Quadratic Programming for Continuous Control of Safety-Critical Multiagent Systems Under Uncertainty

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Abstract—This article studies the control problem for safety-critical multiagent systems based on quadratic programming (QP). Each controlled agent is modeled as a cascade connection of an integrator and an uncertain nonlinear actuation system. In particular, the integrator represents the position–velocity relation, and the actuation system describes the dynamic response of the actual velocity to the velocity reference signal. The notion of input-to-output stability is employed to characterize the essential velocity-tracking capability of the actuation system. The standard QP algorithms for collision avoidance may be infeasible due to uncertain actuator dynamics. Even if feasible, the solutions may be non-Lipschitz because of possible violation of the full rank condition of the active constraints. Also, the interaction between the controlled integrator and the uncertain actuator dynamics may lead to significant robustness issues. Based on the current development of nonlinear control theory and numerical optimization methods, this article first contributes a new feasible-set reshaping technique and a refined QP algorithm for feasibility, robustness, and local Lipschitz continuity. Then, we present a nonlinear small-gain analysis to handle the inherent interaction for guaranteed safety of the closed-loop multiagent system. The proposed method is illustrated by numerical simulation and a physical experiment.

Index Terms—Feasible-set reshaping, quadratic programming (QP), safety-critical systems, small-gain synthesis, uncertain actuator dynamics.

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I. INTRODUCTION

Attaining primary objectives while satisfying motion constraints is an essential yet challenging task for vehicles and robotic systems. This has been one of the most attractive topics in the interdisciplinary literature of controls and robotics in the past decades [1], [2], [3], [4], [5], [6], [7], [8].

Early utilized in constrained optimization [9], [10], barrier functions have been employed to characterize state constraints for nonlinear control systems. Control barrier functions have been developed to enable constrained control designs [11], [12], [13], [14], [15], [16], [17], [18], [19]. The substantial relationship between control barrier functions and control Lyapunov functions [20] opens the door to a systematic development of a multiobjective control theory [21]. In particular, the recent advancement of control barrier functions relaxes the requirement of invariance of every level set (see [22] for zeroing barrier functions) and allows time-varying control bounds and noise [23]. Still, it only assumes an increasing property of the barrier function when the system state is outside the desired safety set. The notions of robust barrier functions [21], [24] and input-to-state safety [25], [26] have been developed to handle perturbations. Moreover, for a control-affine nonlinear system, an appropriately defined control barrier function entails linear inequality constraints on admissible control inputs to keep the system state inside the desired set. See [24] for a discussion on the half-space robustness property of Sontag’s formula [27], and pointwise minimum norm formula [28]. This treatment allows computationally efficient integration of different control strategies to fulfill conflicting constraints.

Quadratic programming (QP) is a powerful tool for the real-time synthesis of controllers by incorporating different specifications simultaneously [29], [30], [31], [32]. For a system subject to motion constraints, a QP algorithm calculates the admissible control input that fulfills the constraints and is as close as possible to the set of control inputs for primary objectives [21]. The study of Lipschitz continuity of QP-based control laws is not only of theoretical interest for well-defined solutions of closed-loop systems but also beneficial to avoiding chattering and other unexpected transient behaviors in practice [21], [24], [33], [34]. The integration of barrier functions and QP algorithms have found various applications, including automotive safety [21], robotic locomotion and manipulation [34], [35].
multirobot systems [36], [37], [38]. See [39] for a recent survey on control barrier functions and QP-based controller synthesis.

The velocity obstacle approach [40], [41] calculates the set of feasible velocities for constrained motions, but the underlying assumption of piecewise continuous speed or acceleration possibly results in limited robustness to dynamic uncertainties. We also recognize the refined designs to address the discontinuity issue [42]. Interested readers may consult the recent paper [43] for a comparative study of control barrier functions and the popular artificial potential fields [44].

This article investigates the safety control problem for a class of mobile agents modeled as a cascade connection of an integrator and an uncertain actuation system. Such a system setup covers a broad class of practical control systems. If the dynamics of the actuation system are neglectable, then the proposed model is reduced to an integrator, which is known as an essential model for safety control. Some other systems, e.g., double-integrators and Euler–Lagrange systems, can also be transformed into our model by introducing appropriate virtual control laws. Interestingly, the identified model of a quadrotor is in the form of our model (see Section VI). Unsurprisingly, dynamic uncertainties challenge the robustness and computational feasibility of QP-based algorithms and may cause the collision avoidance performance to deteriorate (as illustrated in Fig. 1).

This article assumes an essential velocity-tracking capability for the actuation system, which is described by the notion of input-to-output stability (IOS) [45], [46]. A standard QP algorithm may not be feasible in the presence of uncertain actuator dynamics, and its solution may be non-Lipschitz even if it exists. Moreover, the uncertain actuation system leads to an unexpected feedback loop from the constrained position to the velocity-tracking error and possibly destroys the usual half-space robustness.

This article proposes a seamless integration of numerical optimization and nonlinear control to address the major technical difficulty caused by uncertain actuator dynamics. Our first contribution lies in a new feasible-set reshaping technique, which refines a standard QP algorithm for guaranteed feasibility, robustness, and local Lipschitz continuity. Based on the aforementioned treatment, the controlled multiagent system is transformed into an interconnected system composed of two subsystems, one corresponding to the nominal controlled system subject to the velocity-tracking error and the other caused by the uncertain actuator dynamics. We employ gains to represent the interconnections and propose a nonlinear small-gain analysis to guarantee safety.

To the best of our knowledge, some techniques in this article are reported for the first time. The feasible-set reshaping technique ensures (local) Lipschitz continuity of the solution of the refined QP algorithm and would be beneficial to other related problems with continuous motion constraints. The robustness analysis for collision avoidance subject to multiple safety constraints is still valuable when the controlled agents are free of uncertain actuator dynamics. The small-gain analysis takes advantage of the inherent interaction between the nominal system and the uncertain actuator dynamics, and would motivate a new integration methodology for kinematics and dynamics control loops.

The rest of this article is organized as follows. Section II introduces the system setup and gives the collision avoidance problem formulation. In Section III, we employ two examples to discuss the technical difficulty caused by the uncertain actuator dynamics. The main result of a refined QP algorithm with a reshaped feasible set is presented in Section IV. The proof of the main result, given in Section V, is based on several new properties of the refined QP algorithm and local small-gain analysis. In Section VI, we employ numerical simulation and a physical experiment based on quadrotors to illustrate the validity of the proposed method. Section VII concludes this article. Due to space limitations, some proofs are placed in the technical report [47].

Notations: The following notations are given to make the article self-contained.

We use $|x|$ to represent the Euclidean norm of $x \in \mathbb{R}^n$, and use $[A]$ to represent the induced 2-norm of $A \in \mathbb{R}^{m \times n}$. For a nonzero real vector $x$, we denote $\hat{x} = x/|x|$. For a real vector $x$, $[x]_i$ represents the $i$th element. For vectors $x, x' \in \mathbb{R}^n$, $x < x'$ represents that the corresponding elements $[x]_i$ and $[x']_i$ satisfy $[x]_i < [x']_i$. The notations $\leq$, $>$, and $\geq$ are defined in the same way for vectors. For a real vector $x$, $\min\{x\}$ and $\max\{x\}$ denote the smallest and the largest element of $x$, respectively. For a real matrix $A$, $[A]_{i,j}$, $[A]_{i,:}$, and $[A]_{:,j}$ represent the element at the $i$th row and the $j$th column, the $i$th row vector and the $j$th column vector of the matrix $A$, respectively. $\min\{A\}$ denotes a column vector containing the minimum value of each row, and $\max\{A\}$ denotes a column vector containing the maximum value of each row. We use $\otimes$ to represent the Kronecker product, and use $\circ$ to represent the Hadamard product. In particular, for $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{1 \times n}$, $C = A \circ B$ is defined by $[C]_{i,j} = [A]_{i,j} [B]_{1,j}$. For a real symmetric matrix $A \in \mathbb{R}^{n \times n}$, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ take the maximum eigenvalue and the minimum eigenvalue, respectively, and $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$ take the maximum singular value and the minimum singular value, respectively.

For a measurable and locally essentially bounded signal $u : \mathbb{R}_+ \to \mathbb{R}^m$, $\|u\|_{[t_1,t_2]} = \text{ess sup}_{\tau \in [t_1,t_2]} |u(\tau)|$, and $\|u\|_0 = \|u\|_{[0,t]}$. Let $f(t)$ be a real-valued function defined over an open interval $(a,b)$. The upper right Dini derivative of $f(t)$ at $t_0 \in (a,b)$ is defined as $D^+ f(t_0) = \limsup_{t \to t_0^+} (f(t) - f(t_0))/(t - t_0)$.

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $K$ if it is continuous and strictly increasing, and $\alpha(0) = 0$; it is said to be of class $K_{\infty}$ if it is of class $K$ and unbounded. A continuous
function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is said to be of class $\mathcal{KL}$ if, for each fixed $t \in \mathbb{R}^+$, $\beta(s, t)$ is a class $\mathcal{K}$ function and, for each fixed $s \in \mathbb{R}^+$, $\beta(s, \cdot)$ is a decreasing function satisfying $\lim_{t \to \infty} \beta(s, t) = 0$. See [48, Example 4.16] for examples of such functions. A continuous function $\alpha : (-a, b) \to (-c, \infty)$ with constants $a, b, c > 0$ is said to be of class $\mathcal{K}^r$ if it is strictly increasing and $\alpha(0) = 0$. A continuously differentiable function $\mu : (-a, \infty) \to (0, b)$ with constants $a, b > 0$ is said to be of class $\mathcal{M}^r$, denoted by $\mu \in \mathcal{M}^r$, if it is strictly decreasing, strictly convex, $\lim_{s \to \infty} \mu(s) = 0$, and $\lim_{s \to \infty} \partial \mu(s)/\partial s = 0$. $\text{Id}$ denotes the identity function. $\text{sat}(r) := \max\{\min\{r, 1\}, 0\}$ is a saturation function defined for $r \in \mathbb{R}$.

II. PROBLEM FORMULATION AND PRELIMINARIES

This article aims to address the difficulty caused by multiple safety constraints in the control of safety-critical multiagent systems. For this purpose, we focus on the typical system setup, in which each mobile agent is modeled as a cascade connection of an integrator and an uncertain actuation system, and mainly consider position constraints. This section first introduces the class of systems to be studied in this article, and then gives the problem formulation of safety control.

Suppose there are $n_a$ agents and $n_o$ obstacles indexed by

$$\mathcal{N}_a = \{1, \ldots, n_a\}, \quad \mathcal{N}_o = \{n_a + 1, \ldots, n_a + n_o\}$$

respectively. Denote $\mathcal{N}_{ao} = \mathcal{N}_a \cup \mathcal{N}_o$, and $n_{ao} = n_a + n_o$. For each $i \in \mathcal{N}_{ao}$, we use $p_i \in \mathbb{R}^n$ to represent the position of the mobile agent or the obstacle, and use

$$\rho = [p_1, \ldots, p_{n_{ao}}]$$

to represent the positions of all the mobile agents and the obstacles.

For $i \in \mathcal{N}_a$, each agent $i$ is modeled as a cascade connection of a nominal system and an uncertain actuation system. Specifically, the nominal system is represented by

$$\dot{p}_i = v_i$$

(3)

where $v_i \in \mathbb{R}^n$ is the velocity. The velocity $v_i$ is generated by an actuation system, which is in the general form:

$$\dot{z}_i = f(z_i, v_i^*)$$

where $z_i \in \mathbb{R}^m$ is the state of the actuation system, $v_i^* \in \mathbb{R}^n$ represents the velocity reference signal, and $f$ and $g$ are locally Lipschitz functions.

