Validity of the GGE for quantum quenches from interacting to noninteracting models

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Received 7 April 2014
Accepted for publication 3 June 2014
Published 24 July 2014

Online at stacks.iop.org/JSTAT/2014/P07024
doi:10.1088/1742-5468/2014/07/P07024

Abstract. In the majority of analytical verifications of the conjecture that the Generalized Gibbs Ensemble (GGE) describes the large time asymptotics of local observables in quantum quench problems, both post- and pre-quench Hamiltonians are essentially noninteracting. We test this conjecture studying field correlations in the more general case of an arbitrary pre-quench Hamiltonian, while keeping the post-quench one noninteracting. We first show that, in the previously studied special case of a noninteracting pre-quench Hamiltonian, the validity of the conjecture is a consequence of Wick’s theorem. We then show that this conjecture is more generally valid for an arbitrary interacting pre-quench Hamiltonian, but this time as a consequence of the cluster decomposition property of the initial state, which is a fundamental principle for generic physical states. For arbitrary initial states that do not satisfy the cluster decomposition property, the above conjecture is not generally true. As a byproduct of our investigation we obtain an analytical derivation of earlier numerical results for the large time evolution of correlations after a quantum quench of the interaction in the Lieb–Liniger model from a nonzero value to zero.

Keywords: correlation functions, stationary states, quantum quenches, thermalization

ArXiv ePrint: 1403.7431
1. Introduction

One of the fundamental principles of statistical mechanics is that a generic isolated classical system in the thermodynamic limit, prepared in an arbitrary initial state, would evolve so as to maximize its entropy [1, 2]: that is, for large times it would tend to thermal equilibrium described by the microcanonical ensemble, with a total energy equal to the initial one. An obvious exception is provided by integrable systems: i.e. systems that possess a set of integrals of motion equal in number to their degrees of freedom. Maximization of the entropy under all the additional conservation constraints would lead to generalized thermal equilibrium, in which all other constraints are satisfied too.

But what happens in a quantum system? In an isolated system the time evolution is unitary. So, if the system is initially prepared in a pure state it will always remain in a pure state, instead of a statistical ensemble. In fact, the unitary evolution is periodic or quasiperiodic: that is, after a sufficiently large time the system will return to its initial state, or arbitrarily close to it. However, subsystems of the whole system are not isolated and therefore are described by a reduced density matrix that is not pure and may well be equivalent to a statistical ensemble. The same is true for local physical observables, whose expectation values are given by traces over such reduced density matrices. On the other hand, the period of quantum recurrences typically diverges with the system size. So, if we consider first the thermodynamic and then the large time limit in a suitable well-defined manner (see e.g. [3]), it is possible that the system becomes
stationary, always at the level of its subsystems and local physical observables, rather than the whole system or global quantities. Under these clarifications, the question of whether an isolated quantum system thermalizes at large times, when starting from an arbitrary initial state, is a sensible one which has been considered already in a series of theoretical and experimental works [4–14] (see also [15] for a review).

This question is hard to answer in full generality, since the time evolution of a general quantum system cannot be calculated exactly. For one-dimensional integrable quantum systems, on the other hand, this is possible—at least, in principle. In this case it has been proposed [16], in analogy to the classical case, that, provided the system tends to equilibrium in the above sense, its stationary behaviour is described by a Generalized Gibbs Ensemble (GGE) in which all integrals of motion have been taken into account and not only the energy. More explicitly, the GGE is described by a density matrix

\[ \rho_{\text{GGE}} \sim \exp\left(-\sum_j \lambda_j I_j\right), \]

in which a separate Lagrange multiplier \( \lambda_j \) is introduced for each of the integrals of motion \( I_j \) in the same way that, in the canonical ensemble, the temperature is the Lagrange multiplier associated with the constraint of conservation of energy. The values of \( \lambda_j \) are fixed by the condition that the ensemble expectation values of the integrals of motion are equal to their initial values. The conjecture then is that the stationary expectation values of any local physical observable are equal to their ensemble expectation values or, equivalently, that the reduced density matrix of any finite subsystem is equal to the corresponding reduced density matrix of \( \rho_{\text{GGE}} \).

However, this conjecture, as stated above, contains a subtle ambiguity: which are the integrals of motion that should be included in \( \rho_{\text{GGE}} \)? Unlike in classical integrable systems—where the required existence of a set of integrals of motion, equal in number to the degrees of freedom, is a sufficient definition of integrability—in quantum systems there is always an infinite set of integrals of motion which are projections onto each of the eigenstates of the Hamiltonian. This fact, however, is not sufficient for their exact solvability. Obviously, an ensemble with a density matrix that incorporates all projections onto eigenstates as integrals of motion would give correctly the stationary values of all observables of the system (or their long time averaged values, if they do not equilibrate) for any arbitrary initial state. But this is a trivial fact that has no connection with the economy of the maximum entropy principle. Such an ensemble would retain all the information about the initial state, rather than only about a minimal set of integrals of motion. Indeed, assuming for simplicity that the energy spectrum is non-degenerate, the above ensemble is identical to the so-called diagonal ensemble (where the integrals of motion are projections onto equal-energy eigenspaces) and its entropy is always smaller than the entropy of the GGE [17–23]. (When there are degeneracies, it is possible that one should include additional integrals of motion in order to correctly capture the stationary behaviour, as discussed in some particular cases in [24].) The ambiguity in the set of integrals of motion that should be included is resolved when we consider the special characteristics of quantum integrable systems (see e.g. [25]). The most useful definition of quantum integrability (for our purposes) is based on the existence of (an infinite set of) local integrals of motion, which is what guarantees
their exact solvability, through the factorization of their scattering matrix—another characteristic property of quantum integrability. The notion of locality of the integrals of motion (or conserved charges) is appealing in the context of the above conjecture, since if the GGE is supposed to describe the reduced density matrix of any finite subsystem—viewed as an open system in contact with the rest of the system, which plays the role of a bath—then $\rho_{GGE}$ must be given in terms of local quantities defined within the spatial region of the subsystem [3, 26].

1.1. Quantum quenches and the role of the initial state

From the tests of the conjecture that the GGE describes stationary behaviour in integrable systems, which have been performed until now, it turns out that the role of the initial state is crucial to its validity. Obviously, if the initial state is a finite superposition of eigenstates of the Hamiltonian under which it evolves, there would be no equilibration, since the evolution would be periodic even in the thermodynamic limit. A common protocol for setting the system in a ‘proper’ out-of-equilibrium state is the so-called quantum quench [27], i.e. an instantaneous change in the Hamiltonian of the system so that initially it lies in the ground state, as typically chosen, of the Hamiltonian before the quench. The GGE has turned out to be valid for all quantum quench problems studied so far [3, 16, 24, 26, 28–51]. However, the vast majority of these studies refer to physical models that are either noninteracting or interacting but exactly equivalent to noninteracting ones, both before and after the quench. This is mainly due to the fact that, for integrable models with a non-trivial scattering matrix—which therefore cannot be mapped into noninteracting ones—both the derivation of GGE predictions and the study of the time evolution are technically difficult problems. These problems have been solved only in a few cases [48–66]. Testing analytically the validity of the GGE in such models remains a crucial and very challenging open problem which should parallel current numerical efforts in the same direction [63].

The verification of validity of the GGE in noninteracting models displays common characteristics in all cases. The system can be decomposed into an infinite set of non-interacting modes, whose occupation numbers $n_k$ are non-local conserved charges, but which, however, are always a linear combination of the local ones [3, 26]. Under certain conditions, the interference of these modes leads to equilibration for large times. For example, in the cases we will study below, the post-quench dispersion relation of excitations is gapped and the initial state is translationally invariant. There are sufficient but not necessary conditions for equilibration. Indeed, on the one hand, gapless post-quench Hamiltonians also lead, for different reasons, to equilibration [28]. On the other hand, the breaking of translational invariance in the pre- or post-quench Hamiltonians does not necessarily prevent equilibration (the presence of localization in the post-quench Hamiltonian instead is one of the reasons that prevents equilibration, but the GGE may still be applicable for certain observables [67]). As long as equilibration occurs, the stationary values of two-point correlations of fields are typically given by some linear combination of the values of the conserved charges $\langle n_k \rangle_0$. The GGE prediction for the two-point correlations, on the other hand, is given by the same linear combination, but with the GGE values of the charges $\langle n_k \rangle_{GGE}$ instead. Since these are by definition equal to their initial values, the actual stationary correlations and their GGE predictions.
are automatically and trivially equal to each other. Analogously, both the stationary values and the GGE values of all higher-order correlations ($n$-point functions, but in general of any observable) can be expressed in terms of the expectation values of the charges, and of their products in the initial state and in the GGE respectively. The GGE for a noninteracting system is a Gaussian ensemble: this means that, by virtue of Wick’s theorem, the products of the charges are uncorrelated in the GGE (i.e. their GGE expectation values of the products equal the product of the expectation values). The same is also true for the initial state which, being the ground state of a noninteracting Hamiltonian, is also Gaussian. Consequently, even for higher-order correlations the GGE gives always the correct predictions. This argument explains generally the validity of the GGE for noninteracting systems.

