New M(atrix)-models for Commutative and Noncommutative Gauge Theories

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Abstract

We propose a M(atrix) model for $\mathcal{N} = 4$ SU($k$) Super-Yang-Mills theory compactified on $T^4$. In this model it is possible to make $T^4$ noncommutative and it is easy to turn on all 6 components of the noncommutativity on $T^4$. The action of S-duality on the noncommutativity parameters is also manifest. The M(atrix)-model is given by the large $N$ limit of a $\sigma$-model on $T^2$ whose target space is the moduli space of $k$ SU($N$) instantons on $T^3 \times R$. We also propose that the SU($k$) 2+1D Spin(8) theory (the low-energy description of $k$ M2-branes) on $T^3$ corresponds to the large $N$ limit of an integral over the latter instanton moduli space. The identification is based on the fact that Euclidean wrapped M2-branes in toroidally compactified M-theory correspond to instantons in the M(atrix)-model. In the new M(atrix) models, operators with nonzero momentum along $T^3$ (or $T^4$) correspond to insertions of Wilson lines along a 1-cycle that is determined by the momentum. Momentum is conserved in the large $N$ limit.
1 Introduction

The M(atrix)-model for M-theory \cite{1} uses the large $N$ limit of 0+1D $U(N)$ Super-Yang-Mills theory (SYM) with $\mathcal{N} = 16$ supersymmetry (see \cite{4,5,6} for previous appearances of this theory). M(atrix)-theory also suggests a new nonperturbative formulation of various field theories. Thus, the $(2,0)$-theory that is realized as the low-energy limit of $k$ coincident M5-branes is described in M(atrix)-theory by the large $N$ limit of the $\mathcal{N} = 8$ supersymmetric extension of quantum mechanics on the moduli space $M_{N,k}$ of $N U(k)$ instantons on $\mathbb{R}^4 \mathbb{Z}_k \mathbb{R}_\kappa \mathbb{R}$. The M(atrix)-model of 3+1D $\mathcal{N} = 4$ SYM and the 2+1D $Spin(8)$ theory \cite{8,9} can be derived from a limit of the moduli space of $N U(k)$ instantons on $\mathbb{T}^2 \times \mathbb{R}^2$ and $\mathbb{T}^3 \times \mathbb{R} \mathbb{Z}_k \mathbb{R}_\kappa \mathbb{R}$.

In all of those M(atrix)-models the field theory on $\mathbb{R}^{d,1}$ is required to have at least two noncompact directions, one of which is light-like. Alternatively, the theory can have a compactified light-like direction as in \cite{10} but it cannot have all $d$ space-like directions compactified simultaneously. Another question that can only partially be answered in the framework of those M(atrix)-models is how to describe the field theories on a noncommutative space.

Standard gauge field-theories have an extension to noncommutative spaces for which the product of fields is replaced with a noncommutative $\star$-product. These theories are parameterized by an anti-symmetric contravariant 2-tensor (bivector) $\theta^{ij}$. As will be explained in section (5), using the ideas of \cite{7,12,13}, the M(atrix)-models of \cite{10} can easily be extended to describe $\mathcal{N} = 4$ SYM on a noncommutative $\mathbb{R}^{3,1}$ (NCSYM). However, only two out of the 6 components of $\theta^{ij}$ can easily be turned on in this framework.

In these notes we would like to suggest M(atrix)-models for completely compactified, Euclidean field-theories. The theories that we will discuss are $\mathcal{N} = 4$ SYM compactified on $\mathbb{T}^4$ and the $Spin(8)$ theory compactified on $\mathbb{T}^3$. The M(atrix)-models are going to be a $\sigma$-model on $\mathbb{T}^2$ with a certain moduli space of instantons as the target space, and an integral on a certain moduli space of instantons, respectively. In these models for $\mathcal{N} = 4$ SYM all 6 components of the noncommutativity can easily be turned on.

The idea behind these models is as follows. Starting with M-theory compactified on $\mathbb{T}^d$ we can consider Euclidean M2-branes or M5-branes that wrap 3-cycles or 6-cycles of $\mathbb{T}^d$ and
produce instanton effects (see [14] and refs therein). In general, such instanton effects can be separated from other quantum effects because they are accompanied by a characteristic phase dependence on the moduli of the compactification. For example, if we compactify M-theory on $T^3$, the 3-form flux along $T^3$, $\phi \equiv \int_{T^3} C$, is a periodic modulus. An instanton effect that is a result of $k$ Euclidean M2-branes wrapping $T^3$ comes with the characteristic prefactor $e^{ik\phi}$. In addition to the phase, the instanton effect can be calculated from the action of the branes and the specific physical question (e.g. the scattering amplitude of some particles) that we are trying to solve. For example, when all the dimensions of $T^3$ are large compared to $M_p^{-1}$, the instanton contribution can presumably be calculated from the low-energy theory of $k$ coincident M2-branes, i.e. the 3D $Spin(8)$ theory [8, 9].

How is all of that manifested in M(atrix)-theory? The M(atrix)-model of M-theory on $T^3$ is 3+1D $\mathcal{N} = 4$ SYM compactified on $T^3$. The instanton described above corresponds to an instanton of the SYM theory [15] and therefore there must be a map from quantities calculated in the $Spin(8)$ theory to quantities calculated by integrating over the moduli space of instantons.

Thus, the large $N$ limit of the integral over the moduli space of instantons on $T^3 \times \mathbb{R}$, where $\mathbb{R}$ is the time direction, is a M(atrix)-model for the compactified $Spin(8)$ theory. In a similar spirit, one can derive a M(atrix)-model for the compactified $\mathcal{N} = 4$ SYM. The purpose of these notes is to derive these M(atrix)-models and explore them.

The paper is organized as follows. In section (2) we construct the new M(atrix)-models for the $Spin(8)$ theory on $T^3$ and for $\mathcal{N} = 4$ SYM on $T^4$ and argue for the decoupling of other M(atrix)-theory fields from the variables in the moduli spaces of instantons. In section (3) we discuss the flat directions of the field theories and the instanton moduli spaces. In that section we also describe compactification of the field theories with R-symmetry twists. Their M(atrix)-models are related to dipole-theories [16, 17] (and see also [18] for a special case). In section (4) we attempt to construct a map between questions about the compactified theory and questions about the new M(atrix)-models. We propose that field theory momentum should be translated into $\mathbb{Z}_N$ charge in the M(atrix) model and operators that carry momentum should be mapped to insertions of Wilson lines. Finally, in section (5) we describe the extension of the $\mathcal{N} = 4$ SYM M(atrix)-model to a noncommutative $T^4$. 

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For the benefit of readers who wish to skip sections (3)-(4), we note that section (5) can be read immediately after section (2).

