Free field representation of Toda field theories

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Abstract

We study the following problem: can a classical $sl_n$ Toda field theory be represented by means of free bosonic oscillators through a Drinfeld–Sokolov construction? We answer affirmatively in the case of a cylindrical space–time and for real hyperbolic solutions of the Toda field equations. We establish in fact a one–to–one correspondence between such solutions and the space of free left and right bosonic oscillators with coincident zero modes. We discuss the same problem for real singular solutions with non hyperbolic monodromy.
1 Introduction

Toda field theories in 2D have become a favorite research topic in theoretical physics. One reason for this interest lies in the fact that Toda field theories (ToFT’s) based on classical finite dimensional Lie algebras (in this paper we will be dealing only with the latter) underlie many remarkable conformal field theory models – in particular all sort of minimal models –; in fact they possess both the necessary conformal structure, which manifests itself for example in the chiral splitting, and the integrable structure, which is connected with the hidden quantum group symmetry typical of these models. Both structures are simultaneously expressed in an elegant form by the exchange algebra.

Another reason of interest is the connection of the \( sl_2 \) ToFT, i.e. the Liouville theory ([1],[2],[3], with string theory and 2D gravity, together with the evoked possibility that \( sl_n \) ToFT might bear a relation to 2D gravity coupled to conformal matter in much the same way as the latter combination appears in matrix models. The present state of affairs does not even allow us to exclude that there might be a direct connection between matrix models and Liouville or Toda theories formulated on the lattice.

Toda field theories can be regarded as Hamiltonian reductions of WZNW models – this is in itself a vast field of research – or – as we do in this paper – as autonomous field theories characterized by a W–algebras symmetry. They are in fact at the origin of the present interest in W–algebra. The geometrical meaning of W–algebras is still rather obscure. However if we think that the Liouville equation is well–known to be the basis of the uniformization theory of Riemann surfaces, it is not unmotivated to expect geometry to play a deep role in ToFTs and, viceversa, that the latter might lead to significant geometrical developments.

All this sounds pretty appealing to all those who have followed the most recent developments in theoretical physics. On the other hand, even though the research in this field has been intensive ([5],[4],[6],[7]), many questions in Toda field theories are still unanswered. Among the latter we quote in particular the problem of constructing conformal blocks in W–algebra. We feel a more thorough analysis of ToFT’s is necessary. This motivated us in taking up this research, which could be synthesized as the search for an answer to the following question: to what extent can we represent ToFTs by means of free bosonic oscillators ? In this sense this paper is the continuation of [8]. There it was shown that to every ToFT we can associate two (one for each chirality) Drinfeld–Sokolov (DS) linear systems defined in terms of independent free bosonic oscillator fields. By means of these we can construct solutions of the ToFT equations of motion. The question left unanswered in [8] was: Do we construct in this way all the periodic and local solutions of the ToFT equations of motion so that the Poisson brackets for the bosonic oscillators correspond to the canonical Poisson brackets in the ToFT?

In this paper we address the above two problems for an \( sl_n \) ToFT defined in a cylindrical space–time, where space is represented by a circle and time by...
a straight line, therefore with periodic space boundary conditions. The most important result is the following: we construct a one-to-one correspondence between real hyperbolic solutions of ToFT equations and appropriate set of free bosonic oscillator fields of the DS linear systems which preserve the canonical symplectic structure of the ToFT. We realize in this way a parametrization of the classical phase space of the ToFT’s, which lends itself to canonical quantization. In conclusion the above question has the following answer: ToFT’s can be fully represented by means of free bosonic oscillators, with the limitations expressed above.

On the one hand, this conclusion can be considered, as a completion of a program started by Leznov and Saveliev \[9\] and continued in \[8\]. On the other hand the same problem is still open for more complicated topologies (see \[10\] for a discussion over Riemann surfaces).

We partially analyze also the (singular) solutions of parabolic monodromy. For a large family of them it is also possible to define a free field representation. The role of these solutions in quantizing the theory is unclear.

The paper is organized as follows. Since we wish the paper to be as self-contained as possible, in section 2 we review ref.\[8\] and state the problem announced above in a more precise form. In section 3 we show how to find the solution for the Liouville theory. In section 4 we extend the proof to an \(sl_n\) Toda field theory. Section 3 is used as a general guide and presented in a very detailed form, while section 4 is more sketchy. In section 5 we discuss the family of singular solutions of the Liouville equation referred to above.

2 The problem

Let \(\mathcal{G}\) be a simple finite dimensional Lie algebra of rank \(r\), equipped with an invariant scalar product denoted by \((\ , \ )\). We choose a Cartan subalgebra \(\mathcal{H}\) with an orthonormal basis \(\{H_i\}\). We recall the following commutation relations in a Cartan-Weyl basis

\[
\begin{align*}
[H, E_{\pm\alpha}] &= \pm \alpha(H) E_{\pm\alpha} \\
[E_{\alpha}, E_{-\alpha}] &= H_{\alpha}
\end{align*}
\]

Toda field theories are defined by means of a linear system.

\[
(\partial_{\pm} + A_{\pm})T = 0 \tag{1}
\]

where \(x_{\pm} = x \pm t\) are the light cone coordinates, and \(\partial_{\pm} \equiv \partial_{x_{\pm}} = \frac{1}{2}(\partial_x \pm \partial_t)\).

\[
A_+ = \partial_+ \Phi + e^{ad_{\Phi}} E_+ , \quad A_- = -\partial_- \Phi + e^{-ad_{\Phi}} E_-
\]

\[2\] Hyperbolic solution means a solution with hyperbolic monodromy. All regular enough solutions are hyperbolic. For a more precise statement see Appendix A and B.
The field $\Phi$ takes values in the Cartan subalgebra and

$$
\mathcal{E}_+ = \sum_{\alpha \text{ simple}} E_{\alpha} \\
\mathcal{E}_- = \sum_{\alpha \text{ simple}} E_{-\alpha}
$$

The zero curvature condition

$$
F_{+-} = \partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0
$$

yields the equations of motion

$$
\partial_+ \partial_- \Phi = \frac{1}{2} \sum_{\alpha \text{ simple}} e^{2\alpha(\Phi)} H_\alpha
$$

There is a standard way to represent solutions of eq.(2). Given a highest weight vector $|\Lambda^{(r)}\rangle$, we define

$$
\xi^{(r)}(x) = <\Lambda^{(r)}| e^{-\Phi(x)} T(x) \\
\bar{\xi}^{(r)}(x) = T^{-1}(x) e^{-\Phi(x)} |\Lambda^{(r)}> \tag{3}
$$

where $T(x)$ is the transport matrix

$$
T(x) = P \exp \left( - \int_0^x A_x dx \right)
$$

and $A_x = A_+ + A_-$. $T(x)$ is the solution of the equation $(\partial_x + A_x)T = 0$ with the initial condition $T(0) = 1$.

Using the explicit form of $A_+$ and $A_-$ one can easily show that $\xi$ and $\bar{\xi}$ are chiral objects, i.e.

$$
\partial_- \xi = 0 \quad \partial_+ \bar{\xi} = 0
$$

The objects defined by eqs.(3) satisfy the exchange algebra

$$
\{\xi^{(r)}(x) \otimes \xi^{(r')}(y)\} = \xi^{(r)}(x) \otimes \xi^{(r')}(y) [\theta(x - y) r^+ + \theta(y - x) r^-] \\
\{\bar{\xi}^{(r)}(x) \otimes \bar{\xi}^{(r')}(y)\} = [\theta(x - y) r^- + \theta(y - x) r^+] \xi^{(r)}(x) \otimes \bar{\xi}^{(r')}(y) \\
\{\xi^{(r)}(x) \otimes \bar{\xi}^{(r')}(y)\} = -\xi^{(r)}(x) \otimes 1 \cdot r^- \cdot 1 \otimes \bar{\xi}^{(r')}(y) \\
\{\bar{\xi}^{(r)}(x) \otimes \xi^{(r')}(y)\} = -1 \otimes \xi^{(r')}(y) \cdot r^+ \cdot \bar{\xi}^{(r)}(x) \otimes 1 \tag{4}
$$

where $r^\pm$ are the solutions of the classical Yang-Baxter equation

$$
r^+ = t_0 + 2 \sum_{\alpha \text{ positive}} \frac{E_\alpha \otimes E_{-\alpha}}{(E_\alpha, E_{-\alpha})} \\
r^- = -t_0 - 2 \sum_{\alpha \text{ positive}} \frac{E_{-\alpha} \otimes E_\alpha}{(E_{-\alpha}, E_\alpha)} \tag{5}
$$
and

\[ t_0 = \sum_i H_i \otimes H_i \]

This is a consequence of the canonical Poisson bracket defined by

\[ \{ \pi_\Phi(x) \otimes \Phi(y) \} = \delta(x - y) \cdot t_0 \]

\( \pi_\Phi \) being the canonical momentum of \( \Phi \), i.e. \( \pi_\Phi = \partial_t \Phi \).

If the Toda field \( \Phi(x) \) is periodic, the \( \xi, \bar{\xi} \) fields have the monodromy properties

\[ \xi^{(r)}(x + 2\pi) = \xi^{(r)}(x) \cdot T \]
\[ \bar{\xi}^{(r)}(x + 2\pi) = T^{-1} \bar{\xi}^{(r)}(x) \]

where \( T = T(2\pi) \). We also recall the following Poisson brackets

\[ \{ T \otimes T \} = -[r_{12}^+, T \otimes T] \]
\[ \{ \xi^{(r)}(x) \otimes T \} = \xi^{(r)}(x) \otimes T \cdot r^- \]
\[ \{ \bar{\xi}^{(r)}(x) \otimes T \} = -1 \otimes T \cdot r^+ \cdot \bar{\xi}^{(r)}(x) \otimes 1 \]

Finally, from eq.(3), we see that the fields of Toda theory, are reconstructed by means of the formula

\[ e^{-2\lambda^{(r)}(\Phi)(x_+,x_-)} = \xi^{(r)}(x_+) \cdot \bar{\xi}^{(r)}(x_-) \]

where \( \lambda^{(r)} \) is the weight corresponding to \( |\Lambda^{(r)}| > \). These fields are obviously periodic and one can check that they are local, i.e. they Poisson commute at equal time.

