Role of unphysical solution in nucleon QCD sum rules

E. G. Drukarev, M. G. Ryskin, V. A. Sadovnikova,
Petersburg Nuclear Physics Institute
Gatchina, St. Petersburg 188300, Russia

Abstract

We show that at certain values of QCD condensates the nucleon QCD sum rules with “pole+continuum” model for the hadron spectrum obtain an unphysical solution. This provides constrains for the values of condensates to be consistent with existence of a physical solutions. The constrains become much weaker if the radiative corrections are included perturbatively. We demonstrate that the most important dependence of nucleon mass on the quark scalar condensate becomes less pronounced under factorization assumption for the four-quark and six-quark condensates.

1 Introduction

The QCD sum rules invented by Shifman et al. [1] enable to express vacuum characteristics of hadrons in terms of the vacuum expectation values of QCD operators. This approach was employed to description of nucleons in [2] and [3, 4]. The improved analysis was presented later in [5, 6]. The method was used also for description of delta-isobars [5, 6] and of the baryons containing heavier quarks [7, 8]. Further applications of the vacuum nucleon QCD sum rules are reviewed in [9]. The method was expanded also for the cases of finite temperatures [10] and densities [11].

The main tool of the vacuum QCD sum rules for a hadron is the dispersion relation for polarization operator

$$
\Pi(q^2) = i \int d^4x e^{i(qx)} \langle 0 | T j(x) \bar{j}(0) | 0 \rangle,
$$

with $j(x)$ a local operator (“current”) carrying the quantum numbers of the hadron. It is nucleon (proton) in our case. The dispersion relation is considered at large values of $|q^2| (q^2 < 0)$ where $\Pi(q^2)$ can be represented as a power series in $q^{-2}$, with the vacuum expectation values of QCD operators as the coefficients of the expansion. This is known as operator power expansion (OPE) [12].

Recall the main milestones of the QCD sum rules analysis. Following [2] one can write dispersion relations

$$
\Pi^i(q^2) = \frac{1}{\pi} \int \frac{\text{Im} \, \Pi^i(k^2)}{k^2 - q^2} \, dk^2
$$

1
We equal the OPE of the left-hand side (LHS) of Eq. (2) to the contribution of the observable hadrons to its right-hand side (RHS). The latter is usually approximated by the “pole + continuum” model in which the lowest pole is written exactly while the other states are approximated by continuum:

\[
\frac{1}{\pi} \text{Im} \, \Pi^i(k^2) = \lambda_N^2 \delta(k^2 - m^2) + \theta(k^2 - W^2) f^i(k^2).
\] (4)

Of course, the “pole + continuum” model is reasonable only if the contribution of the pole exceeds that of the continuum.

Behavior of both sides of Eq. (2) prompts the choice [1]

\[
f^i(k^2) = \frac{1}{\pi} \text{Im} \, \Pi^i(\text{OPE})(k^2).
\]

Note that in this approach the continuum threshold \(W^2\) does not coincide with the physical continuum threshold. Thus the position of the lowest pole \(m\), its residue \(\lambda_N^2\) and the model continuum threshold \(W^2\) are the unknowns which are expected to be determined by the QCD sum rules equations. The standard next step is the Borel transform, after which Eqs.(2) take the form

\[
\mathcal{L}^i(M^2) = R^i(M^2); \quad L^i(M^2) = R^i(M^2).
\] (5)

Here \(\mathcal{L}^i(R^i)\) are the Borel transforms of the LHS (RHS) of Eq. (2), \(M^2\) is the Borel mass. This approach provided good results for the nucleon mass and for the other nucleon parameters [9].

Note that both \(\mathcal{L}^i(M^2)\) and \(R^i(M^2)\) are calculated in framework of certain models. The OPE expansion for \(\mathcal{L}^i(M^2)\) is increasingly true at large values of \(M^2\). The “pole + continuum” model for \(R^i(M^2)\) is increasingly true at small values of \(M^2\). Important assumption is that there is a region of intermediate values of \(M^2\) where both approximations work and reproduce to some extend the true (unknown) function of \(M^2\). Thus our task is to find the interval of the values of \(M^2\), where the functions \(\mathcal{L}^i(M^2)\) can be approximated by the functions \(R^i(M^2)\) and to find the set of parameters \(m, \lambda_N^2, W^2\) which insure the most accurate approximation of \(\mathcal{L}^i(M^2)\) by \(R^i(M^2)\). The set of values of parameters \(m, \lambda_N^2, W^2\), which minimize the function

\[
\chi^2(m, \lambda_N^2, W^2) = \sum \sum \left( \frac{\mathcal{L}^i(M_j^2) - R^i(M_j^2)}{\mathcal{L}^i(M_j^2)} \right)^2
\]

will be referred to as a solution of the sum rules equations.

Note that it is important to obtain “duality” between the LHS and RHS of Eq.(5) in some interval of the values of \(M^2\), but not at certain point \(M_j^2\). Therefore we will look for the three unknown parameters simultaneously.

