Why Auxiliary Fields Matter: The Strange Case of the 4D, N = 1 Supersymmetric QCD Effective Action (II)

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ABSTRACT

Within a four dimensional N = 1 superspace, we present a new ansatz for the Skyrme term and for the gauged WZNW term embedded into a superaction. We use the new chiral-nonminimal (CNM) formulation for the effective low-energy action of 4D, N = 1 supersymmetric QCD constructed by assigning right-handed components of Dirac fields to chiral multiplets and left-handed components of Dirac fields to nonminimal multiplets. It is noted that such a construction likely allows for a new type of parity violation in low-energy 4D, N = 1 supersymmetric QCD.

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1 Introduction

The theory of supersymmetric effective actions has been found to possess a number of interesting surprises. Over a decade ago, there first appeared in the literature the class of 4D, N = 2 supersymmetric “Kählerian Vector Multiplet” models. This is a non-linear $\sigma$-model constructed from the 4D, N = 2 vector multiplet and a specific member of this class of models (the 4D, N = 2 effective action) has recently been shown to possess information concerning the non-perturbative structure of the 4D, N = 2 supersymmetric Yang-Mills effective theory. This serendipitous circumstance has been the trigger of a huge amount of activity presently among theoretical physicists to study 4D, N = 2 supersymmetric effective actions (see and references therein).

About a decade ago, there was also a period in which the topic of 4D, N = 1 supersymmetric effective actions was more actively pursued. This prior epoch had as its trigger works on the structure of the QCD low-energy effective action. The importance of the higher derivative terms such as the “Skyrme” term had been elucidated for soliton stability. Another major advance occurred with the initial presentation of the “topological” approach to the WZNW term. The works of reference were attempts to describe supersymmetric extensions of various terms that appear in the low-energy QCD effective action, paying attention especially to terms involving spin-0 fields in the non-supersymmetric limit of the effective action. There has also been some discussion of the 4D, N = 1 supersymmetric effective actions (for spin-1 fields) that are related to the low-energy limit of open-string theory.

Yet from our prospective, throughout most of this period, there had (until quite recently) remained one major interesting puzzle that we call the “auxiliary freedom problem” for 4D, N = 1 supersymmetric effective actions. To see this problem it suffices to consider a 4D, N = 1 supersymmetric theory involving only scalar multiplets where, however, the spin-0 components appear in an action with higher than second order derivatives and the spin-1/2 components appear with higher than first order derivatives. This is the form of a generic term in an effective action. We believe it is a desideratum to use 4D, N = 1 superfields. Thus accompanying the spin-0 and spin-1/2 fields there must be a set of auxiliary fields. Since the physical fields’ equations of motion are higher than usual order in derivatives, it is natural to expect that derivative operators appear in the equations of motion for auxiliary fields. In this event, auxiliary fields become propagating!

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2The locally supersymmetric version of these theories made their first appearance in.
In fact, this problem of propagating auxiliary fields in higher derivative 4D, \( N = 1 \) manifestly supersymmetric actions has been one of “the dirty little secrets” bedeviling such theories but almost never discussed and sometimes not even recognized. For example, in their treatment of 4D, \( N = 1 \) supersymmetric “Skyrmions,” Bergshoeff, Nepomechie and Schnitzer (BNS work of [8]) noted that their ansatz for the form of the supersymmetric Skyrme term necessarily implies the presence of propagating \( F \)-fields. Similarly, although Nemeschansky and Rohm (NR work of [8]) did give an ansatz for the form of the 4D, \( N = 1 \) supersymmetric WZNW term, they did not make mention of the fact that their proposal also necessarily contains propagating \( F \)-fields.

We would be remiss, if we did not highlight the work of Karlhede, Lindstrom, Roček and Theodoridis [12] in this regard. To our knowledge, this has for a long time been the only work in the literature where the problem of propagating auxiliary fields in higher derivative 4D, \( N = 1 \) supersymmetric actions has been forthrightly addressed.

While for the effective action of 4D, \( N = 1 \) supersymmetric QCD we are not able to make a definitive argument that propagating auxiliary fields are “bad,” for compactified 4D, \( N = 1 \) superstrings and heterotic strings there is a potential for making such arguments for their low energy effective actions. The point in these theories is that the spectrum of the effective action is strictly controlled by string theory. Propagating auxiliary fields would have to correspond to higher mass (\( m > 0 \)) modes of the string. If the spectrum of the string cannot accommodate the states described by the propagating auxiliary fields, that is reason to rule them out.

The present generally accepted proposal for the superfield description of the 4D, \( N = 1 \) supersymmetric low-energy QCD effective action has numbers of other problems even if one is willing to put aside the question of propagating auxiliary fields. The previous work on 4D, \( N = 1 \) Skyrmions [8] ends in part by concluding “...both terms contain quartic time derivatives and lead to actions that are not bounded from below.” Similarly, the authors find, “In particular, one expects that the \( \text{CP}^1 \) case (\( \gamma = 0 \)) should be special, corresponding to an infinitely thin rigid rod, having zero moment of inertia about the symmetry axis. This expectation is not borne out by our explicit collective coordinate calculation.” In the case of the previous work on the 4D, \( N = 1 \) supersymmetric WZNW term, we find the conclusion “...a Wess-Zumino term for this effective Lagrangian model of supersymmetric QCD exists and has the correct anomalies, although we do not know a way to construct it explicitly.” Furthermore, the sigma model manifold associated with this older formulation of the 4D, \( N = 1 \) supersymmetric WZNW term must necessarily possess a non-compact geometry.

To our thinking, all of these were sufficiently severe problems to raise the question of whether there is something fundamentally lacking in our understanding of higher
derivative 4D, N = 1 superfield theory. By the end of our first effort to study this class of theories \[13\] we stated, “In our opinion, more work is required to reconcile supersymmetry with higher derivative actions in four dimensions.”

Although our concerns along these lines have been constant and of long duration, we have never had concrete alternative suggestions to those in the literature and which might avoid all the problems described above...until recently. In one of our recent investigations \[14\], we have found that there is a mechanism that permits the existence of a previously unknown class of higher derivative 4D, N = 1 manifestly supersymmetric actions involving propagating spin-0 and spin-1/2 fields and no propagating auxiliary fields! This result is the (0,1/2) multiplet analog of the (1,1/2) multiplet result \[12\]. Our mechanism for achieving this result uses 4D chirality in an essential way. We find that given a theory with Dirac particles, we can assign the right-handed spinor components to chiral multiplets \[13\] and the left-handed spinor components to 4D, N = 1 nonminimal scalar multiplets \[16, 17\]. We call these chiral-nonminimal (CNM) models. A CNM model, in and of itself, is not sufficient to suppress the propagation of auxiliary fields. To complete their suppression, we impose strong\footnote{We define weak holomorphy as the condition that all non-polynomial functions which determine actions are holomorphic or chiral functions even though the actions are not supersymmetrically chiral. The 4D, N = 1 nonlinear \(\sigma\)-model involving chiral superfields and special Kähler geometry is an example of weak holomorphy.} holomorphy \[18\], the condition that the higher derivative action itself is holomorphic (i.e. supersymmetrically chiral).

It is the purpose of this present work to establish a new paradigm, the CNM \[14\] approach, to the description of 4D, N = 1 supersymmetric effective actions for dynamical superfield multiplets of spins (0,1/2). We do not present derivations of our proposed CNM formulation from underlying fundamental theories. Instead our emphasis is on the reconciliation that our CNM approach provides between the 4D, N = 1 superfield formalism and the well-known characteristic structures of low-energy QCD. In a sense our discussion may be thought of as the introduction of a new phenomenological model of “super-pion” dynamics. The dynamical bosonic fields include the usual SU(3) pion octet, a second psuedo-scalar SU(3) octet (denoted by \(\Theta\)) isomorphic to the pion octet and a complex scalar SU(3) octet (denoted by \(G_+\)). The dynamical fermions consist of an SU(3) octet of Dirac particles (denoted by \(\ell\)). Supersymmetry requires this spectrum in order to have equal numbers of bosons and fermions. At this stage we defer to the future the issues of supersymmetry breaking, masses, etc. Instead in this work we will concentrate on sorting out various conceptual issues involved in our approach.
In chapter two we introduce the CNM description of scalar multiplet systems. Chiral and nonminmal scalar multiplets are briefly reviewed. The role of holomorphy is defined as essentially restricting the non-polynomial dependence of the effective action solely to chiral superfields. The leading $\sigma$-model terms of the effective action are described. Next the suggestion of [14] is reviewed in order to include higher derivative terms such as the Skyrme and WZNW terms. A simple proof is described that shows that no propagating auxiliary fields arise from the inclusion of the higher derivative sector. The chapter ends with setting up further investigation to study the uniqueness of the form of the higher derivative terms.

In chapter three we review the standard (N = 0) Skyrme and WZNW terms and as well review the only works, known to us, in the literature where an extension to include 4D, N = 1 supersymmetry had been attempted. In particular, the auxiliary field sector of the work by Bergshoeff, Nepomechie and Schnitzer on 4D, N = 1 skyrmions and as well the work by Nemeschansky and Rohm on the 4D, N = 1 supersymmetric extension of the WZNW term are critically reviewed. It is shown (although no prior discussions along this line exist in the literature) that the auxiliary fields of the NR action are actually dynamical. We re-confirm that similar behavior exists for the Skyrme term.

In chapter four the conceptual issues in regards to Kähler geometry are confronted. It is proposed that the nonminimal multiplets, unlike chiral multiplets, are elements of the co-tangent bundle of the manifold described by the chiral sub-sector of the complete CNM model. For the first time it is noted that chiral superfield multiplets in supersymmetric non-linear $\sigma$-models possess an intrinsic definition of spacetime chirality defined by the association of the spinors in the supermultiplets with either the holomorphic or anti-holomorphic co-tangent bundles. It is also noted that the spinors of nonminimal multiplets can be used in addition to those of chiral multiplets to realize holomorphic vector-like models with respect to the intrinsic definition.

In chapter five we discuss issues of irreducible super p-form geometry and the CNM-WZNW action. The discussion begins with a review of the super p-form gauge supermultiplets as an exercise to orient the reader regarding the structure of the simplex of 4D, N = 1 super irreducible p-forms. As another exercise, the 4D, N = 1 supersymmetric 4-form associated with the instanton number density is reviewed. Next a complete supergeometry is constructed for a super 4-form that contains the CNM-WZNW term. It is found that such a super 4-form exists consistent with the “p-form theorems” of the 4D, N = 1 supersymmetric p-form simplex. The CNM-WZNW action is shown to be the superspace integral of one component of the WZNW super 4-form similar to the fact that the superfield instanton index is one component of a
super 4-form.

In chapter six we discuss, in a very preliminary way, some aspects of the component fields described by the CNM approach. The particle spectrum of dynamical fields is explicitly defined from the corresponding superfields. Most amusingly, a mild analogy with the structure of the Glashow-Salam-Weinberg model is noted. This includes the introduction of a mixing angle denote by $\gamma_S$ (analogous to the weak mixing angle) that is restricted to satisfy the condition $\sin(2\gamma_S) \neq 0$.

In chapter seven we set out the issues and begin the task of constructing the gauged version of the 4D, $N = 1$ supersymmetric nonlinear $\sigma$-model, Skyrme and WZNW terms. The usual ($N = 0$) theory is written in such a way so as to facilitate an ansatz for its supersymmetric extension. A proposal is made for the form of the supersymmetric “pull-backs” required to write the complete action.

In chapter eight, we explore the implications of the CNM approach to effective actions for a new re-formulation of its underlying supersymmetric Yang-Mills gauge theory. It is shown that with the use of nonminimal multiplets, there are two distinct definitions of Dirac fields within the context of supersymmetric theories. For one version with gauge group $G$ in the WZ gauge (that used exclusively in the literature), the gauge group outside of the WZ gauge is found to be $G_V \otimes G_A$. For the CNM version with gauge group $G$ in the WZ gauge, the gauge group is found to be $G_c$, the complexified extension.

In our conclusions, we discuss the possible significance of our new results for a disagreement on the structure of the low-energy effective action of heterotic string theory. We also note that a CNM model for the effective action seems to characteristically predict the breaking of parity! Thus, the model is one where P-violation occurs even for the strong interactions. Two appendices containing presentations of related results are included.

2 4D Chirality as the Solution to the Auxiliary Freedom Problem

The underlying fermions, the quarks, of the fundamental QCD theory are Dirac particles. Using the chirality projectors, we can always split a Dirac field into right-handed and left-handed components. On the other hand, the fundamental fermions contained in 4D, $N = 1$ superfields may be considered as either Majorana or Weyl particles. Thus, to embed any theory of Dirac particles into a supersymmetric theory,
one is faced with an initial choice of to what supersymmetric representation should the two 4D chiral components of the Dirac field be assigned. Any brief consultation of the huge body of the literature shows that the state-of-the-art in constructing phenomenologically relevant 4D, N = 1 supersymmetric field theories has been to assign both the right-handed components as well as the charge conjugates of the left-handed components to chiral multiplets [15]. If this was the only off-shell (0, 1/2) supersymmetric representation we would be forced to do this.

As was pointed out a long time ago [16, 17], this is not the case. There exist a number of other off-shell (0, 1/2) supersymmetric representations. For example, there is the linear multiplet [19] containing the axion and dilaton that occur in 4D, N = 1 superspace geometry [20] arising as a limit of heterotic string theory. However, among all the variant representations to the chiral multiplet there is one, the nonminimal multiplet, which is unique. The nonminimal multiplet (and its infinite family of daughters) is the only variant to the chiral multiplet which does not contain component gauge fields. This singles this 4D, N = 1 supersymmetric representation out as being eminently suited to play the role of a matter multiplet in phenomenologically interesting proposals.