Presented in this form all the matter sounds a bit tautological, since we have to already know the solutions of the Toda field equations in order for the above formulas (in particular eq.(8)) to work. In this way we do not get any explicit representation of the space of classical solutions. To achieve this we have to proceed in a different way, i.e. consider separately the two chiral halves of the theory and introduce the so-called associated Drinfeld–Sokolov linear systems [11]:

\[ \partial_+ Q_+ - (P - \mathcal{E}_+) Q_+ = 0, \quad \partial_- Q_+ = 0 \]
\[ \partial_+ Q_- = 0, \quad \partial_- Q_- + Q_- (\mathcal{T} - \mathcal{E}_-) = 0 \]

where \( Q_+(x_+) \) and \( Q_-(x_-) \) takes values in a Lie group whose Lie algebra is \( \mathcal{G} \), and where \( P \) and \( \mathcal{T} \) are periodic chiral and antichiral, respectively, fields which take values in the Cartan subalgebra and have the Poisson brackets

\[ \{ P(x) \otimes P(y) \} = - (\partial_x - \partial_y) \delta(x - y) t_0 \]
\[ \{ P(x) \otimes \mathcal{T}(y) \} = 0 \]
\[ \{ \mathcal{T}(x) \otimes \mathcal{T}(y) \} = (\partial_x - \partial_y) \delta(x - y) t_0 \]
At times we will refer to $P$ and $\bar{P}$ as DS fields. From the solution $Q_+(x)$ and $Q_-(x)$ of eqs. (10) normalised by $Q_+(0) = 1$, $Q_-(0) = 1$ we define a basis $\sigma$, $\bar{\sigma}$

$\sigma^{(r)}(x) = < \Lambda^{(r)}|Q_+(x) >$

$\bar{\sigma}^{(r)}(x) = Q_-(x)|\Lambda^{(r)}>$

This basis has the Poisson bracket algebra

$\{\sigma^{(r)}(x) \otimes \sigma^{(r)}(y)\} = \sigma^{(r)}(x) \otimes \sigma^{(r)}(y)[\theta(x-y)r^+ + \theta(y-x)r^-]$ (14)

$\{\bar{\sigma}^{(r)}(x) \otimes \bar{\sigma}^{(r)}(y)\} = [\theta(x-y)r^- + \theta(y-x)r^+]\sigma^{(r)}(x) \otimes \bar{\sigma}^{(r)}(y)$ (15)

while

$\{\sigma^{(r)}(x) \otimes \bar{\sigma}^{(r)}(y)\} = 0$ (16)

Since $P(x)$ and $\bar{P}(x)$ are periodic we can expand them in Fourier series

$P(x) = \sum_n P_n e^{inx}$, $\bar{P}(x) = \sum_n \bar{P}_n e^{inx}$

An important role in the following is played by the left and right monodromy matrices

$S = Q_+(2\pi)$, $\bar{S} = Q_-(2\pi)$

We will also need $K$ and $\bar{K}$, defined as follows

$K = \sum_{n\neq 0} \frac{iP_n}{n}$, $\bar{K} = \sum_{n\neq 0} \frac{i\bar{P}_n}{n}$

The aim now is to construct periodic local solutions of the the Toda field equations (2), exactly as the formal solutions (8) are, but in terms of the free bosonic fields $P$ and $\bar{P}$. A solution to eqs. (2) is given by

$e^{-2\lambda^{(r)}(\Phi)(x_+, x_-)} = \sigma^{(r)}(x_+)M\bar{\sigma}^{(r)}(x_-)$ (17)

where $M$ is a constant matrix to be determined. Since

$\sigma^{(r)}(x + 2\pi) = \sigma^{(r)}(x)S$, $\bar{\sigma}^{(r)}(x + 2\pi) = \bar{S}\sigma^{(r)}(x)$

to get a periodic solution, we must have

$S M \bar{S} = M$ (18)

In order to satisfy this equation one proceeds to diagonalize the monodromy matrices

$S = gS\kappa gS^{-1}$, $\kappa = e^{2\pi P_0}$

$\bar{S} = \bar{gS}\bar{\kappa}gS^{-1}$, $\bar{\kappa} = e^{-2\pi \bar{P}_0}$ (19)
Then condition (18) will be satisfied if
\[ M = g_S D \bar{g}_S^{-1} \]
\[ \kappa \bar{\kappa} = 1 \]
where \( D \in \exp(\mathcal{H}) \), \( \mathcal{H} \) being the Cartan subalgebra. The second condition simply means that \( P_0 = \bar{P}_0 \) and will eventually be imposed. The diagonal matrix \( D \) has to be chosen in such a way that the fields of the LHS of eq.(18) be local. The solution is
\[ D = \Theta \Theta, \quad \Theta = e^{Q-K}, \quad \bar{\Theta} = e^{\bar{Q}+\bar{K}} \]
and \( Q, \bar{Q} \) are the conjugate variables of \( P_0 \) and \( \bar{P}_0 \), respectively. Therefore
\[ \{ Q \otimes P_0 \} = \frac{1}{\pi} t_0 \]
\[ \{ \bar{Q} \otimes \bar{P}_0 \} = \frac{1}{\pi} t_0 \]
Finally we can write
\[ e^{-2\lambda(r)\Phi(x_+,x_-)}(x_+ \bar{\psi}^{(r)}(x_-)) \]
where we define the new objects (Bloch wave basis)
\[ \psi^{(r)}(x) = \sigma^{(r)}(x)g_S \Theta, \quad \bar{\psi}^{(r)}(x) = \bar{\Theta} \bar{g}_S^{-1} \bar{\sigma}^{(r)}(x) \]
The \( \psi \) and \( \bar{\psi} \) have diagonal monodromy \( \kappa \) and \( \bar{\kappa} \), respectively, and obey the exchange algebra
\[ \{ \psi^{(r)}(x) \otimes \psi^{(r)}(y) \} = -\frac{1}{2} \psi^{(r)}(x) \otimes \psi^{(r)}(y) \left\{ \epsilon(x-y)(r^+ - r^-) - \coth(\pi a d_1 P_0)(r^+ - t_0) - \coth(\pi a d_2 P_0)(r^- + t_0) \right\} \]
\[ \{ \bar{\psi}^{(r)}(x) \otimes \bar{\psi}^{(r)}(y) \} = \frac{1}{2} \left\{ \epsilon(x-y)(r^+ - r^-) + \coth(\pi a d_1 \bar{P}_0)(r^- + t_0) + \coth(\pi a d_2 \bar{P}_0)(r^+ - t_0) \right\} \bar{\psi}^{(r)}(x) \otimes \bar{\psi}^{(r)}(y) \]
and, as long as the \( P_0 \) and \( \bar{P}_0 \) are considered independent, we also have
\[ \{ \psi^{(r)}(x) \otimes \bar{\psi}^{(r)}(y) \} = 0 \]
It is now easy to prove that eq.(21) represents a general solution of eqs.(2) which is both periodic and local provided we reduce the phase space by imposing \( P_0 = \bar{P}_0 \).

It is clear that the free bosonic oscillators \( P_n \) and \( \bar{P}_n \) (together with \( Q \) and \( \bar{Q} \)) provide a parametrization of the space of classical periodic and local solutions of the Toda field equations. One can also verify by means of (21) that (11) implies (6). However there still remain an unanswered question:

**Does the above construction (referred to in the following as the DS construction) exhaust all the periodic and local solutions of the Toda field equations, so**
that the canonical Poisson brackets (6) match the Poisson brackets (11) for the free bosonic oscillators?

The rest of the paper is devoted to answering this question. Before we end this section two remarks are in order:

Remark 1. The DS construction could also proceed in a slightly different manner. We could first identify \( P_0 \) with \( \bar{P}_0 \) and then introduce \( D = e^{Q+\bar{K}-\bar{K}} \) in \( \sigma^{(r)} g S \bar{D} \sigma^{-1} \sigma^{(r)} \) together with \( Q = \bar{Q} \) in eq. (21). All the above conclusions would hold except, of course, eq. (25). This remark will be useful in the next section.

In fact, since in the next section we reconstruct the DS system starting from a periodic solution \( \varphi \), we expect the left and right zero modes to coincide.

Remark 2. The problem studied in the next sections is not completely new in the literature. At least for the Liouville model two fields \( p \) and \( \tilde{p} \) were found in [1] that correspond to the two solutions for the \( p \) field found in the next section. However that result is limited to the open string case, which has particular features.

### 2.1 Description of the method for solving the problem

The method for solving the problem consists of two steps.

The most natural (but, as it will turn out, insufficient) idea is to perform successive field–dependent gauge transformations on the initial linear system (1) so as to reduce it to the DS form – we will see later on that this corresponds to the Gauss decomposition of the transport matrix [9]. Let us consider first the following transformations

\[
V(x) = e^{-\Phi(x)} T(x), \quad \bar{V}(x) = T^{-1}(x) e^{-\Phi(x)}
\]  

(26)

This leads to the equivalent linear systems

\[
\begin{align*}
(\partial_+ + A_+^\nu) V &= 0, \\
(\partial_- + A_-^\nu) V &= 0,
\end{align*}
\]

with

\[
A_+^\nu = 2 \partial_+ \Phi + \mathcal{E}_+ \]

\[
A_-^\nu = e^{-2ad_\Phi} \mathcal{E}_-
\]

(27)

and, similarly, for \( \bar{V} \)

\[
\begin{align*}
\partial_+ \bar{V} &= \bar{V} A_+^\nu, \\
\partial_- \bar{V} &= \bar{V} A_-^\nu,
\end{align*}
\]

with

\[
A_+^\nu = e^{2ad_\Phi} \mathcal{E}_+ \]

\[
A_-^\nu = -2 \partial_- \Phi + \mathcal{E}_-.
\]

(28)

Notice that in terms of \( V \) and \( \bar{V} \), we have

\[
e^{-2\Lambda^{(r)}(\Phi)} = < \Lambda^{(r)} | V \bar{V} | \Lambda^{(r)} >= \xi^{(r)} \cdot \bar{\xi}^{(r)}.
\]

(29)

We will see in the next sections that there are gauge transformations which map \( A_+^\nu \) to zero, and other gauge transformations for the second system which similarly maps \( A_-^\nu \) to zero. We obtain a system which is similar in form to the
DS system. However the two systems do not yet coincide. A further step is necessary: it consists in diagonalizing the monodromy matrix $T$ – actually this is logically the first operation one must carry out. After that we will be able to reconstruct the true DS system. In the next section we will see this method at work for the Liouville model.

