Iterated Extensions and Uniserial Length Categories

Eivind Eriksen

Abstract

In this paper, we study length categories using iterated extensions. We consider the problem of classifying all indecomposable objects in a length category, and the problem of characterizing those length categories that are uniserial. We solve the last problem, and obtain a necessary and sufficient criterion for uniseriality under weak assumptions. This criterion turns out to be known by Amdal and Ringdal already in 1968; we give a new proof that is both elementary and constructive. The first problem is the most fundamental one, and its general solution is “the main and perhaps hopeless purpose of representation theory” according to Gabriel. We solve the problem in the case when the length category is uniserial, using our constructive methods. As an application, we classify all graded holonomic $D$-modules on a monomial curve over the complex numbers, obtaining the most explicit results over the affine line, when $D$ is the first Weyl algebra. Finally, we show that the iterated extensions are completely determined by the noncommutative deformations of its simple factors. This tells us precisely what we can learn about a length category by studying its species; it gives the tangent space of the noncommutative deformation functor, or the infinitesimal deformations, but not the obstructions for lifting these deformations.

Introduction

Let $S = \{S_\alpha : \alpha \in I\}$ be a family of non-zero, pairwise non-isomorphic objects in an Abelian $k$-category $\mathcal{A}$, where $k$ is a field. We consider the minimal full subcategory $\mathcal{A}(S) \subseteq \mathcal{A}$ that contains $S$ and is closed under extensions. The family $S$ is called a family of orthogonal points if $\text{End}(S_\alpha)$ is a division algebra and $\text{Hom}(S_\alpha, S_\beta) = 0$ for all $\alpha, \beta \in I$ with $\alpha \neq \beta$. In this case, $\mathcal{A}(S) \subseteq \mathcal{A}$ is a length category with $S$ as its simple objects.

We use the category $\text{Ext}(S)$ of iterated extensions of $S$ to study the length category $\mathcal{A}(S)$. An iterated extension of $S$ is a couple $(X, C)$ where $X$ is an object in $\mathcal{A}$, and $C$ is a cofiltration $X = C_n \xrightarrow{f_n} C_{n-1} \rightarrow \cdots \rightarrow C_2 \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0 = 0$ where $f_i : C_i \rightarrow C_{i-1}$ is surjective and $K_i = \ker(f_i) \cong S_{\alpha(i)}$ with $\alpha(i) \in I$ for $1 \leq i \leq n$. Hence the assignment $(X, C) \mapsto X$ defines a forgetful functor $\text{Ext}(S) \rightarrow \mathcal{A}(S)$. When we work with the category $\text{Ext}(S)$ of iterated extensions, the order vector $\mathbf{n} = (\alpha(1), \ldots, \alpha(n)) \in I^n$ is an invariant, in addition to the usual invariants in the length category $\mathcal{A}(S)$ such as the length $n$, the simple factors $\{K_1, \ldots, K_n\}$, and their multiplicities.

An important special case is when $\mathcal{A} = \text{Mod}_A$ is the category of modules over an associative $k$-algebra $A$, and $S$ is a subset of the simple $A$-modules. If $S$ is the family of all simple modules, then $\mathcal{A}(S)$ is the category of all modules of finite length. There are also many other interesting applications, for example when $\mathcal{A}$ is the category of graded modules over a graded $k$-algebra, or the category of coherent sheaves over a $k$-scheme. Note that any length category is exact equivalent to an exact subcategory of a module category. Nevertheless, it is often better to work directly in the Abelian category of interest than to use such an embedding.

We say that $\mathcal{A}(S)$ is a uniserial length category if any indecomposable object in $\mathcal{A}(S)$ has a unique composition series, and that at point $S$ in $S$ is $k$-rational if $\text{End}(S) = k$. When $S$ is a
family of \(k\)-rational orthogonal points, we show that \(\mathcal{A}(S)\) is a uniserial length category if and only if the family \(S\) satisfies the condition

\[
(\text{UC}) \quad \sum_{\beta \in I} \dim_k \text{Ext}^1_{\mathcal{A}}(S_\alpha, S_\beta) \leq 1 \quad \text{for all } \alpha \in I
\]

\[
\sum_{\alpha \in I} \dim_k \text{Ext}^1_{\mathcal{A}}(S_\alpha, S_\beta) \leq 1 \quad \text{for all } \beta \in I
\]

It turns out that the condition (\text{UC}) and the characterization of uniserial length categories was known already in the 60’s; see Section 8.3 in Gabriel [7]. As far as we know, it first appeared in Amdal, Ringdal [1], where it is stated without proof. We give an elementary and constructive proof of the result that \(\mathcal{A}(S)\) is uniserial if and only if (\text{UC}) is satisfied, using the category \(\text{Ext}(S)\) of iterated extensions. In fact, after showing that the condition is necessary, we explicitly construct all indecomposable objects in \(\mathcal{A}(S)\) when (\text{UC}) holds, and prove that these objects are uniserial.

**Theorem.** Let \(S = \{S_\alpha : \alpha \in I\}\) be a family of orthogonal \(k\)-rational points in an Abelian \(k\)-category \(\mathcal{A}\). If \(S\) satisfies (\text{UC}), then the indecomposable objects in \(\mathcal{A}(S)\) of length \(n\) are given by \(\{X(\mathbf{a}) : \mathbf{a} \in J\}\), up to isomorphism in \(\mathcal{A}(S)\), where the subset \(J \subseteq I^n\) consists of the vectors \(\mathbf{a}\) such that the following conditions hold:

(i) \(\text{Ext}^1_{\mathcal{A}}(S_{\alpha(i-1)}, S_{\alpha(i)}) \neq 0\) for \(2 \leq i \leq n\)

(ii) If \(\sigma_i \in \text{Ext}^1_{\mathcal{A}}(S_{\alpha(i-1)}, S_{\alpha(i)})\) is non-zero for \(2 \leq i \leq n\), then the matric Massey product \(\langle \sigma_2, \sigma_3, \ldots, \sigma_n \rangle\) is defined and contains zero.

Moreover, the indecomposable objects \(X(\mathbf{a})\) are uniserial, and can be constructed from the family \(S\) and their extensions.

As an application, we show that the category \(\text{grHol}_D\) of graded holonomic \(D\)-modules is uniserial when \(D = \text{Diff}(A)\) is the ring of differential operators on a monomial curve \(A\) defined over the field \(k = \mathbb{C}\) of complex numbers. Moreover, we classify all indecomposable objects in \(\text{grHol}_D\). We build upon the results in Eriksen [4], where we studied this category. We obtain the most explicit result in the case when \(A = k[t]\) and \(D = A_1(k)\) is the first Weyl algebra. The classification is similar in the other cases, since all rings of differential operators on monomial curves are Morita equivalent.

**Theorem.** Let \(D = A_1(k)\) be the first Weyl algebra. Then the category \(\text{grHol}_D\) of graded holonomic \(D\)-modules is uniserial, and the indecomposable \(D\)-modules in \(\text{grHol}_D\) are, up to graded isomorphisms and twists, given by

\[
M(\alpha, n) = D/D (E - \alpha)^n, \quad M(\beta, n) = D/D w(\beta, n)
\]

where \(n \geq 1\), \(\alpha \in J^* = \{\alpha \in k : 0 \leq \text{Re}(\alpha) < 1, \alpha \neq 0\}\), \(\beta \in \{0, \infty\}\), and \(w(\beta, n)\) is the alternating word on \(n\) letters in \(t\) and \(\partial\), ending with \(\partial\) if \(\beta = 0\), and in \(t\) if \(\beta = \infty\).

In the last section, we prove that for a swarm \(S\) of orthogonal points in an Abelian \(k\)-category \(\mathcal{A}\), the iterated extensions of the family \(S\) are completely determined by the noncommutative deformations of its simple factors. Hence the length category \(\mathcal{A}(S)\) is also determined by these deformations. If the noncommutative deformations are unobstructed, then they are determined by the species of \(\mathcal{A}(S)\). This is the case for modules over a hereditary ring, such as the ring \(D\) of differential operators on a monomial curve over the complex numbers. In general, we need both the species of \(\mathcal{A}(S)\), which defines the noncommutative deformations on the tangent level, and the obstructions for lifting these deformations, to determine the iterated extensions in \(\text{Ext}(S)\).
1. Iterated extensions

Let $k$ be a field, let $\mathcal{A}$ be an Abelian $k$-category, and let $S = \{S_\alpha : \alpha \in I\}$ be a fixed family of non-zero, pairwise non-isomorphic objects in $\mathcal{A}$. In this section, we define the category $\Ext(S)$ of iterated extensions of the family $S$, equipped with a forgetful functor $\Ext(S) \to \mathcal{A}(S)$ into the minimal full subcategory $\mathcal{A}(S) \subseteq \mathcal{A}$ that contains $S$ and is closed under extensions, and study its properties.