We can compare the fields of the two different multiplets by looking at the following table.

| Phys. vs. Aux. | P | P | A | A | A |
|----------------|---|---|---|---|---|
| Eng. Dim.      | 1 | 3/2 | 3/2 | 2 | 5/2 |
| Chiral SF      | A | ψ_α | F |
| Nonminimal SF  | B | ζ_α | ρ_α | H, p_α | β_α |

Table I

Chiral versus nonminimal superfields are defined by the respective conditions,

\[ \overline{D}_α \Phi = 0 \quad , \quad \overline{D}^2 Σ = 0 \quad , \]

which in a real sense may be called the “Bianchi Identities” for each multiplet. The simplest actions for describing the dynamics of such superfields are,

\[ S_{WZ} = \int d^4 x d^2θ d^2φ \overline{Φ} Φ \quad , \quad S_{NM} = - \int d^4 x d^2θ d^2φ \overline{Σ} Σ \quad , \]

where the explicit component forms for these (as well as the definition of the components) can be found in our earlier paper [14]. The equations of motion that follow from these actions imply that A and B satisfy massless Klein-Gordon equations, ψ_α.
and $\zeta_\alpha$ satisfy massless Dirac equations and that all remaining fields vanish. The superfield equations of motion that follow from these actions (2.2) as well as the defining conditions in (2.1) above reveal that there exists a type of duality between these two multiplets (see *Superspace* [21]). Let us here define “Poincaré dual pairs.” We will call two fields (or superfields) Poincaré dual pairs if when we exchange the Bianchi identity with the equation of motion for one member of the pair, we arrive at the other member. We’ll describe pairs of fields with this property as possessing “Poincaré duality.” It can be seen that this definition is nothing but a generalization of the electric-magnetic duality of the photon. So for example, the sum of the two actions in (2.2) can be said to possess “Poincaré duality invariance” since it would be meaningless to speak of electric-magnetic duality invariance in a theory with no dynamical spin-1 field. As well it is worth mentioning that the sum also realizes on-shell N = 2 supersymmetry.

|            | Constraint | Equation of Motion |
|------------|------------|--------------------|
| E. & M.    | $dF = 0$   | $d^*F = 0$         |
| Chiral SF  | $\overline{D}_\alpha \Phi = 0$ | $D^2 \Phi = 0$ |
| Nonminimal SF | $D^2 \Sigma = 0$ | $\overline{D}_\alpha \overline{\Sigma} = 0$ |

Table II

Thus, we finally see that our proposal to embed the two different chiral components of a Dirac spinor into chiral and nonminimal superfields is equivalent to imposing the condition that only Poincaré dual pairs $(\Phi, \Sigma)$ should be thought of as the 4D, N = 1 supersymmetric definition of a Dirac spinor.

Although we began our discussion considering the underlying Dirac fields of QCD, it is ultimately only the low-energy 4D, N = 1 supersymmetric QCD effective action in which we wish to implement our proposal of the use of dual pairs. As such, we wish to embed the pion octet into a supersymmetric formulation. Due to supersymmetry, these are also accompanied by their superpartners, the pionini octet, as well as other spin-0 fields. So we introduce Poincaré dual pairs $(\Phi^I, \Sigma^I)$ where $I = 1, \ldots, 8$. The most general non-linear $\sigma$-model term involving these superfields takes the form,

$$S_{\sigma} = \int d^4xd^2\theta d^2\overline{\theta} \hat{\Omega}(\Phi, \overline{\Phi}; \Sigma, \overline{\Sigma})$$

where $\hat{\Omega}$ is a Kähler potential. In order to obtain a slightly more explicit form of the Kähler potential, we assume three constraints; (a.) the flat limit ought to correspond to the free action for these multiplets, (b.) $\hat{\Omega}$ is subject to weak holomorphy and (c.)
the action is no more than quadratic in the nonminimal multiplet. These conditions are sufficient to imply that the Kähler potential has the form

$$\hat{\Omega} = \frac{1}{2} \left[ \mathcal{F}^I H_I(\Phi) + \sum E_I(\Phi) - J_{IJ}(\Phi) \Sigma^I \Sigma^J + (H_{IJ}(\Phi) + K_{IJ}(\Phi)) \Sigma^I \Sigma^J \right] + \text{h.c.}$$

(2.4)

where $E_I(\Phi), H_I(\Phi), H_{IJ}(\Phi), K_{IJ}(\Phi)$ and $J_{IJ}(\Phi)$ are holomorphic. It can be seen that one special choice of these is given by

$$H_I = \partial_I H, \quad J_{IJ} = \frac{1}{2} \partial_I \partial_J H, \quad E_I = K_{IJ} = H_{IJ} = 0,$$

(2.5)

for which the equation $\partial_I H_J = 2J_{IJ}$ is satisfied. The results in (2.4) and (2.5) bear a strikingly similar appearance to Kählerian vector multiplet models. In fact, it seems extremely likely that these two classes of models are related to each other via the RADIO technique [22]. Finally, we will argue that the natural geometric interpretation of the Poincaré dual pairs $(\Phi, \Sigma)$ is that the coordinates of a complex manifold $\mathcal{M}$ are provided by $\Phi$ while $\Sigma$ are elements of $^*T_p(\mathcal{M})$, the dual to the holomorphic vector bundle of the manifold. Since this together with (2.4) and (2.5) obviously defines a very particular type of fibered Kähler geometry, we will call this “specular Kähler geometry.”

The action of equation (2.3) describes a non-linear $\sigma$-model and as such no higher order derivative terms are present. It is well known that such higher order derivative terms are present in the standard QCD low-energy effective action. So in order to introduce such terms here we must go beyond (2.3). As we noted in [14] if we assume that strong holomorphy is satisfied, then the form of the higher derivative terms is completely determined

$$S_{\text{H.D.}} = (\gamma')^3 \left[ \int d^4 x \, d^2 \theta \, \sum_{N=4} \mathcal{L}_{(N)} + \text{h.c.} \right] = (\gamma')^3 \left[ S_{\text{H.D.}}^{(4)} + S_{\text{H.D.}}^{(6)} + \ldots \right],$$

(2.6)

where we introduced an expansion parameter of appropriate dimensions denoted by $\gamma'$. Thus, we demand that all terms in (2.6) are determined by a set of holomorphic tensors $J_{1A_{k_1} \ldots K_1 \ldots}(\Phi)$. Geometrically these are to be thought of as proper holomorphic tensors defined over the various bundles of the Kähler manifold. The $A$ label denotes the different irreducible representations possible for a fixed number of Kähler manifold indices and integers $k_1, \ldots k_N$ denote other “naming” labels. We further introduce $P_{A_{k_1} \ldots k_N}$ as a set of constant tensors carrying non-trivial Lorentz representations. Utilizing the $J$-symbols and $P$-symbols, we write the $\mathcal{L}_{(N)}$’s in the

\footnote{One choice is the QCD cutoff, $\Lambda_{QCD}$. Another is given by $f_\pi$.}
forms
\[
\mathcal{L}(N) \equiv \sum \partial_{\alpha_1}^{k_1} \cdots \partial_{\alpha_n}^{k_n} \mathcal{J}^{A_{i_1} \ldots K_k} (\Phi) (\prod \mathcal{G}_{i_1, j_1}^{1, \ldots, K_1}) \cdots (\prod \tilde{\mathcal{G}}^{K_j}_{\omega_j}) \text{,}
\]
\[
\mathcal{G}_{i_1, j_1}^{1, \ldots, K_1} \equiv (\gamma')^{-3} (\overline{D}^{\delta_i} \Sigma_{l_i}) (\overline{D}^{\beta_j} \Sigma_{h_j}) \text{,}
\]
\[
\tilde{\mathcal{G}}^{K_j}_{\omega_j} \equiv (\gamma')^{-(1+k_j)} (\partial_{\omega_1} \cdots \partial_{\omega_j} \Phi^{K_j}) \text{,}
\tag{2.7}
\]
where \( N \) denotes the number of bosonic fields with derivative operators that appear in \( \mathcal{L}(N) \) (i.e. \( N \) equals twice the number of factors of \( \mathcal{G} \) plus the number of factors of \( \tilde{\mathcal{G}} \) in a given term). It can be seen that if \( \mathcal{G} \) appears other than linearly, these actions possess no purely bosonic terms. Thus to have terms with the property that they are non-trivial in the purely bosonic limit, we need only keep \( \mathcal{G} \) to the first power.

The proof that (2.7) contains no propagating auxiliary field is very simple. As a first step let us concentrate only on possible bosonic terms. First, since it is a chiral superfield \( \overline{D}\mathcal{L}(N) = 0 \) we only need to evaluate it by applying \( D^a D_a \) to obtain component results. Next we have to perform the differentiations. If both \( D \)'s act on the \( \mathcal{J} \)-term, the most we can get is an \( F \) auxiliary field. When evaluated at \( \theta = 0 \), the remaining terms are purely physical fields. Similarly, if both \( D \)'s act on \( \mathcal{G} \), the most we can get are terms quadratic in the \( p_\sigma \) auxiliary field. Finally if both \( D \)'s act on \( \tilde{\mathcal{G}} \), the most we can get is an \( F \) auxiliary field which however has derivatives acting upon it. This term is apparently dangerous until we realize that all spacetime derivatives may be integrated “off” of the \( F \) auxiliary field and onto the remaining factors which are themselves physical fields. The extension to include fermionic terms is a straightforward exercise.

Now we come the the central suggestion of our new formulation of the low-energy 4D, \( N = 1 \) supersymmetric QCD effective action. We propose that it should be written as
\[
\mathcal{S}_{\text{eff}}^{\text{SUSY}} (QCD) = \mathcal{S}_{\sigma} + \mathcal{S}_{\text{H.D.}} \text{.}
\tag{2.8}
\]
This action has two important properties. Holomorphy is completely manifest because the nonlinear functions \( E_I, H_I, J_{I, J}, H_{I, J} \) and \( J_{i_1, j_1}^{A_{i_1} \ldots K_k} \) determine the explicit form of the action. The equations of motion for \textbf{all} auxiliary fields are \textit{algebraic} so that no auxiliary fields propagate.

However, the use of this mechanism for suppression of propagating auxiliary fields comes at a price. The roles of the right-handed \( \psi \)-spinors (contained in the chiral superfields) are completely different from that of the left-handed \( \zeta \)-spinors (contained in the nonminimal superfields). This should be particularly obvious by noting that the factors of \( \overline{D} \Sigma \) in the higher derivative expansion correspond to the \( \zeta \)-spinors.
Supersymmetry itself forbids the symmetrical appearance of the different types of spinors in (2.7). Thus, there is a possibility to realize a breaking of parity in this proposed 4D, N = 1 supersymmetric extension of the QCD effective action.

We end this chapter with a few comments on research that still needs to be undertaken to completely clarify the issue of auxiliary-free higher derivative terms in theories involving 4D, N = 1 scalar multiplets. The first question that comes to mind is whether there are other auxiliary-free higher derivative terms that cannot be expressed in terms of chiral superfield Lagrangians. Along these lines, there are two classes of actions that will be studied in the future. We represent these two classes in the form of two actions

\[ S_{\text{Class-I}} = \int d^4 x \, d^2 \theta \, d^2 \bar{\theta} \left[ (D^\dagger \Sigma^I) (D^a \Sigma^I) \, \mathcal{Y}_{1,2}^{\text{Class-I}} \right] , \quad (2.9) \]

and as well

\[ S_{\text{Class-II}} = \int d^4 x \, d^2 \theta \, d^2 \bar{\theta} \left[ G^{IJ} \, \hat{\alpha}^{\beta \gamma} KL \, \alpha \beta \, \mathcal{Y}_{1,JKL,ab}^{\text{Class-II}} \right] , \quad (2.10) \]

where the \( \mathcal{Y} \)'s are, in complete generality, functions of \( \Phi \) and \( \Sigma \) and any of their derivatives (either bosonic or fermionic). We point out that the split above is somewhat artificial since in fact \( S_{\text{Class-II}} \) is a special case of \( S_{\text{Class-I}} \). However, this split is useful once we note (2.10) together with choosing the function \( \mathcal{Y}_{1,JKL,ab}^{\text{Class-II}} \) according to (where \( m_i \) are integers)

\[ \mathcal{Y}_{1,JKL,ab}^{\text{Class-II}} = \mathcal{Y}_{1,JKL,ab}^{\text{Class-II}}(\Phi, \Sigma, \Sigma, \partial^{m_1} \Phi, \partial^{m_2} \Sigma, \partial^{m_3} \Sigma, \partial^{m_4} \Phi, \partial^{m_5} \Sigma) , \quad (2.11) \]

necessarily implies purely bosonic higher derivative terms in the component level action. What remains, however, is to find the conditions required on \( \mathcal{Y}_{1,JKL,ab}^{\text{Class-II}} \) to insure auxiliary freedom in even in its fermionic sector. A general analysis will carried out in the future.

Finally, we end by noting that the mechanism that we use to construct 4D, N = 1 supersymmetric auxiliary-free higher derivative actions for spin (0.1/2) multiplets bares some resemblance to that used in the (1, 1/2) case [12].
3 Review of N = 0 and ‘Old’ N = 1 Supersymmetric Skyrme and WZNW Terms

The standard Skyrme term has the familiar form

\[ S_{\text{Skyrme}} = \frac{1}{32\pi^2} \int d^4 x \ Tr \left[ (\partial^a U) (\partial^b U^{-1}) (\partial_a U) (\partial_b U^{-1}) \right] \]

\[ = \int d^4 x \ k_{mnr} (\Pi) (\partial^a \Pi^m) (\partial^b \Pi^n) (\partial^c \Pi^r) (\partial^d \Pi^s) \]

where we use the notation of appendix A in reference [14]. On the other hand, the WZNW term can be expressed as

\[ S_{\text{WZNW}} = \int d^4 x \epsilon_{abcd} \mathcal{J}_{mnr} (\Pi) (\partial^a \Pi^m) (\partial^b \Pi^n) (\partial^c \Pi^r) (\partial^d \Pi^s) \]

A long time ago we [13] gave a local set of geometric conditions which seemed to us to be necessary but not sufficient for these actions to correspond to the Skyrme and WZNW terms respectively. We may regard the fields \( \Pi^m \) as the coordinates of some manifold endowed with a metric \( g_{mn} \). We further require that this metric possess a set of Killing vectors \( \xi^m \) such that Killings equation takes the form \( L_{\xi}(g) = 0 \) expressed in terms of Lie differentiation. Necessary conditions for the local tensors \( k_{mnr} \) and \( \mathcal{J}_{mnr} \) to generate Skyrme and WZNW terms via the actions above are;

\[ L_{\xi}(k) = 0 \quad , \quad dL_{\xi}(\mathcal{J}) = 0 \]

where \( d \) in the second equation denotes the exterior derivative with respect to the \( \Pi \)-coordinates. Note that neither Lie nor exterior differentiation requires a Riemannian geometry. It is also obvious that the Skyrme term and WZNW terms are related by Poincaré duality.

Prior to examining the form of our new supersymmetric proposal, we feel that it is useful to review the prior results. We wish to explicit demonstrate the reasons for our long standing concerns about the old proposals. Namely in both of these suggestions, the auxiliary \( F \)-field of the chiral multiplets possess equations of motion that are not purely algebraic.