## 3 The Liouville model

### 3.1 Introduction

The Liouville model is the $sl_2$ ToFT. In such a case we put

$$\Phi(x) = \frac{1}{2} \varphi(x) H, \quad P(x) = p(x) H, \quad Q = q H, \quad \mathcal{E}_\pm = E_\pm$$

etc. In the defining representation, whose highest weight vector is $|\Lambda> = (1\ 0)$, we have

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The equation of motion is

$$\partial_+ \partial_- \varphi = e^{2\varphi} \quad (30)$$

We will also use the notation

$$V = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}, \quad \bar{V} = \begin{pmatrix} \bar{\xi}_{11} & \bar{\xi}_{12} \\ \bar{\xi}_{21} & \bar{\xi}_{22} \end{pmatrix}, \quad T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad (31)$$

Our problem can be formulated as follows:

For any solution $\varphi$ of the Liouville equation construct a couple of real fields $p$ and $\bar{p}$ such that

1) $p$ is chiral, $\bar{p}$ is antichiral;
2) $p$ and $\bar{p}$ are periodic;
3) they satisfy the Poisson brackets:

$$\{p(x), p(y)\} = -(\partial_x - \partial_y)\delta(x-y)$$
$$\{p(x), \bar{p}(y)\} = 0$$
$$\{\bar{p}(x), \bar{p}(y)\} = (\partial_x - \partial_y)\delta(x-y) \quad (32)$$

and such that the zero modes $p_0$ and $\bar{p}_0$ coincide. Moreover, construct $q$ such that

$$\{q, p_0\} = \frac{1}{\pi} \quad (33)$$

and Poisson commute with the remaining degrees of freedom. Verify that $p(x), \bar{p}(x)$ and $q$, inserted into the DS construction give rise to exactly the solution $\varphi$ we started from.
3.2 Reduction to the DS form

Let us gauge away $A_V^\pm$. We consider the transformation

$$V \rightarrow h^{-1}V, \quad A_V^\pm \rightarrow h^{-1}A_V^\pm h + h^{-1}\partial_\pm h,$$

where $h := h(x,t)$ has the form

$$h = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}.$$

Imposing the condition we aim at, we find

$$hA_{-}^\pm = 0 \Rightarrow \partial_- a + e^{2\varphi} = 0$$

In order to solve for $a$ we proceed as follows. We notice that

$$a = -\partial_+ \varphi + \text{(chiral terms)}.$$

We can determine $a$ more precisely by imposing that $A_V^\pm$ remains upper triangular. Since

$$h^{-1}V = \begin{pmatrix} \xi_{11} & \xi_{12} \\ -a\xi_{11} + \xi_{21} & -a\xi_{12} + \xi_{22} \end{pmatrix}$$

we must have

$$-a\xi_{11} + \xi_{21} = 0$$

In this way we find

$$A_+^{(+)} := gA_+^\pm = \begin{pmatrix} -p & 1 \\ 0 & p \end{pmatrix},$$

where

$$p = -\partial_+ \varphi - \xi_{21}\xi_{11}^{-1}.$$

It is easy to verify by a direct computation that $\partial_- p = 0$.

We can do the same for the antichiral sector using a strictly upper triangular matrix $\bar{h}$, and find

$$\bar{h}A_{+}^\pm = 0, \quad A_-^{(-)} := \bar{h}A_-^\pm = \begin{pmatrix} -\bar{p} & 0 \\ 1 & \bar{p} \end{pmatrix},$$

where

$$\bar{p} = \partial_- \varphi - \xi_{12}\xi_{11}^{-1}.$$
is antichiral.

It looks like we have reached our goal, that is we have reconstructed the DS system, but it is not so. For, although the Poisson bracket of $p$ with $p$ and $\bar{p}$ with $\bar{p}$ are the expected ones (32), $p$ and $\bar{p}$ do not Poisson commute. Moreover, by considering the effect of the monodromy matrix, we can quickly verify that $p$ and $\bar{p}$ are not periodic. To obtain the correct answer we have first to diagonalize the monodromy matrix (the further step announced above).

### 3.3 Diagonalizing the monodromy

In section 2 we introduced the initial condition $T(0) = 1$. However there is no a priori reason why we should use precisely this condition. We are free to change it at will. We take advantage of this freedom to diagonalize the monodromy matrix. More precisely, we change the initial data of the linear systems by making the transformation

$$V \rightarrow Vg, \quad \bar{V} \rightarrow g^{-1}\bar{V}$$

with $g$ a unimodular $2 \times 2$ constant matrix. Accordingly, the monodromy will change as

$$T \rightarrow T^g = g^{-1}Tg$$

We would like $T^g$ to be diagonal (or, at least, triangular). This is always possible over the complex numbers, i.e. with $g \in SL_2(\mathbb{C})$. However, given a real solution $\varphi$ of the Liouville equation, we are interested in doing this over the reals, i.e. with $g \in SL_2(\mathbb{R})$, since only in this case will the $p$ and $\bar{p}$ fields be real (see below). Although the answer to this question is known to be positive [12], we have not been able to find an explicit proof of it in the literature. Therefore we think it useful, also in view of the $sl_n$ case, to exhibit an explicit proof that for regular real solutions $\varphi$ the monodromy is hyperbolic, i.e. $tr\ T > 2$. The proof is given in Appendix A; here we assume the result, which allows us to conclude that $T$ can be diagonalized by means of a real $g$ matrix.

For the diagonalizing matrix $g$ let us make the position

$$g = RS$$

with

$$R = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

In this way $r$ satisfies a second order equation and $s$ is determined after $r$. We could have made other choices for $R$ and $S$, the results would be completely equivalent. What is important is that in the hyperbolic case we have two real solutions for $r$, namely

$$r_\pm = \frac{-(\alpha - \delta) \pm \sqrt{\Delta}}{2\beta}$$
where
\[ \Delta = (\alpha + \delta)^2 - 4 \equiv (\text{tr } T)^2 - 4 \]
is the positive discriminant. The corresponding elements \( s_{\pm} \) are given by
\[ s_{\pm} = \pm \frac{\beta}{\sqrt{\Delta}} \]

Thus we distinguish between the two solutions by appending a plus or minus sign: \( g_{\pm} \). Their explicit forms are
\[ g_{\pm} = \begin{pmatrix} 1 & \pm \frac{\beta}{\sqrt{\Delta}} \\ r_{\pm} & \mp \frac{\beta}{\sqrt{\Delta}} r_{\mp} \end{pmatrix} \]
The difference between the two is that \( g_{+1}^T g_{+} = D, \quad D = \text{diag}(\lambda_+, \lambda_-) \), while \( g_{-1}^T g_{-} = D^{-1} \), where \( \lambda_{\pm} \) are the eigenvalues of \( T \). We will clarify later on the significance of these two possibilities. Henceforth, when the labels \( \pm \) are omitted we mean that the equations or statements are true for both choices, i.e. both for + and for −.

After this lengthy discussion of the diagonalizing matrix \( g \), let us return to the main problem. Let us apply the same procedure as in the last subsection, starting from \( V g \) and \( g^{-1} \bar{V} \) instead of \( V \) and \( \bar{V} \), i.e. we determine transformations \( h_g \) and \( \bar{h}_g \) which play the role of \( g \) and \( \bar{g} \), respectively. We end up with
\[ h_g A_{Vg} = 0, \quad \hat{A}_+ := h_g A_{Vg} = \begin{pmatrix} -p & 1 \\ 0 & p \end{pmatrix} \] \[ \bar{h}_g A_{\bar{V}g} = 0, \quad \hat{A}_- := \bar{h}_g A_{\bar{V}g} = \begin{pmatrix} -\bar{p} & 0 \\ 1 & \bar{p} \end{pmatrix} \]
where \( p \) and \( \bar{p} \) are represented by the bosonization formulas
\[ p = \partial_x \log \xi_{11}^g = \partial_x \log (\xi_{11} + r \xi_{12}) , \] \[ \bar{p} = -\partial_x \log \bar{\xi}_{11}^g = -\partial_x \log ((1 + rs) \bar{\xi}_{11} - s \bar{\xi}_{21}) . \]

Here \( \xi_{ij}^g \) (\( \bar{\xi}_{ij}^g \)) represent the elements of the matrix \( V g \) (\( \bar{V} g \)). Remember that each of the above equations is a shorthand for a pair of equations corresponding to the two solutions \( g_{\pm} \), so that we have in fact the pair \( p_{++}, \bar{p}_{++} \) and the pair \( p_{--}, \bar{p}_{--} \). For both choices, i.e. either + and −, \( p \) and \( \bar{p} \) are now periodic fields which satisfy the Poisson brackets \[ \{ \} \]

We notice that the zero modes of the fields \( p \) and \( \bar{p} \) are well defined. Indeed
\[ p_{\pm,0} = \log \lambda_{\pm} \]
\[ \bar{p}_{\pm,0} = -\log \lambda_{\mp} \]

\[^3\text{The Poisson brackets are calculated starting from the brackets} (\text{[12]}), \text{and those given in Appendix C}\]
So, as expected, they are equal, \( p_0 = \bar{p}_0 \) (indeed \( \lambda_- = 1/\lambda_+ \)).

Summarizing, we have (almost) achieved to reconstruct the DS linear system in terms of the interacting field \( \varphi \), since

\[
(\partial_+ + \hat{A}_+) \mathcal{V}_+^g = 0, \quad \partial_- \mathcal{V}_-^g = 0
\]

where \( \hat{A}_\pm \) are given above and

\[
\mathcal{V}_+^g = h_g^{-1} V g, \quad \mathcal{V}_-^g = g^{-1} \bar{V} \bar{h}_g
\]  

Moreover

\[
e^{-\varphi} = < \Lambda | \mathcal{V}_+^g \mathcal{V}_-^g | \Lambda >
\]  

However there are still two undefined points. The first is that \( \mathcal{V}_+^g \) and \( \mathcal{V}_-^g \) cannot yet be identified with \( Q_+ \) and \( Q_- \) as the initial conditions are different. The second point is that we have not yet retrieved the conjugate variable \( q \) to the zero modes (see Remark 1 at the end of the previous section). The two points are related.