An object of $\Ext(S)$ is a couple $(X, C)$, where $X$ is an object of the category $\mathcal{A}$ and $C$ is a cofiltration of $X$ in $\mathcal{A}$ of the form

$$X = C_n \xrightarrow{f_n} C_{n-1} \to \cdots \to C_2 \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0 = 0$$

where $f_i : C_i \to C_{i-1}$ is surjective and $K_i = \ker(f_i) \cong S_{\alpha(i)}$ with $\alpha(i) \in I$ for $1 \leq i \leq n$. The integer $n \geq 0$ is called the length, the objects $K_1, \ldots, K_n$ are called the factors, and the vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ is called the order vector of the iterated extension $(X, C)$.

Let $(X, C)$ and $(X', C')$ be a pair of objects in $\Ext(S)$ of lengths $n, n' \geq 0$. A morphism $\phi : (X, C) \to (X', C')$ in $\Ext(S)$ is a collection $\{\phi_i : 0 \leq i \leq N\}$ of morphisms $\phi_i : C_i \to C_i'$ in $\mathcal{A}$ such that $\phi_{i-1} f_i = f_i' \phi_i$ for $1 \leq i \leq N$, where $N = \max\{n, n'\}$. By convention, $C_i = X$ for all $i > n$ and $C_i' = X'$ for all $i > n'$.

The category $\Ext(S)$ has a dual category defined by filtrations. An object of this category is a couple $(X, F)$, where $X$ is an object of $\mathcal{A}$ and $F$ is a filtration of $X$ in $\mathcal{A}$ of the form

$$0 = F_n \subseteq F_{n-1} \subseteq \cdots \subseteq F_0 = X$$

such that $K_i = F_i - F_i/ F_i \cong S_{\alpha(i)}$ with $\alpha(i) \in I$ for $1 \leq i \leq n$. Given an object $(X, F)$ in the dual category, the corresponding object in $\Ext(S)$ is $(X, C)$, where the cofiltration $C$ is defined by $C_i = X/ F_i$ for $0 \leq i \leq n$, with the natural surjections $f_i : C_i \to C_{i-1}$. Conversely, if the object $(X, C)$ in $\Ext(S)$ is given, then the corresponding filtration of $X$ is given by $F_i = \ker(X \to C_i)$ for $0 \leq i \leq n$, where $X \to C_i$ is the composition $f_{i+1} \circ \cdots \circ f_n : C_n \to C_i$. It is clear from the construction that the dual objects $(X, C)$ and $(X, F)$ have the same length, the same factors, and the same order vector.

We recall that a short exact sequence $0 \to Y \to Z \to X \to 0$ in $\mathcal{A}$ is called an extension of $X$ by $Y$, and that $\Ext^1(\mathcal{A}, X, Y)$ denotes the set of all extensions of $X$ by $Y$, modulo equivalence. The set $\Ext^1(\mathcal{A}, X, Y)$ has a natural $\End(\mathcal{A}, Y) - \End(\mathcal{A}, X)$ bimodule structure, inherited from the bimodule structure on $\Hom(\mathcal{A}, X, Y)$.

As the name suggests, the category $\Ext(S)$ can be characterized in terms of extensions. In fact, for any object $(X, C)$ in $\Ext(S)$ of length $n$ and for any integer $i$ with $2 \leq i \leq n$, the cofiltration $C$ induces a commutative diagram

$$
\begin{array}{ccc}
0 & \to & K_i & \to & C_i & \xrightarrow{f_i} & C_{i-1} & \to & 0 \\
 & | & \downarrow & & \downarrow & & & |
\end{array}
$$

in $\mathcal{A}$, where the rows are exact and $Z = f_i^{-1}(K_{i-1})$. We define $\xi_i \in \Ext^1(\mathcal{A}, C_{i-1}, K_i)$ and $\tau_i \in \Ext^1(\mathcal{A}, K_{i-1}, K_i)$ to be the extensions corresponding to the upper and lower row. By construction, $\xi_i \mapsto \tau_i$ under the map $\Ext^1(\mathcal{A}, C_{i-1}, K_i) \to \Ext^1(\mathcal{A}, K_{i-1}, K_i)$ induced by the inclusion $K_{i-1} \subseteq C_{i-1}$. In particular, $C_2$ is an extension of $C_1 = K_1$ by $K_2$, $C_3$ is an extension of $C_2$ by $K_3$, and in general, $C_{i+1}$ is an extension of $C_i$ by $K_{i+1}$ for $1 \leq i \leq n - 1$. It follows that $X = C_n$ is obtained from the factors $\{K_1, \ldots, K_n\} \subseteq S$ by an iterated use of extensions, and this justifies the name *iterated extensions*.

Let us consider the natural forgetful functor $\Ext(S) \to \mathcal{A}$ given by $(X, C) \mapsto X$, and the full subcategory $\mathcal{A}(S) \subseteq \mathcal{A}$ defined in the following way: An object $X$ in $\mathcal{A}$ belongs to $\mathcal{A}(S)$
if there exists a cofiltration $C$ of $X$ such that $(X, C)$ is an object of $\text{Ext}(S)$. The following lemma proves that $\mathcal{A}(S) \subseteq \mathcal{A}$ is the minimal full subcategory that contains $S$ and is closed under extensions:

**Lemma 1.1.** Let $(X', C'), (X'', C'')$ be iterated extensions of the family $S$. If $X$ is an extension of $X'$ by $X''$ in $\mathcal{A}$, then there is a cofiltration $C$ of $X$ such that $(X, C)$ is an iterated extension of the family $S$. In particular, the full subcategory $\mathcal{A}(S) \subseteq \mathcal{A}$ is closed under extensions.

**Proof.** Let us assume that $(X', C')$ and $(X'', C'')$ are iterated extensions of the family $S$ of lengths $n', n''$. Since $X$ is an extension of $X''$ by $X'$, we can construct a cofiltration of $X$ of length $n = n' + n''$ in the following way: Let $f : X' \to X$ and $g : X \to X''$ be the maps given by the extension $0 \to X' \to X \to X'' \to 0$, let $F'$ be the filtration of $X'$ dual to the cofiltration $C'$, and let $F''$ be the filtration of $X''$ dual to the cofiltration $C''$. We define $F_i = g^{-1}(F''_i)$ for $0 \leq i \leq n''$, and $F_i = f(F'_{i-n'})$ for $n'' \leq i \leq n$. Then $F$ is a filtration of $X$, and $F_{i-1}/F_i \cong \ker(X \to C''_{i-1})/\ker(X \to C''_i) \cong K''_i$ for $0 \leq i \leq n''$, and $F_{i-1}/F_i \cong K'_{i-n''}$ for $n'' \leq i \leq n$. Let $C$ be the cofiltration of $X$ dual to the filtration $F$. Then it follows by construction that $(X, C)$ is an iterated extension of the family $S$ of length $n$. □

We recall that $\mathcal{A}(S) \subseteq \mathcal{A}$ is called an exact Abelian subcategory if the inclusion functor $\mathcal{A}(S) \to \mathcal{A}$ is an exact functor. It is well-known that this is the case if and only if $\mathcal{A}(S)$ is closed in $\mathcal{A}$ under kernels, cokernels and finite direct sums. It is clear that $\mathcal{A}(S)$ is closed under finite direct sums since it closed under extensions. But in general, it is not closed under kernels and cokernels.

**Proposition 1.2.** The full subcategory $\mathcal{A}(S) \subseteq \mathcal{A}$ is an exact Abelian subcategory if and only if the following conditions hold:

(i) $\text{End}_\mathcal{A}(S_\alpha)$ is a division algebra for all $\alpha \in I$

(ii) $\text{Mor}_\mathcal{A}(S_\alpha, S_\beta) = 0$ for all $\alpha, \beta \in I$ with $\alpha \neq \beta$

If this is the case, then $S$ is the set of simple objects in $\mathcal{A}(S)$, up to isomorphism.

