ON THE GLASNER PROPERTY FOR MATRICES
WITH POLYNOMIAL ENTRIES

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Abstract. We obtain a new bound in the uniform version of the
Glasner property for matrices with polynomial entries, improving
that of K. Bulinski and A. Fish (2021). This improvement is based
on a more careful examination of complete rational exponential
sums with polynomials and can perhaps be used in other questions
of the similar spirit.

1. Introduction

1.1. Set-up and motivation. Let \( T = \mathbb{R}/\mathbb{Z} \) be the unit torus which
we identify with the half-open interval \([0, 1)\). Glasner [6] has shown
that for any infinite set \( \mathcal{X} \subseteq T \) and any \( \varepsilon > 0 \) one can find an \( n \in \mathbb{N} \)
such that the dilation \( n\mathcal{X} \) is \( \varepsilon \)-dense in \( T \), that is, for some \( n \in \mathbb{N} \) we have

\[
\sup_{\zeta \in T} \inf_{x \in \mathcal{X}} |\zeta - nx| \leq \varepsilon.
\]

The result has been extended and improved in several directions
including polynomial sequences of dilations \( f(n)\mathcal{X} \) with \( f \in \mathbb{Z}[X] \) and
also to multidimensional analogues for sets \( \mathcal{X} \subseteq T_d \), where

\[ T_d = (\mathbb{R}/\mathbb{Z})^d \]

for some integer \( d \geq 1 \), we refer to [2] for a survey of previous results
and further references.

Here we continue considering the same scenario as in the recent work
of Bulinski and Fish [2] and consider dilations \( A(n)\mathcal{X} \) of a set \( \mathcal{X} \subseteq T_d \)
by an integer polynomial matrix

\[
A(\mathcal{X}) = (a_{r,s}(X))_{r,s=1}^d \in \mathbb{Z}[X]^{d \times d}.
\]

Recently, improving and generalising several previous results, Bulin-
ski and Fish [2] have shown a version of the multidimensional Glasner
theorem [6] with polynomial matrix dilations, where the set \( \mathcal{X} \subseteq T_d \)

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can be finite. Namely, provided that the matrix $A$ satisfies some natural necessary condition (see Theorem 1.1 below), for any $\varepsilon > 0$ there is some $k_{d,A}(\varepsilon)$ such that if $\mathcal{X} \subseteq T_d$ is of cardinality $\#\mathcal{X} \geq k_{d,A}(\varepsilon)$ then for some $n \in \mathbb{N}$ the dilation $A(n)\mathcal{X}$ is $\varepsilon$-dense in $T$, that is, for some $n \in \mathbb{N}$ we have

$$\sup_{\zeta \in T_d} \inf_{x \in \mathcal{X}} \| \zeta - A(n)x \| \leq \varepsilon,$$

where $\| \cdot \|$ is the distance on $T_d$ induced by the Euclidean norm. In fact \cite[Theorem 2.8]{2} gives the following bound

\begin{equation}
(1.2) \quad k_{d,A}(\varepsilon) \leq c(d,e)H^{3d+1}/2+o(1)\varepsilon^{-d(2d+1)e-(2d+1)(2e+1)/2+o(1)} \quad \text{as } \varepsilon \to 0,
\end{equation}

where $e$ and $H$ are the largest degree and absolute value of non-constant coefficients of polynomials $a_{r,s}$ in (1.1), respectively, and $c(d,e)$ depends only on $d$ and $e$.

1.2. New bound. As in \cite{2} we note that we can always assume that the matrix (1.1) satisfies

\begin{equation}
(1.3) \quad a_{r,s}(0) = 0, \quad r, s = 1, \ldots, d.
\end{equation}

As usual, for a vector $u \in \mathbb{R}^d$ we denote by $u^t$ the transposed vector.

**Theorem 1.1.** Suppose that the matrix $A(X)$ as in (1.1) satisfies (1.3) and is such that for any non-zero vectors $u, v \in \mathbb{Z}^d$ we have $u^tA(X)v \neq 0$. Then

$$k_{d,A}(\varepsilon) \leq c(d,e)H^{(3d+1)/2+o(1)}\varepsilon^{-d(2d+1)e-(7d+1)/2+o(1)}, \quad \text{as } \varepsilon \to 0,$$

where $e$ and $H$ are the largest degree and absolute value of the coefficients of polynomials $a_{r,s}(X), \ r, s = 1, \ldots, d$, respectively.

