Between the genus and the Γ-genus of an integral quadratic Γ-form

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Abstract
Let Γ be a finite group and \((V, q)\) be a regular quadratic Γ-form defined over an integral domain \(\mathcal{O}_S\) of a global function field (of odd characteristic). We use flat cohomology to classify the quadratic Γ-forms defined over \(\mathcal{O}_S\) that are locally Γ-isomorphic for the flat topology to \((V, q)\) and compare between the genus \(c(q)\) and the Γ-genus \(c_\Gamma(q)\) of \(q\). We show that \(c_\Gamma(q)\) should not inject in \(c(q)\). The suggested obstruction arises from the failure of the Witt cancellation theorem for \(\mathcal{O}_S\).

1 Introduction
Let \(C\) be a geometrically connected and smooth projective curve defined over the finite field \(\mathbb{F}_q\) (\(q\) is odd). Let \(K = \mathbb{F}_q(C)\) be its function field and let \(\Omega\) denote the set of all closed points of \(C\). For any point \(p \in \Omega\) let \(v_p\) be the induced discrete valuation on \(K\), \(\hat{O}_p\) the complete discrete valuation ring with respect to \(v_p\) and \(\hat{K}_p\) its fraction field. Any Hasse set of \(K\) namely, a non-empty finite set \(S \subset \Omega\), gives rise to an integral domain of \(K\) called a Hasse domain:

\[ \mathcal{O}_S := \{x \in K : v_p(x) \geq 0 \ \forall p \notin S\}. \]

Any \(\mathcal{O}_S\)-scheme is underlined, being omitted in the notation of its generic fiber.

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Let $\Gamma$ be a finite group, faithfully represented by $\rho: \Gamma \hookrightarrow GL(V)$ where $V$ is a projective $O_S$-module of rank $n \geq 1$. We briefly write $\gamma$ instead of $\rho(\gamma)$ when no confusion may occur and assume $|\Gamma|$ is prime to $\text{char}(K)$. Then $\Gamma$ acts on $GL(V)$ on the left by conjugation:
\[
\forall \gamma \in \Gamma, A \in GL(V) : \quad \gamma A = \gamma A \gamma^{-1}.
\]

Let $V$ be equipped with a degree two homogeneous $O_S$-form $q: V \rightarrow O_S$, turning $(V, q)$ into an integral quadratic $O_S$-space, represented by a bilinear map $B_q: V \times V \rightarrow O_S$ such that:
\[
B_q(u, v) = q(u + v) - q(u) - q(v).
\]

We assume $(V, q)$ is $O_S$-regular, i.e., that the induced homomorphism $V \rightarrow V^\vee := \text{Hom}(V, O_S)$ is an isomorphism. We say it is rationally isotropic (or just isotropic), if there exists a non-zero $v \in V$ for which $q(v) = 0$. It is considered a $\Gamma$-form if it satisfies $q \circ \gamma = q$ for all $\gamma \in \Gamma$. Two integral forms $(V, q)$ and $(V', q')$ are said to be $R$-isomorphic where $R$ is an extension of $O_S$ if there exists an $R$-isomorphism $A: V' \cong V$ such that $q \circ A = q'$. $A$ is called an $R$-isometry. Two $\Gamma$-forms are said to be $\Gamma$-isomorphic over $R$ if there exists an $R$-isometry between them which is $\Gamma$-equivariant.

The classification of quadratic forms (without a group action) defined over an integral domain of a function field was initially studied by L. Gerstein ([Ger]) and J. S. Hsia ([Hsia]) in the late seventies. Later on, J. Morales proved in [Mor] that there are only finitely many $\Gamma$-isomorphism classes of $\Gamma$-forms defined over $Z$ of a given discriminant. He also showed that the classical Hasse-Minkowski theorem, stating that two forms defined over a global field $K$ are $K$-isomorphic if and only if they are $\hat{K}_v$-isomorphic at any place $v$, does not hold for $\Gamma$-forms with $\Gamma$-isomorphisms when $K = \mathbb{Q}$. Recently, E. Bayer-Fluckiger, N. Bhaskhar and R. Parimala showed in [BBP] that this principle does hold, however, for $\Gamma$-forms when $K$ is a global function field.

The failure of this local-global principle in the case of integral forms (even without a group action) is measured by the genus of such a form. This term and its generalization to integral $\Gamma$-forms are defined as follows:

**Definition 1.1.** The genus $c(q)$ of an $O_S$-form $q$ is the set of isomorphism classes of $O_S$-forms that are $K$ and $\hat{O}_p$-isomorphic to $q$ for any prime $p \notin S$. The $\Gamma$-genus $c_{\Gamma}(q)$ of a $\Gamma$-form $q$ defined over $O_S$ is the set of $\Gamma$-isomorphism
classes of $\Gamma$-forms defined over $\mathcal{O}_S$ that are $K$ and $O_p$-isomorphic to $q$ for any prime $p \notin S$, and these isomorphisms are $\Gamma$-ones.

We denote by $c^+(q)$ and $c^\Gamma_+(q)$ the genus and the $\Gamma$-genus of $q$ respectively, with respect to proper (i.e., of det $= 1$) isomorphisms only.