In practice, it is essential that the actuation system (also possibly subject to constraints) is inherently stable and admits some velocity-tracking capability. In this article, we assume that for a constant reference signal $v_i^*$, the actual velocity $v_i$ asymptotically converges to $v_i^*$ and for a time-varying $v_i^*$, the tracking error $\tilde{v}_i$ depends on the changing rate of $v_i^*$.

Under the hypothesis that the velocity reference signal $v_i^*$ is locally Lipschitz on the timeline, and denote

$$v_i^{sd}(t) = D^+ v_i^*(t)$$

(9)

the following assumption is made on the uncertain actuation system.

Assumption 1 (Stability property and reference-tracking capability of the actuation system): Consider the actuation system (4). There exist locally Lipschitz functions $\alpha_{z1}, \alpha_{z2}, \gamma_{z_{sd}} \in \mathcal{K}$ and constants $c_0 \geq 1$ and $\lambda > 0$ such that for any $z_i(0)$ and any locally Lipschitz and bounded $v_i^*$,

$$|\tilde{v}_i(t)| \leq \beta_{e}(|z_i(0)|, t) + \gamma_{z_{sd}}^{\text{sd}} (\|v_i^{sd}\|_1)$$

(10)

$$|z_i(t)| \leq \alpha_{z1}(|z_i(0)|) + \alpha_{z2} (\|v_i^{sd}\|_1)$$

(11)

hold for all $t \geq 0$, where $\beta_{e}(s, t) = c_0 e^{-\lambda t}$ for $s, t \in \mathbb{R}^+$. $\text{Remark 1:}$ With the velocity-reference signal $v_i^*$’s Dini derivative as the input and the velocity-tracking error $\tilde{v}_i$ as the output, property (10) employs the notion of IOS to characterize the velocity-tracking capability: $\beta_{e}$ describes the transient performance, and $\tilde{v}_i$ ultimately converges to the range of $\gamma_{z_{sd}}^{\text{sd}} (\|v_i^{sd}\|_1)$. Given bounded velocity reference signal $v_i^*$, property (11) guarantees the boundedness of state $z_i$. See, e.g., [46], [49], and [50] for tutorials of the related notions.

Remark 2: In the presence of input saturation constraints, the IOS property may not be valid globally anymore, and some local IOS property would be practically meaningful. In this case, our
basic idea would still be applicable by taking into account the region where the IOS property is valid in the analysis.

Remark 3: Suppose the dynamics of the actuation system is neglectable, i.e., \( v_i^* = v_i \). In that case, the agent model is reduced to a single integrator, which has been widely studied in the literature of safety control [51]. The double-integrator model [36] can also be transformed into the form of (3)–(4) by introducing a virtual control law to the velocity loop. Interestingly, the experimental data of a quadrotor in Section VI also coincide with model (3)–(4) and satisfies Assumption 1.

Example 1 gives a condition for linear control systems to satisfy Assumption 1.

Example 1 (Reference tracking capability of linear control systems): Consider a linear control system

\[ \dot{z} = Az + Bu^*, \quad v = Cz \]

where \( z \in \mathbb{R}^n \) is the state, \( v \in \mathbb{R}^n \) is the output, \( u^* \in \mathbb{R}^n \) is the input, and \( A, B, \) and \( C \) are real matrices with appropriate dimensions. It is assumed that \( A \) is Hurwitz and satisfies

\[ CA^{-1}B = -I_{n \times n}. \]  

(13)

Define the tracking error \( \tilde{v} = v - v^* \). For any constant input \( v^* \), condition (13) implies that \( \lim_{t \to \infty} \tilde{v}(t) = 0 \).

Suppose that \( v^* \) is continuously differentiable, and denote \( v^{cd}(t) = \dot{\tilde{v}}(t) \). Define \( \zeta = Az + Bu^* \). Then, direct calculation yields

\[ \dot{\zeta} = A\zeta + Bu^{cd} = A\zeta + Bu^* \]

(14)

\[ \tilde{v} = Cz - v^* = CA^{-1}(\zeta - Bu^*) - v^* = CA^{-1}\zeta. \]

(15)

Because \( A \) is Hurwitz, the transformed linear control system (14)–(15) is stable. Property (10) can be proved by directly applying the definition of IOS-Lyapunov function [52]; see Section VI-A for details. \( \diamond \)

In practice, the velocity command signal \( v_i^c \) is usually generated by a higher level planner or a primary controller, which, without loss of generality, are assumed to guarantee the continuous differentiability of \( v_i^c \). We make the following assumption on the velocity command signal \( v_i^c \).

Assumption 2 (Boundedness of the velocity command and its derivative): For each \( i \in \mathcal{N}_o \), \( v_i^c \) is continuously differentiable with respect to time, and there exist positive constants \( \bar{v}^c \) and \( \bar{v}^{cd} \) such that

\[ |v_i^c(t)| \leq \bar{v}^c, \quad |\dot{v}_i^c(t)| \leq \bar{v}^{cd} \]

(16)

for all \( t \geq 0 \).

B. Characterization of Safety

For the multiagent system (3)–(4), define

\[ \hat{p}_{ij} = p_i - p_j \]

(17)

as the relative positions. Then, from (8), we have

\[ \dot{\hat{p}}_{ij} = v_i - v_j = v_i^c - v_j^c + \tilde{v}_i - \tilde{v}_j. \]

(18)

To characterize the safety of agent \( i \) with respect to another agent or an obstacle \( j \), we define

\[ V(\hat{p}_{ij}) = \mu((\hat{p}_{ij}) - D_s) \]

(19)

where \( \mu : (-D_s, \infty) \to \mathbb{R}_+ \) is an \( M \subset C \) function and \( D_s \geq 0 \) is the safety margin. If the velocity reference signals \( v_i^c \) and \( v_j^c \) satisfy

\[ -\hat{p}_{ij}^T(v_i^c - v_j^c) \leq -\alpha_V(V(\hat{p}_{ij}) - \mu(0)) \]

(20)

with \( \alpha_V \in \mathbb{K}^e \), then along the trajectories of (18), we have

\[ \nabla V(\hat{p}_{ij}) \leq \partial_\mu((\hat{p}_{ij}) - D_s) \left( \alpha_V(V(\hat{p}_{ij})) - \mu(0) \right) + \hat{p}_{ij}^T(\tilde{v}_i - \tilde{v}_j) \]

(21)

and thus

\[ V(\hat{p}_{ij}) \geq \alpha_V^{-1}((\tilde{v}_i + |\tilde{v}_j|) + \mu(0)) \Rightarrow \nabla V(\hat{p}_{ij}) \leq 0. \]

(22)

If the initial states satisfy \( V(\hat{p}_{ij}(0)) \leq \mu(D - D_s) \), and the velocity-tracking errors satisfy \( \|\tilde{v}_i\| + \|\tilde{v}_j\| \leq \alpha_V(\mu(D - D_s) - \mu(0)) \), then

\[ V(\hat{p}_{ij}(t)) \leq \mu(D - D_s) \]

(23)

holds for all \( t \geq 0 \), which means safety according to (6).

Remark 4: If the agents are free of uncertain actuator dynamics, then the safety control problem would be readily solvable by applying barrier-function-based QP designs [19] and using the idea of forward invariance [21, Th. 1]. However, for the class of multiagent systems with uncertain actuator dynamics, the velocity-tracking errors may violate the safety constraint; see the discussions in Section III.

III. LIMITATIONS OF STANDARD DESIGNS

This section employs examples to discuss the technical difficulty caused by the uncertain actuator dynamics. It is shown that a standard QP algorithm for collision avoidance may be infeasible due to uncertain actuator dynamics. Even if feasible, its solution may be non-Lipschitz when there are more than one safety constraints. A non-Lipschitz solution may cause unexpected transient response of the uncertain actuation system and destroy the safety of the controlled agent.

A. QP-Based Controller With an Extended Safety Margin

For convenience of notations, denote

\[ V_{ij} = V(\hat{p}_{ij}) \]

(24)

Given the velocity command \( v_i^c \) for the primary control objective, inspired by the QP-based control strategy [26], [36], one may choose the actual velocity reference signal \( v_i^* \) such that safety is ensured and at the same time \( v_i^* \) is as close to the velocity command \( v_i^c \) as possible:

\[ v_i^* = \arg\min_{v^* \in \mathcal{P}_i^c(p)} \frac{1}{2} v_i^{cT} v_i^* - v_i^T v_i^* \]

(25)

where

\[ \mathcal{P}_i^c(p) = \{ v_i^* \in \mathbb{R}^n : A_i(v_i^*) v_i^* + a_i^c(p) \leq 0 \} \]

(26)

is the feasible set with

\[ A_i^c(p) = [-\hat{p}_{i1}, \ldots, -\hat{p}_{i(i-1)}, -\hat{p}_{i(i+1)}, \ldots, -\hat{p}_{i n_o}]^T \]

(27)

\[ a_i^c(p) = [\alpha_V(V_{i1} - \mu_0), \ldots, \alpha_V(V_{i(i-1)} - \mu_0), \ldots, \alpha_V(V_{i(i+1)} - \mu_0), \ldots, \alpha_V(V_{i n_o} - \mu_0)]^T. \]

(28)
Infeasibility of the QP algorithm by extending the safety margin defined in \( \alpha \) is intended to compensate the effect of \( p = 1 \). To avoid singularity, the algorithm requires \( \tilde{p}_{ij} \neq 0 \) for \( i \in N_a \) and \( j \in N_{a_0} \setminus \{i\} \).

However, the velocity-tracking errors \( \tilde{v}_i \) may violate the safety constraint of the standard QP-based design [21], [24]. An intuitive solution is to extend the safety margin [53], which, however, still may not guarantee the feasibility of the QP algorithm if the total number of the constraints is larger than one (see Example 2).

**Example 2 (Possible infeasibility of a standard QP algorithm by extending safety margin to handle velocity-tracking error):** Consider the scenario involving one mobile agent with position \( p_1 \) and two obstacles with positions \( p_2 \) and \( p_3 \) (see Fig. 3). In this example, we consider \( p_2 = [0.5; 0.5], p_3 = [-0.5; 0.5], \) and \( v_i = [1; -1] \). If the velocity-tracking error is negligible, then one may consider

\[
v_i = \text{argmin}_{v_i \in \mathbb{P}_i^*} \frac{1}{2} v_i^T \alpha^* T v_i^* - \frac{1}{2} v_i^T v_i^*
\]

where \( \mathbb{P}_i^* = \{v_i^* \in \mathbb{R}^n : A_i^* v_i^* + a_i^* \leq 0\} \) with \( A_i^* = [-\hat{p}_{12}, -\hat{p}_{13}]^T \) and \( a_i^* = [\hat{p}_{12}^2 - D_1^2; \hat{p}_{13}^2 - D_1^2]^T \).

When the velocity-tracking error is nonzero, one may consider extending the safety margin and redesigning the safety controller with new \( a_i^* = [\hat{p}_{12}^2 - D_1^2; \hat{p}_{13}^2 - D_1^2]^T \). But, such a treatment may result in an empty feasible set in specific cases. For example, in the case of \( D = 0.6 \) and \( D_s = 1 \), when \( p_1 = [0; 0] \), the feasible set \( \mathbb{P}_i^* \) in (26) should satisfy \( \sqrt{2} \leq (\sqrt{2} - 1)[1; 1] \), and thus is empty.

