BIMINIMAL PROPERLY IMMERSED SUBMANIFOLDS IN COMPLETE RIEMANNIAN MANIFOLDS OF NON-POSITIVE CURVATURE

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Abstract. We consider a non-negative biminimal properly immersed submanifold $M$ (that is, a biminimal properly immersed submanifold with $\lambda \geq 0$) in a complete Riemannian manifold $N$ with non-positive sectional curvature. Assume that the sectional curvature $K^N$ of $N$ satisfies $K^N \geq -L(1 + \text{dist}_N(\cdot, q_0)^2)^\alpha$ for some $L > 0$, $2 > \alpha \geq 0$ and $q_0 \in N$. Then, we prove that $M$ is minimal. As a corollary, we give that any biharmonic properly immersed submanifold in a hyperbolic space is minimal. These results give affirmative partial answers to the global version of generalized Chen’s conjecture.

1. Introduction

Theory of harmonic maps has been applied into various fields in differential geometry. Harmonic maps between two Riemannian manifolds are critical points of the energy functional $E(\phi) = \frac{1}{2} \int_M |d\phi|^2 dv_g$, for smooth maps $\phi: (M^m, g) \to (N^n, h)$ from an $m$-dimensional Riemannian manifold into an $n$-dimensional Riemannian manifold, where $dv_g$ denotes the volume element of $g$.

On the other hand, in 1983, J. Eells and L. Lemaire [16] proposed the problem to consider polyharmonic maps of order $k$. In 1986, G.Y. Jiang [19] studied biharmonic maps (that is, polyharmonic maps of order 2) which are critical points of the bi-energy

$$E_2(\phi) := \int_M |	au(\phi)|^2 dv_g,$$

where $\tau(\phi)$ denotes the tension field of $\phi$. The Euler-Lagrange equation of $E_2$ is

$$\tau_2(\phi) := -\Delta^\phi \tau(\phi) - \sum_{i=1}^m R^N(\tau(\phi), d\phi(e_i))d\phi(e_i) = 0,$$

where $\Delta^\phi := \sum_{i=1}^m (\nabla_e \nabla^\phi e_i - \nabla_{\nabla^\phi e_i} e_i)$, $R^N$ is the Riemannian curvature of $N$ i.e., $R^N(X, Y)Z := [\nabla_X, \nabla^Y]Z - \nabla^N_{[X,Y]}Z$ for any vector field $X$, $Y$ and $Z$ on $N$, and $\{e_i\}_{i=1}^m$ is a local orthonormal frame field on $M$.
If an isometric immersion $\phi : (M, g) \to (N, h)$ is biharmonic, then $M$ is called a \textit{biharmonic submanifold} in $N$. In this case, we remark that the tension field $\tau(\phi)$ of $\phi$ is written as $\tau(\phi) = mH$, where $H$ is the mean curvature vector field of $M$.

For biharmonic submanifolds, there is an interesting problem, namely Chen’s conjecture (cf. \cite{9}).

\textbf{Conjecture 1.} Any biharmonic submanifold in $\mathbb{E}^n$ is minimal.

There are many affirmative partial answers to Conjecture 1 (cf. \cite{9}, \cite{10}, \cite{14}, \cite{15}, \cite{17}). Conjecture 1 is solved completely if $M$ is one of the following: (a) a curve \cite{15}, (b) a surface in $\mathbb{E}^3$ \cite{9}, (c) a hypersurface in $\mathbb{E}^4$ \cite{14}, \cite{17}.

Note that, since there is no assumption of \textit{completeness} for submanifolds in Conjecture 1, in a sense it is a problem in \textit{local} differential geometry. Recently, Conjecture 1 was reformulated into a problem in \textit{global} differential geometry as follows (cf. \cite{2}, \cite{23}, \cite{25}, \cite{26}):

\textbf{Conjecture 2.} Any complete biharmonic submanifold in $\mathbb{E}^n$ is minimal.

On the other hand, Conjecture 1 was generalized as follows: \textit{Any biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal} (cf. \cite{3}, \cite{4}, \cite{5}, \cite{6}, \cite{7}). This generalization is also a problem in local differential geometry. Y.-L. Ou and L. Tang \cite{27} gave a counterexample of this conjecture (see \cite{19} for an affirmative answer). With these understandings, it is natural to consider the following conjecture.

