Modified Noether theorem and arrow of time in quantum mechanics

V V Asadov\textsuperscript{1}, O V Kechkin\textsuperscript{1,2}
\textsuperscript{1} Neur OK–III, Lomonosov Moscow State University, Vorob’jovy Gory, 119899 Moscow, Russia
\textsuperscript{2} Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Vorob’jovy Gory, 119899 Moscow, Russia
E-mail: asadov@neurok.com, kechkin@gmail.com

Abstract. Relativistic quantum mechanics is presented with modified Noether theorem. It was shown that Noether charges are related with thermodynamic potentials in such scheme. Broken symmetries generated by thermodynamic mode lead to gravity appearance as effective quantum field.

1. Introduction
Symmetries play a very important role in modern theoretical and mathematical physics. Of course, it is possible to study abstract dynamical systems, which do not possess any symmetry. However, from phenomenology we know that all dynamical systems which describe the real world are characterized by some group of explicit and hidden symmetries definitely \cite{1}–\cite{4}. It can be said, that all realistic field theories, as well as all string theory models are some specific realizations of corresponding symmetry groups. Moreover, quantization procedure of such kind of dynamical systems is understood as some non-trivial map of the starting classical theory which conserves its basic symmetry properties. Actually, in the standard approach to quantization all anomalies (the results of original symmetry breaking) must vanish identically if the correct quantization procedure was really found. Thus, symmetries are considered as native property of the physical world; they characterize the realistic dynamical systems and help us establish a correct formalism for dynamics description. This explains the role of symmetries in scientific activity from the theoretical physics point of view.

In mathematical physics context, symmetries of dynamical equations for a given physical system are important because they can be used for integration of these equations – i.e. for the solution of dynamical problem. The corresponding approach to the solution of ordinary and partial differential equations was introduced by Sofus Lee. Now this activity includes the study of the so-called integrable systems – the systems, which can be understood as the most symmetric ones in the class of dynamical systems of the given type. More precise, integrable systems are characterized by their symmetry groups completely: knowledge of symmetry groups allows one to solve corresponding dynamical equations for arbitrary initial data in case of integrable systems. Of course, this possibility is considered as the most optimistic for these systems with highly non-linear dynamics \cite{5}–\cite{9}.

In our modification of quantum and classical dynamics, symmetries also play a crucial role. We develop our approach using some hidden symmetry of original standard dynamical system
in such way, when the final system also possesses this hidden symmetry, but in fact satisfies the
dynamics of a more general type. Namely, we start from the standard reversible dynamics and,
finally, obtain the dynamics with an arrow of time, – i.e. the irreversible dynamical system,
which can be considered as a thermodynamical modification of the original one. Our main goal
is the unification of the fundamental laws of quantum and classical dynamics with the Second
Law of thermodynamics – one of the main goals of all theoretical and mathematical physics of
our days [10]–[13]. Also the problem of violation of time reversal invariance was considered in
publications [14]–[17].

2. Symmetries and conservation laws in standard quantum and classical dynamics

In this section we give a review of well-known facts related to the symmetries in the standard
classical and quantum dynamics. Here and below we restrict our consideration by the class of
dynamical systems which possess Hamilton’s formalism in their description. More precisely, our
programm is based on some ‘minimal modification’ of such kind of dynamical systems to the form
which is not completely the standard Hamiltonian one. For example, we lead to classical limit
of the modified quantum systems which satisfies the improved system of Hamilton’s equations.
The modification found, generates irreversible effects and gives base for the thermodynamical
formalism.

2.1. Symmetry definition

Let us consider a dynamical system with coordinates \( q^a \) and Hamiltonian \( H \); the dynamical
temporal variable (‘Hamilton’s time’) we denote as \( t \). For the quantum variant of the theory we
will use the wave-function \( \Psi \) in the Schrödinger’s picture; for the corresponding classical limit of
this system in its Hamilton’s form we must add to the set of canonical variables the momentums
\( p^a \). We will understand the symmetry of system of the given type as a transformation in the
unified space of independent and dependent dynamical variables (including all derivatives of
independent variables with respect to dependent ones) which preserve dynamical equations
descrribing this system. The unified space mentioned, consists of \( t, q^a, \Psi, \dot{\Psi}, t, \dot{\Psi}, q^a \) and
derivatives of the wave-function \( \Psi \) of the highest order with respect to \( t \) and \( q^a \) in the quantum
case. For the classical variant of the theory, this space includes \( t, q^a, p^a, q^a, p^a, q^a, p^a \) and also the
derivatives of \( q^a \) and \( p^a \), with respect to \( t \) and \( q^a \), again. Of course, we consider \( t \) and \( q^a \) as
independent dynamical variables. Here \( q^a \) are understood as coordinates of the corresponding
representation of the quantum realization of the theory. A solution of the corresponding
dynamical problem is the set of functions \( q^a = q^a(t) \) and \( p^a = p^a(t) \) in the classical case,
and the wave-function \( \Psi = \Psi(t, q^a) \) in the quantum variant of dynamics.

It is clear that symmetries can be defined alternatively, as such transformations of the
discussed type that map an arbitrary solution of dynamical equations describing the system to a
Corresponding solution of the same system. It is easy to see that the symmetry transformations
form a group with respect to the operation of their successive application. This group includes
the discrete and continuous maps; all these symmetries are important for the study of dynamical
properties of the system under consideration.

2.2. Discrete symmetries

In theoretical physics, the most important discrete symmetries include C (charge), P (spatial
parity) and T (time) inversions. The first of them means the change of sign of all the charges (not
only electric charges) to the opposite ones. The second transformation maps the left orientation
to the right one, and vice versa. The third symmetry is understood as the change of given ‘time
direction’ to the opposite direction (the transformation which includes the map \( t \to -t \)) without
any consequences for the system dynamics. Also, it can be said that T-transformation maps the
initial and final data of the dynamics to each other. For further discussion, we call the situation
where the broken T-symmetry is present in the dynamics of the system the ‘arrow of time’. It is important to note that for the wide class of theories, the rigid symmetry with respect to the combined CPT transformation (the CPT-theorem) was established. In our modification of quantum and classical dynamics presented below, an appearance of the arrow of time without the destruction of fundamental principles of quantum and the corresponding classical theory seems the most important result.

2.3. Noether symmetries
Continuous transformations are related to the transformation parameters that vary continuously on some subspace of the corresponding (multidimensional) parameter set. The most famous and well-known continuous transformations are the contact and Noether symmetries. They are important especially in the study of specific realistic systems that arise in the framework of the conventional theoretical physics. For each Noether symmetry, there exists a corresponding Noether current, which yields the null-divergence condition.

2.4. Noether charges and their conservation
It is a well known fact that any Noether symmetry is accompanied by a corresponding Noether charge – the integral quantity (over the coordinates $q^a$), which does not depend on the Hamilton’s parameter of evolution $t$. This conservation law follows from the null-divergent condition mentioned; this is known as the famous Noether theorem. For our purposes it is important to stress that Noether charges are numbers (not functions of some variables), which can be expressed in terms of initial conditions of the problem. Also, it must be noted that independent Noether symmetries generate the independent sets of Noether charges in the standard quantum or classical system.

In contrast to this conventional situation, in the modified dynamics presented below Noether symmetries do not lead to any conservation laws. Using other words, it can be said that in modified quantum and classical dynamics, developed below, Noether charges become some functions on the set of extra parameters of the theory. If one considers some dependence of extra parameters on the evolutionary parameter $t$, one obtains possibility to study corresponding dynamics of the Noether charges on this Hamilton’s time. In fact, we relate such kind of dependence with thermodynamical mode fixing. Finally, for the general thermodynamic mode the relation between Noether symmetries and conservation laws becomes broken. Actually, in our form of modified dynamics, Noether charges are some specific functions of time, not constant quantities. However, in a special case of thermodynamic mode with, for example, conserved energy, one has the energy integral conservation up to construction. To clarify and explain all these unusual properties of modified quantum and classical systems let us consider them in detail below.

3. Modified quantum dynamics, time dependence of Noether charges and arrow of time
In this section, a general theoretical scheme for the modified quantum dynamics is presented and discussed. This scheme can be considered as a ‘totally complexified’ standard quantum theory. Performing that modification allows one to unify quantum and thermodynamical principles in a consistent form. The thermodynamic behavior of the system is related to dependence of its quantum evolution on temperature regime, which appears in the framework of the modification performed. The most interesting prediction of our extension of the standard quantum theory is the presence of the arrow of time for quantum evolution with physically well-motivated temperature regimes. Such kind of time irreversibility can be used for microscopic explanation of the Second Law of thermodynamics. We give some basics for the unified quantum and thermodynamical principles and show that the entropy production actually takes place in the
modified dynamics of the type presented. The results of this section are based on the publication [10].

3.1. Modified quantum dynamics

We base modified quantum theory on the standard Schrödinger’s equation and on analytic condition

\[ i\hbar \partial_\tau \Psi = \hat{H} \Psi, \quad \partial_\tau \Psi = 0, \]

– i.e. on the system of two linear differential dynamical relations with respect to some evolutionary parameter \( \tau \). Our operator of evolution is considered as a non-Hermitian on \( \hat{H} \neq \hat{H}^+ \) in a general case. We preserve the name ‘Hamiltonian’ for the operator \( \hat{H} \) because it is conjugated to the evolutionary parameter \( \tau \) through the Schrödinger’s equation in the conventional form. It can be said that the theory under consideration is ‘totally complexified’ in view of the general complex nature of the wave-function \( \Psi \), evolutionary parameter \( \tau \) and operator of evolution \( \hat{H} \).

Then, the theory is considered as totally analytical: in addition to the second relation in Eq. (1), we restrict the system by the conditions of analyticity

\[ \partial_\tau \hat{H} = \partial_\tau H = 0 \]

with respect to its Hamiltonian. Let us now give the interpretation for all quantities in the theory. For the extended objects introduced above we use the following decompositions:

\[ \tau = t - \frac{i\hbar}{2} \beta, \quad \hat{H} = \hat{E} - \frac{i\hbar}{2} \hat{\Gamma}. \]

Here, parameters \( t \) and \( \beta \) are understood as ‘usual’ time and inverse temperature, respectively (the temperature is measured in energy units). Then, for the Hermitian operators \( \hat{E} \) and \( \hat{\Gamma} \) we use the interpretation of operators of energy and decay. In the special case of trivial decay operator (when \( \hat{\Gamma} = 0 \)) and infinite temperature (when \( \beta = 0 \)) one deals with the standard quantum dynamics. In this sense the theory under discussion includes the conventional quantum theory, i.e. the correspondence principle is satisfied.

