Cells in the box and a hyperplane

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Abstract. It is well known that a line can intersect at most $2n - 1$ cells of the $n \times n$ chessboard. Here we consider the high-dimensional version: how many cells of the $d$-dimensional $n \times \cdots \times n$ box can a hyperplane intersect? We also prove the lattice analogue of the following well-known fact: if $K, L$ are convex bodies in $\mathbb{R}^d$ and $K \subset L$, then the surface area of $K$ is smaller than that of $L$. 

Keywords. Lattices, polytopes, lattice points in convex bodies

1. Introduction and main result

It is well-known that a line can intersect the interior of at most $2n - 1$ cells of the $n \times n$ chessboard. What happens in high dimensions? This is the question we address here.

Write $Q_n = Q_n^d = [0,n]^d$, $Q_d = Q_1^d$ so $Q_n^d = n Q_n^d$. Let $e_1, \ldots, e_d$ be the standard basis vectors of $\mathbb{R}^d$ and $\mathbb{Z}^d$. For $z = (z_1, \ldots, z_d) \in \mathbb{Z}^d$ define the unit cube $C(z) = \{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d : z_i \leq x_i \leq z_i + 1, \; i \in [d] \}$, which we call a cell in this paper. Here $[d]$ stands for the set $\{1, \ldots, d\}$. For $v \in \mathbb{R}^d$ ($v \neq 0$) let $A(v, t)$ denote the hyperplane $\{ x \in \mathbb{R}^d : vx = t \}$ where $vx$ is the scalar product of the two vectors. Define $N^d(n)$ as the maximal number of cells in $Q_n^d$ that a hyperplane $A(v, t)$ can intersect properly, meaning that $A(v, t) \cap \text{int} C(z) \neq \emptyset$.

It is well-known that $N^2(n) = 2n - 1$. Variants of this result have appeared as olympiad problems in several countries. In a seminal paper [5], József Beck used a slightly stronger version of this fact to answer questions of Dirac, Motzkin, and Erdős. In a companion paper [3] we show that $N^3(n) = \Theta(n^2)$. Here we determine the asymptotic behaviour of $N^d(n)$.

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We need some definitions. We let $|v|$ resp. $|v|_1$ denote the $\ell_2$ and $\ell_1$ norm of the vector $v \in \mathbb{R}^d$. Set
\[
V_d(v) = \frac{|v|_1}{|v|} \max_{t \in \mathbb{R}} \operatorname{vol}_{d-1}(A(v, t) \cap Q^d),
\]
\[
V_d = \max \{ V_d(v) : v \in \mathbb{R}^d, v \neq 0, t \in \mathbb{R} \}.
\]
It is a consequence of the Brunn–Minkowski theorem (cf. [6] and the proof of Lemma 4.1 below) that for fixed $v$ the quantity $\operatorname{vol}_{d-1}(A(v, t) \cap Q^d)$ is maximal when $A(v, t) \cap Q^d$ is the central section of $Q^d$, that is, $A(v, t)$ contains the centre of $Q^d$, which is the point $e/2$ where $e = e_1 + \cdots + e_d$. In this case of course $t = ev/2$. It is known that
\[
1 \leq \operatorname{vol}_{d-1}(A(v, ev/2) \cap Q^d) \leq \sqrt{2};
\]
the upper bound is a famous result of Keith Ball [2], the lower bound is trivial. This implies that
\[
\sqrt{d} \leq V_d \leq \sqrt{2d}.
\]
It is known (see [1] or [2]) that the sequence $V_2, V_3, \ldots$ is increasing, $V_2 = 2$, $V_3 = \frac{9}{4}$, $V_4 = \frac{8}{3}$ etc., and its limit is $\sqrt{6d/\pi}$. We conjectured that the vector $v = e$ gives the maximum in the definition of $V_d$. This has recently been proved by Iskander Aliev [1].

Our main result is

**Theorem 1.1.** $N^d(n) = V_d n^{d-1}(1 + o(1))$.

In Section 3 we give an outline of the proof.

From now on we assume that $v \in \mathbb{R}^d$ is a unit vector, i.e., $|v| = 1$, and $v \geq 0$; the latter causes no loss generality because of symmetry. Define the (open) strip
\[
S(v, t) = \{ x \in \mathbb{R}^d : t - ev < vx < t \}.
\]
Clearly
\[
N^d(n) = \max_{v, t} |S(v, t) \cap Q^d_n \cap \mathbb{Z}^d|.
\]
So we have to determine the number of lattice points in the convex set $S(v, t) \cap Q^d_n$. But this convex set is very thin in one direction (of $v$) and standard methods do not seem to work. In Section 2 we introduce a novel approach to deal with such cases.

Our result extends to any convex body (convex compact set with non-empty interior) $K \subset \mathbb{R}^d$. We define $V(K) = \max\{|v|_1 \operatorname{vol}_{d-1}(K \cap A(v, t)) : v \in \mathbb{R}^d, |v| = 1, t \in \mathbb{R} \}$ and consider the lattice $\frac{1}{n}\mathbb{Z}^d$. Write $N(K, n)$ for the maximal number of cells contained in $K$ that a hyperplane can intersect properly (in the same sense as earlier). A cell in this case is $\frac{1}{n}C(z)$ with $z \in \mathbb{Z}^d$. With this notation $N^d(n) = N(Q^d, n)$. Theorem 1.1 extends to this case as follows.

**Theorem 1.2.** $N(K, n) = V(K)n^{d-1}(1 + o(1))$.

The proof goes along the same lines as that of Theorem 1.1 and is therefore omitted.
2. Inside cells and boundary cells

For a general convex body $K$ in $\mathbb{R}^d$, a metatheorem says that $\text{vol } K$ is approximately equal to $|K \cap \mathbb{Z}^d|$, that is,

$$\text{vol } K \approx |K \cap \mathbb{Z}^d|,$$

valid when $K$ is well positioned with respect to $\mathbb{Z}^d$. But this is not necessarily the case with $S(v, t) \cap Q_n^d$. We are going to well-position it or rather choose a suitable basis of $\mathbb{Z}^d$ in which $S(v, t) \cap Q_n^d$ is well positioned. We start out more generally.