**Proof.** This follows from Theorem 1.2 in Ringel [10], and the comments preceding it. □

Let us use the notation from Ringel [10], and say that an object $X$ in $\mathcal{A}$ is a point if $\text{End}_\mathcal{A}(X)$ is a division ring, and that two points $X, Y$ in $\mathcal{A}$ are orthogonal if $\text{Mor}_\mathcal{A}(X, Y) = 0$ and $\text{Mor}_\mathcal{A}(Y, X) = 0$. Moreover, we shall write $k(X) = \text{End}_\mathcal{A}(X)$ for the division algebra over $k$ associated with a point $X$, and say that $X$ is a $k$-rational point if $k(X) = k$.

2. Length categories

A length category is an Abelian category such that any of its objects has finite length, and such that the isomorphism classes of objects form a set. We recall some well-known facts about length categories; see for instance Gabriel [17]:

(1) The Jordan-Hölder Theorem: Any object $X$ in a length category has a composition series; that is, it has a filtration

$$0 = F_n \subseteq F_{n-1} \subseteq \cdots \subseteq F_1 \subseteq F_0 = X$$
such that $K_i = F_{i-1}/F_i$ is a simple object for $1 \leq i \leq n$. The length $n$ and the simple factors $K_1, \ldots, K_n$ in a composition series are unique, up to a permutation of the simple factors.

(2) The Krull-Schmidt Theorem: Any object $X$ in a length category is a finite direct sum

$$X = X_1 \oplus X_2 \oplus \cdots \oplus X_r$$

of indecomposable objects. The indecomposable direct summands $X_1, \ldots, X_r$ are unique, up to a permutation.

(3) Mitchell’s Embedding Theorem: A length category is exact equivalent to an exact subcategory of $\text{Mod}_A$ for an associative ring $A$.

Let $S$ be a family of orthogonal points in an Abelian $k$-category $A$. It follows from Proposition 1.2 that $A(S)$ is a length category, with $S$ as its simple objects. In fact, any length category which is an Abelian $k$-category is of this type.

Our goal is to classify and explicitly construct the indecomposable objects in the length category $A(S)$. Even though this is a quite hopeless task in general, we prove a classification result in the special case of uniserial length categories in Section 3 and 4. Our philosophy is to start with the family $S$, and use iterated extensions in $\text{Ext}(S)$ to build larger indecomposable modules.

The species of the length category $A(S)$ consists of the family $\{k(S_\alpha) : \alpha \in I\}$ of division algebras of its simple objects, and the family $\{\text{Ext}^1_A(S_\alpha, S_\beta) : \alpha, \beta \in I\}$ of $k(S_\beta)$-$k(S_\alpha)$ bimodules of extensions. If $S$ is a family of orthogonal $k$-rational points, the species of $A(S)$ can be represented by a quiver $\Lambda$, with $I$ as nodes, and with $\dim_k \text{Ext}^1_A(S_\alpha, S_\beta)$ arrows from node $\alpha$ to node $\beta$ for all $\alpha, \beta \in I$. The quiver $\Lambda$ (and more generally, the species) of the length category $A(S)$ contains a lot of information about $A(S)$ and its indecomposable objects.

In fact, we shall show in Section 3 that the iterated extensions in $\text{Ext}(S)$ are completely determined by noncommutative deformations of its simple factors. In the unobstructed case, these deformations are determined by the species of $A(S)$.

### 3. Uniserial length categories

Let $S$ be a family of orthogonal $k$-rational points in an Abelian $k$-category $A$, and let $A(S)$ be the corresponding length category. We denote by $\Lambda$ the quiver of the species of $A(S)$.

We say that an object $X$ in $A(S)$ is uniserial if its lattice of subobjects is a chain. If this is the case, then this chain is the unique decomposition series of $X$. It follows that $X$ is uniserial if and only if any two cofiltrations of $X$ are isomorphic. Any uniserial object in $A(S)$ is indecomposable, but the opposite implication does not hold in general. We say that $A(S)$ is a uniserial category if every indecomposable object in $A(S)$ is uniserial.

**Lemma 3.1.** Let $X$ be an object in $A(S)$, and consider the following conditions:

(i) $X$ is uniserial

(ii) $X$ has a unique minimal subobject $S \subseteq X$

(iii) $X$ is indecomposable

Then we have $(1) \Rightarrow (2) \Rightarrow (3)$. In particular, all conditions are equivalent if and only if $A(S)$ is a uniserial category.

**Proof.** The implication $(1) \Rightarrow (2)$ is obvious. To prove $(2) \Rightarrow (3)$, let $X = Y_1 \oplus Y_2$ be a direct decomposition of $X$ with $Y_1, Y_2 \neq 0$. Then there are minimal subobjects $S_i \subseteq Y_i$ in $A(S)$ for $i = 1, 2$ and this contradicts $(2)$. The last part follows directly from the definition. \[\square\]
The implication (3) ⇒ (1) in Lemma 3.1 clearly holds if X has length n = 2, since an indecomposable object of length 2 is a non-split extension of two objects in S. But already for n = 3, it is easy to find examples where this implication fails:

**Lemma 3.2.** If S contains orthogonal k-rational points S, T with \( \dim_k \text{Ext}^1_A(S, T) \geq 2 \), then there exists an indecomposable, non-uniserial object in \( A(S) \) of length n = 3.

**Proof.** Notice that there exist non-split extensions \( U, V \) of S by T such that \( U \) and \( V \) are not isomorphic in \( A(S) \). In fact, if \( U, V \) are non-split extensions of S by T, then any isomorphism \( u : U \to V \) satisfies \( u(T) \subseteq T \) since \( T \) is the unique minimal subobject of \( U, V \) in \( A(S) \). Therefore, \( u \) induces automorphisms on \( T \), and on \( S \), which are given by multiplication in \( k^* \) since \( S, T \) are \( k \)-rational points. This means that not all non-split extensions are isomorphic in \( A(S) \); otherwise, we would have that \( \dim_k \text{Ext}^1_A(S, T) \leq 1 \). We define \( X = \text{coker}(f) \), where \( f : T \to U \oplus V \) is the diagonal map, and consider the short exact sequence

\[
0 \to T \to U \oplus V \to X \to 0
\]

We see that \( X \) has length n = 3, that \( U, V \subseteq X \) are subobjects in \( A(S) \) of length n = 2, and that \( T \) is the unique minimal subobject of \( X \) in \( A(S) \). In fact, if \( T' \) is another minimal subobject of \( X \), then \( T' \) is not contained in \( U, V \) since they are uniserial. Hence \( U \oplus T' = X = V \oplus T' \), and this implies that \( U \) is isomorphic to \( V \), which is a contradiction. Since \( X \) has a unique minimal subobject in \( A(S) \), it is indecomposable, and it is non-uniserial since

\[
0 \subseteq T \subseteq U \subseteq X \quad \text{and} \quad 0 \subseteq T \subseteq V \subseteq X
\]

are different composition series of \( X \).

**Lemma 3.3.** If S contains points S, T, U such that \( \text{Ext}^1_A(U, S), \text{Ext}^1_A(U, T) \neq 0 \) and \( S, T \) are orthogonal, then there exists an indecomposable object in \( A(S) \) of length n = 3 with S, T as minimal subobjects.

**Proof.** Let \( \xi_1, \xi_2 \neq 0 \) be non-split extensions of U by S, and of U by T, given by short exact sequences

\[
0 \to S \to E_1 \xrightarrow{g_1} U \to 0 \quad \text{and} \quad 0 \to T \to E_2 \xrightarrow{g_2} U \to 0
\]

Define \( X \subseteq E_1 \oplus E_2 \) to be the pullback of \( g_1 \) and \( g_2 \). Then the short exact sequence

\[
0 \to S \oplus T \to X \to U \to 0
\]

represents the direct sum extension \( (\xi_1, \xi_2) \in \text{Ext}^1_A(U, S \oplus T) \). It is clear that \( S, T \) are minimal subobjects of \( X \). Moreover, \( X \) is clearly indecomposable; otherwise, it would have \( S \) or \( T \) as a direct summand, and this is not possible since \( \xi_1, \xi_2 \neq 0 \).

**Lemma 3.4.** If S contains points S, T, U such that \( \text{Ext}^1_A(S, U), \text{Ext}^1_A(T, U) \neq 0 \) and \( S, T \) are orthogonal, then there exists an indecomposable, non-uniserial object in \( A(S) \) of length n = 3.

