Effect of joule heating and MHD on peristaltic blood flow of Eyring–Powell nanofluid in a non-uniform channel

S. K. Asha and G. Sunitha

Department of Mathematics, Karnataka University, Dharwad, India

ABSTRACT
This paper examines the effect of joule heating and MHD on peristaltic blood flow of a Eyring–Powell nanofluid in a non-uniform channel. The transport equations involve the combined effects of Brownian and Thermophoresis diffusion of nanoparticles. The mathematical modelling is carried out by utilizing long wavelength and low Reynolds number assumptions. The energy equation is modelled by taking joule heating effect. The resulting nonlinear system of partial differential equations is solved for the stream function, velocity, pressure gradient, concentration and temperature distributions with the help of the Homotopy Analysis Method. The effect of various physical parameters on the flow characteristics is shown and discussed with the help of graphs. Pressure gradient gives opposite behaviour with increasing values of Eyring–Powell fluid parameters $A$ and $B$. Pressure gradient decreases by increasing Hartman number and joule heating. Nanoparticle concentration increases with increasing thermophoresis parameter, but the reverse trend is observed with the effect of Brownian motion parameter.

1. Introduction
The peristaltic transport of Eyring–Powell nanofluid in a non-uniform channel is an important problem in engineering, industrial and biological applications. It is useful for the description of the progressive wave contraction along a channel or tube. In particular, the mechanism involved in swallowing food through esophagus, urine transport from kidney to bladder through the ureter, movement of chime in the gastrointestinal tract, vasomotion of small blood vessels and the locomotion of some warms, etc. Peristaltic pump is also used in roller and finger pump. Such devices are used to pump the blood, in bypass surgery it is used to circulate the blood in heart lung machine, slurries is to prevent the transport of fluid from coming in contact with mechanical part of the pump. The word peristaltic comes from the Greek word as “peristaltikos” which means clutching and compressing. Latham [1] was the first who investigated the fluid motion in peristaltic pump. After his first investigation, a large number of researchers and scientist focus their attentions to study the peristaltic flows considering Newtonian and non-Newtonian fluids [2–7]. Bhatti et al. [8] have studied the study of variable magnetic field on the peristaltic...
flow of Jeffrey fluid in a non-uniform rectangular duct having compliant walls. Further, nanofluid is a fluid containing nanometer-sized particles called nanoparticle. The nanoparticles used in nanofluids are typically made of metals (Al, Cu), oxides (ceramics, Al₂O₃, CuO); nonmetal like graphite and carbon nanotubes, nanofibres, droplet, nanosheet, etc. Chio [9] was the first to introduce the word nanofluid that represents the fluid in which nanoscale particles (diameter less than 100 nm) are suspended in the base fluid. Recent articles on the nanofluid are cited in [10–13].

Powell and Eyring proposed a new fluid model in 1944 known as Eyring–Powell fluid model [14]. Viscoclastic fluids in physiology and industry have a prominent place. The Eyring–Powell nanofluid is one such model which is advantageous to recover accurate results of viscous nanofluid at low and high shear rates. At low shear rates less than 100 s⁻¹, the best example of Eyring–Powell nanofluid is human blood. Possibly the complex mathematical description of this model in flexible curved channel averts the study of nanoparticles subject to Eyring–Powell fluid [15]. Further investigation of mixed convection peristaltic flow of Eyring–Powell nanofluid with magnetic field in a non-uniform channel by Asha and Sunitha [16]. Nooren and Qasim [17] studied the peristaltic flow of MHD Eyring–Powell fluid in a channel. Recent articles on the peristaltic transport of Eyring–Powell fluid are cited in [18–21]. Peristalsis in connection with nanofluids has application in biomedicines, i.e. cancer treatment, radiotherapy, etc.

Peristaltic blood flow has gained considerable attention because of its major importance in physiological. The velocity of blood flow in the human body is different depending on what type of vessels it is flowing through. Velocity is indirectly related to the cross-sectional area of the vascular bed, so when the cross-sectional area is high (i.e. in capillaries) then the speed of blood flow decreases dramatically. Blood flow follows the laws of hydrodynamics where speed is proportional to pressure difference, fluidity, radius of the tube. When blood flows from heart to various organs it passes through aorta to artery to arterioles to capillaries. Capillaries have very small diameter but as arteriole is divided into many capillaries, the overall diameter increases, so blood volume decreases. So through flow of blood from aorta to capillaries the speed of flow gradually decreases and flow gradually become streamline. Recently, peristaltic flow with magnetic particles grabbed the attention of several researchers due to its emerging applications in magnetic drug targeting [22], pumping of blood, reduction of bleeding during surgery, continue casting process, hyperthermia, magnetic resonance imaging (MRI) and Magneto-therapy, etc. Moreover, in biomedical engineering, ferrofluid is beneficial for an experimental treatment of cancer disease (known as targeted magnetic hyperthermia). Furthermore, magneto-hydrodynamic (MHD) flow and heat transfer towards moving or fixed flat surface have become a vibrant problem in the current field and huge applications in the field of engineering problems, for instance, petroleum engineering, plasma studies, geothermal energy abstractions, aerodynamics and much more. Bhatti et al. [23] have studied the Combine effects of Magnetohydrodynamics (MHD) and partial slip on peristaltic Blood flow of Ree-Eyring fluid with wall properties. Particularly, to regulate the behaviour of boundary layer, numerous artificial approaches have been established in references [24–26]. Joule heating effect cannot be ignored when strong magnetic field is applied. There are many practical uses of joule heating such as conductors in electronics, electric stoves and other electric heaters, power lines, fuses. The study of heat and mass transfer with joule heating on MHD peristaltic blood flow under the influence of hall effect was given by Bhatti et al. [27]. Recent attempts on mixed convection peristaltic flow with joule effect can be seen in [28,29].