Let us recall eqs. (45) and the reconstruction formula for the solutions of the previous section, and notice that if we want the DS systems to coincide the relation among them must be

\[
\mathcal{V}_+^g \mathcal{V}_-^g = Q_+ g_S D g_S^{-1} Q_-
\]

The simplest thing to do is to examine this relation at the origin, where \( Q_+(0) = Q_-(0) = 1 \). So we must have

\[
\mathcal{V}_+^g(0) \mathcal{V}_-^g(0) = g_S D g_S^{-1}
\]

On the other hand, due to the normalization condition \( T(0) = 1 \) on the transport matrix, at the origin we have \( V(0) = e^{-\Phi(0)} \) and \( \bar{V}(0) = e^{-\Phi(0)} \). After some algebra one finds

\[
\mathcal{V}_+^g(0) = \begin{pmatrix} e^{-\varphi_0/2} & s e^{-\varphi_0/2} \\ 0 & e^{\varphi_0/2} \end{pmatrix}, \quad \mathcal{V}_-^g(0) = \begin{pmatrix} (1 + sr)e^{-\varphi_0/2} & 0 \\ -re^{-\varphi_0/2} & (1 + sr)^{-1} e^{\varphi_0/2} \end{pmatrix}
\]

where \( \varphi_0 = \varphi(0) \). Now the matrices \( g_S \) and \( \bar{g}_S \) are unipotent, hence they can be uniquely determined on the basis of eq. (46). Simultaneously the matrix \( D \) is uniquely identified

\[
D = \begin{pmatrix} (1 + sr)e^{-\varphi_0} & 0 \\ 0 & (1 + sr)^{-1} e^{\varphi_0} \end{pmatrix}
\]

One can verify that

\[
\{ p_0, D \} = \frac{1}{\pi} H D
\]

which is what we wanted to prove.

In conclusion, we have reconstructed all the elements of the DS linear system in terms of the periodic solution \( \varphi \) of the Liouville equation.
3.4 Action of the Weyl group

Let us return now to the meaning of the existence of the two pairs \( p_+, \bar{p}_+ \) and \( p_-, \bar{p}_- \) of free fields satisfying all our requirements. First we notice that we can pass from the diagonal matrix \( D \) to \( D^{-1} \) via a Weyl transformation

\[
D^{-1} = wDw^{-1}, \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

(we identify the group element \( w \) with the element of the Weyl group it represents). It follows that \( g_-wg_+^{-1} \) commutes with \( T \), so that \( z = g_+^{-1}g_-w \) commutes with \( D \), hence is diagonal.

By construction, under a monodromy transformation we have

\[
\begin{align*}
\mathcal{V}_+^{g_+} &\rightarrow \mathcal{V}_+^{g_+} D, & \mathcal{V}_-^{g_-} &\rightarrow D^{-1}\mathcal{V}_+^{g_+} \\
\mathcal{V}_+^{g_+} &\rightarrow \mathcal{V}_+^{g_+} D^{-1}, & \mathcal{V}_-^{g_-} &\rightarrow D\mathcal{V}_-^{g_-}
\end{align*}
\]

but we also have

\[
\begin{align*}
\mathcal{V}_+^{g_-} w &\rightarrow \mathcal{V}_+^{g_-} wD, & w^{-1}\mathcal{V}_-^{g_-} &\rightarrow D^{-1}w^{-1}\mathcal{V}_-^{g_-}
\end{align*}
\]

so that the quantities \( \mathcal{V}_+^{g_-} w \) and \( w^{-1}\mathcal{V}_-^{g_-} \) have the same monodromy as \( \mathcal{V}_+^{g_+} \) and \( \mathcal{V}_+^{g_-} \), respectively. Actually we have \( \mathcal{V}_+^{g_-} w = \mathcal{V}_+^{g_-} z \) and \( w^{-1}\mathcal{V}_-^{g_-} = z^{-1}\mathcal{V}_-^{g_-} \) and since \( z \) is diagonal, the free bosonic fields constructed from \( \mathcal{V}_+^{g_-} w \) and \( w^{-1}\mathcal{V}_-^{g_-} \) are in fact equal to \( p_+ \) and \( \bar{p}_+ \). A similar conclusion holds for \( p_- \) and \( \bar{p}_- \).

The outcome of this discussion is that we can pass from one pair of solutions \( p, \bar{p} \) to the other pair through the action of the Weyl group.

4 The \( sl_n \) Toda field theories

In this section we generalize to \( sl_n \) Toda field theories what we have done in the previous section for the Liouville theory. In more detail, we do the following:

**For any hyperbolic solution \( \Phi \) of the Toda field equations we construct a couple of real fields \( P \) and \( \bar{P} \) valued in the Cartan subalgebra such that**

1) \( P \) is chiral, \( \bar{P} \) is antichiral;
2) \( P \) and \( \bar{P} \) are periodic;
3) they satisfy the Poisson brackets \([14]\).

Moreover the zero modes \( P_0 \) and \( \bar{P}_0 \) coincide and we construct the conjugate of them. \( P \) and \( \bar{P} \), inserted in the DS construction, give back the solution we started from. We also show that there are many possible choices of the pair \( P \) and \( \bar{P} \) for any given solution, such choices being mapped into one another by the action of the Weyl group.

Two remarks are in order:
– Contrary to the previous section, what we do in this section is explicitly representation independent.

– A large part of this section applies to Toda field theories based on any simple Lie algebra; however the proof is tailored to $sl_n$ and further work is needed to extend the results to any simple Lie algebra.

Let us pass now to the proofs. Instructed by the $sl_2$ case, we proceed as follows. First we prove that the monodromy matrix is hyperbolic (this is actually done in Appendix B). Then we apply the Cartan decomposition to the transport matrix and extract the $P$ and $\bar{P}$ fields. Next we prove that they have the correct Poisson brackets (11). Finally we discuss the zero modes and their conjugate momenta. As an aside we obtain the result concerning the action of the Weyl group.

4.1 Diagonalizing the Monodromy Matrix

In Appendix B we prove that the monodromy matrix $T$ for hyperbolic solutions of any $\mathcal{G} = sl_n$ Toda field theory is hyperbolic, i.e. has real positive eigenvalues and can be diagonalized within $SL_n(\mathbb{R})$. I.e. we have

$$ T^g = g^{-1}Tg \quad (48) $$

where $T^g$ is diagonal matrix with positive diagonal entries and $g$ is a suitable element of $SL_n(\mathbb{R})$. Here we assume this result and show some simple consequences.

Let us consider an arbitrary finite dimensional representation of the Lie algebra $\mathcal{G}$, which can be lifted to a representation of $SL_n(\mathbb{R})$. The representation space is spanned by vectors $|\mu, m\rangle$ where $\mu$ is a weight and $m$ counts its degeneracy. We have

$$ T^g |\mu, m\rangle = \mu(T^g) |\mu, m\rangle \quad (49) $$

The first consequence we extract from eqs. (48,49) is the Poisson brackets of an arbitrary dynamical variable $\mathcal{A}$, $\{\mathcal{A}, g\}$, in terms of simpler and known Poisson brackets. We easily find

$$ <\nu, n|g^{-1}\{\mathcal{A}, g\}|\mu, m> = \frac{1}{\mu(T^g) - \nu(T^g)} <\nu, n|g^{-1}\{\mathcal{A}, T\}g|\mu, m> \quad (50) $$

This identity fixes $g^{-1}\{\mathcal{A}, g\}$ up to terms belonging to the Cartan subalgebra. In fact, in order to find $\{P \otimes P\}$, $\{\bar{P} \otimes \bar{P}\}$ and $\{P \otimes \bar{P}\}$ we do not need to know explicitly these terms; however they are important in showing that the fields $P$ and $\bar{P}$ have the correct Poisson brackets with the conjugate variable $D$. In order to find a more complete result let us recall the Cartan decomposition of $\mathcal{G}$

$$ \mathcal{G} = \mathcal{N}_+ \oplus \mathcal{H} \oplus \mathcal{N}_- \quad (51) $$
and its Borel subalgebras

$$B_{\pm} = N_{\pm} \oplus \mathcal{H}$$

(52)

By $P_H, P_{N_{\pm}}$ we will denote the projection operators on the corresponding subalgebras.

Next, our first observation is that if $g = g_+ g_- k$ ( $g_{\pm} \in \mathcal{E}_{N_{\pm}}$, $k \in \mathcal{E}_{\mathcal{H}}$) diagonalizes the monodromy matrix, the element $g^{-1} k$ will also diagonalize $T$. Therefore, we can set $g = g_+ g_-$. For this special choice of the diagonalizing element the Poisson bracket

$$g^{-1} \{ A, g \} \cdot g^{-1}$$

is orthogonal to the Cartan subalgebra $\mathcal{H}$. From this observation and from (50) one obtains

$$g^{-1} \{ A, g \} = - \sum_{\mu \neq \nu} \sum_{\alpha > 0} \frac{1}{\mu(T_g) - \nu(T_g)} \cdot \mu(T_g)^{-1} \cdot \nu(T_g)^{-1}$$

(53)

$$\cdot \left( \nu, n | g^{-1} \{ A, T \} g | \mu, m > < \mu, m | E_n | \nu, n > \cdot (g_+ E_{-\alpha} g_+^{-1}, H_p) \cdot H_p + \right. \cdot \left. \sum_{\mu \neq \nu} \frac{1}{\mu(T_g) - \nu(T_g)} \cdot \mu(T_g)^{-1} \cdot \nu(T_g)^{-1} \cdot \langle \nu, n | g^{-1} \{ A, T \} g | \mu, m > < \mu, m | E_{-\alpha} | \nu, n > \cdot E_\alpha \right)$$

where $\alpha$ in the first term runs over the set of all positive roots while the summation in $\alpha$ in the second term is over all (positive and negative) roots.

We shall finish this subsection with the following remark. The matrix $g$ which diagonalizes the monodromy matrix $T$ is not fixed uniquely. This comes from the fact that the Cartan subgroup is invariant under the elements $k w$ where $k$ is in the Cartan subgroup and $w$ belongs to the Weyl group $W$ of the Lie group $G$. The latter is defined as the quotient $N(H)/H$, $N(H)$ being the normalizer of the Cartan subgroup $H$. For $SL(n)$ the Weyl group is isomorphic to the permutation group $S_n$ of $n$ elements.

