ON THE GEOMETRY OF ORBITS OF PATH GROUP ACTIONS INDUCED BY SIGMA-ACTIONS

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Abstract. It is known that a hyperpolar action on a compact symmetric space gives rise to a hyperpolar proper Fredholm (PF) action on a Hilbert space. This action is defined as an action of a path group on a Hilbert space via the gauge transformations and all its orbits are proper Fredholm (PF) submanifolds of the Hilbert space. Those two hyperpolar actions are related by an equivariant Riemannian submersion called the parallel transport map. In this paper we introduce a natural isomorphism of Hilbert spaces, which is equivariant with respect to the gauge transformations and is compatible with the parallel transport map. Using this isomorphism we extend and unify all known computational results of principal curvatures of PF submanifolds in Hilbert spaces. Moreover we study the submanifold geometry of orbits of hyperpolar PF actions induced by sigma-actions and show new examples of austere PF submanifolds and weakly reflective PF submanifolds in Hilbert spaces. Furthermore we show that the isomorphism of Hilbert spaces given in this paper corresponds to a natural isomorphism of affine Kac-Moody symmetric spaces of group type.

1. Introduction

An isometric action of a compact Lie group on a Riemannian manifold $M$ is called polar if there exists a closed connected submanifold $\Sigma$ of $M$ which meets every orbit and is orthogonal to the orbits at every point of intersection. Such a $\Sigma$ is called a section, that is automatically totally geodesic in $M$. If $\Sigma$ is also flat in the induced metric then the action is called hyperpolar ([12]).

If $M$ is a Euclidean space then hyperpolar actions are essentially the isotropy representations of symmetric spaces. More precisely, let $U/L$ be a symmetric space of compact type with canonical decomposition $u = l + p$. Then the adjoint representation of $L$ on $p$ is called the isotropy representation of $U/L$. It follows that the isotropy representation is hyperpolar where any maximal abelian subspace in $p$ is a section. Conversely it was shown that any hyperpolar representation on a Euclidean space is orbit equivalent to the isotropy representation of a symmetric space ([4, 5]). Here two isometric actions are called orbit equivalent if their orbits are identified via a suitable isometry.

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If $M = G/K$ is a symmetric space of compact type, important examples of hyperpolar actions are Hermann actions, that is, the actions by symmetric subgroups of $G$ ([13, 14]). Here a closed subgroup $H$ of $G$ is called symmetric if there exists an involutive automorphism $\theta$ of $G$ which satisfies $G^\theta \subset H \subset G_0^\theta$, where $G^\theta$ denotes the fixed point subgroup and $G_0^\theta$ its identity component. It was shown that any indecomposable hyperpolar actions of cohomogeneity greater than one is orbit equivalent to a Hermann action ([22, 23]). Note that the associated action of $H \times K$ on $G$ defined by $(b, c) \cdot a = bac^{-1}$ is also hyperpolar ([12]).

A special class of Hermann actions is given by sigma-actions ([3]). Let $G$ be a connected compact semisimple Lie group and $\sigma$ an automorphism of $G$. Then the action of $G(\sigma) = \{(b, \sigma(b)) \mid b \in G\}$ on $G$ defined by $(b, \sigma(b)) \cdot a = ba\sigma(b)^{-1}$ is called a $\sigma$-action. If we regard $G$ as the symmetric space $(G \times G)/\Delta G$ where $\Delta G$ is the diagonal, then $G(\sigma)$ is a symmetric subgroup of $G \times G$ with involution $(b, c) \mapsto (\sigma^{-1}(c), \sigma(b))$. The $G(\sigma)$-orbit through the identity is also called the Cartan embedding associated to $(G, \sigma)$.

It is an interesting problem to study the submanifold geometry of orbits of hyperpolar actions. The principal orbits of polar representations are isoparametric submanifolds in the sense of Terng [38]. Thorbergsson [42] conversely showed that an irreducible compact full isoparametric submanifold of a Euclidean space with codimension at least 3 is an orbit of a polar representation. These results were extended to the case of hyperpolar actions on compact symmetric spaces and equifocal submanifolds of symmetric spaces ([41, 2]). Hirohashi, Song, Takagi and Tasaki [15] studied the submanifold geometry of orbits of the isotropy representations of symmetric spaces and showed that in each strata of the stratification of orbit types there exists a unique orbit which is a minimal submanifold of the sphere. Ikawa [16] extended this result to the case of Hermann actions with commuting involutions. Many geometers have studied orbits of hyperpolar actions and shown various kinds of examples of homogeneous submanifolds.

It is also interesting to study hyperpolar actions in infinite dimensions.

Palais and Terng [32, 39] introduced a suitable class of isometric actions on Hilbert spaces, namely proper Fredholm (PF) actions, and showed examples of hyperpolar PF actions which are orbits of the gauge transformations. These examples were later extended by Pinkall and Thorbergsson [33] and reformulated by Terng [40] as follows. Let $G$ be a connected compact Lie group with a bi-invariant metric. Denote by $\mathcal{G} = H^1([0, 1], G)$ the path group of all Sobolev $H^1$-paths from $[0, 1]$ to $G$ and by $V_g = H^0([0, 1], g)$ the Hilbert space of all $H^0$-paths from $[0, 1]$ to the Lie algebra $g$ of $G$. Let $\mathcal{G}$ act on $V_g$ by the affine isometry:

$$g * u = gug^{-1} - g'g^{-1},$$

where $g \in \mathcal{G}$ and $u \in V_g$. For any closed subgroup $L$ of $G \times G$ the subgroup

$$P(G, L) = \{g \in \mathcal{G} \mid (g(0), g(1)) \in L\}$$

acts on $V_g$ by the same formula. It was shown that the $P(G, L)$-action is PF and that if the $L$-action on $G$ defined by $(b, c) \cdot a = bac^{-1}$ is hyperpolar then
the $P(G,L)$-action on $V_g$ is also hyperpolar. Applying this result to the examples of hyperpolar actions on $G$ she showed that $P(G,H \times K)$-actions and $P(G,G(\sigma))$-actions associated to Hermann actions and $\sigma$-actions respectively are hyperpolar. (Palais and Terng [32, 39] considered the case $\sigma = \text{id}$. Pinkall and Thorbergsson [33] considered the case $H = K$.) Note that all orbits of $P(G,L)$-actions are proper Fredholm (PF) submanifolds of the Hilbert space $V_g$ ([32, 39]). There are many interesting examples of PF submanifolds which are orbits of hyperpolar $P(G,L)$-actions (e.g., [39, 40, 41, 25, 26]).

It should be also noted that there is a class of infinite dimensional symmetric spaces closely related to hyperpolar PF actions. Recall that in the finite dimensional case hyperpolar representations are essentially the isotropy representations of symmetric spaces. Terng [40] conjectured that there is an analogous result in infinite dimensions and showed that $P(G,G(\sigma))$-actions are essentially the adjoint actions of affine Kac-Moody groups. Later Heintze, Palais, Terng and Thorbergsson [12] studied involutions of affine Kac-Moody algebras and showed that $P(G,H \times K)$-actions associated to Hermann actions are essentially the isotropy representations of infinite dimensional symmetric spaces induced by those involutions. However they did not give a precise definition of those symmetric spaces due to functional analytic difficulties inherent in affine Kac-Moody groups. Afterward Heintze and Popescu [11, 34] started to study those symmetric spaces in the category of tame Fréchet manifolds ([8]) and showed their fundamental properties. Nowadays they are called affine Kac-Moody symmetric spaces and known as the closest infinite dimensional analogue of finite dimensional symmetric spaces ([6]). According to the theory of those symmetric spaces their isotropy representations (restricted to appropriate subspaces) are essentially equivalent to $P(G,H \times K)$-actions associated to Hermann actions. In particular those of group type are affine Kac-Moody groups and their isotropy representations are essentially equivalent to $P(G,G(\sigma))$-actions.

In the study of $P(G,L)$-actions in general, it is important to consider an equivariant Riemannian submersion $\Phi : V_g \to G$, called the parallel transport map. It was shown that each orbit of the $P(G,L)$-action is the inverse image of an $L$-orbit under $\Phi$. More generally, if $N$ is a closed submanifold of $G$ then its inverse image $\Phi^{-1}(N)$ is a PF submanifold of $V_g$. For a given compact symmetric space $G/K$ with projection $\pi : G \to G/K$ we consider the composition $\Phi_K := \pi \circ \Phi : V_g \to G \to G/K$ which is also an equivariant Riemannian submersion called the parallel transport map over $G/K$. Similarly, if $N$ is a closed submanifold of $G/K$ then its inverse image $\Phi_K^{-1}(N)$ is a PF submanifold of $V_g$. In particular, if $N$ is an orbit of the $H$-action then $\Phi_K^{-1}(N)$ is an orbit of the $P(G,H \times K)$-action. The parallel transport map $\Phi_K$ is also known as a useful tool to study the submanifold geometry in symmetric spaces ([11]).

In [28] the author gave a formula for the principal curvatures of the PF submanifold $\Phi_K^{-1}(N)$ for a curvature-adapted submanifold $N$ of $G/K$ (see also [21, 25]) and showed an explicit formula for the principal curvatures of orbits of $P(G,H \times K)$-actions associated to Hermann actions. Using this formula he studied conditions for those orbits to be austere PF submanifolds of $V_g$. Here a submanifold is called austere ([9]) if for each normal vector $\xi$ the set of principal curvatures with multiplicities in the direction of $\xi$ is invariant under the
multiplication by \((-1)\). Thus austere submanifolds are minimal submanifolds. He considered two conditions:

(A) The orbit \(N = H \cdot (\exp w)K\) is an austere submanifold of \(G/K\).
(B) The orbit \(\Phi_K^{-1}(N) = P(G, H \times K) \ast \tilde{w}\) is an austere PF submanifold of \(V_\theta\).

Here \(\tilde{w}\) denotes the constant path with value \(w \in \mathfrak{g}\). Let \(\theta_K\) and \(\theta_H\) denote the involutions of \(G\) associated to the symmetric subgroups \(K\) and \(H\) respectively. Denote by \(\mathfrak{g} = \mathfrak{t} + \mathfrak{m}\) and \(\mathfrak{g} = \mathfrak{h} + \mathfrak{p}\) the corresponding canonical decompositions. Take a maximal abelian subspace \(\mathfrak{t}\) in \(\mathfrak{m} \cap \mathfrak{p}\) and write \(\Delta(\theta_K, \theta_H)\) for the corresponding root system of \(t\). He showed:

**Theorem ([28]).**

(i) If \(\Delta(\theta_K, \theta_H)\) is a reduced root system then (A) and (B) are equivalent.
(ii) If \(\theta_K = \theta_H\) then (A) and (B) are equivalent.
(iii) If \(\theta_K \circ \theta_H = \theta_H \circ \theta_K\) then (A) implies (B).
(iv) If \(G\) is simple then (A) implies (B).

Here (B) does not imply (A) in the cases (iii) and (iv). In fact, there exists a counterexample.

Applying these results to the examples of austere orbits of Hermann actions he showed many examples of austere PF submanifolds which are orbits of hyperpolar \(P(G, H \times K)\)-actions.

The main purpose of this paper is to extend those results to the case of \(P(G, G(\sigma))\)-actions. Notice that although the \(\sigma\)-action is a special case of a Hermann action, we cannot apply the previous results directly to the present case because \(G(\sigma)\) is not the product of two symmetric subgroups of \(G\). In this paper we introduce an injective homomorphism \(\Omega : H^1([0, 1], G) \to H^1([0, 1], G \times G)\) and a linear isomorphism \(\Upsilon : H^0([0, 1], \mathfrak{g}) \to H^0([0, 1], \mathfrak{g} \oplus \mathfrak{g})\), and show (Theorem 3.2 and Corollary 3.3):

**Theorem I.**

(i) The \(P(G, L)\)-action on \(V_\theta\) is conjugate to the \(P(G \times G, L \times \Delta G)\)-action on \(V_{\theta \oplus \theta}\) via \(\Omega\) and \(\Upsilon\), that is, \(\Omega\) maps \(P(G, L)\) isomorphically onto \(P(G \times G, L \times \Delta G)\) and \(\Upsilon(g \ast u) = \Omega(g) \ast \Upsilon(u)\) holds for \(g \in P(G, L)\) and \(u \in V_\theta\). In particular the \(P(G, G(\sigma))\)-action on \(V_\theta\) is conjugate to the \(P(G \times G, G(\sigma) \times \Delta G)\)-action on \(V_{\theta \oplus \theta}\) via \(\Omega\) and \(\Upsilon\).

(ii) The following diagram commutes:

\[
\begin{array}{ccc}
V_\theta & \xrightarrow{\Upsilon} & V_{\theta \oplus \theta} \\
\phi \downarrow & & \phi_{\Delta G} \downarrow \\
G & \xleftarrow{\rho} & (G \times G)/\Delta G,
\end{array}
\]

where \(\phi_{\Delta G} : V_{\theta \oplus \theta} \to G \times G \to (G \times G)/\Delta G\) denotes the parallel transport map over \((G \times G)/\Delta G\) and \(\rho\) the isomorphism \((a, b) \mapsto ab^{-1}\).

The property (i) allows us to apply the general results of \(P(G, H \times K)\)-actions to \(P(G, G(\sigma))\)-actions. Note that the case \(\sigma = \text{id}\) was essentially observed by Pinkall and Thorbergsson [33, p. 283]. The property (ii) means that \(\Upsilon\) is natural
in the framework of parallel transport maps. It allows us to apply the general results of $\Phi_K$ to $\Phi$.

Using Theorem I we derive a formula for the principal curvatures of the PF submanifold $\Phi^{-1}(N)$ for a curvature-adapted submanifold $N$ of $G$ (Theorem 4.1) and a formula for the principal curvatures of orbits of $P(G, G(\sigma))$-actions (Theorem 6.1). These formulas generalize the results by King-Terng [20] in the case of fibers and by Palais-Terng [32, 39] in the case of $\sigma = \text{id}$. Consequently we unify all known computational results of principal curvatures of PF submanifolds as follows:

Fiber $\Phi_K^{-1}(aK)$ (The author [25]) $\overset{\text{Theorem I}}{\Longrightarrow}$ Fiber $\Phi^{-1}(a)$ (King-Terng [20])

$\uparrow \quad N = \{aK\}$

PF submanifold $\Phi_K^{-1}(N)$ (Koike [21], the author [25, 28]) $\overset{\text{Theorem I}}{\Longrightarrow}$ PF submanifold $\Phi^{-1}(N)$ (Theorem 4.1 of this paper)

$\downarrow \quad N = H \cdot (\exp w)K$

Orbit $P(G, H \times K) \ast \hat{\omega}$ (The author [28]) $\overset{\text{Theorem I}}{\Longrightarrow}$ Orbit $P(G, G(\sigma)) \ast \hat{\omega}$ (Theorem 6.1 of this paper)

$\downarrow \quad H = K$

Orbit $P(G, K \times K) \ast \hat{\omega}$ (Pinkall-Thorbergsson [33]) $\overset{\text{Theorem I}}{\Longrightarrow}$ Orbit $P(G, \Delta G) \ast \hat{\omega}$ (Palais-Terng [32, 39]).

