Generic nonadditivity of quantum capacity in simple channels

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Determining capacities of quantum channels is a fundamental question in quantum information theory. Despite having rigorous coding theorems quantifying the flow of information across quantum channels, their capacities are poorly understood due to super-additivity effects. Studying these phenomena is important for deepening our understanding of quantum information, yet simple and clean examples of super-additive channels are scarce. Here we study a family of channels called platypus channels. Its simplest member, a qutrit channel, is shown to display super-additivity of coherent information when used jointly with a variety of qubit channels. Higher-dimensional family members display super-additivity of quantum capacity together with an erasure channel. Subject to the “spin-alignment conjecture” introduced in the companion paper [1], our results on super-additivity of quantum capacity extend to lower-dimensional channels as well as larger parameter ranges. In particular, super-additivity occurs between two weakly additive channels each with large capacity on their own, in stark contrast to previous results. Remarkably, a single, novel transmission strategy achieves super-additivity in all examples. Our results show that super-additivity is much more prevalent than previously thought. It can occur across a wide variety of channels, even when both participating channels have large quantum capacity.

Introduction. A central aim of quantum information theory is to find out how much information a noisy quantum channel can transmit reliably—to find a quantum channel’s capacity [2, 3]. In fact, a quantum channel has many capacities, depending on what sorts of information are to be transmitted and what additional resources are on hand. The primary capacities of a quantum channel are the classical [4–6], private [7–9], and quantum capacities [9–14]. This paper focuses on unassisted capacities, when no additional resources (such as free entanglement) are available.

The theory of quantum capacities is far richer and more complex than the corresponding classical theory [15, 16]. This richness includes many synergies and surprises: super-additivity of coherent information [17–31], private information [32–34], and Holevo information [35], super-activation of quantum capacity [36–40], and private communication at a rate above the quantum capacity [41, 42]. Over the past two decades, there have been numerous exciting discoveries about these phenomena, but they remain mysterious. As a result, we don’t have a theory of how to best communicate with quantum channels, and can’t answer many of the sorts of questions classical information theory does. For example, in quantum information theory random codes can be suboptimal, and we can only evaluate capacities in special cases [43–51]. Our understanding of error correction in the quantum setting is thus incomplete, whether the data is classical, private, or quantum.

Any quantum channel \( B \) can be expressed as an isometry \( J : A \rightarrow BE \) followed by a partial trace over the environment \( E : B(\rho) = \text{Tr}_E(J\rho J^\dagger) \). Physically, it means that quantum noise arises from sharing the unclonable quantum data with the environment which is subsequently lost (i.e., traced out). Therefore, to understand quantum transmission we must also consider the environment’s view of the channel, known as the complementary channel: \( B^c(\rho) = \text{Tr}_B(JpJ^\dagger) \). Together, the channel and its complement allow us to define the coherent information of a channel \( B \) on an input state \( \rho \) as \( \Delta(B, \rho) := S(B(\rho)) - S(B^c(\rho)) \), where \( S(\sigma) = -\text{tr}(\sigma \log \sigma) \) is the von Neumann entropy of \( \sigma \). Mathematically, the coherent information signifies how much more information about the input is available in system \( B \) than in system \( E \). Operationally, a random coding argument shows that indeed, for any input state \( \rho \), the quantity \( \Delta(B, \rho) \) is an achievable rate for quantum transmission [9, 11–14]. Maximizing over all inputs \( \rho \) gives the channel coherent information \( Q^{(1)}(B) \).

If the channel coherent information is additive, that is, \( Q^{(1)}(B_1 \otimes B_2) = Q^{(1)}(B_1) + Q^{(1)}(B_2) \) for any two channels \( B_1 \) and \( B_2 \), then the theory of quantum capacity will resemble its classical analogue. However, a rich theory
of quantum capacity originates from two distinct notions of nonadditivity: violations of weak additivity and violations of strong additivity.

We first discuss violations of weak additivity. The quantum capacity can be expressed as [9, 11–14, 52]

\[ Q(B) = \lim_{n \to \infty} \frac{1}{n} Q^{(1)}(B^\otimes n), \]  

(1)

where \( B^\otimes n \) is the \( n \)-fold tensor product of \( B \). If \( Q^{(1)}(B^\otimes n) = n Q^{(1)}(B) \) for all \( n \in \mathbb{N} \), we say that \( B \) has weakly additive coherent information, in which case \( Q(B) = Q^{(1)}(B) \). However, there are channels \( B \) for which \( Q^{(1)}(B^\otimes n) > n Q^{(1)}(B) \) holds for some \( n \) [17–21, 23–26, 28–30]. Thus, the \( n \to \infty \) limit is in general required in the above regularized expression for the quantum capacity. When a channel does not have weakly additive coherent information, special quantum codes can outperform the classical-inspired random coding strategy achieved by \( Q^{(1)} \). This unbounded optimization also means that we can rarely determine the quantum capacity of a quantum channel.

The second notion of nonadditivity, violations of strong additivity, can be phrased as follows. For two channels \( B_1 \) and \( B_2 \), we have the general inequality

\[ Q^{(1)}(B_1 \otimes B_2) \geq Q^{(1)}(B_1) + Q^{(1)}(B_2). \]  

(2)

Letting \( B_1 \) be a fixed channel, if equality in (2) holds for all channels \( B_2 \), we say that \( B_1 \) has strongly additive coherent information. In this case, the quantum capacity satisfies \( Q(B_1 \otimes B_2) = Q(B_1) + Q(B_2) \). Note that strong additivity implies weak additivity. Violations of strong additivity imply that two different channels can have strictly superadditive coherent information, or even capacity. As a result, not only do we not know the capacity of most quantum channels, we also do not know when two channels used jointly can have capacity exceeding the sum of the individual channels. A more general notion of a channel’s capability to transmit quantum data thus depends on the details of other resources available [36, 53, 54], and does not necessarily coincide with its capacity; a drastic deviation from the classical theory.

Similar to the quantum capacity, a channel’s private and classical capacities can be defined as the highest rates of faithful transmission of private and classical information, respectively; expressions analogous to (1) are known [5, 6, 8, 9]. Both capacities require regularized expressions [35, 55], and the private capacity can be shown to be non-additive for some channels [32, 56].

For classical capacity, the underlying information quantity is the Holevo information, which was conjectured to be additive for a long time. In fact, strong additivity was proved for certain channels such as entanglement-breaking [44], depolarizing [46], Hadamard [48, 49], and unital qubit channels [45]. As a result, for these channels the classical capacity completely characterizes their ability to faithfully send classical information. Furthermore, the only known proofs of violation of weak additivity of the Holevo information [35, 57, 58] are based on random channel constructions and no explicit example has been found yet [3, 35]. It is still open if the classical capacity can be non-additive. It is furthermore unclear if additivity is more prevalent for classical data transmission, or if proofs are simply harder to come by since the Holevo information involves a more complex optimization compared to coherent information.

The situation for quantum information transmission is quite different. There is a plethora of concrete channels with super-additive coherent information [17–20, 23–30]. The only known class of channels with strongly additive coherent information are the entanglement-breaking channels, but they are somewhat trivial – their quantum capacity is zero. Degradable channels [47, 59] have weakly additive coherent information, and two degradable channels have additive coherent information, yet surprisingly additivity does not imply strong additivity for a channel. Even weakly additive channels like some (anti-)degradable [47] and PPT channels [60] may have super-additive quantum capacity in combination with suitable channels [30, 39, 56]. A common feature in these violations of strong additivity is that one or both of the channels are manifestly noisy, that is, with vanishing or small quantum capacity. Most of these proofs come from a qualitative inability for the channels to transmit quantum data; in addition, nearly noiseless channels are indeed limited in their non-additivity [61].

In this paper, we provide qualitatively new examples of super-additivity of quantum capacity. The phenomenon seems prevalent, does not involve channels engineered to exhibit the effect, and can involve pairs of channels with large quantum capacity. Our findings show an even more complex landscape of non-additivity than hitherto appreciated. Yet, our channels and the proofs are simple, and thus we hope they improve our understanding of the subject.

Main results. Our first main result is that a simple qutrit ‘platypus channel’, defined via eq. (3) below, violates strong additivity of coherent information when used together with a variety of simple and well-known qubit channels such as the erasure, amplitude damping, depolarizing, and even randomly constructed qubit channels. Even more remarkably, the same simple code achieves non-additivity in all cases. Our findings strongly suggest that super-additivity is much more prevalent and generic than previously thought.

Second, as proved in our companion paper [1], platypus channels have weakly additive coherent information if the spin alignment conjecture introduced in [1] holds. As the erasure channel and the amplitude damping channel also have weakly additive coherent information, we have an example of non-additivity of quantum capacity between two weakly additive channels. The only known prior example revolves around superactivation [36], and requires substantial fine-tuning to demonstrate the effect. In contrast, our channel requires no such tuning, and both channels exhibit non-additivity over a wide range of
parameters, including regimes where both channels have substantial capacity themselves.

Third, we show that higher-dimensional platypus channels have similar non-additive behavior. In particular, when used jointly with a higher-dimensional erasure channel, it exhibits super-additivity of quantum capacity unconditionally, i.e., without relying on the spin alignment conjecture. The underlying mechanism at work achieving all of these non-additivity results is qualitatively different from previous results in [36, 39, 56], as explained in the Discussion section.

In the following paragraphs we discuss our main results; see the Supplementary information for additional details. MATLAB and Python code used to obtain the numerical results mentioned above will be made available at [62].

The qutrit platypus channel. The qutrit ‘platypus channel’ \( N_s \) is defined by the following isometry \( F_s : \mathcal{H}_a \to \mathcal{H}_b \otimes \mathcal{H}_c \):

\[
\begin{align*}
F_s[0] &= \sqrt{s}|0\rangle \otimes |0\rangle + \sqrt{1-s}|1\rangle \otimes |1\rangle \\
F_s[1] &= |2\rangle \otimes |0\rangle \\
F_s[2] &= |2\rangle \otimes |1\rangle ,
\end{align*}
\]

where \( 0 \leq s \leq 1/2 \), and the input \( \mathcal{H}_a \), output \( \mathcal{H}_b \) and environment \( \mathcal{H}_c \) have dimension 3, 3, and 2, respectively. This channel [28, 63] is extensively studied in the companion paper [1]. From [1, 28], the channel coherent information is always positive and can be attained on inputs of the form \( \sigma(u) := (1-u)|0\rangle\langle 0| + u|2\rangle\langle 2| \):

\[
Q^{(1)}(N_s) = \max_{u \in [0,1]} \Delta(N_s, \sigma(u)) > 0.
\]

Conditioned on the spin-alignment conjecture (SAC) formulated in [1], the channel coherent information \( Q^{(1)}(N_s) \) can be proved to be weakly additive, and thus \( Q(N_s) = Q^{(1)}(N_s) \). Without the SAC, we have the upper bound \( Q(N_s) \leq \log(1 + \sqrt{1-s}) \).

Violation of strong additivity. We find that \( N_s \) displays super-additivity in the strong sense,

\[
Q^{(1)}(N_s \otimes \mathcal{K}) > Q^{(1)}(N_s) + Q^{(1)}(\mathcal{K}),
\]

when used with just about any small channel \( \mathcal{K} \). Since \( Q^{(1)}(N_s) > 0 \), the additional channel \( \mathcal{K} \) is said to amplify \( Q^{(1)}(N_s) \). We consider various well-known and physically relevant channels \( \mathcal{K} \), such as the qubit erasure channel, \( \mathcal{E}_\lambda(\rho) = (1-\lambda)\rho + \lambda |e\rangle\langle e| \) with erasure probability \( \lambda \in [0,1] \), the qubit amplitude damping channel, \( \mathcal{A}_\gamma(\rho) = N_0\rho N_0^\dagger + N_1\rho N_1^\dagger \) with damping probability \( \gamma \in [0,1] \), and Kraus effects \( N_0 = |0\rangle \langle 0| + \sqrt{1-\gamma}|1\rangle \langle 1| \) and \( N_1 = \sqrt{\gamma}|0\rangle \langle 1| \), and the qubit depolarizing channel, \( \mathcal{D}_p(\rho) = (1-4p/3)\rho + 2p/3I \) with depolarizing parameter \( p \in [0,1] \). For erasure and amplitude damping channels the quantum capacity equals the channel coherent information [43, 47, 64]. The amplification in (4) not only occurs when each of the channels \( \mathcal{E}_\lambda, \mathcal{A}_\gamma, \) and \( \mathcal{D}_p \) has zero coherent information (see Fig. 1), but it persists for a wide range of channel parameters \( 0 \leq s \leq 1/2 \), \( \lambda_{\min} \leq \lambda \leq \lambda_{\max} \), \( \gamma_{\min} \leq \gamma \leq \gamma_{\max} \), and \( p_{\min} \leq p \leq p_{\max} \) (see Supplementary material).