Both the interval of the values of the Borel mass (“Borel window”) and the solution of the sum rules depend on the form of the proton nucleon current \(j(x)\), which is not determined in an unique way. The general form is [2, 13]

\[
j(x; t) = j_1(x) + tj_2(x),
\] (7)
with
\[ j_1(x) = \varepsilon_{abc} [u_a^T(x) C d_b(x)] \gamma_5 u_c(x); \quad j_2(x) = \varepsilon_{abc} [u_a^T(x) C \gamma_5 d_b(x)] u_c(x), \]
where \( u \) and \( d \) are the quark operators, \( a, b, c \) are the color indices, \( T \) denotes a transpose and \( C \) is the charge conjugation matrix, while \( t \) is an arbitrary parameter.

Following [2], we shall use the current determined by Eq.(7) with \( t = -1 \). It can be written (up to a factor 1/2) as [2]
\[ j(x) = \varepsilon_{abc} [u_a^T(x) C \gamma_\mu u_b(x)] \gamma_5 \gamma_\mu d_c(x) \]  
(8)

This choice was shown in [14] to be most relevant for description of nucleons, since the polarization operator \( \Pi(q^2) \) calculated with this current satisfies two main requirements. On the RHS of Eq.(5) the contribution of the nucleon pole exceeds that of the higher states (approximated by continuum). On the LHS of Eq.(5) the higher order terms of \( M^{-2} \) series should drop fast enough to be consistent with the convergence of the OPE. It is important also that there is a gap between the position of the lowest pole \( m^2 \) and the effective threshold \( W^2 \). This choice of the current was advocated in [13]. There are many papers, in which this very current was used - see, e.g., [15].

Several lowest terms of the OPE for the current (8) have been calculated in [2, 5]. The leading term depends on \( q^2 \) as \( q^4 \ln q^2 \). It comes from the free three-quark loop. The higher order OPE terms contain the matrix elements
\[ \langle 0 | \bar{q} q | 0 \rangle, \quad \langle 0 | \frac{\alpha_s}{\pi} G_{\mu \nu}^a G_{\mu \nu}^a | 0 \rangle, \quad \langle 0 | \bar{q} q \bar{q} q | 0 \rangle, \]
etc. Analysis carried out in [5] contained also the most important radiative corrections in which the QCD coupling constant \( \alpha_s \) is multiplied by “large logarithm” \( \ln q^2 \). Corrections of the order \( (\alpha_s \ln q^2)^n \) to the leading OPE terms have been calculated earlier in [16].

The appropriate interval
\[ 0.8 \text{ GeV}^2 < M^2 < 1.4 \text{ GeV}^2 \]  
(9)
(“Borel window”) was found in [5]. The values of the lowest OPE terms at conventional values of the QCD condensates enable to expect the convergence of OPE series. Also, for the solution found in [5] the contribution of the pole exceeds that of the continuum.

The OPE series for \( \Pi^q \) and \( \Pi^I \) start from the terms \( q^4 \ln q^2 \) and \( q^2 \ln q^2 \) correspondingly. Thus, in somewhat straightforward interpretation of OPE only \( \Pi^q \) contains a leading term. This allows to consider the chirality conserving structure \( \Pi^q \) as a more important one. However actually we consider the values of the Borel mass \( M^2 \) of the order of the proton mass. After the Borel transform the leading contributions to \( \Pi^q \) and \( \Pi^I / m \) are of the same order. The chirality violating sum rule for \( \Pi^I \) can be considered as important as that for \( \Pi^q \). The two sum rules were considered on the same terms, requiring the same accuracy for both of them. That’s why the terms corresponding to \( \Pi^q \) and \( \Pi^I \) on the right hand side of Eq.(6) were included with the same weights.

A weak point of this procedure is that the choice of parameters \( m, \lambda^2 \) and \( W^2 \) may be not simple. There can be several local minima of \( \chi^2 \) corresponding to several sets of the parameters.
Note that the values of QCD condensates are known with large uncertainties. The expectation value \( \langle 0 | \bar{q} q | 0 \rangle \) can be determined with the larger accuracy than the other condensates due to the Gell-Mann–Oakes–Renner relation [17]. There is no experimental data on values of four- and six-quark condensates. They can be calculated in the factorization approximation for the case of large number of colors \( N_c \gg 1 \). The accuracy of this approximation for \( N_c = 3 \) is obscure. The value of gluon condensate was initially obtained from the QCD sum rules for \( \rho \) mesons [18]. However the sum rules for other mesons lead to somewhat smaller [19] or larger [20] values, with the latest analysis presented in [21]. Hence it is reasonable to study dependence of the nucleon parameters on the values of the QCD condensates.

This dependence should be investigated together with inclusion of radiative corrections. The latter imitate modification of the values of the condensates providing the contributions \( \sim \alpha_s \) and \( \alpha_s \ln q^2 \) to each of OPE term. Thus, including the radiative corrections into our analysis, we can separate the effects of uncertainties in the QCD condensates values.

