Two-dimensional ternary locally resonant phononic crystals with a comblike coating

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Abstract
Two-dimensional ternary locally resonant phononic crystals can be used for vibration control and noise insulation in the low (even audible) frequency range. They traditionally consist of cylindrical scatterers with uniform coatings in their exterior. An alternative coating profile with a comblike profile is proposed and investigated in this paper. The band structures are calculated using the finite element method. We find that a complete bandgap can be induced at a significantly low frequency, the wavelength of which is more than 20 times the lattice constant. The mechanism for such a change is suggested using an equivalent spring–mass model and analyzing the eigenmodes at the bandgap edges. Numerical results and the results predicted by the spring–mass model are coherent.

Keywords: locally resonant phononic crystals, bandgap, comblike coating, equivalent spring-mass models

(Some figures may appear in colour only in the online journal)

1. Introduction
A growing interest has been focused on periodic, elastic structures, called phononic crystals (PCs) [1]. PCs, composed of two or more different types of materials with different mechanical properties and mass densities, may exhibit bandgaps, within which the propagation of elastic or acoustic waves is prohibited. Two mechanisms may give rise to bandgaps; one is the Bragg scattering, and the other is local resonance [2]. The latter can induce a complete bandgap at a frequency which is lower than that of the former by two orders of magnitude. Therefore, locally resonant PCs may have potential applications in the control of noise or vibration at low frequencies [3].

So far, various kinds of locally resonant PCs have been investigated [2–5], and some equivalent analytical models have been developed to evaluate the resonant frequencies or the lowest bandgap [6–8]. Wang et al [7] examined the bandgap edge modes of ternary locally resonant PCs, which are composed of lead spheres (solid cores) coated with silicon rubber (soft coating) and embedded in hard epoxy (matrix). At the lower edge of the bandgap, the core oscillates as a rigid sphere and the coating acts as springs, while at the upper edge of the bandgap, the core and the matrix oscillate in a reverse phase. Based on the analysis of the edge modes, they developed equivalent spring–mass models to predict the bandgap edges. In their model, only the tensile spring was considered. The tensile spring constant was calculated from the tensile elastic constant of the coating, which was approximated as a plane layer [6] between the cylinder and the matrix. Their results showed that a bandgap at a much lower frequency might be obtained by decreasing the modulus of the coating with the decrease of the bandgap width. However, it is not easy to find a material with a very low modulus. Another option is to decrease the area or volume of the coating by introducing holes.

In this paper, we propose a two-dimensional PC with a comblike coating by breaking the uniform coating into N discrete pieces, as shown in figure 1(a). The sectorial pieces with the central angle of $2\Delta(=\pi/N)$ are equispaced in azimuth and oriented in the radial direction. Each piece (figure 1(c)) can be regarded as a spring in vibration, where $\theta_i$ is the angle formed by the centre line of the $i$th piece and the vibration direction of the core and is given by $\theta_i = 2\pi(i - 1)/N$. The objective is to introduce free surfaces on every sectorial piece and to reduce the effective modulus of the springs, which will lead to a lower resonant frequency.
Table 1. Elastic parameters of the materials.

| Materials     | Core        | Coating    | Matrix     |
|---------------|-------------|------------|------------|
| Mass density, $\rho$ (kg m$^{-3}$) | 8950        | 1020       | 1200       |
| Young’s modulus, $E$ (Pa)         | $2.1 \times 10^{11}$ | $1 \times 10^{5}$ | $3.5 \times 10^{7}$ |
| Poisson’s ratio, $\nu$           | 0.29        | 0.47       | 0.49       |

