Minimum degree conditions for the existence of cycles of all lengths modulo $k$ in graphs

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Abstract

Thomassen, in 1983, conjectured that for a positive integer $k$, every 2-connected non-bipartite graph of minimum degree at least $k + 1$ contains cycles of all lengths modulo $k$. In this paper, we settle this conjecture affirmatively.

Keywords: Cycles, Length modulo $k$, Minimum degree

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1 Introduction

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. In [3], Thomassen conjectured the following.

Conjecture A (Thomassen [3]) For a positive integer $k$, every 2-connected non-bipartite graph of minimum degree at least $k + 1$ contains cycles of all lengths modulo $k$.

In 2018, Liu and Ma proved that this conjecture is true for all even integers $k$, see [2, Theorem 1.9] (for the history and other related results to the conjecture, we also refer the reader to [2]). In this paper, we settle Conjecture A by showing that it is also true for all odd integers $k$. For this purpose, we give the following result, which is our main theorem. Here $E(G)$ denotes the edge set of a graph $G$.

Theorem 1 For a positive integer $k$, every 2-connected graph of minimum degree at least $k + 1$ contains $k$ cycles $C_1, \ldots, C_k$ such that either (i) $|E(C_{i+1})| - |E(C_i)| = 1$ for $1 \leq i \leq k - 1$, or (ii) $|E(C_{i+1})| - |E(C_i)| = 2$ for $1 \leq i \leq k - 1$.

We here prove Conjecture A assuming Theorem 1 for the case where $k$ is odd. It follows from the proof that the condition “non-bipartite” in Conjecture A is not necessary if $k$ is odd.

Proof of Conjecture A for the case where $k$ is odd. Let $k$ be a positive odd integer, and let $G$ be a 2-connected graph of minimum degree at least $k + 1$ (the graph $G$ may be bipartite). We will show

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that $G$ contains cycles of all lengths modulo $k$. By Theorem 1, $G$ contains $k$ cycles satisfying (i) or (ii) in Theorem 1. If the $k$ cycles satisfy (i), then the $k$ cycles clearly have all lengths modulo $k$. So, suppose that the $k$ cycles satisfy (ii). Since $k$ is odd, we may assume that

$$(l, l + 2, \ldots, l + k - 3, l + k - 1, l + k + 1, \ldots, l + 2k - 4, l + 2k - 2)$$

is a sequence of the lengths of the $k$ cycles for some integer $l \geq 3$. Then it follows that

$$l + 2i + 1 \equiv l + k + 2i + 1 \pmod{k} \text{ for } 0 \leq i \leq \frac{k - 3}{2}.$$

Thus the $k$ cycles have all lengths modulo $k$. □

Our proof of Theorem 1 is based on the technique of Liu and Ma [2]. In the next section, we introduce results to prove Theorem 1. In particular, we give sharp degree conditions for the existence of paths with specified end vertices whose lengths differ by one or two (see Theorems 2 and 3 for the detail), which are our key results.

2 Outline of the proof of Theorem 1

In this section, we introduce results for the proof of Theorem 1 according to the flowchart of Figure 1.
We require some terminology and notation. Let $G$ be a graph. The vertex set of $G$ is denoted by $V(G)$. For a vertex $v$ of $G$, $\deg_G(v)$ denotes the degree of $v$ in $G$, and let $\delta(G)$ denote the minimum degree of $G$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of $G$ induced by $S$, and let $G - S = G[V(G) \setminus S]$. For distinct vertices $x$ and $y$ of $G$, $(G, x, y)$ is called a rooted graph. A rooted graph $(G, x, y)$ is 2-connected if

(R1) $G$ is a connected graph of order at least 3 with at most two end blocks, and
(R2) every end block of $G$ contains at least one of $x$ and $y$ as a non-cut vertex.

Note that $(G, x, y)$ is 2-connected if and only if $G + xy$ (i.e., the graph obtained from $G$ by adding the edge $xy$ if $xy \notin E(G)$) is 2-connected. For convenience, we say that a sequence of paths or cycles $H_1, H_2, \ldots, H_k$

• have consecutive lengths if $|E(H_1)| \geq 2$ and $|E(H_{i+1})| - |E(H_i)| = 1$ for $1 \leq i \leq k - 1$;
• satisfy the length condition if $|E(H_1)| \geq 2$ and $|E(H_{i+1})| - |E(H_i)| = 2$ for $1 \leq i \leq k - 1$;
• satisfy the semi-length condition if $|E(H_1)| \geq 2$ and there exists an index $j$ with $1 \leq j \leq k - 1$, which is called a switch, such that

$$|E(H_{i+1})| - |E(H_i)| = 2 \text{ for } 1 \leq i \leq j - 1,$$
$$|E(H_{j+1})| - |E(H_j)| = 1 \text{ and }$$
$$|E(H_{i+1})| - |E(H_i)| = 2 \text{ for } j + 1 \leq i \leq k - 1.$$ 

The first step of the proof of Theorem 1 is to show the following two results concerning degree conditions for the existence of paths satisfying the length condition or the semi-length condition.

**Theorem 2** Let $k$ be a positive integer, and let $(G, x, y)$ be a 2-connected rooted graph. Suppose that $\deg_G(v) \geq 2k$ for any $v \in V(G) \setminus \{x, y\}$. Then $G$ contains $k$ paths from $x$ to $y$ satisfying the length condition.

**Theorem 3** Let $k$ be a positive integer, and let $(G, x, y)$ be a 2-connected rooted graph. Suppose that $\deg_G(v) \geq 2k - 1$ for any $v \in V(G) \setminus \{x, y\}$. Then $G$ contains $k$ paths from $x$ to $y$ satisfying the length condition or the semi-length condition.

The complete graphs of orders $2^k$ and $2^k - 1$ show the sharpness of the degree conditions in Theorems 2 and 3 respectively. Theorem 2 is an improvement of [2, Lemma 3.1].

In Section 3 we prepare lemmas for the proofs of Theorems 2 and 3. We will use Theorem 2 in a part of the proof of Theorem 3 (see also Figure 1). So, we first prove Theorem 2 in Section 4 and then we give the proof of Theorem 3 in Section 5.

In the second step, we divide the proof of Theorem 1 into three cases according as a graph is (I) 2-connected, but not 3-connected, (II) 3-connected and non-bipartite, or (III) bipartite.

For the case (I), we show the following theorem by using Theorems 2 and 3 which is an improvement of [2, Lemma 4.1]. We give the proof in Section 6.
Theorem 4 Let $k$ be a positive integer, and let $G$ be a 2-connected but not 3-connected graph. If $\delta(G) \geq k + 1$, then $G$ contains $k$ cycles satisfying the length condition.

To show the case (II), in Section 5 we also prove the following theorem by using Theorems 2 and 3 and additional lemmas. Here, for a cycle $C$ in a connected graph $G$, $C$ is said to be non-separating if $G - V(C)$ is connected.

Theorem 5 Let $k$ be a positive integer, and let $G$ be a 2-connected graph containing a non-separating induced odd cycle. If $\delta(G) \geq k + 1$, then $G$ contains $k$ cycles, which have consecutive lengths or satisfy the length condition.

The following result is known for the existence of a non-separating induced odd cycle.

Theorem B (see the proof of Theorem 2 in [1]) Every 3-connected non-bipartite graph contains a non-separating induced odd cycle.

Combining Theorems 5 and B, we can obtain the following theorem for the case (II).

Theorem 6 Let $k$ be a positive integer, and let $G$ be a 3-connected non-bipartite graph. If $\delta(G) \geq k + 1$, then $G$ contains $k$ cycles, which have consecutive lengths or satisfy the length condition.

On the other hand, for the case (III), the following theorem is proved by Liu and Ma.

Theorem C ([2, Theorem 1.2]) Let $k$ be a positive integer, and let $G$ be a bipartite graph. If $\delta(G) \geq k + 1$, then $G$ contains $k$ cycles satisfying the length condition.

Consequently, we can obtain Theorem 1 by Theorems 4, 6 and C. In Section 7, we give some remarks on the minimum degree and the connectivity conditions in Theorem 1.

In the rest of this section, we prepare terminology and notation which will be used in the subsequent sections. Let $G$ be a graph. We denote by $N_G(v)$ the neighborhood of a vertex $v$ in $G$. For $S \subseteq V(G)$, we define $N_G(S) = \bigcup_{v \in S} N_G(v) \setminus S$. For $S, T \subseteq V(G)$ with $S \cap T = \emptyset$, $E_G(S, T)$ denotes the set of edges of $G$ between $S$ and $T$, and let $e_G(S, T) = |E_G(S, T)|$. Furthermore, $G[S, T]$ is the graph defined by $V(G[S, T]) = S \cup T$ and $E(G[S, T]) = E_G(S, T)$. Note that $G[S, T]$ is a bipartite subgraph of $G$ with partite sets $S$ and $T$, and we always assume that $G[S, T]$ is such a bipartite graph. For a rooted graph $(G, x, y)$, we define $\delta(G, x, y) = \min\{\deg_G(v) : v \in V(G) \setminus \{x, y\}\}$. In the rest of this paper, we often denote the singleton set $\{v\}$ by $v$, and we often identify a subgraph $H$ of $G$ with its vertex set $V(H)$.

For $S \subseteq V(G)$, a path in $G$ is an $S$-path if it begins and ends in $S$, and none of its internal vertices are contained in $S$. For $S, T \subseteq V(G)$ with $S \cap T = \emptyset$, a path in $G$ is an $(S, T)$-path if one end vertex of the path belongs to $S$, another end vertex belongs to $T$, and the internal vertices do not belong to $S \cup T$. We write a path or a cycle $P$ with a given orientation as $\overrightarrow{P}$. If there exists no fear of confusion, we abbreviate $\overrightarrow{P}$ by $P$. Let $\overrightarrow{P}$ be an oriented path (or cycle). For $u \in V(P)$, the $h$-th successor and the $h$-th predecessor of $u$ on $\overrightarrow{P}$ (if exist) is denoted by $u^{+h}$ and $u^{-h}$, respectively, and
we let $u^+ = u^+1$ and $u^- = u^-1$. For $u, v \in V(P)$, the path from $u$ to $v$ along $P$ (if exist) is denoted by $u \rightarrow P v$. The reverse sequence of $u \rightarrow P v$ is denoted by $v \rightarrow P u$. In the rest of this paper, if $P$ is an $(S, T)$-path in $G$, we always assume that the orientation of $P$ is given from the end vertex belonging to $S$ to the end vertex belonging to $T$ along the edges of $P$.

3 Preliminaries for the proofs of Theorems 2 and 3

In this section, we introduce the concept of a core which was used in the argument of [2] and give some lemmas for the proofs of Theorems 2 and 3.

Let $x$ and $y$ be two distinct vertices of a graph $G$. For a bipartite subgraph $H = G[S, T]$ of $G$ and an integer $l$, $H$ is called an $l$-core with respect to $(x, y)$ if

(C1) $H$ is complete bipartite and $|T| \geq |S| = l + 1 \geq 2$,
(C2) $x \in S$ and $y \notin V(H)$,
(C3) $e_G(v, S) \leq l$ for $v \in V(G) \setminus (V(H) \cup \{y\})$, and
(C4) $e_G(v, T \setminus \{v\}) \leq l + 1$ for $v \in V(G) \setminus (S \cup \{y\})$.

In the rest of this section, we fix the following notation. Let $(G, x, y)$ be a 2-connected rooted graph, and let $H = G[S, T]$ be an $l$-core of $G$ with respect to $(x, y)$. Furthermore, let $C$ be the component of $G - V(H)$ such that $y \in V(C)$. Since $E_H(H - x, C) \neq \emptyset$ by (R2), the following two lemmas (Lemmas 1 and 2) easily follows from (C1). So, we omit the proof.

Lemma 1 If either (i) $l \geq k$ or (ii) $l = k - 1$ and $E_G(T, C) \neq \emptyset$, then $G$ contains $k$ $(x, y)$-paths satisfying the length condition.

Lemma 2 If $l = k - 1$, $E_G(S \setminus \{x\}, C) \neq \emptyset$ and $E(G[T]) \neq \emptyset$, then $G$ contains $k$ $(x, y)$-paths satisfying the semi-length condition.

The following two lemmas (Lemmas 3 and 4) are proved by Liu and Ma, see [2] Lemmas 2.7 and 2.11. Note that the argument in [2] can work for paths satisfying the length condition but also for paths satisfying the semi-length condition.

Lemma 3 Let $s$ be a vertex of $S \setminus \{x\}$ such that $E_G(s, C) \neq \emptyset$. If one of the following (i)–(iii) holds, then $G$ contains $k$ $(x, y)$-paths satisfying the length condition (resp., the semi-length condition).

(i) $G - V(C)$ contains $k - l + 1$ $T$-paths internally disjoint from $V(H)$ and satisfying the length condition (resp., the semi-length condition) [2, Lemma 2.7-2].

(ii) $G - V(C)$ contains $k - l + 1$ $(T, \{x, s\})$-paths internally disjoint from $V(H)$ and satisfying the length condition (resp., the semi-length condition) [2 Lemma 2.7-4].

(iii) $G - V(C)$ contains $k - l + 2$ $(T, S \setminus \{x, s\})$-paths internally disjoint from $V(H)$ and satisfying the length condition (resp., the semi-length condition) [2 Lemma 2.7-3].
Lemma 4 ([2, Lemma 2.11]) If one of the following (i) and (ii) holds, then $G$ contains $k$ $(x, y)$-paths satisfying the length condition (resp., the semi-length condition).

(i) $G$ contains $k - l (T, y)$-paths internally disjoint from $V(H)$ and satisfying the length condition (resp., the semi-length condition).

(ii) $G$ contains $k - l + 1 (S \setminus \{x, y\})$-paths internally disjoint from $V(H)$ and satisfying the length condition (resp., the semi-length condition).