Bergshoeff, Nepomechie and Schnitzer [8] (BNS) have made, to our knowledge, the only explicit proposal for the form of the 4D, N = 1 supersymmetric Skyrme term given as;

\[ S^{\text{BNS}}_{\text{Skyrme}} = \int d^4 x d^2 \theta d^2 \bar{\theta} \left\{ \alpha \left[ (\nabla^2 \Phi_i \nabla^2 \Phi^j) (\nabla^2 \Phi_i \nabla^2 \Phi^j) \right] + \beta \left[ (\nabla^2 \Phi_i \nabla^2 \Phi^j) (\nabla^2 \Phi_i \nabla^2 \Phi^j) \right] \right\} \]
where this action is for a 4D, N = 1 supersymmetric CP\textsuperscript{1} model in particular and we have neglected the normalization since it is irrelevant to our discussion. We wish to solely concentrate upon how the auxiliary fields appear in this action. (The remaining purely bosonic terms can be found in their work.) It is a straightforward calculation to show that F-field dependent terms are given by

\[ S_{\text{Skyrme}}^{BNS} = \int d^4x \left\{ -\alpha \left[ \overline{F}^i F_j (\nabla_a A_i) (\nabla_a \overline{A}^j) + \overline{F}^i F_i (\nabla_a A_j) (\nabla_a \overline{A}^j) \right. \right. \]
\[ \left. \left. + 4 \overline{F}^i F_j \overline{F}^j F_i \right] \right. \] \[ + \beta \left[ \overline{F}^i (\nabla^2 \nabla_a F_i) - F_i (D')^i_j \overline{F}^j \right. \]
\[ \left. + \overline{F}^i \overline{F}^i \right] \}

(3.5)

where the following definitions are to be used,

\[ \overline{F}^i \equiv (\nabla^2 \nabla_a \overline{A}^i) - [(D'), \overline{A}^i] , \quad (D')^i_j \equiv (\nabla^2 A_j) (\nabla_a \overline{A}^i) - \overline{F}^i F_j \]. (3.6)

In evaluating the last two terms of (3.5) we only retain the F-dependent pieces. Some very interesting features can be seen here. Foremost for nonvanishing \( \beta \), the F-field acquires a mass proportional to \( \beta^{-\frac{1}{2}} \). While it is true that for \( \beta = 0 \), the F-field again becomes non-propagating, it is known in the BNS 4D, N = 1 extension of the Skyrme term that for \( \beta = 0 \), the Skyrmion is not stabilized.

Nemenschansky and Rohm [8] (NR) have proposed that the 4D, N = 1 supersymmetric extension of the WZNW term is of the form

\[ S_{\text{WZNW}}^{NR} = \int d^4xd^2\theta d^2\overline{\theta} \left[ \beta_{IJK}(\Phi, \overline{\Phi} )(D^\alpha \Phi^I) (\partial_{\alpha \beta} \Phi^J) (\overline{D}^{\beta \gamma} \overline{\Phi}^K) + \text{h.c.} \right] \], (3.7)

where \( \beta_{IJK} \) is not holomorphic. If we impose the condition of weak holomorphy, then this expression must be modified to

\[ S_{\text{WZNW}}^{\text{mod.}NR} = \int d^4xd^2\theta d^2\overline{\theta} \left[ \mathcal{J}_{IJKL}(\Phi)(D^\alpha \Phi^I) (\partial_{\alpha \beta} \Phi^J) (\overline{D}^{\beta \gamma} \overline{\Phi}^K) (\overline{D}^{\gamma \delta} \overline{D}^\delta \overline{\Phi}^L) + \text{h.c.} \right] \], (3.8)

which looks very similar to our result in ref. [14]. This eliminates all 6-fermion terms also. However, even this modification has other difficulties within the structure of Kähler geometry and the realization of isometries. Furthermore, the problem described immediately below remains even with this modification.

Returning now to (3.7), a large number of the component level terms contained in this action were presented before [8]. However, the critical (for our purposes) auxiliary field terms, denoted by \( L_{\text{aux.}} \), were not explicitly given. Again it is a straightforward
calculation to show that $F$-field dependent terms are given by

$$
S_{\text{WZNW}}^{NR} = \int d^4x \left[ ... - i4(\beta_{1JK} - \beta_{1JK})(\partial^A A^I)(\partial^F F^J)F^K + \text{h.c.} \\
+ i4(\beta_{1JK,L} - \beta_{1JK,K}) F^I(\partial^A A^J) F^K(\partial^\alpha A^L) + \text{h.c.} \right].
$$

(3.9)

We see that for the NR 4D, $N = 1$ extension of the WZNW term, the propagation of the $F$-fields proceeds only by "non-linear $\sigma$-model mixing." Thus, equation (3.9) above explicitly demonstrates the propagation of $F$-fields. Thus, barring miraculous accidents, the $F$-fields become dynamical in NR WZNW term.

In closing this chapter, we note the generality of these results, although we have been investigating within the confines of the 4D, $N = 1$ supersymmetric low-energy effective QCD action, our comments apply to any higher derivative manifestly 4D, $N = 1$ supersymmetric action (as all SUSY effective actions are) whether the application is effective SUSY QCD, MSSM or even SUSY haplon or preon type models.

4 Kähler Geometric Interpretation

At the time we made our introduction of this new approach it was not completely clear what geometric structure could undergird our proposal. In this section we would like to suggest that (as was alluded to in section two), the most natural geometric structure seems to be provided by a Kähler manifold together with its holomorphic co-tangent bundle. Let us explore how this seems consistent.

We first wish to establish a notation for the various geometrical structures associated with a Kähler manifold denoted by $\mathcal{M}$. Let $T_p(\mathcal{M})$ denote the holomorphic vector bundle (tangent plane) at point $p$. Also let $\overline{T}_p(\mathcal{M})$ denote the anti-holomorphic vector bundle at point $p$. Additionally let $^*T_p(\mathcal{M})$ denote the holomorphic co-tangent bundle (dual to $T_p(\mathcal{M})$). Finally let $^*\overline{T}_p(\mathcal{M})$ denote the anti-holomorphic co-tangent bundle (dual to $\overline{T}_p(\mathcal{M})$).

Having established a notation for the various Kähler geometrical objects of interest, we observe that the usual 4D, $N = 1$ supersymmetric non-linear $\sigma$-model implies that its various fields are among the elements of the various geometrical structures according to,

$$
(dA, \psi_\alpha) \in {}^*T_p(\mathcal{M}) \quad , \quad (d\overline{A}, \overline{\psi}_\alpha) \in {}^*\overline{T}_p(\mathcal{M}) \quad ,
$$

$$
(A, \overline{A}) \in \mathcal{M} \quad , \quad \partial/\partial A \in T_p(\mathcal{M}) \quad , \quad \partial/\partial \overline{A} \in \overline{T}_p(\mathcal{M}) \quad .
$$

(4.1)
These equations allow us to define a holomorphic chirality for a theory with chiral multiplets which appear in a non-linear σ-model. Namely, we see that all undotted (“right-handed”) spinors are elements of \( *T_p(\mathcal{M}) \) and all dotted (“left-handed”) spinors are elements of \( *\mathcal{T}_p(\mathcal{M}) \). Thus, by holomorphic chirality we mean that there exists a specific correlation between the space-time chirality of the spinors and the co-tangent bundles (i.e. no undotted spinors are elements of \( *\mathcal{T}_p(\mathcal{M}) \) and vice-versa).

Since \( \Phi^I \) are the coordinates of a Kähler manifold, there must exist some Kähler metric. Let us assume that the Kähler metric which we denote by \( g_{IK} \) possesses a set of isometries generated by the holomorphic co-tangent fields \( \xi^I(\Phi) \) which thus satisfy the Killing equation,

\[
L_{\xi}(g_{IK}) = \xi^K \partial_I g_{IK} + \xi^K \partial_J g_{IK} + (\partial_I \xi^K) g_{LJK} + (\partial_K \xi^K) g_{IKL} = 0 .
\]

Since the chiral superfields are the coordinates for the Kähler manifold, under the action of the isometries these transform according to the rule

\[
\delta_{\xi} \Phi^I = \xi^I(\Phi) = \alpha^{(A)} \xi^{I(A)}(\Phi) .
\]

(4.3)

In the second part of the equation above we have noted that in general there will be numbers of independent such isometries \((A) = 1, ..., m\). For each such isometry, we can introduce a constant parameter \( \alpha^{(A)} \).

On the other hand, the nonminimal superfields may be identified as elements of \( *T_p(\mathcal{M}) \). Thus, their transformation laws are determined to be,

\[
\delta_{\xi} \Sigma^I = (\partial_K \xi^I) \Sigma^K = \alpha^{(A)} (\partial_K \xi_{(A)}^I) \Sigma^K .
\]

(4.4)

Equations (4.3) and (4.4) also give us a very deep insight into why our chiral-nonminimal (CNM) superfield models are fundamentally different from the usual chiral superfield models. The respective transformation laws of the component spinors contained in each multiplet are

\[
\delta_{\xi} \psi^I_\alpha = (\partial_K \xi^I) \psi^K_\alpha , \quad \delta_{\xi} \bar{\psi}_\alpha^I = (\partial_K \xi^I) \bar{\psi}_\alpha^K .
\]

(4.5)

In other words, the undotted physical spinors and the dotted physical spinors in a chiral-nonminimal (CNM) model are elements of \( *T_p(\mathcal{M}) \). This situation can never be realized with the sole use of chiral superfields! For purely chiral multiplet theories, the dotted spinors are elements of \( *\mathcal{T}_p(\mathcal{M}) \).

The results of (4.4) indicate that the physical fields of the nonminimal multiplet satisfy

\[
(B, \bar{\zeta}_\alpha) \in *T_p(\mathcal{M}) , \quad (\bar{B}, \zeta_\alpha) \in *\mathcal{T}_p(\mathcal{M}) .
\]

(4.6)
It can be seen that the geometrical interpretation here is totally different from that of a chiral multiplet. Further, we see that any theory that includes both chiral and nonminimal multiplets can be said to be holomorphically vector-like with respect to the intrinsic definition of space-time chirality associated with the non-linear $\sigma$-model by simply making sure that equal numbers and representations of $\psi_\alpha$ and $\zeta_\alpha$ spinors are present on $\star T_p(M)$ and the conjugate spinors on $\star T_p(M)$.

Let us say a few words about why we assume that the isometries are generated by holomorphic co-tangents. As written (4.3) is a variation for an infinitesimal transformation. To obtain the finite change of variables associated with (4.3) requires exponentiation. In order for this to be well defined, the co-tangent fields $(\xi^I_A)$ must form a ring. This property would be violated if the co-tangent fields depended on both $\Phi$ and $\Sigma$. It is a basic fact of 4D, $N = 1$ supersymmetry that the multiplication of chiral superfields forms a ring. This does not apply to the multiplication of nonminimal superfields. Also in order for the finite transformations associated with the infinitesimal variations in (2.4) to form a group we cannot permit the co-tangent fields to depend on $\Sigma$.

Now we apply the isometry variation to (2.3). That action will be invariant if

$$
\left[ \xi^L \frac{\partial}{\partial \Phi^L} + \xi^\tau \frac{\partial}{\partial \Phi^\tau} + \Sigma^K \left( \frac{\partial \xi^1}{\partial \Phi^K} \frac{\partial}{\partial \Sigma^1} + \frac{\partial \xi^2}{\partial \Phi^K} \frac{\partial}{\partial \Sigma^2} \right) \right] \hat{\Omega} = \eta + \bar{\eta} ,
$$

(4.7)

for some chiral superfield $\eta$.

Of slightly more interest is the effect of the isometry variation on the higher derivative terms. It can be seen that with a slight modification of the definition of $G^K_{i_1 \ldots i_j}$ so as to respect the geometry of the Kähler manifold (i.e. replace the derivatives by appropriate Kähler manifold covariant derivatives), the isometry variation of the Lagrangians takes the form

$$
\delta_\xi L(N) \equiv P_{A \hat{\alpha}_1 \hat{\beta}_1} \cdots L_\xi (J^{A_{i_1} \ldots K_{i_j}}(\Phi))(\prod G^{I_{i_1} J_{i_1} \ldots K_{i_j}})(\prod \hat{G}^{\hat{K}_j}_{\hat{\alpha}_1 \ldots \hat{\beta}_1}) \ ,
$$

(4.8)

where $L_\xi$ denotes the Lie derivative. Thus, requiring the Lie derivative to vanish we find the higher derivative terms are invariant under the isometries of the $\sigma$-model term.

The vanishing of the Lie derivative on $J_{i_1 j_1 k_1}$, however, is too strong a condition. If we assume the weaker condition that its Lie derivative is equal to the exterior derivative of a holomorphic 3-form, then the WZNW term changes by a total divergence under the isometry variation.

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5See discussion in chapter seven.
5 Super P-form Geometric Interpretation

Some years ago [23, 24], the complete simplex of irreducible p-form gauge superfields was derived for 4D, N = 1 superspace. We now wish to show that the CNM formulation of the WZNW action has a natural interpretation in the formalism of 4D, N = 1 super p-forms. Since the component WZNW term is based on a 4-form, the natural starting place for our considerations is a super 4-form, \( \Omega_{ABCD}^{WZNW} \) whose vector-vector-vector-vector component contains the usual component-level WZNW 4-form. As noted in *Superspace* [24], in general for a super p-form, “(field strength) coefficients with more than two spinor indices have too low dimensions to contain component field strengths (or auxiliary fields), and must be constrained to vanish.” We will call this the “p-form theorem.”

The simplex is illustrated in the table below taken from *Superspace*.

| p | \( \hat{A}_p \) | \( d\hat{A}_p \) |
|---|---|---|
| 0 | \( \Phi \) | \( i(\Phi - \Phi) \) |
| 1 | \( V \) | \( i \tilde{D}^2 D_\alpha V \) |
| 2 | \( \varphi_\alpha \) | \( -\frac{1}{2}(D^\alpha \varphi_\alpha + \tilde{D}^\alpha \varphi_\alpha) \) |
| 3 | \( \hat{V} \) | \( \tilde{D}^2 \hat{V} \) |
| 4 | \( \hat{\Phi} \) | 0 |

**Table III**

This table lists for each value of \( p \) the irreducible \( p \)-form and its irreducible super exterior derivative. One of the most interesting features of this table is that it defines the chiral superfield as the irreducible 0-form. The general real superfield \( V \) is associated with the 1-form.

The plan of this chapter is to first recall all known irreducible 4D, N = 1 \( p \)-form theories. Next we use these irreducible \( p \)-forms to study an assortment of super 4-forms that can be constructed by using the super wedge product on the lower order forms. The main purpose of this exercise is to establish a basis for the gauged CNM-WZNW term. Since the non-supersymmetric theory of the gauged WZNW term is also a 4-form, it is natural to expect that the supersymmetric theory should be related to super 4-form geometry.

---

6Interestingly enough, a slightly modified version of the \( p \)-form theorem holds for the case of 4D, \( N = 4 \) supergravity [21] and 10D, \( N = 1 \) supergravity even with lowest order string corrections [27, 28].