2 New M(atrix)-models from instantons

We will derive the M(atrix)-models for the 2+1D Spin(8) theory on $T^3$ and the 3+1D $\mathcal{N} = 4$ SYM on $T^4$ by studying instantons in M-theory on $T^3$ or type-IIB string theory on $T^4$. We will assume that the metric is Euclidean in both cases.

2.1 The 3D Spin(8) theory on $T^3$

M-theory on $T^3$ has a moduli space:

$$(SL(3,\mathbb{Z}) \backslash SL(3,\mathbb{R})/SO(3)) \times (SL(2,\mathbb{Z}) \backslash SL(2,\mathbb{R})/SO(2)).$$

The first factor corresponds to the geometrical shape of the $T^3$ and the second factor parameterizes the volume and 3-form flux such that the combination:

$$\tau = \frac{iM_p^3V + C}{2\pi},$$

(where $V$ is the volume, $M_p$ is the 11D Planck energy scale and $C$ is the 3-form flux) transforms under the generators of $SL(2,\mathbb{Z})$ as $\tau \to \tau + 1$ and $\tau \to -1/\tau$. The M(atrix)-model of M-theory on $T^3$ is $\mathcal{N} = 4 \ U(N)$ SYM on the geometrical dual $\hat{T}^3$ with coupling constant and $\theta$-angle given by the same $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$. The volume of $\hat{T}^3$ is immaterial because $\mathcal{N} = 4$ SYM is a conformally invariant theory.

Various amplitudes in M-theory on $T^3$ receive instanton contributions from Euclidean M2-branes wrapped on $T^3$ (see [14] and refs therein). The amplitude of an instanton effect made out of $k$ M2-branes has the characteristic factor $e^{ikC}$. In the M(atrix)-model, such phases correspond to effects of $k$ Yang-Mills instantons.

Now consider the limit $M_p^3V \to \infty$. In this limit, when the $k$ M2-branes are close to each other, the instanton effect has a factor $e^{2\pi i k \tau} Z$ where $Z$ can be calculated from the partition function of the "$U(k)$" Spin(8) theory [8, 9]. In the M(atrix)-model, this limit corresponds to $g_{YM} \to 0$ and the effect we are looking for comes from a sector with Yang-Mills instanton
number $k$ that carries a prefactor $e^{2\pi i k \tau}$. If we also take the low-energy limit on both sides, then in M-theory gravity decouples and we get the $\text{Spin}(8)$ theory. In M(atrix)-theory the instanton contribution is of the form $e^{2\pi i k \tau}Z'$ where $Z'$ can be calculated from an integral over the moduli space of $k \ SU(N)$ instantons on $\hat{T}^3 \times \mathbb{R}$. Here $\mathbb{R}$ is the time direction.

So, the M(atrix)-model for the “$SU(k)$” $\text{Spin}(8)$ theory compactified on $\mathbb{T}^3$ is an integral over $\mathcal{M}_{N,k}(\hat{T}^3 \times \mathbb{R})$ – the moduli space of $k \ SU(N)$ instantons on the dual $\hat{T}^3$ times $\mathbb{R}$ in the limit $N \to \infty$.

### 2.2 4D $\mathcal{N} = 4$ SYM on $\mathbb{T}^4$

We can use a similar reasoning to find a M(atrix)-model for $\mathcal{N} = 4 \ SU(k)$ SYM on $\mathbb{T}^4$. We start with type-IIB string theory on $\mathbb{T}^4$ which, for simplicity, we take to be rectangular with radii $R_1, \ldots, R_4$. We let the complex type-IIB coupling constant be $\tau$. This is equivalent to M-theory on $\mathbb{T}^3 \times \mathbb{T}^2$ where $\mathbb{T}^2$ has complex structure $\tau$ and area $\tau_2^{-1/3}M_p^{-2}(M_s R_4)^{-4/3}$ ($M_p$ is the 11D Planck scale and $M_s$ is the 10D string scale). $\mathbb{T}^3$ has radii $M_p^{-1}M_s^{1/3}\tau_2^{1/3}R_4^{1/3}R_i$ ($i = 1 \ldots 3$). According to [20], the M(atrix)-model for M-theory on $\mathbb{T}^5$ with these parameters is given by the (type-IIB) little-string-theory (LST) compactified on another $\hat{T}^3 \times \mathbb{T}^2$. We define the LST energy scale to be $\tilde{M}_s$. The $\mathbb{T}^2$ has area:

$$A_2 \equiv \tilde{M}_s^{-2}M_s^4\tau_2 R_1R_2R_3R_4,$$

and $\hat{T}^3$ has radii:

$$L_i \equiv \tilde{M}_s^{-1}R_4^{-1/2}(R_1R_2R_3)^{1/2}R_i^{-1} \quad i = 1 \ldots 3,$$

so that the area of $\hat{T}^3$ is:

$$A_3 \equiv \tilde{M}_s^{-3}R_4^{-3/2}(R_1R_2R_3)^{1/2}.$$

Now we take the limit $R_1, R_2, R_3, R_4 \gg M_s^{-1}$. In this limit:

$$\tilde{M}_s^3 A_3 = O(1), \quad \tilde{M}_s^2 A_2 \gg 1.$$
We wish to extract from this LST the part that describes the M(atrix)-model for 4D \( \mathcal{N} = 4 \) SYM on \( T^4 \). As we have seen, the LST is formulated on \( W \equiv T^2 \times \hat{T}^3 \times \mathbb{R} \) where \( \hat{T}^3 \) is of order \( \tilde{M}_s^{-1} \) and \( T^2 \) is large. We would like to argue that this M(atrix)-model is the partition function of a certain 2D CFT on \( T^2 \) with complex structure \( \tau \). This \( T^2 \) corresponds to the first factor in \( W \). The CFT, we will argue, is a \( \sigma \)-model with target space \( X \equiv \mathcal{M}_{N,k}(\hat{T}^3 \times \mathbb{R}) \) – the moduli space of \( k \) SU(\( N \)) instantons on \( \hat{T}^3 \times \mathbb{R} \).

