The pressure, densities and first order phase transitions associated with multidimensional SOFT

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Abstract

We study theoretical and computational properties of the pressure function for subshifts of finite type on the integer lattice $\mathbb{Z}^d$, multidimensional SOFT, which are called Potts models in mathematical physics. We show that the pressure is Lipschitz and convex. We use the properties of convex functions in several variables to show rigorously that the phase transitions of the first order correspond exactly to the points where the pressure is not differentiable. We give computable upper and lower bounds for the pressure, which can be arbitrary close to the values of the pressure given a sufficient computational power. We apply our numerical methods to confirm Baxter’s heuristic computations for two dimensional monomer-dimer model, and to compute the pressure and the density entropy as functions of two variables for the two dimensional monomer-dimer model.

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1 Introduction

The most celebrated models in statistical mechanics are the Ising models, introduced by Ising in [18], and their generalizations to Potts models [25]. Usually, the one-dimensional Ising or Potts models admit a closed-form analytical solution and do not exhibit the phase transition phenomenon, as in the case of the original work of Ising for ferromagnetism. The importance of Ising models was demonstrated by Onsager’s closed-form solution for the two-dimensional ferromagnetism model in the zero-field case [23], which does exhibit phase transition at exactly one temperature. Unfortunately, there are only a handful of known closed-form solutions for two-dimensional Potts models, including the dimer problem due to Fisher, Kasteleyn and Temperley [8], [20], [29]; residual entropy of square ice by Lieb [22]; hard hexagons by Baxter [5]. See also [6].

Thus, most of the interesting Potts models, in particular all problems in dimension 3 and up, are treated by ad hoc asymptotic expansions or by some kind of numerical solutions, in many circumstances with the help of Monte Carlo simulations, which usually have a heuristic basis. The aim of this paper is to introduce a new mathematical foundation to this subject, which also gives rise to reliable numerical methods, using converging upper

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and lower bounds, for computing the pressure and its derivatives for known quantities in statistical mechanics. In principle, these quantities can be computed to any accuracy given sufficient computing power. The first-order phase transition is manifested by a jump in a corresponding directional derivative of the pressure, which can be detected within the given precision of the computation. Our approach to the phase transition is significantly simpler than the approaches using the Gibbs equilibrium measures corresponding to the pressure, e.g. [1, 7, 19, 27]. In models with one variable, the situation is relatively well understood by physicists. The basic argument of phase transition in Ising model is due to Peierls [24]. For more modern account of the physicist’s approach see [14, pp’ 5 9].

We now introduce the main ideas of this paper as nontechnically as possible. Assume that we have a standard lattice \( \mathbb{Z}^d \), consisting of points in \( d \)-dimensional space \( \mathbb{R}^d \) with integer coordinates, which we call sites. Each site \( i = (i_1, \ldots, i_d)^\top \) is occupied by exactly one particle, or color, out of the set \( \{n\} := \{1, \ldots, n\} \) of \( n \) distinct colors (if we do not insist that every site be occupied, we agree to use color \( n \) for an unoccupied site). In general, one has a local type of restriction on the allowed configurations of the colors, which is called a shubshift of finite type, or SOFT, known as the hard-core model in physics terminology. The exact definition of a SOFT is given in the next section. For an example of SOFT, consider the residual entropy of square ice studied in [22]. This entropy is the exponential growth rate of the number of colorings of increasing sequences of squares in \( \mathbb{Z}^2 \) with \( n = 3 \) colors, subject to the local restriction that no two adjacent sites receive the same color. More generally, we consider a nonempty near neighbor SOFT (NNSOFT), specified by a \( d \)-tuple \( \Gamma = (\Gamma_1, \ldots, \Gamma_d) \), where each \( \Gamma_k \subseteq (\{n\}) \times (\{n\}) \) is a digraph whose set of vertices is the set \( (\{n\}) \) of colors. Two adjacent sites \( i \) and \( i + e_k \), where \( e_k = (\delta_{ik}, \ldots, \delta_{dk})^\top \), are allowed to receive the colors \( p \) and \( q \) respectively only if \( (p, q) \in \Gamma_k \). We denote the set of all allowed colorings in this NNSOFT by \( C_{\Gamma}(\mathbb{Z}^d) \).

We assume for simplicity of the exposition that the Hamiltonian of a particle of color \( i \) is \( u_i \in \mathbb{R} \). If this is not the case, as for the Ising model or the monomer-dimer model, there is a way to reduce such a model to our model by enlarging the number of colors. We show how to carry out this reduction for the monomer-dimer model.

For \( m = (m_1, \ldots, m_d) \in \mathbb{N}^d \), let \( \langle m \rangle \) denote the \( d \)-dimensional box \( \langle m_1 \rangle \times \cdots \times \langle m_d \rangle \). Let \( \phi : \langle m \rangle \to (\{n\}) \) be a coloring \( \langle m \rangle \) with \( n \) colors, i.e., an ensemble of \( \text{vol}(\langle m \rangle) := m_1 \cdots m_d \) particles of \( n \) kinds occupying the sites in \( \langle m \rangle \). Let \( c_i(\phi) \) be the number of sites in \( \langle m \rangle \) colored with color \( i \). Let \( c(\phi) = (c_1(\phi), \ldots, c_n(\phi))^\top \) and \( u = (u_1, \ldots, u_n)^\top \in \mathbb{R}^n \). Then the Hamiltonian of the system \( \phi \) is equal to \( c(\phi)^\top u \). The grand partition function corresponding to the set \( C_{\Gamma}(\langle m \rangle) \) of all colorings \( \phi : \langle m \rangle \to (\{n\}) \) allowed by \( \Gamma \) is given by

\[
Z_{\Gamma}(\langle m \rangle, u) := \sum_{\phi \in C_{\Gamma}(\langle m \rangle)} e^{c(\phi)^\top u}.
\]

It is well-known that \( \log Z_{\Gamma}(\langle m \rangle, u) \) is a convex function. Furthermore, the multisequence \( \log Z_{\Gamma}(\langle m \rangle, u) \), \( m \in \mathbb{N}^d \) is subadditive in each coordinate of \( m \). Hence the following limit exists

\[
P_{\Gamma}(u) := \lim_{m \to \infty} \frac{\log Z_{\Gamma}(\langle m \rangle, u)}{\text{vol}(\langle m \rangle)},
\]

where \( m \to \infty \) means \( m_j \to \infty \) for all \( j \in \{d\} \). This limit is called the pressure function. The value \( h_{\Gamma} := P_{\Gamma}(0) \) is the (free) entropy of the corresponding SOFT, and our previous paper [12] was devoted to the theory of its computation. The function \( P_{\Gamma}(\cdot) : \mathbb{R}^n \to \mathbb{R} \) is a Lipschitz convex function. Hence it is continuous and subdifferentiable everywhere, and differentiable almost everywhere. Assume that \( P_{\Gamma}(\cdot) \) is differentiable at \( u \) with gradient vector \( p(u) = (p_1(u), \ldots, p_n(u))^\top \). Then \( p(u) \) is a probability vector, and \( p_j(u) \) is the relative frequency, or proportion, of color \( i \) corresponding to the pressure \( P_{\Gamma}(u) \). We show that the points \( u \) where \( P_{\Gamma}(u) \) is not differentiable correspond to phase transitions of the first order, i.e., these are points \( u \) where the proportions of the colors are not unique.

Let \( \Pi_n \subseteq \mathbb{R}^n \) denote the simplex of probability vectors. Assume that \( p \in \Pi_n \) is a limiting color proportion vector for some multisequence of configurations in \( C_{\Gamma}(\langle m \rangle) \), \( m \to \infty \). Then
one can define the density entropy $h_{\Gamma}(p)$ as the maximal exponential growth rate of the number of configurations, the maximum being taken over all multisequences whose color proportion vector tends to $p$. (See for example [15] for the special case of the monomer-dimer configurations.) We denote by $\Pi_{\Gamma} \subseteq \Pi_{n}$ the compact set of all limiting color proportion vectors.

Let $P_{\Gamma}^{+} : \mathbb{R}^{n} \to [-h_{\Gamma}, \infty]$ be the conjugate of $P_{\Gamma}(\cdot)$, which is called the Legendre-Fenchel transform in the case of differentiable convex functions [2, 26]. Recall that $P_{\Gamma}^{+}(\cdot)$ is a convex function. We show that for a limiting color proportion vector $p$, that is also a subgradient of the pressure function somewhere, $h_{\Gamma}(p) = -P_{\Gamma}^{+}(p)$. Thus $h_{\Gamma}$ is a concave function on each convex set of such vectors $p$ in $\Pi_{\Gamma}$.

We next show that in many SOFT arising in physical models, the set $\Pi_{\Gamma}$ is convex and the function $h_{\Gamma} : \Pi_{\Gamma} \to \mathbb{R}_{+}$ is concave. A simple example is as follows. Assume that our SOFT given by $\Gamma$ has a friendly color, say $n$. This is, in each digraph $\Gamma_{k}$ the vertices $n$ and $i$ connected in both directions, i.e. $(n, i), (i, n) \in \Gamma_{k}$, for $i = 1, \ldots, n$ and $k = 1, \ldots, d$. Then $\Pi_{\Gamma}$ is convex and $h_{\Gamma} | \Pi_{\Gamma}$ is concave. The hard core model has a friendly color. The monomer-dimer model has essentially a friendly color, which corresponds to the dimer, hence $\Pi_{\Gamma} = \Pi_{d+1}$ and $h_{\Gamma} | \Pi_{d+1}$ is concave. These results can be viewed as generalizations of the result of Hammersley [15].

For numerical computations of the pressure one needs to have lower bounds for the pressure, which converge to the pressure in the limit. (The convergent upper bounds are given by $\log Z_{\Gamma}(m, u) / \text{vol}(m)$, since the multisequence log $Z_{\Gamma}(m, u), m \in \mathbb{N}^{d}$ is subadditive in each coordinate of $m$.) We extend the results in [9, 12] to give lower convergent bounds if at least $d - 1$ digraphs out of $\Gamma_{1}, \ldots, \Gamma_{d}$ are symmetric. (A digraph $\Gamma \subseteq \langle n \rangle \times \langle n \rangle$ is called symmetric, (reversible), if the digedge $(i, j)$ is in $\Gamma$ whenever $(j, i)$ is in $\Gamma$.) This condition holds for most of the known physical models. In this paper we show how to apply the computational methods developed in [12] to the pressure. We demonstrate the applications of our methods to the two dimensional monomer-dimer model on $\mathbb{Z}^{2}$. First we confirm the heuristic computations of Baxter [4]. Second, we find numerically a number of values of the pressure function $P_{2}(v_{1}, v_{2})$ and the value of the density entropy $\hat{h}_{2}(p_{1}, p_{2})$ for dimers with densities $p_{1}, p_{2}$ in the directions $x_{1}, x_{2}$ respectively. In Figures 1 and 2 we give the plots of these functions. These computations go beyond the known computations of [4, 16], where one considers the total density of dimers, (which reduce to the computations of functions of one variable).

We hope to show that the methods of this paper can be applied to other interesting models. We already know that our approach works for the numerical computation of the pressure function for the 2D and 3D Ising models in external magnetic field. (It is similar to the monomer-dimer model we study here.) We plan to study if our numerical computations are precise enough to discover the second order phase transitions, which occurs in multidimensional Ising models.

We now survey briefly the contents of the paper. In Section 2 we describe in details SOFT, NNSOFT and the pressure function. We also show that in the one-dimensional case, the pressure is the logarithm of the spectral radius of a corresponding nonnegative matrix. In Section 3 we show that under certain symmetry (reversibility) assumptions on $d - 1$ digraphs among $\Gamma_{1}, \ldots, \Gamma_{d}$, we have computable converging upper and lower bounds for the pressure. In Section 4 we relate certain properties of a convex function, as differentiability and its conjugate $P_{\Gamma}^{+}(p)$, (the Legendre-Fenchel transform), to the physical quantities associated with a given SOFT, i.e. the corresponding Potts model. In particular we show that the points where the pressure $P_{\Gamma}(\cdot)$ is differentiable correspond to unique color frequency vectors. On the other hand the points were the pressure was not differentiable correspond to the phase transition of first order, since to this value of $u$ correspond at least two different color frequencies. We also relate the density entropy $h_{\Gamma}(p)$ to the conjugate function $P_{\Gamma}^{+}(p)$. In Section 5 we apply the results of Section 4 to a one-dimensional SOFT. The importance of one-dimensional SOFT is due to the fact that our approximations of the pressure are obtained by using the exact results on one-dimensional SOFT. Section 6 we apply some of
our results in Section 4 to the monomer-dimer model in \( \mathbb{Z}^d \). We also relate our results to the works of Hammersley and Baxter \([15, 4]\). This is done by using the fact that the monomer-dimer model in \( \mathbb{Z}^d \) can be realized as SOFT with \( 2d + 1 \) colors \([10, 12]\). As we pointed out in \([12]\) this SOFT does not have symmetric properties, and hence cannot be used for computation. In Section 7 we use the symmetric encoding of the monomer-dimer model developed in \([12]\), to obtain computer upper and lower bounds for the pressure function.

In Section 8 we apply our techniques to the computations of two dimensional pressure and density entropy for the monomer-dimer model in \( \mathbb{Z}^2 \).

2 SOFT, NNSOFT and Pressure

We use the notation \( \langle r \rangle := \{1, \ldots, r\} \) for \( r \in \mathbb{N} := \{1, 2, 3, \ldots\} \), and for \( m = (m_1, \ldots, m_d) \in \mathbb{N}^d \), \( \langle m \rangle := \langle m_1 \rangle \times \cdots \times \langle m_d \rangle \) denotes a box with volume \( \text{vol}(m) := m_1 \cdots m_d \).

Then \( \langle n \rangle^{(m)} \) is the set of all colorings \( \phi : \langle m \rangle \to \langle n \rangle \) of \( \langle m \rangle \) with colors from \( \langle n \rangle \). We denote by \( c(\phi)_i := \#\phi^{-1}(i) \) the number of sites in \( \langle m \rangle \) colored with the color \( i \in \langle n \rangle \), and let \( c(\phi) := (c(\phi)_1, \ldots, c(\phi)_n)^T \).

Similarly, with \( \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\} \), \( \langle n \rangle^{\mathbb{Z}^d} \) is the set of all colorings \( \phi : \mathbb{Z}^d \to \langle n \rangle \) of \( \mathbb{Z}^d \) with colors from \( \langle n \rangle \). Given a \( d \)-digraph \( \Gamma = (\Gamma_1, \ldots, \Gamma_d) \) on \( \langle n \rangle \), let \( C_\Gamma(\mathbb{Z}^d) \subseteq \langle n \rangle^{\mathbb{Z}^d} \) be the set of all \( \Gamma \)-colorings, namely colorings \( \phi : \mathbb{Z}^d \to \langle n \rangle \) such that for each \( i \in \mathbb{Z}^d \) and \( k \in (d) \), \( \langle \phi_i, \phi_{i+e_k} \rangle \in \Gamma_k \), where \( e_k \) is the unit vector with \( k \)th component equal to 1. In ergodic theory, the set \( C_\Gamma(\mathbb{Z}^d) \) is called a nearest neighbor subshift of finite type (NNSOFT).

A general SOFT can be described as follows. Let \( M \in \mathbb{N}^d \) and a nonempty subset \( P \subseteq \langle n \rangle^{(M)} \) be given. Every element \( a \in P \) is viewed as an allowed coloring (configuration) of the box \( \langle M \rangle \) with \( n \) colors. For \( i \in \mathbb{Z}^d \), we define the shifted coloring \( \tau_i(a) \in P \) as the coloring of the shifted box \( \langle M \rangle + i \) that gives to the site \( x + i \) the same color that \( a \) gives to \( x \in \langle M \rangle \). We denote by \( \tau_i(P) \) the set \( \{\tau_i(a) : a \in P\} \), and regard it as the set of allowed colorings of \( \langle M \rangle + i \). A coloring \( \phi \in \langle n \rangle^{\mathbb{Z}^d} \) is called a \( P \)-state if for each \( i \in \mathbb{Z}^d \) the restriction of \( \phi \) to \( \langle M \rangle + i \) is in \( \tau_i(P) \). We denote by \( \langle n \rangle^{\mathbb{Z}^d}(P) \) the set of all \( P \)-states. In ergodic theory, the set \( \langle n \rangle^{\mathbb{Z}^d}(P) \) is called a subshift of finite type (SOFT) \([28]\).

Each NNSOFT \( C_\Gamma(\mathbb{Z}^d) \) is a special kind of SOFT obtained by letting \( M = (2, \ldots, 2) \) and \( P \) be the set of all colorings \( \phi \in \langle n \rangle^{(M)} \) such that \( i, i + e_k \in \langle M \rangle \) imply \( \langle \phi_i, \phi_{i+e_k} \rangle \in \Gamma_k \). Conversely \([9]\), each SOFT \( (n)^{\mathbb{Z}^d}(P) \) can be encoded as an NNSOFT \( C_\Gamma(\mathbb{Z}^d) \), where \( \Gamma = (\Gamma_1, \ldots, \Gamma_d) \) is defined as follows. Take \( N = \#P \) and use a bijection between \( P \) and \( \langle N \rangle \). The digraph \( \Gamma_k \subseteq \langle N \rangle \times \langle N \rangle \) is defined so that for \( a, b \in P \) we have \( (a, b) \in \Gamma_k \) if and only if there is a configuration \( \phi \in \langle n \rangle^{\langle M+e_k \rangle} \) such that the restriction of \( \phi \) to \( \langle M \rangle + e_k \) is \( \tau_{e_k}(b) \). Because of this equivalence, we will be dealing here with NNSOFT only.

In the sequel we will take \( \lim \sup \) and \( \lim \inf \) of real multisquences \( (a_m)_{m \in \mathbb{N}^d} \) as \( m \to \infty \). In order to be clear, we define these here and observe that they are limits of subsequences \([12]\). We also define the limit of real multisquence in terms of \( \lim \sup \) and \( \lim \inf \), which is equivalent to other definitions in the literature.

**Definition 2.1** Let \( (a_m)_{m \in \mathbb{N}^d} \) be a multisquence of real numbers. Then

(a) \( \limsup_{m \to \infty} a_m \) is defined as the supremum (possibly \( \pm \infty \)) of all numbers of the form \( \limsup_{q \to \infty} a_{m_q} \), where \( (m_q)_{q \in \mathbb{N}} \) is a sequence in \( \mathbb{N}^d \) satisfying \( \lim_{q \to \infty} m_q = \infty \), i.e., \( \lim_{q \to \infty} (m_q)_k = \infty \) for each \( k \in (d) \). We define \( \liminf_{m \to \infty} a_m \) similarly.

(b) \( \lim_{m \to \infty} a_m = \alpha \) means \( \limsup_{m \to \infty} a_m = \liminf_{m \to \infty} a_m = \alpha \).

