AN APPLICATION OF NAKAYAMA Functor IN REPRESENTATION STABILITY THEORY

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Abstract. Using the Nakayama functor, we construct an equivalence from a Serre quotient category of a category of finitely generated modules to a category of finite-dimensional modules. We then apply this result to the categories FI_G and VI_q, and answer positively an open question of Nagpal on representation stability theory.

1. Introduction

In a recent paper [9], Nagpal has proved that quite a few representation theoretic and homological properties of the category FI, whose objects are finite sets and morphisms are injections (see [2]), hold for the category VI_q, whose objects are finite-dimensional vector spaces over a finite field F_q and morphisms are linear injections. In that paper, he also asked a question of whether one can establish an equivalence

$$\text{VI}_q\text{-mod/VI}_q\text{-fdmod} \xrightarrow{\sim} \text{VI}_q\text{-fdmod},$$

where VI_q-mod is the category of finitely generated VI_q-modules over a field of characteristic zero, and VI_q-fdmod is its full subcategory of finite-dimensional VI_q-modules; see [9, Question 1.11]. An analogue of this equivalence for FI-modules was proved by Sam and Snowden in [11, Theorem 3.2.1].

The purpose of the present paper is to prove a general result in an abstract setting from which this kind of equivalences can be deduced. In particular, applying our general result, the above equivalence can be obtained for both FI_G and VI_q with G an arbitrary finite group (see Section 4 for definition). Therefore, we give not only an affirmative answer to the above-mentioned open question in [9, Question 1.11], but also a new proof for the case of the category FI when taking G to be trivial in FI_G. Our approach only relies on several abstract homological properties of representations, and hence works for a wider class of categories including FI_G and VI_q as specific examples. Furthermore, via the Nakayama functor, the above equivalence for FI becomes transparent in our approach, compared with the one in [11].

We briefly describe the essential idea of our approach. Let C be a small EI-category (EI means that every endomorphism is an isomorphism) satisfying certain finiteness conditions. One can define the Nakayama functor $\nu$ and inverse Nakayama functor $\nu^{-1}$

$$\begin{array}{ccc}
\text{C-mod} & \xrightarrow{\nu} & \text{C-fdmod} \\
\downarrow \nu^{-1} & & \\
\end{array}$$

between the category C-mod of finitely generated C-modules and the category C-fdmod of finite-dimensional C-modules. Note that $\nu$ and $\nu^{-1}$ form a pair of adjoint functors, and furthermore, they give rise to an equivalence between the category of finitely generated projective C-modules and
the category of finite-dimensional injective modules. Under the assumption that \( \mathcal{C} \) is \textit{locally self-injective} (that is, every finitely generated projective \( \mathcal{C} \)-module is also injective), \( \nu \) is an exact functor, and \( \nu \circ \nu^{-1} \) is isomorphic to the identity functor on \( \mathcal{C}\text{-fdmod} \). Therefore, by a classical result of Gabriel ([4, Proposition III.2.5]), the kernel of \( \nu \) is a localizing subcategory of \( \mathcal{C}\text{-mod} \), and one obtains the following commutative diagram in which \( \bar{\nu} \) and \( \bar{\nu}^{-1} \) are quasi-inverse to each other, \( \text{loc} \) is the localization functor, and \( \text{sec} \) is a section functor:

\[
\begin{array}{ccc}
\text{Ker}(\nu) & \xrightarrow{\text{inc}} & \mathcal{C}\text{-mod} \\
\downarrow & & \downarrow \\
\mathcal{C}\text{-fdmod} & \xrightarrow{\nu^{-1}} & \mathcal{C}\text{-mod} \\
\downarrow & & \downarrow \\
\mathcal{C}\text{-mod} & \xrightarrow{\nu} & \mathcal{C}\text{-fdmod} \\
\end{array}
\]

We then consider \( \text{Ker}(\nu) \). If every morphism in \( \mathcal{C} \) is a monomorphism, then \( \mathcal{C}\text{-fdmod} \subseteq \text{Ker}(\nu) \). We also give two equivalent characterizations such that \( \text{Ker}(\nu) \subseteq \mathcal{C}\text{-fdmod} \). When the category \( \mathcal{C} \) satisfies these conditions (for instances, \( \mathcal{C} \) is a skeleton of \( \text{FI}_G \) or \( \text{VI}_q \)), the above commutative diagram becomes

\[
\begin{array}{ccc}
\text{C}\text{-fdmod} & \xrightarrow{\text{inc}} & \text{C}\text{-mod} \\
\downarrow & & \downarrow \\
\text{C}\text{-mod} & \xrightarrow{\nu^{-1}} & \text{C}\text{-fdmod} \\
\downarrow & & \downarrow \\
\text{C}\text{-mod}/\text{C}\text{-fdmod} & \xrightarrow{\nu} & \text{C}\text{-mod}/\text{C}\text{-fdmod} \\
\end{array}
\]

Thus we get what we want.

The paper is organized as follows. In Section 2, we collect basic results on Nakayama functor. In Section 3 and Section 4, we prove our main result and consider its application to both \( \text{FI}_G \) and \( \text{VI}_q \) in representation stability theory, respectively.