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In the case of the 1-form gauge superfield, the field strength is a super 2-form \( F_{AB} \). This corresponds to the well studied case of supersymmetric Yang-Mills theory. As such no few review comments are necessary in this case. However, it is useful to note that given a constant algebraic tensor \( t_{IJKL} \) that satisfies some restrictions (see discussion above (6.1)), the superfield action given by

\[
S_{4-\Gamma} = \int d^4x d^2\theta d^2\bar{\theta} \, t_{IJKL} \, V^I \Gamma^\alpha J \Gamma^\hat{a} K \Gamma^\hat{a} L , \tag{5.1}
\]

will contain the component level term \( \epsilon^{abcd} t_{IJKL} A_a^I A_b^J A_c^K A_d^L \). This particular superfield action is distinguished by its very close relation with a term in the 3D, \( N = 1 \) superfield Chern-Simons action \[25\]. (We have also written this term in the \( K \)-guage where all superconnections \( \Gamma^A_\alpha \) are non-vanishing.)

The ordinary instanton number density is also associated with a component level 4-form. This suggests that there must be a corresponding super 4-form associated with the instanton density of 4D, \( N = 1 \) supersymmetric Yang-Mills theory. So as an example of an irreducible super 4-form, it is interesting to treat this case. We also recall that the 4D, \( N = 1 \) supersymmetric instanton number is also a chiral action and note that 4D, \( N = 1 \) supersymmetric Yang-Mills theory has one graded commutator of the form

\[
\{ \nabla_\alpha , \nabla_\beta \} = i F_{\alpha \beta} = C_{\alpha \beta} W_{\bar{\beta}} . \tag{5.2}
\]

The field strength superfield \( F_{AB} \) is an irreducible super 2-form if the usual constraints of supersymmetric Yang-Mills theory are enforced. The super 2-form can be used to construct a super 4-form \( Y_{ABCD} \equiv F_{[AB]} F_{[CD]} \). The only non-vanishing components of this super 4-form are given by

\[
Y_{\alpha \beta cd} = -C_{\gamma \delta} C_{\alpha(\gamma} C_{\beta)\delta} W^\gamma W_\delta , \\
Y_{\alpha \beta cd} = i 4 \epsilon_{abcd} W^\epsilon \bar{W}_\epsilon , \\
Y_{\alpha \beta cd} = -i 2 \left[ C_{\alpha \beta} W_{\gamma} F_{\epsilon \delta} + C_{\alpha \gamma} W_{\delta} F_{\epsilon \beta} + C_{\alpha \delta} W_{\beta} F_{\epsilon \gamma} \right] , \\
Y_{ab \beta cd} = \frac{1}{4} \epsilon_{abcd} F_{[ab]} F_{[cd]} ,
\]  

and their complex conjugates. Also in analogy with the super 4-form field strength, we here find \( D_\epsilon Y_{\alpha \beta cd} = 0 \). The super 4-form \( Y_{ABCD} \), like \( F_{ABCD} \), is super closed and can be expressed as the super exterior derivative of the super Chern-Simons 3-form \( Y_{\overline{ABC}} = D_{[A} X_{\overline{BCD}}^{CS} - T_{[AB]} X_{[CD]}^{CS} \) whose non-vanishing components are
given by
\[ X_{\alpha \beta \gamma}^{CS} \equiv -\frac{1}{3} f_{IJK} \Gamma_{(\alpha} | I \Gamma_{| \beta}^{|} J \Gamma_{| \gamma)}^{|} K, \]
\[ X_{\alpha \beta}^{CS} \equiv -\frac{1}{3} f_{IJK} \Gamma_{(\alpha} | I \Gamma_{| \beta)}^{|} J \Gamma_{| \gamma)}^{|} K, \]
\[ X_{\alpha \beta}^{CS} \equiv \Gamma_{(\alpha} F_{I}^{1 | | | I} \Gamma_{| \beta)}^{|} I \Gamma_{| \gamma)}^{|} K, \]
\[ X_{\alpha \beta}^{CS} \equiv \Gamma_{\alpha} F_{I}^{1 | | | I} \Gamma_{| \beta)}^{|} I \Gamma_{| \gamma)}^{|} K, \]
\[ X_{\alpha \beta}^{CS} \equiv \frac{1}{2} \Gamma_{(\alpha} F_{I}^{1 | | | I} \Gamma_{| \beta)}^{|} I \Gamma_{| \gamma)}^{|} K, \]

(5.4)

where \( \Gamma_{A}^{1} \) denotes the superspace Yang-Mills connection superfield.

The instanton density action takes the super-geometric form
\[ S_{Inst} = \left\{ -\frac{1}{4} \int d^4 x \ d^2 \bar{\theta} \ e^{a b c d} C_{(\alpha \beta} \ Y_{\gamma \delta \epsilon \delta) + h.c. \right\}, \]

\[ = \left\{ i \int d^4 x \ d^2 \bar{\theta} \ e^{a b c d} C_{(\alpha \beta} \ F_{\gamma \delta \epsilon \delta) + h.c. \right\}, \]

\[ = \left\{ i \int d^4 x \ d^2 \bar{\theta} \ C^{(\alpha \beta} \ C^{\gamma \delta \epsilon \delta) F_{\gamma \delta \epsilon \delta) + h.c. \right\}, \]

\[ = \left\{ i \int d^4 x \ d^2 \theta \ W^{\alpha} W_{\alpha} + h.c. \right\}, \]

(5.5)

and we see that there is an exact analogy between the forms of the corresponding non-supersymmetric quantities and their supersymmetric extensions!

In the case of the 2-form gauge superfield, the field strength is a super 3-form \( G_{A B C} \) that is known to satisfy the following constraints,
\[ G_{\alpha \beta} = 0, \quad G_{\alpha \beta} = i C_{\alpha \gamma} C_{\beta \gamma} G, \]

(5.6)

where \( D^{2} G = \overline{D}^{2} G = G - \overline{G} = 0 \). The remaining non-vanishing field strengths can be written as,
\[ G_{\alpha b c} = C_{(\alpha (C_{\gamma)} \delta C_{\beta]} D^{\delta} G, \]

\[ G_{a b c} = \epsilon_{a b c d} [ D^{\delta}, \overline{D}^{\delta} ] G. \]

(5.7)

In this last expression we have utilized the definition
\[ \epsilon_{a b c d} \equiv i \frac{1}{2} \left[ C_{\alpha \beta} C_{\gamma \delta} C_{a(\gamma} C_{\beta)} G - C_{\alpha a} C_{\gamma \delta} C_{a(\gamma} C_{\beta)} G \right]. \]

(5.8)

We also know that the field strength can be expressed in terms of an unconstrained chiral spinor superfield \( \varphi^{\alpha} \) (analogous to \( V \) for Yang-Mills theory) via \( G = -\frac{1}{2} \left( D^{\alpha} \varphi_{\alpha} + \)
We note in passing that this is the rigid super-geometrical formulation of the axion multiplet (also known as the linear multiplet) that contains the axion, dilaton and dilatino. The action for this multiplet that contains purely the square of the 3-form field strength is just

\[ S_{2-\text{form}} = -\frac{1}{2} \int d^4x d^2\theta d^2\bar{\theta} G^2. \tag{5.9} \]

In the presence of a vector multiplet, an interesting interaction of the form

\[ S_{2-\text{form-mass}} = \int d^4x d^2\theta \, V G + \text{h.c.} = -\int d^4x d^2\theta \, \varphi^\alpha W_\alpha + \text{h.c.}, \tag{5.10} \]

which contains the bosonic term \( \epsilon^{a\beta\gamma\delta} A_\alpha \partial_\beta b_{\gamma\delta} \) (where \( A_\alpha \) is the component vector in the vector multiplet and \( b_{\gamma\delta} \) is the component tensor in the 2-form multiplet) can be introduced.

In the case of the 3-form gauge superfield, the field strength is a super 4-form \( F_{ABCD} \) that is known to satisfy the following constraints,

\[ F_{\alpha\beta\gamma\delta} = F_{\alpha\beta\gamma\delta} = F_{\alpha\beta\gamma\delta} = 0, \quad F_{\alpha\beta\gamma\delta} = C_{\gamma\delta} C_{\alpha(\gamma} C_{\delta)\delta} \bar{F}, \tag{5.11} \]

where \( \overline{D_\delta F} = 0 \). The remaining non-vanishing field strength superfields take the forms

\[ F_{\alpha\beta\gamma\delta} = -\epsilon_{\alpha\beta\gamma\delta} \overline{D^i F}, \]
\[ F_{\alpha\beta\gamma\delta} = i\epsilon_{\alpha\beta\gamma\delta} \left[ D^2 F - \overline{D^2 F} \right]. \tag{5.12} \]

As was described in the first considerations of irreducible super \( p \)-forms \([23]\), the super 4-form defined by (5.11) and (5.12) is super-closed \( (\delta F)_{ABCD} = 0 \). The action for this multiplet that contains the square of the 4-form field strength is just

\[ S_{4-\text{form}} = \int d^4x d^2\theta d^2\bar{\theta} \bar{F} F, \tag{5.13} \]

and as well a purely topological term is obtained from

\[ S_{3-\text{form-top}} = i \int d^4x d^2\theta \, F + \text{h.c.}. \tag{5.14} \]

The field strength \( F \) can be expressed as \( F = \overline{D^2 \hat{V}} \) in terms of an unconstrained prepotential superfield \( \hat{V} \).

In the case of the 4-form gauge superfield, the field strength is a super 5-form \( \mathcal{H}_{ABCD} \) which we may constrain to be identically zero. In this case, there exist a
super closed 4-form gauge superfield denoted by $h_{ABCD}$ that is not super exact. This 4-form gauge superfield takes the form,

$$h_{\alpha \beta \gamma D} = h_{\alpha \beta D} = h_{\alpha \beta \epsilon \delta} = 0,$$

$$h_{\alpha \beta \epsilon \delta} = C_{\gamma (\alpha} C_{\delta) \beta} \hat{\Phi},$$

$$h_{\alpha \beta \epsilon \delta} = -\epsilon_{a b c d} D_{\alpha} \hat{\Phi},$$

$$h_{a b c d} = i\epsilon_{a b c d} \left[ D^2 \hat{\Phi} - \overline{D}^2 \hat{\Phi} \right].$$

where $D_{\alpha} \hat{\Phi} = 0$.

In closing this section of this chapter, we emphasize that the super 4-forms discussed above bear no direct relation to the super WZNW-form discussed next. The preceding discussion was included as a convenient introduction to the topic of irreducible 4D, $N = 1$ super 4-forms. The most important lesson to draw from this discussion is that whenever ordinary $p$-forms appear in a non-supersymmetric theory, it should be expected that irreducible 4D, $N = 1$ super $p$-forms will appear in the supersymmetric extension of that theory.

The “$p$-form theorem” together with the result above suggests that the first non-trivial component relevant to the irreducible WZNW super 4-form is the component $\Omega_{\alpha \beta \epsilon \delta}^{WZW}$. All lower or equal dimension components must be equal to zero up to an exact super 4-form. We can thus make the natural ansatz

$$\Omega_{\alpha \beta \epsilon \delta}^{WZW} = \frac{1}{4!} \Omega_{[IJKL]}(D_{\alpha} \Sigma^i)(D_{\beta} \Sigma^j)(\partial_c \Phi^K)(\partial_d \Phi^L),$$

which has all the correct symmetries to be identified as the first non-trivial component of WZNW super 4-form. If the coefficient function $\Omega_{[IJKL]}$ is anti-holomorphic, then we find, $D_{\epsilon} \Omega_{\alpha \beta \epsilon \delta}^{WZW} = 0$ exactly like the two cases discussed earlier in the chapter. Clearly this 4D, $N = 1$ super 4-form component has the obvious interpretation as being the anti-holomorphic pull-back of the anti-holomorphic Kähler manifold tensor $\Omega_{[IJKL]}$. It can be noted at this stage that if we were to restrict ourselves solely to chiral superfields, the only anti-holomorphic pull-back would have four vector indices that could be contracted with $\epsilon_{a b c d}$. The resultant superfield could then be used in an action. However, the component level WZNW term is absent after integrating out the Grassmann coordinates.

With a bit of algebra, we find that the general irreducible decomposition with respect to Lorentz symmetry yields,

$$\Omega_{\alpha \beta \epsilon \delta}^{WZW} \equiv \hat{\Omega}^{(W)}_{(\alpha \beta \gamma \delta)} C_{\gamma \delta} + \hat{\Omega}^{(R)}_{(\alpha \beta \gamma)} C_{\gamma \delta} + \hat{\Omega}^{(S)}_{(\alpha \beta \gamma \delta)} C_{\gamma \delta},$$

where $\hat{\Omega}^{(W)}_{(\alpha \beta \gamma \delta)}$, $\hat{\Omega}^{(R)}_{(\alpha \beta \gamma)}$, and $\hat{\Omega}^{(S)}_{(\alpha \beta \gamma \delta)}$ are the Weyl, Ricci, and Spinor components, respectively.
where the labels \((W), (R)\) and \((S)\) are meant to be reminders of the similarity of this decomposition to that of the Riemann curvature tensor into its Weyl, Ricci (traceless) and scalar curvature components. To complete our proposed super 4-form description, we require explicit expressions for \(\Omega^W_{\alpha b c d}\) and \(\Omega^ZW_{\alpha b c d}\). For these we propose,

\[
\Omega^W_{\alpha b c d} = \epsilon_{b c d} \epsilon^\alpha \left[ C_{\beta} \Omega^W_{\alpha \beta (S)} - \frac{1}{2} \delta_{\beta} \Omega^W_{(\alpha) (S)} \right] = \epsilon_{b c d} \epsilon^\alpha \left[ \hat{\Omega}^W_{\alpha (S)} \right], \tag{5.18}
\]

\[
\Omega^ZW_{\alpha b c d} = \epsilon_{b c d} \left[ D^2 \hat{\Omega}^W_{(S)} - \delta_{\alpha} \Omega^W_{(S)} \right]. \tag{5.19}
\]

