Abstract—We consider the capacity of quantum private information retrieval (QPIR) with multiple servers. In the QPIR problem with multiple servers, a user retrieves a classical file by downloading quantum systems from multiple servers each of which containing the whole classical file set, without revealing the identity of the retrieved file to any individual server. The QPIR capacity is defined as the maximum rate of the file size over the whole dimension of the downloaded quantum systems. Assuming the preexisting entanglement among servers, we prove that the QPIR capacity with multiple servers is 1 regardless of the number of servers and files. We propose a rate-one protocol which can be implemented by using only two servers. This capacity-achieving protocol outperforms its classical counterpart in the sense of the capacity, server secrecy, and upload cost.

I. INTRODUCTION

Introduced in the seminal paper by Chor et al. [1], private information retrieval (PIR) problem finds efficient methods to download a file from servers each of which containing the whole classical file set, without revealing the identity of the downloaded file to each server. This problem is trivially solved by requesting all files to the servers, but this method is inefficient. Finding an efficient method is the goal of this problem and it has been extensively studied in many papers [2]–[5]. Moreover, the papers [6]–[10] studied quantum PIR (QPIR) problem where the user downloads quantum systems, instead of classical bits, in order to retrieve a classical file from the servers.

In classical PIR studies, the paper [11] started the discussion of capacities for PIR problem with multiple servers, when the file size is arbitrarily large and therefore the upload cost is negligible with respect to the download cost. When there exist $n$ non-communicating servers each containing the whole set of $f$ files, the paper [11] showed that the capacity is $(1 - 1/n)/(1 - (1/n)^f)$ as the file size $m$ goes to infinity where the capacity is defined by the maximum rate of the file size over the download size. After [11], several PIR capacities are studied such as the PIR capacity with colluding servers [12], [13] or that with coded databases [14]–[16].

On the other hand, the QPIR problem is rarely treated with multiple servers. Though the paper [17] treated the QPIR problem with multiple servers, [7] focused on the communication complexity which evaluates the sum of upload and download costs required to retrieve a bit, instead of the capacity which evaluates the number of retrieved bits per the unit of download cost when the file size goes arbitrarily large. Moreover, there is no study on the capacity of the QPIR problem even for the single server case.

As a quantum generalization of the classical PIR capacity [11], we present the QPIR capacity when a user retrieves a file secretly from $n$ non-communicating servers containing the whole set of $f$ files by downloading quantum states with the assumption that an arbitrary entangled state is shared previously among servers. We define the security of QPIR protocol as the collection of retrieval error probability, user secrecy regarding the identity of the querying file, and server secrecy regarding the information of the files other than the querying file. As the main result, we show that the QPIR capacity is 1 regardless of whether it is of exact/asymptotic security and with/without the restriction that the upload cost is negligible to the download cost. Moreover, our result asserts the QPIR capacity is 1 from the strongest condition that QPIR protocols have no error, perfect secrecy, and negligible upload cost, to the weakest condition that QPIR protocols have no error probability less than 1, no secrecy, and no upload constraint. We concretely propose a capacity-achieving protocol in this paper and prove the converse.

Our capacity-achieving protocol has several remarkable advantages compared to the protocol in [11] (see Table I). First, the rate one of our protocol is greater than the rate $(1 - n^{-1})/(1 - n^{-f})$ of the protocol [11]. Secondly, our protocol is a symmetric QPIR protocol which guarantees not only the...
user secrecy but also the server secrecy, in the sense that the user obtains no information from files other than the retrieved one. This contrasts with the protocol [11] that retrieves some information of not-querying files. Thirdly, there is no benefit to using more servers in our protocol since our protocol achieves the capacity 1 using only two servers. On the other hand, in the protocol [11], the capacity is strictly increasing in the number of servers and an infinite number of servers are needed to achieve the capacity 1. Fourthly, the upload in this protocol is $\log m = n^t$ bits whereas the protocol [11] needs $n f \log((n^f)!/(n^{f-1}!))$-bit upload (See [17, Section 3.1]). Lastly, our protocol is implemented if the file size $m$ is the square of any integer, but the protocol [11] requires exact file size $\log m = n^t$ bits.