\textbf{Conjecture 3.} Any complete biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal.

N. Nakauchi and H. Urakawa gave an affirmative partial answer to Conjecture 3 (cf. \cite{25}, \cite{26}).

An immersed submanifold $M$ in a Riemannian manifold $N$ is said to be \textit{properly immersed} if the immersion is a proper map. K. Akutagawa and the author gave an affirmative partial answer to Conjecture 1 (Conjecture 2 particularly) as follows (cf. \cite{2}, \cite{23}):

\textbf{Theorem 1.1} (\cite{2}). Any biharmonic properly immersed submanifold in $\mathbb{E}^n$ is minimal.

For Conjecture 3, we consider a biharmonic properly immersed submanifold in a complete Riemannian manifold with non-positive sectional curvature.

Recently, E. Loubeau and S. Montaldo \cite{20} introduced a \textit{biminimal immersion} as follows:

\textbf{Definition 1.2} (\cite{20}). An immersion $\phi : (M^m, g) \to (N^n, h)$, $m \leq n$ is called \textit{biminimal} if it is a critical point of the functional

$$E_{2, \lambda}(\phi) = E_2(\phi) + \lambda E(\phi), \quad \lambda \in \mathbb{R},$$

for any smooth variation of the map $\phi_t (-\varepsilon < t < \varepsilon)$, $\phi_0 = \phi$ such that $V = \left. \frac{d\phi}{dt} \right|_{t=0}$ is normal to $\phi(M)$.

The Euler-Lagrange equation of $E_{2, \lambda}$ is

$$[\tau_2(\phi)]^\perp + \lambda[\tau(\phi)]^\perp = 0,$$
where $[\cdot]^\perp$ denotes the normal component of $[\cdot]$. We call an immersion free biminimal if it is biminimal condition for $\lambda = 0$. If $\phi : (M, g) \to (N, h)$ is an isometric immersion, then the biminimal condition is

$$
-\Delta \phi H - \sum_{i=1}^{m} R^{N}(H, d\phi(e_i))d\phi(e_i) + \lambda H = 0,
$$

for some $\lambda \in \mathbb{R}$, and then $M$ is called a biminimal submanifold in $N$. If $M$ is a biminimal submanifold with $\lambda \geq 0$ in $N$, then $M$ is called a non-negative biminimal submanifold in $N$.

**Remark 1.3.** We remark that every biharmonic submanifold is free biminimal one.

Before mentioning our main theorem, we define the following notion.

**Definition 1.4.** For a complete Riemannian manifold $(N, h)$ and $\alpha \geq 0$, if the sectional curvature $K^N$ of $N$ satisfies

$$K^N \geq -L(1 + \text{dist}_N(\cdot, q_0)^2)^{\alpha/2},$$

for some $L > 0$ and $q_0 \in N$.

Then we shall call that $K^N$ has a polynomial growth bound of order $\alpha$ from below.

In this article, our main theorem is the following.

**Theorem 1.5.** Let $(N, h)$ be a complete Riemannian manifold with non-positive sectional curvature. Assume that the sectional curvature $K^N$ has a polynomial growth bound of order less than $2$ from below. Then, any non-negative biminimal properly immersed submanifold in $N$ is minimal.

Since every biharmonic submanifold is free biminimal one, we obtain the following result.

**Corollary 1.6.** Let $(N, h)$ be a complete Riemannian manifold with non-positive sectional curvature. Assume that the sectional curvature $K^N$ has a polynomial growth bound of order less than $2$ from below. Then, any biharmonic properly immersed submanifold in $N$ is minimal.

This result gives an affirmative partial answer to Conjecture 3.

**Remark 1.7.** If $N$ is a complete Riemannian manifold whose non-positive sectional curvature is bounded from below (including a hyperbolic space), then $N$ satisfies the assumption in Corollary 1.6.

For the case of $\lambda < 0$, the author constructed non-minimal biminimal submanifolds in $\mathbb{E}^n$ (cf. [23]).