Thus (5.16-5.19) complete our ansatz for the super 4-form \(\Omega^W_{\alpha b c d}\) and with it we can easily compute its super exterior derivative via the equation,

\[
(d\Omega^W_{\alpha b c d})_{\alpha b c d} = D_{\alpha} \Omega^W_{\alpha b c d} - T_{\alpha b c d} \hat{E} \Omega^W_{\alpha b c d}, \tag{5.20}
\]

which when expressed in terms of the non-vanishing components of \(\Omega^W_{\alpha b c d}\) takes the more explicit forms,

\[
(d\Omega^W_{\alpha b c d})_{\alpha b c d} = D_{\alpha} \Omega^W_{\alpha b c d} + i \Omega^W_{(\alpha) (\beta) (\gamma) b c d} ,
\]

\[
(d\Omega^W_{\alpha b c d})_{\alpha b c d} = \frac{1}{2} \delta_{\alpha} \Omega^W_{\alpha b c d} + D_{\alpha} \Omega^W_{\alpha b c d},
\]

\[
(d\Omega^W_{\alpha b c d})_{\alpha b c d} = D_{\alpha} \Omega^W_{\alpha b c d} - i \Omega^W_{\alpha b c d}, \tag{5.21}
\]

\[
(d\Omega^W_{\alpha b c d})_{\alpha b c d} = \frac{1}{2} \delta_{\alpha} \Omega^W_{\alpha b c d}.
\]

These yield after substitution from (5.10-5.13) the results

\[
(d\Omega^W_{\alpha b c d})_{\alpha b c d} = C_{\alpha} \hat{D}_{\alpha} \hat{\Omega}^W_{(W)} + \frac{1}{4} C_{\alpha} \hat{D}_{\alpha} \hat{\Omega}^W_{(W)} (\hat{\alpha}), + \hat{D}_{\alpha} \hat{\Omega}^W_{(W)} (\hat{\alpha}) \right) ] + \frac{1}{6} C_{\alpha} \hat{D}_{\alpha} \hat{\Omega}^W_{(W)} (\hat{\alpha}) + C_{\alpha} \hat{D}_{\alpha} \hat{\Omega}^W_{(W)} (\hat{\alpha}) \right) ] + \frac{1}{6} C_{\alpha} \hat{D}_{\alpha} \hat{\Omega}^W_{(W)} (\hat{\alpha}) + C_{\alpha} \hat{D}_{\alpha} \hat{\Omega}^W_{(W)} (\hat{\alpha}) \right) ] ,
\]

\[
(d\Omega^W_{\alpha b c d})_{\alpha b c d} = -\epsilon_{\alpha \beta \gamma \delta} k \left[ \partial_{\alpha} \hat{\Omega}^W_{(W)} (\hat{\alpha}) + \partial_{\beta} \hat{\Omega}^W_{(W)} (\hat{\alpha}) + \partial_{\gamma} \hat{\Omega}^W_{(W)} (\hat{\alpha}) + \partial_{\delta} \hat{\Omega}^W_{(W)} (\hat{\alpha}) \right] ,
\]

\[
(d\Omega^W_{\alpha b c d})_{\alpha b c d} = -\frac{1}{6} \epsilon_{\alpha \beta \gamma \delta} k \left[ \hat{D}_{\alpha} \hat{\Omega}^W_{(W)} (\hat{\alpha}) + \hat{D}_{\beta} \hat{\Omega}^W_{(W)} (\hat{\alpha}) + \hat{D}_{\gamma} \hat{\Omega}^W_{(W)} (\hat{\alpha}) + \hat{D}_{\delta} \hat{\Omega}^W_{(W)} (\hat{\alpha}) \right] ,
\]

\[
(d\Omega^W_{\alpha b c d})_{\alpha b c d} = i \epsilon_{\alpha \beta \gamma \delta} \partial_{\alpha} \left[ D^2 \hat{\Omega}^W_{(S)} - \hat{D}^2 \hat{\Omega}^W_{(S)} \right] . \tag{5.22}
\]

The differences between the closed super \(p\)-forms described by \(F_{\alpha b c d}\) or \(Y_{\alpha b c d}\) and the non-closed super \(p\)-form described by \(\Omega^W_{\alpha b c d}\) are now completely obvious.
The super exterior derivative of the former vanish while the super exterior derivative of the latter depends only on its ‘Weyl’ and ‘traceless Ricci’ pieces. For the former

\[(dF)_{a b c d e} = (dY)_{a b c d e} = 0, \text{ while } (d\Omega^{\text{WZNW}})_{a b c d e} \neq 0.\]

A final point to note is that in the CNM formulation, the 4D, N = 1 supersymmetric WZNW term can be expressed as

\[S_{\text{WZNW}}^{\text{H.D.}} = \left[ -\frac{1}{12} \int d^4 x \, d^2 \theta \, \epsilon^{abcd} e^\alpha_{\beta \gamma \delta} \Omega^{\text{WZNW}}_{\alpha \beta \gamma \delta} + \text{h.c.} \right],\]

\[= \left[ -\frac{1}{12} \int d^4 x \, d^2 \theta \, C^{\gamma \delta} C^{\alpha \gamma} C^{\beta \delta} \Omega^{\text{WZNW}}_{\alpha \beta \gamma \delta} + \text{h.c.} \right], \quad (5.23)\]

so that our proposal is the chiral superspace integral of a super 4-form! No such super-geometric interpretation is available for (3.7). Roughly speaking, the result in (5.5) is analogous to that in (5.17) although in the latter case the action is not a surface term. Alternately we see that there is clearly a close super geometrical relation between the instanton density and the CNM-WZNW term.

6 A Component Preview of 4D, N = 1 Supersymmetric Auxiliary-free 4-J Terms

The task of giving the complete component level description of the action in (2.3) is enormous. A major step is simply the component evaluation of \(S_{\sigma}\). This is important in order to be able to derive the auxiliary field equations of motion among other reasons. This will be completed in a future work. In this section, we wish to focus some attention on the component structure of the lowest order terms in \(S_{\text{H.D.}}\).

From equation (2.7) we see that there are in fact three broad classes of such terms. One class consists of terms quadratic in \(\hat{G}\). Such terms are purely proportional to fermions. Another class of terms are linear in \(\hat{G}\) and linear in \(G\). Finally, there terms quadratic in \(G\). (an example of such a term can be seen in (A.8)). It is only the second class for which we will give a brief description. This class of terms includes the Skyrme and WZNW terms.

We begin with a fourth order holomorphic tensor denoted by \(J_{IJKL}(\Phi)\) which most generally satisfies \(J_{IJKL} = J_{IJLK}\). In the applications we wish to show, we satisfy this condition by imposing \(J_{IJKL} = -J_{JIKL}\) and \(J_{IJKL} = -J_{IJLK}\). The
lowest order term we wish to show thus arise as,

$$S^{(4)}_{\text{H.D.}} = \int d^4x \, d^2\theta \, \mathcal{J}_{JIKL} \mathcal{H}(\Phi) \left( \overline{D^\dagger \Sigma^I} \right) \left( \overline{D^\dagger \Sigma^J} \right) \left( \partial^{\gamma \alpha}_\alpha \Phi^K \right) \left( \partial^{\gamma \beta}_\beta \Phi^L \right) + \text{h.c.} \quad (6.1)$$

However, (6.1) has the consequence that it describes more than the WZNW term. As well it contains the Skyrme term. This occurs by noting that an irreducible decomposition of $\mathcal{J}_{JIKL}$ takes the form

$$\mathcal{J}_{JIKL} = \sum_A \mathcal{J}^A_{JIKL}, \quad (6.2)$$

where $A$ denotes the different irreducible representations. (See also A.12.)

The calculation of the component results follows using the by now well established projection technique. We find $S^{(4)}_{\text{H.D.}}$ leads to

$$\int d^4x \, d^2\theta \, \mathcal{J}_{JIKL} \mathcal{H}(\Phi) \left( \overline{D^\dagger \Sigma^I} \right) \left( \overline{D^\dagger \Sigma^J} \right) \left( \partial^{\gamma \alpha}_\alpha \Phi^K \right) \left( \partial^{\gamma \beta}_\beta \Phi^L \right)$$

$$= \int d^4x \left[ \mathcal{J}_{JIKL} \left( \partial^{\alpha \hat{\alpha}} \Phi^K + i p^{\alpha \hat{\alpha}} \right) \left( \partial^{\beta \hat{\beta}} \Phi^L + i p^{\beta \hat{\beta}} \right) \right]$$

$$+ \mathcal{J}_{JIKL} \left( i \partial^{\alpha \hat{\alpha}} \rho^{\alpha \hat{\alpha}} - 2 \beta^{\hat{\beta}} \right) \left( \partial^{\alpha \hat{\alpha}} \Phi^K \right) \left( \partial^{\beta \hat{\beta}} \Phi^L \right)$$

$$+ \mathcal{J}_{JIKL} \left( \partial^{\alpha \hat{\alpha}} \Phi^K \right) \left( \partial^{\gamma \beta}_\beta \Phi^L \right)$$

$$+ i 4 \mathcal{J}_{JIKL} \left( \partial^{\alpha \hat{\alpha}} \Phi^K + i p^{\alpha \hat{\alpha}} \right) \left( \partial^{\gamma \beta}_\beta \Phi^L \right)$$

$$+ \mathcal{J}_{JIKL,M} \psi^{\alpha M} \left( \partial^{\alpha \hat{\alpha}} \Phi^K \right) \left( \partial^{\beta \hat{\beta}} \Phi^L \right)$$

$$+ \mathcal{J}_{JIKL,M} F^M \left( \partial^{\alpha \hat{\alpha}} \Phi^K \right) \left( \partial^{\beta \hat{\beta}} \Phi^L \right)$$

$$+ \mathcal{J}_{JIKL,MN} \psi^{\alpha M} \psi^{\alpha N} \left( \partial^{\alpha \hat{\alpha}} \Phi^K \right) \left( \partial^{\beta \hat{\beta}} \Phi^L \right) \] . \quad (6.3)$$

As can be seen, only the first line of the rhs consists of purely bosonic terms. Let us focus our analysis by only considering these terms.

It is our first observation that if we set the auxiliary field $p^{\alpha \hat{\alpha}}$ to zero, then the purely bosonic terms collapse to

$$\int d^4x \, d^2\theta \, \mathcal{J}_{JIKL} \mathcal{H}(\Phi) \left( \overline{D^\dagger \Sigma^I} \right) \left( \overline{D^\dagger \Sigma^J} \right) \left( \partial^{\gamma \alpha}_\alpha \Phi^K \right) \left( \partial^{\gamma \beta}_\beta \Phi^L \right) |_{\text{phys. fields}}$$

$$= \int d^4x \left[ \mathcal{J}_{JIKL}(A) \left( \partial^{\alpha \hat{\alpha}} B^I \right) \left( \partial^{\beta \hat{\beta}} B^J \right) \left( \partial^{\gamma \alpha}_\alpha A^K \right) \left( \partial^{\gamma \beta}_\beta A^L \right) \right] \quad (6.4)$$

$$= \int d^4x \left[ \mathcal{J}_{JIKL}(A) P^{a b c d} \left( \partial_a B^I \right) \left( \partial_b B^J \right) \left( \partial_c A^K \right) \left( \partial_d A^L \right) \right] ,$$
where \( P_{abcd} \equiv \frac{1}{4} [ \eta_{ab} \eta_{cd} + \eta_{ac} \eta_{bd} + i \epsilon_{abcd} ] \). In going from the second to the third line above, we have made use of the following identities,
\[
\begin{align*}
\eta_{ab} & \equiv C_{\alpha\beta} C_{\alpha\beta}^*, \quad \eta_{ab}^\dagger \equiv C^{\alpha\beta} C_{\alpha\beta}^*, \\
C^{\alpha\beta} C_{\gamma\delta} C_{\gamma\delta}^* C_{\alpha\beta}^* & = \frac{1}{2} [ \eta^\dagger_{ab} \eta_{cd} + \eta^\dagger_{ac} \eta_{bd} - \eta^\dagger_{ad} \eta_{bc} + i \epsilon_{abcd} ] ,
\end{align*}
\]
Due to the symmetry restrictions on \( J_{IJKL} \) imposed above equation (6.1) we may drop the leading term in the definition of \( P_{abcd} \) since upon contraction this gives zero. The next term in \( P_{abcd} \) produces the Skyrme term and the remaining term produces the WZNW term (see also A.12 in appendix A).

Although we defer to the future a detailed discussion of the component results implied by the CNM approach to the 4D, \( N = 1 \) supersymmetric low-energy QCD effective action, there is one aspect that is so amusing that we wish to focus upon it here. The dynamical bosonic fields in our construction correspond to the leading components of \( \Phi^1 \) and \( \Sigma^1 \). For these we may write,
\[
\begin{align*}
\Phi^1 & = A^1(x) = A^1(x) + i \left[ \Pi^1(x) \cos(\gamma_S) + \Theta^1(x) \sin(\gamma_S) \right] , \\
\Sigma^1 & = B^1(x) = B^1(x) + i \left[ -\Pi^1(x) \sin(\gamma_S) + \Theta^1(x) \cos(\gamma_S) \right] ,
\end{align*}
\]
in terms of two real octets of scalar spin-0 fields \( A^1 \) and \( B^1 \) as well as two real octets of pseudoscalar spin-0 fields \( \Pi^1 \) and \( \Theta^1 \). Here we introduce a mixing angle \( \gamma_S \) that is restricted by the form of the supersymmetric WZNW action to satisfy the condition \( \sin(2\gamma_S) \neq 0 \) (see below).

The amusement begins by recalling the structure of the Glashow-Salam-Weinberg model [24]. We also can define an octet of complex scalar spin-0 fields by the definitions,
\[
\begin{align*}
G^1_+ & \equiv \frac{1}{\sqrt{2}} \left[ A^1 + i B^1 \right] , \quad G^1_- \equiv \frac{1}{\sqrt{2}} \left[ A^1 - i B^1 \right] ,
\end{align*}
\]
so that \( G^1_- = [G^1_+]^* \). Finally, we see that the set of spin-0 fields \( (\Pi^1, \Theta^1, G^1_+, G^1_-) \) bares an uncanny resemblance to \( (A_\mu, Z^\mu_0, W^+_\mu, W^-_\mu) \) of the GSW model! The mirth even more increases when we realize that after supersymmetry-breaking there must develop a mass gap between \( \Pi^1 \) and the remaining spin-0 fields. Furthermore, the mixing angle here \( \gamma_S \) is clearly the analog of the weak mixing angle \( \theta_W \).