**Proof.** Let \( \xi_1, \xi_2 \neq 0 \) be non-split extensions of S by U, and of T by U, given by short exact sequences

\[
0 \to U \xrightarrow{f_1} E_1 \to S \to 0 \quad \text{and} \quad 0 \to U \xrightarrow{f_2} E_2 \to T \to 0
\]
Define $X$ be the push-out of $f_1$ and $f_2$. Then the induced short exact sequence
$$0 \to U \to X \to S \oplus T \to 0$$
represents the direct sum extension $(\xi_1, \xi_2) \in \text{Ext}^1_A(S \oplus T, U)$. Clearly, there are natural injections $E_1 \to X$ and $E_2 \to X$, which gives the composition series
$$0 \subseteq U \subseteq E_1 \subseteq X \quad \text{and} \quad 0 \subseteq U \subseteq E_2 \subseteq X$$
of $X$. Hence, $X$ is not uniserial. Moreover, $U$ is the unique minimal subobject of $X$ since $\xi_1, \xi_2 \neq 0$, and it follows that $X$ is indecomposable.

**Proposition 3.5.** Let $S = \{S_\alpha : \alpha \in I\}$ be a family of orthogonal $k$-rational points in an Abelian $k$-category $A$, and let $\Lambda$ be the quiver of the species of the length category $A(S)$. If $A(S)$ is uniserial, then following conditions hold:

(i) For any $\alpha \in I$, there is at most one arrow in $\Lambda$ with source $\alpha$.

(ii) For any $\beta \in I$, there is at most one arrow in $\Lambda$ with target $\beta$.

**Proof.** It follows from Lemma 3.2, Lemma 3.3 and Lemma 3.4 that the length category $A(S)$ is not uniserial if the quiver $\Lambda$ contains a subquiver of one of the forms

$$\alpha \to \beta \to \beta \to \beta' \to \alpha \quad \text{with} \quad \beta \neq \beta' \in \text{the middle quiver} \quad \text{and} \quad \alpha \neq \alpha' \in \text{the right quiver}.$$  

The two conditions in Proposition 3.5 are equivalent to the following condition, which we call the uniseriality criterion and refer to as (UC):

$$(UC) \quad \sum_{\beta \in I} \dim_k \text{Ext}^1_A(S_\alpha, S_\beta) \leq 1 \quad \text{for all} \quad \alpha \in I$$
$$\sum_{\alpha \in I} \dim_k \text{Ext}^1_A(S_\alpha, S_\beta) \leq 1 \quad \text{for all} \quad \beta \in I$$

We claim that the length category $A(S)$ is uniserial if and only if the condition (UC) holds. It turns out that this was known already in the 60’s; see Section 8.3 in Gabriel [7]. As far as we know, this result first appeared in Amdal, Ringdal [1], where it is stated without proof.

We shall give an elementary and constructive proof of this characterization in the next section. The proof is constructive in the sense that we classify and explicitly construct all indecomposable objects in $A(S)$ when (UC) holds, and show that these indecomposable objects are uniserial.

**4. Construction of indecomposable objects**

Let $S$ be a family of orthogonal $k$-rational points in an Abelian $k$-category $A$, and let $A(S)$ be the corresponding length category. We denote by $\Lambda$ the quiver of the species of $A(S)$, and assume that the condition (UC) holds. In this situation, we shall classify and explicitly construct all indecomposable objects in $A(S)$.

We consider the full subcategory $\text{Ext}(S, n, *) \subseteq \text{Ext}(S)$ of iterated extensions $(X, C)$ of length $n$ such that $(\xi_2, \ldots, \xi_n) \neq 0$. Any indecomposable object $X$ in $A(S)$ of length $n$ has a cofiltration $C$ such that $(X, C)$ is an iterated extension in $\text{Ext}(S)$ with $\xi_n \neq 0$. The idea is that many indecomposable objects, though not necessarily all, have a cofiltration $C$ such that $(X, C)$ is in $\text{Ext}(S, n, *)$, and we start by classifying these indecomposable objects.
Lemma 4.1. Let \((X, C)\) be an iterated extension in \(\text{Ext}(S, n, \ast)\). If \(S\) satisfies (UC), then the \(k\)-linear map

\[
\text{Ext}^1_A(C_{i-1}, K) \rightarrow \text{Ext}^1_A(K_{i-1}, K)
\]

induced by the inclusion \(K_{i-1} \subseteq C_{i-1}\) is an isomorphism for all integers \(i\) such that \(2 \leq i \leq n\) and for all simple objects \(K \in S\). In particular, \(\tau_i \neq 0\) for \(2 \leq i \leq n\).

Proof. We show the result by induction on \(n\). Since \(C_1 = K_1\) by definition, the result is clearly true for \(n = 2\). So let \(n \geq 3\), and assume that the result holds for all integers less than \(n\) and all simple objects \(K \in S\). In particular, this implies that

\[
\text{Ext}^1_A(C_{n-2}, K) \rightarrow \text{Ext}^1_A(K_{n-2}, K)
\]

is an isomorphism. Since \(\xi_{n-1} \mapsto \tau_{n-1}\) under this map when \(K = K_{n-1}\), it follows that \(\tau_{n-1} \neq 0\), and in particular, that \(\text{Ext}^1_A(K_{n-2}, K_{n-1}) = k \cdot \tau_{n-1} \cong k\) by (UC). Hence we also have \(\text{Ext}^1_A(C_{n-2}, K) = k \cdot \xi_{n-1} \cong k\). Let us consider the long exact sequence of the functor \(\text{Hom}_A(\ast, K)\) applied to the extension \(\xi_{n-1}\), given by

\[
\cdots \rightarrow \text{Hom}_A(C_{n-1}, K) \rightarrow \text{Hom}_A(K_{n-1}, K) \rightarrow \text{Ext}^1_A(C_{n-2}, K) \rightarrow \text{Ext}^1_A(C_{n-1}, K) \rightarrow \text{Ext}^1_A(K_{n-1}, K) \rightarrow \cdots
\]

For all simple objects \(K \in S\), we claim that \(\text{Hom}_A(K_{n-1}, K) \cong \text{Ext}^1_A(K_{n-2}, K)\) and that \(\text{Hom}_A(K_{n-1}, K) \rightarrow \text{Ext}^1_A(C_{n-2}, K)\) is an isomorphism: If \(K = K_{n-1}\), we have that \(\text{End}_A(K_{n-1}) \cong k\) and that \(\text{Ext}^1_A(K_{n-2}, K_{n-1}) = k \cdot \tau_{n-1} \cong k\). This proves the claim, since \(\text{Ext}^1_A(C_{n-2}, K_{n-1}) = k \cdot \xi_{n-1} \cong k\) by the comments above and the identity on \(K_{n-1}\) maps to \(\xi_{n-1}\) by construction. If \(K\) is not isomorphic to \(K_{n-1}\), we have that \(\text{Hom}_A(K_{n-1}, K) = 0\) and that \(\text{Ext}^1_A(K_{n-2}, K) = 0\) by orthogonality and (UC). This proves the claim, since \(\text{Ext}^1_A(C_{n-2}, K) \cong \text{Ext}^1_A(K_{n-2}, K) = 0\) by the induction hypothesis. We conclude that for all simple objects \(K \in S\), the \(k\)-linear map

\[
\text{Ext}^1_A(C_{n-1}, K) \rightarrow \text{Ext}^1_A(K_{n-1}, K)
\]

is injective, and it is enough to show that it is an isomorphism to conclude the proof. If \(K = K_n\), then \(\xi_n \mapsto \tau_n\) under this map, and by injectivity, it follows that \(\tau_n \neq 0\) since \(\xi_n \neq 0\). Therefore, \(\text{Ext}^1_A(K_{n-1}, K_n) = k \cdot \tau_n \cong k\) by (UC), and the map is an isomorphism. If \(K\) is not isomorphic to \(K_n\), then it follows from (UC) that \(\text{Ext}^1_A(K_{n-1}, K) = 0\), and the map is an isomorphism also in this case.