4.2 Gauss decomposition and $P$, $\bar{P}$ fields.

Let $g$ be one of the diagonalizing elements introduced in the previous subsection. The transport matrices

$$V^g = V g$$
$$\bar{V}^g = g^{-1} \bar{V}$$

satisfy the same linear systems as $V$ and $\bar{V}$ and have diagonal monodromy

$$V^g(x + 2\pi) = V(x) T^g$$
$$\bar{V}^g(x + 2\pi) = (T^g)^{-1} \bar{V}^g(x)$$
$$T^g = g^{-1} T g$$
From the Gauss decomposition
\[ V^g(x) = N^g(x)e^{K^g(x)}M^g_+(x) \]
\[ \bar{V}^g(x) = \bar{N}^g(x)e^{\bar{K}^g(x)}\bar{M}^g_+(x) \] (54)
where \( N^g_- \), \( \bar{N}^g_- \) \( K^g \), \( \bar{K}^g \) \( M^g_+ \), \( \bar{M}^g_+ \) \( \in \mathcal{N}_- \), and from the linear systems (4) one obtains that the matrices
\[ \mathcal{V}^g_+ = e^{K^g(x)}M^g_+(x) \]
\[ \mathcal{V}^g_- = \bar{N}^g_-(x)e^{\bar{K}^g(x)} \] (55)
satisfy the DS systems (9) and (10) with
\[ P^g(x) = \partial_x K^g(x) \]
\[ \bar{P}^g(x) = \partial_x \bar{K}^g(x) \] (56)
where, out of clarity, we have explicitly denoted the dependence on the diagonalizing element \( g \). In the following we will at times understand the label \( g \) in \( P \) and \( \bar{P} \).

## 4.3 Poisson brackets of \( P \) and \( \bar{P} \)

In the previous subsection we have defined the fields \( P \) and \( \bar{P} \). Now we prove that their Poisson on brackets are exactly (11). This subsection is rather technical, but we think it instructive to show explicitly at least in this case what kind of computations are involved.

From (54) it follows that \( \{ P^g \otimes P^g \} \) coincides with the projection on \( \mathcal{H} \otimes \mathcal{H} \) of
\[ \partial_x \partial_y \left( N^g_-(x)^{-1} \otimes N^g_-(y)^{-1} \cdot \{ V^g(x) \otimes V^g(y) \} \cdot \mathcal{V}^g_+(x)^{-1} \otimes \mathcal{V}^g_+(y)^{-1} \right) \] (57)

We will consider separately the various contributions to \( \{ V^g \otimes V^g \} \). One such contribution is given by \( \{ V \otimes V \} \cdot g \otimes g \) and produces the terms
\[
P\mathcal{H} \otimes \mathcal{H} \cdot (Ad(V^g_+(x)) \otimes Ad(V^g_+(y)))(\theta(x-y)(r^+)^g + \theta(y-x)(r^-)^g) + \\
+P\mathcal{H}Ad(V^g_+(x))P\mathcal{N}_- \otimes P\mathcal{H}Ad(V^g_+(y))P\mathcal{N}_-(r^g) + \\
+P\mathcal{H}Ad(V^g_+(x))P\mathcal{N}_- \otimes P\mathcal{H}Ad(V^g_+(y))P\mathcal{N}_-(r^g) \\
\]
where
\[ Ad(V^g_+(x))(a) = V^g_+(x)aV^g_+(x)^{-1}, \quad a \in \mathcal{G} \]
\[ (r^\pm)^g = Ad(g^{-1}) \otimes Ad(g^{-1})(r^\pm) \]
Taking into account that
\[(r^+)^g - (r^-)^g = r^+ - r^-\]
we obtain the following identities
\[
\begin{align*}
\mathcal{P}_H \otimes \mathcal{P}_H (r^+)^g &= \mathcal{P}_H \otimes \mathcal{P}_H (r^-)^g + 2 \sum_i H_i \otimes H_i \\
\mathcal{P}_H \otimes \mathcal{P}_{N_\pm} (r^+)^g &= \mathcal{P}_H \otimes \mathcal{P}_{N_\pm} (r^-)^g \\
\mathcal{P}_{N_\pm} \otimes \mathcal{P}_H (r^+)^g &= \mathcal{P}_{N_\pm} \otimes \mathcal{P}_H (r^-)^g
\end{align*}
\]
from which it then follows that the corresponding contribution to \(\{P^g \otimes P^g\}\) is reduced to the expression
\[
\partial_x \partial_y (x - y) \cdot t_0 + \partial_x \partial_y \mathcal{P}_H \text{Ad} (\mathcal{V}_g^0(x)) \mathcal{P}_{N_-} \otimes \mathcal{P}_H \text{Ad} (\mathcal{V}_g^0(y)) \mathcal{P}_{N_-} (r^g)
\]
while the other two terms give vanishing contributions to this Poisson bracket since they depend only either on \(x\) or on \(y\).

Next we consider the other two contributions
\[
1 \otimes V(y) \cdot \{V(x) \otimes g\} \cdot g \otimes 1 + V(x) \otimes 1 \cdot \{g \otimes V(y)\} \cdot 1 \otimes g
\]
to \(\{V^g \otimes V^g\}\). We remark that
\[
\text{Ad} (\mathcal{V}_g^0(x)) (h) = h, \quad \text{if} \quad h \in \mathcal{H}
\]
Therefore the terms belonging to \(\mathcal{H}\) of the Poisson brackets \((58)\) do not produce contributions to \(\{P^g \otimes P^g\}\). From this and from the main result \((53)\) of the subsection 4.1, we obtain that \((59)\) gives the contribution
\[
\sum_{\lambda \neq \sigma} \sum_{\mu \neq \nu} \sum_{\alpha > 0} \sum_{\beta > 0} (\frac{\nu(T^g)}{\mu(T^g) - \nu(T^g)} + \frac{\sigma(T^g)}{\lambda(T^g) - \sigma(T^g)}) .
\]
\[
< \lambda, l | E_\alpha | \sigma, s > < \mu, m | E_\beta | \nu, n > (E_\alpha, E_{-\alpha})(E_\beta, E_{-\beta}) < \sigma, s \otimes \nu, n | (r^g) | \lambda, l \otimes \mu, m > .
\]
\[
\partial_x \partial_y \mathcal{P}_H \text{Ad} (\mathcal{V}_g^0(x)) \otimes \mathcal{P}_H \text{Ad} (\mathcal{V}_g^0(y)) (E_{-\alpha} \otimes E_{-\beta}) (60)
\]
Finally the term
\[
V(x) \otimes V(y) \cdot \{g \otimes g\}
\]
gives the contribution
\[
\sum_{\lambda \neq \sigma} \sum_{\mu \neq \nu} \sum_{\alpha > 0} \sum_{\beta > 0} (\frac{\nu(T^g)\sigma(T^g) - \lambda(T^g)\mu(T^g)}{(\mu(T^g) - \nu(T^g))(\lambda(T^g) - \sigma(T^g))}) .
\]
\[
< \lambda, l | E_\alpha | \sigma, s > < \mu, m | E_\beta | \nu, n > (E_\alpha, E_{-\alpha})(E_\beta, E_{-\beta}) < \sigma, s \otimes \nu, n | (r^g) | \lambda, l \otimes \mu, m > .
\]
\[ \partial_x \partial_y \mathcal{P}_H \text{Ad} \left( \mathcal{V}_+^q(x) \right) \otimes \mathcal{P}_H \text{Ad} \left( \mathcal{V}_+^q(y) \right) \left( E_{-\alpha} \otimes E_{-\beta} \right) \]  

(61)

From the identity

\[
\frac{\nu(T^g)}{\mu(T^g) - \nu(T^g)} + \frac{\sigma(T^g)}{\lambda(T^g) - \sigma(T^g)} + \frac{\nu(T^g) \sigma(T^g) - \lambda(T^g) \mu(T^g)}{(\mu(T^g) - \nu(T^g))(\lambda(T^g) - \sigma(T^g))} = -1
\]  

(62)

it follows that (60) and (61) cancel the second term of (58) and therefore

\[ \{ P^g(x) \otimes P^g(y) \} = \partial_x \partial_y \epsilon(x - y) \cdot t_0 \]

Analogously in the antichiral sector we obtain

\[ \{ \bar{P}^g(x) \otimes \bar{P}^g(y) \} = -\partial_x \partial_y \epsilon(x - y) \cdot t_0 \]

where \( \bar{g} \) is another diagonalizing element, in general different from \( g \).

Next we show that the Poisson bracket \( \{ P^g \otimes \bar{P}^g \} \) vanishes. Proceeding as in the previous subsection we first obtain that the contribution of the term

\[ 1 \otimes \bar{g}^{-1} \cdot \{ V(x) \otimes \bar{V}(y) \} g \otimes 1 \]

is given by

\[ -\partial_x \partial_y \mathcal{P}_H \text{Ad} \left( \mathcal{V}_+^q(x) \right) \mathcal{P}_{N^-} \otimes \mathcal{P}_H \text{Ad} \left( \mathcal{V}_-^q(y)^{-1} \right) \mathcal{P}_{N^-} \left( (r^-)^{g\bar{g}} \right) \]

(63)

where

\[ (r^-)^{g\bar{g}} = g^{-1} \otimes \bar{g}^{-1} \cdot r^- g \otimes \bar{g} \]

The contribution of

\[ V(x) \otimes \bar{g}^{-1} \cdot \{ g \otimes \bar{V}(y) \} + \{ V(x) \otimes \bar{g}^{-1} \} \cdot g \otimes \bar{V}(y) \]

reads

\[ -\sum_{\lambda, \mu, \nu, n, m, l, \alpha > 0, \beta > 0} \frac{\nu(T^g)}{\mu(T^g) - \nu(T^g)} + \frac{\sigma(T^g)}{\lambda(T^g) - \sigma(T^g)} \cdot \frac{< \lambda, l | E_\alpha | \sigma, s > < \mu, m | E_{-\beta} | \nu, n >}{(E_\alpha, E_{-\alpha})(E_{\bar{\beta}}, E_{-\bar{\beta}})} \cdot \frac{< \sigma, s | \otimes | \nu, n | (r^-)^{g\bar{g}} | \lambda, l > \otimes | \mu, m >}{(E_\alpha, E_{-\alpha})(E_{\bar{\beta}}, E_{-\bar{\beta}})} \cdot \partial_x \partial_y \mathcal{P}_H \text{Ad} \left( \mathcal{V}_+^q(x) \right) \otimes \mathcal{P}_H \text{Ad} \left( \mathcal{V}_-^q(y)^{-1} \right) \left( E_{-\alpha} \otimes E_{\beta} \right) \]

(64)

and the term

\[ V(x) \otimes 1 \cdot \{ g \otimes \bar{g}^{-1} \} \cdot 1 \otimes \bar{V}(y) \]

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produces
\[- \sum \sum_{\lambda \neq \nu} \frac{\nu(T^\gamma)\sigma(T^\gamma) - \lambda(T^\gamma)\mu(T^\gamma)}{(\mu(T^\gamma) - \nu(T^\gamma))(\lambda(T^\gamma) - \sigma(T^\gamma))}.\]

\[
\langle \lambda, l | E_\alpha | \sigma, s \rangle < \mu, m | E_{-\beta} | \nu, n \rangle \\
(E_\alpha, E_{-\alpha})(E_{\beta}, E_{-\beta}) < \sigma, s \otimes < \nu, n | (r^-)^g | \lambda, l \otimes | \mu, m \rangle.
\]

\[
- \partial_x \partial_y \mathcal{P}_H \text{Ad}(V_+^g(x)) \otimes \mathcal{P}_H \text{Ad}\left(V_+^g(y)^{-1}\right) (E_{-\alpha} \otimes E_\beta)
\]

Using again (62) we conclude that \(P^g\) and \(\bar{P}^g\) Poisson commute.