Using this principle, it is possible to identify the energy operator \( \hat{E} \) with the Hermitian Hamiltonian of the standard form of quantum system under modification. Then, the proposed modification, is related to the complex extension of the evolutionary parameter (i.e. to the map \( t \to \tau \)), and to the introduction of the decay operator \( \hat{\Gamma} \) in accordance with Eq. (3). Of course, both operators \( \hat{E} \) and \( \hat{\Gamma} \) must be taken as independent on \( t \) and \( \beta \): this follows from the generalized stationary condition (2). It is clear that the presented theoretical scheme must be specified with respect to the choice of the decay operator \( \hat{\Gamma} \). We propose to use, for this specification, the commutativity condition

\[ [\hat{E}, \hat{\Gamma}] = 0, \]

which simplifies the extremely analytical structure of this theory and also has an evident physical basis.

Namely, let us assume that for the general element \( \Psi \) of solution space of dynamical system (1)–(4) one has:

\[ \Psi = \sum_n C_n \Psi_n, \]
where $C_n = \text{const}$ are the arbitrary complex constants and the eigen-vector $\Psi_n$ satisfies the eigen-problem

$$\hat{H}\Psi_n = \left(E_n - \frac{i\hbar}{2}\Gamma_n\right)\Psi_n.$$  \hfill (6)

Here $E_n$ and $\Gamma_n$ are the eigen-values for operators $\hat{E}$ and $\hat{\Gamma}$, respectively; the condition (4) guarantees the correctness of Eqs. (5)–(6). Then, using the standard quantum formalism (for the non-normalized states), for probability $P_n$ to find the state $\Psi$, given by Eq. (5), in the eigen-state $\Psi_n$, one obtains:

$$P_n = \frac{w_n}{Z},$$  \hfill (7)

where $Z = \sum w_n$, and

$$w_n = r_n \exp \left[-(E_n\beta + \Gamma_n t)\right]$$  \hfill (8)

with $r_n = |C_n|^2$. It is seen that the quantity $Z$ is the norm of the state $\Psi$, the constant parameter $r_n$ is defined by initial data of the problem and plays the role of a weight coefficient in this probability distribution. Also, it is clear that the function $Z$ can be considered as the statistical sum for this system. In the special case of $\beta = 0$, one immediately interprets the dynamical parameter $\beta$ as inverse temperature (in the energy units) if the eigen-values $E_n$ are identified with conventional energy levels of the system.

For interpretation of the set of constants $\Gamma_n$, let us consider a special state with only two nontrivial probabilities $P_0$ and $P_N$, where $P_N$ is the small perturbation over the ‘ground’ probability $P_0$:

$$P_0 + P_N = 1, \quad P_0 \geq P_N.$$  \hfill (9)

Using Eqs. (7)–(8), it is not difficult to show that for such distribution of probabilities

$$P_N \sim \exp (-\Gamma_N t)$$  \hfill (10)

in case of $\beta = \text{const}$ (i.e., for the isothermal evolution). Thus, the parameter $t_N = 1/\Gamma_N$ plays the role of time of life of the state $\Psi$ in the eigen-state $\Psi_N$ in the conventional sense of physics of resonances. Of course, for the general solution (5) of the problem, dynamics of probabilities $P_n$ is more complicated. Let us note that condition (4) guarantees the possibility of simultaneous measurement of the energy levels and times of life for the corresponding states of the system. It is possible to say that in the theory under consideration, times of life are not less important parameters than the physical energies; exactly such situation takes place in the resonance phenomena. Our hypothesis is related with this evident fact: we suppose that the complex analytic extension (1)–(4) of the standard quantum theory describes adequately the world of nonstable particles at least. Below we present the relativistic models, based on the general scheme (1)–(4) and on the explicitly realized conception of the multi-universe world.

3.2. Quantum correlations in modified quantum dynamics

Correlations between different physical variables play a crucial role in the theory under discussion. Here, we give all necessary definitions and discuss the physical sense of the introduced quantities.

First of all, for expectation value of operator $\hat{A}$ in the state $|\Psi>$ we use the conventional Dirac’s formula

$$\hat{A} = \frac{<\Psi|\hat{A}|\Psi>}{<\Psi|\Psi>}. \hfill (11)$$
Of course, the relation

$$Z = \langle \Psi | \Psi \rangle$$

(12)

holds, and some explicit rule for the calculation of all matrix elements and scalar products in
the theory is supposed. We define correlation $A \circ B$ of the quantum observables $\hat{A}$ and $\hat{B}$ as a
natural binary generalization

$$A \circ B = \overline{AB} + \overline{BA} - 2\overline{\hat{A}\hat{B}}$$

(13)

of conventional formula $D_{A}^{2} = \overline{A^{2}} - \overline{A}^{2}$ for dispersion of quantity $\hat{A}$. It is seen that from Eq. (13) it follows the relation

$$A \circ A = 2D_{A}^{2}$$

(14)
in the special case of $\hat{B} = \hat{A}$. Let us stress that we preserve the definition of dispersion, so

that if operators $\hat{A}$ and $\hat{B}$ satisfy the commutation relation $[\hat{A}, \hat{B}] = i\hbar$, than the uncertainty relationship $D_{A}^{2}D_{B}^{2} \geq \hbar^{2}/4$ holds. In the relativistic realization of the theory under consideration

we will study the coherent states; for these states, which are coherent with respect to the

observables $\hat{A}$ and $\hat{B}$ one has the equality

$$D_{A}^{2}D_{B}^{2} = \frac{\hbar^{2}}{4}.$$  

(15)

### 3.3. Symmetries in modified quantum dynamics

Symmetries also play extremely important role in the modified quantum theory (1)--(4). Let us

now formalize a symmetry conception which was given in the section 2. Following the standard

mathematical definition, we name operator $\hat{S}$ the symmetry operator if transformed quantum

state $\hat{S}\Psi$ satisfies the dynamical equations (1) together with the original quantum state $\Psi$. It is

easy to see that any symmetry operator $\hat{S}$ must commute with the Hamiltonian of the system:

$$[\hat{S}, \hat{H}] = 0;$$ 

(16)

this follows from linearity of Eq. (1). From Eq. (4) it follows immediately that the decay

operator $\hat{\Gamma}$ is one of the symmetry operators of the system. From the pure mathematical point

of view, the formal scheme (1)--(4) is defined completely as an extension of the standard quantum

theory if one takes the decay operator $\hat{\Gamma}$ in the form of some function of the energy operator $\hat{E}$, i.

e. $\hat{\Gamma} = f(\hat{E})$. In this special case the relation (4) becomes an equality. However, for relativistic

realizations of the modified quantum dynamics under consideration, such kind of extensions of the

standard theory seems not realistic. It is possible to show that for the nonlinear function

$f(\hat{E})$ the resulting modified field theory contains (infinite) set of terms with highest derivatives

with respect to coordinates of the physical space-time. Below we develop an alternative approach
to the decay operator construction. It is based on the use of the multi-universe conception in

quantum theory and on the possibility of its explicit realization in the framework of the general

scheme (1)--(4).

### 3.4. Thermodynamic modes and quantum observer

In the previous section, isothermal evolution appeared – i.e. the special regime of consideration

of the theory where quantities are specified by the restriction

$$\beta = \text{const.}$$

(17)
Of course, this restriction does not follow from the general theoretical scheme (1)–(4) for the modified quantum dynamics, because in this scheme the dynamical parameters \( t \) and \( \beta \) are completely independent. Thus, the restriction (17) must be understood as some additional condition, which actually specifies all physical quantities calculated on the base of the solution \( \Psi \) of Eq. (1). Another natural extra condition imposed on the quantum evolution is related with the conservation of expectation value of the system energy:

\[
\bar{E} = \text{const.} \tag{18}
\]

This special mode we name as an ‘adiabatic regime’; it is clear that this mode corresponds to some special relation between the dynamical parameters \( t \) and \( \beta \).

Starting with the examples discussed above, let us consider some observable which is described by the operator \( \hat{A} \). Let us consider the expectation value of this observable, i.e. the quantity \( \bar{A} \). It is clear that \( \hat{A} \) is a specific function of the dynamical parameters \( t \) and \( \beta \), \( \hat{A} = \hat{A}(t, \beta) \). Actually, all the coordinates \( q^a \) of the quantum representation were integrated out through the process of the expectation value construction. Thus, the special evolution mode with \( A(t, \beta) = \text{const} \) defines (in not explicit form) the ‘temperature regime’ (or the ‘temperature curve’)

\[
\beta = \beta(t) \tag{19}
\]

such that the relation \( A(t, \beta(t)) = \text{const} \) becomes the equality for any given value of the Hamilton’s time \( t \). Finally, we postulate that the specified temperature regime (19) completes the general scheme (1)–(4); the resulting theory can be studied and compared with phenomenology. Let us note that imposing the temperature regime can be interpreted as introducing the macroscopic observer into the general quantum theory scheme. Actually, fixing some special temperature regime (19), this observer defines the macroscopic experiment conditions. These conditions may have really different nature – this evident fact is reflected by extremely wide class of the temperature curves (19) which can be considered within the general quantum microscopic dynamics (1)–(4).

### 3.5. Time dependence of expectation values of symmetry operators: arrow of time appearance

Now we are ready to state that the modified quantum dynamics given by the general scheme (1)–(4) and the temperature regime (19) fixed, agree with the Second Law of thermodynamics. More precisely, we state that for the wide class of the temperature regimes (19), the modified quantum dynamics (1)–(4) possesses an arrow of time. In this subsection we prove this statement for the physically well-motivated isothermal and adiabatic regimes (17) and (18). In our proof we will operate with the most important expectation values of the theory, defined by its Hamiltonian:

\[
\bar{E} = \sum_n P_n E_n, \quad \bar{\Gamma} = \sum_n P_n \Gamma_n. \tag{20}
\]

For the isothermal evolution (17), a straightforward calculation leads to the following remarkable result:

\[
\frac{d\bar{\Gamma}}{dt} = -D^2 \bar{\Gamma}, \tag{21}
\]

where \( D^2 \bar{\Gamma} \) denotes the squared dispersion of the decay operator \( \bar{\Gamma} \). Thus, for theories with not negative probabilities \( P_n \) the inequality

\[
\frac{d\bar{\Gamma}}{dt} \leq 0 \tag{22}
\]
holds for the arbitrary value of time parameter $t$. Note that this inequality holds for any initial data which specifies the solution $\Psi$ of the dynamical equations (1). Thus, we deal with the irreversible in time evolution, or with the ‘arrow of time’ in the isothermal quantum evolution.