To compare Theorem 1.1 with the bound (1.2) we notice that

$$\frac{3d+1}{2} \leq d(d+1) \quad \text{and} \quad d(2d+1)e + \frac{7d+1}{2} \leq d(d+1)(2e+1)$$

for all $d \geq 1$ and $e \geq 2$.

1.3. Ideas behind the proof. The proof of (1.2) in \cite{2} is based on the classical **Hua bound** (see (2.4) below) of complete rational exponential sums (for a proof see, for example, \cite[Theorem 7.1]{10}). This bound has also been used in \cite{7}. Generally speaking, the Hua bound (2.4) is tight and cannot be improved for arbitrary moduli $q$. However the set of moduli for which it is optimal is rather sparse, which is the idea we exploit here. More precisely, we use more refined information about complete rational exponential sums (see Lemma 2.4) and some well-known results about the arithmetic structure of integers (see (2.3)). This allows us to improve the exponent of (1.2).
Although the improved bound is perhaps still far from the optimal bound, we believe that the technique we employ deserves to be known better and can also be used for some other problems.

2. Preparations

2.1. Notation and conventions. Throughout the paper, the notations $U = O(V)$, $U \ll V$ and $V \gg U$ are equivalent to $|U| \leq cV$ for some positive constant $c$, which throughout the paper may depend on the dimension $d$ and the degree $e$.

We always use $e$ for the largest degree and use $H$ for the absolute value of the coefficients of polynomials $a_{r,s}(X)$, $r,s = 1,\ldots,d$ of the matrix (1.1), which we always assume to satisfy (1.3).

Thus we always suppress the dependence on $d$ and $e$ in the ‘$\ll$’, however we give implicit (albeit not optimised) estimates in terms of $H$.

For any quantity $V > 1$ we write $U = V^{o(1)}$ (as $V \to \infty$) to indicate a function of $V$ which satisfies $V^{-\delta} \leq |U| \leq V^\delta$ for any $\delta > 0$, provided $V$ is large enough. One additional advantage of using $V^{o(1)}$ is that it absorbs $\log V$ and other similar quantities without changing the whole expression.

We identify $T_d$ with the unit cube $[0,1)^d$ and thus treat elements of $T_d$ as real numbers.

Given a vector $a = (a_1,\ldots,a_\nu) \in \mathbb{Z}^\nu$ and $q \in \mathbb{N}$ we write

$$\gcd(a,q) = \gcd(a_1,\ldots,a_\nu,q).$$

For a polynomial $f(X) = f_e X^e + \ldots + f_1 X + f_0 \in \mathbb{Z}[X]$ and $q \in \mathbb{N}$, we define the $q$-content $\text{cont}_q(f)$ of $f$ by

$$\text{cont}_q(f) = \gcd(f_1,\ldots,f_e,q).$$

We say that $f$ is $q$-primitive if $\text{cont}_q(f) = 1$.

Finally, we denote

$$e_q(z) = \exp(2\pi z/q).$$

2.2. Reduction to bounds of complete rational exponential sums. It is shown in the proof of [2, Theorem 2.8] that we can assume that

$$x_i - x_j \in \mathbb{Q}, \quad i,j = 1,\ldots,k.$$ 

Furthermore, since additive shifts of $\mathcal{X}$ do not change the property of $A(n)\mathcal{X}$ being $\varepsilon$-dense it is sufficient to consider the case $\mathcal{X} \subseteq T_d \cap \mathbb{Q}^d$. 

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We now assume that the set
\begin{equation}
\mathcal{X} = \{x_1, \ldots, x_k\} \subseteq T_d \cap \mathbb{Q}^d
\end{equation}
is fixed and for an integer \( q \geq 1 \) we denote by \( h_q \) the number of pairs \((i, j)\), \( 1 \leq i, j \leq k \), such that \( q \) is the smallest integer with \( q(x_i - x_j) \in \mathbb{Z}^d \) (or, alternatively, \( q(x_i - x_j) = 0 \) if considered as elements of \( T_d \)).