We consider the map \(v_n : \text{Ext}(S, n, \ast) \rightarrow I^n\), which maps an iterated extension \((X, C)\) to its order vector \(\alpha\), and say that a vector \(\alpha \in I^n\) is admissible if \(\alpha \in \im(v_n)\). For any iterated extension \((X, C)\) in \(\text{Ext}(S, n, \ast)\), it follows from Lemma \([1]\) that \(\tau_i \neq 0\) for \(2 \leq i \leq n\), and that

\[
\text{Ext}^1_A(K_{i-1}, K_i) = k \cdot \tau_i \cong k \quad \text{and} \quad \text{Ext}^1_A(C_{i-1}, K_i) = k \cdot \xi_i \cong k
\]

This means that if \(\alpha\) is admissible, and \((X, C), (X', C')\) are two iterated extensions in \(\text{Ext}(S, n, c)\) with order vector \(\alpha\), then \(X \cong X'\) in \(A(S)\). In fact, we have that \(\xi_i = c_i \xi_i\) for \(2 \leq i \leq n\) with \(c_i \in k^*\), and it is well-known that the extensions of \(C_{i-1}\) by \(K_i\) in the same \(k^*\) orbit of \(\text{Ext}^1_A(C_{i-1}, K_i)\) are isomorphic in \(A(S)\).

Let \(\alpha \in I^n\) be an admissible vector. Then there is an iterated extension \((X, C)\) in \(\text{Ext}(S, n, \ast)\) with order vector \(\alpha\) and it follows from the comments above that \(X\) is unique, up to isomorphism in \(A(S)\). We shall write \(X(\alpha)\) for this object in \(A(S)\) when \(\alpha\) is admissible. It turns out that \(X(\alpha)\) is an indecomposable and uniserial object in \(A(S)\):
Proposition 4.2. Let \( \alpha \in I^n \) be an admissible vector. If \( S \) satisfies (UC), then \( X(\alpha) \) is indecomposable and uniserial.

Proof. We claim that if \((X,C)\) is an iterated extension in \( \text{Ext}(S,n,*), \) then there is a unique minimal subobject of \( X \). The claim clearly holds if \( n = 2 \), and we shall prove the claim by induction on \( n \). We therefore assume that \( n \geq 3 \), and that the claim holds for all iterated extensions of length less than \( n \). To prove that it holds for iterated extensions of length \( n \), it is enough to prove to \( \phi(K) = K_n \) for any injective homomorphism \( \phi : K \rightarrow X \). We consider the commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & K_n \\
\downarrow & & \downarrow \\
0 & \rightarrow & C_n \rightarrow C_{n-1} \rightarrow 0 \\
\end{array}
\]

given by the cofiltration \( C \), where the horizontal rows represent \( \tau_n \) and \( \xi_n \). Assume that \( \phi(K) \neq K_n \), which implies that \( \phi(K) \cap K_n = 0 \) since \( K_n \) is simple, and consider the induced morphism \( f_n \circ \phi : K \rightarrow C_{n-1} \). We claim that this morphism is injective. In fact, we have that \( \ker((f_n \circ \phi) = \phi^{-1}(K_n) = 0 \). By the induction hypothesis, this means that \( f_n \circ \phi(K) = K_{n-1} \). This implies that \( \phi(K) \subseteq f_n^{-1}(K_{n-1}) \), which is a contradiction since \( \tau_n \neq 0 \) by Lemma 4.1 and it follows that \( \phi(K) = K_n \). We have therefore proven the induction step, which means that \( X \) is indecomposable with a unique minimal submodule in \( A(S) \). Finally, \( X \) is uniserial since \( C_i \) has a unique minimal submodule in \( A(S) \) for \( 2 \leq i \leq n \). In fact, we can see this by applying the argument above to the iterated extension \((C_i,C')\) in \( \text{Ext}(S,i,*) \), where \( C' \) is the cofiltration

\[
C_i \xrightarrow{f_i} C_{i-1} \rightarrow \cdots \rightarrow C_2 \xrightarrow{f_2} C_1 \rightarrow C_0 = 0
\]

obtained by capping \( C \) at \( C_i \).

The next step in the classification, is to characterize the vectors \( \alpha \in I^n \) that are admissible. If \( \alpha \) is admissible, then by definition there is an iterated extension \((X,C)\) in \( \text{Ext}(S,n,*) \) with order vector \( \alpha \), and it follows from Lemma 4.1 that

\[
\text{Ext}^1_A(K_{i-1}, K_i) = k \cdot \tau_i \neq 0
\]

for \( 2 \leq i \leq n \). This means that \( \alpha \) corresponds to a path of length \( n - 1 \) in the quiver \( \Lambda \), with an arrow from node \( \alpha(i-1) \) to node \( \alpha(i) \) for \( 2 \leq i \leq n \).

Conversely, if \( \alpha \in I^n \) is a vector corresponding to a path of length \( n - 1 \) in the quiver \( \Lambda \), such that

\[
\text{Ext}^1_A(K_{i-1}, K_i) \neq 0
\]

for \( 2 \leq i \leq n \), with \( K_i = S_{\alpha(i)} \), then we may choose \( \sigma_i \in \text{Ext}^1_A(K_{i-1}, K_i) \) with \( \sigma_i \neq 0 \). The vector \( \alpha \) is admissible if there is an iterated extensions \((X,C)\) of length \( n \) in \( \text{Ext}(S) \) with \( \tau_i = \sigma_i \) for \( 2 \leq i \leq n \), where \( \tau_i \) is the extension induced by the cofiltration \( C \). This is clearly the case when \( n = 2 \), since \( \sigma_2 \neq 0 \) is a non-split extension

\[
0 \rightarrow K_2 \rightarrow E \rightarrow K_1 \rightarrow 0
\]

of \( K_1 \) by \( K_2 \). In fact, we may choose \( X = E \) and the cofiltration \( C \) of length \( n = 2 \) given by \( E \rightarrow K_1 \rightarrow 0 \), which has \( \tau_2 = \sigma_2 \). However, if \( n \geq 3 \), there are obstructions for the existence of such an iterated extension:
Proposition 4.3. Let \( \alpha \in I^n \) be a vector corresponding to a path of length \( n-1 \) in the quiver \( \Lambda \), and choose a non-zero extension \( \sigma_i \in \text{Ext}^1(\Lambda(K_{i-1}, K_i)) \) for \( 2 \leq i \leq n \). If \( S \) satisfies (UC), then \( \alpha \) is admissible if and only if the matrix Massey product
\[
\langle \sigma_2, \sigma_3, \ldots, \sigma_n \rangle
\]
is defined and contains zero.

Proof. Since \( \mathcal{A}(S) \) is exact equivalent to an exact subcategory of a category of modules over an associative \( k \)-algebra, we may assume that \( \mathcal{A} \) is such a module category without loss of generality. In this case, the result follows from Proposition 4 in Eriksen, Sivveland [6], and the preceding construction.

We remark that the use of an exact embedding of \( \mathcal{A} \) into a module category is a choice of convenience in the proof of Proposition 4.3 and not essential. Matrix Massey products are tied to noncommutative deformations and may be computed directly in many Abelian \( k \)-categories; see Eriksen, Laudal, Sivveland [5].

We say that \( \mathcal{A}(S) \) is a hereditary length category if \( \text{Ext}^2(S, T) = 0 \) for any objects \( S, T \in S \). If this is the case, then the obstruction in Proposition 4.3 vanishes. This is clear from the construction of matrix Massey products.

We claim that any indecomposable object \( X \) in \( \mathcal{A}(S) \) has the form \( X(\alpha) \) for an admissible vectors \( \alpha \in I^n \), and this would complete the classification. To prove the claim, we must show that any indecomposable object \( X \) in \( \mathcal{A}(S) \) has a cofiltration \( C \) such that \( (X, C) \in \text{Ext}(S, n, *) \):

Proposition 4.4. Let \( (X, C) \) be an iterated extension of length \( n \) in \( \text{Ext}(S) \) such that \( X \) is indecomposable. If \( S \) satisfies (UC), then \( \xi_i \neq 0 \) for \( 2 \leq i \leq n \).