In the present paper we analyze the role of the unphysical solution for the nucleon QCD sum rules, which corresponds to the continuum contribution exceeding that of the pole. We mentioned this solution in our earlier papers [22, 23]. Here we show that both physical and unphysical solutions provide local minima of the function \( \chi^2(m, \lambda^2 N, W^2) \) determined by Eq. (6). For the absolute values of condensates, which differ noticeably from the conventional values (still consistent with convergence of the OPE series on the LHS of the sum rules), the minima corresponding to physical solutions may vanish.

We show that inclusion of the radiative corrections modifies the situation. One could expect this, since it was shown in [22] that the radiative corrections affect mostly the value of the nucleon residue \( \lambda^2 N \). After the corrections of the order \( \alpha_s \) are included perturbatively, the physical solution exists for a broader interval of the values of condensates. Also, domination of continuum contribution over that of the pole for the unphysical solution becomes stronger. The unphysical solution becomes "more unphysical”.

Note that the problem of the pole dominance emerged in other QCD sum rules studies. In review on the QCD sum rules analysis of the pentaquark states [24] the authors found that it is very difficult to satisfy the requirements of pole domination, OPE series convergence and stability within the Borel window simultaneously. This contrasts the earlier statements (cited in [24]) that the QCD sum rules support the existence of the pentaquark. However, the authors of [24] do not make a definite statement on the QCD sum rules predictions about the pentaquark states. In the QCD sum rules analysis of the light tetraquark states [25] it was found that the solution with the domination of the pole can be obtained only for small values of the Borel mass \( M^2 \), where the OPE series does not converge. The authors of [25] conclude the QCD sum rules analysis does not support existence of light tetraquark particles. In view of the analysis carried out in the present paper, inclusion of the radiative corrections may become important here.

In Section 2 we analyze the interplay of the physical and unphysical solutions taking into account only the leading radiative corrections. We include corrections of the order \( \alpha_s \) in Sec. 3. We summarize in Sec. 4.
2 Interplay of the physical and unphysical solutions

We start by representing the nucleon QCD sum rules [5] without using the factorization hypothesis for the condensates of the high dimensions. The LHS of the Borel transformed nucleon sum rules (Eq.(5)) with inclusion of the anomalous dimensions (i.e. of the corrections of the order \((\alpha_s \ln q^2)^n\)) can be written as

\[
\mathcal{L}^q = \sum_n \tilde{A}_n(M^2), \quad \mathcal{L}' = \sum_n \tilde{B}_n(M^2).
\]

Here the lower indices show the dimensions of the condensates. If the current \(j\) in Eq.(8), the terms on the right hand sides of Eq. (10) are [2, 5]

\[
\tilde{A}_0 = \frac{M^6 E_2}{L}, \quad \tilde{A}_4 = \frac{c_4 M^2 E_0}{4L}, \quad \tilde{A}_6 = \frac{4}{3} c_6 L, \quad \tilde{A}_8 = -\frac{1}{3} \frac{c_8}{M^2}, \\
\tilde{B}_3 = 2c_3 M^4 E_1, \quad \tilde{B}_7 = -\frac{c_7}{12}, \quad \tilde{B}_9 = \frac{272}{81} \frac{c_9}{M^2}.
\]

Here \(L\) accounts for the leading radiative corrections \(\sim \alpha_s \ln q^2\) [16]

\[
L = \left( \frac{\ln q^2/\Lambda_{QCD}^2}{\ln \mu^2/\Lambda_{QCD}^2} \right)^{4/9},
\]

with the anomalous dimension \(\gamma = 4/9\) (\(L = 1\) if the leading radiative corrections are neglected). After the Borel transform

\[
L = \left( \frac{\ln M^2/\Lambda_{QCD}^2}{\ln \mu^2/\Lambda_{QCD}^2} \right)^{4/9}.
\]

The condensates \(c_i\) are

\[
c_3 = -(2\pi)^2 \langle 0|\bar{q}q|0 \rangle, \quad c_4 = (2\pi)^2 \langle 0|\frac{\alpha_s}{\pi} G^2|0 \rangle, \\
c_6 = (2\pi)^4 \langle 0|\bar{q}\gamma_5 q|0 \rangle, \quad c_7 = -(2\pi)^4 \langle 0|\bar{q}\gamma_5 G^2 q|0 \rangle, \\
c_8 = (2\pi)^4 \langle 0|\bar{q}q\gamma_5 G_{\mu\nu} \frac{\Lambda^a}{2} \sigma_{\mu\nu} q|0 \rangle, \quad c_9 = -(2\pi)^6 \frac{\alpha_s}{\pi} \langle 0|\bar{q}q\gamma_5 q|0 \rangle.
\]