Also for an integer \( M \geq 1 \) we consider the set
\[ \mathcal{B}(M) = \{m \in \mathbb{Z}^d : m \neq 0\} \cap [-M, M]^d. \]

As in [2], our argument is based on the following inequality which is contained in [2, Proposition 2.6], and which in turn follows from [7, Proposition 2].

**Lemma 2.1.** If a set \( \mathcal{X} \) as in (2.1) is such that for any \( n \in \mathbb{N} \) the dilation \( A(n)\mathcal{X} \) is not \( \varepsilon \)-dense in \( T_d \) then for \( M = \lfloor d/\varepsilon \rfloor \) we have
\[
k^2 \ll \varepsilon^{-d} \sum_{m \in \mathcal{B}(M)} \sum_{q=1}^{\infty} \frac{h_q}{q} \left| \sum_{n=1}^{q} e_q (m^t A(n)b_q) \right| + \varepsilon^{-d} M^d k,
\]
where \( b_q \in \mathbb{Z}^d \) some integer vectors with \( \gcd(b_q, q) = 1 \).

Hence, to use Lemma 2.1 we need:
- estimate the coefficients \( h_q \), see the bound (2.2) and Lemma 2.2;
- estimate the content \( \text{cont}_q (m^t A(X)b_q) \) of the polynomials in the exponential sums see Lemma 2.3;
- use bounds of complete rational exponential sums for \( q \)-primitive polynomials. see Corollary 2.5.

### 2.3. Some arithmetic estimates
First we estimate the coefficients \( h_q \) for a given set (2.1). Hence, we have the following trivial identity
\begin{equation}
\sum_{q=1}^{\infty} h_q = k^2.
\end{equation}

Using the argument in the proof of [7, Proposition 1], which is also used in the proof of [2, Proposition 2.7], we immediately obtain the following bound.

**Lemma 2.2.** For any \( q \in \mathbb{N} \) we have \( h_q \leq kq^d \).

**Proof.** Clearly, for each \( i = 1, \ldots, k \) there are at most \( q^d \) vectors \( x \in T_d \) with \( q(x_i - x) \in \mathbb{Z}^d \). \( \square \)

We make use of the following upper bound on \( \text{cont}_q (m^t A(X)b_q) \) in Lemma 2.1, which follows from [2, Corollary 2.4].
Lemma 2.3. For any real $M \geq 1$ uniformly over vectors $m \in B(M)$ and vectors $b \in \mathbb{Z}^d$ with $\gcd(b, q) = 1$, we have

$$\text{cont}_q (m^tA(X)b) \ll (HM)^d.$$ 

Given an integer $\nu \geq 2$, an integer number $n$ is called

- $\nu$-th power free if any prime number $p | n$ satisfies $p^\nu \nmid n$;
- $\nu$-th power full if any prime number $p | n$ satisfies $p^\nu | n$.

We note that 1 is both $\nu$-th power free and $\nu$-th power full for any $\nu$.

For any integer $i \geq 2$ it is convenient to denote

$$F_\nu = \{n \in \mathbb{N} : n \text{ is } \nu\text{-th power full}\} \text{ and } F_\nu(x) = F_\nu \cap [1, x].$$

The classical result of Erdős and Szekeres [5] gives an asymptotic formula for the cardinality of $F_\nu(x)$ which we present here in a very relaxed form as the upper bound

$$\#F_\nu(x) \ll x^{1/\nu}. \tag{2.3}$$

2.4. Bounds of complete rational exponential sums. For $q \in \mathbb{N}$ and $f = (f_1, \ldots, f_e)$, we denote

$$S_{e,q}(f) = \sum_{n=1}^{q} e_q(f_1n + \ldots + f_en^e).$$

Our new tool is the following bound on $|S_{e,q}(f)|$ which is derived in [1] from the classical Weil and Hua bounds, see, for example, [8, Theorem 5.38], combined with, [3, Equation (2.5)] and the results of, [4, 9] giving a slight improvement of the Hua bound (see also [10, Theorem 7.1]).

