VOREX PRODUCTION IN A FIRST ORDER PHASE-TRANSITION AT FINITE TEMPERATURE

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ABSTRACT

We simulate the production of vortices in a first order phase transition at finite temperature. The transition is carried out by randomly nucleating critical bubbles and the effects of thermal fluctuations (which could be relevant for vortex production) are represented by randomly nucleating subcritical bubbles. Our results show that the presence of subcritical bubbles suppresses vortices with clear and prominent profiles, though net number of vortices is consistent with theoretical estimates. We also determine the typical speed of vortices arising due to randomness associated with the phase transition to be about 0.5.

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The study of a system undergoing a phase transition can have many important implications; production of topological defects being one of them [1]. An effective theory describing the transition can be constructed by considering the evolution of its order-parameter field (Φ). Above the transition temperature Φ is zero whereas below the transition temperature Φ assumes non-zero vacuum expectation value. For the case of first order phase transition, bubbles of lower temperature phase nucleate within the supercooled (metastable) high temperature phase as the temperature is lowered through the transition temperature. Associated with the production of such a true vacuum bubble there is a gain in the volume energy and a loss in the surface energy. There is thus a critical size for which the bubble formation is energetically favored. These critical bubbles expand and coalesce with one another to fill the space with the lower temperature phase [2]. Though the energetically unfavorable sub-critical bubbles collapse eventually, they survive long enough and may significantly affect the history of the phase transition, especially the process of vortex formation (see [3]).

A numerical simulation of the dynamical production of vortices through bubble collisions has been carried out in [3]. However, the probability of bubble nucleation was chosen a priori there as the intention was to check the theoretical prediction of 1/4 vortices per bubble (for U(1) global strings). Further, the bubbles which were randomly nucleated, were all critical bubbles appropriate for the zero temperature case. In this paper we extend the investigation of [3] by considering the bubbles at finite temperature. For this case we make an estimate of the nucleation rate (along with the pre-exponential factor) and nucleate bubbles with corresponding probabilities. We carry out the simulation of the phase transition for the case when a U(1) global symmetry is spontaneously broken. We consider the nucleation of critical bubbles at finite temperature and include subcritical bubbles as representing the dominant contribution of thermal fluctuations in the process of vortex formation. Subcritical bubbles have been considered as playing a crucial role in the phase transition by Gleiser, Kolb, and Watkins (see [4]). We would like to mention that we do not attempt to incorporate all possible effects of thermal fluctuations. Such effects could very well affect our results on the estimation of the random speeds of the vortices. However, we believe that an important class of thermal fluctuations is represented by the subcritical bubbles, which we do incorporate, especially from the point of view of estimating the vortex formation probability.

In this work we consider a 2+1 dimensional case and adopt the following Lagrangian:
\begin{align}
L &= \frac{1}{2} \partial_{\mu} \Phi^{\dagger} \partial^{\mu} \Phi - V(\Phi) \\
V(\phi) &= \frac{1}{2} m^2 \phi^2 - \frac{1}{3} \delta \phi^3 + \frac{1}{4} \lambda \phi^4
\end{align}

where \( \Phi = \phi e^{i\theta} \) is a complex scalar field. The coefficients \( m, \delta, \) and \( \lambda \) in the effective potential \( V(\phi) \) are assumed to be temperature dependent renormalized quantities. Depending on the values of these parameters, the potential has minima at

\[ \phi = 0 \quad \text{and at} \quad \phi = \eta = \frac{1}{2\lambda} \left\{ \delta + (\delta^2 - 4\lambda m^2)^{1/2} \right\} \]

If \( V(\eta) < V(0) \), true ground state for the system is given by \( \phi = \eta \) and the global \( U(1) \) symmetry in (1) is spontaneously broken. Topological defects in the form of global vortices appear because of loops with non-trivial winding number in the \( U(1) \) vacuum manifold. If the system is supercooled and still remains in the false vacuum \( \phi = 0 \) below the transition temperature, it tunnels through the barrier to the true vacuum. At zero temperature, the tunneling probability can be calculated by finding the bounce solution which is a solution of three dimensional Euclidean equations of motion [2]. However, at finite temperature, the theory becomes effectively 1+1 dimensional if the temperature is sufficiently high and the tunneling probability is governed by the solutions of two dimensional Euclidean equations of motion [5].