Note that the structure of the condensates \(c_6\) and \(c_9\) is indeed more complicated. For example, \(c_6\) contains the condensates \(\langle 0|q\Gamma^A \bar{q}q\Gamma^A q|0 \rangle\) with \(\Gamma^A\) being the basic \(4 \times 4\) matrices \((\Gamma^A = I, \gamma_\mu, \gamma_5, i\gamma_5 \gamma_\mu, \sigma_{\mu\nu})\). The same is true for \(c_9\). Hence the matrix elements \(\langle 0|\bar{q}q|0 \rangle\) and \(\langle 0|\bar{q}q\gamma_5 q|0 \rangle\) in expressions for \(c_6\) and \(c_9\) in Eq. (13) are somewhat “effective” 4q and 6q condensates. Denote

\[
c_3 = a_0 f_{2q}, \quad c_4 = b_0 f_b, \quad c_6 = a_0^2 f_{4q}, \\
c_7 = a_0 b_0 f_{qg}, \quad c_9 = \frac{\alpha_s}{\pi} a_0^3 f_{6q}.
\]

Here we introduced dimensionless parameters \(f_i\). In the factorization approximation \(f_{4q} = f_{2q}^2, f_{qg} = f_{2q} f_b, f_{6q} = f_{2q}^3\). We shall investigate the dependence of nucleon parameters on QCD condensates modifying the values of \(f_i\). Note also that \(c_8 = \mu_0^2 a_0^2\) with \(\mu_0^2 \approx 0.8\) GeV² [2].
Following [2] we write

\[ a = -(2\pi)^2 \langle 0 | \bar{q} q | 0 \rangle, \quad b = (2\pi)^2 \langle 0 | \frac{\alpha_s}{\pi} G^2 | 0 \rangle. \]  

(15)

For traditional choice of the normalization point \( \mu = 0.5 \text{ GeV} \)

\[ a = a_0 = 0.55 \text{ GeV}^3, \quad b = b_0 = 0.5 \text{ GeV}^4. \]  

(16)

Putting all \( f_i = 1 \) in Eq. (14) we come to the standard nucleon sum rules [2, 5] with

\[ \tilde{A}_0 = \frac{M^6 E_n}{L}, \quad \tilde{A}_4 = \frac{b M^2 E_0}{4 L}, \quad \tilde{A}_6 = \frac{4}{3} a^2 L, \quad \tilde{A}_8 = -\frac{1}{3} \frac{\mu_0^2}{M^2} a^2, \]

\[ \tilde{B}_3 = 2 a M^4 E_1, \quad \tilde{B}_7 = -\frac{ab}{12}, \quad \tilde{B}_9 = \frac{272 \alpha_s}{81} \frac{a^3}{M^2} \]  

(17)

Here

\[ E_n = E_n(x) = 1 - e^{-x} \sum_{k=0}^{n} \frac{x^n}{n!}, \quad n = 0, 1, 2, \quad x = W^2 / M^2, \]

i.e. the contributions of continuum are transferred to the LHS of Eq. (5). The RHS of Eq. (5)

\[ R^g(M^2) = \lambda^2 e^{-m^2 / M^2}, \quad R^f(M^2) = m \lambda^2 e^{-m^2 / M^2} \]  

(18)

contain only the contribution of the nucleon pole with \( \lambda^2 = 32 \pi^4 \lambda_N^2 \).

In the Borel window

\[ \frac{|\tilde{A}_8|}{|\tilde{A}_6|} \approx \frac{1}{4}, \quad \frac{\tilde{B}_9}{\tilde{B}_3} \ll 1, \]  

(19)

with \( \tilde{B}_3' = 2aM^4 \) is just \( \tilde{B}_3 \) determined by Eq.(17) before the threshold contribution is transferred to the LHS. (Note that \( \tilde{A}_4 \) and \( \tilde{B}_7 \) are numerically small due to a small coefficient connected with contributions of the gluon condensate). This is consistent with the hypothesis about convergence of the OPE series.

We can expect that the convergence will not be spoiled by the disregarder terms. Note that the term \( A_8 \) can be viewed as coming from expansion of the expectation value of the operator \( \bar{q}(0)q(x) \) in powers of \( x^2 \). The ratio \( A_8 / A_6 \) is the characteristic scale for further expansion in powers of \( M^{-2} \), caused by expansion in powers of \( x^2 \). The mixed quark-gluon condensates of higher dimensions are expected to have small numerical factors, connected with the gluons, similar to \( c_4 \) and \( c_7 \). Finally, there is the only QCD parameter \( \Lambda_{QCD} \), and there can be contributions containing the factor \( \Lambda_{QCD}^2 / M^2 \ll 1 \).

In framework of the factorization hypothesis the method developed in [2, 5] provides

\[ m = 0.931 \text{ GeV}, \quad \lambda^2 = 1.86 \text{ GeV}^6, \quad W^2 = 2.09 \text{ GeV}^2, \]  

(20)

if the numerical values (16) are employed. We assume \( \Lambda_{QCD} \approx 150 \text{ MeV} \). In the one-loop approximation this corresponds to \( \alpha_s (1 \text{ GeV}^2) \approx 0.37 \), which is consistent with the PDG data [26].
If the radiative corrections are neglected, i.e. $L = 1$ the solution appears to be

$$m = 0.930 \text{ GeV}, \quad \lambda^2 = 1.79 \text{ GeV}^6, \quad W^2 = 2.00 \text{ GeV}^2. \quad (21)$$

One can see that for the solution represented by Eq. (21) the contribution of the pole exceeds that of continuum more than twice.

