Weighted Average Number of Prime \( m \)-tuples lying on an Admissible \( k \)-tuple of Linear Forms

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Abstract

In this paper we find an upper bound for the sum \( \sum_{x<n \leq 2x} 1_{\mathbb{P}}(n+h_{i}) \cdots 1_{\mathbb{P}}(n+h_{m+1}) w_{n} \), where \((h_{1},...,h_{m+1})\) is any \((m+1)\)-tuple of elements in the admissible set \( \mathcal{H} = \{h_{1},...,h_{k}\} \), \( m \geq 1 \) and \( x \) is sufficiently large, with the same weights \( w_{n} \) used in the Maynard’s paper “Dense clusters of primes in subsets”. The estimate will be uniform over positive integer \( k \) with \( m+1 \leq k \leq (\log x)^{1/5} \) and on admissible set \( \mathcal{H} \) with \( 0 \leq h_{1} < ... < h_{k} \leq x \). Moreover, we make explicit the dependence on \( m \). The upper bound will depend on an integral of a smooth function and on the singular series of \( \mathcal{H} \), which naturally arises in this context.

1. INTRODUCTION

Let us fix \( \mathbb{P} \) the set of prime numbers, \( k \) a positive integer and \( \mathcal{H} = \{h_{1},...,h_{k}\} \) an admissible set. In the paper “Small gaps between primes” [4], Maynard proved that there exist infinitely many bounded intervals containing at least \( m \geq 1 \) primes, showing that the weighted sum

\[
S = \sum_{x<n \leq 2x} \left( \sum_{i=1}^{k} 1_{\mathbb{P}}(n+h_{i}) - (m-1) \right) w_{n}
\]

is positive, when \( x \) is sufficiently large. Here \( 1_{\mathbb{P}}(n) \) is the characteristic function of the set of prime numbers and \( w_{n} \) are chosen as non-negative smooth \( k \)-dimensional Selberg sieve weights. In fact, if \( S > 0 \), there must exist an integer \( n \in [x,2x] \) such that the corresponding factor in parentheses is positive, which is equivalent to say that at least \( m \) prime numbers lie on the translates \( n+h_{1},...,n+h_{k} \). Since \( \mathcal{H} \) is fixed and we can vary \( x \), we obtain the aforementioned result.

In the subsequent paper “Dense clusters of primes in subsets” [5], Maynard proved that a uniform version of the sieve method (1.1) can lead to improve the above result, finding a lower bound on the number of integers \( n \in [x,2x] \) for which there are at least \( m \) primes among \( n+h_{1},...,n+h_{k} \). More specifically, he showed that

\[
\#\{n \in [x,2x] : \#(\{n+h_{1},...,n+h_{k}\} \cap \mathbb{P}) \geq m \} \gg \frac{x}{\log^{k} x}.
\]

This is [5] Theorem 3.1. The estimate holds with some uniformity on the parameters \( m \) and \( k \) and on the admissible set \( \mathcal{H} \), which vary with \( x \) in certain ranges.

In order to obtain such result, Maynard estimated various sums involved in a generalization on the sieve method (1.1), which is essentially of the following form

\[
S' = \sum_{x<n \leq 2x} \left( \sum_{i=1}^{k} 1_{\mathbb{P}}(n+h_{i}) - (m-1) - k1_{B}(n) \right) w_{n},
\]

where \( 1_{B} \) is the characteristic function of the set of integers \( B \). Maynard gave estimates on these new particular sums in [5] Proposition 6.1. To find a lower bound for \( S' \), useful for application, he needed to obtain a lower bound on the weighted average number of primes lying on the admissible set \( \mathcal{H} \). Indeed, he showed that

\[
\sum_{x<n \leq 2x} 1_{\mathbb{P}}(n+h)w_{n}(\mathcal{H}) \gg \left( \frac{B}{\varphi(B)} \right)^{k-1} \mathcal{G}_{B}(\mathcal{H})x(\log x)^{\epsilon} \frac{\log k}{k} I_{k}(F),
\]

for every \( h \in \mathcal{H} \) and where \( B \) is a suitable positive integer. Here \( \mathcal{G}_{B}(\mathcal{H}) \) is the singular series associated with the set \( \mathcal{H} \) and

\[
I_{k}(F) = \int_{0}^{\infty} \cdots \int_{0}^{\infty} F^{2}(t_{1},...,t_{k}) dt_{1} \cdots dt_{k},
\]

with \( F \) a smooth function \( F : \mathbb{R}^{k} \rightarrow \mathbb{R} \) depending only on \( k \).
The aim of our paper is to find an upper bound for the generalization of the sum [13] to \( m \)-tuples of primes. For any admissible set of linear functions \( \mathcal{L} = \{L_1(n), ..., L_k(n)\} = \{n + h_1, ..., n + h_k\} \) and for every prime numbers \( p \) and integer \( B \), let us define the function \( \omega(p) \) as

\[
\omega(p) = \begin{cases} 
\#\{1 \leq n \leq p : \prod_{i=1}^{k} L_i(n) \equiv 0 \pmod{p}\} & \text{if } p \nmid B; \\
0 & p \mid B.
\end{cases}
\]

Moreover, we define for every integer \( D \) the singular series attached to \( \mathcal{L} \) as

\[
\mathcal{S}_D(\mathcal{L}) = \prod_{p \mid D} \left(1 - \frac{\omega(p)}{p}\right) \left(1 - \frac{1}{p}\right)^{-k}.
\]

From the admissibility of \( \mathcal{L} \) and the definition of \( \omega(p) \) it follows that \( \mathcal{S}_D(\mathcal{L}) \) converges.

Finally, we put \( W = \prod_{p \leq 2k^2, p
parallel B} p \) and we let \( W_1, ..., W_k \) to be square-free integers each a multiple of \( WB \), such that any prime \( p \nmid WB \) divides exactly \( k - \omega(p) \) of them. Now, we can state the main result.

**Theorem 1.1.** For any \( m \geq 1 \) and \( 0 \leq h_1 < ... < h_k \leq x \), and if \( k \) is a sufficiently large positive integer with \( m + 1 \leq k \leq (\log x)^{1/5} \), we have

\[
(1.4)
\]

\[
\sum_{x < n \leq 2x} 1_p(L_{i_1}(n)) \cdot 1_p(L_{i_{m+1}}(n)) w_n \leq C\sum_{x < n \leq 2x} \left(\frac{W}{\phi(W)}\right)^{k-1} \frac{(WB)^{k-2m-1}}{\phi(WB)^{k-2m-1}} \mathcal{S}_B(\mathcal{L}) \left(\frac{\log k}{k}\right)^{m+1} I_k(F) x(\log x)^k,
\]

for any \( 1 \leq i_1 < ... < i_{m+1} \leq k \), for a certain constant \( C > 0 \) and for any \( x \) large enough. Here, we take \( B \) as a suitable integer such that \( B/\phi(B) = 1 + o(1) \) and we let \( \bar{y} \) to be

\[
\bar{y} = (3m)^m \left(\frac{W_{i_1} \cdot ... \cdot W_{i_m}}{\phi(W_{i_1} \cdot ... \cdot W_{i_m})}\right)^m \prod_{p \mid n(p) \geq 1} \left(1 + \frac{n(p)}{p-1}\right)^{-1} \left(1 - \frac{1}{p}\right)^{-m},
\]

where \( n(p) = \#\{j \in \{1, ..., m\} : p \nmid W_j \} \). Moreover, \( I_k(F) \) is defined as above and the weights \( w_n \) are the same used in [5].

Unfortunately, Theorem 1.1 is of difficult application due to the strong dependence on the factors \( W_{i_1}, ..., W_{i_m} \). However, estimating carefully the product present in \( \bar{y} \) and averaging over all the \( (m+1) \)-tuples \( (h_{i_1}, ..., h_{i_{m+1}}) \) could lead to cancellations and we could obtain a bound free of these terms. Anyway, adding some restrictions we can simplify the result above in the following corollary.