Let $K \subset \mathbb{R}^d$ be a convex body. A cell $C(z)$, $z \in \mathbb{Z}^d$, called inside if $C(z) \subset K$, outside if $C(z) \cap K = \emptyset$, and boundary otherwise. The following result will be useful in other applications as well. It is similar to the well-known fact that the surface area of a convex subset of a convex set $K$ is smaller that the surface area of $K$ itself. To our surprise we could not find it anywhere in the literature.

**Theorem 2.1.** Assume $K, L$ are convex bodies in $\mathbb{R}^d$ and $K \subset L$. Then

$$|\text{boundary cells of } K| \leq |\text{boundary cells of } L|.$$

We prove this theorem in Section 8.

Now we return to the generic convex body $K$. Since $K$ contains all inside cells and is contained in the union of inside and boundary cells, we have

$$|\text{inside cells of } K| \leq \text{vol } K \leq |\text{inside or boundary cells of } K|.$$

It is not hard to check that

$$|\text{inside cells of } K| \leq |K \cap \mathbb{Z}^d| \leq |\text{inside or boundary cells of } K|,$$

implying that

$$|\text{vol } K - |K \cap \mathbb{Z}^d|| \leq |\text{boundary cells of } K|. \quad (2.1)$$

Given a basis $F = \{f_1, \ldots, f_d\}$ of $\mathbb{Z}^d$ we define the $F$-box with parameters $\alpha, \beta \in \mathbb{R}^d$ as

$$B(\alpha, \beta, F) = \left\{ x = \sum_{i=1}^d x_i f_i \in \mathbb{R}^d : \alpha_i \leq x_i \leq \beta_i, \ i \in [d] \right\}.$$

This is a parallelootope. We of course assume that $\alpha_i \leq \beta_i$ for all $i$. The minimal box containing $K$ is denoted by $B(K, F)$; it is the $F$-box $B(\alpha, \beta, F)$ with all $\alpha_i$ maximal and $\beta_i$ minimal under the condition that $K \subset B(\alpha, \beta, F)$. We will make use of the following theorem of Bárány and Vershik [4] (see also [7]).

**Theorem 2.2.** For every convex body $K$ in $\mathbb{R}^d$ there is a basis $F$ such that

$$\text{vol } B(K, F) \ll_d \text{vol } K.$$

The notation $\ll_d$ means, as usual, that the quantity on the LHS is smaller than the one on the RHS times a positive constant that only depends on $d$. When $d$ is clear from
the context, we will use \( \ll \) instead of \( \ll_d \). Of course one can use \( F \)-cells (i.e. basic parallelotopes in the basis \( F \)) and call them inside, outside, and boundary \( F \)-cells with respect to \( K \). Then inequality (2.1) becomes

\[
|\text{vol } K - |K \cap \mathbb{Z}^d| | \leq |\text{boundary } F \text{-cells of } K|.
\]

(2.2)

This inequality extends to any lattice \( \Lambda \) and a basis \( F \) of \( \Lambda \) in the following form:

\[
\left| \frac{1}{\det \Lambda} \text{vol } K - |K \cap \Lambda| \right| \leq |\text{boundary } F \text{-cells of } K|.
\]

(2.3)

We need a non-degeneracy condition on \( K \):

\[
K \cap \mathbb{Z}^d \text{ contains } d + 1 \text{ affinely independent vectors.}
\]

(2.4)

Under this condition and with minimal box \( B(K, F) = B(\alpha, \beta, F) \) we have \( \alpha_i \leq \lfloor \alpha_i \rfloor < \lfloor \beta_i \rfloor \leq \beta_i \) for all \( i \in [d] \). Setting \( \gamma_i = \beta_i - \alpha_i \), \( \text{vol } B(K, F) = \prod_{i=1}^d \gamma_i \). The number of boundary cells of \( B(K, F) \) is easy to estimate: it is at most

\[
2 \sum_{i=1}^d \prod_{j \neq i} (\gamma_j + 2) \ll \sum_{i=1}^d \prod_{j \neq i} \gamma_j = \text{vol } B(K, F) \cdot \left( \frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_d} \right).
\]

Combining the previous theorems we have

**Theorem 2.3.** *Let \( K \) be a convex body in \( \mathbb{R}^d \) satisfying (2.4), and let \( F \) be the basis from Theorem 2.2. Then

\[
|\text{vol } K - |K \cap \mathbb{Z}^d| | \ll \text{vol } K \cdot \left( \frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_d} \right).
\]

The corresponding version for a general lattice \( \Lambda \) says the following. Assume \( K \) is a convex body, \( \Lambda \) a lattice in \( \mathbb{R}^d \), and \( K \) contains \( d + 1 \) affinely independent points from \( \Lambda \). Then there is a basis \( F \) of \( \Lambda \) such that

\[
\left| \frac{1}{\det \Lambda} \text{vol } K - |K \cap \mathbb{Z}^d| \right| \ll \frac{1}{\det \Lambda} \text{vol } K \cdot \left( \frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_d} \right).
\]

(2.5)

Here, just as in Theorem 2.3, the parameters \( \gamma_i \) come from the minimal box \( B(K, F) \).

3. Outline of the proof

In this section we give a sketch of the proof of Theorem 1.1. One main ingredient is Theorem 2.3.

The next section establishes some basic properties of \( A(v, t) \) and \( S(v, t) \). For instance, we show that for fixed \( v \), \( \text{vol}(S(v, t) \cap Q^n) \) is maximal when \( S(v, t) \) is the central strip (Lemma 4.1). Write \( S^*(v, t) = S(v, t) \cap Q_n \) for the strip that maximizes, for fixed \( v \), the number of lattice points in \( S(v, t) \cap Q^n \). We also prove the important but not surprising
fact (Lemma 4.3) that the convex set $S^*(v, t)$ contains an ellipsoid whose half-axes have lengths of order $n$ apart from one that has length $|v|_1/2$.

The lower bound in Theorem 1.1 is simpler and is based on estimating $|S^*(v, t) \cap \mathbb{Z}^d|$ when $v = z/|z|$ with $z \in \mathbb{Z}^d$ a primitive vector. In this case the points of $S^*(v, t) \cap \mathbb{Z}^d$ lie on $|z|_1$ consecutive lattice hyperplanes $A(z, k)$ where $k$ is an integer, and $|A(z, k) \cap \mathbb{Z}^d|$ is estimated using Theorem 2.3 in the form (2.5).

