Global Well-Posedness and Scattering for the Defocusing, Mass-Critical Generalized KdV Equation

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Abstract In this paper we prove that the defocusing, mass-critical generalized KdV initial value problem is globally well-posed and scattering for $u_0 \in L^2(\mathbb{R})$. To prove this, we combine the profile decomposition of Killip et al. (Discrete Contin Dyn Syst Ser A 32(1):191–221, 2012) with an interaction Morawetz estimate constructed from the monotonicity formula of Tao (Discrete Contin Dyn Syst Ser A 18(1):1–14, 2007).

Keywords KdV · Scattering · Concentration compactness

1 Introduction

In this paper we plan to study the global well-posedness theory for the initial value problem for the defocusing, real valued, generalized KdV equation,

$$\partial_t u + \partial_{xxx} u = \partial_x (u^5), \quad u(0) \in L^2_x(\mathbb{R}), \quad x \in \mathbb{R}, t \in \mathbb{R}. \quad (1.1)$$

The set of solutions of (1.1) is invariant under the scaling

$$u_{\lambda}(x, t) = \lambda^{1/2} u(\lambda^3 t, \lambda x) \quad (1.2)$$

in the sense that if $u$ solves (1.1) then so does $u_{\lambda}$ with initial datum

$$u_{\lambda}(0, x) = \lambda^{1/2} u(0, \lambda x). \quad (1.3)$$
Notice that \( \|u_\lambda(0, x)\|_{L^2(Z)} = \|u(0, x)\|_{L^2(Z)} \), so (1.1) is called an \( L^2 \)-critical generalized KdV equation. The \( L^2 \) norm, or mass, is conserved under the flow (1.1).

\[
M(u(t)) = \int_R u(t, x)^2 \, dx = M(u(0)),
\]

(1.4)

Another conserved quantity of (1.1) is the energy

\[
E(u(t)) = \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(t, x) \, dx + \frac{1}{6} \int_{-\infty}^{\infty} u^6(t, x) \, dx = E(u(0)).
\]

(1.5)

**Definition 1.1 (Solution)** A function \( u : I \times R \to R \) on a non-empty interval \( 0 \in I \subset R \) is a (strong) solution to (1.1) if it lies in the class \( C^0_C \times L^2_x \cap L^5_x L^{10}_t (J \times R) \) for any compact \( J \subset I \), and obeys the Duhamel formula

\[
u(t) = e^{-t\partial_x^3} u(0) + \int_0^t e^{-(t-\tau)\partial_x^3} \partial_x(u^5(\tau)) \, d\tau.
\]

(1.6)

We say that \( u \) is a maximal lifespan solution if the solution cannot be extended to any strictly larger interval. We say that \( u \) is a global solution if \( I = R \).

[18] developed a global in time theory for initial data with sufficiently small \( L^2(\mathbb{R}) \) norm. The results turn local for arbitrary \( L^2(\mathbb{R}) \) data, with the time of existence depending on the shape of the initial data \( u_0 \) not just its size. In particular, if \( u_0 \) is a little bit more regular than \( L^2(\mathbb{R}) \), say \( u_0 \in H^s(\mathbb{R}) \) for some \( s > 0 \), then a solution to (1.1) exists on a time interval \([0, T]\) for \( T(\|u_0\|_{H^s(\mathbb{R})}) > 0 \). This combined with (1.5) and Sobolev embedding implies that a solution to (1.1) is global if \( u_0 \in H^1(\mathbb{R}) \).

The size of the integral in (1.6) is controlled by the scattering size.

**Definition 1.2 (Scattering size)**

\[
S_I(u) = \int_R \left( \int_I |u(t, x)|^{10} \, dt \right)^{1/2} \, dx = \|u\|_{L^2_x L^{10}_t (I \times R)}^5.
\]

(1.7)

Associated with the notion of a solution is a corresponding notion of blowup.

**Definition 1.3 (Blowup)** We say that a solution \( u \) to (1.1) blows up forward in time if there exists \( t_1 \in I \) such that

\[
S_{[t_1, \sup(I))}(u) = \infty,
\]

(1.8)

and that \( u \) blows up backward in time if there exists a time \( t_1 \in I \) such that

\[
S_{(\inf(I), t_1]}(u) = \infty.
\]

(1.9)

This precisely corresponds to the impossibility of continuing the solution (in the case of blowup in finite time) or failure to scatter (in the case of blowup in infinite time).

We summarize the results of [18] below.
Theorem 1.1 (Local well-posedness) Given $u_0 \in L^2_x(\mathbb{R})$ and $t_0 \in \mathbb{R}$, there exists a unique solution $u$ to (1.1) with $u(t_0) = u_0$. Let $I \subset \mathbb{R}$ denote the lifespan of $u$. This solution also has the following properties:

1. (Local existence) $I$ is an open neighborhood of $t_0$.
2. (Blowup criterion) If $\sup(I)$ is finite then $u$ blows up forward in time. If $\inf(I)$ is finite then $u$ blows up backward in time.
3. (Scattering) If $\sup(I) = +\infty$ and $u$ does not blow up forward in time, then $u$ scatters forward in time. That is, there exists a unique $u_+ \in L^2_x(\mathbb{R})$ such that
   \[ \lim_{t \to +\infty} \| u(t) - e^{-t\partial_x^3}u_+ \|_{L^2_x(\mathbb{R})} = 0. \] (1.10)

Conversely, given $u_+ \in L^2_x(\mathbb{R})$ there is a unique solution to (1.1) in a neighborhood of $+\infty$ so that (1.10) holds. One can define scattering backward in time in a completely analogous manner.
4. (Small data global existence) If $M(u_0)$ is sufficiently small then $u$ is a global solution which does not blow up either forward or backward in time. Indeed, in this case
   \[ S_{\mathbb{R}}(u) \lesssim M(u)^{5/2}. \] (1.11)

Remark See [4,5] for the analogous result for the nonlinear Schrödinger equation.

The proof uses the oscillatory integral estimates of [17] in a similar manner to [41] for the nonlinear Schrödinger equation. In this paper we will prove

Theorem 1.2 (Spacetime bounds for the mass-critical gKdV) The defocusing mass-critical gKdV problem (1.1) is globally well-posed for arbitrary initial data $u_0 \in L^2(\mathbb{R})$. Furthermore, the global solution satisfies the following spacetime bounds
   \[ \| u \|_{L^5_t L^{10}_x(\mathbb{R} \times \mathbb{R})} \leq C(M(u_0)). \] (1.12)

The function $C : [0, \infty) \to [0, \infty)$.

Remark This result is sharp in the sense that there is no local well-posedness in $H^s$ for any $s < 0$. See [6].

Remark This paper does not consider the focusing problem at all. See [19,23] for more information on this topic and the conjectured result.

This theorem is proved using concentration compactness. This method has proved to be useful to the study of elliptic, parabolic, and hyperbolic partial differential equations. For example the concentration compactness method and the closely related induction on energy method have proved very useful to the study of the energy critical [2,9,15,20,21,25,26,36,42,48] and mass critical [1,3,11–14,22,24,27,33,38,46,47] nonlinear Schrödinger equations.

[19] demonstrated that if a solution to (1.1) blows up in finite time $T_* < \infty$, then there exists a $C_0$ such that at least $C_0$ amount of mass must concentrate in a window of width $c(T_* - t)^{1/2} \| u(t) \|_H^{1/2s}$ for some $s > 0$. 

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[37] proved that a sequence of \( u_n \) bounded in \( L^2_x(\mathbb{R}) \) has the profile decomposition

\[
  u_n = \sum_{j=1}^{J} (\lambda_n^j)^{-1/2} e^{it_n^j} \partial_x^3 \text{Re}[e^{i \frac{\lambda_n^j}{\lambda_n^j} \phi^j(x - x_n^j)}] + w_n^J, \tag{1.13}
\]

where \( (t_n^j, \xi_n^j, x_n^j, \lambda_n^j) \) and \( (t_n^k, \xi_n^k, x_n^k, \lambda_n^k) \) are pairwise asymptotically orthogonal for \( j \neq k \), and

\[
  \lim_{J \to \infty} \limsup_{n \to \infty} \|e^{i \partial_x^3} |\partial_x|^{1/6} w_n^J \|_{L^6_t L^6_x(\mathbb{R}^2)} = 0. \tag{1.14}
\]

[37] called \( (t_n^j, \xi_n^j, x_n^j, \lambda_n^j) \) and \( (t_n^k, \xi_n^k, x_n^k, \lambda_n^k) \) asymptotically orthogonal if

\[
  \lim_{n \to \infty} \left( \ln \frac{\lambda_n^j}{\lambda_n^k} + \lambda_n^j |\xi_n^j| - \xi_n^k | - \frac{|t_n^j - t_n^k|}{(\lambda_n^j)^3} + \frac{3 |(t_n^j - t_n^k)\xi_n^j|}{(\lambda_n^j)^2} + \frac{|\lambda_n^j - \lambda_n^k + 3(t_n^j - t_n^k)(\xi_n^j)^2}{\lambda_n^j} \right) = \infty.
\]

Later, [23] proved a conditional concentration compactness result.

**Theorem 1.3** (Concentration compactness theorem) Assume that the defocusing mass-critical nonlinear Schrödinger equation in one dimension,

\[
  (i \partial_t + \partial_{xx}) v = |v|^4 v \tag{1.15}
\]

has global spacetime bounds

\[
  \int_{\mathbb{R}} \int_{\mathbb{R}} |v(t, x)|^6 dx dt \leq C(M(v(0, x))). \tag{1.16}
\]

Then if theorem 1.2 fails to be true, there exists a critical mass \( 0 < M_c < \infty \) and a solution \( u \) to (1.1) that blows up both forward and backward in time on its lifespan \( I, M(u(t)) = M_c, \) and \( \{u(t) : t \in I\} \subset \{\lambda^{1/2} f(\lambda(x + x_0)) : \lambda \in (0, \infty), x_0 \in \mathbb{R}, f \in K\} \) for some compact \( K \subset L^2_x(\mathbb{R}) \).

(1.16) was later proved to be true.

**Theorem 1.4** (Space-time bounds for the mass-critical nonlinear Schrödinger equation) The space-time bounds (1.16) hold for all solutions of (1.15).

**Proof** [12].

Therefore, to prove theorem 1.2 it suffices to show that the solution to (1.1) described in theorem 1.3 can only be \( u \equiv 0 \). Notice that modulo symmetries in \( x_0 \) and \( \lambda \) the minimal mass blowup solution described in theorem 1.3 lies in a precompact set. Therefore, a
sequence of solutions will have a convergent subsequence modulo symmetries in \(x_0\) and \(\lambda\).

For any \(t \in I\) let \(N(t) \in (0, \infty)\) and \(x(t) \in \mathbb{R}\) be the scale function and spatial function respectively such that

\[
\frac{1}{N(t)^{1/2}} u \left( \frac{x - x(t)}{N(t)} \right) \in K \subset L^2(\mathbb{R}), \quad K \text{ compact.}
\] (1.17)

**Remark** We have some flexibility with regard to the \(N(t), x(t)\) and \(K\) that we choose. This will be discussed in the concentration compactness section.

To rule out the minimal mass blowup solution in theorem 1.3 it suffices to exclude three scenarios:

1. The double rapid cascade.

\[
N(t) \geq 1, \quad N(0) = 1, \quad \int_I N(t)^2 dt \lesssim 1, \quad \lim_{t \uparrow \sup(I)} N(t) = \lim_{t \downarrow \inf(I)} N(t) = +\infty.
\] (1.18) (1.19)

2. The self-similar solution.

\[
N(t) \sim t^{-1/3}, \quad t \in (0, \infty).
\] (1.20)

3. The quasi-soliton solution.

\[
\int_J N(t)^3 dt \sim \mathcal{J}, \quad \int_J N(t)^2 \lesssim \mathcal{J}, \quad E(u(t)) \lesssim 1,
\] (1.21) (1.22)

for some \(\mathcal{J}\) large, \(J \subset I\).

