On some non-periodic groups whose cyclic subgroups are GNA-subgroups

Aleksandr A. Pypka

11.10.2017

Abstract. In this paper we obtain the description of non-periodic locally generalized radical groups whose cyclic subgroups are GNA-subgroups.

Introduction

Let $G$ be a group. Recall that a subgroup $H$ of $G$ is called abnormal in $G$ if $g \in \langle H, H^g \rangle$ for every element $g \in G$. Recall also that a subgroup $H$ of $G$ is self-normalizing in $G$ if $N_G(H) = H$. It is well known that every abnormal subgroup of $G$ is self-normalizing in $G$. Clearly abnormal and self-normalizing subgroups are antipodes of normal subgroups. On the one hand, a subgroup $H$ of $G$ is both normal and abnormal in $G$ iff $H = G$. On the other hand, if $H$ is a normal subgroup of $G$, then $N_G(H) = G$. This reminds shows that the properties of normal subgroups and abnormal (respectively, self-normalizing) subgroups are diametrically opposite.

In the same time, there are subgroups that combine the concepts of normality and abnormality. Recall that a subgroup $H$ of a group $G$ is called pronormal in $G$ if for every element $g \in G$ the subgroups $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$. Thus, every normal and abnormal subgroup of $G$ is pronormal in $G$. Note that the normalizer $N_G(H)$ of pronormal subgroup $H$ is abnormal in $G$ (see, for example, [1]), and hence self-normalizing in $G$.

2010 Mathematics Subject Classification: 20F16, 20F18, 20F19, 20F22.

Key words and phrases: normal subgroup, abnormal subgroup, pronormal subgroup, self-normalizing subgroup, GNA-subgroup, locally nilpotent radical, locally finite radical, (generalized) radical group.
In the paper [6] the authors introduced the following generalization of normal and abnormal subgroups.

**Definition 1.** A subgroup $H$ of a group $G$ is called a $GNA$-subgroup of $G$ if for every element $g \in G$ either $H^g = H$ or $N_K(N_K(H)) = N_K(H)$, where $K = \langle H, g \rangle$.

Clearly every pronormal subgroup is a $GNA$-subgroup. Moreover, example from [6] shows that there are $GNA$-subgroups, which are not pronormal.

In the paper [6], the authors obtained the description of locally finite groups whose all subgroups are $GNA$-subgroups. Later, in the paper [5], it has been obtained the description of locally finite groups whose cyclic subgroups are $GNA$-subgroups.

In this article, we continue to study the influence of $GNA$-subgroups on the group structure. More precisely, we investigate the structure of some non-periodic groups whose cyclic subgroups are $GNA$-subgroups.

First, we recall some definitions. A *locally nilpotent radical* of a group $G$ is a subgroup generated by all normal locally nilpotent subgroups of $G$. We will denote this subgroup by $L_{nr}(G)$. We recall also that a *locally finite radical* of a group $G$ is a subgroup generated by all normal locally finite subgroups of $G$. We will denote this subgroup by $L_{nf}(G)$.

A group $G$ is called *radical* if $G$ has an ascending series whose factors are locally nilpotent. A group $G$ is called *generalized radical* if $G$ has an ascending series whose factors are locally nilpotent or locally finite. Hence a generalized radical group $G$ either has an ascendant locally nilpotent subgroup or an ascendant locally finite subgroup. In the first case, the locally nilpotent radical $L_{nr}(G)$ of $G$ is non-identity. In the second case, it is not hard to see that $G$ contains a non-identity normal locally finite subgroup. Clearly, in every group $G$ the subgroup $L_{fr}(G)$ is the largest normal locally finite subgroup. Thus, every generalized radical group has an ascending series of normal subgroups with locally nilpotent or locally finite factors.

Observe also that a periodic generalized radical group is locally finite, and hence periodic locally generalized radical group is also locally finite.