Remarkably, the amplification of \( Q^{(1)}(N_s) \) by all three channels \( \mathcal{E}_\lambda, \mathcal{A}_\gamma, \) and \( \mathcal{D}_p \) can be achieved by a single input state ansatz for \( N_s \otimes \mathcal{K} \),

\[
\rho(\epsilon, r_1, r_2) = r_1|00\rangle \langle 00| + r_2|01\rangle \langle 01| + (1-r_1-r_2)|\chi_\epsilon\rangle \langle \chi_\epsilon|,
\]

where \( |\chi_\epsilon\rangle = \sqrt{1-\epsilon}|20\rangle + \sqrt{\epsilon}|11\rangle \), and the parameters satisfy the constraints \( \epsilon, r_1, r_2, r_1+r_2 \in [0,1] \). In more detail, we find that \( \Delta^*(N_s \otimes \mathcal{K}_x) := \max_{\epsilon, r_1, r_2} \Delta(N_s \otimes \mathcal{K}_x, \rho(\epsilon, r_1, r_2)) \) exceeds \( Q^{(1)}(N_s) + Q^{(1)}(\mathcal{K}_x) \) where \( \mathcal{K}_x \) is one of \( \mathcal{E}_\lambda, \mathcal{A}_\gamma, \) or \( \mathcal{D}_p \). Since all three channels \( \mathcal{E}_\lambda, \mathcal{A}_\gamma, \) and \( \mathcal{D}_p \) have well known symmetries, one may suspect that the amplification strategy (5) coincides because of these symmetries. We find this not to be the case. Numerics reveal that amplification of \( Q^{(1)}(N_{1/2}) \) using (5) occurs even when \( \mathcal{K} \) is defined in terms of a random qubit channel. Super-additivity occurs both when \( Q^{(1)}(\mathcal{K}) > 0 \) or when the coherent information of \( \mathcal{K} \) itself vanishes.

Unconditional super-additivity of quantum capacity. In the previous section we showed super-additivity of the coherent information of \( N_s \) when used in parallel with other channels such as \( \mathcal{E}_\lambda \) or \( \mathcal{A}_\gamma \). In parallel with the latter channels are known to satisfy \( Q(\mathcal{E}_\lambda) = Q^{(1)}(\mathcal{E}_\lambda) \) and \( Q(\mathcal{A}_\gamma) = Q^{(1)}(\mathcal{A}_\gamma) \).

FIG. 1. Amplification of coherent information for the channel \( N_s \) and various additional channels. We plot \( Q^{(1)}(N_s \otimes \mathcal{K}) - Q^{(1)}(N_s) \) for \( \mathcal{K} = \mathcal{E}_{1/2} \) (solid magenta), \( \mathcal{K} = \mathcal{A}_{1/2} \) (solid blue), and \( \mathcal{K} = \mathcal{D}_{p^*} \) (solid green). Here, \( \mathcal{E}_{1/2} \) and \( \mathcal{A}_{1/2} \) are the symmetric erasure and amplitude damping channels respectively, \( \mathcal{D}_{p^*} \) is the qubit depolarizing channel with \( p^* \approx 0.1893 \), so that all three channels have zero coherent information \( Q^{(1)}(\mathcal{K}) = 0 \). We also plot \( R_s(N_s) - Q^{(1)}(N_s) \) (dashed orange), where \( R_s(\cdot) \) with \( \alpha = 1+2^{-3} \) is the upper bound (UB) on the quantum capacity \( Q(\cdot) \) derived in [65].
\( Q^{(1)}(\mathcal{N}_s) \). Moreover, conditioned on the spin alignment conjecture (SAC) \([1]\), we also have \( Q^{(1)}(\mathcal{N}_s) = Q(\mathcal{N}_s) \). Hence, the super-additivity of \( Q^{(1)} \) in \((4)\) can be elevated to super-additivity of the quantum capacity \( Q \), provided the SAC is true.

We now show that, remarkably, this result can be strengthened to an unconditional super-additivity of quantum capacity. To this end, we consider a channel \( \mathcal{M}_d \) introduced in \([1]\) that generalizes \( N_{1/2} \) to \( d \) input and output dimensions, and \( d-1 \) environment dimensions, with \( d \geq 3 \). The isometry \( G : \mathcal{H}_a \to \mathcal{H}_b \otimes \mathcal{H}_c \) acts on an orthonormal input basis \( \{|i\rangle\}_{k=0}^{d-1} \) as

\[
G(0) = \frac{1}{\sqrt{d-1}} \sum_{j=0}^{d-2} |j \rangle \otimes |j \rangle,
\]

\[
G|i\rangle = |d-1 \rangle \otimes |i \rangle \quad \text{for } i = 1, \ldots, d-1,
\]

and defines the channel \( M_d(.) := \text{tr}_c(G \cdot G^\dagger) \).

Comparing \((6)\) to the isometry \((3)\) for \( N_{1/2} \), we see that \( M_3 = N_{1/2} \), and hence \( M_d \) is indeed a \( d \)-dimensional generalization of \( N_{1/2} \). The coherent information \( Q^{(1)}(\mathcal{M}_d) \) is evaluated in \([1]\), and similar to \( N_{1/2} \) we have \( Q(\mathcal{M}_d) = Q^{(1)}(\mathcal{M}_d) \) modulo (a generalized version of) the spin alignment conjecture. However, we do not make use of this (conjectured) identity here and instead use the following upper bound on the quantum capacity of \( M_d \) derived in \([1]\):

\[
Q(\mathcal{M}_d) \leq \log \left( 1 + \frac{1}{\sqrt{d-1}} \right) \leq \frac{1}{\ln 2} \frac{1}{\sqrt{d-1}}. \tag{7}
\]

This upper bound follows from evaluating the “transposition bound” on the quantum capacity of a quantum channel \([66]\). It is phrased in terms of the diamond norm and can be evaluated using semidefinite programming techniques.

The quantum capacity of \( \mathcal{M}_{d+1} \) is super-additive when used together with the \( d \)-dimensional erasure channel \( \mathcal{E}_{\lambda,d} \) for \( \lambda \in [0,1] \). More precisely, we show that

\[
Q(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}) > Q(\mathcal{M}_{d+1}) + Q(\mathcal{E}_{\lambda,d}) \tag{8}
\]

for suitable \( \lambda \) and \( d \) in two steps: First, using the upper bound \((7)\) on \( Q(\mathcal{M}_d) \) and the fact that the quantum capacity of \( \mathcal{E}_{\lambda,d} \) is given by \( Q(\mathcal{E}_{\lambda,d}) = \max\{1-2\lambda \log d, 0\} \) \([43]\), we obtain an upper bound

\[
u(\lambda, d) := \log \left( 1 + 1/\sqrt{d} \right) + \max\{1-2\lambda \log d, 0\} \tag{9}
\]

on the right-hand side of \((8)\). Second, letting \( \mathcal{H}_a \) and \( \mathcal{H}_{aa'} \) be the input Hilbert spaces for \( \mathcal{M}_{d+1} \) and \( \mathcal{E}_{\lambda,d} \), respectively, we find an input state \( \rho_{aa'} \) with coherent information \( \Delta(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}, \rho_{aa'}) \) exceeding \( u(\lambda, d) \),

\[
Q(\mathcal{M}_{d+1}) + Q(\mathcal{E}_{\lambda,d}) \leq u(\lambda, d) < \Delta(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}, \rho_{aa'}) \leq Q(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}). \tag{10}
\]

This chain of inequalities proves \((8)\).

The input state achieving \((10)\) is \( \rho_{aa'} = \text{Tr}_{rr'}[\psi]_{aa'rr'} \), where for \( w \in [0,1] \) we define

\[
|\psi\rangle_{aa'rr'} = \sqrt{1-w} |0\rangle_r |0\rangle_{a'} + \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_r |i\rangle_{a'} + \sqrt{w} |1\rangle_r |0\rangle_{a'} \tag{11}
\]

and the reference spaces \( \mathcal{H}_r \) and \( \mathcal{H}_{a'} \) have dimensions two and \( d \), respectively. The pure state \( |\psi\rangle_{aa'rr'} \) is a superposition of two orthogonal ‘pieces’ with amplitudes \( \sqrt{1-w} \) and \( \sqrt{w} \), respectively. By itself, the first piece only generates coherent information via \( \mathcal{E}_{\lambda,d} \), as the input of \( \mathcal{M}_{d+1} \) in \( \mathcal{H}_a \) is in a product state with both the input to \( \mathcal{E}_{\lambda,d} \) and the reference. The second piece by itself generates no coherent information, since the joint input system \( \mathcal{H}_a \otimes \mathcal{H}_{a'} \) is unentangled with the reference \( \mathcal{H}_r \otimes \mathcal{H}_{r'} \).

Optimizing over the parameter \( w \in [0,1] \), this superposition of coding strategies results in a coherent information of the joint channel \( \mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d} \) that exceeds the upper bound \( u(\lambda, d) \) on \( Q(\mathcal{M}_{d+1}) + Q(\mathcal{E}_{\lambda,d}) \). We first show this numerically for \( \lambda \in [0.37, 0.57] \) and sufficiently large \( d \). This is summarized in Fig. 2, where we plot the minimal values \( \lambda_{\min}(d) \) (dashed blue) and \( \lambda_{\max}(d) \) (dashed magenta) of \( \lambda \) as a function of \( d \) such that \((8)\) holds numerically for all \( \lambda \in [\lambda_{\min}(d), \lambda_{\max}(d)] \). Note that \( \mathcal{E}_{\lambda,d} \) has positive quantum capacity when \( \lambda < 1/2 \), and hence for suitable \( d \) and \( \lambda \) we obtain super-additivity of quantum capacity \((8)\) for two channels, \( \mathcal{M}_d \) and \( \mathcal{E}_{\lambda,d} \), each with strictly positive \( Q \).

In Fig. 2 we also plot the minimal values \( \lambda_{\min}(d) \) (solid blue) and \( \lambda_{\max}(d) \) (solid magenta) such that the coherent information of \( \mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d} \) is super-additive for all \( \lambda \in [\lambda_{\min}(d), \lambda_{\max}(d)] \). While the interval \([\lambda_{\min}(d), \lambda_{\max}(d)]\) marks the ‘true’ extent of the super-additivity of quantum capacity (modulo the spin alignment conjecture), we stress once again that the super-additivity of quantum capacity within the interval \([\lambda_{\min}(d), \lambda_{\max}(d)]\) is unconditional.

We can further strengthen the numerical results of Fig. 2 by proving analytically that the super-additivity of quantum capacity in \((8)\) indeed holds for all \( \lambda \in (0,1) \) and sufficiently large \( d \). The proof is based on a log-singularity-like argument \([28]\), and applied for any \( \lambda \in (0,1) \), by a suitable choice of the parameter \( w \) in the state \((11)\). Details of this calculation can be found in the Supplementary material.

Discussion. Interestingly, a single ansatz \((11)\) is responsible for super-additivity of \( Q^{(1)} \) when \( \mathcal{N}_s \) is used with a variety of other channels \( \mathcal{E}_s, \mathcal{A}_s, \mathcal{D}_s \), and randomly constructed qubit channels. A higher dimensional version of this ansatz gives rise to super-additivity of quantum capacity when \( \mathcal{M}_d \) is used with \( \mathcal{E}_{\lambda,d} \). The mechanism and extent of this super-additivity is distinct from superactivation, where the private capacity of a zero quantum
capacity channel $\mathcal{N}$ is transformed into quantum capacity when used jointly with an anti-degradable channel $\mathcal{A}$. This transformation has efficiency at most $1/2$, and thus one obtains super-activation when $0 = Q(\mathcal{N}) < P(\mathcal{N})/2$. By contrast, $Q(\mathcal{N}) > P(\mathcal{N})/2 > 0$, thus ruling out the super-activation mechanism as the cause for our super-additivity involving $\mathcal{N}$; our protocol (11) employs a different mechanism.