### 4.4 The variable conjugate to the zero modes.

Let us choose now a fixed diagonalizing element \(g\) for both the chiral and anti-chiral part. Then, by construction, the zero modes \(P^g_0\) and \(\bar{P}^g_0\) coincide. It remains for us to reconstruct the conjugate variable to these zero modes.

Let \(|\Lambda\rangle\) be the highest weight vector of the representation we choose to work on. The Gauss decomposition (54) allow us to express the Toda fields as

\[
e^{-2\lambda(\Phi)} = \langle \Lambda | V(x) V(x) | \Lambda \rangle = \langle \Lambda | V^g_+(x) \bar{V}^g_+(x) | \Lambda \rangle = \langle \Lambda | Q^g_+(x) V^g_+(0) | \Lambda \rangle = \langle \Lambda | Q^g_+(x) V^g_+(0) \bar{Q}^g_+(x) | \Lambda \rangle
\]

where the chiral objects

\[
Q^g_+(x) = V^g_+(x) V^g_+(0)^{-1} \\
Q^g_-(x) = V^g_-(0)^{-1} V^g_+(x)
\]

satisfy the DS systems (51), (52). The Gauss decomposition of the constant matrix

\[
V^g_+(0) V^g_+(0) = g_S D \bar{g}_S^{-1}, \quad g_S \in e^{N_+}, \quad \bar{g}_S \in e^{N_-}, \quad D \in e^H
\]

is obtained from (54)

\[
g_S = e^{K^g(0)} M^g_+ e^{-K^g(0)}, \quad \bar{g}_S = e^{-K^g(0)} \bar{N}^g_+ e^{K^g(0)}, \quad D = e^{K^g(0) + \bar{K}^g(0)}
\]

A calculation a bit involved but straightforward shows that

\[
\{ P(x) \otimes D \} = 2\delta(x) \sum_i H_i \otimes H_i D \\
\{ \bar{P}(x) \otimes D \} = 2\delta(x) \sum_i H_i \otimes H_i D
\]
4.5 The action of the Weyl group

We have already noted that the element which diagonalizes the monodromy matrix is not uniquely fixed. Therefore, as we have done in subsection 4.3, one can choose two different diagonalizing elements \( g, \bar{g} \in SL_n(\mathbb{R}) \) and set

\[
V^g = Vg, \quad \bar{V}^{\bar{g}} = \bar{g}^{-1}\bar{V}, \quad g \cdot \bar{g}^{-1} \in e^H \cdot W
\]

The corresponding free chiral fields \((P^g, \bar{P}^{\bar{g}})\) are obviously periodic. On the other hand the right (left) multiplication of \( V(\bar{V}) \) by elements of the Cartan subgroup shifts the fields \( K^g (\bar{K}^g) \) (see (54), (55)) by a constant and thus does not change the momenta \( P (\bar{P}) \). This observation allows us to state that only the Weyl group acts nontrivially on these fields.

Therefore there are at least \(|W|^2\) possibilities to construct a pair of free chiral fields \((P, \bar{P})\) with the right Poisson brackets. However the number of possible choices reduces to \(|W|\) if one wants, as we do, the Toda field to be reconstructed according to the formula (66). In fact this formula implies that \( g \cdot \bar{g}^{-1} \in e^H \) and therefore, without changing \( P \) and \( \bar{P} \) one can set \( g \cdot \bar{g}^{-1} = 1 \).

5 Parabolic monodromies.

In this section we consider other types of non–regular solutions of the Liouville equation. They correspond to parabolic monodromies. The following discussion does not pretend to cope with the problem of classifying all singular solutions. But it is nevertheless interesting to see how a large class of them can be represented by means of free fields.

We come across these singular solutions in the following way. Let us return to subsection 3.2. There we found two fields, a chiral field \( p \), (37), and an antichiral one \( \bar{p} \), (39), which were almost the correct solution to our problem but not quite, in that they do not Poisson commute. For convenience we relabel them \( p_1 \) and \( \bar{p}_1 \). Now instead of proceeding, as we did in section 3, with the diagonalization of the monodromy, we can remark that there is a gauge freedom left after gauging away \( A^\gamma \) and \( A_\gamma^+ \). We can take advantage of this in order to modify \( p_1 \) and \( \bar{p}_1 \). It turns out, for example, that we can add to \( p_1 \) an object of weight 1 which is a total derivative of \( x_+ \). After some calculations one realizes that there are two other interesting chiral and antichiral fields, precisely

\[
p_2 = -(\partial_+ \varphi + \xi_{22}/\xi_{12}), \quad \bar{p}_2 = -\partial_- \varphi + \xi_{22}/\xi_{21}
\]

We notice that these can also be obtained from the previous ones via the action of the Weyl group.

One then computes all the possible Poisson brackets among the elements of these two pairs and we finds the following result: The pairs \((p_1, \bar{p}_2), (p_2, \bar{p}_1), \) and \((p_2, \bar{p}_2)\), all satisfy the Poisson brackets (92).
This is of course only an intermediate result, since we must address the question of periodicity with the additional problem of deciding which choice is the most suitable one for our purposes. In order to discuss periodicity, we consider the effect of a monodromy operation

\[
\begin{align*}
\xi_{11} &\to \alpha \xi_{11} + \gamma \xi_{12} & \bar{\xi}_{11} &\to \delta \xi_{11} - \beta \bar{\xi}_{21} \\
\xi_{12} &\to \beta \xi_{11} + \delta \xi_{12} & \bar{\xi}_{12} &\to \delta \bar{\xi}_{12} - \beta \bar{\xi}_{22} \\
\xi_{21} &\to \alpha \xi_{21} + \gamma \xi_{22} & \bar{\xi}_{21} &\to -\gamma \bar{\xi}_{11} + \alpha \bar{\xi}_{21} \\
\xi_{22} &\to \beta \xi_{21} + \delta \xi_{22} & \bar{\xi}_{22} &\to -\gamma \bar{\xi}_{12} + \alpha \bar{\xi}_{22}
\end{align*}
\]

Consequently the periodicity properties of the various fields are

\[
\begin{align*}
p_1 &\to p_1 - \frac{\gamma}{\alpha \xi_{11} + \gamma \xi_{11} \xi_{12}} & \bar{p}_1 &\to \bar{p}_1 - \frac{\beta}{\delta \xi_{11} - \beta \xi_{11} \xi_{21}} \\
p_2 &\to p_2 + \frac{\beta}{\delta \xi_{12} + \beta \xi_{11} \xi_{12}} & \bar{p}_2 &\to \bar{p}_2 + \frac{\gamma}{-\gamma \xi_{11} + \alpha \xi_{21} \xi_{11}}
\end{align*}
\]

Therefore for the pair \((p_1, \bar{p}_2)\) we would have periodicity if the condition \(\gamma = 0\) held true, while for the pair \((p_2, \bar{p}_1)\) we would have periodicity with \(\beta = 0\) and finally for the pair \((p_2, \bar{p}_2)\) to have periodicity the condition \(\gamma = \beta = 0\) is required. We will try to impose these conditions as dynamical constraints. Therefore we have to calculate the Poisson brackets of \(\beta\) and \(\gamma\) with the various pairs above. We find

- For the pair \((p_1, \bar{p}_2)\)
  \[
  \{p(x), \gamma\} = \gamma \delta(x), \quad \{\bar{p}(x), \gamma\} = -\gamma \delta(x)
  \]

- For the second pair \((p_2, \bar{p}_1)\)
  \[
  \{p(x), \beta\} = -\beta \delta(x), \quad \{\bar{p}(x), \beta\} = \beta \delta(x)
  \]

- For the last pair \((p_2, \bar{p}_2)\)
  \[
  \{p(x), \beta\} = -\beta \delta(x), \quad \{\bar{p}(x), \beta\} = \beta \delta(x) \\
  \{p(x), \gamma\} = \gamma \delta(x), \quad \{\bar{p}(x), \gamma\} = -\gamma \delta(x)
  \]

- Moreover \(\{\beta, \gamma\} = 0\).

The elements \(\beta\) and \(\gamma\) are preserved by taking the Poisson brackets with the corresponding \(p\)'s, thus we can set them to zero as hamiltonian constraints on the phase space. In this case the above Poisson brackets tell us that the interesting physical quantities, i.e. the fields \(p\), are first class with respect to the constraints.

It remains for us to discuss periodicity. Since in our setting the fields \(p, \bar{p}\) are in principle not periodic, we define the zero modes of the various fields to be the integral over a period, namely we put

\[
p_{i,0} = \int_{0}^{2\pi} p_i(x) \, dx, \quad i = 1, 2
\]
and analogously, for $\bar{p}_i$, $i = 1, 2$. Using the linear systems (27) and (28) we find

\begin{align*}
p_{1,0} &= \log(\alpha + \gamma \frac{\xi_{21}(0)}{\xi_{11}(0)}) \\
p_{2,0} &= \log(\delta + \beta \frac{\xi_{11}(0)}{\xi_{12}(0)}) \\
\bar{p}_{1,0} &= -\log(\delta - \beta \frac{\xi_{21}(0)}{\xi_{11}(0)}) \\
\bar{p}_{2,0} &= -\log(\alpha - \gamma \frac{\xi_{11}(0)}{\xi_{21}(0)})
\end{align*}

Imposing periodicity and equal zero modes yields the following picture.