As far the best of author’s knowledge, the effect of joule heating on MHD peristaltic blood flow of Eyring–Powell nanofluid in a non-uniform channel has not been studied yet. However, Anum et al. [12] studied the mixed convection peristaltic flow of Eyring–Powell nanofluid in a curved channel with compliant walls, but did not consider the effect of joule heating. The prime object of the present attempt is to explore the ideas of MHD and joule heating on the peristaltic blood flow by considering the Eyring–Powell nanofluid. The present study has numerous applications in biomedical engineering. In most medical therapies, MHD with joule heating plays an important role to modulate the blood and to reduce the pain of human body (like in laparoscopic treatment). This paper is organized in the following fashion. In Section 1, the mathematical formulation of the problem is constructed by employing the wave frame and then reduced by assumptions of long wavelength and low Reynolds number approximation. In Section 2, solutions for velocity, temperature and concentration are obtained by using the Homotopy Analysis Method (HAM) [30,31]. This method is a general approximate analytic solution, used to obtain series solutions of linear/nonlinear equations. This method is valid for the nonlinear problem, contains small or large physical parameters. The method provides great freedom in choosing initial approximation and auxiliary linear operators. By means of choosing such approximation and operators, any complicated nonlinear problems can be solved by transforming into linear subproblems. In the last section, the effects of various physical parameters on pressure gradient, frictional force, concentration and temperature are analysed through graphs.
2. Mathematical formulation

Let us consider the peristaltic blood flow of Eyring–Powell nanofluid through a two-dimensional non-uniform channel with the sinusoidal wave propagating towards down its walls. The nanofluid is electrically conducting by an external magnetic field \( \mathbf{B}_0 \) and the effect of joule heating. Here we consider the Cartesian coordinate system \((\tilde{x}, \tilde{y})\) such that \( \tilde{x} \)-axis is considered along the centre line in the direction of wave propagation and \( \tilde{y} \)-axis is transverse. The geometry of the wall surface can be written as

\[
\tilde{h}(\tilde{x}, \tilde{t}) = a(\tilde{x}) + b \sin \left( \frac{2\pi}{\lambda} (\tilde{x} - c\tilde{t}) \right). \tag{1}
\]

Here \( a(\tilde{x}) = a_{20} + k\tilde{x} \) is the channel half-width, \( \lambda \) is the wavelength, \( \tilde{t} \) is the time and \( b \) represents the wave amplitude. The velocity components \( \tilde{U} \) and \( \tilde{V} \) along the \( \tilde{x} \) and \( \tilde{y} \) directions, respectively, in the fixed frame, the velocity field \( \tilde{V} \) is taken as

\[
\tilde{V} = [\tilde{U}(\tilde{x}, \tilde{y}, \tilde{t}), \tilde{V}(\tilde{x}, \tilde{y}, \tilde{t}), 0]. \tag{2}
\]

The shear of non-Newtonian fluid is considered to the study of Eyring–Powell fluid model. The stress tensor of Eyring–Powell fluid model is given by

\[
\tilde{\sigma} = \mu \nabla \tilde{V} + \frac{1}{\beta} \sinh^{-1} \left( \frac{1}{c^*} \nabla^2 \tilde{V} \right), \tag{3}
\]

where \( \mu \) is the coefficient of shear viscosity, \( \beta \) and \( c^* \) are the material constant of Eyring–Powell fluid. The term \( \frac{1}{\beta} \sinh^{-1} \) is approximated using the second-order approximation of the hyperbolic sine function as follows:

\[
\sinh^{-1} \left( \frac{1}{c^*} \nabla^2 \tilde{V} \right) \approx \frac{1}{c^*} \left( 1 - \frac{1}{6} \left( \frac{1}{c^*} \nabla^2 \tilde{V} \right)^2 \right), \quad \left| \frac{1}{c^*} \nabla^2 \tilde{V} \right| \ll 1. \tag{4}
\]

The governing equations for the Eyring–Powell nanofluid can be written as follows [17]:

The continuity equation:

\[
\rho \left( \frac{\partial \tilde{U}}{\partial t} + \tilde{U} \frac{\partial \tilde{U}}{\partial \tilde{x}} + \tilde{V} \frac{\partial \tilde{U}}{\partial \tilde{y}} \right) = 0. \tag{5}
\]

The momentum equation:

\[
\rho \left( \frac{\partial \tilde{U}}{\partial t} + \tilde{U} \frac{\partial \tilde{U}}{\partial \tilde{x}} + \tilde{V} \frac{\partial \tilde{U}}{\partial \tilde{y}} \right) = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \left( \mu + \frac{1}{\beta c^*} \right) \left( \frac{\partial^2 \tilde{U}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{U}}{\partial \tilde{y}^2} \right) - \sigma B_0^2 \tilde{U} - \frac{1}{2\beta c^*} \left( \frac{\partial \tilde{U}}{\partial \tilde{x}} \right)^2 + \left( \frac{\partial \tilde{U}}{\partial \tilde{y}} \right)^2 + 2 \frac{\partial \tilde{U}}{\partial \tilde{x}} \frac{\partial \tilde{U}}{\partial \tilde{y}} \right) \times \left( \frac{\partial^2 \tilde{U}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{U}}{\partial \tilde{y}^2} \right) + \rho f g_k (\tilde{T} - \tilde{T}_0) + \rho f g_k (\tilde{C} - \tilde{C}_0), \tag{6}
\]

The energy equation:

\[
(\rho c)_H \frac{\partial \tilde{C}}{\partial t} + \tilde{U} \frac{\partial \tilde{C}}{\partial \tilde{x}} + \tilde{V} \frac{\partial \tilde{C}}{\partial \tilde{y}} = k^* \left( \frac{\partial^2 \tilde{C}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{C}}{\partial \tilde{y}^2} \right) + (\rho c)_p D_B \left( \frac{\partial \tilde{C}}{\partial \tilde{x}} + \frac{\partial \tilde{C}}{\partial \tilde{y}} \right) \times \left( \frac{\partial^2 \tilde{C}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{C}}{\partial \tilde{y}^2} \right) + \Phi + \sigma B_0^2 \tilde{U}^2 \tag{7}
\]

The concentration equation:

\[
\frac{\partial \tilde{C}}{\partial t} + \tilde{U} \frac{\partial \tilde{C}}{\partial \tilde{x}} + \tilde{V} \frac{\partial \tilde{C}}{\partial \tilde{y}} = D_B \left( \frac{\partial^2 \tilde{C}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{C}}{\partial \tilde{y}^2} \right) + D_T \left( \frac{\partial^2 \tilde{C}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{C}}{\partial \tilde{y}^2} \right), \tag{8}
\]

where \( \tilde{p} \) is the pressure, \( \rho_f \) is the density of the fluid, \( \sigma \) is the electrical conductivity of the fluid, \( B_0 \) is the applied magnetic field, \( \Phi \) is the constant heat addition/absorption, \( g \) is the acceleration due to gravity, \( \kappa \) is the volume expansion coefficient, \( \tilde{T} \) is the temperature of the fluid, \( C \) is the nanoparticle concentration, \( (\rho c)_H \) is the heat capacity of the fluid, \( k^* \) is the thermal conductivity, \( (\rho c)_p \) is the effective heat capacity of the nanoparticle material, \( D_B \) is the Brownian diffusion coefficient, \( D_T \) is thermophoresis diffusion and \( T_m \) is the fluid mean temperature.