The remaining sections are organized as follows. Section 2 contains some necessary definitions and preliminary geometric results. In section 3, we prove our main theorem. In section 4, we show that any complete biharmonic submanifold with Ricci curvature bounded from below in a Riemannian manifold with non-positive sectional curvature is minimal. In section 5, we consider a non-negative biminimal hypersurface in a Riemannian manifold.
2. Preliminaries

Let \( \phi : (M^m, g) \to (N^n, h = \langle \cdot, \cdot \rangle) \) be an isometric immersion from an \( m \)-dimensional Riemannian manifold into an \( n \)-dimensional Riemannian manifold. In this case, we identify \( d\phi(X) \) with \( X \in \mathfrak{X}(M) \) for each \( x \in M \). We also denote by \( \langle \cdot, \cdot \rangle \) the induced metric \( \phi^{-1}h \).

Then, the Gauss formula is given by

\[
\nabla^N_X Y = \nabla_X Y + B(X, Y), \quad X, Y \in \mathfrak{X}(M),
\]

where \( \nabla^N \) and \( \nabla \) are the Levi-Civita connections on \( N \) and \( M \) respectively, and \( B \) is the second fundamental form of \( M \) in \( N \). The Weingarten formula is given by

\[
\nabla^N_X \xi = -A_\xi X + \nabla_\xi^\perp X, \quad X \in \mathfrak{X}(M), \quad \xi \in \mathfrak{X}(M) \perp,
\]

where \( A_\xi \) is the shape operator for a normal vector field \( \xi \) on \( M \), and \( \nabla^\perp \) denotes the normal connection of the normal bundle on \( M \) in \( N \). It is well known that \( B \) and \( A \) are related by

\[
\langle B(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.
\]

For any \( x \in M \), let \( \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_n\} \) be an orthonormal basis of \( N \) at \( x \) such that \( \{e_1, \ldots, e_m\} \) is an orthonormal basis of \( T_xM \). Then, \( B \) is decomposed as

\[
B(X, Y) = \sum_{\alpha=m+1}^{n} B_\alpha(X, Y)e_\alpha, \quad \text{at } x.
\]

The mean curvature vector field \( H \) of \( M \) at \( x \) is also given by

\[
H(x) = \frac{1}{m} \sum_{i=1}^{m} B(e_i, e_i) = \sum_{\alpha=m+1}^{n} H_\alpha(x)e_\alpha, \quad H_\alpha(x) := \frac{1}{m} \sum_{i=1}^{m} B_\alpha(e_i, e_i).
\]

The necessary and sufficient condition for \( M \) in \( N \) to be biharmonic is the following (cf. [24]):

\[
\Delta^\perp H - \sum_{i=1}^{m} B(A_H e_i, e_i) + \left[ \sum_{i=1}^{m} R^N(H, d\phi(e_i)) d\phi(e_i) \right]^\perp = 0,
\]

\[
m \nabla[H]^2 + 4 \text{ trace } A_{\nabla^\perp H} + \left[ \sum_{i=1}^{m} R^N(H, d\phi(e_i)) d\phi(e_i) \right]^T = 0,
\]

where \( \Delta^\perp \) is the (non-positive) Laplace operator associated with the normal connection \( \nabla^\perp \). Similarly, the necessary and sufficient condition for \( M \) in \( N \) to be biminimal is the following:

\[
\Delta^\perp H - \sum_{i=1}^{m} B(A_H e_i, e_i) + \left[ \sum_{i=1}^{m} R^N(H, d\phi(e_i)) d\phi(e_i) \right] = \lambda H.
\]
3. Proof of main theorem

In this section, we prove Theorem [1.5]. We shall show the following lemma.

Lemma 3.1. Let \((M, g)\) be a properly immersed submanifold in a complete Riemannian manifold \((N, h)\) whose sectional curvature \(K^N\) has a polynomial growth bound of order less than 2 from below. Assume that there exists a positive constant \(k > 0\) such that

\[
\Delta|H|^2 \geq k|H|^4 \quad \text{on } M.
\]

Then \(M\) is minimal.

Proof. If \(M\) is compact, applying the standard maximum principle to the elliptic inequality (9), we have that \(H = 0\) on \(M\). Therefore, we may assume that \(M\) is noncompact. Suppose that \(H(x_0) \neq 0\) at some point \(x_0 \in M\). Then, we will lead a contradiction.