The first two scenarios are precluded by an additional regularity argument. Conservation of energy and \(E(u(t)) \lesssim 1\) prevents \(N(t) \not\to \infty\).

As the reader can probably surmise, the quasi-soliton analysis heavily utilizes previous soliton results. [10] proved that there do not exist \(L^2\) soliton solutions for the KP-II equation. For the defocusing generalized KdV problem [44] proved the nonexistence of a soliton solution to the generalized KdV equation by showing that the center of energy moves to the left faster than the center of mass. This analysis was continued by [28] and then by [40]. We utilize the computations in [44] to produce an interaction Morawetz estimate that is similar in flavor to the interaction Morawetz estimate of [14]. This rules out the final scenario, proving theorem 1.2.

**Remark** The reader should consult [29–31] for more information on focusing solitons. See [32] for more information on other blowup solutions.

**Outline of the Paper** In section two we discuss some properties of the linear solution to the Airy equation \((\partial_t + \partial_{xxx})u = 0\) as well as estimates for the nonlinear equation
Most of these estimates can be found in [18, 19, 23]. Section three will discuss the local conservation of the quantities mass and energy, as well as the monotonicity formula of [44]. In section four we will describe the concentration compactness of [23] and then list the three minimal mass blowup scenarios. In section five we will rule out the double rapid cascade. In section six we will rule out the self-similar blowup scenario. In section seven we will rule out the quasi-soliton.

2 Linear Estimates

This section gives a brief overview of the notation that will be used in this paper, and then reviews some linear and nonlinear estimates for the KdV problem. Nothing in this section is original.

Definition 2.1 (Mixed norm spaces) Suppose $I \subset \mathbb{R}$ is an interval. Define

$$L^p_x L^q_t(I \times \mathbb{R}) = \{ F(x, t) : \left( \int_I \left( \int_{\mathbb{R}} |F(x, t)|^q \, dt \right)^{p/q} \, dx \right)^{1/p} < +\infty \}, \quad (2.1)$$

and

$$L^p_t L^q_x(I \times \mathbb{R}) = \{ F(t, x) : \left( \int_I \left( \int_{\mathbb{R}} |F(t, x)|^q \, dx \right)^{p/q} \, dt \right)^{1/p} < +\infty \}. \quad (2.2)$$

Definition 2.2 (Littlewood–Paley projection) Let $P_k$ be the Fourier multiplier,

$$(\hat{P_k} f)(\xi) = \phi(2^{-k}\xi) \hat{f}(\xi), \quad (2.3)$$

where $\phi$ is a smooth function supported on $\frac{1}{4} \leq |\xi| \leq 4$, $\phi \equiv 1$ on $\frac{1}{2} \leq |\xi| \leq 2$, $\sum_{k \in \mathbb{Z}} \phi(2^{-k}\xi) = 1$ when $\xi \neq 0$. We can also define $P_{\geq k}$ and $P_{\leq k}$ in the standard manner.

Definition 2.3 (Japanese bracket) Let

$$\langle x \rangle = (1 + x^2)^{1/2}. \quad (2.4)$$

Definition 2.4 $(p, q, \alpha)$ is an admissible triple if

$$\frac{1}{p} + \frac{1}{2q} = \frac{1}{4}, \quad \alpha = \frac{2}{q} - \frac{1}{p}, \quad 1 \leq p, q \leq \infty, \quad -\frac{1}{4} \leq \alpha \leq 1. \quad (2.5)$$

If $(p, q, \alpha)$ is an admissible triple denote $(p, q, \alpha) \in A$.

Proposition 2.1 (Linear estimates) Let $u$ be a solution of the initial value problem

$$(\partial_t + \partial_x^3)u = F, \quad t \in I, x \in \mathbb{R},$$

$$u(0, x) = u_0. \quad (2.6)$$
Then for any admissible triples \((p_j, q_j, \alpha_j), j = 1, 2\),
\[
\|u\|_{L_t^\infty L_x^2(I \times \mathbb{R})} + \|D_x^{\alpha_j} u\|_{L_t^p L_x^{q_j}(I \times \mathbb{R})} \lesssim \|u_0\|_{L_t^2(\mathbb{R})} + \|D_x^{-\alpha_2} F\|_{L_x^{p_2} L_t^{q_2}(I \times \mathbb{R})}.
\]
(2.7)

**Proof.** See [19].

Next, taking a cue from the analysis of the nonlinear Schrödinger equation (see for example [43]), consider the analogue of the Strichartz spaces in the gKdV case.

**Definition 2.5** Let
\[
\|u\|_{S^0(I \times \mathbb{R})} = \|\partial_x u\|_{L_t^\infty L_x^2(I \times \mathbb{R})} + \|\partial_x |^{-1/4} u\|_{L_x^4 L_t^1(I \times \mathbb{R})}.
\]
(2.8)

Notice that by interpolation, if \((p, q, \alpha)\) is an admissible triple, then
\[
\|D_x^{\alpha} u\|_{L_x^p L_t^q(I \times \mathbb{R})} \lesssim \|u\|_{S^0(I \times \mathbb{R})}.
\]
(2.9)

Then let \(N^0(I \times \mathbb{R})\) be the dual of \(S^0(I \times \mathbb{R})\) with norm
\[
\|F\|_{N^0(I \times \mathbb{R})} = \inf_{F_1 + F_2} \|\partial_x |^{1/4} F_1\|_{L_x^{4/3} L_t^1(I \times \mathbb{R})} + \|\partial_x |^{-1} F_2\|_{L_x^1 L_t^2(I \times \mathbb{R})}.
\]
(2.10)

**Lemma 2.2** (More linear estimates) If \(u\) is a solution to (2.6) then
\[
\|u\|_{S^0(I \times \mathbb{R})} + \|u\|_{L_t^\infty L_x^2(I \times \mathbb{R})} \lesssim \|u_0\|_{L_t^2(\mathbb{R})} + \|F_1\|_{N^0(I \times \mathbb{R})} + \|F_2\|_{L_x^1 L_t^2(I \times \mathbb{R})},
\]
(2.11)

for any \(F = F_1 + F_2\) decomposition.

**Proof.** See [16–19].

**Lemma 2.3** (Dispersive estimate) For \(2 \leq p \leq \infty\),
\[
\|e^{-i\alpha^3} u_0\|_{L_x^p(\mathbb{R})} \lesssim t^{-\frac{2}{3}(\frac{1}{2} - \frac{1}{p})} \|u_0\|_{L_x^p(\mathbb{R})}.
\]
(2.12)

**Proof.** Stationary phase calculation. See for example [39].

The linear estimates give a long-time stability theorem.

**Theorem 2.4** (Long-time stability for the mass-critical gKdV) Let \(I\) be a time interval containing zero and let \(\tilde{u}\) be a solution to
\[
(\partial_t + \partial_{xxx})\tilde{u} = \partial_x (\tilde{u}^5) + e, \quad \tilde{u}(0, x) = \tilde{u}_0(x).
\]
(2.13)

Assume that
\[
\|\tilde{u}\|_{L_t^\infty L_x^2(I \times \mathbb{R})} \leq M, \quad \|\tilde{u}\|_{L_x^8 L_t^1(I \times \mathbb{R})} \leq L,
\]
(2.14)
for some positive constants $M$ and $L$. Let $u_0$ be such that
\[
\|u_0 - \tilde{u}_0\|_{L^2(R)} \leq M'. 
\] (2.15)

Also assume the smallness conditions
\[
\|e^{-\frac{3}{2}t^3} (u_0 - \tilde{u}_0)\|_{L^2_x L^1_t (I \times R)} \leq \epsilon,
\|e\|_{N^0(I \times R)} \leq \epsilon, 
\] (2.16)
for some small $0 < \epsilon < \epsilon_1(M, M', L)$. Then there exists a solution $u$ to (1.1) on $I \times R$ with initial data $u_0$ at time $t = 0$ satisfying
\[
\|u - \tilde{u}\|_{L^2_x L^1_t (I \times R)} \leq C(M, M', \epsilon), \\
\|u^5 - \tilde{u}^5\|_{L^1_t L^2_x (I \times R)} \leq C(M, M', \epsilon), \\
\|u - \tilde{u}\|_{L^\infty_t L^2_x (I \times R)} + \|u - \tilde{u}\|_{S^0(I \times R)} \leq C(M, M', L), \\
\|u\|_{L^\infty_t L^2_x (I \times R)} + \|u\|_{S^0(I \times R)} \leq C(M, M', L). 
\] (2.17)

Proof See [23].

In particular, this theorem implies that if $u^n_0 \to u_0$ strongly in $L^2$, and $u$ is the solution to (1.1) on $I \subset R$ with initial data $u_0$, then for any compact $J \subset I$,
\[
\|u\|_{S^0(J \times R)} \leq C < \infty, 
\] (2.18)
which implies that $u^n \to u$ in $S^0(J \times R)$ and $L^\infty_t L^2_x (J \times R)$, where $u^n$ is the solution to (1.1) with initial data $u^n_0$.

3 Conservation of Mass and Energy

Definition 3.1 (Mass density and mass current) The mass density is given by
\[
\rho(t, x) = u(t, x)^2. 
\] (3.1)
The mass current is given by
\[
j(t, x) = 3u_x(t, x)^2 + \frac{5}{3}u(t, x)^6. 
\] (3.2)

Definition 3.2 (Energy density and energy current) The energy density is given by
\[
e(t, x) = \frac{1}{2} u_x(t, x)^2 + \frac{1}{6} u(t, x)^6. 
\] (3.3)
The energy current is given by

\[ k(t, x) = \frac{3}{2} u_{xx}(t, x)^2 + 10u(t, x)^4u_x(t, x)^2 + \frac{1}{2} u(t, x)^{10}. \quad (3.4) \]

A routine computation verifies (for Schwartz solutions, at least) the pointwise conservation laws

\[ \rho_t + \rho_{xxx} = j_x, \quad \text{(3.5)} \]
\[ e_t + e_{xxx} = k_x. \quad \text{(3.6)} \]

**Remark** If \( u \) is a solution to (1.1) on a compact interval \( J \subset I \) that satisfies (2.18), then theorem 2.4 implies that for any \( s < \infty \), \( u \) may be approximated arbitrarily closely in \( L_t^\infty L_x^2 \) by a solution lying in \( L_t^\infty H^s \). This means that we may integrate by parts as often as we would like, and a norm that is uniform over a sequence of successively closer approximations of \( u \) gives a bound on \( u \). Similar computations are done in the case of the interaction Morawetz estimate for the Schrödinger equation.

**Remark** (3.5) and (3.6) readily imply conservation of mass (1.4) and energy (1.5) respectively.

The pointwise conservation conservation laws have been utilized to prove a number of useful results, see for example [28,40,44]. For example, [44] proved a monotonicity formula which excluded the existence of traveling wave solutions to the defocusing problem (1.1). This result will be used very heavily in section seven, when we exclude the quasi-soliton solution.