4 Proof of Theorem 2

Proof of Theorem 2  We prove it by induction on $|V(G)| + |E(G)|$. Let $(G, x, y)$ be a minimum counterexample with respect to $|V(G)| + |E(G)|$. If $k = 1$, then by (R1) and (R2), we can easily see that $G$ contains an $(x, y)$-path of length at least 2, a contradiction. Thus $k \geq 2$. Since $\delta(G, x, y) \geq 2k$, this implies that $|G| \geq 5$. By symmetry, we may assume that $\deg_G(x) \leq \deg_G(y)$.

Claim 4.1 $G$ is 2-connected.

Proof. Suppose that $G$ is not 2-connected. Then by (R1), $G$ has a cut vertex $c$ and $G - c$ has exactly two components $C_1$ and $C_2$. Since $|G| \geq 5$ ($\geq 4$), we may assume that $|C_1| \geq 2$. Let $G_i = G[V(C_i) \cup \{c\}]$ for $i \in \{1, 2\}$. Then by (R2), for some two distinct vertices $x', y' \in \{x, y\}$, (i) $(G_1, x', c)$ is a 2-connected rooted graph such that $\delta(G_1, x', c) \geq \delta(G, x, y) \geq 2k$, and (ii) $y'$ is contained in a block of $G_2$. Hence by the induction hypothesis, $G_1$ contains $k$ $(x', c)$-paths $\overrightarrow{P_1}, \ldots, \overrightarrow{P_k}$ satisfying the length condition. Then $x'\overrightarrow{P_1}c\overrightarrow{P}_y', \ldots, x'\overrightarrow{P_k}c\overrightarrow{P}_y'$ are $k$ $(x, y)$-paths in $G$ satisfying the length condition, where $\overrightarrow{P}$ denotes the $(c, y')$-path in $G_2$, a contradiction. \(\square\)

Claim 4.2 $xy \notin E(G)$.

Proof. If $xy \in E(G)$, then by Claim 4.1 $(G - xy, x, y)$ is a 2-connected rooted graph such that $\delta(G - xy, x, y) = \delta(G, x, y) \geq 2k$, and hence the induction hypothesis yields that $G - xy$ (and also $G$) contains $k$ $(x, y)$-paths satisfying the length condition, a contradiction. \(\square\)

Case 1. $G - y$ does not contain a cycle of length 4 passing through $x$.

Since $xy \notin E(G)$ by Claim 4.2 in this case, we have

$$e_G(v, N_G(x) \setminus \{v\}) \leq 1 \text{ for } v \in V(G) \setminus \{x, y\}. \quad (4.1)$$

Let $G^*$ be the graph obtained from $G$ by contracting the subgraph induced by $N_G(x) \cup \{x\}$ into a single vertex $x^*$ and then removing multiple edges. Then by (4.1),

$$\deg_{G^*}(v) = \deg_G(v) \text{ for } v \in V(G^*) \setminus \{x^*, y\}. \quad (4.2)$$
By (4.1) and since \( \delta(G, x, y) \geq 2k \geq 4 \), we have \(|G^*| \geq 3\). Hence by Claim 4.1 if \( G^* \) is not 2-connected, then \( x^* \) is the unique cut vertex of \( G^* \) and each block of \( G^* \) is an end block containing \( x^* \).

Now, let \( B^* \) be the block of \( G^* \) which contains \( y \) if \( G^* \) is not 2-connected; otherwise, let \( B^* = G^* \).

Assume that \((B^*, x^*, y)\) is 2-connected. Since \( \delta(B^*, x^*, y) \geq \delta(G, x, y) \geq 2k \) by (4.2), it follows from the induction hypothesis that \( B^* \) contains \( k \) \((x^*, y)\)-paths \( \overrightarrow{P_1}, \ldots, \overrightarrow{P_k} \) satisfying the length condition. By the definition of \( G^* \) and since \( xy \notin E(G) \), we see that for each \( i \) with \( 1 \leq i \leq k \), there is a vertex \( u_i \) of \( N_G(x) \) such that \( u_i u'_i \in E(G) \), where \( u'_i \) is the successor of \( x^* \) along \( \overrightarrow{P_i} \). Therefore, \( ux_i u'_i \overrightarrow{P_i} y, \ldots, ux_k u'_k \overrightarrow{P_k} y \) are \((x, y)\)-paths in \( G \) satisfying the length condition, a contradiction. Thus \((B^*, x^*, y)\) is not 2-connected.

Then we have \( V(B^*) = \{x^*, y\} \). This together with the definition of \( G^* \) and Claim 4.2 implies that \( N_G(y) \subseteq N_G(x) \). Since \( \text{deg}_G(x) \leq \text{deg}_G(y) \), this yields that \( N_G(x) = N_G(y) \). Since \(|G^*| \geq 3\), the definition of \( G^* \) also implies that there is a component \( C \) of \( G - (N_G(x) \cup \{x, y\}) \). Then \( N_G(C) \subseteq N_G(x) \) and \( G[V(C) \cup N_G(C) \cup \{x\}] \) is 2-connected, since \( G \) is 2-connected and \( N_G(x) = N_G(y) \). Hence \( N_G(C) \subseteq N_G(x) \) can be partitioned into two vertex-disjoint non-empty sets \( S \) and \( T \) so that the graph \( C^* \) defined as follows is 2-connected:

\[
V(C^*) = V(C) \cup \{x, S, T\} \quad \text{and} \quad E(C^*) = E(C) \cup \{xs, xt\} \cup \{vS : v \in V(C), E_G(v, S) \neq \emptyset\} \cup \{vT : v \in V(C), E_G(v, T) \neq \emptyset\}.
\]

Then by the definition of \( C^* \), it follows that \((C^* - x, S, T)\) is a 2-connected rooted graph and \( \text{deg}_{C^*-x}(v) = \text{deg}_G(v) \) for \( v \in V(C^*) \setminus \{x, S, T\} \); thus \( \delta(C^* - x, S, T) \geq \delta(G, x, y) \geq 2k \). By the induction hypothesis, \( C^* - x \) contains \( k \) \((S, T)\)-paths \( \overrightarrow{P_1}, \ldots, \overrightarrow{P_k} \) satisfying the length condition. Note that each \( P_i \) has order at least 3, and each \( P_i - \{S, T\} \) is contained in \( C \). For each \( i \) with \( 1 \leq i \leq k \), let \( s_i \) and \( t_i \) be vertices in \( S \) and \( T \), respectively, such that \( s_i s'_i t_i t'_i \in E(G) \), where \( s'_i \) and \( t'_i \) are the successor of \( S \) and the predecessor of \( T \) along \( \overrightarrow{P_i} \), respectively. Then \( xs_1 s'_1 \overrightarrow{P_1} t'_1 t_1 y, \ldots, xss'_k \overrightarrow{P_k} t'_k k y \) are \((x, y)\)-paths in \( G \) satisfying the length condition, a contradiction.

This completes the proof of Case 1.

**Case 2.** \( G - y \) contains a cycle of length 4 passing through \( x \).

By the assumption of Case 2, \( G \) contains a bipartite subgraph \( H = G[S, T] \) such that \( H \) is complete bipartite, \(|T| \geq |S| = l + 1 \geq 2\), \( x \in S \), \( y \notin V(H) \) (i.e., \( H \) satisfies (C1) and (C2)). Let \( C \) be the component of \( G - V(H) \) such that \( y \in V(C) \). Choose \( H \) so that

(a) \(|S|\) is maximum,

(b) \(|T|\) is maximal, subject to (a),

(c) \(|C|\) is maximum, subject to (a) and (b), and

(d) \(|N_G(C) \cap S|\) is minimum, subject to (a), (b) and (c).

Then by the choices (a) and (b), we can obtain the following.

**Claim 4.3** \( H \) is an \( l \)-core of \( G \) with respect to \((x, y)\).
Further, if \( v \), then we can take an (\( \text{Proof.} \) Assume that \( v \)).

We first show (C4). Suppose that there exists a vertex \( v \in V(G) \setminus (S \cup \{y\}) \) such that \( e_G(v, T \setminus \{v\}) \geq l + 2 \). Let \( S' = S \cup \{v\} \) and \( T' = \{v' \in V(G) \setminus (S' \cup \{y\}) : S' \subseteq N_G(v')\} \). Note that \( N_G(v) \cap T \subseteq T' \). Hence \( G[S', T'] \) is a complete bipartite subgraph of \( G \) such that \( |T'| \geq e_G(v, T \setminus \{v\}) \geq l + 2 = |S'| > |S|, \), which contradicts the choice (a). Thus (i) and (ii) hold.

We next show (C3). Suppose that there exists a vertex \( v \in V(G) \setminus (V(H) \cup \{y\}) \) such that \( e_G(v, S) \geq l + 1 \). Since \( |S| = l + 1 \), this implies that \( S \subseteq N_G(v) \). Hence, \( G[S, T \cup \{v\}] \) is a complete bipartite subgraph of \( G \) such that \( |T \cup \{v\}| > |T| \geq |S|, \), which contradicts the choice (b).\( \Box \)

By the choices (c) and (d), we can obtain the following.

**Claim 4.4** If \( E_G(S \setminus \{x\}, C) \neq \emptyset \), then (i) \( e_G(v, T) \leq l \) for \( v \in V(G) \setminus (V(H \cup C) \), and (ii) \( e_G(v, T \setminus \{v\}) \leq l \) or \( E_G(v, C) \neq \emptyset \) for \( v \in T \).

**Proof.** Assume that \( E_G(S \setminus \{x\}, C) \neq \emptyset \), and let \( s \) be a vertex of \( S \setminus \{x\} \) such that \( E_G(s, C) \neq \emptyset \). We now suppose that there is a vertex \( v \in V(G) \setminus (S \cup (V(C)) \) such that \( e_G(v, T \setminus \{v\}) \geq l + 1 \). Further, if \( v \in T \), then we suppose that \( E_G(v, C) = \emptyset \). Note that \( E_G(v, C) = \emptyset \) also holds in the case of \( v \in V(G) \setminus (V(H \cup C)) \), since \( C \) is a component of \( G - V(H) \).

Let \( S' = (S \setminus \{s\}) \cup \{v\} \) and \( T' = \{v' \in V(G) \setminus (S' \cup \{y\}) : S' \subseteq N_G(v')\} \), and let \( H' = G[S', T'] \). Since \( e_G(v, T \setminus \{v\}) \geq l + 1 \), it follows from the definitions of \( S', T' \), and \( H' \) that \( H' \) is a complete bipartite subgraph of \( G \) such that \( |T'| \geq e_G(v, T \setminus \{v\}) \geq l + 1 = |S'| = |S|, \), which contradicts the choice (a) and (b).

Let \( C' \) be the component of \( G - V(H') \) such that \( y \in V(C') \). Note that \( V(C) \subseteq V(C') \subseteq V(G) \setminus V(H') \) since \( S' = (S \setminus \{s\}) \cup \{v\} \) and \( E_G(v, C) = \emptyset \). Hence the choice (c) yields that \( C' = C \). In particular, \( H' \) also satisfies (c). But then, since \( s \in N_G(C) \) and \( v \notin N_G(C) \), we have

\[ |N_G(C') \cap S'| = |N_G(C) \cap ((S \setminus \{s\}) \cup \{v\})| = |N_G(C) \cap S| - 1, \]

which contradicts the choice (d). Thus (i) and (ii) hold.\( \Box \)

Note that \( l \leq k - 1 \) by Lemma 3 and Claim 4.3.

**Claim 4.5** If \( E_G(S \setminus \{x\}, C) \neq \emptyset \), then \( N_G(T) \subseteq V(H \cup C) \).

**Proof.** Assume that \( E_G(S \setminus \{x\}, C) \neq \emptyset \), and let \( s \) be a vertex of \( S \setminus \{x\} \) such that \( E_G(s, C) \neq \emptyset \). Then we can take an \((s, y)\)-path \( \overrightarrow{P} \) in \( G[V(C) \cup \{s\}] \).

We now suppose that \( N_G(T) \nsubseteq V(H \cup C) \), i.e., there exists a component \( D \) of \( G - V(H) \) such that \( D \neq C \) and \( E_G(T, D) \neq \emptyset \).

**Subclaim 4.5.1** Let \( B \) be an end block of \( D \), and let \( b \) be a cut vertex of \( D \) which is contained in \( B \). Then \( E_G(B - b, T \cup \{x, s\}) \neq \emptyset \).
Proof. Suppose that $E_G(B - b, T \cup \{x, s\}) = \emptyset$. Since $G$ is 2-connected, we have $E_G(B - b, S \setminus \{x, s\}) \neq \emptyset$. In particular, $S \setminus \{x, s\} \neq \emptyset$, that is, $l \geq 2$. We define the graph $B^*$ as follows:

$$V(B^*) = V(B) \cup \{S^*\} \quad \text{and} \quad E(B^*) = E(B) \cup \{vS^* : v \in V(B), E_G(v, S \setminus \{x, s\}) \neq \emptyset\}.$$ 

Then $(B^*, S^*, b)$ is a 2-connected rooted graph. Since $l \geq 2$, it also follows that for each $v \in V(B^*) \setminus \{S^*, b\}$,

$$\deg_{B^*}(v) \geq \begin{cases} 
\deg_G(v) \geq 2k & \text{(if } E_G(v, S \setminus \{x, s\}) = \emptyset) \\
\deg_G(v) - (l - 1) + 1 \geq 2(k - l + 2) & \text{(if } E_G(v, S \setminus \{x, s\}) \neq \emptyset) 
\end{cases},$$ 

and thus $\delta(B^*, S^*, b) \geq 2(k - l + 2)$. By the induction hypothesis, $B^*$ contains $k - l + 2$ $(S^*, b)$-paths satisfying the length condition. Then it follows from the definition of $B^*$ that $G - V(C)$ contains $k - l + 2 (S \setminus \{x, s\}, b)$-paths internally disjoint from $V(H)$ and satisfying the length condition. On the other hand, since $E_G(T, D) \neq \emptyset$ and $E_G(T, B - b) = \emptyset$ by the assumption, we can take a $(T, b)$-path in $G[V(D - B) \cup \{b\} \cup T]$. Combining the $(T, b)$-path with the above $k - l + 2 (S \setminus \{x, s\}, b)$-paths, we can get $k - l + 2 (T, S \setminus \{x, s\})$-paths internally disjoint from $V(H)$ and satisfying the length condition, which contradicts Lemma 3(iii). □

Subclaim 4.5.2 $E_G(\{x, s\}, D) = \emptyset$.