The tunneling probability per unit volume (area) per unit time in the high temperature approximation is given by [5] (we use \( \hbar = c = 1 \) in this work)

\[ \Gamma = A e^{-S_2(\phi)/T} \]

where \( S_2(\phi) \) is the two dimensional Euclidean action for a field configuration that satisfies the classical Euclidean equations of motion. The dominant contribution to \( \Gamma \) comes from the least action \( O(2) \) symmetric configuration which is a solution of the following equation.

\[ \frac{d^2 \phi}{dr^2} + \frac{1}{r} \frac{d \phi}{dr} - V'(\phi) = 0 \]

where \( r \equiv r_E = \sqrt{x^2 + t^2_E} \), subscript \( E \) denoting the coordinates in the Euclidean space.
The boundary condition imposed on $\phi$ are

$$
\begin{align*}
\phi &= 0 \quad r \to \infty \\
\frac{d\phi}{dr} &= 0 \quad r = 0
\end{align*}
$$

(6)

Bounce solution of Eq.(5) can be analytically obtained in the ‘thin wall’ limit where

$$
\epsilon = V(0) - V(\eta)
$$

(7)
is much smaller than the barrier height. However, such bubbles are large and not suitable for numerical simulation. We therefore work with thick wall bubbles and choose parameters such that the bubble size is small. The bubble profile then has to be obtained by numerically solving Eq.(5).

We choose following parameters for $V(\phi)$

$$
m^2 = 30, \quad \delta = 26.0, \quad \lambda = 4
$$

(8)

The choice of parameters is governed by the requirements that the bubble size as well as its action be small for appropriate values of the temperature. The condition for high temperature approximation to be valid is that $T >> r_0^{-1}$, where $r_0$ is the radius of the critical bubble in 3 dimensional Euclidean space. From now on we will use the Higgs mass $m_H$ (= 8.37 for above choice of parameters) to define our mass scale. The value of temperature we choose is $T = 0.6$ in these units. For the above choice of parameters, $r_0^{-1}$ is found to be about 0.14 which justifies our use of high temperature approximation.

The solution of Eq. (5) is a bubble which remains static when evolved by the classical equations of motion in the Minkowski space. Expanding bubbles are the ones which are somewhat larger than this bubble and we construct such a critical bubble by first choosing $\delta = 25.9$ and finding the corresponding static bubble. This bubble when evolved by the equations of motion with $\delta = 26.0$ becomes an expanding bubble. Similarly, the subcritical bubble is found by finding the static bubble with $\delta = 26.1$ which collapses when evolved with $\delta = 26.0$.

As we have mentioned earlier, in the high temperature approximation our theory effectively becomes two dimensional. For a theory with one real scalar field in two Euclidean
dimensions the pre-exponential factor arising in the nucleation rate of critical bubbles has been analytically calculated by Voloshin [6]. We can therefore use results of [6] for our case as long as we work within high temperature approximation. The pre-exponential factor obtained from [6] for our case becomes

$$A = \frac{\epsilon}{2\pi T}$$  \hspace{1cm} (9)

It is important to note here that the results of [6] were for a single real scalar field and one of the crucial ingredients used in [6] for calculating the pre-exponential factor was the fact that for a bounce solution the only light modes contributing to the determinant of fluctuations were the deformations of the bubble perimeter. In the present case this is no longer true due to the presence of the Goldstone boson which then also has to be accounted for in the calculation of the determinant. We will, however, not worry about this in the present paper for the following reason. The nucleation rate decides how frequent the bubble production is, and for our case with fixed lattice size a moderate change in the nucleation rate simply amounts to a change in the total duration of time in which all the bubbles are nucleated. Thus as long as the nucleation rate is not changed by many orders of magnitude the net effect is going to be a moderate change in the relative sizes of various bubbles as they coalesce (since the time available to various bubbles for expanding before coalescing will change). Of course, for large enough change in the nucleation rate, even the net number of bubbles produced can change which can then significantly effect our results of estimating the probability of vortex formation. We will assume in this paper that the inclusion of Goldstone bosons does not change the nucleation rate by many orders of magnitude. We are investigating the corrections induced by the Goldstone boson in the pre-exponential factor of the nucleation rate.

