MODELLING SOLUTIONS TO THE KDV-BURGERS EQUATION IN THE CASE OF NON-HOMOGENEOUS DISSIPATIVE MEDIA

ALEXEY SAMOKHIN

Abstract. We study the behavior of the soliton which, while moving in non-dissipative medium encounters a barrier with finite dissipation. The modelling included the case of a finite dissipative layer similar to a wave passing through the air-glass-air as well as a wave passing from a non-dissipative layer into a dissipative one (similar to the passage of light from air to water).

The dissipation predictably results in reducing the soliton amplitude/velocity, but some new effects occur in the case of finite barrier on the soliton path: after the wave leaves the dissipative barrier it retains a soliton form, yet a reflection wave arises as small and quasi-harmonic oscillations (a breather). The breather spreads faster than the soliton as moves through the barrier.

Keywords: KdV- Burgers, non-homogeneous layered media, soliton, reflection, refraction.

1. Introduction

The behavior of solutions to the KdV - Burgers equation is a subject of various recent research, [2]–[3]; the present paper is a continuation of [1]. The aim is to study the behavior of the soliton that, while moving in nondissipative medium encounters a barrier (finite or infinite) with finite dissipation. The modelling included the case of a finite dissipative layer similar to a wave passing through the air→glass→air as well as a wave passing from a nondissipative layer into a dissipative one (similar to the passage of light from air to water).

New results include a numerical model of the wave’s behavior for different types of the media non-homogeneity. The dissipation predictably results in reducing the soliton amplitude, but some new effects occur in the case of finite barrier on the soliton path: after the wave leaves the dissipative barrier it retains, on the whole, a soliton form yet a reflection wave arises as small and quasi-harmonic oscillations (a breather). This breather spreads as the soliton is moving through the barrier, and the breather moves faster than the soliton in the opposite direction.

For the modelling we used the Maple PDETools packet.

The generalized KdV-Burgers equation considered here is of the form

\[ u_t(x,t) = \gamma(x)u_{xx}(x,t) - 2u(x,t)u_x(x,t) + u_{xxx}(x,t). \]  

(1)
It is the simplest model for the medium which is both viscous and dispersive. The viscosity dampens oscillations except for stationary (or travelling wave) solutions.

Note that $\gamma(x) \equiv 0$ corresponds to the KdV equations whose travelling waves solutions are solitons and $\gamma(x) \equiv \text{const} > 0$ corresponds to the KdV-Burgers equation whose travelling waves solutions are shock waves. Tentatively for $\gamma(x) \neq 0$ the situation is similar to the geometric optics: as a ray enters water from the air, one can observe the reflected wave and a decay of the transient wave.

We consider the layered media which consist of layers with both dispersion an dissipation and layers without dissipation. In the latter case the waves are described by the KdV equation, while in the former — by the KdV-Burgers one. A soliton solution of the KdV equation, meeting a layer with dissipation, transforms somewhat similarly to a ray of light in the air crossing a semi-transparent plate. Thus we consider three possibilities for $\gamma(x)$.

1. $\gamma(x) = \alpha(1 - \theta(x))$ is the Heaviside step function (the two-layer case);
2. $\gamma(x) = \alpha(\theta(x - \beta) - \theta(x + \beta))$ is a II-form density of viscosity (the three-layer case);
3. $\gamma(x) = \alpha \cosh^{-2} 2(\beta x)$ is a function with (numerically) compact support (the three-layer case).

We use the following initial value — boundary problem (IVBP) for the KdV-Burgers equation on $\mathbb{R}$:

$$u(x, 0) = 6a^2 \cosh(a(x + s)), \; u(\pm \infty, t) = 0, \; u_x(\pm \infty, t) = 0.$$  

For numerical computations we use $x \in [a, b]$ for appropriately large $a$, $b$ instead of $\mathbb{R}$.

2. Soliton in 2-layer medium

This case models a passage from non-dissipative half-space to a dissipative one. We take $\gamma(x) = 0.5(1 - \theta(x))$ as a dissipation distribution function to present a single boundary separating these half-spaces.

Expect each solution to behave as the one of the KdV at the right half-space and as solution of KdV-B at the left one.

The process of transition from the soliton solution of the KdV to the correspondent solution of the KdV-B proceeds as expected. The transient wave in a dissipative media becomes a solitary shock which loses speed and decays to become nonexistent at $t \to +\infty$; and a reflected wave is seen in the non-dissipative half-space. It is similar to the passage of light from air into water; see figures 1–3.
Figure 1. **Left:** Soliton $u(x, t, a, s) = 6a^2 \cosh^{-2}(4a^3 t + a(x + s))$. Velocity $-4a^2 = -4$, initial shift $s = -6$.

**Right:** Step-like obstacle $\gamma(x) = \alpha(1 - \theta(x))$, $\alpha = 6$

Figure 2. Soliton ($a = 0.5$) passing a step-like obstacle ($\alpha = 6$). **Left:** $t = 6$; **Right:** $t = 8$

Figure 3. **Left:** Soliton ($a = 0.5$) passing a step-like obstacle ($\alpha = 6$), $t = 10$.