As in \([12]\), given an NNSOFT \( C_\Gamma(\mathbb{Z}^d) \) and \( m \in \mathbb{N}^d \), we denote by \( C_\Gamma(\langle m \rangle) \) the set of all colorings \( \phi \in \langle n \rangle^{(m)} \) such that \( i, i + e_k \in \langle m \rangle \) imply \( \langle \phi_i, \phi_{i+e_k} \rangle \in \Gamma_k \). Similarly, we denote by \( C_{\Gamma, \text{top}}(\langle m \rangle) \subseteq C_\Gamma(\langle m \rangle) \) the projection of \( C_\Gamma(\mathbb{Z}^d) \) on \( \langle m \rangle \), i.e., the set of colorings in \( \langle n \rangle^{(m)} \) that can be extended to colorings in \( C_\Gamma(\mathbb{Z}^d) \), and by \( C_{\Gamma, \text{per}}(\langle m \rangle) \subseteq C_{\Gamma, \text{top}}(\langle m \rangle) \) the
set of periodic \( \Gamma \)-colorings with period \( \mathbf{m} \), i.e., the set of colorings in \( \langle n \rangle^{(\mathbf{m})} \) that can be extended to colorings in \( C_{\Gamma}(\mathbb{Z}^d) \) with period \( \mathbf{m} \). For a weight vector \( \mathbf{u} = (u_1, \ldots, u_n)^{\top} \in \mathbb{R}^n \) on the colors, we define

\[
Z_{\Gamma}(\mathbf{m}, \mathbf{u}) := \sum_{\phi \in C_{\Gamma}(\mathbf{m})} e^{\mathbf{c}^{\top} \mathbf{u}},
\]

\[
Z_{\Gamma,\text{top}}(\mathbf{m}, \mathbf{u}) := \sum_{\phi \in C_{\Gamma,\text{top}}(\mathbf{m})} e^{\mathbf{c}^{\top} \mathbf{u}},
\]

\[
Z_{\Gamma,\text{per}}(\mathbf{m}, \mathbf{u}) := \sum_{\phi \in C_{\Gamma,\text{per}}(\mathbf{m})} e^{\mathbf{c}^{\top} \mathbf{u}}.
\]

As usual, a summation over an empty set is understood as 0. Obviously

\[
\#C_{\Gamma}(\langle \mathbf{m} \rangle) = Z_{\Gamma}(\mathbf{m}, 0),
\]

\[
\#C_{\Gamma,\text{top}}(\langle \mathbf{m} \rangle) = Z_{\Gamma,\text{top}}(\mathbf{m}, 0),
\]

\[
\#C_{\Gamma,\text{per}}(\langle \mathbf{m} \rangle) = Z_{\text{per}}(\mathbf{m}, 0).
\]

A function \( f(\mathbf{u}) \geq 0 \) on \( \mathbb{R}^n \) is called log-convex when \( \log f(\mathbf{u}) \) is convex. (The zero function is by definition log-convex.) Recall that the log-convex functions are closed under linear combinations with nonnegative coefficients \([21]\). Since for \( \mathbf{c} \in \mathbb{R}^n \) the function \( e^{\mathbf{c}^{\top} \mathbf{u}} \) is log-convex, the weighted sums \( Z_{\Gamma}(\mathbf{m}, \mathbf{u}), Z_{\Gamma,\text{top}}(\mathbf{m}, \mathbf{u}), Z_{\Gamma,\text{per}}(\mathbf{m}, \mathbf{u}) \) are log-convex functions of \( \mathbf{u} \) for each \( \mathbf{m} \in \mathbb{N}^d \). As in \([12]\) it follows that for a fixed \( \mathbf{u} \), \( \log Z_{\Gamma}(\mathbf{m}, \mathbf{u}) \) and \( \log Z_{\Gamma,\text{top}}(\mathbf{m}, \mathbf{u}) \) are subadditive in each coordinate of \( \mathbf{m} \), and so the limits (2.2) and (2.3) below exist. The quantities

\[
P_{\Gamma}(\mathbf{u}) := \lim_{\mathbf{m} \to \infty} \frac{\log Z_{\Gamma}(\mathbf{m}, \mathbf{u})}{\text{vol}(\mathbf{m})},
\]

\[
P_{\Gamma,\text{top}}(\mathbf{u}) := \lim_{\mathbf{m} \to \infty} \frac{\log Z_{\Gamma,\text{top}}(\mathbf{m}, \mathbf{u})}{\text{vol}(\mathbf{m})},
\]

\[
P_{\Gamma,\text{per}}(\mathbf{u}) := \lim\sup_{\mathbf{m} \to \infty} \frac{\log Z_{\Gamma,\text{per}}(\mathbf{m}, \mathbf{u})}{\text{vol}(\mathbf{m})}
\]

are called the pressure, the topological pressure and the periodic pressure of \( C_{\Gamma}(\mathbb{Z}^d) \), respectively. The special cases \( h_{\Gamma} := P_{\Gamma}(0) \), \( h_{\Gamma,\text{top}} := P_{\Gamma,\text{top}}(0) \), \( h_{\Gamma,\text{per}} := P_{\Gamma,\text{per}}(0) \) are the entropy, the topological entropy and the periodic entropy, respectively, discussed in \([12]\). Clearly

\[-\infty \leq P_{\Gamma,\text{per}}(\mathbf{u}) \leq P_{\Gamma,\text{top}}(\mathbf{u}) \leq P_{\Gamma}(\mathbf{u}) .\]

By the log-convexity of \( Z_{\Gamma}(\mathbf{m}, \mathbf{u}), Z_{\Gamma,\text{top}}(\mathbf{m}, \mathbf{u}), Z_{\Gamma,\text{per}}(\mathbf{m}, \mathbf{u}) \), it follows that \( P_{\Gamma}(\mathbf{u}), P_{\Gamma,\text{top}}(\mathbf{u}), P_{\Gamma,\text{per}}(\mathbf{u}) \) are convex functions on \( \mathbb{R}^n \). (We agree that the constant function \(-\infty \) is convex.) As in \([9]\), one has the equality

\[P_{\Gamma,\text{top}}(\mathbf{u}) = P_{\Gamma}(\mathbf{u}).\]

Since \( \log Z_{\Gamma}(\mathbf{m}, \mathbf{u}) \) is subadditive in each coordinate of \( \mathbf{m} \), it follows that

\[P_{\Gamma}(\mathbf{u}) \leq \frac{\log Z_{\Gamma}(\mathbf{m}, \mathbf{u})}{\text{vol}(\mathbf{m})}.\]

In the one-dimensional case \( d = 1 \), we can express \( P_{\Gamma_1}(\mathbf{u}) \) as the logarithm of the spectral radius (largest modulus of an eigenvalue) of a certain \( n \times n \) matrix as follows.

**Proposition 2.2** Let \( D_{\Gamma_1} = (d_{ij})_{i,j \in \langle n \rangle} \) be the \((0,1)\)-adjacency matrix of \( \Gamma_1 \), and let \( D_{\Gamma_1}(\mathbf{u}) = (d_{ij}(\mathbf{u}))_{i,j \in \langle n \rangle} \) be defined by

\[d_{ij}(\mathbf{u}) := d_{ij} \cdot e^{\frac{1}{2}(\mathbf{e}_i^{\top} \mathbf{u} + \mathbf{e}_j^{\top} \mathbf{u})}.
\]

Let \( \rho_{\Gamma_1}(D(\mathbf{u})) \) be the spectral radius of \( D_{\Gamma_1}(\mathbf{u}) \). Then

\[P_{\Gamma_1}(\mathbf{u}) = \log \rho_{\Gamma_1}(D(\mathbf{u})).\]
Proof: Recall the following characterization of the spectral radius of a nonnegative matrix $M$: for any vector $w$ with positive components, we have $\rho(M) = \lim_{k \to \infty} (w^T M^k w)^{1/k}$ (cf., Proposition 10.1 of [10]). Consider the positive vector

$$1(u) = (e^{pl_i u})_{i \in \Gamma(m)}.$$  

Since $C_{\Gamma_i}((m_1))$ is the set of walks of length $m_1 - 1$ on $\Gamma_1$, we have $Z_{\Gamma_i}((m_1), u) = 1(u)^T D_{\Gamma_i}(u)m_i - 11(u)$. Therefore

$$\log \rho_{\Gamma_i}(D(u)) = \lim_{m_i \to \infty} \frac{\log 1(u)^T D_{\Gamma_i}(u)m_i - 11(u)}{m_i} = \lim_{m_i \to \infty} \frac{\log Z_{\Gamma_i}(m_1, u)}{m_i} = P_{\Gamma_i}(u).$$

3 Main Inequalities for Symmetric NNSOF

In this section we derive bounds for the pressure analogous to those for the entropy in [12, Section 3] under the assumption that some of the digraphs $\Gamma_1, \ldots, \Gamma_d$ are symmetric.

For $d \geq 2$, consider $m = (m_1, \ldots, m_d) \in \mathbb{N}^d$ and $m^- = (m_2, \ldots, m_d)$. We denote by $T(m)$ the discrete torus with sides of length $m_1, \ldots, m_d$, i.e., direct product of cycles of lengths $m_1, \ldots, m_d$. Let $C_{\Gamma_{\text{per},(1)}}(m)$ be the set of $\Gamma$-colorings of the box $\langle m \rangle$ that correspond to $\Gamma$-colorings of $T(m_1) \times \langle m^- \rangle$, i.e., that can be extended periodically in the direction of $e_1$ with period $m_1$ into $\Gamma$-colorings of $\mathbb{Z} \times \langle m^- \rangle$. We can view these colorings as $\Gamma$-colorings of the box $\langle m^- \rangle$, where $\Gamma = (\Gamma_2, \ldots, \Gamma_d)$, for each $k$ the vertex set of $\Gamma_k$ is the set $\Gamma_{\text{per}}^{m_1}$ of closed walks $a = (a_1, \ldots, a_{m_1}, a_1)$ of length $m_1$ on $\Gamma_1$, and $(a, b) \in \Gamma_k$ if and only if $(a_i, b_i) \in \Gamma_k$ for $i = 1, \ldots, m_1$. For this reason, the limit (3.1) below exists and is equal to the pressure $P_{\Gamma}(u)$ of the NNSOF $C_{\Gamma}(\mathbb{Z}^{d-1})$:

$$Z_{\Gamma_{\text{per},(1)}}(m, u) := \sum_{\phi \in C_{\Gamma_{\text{per},(1)}}(m)} e^{\phi^T u},$$

$$P_{\Gamma}(m_1, u) := \lim_{m_1 \to \infty} \frac{\log Z_{\Gamma_{\text{per},(1)}}(m, u)}{\text{vol}(m^-)}, \quad m_1 \in \mathbb{N}. \quad (3.1)$$

Then $P_{\Gamma}(m_1, u)$ is a convex function of $u \in \mathbb{R}^n$. In the degenerate case $m_1 = 0$, we define $Z_{\Gamma_{\text{per},(1)}}((0, m^-), u)$ to be $\# C_{\Gamma}(m^-)$ (regardless of $u$), where $C_{\Gamma}(m^-)$ is the set of $(\Gamma_2, \ldots, \Gamma_d)$-colorings of the box $\langle m^- \rangle$. Then (3.1) is also valid for $m_1 = 0$, where $P_{\Gamma}(0, u) := P_{\Gamma(2, \ldots, d)}(0)$ is the entropy of $C_{\Gamma}(\mathbb{Z}^{d-1})$.

Theorem 3.1 Consider the NNSOF $C_{\Gamma}(\mathbb{Z}^d)$ for $d \geq 2$, and let $P_{\Gamma}(u)$ and $P_{\Gamma}(m_1, u)$ be defined by (2.2) and (3.1), respectively. Assume that $\Gamma_1$ is symmetric. Then for all $p, r \in \mathbb{N}$ and $q \in \mathbb{Z}_+$,

$$\frac{P_{\Gamma}(2r, u)}{2r} \geq P_{\Gamma}(u) \geq \frac{P_{\Gamma}(p + 2q, u) - P_{\Gamma}(2q, u)}{p}. \quad (3.2)$$

Proof: Fix $m^- = (m_2, \ldots, m_d) \in \mathbb{N}^{d-1}$ and let $\Omega_1(m^-)$ be the following transfer digraph on the vertex set $C_{\Gamma}(m^-)$, analogous to the transfer digraph $\Omega_d(m')$ described in [12, Section 1]. Vertices $i$ and $j$ satisfy $(i, j) \in \Omega(m^-)$ if and only if $[i, j] \in C_{\Gamma}(2, m^-)$, where $[i, j]$ is the configuration consisting of $i, j$ occupying the levels $x_1 = 1, 2$ of $\langle (2, m^-) \rangle$, respectively. Let $N = \# C_{\Gamma}(m^-)$ and let $D_{\Gamma}(m^-) = (d_{ij})_{i, j \in C_{\Gamma}(m^-)}$ be the $N \times N$ $(0, 1)$-adjacency matrix of $\Omega_1(m^-)$. Let $D_{\Gamma}(m^-, u) = (d_{ij}(u))_{i, j \in C_{\Gamma}(m^-)}$ be defined by

$$d_{ij}(u) = d_{ij} \cdot e^{\phi_{ij}^T u}, \quad i, j \in C_{\Gamma}(m^-), \quad (3.3)$$

\[
\begin{align*}
&\text{(3.3)}
\end{align*}
\]

\[
\begin{align*}
&\text{(3.3)}
\end{align*}
\]
and let the positive vector $\mathbf{1}(\mathbf{u})$ be defined by

$$
\mathbf{1}((\mathbf{u})^\top)_{i \in C^-_\Gamma((\mathbf{m}^-))}.
$$

Then

$$
\mathbf{1}((\mathbf{u})^\top) D_{\Gamma}((\mathbf{m}^-, \mathbf{u})^{m_1} \mathbf{1}((\mathbf{u})^\top) = Z_{\Gamma}((m_1, \mathbf{m}^-), \mathbf{u}),
$$

and as in the proof of Proposition 2.2

$$
\log \rho(D_{\Gamma}((\mathbf{m}^-, \mathbf{u}))) = \lim_{m_1 \to \infty} \frac{\log \mathbf{1}((\mathbf{u})^\top) D_{\Gamma}((\mathbf{m}^-, \mathbf{u})^{m_1} \mathbf{1}((\mathbf{u})^\top)}{m_1}.
$$

(In particular, $\rho(D_{\Gamma}((\mathbf{m}^-, \mathbf{u})))$ is a log-convex function of $\mathbf{u}$ [21].) It follows that

$$
\frac{\log \rho(D_{\Gamma}((\mathbf{m}^-, \mathbf{u})))}{\text{vol}(\mathbf{m}^-)} = \lim_{m_1 \to \infty} \frac{\log Z_{\Gamma}((m_1, \mathbf{m}^-), \mathbf{u})}{m_1 \text{vol}(\mathbf{m}^-)}. 
$$

(3.4)

Now send $m_2, \ldots, m_d$ to $\infty$, and observe that by (2.2) and (2.5), the right-hand side of (3.4) converges to $P_{\Gamma}(\mathbf{u})$ and bounds it from above for each $\mathbf{m}^-$. Thus we obtain an analog of [9]

$$
\frac{\log \rho(D_{\Gamma}((\mathbf{m}^-, \mathbf{u})))}{\text{vol}(\mathbf{m}^-)} \geq P_{\Gamma}(\mathbf{u}), \quad \mathbf{m}^- \in \mathbb{N}^{d-1}
$$

(3.5)

$$
\lim_{\mathbf{m}^- \to \infty} \frac{\log \rho(D_{\Gamma}((\mathbf{m}^-, \mathbf{u})))}{\text{vol}(\mathbf{m}^-)} = P_{\Gamma}(\mathbf{u}).
$$

(3.6)

Next, we observe that

$$
\text{tr} D_{\Gamma}((\mathbf{m}^-, \mathbf{u})^{q}) = Z_{\Gamma, \text{per}, \{1\}}(\{q, \mathbf{m}^-, \mathbf{u}) , \quad q \in \mathbb{Z}_+,
$$

(3.7)

where $D_{\Gamma}((\mathbf{m}^-, \mathbf{u})^{q}$ is the $N \times N$ identity matrix. Recall that

$$
\text{tr} D_{\Gamma}((\mathbf{m}^-, \mathbf{u})^{q} = \sum_{i=1}^{N} \lambda_i^q, \quad q \in \mathbb{Z}_+,
$$

where $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of $D_{\Gamma}((\mathbf{m}^-, \mathbf{u})$. Since $D_{\Gamma}((\mathbf{m}^-, \mathbf{u})$ is a nonnegative matrix, its spectral radius $\rho(D_{\Gamma}((\mathbf{m}^-, \mathbf{u}) := \max_{i \in \{N\}} |\lambda_i|$ is one of the $\lambda_i$ by the Perron-Frobenius theorem. Since by assumption $\Gamma_1$ is symmetric, $\Omega_1((\mathbf{m}^-)$ and hence $D_{\Gamma}((\mathbf{m}^-, \mathbf{u})$ are symmetric. Therefore $\lambda_1, \ldots, \lambda_N$ are real, and hence $\text{tr} D_{\Gamma}((\mathbf{m}^-, \mathbf{u})^{2q} \geq \rho(D_{\Gamma}((\mathbf{m}^-, \mathbf{u})^{2q}$ for each $r \in \mathbb{N}$. Taking logarithms and using (3.7), we obtain

$$
\frac{\log Z_{\Gamma, \text{per}, \{1\}}(\{2r, \mathbf{m}^-, \mathbf{u})}{2r \text{vol}(\mathbf{m}^-)} \geq \frac{\log \rho(D_{\Gamma}((\mathbf{m}^-, \mathbf{u})}}{\text{vol}(\mathbf{m}^-)}, \quad r \in \mathbb{N}.
$$

(3.8)

Sending $m_2, \ldots, m_d$ to $\infty$ in (3.8) and using (3.1) and (3.6), we deduce the upper bound for $P_{\Gamma}(\mathbf{u})$ in (3.2). To prove the lower bound in (3.2), we note that

$$
\text{tr} D_{\Gamma}((\mathbf{m}^-, \mathbf{u})^{p+2q} = \sum_i \lambda_i^{p+2q} \leq \sum_i |\lambda_i|^{p+2q} = \sum_i |\lambda_i|^p \lambda_i^{2q} 
$$

$$
\leq \sum_i \rho(D_{\Gamma}((\mathbf{m}^-, \mathbf{u})^p \lambda_i^{2q} = \rho(D_{\Gamma}((\mathbf{m}^-, \mathbf{u})^p \text{tr} D_{\Gamma}((\mathbf{m}^-, \mathbf{u})^{2q}
$$

and thus by (3.7)

$$
\rho(D_{\Gamma}((\mathbf{m}^-, \mathbf{u})^p \geq \frac{\text{tr} D_{\Gamma}((\mathbf{m}^-, \mathbf{u})^{p+2q}}{\text{tr} D_{\Gamma}((\mathbf{m}^-, \mathbf{u})^{2q} = \frac{Z_{\Gamma, \text{per}, \{1\}}(\{p+2q, \mathbf{m}^-, \mathbf{u})}{Z_{\Gamma, \text{per}, \{1\}}(\{2q, \mathbf{m}^-, \mathbf{u})}.
$$

(3.9)
Therefore
\[
\frac{\log \rho(D_T(m^-, u))}{\text{vol}(m^-)} \geq \frac{\log Z_{\Gamma, \text{per}, (1)}((p + 2q, m^-), u) - \log Z_{\Gamma, \text{per}, (1)}((2q, m^-), u)}{p \cdot \text{vol}(m^-)}.
\]
Sending \(m^-\) to \(\infty\) and using (3.6) and (3.1) (recall that the latter holds for \(m_1 \in \mathbb{Z}_+\)), we deduce the lower bound in (3.2).

When \(d = 2\), \(P_T(m_1, u)\) is the pressure of the NNSOFT \(C_{\Gamma_2}(Z)\) (recall that \(\mathcal{P}_T(0, u)\) is the entropy \(h_{\Gamma_2}\)). Since this is a 1-dimensional NNSOFT, Proposition 2.2 implies that \(\mathcal{P}_T(m_1, u) = \log \rho(D_{\Gamma_2}(u))\), where \(D_{\Gamma_2}(u)\) is defined as in (2.6). We denote \(\rho(D_{\Gamma_2}(u))\) by \(\theta_2(m_1, u)\), and obtain the following corollary to Theorem 3.1.

**Corollary 3.2** Let \(d = 2\) and assume that \(\Gamma_1\) is symmetric. Then for all \(p, r \in \mathbb{N}\) and \(q \in \mathbb{Z}_+\),
\[
\frac{\log \theta_2(2r, u)}{2r} \geq \mathcal{P}_T(u) \geq \frac{\log \theta_2(p + 2q, u) - \log \theta_2(2q, u)}{p},
\]
where \(\theta_2\) is defined above.

In (3.10) take \(q = 0\) and \(p = 2r\), and send \(r\) to \(\infty\). Clearly the upper and lower bounds then converge to \(\mathcal{P}_T(u)\). Hence \(\mathcal{P}_T(u)\) is computable, as shown in [9] for the entropy \(P_T(0)\). Combining the arguments of the proof of Theorem 3.1 with the arguments of the proof of Theorem 3.4 in [12], we obtain
\[
\mathcal{P}_T(u) \leq \frac{\log \rho(D_T(m^-, u))}{\text{vol}(m^-)}, \quad m_2, \ldots, m_d \text{ even, } \Gamma_2, \ldots, \Gamma_d \text{ symmetric},
\]
where \(D_T(m^-, u)\) is defined in (3.3).

### 4 The Conjugate of Pressure and the Density Entropy

The purpose of this section is to exhibit a striking connection between the conjugate function \(P_T^*(p)\) of the pressure \(P_T(u)\) and the density entropy \(h_T(p)\) for certain probability vectors \(p\).

First we need to recall some properties of convex functions, which can be found in [26]. We adopt the notations of that book.

In this paper we consider only convex functions \(f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}\) that are not identically equal to \(+\infty\). Such convex functions are called *proper* in [26]. Let \(f\) be a proper convex function. Then \(\text{dom } f := \{x \in \mathbb{R}^m : f(x) < \infty\}\), which is called the *effective domain* of \(f\), is a nonempty convex set in \(\mathbb{R}^m\). Let \(L \subseteq \mathbb{R}^m\) be the minimal affine subspace that contains \(\text{dom } f\), and let \(l\) be its dimension. The affine transformation \(A\) that maps \(L\) onto \(\mathbb{R}^l\) maps \(f\) onto a convex set \(C \subseteq \mathbb{R}^l\). We denote the interior of \(C\) by \(\text{int } C\). Then \(\text{ri } (\text{dom } f) := A^{-1}(\text{int } C)\) is called the *relative interior* of \(\text{dom } f\) (note that if \(l = 0\), then \(\text{dom } f\) and \(\text{ri } (\text{dom } f)\) consist of the same single point). A proper convex function \(f\) is Lipschitzian relative to any closed bounded subset of \(\text{ri } (\text{dom } f)\) [26, Thm 10.4]. In particular, \(f\) is continuous on \(\text{ri } (\text{dom } f)\) [26, Thm 10.1].

A proper convex function \(f\) is called *closed* if \(f\) is lower semi-continuous [26, Section 7, p. 52]. In particular, if \(f : \mathbb{R}^m \to \mathbb{R}\) is convex, then \(\text{dom } f = \mathbb{R}^m = \text{ri } (\text{dom } f)\), \(f\) is a continuous function on \(\mathbb{R}^m\), hence closed, and \(f\) is Lipschitzian relative to any closed bounded subset of \(\mathbb{R}^m\).

