JACKSON-TYPE INEQUALITY IN HILBERT SPACES AND ON HOMOGENEOUS MANIFOLDS

ISAAC Z. PESENSON

Abstract. We consider a Hilbert space $H$ equipped with a set of strongly continuous bounded semigroups satisfying certain conditions. The conditions allow to define a family of moduli of continuity $\Omega^r(s, f)$, $r \in \mathbb{N}, s > 0$, of vectors in $H$ and a family of Paley-Wiener subspaces $PW_\sigma$ parametrized by bandwidth $\sigma > 0$. These subspaces are explored to introduce notion of the best approximation $E(\sigma, f)$ of a general vector in $H$ by Paley-Wiener vectors of a certain bandwidth $\sigma > 0$. The main objective of the paper is to prove the so-called Jackson-type estimate $E(\sigma, f) \leq C(\Omega^r(\sigma^{-1}, f) + \sigma^{-r}\|f\|)$ for $\sigma > 1$. Our assumptions are satisfied for a strongly continuous unitary representation of a Lie group $G$ in a Hilbert space $H$. It allows to obtain the Jackson-type estimates on homogeneous manifolds.

1. Introduction and Main Results

One of the main goals of the classical harmonic analysis is to describe relations between frequency content of a function and its smoothness. A famous result in this direction is the so-called Jackson Theorem for functions in $L^2(\mathbb{R})$:

\begin{equation}
\inf_{g \in PW_\sigma(\mathbb{R})} \|f - g\| = \|f - P_\sigma f\| = \mathcal{E}(\sigma, f) \leq C \omega^r(\sigma^{-1}, f), \quad \sigma > 0, \quad r \in \mathbb{N},
\end{equation}

where the Paley-Wiener space $PW_\sigma(\mathbb{R})$, $\sigma > 0$, is the space of functions in $L^2(\mathbb{R})$ whose Fourier transform has support in $[-\sigma, \sigma]$, $P$ is the orthogonal projection of $L^2(\mathbb{R})$ onto $PW_\sigma(\mathbb{R})$, and

$$\omega^r(s, f) = \sup_{0 \leq \tau \leq s} \|\left(T(\tau) - I\right)^r f\|_2,$$

$T(\tau)f(\cdot) = f(\cdot + \tau)$, is the modulus of continuity. Similar estimate also holds true in the case of one-dimensional torus $\mathbb{T}$ if one will replace $PW_\sigma$ by the space of trigonometric polynomials degree $\leq n$. In the case of $\mathbb{R}^d$ and $\mathbb{T}^d$ one defines corresponding modules of continuity by using one-parameter translation groups along single coordinates

\begin{equation}
T_j(\tau)f(x_1, \ldots, x_j, \ldots, x_d) = f(x_1, \ldots, x_j + \tau, \ldots, x_d), \quad f \in L^p, \quad 1 \leq p \leq \infty,
\end{equation}

whose infinitesimal operators are partial derivatives $\partial/\partial x_j$, $1 \leq j \leq d$. The corresponding spaces $PW_\sigma$ and spaces on trigonometric polynomials can be introduced in terms of the Laplace operator $\Delta = \partial^2_1 + \ldots + \partial^2_d$.

The main objective of this paper is to develop a unified approach to Jackson-type estimates in a Hilbert space in which space in which a family of strongly continuous bounded semigroups is given. It allows us to consider an appropriate

2020 Mathematics Subject Classification. 43A85, 41A17;
Key words and phrases. Jackson-type inequality, K-functor, one-parameter groups of operators, Paley-Wiener vectors, modulus of continuity, unitary representations of Lie groups, homogeneous manifolds.
notion of Paley-Wiener vectors and a modulus of continuity in a Hilbert space $H$ space of unitary representation of a Lie group $G$ (see definitions below) and to prove an analog of the Jackson inequality \[ \Omega^r(s, f) \] in a such general setting. We apply these results to function spaces on homogeneous manifolds, i.e. to manifolds which have many symmetries. Our development is extensively using the notion of Peetre’s K-functional.

Note that an approach to a generalization of the classical approximation theory and K-functional to abstract spaces in which a strongly continuous bounded representation of a Lie group is given was outlined without complete proofs in \[ \text{[11]-[23]} \]. The problem of developing approximation theory and K-functional in non-classical settings attracted attention of many mathematicians and in particular was treated in \[ \text{[2, 3, 4, 5, 6, 7, 11, 15, 16, 17, 25]} \].

We consider a Hilbert space $H$ and operators $D_1, D_2, ..., D_d$ which generate strongly continuous uniformly bounded semigroups $T_1(t), T_2(t), ..., T_d(t)$, $\|T(t)\| \leq 1$, $t \geq 0$, (see \[ \text{[1]} \] for the general theory of the one-parameter semigroups). An analog of a Sobolev space is introduced as the space $H^r$ of vectors in $H$ for which the following norm is finite

$$|||f|||_{H^r} = \|f\|_H + \sum_{k=1}^r \sum_{1 \leq j_1, ..., j_k \leq d} \|D_{j_1} ... D_{j_k} f\|_H,$$

where $r \in \mathbb{N}$, $f \in H$. By using the closed graph theorem and the fact that each $D_i$ is a closed operator in $H$, one can show that this norm is equivalent to the norm

$$\|f\|_r = \|f\|_H + \sum_{1 \leq i_1, ..., i_r \leq d} \|D_{i_1} ... D_{i_r} f\|_H, \quad r \in \mathbb{N}.$$ \[ \text{(1.3)} \]

Let $\mathcal{D}(D_i)$ be the domain of the operator $D_i$. For every $f \in H$ we introduce a vector-valued function $Tf : \mathbb{R}^d \rightarrow H$ defined as $Tf(t_1, t_2, ..., t_d) = T_1(t_1)T_2(t_2)...T_d(t_d)f$.

