Bright solitons in a quasi-one-dimensional reduced model of a dipolar Bose–Einstein condensate with repulsive short-range interactions

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Abstract

We study the formation and dynamics of bright solitons in a quasi-one-dimensional reduced mean-field Gross–Pitaevskii equation of a dipolar Bose–Einstein condensate with repulsive short-range interactions. The study is carried out using a variational approximation and a numerical solution. Plots of chemical potential and root mean square (rms) size of solitons are obtained for the quasi-one-dimensional model of three different dipolar condensates of \(^{52}\text{Cr}\), \(^{168}\text{Er}\) and \(^{164}\text{Dy}\) atoms. The results achieved are in good agreement with those produced by the full three-dimensional mean-field model of the condensate. We also study the dynamics of the collision of a train of two solitons in the quasi-one-dimensional model of every condensate above. At small velocities (zero or close to zero) the dynamics is attractive for a phase difference \(\delta = 0\), the solitons coalesce and these oscillate, forming a bound soliton molecule. For a phase difference \(\delta = \pi\) the effect is repulsive. At large velocities the collision is independent of the initial phase difference \(\delta\). This is quasi-elastic and the result is two quasi-solitons.

Keywords: dipolar Bose–Einstein condensate, dimensional reduction, bright solitons

(Some figures may appear in colour only in the online journal)
dipolar interaction (by rotating the magnetic field for magnetic dipoles) and the dipoles aligned in the same direction of the trap. For a quasi-2D dipolar condensate—with positive dipolar strength and atoms with the dipole moments polarized perpendicular to the direction of the trap—stable anisotropic bright solitons were predicted, along with the respective requirements for collapse [15]. The collision of quasi-2D anisotropic bright solitons in a DBEC was performed in [16]. The analysis was carried out using both a time-dependent variational approximation and a full numerical solution. [17] shows the first experimental realization of a quasi-2D anisotropic bright soliton in a dipolar condensate. Recently, the existence of robust 1D and 2D bright solitons was proposed, in a BEC with repulsive dipolar interactions induced by a combination of polarizing fields, oriented perpendicular to the plane in which the condensate is trapped [18]. In a quasi-1D DBEC solitons exist for an arbitrarily large number of atoms [19] because there is no collapse in one-dimensional models with cubic nonlinearity. In a 3D DBEC, considering positive dipolar strength and repulsive atomic interaction, the numerical solution of the full three-dimensional mean-field Gross–Pitaevskii equation (GPE) with confinement in the radial direction and free in the axial one, supports bright solitons below a critical number of atoms when the scattering length is less than the dipolar strength [20]. So, beyond a critical value of atom number the condensate is unstable and should collapse. We show the existence of bright solitons for repulsive contact interaction in a DBEC using a quasi-1D reduced model of the GPE [21]. We also study the dynamics of these solitons. The results of the quasi-1D equation are in good agreement with those produced by the full 3D condensate. Our findings show that the solitons exist provided the value of the scattering length is less than the dipolar strength and there is no collapse in this reduced 1D model. We also found that the collision between two bright solitons is sensitive to the initial phase difference at low velocities (close to zero or zero). At high velocities the collision is quasi-elastic and it is independent of the initial phase difference. In section 2 we present the 1D reduced mean-field model for studying the statics and dynamics of a dipolar BEC without confinement trap. We also include a Gaussian variational approximation of this equation. The numerical and variational results are shown in section 3. Finally, we present a brief summary and discussion of our study in section 4.

2. Analytical consideration

2.1. 1D reduced mean-field model for a dipolar condensate

In a trapped dipolar condensate many static and dynamic properties can be described using the time-dependent non-local 3D GPE. The numerical solution of this equation is a complicated issue because of the anisotropic long-range character of the dipolar interaction. However, in many experimental arrangements with extreme symmetry of the harmonic traps the dynamics of the condensate takes place in reduced dimensions and the system becomes disc- or cigar-shaped. A quasi-1D model of the GPE describes a cigar-shaped condensate. The numerical solution and the variational approximation of such quasi-1D equation is simpler than that of the full 3D equation [21].