The relations between the laboratory and wave frame are introduced through

\[
\begin{align*}
\tilde{x} &= \tilde{x} - c\tilde{t}, \\
\tilde{y} &= \tilde{y}, \\
\tilde{u}(\tilde{x}, \tilde{y}) &= \tilde{U} - c, \\
\tilde{v}(\tilde{x}, \tilde{y}) &= \tilde{V},
\end{align*} \tag{9}
\]

where \( (\tilde{u}, \tilde{v}) \) and \( (\tilde{x}, \tilde{y}) \) indicate the velocity components and coordinates in the wave frame.

Introducing the velocity field in terms of stream functions can be written as \( \tilde{u} = \frac{\partial \tilde{\psi}}{\partial \tilde{y}}, \tilde{v} = -\frac{\partial \tilde{\psi}}{\partial \tilde{x}} \). The corresponding boundary conditions for the above problem is...
given by

$$\dot{\psi} = 0, \quad \ddot{u} = \frac{\partial \ddot{\psi}}{\partial y} = 0, \quad \ddot{T} = \ddot{T}_0, \quad \dot{C} = \dot{C}_0 \quad \text{at} \ y = 0,$$

$$\ddot{\psi} = q, \quad \ddot{u} = \frac{\partial \ddot{\psi}}{\partial y} = -c, \quad \ddot{T} = \dot{T}_1, \quad \dot{C} = \dot{C}_1 \quad \text{at} \ y = h$$

$$= a(x) + b \sin \frac{2\pi}{\lambda}(x). \quad \text{(11)}$$

Introducing the following non-dimensional quantities:

$$\psi = \frac{\dot{\psi}}{ca}, \quad B = \frac{1}{\mu \beta^*}, \quad A = \frac{Bc^2}{2\sigma^2 c^2}, \quad X = \frac{x}{\lambda}, \quad x = \frac{x}{\lambda},$$

$$Y = \frac{\dot{y}}{a}, \quad y = \frac{\dot{y}}{a}, \quad t = \frac{\dot{t}}{\lambda}, \quad p = \frac{\dot{p}}{ca \mu}, \quad v = \frac{\dot{v}}{c},$$

$$\delta = \frac{a}{\lambda}, \quad u = \frac{\dot{u}}{c}, \quad Re = \frac{2\nu ca}{\mu}, \quad M = \sqrt{\frac{\sigma C}{B^0}},$$

$$\beta^* = \frac{k_p}{(\rho c)_f} \mu = \frac{\mu}{\beta^*} = \frac{q}{ca} \mu = \frac{\rho gka(\ddot{T}_1 - \ddot{T}_0)}{c\mu},$$

$$\frac{Q_r}{\mu} = \frac{(\rho c)_p\dot{D} (\ddot{T}_1 - \ddot{T}_0)}{(\rho c)_f \dot{D} m \mu}, \quad Nt = \frac{(\rho c)_p D (\ddot{T}_1 - \ddot{T}_0)}{(\rho c)_f D m \mu}, \quad Q_0 = \frac{(\ddot{T}_1 - \ddot{T}_0)}{1}$$

$$\frac{E_c}{\mu} = \frac{c^2}{(\rho c)_f (\ddot{T}_1 - \ddot{T}_0)}, \quad Br = \frac{Pr E_c}{\dot{T}_m \mu}, \quad \Omega = \frac{\ddot{C}_0 - \ddot{C}_0}{\ddot{C}_1 - \ddot{C}_0},$$

$$\theta = \frac{\ddot{T}_1 - \ddot{T}_0}{\ddot{T}_1 - \ddot{T}_0}, \quad \Omega = \frac{\ddot{C}_0}{\ddot{C}_1}$$

$$+ \alpha \sin 2\pi x, \quad \alpha = \frac{b}{a_20} \quad \text{(12)}$$

The corresponding dimensionless boundary conditions can be written in the following form:

$$\psi = 0, \quad \frac{\partial^2 \psi}{\partial y^2} = 0, \quad \theta = 0, \quad \Omega = 0 \quad \text{at} \ y = 0, \quad \text{or} \ y = 0,$$

$$\psi = F, \quad \frac{\partial \psi}{\partial y} = -1, \quad \theta = 1, \quad \Omega = 1 \quad \text{at} \ y = h$$

$$= 1 + \frac{\lambda kx}{a_20} + \alpha \sin 2\pi x, \quad \text{where the dimensionless time mean flow rate} \ F \text{is the}$$

wave frame is related to the dimensionless time mean flow rate in the laboratory frame as follows:

$$\Theta = F + 1, \quad F = \int_0^h \frac{\partial \psi}{\partial y} \ dy, \quad \text{here} \ \Theta = \frac{Q}{\mu} \quad \text{and} \ F = \frac{Q}{\mu} \quad \text{are the dimensionless time}$$

mean flow in a fixed and wave form respectively.

3. Method of solution

Analysing the governing Equations (13)–(16) and corresponding boundary conditions (17), it is to find that

the solution is an odd function (13) equation and even functions of (15)–(16). Besides, noticing that Equations (13)–(16) contain the term y, it is convenient to express uy(y), y(θ) and Ω(y) by power polynomials. So, we consider the auxiliary linear operators by using HAM. For HAM solutions, we calculate the initial guesses and auxiliary linear operators in the following form:

$$\psi_0(y) = \frac{y(4Fh^3 - Fy^3 + h^4 - hy^2)}{3h^4}, \quad \text{(19)}$$

$$\Theta_0(y) = \frac{y}{h}, \quad \text{(20)}$$

$$\Omega_0(y) = \frac{y}{h}, \quad \text{(21)}$$
The corresponding auxiliary linear operators are
\[ L_\psi = \frac{\partial^5}{\partial y^5}, \quad L_v = \frac{\partial^2}{\partial y^2}, \quad L_\Omega = \frac{\partial^2}{\partial y^2} \] (22)
with property
\[ L_\psi \left[ C_1 + C_2 y + \frac{C_3 y^2}{2} + \frac{C_4 y^3}{6} + \frac{C_5 y^4}{24} \right] = 0, \]
\[ L_v [C_1 + C_2 y] = 0, \quad L_\Omega [C_1 + C_2 y] = 0, \] (23)
where \( C_1, C_2, C_3, C_4 \) and \( C_5 \) are arbitrary constants, the zeroth-order deformation equations are
\[ (1 - q) L_\psi [\psi(y,q) - \psi_0(y,q)] = q H_\psi h_\psi N[\psi(y,q)], \] (24)
\[ (1 - q) L_v [\theta(y,q) - \theta_0(y,q)] = q H_\theta h_\theta N[\theta(y,q)], \] (25)
\[ (1 - q) L_\Omega [\Omega(y,q) - \Omega_0(y,q)] = q H_\Omega h_\Omega N[\Omega(y,q)], \] (26)
where \( q \in [0,1] \) is the embedding parameter, \( h_\psi, h_\theta \) and \( h_\Omega \) are auxiliary parameters, \( L \) is an auxiliary linear operator, \( H_\psi, H_\theta \) and \( H_\Omega \) are auxiliary functions, \( N \) is a nonlinear operator, \( \psi(y,q), \theta(y,q) \) and \( \Omega(y,q) \) are unknown functions, \( \psi_0(y,q), \theta_0(y,q) \) and \( \Omega_0(y,q) \) are initial approximations of \( \psi(y,q), \theta(y,q) \) and \( \Omega(y,q) \).