**B. Introducing a Relaxation Parameter**

An alternative solution is to introduce a relaxation parameter (RP) [21], [24], [36] to the QP algorithm as

\[
\begin{align*}
\begin{bmatrix}
v_i^T \\
\delta_i
\end{bmatrix} &= \text{argmin}_{\delta_i \in \mathbb{P}_i^*(p)} \frac{1}{2} \begin{bmatrix}
v_i^T \\
\delta_i
\end{bmatrix}^T \begin{bmatrix}
\alpha_i^* \\
\delta_i
\end{bmatrix} - \begin{bmatrix}
v_i^T \\
\delta_i
\end{bmatrix}^T \begin{bmatrix}
v_i^* \\
\delta_i
\end{bmatrix} 
\end{align*}
\]

where \( \tilde{p}_i \) is the RP, \( \delta_i \) is intended to compensate the effect of \( \delta_i \) to \( \mathbb{P}_i^*(p) = \{v_i^*; \delta_i : A_i^*(p)v_i^* + a_i^*(p)\delta_i \leq 0, \delta_i \in [0, \delta]\} \) is the feasible set with \( A_i^*(p) = A_i^*(p) \) defined in (27), and without loss of generality,

\[
a_i^*(p) = \frac{1}{\delta} a_i^*(p)
\]

where \( a_i^*(p) \) is defined in (28).

Clearly, if \( \tilde{p}_{ij} \neq 0 \) for \( i \in N_a \) and \( j \in N_{a_0} \setminus \{i\} \), then zero is always an element of \( \mathbb{P}_i^*(p) \), and thus, the QP algorithm is always feasible. However, such a modification does not guarantee the Lipschitz continuity of the solution (see Example 3).

**Example 3 (Lipschitz continuity of the solution to a standard QP algorithm not guaranteed by simply adding an RP):** Still consider the case in Example 2. We use \( (\check{A}_i^*, \check{a}_i^*) \) to represent the nonredundant active constraints of the QP algorithm defined by (30), with \( \check{A}_i^* \) and \( \check{a}_i^* \) being submatrices of \( A_i^* \) and \( a_i^* \), respectively. From [9, Example 2.1.5], the solution to the QP algorithm is

\[
v_i^* = v_i^0 - \check{A}_i^* \left( \check{A}_i^* \check{A}_i^T + \check{a}_i^* \check{a}_i^T \right)^{-1} \check{a}_i^* v_i^0 \delta_i.
\]

Suppose that the nonredundant active constraints are not changed in some domain of \( p \). Then, in this domain, taking the partial derivative of \( v_i^* \) with respect to \( \check{a}_i^* \) yields

\[
\frac{\partial v_i^*}{\partial \check{a}_i^*} = \frac{\delta (2\lambda_3 \lambda_0 - \lambda_1) - 2\lambda_2 \lambda_3 \check{A}_i^T v_i^0 \delta_i}{\check{A}_i^T (2\lambda_3 \lambda_0 - \lambda_1) - 2\lambda_2 \lambda_3 \lambda_0^T \check{A}_i^T \lambda_0^{-1} \check{A}_i^T + (v_i^0 \check{A}_i^T \check{a}_i^*) \lambda_2 + (v_i^0 \check{A}_i^T \lambda_0^{-1} \check{a}_i^*) \lambda_1}
\]

where \( \lambda_0 = \check{A}_i^* \check{A}_i^T, \lambda_1 = (1 + \check{a}_i^* \lambda_0^{-1} \check{a}_i^*), \lambda_2 = \check{A}_i^T \lambda_0^{-1}, \) and \( \lambda_3 = \check{a}_i^* \lambda_0^{-1} \check{A}_i^* \). \( \check{A}_i^* \) consists of the relative positions between the agent and the obstacles. Thus, \( \sigma_{\text{min}}(\check{A}_i^*) \) cannot be guaranteed to be lower bounded by a positive constant. This means that the solution to the QP algorithm may not be Lipschitz. Indeed, [54, Th. 3.1] requires a positive lower bound of \( \sigma_{\text{min}}(\check{A}_i^*) \) to guarantee the Lipschitz continuity of the QP problem. In this numerical example, we consider \( D = 0.6, D_s = 0.68, v_i = [1; -1] \), \( p_1 \in [0; 0], p_2 = [-0.5; 0.5], p_3 = [0.5; 0.5], \delta = 100, \check{A}_i^*(p) = [-\hat{p}_{12}, -\hat{p}_{13}]^T, \) and \( a_i^*(p) = \delta^{-1} [\hat{p}_{12}^2 - D_1^2; \hat{p}_{13}^2 - D_1^2]^T \). Fig. 4 shows how \( p_1 \) influences \( v_i^* \). One may observe the sudden change of \( v_i^* \) when \( p_1 \) is close to zero.

Examples 2 and 3 show that the aforementioned methods do not guarantee the local Lipschitz property of the solution

A constraint of the QP problem is nonredundant if removing it changes the feasible set; it is active at a solution \( v_i^* \) to the QP problem, if its equality holds at \( v_i^* \) [10].
to the QP algorithm with respect to time. However, the local Lipschitz continuity of the solution is viewed to be essential for the robustness of a QP algorithm in practical systems. In the system setup in this article, the non-Lipschitz velocity reference signal may lead to a large gain from the barrier function to the velocity-tracking error and result in the unexpected dynamic response of the actuation system, which may violate the safety of the nominal system. This is also verified by the numerical example in Section VI-B.

IV. NEW QP-BASED DESIGN WITH RELAXATION PARAMETER AND RESHAPED FEASIBLE REGION

In this section, we present our solution to the safety control problem involving multiple controlled mobile agents and multiple stationary obstacles. Our major contribution lies in a new class of QP-based controllers with relaxation parameters and reshaped feasible sets (RPRF) to address the non-Lipschitz issue discussed in Section III. To be specific, positive bases are used to reshape the feasible sets for ensured feasibility and Lipschitz continuity.

The proposed safety controller is in the form of

$$\begin{align*}
\begin{bmatrix}
v^*_s \\
v^*_i \end{bmatrix} = \arg\min_{\begin{bmatrix}
v_i^* \\
\delta_i \end{bmatrix} \in \mathcal{P}_a(p)} \frac{1}{2} \begin{bmatrix}
v^*_i \\
\delta_i \end{bmatrix}^T A_i^s v_i^* + a_i^s(p) \delta_i, \\
\delta_i \end{bmatrix} \leq 0, \quad \delta_i \in [0, \delta],
\end{align*}
$$

(35)

where

$$\begin{align*}
\mathcal{P}_a(p) &= \left\{ \begin{bmatrix}
v_i^* \\
\delta_i \end{bmatrix} : A_i^s v_i^* + a_i^s(p) \delta_i \leq 0, \quad \delta_i \in [0, \delta] \right\}
\end{align*}
$$

(36)

is the reshaped feasible set. Here, $A_i^s \in \mathbb{R}^{n \times n}$ is a constant matrix with $n_p > n$, and satisfies that each row is a unit vector, any $n$ rows are linearly independent, and for any unit vector $\hat{u} \in \mathbb{R}^n$, $\min_{q \in \mathcal{Q}(\hat{u})} \max_{j} \left| A_i^s \right|_{jj} q > 0$

(37)

$$\text{card}(\mathcal{J}(\hat{u})) \geq n
$$

(38)

where $\mathcal{Q}(\hat{u}) = \{ q \in \mathbb{R}^n : q^T \hat{u} > 0 \}$, $\mathcal{J}(\hat{u}) = \{ j = 1, \ldots, n_p : \left[ A_i^s \right]_{jj} \hat{u} \geq c_A \}$ with $c_A$ being a positive constant less than 1, and card takes the cardinality.

We choose $a_i^s : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n_p}$ as

$$a_i^s(p) = \max_{\varphi} \left\{ \varphi \left( A_i^s(p), a_i^s(p), \frac{c_P}{\delta} \right) \right\}
$$

(39)

with

$$\varphi \left( A_i^s, a_i^s, \frac{c_P}{\delta} \right) = A_i^s A_i^{T \delta} \circ a_i^{s T} \circ \left( c_K \left( c_A - A_i^s A_i^{T \delta} \right) \right) \circ \left( A_i^s A_i^{T \delta} \circ a_i^{s T} + \frac{c_P}{\delta} \right)
$$

(40)

where $c_A$ is associated with $A_i^s$ given in the definition of $\mathcal{J}(\hat{u})$, $c_K \geq 1/c_A$ and $c_P$ can be any positive constants, and $\delta$, $A_i^s$, and $a_i$ are defined in (30)–(31). The positive constant $c_A$ is an independent parameter, and the selection of $A_i^s$ and $c_K$ depends on $c_A$.

Condition (37) guarantees that any vector in $\mathbb{R}^n$ can be represented by a strictly positive combination of some row vectors of $A_i^s$ [55, Th. 3.3]. Based on [54, Th. 3.1], it can be proved that such an $A_i^s$ can be used to guarantee Lipschitz continuity of the solution to the QP algorithm. The existence of such an $A_i^s$ is proved in Section V-A as part of the proof of the main result in Theorem 1.

In Example 4, we examine the refined QP algorithm (35) in the same scenario as in Examples 2 and 3.

**Example 4** (Lipschitz continuity of the solution to the QP algorithm guaranteed by adding an (RPRF)): Continued from Example 3. We construct a $\mathcal{P}_a(p)$ in the form of (36) and examine the solution to the QP algorithm (35) with a reshaped feasible set. For any specific odd integer $n_p \geq 5$, a typical $A_i^s \in \mathbb{R}^{n_p \times 2}$ consists of the outward normal vectors to an odd-sided regular polygon, that is, each row of $A_i^s$ is $[\cos(2\pi j/n_p), \sin(2\pi j/n_p)]$ for $j = 1, \ldots, n_p$. Accordingly, $c_A = \cos(2\pi/n_p)$. In the numerical simulation, we choose $n_p = 5$, $c_K = 1$, $c_P = 5/2$, and $c_A = \cos(2\pi/5)$. Fig. 5 shows the original and the reshaped feasible sets when $p_1 = [-0.4; 0.4]$. Fig. 6 shows how $p_1$ influences $|v_i^*|$, which is in accordance with our expectation of Lipschitz continuity.

When the RP $\delta_i$ goes to zero, the robustness with respect to the velocity-tracking error $\hat{v}_i$ is weakened [see (86) in the proof of Proposition 4]. In this case, motivated by [36], we set $v_i^* = 0$. 
Hence, the implemented safety controller is defined as

\[\begin{align*}
\begin{bmatrix} v_i^r(t) \\ \delta_i(t) \end{bmatrix} &= \begin{cases} 
\arg\min_{\{v_i^r, \delta_i\} \in P_i^r(p)} & \frac{1}{2} \left( v_i^r \right)^T \delta_i \left( v_i^r \right) - \left( v_i^r \right)^T \delta_i \\
\text{if } \inf_{\tau \in [0, t]} \delta_i(\tau) \neq 0 \\
0, & \text{if } \inf_{\tau \in [0, t]} \delta_i(\tau) = 0
\end{cases}
\end{align*}\]  

(41)

where \( P_i^r \) is defined in (35).