Then, repeating the calculations in the case of adiabatic evolution (18), one obtains the following result:

$$ \frac{d\bar{\Gamma}}{dt} = -D^2 \left[ 1 - \frac{\left( \bar{E}T - \bar{T} \right)^2}{D^2_\bar{E} D^2_{\bar{T}}} \right], $$

(23)

where $D^2_\bar{E}$ is the squared dispersion for the energy operator. Of course, Eq. (23) is much more complicated than the previous one given by Eq. (21). However, using the Cauchy-Bunyakovsky inequality (for the theory with non-negative probability set), one concludes that the therm in the square brackets in Eq. (23) is not negative. From this fact it follows that the inequality (22) remains true for the case of adiabatic evolution. Thus, the adiabatic quantum evolution is also irreversible – i.e., it also possesses the arrow of time. The presence of the arrow of time, or time irreversibility in the quantum system evolution, means that this evolution can agree with the Second Law of thermodynamics.

To clarify the situation, let us study the dynamics of the entropy function $\Sigma$, defined by the relation

$$ \Sigma = \sum_n P_n \log P_n = \log r - \bar{E}\beta - \bar{\Gamma}t - \log Z, $$

(24)

where $\log r = \sum P_n \log r_n$. Using the straightforward calculations, it is possible to prove that the following remarkable inequality holds for any $t \geq 0$ in the isothermal and adiabatic regimes of evolution:

$$ \frac{d\Sigma_{\text{phys}}}{dt} = -t \frac{d\bar{\Gamma}}{dt} \geq 0. $$

(25)

Here we have used Eq. (22), and the quantity $\Sigma_{\text{phys}}$ is introduced as the renormalized entropy (24):

$$ \Sigma_{\text{phys}} = \Sigma - \log r. $$

(26)

We name this new entropy function as the ‘physical entropy’. It is necessary to stress that for the special statistics with equal values of the weight coefficients $r_n$ one has $\log r = \text{const}$, and from Eq. (25) it follows the same relation for the original entropy function $\Sigma$. Note that exactly such kind of statistics is realized in nature through the microcanonical Gibbs distribution. Thus, the arrow of time detected by the inequality (22) actually corresponds to the Second Law of thermodynamics.

Also, our conclusion shows that for the consistent explanation of the Second Law of thermodynamics in the microscopic framework, one must introduce the non-trivial decay operator into the fundamental quantum system description. Thus, we propose to see the decay processes as the reason for the irreversibility of dynamics of physical systems in the real world.

4. Quasi-classical limit: modified classical dynamics, time dependence of Noether charges and arrow of time

Classical limit of quantum theory must describe the ‘worlds around us’, i.e. the macroscopic reality. Usually, in constructing the classical limit of the specific quantum dynamical system, one neglects the terms, which include the Plank constant. Thus, this limit is related to the
calculation of all necessary theory quantities and all corresponding dynamical relations in the regime with $\hbar \to 0$. However, taking into account the really small value of the Plank constant is not the single condition which can be imposed on the fundamental quantum theory. Actually, the crucial property of the quantum theory – the non-localized nature of the dynamical quantities – does not break at the limit of $\hbar = 0$. Thus, it is possible to develop the theory with non-trivial distribution of probability density and vanishing value of the Plank constant. This type of generalized dynamics, being classical, differs essentially from the dynamics with a completely localized probability density which is usually identified with the adequate candidate to the macroscopical world description. In this section we construct both types of classical limit for the generalized quantum dynamics developed above. It is shown that explanation of irreversible properties in evolution of macroscopic systems can be obtained only in the classical limit of their quantum dynamics with non-locality saved. Thus, the Second Law of thermodynamics provides the true choice between the two variants of classical limits for real systems: one must choose as correct the dynamics with non-trivial (not localized completely) distributions of the classical probability density. The results of this section are based on the publications [10]–[11].

4.1. Definition of modified classical dynamics

Let us start our consideration with the one-component wave-function $\Psi$ (the more complicated quantum systems are discussed below). Such wave function can be represented in the exponential form

$$
\Psi = \exp \left( \frac{iS}{\hbar} \right),
$$

where the complex phase $S$ can be decomposed to its real and imaginary parts $S_1$ and $S_2$ in the following way:

$$
S = S_1 + \frac{i\hbar}{2} S_2.
$$

Our goal is to establish the dynamical relations which follow from the quantum equations (1) at the limit $\hbar \to 0$ (we will use the same notation $S_1$ and $S_2$ for the limit values of the phase components).

To solve the corresponding mathematical problem, let us use the coordinate representation of the original quantum theory; then for the momenta, one has the standard relation $\hat{p}_k = i\hbar \partial_{x_k}$. Study of the general quantum scheme (1)–(4) with $\hat{H}$ realized as some function of the total set of the canonical variables $q_k$ and $\hat{p}_k$ in the limit $\hbar = 0$ leads to the following result for the real part $S_1$ of the complex quantum phase:

$$
S_{1,t} = -E, \quad S_{1,\beta} = 0.
$$

It is seen that the function $S_1$ does not depend on the temperature parameter $\beta$ and satisfies the standard Hamilton-Jacoby equation. Note that the classical function $E$ in Eq. (29) is constructed from the quantum operator $\hat{E} = E(q_k, \hat{p}_k)$ using the standard substitution

$$
\hat{p}_k \to S_{1,q_k}
$$

of the analytical mechanics. To simplify the form of the remaining relations, let us introduce the following new quantity $S_e$:

$$
S_e = S_2 - \beta E.
$$
which is explicitly entropy-like. Then, for the remaining dynamical relations we obtain:

\[ S_{e,t} + E_{.,p} S_{e,q} = \Gamma_*, \quad S_{e,\beta} = 0, \]  (32)
i.e. the function \( S_e \) is also \( \beta \)-independent at the limit \( \hbar \to 0 \). Here the quantity \( \Gamma_* \) means the shifted decay dynamical variable; it is defined as

\[ \Gamma_* = \Gamma + \text{tr} \left( E_{.,pp^T} S_{1,qq^T} \right), \]  (33)

whereas the notation \( p = S_{1,q} \) corresponds to Eq. (30) and is understood in its natural matrix form. Of course, the substitution (30) is performed in all relations, which need this action for their existence. Finally: Eqs. (29) and (32) play the role of the fundamental quantum dynamical relations in the limit \( \hbar \to 0 \). They generalize the conventional Hamilton-Jacoby equation to this case of modified classical dynamics.

Now let us consider the dynamics of soliton-like excitations of probability density \( \rho \) of the theory. This quantity is defined as

\[ \rho = \frac{\exp \left( -S_2 \right)}{Z}, \]  (34)

where, for the norm of the solution, the following explicit formula can be used:

\[ Z = \int dq \exp \left( -S_2 \right). \]  (35)

It is seen that the probability density is normalized to the unit, because \( \int dq \rho \equiv 1 \). Our goal is to study the dynamics of maxima of this function: we will identify classical positions \( q = q_{\text{classic}}(t, \beta) \) of these solitons with the set of maxima using natural arguments. Thus, for definition of the classical position we have:

\[ \max_q \rho(q, t, \beta) = \rho(q_{\text{classic}}(t, \beta), t, \beta). \]  (36)

It is clear that such kind of description of this soliton-like dynamic is incomplete under construction and does not take into account the processes of the soliton deformation. Below we will detect the kinematical terms which describe correctly the solitonic formation and decreasing. Now our goal is to derive the dynamical relations for the quantities \( q = q_{\text{classic}} = q_{\text{classic}}(t) \) and \( p = p_{\text{classic}} \equiv S_{1,q}(t, q_{\text{classic}}(t)) \). Using the necessary differential conditions for the extremal principle (36), after its differentiation with respect to \( t \) and \( \beta \), we obtain:

\[ q, t = E_{.,q} - A_2^{-1} \Gamma_{*,q}, \quad q, \beta = -A_2^{-1} E_{.,q}, \]

\[ p, t = -E_{.,q} - A_1 A_2^{-1} \Gamma_{*,q}, \quad p, \beta = -A_1 A_2^{-1} E_{.,q}. \]  (37)

Eqs. (37) generalize Hamilton’s equations to the case of \( \beta \)-dependence of the canonical variables. Also, these equations contain extra terms related to the matrices \( A_1 \) and \( A_2 \), which are defined as

\[ A_1 = S_{1,qq^T}, \quad A_2 = S_{2,qq^T}; \]  (38)

their kinematical sense is clarified in the next subsection. It is shown that these terms describe the non-locality of the soliton’s distribution in the phase space of the theory. In Eqs. (37), the total derivative with respect to \( q \) is defined by the relation

\[ \dot{q} = \frac{\partial}{\partial q} + A_1 \frac{\partial}{\partial p}. \]  (39)
Finally, to calculate the total \( t \)-derivative of the set of canonical variables, one must use Eqs. (37) and the standard chain rule
\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{\beta} \frac{\partial}{\partial \beta},
\]
where \( \dot{\beta} = d\beta/dt \). Finally, Eqs. (29), (32) give the generalization of the Hamilton-Jacoby equations, whereas the system of equations (37) modify the Hamilton’s system. Together, these two systems of dynamical relations describe the modified classical dynamics which is the limit of the fundamental quantum dynamics (1)–(4) at the limit \( \hbar \to 0 \).