For the upper bound in Theorem 1.1 we fix a maximizer vector $v = v(n)$ and find a basis $F = F^n = \{f_1, \ldots, f_d\}$ of $\mathbb{Z}^d$ using Theorem 2.3. This basis is more suitable than the standard one. The main difficulty is to bound $\frac{1}{v_1} + \cdots + \frac{1}{v_d}$ on the right hand side of the inequality in Theorem 2.3. Here of course $\gamma_i = \gamma_i(n)$ for all $i \in [d]$. The upper bound is easy when $\gamma_i(n) \to \infty$ for all $i \in [d]$. So we assume that $\gamma_i(n)$ is bounded along a subsequence $n'$ for some $i \in [d]$, for $i = 1$, say.

Let $G = G^{n'}$ be the corresponding dual basis, and $g_1(n') \in \mathbb{Z}^d$ be the corresponding dual basis vector. We show next that $g_1(n')$ is also bounded, implying that $g_1(n''') = g$ is a constant (primitive) vector along a further subsequence $n'''$. This means that the lattice points in $S^*(v, t)$ lie on $\gamma$ consecutive lattice hyperplanes orthogonal to $g$. Here $\gamma$ is the floor of $\gamma_1(n''')$, which we can assume to be a constant since $\gamma_1(n''')$ is bounded. It turns further out that $v(n''')$ tends to $g_0 = g/|g|$ because the angle $\phi_{n'''}$ between these two vectors is $\ll \frac{1}{|g|n'''}$.

The next step of the argument is 2-dimensional. Let $\Psi = \Psi_n$ denote the orthogonal projection of $\mathbb{R}^d$ to the 2-plane $\Pi$ spanned by $v(n'')$ and $g$. The projections of the lattice points in $S^*(v, t)$ lie on $\gamma$ parallel lines $\ell_h$ (that are $\frac{1}{|g|}$ apart) see Figure 1. The projected lattice points on the $h$th line belong to a segment $Y_h$ whose length is $|v(n'')|_1/\sin \phi_{n'''}$.

We show (Claim 7.1) that any line orthogonal to $\ell_h$ intersects at most $\gamma + 1$ segments $Y_h$, and, more importantly, any such line intersects at most $\gamma$ segments $Y^*_h$ where $Y^*_h$ is what you get after deleting a short segment (of length $\sqrt{2d}$) from the left end of $Y_h$.

The number of lattice points in $S^*(v, t)$ is the sum of the lattice points in $\Psi^{-1}(Y_h)$, which is close to $\frac{1}{|g|} \psi_{d-1}^{-1} \Psi^{-1}(Y^*_h)$, which is close to $\frac{1}{|g|} \psi_{d-1}^{-1} \Psi^{-1}(Y^*_h)$. Estimating the sum of these volumes finishes the proof.

4. Preparations for the proof of Theorem 1.1

In this section we establish some basic properties of the hyperplane $A(v, t)$ and the strip $S(v, t)$ that give the maximal value of $V_d(v)$. We assume again that $v$ is a unit vector, and suppose without loss of generality that $v \geq 0$, that is, $v_i \geq 0$ for all $i \in [d]$. Actually, we can assume that $v_i > 0$ for each $i$ because the requirement $A(v, t) \cap \text{int} C(z) \neq \emptyset$ remains valid even if $v_i$ is modified a little.

For simpler notation we write $A^*(v, t) = A(v, t) \cap Q_n$ and $S^*(v, t) = S(v, t) \cap Q_n$. These intersections of course depend on $n$, but we suppress this dependence as long as it is not needed. The central section is $A^*(v, t_0)$ where $t_0 = n|v|_1/2$; it contains $en/2$, the centre of $Q_n$. The central strip is $S^*(v, t_2)$ where $t_2 = t_0 + |v|_1/2$; it is centrally
where \( t \) implies that radius and \( d \) gained an a

This shows that max

We still assume that \( v > 0 \) and \( |v| = 1 \). By the Brunn–Minkowski theorem (see [6]) the function \( t \mapsto (\text{vol}_{d-1} A^*(v, t))^{1/(d-1)} \), defined for \( t \in [0, n|v|_1] \), is concave. It is also symmetric with respect to \( t_0 = n|v|_1/2 \), and equals zero at the endpoints of \([0, n|v|_1]\). So its maximum is taken at \( t_0 \), implying that \( A^*(v) = A^*(v, t_0) \). The integral formula

implies that

The volume of the central strip is

where \( t_1 = t_0 - |v|_1/2 \). Concavity implies that on the interval \([t_1, t_0]\),

We next estimate \( D := |v|_1 \text{vol}_{d-1} A^*(v, t_0) - \text{vol} S^*(v, t) \) for \( t \in [t_1, t_0] \):

This shows that max, \( \text{vol} S^*(v, t) \geq V_d(v) \left( 1 - \frac{d}{2n} \right). \)

Here come the properties of \( A(v, t) \) and \( S(v, t) \) that we need. Every \( A^*(v, t) \) is contained an a \( d - 1 \)-dimensional ball of radius \( \ll n \) because \( Q_n \) is contained in a ball of radius \( \sqrt{d} n/2 \). Fix a unit vector \( v \). The maximizer is the slice \( A^*(v, t) \) that properly intersects the maximal number of cells in \( Q_n \) among all \( A^*(v, s) \), \( s \in \mathbb{R} \). The corresponding \( S^*(v, t) \) is also a maximizer.

**Lemma 4.2.** There is a maximizer \( A^*(v, t) \) whose inscribed ball has radius \( \gg n \).
Proof. Recall that \( e = e_1 + \cdots + e_d \) where \( e_1, \ldots, e_d \) form the standard basis of \( \mathbb{R}^d \). We can assume by symmetry that the hyperplane \( A(v, t) \) satisfies \( v > 0 \) and \( t \leq ve/2 \); for each \( i \in [d] \), \( A(v, t) \) contains the (unique) point \( a_i e_i \), and \( a_i > 0 \), of course. We choose \( A(v, t) \) so that \( \min \{ a_i : i \in [d] \} \) is maximal. We claim that this maximum is at least \( n \).