However the successive inclusion of the OPE terms leads to another solution. Taking into account only the condensates with the dimensions $d = 3, 4$ we find a trivial solution $m = \lambda^2 = W^2 = 0$. Inclusion of the condensate with $d = 6$ keeps $m = 0$, $W^2 = 0$ but provides $\lambda^2 = 4/3 a^2 = 0.4 \text{ GeV}^6$. The condensates with $d = 7$ and $d = 8$ require the nonzero values of $m$ and $W^2$, i.e.

$$m = 0.6 \text{ GeV}, \quad \lambda^2 = 0.79 \text{ GeV}^6, \quad W^2 = 1.0 \text{ GeV}^2. \quad (22)$$

We treat this solution as an unphysical one since in the Borel window determined by Eq.(9) the contribution of the pole is smaller than that of the continuum. For example, at the characteristic value $M^2 = 1 \text{ GeV}^2$ of the Borel window the ratio of the continuum and pole contributions is 2 and 1.75 for the $\Pi^0$ and $\Pi^I$ structures correspondingly.

As we shall see below, the unphysical solution manifests itself when the QCD condensates deviate from their conventional values. Note that for all the cases discussed below Eq.(19) is still true, and thus the convergence of the OPE series is not violated. Also (see, e.g. Fig.1 below) the value of $\chi^2$ is small enough. Hence, the only reason for assuming this solution to be unphysical is the domination of the contribution of the continuum over that of the pole.

It is instructive to trace the dependence of nucleon parameters on the value of gluon condensate. At $f_b = 1$, i.e. at the value of the gluon condensate corresponding to Eq. (16) the functional (6) has two local minima corresponding to the solutions (21) and (22). The deeper minimum is provided by the unphysical solution (22). At $f_b < 1$ we still have two minima, and the values of the nucleon parameters do not change much even for $f_b = 0$ due to a relatively small value of the term $\tilde{A}_1$ in Eq. (17). However for $f_b > 0.2$ the deeper minimum corresponds to the unphysical solution (22). Somewhat straightforward employing of the chi-squared method may leave the physical solution unnoticed. The situation is more dramatic for $f_b > 1$. At $f_b > 1.04$ the certain minimum corresponding to the physical solution vanishes and only the unphysical solution survives.

In order to illustrate the role of the unphysical solution in the Borel window we introduce the function

$$m(M^2) = \frac{L^I(M^2)}{L^0(M^2)},$$

with $L^I$ and $L^0$ defined by Eq.(10). Using Eq.(18) we see that for the values of $M^2$ determined by Eq.(9) we can expect $m(M^2) \approx \text{const} = m$ for the solutions of the sum rules equations. Putting $f_b = 0.6$ (to make the difference of the corresponding $\chi^2$ values more visible) we expect the solutions to be close to those described by Eqs.(21),(22) for $f_b = 1$. The functions $m(M^2)$ for $W^2 = 2.09 \text{ GeV}^2$ - see Eq.(21) and for $W^2 = 1.0 \text{ GeV}^2$ - see Eq.(22) are shown in Fig.3. One can see that the unphysical solution exhibits a more stable behavior then a physical one.

In Fig. 3 we show the dependence of $m, \lambda^2$ and $W^2$ on the value of $f_b$. 

7
Note that if there is an unphysical solution which corresponds to the absolute minimum of the functional $\chi^2(m, \lambda^2, W^2)$ defined by Eq.(6), the search for another solution and its interpretation becomes more complicated. It is rather simple if both minima are sharp. However, if the unphysical solution corresponds to a wide minimum, while the second solution corresponds to a shallow one, interpretation of the latter solution requires additional analysis.

Now we come to variation of the values of the quark condensates. Let us first modify the value of the condensate $\langle 0|\bar{q}q|0 \rangle$, keeping the other ones to be unchanged. Hence we put $f_{4q} = f_{6q} = f_b = f_{qg} = 1$ in Eq. (14), changing the value of $f_{2q}$. The dependence of the nucleon mass on $f_{2q}$, somewhat “partial derivative” with respect to $\langle 0|\bar{q}q|0 \rangle$ is shown in Fig. 4. The physical solution exist only if $f_{2q} > 0.99$.

We investigate dependence on the parameters $f_{4q}$ and $f_{6q}$ in the same way. The physical solution disappears if $f_{4q}$ exceeds slightly the value $f_{4q} = 1$ corresponding to the factorization hypothesis. There is no minimum corresponding to a physical solution for $f_{4q} > 1.01$ – Fig. 5. On the contrary, the physical solution is not consistent with small values of $f_{6q}$. It requires $f_{6q} > 0.96$ – Fig. 6.