Lemma 2.4. Write an integer $q \geq 1$ as $q = q_2 \ldots q_e$ such that

- $q_2 \geq 1$ is cube free,
- $q_i$ is $i$-th power full but $(i+1)$-th power free when $3 \leq i \leq e-1$,
- $q_e$ is $e$-th power full,

with $\gcd(q_i, q_j) = 1$, $2 \leq i < j \leq e$. For $f = (f_1, \ldots, f_e) \in \mathbb{Z}^e$ with

$$\gcd (q, f_1, \ldots, f_e) = 1,$$

we have

$$|S_{e,q}(f)| \leq \prod_{i=2}^{e} q_i^{1-1/i} q^{o(1)}.$$ 

We remark, that in [2] the bound

$$|S_{e,q}(f)| \leq q^{1-1/e + o(1)} \tag{2.4}$$
has been used. So our improvement comes from using Lemma 2.4 together with a classification of moduli \( q \) for which it gives an improvement of (2.4).

We now immediately derive from Lemma 2.4 the following more general bound.

**Corollary 2.5.** Write an integer \( q \geq 1 \) as \( q = q_2 \ldots q_e \) such that

- \( q_2 \geq 1 \) is cube free,
- \( q_i \) is \( i \)-th power full but \((i+1)\)-th power free when \( 3 \leq i \leq e-1 \),
- \( q_e \) is \( e \)-th power full,

with \( \gcd(q_i, q_j) = 1 \), \( 2 \leq i < j \leq e \). For \( f = (f_1, \ldots, f_e) \in \mathbb{Z}^e \) with

\[
\gcd(q, f_1, \ldots, f_e) = s,
\]

we have

\[
|S_{e,q}(f)| \leq q^{1+o(1)} \prod_{i=2}^{e} \left( q_i/\gcd(q_i, s) \right)^{-1/i}.
\]

3. **Proof of Theorem 1.1**

3.1. **Preliminary split.** Let \( \mathcal{X} \) be a set as in (2.1) such that for any \( n \in \mathbb{N} \) the dilation \( A(n)\mathcal{X} \) is not \( \varepsilon \)-dense in \( T_d \).

We choose some integer parameter \( R \geq 1 \) and using Lemma 2.1 write

\[
k^2 \ll \varepsilon^{-d}(S_1 + S_2) + \varepsilon^{-d}M^d k,
\]

where

\[
S_1 = \sum_{m \in B(M)} \sum_{q=1}^{R} \frac{h_q}{q} \left| \sum_{n=1}^{q} e_q (m'A(n)b_q) \right|,
\]

\[
S_2 = \sum_{m \in B(M)} \sum_{q=R+1}^{\infty} \frac{h_q}{q} \left| \sum_{n=1}^{q} e_q (m'A(n)b_q) \right|.
\]

3.2. **Bound on** \( S_1 \). To estimate \( S_1 \) we first use Lemma 2.2 and write

\[
S_1 \leq k \sum_{m \in B(M)} \sum_{q=1}^{R} q^{d-1} \left| \sum_{n=1}^{q} e_q (m'A(n)b_q) \right|.
\]
We now note that with \( q_2, \ldots, q_e \) defined as in Corollary 2.5, we have
\[
\left| \sum_{n=1}^{q} e_q \left( m^t A(n) b_q \right) \right| 
\leq q^{1+o(1)} \prod_{i=2}^{e} \left( \frac{q_i}{\gcd(q_i, \cont_q(m^t A(X)b_q))} \right)^{-1/i}
\leq \cont_q \left( m^t A(X)b_q \right)^{1/2} q^{1+o(1)} \prod_{i=2}^{e} q_i^{-1/i}
\]
and using Lemma 2.3, we obtain
\[
(3.3) \quad \left| \sum_{n=1}^{q} e_q \left( m^t A(n) b_q \right) \right| 
\leq (HM)^{d/2} q^{1+o(1)} \prod_{i=2}^{e} q_i^{-1/i}.
\]

We now define the sets
\[
G_\nu(x) = F_\nu(x) \setminus F_\nu(x/2)
\]
where, as in Section 2.3, \( F_\nu(x) \) denotes the set of \( \nu \)-th power full positive integers \( n \leq x \).