**Lemma 3.1** (Monotonicity formula) For a smooth function \( u \) with initial data lying in the appropriate weighted Sobolev spaces,

\[ \frac{d}{dt} (\langle x \rangle_M(t) - \langle x \rangle_E(t)) > 0, \quad \text{(3.7)} \]

where \( \langle x \rangle_M \) denotes the center of mass and \( \langle x \rangle_E \) denotes the center of energy.

\[ \langle x \rangle_M = \frac{1}{M(u)} \int x u(t, x)^2 dx, \quad \langle x \rangle_E = \frac{1}{E(u)} \int x [\frac{1}{2} u_x^2 + \frac{1}{6} u^6] dx. \quad \text{(3.8)} \]

Thus

\[ \left( \int \rho(t, x) dx \right) \left( \int k(t, x) dx \right) - \left( \int e(t, x) dx \right) \left( \int j(t, x) dx \right) > 0. \quad \text{(3.9)} \]

**Proof** See [44].
In fact, the proof of lemma 3.1 in [44] actually showed
\[
\left( \int \rho(t, x)dx \right) \left( \int k(t, x)dx \right) - \left( \int e(t, x)dx \right) \left( \int j(t, x)dx \right) \\
\geq \frac{2}{9} \left( \int u(t, x)^6 dx \right)^2 > 0.
\] (3.10)

[44] showed that (3.10) holds for all \( t \), so in particular (3.10) holds at \( t = 0 \). Since the choice of the initial data is arbitrary, this means that (3.10) holds for any sufficiently smooth \( f \) lying in an appropriate weighted space.

### 4 Concentration Compactness

An important step in the study of the mass critical generalized KdV was the reduction to solutions that are almost periodic modulo symmetries by [23].

**Definition 4.1** (Almost periodic modulo symmetries) A solution \( u \) to (the gKdV problem) with lifespan \( I \) is said to be almost periodic modulo symmetries if there exist functions \( N : I \to \mathbb{R}_+ \), \( x : I \to \mathbb{R} \) such that
\[
\left\{ \frac{1}{N(t)^{1/2}} u(t, \frac{x - x(t)}{N(t)}) : t \in I \right\}
\] (4.1)
is contained in a compact subset of \( L^2(\mathbb{R}) \).

The parameter \( N(t) \) measures the frequency scale of the solution at time \( t \), while \( \frac{1}{N(t)} \) measures the spatial scale. \( x(t) \) is the spatial center function. We can multiply \( N(t) \) by any function \( \alpha(t), 0 < \epsilon < \alpha(t) < \frac{1}{\epsilon} \).

**Theorem 4.1** (Arzela–Ascoli theorem) A family of functions is precompact in \( L^2_x(\mathbb{R}) \) if and only if it is norm bounded and there exists a compactness modulus function \( C \) such that
\[
\int_{|x| \geq C(\eta)} |f(x)|^2 dx + \int_{|\xi| \geq C(\eta)} |\hat{f}(\xi)|^2 d\xi \leq \eta
\] (4.2)
for all functions \( f \) in the family.

Thus there exists a function \( C : \mathbb{R}_+ \to \mathbb{R}_+ \) such for all \( t \in I \) and \( \eta > 0 \),
\[
\int_{|x - x(t)| \geq \frac{C(\eta)}{N(t)}} |u(t, x)|^2 dx + \int_{|\xi| \geq C(\eta)N(t)} |\hat{u}(t, \xi)|^2 d\xi < \eta.
\] (4.3)

\( C \) will be called the compactness modulus function.

Let
\[
L(M) = \sup \{ S_I(u) : u : I \times \mathbb{R} \to \mathbb{R}, \ M(u) \leq M \}.
\] (4.4)
The supremum is taken over all solutions $u : I \times \mathbb{R} \to \mathbb{R}$ obeying $M(u) \leq M$. For $M$ small, a small data result implies $L(M) \lesssim M^{3/2}$. This fact combined with theorem 2.4 implies that failure of theorem 1.2 is equivalent to the existence of a critical mass $M_c \in (0, \infty)$ such that

$$L(M) < \infty \text{ for } M < M_c, \quad L(M) = \infty \text{ for } M \geq M_c, \quad (4.5)$$

**Theorem 4.2** Assume theorem 1.2 fails. Let $M_c$ denote the critical mass. Then there exists a maximal lifespan solution to (1.1) with mass $M(u) = M_c$ which is almost periodic modulo symmetries and blows up both forward and backward in time. Also, $[0, \infty) \subset I$, $N(t) \leq 1$ for $t \geq 0$, and

$$|N'(t)| \lesssim N(t)^4, \quad |x'(t)| \lesssim N(t)^2, \quad (4.6)$$

Moreover, there exists $\delta(u) > 0$ such that for any $t_0 \in I$,

$$\|u\|_{S^0([t_0, t_0 + \frac{\delta}{N(t_0)^3}] \times \mathbb{R})} \lesssim 1, \quad (4.7)$$

and by Gronwall’s inequality and (4.6),

$$\int_{t_0}^{t_0 + \frac{\delta}{N(t_0)^3}} N(t)^3 \, dt \sim 1. \quad (4.8)$$

Therefore, since $u$ blows up both forward and backward in time, there exists $t_0 \in I$ such that

$$\int_{t_0}^{\sup(I)} N(t)^3 \, dt = \int_{\inf(I)}^{t_0} N(t)^3 \, dt = \infty. \quad (4.9)$$

**Proof** See [23]. The proof of theorem 4.2 was conditional on scattering for the one dimensional, defocusing mass-critical nonlinear Schrödinger equation. $\square$

**Theorem 4.3** If $u$ is a solution to the one dimensional, mass-critical nonlinear Schrödinger equation

$$(i \partial_t + \partial_{xx})u = |u|^4 u, \quad (4.10)$$

then

$$\|u\|_{L_{t,x}^6(\mathbb{R} \times \mathbb{R})} \leq C(\|u(0, \cdot)\|_{L^2}). \quad (4.11)$$

**Proof** See [12]. $\square$
Lemma 4.4 (No waste lemma) If $u$ is a minimal mass blowup solution to (1.1) then for any $t \in I$,

$$u(t) = \lim_{T \to \sup(I)} \int_t^T e^{-(t-\tau)\beta_3} \partial_3^3 \partial_x^3 (u^5)(\tau) d\tau = \lim_{T \to \inf(I)} \int_t^T e^{-(t-\tau)\beta_3} \partial_3^3 \partial_x^3 (u^5)(\tau) d\tau,$$

(4.12)

weakly in $L^2_x(\mathbb{R})$.

Proof This follows in a similar manner to the nonlinear Schrödinger equation (see [46]). If $\sup(I) < +\infty$ then $N(T) \to +\infty$ as $T \nearrow \sup(I)$ combined with (4.3) implies that for any test function $f$,

$$\lim_{T \to \sup(I)} \langle e^{-(t-T)\beta_3} u(T), f \rangle = 0. \quad (4.13)$$

The same would be true if $N(T) \to 0$. If $N(T) \sim N(t)$ as $T \to \sup(I)$ then $\sup(I) = +\infty$. The dispersive estimate (2.12) combined with (4.3) implies that in this case also

$$\lim_{T \to \sup(I)} \langle e^{-(t-T)\beta_3} u(T), f \rangle = \lim_{T \to \sup(I)} \langle u(T), e^{i(T-t)\beta_3} f \rangle = 0. \quad (4.14)$$

The last equality follows from the fact that if $\chi$ is a smooth function supported on $|x| \leq 2$, $\chi(x) = 1$ on $|x| \leq 1$,

$$\|e^{i(T-t)\beta_3} \chi(\frac{x}{R}) f\|_{L^\infty_x(\mathbb{R})} \lesssim R^{1/2} \frac{1}{(T-t)^{1/3}} \|f\|_{L^2_x(\mathbb{R})},$$

$$\|e^{i(T-t)\beta_3} (1 - \chi(\frac{x}{R})) f\|_{L^2_x(\mathbb{R})} \lesssim \eta(R),$$

$$\|\chi (\frac{x - x(T)}{R}) u(T)\|_{L^1_x(\mathbb{R})} \lesssim R^{1/2} \|u(T)\|_{L^2_x(\mathbb{R})},$$

and

$$\|(1 - \chi(\frac{x - x(T)}{R})) u(T)\|_{L^2_x(\mathbb{R})} \lesssim \eta(R),$$

where $\eta(R) \searrow 0$ as $R \nearrow \infty$. Then

$$\langle u(T), e^{i(T-t)\beta_3} f \rangle \lesssim \frac{R}{(T-t)^{1/3}} \|f\|_{L^2_x(\mathbb{R})} \|u(T)\|_{L^2_x(\mathbb{R})} + \eta(R) \|u(T)\|_{L^2_x(\mathbb{R})} + \eta(R) \|f\|_{L^2_x(\mathbb{R})},$$

so choosing $R = (T-t)^{1/12}$ proves (4.13).

Then to prove theorem 1.2 it suffices to show that if $u$ is a solution to (1.1) and satisfies (4.1) on its lifespan $I$, then $u \equiv 0$. Selecting $u$ in this form, we can abbreviate $A \lesssim C(u) B$ as $A \lesssim B$. \hfill \Box
It is convenient to consider three separate scenarios. Let

\[ t_0(T) = \inf\{ t \in [0, T] : N(t) = \inf_{\tau \in [0, T]} N(\tau) \}. \] (4.15)

\( N(t) \) attains its infimum on \([0, T]\) since \( N(t) \) is continuous.

**Case 1:** Rapid double cascade.

\[
\limsup_{T \to \sup(I)} \left( \inf_{t \in [0, T]} N(t) \right) \cdot \left( \int_0^T N(t)^2 \, dt \right) \leq C < +\infty.
\] (4.16)

\[
\limsup_{T \to \sup(I)} \frac{\sup_{t \in [t_0(T), T]} N(t)}{N(t_0(T))} = +\infty.
\] (4.17)

**Case 2:** Self-similar solution.

\[
\limsup_{T \to \sup(I)} \left( \inf_{t \in [0, T]} N(t) \right) \cdot \left( \int_0^T N(t)^2 \, dt \right) \leq C < +\infty.
\] (4.18)

\[
\limsup_{T \to \sup(I)} \frac{\sup_{t \in [t_0(T), T]} N(t)}{N(t_0(T))} \leq C < +\infty.
\] (4.19)

**Case 3:** Quasi-soliton.

\[
\limsup_{T \to \sup(I)} \left( \inf_{t \in [0, T]} N(t) \right) \left( \int_0^T N(t)^2 \, dt \right) = +\infty.
\] (4.20)

Observe that if \( u \) is a solution to (1.1) satisfying (4.1), then by conservation of mass, to show that \( u \equiv 0 \) it is enough to show that there is a sequence of times \( t_n \in I \) such that

\[
\frac{1}{N(t_n)^{1/2}} u \left( \frac{x - x(t_n)}{N(t_n)} \right)
\] (4.21)

converges strongly to 0 in \( L^2(\mathbb{R}) \).

Also observe that theorem 2.4 and (4.1) imply that for any sequence \( t_n \in I \), after passing to a subsequence,

\[
\frac{1}{N(t_n)^{1/2}} u \left( t_n, \frac{x - x(t_n)}{N(t_n)} \right)
\] (4.22)

converges strongly in \( L^2 \) to \( u_0 \in L^2(\mathbb{R}) \), and solving (1.1) for \( u \) with initial data \( u_0 \), \( u \) satisfies (4.1).
5 Rapid Double Cascade

We begin with the rapid double cascade described in (4.16)–(4.17). Recall the definition of the rapid double cascade in (4.16) and (4.17).

**Theorem 5.1** There does not exist a minimal mass blowup solution to the mass-critical gKdV in the form of a rapid double cascade.