It is instructive also to investigate the dependence of the nucleon mass on the quark scalar condensate $\langle 0|\bar{q}q|0 \rangle$ assuming the factorization hypothesis. In this case $f_{4q} = f_{6q} = f_{2q}$ and $f_{qg} = f_{3q}$ while $f_b = 1$ in Eq. (14). The result is shown in Fig. 7. There is no minimum corresponding to a physical solution since it vanishes for $f_{2q} > 1.35$.

3 Inclusion of the radiative corrections of the order $\alpha_s$

Here we include the corrections of the order $\alpha_s$ and $\alpha_s \ln q^2$ in the lowest order of perturbation theory. This modifies the contributions to $\Pi_i(q^2)$. For the contributions of the free quark loop $A_0$ and for those of the scalar quark condensate $B_3$ and the four-quark condensate $A_6$ we have now [27, 28]

$$A_0 = -\frac{1}{64\pi^4} Q^4 \ln \frac{Q^2}{\mu^2} \left( 1 + \frac{71}{12} \frac{\alpha_s}{\pi} - \frac{13}{2} \frac{\alpha_s}{\pi} \ln \frac{Q^2}{\mu^2} \right),$$

$$A_6 = \frac{2}{3} \langle 0|\bar{q}q|0 \rangle^2 \frac{Q^2}{Q^2} \left( 1 - \frac{5}{6} \frac{\alpha_s}{\pi} - \frac{1}{3} \frac{\alpha_s}{\pi} \ln \frac{Q^2}{\mu^2} \right),$$

$$B_3 = -\frac{\langle 0|\bar{q}q|0 \rangle}{4\pi^2} Q^2 \ln \frac{Q^2}{\mu^2} \left( 1 + \frac{3}{2} \frac{\alpha_s}{\pi} \right),$$

with $Q^2 = -q^2$. Corrections to the other OPE terms are not included because of the large uncertainties of the values of the condensate.

The numerically large coefficient of the correction of the order $\alpha_s$ to the term $A_0$ caused doubts in convergence of OPE series [29]. In [28] Eqs. (23) and (24) were used for determination of nucleon parameters in framework of the finite energy sum rules. Inclusion of the radiative correction was shown to diminish the value of the nucleon mass, assuming that the threshold value $W^2$ does not change. Similar result was obtained in [30] in framework of the Borel
transformed sum rules. However the authors of [22] demonstrated that the radiative corrections modify mostly the values of $\lambda^2$ and $W^2$, without important influence on the value of nucleon mass.

For the contributions to the Borel transformed sum rules we can write [22]

$$
\tilde{A}_0(M^2, W^2) = M^6 E_2 \left[ 1 + \frac{\alpha_s}{\pi} \left( \frac{53}{13} - \ln \frac{W^2}{\mu^2} \right) \right] - \frac{\alpha_s}{\pi} \left[ M^4 W^2 \left( 1 + \frac{3W^2}{4M^2} e^{-W^2/M^2} + M^6 \mathcal{E} \left( -\frac{W^2}{M^2} \right) \right) \right],
$$

$$
\tilde{A}_6(M^2, W^2) = \frac{4}{3} a^2 \left[ 1 - \frac{\alpha_s}{\pi} \left( \frac{5}{6} + \frac{1}{3} \left( \ln \frac{W^2}{\mu^2} + \mathcal{E} \left( -\frac{W^2}{\mu^2} \right) \right) \right) \right],
$$

$$
\tilde{B}_3(M^2, W^2) = 2a M^4 E_1 \left( 1 + \frac{3}{2} \frac{\alpha_s}{\pi} \right),
$$

with $\mathcal{E}(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^{\alpha}}$.

It was shown in [22] that the results do not change much if we put $\alpha_s(M^2) = \alpha_s(1\text{GeV}^2)$. As we said above, we assume $\alpha_s(1\text{GeV}^2) = 0.37$, which is consistent with the recent data presented in Fig.9.2 of [26]. A somewhat larger value is given in [21]. Anyway, the nucleon parameters depend weakly on the actual value of $\alpha_s$—see Fig.2 of [22]. Hence the same refers to the present analysis.

The physical solution is now

$$
m = 0.94 \text{ GeV}, \quad \lambda^2 = 2.00 \text{ GeV}^6, \quad W^2 = 1.90 \text{ GeV}^2.
$$

The unphysical solution is

$$
m = 0.60 \text{ GeV}, \quad \lambda^2 = 0.56 \text{ GeV}^6, \quad W^2 = 0.70 \text{ GeV}^2,
$$

with the contribution of the pole 3 times smaller than that of the continuum. Hence the radiative corrections made the unphysical solution even more "unphysical". Dependence of the nucleon mass on the values of condensates is shown in Figs. 2a, 4–7. One can see that the physical solution exists in the larger interval of the values of $f_i$ than in the case when only anomalous dimensions are included. The physical solution (described by Eq.(26) for $f_b = f_{4q} = f_{6q} = 1$) provides absolute minimum of the function $\chi^2$ for $f_b < 1.8$, $f_{4q} < 1.35$ and $f_{6q} > 0.2$—see Figs. 2, 5–7. Dependence on the value $f_{2q}$ without and with factorization assumptions is shown in Figs. 4 and 7 correspondingly.