2. Preliminaries

The Nakayama functor is well known in the representation theory of finite-dimensional algebras (see, for example, [11, 13]). Most of the proofs in this section are standard, so we leave the details to the reader.

2.1. Notations. An EI-category is a small category in which every endomorphism is an isomorphism. Let \( \mathcal{C} \) be a skeletal EI-category, and \( I \) be the set of objects of \( \mathcal{C} \). For any \( i, j \in I \), we write \( \mathcal{C}(i, j) \) for the set of morphisms in \( \mathcal{C} \) from \( i \) to \( j \). Recall that there is a partial order on \( I \) defined by \( i \leq j \) if \( \mathcal{C}(i, j) \) is nonempty.

We fix a field \( k \). A left (respectively, right) \( \mathcal{C} \)-module is a covariant (respectively, contravariant) functor \( W \) from \( \mathcal{C} \) to the category of \( k \)-vector spaces.

For any \( i, j \in I \), denote by \( k\mathcal{C}(i, j) \) the vector space with basis \( \mathcal{C}(i, j) \). Denote by \( A = \bigoplus_{i,j \in I} k\mathcal{C}(i, j) \) the category algebra of \( \mathcal{C} \). The associative algebra \( A \) is non-unital if \( I \) is an infinite set. Denote by
\( e_i \in \mathcal{C}(i, i) \) the identity endomorphism of \( i \). We say that a left (respectively, right) \( A \)-module \( V \) is \emph{graded} if \( V = \bigoplus_{i \in I} e_i V \) (respectively, \( V = \bigoplus_{i \in I} V e_i \)). If \( W \) is a left (respectively, right) \( C \)-module, then \( \bigoplus_{i \in I} W(i) \) has a natural structure of a graded left (respectively, right) \( A \)-module. Conversely, any graded left (respectively, right) \( A \)-module naturally defines a left (respectively, right) \( C \)-module. Thus, we shall not distinguish left (respectively, right) \( C \)-modules from graded left (respectively, right) \( A \)-modules.

A (left or right) \( C \)-module is \emph{finitely generated} (respectively, \emph{finite-dimensional}) if it is \emph{generated} (respectively, \emph{finite-dimensional}) as a graded \( A \)-module. We write \( \mathcal{C} \text{-mod} \) (respectively, \( \mathcal{C}_{\text{op}} \text{-mod} \)) for the category of finitely generated left (respectively, right) \( C \)-modules; and \( \mathcal{C} \text{-fdmod} \) (respectively, \( \mathcal{C}_{\text{op}} \text{-fdmod} \)) for the full subcategory of \( \mathcal{C} \text{-mod} \) (respectively, \( \mathcal{C}_{\text{op}} \text{-mod} \)) whose objects are the finite-dimensional left (respectively, right) \( C \)-modules. On the categories of finite-dimensional modules, there is the \emph{standard duality functor} \( D := \text{Hom}_k(-, \mathbb{k}) \):

\[
\mathcal{C} \text{-fdmod} \xrightarrow{D} \mathcal{C}_{\text{op}} \text{-fdmod}
\]

2.2. \textbf{Finiteness conditions and Nakayama functor.} We first recall from [10, Section 2] that the category \( C \) is said to be \emph{inwards finite} if, for each \( j \in I \), there are only finitely many \( i \in I \) such that \( C(i, j) \) is nonempty; and \emph{hom-finite} if \( C(i, j) \) is a finite set for every \( i, j \in I \). Further, the category \( C \) is called \emph{locally noetherian} if every \( C \)-submodule of each finitely generated left \( C \)-module is also finitely generated.

From now on, we always assume that \( C \) is an inwards finite, hom-finite, and locally noetherian EI-category.

Since \( C \) is inwards finite and hom-finite, the right projective \( C \)-module \( e_j A \) is finite-dimensional for each \( j \in I \). Hence every finitely generated right \( C \)-module is finite-dimensional.

Next, we introduce the Nakayama functor on \( C \)-modules.

The category algebra \( A \) of \( C \) is an \( A \)-bimodule which is graded as both a left \( A \)-module and a right \( A \)-module. If \( V \) (respectively, \( W \)) is a left (respectively, right) \( C \)-module, then \( \text{Hom}_C(V, A) \) (respectively, \( \text{Hom}_{C_{\text{op}}}(W, A) \)) is a right (respectively, left) \( A \)-module. If, moreover, \( V \) (respectively, \( W \)) is \emph{finitely generated}, then \( \text{Hom}_C(V, A) \) (respectively, \( \text{Hom}_{C_{\text{op}}}(W, A) \)) is graded, that is,

\[
\text{Hom}_C(V, A) \cong \bigoplus_{i \in I} \text{Hom}_C(V, e_i A) \\
\text{respectively, } \text{Hom}_{C_{\text{op}}}(W, A) \cong \bigoplus_{j \in I} \text{Hom}_{C_{\text{op}}}(W, e_j A);
\]

see [8, VIII.1.15]. (The proof of the claim below (1) in [8, VIII.1.15] remains valid for our algebra \( A \) even though \( A \) might be non-unital.)

