SYMPLECTIC GENERIC COMPLEX STRUCTURES ON FOUR-MANIFOLDS WITH $b_+ = 1$

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We study symplectic structures on Kähler surfaces with $p_g = 0$. We give an example of a projective surface which admits a symplectic structure which is not compatible with any Kähler metric.

1. Introduction

The main purpose of this note is to give a negative answer to a question raised by Tian-Jun Li [Li08]:

**Question 1.1.** Let $X$ be a closed, smooth, oriented four-manifold which underlies a Kähler surface such that $p_g(X) = 0$. Does $X$ admit a symplectic generic complex structure?

A complex structure $J$ on $X$ is called *symplectic generic* if for any symplectic form $\omega$ of $X$ such that $-c_1(X, \omega)$ coincides with the canonical class $K_J$ of $J$, there exists a Kähler form $\omega'$ cohomologous to either $\omega$ or $-\omega$.

One of the main motivations for this question is the fact that, by a result of Biran [Bir99], the existence of a symplectic generic complex structure on any rational four-manifold implies the famous Nagata’s conjecture which states that given very general points $p_1, \ldots, p_\ell \in \mathbb{CP}^2$, with $\ell \geq 9$, any curve $C$ in $\mathbb{CP}^2$ must satisfy

$$\deg C \geq \frac{\sum_{i=1}^{\ell} \text{mult}_{p_i} C}{\sqrt{\ell}}$$

(see [Li08] for more details). That a smooth four-manifold $X$ is said to be *rational* if it is diffeomorphic to either $S^2 \times S^2$ or $\mathbb{CP}^2 \# k\bar{\mathbb{CP}}^2$, for some $k \geq 0$.

On the other hand, if $X$ is the four-manifold underlying a smooth minimal projective surface of general type (i.e., with big and nef canonical line bundle) then there exists a symplectic form inside the class of the canonical line bundle of $X$ (see [Cat09, STY02]). Therefore, if $p_g(X) = 0$, 493
the existence of a symplectic generic complex structure on $X$ would, in particular, imply the existence of a Kähler–Einstein metric with negative curvature on $X$, by the result of Aubin and Yau. For example, Catanese and LeBrun [CL97] showed the existence of a Kähler–Einstein metric with negative curvature on the generic Barlow surface, which is a projective surface of general type homeomorphic to $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$. But the question remains a hard problem in general, as a classification of the projective surfaces with zero genus is still beyond our reach (see the recent survey [BCP10] for an updated account).

Our example is obtained by considering the four-manifold $X = (\Sigma \times S^2) \# \overline{\mathbb{CP}^2}$, where $\Sigma$ is a Riemannian surface of genus one. We show the existence of a symplectic form on $X$ which is not cohomologous to any Kähler form on $X$, with respect to any complex structure $J$. From an algebraic geometric point of view, this corresponds to saying that the Seshardi constant of a suitable ample class on any uniruled projective surface over an elliptic curve is not maximal (e.g., see [Gar06]). In particular, it follows that $X$ does not admit a symplectic generic complex structure.

Moreover, we describe a minimal surface of general type, for which the underlying manifold does not admit a symplectic generic complex structure. The construction relies on a recent result by Bauer and Catanese [BC11].

Note that both these examples have infinite fundamental group.

2. Preliminary results

In this section, we recall some basic definition and well-known facts about the space of symplectic forms on a smooth four-manifold.

Given a closed smooth oriented four-manifold $X$, we consider the positive cone of $X$, which is defined as the set

$$\mathcal{P}_X = \{a \in H^2(X, \mathbb{R}) \mid a^2 > 0\}.$$ 

Moreover, we denote by $\Omega_X$ the space of orientation-compatible symplectic forms on $X$. Let

$$\mathcal{C}_X = \{[\omega] \mid \omega \in \Omega_X \} \subseteq H^2(X, \mathbb{R}),$$

and let $K_\omega = -c_1(X, \omega)$ be the canonical class of $\omega \in \Omega_X$. We denote by $\mathcal{K}_X$ the union of all elements $K_\omega$ in $H^2(X, \mathbb{Z})$, where $\omega \in \Omega_X$. For any $K \in \mathcal{K}_X$, let

$$\mathcal{C}_{(X,K)} = \{[\omega] \in \mathcal{C}_X \mid K_\omega = K\}.$$

Let $\mathcal{E}_X$ be the set of cohomology classes whose Poincaré dual are represented by smoothly embedded spheres of self-intersection $-1$. In particular, $X$ is said to be minimal if $\mathcal{E}_X$ is empty. Moreover, for any $K \in H^2(X, \mathbb{Z})$,
we denote
\[ \mathcal{E}_{(X,K)} = \{ E \in \mathcal{E}_X \mid E \cdot K = -1 \}. \]