Based on those results we study the relation between the following two conditions on the austerity property of orbits:

(a) The orbit $N = G(\sigma) \cdot \exp w$ is an austere submanifold of $G$.

(b) The orbit $\Phi^{-1}(N) = P(G, G(\sigma)) \ast \hat{\omega}$ is an austere PF submanifold of $V_9$.

Let $t$ be a maximal abelian subalgebra of the fixed point algebra $g^\sigma$ and $\Delta(\sigma)$ the corresponding root system of $t$ (cf. Section 5). We prove (Theorem 7.1):

**Theorem II.**

(i) If $\Delta(\sigma)$ is a reduced root system then (a) and (b) are equivalent.

(ii) If $\sigma = \text{id}$ then (a) and (b) are equivalent.

(iii) If $\sigma^2 = \text{id}$ then (a) implies (b).

(iv) If $G$ is simple then (a) implies (b).

Here (b) does not imply (a) in the cases (iii) and (iv). In fact, there exists a counterexample.

This is an analogue of the previous theorem ([28]). In fact it turns out by Theorem I that (a) and (b) are special cases of (A) and (B) respectively and thus (i)–(iii) follow from the previous results. However (iv) is not trivial because the simplicity is not preserved by $\Omega$ and $\Upsilon$. Moreover the converse is not trivial because the counterexample given in the previous paper is not an orbit of a $\sigma$-action. We prove (iv) based on the structure theory of automorphisms of $G$ and show a counterexample to the converse (Theorem 9.2). Moreover we show
a new example of an austere PF submanifold which is an orbit of a $P(G, G(\sigma))$-action (Proposition 9.1). Furthermore we extend the author’s previous results concerning weakly reflective PF submanifolds (Theorem 8.3) and show a new example of a weakly reflective PF submanifold (Proposition 9.4).

Finally we study Theorem I from the viewpoint of affine Kac-Moody symmetric spaces. The following theorem shows that the isomorphisms $\Omega$ and $\Upsilon$ correspond to a natural isomorphism of affine Kac-Moody symmetric spaces of group type (see Theorem 10.5 for details):

**Theorem III.** Let $\hat{G} = \hat{L}(G, \sigma)$ be an affine Kac-Moody symmetric space of group type. Then there is a natural correspondence between:

(i) the isomorphism between $\hat{G}$ and the quotient $\hat{G} \times G/(\hat{G} \times G)^\ast$,

(ii) the conjugacy between hyperpolar PF actions of $P(G, G(\sigma))$ on $V_g$ and of $P(G \times G, G(\sigma) \times \Delta G)$ on $V_g \oplus g$ via $\Omega$ and $\Upsilon$ shown in Theorem I,

(iii) the conjugacy between hyperpolar actions of $G(\sigma)$ on $G$ and of $G(\sigma)$ on $(G \times G)/\Delta G$ via $\rho$ and $\text{id}$.

This paper is organized as follows. In Section 2 we review foundations of PF submanifolds, PF actions and parallel transport maps. In Section 3 we define and investigate the isomorphisms $\Omega$ and $\Upsilon$ and prove Theorem I. In Section 4 we derive a formula for the principal curvatures of the PF submanifold $\Phi^{-1}(N)$ for a curvature-adapted submanifold $N$ of $G$. In Section 5 we study the submanifold geometry of orbits of $\sigma$-actions. In Section 6 we derive an explicit formula for the principal curvatures of $P(G, G(\sigma))$-orbits. In Section 7 we study conditions for $P(G, G(\sigma))$-orbits to be austere PF submanifolds of $V_g$ and prove Theorem II. In Section 8 we extend the previous results concerning weakly reflective PF submanifolds in Hilbert spaces. In Section 9 we show new examples of austere PF submanifolds and weakly reflective PF submanifolds which are orbits of a $P(G, G(\sigma))$-action. In Section 10 we study the relations to affine Kac-Moody symmetric spaces and prove Theorem III.

2. PF submanifolds, PF actions and parallel transport maps

In this section we review foundations of PF submanifolds, PF actions and parallel transport maps.

Let $N$ be a submanifold of a (separable) Hilbert space $V$. Suppose that $N$ has finite codimension in $V$. $N$ is called proper Fredholm (PF) if the end point map $T^1N \to V$, $(p, \xi) \mapsto p + \xi$ restricted to a normal disc bundle of any finite radius is proper and Fredholm ([39]). The proper condition implies that for each $u \in V$ the function $f_u : N \to \mathbb{R}$, $p \mapsto \|p - u\|^2$ satisfies the Palais-Smale condition ([34, 36]). The Fredholm condition implies that the shape operators are compact self-adjoint operators.

Let $G$ be a Hilbert Lie group, acting on a Hilbert space $V$. The action is called proper Fredholm (PF) if the map $\mathcal{L} \times V \to V \times V$, $(l, u) \mapsto (l \cdot u, u)$ is proper and the map $\mathcal{L} \to V$, $l \mapsto l \cdot u$ is Fredholm for each $u \in V$ ([32]). If $\mathcal{L}$ is infinite dimensional and the action is isometric PF, then every $\mathcal{L}$-orbit is a PF submanifold of $V$ ([32, Theorem 7.1.6]).

Let $G$ be a connected compact Lie group with Lie algebra $\mathfrak{g}$. Choose an $\text{Ad}(G)$-invariant inner product $\langle \cdot, \cdot \rangle$ of $\mathfrak{g}$ and equip the corresponding bi-invariant
Riemannian metric with $G$. Denote by $\mathcal{G} = H^1([0, 1], G)$ the path group of all Sobolev $H^1$-paths from $[0, 1]$ to $G$ and by $V_g = H^0([0, 1], g)$ the Hilbert space of all $H^0$-paths from $[0, 1]$ to $g$. Then $\mathcal{G}$ acts on $V_g$ by the affine isometry:

$$g \ast u = gu g^{-1} - g' g^{-1},$$

where $g \in \mathcal{G}$, $u \in V_g$ and $g'$ denotes the weak derivative of $g$. It follows that this action is transitive and PF (10).

Let $L$ be a closed subgroup of $G \times G$. The subgroup $P(G, L) = \{g \in G \mid (g(0), g(1)) \in L\}$ acts on $V_g$ by the same formula. Note that $P(G, L)$ is the inverse image of $L$ under the submersion $\Psi^G : \mathcal{G} \to G \times G$ defined by

$$\Psi^G(g) = (g(0), g(1)).$$

It follows that the $P(G, L)$-action is also PF (10). Thus every orbit of the $P(G, L)$-action is a PF submanifold of $V_g$.

For each $u \in V_g$ we define $g_u \in \mathcal{G}$ as the unique solution to the linear ordinary differential equation

$$g^{-1} g' = u, \quad g(0) = e.$$

The parallel transport map $\Phi : V_g \to G$ is a Riemannian submersion defined by

$$\Phi(u) = g_u(1).$$

By definition $\Phi(\hat{x}) = \exp x$ where $\hat{x}$ denotes the constant path with value $x \in \mathfrak{g}$. Consider the action of $G \times G$ on $G$ by

$$(b, c) \cdot a = bac^{-1}. \quad (2.1)$$

Then $\Phi$ is equivariant via $\Psi^G$, that is,

$$\Phi(g \ast u) = (g(0), g(1)) \cdot \Phi(u)$$

for $g \in \mathcal{G}$ and $u \in V_g$. Moreover it follows that

$$P(G, L) \ast u = \Phi^{-1}(L \cdot \Phi(u))$$

for any closed subgroup $L$ of $G \times G$ (10). More generally, if $N$ is a closed submanifold of $G$ then the inverse image $\Phi^{-1}(N)$ is a PF submanifold of $V_g$ (11, Lemma 5.8).

Let $K$ be a symmetric subgroup of $G$ with Lie algebra $\mathfrak{k}$. Denote by $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ the decomposition into the ($\pm 1$)-eigenspaces of the involution, which is called the canonical decomposition. Restricting the $\text{Ad}(G)$-invariant inner product of $\mathfrak{g}$ to $\mathfrak{m}$ we equip the induced $G$-invariant Riemannian metric with the homogeneous space $G/K$. Then $G/K$ is a compact symmetric space and the natural projection $\pi : G \to G/K$ is a Riemannian submersion with totally geodesic fiber. The composition

$$\Phi_K := \pi \circ \Phi : V_g \to G \to G/K$$

is a Riemannian submersion which is called the parallel transport map over $G/K$. Consider the action of $G$ on $G/K$ by

$$b \cdot (aK) = (ba)K. \quad (2.2)$$
Denote by \( p^G : G \times G \to G \) the projection onto the first component. Then \( \Phi_K \) is equivariant via \( p^G \circ \Psi^G \), that is,
\[
\Phi_K(g \ast u) = g(0) \cdot \Phi_K(u)
\]
for \( g \in P(G, G \times K) \). Moreover we have
\[
P(G, H \times K) * u = \Phi_K^{-1}(H \cdot \Phi_K(u))
\]
for any closed subgroup \( H \) of \( G \). More generally, if \( N \) is a closed submanifold of \( G/K \) then the inverse image \( \Phi_K^{-1}(N) \) is a PF submanifold of \( V_g \).

3. The canonical isomorphisms of path spaces

In this section we introduce isomorphisms of path spaces and show fundamental equivalences among \( P(G, L) \)-actions and parallel transport maps.

Recall that a connected compact Lie group \( G \) with a bi-invariant metric is regarded as the symmetric space \((G \times G)/\Delta G\). In fact the diagonal \( \Delta G \) is a symmetric subgroup of \( G \times G \) with involution \((b, c) \mapsto (c, b)\). The canonical decomposition is given by
\[
g \oplus g = \mathfrak{k} \oplus \mathfrak{m},
\]
where \( \mathfrak{k} = \Delta g = \{ (x, x) \mid x \in g \} \) and \( \mathfrak{m} = (\Delta g)^\perp = \{ (x, -x) \mid x \in g \} \). Consider the diffeomorphism
\[
\rho : (G \times G)/\Delta G \to G, \quad (b, c) \Delta G \mapsto bc^{-1},
\]
whose differential at \((e, e)\Delta G\) is identified with the map
\[
d\rho : (\Delta g)^\perp \to g, \quad (x, -x) \mapsto 2x.
\]
Note that
\[
\langle d\rho(x, -x), d\rho(y, -y) \rangle = 2\langle (x, -x), (y, -y) \rangle.
\]
Note also that \( G \times G \) acts on \( G \) by \( (2.1) \) and acts also on \((G \times G)/\Delta G\) by \( (2.2) \). Clearly \( \rho \) is equivariant with respect to these \( G \times G \)-actions.

There is a natural isomorphism of path spaces corresponding to \( \rho \):

**Definition 3.1.** Define the injective homomorphism \( \Omega : \mathcal{G} \to H^1([0, 1], G \times G) \) by
\[
\Omega(g) = (g(t/2), g(1-t/2)),
\]
and the linear isomorphism \( \Upsilon : V_\theta \to V_{\theta \otimes \theta} \) by
\[
\Upsilon(u) = (\frac{1}{2}u(t/2), -\frac{1}{2}u(1-t/2)).
\]
We call \( \Upsilon \) the canonical isomorphism from \( V_\theta \) to \( V_{\theta \otimes \theta} \). We also call \( \Omega \) the canonical isomorphism (from \( \mathcal{G} \) to \( \Omega(\mathcal{G}) \)) if there is no confusion.

It is easy to see that
\[
\langle \Upsilon(u), \Upsilon(v) \rangle_{L^2} = \frac{1}{2}\langle u, v \rangle_{L^2}.
\]
The maps \( \Omega \) and \( \Upsilon \) have the following equivariant properties:

**Theorem 3.2.**

(i) \( \Omega(P(G, L)) = P(G \times G, L \times \Delta G) \) for a closed subgroup \( L \) of \( G \times G \). In particular the image of \( \Omega \) is \( P(G \times G, G \times G \times \Delta G) \).
(ii) $\Upsilon$ is equivariant via $\Omega$, that is,
$$\Upsilon(g \ast u) = \Omega(g) \ast \Upsilon(u)$$
for $g \in \mathcal{G}$ and $u \in V_g$.

(iii) The following diagrams are commutative:
$$\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\Omega} & H^1([0,1], G \times G) \\
\varphi \downarrow & & \downarrow \pi_{\mathcal{G} \times \mathcal{G}}^\ast \pi_{\mathcal{G} \times \mathcal{G}} \\
G \times G & \xrightarrow{\text{id}} & G \times G
\end{array}$$
and
$$\begin{array}{ccc}
V_\emptyset & \xrightarrow{\Upsilon} & V_{\emptyset \emptyset} \\
\Phi \downarrow & & \downarrow \Phi_{\Delta G} \\
G & \xrightarrow{\rho} & (G \times G)/\Delta G
\end{array}$$

where $\Phi_{\Delta G} : V_{\emptyset \emptyset} \rightarrow G \times G \rightarrow (G \times G)/\Delta G$ denotes the parallel transport map over the symmetric space $(G \times G)/\Delta G$.

Proof. (i): Clearly $\Omega(g)(0) = (g(0), g(1))$ and $\Omega(g)(1) = (g(1/2), g(1/2)) \in \Delta G$. Thus $g \in P(G, L)$ if and only if $\Omega(g) \in P(G \times G, L \times \Delta G)$. Conversely every element of $P(G \times G, L \times \Delta G)$ is obtained in this way. This proves (i).

(ii): We set $\Omega(g) = \tilde{g} = (\tilde{g}_1, \tilde{g}_2)$ and $\Upsilon(u) = \tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ so that $\tilde{g}_1(t) = g(t/2), \tilde{g}_2(t) = g(1-t/2), \tilde{u}_1 = \frac{1}{2} u(t/2), \tilde{u}_2 = -\frac{1}{2} u(1-t/2)$. Then we have
$$\Upsilon(gug^{-1})(t) = \left(\frac{1}{2} g(t/2)u(t/2)g(t/2)^{-1}, -\frac{1}{2} g(1-t/2)u(1-t/2)g(1-t/2)^{-1}\right)$$
$$= (\tilde{g}_1(t)\tilde{u}_1(t)\tilde{g}_1(t)^{-1}, \tilde{g}_2(t)\tilde{u}_2(t)\tilde{g}_2(t)^{-1})$$
$$= (\tilde{g}_1, \tilde{g}_2)(\tilde{u}_1, \tilde{u}_2) = \tilde{g}_1(\tilde{g}_2)^{-1}(t) = \Omega(g)\Upsilon(u)\Omega(g)^{-1}(t).$$
Moreover since $\tilde{g}_1(t) = \frac{1}{2} g(t/2)$ and $\tilde{g}_2(t) = -\frac{1}{2} g(1-t/2)$ we have

$$\Upsilon(g'g^{-1})(t) = \left(\frac{1}{2} g'(t/2)g(t/2)^{-1}, -\frac{1}{2} g'(1-t/2)g(1-t/2)^{-1}\right)$$
$$= (\tilde{g}_1'(t)\tilde{u}_1(t)\tilde{g}_1'(t)^{-1}, \tilde{g}_2'(t)\tilde{u}_2(t)\tilde{g}_2'(t)^{-1}) = (\tilde{g}_1, \tilde{g}_2)'(\tilde{g}_1, \tilde{g}_2)^{-1} = \Omega(g')\Omega(g)^{-1}(t).$$

Therefore we have
$$\Upsilon(g \ast u) = \Upsilon(gug^{-1} - g'g^{-1}) = \Upsilon(gug^{-1}) - \Upsilon(g'g^{-1})$$
$$= \Omega(g)\Upsilon(u)\Omega(g)^{-1} - \Omega(g')\Omega(g)^{-1} = \Omega(g) \ast \Upsilon(u).$$

This proves (ii).