To describe the \( \sigma \)-model we need to describe a metric on \( X \) and a \( B \)-field that takes values in \( H^2(X, \mathbb{R})/H^2(X, \mathbb{Z}) \). Let us first describe the parameters of \( \hat{T}^3 \) more generally. Starting with 4D \( \mathcal{N} = 4 \) SYM on \( T^4 \), the geometrical shape of \( T^4 \) corresponds to a point in the moduli-space \( SL(4, \mathbb{Z})\backslash SL(4, \mathbb{R})/SO(4) \). This is equivalent to \( SO(3, 3, \mathbb{Z})\backslash SO(3, 3, \mathbb{R})/(SO(3) \times SO(3)) \). A point in the latter space can be interpreted as a metric and anti-symmetric 2-form \( B \)-field on \( \hat{T}^3 \). Thus, \( \hat{T}^3 \) comes naturally with a metric and \( B \)-field on it. From this metric and \( B \)-field we can derive a metric and \( B \)-field on \( X \) as follows. Let \( \xi \in X \) denote a particular point in the instanton moduli space. Let \( A(\xi) \) be an \( N \times N \) matrix-valued 1-form on \( \hat{T}^3 \times \mathbb{R} \) that describes the instanton solution up to a gauge transformation. Let \( d_X = 4(k-1)(N-1) \) be the dimension of \( X \) (we assume that the gauge fields are zero at \( \pm \infty \)). The metric on \( X \) can be written as:

\[
\tilde{G}_{ij} d\xi^i d\xi^j = \sum_{1 \leq i, j \leq d_X} \left( \int_{\hat{T}^3 \times \mathbb{R}} g^{\alpha\beta} tr\left\{ \frac{\delta A_\alpha}{\delta \xi^i} \frac{\delta A_\beta}{\delta \xi^j} \right\} \right) d\xi^i d\xi^j,
\]

(4)

where \( g^{\alpha\beta} \) is the metric on \( \hat{T}^3 \times \mathbb{R} \) and one has to gauge fix \( A(\xi) \) such that:

\[
D^\alpha \frac{\delta A_\alpha}{\delta \xi^i} = 0, \quad D^\alpha \equiv \partial^\alpha - i[A^\alpha, \cdot].
\]

The 2-form on \( X \) can be written as:

\[
\tilde{B}_{ij} d\xi^i \wedge d\xi^j = \sum_{1 \leq i, j \leq d_X} \left( \int_{\hat{T}^3 \times \mathbb{R}} B \wedge tr\left\{ \frac{\delta A}{\delta \xi^i} \wedge \frac{\delta A}{\delta \xi^j} \right\} \right) d\xi^i \wedge d\xi^j,
\]

(5)

The idea behind these expressions is as follows (see also [20, 5, 6, 10]). At low-energies and when the size of \( \hat{T}^3 \) is large compared to \( \tilde{M}_s^{-1} \) the LST can be approximated by 5+1D SYM on \( T^2 \times \hat{T}^3 \times \mathbb{R} \). This action contains

\[
\int_{T^2 \times \hat{T}^3 \times \mathbb{R}} \left[ \frac{M_s^2}{4} tr\{F^2\} + B \wedge tr\{F \wedge F\} \right].
\]
When the size of $T^2$ is very large, this action can be reduced to the $\sigma$-model on $X$ with metric $\tilde{G}$ and 2-form $\tilde{B}$. We conjecture that formulas (4-5) continue to hold even when $\tilde{T}^3$ is small. Note also that if the parameters $\xi_i$ are chosen to have periodicities of order 1 then $\tilde{G}$ and $\tilde{B}$ are independent of $\tilde{M}_s$, as they should.

2.3 Remarks

Let us comment on the treatment of singularities in $\mathcal{M}_{N,k}$. In general, these moduli spaces of instantons have singularities at (real) codimension-4 that, roughly speaking, correspond to instantons that collide. A particular treatment of these singularities was suggested in [7] in the context of the M(atrix)-model for the $(2,0)$ and LST. There it was suggested to deform the moduli space to a smooth space by a parameter $\theta$. It was argued that this describes a deformation of the $(2,0)$ theory that breaks Lorentz invariance. This deformation of $\mathcal{M}_{N,k}$ was then interpreted in [12, 13] as the moduli space of instantons on noncommutative $\mathbb{R}^4$. The Lorentz-invariant $(2,0)$-theory or LST can be recovered in the limit $\theta \to 0$. In our case, we can use a similar idea to treat the singularities. We can turn on a noncommutativity on $\tilde{T}^3$, which also forces us to work with a $U(N)$ gauge group. (For another example of how some singularities are smoothed for noncommutative instantons on $T^4$, see [21].)

According to [22], the M(atrix)-model of M-theory on $T^5$ with a nonzero 3-form flux in the light-like direction and two of the $T^5$ directions is given formally by 5+1D SYM on a noncommutative $T^5$ (interpreted as LST as in [20]). The point is that in the large $N$ limit, the effect of the C-field flux should disappear, since it can be gauged away. Thus, in our case, changing $\tilde{T}^3$ to a noncommutative space should not affect the large $N$ limit. In our case, unlike the case of [3], smoothing out the singularities in $\mathcal{M}_{N,k}$ by turning on noncommutativity does not result in a breakdown of Lorentz invariance and the large $N$ limit should, presumably, be unaffected!

Finally, note that the $\text{Spin}(8)$ R-symmetry of the 3D theory and the $\text{Spin}(6)$ R-symmetry of $\mathcal{N} = 4$ SYM are not manifest in the new M(atrix)-models. Presumably, they are restored in the large $N$ limit like Lorentz invariance in M(atrix)-theory [1].
2.4 Decoupling arguments

Let us consider M-theory on $T^3$ again in the limit that the dimensions of $T^3$ are much bigger than $M_p^{-1}$. As we have seen, the sector with $k$ instantonic M2-branes wrapping $T^3$ is described in M(atrix)-theory by the sector with instanton number $k$ in $\mathcal{N} = 4$ SYM. We have therefore argued that the $Spin(8)$ theory that describes $k$ M2-branes at low-energy corresponds to the large $N$ limit of an integral over the moduli space on instantons $\mathcal{M}_{N,k}$ on $T^3 \times \mathbb{R}$. Let us elaborate on why we can restrict to the moduli space of instantons and ignore the other modes of the $\mathcal{N} = 4$ SYM M(atrix)-model.

Let us first recall how the decoupling occurs in the M(atrix)-models of [5, 6]. These models describe the $(2,0)$-theories on $\mathbb{R}^6$ as the large $N$ limit of quantum-mechanics on a certain moduli space of instantons. When the M(atrix)-models are derived from the M(atrix)-model of M-theory with $k$ M5-branes [23], the moduli space of instantons is obtained as the minimum of the potential. The excitations that take the variables out of the moduli space of instantons are very massive.

In our case, we need a related but somewhat different argument. We are considering the limit that $M_p^3 V \to \infty$, where $V$ is the volume of $T^3$. This corresponds to $g_{YM} \to 0$. Let us consider a particular $U(N)$ instanton configuration. The M(atrix)-model modes that we have neglected are the 6 adjoint scalars $X^I$, 4 adjoint Dirac fermions on $T^3 \times \mathbb{R}$ and the fluctuations of the gauge fields transverse to the moduli space. In the limit $g_{YM} \to 0$ we can keep only the quadratic interactions and ignore the cubic and quartic terms in the Lagrangian of $\mathcal{N} = 4$ SYM. Thus, we only need to show that the determinants coming from integrating the quadratic modes in the fields of the instantons cancel between bosons and fermions and this easily follows from supersymmetry.