The following result by Li and Liu [LL01] will play an important role:

**Theorem 2.1.** Let \( X \) be a closed, smooth, oriented four-manifold with \( b_+(X) = 1 \).

Then
\[ \mathcal{C}_X = \{ a \in \mathcal{P}_X \mid a \cdot E \neq 0, \text{ for all } E \in \mathcal{E}_X \}. \]

Moreover,
\begin{enumerate}
  \item If \( K \in \mathcal{K}_X \) is not a torsion class, then \( \mathcal{C}_{(X,K)} \) is contained in one of the two components of \( \mathcal{P}_X \), denoted by \( \mathcal{P}_{(X,K)} \), and
\[ \mathcal{C}_{(X,K)} = \{ a \in \mathcal{P}_{(X,K)} \mid a \cdot E > 0, \text{ for all } E \in \mathcal{E}_{(X,K)} \}. \]

  \item If \( K \in \mathcal{K}_X \) is a torsion class, then \( \mathcal{C}_{(X,K)} = \mathcal{P}_X \).
\end{enumerate}

**Proof.** See [LL01, Theorem 4] and [Li08, Theorem 3.11]. \( \square \)

**Remark 2.1.** Let \( (X,\omega) \) be a symplectic four-manifold with \( b_+(X) = 1 \). It follows from [Li06, Proposition 6.3], that if its canonical class \( K_\omega \) is torsion then \( 2K_\omega = 0 \) in \( H^2(X,\mathbb{Z}) \). In particular, \( K_{-\omega} = K_\omega \) and \( \mathcal{C}_{(X,K_\omega)} \) is not contained in one of the two components of \( \mathcal{P}_X \). On the other hand, if \( K_\omega \) is not a torsion class, then \( K_\omega \neq K_{-\omega} \).

We say that a complex structure \( J \) on \( X \) is **symplectic generic** if \( \mathcal{C}_J \) is a connected component of \( \mathcal{C}_{(X,K_J)} \), where \( \mathcal{C}_J \) denotes the Kähler cone of \( J \) and \( K_J \) is the canonical class of \( J \). In particular, if \( K_J \) is not a torsion class, then \( J \) is symplectic generic if \( \mathcal{C}_{(X,K_J)} = \mathcal{C}_J \).

**Lemma 2.1.** Let \( (X,J) \) be a minimal complex surface with \( b_+(X) = 1 \) and which admits a Kähler class \( [\omega] \in \mathcal{C}_J \). Then \( J \) is a symplectic generic complex structure if and only if any \( J \)-holomorphic curve in \( X \) has non-negative self-intersection.

**Proof.** By the Kähler Nakai–Moishezon criterion [Buc99, Lam99], if the Kähler cone \( \mathcal{C}_J \) is not empty then it coincides with the set of elements in \( \mathcal{P}_{(X,K_J)} \) which are positive on every \( J \)-holomorphic curve with negative self-intersection. Thus, if there is no such curve on \( X \), it follows that \( J \) is a symplectic generic complex structure.

Let us assume now that \( C \) is a \( J \)-holomorphic curve with negative self-intersection. Let \( v = \omega(C) \) and \( m = -C^2 \) and define \( a(t) = [\omega] + tPD(C) \in H^2(X,\mathbb{R}) \) for any \( t \geq 0 \). Then, since
\[ a(t)^2 = [\omega]^2 + 2t\omega(C) + t^2C^2 > 2tv - t^2m, \]

it follows that there exists \( T > v/m \) such that \( a(T) \in \mathcal{P}_{(X,K_J)} \). Since \( X \) is minimal, Theorem 2.1 implies that \( a(T) \) is represented by a symplectic form \( \omega_T \) such that \( K_{\omega_T} = K_J \). On the other hand, \( \omega_T(C) = v - Tm < 0 \),
thus $a(T)$ is not a Kähler class. In particular, $J$ is not a symplectic generic complex structure.