Like super-activation our protocol works robustly [39] when $\mathcal{A} = \mathcal{E}_{\lambda,d}$ and $\lambda$ is varied, but unlike super-activation we find super-amplification, i.e., super-additivity even when both channels $\mathcal{M}_d$ and $\mathcal{E}_{\lambda,d}$ have non-zero quantum capacity. Similar super-additivity of quantum capacity arises in high-dimensional rocket and half-rocket channels when used with zero capacity channels [32, 42]. These noisy channels, carefully constructed to display super-additivity, have quantum capacity well below the dimensional bound $r = Q/\log d \ll 1$. By contrast, $\mathcal{M}_d$ is simply constructed by hybridizing a degradable qubit channel with a useless channel, with the goal to support weak additivity of $Q^{(1)}$. Yet, it exhibits super-additivity of $Q$ even when it has modest input dimension and noise; for instance super-additivity occurs at $d = 5$ and $r > .2$. Our result on $\mathcal{M}_d$ also contrasts with those obtainable by extending super-activation via continuity arguments. The super-activating channels can be perturbed to have positive capacities, but these capacities are necessarily very small. Moreover, super-additivity involving $\mathcal{M}_d$ occurs over a wide range of erasure probabilities that is well beyond what one may expect from such perturbations. For instance, at $d = 10$, $r \approx .075$, and super-additivity holds over erasure probabilities $.43 \leq \lambda \leq .53$, and the erasure channel can have substantial capacity. Using $\mathcal{M}_d$ with a symmetric channel, $\mathcal{S}$, of unbounded dimension leads to super-additivity, $Q(\mathcal{M}_d \otimes \mathcal{S}) \geq Q(\mathcal{M}_d) + Q(\mathcal{S})$ for any $d \geq 7$ where $P(\mathcal{M}_d)/2 > Q(\mathcal{M}_d)$ [1], since $Q(\mathcal{M}_d \otimes \mathcal{S}) > P(\mathcal{M}_d)/2$ [36]. These super-additivity results can be strengthened and simplified further if the SAC is proven. The simplicity of the channels involved in super-additivity here raises the question of whether qualitatively similar constructions are possible for investigating super-additivity of private and classical capacities.

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SUPPLEMENTARY INFORMATION FOR
“GENERIC NONADDITIVITY OF QUANTUM CAPACITY IN SIMPLE CHANNELS”

Here we provide additional information about the results discussed in the main text. We start in Sec. I with preliminaries on quantum channels and their quantum capacity. We then discuss violations of strong additivity of quantum capacity involving the channels \( \mathcal{N}_{a} \) and \( \mathcal{M}_{d} \) in Secs. II and III, respectively.

I. PRELIMINARIES

A. Quantum channels

Let \( \mathcal{H} \) be a Hilbert space of finite dimension \( d \). Let \( \mathcal{H}^{d} \) be the dual of \( \mathcal{H} \), and \( \hat{\mathcal{H}} \cong \mathcal{H} \otimes \mathcal{H}^{d} \) be the space of linear operators acting on \( \mathcal{H} \). Throughout this paper, the ket \( |\psi\rangle \in \mathcal{H} \) denotes a unit vector in \( \mathcal{H} \), and the bra \( \langle \psi| \in \mathcal{H}^{d} \) is the dual vector. We use \( |\psi\rangle \) as a shorthand for \( |\psi\rangle \langle \psi| \).

Let \( \mathcal{H}_{a}, \mathcal{H}_{b}, \) and \( \mathcal{H}_{c} \) be three Hilbert spaces of dimensions \( d_{a}, d_{b}, \) and \( d_{c} \) respectively. An isometry \( E : \mathcal{H}_{a} \rightarrow \mathcal{H}_{b} \otimes \mathcal{H}_{c} \), i.e., a map satisfying \( E^{\dagger}E = I_{a} \) (the identity on \( \mathcal{H}_{a} \)), takes an input Hilbert space \( \mathcal{H}_{a} \) to a subspace of a pair of output spaces \( \mathcal{H}_{b} \otimes \mathcal{H}_{c} \). This isometry generates a quantum channel pair, \((\mathcal{B}, \mathcal{B}^{c})\), i.e., a pair of completely positive (CP) trace-preserving (TP) maps, with superoperators,

\[
\mathcal{B}(X) = \text{Tr}_{c}(EXE^{\dagger}), \quad \text{and} \quad \mathcal{B}^{c}(X) = \text{Tr}_{b}(EXE^{\dagger}),
\]

that take any element \( X \in \mathcal{H}_{a} \) to \( \mathcal{H}_{b} \) and \( \mathcal{H}_{c} \), respectively. Each channel in this pair \((\mathcal{B}, \mathcal{B}^{c})\) may be called the complement of the other. If the input of the isometry \( E \) is restricted to a subspace \( \mathcal{H}_{\tilde{a}} \) of \( \mathcal{H}_{a} \), then such a restricted map is still an isometry on \( \mathcal{H}_{\tilde{a}} \) and defines a pair of channels \((\hat{\mathcal{B}}, \hat{\mathcal{C}})\), where each channel \( \hat{\mathcal{B}} \) and \( \hat{\mathcal{C}} \) is called a sub-channel of \( \mathcal{B} \) and \( \mathcal{C} \) respectively. When focusing on some quantum channel \( \mathcal{B} \), it is common to refer to \( \mathcal{H}_{a}, \mathcal{H}_{b}, \) and \( \mathcal{H}_{c} \) as the channel input, output, and environment respectively.

Any CPTP map (together with its complement) may be written as (12) in terms of a suitable isometry \( E \). Another representation of a CPTP map comes from its Choi-Jamiolkowski operator. To define this operator, consider a linear map \( \mathcal{B} : \mathcal{H}_{a} \rightarrow \mathcal{H}_{b} \), an orthonormal basis \( \{|i\rangle_{a}\} \) on \( \mathcal{H}_{a} \), and a maximally entangled state,

\[
|\phi\rangle = \frac{1}{\sqrt{d_{a}}} \sum_{i=1}^{d_{a}} |i\rangle_{a} \otimes |i\rangle_{a},
\]

on \( \mathcal{H}_{a} \otimes \mathcal{H}_{a} \). The unnormalized Choi-Jamiolkowski operator of \( \mathcal{B} \) is

\[
J^{\mathcal{B}}_{ab} = d_{a}(I_{a} \otimes \mathcal{B})(|\phi\rangle \langle \phi|),
\]

where \( I_{a} \) denotes the identity map acting on \( \mathcal{H}_{a} \). The linear map \( \mathcal{B} \) is CP if and only if the above operator is positive semidefinite, and TP if and only if its partial trace over \( \mathcal{H}_{b} \) is the identity \( I_{a} \) on \( \mathcal{H}_{a} \).

B. Quantum capacity

The quantum capacity \( Q(\mathcal{B}) \) of a quantum channel \( \mathcal{B} : \mathcal{H}_{a} \rightarrow \mathcal{H}_{b} \) is defined as the largest rate at which quantum information can be sent faithfully through the channel. It can be expressed in terms of an entropic quantity as follows. Let \( \rho_{a} \) denote a density operator (unit trace positive semi-definite operator) on \( \mathcal{H}_{a} \) and for any \( \rho_{b} := \mathcal{B}(\rho_{a}) \) and \( \rho_{c} := \mathcal{B}^{c}(\rho_{a}) \). The coherent information (or entropy bias) of a channel \( \mathcal{B} \) at a density operator \( \rho_{a} \) is

\[
\Delta(\mathcal{B}, \rho_{a}) = S(\rho_{b}) - S(\rho_{c}),
\]

where \( S(\rho) = -\text{Tr}(\rho \log \rho) \) (we use log base 2 by default) is the von-Neumann entropy of \( \rho \). The channel coherent information (sometimes called the single-letter coherent information),

\[
Q^{(1)}(\mathcal{B}) = \max_{\rho_{a}} \Delta(\mathcal{B}, \rho_{a}),
\]

is the quantum capacity of \( \mathcal{B} \).
is an achievable rate for sending quantum information across the channel $\mathcal{B}$, and hence $Q(\mathcal{B}) \geq Q^{(1)}(\mathcal{B})$ [9, 11, 14]. The maximum achievable rate is equal to the quantum capacity of $\mathcal{B}$, and given by a multi-letter formula (sometimes called a regularized expression) [9, 11–14],

$$Q(\mathcal{B}) = \sup_{n \in \mathbb{N}} \frac{1}{n} Q^{(1)}(\mathcal{B}^{\otimes n}), \quad (17)$$

where $\mathcal{B}^{\otimes n}$ represent $n \in \mathbb{N}$ parallel (sometimes called joint) uses of $\mathcal{B}$. In contrast, (16) is often called the “single-letter expression.”

C. Superadditivity, amplification and super-amplification

The channel coherent information is super-additive. For any two quantum channels $\mathcal{B}$ and $\mathcal{B}'$ used together, the channel coherent information of the joint channel $\mathcal{B} \otimes \mathcal{B}'$ satisfies an inequality,

$$Q^{(1)}(\mathcal{B} \otimes \mathcal{B}') \geq Q^{(1)}(\mathcal{B}) + Q^{(1)}(\mathcal{B}'), \quad (18)$$

which is known to be strict for some $\mathcal{B}$ and $\mathcal{B}'$ [18–20, 23–28, 32, 36]. We will use the following terminology for special cases of super-additivity to facilitate our discussion.

We say that the coherent information of a channel $\mathcal{B}$ is weakly super-additive if there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} Q^{(1)}(\mathcal{B}^{\otimes n}) > Q^{(1)}(\mathcal{B}). \quad (19)$$

In this case the channel capacity $Q(\mathcal{B})$, equal to the supremum of the left-hand side of (19) over all $n \in \mathbb{N}$, is strictly greater than the channel coherent information, and hence the regularization in (17) is necessary [18–20, 24–28, 61].

If a strict inequality holds in (18) for two different channels $\mathcal{B}$ and $\mathcal{B}'$, we say that the coherent information is strongly super-additive. It is known that the quantum capacity can be strongly super-additive as well, i.e., there are channels $\mathcal{B}$ and $\mathcal{B}'$ such that [36]

$$Q(\mathcal{B} \otimes \mathcal{B}') > Q(\mathcal{B}) + Q(\mathcal{B}'). \quad (20)$$

Due to the potential need for regularization for evaluating the quantum capacity, super-additivity has typically been demonstrated for channels with vanishing or well-bounded capacities in order to certify a strict inequality in (20). A particular example of this is the case $Q(\mathcal{B}) = Q(\mathcal{B}') = 0$, which is called superactivation [36]. If $Q(\mathcal{B}) > 0$ and $Q(\mathcal{B}') = 0$, we call the super-additivity of quantum capacity amplification. If $Q(\mathcal{B}) > 0$ and $Q(\mathcal{B}') > 0$, we call the super-additivity of quantum capacity super-amplification. We use similar terminology for the channel coherent information $Q^{(1)}(\cdot)$ (defined in (16)) of a quantum channel. The main results of this paper are demonstrations of both amplification and super-amplification of coherent information and quantum capacity.

D. Special channel classes

A channel $\mathcal{B}$ is called degradable, and its complement $\mathcal{B}^c$ anti-degradable, if there is another channel $\mathcal{D}$ such that $\mathcal{D} \circ \mathcal{B} = \mathcal{B}^c$ [47, 59]. Sometimes this channel $\mathcal{D}$ is called the degrading map of the degradable channel $\mathcal{B}$. For any two channels $\mathcal{B}'$ and $\mathcal{B}$, each either degradable or anti-degradable, the joint channel $\mathcal{B} \otimes \mathcal{B}'$ has additive coherent information. For a degradable channel $\mathcal{B}$, the coherent information $\Delta(\mathcal{B}, \rho_a)$ is concave in $\rho_a$ [67], and thus $Q^{(1)}(\mathcal{B})$ can be computed with relative ease [68, 69]. As a result the quantum capacity of a degradable channel, which simply equals $Q^{(1)}(\mathcal{B})$, can also be computed efficiently. An anti-degradable channel has no quantum capacity due to the no-cloning theorem.