Set

\[
u(x) := |H(x)|^2 \quad \text{for } x \in M.
\]

For each \(\rho > 0\), consider the function

\[
f(x) = f_\rho(x) := (\rho^2 - r(\phi(x))^2)^2 u(x) \quad \text{for } x \in M \cap \phi^{-1}(B_\rho),
\]

where \(r(\phi(x)) = \text{dist}_N(\phi(x), q_0)\) for some \(q_0 \in N\) and \(B_\rho := \{\rho \in N | r(q) \leq \rho\}\) denote respectively the geodesic distance from \(q_0\) and the closed geodesic ball of radius \(\rho\) centered at \(q_0\). Then, there exists \(\rho_0 > 0\) such that \(x_0 \in M \cap \phi^{-1}(B_{\rho_0})\).

For each \(\rho \geq \rho_0\), \(f\) is a non-negative function which is not identically zero on \(M \cap \phi^{-1}(B_\rho)\). Take any \(\rho \geq \rho_0\) and fix it. Since \(M\) is properly immersed in \(N\), \(M \cap \phi^{-1}(B_\rho)\) is compact. By this fact combined with \(f = 0\) on \(M \cap \phi^{-1}(\phi B_\rho)\), there exists a maximum point \(p \in M \cap \phi^{-1}(B_\rho)\) of \(f\) such that \(f(p) > 0\).

We consider the case that \(\phi(p)\) is not on the cut locus of \(q_0\). We have \(\nabla f = 0\) at \(p\), and hence

\[
\frac{\nabla u(p)}{u(p)} = \frac{2 \nabla r(\phi(p))^2}{\rho^2 - r(\phi(p))^2}.
\]

We also have that \(\Delta f \leq 0\) at \(p\). Combining this with (10), we obtain

\[
\frac{\Delta u(p)}{u(p)} \leq \frac{6\|\nabla r(\phi(p))^2\|^2}{(\rho^2 - r(\phi(p))^2)^2} + \frac{2\Delta r(\phi(p))^2}{\rho^2 - r(\phi(p))^2}.
\]

By a direct computation, we have

\[
\Delta r(\phi(p))^2 = 2 \sum_{i=1}^m (\nabla r(\phi(p)), d\phi(e_i))^2
\]

\[
+ 2r(\phi(p)) \sum_{i=1}^m D^2 r(\phi(p))(d\phi(e_i), d\phi(e_i))
\]

\[
+ 2r(\phi(p)) (\nabla r(\phi(p)), \tau(\phi(p)))
\]

\[
\leq 2m + 2r(\phi(p)) \sum_{i=1}^m D^2 r(\phi(p))(d\phi(e_i), d\phi(e_i))
\]

\[
+ 2mr(\phi(p))|H(p)|^2.
\]
By an elementary argument, we only have to consider the following: there exists a \( c \) in the manifold \( \mathcal{U} \) of \( \rho \).

Letting \( \phi \), at most one minimizing geodesic joining \( q \) because of the assumption (14),

\[
\sum_{i=1}^{m} D^2 r(\phi(p))(d\phi(e_i), d\phi(e_i)) \leq m \sqrt{L(1 + \rho^2)\frac{2}{r(p)}} \coth \left( \sqrt{L(1 + \rho^2)\frac{2}{r(p)}} r(\phi(p)) \right).
\]

Combining (13) and (14), we have

\[
\Delta r(\phi(p))^2 \leq 2m + 2m \sqrt{L(1 + \rho^2)\frac{2}{r(p)}} r(\phi(p)) \coth \left( \sqrt{L(1 + \rho^2)\frac{2}{r(p)}} r(\phi(p)) \right) + 2mr(\phi(p))|H(p)|.
\]

It follows from (9), (11), (12) and (15) that

\[
k u(p) \leq \frac{24mr(\phi(p))^2}{(\rho^2 - r(\phi(p))^2)^2}
+ \frac{4m + 4m \sqrt{L(1 + \rho^2)\frac{2}{r(p)}} r(\phi(p)) \coth \left( \sqrt{L(1 + \rho^2)\frac{2}{r(p)}} r(\phi(p)) \right) + 4mr(\phi(p))|H(p)|}{\rho^2 - r(\phi(p))^2},
\]

and hence

\[
k f(p) \leq 24mr(\phi(p))^2 + 4m(\rho^2 - r(\phi(p))^2)
+ 4m \sqrt{L(1 + \rho^2)\frac{2}{r(p)}} r(\phi(p)) \coth \left( \sqrt{L(1 + \rho^2)\frac{2}{r(p)}} r(\phi(p)) \right) (\rho^2 - r(\phi(p))^2)
+ 4mr(\phi(p)) \sqrt{f(p)}.
\]