2. Numerical results

The band structures and the edge modes at the bandgap edges of the proposed locally resonant PC with a comblike coating are calculated using the finite element software COMSOL. The detailed process can be found in [9]. The proposed ternary locally resonant PC is composed of cylindrical metal cores coated with rubber and embedded in the polymer matrix in a square lattice. The material parameters are listed in table 1. The inner and outer radii of the coating are $r_1$ and $r_2$, respectively, and the lattice constant is $a$. The band structures for the PC with a coating of 16 pieces are shown in figure 2(a). For comparison, the results for the PC with a uniform coating are presented in figure 2(b). Here, the reduced frequency $\Omega = \omega a/(2\pi c_t)$ (with $c_t$ being the transverse wave velocity of the matrix) is used. It is noted that a complete bandgap between the third and fourth bands appears in the frequency range of $0.0282 < \Omega < 0.0572$ for the comblike coating, and $0.0714 < \Omega < 0.130$ for the uniform coating. By introducing the comblike coating, the bandgap is notably lowered, as expected. The wavelength of the wave inside the bandgap is more than 20 times the lattice constant.

The associated edge modes of the amplitude of the displacements are shown in figure 3. At the lower edge of the bandgap (figures 3(a) and (c)), the steel core oscillates holistically, while the matrix remains still. The coating acts as springs linking the core and the matrix. At the upper edge of the bandgap (figures 3(b) and (d)), however, the matrix also oscillates, in a reverse phase to the vibration of the steel core. Thus, similar to that described in [7], equivalent ‘mass–spring–fixture’ and ‘mass–spring–mass’ models can be developed to represent these vibration modes, which will be discussed in detail in the next section. For the comblike coating, each piece plays different roles in vibration: some act as tensile springs, some as shear springs, and the rest as a combination of both. The shear deformation of the comblike coating reduces the effective stiffness, and thus results in a lower bandgap.

The variation of the bandgap edges with the piece number ($N$) is shown in figure 4. The results for the PC with a uniform coating ($N = 0$) are also marked in the figure. With the increase of $N$ in figure 4(a) ($r_1/a = 0.27$), the lower edge of the bandgap first decreases quickly and then tends to a fixed value ($\Omega = 0.0258$). The upper edge of the bandgap first decreases sharply, then increases, and finally decreases in the same manner as the lower edge of the bandgap. The bandgap nearly disappears when $N = 2$, then increases again at $N = 3$ and finally remains almost constant ($0.0258 < \Omega < 0.0523$) with $N$ increasing. To understand the narrowing or disappearing of the bandgap at $N = 2$, we illustrate the eigenmodes of the amplitude of the displacements for PC for the coating of two pieces in figure 5. It is noted that the lower edge mode is similar to that of the uniform coating, while at the upper edge of the bandgap, the steel core and the matrix oscillate in a reverse phase along the direction without the coating pieces. So in this case, different from that of the lower edge, the coating acts as shear springs with low effective stiffness.
Figure 2. Band structures of the PC with (a) a comblike coating of 16 pieces and (b) a uniform coating in a square lattice, where \( r_1/a = 0.27 \) and \( r_2/a = 0.4 \).

Figure 3. Vibration modes of the amplitude of the displacements at the bandgap edges marked in figure 2. Panels (a)–(d) correspond to points S1–S4, respectively.

Figure 4. Variations of bandgap edges with the number of the comblike coating for \( r_2/a = 0.4 \) and \( r_1/a = 0.27 \) (a) or \( r_1/a = 0.33 \) (b). The scattered symbols represent the numerical results, and the solid and dashed lines represent the analytical results. The two dotted lines represent the results for PC with a uniform coating, the modulus of which is half that of rubber. \( \square (\bigtriangleup) \) — Lower edge for a square (triangular) lattice; \( \blacksquare (\blacktriangleup) \) — Upper edge for a square (triangular) lattice. \( \cdots (\cdots) \) — Lower edge evaluated using \( E_r(\theta)C_{11} \); \( \cdots (\cdots) \) — Upper edge evaluated using \( E_r(\theta)C_{11} \).