Proof. Suppose that $E_G(\{x, s\}, D) \neq \emptyset$. We define the graph $D^*$ as follows:

$$V(D^*) = V(D) \cup \{x^*, T\} \quad \text{and} \quad E(D^*) = E(D) \cup \{vx^* : v \in V(D), E_G(v, \{x, s\}) \neq \emptyset\} \cup \{vT : v \in V(D), E_G(v, T) \neq \emptyset\}.$$ 

Then by Subclaim 4.3.1 and since $E_G(T, D) \neq \emptyset$, it follows that $(D^*, x^*, T)$ is a 2-connected rooted graph. By (C3), and since $e_G(v, T) \leq l$ for $v \in V(D)$ by Claim 4.4(i), it also follows that each $v \in V(D^*) \setminus \{x^*, T\}$ satisfies the following:

$$\deg_{D^*}(v) \geq \begin{cases} 
\deg_G(v) - ((l - 1) + l) + 1 \geq 2(k - l + 1) & \text{(if } E_G(v, \{x, s\}) = \emptyset, E_G(v, T) \neq \emptyset) \\
\deg_G(v) - (l + l) + (1 + 1) \geq 2(k - l + 1) & \text{(if } E_G(v, \{x, s\}) \neq \emptyset, E_G(v, T) \neq \emptyset) \\
\deg_G(v) - (l - 1) \geq 2(k - l + 1) & \text{(if } E_G(v, \{x, s\}) = \emptyset, E_G(v, T) = \emptyset) \\
\deg_G(v) - l + 1 \geq 2(k - l + 1) & \text{(if } E_G(v, \{x, s\}) \neq \emptyset, E_G(v, T) = \emptyset) 
\end{cases}.$$ 

Thus $\delta(D^*, x^*, T) \geq 2(k - l + 1)$. By the induction hypothesis, $D^*$ contains $k - l + 1 (T, x^*)$-paths satisfying the length condition. Then it follows from the definition of $D^*$ that $G - V(C)$ contains $k - l + 1 (T, \{x, s\})$-paths internally disjoint from $V(H)$ and satisfying the length condition, which contradicts Lemma 3(iii). □

We divide the rest of the proof of Claim 4.3 into two cases as follows.

Case (i) $|N_G(D) \cap T| \leq 1$. 

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Since \( E_G(T, D) \neq \emptyset \), we have \(|N_G(D) \cap T| = 1\), say \( N_G(D) \cap T = \{t\}\). Since \( G \) is 2-connected, it follows from Subclaim \( 4.5.2 \) that \( E_G(D, S \setminus \{x, s\}) \neq \emptyset \). In particular, \( S \setminus \{x, s\} \neq \emptyset \), that is, \( l \geq 2 \).

We define the graph \( D^* \) as follows:

\[
V(D^*) = V(D) \cup \{t, S^*\} \quad \text{and} \quad E(D^*) = E(D) \cup \{vt : v \in V(D), vt \in E(G)\} \cup \{vS^* : v \in V(D), E_G(v, S \setminus \{x, s\}) \neq \emptyset\}.
\]

Then by Subclaims \( 4.5.1 \) and \( 4.5.2 \) \((D^*, t, S^*)\) is a 2-connected rooted graph. By Subclaim \( 4.5.2 \) and since \( l \geq 2 \), it also follows that for each \( v \in V(D^*) \setminus \{t, S^*\}\),

\[
\deg_{D^*}(v) \geq \begin{cases} 
\deg_G(v) \geq 2k & \text{(if } E_G(v, S \setminus \{x, s\}) = \emptyset) \\
\deg_G(v) - (l - 1) + 1 \geq 2(k - l + 2) & \text{(if } E_G(v, S \setminus \{x, s\}) \neq \emptyset) 
\end{cases},
\]

and thus \( \delta(D^*, t, S^*) \geq 2(k - l + 2) \). By the induction hypothesis, \( D^* \) contains \( k - l + 2 \) \((t, S^*)\)-paths satisfying the length condition. Then it follows from the definition of \( D^* \) that \( G - V(C) \) contains \( k - l + 2 \) \((t, S \setminus \{x, s\})\)-paths internally disjoint from \( V(H) \) and satisfying the length condition, which contradicts Lemma \( 3(\text{iii}) \).

**Case (ii)** \(|N_G(D) \cap T| \geq 2\).

Fix \( t \in N_G(D) \cap T \). Note that \( E_G(D, T \setminus \{t\}) \neq \emptyset \). We define the graph \( D^* \) as follows:

\[
V(D^*) = V(D) \cup \{t, T^*\} \quad \text{and} \quad E(D^*) = E(D) \cup \{vt : v \in V(D), vt \in E(G)\} \cup \{vT^* : v \in V(D), E_G(v, T \setminus \{t\}) \neq \emptyset\}.
\]

Then by Subclaims \( 4.5.1 \) and \( 4.5.2 \) \((D^*, t, T^*)\) is a 2-connected rooted graph. It also follows from Claim \( 4.4(\text{i}) \) and Subclaim \( 4.5.2 \) that for each \( v \in V(D^*) \setminus \{t, T^*\}\),

\[
\deg_{D^*}(v) \geq \begin{cases} 
\deg_G(v) - (l - 1) \geq 2(k - l + 1) & \text{(if } E_G(v, T \setminus \{t\}) = \emptyset) \\
\deg_G(v) - ((l - 1) + l) + 1 \geq 2(k - l + 1) & \text{(if } E_G(v, T \setminus \{t\}) \neq \emptyset) 
\end{cases},
\]

and thus \( \delta(D^*, t, T^*) \geq 2(k - l + 1) \). By the induction hypothesis, \( D^* \) contains \( k - l + 1 \) \((t, T^*)\)-paths satisfying the length condition. Then it follows from the definition of \( D^* \) that \( G - V(C) \) contains \( k - l + 1 \) \(T\)-paths internally disjoint from \( V(H) \) and satisfying the length condition, which contradicts Lemma \( 3(\text{iii}) \).

This completes the proof of Claim \( 4.5 \) \(\Box\)

**Claim 4.6** If \( E_G(T, C - y) = \emptyset \), then \( E_G(S \setminus \{x\}, C) = \emptyset \).

**Proof.** Assume that \( E_G(T, C - y) = \emptyset \) and \( E_G(S \setminus \{x\}, C) \neq \emptyset \). Let \( t \) be a vertex of \( T \), and let \( \alpha = 1 \) if \( ty \in E(G) \); otherwise, let \( \alpha = 0 \). Then by Claim \( 4.5 \) and since \( E_G(T, C - y) = \emptyset \), it follows that \( N_G(t) \subseteq V(H) \cup \{y\} \). By the definition of \( \alpha \) and Lemma \( \text{II}(\text{ii}) \), we have \( l \leq k - \alpha - 1 \). Since \( N_G(t) \cap V(C) \subseteq \{y\} \), by (C4) and Claim \( 4.3(\text{ii}) \), we also have \( e_G(t, T \setminus \{t\}) \leq l + \alpha \). Combining this with (C1), it follows from the inequality \( l \leq k - \alpha - 1 \) that

\[
2k \leq \deg_G(t) = |S| + e_G(t, T \setminus \{t\}) + |E(G) \cap \{ty\}| \leq (l + 1) + (l + \alpha) + \alpha
\leq (k - \alpha) + (k - 1) + \alpha = 2k - 1,
\]
a contradiction. Thus the claim holds. □

We divide the proof of Case 2 into two cases according as $|C| = 1$ or $|C| \geq 2$.

Case 2.1. $|C| = 1$, i.e., $V(C) = \{y\}$.

Claim 4.7 $N_G(x) = N_G(y) = T$.

Proof. Since $V(C) \setminus \{y\} = \emptyset$, it follows from Claim 4.6 that $E_G(y, S \setminus \{x\}) = \emptyset$. Since $xy \notin E(G)$ by Claim 4.2, we get $E_G(y, S) = \emptyset$. This together with $\deg_G(x) = \deg_G(y)$ implies that $N_G(x) = N_G(y) = T$. □

Claim 4.8 $|T| \geq 3$.

Proof. Suppose that $|T| = 2$, say $V(T) = \{t_1, t_2\}$. Then by (C1), $|S| = 2$ also holds, say $S \setminus \{x\} = \{s\}$. Let $G' = G - \{x, y\}$. By Claim 4.7 $G'$ is connected and $\deg_{G'}(v) = \deg_G(v)$ for $v \in V(G') \setminus \{t_1, t_2\}$. If $G'$ is 2-connected, then obviously $(G', t_1, t_2)$ is a 2-connected rooted graph. On the other hand, if $G'$ is not 2-connected, then since $G$ is 2-connected and $N_G(x) = N_G(y) = \{t_1, t_2\}$, $t_1$ and $t_2$ are in different end blocks of $G'$; since $\{t_1, t_2\} \subseteq N_G(s)$, $s$ is the cut vertex of $G'$ contained in both of the two end blocks of $G'$, that is, $(G', t_1, t_2)$ is a 2-connected rooted graph. In either case, $(G', t_1, t_2)$ is a 2-connected rooted graph. Therefore, by the induction hypothesis, we can get $k$ $(t_1, t_2)$-paths $P_1, \ldots, P_k$ in $G'$ satisfying the length condition. Then $xt_1P_1t_2y, \ldots, xt_1P_kt_2y$ are $k$ $(x, y)$-paths in $G$ satisfying the length condition, a contradiction. □

Let $s \in S \setminus \{x\}$ and $t \in T$, and let $G' = G - \{s, t\}$. Then $\deg_{G'}(v) \geq \deg_G(v) - 2 \geq 2(k - 1)$ for $v \in V(G') \setminus \{x, y\}$.

We divide the proof of Case 2.1 into three cases according as the connectivity of $G'$.

Case 2.1.1. $G'$ is 2-connected.

By applying the induction hypothesis to $(G', x, y)$, $G'$ contains $k - 1$ $(x, y)$-paths satisfying the length condition. Let $P$ be the longest path in the $k - 1$ $(x, y)$-paths, and then the $k - 1$ $(x, y)$-paths together with the $(x, y)$-path $x \overrightarrow{P}y - sty$ form $k$ $(x, y)$-paths in $G$ satisfying the length condition, a contradiction.

Case 2.1.2. $G'$ is not connected.

In this case, there exists a component $D$ of $G'$ such that $V(D) \cap (V(H) \cup \{y\}) = \emptyset$, and let $D' = G[V(D) \cup \{s, t\}]$. Since $G$ is 2-connected, it follows that $(D', s, t)$ is a 2-connected rooted graph such that $\deg_{D'}(v) = \deg_{G}(v)$ for $v \in V(D') \setminus \{s, t\} (= V(D))$. Then by the induction hypothesis, $D'$ contains $k$ $(s, t)$-paths satisfying the length condition. By adding $xt$ and $st'y$ to each path, where $t'$ is a vertex of $T \setminus \{t\}$, we can obtain $k$ $(x, y)$-paths in $G$ satisfying the length condition, a contradiction.

Case 2.1.3. $G'$ is connected, but not 2-connected.
By Claims 4.7 and 4.8, $G((V(H) \cup \{y\}) \setminus \{s, t\})$ is 2-connected. Since $G'$ is connected but it is not 2-connected, this implies that there is an end block $B$ of $G'$ with cut vertex $b$ such that $V(B - b) \cap (V(H) \cup \{y\}) \setminus \{s, t\} = \emptyset$. Let $\overrightarrow{P}$ be a path from $b$ to some vertex $a \in (V(H) \cup \{y\}) \setminus \{s, t\}$ in $G'$ internally disjoint from $(V(B \cup H) \cup \{y\}) \setminus \{s, t\}$. Since $N_G(x) = N_G(y) = T$ by Claim 4.7, it follows that $a \notin \{x, y\}$ and thus $a \in V(H) \setminus \{s, t\}$. Note that $N_G(B - b) \subseteq \{s, t, b\}$.

We now show that $t \notin N_G(B - b)$. By way of contradiction, suppose $t \notin N_G(B - b)$, and let $B' = G(V(B) \cup \{s\})$. Then $(B', s, b)$ is a 2-connected rooted graph such that $\deg_{B'}(v) = \deg_G(v)$ for $v \in V(B') \setminus \{s, b\}$. Hence by the induction hypothesis, $B'$ contains $k$ $(s, b)$-paths $\overrightarrow{P_1}, \ldots, \overrightarrow{P_k}$ satisfying the length condition. If $a \in T$, then let $\overrightarrow{P'} = \overrightarrow{bPa}$; if $a \in S$, then we take a vertex $t'$ of $T \setminus \{t\}$, and let $\overrightarrow{P''} = \overrightarrow{bPat'y}$. Then $xts\overrightarrow{P_1bP_1y}, \ldots, xts\overrightarrow{P_kbP_ky}$ form $k$ $(x, y)$-paths in $G$ satisfying the length condition, a contradiction. Thus $t \notin N_G(B - b)$ is proved.