The main result of this paper is the following

**Theorem 1.** Let $G$ be a non-periodic locally generalized radical group. Suppose that $R$ is a locally nilpotent radical of $G$. If every cyclic subgroup of $G$ is a $GNA$-subgroup, then either $G$ is abelian or $G = R\langle b \rangle$, where $R$ is abelian, $b^2 \in R$ and $a^b = a^{-1}$ for each element $a \in R$. Moreover, in the second case, the following conditions hold:

(i) if $b^2 = 1$, then the Sylow 2-subgroup $D$ of $R$ is elementary abelian;
(ii) if \(b^2 \neq 1\), then either \(D\) is elementary abelian or \(D = E \times \langle v \rangle\), where \(E\) is elementary abelian and \((b, v)\) is a quaternion group.

Conversely, if a group \(G\) satisfies the above conditions, then every cyclic subgroup of \(G\) is a GNA-subgroup.

1. Preliminary results

**Lemma 1.** Let \(G\) be a group whose cyclic subgroups are GNA-subgroups.

(i) If \(H\) is a subgroup of \(G\), then every cyclic subgroup of \(H\) is a GNA-subgroup.

(ii) If \(H\) is a normal subgroup of \(G\), then every cyclic subgroup of \(G/H\) is a GNA-subgroup.

**Proof.** It follows from the definition of GNA-subgroups.

In the paper [3], B.H. Neumann proved the following classical result: if the factor-group \(G/\zeta(G)\) is finite, then the derived subgroup \([G, G]\) is also finite. As a corollary, we can come to the following generalization: if the factor-group \(G/\zeta(G)\) is locally finite, then the derived subgroup \([G, G]\) is also locally finite.

**Lemma 2.** Let \(G\) be a generalized radical group. If every cyclic subgroup of \(G\) is a GNA-subgroup, then \(G\) is soluble of class at most 3.

**Proof.** Suppose that the locally finite radical \(Lfr(G) = F\) of \(G\) is non-identity. Then \([F, F]\) is abelian [5, Corollary 14]. It follows that in any case the locally nilpotent radical \(Lnr(G) = R\) of \(G\) is non-identity. We will prove that \(G\) is a radical group. Suppose the contrary. Then \(G\) includes the normal subgroups \(T\) and \(S\) such that \(R \leq T \leq S\), \(T\) is radical, \(S/T\) is locally finite and \(Lnr(S/T) = \langle 1 \rangle\). By [5, Corollary 4], \(R\) is a Dedekind group. Corollary 1 from [5] shows that every subgroup of \(R\) is \(G\)-invariant. Then \(S/C_S(R)\) is abelian (see, for example [7, Theorem 1.5.1]). We observe that \(C_S(R) \cap T \leq R\) (see [4, Lemma 4]). Suppose first that \(R\) is periodic. Then

\[
C_S(R)/(C_S(R) \cap R) = C_S(R)/(C_S(R) \cap T) \cong C_S(R)T/T \leq S/T.
\]

In particular, \(C_S(R)/(C_S(R) \cap R)\) is locally finite. Since \(R\) is periodic and locally nilpotent, \(C_S(R)\) is locally finite. Being locally finite, \(C_S(R)\) is metabelian by [5, Corollary 14]. Since \(S/T\) does not include non-identity normal abelian subgroups, \(C_S(R) \leq T\). We have now

\[
S/T \cong (S/C_S(R))/(T/C_S(R)).
\]
We have remarked above that the factor-group $S/C_S(R)$ is abelian, and therefore $S/T$ is abelian. Contradiction.

Suppose now that $R$ is not periodic. Corollary 4 from [5] shows that $R$ is abelian. Let $V$ be the periodic part of $R$ and put $C = C_S(R)$. By proved above, $C/R \cong C/(C \cap R)$ is locally finite. Also, the inclusion $R \leq \zeta(C)$ implies that $[C, C]$ is a locally finite subgroup. Using [5, Corollary 14], we obtain that $C$ is soluble. It follows that $C_S(R) \leq T$, and using the arguments from above, we again obtain a contradiction. This contradiction shows that $G$ is a radical group.