which reduces the eigenfrequency of the upper edge of the bandgap significantly, and hence the width of the bandgap. For comparison, we calculate the band structure of the PC with a ‘virtual’ uniform coating, of which the elastic modulus is half that of rubber (because the total area of the comblike coating is half that of the uniform coating). The two dotted lines in figure 4 mark the lower/upper edges of the bandgap, and the values are all higher than those for the comblike coating when \( N \geq 3 \). So the descent of the bandgap edges when introducing the comblike coating is not only due to the decrease in the area of the coating, but also a consequence of the shear deformation of the comblike coating. A similar trend in the changes of the bandgap edges with \( N \) is shown in the case with larger steel cores \( (r_1/a = 0.33) \), shown in figure 4(b). The most striking modification occurs when \( N = 2 \), where a small complete bandgap exists. A thinner...
coating exhibits a larger effective stiffness and thus results in the opening of this small bandgap. When \( N \geq 3 \), the complete bandgap tends to appear in the fixed frequency range of \( 0.0346 < \Omega < 0.0846 \).

3. Equivalent spring–mass models

The observations in the preceding section cannot be explained simply on the basis of the original locally resonant model [2] and the effective mass model [7] because of (a) the topological difference of the coating layer, and (b) the introduction of free surfaces. In order to understand the mechanism of the bandgap generation and estimate the bandgap edges, we will propose an equivalent mass–spring model by taking into account both tensile and shear deformation, as well as the effect of the free surfaces.

We first consider the PC with a uniform coating. Following the basic idea described in [7], there exists a standing point that is immovable in the corresponding upper edge mode. Therefore, the mass of the partial coating, which is mainly compressed or stretched, was divided into two parts and added to the mass of the two oscillators. However, in this paper, the mass of the whole coating is divided into two parts; and thus the effective masses of the two oscillators are, respectively, represented by

\[
m_1 = m_{\text{core}} + \alpha m_{\text{coating}} / (1 + \alpha)
\]

and

\[
m_2 = m_{\text{matrix}} + m_{\text{coating}} / (1 + \alpha),
\]

where \( \alpha = m_2 / m_1 = (m_{\text{matrix}} + m_{\text{coating}}) / (m_{\text{core}} + m_{\text{coating}}) \).

These equations are indeed a modification of equation (10) in [7].

The coating is considered as a sum of many slenders (with the central angle \( d\theta \)) along the \( \theta \)-direction, as shown in figure 1(b), where \( \theta \) is the angle between each slender bar and the direction of the wave propagation. Unlike in [7], each slender bar is regarded as a tiny spring with both tensile and shear deformation. The effective tensile stiffness of a slender bar is

\[
dK_t = C_{11} \frac{r_1 d\theta}{r_2 - r_1} \cos^2 \theta,
\]

and the effective shear stiffness is

\[
dK_s = C_{44} \frac{r_1 d\theta}{r_2 - r_1} \sin^2 \theta,
\]

where \( C_{11} = E(1 - \nu) / ((1 + \nu)(1 - 2\nu)) \) and \( C_{44} = E / (2(1 + \nu)) \) are the elastic constants for the isotropic elastic coating with \( E \) and \( \nu \) being the Young’s modulus and Poisson’s ratio. The total effective stiffness of the entire coating may then be obtained by considering the contributions of all these tiny springs. The result is

\[
\hat{K} = \int_0^{2\pi} (dK_t + dK_s) = \frac{\pi r_1 (C_{11} + C_{44})}{r_2 - r_1},
\]

where the shear constant \( C_{44} \) represents the contribution of the shear deformation of the coating. Thus, the eigenfrequencies can be obtained by

\[
\omega_1 = \sqrt{\frac{\hat{K}}{m_1}},
\]

for the lower edge of the bandgap, and

\[
\omega_2 = \sqrt{\frac{\hat{K}}{m_1} + \frac{\hat{K}}{m_2}},
\]

for the upper edge of the bandgap.