Without any reference, we shall use the following well-known fact: for any idempotent \( e \in A \), there hold:

\[
\text{Hom}_C(e_i A, A) \cong e A A \\
\text{and } A \text{Hom}_{C_{\text{op}}}(e A, A) \cong A e A.
\]

\textbf{Lemma 2.1.} If \( V \) is a finitely generated left \( C \)-module, then \( \text{Hom}_C(V, A) \) is a finite-dimensional right \( C \)-module.

\textbf{Proof.} Suppose that \( V \) is generated by \( v_1, \ldots, v_s \) where \( v_i \in V(j_1), \ldots, v_s \in V(j_s) \). Then \( \text{Hom}_C(V, e_i A) = 0 \) if \( C(i, j_1), \ldots, C(i, j_s) \) are all empty sets. Since \( C \) is inwards finite, there are only finitely many \( i \in I \) such that \( \text{Hom}_C(V, e_i A) \neq 0 \). Since \( C \) is hom-finite, each \( \text{Hom}_C(V, e_i A) \) is finite-dimensional. \( \square \)

\textbf{Lemma 2.2.} If \( W \) is a finite-dimensional right \( C \)-module, then \( \text{Hom}_{C_{\text{op}}}(W, A) \) is a finitely generated left \( C \)-module.

\textbf{Proof.} There exists a surjective homomorphism \( e_{j_1} A \oplus \cdots \oplus e_{j_s} A \to W \) for some \( j_1, \ldots, j_s \in I \). Applying the functor \( \text{Hom}_{C_{\text{op}}}(-, A) \) and using (2.2), we obtain an injective homomorphism

\[
\text{Hom}_{C_{\text{op}}}(W, A) \to A e_{j_1} \oplus \cdots \oplus A e_{j_s}.
\]
Since $\mathcal{C}$ is locally noetherian, it follows that $\Hom_{\mathcal{C}\text{-proj}}(W, A)$ is finitely generated. \hfill $\Box$

By Lemmas 2.1 and 2.2 we have a pair of contravariant functors

\[
\begin{align*}
\mathcal{C}\text{-mod} & \cong \mathcal{C}\text{-fdmod} \\
\Hom_{\mathcal{C}}(-, A) & \cong \Hom_{\mathcal{C}\text{-proj}}(-, A)
\end{align*}
\]

**Definition 2.3.** The Nakayama functor $\nu$ of $\mathcal{C}$ (or $A$) is defined to be the following composition:

$$D \circ \Hom_{\mathcal{C}}(-, A) : \mathcal{C}\text{-mod} \to \mathcal{C}\text{-fdmod}.$$ 

The inverse Nakayama functor $\nu^{-1}$ is defined to be the following composition:

$$\Hom_{\mathcal{C}\text{-proj}}(-, A) \circ D : \mathcal{C}\text{-fdmod} \to \mathcal{C}\text{-mod}.$$ 

Let us remark that the functor $\nu$ is a right exact covariant functor, while the functor $\nu^{-1}$ is a left exact covariant functor. But we should warn the reader that the functor $\nu^{-1}$ is, in general, neither the inverse nor a quasi-inverse of $\nu$.

**Lemma 2.4.** The pair $(\nu, \nu^{-1})$ is an adjoint pair of functors:

\[
\begin{align*}
\mathcal{C}\text{-mod} & \cong \mathcal{C}\text{-fdmod} \\
\nu & \cong \nu^{-1}
\end{align*}
\]

**Proof.** Let $V \in \Ob(\mathcal{C}\text{-mod})$ and $U \in \Ob(\mathcal{C}\text{-fdmod})$. Since $V$ is a left $A$-module and $DU$ is a right $A$-module, the tensor product $V \otimes_k DU$ is an $A$-bimodule. One has the following canonical isomorphisms:

$$\Hom_{\mathcal{C}}(D \Hom_{\mathcal{C}}(V, A), U) = \Hom_{\mathcal{C}\text{-proj}}(DU, \Hom_{\mathcal{C}}(V, A)) = \Hom_{A\text{-bimod}}(V \otimes_k DU, A) = \Hom_{\mathcal{C}}(V, \Hom_{\mathcal{C}\text{-proj}}(DU, A));$$

see [44 Exercise XI.6.6]. \hfill $\Box$

### 2.3. Projectives and finite-dimensional injectives

Denote by $\mathcal{C}\text{-proj}$ the full subcategory of $\mathcal{C}\text{-mod}$ whose objects are the finitely generated projective left $\mathcal{C}$-modules. Denote by $\mathcal{C}\text{-fdinj}$ the full subcategory of $\mathcal{C}\text{-fdmod}$ whose objects are the finite-dimensional injective left $\mathcal{C}$-modules.

**Lemma 2.5.** (1) Every finitely generated projective left $\mathcal{C}$-module is a finite direct sum of indecomposable projective left $\mathcal{C}$-modules.

(2) Let $V$ be a finitely generated left $\mathcal{C}$-module. Then $V$ is an indecomposable projective left $\mathcal{C}$-module if and only if $V$ is isomorphic to $Ae$ for some primitive idempotent $e \in e_iAe_i$ with $i \in I$.

**Proof.** The two statement follows from [3] Theorem I.11.18 and [13 Proposition I.8.2], respectively. \hfill $\Box$

**Definition 2.6.** (1) A full subcategory $\mathcal{C}'$ of $\mathcal{C}$ is said to be right-closed if, for any $i, j \in \Ob(\mathcal{C})$, we have $i \in \Ob(\mathcal{C}')$ whenever $i \triangleleft j$ for some $j \in \Ob(\mathcal{C}')$.

