ON THE NUMBER OF INTEGRAL IDEALS OF A NUMBER FIELD

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Abstract. In this paper we study the problem of the first moment of the Dedekind zeta function of a number field \( K \) and improve the error term. As a ready generalization of our proof, we improve the error term in the Piltz divisor problem.

1. Introduction

The question of estimating the first moment of the Dedekind zeta function of a number field \( K \) of degree \( n \) was probably first studied by Landau [7] who gave the classical result

\[
\sum_{m \leq X} M(m) = cX + O(X^{1 - \frac{2}{n} + \epsilon}),
\]

where \( M(m) \) is the number of integral ideals of \( K \) of norm equal to \( m \), and \( c \) is the residue of the Dedekind zeta function \( \zeta_K \) of \( K \) at \( s = 1 \) (refer section 2 below). The case where \( K \) is a quadratic field was considered by Huxley and Watt [4] who showed that

\[
\sum_{m \leq X} M(m) = cX + O\left(X^{\frac{23}{73}}(\log X)^{\frac{1}{315}}\right),
\]

and for \( K \) a cubic field Müller [8] showed that

\[
\sum_{m \leq X} M(m) = cX + O\left(X^{\frac{43}{96} + \epsilon}\right).
\]

Recent results include the following estimate of Bordellés [2],

\[
\sum_{m \leq X} M(m) = cX + \begin{cases} O\left(X^{\frac{n}{2n} + \epsilon}\right) & n = 4 \\ O\left(X^{1 - \frac{4}{2n} + \epsilon}\right) & 5 \leq n \leq 10. \end{cases}
\]

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For general number fields Nowak [9] showed
\[ \sum_{m \leq X} M(m) = cX + \begin{cases} O(X^{1-\frac{\alpha}{n} + \frac{8}{n(n+1)}} (\log X)^{\frac{10}{5n+2}}) & 3 \leq n \leq 6 \\ O(X^{1-\frac{\alpha}{n} + \frac{1}{2n}} (\log X)^{\frac{2}{n}}) & n \geq 7. \end{cases} \]

In 2010 Lao [6] improved upon Nowak’s work for \( n > 9 \). Lao proved the following estimate
\[ \sum_{m \leq X} M(m) = cX + O(X^{1-\frac{3}{n}+\epsilon}) \]
for all number fields of degree \( n \).

Most recently, Takeda [10] showed that
\[ \sum_{m \leq X} M(m) = cX + O(X^{2-\frac{3}{n}+\beta+\epsilon}), \]
for any fixed \( 0 \leq \beta \leq \frac{8}{2n+5} - \epsilon. \)

In this paper, we prove the following

**Theorem 1.1.** Let \( K \) be a number field of degree \( n \) over \( \mathbb{Q} \) and let \( \alpha = \min \{ 1/2, 3/n \} \). Then for every \( \epsilon > 0, \)
\[ \sum_{m \leq X} M(m) = cX + O(X^{1-\alpha+\epsilon}) \quad (1.1) \]
where \( M(m) \) denotes the number of ideals of norm equal to the integer \( m \).

We note here that our result improves the known results for \( n \geq 4 \).

**2. Preliminaries**

Let \( K \) be a number field of degree \( n \) over \( \mathbb{Q} \). Then the Dedekind zeta function, \( \zeta_K \) of \( K \) is defined as
\[ \zeta_K(s) = \sum_{a \neq 0} \frac{1}{(Na)^s}, \]
where the summation runs over all non-zero integral ideals of \( K \) and \( N(a) \) denotes the ideal norm of \( a \). It is possible to write \( \zeta_K \) in another form,
\[ \zeta_K(s) = \sum_{m=1}^{\infty} \frac{M(m)}{m^s}, \]
where \( M(m) \) is the number of integral ideals of \( K \) whose norm is equal to \( m \).

It is known that \( \zeta_K \) converges absolutely for \( \text{Re}(s) > 1 \) and admits to a meromorphic continuation to the entire complex plane with a simple
pole at \( s = 1 \). The residue at this pole is given by the class number formula which states that

\[
c := \text{Res}_{s=1} \zeta_K(s) = \frac{2^r_1(2\pi)^{r_2}hR}{w\sqrt{|D|}},
\]

where \( h, R, w \) are respectively the class number, regulator and the number of roots of unity of \( K \). Furthermore \( \zeta_K \) satisfies a functional equation described as below.

Let \( \Lambda_K \) be given by,

\[
\Lambda_K(s) = |D_K|^\frac{s}{2} \left[ \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right]^{r_1} \left[ 2(2\pi)^{-s}\Gamma(s)\right]^{r_2} \zeta_K(s),
\]

where \( \Gamma(s) \) is the usual \( \Gamma \) function (see below). Then \( \Lambda_K \) defines a meromorphic function with poles at \( s \in \{0, 1\} \) and satisfies the functional equation \( \Lambda_K(s) = \Lambda_K(1-s) \) for all \( s \in \mathbb{C} \).