(iii): Let $g \in \mathcal{G}$. Then we have
$$p^{G \times G} \circ \Psi^{G \times G} \circ \Omega(g) = p^{G \times G} \circ \Psi^{G \times G}(\tilde{g}) = p^{G \times G}(\tilde{g}(0), \tilde{g}(1))$$
$$= \tilde{g}(0) = (\tilde{g}_1(0), \tilde{g}_2(0)) = (g(0), g(1)) = \Psi^G(g).$$

Let $u \in V_g$. Take $h \in \mathcal{G}$ satisfying $u = h \ast \hat{0}$. Then we have
$$\Phi_{\Delta G}(\Upsilon(u)) = \Phi_{\Delta G}(\Omega(h) \ast (\hat{0}, \hat{0})) = (p^{G \times G} \circ \Psi^{G \times G})(h) \cdot \Phi_{\Delta G}(\hat{0}, \hat{0})$$
$$= (h(0), h(1))\Delta G = \rho^{-1}(h(0)h(1)^{-1}) = \rho^{-1}(\Phi(u)).$$

This proves (iii). \hfill \Box

Two isometric actions $A_1$ on $X_1$ and $A_2$ on $X_2$ are said to be conjugate if there exist an isomorphism $\phi : A_1 \rightarrow A_2$ and an isometry $\varphi : X_1 \rightarrow X_2$ satisfying $\varphi(a \cdot p) = \phi(a) \cdot \varphi(p)$ for $a \in A_1$ and $p \in X_1$. In this case we say that these actions are conjugate via $\phi$ and $\varphi$. We allow $\varphi$ to be a diffeomorphism such that $\lambda \varphi$ is an isometry for a suitable $\lambda \in \mathbb{R}$. 
Corollary 3.3. The \( P(G, L) \)-action on \( V_\Psi \) is conjugate to the \( P(G \times G, L \times \Delta G) \)-action on \( V_{\Psi \Phi} \) via \( \Omega \) and \( \Upsilon \). In particular the \( P(G, G(\sigma)) \)-action on \( V_\Psi \) is conjugate to the \( P(G \times G, G(\sigma) \times \Delta G) \)-action on \( V_{\Phi \Psi} \) via \( \Omega \) and \( \Upsilon \).

Corollary 3.3 is a consequence of (i) and (ii) of Theorem 3.2. It allows us to apply the general results of \( P(G, H \times K) \)-actions to \( P(G, G(\sigma)) \)-actions. (iii) of Theorem 3.2 allows us to apply the general results of \( \Phi_K \) to \( \Phi \).

Remark 3.4. Considering the case \( \sigma = id \) in Corollary 3.3 we see that the \( P(G, \Delta G) \)-action on \( V_\Psi \) is conjugate to the \( P(G \times G, \Delta G \times \Delta G) \)-action on \( V_{\Psi \Phi} \). This fact was essentially observed by Pinkall and Thorbergsson [33, Remark in p. 283].

4. Principal curvatures via the parallel transport map \( \Phi \)

In this section we derive a formula for the principal curvatures of the PF submanifold \( \Phi^{-1}(N) \) for a curvature-adapted submanifold \( N \) of \( G \).

Let \( G \) be a connected compact semisimple Lie group with a bi-invariant metric, \( \Phi : V_\Psi \to G \) the parallel transport map and \( N \) a closed submanifold of \( G \). Suppose that \( N \) is \( k \)-curvature-adapted (23), that is, for each \( a \in N \) the following conditions hold:

(i) for every \( v \in T_a N \) the curvature operator \( R_v \) leaves \( T_a N \) invariant,
(ii) for each \( v \in T_a N \) there exists a \( k \)-dimensional abelian subalgebra \( t \) of \( g \) satisfying \( v \in dl_a(t) \subset T_a N \) such that

\[
\{R_{dl_a(\xi)} T_a N\}_{\xi \in l} \cup \{A^N_{dl_a(\xi)}\}_{\xi \in l}
\]

is a commuting family of endomorphisms of \( T_a N \).

Here the curvature operator \( R_v \) is defined by \( R_v(x) = R^G(x, v)v \) where \( R^G \) denotes the curvature tensor of \( G \). If \( a = e \) then \( R_v \) is identified with \( -\frac{1}{4} \text{ad}(v)^2 \).

Clearly \( 1 \)-curvature-adapted submanifolds are just curvature-adapted submanifolds in the original sense (11). We know that all orbits of sigma-actions of cohomogeneity \( k \) are \( k \)-curvature-adapted submanifolds ([7, Corollaries 3.3 and 3.4], [28, Proposition 4.2]).

By left translations we can assume without loss of generality that \( N \) is through \( e \in G \). Choose and fix a \( k \)-dimensional abelian subalgebra \( t \) of \( g \) satisfying the above condition (ii) and consider the real root space decomposition with respect to \( t \):

\[
g = g_0 + \sum_{\alpha \in \Delta^+} g_\alpha,
\]

where

\[
g_0 = \{x \in g \mid \text{ad}(\eta)x = 0 \ \text{for all} \ \eta \in t\},
\]

\[
g_\alpha = \{x \in g \mid \text{ad}(\eta)^2 x = -\langle \alpha, \eta \rangle^2 x \ \text{for all} \ \eta \in t\}.
\]

This is the common eigenspace decomposition of the commuting operators \( \{\text{ad}(\xi)^2\}_{\xi \in l} \). On the other hand, we have the common eigenspace decomposition of the commuting shape operators \( \{A^N_\xi\}_{\xi \in l} \). More precisely there exits a unique finite subset \( \Lambda \) of \( t \) such that ([28, Lemma 4.4])

\[
T_\xi N = \sum_{\lambda \in \Lambda} S_\lambda,
\]
where
\[ S_\lambda = \{ x \in T_e N \mid A_\xi^N(x) = \langle \lambda, \xi \rangle x \text{ for all } \xi \in \mathfrak{t} \}. \]

Since all those operators commute we have
\[ T_e N = \sum_{\lambda \in \Lambda_0} (\mathfrak{g}_0 \cap S_\lambda) + \sum_{\alpha \in \Delta^+} \sum_{\lambda \in \Lambda_0} (\mathfrak{g}_\alpha \cap S_\lambda), \]
\[ T_e^\perp N = \mathfrak{g}_0 \cap T_e^\perp N + \sum_{\alpha \in \Delta^+} (\mathfrak{g}_\alpha \cap T_e^\perp N), \]

where \( \Lambda_0 = \{ \lambda \in \Lambda \mid \mathfrak{g}_0 \cap S_\lambda \} \) and \( \Lambda_\alpha = \{ \lambda \in \Lambda \mid \mathfrak{g}_\alpha \cap S_\lambda \neq \{0\} \}. \) Set
\[ m(0, \lambda) = \dim(\mathfrak{g}_0 \cap S_\lambda), \quad m(\alpha, \lambda) = \dim(\mathfrak{g}_\alpha \cap S_\lambda), \]
\[ m(0, \perp) = \dim(\mathfrak{g}_0 \cap T_e^\perp N), \quad m(\alpha, \perp) = \dim(\mathfrak{g}_\alpha \cap T_e^\perp N). \]

Based on these decompositions we can describe the principal curvatures of the PF submanifold \( \Phi^{-1}(N) \):

**Theorem 4.1.** Let \( G \) be a connected compact semisimple Lie group with a bi-invariant metric, \( N \) a \( k \)-curvature-adapted submanifold of \( G \) through \( e \in G \) and \( \mathfrak{t} \) an abelian subalgebra of \( \mathfrak{g} \) satisfying the above condition (ii). Then for each \( \xi \in \mathfrak{t} \) the principal curvatures of \( \Phi^{-1}(N) \) in the direction of \( \hat{\xi} \) are given by
\[ \{0\} \cup \{ \langle \lambda, \xi \rangle \mid \lambda \in \Lambda_0 \cup \bigcup_{\beta \in \Delta^+} \Lambda_\beta \} \]
\[ \bigcup \left\{ \frac{\langle \alpha, \xi \rangle}{2 \arctan \left( \frac{\langle \alpha, \xi \rangle}{2 \langle \beta, \xi \rangle} \right) + 2m\pi} \mid \alpha \in \Delta^+ \backslash \Delta^+_\xi, \ \lambda \in \Lambda_\alpha, \ m \in \mathbb{Z} \right\}, \]
\[ \bigcup \left\{ \frac{\langle \alpha, \xi \rangle}{2n\pi} \mid \alpha \in \Delta^+ \backslash \Delta^+_\xi, \ \mathfrak{g}_\alpha \cap T_e^\perp N \neq \{0\}, \ n \in \mathbb{Z} \backslash \{0\} \right\}, \]

where we set \( \Delta^+_\xi := \{ \beta \in \Delta^+ \mid \langle \beta, \xi \rangle = 0 \} \) and \( \arctan \left( \frac{\langle \alpha, \xi \rangle}{2 \langle \beta, \xi \rangle} \right) := \frac{\pi}{2} \) if \( \langle \lambda, \xi \rangle = 0 \). The multiplicities are respectively given by
\[ \infty, \quad m(0, \lambda) + \sum_{\beta \in \Delta^+_\xi} m(\beta, \lambda), \quad m(\alpha, \lambda), \quad m(\alpha, \perp). \]

**Proof.** Set \( \tilde{N} := \rho^{-1}(N) \). From Theorem \( \ref{thm:principal-curvature} \) (iii) it suffices to compute the principal curvatures of the PF submanifold \( \Phi^{-1}_{\Delta G}(\tilde{N}) \) of \( V_{\mathfrak{g} \in \mathfrak{g}} \). For each \( x \in \mathfrak{g} \) we denote by \( \tilde{x} = (d\rho)^{-1}(x) = \frac{1}{2}(x, -x) \). Set \( \tilde{t} = d\rho^{-1}(t) \) and \( \tilde{\Delta} = d\rho^{-1}(\Delta) \). The root space decomposition of \( \mathfrak{m} = (\Delta \mathfrak{g})^\perp \) with respect to \( \tilde{t} \) is given by
\[ \mathfrak{m} = \mathfrak{m}_0 + \sum_{\tilde{\alpha} \in \Delta} \mathfrak{m}_{\tilde{\alpha}}, \]

where
\[ \mathfrak{m}_0 = \{ \tilde{y} \in \mathfrak{m} \mid \text{ad}(\tilde{\eta})\tilde{y} = 0 \text{ for all } \tilde{\eta} \in \tilde{t} \}, \]
\[ \mathfrak{m}_{\tilde{\alpha}} = \{ \tilde{y} \in \mathfrak{m} \mid \text{ad}(\tilde{\eta})^2\tilde{y} = -\langle \tilde{\alpha}, \tilde{\eta} \rangle^2 \tilde{y} \text{ for all } \tilde{\eta} \in \tilde{t} \}. \]

On the other hand the common eigenspace decomposition by the commuting shape operators \( \{ A_{\tilde{\xi}} \}_{\tilde{\xi} \in \tilde{t}} \) is
\[ T_{(e, e)} \tilde{N} = \sum_{\mu \in \Gamma} S_\mu, \]
where $\Gamma$ is a finite subset of $\hat{t}$ and
\[
S_\mu = \{ \hat{x} \in T_{(e,e)} N \mid A^N_{\hat{\xi}}(\hat{x}) = \langle \mu, \hat{\xi} \rangle \hat{x} \text{ for all } \hat{\xi} \in \hat{t} \}.
\]

Since the eigenvalues of $A^N_{\hat{\xi}}$ and $A^N_{\hat{\xi}}$ coincide it follows that $\Gamma = \{ 2\hat{\lambda} \mid \lambda \in \Lambda \}$. Thus the assertion follows from the general formula [28, Theorem 5.2] together with $\langle \hat{\alpha}, \hat{\xi} \rangle = \frac{1}{2} \langle \alpha, \xi \rangle$ and $\langle 2\hat{\lambda}, \hat{\xi} \rangle = \langle \lambda, \xi \rangle$.

Considering the case $N = \{ e \}$ we obtain the formula for the principal curvatures of fibers of the parallel transport map $\Phi : V \to G$ ([20, Theorem 4.11]). Here we choose a maximal abelian subalgebra $t$ of $g$ and thus $\dim g_{\alpha} = 2$.

**Corollary 4.2** (King-Terng [20]). The principal curvatures of the fiber $\Phi^{-1}(e)$ at $e \in G$ in the direction of $\hat{\xi} \in \hat{t}$ are given by
\[
\{ 0 \} \cup \left\{ \frac{\langle \alpha, \xi \rangle}{2n\pi} \mid \alpha \in \Delta^+ \setminus \Delta_{\hat{\xi}}^+, \ n \in \mathbb{Z} \setminus \{0\} \right\}.
\]
The multiplicities are respectively given by
\[
\infty, \quad 2.
\]

5. **Submanifold geometry of orbits of $\sigma$-actions**

In this section we study the submanifold geometry of orbits of $\sigma$-actions. The material is based on Ohno [30] in the case of Hermann actions (see also [28]).

Let $G$ be a connected compact semisimple Lie group and $\sigma$ an automorphism of $G$. We choose an $\text{Aut}(G)$-invariant inner product of $g$ and equip the corresponding bi-invariant Riemannian metric with $G$. Take a maximal abelian subalgebra $t$ of the fixed point algebra $g^\sigma$. The $\sigma$-action is the action of $G(\sigma) = \{ (b, \sigma(b)) \mid b \in G \}$ on $G$ and is hyperpolar where $\exp t$ is a section.