**Assumptions.**

Our main assumption is that we consider a Hilbert space $H$ and operators $D_1, D_2, ..., D_d$ which generate strongly continuous uniformly bounded semigroups $T_1(t), T_2(t), ..., T_d(t)$, $\|T(t)\| \leq 1$, $t \geq 0$, such that the following properties hold:

(a) There exists a set $\mathcal{G} \subset H^1 = \bigcap_{i=1}^d \mathcal{D}(D_i)$ which is dense in $H$ and invariant with respect to all $T_i(t)$, $1 \leq i \leq d$, $t \geq 0$.

(b) For every $1 \leq i \leq d$, every $f \in \mathcal{G}$ and all $t = (t_1, ..., t_d)$ in the standard open unit ball $U$ in $\mathbb{R}^d$

$$D_iTf(t_1, ..., t_d) = \sum_{k=1}^d \zeta_{i,k}(t) (\partial_k Tf)(t_1, ..., t_d),$$ \[ \text{(1.4)} \]

where $\zeta_{i,k}(t)$ belong to $C^\infty(U)$, $\partial_k = \frac{\partial}{\partial t_k}$.

(c) The operator $L = -D_1^2 - ... - D_d^2$ is a non-negative self-adjoint operator in $H$ and the domain of $L^{r/2}$, $r \in \mathbb{N}$, with the graph norm $\|f\| + \|L^{r/2}f\|$ coincides with the space $H^r$ with the norm (1.3).

Using the groups $T_1, ..., T_d$, $d \geq n = \dim M$, we define an analog of the modulus of continuity by the formula $\Omega^r(s, f) =$.
Theorem 1.1. The following statements hold:

1. (Bernstein inequality) \( f \in PW_{\sigma} (\mathcal{L}) \) if and only if \( f \in H^\infty = \bigcap_{k=1}^{\infty} H^k \), and the following Bernstein inequalities holds true

\[
\|L^{k/2}f\|_H \leq \sigma^k \|f\|_H \quad \text{for all } k \in \mathbb{N};
\]

2. (Paley-Wiener theorem) \( f \in PW_{\sigma} (\mathcal{L}) \) if and only if for every \( g \in H \) the scalar-valued function of the real variable \( t \mapsto (e^{it\mathcal{L}} f, g) \) is bounded on the real line and has an extension to the complex plane as an entire function of the exponential type \( \sigma \).

Next, we define the best approximation

\[
\mathcal{E}_\mathcal{L}(\sigma, f) = \inf_{g \in PW_\sigma (\mathcal{L})} \|f - g\| = \|f - \mathcal{P}_\sigma f\|,
\]

where \( \mathcal{P}_\sigma \) is the orthogonal projector of \( H \) onto \( PW_\sigma (\mathcal{L}) \) We also using the Schrödinger group \( e^{it\mathcal{L}} \) to introduce the modulus of continuity

\[
\omega^*_\mathcal{L}(t, f) = \sup_{0 \leq \tau \leq t} \| (e^{i\tau \mathcal{L}} - I)^r f \|.
\]

In section 2 in Theorem 2.3 we prove the following Jackson-type estimate which holds for any self-adjoint operator \( \mathcal{L} \)

\[
\mathcal{E}_\mathcal{L}(\sigma, f) \leq C(\mathcal{L}) \omega^*_\mathcal{L}(\sigma^{-1}, f).
\]

Note, that \([16]\) contains the following known fact (see [1])

\[
\omega^*_\mathcal{L}(s, f) \leq c_2 K \left( s^r, f, H, D(\mathcal{L}^{r/2}) \right) \leq C_2(\omega^*_\mathcal{L}(s, f) + \min(s^r, 1)\|f\|),
\]
where $\mathcal{D}(L^{r/2})$ is the domain of the operator $L^{r/2}$ with the graph norm $\|f\| + \|L^{r/2}f\|$.

According to the assumption (d) the graph norm of the domain $\mathcal{D}(L^{r/2})$ of the operator $L^{r/2}$ is equivalent to the norm (1.3) and the spaces $\mathcal{D}(L^{r/2})$ and $H^r$ coincide. It implies, in particular, existence of a constant $C_3 > 0$ such that

$$K \left( s^r, f, H, \mathcal{D}(L^{r/2}) \right) \leq C_3 K \left( s^r, f, H, H^r \right), \quad f \in H.$$ 

Thus in our specific situation we obtain by using (1.11), (1.12), (1.13), and (1.6)

$$\mathcal{E}_L(\sigma, f) \leq C(L) \omega_L^r(\sigma^{-1}, f) \leq C(L)c_2 K \left( \sigma^{-r}, f, H, \mathcal{D}(L^{r/2}) \right) \leq$$

$$C(L)c_2 C_3 K \left( \sigma^{-r}, f, H, H^r \right) \leq C(L)c_2 C_3 C_1 \left( \Omega^r(\sigma^{-1}, f) + \min(\sigma^{-r}, 1)\|f\|_H \right).$$

Now we can formulate our main theorem.

**Theorem 1.2.** If the assumptions (a)-(d) are satisfied then there exists a constant $C > 0$ which is independent on $f \in H$ such that

$$\mathcal{E}_L(\sigma, f) \leq C \left( \Omega^r(\sigma^{-1}, f) + \min(\sigma^{-r}, 1)\|f\| \right),$$

where $\mathcal{E}_L(\sigma, f)$ and $\Omega^r(\sigma^{-1}, f)$ defined in (1.9) and (1) respectively.

**Remark 1.3.** It is important to notice that since $\Omega^r(f, \tau)$ cannot be of order $o(\tau^r)$ when $\tau \to 0$ (unless $f$ is invariant), the behavior of the right-hand side in (1.14) is determined by the first term when $\sigma \to \infty$. In particular, if $f \in H^r$, then due to the inequality

$$\Omega^r(s, f) \leq s^{-k} \Omega^{-k}(s, D_{j_1}...D_{j_k} f), \quad 0 \leq k \leq r,$$

one has the best possible estimate

$$\mathcal{E}_L(\sigma, f) \leq C \Omega^r(\sigma, f) \leq C\sigma^{-r}\|f\|_H.$$ 

2. **Jackson inequality for the Schrödinger group of a self-adjoint operator**

The next lemma shows that the inequality (1.8) can be relaxed. This fact is used in the proof of Lemma 2.2 which suggests a way of construction of Paley-Wiener vectors.