Note also that the variation of the limits of the Borel window changes the values of nucleon parameters only by several percent. For example, solving Eq.(5) in the intervals $0.7 \text{ GeV}^2 < M^2 < 1.8 \text{ GeV}^2$ we find $m = 0.946 \text{GeV}$, while in the interval $1.0 \text{ GeV}^2 < M^2 < 1.6 \text{ GeV}^2$ we obtain $m = 0.953 \text{ GeV}$ for $f_b = f_{2q} = f_{4q} = 1$. There is also an unphysical solution, but the corresponding values of $\chi^2$ are at least ten times larger than for the physical one.
4 Summary

We analyzed dependence of the nucleon mass on the values of QCD condensates in framework of the QCD sum rules. We investigated also the dependence of the residue of the nucleon pole and of the continuum threshold in “pole + continuum” model of the hadron spectrum. These dependences were studied with inclusion of radiative corrections of the order $\alpha_s$ and $\alpha_s \ln q^2$.

We presented three sets of the results for minimization of $\chi^2$ defined by Eq.(6) in the Borel window defined by Eq.(9). They correspond to total neglect of the radiative corrections, to inclusion of the corrections $(\alpha_s \ln q^2)^n$ in all orders and to perturbative inclusion of corrections $\sim \alpha_s$ and $\alpha_s \ln q^2$.

It is shown that even at relatively small deviations of QCD condensates from the standard values the QCD sum rules have an unphysical solution with the contribution of continuum exceeding several times that of the nucleon pole. This contradicts the idea of the “pole + continuum” model for the hadron spectrum.

We showed that (neglecting the radiative corrections) at $f_b < 1$ there is an interplay of the physical and unphysical solutions. In this case the two solutions can be separated. If both unphysical and physical solutions are connected with sharp minima of the dependence of the function $\chi^2$ on nucleon parameters, they can be separated easily. However in the general case, for example, for a wide minimum corresponding to the unphysical solution or for a shallow minimum of another solution the interpretation of the latter one may be obscure.

The strongest limits on the values of condensates corresponding to a physical solution emerge if one includes the radiative corrections $(\alpha_s \ln q^2)^n$, the limits are weaker if the radiative corrections are totally neglected and still weaker if the corrections are included perturbatively – Figs. 1,3–7. Thus for consistent calculations it is reasonable either to include the radiative corrections perturbatively or just to ignore them.

We demonstrated that for the physical solution the value of the nucleon mass is less sensitive to the exact values of the condensates than those of the residue $\lambda^2$ and of the continuum threshold $W^2$. In other words, the uncertainties in the value of condensates influence mostly the magnitudes of $\lambda^2$ and $W^2$.

We found that the nucleon mass depends mostly on the expectation value of the scalar quark operator $\bar{q}q$. Dependence on the condensates of higher dimensions is much weaker. The dependence on the condensate $\langle 0|\bar{q}q|0 \rangle$ becomes weaker if one assumes factorization hypothesis for the four-quark and six-quark condensates.

In the QCD sum rules the nucleon mass obtained a nonzero value due to exchange by noninteracting quark–antiquark pairs between the nucleon current and vacuum. As we have seen perturbative inclusion of interactions taking place at the distances of the order $1/M \sim 1 \text{ GeV}^{-1}$, corresponding to inclusion of the radiative corrections $\sim \alpha_s$ make the solution more stable. The physical solution exists in a broad interval of the values of $\langle 0|\bar{q}q|0 \rangle$ if the radiative corrections are included perturbatively. On the other hand the unphysical solution becomes ”more unphysical”, with a stronger dominance of the contribution of continuum over that of the pole. This may be important for the QCD sum rules analysis of the many-quark systems.
carried out nowadays [24, 25].

We thank B. L. Ioffe for useful comments. We acknowledge the partial support by the RSGSS grant 3628.2008.2.
References