**Proof** Take a sequence of times \( T_n \nearrow \sup(I) \) and let \( t_0^n = t_0(T_n) \). Now let

\[
    u_0^n = \frac{1}{N(t_0^n)^{1/2}} u \left( t_0^n, \frac{x + x(t_0^n)}{N(t_0^n)} \right). \tag{5.1}
\]

By (1.17) and theorem 2.4, \( u_0^n \) has a subsequence that converges strongly in \( L^2(\mathbb{R}) \) to \( u_0 \in L^2(\mathbb{R}) \), and \( u_0 \) is the initial data for a minimal mass blowup solution to the gKdV on a maximal interval \( I \). Also, by definition of \( t_0(T) \), \( N(0) = 1 \), \( N(t) \geq 1 \) on \( I \), and by (4.16) and the dominated convergence theorem,

\[
    \int_I N(t)^2 dt \leq C. \tag{5.2}
\]

(4.17) implies that \( u_0 \) is the initial data of a solution \( u \) that blows up both forward and backward in time. Moreover, \( N(t) \geq 1 \) implies \( |I| \leq C \), and \( u \) blows up in finite time, both forward and backward in time, and thus by (4.7),

\[
    \lim_{t \nearrow \sup(I)} N(t) = \lim_{t \searrow \inf(I)} N(t) = +\infty. \tag{5.3}
\]

Also by (4.6), if \( x(0) = 0 \), \( |x(t)| \lesssim C \) on \( I \), and furthermore the limits

\[
    \lim_{t \nearrow \sup(I)} x(t) = x_+, \quad \lim_{t \searrow \inf(I)} x(t) = x_-, \quad |x_+|, |x_-| \lesssim C. \tag{5.4}
\]

exist. Now define a Morawetz potential. Choose \( \psi \in C^\infty(\mathbb{R}) \), where \( \psi \) is an odd function, \( \psi(x) = x \) for \( 0 \leq x \leq 1 \), \( \psi(x) = \frac{3}{2} \) for \( x > 2 \), and

\[
    0 \leq \phi(x) = \psi'(x). \tag{5.5}
\]

**Remark** [14,34] used a potential similar to this for the nonlinear Schrödinger equation.

For any \( 0 < R < \infty \) let

\[
    M_R(t) = R \int \psi \left( \frac{x}{R} \right) u(t, x)^2 dx. \tag{5.6}
\]
For any \( R > 0, \eta > 0, \)
\[
R \int \frac{x}{R} u(t, x)^2 \, dx = \int_{-R}^{R} x u(t, x)^2 \, dx + O(R \int_{|x|>R} u(t, x)^2 \, dx)
\]
\[
= \int_{x(t)-C(\eta)N(t)}^{x(t)+C(\eta)N(t)} x u(t, x)^2 \, dx + O(R \eta) = x(t)M_c + O(R \eta).
\] (5.7)

Since \( N(t) \to \infty \) as \( t \to \sup(I) \) or \( t \to \inf(I) \), so letting \( \eta \to 0 \) as \( t \to \sup(I) \),
\[
\lim_{t \to \sup(I)} R \int \frac{x}{R} u(t, x)^2 \, dx = M_c \cdot x_+,
\lim_{t \to \inf(I)} R \int \frac{x}{R} u(t, x)^2 \, dx = M_c \cdot x_-.
\] (5.8)

Taking a derivative in time, (3.5) and integrating by parts implies that
\[
\frac{d}{dt} M_R(t) = -\int \frac{x}{R} [3u_x^2 + \frac{5}{3} u^6] \, dx + O(\frac{1}{R^2}) \|u(t)\|^2_{L_x^2(\mathbb{R})}.
\] (5.9)

Since \( |I| \leq C \), by conservation of mass (1.4),
\[
\frac{1}{R^2} \int_I \|u(t)\|^2_{L_x^2(\mathbb{R})} \, dt \lesssim \frac{C}{R^2}.
\] (5.10)

Therefore, for any \( R > 1 \)
\[
\int_I \int_{|x| \leq R} [3u_x^2 + \frac{5}{3} u^6] \, dx \, dt \lesssim C.
\] (5.11)

This bound is uniform in \( R < \infty \), so in particular
\[
\int_I \int [3u_x^2 + \frac{5}{3} u^6] \, dx \, dt \lesssim C.
\] (5.12)

Since \( N(0) = 1 \), (4.7) implies \( |I| \gtrsim 1 \), so by the intermediate value theorem there exists a \( t \in I \) such that
\[
\int [3u_x^2 + \frac{5}{3} u^6](t) \, dx \lesssim C.
\] (5.13)

Conservation of energy then implies \( E(u(t)) = E(u(0)) \lesssim C \) for all \( t \in I \). This implies \( \|u(t)\|_{L_x^6(\mathbb{R})} \) is uniformly bounded, which by (4.3), Hölder’s inequality, and \( N(t) \not\to \infty \) as \( t \not\to \sup(I) \) implies \( \|u(t)\|_{L_x^6(\mathbb{R})} \to 0 \) as \( t \not\to \sup(I) \). By conservation of mass this implies \( u \equiv 0 \). Of course, such a solution does not blow up, which proves theorem 5.1.

\( \square \)
6 Self-Similar Solution

Next we consider the self-similar solution, which differs from the rapid double cascade in that (4.19) holds instead of (4.17). Because of this fact a procedure similar to (5.1)–(5.4) will yield a solution that blows up in finite time in one direction and blows up in infinite time in the other direction, say on the interval \((0, \infty)\).

If \(u\) is a self-similar solution, then (4.18) and (4.19) imply that

\[
\liminf_{t \to \infty} N(t) = 0. \tag{6.1}
\]

Indeed, if \(N(t)\) is uniformly bounded below on the maximal interval of existence \(I\) then (4.19) implies that there exists \(t_0 \in I\) such that \(N(t) \lesssim 1\) for \(t \in I, t \geq t_0\). Since \(u\) blows up both forward and backward in time, (4.6), (4.7), and \(N(t) \lesssim 1\) imply that

\[
\int_{t_0}^{\sup(I)} N(t) \, dt \lesssim \int_{t_0}^{\sup(I)} N(t)^2 \, dt. \tag{6.2}
\]

But this combined with \(N(t) \gtrsim 1\) directly contradicts (4.18), and therefore (6.1) must hold.

Now then, (6.1) and (4.19) imply that \(N(t) \to 0\) as \(t \to \infty\). Now for any integer \(l \geq 0\) let

\[
t_l = \inf \{ t : N(t) = 2^{-l} \}. \tag{6.3}
\]

Translating in time, let \(t_0 = 1\). By (4.18)

\[
2^{-3l}(t_l - 1) \leq C, \tag{6.4}
\]

so for any \(l > 0\), \(t_l \lesssim 2^{3l}\). On the other hand (4.6), (4.19), and the fundamental theorem of calculus imply

\[
2^{-l} \leq \int_{t_{l-1}}^{t_l} |N'(t)| \, dt \lesssim \int_{t_{l-1}}^{t_l} N(t)^4 \, dt \lesssim 2^{-4l}(t_l - t_{l-1}) \leq 2^{-4l} t_l. \tag{6.5}
\]

This implies \(t_l \gtrsim 2^{3l}\) and therefore \(t_l \sim 2^{3l}\), and therefore (4.19) implies that \(N(t) \sim t^{-1/3}\) for any \(t \geq 1\). Possibly after modifying \(C(\eta)\) by a constant, let \(N(t) = t^{-1/3}\) for \(t \in [1, \infty)\).

Now let \(x(0) = 0\). (4.6) combined with the fundamental theorem of calculus implies

\[
|x(t)| \lesssim \int_{1}^{t} \tau^{-2/3} \, d\tau \lesssim t^{1/3}. \]

Therefore, again after modifying \(C(\eta)\) by a constant, for any \(\eta > 0\) there exists \(C(\eta) < \infty\) such that

\[
\int_{|x| \gtrsim \frac{C(\eta)}{N(t)}} u(t, x) x^2 \, dx + \int_{|\xi| \gtrsim C(\eta) N(t)} |\hat{u}(t, \xi)|^2 \, d\xi < \eta. \tag{6.6}
\]
Now take a sequence $t_n \to +\infty$ and let

$$u_0^n = \frac{1}{N(t_n)^{1/2}} u\left(t_n, \frac{x}{N(t_n)}\right). \quad (6.7)$$

Then (1.17) and theorem 2.4 imply that after passing to a subsequence there exists some $u_0 \in L^2(\mathbb{R})$ such that $u_0^n \to u_0$ strongly in $L^2$. Moreover, if $u$ solves (1.1) with $u(1, \cdot) = u_0(\cdot)$, then $u$ exists on the maximal time interval $(0, \infty)$ and satisfies (1.17) with $N(t) = t^{-1/3}$ and $x(t) = 0$.

As in the case of the rapid double cascade it suffices to show that a self-similar solution has finite energy. This implies that $\|u(t)\|_{L^6}$ is uniformly bounded, which combined with (4.3), Hölder’s inequality, and $N(t) = t^{-1/3}$ implies that $\|u(t)\|_{L^2} \to 0$ as $t \searrow 0$. Then by conservation of mass this implies $u \equiv 0$, which certainly does not blow up in finite time. Now by the Sobolev embedding theorem and conservation of mass, $u$ has finite energy if and only if $\|u(t)\|_{\dot{H}^1}$ is bounded.

**Theorem 6.1** (Additional regularity) *If $u$ is a self-similar solution to the mass critical gKdV equation then $u(1) \in \dot{H}_x^1(\mathbb{R})$.***

**Corollary 6.2** (No self-similar solution) *There does not exist a self-similar solution.*

**Proof of theorem 6.1:** This proof is very similar to the additional regularity proof in [24,27,47] for the self-similar blowup solution for the nonlinear Schrödinger equation. The proof has two steps. First, using the double Duhamel formula we prove that a self-similar solution must possess some additional regularity. More precisely, we prove that there exists some $s > 0$ such that

$$\|u\|_{\dot{H}^s_x(\mathbb{R})} \sim t^{-s/3}. \quad (6.8)$$

The second step is to argue by induction to show that in fact $u \in \dot{H}_x^1(\mathbb{R})$. To prove this it is convenient to use the notation

$$\mathcal{M}(A) = \sup_T \|P_{\geq AT^{-1/3}} u\|_{L_x^\infty L_t^2([T,2T] \times \mathbb{R})}, \quad (6.9)$$

$$\mathcal{S}(A) = \sup_T \|P_{\geq AT^{-1/3}} u\|_{S^0([T,2T] \times \mathbb{R})}, \quad (6.10)$$

$$\mathcal{N}(A) = \sup_T \|P_{\geq AT^{-1/3}} \partial_x (u^5)\|_{N^0([T,2T] \times \mathbb{R})}. \quad (6.11)$$

where $S^0$ and $N^0$ are given by definition 2.5, and $P_{\geq AT^{-1/3}}$ is the Littlewood–Paley multiplier

$$P_{\geq AT^{-1/3}} = \sum_{k: 2^k \geq AT^{-1/3}} P_k. \quad (6.12)$$

By proposition 2.1,

$$\mathcal{S}(A) \lesssim \mathcal{M}(A) + \mathcal{N}(A).$$
By conservation of mass, (4.7), and \(N(t) = t^{-1/3}\), there is a uniform bound
\[
\mathcal{M}(A) + \mathcal{S}(A) + \mathcal{N}(A) \lesssim 1. \quad (6.13)
\]
Now suppose that for some \(\sigma > 0\),