We study bright solitons in a dipolar BEC of $N$ atoms, each of mass $m$ by means of the dimensionless dipolar GPE (DGPE). We use as unit of length the oscillator length $l_0 = \sqrt{\hbar/m\omega}$, where $\hbar$ and $\omega$ are the reduced Planck constant and the harmonic trap frequency. The time $t$ is measured in units of $\omega_0^{-1}$ and the energy in units $\hbar\omega_0$. So the mean-field DGPE is

\[
i \frac{\partial \Psi(r, t)}{\partial t} = \left[ -\frac{1}{2} \nabla^2 + \frac{1}{2} \rho^2 + 4\pi a N |\Psi(r, t)|^2 ight. \\
\left. + 3N\mu_0 \int \frac{d^3r'}{(\mathbf{r} - \mathbf{r}')^3} |\Psi(r', t)|^2 \right] \Psi(r, t)
\]

(1)

where $\Psi(r, t)$ is the condensate wave function with normalization $\int d^3r |\Psi(r, t)|^2 = 1$, $\rho^2/2$ is the radial harmonic trap, with $\rho^2(x, y) = x^2 + y^2$. The atomic scattering length is $a$. The constant $a_d = \mu_0 \mu^2 m/12\hbar^2$ is the strength of the dipolar interaction with $\mu_0$ the permeability of free space and $\mu$ the magnetic dipole moment of a single atom. The angle $\theta$ is the angle between the relative position of the dipoles $\mathbf{r} - \mathbf{r}'$ and the polarization direction $\mathbf{z}$.

The wave function propagates in time, e.g. as $\Psi(r, t) = \Psi(r) \exp(-i\mu t)$ [22, 23], where $\mu$ is the chemical potential. Using this ansatz in equation (1) we get the time-independent dipolar GPE

\[
\mu \Psi(r) = \left[ -\frac{1}{2} \nabla^2 + \frac{1}{2} \rho^2 + 4\pi a N |\Psi(r)|^2 ight. \\
\left. + 3N\mu_0 \int \frac{d^3r'}{(\mathbf{r} - \mathbf{r}')^3} |\Psi(r')|^2 \right] \Psi(r).
\]

(2)

From the normalization condition of the wave function and the thermodynamic relation $\mu = \partial E/\partial N$, the energy of the condensate is given by

\[
E(\Psi) = \int d^3r \left[ N|\nabla \Psi(r)|^2 + \frac{N}{2} \rho^2 |\Psi(r)|^2 + 2\pi a N^2 |\Psi(r)|^4 \\
+ \frac{3\mu_0 N^2}{2} \int d^3r' \left( \frac{1 - 3\cos^2 \theta}{(\mathbf{r} - \mathbf{r}')^3} \right) |\Psi(r')|^2 |\Psi(r)|^2 \right].
\]

(3)

For a cigar-shaped DBEC the reduction from three to one dimensions is achieved using the adiabatic approximation [24–26], where we consider that, in a cigar-shaped trap with tight radial binding, the dynamics takes place along the axial direction; and in the transverse direction the condensate wave function remains in the ground state. So the wave function to the mean-field equation (1) can be decomposed as

\[
\Psi(r, t) = \kappa(\rho) \phi(z, t),
\]

with

\[
\phi(z, t) = \left( \frac{1}{a_0^2} \right)^{1/2} \exp \left( -\frac{\rho^2}{2a_0^2} \right) \phi(z, t)
\]

(4)

where $a_0$ is the radial harmonic oscillator length and fulfills $\omega_0 a_0^2 = 1$. 

\[
E = \int d^3r \left[ N|\nabla \phi(z)|^2 + \frac{N}{2} \rho^2 |\phi(z)|^2 + 2\pi a N^2 |\phi(z)|^4 \\
+ \frac{3\mu_0 N^2}{2} \int d^3r' \left( \frac{1 - 3\cos^2 \theta}{(\mathbf{r} - \mathbf{r}')^3} \right) |\phi(r')|^2 |\phi(r)|^2 \right].
\]
We get the quasi-one-dimensional DGPE by substituting this ansatz in the DGPE (1), multiplying by $k'(\rho)$ and integrating over $\rho$ (using the normalization $\int d\rho |\phi^2(\rho)| = 1$). The quasi-1D dipolar contribution can be obtained easily using the Fourier transform of the 3D dipole–dipole energy performing the dimensional reduction in the momentum space and returning to the configuration space [21]. So the quasi-1D DGPE is given by

$$i \frac{\partial \phi(z, t)}{\partial t} = \left[ -\frac{1}{2} \frac{\partial^2}{\partial z^2} + \frac{2aN}{a_0^2} \phi(z, t)^2 \right. + N \int_0^\infty dz' U_D^0(z-z') |\phi(z', t)|^2 \left. \right] \phi(z, t)$$