A vector \(y \in \mathbb{R}^m\) is called a *subgradient* of \(f\) at \(x \in \mathbb{R}^m\) if \(f(z) \geq f(x) + y^T(z - x)\) for all \(z \in \mathbb{R}^m\). The set of all subgradients \(y\) at \(x\) is called the *subdifferential* of \(f\) at \(x\) and is denoted by \(\partial f(x)\). As usual, for any set \(S \subseteq \mathbb{R}^m\), \(\partial f(S)\) denotes \(\cup_{x \in S} \partial f(x)\). Obviously \(\partial f(x)\) is a closed convex set. If \(\partial f(x) \neq \emptyset\), then \(f\) is said to be *subdifferentiable* at \(x\). A proper convex function \(f\) is not subdifferentiable at any \(x \notin \text{dom } f\), but is subdifferentiable
at each \( \mathbf{x} \in \text{ri} (\text{dom } f) \) [26, Thm 23.4]. Recall that \( f \) is differentiable at \( \mathbf{x} \) if there exists a vector \( \nabla f(\mathbf{x}) = \mathbf{y} \in \mathbb{R}^n \) (necessarily unique) such that \( f(\mathbf{x} + \mathbf{w}) = f(\mathbf{x}) + \mathbf{y}^\top \mathbf{w} + o(\|\mathbf{w}\|) \), \( \mathbf{w} \to \mathbf{0} \). The vector \( \nabla f(\mathbf{x}) \) is called the gradient of \( f \) at \( \mathbf{x} \). We denote by \( \partial f \) the set of all points where \( f \) is differentiable, so \( \partial f(\text{diff } f) \) denotes the set of all gradient vectors of \( f \). A proper convex function \( f \) is differentiable at a point \( \mathbf{x} \in \text{dom } f \) if and only if \( \partial f(\mathbf{x}) \) consists of a single point, which is then \( \nabla f(\mathbf{x}) \) [26, Thm 25.1].

Assume that \( f \) is a proper convex function and \( \text{int}(\text{dom } f) \neq \emptyset \). Then \( \partial f \) is a dense subset of \( \text{int}(\text{dom } f) \), \( f \) is differentiable a.e. (almost everywhere) in \( \text{int}(\text{dom } f) \), and \( \nabla f \) is continuous on \( \partial f \) [26, Thm 25.5]. Moreover, for each \( \mathbf{x} \in \text{int}(\text{dom } f) \setminus \partial f \), the convex set \( \partial f(\mathbf{x}) \), which consists of more than one point, can be reconstructed as follows from the values of the gradient function \( \nabla f \) on \( \partial f \). Let \( S(\mathbf{x}) \) consist of all the limits of sequences \( \nabla f(\mathbf{x}_i) \) such that \( \mathbf{x}_i \in \partial f \) and \( \mathbf{x}_i \to \mathbf{x} \). Then \( S(\mathbf{x}) \) is a closed and bounded subset of \( \mathbb{R}^n \) and \( \partial f(\mathbf{x}) = \text{conv } S(\mathbf{x}) \) [26, Thm 25.6, 7.4].

We now recall properties of the conjugate of a convex function \( f \) [26, Section 12], denoted by \( f^* \):

\[
\begin{align*}
f^*(\mathbf{y}) := \sup_{\mathbf{x} \in \mathbb{R}^n} \mathbf{x}^\top \mathbf{y} - f(\mathbf{x}) & \quad \text{for each } \mathbf{y} \in \mathbb{R}^m. \\
\end{align*}
\]

Since we assumed that \( f \) is proper, it follows that \( f^* \) is a proper closed convex function; moreover, if \( f \) is closed then \( f^{**} = f \) [26, Thm. 12.2]. A straightforward argument shows that

\[
\begin{align*}
f^*(\mathbf{y}) = \mathbf{x}^\top \mathbf{y} - f(\mathbf{x}) & \quad \text{for each subgradient } \mathbf{y} \in \partial f(\mathbf{x}). \quad (4.1)
\end{align*}
\]

Recall that if \( f \) is closed then \( \partial f^* \) is the inverse of \( \partial f \) in the sense of multivalued mappings, i.e., \( \mathbf{x} \in \partial f^*(\mathbf{y}) \) if and only if \( \mathbf{y} \in \partial f(\mathbf{x}) \) [26, Cor 23.5.1]. In what follows we need the following result:

**Lemma 1** Let \( f \) be a proper closed convex function on \( \mathbb{R}^m \). Then \( \partial f(\mathbb{R}^m) \) is exactly the set of points in \( \mathbb{R}^m \) where \( f^* \) is subdifferentiable. In particular, \( \text{ri} (\text{dom } f^*) \subseteq \partial f(\mathbb{R}^m) \subseteq \text{dom } f^* \), and the closure of \( \partial f(\mathbb{R}^m) \) is equal to the closure of \( \text{dom } f^* \).

**Proof** Assume that \( f^* \) is subdifferentiable at \( \mathbf{y} \). Then there exists some \( \mathbf{x} \in \mathbb{R}^m \) such that \( \mathbf{x} \in \partial f^*(\mathbf{y}) \). Since \( f \) is closed, it now follows that \( \mathbf{y} \in \partial f(\mathbf{x}) \), so \( \mathbf{y} \in \partial f(\mathbb{R}^m) \). The converse is shown in the same way. The first statement of the “in particular” now follows since \( f^* \) is proper and hence, as noted above, is subdifferentiable in \( \text{ri} (\text{dom } f^*) \) but not outside \( \text{dom } f^* \). The second statement of the “in particular” follows from the first one and from the fact that for a convex set \( S \) such as \( \text{dom } f^* \), \( \text{ri } S \) and \( S \) have the same closure [26, Thm 6.3].

We return to a general NNSOFT \( C_\Gamma(\mathbb{Z}^d) \). We assume throughout that \( C_\Gamma(\mathbb{Z}^d) \neq \emptyset \), for otherwise the pressure function \( P_\Gamma \) is identically \(-\infty\).

**Proposition 4.1** The pressure \( P_\Gamma \) is a convex nonexpansive Lipschitz function on \( \mathbb{R}^n \) with constant at most 1 with respect to the norm \( \|v_1, \ldots, v_n\|_\infty := \max_{i \in \langle n \rangle} |v_i| : |P_\Gamma(\mathbf{u} + \mathbf{v}) - P_\Gamma(\mathbf{u})| \leq \|\mathbf{v}\|_\infty, \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^n. \quad (4.2) \)

In particular, \( P_\Gamma \) is finite throughout \( \mathbb{R}^n \) (this also follows from (2.5)). Therefore it is a proper closed convex function.

**Proof** The convexity of \( P_\Gamma \) was pointed out in Section 1. Let \( \phi \in C_\Gamma((\mathbf{m})) \). Then

\[
\|c(\phi)^\top \mathbf{v}\| \leq \text{vol}(\mathbf{m})\|\mathbf{v}\|_\infty.
\]

Therefore a term-by-term comparison gives

\[
e^{-\text{vol}(\mathbf{m})\|\mathbf{v}\|_\infty} Z_{\Gamma}(\mathbf{m}, \mathbf{u}) \leq Z_{\Gamma}(\mathbf{m}, \mathbf{u} + \mathbf{v}) \leq e^{\text{vol}(\mathbf{m})\|\mathbf{v}\|_\infty} Z_{\Gamma}(\mathbf{m}, \mathbf{u}).
\]

Take logarithms and divide by \( \text{vol}(\mathbf{m}) \) to obtain

\[
\left| \frac{\log Z_{\Gamma}(\mathbf{m}, \mathbf{u} + \mathbf{v})}{\text{vol}(\mathbf{m})} - \frac{\log Z_{\Gamma}(\mathbf{m}, \mathbf{u})}{\text{vol}(\mathbf{m})} \right| \leq \|\mathbf{v}\|_\infty. \quad (4.3)
\]

Letting \( \mathbf{m} \to \infty \), we deduce (4.2).
Since $P_T : \mathbb{R}^n \to \mathbb{R}$ is a convex function, it is differentiable almost everywhere. Consider the set of probability distributions on the set of colors

$\Pi_n := \{\mathbf{p} = (p_1, \ldots, p_n) : p_1, \ldots, p_n \geq 0, p_1 + \cdots + p_n = 1\}$. For $m \in \mathbb{N}$, we denote

$\Pi_n(m) := \{\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{Z}_+^n : c_1 + \cdots + c_n = m\} = m\Pi_n \cap \mathbb{Z}^n$.

Let

$C_T((\mathbf{m}), \mathbf{c}) := \{\phi \in C_T((\mathbf{m})) : \mathbf{c}(\phi) = \mathbf{c}\}, \text{ for all } \mathbf{c} \in \Pi_n(\text{vol}(\mathbf{m}))$.

This is the set of $T$-colorings of $(\mathbf{m})$ with color frequency vector $\mathbf{c}$.

**Definition 4.2** A probability distribution $\mathbf{p} \in \Pi_n$ is called a density point of $C_T(\mathbb{Z}^d)$ when there exist sequences of boxes $(\mathbf{m}_q) \subseteq \mathbb{N}^d$ and color frequency vectors $\mathbf{c}_q \in \Pi_n(\text{vol}(\mathbf{m}_q))$ such that

$\mathbf{m}_q \to \infty, \quad C_T((\mathbf{m}_q), \mathbf{c}_q) \neq \emptyset \quad \forall q \in \mathbb{N}, \quad \text{and} \quad \lim_{q \to \infty} \frac{\mathbf{c}_q}{\text{vol}(\mathbf{m}_q)} = \mathbf{p}$. \hspace{1cm} (4.4)

We denote by $\Pi_T$ the set of all density points of $C_T(\mathbb{Z}^d)$. For $\mathbf{p} \in \Pi_T$ we let

$h_T(\mathbf{p}) := \sup_{\mathbf{m}_q, \mathbf{c}_q} \limsup_{q \to \infty} \frac{\log \# C_T((\mathbf{m}_q), \mathbf{c}_q)}{\text{vol}(\mathbf{m}_q)} \geq 0$, \hspace{1cm} (4.5)

where the supremum is taken over all sequences satisfying (4.4). One can think of $h_T(\mathbf{p})$ as the entropy for color density $\mathbf{p}$, called here the density entropy.

It is straightforward to show (using a variant of the Cantor diagonal argument) that $\Pi_T$ is a closed set. Furthermore, $h_T$ is upper semi-continuous on $\Pi_T$, because it is defined as a supremum.

**Theorem 4.3** Let $P_T^*$ be the conjugate convex function of the pressure function $P_T$. Then

(a) $h_T(\mathbf{p}) \leq -P_T^*(\mathbf{p})$ for all $\mathbf{p} \in \Pi_T$.

(b) $P_T(\mathbf{u}) = \max_{\mathbf{p} \in \Pi_T} (\mathbf{p}^\top \mathbf{u} + h_T(\mathbf{p}))$ for all $\mathbf{u} \in \mathbb{R}^n$. \hspace{1cm} (4.6)

For $\mathbf{u} \in \mathbb{R}^n$, we denote

$\Pi_T(\mathbf{u}) := \arg \max_{\mathbf{p} \in \Pi_T} (\mathbf{p}^\top \mathbf{u} + h_T(\mathbf{p})) = \{\mathbf{p} \in \Pi_T : P_T(\mathbf{u}) = \mathbf{p}^\top \mathbf{u} + h_T(\mathbf{p})\}$. \hspace{1cm} (4.7)

(c) For each $\mathbf{p} \in \Pi_T(\mathbf{u})$, $h_T(\mathbf{p}) = -P_T^*(\mathbf{p})$.

(d) $\Pi_T(\mathbf{u}) \subseteq \partial P_T(\mathbf{u})$. In particular, if $\mathbf{u} \in \text{diff } P_T$, then $\Pi_T(\mathbf{u}) = \{\nabla P_T(\mathbf{u})\}$. Therefore $\partial P_T(\text{diff } P_T) \subseteq \Pi_T$.

(e) Let $\mathbf{u} \in \mathbb{R}^n \setminus \text{diff } P_T$, and let $S(\mathbf{u})$ consist of all the limits of sequences $\nabla P_T(\mathbf{u}_i)$ such that $\mathbf{u}_i \in \text{diff } P_T$ and $\mathbf{u}_i \to \mathbf{u}$. Then $S(\mathbf{u}) \subseteq \Pi_T(\mathbf{u})$.

(f) $\text{conv } \Pi_T(\mathbf{u}) = \text{conv } S(\mathbf{u}) = \partial P_T(\mathbf{u})$. Hence $\partial P_T(\mathbb{R}^n) \subseteq \text{conv } \Pi_T \subseteq \Pi_n$.

(g) $\text{conv } \Pi_T = \text{dom } P_T^*$. 

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This proves (d).

**Proof.** First we show that $P_T(u) \geq p^T u + h_T(p)$ for all $p \in \Pi_T$. Fix $p \in \Pi_T$ and let $m_q, c_q$, $q \in \mathbb{N}$, be sequences satisfying \((4.4)\). We have $Z_T(m_q, u) \geq \#C_T(m_q, c_q)e^{c^T u}$, since the right-hand side is just one term of the sum represented by left-hand side. Take logarithms, divide by $\text{vol}(m_q)$, take $\limsup_{q \to \infty}$ and use the definition of $P_T(u)$ and the limit in \((4.4)\) to deduce $P_T(u) \geq p^T u + \limsup_{q \to \infty} \frac{\log \#C_T(m_q, c_q)}{\text{vol}(m_q)}$. Now take the supremum over all sequences $m_q, c_q$ satisfying \((4.4)\) and use \((4.5)\) to obtain

$$P_T(u) \geq p^T u + h_T(p) \quad \text{for all } p \in \Pi_T \quad (4.8)$$

and thus

$$P_T(u) \geq \sup_{p \in \Pi_T} p^T u + h_T(p). \quad (4.9)$$

On the other hand, \((4.8)\) can be written as $p^T u - P_T(u) \leq -h_T(p)$ for all $p \in \Pi_T$, and then taking the supremum over $u$ gives $P_T(p) \leq -h_T(p) < \infty$ for all $p \in \Pi_T$. Thus we have established (a) as well as

$$\Pi_T \subseteq \text{dom } P_T^\star. \quad (4.10)$$

We now show that for each $u \in \mathbb{R}^n$ there exists $p(u) \in \Pi_T$ satisfying $P_T(u) \leq p(u)^T u + h_T(p(u))$, which together with \((4.9)\) will establish \((4.6)\). Observe first that

$$\#\Pi_n(m) = \frac{(m + n - 1)}{n - 1} = O(m^{n-1}), \quad m \to \infty.$$

Therefore for each $m \in \mathbb{N}^d$,

$$Z_T(m, u) = O(\text{vol}(m)^{n-1}) \max_{c \in \Pi_n(\text{vol}(m))} \#C_T((m), c)e^{c^T u}.$$

Let

$$C(m, u) := \arg \max_{c \in \Pi_n(\text{vol}(m))} \#C_T((m), c)e^{c^T u}. \quad (4.11)$$

Then for $c(m, u) \in C(m, u)$ we have

$$Z_T(m, u) = O(\text{vol}(m)^{n-1})\#C_T((m), c(m, u))e^{c(m, u)^T u}. \quad (4.12)$$

Since $C_T(\mathbb{Z}^d) \neq \emptyset$, for each $m \in \mathbb{N}^d$ and $u \in \mathbb{R}^n$, $\frac{c(m, u)}{\text{vol}(m)}$ is a well-defined point in $\Pi_n$. Choose a sequence $m_q \to \infty$ such that $\frac{c(m_q, u)}{\text{vol}(m_q)}$ converges to some $p(u)$. We have $p(u) \in \Pi_T$ by Definition 4.2. Apply \((4.12)\) to $m_q$, and use the definition of $P_T(u)$ and $h_T(p(u))$ to deduce

$$P_T(u) \leq p(u)^T u + \limsup_{q \to \infty} \frac{\log \#C_T((m_q), c(m_q, u))}{\text{vol}(m_q)} \leq p(u)^T u + h_T(p(u)),$$

which is the desired inequality establishing (b).

By the definition of $\Pi_T(u)$, for each $p \in \Pi_T(u)$ we have $-h_T(p) = p^T u - P_T(u) \leq P_T^\star(p)$. Combining this with (a), we deduce (c).

Let $p \in \Pi_T(u)$ and $v \in \mathbb{R}^n$. Then the maximal characterization \((4.6)\) of $P_T(u + v)$ and \((4.7)\) give

$$P_T(u + v) \geq p^T (u + v) + h_T(p) = p^T v + P_T(u).$$

This proves (d).

We now prove (e). Assume that $u \in \mathbb{R}^n \setminus \text{diff } P_T$ and $p \in S(u)$. Then there exists a sequence $u_i \in \text{diff } P_T$ such that $u_i \to u$ and $\nabla P_T(u_i) \to p$. We have $\{\nabla P_T(u_i)\} = \Pi_T(u_i) \subseteq \Pi_T$ by (d), and since $\Pi_T$ is closed, $p \in \Pi_T$. By definition of $\Pi_T(u)$ we have $P_T(u_i) = \nabla P_T(u_i)^T u_i + h_T(\nabla P_T(u_i))$. When $i \to \infty$ we have firstly $P_T(u_i) \to P_T(u)$ by the continuity of $P_T$, secondly $\nabla P_T(u_i)^T u_i \to p^T u$, and thirdly $\limsup h_T(\nabla P_T(u_i)) \leq h_T(p)$

This proves (e).
by the upper semi-continuity of $h_{\Gamma}$. Therefore $P_\Gamma(u) \leq p^\top u + h_{\Gamma}(p)$. This, the fact that $p \in \Pi_{\Gamma}$, and (4.6) show that $P_\Gamma(u) = p^\top u + h_{\Gamma}(p)$, which by definition means $p \in \Pi_{\Gamma}(u)$.

We show the first first identity of (f). Let $u \in \mathbb{R}^n \setminus \text{diff } P_\Gamma$, and let $S(u)$ be as in (e). By (e) we have $S(u) \subseteq \Pi_{\Gamma}(u)$, and therefore $\partial P_\Gamma(u) = \text{conv } S(u) \subseteq \text{conv } \Pi_{\Gamma}(u)$. Since $\partial P_\Gamma(u)$ is convex, from the first claim of (d) we obtain $\partial P_\Gamma(u) \supseteq \text{conv } \Pi_{\Gamma}(u)$. Hence $\partial P_\Gamma(u) = \text{conv } \Pi_{\Gamma}(u)$. Clearly, $\Pi_{\Gamma}(u) \subseteq \text{conv } \Pi_{\Gamma}$. Hence $\partial P_\Gamma(\mathbb{R}^n) \subseteq \text{conv } \Pi_{\Gamma}$. The second inclusion of the second claim of (f) follows from $\Pi_{\Gamma} \subseteq \Pi_n$, which holds by definition of $\Pi_{\Gamma}$.

We finally show (g). By (4.10) and the convexity of dom $P_{\Gamma}^*$, we have $\Pi_{\Gamma} \subseteq \text{dom } P_{\Gamma}^*$. It is left to show that dom $P_{\Gamma}^* \subseteq \text{conv } \Pi_{\Gamma}$. By Lemma 1 $\partial P_{\Gamma}(\mathbb{R}^n)$ is the set of all points where $P_{\Gamma}^*$ is subdifferentiable. In particular $\text{ri } (\text{dom } P_{\Gamma}^*) \subseteq \partial P_{\Gamma}(\mathbb{R}^n)$, so by (f) we have $\text{ri } (\text{dom } P_{\Gamma}^*) \subseteq \text{conv } \Pi_{\Gamma}$. Apply the closure operator to both sides of this inclusion. On the left we get $\text{cl } (\text{dom } P_{\Gamma}^*)$ by the convexity of dom $P_{\Gamma}^*$ (by [26, Thm 6.3], every convex set $C$ satisfies $\text{cl } (\text{ri } C) = \text{cl } C$). On the right we get $\text{conv } \Pi_{\Gamma}$ because $\Pi_{\Gamma}$ is closed. So we obtain $\text{dom } P_{\Gamma}^* \subseteq \text{cl } (\text{dom } P_{\Gamma}^*) \subseteq \text{conv } \Pi_{\Gamma}$, as required.

As $h_{\Gamma} = P_{\Gamma}(0)$, we obtain from (4.6) the following generalization of [12, (4.12)], which deals with the case of monomer-dimer entropy:

**Corollary 4.4**

$$h_{\Gamma} = \max_{p \in \Pi_{\Gamma}} h_{\Gamma}(p).$$

For each $p \in \Pi_{\Gamma}(\mathbb{R}^n) := \bigcup_{u \in \mathbb{R}^n} \Pi_{\Gamma}(u)$ we have $h_{\Gamma}(p) = -P_{\Gamma}(p)$ by (c) of Theorem (4.3). Since $P_{\Gamma}^*$ is a convex function, we obtain the following generalization of the result Hammersley [15].

**Corollary 4.5** The function $h_{\Gamma}(\cdot) : \Pi_{\Gamma} \to \mathbb{R}_+$ is concave on every convex subset of $\Pi_{\Gamma}(\mathbb{R}^n)$.

To obtain the exact generalization of the result of Hammersley that $\Pi_{\Gamma}$ is convex and $h_{\Gamma}(\cdot) : \Pi_{\Gamma} \to \mathbb{R}_+$ is a concave function on the entire $\Pi_{\Gamma}$, we need additional assumptions on the digraph $\Gamma$, which do hold for the $\Gamma$ that codes the monomer-dimer tilings of $\mathbb{Z}^d$. For $m \in \mathbb{N}^d$, if $\alpha : \langle m \rangle \to < n >$ is a coloring of a box $\langle m \rangle$ and $j \in \mathbb{Z}^d$, then to color the shifted box $\langle m \rangle + j$ by $\alpha$ means to give to $x + j$ the color $\alpha(x)$ for each $x \in \langle m \rangle$. Recall that $C_{\Gamma}(\langle m \rangle)$ denotes the set of all $\Gamma$-allowed colorings $\alpha : \langle m \rangle \to < n >$, that is to say, such that if $x, x + e_i \in \langle m \rangle$, then $\alpha(x), \alpha(y) \in \Gamma_i$.