**Right:** Soliton ($a = 1$) passing a step-like obstacle ($\alpha = 6$), $t = 3.6$

3. **Π - TYPE OBSTACLE**

This case models a passage from non-dissipative half-space to another one passing through a dissipative layer. We take $\gamma(x) = \alpha(\theta(x -$
$\beta - \theta(x + \beta)$ as a \Pi-form density of viscosity (the three-layer case) distribution function to present the layer separating these half-spaces.

Our experiments show that the initial soliton behaves as the one of the KdV at the right half-space and as a diminished soliton at the left one.

The process of transition is natural enough. The transient wave in a dissipative media looses energy and speed to become a lesser and slower soliton at the left non-dissipative half-space; and a reflected wave is seen at the right half-space as it is shown at figures 4–6.

**Figure 4.** Left: \Pi - type viscose layer; $\alpha = 1$, $\beta = 2$. Right: Soliton ($a = 1$) passing a thin viscose \Pi-type layer ($\alpha = 2.5$, $\beta = 2$); $t = 3$

**Figure 5.** Soliton ($a = 1$) passing a thin viscose \Pi-type layer ($\alpha = 2.5$, $\beta = 2$). Left: $t = 5$ Right: $t = 10$

The size of the reflected breather is connected, in particular, to the properties of the barrier $\alpha$ and $\beta$. So it may be of a practical use: for instance, measuring it one can judge whether the layer connecting two details is uniform at different points.
Since the refraction coefficient $k = \frac{V_1}{V_2} = \frac{4a_1^2}{4a_2^2}$ equals the ratio of amplitudes $\frac{6a_1^2}{6a_2^2}$, on figure 6 (right) we see that $k \approx \frac{6}{0.95} \approx 6.3$.

4. **Soliton-like obstacle**

Since the barrier $\gamma(x) = \delta \cosh^{-2}(\delta x)$ has a numerically compact support, the transition process differs only slightly from that for the Π-type obstacle as it is illustrated by figures 7–8.

5. **Other Travelling Wave Solutions**

The KdV equation (for $\gamma = 0$) possesses the travelling wave solutions (TWS) of the form $-6a^2 \tanh(a(Vt + x) + s)^2 + 4a^2 + (1/2)V$.

The soliton proper is such a TWS that tends to zero at $t \to \pm\infty$; thus $V = 4a^2$; and it moves to the left.
Figure 8. **Left**: Soliton \( a = 0.5 \) passing a thin viscose layer \( \gamma(x) = 3 \cosh^{-2}(3x) \), \( t = 1.5 \) (black), \( t = 12 \) (green).

**Right**: Soliton \( a = 1 \) passing a thin viscose layer \( 3 \cosh^{-2}(3x) \), \( t = 20 \). The refraction coefficient \( k \approx \frac{6}{43} \approx 1.4 \).

Take \( V = -1., s = 3.75, a = 0.25 \). The corresponding graph is presented on figure 9 left. The barrier is \( \gamma(x) = 3 \cosh^{-2}(3x) \).

In is case both TWS and the breather move to the right and the breather moves ahead of the TWS; the gap between the TWS and the preceding breather is widening over the time, see figures 9–10.

Figure 9. **Left**: TWS: \( V = -1., s = 3.75, a = 0.25 \), moving to the right, at \( t = 0 \).

**Right**: TWS at \( t = 550; V = -1., s = 3.75, a = 0.25 \)

6. **Conclusion and numeric considerations**

The results may be of a practical use. For once, the form of the reflected wave may be used to estimate the thickness and/or the density of the viscous barrier. A refraction may also be predicted.
The figures in this paper were generated numerically using Maple PDETools package. The mode of operation uses the default Euler method, which is a centered implicit scheme, and can be used to find solutions to PDEs that are first order in time, and arbitrary order in space, with no mixed partial derivatives. This implicit scheme is unconditionally stable for many problems (though this may need to be checked).

Yet note that shocks and points of derivative’s discontinuity is intrinsic for the model considered. Thus the standard procedures used with the default parameters may easily lose stability, which leads to multi-oscillations and a general loss of precision. This problem was dealt with mainly by adapting the spacestep and/or the timestep parameters.

Qualitative estimations of the refraction coefficient, based on the relative decay of the KdV selected conservation laws will be published elsewhere.

REFERENCES

[1] A. Samokhin, On nonlinear superposition of the KdV-Burgers shock waves and the behavior of solitons in a layered medium. // Journal of Differential Geometry and its Applications. In press: available online 9 March 2017. 10p. [http://dx.doi.org/10.1016/j.difgeo.2017.03.001]

[2] R.L. Pego, P. Smereka, M.I. Weinstein. Oscillatory instability of traveling waves for a KdV-Burgers equation // Physica D. 67 (1993), p. 45–65.

[3] A.P. Chugainova, V.A. Shargatov. Stability of non-stationary solutions of a generalized Korteweg-de Vries-Burgers equation// Comp. Math. and Math. Physics. Vol. 55(2), (2015), 253-266. (in Russian)

Dept. of Math., Moscow State Technical University of Civil Aviation

E-mail address: samohinalexey@gmail.com