By an elementary argument, we only have to consider the following: there exists a positive constant \( c \) depending only on \( k, L \) and \( m \) such that

\[
f(p) \leq c(1 + \rho^2)^{\frac{4m + 4}{2}}.
\]

Since \( f(p) \) is the maximum of \( f \), we have

\[
f(x) \leq f(p) \leq c(1 + \rho^2)^{\frac{4m + 4}{2}} \quad \text{for} \quad x \in M \cap \phi^{-1}(\mathcal{B}_\rho),
\]

and hence

\[
|H(x)|^2 = u(x) \leq \frac{c(1 + \rho^2)^{\frac{4m + 4}{2}}}{(\rho^2 - r(\phi(x))^2)^2} \quad \text{for} \quad x \in M \cap \phi^{-1}(\mathcal{B}_\rho), \quad \text{and} \quad \rho \geq \rho_0.
\]

Letting \( \rho \to \infty \) in (16) for \( x = x_0 \), we have that

\[
|H(x_0)|^2 = 0,
\]

because of the assumption \( \alpha < 2 \). This contradicts our assumption that \( H(x_0) \neq 0 \). Therefore \( M \) is minimal.

If \( \phi(p) \) is on the cut locus of \( q_0 \), then we use a meted of Calabi (cf. [8]). Let \( \sigma \) be a minimal geodesic joining \( \phi(p) \) and \( q_0 \). Then for any point \( q' \) in the interior of \( \sigma \), \( q' \) is not conjugate to \( q_0 \). Fix such a point \( q' \). Let \( U_{q'} \subset \mathcal{B}_\rho \) be a conical neighborhood of the geodesic segment of \( \sigma \) joining \( q' \) and \( \phi(p) \) such that, for any \( \phi(x) \in U_{q'} \), there is at most one minimizing geodesic joining \( q' \) and \( \phi(x) \). Let \( \tilde{r}(\phi(x)) = \text{dist}_{\rho'}(\phi(x), q') \) in the manifold \( U_{q'} \). Then we have \( \tilde{r}(\phi(x)) \geq \text{dist}_{\rho}(\phi(x), q') \), \( r(\phi(x)) \leq r(q') +... \)
\[ \bar{r}(\phi(x)), \ r(q') + \bar{r}(\phi(p)) = r(\phi(p)) \] and \( \bar{r} \) is smooth in a neighborhood of \( \phi(p) \). We claim that the function
\[ \tilde{f}(x) = \tilde{f}_\rho(x) := (\rho^2 - \{r(q') + \bar{r}(\phi(x))\})^2 u(x) \quad \text{for} \quad x \in M \cap \phi^{-1}(U_{q'}) \]
also attains a local maximum at the point \( p \). In fact, for any point \( x \in M \cap \phi^{-1}(U_{q'}) \), we have
\[
\tilde{f}(p) = (\rho^2 - \{r(q') + \bar{r}(\phi(p))\})^2 u(p) \\
= (\rho^2 - r(\phi(p))^2) u(p) \\
= f(p) \\
= (\rho^2 - r(\phi(x))^2) u(x) \\
\geq (\rho^2 - \{r(q') + \bar{r}(\phi(x))\})^2 u(x) \\
= \tilde{f}(x).
\]

Therefore the claim is proved and we can take the gradient and the Laplacian of the function \( \tilde{f}(x) \) at \( p \). The same argument as before then shows that
\[
k \tilde{f}(p) \leq 24mr(\rho(\phi(p))^2 + 4m(\rho^2 - r(\phi(p))^2)) \\
+ 4m \sqrt{L(1 + \rho^2)\frac{\bar{r}(\phi(p))}{\rho(\phi(p))} \left( \sqrt{L(1 + \rho^2)\bar{r}(\phi(p))} \right) (\rho^2 - r(\phi(p))^2)} \\
+ 4mr(\rho(\phi(p)) \sqrt{\tilde{f}(p)}).
\]

Take \( q' = q_0 \). By an elementary argument, we only have to consider the following: there exists a positive constant \( c \) depending only on \( k, L \) and \( m \) such that
\[
f(p) = \tilde{f}(p) \leq c(1 + \rho^2)^{\frac{3+k}{1-k}}.
\]
The same argument as before then shows that \( M \) is minimal. \( \Box \)

By the same argument as in Lemma 3.1 we also obtain the following results.