We realize that the above reasoning is crucially based on the Gaussianity of the initial state. Indeed, Wick’s theorem, which was employed in order to show the validity of the GGE, is valid if and only if the initial state is Gaussian in terms of the post-quench quasiparticle modes. The crucial role of Wick’s theorem (in combination with the double limit: thermodynamic followed by large time limit, or time averaging followed by thermodynamic limit) was pointed out in [36] for various quantum quenches in essentially noninteracting models (quantum Ising, XX spin chain and Luttinger model), starting with a Gaussian initial state. Gaussianity of the initial state manifests itself in various different forms, which have been identified as being sufficient conditions for the validity of the GGE: quadratic form of the initial density matrix in terms of the post-quench quasiparticle modes [36] or factorization of the initial expectation values of charge products [68]. Both the former and the latter forms have been shown to be sufficient (but not necessary) conditions for the validity of the GGE also in the more general case of interacting post-quench Hamiltonians [49, 50, 59].

What if, however, the pre-quench Hamiltonian is genuinely interacting, so that the initial state is non-Gaussian, while the post-quench Hamiltonian is kept noninteracting? The present study is concerned with precisely this question, which was also raised in [30]. To compare the stationary values of the higher-order correlation functions with their GGE values, we can express the former as a convolution of the initial correlation functions and expand these as a cumulant expansion, i.e. in connected and disconnected terms. We find that, when performing the spatial integration of these correlation functions, it is only a maximally-disconnected part that survives in the combined thermodynamic and large time limit. This means that all stationary correlations of a higher order can be derived from solely the two-point initial correlations: that is, from information contained solely in the values of the conserved charges. We therefore show that all stationary correlations are exactly equal to their GGE predictions. The proof is based on the *cluster decomposition principle*, which states that the correlations between two subsets of points separated by a distance that tends to infinity become disconnected, i.e. [69]

$$\lim_{R \to \infty} \left\langle \prod_i \phi(x_i) \prod_j \phi(x_j + R) \right\rangle = \left\langle \prod_i \phi(x_i) \right\rangle \left\langle \prod_j \phi(x_j) \right\rangle.$$  

(2)

This property is a fundamental requirement for generic physical states: such as, the ground state of any physical Hamiltonian, the thermal density matrix etc. Therefore,
we conclude that neither Gaussianity nor factorization of charge products in the initial state are necessary conditions for the validity of the GGE. Contrarily, even in the much more general case of a quantum quench from an arbitrary interacting pre-quench Hamiltonian, the GGE is still valid: this time as a consequence of a fundamental property of the initial state correlations. However, in the case of an initial state that does not satisfy the cluster decomposition property, the stationary expectation values of observables would generally retain memory of the initial correlations between conserved charges beyond their maximally-disconnected part, so the GGE would not apply. Our results are consistent with, and generalize, earlier findings [30] obtained by completely different methods.

We stress that the cluster decomposition property is satisfied by the ground states of any physical Hamiltonian: including, but not limited to, local Hamiltonians. Indeed even Hamiltonians with long range interactions may have ground states that satisfy this condition. Furthermore, thermal (mixed) states of physical Hamiltonians also satisfy the same property and even many of the eigenstates are expected to do so. In fact, numerical results [70] confirm that, starting from excited states of interacting Hamiltonians, several stationary observables are well approximated by the GGE.

We demonstrate these ideas in the context of two prototypical theories: a relativistic and a non-relativistic bosonic field theory in one spatial dimension with no interaction after the quench. In the first model, we keep our exposition as general as possible and show that our findings are insensitive to the form of the post-quench dispersion relation (and so insensitive to relativistic invariance too) or other details of the particular quench. We identify the minimal requirements for equilibration to occur and for the rest of our arguments to hold. In this way, it is clear that our results are true for a wide range of physical systems that are equivalent to systems of coupled harmonic oscillators with arbitrary couplings. In the second model, as a byproduct of our investigation, we obtain an analytical derivation of earlier discovered numerical results [37] for the relaxation of density–density correlations in the case of free non-relativistic bosons, starting from an initial state with pointlike interactions (quantum quench in the Lieb–Liniger model from positive to zero interaction). It turns out that the relaxation is a power-law in time and is related to the decay of the initial four-point correlation function at large distances. This is governed by the Luttinger Liquid approximation of the Lieb–Liniger model, allowing us to calculate the exponent of the power law from the Luttinger parameter $K$.

2. Relativistic bosonic field theory

We consider the one-dimensional system of harmonic oscillators described by the Hamiltonian (in momentum space)

$$H = \frac{1}{2} \sum_k \left( \pi_k \pi_{-k} + \omega_k^2 \phi_k^2 \Phi_{-k}^2 \right).$$

As is well-known, such a Hamiltonian may describe a relativistic free field theory, if $\omega_k = \sqrt{k^2 + m^2}$ with particle mass $m$, but we do not need to specify the exact form of
the dispersion relation for our subsequent study. This Hamiltonian will play the role of the post-quench Hamiltonian in our problem. Since it is free, the time-evolved field operator $\phi$ (in the Heisenberg picture) can be written in momentum space as

$$\phi(x,t) = \frac{1}{\sqrt{L}} \sum_k e^{ikx} \tilde{\phi}_k(t) = \frac{1}{\sqrt{L}} \sum_k e^{ikx} \frac{1}{\sqrt{2\omega_k}} \left( a_k e^{-\omega_k t} + a_k^\dagger e^{\omega_k t} \right),$$

(4)

and the conjugate momentum $\pi$ as

$$\pi(x,t) = \frac{1}{\sqrt{L}} \sum_k e^{ikx} \tilde{\pi}_k(t) = \frac{1}{\sqrt{L}} \sum_k e^{ikx} \left( -i \sqrt{\omega_k} \right) \left( a_k e^{-\omega_k t} - a_k^\dagger e^{\omega_k t} \right).$$

(5)

Here $L$ is the system size and we assume periodic boundary conditions, so that the momenta are given by $k = 2\pi n/L$ with $n$ integer (although this assumption is not essential). The creation-annihilation operators can therefore be expressed in the following form, which will be useful later:

$$a_k = \sqrt{\frac{\omega_k}{2}} \phi_k(0) + \frac{i}{\sqrt{2\omega_k}} \pi_k(0), \quad a_k^\dagger = \sqrt{\frac{\omega_k}{2}} \phi_k^*(0) - \frac{i}{\sqrt{2\omega_k}} \pi_k^*(0).$$

(6)

### 2.1. The two-point correlation function

As we anticipated, the calculation of the two-point function is rather insensitive on the initial state and so the derivation parallels the one for a quench between free theories [28, 32]. Explicitly, the equal-time two-point function is

$$C_t^{(2)}(x,y) \equiv \langle \phi(x,t) \phi(y,t) \rangle = \frac{1}{L} \sum_{k_1, k_2} \frac{1}{\sqrt{2\omega_{k_1}}} \frac{1}{\sqrt{2\omega_{k_2}}} e^{ik_1 x + ik_2 y}$$

$$\times \left[ \langle a_{k_1} a_{k_2} \rangle_0 e^{-i(\omega_{k_1} + \omega_{k_2})t} + \langle a_{-k_1} a_{k_2} \rangle_0 e^{i(\omega_{k_1} - \omega_{k_2})t} \right.$$

$$+ \left. \langle a_{k_1} a_{-k_2} \rangle_0 e^{-i(\omega_{k_1} - \omega_{k_2})t} + \langle a_{-k_1} a_{-k_2} \rangle_0 e^{i(\omega_{k_1} + \omega_{k_2})t} \right],$$

(7)

where the index zero means that the expectation values are calculated on the initial state. Assuming that the latter is translationally invariant, the above expectation values are zero unless the momenta $k_1, k_2$ are equal or opposite. Using the canonical commutation relation $[a_k, a^\dagger_{-k}] = \delta_{k,p}$, the non zero initial correlators can be parametrized as follows

$$\langle a_k a_q \rangle_0 = A_k \delta_{k,-q},$$

$$\langle a^\dagger_{-k} a_q \rangle_0 = B_k \delta_{k,-q},$$

$$\langle a_k a^\dagger_{-q} \rangle_0 = (1 + B_{-k}) \delta_{k,-q},$$

$$\langle a^\dagger_{-k} a^\dagger_{-q} \rangle_0 = A_{-k}^* \delta_{k,-q}.$$

(8)
where $A_k$ and $B_k$ are functions that depend on the particular initial state. From the definition of $B_k$ we have

$$B_k = \langle a_k^\dagger a_k \rangle_0 = \langle n_k \rangle_0,$$

which is the momentum occupation number in the initial state.