By the Kähler Nakai–Moishezon criterion and Theorem 2.1, it also follows that a positive answer to Question 1.1 in the case of rational four-manifolds is equivalent to the following conjecture: any integral curve with negative self-intersection on the blow-up of $\mathbb{C}P^2$ at a set of points in very general position is a smooth rational curve with self-intersection $-1$. In fact, both the conjectures are equivalent to the following:

**Conjecture 2.1 (Harbourne–Hirschowitz).** Let $X$ be the blow-up of $\mathbb{C}P^2$ at a set of $n \geq 10$ points in very general position. Then, the closed cone of curves of $X$ is spanned by the smooth rational curves with self-intersection $-1$ and the round positive cone of $x \in N_1(X)$ such that

$$x^2 \geq 0 \quad \text{and} \quad H \cdot x \geq 0$$

for some fixed ample class $H$ on $X$.

### 3. Ruled manifolds

In this section, we show the existence of a smooth uniruled complex manifold, which does admit a symplectic generic complex structure.

**Lemma 3.1.** Let $\Sigma$ be an elliptic curve, and let $p: Y \to \Sigma$ be a minimal ruled surface over $\Sigma$, such that the parity of the intersection pairing on $H^2(Y, \mathbb{Z})$ is odd. Let $X$ be the blow-up of $Y$ at one point $\eta \in Y$. Let $k$ be the canonical class of $X$, and let $e$ be the class of the exceptional divisor.

Then the class $e - 2k$ contains an effective curve.

**Proof.** By Atiyah’s classification [Ati57] of rank 2 vector bundles on an elliptic curve, it follows that $Y = \mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is either the indecomposable vector bundle contained in the sequence

$$0 \to \mathcal{O}_\Sigma \to \mathcal{E} \to \mathcal{O}_\Sigma(p) \to 0,$$

for some $p \in \Sigma$ or $\mathcal{E} = \mathcal{O}_\Sigma \oplus \mathcal{O}_\Sigma(L)$, where $L$ is a line bundle of odd degree $m < 0$.

Let us consider first the case of the indecomposable vector bundle. It is known (e.g., see [CC93]) that in this case $\mathbb{P}(\mathcal{E})$ is isomorphic to the symmetric product $S^2\Sigma$ of the elliptic curve $\Sigma$, i.e., the quotient of $\Sigma \times \Sigma$ by the natural action of $\mathbb{Z}/2\mathbb{Z}$. We will denote by $[x, y] \in S^2\Sigma$ the class of an element $(x, y) \in \Sigma \times \Sigma$. Note that the projection $p: S^2\Sigma \to \Sigma$ is defined by $p([x, y]) = x + y$. Consider the family of curves

$$C_t = \{[x, t + x] \mid x \in \Sigma\}, \quad \text{for any } t \in \Sigma.$$
If \( t \in \Sigma \) is not a two-torsion point, then the curve \( C_t \) is a smooth elliptic curve. Otherwise, \( C_t \) is a non-reduced elliptic curve. Note that, for any \( s, t \in \Sigma \), we have \( C_t = C_s \) if and only if \( t = s \) or \( t = -s \). It follows that \( C_t^2 = 0 \). Moreover, given \( s, t \in \Sigma \), there exist exactly four points \( x \in \Sigma \) such that \( 2x + t = s \). Thus, if \( t \) is a general point in \( \Sigma \), then the general fiber of \( p \) meets \( C_t \) in exactly four points. Let \( f \) be the numerical class of the pull-back of the general fiber of \( p \) in \( X \) and let \( \delta \) be the numerical class of the pull-back of \( C_t \). Then
\[
\delta^2 = C_t^2 = 0, \quad \delta \cdot e = 0 \quad \text{and} \quad \delta \cdot f = 4.
\]

By adjunction, we have that \( k \cdot \delta = -\delta^2 = 0 \). Similarly, we have \( k \cdot e = -1 \) and \( k \cdot f = -2 \). Moreover, since \( e, f \) and \( k \) are a basis of \( H^2(X, \mathbb{Q}) \), it follows easily that \( \delta = 2e - 2k \). For any point \( \eta \in S^2\Sigma \) there exists \( t \in \Sigma \) such that \( \eta \in C_t \). If \( X \) is the blow-up of \( Y \) at \( \eta \) and \( C_t' \) is the proper transform of \( C_t \) in \( X \), then \( C_t' \) is in the class of \( (2 - q)e - 2k \), where \( q \geq 1 \) is the multiplicity of \( C_t \) at \( \eta \). In particular, the class \( e - 2k \) contains an effective curve, as claimed.