This paper was motivated by the following question, as was posed to me by B. Kunyavski˘ı:

**Question 1.2.** Suppose two integral $\Gamma$-forms share the same $\Gamma$-genus and they are $\mathcal{O}_S$-isomorphic (the $\Gamma$-action is forgotten). Would they necessarily be also $\Gamma$-isomorphic ?

Any integral $\Gamma$-form representing a class in $c_\Gamma(q)$ clearly represents a class in $c(q)$ as well. So the map $\psi : c_\Gamma(q) \rightarrow c(q)$ is well-defined, and we may rephrase Question 1.2 as follows: Is $\psi$ always an injection ? Because if the answer is no, and only then, this would mean that there exist two integral $\Gamma$-forms representing two distinct classes in $c_\Gamma(q)$, though being $\mathcal{O}_S$-isomorphic.

After showing in Section 2 that $H^1_{fl}(\mathcal{O}_S, \mathbf{SO}_V^\Gamma_\mathbb{K})$ [properly] classifies the integral $\Gamma$-forms that are locally $\Gamma$-isomorphic to $(V, q)$, where $\mathbf{SO}_V^\Gamma_\mathbb{K}$ stands for the [special] orthogonal group of $(V, q)$, we compare in Section 3 between the genus and the $\Gamma$-genus of $(V, q)$, and give a necessary and sufficient condition for the positive answer to Question 1.2. In order to provide a counter-example, we refer in Section 4 more concretely to the case in which $(O_v^\Gamma)^0$ is the special orthogonal group of another isotropic form $(V', q')$. Based on a result established in [Bit1] and [Bit2], stating that if $q$ is isotropic of rank $\geq 3$, then $c(q) \cong \text{Pic } (\mathcal{O}_S)/2$, we show that if $q'$ is isotropic of rank 2 then it may possess twisted forms which are only stably isomorphic, namely, that become isomorphic over (any) regular non-trivial extension of $V'$. So our suggested obstruction to Question 1.2 arises from the failure of the Witt cancellation theorem over $\mathcal{O}_S$.

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2 The classification via flat cohomology

The following general framework that appears in [CF, §2.2.4] allows us to derive some known facts about the classification of integral forms via flat cohomology, to integral Γ-forms.

Proposition 2.1. Let $R$ be a scheme and $X_0$ be an $R$-form, namely, an object of a fibered category of schemes defined over $R$. Let $\text{Aut}_{X_0}$ be its $R$-group of automorphisms. Let $\mathfrak{F}_{\text{orfs}}(X_0)$ be the category of $R$-forms that are locally isomorphic for some topology to $X_0$ and let $\mathfrak{T}_{\text{ors}}(\text{Aut}_{X_0})$ be the category of $\text{Aut}_{X_0}$-torsors in that topology. The functor

$$\varphi : \mathfrak{F}_{\text{orfs}}(X_0) \to \mathfrak{T}_{\text{ors}}(\text{Aut}_{X_0}) : X \mapsto \text{Iso}_{X_0,X}$$

is an equivalence of fibered categories.

We first implement this Proposition on split torsion $R$-groups. For a non-negative integer $m$ we consider the $R$-group $\mu_m := \text{Spec } R[t]/(t^m - 1)$. The pointed-set $H^1_{\text{fl}}(R, \mu_m)$ classifies $\mu_m$-torsors, namely, $R$-groups that are locally isomorphic to $\mu_m$ for the flat topology. We briefly introduce another description of these elements, as can be found for example in [AG, §5.1].

An $m$-degree $R$-Kummer pair is a couple $\Lambda = (L, h)$ consisting of a rank 1 projective $R$-module $L$ and an isomorphism $h : R \cong L^\otimes m$. It gives rise to a $\mu_m$-torsor $E_\Lambda$ assigning to any extension $R'/R$ the group:

$$E_\Lambda(R') = \{ \varphi \in L^\vee \otimes R' : \varphi^\otimes m = h \}$$

where $L^\vee := \text{Hom}(R, L)$.

In particular, for $m = 2$, we set $X_0$ to be the quadratic $R$-algebra $R \oplus R$ with the standard involution $(r_1, r_2) \mapsto (r_1, -r_2)$, thus $\text{Aut}_{X_0} = \mu_2$. Let $L$ be a fractional ideal of order 2 in $\text{Pic } (R)$. Then any 2-degree Kummer pair $\Lambda = (L, h)$ gives rise to an $R$-algebra $X = R \oplus L$ with multiplication defined by $(0, l_1) \cdot (0, l_2) = (h^{-1}(l_1 \otimes l_2), 0)$, viewed as an $R$-form, whence according to Proposition 2.1 $\Lambda$ corresponds to the $\mu_2$-torsor

$$E_\Lambda = \text{Iso}(R \oplus R, R \oplus L),$$

in which the isomorphism induced by $h$ is $\varphi : (r_1, r_2) \mapsto (r_1, l)$ where $l$ is such that $l \otimes l = h(r_2)$, i.e., $\varphi \otimes \varphi = h$. 
The genus of an integral quadratic form

Let \( \mathcal{O}_V \) be the orthogonal group of \((V, q)\), namely, the \( \mathcal{O}_S \)-group assigning to any \( \mathcal{O}_S \)-algebra \( R \) the group of self isometries of \( q \) defined over \( R \):

\[
\mathcal{O}_V(R) = \{ A \in \text{GL}(V \otimes R) : q \circ A = q \}.
\]