(2) The support of a (left or right) $\mathcal{C}$-module $V$ is the set of all $i \in I$ such that $V(i)$ is nonzero, where $V(i)$ is the image of $i$ under the functor $V$.

By definition, we have the following trivial observation.

**Lemma 2.7.** Let $\mathcal{C}'$ be a right-closed subcategory of $\mathcal{C}$. If $V$ is an injective left $\mathcal{C}$-module whose support is contained in $\Ob(\mathcal{C}')$, then restricting $V$ to $\mathcal{C}'$ gives an injective left $\mathcal{C}'$-module. If $V'$ is an injective left $\mathcal{C}'$-module, then extending $V'$ to $\mathcal{C}$ by zero on $\Ob(\mathcal{C}) \setminus \Ob(\mathcal{C}')$ gives an injective left $\mathcal{C}$-module.

**Lemma 2.8.** Let $U$ be a finite-dimensional left $\mathcal{C}$-module. Then $U$ is an indecomposable injective left $\mathcal{C}$-module if and only if $U$ is isomorphic to $D(eA)$ for some primitive idempotent $e \in e_iAe_i$ with $i \in I$.

**Proof.** This follows from Lemma 2.7 and [13 Proposition I.8.19]. \hfill $\Box$
Corollary 2.9. The functor $\nu$ gives an equivalence of categories
\[ \mathcal{C} \text{-proj} \overset{\sim}{\longrightarrow} \mathcal{C} \text{-fdinj} \]
with a quasi-inverse given by the functor $\nu^{-1}$.

Proof. This follows immediately from (2.2), Lemmas 2.5 and 2.8. \qed

Definition 2.10. We say that $\mathcal{C}$ is locally self-injective if $Ae_i$ is an injective left $\mathcal{C}$-module for every $i \in I$.

Clearly, $\mathcal{C}$ is locally self-injective if and only if every finitely generated projective left $\mathcal{C}$-module is injective.

Lemma 2.11. Suppose that $\mathcal{C}$ is locally self-injective. Then the Nakayama functor $\nu : \mathcal{C} \text{-mod} \rightarrow \mathcal{C} \text{-fdmod}$ is exact.

Proof. Since $Ae_i$ is an injective left $\mathcal{C}$-module for each $i \in I$, the functor $\bigoplus_{i \in I} \text{Hom}_\mathcal{C}(-, Ae_i)$ is exact. It follows from (2.1) and the exactness of $D$ that $\nu : \mathcal{C} \text{-mod} \rightarrow \mathcal{C} \text{-fdmod}$ is exact. \qed

3. Main result

3.1. Injective resolutions of finite-dimensional modules. The partial order on $I$ induces a partial order on the set of objects of any full subcategory of $\mathcal{C}$.

Lemma 3.1. Suppose that the characteristic of $k$ is zero. Let $\mathcal{C}'$ be a right-closed subcategory of $\mathcal{C}$ such that $\text{Ob}(\mathcal{C}')$ is a finite set. Let $\mathcal{C}''$ be the full subcategory of $\mathcal{C}'$ on the objects which are not maximal in $\text{Ob}(\mathcal{C}')$. Let $W$ be a finite-dimensional right $\mathcal{C}$-module whose support is contained in $\text{Ob}(\mathcal{C}')$. Then:

(1) The subcategory $\mathcal{C}''$ of $\mathcal{C}$ is right-closed.

(2) There exists a short exact sequence
\[ 0 \rightarrow W' \rightarrow P \rightarrow W \rightarrow 0 \]
where $P$ is a finite direct sum of right $\mathcal{C}$-modules of the form $eA$ with $e^2 = e \in e_i Ae_i$ and $i \in \text{Ob}(\mathcal{C}')$, and where $W'$ is a finite-dimensional right $\mathcal{C}$-module whose support is contained in $\text{Ob}(\mathcal{C}'')$.

Proof. (1) Suppose $i \in \text{Ob}(\mathcal{C})$ and $i \leq j$ for some $j \in \text{Ob}(\mathcal{C}'')$. Then $j \in \text{Ob}(\mathcal{C}')$, and $j$ is not a maximal object in $\text{Ob}(\mathcal{C}')$. Hence $i \in \text{Ob}(\mathcal{C}')$, and $i$ is not a maximal object in $\text{Ob}(\mathcal{C}')$.

(2) Let
\[ P = \bigoplus_{i \in \text{Ob}(\mathcal{C}')} W_{e_i} \otimes_{e_i Ae_i} e_i A. \]

The algebra $e_i Ae_i$ is the group algebra of the finite group $\text{Aut}_\mathcal{C}(i)$ for each $i \in I$. Since $k$ has characteristic zero, the algebra $e_i Ae_i$ is semisimple. It follows that $W_{e_i}$ is a finite direct sum of irreducible right $e_i Ae_i$-modules, each of which is isomorphic to $eAe_i$ for some primitive idempotent $e \in e_i Ae_i$. One has a canonical isomorphism of right $\mathcal{C}$-modules: $eAe_i \otimes_{e_i Ae_i} e_i A \cong eA$. We see that $P$ is of the required form.