**Corollary 1.2.** If we assume that the elements of the admissible set \( \mathcal{L} = \{n + h_1, ..., n + h_k\} \) verify also that \( h_1, ..., h_k \leq k^2 \), we have

\[
(1.5)
\]

\[
\sum_{x < n \leq 2x} 1_p(L_{i_1}(n)) \cdot 1_p(L_{i_{m+1}}(n)) w_n \leq D^m m^m \left(\frac{B}{\phi(B)}\right)^{k-1} \mathcal{S}_B(\mathcal{L}) x(\log x)^k \left(\frac{\log k}{k}\right)^{m+1} I_k(F),
\]

for any \( 1 \leq i_1 < ... < i_{m+1} \leq k \), for a suitable absolute constant \( D > 0 \) and if \( x \) is sufficiently large.

Comparing the estimate [13] with [13], we see that they are of the same correct order of magnitude. The motivation in proving the Theorem 1.1 and its Corollary relies on the analogy with [1]. In that occasion was considered an admissible set of linear functions \( \mathcal{L} = \{n + h_1, ..., n + h_k\} \), with \( n \in \mathbb{N} \) and with \( h_j - h_i \asymp \sqrt{N} \). The elements of the \( k \)-tuple \( \mathcal{H} = \{h_1, ..., h_k\} \) were allowed to grow with \( N \) and were chosen weights \( w_n \) suitable to such uniform situation. Under these circumstances, Banks, Freiberg and Maynard found an upper bound for the sum

\[
\sum_{x < n \leq 2x} 1_p(n + h_i) 1_p(n + h_j) w_n,
\]

for each pair \( i \neq j \in \{1, ..., k\} \), which, inserted in a sieve method like [12], led them to obtain \( m \)-tuples of primes where each of these primes belongs to a different subset, of a prescribed partition of \( \mathcal{H} \), containing no other prime numbers.

Combining this with an Erdős–Rankin construction, the authors of [1] found information on the percentage of limit points of the sequence of normalized prime gaps in the set of positive real numbers. We are confident that the bound [13] can find applications in the context of the sieve method introduced in [3], joined to the other results proved in [3] Proposition 6.1]. Perhaps, an explicit version of [13] could be useful exactly regarding the study of limit points of the sequence of normalized prime gaps, in a way similar to [1].
The proof of Theorem 1.1 is based on computations and ideas coming from the Maynard’s paper [5]. Therefore, we borrow from it the notations and the main definitions, which we rewrite in section 2 and 3 for completeness, following closely the presentation in [5].

2. Notations

We consider $0 < \theta < 1$ a fixed real constant and $m \geq 1$ a positive integer. All asymptotic notation such as $O(\cdot), o(\cdot), <<, >>$ should be interpreted as referring to the limit $x \to \infty$, and any constants (implied by $O(\cdot)$) may depend on $\theta$ or $m$, but no other variable, unless otherwise noted.

Let $k = \# \mathcal{L} \geq m + 1$ be the size of $\mathcal{L} = \{L_1, \ldots, L_k\}$ an admissible set of integer linear functions of the form $L_i(n) = n + h_i$. Moreover, $B$ will be an integer, and $x, k$ will always to be assumed sufficiently large (in terms of $\theta$ and $m$).

All sums, products and suprema will be assumed to be taken over variables lying in the natural numbers $\mathbb{N} = \{1, 2, \ldots\}$ unless specified otherwise. The exception to this is when sums or products are over a variable $p$, which instead will be assumed to lie in the prime numbers $\mathbb{P} = \{2, 3, \ldots\}$.

Throughout the paper, $\varphi$ will denote the Euler totient function, $\tau_r(n)$ the number of ways of writing $n$ as a product of $r$ natural numbers and $\mu$ the Moebius function. We let $\# \mathcal{A}$ denote the number of elements of a finite set $\mathcal{A}$, and $1_\mathcal{A}(x)$ the indicator function of $\mathcal{A}$ (so $1_\mathcal{A}(x) = 1$ if $x \in \mathcal{A}$, and 0 otherwise). We let $(a, b)$ be the greatest common divisor of integers $a$ and $b$, and $[a, b]$ the least common multiple of integers $a$ and $b$. (For real numbers $x, y$ we also use $[x, y]$ to denote the closed interval. The usage of $[\cdot, \cdot]$ should be clear from the context.)

To simplify notation we will use vectors in a way which is somewhat non-standard. In fact, $d$ will denote a vector $(d_1, \ldots, d_k) \in \mathbb{N}^k$. Given a vector $d$, when it does not cause confusion, we write $d = \prod_{i=1}^k d_i$. Given $d, e$, we will let $[d, e] = \prod_{i=1}^k [d_i, e_i]$ be the product of least common multiples of the components of $d, e$, and similarly let $(d, e) = \prod_{i=1}^k (d_i, e_i)$ be the product of greatest common divisors of the components, and $d|e$ denote the $k$ conditions $d_i|e_i$ for each $1 \leq i \leq k$.

3. Main definitions

We recall that we are given an admissible set $\mathcal{L} = \{L_1, \ldots, L_k\} = \{n + h_1, \ldots, n + h_k\}$ of integer linear functions, an integer $B$ and quantities $R, x$. We assume that $0 \leq h_1 < \ldots < h_k \leq x$ and $k$ is sufficiently large in terms of $m$ and satisfies $m + 1 \leq k \leq (\log x)^{1/5}$. Moreover, we fix $R$ as $R = x^{\theta/3}$.

We define the multiplicative functions $\omega = \omega_\mathcal{L}$ and $\varphi_\omega = \varphi_{\omega_\mathcal{L}}$ and the singular series $\mathcal{S}_D(\mathcal{L})$ for an integer $D$ by

\begin{equation}
\omega(p) = \begin{cases} 
\#\{1 \leq n \leq p : \prod_{i=1}^k L_i(n) \equiv 0 \pmod{p}\} & \text{if } p \nmid B; \\
0 & \text{if } p|B. 
\end{cases}
\end{equation}

\begin{equation}
\varphi_\omega(d) = \prod_{p|d} (p - \omega(p)),
\end{equation}

\begin{equation}
\mathcal{S}_D(\mathcal{L}) = \prod_{p|D} \left(1 - \frac{\omega(p)}{p}\right) \left(1 - \frac{1}{p}\right)^{-k}.
\end{equation}

Since $\mathcal{L}$ is admissible, we have $\omega(p) < p$ for all $p$ and so $\varphi_\omega(n) > 0$ and $\mathcal{S}_D(\mathcal{L}) > 0$ for any integer $D$. Since $\omega(p) = k$ for all $p \nmid \prod_{i \neq j} (h_j - h_i)$ we see the product $\mathcal{S}_D(\mathcal{L})$ converges. We will consider sieve weights $w_n = w_n(\mathcal{L})$, which are defined to be 0 if $\prod_{i=1}^k L_i(n)$ is a multiple of any prime $p \leq 2k^2$ with $p \nmid B$. We let $W = \prod_{p \leq 2k^2, p|B} p$. If $(L_i(n), W) = 1$ for all $1 \leq i \leq k$ we have

\begin{equation}
w_n = \left(\sum_{d_i|L_i(n), \forall i} \lambda_d\right)^2,
\end{equation}

for some real variables $\lambda_d$ depending on $d = (d_1, \ldots, d_k)$. We first restrict our $\lambda_\mathcal{L}$ to be supported on $d$ with $d = \prod_{i=1}^k d_i$ square-free and coprime to $WB$.