Let us consider now the case of a decomposable vector bundle \( E = O_{\Sigma} \oplus O_{\Sigma}(L) \) where \( L \) is a line bundle on \( \Sigma \) of odd degree \( m < 0 \). Then, there exists an holomorphic section \( C \) in \( Y \) such that \( C^2 = m \). If \( \xi \) is the numerical class of the pull-back of \( C \) in \( X \), it follows easily that \( 2\xi = e + mf - k \), where \( f \) is the pull-back of the general fiber of \( p \). In particular, \( e - 2k = 4\xi + (-2mf - e) \) is the class of a (possibly not irreducible) effective curve in \( X \).

**Remark 3.1.** Note that the uniruled surface which is the projectivization of the decomposable vector bundle can be obtained as a deformation of the projectivization of the indecomposable one. Thus, in the proof of the previous lemma, the second case would follow immediately from the first one.

**Lemma 3.2.** A complex surface \( X \) homeomorphic to \( (\Sigma \times S^2)\#4\overline{CP^2} \), is bi-holomorphic to a blow up at a single point of a minimal ruled surface \( Y \) over an elliptic curve, such that the intersection pairing on \( H^2(Y, \mathbb{Z}) \) is odd.

**Proof.** Recall that from the Enriques–Kodaira classification of complex surfaces, it follows that each complex surface with odd \( b_+ \) is Kähler, and that any algebraic surface of non-negative Kodaira dimension and zero holomorphic Euler characteristics is bi-meromorphic to a torus or a bi-elliptic surface. Since \( b_+(X) = 1 \), it follows that \( X \) is Kähler and \( p_g(X) = 0 \). Thus \( X \) is algebraic. Since \( \pi_1(X) = \mathbb{Z}^2 \) and \( \chi(O_X) = 0 \), we conclude that \( X \) has Kodaira dimension \(-\infty\).

By the classification of algebraic surfaces, it follows that if \( Y \) is the minimal model of \( X \), i.e., the surface obtained after blowing-down all the holomorphic \((-1)\) spheres on \( X \), then \( Y \) is a uniruled surface over a Riemannian surface \( \Sigma \). Since \( b_1(Y) = b_1(X) = 2 \), it follows that the genus of \( \Sigma \) is one. Moreover, since \( b_2(X) = 3 \), it follows that \( X \) is the blow-up of a
ruled surface over an elliptic curve at a single point \( p \in Y \). In particular \( X \) has exactly two holomorphic rational curves \( E_1 \) and \( E_2 \) with self-intersection \(-1\); one is the exceptional divisor of the blow-up map and the other is the strict transform of the rational fiber passing through the blown-up point. Assume that the intersection form on \( H^2(Y, \mathbb{Z}) \) has even parity. Let \( C \) be a curve on \( Y \) which passes through \( p \) and which meets the fiber of the fibration \( Y \to \Sigma \) transversally at \( p \). Then the strict transform of \( C \) in \( X \) has odd self-intersection and it does not intersect \( E_2 \). Thus, after contracting \( E_2 \) we obtain a surface \( Y' \) such that the intersection form on \( H^2(Y', \mathbb{Z}) \) has odd parity. After replacing \( Y \) by \( Y' \), we may assume that \( H^2(Y, \mathbb{Z}) \) has odd parity. □

**Lemma 3.3.** Let \( \pi: Y \to \Sigma \) be a ruled projective surface over an elliptic curve \( \Sigma \), such that \( H^2(Y, \mathbb{Z}) \) has odd parity. Let \( X \) be the blow up of \( Y \) at a single point. Let \( k \) be the class of the canonical class of \( X \) and let \( e_1, e_2 \) be the classes of the two rational curves of self-intersection \(-1\) on \( X \).

Then \( \mathcal{E}_{(X,k)} = \{ e_1, e_2 \} \).