(5)

where the 1D potential in configuration space is

$$U_D^0(z-z') = \frac{1}{2\pi} \int_0^\infty dq_z \exp(iq_zz) V_{ID}(q_z)$$

(6)

and the 1D potential in Fourier space is

$$V_{ID}(q_z) = 2a_{dd} \int_0^\infty dq_z \frac{3q_z^2}{q_0^2 + q_z^2} \left(1 - \frac{q_0^2q_z^2}{2}\right)$$

(7)

2.2. Variational approximation

An understanding of the existence of bright solitons in the cigar-shaped DBEC can be obtained from a variational approximation to equation (5) with the following time-independent Gaussian ansatz

$$\phi(z) = \left( \frac{1}{w_z^2\pi} \right)^{1/4} \exp\left( -\frac{z^2}{2w_z^2} \right)$$

(8)

where $w_z$ is the variational width along the $z$ direction. However, calculation of energy using this wave function in the quasi-1D model involves integrals of the non-trivial dipolar contribution (6). Instead, it is easier to obtain the 3D Gaussian energy and consider the quasi-1D energy as a special case of this. Thus, using the conventional 3D Gaussian ansatz [5, 20, 21]

$$\Psi(\rho, z) = \left( \frac{1}{w_{\rho}w_z\pi^{3/2}} \right)^{1/2} \exp\left( -\frac{\rho^2}{2w_{\rho}^2} - \frac{z^2}{2w_z^2} \right)$$

(9)

The energy in the quasi-1D model $E_{ID}$ can be obtained using the radial harmonic oscillator length $a_0$ such that $w_{\rho} = a_0$ in the dipolar contribution and neglecting the derivatives with respect to $w_{\rho}$ in the kinetic energy term [21]. These considerations lead to

$$E_{ID} = \frac{N}{4w_z^2} + \frac{N^2}{\sqrt{2\pi}a_0^2w_z}\left[ a-a_{dd}f(\kappa) \right]$$

(10)
where $\kappa_0 = a_0/\omega_z$. In a extreme quasi-1D DBEC, the axial width is much larger than the transverse oscillator length, as well as $k_0 \rightarrow 0$ then $f(k_0) \rightarrow 1$ and the dipolar energy becomes $-N^2a_{dd}/(\sqrt{2}\pi a_0^2\omega_z)$. Therefore the variational approximation indicates that the dipolar interaction turns into a contact interaction. So the total interaction (contact and dipolar) is an effective contact interaction such that

$$a_{\text{eff}} = a - a_{dd}. \quad (12)$$

This effective scattering length becomes attractive when $a_{dd}>a$, i.e. it is possible to have bright solitons, even for repulsive scattering length ($a > 0$). This interesting scenario was shown for the first time in a 3D dipolar condensate [20].

The bright solitons can be regarded as stable stationary states and these are given by the minimum of the energy. So by a minimization of 1D ground state energy (11) by $\partial E_1/\partial w_z = 0$ the equation for the width of the condensate $w_z$ [21] is

$$\frac{1}{2\omega_z^2} + \frac{N[a - a_{dd}h(k_0)]}{\sqrt{2}\pi a_0^2\omega_z^2} = 0 \quad (13)$$

where the function $h(k_0)$ can be written as

$$h(x) = \frac{1 + 10k_0^2 - 2k_0^4 - 9k_0^2d(k_0)}{(1-k_0^2)^2} \quad (14)$$

3. Numerical and variational results

The 3D and quasi-1D GPEs are solved numerically by the split-step Crank–Nicolson method in Cartesian coordinates [29]. The 3D dipolar contribution is evaluated in the momentum space using the convolution theorem as

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**Figure 3.** Chemical potential $\mu$ and rms size $\langle z \rangle$ versus the scattering length $a$ of a three- and a quasi-1D dipolar BEC of 1000 atoms. In the same way is presented the rms size $\langle \rho \rangle$ for the 3D dipolar condensate. (a) and (b) $^{52}$Cr ($a_{dd} = 15a_0$). (c) and (d) $^{164}$Dy ($a_{dd} = 66.6a_0$). (e) $^{164}$Dy ($a_{dd} = 132.7a_0$). Numerical (N) and variational (V) solution. The oscillator length used is 1 $\mu$m.
The Fourier transform of the 3D dipolar potential \( F_U \) is given analytically by \([5, 30, 31]\).