Assume that, on the contrary, \( a_1 = \min \{ a_i : i \in [d] \} < n - 1 \). If \( A(v, t) \) intersects the cell \( C(z) \in Q_n \), then the hyperplane \( A(v, t) + e_1 \) intersects the cell \( C(z) + e_1 \) which lies in \( Q_n \), so it intersects at least as many cells as \( A(v, t) \). It is easy to check that for each \( i \in [d] \), \( A(v, t) + e_1 \) contains the (unique) point \( a'_i e_i \) with \( a'_i > a_i \), a contradiction.

Then the \( d - 1 \)-dimensional ball inscribed in \( A^*(v, t) \) has radius at least \( n/d \), as one can see easily.

We now fix this maximizer \( A^*(v, t) \) together with \( S^*(v, t) \).

**Lemma 4.3.** The maximizer \( S^*(v, t) \) contains an ellipsoid with all half-axes length \( \gg n \) apart from one whose length is \( |v|_1/2 \), which is between \( 1/2 \) and \( \sqrt{d}/2 \).

Proof. The middle section \( A^*(v, t - ev/2) \) of \( S^*(v, t) \) contains a \( d - 1 \)-dimensional ball of radius \( \gg n \). This follows from Lemma 4.2 for \( n \) large. The width of the strip in direction \( v \) is \( |v|_1 \).

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5. Lattice points in \( A^*(z, h) \)

Given a primitive vector \( z \in \mathbb{Z}^d \) we are going to estimate the number of lattice points in \( A^*(z, h) \) where \( h \in \mathbb{Z} \). We will need a more general setting so assume \( K \) is a convex subset of \( A^*(z, h) \) and we will estimate \( |K \cap \mathbb{Z}^d| \). As \( A^*(z, h) \) is \( d - 1 \)-dimensional, condition (2.4) requires having \( d \) affinely independent points in \( K \cap \mathbb{Z}^d \).

**Lemma 5.1.** If \( K \) does not satisfy the non-degeneracy condition (2.4), then

\[
|K \cap \mathbb{Z}^d| \ll n^{d-2}.
\]

Proof. Under the above conditions the lattice points in \( K \) lie on a hyperplane in \( A(z, t) \), that is, a \( d - 2 \)-dimensional affine (lattice) subspace. One can project \( K \) orthogonally to a facet of \( Q^n \) so that distinct lattice points project to distinct (lattice) points. An induction argument on dimension finishes the proof.

**Lemma 5.2.** If \( K \) satisfies the non-degeneracy condition (2.4), then

\[
|K \cap \mathbb{Z}^d| \ll \frac{1}{|z|} \text{vol}_{d-1} K.
\]

Proof. We can apply the general lattice version of Theorem 2.3, i.e., (2.5). The lattice now is \( \Lambda = A(z, h) \cap \mathbb{Z}^d \), it is \( d - 1 \)-dimensional and its determinant equals \( |z| \), the \( \ell_2 \) norm of \( z \). So there is a basis \( F = \{ f_1, \ldots, f_{d-1} \} \) of \( \Lambda \) such that \( \text{vol}_{d-1} B(K, F) \ll \text{vol}_{d-1} K \). Here \( B(K, F) \) is the minimal box in \( \Lambda \) containing \( K \), and so it is of the form
\[ x = \sum_{i=1}^{d-1} x_i f_i : \alpha_i \leq x_i \leq \beta_i, \ i \in [d-1] \] with suitable \( \alpha_i, \beta_i \). Because of the non-degeneracy assumption, \( \gamma_i := \beta_i - \alpha_i \geq 1 \). Theorem 2.3 shows now that

\[
\left| \frac{1}{|z|} \text{vol}_{d-1} K - |K \cap \mathbb{Z}^d| \right| \ll \frac{1}{|z|} \text{vol}_{d-1} K \cdot \left( \frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_{d-1}} \right).
\]

As \( \gamma_i \geq 1 \) for all \( i \); this implies the statement.

We assume now that \( K \subset A^*(z, h) \) contains a \( d - 1 \)-dimensional ball of radius \( c_1 n \) where \( c_1 > 0 \) is a constant depending only on \( d \). Of course \( K \) lies in a \( d - 1 \)-dimensional ball of radius \( \sqrt{d} n/2 \) because \( Q_n \) lies in the \( d \)-dimensional ball of the same radius and centre \( en/2 \).

**Lemma 5.3.** Assume further that \( K \) contains \( d \) affinely independent points from \( \mathbb{Z}^d \). Then

\[
|K \cap \mathbb{Z}^d| = \frac{1}{|z|} \text{vol}_{d-1} K \cdot \left( 1 + |z|O\left( \frac{1}{n} \right) \right),
\]

where the implicit constant depends only on \( d \).

**Proof.** We assume \( z \geq 0 \) because of symmetry. Again there is a basis \( F = \{ f_1, \ldots, f_{d-1} \} \) of \( \Lambda \) such that \( \text{vol}_{d-1} B(K, F) \ll \text{vol}_{d-1} K \ll n^{d-1} \) where \( B(K, F) \) is the minimal box in \( \Lambda \) containing \( K \) which is of the form \( \left\{ x = \sum_{i=1}^{d-1} x_i f_i : \alpha_i \leq x_i \leq \beta_i, \ i \in [d-1] \right\} \) with suitable \( \alpha_i, \beta_i \). Set \( \gamma_i = \beta_i - \alpha_i \) again and note that \( \text{vol}_{d-1} B(K, F) = |z| \prod_{i=1}^{d-1} \gamma_i \).

**Claim 5.1.** \( n \ll |\gamma_i| f_i | \ll n \) for every \( i \in [d-1] \).

**Proof.** Let \( E \) be the largest volume \( (d - 1) \)-dimensional ellipsoid contained in \( B(K, F) \) and define \( E^* \) as the blown-up copy of \( E \) from its centre by the factor \( d - 1 \). Then \( B(K, F) \) is contained in \( E^* \) by the well-known Loewner–John theorem. The volume of \( E^* \) is \( \ll n^{d-1} \) and \( E^* \) contains the ball of radius \( c_1 n \). This implies that each axis of \( E^* \) has length \( \gg_d n \), which implies in turn that each axis has length \( \ll_d n \). Then the diameter of \( E^* \) is \( \ll n \), and then so is the diameter of \( B(K, F) \) as well. Thus every edge of the parallelootope \( B(K, F) \) has length \( \ll n \). These edges are of the form \( \gamma_i f_i \), so \( \gamma_i |f_i| \ll n \) follows.