Then $C_G(R) \leq R$ [4, Lemma 4]. By [5, Corollary 4], $R$ is a Dedekind group, in particular, $R$ is metabelian. Corollary 1 from [5] shows that $G$ is soluble if $G/C_G(R)$ is soluble (see, e.g., Theorem 1.5.1 in [7]). The inclusion $C_G(R) \leq R$ implies that $G/R$ is soluble, so that $G$ is soluble and $\text{scl}(G) \leq 3$. \hfill $\Box$

**Corollary 1.** Let $G$ be a locally generalized radical group. If every cyclic subgroup of $G$ is a $GNA$-subgroup, then $G$ is soluble of class at most 3.

**Lemma 3.** Let $G$ be a group and $A$ be a normal abelian subgroup of $G$. Suppose that $G = A(b)$ where $b^2 \in A$ and $a^b = a^{-1}$ for each element $a \in A$. If the subgroup $\langle b \rangle$ is a $GNA$-subgroup, then

(i) if $b^2 = 1$, then the Sylow 2-subgroup $D$ of $A$ is elementary abelian;

(ii) if $b^2 \neq 1$, then either $D$ is elementary abelian or $D = E \times \langle v \rangle$ where $E$ is elementary abelian and $\langle b, v \rangle$ is a quaternion group.

**Proof.** Suppose that $a \in C_A(b)$, then $a^b = a$. On the other hand, by our conditions, $a^b = a^{-1}$, that is $a^{-1} = a$ and $1 = a^2$. Thus $C_A(b)$ is an elementary abelian 2-subgroup. If $c = b^2 \neq 1$, then $c \in C_A(b)$, and by proved above, $1 = c^2 = b^4$. Conversely, if $|a| = 2$, then $a \in C_A(b)$.

Note that if $a \in \langle b \rangle$, then $\langle b \rangle = \langle b \rangle$. Let $a$ be an arbitrary element of $A$. Then $b^{-1}a^{-1}ba = aa = a^2$, and $b^2 = a^{-1}ba = ba^2$. Furthermore, $b^{-1}ab = a^{-1}$ and $ab = ba^{-1}$. Then we have

$$(ba)(ba) = b(ab)a = b(ba^{-1})a = b^2.$$ 

Since this is valid for arbitrary element $a$, we obtain $(ba)^2 = b^2$.

Since $\langle b \rangle$ is a $GNA$-subgroup, we have two possibilities: either $\langle b \rangle$ or $N_K(\langle b \rangle) = N_K(N_K(\langle b \rangle))$, where $K = \langle \langle b \rangle, a \rangle = \langle b, a \rangle$, $a \in A$. In the first case, we obtain that a subgroup

$$\langle b \rangle = \langle b \rangle = \langle b^2 \rangle = \langle b, a^2 \rangle$$
is a 2-subgroup, in particular, $a^2$ (and hence $a$) is a 2-element. In the second case, we again obtain that a subgroup

$$\langle b \rangle = N_K(\langle b \rangle) = N_K(N_K(\langle b \rangle))$$

is a 2-subgroup.

Suppose first $|b| = 2$. Then $\langle b \rangle \cap A = \langle 1 \rangle$. Assume that $A$ has an element $u$ of order 4. By proved above $u^{-1}bu = bu^2$. Since $|u^2| = 2$, $u^2 \in C_A(b)$. It follows that $\langle b, u^2 \rangle$ is abelian. On the other hand, $\langle b \rangle \neq \langle b \rangle u$. On the other hand $N_K(\langle b \rangle) = \langle b, u^2 \rangle \neq \langle b \rangle$, $K = \langle (b, u) = \langle b, u \rangle$. So that $N_K(\langle b \rangle) \neq N_K(N_K(\langle b \rangle))$, and we obtain a contradiction. This contradiction shows that a Sylow 2-subgroup of $A$ is elementary abelian.