4.2. Correlations and double quasi-classical limit

The limit \( \hbar \to 0 \) leads to the introduction of the dynamical variables like the quantity \( A = A(q, p) = A(q, S_1, q) \) which arise from the initial quantum operator \( \hat{A} = A(q, \hat{p}) \). In the limit under consideration, the expectation value \( \bar{A} \) is given by the conventional relation
\[
\bar{A} = \int dq \rho A,
\]
where the probability density is defined by Eqs. (34)–(35) with the limit value of the function \( S_2 \). The main statement of this subsection is that the relations
\[
q \circ q^T \approx A_2^{-1}, \quad p \circ q^T \approx A_1 A_2^{-1}, \quad p \circ p^T \approx A_1 A_2^{-1} A_1
\]
hold in the kinematics of the theory if one considers the regime with
\[
A_2^{-1} \approx 0.
\]
Note that the system (37) transforms into conventional Hamilton’s equations in the limit \( A_2^{-1} \to 0 \). Actually, all the additional terms there have a correlation nature, whereas this limit is exactly the limit of infinitesimal correlations. It is interesting to note that not only the coordinate-coordinate correlations affect the system dynamics, but also the coordinate-momentum and momentum-momentum ones. Thus, the dynamics ‘feels’ the total set of correlations of the canonical variables of the theory. It is seen that conventional Hamilton’s dynamics arises in the double limit with \( \hbar \to 0 \) and \( A_2^{-1} \to 0 \). We name this double limit as the ‘hard’ one, whereas the ‘soft’ limit will refer with the limit procedure \( \hbar \to 0 \) only. Let us summarize these
\[
\text{soft classical limit : } \quad \hbar \to 0 \quad \text{correlations } \sim A_2^{-1} \neq 0,
\]
\[
\text{hard classical limit : } \quad \hbar \to 0 \quad \text{correlations } \sim A_2^{-1} \to 0.
\]
There is another way to detect the correlation nature of the quantity \( A_2^{-1} \), which is crucial for the hard classical limit construction. Namely, let us consider the decomposition of the function \( S_2 \) in vicinity of the classical position \( q_{\text{classic}} \) at the given values of the time \( t \) and inverse temperature \( \beta \). First of all, for the first differential of this function we have \( dS_2 = 0 \) here in view of the necessary condition corresponding to the maximum principle given by Eq. (36). Then, the sufficient condition can be guaranteed by the second differential \( d^2 S_2 \) positive definition. It is easy to see that \( d^2 S_2 = dq^T A_2 dq \) in view of the second relation in Eq. (38). Thus, \( d^2 S_2 \) becomes the infinite positive term at the limit \( A_2^{-1} \to +0 \). Finally, using the relations (34)–(35) between functions \( S_2 \) and \( \rho \), it is possible to show that
\[
\lim_{A_2^{-1} \to 0} \rho = \delta (q - q_{\text{classic}}).
\]
Thus, the limit procedure with \( A_2^{-1} \to 0 \) corresponds to the localization of the soliton-like probability density distribution and arising of the delta-functional localization at the limit \( A_2^{-1} = 0 \).
4.3. Modified Newtonian force: MOND and General Relativity effects out of correlations

In this subsection we study the one special system to demonstrate the role of correlations in the soft variant of classical dynamics. We would like to show that the correlation terms modify the dynamics and can explain, for example, the classical effects in the General Relativity. Thus, it is possible to say that the non-locality generates additional forces which can lead to arising of effective gravity.

Actually, let us study the Second Law of classical mechanics in the soft limit of the modified quantum theory. It is given by the standard relation

$$\frac{dp}{dt} = F_m,$$  \hspace{1cm} (46)

where the total force $F_m$ can be obtained from Eqs. (37) for the momentum $p$, and the relation (40) for the total $t$-derivative must be taken into account. The result reads:

$$F_m = -E_q - A_1 A_2^{-1} \left( \Gamma_{*;q} + \frac{d\beta}{dt} E_q \right);$$  \hspace{1cm} (47)

from this it is seen that the modified force value $F_m$ differs from the conventional one $F = -E_q$ exactly by the terms of the correlation type (compare Eq. (47) with Eqs. (42)). Also, it is clear that in the hard type of the classical limit these terms vanish, and we recover the standard Newton’s relation for the potential force. This situation takes place in a general case of the theory under consideration.

Now let us specify the theory and take its energy function as the Kepler’s one,

$$E = \sum_{k=1}^{3} \frac{p_k^2}{2m} - \frac{Gm}{r},$$  \hspace{1cm} (48)

whereas for the decay function we will take its trivial value, i.e. we will study the system with $\Gamma = 0$. Then, for the shifted decay variable (33) we will obtain the relation

$$\Gamma_{*} = \text{tr} \left( E_{,pp} S_{1,qq} \right) = \frac{1}{m} \Delta S_1,$$  \hspace{1cm} (49)

which defines the force $F_m$ completely. The explicit form for its $k$-th component reads:

$$F_{mk} = F_{0k} - \frac{d}{dt} \left[ \left( A_2^{-1} \right)_{kl} \Delta S_{1,x_l} \right] - \frac{1}{m} \left( A_1 A_2^{-1} \right)_{kl} \Delta S_{1,x_l},$$  \hspace{1cm} (50)

where

$$F_{0k} = -\frac{Gm}{r^2} \frac{x_k}{r}$$  \hspace{1cm} (51)

is the $k$-th component of the classical Newtonian gravity force. Our goal is to calculate the extra terms in Eq. (50) for the case of some natural distribution of probability density, and to study the asymptotic behavior of these force corrections. For this system, the modified Hamilton-Jacoby equation (29), (32) leads to the equations

$$\Sigma_{,t} + \frac{1}{2} \sum_{k=1}^{3} \Sigma_{,x_k}^2 - \frac{GM}{r} = 0, \quad S_{e,t} + \sum_{k=1}^{3} \Sigma_{,x_k} S_{e,x_k} = \Delta \Sigma,$$  \hspace{1cm} (52)

where the new function $S_1 = m \Sigma$ has been introduced. Here and below we consider the isothermal evolution ($\beta = \text{const}$), which is natural in the framework of planet dynamics. Possible
influence of the temperature regime to the gravity effects will be discussed below – here we are interested in the correlation ones only.

To establish the correlation modification of the Newtonian gravitational force, let us find the solution of Eq. (52) of the spherical type. Using separation of variables, we explore for the corresponding ansatz the following relation:

\[ \Sigma = \Sigma (t, r) = -\frac{\alpha t}{2} + f(r), \quad (53) \]

where \( \alpha = \text{const} \) (below we consider the case of \( \alpha > 0 \) for simplicity). From the first equation of (52) one obtains that

\[ f^{-1}(r) = \left( \alpha + \frac{2GM}{r} \right)^{\frac{1}{2}}. \quad (54) \]

We would like to calculate the total force \( F_m \) which acts on the soliton-like distribution of the probability density. Of course, we interpret this localized object as a planet with the mass \( m \), which moves under the action of the static mass \( M \) – i.e. of the Sun.

To perform the calculation, let us suppose, for simplicity that the spatial distribution of probability density is spherically symmetric with the center in the classical position of the planet at the time under consideration. We will not study the deformation of this soliton shape during the dynamics; it is defined by Eq. (52) completely. Taking into account Eqs. (42), for the spherically symmetric soliton we put:

\[ \left( A_2^{-1} \right)_{kl} = \epsilon \delta_{kl}, \quad (55) \]

where \( \epsilon \) is some positive parameter. We will study the highly localized probability density case, where

\[ \epsilon \to +0, \quad (56) \]

and the soliton almost coincides with the classical object position. In other words, we consider the weak correlation case which is not exactly the dynamics of a single point mass. Using Eqs. (54) and (55), for the force \( F_m \) components up to \( \sim \epsilon \) terms one obtains the following result:

\[ F_{mk} = F_{0k} - \frac{4\epsilon m}{r^3} \left( \alpha + \frac{3GM}{r} \right)^2 \frac{x_k}{r}. \quad (57) \]

Eq. (57) gives the modified force acting on the moving mass \( m \) in the weak and spherically symmetric correlation case.

It is important to note that the force (57) has the asymptotics

\[ F_{mk} = F_{0k} - \frac{4\epsilon m \alpha}{r^3} \frac{x_k}{r}, \quad (58) \]

at the spatial infinity \( r \to +\infty \). It is seen that the additional term is proportional to \( 1/r^3 \) in this asymptotics – exactly as the Einstein’s term in the General Relativity. From this follows the possibility to base the explanation of classical effects of General Relativity on the pure correlation conception. For another asymptotical regime with \( 0 \ll r \ll 2GM/\alpha \), which can be realized in the case of \( \alpha \ll 2GM \), one obtains the following result:

\[ F_{mk} = F_{0k} - \frac{18\epsilon GMm}{r^4} \frac{x_k}{r}. \quad (59) \]
Thus, the force (57) is proportional to the squared acceleration $a_0 = F_0/m$ for the large, but not infinite distances. Finally, the force (57) can be considered as the fundamental candidate to the role of the Modified Newtonian Dynamics (MOND) force. Actually, the MOND force possesses exactly the discussed asymptotical properties [20]–[22]. It is important to stress that the modified Newtonian force (57) was obtained using the fundamental quantum principles and consistent procedure of constructing the (soft) classical limit.

4.4. Symmetries in modified classical dynamics

Now let us come back to the general formalism of the theory to study the symmetries and irreversible effects in the soft classical limit of the modified quantum dynamics.

Let us start with the system symmetries. The statement is that the quantum symmetry relation \( (16) \) with commutation of the symmetry operator $\hat{S}$ and the Hamiltonian $\hat{H}$ transforms to condition of vanishing of the Poisson bracket

\[
\{ S, H \} = 0
\]

for the symmetry variable $S = S(q, p)$ and the Hamilton’s function $H = H(q, p)$, calculated on the soft limit with $\hbar \to 0$. In particular, we obtain the relation of this type for the decay function $\Gamma$. From this, it follows also the zero value of Poisson bracket for the decay and energy dynamical variables:

\[
\{ \Gamma, E \} = 0.
\]

Of course, in Eq. (61) the classical quantity $\Gamma = \Gamma(q, p) = \Gamma(q, S_1(q))$ is obtained from the corresponding quantum one $\hat{\Gamma} = \Gamma(\hat{q}, \hat{p})$ using the straightforward calculation at the soft limit $\hbar \to 0$ of the theory (the same situation takes place for the energy function $E$ and all other observables). Thus, all quantum symmetries transform to the corresponding classical ones through the consistent procedure of constructing the classical limit.

4.5. Arrow of time in modified classical dynamics: phase space evolution

The same situation holds for the irreversible properties of the dynamics of classical systems which arise in the original quantum theory (1)–(4) in its soft classical limit. Namely, all conclusions of the subsection 3.5 are preserved when the soft classical limit is taken. The only differences are related to the definition of the expectation values of quantities, which must be used for detecting the arrow of time. For example, the limit of Eq. (20) at $\hbar \to 0$ reads:

\[
\bar{E} = \int dq \rho E \quad \bar{\Gamma} = \int dq \rho \Gamma;
\]

i.e. the expectation values of classical variables have the conventional definition in terms of the probability density $\rho$. Then, performing the same calculations, as in the subsection 3.5., we conclude that $d\bar{\Gamma}(t)/dt \leq 0$ for arbitrary initial data in the isothermal and adiabatic regimes of evolution. This conclusion is based on Eqs. (21) and (23), which preserve their form in the soft classical dynamics. Moreover, after the introduction of physical entropy

\[
\Sigma_{\text{phys}} = \int dq \rho \log \rho - \log r
\]

we obtain the non-decreasing entropy law $d\Sigma_{\text{phys}}/dt \geq 0$ for any value of time $t \geq 0$. The corresponding results for the quantum dynamics are given by Eqs. (25)–(26), which remain true in the classical case.
Finally, we see that the soft classical dynamics preserves, exactly, the irreversible properties of the quantum one. In both quantum and classical regimes, the arrow of time effects are defined by the correlation structures, see Eqs (21) and (23). It is possible to say that the arrow of time for macro-world is the direct consequence of the micro-world arrow of time in the modified theory under discussion. We would like to note that the phenomenology related to the macroscopic physics is based on the Second Law of thermodynamics, i.e. the macroscopic arrow of time exists definitely. Thus, if our theory is realistic, the microscopic dynamics must also possess the irreversible properties, i.e. it must be based on the Hamiltonian with the non-trivial decay operator $\hat{\Gamma}$.