From all the above, we obtain

$$C^{(2)}_t(x, y) = \frac{1}{L} \sum_k \frac{1}{2\omega_k} e^{i\langle x-y \rangle_0} \langle a_k a_{-k} \rangle_0 e^{-2\omega_k t} + \langle a_{-k}^\dagger a_{-k} \rangle_0 + \langle a_k^\dagger a_k \rangle_0$$

$$+ \langle a_{-k}^\dagger a_k^\dagger \rangle_0 e^{i2\omega_k t}. \tag{10}$$

In the thermodynamic limit $L \to \infty$, the sum in the above expression becomes an integral over continuous momenta

$$C^{(2)}_t(x, y) = \int \frac{dk}{2\pi} \frac{1}{2\omega_k} e^{i\langle x-y \rangle_0} \langle a_k a_{-k} \rangle_0 e^{-2\omega_k t} + \langle a_{-k}^\dagger a_{-k} \rangle_0 + \langle a_k^\dagger a_k \rangle_0$$

$$+ \langle a_{-k}^\dagger a_k^\dagger \rangle_0 e^{i2\omega_k t}. \tag{11}$$

Finally we take the long time limit $t \to \infty$ of the above expression. For a massive post-quench dispersion relation, $\omega_k = \sqrt{k^2 + m^2}$ with $m \neq 0$, the stationary phase method shows that the oscillating time-dependent terms in the above integral vanish; therefore, the two-point function becomes stationary. Notice that this argument is not sensitive to the particular quench we consider; it only requires a gapped post-quench dispersion relation with a single smooth local minimum at $k=0$, but not necessarily relativistic. In fact even for a gapless dispersion relation, the model will generically equilibrate [28, 71], even though the stationary phase argument does not straightforwardly apply. The difference is that, in this case, equilibration refers to the correlations of the vertex operators, which are the real physical observables in the gapless case.

As long as the above condition is satisfied, the two-point function equilibrates. We will now express its stationary value in terms of $\langle n_k \rangle_0$. Having recognized which terms of (11) survive in the large time limit, we will keep them in their finite volume form as sums over discrete momenta, since in the next step we obtain a $\delta(0)$ term which makes sense only within the finite volume expression. We therefore find

$$C^{(2)}_\infty(x, y) = \frac{1}{L} \sum_k \frac{1}{2\omega_k} e^{i\langle x-y \rangle_0} (1 + \langle n_{-k} \rangle_0 + \langle n_k \rangle_0). \tag{12}$$

or returning to the thermodynamic limit

$$C^{(2)}_\infty(x, y) = \int \frac{dk}{2\pi} \frac{1}{2\omega_k} e^{i\langle x-y \rangle_0} (1 + \langle n_{-k} \rangle_0 + \langle n_k \rangle_0). \tag{13}$$

We will now calculate the GGE prediction for the two-point correlation function. The GGE is given, always in the thermodynamic limit, by the density matrix

$$\rho_{GGE} = Z^{-1} \exp \left( - \int \frac{dk}{2\pi} \lambda_k n_k \right), \tag{14}$$

doi:10.1088/1742-5468/2014/07/P07024
where
\[
Z = \text{Tr} \left\{ \exp \left( - \int \frac{dk}{2\pi} \lambda_k n_k \right) \right\},
\]
and the Lagrange multipliers \( \lambda_k \) are defined through the requirement that the values of the charges \( n_k \) in the GGE are equal to their initial values
\[
\langle n_k \rangle_{\text{GGE}} = \langle n_k \rangle_0.
\]
The GGE value of the two-point function is then
\[
C_{\text{GGE}}^{(2)}(x, y) \equiv \langle \phi(x) \phi(y) \rangle_{\text{GGE}} = \int \frac{dk}{2\pi} \frac{1}{2\omega_k} e^{ik(x-y)} \left( \langle a_k^\dagger a_k \rangle_{\text{GGE}} + \langle a_k a_k^\dagger \rangle_{\text{GGE}} \right) = \int \frac{dk}{2\pi} \frac{1}{2\omega_k} e^{ik(x-y)} (1 + \langle n_{-k} \rangle_{\text{GGE}} + \langle n_k \rangle_{\text{GGE}}),
\]
since as expressed in (14) the GGE is diagonal in the momentum basis.

The last result is obviously equal to (12), by virtue of the defining condition (16) of the GGE. Note that, as long as the two-point function equilibrates for long times, its asymptotic value is automatically given by the GGE prediction with no further assumption, simply because the only information of the initial state on which it depends are the values of the charges which are fixed in the GGE by definition. We also stress that the only assumption required for the equilibration—and therefore also for the verification of the equality between (18) and (12)—was that the post-quench dispersion relation has a gap and a single minimum (note however that, as already mentioned, this condition can be released); no further information about the initial state is required (we also assumed the generally applicable property of translational invariance). By the generally assumed symmetry of the initial state and the evolving Hamiltonian under parity (coordinate space reflections), we have \( \langle n_{-k} \rangle_0 = \langle n_k \rangle_0 \), but we did not need to use this fact. We therefore arrive at the general conclusion that the stationary expression of the two-point function, in the case of noninteracting evolution, will always be trivially described by a GGE, with the momentum occupation numbers as conserved charges, for any initial state. This statement strictly holds only as long as the single particle spectrum is non-degenerate; in the opposite case, particular care should be taken in order to fix the appropriate set of integrals of motion, as discussed in [24].

Before we proceed further, we will derive a direct relation between the large time asymptotic value of the field correlations and the initial ones in their coordinate space form. From (6) the operator \( n_k + n_{-k} \) that appears in (12) can be written in terms of the field \( \phi \) and its time derivative \( \phi = \pi \) as
\[
n_k + n_{-k} = a_k^\dagger a_k + a_{-k}^\dagger a_{-k} = \frac{1}{2} \left[ \omega_k \left( \bar{\phi}_k \bar{\phi}_k + \bar{\phi}_k \bar{\phi}_k \right) + i \left( \bar{\phi}_k \bar{\pi}_k + \bar{\phi}_k \bar{\pi}_{-k} - \bar{\pi}_k \bar{\phi}_k - \bar{\pi}_{-k} \bar{\phi}_k \right) \right] + \left( \bar{\pi}_{-k} \bar{\pi}_k + \bar{\pi}_k \bar{\pi}_{-k} \right) / \omega_k \right] = \omega_k \bar{\phi}_k \bar{\phi}_{-k} + \bar{\pi}_k \bar{\pi}_{-k} / \omega_k - 1, \quad (18)
\]
where in the last step we used the commutation relations \([ \bar{\phi}_k, \bar{\pi}_q ] = i\delta_{k-q} \) and \([ \bar{\phi}_k, \bar{\phi}_q ] = [ \bar{\pi}_k, \bar{\pi}_q ] = 0 \). Therefore, substituting in equation (12) we have

\[
doi:10.1088/1742-5468/2014/07/P07024
\]
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\begin{equation}
C^{(2)}(x, y) = \frac{1}{2} \int \frac{dk}{2\pi} e^{ik(x-y)} \left( \langle \tilde{\phi}_k \tilde{\phi}_{-k} \rangle_0 + \frac{1}{\omega_k^2} \langle \tilde{\phi}_k \tilde{\phi}_{-k} \rangle_0 \right) + \int_{-\infty}^{+\infty} ds H(s) D_0^{(2)}(x-y-s),
\end{equation}

where we defined

\begin{equation}
H(x) \equiv \int \frac{dk}{2\pi} \frac{e^{ikx}}{\omega_k^2},
\end{equation}

\begin{equation}
C_0^{(2)}(x) \equiv C_0^{(2)}(0, x),
\end{equation}

and

\begin{equation}
D_0^{(2)}(x) \equiv \langle \pi(0, 0) \pi(x, 0) \rangle = \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} \langle \phi(0, t_1) \phi(x, t_2) \rangle \bigg|_{t_1=t_2=0}.
\end{equation}

These formulas will be useful later in the comparison of the stationary four-point function with its GGE prediction.

2.2. The four-point correlation function

We now proceed to the calculation of the four-point function which, according to the discussion in section 1.1, is the first non-trivial test of the conjecture that the GGE describes the stationary behaviour after a quantum quench. Our main objective is to check whether the large time four-point function retains such memory of the initial four-point function that cannot be derived from the initial two-point function. If this is true, the large time four-point function will not be described by the GGE (14) because, in the latter, Wick’s theorem is valid. Therefore, the predicted four-point function depends solely on the two-point function (or in other words on \( \langle n_k \rangle_0 \) and not on any additional information about the initial state. The calculation below is largely inspired by one for a very specific case (the quench of a Bose gas from zero to infinite interaction [46]), but as we shall see, the reason for its general validity is a deeper physical requirement on the initial state, i.e. the cluster decomposition principle.