For the choice of parameters in Eq.(8), a plot of the potential is shown in Fig. 1. We have added a constant to $V(\phi)$ while plotting to make $V(\eta) = 0$. The static bubble (solution of Eq. (5)) has an outer radius of about 12.0 and is shown by the solid line in Fig.2. The expanding critical bubble has radius $\approx 12.1$ (shown by the dashed curve in Fig.2) while the subcritical bubble has radius $\approx 11.9$ (dotted curve in Fig.2). The radius of a bubble is determined by the distance from the bubble center where $\phi$ is very small (appropriate to the lattice cutoff). On the other hand the radius $r_0$ of the critical three dimensional bubble was determined by the distance at which $\phi$ drops significantly (to about $1/e$ of the value of $\phi$ at the center of the bubble). The values of the nucleation
rate \ \Gamma \ \text{(per unit volume per unit time)} \ \text{for the static, critical and subcritical bubbles are respectively,} \ 1.98 \times 10^{-4}, \ 1.67 \times 10^{-4} \ \text{and} \ 2.32 \times 10^{-4} \ \text{(corresponding to temperature} \ T = 0.6).

In our simulation critical bubbles represent the dominant class of expanding bubbles, and subcritical bubbles represent the dominant class of bubbles which collapse. These bubbles are nucleated at any time during the simulation with a uniform probability per unit time per unit volume (governed by the respective nucleation rates as given above). The location of the centers of the bubbles are also chosen at random. If \ \phi \ \text{is very small in the region of interest (so that there is no bubble there or in the immediate neighborhood), the false vacuum is replaced by the bubble profile with a randomly chosen value of the Higgs phase. Figs 3a-3b show nucleation of few scattered bubbles. In plotting the Higgs phase, the length of the arrows are chosen to be large for large} \ \phi \ \text{and the direction of the arrow denotes} \ \theta. \ \text{Arrows are not plotted when} \ \phi \ \text{is very small.}

After nucleation, bubbles are evolved by time dependent equations of motion in the Minkowski space.

\[
\Box \Phi_i = -\frac{\partial V(\Phi)}{\partial \Phi_i}, \quad i = 1, 2. \tag{10}
\]

with \ \frac{\partial \Phi}{\partial t} = 0 \ \text{at} \ t = 0. \ \text{Here} \ \Phi = \Phi_1 + i\Phi_2 \ \text{and} \ \Box \ \text{is the d’Alembertian operator in} \ 2+1 \ \text{dimensions. As we have noted before, the critical bubbles expand and coalesce with other bubbles while the subcritical bubbles eventually collapse though they may oscillate for a while.}

The bubble evolution was numerically implemented by a stabilized leapfrog algorithm of second order accuracy both in space and in time with the d’Alembertian operator approximated by a diamond shaped grid [3]. Lattice spacing in the spatial directions was chosen to be \ \Delta x = .084 \ \text{and the spacing in the time direction was chosen to be} \ \Delta t = .059. \ \text{This satisfies the criterion for the stability of the numerical evolution that the Courant number} \ C \equiv \frac{\Delta t}{\Delta x} \leq \frac{1}{\sqrt{d}}, \ \text{where} \ d \ \text{is the number of spatial dimensions (2 in our case). The simulations were performed on the Cray-2 supercomputer at the Minnesota Supercomputer Institute.}

Our simulation resulted in the production of 25 critical and 43 subcritical bubbles, the ratio of their numbers being about 0.58. The expected ratio form their nucleation
rates is 0.72. The last critical bubble was nucleated at \( t = 19.97 \) while the last subcritical bubble was nucleated at \( t = 41.05 \). We performed the simulation up to \( t = 67.35 \). Energy was well conserved in the early stages (to within about 4% up to \( t \approx 47 \)). During the late stages of evolution and well after the defects were formed, some of the expanding bubbles partly escaped from the lattice leading to the leakage of energy out of the region.

A snapshot of the process of bubble expansion and coalescence is illustrated in Fig. 4a and in Fig. 4b. A plot of the Higgs phase at the close of the simulation is shown in Fig. 4c. Almost all the bubbles have collided by this time with most of the region consisting of true vacuum and of vortices.

Vortices are located by looking for loops along which the phase of the Higgs field has a non-zero winding number. We would like to emphasize that there were no defects present at the start of the simulation. Vortices and antivortices appeared in the course of bubble evolution if the randomly chosen Higgs phases of the colliding bubbles trapped a nonzero winding number. To illustrate these points in more detail, let us concentrate on a pair of vortices shown in Fig. 5a which is a plot of \( \eta - \phi \). From the phase plot in Fig. 5b one can see that one of them is a vortex near the coordinates \((x = 244, y = 119)\) and the other one is an antivortex near \((x = 245, y = 98)\). These defects generally pick up large speeds from unbalanced momenta of colliding bubble walls at the time of formation or due to asymmetric distribution of the Higgs phase, see [3]. Further, during their evolution they may be subjected to random changes in momentum due to the energy emitted by the decaying portions of walls. We plot the positions of the vortex and the antivortex of Figs. 5a-5b respectively in Figs. 6a-6b showing these effects. The average speed for these vortices is 0.55 (in c=1 units).