**Proof.** Let \( e \) be a class in \( H_2(X, \mathbb{Z}) \) which can be represented by a smoothly embedded sphere in \( X \) such that \( e^2 = -1 \). Then \( e \) belongs to the kernel of \( \pi_*: H_2(X, \mathbb{Z}) \to H_2(\Sigma, \mathbb{Z}) \). This kernel is spanned by \( e_1 \) and \( e_2 \) and we deduce \( e = \pm (ne_1 + (n-1)e_2) \) for some integer \( n \). At the same time \( e_1 \cdot k = e_2 \cdot k = -1 \), since \( e_1, e_2 \) are the classes of exceptional curves on \( X \). Thus, if \( e \in \mathcal{E}_{(X,k)} \), then \( e \cdot k = -1 \) which implies \( e = e_1 \) or \( e = e_2 \). □

**Theorem 3.1.** Let \( \Sigma \) be a Riemann surface of genus 1, let \( \Sigma \times S^2 \) be the trivial \( S^2 \)-bundle on \( \Sigma \) and let \( X = \left( \Sigma \times S^2 \right) \# \overline{\mathbb{CP}^2} \).

Then, for any complex structure \( J \) on \( X \), there exists a symplectic form \( \omega \) on \( X \) such that \( \omega \) is not Kähler with respect to \( J \). Moreover, \( X \) does not admit any symplectic generic complex structure.

**Proof.** Let \( J \) be a complex structure on \( X \), let \( k \) be the canonical class of \((X,J)\) and let \( e \) be the class of the exceptional divisor \( E \) of the contraction \( X \to Y \), whose existence is guaranteed by Lemma 3.2. Let \( a \) be the first Chern class of an ample line bundle on \( X \). By Lemma 3.1, it follows that \( v = a \cdot (e - 2k) > 0 \). Let

\[
a(t) = a + t(e - 2k) \in H^2(X, \mathbb{R}), \quad \text{for all } t > 0.
\]

In particular, \( a(t) \cdot (e - 2k) = v - t \) and \( a(v)^2 = a^2 + v^2 > 0 \). Thus, there exists \( T > v \) such that \( a(T)^2 > 0 \). Moreover, if \( E \in \mathcal{E}_{(X,k)} \), then

\[
a \cdot E > 0 \quad k \cdot E = -1 \quad \text{and by Lemma 3.3} \quad e \cdot E \geq -1.
\]

Thus, \( a(t) \cdot E > 0 \) for all \( t > 0 \). Since \( b_1(X) = 1 \), Theorem 2.1 implies that the class \( a(T) \) is represented by a symplectic form \( \omega \), such that \( K_\omega = k \).

On the other hand, by Lemma 3.1, the class \( e - 2k \) is represented by a \( J \)-holomorphic curve \( C \) such that \( a(T) \cdot C < 0 \), since \( T > v \). Thus, the class
a(T) does not contain a Kähler form. In particular, $J$ is not a symplectic generic complex structure.

4. Non-ruled manifolds

In this section, we study Question 1.1 in the case of smooth minimal four-manifolds with non-negative Kodaira dimension.

Question 4.1. Let $X$ be a minimal four-manifold which underlies a Kähler surface such that $p_g(X) = 0$. Does $X$ admit a symplectic generic complex structure?

In particular, we show that the question has positive answer in the case of zero Kodaira dimension and we provide an example of a minimal surface of general type which does not admit a symplectic generic complex structure.

By the Seiberg–Witten theory, the Kodaira dimension of a Kähler surface is preserved under diffeomorphism [FM97]. As noted in [Li08], any uniruled four-manifold, i.e., a manifold which underlies a Kähler surfaces of Kodaira dimension $-\infty$, admits a symplectic generic complex structure.

We first consider the case of zero Kodaira dimension:

Proposition 4.1. Let $X$ be a four-manifold which underlies a Kähler surface such that $p_g(X) = 0$ and $\text{kod}(X) = 0$.

Then $X$ admits a symplectic generic complex structure.

Proof. By the classification of algebraic surfaces, it follows that the canonical class of $X$ is numerically trivial. Thus, by the adjunction formula, the only holomorphic curves of negative self-intersection, are smooth rational curves $C$ such that $C^2 = -2$. In particular, Lemma 2.1 implies that it is sufficient to show that there exists a complex structure on $X$ which does not admit any of these curves.

By the classification of algebraic surfaces, we just need to consider two cases: Enriques surfaces and bi-elliptic surfaces. The moduli space of Enriques surfaces is irreducible and by a result of Barth and Peters [BP83, Proposition 2.8], the generic Enriques surface does not contain any smooth rational curve of self-intersection $-2$.