The term \( F\{ |\Psi|^2 \} (q) \) is evaluated numerically using the fast Fourier transform (FFT) in Cartesian coordinates \([32]\). The inverse Fourier transform is obtained by means of the FFT. The quasi-1D dipolar contribution (6) is also calculated using FFT.

In figures 1(a) and (b) we plot the numerical and variational results of the profile of \( |\phi(z,t)|^2 \) from quasi-1D reduced mean-field DGPE (5) and the Gaussian ansatz (8) respectively. We also shown the numerical results of the full 3D equation (1). We use two condensates each of 1000 atoms with dipolar strengths \( a_{dd} = 15a_0 \) (\( a = 10a_0 \)) and \( a_{dd} = 100a_0 \) (\( a = 75.6a_0 \)), where \( a_0 \) is the Bohr radius. In both cases, variational and numerical results of the quasi-1D equation are in good agreement with those of the full 3D model. In figure 2 we compare numerical and variational results of chemical potential \( \mu \) and root mean square (rms) \( \langle z \rangle \) as functions of the scattering length using the quasi-1D model for a dipolar strength \( a_{dd} = 100a_0 \) with the numerical results of full 3D equation obtained from \([20]\).

After having established the appropriateness of the 1D reduced equation we plot in figures 3(a)–(f) the respective

\[
\int d\mathbf{r}' U_{dd}(\mathbf{r}-\mathbf{r}') |\Psi(\mathbf{r}')|^2 = \mathcal{F}^{-1}[\mathcal{F}\{ U_{dd} \}(q) \mathcal{F}\{ |\Psi|^2 \} (q)](\mathbf{r})
\]

with \( U_{dd}(\mathbf{r}-\mathbf{r}') = 3a_{dd}(1-3 \cos^2 \theta) |\mathbf{r}-\mathbf{r}'|^3 \). The Fourier transform of the 3D dipolar potential \( \mathcal{F}\{ U_{dd} \}(q) \) is given analytically by \([5,30,31]\).

\[
\mathcal{F}\{ U_{dd} \}(q) = 4 \pi a_{dd} \left( \frac{q^2}{q_0^2} - 1 \right).
\]

The profile of \( |\phi(z,t)|^2 \) versus \( z \) and \( t \) for a train of two bright solitons in a quasi-1D dipolar condensate of \(^{52}\text{Cr} \) \((a_{dd} = 15a_0)\) atoms with atomic scattering length \( a = 10a_0 \), 2000 atoms and velocity \( v = 0 \). (a) and (b) correspond to the phase difference \( \delta = 0 \). (c) and (d) correspond to \( \delta = \pi \). The oscillator length used is 1 \( \mu m \).

Figure 4. (a) and (b) The profile of \( |\phi(z,t)|^2 \) versus \( z \) and \( t \) for a train of two bright solitons in a quasi-1D dipolar condensate of \(^{168}\text{Er} \) \((a_{dd} = 66.6a_0)\) atoms with atomic scattering length \( a = 50a_0 \), 2000 atoms and velocity \( v = 3 \text{ mm s}^{-1} \). The oscillator length used is 1 \( \mu m \).

Figure 5. (a) and (b) The profile of \( |\phi(z,t)|^2 \) versus \( z \) and \( t \) for a train of two bright solitons in a quasi-1D dipolar condensate of \(^{168}\text{Er} \) \((a_{dd} = 66.6a_0)\) atoms with atomic scattering length \( a = 50a_0 \), 2000 atoms and velocity \( v = 3 \text{ mm s}^{-1} \). The oscillator length used is 1 \( \mu m \).
Figure 6. The profile of $|\Phi(z, t)|^2$ versus $z$ and $t$ of two bright solitons in a quasi-1D dipolar condensate of $^{164}$Dy ($a_{dd} = 132.7\,\mu$m) atoms with atomic scattering length $a = 120 a_0$, 2000 atoms and velocity $v = 0.2\,\text{mm}\,\text{s}^{-1}$. (a) and (b) correspond to $\delta = 0$. (c) and (d) correspond to $\delta = \pi$. The oscillator length used is 1 $\mu$m.