The multiplication map $\rho : P \rightarrow W$ is a homomorphism of right $A$-modules. Let $W'$ be the kernel of $\rho$. Since $P$ is finite-dimensional and has support contained in $\text{Ob}(\mathcal{C}')$, the same is true for $W'$.

For each $i \in \text{Ob}(\mathcal{C}')$, since $\rho$ maps $W_{e_i} \otimes_{e_i Ae_i} e_i Ae_i$ bijectively to $W_{e_i}$, we see that $\rho$ is surjective. Moreover, if $i$ is maximal in $\text{Ob}(\mathcal{C}')$, then $W_{e_i} \otimes_{e_i Ae_i} e_i Ae_i = 0$ if $j \in \text{Ob}(\mathcal{C}')$ and $j \neq i$. Therefore if $i$ is maximal in $\text{Ob}(\mathcal{C}')$, then $\rho$ maps $P_{e_i}$ bijectively to $W_{e_i}$. It follows that $W'$ has support contained in $\text{Ob}(\mathcal{C}')$. \qed
Lemma 3.2. Suppose that the characteristic of $k$ is zero. Let $U$ be a finite-dimensional left $\mathcal{C}$-module. Then there exists an exact sequence

$$0 \to U \to I_0 \to I_1 \to \cdots \to I_n \to 0$$

of left $\mathcal{C}$-modules such that $I_0, I_1, \ldots, I_n$ are finite-dimensional injective left $\mathcal{C}$-modules.

Proof. Let $W$ be the right $\mathcal{C}$-module $DU$. Let $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ on the objects $i$ such that $i \leq j$ for some $j$ in the support of $W$. It is clear that $\mathcal{C}_0$ is right-closed. Since the support of $W$ is a finite set and $\mathcal{C}$ is inwards finite, the set $\text{Ob}(\mathcal{C}_0)$ is finite. Let $\mathcal{C}_1$ be the full subcategory of $\mathcal{C}_0$ on the objects which are not maximal in $\mathcal{C}_0$. Then $\mathcal{C}_1$ is also a right-closed subcategory of $\mathcal{C}$ with $\text{Ob}(\mathcal{C}_1)$ finite. By Lemma 3.1, there is a short exact sequence

$$0 \to W_1 \to P_0 \to W \to 0$$

of right $\mathcal{C}$-modules such that:

- $P_0$ is a finite direct sum of right $\mathcal{C}$-modules of the form $eA$ for some idempotent $e \in e_iAe_i$ with $i \in \text{Ob}(\mathcal{C}_0)$;
- $W_1$ is finite dimensional and its support is contained in $\text{Ob}(\mathcal{C}_1)$.

Let $\mathcal{C}_2$ be the full subcategory of $\mathcal{C}_1$ on the objects which are not maximal in $\mathcal{C}_1$. We now apply Lemma 3.1 again to obtain a short exact sequence

$$0 \to W_2 \to P_1 \to W_1 \to 0$$

of right $\mathcal{C}$-modules such that:

- $P_1$ is a finite direct sum of right $\mathcal{C}$-modules of the form $eA$ for some idempotent $e \in e_iAe_i$ with $i \in \text{Ob}(\mathcal{C}_1)$;
- $W_2$ is finite dimensional and its support is contained in $\text{Ob}(\mathcal{C}_2)$.

Recursively, we obtain the subcategories $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots$ and a projective resolution

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to W \to 0 \tag{3.1}$$

where each $P_s$ is a finite direct sum of right $\mathcal{C}$-modules of the form $eA$ for some idempotent $e \in e_iAe_i$ with $i \in \text{Ob}(\mathcal{C}_s)$. Since $|\text{Ob}(\mathcal{C}_0)| > |\text{Ob}(\mathcal{C}_1)| > |\text{Ob}(\mathcal{C}_2)| > \cdots$, the projective resolution (3.1) is of finite length. By Lemma 2.8 we see that applying the functor $D$ to (3.1) gives the required exact sequence. □

3.2. An equivalence of categories. For the adjoint pair $(\nu, \nu^{-1})$ in Lemma 2.4 we have the following result.

Proposition 3.3. Suppose that the characteristic of $k$ is zero and $\mathcal{C}$ is locally self-injective. Then the counit $\nu \circ \nu^{-1} \to \text{id}$ is an isomorphism.

Proof. We need to prove that the homomorphism $\nu(\nu^{-1}(U)) \to U$ is an isomorphism for each object $U$ of $\mathcal{C}$-fdmod. By Lemma 3.2 there is a finite injective resolution of $U$ by finite-dimensional injective left $\mathcal{C}$-modules; we shall use induction on the minimal length $n$ among all such resolutions. For $n = 0$, the proposition follows from Corollary 2.9.