**Proposition 3.2.** Let \((M, g)\) be a properly immersed submanifold in a complete Riemannian manifold \((N, h)\) whose sectional curvature \(K^N\) has a polynomial growth bound of order less than 2 from below. Assume that there exists a positive constant \( k > 0 \) such that
\[
\Delta |B|^2 \geq k |B|^4 \quad \text{on} \quad M,
\]
where \(|B|\) is the norm of the second fundamental form. Then \( M \) is totally geodesic.

**Proof.** In general, we have \( m|H|^2 \leq |B|^2 \). By using this inequality, the same argument as in Lemma 3.1 shows the proposition. \( \Box \)

**Proposition 3.3.** Let \((M, g)\) be a properly immersed submanifold in a complete Riemannian manifold \((N, h)\) whose sectional curvature \(K^N\) has a polynomial growth bound of order less than 2 from below. Let \( u \) be a smooth non-negative function on \( M \). Assume that there exists a positive constant \( k > 0 \) such that
\[
\Delta u \geq ku^2 \quad \text{on} \quad M.
\]
If the mean curvature is bounded from above by a constant \( C \), then \( u = 0 \) on \( M \).

**Proof.** The same argument as in Lemma 3.1 shows the proposition. \( \Box \)

From the equation of (8), we obtain the following lemma.
Lemma 3.4. Let \((M, g)\) be a non-negative biminimal submanifold (that is, a biminimal submanifold with \(\lambda \geq 0\) ) in a Riemannian manifold with non-positive sectional curvature. Then, the following inequality for \(|H|^2\) holds

\[
\Delta |H|^2 \geq 2m |H|^4.
\]

Proof. The equation of (23) implies that, at each \(x \in M\),

\[
\Delta |H|^2 = 2 \sum_{i=1}^{m} \langle \nabla_{e_i} H, \nabla_{e_i} H \rangle + 2 \langle \Delta \frac{1}{2} H, H \rangle
\]

\[
= 2 \sum_{i=1}^{m} \langle \nabla_{e_i} H, \nabla_{e_i} H \rangle + 2 \sum_{i=1}^{m} \langle B(A_{He_i}, e_i), H \rangle
\]

\[
- 2 \left( \sum_{i=1}^{m} R^N(H, d\phi(e_i))d\phi(e_i), H \right) + 2 \lambda \langle H, H \rangle
\]

\[
\geq 2 \sum_{i=1}^{m} \langle A_{He_i}, A_{He_i} \rangle.
\]

The last inequality follows from \(K^N \leq 0, \lambda \geq 0\) and (5). When \(H(x) \neq 0\), set \(e_n := \frac{H(x)}{|H(x)|}\). Then, \(H(x) = H_n(x)e_n\) and \(|H(x)|^2 = H_n(x)^2\). From (20), we have at \(x\)

\[
\Delta |H|^2 \geq 2 H_n^2 \sum_{i=1}^{m} \langle A_{He_i}, A_{He_i} \rangle
\]

\[
= 2 |H|^2 |B_n|^2
\]

\[
\geq 2m |H|^4.
\]

Even when \(H(x) = 0\), the above inequality (19) still holds at \(x\). This completes the proof. \(\square\)

From the inequality (19), we obtain the following propositions.

Proposition 3.5. Let \((M, g)\) be a compact non-negative biminimal submanifold in a Riemannian manifold with non-positive sectional curvature. Then, \(M\) is minimal.

Proof. Applying the standard maximum principle to the elliptic inequality (19), we have that \(H = 0\) on \(M\). \(\square\)

Proposition 3.6. Let \((M, g)\) be a non-negative biminimal submanifold in a Riemannian manifold with non-positive sectional curvature. If the mean curvature is constant, then \(M\) is minimal.

Proof. Since \(\Delta |H|^2 = 0\), by using (19), we obtain the proposition. \(\square\)

We shall show our main theorem (cf. Theorem 1.5).