Substituting (6.6) into (6.4) and keeping only the purely \( \Pi \) dependent terms yields,
\[
\begin{align*}
& \int d^4x d^2 \theta \ J_{IJKL}(\Phi) \left( \bar{D}_{\alpha}^i \Sigma^1 \right) \left( \bar{D}_{\beta}^j \Sigma^1 \right) \left( \bar{D}_{\gamma}^k \Phi^1 \right) \left( \bar{D}_{\delta}^l \Phi^1 \right) |_{\text{pion}} \\
& = \frac{1}{4} \sin^2(2\gamma_S) \int d^4x J_{IJKL}(\Pi) P_{abcd} \left[ (\bar{\partial}_a \Pi^1) (\bar{\partial}_b \Pi^1) (\bar{\partial}_c \Pi^K) (\bar{\partial}_d \Pi^L) \right] .
\end{align*}
\]

\footnote{In our work of ref. [14] we simply assumed maximal mixing for the sake of simplicity.}
The remaining dynamical fields in (2.8), the pionino fields, can be described by an SU(3) octet of Dirac spinors \( \ell^I (\alpha) \equiv (\psi^I_\alpha, \zeta^I_\alpha) \) or using a 4-component spinor notation

\[
\ell^I (x) \equiv \begin{pmatrix} \psi^I_\alpha \\ \zeta^I_\alpha \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \psi^I_\alpha = \frac{1}{2} (I + \gamma^5) \ell^I, \quad \zeta^I_\alpha = \frac{1}{2} (I - \gamma^5) \ell^I.
\]

Using the definitions in (6.6, 6.7, 6.9), the action in (6.3) may be rewritten in terms of the Dirac spinor \( \ell^I \) and the bosons \( (\Pi^I, \Theta^I, G^I_+, G^I_-) \).

7 Toward the Gauged Auxiliary-free 4D, N = 1 Supersymmetric QCD Effective Action

The problem of deriving the complete gauged version of the CNM-WZNW term will be addressed in a future work. In this section, however, we wish to establish some fundamental structures that will likely be required for its complete construction. The topic of gauging isometries in 4D, N = 1 supersymmetric non-linear \( \sigma \)-models began sometime ago [30]. However, the superfield methods needed to tackle the present problem were established by Hull, Karlhede, Lindstrom and Rocek [31]. It is not our purpose here to review their formalism, however, we do note that even the gauging of the terms only involving chiral superfields in the Kähler potential of (2.4), is non-trivial. Before writing their result, we note that equation (4.3) can also be written as

\[
\delta \xi^I |_{(A)} = \alpha^{(A)} [\xi^K |_{(A)} (\frac{\partial}{\partial \Phi^K}) , \Phi^J] \equiv -iL_\alpha \Phi^J.
\]

Now we simply note their result for gauging the pure chiral terms reads,

\[
\int d^4x d^2 \theta d^2 \bar{\theta} \left[ K(\Phi, \bar{\Phi}) + \left( \frac{e^{LV} - 1}{L_V} \right) X_{(A)V(A)} \right],
\]

where we have used the following definitions,

\[
K(\Phi, \bar{\Phi}) \equiv \frac{i}{2} [ \bar{\Phi}^I H_1(\Phi) + \text{h.c.} ] , \quad \xi_{(A)}^I (\frac{\partial}{\partial \Phi^J}) K = -iX_{(A)} + \eta_{(A)},
\]

and the second equation above acts as the definition of the real superfield function \( X_{(A)}(\Phi, \bar{\Phi}) \) required to write the gauging (note that in (4.7) we make the replacement \( \eta \rightarrow \eta_{(A)} \)) of the isometry. One point of interest can be seen in that this action is an infinite powers series in the vector fields \( V^{(A)} \). This is in accord with a theorem stated in [13]. Namely whenever a global “symmetry” exists that only leaves an action invariant up to surface terms, the coupling to gauge field must be other than through
simple minimal covariantization. (This is the same reason that minimal coupling does not correctly gauge the ordinary WZNW term.) For the purely chiral superfield terms of (2.4), equation (4.7) tells us that the isometries generate surface terms proportional to $\eta_{(A)} + \bar{\eta}_{(A)}$. There is something very intriguing about (7.2-7.3). Let us assume that the superfields $\eta$ are identically zero, then (7.2) can be re-written as

$$\int d^4x d^2\theta d^2\bar{\theta} K(c^{LV} \Phi, \overline{\Phi}),$$

noting that $L_V$ is defined by (7.1) with the replacement $\alpha \to V$. This result looks exactly like what one would obtain in a Yang-Mills theory! We may think of the operator superfield

$$\exp[V^{(A)} \xi^K_{(A)}(\partial_{\Phi^K})] \equiv \exp[V^{(A)} \xi^K_{(A)} \partial_K],$$

as we would any similar operator in a 4D, $N = 1$ supersymmetric Yang-Mills theory. This is also supported by noting that the algebra of $\xi^K_{(A)} \partial_K$ must be closed thus this acts just as the matrix generators of some compact Lie algebra. With this recognition, we see that the transformation laws in (4.3) and (4.4) may be derived as the infinitesimal versions of the equations

$$(\Phi^I)' = \exp[\alpha^{(A)} \xi^K_{(A)} \partial_K] \Phi^I,$$

$$(\Sigma^I \partial_I)' = \exp[-i\Lambda^{(A)} \xi^K_{(A)} \partial_K] (\Sigma^I \partial_I) \exp[\alpha^{(A)} \xi^K_{(A)} \partial_K].$$

This last observation suggests that we re-examine the issue of gauging isometries in superspace. We are further motivated to look at this question particularly in light of the existence of Kähler manifold covariant derivatives that were shown to exist sometime ago [32].

Given the operator in (7.5) it is natural to define a group of gauge transformations acting on the superfield $V^{(I)}$ via the equation

$$(\exp[V^{(A)} \xi^K_{(A)} \partial_K])' = \exp[-i\Lambda^{(A)} \xi^K_{(A)} \partial_K] \exp[V^{(A)} \xi^K_{(A)} \partial_K] \exp[i\Lambda^{(A)} \xi^K_{(A)} \partial_K].$$

Thus there appears to be no impediment that prevents us from using the operator in (7.5) as the starting point in the construction of a Kähler manifold covariant derivative that is modeled exactly like the usual 4D, $N = 1$ supersymmetric Yang-Mills covariant derivative. (This construction differs somewhat from that used by Koller.)

Once we notice these points, the covariantization of the nonminimal multiplet terms becomes trivial. We use the same prescription as for the chiral multiplets. Alternately, once we possess Kähler manifold covariant derivatives, we can also accomplish the gauging of isometries simply by replacing the derivatives in (2.1) by
their appropriate Kähler manifold covariant derivatives. This superspace minimal coupling procedure can be used on all of the higher derivative terms too...with the exception of the CNM-WZNW term to which we now turn.

The component level ungauged WZNW term has the familiar form,

\[ S_{\text{WZNW}} = C_0 \int d^4x \int_0^1 dy \, \text{Tr} \left[ (\hat{U}^{-1} \partial_y \hat{U}) \, \hat{W}_4 \right] , \]

\[ \hat{W}_4 = \epsilon^{abcd} (\partial_a \hat{U}^{-1}) (\partial_b \hat{U}) (\partial_c \hat{U}^{-1}) (\partial_d \hat{U}) \equiv (d\hat{U}^{-1}) (d\hat{U}) (d\hat{U}^{-1}) (d\hat{U}) , \]

\[ C_0 = -iN_C [2 \cdot 5!]^{-1} , \]  

(7.8) expressed in term of the \( y \)-extended SU(3) group element \( \hat{U} = \exp[if^{-1}y]\Pi \). For the rest of the gauged WZNW action we simply note that \( U \equiv \hat{U}(y = 1) \).

Using the results of ref. [11] as our starting point, we find that the form of the gauged WZNW action that insures the conservation of the vector current can be expressed as

\[ S_{\text{gauged}}^{\text{WZNW}}(U, A_L, A_R) = C_0 \left[ S_{\text{WZNW}}(U) + S_{\text{MC}^3}^{\text{WZNW}}(U, A_L, A_R) \right. \]

\[ + S_{\text{MC}^2}^{\text{WZNW}}(U, A_L, A_R) + S_{\text{MC}^1}^{\text{WZNW}}(U, A_L, A_R) \]

\[ + S_{\text{MC}^0}^{\text{WZNW}}(U, A_L, A_R) + S_{\text{AF}^{\text{WZNW}}}(U, A_L, A_R) \]

\[ \left. + S_{\text{AF}^2}^{\text{WZNW}}(U, A_L, A_R) \right] . \]  

(7.9)

The explicit form of \( S_{\text{WZNW}} \) has already been given, so below we give the remaining terms

\[ S_{\text{MC}^3}^{\text{WZNW}}(U, A_L, A_R) = -\int d^4x Tr \left[ (dU) (dU^{-1}) (dU) U^{-1} A_L \right] \]

\[ + \int d^4x Tr \left[ (dU^{-1}) (dU) (dU^{-1}) U A_R \right] , \]  

(7.10)

\[ S_{\text{MC}^2}^{\text{WZNW}}(U, A_L, A_R) = -\frac{1}{2} \int d^4x Tr \left[ (dU) U^{-1} A_L (dU) U^{-1} (dU) A_L \right] \]

\[ + \frac{1}{2} \int d^4x Tr \left[ U^{-1} (dU) A_R U^{-1} (dU) A_R \right] \]

\[ + \int d^4x Tr \left[ (dU) (dU^{-1}) U^{-1} A_L U A_R \right] \]

\[ - \int d^4x Tr \left[ (dU) (dU^{-1}) A_R U^{-1} A_L U \right] , \]  

(7.11)
\[ S_{WZWN}^{MC_1}(U, A_L, A_R) = -\int d^4 x Tr \left[ (dU) \left( U^{-1} A_L A_L + A_R A_R U^{-1} \right) \right] + \int d^4 x Tr \left[ (dU) U^{-1} A_L U A_R U^{-1} A_L \right] + \int d^4 x Tr \left[ (dU) A_R U^{-1} A_L U A_R U^{-1} \right] + \int d^4 x Tr \left[ (dU^{-1}) A_L A_R U A_R \right] - \int d^4 x Tr \left[ (dU) A_R A_R U^{-1} A_L \right] , \]  
\[ (7.12) \]

\[ S_{WZWN}^{MC_0}(U, A_L, A_R) = -\int d^4 x Tr \left[ U^{-1} A_L A_L U A_R - U A_R A_R U^{-1} A_L \right] - \int d^4 x Tr \left[ U^{-1} A_L U A_R U^{-1} A_L U A_R \right] , \]  
\[ (7.13) \]

\[ S_{WZWN}^{AF}(U, A_L, A_R) = +\int d^4 x Tr \left[ (dU) U^{-1} \left( A_L F_L + F_L A_L \right) \right] - \int d^4 x Tr \left[ (dU^{-1}) U \left( A_R F_R + F_R A_R \right) \right] + \int d^4 x Tr \left[ (dU) F_R U^{-1} A_L - (dU^{-1}) F_L U^{-1} A_R \right] , \]  
\[ (7.14) \]

\[ S_{WZWN}^{A^2F}(U, A_L, A_R) = +\int d^4 x Tr \left[ F_L \left( A_L U A_R U^{-1} - U A_R U^{-1} A_L \right) \right] + \int d^4 x Tr \left[ F_R \left( U^{-1} A_L U A_R - A_R U^{-1} A_L U \right) \right] . \]  
\[ (7.15) \]

This particular decomposition is motivated only by the fact that it makes an especially simple starting point in an effort to embed the entirety of \( S_{WZWN}^{gauged} \) within a 4D, \( N = 1 \) supersymmetric action. For example, the terms \( S_{WZWN}^{AF} \) and \( S_{WZWN}^{A^2F} \) must depend linearly on the Yang-Mills field strength supertensor \( W^{(A)}_\alpha \) in a supersymmetric extension of these terms.

An important point to realize regarding (6.4) is that we should have in mind the similarity relations between superfield pull-back quantities and component field
pull-back quantities. For example,

\[(\overline{D}^i \Sigma^j)(\overline{D}^b \Sigma^c)(\partial_{\gamma} \Phi^K)(\partial_{\delta} \Phi^L) \sim \epsilon^{abcd}(\partial_{\delta} \phi^j)(\partial_{\delta} \phi^k)(\partial_{\delta} \phi^l) \]  \quad (7.16)

In order to write the gauged version of the WZNW action, we require lower order pull-back quantities. For these we require ansätze and we propose these as given in the table below.

| Comp. Field. Pull-back | Superfield Pull-back |
|------------------------|----------------------|
| \(\epsilon^{abcd}(\partial_{\delta} \phi^j)(\partial_{\delta} \phi^k)(\partial_{\delta} \phi^l)\) | \((\overline{D}^i \Sigma^j)(\partial_{\alpha} \Phi^K)(\partial_{\alpha} \Phi^L)\) |
| \(\epsilon^{abcd}(\partial_{\delta} \phi^k)\) | \((\partial_{\alpha} \Phi^K)(\partial_{\delta} \phi^l)\) |
| \((\partial_{\delta} \phi^l)\) | \((D_{\alpha} \overline{\Sigma}^L)\) |

Table IV

This table is not meant to be exhaustive. However, these particular choices have some very interesting properties.

Motivated by these choices and the discussion of super p-form geometry in chapter five, we present the following ansätze for the 4D, N = 1 superfield generalizations of the terms in equations (7.10-7.15),

\[S^{MC^3}_{CMN-WZNW} = \int d^4x \, d^2\theta \, d^2\overline{\theta} \left[ g_{(I)JKL}(\Phi, \overline{\Phi}, V) \, V^{(I)}(\overline{D}^i \Sigma^j)(\partial_{\alpha} \Phi^K)(\partial_{\alpha} \Phi^L) + h. c. \right] \quad (7.17)\]

\[S^{MC^2}_{CMN-WZNW} = \int d^4x \, d^2\theta \, d^2\overline{\theta} \left[ g_{(I)(J)KL}(\Phi, \overline{\Phi}, V) \, V^{(I)}(\Gamma^{\alpha}(J)(\partial_{\alpha} \Phi^K)(\partial_{\alpha} \Phi^L) + h. c. \right] \quad (7.18)\]

\[S^{MC^1}_{CMN-WZNW} = \int d^4x \, d^2\theta \, d^2\overline{\theta} \left[ g_{(I)(J)(K)L}(\Phi, \overline{\Phi}, V) \, V^{(I)}(\Gamma^{\alpha}(J)(\Gamma^{\beta}(K)(\partial_{\alpha} \Phi^K)(\partial_{\alpha} \Phi^L) + h. c. \right] \quad (7.19)\]

\[S^{MC^0}_{CMN-WZNW} = \int d^4x \, d^2\theta \, d^2\overline{\theta} \left[ g_{(I)(J)(K)(L)}(\Phi, \overline{\Phi}, V) \, V^{(I)}(\Gamma^{\alpha}(J)(\Gamma^{\beta}(K)(\Gamma^{\gamma}(L)) + h. c. \right] \quad (7.20)\]

\[S^{AF}_{CMN-WZNW} = \int d^4x \, d^2\theta \, d^2\overline{\theta} \left[ g_{(I)(J)KL}(\Phi, \overline{\Phi}, V) \, V^{(I)}W^{\alpha}(J)(\partial_{\alpha} \Phi^K) + h. c. \right] \quad (7.21)\]
\[ S_{CNM-WZNW}^{AF} = \int d^4x \, d^2\theta \, d^2\bar{\theta} \left[ g^{(I)}(J)(K)(\Phi, \bar{\Phi}, V) \, V^{(I)} \, W^{(J)} \Gamma^{(K)} \right. \\
\left. + \text{h. c.} \right], \quad (7.22) \]

where the non-polynomial functions \( g(\Phi, \bar{\Phi}, V) \) are yet to be determined. Let us end this chapter on a cautionary note. We have been guided by the fact that the actions above represents a “minimal choice” which has the property of producing the minimal number of component level terms. In fact, it seems that the terms above maintain the auxiliary-freedom for the gauged CNM-WZNW term (at least in the WZ gauge). There are ambiguities in making these ansätze. Any of the factors involving the chiral and nonminimal multiplets can be replaced by terms where a different scalar multiplet is used (e.g. in (7.17) \( \Sigma \to \Phi \) etc.). However, such replacements are not guaranteed to maintain auxiliary-freedom. This can be seen most vividly in (7.17). Replacing both nonminimal multiplets by chiral multiplets leads to an action that is isomorphic to the NR-WZNW term!