The equal time four-point function is

\begin{equation}
C_t^{(4)}(x_1, x_2, x_3, x_4) \equiv \langle \phi(x_1; t) \phi(x_2; t) \phi(x_3; t) \phi(x_4; t) \rangle
\end{equation}

\begin{equation}
= \frac{1}{L^2} \sum_{k_1,k_2,k_3,k_4} \frac{1}{4\prod_{i=1}^{4} \omega_{k_i}} e^{\sum_{i=1}^{4} k_i x_i} \sum_{\sigma \in \{\sigma\}} \left\langle \prod_{i=1}^{4} a_{-\sigma_{k_i}^i}^{(\sigma)} \right\rangle_0 e^{\sum_{i=1}^{4} \sigma \omega_{k_i} t},
\end{equation}

where we used the compact notation \( a_{k}^{(+)} \equiv a_k^+ \) and \( a_{k}^{(-)} \equiv a_k \). Taking first the thermodynamic limit, in which the momentum sums become integrals, and then the large time limit, we observe that—provided that the stationary phase argument applies as above for a gapped post-quench Hamiltonian—only terms with no oscillating phase survive,
i.e. those satisfying the condition $\sum_{i=1}^{4}\sigma\omega_{k_i}=0$ with $\sigma_i=\pm 1$. The latter is satisfied only by terms with an equal number of $a$ and $a^\dagger$ operators (which are $4!/(2! 2!)=6$ in number, out of $2^4=16$) and more specifically those for which the momenta of the $a$ operators match with those of the $a^\dagger$ operators in pairs of equal or opposite values. This is because non-polynomial equations of the form $\sum_{i=1}^{4}\sigma\omega_{k_i}=0$ have only sparse solutions for discretized momenta $k=2\pi n/L$, that result in negligible contributions in the thermodynamic limit, unless $\sum_{i=1}^{4}\sigma=0$, in which case the trivial solutions—in which the $k_i$’s appear in pairs of equal or opposite values—give finite (non-vanishing) contributions to the thermodynamic limit. Obviously, in the above we make use of the fact that the dispersion relation is an even function, i.e. $\omega_{-k}=\omega_k$. For example, the equation $\omega_{k_1}+\omega_{k_2}=\omega_{k_3}+\omega_{k_4}$ is satisfied by the terms that contain the expectations $\langle a_{k_1}a_{k_2}a_{-k_3}a_{-k_4}\rangle_0$ and $\langle a_{-k_1}a_{-k_2}a_{k_3}a_{k_4}\rangle_0$ under the condition that $k_1=\pm k_3$ and $k_2=\pm k_4$ or $k_1=\pm k_4$ and $k_2=\pm k_3$, while the equation $\omega_{k_1}+\omega_{k_2}+\omega_{k_3}=\omega_{k_4}$ does not have any such simple solutions. The corresponding terms give vanishing contributions in the thermodynamic and large time limit. Note that we should take first the thermodynamic and then the large time limit, since in order to apply the stationary phase method, the sums should have been first written as integrals. However, as in the case of the two-point function, now that we have recognized which types of terms survive in the large time limit, we will keep them in their finite volume form: in the subsequent algebra some contractions result in $\delta^2$ terms that make sense only within the finite volume expressions.

According to the above we have

$$C_{\infty}^{(4)}(x_1, x_2, x_3, x_4) = \frac{1}{L^2} \sum_{k_1,k_2,k_3,k_4} \frac{1}{4!} e^{i\sum_{i=1}^{4} k_i x_i} e^{i\sum_{i=1}^{4} \delta_{k_i}}$$

Using the canonical commutation relations we can bring each of the operator products above, in the order $a^\dagger a$ $a^\dagger a$ so that we can focus only on this term and recover the others at the end by considering permutations of the indices 1, 2, 3, 4. This re-ordering procedure introduces additional lower-order terms of the form $a^\dagger a$ which will be taken into account below. Due to the translational invariance of the initial state that enforces $\sum_{i=1}^{4} k_i=0$ for the initial expectation values, only one out of the four pairing combinations for each term survives in the thermodynamic limit. Therefore, we have
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\[ \frac{1}{L^2} \sum_{k_1, k_2, k_3, k_4} \frac{1}{4 \prod_{i=1}^{4} \omega_{k_i}} e^{\sum_{i=1}^{4} k_{x_i} \left( a_{-k_i}^{+} a_{-k_i} a_{k_i}^{+} a_{k_i} \right)_{0}} (\delta_{k_1 k_2} + \delta_{k_1 k_3} + \delta_{k_2 k_4} + \delta_{k_3 k_4}) \]

\[ = \frac{1}{L^2} \sum_{k_1, k_2, k_3, k_4} \frac{1}{4 \prod_{i=1}^{4} \omega_{k_i}} e^{\sum_{i=1}^{4} k_{x_i} \left( a_{-k_i}^{+} a_{s_{k_1}} a_{s_{k_2}} a_{k_i} \right)_{0}} \delta_{k_1 k_2} \delta_{k_3 k_4} \]

\[ = \frac{1}{L^2} \sum_{k, p} \frac{1}{4 \omega_{k} \omega_{p}} e^{ik(x_2 - x_1) + ip(x_4 - x_3)} \left( \langle n_k n_p \rangle_0 \right) , \]

and

\[ \frac{1}{L^2} \sum_{k_1, k_2, k_3, k_4} \frac{1}{4 \prod_{i=1}^{4} \omega_{k_i}} e^{\sum_{i=1}^{4} k_{x_i} \left( a_{-k_i}^{+} a_{-k_i} a_{k_i}^{+} a_{k_i} \right)_{0}} (\delta_{k_1 k_4} + \delta_{k_1 k_3} + \delta_{k_2 k_4} + \delta_{k_2 k_3}) \]

\[ = \frac{1}{L^2} \sum_{k_1, k_2, k_3, k_4} \frac{1}{4 \prod_{i=1}^{4} \omega_{k_i}} e^{\sum_{i=1}^{4} k_{x_i} \left( a_{-k_i}^{+} a_{s_{k_1}} a_{s_{k_2}} a_{k_i} \right)_{0}} \delta_{k_1 k_4} \delta_{k_3 k_2} \]

\[ = \frac{1}{L^2} \sum_{k, p} \frac{1}{4 \omega_{k} \omega_{p}} e^{ik(x_1 - x_3) + ip(x_2 - x_3)} \left( \langle n_k n_p \rangle_0 + \langle n_k \rangle_0 \right) \]

Finite volume corrections have been omitted in the above expressions. Taking into account all other \( (2 \times 6 = 12) \) terms in (25), which can be obtained by re-ordering and permutation of indices, and symmetrizing the terms containing \( \langle n_k n_p \rangle_0 \) under the exchange \( k \leftrightarrow p \) (since \( \langle n_k n_p \rangle_0 = \langle n_p n_k \rangle_0 \)), we finally find

\[ C_{\infty}^{(4)}(x_1, x_2, x_3, x_4) = \sum_{\text{all perm.s} \ of \ 1,2,3,4} \left[ \frac{1}{2} \sum_{k, p} \frac{1}{4 \omega_{k} \omega_{p}} e^{ik(x_2 - x_1) + ip(x_4 - x_3)} \left( \langle n_k n_p \rangle_0 \right) \right] \]

\[ + \frac{1}{2} \left( \frac{1}{L} \sum_{k} \frac{1}{2 \omega_{k}} e^{ik(x_2 - x_1)} \langle n_k \rangle_0 \right) \left( \frac{1}{L} \sum_{p} \frac{1}{2 \omega_{p}} e^{ip(x_4 - x_3)} \right) \]

\[ + \sum_{\text{perm.s \ of \ 2\to3, 2\to4}} \left[ \frac{1}{L} \sum_{k} \frac{1}{2 \omega_{k}} e^{ik(x_2 - x_1)} \right] \left[ \frac{1}{L} \sum_{p} \frac{1}{2 \omega_{p}} e^{ip(x_4 - x_3)} \right] \]

\[ = \frac{1}{2} \frac{1}{L^2} \sum_{k, p} \frac{1}{4 \omega_{k} \omega_{p}} \left( \sum_{\text{all perm.s} \ of \ 1,2,3,4} e^{ik(x_2 - x_1) + ip(x_4 - x_3)} \right) \left( \langle n_k n_p \rangle_0 + \langle n_k \rangle_0 + \frac{1}{4} \right) , \]

or, in the thermodynamic limit
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\[ C_{4}^{(k)}(x_1, x_2, x_3, x_4) = \frac{1}{2} \int \frac{dk \, dp}{(2\pi)^2 \frac{1}{4\wp \omega_p}} F(k, p; x_1, x_2, x_3, x_4) \left( \langle n_k n_p \rangle_0 + \langle n_k \rangle_0 + \frac{1}{4} \right), \]  
\( (26) \)

where we have set

\[ F(k, p; x_1, x_2, x_3, x_4) \equiv \sum_{\text{all perm.s of } 1, 2, 3, 4} e^{ik(x_2 - x_1) + ip(x_4 - x_3)}. \]  
\( (27) \)

The function \( F(k, p; x_1, x_2, x_3, x_4) \) is even in both \( k \) and \( p \) and symmetric under their interchange. Therefore, we can replace the expression \( \left( \langle n_k n_p \rangle_0 + \langle n_k \rangle_0 + \frac{1}{4} \right) \) in the sum by

\[ \frac{1}{4} \left( \langle n_k + n_{-k} + 1 \rangle (n_p + n_{-p} + 1) \right)_0 = \frac{1}{4} \left( \langle n_k n_p \rangle_0 + \langle n_{-k} n_p \rangle_0 + \langle n_k n_{-p} \rangle_0 + \langle n_{-k} n_{-p} \rangle_0 + \langle n_k \rangle_0 + \langle n_{-k} \rangle_0 + \langle n_p \rangle_0 + \langle n_{-p} \rangle_0 + 1 \right); \]  
\( (28) \)

to obtain the alternative form

\[ C_{4}^{(k)}(x_1, x_2, x_3, x_4) = \frac{1}{8} \int \frac{dk \, dp}{(2\pi)^2 \frac{1}{4\wp \omega_p}} F(k, p; x_1, x_2, x_3, x_4) \times \left( \langle n_k + n_{-k} + 1 \rangle (n_p + n_{-p} + 1) \right)_0. \]  
\( (29) \)

