 (1986), no. 1, 51–67.

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