Electromagnetic finite-size effects beyond the point-like approximation

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Abstract. We present a model-independent and relativistic approach to analytically derive electromagnetic finite-size effects beyond the point-like approximation. The key element is the use of electromagnetic Ward identities to constrain vertex functions, and structure-dependence appears via physical form-factors and their derivatives. We apply our general method to study the leading finite-size structure-dependence in the pseudoscalar mass (at order 1/L³) as well as in the leptonic decay amplitudes of pions and kaons (at order 1/L²). Knowledge of the latter is essential for Standard Model precision tests in the flavour physics sector from lattice simulations.

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1 Introduction

Lattice quantum chromodynamics (QCD) allows for systematically improvable Standard Model (SM) precision tests from numerical simulations performed in a finite-volume (FV), discretised Euclidean spacetime. In order to reach (sub-)percent precision in lattice predictions, also strong and electromagnetic isospin breaking corrections have to be included. The latter are encoded via quantum electrodynamics (QED), but the inclusion of QED in a FV spacetime is complicated because of Gauss’ law [1]. This problem is related to zero-momentum modes of photons and the absence of a QED mass-gap. Several prescriptions of how to include QED in a finite volume have been formulated and the one used here is QED$_\text{L}$ where the spatial zero-modes are removed on each time-slice. The long-range nature of QED in addition enhances the FV effects (FVEs), which typically leads to power-law FVEs that are larger than the exponentially suppressed ones for single-particle matrix elements in QCD alone.

The FVEs for a QCD+QED process depend on properties of the involved particles, including masses and charges, but also structure-dependent quantities such as electromagnetic form-factors and their derivatives. In order to analytically capture the finite-volume scaling fully, one cannot neglect hadron structure, and in the following we develop a relativistic and model-independent method to go beyond the point-like approximation at order $e^2$ in QED$_\text{L}$.

We consider a space-time with periodic spatial extents $L$ but with infinite time-extent. To exemplify the method, we first consider the pseudoscalar mass in Sec. 2, and then proceed to leptonic decays in Sec. 3. The discussion is based on the results in Ref. [2], and the reader is referred there for further technical details.

2 Pseudoscalar Mass

To study the finite-size scaling in the mass $m_P(L)$ of a charged hadronic spin-0 particle $P$, we first define the full QCD+QED infinite-volume two-point Euclidean correlation function

$$C_2^\infty(p) = \int d^4x \langle 0 | T[\phi(x)\phi^\dagger(0)] | 0 \rangle e^{-ipx}. \quad (1)$$

Here $\phi$ is an interpolating operator coupling to $P$, and $p = (p_0, \mathbf{p})$ is the momentum. We denote the finite-volume counterpart of this correlator $C_2^L(p)$, but for the moment only consider $C_2^\infty(p)$. This can be diagrammatically represented as

$$C_2^\infty(p) = \begin{array}{c} \phi \\ \end{array} \begin{array}{c} \phi \\ \end{array} = Z_P \cdot D(p) \cdot Z_P, \quad D(p) = \frac{Z(p^2)}{p^2 + m_P^2}, \quad Z_P = \langle 0 | \phi(0) | P, \mathbf{p} \rangle, \quad (2)$$

where the double-line represents the QCD+QED propagator $D(p)$, the $\phi$-blob is the overlap between $\phi$ and $P$ and $Z(p^2) = 1 + O(p^2 + m_P^2)$ is the residue of the propagator. Expanding $C_2^\infty(p)$ in (2) around $e = 0$ yields

$$\begin{array}{c} \phi \\ \end{array} \begin{array}{c} \phi \\ \end{array} = \begin{array}{c} \phi_0 \\ \end{array} \begin{array}{c} \phi_0 \\ \end{array} + \begin{array}{c} \phi_0 \\ \end{array} \begin{array}{c} C \\ \end{array} \begin{array}{c} \phi_0 \\ \end{array} + O(e^4), \quad (3)$$
where quantities with subscript 0 are evaluated in QCD alone. The grey blob is the Compton scattering kernel defined via

$$C = C_{\mu\nu}(p, k, q) = \int d^4x d^4y d^4z e^{i p x + i k y + i q z} \frac{\langle 0 | T[\phi(0) J_\mu(x) J_\nu(y) \phi^\dagger(z)] | 0 \rangle}{Z_{\bar{p}0}^2 D_0(p) D_0(p + k + q)}.$$ (4)