From the above relations we see that the large time asymptotic of the four-point function does depend on the initial correlations \( \langle n_k n_p \rangle_0 \). For an arbitrary initial state, the latter are generally independent of \( \langle n_p \rangle_0 \) and therefore the GGE, which contains only information about \( \langle n_k \rangle_0 \) and not about \( \langle n_k n_p \rangle_0 \), would not predict correctly the large time four-point function. Indeed, in the GGE—since it is Gaussian and therefore Wick’s theorem applies—the prediction for the four-point function is also disconnected: that is,

\[ C_{4}^{(k)}_{\text{GGE}}(x_1, x_2, x_3, x_4) = C_{4}^{(2)}_{\text{GGE}}(x_1, x_2) C_{4}^{(2)}_{\text{GGE}}(x_3, x_4) + [2 \leftrightarrow 3] + [2 \leftrightarrow 4] \]

\[ = \int \frac{d{kd}p}{(2\pi)^2 \frac{1}{4\wp \omega_p}} e^{ik(x_1 - x_3) + ip(x_4 - x_2)} \left( \langle n_{-k} \rangle_{\text{GGE}} + \langle n_k \rangle_{\text{GGE}} + 1 \right) \]

\[ \times \left( \langle n_{-p} \rangle_{\text{GGE}} + \langle n_p \rangle_{\text{GGE}} + 1 \right) + [2 \leftrightarrow 3] + [2 \leftrightarrow 4]. \]  
\( (30) \)

However, the situation may be different for initial states prepared by performing a quantum quench—i.e. ground states of some Hamiltonian—since we have not yet used all properties of ground state expectation values. A fundamental such property is the cluster decomposition principle, which states that, at large separations between two subsets of physical operators of an \( n \)-point function the latter becomes disconnected and—as we will soon show—is responsible for the validity of the conjecture that the GGE describes the stationary behaviour.

2.2.1. Non-interacting pre-quench Hamiltonian. Before we proceed to the general case, let us start with the special case of a noninteracting pre-quench Hamiltonian. In this case the initial state is Gaussian and by Wick’s theorem we have that the expectation values of the conserved charges factorize

\[ \langle n_k n_p \rangle_0 = \langle n_k \rangle_0 \langle n_p \rangle_0. \]  
\( (31) \)
Therefore, the correlations $\langle n_k n_p \rangle_0$ do not contain any more information than $\langle n_k \rangle_0$ and the GGE is capable of describing the large time asymptotics of the four-point function. Indeed, in view of the factorization property (32), the two expressions (30) and (31) are identical. The same is true for all higher-order correlation functions too: as long as the post-quench dispersion relation satisfies the previously mentioned condition for equilibration, the correlation functions of any order tend for large times to stationary values which are related to the initial expectation values of products of occupation number operators $\prod_i n_{k_i}$ by Wick’s theorem, these equal to the products of the expectation values $\prod_i \langle n_{k_i} \rangle_0$. According to the above discussion this means that the GGE predicts correctly their values. We therefore see that for noninteracting pre-quench Hamiltonians, the GGE is valid, provided that equilibration occurs: this is a direct consequence of Wick’s theorem.

2.2.2. Interacting pre-quench Hamiltonian. We now consider the more general case of an arbitrarily interacting pre-quench Hamiltonian. In this case the initial state is non-Gaussian—Wick’s theorem does not apply. To take advantage of the cluster decomposition principle, we should first express the initial correlations $\langle n_k n_p \rangle_0$ in terms of the fields $\phi(x)$ and $\pi(x)$ whose initial correlations are known from the ground state properties of the pre-quench theory. From (18) we know that $n_k + n_{-k} + 1 = \omega_k \tilde{\phi}_{k} \tilde{\phi}_{-k} + \tilde{\pi}_{k} \tilde{\pi}_{-k} / \omega_k$, therefore (26) or (30) becomes

$$
C_{x_1, x_2, x_3, x_4}^{(4)} = \frac{1}{32 L^2} \sum_{k,p} F(k, p; x_1, x_2, x_3, x_4) \left( \langle \tilde{\phi}_{k} \tilde{\phi}_{p} \tilde{\phi}_{k} \tilde{\phi}_{-p} \rangle_0 + \frac{1}{\omega_k^2} \langle \tilde{\pi}_{k} \tilde{\pi}_{-k} \tilde{\phi}_{p} \tilde{\phi}_{-p} \rangle_0 \right).
$$

(32)

Notice that the large time value of the four-point correlation function depends solely on four-point initial correlations, not on lower order correlations of the initial state. Let us first focus on the first term of this sum, and write it in coordinate space

$$
\frac{1}{32 L^2} \sum_{k,p} F(k, p; x_1, x_2, x_3, x_4) \left( \langle \tilde{\phi}_{k} \tilde{\phi}_{p} \tilde{\phi}_{-k} \tilde{\phi}_{-p} \rangle_0 \right) = \frac{1}{32 L^2} \int_{-L/2}^{+L/2} dx_1 dx_2 dx'_3 dx'_4
$$

$$
\times \sum_{k,p} e^{i(x'_2 - x'_3) + \omega(p - k)} F(k, p; x_1, x_2, x_3, x_4) \phi(x'_1) \phi(x'_2) \phi(x'_3) \phi(x'_4) \delta(x'_2 - x'_1 + x_2 - x_1)
$$

$$
= \frac{1}{32 L^2} \int_{-L/2}^{+L/2} dx_1 dx_2 dx'_3 dx'_4 \sum \delta(x'_2 - x'_1 + x_2 - x_1)
$$

$$
\times \delta(x'_4 - x'_3 + x_4 - x_3) C_0^{(4)}(s, s + x_1 - x_2, r, r + x_3 - x_4)
$$

$$
= \frac{1}{32 L^2} \sum \int_{-L/2}^{+L/2} ds \delta C_0^{(4)}(s, s + x_1 - x_2, r, r + x_3 - x_4). \tag{33}
$$
The last correlator can be decomposed, completely generally, in two parts: the disconnected and the connected one. The disconnected part corresponds to the sum of all two-point correlations between pairs of the four points in any combination, while the connected part corresponds to all the rest correlations that are present in the four-point function, i.e.

\[
C_0^{(4)}(s, s + x_1 - x_2, r, r + x_3 - x_4) = C_0^{(2)}(s, s + x_1 - x_2) C_0^{(2)}(r, r + x_3 - x_4)
+ C_0^{(2)}(s, r) C_0^{(2)}(s + x_1 - x_2, r + x_3 - x_4)
+ C_0^{(2)}(s, r + x_3 - x_4) C_0^{(2)}(s + x_1 - x_2, r)
+ C_0^{(4)}(s, s + x_1 - x_2, r, r + x_3 - x_4)
= C_0^{(2)}(x_1 - x_2) C_0^{(2)}(x_3 - x_4) + C_0^{(2)}(r - s) C_0^{(2)}(r - s + x_3 - x_4 + x_2 - x_1)
+ C_0^{(2)}(r - s + x_3 - x_4) C_0^{(2)}(r - s + x_2 - x_1)
+ C_0^{(4)}(0, x_1 - x_2, r - s, r - s + x_3 - x_4),
\]

(34)

where \(C_0^{(2)}\) and \(C_0^{(4)}\) are the initial two-point correlation function and connected four-point correlation function respectively. For ground states of noninteracting Hamiltonians, by application of Wick’s theorem, the four-point function is exactly equal to the disconnected part, therefore the connected one vanishes.

We now substitute the above expansion into (34). The first term in the last sum does not depend on \(r\) and \(s\) and therefore the integration over these variables results simply in an \(L^2\) factor that cancels the overall \(L^{-2}\) prefactor. The other three terms are functions of \(r - s\) and therefore from the integration over \(r + s\) we obtain a single \(L\) factor, while after performing the remaining integration over \(r - s\) these terms scale slower than the first one. This is because the integrands of those terms are decaying functions of the distance \(r - s\). In particular the decay of the connected term \(C_0^{(4)}(0, x, s + y)\) at large \(|s|\) is ensured by the cluster decomposition property, from which we know that, at large distances between any two pairs of the four points, the connected part tends to zero.

The above observations mean that only the first term gives a finite contribution in the thermodynamic limit \(L \to \infty\), while the rest give only finite size corrections. We can work similarly for the other correlators \(\langle \phi_{\mathbf{k}}(\mathbf{x}_1)\phi_{\mathbf{p}}(\mathbf{y}_2)\rangle_0\) and \(\langle \phi_{\mathbf{k}}(\mathbf{x}_1)\phi_{\mathbf{p}}(\mathbf{y}_2)\rangle_0\), however, there is a difference when we pass to the coordinate space form of these correlators. For example, for the last one we have

\[
\frac{1}{32} \frac{1}{L^2} \sum_{k,p} \int_{-L/2}^{+L/2} d\mathbf{x}'_1 d\mathbf{x}'_2 d\mathbf{x}'_3 d\mathbf{x}'_4 \sum_{k,p} e^{i\mathbf{k} \cdot (\mathbf{x}_2 - \mathbf{x}_1)} F(k, p; x_1, x_2, x_3, x_4) \frac{1}{\omega_k^2 \omega_p^2} \langle \tilde{\pi}_k \tilde{\pi}_p \tilde{\pi}_p \tilde{\pi}_p \rangle_0
= \frac{1}{32} \frac{1}{L^4} \int_{-L/2}^{+L/2} d\mathbf{x}'_1 d\mathbf{x}'_2 d\mathbf{x}'_3 d\mathbf{x}'_4 \sum_{k,p} e^{i\mathbf{k} \cdot (\mathbf{x}_2 - \mathbf{x}_1)} F(k, p; x_1, x_2, x_3, x_4) \\
\times \frac{1}{\omega_k^2 \omega_p^2} \langle \pi(\mathbf{x}'_1) \pi(\mathbf{x}'_2) \pi(\mathbf{x}'_3) \pi(\mathbf{x}'_4) \rangle_0 \\
= \frac{1}{32} \frac{1}{L^2} \int_{-L/2}^{+L/2} ds dr s' dr' \sum_{\text{all perm.s of } x_1, x_2, x_3, x_4} H(s') H(r') \\
\times \langle \pi(s) \pi(s' + s + x_1 - x_2) \pi(r) \pi(r' + r + x_3 - x_4) \rangle_0,
\]