The above proposed model is different from the one developed in [7] as it excludes the shear deformation of the coating. The results predicted by these two models are listed in table 2 with the numbers in the brackets being the relative errors. Comparison of the numerical results from FEM and predicted results from these two models show that both models can give satisfactory predicted results for both the lower and upper edges of the bandgap with small relative errors. The small mismatch between these two predicted results is mainly due to the fact that \( C_{11} \gg C_{44} \). That is to say, the two models give very similar results only in this particular case.

The above proposed model is developed for the PC with a perfect coating. Can it therefore be used for the PC with a comblike coating? To answer this question, we applied it directly to evaluate the lowest bandgap for the PC with a comblike coating. In this case, equation (5) should be rewritten as

\[
\hat{K} = \sum_{i=1}^{N} \int_{\theta_i - \Delta}^{\theta_i + \Delta} (dK_t + dK_s).
\]

Particular attention should be paid to the evaluation of the upper edge for \( N = 2 \), in which case the vibration mode is shown in figure 5(b), and thus \( \theta_i = \pi(i - 1) + \pi / 2 \). Evaluation of equation (8) with substantiation of equations (3) and (4) yields

\[
\hat{K} = \frac{\pi r_1 (C_{11} + C_{44})}{2(r_2 - r_1)},
\]

for \( N \neq 2 \), and

\[
\hat{K} = \begin{cases} 
\frac{1}{r_2 - r_1} \left[ C_{11} + C_{44} \pi - (C_{11} - C_{44}) \right] & \text{upper edge of the bandgap} \\
\frac{1}{r_2 - r_1} \left[ C_{11} + C_{44} \pi + (C_{11} - C_{44}) \right] & \text{lower edge of the bandgap}
\end{cases}
\]
for \( N = 2 \). The difference between the lower and upper edges for \( N = 2 \) is directly induced by the different roles of the coating in the vibration mode. It is noted that \( \tilde{K} \), and therefore the eigenfrequencies, are independent of \( \bar{N} \) except when \( N = 2 \). The predicted results from this model are shown as the thin-dashed and thin-solid lines in figure 4, which are unfortunately higher than the FEM results for the comblike coating.

We note that the area of the free surfaces increases as \( N \) increases. Therefore, its influence on the total effective stiffness \( \tilde{K} \) might be the key to understanding the variation of bandgap edges with \( N \). It is known that the normal stress is zero on any free surface. Therefore, a thin 'surface layer' near the free surface is in the plane-stress state in each piece; while the central part is close to the plane-strain state. However, in the above model, the tensile stiffness \( (K_i) \) of a slender bar is calculated through \( C_{11} \) (see equation (5)) by assuming the plane-strain state in the entire piece. This leads to the overestimation of the tensile spring constant of the comblike coating and thus yields the higher eigenfrequencies of the bandgap edges. In fact, for a uniform strain field along the \( r \)-direction, the stress in each piece is inhomogeneous along the \( \theta \)-direction. Therefore, this should be a two-dimensional problem. For simplicity, we will replace it with a one-dimensional problem. In this sense, an improved one-dimensional model will be developed to evaluate the total effective stiffness of the coating.

The effective shear stiffness, immune to the free surface, is still given by equation (4). We will develop an improved model to calculate the effective tensile spring stiffness. For simplicity, we model each sectorial piece (figure 1(c)) as a strip infinitely long in the \( z \)-direction (\( \varepsilon_z \equiv 0 \)) with a \( l \times 2h \) rectangular cross-section, as shown in figure 6. To determine the geometry map from a sectorial piece to a rectangular strip, we consider the following facts: the comblike coating tends to be a perfect coating when \( \Delta \to \pi \), in which case the whole coating, and therefore the rectangular strip, should be in the plane-strain state [6]. That is to say, \( h \to \infty \) when \( \Delta \to \pi \). On the other hand, when \( \Delta \to 0 \), each sectorial piece is very thin, in which case we should have \( h \sim O(r_1\Delta) \). Here, we suggest the following relation between \( (r_1, \Delta) \) and \((l, h)\):

\[
l = (r_2 - r_1), \quad h = r_1[(1 - \Delta/\pi)^{-2} - 1],
\]

which satisfies the above requirements. Then, the deformation of the sectorial piece in the \( r \)-direction can be simplified as one-dimensional tension or compression of the rectangular strip along the \( x \)-axis.