4.6. Arrow of time in modified classical dynamics: soliton trajectory evolution

At the end of this section we would like to show the appearance of the arrow of time in the classical trajectory description. This important dynamical property cannot be related to behavior of expectation values of the dynamical variables, because the trajectory does not coincide with the total probability stream. However, irreversibility effects can be detected surprisingly in the system dynamics using this incomplete dynamical data too.

Namely, let us consider the system with the energy function

$$E = \frac{p^2}{2m} + \alpha r^\delta,$$

where $\alpha$ and $\delta$ are constant parameters. Let us study dynamics of the classical quantity

$$\Gamma_* = \Gamma + \text{tr} \left( E_{pp^T} A_1 \right).$$

The statement is that the inequality

$$\frac{d\Gamma_*(t)}{dt} \leq 0$$

holds for the systems with the parameters $\alpha$ and $\delta$ satisfying the restriction

$$\alpha (\delta + 1) \geq 0$$

for arbitrary initial data. Of course, Eqs. (65) and (66) are considered on the classical trajectory with $q = q_{\text{classic}}(t)$ and $p = p_{\text{classic}}$. Note that the special Coulomb case, which is characterized by $\delta = -1$, satisfies the restriction (67) identically. Note that this special case is actually important for different applications.

5. Modified relativistic dynamics, time structure and arrow of time

For application of the modified quantum theory to fundamental physics, it is necessary to develop the relativistic realizations of the general scheme (1)–(4). In this section we formulate a concept of constructing the specific relativistic quantum models of the proposed type and give a universal precept for introducing a decay operator for the system. This precept is related to the famous multi-world concept in quantum theory. In fact, we realize this concept in the framework of general quantum formalism and show that the resulting relativistic theory possesses the arrow of time. The results of this section are based on the publications [10]–[11].
5.1. Extra time dimension conception
We propose to construct the relativistic models which satisfy the relations (1)–(4) in the explicitly relativistic form. Namely, our idea is to work with the wave functions
\[ \Psi = \Psi (\tau, x^\mu), \] (68)
which realizes Poincare group representation, and depends on the complex Hamilton’s time \( \tau \) and the four-dimensional coordinates \( x^\mu \) (where \( \mu = 0, 1, 2, 3 \)) as on the kinematically independent variables. We propose to consider that
\[ \tau = \text{inv}, \] (69)
i.e. that Hamilton’s time is the Poincare-invariant parameter of evolution. In the framework of this general approach, the Hamiltonian \( \hat{H} \) must act on the state vector \( \Psi \) in such a way that the vector \( \hat{H}\Psi \) transforms exactly as \( \Psi \). Actually, in this case, the dynamical equations (1) become Poincare-invariant relations identically.

We would like to stress that the real part \( t \) of the complex evolutionary parameter \( \tau \) cannot be related to the Minkowskian time \( x^0 \) on the kinematical level, i.e.
\[ x^0 \neq ct. \] (70)
In this sense, our realization of the modified quantum theory and its classical limit are some dynamical models with double time. Below we study how it is possible to reduce the extra time parameter in the theory of such type. Briefly, we postulate, additionally, a new extremal principle: all extra variables must be fixed on their values which realize maximal value for the probability density for any given value of the space-time coordinates \( x^\mu \). Here it is important to note that the final theoretical scheme is free of the ‘second time’ \( t \) effectively – it is a conventional four-dimensional quantum or classical theory.

5.2. Double universe case and arrow of time
It is clear that the main problem in construction of some specific relativistic realization of the modified quantum theory (1)–(4) is related to the search of the decay operator \( \hat{\Gamma} \), which commutes with the energy \( \hat{E} \), does not destroy the formal structure of the theory, and possesses a clear physical sense. For example, the non-linear functions \( f(\hat{E}) \) cannot be taken as realizations of \( \hat{\Gamma} \), because the resulting theory will be related to the higher order derivatives with respect to the four-dimensional coordinates \( x^\mu \), than the starting theory with trivial \( \hat{\Gamma} \).

To solve this problem in the universal form, we propose the following procedure of the system ‘multiplication’. Let us consider two identical quantum subsystems with the wave functions \( \Psi_1 \) and \( \Psi_2 \), and the coinciding Hamiltonians \( \hat{H}_1 = \hat{H}_2 = \hat{E} \). We unify these subsystems into a total one using the following embedding relations:
\[ \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \quad \hat{\mathcal{H}} = \hat{\mathcal{E}} - \frac{i\hbar}{2} \hat{\mathcal{S}}, \] (71)
where
\[ \hat{\mathcal{E}} = \begin{pmatrix} \hat{E} & 0 \\ 0 & \hat{E} \end{pmatrix}, \quad \hat{\mathcal{S}} = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}, \] (72)
and
\[ \gamma_1 < \gamma_2 = \text{const.} \] (73)
Thus, we suppose to unify the two identical subsystems using the direct sum formalism. This approach leads to arising of the two fundamental constants $\gamma_1$ and $\gamma_2$ in the theory (in fact, it can be shown that only their difference $\gamma_2 - \gamma_1$ has the fundamental sense).

The main statement related to the realization (71)–(73) of the general scheme (1)–(4) is that for the resulting theory the fundamental inequality

$$\frac{d\hat{\mathcal{I}}(t)}{dt} \leq 0$$

(74)

actually holds. Thus, this theory possesses the arrow of time and is due to the fundamental thermodynamical framework.

5.3. Example: parity symmetry violation

Let us illustrate the results of the previous subsection by the following explicit example. First of all, let us take

$$-\gamma_1 = \gamma_2 = \gamma > 0;$$

(75)

then

$$\hat{\mathcal{I}} = \gamma \hat{\Pi},$$

(76)

where

$$\hat{\Pi} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

(77)

In fact, the restriction (75) is not sufficient – this can be obtained using some simple ‘renormalization’ of the theory quantities. It is easy to see that $\Pi^2 = 1$; thus, this theory contains the decay operator of the parity type. Let us use here the notation $\Psi_1 = \Psi_+$ and $\Psi_2 = \Psi_-$ for the odd and even eigenstates of the parity operator, respectively. Then the eigenvalue problem in the theory under consideration will take the following simple form:

$$\hat{E} \Psi_n = E_n \Psi_n, \quad \Pi \Psi_n = \pm \Psi_n.$$ 

(78)

Of course, the relation (74) takes place in this theory. We state, however, that the additional remarkable identity

$$\lim_{t \to \pm \infty} \hat{\mathcal{I}}(t) = \mp \gamma$$

(79)

is satisfied for this system for arbitrary initial data and for any temperature mode $\beta = \beta(t)$ which allows the limit temperature values $\beta_\pm = \lim_{t \to \pm \infty} \beta(t)$. Thus, the global evolution of this system (from $t \to -\infty$ to $t \to +\infty$) has the ‘kink nature’. Actually, this evolution leads to the map of the subsystem with $-\gamma$-value of the decay operator to the subsystem with its $+\gamma$-value. In fact, it is possible to find this kink in its explicit form here, using special initial data with $P_{n+} = P_{n-}$ (and, thus, with $Z_+ = Z_-$) at $t = 0$. In this special case, one obtains the following relation for the expectation of the decay operator:

$$\hat{\mathcal{I}}(t) = -\gamma \tanh (\gamma t).$$ 

(80)

It is seen that Eq. (80) gives kink exactly; this kink relates to the subspaces $\Psi_\pm$ as attractor states at $t \to \pm \infty$. Thus, the global evolution of the system has the sense of transformation of the ‘even world’ to the ‘odd’ one (or vice versa).
5.4. Multi-universe conception

It is clear that the approach of introducing a decay operator into the theory proposed in the previous subsection, can be generalized naturally to the case of an arbitrary number of identical realizations of the given quantum system. Namely, let us consider the theory with the wave function

\[ \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \vdots \end{pmatrix}, \tag{81} \]

and the energy and decay operators given by the following matrices:

\[ \hat{\mathcal{E}} = \begin{pmatrix} \hat{E} & 0 & 0 & \cdots \\ 0 & \hat{E} & 0 & \cdots \\ 0 & 0 & \hat{E} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \hat{\Theta} = \begin{pmatrix} \gamma_1 & 0 & 0 & \cdots \\ 0 & \gamma_2 & 0 & \cdots \\ 0 & 0 & \gamma_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{82} \]

where

\[ \gamma_1 < \gamma_2 < \gamma_3 < \cdots = \text{const.} \tag{83} \]

The statement is, that the fundamental inequality (74) holds in this theory for the well physically-motivated temperature modes again. For example, it is easy to check Eq. (74) for the isothermal regime \( \beta = \text{const.} \).

Thus, the proposed scheme allows one to transform the reversible theory with the trivial decay operator \( \hat{\Gamma} = 0 \) and some fixed energy operator \( \hat{E} \) to the irreversible in time quantum system with the energy and decay operators, given by Eq. (82). This new quantum ‘super system’ describes the dynamics of the quantum reality between ‘instant worlds’ represented by the subsystems

\[ \begin{pmatrix} \Psi_1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \Psi_2 \\ 0 \\ \vdots \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ \Psi_3 \\ \vdots \end{pmatrix}, \ldots. \tag{84} \]

The arrow of time is ‘directed’ to some attractor in this ‘multi-world’ quantum system; for the isothermal evolution, the final ‘instant world’ is characterized by the minimal operator \( \hat{\Theta} \) eigenvalue. Thus, for this simplest type of a temperature mode, the final state is the first one in the states row (84). In fact, Eqs. (81)–(84) give realization of the multi-world conception in the quantum theory in its explicit form. Any evolution here is related to some ‘migration’ of this system through its ‘equivalent realizations’ (84) from the probability point of view.