(35)
where the function $H(x)$ has been defined in (20). This function is not a Dirac $\delta$-function—as it was in the case of $\langle \tilde{\phi}_k \tilde{\phi}_{-k} \tilde{\phi}_p \tilde{\phi}_{-p} \rangle_0$. Therefore, we cannot reduce the number of integrals from four to two, as before. However, since the post-quench dispersion relation is already assumed to be gapped, from (20) we can see that $H(x)$ should typically be a function that decays exponentially with the distance over a scale $m^{-1}$, where $m$ is the gap. Therefore integration over the coordinate variables $s'$ and $r'$ is restricted within a range of the order $m^{-1}$. This means that, for the $L \to \infty$ behaviour of (36) one can make observations analogous to those for (34). First we expand $\langle \pi(s) \pi(s' + x_1 - x_2) \pi(r) \pi(r' + x_3 - x_4) \rangle_0$ as

$$D_0^{(4)}(s, s' + s + x_1 - x_2, r, r' + r + x_3 - x_4)$$

$$= D_0^{(2)}(s, s' + s + x_1 - x_2) D_0^{(2)}(r, r' + r + x_3 - x_4)$$

$$+ D_0^{(2)}(s, r) D_0^{(2)}(s' + s + x_1 - x_2, r' + r + x_3 - x_4)$$

$$+ D_0^{(2)}(s, r') D_0^{(2)}(s' + s + x_1 - x_2, r)$$

$$+ D_0^{(4)}_{\text{comm}}(s, s' + x_1 - x_2, r, r' + x_3 - x_4)$$

$$= D_0^{(2)}(x_1 - x_2 + s') D_0^{(2)}(x_3 - x_1 + r')$$

$$+ D_0^{(2)}(r - s) D_0^{(2)}(r' - s' + r - s + x_3 - x_4 + x_2 - x_1)$$

$$+ D_0^{(2)}(r' + x_3 - x_4) D_0^{(2)}(r' - r' - s' - s + x_2 - x_1)$$

$$+ D_0^{(4)}_{\text{comm}}(0, x_1 - x_2 + s' - s, r - s, r' - s + x_3 - x_4). \quad (36)$$

Substituting into (36) we see that, for the same reasons as before, it is only the first term in the above expansion that gives a finite contribution in the thermodynamic limit. Therefore we have

$$\frac{1}{32} \sum_{k, p} \sum_{x_1, x_2, x_3, x_4} \int ds' H(s') D_0^{(2)}(x_1 - x_2 + s') \int dr' H(r') D_0^{(2)}(x_3 - x_4 + r'). \quad (37)$$

Summing all terms together we find that

$$C_{\infty}^{(4)}(x_1, x_2, x_3, x_4) = \frac{1}{32} \sum_{\text{all perm.s}} \left( C_0^{(2)}(x_1 - x_2 + \int ds' H(s') D_0^{(2)}(x_1 - x_2 + s') \right)$$

$$\times \left( C_0^{(2)}(x_3 - x_4 + \int dr' H(r') D_0^{(2)}(x_3 - x_4 + r')). \quad (38)$$

and using our result (19) for the two-point correlation function

$$C_{\infty}^{(4)}(x_1, x_2, x_3, x_4) = \frac{1}{8} \sum_{\text{all perm.s}} C_0^{(2)}(x_1 - x_2) C_0^{(2)}(x_3 - x_4)$$

$$= C_0^{(2)}(x_1 - x_2) C_0^{(2)}(x_3 - x_4) + [2 \leftrightarrow 3] + [2 \leftrightarrow 4]. \quad (39)$$

doi:10.1088/1742-5468/2014/07/P07024
Comparing with the GGE expression (31), we finally conclude that the stationary four-point function \( C_{\infty}^{(4)}(x_1, x_2, x_3, x_4) \) is equal to GGE prediction, also for this more general choice of the initial state.

### 2.3. Higher order correlations

We can now outline the generalization of the above proof to the case of the \( 2n \)-point function. The stationary value of the latter \( C_{\infty}^{(2n)}(\{x_i\}) \) would turn out to depend on

\[
\left\langle \prod_{j=1}^{n} (n_k + n_{-k} + 1) \right\rangle_0
\]

which in turn depends on \( \left\langle \prod_{j=1}^{n} \tilde{\phi}^{(\sigma_j)}(\tilde{k}_j) \tilde{\phi}^{(-\sigma_j)}(-\tilde{k}_j) \right\rangle_0 \), where \( \sigma_j = 0, 1 \) and \( \tilde{\phi}_{k_j}^{(0)} \equiv \tilde{\phi}_{k_j}, \tilde{\phi}_{k_j}^{(1)} \equiv \tilde{\phi}^{(0)}_{k_j} \). The initial correlator \( \left\langle \prod_{j=1}^{n} \tilde{\phi}(s_j + x_{2j-1}) \tilde{\phi}(s_j + x_{2j}) \right\rangle_0 \) can be expressed in terms of \( \left\langle \prod_{j=1}^{n} \phi(s_j + x_{2j-1}) \phi(s_j + x_{2j}) \right\rangle_0 \) which in the thermodynamic limit \( L \to \infty \), due to the cluster decomposition property of the initial state, tends to the maximally-disconnected form \( \prod_{j=1}^{n} \left\langle \phi^{(0)}(x_{2j-1}) \phi^{(0)}(x_{2j}) \right\rangle_0 \). Working similarly for the other initial correlators, we can reconstruct the GGE prediction.

### 3. Non-relativistic free bosons

In this second part of our study, we will check the validity of the GGE for a quantum quench of the interaction \( c \) in the Lieb–Liniger model from arbitrary initial \( c > 0 \) to \( c = 0 \), a quench already studied in great detail in [37]. We first show that, evolving under this free Hamiltonian, the verification of the conjecture of validity of the GGE for the \( g_1 \) function (the equal time two-point function) is trivial and tautological. This is not true for the \( g_2 \) function (the non-local pair correlation function).

The initial state \( |\Omega\rangle \) is the ground state of the pre-quench Hamiltonian

\[
H_0 = \int_0^L dx \left( \partial_x \Psi^\dagger(x) \partial_x \Psi(x) + c \Psi^\dagger(x) \Psi^\dagger(x) \Psi(x) \Psi(x) \right),
\]

while the evolution is described by the free boson Hamiltonian

\[
H = \int_0^L dx \partial_x \Psi^\dagger(x) \partial_x \Psi(x) = \sum_{k=-\infty}^{+\infty} k^2 \Psi_k^\dagger \Psi_k,
\]

where

\[
\Psi_k = \int_0^L \frac{dx}{\sqrt{L}} e^{-ikx} \Psi(x), \quad \Psi(x) = \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{+\infty} e^{ikx} \Psi_k.
\]

The boson mass is set to \( m = 1/2 \), the system size is \( L \) and periodic boundary conditions have been assumed, so that \( k = 2\pi n / L \) with \( n \) integer. The time evolution of the mode operators is

\[
\Psi_k(t) = e^{iHt} \Psi_k(0) e^{-iHt} = \Psi_k(0) e^{-ik^2t}.
\]
3.1. The $g_1$ function

The $g_1$ function is

$$g_1(x; t) = \langle \Omega | \Psi (0; t) \Psi (x; t) | \Omega \rangle = \frac{1}{L} \sum_{k = -\infty}^{+\infty} e^{i k x} \langle \Omega | \Psi_k^\dagger (0) \Psi_k (0) | \Omega \rangle = \frac{1}{L} \sum_{k = -\infty}^{+\infty} e^{i k x} \langle n_k \rangle_0 ,$$

(44)

where $n_k \equiv \Psi_k^\dagger \Psi_k$ are the mode occupation number operators, i.e. the conserved charges. We see that $g_1(x; t)$ is actually time independent and automatically described by the GGE, since it is a linear combination of the values of $n_k$ in the initial state which, by definition of the GGE, are equal to their GGE values. Explicitly

$$g_{1,GGE}(x) = \langle \Psi (0) | \Psi (x) \rangle_{GGE} = \frac{1}{L} \sum_{k = -\infty}^{+\infty} e^{i k x} \frac{\text{Tr} \left\{ \Psi_k^\dagger \Psi_k e^{-\sum_n \lambda_n n_n} \right\}}{\text{Tr} \{ e^{-\sum_n \lambda_n n_n} \}} = \frac{1}{L} \sum_{k = -\infty}^{+\infty} e^{i k x} \langle n_k \rangle_{GGE} .$$

(45)

Therefore, since $\langle n_k \rangle_{GGE} = \langle n_k \rangle_0$, we have

$$g_{1,GGE}(x) = g_1(x; t \to \infty) = g_1(x; 0) .$$

(46)

This result was previously obtained in [37].