Given a prime $p \nmid WB$, let $1 \leq r_{p,1} < \cdots < r_{p,\omega(p)} \leq p$ be the $\omega(p)$ residue classes for which $\prod_{i=1}^k L_i(n)$ vanishes modulo $p$. For each such prime $p$, we fix a choice of indices $j_{p,1}, \ldots, j_{p,\omega(p)} \in \{1, \ldots, k\}$ such that $j_{p,i}$ is the smallest index such that

\begin{equation}
L_{j_{p,i}}(r_{p,i}) \equiv 0 \pmod{p}
\end{equation}

for each $i \in \{1, \ldots, \omega(p)\}$. All the functions $L_i$ are linear and, since $\mathcal{L}$ is admissible, none of the $L_i$ are a multiple of $p$. This means that for any $L \in \mathcal{L}$ there is at most one residue class for which $L$
vanishes modulo $p$. Thus the indices $j_p, 1, \ldots, j_p, \omega(p)$ we have chosen must be distinct. We now restrict the support of $\lambda_d$ to $(d, p) = 1$ for all $j \notin \{j_p, 1, \ldots, j_p, \omega(p)\}$.

We see these restrictions are equivalent to the restriction that the support of $\lambda_d$ must lie in the set
\begin{equation}
D_k = D_k(L) = \{d \in \mathbb{N}^k : \mu^2(d) = 1, (d_j, W_j) = 1, \forall j\},
\end{equation}
where $W_j$ are square-free integers each a multiple of $WB$, and any prime $p \nmid WB$ divides exactly
\begin{equation}
k - \omega(p)\end{equation}
of the $W_j$ (such $p|W_j$ if $j \notin \{j_p, 1, \ldots, j_p, \omega(p)\}$).

The key point of these restrictions is so that different components of different $d$ occurring in our sieve weights will be relatively prime. Indeed, let $d$ and $d'$ both occur in the sum (3.4). If $p|d_i$ then $p|L_i(n)$, and so $i$ must be the chosen index for the residue class $n \pmod{p}$. But if we also have $p|d_j$ then similarly $j$ must be the chosen index for this residue class, and so we must have $i = j$. Hence $(d, d_j) = 1$ for all $i \neq j$.

We define $\lambda_d$ in terms of variables $y_r$ supported on $r \in D_k$ by
\begin{equation}
\lambda_d = \mu(d)d \sum_{d|r} \frac{y_r}{\varphi(r)},
\end{equation}
where $F : \mathbb{R}^k \to \mathbb{R}$ is a smooth function given by
\begin{equation}
F(t_1, \ldots, t_k) = \psi\left(\sum_{i=1}^k t_i\right) \prod_{i=1}^k \frac{\psi(t_i/U_k)}{1 + T_k t_i}, \quad T_k = k \log k, \quad U_k = k^{-1/2}.
\end{equation}
Here $\psi : [0, \infty) \to [0, 1]$ is a fixed smooth non-increasing function supported on $[0, 1]$ which is 1 on $[0, 9/10]$. In particular, we note that this choice of $F$ is non-negative, and that the support of $\psi$ implies that
\begin{equation}
\lambda_d = 0 \quad \text{if} \quad d = \prod_{i=1}^k d_i > R.
\end{equation}
We will find it useful to also consider the closely related functions $F_1$ and $F_2$ which will appear in our error estimates, defined by
\begin{equation}
F_1(t_1, \ldots, t_k) = \prod_{i=1}^k \frac{\psi(t_i/U_k)}{1 + T_k t_i}, \quad F_2(t_1, \ldots, t_k) = \sum_{1 \leq j \leq k} \left[ \psi\left(\frac{t_j/2}{1 + T_k t_j}\right) \prod_{1 \leq i \leq k, i \neq j} \frac{\psi(t_i/U_k)}{1 + T_k t_i} \right].
\end{equation}
Finally, by Moebius inversion, we see that (3.7) implies that for $r \in D_k$
\begin{equation}
y_r = \mu(r)\varphi(r) \sum_{d | r} \sum_{d | d} \lambda_d = \mu(r)\varphi(r) \sum_{d | d} \lambda_d.
\end{equation}

4. PROOF OF THEOREM 1.1

The main aim of this section is to prove the estimate (1.4), which is the heart of Theorem 1.1. We start considering the following multidimensional Selberg bound:
\begin{equation}
1_p(n + h_{i_1}) \cdots 1_p(n + h_{i_m}) \leq \frac{1}{\lambda^2_{(1, \ldots, 1)}} \left( \sum_{e | n + h_{i_1}, \ldots, e | n + h_{i_m}} \tilde{\lambda}_e \right)^2,
\end{equation}
where $\tilde{\lambda}_e$ is a real function, with $\tilde{\lambda}_{(1, \ldots, 1)} \neq 0$, supported on the set
\begin{equation}
E_m = \{e \in \mathbb{N}^m : e < R^h, \mu^2(e) = 1, \text{ and } (e_j, W_{i_j}) = 1, \forall j = 1, \ldots, m\}.
\end{equation}
Inserting the upper bound (1.1) in the sum (1.4), which now we consider restricted on the arithmetic progression $n \equiv v_0 \pmod{W}$ with $v_0$ such that $(\prod_{i=1}^k L_i(v_0), W) = 1$, expanding $w_n$ using (3.3) and swapping the order of summation, we find
\begin{equation}
\sum_{n \equiv v_0 \pmod{W}} 1_p(n + h_{i_1}) \cdots 1_p(n + h_{i_m+1}) w_n(L) \leq \frac{1}{\lambda^2_{(1, \ldots, 1)}} \sum_{e, e' \in E_m} \tilde{\lambda}_e \tilde{\lambda}_{e'} \sum_{d, d' \in D_k} \lambda_d \lambda_{d'} \sum_{n \equiv v_0 \pmod{W}} 1_p(n + h_{i_{m+1}}).
\end{equation}
We note that the sum over \( d \) and \( d' \) is restricted to have \( d_{ij} = d'_{ij} = 1 \), for every \( j = 1,...,m+1 \), otherwise the sum on the left hand side of (4.3) contributes to 0, by the support of our weights.

We have no contribution unless \( (d_{ij}, d_{ij}') = (e_{ij}, e_{ij}') = 1, \forall i \neq j \) and \( (d_{ij}, e_{ij}) = 1, \forall i, j \). In fact, if we had \( p(d_{ij}', e_{ij}') \) or \( p(e_{ij}', e_{ij}) \) or again \( p(d_{ij}, e_{ij}') \), for suitable \( i, j \), then we would find two different indices \( a, b \) for which \( p(n + h_a) \) and \( p(n + h_b) \). By the support of our variables, it implies that \( a \) and \( b \) should be the chosen indices for the residue class \( n \) (mod \( p \)) and therefore they would be equal. Again by the support of \( \lambda_d \) and of \( \lambda_{d'} \), we have that \( (d_{ij}, W) = (e_{ij}', W) = 1 \), for any \( i \) and \( j \). We see that we can combine the congruence conditions by the Chinese remainder theorem, and the inner sum in the second line of (4.3) becomes

\[
\sum_{x<n\leq 2x} \sum_{n\equiv a \pmod{q}} 1_P(n + h_{i_{m+1}}) = \sum_{x<n\leq 2x} \frac{1_P(n + h_{i_{m+1}})}{\varphi(W)\varphi([d, d']')\varphi([e, e']')} + O(E_q),
\]

for some \( a \) coprime with \( q = W \prod \pi_j [d, d] \prod \pi_j [e, e'] \) and where

\[
E_q = \max_{(a,q)=1} \left| \sum_{x<n\leq 2x} \sum_{n\equiv a \pmod{q}} 1_P(n + h_{i_{m+1}}) - \sum_{x<n\leq 2x} \frac{1_P(n + h_{i_{m+1}})}{\varphi(q)} \right|.
\]

Thus, the principal contribution in the estimate of (4.3) comes from

\[
\frac{1}{\lambda_2^{(1,...,1)}} \frac{1}{\varphi(W)} \sum_{x<n\leq 2x} \sum_{e,e'\in E_m} \sum_{d,d'\in D_k} \lambda_e e' \lambda d d' \varphi([e, e']') \sum_{d_j = d_{ij}' = 1, \forall j = 1,...,m+1} \varphi([d, d']').
\]