Now let $B'' = G[V(B) \cup \{t\}]$. Then $(B'', t, b)$ is a 2-connected rooted graph such that $\deg_{B''}(v) \geq \deg_G(v) - 1 > 2(k - 1)$ for $v \in V(B'') \setminus \{t, b\}$. By the induction hypothesis, $B''$ contains $k - 1$ $(t, b)$-paths $\overrightarrow{Q_1}, \ldots, \overrightarrow{Q_{k-1}}$ satisfying the length condition. If $a \in T$, then there exists a vertex $t' \in T \setminus \{t, a\}$ and thus $xt\overrightarrow{Q_1bP_1y}, \ldots, xt\overrightarrow{Q_{k-1}bP_{k-1}y}$ and $xt\overrightarrow{Q_{k-1}bP_{k-1}y}$ form $k$ $(x, y)$-paths in $G$ satisfying the length condition, a contradiction. Thus $a \in S$. Then there exist two distinct vertices $t_1, t_2 \in T \setminus \{t\}$, and hence $xt\overrightarrow{Q_1bP_1at_1y}, \ldots, xt\overrightarrow{Q_{k-1}bP_{k-1}at_1y}$ and $xt\overrightarrow{Q_{k-1}bP_{k-1}at_1y}$ form $k$ $(x, y)$-paths in $G$ satisfying the length condition, a contradiction.

This completes the proof of Case 2.1.

Case 2.2. $|C| \geq 2$.

Claim 4.9 $E_G(T, C - y) \neq \emptyset$.

**Proof.** By way of contradiction, suppose that $E_G(T, C - y) = \emptyset$. By Claim 4.6, $E_G(S \setminus \{x\}, C) = \emptyset$. Hence $N_G(C - y) \subseteq \{x, y\}$. Let $C' = G[V(C) \cup \{x\}]$. Since $G$ is 2-connected, $(C', x, y)$ is a 2-connected rooted graph such that $\deg_{C'}(v) = \deg_G(v)$ for $v \in V(C') \setminus \{x, y\}$. By the induction hypothesis, $C'$ (and also $G$) contains $k$ $(x, y)$-paths satisfying the length condition, a contradiction. Thus $E_G(T, C - y) \neq \emptyset$. □

By (C1), for a vertex $t$ of $T$, $H$ contains 2 $(x, t)$-paths of lengths 1 and 3, respectively. Since $E_G(T, C - y) \neq \emptyset$ by Claim 4.9, this implies that

$$k \geq 3. \quad (4.3)$$

In the rest of this proof, we say that an end block $B$ of $C$ is feasible if $y \notin V(B) \setminus \{b\}$, where $b$ is the cut vertex of $C$ contained in $B$.

Claim 4.10 (i) $C$ is not 2-connected, and (ii) if $B$ is a feasible end block of $C$ with cut vertex $b$, then $E_G(T, B - b) = \emptyset$.

**Proof.** Suppose that either $C$ is 2-connected or there is a feasible end block $B$ of $C$ with cut vertex $b$ such that $E_G(T, B - b) \neq \emptyset$. In the former case, we define $B' = C$ and $b' = y$. Note that, in this
Claim 4.12. (i) Let $k$ satisfy the length condition. Combining these

By Claim 4.10(i), $(\{b, y\} = \emptyset)$

Now we define the graph $B^*$ as follows:

$$V(B^*) = V(B') \cup \{T\} \text{ and } E(B^*) = E(B') \cup \{vT : v \in V(B'), E_G(v, T) \neq \emptyset\}.$$  

Then $(B^*, T, b')$ is a 2-connected rooted graph. By (C3) and (C4), it also follows that for each $v \in V(B^*) \setminus \{T, b'\},$

$$\deg_{B^*}(v) \geq \begin{cases} 
\deg_G(v) - l \geq 2(k - l) & \text{(if } E_G(v, T) = \emptyset) \\
\deg_G(v) - (l + (l + 1)) + 1 \geq 2(k - l) & \text{(if } E_G(v, T) \neq \emptyset) 
\end{cases},$$

and thus $\delta(B^*, T, b') \geq 2(k - l).$ By the induction hypothesis, $B^*$ contains $k - l$ $(T, b')$-paths satisfying the length condition. Therefore, by the definition of $B^*$ and by adding a $(b', y)$-path in $C$ to each of the $k - l$ paths, we can obtain $k - l$ $(T, y)$-paths in $G$ internally disjoint from $V(H)$ and satisfying the length condition, which contradicts Lemma 3.11. \qed

Claim 4.11. If $B$ is a feasible end-block of $C$ with cut vertex $b$, then $E_G(S \setminus \{x\}, B - b) \neq \emptyset.$

Proof. Suppose that $E_G(S \setminus \{x\}, B - b) = \emptyset.$ By Claim 4.10(ii) and the 2-connectivity of $G,$ we have $N_G(B - b) = \{x, b\}. Let B' = G[V(B) \cup \{x\}]. Then $(B', x, b)$ is a 2-connected rooted graph such that $\deg_{B'}(v) = \deg_G(v)$ for $v \in V(B') \setminus \{x, b\}. By the induction hypothesis, $B'$ contains $k$ $(x, b)$-paths satisfying the length condition. Combining these $k$ paths and a $(b, y)$-path in $C,$ we can obtain $k$ $(x, y)$-paths in $G$ satisfying the length condition, a contradiction. \qed

Claim 4.12. (i) $l = 1,$ and (ii) if $B$ is a feasible end block of $C$ with cut vertex $b,$ then $N_G(B - b) = S \cup \{b\}.$

Proof. By Claim 4.10(i), $C$ contains a feasible end block. Let $B$ be an arbitrary feasible end block of $C$ with cut vertex $b,$ and we show that $l = 1$ and $N_G(B - b) = S \cup \{b\}.$ Suppose that either $l \geq 2,$ or $l = 1$ but $N_G(B - b) \neq S \cup \{b\}. We define the graph $B^*$ as follows:

$$V(B^*) = V(B) \cup \{S^*\} \text{ and } E(B^*) = E(B) \cup \{vS^* : v \in V(B), E_G(v, S \setminus \{x\}) \neq \emptyset\}.$$  

Then by Claim 4.11 $(B^*, S^*, b)$ is a 2-connected rooted graph.

We first consider the case of $l \geq 2.$ Then by Claim 4.10(ii) and (C3), and since $l \geq 2,$ it follows that for each $v \in V(B^*) \setminus \{S^*, b\},$

$$\deg_{B^*}(v) \geq \begin{cases} 
\deg_G(v) \geq 2(k - l + 1) & \text{(if } E_G(v, S) = \emptyset) \\
\deg_G(v) - 1 \geq 2(k - l + 1) & \text{(if } E_G(v, S \setminus \{x\}) = \emptyset \text{ and } E_G(v, x) \neq \emptyset) \\
\deg_G(v) - l + 1 \geq 2(k - l + 1) & \text{(if } E_G(v, S \setminus \{x\}) \neq \emptyset \text{ and } E_G(v, x) = \emptyset) \\
\deg_G(v) - l + 1 - 1 \geq 2(k - l + 1) & \text{(if } E_G(v, S \setminus \{x\}) \neq \emptyset \text{ and } E_G(v, x) \neq \emptyset) 
\end{cases},$$

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and thus \( \delta(B^*, S^*, b) \geq 2(k - l + 1) \). By the induction hypothesis, \( B^* \) contains \( k - l + 1 \) \((S^*, b)\)-paths satisfying the length condition. Then by the definition of \( B^* \) and by adding a \((b, y)\)-path in \( C \) to each of the \( k - l + 1 \) paths, we can obtain \( k - l + 1 \) \((S \setminus \{x\}, y)\)-paths in \( G \) internally disjoint from \( V(H) \) and satisfying the length condition, which contradicts Lemma 4(ii).

We next consider the case of \( l = 1 \). In this case, \( N_G(B - b) = (S \setminus \{x\}) \cup \{b\} \) since \( E_G(S \setminus \{x\}, B - b) \neq \emptyset \) (by Claim 4.11) and \( N_G(B - b) \neq S \cup \{b\} \). Hence it follows that \( \deg_{B^*}(v) = \deg_{G}(v) \) for \( v \in V(B^*) \setminus \{S^*, b\} \). By the induction hypothesis, \( B^* \) contains \( k \) \((S^*, b)\)-paths satisfying the length condition. Therefore, by the definition of \( B^* \), we can easily see that \( G \) contains \( k \) \((x, y)\)-paths in \( G \) satisfying the length condition, a contradiction. \( \square \)

By Claim 4.12(i), \(|S| = 2\), say \( S \setminus \{x\} = \{s\} \). On the other hand, by Claim 4.12(ii), if \( B \) is a feasible end block of \( C \) with cut vertex \( b \) and \( v \) is a vertex of \( S \), then \((G[V(B) \cup \{v\}], v, b)\) is a 2-connected rooted graph such that \( \deg_{G[V(B) \cup \{v\}]}(v') \geq \deg_{G}(v') - 1 \geq 2k - 1 \) for \( v' \in V(B) \setminus \{b\} \).

This together with the induction hypothesis implies that 

\[
G[V(B) \cup \{v\}] \text{ contains } k - 1 \text{ \((v, b)\)-paths satisfying the length condition}
\]

for any feasible end block \( B \) of \( C \) with cut vertex \( b \) and any vertex \( v \in S \).

Now let 

\[
U = N_G(T) \cap (V(C) \setminus \{y\}).
\]

Note that by Claim 4.9 \( U \neq \emptyset \). Let

\( B_1, \ldots, B_h \) be all the feasible end blocks of \( C \) with cut vertices \( b_1, \ldots, b_h \), respectively

(note that by Claim 4.10(i), such blocks exist). We further let

\[
C' = C - \bigcup_{1 \leq i \leq h} (V(B_i) \setminus \{b_i\}).
\]

Note that \( C' \) is connected and that by Claim 4.12(ii), \( C' \) contains all the vertices of \( U \cup \{b_1, b_2, \ldots, b_h\} \).

In the rest of this proof, let \( \overrightarrow{P_1} \) and \( \overrightarrow{P_2} \) be 2 \((x, b_1)\)-paths in \( G[V(B_1) \cup \{x\}] \) satisfying the length condition (note that by (4.3) and (4.4), such two paths exist).

**Claim 4.13** There exists an end block \( B_y \) of \( C \) with cut vertex \( b_y \) such that \( y \in V(B_y) \setminus \{b_y\} \).

**Proof.** Suppose that \( B_1, \ldots, B_h \) are all the end blocks of \( C \). Since \( y \in V(C') \) and \( U \subseteq V(C') \setminus \{y\} \), and since the block-cut tree of \( C \) has order at least 3, there exist two vertex-disjoint paths \( \overrightarrow{P} \) and \( \overrightarrow{Q} \) in \( C' \) such that \( \overrightarrow{P} \) is a path from \( b_i \) to some vertex \( u \in U \) and \( \overrightarrow{Q} \) is a path from \( b_j \) to \( y \) for some \( i, j \) with \( i \neq j \), say \( i = 1 \) and \( j = 2 \).

On the other hand, it follows from (4.4) that \( G[V(B_2) \cup \{s\}] \) contains \( k - 1 \) \((s, b_2)\)-paths \( \overrightarrow{Q_1}, \ldots, \overrightarrow{Q_{k-1}} \) satisfying the length condition. By the definition of \( U \), we can take a vertex \( t \) of \( T \) such that \( tu \in E(G) \). Then

\[
x \overrightarrow{P_1} b_1 \overrightarrow{uts} \overrightarrow{Q_1} b_2 \overrightarrow{Q} y, \quad x \overrightarrow{P_1} b_1 \overrightarrow{uts} \overrightarrow{Q_2} b_2 \overrightarrow{Q} y, \quad \ldots, \quad x \overrightarrow{P_1} b_1 \overrightarrow{uts} \overrightarrow{Q_{k-1}} b_2 \overrightarrow{Q} y, \quad x \overrightarrow{P_2} b_1 \overrightarrow{uts} \overrightarrow{Q_{k-1}} b_2 \overrightarrow{Q} y
\]
are \( k (x, y) \)-paths in \( G \) satisfying the length condition, a contradiction. \( \square \)

Let \( B_y \) and \( b_y \) be the same ones as in Claim 4.13. Then \( B_1, \ldots, B_h \) and \( B_y \) are all the end blocks of \( C \).

**Claim 4.14** For each \( v \in V(C) \setminus \{ y \} \), either \( e_G(v, H) \leq 2 \) or \( v \) is a cut vertex of \( C \) separating \( y \) and all feasible end blocks of \( C \).

**Proof.** Suppose that there exist a vertex \( v \in V(C) \setminus \{ y \} \) such that \( e_G(v, H) \geq 3 \) and a feasible block \( B_i \), say \( i = 1 \), such that \( C - v \) has a \((b_1, y)\)-path \( \overrightarrow{Q} \) internally disjoint from \( B_i \) (note that \( v \in V(C') \)), since every vertex of \( \bigcup_{1 \leq j \leq h}(V(B_j) \setminus \{ b_j \}) \) is adjacent to at most two vertices of \( H \) by Claim 4.12. Since \( l = 1 \) by Claim 4.12(i), (C3) and (C4) ensure that \( v \) is adjacent to exactly two distinct vertices in \( T \), say \( t_1 \) and \( t_2 \). By (4.4), \( G[V(B_1) \cup \{ s \}] \) contains \( k - 1 \) \((s, b_1)\)-paths \( \overrightarrow{Q_1}, \ldots, \overrightarrow{Q_{k-1}} \) satisfying the length condition. Then the \( k - 1 \) paths \( x_1t_1b_1Q_1y, x_2t_2v_1Q_1b_1Q_2y, \ldots, x_{k-1}t_{k-1}b_1Q_1y \) form \( k \) \((x, y)\)-paths in \( G \) satisfying the length condition, a contradiction. \( \square \)

By adding a \((b_1, b_y)\)-path in \( C' \) to each of \( \overrightarrow{P_1} \) and \( \overrightarrow{P_2} \), we can get two \((x, b_y)\)-paths \( \overrightarrow{P_1} \) and \( \overrightarrow{P_2} \) in \( G'[(V(C) \cup \{ x \}) \setminus V(B_y - b_y)] \) satisfying the length condition.