Besides anti-degradable channels, the only other known class of zero-quantum-capacity channels are entanglement binding or positive under partial-transpose (PPT) channels [60]. A channel is PPT if its Choi-Jamiolkowski operator (14) is positive under partial transpose.
II. VIOLATION OF STRONG ADDITIVITY OF QUANTUM CAPACITY INVOLVING $\mathcal{N}_s$

A. The $\mathcal{N}_s$ channel

Let $\mathcal{H}_a, \mathcal{H}_b,$ and $\mathcal{H}_c$ have dimensions $d_a = d_b = 3,$ and $d_c = 2.$ Consider an isometry $F_s : \mathcal{H}_a \rightarrow \mathcal{H}_b \otimes \mathcal{H}_c$ with $0 \leq s \leq 1/2$ of the form

$$F_s[0] = \sqrt{s} |0 \rangle \otimes |0\rangle + \sqrt{1 - s} |1\rangle \otimes |1\rangle, \quad F_s[1] = |2\rangle \otimes |0\rangle, \quad F_s[2] = |2\rangle \otimes |1\rangle.$$  \hfill (21)

This isometry was introduced previously by one of us in [28] with |1⟩ and |2⟩ in $\mathcal{H}_a$ exchanged. Furthermore, the channel defined via (21) is unitarily equivalent to a quantum channel introduced in [63] and further studied in [70] (see [1] for a more detailed discussion). The isometry (21) gives rise to a complementary pair of channels $\mathcal{N}_s : \mathcal{H}_a \rightarrow \mathcal{H}_c$ and $\mathcal{N}_s^c : \mathcal{H}_a \rightarrow \mathcal{H}_b$. The channel maps an input operator $\rho = \sum_{ij} \rho_{ij} |i\rangle \langle j|$ to

$$\mathcal{N}_s(\rho) = \begin{pmatrix} s\rho_{00} & 0 & \sqrt{s}\rho_{01} \\ 0 & (1 - s)\rho_{00} & \sqrt{1 - s}\rho_{02} \\ \sqrt{s}\rho_{10} & \sqrt{1 - s}\rho_{20} & \rho_{11} + \rho_{22} \end{pmatrix}.$$ \hfill (22)

In [1], the capacities of $\mathcal{N}_s$ and $\mathcal{N}_s^c$ are studied in detail. It is proved that both $\mathcal{N}_s$ and $\mathcal{N}_s^c$ are neither degradable nor antidegradable, and neither channel belongs to any class of channels with known quantum, private, or classical capacity. Surprisingly, the capacities can still be found through a variety of techniques (see [1] for details). In summary, the quantum, private, and classical capacities of $\mathcal{N}_s^c$ are all equal to 1:

$$Q(\mathcal{N}_s^c) = P(\mathcal{N}_s^c) = C(\mathcal{N}_s^c) = 1.$$  \hfill (23)

Proof for these equalities relies on identifying a perfect channel from the two-dimensional input subspace $\mathcal{H}_a$, spanned by |1⟩, |2⟩, to the two dimensional output $\mathcal{H}_c$ of $\mathcal{N}_s^c$. The private and classical capacity of $\mathcal{N}_s$ likewise are equal to 1, and the underlying information quantities turn out to be additive: $P^{(1)}(\mathcal{N}_s) = \chi(\mathcal{N}_s) = P(\mathcal{N}_s) = C(\mathcal{N}_s) = 1$. Proof for these equalities shows a lower bound on $P^{(1)}(\mathcal{N}_s)$ that matches an upper bound on $C(\mathcal{N}_s)$. The lower bound on $P^{(1)}(\mathcal{N}_s)$ uses an explicit ensemble of two orthogonal inputs that remain orthogonal at the channel output $\mathcal{H}_b$, but become indistinguishable at the complementary output $\mathcal{H}_c$. The upper bound on $C(\mathcal{N}_s)$ uses an explicit solution to a semidefinite programming upper bound on $C$ proved in [70]. The coherent information of $\mathcal{N}_s$ is the solution of a one-parameter concave maximization problem over a bounded interval. For any $0 \leq s \leq 1/2$,

$$Q^{(1)}(\mathcal{N}_s) = \max_{\rho_u(u)} \Delta(\mathcal{N}_s, \rho_u(u)),$$ \hfill (24)

where $\rho_u(u)$ is a one-parameter density operator of the form $\rho_u(u) = (1 - u)|0\rangle \langle 0| + u|2\rangle \langle 2|$ with $0 \leq u \leq 1$. Moreover, provided the spin alignment conjecture stated in [1] is true, the coherent information $Q^{(1)}(\mathcal{N}_s)$ is also additive, and $Q^{(1)}(\mathcal{N}_s) = Q(\mathcal{N}_s)$. Proof for this equality has two steps. First step identifies that an input restricted to a degradable sub-channel of $\mathcal{N}_s$ achieves the single-letter coherent information $Q^{(1)}(\mathcal{N}_s)$. The second step shows that inputs restricted to tensor products of this degradable sub-channel achieve the multi-letter coherent information of $\mathcal{N}_s$ conditioned on what we call the spin-alignment conjecture. Together, the results in [1] show that the coherent information, the private information, and the Holevo information of $\mathcal{N}_s$ are all weakly additive.

Operationally, a channel having a weakly additive information quantity means that a more complex coding strategy (defined by typical subspaces of multi-letter entangled input states) does not increase the rate. It is hence surprising that, for coherent information, the qutrit channel $\mathcal{N}_s$ exhibits strong super-additivity when it is used jointly with a different channel $\mathcal{K}$. Even more surprisingly, this additional channel $\mathcal{K}$ can be just about any small channel! For example, we found super-additivity when $\mathcal{K}$ is chosen from well-known and physically relevant channels, such as the amplitude damping channel, erasure channel, and depolarizing channel on a qubit, over large ranges of the corresponding noise parameters. Alternatively, this additional channel can be randomly sampled and a substantial fraction have super-additive coherent information jointly with $\mathcal{N}_1/2$. Note that $Q(\mathcal{N}_s) > 0$ for all $s \in [0,1/2]$ (see Section 4 and Fig. 1 in [1]), and hence the additional channel $\mathcal{K}$ amplifies the coherent information of $\mathcal{N}_s$ (in the terminology introduced in Sec. 1C above):

$$Q^{(1)}(\mathcal{N}_s \otimes \mathcal{K}) > Q^{(1)}(\mathcal{N}_s) + Q^{(1)}(\mathcal{K}).$$ \hfill (25)
Finally, as many of the additional channels have known quantum capacity, some positive and some vanishing, the inequality (25) can be lifted to super-amplification and amplification of quantum capacity subject to the spin alignment conjecture [1].

We will describe the aforementioned results in detail in the rest of this section. The effect can be demonstrated for a large range of $s$ and many choices of $K$; to facilitate the discussion, we organize the presentation as follows. In Section II B we focus on three specific choices of $K$ that have vanishing coherent information and demonstrate amplification of coherent information with $\mathcal{N}$ for a large range of the parameter $s$. In Section II C, we instead fix $s = 1/2$ and use the qubit erasure channel as the additional channel $\mathcal{K}$, varying the noise parameter of the latter over the full range. We show amplification of coherent information in a very large interval of this noise parameter, including regimes when $Q^{(1)}(\mathcal{K}) = Q(\mathcal{K}) > 0$, hence demonstrating super-amplification. We examine the novel mechanism behind the (super-)amplification. In Section II D, we return to the full family of $\mathcal{N}$, and present large regions of joint noise parameters for (super-)amplification. Furthermore, two of our extended families of $\mathcal{K}$ have quantum capacities equal to the coherent information. Therefore, conditioned on the spin alignment conjecture, all aforementioned results on (super-)amplification hold also for quantum capacity. In Section II E, we describe the prevalence of amplification and super-amplification over randomly chosen $\mathcal{K}$.

B. Amplification by channels with zero coherent information or quantum capacity

In this section, we illustrate amplification in (25) by choosing the additional channel $\mathcal{K}$ to be the erasure channel $\mathcal{E}_\lambda$, the amplitude damping channel $\mathcal{A}_\gamma$, and the depolarizing channel $\mathcal{D}_\epsilon$. In each case, a non-negative parameter $x$ determines the noise level; in this section, we fix the noise level $x$ for each channel so that the coherent information vanishes, $Q^{(1)}(\mathcal{K}_x) = 0$. Since the erasure channel and the amplitude damping channel are either degradable or anti-degradable for all values of the noise level $x$, their coherent information is additive and equal to the quantum capacity, $Q^{(1)}(\mathcal{K}_x) = Q(\mathcal{K}_x)$. Hence, in these cases we also have $Q(\mathcal{K}_x) = 0$, which illustrates amplification of quantum capacity subject to the spin alignment conjecture.

We now describe the assisting channels in more detail, and compute both sides of (25) numerically for the whole interval $s \in [0,1/2]$ with the three chosen additional channels. These numerical results are summarized in Fig. 3. We also present a common structure of the joint input state achieving (25).

a. Erasure channel. The general erasure channel has input space $\mathcal{H}_u'$ and output and environment spaces $\mathcal{H}_u = \mathcal{H}_u' \oplus \mathbb{C} \langle e \rangle$, where $\langle e \rangle$ is an ‘erasure flag’ orthogonal to the input space $\mathcal{H}_u'$. The erasure channel acts as follows: with erasure probability $\lambda$ the channel erases the input and replaces it with the erasure flag $|e\rangle$, while with probability $1 - \lambda$ the channel transmits the input unaltered. When $\mathcal{H}_u'$ is 2-dimensional, the erasure channel pair $(\mathcal{E}_\lambda, \mathcal{E}_\lambda')$ can be generated from a channel isometry $E_\lambda$ defined via

$$E_\lambda |0\rangle = \sqrt{1 - \lambda} |0e\rangle + \sqrt{\lambda} |e0\rangle, \quad E_\lambda |1\rangle = \sqrt{1 - \lambda} |1e\rangle + \sqrt{\lambda} |e1\rangle. \quad (26)$$

Note that the complementary channel $\mathcal{E}_\lambda'$ is an erasure channel with erasure probability $1 - \lambda$, i.e., $\mathcal{E}_\lambda' = \mathcal{E}_1 - \mathcal{E}_\lambda$, and the erasure channel is degradable for $\lambda \in [0,1/2]$ and anti-degradable for $\lambda \in [1/2,1]$. As a result, the channel coherent information coincides with the quantum capacity for all $\lambda$, taking on the simple form [43]

$$Q^{(1)}(\mathcal{E}_\lambda) = Q(\mathcal{E}_\lambda) = \max\{1 - 2\lambda, 0\}. \quad (27)$$

We fix $\lambda = 1/2$ so that $Q^{(1)}(\mathcal{E}_\lambda) = Q(\mathcal{E}_{1/2}) = 0$ and we numerically maximize the coherent information $Q^{(1)}(\mathcal{N}_s \otimes \mathcal{E}_{1/2})$ for $s \in [0,1/2]$. We find that its value exceeds $Q^{(1)}(\mathcal{N}_s)$ for all $s \in (0,1/2)$, as shown by the solid magenta line in Fig. 3. An exhaustive search over the full input space reveals the following structure of the joint input $\rho_{aa'}$ that attains the maximum joint coherent information:

$$\rho_{aa'} \equiv \rho_{aa'}(\epsilon, r_1, r_2) = r_1|00\rangle_{aa'} + r_2|01\rangle_{aa'} + (1 - r_1 - r_2)|\chi\rangle_{aa'}, \quad (28)$$

where $\mathcal{H}_a$ and $\mathcal{H}_a'$ denote the input spaces of $\mathcal{N}_s$ and $\mathcal{E}_{1/2}$, respectively, $|\chi\rangle_{aa'} = \sqrt{1 - \epsilon}|20\rangle_{aa'} + \sqrt{\epsilon}|11\rangle$, and the parameters $r_1, r_2, \epsilon \in [0,1], r_1 + r_2 \leq 1$ are optimized. In Section II C we further analyze the coherent information for $\mathcal{N}_{1/2} \otimes \mathcal{E}_\lambda$ for varying $\lambda \in [0,1]$ and discuss the mechanism for the amplification.

b. Amplitude damping channel. The amplitude damping channel is defined as follows. Let $\mathcal{H}_d', \mathcal{H}_d$, and $\mathcal{H}_d'$ be two-dimensional Hilbert spaces. Consider an isometry $A_d: \mathcal{H}_d' \mapsto \mathcal{H}_d \otimes \mathcal{H}_d'$ of the form

$$A_d |0\rangle = |00\rangle, \quad A_d |1\rangle = \sqrt{\gamma} |01\rangle + \sqrt{1 - \gamma} |10\rangle. \quad (29)$$
The isometry above defines a channel pair \((A_\gamma, A_{1-\gamma})\), where \(A_\gamma : \hat{H}_{a'} \rightarrow \hat{H}_{a'}\) is a qubit amplitude damping channel with damping probability \(\gamma\), and \(A_{1-\gamma} = A_\gamma\). When \(\gamma = 1/2\), \(A_\gamma = A_{1-\gamma}\) so the two channels are symmetric. The channel coherent information is attained on \(\rho(z) = (1 - z)[0] + z[1]\) [64], and hence

\[
Q^{(1)}(A_\gamma) = \max_{0 \leq z \leq 1} \Delta(A_\gamma, \rho(z)).
\] (30)

The channel \(A_\gamma\) is degradable for \(\gamma \in [0, 1/2]\), and anti-degradable for \(\gamma \in [1/2, 1]\). As a result, for all \(\gamma \in [0, 1]\) the coherent information (30) is additive,

\[
Q(A_\gamma) = Q^{(1)}(A_\gamma).
\] (31)

This capacity is zero in the antidegradability regime, \(Q(A_\gamma) = 0\) for \(\gamma \geq 1/2\). Similar to the earlier analysis involving the erasure channel, we choose \(\gamma = 1/2\), and numerically maximize \(Q^{(1)}(N_s \otimes A_{1/2})\). Again, we find that \(Q^{(1)}(N_s \otimes A_{1/2}) > Q^{(1)}(N_s)\) for all \(s \in (0, 1/2)\), as shown by the solid magenta line in Fig. 3. The optimal input achieving the maximal coherent information has the same form given by (28), just as in the case when \(N_s\) is used with the erasure channel.

c. Depolarizing channel. The qubit depolarizing channel has input and output space \(\mathcal{H}_{a'} = \mathcal{H}_{b'} = \mathbb{C}^2\) and environment space \(\mathcal{H}_e = \mathbb{C}^4\). The depolarizing channel acts as follows: with probability \(4p/3\) the channel replaces the input by the maximally mixed state \(I_{a'}/2\), or in Kraus representation,

\[
D_p(\rho) = (1 - p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z),
\] (32)

where \(0 \leq p \leq 3/4\) and \(X, Y, Z\) are the Pauli matrices. The depolarizing channel is anti-degradable for \(p \geq 1/4\), and hence \(Q(D_p) = 0\) in that regime. For \(p \in (0, 1/4)\), the quantum capacity of \(D_p\) is unknown, and only lower and upper bounds on \(Q(D_p)\) are available [18–20, 24, 25, 61, 71–74]. For our discussion, we fix \(p^* = 0.1893\), which is known as the “hashing point”, defined to be the smallest \(p\) such that \(Q^{(1)}(D_p) = 0\). Note that, in contrast to the additional channels used earlier, \(D_{p^*}\) does have positive quantum capacity, \(Q(D_{p^*}) > 0\) [17, 18], but we do not know its exact value.