It is worth noting that the converse proof of the QPIR capacity is much simpler than that of the PIR capacity [11].

The remaining of this paper is organized as follows. Section II presents the main theorem of this paper, after the formal definition of the QPIR problem, QPIR protocol, and QPIR capacity. Section III constructs the capacity-achieving protocol and Section IV proves the converse bound. Section V is the conclusion of this paper.

Fig. 1. Quantum private information retrieval protocol with multiple servers. The composite system of the servers is initialized to an arbitrary entangled state.

**Notation:** For any set $\mathcal{T}$, $|\mathcal{T}|$ is the cardinality of the set $\mathcal{T}$ and $\mathcal{B}$ (or $\mathcal{L}$) is the identity operator on $\mathcal{T}$. For any matrix $\mathcal{B}$, $\mathcal{B}^\dagger$ is the complex conjugate of $\mathcal{B}$ and $\mathcal{B}^\dagger := \mathcal{B}^T$. We use the terms a quantum system, a quantum operation, and a quantum state to denote a finite-dimensional Hilbert space, a trace-preserving completely-positive map, and a density matrix (or a unit vector). For any quantum system $\mathcal{A}$, $\mathcal{S}(\mathcal{A})$ is the set of quantum states on $\mathcal{A}$. $\mathbb{Z}_d$ is the ring of integer and for any integer $d$, $\mathbb{Z}_d := \mathbb{Z}/d\mathbb{Z}$. For any random variable $R$, the expectation for $R$ is written as $E_R$.

II. PROBLEM STATEMENT AND MAIN THEOREM

In this section, we introduce the QPIR problem with multiple servers. QPIR protocol and its capacity are formally defined in this section. Moreover, we give the main theorem of the paper.

The QPIR problem with multiple servers (hereinafter the QPIR problem) is described as follows. Consider a user and non-communicating $n$ servers $\text{serv}_1, \ldots, \text{serv}_n$ each of which containing the whole set of uniformly and independently distributed $f$ files $W_1, \ldots, W_f \in \{0, \ldots, m-1\}$ for integers $n, f, m \geq 2$. The goal of QPIR problem is that the user retrieves the $K$-th file $W_K$ without revealing the query index $K$ to any individual server and without obtaining any information about the files other than $W_K$. In this problem, we assume that each server $\text{serv}_i$ possesses a quantum system $\mathcal{A}_i^0$ and the $n$ servers share any entangled state $\rho_{\text{prev}} \in \mathcal{S}(\bigotimes_{i=1}^n \mathcal{A}_i^0)$ in the beginning.

To solve the QPIR problem, the user chooses a random variable $R_{\text{user}}$ in a set $\mathcal{R}_{\text{user}}$ and encodes the queries for retrieving $W_K$ by user encoder $\text{Enc}_{\text{user}}$:

$$\text{Enc}_{\text{user}}(K, R_{\text{user}}) = (Q_1, \ldots, Q_n) \in \mathcal{Q}_1 \times \cdots \times \mathcal{Q}_n,$$

where $\mathcal{Q}_i$ is the set of query symbols to the $t$-th server for any $t \in \{1, \ldots, n\}$. The $n$ queries $Q_1, \ldots, Q_n$ are sent to the servers $\text{serv}_1, \ldots, \text{serv}_n$, respectively. After receiving the $t$-th query $Q_t$, each server $\text{serv}_i$ applies a quantum operation $\Lambda_i$ from $\mathcal{A}_i^0$ to $\mathcal{A}_i$ depending on $Q_t, W_1, \ldots, W_f$ and sends the quantum system $\mathcal{A}_i$ to the user. With server encoder $\text{Enc}_{\text{serv}_i}$, the operation $\Lambda_i$ is written as