From Equations (24)–(26), we can easily find the system of equations with their relevant boundary conditions. According to the methodology of the HAM, the solutions are given by
\[ \theta(y,q) = \theta_0 + h_\theta \theta_1 + h_\theta^2 \theta_2 + \cdots \] (27)
\[ \Omega(y,q) = \Omega_0 + h_\Omega \Omega_1 + h_\Omega^2 \Omega_2 + \cdots \] (28)
\[ \psi(y,q) = \psi_0 + h_\psi \psi_1 + h_\psi^2 \psi_2 + \cdots \] (29)
The solutions for velocity \( (u = \frac{\partial \psi}{\partial y}) \), temperature and concentration are evaluated using Equations (27)–(29) we get
\[ u(y,q) = \left( \frac{1}{3} + \frac{4F}{3h} \right) - \left( \frac{4}{3h^3} + \frac{4F}{3h^2} \right) y^3 \]
\[ - h_\psi^2 a_{11} + h_\psi^3 a_{12} + h_\psi^4 a_{13} + h_\psi^6 a_{14} + h_\theta h_\psi^3 a_{15} + h_\theta h_\Omega^3 a_{16}, \] (30)
\[ \theta(y,q) = \frac{y}{h} + 2h_\theta b_{11} + h_\theta^2 b_{12} + + Pr Nb \frac{h_\theta^2 h_\Omega}{h} \left( 1 + \frac{Nt}{Nb} \right) b_{13} + Pr Nth_\theta^3 b_{14} + h_\theta h_\psi^2 b_{15} + Pr Qa h_\psi \frac{y^2}{2}, \] (31)
\[ \Omega(y,q) = \frac{y}{h} + h_\Omega^2 \left( 1 + \frac{Nt}{Nb} \right) \frac{y}{h} + h_\Omega^3 \left( 1 + \frac{Nt}{Nb} \right) \frac{y}{h} + h_\Omega^4 \left( 1 + \frac{Nt}{Nb} \right) \frac{y}{h} \]
\[ + h_\Omega^5 \frac{y}{h} \left( 1 + \frac{Pr Nb}{h^2} + \frac{Pr Nt}{h^2} + Pr Q_0 \right) \left( \frac{y^2}{2} + 1 \right) \]
\[ \times \left( \frac{y^4}{2} + \frac{Pr Nb}{h^2} \right) \left( \frac{y^3}{3h^3} + \frac{Pr Nt}{h^2} \right) \frac{y^5}{20} \] (32)
The volumetric flow rate in the wave frame is defined by
\[ Q(x,t) = \int_0^h u(y,q) dy. \] (33)
The pressure gradient can be calculated from the volumetric flow rate and after some simplification they can be written as
\[
\frac{\partial p}{\partial x} = \begin{pmatrix}
Q - \left( \frac{1}{3} + \frac{4F}{3h} \right) y & -h_\psi^2 c_{11} - h_\psi^3 c_{12} - h_\psi^4 c_{13} - h_\psi^6 c_{14} \\
-h_\psi^2 c_{13} - h_\psi^3 c_{15} - h_\psi^4 c_{16} & M^2 c_{17} + h_\psi^2 c_{18} + h_\psi^3 c_{19} + h_\psi^4 c_{20} + h_\psi^6 c_{21} \\
-h_\psi^2 (1 + B) h_\psi^2 c_{22} & -h_\psi^2 (1 + B) h_\psi^2 c_{23} + h_\psi^4 c_{24} + h_\psi^6 c_{25}
\end{pmatrix}.
\] (34)
The pressure rise \( \Delta P_L(t) \) and the frictional force \( \Delta F_L(t) \) at the wall in the non-uniform channel of length \( L \) and it can be defined in their non-dimensional form.
\[ \Delta P_L(t) = \int_0^{L/\lambda} \frac{\partial p}{\partial x} dx, \] (35)
\[ \Delta F_L(t) = \int_0^{L/\lambda} \left( -h_\psi \frac{\partial p}{\partial x} \right) dx. \] (36)

4. Results and discussion

This section prepared to describe the results of our problem graphically using MATHEMATICA program. It consists of four parts: the first part illustrates the impact of some physical parameters on the pressure gradient, the second part illustrates the frictional force. However in the third part is the evaluation of temperature distribution with different parameters discussed and the fourth part illustrates the concentration with different parameters.

4.1. Pressure gradient distribution

Peristaltic pumps are used to pump the blood, in bypass surgery it is used to circulate the blood in heart lung machine. Peristaltic pumping mechanism is used to transport various biological fluids in diagnostic problems to analyse the pumping characteristic. Figures 1 and 2 are plotted to show the behaviour
of different parameters on the pressure gradient distribution. Figure 1(a) and (b) are plotted to show the behaviour of fluid parameters $A$ and $B$ on pressure gradient. It is observed that the pressure gradient has the opposite behaviour on these two fluid parameters. The pressure gradient decreases with an increase in the Eyring–Powell fluid parameter $A$, which appears with the nonlinear term of the governing momentum, in the wider part of channel is relatively small $x \in [0, 0.5]$ and $x \in [0.8, 1.0]$ the pressure gradient is relatively small and the flow can easily pass without the imposition of large pressure gradient. However, in the narrow part of the channel $x \in [0.5, 0.8]$ as much larger pressure gradient is required to maintain the same flux to pass through it. Moreover, it is seen that pressure gradient ($> 0$) appears to increase when fluid parameters $B$ is increased. Figure 1(c) and (d) show the effect of Grashof number $Gr$ and local nanoparticle Grashof number $Qr$ on pressure gradient. Here it is observed that the pressure gradient increases as Grashof number $Gr$ and local nanoparticle Grashof number $Qr$ increases. Figure 1(e) shows the effect thermophoresis parameter $Nt$ on pressure gradient. The pressure gradient appears to increase when the thermophoresis parameter $Nt$ increased. Whereas, in Figure 1(f), the pressure gradient decreases when Brownian motion parameter $Nb$ increased.