Consider the root space decomposition
\[
g = g(0) + \sum_{\alpha \in \Delta} g(\alpha),
\]
where
\[
g(0) = \{ z \in g^C \mid \text{ad}(z) = 0 \text{ for all } \eta \in t \},
g(\alpha) = \{ z \in g^C \mid \text{ad}(\eta)z = \sqrt{-1}\langle \alpha, \eta \rangle z \text{ for all } \eta \in t \},
\]
and $\Delta = \Delta(\sigma)$ is a root system of $t$. The real form is
\[
g = g_0 + \sum_{\alpha \in \Delta^+} g_{\alpha},
\]
where
\[
g_0 = g(0) \cap g, \quad g_{\alpha} = (g(\alpha) + g(-\alpha)) \cap g.
\]
These are expressed as
\[
g_0 = \{ x \in g \mid \text{ad}(\eta)x = 0 \text{ for all } \eta \in t \},
g_{\alpha} = \{ x \in g \mid \text{ad}(\eta)^2x = -\langle \alpha, \eta \rangle^2 x \text{ for all } \eta \in t \}.
\]
We set $m(\alpha) := \dim g_{\alpha}$.
Consider the eigenspace decomposition of $\sigma$

$$\mathfrak{g}^C = \sum_{\epsilon \in U(1)} \mathfrak{g}(\epsilon),$$

where

$$\mathfrak{g}(\epsilon) = \{ z \in \mathfrak{g}^C \mid \sigma(z) = \epsilon z \}.$$  

Since $\sigma$ commutes with $\text{ad}(\eta)$ we have

$$\mathfrak{g}^C = \sum_{\epsilon \in U(1)} \mathfrak{g}(0, \epsilon) + \sum_{\alpha \in \Delta} \sum_{\epsilon \in U(1)} \mathfrak{g}(\alpha, \epsilon),$$

where

$$\mathfrak{g}(0, \epsilon) = \mathfrak{g}(0) \cap \mathfrak{g}(\epsilon), \quad \mathfrak{g}(\alpha, \epsilon) = \mathfrak{g}(\alpha) \cap \mathfrak{g}(\epsilon).$$

The real form is given by

$$\mathfrak{g} = \sum_{\epsilon \in U(1) \geq 0} \mathfrak{g}_{0, \epsilon} + \sum_{\alpha \in \Delta^+} \sum_{\epsilon \in U(1)} \mathfrak{g}_{\alpha, \epsilon},$$

where $U(1)_{\geq 0} = \{ \epsilon \in U(1) \mid \text{Im}(\epsilon) \geq 0 \}$ and

$$\mathfrak{g}_{0, \epsilon} = (\mathfrak{g}(0, \epsilon) + \mathfrak{g}(0, \epsilon^{-1}) \cap \mathfrak{g},$$

$$\mathfrak{g}_{\alpha, \epsilon} = (\mathfrak{g}(\alpha, \epsilon) + \mathfrak{g}(-\alpha, \epsilon^{-1}) \cap \mathfrak{g}.$$  

We set $m(\alpha, \epsilon) = \dim \mathfrak{g}_{\alpha, \epsilon}.$

**Proposition 5.1.** Let $w \in \mathfrak{g}.$ Then the tangent space and the normal space of the orbit $N = G(\sigma) \cdot a$ through $a = \exp w$ are expressed as follows:

$$T_a N = d_{a}(\sum_{\epsilon \in U(1)_{\geq 0} \epsilon \neq 1} \mathfrak{g}_{0, \epsilon} + \sum_{\alpha \in \Delta^+ \epsilon \in U(1)} \mathfrak{g}_{\alpha, \epsilon}),$$

$$T^a_{-1} N = d_{a}(\mathfrak{t} + \sum_{\alpha \in \Delta^+} \sum_{\epsilon \in U(1)} \mathfrak{g}_{\alpha, \epsilon}).$$

Moreover (5.2) is the common eigenspace decomposition of the family of shape operators $\{ A_{d_{a}(\xi)}^N \}_{\xi \in \mathfrak{t}^\perp}.$ In fact

$$d_{a}(\mathfrak{g}_{0, \epsilon}) : \text{ the eigenspace associated with the eigenvalue } 0,$$

$$d_{a}(\mathfrak{g}_{\alpha, \epsilon}) : \text{ the eigenspace associated with the eigenvalue } -\frac{\langle \alpha, \epsilon \rangle}{2} \cot \frac{\langle \alpha, \alpha \rangle + \text{arg} \epsilon}{2}.$$

**Proof.** Recall that $K := G$ and $H := G(\sigma)$ are symmetric subgroups of $U := G \times G$ with involution $\theta_K : (b, c) \mapsto (c, b)$ and $\theta_H : (b, c) \mapsto (\sigma^{-1}(c), \sigma(b))$ respectively. Their canonical decompositions are respectively given by

$$u = \mathfrak{t} + \mathfrak{m} \quad \text{and} \quad u = \mathfrak{h} + \mathfrak{p},$$

where $u = \mathfrak{g} \oplus \mathfrak{g}, \quad \mathfrak{t} = (\Delta \mathfrak{g})^\perp, \quad m = \{ (x, \sigma(x)) \mid x \in \mathfrak{g} \}$ and $p = \{ (x, -\sigma(x)) \mid x \in \mathfrak{g} \}.$ For each $x \in \mathfrak{g}$ we set $\tilde{x} = (d\rho)^{-1}(x) = \frac{1}{2}(x, -x).$ Note that $\mathfrak{t} = (d\rho)^{-1}(\mathfrak{t})$ is a maximal abelian subspace in $\mathfrak{m} \cap \mathfrak{p}.$ Set $\tilde{\Delta} = d\rho^{-1}(\Delta).$ The root space decomposition of $u$ with respect to $\mathfrak{t}$ is given by

$$u = u(0) + \sum_{\tilde{\alpha} \in \Delta} u(\tilde{\alpha}),$$

and
where

\[ u(0) = \{(z_1, z_2) \in u^C | \text{ad}(\tilde{\eta})(z_1, z_2) = 0 \text{ for all } \tilde{\eta} \in \mathfrak{t}\}, \]

\[ u(\tilde{\alpha}) = \{(z_1, z_2) \in u^C | \text{ad}(\tilde{\eta})(z_1, z_2) = \langle \tilde{\alpha}, \tilde{\eta}\rangle(z_1, z_2) \text{ for all } \tilde{\eta} \in \mathfrak{t}\}. \]

Clearly \( u(0) = g(\mathfrak{g}) \oplus g(0) \) and \( u(\tilde{\alpha}) = g(\alpha) \oplus g(\alpha) \). Consider the eigenspace decomposition of the composition \( \theta_K \circ \theta_H : (b, c) \mapsto (\sigma(b), \sigma^{-1}(c)) \):

\[ u^C = \sum_{\epsilon \in U(1)} u(\epsilon), \]

where

\[ u(\epsilon) = \{(z_1, z_2) \in u^C | \theta_K \circ \theta_H(z_1, z_2) = \epsilon(z_1, z_2)\}. \]

Clearly \( u(\epsilon) = g(\epsilon) \oplus g(\epsilon^{-1}) \). Similarly to (5.1) we have

\[ u = \sum_{\epsilon \in U(1) \geq 0} u_{0, \epsilon} + \sum_{\alpha \in \Delta^+ \epsilon \in U(1)} u_{\alpha, \epsilon} \]

and therefore we get

\[ m = \sum_{\epsilon \in U(1) \geq 0} m_{0, \epsilon} + \sum_{\alpha \in \Delta^+ \epsilon \in U(1)} m_{\alpha, \epsilon}, \]

where \( m_{0, \epsilon} = u_{0, \epsilon} \cap m \) and \( m_{\alpha, \epsilon} = u_{\alpha, \epsilon} \cap m \). It is easy to see that \( m_{0, \epsilon} = \{(x, -x) | x \in g_{0, \epsilon}\} \) and \( m_{\alpha, \epsilon} = \{(x, -x) | x \in g_{\alpha, \epsilon}\} \). Thus \( dp(m_{0, \epsilon}) = g_{0, \epsilon} \) and \( \rho(m_{\alpha, \epsilon}) = g_{\alpha, \epsilon} \). Hence the assertion follows from the results of Hermann actions ([30] p. 12), [28] Section 3) together with \( \langle \tilde{\alpha}, \tilde{\xi}\rangle = \frac{1}{2}\langle \alpha, \xi\rangle \) and \( \langle \tilde{\alpha}, \tilde{w}\rangle = \frac{1}{2}\langle \alpha, w\rangle \). \( \square \)

If \( \sigma \) is involutive then we have the \((\pm 1)\)-eigenspace decomposition

\[ g = g^+ + g^-, \]

where \( g^\pm = \{x \in g | \sigma(x) = \pm x\} \). We set

\[ g^+_\alpha := g_\alpha \cap g^+ \quad \text{and} \quad g^-_\alpha := g_\alpha \cap g^- . \]

Since \( g_{\alpha, 1} = g^+_\alpha \) and \( g_{\alpha, -1} = g^-_\alpha \) we obtain:

**Corollary 5.2.** Suppose that \( \sigma^2 = \text{id} \). Then the tangent space and the normal space of the orbit \( N = G(\sigma) \cdot a \) through \( a = \exp w \) are expressed as follows:

\[ T_aN = dl_a( g^-_0 + \sum_{\alpha \in \Delta^+ \langle \alpha, w\rangle \notin 2\pi Z} g^+_\alpha + \sum_{\alpha \in \Delta^+ \langle \alpha, w\rangle + \pi \notin 2\pi Z} g^-_\alpha ), \quad (5.4) \]

\[ T^+_aN = dl_a( t + \sum_{\alpha \in \Delta^+ \langle \alpha, w\rangle \in 2\pi Z} g^+_\alpha + \sum_{\alpha \in \Delta^+ \langle \alpha, w\rangle + \pi \in 2\pi Z} g^-_\alpha ). \quad (5.5) \]

Moreover (5.4) is the common eigenspace decomposition of \( \{ A_{dl_a(\xi)}^N \}_{\xi \in V} \). In fact

\[ dl_a(g^-_0) : \text{the eigenspace associated with the eigenvalue } 0, \]

\[ dl_a(g^+_\alpha) : \text{the eigenspace associated with the eigenvalue } -\frac{\langle \alpha, \xi\rangle}{2} \cot \frac{\langle \alpha, w\rangle}{2}, \]

\[ dl_a(g^-_\alpha) : \text{the eigenspace associated with the eigenvalue } \frac{\langle \alpha, \xi\rangle}{2} \tan \frac{\langle \alpha, w\rangle}{2}. \]
Corollary 5.3. Suppose that $\sigma = \text{id}$. Then the tangent space and the normal space of the orbit $N = \Delta G \cdot a$ through $a = \exp w$ are expressed as follows:

$$T_a N = dl_a \left( g_0 + \sum_{\alpha \in \Delta^+ \backslash \{\alpha, w\} \notin 2\pi \mathbb{Z} \atop (\alpha, w) \in 2\pi \mathbb{Z}} g_\alpha \right),$$  

$$T_a^\perp N = dl_a \left( t + \sum_{\alpha \in \Delta^+ \backslash \{\alpha, w\} \notin 2\pi \mathbb{Z} \atop (\alpha, w) \in 2\pi \mathbb{Z}} g_\alpha \right).$$

Moreover (5.6) is the common eigenspace decomposition of $\left\{ A_{dl_a(\xi)}^N \right\}_{\xi \in \hat{t}}$. In fact

$$dl_a(\mathfrak{g}_0): \text{ the eigenspace associated with the eigenvalue } 0,$$

$$dl_a(\mathfrak{g}_\alpha): \text{ the eigenspace associated with the eigenvalue } -\frac{\langle \alpha, \xi \rangle}{2} \cot \frac{\langle \alpha, w \rangle}{2}.$$

6. Principal curvatures of orbits of $P(G, G(\sigma))$-actions

In this section we derive an explicit formula for the principal curvatures of orbits of $P(G, G(\sigma))$-actions.

As in the last section we let $G$ be a connected compact semisimple Lie group with a bi-invariant metric induced from an Aut($G$)-invariant inner product of $\mathfrak{g}$ and $\sigma$ an automorphism of $G$. Take a maximal abelian subalgebra $t$ of $\mathfrak{g}$. Then $\exp t$ is a section of the $\sigma$-action and $\hat{t} = \{ \hat{x} \mid x \in t \}$ is a section of the $P(G, G(\sigma))$-action where $\hat{x}$ denotes the constant path with value $x \in \mathfrak{g}$ ([10, Theorem 1.2]). Take $w \in t$ and set

$$U(1)_{\alpha}^\top = \{ \epsilon \in U(1) \mid g_{\alpha, \epsilon} \neq \{0\}, \langle \alpha, w \rangle + \arg \epsilon \notin 2\pi \mathbb{Z} \},$$

$$U(1)_{\alpha}^\perp = \{ \epsilon \in U(1) \mid g_{\alpha, \epsilon} \neq \{0\}, \langle \alpha, w \rangle + \arg \epsilon \in 2\pi \mathbb{Z} \}.$$

Theorem 6.1. The principal curvatures of $P(G, G(\sigma)) \ast \hat{w}$ in the direction of $\hat{\xi} \in \hat{t}$ are given by

$$\{0\} \cup \left\{ \frac{\langle \alpha, \xi \rangle}{\langle \alpha, w \rangle - \arg \epsilon + 2m\pi} \left| \alpha \in \Delta^+ \backslash \Delta^+_\xi, \epsilon \in U(1)_{\alpha}^\top, m \in \mathbb{Z} \right. \right\}$$

$$\cup \left\{ \frac{\langle \alpha, \xi \rangle}{2n\pi} \left| \alpha \in \Delta^+ \backslash \Delta^+_\xi \text{ satisfying } U(1)_{\alpha}^\perp \neq \emptyset, n \in \mathbb{Z} \backslash \{0\} \right. \right\}.$$

The multiplicities are respectively given by

$$\infty, m(\alpha, \epsilon), \sum_{\epsilon \in U(1)_{\alpha}^\perp} m(\alpha, \epsilon).$$

If the orbit is principal then the term $\frac{\langle \alpha, \xi \rangle}{2m\pi}$ vanishes.