$$\Lambda_i = \text{Enc}_{\text{serv}_i}(Q_t, W_1, \ldots, W_f),$$

and the received state of the user is written as

$$\rho^{W,Q} := \Lambda_1 \otimes \cdots \otimes \Lambda_n(\rho_{\text{prev}}) \in \mathcal{S} \left( \bigotimes_{i=1}^n \mathcal{A}_i \right),$$

where $W := (W_1, \ldots, W_f)$ and $Q := (Q_1, \ldots, Q_n)$. Next, the user retrieves the file $W_K$ by a decoder which is defined depending on $K, Q$ as a set $\text{Dec}(K, Q) := \{Y_0, \ldots, Y_{m-1}\}$ of positive semidefinite matrices on $\bigotimes_{i=1}^n \mathcal{A}_i$ satisfying $\sum_{i=0}^{m-1} Y_i \leq 1$. Then, the set $\text{Dec}(K, Q) \cup \{1 - \sum_{i=0}^{m-1} Y_i\}$ is a positive operator-valued measurement (POVM), where $Y_i$ corresponds to the measurement outcome $i$ and $1 - \sum_{i=0}^{m-1} Y_i$ corresponds to retrieval failure. By this measurement, the user retrieves the $K$-th file $W_K$ and the error probability is

$$1 - \text{Tr} \rho^{W, Q} Y_{W_K}.$$
2) Security: A QPIR protocol has two kinds of security parameters, the error probability and secrecy parameters. The error probability of the protocol $\Psi^{(m)}_{\text{QPIR}}$ is written as

$$P_{err}(\Psi^{(m)}_{\text{QPIR}}) := \mathbb{E}_{W,K,Q}(1 - Tr_{W,K}^{W}Q^{W}_{K})$$

where the distribution of $W$ is uniform and independent of the file $W_i$ for any $i \in \{1, \ldots, f\}$.

The secrecy parameters are defined as follows. For any $t \in \{1, \ldots, n\}$, let $\text{user}(\Psi^{(m)}_{\text{QPIR}})$ and $\text{serv}_t(\Psi^{(m)}_{\text{QPIR}})$ be the information of the user and the server $\text{serv}_t$ at the end of the protocol, respectively. We call the protocol $\Psi^{(m)}_{\text{QPIR}}$ has server secrecy at level $\beta \geq 0$ and user secrecy at level $\gamma \geq 0$ if

$$I(W_{K}; \text{user}(\Psi^{(m)}_{\text{QPIR}})) \leq \beta,$$

$$I(K; \text{serv}_t(\Psi^{(m)}_{\text{QPIR}})) \leq \gamma \text{ for any } t \in \{1, \ldots, n\}.$$  

(2) and (3)

where $I$ denotes the mutual information and $W_{K^c} := (W_1, \ldots, W_{K-1}, W_{K+1}, \ldots, W_t)$. If $\beta = 0$, Eq. (2) implies that the files other than $W_K$ are independent of the user information. Similarly, if $\gamma = 0$, Eq. (3) implies that the query index $K$ is independent of any individual server information.

3) Costs, rate, and capacity: Given the QPIR protocol $\Psi^{(m)}_{\text{QPIR}}$, the upload cost, download cost, and rate are defined by

$$U(\Psi^{(m)}_{\text{QPIR}}) := \prod_{t=1}^{n} |A_t|; \quad D(\Psi^{(m)}_{\text{QPIR}}) := \prod_{t=1}^{n} \dim A_t; \quad \text{and } R(\Psi^{(m)}_{\text{QPIR}}) := (\log m)/\log D(\Psi^{(m)}_{\text{QPIR}}).$$