The pointed-set \( H^1_{fl}(\mathcal{O}_S, \mathcal{O}_V) \) classifies the integral quadratic forms of rank \( n \) (cf. [Knu IV.5.3.1]), being all locally isomorphic for the flat topology to \((V, q)\). We may generalize this to \( \Gamma \)-forms as follows: Suppose \((V, q)\) is a \( \Gamma \)-form. Then restricting \( \varphi \) in Proposition 2.1 to the \( \mathcal{O}_S \)-group of \( \Gamma \)-automorphisms \( \mathcal{O}_V^\Gamma \) for the flat topology, the corresponding forms are the integral \( \Gamma \)-forms that are locally \( \Gamma \)-isomorphic to \((V, q)\) in the flat topology. Modulo \( \mathcal{O}_S \)-isomorphisms we get \( H^1_{fl}(\mathcal{O}_S, \mathcal{O}_V^\Gamma) \).

As \( 2 \in \mathcal{O}_S^\times \) and \((V, q)\) is \( \mathcal{O}_S \)-regular, \( \mathcal{O}_V \) is smooth and its connected component, the special orthogonal group of \((V, q)\), is \( \text{SO}_V = \ker[\mathcal{O}_V^{\text{det}} \to \mu_2] \), containing only the proper isometries (cf. [Con Thm. 1.7, Cor. 2.5]). The push-forward homomorphism \( \text{det}_*: H^1_{fl}(\mathcal{O}_S, \mathcal{O}_V) \to H^1_{fl}(\mathcal{O}_S, \mu_2) \) induced by flat cohomology, assigns to any quadratic form \((V', q')\) (taken up to an \( \mathcal{O}_S \)-isomorphism) the class of its discriminant module \( D(q') = D(V', q') = (\wedge^n V', \text{det}(q')) \) (see [Knu IV.4.6, 5.3.1]). Let \( \text{SO}_V^\Gamma \) be its \( \Gamma \)-invariant subgroup. Our assumption that \( |\Gamma| \in \mathcal{O}_S^\times \) guarantees that \( \text{SO}_V^\Gamma \) remains smooth (cf. [CGP2 Proposition A.8.10(2)]). Any representative \((V', q')\) of a class in \( H^1_{fl}(\mathcal{O}_S, \text{SO}_V^\Gamma) \) represents a class in \( H^1_{fl}(\mathcal{O}_S, \mathcal{O}_V^\Gamma) \), though \( H^1_{fl}(\mathcal{O}_S, \text{SO}_V^\Gamma) \) should not embed in \( H^1_{fl}(\mathcal{O}_S, \mathcal{O}_V^\Gamma) \) (notice that these pointed-sets do not have to be groups). More precisely, any class in \( H^1_{fl}(\mathcal{O}_S, \text{SO}_V^\Gamma) \) is represented by a triple \((V', q', \theta')\) where \( \theta' \) is the trivialization of \( D(q') \), namely, an isomorphism \( \theta': D(q') \otimes S \cong D(q) \) where \( S \) is some 2-degree flat extension of \( \mathcal{O}_S \). Any proper \( \Gamma \)-isometry \( A : V' \cong V : q \circ A = q' \) induces an isomorphism \( D(A) : D(q') \otimes S \cong D(q) \). The question is whether these additional data \( D(q') \) and \( D(A) \) (when \( q' \) is a \( \Gamma \)-form) are also \( \Gamma \)-equivariant.

In order to answer this question we may consider as mentioned before the \( \mathcal{O}_S \)-module \( D(q') \), being of rank 1 and isomorphic to \( \mathcal{O}_S \) over some at most 2-degree étale extension, as a 2-degree Kummer pair \((L = D(q'), h)\), giving rise to the \( \mu_2 \)-torsor \( E_{q'} \) for which \( E_{q'}(\mathcal{O}_S) = \{ \varphi \in L^\times : \varphi \otimes \varphi = h \} \). Since \( q' \) is a \( \Gamma \)-form, \( \Gamma \) admits the following commutative diagram:

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\mu_2} & \mathcal{O}_V^\Gamma(\mathcal{O}_S) \\
\downarrow & & \downarrow \text{det} \\
& \mu_2(\mathcal{O}_S) & 
\end{array}
\]
from which we see that $\Gamma$ acts on $E'_q(O_S)$ through its determinant in $\mu_2(O_S)$. But $\mu_2(O_S)$ is the automorphism group of $O_S \oplus O_S$ with respect to its standard involution $\tau = (\text{id}, -\text{id})$ and $E'_q(O_S)$ is stable (not fixed point-wise) under $\tau$, as $-l \otimes -l = l \otimes l$. Furthermore, the correspondence $A \leadsto D(A)$ is functorial, thus referring to any $\gamma \in \Gamma$ as to an isometry one has

$$\gamma A \gamma^{-1} = A \iff D(\gamma)D(A)D(\gamma)^{-1} = D(A),$$

i.e. $D(A)$ is $\Gamma$-invariant as well. So Proposition 2.1 is also applied to the proper classification.

**Corollary 2.2.** Given an integral $\Gamma$-form base-point $(V, q)$, the pointed set $H^1_{fl}(O_S, O_S^G) \left[ H^1_{fl}(O_S, SO^G) \right]$ [properly] classifies the integral $\Gamma$-forms that are locally $\Gamma$-isomorphic to $(V, q)$ for the flat topology.