The profile of $|\Phi(z, t)|^2$ of two bright solitons in a quasi-1D dipolar condensate of $^{164}$Dy ($a_{dd} = 132.7\,\mu$m) atoms with atomic scattering length $a = 120 a_0$, 2000 atoms and velocity $v = 0.2\,\text{mm}\,\text{s}^{-1}$. (a) and (b) correspond to $\delta = 0$. (c) and (d) correspond to $\delta = \pi$. The oscillator length used is 1 $\mu$m.

Root mean square sizes and the chemical potential for a 3D and a quasi-1D dipolar condensate. We did this for three different dipolar BECs of $^{52}$Cr ($a_{dd} = 15\,a_0$), $^{166}$Er ($a_{dd} = 66.6\,a_0$) and $^{164}$Dy ($a_{dd} = 132.7\,a_0$) atoms in the same regime of repulsive atomic interaction ($a > 0$), each with 1000 atoms. The three dipolar BECs in 3D with attractive dipolar interaction and repulsive scattering length, should be stable only for a scattering length greater than a critical value as we plot in the figures 3(a)–(f). This critical value is larger for $^{164}$Dy atoms compared to that for $^{52}$Cr and $^{166}$Er atoms.

Although the numerical solution of the quasi-1D reduced model is simpler than the full 3D DGPE, this is still complicated because of the long-range anisotropic dipolar interaction. The variational approximation provides results for the rms size and the chemical potential in good agreement with the numerical solution of the quasi-1D and full 3D GPE. The Gaussian ansatz to the reduced model is relatively simple and it could be used as an approximate solution.

Now we investigate the collision between two solitons. The numerical solution of the quasi-1D mean-field dipolar GPE enables the dynamical study of bright solitons. To investigate the collision between two bright solitons we apply the following procedure. The first step is the creation of one soliton with a number of atoms $N = 1000$, employing imaginary-time propagation ($t \to -i t$) in the Crank–Nicolson method. Afterwards two solitons are placed at positions $z_0$ with $z_0$ the initial separation between these. Then these are advanced by real-time propagation of reduced model (5) with $N = 2000$. To introduce the dynamics in the system, the two solitons are superposed with a phase difference $\delta$. Here we consider the following superposition [33]

$$\Phi(z, t) = e^{i\delta} |\phi(z-z_0, t)| + |\phi(z+z_0, t)|. \quad (17)$$

Similarly the dynamics can be obtained for a constant velocity $v$ different from zero, where the soliton placed on the right hand is multiplied by a phase factor $\exp(ivz)$, while the soliton on the left hand is multiplied by a phase factor $\exp(-ivz)$. So the superposition is given by [16, 20]

$$\Phi(z, t) = e^{ivz} |\phi(z-z_0, t)| + e^{-ivz} |\phi(z+z_0, t)|. \quad (18)$$

To keep the total number of atoms as a constant we need to normalize the new wave function of the two solitons $\Phi(z, t)$ to 2 because it contains twice the number of atoms of a single soliton $\phi(z, t)$. We present results on the effect of the phase difference between the solitons $\delta$, for velocities $v = 0$ and $v \neq 0$ in a train with two equal solitons in the three quasi-1D dipolar condensates of $^{52}$Cr, $^{166}$Er and $^{164}$Dy atoms.

In a dipolar condensate of 2000 atoms of $^{52}$Cr ($a_{dd} = 15\,a_0$) with scattering length $a = 10\,a_0$, velocity $v = 0$ and phase difference $\delta = 0$ we plot the time evolution of the train of two such solitons by means of the profile of $|\Phi(z, t)|^2$ versus $z$ and $t$. In figures 4(a) and (b) we show that due to the dipolar attraction, the solitons come close, coalesce and oscillate forming a bound soliton molecule around $z = 0$. We have initial positions at $z = \pm 15.75\,\mu$m and an interval of time 400 ms. For a phase difference $\delta = \pi$ in an interval of time 400 ms the two solitons repel and stay away from each other, figures 4(c) and (d). These moved from positions $z = \pm 13.25\,\mu$m to $z = \pm 23.75\,\mu$m. In this short time, because of the long-range character of the dipolar interaction, the solitons still remain interacting. The final solitons are different to the initial ones. Our numerical results