3.2. The $g_2$ function

The $g_2$ function is

$$g_2(x; t) = \langle \Omega | \Psi (x; t) \Psi (0; t) \Psi (x; t) \Psi (0; t) | \Omega \rangle = L^{-2} \sum_{k_1, k_2, k_3} e^{i(k_1 - k_2)x + i(k_1^2 + k_2^2 - k_3^2 - (k_1 + k_2 - k_3)^2)} t$$

$$\times \langle \Omega | \Psi_k^\dagger (0) \Psi_k^\dagger (0) \Psi_{k_1 + k_2 - k_3} (0) | \Omega \rangle$$

$$= L^{-2} \sum_{k_1, k_2, k_3} e^{i(k_1 - k_2)x - i(k_1^2 + k_2^2 - k_3^2 - (k_1 + k_2 - k_3)^2)} t$$

$$\times \langle \Omega | \Psi_k^\dagger (0) \Psi_k^\dagger (0) \Psi_{k_1 + k_2 - k_3} (0) | \Omega \rangle = L^{-4} \sum_{k_1, k_2, k_3} e^{i(k_1 - k_2)x - i(k_1^2 + k_2^2 - k_3^2 - (k_1 + k_2 - k_3)^2)} t$$

$$\times \langle \Omega | \Psi (x_1; 0) \Psi (x_2; 0) \Psi (x_3; 0) \Psi (x_4; 0) | \Omega \rangle$$

(47)

$$= L^{-3} \sum_{k_1, k_2, k_3} e^{i(k_1 - k_2)x - i(k_1^2 + k_2^2 - k_3^2 - (k_1 + k_2 - k_3)^2)} t$$

$$\times \langle \Omega | \Psi (x_1; 0) \Psi (x_2; 0) \Psi (x_3; 0) \Psi (0; 0) | \Omega \rangle$$

(48)
\[ \begin{align*}
&= L^{-2} \sum_{q,k} e^{iq(x-2kt)} \int_0^\infty dx_1 dx_2 e^{-iq_1 - ikx_2} \\
&\quad \times \langle \Omega \mid \Psi^\dagger(x_1;0) \Psi^\dagger(x_2;0) \Psi(x_1 + x_2;0) \Psi(0;0) \mid \Omega \rangle \\
&= L^{-1} \sum_{k} \int_0^L dz e^{-ikz} \langle \Omega \mid \Psi^\dagger(x-2kt;0) \Psi^\dagger(z;0) \\
&\quad \times \Psi(x-2kt+z;0) \Psi(0;0) \mid \Omega \rangle. 
\end{align*} \]

where \( k_0 \equiv k_x - k_y \) and we made use of the translational invariance of the system. The last expression has been derived also in [37]. If we decompose the initial four-point correlation function of the last line into disconnected and connected pieces, we have

\[ \begin{align*}
\langle \Omega \mid \Psi^\dagger(x-2kt;0) \Psi^\dagger(z;0) \Psi(x-2kt+z;0) \Psi(0;0) \mid \Omega \rangle \\
&= \langle \Omega \mid \Psi^\dagger(x-2kt;0) \Psi(x-2kt+z;0) \mid \Omega \rangle \langle \Omega \mid \Psi^\dagger(z;0) \Psi(0;0) \mid \Omega \rangle \\
&\quad + \langle \Omega \mid \Psi^\dagger(x-2kt;0) \Psi(0;0) \mid \Omega \rangle \langle \Omega \mid \Psi^\dagger(z;0) \Psi(x-2kt+z;0) \mid \Omega \rangle \\
&\quad + \langle \Omega \mid \Psi^\dagger(x-2kt;0) \Psi^\dagger(z;0) \Psi(x-2kt+z;0) \Psi(0;0) \mid \Omega \rangle_{\text{conn}} \\
&= \langle \Omega \mid \Psi^\dagger(0;0) \Psi(z;0) \mid \Omega \rangle \langle \Omega \mid \Psi^\dagger(z;0) \Psi(0;0) \mid \Omega \rangle \\
&\quad + \langle \Omega \mid \Psi^\dagger(x-2kt;0) \Psi(0;0) \mid \Omega \rangle \langle \Omega \mid \Psi^\dagger(0;0) \Psi(x-2kt;0) \mid \Omega \rangle \\
&\quad + \langle \Omega \mid \Psi^\dagger(x-2kt;0) \Psi^\dagger(z;0) \Psi(x-2kt+z;0) \Psi(0;0) \mid \Omega \rangle_{\text{conn}} \\
&= |g_1(z;0)|^2 + |g_1(x-2kt;0)|^2 + G_{\text{conn}}^{(4)}(x-2kt, z, x-2kt+z, 0;0),
\end{align*} \]

where \( G_{\text{conn}}^{(4)} \) is the connected part of the four-point function. Once again we used the fact that the initial state is translationally invariant. We now find

\[ \begin{align*}
g_2(x,t) &= \frac{1}{L} \sum_{k} \int_0^L dze^{-ikz} \left( |g_1(z;0)|^2 + |g_1(x-2kt;0)|^2 \\
&\quad + G_{\text{conn}}^{(4)}(x-2kt, z, x-2kt+z, 0;0) \right) \\
&= (g_1(0;0))^2 + |g_1(x;0)|^2 \\
&\quad + \frac{1}{L} \sum_{k} \int_0^L dze^{-ikz} G_{\text{conn}}^{(4)}(x-2kt, z, x-2kt+z, 0;0),
\end{align*} \]

where in the first term we performed first the \( k \)-summation and then the \( z \)-integration, while in the second we performed them in reverse order.

Next we take first the infinite system size limit and then the large time limit of the last term. For \( L \to \infty \) the sum becomes an integral and the above expression becomes

\[ \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \int_{-\infty}^{+\infty} dze^{-ikz} G_{\text{conn}}^{(4)}(x-2kt, z, x-2kt+z, 0;0) \equiv I(x,t). \]

The function \( G_{\text{conn}}^{(4)}(x-2kt, z, x-2kt+z, 0;0) \) is regular as \( z \to 0 \) and \( k \to x/2t \), because the initial four-point function itself is regular when each of its coordinates tends to

doi:10.1088/1742-5468/2014/07/P07024
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zero. This is true for any initial state that is the ground state of the Lieb–Liniger model for some $c > 0$. On the other hand, the large distance behaviour of the connected four-point function is given by the harmonic fluid or Luttinger liquid approximation, which states that $G^{(4)}_{\text{conn}}$ decays as a power law (along with oscillating subleading corrections). The exponent of this power law is given in terms of the Luttinger parameter $K$, which is related to the interaction $c$ of the Lieb–Liniger model and varies monotonically from $+\infty$ to 1 as $c$ varies from 0 to $\infty$. The explicit expression for the connected four-point function is derived in the appendix A, using Conformal Field Theory methods. From equation (A.3) we find that

$$G^{(4)}_{\text{conn}}(x-2kt, z, x-2kt+z, 0;0)$$

Replacing $G^{(4)}_{\text{conn}}$ by this asymptotic expression in the integral does not affect its large time behaviour, since both the exact and the asymptotic expressions are integrable at $z \to 0$ and $k \to x/2t$ (because $K$ varies from $+\infty$ to 1 so that $1/2K < 1/2$) and the contribution of the region around these points is subleading for $t \to \infty$. The large time behaviour can be easily derived from the above scaling form. Indeed, performing the change of integration variables $k \to -k'/ (2\sqrt{t}) + x/(2t)$ and $z \to z' \sqrt{t}$ we have

$$I(x, t \to \infty) \sim \rho_0^2 \int_{-\infty}^{+\infty} \frac{dk'}{2\pi} \int_{-\infty}^{+\infty} \frac{dz'}{2\pi} e^{ik'z'/2} e^{-i\frac{x}{t}\sqrt{t}}$$

$$\times \left| \frac{1}{z'^2} - \frac{1}{k'^2} \right|^{1/(2K)} - \frac{1}{|z'|^{1/K}} - \frac{1}{|k'|^{1/K}}$$

For $t \to \infty$ the exponential $e^{-i\frac{x}{t}\sqrt{t}}$ tends to 1, so that the value of $x$ is completely irrelevant in this limit. Therefore we find

$$I(x, t \to \infty) \sim A(K) \rho_0^2 t^{-1/(2K)} \to 0,$$

where

$$A(K) \equiv \int_{-\infty}^{+\infty} \frac{dk'}{2\pi} \int_{-\infty}^{+\infty} dz' e^{ik'z'/2} \left| \frac{1}{z'^2} - \frac{1}{k'^2} \right|^{1/(2K)} - \frac{1}{|z'|^{1/K}} - \frac{1}{|k'|^{1/K}},$$

which is a convergent integral for all $1 < K < \infty$. This scaling law is in perfect agreement with earlier numerical studies [37], in which it was shown that $I(x, t \to \infty)$ decays with time as a power law with an exponent that is a decreasing function of the initial interaction $c$ and tends to 1/2 as $c \to \infty$.