8 Contemplations on Holomorphy and Supersymmetric Yang-Mills Theory

In the past [17], we voiced concern over the singular direction of investigations of supersymmetric phenomenologically interesting models. In particular, the universal use of models wherein matter was solely represented by chiral Wess-Zumino superfields seemed to us a very imprudent and incomplete course to pursue. Although our comments may have been viewed as vague misgivings, some implications of this present work permit us to give a much sharper and clearer discussion of our concerns. These concerns will be articulated in this chapter.

Holomorphy, as we have interpreted it in this work, is a concept that applies to the complete effective action in (2.8). The isometry group of the Kähler manifold suggested a new and intrinsic definition of space-time “vector-like” effective actions where spinors of both space-time chirality are found to be elements of \(*T_p(\mathcal{M})\). Presumably in a fundamental supersymmetric version of QCD, there is no such Kähler geometry, so at first there would seem to be no way that the new definition of a “vector-like” theory might apply. This is not quite true. In a fundamental theory, the role of the isometry group of the Kähler manifold is taken over by the actual supersymmetric Yang-Mills gauge group. If we write the transformation laws of 4D, \(K\)-gauge \( N = 1 \) Yang-Mills supercovariant derivative \( \nabla_A \equiv (\nabla_\alpha, \nabla_{\dot{\alpha}}, \nabla_\theta) \) in the usual
with the usual hermitian parameter superfields $K$, it would seem impossible for the concept of a “holomorphic” Yang-Mills gauge group (analogous to the Kähler manifold isometry group) to arise. Covariantly chiral superfields $\Phi$ certainly transform as

$$
\Phi' = e^{iK} \Phi, \quad \nabla_\alpha \Phi = 0 .
$$

(8.2)

However, the “true” gauge group of supersymmetric Yang-Mills theory is the $\Lambda$-gauge group. With respect to the $\Lambda$-gauge group we define a Yang-Mills covariant $\nabla_\Lambda \equiv (\nabla_\alpha, \overline{D}_{\dot{\alpha}}, \nabla_{\dot{\alpha}})$ and chiral superfields $\Phi$ transform as

$$
\Phi' = e^{i\Lambda} \Phi, \quad \overline{D}_{\dot{\alpha}} \Phi = 0 .
$$

(8.3)

With respect to supersymmetric Yang-Mills theory in superspace, there are three Yang-Mills type fiber bundles over the supermanifold, one associated with each $K$, $\Lambda$ and $\overline{\Lambda}$. For a given Yang-Mills group denoted by $G$, let us denote the three fiber bundles by $\mathcal{F}_K(G)$, $\mathcal{F}_\Lambda(G)$ and $\overline{\mathcal{F}}_\Lambda(G)$ respectively. It is abundantly clear that $\mathcal{F}_\Lambda(G)$ is holomorphic and $\overline{\mathcal{F}}_\Lambda(G)$ is anti-holomorphic.

It has been well known that the $K$-gauge is an artifact, so we should really focus our attention on $\mathcal{F}_\Lambda(G)$ and $\overline{\mathcal{F}}_\Lambda(G)$. Since for the spinors in the chiral and conjugate chiral multiplet their respective transformations yield

$$
(\psi_\alpha)' = (\nabla_\alpha \Phi)' = e^{i\hat{\Lambda}} (\nabla_\alpha \Phi) = e^{i\hat{\Lambda}} \psi_\alpha ,
$$

$$
(\overline{\psi}_{\dot{\alpha}})' = (\nabla_{\dot{\alpha}} \overline{\Phi})' = e^{-i\overline{\Lambda}} (\nabla_{\dot{\alpha}} \overline{\Phi}) = e^{-i\overline{\Lambda}} \overline{\psi}_{\dot{\alpha}} ,
$$

(8.4)

(where $\hat{\Lambda}$ and $\overline{\Lambda}$ denote the $\theta = 0$ part of the superfield) we find

$$
\psi_\alpha \in \mathcal{F}_\Lambda(G) , \quad \overline{\psi}_{\dot{\alpha}} \in \overline{\mathcal{F}}_\Lambda(G) .
$$

(8.5)

Thus, there exists the same type of correlation between the space-time chirality and a holomorphic or anti-holomorphic structure. The only way to break this correlation is to introduce non-minimal multiplets.

Covariant nonminimal superfields $\Sigma$ transform as

$$
\Sigma' = e^{iK} \Sigma , \quad \nabla^2 \Sigma = 0 ,
$$

(8.6)

or with respect to the $\Lambda$-gauge group as

$$
\Sigma' = e^{i\Lambda} \Sigma , \quad \overline{D}^2 \Sigma = 0 .
$$

(8.7)
The physical spinors in the nonminimal multiplets transform as

\[
\begin{align*}
(\zeta_\alpha)' &= (\nabla_\alpha \Sigma)' = e^{i \hat{\Lambda}} (\nabla_\alpha \Sigma) = e^{i \hat{\Lambda}} \zeta_\alpha , \\
(\zeta_\alpha)' &= (\nabla_\alpha \Sigma)' = e^{-i \hat{\Lambda}} (\nabla_\alpha \Sigma) = e^{-i \hat{\Lambda}} \zeta_\alpha ,
\end{align*}
\]  

(8.8)

so that we immediately see

\[
\begin{align*}
\zeta^\alpha \in & \ F_{\Lambda}(G) , \\
\zeta^\alpha \in & \ F_{\Lambda}(G).
\end{align*}
\]  

(8.9)

If these notions are correct, the reader may then wonder, “In what sense is the standard construction, used throughout the literature, vector-like?” From our view the simplest answer to this is that the standard construction is vector-like with respect to \(F_K(G)\). However, the inability to write an auxiliary-free WZNW term with respect to this structure raises our concern that there may be subtle difficulties in such an approach.

Another way to formulate these issues is based on the following argument. Let \(\Psi\) denote a Dirac spinor field. Our discussion indicates that there are two ways in which to construct such an object from 4D, N = 1 superfields. One definition, we call the C-definition (chiral superfield), is given by

\[
\Psi_C(x) \equiv \begin{pmatrix} D_\alpha \Phi^+ \mid \\ \overline{D_\alpha \Phi^-} \end{pmatrix} ,
\]  

(8.10)

with \(\Phi^+ \neq \Phi^-\). If \(\Phi^+ = \Phi^-\), then \((\Psi_C)^* = \sigma^1 \Psi_C\) which implies that \(\Psi_C\) defines a Majorana fermion. The undotted components of a Dirac spinor reside in \(\Phi^+\) while the charge conjugates of the dotted components reside in \(\Phi^-\). The other definition, referred to as the CNM-definition (chiral-nonminimal superfield), is given by

\[
\Psi_{CNM}(x) \equiv \begin{pmatrix} D_\alpha \Phi \mid \\ \overline{D_\alpha \Sigma} \end{pmatrix} .
\]  

(8.11)

In (8.10) and (8.11) it is to be understood that the \(\theta \to 0\) limit must be taken on the superfields. The coupling of these Dirac fields to a U(1) gauge superfield \(V\) (we consider U(1) for the sake of simplicity) is given by the respective Lagrangians

\[
\begin{align*}
\mathcal{L}_C &= \overline{\Phi^+} e^V \Phi^+ + \Phi^- e^{-V} \overline{\Phi^-} , \\
\mathcal{L}_{CNM} &= \overline{\Phi} e^V \Phi - \overline{\Sigma} e^V \Sigma .
\end{align*}
\]  

(8.12)
Now an interesting point is to compare how the U(1) gauge symmetry is realized on the two different definitions for a Dirac field.

\[
(\Psi_C)' = \exp \left[ i \frac{1}{2} (\hat{\Lambda} + \bar{\Lambda}) + i \frac{1}{2} \sigma^3 (\hat{\Lambda} - \bar{\Lambda}) \right] \Psi_C ,
\]

\[
(\Psi_{CNM})' = \exp \left[ i \hat{\Lambda} \right] \Psi_{CNM} .
\] (8.13)

This equation illustrates the point that for Dirac spinors \(\Psi_{CNM}\) or \(\Psi_C\) contained in a 4D, N = 1 supersymmetric theory (outside of the Wess-Zumino gauge) the gauge group is not \(G\) as in component theories, but is either \(G_c\) (the complexification of \(G\)) or \(G_V \otimes G_A\). In a Wess-Zumino gauge \(\hat{\Lambda}(x) = \bar{\Lambda}(x)\) and these two definitions for the U(1) transformation law of a Dirac particle coincide. It is completely clear that the group of gauge transformations only for \(\Psi_{CNM}\) form a holomorphic U(1) group. For \(\Psi_C\), the term dependent on \(i(\hat{\Lambda} - \bar{\Lambda})\) is a \(\gamma^5\)-rotation (\(R\)-symmetry transformation)!

This raises a real possibility of having an auxiliary-field anomaly\(^8\) [33].

### 9 Conclusion

We believe our recent realizations have a significance for a disagreement that occurred some years ago. At that time, we made the first explicit suggestion [27, 28] as to how the string corrections associated with the presence of the Lorentz Chern-Simons form modify the geometry of 10D, N = 1 superspace supergravity. Later a different suggestion [35] BPT-FFP appeared in the literature and there ensued a vigorous disagreement over which approach was “the correct one.” Although most theorists conversant in this matter concluded that the approach of [35] was correct, we remain absolutely convinced that our suggestion of [27] is indeed the correct one. We would like to note some extremely interesting analogies regarding the disagreements of the works of [8] versus [14] and those of [27, 28] versus [35].

Foremost, note that our equation of (2.8) is “spectrum stable.” By this we mean that independent of the order to which we expand in \(\gamma'\), the spectrum of dynamical fields in the action remains unchanged. This condition is violated if we replace the leading term in \(S_{H,D}\) by the BNS and NR actions. To lowest order in \(\gamma'\) only the \(A\) and \(B\) fields propagate and at higher order the \(F\) fields propagate.

Let us now contrast this with the results of [27, 28] versus [35]. In our previous work on the 10D, N = 1 superspace geometry associated with the low-energy effective action of the heterotic strings [27], both at zero order and first order in \(\gamma'\), the

\(^8\)This is more often referred to as the “Konishi anomaly” [34] although the first discussion of this phenomenon in the physics literature was in ref. [33].
spectrum is unchanged. By way of comparison, in the superspace geometry associated with the low-energy effective action of the heterotic strings according to \[35\], the spectrum at zero order in $\gamma'$ is different from that at first order in $\gamma'$. In particular, the approach of \[35\] requires for its mathematical consistency a propagating spin-connection. However, this spin-connection only propagates when the first order $\gamma'$-terms are included. At zero order in $\gamma'$, the BPT-FFP spin-connection has an algebraic equation of motion. This is just like the explicit example we showed by studying the behavior of the $F$-field in either the BNS or NR higher derivative actions.

Thus by analogy, we assert that the work of \[35\] is the 10D, $N = 1$ superspace geometry associated with a higher dimensional generalization of an action in the same class as the BNS or NR actions. This would mean that the mathematical correctness of the BTP-FFP approach (which we previously doubted) is no guarantee of uniqueness! Also by analogy, we assert that our work in reference \[27\] is the higher dimensional generalization of the class of actions we have discussed here as well as in \[14\]. We are well aware of the claims by the proponents of the BTP-FFP approach that there is an “obstruction” \[30\] at second order which precludes the higher extension of our first order results. We now appeal to history. We have always felt that the “obstruction” is really a statement about what assumptions are being made in the attempt to find a solution to the Chern-Simons modified 10D, $N = 1$ superspace geometry. Note this is analogous to what we have found for the auxiliary-free higher derivative 4D, $N = 1$ supersymmetric actions. If we assume that only chiral multiplets describe the matter fields, we are inevitably led to propagating $F$-fields. If we release this assumption (i.e. assign matter to both chiral and nonminimal multiplets) then there exists a mechanism for finding spectrum stable theories. Ten years elapsed between the work of \[27, 28\] and \[14\]. It remains to be seen what is the precise and subtle mechanism that would allow the existence of a spectrum stable 10D, $N = 1$ Lorentz Chern-Simons modified supergeometry that describes the low-energy effective action of the heterotic (and as well type-II) superstrings.\[9\]

Perhaps the most interesting point of our suggestion of an auxiliary-free 4D, $N = 1$ supersymmetric low-energy effective QCD action (and a feature shared by the chiral-nonminimal models in references \[14, 17\]) is that these model provide a new way in which parity non-conservation may be realized. In all models prior to the construction of these, the mechanism for parity breaking was the inequality (either of matter fields, gauge fields or both) of the realization of the left Yang-Mills gauge group versus the right Yang-Mills gauge group on physical fields. The models in our

\[9\] We are hopeful that it will not take another decade to find this mechanism.
previous [14, 17] and present works show that even if the left Yang-Mills gauge group is equal to the right Yang-Mills gauge group on propagating fields, parity can still be broken in some circumstances by simply assigning right-handed spin-1/2 particles and left-handed spin-1/2 particles to different supersymmetry representations (i.e. chiral versus nonminimal multiplets). Looking at the fields in Table I, we see that the auxiliary fields $\rho_\alpha$, $p_\alpha$ and $\beta_\alpha$ transform covariantly under the left gauge group but have no analogs that transform covariantly under the right gauge group. Thus when utilized in a fundamental theory, the $(\Phi, \Sigma)$ Poincaré dual pair implies a broken parity symmetry even though parity is realized on the propagating fields in the multiplets. To our knowledge our present model is the first one where the breaking of parity is required by a theoretical reason (i.e. auxiliary freedom) of the effective 4D, $N = 1$ supersymmetric QCD low-energy action.