**Claim 4.15** \(|B_y| = 2 \) (i.e., \( V(B_y) = \{ y, b_y \} \)).

**Proof.** Suppose that \(|B_y| \geq 3 \), that is, \( B_y \) is 2-connected. For each vertex \( v \) of \( V(B_y) \setminus \{ y, b_y \} \), \( v \) is not a cut vertex of \( C \) separating \( y \) and all feasible end blocks of \( C \), and hence by Claim 4.14 we have \( e_G(v, H) \leq 2 \). This implies that \((B_y, b_y, y)\) is a 2-connected rooted graph such that \( \deg_{B_y}(v) \geq \deg_G(v) - 2 \geq 2(k - 1) \) for \( v \in V(B_y) \setminus \{ y, b_y \} \). By the induction hypothesis, \( B_y \) contains \( k - 1 \) \((b_y, y)\)-paths satisfying the length condition. Concatenating these \( k - 1 \) paths with \( P'_1 \) and \( P'_2 \), we can obtain \( k \) \((x, y)\)-paths in \( G \) satisfying the length condition, a contradiction. \( \square \)

By Claim 4.15 and since \( G \) is 2-connected, \( E_G(y, H) \neq \emptyset \). Since \( xy \notin E(G) \), there exists a vertex \( a \) of \( V(H) \setminus \{ x \} \) such that \( ay \in E(G) \).

**Claim 4.16** \( h = 1 \).

**Proof.** Suppose that \( h \geq 2 \). By (4.4), \( G[V(B_2) \cup \{ s \}] \) contains \( k - 1 \) \((b_2, s)\)-paths \( \overrightarrow{Q_1}, \ldots, \overrightarrow{Q_{k-1}} \) satisfying the length condition. Let \( \overrightarrow{R} \) be a \((b_1, b_2)\)-path in \( C' \). Note that by Claim 4.13 \( y \notin V(R) \). We also let \( \overrightarrow{R'} \) be an \((s, a)\)-path in \( H - x \). Then

\[
\begin{align*}
x\overrightarrow{P_1}b_1\overrightarrow{R}b_2\overrightarrow{Q_1}\overrightarrow{R}a, \quad &x\overrightarrow{P_1}b_1\overrightarrow{R}b_2\overrightarrow{Q_2}\overrightarrow{R}a, \quad \ldots, \quad x\overrightarrow{P_1}b_1\overrightarrow{R}b_2\overrightarrow{Q_{k-1}}s\overrightarrow{R}a, \\
x\overrightarrow{P_2}b_1\overrightarrow{R}b_2\overrightarrow{Q_1}\overrightarrow{R}a, \quad &x\overrightarrow{P_2}b_1\overrightarrow{R}b_2\overrightarrow{Q_2}\overrightarrow{R}a, \quad \ldots, \quad x\overrightarrow{P_2}b_1\overrightarrow{R}b_2\overrightarrow{Q_{k-1}}s\overrightarrow{R}a
\end{align*}
\]

are \( k \) \((x, y)\)-paths in \( G \) satisfying the length condition, a contradiction. \( \square \)

By Claim 4.16 \( B_1 \) and \( B_y \) are all the end blocks of \( C \). Therefore, \( C \) has a unique block \( W \) of \( C \) such that \( W \neq B_y \) and \( b_y \in V(W) \). Then by (C3), (C4), Claim 4.12(i) and 4.15 \( \deg_W(b_y) \geq 2k - (2l + 1) - |\{b_yy\}| \geq 2 \), which implies that \( W \) is 2-connected.
We show that $W \neq B_1$. Assume not. Then by the definition of $U$, Claims 4.12(ii), 4.15 and 4.16 we have $U \subseteq \{b_y\}$. Then by the definition of $U$, we have $N_G(t) \cap V(C) \subseteq \{y, b_y\}$ for $t \in T$. Let $t$ be an arbitrary vertex of $T$. Since $E_G(S \setminus \{x\}, C) \neq \emptyset$ by Claim 4.12(ii), it follows from Claim 4.15 that $N_G(t) \subseteq V(H \cup C)$. Combining this with the above, we have $N_G(t) \subseteq V(H) \cup \{y, b_y\}$. Then by (C1), (C4), (4.3) and Claim 4.12(i),

$$6 \leq 2k \leq \deg_G(t) = |S| + e_G(t, T \setminus \{t\}) + |E(G) \cap \{ty, tb_y\}| \leq 2l + 4 = 6.$$ 

Thus the equality holds. This yields that $k = 3$ and also that $e_G(t, T \setminus \{t\}) = l + 2$ and $ty, tb_y \in E(G)$. Since $t$ is an arbitrary vertex of $T$, it follows that $G[T]$ contains an edge $t_1t_2$, $t_1b_y \in E(G)$ and $t_2y \in E(G)$. Then $xP_1b_y, xP_2b_2y$ and $xP_2b_yt_1t_2y$ are $3 (= k)$ $(x, y)$-paths in $G$ satisfying the length condition, a contradiction. Thus $W \neq B_1$. In short, $W$ is 2-connected and $W \neq B_1$.

Let $w$ be a cut vertex of $C$ which is contained in $W$ such that $w \neq b_y$. Note that $w$ and $b_y$ are all the cut vertices of $C$ which is contained in $W$. Then each vertex $v$ of $V(W) \setminus \{w, b_y\}$ is not a cut vertex of $C$ separating $y$ and all feasible end blocks of $C$, since $W$ is 2-connected. Hence by Claim 4.14 we have $e_G(v, H) \leq 2$ for $v \in V(W) \setminus \{w, b_y\}$. This implies that $(W, w, b_y)$ is a 2-connected rooted graph such that $\deg_G(v, t) \geq \deg_G(v) - 2 \geq 2(k-1)$ for $v \in V(W) \setminus \{w, b_y\}$. By the induction hypothesis, $W$ contains $k - 1$ $(w, b_y)$-paths satisfying the length condition. Concatenating these $k - 1$ $(w, b_y)$-paths with $P_1$ and $P_2$, the edge $b_yy$ and a $(b_1, w)$-path in $C$, we can obtain $k$ $(x, y)$-paths in $G$ satisfying the length condition, a contradiction.

This completes the proof of Theorem 2. \qed

5 Proof of Theorem 3

In this section, we give the proof of Theorem 3. The direction is the same as the proof of Theorem 2 and the argument is also similar. Therefore we mainly describe the difference from the proof of Theorem 2. In the following proof of Theorem 3, the claims without proof are obtained by the same arguments as in the proof of Theorem 2 (note that the numberings of the claims correspond to the ones of the claims in the proof of Theorem 2)

Proof of Theorem 3 We prove it by induction on $|V(G)| + |E(G)|$. Let $(G, x, y)$ be a minimum counterexample with respect to $|V(G)| + |E(G)|$. If $k = 1$, then by (R1) and (R2), we can easily see that $G$ contains an $(x, y)$-path of length at least 2, a contradiction. Thus $k \geq 2$. Since $\delta(G, x, y) \geq 2k - 1$, this implies that $|G| \geq 4$. By symmetry, we may assume that $\deg_G(x) \leq \deg_G(y)$.

Claim 5.1 $G$ is 2-connected.

Claim 5.2 $xy \notin E(G)$.

Case 1. $G - y$ does not contain a cycle of length 4 passing through $x$. 16
Since \( xy \notin E(G) \) by Claim 5.2 in this case, we have
\[
e_G(v, N_G(x) \setminus \{v\}) \leq 1 \text{ for } v \in V(G) \setminus \{x, y\}. \tag{5.1}
\]
Let \( G^* \) be the graph obtained from \( G \) by contracting the subgraph induced by \( N_G(x) \cup \{x\} \) into a single vertex \( x^* \) and then removing multiple edges. Then by (5.1),
\[
deg_{G^*}(v) = \deg_G(v) \text{ for } v \in V(G^*) \setminus \{x^*, y\}. \tag{5.2}
\]
Assume for the moment that \( |G^*| = 2 \). This implies that \( V(G) = \{x, y\} \cup N_G(x) \). Since \( e_G(v, N_G(x) \setminus \{v\}) \leq 1 \) for \( v \in N_G(x) \), each vertex \( v \) of \( N_G(x) \) satisfies \( 3 \leq 2k - 1 \leq \deg_G(v) \leq e_G(v, N_G(x) \setminus \{v\}) + |\{xv\}| + |E(G) \cap \{yv\}| \leq 3 \). Thus the equality holds. This yields that \( k = 2 \) and also that \( G[N_G(x)] \) contains an edge \( v_1v_2 \) and \( v_1y, v_2y \in (G) \). Therefore, \( xv_1y \) and \( xv_1v_2y \) are 2 (= \( k \)) \( (x, y) \)-paths satisfying the semi-length condition, a contradiction. Thus \( |G^*| \geq 3 \).

By (5.1), (5.2) and since \( |G^*| \geq 3 \), we can prove the rest of Case 1 by the same way as in the paragraphs following (4.2) in Case 1 of the proof of Theorem 2.

**Case 2.** \( G - y \) contains a cycle of length 4 passing through \( x \).

By the assumption of Case 2, \( G \) contains a bipartite subgraph \( H = G[S, T] \) such that \( H \) is complete bipartite, \( |T| \geq |S| =: l + 1 \geq 2, x \in S, y \notin V(H) \) (i.e., \( H \) satisfies (C1) and (C2)). Let \( C \) be the component of \( G - V(H) \) such that \( y \in V(C) \). Choose \( H \) so that
(a) \( |S| \) is maximum,
(b) \( |T| \) is maximal, subject to (a),
(c) \( |C| \) is maximum, subject to (a) and (b), and
(d) \( |N_G(C) \cap S| \) is minimum, subject to (a), (b) and (c).

**Claim 5.3** \( H \) is an \( l \)-core of \( G \) with respect to \( (x, y) \).

**Claim 5.4** If \( E_G(S \setminus \{x\}, C) \neq \emptyset \), then (i) \( e_G(v, T) \leq l \) for \( v \in V(G) \setminus V(H \cup C) \), and (ii) \( e_G(v, T \setminus \{v\}) \leq l \) or \( E_G(v, C) \neq \emptyset \) for \( v \in T \).

Note that \( l \leq k - 1 \) by Lemma 1 and Claim 5.3.

**Claim 5.5** If \( E_G(S \setminus \{x\}, C) \neq \emptyset \), then \( N_G(T) \subseteq V(H \cup C) \).

**Claim 5.6** If \( E_G(T, C - y) = \emptyset \), then \( E_G(S \setminus \{x\}, C) = \emptyset \).

**Proof.** Assume that \( E_G(T, C - y) = \emptyset \) and \( E_G(S \setminus \{x\}, C) \neq \emptyset \). Let \( t \) be a vertex of \( T \), and let \( \alpha = 1 \) if \( ty \in E(G) \); otherwise, let \( \alpha = 0 \). Then by Claim 5.3 and since \( E_G(T, C - y) = \emptyset \), it follows that \( N_G(t) \subseteq V(H) \cup \{y\} \). By the definition of \( \alpha \) and Lemma 1(ii), we have \( l \leq k - \alpha - 1 \). Since
Since $e_N$ implies that (i) Claim 5.10, thus the equality holds in the above inequality. This implies that $\alpha = 1$, i.e., $C$ the cut vertex of $k$

Therefore, we can easily find $\alpha = 1$, and so Lemma 2 implies that $G$ contains $k (x, y)$-paths satisfying the semi-length condition, a contradiction. Therefore $\alpha = 1$, i.e., $ty \in E(G)$. Then the equality $e_N(t, T \setminus \{t\}) = l + \alpha = l + 1 = k - 1$ gives $|T| \geq k$. Now, since $t$ is an arbitrary vertex of $T$, these arguments imply the following:

$G[T]$ contains an edge $t_1t_2$ such that $t_1y, t_2y \in E(G)$ and, $|S| = k - 1$ and $|T| \geq k$.

Therefore, we can easily find $k (x, y)$-paths in $G[V(H) \cup \{y\}]$ satisfying the semi-length condition, a contradiction. Thus the claim holds. □

We divide the proof of Case 2 into two cases according as $|C| = 1$ or $|C| \geq 2$.

**Case 2.1.** $|C| = 1$, i.e., $V(C) = \{y\}$.

**Claim 5.7** $N_C(x) = N_C(y) = T$.

**Claim 5.8** $|T| \geq 3$.

Note that if $P_1, \ldots, P_{k-1}$ are $k - 1$ paths satisfying the semi-length condition and $Q$ is another path of length $|E(P_{k-1})| + 2$, then $P_1, \ldots, P_{k-1}, Q$ are $k$ paths satisfying the semi-length condition. Therefore we can prove the rest of Case 2.1 by the same way as in the paragraphs following Claim 4.8 in Case 2.1 of the proof of Theorem 2.

**Case 2.2.** $|C| \geq 2$.

**Claim 5.9** $E_G(T, C - y) \neq \emptyset$.

By (C1), for a vertex $t$ of $T$, $H$ contains 2 $(x, t)$-paths of lengths 1 and 3, respectively. Since $E_G(T, C - y) \neq \emptyset$ by Claim 5.9, this implies that

$$k \geq 3. \tag{5.3}$$

In the rest of this proof, we say that an end-block $B$ of $C$ is **feasible** if $y \notin V(B) \setminus \{b\}$, where $b$ is the cut vertex of $C$ contained in $B$.