6. Modified Dirac’s system: relation between different Noether symmetries

In this section we present one specific relativistic quantum theory of the modified type presented above. This theory describes the spinor field in four dimensions with the additional complex parameter of evolution. It is shown that spectrum of this model consists of particles, antiparticles...
and tachyons. We propose to use the tachyon modes as the source of dark mass in this system, and calculate the tachyon input to the mass expectation value for the given quantum state. We study the Noether structures of the theory, and establish some relation between the gauge charge and mass expectation value. Also, the spinor model with analytic dependence of wave function on the full set of complex coordinates is introduced. This totally complexified theory possesses additional relations between the gauge and energy-momentum charges, which makes it especially attractive for the following applications. The results of this section are based on the publications [12]–[13]. Also, the integrals of motion for systems with nonstationary conditions were discussed in papers [18]–[19].

6.1. Definition of modified Dirac’s system
First of all, let us note that the conventional Dirac’s equation for spinor field \( \Psi \) with the mass \( m \) can be written in the form of eigenvalue problem

\[
\hat{E} \Psi = mc^2 \Psi \tag{85}
\]

with respect to the energy operator

\[
\hat{E} = -c \left( \gamma^4 \right)^{-1} \gamma^\mu \hat{p}_\mu. \tag{86}
\]

Here the matrix \( \gamma^4 \) can be taken in one of the two following forms:

\[
\gamma^4 = \left\{ \begin{array}{c} -I \\ \gamma^5 \end{array} \right\}, \tag{87}
\]

which gives rise to two possible branches of the theory – for normal and abnormal spinors, respectively, whereas \( \hat{p}_\mu = i\hbar \partial_{x^\mu} \) is the standard momentum operator. From Eq. (85) it follows that the observable

\[
\hat{m} = \frac{1}{c^2} \hat{E} \tag{88}
\]

plays the role of the mass operator in the theory. It is not difficult to prove that the equality

\[
\hat{m}^2 c^2 = \hat{p}^2 = \eta^{\mu\nu} \hat{p}_\mu \hat{p}_\nu, \tag{89}
\]

where \( \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1) \) is the inverse Minkowskian metrics, holds for the both realizations (87) of the matrix \( \gamma^4 \). This equality has exactly the form of the famous relation in special relativity between the squared four-dimensional momentum and the mass of the particle. Thus, we have an additional argument to interpret the operator \( \hat{m} \), given by Eq. (88), as the mass operator for the spinor system.

Now we are ready to define the modified Dirac’s system, in complete accordance to the general scheme (1)–(4), and conception of the relativistic models construction developed in the previous section. Namely, let us define the theory of this type as the system with the energy operator \( \hat{E} \) given by Eq. (86), and with the decay operator \( \Gamma = 0 \). Then, the dynamical relations (1) take the following form:

\[
i\hbar \Psi_{*, \tau} = -c \left( \gamma^4 \right)^{-1} \gamma^\mu \hat{p}_\mu \Psi, \quad \Psi_{*, \tau*} = 0. \tag{90}
\]

Let us note that this theory does not contain any mass parameter fixed – this is a massless five-dimensional dynamical model with one complex and four real coordinates. From the general point of view, the problems (85) and (90) give the stationary and non-stationary Schrödinger’s
equations for the quantum theory under consideration. From this, it follows that the generalized system (90) contains superposition of the total set of special systems like the system (85), i.e. we deal with some ‘complete spinor world’ in a framework of the modified spinor theory (90).

For the operator \( \hat{A} \) in this theory, the matrix element between the states \( |\Psi_1> \) and \( |\Psi_2> \) is defined by the relation

\[
<\Psi_1|\hat{A}|\Psi_2> = \int d^4x \bar{\Psi}_1 \hat{A}\gamma^4 \Psi_2.
\] (91)

Taking \( \hat{A} = I \) and \( |\Psi_1> = |\Psi_2> = |\Psi> \) in Eq. (91), we obtain the definition of the statistical sum \( Z \) (and the norm of the state \( |\Psi> \)) in this theory:

\[
Z = <\Psi|\Psi> = \int d^4x \bar{\Psi}\gamma^4 \Psi.
\] (92)

For the probability density \( \rho \) we obtain immediately the following relation:

\[
\rho = \frac{\bar{\Psi}\gamma^4 \Psi}{Z};
\] (93)

this result will be discussed additionally in the last subsection of this section. It will be shown that the definitions (91)–(93) are consistent with the fundamental formalism of Noether symmetries in this theory. Here we would like to note that the rule (91) for calculating matrix elements and the mass operator definition given by Eqs. (86) and (88) guarantee the Hermitian property of this mass operator. Namely, one can prove using straightforward calculation that the equality

\[
<\Psi_1|\hat{m}|\Psi_2> = <\Psi_2|\hat{m}|\Psi_1>^*
\] (94)

holds in this theory. This means that \( \hat{m} = \hat{m}^+ \). This fact gives another argument to the mathematical correctness of the general scheme presented in addition to the one related to Eq. (89). In the following subsections we study the generalized quantum system (90) and its possible applications in details.

6.2. Mass generation mechanism

To perform the Fourier analysis of the theory (90), let us change the coordinate representation to the momentum one. Then, for the wave function in the momentum representation, we have the superposition formula

\[
\Psi(\tau, p) = \sum_{k=1}^{4} C_{pk} \Psi_k(\tau, p),
\] (95)

where \( C_{pk} = \text{const.} \), and the Fourier modes \( \Psi_k(\tau, p) \) are defined as

\[
\Psi_k(\tau, p) = \exp\left(-\frac{i\epsilon_{pk}\tau}{\hbar}\right) \psi_{pk}.
\] (96)

Here the wave functions \( \psi_{p,k} \), which are stationary with respect to the complex evolutionary parameter \( \tau \), are solutions of the eigenvalue problem

\[
\hat{m} \psi_{pk} = c^{-1} \epsilon_{pk} \psi_{pk}
\] (97)
with $c^2_{pk} = p^2$. In fact, Eq. (97) coincides with Eq. (85) up to the notations change; thus, we have decomposed the solution $\Psi$ of the dynamical system (90) to the superposition of the standard type Dirac's modes. This conclusion can be done immediately for the modes with $p^2 > 0$ and $p^0 > 0$, which describe the Dirac particles in the theory, and also for anti-particles with $p^2 > 0$ and $p^0 < 0$. Then, for the massless objects with $p^2 = 0$ this statement holds again – all these three special cases are described by the original Dirac's theory (85)–(86). It is interesting to note that the tachyon modes with $p^2 < 0$ are included into the spectrum of the modified spinor theory too. Thus, the theory under consideration describes the quantum superposition of all the spinor particles, antiparticles and tachyons.

These three types of the spinor objects split the momentum space into three subspaces: the cone of future $V_I$ with $p^2 > 0$ and $p^0 > 0$, the cone of past $V_{II}$ with $p^2 > 0$ and $p^0 < 0$, and the rest part of the momentum space $V_{III}$ with $p^2 < 0$. Using Eqs. (92) and (95)–(97), for the statistical sum $Z$ one obtains the following explicit form:

$$Z = \int_{V_I} d^4p \left[ - \left( |C_{p1}^I|^2 + |C_{p2}^I|^2 \right) \exp(-c\epsilon_p\beta) + \left( |C_{p3}^I|^2 + |C_{p4}^I|^2 \right) \exp(c\epsilon_p\beta) \right] + (98)$$

$$+ \int_{V_{II}} d^4p \left[ - \left( |C_{p1}^{II}|^2 + |C_{p2}^{II}|^2 \right) \exp(c\epsilon_p\beta) + \left( |C_{p3}^{II}|^2 + |C_{p4}^{II}|^2 \right) \exp(-c\epsilon_p\beta) \right] -$$

$$- i \int_{V_{III}} d^4p \left[ \left( C_{p1}^{III} + C_{p2}^{III} \right) \exp(ic\epsilon_p\beta) - c.c. \right],$$

where the quantity $\epsilon_p = |p^2|^{1/2}$ was introduced, and we have taken the $\gamma^5$-variant of the theory for definiteness (see Eq. (87)). It is seen that all the spinor modes generate their input to the statistical sum of the theory. From this fact and using the formula

$$\bar{m} = \frac{<\bar{\Psi}|\bar{m}|\bar{\Psi}>}{<\bar{\Psi}|\bar{\Psi}>} = c^2 (\log |Z|)_{\bar{C},\bar{\beta}}, \tag{99}$$

which can be proved directly, one concludes that all the spinor modes give their input to the expectation value $\bar{m}$ of the mass of this quantum system. Together, Eqs. (98) and (99) represent the mass generation mechanism in the theory (90), which does not contain any mass parameter fixed. Of course, this mass generation mechanism is based on the presence of an extra dimension $\tau$ of special type in the theory. We will illustrate this mechanism by one interesting specific example in the next subsection.

### 6.3. Tachyons in theory: invisibility and dark matter source

It is important to note that

$$<\psi_k(p)|\psi_k(p)> = 0 \tag{100}$$

if $p^2 < 0$; this relation follows from Eq. (97) for the case of imaginary value of $\epsilon_{pk}$. Eq. (100) means that every tachyonic mode itself has no effect on the mass expectation value $\bar{m}$ and it is possible to speak about tachyons as about ‘invisible matter’. However, from Eq. (98) it follows that the nontrivial combinations of tachyons with different polarizations generate the non-vanishing term in the function $Z$. Actually, this term is related to the integration over the subspace $V_{III}$ of the momentum space. Thus, we have a possibility to identify the tachyons with invisible matter with non-trivial mass (see Eq. (99) which relates $\bar{m}$ with $Z$). Finally, the theory can be considered as describing the ‘spinor world’ (or the ‘spinor universe’) with the sector of visible matter which consists of particles and antiparticles ($p^2 \geq 0$), and invisible sector constructed of tachyons ($p^2 < 0$). It is possible to introduce the ‘defect of mass’ $\Delta \bar{m}$,

$$\Delta \bar{m} = \bar{m} - \bar{m}_0, \tag{101}$$
which can be identified with the dark mass in this system. Here the quantity $\bar{m}_0$ is understood as the mass expectation value, calculated over the visible sector of the theory states. Of course, tachyons are the reason of the dark matter in the modified spinor dynamics predictions.