Suppose now that $c = b^2 \neq 1$. Let $D$ be a Sylow 2-subgroup of $A$. Since the subgroup $\langle c \rangle$ is normal in $G$, its image in the factor-group $G/\langle c \rangle$ is a $GNA$-subgroup. As proved above, $D/\langle c \rangle$ is an elementary abelian 2-subgroup. Then either $D$ is elementary abelian or $D$ has an element $v$ of order 4 such that $v^2 = c = b^2$. Consider the last situation. Since $v$ has a maximal order among all the elements of $D$, $D = E \times \langle v \rangle$. Since $\langle v \rangle$ is $\langle b \rangle$-invariant, we have

$$|\langle b \rangle \langle v \rangle| = (|\langle b \rangle||\langle v \rangle|)/(\langle b \rangle \cap \langle v \rangle) = 8.$$

Furthermore, as proved above, $v^{-1}bv = bu^2 = bb^2 = b^3$. Hence $\langle b, v \rangle$ is a product of two normal cyclic subgroups of order 4. It follows that $\langle b, v \rangle$ is a quaternion group. $\square$

**Corollary 2.** Let $G$ be a group and $A$ be a normal abelian non-periodic subgroup of $G$. Suppose that $G = A\langle b \rangle$ where $b^2 \in A$, and $a^b = a^{-1}$ for each element $a \in A$. Then $G$ has a subgroup, which is not a $GNA$-subgroup.

**Proof.** Indeed, let $h$ be an element of $A$ of infinite order. Put $H = \langle h^4 \rangle$. Then $H$ is normal in $G$, the element $hH$ has order 4, and $\langle hH \rangle \cap \langle bH \rangle = H$. Lemma 3 shows that the subgroup $\langle b, h^4 \rangle$ can not be a $GNA$-subgroup. $\square$

**Lemma 4.** Let $G$ be a non-periodic finitely generated soluble group. Suppose that $R$ is a locally nilpotent radical of $G$. If every cyclic subgroup of $G$ is a $GNA$-subgroup, then either $G$ is abelian or $G = R\langle b \rangle$ where $R$ is abelian, $b^2 \in R$, and $a^b = a^{-1}$ for each element $a \in R$.

**Proof.** By [5, Corollary 4], $R$ is a Dedekind group. Corollary 1 from [5] shows that every subgroup of $R$ is $G$-invariant. Then $G/C_G(R)$ is abelian (see, for example [7, Theorem 1.5.1]). The inclusion $C_G(R) \leq R$
[4, Lemma 4] implies that $G/R$ is abelian. Being abelian and finitely generated $G/R$ is finitely presented. It follows that $R$ has the elements $x_1, \ldots, x_k$ such that $R = \langle x_1 \rangle^G \ldots \langle x_k \rangle^G$ (see, for example, [2, p. 421]). Since every subgroup of $R$ is $G$-invariant, $\langle x_j \rangle^G = \langle x_j \rangle$, $1 \leq j \leq k$. It follows that $R$ is finitely generated. If we suppose that $R$ is periodic, then $R$ is finite. The inclusion $C_G(R) \leq R$ [4, Lemma 4] implies that $G/R$ is also finite, and hence $G$ is finite. This contradiction proves that $R$ is non-periodic.

Then Corollary 2 and 3 from [5] shows that $R$ is abelian. Suppose that there exists an index $s$ such that $u_s \notin \zeta(G)$. Then there exists an element $g$ such that $v_m^g = v_m^r \neq v_m$ where $r$ is a certain positive integer. Consider the element $u_1v_m$. We have