Here we write \( \Sigma' \) for the summation with all the restrictions stated above. We note that by the Brun–Titchmarsh theorem [6, Theorem 3.9]

\[
\sum_{x<n\leq 2x} 1_P(n + h_{i_{m+1}}) \ll \frac{x}{\log x}.
\]

On the other hand, the error term is

\[
\ll \frac{1}{\lambda_2^{(1,...,1)}} \sum_{e,e'\in E_m} \sum_{d,d'\in D_k} \lambda_e e' \lambda d d' \varphi([e, e']') \sum_{d_j = d_{ij}' = 1, \forall j = 1,...,m+1} \varphi([d, d']').
\]

To work with (4.5) and (4.7), we make some change of variables:

\[
y_{r_0} = \mu(r_0) \varphi(r_0) \sum_{d|r_0} \frac{\lambda_d \lambda_e}{\varphi(d) \varphi(e)},
\]

\[
y_{r_0} = \mu(r_0) \varphi(r_0) \sum_{e|r_0} \lambda e e \varphi(e),
\]

\[
y_{r}^{(m)} = \mu(r) \varphi(r) \sum_{e|r} \frac{\lambda d}{\varphi(d)},
\]

\[
\frac{1}{\lambda_2^{(1,...,1)}} \sum_{e,e'\in E_m} \lambda_e e' \lambda d d' \varphi([e, e']') \sum_{d_j = d_{ij}' = 1, \forall j = 1,...,m+1} \varphi([d, d']') \ll \frac{x}{W(\log x)^{2k}},
\]

if \( x \) is sufficiently large.
Proof. By the Moebius inversion formula we can write

\[ \tilde{\lambda}_{e} = \varphi(e)\mu(e) \sum_{e|r_0 \in E_m} \frac{y_{r_0}}{\varphi(r_0)}. \]

From this we easily deduce that

\[ |\tilde{\lambda}_{e}| \leq \varphi(e) \sum_{e|r_0 \in E_m} \frac{y_{r_0}H^2(r_0)}{\varphi(r_0)} \leq \sum_{k \in E_m} \frac{ykH^2(k)}{\varphi(k)} = \tilde{\lambda}(1, \ldots, 1), \]

if we choose \( y_{r_0} \) to be a positive constant, when \( r_0 \in E_m \). Using the estimate for \( |\lambda_d| \) given by [5] Lemma 8.5 (i) jointly with (4.13), we can estimate (4.17) with

\[ \ll (\log x)^{2k} \sum_{q \leq W^{2/3}, (q, B) = 1} \mu^2(q)\tau_{3k}(q)E_q. \]

Note that \( W^{2}R^{2/3} = W R^{8/3} < x^9 \). We now take \( \theta = 1/3 \). By the Landau–Page theorem (see, for example, [2, Chapter 14]) there is at most one modulus \( q_0 \leq \exp(2c_1\sqrt{\log x}) \), such that there exists a primitive character \( \chi \) modulo \( q_0 \) for which \( L(s, \chi) \) has a real zero larger than \( 1 - c_2(\log x)^{-1/2} \) (for suitable fixed constants \( c_1, c_2 > 0 \)). If this exceptional modulus \( q_0 \) exists, we take \( B = 1 \) to be the largest prime factor of \( q_0 \), and otherwise we take \( B = 1 \). For all \( q \leq \exp(2c_1\sqrt{\log x}) \) with \( q \neq q_0 \) we then have the effective bound (see, for example, [2, Chapter 20])

\[ \varphi(q)^{-1} \sum_{\chi} |\psi(x, \chi)| \ll x \exp(-3c_1\sqrt{\log x}), \]

where the summation is over all primitive \( \chi \) (mod \( q \)) and \( \psi(x, \chi) = \sum_{n \leq x} \chi(n)\Lambda(n) \).

Following a standard proof of the Bombieri–Vinogradov Theorem (see [2, Chapter 28], for example), we have

\[ \sum_{q < x^{1/3}, (q, B) = 1} \sup_{\pi(x, q, a) = 1} |\pi(x, q, a) - \frac{\pi(x)}{\varphi(q)}| \ll x \exp(-c_1\sqrt{\log x}) + x \sum_{q < \exp(2c_1\sqrt{\log x})} \sum_{(q, B) = 1} \sum_{\chi} |\psi'(x, \chi)| \]

for a certain \( \varepsilon > 0 \), which shows that

\[ \sum_{q < x^{1/3}, (q, B) = 1} \mu^2(q)E_q \ll \frac{x}{(\log x)^{100k^2}}, \]

if \( x \) is sufficiently large. Now, using that trivially \( E_q \ll x/\varphi(q) \) and applying Cauchy–Schwarz, we find that (4.14) is

\[ \ll (\log x)^{2k} \left( \sum_{q < x^{1/3}, (q, B) = 1} \mu^2(q)\tau_{3k}(q)E_q \right)^{1/2} \left( \sum_{q < x^{1/3}, (q, B) = 1} \mu^2(q)E_q \right)^{1/2} \]

\[ \ll (\log x)^{2k} \sqrt{\frac{x}{(\log x)^{100k^2}}} \left( \sum_{q < x^{1/3}, (q, B) = 1} \frac{\mu^2(q)\tau_{3k}(q)}{\varphi(q)} \right)^{1/2} \left( \frac{x}{(\log x)^{100k^2}} \right)^{1/2}. \]

This concludes the proof of Lemma 4.1, since

\[ \sum_{q < x^{1/3}, (q, B) = 1} \frac{\mu^2(q)\tau_{3k}(q)}{\varphi(q)} \leq \prod_{p \leq x^{1/3}} \left( 1 + \frac{9k^2}{p - 1} \right) \leq e^{9k^2C} \log^{9k^2} x \ll (\log x)^{10k^2}, \]

for a suitable constant \( C > 0 \), by Mertens’s Theorem [3 Theorem 2.7], if \( x \) is sufficiently large.

We note that, if \( q_0 \) exists it must be square-free apart from a possible factor of at most 4, and must satisfy \( q_0 \gg (\log x)/\log \log x \) \(^2\). In this case we find \( \log \log x \ll B \ll \exp(c_1\sqrt{\log x}) \). Thus, whether or not \( q_0 \) exists, we have \( B/\varphi(B) = 1 + o(1) \).

After the change of variables we are left with the estimate of (1.5). The double sum is equal to

\[ \sum_{e, e' \in E_m} \frac{\varphi(e)\varphi(e')}{\varphi(e, e')} \sum_{d, d' \in D_k} \frac{\varphi(d)\varphi(d')}{\varphi(d, d')} \sum_{d \mid r_0} \frac{y_{r_0}}{\varphi_{\omega}(r_0)} \sum_{d \mid s_0} \frac{y_{s_0}}{\varphi_{\omega}(s_0)} \]

\[ \sum_{d_1 = d_2, \ldots, d_{m+1}} \sum_{e_{ij} \in E_{m+1}} \sum_{r, s \in E_{m+1}} \frac{y_{r, s}}{\varphi_{\omega}(r, s)}. \]
Now we restrict $D_k$ asking that $(d_i, W'_i) = 1$, for all $i \in \{1, \ldots, k\}$ but $i \neq i_1, \ldots, i_{m+1}$, where we put

$$W'_i = \prod_{p | W_i(h_{i_1}-h_i)\ldots(h_{i_{m+1}}-h_i)} p.$$

In fact, in the sum in the first line of (4.3) we have no contribution from the $n + h_i$ not primes. Therefore, if $p | d_i$, for a certain $i$, than the sum defining $w_n$ requires that $p | n + h_i$. However, if we also had $p | h_j - h_i$, for a certain $j$, then this would imply $p | n + h_j$, and this is not possible because by the support of $\lambda_d$ we have $d_i < R$. For the sake of simplicity, we will work with the weaker conditions

$$(d_i, W'_i) = 1, \forall i \in \{1, \ldots, k\};$$

$$(d_i, W'_i) = 1, \forall i \in \{1, \ldots, k\}.$$
Lemma 4.2. We have
\begin{equation}
|y_{r,r_0}| \ll y_{r_0}y_{r_0}^{(m)} \prod_{p \mid r_0} \frac{p}{p-1} \prod_{p \mid r,p \mid W_{i_1}} \frac{p}{p-1} \cdots \prod_{p \mid r,p \mid W_{i_m}} \frac{p}{p-1}.
\end{equation}