\[
\text{Theorem 6.3} \quad \text{There is a uniform bound}
\]
\[
\mathcal{M}(A) + \mathcal{S}(A) + \mathcal{N}(A) \lesssim A^{-\sigma}. \quad (6.14)
\]
Then by the product rule and lemma 2.2,
\[
\| P_{> AT^{-1/3}} \partial_x (u^5) \|_{N^0([T, 2T] \times \mathbb{R})} \lesssim \| \partial_x P_{> AT^{-1/3}} u \|_{L_\infty^\infty L_1^2([T, 2T] \times \mathbb{R})} \| P_{> AT^{-1/3}} u \|_{L_2^2 L_1^0([T, 2T] \times \mathbb{R})}^4 \]
\[
+ A^{-1} T^{1/3} \| \partial_x P_{> AT^{-1/3}} u \|_{L_\infty^\infty L_1^2([T, 2T] \times \mathbb{R})} \| P_{\leq AT^{-1/3}} u \|_{L_2^2 L_1^0([T, 2T] \times \mathbb{R})}^4 \]
\[
\lesssim A^{-5\sigma} + A^{-1} T^{1/3} A^{-\sigma} \left( \sum_{N \leq AT^{-1/3}} N^{1/4} \langle NT^{1/3} \rangle^{-\sigma} \right)^4 \lesssim A^{-5\sigma} + A^{-1-\sigma}. \quad (6.15)
\]
Now by (1.6) for any \(T > 1\),
\[
\langle P_{> A} u(1), P_{> A} u(1) \rangle = \langle P_{> A} u(1), P_{> A} e^{(1-T)\hat{\alpha}_3} u(T) \rangle + \langle P_{> A} u(1), P_{> A} \int_1^T e^{(1-\tau)\hat{\alpha}_3} \partial_x (u^5) d\tau \rangle \big|_{L_2^2}.
\]
so by lemma 4.4,
\[
\langle P_{> A} u(1), P_{> A} u(1) \rangle = \lim_{T \to \infty} \langle P_{> A} u(1), P_{> A} \int_1^T e^{(1-\tau)\hat{\alpha}_3} \partial_x (u^5) d\tau \rangle \big|_{L_2^2}. \quad (6.16)
\]
Then by proposition 2.1 and (6.15)–(6.17),
\[
\| P_{> A} u(1) \|_{L_2^2} \lesssim \sum_{k \geq 0} \| P_{> A} u(5) \|_{L_1^1 L_2^2([2^k, 2^{k+1}] \times \mathbb{R})} \lesssim \sum_{k \geq 0} \| P_{> A} u \|_{L_2^2 L_1^0([2^k, 2^{k+1}] \times \mathbb{R})}^5 \]
\[
+ \sum_{k \geq 0} \| P_{> A} u \|_{L_\infty^\infty L_1^2([2^k, 2^{k+1}] \times \mathbb{R})} \| P_{\leq A} u \|_{L_2^2 L_1^0([2^k, 2^{k+1}] \times \mathbb{R})}^4 \]
\[
\lesssim A^{-5\sigma} \sum_{k \geq 0} 2^{-5k\sigma} + A^{-1-\sigma} \sum_{k \geq 0} 2^{-\frac{4k\sigma}{3}} \lesssim A^{-5\sigma} + A^{-1-\sigma}. \quad (6.18)
\]
Now we exploit the fact that the scaling in (1.2) rescales a self-similar solution with \(N(t) = t^{-1/3}\) to another self-similar solution with \(N(t) = t^{-1/3}\). Thus applying (1.2)
for some $\lambda > 0$ rescales the self-similar solution to a new self-similar solution with
\[
u_\lambda(1) = \frac{1}{\lambda^{1/2}} u(\frac{1}{\lambda^3}, \frac{x}{\lambda}).
\]
(6.23)
Repeating (6.20)–(6.22) and then rescaling back to $u$ then implies
\[
\mathcal{M}(A) \lesssim A^{-5\sigma} + A^{-1-\sigma},
\]
which by (6.12) implies
\[
\mathcal{M}(A) + \mathcal{S}(A) + \mathcal{N}(A) \lesssim A^{-5\sigma} + A^{-1-\sigma}.
\]
(6.25)
Iterating this argument finitely many times proves that $u(1) \in H^1$, which under the assumption that theorem 6.3 is true completes the proof of theorem 6.1.

\[\square\]

**Proof of theorem 6.3:** We start by showing
\[
\lim_{A \to \infty} \mathcal{M}(A) + \mathcal{S}(A) + \mathcal{N}(A) = 0.
\]
(6.26)
(4.3) directly implies $\lim_{A \to \infty} \mathcal{M}(A) = 0$. Next, by (6.13) we can partition $[T, 2T]$ into finitely many pieces $I_j$ such that $\|u\|_{L^5_t L^{10}_x(I_j \times \mathbb{R})} \leq \epsilon$. Then
\[
\| P_{> \frac{AT^{-1/3}}{32}} \partial_x u \|_{L_x^\infty L_t^2(I_j \times \mathbb{R})} \lesssim \epsilon^4 \| P_{> \frac{AT^{-1/3}}{32}} \partial_x u \|_{L_x^\infty L_t^2(I_j \times \mathbb{R})}.
\]
(6.27)
Therefore, by proposition 2.1,
\[
\| P_{> \frac{AT^{-1/3}}{32}} u \|_{S^0(I_j \times \mathbb{R})} \lesssim \mathcal{M}(A) + \epsilon^4 \| P_{> \frac{AT^{-1/3}}{32}} u \|_{S^0(I_j \times \mathbb{R})}.
\]
(6.28)
Thus,
\[
\lim_{A \to \infty} \mathcal{S}(A) = 0.
\]
(6.29)
Finally, since
\[
\| P_{> \frac{AT^{-1/3}}{32}} \partial_x u \|_{L_x^{5/4} L_t^{10/9}([T,2T] \times \mathbb{R})} \lesssim \| P_{> \frac{AT^{-1/3}}{32}} \partial_x u \|_{L_x^\infty L_t^2([T,2T] \times \mathbb{R})} \| u \|_{L_x^4 L_t^{10}([T,2T] \times \mathbb{R})} \lesssim \mathcal{S}(\frac{A}{32}),
\]
(6.30)
we have proved
\[
\lim_{A \to \infty} \mathcal{N}(A) = 0.
\]
(6.31)
Remark Note that (6.27)–(6.31) also imply that

\[ \mathcal{M}(A) \lesssim A^{-\sigma} \Rightarrow \mathcal{M}(A) + S(A) + \mathcal{N}(A) \lesssim A^{-\sigma}. \]  

(6.32)

The proof that \( \mathcal{M}(A) \lesssim A^{-\sigma} \) uses the double Duhamel argument. To do this we will analyze the size of each projection \( P_k \) in \( P_{>A} = \sum_{k:2^k \geq A} P_k \) separately. Similar to (6.19), lemma 4.4 implies

\[ \langle P_k u(1), P_k u(1) \rangle_{L^2} = \lim_{T \rightarrow \infty} \langle P_k u(1), \int_1^T e^{(1-\tau)\partial_x^3} \partial_x P_k(u^5) d\tau \rangle_{L^2}. \]  

(6.33)

Applying lemma 4.4 again, this time for times \( t < 1 \) gives the double Duhamel formula

\[ \| P_k u(1) \|_{L^2}^2 = \int_1^1 \int_1^\infty \langle e^{-\tau \partial_x^3} P_k \partial_x (u^5), e^{-(1-\tau)\partial_x^3} P_k \partial_x (u^5) \rangle dtd\tau \]

\[ = \int_0^1 \int_0^\infty \langle e^{(t-\tau)\partial_x^3} P_k \partial_x (u^5), P_k \partial_x (u^5) \rangle dtd\tau. \]  

(6.34)

If \( A + B = C \) then elementary linear algebra implies \( \langle A + B, D \rangle \lesssim |A|^2 + \langle B, D \rangle \). Let

\[ A = \int_{2^{7(k-k_0)}}^{2^{7(k-k_0)}} e^{(1-\tau)\partial_x^3} \partial_x P_k(u^5) dt, \]

(6.35)

\[ B = \int_{2^{7(k-k_0)}}^{2^{7(k-k_0)}} e^{(1-\tau)\partial_x^3} \partial_x P_k(u^5) dt, \]

(6.36)

and

\[ D = \int_0^1 e^{(1-\tau)\partial_x^3} \partial_x P_k(u^5) dt. \]

(6.37)

Utilizing the dispersive estimate (2.12),

\[ \int_0^1 \int_0^\infty \langle e^{(t-\tau)\partial_x^3} P_k \partial_x (u^5), P_k \partial_x (u^5) \rangle dtd\tau \]

\[ \lesssim 2^k \| P_k(u^5) \|_{L_{t,x}^1([0,1] \times \mathbb{R})} \left( \int_{2^{7(k-k_0)}}^{\infty} \frac{1}{t^{1/3}} \| P_k(u^5) \|_{L_{x}^2} dt \right). \]  

(6.38)

By (6.13),

\[ 2^k \| P_k(u^5) \|_{L_{t,x}^1([T,2T] \times \mathbb{R})} \lesssim 2^k T^{1/4} \| P_{>k} u \|_{L_{t,x}^2([T,2T] \times \mathbb{R})}^{9/2} \| u \|_{L_{x}^2}^{1/2} \| P_{\leq k} u \|_{L_{t,x}^2([T,2T] \times \mathbb{R})}^{7/2} \]

\[ \quad + 2^k T^{1/4} \| P_{\leq k} u \|_{L_{x}^2([T,2T] \times \mathbb{R})}^{1/2} \| u \|_{L_{x}^2}^{1/2} \| P_{>k} u \|_{L_{t,x}^2([T,2T] \times \mathbb{R})} \]

\[ \lesssim T^{1/4} 2^{k/2}. \]  

(6.39)
Therefore,

\[ 2^k \int_{2^7(k-k_0)}^{\infty} \frac{1}{t^{1/3}} \| P_k(u^5) \|_{L^1_x} dt \lesssim 2^{-k/3}, \]  

(6.41)

and

\[ \langle B, D \rangle = (6.38) \lesssim 2^{-k/12}. \]  

(6.42)

It only remains to estimate \(|A|^2\), which will utilize a long time Strichartz estimate. Define \(\alpha(k)\) to be a frequency envelope that bounds \(\| P_k u(1) \|_{L^2} \): take \(\delta = \frac{1}{40}\) and let

\[ \alpha(k) = \sum_j 2^{-\delta|j-k|} \| P_j u(1) \|_{L^2(R)}. \]  

(6.43)

By (6.26) choose \(\epsilon > 0\) very small, \(k_0(\epsilon)\) sufficiently large so that

\[ \mathcal{M}(2^{k_0/2}) + \mathcal{S}(2^{k_0/2}) + \mathcal{N}(2^{k_0/2}) < \epsilon, \]  

(6.44)

\[ 2^{-k_0} < \epsilon^{200}, \]  

(6.45)

and

\[ \sum_{k > k_0/2} \alpha(k)^2 \leq \epsilon^2. \]  

(6.46)

**Theorem 6.4** (Long time Strichartz estimate) For \(k \geq k_0\),

\[ \| P_k u \|_{S^0([1,2^{7(k-k_0)}] \times R)} \lesssim \alpha(k), \]  

(6.47)

and for \(j > 7(k-k_0)\),

\[ \| P_k u \|_{S^0([2^j,2^{j+1}] \times R)} \lesssim 2^{\frac{1}{20}(j-7(k-k_0))} \alpha(k). \]  

(6.48)

**Proof** The proof uses a bootstrap argument. Let \(A\) be the set of \(T \in [1, \infty)\) such that for a large, fixed constant \(C\) we have the uniform bound

\[ \| P_k u \|_{S^0([1,2^{7(k-k_0)}] \cap [1,T] \times R)} + \| P_k u \|_{L^\infty_T L^2_x([1,2^{7(k-k_0)}] \cap [1,T] \times R)} \leq \frac{C}{2} \alpha(k), \]  