$$ (u_1v_m)^g = u_1^g v_m^g = u_1 v_m^r \neq u_1 v_m. $$

We remark that $u_1v_m$ has infinite order. By [5, Corollary 1], a subgroup $\langle u_1v_m \rangle$ is $G$-invariant. Then the fact that $g \notin C_G(u_1v_m)$ implies $\langle u_1v_m \rangle^g = \langle u_1v_m \rangle^{-1} = u_1^{-1}v_m^{-1}$. On the other hand, we have $(u_1v_m)^g = u_1 v_m^r$, which implies that $u_1 = u_1^{-1}$. Contradiction. So, there exists an index $j$ such that $u_j \notin \zeta(G)$. Without loss of generality we can suppose that $j = 1$. Let $b$ be an element of $G$ such that $G = \langle b \rangle C_G(\langle u_1 \rangle)$. Then $u_1^b = u_1^{-1}$, and $b^2 \in C_G(\langle u_1 \rangle)$. Suppose now that there exists an index $s$, $1 < s \leq n$, such that $[b, u_s] = 1$. Then

$$ (u_1u_s)^b = u_1^b u_s^b = u_1^{-1} u_s \neq u_1 u_s. $$

On the other hand, an infinite cyclic subgroup $\langle u_1 u_s \rangle$ is $G$-invariant by [5, Corollary 1]. Then it follows that

$$ (u_1 u_s)^b = (u_1 u_s)^{-1} = u_1^{-1} u_s^{-1}. $$

Hence $u_s = u_1^{-1}$, and we obtain a contradiction. This contradiction shows that $u_j^b = u_j^{-1}$ for all $j$, $1 \leq j \leq n$. Using the same arguments we can
prove that \(v_j^b = v_j^{-1}\) for all \(j, 1 \leq j \leq t\). It follows that \(a^b = a^{-1}\) for all elements \(a \in R\).

With the help of similar arguments we can prove that

\[
C_G(\langle u_1 \rangle) = C_G(R) = R.
\]

Hence \(G = R(b)\) and \(b^2 \in R\).

**Corollary 3.** Let \(G\) be a non-periodic locally generalized radical group. Suppose that \(R\) is a locally nilpotent radical of \(G\). If every cyclic subgroup of \(G\) is a \(GNA\)-subgroup, then either \(G\) is abelian or \(G = R(b)\) where \(R\) is abelian, \(b^2 \in R\), and \(a^b = a^{-1}\) for each element \(a \in R\).

**Proof.** By Corollary 1, \(G\) is soluble. Suppose that \(G\) is not abelian. Then \(G\) includes a non-periodic finitely generated non-abelian subgroup \(K\). By Lemma 4, \(K = \text{Lnr}(K)(b)\), where \(\text{Lnr}(K)\) is abelian, \(b^2 \in \text{Lnr}(K)\), \(b^4 = 1\), and \(a^b = a^{-1}\) for each element \(a \in \text{Lnr}(K)\).

Choose in \(G\) a local family \(\mathfrak{L}\) of finitely generated subgroups containing \(K\), and let \(L \in \mathfrak{L}\). Using again Lemma 4 we obtain that \(L = \text{Lnr}(L)(b_1)\), where \(\text{Lnr}(L)\) is abelian, \(b_1^2 \in \text{Lnr}(L)\), \(b_1^4 = 1\), and \(a^{b_1} = a^{-1}\) for each element \(a \in \text{Lnr}(L)\). Since \(K\) is not locally nilpotent, \(\text{Lnr}(L) \cap K \neq K\). On the other hand,

\[
|K : \text{Lnr}(L) \cap K| \leq |L : \text{Lnr}(L)| = 2,
\]

so that \(\text{Lnr}(K) = \text{Lnr}(L) \cap K\). In particular, \(b \not\in \text{Lnr}(L)\). It follows that \(b = b_1u\) for some element \(u \in \text{Lnr}(L)\). As in the proof of Lemma 3, we can show that \(b^2 = (b_1u)^2 = b_1^2\). So, instead of \(b_1\) we can put \(b\). In other words, if \(L\) is an arbitrary subgroup of the family \(\mathfrak{L}\), then \(L = \text{Lnr}(L)(b)\), where \(\text{Lnr}(L)\) is abelian, \(b^2 \in \text{Lnr}(L)\), \(b^4 = 1\), and \(a^b = a^{-1}\) for each element \(a \in \text{Lnr}(L)\). Since \(\mathfrak{L}\) is a local family, \(G = \text{Lnr}(G)(b)\), where \(\text{Lnr}(G)\) is abelian, \(b^2 \in \text{Lnr}(G)\), \(b^4 = 1\) and \(a^b = a^{-1}\) for each element \(a \in \text{Lnr}(G)\).