Proof. Inserting the Moebius inversion of (4.9) and (4.10) into (4.8), we may rewrite
\begin{equation}
y_{r,r_0} = \mu(r_0) \varphi(r_0) \sum_{r \mid f} \mu(d) \sum_{r_0 \mid f_0} \frac{y_{r}^{(m)}}{\varphi(f)} \varphi(f_0).
\end{equation}
Swapping the order of summation, it becomes
\begin{equation}
\mu(r_0) \varphi(r_0) \sum_{r_0 \mid f_0} \frac{y_{r}^{(m)}}{\varphi(f)} \varphi(f_0) \sum_{r \mid f} \mu(d).
\end{equation}
Using the fact that each $y_{r_0}$ is constant and the function $F$ is non-increasing by (4.8), we have
\begin{equation}
|y_{r,r_0}| \ll \mu^2(r_0) \varphi(r_0) y_{r_0} y_{r_0}^{(m)} \sum_{r_0 \mid f_0 \in E_m} \sum_{p \mid (f,f_0) \not= p(r_0)} \frac{1}{\varphi(f) \varphi(f_0)},
\end{equation}
because the inner sum in (4.25) is 0 unless every prime dividing one of $f$, $f_0$ but not the other is a divisor of $r_0$. In this case the sum is ±1. We let $f_i = r_i f_j g_i$, with $f_i = \frac{f_0}{r_0 r_0 j_0} g_i$ and $g_i \mid r_0, \forall i = 1, ..., k$. Moreover, we let $f_0_j = r_j f_j g_0_j$, with $f_0_j = \frac{f_0}{j_0 r_0 j_0}$ and $g_j \mid r, \forall j = 1, ..., m$. We see the constraint
\begin{equation}
p \mid f / f_0 \Rightarrow p \mid r_0
\end{equation}
means that $f'_0 = \prod_{j=1}^m f_0_j = \prod_{j=1}^m f_j = f'$. Therefore, we can bound the double sum in (4.26) with
\begin{equation}
\begin{align*}
\frac{1}{\varphi(r_0) \varphi(r)} & \sum_{f \in D_k} \tau_m(f') \frac{1}{\varphi(f') \varphi(f)} \\
& \sum_{g \mid r_0, \forall i = 1, ..., k} \frac{1}{\varphi(f)} \\
& \sum_{g_0 \mid E_m} \frac{1}{\varphi(g_0)} \\
& \leq \frac{1}{\varphi(r_0) \varphi(r)} \prod_{p \mid W_B} \left(1 + \frac{m \omega(p)}{(p-1)(p-\omega(p))} \right) \prod_{p \mid r_0} \left(1 + \frac{\omega(p)}{p-\omega(p)} \right) \\
& \prod_{p \mid r,p \mid W_{i_1}} \left(1 + \frac{1}{p-1}\right) \cdots \prod_{p \mid r,p \mid W_{i_m}} \left(1 + \frac{1}{p-1}\right).
\end{align*}
\end{equation}
The first product is $O(1)$ since it is over primes $p > 2k^2$ and $k > m$. Thus, we have
\begin{equation}
|y_{r,r_0}| \ll y_{r_0} y_{r_0}^{(m)} \mu^2(r_0) \frac{\varphi(r_0)}{\varphi(r)} \prod_{p \mid r_0} \left(1 + \frac{\omega(p)}{p-\omega(p)} \right) \prod_{p \mid r,p \mid W_{i_1}} \left(1 + \frac{1}{p-1}\right) \cdots \prod_{p \mid r,p \mid W_{i_m}} \left(1 + \frac{1}{p-1}\right).
\end{equation}
We note that $y_{r_0} y_{r_0}^{(m)}$ is multiplied by
\begin{equation}
\frac{\mu^2(r_0) \prod_{p \mid r_0} (p - \omega(p))}{\prod_{p \mid r_0} (p-1) \prod_{p \mid r,p \mid W_{i_1}} (p-\omega(p))} \prod_{p \mid r_0} \left(1 + \frac{1}{p-1}\right) \cdots \prod_{p \mid r,p \mid W_{i_m}} \left(1 + \frac{1}{p-1}\right)
\end{equation}
because $(r, r_0) = 1$. \hfill \Box

By Lemma 4.2, the last sum in (4.22) may be bounded by
\begin{equation}
\ll \sum_{r_0 \in E_m} \prod_{p \mid r_0} \left(\frac{2}{p + O(k)}\right)^{1 - \frac{1}{p}} \sum_{r_0, r, \forall j = 1, ..., m+1} \prod_{p \mid r} \left(\frac{y_{r}^{(m)}}{p + O(k)} h(r) \right)^2,
\end{equation}
where we have put
\begin{equation}
h(r) = \prod_{p \mid r,p \mid W_{i_1}} \left(1 - \frac{1}{p}\right) \cdots \prod_{p \mid r,p \mid W_{i_m}} \left(1 - \frac{1}{p}\right).
\end{equation}
Now we want to find an estimate on \( y_r^{(m)} \). We provide it in the next lemma.

**Lemma 4.3.** We have

\[
(4.32) \quad y_r^{(m)} \leq \left( \frac{\varphi(r)}{r} \right)^m \left( \frac{WB}{\varphi(WB)} \right)^{k-m-1} \Theta_{WB}(L)(\log R)^m \left( (\log R)H' + T_k(\log \log R)^2H'' \right),
\]

where \( H' \) and \( H'' \) are the integrals in \( dt_1 \cdots dt_{m+1} \) of \( F_1 \) and \( F_2 \) respectively, which are evaluated in \( (\log r_i)/(\log R) \) in every positions \( i \neq i_1, \ldots, i_{m+1} \) and \( t_i \) elsewhere.

**Proof.** Substituting (3.47) in (4.10), we get

\[
(4.33) \quad y_r^{(m)} = \mu(r)\varphi_\omega(r) \sum_{d_j=1, \forall j=1, \ldots, m+1} \frac{\lambda_d}{\varphi(d)} = \mu(r)\varphi_\omega(r) \sum_{d_j=1, \forall j=1, \ldots, m+1} \frac{\lambda_d}{\varphi(d)} \sum_{e \mid r} \frac{y_e}{\varphi_\omega(e)}
\]

\[
= \mu(r)\varphi_\omega(r) \sum_{d_j=1, \forall j=1, \ldots, m+1} \frac{\lambda_d}{\varphi(d)} \sum_{e \mid r} \frac{y_e}{\varphi_\omega(e)} = \mu(r)\varphi_\omega(r) \sum_{d_j=1, \forall j=1, \ldots, m+1} \frac{\lambda_d}{\varphi(d)} \sum_{e \mid r} \frac{y_e}{\varphi_\omega(e)} \prod_{f \mid r} S'_{p\mid f}(e, r).
\]

Here we define \( S'_{p\mid f}(e, r) = 1 \), when \( p \mid e_j \) for \( j \in \{i_1, \ldots, i_{m+1}\} \), otherwise \( p \notmid e_j/r_j \) and we put

\[
S'_{p\mid f}(e, r) = \sum_{f \mid r} \frac{f \mu(f)}{\varphi(f)} = \begin{cases} 1 & \text{if } p \notmid W_j', \\ \frac{1}{p} & \text{if } p \mid W_j'/W_j. \end{cases}
\]