**Lemma 2.1.** If $L$ is a self-adjoint operator in a Hilbert space $H$ then a vector $f$ belongs to the subspace $PW_\sigma(L)$, $\sigma > 0$, if and only if there exists a constant $C = C(f, \sigma) > 0$ such that for all $k \in \mathbb{N}$

$$\|L^{k/2}f\| \leq C\sigma^k\|f\|.$$

**Proof.** If $f \in PW_\sigma(L)$ then by (1.8) the inequality (2.1) holds with $C = 1$. Conversely, if for an $f \in H$ the inequality (2.1) holds for some $C = C(f, \sigma)$ then for any complex number $z$ we have

$$\|e^{zL}f\| = \left\| \sum_{m=0}^\infty (z^m L^m f)/m! \right\| \leq C \sum_{m=0}^\infty |z|^m \sigma^m/m! = Ce^{\|z\|^\sigma}.$$

It implies that for any functional $\psi^* \in E^*$ the scalar function $(e^{zL}f, \psi^*)$ is an entire function of exponential type $\sigma$ which is bounded on the real axis by the constant $\|\psi^*\|\|f\|$. An application of the classical Bernstein inequality gives
\[
\| \langle e^{it\mathcal{L}}f, \psi^* \rangle \|_{C(\mathbb{R}^1)} = \left\| \left( \frac{d}{dt} \right)^k \langle e^{it\mathcal{L}}f, \psi^* \rangle \right\|_{C(\mathbb{R}^1)} \leq \sigma^k \|\psi^*\| |f|.
\]

From here for \( t = 0 \) we obtain
\[
|\langle \mathcal{L}^k f, \psi^* \rangle| \leq \sigma^k \|\psi^*\| |f|.
\]
Choice of \( \psi^* \in E^* \) such that \( \|\psi^*\| = 1 \) and \( \langle \mathcal{L}^k f, \psi^* \rangle = \|\mathcal{L}^k f\| \) gives the inequality \( \|\mathcal{L}^k f\| \leq \sigma^k |f|, \ k \in \mathbb{N} \), which implies Theorem.

**Lemma 2.2.** If \( \mathcal{L} \) is a self-adjoint operator in a Hilbert space \( \mathbf{H} \) and \( p \in L_1(\mathbb{R}) \) is an entire function of exponential type \( \sigma \) then for any \( f \in \mathbf{H} \) the vector
\[
P_\sigma f = \int_{-\infty}^{\infty} p(t)e^{i\sigma t} f dt
\]
belongs to \( PW_\sigma(\mathcal{L}) \).

**Proof.** For \( g = P_\sigma f, f \in \mathbf{H} \), and for every real \( \tau \) we have
\[
e^{i\tau t}\mathcal{E} g = \int_{-\infty}^{\infty} p(t)e^{i(t+\tau)\mathcal{E}} f dt = \int_{-\infty}^{\infty} p(t-\tau)e^{i\sigma t} f dt.
\]
Using this formula we can extend the abstract function \( e^{i\tau \mathcal{E}} g \) to the complex plane as
\[
e^{iz\mathcal{E}} g = \int_{-\infty}^{\infty} p(t-z)e^{i\sigma t} f dt.
\]
One has
\[
\|e^{iz\mathcal{E}} g\| \leq \|f\| \int_{-\infty}^{\infty} |p(t-z)| dt.
\]
Since by assumption \( p \in L_1(\mathbb{R}) \) is an entire function of exponential type \( \sigma \) we have for \( z = x + iy \) and \( u = t - x \)
\[
\int_{-\infty}^{\infty} |p(t-z)| dt = \int_{-\infty}^{\infty} |p(u-iy)| du \leq e^{\sigma|y|} \|p\|_{L_1(\mathbb{R})},
\]
because
\[
p(u-iy) = \sum_{m=0}^{\infty} \frac{(-iy)^m}{m!} p^{(m)}(u),
\]
and according to the classical Bernstein inequality
\[
\|p^{(m)}\|_{L_1(\mathbb{R})} \leq \sigma^m \|p\|_{L_1(\mathbb{R})}.
\]
Thus
\[
\|e^{iz\mathcal{E}} g\| \leq \|f\| \int_{-\infty}^{\infty} |p(t-z)| dt \leq \|f\| e^{\sigma|y|} \|h\|_{L_1(\mathbb{R})}.
\]
It shows that for every vector \( h \in \mathbf{H} \) the function \( \langle e^{iz\mathcal{E}} g, h \rangle \) is an entire function and
\[
|\langle e^{iz\mathcal{E}} g, h \rangle| \leq \|h\| \|f\| e^{\sigma|y|} \|f\|_{L_1(\mathbb{R})}.
\]
In other words the \( \langle e^{iz\mathcal{E}} g, h \rangle \) is an entire function of the exponential type \( \sigma \) which is bounded on the real line and another application of the classical Bernstein inequality in the norm \( C(\mathbb{R}) \) gives the inequality
\[
\left| \left( \frac{d}{dt} \right)^k \langle e^{it\mathcal{E}} g, h \rangle \right| \leq \sigma^k \sup_{t \in \mathbb{R}} |\langle e^{it\mathcal{E}} g, h \rangle|.
\]
Since
\[
\left( \frac{d}{dt} \right)^k \langle e^{it\mathcal{L}} g, h \rangle = \langle e^{it\mathcal{L}} (i\mathcal{L})^k g, h \rangle
\]
we obtain for \( t = 0 \)
\[
|\langle \mathcal{L}^k g, h \rangle| \leq \sigma^k \| h \| \| f \| \int_{-\infty}^{\infty} |p(\tau)|d\tau.
\]
Choosing \( h \) such that \( \| h \| = 1 \) and \( \langle \mathcal{L}^k g, h \rangle = \| \mathcal{L}^k g \| \) we obtain the inequality
\[
\| \mathcal{L}^k g \| \leq \sigma^k \| f \| \int_{-\infty}^{\infty} |p(\tau)|d\tau, \quad k \in \mathbb{N},
\]
which according to Lemma 2.1 implies that \( g \) belongs to \( PW_\sigma(\mathcal{L}) \). Lemma is proven. \( \square \)