The numerically optimized coherent information of the joint channel \(N_s \otimes D_{p^*}\) is presented in Fig. 3, with the solid green line showing that \(Q^{(1)}(N_s \otimes D_{p^*}) > Q^{(1)}(N_s)\) for \(s \in (0.45, 0.5]\). The joint input achieving the maximal coherent information for \(N_s \otimes D_{p^*}\) again has the same form (28) as the optimal states for the qubit amplitude damping and qubit erasure channel.

C. (Super-)amplification of coherent information of \(N_{1/2}\) and erasure channel

In this subsection, we fix \(s = 1/2\) and consider the joint optimization in \(Q^{(1)}(N_{1/2} \otimes K_x)\) for \(K_x\) being the continuous families of erasure and amplitude damping channels. We find both amplification and super-amplification of coherent information over large ranges of the noise parameter \(x\). As \(K_x\) has additive coherent information, our results also imply amplification and super-amplification of quantum capacity, conditioned on the spin alignment conjecture. We focus on \(N_{1/2}\) used jointly with the erasure channel, since this allows for a better comparison with earlier results on super-additivity of quantum capacity. However, we note that similar results also hold for the amplitude damping channel.

To demonstrate this (super-)amplification for \(N_{1/2} \otimes E_\lambda\), consider an input \(\rho_{aa'}(\epsilon, r_1, r_2)\) of the form given in (28), evaluate the coherent information on the input, and maximized over the parameters \(\epsilon, r_1,\) and \(r_2\):

\[
\Delta^*(N_{1/2} \otimes E_\lambda) = \max_{\epsilon, r_1, r_2} \Delta(N_{1/2} \otimes E_\lambda, \rho_{aa'}(\epsilon, r_1, r_2)).
\] (33)

Consider the quantity

\[
\delta(\lambda) := \Delta^*(N_{1/2} \otimes E_\lambda) - \left(Q^{(1)}(N_{1/2}) + Q^{(1)}(E_\lambda)\right)
\] (34)

whose strict positivity is a witness for amplification or superamplification of the coherent information of \(N_{1/2}\) and \(E_\lambda\) because \(\Delta^*(N_{1/2} \otimes E_\lambda) \leq Q^{(1)}(N_{1/2} \otimes E_\lambda)\). Our numerical optimization shows that \(\delta(\lambda)\) is positive when \(\lambda_{\min} \leq \lambda \leq \lambda_r\) where \(\lambda_0 \approx .41\) and \(\lambda_r \approx .663\). The quantity \(\delta(\lambda)\) is increasing for \(\lambda_{\min} \leq \lambda \leq 1/2\) and decreasing for \(1/2 \leq \lambda \leq \lambda_r\), reaching a maximum value of \(\approx .033\) at \(\lambda = 1/2\). In addition, a log-singularity-based argument [28] (see also Section IID.1) shows that \(\delta(\lambda) > 0\) for \(\lambda < \lambda_{\min}\) where \(\lambda \approx .723\). Fig. 4 depicts \(\delta(\lambda)\) for all \(\lambda\) of interest. In summary, we
find amplification of coherent information of the joint channel $N_{1/2} \otimes E_\lambda$ for the large range $0.41 \lesssim \lambda \lesssim 0.72$ spanning both the degradable and antidegradable regimes of the erasure channel. For $\lambda < 1/2$, this finding demonstrates super-amplification of coherent information (complementary to the previous subsection).

Using the same arguments as before, the above holds also for quantum capacity subject to the spin-alignment conjecture. This is a striking result: two channels $N_{1/2}$ and $E_\lambda$, each with non-zero quantum capacity, together have larger quantum capacity than the sum of each.

The mechanism behind this super-amplification of quantum capacity in our work appears to be novel. In the pioneering example of super-additivity of quantum capacities [36], a (zero-capacity) symmetric channel is employed to transform the private capacity of another channel into the quantum capacity of the joint channel, at the cost of a factor of 1/2. In particular, PPT channels with private capacity can be used to establish p-bits [41, 75]. A p-bit is a unit of private information and consists of a key and a shield system on each side. The private correlation resides in the key systems; the shield systems are correlated with both the key systems and the environment, in a way that protects the key systems from the environment. Furthermore, if the shield systems are brought to one of the two parties, it can be decorrelated from the key systems, turning the key into entanglement. The protocol by Smith and Yard generates p-bits with a PPT channel and transmits the sender’s shield systems through the 50% erasure channel. If the shield is transmitted, which happens half the time, the p-bit can be transformed into entanglement. If the shield is erased, the coherent information generated is 0. This is the origin of the factor of 1/2. Amplification of quantum capacities is also demonstrated in [32, 42] using the same Smith-Yard mechanism (note that an additional suppression of private capacity and other tricks are used in [32] to attain an amplification bigger than half of the private capacity).

Even though the private capacity of $N_\alpha$ is larger than its quantum capacity, we actually have $Q(N_\alpha) > \frac{1}{4} P(N_\alpha)$. It is therefore not advantageous to use the Smith-Yard protocol because of the associated factor of 1/2. We may expect that, in this case, the best strategy to optimize the coherent information is to put independent states into the erasure channel and $N_\alpha$. If this strategy were optimal we would have additive coherent information across $N_\alpha$ and $E_\lambda$.

Super-amplification of capacity here follows a more delicate mechanism. Examining the form of the input state (28), it appears to be carrying out the Smith-Yard mechanism and the independent strategies in superposition, while retaining a coherent memory of which strategy was used. The coherence between these two strategies is substantial, providing additional coherent information that is otherwise unattainable. This exhibits the strengths of the two individual strategies without their corresponding drawbacks.
FIG. 4. The quantity $\delta(\lambda)$ defined in (34), as a function of $\lambda$. Since $\delta(\lambda)$ is a lower bound for $Q^{(1)}(N_{1/2} \otimes E_{\lambda}) - (Q^{(1)}(N_{1/2}) + Q^{(1)}(E_{\lambda}))$, the coherent information of $N_{1/2}$ and $E_{\lambda}$ is (super)-amplified whenever $\delta(\lambda) > 0$, which holds when $\lambda_{\min} \leq \lambda \leq \lambda_{\max}$.

We also show similar results for the amplitude damping channel: the coherent information (and quantum capacity, subject to the SAC) of $N_{s}$ are (super-)amplified by the amplitude damping channel $A_\gamma$ with similar magnitude and over a similar range of the noise parameter $\gamma$. Since this is again achieved by the same input (28), the (super-)amplification is caused by the same mechanism as detailed above.

D. (Super)-amplification of coherent information of $N_{s}$

In this subsection, we study the (super-)amplification of coherent information when $N_{s}$ is used jointly with an additional channel $K_x$, which is either an erasure channel $E_{\lambda}$, an amplitude damping channel $A_\gamma$, or a depolarizing channel $D_p$. We let both parameters $s$ and $x$ vary and chart out the two-dimensional regions $(s, x)$ in which (super-)amplification of coherent information can be demonstrated.

As an overview, our analysis will be similar to those presented in Section II C, except now the two channels of interest are $N_{s}$ and $K_{x}$. We fix an arbitrary $s$ and find the range of $x$ for which we can verify (super-)amplification, using both numerical methods and log-singularity based arguments. The verification relies on the positivity of a quantity $\delta(N_{s}, K_{x})$ (defined below in (36)) that generalizes $\delta(\lambda)$ in (34). This generalized quantity depends on both parameters $s$ and $x$; as a function of $x$ for fixed $s$, it behaves similarly to $\delta(\lambda)$ as depicted in Fig. 4.

In more detail, recall that $H_a$ and $H_a'$ denote the input spaces of $N_{s}$ and $K_{x}$ respectively. Consider again the density operator $\rho_{aa'}(\epsilon, r_1, r_2)$ given in (28), and the coherent information of $N_{s} \otimes K_{x}$ optimized over the parameters of $\rho_{aa'}$:

$$\Delta^*(N_{s} \otimes K_{x}) = \max_{\epsilon, r_1, r_2} \Delta(N_{s} \otimes K_{x}, \rho_{aa'}(\epsilon, r_1, r_2)).$$  \hspace{1cm} (35)$$

We can generalize $\delta(\lambda)$ in (34) to the following quantity:

$$\delta(N_{s}, K_{x}) = \Delta^*(N_{s} \otimes K_{x}) - \left( Q^{(1)}(N_{s}) + Q^{(1)}(K_{x}) \right).$$  \hspace{1cm} (36)$$

As before, since $\Delta^*(N_{s} \otimes K_{x}) \leq Q^{(1)}(N_{s} \otimes K_{x})$, the positivity of $\delta(N_{s}, K_{x})$ verifies the (super)-amplification of coherent information of $N_{s}$ and $K_{x}$. For each $K_{x}$, the quantity $\delta(N_{s}, K_{x})$ is found to be positive for an interval of $x$ denoted
as \([x_{\min}(s), x_{\max}(s)]\). Within this range, \(\delta(N_s, K_s)\) increases with \(x\) reaching a substantial maximum and vanishes numerically, and a log-singularity based argument is applied to verify positivity until \(x = x_{\max}(s)\). Repeating this analysis for every \(s\) gives the region \((s, x)\) for (super)-amplification. We are now ready to apply this framework to the three families of additional channel of interest.

1. (Super)-amplification with erasure channel

We numerically find that the threshold noise rate \(\lambda_{\min}(s) < 1/2\) for all \(s \in (0, 1/2]\). Therefore, the erasure channel super-amplifies the coherent information for all such \(s\). The maximum of \(\delta(N_s, E_\lambda)\) is attained at \(\lambda = 1/2\). The numerical value of \(\delta(N_s, E_\lambda)\) vanishes at some \(\lambda > 1/2\), and so we switch to a method based on log-singularity [28] to obtain \(\lambda_{\max}(s)\).

We now review the ideas in [28]. Let \(\sigma(\epsilon)\) denote a density operator that depends on a real parameter \(\epsilon\). If some eigenvalues of \(\sigma(\epsilon)\) increase linearly from zero to leading order in \(\epsilon\), the von Neumann entropy, \(S(\sigma(\epsilon)) = -\text{Tr}(\sigma(\epsilon) \log \sigma(\epsilon))\), increases by an amount \(\sim r |\epsilon| \log |\epsilon|\) to leading order in \(\epsilon\) for some \(r > 0\). This increase is discontinuous at \(\epsilon = 0\); for small \(\epsilon\), we call the increase a log-singularity, and the value \(r\) is called the rate of the singularity. We also say that \(S(\sigma(\epsilon))\) has an \(\epsilon\)-log-singularity of rate \(r\). These ideas are easily generalized to linear functions of entropies.