Now, we let \( e_j = r_j s_j t_j \) for each \( j \neq i_1, \ldots, i_{m+1} \), where \( s_j \) is the product of the primes dividing \( e_j/r_j \) but not \( W_j' \) and \( t_j \) is the product of primes dividing both \( e_j/r_j \) and \( W_j'/W_j \). We put \( s_j = t_j = 1 \) for every \( j \in \{1, \ldots, m + 1\} \) and consider the relative \( e_j \) separately. For \( e \in D_k \) the product \( \prod_{f \mid r} S'_{p\mid f}(e, r) \) is then \( \mu(s)/\varphi(s) \). We let \( r' \) the vector \( r \) in which \( r_j \) is replaced with \( e_j \), for every component \( i_1, \ldots, i_{m+1} \). By [8, Lemma 8.2], we obtain the following relation

\[
(4.34) \quad y_e = y_r + O(T_k Y_{r'} \frac{\log(st)}{\log R}),
\]

where

\[
Y_{r'} = \frac{W^k B^k \Theta_{WB}(L)}{\varphi(WB)^k} F_2 \left( \frac{\log r_1'}{\log R}, \ldots, \frac{\log r_m'}{\log R} \right),
\]

with \( F_2 \) as in (3.10). Inserting this in the last line of (4.33), we obtain

\[
(4.35) \quad y_r^{(m)} = \frac{r}{\varphi(r)} \sum_{e_i \cdots e_{i+1}} \frac{y_{e'}}{\varphi_\omega(e)} \sum_{s, t} \frac{\mu(s)}{\varphi_\omega(st)} + O \left( \frac{T_k}{\log R} \varphi(r) \sum_{e_i \cdots e_{i+1}} \frac{Y_{r'} \log(st)}{\varphi(st)} \right),
\]

where now we indicate with \( e \) the product \( e_1 \cdots e_{i+1} \) and the inner sum is over \( s \in D_k, t \in D_k \) subject to \( s_j = t_j = 1 \), for any \( j = 1, \ldots, m + 1 \), with \( (s, t) = (st, re_1 \cdots e_{i+1}) = 1 \) and \( t_j | W_j'/W_j \).

We concentrate first on the main term. We clearly have

\[
\sum_{s, t} \frac{\mu(s)}{\varphi(st)} = \prod_{p \mid s, t} \frac{\mu(p)}{\varphi(p)},
\]

where \( \Sigma' \) means \( s_j = t_j = 1 \), for any \( j \in \{1, \ldots, m + 1\} \), \( (s, t) = 1 \), \( (s, W_i' | re) = (t, W_i | re) = 1 \) and \( t_i | W_i'/W_i \), for every \( i = 1, \ldots, k \). Therefore, we may bound (4.36) with

\[
(4.37) \quad \leq \prod_{p \mid W_i \cdots W_{i+1}} \left( 1 - \frac{\omega(p) - l(p)}{(p-1)(p - \omega(p))} \right) \prod_{p \mid W_i \cdots W_{i+1}} \left( 1 - \frac{\omega(p) - 1}{(p-1)(p - \omega(p))} + \frac{1}{p - \omega(p)} \right),
\]

where \( l(p) = \#\{j \in \{1, \ldots, m + 1\} : p \mid W_j \} \).

In fact, if \( p \mid W_i' \) there are no components of \( s, t \) which can be a multiple of \( p \). If \( p \nmid W_i' \) we have exactly \( \omega(p) \) indices \( i \) for which \( p \mid W_i \). In the case in which \( p \mid W_i \), we have exactly
ω(p) − 1 indices such that p ∤ W′, since we should not consider the chosen index for the residue classes −h1 ≡ ... ≡ −hm+1 (mod p). On the other hand, when p ∤ (W1, ..., Wm+1), among such ω(p) indices we might count those i such that p ∤ W′ for each i. We note that if i ∖ {j} = 1, we find at least ω(p) − l(p) ≥ 0 components of s that can be a multiple of p.

If p ∤ (W1, ..., Wm+1), then no components of t can be a multiple of p, since p ∤ W′/W for each i. On the other hand, if p|(W1, ..., Wm+1) and p ∤ WBr, then exactly one component of t can be a multiple of p, which is the unique i such that p|W′/W. Finally, since (s, t) = 1, no component of s can be a multiple of p if t is it.

We can split (4.37) further to:

\[
\prod_{p|(W_i, ..., W_{m+1})} \left(1 - \frac{\omega(p) - l(p)}{(p-1)(p-\omega(p))}\right) \prod_{p|e} \left(1 - \frac{\omega(p) - l(p)}{(p-1)(p-\omega(p))}\right)^{-1} \prod_{p|W_{Br}} \frac{p}{p-1},
\]

since (e, r(W1, ..., Wm+1)) = 1 and

\[
1 - \frac{\omega(p) - 1}{(p-1)(p-\omega(p))} + \frac{1}{p-\omega(p)} = \frac{p}{p-1}.
\]

Observe that the first product in (4.38) is ≪ 1, since it is in particular over primes p ∤ WB. Inserting (4.38) in (4.35), the main term becomes

\[
\prod_{p|W_{Br}} \frac{p}{p-1},
\]

where

\[
g(e) = \varphi(e) \prod_{p|e} \left(1 - \frac{\omega(p) - l(p)}{(p-1)(p-\omega(p))}\right)
\]

and it is easily to see that g(p) = p + O(k). Substituting (4.37) in place of \(y_{t,e}\), we may write (4.39) as

\[
\prod_{p|W_{Br}} \frac{p}{p-1},
\]

where we indicate with \(\bar{F}\) the function F evaluated in (log r_i)/(log R) in every position i ≠ i1, ..., im+1 and (log e_i)/(log R) in each i position. We estimate the sum in (4.40) by [5, Lemma 8.4], taking the quantity \(\Omega_C\) in the lemma as O(kT^2_k), and we end up with the following bound

\[
\prod_{p|W_{Br}} \frac{p}{p-1},
\]

where \(H\) and \(H'\) are the integrals in dt_i1 · · · dt_im+1 of F and F1 respectively, which are evaluated in (log r_i)/(log R) in every positions i ≠ i1, ..., im+1 and t_i, otherwise. Moreover, we have defined

\[
n(p) = \#\{j \in \{1, ..., m, m+1\} : p \mid rW_i\},
\]

We note that

\[
\prod_{p|W_{Br}} \frac{p}{p-1} \prod_{p|W_{Br}} \frac{p}{p-1} = \frac{\varphi(WB)}{WB} \prod_{p|W_{Br}} \frac{p}{p-1},
\]

since n(p) ≤ m + 1 and the product is over primes \(p > 2k^2\). Furthermore, we manage the first product in (4.41) as

\[
\prod_{p|W_{Br}} \frac{p}{p-1} = \frac{\varphi(WB)}{WB} \prod_{p|W_{Br}} \frac{p}{p-1},
\]

since n(p) ≤ m + 1 and the product is over primes \(p > 2k^2\). Furthermore, we manage the first product in (4.41) as
because \((WB, r) = 1\) and \(WB | (W_1, \ldots, W_{m+1})\), and the second product in (4.41) as

\[
\prod_{p | r (W_1, \ldots, W_{m+1})} \left( 1 - \frac{1}{p} \right)^{m+1} = \prod_{p | r} \left( 1 - \frac{1}{p} \right)^{m+1} \prod_{p \not| r} \left( 1 - \frac{1}{p} \right)^{m+1}
\]

\[
= \left( \frac{\varphi(r)}{r} \right)^{m+1} \prod_{p | (W_1, \ldots, W_{m+1})} \left( 1 - \frac{1}{p} \right)^{m+1} .
\]

In particular, we observe that

\[
\frac{\varphi(WB)}{WB} \prod_{p | (W_1, \ldots, W_{m+1})} \left( 1 - \frac{1}{p} \right)^{m+1} \frac{p}{p-1} \ll \left( \frac{\varphi(WB)}{WB} \right)^{m+1} .
\]

Collecting our estimates, we deduce that the main term in (4.35) is

(4.43)

\[
\ll \frac{r}{\varphi(r)} \left( \frac{\varphi(r)}{r} \right)^{m+1} \left( \frac{WB}{\varphi(WB)} \right)^k \left( \frac{\varphi(WB)}{WB} \right)^{m+1} \mathcal{E}_{WB}(\mathcal{L})(\log R)^{m+1} \left( H + O \left( k^2 T_k^2 \log \log R \right) H' \right)
\]

\[
\ll \left( \frac{\varphi(r)}{r} \right)^m \left( \frac{WB}{\varphi(WB)} \right)^{k-m-1} \mathcal{E}_{WB}(\mathcal{L})(\log R)^{m+1} H',
\]

because clearly \(F(t_1, \ldots, t_k) \leq F_s(t_1, \ldots, t_k)\), for every \(k\)-tuples \((t_1, \ldots, t_k)\).