**Claim 5.10** (i) $C$ is not 2-connected, and (ii) if $B$ is a feasible end block of $C$ with cut vertex $b$, then $E_G(T, B - b) = \emptyset$.

**Claim 5.11** If $B$ is a feasible end block of $C$ with cut vertex $b$, then $E_G(S \setminus \{x\}, B - b) \neq \emptyset$. 

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Claim 5.12  (i) \( l = 1 \), and (ii) if \( B \) is a feasible end block of \( C \) with cut vertex \( b \), then \( N_G(B - b) = S \cup \{ b \} \).

By Claim 5.12(i), \( |S| = 2 \), say \( S \setminus \{ x \} = \{ s \} \). On the other hand, by Claim 5.12(ii), if \( B \) is a feasible end block of \( C \) with cut vertex \( b \) and \( v \) is a vertex of \( S \), then \( (G[V(B) \cup \{ v \}], v, b) \) is a 2-connected rooted graph such that \( \deg_{G[V(B) \cup \{ v \}]}(v') \geq \deg_{G}(v') - 1 \geq 2(k - 1) \) for \( v' \in V(B) \setminus \{ b \} \). Therefore, by Theorem 2

\[
G[V(B) \cup \{ v \}] \text{ contains } k - 1 \text{ } (v, b)\text{-paths satisfying the length condition }
\]

for any feasible end block \( B \) of \( C \) with cut vertex \( b \) and any vertex \( v \in S \).

Notice that these paths satisfy the length condition, not the length condition or the semi-length condition. Now let

\[
U = N_G(T) \cap (V(C) \setminus \{ y \})
\]

Note that by Claim 5.9 \( U \neq \emptyset \). Let

\[
B_1, \ldots, B_h \text{ be all the feasible end blocks of } C \text{ with cut vertices } b_1, \ldots, b_h, \text{ respectively}
\]

(note that by Claim 5.10(i), such blocks exist). We further let

\[
C' = C - \bigcup_{1 \leq i \leq h} (V(B_i) \setminus \{ b_i \})
\]

Note that \( C' \) is connected and that by Claim 5.12(ii), \( C' \) contains all the vertices of \( U \cup \{ b_1, b_2, \ldots, b_h \} \).

In the rest of this proof, let \( P_1, \ldots, P_{k-1} \) be \( k - 1 \) (\( \geq 2 \)) \((x, b_1)\)-paths in \( G[V(B_1) \cup \{ x \}] \) satisfying the length condition (note that by (5.3) and (5.4), such two paths exist).

Claim 5.13  There exists an end block \( B_y \) of \( C \) with cut vertex \( b_y \) such that \( y \in V(B_y) \setminus \{ b_y \} \).

Let \( B_y \) and \( b_y \) be the same ones as in Claim 5.13 Then \( B_1, \ldots, B_h \) and \( B_y \) are all the end blocks of \( C \).

Claim 5.14  For each \( v \in V(C) \setminus \{ y \} \), either \( e_C(v, H) \leq 2 \) or \( v \) is a cut vertex of \( C \) separating \( y \) and all feasible end blocks of \( C \).

By adding a \((b_1, b_y)\)-path in \( C' \) to each \( P_i \), we can get \( k - 1 \) (\( \geq 2 \)) \((x, b_y)\)-paths \( \overrightarrow{P_1}, \ldots, \overrightarrow{P_{k-1}} \) in \( G[(V(C) \cup \{ x \}) \setminus (V(B_y) - b_y)] \) satisfying the length condition. Note that if \( Q_1, \ldots, Q_{k-1} \) are \( k - 1 \) \((b_y, y)\)-paths in \( B_y \) satisfying the semi-length condition, then

\[
xP_1b_yQ_1y, xP_1b_yQ_2y, \ldots, xP_1b_yQ_{k-1}y, xP_2b_yQ_{k-1}y
\]

are \( k \) \((x, y)\)-paths satisfying the semi-length condition. Therefore, the following claim is also obtained by the same argument as in the proof of Claim 4.15.

Claim 5.15  \(|B_y| = 2 \) (i.e., \( V(B_y) = \{ y, b_y \} \)).
By Claim 5.15 and since \( G \) is 2-connected, \( E_G(y, H) \neq \emptyset \). Since \( xy \not\in E(G) \), there exists a vertex \( a \) of \( V(H) \setminus \{x\} \) such that \( ay \in E(G) \).

Furthermore, if \( N_G(y) \cap N_G(b_y) \cap V(H - x) \neq \emptyset \), say \( v \in N_G(y) \cap N_G(b_y) \cap V(H - x) \), then \( xP_1b_2y, \ldots, xP_{k-1}b_2y \) and \( xP_{k-1}b_2vy \) are \( k \) \((x, y)\)-paths satisfying the semi-length condition, a contradiction. Thus we also have
\[
N_G(y) \cap N_G(b_y) \cap V(H - x) = \emptyset. \tag{5.5}
\]

**Claim 5.16** \( h = 1 \).

By Claim 5.16, \( B_1 \) and \( B_y \) are all the end blocks of \( C \). Therefore, \( C \) has a unique block \( W \) of \( C \) such that \( W \neq B_y \) and \( b_y \in V(W) \).

**Claim 5.17** \( W \neq B_1 \).

**Proof.** Suppose that \( W = B_1 \). Then by the definition of \( U \), Claims 5.12(ii), 5.15 and 5.16 we have \( U \subseteq \{b_y\} \). By the definition of \( U \), we have \( N_G(t) \cap V(C) \subseteq \{y, b_y\} \) for \( t \in T \). Let \( t \) be an arbitrary vertex of \( T \). Since \( E_G(S \setminus \{x\}, C) \neq \emptyset \) by Claim 5.12(ii), it follows from Claim 5.5 that \( N_G(t) \subseteq V(H \cup C) \). Combining this with the above, we have \( N_G(t) \subseteq V(H \cup \{y, b_y\}) \). Moreover, by (5.5), we also have \( |E(G) \cap \{ty, tb_y\}| \leq 1 \). Therefore, by (C1), (C4), Claim 5.12(i) and (5.3),
\[
5 \leq 2k - 1 \leq \deg_G(t) = |S| + e_G(t, T \setminus \{t\}) + |E(G) \cap \{ty, tb_y\}| \leq (l + 1) + (l + 1) + 1 = 2l + 3 = 5.
\]

Thus the equality holds. This yields that \( k = 3 \) and also that \( e_G(t, T \setminus \{t\}) = l + 1 = 2 \) and \( |E(G) \cap \{ty, tb_y\}| = 1 \). Since \( t \) is an arbitrary vertex of \( T \), we have
\[
e_G(t, T \setminus \{t\}) = 2 \quad \text{and} \quad |E(G) \cap \{ty, tb_y\}| = 1 \quad \text{for} \quad t \in T. \tag{5.6}
\]

Assume first that \( N_G(y) \cap T \neq \emptyset \). We may assume that \( a \in N_G(y) \cap T \) (see the paragraph following Claim 5.15). Let \( t_1, t_2 \in N_G(a) \cap T \) with \( t_1 \neq t_2 \) (note that by (5.6), such two vertices exist). If \( t_ib_y \in E(G) \) for some \( i \) with \( i \in \{1, 2\} \), then \( xP_1b_2y, xP_2b_2y \) and \( xP_2b_2y_iy \) are \( k \) \((x, y)\)-paths satisfying the length condition, a contradiction. Thus \( t_ib_y \notin E(G) \) for \( i \in \{1, 2\} \), and hence by (5.6), \( t_y \in E(G) \) for \( i \in \{1, 2\} \). Then \( xay, xat_1y \) and \( xat_1st_2y \) are \( k \) \((x, y)\)-paths satisfying the semi-length condition, a contradiction. Assume next that \( N_G(y) \cap T = \emptyset \). Then \( a \in S \), i.e., \( a = s \) since \( a \in V(H) \setminus \{x\} \). Since \( N_G(y) \cap T = \emptyset \), \( tb_y \in E(G) \) for \( t \in T \). Hence by taking a vertex \( t \) of \( T \), it follows that \( xP_1b_2y, xP_2b_2y \) and \( xP_2b_2ytsy \) are \( k \) \((x, y)\)-paths satisfying the length condition, a contradiction. \( \Box \)

Moreover, we can show that the following holds.

**Claim 5.18** \( W \) is 2-connected.
Proof. It suffices to show that $\deg_W(b_y) \geq 2$. By way of contradiction, suppose that $\deg_W(b_y) \leq 1$. Then by (C3), (C4), (5.3) and Claim [b.12]1),

$$5 \leq 2k - 1 \leq \deg_G(b_y)$$

$$= e_G(b_y, S) + e_G(b_y, T) + \deg_W(b_y) + |b_y| = l + (l + 1) + 1 + 1 = 2l + 3 = 5.$$ 

Thus the equality holds. This yields that $k = 3$ and also that $e_G(b_y, S) = 1$ and $e_G(b_y, T) = 2$.

Let $t_1, t_2 \in N_G(b_y) \cap T$ with $t_1 \neq t_2$. If $b_y x \in E(G)$, then $xb_y, xt_1b_y, xt_1st_2b_y$ are $3 (= k)$ $(x, y)$-paths satisfying the semi-length condition, a contradiction. Thus $b_y x \notin E(G)$, and hence the equality $e_G(b_y, S) = 1$ implies that $b_y s \in E(G)$. Then by (5.3), $a \in T$, and hence $xP_1^b(b_y, xP_2^b(b_y, y)$ and $xP_3^b(b_y, y)$ are $3 (= k)$ $(x, y)$-paths satisfying the length condition, a contradiction. Thus $\deg_W(b_y) \geq 2$.

Since $W \neq B_1$ and $W$ is 2-connected by Claims [5.17] and [5.18] we can prove the rest of Case 2.2 by the same way as in the last paragraph of Case 2.2 in the proof of Theorem [2].

This completes the proof of Theorem [5]. □

6 Proofs of Theorems 4 and 5

In this section, for a positive integer $k$, we let

$$\varphi (= \varphi(k)) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even} \end{cases}.$$ 

Now we first show Theorem 4.

Proof of Theorem 4. By the definition of $\varphi$, we have $k = 2l - 1 + \varphi$ for some $l \geq 1$. Then by the degree condition, $\delta(G) \geq k + 1 = 2l + \varphi = 2(l + \varphi) - \varphi$. Since $G$ is 2-connected but not 3-connected, there exists a separation $(A, B)$ of $G$ of order two, say $A \cap B = \{x, y\}$. Then it is easily seen that each of $(G[A], x, y)$ and $(G[B], x, y)$ is a 2-connected rooted graph and,

$$\delta(G[A], x, y) \geq \delta(G) \geq 2(l + \varphi) - \varphi \geq 2(l + \varphi) - 1$$

and also $\delta(G[B], x, y) \geq 2(l + \varphi) - \varphi$.

Therefore, by applying Theorem [2] to each of $(G[A], x, y)$ and $(G[B], x, y)$, it follows that $G[A]$ (resp., $G[B]$) contains $l + \varphi$ $(x, y)$-paths $P_1, \ldots, P_{l+\varphi}$ (resp., $Q_1, \ldots, Q_{l+\varphi}$) satisfying the length condition or the semi-length condition. In particular, Theorem [2] guarantees that both of $P_1, \ldots, P_{l+\varphi}$ and $Q_1, \ldots, Q_{l+\varphi}$ satisfy the length condition if $\varphi = 0$.

Now, suppose that either $P_1, \ldots, P_{l+\varphi}$ or $Q_1, \ldots, Q_{l+\varphi}$ satisfy the length condition. By the symmetry, we may assume that $P_1, \ldots, P_{l+\varphi}$ satisfy the length condition. By applying Theorem [2] to $(G[B], x, y)$, we can take other $l (x, y)$-paths $Q_1, \ldots, Q_{l+\varphi}$ satisfying the length condition, since $\delta(G[B], x, y) \geq 2(l + \varphi) - \varphi \geq 2l$. Then Table [1] gives $l + (l + \varphi - 1) = 2l - 1 + \varphi = k$ cycles in $G$.

1Consider in order from the first column of the left in Table [1]
satisfying the length condition. Therefore, we may assume that neither \( P_1, \ldots, P_{l+\varphi} \) nor \( Q_1, \ldots, Q_{l+\varphi} \) satisfy the length condition, and then both of \( P_1, \ldots, P_{l+\varphi} \) and \( Q_1, \ldots, Q_{l+\varphi} \) satisfy the semi-length condition. Note that \( \varphi = 1 \).

Let \( p \) and \( q \) be the switches of \( P_1, \ldots, P_{l+\varphi} \) (\( = P_{l+1} \)) and \( Q_1, \ldots, Q_{l+\varphi} \) (\( = Q_{l+1} \)), respectively. Then Table 2 gives \( 2l = 2l - 1 + \varphi = k \) cycles in \( G \) satisfying the length condition.