We now apply the above ideas on log-singularity to show \(\delta(N_s, E_\lambda) > 0\) for certain values of \(\lambda\). First note that \(\rho_{aa'}(0, r_1, 0) = (r_1[0] + (1 - r_1)[2])_a \otimes [0]_{a'}\). Following (24) there is a choice of \(r_1\) (call it \(u\)) so that \(r_1[0] + (1 - r_1)[2]\) attains the coherent information of \(N_s\). Furthermore, for \(\lambda > 1/2\), following (27), \([0]\) attains the coherent information of \(E_\lambda\). Therefore, \(\Delta(N_s \otimes E_\lambda, \rho_{aa'}(0, u, 0)) = Q_1^1(N_s) + Q_1^1(E_\lambda)\) and the difference

\[
\Delta(N_s \otimes E_\lambda, \rho_{aa'}(\epsilon, u, 0)) - \Delta(N_s \otimes E_\lambda, \rho_{aa'}(0, u, 0))
\]

is a lower bound on \(\delta(N_s, E_\lambda)\). Note that

\[
\Delta(N_s \otimes E_\lambda, \rho_{aa'}(\epsilon, u, 0)) = S(\rho_{bb'}(\epsilon, u, 0)) - S(\rho_{cc'}(\epsilon, u, 0))
\]

where

\[
\rho_{bb'}(\epsilon, u, 0) = (N_s \otimes E_\lambda)(\rho_{aa'}(\epsilon, u, 0)), \quad \rho_{cc'}(\epsilon, u, 0) = (N_s^c \otimes E_\lambda^c)(\rho_{aa'}(\epsilon, u, 0)).
\]

Using a suitable basis of \(H_{bb'}\) and \(H_{cc'}\), one can show that

\[
\rho_{bb'}(\epsilon, u, 0) = \text{diag}(s(1 - u)(1 - \lambda), 0, s(1 - u)\lambda)
\]

\[
\oplus \text{diag}((1 - s)(1 - u)(1 - \lambda), 0, (1 - s)(1 - u)\lambda)
\]

\[
\rho_{cc'}(\epsilon, u, 0) = \text{diag}(s(1 - u)\lambda, (s(1 - u) + uc)(1 - \lambda)),
\]

\[
\oplus \text{diag}(0, (1 - s(1 - u) - uc)(1 - \lambda))
\]

\[
\oplus \left( \begin{array}{c}
\epsilon u \lambda \\
\epsilon u \lambda \sqrt{\epsilon(1 - \epsilon)} \\
\epsilon u \lambda \sqrt{\epsilon(1 - \epsilon)} (1 - s(1 - u) - uc) \lambda
\end{array} \right).
\]

where \(\text{diag}(d_1, d_2, \ldots)\) represents a diagonal square matrix of diagonal entries \(d_1, d_2, \ldots\), and the operation \(\oplus\) performs the direct sum of matrices. From the above expressions, one can conclude that \(S(\rho_{bb'}(\epsilon, u, 0))\) has an \(\epsilon\)-log-singularity of rate \(u(1 - \lambda)\) and \(S(\rho_{cc'}(\epsilon, u, 0))\) has an \(\epsilon\)-log-singularity of rate \(u\lambda(1 - u)(1 - s)/(1 - s(1 - u))\). The coherent information \(\Delta(N_s \otimes E_\lambda, \rho_{aa'}(\epsilon, u, 0))\) thus has an \(\epsilon\)-log-singularity of positive rate if

\[
u(1 - \lambda) > u\lambda(1 - u)(1 - s)/(1 - s(1 - u))
\]

or equivalently,

\[
\lambda < \frac{1 - s + us}{2 - 2s - u + 2us} =: \lambda_{\max}(s),
\]

in which case (37) is strictly positive for small \(\epsilon > 0\). This implies \(\delta(N_s, E_\lambda) > 0\) for \(0 < s \leq 1/2\) and \(1/2 \leq \lambda < \lambda_{\max}(s)\), demonstrating amplification of coherent information of \((N_s, E_\lambda)\).

We summarize our findings in this subsection by plotting \(\lambda_{\min}(s)\) and \(\lambda_{\max}(s)\) as a function of \(s\) in Fig. 5; each point \((s, \lambda)\) between these two curves in the orange-shaded region corresponds to a pair of channels \(N_s, E_\lambda\) with provable (super)-amplification of coherent information.
Numerically, $\delta(N_s, A_γ)$ is positive for $γ_{\min}(s) ≤ γ ≤ γ_{\max}(s)$ (see Fig. 6). Since $γ_{\min}(s) < 1/2$, the coherent information of any $N_s$ can be super-amplified by some amplitude damping channel (using similar arguments as before). The numerical value of $\delta(N_s, A_γ)$ first increases with $γ$, reaches a maximum, and then decreases and vanishes. A log-singularity based argument provides the value of $γ_{\max}(s)$ [28],

$$γ_{\max}(s) = \frac{1}{1 + k} \quad \text{with} \quad k = \frac{(1 - s)(1 - u)}{u + (1 - s)(1 - u)}, \quad (43)$$

and $u$ is chosen so that $(1 - u)|0⟩ + u|1⟩$ attains the coherent information of $A_γ$ (see (30)). We have $γ_{\max}(s) ≥ 1/2$ for all $s ∈ (0, 1/2]$, and hence the channel $A_γ$ amplifies the coherent information of $N_s$ for $1/2 ≤ γ ≤ γ_{\max}(s)$.

3. (Super)-amplification with depolarizing channel

Finally, we study (super)-amplification of coherent information between $N_s$ and the depolarizing channel $D_p$. Numerically, $\delta(N_s, D_p)$ is positive for some $p$ only if $s ∈ [s_{\min}, 1/2]$, where $s_{\min} ≃ 0.4539$. For each $s$ in this interval, $\delta(N_s, D_p) > 0$ for $p_{\min}(s) ≤ p ≤ p_{\max}(s)$ (see Fig. 7). Since $p_{\min}(s) ≤ p^* ≤ p_{\max}(s)$, where $p^* ≃ 0.1893$ is the smallest $p$ for $Q^{(1)}(D_p) = 0$, both super-amplification and amplification of coherent information are demonstrated for each $s ∈ [s_{\min}, 1/2]$. We note that $\delta(N_s, D_p)$ is increasing for $p_{\min}(s) ≤ p ≤ p^*$ and decreasing for $p^* ≤ p ≤ p_{\max}(s)$, reaching a maximum at $p = p^*$.

E. Universal amplification

In the sections above we showed how one single coding strategy (the input state in (28)) achieves amplification of quantum capacity of $N_s$ when used in combination with an amplitude damping, erasure, or depolarizing channel.
However, our results raise the following rather natural question: to what extent is the amplification mechanism making use of the symmetries of the channel used in conjunction with $N$? In particular, both the depolarizing and erasure channel have full unitary covariance at the input; an amplification code ansatz tailored to the amplitude damping channel could thus make use of this covariance property and “coincidentally” enable the amplification effect for erasure and depolarizing noise. In this section, we show that this is not the case: we provide numerical results demonstrating that the code ansatz (28) successfully amplifies the coherent information of $N_{1/2}$ when used together with a random qubit channel.

Before explaining this result in more detail, we briefly summarize the linear representation of qubit-qubit channels based on the “Bloch vector” as used in, e.g., [76]. This representation of quantum channels uses the Bloch vector representation for qubit states: a qubit state $\rho$ is written as $\rho = \frac{1}{2}(I + s_x X + s_y Y + s_z Z)$, where $s = (s_x, s_y, s_z) \in \mathbb{R}^3$ is the Bloch vector with $\|s\| \leq 1$. The state $\rho$ is pure if and only if $\|s\| = 1$. With the identification $\rho \mapsto r(\rho) = \frac{1}{2}(1, s)$, any linear map $F: M_2(\mathbb{C}) \to M_2(\mathbb{C})$ from the space $M_2(\mathbb{C})$ of complex $(2 \times 2)$-matrices into itself can be represented as a $(4 \times 4)$-matrix $F \in M_4(\mathbb{C})$ acting on the vector representation $r(\rho)$ of $\rho$. Assuming that $F$ is a quantum channel and hence trace-preserving and completely positive, the linear representation $F$ has the following form:

$$F = \begin{pmatrix} 1 & 0 \\ t & T \end{pmatrix},$$

(44)

where $t \in \mathbb{R}^3$ and $T \in M_3(\mathbb{R})$ is a matrix whose eigenvalues satisfy a certain algebraic condition ensuring complete positivity of $F$ [76]. Every qubit-qubit quantum channel $F: M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is unitarily equivalent to a channel $R$ with matrix representation (44) in “normal form”

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_2 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{pmatrix},$$

(45)

again, the vectors $t = (t_1, t_2, t_3)$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ need to satisfy certain conditions for $R$ to be completely positive. Note that a channel is unital if and only if $t = 0$ in (44); in this case, a channel in normal form (45) corresponds to a Pauli channel, i.e., $R(\rho) = p_0 \rho + p_1 X \rho X + p_2 Y \rho Y + p_3 Z \rho Z$ for some probability distribution $(p_0, p_1, p_2, p_3)$, with each $p_i$ a function of $\lambda$ [76].
Having introduced the representation (44) of qubit-qubit channels and their normal form (45), we can now outline our numerical procedure to demonstrate the universal character of the amplification strategy (28) for $N_{1/2}$:

1. Randomly sample $t = (t_1, t_2, t_3)$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, ensuring that the map $R$ defined through its matrix $R$ as in (45) is completely positive and hence a quantum channel.

2. Form the 1-parameter family of channels $R_x = (1-x)\text{id} + xR$, where $x \in [0,1]$.

3. Choose $x^*$ to be the “hashing point” of $R_x$, i.e., the smallest $x \in [0,1]$ such that $Q^{(1)}(R_x) = 0$; this is easily achieved as $x \to Q^{(1)}(R_x)$ is typically monotonically decreasing.

4. Evaluate the coherent information $\Delta(N_{1/2} \otimes R_{x^*}, \rho)$, where $\rho$ is the amplification code ansatz from (28), and test whether

$$Q^{(1)}(N_{1/2} \otimes R_{x^*}) \geq \Delta(N_{1/2} \otimes R_{x^*}, \rho) > Q^{(1)}(N_{1/2}) = Q^{(1)}(N_{1/2}) + Q^{(1)}(R_{x^*}).$$  (46)

We find that in a substantial fraction of test runs, the above procedure produces a qubit-qubit channel $R_{x^*}$ for which the strict inequality in (46) indeed holds. Super-additivity typically persists in a non-trivial interval around the hashing point $x^*$. In particular, we also find super-additivity of coherent information for $N_{1/2} \otimes R_x$ for some $x < x^*$ such that $Q^{(1)}(R_x) > 0$. The channel $R$ used in the definition of $R_x$ is randomly sampled, and hence has no special structure other than the normal form (45). This demonstrates the universal character of our amplification mechanism (facilitated by the code ansatz (28) and explained in Sec. II C) in a clean and compelling way.

As a concrete example, consider the channel $R$ with corresponding matrix representation

$$R = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0.0078 & 0.1253 & 0 & 0 \\
0.4231 & 0 & 0.1302 & 0 \\
0.6556 & 0 & 0 & 0.0924
\end{pmatrix}. \quad (47)$$

The hashing point of the channel family $R_x = (1-x)\text{id} + xR$ lies in the interval $[0.3061, 0.3265]$, with $Q^{(1)}(R_{x^*}) = 0$ for $x^* = 0.3265$. At this noise level we achieve the following coherent information using the ansatz (28) for the input state $\rho$:

$$\Delta(N_{1/2} \otimes R_{x^*}, \rho) = 0.7175 > 0.6942 = Q^{(1)}(N_{1/2}) + Q^{(1)}(R_{x^*}).$$  (48)
This super-additivity of coherent information achieved by the ansatz (28) persists in the interval $x \in [0.246, 0.365]$. The normal form (45) is not restrictive and chosen simply for convenience; the amplification of coherent information can also be observed when randomly sampling a channel of the more general form (44). Finally, we note that the amplification of coherent information can typically be enhanced further when searching over all possible codes for the joint channel $\mathcal{N}_{1/2} \otimes \mathcal{R}_\sigma^*$; this shows that, while our amplification mechanism (28) is in a certain sense universal, it is by no means optimal in general. For example, for the concrete channel $\mathcal{R}_x$ defined using (47), an even higher coherent information of $\Delta(N_{1/2} \otimes \mathcal{R}_x^*, \tau) = 0.7241$ is achieved by the input state $\tau_{aa'} = p[0]_a \otimes \sigma_{a'} + (1-p)[\psi]_{aa'}$ with $p = 0.5281$ and
\[
[\sigma_{a'} = \begin{pmatrix} 0.7434 & 0.0032 - 0.1702i \\ 0.0032 + 0.1702i & 0.2566 \end{pmatrix}, \quad |\psi\rangle_{aa'} = (0.4805 + 0.1806i)|1\rangle_a|0\rangle_{a'} + (0.3184 - 0.1260i)|1\rangle_a|1\rangle_{a'}, \\
+ (-0.2938 - 0.5865i)|2\rangle_a|0\rangle_{a'} + 0.4348|2\rangle_a|1\rangle_{a'}].
\]
For this example channel $\mathcal{R}_x = (1-x)\text{id} + x\mathcal{R}$ with $\mathcal{R}$ as in (47), optimizing over all codes extends the range of super-additivity of coherent information to $x \in [0.228, 0.383]$. For details of these computations, see the code repository [62] for this paper.

III. VIOLATION OF STRONG ADDITIVITY OF QUANTUM CAPACITY INVOLVING $\mathcal{M}_d$

In the last section, we studied the coherent information and the quantum capacity of $\mathcal{N}_s$ when used in parallel with other channels. In this section, we carry out a similar analysis for another family $\mathcal{M}_d$ of channels generalizing $\mathcal{N}_{1/2}$ (fixing $s = 1/2$) to arbitrary input dimension $d$. For the additional channel, we focus on $\mathcal{E}_{\lambda,d}$, the $d$-dimensional generalization of the erasure channel with erasure probability $\lambda$. We obtain the following results:

(1) Similar to the analysis for $\mathcal{N}_s$, we yet again find super-additivity of the channel coherent information for all $d \geq 3$ and erasure probability $\lambda = 1/2$. Assuming the validity of the spin-alignment conjecture, this super-additivity can be elevated to the level of quantum capacity.