**Definition 4.6** For a given digraph $\Gamma = (\Gamma_1, \ldots, \Gamma_d)$ on the vertex set $\langle n \rangle$, a set $F = \bigcup_{m \in \mathbb{N}^d} C_{\Gamma}(\langle m \rangle)$, where $C_{\Gamma}(\langle m \rangle) \subseteq C_{\Gamma}(\langle m \rangle)$ for each $m \in \mathbb{N}^d$, is called friendly if the following condition holds: whenever a shifted box is cut in two and each part is colored by a coloring in $F$, then the combined coloring also belongs to $F$. More precisely, let $m, n \in \mathbb{N}^d$ and $j \in \mathbb{Z}^d$ be such that $\langle m \rangle \cap \langle n \rangle + j = \emptyset$, and such that $T := m \cup (n + j)$ is a box $\langle k \rangle + i$ for some $k \in \mathbb{N}^d$ and $i \in \mathbb{Z}^d$. Let $\alpha \in C_{\Gamma}(\langle m \rangle)$, $\beta \in C_{\Gamma}(\langle n \rangle)$, and let $\gamma : T \to < n >$ color $\langle m \rangle$ by $\alpha$ and $\langle n \rangle + j$ by $\beta$. Then the coloring $\delta : \langle k \rangle \to < n >$ defined by $\delta(x) = \gamma(x + i)$ belongs to $C_{\Gamma}(\langle k \rangle)$.

The digraph $\Gamma$ is called friendly if there exist a friendly set $F = \bigcup_{m \in \mathbb{N}^d} C_{\Gamma}(\langle m \rangle)$ and a constant vector $b \in \mathbb{N}^d$ such that if any box $\langle m \rangle$ is padded with an envelope of width $b_i$ in the direction of $e_i$, then each $\Gamma$-allowed coloring of $\langle m \rangle$ can be extended in the padded part to a coloring in $F$. More precisely, for each $m \in \mathbb{N}^d$ and each $\alpha \in C_{\Gamma}(\langle m \rangle)$, there exists a coloring in $C_{\Gamma}(\langle m + 2b \rangle)$ that colors $\langle m \rangle + b$ by $\alpha$.

**Example 4.7** Let $\Gamma = (\Gamma_1, \ldots, \Gamma_d)$ be a coloring digraph with vertex set $\langle n \rangle$. Then $\Gamma$ is a friendly digraph with $b = 1$ if one of the following conditions holds:
(a) $\Gamma$ has a friendly color $f \in \langle n \rangle$, i.e., for each $i \in \langle d \rangle$ we have $(f,j),(j,f) \in \Gamma_i$ for all $j \in \langle n \rangle$ (we can take $\bar{C}_\Gamma(\langle m \rangle)$ to be those $\Gamma$-allowed colorings of $\langle m \rangle$ whose boundary points are colored with $f$). This example is useful for the hard-core model with $n=2$ and $\Gamma_i = \{(1,1),(1,2),(2,1)\}$, $f=1$.

(b) $\Gamma$ is the digraph associated with the monomer-dimer covering as defined in (6.1) (we can take $\bar{C}_\Gamma(\langle m \rangle)$ to be the set of tilings of $\langle m \rangle$ by monomers and dimers, i.e., the coverings in which no dimer protrudes out of $\langle m \rangle$, as in Hammersley).

The following theorem strengthens Theorem 4.3 and generalizes the results of Hammer-sley in case $\Gamma$ is a friendly digraph.

**Theorem 4.8** Let $\Gamma = (\Gamma_1, \ldots, \Gamma_d)$ be a friendly coloring digraph. Then

(a) $\Pi_\Gamma$ is convex. Hence $\Pi_\Gamma = \text{dom} P_\Gamma^*$.

(b) $h_\Gamma(\cdot) : \Pi_\Gamma \to \mathbb{R}_+$ is concave.

(c) For each $u \in \mathbb{R}^n$, $\Pi_\Gamma(u) = \partial P_\Gamma(u)$.

(d) For each $u \in \mathbb{R}^n$, $h_\Gamma(\cdot)$ is an affine function on $\partial P_\Gamma(u)$.

(e) $h_\Gamma(p) = -P_\Gamma^*(p)$ for each $p \in \Pi_\Gamma$.

**Proof** We first prove (a). Let $\alpha \in \bar{C}_\Gamma(\langle m \rangle)$, let $c(\alpha) = (c_1, \ldots, c_n) \in \Pi_n(\text{vol}(m))$ be the color frequency vector of $\alpha$, and let $p := \frac{1}{\text{vol}(m)} c(\alpha)$. We assert that $p \in \Pi_\Gamma$. For $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$, we define $\mathbf{k} \cdot \mathbf{m} := (k_1 m_1, \ldots, k_d m_d)$ and view $\langle \mathbf{k} \cdot \mathbf{m} \rangle$ as a box composed of $\text{vol}(\mathbf{k})$ boxes isomorphic to $\langle \mathbf{m} \rangle$, i.e., as $\langle \mathbf{m} \rangle$ duplicated by a factor of $\mathbf{k}$. We color each of these boxes by $\alpha$, obtaining a coloring $\alpha(\mathbf{k} \cdot \mathbf{m})$ of $\langle \mathbf{k} \cdot \mathbf{m} \rangle$. Clearly, $p = \frac{1}{\text{vol}(\mathbf{k} \cdot \mathbf{m})} c(\alpha(\mathbf{k} \cdot \mathbf{m}))$. Since $\alpha \in \bar{C}_\Gamma(\langle m \rangle)$, it follows that $\alpha(\mathbf{k} \cdot \mathbf{m})$ belongs to $\bar{C}_\Gamma(\langle \mathbf{k} \cdot \mathbf{m} \rangle)$, so in particular is $\Gamma$-allowed. Choosing a sequence $\mathbf{k}_q \to \infty$, we deduce that $p = \lim_{q \to \infty} \frac{1}{\text{vol}(\mathbf{k}_q \cdot \mathbf{m})} c(\alpha(\mathbf{k}_q \cdot \mathbf{m}))$. Hence $p \in \Pi_\Gamma$ according to (4.4), as asserted.

Let $\beta \in \bar{C}_\Gamma(\langle n \rangle)$. By the above argument we also have $q := \frac{1}{\text{vol}(\mathbf{n})} c(\beta) \in \Pi_\Gamma$. We assert that all $i,j \in \mathbb{N}$ satisfy $\frac{1}{i+j} p + \frac{1}{i+j} q \in \Pi_\Gamma$. Let $\alpha(\mathbf{n} \cdot \mathbf{m})$ and $\beta(\mathbf{m} \cdot \mathbf{n})$ be defined as above. Notice that $\langle \mathbf{n} \cdot \mathbf{m} \rangle$ is isomorphic to $\langle \mathbf{m} \cdot \mathbf{n} \rangle$. By the above argument, $\alpha(\mathbf{n} \cdot \mathbf{m}) \in \bar{C}_\Gamma(\langle \mathbf{n} \cdot \mathbf{m} \rangle)$ and $\beta(\mathbf{m} \cdot \mathbf{n}) \in \bar{C}_\Gamma(\langle \mathbf{m} \cdot \mathbf{n} \rangle)$. We define $\mathbf{k} := (m_1 n_1, \ldots, m_{d-1} n_{d-1}, (i+j)m_d n_d)$ and view the box $\langle \mathbf{k} \rangle$ as composed of $i+j$ boxes isomorphic to $\langle \mathbf{m} \cdot \mathbf{n} \rangle$ aligned side-by-side along the direction $e_d$. Color the first $i$ of these boxes by $\alpha(\mathbf{m} \cdot \mathbf{n})$ and the last $j$ by $\beta(\mathbf{n} \cdot \mathbf{m})$, obtaining a coloring $\gamma$ of $\langle \mathbf{k} \rangle$, which satisfies $\frac{1}{\text{vol}(\mathbf{k})} c(\gamma) = \frac{1}{i+j} p + \frac{1}{i+j} q$. Also $\gamma \in \bar{C}_\Gamma(\langle \mathbf{k} \rangle)$, so in particular $\gamma$ is $\Gamma$-allowed. By the above argument we obtain that $\frac{1}{i+j} p + \frac{1}{i+j} q \in \Pi_\Gamma$, as asserted. Since $\Pi_\Gamma$ is closed we deduce that $ap + (1-a)q \in \Pi_\Gamma$ for all $a \in [0,1]$.

Let $\Pi_\Gamma$ be the convex hull of all points of the form $\frac{1}{\text{vol}(\mathbf{m})} c(\alpha)$ for some $\mathbf{m}$ and some $\alpha \in \bar{C}_\Gamma(\langle \mathbf{m} \rangle))$. By the argument above we have $\Pi_\Gamma \subseteq \Pi_\Gamma$. Let $p \in \Pi_\Gamma$. By Definition 4.2 there exist sequences $\mathbf{m}_q \to \infty$ and color frequency vectors $c_q \in \Pi_n(\text{vol}(\mathbf{m}_q))$ satisfying (4.4). Let $\alpha_q \in C_\Gamma(\langle \mathbf{m}_q \rangle), c_q)$). By Definition 4.6, $\alpha_q$ can be extended to a coloring $\tilde{\alpha}_q \in \bar{C}_\Gamma(\langle \mathbf{m}_q + 2b \rangle, c_q)$ for some $\tilde{c}_q$. Since $b$ is constant and $\mathbf{m}_q \to \infty$, we have $\lim_{q \to \infty} \frac{1}{\text{vol}(\mathbf{m}_q + 2b)} \tilde{c}_q = p$. Since $\frac{1}{\text{vol}(\mathbf{m}_q + 2b)} \tilde{c}_q \in \Pi_\Gamma$, we have $p \in \text{cl} \Pi_\Gamma$. Thus $\Pi_\Gamma \subseteq \Pi_\Gamma \subseteq \text{cl} \Pi_\Gamma$. Applying the closure operator, we deduce $\Pi_\Gamma = \text{cl} \Pi_\Gamma$, and since $\Pi_\Gamma$ is convex, so is $\Pi_\Gamma$. The equality $\Pi_\Gamma = \text{dom} P_\Gamma^*$ follows from part (g) of Theorem 4.3.

We now prove (b). Choose any $\varepsilon > 0$. Let $p \in \Pi_\Gamma$. By Definition 4.2 there exist sequences $\mathbf{m}_q \to \infty$ and color frequency vectors $c_q \in \Pi_n(\text{vol}(\mathbf{m}_q))$ satisfying (4.4). By (4.5) we may assume by selecting appropriate subsequences that the following limit exists and satisfies $\lim_{q \to \infty} \frac{\log \# C_\Gamma(\langle \mathbf{m}_q \rangle, c_q)}{\text{vol}(\mathbf{m}_q)} \geq h_\Gamma(p) - \varepsilon$. Each coloring $\alpha_q \in C_\Gamma(\langle \mathbf{m}_q \rangle, c_q)$ can be
extended to some coloring $\tilde{c}_q \in \tilde{C}_\Gamma(\langle m_q + 2b \rangle)$. Denote by $c(\tilde{c}_q)$ the color frequency vector of $\tilde{c}_q$. Since $b$ is constant and $m_q \to \infty$, we have $\lim_{q \to \infty} \frac{\text{vol}(m_q + 2b)}{\text{vol}(m_q)} c(\tilde{c}_q) = p$. Let $C_q \subseteq \Pi_n(\text{vol}(m_q + 2b))$ be the set of all possible color frequency vectors of all extensions of the colorings of $C_\Gamma(\langle m_q \rangle, c_q)$ to $\tilde{C}_\Gamma(\langle m_q + 2b \rangle)$. Clearly $\#C_q \leq (\text{vol}(m_q + 2b))^{n-1} = O(\text{vol}(m_q + 2b)^{n-1})$, $q \to \infty$. For each $c \in \Pi_n(\text{vol}(m))$, let $\tilde{C}_\Gamma(\langle m \rangle, c)$ be the set of those colorings in $\tilde{C}_\Gamma(\langle m \rangle)$ that have color frequency vector $c$. From the above it follows that

$$\#C_\Gamma(\langle m_q \rangle, c_q) \leq \sum_{c \in C_q} \#\tilde{C}_\Gamma(\langle m_q + 2b \rangle, c) \leq \#\tilde{C}_\Gamma(\langle m_q + 2b \rangle, \tilde{c}_q) #C_q$$

$$= \#\tilde{C}_\Gamma(\langle m_q + 2b \rangle, \tilde{c}_q) O(\text{vol}(m_q + 2b)^{n-1})$$

for some $\tilde{c}_q \in C_q$. Taking logarithms, dividing by $\text{vol}(m_q + 2b)$, and noting that $\lim_{q \to \infty} \frac{\text{vol}(m_q + 2b)}{\text{vol}(m_q)} = 1$, we deduce that

$$\lim_{q \to \infty} \frac{\tilde{c}_q}{\text{vol}(m_q + 2b)} = p \quad \text{and} \quad \lim_{q \to \infty} \frac{\log \#\tilde{C}_\Gamma(\langle m_q + 2b \rangle, \tilde{c}_q)}{\text{vol}(m_q)} \geq h_\Gamma(p) - \varepsilon.$$

Thus for $p, q \in \Pi_\Gamma, \varepsilon > 0$ we have sequences $m_q := (m_{1,q}, \ldots, m_{d,q}), n_q := (n_{1,q}, \ldots, n_{d,q}) \in \mathbb{N}^d, q \in \mathbb{N}$, with $m_q, n_q \to \infty$ such that the following two conditions hold:

$$\tilde{C}_\Gamma(\langle m_q \rangle, c_q), \tilde{C}_\Gamma(\langle n_q \rangle, d_q) \neq \emptyset, q \in \mathbb{N}, \lim_{m_q \to \infty} \frac{1}{\text{vol}(m_q)} c_q = p, \lim_{n_q \to \infty} \frac{1}{\text{vol}(n_q)} d_q = q.$$

$$\lim_{q \to \infty} \frac{\log \#\tilde{C}_\Gamma(\langle m_q \rangle, c_q)}{\text{vol}(m_q)} \geq h_\Gamma(p) - \varepsilon, \lim_{q \to \infty} \frac{\log \#\tilde{C}_\Gamma(\langle n_q \rangle, d_q)}{\text{vol}(n_q)} \geq h_\Gamma(q) - \varepsilon.$$

For $i, j \in \mathbb{N}$ we show that

$$h_\Gamma(\frac{i}{i+j} p + \frac{j}{i+j} q) \geq \frac{i}{i+j} h_\Gamma(p) + \frac{j}{i+j} h_\Gamma(q) - \varepsilon. \quad (4.13)$$

Observe first that for any $m, n \in \mathbb{N}^d$ and $c \in \Pi_n(\text{vol}(m))$ one has the inequality:

$$\#\tilde{C}_\Gamma(\langle m \cdot n \rangle, \text{vol}(n)c) \geq (\#\tilde{C}_\Gamma(\langle m \rangle, c))^\text{vol}(n). \quad (4.14)$$

Indeed, view as above, the box $\langle n \cdot m \rangle$ as a disjoint union of $\text{vol}(n)$ boxes $\langle m \rangle$. Color each box $\langle m \rangle$ in some color in the set $\tilde{C}_\Gamma(\langle m \rangle, c)$. Such a coloring is a member of $\tilde{C}_\Gamma(\langle n \cdot m \rangle, \text{vol}(n)c)$. Hence $(4.14)$ holds.

Let $k_{1,q} := (m_{1,q}, n_{1,q}, \ldots, m_{d-1,q}, n_{d-1,q}), k_q := (k_{1,q}, (i+j)m_{d,q}n_{d,q})$. View $\langle k_q \rangle$ composed of $(i+j)$ boxes $\langle m_q \cdot n_q \rangle$. The above arguments show that

$$\#\tilde{C}_\Gamma(\langle k_q \rangle, \text{vol}(k_{1,q})(ic_q + jd_q)) \geq$$

$$\#\tilde{C}_\Gamma(\langle (k_{1,q}, m_{d,q}n_{d,q}), \text{vol}(k_{1,q})ic_q \rangle \#\tilde{C}_\Gamma(\langle (k_{1,q}, jm_{d,q}n_{d,q}), \text{vol}(k_{1,q})jd_q \rangle \geq$$

$$\#\tilde{C}_\Gamma(\langle m_q, c_q \rangle)^{\text{vol}(n_q)} \#\tilde{C}_\Gamma(\langle n_q, d_q \rangle)^{\text{vol}(m_q)}.$$

Since $\tilde{C}_\Gamma(\langle k_q \rangle, \text{vol}(k_{1,q})(ic_q + jd_q)) \supseteq \tilde{C}_\Gamma(\langle k_q \rangle, \text{vol}(k_{1,q})ic_q + jd_q)$ by considering the first coloring sequence in this inclusion, and using the maximal characterization of $h_\Gamma(\frac{i}{i+j} p + \frac{j}{i+j} q)$, we deduce $(4.13)$. Since $\varepsilon$ was an arbitrary positive number we deduce $(4.13)$ with $\varepsilon = 0$. Since $h_\Gamma$ is upper semi-continuous we deduce the inequality $h_\Gamma(ap + (1-a)q) \geq ah_\Gamma(p) + (1-a)h_\Gamma(q)$ for any $a \in [0,1]$.

We now prove the claims (c-d). Assume first that Let $u \in \text{diff} P_\Gamma$. Then $\Pi_\Gamma(u) = \{ \nabla P_\Gamma(u) \} = \partial P_\Gamma(u)$ and our assertions trivially hold. Assume next that the assumptions of part (e) of Theorem 4.3 hold. Recall that $S(u) \subseteq \Pi_\Gamma(u)$ and conv $S(u) = \partial P_\Gamma(u) \supseteq \Pi_\Gamma(u)$. 

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Let \( p_i \in S(u), i = 1, \ldots, j \). So \( P_\Gamma(u) = p_i^\top u + h_\Gamma(p_i), i = 1, \ldots, j \). Since \( \Pi_\Gamma \) is convex, we obtain that for any \( a = (a_1, \ldots, a_j) \in \Pi_j \) \( p := \sum_{i=1}^j a_ip_i \in \Pi_\Gamma \). As \( h_\Gamma \) concave we deduce

\[
P_\Gamma(u) = \sum_{i=1}^j a_ip_i^\top u + h_\Gamma(p_i) \leq p^\top u + h_\Gamma(p).
\]

The maximal characterization (4.6) yields that \( P_\Gamma(u) = p^\top u + h_\Gamma(p) \). So \( p \in \Pi_\Gamma(u) \) and \( h_\Gamma(p) = \sum_{i=1}^j a_ih_\Gamma(p_i) \). This proves (c-d).

We now prove (e). Recall that \( p \in \partial P_\Gamma(\mathbb{R}^n) \) if and only if \( p \in \partial P_\Gamma(u) \) for some \( u \in \mathbb{R}^n \). Use part (c) of this Theorem and part (c) of Theorem 4.3 to deduce the equality \( h_\Gamma(p) = -P_\Gamma^*(p) \). If \( \Pi_\Gamma \) consists of one point then \( \Pi_\Gamma = \partial P_\Gamma(\mathbb{R}^n) \) and (e) trivially holds. Assume that \( \Pi_\Gamma \) consists of more than one point. Since \( \partial P_\Gamma(\mathbb{R}^n) \supseteq \mathrm{ri} \, (\mathrm{dom} P_\Gamma^*) \neq \emptyset \), use the second part of (a) of this Theorem to deduce \( h_\Gamma(p) = -P_\Gamma^*(p) \) for each \( p \in \mathrm{ri} \, (\Pi_\Gamma) \). Suppose that \( q \in \Pi_\Gamma \setminus \mathrm{ri}(\Pi_\Gamma), p \in \mathrm{ri}(\Pi_\Gamma) \). Let

\[
f(a) := -h_\Gamma(aq + (1-a)p), g(a) := P_\Gamma^*(aq + (1-a)p), \text{ for } a \in [0,1].
\]

Since \( aq + (1-a)p \in \mathrm{ri}(\Pi_\Gamma) \) for \( a \in [0,1] \) it follows that \( f(a) = g(a) \) for \( a \in [0,1] \). Since \( P_\Gamma^* \) is a proper closed function, it is lower semi-continuous. Hence \( \Pi_\Gamma(q) = g(1) \leq \liminf_{a \to 1} g(a) \). Since \( g \) is a convex function on \([0,1]\) it follows that \( \liminf_{a \to 1} g(a) = \lim_{a \to 1} g(a) \leq g(1) \).

Hence \( g(1) = \lim_{a \to 1} g(a) \). Recall that \( h_\Gamma \) is a concave upper semi-continuous on \( \Pi_\Gamma \). Hence \( -h_\Gamma \) is a convex lower semi-continuous function on \( \Pi_\Gamma \). Hence \( -h_\Gamma(q) = f(1) = \lim_{a \to 1} f(a) \). Therefore \( f(1) = g(1), \text{ i.e. } h_\Gamma(q) = P_\Gamma^*(q) \).

We now list several facts which are consequences of Theorem 4.3. Given \( u \in \mathbb{R}^n \), then by (d) each \( p \in \Pi_\Gamma(u) \), namely each \( p \) achieving the maximum in (4.6), is a possible density of the \( n \) colors in an allowable configuration from \( C_\Gamma(\mathbb{Z}^2) \) with the potential \( u \). That is, the relative frequency of color \( i \) is equal to \( p_i \). Each \( u \) where \( P_\Gamma \) is differentiable, there exists a unique density of the \( n \) colors. Assume that \( P_\Gamma \) is not differentiable at \( u \). Then \( \partial P_\Gamma \) consists of more than one point. Let \( S(u) \) be defined as in (e). Since \( \partial P_\Gamma(u) = \text{conv } S(u) \), \( S(u) \) consists of more than one point. Hence by (e) \( \Pi_\Gamma(u) \) consists of more than one point, that is to say, there is more than one density for \( u \). In this case \( u \) is called a point of phase transition, sometimes called a phase transition point of the first order.