On the other hand, the parallelootope \( B(K, F) \) contains the ball of radius \( c_1 n \) so its edges have length at least \( c_1 n \), showing that \( n \ll |\gamma_i| f_i | \).

We remark that in view of the claim,

\[
n^{d-1} \gg \prod \gamma_i \prod |f_i| = \frac{1}{|z|} \text{vol}_{d-1} B(K, F) \cdot \prod |f_i| \gg \frac{1}{|z|} n^{d-1} \prod |f_i|,
\]

implying \( \prod |f_i| \ll |z| \) and so \( |f_i| \ll |z| \) as \( |f_i| \geq 1 \) for all \( i \).
As \( \mathbb{Z}^d \cap K \) contains \( d \) affinely independent vectors, Theorem 2.3, or rather its lattice version (2.5), applies. Using \(|f_1| \ll |z|\) we see that

\[
\frac{1}{|z|} \operatorname{vol}_{d-1} K - |K \cap \mathbb{Z}^d| \ll \frac{1}{|z|} \operatorname{vol}_{d-1} K \cdot \left( \frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_{d-1}} \right)
\]

\[
\ll \frac{1}{|z|} \operatorname{vol}_{d-1} K \cdot \left( \frac{|f_1|}{n} + \cdots + \frac{|f_{d-1}|}{n} \right) \ll \frac{1}{n} \operatorname{vol}_{d-1} K.
\]

So we have indeed

\[
|\mathbb{Z}^d \cap K| = \frac{1}{|z|} \operatorname{vol}_{d-1} K \cdot \left( 1 + |z|O\left( \frac{1}{n} \right) \right).
\]

Set \( z_0 = z/|z| \) and define

\[
M_d(z, n) = \max_t |\mathbb{Z}^d \cap S^*(z_0, t)|.
\]

The lattice points in a maximizer \( S^*(z_0, t) \) (in the sense used in Lemma 4.2) are all contained in \(|z|_1\) consecutive lattice hyperplanes of the form \( A(z, h) \). Consequently,

\[
M_d(z, n) = \max_{k \in \mathbb{Z}} \sum_{h=1}^{|z|_1} |\mathbb{Z}^d \cap A^*(z, k-h)|.
\] (5.1)

**Theorem 5.1.** For any primitive vector \( z \in \mathbb{Z}^d \) there is \( n_0(z) \in \mathbb{Z} \) such that for all \( n > n_0(z) \),

\[
M_d(z, n) = n^{d-1} V_d(z_0) + O(n^{d-2}),
\]

where the implied constant depends only on \( d \).

**Proof.** We will use Lemma 5.3 with \( K = A^*(z, k-h) \). By Lemma 4.2 the maximizer \( A^*(z, k) \) contains a ball of radius \( \gg n \). It also contains \( d \) affinely independent lattice points if \( n \) is large enough (depending on \( z \)). The same applies to all \( A^*(z, k-h) \) with \( h \in [z] \) because for large \( n \) the slice \( A^*(z, k-h) \) is very close to \( A^*(z, k) \). We can use Lemma 5.3 in (5.1) to get

\[
\sum_{h=1}^{|z|_1} |\mathbb{Z}^d \cap A^*(z, k-h)| = \sum_{h=1}^{|z|_1} \frac{1}{|z|} \operatorname{vol}_{d-1} A^*(z, k-h) \cdot \left( 1 + |z|O\left( \frac{1}{n} \right) \right).
\]

As we have seen, \( \operatorname{vol}_{d-1} A^*(z, k-h) \) is at most the \( d-1 \)-dimensional volume of the central slice \( A^*(z) = A^*(z, t_0) \). So the sum of \( \operatorname{vol}_{d-1} A^*(z, k-h) \) for \(|z|_1\) consecutive slices is at most \(|z| \operatorname{vol}_{d-1} A^*(z)\). This sum is maximal when the slices are as close to the central slice as possible. This follows from the concavity of the function \( t \mapsto (\operatorname{vol}_{d-1} A^*(z, t))^{1/(d-1)} \). The sum of these central slices is estimated as in the proof of Lemma 4.1. We omit the details. \( \blacksquare \)

**Corollary 5.1.** \( N^d(n) \geq V_d n^{d-1}(1 + o(1)) \).
Proof. Denote by $A^0(v)$ the central section $A(v, t) \cap Q^d$. Since the function $v \mapsto |v|_1 \text{vol}_{d-1} A^0(v)$ (for unit vectors in $\mathbb{R}^d$) is continuous, for any $\varepsilon > 0$ we can choose a primitive vector $z \in \mathbb{Z}^d$ such that $V_d(z_0) \geq V_d - \varepsilon/2$ where $z_0 = z/|z|$. Then for all large enough $n$,

$$M_d(z, n) \geq n^{d-1} V_d(z_0) + O(n^{d-2}) \geq n^{d-1}(V_d - \varepsilon/2) + O(n^{d-2}) \geq n^{d-1}(V_d - \varepsilon).$$

6. Proof of the upper bound in Theorem 1.1

Let $S_n = S^*(v, t)$ be the maximizer for $N_d(n)$; of course $v = v(n)$ and $t = t(n)$ but we suppress this dependence as long as possible. We are to show that for every $\varepsilon > 0$,

$$|S_n \cap \mathbb{Z}^d| \leq (V_d + \varepsilon)n^{d-1} \quad (6.1)$$

for all large enough $n$. Fix $\varepsilon > 0$.

We claim first that $S_n$ satisfies the non-degeneracy condition (2.4). Otherwise $S_n \cap \mathbb{Z}^d$ is contained in a hyperplane of normal $w$ with $w_i \neq 0$ for some $i \in [d]$, $i = d$ say. Projecting the points of $S_n \cap \mathbb{Z}^d$ to the hyperplane $x_d = 0$ we get lattice points on a facet of $Q_n$, and distinct points project to distinct points. No facet contains more than $(n + 1)^{d-1}$ lattice points, so $|S_n \cap \mathbb{Z}^d| \leq (n + 1)^{d-1}$, which is smaller than $M_d(n) \geq \sqrt{d} n^{d-1} + O(n^{d-2})$. The last inequality follows from Corollary 5.1 and from $V_d \geq \sqrt{d}$.