Here $k$ and $q$ are incoming photon momenta and $J_\mu(x)$ is the electromagnetic current. Note that the unphysical dependence on the arbitrary interpolating operator $\phi$ must cancel for any physical quantity, and when the external legs in $C_{\mu\nu}(p, k, q)$ go on-shell the kernel is nothing but the physical forward Compton scattering amplitude. Using (3) the electromagnetic mass-shift of the meson is readily obtained in terms of an integral over the photon loop-momentum $k$. One may follow an equivalent procedure for the finite-volume correlation function $C_L(p)$, where the integral over the spatial momentum $k$ is replaced by a sum. The leading electromagnetic FVEs in the mass, $\Delta m^2_p(L)$, are thus given by the sum-integral difference

$$\Delta m^2_p(L) = -\frac{e^2}{2} \lim_{p_0 \to -m_0} \left\{ \frac{1}{L^2} \sum' k - \int d^3k \frac{C_{\mu\nu}(p, k, -k)}{(2\pi)^3} \right\} \bigg|_{p_0 = 0},$$ (5)

where the rest-frame $p = 0$ was chosen for convenience and the primed sum indicates the omission of the photon zero-mode $k = 0$ in QED$_L$. The analytical dependence on $1/L$ including structure-dependence can now be obtained from this formula through a soft-photon expansion of the integrand, i.e. an expansion order by order in $|k|$ which is directly related to the expansion in $1/L$ via $|k| = 2\pi |n|/L$ where $n$ is a vector of integers. The first step is to decompose $C_{\mu\nu}(p, k, q)$ into irreducible electromagnetic vertex functions $\Gamma_1$ and $\Gamma_2$ according to

$$C = \Gamma_1 \Gamma_1 + \Gamma_1 \Gamma_1 \Gamma_2 \Gamma_2.$$ (6)

The vertex functions depend in general on the structure of the particle, as can be seen from e.g. the form-factor decomposition

$$\Gamma_1 = \Gamma_\mu(p, k) = (2p + k)_\mu F(k^2, (p + k)^2, p^2) + k_\mu G(k^2, (p + k)^2, p^2),$$ (7)

where $F(k^2, (p + k)^2, p^2)$ and $G(k^2, (p + k)^2, p^2)$ are structure-dependent electromagnetic form-factors depending on three virtualities. This means that $F$ and $G$ contain off-shell effects, but we stress that these non-physical quantities always cancel in the FVEs. The cancellation occurs since the vertex functions $\Gamma_{1,2}$ are related to each other and the propagator $D_0(p)$ via Ward identities. An example of an off-shell relation is $F(0, p^2, -m_{P0}^2) = Z_0(p^2)^{-1}$. The derivatives of $Z_0(p^2)$ are already known in the literature as $\delta D^{(0)}(0) [3]$ and $z_n [4]$, but these could in principle be set to zero as they always cancel in the final results. The Ward identities further yield $G$ as a function of $F$. The form-factor $F$ also contains physical information, and for our purposes it suffices to know that $F^{(1,0,0)}(0, -m_{P0}^2, -m_{P0}^2) = F'(0) = -(r^2_p)/6$, where $r^2_p$ is the physical electromagnetic charge radius of $P$ which is well-known experimentally [5].