If $X$ is a bi-elliptic surface, then $X = \Sigma_1 \times \Sigma_2/G$, where $\Sigma_1$ and $\Sigma_2$ are Riemannian surfaces of genus one and $G$ is an abelian group acting by complex multiplication on $\Sigma_1$ and by translation on $\Sigma_2$. In particular, since the universal cover of $X$ is $\mathbb{C}^2$, it follows that $X$ does not contain any rational curve. Thus, $X$ does not admit any negative self-intersection curve.

By Lemma 2.1, it follows that any complex structure on $X$ is symplectic generic.

If $X$ is a minimal surface of general type with $p_g(X) = 0$, it is well known that $q = 0$ and $1 \leq K_X^2 \leq 9$. Thus, the moduli space of $X$ is
a union of finitely many irreducible varieties. Nevertheless, it is still not clear what the topology for these surfaces is (see [BCP10] for a recent survey). As stated in the introduction, if $X$ is the four-manifold underlying the surface $X$, a positive answer to Question 4.1 would imply the existence of a complex structure on $X$ which admits a Kähler-Einstein metric. By the results in [Bar84, LP07, PPS09a, PPS09b], it follows that there exist a surface of general type which is homeomorphic to $\mathbb{CP}^2 \# k \mathbb{CP}^2$, for $5 \leq k \leq 8$.

It follows by [CL97, RS09] that, on any of these surfaces, there exists a complex structure which admits a Kähler–Einstein metric with negative curvature.

In general, if $X$ is a minimal surface of general type with $p_g(X) = 0$, then $\chi(\mathcal{O}_X) = 1$ and by Noether’s formula we have

$$b_2(X) = \chi(X) - 2 = 12\chi(\mathcal{O}_X) - K_X^2 - 2 = 10 - K_X^2.$$  

Thus, if $K_X^2 = 9$, then any class in $\mathcal{P}_X$ is the multiple of an ample class and the answer to Question 4.1 is obvious.

Let us consider now the case of a surface of general type $S$ with $p_g(X) = 0$ and $K_X^2 = 8$. All the known examples have infinite fundamental group and their universal cover is the bidisk $\Delta_1 \times \Delta_2 \subseteq \mathbb{C}^2$ [BCP10], so we assume that $S$ is of this type. Denote by $w_1$ and $w_2$ two semi-positive $(1,1)$-forms on $\Delta_1 \times \Delta_2$ obtained via pullbacks of Poincaré metrics from the projections of the bidisk to its factors. For any $a, b > 0$ the form $aw_1 + bw_2$ is Kähler on the bidisk and is invariant under the action of $\pi_1(X)$. Thus, it descends to a Kähler form $w_{a,b}$ on $S$. Since $b_2(X) = 2$, it follows that for $a, b > 0$ the forms $w_{a,b}$ span one of the two connected components of $\mathcal{P}_X$, and so the complex structure on $X$ is symplectic generic.

On the other hand, the results in [BC11] immediately imply the existence of a minimal surface of general type which does not admit a symplectic generic complex structure. Burniat showed the existence of a minimal surface $X$ of general type such that $K_X^2 = 6$, $p_g(X) = 0$, and which is a $(\mathbb{Z}/2\mathbb{Z})^2$-cover of $\mathbb{CP}^2$ blown-up at three points. We will call such a surface a Burniat surface.

**Theorem 4.1.** Let $X$ be a four-manifold which underlies a Burniat surface. Then $X$ does not admit a symplectic generic complex structure.

**Proof.** By [BC11, Theorem 0.2], any complex structure $J$ on $X$ is a Burniat surface. In particular, $X$ admits a $J$-holomorphic curve $C$ of negative self-intersection, which maps to a $(-1)$-curve on the blow-up of $\mathbb{CP}^2$ at three points. More specifically, $C$ is an elliptic curve of self-intersection $-1$. Thus, by Lemma 2.1, it follows that $J$ is not symplectic generic. \qed

Note that a Burniat surface has infinite fundamental group. We do not know any complex surface with $p_g = 0$, finite fundamental group and which does not admit a symplectic generic complex structure.
Recall finally that there exist a wide class of minimal elliptic surfaces of Kodaira dimension 1 and with $p_g = 0$. These surfaces have topological Euler characteristic equal to 12, the base of the corresponding elliptic fibration is $\mathbb{C}P^1$, and the fibration can have any number of multiple fibers greater than 1. It would be interesting to show that all such surfaces admit a symplectic generic complex structure.

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