The above result means that the connected part vanishes for large times and what remains is the disconnected parts

$$\lim_{t \to \infty} g_2(x; t) = (g_1(0;0))^2 + |g_1(x;0)|^2.$$
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\[ g_{2,\text{GGE}}(x) = \left\langle \Psi^\dagger(x) \Psi^\dagger(0) \Psi(x) \Psi(0) \right\rangle_{\text{GGE}} = \left\langle \Psi^\dagger(x) \Psi(x) \right\rangle_{\text{GGE}} \left\langle \Psi^\dagger(0) \Psi(0) \right\rangle_{\text{GGE}} + \left\langle \Psi^\dagger(x) \Psi(0) \right\rangle_{\text{GGE}} \left\langle \Psi^\dagger(0) \Psi(x) \right\rangle_{\text{GGE}} \]

\[ = \left( g_{1,\text{GGE}}(0) \right)^2 + \left| g_{1,\text{GGE}}(x) \right|^2, \]

and using (47), we see that the two expressions are in agreement—the GGE is correct for the \( g_2 \) correlation function.

Notice that the crucial point in the above argument was the fact that the connected four-point function \( G^{(4)}_{\text{conn}} \) decays to zero at large distances. As in the previous problem we studied, this is generally satisfied for any initial state prepared by performing a quantum quench i.e. any state that is the ground state of some physical Hamiltonian, due to the cluster decomposition principle. Using the same change of integration variables as above, we conclude that \( I(x, t) \) tends for large times to zero, in the same way that \( G^{(4)}_{\text{conn}} \) decays to zero when each of its coordinates tends to infinity. Therefore the validity of the GGE is once again a consequence of the cluster decomposition principle.

4. Conclusions

We have shown that, for a quantum quench problem whose evolution is governed by a noninteracting Hamiltonian, the existence of a (local) stationary state, described by a GGE, can be exclusively attributed to the cluster decomposition properties of the initial state. If the cluster decomposition holds for the initial state, we have analytically shown that, in the thermodynamic and large time limit, the GGE describes multi-point correlation functions of the fields. Inversely, when cluster decomposition does not hold, additional non-vanishing terms arise which do not comply with Wick’s theorem, which is tautologically valid in the GGE of noninteracting models. While the calculation has been performed for two specific free bosonic theories (gapped ‘relativistic’ bosons and free non-relativistic bosons), the line of the derivation is completely general and should be applicable to arbitrary local noninteracting systems. Notice that, even though we tested the validity of the GGE only for equal-time correlations, a general theorem [41] ensures that, when this happens, different-time stationary correlations are also described by the GGE.

We also point out another interesting byproduct of this work. While the connected part of multi-point correlations in the initial state does not contribute to the stationary value, it does, however, contribute to the approach to the GGE. As explained in the introduction, in order to avoid the quantum revivals, the thermodynamic limit should be taken before the large time limit. Then the extra terms (including the connected part of multi-point correlations) determine the time decaying part of observables, as in the calculation of section 3. On the other hand, when it is already known that the system equilibrates, the stationary behaviour can be derived by first time-averaging over infinite time and then taking the thermodynamic limit. In this case the extra terms determine the finite size part of observables that decays with the system size, as in the discussion of section 2. If specialized to finite but large systems, similar arguments can
be used to understand revival properties of some observables, a topic of intense recent interest [37, 72, 73].

It seems reasonable to apply the present explanation of the validity of the GGE to the general case of a genuinely interacting integrable post-quench Hamiltonian. Let us think about this scenario in more detail. It is true that, even in this more general case, the system can be decomposed into momentum (or rapidity) modes, whose occupation number operators are conserved and are linear combinations of the local conserved charges. The creation and annihilation operators of these modes evolve in time, exactly as in the noninteracting case, and satisfy generalized canonical commutation relations, which involve the two-particle scattering matrix of the model (in the context of relativistic Integrable Field Theory, this is known as the Zamolodchikov–Faddeev algebra [55]). Using these generalized canonical commutation relations, it is possible to derive a generalized version of Wick’s theorem, reducing all higher-order correlations of the creation/annihilation operators to their lowest (two-particle) correlations. In the context of quantum quenches, this allows higher-order correlations to be expressed solely in terms of the initial values of the charges—as in the noninteracting case. However, unlike the noninteracting case, the local physical observables (fields or vertex operators) are given in terms of the mode creation/annihilation operators through complicated nonlinear expressions (typically series expansions involving all of their Form Factors, i.e. the matrix elements of the observables in the energy eigenstates). Therefore, at this point it is technically difficult to make a connection with the above reasoning and take advantage of the cluster decomposition principle for the initial field correlations.

Acknowledgments

We are grateful to J Cardy and M Collura for helpful discussions. This work was supported by the ERC under Starting Grant 279391 EDEQS (PC and SS).

Appendix A. The four-point function in the Luttinger model

In this appendix, we will calculate the connected part of the four-point function of the Luttinger model, that gives an effective description of the Lieb–Liniger model at large distances. The calculation is based on the Conformal Field Theory method (for details and notation, see [74]). The general four-point function is

\[
\langle \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi(x_3) \Psi(x_4) \rangle = \rho_0^2 \sum_{m_1, m_2, m_3 = -\infty}^{+\infty} e^{2\pi i p_0 (m_1 x_1 + m_2 x_2 - m_3 x_3 - (m_1 + m_2 - m_3) x_4)}
\]

\[
\times \langle A_{2m_1 - 1}(x_1) A_{2m_2 - 1}(x_2) A_{-2m_3 + 1}(x_3) A_{-2(m_1 + m_2 - m_3) + 1}(x_4) \rangle
\]

\[
= \rho_0^2 \langle A_{0, -1}(x_1) A_{0, -1}(x_2) A_{0, +1}(x_3) A_{0, +1}(x_4) \rangle + \cdots
\]

where \( A_{m,n} = e^{im\theta} e^{in\phi} = e^{i\beta(m-n)\phi} e^{i\delta(m,n)\phi} \) are the vertex operators with \( \beta(m,n) = m\sqrt{K} / 2 + n / (2\sqrt{K}) \), \( K \) is the Luttinger parameter (related to the interaction \( c \) of the Lieb–Liniger model), and \( \rho_0 \) is the boson density. The dots denote
oscillating terms that decay with the distance faster than the term we kept and are therefore negligible. The leading term is

\[
\langle A_{0,-1}(x_1) A_{0,-1}(x_2) A_{0,+1}(x_3) A_{0,+1}(x_4) \rangle
= \left( \frac{2\pi}{L} \right)^4 q^2 \left| z_1 z_2 z_3 z_4 \right|^2 \langle V_q(z_1) V_q(z_2) V_q(z_3) V_q(z_4) \rangle \\
\times \langle \overline{V}_q(z_1) \overline{V}_q(z_2) \overline{V}_q(z_3) \overline{V}_q(z_4) \rangle
= \left( \frac{2\pi}{L} \right)^4 q^2 \left| z_1 z_2 z_3 z_4 \right|^2 \left( \frac{z_{1234}}{z_{1324} z_{2324}} \right)^2 \left( \frac{z_{1234}}{z_{1324} z_{2324}} \right)^2
= \left( \frac{2\pi}{L} \right)^{1/K} \left| z_1 z_2 z_3 z_4 \right|^2 \left( \frac{z_{1234}}{z_{1324} z_{2324}} \right)^2 \left( \frac{z_{1234}}{z_{1324} z_{2324}} \right)^2
= \left( \frac{d(x_1-x_2|L) d(x_3-x_4|L) d(x_1-x_3|L) d(x_2-x_4|L) d(x_2-x_3|L)}{d(x_1-x_2|L) d(x_3-x_4|L) d(x_1-x_3|L) d(x_2-x_4|L) d(x_2-x_3|L)} \right)^{1/(2K)},
\]

(A.1)

where \( V_q(z) = e^{i q \phi L(z)} \); are the normal-ordered vertex operators, the expectation values are evaluated in a cylindrical geometry (due to the periodic boundary conditions) i.e. \( z_i = e^{2\pi i w_i/L} \) with \( w_i = \tau_i + i z_i \) (in our case the imaginary times are zero, \( \tau_i = 0 \)) and \( q = 1/(2\sqrt{K}) \). The function \( d(x|L) = \sin(\pi x/L) |L/\pi | \) is the cord function and in the infinite size limit \( L \to \infty \) becomes \( d(x|L) \to |x| \). In this limit we therefore have

\[
\langle \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi(x_3) \Psi(x_4) \rangle = \rho_0^2 \left| \frac{(x_1-x_2)(x_3-x_4)}{(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)} \right|^{1/(2K)} + \cdots \]  

(A.2)

Finally, the connected part of the above is

\[
G^{(4)}_{\text{conn}}(x_1, x_2, x_3, x_4;0) = \langle \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi(x_3) \Psi(x_4) \rangle
- \langle \Psi^\dagger(x_1) \Psi(x_3) \rangle \langle \Psi^\dagger(x_2) \Psi(x_4) \rangle
- \langle \Psi^\dagger(x_1) \Psi(x_4) \rangle \langle \Psi^\dagger(x_2) \Psi(x_3) \rangle
= \rho_0^2 \left| \frac{(x_1-x_2)(x_3-x_4)}{(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)} \right|^{1/(2K)}
- \frac{1}{\left| (x_1-x_3)(x_2-x_4) \right|^{1/(2K)}} - \frac{1}{\left| (x_1-x_4)(x_2-x_3) \right|^{1/(2K)}} + \cdots
\]

(A.3)

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