Using our definitions of the vertex functions in $C_{\mu\nu}(p, k, q)$ in (5) we obtain the FVEs

$$\Delta m^2_p(L) = e^2 m_p^2 \left\{ \frac{c_2}{4\pi^2 m_p L} + \frac{c_1}{2\pi (m_p L)^2} + \frac{\langle r^2_p \rangle}{3m_p L^3} + \frac{C}{(m_p L)^3} + O\left[ \frac{1}{(m_p L)^3} \right] \right\},$$ (8)
where the $c_j$ are finite-volume coefficients specific to QED$_L$ arising from the sum-integral difference in (5). These are discussed in detail in Ref. [2]. Here we see the charge radius $r_p^2$ appearing at order $1/L^3$ and its coefficient agrees with that derived within non-relativistic scalar QED [6]. However, there is an additional structure-dependent term $C$ related to the branch-cut of the forward, on-shell Compton amplitude. This contribution can be found, in other forms, also in Refs. [3, 7], and only arises because of the QED$_L$ prescription with the subtracted zero-mode. Its value is currently unknown but one can show $C > 0$ [2], meaning that it cannot cancel the charge radius contribution. Note that all unphysical off-shell contributions from the form-factors $F$ and $G$ have vanished.

### 3 Leptonic Decays

Leptonic decay rates of light mesons are of the form $P^- \to \ell^- \bar{\nu}_\ell$, where $P$ is a pion or kaon, $\ell$ a lepton and $\nu_\ell$ the corresponding neutrino. These are important for the extraction of the Cabibbo-Kobayashi-Maskawa matrix elements $|V_{ud}|$ and $|V_{ud}|$ [8, 9]. The leading virtual electromagnetic correction to this process yields an infrared (IR) divergent decay rate $\Gamma_0$. One must therefore add the real radiative decay rate $\Gamma_1(\Delta E)$ for $P^- \to \ell^- \bar{\nu}_\ell \gamma$, where the photon has energy below $\Delta E$, to cancel the IR-divergence in $\Gamma_0$. The IR-finite inclusive decay rate is thus $\Gamma(P^- \to \ell^- \bar{\nu}_\ell \gamma)$, and following the lattice procedure first laid out in Ref. [8] we may write

$$\Gamma_0 + \Gamma_1(\Delta E, \gamma; x) = \lim_{L \to \infty} [\Gamma_0(L) - \Gamma_0\text{uni}(L)] + \lim_{L \to \infty} [\Gamma_0\text{uni}(L) + \Gamma_1(L, \Delta E, \gamma)] \quad \text{(9)}$$

Here, Ref. [8] chose to add and subtract the universal finite-volume decay rate $\Gamma_0\text{uni}(L)$, calculated in point-like scalar QED in Ref. [4], to cancel separately the IR-divergences in $\Gamma_0$ and $\Gamma_1$. In the following we are interested in only the first term in brackets. The subtracted term $\Gamma_0\text{uni}(L)$ cancels the FVEs in $\Gamma_0(L)$ through order $1/L$, and hence $\Gamma_0(L) - \Gamma_0\text{uni}(L) \sim O(1/L^2)$. Structure-dependence enters at order $1/L^2$. With the goal of systematically improving the finite-volume scaling order by order including structure-dependence, we replace the universal contribution by

$$\Gamma_0\text{uni}(L) \rightarrow \Gamma_0(\text{uni})(L) = \Gamma_0\text{uni}(L) + \sum_{j=2}^n \Delta \Gamma_0^{(j)}(L),$$

where $n \geq 2$ and $\Delta \Gamma_0^{(j)}(L)$ contains the FVEs at order $1/L^j$. This means that the finite-volume residual instead scales as $\Gamma_0(L) - \Gamma_0(\text{uni})(L) \sim O(1/L^{n+1})$. We may parametrise $\Gamma_0(\text{uni})(L)$ in terms of a finite-volume function $Y_0^{(n)}(L)$ according to

$$\Gamma_0(\text{uni})(L) = \Gamma_0\text{tree}(L) \left[ 1 + \frac{\alpha}{4\pi} Y_0^{(n)}(L) \right] + O\left(\frac{1}{L^{n+1}}\right),$$

where $\Gamma_0\text{tree}$ is the tree-level decay rate.