Let us look more closely at the vortex-antivortex pair shown in Figs. 5a-5b. The vortex and antivortex should move closer due to attractive forces and eventually annihilate each other. However, we can see from Figs. 6a-6b that these objects are actually moving away from each other. This is an interesting situation where the random velocities of the vortex and antivortex are able to dominate over the attractive forces between them (which will be small due to large separation between the vortex and the antivortex).

Not all of the defects produced in our simulation are as clear as the ones depicted in Fig. 5. Sometimes they are close to other defects and are difficult to resolve. In other situations they are close to the false vacuum and we discard them on the grounds of stability. We find a total of 16 defects (two are connected to walls) formed in our simulation over a
region of size $335 \times 335$. As the number of bubbles (critical and subcritical) is 68, this gives a probability of vortex formation about 0.23 in approximate agreement with the theoretical predictions of the vortex formation probability (1/4 per correlation domain). We would like to mention here that the number of vortices observed per bubble in [3] was larger than the theoretical prediction in the early stages. [Though, in [3], several vortex-antivortex pairs annihilated later leaving the final numbers consistent with the theoretical predictions.] However there is a crucial difference between the simulation of [3] and the present work. In [3] all the bubbles were critical bubbles whereas in the present simulation more than half of the bubbles are subcritical bubbles which affect the vortex formation in a very different manner. For example, a subcritical bubble either leads to only a short lived vortex which eventually escapes into the false vacuum [3], or it leads to the formation of a vortex-antivortex pair which annihilate each other, see [7]. We mention that recently an experimental investigation of the string formation probability has been carried out in nematic liquid crystals with results in good agreement with the theoretical predictions, see [8].

We conclude by emphasizing the main aspects of our work. Here we estimate the nucleation rate for a given choice of parameters and nucleate critical as well as subcritical bubbles with their respective nucleation probabilities. The result is that most of the bubbles are subcritical bubbles which suppress direct collisions between critical bubbles. Therefore very few of the vortices (only 3-4) we see here are reasonably isolated and have clear prominent profiles. These should be the ones produced by the collisions of critical bubbles only. Most of the vortices are vortex-antivortex pairs with very small separations and seem to be resulting from the collision of subcritical bubbles with the critical ones. Counting all such vortices, the vortex production is in agreement with the theoretical estimates. However, the presence of subcritical bubbles seems to suppress the production of prominent and clearly separated vortices (for 25 critical bubbles we get only about 3-4 prominent vortices). One may expect that these are the only ones which will eventually survive and the vortex-antivortex pairs which are almost overlapping will all annihilate. Hence the defect production will be suppressed due to the presence of subcritical bubbles. We also estimate typical speed of such objects imparted by various random processes operating during a phase transition to be about .5 which is in agreement with the results in earlier investigation [3].

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FIGURE CAPTION

Figure 1: Plot of the potential. $V(\Phi)$ is plotted in the units of $V(\eta)$ and $\phi$ in the units of $\eta$, $\eta$ being the vacuum expectation value of $\Phi$. A constant has been added to the potential so that $V(\eta) = 0$.

Figure 2: Profiles of bubbles. The solid curve shows the static bubble. The critical bubble is slightly larger and is shown by the dashed curve on the outside of the solid curve. The subcritical bubble is slightly smaller than the static bubble and is shown by the dotted curve on the inside of the solid curve. The length scale is in the Higgs mass units.

Figure 3: (a) shows the profiles of few scattered bubbles at $t = 2.93$. The bubbles are randomly nucleated. (b) shows the Higgs phase plot for the bubbles at the same stage.

Figure 4: (a) Plot of $\eta - \phi$ showing the coalescence of bubbles at $t = 41.00$. (b) Higgs phase plot at the same stage. (c) Higgs phase plot at the close of the simulation.

Figure 5: (a) Profiles of a vortex - antivortex pair. Note the deformed configurations of $\phi$. (b) Higgs phase plot for the same pair.

Figure 6: (a) Trajectory of the vortex in Fig. 5 form $t = 52.71$ to $t = 67.35$. Arrowheads show the locations of the vortex at successive time steps $\Delta t = 2.93$. (b) Same for the antivortex of Fig. 5. Random motion of the pair is clear as is their motion away from each other on the average. Average speed of all these vortices is $\simeq 0.55c$. 