### 3 The [proper] genus and [proper] $\Gamma$-genus

Consider the ring of $S$-integral adèles $A_S := \prod_{p \in S} \hat{K}_p \times \prod_{p \notin S} \hat{O}_p$, being a subring of the adèles $A$. The $S$-class set of an affine and flat $O_S$-group $G$ is the set of double cosets:

$$\text{Cl}_S(G) := G(A_S) \setminus G(A) / G(K)$$

(where for any prime $p$ the geometric fiber $G_p$ of $G$ is taken). According to Nisnevich ([Nis, Theorem I.3.5]) $G$ admits the following exact sequence of pointed sets

$$1 \to \text{Cl}_S(G) \to H^1_{fl}(O_S, G) \to H^1(K, G) \times \prod_{p \notin S} H^1_{fl}(\hat{O}_p, G_p)$$

in which the left exactness reflects the fact that $\text{Cl}_S(G)$ is the genus of $G$, namely the set of (classes of) $G$-torsors that are generically and locally at $p \notin S$ isomorphic to $G$. If $G$ admits the property

$$\forall p \notin S: \quad H^1_{fl}(\hat{O}_p, G_p) \hookrightarrow H^1_{fl}(\hat{K}_p, G_p),$$

then due to Corollary 3.6 in [Nis] the Nisnevich’s sequence (3.1) simplifies to

$$1 \to \text{Cl}_S(G) \to H^1_{fl}(O_S, G) \to H^1(K, G),$$

which indicates that any two $G$-torsors belong to the same genus if and only if they are $K$-isomorphic.
Remark 3.1. Since Spec $\mathcal{O}_S$ is normal, i.e., is integrally closed locally everywhere (due to the smoothness of $C$), any finite étale covering of $\mathcal{O}_S$ arises by its normalization in some separable unramified extension of $K$ (see [Len, Theorem 6.13]). Consequently, if $G$ is a finite $\mathcal{O}_S$-group, then $H^1_{\text{ét}}(\mathcal{O}_S, G)$ is embedded in $H^1(K, G)$.

Lemma 3.2. Let $\varphi : G \to G'$ be a monomorphism of smooth affine $\mathcal{O}_S$-groups and let $\tilde{Q}$ be the sheafification of $Q := \text{coker}(\varphi)$. Then

- The map $G'(\mathcal{O}_S) \to \tilde{Q}(\mathcal{O}_S)$ is surjective iff $\ker[\text{Cl}_S(G) \xrightarrow{\psi} \text{Cl}_S(G')] = 1$.

If, moreover, $G$ is locally of finite presentation, $G$ and $G'$ admit property (3.2), $Q$ is a finite $\mathcal{O}_S$-group and $G'(K) \to Q(K)$ is surjective, then $\psi$ is surjective.

Proof. As a pointed set, $\text{Cl}_S(G)$ is bijective to the first Nisnevich’s cohomology set $H^1_{\text{Nis}}(\mathcal{O}_S, G)$ (cf. [Nis, I. Theorem 2.8]), classifying $G$-torsors for the Nisnevich’s topology. But Nisnevich’s covers are étale, so $\text{Cl}_S(G)$ is a subset of $H^1_{\text{ét}}(\mathcal{O}_S, G)$. The monomorphism $\varphi$ does not have to be a closed immersion hence $\tilde{Q}$ may not be representable, and so we may not be able to apply flat cohomology on the obtained short exact sequence of $\mathcal{O}_S$-schemes. Restricting, however, to the small site of flat extensions of $\mathcal{O}_S$, we have $\tilde{G}(R) \subseteq \tilde{G}'(R)$ for any such extension $R/\mathcal{O}_S$, where $\tilde{G}$ and $\tilde{G}'$ stand for the sheafifications of $G$ and $G'$, respectively. Then flat cohomology applied to the exact sequence of flat sheaves

\[(3.4) \quad 1 \to \tilde{G} \xrightarrow{i} \tilde{G}' \xrightarrow{\psi} \tilde{Q} \to 1\]

yields a long exact sequence in which $H^1_{\text{fl}}(\mathcal{O}_S, \tilde{G}) = H^1_{\text{fl}}(\mathcal{O}_S, *) = H^1_{\text{ét}}(\mathcal{O}_S, *)$ for both smooth groups $* = G'$ and $G$, whence $\tilde{G}'(\mathcal{O}_S) = G'(\mathcal{O}_S) \to \tilde{Q}(\mathcal{O}_S)$ is surjective if and only if $\ker[H^1_{\text{ét}}(\mathcal{O}_S, G) \xrightarrow{\psi} H^1_{\text{ét}}(\mathcal{O}_S, G')] = 1$, being equivalent by restriction to $\text{Cl}_S(G)$, to $\ker[\text{Cl}_S(G) \xrightarrow{\psi} \text{Cl}_S(G')] = 1$, since any twisted form in $H^1_{\text{ét}}(\mathcal{O}_S, G)$ can be $\mathcal{O}_S$-isomorphic to $G$ only if it belongs to $\text{Cl}_S(G)$.