Proof. By Corollary 3.3 the $P(G, G(\sigma))$-action on $V_\mathfrak{g}$ is conjugate the $P(G \times G, G(\sigma) \times \Delta G)$-action on $V_{\mathfrak{g} \oplus \mathfrak{g}}$. Since the $G(\sigma)$-action on $(G \times G)/\Delta G$ is a Hermann action the assertion follows from [28, Theorem 6.1] together with $\langle \hat{\alpha}, \hat{\xi} \rangle = \frac{1}{2} \langle \alpha, \xi \rangle$ and $\langle \hat{\alpha}, \hat{\xi} \rangle = \frac{1}{2} \langle \alpha, w \rangle$. (It can be also proven by applying Theorem 4.1 to Proposition 5.1.)
Corollary 6.2. Suppose that $\sigma^2 = \text{id}$. Then the principal curvatures of $P(G, G(\sigma))^\ast \hat{w}$ in the direction of $\hat{\xi} \in \hat{\mathfrak{t}}$ are given by

$$
\{0\} \cup \left\{ \frac{\langle \alpha, \xi \rangle}{-\langle \alpha, w \rangle + 2\pi m} \mid \alpha \in \Delta^+ \setminus \Delta^+_\xi, \ g^+_\alpha \neq \{0\}, \ \langle \alpha, w \rangle \notin 2\pi\mathbb{Z}, \ m \in \mathbb{Z} \right\}
\cup \left\{ \frac{\langle \alpha, \xi \rangle}{-\langle \alpha, w \rangle - \pi + 2\pi m} \mid \alpha \in \Delta^+ \setminus \Delta^+_\xi, \ g^-_\alpha \neq \{0\}, \ \langle \alpha, w \rangle + \pi \notin 2\pi\mathbb{Z}, \ m \in \mathbb{Z} \right\}
\cup \left\{ \frac{\langle \alpha, \xi \rangle}{2\pi n} \mid \alpha \in \Delta^+ \setminus \Delta^+_\xi, \ g^+_\alpha \neq \{0\}, \ \langle \alpha, w \rangle \in 2\pi\mathbb{Z}, \ n \in \mathbb{Z}\setminus\{0\} \right\}.
$$

The multiplicities are respectively given by

$$
\infty, \ \dim g^+_\alpha, \ \dim g^-_\alpha, \ \dim g^+_\alpha + \dim g^-_\alpha.
$$

If the orbit is principal then the term $\frac{\langle \alpha, \xi \rangle}{2\pi n}$ vanishes.

Corollary 6.3. Suppose that $\sigma = \text{id}$. Then the principal curvatures of $P(G, \Delta G)^\ast \hat{w}$ in the direction of $\hat{\xi} \in \hat{\mathfrak{t}}$ are given by

$$
\{0\} \cup \left\{ \frac{\langle \alpha, \xi \rangle}{-\langle \alpha, w \rangle + 2\pi m} \mid \alpha \in \Delta^+ \setminus \Delta^+_\xi, \ \langle \alpha, w \rangle \notin 2\pi\mathbb{Z}, \ m \in \mathbb{Z} \right\}
\cup \left\{ \frac{\langle \alpha, \xi \rangle}{2\pi n} \mid \alpha \in \Delta^+ \setminus \Delta^+_\xi, \ \langle \alpha, w \rangle \in 2\pi\mathbb{Z}, \ n \in \mathbb{Z}\setminus\{0\} \right\}.
$$

The multiplicities are respectively given by

$$
\infty, \ \dim g^-_\alpha, \ \dim g^-_\alpha.
$$

If the orbit is principal then the term $\frac{\langle \alpha, \xi \rangle}{2\pi n}$ vanishes.

Remark 6.4. Corollary 6.3 generalizes a result by Palais and Terng in the case of principal $P(G, \Delta G)$-orbits ([32, Section 5.8] and [39, p. 24]).

7. The austere property

In this section we study conditions for $P(G, G(\sigma))$-orbits to be austere PF submanifolds of $V_\mathfrak{g}$.

Let $G$ be a connected compact semisimple Lie group with a bi-invariant metric induced from an Aut($G$)-invariant inner product on $\mathfrak{g}$ and $\sigma$ be an automorphism of $G$. Consider two conditions for $w \in \mathfrak{g}$:

(a) The orbit $N = G(\sigma) \cdot \exp w$ is an austere submanifold of $G$.

(b) The orbit $\Phi^{-1}(N) = P(G, G(\sigma))^\ast \hat{w}$ is an austere PF submanifold of $V_\mathfrak{g}$.

Take a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}^\sigma$ and denote by $\Delta = \Delta(\sigma)$ the corresponding root system of $\mathfrak{t}$. We show (Theorem II in Introduction):

Theorem 7.1.

(i) If $\Delta(\sigma)$ is a reduced root system then (a) and (b) are equivalent.

(ii) If $\sigma = \text{id}$ then (a) and (b) are equivalent.

(iii) If $\sigma^2 = \text{id}$ then (a) implies (b).

(iv) If $G$ is simple then (a) implies (b).
Here (b) does not imply (a) in the cases (iii) and (iv). In fact, there exists a counterexample.

As mentioned in the Introduction the above (i)–(iii) follow from Theorem 3.2 and the previous result ([28]). Thus in this section we prove (iv). (A counterexample to the converse will be shown in Theorem 9.2.) To do this we need two lemmas. The first one is well-known (cf. [24, p. 44], [12, Theorem 3.9]):

**Lemma 7.2.** Suppose that $G$ is simple. Then there exist $a \in G$ and a diagram automorphism $\tau$ of $G$ which has order 1, 2 or 3 such that $\sigma = \tau \circ \text{Ad}(a)$.

**Lemma 7.3.** Suppose that there exist an automorphism $\tau$ of $G$ and $a \in G$ such that $\sigma = \tau \circ \text{Ad}(a)$. Then

(i) the $G(\sigma)$-action is conjugate to the $G(\tau)$-action,

(ii) the $P(G, G(\sigma))$-action is conjugate to the $P(G, G(\tau))$-action.

**Proof.** (i) Since $G(\sigma) = (a, e)^{-1}G(\tau)(a, e)$ it follows that the isometry $l_a : G \to G$ is equivariant via the isomorphism $\text{Ad}(a, e) : G(\sigma) \to G(\tau)$. This proves (i).

(ii) From the standard arguments in the theory of linear ordinary differential equations there exists a unique $g \in P(G, G \times \{e\})$ satisfying $g(0) = a$. Since $\Psi$ is a group homomorphism it follows that the diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\text{Ad}(g)} & G \\
\Psi \downarrow & & \Psi \downarrow \\
G \times G & \xrightarrow{\text{Ad}(a, e)} & G \times G
\end{array}
$$

commutes. Since $P(G, L)$ is the inverse image of $L$ under $\Psi$ it follows that $\text{Ad}(g)$ maps $P(G, G(\sigma))$ isomorphically onto $P(G, G(\tau))$. Moreover the isometry $g_* : V_{g} \to V_{g}$ is equivariant via the isomorphism $\text{Ad}(g) : P(G, G(\sigma)) \to P(G, G(\tau))$. This proves (ii). \qed

We are now in position to prove (iv) of Theorem 7.1.

**Proof of Theorem 7.1 (iv).** From Lemmas 7.2 and 7.3 we can assume without loss of generality that $\sigma$ is a diagram automorphism of $G$ and has order 1, 2 or 3. If $\sigma$ has order 1 then the assertion follows from Theorem 7.1 (ii). If $\sigma$ has order 2 then the assertion follows from Theorem 7.1 (iii). If $\sigma$ has order 3 then $g = o(8)$ and $\sigma$ is the so-called triality automorphism. Take a maximal abelian subalgebra $t$ of $g^\sigma = g_2$. Then the root system $\Delta$ is of type $G_2$ and the assertion follows from Theorem 7.1 (i). \qed

**Corollary 7.4.** If $\sigma$ is inner then (a) and (b) are equivalent.

**Proof.** Since $\sigma$ is inner there exists a maximal abelian subalgebra $t$ of $g$ which is fixed by $\sigma$, that is, $t \subset g^\sigma$. Thus the corresponding root system $\Delta$ of $t$ is reduced and the assertion follows from (i) of Theorem 7.1. \qed

**Example 7.5.** Ikawa [17] classified austere orbits of $\sigma$-actions when $\Delta$ is irreducible and $\sigma$ is involutive. Recently Kimura and Mashimo [19] classified Cartan embeddings which are austere submanifolds when $G$ is simple. Applying Theorem 7.1 to their results we obtain many examples of $P(G, G(\sigma))$-orbits which are austere PF submanifolds of $V_g$. 
8. The weakly reflective property

In this section we extend the author’s previous results concerning weakly reflective PF submanifolds in Hilbert spaces.

Recall that a submanifold $N$ of a Riemannian manifold $M$ is called weakly reflective \((\text{[18]})\) if for each normal vector $\xi$ at each $p \in N$ there exists an isometry $\nu_\xi$ of $M$ satisfying

$$
\nu_\xi(p) = p, \quad \nu_\xi(N) = N, \quad d\nu_\xi(\xi) = -\xi.
$$

(8.1)

It follows that weakly reflective submanifolds are austere submanifolds.

The author \([26]\) extended the concept of weakly reflective submanifolds to the class of PF submanifolds in Hilbert spaces and studied the relation between the following conditions:

\begin{itemize}
  \item[(C)] $N$ is a weakly reflective submanifold of $G/K$.
  \item[(D)] $\Phi_K^{-1}(N)$ is a weakly reflective PF submanifold of $V_\mathfrak{g}$.
\end{itemize}

Here $N$ is a closed submanifold of a compact symmetric space $G/K$. He showed (\([26, \text{Theorem 8}]\)):

**Theorem 8.1** (\([26]\)). Let $G$ be a connected compact semisimple Lie group and $K$ a symmetric subgroup of $G$. Suppose that the bi-invariant Riemannian metric of $G$ is induced by an $\text{Aut}(G)$-invariant inner product of $\mathfrak{g}$ and $G$ acts effectively on $G/K$. Then (C) implies (D).

It is interesting to remark that here $G$ need not be simple and $N$ need not be an orbit of a Hermann action, unlike in the austere case \([28]\). Applying this theorem to examples of weakly reflective submanifold in $G/K$ he obtained many examples of weakly reflective PF submanifolds in $V_\mathfrak{g}$. We do not know whether (D) implies (C) or not. If the symmetric space $G/K$ is irreducible (or more generally, $G/K$ is a compact isotropy irreducible Riemannian homogeneous space) we can characterize the weakly reflective PF submanifold $\Phi_K^{-1}(N)$ (\([27]\)).

In \([26]\) he also studied the relation between the following conditions:

\begin{itemize}
  \item[(c)] $N$ is a weakly reflective submanifold of $G$.
  \item[(d)] $\Phi^{-1}(N)$ is a weakly reflective PF submanifold of $V_\mathfrak{g}$.
\end{itemize}

Here $N$ is a closed submanifold of a connected compact Lie group $G$. The following theorem claims that (c) implies (d) under strong conditions (\([26, \text{Theorem 7}]\)):

**Theorem 8.2** (\([26]\)). Let $G$ be a connected compact semisimple Lie group with a bi-invariant metric. Suppose that $N = L \cdot e$ is the orbit through the identity where $L$ is a closed subgroup of $G \times G$ acting on $G$ by \([21]\). Suppose also that $N$ is a weakly reflective submanifold of $G$ such that for each $\xi \in T_e^\perp N$ the isometry $\nu_\xi$ satisfying (8.1) can be taken from $\text{Aut}(G)$. Then $\Phi^{-1}(N) = P(G, L) \ast \hat{0}$ is a weakly reflective PF submanifold of $V_\mathfrak{g}$.

The following theorem greatly extends the above theorem:

**Theorem 8.3.** Let $G$ be a connected compact semisimple Lie group with a bi-invariant Riemannian metric induced by an $\text{Aut}(G)$-invariant inner product of $\mathfrak{g}$. Then (c) implies (d).
Proof. Set $N' := \rho^{-1}(N)$. Since $N'$ is weakly reflective it follows from Theorem 8.1 that $\Phi_{\Delta G}^{-1}(N')$ is also weakly reflective. Thus by Theorem 3.2 (iii) the assertion follows.

Here $G$ need not be simple and $N$ need not be an orbit of $\sigma$-action, unlike in the austere case (Theorem 7.1).

Example 8.4. Kimura and Mashimo [19] gave examples of Cartan embeddings which are weakly reflective submanifolds. Applying Theorem 8.3 to the results we obtain $P(G, G(\sigma))$-orbits which are weakly reflective PF submanifold of $V_g$.

Remark 8.5. Taketomi [37] introduced a generalized concept of weakly reflective submanifolds, namely arid submanifolds. The results in this section are also valid in the case of arid submanifolds (see also [25]).

9. Examples and Counterexamples

In this section we show new and concrete examples of austere PF submanifolds and weakly reflective PF submanifolds which are orbits of a $P(G, G(\sigma))$-action. In particular we show a counterexample to the converse of (iii) and (iv) of Theorem 7.1. Throughout this section we will consider the pair $(G, \sigma) = (SU(2m + 1), \text{the complex conjugation})$.

This is the only example of a $\sigma$-action whose root system $\Delta(\sigma)$ is non-reduced ([17, p. 558]). The canonical decomposition $g = g^+ + g^-$ is given by $g = su(2m + 1)$, $g^+ = g^{\sigma} = so(2m + 1)$ and

$$g^- = \sqrt{-1}\{X \in \text{sym}(2m + 1, \mathbb{R}) \mid \text{tr} X = 0\}.$$ 

We define the $\text{Aut}(G)$-invariant inner product of $g$ by

$$\langle X, Y \rangle = -\frac{1}{2} \text{tr}(XY) \quad \text{where } X, Y \in g.$$ 

A maximal abelian subalgebra of $g^+$ is

$$t = \left\{ \begin{bmatrix} X_1 & \cdots & X_m \\ \vdots & \ddots & \vdots \\ X_m & & 0 \end{bmatrix} \in g^+ \mid X_i = \begin{bmatrix} 0 & -x_i \\ x_i & 0 \end{bmatrix}, \quad x_i \in \mathbb{R} \right\}.$$ 

For each $i = 1, \ldots, m$ we set

$$e_i = \begin{bmatrix} J \end{bmatrix}_i \quad \text{where } J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$ 

Then $\{e_i\}_{i=1}^m$ is an orthogonal basis of $t$. We set

$$u = \left\{ \begin{bmatrix} Y_1 & \cdots & Y_m \\ \vdots & \ddots & \vdots \\ Y_m & & y \end{bmatrix} \in g^- \mid Y_i = \sqrt{-1}\begin{bmatrix} y_i & 0 \\ 0 & y_i \end{bmatrix}, \quad y_i \in \mathbb{R}, \quad y = -2\sqrt{-1}(y_1 + \cdots + y_m) \right\}.$$ 

We denote by $g_{2e_i}$ the subspace of $g^-$ consisting of matrices
where \( P = \sqrt{-1} \begin{bmatrix} p & q \\ q & -p \end{bmatrix}, p, q \in \mathbb{R} \)

and by \( \mathfrak{g}_i \) the subspace of \( \mathfrak{g} \) consisting of matrices

\[
\begin{bmatrix}
  i \\
  v \\
  -\bar{v}
\end{bmatrix}
\]

where \( v = \begin{bmatrix} z \\ w \end{bmatrix} \in \mathbb{C}^2 \)

lies in the \((2m + 1)\)-column.

For each \( 1 \leq i < j \leq m \) we denote by \( \mathfrak{g}_{ei, \pm ej} \) the subspace of \( \mathfrak{g} \) consisting of matrices

\[
\begin{bmatrix}
  i \\
  j \\
  A \\
  -\bar{A}
\end{bmatrix}
\]

where \( A = \begin{bmatrix} \alpha & \pm \beta \\ \beta & \mp \alpha \end{bmatrix} \), \( \alpha, \beta \in \mathbb{C} \).