(2) An additional feature offered by $\mathcal{M}_d$ is a nice upper bound for the quantum capacity of $\mathcal{M}_d$ as a (decreasing) function of $d$, proved in the companion paper [1]. This bound allows superadditivity of quantum capacity to be proved unconditionally (without the need for the spin alignment conjecture), and for the full range of $\lambda$! More specifically, superadditivity of quantum capacity can be proved for:

(a) erasure probability $\lambda \in [0, 0.37, 0.57]$ and moderate values of $d$ (depending on $\lambda$);
(b) any $0 < \lambda < 1$ for sufficiently large $d$ using a log-singularity argument.

A. The $\mathcal{M}_d$ channel and the qudit erasure channel $\mathcal{E}_{\lambda,d}$

The isometry $F_x$ in (21) with $s = 1/2$ has a higher-dimensional generalization $G$ taking $\mathcal{H}_a$ to $\mathcal{H}_b \otimes \mathcal{H}_c$, where $\mathcal{H}_a$, $\mathcal{H}_b$ and $\mathcal{H}_c$ have dimensions $d, d$, and $d - 1$, respectively. This generalization acts as
\[
G|0\rangle = \frac{1}{\sqrt{d-1}} \sum_{j=0}^{d-2} |j\rangle \otimes |j\rangle, \quad G|i\rangle = |d-1\rangle \otimes |i-1\rangle,
\]
for $1 \leq i \leq d - 1$. For $d = 3$ this is the isometry from Sec. II A, i.e., $G = F_{1/2}$. When $d \geq 3$, the isometry $G$ generates a pair of channels $(\mathcal{M}_d, \mathcal{M}_d^c)$ generalizing the pair $(\mathcal{N}_{1/2}, \mathcal{N}_{1/2}^c)$. We prove in Sections 3.2 and 8.2 of [1] that $Q(\mathcal{M}_d^c) = P(\mathcal{M}_d^c) = C(\mathcal{M}_d^c) = \log(d-1)$ and $P(\mathcal{M}_d) = C(\mathcal{M}_d) = 1$.

We mostly focus on the channel coherent information and the quantum capacity here. In Section 4 of [1], we show that for any $d \geq 3$,
\[
Q^{(1)}(\mathcal{M}_d) = \max_{\rho_a(u)} \Delta(\mathcal{M}_d, \rho_a(u)),
\]
where $\rho_a(u) = (1-u)|0\rangle + u|i\rangle$, $0 \leq u \leq 1$ and $i$ is any fixed integer between $1$ and $d - 1$. The channel coherent information described above can be further evaluated:
\[
Q^{(1)}(\mathcal{M}_d) = \max_{0 \leq u \leq 1} \left( h(u) + (1-u) \log(d-1) + g(u, d-1) \right),
\]
where \( h(\cdot) \) is the binary entropy function, and

\[
g(u, d) = (d - 1) \eta\left(\frac{1 - u}{d}\right) + \eta\left(\frac{1 - u}{d} + u\right),
\]

with \( \eta(x) := x \log x \).

To see this, we first record the following useful identity, which is valid for \( x \in [0, 1] \) and any two states \( \rho, \sigma \) with orthogonal support:

\[
S((1 - x)\rho + x\sigma) = h(x) + (1 - x)S(\rho) + xS(\sigma),
\]

(54)

Setting \( \rho_a(u) = (1 - u)[0] + u[1] \), we use (50) to compute the action of the channel \( \mathcal{M}_d \) on the input state \( \rho_a(u) \):

\[
\mathcal{M}_d(\rho_a(u)) = (1 - u) \frac{1}{d - 1} \sum_{j=0}^{d-2} [j]_b + u[d - 1]_b,
\]

(55)

which using (54) has entropy

\[
S(\mathcal{M}_d(\rho_a(u))) = h(u) + (1 - u) \log(d - 1).
\]

(56)

On the other hand, the complementary channel \( \mathcal{M}_d^c \) acting on \( \rho_a(u) \) yields a state

\[
\mathcal{M}_d^c(\rho_a(u)) = (1 - u) \frac{1}{d - 1} \sum_{j=0}^{d-2} [j]_c + u[0]_c = \left(\frac{1 - u}{d - 1} + u\right)[0]_c + \frac{1 - u}{d - 1} \sum_{j=1}^{d-2} [j]_c.
\]

(57)

with entropy

\[
S(\mathcal{M}_d^c(\rho_a(u))) = -\eta\left(\frac{1 - u}{d - 1} + u\right) - (d - 2)\eta\left(\frac{1 - u}{d - 1}\right).
\]

(58)

The difference of (56) and (58) yields the expression for the channel coherent information in (52).

Subject to the spin-align conjecture in [1],

\[
Q(\mathcal{M}_d) = Q(\mathcal{M}_d) = Q^{(1)}(\mathcal{M}_d).
\]

(59)

Both \( \mathcal{N}_{1/2} \) and \( \mathcal{M}_d \) have non-zero quantum capacity (see Sec. 4 of [1]). We also have an upper bound of the quantum capacity from Section 7.2 of [1]:

\[
Q(\mathcal{M}_d) \leq \log\left(1 + \frac{1}{\sqrt{d - 1}}\right) \leq \frac{1}{\ln 2} \frac{1}{\sqrt{d - 1}}.
\]

(60)

In the following we consider the \( d \)-dimensional generalization of the erasure channel, \( \mathcal{E}_{\lambda,d} \), which replaces a \( d \)-dimensional input by an erasure symbol \( [e] \) (orthogonal to the input space) with probability \( \lambda \) and transmits the input noiselessly with probability \( 1 - \lambda \). Similar to the qubit case, \( \mathcal{E}_{\lambda,d} = \mathcal{E}_{1-\lambda,d} \), and \( \mathcal{E}_{\lambda,d} \) is degradable for \( 0 \leq \lambda \leq 1/2 \) and anti-degradable for \( 1/2 \leq \lambda \leq 1 \). The quantum capacity of \( \mathcal{E}_{\lambda,d} \) is given by an analogue of (27) [43],

\[
Q^{(1)}(\mathcal{E}_{\lambda,d}) = Q(\mathcal{E}_{\lambda,d}) = \max\{(1 - 2\lambda) \log d, 0\}.
\]

(61)

**B. Amplification of \( \mathcal{M}_{d+1} \) with the qudit erasure channel \( \mathcal{E}_{\lambda,d} \)**

Let \( \mathcal{H}_r, \mathcal{H}_{r'}, \mathcal{H}_a \) and \( \mathcal{H}_{a'} \) be Hilbert spaces of dimension 2, \( d \), \( d + 1 \), and \( d \), respectively, with \( \mathcal{H}_a \) and \( \mathcal{H}_{a'} \) being the input spaces for \( \mathcal{M}_{d+1} \) and \( \mathcal{E}_{\lambda,d} \), respectively. Consider a pure state \( |\psi\rangle \in \mathcal{H}_a \otimes \mathcal{H}_r \otimes \mathcal{H}_{a'} \otimes \mathcal{H}_{r'} \) of the following form:

\[
|\psi\rangle_{ara'r'} = \sqrt{1 - w} |0\rangle_r |0\rangle_a + \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_r |i\rangle_{a'} + \sqrt{w} |1\rangle_r |0\rangle_{a'}. \]

(62)

Taking the partial trace of the projector \( |\psi\rangle_{ara'r'} \) over \( \mathcal{H}_r \) and \( \mathcal{H}_{r'} \) results in a density operator \( \rho_{aa'} \) on the input space of \( \mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda}^d \).
We now show that the coherent information \( \Delta(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}, \rho_{aa'}) \) is given by the following expression:

\[
\Delta(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}, \rho_{aa'}) = h(w) + (1 - w) \log d + \lambda f(w, d),
\]

where \( f(w, d) = \eta \left( \frac{1 - w}{d^2} + w \right) + (d^2 - 1) \eta \left( \frac{1 - w}{d^2} \right). \)

(63)

(64)

To show this, we recall that the erasure flag of an erasure channel \( \mathcal{E}_{\lambda,d} \) is orthogonal to the input/output space, which allows us to apply (54) to output states of this channel. Furthermore, the complementary channel of an erasure channel is again an erasure channel, \( \mathcal{E}_{\lambda,d}' = \mathcal{E}_{1-\lambda,d} \).

We first compute the mixed input state \( \rho_{aa'} = \text{tr}_{rr'}[\psi]_{aa'rr'} \) on \( \mathcal{H}_a \otimes \mathcal{H}_{a'} \) from (62):

\[
\rho_{aa'} = (1 - w)[0]_a \otimes \frac{1}{d} \mathds{1}_{a'} + \frac{w}{d} \sum_{i,j=1}^{d} |i, j \rangle \langle j, i - 1|_{aa'}.
\]

(65)

Acting with the erasure channel \( \mathcal{E}_{\lambda,d} \) on the \( a' \)-system gives a state

\[
\sigma_{ab'} = (\text{id}_a \otimes \mathcal{E}_{\lambda,d})(\rho_{aa'}) = (1 - \lambda)\rho_{ab'} + \lambda \rho_{a} \otimes [e]_{b'},
\]

(66)

where we relabeled \( a' \rightarrow b' \) for \( \rho \), and \([e]_{b'}\) denotes the pure erasure flag. The action of \( \mathcal{M}_{d+1} \) on the \( a \)-system of \( \rho_{ab'} \) produces a state

\[
\tau_{bb'} = (\mathcal{M}_{d+1} \otimes \text{id}_{b'})(\rho_{ab'}) = (1 - w)\mathcal{M}_{d+1}([0]_a) \otimes \frac{1}{d} \mathds{1}_{b'} + \frac{w}{d} \sum_{i,j=1}^{d} \mathcal{M}_{d+1}(|i\rangle \langle j|_a) \otimes |i - 1\rangle \langle j - 1|_{bb'}.
\]

(68)

\[
= (1 - w)\frac{1}{d} \hat{\mathds{1}}_b \otimes \frac{1}{d} \mathds{1}_{b'} + \frac{w}{d} \mathds{1}_{b} \otimes \frac{1}{d} \mathds{1}_{b'},
\]

(69)

where we introduced the notation \( \hat{\mathds{1}}_b \) for the identity operator on the \( d \)-dimensional subspace of \( \mathcal{H}_b \) spanned by \( \{|i\rangle_b\}_{i=0}^{d-1} \) (recall that \( \dim \mathcal{H}_b = d + 1 = \dim \mathcal{H}_a \)) and \( \mathds{1}_b \) for the identity operator on the \( d \)-dimensional subspace of \( \mathcal{H}_{b'} \) spanned by \( \{|i\rangle_{b'}\}_{i=0}^{d-1} \). Taking the partial trace over \( b' \) in (69) yields the state

\[
\tau_b = \mathcal{M}_{d+1}(\rho_a) = (1 - w)\frac{1}{d} \hat{\mathds{1}}_b + w|d\rangle_b \langle d|.
\]

(70)

We then have \( \sigma_{bb'} = (\mathcal{M}_{d+1} \otimes \text{id}_{b'})(\sigma_{ab'}) = (1 - \lambda)\tau_{bb'} + \lambda \tau_b \otimes [e]_{b'}, \) whose entropy we can compute using (54):

\[
S(\sigma_{bb'}) = h(\lambda) + (1 - \lambda)S(\tau_{bb'}) + \lambda S(\tau_b)
\]

\[
= h(\lambda) + (1 - \lambda)[h(w) + (1 - w)2\log d + w\log d] + \lambda[h(w) + (1 - w)\log d]
\]

\[
= h(\lambda) + h(w) + (2 - \lambda - w)\log d.
\]

(71)

(72)

(73)

For the complementary channel \( \mathcal{M}_{d+1}^c \otimes \mathcal{E}_{\lambda,d}^c \) we follow a similar strategy, computing first the action of the erasure channel on the \( a' \)-system of the input state \( \rho \):

\[
\sigma_{ac'} = (\text{id}_a \otimes \mathcal{E}_{\lambda,d}^c)(\rho_{ac'}) = \lambda \rho_{ac'} + (1 - \lambda)\rho_{a} \otimes [e]_{c'}.
\]

(74)

Let now \( |\phi\rangle_{cc'} \) be a maximally entangled state between \( \mathcal{H}_{c'} \) and the \( d \)-dimensional subspace of \( \mathcal{H}_c \) spanned by \( \{|i\rangle_b\}_{i=0}^{d-1} \). The channel \( \mathcal{M}_{d+1} \) acting on the \( a \)-system of \( \rho_{ac'} \) yields a state

\[
\tau_{cc'} = (\mathcal{M}_{d+1}^c \otimes \text{id}_{c'})(\rho_{ac'}) = (1 - w)\mathcal{M}_{d+1}^c([0]_a) \otimes \frac{1}{d} \mathds{1}_{c'} + \frac{w}{d} \sum_{i,j=1}^{d} \mathcal{M}_{d+1}^c(|i\rangle \langle j|_a) \otimes |i - 1\rangle \langle j - 1|_{cc'}.
\]