For the modulus of continuity introduced in (1.10) the following inequalities hold:
\[
(2.2) \quad \omega^m_\mathcal{L} (s, f) \leq s^k \omega^m_\mathcal{L} (s, \mathcal{L}^k f), \quad 0 \leq k \leq m,
\]
and
\[
(2.3) \quad \omega^m_\mathcal{L} (as, f) \leq (1 + a)^m \omega^m_\mathcal{L} (s, f), \quad a \in \mathbb{R}_+.
\]
The first one follows from the identity
\[
(2.4) \quad (e^{is\mathcal{L}} - 1)^k f = \int_0^s \ldots \int_0^s e^{i(\tau_1 + \ldots + \tau_k)} \mathcal{L}^k f d\tau_1 \ldots d\tau_k,
\]
where \( I \) is the identity operator and \( k \in \mathbb{N} \). The second one follows from the property
\[
\omega^k_\mathcal{L} (s_1 + s_2, f) \leq \omega^k_\mathcal{L} (s_1, f) + \omega^k_\mathcal{L} (s_2, f)
\]
which is easy to verify. The next Theorem and its proof are motivated by Theorem 5.2.1 in [14].

**Theorem 2.3.** Let \( \mathcal{L} \) be a self-adjoint operator in a Hilbert space \( \mathbf{H} \). For a given natural \( m \) there exists a constant \( c = c(m) > 0 \) such that for all \( \sigma > 0 \) and all \( f \) in \( \mathbf{H} \)
\[
(2.5) \quad \mathcal{E}_\mathcal{L}(\sigma, f) \leq c \omega^m_\mathcal{L} (1/\sigma, f).
\]
Moreover, for any \( 1 \leq k \leq m \) there exists a \( C = C(m, k) > 0 \) such that for any \( f \in \mathcal{D}(\mathcal{L}^k) \) one has
\[
(2.6) \quad \mathcal{E}_\mathcal{L}(\sigma, f) \leq \frac{C}{\sigma^k} \omega^m_\mathcal{L} (1/\sigma, \mathcal{L}^k f), \quad 0 \leq k \leq m.
\]

**Proof.** Let
\[
(2.7) \quad \rho(t) = a \left( \frac{\sin(t/n)}{t} \right)^n
\]
where \( n = 2(m + 3) \) and
\[
a = \left( \int_{-\infty}^{\infty} \left( \frac{\sin(t/n)}{t} \right)^n dt \right)^{-1}.
\]

With such choice of \( a \) and \( n \) function \( \rho \) will have the following properties:
1. \( \rho \) is an even nonnegative entire function of exponential type one;
2. \( \rho \) belongs to \( L_1(\mathbb{R}) \) and its \( L_1(\mathbb{R}) \)-norm is 1;
(3) the integral

(2.8) \[ \int_{-\infty}^{\infty} \rho(t)|t|^m dt \]

is finite.

Next, we observe the following formula

(2.9) \[ (-1)^{m+1}(e^{i\sigma L} - I)^m f = \sum_{j=0}^{m} (-1)^{m-j} C_j e^{js(i\sigma L)} f - f, \]

where \( b_1 + b_2 + \ldots + b_m = 1 \). Consider the vector

(2.10) \[ Q_{\rho}^{\sigma,m}(f) = \int_{-\infty}^{\infty} \rho(t) \left\{ (-1)^{m+1}(e^{i\sigma L} - I)^m f + f \right\} dt. \]

According to (2.10) we have

\[ Q_{\rho}^{\sigma,m}(f) = \int_{-\infty}^{\infty} \rho(t) \sum_{j=1}^{m} b_j e^{j(i\sigma L)} f dt. \]

Changing variables in each of integrals

\[ \int_{-\infty}^{\infty} \rho(t)e^{j(i\sigma L)} f dt, \quad 1 \leq j \leq m, \]

we obtain the formula

\[ Q_{\rho}^{\sigma,m}(f) = \int_{-\infty}^{\infty} \Phi(t)e^{i\sigma L} f dt, \]

where

\[ \Phi(t) = \sum_{j=1}^{m} b_j \left( \frac{\sigma}{j} \right) \rho \left( \frac{\sigma}{j} \right), \quad b_1 + b_2 + \ldots + b_m = 1. \]

Since the function \( \rho(t) \) has exponential type one every function \( \rho(t \sigma/j) \) has the type \( \sigma/j \) and because of this the function \( \Phi(t) \) has exponential type \( \sigma \). It also belongs to \( L_1(\mathbb{R}) \) and as it was just shown it implies that the vector \( Q_{\rho}^{\sigma,m}(f) \) belongs to \( PW_{\sigma}(\mathcal{L}) \). Now we estimate the error of approximation of \( Q_{\rho}^{\sigma,m}(f) \) to \( f \). Since by\n
we obtain by using

\[ E_{\mathcal{L}}(\sigma, f) \leq \|f - Q_{\rho}^{\sigma,m}(f)\| \leq \int_{-\infty}^{\infty} \rho(t) \left\| (e^{i\sigma L} - I)^m f \right\| dt \leq \int_{-\infty}^{\infty} \rho(t) \omega_{\mathcal{L}}^{m}(t/\sigma, f) dt \leq c\omega_{\mathcal{L}}^{m}(1/\sigma, f), \quad c = \int_{-\infty}^{\infty} \rho(t)(1 + |t|)^m dt. \]