Now Theorem 2.3 gives

$$\left| \text{vol} S_n - |S_n \cap \mathbb{Z}^d| \right| \ll \text{vol} S_n \cdot \left( \frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_{d-1}} \right). \quad (6.2)$$

Here of course $\alpha_i = \alpha_i(n)$, $\beta_i = \beta_i(n)$ and $\gamma_i = \gamma_i(n) = \beta_i(n) - \alpha_i(n)$. A simple case is when there is a sequence $n'$ of positive integers such that $\lim \gamma_i(n') = \infty$ for every $i \in [d]$. For simplicity of writing we use $n$ instead of $n'$. Then (6.2) implies that

$$|S_n \cap \mathbb{Z}^d| = \text{vol} S_n \cdot (1 + o(1)) \leq V_d n^{d-1}(1 + o(1)),$$

so (6.1) holds true indeed.

Assume next that there is a subsequence $n'$ of the previous subsequence such that $\gamma_i(n')$ is bounded for some $i \in [d]$, $i = 1$ say. We write again $n$ instead of $n'$. Let $G^n = \{g^n_1, \ldots, g^n_d\}$ be the dual basis of $F = F^n$. Set

$$\alpha(n) = \min \{g^n_1 x : x \in S_n\} \quad \text{and} \quad \beta(n) = \max \{g^n_1 x : x \in S_n\}.$$

Of course $\beta(n) - \alpha(n) = \gamma_1(n)$ and $\gamma_1(n)$ is bounded. So along another subsequence (to be denoted invariably by $n$) $\lim(\beta(n) - \alpha(n)) = \gamma$ for some $\gamma \geq 0$.

We claim now that the corresponding dual basis vector $g^n_1$ is also bounded. This is simple again: otherwise the width of $S_n$ in direction $g^n_1$ is $\gamma/|g^n_1|$, which tends to zero as $n \to \infty$. But $S_n$ contains a ball of radius $\gg 1$ (by Lemma 4.3), a contradiction. This implies that along a further subsequence, $g^n_1$ is equal to a fixed primitive vector, $g$, say.
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Define the strip

\[T_n = \{ x \in \mathbb{R}^d : \alpha(n) \leq g x \leq \beta(n) \}.\]

Then \( S_n \cap \mathbb{Z}^d \subset T_n \) because of the definition of \( \alpha(n) \) and \( \beta(n) \). Set \( g_0 = g/|g| \). Let \( \phi_n \) be the angle between \( g \) and \( v(n) \), so \( \cos \phi_n = v(n)g_0 \). Define \( \Psi : \mathbb{R}^d \to \Pi_n \) as the orthogonal projection to the 2-dimensional plane spanned by \( v(n) \) and \( g \). Note that here we can assume \( g \neq v(n) \) since a minute change of \( v(n) \) does not influence what cells the hyperplane \( A(v(n), t) \) intersects.

**Claim 6.1.** Along the present subsequence, \( \phi_n \ll \frac{\gamma}{|g|n} \) and so \( v(n) \to g_0 \).

**Proof.** We drop the subscript \( n \) whenever possible. \( \Psi(Q_n) \) is a centrally symmetric convex polygon. The \( \Psi \)-image of the lattice hyperplane \( A(g, [\alpha(n)] + h) \) is the line \( \ell_h \) on \( \Pi_n \), represented by a horizontal line in Figure 1, \( h = 0, 1, \ldots, \gamma \). Here we take the upper integer part of \( \alpha(n) \) because we need lattice hyperplanes. We should also take \( h = 0, 1, \ldots, \lfloor \gamma \rfloor \) because \( \gamma \) may not be an integer. But for simplicity we keep writing \( \gamma \) now and in what follows.

![Fig. 1. v(n) tends to g0.](image1)

The \( \Psi \)-image of the two hyperplanes bounding \( S_n = S(v(n), t(n)) \) are the lines \( \ell^+ \) and \( \ell^- \) in Figure 2. Their distance is \( |v|_1 \). The length of the segments \( \ell^+ \cap \Psi(Q^n) \) and \( \ell^+ \cap \Psi(Q^n) \) is \( \gg n \) because \( S_n \) contains the ellipsoid from Lemma 4.3 and \( S_n \subset T_n \). So with \( \phi = \phi_n \),

\[
\sin \phi \ll \frac{\gamma}{n|g|}.
\]

![Fig. 2. Projection onto \( \Pi \).](image2)
Define now for \( h = 0, 1, \ldots, \gamma \) the \( d - 1 \)-dimensional convex polytope
\[
P^n_h = S_n \cap A(g, [\alpha(n)] + h).
\]

Every lattice point in \( S_n \) belongs to some \( P^n_h \). The proof of the upper bound on \( M_d(n) \) is based on estimating \( \sum_{j=0}^\gamma |P^n_h \cap \mathbb{Z}^d| \).

Define a map \( \Phi = \Phi_h : \mathbb{R}^d \to \mathbb{R}^d \) by \( \Phi(x) = x/n \). Then \( \Phi(P^n_h) \) is a convex compact set in \( Q^d \) for all \( h \in \{0, 1, \ldots, \gamma\} \). We use the Blaschke selection theorem (see for instance [6]): along a subsequence (denoted by \( n \) again) \( \Phi(P^n_h) \) tends to a convex polytope \( P_h \) for \( h \in \{0, 1, \ldots, \gamma\} \). Note also that each \( P_h \) lies in \( A(g, t) \cap Q^d \) for some fixed \( t \).

Let \( I \) denote the set of \( h \in \{0, \ldots, \gamma\} \) with \( \text{vol}_{d-1} \Psi(P_h) > C_0 \) where \( C_0 > 0 \) will be specified later. Write \( J_1 \) resp. \( J_0 \) for those \( h \notin I \) for which \( P^n_h \) does (resp. does not) contain \( d \) affinely independent vectors from \( \mathbb{Z}^d \). We are going to estimate \( |P^n_h \cap \mathbb{Z}^d| \) separately for \( h \in I \), in \( J_0 \) and in \( J_1 \).

When \( h \in J_0 \), Lemma 5.1 applies and gives \( |P^n_h \cap \mathbb{Z}^d| \ll n^{d-2} \). The total contribution of such \( P^n_h \)’s to \( |S_n \cap \mathbb{Z}^d| \) is at most \( \ll |J_0|n^{d-2} \).