3 Flat directions and twists

In section (2) we suggested various correspondences between partition functions of field-theories and integrals or $\sigma$-models over moduli-spaces of instantons. However, the correspondence is incomplete because the partition functions of the field theories are ill-defined. The integrals defining the partition functions have noncompact bosonic zero modes. For the
Spin(8) theory there are 8\(k\) such modes corresponding to the separation of the M2-branes. For \(\mathcal{N} = 4\) SYM there are 6\(k\) such modes. Moreover, there are fermionic zero modes as well. Our task in this section is to understand how to treat those zero modes in the corresponding M(atrix)-models.

### 3.1 Regularizing the partition function

It is not difficult to trace the zero modes of the field-theory into the M(atrix)-model. For concreteness, let us concentrate on the Spin(8) theory. Let us start with a single Euclidean M2-brane in M-theory on \(T^3\) as in section (2). It has 8 bosonic zero modes corresponding to translations in transverse directions. Two out of the eight correspond to translations in the light-like \(x^+\) and \(x^-\) directions. In M(atrix)-theory, the \(x^+\) coordinate is periodic with period \(2\pi R_\parallel\) and so this zero mode gives an overall finite \(2\pi R_\parallel\) factor to the partition function of the M2-brane. The \(x^-\) direction corresponds in M(atrix)-theory to the time direction. The M(atrix)-model configuration was given by a \(U(N)\) instanton on \(\hat{T}^3 \times \mathbb{R}\). So the \(x^-\) zero mode corresponds to translation of the position of the instanton in the \(\mathbb{R}\), i.e. time, direction. The other 6 zero modes correspond to the 6 scalars of \(\mathcal{N} = 4\) SYM.

Now consider two Euclidean M2-branes in M-theory on \(T^3\). There are the overall center of mass flat directions that can be treated as before but there are also flat directions corresponding to the relative separation of the instantons. The 6 flat directions that describe separation of the M2-branes in transverse directions (orthogonal to \(x^+\) and \(x^-\)) correspond to breaking the \(U(N)\) gauge-group by VEVs of the 6 adjoint scalars of SYM down to \(U(N_1) \times U(N_2)\) such that \(N = N_1 + N_2\) and then embedding one instanton in \(U(N_1)\) and the other in \(U(N_2)\). Turning on the VEVs of the scalars is undesirable for us, since we wish to argue that we can restrict to the SYM instanton moduli space variables. In order to suppress the excitations of the 6 adjoint scalars, we have to put the instantons in some physical context.

We can imagine that we are trying to calculate the instanton contribution to the scattering of gravitons or to some terms in the low-energy effective action. For example, as shown in [24] (see also [25]), there is an \(R^4\) correction to the low-energy Wilsonian effective action of M-theory on \(T^3\) that receives contributions from such instantons. In a scattering amplitude that is calculated from this Wilsonian effective action there can be two types of contributions
from such terms at instanton number 2. One is an “irreducible” term where there is a single $R^4$ vertex and we take the instanton number 2 contribution to that vertex. The second type is a “reducible” contribution where there are, say, two $R^4$ vertices connected by graviton propagators and each vertex has an instanton number 1 contribution. It is clear that the terms we would like to consider are those of the irreducible type. On the field-theory side, they are calculated from the partition function of the $SU(2) \text{Spin}(8)$ theory. The overall center of mass flat direction decouples in that partition function. The flat direction that corresponds to separation of the two M2-branes seems to give an infinite contribution, but there are also fermionic zero modes that, if treated properly, cancel the infinite contribution from the bosonic flat directions. An explicit calculation was performed in [25] for the case of the D(-1)-instanton. The explicit calculation from the $R^4$ terms in type-IIB string theory was shown to agree with the result of $\frac{5}{4}$ calculated from the $SU(2)$ SYM partition function in [26, 27]. It is tempting to identify this “irreducible” contribution on the M-theory (compactified on $T^3$) side with the integral over the moduli space of instantons on $T^3 \times \mathbb{R}$ on the M(atrix)-model side, where we set the 6 adjoint scalar fields to zero. On the M-theory side, the $\text{Spin}(8)$ theory is the correct description of the dynamics of the M2-branes when the separation between the M2-branes is much smaller than the 11D Planck length $M_p^{-1}$. On the M(atrix)-model side this means that we should indeed set the adjoint scalar fields to zero. (In principle we might want to neglect only the quartic term but keep the quadratic term in the 6 scalar fields, but the contribution of the fermions cancels the quadratic integral of the bosons.)

Ideally, we would be led to believe that the partition function of the $\text{Spin}(8)$ theory (which is finite once the fermionic zero modes are properly treated to cancel the infinite integration range over the flat directions) is equal to the integral over the moduli space of instantons on $T^3 \times \mathbb{R}$. The latter integral contains an unbound flat direction that corresponds to large separation of the instantons in the $\mathbb{R}$ direction. On the M-theory side this corresponds to separation of the M2-branes in the $x^-$ direction. Once the zero modes are treated correctly the integral over the instanton moduli space is finite. In fact, the $\text{Spin}(8)$-theory partition function and the moduli space integral need not be equal, but their ratio will determine the overall normalization. We will return to this point in subsection (4.3).
3.2 Twisting

Another way to get rid of the $8k$ flat directions is to use twisted boundary conditions. This is a well-known way to avoid zero-modes (see [28]). In the context of M-theory, the compactification described below was used in [29] to propose a definition for a partition function of gravity. It will be further explored in [30]. The idea is to modify $T^3 \times R^8$ (for the moment, we either think of space-time as Euclidean or we set one of the $T^3$ directions to be time-like) into an $R^8$-fibration over $T^3$ such that when we go around a cycle of $T^3$ we perform a $Spin(8)$ rotation of the transverse $R^8$. Locally this space is flat but globally it differs from $T^3 \times R^8$ because of the $Spin(8)$ twists. For generic twists, an M2-brane that wraps the $T^3$ has minimal volume when it is at the origin of each of the $R^8$-fibers. If it wishes to escape the origin it must increase its volume so the flat directions are gone. The fermionic zero modes also become massive, in general. In the M(atrix)-model setting we can only use $Spin(6)$ twists since we wish to preserve the light-like directions $x^+$ and $x^-$. The $Spin(6)$ elements must all commute with each other and we can choose them in a subgroup $U(1)^3 \subset Spin(6)$. We will denote the $T^3$ directions by $x_1, x_2, x_3$. Let the corresponding M(atrix)-theory fields be $X_1, X_2, X_3$. They are adjoint scalars. $x_4, \ldots, x_9$ will denote the 6 transverse directions. Let

$$z_1 = x_4 + ix_5, \quad z_2 = x_6 + ix_7, \quad z_3 = x_8 + ix_9,$$

and let the corresponding M(atrix) theory fields be $Z_1, Z_2, Z_3$. Let the twists be given by the identification:

$$(x_1, x_2, x_3, z_1, z_2, z_3) \sim (x_1 + 2\pi n_1 R_1, x_2 + 2\pi n_2 R_2, x_3 + 2\pi n_3 R_3, e^{i \sum_{a=1}^3 n_a \alpha_{1a} z_1}, e^{i \sum_{a=1}^3 n_a \alpha_{2a} z_2}, e^{i \sum_{a=1}^3 n_a \alpha_{3a} z_3}).$$

Here $\alpha_{ij}$ is a matrix of phases.