We now return to the error term in (4.35). We use \(\log(st) \ll \sqrt{s}(1 + \log t)\) and we drop the requirement \((s, t) = 1\). In this way, the sum over \(s\) factorizes as an Euler product and we get

\[
\sum_{s \in \mathcal{D}_k} \frac{\sqrt{s}}{\varphi(s) \varphi_\omega(s)} \leq \prod_{p > 2k^2} \left( 1 + \frac{\sqrt{p}}{(p-1)(p-\omega(p))} \right) \ll 1.
\]

We are summing over square-free \(t\) with \((t, WBre) = 1\) and

\[
t | \Delta = \prod_{i = 1, \ldots, k, i \neq 1, \ldots, m+1} (h_{i_1} - h_i, \ldots, h_{i_m+1} - h_i).
\]

For every such \(t\) there is at most one possible \(t\). In fact, two cases may happen. If \(p | \Delta\) and \(p | (W_1, \ldots, W_{m+1})\), there exists a unique index \(i \in \{1, \ldots, k\} \setminus \{i_1, \ldots, i_{m+1}\}\) for which \(p | W_i' / W_i\). It was the chosen index for the residue classes \(-h_i \equiv \cdots \equiv -h_{i_{m+1}} \pmod{p}\). When this holds for every \(p\) dividing \(t\), it gives rise to a unique vector \(t\). If otherwise \(p | \Delta\) and \(p \nmid (W_1, \ldots, W_{m+1})\), there exists an index \(j \in \{1, \ldots, m + 1\}\) such that \(j\) was the chosen index for the residue class \(-h_{i_j} \pmod{p}\). In this case there is not any vector \(t\). Thus, the sum over \(t\) contributes at most

\[
\sum_{t \in \mathcal{D}_k : t | \Delta} \frac{1 + \sum_{p | t} \log p}{\varphi_\omega(t)} \ll \left( 1 + \sum_{p > 2k^2, p | \Delta} \frac{\log p}{p} \right) \prod_{p > 2k^2, p | \Delta} \left( 1 + \frac{1}{\varphi_\omega(p)} \right) \ll (\log \log \Delta)^2 \ll (\log \log R)^2 .
\]

Therefore, the error term in (4.35) becomes

(4.44)

\[
\ll \frac{T_k (\log \log R)^2}{\log R} \frac{r}{\varphi(r)} \sum_{e_{i_1}, \ldots, e_{i_{m+1}}} \frac{Y_{e'}}{\varphi_\omega(e')},
\]

relaxing the constraint \((e_{i_j}, rW_{i_j}) = 1\) to \((e_{i_j}, WBre) = 1\). Substituting the definition of \(Y_{e'}\), we should estimate

(4.45)

\[
\ll \frac{T_k (\log \log R)^2}{\log R} \frac{r}{\varphi(r)} \frac{WB^k \mathcal{E}_{WB}(\mathcal{L})}{\varphi(WB)^k} \sum_{e_{i_1}, \ldots, e_{i_{m+1}}} \frac{\tilde{F}_2}{\varphi_\omega(e')},
\]
where we have indicated with $\tilde{F}_2$ the function $F_2$ evaluated in $(\log r_i)/(\log R)$ in every position $i \neq i_1, \ldots, i_{m+1}$ and $(\log e_i)/(\log R)$ in each $i_j$ position. Now, since we can write

$$F_2(t_1, \ldots, t_k) = \sum_{j=1}^{k} G_j(t_j) \prod_{\substack{1 \leq i \leq k \backslash i \neq j}} G_i(t_i),$$

for every $k$-tuples $(t_1, \ldots, t_k)$, for certain functions $G_1, \ldots, G_k$, we apply a simple variation of the Lemma 8.4 in \[5\], in which we allow for different functions $G_i$ instead of a single one, but verifying the same conditions present in the Lemma. In this way, we can bound (4.48) with

$$(4.49) \quad \ll \frac{T_k(\log R)^2}{\log R} \frac{r}{\varphi'(r)} (WB)^k \mathcal{S}_{WB}(L) \left( \varphi(WB) \right)^{m+1} \prod_{p \mid WB} \left( 1 + \frac{m+1}{p - \omega(p)} \right) \left( 1 - \frac{1}{p} \right)^{m+1} (\log R)^{m+1} H'',$$

where $H''$ is the integral in $dt_{i_1} \cdots dt_{i_{m+1}}$ of $F_2$, which is evaluated in $(\log r_i)/(\log R)$ in every positions $i \neq 1, \ldots, i_{m+1}$ and $t_{i_j}$ otherwise. Clearly, (4.49) is equal to

$$(4.50) \quad T_k(\log R)^2 \left( \log R \right)^m \left( \frac{\varphi(r)}{r} \right)^m \left( \frac{WB}{\varphi(WB)} \right)^{k-m-1} \mathcal{S}_{WB}(L)H'',$$

because $(WB,r) = 1$ and the product is $\ll 1$, since it is over primes $p > 2k^2$. This concludes the estimate of the error term in (4.32) and the proof of the Lemma 3.4

Now we return to (4.30) and we firstly estimate the second sum which, after inserting (4.32), becomes:

$$(4.51) \quad \ll \left( \frac{WB}{\varphi(WB)} \right)^{2(k-m-1)} \mathcal{S}_{WB}(L)(\log R)^{2(m+1)} \sum_{r_j=1, \forall j=1, \ldots, m+1}^{D_k} \frac{(H')^2}{\prod_{p \mid r}(p + O(k))} + \left( \frac{WB}{\varphi(WB)} \right)^{2(k-m-1)} \mathcal{S}_{WB}(L)(\log R)^{2m} T_k(\log R)^4 \sum_{r_j=1, \forall j=1, \ldots, m+1}^{D_k} \frac{(H')^2}{\prod_{p \mid r}(p + O(k))},$$

because

$$\frac{1}{h(r)^2} \left( \frac{\varphi(r)}{r} \right)^{2m} = \left( \frac{\varphi(r)}{r} \right)^{2m} \prod_{p \mid r} \left( 1 - \frac{p}{p-1} \right)^2 \prod_{p \mid WB} \left( 1 - \frac{p}{p-1} \right)^{k-m-1} \leq 1.$$

We start working with the first sum in (4.51). By [5] Lemma 8.4 we obtain the following bound

$$(4.52) \quad \ll L_k(F_1)(\log R)^{k-m-1} \prod_{p \mid WB} \left( 1 + \frac{\omega(p) - m - 1}{p + O(k)} \right) \left( 1 - \frac{1}{p} \right)^{k-m-1},$$

and we note that the two products over the primes are respectively $\ll \mathcal{S}_{WB}(L)$ and $\left( \frac{\varphi(WB)}{WB} \right)^{k-m-1}$.