For \( h \in J_1 \), Lemma 5.2 shows that \( |P^n_h \cap \mathbb{Z}^d| \leq C_d \frac{1}{|g|} \text{vol}_{d-1} P^n_h \). Here \( C_d > 0 \) is the constant implicit in the \( \ll \) notation. The total contribution of such \( P^n_h \)’s to \( |S_n \cap \mathbb{Z}^d| \) is at most \( \ll |J_1| C_d C_0 n^{d-1} \).

For \( h \in I \), let \( E^n_h \) be the ellipsoid of largest volume inscribed in \( P^n_h \) with half-axes of length \( a_1, \ldots, a_{d-1} \). The Loewner–John theorem implies that
\[
\text{vol}_{d-1} E^n_h \geq (d - 1)^{-(d-1)} \text{vol}_{d-1} P^n_h \geq C_0 \left( \frac{n}{d-1} \right)^{d-1}.
\]

Also \( \text{vol}_{d-1} E^n_h = \kappa_{d-1} \prod_{i=1}^{d-1} a_i \) where \( \kappa_{d-1} \) is the volume of the \( d - 1 \)-dimensional unit ball. As \( a_i \leq \sqrt{d} n \) for all \( i \), the minimal \( a_i \) satisfies \( a_i \gg C_0 n \). So \( P^n_h \) contains a ball of radius \( \gg C_0 n \). It is also clear that for large enough \( n \), \( P^n_h \) contains \( d \) affinely independent points from \( \mathbb{Z}^d \). So we can apply Lemma 5.3: for \( h \in I \),
\[
|P^n_h \cap \mathbb{Z}^d| \leq \frac{1}{|g|} \text{vol}_{d-1} P^n_h \cdot \left( 1 + |g|O\left( \frac{1}{n} \right) \right),
\]
showing that the total contribution of those \( P^n_h \)’s to \( |S_n \cap \mathbb{Z}^d| \) is at most
\[
\frac{1}{|g|} \sum_{i \in I} \text{vol}_{d-1} P^n_h \cdot \left( 1 + |g|O\left( \frac{1}{n} \right) \right).
\]

Lemma 6.1. With the previous notation,
\[
\frac{1}{|g|} \sum_{h=0}^\gamma \text{vol}_{d-1} P^n_h \leq V_d(g)n^{d-1}(1 + o(1)).
\]

We postpone the proof to the next section. We show now how to complete the proof of Theorem 1.1 using this lemma.
The number of lattice points in $S_n$ is $V_d(g)n^{d-1}(1+o(1)) \leq V_d n^{d-1} + \frac{1}{2} \varepsilon n^{d-1}$ if $n$ is large enough plus an error term of the form
\[ |J_0|n^{d-2} + |J_1|C_d C_0 n^{d-1} \]
times a constant depending only on $d$. Here $|J_1|, |J_0| \leq \gamma$, and $g$ and $\gamma$ are fixed. So if we choose $C_0 > 0$ small enough the error term becomes smaller than $\frac{1}{2} \varepsilon n^{d-1}$.

\[ \text{Claim 7.1. Each vertical line intersects at most } |g|_1 + 1 \text{ segments } Y_h \text{ and at most } |g|_1 \text{ segments } Y_h^*, h = 0, 1, \ldots, \gamma. \]

Proof. This is elementary plane geometry using the fact that $v(n)$ and $g_0$ are very close to each other. We assume $v(n) > 0$; then $g \geq 0$ as well and $|v(n)|_1 = v(n)e$, $|g|_1 = ge$. Assume $\ell^-$ intersects $\ell_h$ in a point $a$, and $\ell^+$ intersects $\ell_h$ resp. $\ell_{h+|g|_1}$ in points $b$ and $c$, and let $e$ denote the orthogonal projection of $c$ to $\ell_h$. We consider $a, b, e$ as real numbers on the $x$-axis. The length of $Y_h$ is $b - a = ve/\sin \phi$, and $b - e = |g|_1/(|g| \tan \phi) = g_0 e/\tan \phi$ and
\[
e - a = \frac{ve - g_0 e}{\sin \phi} = \frac{1}{\sin \phi} (ve - g_0 e \cos \phi)
\]
\[
= \frac{1}{\sin \phi} [(v - g_0)e + g_0 e (1 - \cos \phi)]
\]
\[
\leq \frac{1}{\sin \phi} \left( \frac{\sqrt{2} \sin \phi}{\sqrt{1 + \cos \phi}} \sqrt{d} + \frac{\sin^2 \phi}{1 + \cos \phi} \right) < \sqrt{2d},
\]
as one can check easily. This implies that $Y_h^*$ is contained in the interval $[e, b]$. Moreover, a vertical line intersecting the segment $[a, e]$ intersect $Y_h, Y_{h+1}, \ldots, Y_{h+|g|_1}$ but no other $Y_i$. And a vertical line intersecting $(e, b)$ intersects $Y_h, \ldots, Y_{h+|g|_1-1}$ but no other $Y_i$. \[ \]

The claim implies what we need. Note that $P^n_h = \Psi^{-1}(Y_h) \cap Q^n$, and define $P^n_h^* = \Psi^{-1}(Y_h^*) \cap Q^n$. Then $P^n_h^* \subset P^n_h$ and evidently
\[ \text{vol}_{d-1} P^n_h^* - \text{vol}_{d-1} P^n_h = O(n^{d-2}). \]

Recalling that $\Phi(x) = x/n$ we have
\[
\sum_{h=0}^{\gamma} \text{vol}_{d-1} P^n_h^* = n^{d-1} \sum_{h=0}^{\gamma} \text{vol}_{d-1} \Phi(P^n_h^*). \]
The sets \( \Phi(P_{n+*}^h) \) tend to a set \( P^h \subset A(g, t) \cap Q^d \) for the same \( t \) as before, so \( \text{vol}_{d-1} \Phi(P_{n+*}^h) = n^{d-1} \text{vol}_{d-1} P^h \cdot (1 + o(1)) \). The sets \( P^h \) for \( h = 0, 1, \ldots, \gamma \) cover \( A(g, t) \cap Q^d \) at most \( |g|_1 \) times. So their total \( d - 1 \)-volume is at most \( |g|_1 \text{vol}_{d-1} A(g, t) \cap Q^d \). Thus

\[
\sum_{i=0}^\gamma \text{vol}_{d-1} P^h_n \leq n^{d-1} \sum_{i=0}^\gamma \text{vol}_{d-1} \Phi(P_{n+*}^h) + O(n^{d-2})
\]

\[
\leq n^{d-1} \sum_{i=0}^\gamma \text{vol}_{d-1} P^h \cdot (1 + o(1))
\]

\[
\leq n^{d-1} |g|_1 \text{vol}_{d-1} A(g, t) \cap Q^d (1 + o(1)).
\]