(6.49)

and for \(j > 7(k-k_0)\),

\[ \| P_k u \|_{S^0([2^j,2^{j+1}] \cap [1,T] \times R)} + \| P_k u \|_{L^\infty_T L^2_x([2^j,2^{j+1}] \cap [1,T] \times R)} \leq \frac{C}{2} 2^{\frac{1}{20}(j-7(k-k_0))} \alpha(k). \]  

(6.50)
\[1 \in A \text{ so } A \text{ is nonempty. By the dominated convergence theorem } A \text{ is closed, so it remains to show that } A \text{ is open. Suppose } A = [1, T_0]. \text{ Then an argument similar to (6.27)–(6.32) shows that there exists some } T_0 < T < 2T_0 \text{ such that}\]

\[\| P_k u \|_{S^0([1,2^{7(k-k_0)}] \cap [1,T] \times \mathbb{R})} + \| P_k u \|_{L^\infty_t L^2_x([1,2^{7(k-k_0)}] \cap [1,T] \times \mathbb{R})} \leq C \alpha(k), \quad (6.51)\]

and for \( j > 7(k-k_0), \)

\[\| P_k u \|_{S^0([2^j, 2^{j+1}] \cap [1,T] \times \mathbb{R})} + \| P_k u \|_{L^\infty_t L^2_x([2^j, 2^{j+1}] \cap [1,T] \times \mathbb{R})} \leq C 2^{\frac{1}{10}(j-7(k-k_0))} \alpha(k). \quad (6.52)\]

First, by proposition 2.1,

\[\| e^{-(t-1)\delta^2} P_k u(1) \|_{S^0([1,2^{7(k-k_0)}] \times \mathbb{R})} \lesssim \alpha(k), \quad (6.53)\]

and

\[\| P_k \partial_x (u^5) \|_{N^0([1,2^{7(k-k_0)}] \cap [1,T] \times \mathbb{R})} \lesssim 2^{5k/6} \sum_{k_1 \geq k} \| P_{k_1} u \|_{L^6_t L^6_x([1,2^{7(k-k_0)}] \cap [1,T] \times \mathbb{R})} \quad (6.54)\]

\[+ \left(2^{5/6} \| P_{k-5 \leq \cdot \leq k+5} u \|_{L^\infty_t L^2_x([1,2^{7(k-k_0)-5}] \cap [1,T] \times \mathbb{R})} \right) \| P_{\leq k} u \|_{L^{12}_t L^{12}_x([1,2^{7(k-k_0)-5}] \cap [1,T] \times \mathbb{R})} \quad (6.55)\]

\[\| P_{\leq k} u \|_{L^{12}_t L^{12}_x([2^{7(k-k_0)-5}, 2^{7(k-k_0)}] \cap [1,T] \times \mathbb{R})}. \quad (6.56)\]

By (6.44) and the bootstrap assumption, when \( l \geq k_0, \)

\[\| P_l u \|_{L^{12}_t L^{12}_x([1,2^{7(k-k_0)}] \cap [1,T] \times \mathbb{R})} \lesssim 2^{l/24} \| P_l u \|_{S^0([1,2^{7(l-k_0)}] \cap [1,T] \times \mathbb{R})} \quad (6.57)\]

\[+ 2^{l/24} \| P_l u \|_{S^0([2^{7(l-k_0)}, 2^{7(k-k_0)}] \cap [1,T] \times \mathbb{R})} \lesssim C \alpha(l) 2^{l/24} + 2^{l/24} \epsilon(k-l). \quad (6.58)\]

For \( k_0/2 \leq l \leq k_0, \) (6.44) implies

\[\| P_l u \|_{L^{12}_t L^{12}_x([1,2^{7(k-k_0)}] \cap [1,T] \times \mathbb{R})} \lesssim 2^{l/24} \epsilon(k-k_0)^{5/24}. \quad (6.59)\]

Finally for \( l \leq k_0/2, \) (6.13) implies

\[\| P_l u \|_{L^{12}_t L^{12}_x([1,2^{7(k-k_0)}] \cap [1,T] \times \mathbb{R})} \lesssim 2^{l/24} (k-k_0)^{5/24}. \quad (6.60)\]
By the bootstrap assumption and (6.46),
\[
2^{5k/6} \sum_{k_1 \geq k} \| P_{k_1} u \|^5_{L^4_{t,x}([1,2^{7(k-k_0)}] \cap [1,T] \times \mathbb{R})} \lesssim C^5 2^{5k/6} \sum_{k_1 \geq k} \alpha(k_1)^5 2^{-5k_1/6} \lesssim C^5 \alpha(k) e^4.
\]
(6.61)

Also by the bootstrap assumption and (6.57)–(6.60),
\[
2^{5k/6} \| P_k - 5 \leq \cdot \leq k + 5 \| u \|^4_{L^8 L^4_t([1,2^{7(k-k_0-5)}] \cap [1,T] \times \mathbb{R})} \| P_{k \leq k} u \|^4_{L^8 L^4_t([1,2^{7(k-k_0-5)}] \cap [1,T] \times \mathbb{R})} \lesssim C^2 \alpha(k) (\sum_{k_0 \leq l \leq k} 2^{l/24} (C \alpha(l) + \epsilon(k - l)))^4
\]
\[
+ 2^{-k/6} \alpha(k) (\sum_{k_0/2 \leq l \leq k_0} 2^{l/24} \epsilon(k - k_0)^{5/24})^4
\]
\[
+ 2^{-k/6} \alpha(k) (\sum_{l \leq k_0/2} 2^{l/24} (k - k_0)^{5/24})^4 \lesssim C^5 \alpha(k) e^4,
\]
(6.63)
and
\[
2^{5k/6} \| P_k - 5 \leq \cdot \leq k + 5 \| u \|^4_{L^8 L^4_t([1,2^{7(k-k_0-5)}] \cap [1,T] \times \mathbb{R})} \| P_{k \leq k} u \|^4_{L^8 L^4_t([1,2^{7(k-k_0-5)}] \cap [1,T] \times \mathbb{R})} \lesssim C^5 \alpha(k) e^4.
\]
(6.64)

For \( \epsilon > 0 \) sufficiently small, proposition 2.1, (6.53), and (6.61)–(6.64) imply
\[
\| P_k u \|^5_{L^4_{t,x}([1,2^{7(k-k_0)}] \cap [1,T] \times \mathbb{R})} \| P_{k \leq k} u \|^5_{L^4_{t,x}([1,2^{7(k-k_0)}] \cap [1,T] \times \mathbb{R})} \lesssim \alpha(k)
\]
\[
+ C^5 \alpha(k) e^4 \lesssim 2 \alpha(k),
\]
(6.65)
for \( \epsilon > 0 \) sufficiently small.

Now take \( j > 7(k - k_0) \). Again by proposition 2.1,
\[
\| P_k u \|^5_{L^4_{t,x}([2^j,2^{j+1}] \times \mathbb{R})} \lesssim 2^{5k/6} \| P_k (u^5) \|^5_{L^{6/5}_{t,x}([2^j,2^{j+1}] \times \mathbb{R})}.
\]
(6.66)

Combining the bootstrap assumption with (6.61)–(6.64) and (6.44),
\[
2^{5k/6} \| P_k (u^5) \|^5_{L^{6/5}_{t,x}([2^j,2^{j+1}] \cap [1,T] \times \mathbb{R})} \lesssim 2^k \| P_k - 5 \leq \cdot \leq k + 5 \| u \|^4_{L^8 L^4_t([2^j,2^{j+1}] \cap [1,T] \times \mathbb{R})} \| P_{k \leq k} u \|^4_{L^8 L^4_t([2^j,2^{j+1}] \cap [1,T] \times \mathbb{R})}
\]
\[
+ 2^{5k/6} \sum_{k_1 > k} \| P_{k_1} u \|^5_{L^4_{t,x}([2^j,2^{j+1}] \cap [1,T] \times \mathbb{R})}
\]
(6.68)
\[ \lesssim C^2 (\epsilon - (k-k_0))^{10} \alpha(k) \epsilon^4 + C 2^{5k/6} \epsilon^4 \sum_{k \leq k_1} 2^{(j-7(k-k_0)) / 10} 2^{-5k_1/6} \alpha(k_1) \]

\[ \lesssim C^2 (\epsilon - (k-k_0))^{10} \alpha(k) \epsilon^4. \quad (6.69) \]

If \([2^j, 2^{j+1}] \cap [1, T]\) is nonempty, \([1, 2^j] \subset [1, T]\), so by proposition 2.1, (6.65), and

\[ \| P_k u \|_{L^\infty_t L^2_x ([2^j, 2^{j+1}] \cap [1, T] \times \mathbb{R})} + \| P_k u \|_{S^0 ([2^j, 2^{j+1}] \cap [1, T] \times \mathbb{R})} \]

\[ \lesssim \| P_k u (2^j) \|_{L^2_x (\mathbb{R})} + 2^{5k/6} \| P_k (u^5) \|_{L^{6/5} ([2^j, 2^{j+1}] \cap [1, T] \times \mathbb{R})} \]

\[ \lesssim \alpha(k) + \sum_{k \leq l \leq j} C^{2(j-7(k-k_0)) / 10} \alpha(k) \epsilon^4 + C 2^{(j-7(k-k_0)) / 10} \alpha(k) \epsilon^4, \quad (6.71) \]

so

\[ \| P_k u \|_{L^\infty_t L^2_x ([2^j, 2^{j+1}] \cap [1, T] \times \mathbb{R})} + \| P_k u \|_{S^0 ([2^j, 2^{j+1}] \cap [1, T] \times \mathbb{R})} \]

\[ \lesssim C^2 \frac{1}{10} (j-7(k-k_0)) \alpha(k) \epsilon^4 + \alpha(k). \quad (6.72) \]

Choosing \( \epsilon > 0 \) sufficiently small implies that the bounds for \( C \) imply the bounds for \( C \), which proves that \( A \) is open, proving that \( A = [1, \infty) \). \( \square \)

Notice that the proof of theorem 6.4 implies (see (6.61)–(6.64)) that

\[ |A|^2 \lesssim C^{10} \alpha(k)^2 \epsilon^8. \quad (6.73) \]

Combining (6.42) with (6.73),

\[ \| P_k u (1) \|_{L^2_x (\mathbb{R})} \lesssim C^{10} \alpha(k)^2 \epsilon^8 + 2^{-k/12}. \quad (6.74) \]

Let \( \beta(k) \) be another frequency envelope.

\[ \beta(k) = \sum_j 2^{-\frac{4}{7} j - k} \| P_j u (1) \|_{L^2_x (\mathbb{R})}. \quad (6.75) \]

(6.74) and conservation of mass imply

\[ \sum_j 2^{-\delta |j-k|/2} \| P_j u (1) \|_{L^2_x (\mathbb{R})} \lesssim C^5 \epsilon^4 \sum_j 2^{-\delta |j-k|/2} \sum_{j_1} 2^{-\delta |j-j_1|} \| P_{j_1} u (1) \|_{L^2_x (\mathbb{R})} \]

\[ + \sum_j 2^{-\delta |j-k|/2} (1 + 2^j)^{-1/24}, \quad (6.76) \]

which implies that for \( \epsilon > 0 \) sufficiently small,

\[ \beta(k) \lesssim C^5 \epsilon^4 \beta(k) + 2^{-\delta k/2} \Rightarrow \beta(k) \lesssim 2^{-\delta k/2}. \quad (6.77) \]
Repeating the rescaling argument (6.20)–(6.23) then implies
\[ M(2^k) \lesssim 2^{-\delta k/2}. \] (6.78)

Thus by (6.32) the proof of theorem 6.4, and also the exclusion of the self-similar solution, is complete. \( \square \)

7 Quasi-soliton

We turn to the third and final scenario, the quasi-soliton, (4.20). Unlike the rapid double cascade and the self-similar solution, a solution satisfying (4.20) need not have a rescaled limit that converges to a finite time blowup solution. For example, for the focusing problem, the traveling wave
\[ u(t, x) = Q(x - t) \] (7.1)
satisfies (4.20). Therefore, excluding the quasi-soliton will instead utilize the monotonicity formula in lemma 3.1.