Let $0 \to U \to I_0 \to I_1 \to \cdots \to I_n \to 0$ be a resolution of $U$ by finite-dimensional injective left $\mathcal{C}$-modules and $n$ be minimal length with this property. Let $U'$ be the cokernel of $U \to I_0$. One gets a short exact sequence

$$0 \to U \to I_0 \to U' \to 0.$$
Since \( \nu^{-1} \) is left exact and \( \nu \) is exact by Lemma 2.11 the following diagram commutes and has exact rows:

\[
\begin{array}{cccc}
0 & \longrightarrow & \nu(\nu^{-1}(U)) & \longrightarrow & \nu(\nu^{-1}(I_0)) & \longrightarrow & \nu(\nu^{-1}(U')) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & U & \longrightarrow & I_0 & \longrightarrow & U'.
\end{array}
\]

Observe that the middle vertical map is an isomorphism by Corollary 2.9 and the right vertical map is an isomorphism by induction hypothesis. Therefore the left vertical map is also an isomorphism. \( \square \)

The kernel \( \text{Ker}(\nu) \) of the functor \( \nu : \mathcal{C}\text{-mod} \to \mathcal{C}\text{-fdmod} \) is the full subcategory of \( \mathcal{C}\text{-mod} \) on the objects \( V \) such that \( \nu(V) = 0 \). When \( \nu \) is an exact functor, \( \text{Ker}(\nu) \) is a Serre subcategory of \( \mathcal{C}\text{-mod} \) (that is, closed under submodules, quotients and extensions) and its Serre quotient category is denoted by \( \mathcal{C}\text{-mod}/\text{Ker}(\nu) \).

To establish the main result of this paper, we recall the definition of section functors and a classical result of Gabriel.

**Definition 3.4.** We say an exact functor \( F : \mathcal{A} \to \mathcal{B} \) between two abelian categories admits a section functor if \( F \) has a right adjoint \( S : \mathcal{B} \to \mathcal{A} \) such that the counit \( F \circ S \to \text{id}_\mathcal{B} \) is an isomorphism.

**Lemma 3.5.** [4, Proposition III.2.5] Let \( F : \mathcal{A} \to \mathcal{B} \) be an exact functor between abelian categories which admits a section functor. Then the kernel \( \text{Ker}(F) \) is a localizing subcategory of \( \mathcal{A} \), and \( F \) induces an equivalence \( \mathcal{F} : \mathcal{A}/\text{Ker}(F) \to \mathcal{B} \).

Now we can prove the following main theorem.

**Theorem 3.6.** Suppose that the characteristic of \( k \) is zero, and suppose that \( \mathcal{E} \) is locally self-injective. Then the functor \( \nu \) induces an equivalence of categories

\[
\overline{\gamma} : \mathcal{E}\text{-mod}/\text{Ker}(\nu) \longrightarrow \mathcal{E}\text{-fdmod}.
\]

**Proof.** Lemma 2.11 and Proposition 3.3 tell us that the Nakayama functor \( \nu \) is exact and admits a section functor \( \nu^{-1} \). The conclusion now follows from Lemma 3.5. \( \square \)

### 3.3. Observations on \( \text{Ker}(\nu) \)

In this subsection we compare the subcategory \( \mathcal{E}\text{-fdmod} \) with the subcategory \( \text{Ker}(\nu) \), and prove that they coincide under certain conditions. In this case, Theorem 3.6 says that the Serre quotient category \( \mathcal{E}\text{-mod}/\mathcal{E}\text{-fdmod} \) is equivalent to \( \mathcal{E}\text{-fdmod} \). Therefore the Serre quotient category has enough injective objects, and every finitely generated object in it has finite length and finite injective dimension. Moreover, \( \mathcal{E}\text{-mod} \) can be regarded as an extension of \( \mathcal{E}\text{-fdmod} \) by itself.

Firstly, we give a sufficient condition such that \( \mathcal{E}\text{-fdmod} \subseteq \text{Ker}(\nu) \), which should be easy to check in practice.

**Definition 3.7.** We say that a left \( \mathcal{E} \)-module \( V \) is torsion-free if, for every morphism \( f \) in \( \mathcal{E} \), say \( f \in \mathcal{E}(i, j) \), the induced map \( f_* : V(i) \to V(j) \) is injective.

**Lemma 3.8.** The following statements are equivalent:

1. Every morphism in \( \mathcal{E} \) is a monomorphism, that is, if one has \( f \in \mathcal{E}(j, k) \) and \( g, h \in \mathcal{E}(i, j) \) such that \( fg = fh \), then \( g = h \).
2. Every finitely generated projective left \( \mathcal{E} \)-module is torsion-free.

**Proof.** (1)\( \Rightarrow \) (2) It suffices to prove that \( Ae_i \) is torsion-free for each \( i \in I \). Suppose \( f \in \mathcal{E}(j, k) \). Since \( f \) is a monomorphism, the map \( f_* : e_j Ae_i \to e_k Ae_i \) sends the basis \( \mathcal{E}(i, j) \) of \( e_j Ae_i \) bijectively onto a subset of the basis \( \mathcal{E}(i, k) \) of \( e_k Ae_i \).

(2)\( \Rightarrow \) (1) Suppose one has \( f \in \mathcal{E}(j, k) \) and \( g, h \in \mathcal{E}(i, j) \) such that \( fg = fh \). Since \( Ae_i \) is torsion-free, the map \( f_* : e_j Ae_i \to e_k Ae_i \) is injective. But \( f_*(g - h) = 0 \), so \( g = h \). \( \square \)
**Corollary 3.9.** Suppose that the partially ordered set I has no maximal element. If every morphism in \( \mathcal{C} \) is a monomorphism, then \( \mathcal{C}\text{-fdmod} \subseteq \text{Ker}(\nu) \).