In order to describe the inhomogeneous distribution of the stress, \( \sigma_x(y) \), under a uniform strain in the \( x \)-direction, \( \varepsilon_x \), we assume that the Young’s modulus varies as a function of \( y \), which is denoted by \( E_r(y) \) and sketched in figure 6. As mentioned previously, the free surfaces \( (y = \pm h) \) are in the plane-stress state, i.e. \( \sigma_x = \tau_{yx} = \tau_{yz} = 0 \), which also implies that \( \tau_{zy} = \tau_{yx} = 0 \). The deformation along the \( z \)-direction is obviously uniform; therefore we have \( \tau_{xc} = \tau_{xc} = 0 \). Thus, the stress-strain relation near the free surface is

\[
\sigma_x = E'\varepsilon_x, \quad \text{(at } y = \pm h),
\]

where \( E' = E_r(\pm h) = E/(1 - \nu^2) \). The central part far from the free surface is close to the plane-strain state. Therefore, \( E_r(0) \) should be between \( E' \) and \( C_{11} \). If the strip is very thin \((h/l \to 0)\), the whole strip is in the plane-strain state, and thus \( E_r(y) \equiv E' \). If the strip is very thick \((h/l \to \infty)\), the central part of the whole strip is nearly in the plane-strain state with the stress-strain relation given by

\[
\sigma_x = C_1\varepsilon_x.
\]

These facts imply that \( E_r(y) \) should satisfy the following constraint conditions:

\[
\begin{align*}
E_r(\pm h) &= E', \\
E_r(y) &= E' \quad \text{when } h/l \to 0, \\
E_r(y_0) &= C_{11} \quad \text{when } h/l \to \infty,
\end{align*}
\]

where \( y_0 \) is an arbitrary finite value. Based on the above analysis, we interpolate \( E_r(\theta) \) by

\[
E_r(\theta) = E' + (C_{11} - E')\left(1 - |\frac{\theta - \theta_i}{\pi}|^2 - 1\right)
\times\left(1 - e^{-r_i(1-\Delta/\pi)^{-2}-1}/l\right)
\]

in the cylindrical coordinate.

To verify the rationality of the proposed model, we calculate the strain energy of two configurations. One is a sectorial piece (figure 1(c)) subjected to an elongation along the \( r \)-direction. The strain energy is calculated by FEM. The other is the corresponding rectangular strip under the same elongation along the \( x \)-axis. The strain energy is calculated with equation (15). The results for sectorial pieces with different central angles and those for their corresponding rectangular strips are illustrated in figure 7. An agreement is found between these two configurations, especially when the coating has more pieces. Finally, the associated effective tensile stiffness of each tiny spring is computed by

\[
dK'_r = E_r(\theta)\frac{r_1d\theta}{r_2 - r_1}\cos^2\theta.
\]

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Table 2. Normalized bandgap edges \((\Omega = \omega a/(2\pi c_0))\) predicted by equivalent models for PC with a uniform coating under different geometrical parameters.

| Geometry       | \( r_1/a = 0.27 \) and \( r_2/a = 0.4 \) | \( r_1/a = 0.33 \) and \( r_2/a = 0.4 \) |
|----------------|------------------------------------------|------------------------------------------|
| Bandgap edges  | Lower edge                                | Upper edge                                |
| Present model (equation (5)) | 0.0705 (1.3%) | 0.134 (3.1%) |
| FEM results    | 0.0905 (1.6%) | 0.209 (1.4%) |
| Model in [7]   | 0.0725 (1.5%) | 0.135 (3.8%) |

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Figure 6. Explanation of the calculation method of parameters in the proposed model for a $l \times 2h$ rectangular strip with $l = (r_2 - r_1)$ and $h = r_1[(1 - \Delta/\pi)^{-2} - 1]$, which corresponds to the sectorial piece in figure 1(c).