\[ |\Phi(z, t)|^2 \]

\[ \Phi(z, t) = e^{i\delta} |\phi(z-z_0, t)| + |\phi(z+z_0, t)|. \quad (17) \]

\[ \Phi(z, t) = e^{ivz} |\phi(z-z_0, t)| + e^{-ivz} |\phi(z+z_0, t)|. \quad (18) \]
show that we need to increase the time evolution by three or four times to get the final solitons very similar to the initial ones. In figures 5(a) and (b) we illustrate the collision between two bright solitons in a dipolar condensate of 2000 atoms of $^{168}$Er ($a_d = 66.6a_0$) with scattering length $a = 50a_0$. The collision is insensitive to the initial phase difference $\delta = 0$ or $\delta = \pi$ when the velocity is $v = 3$ mm s$^{-1}$. The solitons come towards each other and interact at $z = 0$. Then these are separated and continue practically unchanged. The solitons are placed at $z = \pm 38.4$ $\mu$m at $t = 0$ and each of these is advanced with a constant velocity $v = 3$ mm s$^{-1}$ towards centre $z = 0$. The real-time simulation of quasi-1D model (5) is terminated when the solitons reach approximately $z = \pm 38.4$ $\mu$m at time about $t = 24$ ms. The final solitons are not equal to the initial ones. So the interaction is quasi-elastic and the final result is two quasi-solitons.

Finally, we study the effect of the phase difference $\delta = 0$ and $\delta = \pi$ for the collision of two bright solitons in a dipolar condensate of 2000 atoms of $^{166}$Dy ($a_d = 132.7a_0$), with scattering length $a = 120a_0$ and velocity $v = 0.2$ mm s$^{-1}$. As we show in figures 6(a)–(d), the collision is sensitive to the initial phase difference. In an interval of time 400 ms with initial positions $z = \pm 37.5$ $\mu$m and $\delta = 0$ the solitons come towards each other, interact and the dipolar attraction along with the attractive phase difference allow the solitons together, thus forming a bound soliton molecule around $z = 0$, figures 6(a) and (b). This happens in a similar way as when the velocity is zero in figures 4(a) and (b). For a phase difference $\delta = \pi$ in an interval of time 400 ms the two solitons come towards each other and interact but (unlike when the velocity is zero in figures 4(c) and (d)) do not coalesce because of the repulsive effect of the phase difference, figures 6(c) and (d). So these repel and stay away from each other. The solitons were placed at initial positions $z = \pm 40$ $\mu$m and these reach the final state at $z = \pm 44.5$ $\mu$m. At 400 ms the long-range dipolar interaction still maintains the interaction between the solitons and their shapes are not the same as the initial ones. To recover the initial solitons we need to increase the system’s time evolution.

4. Summary and discussion

To find bright solitons numerically in a 3D dipolar condensate using the mean-field non-local Gross–Pitaevskii equation is a complicated issue because of the anisotropic long-range character of the dipolar interaction. Nevertheless, the reduced quasi-1D equation provides an alternative to the complete 3D equation. We show the existence of bright solitons in a quasi-1D reduced model of a dipolar BEC as a result of balance between the repulsive short-range contact interaction and the anisotropic long-range dipolar attraction. We show this with plots of the rms size and the chemical potential for three different condensates of $^{52}$Cr, $^{168}$Er and $^{166}$Dy atoms. Our findings show that bright solitons exist for any value of the scattering length less than the dipolar strength. There is no collapse in this reduced model in contrast to the 3D full equation. The Gaussian variational approximation in the quasi-1D model is relatively simple and it provides results for the stationary cigar-shaped DBEC in good agreement with the numerical solution of the GPE. The collision between two bright solitons at small velocities shows that, for a phase difference $\delta = 0$, we have a bound soliton molecule due to the dipolar attraction. When the phase difference is $\delta = \pi$ the solitons repel and move away. However, the long-range character of the dipolar contribution still maintains the interaction between the solitons. At high velocities the collision is quasi-elastic and it is independent of the initial phase difference $\delta$.

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