Proof of Theorem 1.5. By using Lemma 3.1, we obtain the inequality (19). Therefore, by using Lemma 3.1, we obtain Theorem 1.5. \(\square\)
4. Another result for Conjecture 3

In this section, we show that any complete biharmonic submanifold with Ricci curvature bounded from below in a Riemannian manifold with non-positive sectional curvature is minimal.

We recall the generalized maximum principle developed in Cheng-Yau [12].

Lemma 4.1 ([12]). Let \((M, g)\) be a complete Riemannian manifold whose Ricci curvature is bounded from below. Let \(u\) be a smooth non-negative function on \(M\). Assume that there exists a positive constant \(k > 0\) such that
\[
\Delta u \geq ku^2 \quad \text{on} \quad M.
\]
Then, \(u = 0\) on \(M\).

By using Lemma 4.1 and the inequality (19), we obtain the following proposition.

Proposition 4.2. Let \((M, g)\) be a complete non-negative biminimal submanifold in a Riemannian manifold with non-positive sectional curvature. If the Ricci curvature of \(M\) is bounded from below, then \(M\) is minimal.

Proposition 4.2 implies the following result.

Corollary 4.3. Let \((M, g)\) be a complete biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature. If the Ricci curvature of \(M\) is bounded from below, then \(M\) is minimal.

This result gives an affirmative partial answer to Conjecture 3.

5. Non-negative biminimal hypersurfaces in Riemannian manifolds

In this section, we consider a non-negative biminimal hypersurface \((M^n, g)\) in a Riemannian manifold \((N^{m+1}, h)\). In this case, we can denote that \(H = H\xi\), where \(H\) and \(\xi\) are the mean curvature and a unit normal vector field along \(\phi\) respectively. We have the following lemma.

Lemma 5.1. Let \((M, g)\) be a non-negative biminimal hypersurface (that is, a biminimal hypersurface with \(\lambda \geq 0\)) in a Riemannian manifold with non-positive Ricci curvature. Then, the following inequality for \(|H|^2\) holds
\[
\Delta |H|^2 \geq 2m |H|^4.
\]
Proof. Since \(\sum_{i=1}^{m} (R^N(H, d\phi(e_i))d\phi(e_i), H) = H^2\text{Ric}^N(\xi, \xi) \leq 0\), the same argument as in Lemma 3.4 shows this lemma.

By using this lemma, we obtain the following results.

Proposition 5.2. Let \((M, g)\) be a compact non-negative biminimal hypersurface in a Riemannian manifold with non-positive Ricci curvature. Then, \(M\) is minimal.

Proof. Applying the standard maximum principle to the elliptic inequality (23), we have that \(H = 0\) on \(M\).

Proposition 5.3. Let \((M, g)\) be a non-negative biminimal hypersurface in a Riemannian manifold with non-positive Ricci curvature. If the mean curvature is constant, then \(M\) is minimal.
Proof. Since $\Delta |\mathbf{H}|^2 = 0$, by using (23), we obtain the proposition. □

**Theorem 5.4.** Let $(N, h)$ be a complete Riemannian manifold with non-positive Ricci curvature. Assume that the sectional curvature $K^N$ has a polynomial growth bound of order less than 2 from below. Then, any non-negative biminimal properly immersed hypersurface in $N$ is minimal.

Proof. By using Lemma 5.1 we obtain the inequality (23). Therefore, by using Lemma 5.1 we obtain the theorem. □

Since every biharmonic submanifold is free biminimal one, we obtain the following result.

**Corollary 5.5.** Let $(N, h)$ be a complete Riemannian manifold with non-positive Ricci curvature. Assume that the sectional curvature $K^N$ has a polynomial growth bound of order less than 2 from below. Then, any biharmonic properly immersed hypersurface in $N$ is minimal.

By using Lemma 4.1 and Lemma 5.1 we also obtain the following result.

**Proposition 5.6.** Let $(M, g)$ be a complete non-negative biminimal hypersurface in a Riemannian manifold with non-positive Ricci curvature. If the Ricci curvature of $M$ is bounded from below, then $M$ is minimal.

Since every biharmonic submanifold is free biminimal one, we obtain the following result.

**Corollary 5.7.** Let $(M, g)$ be a complete biharmonic hypersurface in a Riemannian manifold with non-positive Ricci curvature. If the Ricci curvature of $M$ is bounded from below, then $M$ is minimal.

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