Figure 2(g) captures the effects of Hartman number $M$ on the pressure gradient. Here it is observed that the pressure gradient is decreasing with the increase of $M$.
which is very helpful during heart surgery. Figure 2(h) includes flow rate $Q$ the pressure gradient appears to decrease when $Q$ is increased and it is observed that the pressure is maximum at zero flow rate. In Figure 2(i), the effect of amplitude ratio $\phi$ on pressure gradient is plotted. The pressure gradient appears to increase when the amplitude ratio $\phi$ increased. In Figure 2(j), the effect of flow rate $Q$ and Hartman number $M$ on pressure gradient is plotted. The pressure gradient decreases when the flow rate decreases and Hartman number also decreases.

### 4.2. Frictional force distribution

Figure 3 is plotted to show the behaviour of different parameters on the frictional force distribution. Figure 3(a) shows that the frictional force decreases with an increase in the Eyring–Powell fluid parameter $A$. Figure 3(b) shows that as Eyring–Powell fluid parameter $B$ increases the frictional force increases. Figure 3(c) captures the effects of Hartman number $M$ on the frictional force. Here it is observed that frictional force is a decreasing function of $M$.

### 4.3. Temperature distribution

Figures 4–5 are plotted to show the behaviour of different parameters on the temperature distribution $\theta$. Figure 4(a) shows that the temperature decreases with an increase in the Eyring–Powell fluid parameter $A$. Figure 4(b) shows the effect of Eyring–Powell fluid parameter $B$ on frictional force. It is observed that the temperature appears to increase when $B$ is increased. Figure 4(c) shows that the temperature increases with increasing of Hartman number. In Figure 4(d) is the effect joule heating $Q_0$. The temperature appears to increase when joule heating effect $Q_0$ increased.

Figure 5(e) shows that the temperature increases with an increase in the Brownian motion parameter $Nb$. Figure 5(f) captures the effect of thermophoresis parameter $Nt$ on the temperature, the temperature increases when the thermophoresis parameter increases. Figure 5(g) shows that as Grashof number $Gr$ increases the temperature increases. Figure 5(h) includes the effect of local nanoparticle Grashof number $Qr$. Here temperature appears to decrease when $Qr$ is increased. Figure 5(i) captures the effects of Brinkman number $Br$ on the temperature. Here it is observed that temperature increases when Brinkman number $Br$ increased. Figure 5(j) shows that the temperature increases with an increase in the Prandtl number $Pr$.

### 4.4. Concentration distribution

Figure 6 describes the variation of the concentration profile for several values of the Hartman number $M$, joule heating effect $Q_0$, thermophoresis parameter $Nt$ and the Brownian motion parameter $Nb$, Brinkman number $Br$ and Prandtl number $Pr$. In Figure 6(a), we observe that the concentration profile increases with...
Figure 3. Frictional force $\Delta F(t)$ versus $Q$ when $t = 0.1$, $x = 0.2$, $\lambda = 10$, $k = 0.1$, $\alpha = 2.0$, $Pr = 6.9$, $Nt = 0.4$, $Nb = 0.4$, $Gr = 0.3$, $Qr = 0.3$, $Br = 2.0$, $Q_0 = 2.0$, $\varphi = 0.6$, $Q = 0.25$. (a) $B = 2.0$, $M = 0.5$. (b) $A = 0.001$, $M = 0.5$. (c) $A = 0.001$, $B = 2.0$.

Figure 4. Temperature $\theta$ versus $y$ when $t = 0.1$, $x = 0.2$, $\lambda = 10$, $k = 0.1$, $\alpha = 2.0$, $Pr = 6.9$, $Nt = 0.4$, $Nb = 0.4$, $Gr = 0.3$, $Qr = 0.3$, $Br = 2.0$, $\varphi = 0.6$, $Q = 0.25$. (a) $B = 2.0$, $M = 0.5$, $Q_0 = 2.0$. (b) $A = 0.001$, $M = 0.5$, $Q_0 = 2.0$. (c) $A = 0.001$, $B = 2.0$, $Q_0 = 2.0$. (d) $A = 0.001$, $B = 2.0$, $M = 0.5$. 
an increase in Hartman number $M$. Figure 6(b) captures the effect of joule heating $Q_0$ on the concentration profile, the concentration increases when the joule heating effect $Q_0$ increases. Figure 6(c) shows that the concentration profile decreases with an increase in $Nb$. Figure 6(d) captures the effect of thermophoresis parameter $Nt$ on the concentration profile, the concentration increases when the thermophoresis parameter increased. Figure 6(e) includes the effect of Brinkman number $Br$ on the concentration. Here it is observed that the concentration increases when Brinkman number $Br$ increased. Figure 6(f) shows that the concentration increases with an increase in the Prandtl number $Pr$.

5. Concluding remarks

Here we analysed the effect of joule heating on peristaltic blood flow of Eyring–Powell nanofluid in a non-uniform channel, is modelled in the presence of MHD, is studied under the assumptions of long wavelength and low Reynolds number. The results are discussed through graphs. We have the following main observations of the present analysis.

- Both pressure gradient and temperature distribution give opposite results with increasing values of Eyring–Powell fluid parameters $A$ and $B$. 

Similar behaviour of pressure gradient and temperature profiles is observed with an increase in thermal Grashof number $Gr$. Also opposite behaviour of pressure gradient and temperature in the presence of local nanoparticle Grashof number $Qr$.

Both pressure gradient and concentration profiles give opposite results with increasing values of $Nt$ and $Nb$. Also the temperature increases with an increase in Brownian motion parameter $Nb$ and thermophoresis parameter $Nt$.

The pressure gradient decreases with increasing Hartman number $M$ and flow rate $Q$.

Pressure gradient decreases by increasing amplitude ratio $\phi$.

Frictional force gives the opposite behaviour with increasing values of Eyring–Powell fluid parameters $A$ and $B$, also frictional force decreases with an increasing Hartman number $M$.

Similar behaviour of temperature and concentration profiles increases in the presence of joule heating effect $Q_0$, Hartman number $M$, Brinkman number $Br$ and Prandtl number $Pr$.

Acknowledgements

We would like to thank the editor and the reviewer's for their constructive comments that significantly improved this manuscript.
Disclosure statement
No potential conflict of interest was reported by the authors.