[1] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B 147, 385 (1979).
[2] B. L. Ioffe, Nucl. Phys. B 188, 317 (1981); E 191, 591 (1981).
[3] Y. Chung, H. G. Dosch, M. Kremer and D. Schall, Phys. Lett. B 102, 175 (1981).
[4] Y. Chung, H. G. Dosch, M. Kremer and D. Schall, Nucl. Phys. B 197, 55 (1982).
[5] B. L. Ioffe and A. V. Smilga, Nucl. Phys. B 232, 109 (1984).
[6] H. G. Dosch, M. Jamin and S. Narison, Phys. Lett B 220, 251 (1989).
[7] E. Bagan, M. Chabab, H G. Dosch and S. Narison, Phys. Lett B. 301, 243 (1993).
[8] E. Bagan, H G. Dosch, P. Gosdzinsky, S. Narison and J. M. Richard, Z. Phys. C. 64, 57 (1994).
[9] B. L. Ioffe, ArXiv hep-ph: 0810.4234.
[10] C. Adami and I. Zahed, Phys. Rev. D 45, 4312 (1992).
[11] E. G. Drukarev, M. G. Ryskin, V. A. Sadovnikova, Prog. Part. Nucl. Phys. 47, 73 (2001).
[12] K. G. Wilson, Phys. Rev. 179, 1499 (1969).
[13] D. Espiru, P. Pascual and R. Tarrach, Nucl. Phys. B 214, 285 (1983).
[14] B. L. Ioffe, Z. Phys. C 18, 67 (1983).
[15] L. J. Reinders, H. Rubinstein, and S. Yazaki, Phys. Rep.127, 1 (1987).
[16] M. E. Peskin, Phys. Lett. B 88, 126 (1979).
[17] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).
[18] A. I. Vainshtein, V. I. Zakharov and M. A. Shifman, Sov. Phys. JETP Letters, 27, 50 (1978).
[19] B. L. Ioffe and K. N. Zyablyuk, Eur. Phys. J. C 27, 229 (2003); K. N. Zyablyuk, JHEP 0301, 081 (2003); A. V. Samsonov, hep-ph/047199.
[20] R. A. Bertlmann, C. A. Dominguez, M. Loewe, M. Perrotet and E. de Rafael, Z. Phys. C. 39, 231 (1988), B. V. Geshkenbein, Phys. Atom. Nucl. 59, 289 (1996).
[21] B. L. Ioffe, Prog. Part. Nucl. Phys. 56, 232 (2006).
[22] V. A. Sadovnikova, E. G. Drukarev, and M. G. Ryskin, Phys. Rev. D 72, 114015 (2005).
[23] V. A. Sadovnikova, E. G. Drukarev, and M. G. Ryskin, Phys. At. Nucl. 71, 1431 (2008); Yad. Fiz. 71, 1459 (2008).

[24] R. D. Matheus, F. S. Navarra and M. Nielsen, Braz. J. Phys. 36, 1397 (2006).

[25] R. D. Matheus, F. S. Navarra, M. Nielsen and R. Rodrigues da Silva, Phys. Rev. D 76, 056005 (2007).

[26] Particle Data Group, Phys. Lett. B, 667, 1 (2008).

[27] M. Jamin, Z. Phys. C 37, 635 (1988).

[28] A. A. Ovchinnikov, A. A. Pivovarov, and L. R. Surguladze, Int. J. Mod. Phys. A 6, 2025 (1991).

[29] D. B. Leinweber, Ann. Phys. (NY) 254, 328 (1997).

[30] H. Shiomi and T. Hatsuda, Nucl. Phys. A 594, 294 (1995).
5 Figure captions

Fig.1 Dependence of the nucleon mass and of $\chi^2$ per degree of freedom (assuming 1% error bar) on value of the gluon condensate $f_b$. The solid lines correspond to the physical solution. The dashed lines correspond to the unphysical solution. For convenience the values of $\chi^2$ are reduced by the factor of 5, $m_0 = 0.93$ GeV$^2$ is the value of $m$ corresponding to physical solution for $f_b = 1$. For $f_b > 1.04$ only the latter one exists.

Fig.2 Behavior of the function $m(M^2)$ for the physical (solid line) and unphysical (dashed line) in the case $f_b = 0.6$.

Fig.3 Dependence of the nucleon parameters on gluon condensate. Figs. a,b,c show the mass $m$, residue $\lambda^2$ and threshold $W^2$ correspondingly. Dashed curves show the case with all the radiative corrections neglected, dotted curves correspond to inclusion of the anomalous dimensions. Solid curves are for perturbative inclusion of the corrections $\sim \alpha_s$. The horizontal dashes denote the transitions from the physical solutions to unphysical ones.

Fig.4 Dependence of the nucleon mass on the parameter $f_{2q}$. Only the term $\tilde{B}_3$ in Eqs. (11) and (25) is modified. The notations are the same as in Fig. 3.

Fig.5 Dependence of the nucleon mass on the parameter $f_{4q}$. Only the term $\tilde{A}_6$ in Eqs. (11) and (25) is modified. The notation are the same as in Fig. 3.

Fig.6 Dependence of the nucleon mass on the parameter $f_{6q}$. Only the term $\tilde{B}_9$ in Eq. (11) is modified. The notations are the same as in Fig. 3.

Fig.7 Dependence of the nucleon mass on the parameter $f_{2q}$ under the factorization hypothesis $f_{4q} = f_{2q}^2$, $f_{6q} = f_{2q}^3$. The notations are the same as in Fig. 3.
Figure 1:

Figure 2:
Figure 3:
Figure 4:

Figure 5:
Figure 6:

Figure 7: