ESTIMATES FOR GREEN FUNCTIONS OF STOKES SYSTEMS IN TWO DIMENSIONAL DOMAINS

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Abstract. We prove the existence and pointwise bounds of the Green functions for stationary Stokes systems with measurable coefficients in two dimensional domains. We also establish pointwise bounds of the derivatives of the Green functions under a regularity assumption on the $L_1$-mean oscillations of the coefficients.

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ and $\mathcal{L}$ be a differential operator in divergence form acting on column vector valued functions $u = (u^1, u^2)^\top$ as follows:

$$\mathcal{L}u = D_\alpha (A^{\alpha\beta} D_\beta u),$$

where the coefficients $A^{\alpha\beta}$ are $2 \times 2$ matrix-valued functions on $\Omega$, which satisfy the strong ellipticity condition (2.1). The Green function of the operator $\mathcal{L}$ is a pair $(G, \Pi) = (G(x, y), \Pi(x, y))$, where $G$ is a $2 \times 2$ matrix-valued function and $\Pi$ is a $1 \times 2$ vector-valued function, such that if $(u, p)$ is a weak solution of the problem

$$\begin{cases}
\text{div } u = g - (g)_\Omega & \text{in } \Omega, \\
\mathcal{L}^* u + \nabla p = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}$$

with bounded data, then the solution $u$ is given by

$$u(y) = -\int_{\Omega} G(x, y)^\top f(x) \, dx + \int_{\Omega} \Pi(x, y)^\top g(x) \, dx. \quad (1.1)$$

Here, $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$ and $(g)_\Omega = \frac{1}{|\Omega|} \int_{\Omega} g \, dx$. For a more precise definition of the Green function, see Section 2.3. We sometimes call this the Green function for the flow velocity of $\mathcal{L}$ because of the representation formula (1.1) for the flow velocity $u$.

In this paper, we prove that if the divergence equation is solvable in a bounded domain $\Omega \subset \mathbb{R}^2$ with an exterior measure condition (3.1), then there exists a unique Green function $(G, \Pi)$ of $\mathcal{L}$ having the logarithmic pointwise bound

$$|G(x, y)| \leq C \left( 1 + \log \left( \frac{\text{diam}(\Omega)}{|x - y|} \right) \right), \quad \forall x, y \in \Omega, \; x \neq y.$$

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For further details, see Theorem 3.2. We emphasize that we do not impose any regularity assumptions on the coefficients $A^{\alpha\beta}$ of $L$. Moreover, the assumption on the domain is sufficiently general to allow $\Omega$ to be, for example, a John domain with the exterior measure condition (3.1). Hence, the class of domains we consider includes Lipschitz domains, Reifenberg flat domains, and Semmes-Kenig-Toro (SKT) domains. We also prove the following $L_\infty$-estimate away from $\partial \Omega$:

$$\text{ess sup}_{B_{\frac{3}{4}|x-y|}(x)} (|DG(\cdot, y)| + |\Pi(\cdot, y)|) \leq C|x-y|^{-1} \quad (1.2)$$

under the assumption that $A^{\alpha\beta}$ are of partially Dini mean oscillation (i.e., they are merely measurable in one direction and have Dini mean oscillations in the other direction). For further details, see Theorem 3.5. The above estimate holds globally, i.e.,

$$|DG(x, y)| + |\Pi(x, y)| \leq C|x-y|^{-1}, \quad \forall x, y \in \Omega, \quad x \neq y, \quad (1.3)$$

when $A^{\alpha\beta}$ are of Dini mean oscillation in all the directions and $\Omega$ has a $C^1$ Dini boundary; see Theorem 3.7. As far as the existence of the Green function is concerned, the coefficients $A^{\alpha\beta}$ need only be measurable. Stokes systems with irregular coefficients of this type are partly motivated by the study of inhomogeneous fluids with density dependent viscosity and multiple fluids with interfacial boundaries; see [17, 19, 1, 13]. Moreover, they can be employed to describe the motion of a laminar compressible viscous fluid; see [24].

Green functions play an important role in the study of boundary value problems, in particular, in establishing the existence, uniqueness, and regularity of solutions to PDEs. We refer the reader to [11, 30], where the authors utilized Green function estimates for the existence and non-tangential maximal function estimates of harmonic functions satisfying certain boundary conditions. In [29, 3], the authors used the Green function for the uniqueness of solutions to elliptic equations. Regarding the classical Stokes system, we refer to [15, 28, 23, 26] for the usage of Green functions in establishing the existence of solutions with non-tangential or $L_p$-estimates. By using our results in this paper, one may study the problems in the aforementioned papers for Stokes systems with variable coefficients in two dimensional irregular domains.

There is a large body of literature concerning Green functions of Stokes systems. With respect to the classical Stokes system

$$\Delta u + \nabla p = f,$$

we refer the reader to Ladyzhenskaya [18], Maz’ya-Plamenevskiĭ [20, 21], Fabes-Kenig-Verchota [15], and D. Mitrea-I. Mitrea [25]. In [18], the author provided an explicit formula for the fundamental solution in two and three dimensions. In [20, 21], the authors established the existence and pointwise estimate of the Green function of a Dirichlet problem in a piecewise smooth domain in $\mathbb{R}^3$. The corresponding results were obtained in [15] and [25] on Lipschitz domains in $\mathbb{R}^d$, where $d \geq 3$ and $d \geq 2$, respectively. For Green functions of mixed problems, one can refer to the work of Maz’ya-Rossmann [22] in three dimensional polyhedral domains and Ott-Kim-Brown [27] in two dimensional Lipschitz domains. Regarding Stokes systems with variable coefficients

$$L u + \nabla p = f,$$
we refer the reader to [9, 10, 7]. In [9], the authors established the existence and pointwise estimate of the Green function of a Dirichlet problem in a bounded $C^1$ domain when $d \geq 3$ and the coefficients of $\mathcal{L}$ have vanishing mean oscillations. The corresponding results were obtained in [10] on the whole space and a half space when coefficients are merely measurable in one direction and have small mean oscillations in the other directions (partially BMO). In [7], the authors constructed Green function for a conormal derivative problem when coefficients are variably partially BMO. We also refer the reader to [16] for Green functions of Stokes systems with oscillating periodic coefficients.

Note that all of the above mentioned results for Green functions of Stokes systems with variable coefficients are limited to the case that $d \geq 3$. In this paper, as mentioned as an interesting problem in [27], we extend and apply the method used in the construction of Green function of the classical Stokes system to Stokes systems with non-constant coefficients when $d = 2$. Because we are unable to find any literature dealing with Green functions of Stokes systems with variable coefficients in two dimensional domains, we anticipate that our results fill a gap in the literature for the two dimensional case. The only literature we have found is a recent paper [6], where the authors treated a Green function for the representation formula of the pressure when $d \geq 2$ and coefficients are of (partially) Dini mean oscillation. Indeed, the method employed in this paper is applicable to higher dimensional cases, but it is questionable whether one can obtain the same generality achieved in [9, 10, 7] by using the approach in this paper. In particular, to establish the existence of Green functions for $d \geq 3$ following the steps in this paper, one needs global $W^1_q$-estimates, $q > d$, which require stronger regularity assumptions on the coefficients and on the boundary of the domain than those, for instance, in [7]. On the other hand, for the two dimensional case with variable coefficients, one may consider applying the method used in [9, 10, 7], which is based on local $C^\alpha$ and $L^\infty$-estimates, as well as a global $W^1_2$-estimate. Here, by the global $W^1_2$-estimate we mean

$$\|Du\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)} \leq C \left( \|f\|_{L^2(\Omega)} + \|f^\alpha\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} \right),$$

where $d \geq 3$, $2^# = \frac{2d}{d+2}$, and $(u, p)$ is a weak solution of

$$\begin{cases}
\mathcal{L}u + \nabla p = f + D_\alpha f, & \text{in } \Omega, \\
\text{div } u = g & \text{in } \Omega,
\end{cases}$$

with some boundary condition. The estimate (1.4) is optimal in the sense that the constant $C$ does not depend on the size of $\Omega$ when $\Omega$ is a ball. However, if $d = 2$, such an estimate is not true. Indeed, the estimate (1.4) holds with $q > 1$ in place of $2^# = 1$, which is not optimal in the aforementioned sense, nor is well suited to providing necessary estimates of the Green function when $d = 2$. Thus, to apply the method used in the higher dimensional case to the two dimensional case, we need some modifications, which seem inevitable because the higher dimensional Green function and the two dimensional Green function have different types of pointwise bounds. Rather than modifying the method for $d \geq 3$, in this paper we take a straightforward approach so that we directly derive the Green function. We recall that in the higher dimensional case, the Green functions are obtained by an approximation argument.
Some remarks are in order regarding the approach in this paper. In fact, there are several paths to constructing Green functions with logarithmic growth for Stokes systems and elliptic systems in two dimensional domains. In many references, for instance [12, 27], the construction of Green functions relies on the existence of a solution with gradient estimates in the weak Lebesgue space $L_{2,\infty}(\Omega)$. In this paper, we derive the gradient estimates by adapting the idea of Dolzmann-Müller [12], where the authors constructed Green functions for elliptic systems with the zero Dirichlet boundary condition on bounded domains in $\mathbb{R}^d$ ($d \geq 2$) with a $C^1$ or Lipschitz boundary. For the $L_{2,\infty}$-estimate, we utilize $W_1^q$-estimate and solvability for the Stokes system together with real interpolation, where the $W_1^q$-estimate follows from the reverse Hölder’s inequality. In [27], by using complex interpolation the authors derived the $L_{2,\infty}$-estimate for the Green function of the classical Stokes system with a mixed boundary condition in a Lipschitz domain.

For another approach to constructing Green functions, we refer the reader to [14], where the authors construct Green functions for elliptic systems in a (possibly unbounded) domain in $\mathbb{R}^2$ by integrating parabolic Green functions in $t$ variable.

The estimates of Green functions are closely related to the regularity theory of solutions. In particular, for the bounds of the derivatives of the Green function such as (1.2) and (1.3), solutions of the system are required to have bounded gradients, which are not available for Stokes systems and elliptic systems with measurable coefficients. For this reason, we need to impose certain regularity assumptions on the coefficients and domains. In this paper, for the estimates (1.2) and (1.3), we utilize the results given in [5, 4], where the authors proved $W_1^\infty$ and $C^1$-estimates for Stokes systems with coefficients having (partially) Dini mean oscillations.

The remainder of this paper is organized as follows. We introduce some notation and definitions in the next section. In Section 3 we state the main theorems. In Section 4 we present some auxiliary results, and in Section 5 we provide the proofs of the main theorems.

2. Preliminaries

2.1. Notation. Throughout the paper we denote by $\Omega$ a bounded domain in the Euclidean space $\mathbb{R}^2$. For any $x \in \mathbb{R}^2$ and $r > 0$, we write $\Omega_r(x) = \Omega \cap B_r(x)$, where $B_r(x)$ is the usual Euclidean disk of radius $r$ centered at $x$. For $q \in [1, \infty]$, we denote by $W_1^q(\Omega)$ the usual Sobolev disk of radius $r$ centered at $x$. For $q \in [1, \infty]$, we denote by $W_1^q(\Omega)$ the usual Sobolev space and $\tilde{W}_1^q(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W_1^q(\Omega)$. We also define the weak $L_q$ space, denoted by $L_{q,\infty}(\Omega)$, as the set of all measurable functions on $\Omega$ having a finite quasi-norm

$$
\|u\|_{L_{q,\infty}(\Omega)} = \sup_{t>0} t \left| \{ x \in \Omega : |u(x)| > t \} \right|^{1/q}.
$$

We define

$$
\tilde{L}_q(\Omega) = \{ u \in L_q(\Omega) : (u)_\Omega = 0 \},
$$

$$
\tilde{L}_{q,\infty}(\Omega) = \{ u \in L_{q,\infty}(\Omega) : (u)_\Omega = 0 \},
$$

$$
\tilde{W}_1^q(\Omega) = \{ u \in W_1^q(\Omega) : (u)_\Omega = 0 \},
$$

where $(u)_\Omega$ is the average of $u$ over $\Omega$, i.e.,

$$
(u)_\Omega = \int_{\Omega} u \, dx = \frac{1}{|\Omega|} \int_{\Omega} u \, dx.
$$
We recall that
\[ \|u + v\|_{L^q(\Omega)} \leq 2\left(\|u\|_{L^q(\Omega)} + \|v\|_{L^q(\Omega)}\right) \quad \text{for all } u, v \in L^q(\Omega) \]
and
\[ L^q(\Omega) \subset L^s(\Omega) \subset L^r(\Omega) \quad \text{for } s < q. \]

We say that a measurable function \( \omega : (0, a] \to [0, \infty) \) is a Dini function provided that there are constants \( c_1, c_2 > 0 \) such that
\[ c_1 \omega(t) \leq \omega(s) \leq c_2 \omega(t) \quad \text{whenever } 0 < \frac{t}{2} \leq s \leq t \leq a \]
and that \( \omega \) satisfies the Dini condition
\[ \int_0^a \frac{\omega(t)}{t} \, dt < \infty. \]

**Definition 2.1.** Let \( f \in L^1(\Omega) \).

(a) We say that \( f \) is of **partially Dini mean oscillation** in (the interior of) \( \Omega \) if there exists a Dini function \( \omega : (0, 1] \to [0, \infty) \) such that for any \( x = (x_1, x_2) \in \Omega \) and \( r \in (0, 1] \) satisfying \( B_{2r}(x) \subset \Omega \), we have
\[ \int_{B_r(x)} \left| f(y) - \int_{B_r'(x_2)} f(y_1, s) \, ds \right| \, dy \leq \omega(r), \]
where \( B'_r(x_2) = \{ t \in \mathbb{R} : |t - x_2| < r \} \).

(b) We say that \( f \) is of **Dini mean oscillation** in \( \Omega \) if there exists a Dini function \( \omega : (0, 1] \to [0, \infty) \) such that for any \( x \in \overline{\Omega} \) and \( r \in (0, 1] \), we have
\[ \int_{\Omega_r(x)} \left| f(y) - \int_{\Omega_r(x)} f(z) \, dz \right| \, dy \leq \omega(r). \]

We define a **C\(^1\).Dini** domain by locally the graph of a \( C^1 \) function whose derivatives are uniformly Dini continuous.

**Definition 2.2.** We say that \( \Omega \) has a **C\(^1\).Dini** boundary if there exist a constant \( R_0 \in (0, 1] \) and a Dini function \( \varrho_0 : (0, 1] \to [0, \infty) \) such that the following holds: For any \( z = (z_1, z_2) \in \partial \Omega \), there exist a \( C^1 \) function \( \chi : \mathbb{R} \to \mathbb{R} \) and a coordinate system depending on \( z \), such that in the new coordinate system we have
\[ |\chi'(z_2)| = 0, \quad \Omega_{R_0}(z) = \{ x = (x_1, x_2) \in B_{R_0}(z) : x_1 > \chi(x_2) \}, \]
and
\[ \varrho_\chi(r) \leq \varrho_0(r) \quad \text{for all } r \in (0, R_0), \]
where \( \varrho_\chi \) is the modulus of continuity of \( \chi' \), i.e.,
\[ \varrho_\chi(r) = \sup \{ |\chi'(s) - \chi'(t)| : s, t \in \mathbb{R}, |s - t| \leq r \}. \]

2.2. **Stokes system.** Let \( \mathcal{L} \) be a strongly elliptic operator of the form
\[ \mathcal{L}u = D_\alpha (A^{\alpha\beta} D_\beta u), \]
where the coefficients \( A^{\alpha\beta} = A^{\alpha\beta}(x) \) are \( 2 \times 2 \) matrix-valued functions on \( \Omega \) with entries \( A^{\alpha\beta}_{ij} \) satisfying the strong ellipticity condition, i.e., there is a constant \( \lambda \in (0, 1] \) such that
\[ |A^{\alpha\beta}(x)| \leq \lambda^{-1}, \quad \sum_{\alpha, \beta = 1}^2 A^{\alpha\beta}(x) \xi_\beta \cdot \xi_\alpha \geq \lambda \sum_{\alpha = 1}^2 |\xi_\alpha|^2 \quad (2.1) \]
for any \( x \in \Omega \) and \( \zeta_\alpha \in \mathbb{R}^2 \), \( \alpha \in \{1, 2\} \). We do not assume that the coefficients \( A^{\alpha \beta} \) are symmetric. The adjoint operator \( \mathcal{L}^* \) is given by

\[
\mathcal{L}^* u = D_\alpha ((A^{\alpha \beta})^\top D_\beta u),
\]

where \((A^{\alpha \beta})^\top\) is the transpose of the matrix \( A^{\alpha \beta} \) for each \( \alpha, \beta \in \{1, 2\} \).

Let \( f \in L_{q_1}(\Omega)^2 \) and \( f_\alpha \in L_q(\Omega)^2 \), where \( q, q_1 \in (1, \infty) \) and \( q_1 \geq 2q/(q + 2) \). We say that \((u, p) \in \dot{W}^1_q(\Omega)^2 \times L_q(\Omega)\) is a weak solution of the problem

\[
\mathcal{L} u + \nabla p = f + D_\alpha f_\alpha \quad \text{in } \Omega,
\]

if

\[
\int_{\Omega} A^{\alpha \beta} D_\beta u \cdot D_\alpha \phi \, dx + \int_{\Omega} p \, \text{div} \phi \, dx = - \int_{\Omega} f \cdot \phi \, dx + \int_{\Omega} f_\alpha \cdot D_\alpha \phi \, dx
\]

holds for any \( \phi \in \dot{W}^1_{q/(q-1)}(\Omega)^2 \). Similarly, we say that \((u, p) \in \dot{W}^1_q(\Omega)^2 \times L_q(\Omega)\) is a weak solution of

\[
\mathcal{L}^* u + \nabla p = f + D_\alpha f_\alpha \quad \text{in } \Omega,
\]

if

\[
\int_{\Omega} A^{\alpha \beta} D_\beta \phi \cdot D_\alpha u \, dx + \int_{\Omega} p \, \text{div} \phi \, dx = - \int_{\Omega} f \cdot \phi \, dx + \int_{\Omega} f_\alpha \cdot D_\alpha \phi \, dx
\]

holds for any \( \phi \in \dot{W}^1_{q/(q-1)}(\Omega)^2 \).

2.3. Green function for the flow velocity. The following is the definition of the Green function of Stokes system. Here, \( G = G(x, y) \) is a \( 2 \times 2 \) matrix-valued function and \( \Pi = \Pi(x, y) \) is a \( 1 \times 2 \) vector-valued function.

**Definition 2.3.** We say that a pair \((G, \Pi)\) is the Green function (for the flow velocity) of \( \mathcal{L} \) in \( \Omega \) if it satisfies the following properties.

(i) For any \( y \in \Omega \) and \( r > 0 \),

\[
G(\cdot, y) \in \dot{W}^1_{q_1}(\Omega)^{2 \times 2}, \quad \Pi(\cdot, y) \in \dot{L}^1_{q_1}(\Omega)^2.
\]

(ii) For any \( y \in \Omega \), \((G(\cdot, y), \Pi(\cdot, y))\) satisfies

\[
\begin{cases}
\text{div } G(\cdot, y) = 0 & \text{in } \Omega, \\
\mathcal{L} G(\cdot, y) + \nabla \Pi(\cdot, y) = -\delta_y I & \text{in } \Omega,
\end{cases}
\]

in the sense that for \( k \in \{1, 2\} \) and \( \phi \in \dot{W}^1_{q_1}(\Omega)^2 \cap C(\Omega)^2 \), we have

\[
\text{div } G^k(\cdot, y) = 0 \quad \text{in } \Omega
\]

and

\[
\int_{\Omega} A^{\alpha \beta} D_\beta G^k(\cdot, y) \cdot D_\alpha \phi \, dx + \int_{\Omega} \Pi^k(\cdot, y) \text{div} \phi \, dx = \phi^k(y),
\]

where \( G^k(\cdot, y) \) is the \( k \)-th column of \( G(\cdot, y) \).

(iii) If \((u, p) \in \dot{W}^1_q(\Omega)^2 \times L_q(\Omega)\) is a weak solution of the problem

\[
\begin{cases}
\text{div } u = g & \text{in } \Omega, \\
\mathcal{L}^* u + \nabla p = f + D_\alpha f_\alpha & \text{in } \Omega,
\end{cases}
\]

(2.2)
where \( f, f_\alpha \in L_\infty(\Omega)^2 \) and \( g \in \tilde{L}_\infty(\Omega) \), then for a.e. \( y \in \Omega \), we have

\[
    u(y) = -\int_\Omega G(x, y)^T f(x) \, dx \\
    + \int_\Omega D_\alpha G(x, y)^T f_\alpha(x) \, dx + \int_\Omega \Pi(x, y)^T g(x) \, dx,
\]

where \( G(x, y)^T \) and \( \Pi(x, y)^T \) are the transposes of \( G(x, y) \) and \( \Pi(x, y) \).

The Green function of the adjoint operator \( \mathcal{L}^* \) is defined similarly.

Remark 2.4. The \( W_2^1 \)-solvability of Stokes system (Lemma 4.1) and the property (iii) in the above definition ensure the uniqueness of the Green function in the following sense. Let \( (\tilde{G}, \tilde{\Pi}) \) be another Green function satisfying the properties in assumption stating that the divergence equation is solvable, which is valid on, for instance, a John domain; see \cite{2} Theorem 4.1. As mentioned in Section 2, \( \Omega \) is a bounded domain in \( \mathbb{R}^2 \).

Assumption 3.1. There exists a constant \( K_0 > 0 \) such that the following holds: For any \( g \in L_2(\Omega) \), there exists \( u \in W_2^1(\Omega)^2 \) such that

\[
    \text{div } u = g \quad \text{in } \Omega, \quad \|Du\|_{L_2(\Omega)} \leq K_0\|g\|_{L_2(\Omega)}.
\]

Theorem 3.2. Let \( \Omega \) satisfy

\[
    |B_R(x_0) \setminus \Omega| \geq \theta R^2, \quad \forall x_0 \in \partial \Omega, \quad \forall R \in (0, 1]. \tag{3.1}
\]

Then under Assumption 3.1, there exist Green functions \( (G, \Pi) \) of \( \mathcal{L} \) and \( (G^*, \Pi^*) \) of \( \mathcal{L}^* \). Moreover, \( G \) and \( G^* \) are continuous in \( \{(x, y) \in \Omega \times \Omega : x \neq y\} \) and satisfy

\[
    G(x, y) = G^*(y, x)^T \quad \text{for all } x, y \in \Omega, \quad x \neq y. \tag{3.2}
\]

Furthermore, we have the following estimates.

(a) For any \( y \in \Omega \), we have

\[
    \|DG(\cdot, y)\|_{L_{\infty,2}(\Omega)} + \|\Pi(\cdot, y)\|_{L_{\infty,2}(\Omega)} \leq C. \tag{3.3}
\]

(b) There exists \( q_0 = q_0(\lambda, \theta, K_0) > 2 \) such that for any \( x \in \mathbb{R}^2, y \in \Omega, 0 < R < \text{diam}(\Omega) \) satisfying \( |x - y| > R \), we have

\[
    \|DG(\cdot, y)\|_{L_{q_0}(\Omega_R/2(x))} + \|\Pi(\cdot, y)\|_{L_{q_0}(\Omega_R/2(x))} \leq CR^{-\gamma}, \tag{3.4}
\]

and

\[
    [G(\cdot, y)]_{C^0(\Omega_R/2(x))} \leq CR^{-\gamma}, \tag{3.5}
\]

where \( \gamma = 1 - 2/q_0 \).

(c) For any \( x, y \in \Omega \) with \( x \neq y \), we have

\[
    |G(x, y)| \leq C \left(1 + \log \left(\frac{\text{diam}(\Omega)}{|x - y|}\right)\right). \tag{3.6}
\]
In the above, the constants \( C \) depend only on \( \lambda, \theta, K_0, \) and \( \text{diam}(\Omega) \).

**Remark 3.3.** In Theorem 3.2 because \( G(\cdot, y) \) satisfies the zero Dirichlet boundary condition, we have a better estimate than \( 3.4 \) near the boundary of \( \Omega \). Indeed, by (3.1) and \( G(\cdot, y) \equiv 0 \) on \( \partial \Omega \), it is easily seen that for any \( x, y \in \Omega \) with \( \text{dist}(x, \partial \Omega) \leq |x - y|/4 \), we have

\[
|G(x, y)| \leq C|\text{dist}(x, \partial \Omega)|^\gamma /|x - y|^{\gamma},
\]

where \( C = C(\lambda, \theta, K_0, \text{diam}(\Omega)) \).

**Remark 3.4.** Let \( (u, p) \in \tilde{W}^1_2(\Omega)^2 \times L_2(\Omega) \) be a weak solution of the problem

\[
\begin{align*}
\text{div} u &= 0 \quad \text{in } \Omega, \\
Lu + \nabla p &= f + D_\alpha f_\alpha \quad \text{in } \Omega,
\end{align*}
\]

where \( f, f_\alpha \in L_2(\Omega)^2 \). Then by using (3.2) and the counterpart of (iii) in Definition 2.3 for \( (G^*, \Pi^*) \), we have

\[
u(y) = - \int_\Omega G(y, x) f(x) \, dx + \int_\Omega D_\alpha G(y, x) f_\alpha (y) \, dy
\]

for a.e. \( y \in \Omega \).

In the theorem below, we prove an interior \( L^\infty \)-estimate for \( (DG, \Pi) \) when the coefficients of \( \mathcal{L} \) are of partially Dini mean oscillation.

**Theorem 3.5.** Let \( \Omega \) satisfy (3.1) and \( (G, \Pi) \) be the Green function of \( \mathcal{L} \) in \( \Omega \) constructed in Theorem 3.2 under Assumption 3.1. Suppose that the coefficients \( A^{\alpha\beta} \) of \( \mathcal{L} \) are of partially Dini mean oscillation in \( \Omega \) satisfying Definition 2.1 (a) with a Dini function \( \omega = \omega_A \). Then for any \( x, y \in \Omega \) with \( 0 < |x - y| \leq \frac{1}{2} \text{dist}(y, \partial \Omega) \), we have

\[
\text{ess sup}_{B_{|x-y|/4}(x)} \left( |DG(\cdot, y)| + |\Pi(\cdot, y)| \right) \leq C|x - y|^{-1},
\]

(3.7)

where \( C = C(\lambda, \theta, K_0, \text{diam}(\Omega), \omega_A) \).

**Remark 3.6.** In Theorem 3.5 if we assume further that \( A^{\alpha\beta} \) are of Dini mean oscillation with respect to all the directions in \( \Omega \) satisfying Definition 2.1 (b) as in Theorem 3.6 below (but without the \( C^{1, \text{Dini}} \) regularity assumption on the boundary in Theorem 3.7), we obtain an estimate as in (3.9) below, but only in the interior of the domain. Indeed, by Definition 2.3 (ii) and (5.12), we see that \( DG(\cdot, y) \) and \( \Pi(\cdot, y) \) are continuous in \( \Omega \setminus \{y\} \). Hence, “ess sup” in (3.7) can be replaced by “sup”. Therefore, for any \( x, y \in \Omega \) with \( 0 < |x - y| \leq \frac{1}{4} \text{dist}(y, \partial \Omega) \), we have

\[
|D_x G(x, y)| + |\Pi(x, y)| \leq C|x - y|^{-1},
\]

where \( C = C(\lambda, \theta, K_0, \text{diam}(\Omega), \omega_A) \).

In the next theorem, we prove a global pointwise bound for \( (DG, \Pi) \) when the coefficients of \( \mathcal{L} \) are of Dini mean oscillation and \( \Omega \) has a \( C^{1, \text{Dini}} \) boundary.

**Theorem 3.7.** Let \( \Omega \) have a \( C^{1, \text{Dini}} \) boundary as in Definition 2.2. Let \( (G, \Pi) \) be the Green function of \( \mathcal{L} \) in \( \Omega \) constructed in Theorem 3.2. Suppose that the coefficients \( A^{\alpha\beta} \) of \( \mathcal{L} \) are of Dini mean oscillation in \( \Omega \) satisfying Definition 2.1 (b) with a Dini function \( \omega = \omega_A \). Then for any \( y \in \Omega \) and \( R > 0 \), we have

\[
(G(\cdot, y), \Pi(\cdot, y)) \in C^1(\Omega \setminus B_R(y))^{2 \times 2} \times C(\Omega \setminus B_R(y))^{2 \times 2}.
\]

(3.8)
Moreover, for any \( x, y \in \Omega \) with \( x \neq y \), we have
\[
|D_xG(x,y)| + |\Pi(x,y)| \leq C|x-y|^{-1},
\]
where \( C = C(\lambda, \text{diam}(\Omega), \omega_A, R_0, \varrho_0) \).

**Remark 3.8.** In the above theorem, the existence of the Green function follows from Theorem 3.2 because \( \Omega \) satisfies Assumption 3.1 and (3.1). Indeed, by [2, Theorem 4.1], \( \Omega \) satisfies Assumption 3.1 with \( K_0 = K_0(\text{diam}(\Omega), R_0, \varrho_0) \) because \( \Omega \) is a John domain as in [2, Definition 2.1] with respect to \( (x_0, L) = (x_0, L)(R_0, \varrho_0) \). Moreover, \( \Omega \) satisfies (3.1) with \( \theta = \theta(R_0, \varrho_0) \) owing to the properties in Definition 2.2.

### 4. Auxiliary results

In this section, we prove some auxiliary results. We do not impose any regularity assumptions on the coefficients \( A^{\alpha\beta} \) of the operator \( L \). The following lemma concerns the solvability of the Stokes system in \( W^2_2(\Omega)^2 \times \tilde{L}_2(\Omega) \).

**Lemma 4.1.** Let \( f_\alpha \in L_2(\Omega)^2 \) and \( g \in \tilde{L}_2(\Omega) \). Then under Assumption 3.1 there exists a unique \((u, p) \in W^2_2(\Omega)^2 \times \tilde{L}_2(\Omega)\) satisfying
\[
\begin{cases}
\text{div } u = g & \text{in } \Omega, \\
Lu + \nabla p = D_\alpha f_\alpha & \text{in } \Omega.
\end{cases}
\]
Moreover, we have
\[
\|Du\|_{L_2(\Omega)} + \|p\|_{L_2(\Omega)} \leq C\left(\|f_\alpha\|_{L_2(\Omega)} + \|g\|_{L_2(\Omega)}\right),
\]
where \( C = C(\lambda, K_0) \).

**Proof.** See, for instance, [9, Lemma 3.2], where the authors proved the solvability of the Stokes system \( \Box \) with \( f + D_\alpha f_\alpha \) in place of \( D_\alpha f_\alpha \), and the \( L_2 \)-estimate
\[
\|Du\|_{L_2(\Omega)} + \|p\|_{L_2(\Omega)} \leq C\left(\|f\|_{L_2(\Omega)} + \|f_\alpha\|_{L_2(\Omega)} + \|g\|_{L_2(\Omega)}\right),
\]
where \( C = C(\lambda, K_0, |\Omega|) \). From the proof of [9, Lemma 3.2], it is easily seen that if \( f \equiv 0 \), then the constant \( C \) depends only on \( \lambda \) and \( K_0 \). We omit the details. \( \Box \)

**Lemma 4.2.** Let \( \Omega \) satisfy
\[
|B_R(x_0) \setminus \Omega| \geq \theta R^2, \quad \forall x_0 \in \partial \Omega, \quad \forall R \in (0, 1].
\]
Let \( f_\alpha \in L_\infty(\Omega)^2 \), \( g \in \tilde{L}_\infty(\Omega) \), and \((u,p) \in \tilde{W}^2_2(\Omega)^2 \times \tilde{L}_2(\Omega)\) be the weak solution of \( \Box \) derived from Lemma 4.1 under Assumption 3.1. Then for \( x_0 \in \Omega \) and \( R \in (0, 1] \) satisfying either
\[
B_R(x_0) \subset \Omega \quad \text{or} \quad x_0 \in \partial \Omega,
\]
we have
\[
\left(|Du|^2 + |p|^2\right)^{1/2}_{B_R(x_0)} \leq C\left(|D\bar{u}|^{q_0} + |\bar{p}|^{q_0}\right)^{1/2}_{B_R(x_0)} + C\left(|f_\alpha|^2 + |g|^2\right)^{1/2}_{B_R(x_0)},
\]
where \( q_0 \in (1, 2) \) and \( C = C(\lambda, \theta, K_0, q_0) \). Here, \( \bar{u}, \bar{p}, \bar{f}_\alpha, \) and \( \bar{g} \) are the extensions of \( u, p, f_\alpha \), and \( g \) to \( \mathbb{R}^2 \) so that they are zero on \( \mathbb{R}^2 \setminus \Omega \).

**Proof.** For the proof of the lemma, we refer the reader to that of [13, Lemma 3.5], where the authors proved the same inequality for the Stokes system with measurable coefficients in a Reifenberg flat domain. We note that Reifenberg flat domains satisfy 3.3 and Assumption 3.1. The argument in the proof of [13, Lemma 3.5] is sufficiently general to allow the domain \( \Omega \) to satisfy (4.3) and Assumption 3.1. \( \Box \)
We obtain the following reverse Hölder’s inequality.

**Lemma 4.3.** Let $\Omega$ satisfy $[13]$. Let $f_\alpha \in L_\infty(\Omega)^2$, $g \in \tilde{L}_\infty(\Omega)$, and $(u, p) \in W^1_q(\Omega)^2 \times L_q(\Omega)$ be the weak solution of $[14]$ derived from Lemma $4.1$ under Assumption $5.1$. Then there exists $\varepsilon_0 = \varepsilon_0(\lambda, \theta, K_0) \in (0, 1)$ such that for $q \in [2, 2 + \varepsilon_0]$, $x_0 \in \mathbb{R}^2$, and $R \in (0, 1)$, we have

$$
(\|Du\|_{B_0}^q + \|p\|_{B_0}^q)^{1/q} \leq C(\|D\tilde{u}\|_{B_R}^q + \|\tilde{p}\|_{B_R}^q)^{1/2} + C(\|\bar{f}_\alpha\|_{B_0}^q + \|\bar{g}\|_{B_0}^q)^{1/q},
$$

where $C = C(\lambda, \theta, K_0, q)$. Here, $\bar{u}$, $\bar{p}$, $\bar{f}_\alpha$, and $\bar{g}$ are the extensions of $u$, $p$, $f_\alpha$, and $g$ to $\mathbb{R}^2$ so that they are zero on $\mathbb{R}^2 \setminus \Omega$.

**Proof.** By using Lemma $[12]$ and Gehring’s lemma, one can easily prove the lemma; see $[13]$, Lemma $3.8$. We omit the details here. \hfill $\square$

In the lemma below, we prove the solvability of Stokes system in $\tilde{W}^1_q(\Omega)^2 \times \tilde{L}_q(\Omega)$ when $q$ is close to $2$.

**Lemma 4.4.** Let $\Omega$ satisfy $[13]$. Assume that Assumption $[7.1]$ holds, and let

$$
q \in \left[2 - \frac{\varepsilon_0}{1 + \varepsilon_0}, 2 + \varepsilon_0\right],
$$

where $\varepsilon_0 = \varepsilon_0(\lambda, \theta, K_0) \in (0, 1)$ is the constant from Lemma $4.3$. Then for $f_\alpha \in L_q(\Omega)^2$ and $g \in \tilde{L}_q(\Omega)$, there exists a unique $(u, p) \in W^1_q(\Omega)^2 \times L_q(\Omega)$ satisfying $[4.1]$. Moreover, we have

$$
\|Du\|_{L_q(\Omega)} + \|p\|_{L_q(\Omega)} \leq C(\|f_\alpha\|_{L_q(\Omega)} + \|g\|_{L_q(\Omega)}),
$$

where $C = C(\lambda, \theta, K_0, \text{diam}(\Omega), q)$.

**Proof.** Consider the following three cases:

$$
q = 2, \quad q \in \left[2 - \frac{\varepsilon_0}{1 + \varepsilon_0}, 2\right], \quad q \in (2, 2 + \varepsilon_0].
$$

The first case follows from Lemma $4.1$. The second case is a simple consequence of the last case combined with the duality argument; see the proof of $[13]$, Theorem $2.4$. Hence, here we only prove the case with $q \in (2, 2 + \varepsilon_0]$. First, we assume that $f_\alpha \in L_\infty(\Omega)^2$ and $g \in \tilde{L}_\infty(\Omega)$. By Lemma $4.1$, there exists a unique $(u, p) \in W^1_q(\Omega)^2 \times L_q(\Omega)$ satisfying $[4.1]$ and $[14]$.

Thus from Lemma $4.3$, we see that $(u, p)$ belongs to $W^1_q(\Omega)^2 \times \tilde{L}_q(\Omega)$, and that

$$
\|Du\|_{L_q(\Omega)} + \|p\|_{L_q(\Omega)} \leq C(\|f_\alpha\|_{L_q(\Omega)} + \|g\|_{L_q(\Omega)}),
$$

for any $x_0 \in \overline{\Omega}$, where $C = C(\lambda, \theta, K_0, \text{diam}(\Omega), q)$. By applying a covering argument, we obtain the desired estimate.

To complete the proof, let $f_\alpha \in L_q(\Omega)^2$ and $g \in \tilde{L}_q(\Omega)$. For $k \in \{1, 2, \ldots\}$, we define $f_{\alpha,k} = (f^1_{\alpha,k}, \ldots, f^d_{\alpha,k})^T$ and $g_k$ by

$$
f^i_{\alpha,k} = \max\{-k, \min\{f^i_{\alpha,k}\}, g_k = \max\{-k, \min\{g,k\}\}.\]
Finally, by taking the limit of the system (4.4), it can easily be seen that $(\tilde{u}, \tilde{p})$ satisfies (4.6). Moreover, we have

\[
\begin{aligned}
\nabla \tilde{u} &+ \nabla \tilde{p} = D_\alpha f_{\alpha,k} \quad \text{in } \Omega,
\end{aligned}
\]

and

\[
\|D\tilde{u}\|_{L_q(\Omega)} + \|\tilde{p}\|_{L_q(\Omega)} \leq C(\|f_{\alpha,k}\|_{L_q(\Omega)} + \|g\|_{L_q(\Omega)})
\]

where $C = C(\lambda, \theta, K_0, \text{diam}(\Omega), q)$. Note that $f_{\alpha,k} \to f_\alpha$ and $g_k \to g$ in $L_q(\Omega)$ as $k \to \infty$. Then by (4.5), \{(u_k, p_k)\} is a Cauchy sequence in $W^1_q(\Omega)^2 \times L_q(\Omega)$, and thus, there exists $(u, p) \in W^1_q(\Omega)^2 \times L_q(\Omega)$ such that $(u_k, p_k) \to (u, p)$ in $W^1_q(\Omega)^2 \times L_q(\Omega)$ and

\[
\|D\tilde{u}\|_{L_q(\Omega)} + \|p\|_{L_q(\Omega)} \leq C(\|f_\alpha\|_{L_q(\Omega)} + \|g\|_{L_q(\Omega)}).
\]

Finally, by taking the limit of the system (4.4), it can easily be seen that $(u, p)$ satisfies (4.1). Thus, the lemma is proved. □

**Remark 4.5.** One can extend the result in Lemma 4.4 to a system

\[
\begin{aligned}
\text{div } u &= g \quad \text{in } \Omega, \\
D_\alpha f_\alpha &= f + D_\alpha f_\alpha \quad \text{in } \Omega,
\end{aligned}
\]

when $q \in (2, 2 + \varepsilon_0]$ and $f \in L_{2q/(2+q)}(\Omega)^2$. Indeed, if we fix $R > 0$ such that $\Omega \subset B_R = B_R(0)$, then by [8, Lemma 3.1], there exist $F_\alpha \in \dot{W}^1_{2q/(2+q)}(B_R)^2$, $\alpha \in \{1, 2\}$, satisfying

\[
D_\alpha F_\alpha = f_\chi_{\Omega} \quad \text{in } B_R,
\]

\[
\|F_\alpha\|_{L_q(B_R)} + \|DF_\alpha\|_{L_{2q/(2+q)}(B_R)} \leq C(q)\|f\|_{L_{2q/(2+q)}(\Omega)},
\]

where $\chi_{\Omega}$ is the characteristic function. Thus, by Lemma 4.4 applied to (4.6) with $D_\alpha(F_\alpha + f_\alpha)$ in place of $f + D_\alpha f_\alpha$, there exists a unique $(u, p) \in \dot{W}^1_q(\Omega)^2 \times L_q(\Omega)$ satisfying (4.6). Moreover, we have

\[
\|D\tilde{u}\|_{L_q(\Omega)} + \|p\|_{L_q(\Omega)} \leq C(\|F_\alpha + f_\alpha\|_{L_q(\Omega)} + \|g\|_{L_q(\Omega)})
\]

\[
\leq C(\|f\|_{L_{2q/(2+q)}(\Omega)} + \|F_\alpha\|_{L_q(\Omega)} + \|g\|_{L_q(\Omega)}),
\]

where $C = C(\lambda, \theta, K_0, \text{diam}(\Omega), q)$.

We finish this section by establishing a weak $L_2$-estimate.

**Lemma 4.6.** Let $\Omega$ satisfy (1.2). Let $f_\alpha \in L_{2,\infty}(\Omega)^2$ and $g \in \tilde{L}_{2,\infty}(\Omega)$. Then under Assumption 3.7 there exists a unique $(u, p)$ belonging to

\[
\bigcap_{q \in [1,2]} \dot{W}^1_q(\Omega)^2 \times L_q(\Omega)
\]

and satisfying (4.1). Moreover, $(Du, p) \in L_{2,\infty}(\Omega)^{2 \times 2} \times \tilde{L}_{2,\infty}(\Omega)$ with the estimate

\[
\|Du\|_{L_{2,\infty}(\Omega)} + \|p\|_{L_{2,\infty}(\Omega)} \leq C(\|f_\alpha\|_{L_{2,\infty}(\Omega)} + \|g\|_{L_{2,\infty}(\Omega)}),
\]

where $C = C(\lambda, \theta, K_0, \text{diam}(\Omega), q)$. 

Similarly, define $g^+ \in L_{q_1}(\Omega)$ and $g^- \in L_{q_2}(\Omega)$. Then we have

\begin{align}
\|f^+\|_{L_{q_1}(\Omega)}^q & = \left( \int_0^\infty \|f^+(t)\|^q \, dt \right)^{\frac{1}{q}} \\
\|f^-\|_{L_{q_2}(\Omega)}^q & = \left( \int_0^\infty \|f^-(t)\|^q \, dt \right)^{\frac{1}{q}} \\
\|g^+\|_{L_{q_1}(\Omega)}^q & = \left( \int_0^\infty \|g^+(t)\|^q \, dt \right)^{\frac{1}{q}} \\
\|g^-\|_{L_{q_2}(\Omega)}^q & = \left( \int_0^\infty \|g^-(t)\|^q \, dt \right)^{\frac{1}{q}}.
\end{align}

(4.9)

Indeed, for example, the first inequality of (4.9) follows from

\begin{align}
\|f^+\|_{L_{q_1}(\Omega)}^q & = \left( \int_0^\infty \|f^+(t)\|^q \, dt \right)^{\frac{1}{q}} \\
& \leq q_1 k^{-2} \int_0^t t^{q_1-1} \, dt \cdot \|f_\alpha\|_{L_{2,\infty}(\Omega)}^2 + q_1 \int_t^\infty t^{q_1-3} \, dt \cdot \|f_\alpha\|_{L_{2,\infty}(\Omega)}^2 \\
& = \frac{2}{2 - q_1} k^{q_1-2} \|f_\alpha\|_{L_{2,\infty}(\Omega)}^2.
\end{align}

By Lemma 4.4 there exists a unique $(u^+, p^+) \in \tilde{W}_{q_1}^1(\Omega)^2 \times L_{q_1}(\Omega)$ satisfying (4.1) with $(f^+, g^+)$ in place of $(f_\alpha, g)$. Moreover, we have

$$
\|Du^+\|_{L_{q_1}(\Omega)} + \|p^+\|_{L_{q_1}(\Omega)} \leq C \left( \|f^+\|_{L_{q_1}(\Omega)} + \|g^+\|_{L_{q_1}(\Omega)} \right),
$$

where $C = C(\lambda, \theta, K_0, \text{diam}(\Omega))$. Using this together with (4.9) and (4.10), we obtain for $t > 0$ that

\begin{align}
|\{ x \in \Omega : |D u^+| + |p^+| > t/2 \}| & \leq 2^{q_1} k^{q_1-2} \int_\Omega (|D u^+| + |p^+|)^{q_1} \, dx \\
& \leq C t^{q_1-2} k^{q_1-2} \left( \|f^+\|_{L_{2,\infty}(\Omega)}^2 + 2^{q_1} \|g^+\|_{L_{2,\infty}(\Omega)}^2 \right).
\end{align}
Similarly, there exists a unique \((u^-, p^-) \in W_{1,q}^2(\Omega)^2 \times \tilde{L}_{q_2}(\Omega)\) satisfying (4.1) with \((f_\alpha, g)\) in place of \((f_\alpha, g, p^-)\), and

\[\|f_\alpha\|_{L_{2,\infty}(\Omega)}^2 + \|g\|_{\tilde{L}_{2,\infty}(\Omega)}^2 \leq \frac{C}{t^{2-q_1kq_2} + t^{2-q_2kq_2}} \left( \frac{f_\alpha}{2} + \frac{g}{2} \right)_{L_{2,\infty}(\Omega)}^2 \]

Combining these together and using the fact that \((u, p) = (u^+, p^+) + (u^-, p^-)\), we obtain

\[\|\psi\|_{L_{2,\infty}(\mathbb{R}^2)} \leq \frac{1}{2\sqrt{\pi}} \]

and

\[\text{div}_x \psi(x - y) = \delta_y(x)\]

in the sense that

\[\int_{\mathbb{R}^2} \psi(x - y) \cdot \nabla \phi(x) \, dx = -\phi(y), \quad \forall y \in \mathbb{R}^2, \quad \forall \phi \in C_0^\infty(\mathbb{R}^2).\]

For each \(y \in \Omega\) and \(\alpha, k \in \{1, 2\}\), we set

\[f_{\alpha,y,k}(x) = -\psi^\alpha(x - y) e_k,\]

where \(e_k\) is the \(k\)-th unit vector in \(\mathbb{R}^2\). By Lemma 4.6 there exists \((v, \pi) = (v_{y,k}, \pi_{y,k})\) belonging to

\[\bigcap_{q \in [1,2]} \tilde{W}_q^1(\Omega)^2 \times \tilde{L}_q(\Omega)\]

and satisfying

\[
\begin{array}{ll}
\text{div} \, v = 0 & \text{in } \Omega, \\
\mathcal{L} v + \nabla \pi = D_\alpha f_{\alpha,y,k} & \text{in } \Omega.
\end{array}
\]
Moreover, there is a version \( \tilde{v} = \tilde{v}_{y,k} \) of \( v \) such that \( \tilde{v} = v \) a.e. in \( \Omega \) and \( \tilde{v} \) is continuous in \( \Omega \setminus \{y\} \). Indeed, by (5.2) and (5.4), we see that \( (\eta v, \eta \pi) \) satisfies
\[
\begin{cases}
\text{div}(\eta v) = \nabla \eta \cdot v & \text{in } \Omega, \\
\mathcal{L}(\eta v) + \nabla(\eta \pi) = F + D_{\alpha}F_{\alpha} & \text{in } \Omega,
\end{cases}
\]
where we set
\[
F = A^{\alpha\beta}D_{\beta}vD_{\alpha}\eta + \pi \nabla \eta, \quad F_{\alpha} = A^{\alpha\beta}D_{\beta}y_{\eta}.
\]
Here, \( \eta \) is a smooth function on \( \mathbb{R}^2 \) satisfying
\[
\eta \equiv 0 \text{ on } B_{r/2}(y), \quad \eta \equiv 1 \text{ on } \mathbb{R}^2 \setminus B_r(y), \quad r > 0.
\]
Since \( F \in L_{q_1}(\Omega)^2, F_{\alpha} \in L_{q_0}(\Omega)^2, \) and \( \nabla \eta \cdot v \in L_{q_0}(\Omega) \), by Remark 4.5 we have
\[
(v, \pi) \in \dot{W}^{1,q_1}(\Omega)^2 \times L_{q_0}(\Omega),
\]
which implies that
\[
(v, \pi) \in \dot{W}^{1,q_1}(\Omega \setminus B_r(y))^2 \times L_{q_0}(\Omega \setminus B_r(y)), \quad \forall r > 0.
\]
Thus, by the Morrey-Sobolev embedding, there is a version \( \tilde{v} = \tilde{v}_{y,k} \) of \( v \) which is continuous in \( \Omega \setminus \{y\} \). We define a pair \( (G, \Pi) \) by
\[
G^{j,k}(x, y) = \tilde{v}_{y,k}^j(x) \quad \text{and} \quad \Pi^{j,k}(x, y) = \pi_{y,k}^j(x).
\]
Here, \( G \) is a \( 2 \times 2 \) matrix-valued function and \( \Pi \) is a \( 1 \times 2 \) vector-valued function on \( \Omega \times \Omega \).

In the remainder of this step, we prove that \( (G, \Pi) \) satisfies the properties (i) – (iii) in Definition 2.3 so that \( (G, \Pi) \) is the Green function (for the flow velocity) of \( \mathcal{L} \) in \( \Omega \). Clearly, the property (i) holds. To see the property (ii), let \( k \in \{1, 2\} \) and \( \phi \in \dot{W}^{1,\infty}(\Omega)^2 \cap C(\Omega)^2 \). Since \( \text{div } v_{y,k} = 0 \) in \( \Omega \), the \( k \)-th column \( G^k(\cdot, y) \) of \( G(\cdot, y) \) satisfies \( \text{div } G^k(\cdot, y) = 0 \) in \( \Omega \). Notice from (5.2) that
\[
\int_{\Omega} f_{\alpha, y,k} \cdot D_{\alpha} \phi \, dx = -\int_{\Omega} \psi(\alpha(x, y)) e_k \cdot D_{\alpha} \phi \, dx = -\int_{\Omega} \psi(x, y) \cdot \nabla \phi^k \, dx = \phi^k(y).
\]
From this equality and (5.2), it follows that
\[
\int_{\Omega} A^{\alpha\beta}D_{\beta}G^k(\cdot, y) \cdot D_{\alpha} \phi \, dx + \int_{\Omega} \Pi^k(\cdot, y) \text{div } \phi \, dx = \phi^k(y).
\]
To show the property (iii), let \( (u, p) \in \dot{W}^1_2(\Omega)^2 \times \dot{L}_2(\Omega) \) be a weak solution of the adjoint problem (2.2). By Lemma 1.3 we see that
\[
(u, p) \in \dot{W}^1_{q_1}(\Omega)^2 \times \dot{L}_{q_1}(\Omega) \quad \text{for some } q_1 > 2,
\]
and thus, by the Morrey-Sobolev embedding, there is a version of \( u \), denoted by \( \tilde{u} \), which is Hölder continuous in \( \overline{\Omega} \). From (5.6) with \( \tilde{u} \) in place of \( u \) and the fact that
\[
(G(\cdot, y), \Pi(\cdot, y)) \in \dot{W}^1_1(\Omega)^2 \times \dot{L}_q(\Omega), \quad \forall q \in [1, 2),
\]
we see that \( \tilde{u} \) and \( G^k(\cdot, y) \) are legitimate test functions to (5.2) and (5.4), respectively. By testing (5.4) and (2.2) with \( \tilde{u} \) and \( G^k(\cdot, y) \), respectively, we conclude that

\[
\tilde{u}^k(y) = - \int_{\Omega} G^k(\cdot, y) \cdot f \, dx + \int_{\Omega} D_\alpha G^k(\cdot, y) \cdot f_\alpha \, dx + \int_{\Omega} \Pi^k(\cdot, y) g \, dx
\]

for all \( y \in \Omega \). Since \( u = \tilde{u} \) a.e. in \( \Omega \), the above identity implies (2.3). Thus, the property (iii) holds. Therefore, the pair \((G, \Pi)\) is the Green function of \( L \) in \( \Omega \).

**Step 2.** In this step, we prove the assertions \((a) - (c)\) in Theorem 3.2. The assertion \((a)\) follows immediately from (4.8) and (5.1).

To prove the assertion \((b)\), we first claim that, for any \( x \in \mathbb{R}^2 \), \( y \in \Omega \), and \( 0 < R < \text{diam}(\Omega) \), the inequality holds: \( |\gamma_{\alpha} - \partial_{\alpha}(\cdot)\|_{L^q(\Omega_R(x))} \leq CR^{-1} \left( \|DG(\cdot, y)\|_{L^q(\Omega_R(x))} + \|\Pi(\cdot, y)\|_{L^q(\Omega_R(x))} \right) \)

where \( C = C(\lambda, \theta, K_0, \text{diam}(\Omega)) \). We consider the following two cases:

1. \( B_r(x) \subset \Omega \), \( B_r(x) \cap \partial \Omega \neq \emptyset \),

   where \( r = R/4 \).

   - **i.** \( B_r(x) \subset \Omega \). Let \( \eta_1 \) be a smooth function on \( \mathbb{R}^2 \) satisfying

     \[
     0 \leq \eta_1 \leq 1, \quad \eta_1 \equiv 1 \text{ on } B_{r/2}(x), \quad \text{supp } \eta_1 \subset B_r(x), \quad |\nabla \eta_1| \leq 4r^{-1}.
     \]

     Then (5.5) holds with \( v - (v)_{B_r(x)} \) and \( \eta_1 \) in place of \( v \) and \( \eta \). Hence by (4.7) and the Poincaré inequality, we have

     \[
     \|Dv\|_{L^q(B_{r/2}(x))} + \|\eta\|_{L^q(B_{r/2}(x))} \leq C \left( \|F\|_{L^q(\Omega)} + \|F_{\alpha}\|_{L^q(\Omega)} + \|\nabla \eta_1 \cdot (v - (v)_{B_r(x)})\|_{L^q(\Omega)} \right) \leq Cr^{-1} \left( \|Dv\|_{L^q(B_r(x))} + \|\eta\|_{L^q(B_r(x))} \right),
     \]

     which gives (5.7).

   - **ii.** \( B_r(x) \cap \partial \Omega \neq \emptyset \). We take \( x_0 \in B_r(x) \cap \partial \Omega \) such that \( |x - x_0| = \text{dist}(x, \partial \Omega) \), and observe that

     \[
     B_{r/2}(x) \subset B_{3r/2}(x_0) \subset B_{3r}(x_0) \subset B_{4r}(x) = B_R(x).
     \]

     Let \( \eta_2 \) be a smooth function on \( \mathbb{R}^2 \) satisfying

     \[
     0 \leq \eta_2 \leq 1, \quad \eta_2 \equiv 1 \text{ on } B_{3r/2}(x_0), \quad \text{supp } \eta_2 \subset B_{3r}(x_0), \quad |\nabla \eta_2| \leq 4r^{-1}.
     \]

     Since (5.5) holds with \( \eta_2 \) in place of \( \eta \), by using (4.7) and (5.8), we get

     \[
     \|Dv\|_{L^q(\Omega_{r/2}(x))} + \|\eta\|_{L^q(\Omega_{r/2}(x))} \leq C \left( \|F\|_{L^q(\Omega)} + \|F_{\alpha}\|_{L^q(\Omega)} + \|\nabla \eta_2 \cdot v\|_{L^q(\Omega)} \right) \leq Cr^{-1} \left( \|Dv\|_{L^q(\Omega_R(x))} + \|\eta\|_{L^q(\Omega_R(x))} \right),
     \]

     where we used the boundary Poincaré inequality together with (3.1) in the last inequality. This gives the inequality (5.7).
Similarly, we have

\[ x_i \]

we obtain that

\[ \text{By combining these together, and using (3.3) and (5.7) with a covering argument, we have} \]

\[ \|DG(\cdot, y)\|_{L^q(\Omega_R(x))} = q_i \left( \int_0^{M/R} + \int_{M/R}^{\infty} \right) t^{q_i-1} |\{z \in \Omega_R(x) : |DG(\cdot, y)| > t\}| dt \]

\[ \leq CR^{2-q_i} M^{q_i}. \]

Similarly, we have

\[ \|\Pi(\cdot, y)\|_{L^q(\Omega_R(x))} \leq CR^{2-q_i} \|\Pi(\cdot, y)\|_{L^q(\Omega_R(x))}. \]

By combining these together, and using (3.3) and (5.7) with a covering argument, we obtain that

\[ \|DG(\cdot, y)\|_{L_{q_i}(\Omega_{R/2}(x))} + \|\Pi(\cdot, y)\|_{L_{q_i}(\Omega_{R/2}(x))} \leq CR^{-\gamma} \left( \|DG(\cdot, y)\|_{L^q(\Omega_R(x))} + \|\Pi(\cdot, y)\|_{L^q(\Omega_R(x))} \right) \]

\[ \leq CR^{-\gamma}, \]

where \( \gamma = 2 - 2/q_i = 1 - 2/q_0 \) and \( C = C(\lambda, \theta, K_0, \text{diam}(\Omega)) \). This proves (3.4).

We extend \( G(\cdot, y) \) by zero on \( \mathbb{R}^2 \setminus \Omega \). Then using Morrey’s inequality and the above inequality, we see that

\[ [G(\cdot, y)_{C^\gamma(B_{R/2}(x))} \leq C \|DG(\cdot, y)\|_{L_{q_0}(B_{R/2}(x))} \leq CR^{-\gamma}. \]

This implies (3.5), and thus the assertion (b) is proved. Note that by the above inequality, we have

\[ \left|G(z_0, y) - \int_{B_{R/2}(x)} G(z, y) \, dz\right| \leq C_0 \quad (5.9) \]

for any \( x, z_0 \in \mathbb{R}^2, y \in \Omega, \) and \( 0 < R < \text{diam}(\Omega) \) satisfying \( |x - y| > R \) and \( z_0 \in B_{R/2}(x), \) where \( C_0 = C_0(\lambda, \theta, K_0, \text{diam}(\Omega)) \).

We now turn to the assertion (c). Let \( x, y \in \Omega \) with \( x \neq y \), and set

\[ \rho = \frac{1}{4}|x - y|. \]

Without loss of generality, we may assume that \( x = 0 \) and \( y = (-4\rho, 0) \). We choose a positive integer \( k \geq 1 \) satisfying

\[ 2^k \rho < \frac{\text{diam}(\Omega)}{2} \leq 2^{k+1} \rho. \quad (5.10) \]

For \( i \in \{0, \ldots, k\} \), let \( x_i = (\alpha_i, 0) \in \mathbb{R}^2, \) where

\[ \alpha_0 = 0, \quad \alpha_i = \alpha_{i-1} + 2^i \rho = 2^i(2^i - 1), \quad i = 1, \ldots, k, \]

and observe that

\[ B_{2^i \rho}(x_i) \cap B_{2^{i-1} \rho}(x_{i-1}) \neq \emptyset, \quad |x_i - y| > 2^{i+1} \rho, \quad i \in \{1, \ldots, k\}. \]

For each \( i \in \{1, \ldots, k\} \), we choose \( z_i \in B_{2^i \rho}(x_i) \cap B_{2^{i-1} \rho}(x_{i-1}) \) and write

\[ \left|\int_{B_{2^{i-1} \rho}(x_{i-1})} G(z, y) \, dz\right| \leq \left|\int_{B_{2^{i-1} \rho}(x_{i-1})} G(z, y) \, dz - G(z_i, y)\right| \]
Thanks to the estimate (5.9), this inequality implies that
\[
\left| \int_{B_{2^{-i+1}, \rho}(x_i-1)} G(z, y) \, dz \right| \leq 2C_0 + \left| \int_{B_{2^{-i}, \rho}(x_i)} G(z, y) \, dz \right|, \quad i \in \{1, \ldots, k\},
\]
and thus, by iterating we see that
\[
\left| \int_{B_{\rho}(x)} G(z, y) \, dz \right| \leq 2kC_0 + \left| \int_{B_{2k, \rho}(x_k)} G(z, y) \, dz \right|.
\]
From this and (5.9) we have
\[
|G(x, y)| \leq \left| G(x, y) - \int_{B_{\rho}(x)} G(z, y) \, dz \right| + \left| \int_{B_{\rho}(x)} G(z, y) \, dz \right| \\
\leq (2k + 1)C_0 + \left| \int_{B_{2k, \rho}(x_k)} G(z, y) \, dz \right| \\
\leq C \left( \log \left( \frac{\text{diam}(\Omega)}{\rho} \right) + \|G(\cdot, y)\|_{L_1(\Omega)} \right),
\]
where the last inequality is due to the fact that (using (5.10))
\[
1 \leq k < \frac{1}{\log 2} \log \left( \frac{\text{diam}(\Omega)}{\rho} \right), \quad |B_{2k, \rho}| \geq \left( \frac{\text{diam}(\Omega)}{4} \right)^2.
\]
Notice from Hölder’s inequality, the Sobolev inequality, and (3.3) that
\[
\|G(\cdot, y)\|_{L_1(\Omega)} \leq C\|G(\cdot, y)\|_{L_2(\Omega)} \leq C\|DG(\cdot, y)\|_{L_1(\Omega)} \\
\leq C\|DG(\cdot, y)\|_{L_2, \infty(\Omega)} \leq C,
\]
where \( C = C(\lambda, \theta, K_0, \text{diam}(\Omega)) \). Therefore, from (5.11) combined with the above inequalities we arrive at (5.6).

**Step 3.** In this step, we prove the identity (3.2). For each \( x \in \Omega \), we define the Green function \((G^*(\cdot, x), \Pi^*(\cdot, x))\) of the adjoint operator \(L^*\) in the same manner that \((G(\cdot, y), \Pi(\cdot, y))\) is defined for the operator \(L\). More precisely, we find a unique solution
\[
(w_{x,l}, \tau_{x,l}) \in \bigcap_{q \in [1, 2]} \mathcal{W}^1_q(\Omega)^2 \times \tilde{L}_q(\Omega)
\]
to the system
\[
\begin{aligned}
\text{div } w_{x,l} &= 0 \quad \text{in } \Omega, \\
L^* w_{x,l} + \nabla \tau_{x,l} &= D_{\alpha} f_{\alpha,x,l} \quad \text{in } \Omega,
\end{aligned}
\]
where \(f_{\alpha,x,l}\) is the function as in (5.3). Then we set \((w_{x,l}, \tau_{x,l})\) to be the \(l\)-th column of \((G^*(\cdot, x), \Pi^*(\cdot, x))\). Using the arguments in **Steps 1 and 2**, we find that \((G^*, \Pi^*)\) satisfies the corresponding properties to those of \((G, \Pi)\).

Let \(x, y \in \Omega\) with \(x \neq y\), and denote \(r = |x - y|/2\). Let \(\zeta\) be a smooth function in \(\mathbb{R}^2\) satisfying
\[
\zeta \equiv 0 \text{ on } B_{r/2}(x), \quad \zeta \equiv 1 \text{ on } \mathbb{R}^2 \setminus B_r(x).
\]
Observe that \( \zeta G^*(\cdot, x) \) and \((1 - \zeta)G^*(\cdot, x)\) can be applied to \([3,4]\) as test functions. By testing the \(l\)-th columns of those functions to \([3,4]\), and using the continuity of \(G^*(\cdot, x)\) in \(\Omega \setminus \{x\}\) and the fact that

\[
G^*(\cdot, x) = \zeta G^*(\cdot, x) + (1 - \zeta)G^*(\cdot, x),
\]
we have

\[
\int_{\Omega} A_{ij}^{\alpha \beta} D_{\beta} G^{jk}(\cdot, y) D_{\alpha} (G^*)^{il}(\cdot, x) \, dz = (G^*)^{kl}(y, x).
\]

Similarly, we obtain

\[
\int_{\Omega} A_{ij}^{\alpha \beta} D_{\beta} G^{jk}(\cdot, y) D_{\alpha} (G^*)^{il}(\cdot, x) \, dz = G^{ilk}(x, y).
\]

By combining these together, we see that

\[
G^{lk}(x, y) = (G^*)^{kl}(y, x),
\]
which gives \([3,4]\). Finally, by the above identity and the continuity of \(G^*(\cdot, x)\), it holds that \(G(x, \cdot)\) is continuous in \(\Omega \setminus \{x\}\). By using this and the continuity of \(G(\cdot, y)\) in \(\Omega \setminus \{y\}\), we conclude that \(G\) is continuous in \(\{(x, y) \in \Omega \times \Omega : x \neq y\}\). Thus, the theorem is proved. \(\square\)