Let us present one exact tachyonic solution with the explicit mass formula. This solution is defined by the parameters, which satisfy the relation

$$-i \left( C_{p1}^{III} C_{p3}^{III} - C_{p2}^{III} C_{p4}^{III} \right) = r \exp(-i\alpha) \delta(p - p_0),$$

where $r$ and $\alpha$ are the arbitrary real constants, and $\epsilon_0 = \sqrt{|p_0^2|}$. Performing the calculations, one obtains the following remarkable result:

$$\bar{m}_{tach} = \frac{\epsilon_0}{c} \tan (\alpha \epsilon_0 \beta + \alpha).$$

Thus, for this simplest nontrivial tachyonic state, expectation value for the mass depends on the temperature parameter $\beta$, i.e. $\bar{m}_{tach} = \bar{m}_{tach}(\beta)$. This is the illustration of the general fact that the dark mass which is generated by the tachyons has temperature dependence, as it follows from Eqs. (98)–(99). Note, that the standard Dirac’s modes with $p^2 \geq 0$ has not such kind of thermodynamic properties, because for such pure states $\bar{m} = \epsilon_0 = \text{const}$. Thus, our theory gives an important prediction related to the dark mass problem: the dark mass of the system must depend on the temperature of this system. Alternative approaches to the dark matter problem can be found in [23]–[25].

### 6.4. Relations between Noether currents in totally analytic theory

In this subsection we come back to the study of analytical properties of the modified spinor dynamics. First of all: Eqs. (90) can be rewritten in the following equivalent form:

$$\gamma^M \Psi, x_M = 0, \quad \Psi, \beta = -\frac{i\hbar}{2} \Psi, x^4,$$

where the new coordinate notation $x^4 = ct$ was introduced. Then, the first relation in (104) can be derived as the Euler-Lagrange equation from the five-dimensional action

$$S_5 = \int d^5x L_5,$$

where the five-dimensional Lagrangian is given by the relation

$$L_5 = \frac{i}{2} \left( \overline{\Psi} \gamma^M \Psi, x^M - \overline{\Psi, x^M} \gamma^M \Psi \right).$$

Thus, we have established the Lagrangian structure in the theory under consideration, which can be used for study of canonical structures and relations between them.

Of course, only the coordinate $x^4$ possesses a ‘partner’ (the temperature parameter $\beta$) in this system; this fact gives rise to the second relation in Eq. (104). Together with another fact that $t = x^4/c$ is the real part of complex Hamilton’s parameter of evolution, we will define the set of Noether charges with respect to split of the (formal) five-dimensional space-time $x^M$ to the original four-dimensional space-time $x^\mu$ and the evolutionary variable $x^4$. Namely, we define the Noether charges as quantities, which do not depend explicitly on the time coordinate $x^4$, and which are constructed as integrals over the coordinates $x^\mu$ (with the differential volume element $d^4x = dx^0 dx^1 dx^2 dx^3$). Performing the standard analysis, we first obtain the gauge charge $Q$,

$$Q = \int d^4x J^4,$$
which is related via integration to the 4-th component of the gauge current four-dimensional vector field

\[ J^N = \bar{\Psi} \gamma^N \Psi. \]  

(108)

The statement is, that the identities

\[ \partial_x^N J^N = 0, \quad \partial_x^4 Q = 0 \]  

(109)

hold (and that the first relation in Eq. (109) is the consequence of the second one there). Then, the second Noether charge is the energy-momentum five-dimensional vector \( P_M \), which is defined by the relation

\[ P_M = \int d^4x T_M^N, \]  

(110)

where \( T_M^N \) is the 4-th component of the five-dimensional tensor of energy-momentum:

\[ T_M^N = \frac{i}{2} \left( \bar{\Psi} \gamma^N \Psi, x^M - \bar{\Psi}, x^M \gamma^N \Psi \right). \]  

(111)

Again, the statement is that the equalities

\[ \partial_x^N T_M^N = 0, \quad \partial_x^4 P_M = 0 \]  

(112)

are satisfied on-shell, and that the second relation follows from the first one in Eq. (112). These properties of the modified spinor dynamics are common with the standard Dirac’s theory. The unusual facts are related to the ‘hidden’ \( \beta \)-dependence of all quantities, which lead to surprising relations between some of them here. Namely, it is easy to prove that the relations

\[ T_A^N = -\frac{1}{\hbar c} \partial_\beta J^N, \quad P_A = -\frac{1}{\hbar c} \partial_\beta Q \]  

(113)

are satisfied in this theory. For example, the second of them allows one to calculate the conserving quantity \( P_A \) using the gauge charge \( Q \) through its \( \beta \)-dependence. Note that Eq. (112) can be obtained from Eq. (109) using the differentiation over the temperature parameter \( \beta \) as direct consequence. Finally, it can be said, that in this theory the symmetry with respect to the \( x^A \)-direction follows from the gauge symmetry. In standard theory, such kind of symmetries are completely independent.

At the end of this section let us generalize the modified spinor dynamics to the totally analytic case. It will be shown that this generalization is supported by the symmetry structure of the theory. Namely, let us complexify the coordinates \( x^\mu \). Then, the set of five-dimensional coordinates can be represented as

\[ X^M = x^M - \frac{i\hbar}{2} \beta^M, \]  

(114)

and the theory wave-function \( \Psi = \Psi(X^M) \) will be defined by the dynamical relation

\[ \gamma^M \Psi, X^M = 0. \]  

(115)

Performing the Noether analysis will lead to the same set of the Noether charges. The remarkable fact is, that, now the relation between translation and gauge symmetries has the covariant form:

\[ T_M^N = -\frac{1}{\hbar c} \partial_\beta J^N, \quad P_M = -\frac{1}{\hbar c} \partial_\beta Q. \]  

(116)

Thus, the gauge symmetry generates all the translation shifts (in the directions \( x^M \)), and the total set of corresponding conserving charges \( P_N \) can be calculated using the \( \beta^N \)-dependence of the single gauge charge \( Q \).
7. Perspectives: dynamical reduction mechanism and effective gravity appearing

Double time structure provides the evident problems with the interpretation of the theory. Actually, in the conventional physics, time is a one-dimensional real parameter, which is identified with the Minkowaskian temporal coordinate $x^0$. In fact, the standard theoretical framework admits only the four-dimensional coordinates $x^\mu$ in the role of independent space-time parameters. This means that all possible extra (Kaluza-Klein type) dimensions must be compactified; the resulting physical theory must be expressed in terms of the four-dimensional Minkowskian coordinates only. In accordance with this ‘hard’ paradigm, the extra times (as well as all other extra or hidden parameters) must be fixed as functions of the coordinates $x^\mu$ using well-motivated compactification principle. Finally, the reduced four-dimensional theory will contain a set of the additional four-dimensional fields, which are the fields of compactification under discussion.

These compactification fields can describe different physical interactions in the theory, which can be taken as originally free ones. In this section we propose to relate the fields, generated by extra parameters of the theory, with gravity in the reduced four-dimensional system. The gravity arisen is not fundamental, but the effective field only. The effective metrics is expressed in terms of the solution $\Psi$ of the quantum dynamical relations of the theory. We start our consideration with the modified theory of the scalar field, and continue our study to the more interesting case of the totally analytic quantum theory of the spinor field. In fact, we present two approaches for constructing the effective gravity in the system. In the framework of the first one, gravity effects can be calculated on the soliton trajectory only. In the second approach, we construct gravity as the physical field on the full four-dimensional physical space-time. The results of this section are based on the publications [10]–[11].

7.1. Modified Klein–Gordon system and metrics generation

Let us start our effective gravity construction from the study of the case of a modified scalar theory. This simplest four-dimensional relativistic theory arises, if one takes

$$\hat{E} = \eta^{\mu\nu} \hat{p}_\mu \hat{p}_\nu / 2m,$$

where $m = \text{const}$ is some fixed parameter of the mass type, and $\hat{\Gamma} = 0$. Then, for the modified Hamilton’s equations (37) one obtains:

$$\frac{dx^\mu}{dt} = \frac{1}{m} \left\{ \eta^{\mu\nu} p_\nu - \eta^{\nu\lambda} \left[ (A^{-1}_2)^{\mu\sigma}_\nu S_{1,\sigma\nu\lambda} + \beta \left( A^{-1}_2 A_1 \right)^{\mu\nu}_\sigma p_\lambda \right] \right\},$$

$$\frac{dp_\mu}{dt} = -\frac{1}{m} \eta^{\nu\lambda} \left[ (A_1 A^{-1}_2)^{\sigma}_\mu S_{1,\sigma\nu\lambda} + \beta \left( A^{-1}_2 A^{-1}_1 \right)^{\mu\nu}_\sigma p_\lambda \right].$$

(118)

It is easy to see that the trajectory, which corresponds to Eq. (118), becomes the straight line in the case of vanishing correlations. Moreover, in this hard classical limit, the relation $p_\mu = \eta_{\mu\nu} m \frac{dx^\nu}{dt}$ (which follows from the first relation in Eq. (118)), allows one to identify the Hamilton’s time $t$ with the proper time of the moving point mass $m$ in this theory.

To establish the effective gravity in the system, let us compare the relations (118) with conventional equations of motion of the point mass $m$ propagating in the gravitational field with the metrics $g_{\mu\nu}$:

$$\frac{dx^\mu}{dt} = \frac{1}{m} g^{\mu\nu} p_\nu, \quad \frac{dp_\mu}{dt} = -\frac{1}{2m} g^{\nu\lambda}_\mu p_\nu p_\lambda.$$  

(119)

We will study the case of weak correlations, i.e. the dynamics of the well-localized soliton in the framework of Eq. (118). It is clear that we must compare it with the regime of the weak gravity in Eq. (119), where it is convenient to put

$$g_{\mu\nu} = \eta_{\mu\nu} + \xi_{\mu\nu}$$

(120)
with \( g_{(1)\mu\nu} \to 0 \). We state that it is possible to identify Eq. (118) at \( A_2^{-1} \to 0 \) with the system (119)–(120), using some highly non-trivial temperature mode \( \beta = \beta(t) \). The result has the especially simple form in the reference frame, which coincides with initial rest frame of the mass. For the infinitesimal potential \( \xi_{00} \) the result of this identification reads:

\[
\xi_{00} = \left( A_2^{-1} \right)^{0\mu} \left[ \frac{1}{mc} \eta^{\nu\lambda} S_{1,\nu\mu\lambda} + \dot{\beta} (A_1)_{\mu0} \right].
\]

Note that this potential defines the non-relativistic limit of the General Relativity and provides the special interest for different application.

Thus, uncertainty in the position and momentum value of the point mass can be interpreted by physical observer as a presence of some non-trivial gravity in the system. Note that this gravity is calculated on the classical trajectory only, i.e. on the trajectory of maximums of the probability density. Thus, this approach cannot give information about the effective gravity as about the field defined over the whole four-dimensional physical space-time.