The QPIR capacity is defined with constraints on the security parameters and upload cost. The asymptotic security-constrained capacity and exact security-constrained capacity are defined with error constraint $\alpha \in [0, 1]$, server secrecy constraint $\beta \in [0, \infty]$, user secrecy constraint $\gamma \in [0, \infty]$, and upload constraint $\theta \in [0, \infty]$ by

$$C^{\alpha,\beta,\gamma,\theta}_{\text{asymp}} := \sup_{\{m_i\}_{i=1}^{\ell}} \left\{ \liminf_{\ell \to \infty} \frac{\log U(\Psi^{(m)}_{\text{QPIR}})}{\log D(\Psi^{(m)}_{\text{QPIR}})} \leq \theta \right\},$$

$$C^{\alpha,\beta,\gamma,\theta}_{\text{exact}} := \sup_{\{m_i\}_{i=1}^{\ell}} \left\{ \limsup_{\ell \to \infty} \frac{\log U(\Psi^{(m)}_{\text{QPIR}})}{\log D(\Psi^{(m)}_{\text{QPIR}})} \leq \theta \right\},$$

where the suprema are taken for sequences $\{m_i\}_{i=1}^{\ell}$ such that $\lim_{\ell \to \infty} m_\ell = \infty$ and sequences $\{\Psi^{(m)}_{\text{QPIR}}\}_{i=1}^{\infty}$ of QPIR protocols. It is trivial from the definition that for any $\alpha \in [0, 1], \theta \in [0, \infty], \beta \in [0, \infty]$, and $\gamma \in [0, \infty]$, $C^{\alpha,\beta,\gamma,\theta}_{\text{exact}} \leq C^{\alpha,\beta,\gamma,\theta}_{\text{asymp}} \leq C^{\alpha,\beta,\gamma,\theta}_{\text{asymp}}$. (4)

The capacities with upload constraints $\theta = 0$ and $\theta = \infty$ are called capacities with negligible upload cost and without upload constraint, respectively.

A. Main Result

The main theorem of this paper is given as follows.

**Theorem II.1.** For any $\alpha \in [0, 1]$ and $\beta, \gamma, \theta \in [0, \infty]$, the capacity of the quantum private information retrieval with $f$ files and $n \geq 2$ servers sharing preexisting entanglement is

$$C^{\alpha,\beta,\gamma,\theta}_{\text{exact}} = C^{\alpha,\beta,\gamma,\theta}_{\text{asymp}} = 1.$$  

**Proof.** In Sections III and IV, we will prove $C^{0,0,0,0}_{\text{exact}} \geq 1$ and $C^{\alpha,\beta,\gamma,\theta}_{\text{asymp}} \leq 1$ for any $\alpha \in (0, 1)$, respectively. Then, the inequality (4) implies Theorem II.1. \qed

Note that the capacity does not depend on the number of files $f$ and the number of servers $n$ if $n \geq 2$. This contrasts to the classical PIR capacity [11] which is strictly increasing for $f$ and $n$.

III. CONSTRUCTION OF PROTOCOL

In this section, we construct a rate-one two-server QPIR protocol with exact security and negligible upload cost when the file size $m$ is the square of an arbitrary integer $\ell$, i.e., $m = \ell^2$. Then, by taking $m_\ell = \ell^2$, the sequence $\{\Psi^{(m)}_{\text{QPIR}}\}_{\ell=1}^{\infty}$ of our protocols achieves rate 1 with exact security and negligible upload cost, which implies

$$C^{0,0,0,0}_{\text{exact}} \geq 1.$$  

(5)

In the following, after giving preliminaries on quantum operations and states in Section III-A, we present the QPIR protocol in Section III-B.

A. Preliminaries

For an arbitrary integer $\ell \geq 2$, let $A$ be an $\ell$-dimensional Hilbert space spanned by an orthonormal basis $\{|0\rangle, \ldots, |\ell - 1\rangle\}$. Define a maximally entangled state $|\Phi\rangle$ on $A \otimes A$ by

$$|\Phi\rangle := \frac{1}{\sqrt{\ell}} \sum_{i=0}^{\ell-1} |i\rangle \otimes |i\rangle.$$  

For any $x, y \in \mathbb{Z}_\ell$, define generalized Pauli operations on $A \otimes A$

$$X := \sum_{i=0}^{\ell-1} |i+1\rangle \langle i|, \quad Z := \sum_{i=0}^{\ell-1} \omega^i |i\rangle \langle i|, \quad A(a, b) := X^a Z^b,$$
relations begin the protocol.