Then using the dyadic partition of the whole domain of possible values of \( q_2, \ldots, q_e \) as in Corollary 2.5 in (3.2). That is, we fix a family \( \Omega \) of at most \( O((\log R)^{e-1}) \) vectors of real parameters \( (Q_2, \ldots, Q_e) \) with
\[
(3.4) \quad Q_2 \ldots Q_e \leq R
\]
and cover the whole range where \( q_2, \ldots, q_e \) may vary, by rectangular boxes with
\[
q_2 \sim Q_2, \ldots, q_e \sim Q_e
\]
for \( (Q_2, \ldots, Q_e) \in \Omega \). Recalling the arithmetic structure of \( q_2, \ldots, q_e \) we see that in fact
\[
q_3 \in G_3 (Q_3), \ldots, q_e \in G_e (Q_e).
\]
We now see that there are some real numbers \( Q_2, \ldots, Q_e \) with (3.4) for which we have
\[
S_1 \ll k (\log R)^{e-1} \sum_{m \in \mathcal{B}(M)} \sum_{q_2, \ldots, q_e} (q_2 \ldots q_e)^{d-1}
\sum_{q_h \in \mathcal{G}_h (Q_h), 3 \leq h \leq e, \gcd(q_i, q_j) = 1, 2 \leq i < j \leq e} \left| \sum_{n=1}^{q_2 \ldots q_e} e_{q_2 \ldots q_e} \left( m^t A(n) b_{q_2 \ldots q_e} \right) \right|.
\]
Hence, recalling (3.4) and applying the bound (3.3), we obtain
\[ S_1 \leq kH^{d/2}M^{3d/2}R^{d+o(1)}Q_2^{1/2} \prod_{i=3}^{e} \left( \#G_i(Q_i)Q_i^{-1/i} \right). \]

Finally, using the bound (2.3), we obtain
\begin{equation}
S_1 \leq kH^{d/2}M^{3d/2}R^{d+o(1)}Q_2^{1/2} \leq kH^{d/2}M^{3d/2}R^{d+1/2+o(1)}.
\end{equation}

3.3. Bound on $S_2$. To estimate $S_2$ we use the bound of Corollary 2.5 in the following crude form $|S_{e,q}(f)| \leq q^{1-1/e+o(1)}s^{1/e}$ (which is exactly the bound used in [2]). However it is technically more convenient (but does not affect the final result) to use a slightly more precise bound
\begin{equation}
S_{e,q}(f) \ll q^{1-1/e}s^{1/e}
\end{equation}
without $o(1)$ in the exponent, see [4,9] for explicit evaluations of the implied constant.

The bound (3.6), together with (2.2) and Lemma 2.3 implies
\begin{equation}
S_2 \leq (HM)^{1/e} \sum_{m \in B(M)} \sum_{q=R+1}^{\infty} h_q q^{-1/e} \ll k^2 H^{1/e} M^{d+1/e} R^{-1/e}.
\end{equation}

3.4. Concluding the proof. Substituting the bounds (3.5) and (3.7) in (3.1), and recalling the value of $M$ in Lemma 2.1, we obtain
\begin{equation}
k^2 \ll kH^{d/2} \varepsilon^{-5d/2} R^{d+1/2+o(1)} + k^2 H^{1/e} \varepsilon^{-2d-1/e} R^{-1/e}
\end{equation}
(clearly the term $\varepsilon^{-d}M^d$ can be absorbed in the first term). Taking
\begin{equation}
R = CH\varepsilon^{-2de-1}
\end{equation}
for a sufficiently large constant $C$ (which depends only on $d$ and $e$), we see that the contribution from the second term in (3.8) (together with the implied constant) does not exceed $k^2/2$. Hence, for the choice of $R$ as in (3.9) we have
\[ k^2 \leq kH^{d/2} \varepsilon^{-5d/2} R^{d+1/2+o(1)} = kH^{(3d+1)/2+o(1)} \varepsilon^{-2d^2e-de-7d/2-1/2+o(1)}, \]
which concludes the proof.
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