For the purposes of this paper, it is necessary to explicate certain properties of the \( \Gamma \) function. The \( \Gamma \) function is a complex analytic function defined as an integral

\[
\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx.
\]

The integral is convergent for \( \text{Re}(s) > 1 \) and the \( \Gamma \) function satisfies the functional equation \( s\Gamma(s) = \Gamma(s+1) \). Furthermore the \( \Gamma \) function admits a meromorphic continuation to the entire complex plane with poles at all non positive integers. The absolute value of the \( \Gamma \) function is given by the formula \([1]\),

\[
|\Gamma(x + iy)| = |\Gamma(x)| \left\{ \prod_{n=0}^{\infty} \left( 1 + \frac{y^2}{(n+x)^2} \right) \right\}^{-1}.
\]

In particular, if \( x \) is not a pole of the \( \Gamma \) function, we have

\[
|\Gamma(x + iy)| = |\Gamma(x)| \left\{ \prod_{n=0}^{\infty} \left( 1 + \frac{y^2}{(n+x)^2} \right) \right\}^{-1}.
\]

3. Main Results

In the remainder of the paper, \( s \) shall denote a complex variable and \( \sigma, t \) shall denote the real and imaginary parts respectively of \( s \). In order to get better estimates for the error term, we need to push the line of integration further to the left. To get estimates on the Dedekind zeta function on the region \( \text{Re}(s) < 1/2 \) we make use of the functional equation.

\[\text{1There are at least four other equivalent definitions of the } \Gamma \text{ function.}\]
Fix a number field $K$ of degree $n$ over $\mathbb{Q}$ and let $\Lambda_K$ be as in equation (2.1). We know that $\Lambda_K(s) = \Lambda_K(1 - s)$ for all $s \in \mathbb{C}$. In particular $|\Lambda_K(s)| = |\Lambda_K(1 - s)|$, which gives us

\[
|D_K|^\frac{2}{n} \left[ \pi^{-\frac{n}{2}} \Gamma \left( \frac{s}{2} \right) \right]^{r_1} [2(2\pi)^{-s} \Gamma(s)]^{r_2} \zeta_K(s) = \left| D_K \right|^\frac{2}{n} \left[ \pi^{-\frac{n}{2}} \Gamma \left( \frac{1-s}{2} \right) \right]^{r_1} [2(2\pi)^{-1-s} \Gamma(1-s)]^{r_2} \zeta_K(1-s).
\]

Therefore we get

\[
|\zeta_K(s)| = \frac{|D_K|^\frac{2}{n} \left[ \pi^{-\frac{n}{2}} \Gamma \left( \frac{1-s}{2} \right) \right]^{r_1} [2(2\pi)^{-1-s} \Gamma(1-s)]^{r_2} \zeta_K(1-s)}{|D_K|^\frac{2}{n} \left[ \pi^{-\frac{n}{2}} \Gamma \left( \frac{s}{2} \right) \right]^{r_1} [2(2\pi)^{-s} \Gamma(s)]^{r_2} \zeta_K(s)}.
\]

\[
= C_K^{s-\sigma} \left[ \frac{\Gamma \left( \frac{1-s}{2} \right)}{\Gamma \left( \frac{s}{2} \right)} \right]^{r_1} \left[ \frac{\Gamma(1-s)}{\Gamma(s)} \right]^{r_2} \zeta_K(1-s)
\]

\[
= C_K^{s-\sigma} \left( \frac{s}{1-s} \right)^{r_1+r_2} \left[ \frac{\Gamma \left( \frac{3-s}{2} \right)}{\Gamma \left( \frac{s+2}{2} \right)} \right]^{r_1} \left[ \frac{\Gamma(2-s)}{\Gamma(s+1)} \right]^{r_2} |\zeta_K(1-s)|
\]

(3.2)

where $C_K = 2^{r_2} \pi^{\frac{n}{2}} |D_K|^{-1/2}$.

Fix an $\epsilon > 0$. Let $T \geq 1$ be a real number (it is a parameter to be specified at a later stage). We are primarily interested in the following three contours.

The first contour $\gamma_1$ is the straight line joining $-\epsilon + iT$ to $-\epsilon - iT$, the second contour $\gamma_2$ is the horizontal line from $-\epsilon + iT$ to $1 + \epsilon + iT$, and similarly the third contour $\gamma_3$ is the horizontal line joining $1 + \epsilon - iT$ to $-\epsilon - iT$.