It is possible to derive the M(atrix)-model for such a compactification using the rules of toroidal compactifications [1, 19, 15]. A M(atrix)-model for a similar theory was discussed in [18, 32, 16] and we will repeat the arguments here. To compactify M(atrix)-theory on $T^3$,

$^1$Two possible ways to use the whole $Spin(8)$ twists will be explored in [30]. One is to include $SO(1,1)$ twists which is a symmetry only at the limit $N = \infty$. Another way is to use the IKKT model[31].
we need to pick 3 $U(\infty)$ matrices, $\Omega_1, \Omega_2, \Omega_3$ and require that the matrices satisfy [1, 19, 15]:

$$\Omega_j^{-1} X_i \Omega_j = X_i + 2\pi \delta_{ij} R_i, \quad \Omega_j^{-1} Z_a \Omega_j = e^{i\alpha_{ja}} Z_a.$$ 

The solution is to think about $\Omega_j$ and the $X_i$'s as operators on the Hilbert space of functions on a dual $\hat{T}^3$ of radii $\frac{1}{2\pi R_i}$. Let $0 \leq \sigma_j \leq \frac{1}{R_j}$ (with $j = 1, 2, 3$) be the 3 periodic coordinates. $\Omega_j$ is taken to be diagonal so that

$$\langle \sigma_1 \sigma_2 \sigma_3 | \Omega_j | \sigma'_1 \sigma'_2 \sigma'_3 \rangle = e^{2\pi i \sigma_j} \delta(\sigma_1 - \sigma'_1) \delta(\sigma_2 - \sigma'_2) \delta(\sigma_3 - \sigma'_3)$$

and $X_j = i\partial_{\sigma_j} - A_j(\vec{\sigma})$ where $A_j$ is a gauge-field on the dual $\hat{T}^3$. The twist requires us to find operators $Z_a$ such that:

$$\Omega_j^{-1} Z_a \Omega_j = e^{i\alpha_{ja}} Z_a.$$ 

The solution is an operator $Z_a$ with matrix elements of the form:

$$\langle \sigma_1 \sigma_2 \sigma_3 | Z_a | \sigma'_1 \sigma'_2 \sigma'_3 \rangle = \delta(\alpha_{1a} + \sigma_1 - \sigma'_1) \delta(\alpha_{2a} + \sigma_2 - \sigma'_2) \delta(\alpha_{3a} + \sigma_3 - \sigma'_3) \Phi_a(\vec{\sigma}),$$

where $\Phi_a$ are arbitrary local fields on $\hat{T}^3$. If we did not have the twist, the Lagrangian would be 3+1D SYM with $N = 8$ supersymmetry, as in [1, 19, 15]. The effect of the twist is to make the fields $\Phi_a(\vec{\sigma}, t)$ nonlocal. Instead of transforming in the adjoint of the local $U(N)$ gauge group at the point $(\vec{\sigma}, t)$, they transform in the representation $(N, \overline{N})$ of $U(N)(\vec{\sigma}, t) \times U(N)(\vec{\sigma} + \vec{L}_a, t)$ (where $U(N)(\vec{\sigma}, t)$ is the restriction of the gauge group to the point $(\vec{\sigma}, t)$ and

$$\vec{L}_a \equiv \left( \frac{\alpha_{1a}}{2\pi R_1}, \frac{\alpha_{2a}}{2\pi R_2}, \frac{\alpha_{3a}}{2\pi R_3} \right).$$

For example, the covariant derivative is defined as:

$$D_i \Phi_a \equiv \partial_i \Phi_a(\vec{\sigma}) - iA_i(\vec{\sigma}) \Phi_a(\vec{\sigma}) + i\Phi_a(\vec{\sigma}) A_i(\vec{\sigma} + \vec{L}_a).$$  \hspace{1cm} (6)$$

Such theories where discussed in [17], where they were obtained by studying T-duality in gauge-theories on Noncommutative spaces, and in [33] from pinned-branes. They will be referred to as “dipole-theories.”

Let us check that the M2-branes cannot separate, for generic twists, $\vec{L}_a$. To see this note that the $U(1) \subset U(N)$ does not decouple in the dipole-theories. In the presence of an instanton, and for generic twists, the equation $D_i \Phi_a = 0$, with $D_i \Phi_a$ defined in (6) has no nonzero solution. Note that:

$$[D_i, D_j] \Phi_a = -iF_{ij}(\vec{\sigma}) \Phi_a(\vec{\sigma}) + i\Phi_a(\vec{\sigma}) F_{ij}(\vec{\sigma} + \vec{L}_a),$$
and in general \( \det F_{ij}(\vec{\sigma}) \neq \det F_{ij}(\vec{\sigma} + \vec{L}_a) \). Separating the M2-branes requires \( \Phi_a \) to be nonzero.

In the M(atrix)-model, we should keep the quadratic terms in \( \Phi_a \) and the fermions. We can neglect the quartic terms because, as we argued above, the fluctuations in \( \Phi_a \) are much smaller than \( M_p^{-1} \). The quadratic terms can be integrated to give factors of \( \det D^i(\vec{L}_a)D_i(\vec{L}_a) \) (as a function of the point in the instanton moduli space). The fermion contribution does not cancel this because, generically, supersymmetry is broken by the twists.

### 4 Partition functions and Operators

We now discuss in more detail the “dictionary” that translates questions about the field theories to questions about their M(atrix)-models. For simplicity we will restrict ourselves to the \( \text{Spin}(8) \) theory.

In section (3) we suggested that the \( \text{Spin}(8) \) \( SU(k) \) theory compactified on \( T^3 \) is related to the large \( N \) limit of the integral over \( \mathcal{M}_{N,k} \), but following the discussion in section (3), we need to mod out the overall translation modes along \( T^3 \). We therefore define \( \hat{\mathcal{M}}_{N,k} \) – the “reduced” moduli space of \( k \) \( SU(N) \) instantons on \( T^3 \times \mathbb{R} \). By “reduced” moduli space we mean the following. Let \( \mathcal{M}_{N,k} \) be the moduli space of \( k \) instantons of \( SU(N) \). The \( T^3 \times \mathbb{R} \), considered as an abelian group, acts on \( \mathcal{M}_{N,k} \) by translations. The reduced space is the space of orbits of this \( T^3 \times \mathbb{R} \), which we will denote by \( \hat{\mathcal{M}}_{N,k} \). By an “integral” over the moduli space, we mean the dimensional reduction to 0D of the 0+1D Quantum-Mechanics with 8 supersymmetries over the moduli space of instantons.