Then we obtain the root space decomposition

\[
\mathfrak{g} = t + u + \sum_{i=1}^{m} \mathfrak{g}_{2ei} + \sum_{i=1}^{m} \mathfrak{g}_i + \sum_{1 \leq i < j \leq m} \mathfrak{g}_{ei+ej} + \sum_{1 \leq i < j \leq m} \mathfrak{g}_{ei-ej}.
\]

The root system \( \Delta = \{e_i, 2e_i\} \cup \{e_i \pm e_j\}_{i < j} \) is of type \( BC \). The dimensions of those spaces are

\( m, m, 2, 4, 4, 4, \)

respectively. This decomposition is refined as

\[
\mathfrak{g}^+ = t + \sum_{i=1}^{m} \mathfrak{g}_{ei}^+ + \sum_{1 \leq i < j \leq m} \mathfrak{g}_{ei+ej}^+ + \sum_{1 \leq i < j \leq m} \mathfrak{g}_{ei-ej}^+,
\]

\[
\mathfrak{g}^- = \mathfrak{g}^-_0 + \sum_{i=1}^{m} \mathfrak{g}_{2ei}^- + \sum_{i=1}^{m} \mathfrak{g}_i^- + \sum_{1 \leq i < j \leq m} \mathfrak{g}_{ei+ej}^- + \sum_{1 \leq i < j \leq m} \mathfrak{g}_{ei-ej}^-,
\]

where \( \mathfrak{g}^-_0 = u \).

From now on we take \( w \in t \) and show examples of orbits \( G(\sigma) \cdot \exp w \) and \( P(G, G(\sigma)) \cdot \hat{w} \) which are austere or weakly reflective. Note that since the actions are hyperpolar it suffices to consider normal vectors \( \{ dl_{\exp w}(\xi) \}_{\xi \in t} \) and \( \{ \hat{\xi} \}_{\xi \in \hat{t}} \) respectively (cf. [18, Lemma 4.3 and p. 458], [28, Lemma 7.2]).

In the following proposition, the assertion (i) essentially follows from [16, Theorem 2.18]. However we give a direct proof here because his proof is incorrect when \( \Delta \) is of type \( BC_1 \). (It can be corrected by using the property of multiplicities of roots [28, Lemma 8.1].)
Proposition 9.1. Let $G$, $\sigma$ be as above. Set

$$w := \pi \sum_{i=1}^{m} e_i.$$ 

Then

(i) the orbit $N = G(\sigma) \cdot \exp w$ is an austere submanifold of $G$,

(ii) the orbit $\Phi^{-1}(N) = P(G, G(\sigma)) \ast \hat{w}$ is an austere PF submanifold of $V_g$.

Proof. (i) Set $a = \exp w$. By Corollary 5.2 we have

$$T_a N = \{ \theta_0 + \sum_{i=1}^{m} \theta_{e_i}^+ + \sum_{1 \leq i < j \leq m} \theta_{e_i + e_j}^+ + \sum_{i=1}^{m} \theta_{e_i}^- + \sum_{1 \leq i < j \leq m} \theta_{e_i - e_j}^- \} ,$$

$$T_a^\perp N = \{ t + \sum_{1 \leq i < j \leq m} \theta_{e_i - e_j}^+ + \sum_{i=1}^{m} \theta_{2e_i} + \sum_{1 \leq i < j \leq m} \theta_{e_i + e_j}^- \} ,$$

and the principal curvatures of $N$ in the direction of $dl_a(\xi)$ are expressed as the inner product of $\xi$ and vectors

$$0, \ -\frac{1}{2} e_i, \ 0, \ \frac{1}{2} e_i, \ 0.$$

Since the set of these vectors are invariant under the multiplication by $(-1)$ it follows that $N$ is an austere submanifold of $G$.

(ii) The assertion follows by applying Theorem 7.1 (iii) to (i). To describe the principal curvatures explicitly, we give a direct proof here. By corollary 6.2 the principal curvatures of $P(G, G(\sigma)) \ast \hat{w}$ in the direction of $\hat{\xi}$ is expressed as the inner product of $\xi$ and vectors

$$\{0\}, \ \left\{ \frac{1}{-\pi + 2m\pi} e_i \right\}_{m \in \mathbb{Z}} , \ \left\{ \frac{1}{-\pi + 2m\pi} (e_i + e_j) \right\}_{m \in \mathbb{Z}}, \ \left\{ \frac{1}{-\frac{3}{2}\pi + 2m\pi} e_i \right\}_{m \in \mathbb{Z}} , \ \left\{ \frac{1}{-\pi + 2m\pi} (e_i - e_j) \right\}_{m \in \mathbb{Z}} , \ \left\{ \frac{1}{2n\pi} \alpha \right\}_{\alpha = e_i - e_j, \ 2e_i, \ e_i + e_j}_{\alpha \in \mathbb{Z} \backslash \{0\}} .$$

Clearly the set $\left\{ \frac{1}{2n\pi} \alpha \right\}_{\alpha \in \mathbb{Z} \backslash \{0\}}$ is austere (i.e. invariant under the multiplication by $(-1)$) for each $\alpha$. By the equality

$$\frac{1}{-\pi + 2m\pi} (e_i \pm e_j) = (-1) \times \frac{1}{-\pi + 2(-m + 1)\pi} (e_i \pm e_j), \quad (9.1)$$

the set $\left\{ \frac{1}{\pi + 2m\pi} (e_i \pm e_j) \right\}_{m \in \mathbb{Z}}$ is austere. By the equality

$$\frac{1}{-\frac{3}{2}\pi + 2m\pi} e_i = (-1) \times \frac{1}{-\frac{3}{2}\pi + 2(-m + 1)\pi} e_i, \quad (9.2)$$

the set $\left\{ \frac{1}{-\frac{3}{2}\pi + 2m\pi} e_i \right\}_{m \in \mathbb{Z}}$ is austere. These show that the orbit $P(G, G(\sigma)) \ast \hat{w}$ is an austere PF submanifold of $V_g$. \qed

The following theorem shows a counterexample to the converse of (3) and (4) of Theorem 7.1.
**Theorem 9.2.** Let $G, \sigma$ be as above. Set

$$w := \frac{\pi}{4} \sum_{i=1}^{m} e_i.$$  

Then

(i) the orbit $N = G(\sigma) \cdot \exp w$ is a minimal submanifold of $G$, but not an austere submanifold of $G$,

(ii) the orbit $\Phi^{-1}(N) = P(G, G(\sigma)) \ast \hat{w}$ is an austere PF submanifold of $V_9$.

**Remark 9.3.** This counterexample is different from the one given in [28, Section 9]. In fact the previous one is not an orbit of a $\sigma$-action. The symmetric triads corresponding to those actions are of the same type (II-BC, see [16, 17]). However their multiplicities are different.

**Proof of Theorem 9.2**

(i) Set $a = \exp w$. By Corollary 5.2 we have

$$T_a N = d l_a ( g_0 + \sum_{i=1}^{m} g_{e_i}^+ + \sum_{1 \leq i < j \leq m} g_{e_i + e_j}^+$$

$$+ \sum_{i=1}^{m} g_{e_i}^- + \sum_{1 \leq i < j \leq m} g_{e_i - e_j}^- )$$

$$T_a^+ N = d l_a ( t + \sum_{1 \leq i < j \leq m} g_{e_i - e_j}^+),$$

and the principal curvatures of $N$ in the direction of $dl_a(\xi)$ are expressed as the inner product of $\xi$ and vectors

$$0, -\frac{\sqrt{2} + 1}{2} e_i, -\frac{1}{2} (e_i + e_j), e_i, \frac{\sqrt{2} - 1}{2} e_i, \frac{1}{2} (e_i + e_j), 0.$$  

Since the sum of these vectors are zero, $N$ is a minimal submanifold. However since the set of those vectors is not invariant under the multiplication by $(-1)$ it is not austere.

(ii) By Corollary 6.2 the principal curvatures of $P(G, G(\sigma)) \ast \hat{w}$ in the direction of $\xi$ are the inner product of $\xi$ and vectors

$$\{0\}, \left\{ \frac{1}{-\pi + 2m\pi} e_i \right\}_{m \in \mathbb{Z}}, \left\{ \frac{1}{-\pi + 2m\pi} (e_i + e_j) \right\}_{m \in \mathbb{Z}},$$

$$\left\{ \frac{1}{-\frac{3}{2} \pi + 2m\pi} 2e_i \right\}_{m \in \mathbb{Z}}, \left\{ \frac{1}{-\frac{3}{2} \pi + 2m\pi} e_i \right\}_{m \in \mathbb{Z}}, \left\{ \frac{1}{-\frac{3}{2} \pi + 2m\pi} (e_i + e_j) \right\}_{m \in \mathbb{Z}},$$

$$\left\{ \frac{1}{-\frac{3}{2} \pi + 2m\pi} (e_i - e_j) \right\}_{m \in \mathbb{Z}}, \left\{ \frac{1}{-2n\pi} (e_i - e_j) \right\}_{n \in \mathbb{Z} \setminus \{0\}}.$$  

By the similar arguments as in Proposition 9.1 the sets $\left\{ \frac{1}{-\pi + 2m\pi} (e_i - e_j) \right\}_{m \in \mathbb{Z}}$, $\left\{ \frac{1}{-\pi/2 + 2m\pi} (e_i + e_j), \frac{1}{-3\pi/2 + 2m\pi} (e_i + e_j) \right\}_{m \in \mathbb{Z}}$ are austere. Moreover we have

$$\left\{ \frac{1}{-\frac{3}{2} \pi + 2m\pi} 2e_i \right\}_{m \in \mathbb{Z}} = \left\{ \frac{1}{-\frac{3}{4} \pi + 2m\pi} e_i \right\}_{m \in \mathbb{Z}} \bigcup \left\{ \frac{1}{-\frac{3}{4} \pi + 2m\pi} e_i \right\}_{m \in \mathbb{Z}}.$$
This together with the equalities
\[
\begin{align*}
\frac{1}{-\frac{\pi}{4} + 2m\pi} e_i &= (-1) \cdot \frac{1}{-\frac{\pi}{4} + 2(-m + 1)\pi} e_i, \\
\frac{1}{-\frac{5\pi}{4} + 2m\pi} e_i &= (-1) \cdot \frac{1}{-\frac{3\pi}{4} + 2(-m + 1)\pi} e_i,
\end{align*}
\]
shows that the set \( \{ -\frac{\pi}{4} + 2m\pi e_i, \frac{1}{-\frac{5\pi}{4} + 2m\pi} e_i, \frac{1}{-\frac{3\pi}{4} + 2m\pi} 2e_i \}_{m \in \mathbb{Z}} \) is austere. These show that the orbit \( P(G, G(\sigma)) \ast \dot{w} \) is an austere PF submanifold of \( V_g \).

The following proposition shows that the austere examples given in Proposition 9.1 are actually weakly reflective. Here (i) is based on a result by Ohno \[29, \text{Theorem 5}\] (see also Kimura-Mashimo \[19, \text{Proposition 5.2}\]).

**Proposition 9.4.** Let \( G, \sigma \) be as above. Set
\[
w := \frac{\pi}{2} \sum_{i=1}^{m} e_i.
\]
Then
\[\begin{align*}
(\text{i}) & \text{ the orbit } N = G(\sigma) \cdot \exp w \text{ is a weakly reflective submanifold of } G, \\
(\text{ii}) & \text{ the orbit } \Phi^{-1}(N) = P(G, G(\sigma)) \ast \dot{w} \text{ is a weakly reflective PF submanifold of } V_g.
\end{align*}\]

**Proof.** (i) It is easy to see that
\[
a = \exp w = \begin{bmatrix} J & \cdots & J \\ \vdots & \ddots & \vdots \\ J & \cdots & J \end{bmatrix} \quad \text{where } J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\]
Thus
\[
dl_a(t) = \begin{bmatrix} R_1 & \cdots & R_m \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \quad \text{where } R_i = \begin{bmatrix} r_i & 0 \\ 0 & r_i \end{bmatrix}.
\]
Define an isometry \( \nu : G \to G \) by
\[
\nu = (b, \sigma(b)) \quad \text{where } b := \begin{bmatrix} \sqrt{-1}L & \cdots & \sqrt{-1}L \\ \vdots & \ddots & \vdots \\ \sqrt{-1}L & \cdots & \sqrt{-1}L \\ 1 & \cdots & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]
Then \( \nu \in G(\sigma) \) and we have
\[\nu(a) = a, \quad \nu(G(\sigma) \cdot a) = G(\sigma) \cdot a \quad \text{and} \quad d\nu(dl_a \xi) = -dl_a \xi\]
for any \( \xi \in \mathfrak{t} \). By homogeneity it follows that \( G(\sigma) \cdot \exp w \) is a weakly reflective submanifold of \( G \).

(ii) The assertion follows by applying Theorem \[8.3\] to (i). To express an isometry with respect to normal vectors explicitly we give a direct proof here.
Let \( q \in G \) be
\[
q(t) = \begin{bmatrix}
Q(t) & \cdots & Q(t)
\end{bmatrix}
\]
where \( Q(t) = \sqrt{-1} \begin{bmatrix}
\sin \pi t & \cos \pi t \\
\cos \pi t & -\sin \pi t
\end{bmatrix} \)
and set \( \hat{\nu} := (q^*) \).

Clearly \( (q(0), q(1)) \in G(\sigma) \) and thus \( q \in P(G, G(\sigma)) \). Hence
\[
\nu(P(G, G(\sigma)) * \hat{w}) = P(G, G(\sigma)) * \hat{w}.
\]
Moreover it follows from matrix computations that
\[
q^* \hat{w} = \hat{w} \quad \text{and} \quad d(q^*) \hat{\xi} = -\hat{\xi},
\]
where \( d(q^*) = \text{Ad}(q) \). Thus by homogeneity it follows that \( P(G, G(\sigma)) * \hat{w} \) is a weakly reflective PF submanifold of \( V_g \).

10. Relations to affine Kac-Moody symmetric spaces

In this section we study the relation between the canonical isomorphisms of path spaces defined in Section 3 and isomorphisms of affine Kac-Moody symmetric spaces.

First we review foundations of affine Kac-Moody symmetric spaces following [10].

Let \( G \) be a simply connected compact simple Lie group with Lie algebra \( g \) and \( \sigma \) an automorphism of \( G \). The differential of \( \sigma \) is still denoted by \( \sigma \). Denote by \( \langle \cdot, \cdot \rangle \) the inner product of \( g \) which is the negative of the Killing form of \( g \).