Proof. The result clearly holds if \( n = 2 \), and we shall prove the claim by induction on \( n \). We therefore assume that \( n \geq 3 \), and that the claim holds for all iterated extensions of length less than \( n \). For an iterated extension \( (X, C) \) of length \( n \) in \( \text{Ext}(S) \), where \( X \) is indecomposable, there is a non-split, short exact sequence \( 0 \rightarrow K_n \rightarrow X \rightarrow C_{n-1} \rightarrow 0 \), and \( C_{n-1} \) has a direct decomposition
\[
C_{n-1} = Y_1 \oplus \cdots \oplus Y_q
\]
such that \( Y_j \) is an indecomposable object in \( \mathcal{A}(S) \) of length \( n_j \) for \( 1 \leq j \leq q \). If \( q = 1 \), then \( \xi_i \neq 0 \) for \( 2 \leq i \leq n - 1 \) by the induction hypothesis, and \( \xi_n = 0 \) since \( X \) is indecomposable. Hence we have proved the induction step if \( q = 1 \). Next, we suppose that \( q > 1 \), and show that this leads to a contradiction: Choose a cofiltration of \( Y_j \) given by
\[
Y_j = C_{j,n_j} \xrightarrow{f_{j,n_j}} C_{j,n_j-1} \rightarrow \cdots \rightarrow C_{j,1} \xrightarrow{f_{j,1}} C_{j,0} = 0
\]
for \( 1 \leq j \leq q \), with \( K_{ji} = \ker(f_{ji}) \cong S_{\alpha(j,i)} \) for \( 1 \leq i \leq n_j \) and for some \( \alpha(j,i) \in I \). Since
\[
\text{Ext}^1(\Lambda(C_{n-1}, K_n)) \cong \bigoplus_j \text{Ext}^1(\Lambda(Y_j, K_n))
\]
we have \( \xi_n = (\xi_{n,1}, \ldots, \xi_{n,q}) \), and we claim that \( \xi_{n,j} \neq 0 \) for \( 1 \leq j \leq q \). In fact, if \( \xi_{n,j} = 0 \) for some \( j \), then the short exact sequence
\[
0 \rightarrow K_n \rightarrow f_{n-1}^{-1}(Y_j) \rightarrow Y_j \rightarrow 0
\]
splits, hence there is a section of \( X \rightarrow Y_j \) making \( Y_j \) a direct summand of \( X \). This is a contradiction, and therefore we must have \( \xi_{n,j} \neq 0 \) for \( 1 \leq j \leq q \). Since \( Y_j \) is indecomposable of length \( n_j < n \) for \( 1 \leq j \leq q \), it follows from the induction hypothesis that the extensions
\( \xi_{ij} \in \text{Ext}^1_A(C_{j,i-1}, K_{ji}) \neq 0 \) for \( 2 \leq i \leq n_j \). Let \( X_j = f_n^{-1}(Y_j) \subseteq X \), with induced surjective morphism \( f_n : X_j \to Y_j = C_{j,n_j} \). Then \( (X_j, C_{j,*}) \) is an iterated extension in \( \text{Ext}(S, n_j, *) \), given by

\[
X_j = C_{j, n_j + 1} \xrightarrow{f_n} N_j = C_{j, n_j} \to \cdots \to C_{j, 1} \to C_{j, 0} = 0
\]

Therefore, it follows from Lemma 1.1 that \( \text{Ext}^1_A(N_j, K_n) \to \text{Ext}^1_A(K_{j,n_j}, K_n) \) is an isomorphism and that the image \( \tau_{n,j} \) of \( \xi_{n,j} \) is non-zero for \( 1 \leq j \leq n \). Hence, we must have that

\[
\alpha(1, n_1) = \alpha(2, n_2) = \cdots = \alpha(q, n_q)
\]

is the unique node in \( \Lambda \) with an arrow to node \( \alpha(n) \). In a similar manner, it follows that \( \text{Ext}^1_A(C_{j, n_j - i}, K_{j, n_j - i + 1}) \to \text{Ext}^1_A(K_{j, n_j - i}, K_{j, n_j - i + 1}) \) is an isomorphism and therefore that \( \tau_{j,n_j - i - 1} \neq 0 \) for \( 1 \leq j \leq q \) and \( 1 \leq i \leq \min(n_1, \ldots, n_q) - 1 \). Hence we must have that

\[
\alpha(1, n_1 - i) = \alpha(2, n_2 - i) = \cdots = \alpha(q, n_q - i)
\]

Without loss of generality, we may assume that \( n_1 \leq n_2 \leq \cdots \leq n_q \). From the argument above, it follows that \( Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y_q \). For any injective morphism \( Y_1 \to Y_1 \oplus Y_2 \) that has the form \( y_1 \mapsto (y_1, C y_1) \) with \( C \in k \), there is an induced injective map \( i : Y_1 \to C_{n - 1} = N_1 \oplus \cdots \oplus N_q \), and we may consider the commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{K_n} & X & \xrightarrow{f_n} & C_{n-1} & \xrightarrow{i} & 0 \\
0 & \xrightarrow{K_n} & f_n^{-1}(i(Y_1)) & \xrightarrow{i} & Y_1 & \xrightarrow{} & 0
\end{array}
\]

where the first row is the extension \( \xi_n = (\xi_{n,1}, \ldots, \xi_{n,q}) \). We claim that we may choose \( C \) such that second row is a split extension. In fact, this follows from the fact that there are \( c, c_i \in k^* \) such that \( \tau_{n,1} = c \cdot \tau_{n,2} \) and

\[
\tau_{i,n_1 - i + 1} = c_i \cdot \tau_{2,n_2 - i + 1}
\]

for \( 1 \leq i \leq n_1 - 1 \). It follows that \( Y_1 \) is a direct summand in \( X \), and this contradicts the assumption \( q > 1 \).

**Theorem 4.5.** Let \( S = \{ S_\alpha : \alpha \in I \} \) be a family of orthogonal \( k \)-rational points in an Abelian \( k \)-category \( A \). If \( S \) satisfies \((UC)\), then the indecomposable objects in \( \text{Ext}(S, n) \) of length \( n \) are given by \( \{ X(\alpha) : \alpha \in J \} \), up to isomorphism in \( \text{Ext}(S) \), where the subset \( J \subseteq I^n \) consists of the vectors \( \alpha \) such that the following conditions hold:

(i) \( \text{Ext}^1_A(S_{\alpha(i-1)}, S_{\alpha(i)}) \neq 0 \) for \( 2 \leq i \leq n \)

(ii) If \( \sigma_i \in \text{Ext}^1_A(S_{\alpha(i-1)}, S_{\alpha(i)}) \) is non-zero for \( 2 \leq i \leq n \), then the matric Massey product \( \langle \sigma_2, \sigma_3, \ldots, \sigma_n \rangle \) is defined and contains zero.

Moreover, the indecomposable objects \( X(\alpha) \) are uniserial, and can be constructed from the family \( S \) and their extensions.

**Proof.** This follows from the results in this section. The explicit construction of \( X(\alpha) \) is obtained by iteratively constructing \( C_i \) as an extension of \( C_{i-1} \) by \( K_i \) for \( 2 \leq i \leq n \).

**Corollary 4.6.** Let \( S = \{ S_\alpha : \alpha \in I \} \) be a family of orthogonal \( k \)-rational points in an Abelian \( k \)-category \( A \). Then \( \text{Ext}(S) \) is uniserial if and only if \((UC)\) holds.

**Proof.** This follows from Proposition 3.5 and Theorem 4.5.
There is a more general form of Corollary 4.6, where the points in $S$ are not assumed to be $k$-rational; see Gabriel [7]. We have chosen to work with $k$-rational points out of convenience, and also because all points are $k$-rational in the applications we have in mind. However, it would be possible to prove the general form of Theorem 4.5 and Corollary 4.6 using the methods of this paper.

5. Graded holonomic D-modules on monomial curves

Let $\Gamma \subseteq \mathbb{N}_0$ be a numerical semigroup, generated by positive integers $a_1, \ldots, a_r$ without common factors, and let $A = k[\Gamma] \cong k[t^{a_1}, \ldots, t^{a_r}]$ be its semigroup algebra over the field $k = \mathbb{C}$ of complex numbers. We call $A$ a monomial curve since $X = \text{Spec}(A)$ is the affine monomial curve $X = \{(t^{a_1}, t^{a_2}, \ldots, t^{a_r}) : t \in k\} \subseteq \mathbb{A}^r_k$.