Funding
This work was supported by the University Grant Commission (IN): [Grant Number 201718-NFST-KAR-01215 Dated: 01 April 2017]; University Grants Commission: [Grant Number F510/3/DRS-III/2016(SAP-I) Dated:29-Feb-2016].

References
[1] Latham TW. Fluid motion in a peristaltic pump. MS Thesis, M.I.T. Cambridge. (1966).
[2] Jaffrin MY, Shapiro AH. Peristaltic pumping. Ann Rev Fluids Mech. 1971;3:13–37.
[3] Hayat T, Tanveer A, Yasmin H, et al. Effects of convective conditions and chemical reaction on peristaltic flow of Eyring–Powell fluid. Appl Bionics Biomech. 2014;1:221–233.
[4] Khan AA, Masood F, Ellahi R, et al. Mass transport on chemicalized fourth-grade propagating peristaltically through a curved channel with magnetic effects. J Mol Liq. 2018;258:186–195.
[5] Bhatti MM, Zeeshan A, Ellahi R, et al. Mathematical modelling of heat and mass transfer effects on MHD peristaltic propulsion of two-phase flow through Darcy–Brinkman–Forchheimer porous medium. Adv Powder Technol. 2018;29(5):1189–1197.
[6] Zeeshan A, Ijaz N, Abbas T, et al. The sustainable characteristic of BIO-Bi-phase flow of peristaltic transport of MHD Jeffrey fluid in the human body. Sustainability. 2018;10:1–17.
[7] Asha SK, Sunita G. Effect of couple stress in peristaltic transport of blood flow by homotopy analysis method. AJST. 2017;12:6958–6964.
[8] Bhatti MM, Ellahi R, Zeeshan A. Study of variable magnetic field on the peristaltic flow of Jeffrey fluid in a non-uniform rectangular duct having compliant walls. J Mol Liq. 2016;222:101–108.
[9] Choi SUS. Enhancing thermal conductivity of the fluids with nanoparticles. ASME Fluids Eng Div. 1995;231:99–105.
[10] Bhatti MM, Zeeshan A, Ellahi R. Simultaneous effects of coagulation and variable magnetic field on peristaltically induced motion of Jeffrey nanofluid containing gyrotactic microorganism. Microvac Res. 2017;110:32–42.
[11] Ellahi R, Alamri SZ, Basit A, et al. Structural impact of kerosene-Al2O3 nanoliquid on MHD Poiseuille flow with variable thermal conductivity. application of cooling process. J Mol Liq. 2018;264:607–615.
[12] Tanveer A, Hayat T, Alsaadi F, et al. Mixed convection peristaltic flow of Eyring–Powell nanofluid in a curved channel with compliant walls. Comput Biol Med. 2017;82:71–79.
[13] Prakash J, Sharma A, Tripathi D. Thermal radiation effects on electroosmosis modulated of ionic nanofluids in biomicrofluidics channel. J Mol Liq. 2018;249:843–855.
[14] Powell RE, Erying H. Mechanism for the relaxation theory of viscosity. Nature. 1944;154:427–428.
[15] Abbasi FM, Alsaedi A, Hayat T. Peristaltic transport of Eyring–Powell fluid in a curved channel. J Aerosp Eng 2014;27(6):1–7.
[16] Asha SK, Sunita G. Mixed convection peristaltic flow of a Eyring–Powell nanofluid with magnetic field in a non-uniform channel. JAMS. 2018;2(8):332–334.
[17] Noreen S, Qasim M. Peristaltic flow of MHD Eyring–Powell fluid in a channel. Eur Phys J Plus. 2013;128(91):1–10.
[18] Hayat T, Irfan Shah S, Ahmad B, et al. Effect of slip on peristaltic flow of Powell–Eyring fluid in a symmetric channel. Appl Bionics Biomech. 2014;1:69–79.
[19] Abbasi FM, Hayat T, Alsaedi A. Numerical analysis for peristaltic motion of MHD Eyring–Prandtl fluid in an inclined symmetric cannal with inclined magnetic field. J Appl Fluid Mech. 2016;9:389–396.
[20] Hina S, Mustafa M, Hayat T, et al. Peristaltic flow of Powell–Eyring in curved channel with heat transfer: a useful application in biomedicine. Comput Methods Programs Biomed. 2016;135:89–100.
[21] Hayat A, Abdulhadi A. Influence of magnetic field on peristaltic transport for Eyring–Powell fluid in a symmetric channel during a porous medium. Math Theor Model. 2017;9:9–22.
[22] Ali Abbas M, Bai YQ, Rashidi MM, et al. Application of drug delivery in magnetohydrodynamics peristaltic blood flow of nanofluid in a non-uniform channel. J Mech Med Biol. 2016;16(04):1650052.
[23] Bhatti MM, Ali Abbas M, Rashidi MM. Combine effects of magnetohydrodynamics (MHD) and partial slip on peristaltic blood flow of Ree–Eyring fluid with wall properties. JESTECH. 2016;19(3):1497–1502.
[24] Zeeshan A, Bhatti MM, Akbar NS, et al. Hydromagnetic blood flow of sisko fluid in a non-uniform channel induced by peristaltic wave. CTP. 2017;68(1):103–110.
[25] Ranjit NK, Shit GC, Tripathi D. Joule heating and zeta potential effect on peristaltic blood flow through porous micro-vessels altered by electrohydrodynamics. Microvasc Res. 2018;117:74–89.
[26] Akbar NS, Nadeem S. Characteristics of heating scheme and mass transfer on the peristaltic flow for an Eyring–Powell fluid in an endoscope. Int Commun Heat Mass. 2015;55:375–383.
[27] Bhatti MM, Rashidi MM. Study of heat and mass transfer with joule heating on magnetohydrodynamic (MHD) peristaltic blood flow under the influence of hall effect. Propul Power Res. 2017;6(3):177–185.
[28] Abbasi FM, Hayat T. Effects of inclined magnetic field and joule heating in mixed convective peristaltic transport of non-Newtonian fluids. Bull Pol Ac Tech. 2015;63(3):501–514.
[29] Hayat T, Aslam N, Rafiq M, et al. Hall and joule heat-effect on unsteady peristaltic motion of MHD Eyring–Prandtl fluid in an inclined symmetric channel with inclined magnetic field. J Phys D Appl Phys. 2017;50:1–10.
[30] Hayat T, Aslam N, Rafiq M, et al. Hall and joule heat-effect on peristaltic motion of MHD Eyring–Prandtl fluid in an inclined symmetric channel with inclined magnetic field. J Phys D Appl Phys. 2017;50:1–10.
[31] Hina S, Mustafa M, Hayat T, et al. Peristaltic flow of Powell–Eyring in curved channel with heat transfer: a useful application in biomedicine. Comput Methods Programs Biomed. 2016;135:89–100.