Moreover, it can be confirmed by simple calculation that

For any unitary $U := \sum_{i,j=0}^{\ell-1} t_{ij} |j\rangle \langle i|$, we define the state $|T\rangle$ in $A \otimes A$ by

$|T\rangle := \sum_{j=0}^{\ell-1} t_{ij} |j\rangle \otimes |j\rangle$.

With this notation, the maximally entangled state is written as $|\Phi\rangle = (1/\sqrt{\ell}) |\ell\rangle \langle \ell|$. Since $T^\dagger = \sum_{j=0}^{\ell-1} t_{ij} |i\rangle \langle j|$, it holds $|T\rangle = (T \otimes I)|\ell\rangle \langle \ell| = (I \otimes T^\dagger)|\ell\rangle \langle \ell|$. Moreover, for any unitaries $U, V$ on $A$,

$$(U \otimes V)|T\rangle = |UTV^\dagger\rangle,$$

$$(U \otimes U)|\ell\rangle = |UU^\dagger\rangle = |\ell\rangle.$$ (6)

With the basis given in the following proposition, we construct the measurement in our QPIR protocol.

**Proposition III.1.** The set

$$B := \{(A(a,b) \otimes I)|\Phi\rangle \ | a, b \in \mathbb{Z}_\ell\}$$

is an orthonormal basis of $A \otimes A$.

**Proof.** Since $(A(a,b) \otimes I)$ is a unitary matrix for any $a, b \in \mathbb{Z}_\ell$, all elements in $B$ are unit vectors. Then, it is enough to show that every different two vectors in $B$ are mutually orthogonal: for any different $(a, b), (c, d) \in \mathbb{Z}_\ell^2$,

$$(A(a,b) \otimes I)|\Phi\rangle^\dagger (A(c,d) \otimes I)|\Phi\rangle = 0.$$ (7)

Since $A(a,b)^\dagger A(c,d) = \omega^{h(a-c)} A(c-a, d-b)$, the left-hand side of (7) is written as

$$\omega^{h(a-c)} |\Phi\rangle (A(c-a, d-b) \otimes I)|\Phi\rangle.$$

Moreover, it can be confirmed by simple calculation that

$$|\Phi\rangle (A(a,b) \otimes I)|\Phi\rangle = \delta_{(a,b),(0,0)} \quad \forall a, b \in \mathbb{Z}_\ell.$$ (7)

Therefore, (7) holds which implies the proposition. \qed

**B. Rate-one QPIR protocol**

In this section, we propose a rate-one two-server QPIR protocol with exact security and negligible upload cost. This protocol is constructed from the idea of the classical two-server PIR protocol in [1] Section 3.1.

In this protocol, a user retrieves a file $W_K$ from two servers $\text{serv}_1$ and $\text{serv}_2$ each containing the whole set of files $W_1, \ldots, W_f \in \{0, \ldots, \ell^2 - 1 := m_\ell - 1\}$ for an arbitrary integer $\ell$. By identifying the set $\{0, \ldots, \ell^2 - 1\}$ with the module $\mathbb{Z}_\ell^2$, the files $W_1, \ldots, W_f$ are considered to be elements of $\mathbb{Z}_\ell^2$. We assume that $\text{serv}_1$ and $\text{serv}_2$ possess the $\ell$-dimensional quantum systems $A_1$ and $A_2$, respectively, and the maximally entangled state $|\Phi\rangle$ in $A_1 \otimes A_2$ is shared at the beginning of the protocol.