1. With $\gamma = 0$ we have $p_1$, $\bar{p}_2$ periodic. Their zero modes are equal, respectively, to $\log \alpha$ and $-\log \alpha$. Setting them equal implies $\alpha = 1$, so that the monodromy must be

$$T = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$$

2. with $\beta = 0$ we have $(p_2, \bar{p}_1)$ to be the periodic pair. The zero modes are $\log \delta$ and $-\log \delta$, so that $p_{2,0} = \bar{p}_{1,0}$ implies

$$T = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$

3. Setting simultaneously $\beta = 0$, $\gamma = 0$ all the fields become periodic and the pair $p_{2,0}$, $\bar{p}_{2,0}$ is admissible. Equality of the zero modes translates into $\log \delta = -\log \alpha$ which implies $\delta = 1/\alpha$, so that for the monodromy we have

$$T = \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix}$$

The last case, with diagonal real monodromy, is just an uninteresting subcase of the solutions discussed in section 3. We disregard this case.

In the first two cases we have parabolic monodromy (and vanishing zero modes). On the basis of the discussion in section 3, all the solutions corresponding to the latter cases are necessarily singular. Moreover they are not expected to be reconstructed through a DS system of the type (9, 10). In conclusion we have obtained a parametrization of such family of singular solutions in terms of free bosonic oscillators (with a restricted phase space). We do not discuss here, for these solutions, the locality property and the conjugate to the zero modes.

In Appendix D we present the same construction for the $sl_n$ case.

**Appendix A. Hyperbolicity of the Monodromy: $sl_2$**

The monodromy of the Liouville theory ensuing from regular solutions (see below for the precise meaning) is hyperbolic. The proof goes as follows. We exploit
the representation of $T$ as a path-ordered exponential, but instead of using
the simplest contour, namely the $t = 0$ circle, we use the following (closed) path

$$\gamma = \begin{cases} 
(\tau, \tau) & \tau \in [0, \pi] \\
(\tau, 2\pi - \tau) & \tau \in [\pi, 2\pi]
\end{cases}$$

which in the light-cone coordinates has the form

$$\gamma = \begin{cases} 
(2\tau, 0) & \tau \in [0, \pi] \\
(2\pi, 2\tau - 2\pi) & \tau \in [\pi, 2\pi]
\end{cases}$$

Since this corresponds to a shift of $2\pi$ first in $x_+$ and then in $x_-$, we have

$$T = \Psi_- \Psi_+$$

with

$$\Psi_+ = \mathbf{P} \exp \left(-\int_0^{2\pi} A_+(x_+, x_- = 0) \, dx_+ \right)$$
$$\Psi_- = \mathbf{P} \exp \left(-\int_0^{2\pi} A_+(x_+ = 2\pi, x_-) \, dx_- \right)$$

Thus $\Psi_+$ is upper triangular, while $\Psi_-$ is lower triangular. Finding the explicit
form of $\Psi_\pm$ amounts to solving (1) on the appropriate paths. The result is

$$\Psi_+ = \begin{pmatrix} 
e^{-\frac{1}{2}(\phi_1 - \phi_0)} & \ne^{-\frac{1}{2}(\phi_0 + \phi_1)} \frac{\int_0^{2\pi} e^{2\phi(x_+, x_- = 0)} \, dx_+}{e^{\frac{1}{2}(\phi_1 - \phi_0)}} & \\
0 & \frac{\int_0^{2\pi} e^{2\phi(x_+ = 2\pi, x_-)} \, dx_-}{e^{-\frac{1}{2}(\phi_2 - \phi_1)}} \end{pmatrix}$$
$$\Psi_- = \begin{pmatrix} \ne^\frac{1}{2}(\phi_2 - \phi_1) & 0 \\\n\ne^{-\frac{1}{2}(\phi_2 + \phi_1)} \frac{\int_0^{2\pi} e^{2\phi(x_+ = 2\pi, x_-)} \, dx_-}{e^{-\frac{1}{2}(\phi_2 - \phi_1)}} & \end{pmatrix}$$

From this it is clear that in both cases the diagonal elements are strictly positive
while the off-diagonal ones are negative. We have indicated with $\phi_0$, $\phi_1$, $\phi_2$ the
values taken by $\phi$ at the vertices of the triangle defined by the path. Notice that
periodicity implies that $\phi_2 = \phi_0$. For the trace we have

$$\text{tr} \, T = 2 \cosh \frac{1}{2}(\phi_2 - 2\phi_1 + \phi_0) + (\text{positive contribution})$$
$$> 2$$

thus proving that in this case the monodromy $T$ is indeed hyperbolic. This
statement has been proved assuming that there are no singularities, but it is
easy to see that extending this method to more complicated zig-zag paths we can
find a path that avoids isolated singularities and remains homotopic to the initial
one. However we must exclude singularities on the $t = 0$ axis and accumulating
singularities, continuous singular lines and all the like. This accounts for the
distinction between ‘regular’ and ‘non regular’ solutions throughout the article.
Appendix B. Hyperbolicity of the Monodromy: \( sl_n \)

A matrix is hyperbolic when all its eigenvalues are strictly positive. Here we prove that for regular solutions the monodromy matrix for any \( sl_n \) Toda field theory is hyperbolic. We proceed as in the previous Appendix. We represent the monodromy matrix in \( sl(n) \) Toda theories

\[ T = P \exp \left( \int_{0}^{2\pi} A_x(x,0)dx \right) \]

as

\[ T = T_-(2\pi)T_+(2\pi) \]

with

\[
T_+(x_+) = P \exp \left( -\int_{0}^{x_+} A_+(\zeta_+,0) d\zeta_+ \right)
\]

\[
T_-(x_-) = P \exp \left( -\int_{0}^{x_-} A_+(2\pi,\zeta_-) dx_- \right)
\]

Then we use the linear system (1) to obtain a system of differential equations for the order \( k (k = 1 \ldots n) \) minors of the matrices \( T^{-1}_\pm(x_\pm) \)

\[ \partial_+ (T^{-1}_+)^{j_1 \ldots j_k}_{i_1 \ldots i_k} = (\lambda_{j_1} + \ldots + \lambda_{j_k}) \left( \partial_+ \Phi \right) \cdot (T^{-1}_+)^{j_1 \ldots j_k}_{i_1 \ldots i_k} + e^{\alpha_{j_1-1}(\Phi)} \cdot (T^{-1}_+)^{j_1-1 \ldots j_k}_{i_1 \ldots i_k} + \ldots + e^{\alpha_{j_k-1}(\Phi)} \cdot (T^{-1}_+)^{j_1 \ldots j_k-1}_{i_1 \ldots i_k} \quad (70) \]

\[ \partial_- (T^{-1}_-)^{j_1 \ldots j_k}_{i_1 \ldots i_k} = -(\lambda_{j_1} + \ldots + \lambda_{j_k}) \left( \partial_- \Phi \right) \cdot (T^{-1}_-)^{j_1 \ldots j_k}_{i_1 \ldots i_k} + e^{\alpha_{j_1}(\Phi)} \cdot (T^{-1}_-)^{j_1+1 \ldots j_k}_{i_1 \ldots i_k} + \ldots + e^{\alpha_{j_k}(\Phi)} \cdot (T^{-1}_-)^{j_1 \ldots j_k+1}_{i_1 \ldots i_k} \quad (71) \]

where \( \lambda_1 \ldots \lambda_n \) are the weights of the defining representations of \( sl(n) \). Taking into account the initial conditions

\[ T_\pm(0) = 1 \]

we obtain

\[ t^i_+(x_+) = e^{\lambda_j(\Phi(x_+,0)) - \lambda_i(\Phi(0,0))} \cdot \int_{0}^{x_+} d\zeta_{j-1} e^{2\alpha_{j-1}(\Phi(\zeta_{j-1},0))} \int_{0}^{\zeta_{j-1}} d\zeta_{j-2} e^{2\alpha_{j-2}(\Phi(\zeta_{j-2},0))} \ldots \int_{0}^{\zeta_{i+1}} d\zeta_i e^{2\alpha_{i}(\Phi(\zeta_{i},0))} \]

for \( i < j \)

\[ t^{ij}_+ = e^{\lambda_j(\Phi(x_+,0)) - \lambda_i(\Phi(0,0))} \]

and

\[ t^{ij}_+ = 0 \]
for \( i > j \). Moreover

\[
 t_{ij}^{ij} = e^{\lambda_i(\Phi(2\pi,0)) - \lambda_j(\Phi(2\pi,x_-))} ,
\]

\[
 \cdot \int_{I-I}^{x_-} d\zeta e^{2\alpha_{ij} (\Phi(2\pi,\zeta_j))} \int_{l}^{\zeta_j} d\zeta_j + 1 e^{2\alpha_{j+1} (\Phi(2\pi,\zeta_{j+1}))} \ldots \int_{l}^{\zeta_{i-2}} d\zeta_{i-2} e^{2\alpha_{i-1} (\Phi(2\pi,\zeta_{i-1}))}
\]

for \( i > j \)

\[
 t_{ij}^{ii} = e^{\lambda_j(\Phi(2\pi,0)) - \lambda_i(\Phi(2\pi,x_-))}
\]

and

\[
 t_{ij}^{ij} = 0
\]

for \( i < j \). Hence \( t_{ij}^{ij} \geq 0 \).

In fact a much stronger result is valid, namely, if \( I = (i_1, i_2, \ldots, i_k) \) \( J = (j_1, j_2, \ldots, j_k) \) are ordered multiindices \( (i_1 < i_2 < \ldots < i_k, j_1 < j_2 < \ldots < j_k) \), the corresponding minors

\[
 (T_+^{-1})^{IJ} = (T_-^{-1})^{i_1 \ldots i_k}_{j_1 \ldots j_k}
\]

are nonnegative. To show this we first note that \( (T_+^{-1})^{IJ} \neq 0 \) only if \( i_l \leq j_l \) \( (l = 1, 2, \ldots, k) \). A similar result is also valid for the minors of \( T_-^{-1}, (T_-^{-1})^{IJ} \neq 0 \) only if \( i_l \geq j_l \) \( (l = 1, 2, \ldots, k) \). Moreover the following factorization property holds

\[
 (T_+^{-1})^{i_1 \ldots j_k}_{i_1 \ldots i_k} = (T_+^{-1})^{i_1 \ldots j_l}_{i_1 \ldots i_l} t_{j_l i_l}^{i_l} (T_-^{-1})^{j_l+1 \ldots j_k}_{i_{l+1} \ldots i_k}
\]

for \( i_l = j_l \)

The differential equations \((70)\) and \((71)\) and this identity allow us to express \((T_+^{-1})^{IJ}\) as a sum of path ordered integrals all the integrands being nonnegative functions. Therefore we conclude that \((72)\) are nonnegative.