**2. Proof of main result**

**Proof of Theorem 1.** The necessity follows from Lemma 3 and Corollary 3.

Conversely, let a group \(G\) satisfies the theorem conditions and let \(x\) be an arbitrary element of \(G\). If \(x \in R\), then \(\langle x \rangle\) is normal in \(G\), in particular, \(\langle x \rangle\) is a \(GNA\)-subgroup. Suppose that \(x \not\in R\). Then \(x = bu\) for some element \(u \in R\). In this case, \(G = R(\langle x \rangle)\). As in the proof of Lemma 3, we can show that \(x^2 = (bu)^2 = b^2\). Since \(R\) is abelian, \(a^x = a^b = a^{-1}\) for each element \(a \in R\).
Let $g$ be an arbitrary element of $G$, then $g = x^k a$ for some element $a \in R$. It follows that $g^{-1} x g = a^{-1} x a$. We have $x^{-1} a^{-1} x a = a a = a^2$, and $a^{-1} x a = x a^2$. Furthermore, $x^{-1} a x = a^{-1}$, and $a x = x a^{-1}$. Then we have $(x a)(x a) = x(ax) a = x(xa^{-1}) a = x^2$.

Consider $\langle x \rangle^a$. We have $\langle x \rangle^a = \langle xa^2 \rangle = \langle x, a^2 \rangle$. In particular, it shows that $\langle x \rangle^a$ is a 2-subgroup. In turn, it follows that $a^2$ is a 2-element, so that $a$ is also a 2-element. Then $a =vc$ where $c^2 = 1$. A subgroup $\langle b, v \rangle$ is a quaternion group, so that $\langle b \rangle$ is $\langle v \rangle$-invariant. It follows that $\langle x \rangle$ is $\langle v \rangle$-invariant. Since $c^2 = 1$, $[c, x] = 1$. It follows that $\langle x \rangle^a = \langle x \rangle$, which shows that $\langle x \rangle$ is a GNA-subgroup.

The following result follows directly from Theorem 1 and Corollary 2.

**Corollary 4.** Let $G$ be a non-periodic locally generalized radical group. Then every subgroup of $G$ is a GNA-subgroup if and only if $G$ is abelian.

**References**

[1] M.S. Ba, Z.I. Borevich, *On arrangement of intermediate subgroups*, Rings and Linear Groups, Kubansk. Univ., Krasnodar (1988), 14–41.

[2] P. Hall, *Finiteness conditions for soluble groups*, Proc. London Math. Soc. 4 (1954), 419–436.

[3] B.H. Neumann, *Groups with finite classes of conjugate elements*, Proc. London Math. Soc. 1 (1951), 178–187.

[4] B.I. Plotkin, *Radical groups*, Math. Sb. 37 (1955), 507–526.

[5] A.A. Pypka, *On locally finite groups whose cyclic subgroups are GNA-subgroups*, Algebra and Discrete Math., to appear.

[6] A.A. Pypka, N.A. Turbay, *On GNA-subgroups in locally finite groups*, Proc. of Francisk Scorina Gomel state university 93 (2015), no. 6, 97–100.

[7] R. Schmidt, *Subgroup lattices of groups*, Walter de Gruyter, Berlin, 1994.

**Contact information**

**A.A. Pypka**
Department of Geometry and Algebra, Faculty of Mechanics and Mathematics, Oles Honchar Dnipro National University, Gagarin ave., 72, Dnipro, 49010, Ukraine

E-Mail: pypka@ua.fm
URL: