WELL-POSEDNESS AND STABILITY IN THE PERIODIC CASE
FOR THE BENNEY SYSTEM

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Abstract. We establish local well-posedness results in weak periodic function
spaces for the Cauchy problem of the Benney system. The Sobolev space
$H^{1/2} \times L^2$ is the lowest regularity attained and also we cover the energy space
$H^1 \times L^2$, where global well-posedness follows from the conservation laws of the
system. Moreover, we show the existence of smooth explicit family of periodic
traveling waves of dnoidal type and we prove, under certain conditions, that
this family is orbitally stable in the energy space.

1. Introduction

In this paper we consider the system introduced by Benney in [10] which models
the interaction between short and long waves, for example in the theory of resonant
water wave interaction in nonlinear medium:

\begin{equation}
\begin{aligned}
     iu_t + u_{xx} &= u v + \beta |u|^2 u, & (x, t) \in \mathcal{M} \times \Delta T \\
     v_t &= (|u|^2)_x, \\
     u(x, 0) &= u_0(x), \\
     v(x, 0) &= v_0(x),
\end{aligned}
\end{equation}

where $u = u(x, t)$ is a complex valued function representing the enveloped of short
waves, and $v = v(x, t)$ is a real valued function representing the long wave. Here $\beta$
is a real parameter, $\Delta T$ is the time interval $[0, T]$ and $\mathcal{M}$ is the real line $\mathbb{R}$ or the
one dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

We let $H^s(\mathcal{M})$ by denoting the classical Sobolev space with the norm

\[ \|f\|_s = \left( \int_{-\infty}^{\infty} (1 + |\xi|)^{2s} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \]
as $\mathcal{M} = \mathbb{R}$,

and

\[ \|f\|_s = \left( \sum_{n \in \mathbb{Z}} (1 + |n|)^{2s} |\hat{f}(n)|^2 dx \right)^{1/2} \]
as $\mathcal{M} = \mathbb{T}$,

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where \( \hat{f}(\xi) \) and \( \hat{f}(\eta) \) denote the Fourier transform and Fourier coefficient of \( f \), respectively. We consider the initial data \((u_0, v_0)\) in the space \( H^s(\mathcal{M}) \times H^s(\mathcal{M}) \) with the induced norm
\[
\|(f, g)\|_{r \times s} := \|f\|_r + \|g\|_s.
\]

The following quantities
\[
E_1[u(\cdot, t)] = \int_I |u(x, t)|^2dx,
\]
\[
E_2[u(\cdot, t), v(\cdot, t)] = \int_I \left[|v(x, t)|^2 + |u(x, t)|^2 + \frac{1}{2}|u(x, t)|^4\right]dx
\]
and
\[
E_3[u(\cdot, t), v(\cdot, t)] = \int_I \left[|v(x, t)|^2 + 2\text{Im}(u(x, t)\overline{\psi}_x(x, t))\right]dx
\]
with the interval \( I = (-\infty, +\infty) \) if \( \mathcal{M} = \mathbb{R} \) and \( I = [0, 1] \) if \( \mathcal{M} = \mathbb{T} \) are invariants by the flux of the system (1.1); i.e., the natural energy space for the system is \( H^1(\mathcal{M}) \times L^2(\mathcal{M}) \).

1.1. Some results in the continuous case. When \( \mathcal{M} = \mathbb{R} \) the local well-posedness for (1.1) for data \((u_0, v_0)\in H^{(s+1/2)}(\mathbb{R}) \times H^s(\mathbb{R}) \) with indices \( s \geq 0 \) was established in the works [3], [16] and [24]. Furthermore, in [24] also was proved global well-posedness in \( H^{(s+1/2)}(\mathbb{R}) \times H^s(\mathbb{R}) \) for \( s = 0 \) if \( \beta = 0 \) and for \( s \in \mathbb{Z}^+ \) and any real \( \beta \) by using the conservation laws (1.2), (1.3) and (1.4).

Recently, in [15] Corcho showed that for \( \beta < 0 \) (focusing case) and for data \((u_0, v_0)\in H^r(\mathbb{R}) \times H^s(\mathbb{R}) \), with \( 0 \leq 3r + 1 < 1 \) and \( r(2s + 3) + 1 \geq 0 \), this problem is ill-posed in the following sense: the data-solution mapping fails to be uniformly continuous on bounded sets of \( H^r(\mathbb{R}) \times H^s(\mathbb{R}) \).

Concerning to the existence and stability of solitary waves solutions for (1.1) of the general form
\[
\begin{cases}
  u(x, t) = e^{i\omega t}e^{ic(x-ct)/2}\phi_s(x - ct), \\
  v(x, t) = \psi_s(x - ct),
\end{cases}
\]

where \( \phi_s, \psi_s : \mathbb{R} \to \mathbb{R} \) are smooth, \( c > 0, \omega \in \mathbb{R}, \) and \( \phi_s(\xi), \psi_s(\xi) \to 0 \) as \( |\xi| \to \infty \), Laurençot in [21] studied for \( \beta = 0 \), the nonlinear stability of the orbit
\[
\Omega(\Phi, \Psi) = \{(e^{i\theta}\Phi(\cdot + x_0), \Psi(\cdot + x_0)); \ (\theta, x_0) \in [0, 2\pi) \times \mathbb{R}\},
\]
in \( H^1(\mathbb{R}) \times L^2(\mathbb{R}) \) by the flow generated by (1.1). Here we have that \( \Phi(\xi) = e^{ic\xi/2}\phi_s(\xi), \Psi(\xi) = \psi_s(\xi), \) and
\[
\phi_s(\xi) = \sqrt{2c}\sigma\text{sech}(\sqrt{\sigma}\xi), \quad \psi_s(\xi) = -\frac{1}{c}\phi_s^2(\xi)
\]
\( \sigma = \omega - \frac{c^2}{4} > 0. \)

1.2. Main results in the periodic case. In the present work we focus the attention on the case \( \mathcal{M} = \mathbb{T} \) and we study the following problems:

- well-posedness in Sobolev spaces with low regularity,
- existence and nonlinear stability of periodic travelling waves.
for the periodic initial values \((u_0, v_0)\) belonging into the space \(H^r(T) \times H^s(T)\), also denoted by \(H^r_{\text{per}} \times H^s_{\text{per}}\).

As follows we define the concepts of well-posedness and the stability that will be use in this work.

**Definition 1.1 (Well-posedness and Ill-posedness).** We say that the system \((1.1)\) is locally well-posed, in time, in the space \(H^r_{\text{per}} \times H^s_{\text{per}}\) if the following conditions hold:

(a) for every \((u_0, v_0)\) in the space \(H^r_{\text{per}} \times H^s_{\text{per}}\) there exists a positive time \(T = T(\|u_0\|_r, \|v_0\|_s)\) and a distributional solution \((u, v) : \mathbb{T} \times \Delta T \rightarrow \mathbb{C} \times \mathbb{R}\)

which is in the space \(C(\Delta T; H^r_{\text{per}} \times H^s_{\text{per}})\);

(b) the data-solution mapping \((u_0, v_0) \mapsto (u, v)\) is uniformly continuous from \(H^r_{\text{per}} \times H^s_{\text{per}}\) to \(C(\Delta T; H^r_{\text{per}} \times H^s_{\text{per}})\);

(c) there is an additional Banach space \(X\) such that \((u, v)\) is the unique solution to the Cauchy problem in \(X \cap C(\Delta T; H^r_{\text{per}} \times H^s_{\text{per}})\).

Moreover, we say that the problem is ill-posed if, at least, one of the above conditions fails.

Before stating our well and ill posedness results we will give some useful notations. Let \(\eta\) be a function in \(C^\infty_0(\mathbb{R})\) such that \(0 \leq \eta(t) \leq 1\),

\[
\eta(t) = \begin{cases} 
1 & \text{if } |t| \leq 1, \\
0 & \text{if } |t| \geq 2, 
\end{cases}
\]

and \(\eta_0(t) = \eta(t^2)\). We denote by \(\lambda \pm\) a number slightly larger, respectively smaller, than \(\lambda\) and by \(\langle t, \xi \rangle = 1 + |\xi|\). The characteristic function on the set \(A\) is denoted by \(\chi_A\). Furthermore, we will work with the auxiliary periodic Bourgain space \(X^s_{\text{per}}\) defined as follows: first we denote by \(X\) the space of functions \(f : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{C}\) such that

(i) \(f(x, \cdot) \in \mathcal{S}(\mathbb{R})\) for each \(x \in \mathbb{T}\);

(ii) \(f(\cdot, t) \in C^\infty(\mathbb{T})\) for each \(t \in \mathbb{R}\).

For \(s, b \in \mathbb{R}\), the spaces \(H^s_{\text{per}}\) and \(X^s_{\text{per}}\) are the completion of \(X\) with respect to the norms

\[
\|f\|_{H^s_{\text{per}}} = \left( \sum_{n \in \mathbb{Z}} \int_{-\infty}^{+\infty} (1 + |n|)^{2s} (1 + |\tau|)^{2b} |\hat{f}(n, \tau)|^2 d\tau \right)^{\frac{1}{2}}
\]

and

\[
\|f\|_{X^s_{\text{per}}} = \|S(-t)f\|_{H^s_{\text{per}}} = \left( \sum_{n \in \mathbb{Z}} \int_{-\infty}^{+\infty} (1 + |n|)^{2s} (1 + |\tau + n^2| + |\tau + n^2|)^{2b} |\hat{f}(n, \tau)|^2 d\tau \right)^{\frac{1}{2}},
\]

respectively, where \(S(t) = e^{it\partial_x^2}\) is the corresponding Schrödinger generator (unitary group) associated to the linear problem,

\[
\begin{cases} 
iu_{tt} + u_{xx} = 0 \\
u(x, 0) = g(x).
\end{cases}
\]
For any \( r, s \in \mathbb{R} \) and \( b_1, b_2 > 1/2 \), we have the embedding \( X_{per}^{r,b_1} \hookrightarrow C(\mathbb{R}; H_{per}^r) \) and \( H_{per}^{b_2} \hookrightarrow C(\mathbb{R}; H_{per}^s) \). For the case \( b = 1/2 \) the embedding can be guaranteed by considering the following slightly modifications of the Bourgain spaces:

\[
\|f\|_{X_{per}^{r,b_1}} := \|f\|_{X_{per}^{r,1/2}} + \|\langle n \rangle^r \hat{f}(n,\tau)\|_{\ell^2_n L^1_\tau}
\]

and

\[
\|f\|_{Y_{per}^{s}} := \|f\|_{H_{per}^{1/2}} + \|\langle n \rangle^s \hat{f}(n,\tau)\|_{\ell^2_n L^1_\tau}
\]

Concerning local well-posedness we obtain the following result:

**Theorem 1.2 (Local Well-Posedness).** For any \((u_0, v_0) \in H_{per}^r \times H_{per}^s\) with \( r, s \) verifying the condition

\[
\max\{0, r - 1\} \leq s \leq \min\{r, 2r - 1\},
\]

there exist a positive time \( T = T(\|u_0\|_r, \|v_0\|_s) \) and a unique solution \((u(t), v(t))\) of the initial value problem \((1.1)\), satisfying

(a) \((\eta_T(t)u, \eta_T(t)v) \in X_{per}^{r}Y_{per}^{s}\);

(b) \((u,v) \in C(\triangle T; H_{per}^r \times H_{per}^s)\).

Moreover, the map \((u_0, v_0) \mapsto (u(t), v(t))\) is locally uniformly continuous from \(H_{per}^r \times H_{per}^s\) into \(C(\triangle T; H_{per}^r \times H_{per}^s)\).

The proof of Theorem 1.2 is based on the Banach fixed point theorem applied on the integral formulation of the system combined with new sharp periodic bilinear estimates, in adequate mixed Bourgain spaces \(X_{per}^{r,b_1} \times H_{per}^{b_2} H_{per}^s\), for the coupling terms \(uv\) and \(\partial_x(|u|^2)\).

Also we find a region which the Cauchy problem is not locally well-posed, more precisely we prove the following theorem:

**Theorem 1.3.** Let \( \beta \neq 0 \). Then for any \( r < 0 \) and \( s \in \mathbb{R} \), the initial value problem \((1.1)\) is locally ill-posed for data in \(H_{per}^r \times H_{per}^s\).

**Figure 1.** Well-posedness results for periodic Benney system. The region \( W \), limited by the lines \( L_1 : s = 2r - 1 \), \( L_2 : s = r \) and \( L_3 : s = r - 1 \), contain the indices \((r,s)\) where the local well-posedness is achieved in Theorem 1.2.
Regarding the stability of periodic travelling waves, namely, solutions for (1.1) of the form

\begin{align}
\begin{cases}
  u(t, x) = e^{-i\omega t} e^{ic(x-ct)/2} \varphi_{\omega, c}(x-ct) \\
  v(x, t) = n_{\omega, c}(x-ct)
\end{cases}
\end{align}

(1.13)

where \(\varphi_{\omega, c}, n_{\omega, c}\) are real smooth, \(L\)-periodic functions, \(c > 0\), and \(\omega < 0\), we have the following definition.

**Definition 1.4 (Non-Linear Stability).** The periodic traveling wave \(\Phi(\xi) = e^{i\xi/2} \varphi_{\omega, c}(\xi)\), \(\Psi(\xi) = n_{\omega, c}(\xi)\), is orbitally stable in \(H^1_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L])\) if for all \(\varepsilon > 0\), there exists \(\delta > 0\), such that if \(||(u_0, v_0) - (\Phi, \Psi)||_{H^1_{\text{per}} \times L^2_{\text{per}}} < \delta\) and \((u(t), v(t))\) is the solution of (1.1) with \((u(0), v(0)) = (u_0, v_0)\), then

\[
\inf_{s \in [0, 2\pi]} \inf_{r \in \mathbb{R}}||(u(t), v(t)) - (e^{i\beta \Phi(r+s)} \Psi(r+s), \Phi(r+s))||_{H^1_{\text{per}} \times L^2_{\text{per}}} < \varepsilon, \quad t \in \mathbb{R}.
\]

Otherwise \((\Phi, \Psi)\) is called orbitally unstable.

We will show below that there exist a smooth explicit family of profiles solutions of minimal period \(L\),

\( (\omega, c) \in A_\beta \rightarrow (\varphi_{\omega, c}, n_{\omega, c}) \in H^r_{\text{per}}([0, L]) \times H^s_{\text{per}}([0, L]), \)

where \(A_\beta = \{(x, y) : y > 0, 1 > \beta y, \text{ and } x < -\frac{2x^2}{\beta}\} \) and which depends of the Jacobian elliptic function \(dn\) called dnoidal, more precisely,

\begin{align}
\varphi_{\omega, c}(\xi) &= \frac{\sqrt{c}}{1-\beta c} \eta_1 dn \left( \frac{\sqrt{c}}{\sqrt{1-\beta c}} \xi; \kappa \right) \\
n_{\omega, c}(\xi) &= -\frac{\eta_1^2}{1-\beta c} dn^2 \left( \frac{\sqrt{c}}{\sqrt{1-\beta c}} \xi; \kappa \right)
\end{align}

(1.14)

with \(\eta_1 = \eta_1(\omega, c)\) and \(\kappa = \kappa(\omega, c)\), being smooth functions of \(\omega\) and \(c\).

So, by following Angulo [4] and Grillakis et al. [17, 18], we obtain the following stability theorem.

**Theorem 1.5 (Stability Theory).** Let \((\omega, c) \in A_\beta\) such that for \(c > 0\) there is \(q \in \mathbb{N}\) satisfying \(4\pi q/c = L\). Define \(\sigma = -\omega - \frac{c}{q}\). Then \(\Phi(\xi) = e^{i\xi/2} \varphi_{\omega, c}(\xi)\), \(\Psi(\xi) = n_{\omega, c}(\xi)\), with \(\varphi_{\omega, c}, n_{\omega, c}\) given in (1.14), is orbitally stable in \(H^1_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L])\) by the periodic flow generated by (1.1):

\begin{enumerate}
\item for \(\beta \leq 0\),
\item for \(\beta > 0\) and \(8\beta \sigma - 3c(1 - \beta c)^2 \leq 0\).
\end{enumerate}

2. **LOCAL THEORY**

We prove Theorem 1.2 using the standard technique, that is: we use the Duhamel integral formulation for the system (1.1) combined with the Banach fixed point theorem in adequate Bourgain spaces \(X^r_{\text{per}} \times Y^s_{\text{per}}\) with the objective to get the desired solution. The main difficulty is the necessity to prove two new mixed periodic bilinear estimates, which we will prove in the following sections.

2.1. **Sharp Periodic Bilinear Estimates.** We begin recalling the following elementary inequalities, which will be used in the proof of the next main estimates.

**Lemma 2.1.** Let \(\theta_1, \theta_2 > 0\) with \(\theta_1 + \theta_2 > 1\) and \(\lambda > 1/2\). Then, there are a positive constants \(C_1\) and \(C_2\) such that
For this purpose, in region $(2.20)$

\[ \sum_{n \in W} \frac{1}{(a+b)^{2n+1}} \leq C_2, \text{ with } a, b \in \mathbb{R}. \]

**Proof.** For details of the proof we refer, for instance, the works [19] and [7]. \(\Box\)

**Lemma 2.2.** Let \(0 < \theta < 1/4\). Then, the following estimates

\[ \|uv\|_{X_{per}^{r,-1/2}} \lesssim \|u\|_{X_{per}^{r+1/2-	heta}} \|v\|_{H_{per}^{1/2} H_{per}^{1/2}} + \|u\|_{X_{per}^{r+1/2}} \|v\|_{H_{per}^{1/2-	heta} H_{per}^{1/2}} \]

\[ \| \langle n \rangle^{\tau} \hat{u}(n, \tau) \|_{L^s_\alpha} \lesssim \|u\|_{X_{per}^{r+1/2-	heta}} \|v\|_{H_{per}^{1/2} H_{per}^{1/2}} + \|u\|_{X_{per}^{r+1/2}} \|v\|_{H_{per}^{1/2-	heta} H_{per}^{1/2}} \]

hold provided \(r \geq 0\) and \(\max\{0, r-1\} \leq s\).

**Proof.** First we prove (2.15). We define \(f(n, \tau) := \langle \tau + n^2 \rangle^{b_1} \hat{u}(n, \tau)\) and \(g(n, \tau) := \langle \tau \rangle^{b_2} \hat{v}(n, \tau)\). Then, using duality arguments we obtain

\[ \|uv\|_{X_{per}^{r,-1/2}} = \sup \{W(\varphi) : \|\varphi\|_{L^s_\alpha} \leq 1\}, \]

where,

\[ W(\varphi) = \sum_{(n, n_1) \in \mathbb{Z}^2} \int_{\mathbb{R}^2} \langle \tau + n^2 \rangle^{1/2} \langle n_1 \rangle^{r} f(n_1, \tau_1) g(n - n_1, \tau - \tau_1) \varphi(n, \tau) d\tau d\tau_1. \]

We will divide the space \(\mathbb{Z}^2 \times \mathbb{R}^2 = A_0 \cup A_1 \cup A_2\) and we separate the integral \(W\) as follows:

\[ \int_{\mathbb{R}^2} \int_{\mathbb{Z}^2} \sum_{(n, n_1, \tau, \tau_1) \in A_j} \langle \tau + n^2 \rangle^{1/2} \langle n_1 \rangle^{r} f(n_1, \tau_1) g(n - n_1, \tau - \tau_1) \varphi(n, \tau) d\tau d\tau_1, \]

for \(j = 0, 1, 2\). It is easy to see that to obtain (2.15) is sufficient to prove that whenever \(r, s \geq 0\) and \(r - s \leq 1\) the estimate

\[ W_j(\varphi) \lesssim \|f\|_{L^r_\alpha} \|g\|_{L^r_\alpha} \|\varphi\|_{L^s_\alpha} \|\varphi\|_{L^s_\alpha}, \]

holds with \(b_1 = 1 - \theta\) and \(b_2 = 1/2\) or with \(b_1 = 1/2\) and \(b_2 = 1/2 - \theta\). Indeed, next we will prove the following estimates:

\[ W_j(\varphi) \lesssim \|u\|_{X_{per}^{r+1/2}} \|v\|_{H_{per}^{1/2-	heta} H_{per}^{1/2}} \|\varphi\|_{L^2}, \text{ for } j = 0, 1, \]

\[ W_2(\varphi) \lesssim \|u\|_{X_{per}^{r+1/2-	heta}} \|v\|_{H_{per}^{1/2} H_{per}^{1/2}} \|\varphi\|_{L^2}. \]

For this purpose, in region \(A_0\) we integrate first over \((n_1, \tau_1)\), in region \(A_1\) we integrate first over \((n, \tau)\) and in region \(A_2\) we integrate first over \((n_2, \tau_2) = (n - n_1, \tau - \tau_1)\); then using Cauchy-Schwarz inequality we easily see that it remains only to bound uniformly the following three expressions:
where in the last inequality we have used that $0 < \theta < \frac{1}{2}$. \hfill (2.23)

Then, we put

$$\tilde{\mathcal{W}}_0 := \sup_{n, \tau} \frac{\langle n \rangle^{2r}}{\langle \tau + n^2 \rangle} \sum_{n_1} \int_{A_0} \frac{d\tau_1}{(\tau_1 + n_1^2)(\tau_2^1)^{1-2\theta}(n_1)^{2r}(n_2)^{2s}}$$

(2.21)

Now we define the regions $A_0, A_1$ and $A_2$. We use the notation

$$\mathcal{L} := \max \left\{ |\tau + n^2|, |\tau_1 + n_1^2|, |\tau_2| \right\}.$$  

and first we introduce the subsets:

$$A_{0,1} := \{ (n, n_1, \tau, \tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2 : |n| \leq 100 \},$$

(2.25)

$$A_{0,2} := \{ (n, n_1, \tau, \tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2 : |n| > 100 \text{ and } |n| \leq 2|n_1| \},$$

$$A_{0,3} := \{ (n, n_1, \tau, \tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2 : |n| > 100, |n_1| < |n|/2 \text{ and } \mathcal{L} = |\tau + n^2| \}.$$  

Then, we put

$$A_0 := A_{0,1} \cup A_{0,2} \cup A_{0,3},$$

(2.26)

$$A_1 := \{ (n, n_1, \tau, \tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2 : |n| > 100, |n_1| < |n|/2 \text{ and } \mathcal{L} = |\tau_1 + n_1^2| \},$$

$$A_2 := \{ (n, n_1, \tau, \tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2 : |n| > 100, |n_1| < |n|/2 \text{ and } \mathcal{L} = |\tau_2| \}.$$  

For later use, we recall that the dispersive relation of this bilinear estimate is:

$$\tau + n^2 - (\tau_1 + n_1^2) - \tau_2 = n^2 - n_1^2,$$

where $\tau - \tau_1 = \tau_2$ and $n - n_1 = n_2$.

We begin with the analysis of (2.21). In the region $A_{0,1}$, using that $|n| \lesssim 1$ and $r, s \geq 0$ we have

$$\tilde{\mathcal{W}}_{0,1} := \sup_{n, \tau} \frac{\langle n \rangle^{2r}}{\langle \tau + n^2 \rangle} \sum_{n_1} \int_{A_{0,1}} \frac{d\tau_1}{(\tau_1 + n_1^2)(\tau_2^1)^{1-2\theta}(n_1)^{2r}(n_2)^{2s}}$$

$$\lesssim \sup_{n, \tau} \frac{1}{\langle \tau + n^2 \rangle} \sum_{n_1} \int_{-\infty}^{+\infty} \frac{d\tau_1}{(\tau_1 + n_1^2)(\tau_2^1)^{1-2\theta}(n_1)^{2r}(n_2)^{2s}}$$

(2.28)

$$\lesssim \sup_{\tau} \sum_{n_1} \frac{1}{(\tau + n_1^2)^{1-2\theta}} \lesssim 1,$$

where in the last inequality we have used that $0 < \theta < 1/4$ combined with Lemma 2.1.
In the region $A_{0.2}$, we have that $\langle n \rangle^{2r} \lesssim \langle n_1 \rangle^{2r}$. Thus, similarly to the previous case, we get

$$
\tilde{W}_{0.2} := \sup_{n, \tau} \frac{\langle n \rangle^{2r}}{\langle \tau + n^2 \rangle} \sum_{n_1} \int_{A_{0.2}} \frac{d\tau_1}{(\tau_1 + n_1^2) (\tau_2) (n_1)^{2r} (n_2)^{2s}}
$$

(2.29)

$$
\lesssim \sup_{n, \tau} \frac{1}{(\tau + n^2)} \sum_{n_1} \int_{A_{0.2}} \frac{d\tau_1}{(\tau_1 + n_1^2) (\tau_2) (n_1)^{1-2\theta}}
$$

$$
\lesssim \sup_{\tau} \sum_{n_1} \frac{1}{(\tau + n_1^2)^{1-2\theta}} \lesssim 1.
$$

In the region $A_{0.3}$ we have that $|n_1| < |n|/2$ and $|n| > 100$, which imply that $|n - n_1| \sim |n + n_1| \sim |n|$. Moreover, the dispersive relation (2.27) says that

$$
\mathcal{L} = |\tau + n^2| \gtrsim |n^2 - n_1^2| = |n - n_1||n + n_1| \sim |n|^2.
$$

Therefore,

$$
\tilde{W}_{0.3} := \sup_{n, \tau} \frac{\langle n \rangle^{2r}}{\langle \tau + n^2 \rangle} \sum_{n_1} \int_{A_{0.3}} \frac{d\tau_1}{(\tau_1 + n_1^2) (\tau_2) (n_1)^{2r} (n_2)^{2s}}
$$

(2.30)

$$
\lesssim \sup_{n, \tau} \frac{\langle n \rangle^{2r-2s}}{\langle \tau + n^2 \rangle} \sum_{n_1} \int_{A_{0.3}} \frac{d\tau_1}{(\tau_1 + n_1^2) (\tau_2) (n_1)^{1-2\theta}}
$$

$$
\lesssim \sup_{n, \tau} \sum_{n_1} \frac{1}{(\tau + n_1^2)^{1-2\theta}} \lesssim 1,
$$

since $r \geq 0$, $r - s \leq 1$ and $0 < \theta < 1/4$.

Putting together the estimates (2.28), (2.29) and (2.30) we conclude that

$$
|\tilde{W}_0| \leq |\tilde{W}_{0.1}| + |\tilde{W}_{0.2}| + |\tilde{W}_{0.3}| \lesssim 1,
$$

obtaining the desired bounded for (2.21).

Next we estimate the contribution of (2.22). In the region $A_1$, we know that $|n_1| < |n|/2$, $|n| > 100$ and $L = |\tau_1 + n_1^2|$. So, $|n_2| \sim |n|$ and the dispersive relation (2.27) implies that $|\tau_1 + n_1^2| \gtrsim n^2$. Thus,

$$
\tilde{W}_1 = \sup_{n_1, \tau_1} \frac{1}{(n_1)^{2r} (\tau_1 + n_1^2)} \sum_{n} \int_{A_1} \frac{\langle n \rangle^{2r}}{(\tau + n^2) (\tau_2) (n_2)^{2s}} d\tau
$$

$$
\lesssim \sup_{\tau_1} \sum_{n} \int_{A_1} \frac{\langle n \rangle^{2r-2s-2}}{(\tau + n^2) (\tau_2) (n_2)^{1-2\theta}} d\tau
$$

$$
\lesssim \sup_{\tau_1} \sum_{n} \frac{1}{(\tau_1 + n_1^2)^{1-2\theta}} \lesssim 1,
$$

since $r \geq 0$, $r - s \leq 1$ and $0 < \theta < 1/4$.

Finally, we bound (2.23) by noting that, in the region $A_2$ it holds $|n| > 100$, $|n_1| < |n|/2$ and $L = |\tau_2|$. Then, $|n_2| \sim |n|$ and the dispersive relation (2.27) yield
that \(|τ_2| ≥ n^2\). Using these conditions and that \(r ≥ 0\), \(r - s ≤ 1\) we obtain

\[
\tilde{W}_2 = \sup_{n_2, τ_2} \frac{1}{(n_2)^2r} \left( \sum_{n} \frac{\langle n \rangle^{2r}}{(n + n^2)^{r_1 + n_1^2})^{1 - 2θ}(n_1)^{2r}} \right) dτ
\]

\[
≤ \sup_{n_2, τ_2} \sum_{n} \int_{-∞}^{+∞} \frac{\langle n \rangle^{2r - 2s - 2}}{(n + n^2)^{r_1 + n_1^2})^{1 - 2θ}} dτ
\]

\[
= \sup_{n_2, τ_2} \left\{ \sum_{n \in H_1} \frac{1}{2(n^2 + \frac{τ^2}{2n^2} - \frac{r^2}{2})} \right\}^{1 - 2θ} + \sum_{n \in H_2} \left( \frac{1}{2(n^2 + \frac{τ^2}{2n^2} - \frac{r^2}{2})} \right)^{1 - 2θ}
\]

where

\[
H_1 := \left\{ n \in \mathbb{Z} : \left| n + \frac{τ^2}{2n^2} - \frac{n^2}{2} \right| < 2 \right\} \quad \text{and} \quad H_2 := \left\{ n \in \mathbb{Z} : \left| n + \frac{τ^2}{2n^2} - \frac{n^2}{2} \right| ≥ 2 \right\}.
\]

Now we note that \(\#H_1 ≤ 4\) and for any \(n \in H_2\) we have

\[
\langle n \rangle \langle n + \frac{τ^2}{2n^2} - \frac{n^2}{2} \rangle \geq \langle n \rangle \langle n + \frac{τ^2}{2n^2} - \frac{n^2}{2} \rangle,
\]

since \(|n_2| \sim |n|\). Then, by Hölder’s inequality

\[
\sum_{n \in H_1} \frac{1}{(n^2 + \frac{τ^2}{2n^2} - \frac{n^2}{2})} \left( \sum_{n \in H_2} \frac{1}{(n^2 + \frac{τ^2}{2n^2} - \frac{n^2}{2})} \right)^{1 - 2θ}
\]

\[
≤ 4 + \left( \sum_{n \in H_2} \frac{1}{\langle n \rangle^{2(1 - 2θ)}} \right)^{1/2} \left( \sum_{n \in H_2} \frac{1}{\langle n \rangle^{2(1 - 2θ)}} \right)^{1/2} \lesssim 1,
\]

since and \(0 < θ < 1/4\). This completes the proof of (2.15).

Next, we prove (2.10). We let \(a \in (1/2, 3/4 - θ)\). By using Cauchy-Schwarz inequality, we have that

(2.31) \[ \left\| \hat{n}^{-1}(n, r) \right\|^2 \leq \sum_{n} \langle n \rangle^{2r} \left\{ \int_{-∞}^{+∞} \frac{dτ}{(τ + n^2)^{2(1 - a)}} \right\} \]

Now, we separate \(\mathbb{Z}^2 \times \mathbb{R}^2\) in the same regions used to estimate (2.15) and we note that, except in the region \(A_{0.3}\), the right-hand of (2.1) can be estimated in the same way that (2.15). To see this, we observe that the integral \(\int_{-∞}^{+∞} \frac{dτ}{(τ + n^2)^{2(1 - a)}}\) is convergent and we replace the term \(\langle τ + n^2 \rangle\) by \(\langle τ + n^2 \rangle^{2(1 - a)}\) in (2.21), (2.22) and (2.23), then we follows the same steps to bound the corresponding expressions in each region, using that the condition \(2(1 - a) + (1 - 2θ) - 1 > 1/2\) holds for \(a \in (1/2, 3/4 - θ)\).

Now we proceed with the estimate of the right-hand of (2.1) in \(A_{0.3}\). Here, by using the fact that \(|τ + n^2| ≥ |n|^2\) we have that

(2.32) \[ \int_{A_{0.3}} \frac{dτ}{(τ + n^2)^{2a}} \lesssim \langle n \rangle^{2(1 - 2a)}. \]
Then, using (2.32), we have
\[ \left\| \left( n \right)^{r} \hat{w}(n, \tau) \chi_{A_{0,3}} \right\|_{L^2_x L^1_t}^2 \lesssim \tilde{W}_{0,3} \| u \|_{X^{r,1/2}}^2 \| v \|_{H^{1/2-\theta} H^\theta_x}^2, \]
where
\[ \tilde{W}_{0,3} = \sup_{n, \tau} \left( n \right)^{2r-2s+2-4\alpha} \frac{\sum_{n_1} \int_{n_1}^{+\infty} \frac{d\tau_1}{(\tau + n_1^2)(\tau_2 + n_1^2)^{1-2\theta}}}{(\tau + n^2)^{2(1-\alpha)}} \lesssim \left( \frac{n}{\tau + n^2} \right)^{2(1-\alpha)} \sum_{n_1} \frac{d\tau_1}{(\tau_1 + n_1^2)(\tau_2 + n_1^2)^{1-2\theta}} \lesssim 1, \]
Similarly to the estimate made in (2.30) we obtain
\[ \tilde{W}_{0,3} \lesssim \sup_{n, \tau} \left( n \right)^{2r-2s+2-4\alpha} \sum_{n_1} \int_{-\infty}^{+\infty} \frac{d\tau_1}{(\tau + n_1^2)(\tau_2 + n_1^2)^{1-2\theta}} \lesssim 1, \]
so that \( 0 < \theta < 1/4 \) and \( r - s \leq 1 \). Finally, combining (2.32) and (2.35) we get
\[ \left\| \left( n \right)^{r} \hat{w}(n, \tau) \chi_{A_{0,3}} \right\|_{L^2_x L^1_t} \lesssim \| u \|_{X^{r,1/2}} \| v \|_{H^{1/2-\theta} H^\theta_x}, \]
as desired. Then, we finished the proof of Lemma 2.2. \( \square \)

The next result shows that the conditions obtained above for indices \( r \) and \( s \) are necessary.

**Proposition 2.3.** For any real numbers \( b_1 \) and \( b_2 \), the veracity of the inequality
\[ \| uv \|_{X^{r,-1/2}} \lesssim \| u \|_{X^{r,b_1}} \| v \|_{H^{b_2} H^\theta_x} \]
implies that \( \max \{ 0, r - 1 \} \leq s \).

**Proof.** Firstly, we fix \( N \gg 1 \) a large integer and define de sequences
\[ \alpha_1(n) = \begin{cases} 1 & \text{if } n = N, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \beta_1(n) = \begin{cases} 1 & \text{if } n = -2N, \\ 0 & \text{otherwise}. \end{cases} \]
Let \( u_{1, N}(x, t) \) and \( v_{1, N}(x, t) \) be given by \( \hat{u}_{1, N}(n, \tau) = \alpha_1(n) \chi_{[-1,1]}(\tau + n^2) \) and \( \hat{v}_{1, N}(n, \tau) = \beta_1(n) \chi_{[-1,1]}(\tau) \). Taking into account the dispersive relation
\[ \tau + n^2 - (\tau_1 + n_1^2) = n^2 - n_1^2, \]
we can easily compute that
\[ \| u_{1, N} v_{1, N} \|_{X^{r,-1/2}} \sim N^r, \quad \| u_{1, N} \|_{X^{r,b_1}} \sim N^r \quad \text{and} \quad \| v_{1, N} \|_{H^{b_2} H^\theta_x} \sim N^s \]
Hence, from the bound \( \| u_{1, N} v_{1, N} \|_{X^{r,-1/2}} \lesssim \| u_{1, N} \|_{X^{r,b_1}} \| v_{1, N} \|_{H^{b_2} H^\theta_x} \) we must have
\( N^r \lesssim N^{r+s} \) for \( N \gg 1 \), which implies that \( s \geq 0 \).

Secondly, we define the sequences
\[ \alpha_2(n) = \begin{cases} 1 & \text{if } n = N, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \beta_2(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise}. \end{cases} \]
Let \( \hat{u}_{2, N}(n, \tau) = \alpha_2(n) \chi_{[-1,1]}(\tau + n^2) \) and \( \hat{v}_{2, N}(n, \tau) = \beta_2(n) \chi_{[-1,1]}(\tau) \). Again, it is easy to see that
\[ \| u_{2, N} v_{2, N} \|_{X^{r,-1/2}} \sim N^{r-1}, \quad \| u_{2, N} \|_{X^{r,b_1}} \sim 1 \quad \text{and} \quad \| v_{2, N} \|_{H^{b_2} H^\theta_x} \sim N^s \]
Hence, the bound \( \|u_{2N}v_{2N}\|_{X^{r-1/2}} \lesssim \|u_{2N}\|_{X^{r,s}} \|v_{2N}\|_{H^{\alpha 2}_x H^s_x} \) implies \( N^{r-1} \lesssim N^s \) for \( N \gg 1 \), so we must have \( r-1 \leq s \).

\[ \square \]

**Lemma 2.4.** Let \( 0 < \theta < 1/4 \). Then, the following estimates

\[
\| \partial_x (\bar w) \|_{H^{-1/2}_x H_{\text{per}}^r} \lesssim \|u\|_{X^{r-1/2-\theta}_x H_{\text{per}}^r} \|w\|_{X^{r-1/2-\theta}_x H_{\text{per}}^r} + \|u\|_{X^{r-1/2-\theta}_x H_{\text{per}}^r} \|w\|_{X^{r-1/2-\theta}_x H_{\text{per}}^r}
\]

\[
\left\| \langle n \rangle \frac{\partial_x \langle \bar w \rangle(n, \tau)}{\langle \tau \rangle} \right\|_{L^2}$ \( \leq \|u\|_{X^{r-1/2-\theta}_x H_{\text{per}}^r} \|w\|_{X^{r-1/2-\theta}_x H_{\text{per}}^r} + \|u\|_{X^{r-1/2-\theta}_x H_{\text{per}}^r} \|w\|_{X^{r-1/2-\theta}_x H_{\text{per}}^r}
\]

hold provided \( 0 \leq s \leq \min \{2r-1, r \} \).

**Proof.** The proof is similar to Lemma 2.2. Here, the relevant dispersive relation is given by

\[
(\tau_1 + n_1^2) + (\tau_2 - n_2^2) - \tau = n_1^2 - n_2^2,
\]

where \( \tau_2 = \tau - \tau_1 \) and \( n_2 = n - n_1 \).

To prove (2.36), by duality arguments, it suffices to bound uniformly the following expressions:

\[
Z_0 = \sup_{n_1, \tau_1} \frac{1}{\langle n \rangle^{2r} \langle \tau_1 + n_1^2 \rangle} \sum_n C_0 \int \frac{|n|^2 \langle n \rangle^{2s}}{\langle \tau \rangle (\tau_2 - n_2^2)^{1-2\theta} (n_2^{2r})^{2r}} d\tau,
\]

\[
Z_1 = \sup_{n, \tau} \frac{|n|^2 \langle n \rangle^{2s}}{\langle \tau \rangle} \sum_{n_1} \int \frac{\tau_1}{\langle \tau_1 + n_1^2 \rangle (\tau_2 - n_2^2)^{1-2\theta} (n_1^{2r})^{2r}} d\tau,
\]

\[
Z_2 = \sup_{n_2, \tau_2} \frac{1}{\langle n_2 \rangle^{2r} \langle \tau_2 - n_2^2 \rangle} \sum_n C_2 \int \frac{|n|^2 \langle n \rangle^{2s}}{\langle \tau \rangle (\tau_1 + n_1^2)^{1-2\theta} (n_1^{2r})^{2r}} d\tau,
\]

where \( C_0, C_1 \) and \( C_2 \) are defined as follows. We denote by

\[
\mathcal{L} := \max \{|\tau|, |\tau_1 + n_1^2|, |\tau_2 - n_2^2|\}
\]

and then we define the following sets:

\[
C_{0.1} := \{(n, \tau, n_1, \tau_1) : |n| \leq 100\},
\]

\[
C_{0.2} := \{(n, \tau, n_1, \tau_1) : |n| > 100, \frac{|n_1|}{2} \leq |n_1| \leq 2|n_2|\},
\]

\[
C_{0.3} := \{(n, \tau, n_1, \tau_1) : |n| > 100, |n_1| < \frac{|n_1|}{2} \text{ or } |n_2| < \frac{|n_1|}{2} \text{ and } \mathcal{L} = |\tau_1 + n_1^2|\}.
\]

Now we put

\[
C_0 := C_{0.1} \cup C_{0.2} \cup C_{0.3},
\]

\[
C_1 := \{(n, \tau, n_1, \tau_1) : |n| > 100, |n_1| < \frac{|n_1|}{2} \text{ or } |n_2| < \frac{|n_1|}{2} \text{ and } \mathcal{L} = |\tau|\},
\]

\[
C_2 := \{(n, \tau, n_1, \tau_1) : |n| > 100, |n_1| < \frac{|n_1|}{2} \text{ or } |n_2| < \frac{|n_1|}{2} \text{ and } \mathcal{L} = |\tau_2 - n_2^2|\}.
\]
Now, we bound (2.39). In the region \( C_{0,1} \), it holds \( |n| \leq 100 \). Hence,

\[
Z_{0,1} := \sup_{n_1, \tau_1} \frac{1}{\langle n_1 \rangle^{2r} (\tau_1 + n_1^2)} \sum_n \int_{C_{0,1}} \frac{|n|^2 \langle n \rangle^{2s}}{\langle \tau \rangle (\tau - n_2^2)^{1-2\theta}} d\tau 
\]

\[
\lesssim \sup_{n_1, \tau_1} \sum_{|n| \leq 100}^{+\infty} \int_{-\infty}^{\infty} \frac{d\tau}{\langle \tau \rangle (\tau - n_2^2)^{1-2\theta}}
\]

\[
\lesssim \sup_{n_1, \tau_1} \sum_{|n| \leq 100} 1 \langle \tau_1 + (n - n_1)^2 \rangle^{1-2\theta} \lesssim 1,
\]

since \( r \geq 0 \) and \( 1 - 2\theta > 0 \).

In the region \( C_{0,2} \), we have that \( |n_1| \sim |n_2| \). Hence,

\[
Z_{0,2} := \sup_{n_1, \tau_1} \frac{1}{\langle n_1 \rangle^{2r} (\tau_1 + n_1^2)} \sum_n \int_{C_{0,2}} \frac{|n|^2 \langle n \rangle^{2s}}{\langle \tau \rangle (\tau - n_2^2)^{1-2\theta}} d\tau 
\]

\[
\lesssim \sup_{n_1, \tau_1} \frac{\langle n_1 \rangle^{2s-4r+2}}{\langle \tau_1 + n_1^2 \rangle} \sum_n^{+\infty} \int_{-\infty}^{\infty} \frac{d\tau}{\langle \tau \rangle (\tau - n_2^2)^{1-2\theta}}
\]

\[
\lesssim \sup_{n_1, \tau_1} \sum_n 1 \langle \tau_1 + (n - n_1)^2 \rangle^{1-2\theta} \lesssim 1,
\]

for \( 0 \leq s \leq 2r - 1 \) and \( 0 < \theta < 1/4 \).

In the region \( C_{0,3} \), the dispersion relation (2.38) and the assumptions \( |n_1| \sim |n_2| \), \( |n| \geq 100 \) and \( \mathcal{L} = |\tau_1 + n_1^2| \) imply that \( |\tau_1 + n_1^2| \gtrsim (\max(|n_1|, |n_2|))^2 \). Then,

\[
Z_{0,3} := \sup_{n_1, \tau_1} \frac{1}{\langle n_1 \rangle^{2r} (\tau_1 + n_1^2)} \sum_n \int_{C_{0,3}} \frac{|n|^2 \langle n \rangle^{2s}}{\langle \tau \rangle (\tau - n_2^2)^{1-2\theta}} d\tau 
\]

\[
\lesssim \sup_{n_1, \tau_1} \sum_n^{+\infty} \int_{-\infty}^{\infty} \frac{(\max(|n_1|, |n_2|))^{2s-2r}}{\langle \tau \rangle (\tau - n_2^2)^{1-2\theta}} d\tau
\]

\[
\lesssim \sup_{n_1, \tau_1} \sum_n 1 \langle \tau_1 + (n - n_1)^2 \rangle^{1-2\theta} \lesssim 1,
\]

for \( 0 \leq s \leq r \) and \( 0 < \theta < 1/4 \). Then, the inequality \( |Z_0| \leq |Z_{0,1}| + |Z_{0,2}| + |Z_{0,3}| \lesssim 1 \) yields the desired estimate for \( Z_0 \).
The contribution of (2.10) can be estimated as follows. In the region $C_1$, we have that $|n| \sim \max\{|n_1|, |n_2|\}$ and $|\tau| \geq (\max\{|n_1|, |n_2|\})^2$. Thus,

\[
Z_1 \leq \sup_{n, \tau} \frac{\langle n \rangle^{2s+2}}{\langle \tau \rangle} \sum_{n_1} \int C_1 \frac{1}{(\tau + n_1^2)^{1-2\theta}(\tau - n_2^2)^{1-2\theta}(\tau)\langle n \rangle^{2r} d\tau}
\]

\[
\lesssim \sup_{n, \tau} \sum_{n_1} \int_{-\infty}^{\infty} \frac{(\max\{|n_1|, |n_2|\})^{2s-2r}}{(\tau + n_1^2)^{1-2\theta}(\tau - n_2^2)^{1-2\theta} d\tau}
\]

\[
\lesssim \sup_{n, \tau} \sum_{n_1} \frac{1}{(\tau + n_1^2 - n_2^2)^{1-2\theta}}
\]

\[
\lesssim \sup_{n, \tau} \sum_{n_1} \frac{1}{\langle 2n \rangle^{1-2\theta}} \lesssim 1,
\]

for $0 \leq s \leq r$ and $0 < \theta < 1/4$, using the same arguments to estimate $\tilde{W}_2$ in Lemma 2.2.

On the other hand, the expression (2.11) can be controlled by using that in the region $C_2$ hold $|n| \sim \max\{|n_1|, |n_2|\}$ and $|\tau - n_2^2| \geq (\max\{|n_1|, |n_2|\})^2$. Then,

\[
Z_2 = \sup_{n_2, \tau_2} \frac{1}{\langle n_2 \rangle^{2r}(\tau_2 - n_2^2)} \sum_{n} \int C_2 \frac{|n|^2\langle n \rangle^{2s}}{(\tau)\langle n_2 \rangle^{1-2\theta}(\tau_2)\langle n_2 \rangle^{2r} d\tau}
\]

\[
\lesssim \sup_{n_2, \tau_2} \sum_{n} \int_{-\infty}^{+\infty} \frac{(\max\{|n_1|, |n_2|\})^{2s-2r}}{(\tau)\langle n_2 \rangle^{1-2\theta} d\tau}
\]

\[
\lesssim \sup_{n_2, \tau_2} \sum_{n} \frac{1}{\langle n_2 \rangle^{1-2\theta}} \lesssim 1,
\]

for $s \leq r$ and $0 < \theta < 1/4$. Collecting all the estimates above we obtain the claimed estimate (2.39).

The proof of (2.37) follows from a similar way to the proof of (2.10). □

Now we exhibit examples showing the necessity of the conditions for $r$ and $s$ used in Lemma 2.4.

**Proposition 2.5.** For any real numbers $b_1$ and $b_2$ the veracity of the inequality

\[
\|\partial_x (u\tilde{w})\|_{H_{-1/2}^r H_{-\theta}^s} \lesssim \|u\|_{X^{r,b_1}} \|w\|_{X^{r,b_2}}
\]

implies that $s \leq \min\{2r - 1, r\}$.

**Proof.** For a fixed large integer $N \gg 1$, we define the following sequences:

\[
\alpha_1(n) = \begin{cases} 1 & \text{if } n = N, \\ 0 & \text{otherwise} \end{cases} \text{ and } \beta_1(n) = \begin{cases} 1 & \text{if } n = -N, \\ 0 & \text{otherwise}. \end{cases}
\]

Putting $\tilde{w}_1(n, \tau) = \alpha_1(n)\chi_{[-1,1]}(\tau + n^2)$ and $\tilde{w}_1(n, \tau) = \beta_1(n)\chi_{[-1,1]}(\tau + n^2)$, a simple calculation using the dispersive relation (2.38) gives that

\[
\|(u_1\tilde{w}_1)_x\|_{H_{-1/2}^r H_{-\theta}^s} \sim N^{s+1} \quad \text{and} \quad \|u_1\|_{X^{r,b_1}} \sim N^{r} \sim \|w_1\|_{X^{r,b_2}}.
\]

Hence, the inequality $\|(u_1\tilde{w}_1)_x\|_{H_{-1/2}^r H_{-\theta}^s} \lesssim \|u_1\|_{X^{r,b_1}} \|w_1\|_{X^{r,b_2}}$ implies

\[
N^{s+1} \leq N^{2r}, \text{ for } N \gg 1 \iff s \leq 2r - 1.
\]
Finally, we define
\[
\alpha_2(n) = \begin{cases} 
1 & \text{if } n = 0, \\
0 & \text{otherwise}
\end{cases}
\quad \text{and} \quad \beta_2(n) = \begin{cases} 
1 & \text{if } n = N, \\
0 & \text{otherwise}.
\end{cases}
\]
and we put \( \hat{\omega}_2(n, \tau) = \alpha_2(n) \chi_{[-1,1]}(\tau + n^2) \) and \( \hat{\omega}_2(n, \tau) = \beta_2(n) \chi_{[-1,1]}(\tau + n^2) \).
Then, by similar calculations as in the previous case we obtain
\[
\| (u_2 \hat{\omega}_2)_x \|_{H^{s-1/2}_{r,\text{per}}} \sim N^s, \quad \| u_2 \|_{X^{r,b_1}} \sim 1 \quad \text{and} \quad \sim \| u_2 \|_{X^{r,b_2}} \sim N^r.
\]
Again, the inequality \( \| (u_2 \hat{\omega}_2)_x \|_{H^{s-1/2}_{r,\text{per}}} \lesssim \| u_2 \|_{X^{r,b_1}} \| u_2 \|_{X^{r,b_2}} \) implies
\[
N^s \leq N^r, \quad \text{for } N \gg 1 \iff s \leq r.
\]
Thus, we finished the proof. \( \square \)

2.2. Proof of Local Theorem. The next lemmas will be useful in the proof of Theorem 1.2.

**Lemma 2.6.** For any \( s \in \mathbb{R}, \delta \in (0,1], 0 < \mu < 1/2 \) and \(-1/2 < b_1 \leq b_2 < 1/2\) we have
\[
\begin{align*}
(a) \quad & \| \eta_1(s) F \|_{X^{s+1/2}_{r,\text{per}}} \leq C\delta^{-\mu} \| F \|_{X^{s+1/2}_{r,\text{per}}} \quad \text{and} \quad \| \eta_1(s) F \|_{H^{1/2}_{r,\text{per}}} \leq C\delta^{-\mu} \| F \|_{H^{1/2}_{r,\text{per}}}, \\
(b) \quad & \| \eta_1(s) F \|_{X^{s,b_1}_{r,\text{per}}} \leq C\delta^{b_2-b_1} \| F \|_{X^{s,b_2}_{r,\text{per}}} \quad \text{and} \quad \| \eta_1(s) F \|_{H^{1/2}_{r,\text{per}}} \leq C\delta^{b_2-b_1} \| F \|_{H^{1/2}_{r,\text{per}}}.
\end{align*}
\]

**Proof.** The proof of this result can be found, for instance, in [24] and [17]. \( \square \)

**Lemma 2.7** (Trilinear Estimate). For any \( s \geq 0 \), we have
\[
\| uv \omega \|_{X^{s}_{r,\text{per}}} \lesssim \| u \|_{X^{s,3/8}_{r,\text{per}}} \| v \|_{X^{s,3/8}_{r,\text{per}}} \| w \|_{X^{s,3/8}_{r,\text{per}}}
\]

**Proof.** See [13] and [7]. \( \square \)

Now we give the sketch of the proof of local theorem. First, we let \((u_0,v_0) \in H^s_{\text{per}} \times H^s_{\text{per}}\) where \( r \) and \( s \) satisfying
\[
\max\{0, r - 1\} \leq s \leq \min\{r, 2r - 1\}
\]
and we consider the operator \( \Phi = (\Phi_1, \Phi_2) \), with
\[
\begin{align*}
\Phi_1(u,v) &= \eta(t) u_0 - i\eta(t) \int_0^t e^{i(t-t')\partial_x^2} \left( (\eta \delta u \eta \delta v)(t') + \eta \delta u |\eta \delta u|^2(t') \right) dt', \\
\Phi_2(u,v) &= \eta(t) v_0 + \eta(t) \int_0^t \partial_x (|\eta \delta u|^2)(t') dt',
\end{align*}
\]
defined on the ball
\[
\mathcal{B}[a,b] = \left\{ (u,v) \in X_{r,\text{per}} \times Y_{r,\text{per}} : \| u \|_{X_{r,\text{per}}} \leq a \quad \text{and} \quad \| v \|_{Y_{r,\text{per}}} \leq b \right\}.
\]
Then, by Lemmas 2.2, 2.4, 2.6 and 2.7 we have
\[
\| \Phi_1(u,v) \|_{X^{r}_{\text{per}}} \leq C_0 \| u_0 \|_{H^s_{\text{per}}} + C \left( \| \eta \delta u \|_{X^{r-1/2-s}_{\text{per}}} \| \eta \delta v \|_{H^{1/2}_{\text{per}}} + \right.
\]
\[
+ \| \| \eta \delta u \|_{X^{r-1/2}_{\text{per}}} \| \eta \delta v \|_{H^{1/2-s}_{\text{per}}} \| \eta \delta u \|_{X^{r,3/8}_{\text{per}}} \right)
\]
\[
\leq C_0 \| u_0 \|_{H^s_{\text{per}}} + C\delta (ab + a^3)
\]
and 
\[ \|\Phi_2(u, v)\|_{Y_{r,s}^{\per}} \leq C_0 \|v_0\|_{H_{r,s}^{\per}} + C \left( \|\eta u\|_{X_{r,1/2}^{\per}} + \|\eta^2 u\|_{X_{r,1/2}^{\per}} \right) \]
\[ \leq C_0 \|v_0\|_{H_{r,s}^{\per}} + C\delta^2, \]
with \( \epsilon \) enough small.

Now we put \( a = 2C_0\|u_0\|_{H_{r,s}^{\per}} \) and \( b = 2C_0\|v_0\|_{H_{r,s}^{\per}} \) and then we let \( \delta \) such that \( \delta^2 \leq \min \left\{ \frac{1}{2C(a + b^2)}, \frac{1}{2C^2}\right\} \). Thus, we have that \( \Phi(B[a, b]) \subset B[a, b] \). The contraction condition
\[ \|\Phi(u, v) - \Phi(u_0, v_0)\|_{Y_{r,s}^{\per}} \leq C(a, b)\delta^\theta\|\eta u - u_0\|_{X_{r,1/2}^{\per}} \]
where \( \|f\|_{X_{r,s}^{\per}} := \|f\|_{X_{r,s}^{\per}} \) and \( C(a, b) \) is a positive constant depending only on \( a \) and \( b \), follows similarly. This shows that the map \( \Phi \) is a contraction on \( B[a, b] \). There we obtain a unique fixed point which solves the system for \( T < \delta \) and we finish the proof.

Remark 2.8. We note that global well-posedness in \( H^1_{r,s} \times L^2_{r,s} \) follows directly of the local theorem for \( (r, s) = (1, 0) \) combined with the conservation laws \( \|u\|_{Y_{r,s}^{\per}} \) and \( \|\eta u\|_{X_{r,1/2}^{\per}} \).

3. ILL-POSEDNESS

In this section we will show that the solution of \( \|11\) cannot depend uniformly continuously on its initial data for \( r < 0 \) and \( s \in \mathbb{R} \). We will use the same argument given in \( \|14\).

3.1. PROOF OF THEOREM \( \|13\) It is easy to check that
\[ u_{N,a}(t, x) = a \exp(iNx) \exp(-it(N^2 + (\gamma + \beta)a^2)) \]
\[ v_{N,a}(t, x) = \gamma a^2, \]
where \( a \in \mathbb{R} \) and \( N \) is any positive integer, solves \( \|11\) with initial data \( u_0(x) = a \exp(iNx) \) and \( v_0(x) = \gamma a^2 \). Moreover, for \( a = \alpha(1 + N^2)^\tau \), where \( \alpha \) is a real constant, and \( |\gamma| = (1 + N^2)^r \) we have
\[ \|u_0(x)\|_{H^r} \leq c\alpha^2, \quad \text{and} \quad \|v_0(x)\|_{H^r} \leq c\alpha^4 \]
where \( c \) is a constant. Let \( a_1 = \alpha_1(1 + N^2)^\tau \) and \( a_2 = \alpha_2(1 + N^2)^\tau \). For the Sobolev norm of the difference of two initial data, we have
\[ \|u_{N,a_1}(0) - u_{N,a_2}(0)\|_{H^r} = c|\alpha_1 - \alpha_2|^2 \rightarrow 0, \quad \text{as} \quad \alpha_1 \rightarrow \alpha_2 \]
and 
\[ \|v_{a_1}(0) - v_{a_2}(0)\|_{H^r} = |\gamma|^2|\alpha_1^2 - \alpha_2^2|^2(1 + N^2)^{-2r} = |\alpha_1^2 - \alpha_2^2|^2, \quad \text{as} \quad \alpha_1 \rightarrow \alpha_2. \]
On the other hand we have
\[ \|u_{N,a_1}(t, x) - u_{N,a_2}(t, x)\|_{H^r} = \sum_{n=-\infty}^{+\infty} (1 + |n|^2)^r|\hat{u}_{N,a_1}(n) - \hat{u}_{N,a_2}(n)|^2 \]
\[ = (1 + N^2)^r|a_1e^{-it(N^2 + (\gamma + \beta)a_1^2)} - a_2e^{-it(N^2 + (\gamma + \beta)a_2^2)}|^2 \]
\[ = |\alpha_1 - \alpha_2e^{it(\gamma + \beta)(\alpha_1^2 - \alpha_2^2)(1 + N^2)^{-r}}|^2 \]
Let $r < 0$, and $\alpha_1$ and $\alpha_2$ are such that $\beta(\alpha_1^2 - \alpha_2^2)(1 + N^2)^{-r} = \delta(1 + N^2)^{\nu}$, where $\nu > 0$, and $\nu + r < 0$. Then for $t = \frac{\delta^{-1}(1 + N^2)^{-\nu}}{2}$ we have
\[ ||u_{N,a_1}(t,x) - u_{N,a_2}(t,x)||_{H^r}^2 \geq c(\alpha_1^2 + \alpha_2^2). \]
Note that $t$ can made arbitrary small, by choosing $N$ sufficiently large.

4. Existence of periodic travelling wave solutions

We are interested in this section in finding explicit solutions for (1.1) in the form
\[
\begin{cases}
  u(t,x) = e^{-i\omega t}e^{i\frac{c}{2}(x-ct)}\varphi_{\omega,c}(x-ct), \\
  v(t,x) = n_{\omega,c}(x-ct),
\end{cases}
\]
where $\varphi_{\omega,c}$ and $n_{\omega,c}$ are smooth and $L$-periodic functions, $c > 0$, $\omega \in \mathbb{R}$ and suppose that there is a $q \in \mathbb{N}$ such that $\frac{4\pi q}{c} = L$. So, putting (4.46) into (1.1) we obtain
\[
\begin{cases}
  \varphi_{\omega,c}'' + (\omega + \frac{c^2}{4})\varphi_{\omega,c} = \varphi_{\omega,c}n_{\omega,c} + \beta\varphi_{\omega,c}^3, \\
  -cn_{\omega,c}' = 2\varphi_{\omega,c}\varphi_{\omega,c}'.
\end{cases}
\]
If $n_{\omega,c} = \gamma\varphi_{\omega,c}^2$, then from the second equation in (4.47) we have $\gamma = -\frac{1}{c}$. Substituting $n_{\omega,c}$ in the first equation in (4.47), it follows that $\varphi_{\omega,c}$ satisfies
\[
\varphi_{\omega,c}'' + \left(\omega + \frac{c^2}{4}\right)\varphi_{\omega,c} = \left(\beta - \frac{1}{c}\right)\varphi_{\omega,c}^3.
\]
If $1 - \beta c > 0$ and $\varphi_{\omega,c} = \left(\frac{c}{1-\beta c}\right)^\frac{1}{2}\varphi_{\omega,c}$, then $\varphi_{\omega,c}$ satisfies the equation
\[
\phi_{\omega,c}'' - \sigma\phi_{\omega,c} + \phi_{\omega,c}^3 = 0,
\]
where $\sigma = -\omega - \frac{c^2}{4}$. So, by following Angulo’s arguments in ([3], [4]) we have from (4.49) that $\varphi_{\omega,c}$ satisfies the first-order equation
\[
[\phi_{\omega,c}']^2 = \frac{1}{2}P_{\phi}(\phi),
\]
where $P_{\phi}(t) = -t^4 + 2\sigma t^2 + 2B_{\phi}$ and $B_{\phi}$ is an integration constant. Let $-\eta_1 < -\eta_2 < \eta_2 < \eta_1$ are the zeros of the polynomial $P_{\phi}(t)$. Then
\[
[\phi_{\omega,c}']^2 = \frac{1}{2}(\eta_1^2 - \phi_{\omega,c}^2)(\phi_{\omega,c}^2 - \eta_2^2).
\]
The solution of (4.51) is
\[
\phi_{\omega,c} = \eta_1 dn\left(\frac{\eta_1}{\sqrt{2}}; \kappa\right),
\]
where
\[
\eta_1^2 + \eta_2^2 = 2\sigma, \quad \kappa^2 = \frac{\eta_1^2 - \eta_2^2}{\eta_1^2}, \quad 0 < \eta_2 < \eta_1.
\]
Define the function in variable $\kappa$, $0 < \kappa < 1$,
\[
K = K(\kappa) = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - \kappa^2t^2)}} 
\]
called the complete elliptic integral of the first kind. Since \( dn \) has fundamental period \( 2K(\kappa) \), it follows that \( \phi_{\omega,c} \) has fundamental period

\[
T_{\phi_{\omega,c}} = \frac{2\sqrt{2}}{\eta_1}K(\kappa)
\]

Analogously as in [3] we obtain the following result.

**Theorem 4.1.** Let \( L \) be fixed but arbitrary positive constant and \( 1 - \beta c > 0 \), and \(-\omega - \frac{c^2}{T} > 0\). Let \( \sigma_0 > \frac{2\pi}{T} \) and \( \eta_{2,0} = \eta_2(\sigma_0) \in (0, \sqrt{2\omega}) \) is the unique such that \( T_\phi = L \). Then hold the following assertions:

1. There exists an interval \( I(\sigma_0) \) around \( \sigma_0 \), an interval \( B(\eta_{2,0}) \) around \( \eta_{2,0} \), and a unique smooth function \( \Lambda : I(\sigma_0) \to B(\eta_{2,0}) \), such that

\[
\Lambda(\sigma_0) = \eta_{2,0} \quad \text{and} \quad \frac{2\sqrt{2}}{\sqrt{2\sigma - \eta_2^2}}K(\kappa) = L,
\]

where \( \sigma \in I(\sigma_0) \), \( \eta_2 = \Lambda(\sigma) \).

2. Solutions \((\varphi_{\omega,c}, n_{\omega,c})\) of (4.47) given by

\[
\begin{align*}
\varphi_{\omega,c} &= \sqrt{\frac{\epsilon}{1 - \beta c}} \eta_1 dn \left( \frac{\eta_1}{\sqrt{2} \epsilon}; \kappa \right), \\
n_{\omega,c} &= -\frac{\eta_1}{1 - \beta c} dn^2 \left( \frac{\eta_1}{\sqrt{2} \epsilon}; \kappa \right),
\end{align*}
\]

with \( \eta_1 = \eta_1(\sigma) \), \( \eta_2 = \eta_2(\sigma) \), \( \eta_1^2 + \eta_2^2 = 2\sigma \), and have the fundamental period \( L \) and satisfies (4.47). Moreover, the mapping

\[
\sigma \in I(\sigma_0) \to (\varphi_{\omega,c}, n_{\omega,c})
\]

is a smooth function.

3. \( I(\sigma_0) \) can be chosen as \( \left( \frac{2\pi}{L}, +\infty \right) \).

4. The mapping \( \sigma \to \kappa(\sigma) \) is a strictly increasing function.

5. Stability of travelling waves

In this section we consider the stability of the orbit

\[
\Omega(\Phi, \Psi) = \{(e^{it}\Phi(\cdot + x_0), \Psi(\cdot + x_0)) : (\theta, x_0) \in [0, 2\pi) \times \mathbb{R}\},
\]

in \( H^1_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L]) \) by the periodic flow generated by (1.1), where we have that \( \Phi(\xi) = e^{i\epsilon_2} \varphi_{\omega,c}(\xi), \Psi(\xi) = n_{\omega,c}(\xi), \) with \( \varphi_{\omega,c}, n_{\omega,c} \) given in (4.54). Let \( X \) be the space \( X = H^1_{\text{complex}}([0, L]) \times L^2_{\text{real}}([0, L]) \), with real inner product

\[
(u_1, u_2) = \mathbb{R} \int_0^L (\epsilon_1 \eta_1 + \epsilon_1 \eta_1 + \epsilon_2 \eta_2) dx.
\]

Let \( T_1, T_2 \) be one-parameter groups of unitary operators on \( X \) defined by

\[
T_1(s) \bar{u}(\cdot) = \bar{u}(\cdot + s), \quad T_2(r) \bar{u}(\cdot) = (e^{-ir\epsilon}(\cdot), n(\cdot))
\]

for \( \bar{u} \in X, s, r \in \mathbb{R} \). Obviously

\[
T_1'(0) = \begin{pmatrix} -\partial_x & 0 \\ 0 & -\partial_x \end{pmatrix}, \quad T_2'(0) = \begin{pmatrix} -i & 0 \\ 0 & 0 \end{pmatrix}.
\]

Note that the equation (1.1) is invariant under \( T_1 \) and \( T_2 \). If

\[
\Phi_{\omega,c}(x) = (\epsilon_{\omega,c}(x), n_{\omega,c}(x))
\]
where \( \varepsilon_{\omega,c}(x) = e^{\Phi_{\omega,c}(x)} \), then from Theorem 4.1 we obtain that
\[
T_1(ct)T_2(\omega t)\Phi_{\omega,c}(x)
\]
is a travelling wave solution of (4.47) with \( \varphi_{\omega,c}(x), n_{\omega,c}(x) \) defined by (4.54).

Now, it is easy to verify that \( E_2(\vec{u}) \) is invariant under \( T_1 \) and \( T_2 \)
\[
E(T_1(s)T_2(r)\vec{u}) = E(\vec{u}).
\]

We also have
\[
E(\vec{u}(t)) = E(\vec{u}(0)),
\]
and that equation (1.1) can be written as the following Hamiltonian system
\[
\frac{d\vec{u}}{dt} = JE'(\vec{u})
\]
where \( \vec{u} = (u, v) \) and \( J \) is a skew-symmetric linear operator defined by
\[
J = \begin{pmatrix}
    -i & 0 \\
    0 & 2\partial_x
\end{pmatrix}
\]
and
\[
E'(u,v) = \begin{pmatrix}
    -u_{xx} + uv + \beta|u|^2u \\
    \frac{1}{2}|\vec{u}|^2
\end{pmatrix}
\]
is the Frechet derivative of \( E \). Define \( B_1 \) and \( B_2 \) such that \( T'_1(0) = JB_1, \ T'_2(0) = JB_2 \), then
\[
Q_1(\vec{u}) = \frac{1}{2}(B_1\vec{u}, \vec{u}) = -\frac{1}{4}\int_0^L v^2dx + \frac{1}{2}Im\int_0^L u_x\overline{u}dx,
\]
\[
Q_2(\vec{u}) = \frac{1}{2}(B_2\vec{u}, \vec{u}) = \frac{1}{2}\int_0^L |\vec{u}|^2dx.
\]

It is easy to verify that
\[
Q_1(T_1(s)T_2(r)\vec{u}) = Q_1(\vec{u}), \ Q_2(T_1(s)T_2(r)\vec{u}) = Q_2(\vec{u})
\]
\[
Q_1(\vec{u}(t)) = Q_1(\vec{u}(0)), \ Q_2(\vec{u}(t)) = Q_2(\vec{u}(0))
\]
and
\[
Q'_1(u,v) = \begin{pmatrix}
    -iu_x \\
    -\frac{1}{4}v
\end{pmatrix}, \ Q'_2(u,v) = \begin{pmatrix}
    u \\
    0
\end{pmatrix}.
\]

From (4.47) we have
\[
E'(\Phi_{\omega,c}) - cQ'_1(\Phi_{\omega,c}) - \omega Q'_2(\Phi_{\omega,c}) = 0.
\]

Define an operator from \( X \) to \( X^* \)
\[
H_{\omega,c} = E''(\Phi_{\omega,c}) - cQ''_1(\Phi_{\omega,c}) - \omega Q''_2(\Phi_{\omega,c})
\]
and the function \( d(\omega, c) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) by
\[
d(\omega, c) = E(\Phi_{\omega,c}) - cQ_1(\Phi_{\omega,c}) - \omega Q_2(\Phi_{\omega,c}).
\]
The operator \( H_{\omega,c} \) is self-adjoint. The spectrum of \( H_{\omega,c} \) consists of the real numbers \( \lambda \) such that \( H_{\omega,c} - \lambda I \) is not invertible. From (4.47) we have
\[
T'_1(0)\Phi_{\omega,c} \in \text{Ker}H_{\omega,c}, \ T'_2(0)\Phi_{\omega,c} \in \text{Ker}H_{\omega,c}.
\]

Let \( Z = \{k_1T'_1(0)\Phi_{\omega,c} + k_2T'_2(0)\Phi_{\omega,c}, \ k_1, k_2 \in \mathbb{R}\} \). By (5.63), \( Z \) is in the kernel of \( H_{\omega,c} \).
Theorem 5.2. \( L \theta \lambda \) in the specific form \( L \) eigenvalue of \( L \) is \( 0 \) of (4.54). Then the first three eigenvalues of operator \( L \).

\[
\langle H_{\omega,c}, \tilde{u}, \tilde{u} \rangle < 0 \quad \text{for} \quad \tilde{u} \in N
\]

and \( P \) is a closed subspace such that \( \langle H_{\omega,c}, \tilde{u}, \tilde{u} \rangle \geq \delta ||\tilde{u}||_X^2 \), for \( \tilde{u} \in P \) with some constant \( \delta > 0 \) independent of \( \tilde{u} \).

Our stability results is based in the following general theorem in [13].

**Theorem 5.1.** (Abstract Stability Theorem) Assume that there exists three functionals \( E, Q_1, Q_2 \) satisfying (5.53)-(5.59). Let \( n(H_{\omega,c}) \) be the number of negative eigenvalues of \( H_{\omega,c} \). Assume \( d(\omega, c) \) is non-degenerated at \( (\omega, c) \) and let \( p(d'') \) be the number of positive eigenvalues of \( d'' \). If \( p(d'') = n(H_{\omega,c}) \), then the periodic travelling wave \( \varphi_{\omega,c}(x) \) is orbitally stable.

The idea of the proof of Theorem 5.1 is to apply the general Theorem 5.1

Initially we identify the quadratic form associated to \( H_{\omega,c} \). Let \( \tilde{Z} = (e^{i\tilde{z} \cdot z_1, z_2}) \), with \( z_1 = y_1 + iy_2, y_1 = Rez_1, y_2 = Imz_1 \). By direct computation, we get

\[
\langle H_{\omega,c}, \tilde{z}_1, \tilde{z}_1 \rangle = \langle L_1 y_1, y_1 \rangle + \langle L_2 y_2, y_2 \rangle + \frac{c}{2} \int_0^L \left( \bar{z}_2 + \frac{2}{c} \varphi_{\omega,c} y_1 \right)^2 dx
\]

where

\[
L_1 = -\partial_x^2 - \left( \frac{c^2}{4} + \omega \right) + 3 \left( \beta - \frac{1}{c} \right) \varphi_{\omega,c}^2
\]

\[
L_2 = -\partial_x^2 - \left( \frac{c^2}{4} + \omega \right) + \left( \beta - \frac{1}{c} \right) \varphi_{\omega,c}^2.
\]

From (4.7) we also have \( L_1(\partial_x \varphi_{\omega,c}) = 0 \) and \( L_2 \varphi_{\omega,c} = 0 \). Consider the following periodic eigenvalue problems for \( i = 1, 2 \),

(5.64)

\[
\begin{cases}
L_i f = \lambda f \\
\partial_x f = f(L), \quad f'(0) = f'(L),
\end{cases}
\]

The problem (5.64) determines a countable infinite set of eigenvalues \( \{\lambda_n\} \) with \( \lambda_n \to \infty \), so from the Oscillation Theorem 22 we have that they are distributed in the specific form \( \lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4, \ldots \).

For the eigenvalue problems (5.64) we have the same results.

**Theorem 5.2.** Let \( \sigma \in [\frac{2\pi}{L}, +\infty) \) and \( \langle \varphi_{\omega,c}, n_{\omega,c} \rangle \) be the travelling wave solutions of (4.47). Then the first three eigenvalues of operator \( L_1 \) are simple, 0 is the second eigenvalue of \( L_1 \) with eigenfunction \( \partial_x \varphi_{\omega,c} \). The first eigenvalue of the operator \( L_2 \) is 0, which is simple.

**Proof.** Since \( L_2 \varphi_{\omega,c} = 0 \) and \( \varphi_{\omega,c} \) has no zeros on \([0, L]\), then zero is the first eigenvalue of \( L_2 \). Now since \( L_1 \partial_x \varphi_{\omega,c} = 0 \) and \( \partial_x \varphi_{\omega,c} \) has two zeros on \([0, L]\), then it follows that eigenvalue zero of \( L_1 \) is either \( \lambda_1 \) or \( \lambda_2 \). Let \( \psi = f(\theta x) \), where \( \theta^2 = \frac{2}{n^2} \). From equality \( \kappa^2 s^2(x) + d n^2(x) = 1 \) and (5.64), we obtain that \( \psi \) satisfies the equation

(5.65)

\[
\psi'' + (\rho - 6\kappa^2 s^2(x))\psi = 0,
\]
From Floquet theory, it follows that \((−\infty, \rho_0)\) and \((\rho_1, \rho_2)\) are instability intervals associated to the Lame’s equation. Therefore the eigenvalues \(\rho_0, \rho_1\) and \(\rho_2\) of (5.66) are simple and the rest of eigenvalues \(\rho_3 \leq \rho_4, \ldots\) satisfies \(\rho_3 = \rho_4, \rho_5 = \rho_6, \ldots\). The eigenvalues \(\rho_0, \rho_1, \rho_2\) and its corresponding eigenfunctions \(\psi_0, \psi_1, \psi_2\) are

\[
\begin{align*}
\rho_0 &= 2(1 + \kappa^2 - \sqrt{1 - \kappa^2 + \kappa^4}), \quad \psi_0 = 1 - (1 + \kappa^2 - \sqrt{1 - \kappa^2 + \kappa^4}) sn^2(x) \\
\rho_1 &= 4 + \kappa^2, \quad \psi_2 = sn(x)cn(x) \\
\rho_2 &= 2(1 + \kappa^2 + \sqrt{1 - \kappa^2 + \kappa^4}), \quad \psi_2 = 1 - (1 + \kappa^2 + \sqrt{1 - \kappa^2 + \kappa^4}) sn^2(x)
\end{align*}
\]

Since \(\rho_0 < \rho_1\) for every \(\kappa^2 \in (0, 1)\), then from (5.66) we have

\[3\lambda_0 = \frac{\eta_1^2}{2} (\kappa^2 - 2 - 2\sqrt{1 - \kappa^2 + \kappa^4}) < 0\]

Therefore \(\lambda_0\) is negative eigenvalue of \(L_1\) with eigenfunction \(\chi_0(x) = \psi_0(\frac{x}{\delta})\). Similarly

\[3\lambda_2 = \frac{\eta_1^2}{2} (\kappa^2 - 2 + 2\sqrt{1 - \kappa^2 + \kappa^4}) > 0\]

and \(\lambda_2\) is the positive eigenvalue of \(L_1\) with eigenfunction \(\chi_2(x) = \psi_2(\frac{x}{\delta})\). Thus

\[\lambda_1 = \frac{\eta_1^2 (\rho_1 - 6) + 2\sigma}{6} = \frac{\eta_1^2}{6} (4 + \kappa^2 - 6 + 2 - \kappa^2) = 0\]

is the second eigenvalue of \(L_1\). This complete the proof of the theorem.  

\[\square\]

**Remark 5.3.** The main properties of the spectrum of \(L_1\), namely, there is exactly a negative eigenvalue and zero is simple, can also be obtained via positive properties of the Fourier transform of the solution \(\varphi_{\omega,c}\) (see Angulo&Natali \([6]\) and Angulo \([5]\)).

So, from Theorem 5.2 we obtain immediately the following two results.

**Lemma 5.4.** For any real function \(y_1 \in H^1\) satisfying \(\langle y_1, \chi_0 \rangle = \langle y_1, \partial_x \varphi_{\omega,c} \rangle = 0\) there exists a positive constant \(\delta_1 > 0\) such that \(L_1 y_1 \geq \delta_1 ||y_1||_{H^1}^2\).

**Lemma 5.5.** For any real function \(y_2 \in H^1\) satisfying \(\langle y_2, \varphi_{\omega,c} \rangle = 0\) there exists a positive constant \(\delta_2\) such that \(L_2 y_2 \geq \delta_2 ||y_2||_{H^1}^2\).

**Proof.** [Theorem 1.5] Choose

\(y^- = \chi_0, y^- = 0, z^- = -\frac{2}{\varepsilon} \varphi_{\omega,c} \chi_0\) and \(\Psi^- = (y^-_1, y^-_2, z^-)\) then

\[\langle H_{\omega,c} \Psi^-, \Psi^- \rangle = \lambda_0 \langle \chi_0, \chi_0 \rangle < 0\]

So \(H_{\omega,c}\) has a negative eigenvalue. Note that the following vectors

\[\Psi_{0,1} = (\partial_x \varphi_{\omega,c}, 0, -\frac{2}{\varepsilon} \varphi_{\omega,c} \partial_x \varphi_{\omega,c}), \quad \Psi_{0,2} = (0, \varphi_{\omega,c}, 0)\]

are in the kernel of operator \(H_{\omega,c}\). Define the following subspaces associated to \(H_{\omega,c}\):

\[Z = \{ k_1 \Psi_{0,1} + k_1 \Psi_{0,2} : k_1, k_2 \in \mathbb{R} \}\]

\[N = \{ k \Psi^- : k \in \mathbb{R} \}\]

\[P = \{ \vec{p} \in X : \vec{p} = (p_1, p_2, p_3), \langle p_1, \chi_1 \rangle = \langle p_1, \partial_x \varphi_{\omega,c} \rangle = \langle p_2, \varphi_{\omega,c} \rangle = 0 \} \]
For any \( \vec{u} \in X, \vec{v} = (y_1, y_2, y_3) \) choose
\[
\begin{align*}
    a &= \langle y_1, \chi_0 \rangle, \\
    b_1 &= \langle \partial_x \varphi_{\omega,c}, y_1 \rangle, \\
    b_2 &= \langle \varphi_{\omega,c}, y_2 \rangle
\end{align*}
\]
then \( \vec{u} \) uniquely can be represented by
\[
    a = a \Psi^- + b_1 \Psi_{0,1} + b_2 \Psi_{0,2} + \vec{p},
\]
where \( \vec{p} \in P \).

For any \( \vec{p} \in P \), by Lemmas 5.4 and 5.5 we have
\[
    \langle H_{\omega,c}, \vec{p}, \vec{p} \rangle \geq 3|\|p_3\|_{L^2}^2 + \frac{\delta}{4} \int_{0}^{L} \left( p_3 + \frac{2}{c} \varphi_{p_1} \right)^2 dx
\]
Next we consider the following two cases:
1. If \( \|p_3\|_{L^2} \geq \frac{8|\|\varphi_{\omega,c}\|_{L^\infty}}{c} \|p_1\|_{L^2} \), then
\[
    \frac{c}{2} \int_{0}^{L} \left( p_3 + \frac{2}{c} \varphi_{\omega,c} p_1 \right)^2 dx \geq \frac{c}{2} \left[ |\|p_3\|_{L^2}^2 - \frac{4}{c} |\|\varphi_{\omega,c}\|_{L^\infty} \right] \left| \|p_1\|_{L^2} \right| \int_{0}^{L} |\|p_3\|_{L^2}^2
\]
2. If \( \|p_3\|_{L^2} \leq \frac{8|\|\varphi_{\omega,c}\|_{L^\infty}}{c} \|p_1\|_{L^2} \), then
\[
    \delta_1 |\|p_1\|_{H^1}^2 \geq \frac{\delta}{2} |\|p_1\|_{H^1}^2 + \frac{\delta}{2} |\|p_2\|_{H^1}^2,
\]
Thus, for any \( \vec{p} \in P \), it follows that
\[
    \langle H_{\omega,c}, \vec{p}, \vec{p} \rangle \geq 3|\|p_3\|_{L^2}^2 + \frac{\delta}{4} |\|p_1\|_{H^1}^2 + \frac{\delta}{4} |\|p_2\|_{H^1}^2,
\]
where \( \delta > 0 \) is independent of \( \vec{p} \). This proved that Assumption above is holds, and \( u(H_{\omega,c}) = 1 \).

Now we shall verify that \( p(d') = 1 \). We have
\[
    d_c(\omega, c) = -Q_1(\Phi_{\omega,c}) = \frac{1}{4(1 - \beta c)^2} \int_{0}^{L} \varphi_{\omega,c}^4 dx - \frac{c^2}{4(1 - \beta c)} \int_{0}^{L} \varphi_{\omega,c}^2 dx
\]
\[
    d_\omega(\omega, c) = -Q_2(\Phi_{\omega,c}) = -\frac{c}{2(1 - \beta c)} \int_{0}^{L} \varphi_{\omega,c}^2 dx.
\]
From equalities
\[
    \int_{0}^{L} \varphi_{\omega,c}^2 dx = \frac{8KE}{L}, \int_{0}^{L} \varphi_{\omega,c} dx = \frac{64}{L^3} V(\kappa)
\]
where \( E = E(\kappa) = \int_{0}^{1} \sqrt{\frac{1 - \kappa^2}{1 - t^2}} dt \) is the complete elliptic integral of the second kind and \( V(\kappa) = \frac{2\beta^2}{3} K^4 + \frac{4}{7}(2 - \kappa^2) K^2 E, \) we obtain
(5.67)
\[
    \begin{align*}
    d_{\omega c} &= \frac{4c}{L(1 - \beta c)}(K'(\kappa) E(\kappa) + K(\kappa) E'(\kappa)) \kappa'(\sigma), \\
    d_{\omega c} &= -\frac{4c}{L(1 - \beta c)} K(\kappa) E(\kappa) + \frac{c}{2} d_{\omega c}, \quad d_{c c} = -\frac{6c}{L(1 - \beta c)} V'(\kappa) \kappa'(\sigma) + \frac{c}{2} d_{\omega c}
\end{align*}
\]
\[
    d_{cc} = \frac{32\beta}{L^3(1 - \beta c)^3} V(\kappa) - \frac{8c}{L^3(1 - \beta c)^2} V'(\kappa) \kappa'(\sigma) - \frac{2\beta(c - \beta c)}{L(1 - \beta c)^2} K(\kappa) E(\kappa) + \frac{c^2}{4} d_{\omega c}.
\]
Thus
\[ d_{cc}d_{\omega\omega} - d_{cw}d_{\omega c} = -\frac{64}{L^4(1-\beta c)^3} V'(\kappa)\kappa'(\sigma)K(\kappa)E(\kappa) + \frac{1}{L^2(1-\beta c)^3} \left[ \frac{32\kappa}{L^2(1-\beta c)^3} V(\kappa) - 2cK(\kappa)E(\kappa) \right] d_{\omega\omega}. \]

We have,
\[ V'(\kappa) = \frac{2K^2E}{\kappa(1-\kappa^2)} [(2 - \kappa^2)E - (1 - \kappa^2)K], \quad \text{and} \quad \frac{V}{L^2} = \frac{\sigma(\kappa^2 - 1)}{12(2 - \kappa^2)} K^2 + \frac{\sigma}{6} KE. \]

Using the above estimates, we obtain
\[ d_{cc}d_{\omega\omega} - d_{cw}d_{\omega c} = \frac{4\kappa}{L^2(1-\beta c)^2} \left\{ -32\frac{K^2E}{2(1-\beta c)} \right\} + \left. \frac{8\beta\sigma(\kappa^2 - 1)}{3(1-\kappa^2)} K + \frac{c(16\beta\sigma - 6c(1 - \beta c)^2)}{3} E \right\} \frac{E^2}{\kappa(1-\kappa^2)} - \frac{K^2}{\kappa}. \]

From Theorem 4.1-(4), we have that \( \kappa' > 0 \). Therefore the sign of \( \text{det}(d'') \) is determined by the sign of
\[ B(c,\omega,\kappa,\beta) = \left\{ -\frac{8\beta\sigma}{2(1-\kappa^2)} [(2 - \kappa^2)E^3 - 2(1 - \kappa^2)KE^2] \right. \]
\[ + \left. \frac{8\beta\sigma(\kappa^2 - 1)}{3(1-\kappa^2)} K + \frac{c(16\beta\sigma - 6c(1 - \beta c)^2)}{3} E \right\} \frac{E^2}{\kappa(1-\kappa^2)} - \frac{K^2}{\kappa}. \]

From the relation
\[ (5.68) \quad 0 < \frac{(1 - \kappa^2)K}{(2 - \kappa^2)E} < \frac{1}{2} \]
we get that the first term of \( B(c,\omega,\kappa,\beta) \) is negative. Now we consider three cases for \( \beta \).

1. Obviously if \( \beta = 0 \), then \( \text{det}(d'') \) < 0.
2. For \( \beta < 0 \), using (5.68), we get
\[ \frac{8\beta\sigma(\kappa^2 - 1)}{3(1-\kappa^2)} K + \frac{c(16\beta\sigma - 6c(1 - \beta c)^2)}{3} E = \]
\[ = -\frac{8c\beta\sigma E}{3} \left[ \frac{(1 - \kappa^2)K}{(2 - \kappa^2)E} - 2 + \frac{6c(1 - \beta c)^2}{8c\beta} \right] < 0 \]
and \( \text{det}(d'') < 0. \)
3. If \( \beta > 0 \) and \( 8\beta\sigma - 3c(1 - \beta c)^2 \leq 0 \), then all terms of \( B(c,\omega,\kappa,\beta) \) are negatives and \( \text{det}(d'') < 0. \)

Thus under above three conditions, \( d''(\omega, c) \) has exactly one positive and one negative eigenvalues and \( p(d'') = 1 \). This finishes the proof of the Theorem. \( \square \)

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