The relation between the compactified \( \text{Spin}(8) \) theory and the integral over the instanton moduli space should be interpreted as follows. For every operator in the \( \text{Spin}(8) \) theory there should exist an appropriate function of the moduli space such that insertions of the \( \text{Spin}(8) \) operators into the partition function correspond to insertion of the functions into the integral.

There is no obvious reason to expect that the partition function of the \( \text{Spin}(8) \) theory itself should be equal to the partition function of its M(atrix)-model. Thus, we expect to have a fixed numerical coefficient \( C_{N,k} \) that is the ratio of the partition function of the \( \text{Spin}(8) \) theory and the integral over the \( \hat{\mathcal{M}}_{N,k} \). Once this \( C_{N,k} \) is known we can begin to
map operators on the field theory side to insertions in the M(atrix)-model integral. We will make some remarks about $C_{N,k}$ in subsection (4.3), but let us start with the operators.

4.1 Momentum and the $Z_N$ symmetry

Suppose we wish to calculate an expectation value:

$$\langle O_1(p_1)O_2(p_2)\cdots O_n(p_n) \rangle$$

in the $Spin(8)$ theory. Here $O_i$ are operators, for example components of the energy-momentum tensor or of the $Spin(8)$ current and $p_i$ are discrete momenta along $T^3$. The momenta belong to a lattice that determines the dual $\hat{T}^3$.

We conjecture that the expectation value above can be calculated from the matrix model as:

$$\lim_{N \to \infty} C_{N,k} \int_{\hat{M}_{N,k}} \hat{O}_1(p_1, \xi) \cdots \hat{O}_n(p_n, \xi)[D\xi],$$

where $\hat{O}(p, \xi)$ can be determined from $O(p)$ and $\xi \in \hat{M}_{N,k}$. We will not give a complete prescription to determine $\hat{O}$ from $O$ but we will suggest a treatment of the momentum label, $p$.

Recall that in the M(atrix)-model for M-theory on $T^3$, momentum becomes electric-flux on the dual $\hat{T}^3$. Let us present $\hat{T}^3$ as $R^3$ divided by a lattice $\Gamma$. The momentum $p$ corresponds to a vector $\vec{v}(p) \in \Gamma$ and so does electric flux. The operators that “have” electric flux are the Wilson lines. Let $\bar{\xi} \in M_{N,k}$ be a particular instanton configuration and let $\bar{x} \in \hat{T}^3$ be a point and $t \in R$ by a time coordinate. Let $W(p, \bar{x}, t, \bar{\xi})$ be the Wilson line that corresponds to a path that is a straight line from $\bar{x}$ to $\bar{x} + \bar{v}(p)$ in $\hat{T}^3$. $W$ is defined up to a gauge transformation.

We conjecture that $\hat{O}(p, \bar{\xi})$ has the form:

$$\int d^3x \, dt \, tr\{\bar{O}(\bar{x}, t, \bar{\xi})W(p, \bar{x}, t, \bar{\xi})\},$$

where $\bar{O}$ is some local operator that is independent of $p$.

Note that because we integrate over $\hat{T}^3 \times R$, $\hat{O}(p, \bar{\xi})$ depends only on the equivalence class of $\bar{\xi}$ under translations, i.e. on $\xi$. 

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Is momentum conserved according to this definition? The answer is yes, but only in the large \(N\) limit!

To see this note that as defined, \(\hat{O}\) has a \(\mathbb{Z}_N\) ambiguity. This is because an instanton configuration defines an \(SU(N)/\mathbb{Z}_N\) gauge configuration (since gauge fields are in the adjoint representation) but the Wilson-line traces are defined in the fundamental representation. Therefore, the Wilson lines have an ambiguity that corresponds to making a \(\mathbb{Z}_N\) gauge transformation in the center of \(SU(N)\) as we go along any of the 1-cycles of \(\hat{T}^3\). The integral in (7) will be independent of this \(\mathbb{Z}_N\) ambiguity only if the sum of the overall momenta of all the operators is an \(N\)-multiple of a \(\Gamma\)-lattice vector. In the large \(N\) limit this could only be zero.

To summarize, the definition (8) satisfies:

- The nontrivial classes of 1-cycles on \(\hat{T}^3\) are naturally labeled by the momentum, \(p\), on the original \(T^3\). Therefore, mapping an operator that carries \(p\) units of momentum to an operator that contains a Wilson line along an appropriate 1-cycle is natural.

- By introducing Wilson lines in the fundamental representation we have introduced a \(\mathbb{Z}_N^3\) symmetry. The \(\mathbb{Z}_N^3\) quantum numbers of \(\hat{O}(p, \xi)\) should equal \(p\) (mod \(N\)).

- Momentum is conserved up to multiples of \(N\) because of the \(\mathbb{Z}_N^3\) ambiguity in determining the Wilson line traces.

- By integrating over the whole \(\hat{T}^3 \times \mathbb{R}\) the operator becomes independent of the center-of-mass position of the instanton configuration, as is required by our prescription of integrating over the reduced moduli space.

If the above conjecture is correct then it is enough to determine the local variables, \(\hat{\mathcal{O}}\), by determining the mapping of operators with zero momentum. We will therefore turn now to discuss the zero modes of the energy momentum tensor and \(R\)-symmetry currents.

### 4.2 Energy momentum tensor

The \(M\)(atrix)-variable, \(\hat{\mathcal{O}}\), corresponding to an energy momentum tensor insertion, \(T_{\mu\nu}(\bar{0})\), with zero momentum can be deduced by studying the variation in the volume form on \(\hat{\mathcal{M}}_{N,k}\) as a result of an infinitesimal change in the metric on \(\hat{T}^3\). We will not do this here, but
instead we will sketch the procedure for a more complicated operator insertion.

The Spin(8) theory has a Noether current operator, \( J^a_\mu \) (\( \mu = 0, 1, 2 \) and \( a = 1 \ldots 28 \), the dimension of Spin(8)) corresponding to the global Spin(8) symmetry. The construction in section (2) makes only an \( SU(4) = Spin(6) \subset Spin(8) \) manifest. We will now map the integral, \( \int_{T^3} J^a_\mu(x) d^3x, \) of the current corresponding to that Spin(6).

For simplicity, let us consider a rectangular \( T^3 = S^1 \times S^1 \times S^1 \). Let the radii of the \( S^1 \)'s be \( R_1, R_2, R_3 \) and let us map the component of the current along the first \( S^1 \). Suppose we add an infinitesimal term, \( \epsilon J^a_\mu \), to the Lagrangian of the 3D Spin(8) theory. In M-theory, this can be realized as a geometrical twist in the boundary conditions for \( S^1 \). Thus, when we go once around \( S^1 \), we also perform a Spin(6) rotation in the transverse \( R^6 \) (keeping \( R^1,1 \) that contains the light-like direction intact).