Appendix
The values of $a_{11}, a_{12}, a_{14}, a_{15}, a_{16}$ in Equation (30) are given as

$$a_{11} = \left( \frac{1}{3} + \frac{4F}{3h} \right) - M^2 \left( \frac{1}{3} + \frac{4F}{3h} \right) y - \frac{Py^2}{2}$$
\[-\left(4(1 + B)\left(\frac{1}{3h} + \frac{F}{3h^2}\right) - \frac{Gr}{24h} - \frac{Qr}{24h}\right)y^3\]

\[-576A\left(\frac{1}{3h^3} + \frac{F}{3h^4}\right)^2y^7.\]

\[\alpha_{12} = \left(1 + B\right)\left(\frac{1}{3} + \frac{4F}{3h}\right) - M^2\left(\frac{1}{3} + \frac{4F}{3h}\right) - Py^2\]

\[-4(1 + B)\left(\frac{1}{3h} + \frac{F}{3h^2}\right) - \frac{Gr}{24h} - \frac{Qr}{24h}\right)y^3\]

\[-576A\left(\frac{1}{3h^3} + \frac{F}{3h^4}\right)^2y^7.\]

\[\alpha_{14} = \frac{AM^2}{4}\left(\frac{1}{3} + \frac{4F}{3h}\right)^2y^3 + 3AM^2P\left(\frac{1}{3} + \frac{4F}{3h}\right)\]

\[\times \left(1 + B\right)\left(\frac{1}{3h} + \frac{F}{3h^2}\right) - \frac{Gr}{24h} - \frac{Qr}{24h}\right)y^4\]

\[+ \left(A^2PM^2\left(\frac{1}{3} + \frac{4F}{3h}\right) + 6AM^4\left(\frac{1}{3} + \frac{4F}{3h}\right)^2\right)\]

\[\times \left(1 + B\right)\left(\frac{1}{3h} + \frac{F}{3h^2}\right) - \frac{Gr}{24h} - \frac{Qr}{24h}\right)y^3.\]

\[-12AP\left(\frac{1}{3h} + \frac{F}{3h^2}\right) - \frac{Gr}{24h} - \frac{Qr}{24h}\right)y^5\]

\[+ 72AM^2\left(\frac{1}{3} + \frac{4F}{3h}\right)\left(\frac{1}{3h} + \frac{F}{3h^2}\right) - \frac{Gr}{24h} - \frac{Qr}{24h}\right)y^5\]

\[-\frac{Qr}{24h}\right)y^5 + 288AP\left(1 + B\right)\left(\frac{1}{3h} + \frac{F}{3h^2}\right) - \frac{Gr}{24h} - \frac{Qr}{24h}\right)y^4\]

\[-\frac{Qr}{24h}\right)y^6 + 576A\left(1 + B\right)\left(\frac{1}{3h} + \frac{F}{3h^2}\right) - \frac{Gr}{24h} - \frac{Qr}{24h}\right)y^6\]

\[-\frac{Qr}{24h}\right)y^7 + 144\left(\frac{1}{3} + \frac{4F}{3h}\right)^3\left(\frac{1}{3} + \frac{3F}{3h}\right)^2y^7.\]

\[+ 12AP\left(1 + B\right)\left(\frac{1}{3h} + \frac{F}{3h^2}\right) - \frac{Gr}{24h} - \frac{Qr}{24h}\right)y^3\]

\[+ 72AM^2\left(\frac{1}{3} + \frac{4F}{3h}\right)\left(\frac{1}{3h} + \frac{F}{3h^2}\right) - \frac{Gr}{24h} - \frac{Qr}{24h}\right)y^3.\]

\[-\frac{Gr}{24h}\right) + 114061AP\left(\frac{1}{3h} + \frac{F}{3h^2}\right)\]

\[(1 + B)\left(\frac{1}{3h} + \frac{F}{3h^2}\right) - \frac{Gr}{24h} - \frac{Qr}{24h}\right)y^6\]

\[+ 57984A^2\left(\frac{1}{3} + \frac{4F}{3h}\right)\left(1 + B\right)\left(\frac{1}{3h} + \frac{F}{3h^2}\right)\]

\[-\frac{Gr}{24h} - \frac{Qr}{24h}\right)y^10.\]

\[+ 165888A^2M^2\left(\frac{1}{3} + \frac{4F}{3h}\right)\]

\[\times \left(\frac{1}{3h} + \frac{F}{3h^2}\right)\left(1 + B\right)\left(\frac{1}{3h} + \frac{F}{3h^2}\right) - \frac{Gr}{24h} - \frac{Qr}{24h}\right)y^6\]

\[+ 19795968A^2\left(\frac{1}{3} + \frac{4F}{3h}\right)^2\left(1 + B\right)\left(\frac{1}{3h} + \frac{F}{3h^2}\right)\]

\[-\frac{Gr}{24h} - \frac{Qr}{24h}\right)y^15.\]

\[+ 2156544A^3\left(\frac{1}{3h} + \frac{F}{3h^2}\right)^2y^14.\]

\[-\frac{Gr}{24h} - \frac{Qr}{24h}\right)y^11 + 165888A^2M^2\left(\frac{1}{3} + \frac{4F}{3h}\right)\]

\[\times \left(\frac{1}{3h} + \frac{F}{3h^2}\right)^2y^13.\]

\[+ 2156544A^3\left(\frac{1}{3h} + \frac{F}{3h^2}\right)^2y^14.\]

\[\times \left(\frac{1}{3h} + \frac{F}{3h^2}\right)^2y^15.\]

\[+ 2156544A^3\left(\frac{1}{3h} + \frac{F}{3h^2}\right)^2y^14.\]

\[-\frac{Gr}{24h} - \frac{Qr}{24h}\right)y^{15.}\]

\[+ 63700992A^4\left(\frac{1}{3h} + \frac{F}{3h^2}\right)^2y^{16.}\]

\[-\frac{Gr}{24h} - \frac{Qr}{24h}\right)y^{335} + 63700992A^4\left(\frac{1}{3h} + \frac{F}{3h^2}\right)^2y^{16.}\]

\[a_{13} = \left(1 + B\right)^2\left(\frac{1}{3} + \frac{4F}{3h}\right) - M^2(1 + B)\left(\frac{1}{3} + \frac{4F}{3h}\right)y^3\]

\[-P(1 + B) + M^2(1 + B)\left(\frac{1}{3} + \frac{4F}{3h}\right)y^2\]