If \( f \in D(\mathcal{L}^k) \) then by using\n
we have

\[ E_{\mathcal{L}}(\sigma, f) \leq \int_{-\infty}^{\infty} \rho(t) \omega_{\mathcal{L}}^{m}(t/\sigma, f) dt \leq \frac{\omega_{\mathcal{L}}^{m-k}(1/\sigma, \mathcal{L}^k f)}{\sigma^k} \int_{-\infty}^{\infty} \rho(t)|t|^k(1 + |t|)^{m-k} dt \leq C \frac{\omega_{\mathcal{L}}^{m-k}(1/\omega, \mathcal{L}^k f)}{\sigma^k}, \]
where

\[ C = \int_{-\infty}^{\infty} \rho(t)|t|^k(1 + |t|)^{m-k}dt, \]

is finite by the choice of \( \rho \). The inequalities (2.5) and (2.6) are proved. \( \square \)

3. UNITARY REPRESENTATIONS OF LIE GROUPS

A strongly continuous unitary representation of a Lie group \( G \) in a Hilbert space \( H \) is a homomorphism \( T : G \rightarrow U(H) \) where \( U(E) \) is the group of unitary operators of \( H \) such that \( T(g)f, \ g \in G, \) is continuous on \( G \) for any \( f \in H \). The Garding space \( G \) is defined as the set of vectors \( h \) in \( H \) that have the representation \( h = \int_G \varphi(g)T(g)f \, dg, \) where \( f \in H, \ \varphi \in C_0^\infty(G), \ dg \) is a left-invariant measure on \( G \). Every element \( X \) which belongs to the corresponding Lie algebra \( g \) can be identified with a right-invariant vector field

\[ X\varphi(g) = \lim_{t \to 0} \frac{\varphi(e^{tX}g) - \varphi(g)}{t}. \]

The correspondence \( X \to D(X) \) is a representation of \( g \) by operators which act on \( G \) by the formula

\[ D(X)h = -\int_G X\varphi(g)T(g)f \, dg. \]

If \( X_1, \ldots, X_d \) is a basis in \( g \) and \( D_i = D(X_i), \ 1 \leq i \leq d, \) we introduce an analog of a Sobolev space as the subspace \( H^r \subset H \) which is the common domain of all the operators \( D_{j_1} \ldots D_{j_k}, \ 1 \leq j_1, \ldots, j_k \leq d, \ 1 \leq k \leq r, \) with the norm

\[ |||f|||_{H^r} = ||f||_H + \sum_{k=1}^{r} \sum_{1 \leq j_1, \ldots, j_k \leq d} ||D_{j_1} \ldots D_{j_k}f||_H, \]

which is equivalent to the norm

\[ ||f||_r = ||f||_H + \sum_{1 \leq i_1, \ldots, i_r \leq d} ||D_{i_1} \ldots D_{i_r}f||_H, \quad r \in \mathbb{N}. \]

It is known that \( \mathcal{G} \subset \bigcap_{r \in \mathbb{N}} H^r = H^\infty \) is invariant with respect to all operators \( D(X), \ X \in g, \) and dense in every \( H^r \). We consider the following analog of the Laplace operator \( \{12, 13\} \)

\[ L_G = -D_1^2 - D_2^2 - \ldots - D_d^2, \]

which is defined on the Garding space \( \mathcal{G} \). Since \( L_G \) is symmetric and the differential operator \( -\sum_{i=1}^{d} X_i^2 \) is elliptic on the group \( G \) the Theorem 2.2 in \( \{13\} \) implies that \( L_G \) is essentially self-adjoint, which means \( \mathcal{T}_G = L_G^* \). In other words, the closure \( \mathcal{L}_G = L \) of \( L_G \) from \( \mathcal{G} \) is a self-adjoint operator. Obviously, \( L \geq 0 \). We introduce the self-adjoint operator \( \Lambda = I + L \geq 0 \).

It was shown in \( \{25\} \), Lemma 2.1 and Theorem 2.2, (see also \( \{21, 22\} \)) that in the case of a unitary representation \( T \) of a Lie group \( G, \ dim \ G = d, \) all our Assumptions (a)-(d) are satisfied for one-parameter groups \( T_1(t), \ldots, T_d(t) \), where for a basis \( X_1, \ldots, X_d, \) of the Lie algebra of \( G \) one has

\[ T_j(t) = T(e^{tX_j}), \quad t \in \mathbb{R}, \quad 1 \leq j \leq d, \]

and where \( e^{tX_j} \in G, \ t \in \mathbb{R}, \) is the one parameter subgroup in the direction of \( X_j \). In particular, a proof of the following important fact is given in Appendix 4.
**Theorem 3.1.** The space $H^r$ with the norm (3.7) is isomorphic to the domain of $\Lambda^{r/2}$ with the norm $\|\Lambda^{r/2}f\|_H$.

An important class of representations of Lie groups appears in connection with homogeneous manifolds. In what follows we introduce some very basic notions about unitary representations of Lie groups in function spaces on homogeneous manifolds [8], [26].

Let $M, \dim M = m$, be a connected $C^\infty$-manifold. It says that a Lie group $G$ effectively acts on $M$ as a group of diffeomorphisms if

1) every element $g \in G$ can be identified with a diffeomorphism

$$g : M \to M$$

of $M$ onto itself and

$$g_1g_2 \cdot x = g_1 \cdot (g_2 \cdot x), g_1, g_2 \in G, x \in M,$$

where $g_1g_2$ is the product in $G$ and $g \cdot x$ is the image of $x$ under $g$.

2) the identity $e \in G$ corresponds to the trivial diffeomorphism

$$e \cdot x = x,$$

3) for every $g \in G, g \neq e$, there exists a point $x \in M$ such that $g \cdot x \neq x$.

A group $G$ acts on $M$ transitively if in addition to 1)- 3) the following property holds

4) for any two points $x, y \in M$ there exists a diffeomorphism $g \in G$ such that

$$g \cdot x = y.$$

A homogeneous manifold $M$ is an $C^\infty$-compact manifold on which transitively acts a Lie group $G$. In this case $M$ is necessary of the form $G/K$, where $K$ is a subgroup of $G$. The notation $L_2(M)$ is used for the usual Hilbert spaces $L_2(M, dx)$, where $dx$ is an invariant (with respect to $G$-action) measure. It is known that the correspondence

$$g \to T(g), \quad T(g)f(x) = f(g \cdot x),$$

where $g \in G, \; x \in M, \; f \in L_2(M)$ is a unitary representation of $G$ in $L_2(M)$.

**Example 1. A compact homogeneous manifold.** The situation on a unit sphere is typical for at least all two-point homogeneous compact manifolds. Consider the unit sphere

$$S^n = \{x = (x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} : \|x\|^2 = x_1^2 + x_2^2 + ... + x_{n+1}^2 = 1\}.$$

Let $e_1, ..., e_{n+1}$ be the standard orthonormal basis in $\mathbb{R}^{n+1}$. If $SO(n+1)$ and $SO(n)$ are the groups of rotations of $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$ respectively then $S^n = SO(n+1)/SO(n)$. On $S^n$ we consider vector fields $X_{i,j} = x_j \partial_{x_i} - x_i \partial_{x_j}, \; i < j$, which are generators of one-parameter groups of rotations $\exp \tau X_{i,j} \in SO(n + 1)$ in the plane $(x_i, x_j)$. These groups are defined by the formulas for $\tau \in \mathbb{R}$,

$$\exp \tau X_{i,j} \cdot (x_1, ..., x_{n+1}) = (x_1, ..., x_j \cos \tau - x_j \sin \tau, ..., x_i \sin \tau + x_j \cos \tau, ..., x_{n+1}).$$

Clearly, there are $d = \frac{1}{2}n(n - 1)$ such groups. Let $T_{i,j}(\tau)$ be a one-parameter group which is a representation of $\exp \tau X_{i,j}$ in the space $L_2(S^n)$. It acts on $f \in L_2(S^n)$ by the following formula

$$T_{i,j}(\tau)f(x_1, ..., x_{n+1}) = f(x_1, ..., x_i \cos \tau - x_j \sin \tau, ..., x_i \sin \tau + x_j \cos \tau, ..., x_{n+1}).$$
The infinitesimal operator of this group will be denoted as $D_{i,j}$. The operator $L = -\sum_{i<j} D_{i,j}$ is the regular Laplace-Beltrami operator on $S^n$ and spaces $PW_\sigma(S^n)$ are comprised of appropriate linear combinations of spherical harmonics.

**Remark 3.2.** This example explains reasons why $d$ is typically greater than $n = \dim M$. In this case it happens because vector fields $D_j$ can vanish along low dimensional submanifolds. For example, on $S^2 \subset \mathbb{R}^3$ one needs three fields $X_{1,2}, X_{1,3}, X_{2,3}$ since they vanish at the poles $(0,0,\pm 1), (0,\pm 1,0), (\pm 1,0,0)$ respectively.

**Example 2. A non-compact homogeneous manifold.**

Consider the upper half of the hyperboloid

$$\mathbb{H}^n_+ = \{x = (x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} : -x_1^2 - x_2^2 - ... - x_n^2 + x_{n+1}^2 = 1, x_{n+1} > 0\}.$$ 

Let $e_1, ..., e_{n+1}$ be the standard orthonormal basis in $\mathbb{R}^{n+1}$. If $SH(n+1)$ is the group of hyperbolic rotations which means it preserves the form

$$|x, y| = -x_1 y_1 - ... - x_n y_n + x_{n+1} y_{n+1}$$

then $\mathbb{H}^n_+ = SH(n+1)/SO(n)$. On $\mathbb{H}^n_+$ we consider the vector fields $X_{i,j} = x_j \partial_{x_i} - x_i \partial_{x_j}$, $i < j < n+1$, which generate euclidean rotation groups in the planes $(x_i, x_j)$, $i < j < n+1$, and the fields $X_{i,n+1} = x_{n+1} \partial_{x_i} + x_i \partial_{x_{n+1}}$ which are generators of the hyperbolic groups of rotations in the planes $(x_i, x_{n+1})$. These groups are defined by the formulas for $\tau \in \mathbb{R}$,

$$\exp \tau X_{i,j} \cdot (x_1, ..., x_{n+1}) = (x_1, ..., x_i \cos \tau - x_j \sin \tau, ..., x_i \sin \tau + x_j \cos \tau, ..., x_{n+1}),$$

$$\exp \tau X_{i,n+1} \cdot (x_1, ..., x_{n+1}) = (x_1, ..., x_i \cosh \tau - x_{n+1} \sinh \tau, ..., x_i \sinh \tau + x_{n+1} \cosh \tau).$$

Strictly continuous one-parameter groups of operators $T_{i,j}(\tau)$ which are representations of $\exp \tau X_{i,j}$ in the space $L_2(\mathbb{H}^n_+)$ can be used to construct corresponding modulus of continuity $\Omega'(\sigma, f)$. Their infinitesimal operators $D_{i,j}$ are just operators $X_{i,j}$ in the space $L_2(\mathbb{H}^n_+)$ and $L = -\sum_{i<j\leq n+1} D_{i,j}$ is an elliptic self-adjoint non-negative operator in $L_2(\mathbb{H}^n_+)$ which has continuous spectrum. As well as we know, the spectral resolution of this operator is unknown. However, the abstract Definition 4 and notion of best approximation still make sense.