1) **Protocol:** The QPIR protocol for querying the $K$-th file $W_K$ is described as follows.

**Step 0.** The maximally entangled state $|\Phi\rangle$ in $A_1 \otimes A_2$ is shared between two servers, i.e., $\rho_{\text{prev}} := |\Phi\rangle \langle \Phi|$. 

**Step 1.** Depending on the query index $K$, the user chooses $R_{\text{user}}$ uniformly randomly from the power set $2^{\{1, \ldots, f\}}$ of $\{1, \ldots, f\}$. Let $Q_1 := R_{\text{user}}$ and

$$(Q_2 := \left\{ \begin{array}{ll} Q_1 \setminus \{K\} & \text{if } K \in Q_1, \\ Q_1 \cup \{K\} & \text{otherwise.} \end{array} \right. )$$

**Step 2.** The user queries $Q_1$ and $Q_2$ to $\text{serv}_1$ and $\text{serv}_2$, respectively.

**Step 3.** $\text{serv}_1$ calculates $H_1 := \sum_{i \in Q_1} W_i \in \mathbb{Z}_\ell^2$ and applies $A(H_1)$ on the quantum system $A_1$. Similarly, $\text{serv}_2$ calculates $H_2 := \sum_{i \in Q_2} W_i$ and applies $A(H_2)$ to the quantum system $A_2$. That is,

$$\text{Enc}_{\text{serv}_1}(Q_1, W_1, \ldots, W_f) = A(H_1),$$

$$\text{Enc}_{\text{serv}_2}(Q_2, W_1, \ldots, W_f) = A(H_2),$$

$$\rho^{W,Q} = (A(H_1) \otimes A(H_2))|\Phi\rangle \langle \Phi|(A(H_1) \otimes A(H_2))^\dagger.$$

**Step 4.** $\text{serv}_1$ and $\text{serv}_2$ send the quantum systems $A_1$ and $A_2$ to the user, respectively.

**Step 5.** The user performs a POVM

Dec($K, Q$) = \{Y_{(a,b)} \ | a, b \in \mathbb{Z}_\ell\}

to the received state $\rho^{W,Q}$, where each POVM element $Y_{(a,b)}$ for the outcome $(a, b)$ is defined by

$$Y_{(a,b)} := (A(a,b) \otimes I)|\Phi\rangle \langle \Phi|(A(a,b)^\dagger \otimes I)$$

if $K \in Q_1$, and

$$Y_{(a,b)} := (A(-a,-b) \otimes I)|\Phi\rangle \langle \Phi|(A(-a,-b)^\dagger \otimes I)$$

otherwise.

2) **Security:** This protocol has no error as explained in the following. Note that $H_1 = H_2 + W_K$ if $K \in Q_1$, and $H_1 = H_2 - W_K$ otherwise. After Step 3, the state on $A_1 \otimes A_2$ is

$$A(H_1) \otimes A(H_2)|\Phi\rangle$$

$$= \frac{\omega^{b_{W_K} a}}{\sqrt{\ell}} (A(\pm W_K) \otimes I)(A(H_2) \otimes A(H_2))|\ell\rangle.$$ (8)

$$= \frac{\omega^{b_{W_K} a}}{\sqrt{\ell}} (A(\pm W_K) \otimes I)|\ell\rangle.$$ (9)

$$= \omega^{b_{W_K} a} (A(\pm W_K) \otimes I)|\Phi\rangle,$$

where $H_2 = (a_{H_2}, b_{H_2})$ and $W_K = (a_{W_K}, b_{W_K}) \in \mathbb{Z}_\ell^2$. The equality (8) is derived from $A(H_1) = A(\pm W_K + H_2) = \omega^{b_{W_K} a} A(\pm W_K) A(H_2)$ and the equality (9) is from (8). Therefore, in Step 5, the measurement outcome is $W_K \in \mathbb{Z}_\ell^2$ with probability 1.