Since we are interested in the leading structure-dependent contribution we consider $Y_0^{(2)}(L)$. In order to derive it, we define the QCD+QED correlation function

$$C^{\ell \bar{\nu}_\ell}_W(p, p_\ell) = \int d^4z \epsilon^{\mu \nu \rho \sigma} \langle \ell^-, p_\ell, r; \nu_\ell, p_{\nu}; s | T[O_W(0) \phi^+(z)] | 0 \rangle,$$

where $p_\ell = (p^0_\ell, p_\ell)$ is the momentum of the on-shell lepton of mass $m_\ell$, $p_{\nu} = (p^0_{\nu}, p_{\nu})$ is the momentum of the massless neutrino and $O_W(0)$ is the four-fermion operator of the decay in
question. We may diagrammatically represent this in a similar way as for the mass according to

\[ C_W(p, p_f) = \phi \begin{array}{c} \phi_0 \\ \Delta \end{array} + \phi_0 \begin{array}{c} \phi_0 \\ W \end{array} \]  

(13)

The grey blob containing \( W \) is of order \( \epsilon^2 \) and can be separated, just like the Compton amplitude, into several irreducible vertex functions. The exact definitions of these vertex functions are quite involved and can be found in Ref. \[2\], but several comments can be made. First of all, the vertex functions are related to various structure-dependent form-factors containing both on-shell and off-shell information. Again, the off-shellness must cancel. The vertex functions also contain physical structure-dependent information (similar to how \( \Gamma_1 \) depends on the charge radius) and for \( Y^{(2)}(L) \) this is the axial-vector form-factor \( F_A(-m_L^2) = F_A^p \) from the real radiative decay \( P^- \to e^-\bar{v}_e\gamma \).

By performing the amputation on the external meson leg in (12) to obtain the matrix element needed for the decay rate in (11), one finds the finite-volume function \( Y^{(2)}(L) \) to be

\[ Y^{(2)}(L) = \frac{3}{4} + 4 \log \left( \frac{m_L}{m_W} \right) + 2 \log \left( \frac{m_W L}{4 \pi} \right) + \frac{c_3 - 2 (c_3(v_L) - B_1(v_L))}{2\pi} - \\
- 2 A_1(v_L) \left[ \log \left( \frac{m_p L}{2\pi} \right) + \log \left( \frac{m_L}{2\pi} \right) - 1 \right] - \frac{1}{m_p L} \left[ (1 + r_L^2 c_2 - 4 r_L^2 c_2(v_L)) \right] + \\
+ \frac{1}{(m_p L)^2} \left[ \frac{F_A^p}{f_p} 4\pi m_p \left[ (1 + r_L^2 c_1 - 4 r_L^2 c_1(v_L)) \right] + \frac{8\pi [(1 + r_L^2 c_1 - 2 c_1(v_L))]}{(1 - r_L^2)} \right]. \]

(14)

Here, \( r_L = m_L/m_p \), \( v_L = p_L/E_L \) the lepton velocity in terms of the energy \( E_L \), and \( m_W \) the \( W \)-boson mass. Also, \( c_k, A_1(v_L), B_1(v_L) \) and \( c_j(v_L) \) are finite-volume coefficients defined in Ref. \[2\]. Note that no unphysical quantities appear. At order \( 1/L^2 \), there is one structure-dependent contribution proportional to \( F_A^p \) and the other term is purely point-like. This result is in perfect agreement with Ref. \[4\] for the universal terms up to \( O(1/L) \), which we derived in a completely different approach. The numerical impact of the \( 1/L^2 \)-corrections is studied in Ref. \[2\].

4 Conclusions

We have presented a relativistic and model-independent method to derive electromagnetic FVEs beyond the point-like approximation. We are currently working to obtain the leading FVEs for semi-leptonic kaon decays, relevant for future precision tests in the SM flavour physics sector.

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