**Example 3.** Schrödinger representation of the Heisenberg group.

The $(2n+1)$-dimensional Heisenberg group $\mathbb{H}_{2n+1}$ has a unitary representation in the space $L_2(\mathbb{R}^n)$

$$T(p, q, x) = e^{i(t+(q,x))} f(x+p), \quad p, q, x \in \mathbb{R}^n, \quad t \in \mathbb{R}.$$ 

One can consider the following set of infinitesimal operators where $i = \sqrt{-1}$:

$$D_j = \partial_j, \quad 1 \leq j \leq n; \quad D_j = i x_j, \quad n+1 \leq j \leq 2n; \quad D_{2n+1} = i.$$

In this case every $T_j(\tau), 1 \leq j \leq n$, is a translation along variable $x_j$,

$$T_j(\tau) f(x_1, ..., x_n) = e^{irx_j} f(x_1, ..., x_n), \quad n+1 \leq j \leq 2n,$$

and $T_{2n+1}(\tau) f(x_1, ..., x_n) = e^{ir} f(x_1, ..., x_n)$. The operator $L$ is the shifted $n$-dimensional linear oscillator

$$L = -\Delta + |x|^2 + 1,$$

where

$$\Delta = \sum_{j=1}^n \partial_j^2, \quad |x|^2 = \sum_j x_j^2, \quad x = (x_1, ..., x_n).$$
It is known that the spectrum of this operator is discrete and its eigenfunctions are products of one-dimensional Hermite functions. One can easily describe corresponding Paley-Wiener spaces and to construct corresponding modulus of continuity by using groups of operators \( T_j, \ 1 \leq j \leq 2n + 1 \).

4. Appendix. Proof of Theorem 3.1

**Theorem 4.1.** The space \( \mathbf{H}^r \) with the norm \( \| \Lambda^{r/2} f \|_\mathbf{H} \) is isomorphic to the domain of \( \Lambda^{r/2} \) with the norm \( \| \Lambda^{r/2} f \|_\mathbf{H} \).

**Proof.** In the case \( r = 2k \), the inequality

\[
\| f \|_{\mathbf{H}^{2k}} \leq C(k) \| \Lambda^k f \|_\mathbf{H}
\]

is shown in [12], Lemma 6.3. The reverse inequality is obvious. We consider now the case \( r = 2k + 1 \). If \( f \in \mathbf{H}^2 = \mathcal{D}(\Lambda) \), then since \( \mathcal{D}(\Lambda) \subset \mathcal{D}(\Lambda^{1/2}) \) we have

\[
\| f \|^2_{\mathbf{H}} + \sum_j \| D_j f \|^2_{\mathbf{H}} = \langle f, f \rangle + \sum_j \langle D_j f, D_j f \rangle = \langle f, f \rangle + \left( -\sum_j D_j^2 f, f \right) =
\]

\[
\left( f - \sum_j D_j f, f \right) = \langle \Lambda f, f \rangle = \| \Lambda^{1/2} f \|^2_{\mathbf{H}}.
\]

These equalities imply that \( \mathbf{H}^1 \) is isomorphic to \( \mathcal{D}(\Lambda^{1/2}) \). Our goal is to to prove existence of an isomorphism between \( \mathbf{H}^{2k+1} \) and \( \mathcal{D}(\Lambda^{k+1/2}) \). It is enough to establish equivalence of the corresponding norms on the set \( \mathbf{H}^{4k+2} = \mathcal{D}(\Lambda^{2k+1}) \) since the latest is dense in \( \mathbf{H}^{2k+1} \). If \( f \in \mathbf{H}^{4k+2} \subset \mathbf{H}^{2k} \) then \( D_j f \in \mathbf{H}^{4k+1} \subset \mathbf{H}^{2k} \) and \( \Lambda^k f = \sum_{m \leq k} \sum D_{j_1}^2 \ldots D_{j_m}^2 f \). Thus if \( f \in \mathbf{H}^{4k+2} \) then

\[
\| D_{j_1} \ldots D_{j_{2k+1}} f \|_\mathbf{H} \leq C \| \Lambda^k D_{j_{2k+1}} f \|_\mathbf{H} = \left\| \sum_{m \leq k} \sum D_{j_1}^2 \ldots D_{j_m}^2 D_{j_{2k+1}} f \right\|_\mathbf{H}.
\]

Multiple applications of the identity \( D_i D_j - D_j D_i = \sum_k \delta_{ij} D_k \) which holds on \( \mathbf{H}^2 \) lead to the inequality \( \| D_{j_1} \ldots D_{j_{2k+1}} f \|_\mathbf{H} \leq C \left( \| D_{j_{2k+1}} \Lambda^k f \|_\mathbf{H} + \| R f \|_\mathbf{H} \right) \), where \( R \) is a polynomial in \( D_1, \ldots, D_d \) whose degree \( \leq 2k \). According to (4.1) and (4) we have that

\[
\| D_{j_{2k+1}} \Lambda^k f \|_\mathbf{H} \leq \| \Lambda^{1/2} \Lambda^k f \|_\mathbf{H} = \| \Lambda^{k+1/2} f \|_\mathbf{H}
\]

and also \( \| R f \|_\mathbf{H} \leq \| f \|_{\mathbf{H}^{2k}} \leq C(k) \| \Lambda^k f \|_\mathbf{H} \). Since \( \| \Lambda^k f \|_\mathbf{H} \) is not decreasing with \( k \) we get the following estimate

\[
\| D_{j_1} \ldots D_{j_{2k+1}} f \|_\mathbf{H} \leq C(k) \| \Lambda^{k+1/2} f \|_\mathbf{H}, \ \ f \in \mathbf{H}^{4k+2}.
\]