**Proposition 4.9** Let \( e := (1, \ldots, 1)^\top \in \mathbb{R}^n \). Then for all \( t \in \mathbb{R} \)

\[
P_\Gamma(u) = t + P_\Gamma(u - te).
\]

**Proof** Recall the definition of \( Z_\Gamma(m, u) \) given in (2.1). Clearly \( c(\phi)^\top e = \text{vol}(m) \). Hence

\[
Z_\Gamma(m, u) = \sum_{\phi \in C_\Gamma(m)} e^{c(\phi)^\top u} = \sum_{\phi \in C_\Gamma(m)} e^{c(\phi)^\top (e + c(\phi)^\top (u - te))} = e^{\text{vol}(m)} Z_\Gamma(m, u - te),
\]

which implies the proposition.

Thus to study \( P_\Gamma \), we may restrict attention to those potentials \( u = (u_1, \ldots, u_n)^\top \) that satisfy \( u_n = 0 \). (i.e. we reduce the number of variables in the function \( P_\Gamma \) to \( n - 1 \).) We show that the same holds for \( \partial P_\Gamma \) and \( \nabla P_\Gamma \). For \( u \in \mathbb{R}^n \), we use the notation

\[
\overline{\pi} := (u_1, \ldots, u_{n-1})^\top
\]

for the projection of \( u \) on the first \( n-1 \) coordinates, and extend it naturally to sets \( \overline{U} := \{ \overline{\pi} : u \in U \} \) for \( U \subseteq \mathbb{R}^n \). In the other direction, for \( x \in \mathbb{R}^{n-1} \), we use the notation

\[
\iota(x) := (x_1, \ldots, x_{n-1}, 1 - x_1 - \cdots - x_{n-1})^\top
\]
for the unique lifting of \( x \) to the hyperplane \( 
abla \Sigma_n := \{ x \in \mathbb{R}^n : x^T e = 1 \} \), and
\[
\eta_0(x) := (x_1, \ldots, x_{n-1}, 0).
\]

A straightforward computation shows that
\[
q^T (z - z_n e) = \ell(q)^T z - z_n \quad \forall q \in \mathbb{R}^{n-1}, z \in \mathbb{R}^n. \tag{4.16}
\]

We define the convex function \( \tilde{P}_T(\cdot) \) on \( \mathbb{R}^{n-1} \) by
\[
\tilde{P}_T(x) := P_T(\eta_0(x)) \quad x \in \mathbb{R}^{n-1}. \tag{4.17}
\]

By taking \( t = u_n \) in Proposition 4.9, we obtain
\[
P_T(u) = u_n + \tilde{P}_T(u - u_n e). \tag{4.18}
\]

We now obtain a straightforward generalization of the density theorem proved in [12] for monomer-dimer tilings.

**Theorem 4.10** Let \( \tilde{P}_T(\cdot) \) be defined on \( \mathbb{R}^{n-1} \) by (4.17). Then
\[
\frac{\partial P_T(u)}{\partial u} = \frac{\partial \tilde{P}_T(u - u_n e)}{\partial u} \quad \forall u \in \mathbb{R}^n, \tag{4.19}
\]
\[
\frac{\partial P_T(u)}{\partial \Sigma} = \frac{\partial \tilde{P}_T(u - u_n e)}{\partial \Sigma}. \tag{4.20}
\]

Furthermore, \( P_T \) is differentiable at \( u \) if and only if \( \tilde{P}_T \) is differentiable at \( u - u_n e \). If \( \tilde{P}_T \) has all \( n-1 \) partial derivatives at \( u - u_n e \), then \( P_T \) is differentiable at \( u \) and
\[
\nabla P_T(u) = \left( \frac{\partial \tilde{P}_T(u - u_n e)}{\partial u_1}, \ldots, \frac{\partial \tilde{P}_T(u - u_n e)}{\partial u_{n-1}}, 1 - \sum_{i=1}^{n-1} \frac{\partial \tilde{P}_T(u - u_n e)}{\partial u_i} \right). \tag{4.21}
\]

**Proof** Assume that \( p \in \partial P_T(u) \). By definition of \( \partial P_T(u) \) we have
\[
P_T(u + v) \geq p^T v + P_T(u) \quad \forall v. \tag{4.22}
\]

Choose \( v \) such that \( v_n = 0 \). Then, using (4.18) for \( P_T(u + v) \) and \( P_T(u) \) in (4.22), we obtain
\[
\tilde{P}_T(u - u_n e + v) \geq \tilde{P}_T(u - u_n e) \quad \forall v,
\]
which by definition means \( p \in \partial \tilde{P}_T(u - u_n e) \).

Conversely, assume that \( q \in \partial \tilde{P}_T(u - u_n e) \), which means that
\[
\tilde{P}_T(u - u_n e + v) \geq q^T v + \tilde{P}_T(u - u_n e) \quad \forall v. \tag{4.23}
\]

Now for arbitrary \( z \), choose \( v = z - z_n e \) in (4.23), and use (4.16) once and (4.18) twice in the resulting inequality to obtain
\[
P_T(u + z) - u_n - z_n \geq \ell(q)^T z - z_n + P_T(u) - u_n \quad \forall z,
\]
which means by definition that \( q \in \partial P_T(u) \), and therefore \( q \in \partial P_T(u) \). We have proved (4.19).

We now show (4.20). It follows easily from (4.19) that \( \partial P_T(\mathbb{R}^n) \subseteq \partial \tilde{P}_T(\mathbb{R}^{n-1}) \). To show the reverse inclusion, let \( q \in \partial \tilde{P}_T(w) \), and let \( u = \eta_0(w) \) so that \( w = u - u_n e \). By (4.19) we have \( q \in \partial P_T(u) \), as required.

Assume that \( P_T \) is differentiable at \( u \). Then \( \partial P_T(u) \) is a singleton, and therefore \( \partial P_T(u) \) is a singleton. By (4.19) \( \partial \tilde{P}_T(u - u_n e) \) is a singleton, and therefore \( \tilde{P}_T \) is differentiable at \( u - u_n e \).
Conversely assume that $\hat{P}_r$ is differentiable at $\mathbf{u} - u_r\mathbf{e}$, and therefore $\partial \hat{P}_r(\mathbf{u} - u_r\mathbf{e})$ is a singleton $\{\mathbf{q}\}$. We assert that $\partial P_r(\mathbf{u}) = \{\mathbf{q}\}$. Indeed, if $\mathbf{x} \in \partial P_r(\mathbf{u})$, then $\mathbf{x} \in \partial \hat{P}_r(\mathbf{u}) = \hat{P}_r(\mathbf{u} - u_r\mathbf{e}) = \{\mathbf{q}\}$, where the first equality is by (4.19), and so $\mathbf{x} = \mathbf{q}$. Since $\mathbf{x} \in \partial P_r(\mathbf{u}) \subseteq \Pi_r \subseteq \Sigma_r$, by (f) of Theorem 4.3, it follows that $\mathbf{x} = \iota(\mathbf{q})$, proving the assertion since $\partial P_r(\mathbf{u}) \neq \emptyset$. Therefore $P_r$ is differentiable at $\mathbf{u}$.

The last statement of the theorem follows from the previous statement and the fact [26, Thm 25.2] that a convex function $f : \mathbb{R}^k \to \mathbb{R}$ is differentiable at a point $\mathbf{a}$ if merely the $k$ partial derivatives $\frac{\partial f}{\partial x_i}$, $i = 1, \ldots, k$ exist at $\mathbf{a}$.

We can reformulate Theorem 4.3 for $\hat{P}_r$.

**Theorem 4.11** Let $\hat{P}_r^*$ be the conjugate convex function of $\hat{P}_r$. Then

(a) $h_r(\mathbf{p}) \leq -\hat{P}_r^*(\mathbf{p})$ for all $\mathbf{p} \in \Pi_r$.

(b) $\hat{P}_r(\mathbf{x}) = \max_{\mathbf{p} \in \Pi_r}[\mathbf{p}^\top \mathbf{x} + h_r(\mathbf{p})]$ for all $\mathbf{x} \in \mathbb{R}^{n-1}$. (4.24)

For $\mathbf{x} \in \mathbb{R}^{n-1}$, let $\mathbf{q}(\mathbf{x}) \in \Pi_r(\iota(x))$, i.e., $\mathbf{q}(\mathbf{x})$ is any vector satisfying

$\iota(\mathbf{q}(\mathbf{x})) \in \Pi_r$, and $\hat{P}_r(\mathbf{x}) = \iota(\mathbf{q}(\mathbf{x})) \top \iota(x) + h_r(\iota(\mathbf{q}(\mathbf{x}))) = \mathbf{q}(\mathbf{x}) \top \mathbf{x} + h_r(\iota(\mathbf{q}(\mathbf{x})))$. (4.25)

(c) $h_r(\iota(\mathbf{q}(\mathbf{x}))) = -\mathbf{q}^\top(\mathbf{q}(\mathbf{x})).$

(d) $\mathbf{q}(\mathbf{x}) \in \partial \hat{P}_r(\mathbf{x})$. In particular, if $\mathbf{x} \in \text{diff} \hat{P}_r$, then $\mathbf{q}(\mathbf{x}) = \nabla \hat{P}_r(\mathbf{x})$. Therefore $\partial \hat{P}_r(\text{diff} \hat{P}_r) \subseteq \Pi_r$.

(e) Let $\mathbf{x} \in \mathbb{R}^{n-1} \setminus \text{diff} \hat{P}_r$, and let $\mathbf{S}(\mathbf{x})$ consist of all the limits of sequences $\nabla \hat{P}_r(\mathbf{x}_i)$ such that $\mathbf{x}_i \in \text{diff} \hat{P}_r$ and $\mathbf{x}_i \to \mathbf{x}$. Then $S(\mathbf{x}) \subseteq \Pi_r(\iota(\mathbf{x})).$

(f) $\text{conv } \Pi_r = \text{dom } \hat{P}_r^*$.

**Proof** From (4.6) we have $P_r(\mathbf{u}) \geq \mathbf{p}^\top \mathbf{u} + h_r(\mathbf{p})$ for all $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{p} \in \Pi_r$. Fix $\mathbf{p} \in \Pi_r$, and let $\mathbf{x} \in \mathbb{R}^{n-1}$ and $\mathbf{u} = \iota_0(\mathbf{x})$. Then $\hat{P}_r(\mathbf{x}) = P_r(\mathbf{u}) \geq \mathbf{p}^\top \mathbf{x} + h_r(\mathbf{p})$, so $-h_r(\mathbf{p}) \geq \mathbf{p}^\top \mathbf{x} - \hat{P}_r(\mathbf{x})$. Now take the supremum over $\mathbf{x} \in \mathbb{R}^{n-1}$ to obtain $-h_r(\mathbf{p}) \geq \hat{P}_r^*(\mathbf{p})$, which is (a). Substitute $\mathbf{u} = \iota_0(\mathbf{x})$ in (4.6) to deduce (4.24). We now show (c). By (4.25) and the definition of $\hat{P}_r^*$ it follows that $-h_r(\iota(\mathbf{q}(\mathbf{x}))) = \mathbf{q}(\mathbf{x}) \top \mathbf{x} - \hat{P}_r(\mathbf{q}(\mathbf{x})) \leq \hat{P}_r^*(\mathbf{q}(\mathbf{x}))$. Combining this with the opposite inequality (a), we deduce (c). To prove (d), let $\mathbf{x}, \mathbf{z} \in \mathbb{R}^{n-1}$. Since $\iota(\mathbf{q}(\mathbf{x})) \in \Pi_r$, (4.24) applied to $\mathbf{x} + \mathbf{z}$ and (4.25) give $\hat{P}_r(\mathbf{x} + \mathbf{z}) \geq \mathbf{q}(\mathbf{x}) \top (\mathbf{x} + \mathbf{z}) + h_r(\iota(\mathbf{q}(\mathbf{x}))) = \mathbf{q}(\mathbf{x}) \top \mathbf{z} + \hat{P}_r(\mathbf{x})$. This proves (d). To prove (e), let $\mathbf{x} \in \mathbb{R}^{n-1} \setminus \text{diff} \hat{P}_r$ and let $\mathbf{q} \in \mathbf{S}(\mathbf{x})$. Here exists a sequence $\mathbf{x}_i \in \text{diff} \hat{P}_r$ such that $\mathbf{x}_i \to \mathbf{x}$ and $\nabla \hat{P}_r(\mathbf{x}_i) \to \mathbf{q}$. By the “furthermore” part of Theorem 4.10 we have $\iota_0(\mathbf{x}_i) \in \text{diff} \hat{P}_r$ and $\iota_0(\mathbf{x}_i) \in \mathbb{R}^n \setminus \text{diff} \hat{P}_r$. By (4.21) and the continuity of $\iota$ we have $\nabla \hat{P}_r(\iota_0(\mathbf{x}_i)) = \iota(\nabla \hat{P}_r(\mathbf{x}_i)) \to \iota(\mathbf{q})$. This shows that $\iota(\mathbf{q}) \in \mathcal{S}(\iota(\mathbf{x}_i)) \subseteq \Pi_r(\iota(\mathbf{x}_i))$, where the inclusion is by (e) of Theorem 4.3. It follows that $\mathbf{q} = \iota(\mathbf{q}) \in \Pi_r(\iota(\mathbf{x}_i))$, as required. Now we prove (f). By (a) we have $\hat{P}_r(\mathbf{p}) \leq -h_r(\mathbf{p}) < \infty$ for all $\mathbf{p} \in \Pi_r$, so $\Pi_r \subseteq \text{dom } \hat{P}_r$. Applying the closure operator, we obtain $\text{dom } \hat{P}_r \subseteq \text{cl (dom } \hat{P}_r^*) = \text{cl (ri (dom } \hat{P}_r^*)) \subseteq \text{conv } \Pi_r$ as in the proof of (g) of Theorem 4.3. ■
Since a probability vector \( p \in \Pi_n \) is determined completely by its projection \( p \) on the first \( n-1 \) components, we can view the function \( h_\Gamma : \Pi_\Gamma \to \mathbb{R}_+ \) as a function on \( \overline{\Pi_\Gamma} \). Formally, let
\[
\tilde{h}_\Gamma(q) := h_\Gamma(u(q)) \text{ for all } q \in \overline{\Pi_\Gamma}.
\]

\[(4.26)\]

5 \( P_\Gamma \) and density entropies for one dimensional SOFT

In this section we apply the results of Section 4 to one dimensional SOFT. In this case \( \Gamma \) is given by a digraph \( \Gamma := \Gamma_1 \subseteq \langle n \rangle \times \langle n \rangle \).

**Theorem 5.1** Let \( \Gamma \subseteq \langle n \rangle \times \langle n \rangle \) be a digraph on \( n \) vertices, with at least one strongly connected component. Then \( P_\Gamma(u) = \log \rho(D(\Gamma, u)) \), where the nonnegative matrix \( D(\Gamma, u) \) is irreducible.

If \( \Gamma \) is strongly connected, or more generally \( \Gamma \) has one connected component, then \( P_\Gamma \) is an analytic function on \( \mathbb{R}_n^+ \), \( \Pi_\Gamma \) is a closed convex set of probability vectors equal to \( \text{dom } P_\Gamma^* \), \( h_\Gamma \) is concave and continuous on \( \text{ri } \Pi_\Gamma \), and coincides there with \(-P_\Gamma^* \). In particular, for any \( u \in \mathbb{R}_n^+ \)

\[
h_\Gamma(p(u)) = -p(u) + \log \rho(D(\Gamma, u)), \quad \text{where } p(u) := \frac{\nabla \rho(D(\Gamma, u))}{\rho(D(\Gamma, u))}.
\]

Furthermore
\[
h_\Gamma = \max_{p \in \Pi_\Gamma^*} h_\Gamma(p) = h_\Gamma\left(\frac{\nabla \rho(D(\Gamma, 0))}{\rho(D(\Gamma, 0))}\right) = \log \rho(\Gamma).
\]

Assume that \( \Gamma \) has \( k > 1 \) connected components \( \Delta_1, \ldots, \Delta_k \). Then \( P_\Gamma(u) = \max(P_{\Delta_1}(u), \ldots, P_{\Delta_k}(u)) \), where each \( P_{\Delta_i} \) is an analytic function on \( \mathbb{R}_n^+ \).

**Proof** Proposition 2.2 yields the equality \( P_\Gamma(u) = \log \rho(D(\Gamma, u)) \). Assume first that \( \Gamma \) is strongly connected, which is equivalent to the assumption that the adjacency matrix \( D(\Gamma) = (d_{ij})_{i,j \in \langle n \rangle} \) is irreducible. From the definition of \( D(\Gamma, u) = (d_{ij} \cdot e_j^T e_i u + e_i u)_{i,j \in \langle n \rangle} \) it follows that \( D(\Gamma, u) \) is a irreducible matrix for each value \( u \in \mathbb{R}_n^+ \). Then \( \rho(D(\Gamma, u)) \) is a simple root of the characteristic equation \( p(z, u) := \det(z I - D(\Gamma, u)) = 0 \) for each \( u \in \mathbb{R}_n^+ \). Since the coefficients of \( p(z, u) \) are analytic in \( u \), where \( u \in C_n^\times \), the implicit function theorem implies that \( \rho(D(\Gamma, u)) \) is analytic function in a neighborhood of \( \mathbb{R}_n^+ \) of \( C_n^\times \). Since \( \rho(D(\Gamma, u)) \) is positive on \( \mathbb{R}_n^+ \) it follows that \( \log \rho(D(\Gamma, u)) \) has an analytic extension to some neighborhood of \( \mathbb{R}_n^+ \) in \( C_n^\times \). Hence \( P_\Gamma \) is analytically on \( \mathbb{R}_n^+ \). In particular, \( P_\Gamma \) is differentiable on \( \mathbb{R}_n^+ \). Theorem 4.3 and Lemma 1 yield that \( \text{dom } P_\Gamma^* = \text{conv } \Pi_\Gamma \subseteq \Pi_\Gamma \supseteq \partial P_\Gamma(\mathbb{R}_n^+) \supseteq \text{ri } (\text{dom } P_\Gamma^*) \). Since \( \Pi_\Gamma \) is closed we obtain that \( \Pi_\Gamma = \text{cl } (\text{dom } P_\Gamma^*) = \text{conv } \Pi_\Gamma \), hence \( \Pi_\Gamma \) is convex. According to Theorem 4.3 cl (dom \( P_\Gamma^* \)) = dom \( P_\Gamma^* \).

Since \( P_\Gamma \) is differentiable, Theorem 4.3 yields that \( h_\Gamma(p) = -P_\Gamma^*(p) \) for \( p \in \partial P_\Gamma(\mathbb{R}_n^+) \).

As \( \partial P_\Gamma(\mathbb{R}_n^+) \supseteq \text{ri } (\text{dom } P_\Gamma^*) \) we deduce that \( h_\Gamma = -P_\Gamma^* \) on \( \text{ri } (\text{dom } P_\Gamma^*) \). Since \( P_\Gamma^* \) is a convex continuous function on \( \text{ri } (\text{dom } P_\Gamma^*) \), it follows that \( h_\Gamma \) is a concave continuous function on \( \text{ri } (\text{dom } P_\Gamma^*) \).

As \( P_\Gamma(u) = \log \rho(D(\Gamma, u)) \) it follows that \( \nabla P_\Gamma(u) = \frac{\nabla \rho(D(\Gamma, u))}{\rho(D(\Gamma, u))} \). Hence (5.1) holds.

Clearly, \( \rho(D(\Gamma, 0)) = \rho(\Gamma) \) and (5.2) follows.

Assume now that \( \Gamma \) is not strongly connected digraph. Rename the vertices of \( \Gamma \) such that \( D(\Gamma) \) is its normal form [13, XIII.4]. That is, \( D(\Gamma) \) is a block lower triangular form matrix, where each submatrix on a diagonal block is either a nonzero irreducible matrix or \( 1 \times 1 \) zero matrix. Then each nonzero irreducible submatrix corresponds to a strongly irreducible component of \( \Gamma \). Let \( \Delta_1, \ldots, \Delta_k \) be the \( k \geq 1 \) irreducible components of \( \Gamma \). Since \( D(\Gamma, u) \) is also in its normal form it follows that \( \rho(D(\Gamma, u)) = \max_{i \in [1,k]} \rho(D(\Delta_i, u)) \). Note that \( \log \rho(D(\Delta_i, u)) = P_{\Delta_i}(u) \) for \( i = 1, \ldots, k \).

Assume first that \( k = 1 \) and \( \Delta_1 \neq \Gamma \). Rename the vertices of \( \Gamma \) such that \( \langle m \rangle \) is the set of vertices of \( \Delta_1 \), where \( 1 \leq m < n \). Let \( \bar{u} = (u_1, \ldots, u_m)^T \). Then \( P_\Gamma(u) = P_{\Delta_1}(\bar{u}) \) and the theorem follows in this case.
Assume finally that \( k > 1 \). The above arguments show that each \( P_{\Delta_j} \) is an analytic function in \( u \), which does not depend on a variable \( u_j \) if \( j \) is not a vertex of \( \Delta_i \). 

Assume that \( \Gamma \) is strongly connected and we want to compute \( p(u) = \frac{\nabla \rho(D(\Gamma, u))}{\rho(D(\Gamma, u))} \). We give the following simple formula for \( p(u) \) which is known to the experts.