(75)

\[
= (1 - w)\frac{1}{d} \hat{\mathds{1}}_c \otimes \frac{1}{d} \mathds{1}_{c'} + \frac{w}{d} \sum_{i,j=1}^{d} |i - 1, j - 1\rangle \langle i - 1, j - 1|_{cc'}.
\]

(76)

\[
= (1 - w)\frac{1}{d} \hat{\mathds{1}}_c \otimes \frac{1}{d} \mathds{1}_{c'} + \frac{w}{d^2} \langle \phi |_{cc'}
\]

(77)

\[
\frac{1 - w}{d^2} P_{cc'} + \left( w + \frac{1 - w}{d^2} \right) \langle \phi |_{cc'},
\]

(78)
where \( P_{cc'} := \hat{1}_c \otimes 1_{c'} - [\phi]_{cc'} \) is a projector onto a \((d^2 - 1)\)-dimensional subspace of \( \mathcal{H}_c \otimes \mathcal{H}_{c'} \), orthogonal to the support of \([\phi]_{cc'}\). Thus the last equality above essentially splits \( \tau_{cc'} \) into a sum of two density operators with orthogonal support. The marginal of \( \tau_{cc'} \) is given by \( \tau_c = \frac{1}{\tau} \hat{1}_c \), and so the entropy of \( \sigma_{cc'} = \lambda \tau_{cc'} + (1 - \lambda) \tau_c \) equals

\[
S(\sigma_{cc'}) = h(\lambda) + \lambda S(\tau_{cc'}) + (1 - \lambda) S(\tau_c)
\]

where

\[
S(\sigma_{cc'}) = h(\lambda) - \lambda \left[ -h\left(\frac{1-w}{d^2}\right) + (d^2 - 1) \eta\left(\frac{1-w}{d^2}\right) \right] + (1 - \lambda) \log d
\]

\[
= h(\lambda) - \lambda f(w, d) + (1 - \lambda) \log d,
\]

with \( f(w, d) \) as defined in (64).

Using (73) and (81), we finally arrive at

\[
\Delta(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}, \rho_{aa'}) = S(\sigma_{bb'}) - S(\sigma_{cc'}) = h(w) + (1 - w) \log d + \lambda f(w, d),
\]

which is what we set out to prove. In the sequel, we let \( \Delta^*(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}) \) denote the value of \( \Delta(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}, \rho_{aa'}) \) maximized over \( w \in [0, 1] \).

1. **(Super)-amplification of coherent information**

Similar to our study in Section II.D, we focus on the quantity

\[
\delta(\mathcal{M}_{d+1}, \mathcal{E}_{\lambda,d}) = \Delta^*(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}) - Q^{(1)}(\mathcal{M}_{d+1}) - Q^{(1)}(\mathcal{E}_{\lambda,d}).
\]

Since \( \Delta^*(\mathcal{M}_{d+1}, \mathcal{E}_{\lambda,d}) \leq Q^{(1)}(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}) \), the positivity of \( \delta(\mathcal{M}_{d+1}, \mathcal{E}_{\lambda,d}) \) implies the (super)-amplification of the coherent information of \( \mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d} \).

For each \( d \geq 2 \), we perform the optimization for \( \Delta^*(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}) \) and compute \( Q^{(1)}(\mathcal{M}_{d+1}) \) numerically to determine if \( \delta(\mathcal{M}_{d+1}, \mathcal{E}_{\lambda,d}) \) is positive. We find that \( \delta(\mathcal{M}_{d+1}, \mathcal{E}_{\lambda,d}) \) starts to turn positive at some \( \lambda_{\text{min}}(d) \ll 1/2 \), increases to a maximum at \( \lambda = 1/2 \), and then decreases and vanishes as \( \lambda \) increases to some \( \lambda_{\text{max}}(d) \). These boundaries \( \lambda_{\text{min}}(d) \) and \( \lambda_{\text{max}}(d) \) are plotted as functions of \( d \) in Fig. 8, while the function \( \delta(\mathcal{M}_{d+1}, \mathcal{E}_{\lambda,d}) \) is plotted for some typical values of \( d \) in Fig. 9. In addition, \( \delta(\mathcal{M}_{d+1}, \mathcal{E}_{\lambda,d}) \) can be proved to be positive for any \( 0 < \lambda < 1 \) provided \( d \) is sufficiently large, see next subsection.

Subject to the spin alignment conjecture, \( Q(\mathcal{M}_{d+1}) = Q^{(1)}(\mathcal{M}_{d+1}) \). From (61), \( Q^{(1)}(\mathcal{E}_{\lambda,d}) = Q(\mathcal{E}_{\lambda,d}) = 0 \). Hence, the above (super)-amplification of the coherent information can be lifted to that of quantum capacity:

\[
Q(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}) \geq Q^{(1)}(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}) > Q(\mathcal{M}_{d+1}) + Q(\mathcal{E}_{\lambda,d}).
\]

2. **Unconditional amplification of quantum capacity**

We now present a direct proof of amplification and super-amplification of quantum capacity for \( \mathcal{M}_{d+1} \) and \( \mathcal{E}_{\lambda,d} \) without conditioning on the SAC. The sum of the separate capacities of \( \mathcal{M}_{d+1} \) and \( \mathcal{E}_{\lambda,d} \) can be bounded from above using (60) and (61):

\[
Q(\mathcal{M}_{d+1}) + Q(\mathcal{E}_{\lambda,d}) \leq \log \left( 1 + \frac{1}{\sqrt{d}} \right) + \max\{(1 - 2\lambda) \log d, 0\} =: u(\lambda, d).
\]

We define a quantity

\[
\mu(w, d, \lambda) := \Delta(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}, \rho_{aa'}) - u(\lambda, d)
\]

so that when \( \mu(w, d, \lambda) \) is positive, we obtain the following chain of inequalities,

\[
Q(\mathcal{M}_{d+1}) + Q(\mathcal{E}_{\lambda,d}) \leq u(\lambda, d) < \Delta^*(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}) \leq Q(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}),
\]

thus establishing amplification or super-amplification of the quantum capacity of \( \mathcal{M}_{d+1} \) and \( \mathcal{E}_{\lambda,d} \).

Our numerical optimization shows that \( \mu(w, d, \lambda) > 0 \) for \( \lambda \in [0.37, 0.57] \) and \( d = d(\lambda) \) chosen sufficiently large. For \( \lambda \in [0.37, 0.5] \) (and suitably chosen \( d \)) both channels \( \mathcal{M}_{d+1} \) and \( \mathcal{E}_{\lambda,d} \) have strictly positive quantum capacity, and hence we find super-amplification of quantum capacity in this parameter regime. For example, this occurs for
0.4476 < \lambda < 0.5210 and \ d \geq 8. The smallest dimension for which amplification of quantum capacity can be certified is \ d = 4 at \ \lambda = 1/2. This can be seen in Fig. 8, in which we numerically find \ \lambda_{\text{Q}}^\text{min}(d), \lambda_{\text{Q}}^\text{max}(d) \ for \ each \ \ d \ so \ that \ \text{max}_w \ \mu(w, d, \lambda) > 0 \ \text{for} \ \lambda_{\text{Q}}^\text{min}(d) \leq \lambda \leq \lambda_{\text{Q}}^\text{max}(d). \ In \ addition, \ we \ find \ that \ \text{max}_w \ \mu(w, d, \lambda) \ \text{depends \ on} \ \lambda \ \text{similarly \ as \ for} \ \delta(M_{d+1} \otimes E_{\lambda,d}) \ (\text{see \ Fig. \ 9}).

Finally, we present an analytic proof of amplification and superamplification of the quantum capacity of \ M_{d+1} \otimes E_{\lambda,d} \ for \ the \ entire \ range \ of \ erasure \ probability \ 0 < \lambda < 1 \ using \ a \ log-singularity-like \ argument \ [28]. \ We \ will \ show \ that \ \mu(w, d, \lambda) \ > 0 \ \text{for sufficient small} \ \lambda \ \text{when} \ d \ \text{is sufficiently large. \ We \ consider \ the \ three \ cases} \ 0 < \lambda < 1/2, \ \lambda = 1/2, \ \text{and} \ 1/2 < \lambda \leq 1 \ \text{separately. \ For \ the \ first \ two \ cases, \ when} \ 0 < \lambda \leq 1/2, \ \text{the quantity} \ \mu(w, d, \lambda) \ \text{takes \ the form}

$$\mu(w, d, \lambda) \simeq c_0 + c_1 \log d + O \left( \frac{1}{\sqrt{d}} \right), \ (88)$$

where \ c_0 = (1 - \lambda)h(w) \ \text{and} \ c_1 = -w(1 - 2\lambda). \ \text{When} \ 0 < \lambda < 1/2, \ \text{since} \ 0 < w < 1, \ \text{both} \ c_0 > 0 \ \text{and} \ c_1 < 0. \ \text{Furthermore,} \ c_0 > 0 \ \text{has a} \ w\text{-log-singularity \ of \ rate} \ 1 - \lambda \ \text{while} \ c_1 \ \text{does \ not, \ so,} \ \mu(w, d, \lambda) > 0 \ \text{for \ sufficient \ small} \ w. \ \text{For \ example, \ this \ is \ achieved \ by \ the \ choice}

$$w = \exp \left( 1 - 2 \frac{|1 - 2\lambda|}{1 - \lambda} \log d \right), \ (89)$$

with \ \text{exp \ taken \ to \ base} \ 2. \ \text{For} \ \lambda = 1/2 \ \text{we \ have} \ c_1 = 0, \ \text{and \ letting} \ w = 1/2 \ \text{gives} \ \mu(w, d, \lambda) \simeq 1/2 > 0.

When \ 1/2 < \lambda < 1,

$$\mu(1 - w, d, \lambda) \simeq c_0 - c_1 \log_2 d + O \left( \frac{1}{\sqrt{d}} \right). \ (90)$$

As \ argued \ before, \ we \ then \ have \ \mu(1 - w, d, \lambda) > 0 \ \text{for \ small \ enough} \ w; \ \text{the \ choice \ in} \ (89) \ \text{once \ again \ works \ here.} \ \text{Note \ that \ this \ analysis \ also \ shows \ (super)-amplification \ for \ coherent \ information \ for \ the \ same \ range \ of} \ \lambda \ \text{mentioned \ in \ Section \ III B 1.}

We \ conclude \ with \ the \ following \ observation: \ For \ \lambda = 1/2 \ \text{and} \ \text{large} \ d, \ \text{the \ sum \ of \ the \ capacities \ of \ the \ two \ channels \ becomes \ arbitrarily \ small, \ since} \ Q(E_{1/2,d}) = 0 \ \text{due \ to} \ (61), \ \text{and} \ Q(M_{d+1}) \ \text{has \ the \ vanishing \ upper \ bound} \ (60). \ \text{Meanwhile, \ the \ coherent \ information} \ \Delta(M_{d+1} \otimes E_{1/2,d}, \rho_{aa'}) \ \text{tends \ to} \ h(w)/2 \ \text{as} \ d \to \infty. \ \text{At} \ w = 1/2, \ \text{this} \ \text{lower \ bound \ is} \ \text{exactly \ equal \ to \ 1/2.} \ \text{Thus, \ the \ channels \ used \ jointly \ retain \ positive \ coherent \ information \ and \ quantum \ capacity \ at \ least} \ 1/2. \ \text{This \ phenomenon \ can \ be \ considered \ as \ a \ form \ of \ “near-super-activation.”}
FIG. 8. Plot of the region of super-additivity of coherent information and quantum capacity of the quantum channel $\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}$. The solid lines are the minimal values $\lambda_{\text{min}}(d)$ (blue) and maximal values $\lambda_{\text{max}}(d)$ (magenta) between which $\delta(\mathcal{M}_{d+1}, \mathcal{E}_{\lambda,d}) > 0$, that is, $\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}$ has super-additive coherent information. The dashed lines are the minimal values $\lambda_{Q\text{min}}(d)$ (blue) and maximal values $\lambda_{Q\text{max}}(d)$ (magenta) between which $\mu(w, d, \lambda) > 0$, that is, $\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}$ has super-additive quantum capacity. This figure is identical to Fig. 2 in the main text and reproduced here for convenience.

FIG. 9. Plotting $\delta(\mathcal{M}_{d+1}, \mathcal{E}_{\lambda,d})$ (green) and $\Delta^*(\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}) - u(\lambda, d)$ (orange) for $d = 10$ (left) and $d = 50$ (right). When these quantities are positive, $\mathcal{M}_{d+1} \otimes \mathcal{E}_{\lambda,d}$ has (super)-amplification of coherent information and quantum capacity respectively.