So indeed

\[
\frac{1}{|g|} \sum_{i=0}^\gamma \text{vol}_{d-1} P^h_n \leq \frac{|g|_1}{|g|} \text{vol}_{d-1} A(g, t) \cap Q^d (1 + o(1))
\]

\[
= V_d(g_0) (1 + o(1))
\]

because \( \frac{|g|_1}{|g|} \text{vol}_{d-1} A(g, t) \cap Q^d \) \( \leq V_d(g_0) \) by the definition of \( V_d(g_0) \).

\[
\end{proof}

8. Proof of Theorem 2.1

We construct a homotopy \( t \mapsto K_t \) where \( t \in [0, 1] \), \( K_t \) is a convex body in \( \mathbb{R}^d \) satisfying \( K_0 = K \), \( K_1 = L \) and the monotonicity condition \( K_t \subset K_s \) for \( t < s \). By monotonicity, boundary cells of \( K_t \) may become inside cells for \( K_s \), and the point in the argument is that whenever a boundary cell is lost, another one emerges.

The simplest homotopy is \( K_t = (1-t)K + tL \), and this works under the following non-degeneracy condition:

(*) whenever \( w \in \partial K_t \cap \mathbb{Z}^d \), then \( w \notin K_s \) for \( s < t \) and \( w \in \text{int} K_s \) for all \( s > t \), and \( K_t \) has an outer normal \( u \) at \( w \in \partial K_t \) with no coordinate zero.

Under this condition the proof is easy. As \( t \) increases, a cell \( C(z) \), say, is boundary for \( K_t \) with \( t < t_0 \) just slightly smaller than \( t_0 \) but \( C(z) \subset K_{t_0} \) and so it becomes inside for \( t > t_0 \). Then there is a vertex \( w \) of \( C(z) \) such that \( w \notin K_t \) for \( t < t_0 \), but of course \( w \in K_{t_0} \) and even \( w \in \partial K_{t_0} \). Let \( H \) be a supporting hyperplane to \( K_{t_0} \) at \( w \) whose outer normal has no zero coordinate. Then \( w \in K_{t_0} \) and \( C(z) \) and \( K_{t_0} \) are on the same side of \( H \).

There is a unique cell \( C(z') \) (unique because of condition (\( * \))) on the other side of \( H \) with \( w \in C(z') \). This unique cell was outside for \( K_t \) with \( t < t_0 \) and becomes boundary for \( K_t \) for \( t \in [t_0, t_0 + \delta] \) for a suitable small \( \delta > 0 \). So when the boundary cell \( C(z) \) is lost at \( t_0 \), another boundary cell appears. Note that \( C(z) \cap H = C(z') \cap H = \{w\} \).

We still have to check that the same cell \( C(z') \) cannot appear twice. So assume the contrary, that is, there is another cell \( C(z^*) \) that is boundary for \( K_t \), for \( t \) slightly smaller
than \( t_0 \) but \( C(z^*) \subset K_{t_0} \) and \( C(z^*) \) has a vertex \( w^* \) with \( w^* \notin K_t \) for \( t < t_0 \) but \( w^* \in \partial K_{t_0} \).

We cannot have \( w = w^* \) here since that would imply \( C(z) = C(z^*) \). Then \( w \) and \( w^* \) are distinct vertices of \( C(z') \) and the segment \( [w, w^*] \) is on the boundary of both \( C(z') \) and \( K_{t_0} \). Then \( [w, w^*] \cap H = \{w\} \) for the previous hyperplane \( H \) supporting \( K_{t_0} \) at \( w \) with no zero coordinate, so \( w^* \notin K_{t_0} \) cannot hold.

To guarantee the non-degeneracy condition we proceed first by assuming that \( K \subset \text{int} L \) and that both \( K \) and \( L \) have smooth boundaries such that for every unit vector \( u \) there is a single point on \( \partial K \) resp. on \( \partial L \) where the outer normal to \( K \) and \( L \) is \( u \). If this were not the case, we can replace \( K, L \) by suitable \((\text{and very close to} \ K \text{ and} \ L)\) convex bodies satisfying these conditions and having the same inside and boundary cells. With the new \( K \) and \( L \) the homotopy \( K_t = (1-t)K + tL \) has the property that for every unit vector \( u \) there is a single point on \( \partial K_t \) where the outer normal to \( K_t \) is \( u \). To see that this is indeed the case, let \( x_K \) and \( x_L \) be the unique points on the boundary of \( K \) and \( L \) with outer normal \( u \). Then the maximum of \( \{ux : x \in K_t\} \) is reached at the unique point \( (1-t)x_K + tx_L \in K_t \), and the outer normal to \( K_t \) there is \( u \).

This condition also guarantees that \( K_t \) has no line segment on its boundary. Assume that, on the contrary, \( \partial K_t \) contains a line segment and let \( u \) be the outer normal to the tangent hyperplane to \( K_t \) containing this segment. Then there is no unique point with outer normal \( u \) as every point on the segment has outer normal \( u \).

Let us see finally that \( K_t \) satisfies condition \((*)\). Assume the cell \( C(z) \) is boundary for \( K_t \) for \((t_0 - \delta, t_0)\) and is inside for \( K_{t_0} \). Then there is a vertex \( w \) of \( C(z) \) on \( \partial K_{t_0} \) with outer normal \( u = (u_1, \ldots, u_d) \) at \( w \) to \( K_{t_0} \). Assume some coordinate of \( u \) is equal to zero, say \( u_1 = 0 \). Either \( w + e_1 \) or \( w - e_1 \) is in \( C(z) \), say \( w + e_1 \). Then the segment \( [w, w + e_1] \) lies both in \( K_{t_0} \) and in \( C(z) \), and actually in the boundary of both because the hyperplane \( \{x : ux = uw\} \) is tangent to both \( K_{t_0} \) and \( C(z) \). \( \blacksquare \)

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