We remark here that the above contour was used by Takeda to study the Piltz divisor problem [10].

**Lemma 3.1.** Fix a number field $K$ and an $\epsilon > 0$, and let $\gamma_1, \gamma_2, \gamma_3$ be as above. Let $s \in \gamma_1 \cup \gamma_2 \cup \gamma_3$, then there exists a constant $C$ which does not depend on $s$, such that

\[
|\zeta_K(s)| \leq C \left| \zeta_K(1-s) \right|.
\]

(3.3)

**Proof.** In light of equation (3.2) it is enough to show that the extra factors appearing on the RHS can be bounded above independent of $s$. \[\]
Firstly, since \( s \in \gamma_1 \cup \gamma_2 \cup \gamma_3 \), we have \(-\epsilon \leq \sigma \leq 1 + \epsilon\) and therefore \( C_K^{1-\sigma} \leq C_K^{1}\).

Secondly, observe that in \(3.2\), the arguments of the \(\Gamma\) factors have positive real part away from 0. Therefore from equation \(2.2\) we see that

\[
\left| \frac{\Gamma \left( \frac{3-s}{2} \right)}{\Gamma \left( \frac{s+2}{2} \right)} \right|^{r_1} \left| \frac{\Gamma \left( 2-s \right)}{\Gamma \left( s+1 \right)} \right|^{r_2} \leq C_1 \left| \frac{\Gamma \left( \frac{3-\sigma}{2} \right)}{\Gamma \left( \frac{\sigma+2}{2} \right)} \right|^{r_1} \left| \frac{\Gamma \left( 2-\sigma \right)}{\Gamma \left( \sigma+1 \right)} \right|^{r_2} \leq C_2
\]

where \(C_1, C_2\) are constants that does not depend on \(s\).

Thirdly the factor \((s_1-s)^{r_1+r_2}\) can also be bounded above independent of \(s\), as \(s\) is away from 1. Hence we have the lemma. □

Lemma \(3.1\) will be used in the sequel. Now we are ready to prove the main result.

4. Proof of Theorem \(1.1\)

Fix \(X > 1\) and \(\epsilon > 0\) and let \(\gamma_1, \gamma_2, \gamma_3\) be as in lemma \(3.1\).

Our primary ingredient for this proof is the Perron’s formula \([5]\), which states that,

\[
\sum_{m \leq X} M(m) = \int_{1+\epsilon-iT}^{1+\epsilon+iT} \zeta_K(s) \frac{X^s}{s} ds + O \left( \frac{X^{1+\epsilon}}{T} \right) \quad (4.1)
\]

for any \(1 \leq T \leq X\).

Let \(f(s) = \zeta_K(s) \frac{X^s}{s}\). Then \(f(s)\) is a meromorphic function on \(\mathbb{C}\) with poles at \(s = 0\) and \(s = 1\). Also, \(\text{Res}_{s=1} f(s) = cX\) where \(c = \text{Res}_{s=1} \zeta_K(s)\), and \(\text{Res}_{s=0} f(s) = \zeta_K(0)\).

Consider the contour \(C\) give by the rectangle with vertices \(\{1 + \epsilon - iT, 1 + \epsilon + iT, -\epsilon + iT, -\epsilon - iT\}\) in the anticlockwise direction. Therefore by Cauchy’s residue formula we have

\[
\int_{C} f(s) = \text{Res}_{s=1} f(s) + \text{Res}_{s=0} f(s) = cX + \zeta_K(0).
\]

Combining with equation \(4.1\) we get

\[
\sum_{m \leq X} M(m) = cX + \zeta_K(0) + \int_{\gamma_1} f(s) ds + \int_{\gamma_2} f(s) ds
\]

\[
+ \int_{\gamma_3} f(s) ds + O \left( \frac{X^{1+\epsilon}}{T} \right) \quad (4.2)
\]

Let

\[
I_1 = \int_{\gamma_1} |f(s)| ds = \int_{-T}^{T} |\zeta_K(-\epsilon + it)| X^{-\epsilon} |s|^{-1} dt.
\]
We have from lemma 3.1 above that,

\[ I_1 \ll \int_{-T}^{T} C|\zeta_K(1 + \epsilon - it)|X^{-\epsilon}|s|^{-1}dt. \]

In the line \( Re(s) = 1 + \epsilon \), \( \zeta_K \) is uniformly bounded and therefore we get

\[ I_1 \ll \int_{-T}^{T} X^{-\epsilon}|s|^{-1}dt \]

Since \( |s| \geq \epsilon > 0 \), we have

\[ I_1 \ll TX^{-\epsilon}. \quad (4.3) \]