**Proposition 5.2** Let \( \Gamma \subseteq \langle n \rangle \times \langle n \rangle \) be a strongly connected digraph on \( n \) vertices. Let \( D(\Gamma, u) \) be the nonnegative matrix given in Proposition 2.2. Let \( x(u) = (x_1(u), \ldots, x_n(u))^T \), \( y(u) = (y_1(u), \ldots, y_n(u))^T \) be positive eigenvectors of \( D(\Gamma, u), D(\Gamma, u)^T \) respectively, normalized by the condition \( y(u)^T x(u) = 1 \). Then

\[
\nabla P_1(u) = \frac{\nabla \rho(D(\Gamma, u))}{\rho(D(\Gamma, u))} = (y_1(x_1(u), \ldots, y_n(x_n(u)) \text{ for each } u \in \mathbb{R}^n). \quad (5.3)
\]

**Proof** Let \( D(u) := D(\Gamma, u), \rho(u) := \rho(D(\Gamma, u)) \). Since \( \rho(u) > 0 \) is a simple root of \( \det(zI - D(u)) \) it follows that one can choose \( x(u), y(u) \) to be analytic on \( \mathbb{R}^n \) in \( u \). (For example first choose \( x(u), y(u) \in \mathbb{R}^n \) to be the unique left and right eigenvectors of \( D(u) \) of length 1. Then let \( y(u) = \frac{1}{y(u)^T x(u)} y(u) \).) Let \( \partial_i \) be the partial derivative with respect to \( u_i \). Then

\[
y(u)^T x(u) = 1 \forall u \in \mathbb{R}^n \Rightarrow \partial_i y(u)^T x(u) + y(u)^T \partial_i x(u) = 0, \text{ for } i = 1, \ldots, n.
\]

Observe next that \( y(u)^T D(u) x(u) = \rho(u) \). Taking the partial derivative with respect to \( u_i \) and using the formula (2.6) for the entries of \( D(u) \) we obtain

\[
\partial_i \rho(u) = \partial_i y(u)^T D(u) x(u) + y(u)^T D(u) \partial_i x(u) + y(u)^T \partial_i D(u) x(u).
\]

\[
\rho(u)(\partial_i y(u)^T x(u) + y(u)^T \partial_i x(u)) + \rho(u) y_i(u) x_i(u) = \rho(u) y_i(u) x_i(u).
\]

This proves (5.3).

We now apply the above results to the following simple digraph on two vertices:

Identify the red color with the state 1 and the blue color with the state 2, which is usually identified with the state 0. Then \( C_\Gamma(Z) \) consists of all coloring of the lattice \( Z \) in blue and red colors such that no two red colors are adjacent. This is the simplest hard core model in statistical mechanics. The adjacency matrix \( D(\Gamma) \) is the following \( 2 \times 2 \) matrix \( \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \).

Let \( u = (s, t)^T \). Then \( \nabla P_1(u) = (p_1(u), p_2(u)) \in \Pi_2 \) it follows that \( p_2(u) = 1 - p_1(u) \).

It is enough to consider \( u = (s, 0) \) and \( p_1(s) = \frac{dP_1((s, 0)^T)}{ds} \). So \( p := p_1(s) \) is the density of 1 in all the configurations of infinite strings of 0, 1, where no two 1 are adjacent. Clearly

\[
D(\Gamma, u) = \begin{pmatrix} 0 & e^s \\ e^s & 1 \end{pmatrix}.
\]

Hence

\[
\rho(u) = \frac{1 + \sqrt{1 + 4e^s}}{2}, \quad p_1(s) = \frac{2e^s}{(1 + \sqrt{1 + 4e^s})\sqrt{1 + 4e^s}} = \frac{1}{2}(1 - \frac{1}{\sqrt{1 + 4e^s}}) \in (0, \frac{1}{2}).
\]

Note that \( p_1(s) \) is increasing on \( \mathbb{R} \), and \( p_1(-\infty) = 0, p_1(\infty) = \frac{1}{2} \). Hence \( \Pi_\Gamma = \text{conv} \{ (0, 1)^T, (1, 1)^T \} \) and \( \partial P_1(\mathbb{R}^2) = \pi \Pi_\Gamma \). As \( P_1(0) = h_\Gamma = \log \frac{1 + \sqrt{e^s}}{2} \) it follows that the value \( p^* := p_1(0) = \frac{2}{(1 + \sqrt{e^s})\sqrt{e^s}} = .2763932024 \) is the density \( p^* \) of 1’s for which \( h_\Gamma = h_\Gamma((p^*, 1 - p^*)) \).
To find the formula for $\tilde{h}_\Gamma(p) = h_\Gamma((p, 1 - p))$ first note that if $p = p_1(s)$ then

$$\sqrt{1 + 4e^s} = \frac{1}{1 - 2p}, \quad s(p) = \log \frac{p(1 - p)}{(1 - 2p)^2}.$$ 

Then

$$\tilde{h}_\Gamma(p) = \log \frac{1 - p}{1 - 2p} - p \log \frac{p(1 - p)}{(1 - 2p)^2}, \quad p \in (0, \frac{1}{2}).$$

Our computations of $P_\Gamma$, for $d \geq 2$, are based on upper and lower bounds, for example as given in Corollary 3.2. We claim that the function $\frac{\log \theta_2(m, u)}{m}$ can be viewed as the pressure function of certain corresponding one dimensional subshift of finite type given.

Consider for the simplicity of the exposition two dimensional SOFT given by $\Gamma = (\Gamma_1, \Gamma_2)$, where $\Gamma_1$ is a symmetric digraph. Let $\Delta$ be the transfer digraph induced by $\Gamma_2$ between the allowable $\Gamma_1$ coloring of the circle $T(m)$. Then $V := C_{\Gamma_1, \text{per}}(m)$ are the set of vertices of $\Delta$. For any $\alpha, \beta \in C_{\Gamma_1, \text{per}}(m)$ the directed edge $(\alpha, \beta)$ is in $\Delta$ if and only if the configuration $[(\alpha, \beta)]$ is an allowable configuration on $C_{\Gamma_1}(m, 2)$. Note that the adjacency matrix $D(\Delta) = (d_{\alpha\beta})_{\alpha, \beta \in C_{\Gamma_1, \text{per}}(m)}$ is $N \times N$ matrix, where $N := \#C_{\Gamma_1, \text{per}}(m)$. Then the one dimensional SOFT is $C_{\Gamma_1}(T(m)) \times \mathbb{Z}$: all $\Gamma$ allowable coloring of the infinite torus in the direction $e_2$ with the basis $T(m)$. The pressure corresponding to this one dimensional SOFT is denoted by $\tilde{P}_\Delta(u)$. It is given by the following formula: Let

$$\tilde{D}(\Delta, u) = (d_{\alpha\beta}(u))_{\alpha, \beta \in C_{\Gamma_1, \text{per}}(m)}, \quad \tilde{d}_{\alpha\beta}(u) = d_{\alpha\beta} e^{\frac{1}{2}(\epsilon(\alpha) + \epsilon(\beta))} u.$$ 

Then

$$\tilde{P}_\Delta(u) := \frac{\log \rho(\tilde{D}(\Delta, u))}{m}. \quad (5.5)$$

The reason we divide $\log \rho(\tilde{D}(\Delta, u))$ by $m$, is to have the normalization

$$\tilde{P}_\Delta(u + te) = \tilde{P}_\Delta(u) + t \quad \text{for any } t \in \mathbb{R}.$$ 

It is straightforward to show that $\frac{\log \theta_2(m, u)}{m} = \tilde{P}_\Delta(u)$. Assume that $\Delta$ has one irreducible component. Then the arguments of the proof of Proposition 5.2 yield that $\tilde{P}_\Delta(u)$ is analytic on $\mathbb{R}^n$. Furthermore

$$\nabla \tilde{P}_\Delta(u) = (y(u)^\top (\partial_1 \tilde{D}(\Delta, u)) x(u), \ldots, y(u)^\top (\partial_n \tilde{D}(\Delta, u)) x(u)), \quad (5.6)$$

for any $u \in \mathbb{R}^n$. Here $x(u)$ and $y(u)$ are the nonnegative eigenvectors of $D(\Delta, u)$ and $D(\Delta, u)^\top$, respectively, normalized by the condition $y(u)^\top x(u) = 1$. Then $\nabla \tilde{P}_\Delta(u) \in \Pi_n$ corresponds to the limiting densities of the $n$ kind of particles in this one dimensional SOFT.

In the numerical computations, as in the next section, we use one dimensional subshifts to estimate the pressure $P_\Gamma$ from above or below as described for example in Corollary 3.2. To estimate the partial derivatives of $P_\Gamma$, one can find the partial derivatives of the pressure corresponding to the one dimensional subshift approximation using Proposition 5.2. Since $P_\Gamma(u)$ is convex in each variable we can estimate each partial derivative from above and below by finite differences. However, these estimates are not as good as taking the derivatives of the one dimensional subshift approximation to $P_\Gamma(u)$.

6 The monomer-dimer model in $\mathbb{Z}^d$

A dimer is a union of two adjacent sites in the grid $\mathbb{Z}^d$, and a monomer is a single site. By a tiling of a set $S \subseteq \mathbb{Z}^d$ we mean a partition of $S$ into monomers and dimers. By a cover of $S$ we mean a tiling of a superset of $S$ with each monomer contained in $S$ and each dimer meeting $S$; in other words, dimers are allowed to protrude halfway out of $S$. Usually our set $S$ will be a box or the entire $\mathbb{Z}^d$; in the case of a torus we only speak of tilings. As
mentioned in [12], the set of monomer-dimer tilings of \( \mathbb{Z}^d \) can be encoded as an NNSOFT \( C_T(\mathbb{Z}^d) \) as follows. We color \( \mathbb{Z}^d \) with the \( 2d + 1 \) colors \( 1, \ldots, 2d + 1 \): a dimer in the direction of \( \mathbf{e}_i \) occupying the adjacent sites \( i, i + \mathbf{e}_k \) is encoded by the color \( k \) at \( i \) and the color \( k + d \) at \( i + \mathbf{e}_k \); a monomer at \( i \) is encoded by the color \( 2d + 1 \) at \( i \). This imposes restrictions on the coloring, which are expressed by the \( d \)-digraph \( \Gamma = (\Gamma_1, \ldots, \Gamma_d) \) on the set of vertices \((2d + 1), \) where

\[
(p, q) \in \Gamma_k \iff (p = k, q = k + d) \text{ or } (p \neq k, q \neq k + d).
\] (6.1)

It is easy to check that this gives a bijection between the monomer-dimer tilings of \( \mathbb{Z}^d \) and \( C_T(\mathbb{Z}^d) \). Let \( P_T(u), u \in \mathbb{R}^{2d+1} \) be the pressure function for the monomer-dimer model in \( \mathbb{Z}^d \). Since each dimer in the direction \( \mathbf{e}_k \) corresponds to the colors \( k \) and \( k + d \) it follows that \( P_T(u) \) is effectively a function of \( d + 1 \) variables. To show that we define the following linear transformations

**Definition 6.1** (a) Let \( T, T_1 : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{2d+1} \) be the linear transformations

\[
T(w_1, \ldots, w_{2d+1}) = \left(\frac{w_1}{2}, \ldots, \frac{w_d}{2}, \frac{w_1}{2}, \ldots, \frac{w_d}{2}, w_{d+1}\right),
\]

\[
T_1(w_1, \ldots, w_{d+1}) = (w_1, \ldots, w_d, w_1, \ldots, w_d, w_{d+1}).
\]

(b) Let \( Q : \mathbb{R}^{2d+1} \rightarrow \mathbb{R}^{d+1} \) be the linear transformation given by \( Q(u_1, \ldots, u_{2d+1}) = (u_1 + u_{d+1}, \ldots, u_d + u_{2d}, u_{2d+1}) \).

(c) Let \( Q_d : \mathbb{R}^d \rightarrow \mathbb{R} \) be the linear transformation \( (v_1, \ldots, v_d) \mapsto v_1 + \cdots + v_d \).

**Theorem 6.2** Let \( \Gamma = (\Gamma_1, \ldots, \Gamma_d) \)-coloring, with \( 2d + 1 \) colors given by (6.1) For \( u \in \mathbb{R}^{2d+1} \) let \( P_T(u) \) denote the pressure function. Then

(a) \( P_T(u) = T(P_T(u)) \).

(b) \( \partial P_T(\mathbb{R}^{2d+1}) \subseteq T(\mathbb{R}^{d+1}) \).

(c) \( \Pi_T = T(\Pi_{d+1}) \). Hence \( \partial P_T(\mathbb{R}^{2d+1}) \subseteq T(\Pi_{d+1}) \).

(d) The function \( h_T : \Pi_{d+1} \rightarrow \mathbb{R}_+ \) is a concave function.

**Proof** (a) Since the colors \( i \) and \( i + d \) describe the two halves of a dimer in the direction \( \mathbf{e}_i \) for \( i = 1, \ldots, d \), we have the identity \( P_T(u) = P_T(TQu) \).

(b) Let \( p = (p_1, \ldots, p_{2d+1}) \in \partial(P_T(u)) \). In case \( u \in \text{diff} P_T \), the Chain Rule applied to the identity in (a) yields the equalities \( p_i = p_{i+d}, i = 1, \ldots, d \). In case \( u \in \partial(P_T(u)) \setminus \text{diff} P_T \), this follows from the fact that \( \partial P_T(u) = \text{conv } S(u) \) as in the beginning of Section 4.

(c) Since the color \( i \) appears with color \( i + d \), it follows that \( p_i = p_{i+d} \) for \( i = 1, \ldots, d \). Hence \( \Pi_T \subseteq T\Pi_{d+1} \). It is left to show that any \( p = (p_1, \ldots, p_d, p_1, \ldots, p_d, p_{2d+1}) \in \Pi_{2d+1} \) is in \( \Pi_T \). Equivalently, the probability vector \( r := (2p_1, \ldots, 2p_d, p_{d+1}) \) is the density vector of the dimer-monomer covering of \( \mathbb{Z}^d \). For \( d = 1 \), this result is straightforward, e.g. [12]. So assume that \( d > 1 \). Suppose first that all the coordinates of \( r \) are rational and positive: \( r = \left(\frac{i_1}{m}, \ldots, \frac{i_d}{m}, \frac{i_{d+1}}{m} \right) \), where \( m \) is a positive integer. Consider the sequence \( m_q = (2qm, 2qm, \ldots, 2qm) \in \mathbb{N}^d, q \in \mathbb{N} \). Partition the cube \( (m_q') \) to \( d + 1 \) boxes with a basis \( (m_q', 2qm, \ldots, 2qm) \in \mathbb{N}^{d-1} \): \( (m_q', 2qm_{d+1}) \), \( j = 1, \ldots, d + 1 \). Tile the boxes \( (m_q', 2qm_{d+1}) \) with the dimers in the direction \( \mathbf{e}_j \) for \( j = 1, \ldots, d \), and the last box \( (m_q', 2qm_{d+1}) \) with monomers. Then \( p = Tr \in \Pi_T \). Since \( \Pi_T \) is closed we deduce that \( \Pi_T \supseteq T\Pi_{d+1} \). Hence \( \Pi_T = T\Pi_{d+1} \).

In case \( u \in \text{diff} P_T \), then \( \nabla P_T(u) \in \Pi_T = T\Pi_{d+1} \). In case \( u \in \partial(P_T(u)) \setminus \text{diff} P_T \), clearly \( S(u) \subseteq T\Pi_{d+1} \). Hence \( \partial P_T(u) = \text{conv } S(u) \subseteq T\Pi_{d+1} \). Therefore \( \partial P_T(\mathbb{R}^{2d+1}) \subseteq T(\Pi_{d+1}) \).
(d) According to part (b) of Example 4.7, the graph \( \Gamma \), corresponding to the monomer-dimer model, is friendly, (as explained in [12, §4]). Part (b) of Theorem 4.8 yields that \( h_\Gamma \) is concave \( \Pi_\Gamma = T \Pi_d \).

Define

\[
R_d(w) := P_\Gamma(T_1(w)).
\]

In analogy with Proposition 4.1 and Proposition 4.9, \( R_d(w) : \mathbb{R}^{d+1} \to \mathbb{R} \) is convex Lipschitz function which satisfies the conditions

\[
|R_d(w + z) - R_d(w)| \leq \|z\|_{max}, \quad w, z \in \mathbb{R}^{d+1},
\]

\[
R_d(w) = t + R_d(w - te) \quad w \in \mathbb{R}^{d+1}, t \in \mathbb{R}.
\]

We now derive the properties \( R_d(w) \) that are analogous to the properties of \( P_\Gamma(u) \) discussed in Section 4. First we view \( R_d \) as the restriction of \( P_\Gamma \) to the \((d + 1)\)-dimensional subspace \( T \mathbb{R}^{d+1} \). Observe that \( Q \Pi_{d+1} = QT \Pi_{d+1} = \Pi_{d+1} \). Note that the vector \( r = (r_1, \ldots, r_d, r_{d+1})^T \in \Pi_{d+1} \) can be defined intrinsically, where \( r_i \) the dimer density in the direction \( e_i \) for \( i = 1, \ldots, d \), and \( r_{d+1} \) is the monomer density in the lattice \( \mathbb{Z}^d \). Let

\[
H_d(r) := h_\Gamma(T \sigma), \quad \text{for any } r \in \Pi_{d+1}.
\]

We view \( H_d(r) \) as the anisotropic dimer-monomer entropy of density \( r \).

It is straightforward to show that \( R_d \) satisfies an analogous theorem to Theorem 4.3. In particular, \( R_d(w) = \max r \in \Pi_{d+1} (r^T w + H_d(r)) \). For \( w \in \mathbb{R}^{d+1} \), we denote

\[
\Pi_{d+1}(w) := \max r \in \Pi_{d+1} (r^T w + H_d(r)) = \{ r \in \Pi_{d+1} : R_d(w) = r^T w + H_d(r) \}.
\]

Because of the equality (6.3), we can use the analogous results to Theorems 4.10 and 4.11. More precisely, for \( v \in \mathbb{R}^d \), let \( P_d(v) \) be defined as in (4.17), i.e.,

\[
P_d(v) = R_d((u_0(v))) = P_\Gamma((u_0)), \quad u_0^T = (v^T, v^T, 0), v^T = (v_1, \ldots, v_d) \in \mathbb{R}^d.
\]

In other words, the two halves of a dimer in the direction of \( e_k \) are given the positive weight \( x_k = e^v \) each, and a monomer is given the weight \( 1 = e^0 \). Then \( Z_{\text{per}}(m, v) := Z_{\text{per}}(m, u_0) \), is the grand partition monomer-dimer (counting) function in which we sum over all monomer-dimer tilings of the torus \( T(m) \), and each tiling having exactly \( \mu \) dimers in the direction \( e_i \) for \( i = 1, \ldots, d \) plus monomers contributes \( \prod_{i=1}^d e^{x_i} \). As in [12], the function \( Z(m, v) := Z \Gamma(m, u_0) \) does not exactly count the weighted monomer-dimer covers of \( m \), because protruding dimers have only half of their weight counted. This can be easily taken care of as in [12], and the pressure \( P_d(v) \) is a convex function of \( v \in \mathbb{R}^d \).

**Lemma 2** Let \( v^T = (v_1, \ldots, v_d) \in \mathbb{R}^d \) and let \( \sigma : \langle d \rangle \to \langle d \rangle \) be a permutation. Then \( P_d((v_1, \ldots, v_d)^T) = P_d((v_{\sigma(1)}, \ldots, v_{\sigma(d)})^T) \); in other words, \( P_d(v) \) is a symmetric function of \( v_1, \ldots, v_d \). Similarly for \( Z(m, v) \).

**Proof** By applying the automorphism of \( \mathbb{N}^d \) given by

\[
(m_1, \ldots, m_d) \mapsto (m_{\sigma(1)}, \ldots, m_{\sigma(d)})
\]

we obtain the equality

\[
Z((m_1, \ldots, m_d), (v_1, \ldots, v_d)^T) = Z((m_{\sigma(1)}, \ldots, m_{\sigma(d)}), (v_{\sigma(1)}, \ldots, v_{\sigma(d)})^T),
\]

and the result follows from (2.1).

Then each for each \( r \in \Pi_{d+1}(i_0(v)) \) we have \( r \in \partial P_d(v) \). We define \( \Delta_d := \Pi_{d+1} \) to be the projection of \( \Pi_{d+1} \) on the first \( d \) coordinates. Let

\[
h_d(r) = H_d(r), \quad r \in \Pi_{d+1}.
\]

We can repeat the proof Theorem 4.11 to obtain:
**Theorem 6.3** Let $P_d^*$ be the conjugate convex function of the pressure function $P_d$. Then

(a) $\tilde{h}_d(q) \leq -P_d^*(q)$ for all $q \in \Delta_d$.

(b) $P_d(v) = \max_{q \in \Delta_d} (q \cdot v + \tilde{h}_d(q))$ for all $v \in \mathbb{R}^d$. \hspace{1cm} (6.8)

For $v \in \mathbb{R}^d$, we denote

$$\Delta_d(v) := \arg \max_{q \in \Delta_d} (q \cdot v + \tilde{h}_d(q)),$$

that is to say $q(v) \in \Delta_d(v)$ if and only if

$$q(v) \in \Delta_d \text{ and } P_d(v) = q(v) \cdot v + \tilde{h}_d(q(v)). \hspace{1cm} (6.9)$$

(c) $\tilde{h}_d(q(v)) = -P_d^*(q(v))$.

(d) $\Delta_d(v) \subseteq \partial P_d(v)$. In particular, if $v \in \text{diff} P_d$, then $\Delta_d(v) = \{\nabla P_d(v)\}$. Therefore $\partial P_d(\text{diff} P_d) \subseteq \Delta_d$.

(e) Let $v \in \mathbb{R}^d \setminus \text{diff} P_d$, and let $S(v)$ consist of all the limits of sequences $\nabla P_d(v_i)$ such that $v_i \in \text{diff} P_d$ and $v_i \to v$. Then $S(v) \subseteq \Delta_d(v)$.