Figure 7. Strain energy versus the central angle for one sectorial piece and its corresponding rectangular strip under the same elongation.

Then the total effective stiffness in equation (8) is modified as

$$
\tilde{K} = \sum_{i=1}^{N} \int_{\theta_i - \Delta}^{\theta_i + \Delta} (dK'_i + dK_c) = \frac{\pi r_1}{2(r_2 - r_1)} (E' + C_{44}) + (C_{11} - E') (1 - e^{-r_1((1 - \Delta/\pi)^{-2} - 1)/l})
	imes \sum_{i=1}^{N} \int_{\theta_i - \Delta}^{\theta_i + \Delta} \left[ 1 - \frac{(1 - |\theta - \theta_i|/\pi)^2 - 1}{(1 - \Delta/\pi)^2 - 1} \right] \cos^2 \theta \, d\theta.
$$

Unlike equation (9), the modified total effective stiffness varies with the piece number $N$.

By substituting equation (18) into equations (6) and (7), we can evaluate the eigenfrequencies at the bandgap edges, as shown in figure 4, where the thick-dashed and thick-solid lines represent the lower and upper edges of the bandgap, respectively. The agreement between the predicted results and the numerical ones is satisfactory.

When $N$ is large enough, the second part in equation (16) approaches zero and we have

$$
E_r(\theta) \approx E'.
$$

Then, the total effective stiffness is simplified by eliminating the second part in equation (18), and we get

$$
\tilde{K} = \frac{\pi r_1 (E' + C_{44})}{2(r_2 - r_1)},
$$

which is independent of $N$. Thus, the eigenfrequencies of the bandgap edges approach constants or ‘fixed values’ when $N \to \infty$, as mentioned earlier.

Moreover, the numerical results for the PCs in a triangular lattice are also plotted in figure 4. The steel core, the coating and their filling ratio are exactly the same as those in the square lattice. The results confirm that the locally resonant gap does not vary with respect to the lattice symmetries [2]. In other words, the proposed model can evaluate the lowest bandgap for two-dimensional locally resonant PCs with a comblike coating, regardless of the lattice symmetry.

Another way to develop an equivalent spring–mass model is to model the comblike coating as an ‘effective homogeneous perfect coating’ of which the effective bulk and shear moduli [10] are obtained based on the equivalence of the strain energy.

As indicated before, the shear modulus is uniform in a sectorial piece and immune to the free surfaces. Therefore,
the effective shear modulus is simply given by
\[ \bar{\mu} = C_{44}/2. \] (21)

The derivation of the effective bulk modulus, \( \bar{k} \), is cumbersome. We present the details in the Appendix. The final result is
\[ \bar{k} = \frac{N h (r_2 + r_1)}{4\pi r_2^2} \left[ E' + (C_{11} - E')(1 - \frac{h}{l})^2/2 \right] + \frac{r_2^2 - r_1^2}{2r_2^2} \bar{\mu}, \] (22)
with \( l \) and \( h \) given by equation (11).

The effective bulk modulus, normalized by the shear modulus of the coating, is shown in figure 8. As we can see, it decreases monotonously as the number of pieces increases, as a main consequence of the decrease of the Young’s modulus in equation (15). Meanwhile, replacing \( C_{11} \) with \( (\bar{k} + \bar{\mu}) \) and \( C_{44} \) with \( \bar{\mu} \) in equation (5), equations (6) and (7) can be used to evaluate the bandgap edges, as shown in table 3. The numbers in the brackets show the relative errors compared to the FEM results. Predicted results with these effective moduli using the model in [7] as well as those predicted by the modulus also be in other shapes, such as a parallelogram or trapezoid. Distinguished features may be expected with various comblike coatings.