For convenience of discussions later, we denote

\[ T_i = \min \{ t \in \mathbb{R}_+ : \delta_i(t) = 0 \} \]

(42)

as the braking time of mobile agent \( i \), and set \( T_i = 0 \) for \( i \in \mathcal{N}_o \). We use

\[ \mathcal{N}_s(t) = \{ i = 1, \ldots, n_o : \delta_i(t) = 0 \} \]

(43)

to represent the set of the breaking agents, and denote

\[ V_m = \max_{i \in \mathcal{N}_s \cup \mathcal{N}_o \setminus \{ i \}} V_{ij} \]

(44)

\[ V_R = \max_{i \in \mathcal{N}_s \cup \mathcal{N}_o \setminus \{ i \}} V_{ij} \]

(45)

\[ V_S = \max_{i \in \mathcal{N}_s \cup \mathcal{N}_o \setminus \{ i \}} V_{ij} \]

(46)

\[ \tilde{v}_m = \arg\max_{x_i \in \mathcal{N}_s \cup \mathcal{N}_o} |x| \]

(47)

\[ z_m = \arg\max_{x_i \in \mathcal{N}_s \cup \mathcal{N}_o} |x| \]

(48)

Our main result is given by Theorem 1.

**Theorem 1:** Under Assumptions 1 and 2, consider the multi-agent system (3)-(4) and the safety controller (35)-(41). There exist \( \mu \in M^-C, \alpha_C \in K^C \), and positive constants \( \delta, D, D_s, \tilde{v}, \tilde{v}_m, \tilde{z}_m, V_{m}, V_{m,0}, \tilde{v}_{m,0} \), and \( V_{m,0} \), such that for any initial state satisfying \( |z_m(0)| \leq \tilde{z}_{m,0} \) and \( V_m(0) \leq V_{m,0} \), property (6) holds for all \( t \geq 0 \).

The proof of Theorem 1 is given in Section V.

**Remark 5:** For the special case involving a single agent and a single obstacle as shown in Fig. 1, it can be directly verified that the standard QP algorithm (25) satisfies all the conditions given by [54, Th. 3.1], and thus, the solution is Lipschitz. In this case, adding RPs and reshaping the feasible set are unnecessary. The idea of considering the controlled multiagent system as an interconnected system in Section IV is still valid even if the obstacle is moving. A detailed discussion can be found in [47].

**V. PROPERTIES OF THE PROPOSED DESIGN AND PROOF OF THE MAIN RESULT**

This section proves Theorem 1 by observing new properties of the proposed refined QP-based controller. We first show the existence of \( A_i^r \) for feasible-set reshaping (see Section V-A) and prove that the reshaped feasible set is a subset of the feasible set with RPs (see Section V-B). Then, we use gains to describe the interconnections between the controlled nominal systems and the uncertain actuation systems (see Sections V-C and V-D) and present a small-gain analysis to guarantee the safety of the controlled multiagent system (see Section V-E).

### A. Existence of \( A_i^r \) for Feasible-Region Reshaping

**Proposition 1:** There exist an \( A_i^r \in \mathbb{R}_+^{n \times n} \) with \( n_p > n \), any \( n \) rows of which are linearly independent, such that for any unit vector \( \bar{u} \in \mathbb{R}^n \), condition (37)-(38) is satisfied.

Due to space limitation, the proof of Proposition 1 is given in the technical report [47].

### B. Reshaped Feasible Region Belonging to the Original Feasible Region

The following proposition shows that the reshaped feasible set \( P_i^r \) is a subset of the relaxed feasible set \( P_i^r \).

**Proposition 2:** Consider \( P_i^r(p) \) defined by (31) and \( P_i^r(p) \) defined by (36). We have \( P_i^r(p) \subseteq P_i^r(p) \).

**Proof:** With \( v_i^r \in \mathbb{R}_+^{n} \) and \( \delta_l \in [0, \delta] \), we define

\[ \mathcal{H}^r_{ik}(p) = \{ v_i^r : \delta_l : [A_i^r(p)]_{k, :} v_i^r + |a_i^r(p)| k \delta_l \leq 0 \} \]

(49)

\[ \mathcal{H}^r_{ik}(p) = \{ v_i^r : \delta_l : [A_i^r(p)]_{k, :} v_i^r + \varphi([A_i^r(p)]_{k, :}, [a_i^r(p)] k, c_P/\delta) \delta_l \leq 0 \} \]

(50)

for \( k = 1, \ldots, n_o - 1 \), where \( \varphi \) is defined in (40).

Obviously, \( \mathcal{H}^r_{ik}(p) \) corresponds to the \( k \)th constraint of the feasible set \( P_i^r(p) \), and thus

\[ P_i^r(p) = \bigcap_{k=1}^{n_o-1} \mathcal{H}^r_{ik}(p) \]

(51)

The definition of \( P_i^r \) in (36) implies that

\[ A_i^r v_i^r + a_i^r(p) \delta_l \leq 0 \]

(52)

holds for any \( [v_i^r ; \delta_l] \in P_i^r(p) \). We rewrite \( \varphi \) defined in (40) as

\[ \varphi(A_i^r(p), a_i^r(p), c_p/\delta) = \begin{bmatrix} \varphi^T([A_i^r(p)]_{1, :}, [a_i^r(p)]_1, c_p/\delta) \\ \vdots \\ \varphi^T([A_i^r(p)]_{n_o-1, :}, [a_i^r(p)]_{n_o-1}, c_p/\delta) \end{bmatrix} \]

(53)

which together with the definition of \( \delta_l \) in (39) implies that

\[ \varphi([A_i^r(p)]_{k, :} v_i^r + a_i^r(p) k, c_p/\delta) \leq \delta_l \]

(54)

for all \( k = 1, \ldots, n_o - 1 \). Then, using \( \delta_l \in [0, \delta] \) and combining (52) and (54), we have \( P_i^r(p) \subseteq \mathcal{H}^r_{ik}(p) \) and thus

\[ P_i^r(p) \subseteq \bigcap_{k=1}^{n_o-1} \mathcal{H}^r_{ik}(p) \]

(55)

Based on (51) and (55), we have

\[ \mathcal{H}^r_{ik}(p) \subseteq \mathcal{H}^r_{ik}(p) \]

(56)

Now we prove \( \mathcal{H}^r_{ik}(p) \subseteq \mathcal{H}^r_{ik}(p) \).

For \( k = 1, \ldots, n_o - 1 \), denote

\[ \mathcal{Y}([A_i^r(p)]_{k, :}) = \{ q : q \in \mathcal{J}([A_i^r(p)]_{k, :}) \} \]

(57)

Then, from the conditions (37) and (38) for the definition of \( A_i^r, [A_i^r(p)]_{k, :} \) is a positive combination of some \( n \) elements of \( \mathcal{Y}([A_i^r(p)]_{k, :}) \) [55, Th. 3.3]. We use these elements to form the rows of matrix \( C_{ik}^r \) and define

\[ \mathcal{H}^r_{ik}(p) = \{ [v_i^r : \delta_l] : C_{ik}^r v_i^r + C_{ik}^r[A_i^r(p)]_{k, :} a_i^r(p) \delta_l \leq 0 \} \]

(58)
Using the definitions of $\varphi$ in (40) and $\mathcal{H}_{ik}^*(p)$ in (50), we have

$$\mathcal{H}_{ik}^*(p) \subseteq R_{ik}^*(p).$$

(59)

Since any $n$ rows of $A_i^r$ are linearly independent, the properties of positive combinations guarantee $[A_i^r(p)]_{k,:} C_{ik}^{-1}_{ik} \geq 0$. Multiplying both sides of the constraint inequality of $R_{ik}^*(p)$ in (58) by $[A_i^r(p)]_{k,:} C_{ik}^{-1}_{ik}$ yields

$$[A_i^r(p)]_{k,:} C_{ik}^{-1}_{ik} \left( C_{ik} v_i^* + C_{ik}[A_i^r(p)]_{k,:}^T [a_i^r(p)]_{k,:} \delta_i \right) = [A_i^r(p)]_{k,:} v_i^* + [a_i^r(p)]_{k,:} \delta_i \leq 0.$$  (60)

Clearly, the last inequality is in accordance with the constraint inequality of $\mathcal{H}_{ik}^*(p)$ defined in (49), which means $R_{ik}^*(p) \subseteq \mathcal{H}_{ik}^*(p)$. This together with property (59) implies $R_{ik}^*(p) \subseteq \mathcal{H}_{ik}^*(p)$. Recall (56). This completes the proof of Proposition 2.

C. Robustness of the Nominal System Safety

The following proposition gives the range of the RP $\delta_i$, which is to be used for the robustness analysis later.

Proposition 3: For any $\tilde{p}_{ij} \neq 0$ with $i \in \mathcal{N}_a$ and $j \in \mathcal{N}_ao \setminus \{i\}$, the solution $[v_i^*: \delta_i]$ to the QP algorithm (35)–(40) has the following properties.

1) If $\delta_i = 0$, then $v_i^* = 0$.

2) If $\delta_i > 0$, then $a_i^r(p)$ is bounded.

3) Choosing $\delta$ large enough can guarantee that $\delta_i > 0 \Rightarrow \delta_i \geq \delta/2$.

Proof: The properties are proved one by one.

Property 1: For any specific $\delta_i \geq 0$ and any $\tilde{p}_{ij} \neq 0$ with $i \in \mathcal{N}_a$ and $j \in \mathcal{N}_ao \setminus \{i\}$, define

$$P_i^*(p) = \{v_i^*: A_i^r v_i^* \leq -a_i^r(p)\delta_i\}.$$  (61)

Then, the definition of $A_i^r$ (below (36)) guarantees that $P_i^*$ is a bounded, closed, convex polyhedron.\(^3\)

Suppose that there exists some $d \neq 0$ such that $A_i^r d \leq 0$.

Then, for any $v_i^* \in P_i^*(p)$ and any $\epsilon > 0$,

$$A_i^r(v_i^* + \epsilon d) = A_i^r v_i^* + A_i^r \epsilon d \leq -a_i^r(p)\delta_i$$

i.e., $(v_i^* + \epsilon d) \in P_i^*(p)$. This means that $P_i^*(p)$ is not bounded.

By contradiction, $v_i^*$ is the only solution of the constraint $A_i^r v_i^* \leq 0$. Property 1 is proved.