5.2. Proof of Theorem 3.5 To prove the theorem, we use the following interior \(L^\infty\)-estimate.

**Lemma 5.1.** Let \(R \in (0, 1]\). Suppose that the coefficients \(A^{\alpha \beta}\) of \(L\) are of partially Dini mean oscillation in \(B_R = B_R(0)\) satisfying Definition 2.7 (a) with a Dini function \(\omega = \omega_A\). If \((u, p) \in W_2^1(B_R)^2 \times L_2(B_R)\) satisfies

\[
\begin{align*}
\text{div} \, u &= 0 \quad \text{in } B_R, \\
L u + \nabla p &= 0 \quad \text{in } B_R,
\end{align*}
\]
then we have

\[
(u, p) \in W_\infty^1(B_R/2)^2 \times L_\infty(B_R/2)
\]

with the estimate

\[
\|Du\|_{L_\infty(B_R/2)} + \|p\|_{L_\infty(B_R/2)} \leq CR^{-2}\left(\|Du\|_{L_1(B_R)} + \|p\|_{L_1(B_R)}\right),
\]

where \(C = C(\lambda, \omega_A)\). If we assume further that \(A^{\alpha \beta}\) are of Dini mean oscillation with respect to all direction in \(B_R\) satisfying Definition 2.7 (b), then we have

\[
(u, p) \in C^{1}(B_R/2)^2 \times C(B_R/2).
\]

**Proof.** The lemma follows from \([5]\) Theorems 2.2 and 2.3, and Eq. (4.16)] together with scaling and covering arguments. For more details, see \([6]\). \(\square\)

Let \(x, y \in \Omega\) with \(0 < |x - y| \leq \frac{1}{2} \text{dist}(y, \partial \Omega)\). Set \(R = |x - y|/2\). Since \(y \notin B_R(x)\) and \(B_R(x) \subset \Omega\), by the property \((ii)\) in Definition 2.8 \((G^k(\cdot, y), \Pi^k(\cdot, y))\) satisfies

\[
\begin{align*}
\text{div} \, G^k(\cdot, y) &= 0 \quad \text{in } B_R(x), \\
L G^k(\cdot, y) + \nabla \Pi^k(\cdot, y) &= 0 \quad \text{in } B_R(x).
\end{align*}
\]

By Lemma 5.1 and a covering argument (in case \(R > 1\)), we have

\[
\begin{align*}
\|DG(\cdot, y)\|_{L_\infty(B_R/2)} + \|\Pi(\cdot, y)\|_{L_\infty(B_R/2)} \\
&\leq CR^{-2}\left(\|DG(\cdot, y)\|_{L_1(B_R(x))} + \|p\|_{L_1(B_R(x))}\right),
\end{align*}
\]

(5.13)
where $C = C(\lambda, \text{diam}(\Omega), \omega_A)$. We denote $M = \|DG(\cdot, y)\|_{L^2_{2,\infty}(B_R(x))}$, and observe that
\begin{align*}
\|DG(\cdot, y)\|_{L^1_1(B_R(x))} &= \left( \int_0^{M/R} + \int_{M/R}^\infty \right) |\{z \in B_R(x) : |D_zG(z, y)| > t\}| dt \\
&\leq CRM. \tag{5.14}
\end{align*}
Similarly, we have
\[ \|\Pi(\cdot, y)\|_{L^1_1(B_R(x))} \leq CR\|\Pi(\cdot, y)\|_{L^2_{2,\infty}(B_R(x))}. \]
By combining these together, we get from (5.13) and (3.3) that
\[ \|DG(\cdot, y)\|_{L^\infty(B_{R/2}(x))} + \|\Pi(\cdot, y)\|_{L^\infty(B_{R/2}(x))} \leq CR^{-1}, \]
where $C = C(\lambda, \theta, K_0, \text{diam}(\Omega), \omega_A)$. This gives the desired estimate. Thus the theorem is proved. \hfill \square

5.3. Proof of Theorem 3.7. To prove the theorem, we use the following estimate on a $C^{1,\text{Dini}}$ domain.

**Lemma 5.2.** Let $\Omega$ have a $C^{1,\text{Dini}}$ boundary as in Definition 2.2. Suppose that the coefficients $A^{ij}$ of $\mathcal{L}$ are of Dini mean oscillation in $\Omega$ satisfying Definition 2.3 (b) with a Dini function $\omega = \omega_A$. Let $x_0 \in \Omega$ and $0 < R < \text{diam}(\Omega)$. If $(u, p) \in W^1_2(\Omega_R(x_0))^2 \times L_2(\Omega_R(x_0))$ satisfies
\[ \begin{aligned}
\text{div} \, u &= 0 \quad \text{in} \quad \Omega_R(x_0), \\
\mathcal{L}u + \nabla p &= 0 \quad \text{in} \quad \Omega_R(x_0), \\
u &= 0 \quad \text{on} \quad \partial \Omega \cap B_R(x_0),
\end{aligned} \]
then we have
\[ (u, p) \in C^1(\Omega_{R/2}(x_0))^2 \times C(\Omega_{R/2}(x_0)) \]
and
\[ \|Du\|_{L^\infty(\Omega_{R/2}(x_0))} + \|p\|_{L^\infty(\Omega_{R/2}(x_0))} \leq CR^{-3}\|u\|_{L^1(\Omega_R(x_0))} + CR^{-2}\|Du\|_{L^1(\Omega_R(x_0))} + \|p\|_{L^1(\Omega_R(x_0))}, \]
where $C = C(\lambda, \text{diam}(\Omega), \omega_A, R_0, \theta_0)$.

**Proof.** The lemma follows from [4, Theorem 1.4 and Eq. (2.27)] with a localization argument. For more details, see [3]. \hfill \square

Let $x, y \in \Omega$ with $x \neq y$ and $R = |x - y|/2$. From the property (ii) in Definition 2.3 we see that
\[ \begin{aligned}
\text{div} \, G^k(\cdot, y) &= 0 \quad \text{in} \quad \Omega_R(x), \\
\mathcal{L}G^k(\cdot, y) + \nabla \Pi^k(\cdot, y) &= 0 \quad \text{in} \quad \Omega_R(x), \\
G^k(\cdot, y) &= 0 \quad \text{on} \quad \partial \Omega \cap B_R(x).
\end{aligned} \]
Then by Lemma 5.2 we have
\[ (G(\cdot, y), \Pi(\cdot, y)) \in C^1(\Omega_{R/2}(x))^{2 \times 2} \times C(\Omega_{R/2}(x))^{2}, \]
which shows (3.3). To prove the estimate (3.9), we consider the following two cases:
\[ \partial \Omega \cap B_R(x) = \emptyset, \quad \partial \Omega \cap B_R(x) \neq \emptyset. \]
i. \( \partial \Omega \cap B_R(x) = \emptyset \). Since \( \Omega_R(x) = B_R(x) \), by following the proof of Theorem 3.5 we have
\[
\|DG(\cdot, y)\|_{L_\infty(B_{R/2}(x))} + \|\Pi(\cdot, y)\|_{L_\infty(B_{R/2}(x))} \leq CR^{-1},
\]
where \( C = C(\lambda, \text{diam}(\Omega), \omega_A, R_0, \varrho_0) \). Together with the continuity of \( DG(\cdot, y) \) and \( \Pi(\cdot, y) \), this implies \( \text{(3.9)} \).

ii. \( \partial \Omega \cap B_R(x) \neq \emptyset \). In this case, by Lemma 5.2 we have
\[
\|DG(\cdot, y)\|_{L_\infty(\Omega_{R/2}(x))} + \|\Pi(\cdot, y)\|_{L_\infty(\Omega_{R/2}(x))} \leq CR^{-3}\|G(\cdot, y)\|_{L_1(\Omega_R(x))}
+ CR^{-2}(\|DG(\cdot, y)\|_{L_1(\Omega_R(x))} + \|\Pi(\cdot, y)\|_{L_1(\Omega_R(x))}),
\]
where \( C = C(\lambda, \text{diam}(\Omega), \omega_A, R_0, \varrho_0) \). Note that (see \( \text{(5.14)} \))
\[
\|DG(\cdot, y)\|_{L_1(\Omega_R(x))} + \|\Pi(\cdot, y)\|_{L_1(\Omega_R(x))}
\leq CR(\|DG(\cdot, y)\|_{L_2, \infty(\Omega)} + \|\Pi(\cdot, y)\|_{L_2, \infty(\Omega)}) \leq CR,
\]
where the last inequality is due to \( \text{(5.3)} \). Fix a point \( z_0 \in \partial \Omega \cap B_R(x) \). Since \( G(z_0, y) = 0 \), we obtain by \( \text{(5.5)} \) that
\[
|G(z, y)| = |G(z, y) - G(z_0, y)| \leq CR^{-1}|z - z_0| \leq C
\]
for all \( z \in \Omega_R(x) \), where \( C = C(\lambda, \text{diam}(\Omega), R_0, \varrho_0) \). This implies
\[
\|G(\cdot, y)\|_{L_1(\Omega_R(x))} \leq CR^2,
\]
and thus, using \( \text{(5.15)} \) and \( \text{(5.10)} \), we conclude that
\[
\|DG(\cdot, y)\|_{L_\infty(\Omega_{R/2}(x))} + \|\Pi(\cdot, y)\|_{L_\infty(\Omega_{R/2}(x))} \leq CR^{-1}.
\]
Finally, by the continuity of \( DG(\cdot, y) \) and \( \Pi(\cdot, y) \), we get the desired estimate \( \text{(3.9)} \).

The theorem is proved. \( \square \)

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