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### Appendix

In this appendix, we show the process to calculate the effective bulk modulus of the comblike coating based on the energy method. We consider the equivalence of the strain energy between two states: one is a comblike coating with \( N \) pieces subjected to an elongation along the \( r \)-direction; the other is the corresponding ‘effective homogeneous perfect coating’ with the effective bulk modulus \( \bar{k} \) and the effective shear modulus \( \bar{\mu} \) under the same elongation along the \( r \)-direction. The effective shear modulus \( \bar{\mu} \) is given in equation (21). Next, we determine the effective bulk modulus \( \bar{k} \).

Suppose that the comblike coating sustains an elongation \( \delta \) along the \( r \)-direction. Its strain energy can be calculated by the sum of the strain energy of \( N \) identical \( l \times 2h \) rectangular strips (figure 6) under the elongation of \( \delta \) along the \( x \)-axis. The strain energy density in each strip is
\[ W_1 = N \int_0^l \int_{-h}^h w_1 \mathrm{d}x \, \mathrm{d}y \]
Then the strain energy of the comblike coating is
\[ W_1 = N \int_0^l \int_{-h}^h w_1 \mathrm{d}x \, \mathrm{d}y \] (A2)
Deformation of an ‘effective homogeneous perfect coating’ under the elongation \( \delta \) along the \( r \)-direction is an axisymmetric elastic problem of which the displacement field can be generally written as \( u_r = D_1 r + D_2/r \) [11], where \( D_1 \) and \( D_2 \) are determined by the boundary condition. Suppose that the inner boundary of the coating is fixed, and the outer boundary is subjected to a displacement \( \delta \). We can have

### Table 3. Normalized bandgap edges (\( \Omega = \omega a/(2\pi c_0) \)) for PC with a comblike coating evaluated using the effective bulk and shear moduli (\( r_1/a = 0.27 \) and \( r_2/a = 0.4 \)).

| Piece number | \( N = 4 \) | \( N = 32 \) |
|--------------|-------------|-------------|
| Bandgap edges | Lower edge | Upper edge | Lower edge | Upper edge |
| FEM results | 0.0342 | 0.0106 | 0.0267 | 0.0542 |
| Present model (equation (16)) | 0.0227 (34%) | 0.0448 (35%) | 0.0203 (24%) | 0.0401 (26%) |
| Present model (equation (5)) | 0.0204 (40%) | 0.0403 (41%) | 0.0175 (34%) | 0.0345 (36%) |

\[ a \ C_{11} \text{ and } C_{44} \text{ in equation (5)} \] are replaced by \( (\bar{k} + \bar{\mu}) \) and \( \bar{\mu} \), respectively.
\( D_1 = \delta r_2/(\Delta^2 - r_1^2) \) and \( D_2 = -D_1 r_2^2 \). Then the strain energy density in the ‘effective homogeneous perfect coating’ is

\[
\begin{align*}
w_2 &= \int_0^{r_2} \sigma_r \, d\varepsilon_r + \int_0^{r_2} \sigma_\theta \, d\varepsilon_\theta \\
&= \frac{\delta^2 r_2^2}{(\Delta^2 - r_1^2)^2} \left[ 4\bar{k} + 2\bar{\mu}(r_1^4/r^4 - 1) \right]
\end{align*}
\] (A3)

and the total strain energy is

\[
W_2 = \int_{r_1}^{r_2} \int_{r_1}^{r_2} w_2 r \, dr \, d\theta = \frac{\pi \delta^2 r_2^2}{r_2^2 - r_1^2} \left[ 4\bar{k} - 2\bar{\mu}(1 - r_1^2/r_2^2) \right].
\] (A4)

Equating the strain energy \( W_1 \) and \( W_2 \), we get the effective bulk modulus as

\[
\bar{k} = \frac{Nh(r_2 + r_1)}{4\pi r_2^2} \left[ E' + (C_{11} - E')(1 - e^{-h/\ell})/2 \right] + \frac{r_2^2 - r_1^2}{2r_2^2} \bar{\mu}.
\] (A5)

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