(f) $\text{conv } \Delta_d = \text{dom } P_d^*$.

Thus, the first order phase transition occurs at the points $v$ where $P_d$ is not differentiable.

As in [15, 4, 17] we consider the total dimer density $q := q_1 + \cdots + q_d$. This is equivalent to the equalities $v_1 = \cdots = v_d = v = \log s$, where $s > 0$ is the weight of a half a dimer in any direction. We define $\text{pres}_d(v) := P_d((v, \ldots, v)\top) = P_d(v\top) : \mathbb{R} \to \mathbb{R}$. Then $\text{pres}_d$ is a nondecreasing convex Lipschitz function satisfies $|\text{pres}_d(u) - \text{pres}_d(v)| \leq |u - v|$.

**Proposition 6.4** For each $d \in \mathbb{N} Q_d(\Delta_d) = [0, 1]$. Let

$$\hat{h}_d(p) := \max_{q \in \Delta_d, q \cdot p = p} \tilde{h}_d(q), \text{ for each } p \in [0, 1]. \hspace{1cm} (6.10)$$

Then

$$\text{pres}_d(v) = \max_{p \in [0, 1]} pv + \hat{h}_d(p). \hspace{1cm} (6.11)$$

Furthermore, $\hat{h}_d(p)$ is the $p$-dimer entropy as defined in [15] or [12].

**Proof** Let $p \in [0, 1]$ be the limit density of dimers, abbreviated here as $p$-dimer density, as $(m) \to \infty$ as discussed in [12]. We recall the definition of the $p$-dimer density in terms of quantities defined in §4. (See in particular Definition 4.2.)

For each $m \in \mathbb{N}^d$ and a nonnegative integer $a \in [0, \text{vol}(m)]$ define

$$C_R((m), a) := \bigcup_{c = (c_1, \ldots, c_{2d+1}) \in \mathbb{N}_{2d+1}, c_{2d+1} = a} C_R((m), c). \hspace{1cm} (6.12)$$

So $C_R((m), a)$ is roughly equal to the set of all covering of the box $(m) \subset \mathbb{Z}^d$ with monomer-dimers, such that the number of monomers is $a$. (It may happen that some of the dimers protruding "out of" the box $(m)$, see [12].) Then $p \in [0, 1]$ is dimer density if there exists a sequence of boxes $(m_q) \subset \mathbb{N}^d$ and a corresponding sequence of nonnegative integers $a_q \in [0, \text{vol}(m_q)]$, such that

$$m_q \to \infty, C_R((m_q), a_q) \neq \emptyset \forall q \in \mathbb{N}, \text{ and } \lim_{q \to \infty} \frac{a_q}{\text{vol}(m_q)} = 1 - p. \hspace{1cm} (6.13)$$
From the definition of the density set $\Delta_d$ of the dimers it follows that $p$ is a dimer density if and only $p = Q_d q$ for some $q \in \Delta_d$. Since $\Delta_d = \Pi_{d+1}$, it follows that $Q_d(\Delta_d) = [0, 1]$.

For each $p \in [0, 1]$ let

$$h_d(p) := \sup_{m_q, a_q} \limsup_{q \to \infty} \frac{\log \# C_r^\top(\langle m_q \rangle, a_q)}{\text{vol}(m_q)} \geq 0,$$  \hspace{1cm} (6.14)

where the supremum is taken over all the sequences satisfying (6.13). Then $h_d(p)$ is the $p$-dimer entropy as defined in [12]. Let $h_{\text{pres}}(p)$ be defined as in (6.10). We claim

$$h_d(p) = \hat{h}_d(p) \text{ for all } p \in [0, 1].$$  \hspace{1cm} (6.15)

Observe first that $C_r^\top(\langle m \rangle, c) \subseteq C_r^\top(\langle m \rangle, c_{2d+1})$ for any $c = (c_1, \ldots, c_{2d+1}) \in \Pi_{2d+1}(\text{vol}(m))$. The definition of $\hat{h}_d(q)$ and $h_d(p)$ implies straightforward the inequality $\hat{h}_d(q) \leq h_d(Q_d q)$. Hence $\hat{h}_d(q) \leq h_d(p)$.

(6.12) yields the inequality

$$\# C_r^\top(\langle m \rangle, a) \leq \left(\frac{\text{vol}(m) + 2d}{2d}\right) \times \max_{c = (c_1, \ldots, c_{2d+1}) \in \Pi_{2d+1}(\text{vol}(m)), c_{2d+1} = a} \# C_r^\top(\langle m \rangle, c).$$

Use the arguments of the proof of part (b) Theorem 4.3 to deduce the existence of $q \in \Delta_d$, such that $Q_d q = p$ and $h_d(p) \leq h_d(q)$. Hence $h_d(p) \leq \hat{h}_d(p)$ and therefore $h_d(p) = \hat{h}_d(p)$.

To show (6.11), take $v = v\text{e}$ in (6.8). We have $q^\top v = pv$, where $p = q^\top \text{e}$. In (6.8) take the maximum in two stages. The first stage is for fixed $p$, and the second stage over all $p$.

The results of [17] yield that $\text{pres}_d(v)$ is analytic. Since $\text{pres}_d(v)$ is also convex and not affine it follows that $\text{pres}_d(v)$ can not be constant on any interval $(a, b)$. Hence $p(v) := \text{pres}_d'(v)$ is increasing on $\mathbb{R}$ with $p(-\infty) = 0$ (no dimers) and $p(\infty) = 1$ (only dimers). Therefore the analytic function $p : \mathbb{R} \to (0, 1)$ has an increasing analytic inverse $v(p) : (0, 1) \to \mathbb{R}$. Recall that $\text{pres}_d'(p)$ is a convex function of $p$. Moreover

$$\frac{dp\text{res}_d}{dp} = v(p) + p \frac{dv(p)}{dp} - \frac{dp\text{res}_d}{dp} \frac{dv}{dp} = v(p) + p \frac{dv(p)}{dp} - p \frac{dv(p)}{dp} = v(p).$$

As $v(p)$ is an increasing function of $p$ it follows that $\text{pres}_d'(p)$ is a strictly convex function on $(0, 1)$. The corresponding dimer density entropy $\text{pres}_d'(p) = -h_d(p)$ is a strictly concave function. This is an improvement of the result of Hammersley [15] which showed that $h_d(p)$ is a concave function on $(0, 1)$. [11, Corollary 3.2] claims a stronger result, namely $h_d(p) + \frac{1}{2}(p \log p + (1-p) \log(1-p))$ is a concave function on $[0, 1]$. (Observe that $p \log p + (1-p) \log(1-p)$ is a strict convex function on $[0, 1]$.)

Since $\text{pres}_d$ is differentiable it follows that $\text{pres}_d'(p) = pv = \text{pres}_d'(v)$.

Hence we obtain the well known formula, e.g. [4]

$$h_d(p(v)) = \text{pres}_d(v) - pv \text{e}, \text{ where } p(v) = \text{pres}_d'(v) \text{ for all } v \in \mathbb{R}. \hspace{1cm} (6.16)$$

Note that $h_d(0) := \lim_{p \searrow 0} h_d(p) = 0$ and $h_d(1) := \lim_{p \nearrow 1} h_d(p)$ is the $d$-dimensional dimer-entropy.

7 Symmetric encoding of the monomer-dimer model

The disadvantage of the encoding (6.1) is that the $\Gamma_k$ are not symmetric, so we cannot apply the results of Section 3 directly. However, as pointed out in [12], there is a hidden symmetry, which enables us to obtain results analogous to those of Section 3. We now adapt the arguments of [12, Section 6] to $P_d(v)$, the pressure corresponding to the weighted monomer-dimer coverings.
For $d \in \mathbb{N}$, $K \subseteq \langle d \rangle$ and $\mathbf{m} \in \mathbb{N}^d$, we denote by $\langle \mathbf{m} \rangle_K$ the projection of $\langle \mathbf{m} \rangle$ on the coordinates with indices in $K$. Let $C_{\text{per, } K}(\mathbf{m})$ be the set of monomer-dimer covers of $T(\mathbf{m}_K) \times \langle \mathbf{m} \rangle_{\langle d \rangle \setminus K}$, and $Z_{\text{per, } K}(\mathbf{m}, \mathbf{v})$ the corresponding weighted sum. Thus $C_{\text{per, } \langle d \rangle}(\mathbf{m}) = C_{\text{per}}(\mathbf{m})$ and $Z_{\text{per, } \langle d \rangle}(\mathbf{m}, \mathbf{v}) = Z_{\text{per}}(\mathbf{m}, \mathbf{v})$. Note that by the isotropy of our $\Gamma$, $\#C_{\text{per, } K}(\mathbf{m})$ is invariant under permutations of the components of $\mathbf{m}$ if $K$ undergoes a corresponding change. Similarly for $Z_{\text{per, } K}(\mathbf{m}, \mathbf{v})$, if $K$ and $\mathbf{v}$ undergo a corresponding change.

In order to analyze $C_{\text{per, } \langle d \rangle}(\mathbf{m})$, we focus on the dimers in the cover lying along the direction $\mathbf{e}_d$. More precisely, with $\mathbf{m}' = (m_1, \ldots, m_{d-1})$, we consider $\langle \mathbf{m}' \rangle \times T(m_d)$ as consisting of $m_d$ levels isomorphic to $\langle \mathbf{m}' \rangle$. A subset $S$ of the sites on level $q$ is covered by dimers joining levels $q - 1$ and $q$ (with level 0 understood as level $m_d$); a subset $T$ disjoint from $S$ is covered by dimers joining levels $q$ and $q + 1$ (with level $m_d + 1$ understood as level 1); and the remainder $U$ of level $q$ is covered by monomers and dimers lying entirely within level $q$. We are interested in counting the coverings of $U$ subject to various restrictions.

With that in mind, for $\mathbf{m}' \in \mathbb{N}^{d-1}$ we define an undirected graph $G(\mathbf{m}')$ whose vertices are the subsets of $\langle \mathbf{m}' \rangle$, in which subsets $S$ and $T$ are adjacent if and only if $S \cap T = \emptyset$. When $S \cap T = \emptyset$ we also define, using $U = \langle \mathbf{m}' \rangle \setminus (S \cup T)$, and $\mathbf{v}' = (v_1, \ldots, v_{d-1})^\top$,

\[
\tilde{a}_{ST}(\mathbf{v}') = \text{sum of weighted monomer-dimer tilings of } U \\
\tilde{b}_{ST}(\mathbf{v}') = \text{sum of weighted monomer-dimer tilings of } U \text{ viewed as a subset of } T(\mathbf{m}') \\
\tilde{p}_{ST}(\mathbf{v}') = \text{sum of weighted monomer-dimer covers of } U, \text{ viewed as a subset of } T(m_1) \times \langle (m_2, \ldots, m_{d-1}) \rangle, \text{ each monomer within } U, \text{ and each dimer meeting } U \text{ but not } S \cup T. \\
\tilde{c}_{ST}(\mathbf{v}') = \text{sum of weighted monomer-dimer covers of } U, \text{ each monomer within } U, \text{ and each dimer meeting } U \text{ but not } S \cup T.
\]

In the tilings/covers counted by $\tilde{a}_{ST}(\mathbf{v}')$, $\tilde{b}_{ST}(\mathbf{v}')$, $\tilde{p}_{ST}(\mathbf{v}')$, $\tilde{c}_{ST}(\mathbf{v}')$, each monomer lies within $U$ and each dimer meets $U$ but not $S \cup T$. In $\tilde{a}_{ST}(\mathbf{v}')$, each dimer occupies two sites of $U$ that are adjacent in $\langle \mathbf{m}' \rangle$. In $\tilde{b}_{ST}(\mathbf{v}')$, each dimer occupies two sites of $U$ that are adjacent in $T(\mathbf{m}')$, so is allowed to “wrap around”. In $\tilde{p}_{ST}(\mathbf{v}')$, the dimers in the direction of $\mathbf{e}_1$ are allowed to “wrap around” and the other dimers are allowed to “protrude out” of $\langle (m_2, \ldots, m_{d-1}) \rangle$. In $\tilde{c}_{ST}(\mathbf{v}')$, the dimers may “protrude” out of $\langle \mathbf{m}' \rangle$. The weight of each monomer-dimer cover is a product of the weights of dimers and “half” dimers appearing in the cover. If a dimer in the direction of $\mathbf{e}_k$ is entirely within $U$, then its weight is $e^{2v_k}$. If a dimer “protrudes out” in the direction of $\mathbf{e}_k$, then its weight is $e^{v_k}$. Therefore

\[
\tilde{a}_{ST}(\mathbf{v}') \leq \tilde{b}_{ST}(\mathbf{v}') \leq \tilde{p}_{ST}(\mathbf{v}') \leq \tilde{c}_{ST}(\mathbf{v}').
\]

By definition, if $U = \emptyset$, then $\tilde{a}_{ST}(\mathbf{v}') = \tilde{b}_{ST}(\mathbf{v}') = \tilde{p}_{ST}(\mathbf{v}') = \tilde{c}_{ST}(\mathbf{v}') = 1$. Notice that when $d = 2$, there is no distinction between $\tilde{b}_{ST}(\mathbf{v}')$ and $\tilde{p}_{ST}(\mathbf{v}')$.

We define matrices $A(\mathbf{m}', \mathbf{v}) = (a_{ST}(\mathbf{v}))(S,T \subseteq \langle \mathbf{m}' \rangle)$, $B(\mathbf{m}', \mathbf{v}) = (b_{ST}(\mathbf{v}))(S,T \subseteq \langle \mathbf{m}' \rangle)$, $P(\mathbf{m}', \mathbf{v}) = (p_{ST}(\mathbf{v}))(S,T \subseteq \langle \mathbf{m}' \rangle)$, $C(\mathbf{m}', \mathbf{v}) = (c_{ST}(\mathbf{v}))(S,T \subseteq \langle \mathbf{m}' \rangle)$ with rows and columns indexed by subsets of $\langle \mathbf{m}' \rangle$ as follows:
\[
A(m', v)_{ST} = \begin{cases} 
\tilde{a}_{ST}(v')e^{(S+T)v_d} & \text{if } S \cap T = \emptyset \\
0 & \text{if } S \cap T \neq \emptyset
\end{cases}
\]
\[
B(m', v)_{ST} = \begin{cases} 
\tilde{b}_{ST}(v')e^{(S+T)v_d} & \text{if } S \cap T = \emptyset \\
0 & \text{if } S \cap T \neq \emptyset
\end{cases}
\]
\[
P(m', v)_{ST} = \begin{cases} 
\tilde{p}_{ST}(v')e^{(S+T)v_d} & \text{if } S \cap T = \emptyset \\
0 & \text{if } S \cap T \neq \emptyset
\end{cases}
\]
\[
C(m', v)_{ST} = \begin{cases} 
\tilde{c}_{ST}(v')e^{(S+T)v_d} & \text{if } S \cap T = \emptyset \\
0 & \text{if } S \cap T \neq \emptyset
\end{cases}
\]

Thus \(A(m', v), B(m', v), P(m', v), C(m', v)\) are symmetric matrices—here is the “hidden symmetry” referred to above. Clearly

\[0 \leq A(m', v) \leq B(m', v) \leq P(m', v) \leq C(m', v)\]

(where the inequalities indicate componentwise comparisons). We use the notation \(\alpha(m', v), \beta(m', v), \pi(m', v), \gamma(m', v)\) for the spectral radii of these matrices, respectively, so that

\[\alpha(m', v) \leq \beta(m', v) \leq \pi(m', v) \leq \gamma(m', v)\]

Note that by Kingman’s theorem [21] all the spectral radii are log-convex in \(v\).

The four matrices have the same zero-nonzero pattern, namely the adjacency matrix of the graph \(G(m')\). If the graph is connected, we say that the matrix is irreducible; if in addition the greatest common divisor of the lengths of all its cycles is 1, equivalently for sufficiently high powers of the matrix all entries are strictly positive, we say that the matrix is primitive.

**Proposition 7.1** Let \(2 \leq d \in \mathbb{N}\) and \(m = (m', m_d) \in \mathbb{N}^d\). Then

\((a)\) \(\text{tr } A(m', v)^{md} = \text{the sum of the weighted monomer-dimer tilings of } \langle m' \rangle \times T(m_d)\)

\((b)\) \(\text{tr } B(m', v)^{md} = Z_{per}(m, v)\)

\((c)\) \(\text{tr } P(m', v)^{md} = Z_{per, \{1,d\}}(m, v)\)

\((d)\) \(\text{tr } C(m', v)^{md} = Z_{per, \{d\}}(m, v)\)

\((e)\) if \(m_d \geq 2\), if column vector \(x(v) = (x_S(v))_{S \subseteq \langle m' \rangle}\) is given by \(x_S(v) = \tilde{b}_{S0}(v')e^{Sv_d}\), then \(x(v)^\top B(m', v)^{md-2}x(v) = Z_{per, \{d-1\}}(m, v)\), if column vector \(y(v) = (y_S(v))_{S \subseteq \langle m' \rangle}\) is given by \(y_S(v) = \tilde{c}_{S0}(v')e^{Sv_d}\), then \(y(v)^\top C(m', v)^{md-2}y(v) = Z(m, v)\), and if column vector \(z(v) = (z_S(v))_{S \subseteq \langle m' \rangle}\) is given by \(z_S(v) = \tilde{p}_{S0}(v')e^{Sv_d}\), then \(z(v)^\top P(m', v)^{md-2}z(v) = Z_{per, \{1\}}(m, v)\);

\((f)\) the matrices \(A(m', v), B(m', v), P(m', v), C(m', v)\) are primitive.

**Proof** We begin with proving (b), observing that (a), (c), (d) and (e) are similar. Assume first that \(m_d = 1\), and let \(\phi \in C_{per}(m)\). Since \(\phi\) can be extended periodically in the direction of \(e_d\) with period 1, it can be viewed as an element of \(C_{per}(m')\). Therefore \(\#C_{per}(m) = \#C_{per}(m')\) and moreover, \(Z_{per}(m, v) = Z_{per}(m', v')\) \((v_d\) does not matter since no dimer lies in the direction of \(e_d\)). We have \(\text{tr } B(m', v) = \sum_{S \subseteq \langle m' \rangle} b_{SS}(v)\). Only the term \(S = \emptyset\) contributes to the sum, and for this term we have \(U = \langle m' \rangle\) and \(b_{\emptyset} = \)
$Z_{\text{per}}(\mathbf{m}', \mathbf{v}') = Z_{\text{per}}(\mathbf{m}, \mathbf{v})$. Hence $\text{tr} B(\mathbf{m}', \mathbf{v}) = Z_{\text{per}}(\mathbf{m}, \mathbf{v})$. Now assume that $m_d > 1$, and consider a closed walk $S_1, S_2, \ldots, S_{m_d}, S_1$ of length $m_d$ in $G(\mathbf{m}')$. For each $p' \in S_q$ place a dimer in the direction of $e_d$ occupying the sites $(p', q)$ and $(p', q+1)$ (with $m_d + 1$ wrapping around to 1). We want to extend these dimers to a monomer-dimer tiling of $T(\mathbf{m}') \times T(m_d) = T(\mathbf{m})$, i.e., to a member of $C_{\text{per}}(\mathbf{m})$, by monomers and by dimers not in the direction of $e_d$, i.e., lying within the levels $1, \ldots, m_d$. The weighted number of choices of such monomers and dimers to fill the remainder of level $q$ is given by $b_{s_{q-1} s_q}(v')$, and together with the weight of the dimers in the direction of $e_d$ intersecting level $q$ it becomes $b_{s_{q-1} s_q}(v)$. Therefore the weighted number of extensions to a member of $C_{\text{per}}(\mathbf{m})$, i.e., the corresponding term of $Z_{\text{per}}(\mathbf{m}, \mathbf{v})$, is $b_{S_1 S_2}(v)b_{S_2 S_3}(v) \cdots b_{S_{m_d-1} S_{m_d}}(v)b_{S_{m_d} S_1}(v)$. Conversely, each term of $Z_{\text{per}}(\mathbf{m}, \mathbf{v})$ is obtained in this way. Hence $Z_{\text{per}}(\mathbf{m}, \mathbf{v})$ is the sum of all the products of the above form, namely $\text{tr} B(\mathbf{m}', \mathbf{v})^{m_d}$.