---

| Both of \( P_1, \ldots, P_{l+\varphi} \) and \( Q_1, \ldots, Q_{l+\varphi} \) satisfy the length condition | |
|---|---|
| \( x \bar{P}_1 y \bar{Q}_1 x \) | \( x \bar{P}_2 y \bar{Q}_1 x \) |
| \( x \bar{P}_1 y \bar{Q}_2 x \) | \( x \bar{P}_3 y \bar{Q}_1 x \) |
| \( \vdots \) | \( \vdots \) |
| \( x \bar{P}_1 y \bar{Q}_l x \) | \( x \bar{P}_{l+\varphi} y \bar{Q}_l x \) |
| \( l \) cycles | \( l + \varphi - 1 \) cycles |

Table 1:

---

| Both of \( P_1, \ldots, P_{l+\varphi} \) and \( Q_1, \ldots, Q_{l+\varphi} \) satisfy the semi-length condition and \( \varphi = 1 \) | |
|---|---|
| \( x \bar{P}_1 y \bar{Q}_1 x \) | \( x \bar{P}_2 y \bar{Q}_1 x \) | \( x \bar{P}_{p+1} y \bar{Q}_{q+2} x \) | \( x \bar{P}_{p+2} y \bar{Q}_{l+1} x \) |
| \( x \bar{P}_1 y \bar{Q}_2 x \) | \( x \bar{P}_3 y \bar{Q}_1 x \) | \( x \bar{P}_{p+1} y \bar{Q}_{q+1} x \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( x \bar{P}_1 y \bar{Q}_q x \) | \( x \bar{P}_q y \bar{Q}_q x \) | \( x \bar{P}_{p+1} y \bar{Q}_{l+1} x \) | \( x \bar{P}_{p+2} y \bar{Q}_{l+1} x \) |
| \( q \) cycles | \( p - 1 \) cycles | \( 1 \) cycle | \( l - q \) cycles | \( l - p \) cycles |

Table 2:

---

We next show Theorem 5. In the proof, we also use the following two lemmas (Lemmas 5 and 6).

**Lemma 5 ([2, Lemma 5.1])** Let \( G \) be a connected graph such that \( \delta(G) \geq 4 \). If \( G \) contains a non-separating induced odd cycle, then \( G \) contains a non-separating induced odd cycle \( \bar{C} \) satisfying one of the following (1) and (2).

(1) \( |V(C)| = 3 \), or

(2) for every non-cut vertex \( v \) of \( G - V(C) \), \( e_G(v, C) \leq 2 \), and the equality holds if and only if \( vu^+, vu^- \in E(G) \) for some \( u \in V(C) \).

**Lemma 6** Let \( k \) and \( l \) be positive integers such that \( k = 2l - 1 + \varphi(k) \). Let \( G \) be a graph and \( \bar{C} \) be an odd cycle in \( G \), say \( |V(C)| = 2m + 1 \) for some \( m \geq 1 \). We further let \( x \in V(G) \setminus V(C) \) and \( u \in V(C) \). If one of the following (i)–(iii) holds, then \( G \) contains \( k \) cycles having consecutive lengths.
(i) \( \varphi(k) = 0 \) and \( G \) contains \( l \) \((u, \{u^+, u^−\})\)-paths internally disjoint from \( V(C) \) and satisfying the length condition or the semi-length condition.

(ii) \( \varphi(k) = 1 \) and \( G \) contains \( l \) \((u, \{u^+, u^−\})\)-paths internally disjoint from \( V(C) \) and satisfying the length condition.

(iii) \( m \geq 2, xu^+, xu^- \in E(G) \) and \( G \) contains \( l−1 \) \((x, u^+)\)-paths internally disjoint from \( V(C) \cup \{x\} \) and satisfying the length condition.

Proof of Lemma 2 To show (i) and (ii), suppose that \( G \) contains \( l \) \((u, \{u^+, u^−\})\)-paths \( \overrightarrow{P_1}, \ldots, \overrightarrow{P_l} \) internally disjoint from \( V(C) \) and satisfying the length condition or the semi-length condition. We further suppose that if \( \varphi = 1 \), then \( P_1, \ldots, P_l \) satisfy the length condition. For each \( i \) with \( 1 \leq i \leq l \), let \( v_i \) be the end vertex of \( P_i \) such that \( v_i \in \{u^+, u^−\} \), and let \( \overrightarrow{Q_i} \) and \( \overrightarrow{R_i} \) denote the paths in \( C \) from \( v_i \) to \( u \) such that \( |E(Q_i)| = m \) and \( |E(R_i)| = m + 1 \). If \( P_1, \ldots, P_l \) satisfy the length condition, then Table 3 gives \( 2l \geq 2l - 1 + \varphi = k \) cycles having consecutive lengths (see also Figure 2). Thus we may assume that \( P_1, \ldots, P_l \) do not satisfy the length condition but satisfy the semi-length condition. In particular, by our assumption, we have \( \varphi \neq 1 \), i.e., \( \varphi = 0 \). Let \( j \) be the switch of \( P_1, \ldots, P_l \), and then Table 4 gives \( 2l - 1 = 2l - 1 + \varphi = k \) cycles having consecutive lengths. Thus (i) and (ii) are proved.

We next show (iii). Suppose that \( m \geq 2, xu^+, xu^- \in E(G) \) and \( G \) contains \( l - 1 \) \((x, u^+)\)-paths \( \overrightarrow{P_1}, \ldots, \overrightarrow{P_l} \) internally disjoint from \( V(C) \cup \{x\} \) and satisfying the length condition. Since \( |C| = 2m + 1 \geq 5 \), we can let \( \overrightarrow{Q_{u^+}} \) and \( \overrightarrow{R_{u^+}} \) (resp., \( \overrightarrow{Q_{u^-}} \) and \( \overrightarrow{R_{u^-}} \)) be the paths in \( C \) from \( u^+ \) (resp., from \( u^+ \) to \( u^- \)) such that \( |E(Q_{u^+})| = m - 1 \) and \( |E(R_{u^+})| = m + 2 \) (resp., \( |E(Q_{u^-})| = m \) and \( |E(R_{u^-})| = m + 1 \)). Hence Table 5 gives \( 2l \geq 2l - 1 + \varphi = k \) cycles having consecutive lengths (see also Figure 3). Thus (iii) is also proved. \( \square \)

| \( P_1, \ldots, P_l \) satisfy the length condition |
|------------------|
| \( u \overrightarrow{P_1} v_1 \overrightarrow{Q_1} u, \quad u \overrightarrow{P_1} v_1 \overrightarrow{R_1} u \) |
| \( u \overrightarrow{P_2} v_2 \overrightarrow{Q_2} u, \quad u \overrightarrow{P_2} v_2 \overrightarrow{R_2} u \) |
| \vdots |
| \( u \overrightarrow{P_l} v_l \overrightarrow{Q_l} u, \quad u \overrightarrow{P_l} v_l \overrightarrow{R_l} u \) |
| \( 2l \) cycles |

Table 3:

\( \overrightarrow{P_1}, \overrightarrow{Q_i} \) and \( \overrightarrow{R_i} \)
Hence by Lemma 5, if $m$ satisfies (1) or (2) in Lemma 5. Now, suppose that $m \geq 2$, $xu^+, xu^- \in E(G)$ and $P_1, \ldots, P_{l-1}$ satisfy the semi-length condition.

Table 4:

$$
\begin{array}{|c|c|}
\hline
m \geq 2, & xu^+, xu^- \in E(G) \\
\hline
\ \ u \xrightarrow{P_1} v_1 \xrightarrow{Q_1} u, \ u \xrightarrow{P_1} v_1 \xrightarrow{R_1} u & \ \ u \xrightarrow{P_{j+1}} v_{j+1} \xrightarrow{R_{j+1}} u \\
\ \ u \xrightarrow{P_2} v_2 \xrightarrow{Q_2} u, \ u \xrightarrow{P_2} v_2 \xrightarrow{R_2} u & \ \ u \xrightarrow{P_{j+2}} v_{j+2} \xrightarrow{Q_{j+2}} u, \ u \xrightarrow{P_{j+2}} v_{j+2} \xrightarrow{R_{j+2}} u \\
\vdots & \vdots \\
\ \ u \xrightarrow{P_{j-1}} v_{j-1} \xrightarrow{Q_{j-1}} u, \ u \xrightarrow{P_{j-1}} v_{j-1} \xrightarrow{R_{j-1}} u & \ \ u \xrightarrow{P_{j+3}} v_{j+3} \xrightarrow{Q_{j+3}} u, \ u \xrightarrow{P_{j+3}} v_{j+3} \xrightarrow{R_{j+3}} u \\
\ \ u \xrightarrow{P_j} v_j \xrightarrow{Q_j} u, \ u \xrightarrow{P_j} v_j \xrightarrow{R_j} u & \ \ u \xrightarrow{P_l} v_l \xrightarrow{Q_l} u, \ u \xrightarrow{P_l} v_l \xrightarrow{R_l} u \\
\hline
\end{array}
$$

Table 5:

$$
\begin{array}{|c|c|}
\hline
2j \text{ cycles} & 2l - 2j - 1 \text{ cycles} \\
\hline
\ \ x \xrightarrow{P_1} u^{+m} \xrightarrow{Q_1} u^+ x, \ x \xrightarrow{P_1} u^{+m} \xrightarrow{Q_u} u^- x & \ \ x \xrightarrow{P_{l-1}} u^{+m} \xrightarrow{R_u} u^- x, \ x \xrightarrow{P_{l-1}} u^{+m} \xrightarrow{R_u} u^+ x \\
\ \ x \xrightarrow{P_2} u^{+m} \xrightarrow{Q_u} u^+ x, \ x \xrightarrow{P_2} u^{+m} \xrightarrow{Q_u} u^- x & \ \ x \xrightarrow{P_{l-1}} u^{+m} \xrightarrow{R_u} u^- x, \ x \xrightarrow{P_{l-1}} u^{+m} \xrightarrow{R_u} u^+ x \\
\vdots & \vdots \\
\ \ x \xrightarrow{P_{l-1}} u^{+m} \xrightarrow{Q_u} u^+ x, \ x \xrightarrow{P_{l-1}} u^{+m} \xrightarrow{Q_u} u^- x & \ \ x \xrightarrow{P_{l-1}} u^{+m} \xrightarrow{R_u} u^- x, \ x \xrightarrow{P_{l-1}} u^{+m} \xrightarrow{R_u} u^+ x \\
\hline
\end{array}
$$

Figure 3: The paths $P_l, Q_{u^+}, Q_{u^-}, R_{u^+}$ and $R_{u^-}$

We are now ready to prove Theorem 5.

Proof of Theorem 5. If $k = 1$, then the assertion clearly holds, since $G$ is 2-connected. If $k = 2$, take an edge $xy$ of $G$ and find 2 $(x, y)$-paths satisfying the length condition or the semi-length condition by using Theorem 3 and then the edge $xy$ and the 2 $(x, y)$-paths induce 2 cycles which satisfy the length condition or have consecutive lengths. Thus we may assume $k \geq 3$.

Let $k = 2l - 1 + \varphi$ for some $l \geq 2$. Then by the degree condition, we have $\delta(G) \geq k + 1 = (2l + \varphi) \geq 4$. Hence by Lemma 5, $G$ contains a non-separating induced odd cycle $\overrightarrow{C}$ in $G$, say $|V(C)| = 2m + 1$ for $m \geq 1$, such that $C$ satisfies (1) or (2) in Lemma 5. Now, suppose that $G$ does not contain...
k cycles having consecutive lengths, and then we will show that G contains k cycles satisfying the length condition.

**Claim 6.1** \(|V(C)| \geq 5\), that is, \(m \geq 2\).

**Proof.** Suppose that \(|V(C)| = 3\), that is, \(m = 1\). Let \(u \in V(C)\). Consider the graph \(G^*\) obtained from \(G\) by contracting \(u^+\) and \(u^-\) into a vertex \(u^*\). Then \(G^*\) is a 2-connected graph and \(\delta(G^*) \geq \delta(G) - 1 \geq k = 2l - 1 + \varphi \geq 2l - 1\). Hence by Theorem 3, \(G^*\) contains \(l\) \((u, u^*)\)-paths satisfying the length condition or the semi-length condition. In particular, if \(\varphi = 1\), then by Theorem 2, we may assume that the \(l\) \((u, u^*)\)-paths satisfy the length condition. Note that each of the \(l\) paths does not contain the edge \(uv\), since the length is at least \(2\). Then it follows from the definition of \(G^*\) that \(G\) contains \(l\) \((u, \{u^+, u^-\})\)-paths internally disjoint from \(V(C)\) and satisfying the length condition or the semi-length condition; in particular, they satisfy the former condition when \(\varphi = 1\). This together with Lemma [ii] leads to a contradiction. \(\square\)

By Claim 6.1, \(C\) satisfies (2) in Lemma 5, i.e., every non-cut vertex \(v\) of \(G - V(C)\) satisfies
\[
\deg_{G - V(C)}(v) \geq \delta(G) - 2 \geq k - 1 = 2l - 1 + \varphi. \tag{6.1}
\]

**Fact 6.2** If \(B\) is an end block of \(G - V(C)\), then \(B\) is 2-connected.

**Proof.** Since \(k \geq 3\), Claim 6.1 implies that \(|B| \geq 3\), and hence \(B\) is 2-connected. \(\square\)

**Claim 6.3** Let \(B\) be an end block of \(G - V(C)\) with cut vertex \(b\) when \(G - V(C)\) is not 2-connected; otherwise, let \(B = G - V(C)\) and \(b\) be an arbitrary vertex of \(G - V(C)\). Further, let \(x \in V(B - b)\) and \(u \in V(C)\). If one of the following (i) and (ii) holds, then we have \(E_G(\{u^+, u^-\}, V(G - C) \setminus V(B - b)) = \emptyset\).

(i) \(xu \in E(G)\) and every vertex of \(B - b\) is adjacent to at most one vertex of \(C\).

(ii) \(xu^+, xu^- \in E(G)\).

**Proof.** Suppose that (i) or (ii) holds and that \(E_G(\{u^+, u^-\}, V(G - C) \setminus V(B - b)) \neq \emptyset\). By the symmetry of \(u^+m\) and \(u^-m\), we may assume that \(E_G(u^+m, V(G - C) \setminus V(B - b)) \neq \emptyset\). Let \(y\) be a vertex of \(V(G - C) \setminus V(B - b)\) such that \(yu^+m \in E(G)\).