\[\times \left(\frac{1}{3} + \frac{4F}{3h}\right)\left(1 + B\right)\left(\frac{1}{3h} + \frac{F}{3h^2}\right)\]

\[-\frac{Gr}{24h} - \frac{Qr}{24h}\right)y^3 + PM^2y^4.\]

\[+ M^4\left(\frac{1}{3} + \frac{4F}{3h}\right)y^2\]

\[-\frac{Gr}{24h} - \frac{Qr}{24h}\right)y^3 + PM^2y^4.\]

\[+ M^4\left(\frac{1}{3} + \frac{4F}{3h}\right)y^2\]

\[-\frac{Gr}{24h} - \frac{Qr}{24h}\right)y^3.\]

\[\alpha_{15} = \frac{Gr}{24h} + \frac{Pr}{24h} + \frac{Pr}{24h} + \frac{Pr}{24h}\]

\[+ BrM^2\left(\frac{1}{3} + \frac{4F}{3h}\right)y^2 - BrGr\left(\frac{1}{3} + \frac{4F}{3h}\right)\]

\[\times \left(\frac{1}{3h} + \frac{F}{3h^2}\right)^2y^7.\]

\[\alpha_{16} = Qr\left(\frac{1 + Nq}{Nq}\right)\]
\[ c_{12} = \frac{Qr}{Nt} \left( 1 + \frac{h}{2} \right)^2 \]

The values of \( b_{11}, b_{12}, b_{13}, b_{14}, b_{15} \) in Equation (33) are given as

\[ b_{11} = \frac{y^2}{2h^2} + \frac{Pr Nt y^2}{2h^2} + \frac{Pr Qo y^2}{2} \]
\[ + BrM^2 \left( \frac{1}{3} + \frac{4F}{3h} \right) \frac{y^2}{2} + 2BrM^2 \left( \frac{1}{3} + \frac{F}{3h^3} \right) \frac{y^8}{7} \]
\[ - BrM^2 \left( \frac{1}{3} + \frac{4F}{3h} \right) \frac{y^5}{28} \]

\[ b_{12} = \frac{y^2}{2h^2} + \frac{Pr Nt y^2}{2h^2} + \frac{Pr Qo y^2}{2} \]
\[ + BrM^2 \left( \frac{1}{3} + \frac{4F}{3h} \right) \frac{y^2}{2} + 8BrM^2 \left( \frac{1}{3} + \frac{F}{3h^3} \right) \frac{y^9}{31} \]
\[ - BrM^2 \left( \frac{1}{3} + \frac{4F}{3h} \right) \frac{y^6}{120} \]

\[ b_{13} = \frac{y^2}{2h^2} + \frac{Pr Nt y^2}{6h^2} + \frac{Pr Nt y^2}{6h^2} + \frac{Pr Qo y^3}{6} \]
\[ + BrM^2 \left( \frac{1}{3} + \frac{4F}{3h} \right) \frac{y^3}{6} + 8BrM^2 \left( \frac{1}{3} + \frac{F}{3h^3} \right) \frac{y^9}{31} \]
\[ - BrM^2 \left( \frac{1}{3} + \frac{4F}{3h} \right) \frac{y^6}{120} \]

\[ b_{14} = \frac{y^2}{2h^2} + \left( \frac{Pr Nt y^2}{h^2} + \frac{Pr Qo}{h^2} + BrM^2 \left( \frac{1}{3} + \frac{4F}{3h} \right) \frac{y^9}{31} \right) \]
\[ \times y^2 + (BrM^2 \left( \frac{1}{3} + \frac{4F}{3h} \right) \frac{y^9}{31}) \]
\[ + 32 \left( BrM^2 \left( \frac{1}{3} + \frac{F}{3h^3} \right) \right)^2 \frac{y^16}{1446} + 2 \left( BrM^2 \left( \frac{1}{3} + \frac{F}{3h^3} \right) \right)^2 \frac{y^2}{h} \]
\[ + 32 \left( Pr Nt y^2 \right) \frac{y^2}{6h^2} + 2Pr Nt y^2 \frac{y^2}{h} \]
\[ + Qo \frac{y^2}{h} \frac{y^2}{6h^2} + 8BrM^2 \left( \frac{1}{3} + \frac{F}{3h^3} \right) \frac{y^6}{120} \]
\[ + (Pr Nt + Qo) + BrM^2 \left( \frac{1}{3} + \frac{4F}{3h} \right) \frac{y^6}{120} \]
\[ \times \left( BrM^2 \left( \frac{1}{3} + \frac{4F}{3h} \right) \frac{y^6}{84} + 16 \left( Pr Nt + Qo \right) + BrM^2 \left( \frac{1}{3} + \frac{4F}{3h} \right) \frac{y^6}{84} \right) \]
\[ \left( BrM^2 \left( \frac{1}{3} + \frac{F}{3h^3} \right) \frac{y^6}{315} \right) \]

\[ b_{15} = \frac{8A^2}{63} \left( \frac{1}{3} + \frac{F}{3h^3} \right) \frac{y^8}{84} + (1 + B) \left( \frac{1}{3} + \frac{F}{3h^3} \right) \frac{y^8}{84} \]
\[
\frac{M^2}{15} \left( \frac{1}{h^3} + \frac{F_3}{h^3} + \frac{Gr}{120h} + \frac{Qr}{120h} \right) \left( \frac{1}{h^3} + \frac{F_3}{h^3} + \frac{Gr}{120h} + \frac{Qr}{120h} \right) \right)^2 y^{10} \frac{9}{9} \\
+ \frac{23328}{63282025} A^2 \left( \frac{1}{h^3} + \frac{F_3}{h^3} \right) y^{18} - \frac{2}{3} A \left( \frac{1}{h^3} + \frac{F_3}{h^3} \right) \\
\left( 1 + B \right) \left( \frac{1}{h^3} + \frac{F_3}{h^3} \right) + M^2 \left( \frac{1}{h^3} + \frac{F_3}{h^3} + \frac{Gr}{120h} + \frac{Qr}{120h} \right) \\
+ \frac{Qr}{120h} \right) y^{6} \frac{9}{15} - \frac{288}{103415} A^2 \left( \frac{1}{h^3} + \frac{F_3}{h^3} \right) y^{13} \\
+ \frac{648}{144781} A \left( 1 + B \right) \left( \frac{1}{h^3} + \frac{F_3}{h^3} + \frac{M^2}{15} \left( \frac{1}{h^3} + \frac{F_3}{h^3} \right) y^{4} \\
+ \frac{Gr}{120h} + \frac{Qr}{120h} \right) \left( \frac{1}{h^3} + \frac{F_3}{h^3} \right) y^{4}. 
\]