To prove (f), we note that $A(\mathbf{m}', \mathbf{v})$ is irreducible, since whenever $S \cap T = \emptyset$, $U$ can be tiled by monomers and therefore each subset of $(\mathbf{m}')$ is adjacent to $\emptyset$ in $G(\mathbf{m}')$. Furthermore, $A(\mathbf{m}', \mathbf{v})$ is primitive since the graph has a cycle of length 1 from $\emptyset$ to $\emptyset$. Since $A(\mathbf{m}', \mathbf{v}) \leq B(\mathbf{m}', \mathbf{v}) \leq P(\mathbf{m}', \mathbf{v}) \leq C(\mathbf{m}', \mathbf{v})$, it follows that $B(\mathbf{m}', \mathbf{v}), P(\mathbf{m}', \mathbf{v})$ and $C(\mathbf{m}', \mathbf{v})$ are also primitive. \end{proof}

For the next lemma, we define $C_0(\mathbf{m})$ as the set of colorings of $(\mathbf{m})$ corresponding to its monomer-dimer tilings (so no dimer protrudes out of $(\mathbf{m})$), and the corresponding weighted sum

$$Z_0(\mathbf{m}, \mathbf{v}) = \sum_{\phi \in C_0(\mathbf{m})} e^{\mathbf{c}(\phi) \cdot \mathbf{u}} \quad \mathbf{u}^\top = (\mathbf{v}^\top, \mathbf{v}^\top, 0).$$

**Lemma 3** Let $2 \leq d \in \mathbb{N}$ and $\mathbf{m}' \in \mathbb{N}^{d-1}$, $\mathbf{v} \in \mathbb{R}^d$. Then

$$\lim_{m_d \to \infty} \frac{\log Z_0((\mathbf{m}', m_d), \mathbf{v})}{m_d} = \log \alpha(\mathbf{m}', \mathbf{v}) \quad (7.1)$$

$$\lim_{m_d \to \infty} \frac{\log Z_0,_{\text{per},(d-1)}((\mathbf{m}', m_d), \mathbf{v})}{m_d} = \log \beta(\mathbf{m}', \mathbf{v}) \quad (7.2)$$

$$\lim_{m_d \to \infty} \frac{\log Z_{\text{per},(1)}((\mathbf{m}', m_d), \mathbf{v})}{m_d} = \log \gamma(\mathbf{m}', \mathbf{v}) \quad (7.3)$$

$$\lim_{m_d \to \infty} \frac{\log Z((\mathbf{m}', m_d), \mathbf{v})}{m_d} = \log \nu(\mathbf{m}', \mathbf{v}) \quad (7.4)$$

**Proof** From Part (a) of Proposition 7.1 $Z_0((\mathbf{m}', m_d), \mathbf{v}) \leq \text{tr} A(\mathbf{m}', \mathbf{v})^{m_d}$, and therefore

$$\limsup_{m_d \to \infty} \frac{\log Z_0((\mathbf{m}', m_d), \mathbf{v})}{m_d} \leq \limsup_{m_d \to \infty} \frac{\log \text{tr} A(\mathbf{m}', \mathbf{v})^{m_d}}{m_d} = \log \alpha(\mathbf{m}', \mathbf{v}). \quad (7.5)$$

The equality in (7.5) follows from a characterization of $\rho(M)$ for a square matrix $M \geq 0$, namely $\rho(M) = \limsup_{k \to \infty} (\text{tr} M^k)^{\frac{1}{k}}$ (see for example Proposition 10.3 of [10]). Since $-\log Z_0((\mathbf{m}', m_d)$ is subadditive in $m_d$, the first lim sup in (7.5) can be replaced by a lim. In order to prove the reverse inequality and thus (7.1), observe that each monomer-dimer tiling of $(\mathbf{m}') \times T(m_d)$ extends to a monomer-dimer tiling in $C_0(\mathbf{m}', m_d + 1)$ having the same weight (replace each dimer occupying $(\mathbf{m}', 1)$ and $(\mathbf{m}', m_d)$ by a monomer occupying $(\mathbf{m}', 1)$ and a dimer occupying $(\mathbf{m}', m_d)$ and $(\mathbf{m}', m_d + 1)$, and tile the rest with monomers). Hence $Z_0((\mathbf{m}', m_d + 1), \mathbf{v}) \geq \text{tr} A(\mathbf{m}', \mathbf{v})^{m_d}$ by Part (a) of Proposition 7.1. Therefore, since $-\log Z_0((\mathbf{m}', m_d), \mathbf{v})$ is subadditive in $m_d$ and thus the limits below exist, we obtain

$$\lim_{m_d \to \infty} \frac{\log Z_0((\mathbf{m}', m_d), \mathbf{v})}{m_d} = \lim_{m_d \to \infty} \frac{\log Z_0((\mathbf{m}', m_d + 1), \mathbf{v})}{m_d}$$

$$\geq \limsup_{m_d \to \infty} \frac{\log \text{tr} A(\mathbf{m}', \mathbf{v})^{m_d}}{m_d} = \log \alpha(\mathbf{m}', \mathbf{v}).$$

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To prove (7.2), (7.3), (7.4), we use the fact mentioned in the proof of Proposition 2.2 that if $M \geq 0$ and $w$ is a column vector with positive entries, then $\rho(M) = \lim_{k \to \infty} (w^\top M^k w)$. Applying this to $M = B(m', v), P(m', v), C(m', v)$ and using Part (c) of Proposition 7.1 with $w = x(v), z(v), y(v)$ defined there proves (7.2), (7.3), (7.4). 

Now we introduce the following notation. For $m \in \mathbb{N}^d$ and $k \in (d)$, $m^{-k} := (m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_d) \in \mathbb{N}^{d-1}$. As special cases we have the previous notation $m' = m^{-d}$ and $m^- = m^{-1}$. For $v = (v_1, \ldots, v_d)^\top \in \mathbb{R}^d$ we use the notation $v^k := (v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_d, v_k)^\top$. Note that $v^d = v$. Part (b) of Proposition 7.1 implies

$$Z_{\text{per}}(m, v) = \text{tr} B(m', v)^{m_d} = \text{tr} B(m^{-k}, v^k)^{m_k}. \tag{7.6}$$

**Proposition 7.2** Let $m \in \mathbb{N}^d, v \in \mathbb{R}^d$, and assume that $m_d$ is even. Then each $k \in (d-1)$ satisfies

$$\frac{\log \beta(m^{-d}, v)}{\text{vol}(m^{-d})} \leq \frac{\log 2}{m_k} + \frac{\log \beta(m^{-k}, v^k)}{\text{vol}(m^{-k})}. \tag{7.7}$$

**Proof** We have

$$\beta(m^{-d}, v)^{m_d} \leq \text{tr} B(m^{-d}, v)^{m_d} = \text{tr} B(m^{-k}, v^k)^{m_k} \leq 2^{\text{vol}(m^{-k})} \beta(m^{-k}, v^k)^{m_k}.$$

The first inequality above follows since $\beta(m^{-d}, v)$ is one of the eigenvalues of $B(m^{-d}, v)$, which are all real, and $m_d$ is even; the next equality from (7.6); and the last inequality from the fact that $B(m^{-k}, v^k)$ has $2^{\text{vol}(m^{-k})}$ eigenvalues, all real, whose moduli are at most $\beta(m^{-k}, v^k)$. Taking logarithms and dividing by $\text{vol}(m)$, we deduce (7.7). 

We define

$$\overline{P}_{d-1}(m_1, v) := \lim_{m \to \infty} \frac{\log Z_{\text{per}, \{1\}}((m_1, m^-), v)}{\text{vol}(m^-)}, \quad m_1 \in \mathbb{N} \tag{7.8}$$

$$\overline{P}_{d-1}(0, v) := \text{vol}. \tag{7.9}$$

Notice that for $m_1 \in \mathbb{N}, \overline{P}_{d-1}(m_1, v)$ is the same as $\overline{P}_{d}(m_1, u)$ defined in (3.1), where $\Gamma$ is given by (6.1). For this reason the limit $\overline{P}_{d-1}(m_1, v)$ exists. The following theorem is an analog of Theorem 3.1 and (3.11).

**Theorem 7.3** Let $2 \leq d \in \mathbb{N}, p, r \in \mathbb{N}, q \in \mathbb{Z}_+, v \in \mathbb{R}^d$. Then

$$\frac{\overline{P}_{d-1}(2r, v)}{2r} \geq \frac{\text{P}_{d}(v)}{p} \geq \frac{\overline{P}_{d-1}(p + 2q, v) - \overline{P}_{d-1}(2q, v)}{p}. \tag{7.10}$$

Let $m' = (m_1, \ldots, m_{d-1}) \in \mathbb{N}^{d-1}$ and assume that $m_1, \ldots, m_{d-1}$ are even. Then

$$P_{d}(v) \leq \frac{\log \beta(m', v)}{\text{vol}(m')} \tag{7.11}$$

**Proof** Since $\#C_0(m + 21) \geq \#C(m)$ as explained in [12], it follows that $Z_0(m + 21, v) \geq Z(m, v)$. Hence, as in [12, formula (4.6) and (6.19)] and by Lemma 3,

$$P_{d}(v) = \lim_{m' \to \infty} \frac{\log \alpha(m', v)}{\text{vol}(m')} = \lim_{m' \to \infty} \frac{\log \beta(m', v)}{\text{vol}(m')} = \lim_{m' \to \infty} \frac{\log \gamma(m', v)}{\text{vol}(m')} \tag{7.12}.$$
Note that (7.7) with \( k = 1 \) states that

\[
\frac{\log \beta(m', v)}{\text{vol}(m')} \leq \frac{\log 2}{m_1} + \frac{\log \beta(m^-, v^1)}{\text{vol}(m^-)}.
\]

Using it \( d - 1 \) times along with \( s = m'_1, m'_2, m'_3, \ldots \), etc., we obtain

\[
\frac{\log \beta(s, (v_2, v_3, \ldots, v_d, v_1)\top)}{\text{vol}(s)} \leq \frac{\log 2}{s_1} + \frac{\log \beta(m'_1, (v_3, \ldots, v_d, v_1, v_2)\top)}{\text{vol}(m'_1)} \leq \ldots
\]

\[
\frac{\log 2}{s_1} + \frac{\log 2}{s_2} + \ldots \leq \sum_{j=1}^{d-1} \frac{\log 2}{s_j} + \frac{\log \beta(m', v)}{\text{vol}(m')}.
\]

Letting \( s \to \infty \) and using (7.12) and Lemma 2 for the left-hand side, we deduce (7.11).

We now demonstrate the lower bound in (7.10). Let \( m^- \in \mathbb{N}^{d-1}, s \in \mathbb{N}, q \in \mathbb{Z}_+ \). Assume first that \( q \in \mathbb{N} \). Since \( \gamma(m^-, v^1) = \rho(C(m^-, v^1)) \) and \( C(m^-, v^1) \) is symmetric, it follows as in the arguments for (3.9) and by the analog of (7.6) for \( C(m^-, v^1) \) that

\[
\gamma(m^-, v^1)^s \geq \frac{\text{tr} C(m^-, v^1)^{s+2q}}{\text{tr} C(m^-, v^1)^{2q}} = \frac{Z_{\text{per}, \{1\}}(s+2q, m^-, v)}{Z_{\text{per}, \{1\}}(2q, m^-, v)}. \tag{7.13}
\]

Taking logarithms, dividing by \( \text{vol}(m^-) \), letting \( m^- \to \infty \), and using (7.12), Lemma 2 and the definition of \( P_{d-1}(m_1, v) \), we deduce the lower bound in (7.10) for the case \( q \in \mathbb{N} \). If \( q = 0 \), we have to replace the denominators in (7.13) by \( \text{tr} I = 2^{\text{vol}(m^-)} \), and the lower bound in (7.10) is verified by (7.9).

We now prove the upper bound of (7.10). Let \( v^1 = (v_2, \ldots, v_{d-1}, v_1)\top \). For each \( m' \in \mathbb{N}^{d-1} \) we have

\[
\gamma(m', v^1)^{2r} \leq \text{tr} C(m', v^1)^{2r} = Z_{\text{per}, \{d\}}((m', 2r), v^1) = Z_{\text{per}, \{1\}}((2r, m'), v),
\]

where the inequality above is true because the eigenvalues of the symmetric matrix \( C(m', v^1) \) are real and \( \gamma(m', v^1) \) is one of them, the first equality follows from Part (d) of Proposition 7.1, and the last equality from (6.6). Therefore

\[
\frac{\log \gamma(m', v^1)}{\text{vol}(m')} \leq \frac{\log Z_{\text{per}, \{1\}}((2r, m'), v)}{2r \text{vol}(m')}.
\]

and letting \( m' \to \infty \), we deduce the upper bound of (7.10) by (7.12), Lemma 2 and the definition of \( P_{d-1}(m_1, v) \).

In view of (7.12) we assume that \( \frac{\log \beta(m', v)}{\text{vol}(m')} \) is a good approximation to \( P_d(v) \), and its partial derivative \( \frac{\partial \beta(m', v)}{\partial v_i} \) is a good approximation to \( q_i := \frac{\partial P_d(v)}{\partial v_i} \), the density of dimers in the direction of \( e_i \).

8 Numerical computations for the monomer-dimer model in \( \mathbb{Z}^2 \)

In this section we explain in detail our computations for two dimensional pressure \( P_2(v) = P_2(v_1, v_2) \) along the lines outlined in Sections 6-7. Our computations based on our ability to compute the spectral radius of the transfer matrix corresponding to the monomer dimer tiling of the torus \( (\mathbb{Z}/m) \times \mathbb{Z} \). This is a two dimensional integer lattice corresponding to a
circle of circumference $m$ times the real line. In the notation of Section 7 this lattice is given by $T(m) \times \mathbb{Z}$. This transfer matrix is denoted by $B(m, v)$. Let $x = e^{sv}, y = e^{sv}$. The weight of the dimer in direction $X$, i.e. the horizontal dimer that lies entirely on the circle $T(m)$, is $x^2$. The weight of the dimer in the direction $Y$, i.e. the vertical dimer that lies on two adjacent circles, is $y^2$. The matrix $B(m, v)$ is of order $2^m$, corresponding to all subsets of $\langle m \rangle$. Denote by $2^{\langle m \rangle}$ the set of all subsets of $\langle m \rangle$. For $S \in 2^{\langle m \rangle}$ denote by $\# S$ the cardinality of the set $S$. Then $B(m, v) = [y^{\# S + \# T}f(x, S, T)]_{S,T \in 2^{\langle m \rangle}}$. Here $f(x, S, T) = 0$ if $S \cap T \neq \emptyset$. For $S \cap T = \emptyset$ the function $f(x, S, T)$ is a polynomial in $x$, which is the sum of the following monomials. Consider the set $F := \langle m \rangle \setminus S \cup T$ viewed as a subset of the torus $T(m)$. Let $\mathcal{F}$ be a tiling of $F$ with monomers and dimers. A dimer $[i, i+1]$, occupying spaces $i, i+1$, can be in $\mathcal{F}$, if and only if $i$ and $i+1$ are in $F$, where $m$ and $m+1 \equiv 1$ are adjacent. To each tiling $\mathcal{F}$ corresponds a monomial $x^{2l}$, where $l$ is the number of dimers in the tiling $\mathcal{F}$ of $E$. Then $f(x, S, T)$ is the sum of all monomials corresponding to all tilings of $F$. Note that if $S \cap T = \emptyset$ and $S \cup T = \langle m \rangle$ then $f(x, S, T) = 1$. Furthermore $f(x, S, T) = f(x, T, S)$. Hence the transfer matrix $B(m, v)$ is a nonnegative symmetric matrix. The quantity $\bar{P}_1(m, v)$, defined by (7.8), is given as the logarithm of the spectral radius of $B(m, v)$. In numerical computations, we view $\frac{\bar{P}_1(m, v)}{m}$ as an approximation to the pressure $P_2(v)$. More precisely, one has the upper and lower bounds on the pressure which are given by (7.10).

As in [12], the matrix $B(m, v)$ has an automorphism group of order $2m$, obtained by rotating the discrete torus $T(m)$ and reflecting it. Thus, to compute the spectral radius of $B(m, v)$, it is enough to compute the spectral radius of the nonnegative symmetric matrix $\bar{B}(m, v)$ whose order is slightly higher than $\frac{2^{m-1}}{m}$. See for details [12, Section 7]. [12, Table 1, page 517] gives the dimensions of $\bar{B}(m, v)$ for $m = 4, \ldots, 17$. We were able to carry out some computations on a desk top computer up to $m = 17$.

We first apply our techniques to examine the Baxter computations in [4], Baxter computes essentially the values of the pressure $\text{pres}_2(v) := P_2(v, v)$ and the corresponding density of the dimers $p(v) := \frac{\partial}{\partial v}\text{pres}_2(v)$. Recall that the corresponding density entropy $h_2(p(v))$ is given by $\text{pres}_2(v) - vp(v)$ (6.16). Note the following correspondence between the variables in [4] and our variables given in Section 6:

$$ s = e^v, \quad \frac{\kappa}{s} = e^{-v + \text{pres}_2(v)}, \quad \rho = \frac{p}{2}. $$

The case $s = v = \infty$ corresponds to the dimer tilings of $\mathbb{Z}^2$. In this case $p = p(\infty) = 1$ and $h_2(1)$ has a known closed formula due to Fisher [8] and Kasteleyn [20]

$$ h_2(1) = \frac{1}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r + 1)^2} = 0.29156090 \ldots $$

As in [4] we consider the following 18 values of $s$

$$ s^{-1} = 0.02, 0.05, 0.10, 0.20, 0.30, 0.40, 0.50, 0.60, 0.80, 1.00, 1.50, 2.00, 2.50, 3.00, 3.50, 4.00, 4.50, 5.00. $$

We computed the upper and the lower bounds for $\text{pres}_2(\log s)$ for the above values of $s$, using inequalities (7.10) for $v = (\log s, \log s)$ and $m = 2, \ldots, 17$. In these computations we observed that the sequence $\frac{\bar{P}_1(2r, (\log s, \log s))}{2r}$ is decreasing for $r = 1, \ldots, 8$. So our upper bound was given by $\frac{\bar{P}_1(16, (\log s, \log s))}{16}$ for all 18 values of $s$. The lower bound was given by $\frac{\bar{P}_1(16, (\log s, \log s)) - \bar{P}_1(14, (\log s, \log s))}{2}$ for $s^{-1} = 0.02, \ldots, 0.3$ and by $\bar{P}_1(17, (\log s, \log s)) - \bar{P}_1(16, (\log s, \log s))$ for other values of $s$.

The values of Baxter for the pressure were all but two values between the upper and the lower bounds. In the two exceptional values $s^{-1} = 1.5, 2.0$ Baxter’s result were off by 1 in the last 10th digit. As in Baxter computations, the difference between the upper and lower bounds grows bigger as the value of $s$ increases. That is, it is harder to compute the precise
value of the pressure and its derivative in configurations where the density of dimers is high. This points to the phase transition in the case where $\mathbb{Z}^2$ is tiled by dimers only [3, p'133]. The pressure value for $s^{-1} = 0.02$ computed by Baxter has 8 values. Our upper and lower bounds give 4 digits of precision of the pressure. For the value $s = 1.0$ our computations confirm the first 9 digits of 10 digit Baxter computation. (This value of the pressure is equal to the monomer-dimer entropy $h_2$ discussed in [12].) For the values $s^{-1} = 2.0, \ldots, 5.0$ our computations gives at least 12 digits of the pressure.

We also computed the approximate value of the dimer density $p(\log s) = \text{pres}_2'(\log s)$ using the following two methods. The first approximation was obtained by computing the exact derivative of $\frac{P_t(\nu, (v_1, v_2))}{14}$ for $m = 2, \ldots, 14$. The second approximation was obtained by computing the ratio $\frac{P_t(\nu + \epsilon, (v_1 + \epsilon, v_2 + \cdot)) - P_t(\nu, (v_1, v_2))}{\epsilon}$ for $\epsilon = 10^{-5}$ and $m = 2, \ldots, 14$. It turned out that the values of the numerical derivatives for $m = 14$ agrees with most values of Baxter computations up to 5 digits, while the values of the exact derivatives agrees only up 2 digits with Baxter computations. Note that to compute the value of $h_2(p(\log s))$ we need the values of $\text{pres}_2(\log s)$ and $p(\log s)$ (6.16).

We next computed the approximate values of the pressure $\text{pres}_2((v_1, v_2))$ and its partial numerical derivatives for $18^2 = 324$ values. The 18 values of $v_1$ and $v_2$ were chosen in the interval $(-1.61, 4.)$. (These values correspond to the 18 values of $\log s$ considered by Baxter.) For the lower bound and upper bounds we chose the values of

$$\frac{P_t(14, (v_1, v_2)) - P_t(12, (v_1, v_2))}{2} \quad \text{and} \quad \frac{P_t(14, (v_1, v_2))}{14}$$

respectively. (8.1)

Follows below the graph of $\frac{P_t(14, (v_1, v_2))}{14}$ and the approximate values of $\bar{h}_2((p_1, p_2))$, where $p_1, p_2$ are the densities of the dimers in the direction $x_1, x_2$ respectively. The approximate values of $\bar{h}_2$ obtained by using the formula

$$\bar{h}_2((p_1, p_2)) \approx \frac{P_t(14, (v_1, v_2))}{14} - p_1v_1 - p_2v_2,$$

$$p_1 = \frac{\bar{P}_t(14, (v_1 + t, v_2)) - P_t(14, (v_1, v_2))}{14t},$$

$$p_2 = \frac{\bar{P}_t(14, (v_1, v_2 + t)) - P_t(14, (v_1, v_2))}{14t},$$

In our computation $t = 10^{-4}$. For more detailed graphs with $42^2 = 1764$ points see http://www2.math.uic.edu/~friedlan/Pressure17Jun09.pdf

The graph of the pressure $P_t((x_1, x_2))$ is convex and the graph of the density entropy $\bar{h}_2((x_1, x_2))$ is concave. Both graphs look is symmetric with respect to the line $x_1 = x_2$. In reality this is not the case, since $P_t(m, v_1, v_2)$ is the pressure of an infinite torus with a basis $m$. So in direction $x_1$ we have at most $\lceil \frac{m}{2} \rceil$ dimers, while in the direction $x_2$ we can have an infinite number of dimers. For $m \geq 10$ the difference $\frac{P_t(m, v_1, v_2) - \bar{P}_t(m, v_2, v_1)}{m}$ is less than $10^{-3}$, which explains the symmetry of our graphs. Note that in Figure 2 the densities $p_1, p_2$ satisfy the condition $p_1, p_2 \in [0, 1], p_1 + p_2 \in [0, 1]$. The entropy $\bar{h}_2$ is in the interval $[0, 0.67]$.

We also got similar graphs for the lower bound given in (8.1) and the corresponding analog of the approximation of $\bar{h}_2((p_1, p_2))$ given by (8.2). These graphs were very similar to the graphs of $\frac{P_t(14, (v_1, v_2))}{14}$ and the approximation of $\bar{h}_2((p_1, p_2))$ given by (8.2).

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