Property 2: Recall the definitions of $P_i^*$, $P_i^r$, and $\mathcal{H}_{ik}^*$ in (31), (36), and (49), respectively. From the proof of Proposition 2, we have

$$P_i^*(p) \subseteq P_i^r(p) \subseteq \mathcal{H}_{ik}^*(p).$$

(63)

For $k = 1, \ldots, n_{ao} - 1$, denote

$$\mathcal{V}([-A_i^r(p)]_{k,:}) = \{[A_i^r]_{q,:} : q \in \mathcal{J}([-A_i^r(p)]_{k,:})\}. $$

(64)

Then, from the properties of $A_i^r$ given by (37) and (38), $[-A_i^r(p)]_{k,:}$ is a positive combination of some $n$ elements of $\mathcal{V}([-A_i^r(p)]_{k,:})$. We define matrix $C_{ik}^{-1}$, with these $n$ elements as the rows, define vector $a_i^r_{ik}$ with the elements of $a_i^r(p)$ corresponding to these $n$ rows, and define

$$\hat{\mathcal{H}}_{ik}^*(p) = \{[v_i^*: \delta_i] : C_{ik}^{-1} v_i^* + a_i^r_{ik} \delta_i \leq 0\}. $$

(65)

Using the definition of $P_i^r$ in (36), we have

$$P_i^r(p) \subseteq \hat{\mathcal{H}}_{ik}^*(p).$$

(66)

Combining (63) and (66) implies

$$P_i^r(p) \subseteq \mathcal{H}_{ik}^*(p) \cap \mathcal{H}_{ik}^*(p).$$

(67)

Since any $n$ rows of $A_i^r$ are linearly independent, using the property of positive combinations, we have

$$-[-A_i^r(p)]_{k,:} C_{ik}^{-1} = [a_i^r(p)]_{k,:} \delta_i \leq 0.$$  (68)

Multiplying both sides of the constraint inequality of $\mathcal{H}_{ik}^*(p)$ in (65) by the nonnegative $k_s$ defined in (68) yields

$$-[-A_i^r(p)]_{k,:} C_{ik}^{-1} C_{ik} v_i^* + k_s a_i^r_{ik} \delta_i = -[A_i^r(p)]_{k,:} v_i^* + k_s a_i^r_{ik} \delta_i \leq 0.$$  (69)

Also, the definition of $\mathcal{H}_{ik}^*(p)$ in (50) implies that

$$[A_i^r(p)]_{k,:} v_i^* \leq -[a_i^r(p)]_{k,:} \delta_i.$$  (70)

Properties (69) and (70) together guarantee that any element $[v_i^*: \delta_i]$ of $\mathcal{H}_{ik}^*(p) \cap \mathcal{H}_{ik}^*(p)$ satisfies

$$k_s a_i^r_{ik} \delta_i \leq [a_i^r(p)]_{k,:} v_i^* \leq -[a_i^r(p)]_{k,:} \delta_i.$$  (71)

If $k_s a_i^r_{ik} > -[a_i^r(p)]_{k,:}$, then the only element of $\mathcal{H}_{ik}^*(p) \cap \mathcal{H}_{ik}^*(p)$ is 0, and thus, $P_i^r(p) = \{0\}$. This contradicts with $\delta_i > 0$. Thus, we have

$$[a_i^r(p)]_{k,:} \leq -k_s a_i^r_{ik}.$$  (72)

Based on the definition of $\varphi$ in (40), we rewrite

$$\varphi(A_i^r, a_i^r, cP/\delta) = -\frac{cP}{\delta} \text{sat} \left( cK \left( cA - \left[ A_i^r A_i^r]^T \right]_{j,k} \right) \right)$$

and then, we have

$$\varphi(A_i^r, a_i^r, cP/\delta) \geq \min \left\{ \left[ A_i^r A_i^r]^T \right]_{j,k} \left[ a_i^r(p) \right]_{k,:} \left[ -\frac{cP}{\delta} \right] \right\}$$

$$\geq \frac{1}{\delta} \min \{\alpha_V(-\mu(0)), -cP\}.$$  (73)

This, together with (72) implies

$$[a_i^r(p)]_{k,:} \leq -k_s a_i^r_{ik} \leq -k_s \alpha_V(-\mu(0)),$$

$$\leq \frac{1}{\delta} \sqrt{n} |k_s| \max \{\alpha_V(-\mu(0)), cP\}.$$  (75)

where $\alpha_V$ is the $n$-dimensional column vector of all ones. On the other hand, the definitions of $a_i^r(p)$ in (28) and $a_i^r(p)$ in (32) imply

$$[a_i^r(p)]_{k,:} \leq \frac{1}{\delta} \alpha_V(\bar{V}_{ik} - \mu(0)) \geq \frac{1}{\delta} \alpha_V(-\mu(0)).$$  (76)

Combining (75) and (76) yields

$$\frac{1}{\delta} \alpha_V(-\mu(0)) \leq [a_i^r(p)]_{k,:}$$

$$\leq \frac{1}{\delta} \sqrt{n} \max \{\alpha_V(-\mu(0)), cP\}.$$  (77)

The boundedness of $[a_i^r(p)]_{k,:}$ for all $k = 1, \ldots, n_{ao} - 1$ guarantees the boundedness of $a_i^r(p)$. This completes the proof of Property 2.

Property 3: If $[v_i^*: \delta] \in P_i^r(p)$, then Property 3 is obvious. Now, we consider the case of $[v_i^*: \delta] \notin P_i^r(p)$. We use $M_i = [A_i^r, a_i^r]$ to represent the nonredundant active constraints of the

---

\(^3\)A polyhedron is the intersection of a finite number of half-spaces and hyperplanes [10].
QP algorithm, with $\hat{A}_i^r$ and $\hat{a}_i^r$ being submatrices of $A_i^r$ and $a_i^r$, respectively. From [9, Example 2.1.5], the solution to the QP algorithm is

$$
\begin{bmatrix}
v_i^r \\
\delta_i
\end{bmatrix} = \left( I - M_i^T (M_iM_i^T)^{-1} M_i \right) \begin{bmatrix} v_i^r \\
\delta_i
\end{bmatrix}. 
$$

(78)

Then, using $M_i = [\hat{A}_i^r, \hat{a}_i^r]$, we have

$$
\delta_i = \frac{\delta - \hat{a}_i^r^T (\hat{A}_i^r \hat{A}_i^r + \hat{a}_i^r \hat{a}_i^r)^{-1} \begin{bmatrix} \hat{a}_i^r \delta + \hat{A}_i^r v_i^r \end{bmatrix}}{1 + \hat{a}_i^r^T (\hat{A}_i^r \hat{A}_i^r)^{-1} \hat{a}_i^r} 
$$

(79)

where Sherman–Morrison–Woodbury formula [56] is used for the second equality. Since $\hat{a}_i^r$ is a submatrix of $a_i^r$, according to Property 2 of Proposition 3, there exists a positive constant $\tau_a$ such that

$$|\hat{a}_i^r| \leq \frac{\tau_a}{\delta}. 
$$

(80)

Since any $n$ rows of $A_i^r$ are chosen to be linearly independent as required after (36), there exists a positive constant $c_M$ such that

$$
\sup_{S_\alpha \in \{2^{2^n}\} \cap \{0\}} \left\{ \frac{1}{\delta} \max \left\{ \frac{1}{(A_i^r)^T S_{\alpha}, \ldots, (A_i^r)^T S_{\alpha})^{-1} \right\} \right\} \leq c_M 
$$

(81)

where $S_\alpha = \{1, 2, \ldots, n_\alpha\}$, $2^{2^n}$ is the power set of $S_\alpha$, and $2^{2^n}$ denotes the set of subsets of $2^{2^n}$ with cardinality not larger than $n$. Combining (80)–(81), we have

$$
\delta_i \geq \frac{\delta - \hat{v}^c_{M_\alpha^r} |\hat{a}_i^r|}{1 + c_M^2 |\hat{a}_i^r|^2} \geq \frac{\delta - \hat{v}^c_{M_\alpha^r} \tau_a \delta^{-1}}{1 + c_M^2 \tau_a^2 \delta^{-2}} = \frac{\delta^2 - \hat{v}^c_{M_\alpha^r} \tau_a}{\delta^2 + c_M^2 \tau_a^2}. 
$$

(82)

If

$$
\delta^2 \geq 2\hat{v}^c_{M_\alpha^r} \tau_a + c_M^2 \tau_a^2 
$$

(83)

then (82) directly implies $\delta_i \geq \delta/2$. This completes the proof of Property 3.

Based on Proposition 3, the following proposition claims a robust safety property of the controlled nominal systems in the presence of velocity-tracking errors.

**Proposition 4:** Consider the mobile agent defined by (8), and the controller defined by (35)–(41). Given any class $\mathcal{K}_{\infty}$ function $\gamma^\hat{v}$ and any positive constant $c_\sigma$, choose

$$
\alpha_{\hat{V}}(s) = 4(1 + c_\sigma) (\gamma^\hat{v})^{-1}(s) 
$$

(84)

for $s \geq 0$. Then, $\alpha_{\hat{V}}$ is of class $\mathcal{K}^\sigma$, and the following properties hold.

1. For any $\hat{p}_{ij} \neq 0$ with $i \in N_a$ and $j \in N_{ao} \setminus \{i\}$, the QP algorithm (35)–(40) is feasible and has a unique solution.
2. There exist $\beta_V \in \mathcal{K}_L, \theta > 0$ and $d_V^c = (1 + \theta) \mu_0$ such that for any $V_{ij}(0) \in \mathbb{R}_+$ and any piecewise continuous and bounded $\hat{v}_i$, $\hat{v}_j$ satisfying $\max_{k=i,j} \|\hat{v}_k\|_L \leq (\gamma^\hat{v})^{-1}(\mu(-D_s) - d_V^c)$

$$
V_{ij}(t) \leq \beta_V (V_{ij}(0), t) + \gamma^\hat{v} \left( \max_{k=i,j} \|\hat{v}_k\|_L \right) + d_V^c 
$$

(85)

for all $0 \leq t < \max\{T_i, T_j\}$.

**Proof:** The properties are proved one by one.

**Property 1:** For any $\hat{p}_{ij} \neq 0$ with $i \in N_a$ and $j \in N_{ao}$, the feasible set $P^\hat{v}(p)$ defined in (36) forms a convex, closed set [10, Sec. 2.2.4]. Since zero is always an element of $P^\hat{v}(p)$, the QP algorithm is feasible. The uniqueness of the solution is guaranteed by the projection theorem [9, Proposition B.11].

**Property 2:** Recall $\rho_{ij} = p_i - p_j$ in (17) and $V_{ij} = V(\rho_{ij})$ in (24). Taking the derivative of $V_{ij}$ along the trajectories of (8), for any $[v_i^r; \delta_i] \in P_i^\hat{v}(p)$, if $V_{ij} \leq \mu(-D_s)$, we have

$$
\nabla V_{ij}\hat{p}_{ij} = \nabla V_{ij} \begin{bmatrix} v_i + \hat{v}_i - v_j - \hat{v}_j \end{bmatrix} = \alpha_{\hat{V}}(V_{ij}) \left( -\hat{p}_{ij}^T (v_i - v_j) - \hat{p}_{ij}^T (\hat{v}_i - \hat{v}_j) \right) 
$$

(86)

$$
\leq \alpha_{\hat{V}}(V_{ij}) \left( -\frac{\delta_i}{\delta} \alpha_{\hat{V}}(V_{ij} - \mu_0) - \hat{p}_{ij}^T (\hat{v}_i - \hat{v}_j) \right) 
$$

(87)

where $\hat{p}_{ij} = \hat{p}_{ij}^T / |\hat{p}_{ij}|$.

$$
\delta_{ij} = \begin{cases} \delta_i + \delta_j, & \text{for } j \in N_a \setminus \{i\} \\
\delta_i, & \text{for } j \in N_a. 
\end{cases} 
$$

(88)

For the inequality in (86), we used the constraint of the feasible set $P_i^\hat{v}(p)$ in (30) and the property that $P_i^\hat{v}(p) \subseteq P_i^\hat{v}(p)$ given by Proposition 2.

With $\alpha_{\hat{V}}$ defined by (84), for any $\hat{p}_{ij}$ satisfying $V(\hat{p}_{ij}) \geq \mu_0$, it holds that

$$
-\alpha_{\hat{V}}(V_{ij} - \mu_0) = -4(1 + c_\sigma) (\gamma^\hat{v})^{-1}(V_{ij} - \mu_0). 
$$

(89)

From Property 2 of Proposition 3, for all $0 \leq t < \max\{T_i, T_j\}$, it holds that

$$
\frac{\delta_{ij}}{\delta} \geq \frac{1}{2}. 
$$

(90)

Then, (86), (89), and (90) together imply that

$$
\nabla V_{ij}\hat{p}_{ij} \leq 2\alpha_{\hat{V}}(V_{ij}) \left( -1 + c_\sigma (\gamma^\hat{v})^{-1}(V_{ij} - \mu_0) + \max_{k=i,j} \|\hat{v}_k\|_L \right) 
$$

(91)

as long as $\mu_0 \leq V_{ij} \leq \mu(-D_s)$.

Denote $d_V^c = (1 + \theta) \mu_0$. In the case of

$$
\gamma^\hat{v} \left( \max_{k=i,j} \|\hat{v}_k\|_L \right) + d_V^c \leq V_{ij} \leq \mu(-D_s) 
$$

(92)

it can be easily verified that

$$
V_{ij} - \mu_0 \geq \gamma^\hat{v} \left( \max_{k=i,j} \|\hat{v}_k\|_L \right) \text{, } V_{ij} - \mu_0 \geq \frac{\theta}{1 + \theta} V_{ij}. 
$$

(93)

Substituting (93) into (91), we have that

$$
\nabla V_{ij}\hat{p}_{ij} \leq -2c_0\alpha_{\hat{V}}(V_{ij}) (\gamma^\hat{v})^{-1} \left( \frac{\theta}{1 + \theta} V_{ij} \right) 
$$

(94)

holds in the case of (92).

Based on the aforementioned discussion, if $\alpha_{\hat{V}} \in \mathcal{K}$, then there exists a $\beta_{\hat{V}} \in \mathcal{K}_L$ for property (85) [46].

Now, we show $\alpha_{\hat{V}} \in \mathcal{K}$. Using the properties of $\mathcal{M}^{-C}$ functions and properties of limits of composition functions [57, Sec. 8.16], we have $\lim_{s \to \infty} \mu(s) = 0$, $\lim_{s \to 0} \mu^{-1}(s) = \infty$, etc.
\[ \frac{\partial \mu}{\partial s} \bigg|_{s=0} = 0, \quad \text{and} \quad \lim_{s \to 0} \frac{\partial \mu^{-1}(s)}{\partial s} = 0. \]

This validates the right continuity of \( \alpha_\mu \) at the origin. Since \( \mu \) is strictly decreasing, for any \( s_2 > s_1 > 0 \), it holds that \( \mu^{-1}(s_1) > \mu^{-1}(s_2) > 0 \). Also recall that \( \mu \) is strictly convex. Using [10, eq. 3.3], we have

\[
\mu(\mu^{-1}(s_1)) > \mu(\mu^{-1}(s_2)) - \alpha_\mu(s_2)(\mu^{-1}(s_1) - \mu^{-1}(s_2)) \tag{95}
\]

\[
\mu(\mu^{-1}(s_2)) > \mu(\mu^{-1}(s_1)) - \alpha_\mu(s_1)(\mu^{-1}(s_2) - \mu^{-1}(s_1)) \tag{96}
\]

Then, it can be verified that

\[
\alpha_\mu(s_2) > \frac{s_2 - s_1}{\mu^{-1}(s_1) - \mu^{-1}(s_2)} > \alpha_\mu(s_1) \tag{97}
\]

and thus, \( \alpha_\mu \) is strictly increasing. This proves \( \alpha_\mu \in \mathcal{K} \). \( \blacksquare \)

**Remark 6**: Proposition 4 employs the gain \( \gamma_V^V \) to represent the influence of \( \max\{|\tilde{v}_i|, |\tilde{v}_j|\} \) on \( \bar{V}(\tilde{p}_{ij}) \), and accordingly, the robustness of \( \bar{V}(\tilde{p}_{ij}) \) with respect to \( \max\{|\tilde{v}_i|, |\tilde{v}_j|\} \). With \( \tilde{v}_t \) considered as the control input, Proposition 4 also shows how to choose \( \alpha_\nu \) for a desired gain \( \gamma_V^V \).

### D. Response of the Uncertain Actuation System

In this section, we study the dynamic response of the actuation system with the velocity reference signal generated by the refined QP-based controller.

**Proposition 5**: Under Assumptions 1 and 2, consider the multiagent system modeled by (30)-(34) and the controller defined by (35)-(41). Then, the following properties hold.

1. For \( t \in [0, T_i) \), whenever the closed-loop signals are defined, the solution to the QP algorithm (35)-(40) is Lipschitz with respect to \( \tilde{a}_t \) and \( \tilde{v}_t \).

2. There exists a class \( \mathcal{K} \) function \( \alpha_\nu \) such that

\[
|\tilde{v}_t(t)| \leq \alpha_\nu(V_R(t)) + \tilde{v}_t \tag{98}
\]

whenever \( V_R(t) \) is defined by (45).

3. There exist \( \delta_\nu \in \mathcal{KL}, \gamma_\nu^V \in \mathcal{K} \), and a constant \( d_\nu \in \mathbb{R}^+ \), such that for all \( z_i(0) \in \mathbb{R}^m \), whenever the closed-loop signals are well defined

\[
|\tilde{v}_i(t)| = \max \left\{ \beta_\nu(|z_i(0)|, t) + \gamma_\nu(V_R(t)) + d_\nu, \gamma_\nu^V(\tilde{v}_i(t)) \right\} \tag{99}
\]

\[
\text{for } t \in [0, T_i), \quad \text{and } \bar{v}_t = e^{-\tilde{a}_t(t-T_i)} z_i(T_i), \text{ for } t \in [T_i, \infty). \]

**Proof**: Before the proofs, we rewrite the QP algorithm defined by (35)-(40) as

\[ u = \arg\min_{u \in \{x \in \mathbb{R}^n+1 : M_F x \leq 0, |x|_{n+1} \geq 0\}} 0.5 u^T u + ru \tag{100} \]

where

\[ u = [v_t^i; \delta], \quad M_F = [A_t^i, \tilde{a}_t], \quad r = -[v_t^i; \delta]. \tag{101} \]

As for the proof of Property 3 of Proposition 3, we define \( M_t = [A_t^i, \tilde{a}_t] \) to represent the nonredundant active constraints of the QP algorithm, where \( A_t^i \) and \( \tilde{a}_t \) are submatrices of \( A_t \) and \( \tilde{a} \), respectively.

Now we prove properties of Proposition 5 one by one.

**Property 1**: For \( t \in [0, T_i) \), we prove the Lipschitz continuity of the solution to the QP algorithm by using [54, Th. 3.1] as follows.

1. As already proved for Property 1 of Proposition 4, the QP algorithm has a unique solution.

2. Since \( \tilde{a}_t \) is a submatrix of \( a_t \), according to Property 2 of Proposition 3, \( \tilde{a}_t \) is bounded; since \( A_t^i \) is composed of unit vectors, \( M_t \) is bounded.

3. We use \( \tilde{v} \) to represent the number of nonredundant active constraints of the QP algorithm. Using the definition of \( T_i \), we can easily verify that \( \tilde{v} \leq n \) for all \( t \in [0, T_i) \). Since any \( n \) rows of \( A_t^i \) are linearly independent and \( \tilde{v} \leq n, \tilde{A}_t^i \), and \( M_t \) are full row rank, which means that there exists a \( \alpha_\nu > 0 \) such that \( \gamma_\nu^V(\tilde{A}_t^i A_t^i)^{-1} \geq \alpha_\nu \). As a direct consequence \( |M_t^T \alpha| \geq \lambda_{\min}(M_t M_t^T) |\alpha| \geq \lambda_{\min}(\tilde{A}_t^i A_t^i) |\alpha| \geq \alpha_\nu |\alpha| \) for all \( \lambda \).

Then, with all the conditions required by [54, Th. 3.1] satisfied, we can prove the Lipschitz continuity of the solution with respect to \( M_F \) and \( r \). Since \( \alpha_t \) and \( A_t^i \) are constant, Property 1 can be proved based on the definitions of \( M_F \) and \( r \) in (101).

**Property 2**: When \( t \geq T_i \), controller (41) guarantees that \( v_t^i \equiv 0 \). Then, when \( t \in [0, T_i) \), we consider the cases of \( [v_t^i; \delta] \notin P_t \) and \( [v_t^i; \delta] \in P_t \) separately. In the case of \( [v_t^i; \delta] \notin P_t \), we have \( v_t^i = \tilde{v}_t \).

Now we consider the case of \( [v_t^i; \delta] \notin P_t \). Define

\[ q_i(v_t^i) = \arg\min_{u \in \{x \in \mathbb{R}^n+1 : M_F x \leq 0, |x|_{n+1} \geq 0\}} 0.5 u^T u - v_t^i u. \tag{102} \]

We use \( A_t^i \) and \( \tilde{a}_t \) to represent the nonredundant active constraints of (102). Clearly, \( A_t^i \) and \( \tilde{a}_t \) are submatrices of \( A_t \) and \( a_t \), respectively. Using the projection theorem [9, Proposition B.11], \( q_i(v_t^i) \) is continuous and nonexpansive, which means that

\[ |q_i(v_t^i) - q_i(0)| \leq |v_t^i|. \tag{103} \]

From (102), we also have \( v_t^i = q_i(v_t^i) \). Then, following [9, Example 2.1.5], we have

\[ |v_t^i| \leq |v_t^i| + |q_i(0)| = \tilde{v}_t^i + \left| A_t^i (A_t^i)^{-1} \tilde{a}_t \right| \tag{104} \]

Applying the Lagrange multiplier algorithm and Karush–Kuhn–Tucker optimality conditions [9], we have

\[ \tilde{a}_t \leq A_t^i \tilde{a}_t \tag{105} \]

Denote \( \tilde{a}_i = \max\{\tilde{a}_i^t\}_{t=1}^{\infty} \). It can be easily verified that

\[
(\tilde{a}_i - \tilde{a}_i^t)^T (A_t^i A_t^i)^{-1} (\tilde{a}_i - \tilde{a}_i^t) \geq 0 \tag{106}
\]

and thus,

\[ (A_t^i A_t^i)^{-1} \tilde{a}_i \leq (\tilde{a}_i - \tilde{a}_i^t)^T (A_t^i A_t^i)^{-1} \tilde{a}_i \tag{107} \]

where \( \tilde{a}_i = \max\{\tilde{a}_i^t\}_{t=1}^{\infty} \). This, together with (104) implies

\[ |v_t^i| \leq |v_t^i| + \tilde{v}_t + c_M \sqrt{n} \max\{\tilde{a}_i^t\} \tag{108} \]
\[
\begin{align*}
\leq & \bar{v}^c + c_M \sqrt{n} \max_{j \in \{1, \ldots, n_p : [A^*_V]_i, j \text{ is a row of } A^*_V \}} \max_{k \in \{1, \ldots, n_{aa} - 1 \}} \times [\varphi(A^*_V, a^*_\delta, c_p)]_{j,k} \\
& \leq \bar{v}^c + c_M \sqrt{n} \alpha \sqrt{(V_R - \mu_0)} \leq c_M \sqrt{n} \alpha \sqrt{(V_R) + \bar{v}^c} 
\end{align*}
\]

where we use (108) for the first inequality, use the definition of $c_M$ in (81) and Property 3 of Proposition 3 for the second inequality, and use the equivalent representation of $\varphi$ in (73) for the fourth inequality.

Thus, Property 2 is proved by defining $\alpha_V^*(s) = c_M \sqrt{n} \alpha \sqrt{(s)}$.

**Property 3:** When $t \geq T_i$, controller (41) gives $v_i^* = 0$. Then, using the definition of $\bar{v}_i$ in (7) and property (10), we have

\[
[\bar{v}_i(t)] \leq c_\epsilon e^{-\lambda(t - T_i)} |z_i(T_i)|
\]

for all $t \geq T_i$.