The first order of business is to extract a finite energy solution from a rescaled limit \( u(t_n) \), where \( t_n \in I \) is a sequence of times and \( I \) is the maximal interval of existence of the quasi-soliton. The argument will be similar to the argument in (5.6)–(5.13), with one important difference. In the case of the rapid double cascade, (4.16) meant that after rescaling so that \( \inf_{t \in [0, T_n]} N(t) = 1 \),
\[ \int_0^{T_n} N(t)^2 dt \leq C. \] (7.2)

Then (5.11) implied that \( E(u(t)) \lesssim 1 \).

In this case (4.20) shows that after rescaling so that \( \inf_{t \in [0, T_n]} N(t) = 1 \), \( \int_0^{T_n} N(t)^2 dt \to \infty \), and thus the estimate corresponding to (5.11) need not imply that \( E(u(t)) \lesssim 1 \). However, an estimate like (5.11) does imply that \( E(u(t)) \) is bounded by a suitable average of \( N(t)^2 \) on some interval \( I \), where \( \int_I N(t)^3 dt \) can be an arbitrarily large, fixed number.

Remark In order for such an average to have any meaning, we cannot take an average over the entire maximal interval of existence of the quasi-soliton.

Turning to the details, (4.6) implies that there exists a fixed constant \( C \) such that \( |x'(t)| \leq \frac{C}{T} N(t)^2 \). Let
\[ R(T) = C \left( \int_0^T N(t)^2 dt \right). \] (7.3)

(4.3) and (4.20) and imply
\[ \sup_{t \in [0, T]} \int_{|x| \geq R(T)} u(t, x)^2 dx \to 0, \] (7.4)
as $T \to \infty$. As in (5.6) let

$$M_R(t) = R \int \psi \left( \frac{x}{R} \right) u(t, x)^2 dx.$$  \hspace{1cm} (7.5)

Then

$$M_R(T) - M_R(0) \lesssim R,$$  \hspace{1cm} (7.6)

and

$$\frac{d}{dt} M_R(t) = - \int \phi \left( \frac{x}{R} \right) \left[ 3u_x^2 + \frac{5}{3} u^6 \right] dx + \frac{1}{R^2} \int \phi'' \left( \frac{x}{R} \right) u^2 dx.$$  \hspace{1cm} (7.7)

For any $t_0 \in [0, T]$, (4.6) implies that for some small $\delta > 0$,

$$\int_{t_0}^{t_0 + \frac{\delta}{N(t)^3}} 1 dt = \frac{\delta}{N(t_0)^3} \lesssim \frac{1}{\delta^2} \left( \int_{t_0}^{t_0 + \frac{\delta}{N(t)^3}} N(t)^2 dt \right).$$  \hspace{1cm} (7.8)

Therefore,

$$\int_0^T 1 dt \lesssim \left( \int_0^T N(t)^2 dt \right)^3,$$  \hspace{1cm} (7.9)

which by conservation of mass and (7.3) implies

$$\frac{1}{R^2} \int_0^T \int u(t, x)^2 dx dt \lesssim \int_0^T N(t)^2 dt.$$  \hspace{1cm} (7.10)

Therefore, by the fundamental theorem of calculus, (7.3), (7.6), (7.7), and (7.10),

$$\int_0^T \int \phi \left( \frac{x}{R} \right) \left[ 3u_x^2 + \frac{5}{3} u^6 \right] dx dt \lesssim \int_0^T N(t)^2 dt.$$  \hspace{1cm} (7.11)

Now fix $\mathcal{J} > 0$ large.

**Lemma 7.1** For any $T < \infty$ there exists $I(T) \subset [0, T]$ such that

$$\int_I N(t)^3 dt = \mathcal{J}, \quad \int_I \int_{|x| \leq R(T)} \left[ 3u_x^2 + \frac{5}{3} u^6 \right] dx dt \lesssim \int_I N(t)^2 dt,$$  \hspace{1cm} (7.12)

and furthermore the bound is uniform in $T$.

**Proof** Suppose $[0, T]$ is an interval and there exists an integer $K$ such that

$$\int_0^T N(t)^3 dt = K \mathcal{J}.$$  \hspace{1cm} (7.13)

Partition \([0, T]\) into disjoint intervals \(I_j\) such that \(\int_{I_j} N(t)^3 dt = J\). By (7.11),

\[
\sum_j \int_{I_j} \int_{|x| \leq R(T)} [3u_x^2 + \frac{5}{3}u^6] dx dt \lesssim \sum_j \int_{I_j} N(t)^2 dt,
\]

(7.14)

so there exists at least one \(j\) such that

\[
\int_{I_j} \int_{|x| \leq R(T)} [3u_x^2 + \frac{5}{3}u^6] dx dt \lesssim \int_{I_j} N(t)^2 dt.
\]

(7.15)

Let this \(I_j = I(T)\).

\(\square\)

**Lemma 7.2** There exists \(t_0(T) \in I(T)\) with

\[
N(t_0) \lesssim \left( \frac{1}{J} \int_I N(t)^2 dt \right)^{-1},
\]

(7.16)

and

\[
\int_{|x| \leq R(T)} [3u_x^2 + \frac{5}{3}u^6](t_0) dx \lesssim N(t_0)^2.
\]

(7.17)

**Proof** The contribution of the large \(N(t)\)'s is small. By (7.12),

\[
\int_{I(T): N(t) \geq 10 \left( \frac{1}{J} \int_I N(t)^2 dt \right)^{-1}} N(t)^2 dt \leq \frac{1}{10(\frac{1}{J} \int_I N(t)^2 dt)^{-1}} \int_I N(t)^3 dt
\]

\[
\leq \frac{1}{10} \left( \int_I N(t)^2 dt \right).
\]

(7.18)

Therefore

\[
\int_I \int_{|x| \leq R(T)} [3u_x^2 + \frac{5}{3}u^6] dx dt \lesssim \int_{I(T): N(t) \leq 10 \left( \frac{1}{J} \int_I N(t)^2 dt \right)^{-1}} N(t)^2 dt.
\]

(7.19)

Then by the intermediate value theorem there exists \(t_0 \in I(T)\) such that \(N(t_0) \leq 10(\frac{1}{J} \int_I N(t)^2 dt)^{-1}\) such that

\[
\int_{|x| \leq R(T)} [3u_x^2 + \frac{5}{3}u^6] dx \lesssim N(t_0)^2.
\]

(7.20)

\(\square\)

This implies that the sequence

\[
\phi \left( \frac{x}{R(T)} \right) \frac{1}{N(t_0(T))^{1/2}} u \left( \frac{x - x(t_0(T))}{N(t_0(T))} \right)
\]

(7.21)
has a subsequence that converges strongly in $L^2$ to $u_0 \in H^1$, $E(u_0) \lesssim 1$, and $u_0$ is the initial data for a solution that satisfies (1.18). Furthermore $\int_{I(T)} N(t)^2 \, dt \lesssim \frac{1}{N(t_0)} \int_{I(T)} N(t)^3 \, dt$, so after making the rescaling in (7.21) and taking the limit,

$$\int_{I} N(t)^2 \, dt \lesssim \int_{I} N(t)^3 \, dt \sim J. \quad (7.22)$$

By Hölder’s inequality,

$$J^2 \sim \left( \int_{I} N(t)^3 \, dt \right)^2 \lesssim \left( \int_{I} N(t)^2 \, dt \right) \left( \int_{I} N(t)^4 \, dt \right). \quad (7.23)$$

so

$$\int_{I} N(t)^4 \, dt \gtrsim J. \quad (7.24)$$

Now we will use the monotonicity formula in lemma 3.1 as well as (3.5) and (3.6) to construct an interaction Morawetz-type estimate. The monotonicity formula was used in [44] to exclude the existence of a traveling wave solution to the defocusing problem (1.1). See [28] for more discussion of solitons.

A chief technical difficulty in this interaction Morawetz estimate lies in the fact that $N(t)$ is free to move around, and in particular, after rescaling we can have $N(t) \leq 1$. Notice that (4.3) implies that the support for most of the mass of $u$ lies in an interval of length $\sim \frac{1}{N(t)}$. Thus we will utilize a construction similar to the one in [12] to obtain the result. See [7, 8, 35, 45] for the interaction Morawetz estimate in the nonlinear Schrödinger equation case.

**Theorem 7.3** (No quasi-soliton) There does not exist a minimal mass blowup solution to (1.1) satisfying (7.22), (7.24), $E(u(0)) \lesssim 1$ for $J$ sufficiently large.

**Proof** Define two large constants $R$ and $R_1$, where $1 << R_1 << R$. Then let $\chi_a \in C_0^\infty(\mathbb{R})$ be an even function, where $\chi_a(x) = 1$ for $|x| \leq a$, $\chi_a(x) = 0$ for $|x| \geq a + R_1$, and $a \geq R$. Now let

$$\phi(x, y) = \frac{1}{R^2} \int_R^{2R} \int \chi_a^2(x-t) \chi_a^2(y-t) \, dt \, da. \quad (7.25)$$

Then by a change of variables,

$$\phi(x, y) = \frac{1}{R^2} \int_R^{2R} \int \chi_a^2(x-y-s) \chi_a^2(s) \, ds \, da = \phi(x-y)$$

$$= \frac{1}{R^2} \int_R^{2R} \int \chi_a^2(y+s-x) \chi_a^2(-s) \, ds \, da = \phi(y-x). \quad (7.26)$$
Then let $\psi(x)$ be the odd function

$$
\psi(x - y) = \int_{0}^{x-y} \phi(t) dt.
$$

(7.27)

Now we produce an interaction Morawetz estimate. Now recalling (3.1) and (3.3), let

$$
M(t) = \int \int \psi((x - y)\tilde{N}(t))\rho(t, y)e(t, x)dxdy,
$$

(7.28)

where $\tilde{N}(t) = N(t)$ for $N(t) \leq \alpha$ and $\tilde{N}(t) = \alpha$ when $N(t) \geq \alpha$, where $\alpha > 0$ is a small quantity. Observe that $\tilde{N}(t) \leq N(t)$, and by the conservation of energy and mass, $|M(t)| \lesssim R$. Next, by (3.5), (3.6), and integrating by parts,

$$
\frac{d}{dt} M(t) = \tilde{N}(t) \int \int \phi((x - y)\tilde{N}(t))[-\rho(t, y)k(t, x) + j(t, y)e(t, x)]dxdy
$$

(7.29)

$$
+ \tilde{N}(t)^3 \int \int \rho(t, y)e(t, x)\phi''((x - y)\tilde{N}(t))dxdy
$$

(7.30)

$$
- \tilde{N}(t)^3 \int \int \rho(t, y)e(t, x)\phi''((x - y)\tilde{N}(t))dxdy
$$

(7.31)

Now by (7.26),

$$
(7.29) = -\tilde{N}(t) \int_{R}^{2R} \int \int \chi_{a}^{2}(x\tilde{N}(t) - s)\chi_{a}^{2}(y\tilde{N}(t) - s)\rho(t, y)k(t, x)dxdydsda
$$