We studied the positively graded algebra $D$ of differential operators on the monomial curve $A = k[\Gamma]$ in Eriksen [3], and the category $\text{grHol}_D$ of graded holonomic left $D$-modules in Eriksen [4]. We recall that any $D$-module $M$ satisfies the Bernstein inequality $d(M) \geq 1$, that $M$ is holonomic if $d(M) = 1$, and that this condition holds if and only if $M$ has finite length; see Proposition 4 and Proposition 5 in Eriksen [4]. This implies that $\text{grHol}_D$ is a length category, and its simple objects are given by

$$\{M_0[w] : w \in \mathbb{Z}\} \cup \{M_\alpha[w] : \alpha \in J^*, w \in \mathbb{Z}\} \cup \{M_\infty[w] : w \in \mathbb{Z}\}$$

where $J^* = \{\alpha \in \mathbb{C} : 0 \leq \Re(\alpha) < 1, \alpha \neq 0\}$; see Theorem 10 in Eriksen [4]. Moreover, the graded extensions of the simple objects are given by

$$\text{Ext}_D^1(M_\alpha[w], M_\beta[w'])_0 = \begin{cases} k\xi \cong k, & (\alpha, \beta) = (0, \infty), (\infty, 0) \text{ and } w = w' \\ k\xi \cong k, & \alpha = \beta \in J^* \text{ and } w = w' \\ 0, & \text{otherwise} \end{cases}$$

for simple graded $D$-modules $M_\alpha[w], M_\beta[w']$ with $\alpha, \beta \in J^* \cup \{0, \infty\}$ and $w, w' \in \mathbb{Z}$; see Proposition 12 in Eriksen [4].

**Proposition 5.1.** The family $S = \{M_\alpha[w] : \alpha \in J^* \cup \{0, \infty\}, w \in \mathbb{Z}\}$ is the family of simple objects in $\text{grHol}_D$, and it is a family of orthogonal $k$-rational points that satisfies (UC). In particular, the category $\text{grHol}_D$ of graded holonomic $D$-modules is a uniserial category.

**Proof.** Since $k = \mathbb{C}$ is algebraically closed, it follows from the main theorem in Quillen [9] that $\text{End}_D(M_\alpha[w]) = k$ for all $\alpha \in J^* \cup \{0, \infty\}, w \in \mathbb{Z}$. Moreover, the comments above show that $S$ is the family of simple objects in $\text{grHol}_D$, and therefore a family or orthogonal $k$-rational points, which satisfies (UC).

It is, in principle, possible to construct all indecomposable objects in $\text{grHol}_D$ using the constructive proof of Theorem 4.5. As an illustration, we shall classify the indecomposable objects in the case $A = k[t]$, which is the unique smooth monomial curve. The classification would be similar in the other cases, since all rings of differential operators on monomial curves are Morita equivalent. However, the indecomposable objects would be defined by more complicated equations in the singular cases.

Note that when $A = k[t]$, the ring $D$ of differential operators on $A$ is the first Weyl algebra $A_1(k) = k[t](\partial)$, with generators $t$ and $\partial = d/dt$, and relation $[\partial, t] = 1$. Let us write $E = t\partial$ for the Euler derivation in $D$. The simple objects in $\text{grHol}_D$, up to graded isomorphisms and twists, are given by $M_0 = D/D\partial$, $M_\infty = D/Dt$ and $M_\alpha = D/D(E - \alpha)$ for $\alpha \in J^*$. 

Theorem 5.2. Let $D = A_1(k)$ be the first Weyl algebra. The indecomposable graded holonomic $D$-module, up to graded isomorphisms and twists, are given by

$$M(\alpha, n) = D/D(E - \alpha)^n, \quad M(\beta, n) = D/D\omega(\beta, n)$$

where $n \geq 1$, $\alpha \in J^*$, $\beta \in \{0, \infty\}$, and $\omega(\beta, n)$ is the alternating word on $n$ letters in $t$ and $\partial$, ending with $\partial$ if $\beta = 0$, and in $t$ if $\beta = \infty$.

Proof. Let us write $I = J^* \cup \{0, \infty\}$, such that $S = \{M_\alpha[w] : (\alpha, w) \in I \times \mathbb{Z}\}$ is the family of simple objects in $\text{grHol}_D$. It follows from the computation of the graded extensions above that for any length $n \geq 1$ and any $(\alpha, w) \in I \times \mathbb{Z}$, there is a unique path

$$(\alpha, w) = (\alpha(1), w_1) \to (\alpha(2), w_2) \to \cdots \to (\alpha(n), w_n)$$

in $\Lambda$ such that $\text{Ext}_D^i(M(\alpha_{i-1})[w_{i-1}], M(\alpha_i)[w_i])_0 \neq 0$ for $2 \leq i \leq n$. The corresponding vector is admissible since $D = A_1(k)$ is a hereditary graded ring; see for instance Coutinho [2]. Note that if $\alpha \in J^*$, then $\alpha(i) = \alpha$ and $w_i = w$ for $1 \leq i \leq n$, and if $\alpha \in \{0, \infty\}$, then we have

$$(\alpha(i), w(i)) = \begin{cases} 
(\alpha, w), & \text{if } i \text{ is odd} \\
(0, w - 1), & \text{if } i \text{ is even, } \alpha = \infty \\
(\infty, w + 1), & \text{if } i \text{ is even, } \alpha = 0 
\end{cases}$$

for $1 \leq i \leq n$. The rest follows from the fact that $\text{Ext}^1(C_{i-1}, K_i) \to \text{Ext}^1(K_{i-1}, K_i)$ is an isomorphism by Lemma [4,1] and that the graded extensions obtained using factorization in $D$, such as

$$0 \to D/D(E - \alpha) \to D/(E - \alpha)^n \to D/D(E - \alpha)^{n-1} \to 0$$

for $\alpha \in J^*$ and $n \geq 2$, are non-split. \hfill $\square$

6. Iterated extensions and noncommutative deformations

Let $S = \{S_\alpha : \alpha \in I\}$ be a family of orthogonal points in an Abelian $k$-category $A$, and let $A(S)$ be the corresponding length category. In this section, we consider the noncommutative deformations of finite subfamilies of $S$; see Laudal [3] and also Eriksen, Laudal, Sivveland [5], and show that they determine the iterated extensions in $\text{Ext}(S)$. This will shed light on some of the results for uniserial length categories in this paper, and in particular Proposition [4,3]. It will also provide a useful tool for future study of length categories that are more complicated than the uniserial ones.

Let $(X, C)$ be an iterated extension in $\text{Ext}(S)$ of length $n$ with order vector $\alpha$. We define the extension type of $(X, C)$ to be the ordered quiver $\Gamma$ with nodes $\{\alpha(1), \alpha(2), \ldots, \alpha(n)\}$ and edges $\gamma_{i-1,i}$ from node $\alpha(i-1)$ to node $\alpha(i)$ for $2 \leq i \leq n$. The quiver is ordered in the sense that there is a total order $\gamma_1 < \gamma_2 < \cdots < \gamma_{n-1,n}$ on the edges in $\Gamma$. Clearly, the extension type $\Gamma$ is uniquely defined by the order vector $\alpha$, and isomorphic iterated extensions have the same extension type. We denote by $\mathcal{E}(S, \Gamma)$ the set of isomorphism classes of iterated extensions of the family $S$ with extension type $\Gamma$.

To fix notation, we shall give the set $J = \{\alpha(1), \alpha(2), \ldots, \alpha(n)\} \subseteq I$ of nodes in $\Gamma$ a total order, and write

$$S(\Gamma) = \{S_\alpha : \alpha \in J\} = \{X_1, \ldots, X_r\}$$

for the associated subfamily of $S$, considered as an ordered set with $X_1 < X_2 < \cdots < X_r$. Note that for $1 \leq l \leq r$, we have that $X_l = S_{\alpha(i)}$ for at least one value of $i$ with $1 \leq i \leq n$, and that $r \leq n$, with $r < n$ if there are repeated factors.
The path algebra $k[\Gamma]$ of the ordered quiver $\Gamma$ is the $k$-algebra with base consisting of paths $\gamma_{i-1,i} \cdots \gamma_{j-1,j}$ of length $j - i + 1$ for $2 \leq i \leq j \leq n$. The product of two paths $\gamma \cdot \gamma'$ is given by juxtaposition when the last arrow $\gamma_{j-1,j}$ in the first path $\gamma$ is the predecessor of the first arrow $\gamma_{j,j+1}$ in the second path $\gamma'$ in the total ordering, and otherwise the product $\gamma \cdot \gamma' = 0$. We consider $e_i$ as a path of length 0 for $1 \leq i \leq r$. For example, an iterated extension of length three with $\alpha(1) < \alpha(2) < \alpha(3)$ has $r = n = 3$, and its extension type $\Gamma$ has path algebra

$$k[\Gamma] = \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix} = \begin{pmatrix} k \cdot e_1 & k \cdot \gamma_{12} & k \cdot \gamma_{12} \gamma_{13} \\ 0 & k \cdot e_2 & k \cdot \gamma_{23} \\ 0 & 0 & k \cdot e_3 \end{pmatrix}$$

We recall that the category $a_r$ of Artinian $r$-pointed algebras consists of Artinian $k$-algebras $R$ with $r$ simple modules fitting into a diagram $k^r \to R \to k^r$, where the composition is the identity. It is clear that if $\Gamma$ is the extension type of an iterated extension of $r$ objects in $S$ of length $n$, then the path algebra $k[\Gamma]$ is an algebra in $a_r(n)$, where $a_r(n)$ is the full subcategory of $a_r$ consisting of algebras $R$ such that $I(R)^n = 0$, with $I(R) = \ker(R \to k^r)$. Noncommutative deformation functors are defined on the category $a_r$, and noncommutative deformations are parameterized by $r$-pointed Artinian algebras; see Chapter 3 of Eriksen, Laudal, Sjøvland \cite{5} for details.