Now, we consider the case of $t \in [0, T_i)$. Property 1 already shows the Lipschitz continuity of (41) with respect to $a^*_\alpha$ and $v^*_\epsilon$. To prove Property 3, we first show the existence of two class $K$ functions $\alpha_V^*$ and $\alpha_\phi^*$, and a positive constant $c_V^*$ such that

\[
|v_i^{ed}(t)| \leq \hat{c}_w |\bar{v}_m(t)| + \alpha_V^*(V_R(t)) + c_V^*
\]

for all $t \in [0, T_i)$. Using the weak triangular inequality in [45], properties (10) and (111) together guarantee the first case in (99) with

\[
d_V^*(t) = (1 + k_\epsilon)\gamma^*_V \circ \rho_V^{ed} \circ (\rho_V^{ed})^{-1}
\]

\[
\gamma_V^*(s) = (1 + k_\epsilon)\gamma^*_V \circ \rho_V^{ed} \circ (\alpha_V^*(s) + c_V^*) - d_V^*
\]

\[
\beta_\phi^*(s, t) = (1 + k_\epsilon)\beta_\phi(s, t)
\]

\[
\hat{c}_w = (1 + k_\epsilon)\gamma^*_V \circ \rho_V^{ed} \circ (\rho_V^{ed} - \text{Id})^{-1} \circ \alpha_V^*
\]

where $\rho_V^{ed}$ is a function of class $K_\infty$ such that $(\rho_V^{ed} - \text{Id})$ is of class $K_\infty$.

In the case of $|v_i^*; \epsilon| \notin \mathcal{P}_\epsilon$, $v^*_\epsilon$ is the solution to the QP algorithm, and thus, $|v_i^{ed}| \leq \bar{v}^c$. In this case, property (111) is obvious.

Now we consider the case of $|v_i^*; \epsilon| \notin \mathcal{P}_\epsilon$. Based on [54, Th. 3.1], there exists a positive constant $c_F \geq 1$ such that

\[
|v_i^{ed}| \leq c_F \bar{v}^c + 2c_F^2 \sqrt{\bar{v}^c + \delta_\epsilon^2} \left|D_a^*(p(t))\right|
\]

If there exist two class $K$ functions $\alpha_V^*$ and $\alpha_\phi^*$ such that

\[
\left|D_a^*(p(t))\right| \leq \alpha_V^*(V_R) + \alpha_\phi^*(|\bar{v}_m|)
\]

then property (111) is established.

Now we prove that inequality (117) holds for all $|v_i^*; \epsilon| \notin \mathcal{P}_\epsilon$. Taking derivatives of $A^*_V$ and $a^*_\alpha$ with respect to $t$, we obtain

\[
\frac{\partial A^*_V}{\partial t} = \left[A^d_{i,1}, \ldots, A^d_{i,(t-1)}, A^d_{i(t+1)}, \ldots, A^d_{i,n_{aa}}\right]^T
\]

\[
\frac{\partial a^*_\alpha}{\partial t} = \left[a^d_{i,1}, \ldots, a^d_{i,(t-1)}, a^d_{i(t+1)}, \ldots, a^d_{i,n_{aa}}\right]^T
\]

where $A^d_{ij} = -(I - \tilde{\rho}_{ij} \tilde{x}_{ij}^T) (v_i - v_j) \tilde{p}_{ij}^{-1}$, and $a^d_{ij} = -\alpha_\phi^*(V_{ij}) \tilde{p}_{ij}^2 (v_i - v_j) (\partial \alpha_V^*(V_{ij} - \mu_0))/ (\partial (V_{ij} - \mu_0))$ with $j \in \mathcal{N}_{aa} \setminus \{i\}$. Using the definition of velocity-tracking error in (7), the maximal velocity of agents satisfies

\[
\max_{i \in \mathcal{N}_a} |v_i| = |\bar{v}_m| + \max_{i \in \mathcal{N}_a} |v_i^*_1| = |\bar{v}_m| + \max_{i \in \mathcal{N}_a \setminus \mathcal{N}_a} |v_i^*_1|
\]

\[
\leq |\bar{v}_m| + \alpha_V^*(V_R) + \bar{v}^c.
\]

The definition of $\alpha_\phi^*$ shows the lower bound of $\alpha_V^*(V_{ij} - \mu_0)$ and implies that there exist a class $K$ function $\alpha_\phi^*$ and a constant $c^*_\phi \in \mathbb{R}_+$ such that

\[
\left|\frac{\partial \alpha_V^*(V_{ij} - \mu_0)}{\partial (V_{ij} - \mu_0)}\right| \leq \alpha_\phi^*(V_R) + c^*_\phi.
\]

Then, (118), (119), and (121) imply

\[
\left|\frac{\partial A^*_V}{\partial t}\right| \leq 2\sqrt{n_{aa} - \max_{i \in \mathcal{N}_a} |v_i|}
\]

\[
\left|\frac{\partial a^*_\alpha}{\partial t}\right| \leq \frac{2\sqrt{n_{aa} - 1}}{\delta_\epsilon} \alpha_\phi^*(V_R) (\alpha_V^*(V_R) + c^*_\phi) \max_{i \in \mathcal{N}_a} |v_i|
\]

where $\alpha_\phi^*$ is a class $K$ function defined in (88), and

\[
\alpha_\phi^*(s) = \begin{cases} \frac{1}{\mu(s) + D_\phi}, & \text{for } s > 0 \\ 0, & \text{for } s = 0 \end{cases}
\]

Clearly, $\alpha_\phi^*(s)$ is a class $K$ function. Each row of $A^*_V$ is a unit vector, and $v_i^*_1$ is bounded when $|v_i^*; \epsilon| \notin \mathcal{P}_\epsilon$ and $t \in [0, T_i)$ [see (77)]. Using the definition of $\varphi$ in (40), we have that $\varphi$ is locally Lipschitz with respect to $A^*_V$ and $a^*_\alpha$, and thus, there exists a constant $c_\varphi > 0$ satisfying

\[
|D_a^*(p(t))| \leq c_\varphi \left(\left|\frac{\partial A^*_V}{\partial t}\right| + \left|\frac{\partial a^*_\alpha}{\partial t}\right|\right)
\]

for all $t \in [0, T_i)$. Substituting (120), (122), and (123) into the aforementioned inequality, we can prove that (117) holds with

\[
\alpha_\phi^*(s) = \left(\alpha_V^*(s) + \bar{v}^c + \frac{\mu(s)}{4k_\varphi}\right) \alpha_A^d(s)
\]

(126)

(127)

where $k_\varphi$ is a positive constant and $\alpha_A^d(s) = 2c_\varphi (n_{aa} - 1)^{1/2} (\alpha_\phi(s) + \delta_\epsilon^{2-1} \alpha_\phi(s)) (\alpha_\phi(s) + c^*_\phi)$. From (116), (117), and (127), we have that (111) holds for all $t \in [0, T_i)$. This together with (110) proves Property 3.

This ends the proof of Proposition 5.

- E. Small-Gain Analysis for Safety of the Multiagent System

Propositions 4 and 5 naturally result in two interconnected subsystems, each of which admits a gain property (see Fig. 7). We propose a small-gain analysis to guarantee the safety of the closed-loop system.
provided that \( \| V_m \|_t \leq \mu (D - D_s) \). With (134), this means that
\[
\| \tilde{v}_m \|_t \leq \beta V^C (|z_m(0)|, 0) + \gamma V^C (Id + \epsilon_1^{-1}) (d_1 + \Delta V) + d V^C.
\]

It can be verified that, \( \| V_m \|_t \) and \( \| \tilde{v}_m \|_t \) are monotone with respect to \( \mu(0), \tilde{v}_m^d \) and the upper bounds of the initial states and the external inputs. Thus, all the conditions in the proof can be satisfied by appropriately choosing these values, and the boundedness of \( V_m \) and \( \tilde{v}_m \) is proved. Then, directly applying properties (98) and (11), the maximal state \( z_m \) of the actuation systems is bounded. This also implies that \( T_{max} \geq \min_{k \in \mathbb{N}_k} T_k \).

Case (B): \( t \geq \min_{k \in \mathbb{N}_k} T_k \).

In this case, \( \mathcal{N}_S \) is not empty and the cardinality of \( \mathcal{N}_S \) is nondecreasing with respect to \( t \).

In accordance with the definitions of \( V_R \) and \( V_S \) in (45) and (46), we define
\[
\hat{v}_R = \arg \max_{x \in \mathcal{F}_i \cap \mathcal{N}_K \cap \mathcal{N}_S} |x|, \quad \hat{v}_S = \arg \max_{x \in \mathcal{F}_i \cap \mathcal{N}_K \cap \mathcal{N}_S} |x|
\]
to represent the maximal velocity-tracking error of the agents belonging to \( \mathcal{N}_S \) and \( \mathcal{N}_S \), respectively. The definitions of \( V_m, V_R, V_S, \tilde{v}_m, \hat{v}_R \), and \( \hat{v}_S \) mean that
\[
V_m = \max \{ V_R, V_S \}, \quad |\tilde{v}_m| = \max \{ |\hat{v}_R|, |\hat{v}_S| \}
\]
Thus, the boundedness of \( V_m \) and \( \tilde{v}_m \) is guaranteed if we can prove the boundedness of \( V_R, V_S, \hat{v}_R, \) and \( \hat{v}_S \).

Denote
\[
T_S = \max_{i \in \mathbb{N}_S} T_i.
\]

B1) Boundedness of \( \tilde{v}_S \) and \( V_S \). For any \( i \in \mathbb{N}_S \) and \( t \geq T_S \), we have \( v_i^d(t) \equiv 0 \), and the following property holds based on property (10):
\[
|\tilde{v}_i(t)| \leq c_6 |z_i(T_S)| e^{-\lambda(t-T_S)}
\]
which implies
\[
|\tilde{v}_S(t)| \leq c_6 |z_m(T_S)| e^{-\lambda(t-t_S)} \leq c_6 |z_m(T_S)|.
\]
This means the boundedness of \( \tilde{v}_S \).

For any \( i \in \mathbb{N}_S \) and any \( j \in \mathcal{N}_S \cup \mathcal{N}_o \setminus \{i\} \), using (3) and (19), we have
\[
|V_{ij}(t)| = \mu \left( \tilde{p}_{ij}(T_S) + \int_{T_S}^t (v_i(\tau) - v_j(\tau))d\tau \right) - D_s
\]
\[
\leq \mu \left( \mu^{-1}(V_{ij}(T_S)) - \int_{T_S}^t |\tilde{v}_i(\tau)| + |\tilde{v}_j(\tau)|d\tau \right)
\]
\[
\leq \mu \left( \mu^{-1}(V_{ij}(T_S)) - c_6 e^{-\lambda(t-S)} |z_i(T_S)| + |z_j(T_S)| \right)
\]
for all \( t \geq T_S \). We use (7) for the first inequality and use (10) for the last inequality. Thus,
\[
|V_S(t)| \leq \mu \left( \mu^{-1}(V_S(T_S)) - 2c_6 e^{-\lambda(t-T_S)} |z_m(T_S)| \right)
\]
for all \( t \geq T_S \). This proves the boundedness of \( V_S \).
B2) Boundedness of $\hat{v}_R$ and $V_R$. Properties (85) and (99) imply that
\[
|\hat{v}_R(t)| \leq \max \left\{ \beta_V^G (\|z_m(T_S)\|, t - T_S) + \gamma_V^G (\|V_R\|_{\mathcal{H}_2}), d_V^G \right\}
\]
(149)
\[
|V_R(t)| \leq \beta_V (V_R(T_S), t - T_S) + d_V^G + \gamma_V^G (\max \{\|\hat{v}_R\|_{\mathcal{H}_2}, \|\tilde{v}_s\|_{\mathcal{H}_2}\})
\]
(150)
for all $t \geq T_S$. From (146), we have
\[
|\tilde{v}_s|_{\mathcal{H}_2} \leq c_v|z_m(T_S)|.
\]
(151)
Combining (150) and (151), we have
\[
|V_R(t)| \leq \beta_V (V_R(T_S), t - T_S) + \gamma_V^G (\|\hat{v}_R\|_{\mathcal{H}_2}) + d_V^G
\]
\[
= \beta_V (V_R(T_S), t - T_S) + \gamma_V^G (\|\hat{v}_R\|_{\mathcal{H}_2}) + d_V^G
\]
(152)
for all $t \geq T_S$.