It is also easy to verify that if \( e^{\alpha_i(\Phi)} \) is not identically zero on the lines \( 0 \leq x_+ \leq 2\pi, x_- = 0 \) and \( x_+ = 2\pi, 0 \leq x_- \leq 2\pi \), all the minors \((T_+^{-1})^{IJ}\) \((T_-^{-1})^{IJ}\) are positive for \( i_l \leq j_l \) \( (i_l \geq j_l) \). Therefore the minors

\[
 (T_+^{-1})^{i_1 \ldots j_k}_{i_1 \ldots j_k} = \sum_{r_{j_l} \geq \max(i_l, j_l)} (T_+^{-1})^{r_{i_1} \ldots r_{j_k}}_{i_1 \ldots i_k} (T_-^{-1})^{j_l+1 \ldots j_k}_{i_{l+1} \ldots i_k}
\]

are positive.

To prove that all the eigenvalues of \( T^{-1} \) are positive we introduce the matrices \( T_k \) of order \( \binom{n}{k} \times \binom{n}{k} \) with entries

\[
 (T_k)_{IJ} = (T^{-1})^{IJ} > 0
\]

Let us denote by \( \mu_1, \mu_2, \ldots, \mu_n \) the eigenvalues of \( T^{-1} \) ordered as follows

\[
 |\mu_1| \geq |\mu_2| \geq \ldots \geq |\mu_n|
\]

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Hence the eigenvalues of $T_k$ are

$$
\mu_1, \mu_2, \ldots, \mu_k
$$

$$
i_1 < i_2 < \ldots < i_k
$$

The Perron-Frobenius theorem (see [13], vol. II, p.53 and 105) states that a matrix with positive entries has at least one positive eigenvalue which exceeds the moduli of all the other eigenvalues. Applying this theorem to the matrices $T_k$ we conclude that $\mu_1, \mu_1\mu_2, \ldots, \mu_1\mu_2\ldots\mu_n = det T^{-1} = 1$ are positive numbers and therefore all the eigenvalues of $T^{-1}$ (and so also of $T$) are real positive numbers.

The distinction made at the end of Appendix A between regular or non regular solutions holds here as well.

**Appendix C. Poisson algebra of $V$ and $\bar{V}$**

The Poisson algebra of $V$ and $\bar{V}$ we need throughout the paper can be calculated form (6) with standard methods

$$
\{V(x) \circ V(y)\} = \theta(x-y)V(x) \otimes V(y)
$$

$$
\cdot \left( r - V^{-1}(y) \otimes V^{-1}(y) \cdot (r-t_0) \cdot V(y) \otimes V(y) \right)
$$

$$
+ \theta(y-x)V(x) \otimes V(y)
$$

$$
\cdot \left( r - V^{-1}(x) \otimes V^{-1}(x) \cdot (r+t_0) \cdot V(x) \otimes V(x) \right)
$$

$$
\{V(x) \circ \bar{V}(y)\} = \theta(x-y)V(x) \otimes 1
$$

$$
\cdot \left( -r + \bar{V}(y) \otimes \bar{V}(y) \cdot (r+t_0) \cdot \bar{V}^{-1}(y) \otimes \bar{V}^{-1}(y) \right)
$$

$$
+ \theta(y-x)V(x) \otimes 1
$$

$$
\cdot \left( -r + V^{-1}(x) \otimes V^{-1}(x) \cdot (r+t_0) \cdot V(x) \otimes V(x) \right)
$$

$$
\{\bar{V}(x) \circ \bar{V}(y)\} = \left( r - \bar{V}(y) \otimes \bar{V}(y) \cdot (r+t_0) \cdot \bar{V}^{-1}(y) \otimes \bar{V}^{-1}(y) \right)
$$

$$
\cdot \bar{V}(x) \otimes \bar{V}(y)\theta(x-y)
$$

$$
+ \left( r - \bar{V}(x) \otimes \bar{V}(x) \cdot (r-t_0) \cdot \bar{V}^{-1}(x) \otimes \bar{V}^{-1}(x) \right)
$$

$$
\cdot \bar{V}(x) \otimes \bar{V}(y)\theta(y-x)
$$

**Appendix D. Explicit construction of the DS fields: $sl_n$**

In this Appendix we do for $sl_n$ the same construction as in section 5 where we constructed the $p$ and $\bar{p}$ fields corresponding to singular solutions of the Liouville equations.

After the gauge transformation

$$
V = G_1 G_2 \ldots G_{n-1} V_+
$$

$$
G_i = e^{X_{i+1}E_{i+1}} e^{X_{i+2}E_{i+2}} \ldots e^{X_{n}E_{n}}
$$

(76)
the transformed transport matrix $V_+$ satisfies DS type linear system (9) iff $X_{ij}, i > j,$ satisfy the equations:

\[
\begin{align*}
\partial_+ X_{ij} &= 2X_{ij} \sum_{k=j}^{i-1} \alpha_k(\partial_+ \Phi) - X_{ij}X_{jj-1} + X_{ij}X_{j+1j} + X_{ij-1} - X_{i+1j} \\
\partial_- X_{ij} &= -e^{2\alpha_{i-1} \Phi} X_{i-1j} \\
X_{ii} &= 1
\end{align*}
\]

(77)
The Toda equations appear both as integrability conditions of (77) and in the chirality of the fields

\[ P = -2\partial_+ \Phi - \sum_{i=1}^{n-1} X_{i+1i} H_{\alpha_i} \]

(78)

Similarly in order to get the antichiral fields we consider the gauge transformation

\[
\begin{align*}
\bar{V} &= V \tilde{G}_{n-1} \tilde{G}_{n-2} \cdots \tilde{G}_1 \\
\tilde{G}_i &= e^{X_{ii+1} E_{ii+1} E_{ii+2} \cdots E_{in}}
\end{align*}
\]

and the new transport matrix $V_-$ satisfy (10) iff $\bar{X}_{ij}, i < j,$ satisfy the following system of differential equations

\[
\begin{align*}
\partial_+ \bar{X}_{ij} &= e^{2\alpha_{j-1} \Phi} X_{ij-1} \\
\partial_- \bar{X}_{ij} &= 2\bar{X}_{ij} \sum_{k=i}^{j-1} \alpha_k(\partial_- \Phi) + \bar{X}_{i-1j} \bar{X}_{ij} - \bar{X}_{ii+1} \bar{X}_{ij} - \bar{X}_{i-1j} + \bar{X}_{ij+1} \\
\bar{X}_{ii} &= 0
\end{align*}
\]

(79)
The Toda equations again appear both as integrability conditions of (79) and in the chirality of the fields

\[ \bar{P} = 2\partial_- \Phi - \sum_{i=1}^{n-1} \bar{X}_{ii+1} H_{\alpha_i} \]

(80)

Let us now see how this construction works in the example of the defining representation of $sl(n)$. Denote by $\xi_{ij} (\hat{\xi}_{ij})$ the matrix elements of the transport matrices $V (\bar{V})$ and introduce the notation

\[
\begin{align*}
V^{\pi_{j_1 \cdots j_k}}_{i_1 \cdots i_k} &= \det \begin{pmatrix} \xi_{i_1 j_1} & \cdots & \xi_{i_1 j_k} \\ \cdots & \cdots & \cdots \\ \xi_{i_k j_1} & \cdots & \xi_{i_k j_1} \end{pmatrix}, & \bar{V}^{\sigma_{j_1 \cdots j_k}}_{i_1 \cdots i_k} &= \det \begin{pmatrix} \hat{\xi}_{i_1 j_1} & \cdots & \hat{\xi}_{i_1 j_k} \\ \cdots & \cdots & \cdots \\ \hat{\xi}_{i_k j_1} & \cdots & \hat{\xi}_{i_k j_1} \end{pmatrix}
\end{align*}
\]

(81)

For arbitrary permutations $\pi$ and $\sigma$ the matrices $V^\pi_+$ and $V^\sigma_-$ with entries

\[
\begin{align*}
\hat{\xi}^\pi_{ij} &= \frac{\bar{V}^{\pi_{j_1 \cdots j_k}}_{i_1 \cdots i_k}}{\bar{V}^{\pi_{j_1 \cdots j_{k-1} i}}_{i_1 \cdots i_{k-1}}}, & \hat{\xi}^\sigma_{ij} &= \frac{\bar{V}^{\sigma_{j_1 \cdots j_k}}_{i_1 \cdots i_k}}{\bar{V}^{\sigma_{j_1 \cdots j_{k-1} i}}_{i_1 \cdots i_{k-1}}}
\end{align*}
\]

(82)

\textsuperscript{4}We use the notation $\pi(i) = \pi_i$, etc.
satisfy the linear systems (9) and (10). The chiral (antichiral) momenta \( P^\pi = \text{diag}(p^\pi_1, \ldots, p^\pi_n) \) (\( \bar{P}^\sigma = \text{diag}(\bar{p}^\sigma_1, \ldots, \bar{p}^\sigma_n) \)) are given by

\[
p_i(x)^\pi = \partial_x \log \hat{\xi}_{ii}^\pi, \quad \bar{p}_i^\sigma(x) - \partial_x \log \hat{\xi}_{ii}^\sigma
\]  

(83)

Then using the Poisson brackets of the theory (see Appendix C) one obtains

\[
\{ p_k^\pi(x), p_l^\pi(y) \} = -\{ \bar{p}_k^\pi(x), \bar{p}_l^\pi(y) \} = \partial_x \partial_y \epsilon(x-y) \sum_{j=1}^{n-1} \lambda_{\pi_k}(H_j) \lambda_{\pi_l}(H_j)
\]  

(84)

On the other hand we have to impose also

\[
\{ p_k^\pi(x), \bar{p}_l^\pi(y) \} = 0
\]  

(85)

This has three classes of solutions

- The pairs \((\pi, \sigma)\) with \(\pi_i = \sigma_{n-i+1}\)

- The pairs \((\pi, \sigma)\) with \(\pi_i = n - i + 1\)

and \(\sigma\) arbitrary.

- The pairs \((\pi, \sigma)\) \(\sigma_i = n - i + 1\)

and \(\pi\) arbitrary.

One can prove that, among these solutions, there exist those for which we can impose, via hamiltonian constraints, that the monodromy matrix be upper triangular (or lower triangular) and the diagonal elements be such that be nondiagonalizable. These solutions are necessarily singular. For them it is possible to define a free field representation.

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