We have described the M(atrix)-models for such twisted compactifications in subsection (3.2). We have argued that the modification to the integral over the moduli space of instantons, because of the twists, is given by integrating the quadratic term in the adjoint bosons and fermions. The bosons give \( \det^{-1}(D^i(\bar{L})D_i(\bar{L})) \), where \( D^i(\bar{L}) \) is the \( \bar{L} \)-dependent covariant derivative, defined in (3) and the fermions give \( \det(\bar{D}^i(\frac{\bar{L}}{2})\bar{D}_i(\frac{\bar{L}}{2})) \). For \( \bar{L} = 0 \) the fermions have zero-modes that become the superpartners of coordinates on the moduli space of instantons. For generic \( \bar{L} \neq 0 \), there are no zero modes as we have seen in (3.2). Expanding the expressions for small \( \bar{L} \) should in principle determine the operator insertion.

### 4.3 The partition function of the Spin(8) theory

We will now return to the point that was left open at the beginning of the section, namely the coefficient \( C_{N,k} \) – the ratio between the Spin(8) partition function and its M(atrix)-model’s partition function. The integral over the moduli space of instantons, with the fermions, becomes the integral of the Euler density:

\[
J_{N,k} \equiv \int_{\mathcal{M}_{N,k}} e(\mathcal{M}_{N,k}),
\]

We wish to compare it to the partition function of the Spin(8) theory and find the numerical relation between the two quantities.
The partition function of the \textit{Spin(8)} theory compactified on \(T^3\) is given by:

\[ I_k = \sum_{d|k} \frac{1}{d}, \]

where the sum is over all divisors \(d\) of \(k\).

This can be argued as follows. Recall that the partition function for the D-instanton action (10D \(SU(k)\) Super-Yang-Mills dimensionally reduced to 0D) is given by \(\sum_{d|k} \frac{1}{d^2}\). For \(k = 2\) this was calculated in \([26, 27]\) and it was then derived from the conjectured type-IIB \(R^4\)-coupling \([25]\) (these couplings were proven in \([34]\), and see also \([35]\).) A similar calculation can be performed for the effective \(R^4\) coupling in M-theory compactified on \(T^3\). The result is \([24]\):

\[ \sum_{k=0}^{\infty} q^k I_k = \log \prod_{n=1}^{\infty} \frac{1}{1 - q^n}, \]

from which the expression for \(I_k\) follows.

We will not calculate the integral \(J_{N,k}\) for \(T^3 \times R\) in these notes but we will note the following special limit. If we replaced \(T^3\) with \(R^3\), we could then borrow the known results for the integral over the instanton moduli space on \(R^4\). In \([36]\) the following result is given:

\[ \lim_{N \to \infty} J_{N,k}(R^4) = 2^{3-2k} \pi^{6k-13/2} \sqrt{N} k^{3/2} \sum_{d|k} \frac{1}{d^2}. \]

In order to compare it to the partition function of the \textit{Spin(8)} theory we need to decompactify the \(\hat{T}^3\) that appears on the M(atrix)-theory side. This can be done along the following steps. The \textit{Spin(8)} theory compactified on \(S^1 \times R^2\) with the \(S^1\) of radius \(r\) can be described at low-energies by 2D SYM with \(\mathcal{N} = (8, 8)\) supersymmetry and coupling constant that is proportional to \(r^{-1/2}\). Replacing \(\hat{T}^3\) by \(\hat{T}^2 \times R\) on the M(atrix)-theory side corresponds to replacing the \textit{Spin(8)} theory with 2D SYM compactified on \(T^2\) and replacing \(\hat{T}^3\) by \(R^3\) corresponds to replacing the partition function of the \textit{Spin(8)}-theory on \(T^3\) with the D-instanton integral of 10D SYM reduced to 0D. This was calculated in \([37]\) to give \(\sum_{d|k} \frac{1}{d^2}\) (see also \([26, 27]\)). So the proportionality coefficient in this case would be \(C_{N,k} = 2^{2k-3} \pi^{6k-13/2} N^{-1/2} k^{-1/2}\).

5 Extension to noncommutative spaces

\(\mathcal{N} = 4\) SYM on a noncommutative \(R^{3,1}\) (NCSYM) can be realized in string-theory by turning on a strong NSNS 2-form \(B\)-field on the brane \([22, 38, 39]\). The NCSYM theories are labeled
by an anti-symmetric contravariant 2-tensor (bivector) $\theta^{ij}$. The standard M(atrix)-models for $\mathcal{N} = 4$ SYM can be extended to describe NCSYM, but not all the components of $\theta^{ij}$ can easily be turned on, as we will explain below. On the other hand, all 6 components of $\theta^{ij}$ on $T^4$ can easily be turned on in the new M(atrix)-model.

5.1 Noncommutativity in the standard M(atrix)-models

In [10], the M(atrix)-models for $\mathcal{N} = 4$ SYM were derived from the M(atrix)-models of the $(2,0)$ theory [4, 5]. These M(atrix)-models where a special limit of the moduli space of instantons on $T^2 \times R^2$. In this limit the $T^2$ becomes small and the moduli space can be described as the moduli space of holomorphic curves in $T^2 \times R^2$.

In [7], a “noncommutative” extension of the 5+1D $(2,0)$-theory was suggested. It was parameterized by a 2-form $C_{ij}$ that is anti-self-dual in 4 space-like directions. Its M(atrix)-model was described as Quantum-Mechanics on a certain deformation of $\mathcal{M}_{N,k}(R^4)$. This target space was later identified as the extension of $\mathcal{M}_{N,k}$ to instantons on a noncommutative $R^4$ [13, 12].

Given these constructions, it is not hard to extend the results of [10] to NCSYM as long as 4 out of the 6 components of $\theta^{ij}$ are set to zero. When we discuss the “standard” M(atrix)-model of $\mathcal{N} = 4$ SYM on $R^{3,1}$, we have to pick two light-like directions whose coordinates are denoted by $x_\pm \equiv x_0 \pm x_3$ (see [40, 41] for a review of M(atrix)-theory). Let the coordinates of the other two directions be denoted by $x_1, x_2$. The dual light-like momenta are $p^+ = N/R_\parallel$ and $p^-$ (that equals the Hamiltonian). In this notation only $\theta^{1+}, \theta^{2+}$ can be turned on. The resulting M(atrix)-model is a limit of the moduli space of instantons on a noncommutative $T^2 \times R^2$. The noncommutativity is given by an anti-self-dual 2-tensor with one direction along $T^2$ and the other along $R^2$.