**Proof.** Let \( V \) be a finite-dimensional left \( \mathcal{C} \)-module. We need to show that \( \text{Hom}_{\mathcal{C}}(V, A_{e_i}) = 0 \) for every \( i \in I \). By Lemma 3.8, the \( \mathcal{C} \)-module \( A_{e_i} \) is torsion-free, so it does not contain any nonzero finite-dimensional \( \mathcal{C} \)-submodules. Hence, the image of any homomorphism from \( V \) to \( A_{e_i} \) is zero. \( \square \)

The following proposition gives two equivalent characterizations such that the reverse inclusion \( \text{Ker}(\nu) \subseteq \mathcal{C}\text{-fdmod} \) holds. Surprisingly, the answer to this question is closely related to classification of injective modules in \( \mathcal{C}\text{-mod} \).

**Proposition 3.10.** Suppose that the characteristic of \( k \) is zero and \( \mathcal{C} \) is locally self-injective. Then the following statements are equivalent:

1. \( \text{Ker}(\nu) \subseteq \mathcal{C}\text{-fdmod} \).
2. If \( V \) is a finitely generated, infinite-dimensional left \( \mathcal{C} \)-module, then there exists \( i \in I \) such that \( \text{Hom}_{\mathcal{C}}(V, A_{e_i}) \neq 0 \).
3. The category \( \mathcal{C}\text{-mod} \) has enough injectives, and every finitely generated injective left \( \mathcal{C} \)-module is isomorphic to a direct sum of a finite-dimensional injective left \( \mathcal{C} \)-module and a finitely generated projective left \( \mathcal{C} \)-module.

**Proof.** The equivalence between (1) and (2) is clear. Note that if \( \mathcal{C} \) has only finitely many objects, then \( \mathcal{C}\text{-fdmod} = \mathcal{C}\text{-mod} \), and there does not exist finitely generated, infinite-dimensional \( \mathcal{C} \)-modules. Therefore, in this case both (1) and (2) hold trivially.

(3) \( \Rightarrow \) (2) Let \( V \) be a finitely generated, infinite-dimensional left \( \mathcal{C} \)-module. By the assumption, there exists an injection \( V \rightarrow P \oplus T \), where \( P \) is a finitely generated projective \( \mathcal{C} \)-module, and \( T \) is a finite-dimensional injective \( \mathcal{C} \)-module. Note that \( P \) is nonzero since \( V \) is infinite-dimensional. Furthermore, the composition map \( V \rightarrow P \oplus T \rightarrow P \) cannot be 0 where the second one is the projection because of the same reason. This observation implies (2).

(2) \( \Rightarrow \) (3) Suppose that (2) (and hence (1)) holds. Let us first prove the following statement:

\[ \text{(★) If } F \text{ is a finitely generated left } \mathcal{C} \text{-module which has no nonzero finite-dimensional } \mathcal{C} \text{-submodules, then there is an injective homomorphism } F \rightarrow P, \text{ where } P \text{ is a finitely generated projective left } \mathcal{C} \text{-module.} \]

As in [6] Proposition 7.5, we use induction on the dimension of \( \nu(F) \). If \( \nu(F) = 0 \), then, by (1), \( F \) is finite-dimensional and so \( F = 0 \). Now, suppose \( \nu(F) \neq 0 \). By assumption, \( F \) is infinite-dimensional. It then follows from (2) that there exists \( i \in I \) and a nonzero homomorphism \( f : F \rightarrow A_{e_i} \). Let \( W \) be the image of \( F \) under \( f \). Then we get a short exact sequence of finitely generated left \( \mathcal{C} \)-modules

\[ 0 \rightarrow U \rightarrow F \rightarrow W \rightarrow 0, \]

which induces another short exact sequence of finite-dimensional left \( \mathcal{C} \)-modules:

\[ 0 \rightarrow \nu(U) \rightarrow \nu(F) \rightarrow \nu(W) \rightarrow 0. \]

Note that \( \text{Hom}_{\mathcal{C}}(W, A_{e_i}) \neq 0 \) since there is an inclusion from \( W \) into \( A_{e_i} \). In particular, \( \nu(W) \neq 0 \). Thus the dimension of \( \nu(U) \) is strictly less than that of \( \nu(F) \). By induction hypothesis, (★) holds for \( U \). Hence (★) holds for \( F \).

Now let \( V \) be any finitely generated left \( \mathcal{C} \)-module. Let \( E \) be the maximal finite-dimensional \( \mathcal{C} \)-submodule of \( V \). By Lemma 3.2, there is an injection \( E \rightarrow T \) where \( T \) is a finite-dimensional injective left \( \mathcal{C} \)-module. Let \( F = V/E \). Then \( F \) is a finitely generated left \( \mathcal{C} \)-module which has no nonzero finite-dimensional \( \mathcal{C} \)-submodules, so by (★), there is an injection \( F \rightarrow P \) where \( P \) is a finitely generated projective left \( \mathcal{C} \)-module. It follows that there is an injection \( V \rightarrow T \oplus P \). Therefore the category \( \mathcal{C}\text{-mod} \) has enough injectives.
As in the proof of [9] Lemma 2.4], we suppose that the module \( V \) is injective and \( E' \) is a maximal essential extension of \( E \) in \( V \). Then \( E' \) is injective by [7] Proposition X.5.4]. Moreover, \( E' \) must be finite-dimensional, for otherwise it cannot be an essential extension of \( E \). Hence \( E = E' \), and so \( E \) is injective. Therefore, \( V \) is isomorphic to \( E \oplus F \). Hence \( F \) is also injective. By (\( \ast \)), it follows that \( F \) is projective. \( \square \)

4. On an open question of Nagpal

We now discuss the application of Theorem 4.6 to the categories \( \text{FI}_G \) and \( \text{VI}_q \) studied in representation stability theory. In particular, we affirmatively answer the following open question of Nagpal [9] Question 1.11):

**Question:** Can one establish an equivalence

\[
\text{VI}_q\text{-mod}/\text{VI}_q\text{-fdmod} \sim \sim \rightarrow \text{VI}_q\text{-fdmod}.
\]

Let \( G \) be a finite group. The category \( \text{FI}_G \) was defined independently in [5], Example 3.8] and [12]; let us first recall its definition. The objects of \( \text{FI}_G \) are the finite sets. The morphisms in \( \text{FI}_G \) from a finite set \( X \) to a finite set \( Y \) are the pairs \((f, c)\) where \( f: X \rightarrow Y \) is an injection and \( c: X \rightarrow G \) is an arbitrary map. If \((f, c)\) is a morphism from \( X \) to \( Y \), and \((f', c')\) is a morphism from \( Y \) to \( Z \), then their composite is defined to be the morphism \((f'', c'')\), where

\[
f''(x) = f'(f(x)), \quad c''(x) = c'(f(x))c(x), \quad \text{for each } x \in X.
\]

When \( G \) is the trivial group, the category \( \text{FI}_G \) is the category \( \text{FI} \) of finite sets and injections.

Now, we recall the definition of the category \( \text{VI}_q \). Let \( \mathbb{F}_q \) be a finite field with \( q \) elements. The objects of \( \text{VI}_q \) are finite-dimensional vector spaces over \( \mathbb{F}_q \), and morphisms are the injective linear maps. The composite of morphisms in \( \text{VI}_q \) is just the usual composition of maps. The category \( \text{VI}_q \) has been studied by many authors; see for example [9] and the references therein.

In the following lemma we collect some important results on \( \text{FI}_G \) and \( \text{VI}_q \), which will be used to establish an equivalence between the Serre quotient category and the category of finite-dimensional modules.

**Lemma 4.1.** Suppose that \( \mathbb{k} \) is of characteristic zero and \( \mathcal{C} \) is a skeleton of \( \text{FI}_G \) or \( \text{VI}_q \). Then

1. \( \mathcal{C} \) is locally noetherian over \( \mathbb{k} \).
2. Every finitely generated projective left \( \mathcal{C} \)-module is injective.
3. Every morphism in \( \mathcal{C} \) is a monomorphism.
4. If \( V \) is a finitely generated, infinite-dimensional left \( \mathcal{C} \)-module, then there exists some \( i \in I \) such that \( \text{Hom}_A(V, \mathcal{A}e_i) \neq 0 \).

**Proof.** (1) is established in [5] Theorem 3.7, Example 3.8, Example 3.10]. (2) is verified in [6] Theorem 1.5]. (3) follows from the definitions of \( \text{FI}_G \) and \( \text{VI}_q \). (4) follows from both [5] Lemma 7.2, Lemma 7.3] for \( \text{FI}_G \) and from [9] Theorem 4.34] for \( \text{VI}_q \). \( \square \)

Now, we are ready to prove the following result for \( \text{FI}_G \) and \( \text{VI}_q \), which answers positively the above-mentioned question.

**Theorem 4.2.** Suppose that the characteristic of \( \mathbb{k} \) is zero and \( \mathcal{C} \) is a skeleton of \( \text{FI}_G \) or \( \text{VI}_q \). Then the Nakayama functor \( \nu \) induces an equivalence of categories

\[
\mathcal{C}\text{-mod}/\text{Ker}(\nu) \sim \sim \rightarrow \mathcal{C}\text{-fdmod}.
\]

**Proof.** It is clear that \( \mathcal{C} \) is an EI-category which is inwards finite and hom-finite, and Lemma 4.1(1) tells us that \( \mathcal{C} \) is a locally noetherian category. Clearly, the set of objects of \( \mathcal{C} \) has no maximal element. By Lemma 4.1(2) and Theorem 3.6] the Nakayama functor \( \nu \) induces an equivalence of categories from \( \mathcal{C}\text{-mod}/\text{Ker}(\nu) \) to \( \mathcal{C}\text{-fdmod} \). We claim \( \text{Ker}(\nu) = \mathcal{C}\text{-fdmod} \). Indeed, by Corollary 3.9
and Lemma 4.1(3), one has $C_{\text{fdmod}} \subseteq \text{Ker}(\nu)$; and by Lemma 4.1(4) and Proposition 3.10 one gets $\text{Ker}(\nu) \subseteq C_{\text{fdmod}}$. □

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