Let

\[ I_2 = \int_{\gamma_2} |f(s)|ds = \int_{-\epsilon}^{1+\epsilon} |\zeta_K(\sigma + iT)|X^{\sigma}|s|^{-1}d\sigma. \]

Since \( |s| \geq T \), we have,

\[ I_2 \ll \int_{-\epsilon}^{1/2} |\zeta_K(\sigma + iT)|X^{\sigma}T^{-1}d\sigma + \int_{1/2}^{1+\epsilon} |\zeta_K(\sigma + iT)|X^{\sigma}T^{-1}d\sigma. \]

Again from lemma 3.1 we see that

\[ I_2 \ll \int_{-\epsilon}^{1/2} |\zeta_K(1 - \sigma - iT)|X^{\sigma}T^{-1}d\sigma + \int_{1/2}^{1+\epsilon} |\zeta_K(\sigma + iT)|X^{\sigma}T^{-1}d\sigma \]

\[ = \int_{1/2}^{1+\epsilon} (|\zeta_K(\sigma - iT)|X^{1-\sigma}T^{-1} + |\zeta_K(\sigma + iT)|X^{\sigma}T^{-1}) d\sigma. \]

For \( 1/2 \leq \sigma \leq 1 \) and \( |t| > 1 \) there is the well known bound of Heath-Brown \[ 3 \]

\[ |\zeta_K(\sigma + it)| \ll |t|^n(1-\sigma), \]

along with the fact that \( X^{1-\sigma} \leq X^\sigma \) gives us

\[ I_2 \ll \int_{1/2}^{1+\epsilon} T^n(1-\sigma)^{-1}X^\sigma d\sigma \]

\[ = T^{n-1} \int_{1/2}^{1+\epsilon} X^{\sigma}T^{-\frac{n}{2}\sigma} d\sigma. \]

\[ \leq T^{n-1} \max_{1/2 \leq \sigma \leq 1+\epsilon} X^{\sigma}T^{-\frac{n}{2}\sigma} \]

\[ \leq X^{1+\epsilon}T^{-1} + X^{1/2}T^{n-1}. \]
If
\[ I_3 = \int_{\gamma_3} |f(s)| ds, \]
proceeding as above we have
\[ I_2 + I_3 \ll X^{1+\epsilon} T^{-1} + X^\frac{3}{2} T^\frac{1}{2n} - 1. \] \hspace{1cm} (4.4)

If we set \( T = X^\alpha \) for some \( 0 \leq \alpha \leq 1 \), combining equations 4.3, 4.4 and 4.2, we see that
\[ \sum_{m \leq X} M(m) = cX + O(X^{1-\alpha+\epsilon}) + O(X^{1/2 + \frac{3}{2n} - \alpha}). \]

If we choose \( \alpha = \min \left\{ \frac{1}{2}, \frac{3}{2n} \right\} \), we get
\[ \sum_{m \leq X} M(m) = cX + O(X^{1-\alpha+\epsilon}) \]
which completes the proof of the theorem.

5. An Auxiliary result

The proof of theorem 1.1 admits an easy generalization which helps obtain a better estimate with respect to the Piltz divisor problem.

Let \( K \) be a number field of degree \( n \) as above, and let \( I^m_K(x) \) denote the number of \( m \)-tuples of integral ideals of \( K \), \((a_1, \ldots, a_m)\) such that \( N(a_1) \cdots N(a_m) \leq X \). The function \( I^m_K \) is called the Piltz divisor function. It is well known that
\[ I^m_K \sim \text{Res}_{s=1} \left( \zeta^m_K(s) \frac{X^s}{s} \right). \]

Recently, Takeda [10] showed that
\[ \Delta^m_K(X) = O \left( X^{2mn + \frac{3}{2mn} + \epsilon} D^\frac{2mn}{1 + \epsilon} \right), \]
where \( \Delta^m_K = I^m_K - \text{Res}_{s=1} \left( \zeta^m_K(s) \frac{X^s}{s} \right) \)

If in the proof of theorem 1.1 we replace \( f(s) \) as \( f(s) = \zeta^m_K(s) \frac{X^s}{s} \), and choose \( \alpha = \min \left\{ \frac{1}{2}, \frac{3}{2mn} \right\} \), we prove

**Theorem 5.1.** Let \( \Delta^m_K \) be as above, then for every \( \epsilon > 0 \), we have
\[ \Delta^m_K = O(X^{1-\alpha+\epsilon}), \]
where \( \alpha = \min \left\{ \frac{1}{2}, \frac{3}{2mn} \right\} \).

We remark here that the above bound on \( \Delta^m_K \) matches the one predicted by the Lindelöf hypothesis whenever \( mn \leq 6 \) (refer eqn. 1.6 in [10]).
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