To obtain the M(atrix)-model for NCSYM one has to proceed along the lines of [10] and take the limit of a small $T^2$. In this limit, the commutative moduli space reduces to a certain moduli-space of holomorphic curves inside $T^2 \times R^2$. In the noncommutative case one also obtains a moduli space of curves, but the space $T^2 \times R^2$ has to be deformed in the following way [21] (see also [12] for general properties of instantons on noncommutative tori). Let $w$ be a holomorphic coordinate on $T^2$ and let $z$ be a holomorphic coordinate on $R^2$. In the
commutative case, the periodic identification is \( w \sim w + n + m\tau \), \((n, m \in \mathbb{Z} \text{ and } \tau \text{ is the complex structure of } \mathbb{T}^2)\). In the noncommutative case, the identification is deformed into:

\[
(w, z) \sim (w + n + m\tau, z - \frac{2\pi i}{\tau_2}(\theta^1 + i\theta^2)(n + m\tau)), \quad n, m \in \mathbb{Z}.
\]

Thus, completing a cycle around \( \mathbb{T}^2 \) has to be accompanied by a translation along \( \mathbb{R}^2 \). It is not clear how to turn on other components of \( \theta \) in the M(atrix)-model.

5.2 Noncommutativity in the new M(atrix)-models

In contrast to the discussion in the previous subsection, in the new M(atrix)-models all 6 noncommutativity parameters enter on an equal footing. To turn on noncommutativity we need to turn on a strong NSNS 2-form \( B \)-field along \( \mathbb{T}^4 \) and take a scaling limit in which the size of the \( \mathbb{T}^4 \) shrinks to zero but the \( B \)-field flux remains finite \([39]\).

Alternatively, we can start with the definition of \([38, 22]\) of the NCSYM limit on tori. We therefore start with type-IIB string theory on \( \mathbb{T}^4 \) with radii \( R_1, \ldots, R_4 \) such that (in the notation of subsection \((2.2)\)) \( M_s R_i \to 0 \). We turn on an NSNS 2-form \( B \)-field with a finite flux along \( \mathbb{T}^4 \). We then need to look for an instanton that has the charge of \( N \) D(−1)-branes. According to the arguments of \([38]\), in this limit, the low-energy description is \( \mathcal{N} = 4 \) SYM on the T-dual \( \mathbb{T}^4 \) with noncommutativity that is set by the \( B \)-field fluxes.

To find the M(atrix)-model, we follow the steps described in subsection \((2.2)\) and obtain LST on \( \mathbb{T}^2 \times \hat{T}^3 \). Looking at the formulas \((1)-(4)\) we see that the size of \( \hat{T}^3 \) is finite, in little-string units, but the size of \( \mathbb{T}^2 \) shrinks to zero. The 6 components of the NSNS 2-form \( B \)-field become the components of an external NSNS 2-form \( B \)-field for the LST with one index along \( \mathbb{T}^2 \) and another along \( \hat{T}^3 \). Since the \( \mathbb{T}^2 \) is small we need to perform another T-duality to make it big. The \( B \)-field fluxes become components of the metric with one index along \( \mathbb{T}^2 \) and another along \( \hat{T}^3 \). More precisely, they become Dehn twists in the 1-cycles of the \( \hat{T}^3 \) as we go along 1-cycles of \( \mathbb{T}^2 \). At the end of the duality transformations, the number of D(−1)-branes becomes the instanton number on \( \hat{T}^3 \times \mathbb{R} \), as is clear from the case when no fluxes are turned on.

We therefore end up with a similar \( \sigma \)-model to the one we had in subsection \((2.2)\) except that we do not mod out by the translations along \( \hat{T}^3 \) and we need to introduce boundary
conditions along $T^2$ that correspond to translations along $\hat{T}^3$ as we go around a 1-cycle of $T^2$. These boundary conditions are described by 6 parameters, 3 for each 1-cycle of $T^2$. These parameters are proportional to the 6 noncommutativity parameters along the original $T^4$. The need to retain the translations along $\hat{T}^3$ is probably related to the non-decoupling of the center of $U(N)$ in NCSYM (see [39]).

5.3 S-duality

One of the pleasing features of the “standard” M(atrix)-model of $\mathcal{N} = 4$ SYM is that S-duality is manifest [10]. S-duality can be extended to NCSYM and takes $\theta^{ij}$ to its dual tensor $\epsilon^{ijkl} \theta^{kl}$ [13, 14, 15]. On a Euclidean space the S-dual theory is well-defined, but on $\mathbb{R}^{3,1}$ there are complications related to space-time noncommutativity [13, 14, 15] and more degrees of freedom are required to make the theory consistent [13, 15] (and see also the related discussions in [16–53]).

In the new M(atrix)-models S-duality is also manifest. It is just the $SL(2, \mathbb{Z})$ duality of the base $T^2$ on which the $\sigma$-model is defined. The S-duality that acts on the coupling as $\tau \rightarrow -1/\tau$ exchanges the boundary conditions along the short cycle of $T^2$ with the boundary conditions along the long cycle of $T^2$. Following the steps of (2.2), this can be seen to agree with taking $\theta^{ij}$ to its dual tensor.

6 Summary

We have proposed that $\mathcal{N} = 4$ $SU(k)$ SYM compactified on $T^4$ with coupling constant $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$ has a manifestly S-dual M(atrix)-model given by the large $N$ limit of a $\sigma$-model compactified on $T^2$ with complex structure $\tau$. The target space of the $\sigma$-model is the moduli space $\mathcal{M}_{N,k}$ of $k$ $SU(N)$ instantons on $\hat{T}^3 \times \mathbb{R}$. The parameters (moduli) of the $\sigma$-model are determined from the shape of $T^4$ (see section (2)). The M(atrix)-model for $\mathcal{N} = 4$ $U(k)$ SYM on a noncommutative $T^4$ corresponds to a similar $\sigma$-model but with modified boundary conditions along $T^2$. The modification is that as we go around a 1-cycle of $T^2$ the instanton configuration is shifted by a translation along $\hat{T}^3$.

We have also proposed that the M(atrix)-model of the $Spin(8)$ theory (the CFT associ-
ated with $k$ M2-branes) compactified on $T^3$ is an integral over $\mathcal{M}_{N,k}$.

We conjectured that operator insertions in the field-theory that carry a specific momentum along $T^3$ correspond to Wilson line insertions in the M(atrix)-model where the Wilson line is calculated along a cycle on the dual $\hat{T}^3$ that is related to the momentum. Momentum conservation is achieved in the large $N$ limit due to the $\mathbb{Z}_N$ ambiguity of the Wilson line.

These results were derived directly from M(atrix)-theory but it might be interesting to derive them also from the AdS/CFT correspondence, along the lines of [54, 55, 36]. For example, the $\mathcal{N} = 4$ SYM appears on a Euclidean D3-brane wrapping $T^4$ in $\text{AdS}_3 \times S^3 \times T^4$ and the latter is believed to be dual to the large $N$ limit of a 2D CFT [56, 57, 58]. It would be interesting to study the instantons in that picture.

**Acknowledgments**

I am indebted to Nikita Nekrasov for helpful discussions about Euler numbers of instanton moduli spaces. This research is supported by NSF grant number PHY-9802498.

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