Now suppose $G$ is locally of finite presentation and $Q$ is representable as a finite $\mathcal{O}_S$-group. Then $\tilde{Q}$ is smooth as well (cf. [SGA3, VI, Proposition 9.2 xii]) and so given furthermore that $G$ and $G'$ admit property (3.2) and $G'(K) \to Q(K)$ is surjective, étale cohomology applied to sequence (3.4) over $\mathcal{O}_S$ and over $K$ extends the exactness of sequence (3.3) to the
commutative diagram

\[
\begin{array}{ccc}
\text{Cl}_S(G) & \xrightarrow{\psi} & \text{Cl}_S(G') \\
\downarrow s & & \downarrow s' \\
1 & \xrightarrow{d} & H^1(\mathcal{O}_S, G) \\
\downarrow m & & \downarrow m' \\
1 & \xrightarrow{d} & H^1(K, G') \\
\end{array}
\]

in which \( m'' \) is injective as \( Q \) is finite, by Remark 3.1. We then get the surjectivity of \( \psi \):

\[
x' \in \text{Cl}_S(G') \Rightarrow m''(d'(i'(x'))) = d'(m'(i'(x'))) = 0 \Rightarrow d(i'(x')) = 0
\]
\[
\Rightarrow \exists y \in H^1(\mathcal{O}_S, G) : \psi(y) = i'(x') \Rightarrow m'(\psi(y)) = h(m(y)) = 0
\]
\[
\Rightarrow m(y) = 0 \Rightarrow \exists x \in \text{Cl}_S(G) : \psi(i(x)) = i'(x') \Rightarrow \psi(x) = x'.
\]

We return now to our integral \( \Gamma \)-form \((V, q)\) for which \( \text{Cl}_S(\mathcal{O}_V^\Gamma) = c_{\Gamma}(q) \) and \( \text{Cl}_S((\mathcal{O}_V^\Gamma)^0) = c_{\Gamma}(q) \). Since \( \mathcal{O}_V^\Gamma \) and \( (\mathcal{O}_V^\Gamma)^0 \) are smooth, their flat cohomology sets over \( \text{Spec} \mathcal{O}_S \) coincide with the \( \acute{e}tale \) ones (cf. \( \text{SGA4, VIII Corol-} \)). According to Lemma 3.2, if \( \mathcal{O}_V(\mathcal{O}_S) \rightarrow (\mathcal{O}_V/\mathcal{O}_V^\Gamma)(\mathcal{O}_S) \) is surjective then \( \ker[c_{\Gamma}(q) \xrightarrow{\psi} c(q)] = 1 \), which means that for any \([q'] \in c_{\Gamma}(q)\), if \( q' \) is not \( \Gamma \)-isomorphic to \( q \) then neither is it \( \mathcal{O}_S \)-isomorphic to it. This still does not imply that \( c_{\Gamma}(q) \) injects into \( c(q) \) since both are not necessarily groups, so there may be two classes of forms other than \([q]\), which are distinct in \( c_{\Gamma}(q) \), yet are \( \mathcal{O}_S \)-isomorphic. We summarize by

**Proposition 3.3.** Question \( \text{[1.3]} \) is answered in the affirmative if and only if \( \mathcal{O}_{V'}(\mathcal{O}_S) \rightarrow (\mathcal{O}_{V'}/\mathcal{O}_{V'}^\Gamma)(\mathcal{O}_S) \) is surjective for any \([V', q'] \in c(q)\).

**Lemma 3.4.** The group scheme \( (\mathcal{O}_V^\Gamma)^0 \) is reductive.

**Proof.** The group scheme \( \mathcal{S}O_V \) is reductive as it is smooth, affine and all its fibers are reductive. As mentioned before, its affine subgroup \( \mathcal{S}O_V^\Gamma \) is smooth as well, so we may refer to its neutral component \( (\mathcal{S}O_V^\Gamma)^0 \) as defined in \( \text{SGA3, VIB, Théorème 3.10} \). The reduction \( (\mathcal{S}O_V^\Gamma)_p \) defined over the residue field \( k_p \) at any prime \( p \) is reductive, hence according to \( \text{CGP2, Proposition A.8.12} \) its subgroup \( ((\mathcal{S}O_V^\Gamma))_p = (\mathcal{S}O_V^\Gamma)_p^0 \) remains reductive, so that \( (\mathcal{S}O_V^\Gamma)^0 \) is reductive. The latter being a smooth, open and connected subgroup of \( \mathcal{O}_V^\Gamma \) coincides with \( (\mathcal{O}_V^\Gamma)^0 \), thus it is reductive. \( \square \)
Remark 3.5. The group \( (\mathcal{O}_V^\Gamma)_0 \) admits property (3.2) by Lang’s Theorem (recall that all residue fields are finite). Being reductive over Spec \( \mathcal{O}_S \) (Lemma 3.4), this property holds for \( \mathcal{O}_V^\Gamma \) as well if \( \mathcal{O}_V^\Gamma/(\mathcal{O}_V^\Gamma)_0 \) is representable as a finite \( \mathcal{O}_S \)-group (see the proof of Proposition 3.14 in [CGP1]).

Following sequence (3.3) and Remark 3.5 we then get:

**Corollary 3.6.** \( c_1^+(q) \cong \ker[H_3^\Gamma(\mathcal{O}_S, (\mathcal{O}_V^\Gamma)_0) \to H^1(K, (\mathcal{O}_V^\Gamma)_0)] \).