Let $\Gamma$ be an extension type, and write $S(\Gamma) = \{X_1, \ldots, X_r\}$ for the corresponding ordered subfamily of $S$. We consider the noncommutative deformation functor

$$\text{Def}_{S(\Gamma)} : a_r \to \text{Sets}$$

of the finite family $S(\Gamma)$ in the Abelian category $A$. We shall assume, without loss of generality, that $A$ is the category of right modules over an associative $k$-algebra $A$ in the rest of this section. This is a choice of convenience, as noncommutative deformations can be computed directly in many other Abelian $k$-categories; see Eriksen, Laudal, Sjøvland \cite{5}.

**Proposition 6.1.** There is a bijective correspondence between the noncommutative deformations in $\text{Def}_{S(\Gamma)}(k[\Gamma])$ and the set $E(S, \Gamma)$ of equivalence classes of iterated extensions of the family $S$ with extension type $\Gamma$.

**Proof.** Let us write $S(\Gamma) = \{X_1, \ldots, X_r\}$, and let $\alpha$ be the order vector corresponding to the extension type $\Gamma$. There is a unique $s$ with $1 \leq s \leq r$ such that $S_{\alpha(1)} = X_s$. Any noncommutative deformation $X_\Gamma \in \text{Def}_{S(\Gamma)}(k[\Gamma])$ has the form $X_\Gamma = (k[\Gamma]_{ij} \otimes_k X_j)$ as a left $k[\Gamma]$-module by flatness, with a right multiplication of $A$. Let $X_\Gamma(s) = e_s \cdot X_\Gamma \subseteq X_\Gamma$, which is closed under right multiplication with $A$. A path in $e_s \cdot k[\Gamma]$ is called leading if it has the form $\gamma_{12} \gamma_{23} \cdots \gamma_{i-1,i}$ and non-leading otherwise. By convention, we consider the path $e_s$ as leading, and define

$$X_\Gamma^{NL}(s) = \bigoplus_{\gamma} \gamma \cdot X_\Gamma(s)$$

where the sum is taken over all non-leading paths $\gamma$ in $e_s \cdot k[\Gamma]$. Notice that $X_\Gamma^{NL}(s) \subseteq X_\Gamma(s)$ is closed under right multiplication by $A$. We define $X = X_\Gamma(s) / X_\Gamma^{NL}(s)$, which has an induced right $A$-module structure. As a $k$-linear space, we have that

$$X \cong \bigoplus_{1 \leq i \leq n} (\gamma_{12} \gamma_{23} \cdots \gamma_{i-1,i}) \otimes_k S_{\alpha(i)}$$

with $S_{\alpha(i)} = X_l$ for some $l$ with $1 \leq l \leq r$, and we claim that there is a cofiltration $C$ of $X$ such that $(X, C)$ is an iterated extension of $S$ with extension type $\Gamma$. In fact, we may choose the cofiltration $C$ dual to the filtration $F$ given by

$$F_j = \bigoplus_{j+1 \leq i \leq n} (\gamma_{12} \gamma_{23} \cdots \gamma_{i-1,i}) \otimes_k S_{\alpha(i)}$$
for $0 \leq j \leq n$, where $F_j \subseteq X$ is closed under right multiplication with $A$. Conversely, if $(X, C)$ is a iterated extension of $S$ with extension type $\Gamma$, then it follows from the construction in Section 3 of Eriksen, Siqveland [6] that

$$X \cong K_n \oplus K_{n-1} \oplus \cdots \oplus K_2 \oplus K_1$$

as a $k$-linear vector space, with $K_i \cong S_{\alpha(i)}$ and with right multiplication of $A$ given by

$$(m_n, \ldots, m_2, m_1)a = (m_n \cdot a + \sum_{i=1}^{n-1} \psi^{i,n}_{a}(m_i), \ldots, m_2 \cdot a + \psi^{1,2}_{a}(m_1), m_1 \cdot a)$$

for $m_i \in K_i$, $a \in A$. Let $I_l = \{i: \alpha(i) = l\}$ for $1 \leq l \leq r$. Then the right multiplication of $A$ on $X_\Gamma = (k[\Gamma] \otimes_k X_j)$ given by

$$(e_l \otimes m_l) \cdot a = e_l \otimes (m_l \cdot a) + \sum_{i \in I_l} \sum_{i+1 \leq j \leq n} (\gamma_{i,i+1} \gamma_{i+1,i+2} \cdots \gamma_{j-1,j}) \otimes \psi^{ij}_{a}(m)$$

for $1 \leq l \leq r$, $a \in A$, $m_l \in X_l$ defines a noncommutative deformation $X_\Gamma \in \text{Def}_{S(\Gamma)}(k[\Gamma])$. \(\square\)

We say that $S(\Gamma)$ is a swarm if $\dim_k \text{Ext}^1_{k}(X_i, X_j)$ if finite for $1 \leq i, j \leq r$. We shall assume that this is the case in the rest of this section. In this case, the noncommutative deformation functor $\text{Def}_{S(\Gamma)}$ has a universal object $(H, X_H)$ by general results; see Eriksen, Laudal, Siqveland [5], where $H$ is the pro-representing hull in the pro-category $\widehat{\mathfrak{a}}_r$, and $X_H \in \text{Def}_{S(\Gamma)}(H)$ is the versal family. We write $X(S, \Gamma) = \text{Mor}(H, k[\Gamma])$ for the set of morphisms $\phi : H \rightarrow k[\Gamma]$ in $\widehat{\mathfrak{a}}_r$. Note that the natural map $X(S, \Gamma) \rightarrow \text{Def}_{S(\Gamma)}(k[\Gamma])$ given by $\phi \mapsto M_\phi = \text{Def}_{S(\Gamma)}(\phi)(M_H)$ is surjective by the versal property.

**Lemma 6.2.** The set $X(S, \Gamma) = \text{Mor}(H, k[\Gamma])$ is an affine algebraic variety.

**Proof.** Since $k[\Gamma]$ is an algebra in $\mathfrak{a}_r(n)$, any morphism $\phi : H \rightarrow k[\Gamma]$ in $\widehat{\mathfrak{a}}_r$ can be identified with $\phi_n : H_n \rightarrow k[\Gamma]$ since $\phi(I(H)^n) = 0$. To prove that $X(S, \Gamma) = \text{Mor}(H_n, k[\Gamma])$ is an affine algebraic variety, it is enough to notice that $H_n$ is a quotient of $T_n^1$, that $\text{Mor}(T_n^1, k[\Gamma])$ is isomorphic to affine space $k^N$, where

$$N = \sum_{i,j} \text{dim}_k \text{Ext}^1_{k}(X_i, X_j) = \text{dim}_k (I(k[\Gamma])/(I(k[\Gamma])^2))_{ij}$$

and that $\text{Mor}(H_n, k[\Gamma]) \subseteq \text{Mor}(T_n^1, k[\Gamma])$ is a closed subset in the Zariski topology, with equations given by the obstructions $f_{ij}(l)^n \in T_n^1$. \(\square\)

**Corollary 6.3.** The set $\mathcal{E}(S, \Gamma)$ of equivalence classes of iterated extensions of the family $S$ with extension type $\Gamma$ is a quotient of the affine algebraic variety $X(S, \Gamma)$, determined by the noncommutative deformations of $S(\Gamma) = \{X_1, \ldots, X_r\}$.

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Eivind Eriksen
BI Norwegian Business School,
Department of Economics,
N-0442 Oslo, Norway

eivind.eriksen@bi.no