The loop algebra \( L(g, \sigma) = \{ u : \mathbb{R} \to g \mid u \in C^\infty, u(t + 2\pi) = \sigma(u(t)) \text{ for all } t \} \) is a Lie algebra with pointwise bracket. We equip the inner product \( \langle u, v \rangle_{L^2} = \int_0^{2\pi} \langle u(t), v(t) \rangle dt \) with \( L(g, \sigma) \). Denote by \( \omega_\lambda \) the cocycle defined by \( \omega_\lambda(u, v) = \lambda \langle u', v \rangle_{L^2} \) for \( \lambda \in \mathbb{R} \setminus \{0\} \). An affine Kac-Moody algebra is a Lie algebra \( \tilde{L}(g, \sigma) := L(g, \sigma) + \mathbb{R}c + \mathbb{R}d \), where the bracket is defined by
\[
[u, v] = [u, v] + \omega_\lambda(u, v)c,
\]
\[
[d, u] = u',
\]
\[
[c, x] = 0,
\]
where \( u, v \in L(g, \sigma) \) and \( x \in \tilde{L}(g, \sigma) \). It has the center \( \mathbb{R}c \) and the derived algebra \( \tilde{L}(g, \sigma) := L(g, \sigma) + \mathbb{R}c \). If \( \sigma_1, \sigma_2 \in \text{Aut} g \) are conjugate by an inner automorphism then the corresponding affine Kac-Moody algebras are isomorphic. Thus we can assume that \( \sigma \) has finite order. We define the Lorentzian inner product on \( \tilde{L}(g, \sigma) \) by
\[
\langle u + \alpha c + \beta d, v + \gamma c + \delta d \rangle = \langle u, v \rangle_{L^2} + \alpha \delta + \beta \gamma.
\]
Clearly \( c, d \perp L(g, \sigma), \|c\| = \|d\| = 0 \) and \( \langle c, d \rangle = 1 \). It follows that \( \langle [x, y], z \rangle = \langle x, [y, z] \rangle \) for \( x, y, z \in L(g, \sigma) \).
The twisted loop group

\[ L(G, \sigma) = \{ g : \mathbb{R} \to G \mid g \in C^\infty, \quad g(t + 2\pi) = \sigma(g(t)) \text{ for all } t \} \]

with pointwise multiplication is a Fréchet Lie group with Lie algebra \( L(\mathfrak{g}, \sigma) \).
The cocycle \( \omega \) defines a left-invariant closed 2-form on \( L(G, \sigma) \) and moreover defines a central extension \( \tilde{L}(G, \sigma) \) of \( L(G, \sigma) \) by the circle \( S^1 \) for discrete values of \( \lambda \) (\[33\]). \( \tilde{L}(G, \sigma) \) has Lie algebra \( \tilde{L}(\mathfrak{g}, \sigma) \). There exists a unique \( \lambda_0 \) such that \( \tilde{L}(G, \sigma) \) is simply connected. An affine Kac-Moody group \( \hat{L}(G, \sigma) \) is a Fréchet Lie group defined by

\[ \hat{L}(G, \sigma) := S^1 \times \tilde{L}(G, \sigma). \]

Here the \( S^1 \)-action on \( \tilde{L}(G, \sigma) \) is induced by the action on \( L(G, \sigma) \) by shifting the parameter of loops. \( \hat{L}(G, \sigma) \) is a 2-torus bundle over \( L(G, \sigma) \) and has Lie algebra \( \hat{L}(\mathfrak{g}, \sigma) \). We equip the bi-invariant Lorentzian metric on \( \hat{L}(G, \sigma) \). Then \( \hat{L}(G, \sigma) \) is a symmetric space where a reflection at the identity is given by \( g \mapsto g^{-1} \).

For an involutive automorphism \( \hat{\rho} \) of \( \hat{G} = \hat{L}(G, \sigma) \) we consider the quotient \( \hat{G}/K \) by the fixed point subgroup \( K = \hat{G}^\rho \). The differential of \( \hat{\rho} \) is still denoted by \( \hat{\rho} \). The Lie algebra \( \hat{\mathfrak{g}} = \hat{L}(\mathfrak{g}, \sigma) \) is decomposed into the \((\pm 1)\)-eigenspaces \( \hat{\mathfrak{g}} = \mathfrak{k} + \mathfrak{m} \). Restricting the inner product on \( \hat{\mathfrak{g}} \) to \( \mathfrak{m} \) we equip the \( \hat{G} \)-invariant metric with \( \hat{G}/K \). Then \( \hat{G}/K \) is a symmetric space where a reflection at \( eK \) is given by \( \hat{g}K \mapsto \hat{\rho}(\hat{g})K \).

From the structure theory of involutions of affine Kac-Moody algebras (\[12\] \[10\] \[11\]) there are essentially two kinds of involutions, namely

\begin{enumerate}
  \item \( \hat{\rho} \) satisfies \( \hat{\rho}(c) = c, \quad \hat{\rho}(d) = d \) and \( \hat{\rho}(u)(t) = \rho(u(t)) \) where \( \rho \in \text{Aut} \mathfrak{g} \),
  \begin{align*}
    \rho^2 &= \text{id} \quad \text{and} \quad \sigma \rho = \rho \sigma , \\
    \rho^2 &= \text{id} \quad \text{and} \quad \sigma \rho = \rho \sigma^{-1}.
  \end{align*}
\end{enumerate}

We will always consider the latter one, called the involution of the second kind, so that the extension from \( L(G, \sigma) \) to \( \hat{L}(G, \sigma) \) is not canceled in the quotient.

By definition an affine Kac-Moody symmetric space is either an affine Kac-Moody group \( \hat{G} \) (the group type) or the symmetric space \( \hat{G}/K \) with respect to an involution \( \hat{\rho} \) of the second kind. Note that \( \hat{G} \) can be written as the quotient \( \hat{G} \times G/(G \times G)^\rho \) where \( G \times G = \hat{L}(G \times G, \sigma \times \sigma^{-1}) \) is a slight generalization of an affine Kac-Moody group and \( \hat{\rho} \) the involution of the second kind defined by

\[ \hat{\rho}(c) = -c, \quad \hat{\rho}(d) = -d, \quad \hat{\rho}(u)(t) = (v(-t), u(-t)). \quad (10.1) \]

It was shown that \( \hat{G} \) and \( \hat{G}/K \) are tame Fréchet manifolds, where an inverse function theorem is available (\[35\]). The unique existence theorem of the Levi-Civita connection and the conjugacy theorem of finite dimensional maximal flats are verified for affine Kac-Moody symmetric spaces (\[34\]). The concept of duality of symmetric spaces is extended to affine Kac-Moody symmetric spaces based on the theory of complex Kac-Moody groups (\[35\]). The classification of affine Kac-Moody symmetric spaces is essentially equivalent to the classification of involutions of affine Kac-Moody algebras up to conjugation (\[11\]).

Next we review their close relations to hyperpolar PF actions.

Let \( \pi : \hat{L}(G, \sigma) \to L(G, \sigma) \) denote the projection. For each \( \hat{g} \in \hat{L}(G, \sigma) \) we write \( g = \pi(\hat{g}) \). The adjoint action of \( \hat{L}(G, \sigma) = S^1 \times \hat{L}(G, \sigma) \) on \( \hat{g} = \hat{L}(\mathfrak{g}, \sigma) \) is
defined by \((35)\)
\[
\text{Ad}(\tilde{g})c = c,
\]
\[
\text{Ad}(\tilde{g})d = d - g'g^{-1} - \frac{1}{2}\|g'g^{-1}\|^2 c,
\]
\[
\text{Ad}(\tilde{g})u = gug^{-1} + \langle g'g^{-1}, gug^{-1} \rangle
\]
for \(\tilde{g} \in \tilde{L}(G, \sigma)\) and
\[
\text{Ad}(e^{is}) = c, \quad \text{Ad}(e^{is}) = d, \quad \text{Ad}(e^{is})u = u_s
\]
for \(e^{is} \in S^1\). Here \(u_s(t) := u(s + t)\). For the involution \(\hat{\rho}\) of the second kind the canonical decomposition \(\hat{g} = \hat{\mathfrak{f}} + \hat{\mathfrak{m}}\) is given by
\[
\hat{\mathfrak{f}} = \{u \in L(\mathfrak{g}, \sigma) \mid \rho(u(-t)) = u(t)\},
\]
\[
\hat{\mathfrak{m}} = \{u + \alpha c + \beta d \mid u \in L(\mathfrak{g}, \sigma), \rho(u(-t)) = -u(t), \alpha, \beta \in \mathbb{R}\}.
\]
The adjoint action of \(\hat{G}\) on \(\hat{\mathfrak{g}}\) induces the action of \(\hat{K}\) on \(\hat{\mathfrak{m}}\), which is called the isotropy representation of \(\hat{G}/\hat{K}\). In the group case we define the isotropy representation of \(\hat{G}\) to be the induced action of \(L(G, \sigma)\) on \(\hat{\mathfrak{g}}\).

Since the adjoint action preserves the inner product and the \(d\)-coefficient it leaves invariant the two-sheeted hyperboloid \(\{x \in \tilde{L}(\mathfrak{g}, \sigma) \mid \langle x, x \rangle = -1\}\), the hyperplane \(\{u + \alpha c + \beta d \mid u \in L(\mathfrak{g}, \sigma)\}\) and hence their intersection
\[
\text{Hor}(\hat{\mathfrak{g}}) = \left\{d + u - \frac{\|u\|^2 + 1}{2} c \mid u \in L(\mathfrak{g}, \sigma)\right\},
\]
which is geometrically interpreted as a horosphere of codimension 2. For \(x = d + u - \frac{\|u\|^2 + 1}{2} c\) we have
\[
(e^{is}, \tilde{g}) \cdot x = \left(d + g \ast u - \frac{\|g \ast u\|^2 + 1}{2} c\right)_s,
\]
where \(g \ast u = gug^{-1} - g'g^{-1}\) is the gauge transformation. Thus via the isometry
\[
\Gamma : L(\mathfrak{g}, \sigma) \to \text{Hor}(\hat{\mathfrak{g}}), \quad u \mapsto d + u - \frac{\|u\|^2 + 1}{2} c
\]
\(\tilde{L}(G, \sigma)\) acts on \(L(\mathfrak{g}, \sigma)\) by the gauge transformations.

Recall that two isometric actions of \(A_1\) on \(X_1\) and of \(A_2\) on \(X_2\) are called essentially equivalent \((35\ p.\ 167)\) if there exist an injective homomorphism \(\phi : A_1 \to A_2\) and an injective isometry \(\varphi : X_1 \to X_2\) which have dense images and satisfy \(\varphi(a \cdot p) = \phi(a) \cdot \varphi(p)\) for \(a \in A_1\) and \(p \in X_1\). For \(r > 0\) we set \(G^r = H^1([0, r], G), V^r_g = H^0([0, r], \mathfrak{g})\) and
\[
P(G, L)^r = \{g \in G^r \mid (g(0), g(r)) \in L\}
\]
for a closed subgroup \(L\) of \(G \times G\). Similarly we can define the \(P(G, L)^r\)-action on \(V^r_g\) by gauge transformations and the parallel transport map \(\Phi^r : V^r_g \to G\).

The following two propositions show the close relation between affine Kac-Moody symmetric spaces and hyperpolar PF actions \((10\ p.\ 148),\ (12\ Proposition\ 4.14)\). In connection with the formulation of our results we give their proofs here.
Proposition 10.1 (Terng [10]). Let \( \hat{G} = \hat{L}(G, \sigma) \) be an affine Kac-Moody symmetric space of group type. Then the isotropy representation restricted to \( \text{Hor}(\hat{g}) \) is essentially equivalent to the \( P(G, G(\sigma))^{2\pi} \)-action on \( V_{g}^{2\pi} \).

Proof. The completion of \( L(G, \sigma) \) with respect to the \( H^{1} \)-metric is
\[
\{ g : \mathbb{R} \to G \mid g \in H^{1}, \ g(t + 2\pi) = \sigma(g(t)) \text{ for all } t \} \\
\cong \{ g : [0, 2\pi] \to G \mid g \in H^{1}, \ g(2\pi) = \sigma(g(0)) \}.
\]
Moreover the completion of \( L(\mathfrak{g}, \sigma) \) with respect to the \( H^{0} \)-metric is
\[
\{ u : \mathbb{R} \to \mathfrak{g} \mid u \in H^{0}, \ u(t + 2\pi) = \sigma(u(t)) \text{ for all } t \} \\
\cong \{ u : [0, 2\pi] \to \mathfrak{g} \mid u \in H^{0} \}.
\]
This proves the proposition.

Proposition 10.2 (Heintze-Palais-Terng-Thorbergsson [12]). Let \( \hat{G}/\hat{K} \) be an affine Kac-Moody symmetric space. Then the isotropy representation restricted to \( \text{Hor}(\hat{g}) \cap \hat{m} \) is essentially equivalent to the \( P(G, G^{\sigma \rho})^{\pi} \)-action on \( V_{g}^{\pi} \). Here the inner product of \( V_{g}^{\pi} \) is defined by \( \langle u, v \rangle := 2 \int_{0}^{\pi} \langle u(t), v(t) \rangle dt \).

Proof. The completion of \( \hat{K} \) with respect to the \( H^{1} \)-metric is
\[
\{ g : \mathbb{R} \to G \mid g \in H^{1}, \ g(t + 2\pi) = \sigma(g(t)), \ \rho(g(-t)) = g(t) \} \\
\cong \{ g : [0, 2\pi] \to G \mid g \in H^{1}, \ g(2\pi) = \sigma(g(0)), \ \rho(\sigma^{-1}g(2\pi - t)) = g(t) \} \\
\cong \{ g : [0, \pi] \to G \mid g \in H^{1}, \ \rho(\sigma^{-1}g(0)) = \sigma(g(0)), \ \rho(\sigma^{-1}g(\pi)) = g(\pi) \} \\
= \{ g : [0, \pi] \to G \mid g \in H^{1}, \ \sigma^{-1}\rho\sigma^{-1}g(0) = g(0), \ \rho\sigma^{-1}g(\pi) = g(\pi) \} \\
= \{ g : [0, \pi] \to G \mid g \in H^{1}, \ \rho g(0) = g(0), \ \sigma\rho(g(\pi)) = g(\pi) \}.
\]
The completion of \( \Gamma^{-1}(\hat{m}) \) with respect to the \( H^{0} \)-metric is
\[
\{ u : \mathbb{R} \to \mathfrak{g} \mid u \in H^{0}, \ u(t + 2\pi) = \sigma(u(t)), \ \rho(u(-t)) = -u(t) \} \\
\cong \{ u : [0, 2\pi] \to \mathfrak{g} \mid u \in H^{0}, \ \rho(\sigma^{-1}u(2\pi - t)) = -u(t) \} \\
\cong \{ u : [0, \pi] \to \mathfrak{g} \mid u \in H^{0} \}.
\]
This proves the proposition.

Finally we focus on the case of group type and show our results.

Proposition 10.3. Let \( G \times G/(\overline{G} \times G) \hat{\rho} \) be the affine Kac-Moody symmetric space isomorphic to \( \hat{G} \). Then the isotropy representation restricted to the horosphere is essentially equivalent to the \( P(G \times G, G(\sigma) \times \Delta G)^{\pi} \)-action on \( V_{g\hat{g}}^{\pi} \).