If \( \mathcal{O}_V^\Gamma/(\mathcal{O}_V^\Gamma)_0 \) is a finite \( \mathcal{O}_S \)-group then \( c_1^T(q) \cong \ker[H_1^\Gamma(\mathcal{O}_S, \mathcal{O}_V^\Gamma) \to H^1(K, \mathcal{O}_V^\Gamma)] \).

**Corollary 3.7.** If \( (V, q) \) of rank \( \geq 3 \) is regular and isotropic then \( c^+(q) = c(q) \).

**Proof.** Any representative \((V', q')\) of a class in \( c(q) \) is \( \mathcal{O}_S \)-regular, thus \( \mathcal{O}_{V'}/\mathcal{O}_{S,V'} \cong \mu_2 \) (cf. [Con] Thm. 1.7), notice that \( \mathbb{Z}/2 := \text{Spec} \mathbb{Z}[t]/(t(t-1)) \) is isomorphic to \( \mu_2 \), as \( 2 \in \mathcal{O}_S^\times \). Moreover, being \( K \)-isomorphic to \( q \) (due to Corollary 3.6), \( q' \) is isotropic as well, therefore as rank\((V) \geq 3 \), the map \( \mathcal{O}_{V'}(\mathcal{O}_S) \to \mu_2(\mathcal{O}_S) = \mu_2(K) = \{ \pm 1 \} \) is surjective (cf. [Bit2] Lemma 4.3).

So setting \( G = \mathcal{O}_{S,V'} \) and \( G' = \mathcal{O}_{V,V'} \) in Lemma 3.2 we get that \( \ker[c_1^+(q') \to c(q')] = 1 \) and \( \psi \) is surjective. This holds as mentioned before to any \([q'] \in c(q)\), which amounts to \( \psi \) being the identity. \( \square \)

**Remark 3.8.** When \( |S| = 1 \), i.e., \( S = \{ \infty \} \) where \( \infty \) is an arbitrary closed point of \( K \), Lemma 3.7 is automatic for any regular \( q \) of rank \( \geq 3 \), since in that case \((V, q)\) must be isotropic (see the proof of [Bit1] Prop. 4.4).

**Lemma 3.9.** Suppose that \( \mathcal{O}_V^\Gamma \cong (\mathcal{O}_V^\Gamma)_0 \rtimes Q \) where \( Q \) is a finite \( \mathcal{O}_S \)-group. Then \( \ker[c_1^+(q) \to c_T(q)] = 1 \) and \( \psi \) is surjective.

**Proof.** As \( Q \) embeds as a semi-direct factor in \( \mathcal{O}_V^\Gamma \), \( \mathcal{O}_V^\Gamma(\mathcal{O}_S) \) surjects onto \( Q(\mathcal{O}_S) \) and \( \mathcal{O}_V^\Gamma(K) \) onto \( Q(K) \). Given furthermore that \( Q \) is a finite \( \mathcal{O}_S \)-group, both \( (\mathcal{O}_V^\Gamma)_0 \) and \( \mathcal{O}_V^\Gamma \) admit property (3.2) (see Remark 3.5), so all conditions in Lemma 3.2 are satisfied and the assertion follows. \( \square \)

**Lemma 3.10.** If rank\((V) = 2 \) then \( c_T(q) \subseteq c(q) \), i.e., Question 1.2 is then answered positively.

**Proof.** If rank\((V) = 2 \) then \( \mathcal{O}_{S,V} \) is a one dimensional torus. We first claim that \( \mathcal{O}_V^\Gamma \) cannot be \( \mathcal{O}_{S,V} \). Indeed, since \( q \) is a \( \Gamma \)-form, \( \Gamma \) embeds by definition in \( \mathcal{O}_V^\Gamma(\mathcal{O}_S) \). As \( (\mathcal{O}_V^\Gamma/\mathcal{O}_{S,V})(\mathcal{O}_S) = \mu_2(\mathcal{O}_S) \) does not commute with \( \mathcal{O}_V^\Gamma(\mathcal{O}_S) \), \( \Gamma \) cannot have a non-trivial image in it and still stabilizing \( \mathcal{O}_{S,V} \), i.e., it must embed in \( \mathcal{O}_{S,V}(\mathcal{O}_S) \) only. But then being finite, the \( \Gamma \)-image must
be the group \( \{ \pm I_2 \} \), which does not kill \( \mu_2(\mathcal{O}_S) \). So either \( \mathcal{O}_V^\Gamma = \mathcal{O}_V \) for which the assertion is trivial, or \( \mathcal{O}_V^\Gamma \) is a finite group, for which \( c_{\Gamma}(q) \cong \ker[H^1_{\text{et}}(\mathcal{O}_S, \mathcal{O}_V^\Gamma) \to H^1(K, \mathcal{O}_V^\Gamma)] \) is trivial according to Remark 3.1. □

As mentioned in the proof of Lemma 3.10, the inequality \( c_{\Gamma}(q) \subsetneq c(q) \) for \( \text{rank}(V) = 2 \) may occur only when \( \mathcal{O}_V^\Gamma \) is finite. For example, suppose \( \mathcal{O}_V^\Gamma = \mu_m \). Then étale cohomology applied to the related Kummer exact sequence of smooth \( \mathcal{O}_S \)-groups

\[
1 \to \mu_m \to \mathbb{G}_m \to \mathbb{G}_m \to 1
\]

yields the exactness of

\[
1 \to \mathcal{O}_S^\times/(\mathcal{O}_S^\times)^m \to H^1_{\text{et}}(\mathcal{O}_S, \mu_m) \to m\text{Pic}(\mathcal{O}_S) \to 1
\]

where the right non-trivial term stands for the \( m \)-torsion part of \( \text{Pic}(\mathcal{O}_S) \). According to Remark 3.1 since \( \text{Spec} \mathcal{O}_S \) is normal, both \( \mathcal{O}_S^\times/(\mathcal{O}_S^\times)^m \) and the preimage of \( m\text{Pic}(\mathcal{O}_S) \) in \( H^1_{\text{et}}(\mathcal{O}_S, \mu_m) \) are embedded in \( K^\times/(K^\times)^m \cong H^1(K, \mu_m) \). The following example demonstrates this embedding, which yields the above inequality.