Here, we define

$$L_k(F_1) = \int_{\{t_i \geq 0: i \neq \{i_1, \ldots, i_{m+1}\}\}} \int \cdots \int F_1(t_1, \ldots, t_k) dt_{i_1} \cdots dt_{i_{m+1}} \quad dt_{i_1} \cdots dt_{i_{m+1}},$$

where $dt_{i_1} \cdots dt_{i_{m+1}}$ means that we are differentiating only by the $dt_i$’s for $i \neq i_1, \ldots, i_{m+1}$. By the symmetry of $F_1$ with respect to each variable, we may rewrite $L_k(F_1)$ in the following simpler form

$$L_k(F_1) = \int_0^\infty \cdots \int_0^\infty \left( \int_0^\infty \cdots \int_0^\infty F_1(t_1, \ldots, t_k) dt_{i_1} \cdots dt_{i_{m+1}} \right)^2 dt_{i_{m+1}} \cdots dt_{i_{m+1}}.$$
Therefore, the second sum in (4.30) is

\[(4.51) \quad \ll (\log R)^{k+m+1} \mathcal{C}_{WB}(L) \left( \frac{WB}{\varphi(WB)} \right)^{k-m-1} \left( L_k(F_1) + \frac{T^2_k(\log \log R)^4}{(\log R)^2} L_k(F_2) \right).\]

Arguing in the same way as in [5, Lemma 8.6], we find the following estimates

\[L_k(F_1) \ll \frac{(\log k)^{m+1}}{k^{m+1}} I_k(F),\]

and

\[L_k(F_2) \ll k^2 L_k(F_1) \ll \frac{(\log k)^{m+1}}{k^{m-1}} I_k(F).\]

In conclusion, we obtain

\[(4.52) \quad \sum_{r \in \mathcal{D}_k} \frac{(y_r^{(m)})^2}{\prod_{p \mid r} (p + O(k)) h(r)^2} \ll \frac{(\log k)^{m+1}}{k^{m+1}} I_k(F) (\log R)^{k+m+1} \mathcal{C}_{WB}(L) \left( \frac{WB}{\varphi(WB)} \right)^{k-m-1}.\]

Before going on, let us explicit the constant \(y_{r_0}\). To this aim we compute the sum \(\sum_{r_0 \in \mathcal{E}_m} \mu^2(r_0)/\varphi(r_0)\).

By a trivial application of [5, Lemma 8.4], we get

\[(4.53) \quad \sum_{r_0 \in \mathcal{E}_m} \frac{\mu^2(r_0)}{\varphi(r_0)} \geq \sum_{(r_{ij}, W_{ij})=1, \forall j} \frac{\mu^2(r_{i0}, \ldots, r_{0m})}{\varphi(r_{i0}, \ldots, r_{0m})} \prod_{j=1}^m \left[ 1 + \frac{n(p)}{p-1} \right] \left( 1 - \frac{1}{p} \right)^m,
\]

where \(n(p) = \# \{ j \in \{1, \ldots, m\} : p \mid W_{ij} \}\). We indicate the last product in (4.53) as \(P(W_{i1}, \ldots, W_{im})\) and we note that it converges, since for all the primes \(p \mid \prod_{j \neq i} (h_j - h_i)\) we have \(p \mid W_i\), for every \(i = 1, \ldots, k\). Anyway, it seems difficult to prove that \(P(W_{i1}, \ldots, W_{im}) \gg 1\), even if we believe this is the case.

Now we can choose \(y_{r_0}\) as

\[y_{r_0} = (3m)^m \left( \frac{(W_{i1}, \ldots, W_{im})}{\varphi(W_{i1}, \ldots, W_{im})} \right)^m P(W_{i1}, \ldots, W_{im})^{-1},\]

so that we immediately find \(\tilde{\lambda}_{(1, \ldots, 1)} \gg (\log R)^m\). We call the value of \(y_{r_0}\) as \(\bar{y}\). We note that for small values of \(m\) the expression of \(\bar{y}\) is easy and computable. For example, one can prove that

\[\bar{y} \ll \begin{cases} \frac{W_{i1}}{\varphi(W_{i1})} & \text{if } m = 1; \\ \frac{\varphi(W_{i1}, W_{i2})}{\varphi(W_{i1}) \varphi(W_{i2})} & \text{if } m = 2. \end{cases}\]

Regarding the first sum in (4.30), by \(m\) applications of [5, Lemma 8.4], we find:

\[(4.54) \quad \sum_{r_0 \in \mathcal{E}_m} \frac{y_{r_0}^2 \mu^2(r_0)}{\prod_{p \mid r_0} (p + O(k)) (1 - \frac{1}{p})^2} \leq \bar{y}^2 \prod_{j=1}^m \sum_{(r_{ij}, W_{ij})=1} \frac{\mu^2(r_{0j})}{\varphi(r_{0j})} \prod_{p \mid r_{0j}} (p + O(k)) \left( 1 - \frac{1}{p} \right).\]

We note that each product over primes, in the last line of (4.54), is bounded. Moreover, we can use the following estimate

\[\frac{\varphi(W_{i1}) \ldots \varphi(W_{im})}{W_{i1} \ldots W_{im}} \leq \left( \frac{\varphi(WB)}{WB} \right)^m.\]

Thus, (4.54) reduces to

\[(4.55) \quad \leq C_1^m \bar{y}^2 (\log R)^m \left( \frac{\varphi(WB)}{WB} \right)^m.\]
for a suitable constant $C_1 > 0$. Using the estimates (4.52) and (4.55) we find
\[
\sum_{r \in \mathcal{D}_k} \frac{y_{r,r_0}^2}{\prod_{p|r_0} (p + O(k))} \leq C_2 y^2 \left( \frac{\log k}{k} \right)^{m+1} I_k(F)(\log R)^{k+2m+1} \mathcal{E}_{WB}(\ell) \left( \frac{WB}{\varphi(WB)} \right)^{k-2m-1},
\]
for a certain $C_2 > 0$. Collecting the results, we get that (4.56) can be estimated by
\[
\leq C_3 y^2 \left( \frac{\log k}{k} \right)^{m+1} I_k(F)(\log x)^k \mathcal{E}_{WB}(\ell) \left( \frac{WB}{\varphi(WB)} \right)^{k-2m-1},
\]
for a suitable constant $C_3 > 0$. This is also the final bound of (4.3). In fact, by Lemma 4.1 and by using [3] Lemma 8.1 (i) and [5] Lemma 8.6 to take into consideration the size of $\mathcal{E}_{WB}(\ell)$ and of $I_k(F)$, we easily see that the error term coming from (4.7) is negligible compared to (4.57).

Finally, we recall that we have to sum the bound (4.57) over all the residue classes $r_0 \pmod{W}$, which is equivalent to multiply it by
\[
\varphi(W) = W \prod_{p|W} \left( 1 - \frac{\omega(p)}{p} \right) \left( 1 - \frac{1}{p} \right)^{-k} \left( \frac{\varphi(W)}{W} \right)^k = W \mathcal{E}_{WB}(\ell)^{-1} \mathcal{E}_B(\ell) \left( \frac{\varphi(W)}{W} \right)^k.
\]
In this way, we can estimate our main sum in (1.4) with
\[
\leq C_4 y^2 \left( \frac{\log k}{k} \right)^{m+1} I_k(F) x(\log x)^k \left( \frac{\varphi(W)}{W} \right)^{k-1} \left( \frac{WB}{\varphi(WB)} \right)^{k-2m-1} \mathcal{E}_B(\ell),
\]
for a certain $C_4 > 0$. When $h_1, \ldots, h_k \leq k^2$ (in fact, it suffices that for every prime $p | WB$ happens that $p \nmid (h_j - h_i)$, for all $i \neq j$) then all the factors $W_i = WB$ and consequently $\bar{y}$ reduces to
\[
\bar{y} \ll (3m)^m \left( \frac{WB}{\varphi(WB)} \right)^m.
\]
This leads to a simplification of the expression (4.58) of the form
\[
\leq C^m m^m \left( \frac{\log k}{k} \right)^{m+1} I_k(F)(\log x)^k \left( \frac{B}{\varphi(B)} \right)^{k-1} \mathcal{E}_B(\ell),
\]
for a suitable $C > 0$, since $(W, B) = 1$. The proofs of Theorem 1.1 and Corollary 1.2 are completed.

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