The use of our holomorphic auxiliary-free 4D, $N = 1$ supersymmetric low-energy effective QCD action as a model for the real world offers a rather stark trade-off. In the pursuit of the goal of achieving auxiliary-freedom (which is accomplished) we are forced to use a class of models in which the supersymmetric version of the strong interaction breaks parity in a new way. We approach this radical new idea with caution but it certainly gives us interesting new questions to explore. Only time will tell if this notion, like that of supersymmetry itself, is one that is pleasing to Nature.

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Appendix A: Chiral Superfield Maurer-Cartan Forms

We begin by interchanging the definition of left and right Maurer-Cartan forms (from that used in [14]) motivated by the fact that this change simplifies later notation in the gauged WZNW term. Our new definition of left \((L_m^i(\Pi))\) and right \((R_m^i(\Pi))\) Maurer-Cartan forms are given by the equations

\[
U^{-1}\partial_\alpha U = if_{\pi}^{-1}(\partial_\alpha \Pi^m) R_m^i(\Pi) \lambda_i , \quad (\partial_\alpha U)U^{-1} = if_{\pi}^{-1}(\partial_\alpha \Pi^m) L_m^i(\Pi) \lambda_i .
\]  

(A.1)

These definitions allow \(L_m^i(\Pi)\) and \(R_m^i(\Pi)\) to be calculated as power series in \(\Pi^i\) from [13]

\[
R_m^i(\Pi) \equiv (C_2)^{-1} \text{Tr}\left[ T^i \left( \frac{1 - e^{-\Delta}}{\Delta} \right) T_m \right] ,
L_m^i(\Pi) \equiv (C_2)^{-1} \text{Tr}\left[ T^i \left( e^{\Delta} - 1 \right) T_m \right] ,
\]

(A.2)

where \(\Delta T_m \equiv if_{\pi}^{-1}[\Pi, T_m]\), \(\Delta^2 T_m = \Delta \Delta T_m\), etc. and the constant \(C_2\) is determined so that \(L_m^i(0) = R_m^i(0) = \delta_m^i\).

Next to extend these definitions to chiral superfields, we first define group elements by

\[
U(\Phi) \equiv \exp\left[ \frac{\Phi}{f_{\pi} \cos(\gamma_S)} \right] , \quad \Phi \equiv \Phi^I \lambda_I .
\]

(A.3)

Note that since the chiral superfield is complex, the quantity \(U\) lies in the complexification of the group associated with the Lie algebra generated by \(\lambda_I\). Thus, \(U\) is not a unitary matrix. This definition of \(U\) also has the property that when we set all of the fields with the exception of \(\Pi\) to zero, then \(U\) reduces back to the usual representation involving the SU(3) pion multiplet. We define our chiral superfield Maurer-Cartan forms \(R^K_I(\Phi)\) and \(L^K_I(\Phi)\) by

\[
U^{-1}D_\alpha U = \left[ f_{\pi} \cos(\gamma_S) \right]^{-1} \left( D_\alpha \Phi^I \right) R^K_I(\Phi) \lambda_K ,
(D_\alpha U)U^{-1} = \left[ f_{\pi} \cos(\gamma_S) \right]^{-1} \left( D_\alpha \Phi^I \right) L^K_I(\Phi) \lambda_K .
\]

(A.4)

These lead to the same expressions for \(R^K_I(\Phi)\) and \(L^K_I(\Phi)\) as in (A.2) except with the replacements \(\Pi \to \Phi\) and \(\Delta T_1 \to [f_{\pi} \cos(\gamma_S)]^{-1}[\Phi, T_1]\). Since the multiplication of chiral superfields is closed we also observe,

\[
U^{-1}\overline{D}_\dot{\alpha} U = 0 , \quad U \overline{D}_\dot{\alpha} U^{-1} = 0 \rightarrow \overline{D}_\dot{\alpha} R^K_I = \overline{D}_\dot{\alpha} L^K_I = 0 .
\]

(A.5)
Using the real parameter $y$, we again define an extended group element $\hat{U}$ through the relation $\hat{U} = \exp[iy[f_{\pi \cos(\gamma_S)}]^{-1}\Phi]$. This further implies that the Vainberg technique [39] can be used in 4D, $N = 1$ superspace. If we start with the Novikov-Witten observation [10] and write the component action in the form,

$$S_{WZNW} = \int d^4x L_{WZNW} = -iN_C [2 \cdot 5!]^{-1} \int d^4x \int_0^1 dy \text{Tr} \left[ (\hat{U}^{-1} \partial_y \hat{U}) \hat{W}_4 \right],$$

$$\hat{W}_4 = \epsilon^{abcd} (\partial_x \hat{U})^{-1} (\partial_y \hat{U})^{-1} (\partial_z \hat{U})^{-1} (\partial_z \hat{U}).$$

or directly using the elements of the pion octet as

$$S_{WZNW} = \int d^4x \epsilon^{abcd} J_{mnr} (\Phi)(\partial_x \Phi^m)(\partial_y \Phi^n)(\partial_z \Phi^r)(\partial_z \Phi^s),$$

$$J_{mnr}(\Phi) = -N_C [8 \cdot 5! f_{\pi}^5]^{-1} \epsilon^{abcd} f_{abcd}^{\frac{1}{2}} \epsilon^{ijkl} \epsilon^{mnop} T \lambda_k \lambda_l \lambda_m \lambda_n \int_0^1 dy y^4 \int d^4x \hat{L}^F \hat{L}^E \hat{L}^A \hat{L}^B \hat{L}^C \hat{L}^D,$$

with $\hat{L}^m i \equiv L^m i (y \Phi)$ and $f_{abk}$ denoting the structure constants of the group, we can simply make the replacements $U(\Phi) \rightarrow U(\hat{\Phi}), \hat{U}(\Phi) \rightarrow \hat{U}(\Phi), \text{etc.}$ in all the quantities in (A.6) and (A.7). Under this circumstance all such expressions become superfields!

$$S_{WZNW} = -i N_C \int d^4x \epsilon^{abcd} J_{mnr} (\Phi)(\partial_x \Phi^m)(\partial_y \Phi^n)(\partial_z \Phi^r)(\partial_z \Phi^s),$$

$$J_{mnr}(\Phi) = [8 \cdot 5! f_{\pi}^5 \cos^5(\gamma_S)]^{-1} f_{abcd}^{\frac{1}{2}} \epsilon^{ijkl} \epsilon^{mnop} T \lambda_k \lambda_l \lambda_m \lambda_n \int_0^1 dy y^4 \int d^4x \hat{L}^F \hat{L}^E \hat{L}^A \hat{L}^B \hat{L}^C \hat{L}^D,$$

One might think that the action in (A.8) is suitable to act as the 4D, $N = 1$ WZNW term. The only problem with this interpretation is that the leading component of a superfield is not a super invariant and the component level WZNW term is the leading component here. However, the discussion above does show that the holomorphic tensor $J_{ijkl}(\Phi)$ exists as a simple generalization of the component level result.

We can further use this result to fix the normalization of our previous work. Namely, the correct normalization of the CNM-WZNW action described in [14] is

$$S_{WZNW} = i 4N_C \left[ \frac{\cos^5(\gamma_S)}{\sin^5(2\gamma_S)} \right] \int d^4x d^2\theta J_{WZNW}^{ijkl}(\Phi) (D_\alpha \Phi^K)(D_\beta \Phi^L) + \text{h.c.},$$

As noted before, the action of (A.9) also contains the Skyrme term. Since we have developed all the necessary “technology,” it is also simple for us to complete
the discussion of its embedding. In (3.1) we replace every field by the appropriate superfield on the first line and calculate using the chiral superfield Maurer-Cartan forms. A brief calculation reveals that the super Skyrme term takes the form

\[ S_{\text{Skyrme}} = 4 \left[ \cos^4(\gamma_S) \right] \int d^4x \, d^2\theta \, \mathcal{J}_{\text{Skyrme}}^{\text{Skyrme}}(\Phi) \left( \overline{\mathcal{D}}^\gamma \Sigma^I \right) \left( \mathcal{D}^\gamma \Sigma^J \right) (\partial_{\gamma} \Phi^K) (\partial_{\gamma} \Phi^L) \]

\[ + \text{ h.c.} \]  

(A.10)

where \( \mathcal{J}_{\text{Skyrme}} \) expressed in terms of chiral superfield Maurer-Cartan forms is given by

\[ \mathcal{J}_{\text{Skyrme}}^{\text{Skyrme}}(\Phi) = \left( \frac{1}{32 e^2} \right) \left[ f_4^4 \cos^4(\gamma_S) \right]^{-1} f_{MN}^A f_{RS}^A L_I^M L_J^N L_K^R L_L^S . \]  

(A.11)

Thus to include both Skyrme and WZNW terms we write (6.1) where \( \mathcal{J}_{IJKL} \) is identified as

\[ \mathcal{J}_{IJKL}(\Phi) = 4 \left[ \cos^4(\gamma_S) \right] \left[ \mathcal{J}_{\text{Skyrme}}^{\text{Skyrme}}(\Phi) + i N_C \cos(\gamma_S) \mathcal{J}_{\text{WZNW}}^{\text{WZNW}}(\Phi) \right] . \]  

(A.12)

As expected the Skyrme and WZNW terms are seen to be the “real” and “imaginary” part of a single quantity. We put quotes around real and imaginary since the \( \mathcal{J} \) functions are all holomorphic.

One final item of interest is more explicit information on the holomorphic isometry vectors. If we let SU_3 act by right multiplication on \( U(\Phi) \) with the usual SU_3 matrices and SU_3 act by left multiplication on \( U(\Phi) \) with the usual SU_3 matrices, then we may write

\[ \delta U(\Phi) = i \left[ \alpha^{(I)} U(\Phi) \lambda_I - \overline{\alpha}^{(I)} \lambda_I U(\Phi) \right] , \]  

(A.13)

where \( \alpha^{(I)} \) and \( \overline{\alpha}^{(I)} \) are the parameters of the transformations. Using (A.4) these lead to the following two equivalent expressions for the isometry vectors,

\[ \xi^I(\Phi) \equiv i [f_\pi \cos(\gamma_S)] \left\{ \alpha^{(I)} \delta^K_I - (C_2)^{-1} \overline{\alpha}^{(I)} \text{Tr}[U^{-1} \lambda_I U \lambda^K] \right\} (R^{-1})_{K}^{I} , \]

\[ \equiv -i [f_\pi \cos(\gamma_S)] \left\{ \overline{\alpha}^{(I)} \delta^K_I - (C_2)^{-1} \alpha^{(I)} \text{Tr}[U \lambda_I U^{-1} \lambda^K] \right\} (L^{-1})_{K}^{I} . \]  

(A.14)

It can be seen that we also have the identities,

\[ U(\Phi) \lambda_I U(-\Phi) = \left\{ \exp(\Delta) \right\} \lambda_I , \quad U(-\Phi) \lambda_I U(\Phi) = \left\{ \exp(-\Delta) \right\} \lambda_I \]  

(A.15)
Appendix B: 4D, N = 1 Superspace Formulation of Donaldson-Nair-Schiff Models

It was recently proposed that the exists a class of 4D, N = 1 supersymmetric models with some remarkable quantum properties [37]. In this extremely brief appendix, we wish to point out that the nonminimal multiplet seems ideally suited for constructing the supersymmetric extension of DNS models. Before we begin, let us note that the DNS model as described in reference [37] is constructed in Atiyah-Ward space. As noted in a previous work [38], all the normal machinery of 4D, N = 1 superspace can be extended into an Atiyah-Ward superspace. Therefore we will actually give all of our arguments within the context of 4D, N = 1 superspace. To carry out the same constructions in Atiyah-Ward superspace merely amounts to a few minor changes of notation.

There several elements that we need for our construction. First, we introduce the ‘strung-out’ version of supergravity that is obtained as the limit of the heterotic string. This theory has been discussed in complete detail in reference [20] and this allows us to introduce supergravity supercovariant derivatives ($\nabla_\alpha, \nabla_{\bar{\alpha}}, \nabla_a$). As explained in ref. [20], this supergravity supercovariant derivative gauges $R$-symmetry with a composite connection constructed from $G = \exp[\varphi]$ (the field strength of the axion-dilaton multiplet) and its derivatives. We note that the super 2-form $B_{\alpha \beta}$ can be totally constructed in terms of a chiral spinor prepotential [23] that we denote by $\varphi^\alpha$. With this information the steps to constructing a supersymmetric DNS model are clear.

We introduce Poincaré dual pairs $(\Phi^I, \Sigma^J)$ and replace the $\sigma$-model action in (2.3) by

$$S_{DNS}^\sigma = \int d^4x d^2\theta d^2\bar{\theta} \ E^{-1} f(G) \tilde{\Omega}(\Phi, \bar{\Phi}; \Sigma, \bar{\Sigma}) \ , \quad (B.1)$$

where $f(G)$ is a function that must be determined by further considerations. We next need a term that is very similar to a 2D WZNW term. Were we in a 2D theory, twisted chiral multiplets might be used. However, in 4D the alternative is to use the nonminimal multiplets. We can construct this term by first introducing a holomorphic second rank anti-symmetric tensor in the space of the Poincaré dual pairs that we denote by $b_{IJ}(\Phi)$. This quantity is then used to write the following holomorphic action,

$$S_{WZW}^{DNS} = \int d^4x d^2\theta \mathcal{E}^{-1} \varphi^\alpha (\nabla_\alpha \Phi^I) (\nabla_{\bar{\alpha}} \Sigma^J) b_{IJ}(\Phi) \ . \quad (B.2)$$

In this expression we have used the notation $\mathcal{E}^{-1}$ to denote the usual chiral density measure. The chirality of the integrand in this expression depends on some unusual
properties of the “heterotic 4D, N = 1 superspace supergravity derivative.” In particular when acting on a scalar with vanishing $U(1)\mathcal{R}$-charge we find

$$[\nabla_\alpha, \nabla_\beta] \Phi^I = 0,$$

(B.3)

which is the critical condition needed to prove the chirality of the integrand. If we work in ordinary 4D, N = 1 superspace, we must add the hermitian conjugate to this last action. However, in Atiyah-Ward superspace everything is real and so the term above may be used as it stands. However, whether we use this action or its “dotted” analog has an important consequence. The duality or anti-duality of the spacetime 2-form gauge field that appears after reducing this to components is correlated with whether we use (B.2) our its “dotted” analog.

The complete 4D, N = 1 supersymmetric DNS model is just the sum of (B.1) and (B.2). Its construction demonstrates that without the existence of the nonminimal multiplet, there is no way to describe a supersymmetric extension of the DNS model. In other words, the DNS model does not even have a supersymmetric extension if we only utilize chiral multiplets!
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