(7.32)

$$
+ \tilde{N}(t) \int_{R}^{2R} \int \int \chi_{a}^{2}(x\tilde{N}(t) - s)\chi_{a}^{2}(y\tilde{N}(t) - s)j(t, y)e(t, x)dxdydsda.
$$

(7.33)

Now to simplify some notation, let $\tilde{\chi}_{a}(x)$ be the characteristic function of the interval $[a, a + R_{1}]$,

$$
\tilde{\chi}_{a}(x) = 1 \text{ on } [a, a + R_{1}], \quad \tilde{\chi}_{a}(x) = 0 \text{ elsewhere.}
$$

(7.34)

We will suppress the $a$ for the moment and take $\chi_{a} = \chi$ for some $a$. Decompose

$$
\int \chi^{2}[F(u)]dx = \int F(\chi u)dx + \int [\chi^{2}F(u) - F(\chi u)]dx,
$$

(7.35)

where $F(u)$ is one of the terms in $j(t, x)$ or $k(t, x)$, i.e. $u_{x}^{2}, u_{x}^{2}u^{4},$ etc. Computing the commutators,
\[ u_{xx}^2 : \text{Integrating by parts,} \]
\[
\int \chi^2 u_{xx}^2 \, dx = \int \chi u_{xx} \{ \partial_{xx} (\chi u) - 2 \chi_x u_x - \chi_{xx} u \} \, dx
\]  
\[= \int \chi u_{xx} \partial_{xx} (\chi u) \, dx - \int \chi_x \chi \partial_x (u_x^2) \, dx - \int \chi_{xx} \chi u_{xx} u \, dx \]  
\[= \int \partial_{xx} (\chi u)^2 \, dx - 2 \int \chi_x u_x \partial_{xx} (\chi u) \, dx - \int \chi_{xx} u \cdot \partial_{xx} (\chi u) \, dx + \frac{1}{2} \int \partial_{xx} (\chi_{xx} \chi) u^2 \, dx. \]  
\[ (7.36) \]

Now, by the product rule and integrating by parts,

\[2 \int \chi_x u_x \partial_{xx} (\chi u) \, dx = 2 \int \chi_x u_x u_{xx} \chi \, dx + 4 \int \chi_x^2 u_x^2 \, dx + 2 \int \chi_x \chi_{xx} u_x u \, dx \]
\[= -\frac{1}{2} \int \partial_{xx} (\chi^2) u_x^2 \, dx + 4 \int \chi_x^2 u_x^2 \, dx - \frac{1}{2} \int \partial_{xx} (\chi_x^2) u^2 \, dx, \]

and

\[\int \chi_{xx} \chi u_{xx} u \, dx = \int \chi_{xx} \chi u_x^2 \, dx + \int \partial_x (\chi_{xx} \chi) u_x u \, dx = \int \chi_{xx} \chi u_x^2 \, dx - \frac{1}{2} \int \partial_{xx} (\chi_{xx} \chi) u^2 \, dx. \]

Therefore,

\[ (7.38) = \int \partial_{xx} (\chi u)^2 \, dx + \frac{1}{R_1^2} \int O(u_x^2 \chi^2) \, dx + \frac{1}{R_1^4} \int O(\chi^2 u^2) \, dx. \]  
\[ (7.39) \]

\[ u_x^2 : \text{Also integrating by parts,} \]
\[
\int \chi^2 u_x^2 \, dx = \int \chi u_x \partial_x (\chi u) - \int \chi u_x \chi_{xx} u
\]  
\[= \int \partial_x (\chi u)^2 + \frac{1}{2} \int \partial_{xx} (\chi^2) u^2 - \int \chi_x u \partial_x (\chi u) = \int \partial_x (\chi u)^2
\]
\[+ \frac{1}{R_1^2} O(\int \chi^2 u^2). \]  
\[ (7.41) \]

\[ u_x^2 u^4 : \]
\[
\int \chi^2 u_x^2 u^4 = \int \partial_x (\chi u) \chi_{xx} u^4 - \int \chi_x u \chi_{xx} u^4
\]  
\[= \int \partial_x (\chi u)^2 u^4 - \int \chi_x u^5 \partial_x (\chi u) + \frac{1}{2} \int \partial_{xx} (\chi^2) u^6. \]  
\[ (7.43) \]
Integrating by parts and using the product rule,
\[
\int \chi_x u^5 \partial_x (\chi u) \, dx = \int \chi_x^2 u^6 \, dx + \int \chi_x \chi u^5 u_x \, dx = \int \chi_x^2 u^6 \, dx - \frac{1}{12} \int \partial_{xx} (\chi_x^2) u^6 \, dx.
\]

Therefore,
\[
(7.43) \geq \int \partial_x (\chi u)^2 (\chi u)^4 + \frac{1}{R^2} O(\int \chi^2 u^6).
\]

\[u^6:\]
\[
\int \chi^2 u^6 \, dx = \int (\chi u)^6 \, dx + \int (1 - \chi^4) (\chi u)^2 u^4.
\]

Now by (3.10), if \(v = \chi a u\),
\[
\frac{3}{2} (\int v^2)(\int v_x^2) - \frac{3}{2} (\int v_x^2)^2 + 10 (\int v_x^6) (\int v^2) + \frac{1}{2} (\int v^{10}) (\int v^2) \geq 2 (\int v^6)^2.
\]

Next, for \(R\) sufficiently large, by Hölder’s inequality and (4.3),
\[
\frac{2}{9R} \int \int \chi^6 a \left( \frac{\chi \bar{\nabla} (t, t)x}{R} - s \right) u(t, x)^6 \, dx \, ds \geq \int_{|x - \chi (t)| \leq \frac{c_0}{\bar{\nabla}}(t, t)} u(t, x)^6 \, dx \gtrsim N(t)^4.
\]

uniformly in \(a\).

Now we estimate the contribution of the errors. Let \(1_A(x)\) be the indicator function of a set \(A\).
\[
\frac{1}{R} \int_R^{2R} 1_{[a, a + R]} \, da \leq \frac{R_1}{R} 1_{[R, 3R]}.
\]

Let \(J_t = [t_0, t_0 + \frac{\delta}{N(t_0)^3}]\). By Hölder’s inequality, the Sobolev embedding theorem, (2.8), and (4.7),
\[
\|u\|^6_{L^6_t ([t_0, t_0 + \frac{\delta}{N(t_0)^3}]) \times \mathbb{R}} \lesssim \|u\|_{N(t_0)}^6_{L^6_t ([t_0, t_0 + \frac{\delta}{N(t_0)^3}]) \times \mathbb{R}}
\]
\[
+ \|u_{\leq N(t_0)}\|^6_{L^6_t ([t_0, t_0 + \frac{\delta}{N(t_0)^3}]) \times \mathbb{R}} \lesssim \frac{1}{N(t_0)} + N(t_0)^2 \frac{1}{N(t_0)^3} \sim \int_{t_0}^{t_0 + \frac{\delta}{N(t_0)^3}} N(t)^2 \, dt.
\]
Therefore, by conservation of energy, (7.49), and (7.50),

\[
\int_{J_l} \int \int \int \int \chi_a(x \tilde{N}(t) - s)^2 (1 - \chi_a(y \tilde{N}(t) - s)^4) \chi_a(y \tilde{N}(t) - s)^2 u(t, y)^6 e(t, x) dxdy ds dt \lesssim \frac{R_1}{R} \int_{J_l} N(t)^2 dt.
\]

(7.51)

Next, by Hölder’s inequality in space and time,

\[
\frac{\tilde{N}(t_0)^3}{R_1^2} \int_{J_l} \int \int \int \chi_a(x \tilde{N}(t) - s)^2 \tilde{\chi}_a(y \tilde{N}(t) - s)^2 \rho(t, y) u_x(t, x)^2 dxdy ds dt \lesssim \frac{R_1}{R} \tilde{N}(t_0)^2.
\]

(7.52)

The last inequality follows from conservation of energy, (2.7), and (4.7),

\[
\|u\|_{L^4_x L^\infty_t (J_l \times \mathbb{R})} \lesssim \|\partial_x u\|_{S^0(J_l \times \mathbb{R})}^{1/4} \|u\|_{S^0(J_l \times \mathbb{R})}^{3/4},
\]

(7.54)

\[
\|u_x\|_{L^\infty_x L^2_t (J_l \times \mathbb{R})} \lesssim \|u\|_{S^0(J_l \times \mathbb{R})}.
\]

(7.55)

Next, by conservation of mass

\[
\frac{\tilde{N}(t_0)^5}{R_1^4} \int_{J_l} \int u(t, x)^2 u(t, y)^2 dxdy \lesssim \frac{\tilde{N}(t_0)^2}{R_1^4}.
\]

(7.56)

Finally, by conservation of mass and (7.50),

\[
\int_{J_l} \frac{\tilde{N}(t)^3}{R_1^2} \int u(t, x)^2 u(t, y)^6 dxdy dt \lesssim \frac{1}{R_1^2} \tilde{N}(t_0)^2.
\]

(7.57)

This takes care of the commutator terms in (7.39), (7.41), (7.44), and (7.45). By the fundamental theorem of calculus, the above computations, and (7.24), if we take say \(R_1 = R^{2/3}\),

\[
\int_{J_l} N(t)^4 \tilde{N}(t) dt \lesssim \frac{1}{R^{1/6}} \int_{J_l} N(t)^2 \tilde{N}(t) dt + R \int \left| \frac{\tilde{N}'(t)}{\tilde{N}(t)} \right|_{|x-y| \leq \frac{R}{\tilde{N}(t)}} u_x^2 u_x^2 dxdy dt.
\]

(7.58)
Then as in (7.52) and (7.53),
\[ R \int_{J_l} \tilde{N}(t)^3 \int_{|x-y| \leq \frac{R}{N(t_0)}} u_x^2 u_y^2 \, dx \, dy \, dt \lesssim \frac{R^2}{N(t_0)^{3/2}} \tilde{N}(t_0)^2 \sim R^2 \int_{J_l} \tilde{N}(t)^2 N(t)^{3/2} \, dt. \]  
(7.59)

Since \( \tilde{N}(t) \leq \alpha \) and \( \tilde{N}(t) \leq N(t) \), by (7.22),
\[ R^2 \int_I \tilde{N}(t)^2 N(t)^{3/2} \lesssim R^2 \alpha^{3/2} \mathcal{J}, \]  
(7.60)

and
\[ \frac{1}{R^{1/6}} \int_I \tilde{N}(t) N(t)^2 \, dt \lesssim \frac{1}{R^{1/6}} \alpha \mathcal{J}. \]  
(7.61)

Next,
\[ \int_{I: N(t) \leq \alpha} N(t)^4 \, dt \leq \alpha^2 \mathcal{J}. \]  
(7.62)

Since (7.23) implies that \( \int_I N(t)^4 \, dt \gtrsim \mathcal{J} \), the fundamental theorem of calculus, (7.29)–(7.31), and the error computations in (7.49)–(7.62) imply that
\[ \alpha \mathcal{J} \lesssim \alpha \int_I N(t)^4 \, dt \sim \int_I N(t)^4 \tilde{N}(t) \, dt \lesssim R + \frac{1}{R^{1/6}} \alpha \mathcal{J} + R^2 \alpha^{3/2} \mathcal{J}. \]  
(7.63)

Choose \( R \) sufficiently large so that \( \frac{1}{R^{1/6}} \ll 1 \). Then choose \( \alpha(R) \) sufficiently small so that \( \alpha^{3/2} R^2 \ll \frac{1}{R^{1/6}} \). Then for \( \mathcal{J} \) sufficiently large, we have a contradiction. \( \square \)

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