Proof. Recall that the involution \( \hat{\rho} \) was defined by (10.1). We need another involution \( \hat{\tau} \) of the second kind defined by
\[
\hat{\tau}(c) = -c, \quad \hat{\tau}(d) = -d, \quad \hat{\tau}(u, v)(t) = (\sigma^{-1}v(-t), \sigma u(-t)). \tag{10.2}
\]
Note that \( \hat{\rho} \) and \( \hat{\tau} \) are conjugate and thus the corresponding quotients are isomorphic. Then by the similar argument as in Proposition 10.2 it follows that the isotropy representation of \( G \times G/(\overline{G} \times G) \hat{\tau} \) restricted to the horosphere is essentially equivalent to the \( P(G \times G, G(\sigma) \times \Delta G)^{\pi} \)-action on \( V_{g\hat{g}}^{\pi} \).
By the same way as in Section 3, we define the canonical isomorphisms $\Omega : G^{2\pi} \to \hat{G}^\pi$ and $\Upsilon : V_g^{2\pi} \to V_{\hat{g}\oplus g}^{\pi}$ by
\[
\Omega(g) = (g(t), g(2\pi - t)), \quad \Upsilon(u) = (u(t), -u(2\pi - t)).
\]

**Corollary 10.4.** Let $\hat{G} = \hat{L}(G, \sigma)$ be an affine Kac-Moody symmetric space of group type and $G \times G / (G \times G)\hat{\tau}$ the quotient isomorphic to $\hat{G}$. Then their isotropy representations restricted to the horospheres are essentially equivalent to the $P(G, G(\sigma))^{2\pi}$-action on $V_g^{2\pi}$ and $P(G \times G, G(\sigma) \times \Delta G)^\pi$-action on $V_{\hat{g}\oplus g}^{\pi}$, respectively and these are conjugate via the canonical isomorphisms $\Omega$ and $\Upsilon$.

**Proof.** The first half of the assertion follow from Propositions 10.1 and 10.3. The second half follows from Corollary 3.3. \qed

Corollary 10.4 implies that there is a correspondence between
(i) the isomorphism between $\hat{G}$ and $G \times G / (G \times G)\hat{\tau}$,
(ii) the conjugacy between hyperpolar PF actions of $P(G, G(\sigma))^{2\pi}$ on $V_g^{2\pi}$ and of $P(G \times G, G(\sigma) \times \Delta G)^\pi$ on $V_{\hat{g}\oplus g}^{\pi}$ via $\Omega$ and $\Upsilon$.

Moreover Theorem 3.2 (iii) implies a correspondence between (ii) and (iii) the conjugacy between hyperpolar actions of $G(\sigma)$ on $G$ and of $G(\sigma)$ on $(G \times G) / \Delta G$ via $\rho$.

The goal of this section is to show these correspondences explicitly.

We denote by
\[
\Lambda : \hat{G} \times G \to \hat{G}
\]
the surjective smooth map corresponding to
\[
d\Lambda : \hat{g} \oplus g \to \hat{g}, \quad (u(t), v(t)) + \alpha c + \beta d \mapsto (u(t) - \sigma^{-1}v(-t)) + \alpha c + \beta d.
\]

Then the inverse image $\Lambda^{-1}(\hat{\epsilon})$ of the identity $\hat{\epsilon}$ is $(\hat{G} \times G)\hat{\tau}$. Thus it induces the isomorphism
\[
\Lambda : \hat{G} \times G / (\hat{G} \times G)\hat{\tau} \to \hat{G}.
\]

This is the isomorphism indicated in (i).

The canonical decomposition $\hat{g} \oplus g = \hat{\epsilon} \oplus \hat{m}$ with respect to $\hat{\tau}$ is given by
\[
\hat{\epsilon} = \{(u(t), \sigma(u(-t))) \mid u \in L(g, \sigma)\},
\]
\[
\hat{m} = \{(u(t), -\sigma u(-t)) + \alpha c + \beta d \mid u \in L(g, \sigma), \alpha, \beta \in \mathbb{R}\}.
\]

There are natural isomorphisms
\[
\alpha : L(g, \sigma) \to \hat{\epsilon}, \quad u(t) \mapsto (u(t), \sigma(u(-t)))
\]
and
\[
\beta : \hat{g} \to \hat{m}, \quad u(t) + \alpha c + \beta d \mapsto (u(t), -\sigma(u(-t))) + \alpha c + \beta d,
\]
which satisfy $d\Lambda \circ \alpha = 0$ and $d\Lambda \circ \beta = 2\text{id}$. Then $\alpha$ gives rise to the isomorphism between the isotropy subgroups
\[
\alpha : L(G, \sigma) \to (\hat{G} \times G)\hat{\tau}, \quad g(t) \mapsto (g(t), \sigma(g(-t))).
\]

If we define the inner product of $\hat{g} \oplus g$ by
\[
((u_1, u_2) + \alpha c + \beta d, (v_1, v_2) + \gamma c + \delta d) = \frac{1}{2}((u_1, u_2)_{L^2} + (v_1, v_2)_{L^2}) + \alpha \delta + \beta \gamma
\]
then $\beta$ induces the isometry
$$\beta : \text{Hor}(\hat{g}) \to \text{Hor}(\hat{g} \oplus \hat{g}) \cap \hat{m},$$
and we have
$$\beta : L(\mathfrak{g}, \sigma) \to \Gamma^{-1}(\hat{m}), \quad u(t) \mapsto (u(t), -\sigma(u(-t))).$$
Since $\sigma(g(-t)) = g(2\pi - t)$ and $\sigma(u(-t)) = u(2\pi - t)$ the diagrams
$$L(G, \sigma) \xrightarrow{\alpha} \hat{G} \times \hat{G} \xrightarrow{\Omega} P(G, G(\sigma))^{2\pi} \xrightarrow{\rho^G \times G(\psi_{G^G})^\pi} P(G \times G, G(\sigma) \times \Delta G)^\pi$$
and
$$L(\mathfrak{g}, \sigma) \xrightarrow{\beta} \Gamma^{-1}(\hat{m}) \xrightarrow{\tau} V_{\phi^{\mathfrak{g}}}^{\pi} \xrightarrow{\Phi^{2\pi}} V_{\phi^{\mathfrak{g}}}^{\pi}$$
are commutative, where the vertical arrows denote the injective maps with dense images given in Propositions 10.1 and 10.2. This shows the correspondence of (i) and (ii) explicitly.

In Theorem 3.2 we essentially showed that the diagrams
$$P(G, G(\sigma))^{2\pi} \xrightarrow{G(\sigma)} P(G \times G, G(\sigma) \times \Delta G)^\pi \xrightarrow{\rho^G \times G(\psi_{G^G})^\pi} P(G \times G, G(\sigma) \times \Delta G)^\pi \xrightarrow{\rho^G \times G(\psi_{G^G})^\pi} P(G \times G, G(\sigma) \times \Delta G)^\pi$$
and
$$V_{\phi^{\mathfrak{g}}}^{\pi} \xrightarrow{\Phi^{2\pi}} (G \times G)/\Delta G,$$
are commutative. This shows the correspondence between (ii) and (iii) explicitly.

These discussions are summarized as follows:

**Theorem 10.5.** Let $\hat{G} = \hat{L}(G, \sigma)$ be an affine Kac-Moody symmetric space of group type. Then there is a natural correspondence between:

(i) the isomorphism $\Lambda$ between $\hat{G}$ and $\hat{G} \times \hat{G}/(\hat{G} \times \hat{G})^\pi$,

(ii) the conjugacy between hyperpolar PF actions of $P(G, G(\sigma))^{2\pi}$ on $V_{\phi^{\mathfrak{g}}}^{\pi}$ and of $P(G \times G, G(\sigma) \times \Delta G)^\pi$ on $V_{\phi^{\mathfrak{g}}}^{\pi}$ via $\Omega$ and $\tau$ (Corollary 3.3),

(iii) the conjugacy between the actions of $G(\sigma)$ on $G$ and of $G(\sigma)$ on $(G \times G)/\Delta G$ via $\rho$.

Here the correspondence of (i) and (ii) (resp. (ii) and (iii)) means that the diagrams (10.3) and (10.4) (resp. (10.5) and (10.6)) are commutative.
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References

[1] J. Berndt, L. Vanhecke, Curvature-adapted submanifolds, Nihonkai Math. J. 3 (1992), no. 2, 177–185.
[2] U. Christ, Homogeneity of equifocal submanifolds, J. Differential Geom. 62 (2002), no. 1, 1–15.
[3] L. Conlon, The topology of certain spaces of paths on a compact symmetric space, Trans. Amer. Math. Soc. 112 (1964), 228–248.
[4] J. Dadok, Polar coordinates induced by actions of compact Lie groups, Trans. Amer. Math. Soc. 288 (1985), no. 1, 125–137.
[5] J.-H. Eschenburg, E. Heintze, Polar representations and symmetric spaces, J. Reine Angew. Math. 507 (1999), 93–106.
[6] W. Freyn, Affine Kac-Moody symmetric spaces, arXiv:1109.2837 (2011).
[7] O. Goertsches, G. Thorbergsson, On the geometry of the orbits of Hermann actions, Geom. Dedicata 129 (2007), 101–118.
[8] R. S. Hamilton, The inverse function theorem of Nash and Moser, Bull. Amer. Math. Soc. (N.S.) 7 (1982), no. 1, 65–222.
[9] R. Harvey and H. B. Lawson, Jr., Calibrated geometries, Acta Math., 148 (1982), 47–157.
[10] E. Heintze, Toward symmetric spaces of affine Kac-Moody type, Int. J. Geom. Methods Mod. Phys. 3 (2006), no. 5–6, 881–898.
[11] E. Heintze, C. Groß, Finite order automorphisms and real forms of affine Kac-Moody algebras in the smooth and algebraic category, Mem. Amer. Math. Soc. 219 (2012), no. 1030, viii+66 pp.
[12] E. Heintze, R. Palais, C.-L. Terng, G. Thorbergsson, Hyperpolar actions on symmetric spaces, Geometry, topology, & physics, 214–245, Conf. Proc. Lecture Notes Geom. Topology, IV, Int. Press, Cambridge, MA, 1995.
[13] R. Hermann, Variational completeness for compact symmetric spaces. Proc. Amer. Math. Soc. 11 (1960), 544–546.
[14] R. Hermann, Totally geodesic orbits of groups of isometries, Nederl. Akad. Wetensch. Proc. Ser. A 65 = Indag. Math. 24 1962 291–298.
[15] D. Hirohashi, H. Tasaki, H. Song, R. Takagi, Minimal orbits of the isotropy groups of symmetric spaces of compact type, Differential Geom. Appl. 13 (2000), no. 2, 167–177.
[16] O. Ikawa, The geometry of symmetric triad and orbit spaces of Hermann actions, J. Math. Soc. Japan 63 (2011), no. 1, 79–136.
[17] O. Ikawa, σ-actions and symmetric triads, Tohoku Math. J. (2) 70 (2018), no. 4, 547–565.
[18] O. Ikawa, T. Sakai, H. Tasaki, Weakly reflective submanifolds and austere submanifolds. J. Math. Soc. Japan 61 (2009), no. 2, 437–481.
[19] T Kimura, K. Mashimo, Classification of Cartan embeddings which are austere submanifolds, Hokuakdo Math. J. 51 (2022), no. 1, 1–23.
[20] C. King, C.-L. Terng, Minimal submanifolds in path space, Global analysis in modern mathematics, 253–281, Publish or Perish, Houston, TX, 1993.
[21] N. Koike, On proper Fredholm submanifolds in a Hilbert space arising from submanifolds in a symmetric space, Japan. J. Math. (N.S.) 28 (2002), no. 1, 61–80.
[22] A. Kollross, A classification of hyperpolar and cocompactness one actions, Trans. Amer. Math. Soc. 354 (2002), no. 2, 571–612.
[23] A. Kollross, Hyperpolar actions on reducible symmetric spaces, Transform. Groups 22 (2017), no. 1, 207–228.
[24] O. Loos, Symmetric Spaces, II: Compact Spaces and Classification. W. A. Benjamin, Inc., New York-Amsterdam 1969 viii+183 pp.
[25] M. Morimoto, Austere and arid properties for PF submanifolds in Hilbert spaces, Differential Geom. Appl., 69 (2020) 101613, 24 pp.
[26] M. Morimoto, On weakly reflective PF submanifolds in Hilbert spaces, Tokyo J. Math. 44 (2021), no. 1, pp. 103–124.
[27] M. Morimoto, On weakly reflective submanifolds in compact isotropy irreducible Riemannian homogeneous spaces, Tokyo J. Math. 44 (2021), no. 2, 467–476.
[28] M. Morimoto, Curvatures and austere property of orbits of path group actions induced by Hermann actions, Transform. Groups, to appear.
[29] S. Ohno, A sufficient condition for orbits of Hermann actions to be weakly reflective, Tokyo J. Math. 39 (2016), no. 2, 537-564.
[30] S. Ohno, Geometric properties of orbits of Hermann actions, [arXiv:2101.00765]
[31] R. S. Palais, Morse theory on Hilbert manifolds, Topology 2 (1963), 299–340.
[32] R. S. Palais, C.-L. Terng, Critical Point Theory and Submanifold Geometry, Lecture Notes in Mathematics, 1353. Springer-Verlag, Berlin, 1988.
[33] U. Pinkall, G. Thorbergsson, Examples of infinite dimensional isoparametric submanifolds, Math. Z. 205 (1990), no. 2, 279–286.
[34] B. Popescu, Infinite dimensional symmetric spaces, Thesis, University of Augsburg (2005).
[35] A. Pressley, G. Segal, Loop Groups, Oxford Mathematical Monographs. Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1986.
[36] S. Smale, Morse theory and a non-linear generalization of the Dirichlet problem, Ann. of Math. (2) 80 (1964), 382–396.
[37] Y. Taketomi, On a Riemannian submanifold whose slice representation has no nonzero fixed points, Hiroshima Math. J. 48 (2018), no. 1, 1–20.
[38] C.-L. Terng, Isoparametric submanifolds and their Coxeter groups, J. Differential Geom. 21 (1985), no. 1, 79–107.
[39] C.-L. Terng, Proper Fredholm submanifolds of Hilbert space. J. Differential Geom. 29 (1989), no. 1, 9–47.
[40] C.-L. Terng, Polar actions on Hilbert space. J. Geom. Anal. 5 (1995), no. 1, 129–150.
[41] C.-L. Terng, G. Thorbergsson, Submanifold geometry in symmetric spaces. J. Differential Geom. 42 (1995), no. 3, 665–718.
[42] G. Thorbergsson, Isoparametric foliations and their buildings, Ann. of Math. (2) 133 (1991), no. 2, 429–446.

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