Properties (149) and (152) result in an interconnection between $\hat{v}_R$ and $V_R$, with $\alpha_V$ quite similar to the one between $\tilde{v}_s$ and $V_s$ shown in Fig. 7. The boundedness of $\hat{v}_R$ and $V_R$ can be proved following the small-gain analysis as for Case (A). The boundedness of $\hat{v}_R$, $\tilde{v}_s$, and $z_m$ can be proved based on the analysis in Cases (B1) and (B2). The boundedness of the closed-loop signals on $[0, T_{\max})$ means $T_{\max} = +\infty$.

F. Tuning the Safety Controllers

This section provides a brief step-by-step guideline of choosing controller parameters to satisfy the conditions required in the aforementioned proofs. See the technical report [47] for details.

1) Step 1—Choose the Functions Defining the Controller With Coefficients to Be Determined Later: For $\hat{v}_R$ and $\alpha_V$ defined by (19) and (84) and the Property 3 of Proposition (3), we choose $\mu(s) = 1/(s + D_s)$, $c_0 = 1/4$, $\gamma_V^G(s) = k_{SV}^G s$ with $k_{SV}^G > 0$, and $\delta$ large enough.

2) Step 2—Calculate the Gains of the Subsystems: Choosing $k_{SV}^G = 1/4$, $\rho_{SV}^G(s) = 2 s$, and $k_0 = 1$, we have the gains $\gamma_V^G$, $\beta_V^G$, and $\gamma_{SV}^G$.

3) Step 3—Fine-Tune the Controller Parameters to Satisfy the Small-Gain-Like Condition: Choose $\epsilon_1(s) = s$ and $\epsilon_2(s) = s$. The conditions (128) and (129) are satisfied by choosing $1/k_{SV}^G, 1/D_s, \hat{v}_s$ small enough. Then, safety is guaranteed by choosing $D_s$ large enough.

This completes the proof of Theorem 1.

VI. SIMULATION AND EXPERIMENT

In this section, we consider two safety control scenarios for quadrotors. Numerical simulation and a physical experiment are used to verify our approach.

The experimental system for model identification and algorithm verification is composed of quadrotors (Crazyflie 2.0 with original onboard attitude controller), an optical motion capture system (OptiTrack), and a laptop computer running the Robot Operating System (ROS). The motion capture system measures the real-time positions and velocities of the quadrotors. Data transmission from the laptop computer to the quadrotors is based on Crazyradio PA, and that from the motion capture system to the laptop computer is through a TCP/IP Ethernet.

A. Identified Model of a Quadrotor and Verification of Assumption 1

The quadrotor velocity is controlled by a PID controller designed by MATLAB Control System Designer and implemented by ROS. We perform system identification for the velocity-controlled quadrotor using MATLAB System Identification Toolbox based on indoor experiment data.

The identified model with $u^*$ as the input and the actual velocity $v$ as the output is in the form of (12) where
\[
A = \text{diag}\{A_{11}, A_{22}, A_{33}, A_{44}\}
\]
(153)
\[
B_T = \begin{bmatrix} B_{T1}^T & B_{T2}^T & 0 & 0 \\ 0 & 0 & B_{T3}^T & B_{T4}^T \\ \end{bmatrix}
\]
(154)
\[
C = \begin{bmatrix} C_{11} & 0 & C_{13} & 0 \\ 0 & C_{22} & 0 & C_{24} \end{bmatrix}
\]
(155)
with $A_{11} = [-1.58, -2.92, -2.92, -1.58]$, $A_{22} = [-2.68, 7.18, -2.68, -2.68]$, $A_{33} = [-2.56, 2.56, 0.65]$, $A_{44} = [-2.56, 6.86, -0.65, -2.56]$, $B_{T1}^T = [1.65, 0.65]$, $B_{T2}^T = [1.5, 0.92]$, $B_{T3}^T = [1.5, 0.84]$, $B_{T4}^T = [1.5, -2.29]$, $C_{11} = [0.78, -1.98]$, $C_{13} = [2.13, 2.41]$, $C_{22} = [-2.2, -2.82]$, and $C_{24} = [-1.51, -0.99]$. Choose $Q = A + A^T$ and $P = I_{8 \times 8}$. Then, $PA + A^TP = -Q$. Define $V(\zeta) = \zeta^T P \zeta$ with $\zeta = Az + Bv^*$. Taking the derivative of $V(\zeta)$ along the trajectories of the system (14)–(15), we have
\[
\nabla V(\zeta) = \zeta^T (A^T P + P A) \zeta + 2 \zeta^T P B v^d u^d
\]
\[
= -\xi |\zeta|^2 - \zeta^T (Q - \xi I) \zeta + 2 \zeta^T P B v^d u^d
\]
\[
\leq -\xi |\zeta|^2 - |\zeta| \left( \mu_{\min}(Q - \xi I) \sqrt{\lambda_{\max}^{-1}(P)} V - 2 |PB| |v^d| \right)
\]
(156)
where $\zeta$ is a constant satisfying $0 \leq \zeta \leq \mu_{\min}(Q)$. It is a direct consequence that
\[
V(\zeta(t)) \geq \left( \frac{2 \lambda_{\max}^{-1}(P) |PB| |v^d|}{\lambda_{\min}(Q - \xi I)} \right)^2 \nabla V(\zeta) \leq -\xi |\zeta|^2.
\]
(157)
Directly applying the definitions of input-to-state stability (ISS) and ISS-Lyapunov function [46], there exists a $\beta \in K\lambda$ such that
\[
V(\zeta(t)) \leq \max \left\{ \beta(|\zeta(0)|, t), \left( \frac{2 \lambda_{\max}^{-1}(P) |PB| |v^d|}{\lambda_{\min}(Q - \xi I)} \right)^2 \right\}
\]
(158)
for all $t \geq 0$.

Applying the norm inequality to (15), we have
\[
|\tilde{v}|^2 \leq \lambda_{\max} \left( C_{\zeta}^T C_{\zeta} \right) |\zeta|^2 \leq \lambda_{\max} \left( \frac{C_{\zeta}^T C_{\zeta}}{\lambda_{\min}(P)} \right) V(\zeta)
\]
(159)
with $C_{\zeta} = CA^{-1}$. 

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Properties (157) and (158) together imply
\[ |\hat{\hat{v}}(t)| \leq 2 \sqrt{\frac{\lambda_{\text{max}}(P)\lambda_{\text{max}}(C_r^T C_r)}{\lambda_{\text{min}}(P)\lambda_{\text{min}}(Q - \xi I)}} |\tilde{v}^\text{ref}| \]
\[ + \sqrt{\frac{\lambda_{\text{max}}(C_r^T C_r)}{\lambda_{\text{min}}(P)}} \beta(|\zeta(0)|, t). \]
(160)

Clearly, (160) is in the form of (10). Property (11) can also be easily validated based on the aforementioned Lyapunov formulation. Assumption 1 is verified.

B. Numerical Simulation: Collision Avoidance in the Case of One Mobile Agent and Two Obstacles

In this section, we consider the scenario in Examples 3 and 4. The identified quadrotor model given in Section VI-A is used for the numerical simulation.

The QP algorithm with RPRF defined by (35)–(41) is compared with the QP algorithm with RP defined by (30)–(32), to show the effectiveness of the proposed method.

We consider constant velocity command:
\[ v_c^i(t) \equiv [2; -2]. \]
(161)

For both of the algorithms, we choose \( D = 0.6, \ D_s = 0.65, \ \delta = 100 \)
\[ \mu(s) = \frac{1}{s + D_s}, \ V(\hat{\tilde{p}}) = |\tilde{p}|^{-1}, \ \alpha_V(s) = 0.75 s. \]
(162)

For the QP algorithm with RPRF, we also choose \( c_K = 1, \ c_P = 2\sqrt{2}, \) and \( [A_1^T]_{jj} = [\cos(2\pi j/11), \sin(2\pi j/11)] \) for \( j = 1, \ldots, n_p \) with \( n_p = 11 \). Accordingly, \( c_A = \cos(2\pi/11) \).

Figs. 8 and 9 show the trajectories of the controlled mobile agent and the velocity reference signals with the two algorithms. Due to the uncertain actuator dynamics, the velocity reference signal generated by the QP algorithm with RP leads to an unexpected response and causes collision. The QP algorithm with RPRF avoids collision.

C. Physical Experiment: Collision Avoidance in the Case of Three Quadrotors

To verify the proposed method in practice, we consider a scenario of three quadrotors swapping positions.

The primary controller is trajectory tracking. The reference trajectories of the agents are generated by sinusoidal functions:
\[ p_r^1(t) = [-\cos(2\pi t/15); -\cos(2\pi t/15); 1] \]
(164)
\[ p_r^2(t) = [\cos(2\pi t/15) - 0.23; \cos(2\pi t/15) + 0.23; 1] \]
(165)
\[ p_r^3(t) = [\cos(2\pi t/15) + 0.23; \cos(2\pi t/15) - 0.23; 1]. \]
(166)

The velocity commands are generated by feedforward-feedback controllers:
\[ v_c^i(t) = -1.7(p_i(t) - p_i^r(t)) + \hat{p}_i^r(t) \]
(167)
for \( i = 1, 2, 3. \)

For both of the algorithms, we choose \( D = 0.2, \ D_s = 0.3, \ \delta = 100 \)
\[ \mu(s) = (s + D_s)^{-1}, \ V(\hat{\tilde{p}}) = |\tilde{p}|^{-1}, \ \alpha_V(s) = 0.3 s. \]
(168)
For the QP algorithm with RPRF, we also choose $c_K = 1$, $c_P = 1$, and $[A_j]_{1:2} = [\cos(2\pi/j/n_p), \sin(2\pi/j/n_p)]$ for $j = 1, \ldots, n_p$ with $n_p = 11$. Accordingly, $c_A = \cos(2\pi/11)$.

Figs. 10 and 11 show the trajectories of the controlled mobile agents and the norms of the velocity reference signals with the two algorithms. Although collision avoidance is achieved in both of the cases, one may recognize a sudden change of agent I’s velocity reference signal at $t = 4.1$ when the QP-based algorithm without feasible-set reshaping is applied.

VII. CONCLUSION

In this article, we developed a systematic solution to the control of safety-critical multiagent systems subject to uncertain actuator dynamics. The major contribution lies in a seamless integration of a new QP-based design with reshaped feasible set and a nonlinear small-gain analysis. In particular, the new feasible-set reshaping technique has been proved to be quite useful for feasibility of the refined QP algorithm and Lipschitz continuity of its solution. The nonlinear small-gain analysis takes advantage of the interconnection between the controlled nominal system and the uncertain actuation system for ensured safety.

We expect that the proposed new techniques will be beneficial for solving fundamentally challenging control problems for more general systems, e.g., higher order systems, control-affine systems, and nonholonomic systems, and control systems subject to information constraints, e.g., partial-state feedback and sampled-data feedback. It is also of theoretical and practical interest to study distributed and coordinated implementations of the algorithms.

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