**Example 3.11.** Consider the elliptic curve \( C = \{ Y^2 Z = X^3 + X Z^2 \} \) defined over \( \mathbb{F}_{11} \). Removing the closed point \( \infty = (0 : 1 : 0) \) results in an affine curve with coordinate ring:

\[
C^\text{af} = \{ y^2 = x^3 + x \}
\]

and: \( \mathcal{O}_S = \mathbb{F}_{11}[C^\text{af}] \).

Let \( V = \mathcal{O}_S^2 \) be generated by the standard basis over \( \mathcal{O}_S \) endowed with the form \( q \) represented by \( B_q = 1/2 \). Then

\[
\text{SO}_V = \text{SO}_S = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} : x^2 + y^2 = 1 \right\}
\]

is a one dimensional \( \mathcal{O}_S \)-torus and as \(-1 \) is not a square, it is isomorphic to the non-split norm torus \( N := R_{\mathcal{O}_S(i)/\mathcal{O}_S}(\mathbb{G}_m) \), fitting into the exact sequence of \( \mathcal{O}_S \)-tori:

\[
1 \to N \to R := R_{\mathcal{O}_S(i)/\mathcal{O}_S}(\mathbb{G}_m) \xrightarrow{\text{det}} \mathbb{G}_m \to 1.
\]

Then since \( R(\mathcal{O}_S) \xrightarrow{\text{det}} \mathcal{O}_S^\times = \mathbb{F}_{11}^\times \) is surjective (\( x^2 + y^2 \) gets any value in \( \mathbb{F}_{11}^\times \)), étale cohomology together with the Shapiro’s Lemma gives rise to the exact sequence:

\[
1 \to H^1_{\text{et}}(\mathcal{O}_S, N) \to \text{Pic}(\mathcal{O}_S(i)) \to \text{Pic}(\mathcal{O}_S)
\]
from which we see that \( c^+(q) = \text{Cl}(\mathbf{SO}_V) = H^1_\text{et}(\mathcal{O}_S, \mathbf{SO}_V) \cong \mathbb{N} \) (cf. [Bit1, Prop. 4.2]) is far from being trivial (say, by the Hasse-Weil bound: \(|\text{Pic}(\mathcal{O}_S)| = |\text{C}^{\text{af}}(\mathbb{F}_{11})| < 11 + 1 + 2\sqrt{11} < 19\) while: \(|\text{Pic}(\mathcal{O}_S(i))| = |\text{C}^{\text{af}}(\mathbb{F}_{11}(i))| > 121 + 1 - 2\sqrt{121} = 100\), see in Example 4.2). As \( \mathcal{O}_V/\mathbf{SO}_V \cong \mu_2 \) and \( \mathcal{O}_V(i) \) is surjective by \( \text{diag}(1,-1) \mapsto -1 \), setting \( G = \mathbf{SO}_V \) and \( G' = \mathcal{O}_V \) in Lemma 3.2 we get that \( c(q) = c^+(q) \), thus is not trivial as well.

Now let \( \Gamma = S_3 = \langle \tau,\sigma \rangle \) be represented in \( V \) by:

\[
\tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma \mapsto \begin{pmatrix} 5 & -8 \\ 8 & 5 \end{pmatrix}.
\]

One can easily check that \( q \) is a \( \Gamma \)-form and that

\[
\mathbf{SO}_V^\Gamma = \left\{ \left( \begin{array}{cc} x & 0 \\ 0 & x \end{array} \right) : x^2 = 1 \right\} \cong \mu_2.
\]

For \( L = \langle x, y \rangle \in \text{Pic}(\mathcal{O}_S) \) one has:

\[
L \otimes L = \langle x^2, xy, y^2 \rangle = \langle x^2, xy, x^3 + x \rangle \subseteq \langle x \rangle.
\]

But \( x = y^2 - x^3 \in L \otimes L, \) thus \( L \otimes L = \langle x \rangle. \) So the 2-degree Kummer pair \( (L,h) \) gives rise to the \( \mu_2 \)-torsor \( \mathcal{O}_S \oplus L \) being isomorphic to \( \mathcal{O}_S^2 \) over \( \mathcal{O}_S[1/\sqrt{x}] \) which is not contained in \( K \). The same happens for the other \( \mu_2 \)-torsors, i.e., \( c_T(q) \cong \ker[H^1_\text{et}(\mathcal{O}_S, \mu_2) \to H^1(K, \mu_2)] = 1 \subset c(q).\)