We first consider the case where (i) holds. Then by the assumption, every vertex \(v\) of \(B - b\) satisfies
\[
\deg_B(v) = \deg_{G - V(C)}(v) \geq \delta(G) - 1 \geq k = 2l - 1 + \varphi \geq 2l - 1.
\]

Then it follows from Fact 6.2 and Theorem 3 that \(B\) contains \(l\) \((x, b)\)-paths satisfying the length condition or the semi-length condition. In particular, if \(\varphi = 1\), then by Theorem 2, we may assume that the \(l\) \((x, b)\)-paths satisfy the length condition. By adding a \((b, y)\)-path in \(G - (V(C) \cup V(B - b))\) to each of the \(l\) \((x, b)\)-paths, we can get \(l\) \((x, y)\)-paths in \(G - V(C)\) satisfying the length condition or the semi-length condition; in particular, they satisfy the former condition when \(\varphi = 1\). Since \(xu, yu^+m \in E(G)\), this together with Lemma [ii] or [iii] leads to a contradiction.
We next consider the case where (ii) holds. Then it follows from Lemma 6.1, Fact 6.2 and Theorem 2 that $B$ contains $l - 1 \ (x,b)$-paths satisfying the length condition. By adding a $(b,y)$-path in $G - (V(C) \cup V(B - b))$ to each of the $l - 1 \ (x,b)$-paths, we can get $l - 1 \ (x,y)$-paths in $G - V(C)$ satisfying the length condition. Since $xu^+, xu^-, yu^+ \in E(G)$ and since $m \geq 2$ by Claim 6.1, this together with Lemma 6.11 leads to a contradiction. □

Claim 6.4 $G - V(C)$ is not 2-connected.

Proof. Suppose that $B := G - V(C)$ is 2-connected. Choose a vertex $x$ of $B$ so that $e_G(x,C)$ is as large as possible. Then $e_G(x,C) = 1$ or 2 and $e_G(v,C) \leq 2$ for $v \in V(B)$ since $G$ is 2-connected and $C$ satisfies (2) in Lemma 5. Furthermore, if $e_G(x,C) = 2$, then there is a vertex $u$ of $C$ such that $xu^+, xu^- \in E(G)$; otherwise, let $u$ be the unique vertex of $C$ such that $xu \in E(G)$. Note that if $e_G(x,C) \neq 2$, then the choice of $x$ implies that every vertex of $B$ is adjacent to at most one vertex of $C$. Now, since $\delta(G) \geq k + 1 \geq 4$ and $C$ is an induced cycle, we have $E_G(u^+, B) \neq \emptyset$, and so let $b$ be a vertex of $B$ such that $u^+b \in E(G)$. Since $m \geq 2$ by Claim 6.1, it follows from the definition of $u$ that $b \neq x$. These imply that (i) or (ii) of Claim 6.3 holds for the graph $B$ and the vertices $b, x, u$, but $u^+b \in E_G(\{u^+, u^-, m\}, V(G - C) \setminus V(B - b))$, a contradiction. □

By Claim 6.4 $G - V(C)$ has two end blocks $B_1$ and $B_2$ with cut vertices $b_1$ and $b_2$, respectively.

Claim 6.5 There is a vertex $x_i$ of $B_i - b_i$ such that $e_G(x_i, C) = 2$ for $i \in \{1, 2\}$.

Proof. Let $B = B_i$ and $b_i = b$ and suppose that every vertex of $B - b$ is adjacent to at most one vertex of $C$. Since $G$ is 2-connected, there is a vertex $u$ of $C$ such that $E_G(u, B - b) \neq \emptyset$, i.e., some vertex $x$ of $B - b$ is adjacent to $u$. Hence (i) of Claim 6.3 holds for the graph $B$ and the vertices $b, x, u$, and so we have $E_G(u^+, V(G - C) \setminus V(B - b)) = \emptyset$, that is, $N_G(u^+) \subseteq V(C) \cup V(B - b)$. Since $\delta(G) \geq 4$ and $C$ is an induced cycle, this in particular implies that $E_G(u^+, B - b) \neq \emptyset$. Then again by applying (i) of Claim 6.3 for the vertex $u^+$ and a neighbor of $u^+$ in $B - b$ instead of $u$ and $x$, respectively, we have $E_G(u^{+2m}, V(G - C) \setminus V(B - b)) = \emptyset$ and thus $N_G(u^{+2m}) \subseteq V(C) \cup V(B - b)$. Repeating this argument we get $N_G(C) \subseteq V(B - b)$ since $|V(C)|$ is odd. This implies that $b$ is a cut vertex of $G$, which contradicts the 2-connectivity of $G$. Thus $e_G(x, C) \geq 2$ for some $x \in V(B - b)$. Since $C$ satisfies (2) in Lemma 5 we have $e_G(x, C) = 2$. □

By Claim 6.5 and since $C$ satisfies (2) in Lemma 5 each $B_i - b_i$ contains a vertex $x_i$ such that $x_iu^+, x_iu^- \in E(G)$ for some $u_i \in V(C)$.

Assume for the moment that $u_1 = u_2$. Then Claim 6.3 yields that $E_G(u_1^{+m}, V(G - C) \setminus V(B_1 - b_1)) = \emptyset$. On the other hand, since $u_1^{+m} = u_2^{+m}$, it also follows from Claim 6.3 that $E_G(u_1^{+m}, V(G - C) \setminus V(B_2 - b_2)) = \emptyset$. Therefore we get $E_G(u_1^{+m}, V(G - C)) = \emptyset$. But, since $C$ is an induced cycle, this implies that $\deg_G(u_1^{+m}) = \deg_G(u_1^{-m}) = 2 < 4$, a contradiction. Thus $u_1 \neq u_2$.

Assume next that $u_1^+ = u_2$. Then $u_1^{-m} = u_2^{+m}$. By using Claim 6.3 and arguing as in the above, we have $E_G(u_1^{-m}, V(G - C)) = \emptyset$, which contradicts that $\deg_G(u_1^{-m}) \geq 4$. Thus $u_1^+ \neq u_2$. Similarly, we have $u_1^- \neq u_2$. Consequently, we get $u_2 \in V(u_1^{+2}C u_1^{-2})$. 

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Since $|V(C)| \geq 5$, without loss of generality, we may assume that

$$u^+ \neq u^-.$$

We will show that there exist $2l - 3 + \varphi$ $(x_1, x_2)$-paths $\overrightarrow{P_1}, \ldots, \overrightarrow{P_{2l-3+\varphi}}$ in $G - V(C)$ satisfying the length condition. In order to show it, we divide the proof into two cases.

**Case 1.** $\varphi(k) = 1$.

Since $\delta(B_i, x_i, b_i) \geq 2(l - 1) + \varphi = 2l - 1$ for $i \in \{1, 2\}$, it follows from Fact 6.2 and Theorem 3 that $B_1$ contains $l$ $(x_1, b_1)$-paths $\overrightarrow{Q_1}, \ldots, \overrightarrow{Q_l}$ satisfying the length condition or the semi-length condition, and $B_2$ contains $l$ $(b_2, x_2)$-paths $\overrightarrow{R_1}, \ldots, \overrightarrow{R_l}$ satisfying the length condition or the semi-length condition.

Now, suppose that either $Q_1, \ldots, Q_l$ or $R_1, \ldots, R_l$ satisfy the length condition. By the symmetry, we may assume that $Q_1, \ldots, Q_l$ satisfy the length condition. By applying Theorem 2 to $(B_2, b_2, x_2)$, we can take other $l - 1$ $(b_2, x_2)$-paths $\overrightarrow{R'_1}, \ldots, \overrightarrow{R'_{l-1}}$ satisfying the length condition, since $\delta(B_2, b_2, x_2) \geq 2(l - 1)$. Concatenating $Q_1, \ldots, Q_l$ and $R'_1, \ldots, R'_{l-1}$ with a $(b_1, b_2)$-path in $G - V(C)$, we can obtain $2l - 2 = (2l - 3 + \varphi)$ $(x_1, x_2)$-paths $\overrightarrow{P_1}, \ldots, \overrightarrow{P_{2l-2}}$ in $G - V(C)$ satisfying the length condition. Therefore, we may assume that neither $Q_1, \ldots, Q_l$ nor $R_1, \ldots, R_l$ satisfy the length condition, and then both of $Q_1, \ldots, Q_l$ and $R_1, \ldots, R_l$ satisfy the semi-length condition.

Let $q$ and $r$ be the switches of $Q_1, \ldots, Q_l$ and $R_1, \ldots, R_l$, respectively, and let $P$ be a $(b_1, b_2)$-path in $G - V(C)$. Then by considering the paths

$$x_1Q_ib_1Pb_2R_ix_2 (1 \leq i \leq r), \quad x_1Q_ib_1Pb_2R_{i+1}x_2 (2 \leq i \leq q),$$

$$x_1Q_{q+1}ib_1Pb_2R_{r+1}x_2, \quad x_1Q_{q+1}ib_1Pb_2R_ix_2 (r + 2 \leq i \leq l), \quad x_1Q_ib_1Pb_2R_{l}x_2 (q + 2 \leq i \leq l),$$

we can obtain $2l - 2 = (2l - 3 + \varphi)$ $(x_1, x_2)$-paths $\overrightarrow{P_1}, \ldots, \overrightarrow{P_{2l-2}}$ in $G - V(C)$ satisfying the length condition.

**Case 2.** $\varphi(k) = 0$.

Since $\delta(B_i, x_i, b_i) \geq 2(l - 1)$ for $i \in \{1, 2\}$, it follows from Fact 6.2 and Theorem 2 that $B_1$ contains $l - 1$ $(x_1, b_1)$-paths satisfying the length condition and $B_2$ contains $l - 1$ $(b_2, x_2)$-paths satisfying the length condition. Concatenating them with a $(b_1, b_2)$-path in $G - V(C)$, we can obtain $2l - 3 = (2l - 3 + \varphi)$ $(x_1, x_2)$-paths $\overrightarrow{P_1}, \ldots, \overrightarrow{P_{2l-3}}$ in $G - V(C)$ satisfying the length condition.

Thus, in both cases, Table 5 gives $2l - 1 + \varphi = k$ cycles satisfying the length condition (see also Figure 4).

This completes the proof of Theorem 5. $\Box$
$P_1, \ldots, P_{2l-3+\nu}$ satisfy the length condition

\[
x_1 \xrightarrow{P_1} x_2 u_2 \xrightarrow{C} u_1^+ x_1 \\
x_1 \xrightarrow{P_2} x_2 u_2 \xrightarrow{C} u_1^+ x_1 \\
\vdots \\
x_1 \xrightarrow{P_{2l-3+\nu}} x_2 u_2 \xrightarrow{C} u_1^+ x_1
\]

$2l - 3 + \nu$ cycles

\[
x_1 \xrightarrow{P_{2l-3+\nu}} x_2 u_2 \xrightarrow{C} u_1^+ x_1 \\
x_1 \xrightarrow{P_{2l-3+\nu}} x_2 u_2 \xrightarrow{C} u_1^+ x_1
\]

2 cycles

Table 6:

7 Concluding remarks

In this paper, we have shown that the Thomassen’s conjecture on the existence of cycles of any length modulo a given integer $k$ (Conjecture A) is true by giving degree conditions for the existence of a specified number of cycles whose lengths differ by one or two (Theorem 1).

The complete graph of order $k + 1$, in a sense, shows the sharpness of the lower bound on the minimum degree condition in Theorem 1. On the other hand, we believe that the assumption of 2-connectivity in Theorem 1 is not necessary. In fact, Liu and Ma conjectured that Theorem 1 also holds even if we drop the connectivity condition (see [2, Conjecture 6.2]). To approach the conjecture, the following improvements of Theorems 2 and 3 will be helpful.

**Problem 1** Let $k$ be a positive integer, and let $(G, x, y)$ be a 2-connected rooted graph such that $|V(G)| \geq 4$, and $z \in V(G)$ (possibly $z = x$ or $z = y$). Suppose that $\deg_G(v) \geq 2k$ for any $v \in V(G) \setminus \{x, y, z\}$. Then $G$ contains $k$ paths from $x$ to $y$ satisfying the length condition.

**Problem 2** Let $k$ be a positive integer, and let $(G, x, y)$ be a 2-connected rooted graph such that $|V(G)| \geq 4$, and $z \in V(G)$ (possibly $z = x$ or $z = y$). Suppose that $\deg_G(v) \geq 2k - 1$ for any $v \in V(G) \setminus \{x, y, z\}$. Then $G$ contains $k$ paths from $x$ to $y$ satisfying the length condition or the semi-length condition.

In fact, if Problems 1 and 2 are true, then by applying them to an end block $B$ with cut vertex $z$ in a given graph of minimum degree at least $k + 1$, and by arguing as in the proofs of Theorems 4 and 5 we can show that $B$ contains $k$ cycles satisfying the length condition when $B$ is 2-connected but not 3-connected; $B$ contains $k$ cycles, which have consecutive lengths or satisfy the length condition when $B$ is 3-connected and non-bipartite. Therefore, Problems 1 and 2 leads to the improvement of Theorem 1.

The above problems also concern with another Thomassen’s conjecture on the existence cycles of any even length modulo a given integer $k$.
Conjecture D (Thomassen [3])  For a positive integer $k$, every graph of minimum degree at least $k + 1$ contains cycles of all even lengths modulo $k$.

It is known that this conjecture is true for all even integers $k$ (see [2, Theorem 1.9]). The improvement of Theorem 1 implies that it is also true for all odd integers $k$ (note that Theorem 1 implies that Conjecture D is true for the case where $k$ is odd and a given graph is 2-connected).

Problems 1 and 2 can be proven by arguing as in the proofs of Theorems 2 and 3 and by a tedious case-by-case analysis (in fact, we have checked Problem 1 is true in a private discussion, but it is unpublished). Therefore, we think that giving a short proof for the above problems might be interesting and helpful for future work of this research area.

References

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