### 7.2. Reduction of extra parameters and source of quantum gravity: basic conception

Now let us come back to the modified spinor dynamics with the motion equations (90). Our goal is the reduction principle, which allows one to extract the additional variables \( x^4 \) and \( \beta^M \) from all observables of the theory. Thus, we are interested in establishing the relations

\[
x^4 = x^4(\mu), \quad \beta^M = \beta^M(\mu),
\]

with some specified right sides. The reduction postulate, which we propose reads: the functions (121) must realize a maximum of the probability density at a given space time point \( x^\mu \):

\[
\max_{x^4, \beta^M} \rho \left( x^\mu, x^4, \beta^M \right) = \rho \left( x^\mu, x^4(\mu), \beta^M(\mu) \right).
\]

Thus, this principle fixes all extra parameters in the framework of the maximal probability arguments. Exactly the same situation takes place in the Hamilton equations construction, see Eq. (36). Performing calculations, like the ones leading to Eq. (37), we obtain the following defining relations for the functions (121):

\[
x^4_{\mu, \nu} = -\frac{1}{\rho_{\mu, x^4 x^4}} \left[ \rho_{\mu, x^4 x^\nu} - (Q^{-1})_{MN} \rho_{\mu, x^4 x^\nu} \left( \rho_{\nu, x^4 x^M} - \frac{\rho_{\nu, x^4 x^N} \rho_{x^4 x^M}}{\rho_{x^4 x^4}} \right) \right],
\]

\[
\beta^M_{\mu, \nu} = -\left( Q^{-1} \right)_{MN} \rho_{\nu, x^4 x^\nu} \left( \rho_{\nu, x^4 x^M} - \frac{\rho_{\nu, x^4 x^N} \rho_{x^4 x^M}}{\rho_{x^4 x^4}} \right),
\]

where the \( 5 \times 5 \)-matrix \( Q \) is defined in its components as

\[
Q_{MN} = \rho_{\nu, x^4 x^M} - \frac{\rho_{\nu, x^4 x^N} \rho_{x^4 x^M}}{\rho_{x^4 x^4}}.
\]

Note, that the parameters \( \beta^M \) describe the anisotropy in the probability density distribution related to the given solution (as imaginary parts of the complex coordinates, see Eq. (114)). Thus, the fields (121) specified by Eqs. (123)–(124) give corresponding localized anisotropy fields in the theory. In the standard approach, the fundamental anisotropy properties of the system can be related to gravity. Actually, metric coefficients \( g_{\mu\nu} = g_{\mu\nu}(x^4) \) can be used for the adequate anisotropy of space-time description. Our main idea is to try to calculate an effective metrics which exactly corresponds to the anisotropy functions defined by Eq. (124). Such gravity will describe the isotropy breaking in the theory. The relations (121) can be also interpreted as thermodynamic mode of the generalized type, see Eq. (19) for comparison.
7.3. Relationship between thermodynamic mode and metrics

The simplest way to detect the effective gravity in a system is related to the study of the squared mass operator behavior under the reduction process (we present a simplified approach here). First of all, let us define the reduced spinor field \( \psi(x^\mu) = \Psi[X_N(x^\mu)] \).

Let us also introduce the following convenient quantities, which are defined by the reduction relations (124) completely:

\[
X^\lambda_{\nu, x^\mu} = \delta^\lambda_\nu - \frac{i\hbar c}{2} \beta^\lambda_{, x^\mu} \equiv A^\lambda_\mu, \quad X^4_{\nu, x^\mu} = x^4_{, x^\mu} - \frac{i\hbar c}{2} \beta^4_{, x^\mu} \equiv B^\mu.
\]

Then, performing straightforward calculations, one obtains the following important relation:

\[
\psi_{, x^\mu} = D^\lambda_\mu \Psi_{, x^\lambda},
\]

where

\[
D^\lambda_\mu = A^\lambda_\mu - \left(\gamma^4\right)^{-1} \gamma^\lambda B^\mu.
\]

Using this, it is possible to establish the reduction formula for the squared operator of momentum of the system (or its squared mass operator). The result reads:

\[
\eta^\sigma_\lambda \hat{p}_\sigma \hat{p}_\lambda \Psi = \eta^\sigma_\lambda \left(D^{-1}\right)^\nu_\sigma \left(D^{-1}\right)^\mu_\lambda \hat{p}_\mu \hat{p}_\nu \psi + i\hbar \eta^\sigma_\lambda \left(D^{-1}\right)^\nu_\lambda \left(D^{-1}\right)^\mu_{, x^\nu} \hat{p}_\mu \psi.
\]

Our introduction of the effective metrics \( g_{\mu\nu} \) in this system is related to appropriate interpretation of Eq. (129). Namely, its left side gives distribution of the squared mass operator of the ‘whole’ system:

\[
\eta^\sigma_\lambda \hat{p}_\sigma \hat{p}_\lambda \Psi = \bar{m}^2 c^2 \Psi;
\]

we will name this operator as the ‘total’ one here. Then, the first term on the right side of Eq. (129) can be rewritten as

\[
\bar{g}^{\mu\nu} \hat{p}_\mu \hat{p}_\nu \psi = \bar{m}^2 \text{red} c^2 \psi,
\]

where the metric-type operator \( \bar{g}^{\mu\nu} \) can be expressed in terms of the matrices inverse with respect to the ones given by Eq. (128):

\[
\bar{g}^{\mu\nu} = \frac{1}{2} \eta^{\lambda\sigma} \left\{ \left(D^{-1}\right)^\nu_\sigma, \left(D^{-1}\right)^\mu_\lambda \right\}.
\]

We identify \( \bar{m}^2 \text{red} \) with the squared mass operator for the reduced wave function \( \psi \), given by Eq. (125).

In this context, the operator (132) becomes identified with the inverse metric operator in the system under consideration. Its expectation value \( \bar{g}^{\mu\nu} \) can be calculated using the general formalism presented in subsection 6.1. Finally, the third term in Eq. (129) can be identified with additional squared mass term, related to the compactification process. We will not discuss it here leaving the corresponding discussion to other works.
7.4. Cosmological solution

We relate the spatially homogeneous and isotropic solutions of the Eq. (90) with cosmological models in effective gravity approach presented in the previous subsections. Here we study these solutions and discuss their remarkable properties. First of all, let us describe them. We will work with subspace of solution space of the theory with

$$\Psi_{x_i} = 0$$  (133)

(where \(i = 1, 2, 3\)), which realizes homogeneity and anisotropy in the sense of three-dimensional space. Thus, the corresponding wave-function \(\Psi\) will depend on the ‘time-like’ parameters \(x^0\) and \(\tau\) only. Let us consider the \(\gamma^5\)-variant of the theory; then, the general solution of Eq. (90) reads:

$$\Psi = \Psi_+ + \Psi_-,$$  (134)

where

$$\Psi_\pm = \frac{1}{\sqrt{2}} \left( f_\pm + i f_\pm \right),$$  (135)

and the 2-columns \(f_\pm\) are defined as

$$f_\pm = f_\pm \left( x^0 \mp c\tau \right).$$  (136)

It is clear that the special solutions \(\Psi_\pm\) represent the solitons running forward and backward in respect to Minkowskian time \(x^0\), when the Hamilton’s time \(x^4\) increases. Such solutions contain particles and antiparticles, respectively, and can be named as ‘universe’ and ‘anti-universe’ in the framework of natural speculation. The total Universe is a superposition of universe and anti-universe in this picture. Then, the observer living in the universe (i.e., constructed of the particles) will not see the anti-universe in his observations, if the anti-universe is described by the wave function of the highly-localized probability density far from the time of the universe and anti-universe solitons ‘collision’. This speculation gives explanation to the particle/antiparticle asymmetry in the real Universe.

Then, it is very important to stress that the cosmological solution (134)–(136) of the quantum motion equations (90) possesses coherent states. These states are described by the special solitons with

$$f_\pm = c_\pm \exp \left[ - \frac{(x^0 \mp c\tau)^2}{4D_{x^0}^2} \right],$$  (137)

where \(c_\pm\) are the complex constant 2-columns, and \(D_{x^0}\) is the real positive constant parameter. It is possible to prove that Eq. (137) gives the Gauss wave-packet with dispersion \(D_{x^0}\) of the probability density near its maximal value at the average value \(\bar{x}^0 = \pm x^4\) of the Minkowskian time \(x^0\). Thus, \(\pm x^4\) corresponds to the center of this distribution; i.e. the Hamilton’s time \(x^4\) can be interpreted as the ‘averaged’ Minkowskian time up to the sign here.

Now let us consider the Universe with vanishing anti-universe mode, i.e. let us take \(\Psi = \Psi_+\). Performing the calculations, one obtains the following results:

$$\tilde{m} = \left( \frac{\hbar}{2D_{x^0}} \right)^2 \beta$$  (138)

– the explicit mass formula, and the uncertainty relation

$$D_{mc^2}D_{x^0} = \frac{\hbar}{2}$$  (139)
with equality realized. Thus, the family (125) of the Gauss soliton solutions actually describes the coherent states of the theory. These states minimize the uncertainty relation between the mass and Minkowskian time. Then, Eq. (138) shows that the established mass formula $\bar{m} = \bar{m}(\beta)$ is completely defined by the nontrivial $\sim \beta$ term, which is inverse proportional to the squared dispersion $D^2_x$ of the probability distribution. It is possible to say that the mass of the state is generated by the nontrivial time dispersion of the probability density to find the Dirac particle at a given moment of the physical time. It is seen that $\bar{m}(0) = 0$, so the mass expectation value vanishes at the infinite temperature for the solitons under study. This seems natural in the framework of the light dominated period of the Universe cosmological evolution scenario.

8. Conclusions
Let us summarize the main results of this publication.

(i) Quantum Mechanics modification is done via the non-Hermitian Hamiltonian and complex parameter of evolution;
(ii) Decay operator is taken as the symmetry operator, this maintains the arrow of time appearing at the quantum level;
(iii) Double limit defines the transition to the classical dynamics with the Planck constant and correlations as infinitesimal parameters. Particles can be presented as solitons in the theory;
(iv) Correlations in the soft limit generate the General Relativity type of correction for Newtonian gravity force;
(v) Soft limit with the non-zero correlations keep the arrow of time in the classical dynamics;
(vi) Relativistic extension based on the Poincare group includes the invariant complex parameters of evolution and relativistic Hamiltonian as the Poincare group generator;
(vii) Wave function multiplication approach leads to appearance of the arrow of time in the relativistic version of the theory. Dynamic parity violation is one of the consequences of the arrow of time appearance;
(viii) Dirac system can be modified as well. No mass introduction required in this theory - the system generates the mass itself;
(ix) Gravity can be constructed as an effective field in the modified relativistic theory. To construct the metrics, one must compactify all extra parameters of the theory.

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