Now, since for \( f \in \mathbf{H}^{4k+2} \) we have \( D_{j_1} \ldots D_{j_{2k+1}} f \in \mathbf{H}^{2k+2} \subset \mathbf{H}^1 = \mathcal{D}(\Lambda^{1/2}) \), and the equality \( \Lambda^k f = \sum_{m \leq k} \sum D_{j_1}^2 \ldots D_{j_m}^2 f \), holds we obtain, by using (4)

\[
\| \Lambda^{k+1/2} f \|_\mathbf{H} = \| \Lambda^{1/2} \sum_{m \leq k} \sum D_{j_1}^2 \ldots D_{j_m}^2 f \|_\mathbf{H} \leq C \| f \|_{\mathbf{H}^{2k+1}}, \ \ C = C(k).
\]

Theorem is proved. \( \square \)
Corollary 4.1. If $T$ is a strongly continuous unitary representation of a Lie group in a Hilbert space $\mathcal{H}$ and $X_1, \ldots, X_d$ is a basis in the corresponding algebra $\mathfrak{g}$, then for $T_j(t) = T(\exp tX_j)$, $1 \leq j \leq d$, and their generators $D_j$, $1 \leq j \leq d$, the item (d) in the Assumptions is satisfied.

References

1. P. Butzer, H. Berens, Semi-Groups of operators and approximation, Springer, Berlin, 1967.
2. Dai, F., Some equivalence theorems with K-functionals, J. Appr. Theory. 121 (2003) 143-157.
3. Dai, Feng; Xu, Yuan, Moduli of smoothness and approximation on the unit sphere and the unit ball, Adv. Math. 224 (2010), no. 4, 1233-1310.
4. Dai, Feng; Xu, Yuan, Approximation theory and harmonic analysis on spheres and balls, Springer Monographs in Mathematics. Springer, New York, 2013. xviii+440 pp. ISBN: 978-1-4614-6659-8; 978-1-4614-6660-4
5. Z. Ditzian, Approximation on Banach spaces of functions on the sphere, J. Approx. Theory 140 (2006), no. 1, 31–45.
6. Z. Ditzian, Jackson-type inequality on the sphere, Acta Math. Hungar. 102 (2004), no. 1-2, 1-35.
7. Feichtinger, Hans G.; Fuhr, Hartmut; Pesenson, Isaac Z., Geometric space-frequency analysis on manifolds, J. Fourier Anal. Appl. 22 (2016), no. 6, 1294-1355.
8. Helgason, S., Geometric Analysis on Symmetric Spaces, Mathematical Surveys and Monographs 39, American Mathematical Society (1994)
9. S. Krein, I. Pesenson, Interpolation Spaces and Approximation on Lie Groups, The Voronezh State University, Voronezh, 1990.
10. S. Krein, Y. Petunin, E. Semenov, Interpolation of linear operators, Translations of Mathematical Monographs, 54. AMS, Providence, R.I., 1982.
11. V. Kumar, M. Ruzhansky, A note on K-functional, Modulus of smoothness, Jackson theorem and Nikolski-Stechkin inequality on Damek-Ricci spaces, arXiv:2020.
12. E. Nelson, Analytic vectors, Ann. of Math., 70(3), (1959), 572-615.
13. E. Nelson, W. Stinespring, Representation of elliptic operators in an enveloping algebra, Amer. J. Math. 81, (1959), 547-560.
14. S. Nikol’skii, Approximation of functions of several variables and imbedding theorems, Springer, Berlin, 1975.
15. Nursultanov, Erlan; Ruzhansky, Michael; Tikhonov, Sergey, Nikolski inequality and Besov, Triebel-Lizorkin, Wiener and Beurling spaces on compact homogeneous manifolds, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 16 (2016), no. 3, 981-1017.
16. Nursultanov, E. D.; Ruzhansky, M. V.; Tikhonov, S. Yu., Nikolski inequality and functional classes on compact Lie groups, Translation of Funktsional. Anal. i Prilozhen. 49 (2015), no. 3, 83-87.
17. El Ouadih, S., An equivalence theorem for a K-functional constructed by Beltrami-Laplace operator on symmetric spaces, J. Pseudo-Differ. Oper. Appl. (2020). https://doi.org/10.1007/s11868-020-00326-2
18. I. Pesenson, Interpolation spaces on Lie groups, (Russian) Dokl. Akad. Nauk SSSR 246 (1979), no. 6, 1298–1303.
19. I. Pesenson, Nikolski- Besov spaces connected with representations of Lie groups, (Russian) Dokl. Akad. Nauk SSSR 273 (1983), no. 1, 45–49.
20. I. Pesenson, The Best Approximation in a Representation Space of a Lie Group, Dokl. Acad. Nauk USSR, v. 302, No 5, pp. 1055-1059, (1988) (Engl. Transl. in Soviet Math. Dokl., v.38, No 2, pp. 384-388, 1989.)
21. I. Pesenson, On the abstract theory of Nikolski-Besov spaces, (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. 1988, no. 6, 59-68; (Engl. Translation in Soviet Math. 85-92)
22. I. Pesenson, Approximations in the representation space of a Lie group, (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. 1990, no. 7, 43-50; translation in Soviet Math. (Iz. VUZ) 34 (1990), no. 7, 49–57.
23. I. Pesenson, Lagrangian splines, spectral entire functions and Shannon-Whittaker theorem on manifolds, Temple University Research Report 95-87 (1995),1-28.
24. I. Pesenson, Sampling of Paley-Wiener functions on stratified groups, J. Four. Anal. Appl. 4 (1998), 269-280.
25. I. Pesenson, *Sobolev, Besov and Paley-Wiener vectors in Banach and Hilbert spaces*, Functional analysis and geometry: Selim Grigorievich Krein centennial, 251-272, Contemp. Math., 733, Amer. Math. Soc., Providence, RI, 2019. [arXiv:1708.07410](https://arxiv.org/abs/1708.07410).

26. Vilenkin, N.J., *Special Functions and the Theory of Group Representations*, Translations of Mathematical Monographs Vol. 22, American Mathematical Society (1978).

Department of Mathematics, Temple University, Philadelphia, PA 19122

Email address: pesenson@temple.edu