### 4 An explicit obstruction

The criterion exhibited in Proposition 3.3 for Question 1.2 to be answered in the affirmative, namely, \( \mathcal{O}_{V',}(\mathcal{O}_S) \to (\mathcal{O}_{V'},/\mathbf{SO}_{V'})/(\mathcal{O}_S) \) is surjective for any \( [(V', q')] \in c(q) \), is somewhat vague. We would like to refer to the case in which \( (\mathbf{O}_V^\Gamma)^0 \) is the special orthogonal group of another isotropic \( \Gamma \)-form. It is shown in [Bit1] Proposition 4.4 for \( |S| = 1 \) and more generally in [Bit2] Theorem 4.6 for any finite \( S \), that if \( \text{rank}(V) \geq 3 \) (\( q \) is isotropic), then \( c(q) \cong \text{Pic}(\mathcal{O}_S)/2 \). For \( \text{rank}(V) = 2 \), however, this genus might be larger. This means that there may be two integral forms of rank 2 that are only *stably isomorphic*, i.e., become isomorphic after being extended by any non-trivial regular common extension. This failure of the Witt Cancellation Theorem over \( \mathcal{O}_S \), invokes a case in which Question 1.2 is answered negatively, namely, when \( \text{rank}(V) \geq 3 \) and \( (\mathbf{O}_V^\Gamma)^0 \) is the special orthogonal group of another integral \( \Gamma \)-form \( (V', q') \) of rank 2, whose genus decreases over \( V \).
Proposition 4.1. Let \((V, q)\) be a regular \(\Gamma\)-form of rank \(\geq 3\) such that \((\mathcal{O}_V^\Gamma)^0\) is the special orthogonal group of an isotropic form of rank 2, being a semi-direct factor in \(\mathcal{O}_V^\Gamma\), while the quotient is a finite \(\mathcal{O}_S\)-group. If \(-1 \in (\mathbb{F}_q^\times)^2\) and \(\text{exp}(\text{Pic}(\mathcal{O}_S)) > 2\) then \(c_\Gamma(q)\) does not inject into \(c(q)\), i.e., Question \ref{question} is then answered negatively.

Proof. Given that \(\text{rank}(V') = 2\) and \(-1 \in (\mathbb{F}_q^\times)^2\), \(\mathcal{O}_{V'}^0 \cong \mathbb{G}_m\), so one has:

\[
c_\Gamma^+(q) = c_\Gamma^+(q') \cong \ker[H^1_{\text{et}}(\mathcal{O}_S, \mathbb{G}_m) \to H^1(K, \mathbb{G}_m)].
\]

This kernel is isomorphic due to Shapiro’s Lemma and Hilbert’s 90 Theorem to Pic \((\mathcal{O}_S)\). Since \((\mathcal{O}_V^\Gamma)^0\) is a normal semi-direct factor in \(\mathcal{O}_V^\Gamma\) and the quotient is a finite \(\mathcal{O}_S\)-group, by Lemma \ref{lemma} \(\ker[c_\Gamma^+(q) \to c_\Gamma(q)] = 1\), so \(\ker[c_\Gamma(q) \to c(q)]\) cannot vanish, because if it would, the composition, being a morphism of abelian groups

\[
c_\Gamma^+(q) \cong \text{Pic}(\mathcal{O}_S) \to c(q) \cong \text{Pic}(\mathcal{O}_S)/2
\]

would be injective, which is impossible whenever \(\text{exp}(\text{Pic}(\mathcal{O}_S)) > 2\). \(\square\)

Example 4.2. Let \(C\) be an elliptic \(\mathbb{F}_q\)-curve such that \(-1 \in (\mathbb{F}_q^\times)^2\) and \(\text{exp}(C(\mathbb{F}_q)) > 2\). Let \(\infty\) be an \(\mathbb{F}_q\)-rational point. Then \(\text{Pic}(\mathcal{O}_S) \cong C(\mathbb{F}_q)\) (cf. e.g., [Bit1, Example 4.8]). Let \((V, q)\) be the quadratic space generated by the standard basis and represented by \(B_q = 1\) for \(n \geq 4\). Then as mentioned before, \(c(q) \cong \text{Pic}(\mathcal{O}_S)/2 \cong C(\mathbb{F}_q)/2\). Let the permutations in \(\Gamma = S_{n-2}\) be canonically represented by monomial matrices in the lower right \((n-2) \times (n-2)\) block of \(\mathcal{O}_V \subset \text{GL}(V)\):

\[
\Gamma \hookrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & S_{n-2} \end{pmatrix}
\]

turning \(q\) into a \(\Gamma\)-form. Now if \(n\) is even, then \(\text{SO}_V^\Gamma \cong \mathbb{G}_m \times \{\pm I_{n-2}\}\). Otherwise, if \(n\) is odd, then \(\text{SO}_V^\Gamma\) is a semi-direct product of \(\mathbb{G}_m \times \{I_{n-2}\} \cong \mathbb{G}_m\) and \(\text{diag}(1, -1) \times \{-I_{n-2}\} \cong \mu_2\). In both cases, \((\mathcal{O}_V^\Gamma)^0 = (\text{SO}_V^\Gamma)^0 \cong \mathbb{G}_m\) is a normal semi-direct factor in \(\mathcal{O}_V^\Gamma\) and the quotient is a finite \(\mathcal{O}_S\)-group, therefore according to Proposition 4.1 \(c_\Gamma(q)\) cannot inject into \(c(q)\).

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