The exact user secrecy follows from that of the protocol [1] Section 3.1. At the end of the protocol, the only information obtained by $\text{serv}_1$ ($\text{serv}_2$) is $H_1$ ($H_2$). Since $H_1$ ($H_2$) is independent of the index $K$, the exact user secrecy is obtained.
The exact server secrecy follows from the converse bound in Section IV. The converse bound restricts that the size of the transmitted information from the servers to the user cannot exceed the entire dimension of the downloaded quantum system which is $\ell^2$ in our protocol. If our QPIR protocol leaked the server information, the above restriction is violated since the file size is already $\ell^2$ and the server information leakage can be considered as another information transmission from the servers to the user. Therefore, the exact server secrecy is obtained.

3) Upload cost, download cost, and rate: The upload cost is $U(\Psi_{QPIR}^{(m)}) = 2^f$ because $Q_1, Q_2 \in \{1,\ldots,f\}$ are uploaded and each element of the power set $2^{\{1,\ldots,f\}}$ is expressed by $f$ bits. The download cost is $D(\Psi_{QPIR}^{(m)}) = \dim A_1 \otimes A_2 = \ell^2 = m \ell$, therefore, the rate is

$$R(\Psi_{QPIR}^{(m)}) = \frac{\log m \ell}{\log D(\Psi_{QPIR}^{(m)})} = 1,$$

and $(\log U(\Psi_{QPIR}^{(m)}))/\log D(\Psi_{QPIR}^{(m)})$ goes to zero as $m \ell \to \infty$.

IV. CONVERSE

In this section, we prove the converse bound

$$C_{\text{asymp}}^{\alpha,\infty,\infty} \leq 1 \quad (10)$$

for any $\alpha \in [0,1)$. By replacing the notation of $\rho_{W,Q}$ defined in [8], let $\rho_{w,z}$ be the quantum state on the composite system $\bigotimes_{i=1}^n A_i$, where $w$ is the file to be retrieved and $z := (w',q)$ for the collection $w'$ of other $m-1$ files and the collection $q$ of queries. Applying (18) (4.66)) to the choice $\sigma_z = (1/m) \sum_{w=0}^{m-1} \rho_{w,z}$ for any $z$, we have

$$\log \frac{1 - P_{\text{err}}(\Psi_{QPIR}^{(m)})} {\log D(\Psi_{QPIR}^{(m)})} \leq \sum_{w=0}^{m-1} \sum_{z=0}^{m-1} \text{Tr} \rho_{w,z} \sigma_z^s \leq \sum_{z=0}^{m-1} \text{Tr} \sigma_z^s \leq \max \text{Tr} \sigma_z^s$$

for $s \in (0,1)$ and $d = \prod_{i=1}^n \dim A_i$. Here, since $s \rightarrow s^{-1}$ is concave, the maximum $\max \prod_{i=1}^n \frac{1}{s}$ is realized by the uniform distribution, which shows the equation (4). Given a sequence of QPIR protocols $\{\Psi_{QPIR}^{(m)}\}_{\ell=1}^\infty$, if $\Psi_{QPIR}^{(m)}$ has the QPIR rate greater than 1 for any sufficiently large $\ell$, $D(\Psi_{QPIR}^{(m)})/m \ell = \prod_{i=1}^n \dim A_i/m \ell$ goes to 0. Hence, $1 - P_{\text{err}}(\Psi_{QPIR}^{(m)})$ approaches zero, which implies (10).

V. CONCLUSION

We have studied the capacity of QPIR problem with multiple servers sharing preexisting entanglement. Considering not only the user secrecy but also the server secrecy, we defined two kinds of QPIR capacity, asymptotic and exact security-constrained capacities with upload constraint, and proved that both QPIR capacities are 1 for any security constraints and any upload constraint. We constructed a capacity-achieving rate-one protocol by using two servers when the file size is the square of an arbitrary integer. The converse is proved by focusing on the downloading step of QPIR protocols.

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