Conjugation spaces and edges of compatible torus actions

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Dedicated to Hans Duistermaat on the occasion of his 65th birthday.

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1 Introduction

Duistermaat introduced the real locus of a Hamiltonian manifold [Du]. In this and in others’ subsequent works [BGH, Go, GH, HH, Ho, OS, Sd], it has been shown that many of the techniques developed in the symplectic category can be used to study real loci, so long as the coefficient ring is restricted to the integers modulo 2. As we will see, these results seem not necessarily to depend on the ambient symplectic structure, but rather to be topological in nature. This observation prompts the definition of conjugation space in [HHP]. We now give a brief survey of the results in symplectic geometry that motivated the definition of a conjugation space.

A symplectic manifold is a manifold $M$ together with a 2-form $\omega \in \Omega^2(M)$ that is closed ($d\omega = 0$) and non-degenerate (for each $X \in T_pM$ there exists $Y \in T_pM$ such that $\omega_p(X,Y) \neq 0$). Let $G$ be a compact Lie group acting on $M$ preserving $\omega$, $\mathfrak{g}$ the Lie algebra of $G$, $\mathfrak{g}^*$ its dual, and $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ the natural pairing. For each $X \in \mathfrak{g}$, we let $X^\#$ denote the vector field on
$M$ generated by the one-parameter subgroup $\exp(tX)$. We say that the $G$-action on $M$ is Hamiltonian if there is a moment map

$$\Phi : M \to \mathfrak{g}^*$$

that satisfies

1. $\iota_X \omega = d\langle \Phi, X \rangle$ for all $X \in \mathfrak{g}$; and
2. $\Phi$ is equivariant with respect to the given $G$ action on $M$ and the coadjoint action on $\mathfrak{g}^*$.

The function $\Phi_X = \langle \Phi, X \rangle$ is called the Hamiltonian function for the vector field $X^\#$.

When $G = T$ is a torus, the second condition on $\Phi$ requires that it be a $T$-invariant map. In this special case, we have

**Theorem 1.1** ([A], [GS]) If $M$ is a compact Hamiltonian $T$-space, then $\Phi(M)$ is a convex polytope. It is the convex hull of $\Phi(M^T)$, the images of the $T$-fixed points.

More generally, there are results first of Kirwan and then many others for non-abelian groups.

A by-product of Atiyah’s proof of Theorem 1.1 is that any of the Hamiltonian functions $\Phi_X$ is a perfect Morse function on $M$, in the sense of Bott, and for generic $X$, the critical set is $M^T$. More precisely,

$$H^*(M; \mathbb{R}) = \sum_{i=1}^{N} H^{*-d_i}(F_i; \mathbb{R}),$$

(1)

where the $F_i$ are the connected components of $M^T$ and $d_i$ is the Morse-Bott index of $F_i$. This statement is also true over $\mathbb{Z}$ provided that the cohomology of each $F_i$ is torsion-free, or when the torus action satisfies some additional hypotheses.

Duistermaat introduced the concept of real locus to this framework [Du]. Let $M$ be a Hamiltonian $T$-space, and $\tau : M \to M$ an anti-symplectic involution that is compatible with the action; that is, it satisfies

$$\tau(t \cdot p) = t^{-1} \cdot \tau(p),$$

for all $t \in T$ and $p \in M$. Then if it is non-empty, the submanifold $M^\tau$ of $\tau$-fixed points is a Lagrangian submanifold of $M$ called the real locus of the involution.

The primary example of such an involution is the one induced by complex conjugation on a complex projective variety defined over $\mathbb{R}$. For example, if $M = \mathbb{C}P^n$ equipped with the Fubini-Study symplectic form and the standard $T^n$ action, then the real locus for complex conjugation consists of the real points $\mathbb{R}P^n$, whence the name real locus. The main results in [Du] generalize Theorem 1.1 and Atiyah’s Morse theoretic results.
Theorem 1.2 ([Du]) If $M$ is a compact Hamiltonian $T$-space, and $\tau$ a compatible involution, then

1. The real locus has full moment image: $\Phi(M^\tau) = \Phi(M)$ is a convex polytope; and

2. Components $\Phi_X$ of the moment map are perfect Morse functions on $M^\tau$, in the sense of Bott, and for generic components the critical set is $M^\tau \cap M^T$, when the coefficients are taken in $\mathbb{Z}_2$.

We have the following immediate corollary, a real locus version of Equation (1), that generalizes classical results on real projective space and real flag varieties.

Corollary 1.3 If $M$ is a compact Hamiltonian $T$-space, and $\tau$ a compatible involution, then

$$H^\ast(M^\tau; \mathbb{Z}_2) = \sum_{i=1}^{N} H^{\ast - d_i}(F_i)^\tau; \mathbb{Z}_2).$$

(2)

where the $F_i$ are the connected components of $M^T$ and $d_i$ is the Morse-Bott index of $F_i$ (in $M$).

Duistermaat’s work began a flurry of activity on properties of real loci. We provide a brief account here; a more detailed record is available in [Sj]. Davis and Januszkiewicz studied the real loci of toric varieties in their own right [DJ], independent of Duistermaat’s work. The first author and Knutson analyze a large class of examples of real loci in their account of planar and spacial polygon spaces [HK]. O’Shea and Sjamaar generalized Kirwan’s non-abelian convexity results to real flag manifolds and real loci [OS]. This has recently been extended by Goldberg [Go].

Schmid and independently Biss, Guillemin and the second author generalized (2) to the equivariant setting: the idempotents $T_2 = \{ t \in T \mid t^2 = 1 \}$ act on the real locus, and many results in $T$-equivariant symplectic geometry may be generalized to $T_2$ equivariant geometry of real loci (with coefficients restricted to $\mathbb{Z}_2$) [BGH, Sd]. This work yields an explicit description of the $T_2$-equivariant cohomology for the fixed set of the Chevalley involution on certain coadjoint orbits, and on the real locus of a toric variety, using localization methods. These results were strengthened to include the fixed set of the Chevalley involution on all coadjoint orbits in [HHP].

Following this, Goldin and the second author [GH] proved that there is a natural involution on an abelian symplectic reduction of a symplectic manifold with involution. Moreover, the $T_2$ equivariant cohomology of the original real locus surjects onto the ordinary cohomology of the real locus of the symplectic reduction. This includes a comprehensive description of toric varieties and their real loci from yet a third perspective.
In all of these papers, a common theme is that there is a degree-halving isomorphism

$$H^{2*}(M; \mathbb{Z}_2) \to H^{*}(M^\tau; \mathbb{Z}_2).$$

As we now describe, this can be seen as part of a purely topological framework, that of a conjugation space, introduced in [HHP]. The remainder of the article is organized as follows. In Section 2, we review the definitions and properties of conjugation spaces. Our main theorem gives a criterion for recognizing when a topological space is a conjugation space; this is stated in Section 3, along with two noteworthy corollaries. We then prove some basic facts in Section 4, and prove the main theorems in Section 5.

**NOTE:** For the remainder of the paper, the cohomology is taken coefficients in the field $\mathbb{Z}_2$: $H^{*}(X) = H^{*}(X; \mathbb{Z}_2)$.

### 2 A review of conjugation spaces

Let $X$ be a $G$-space $X$ for a topological group $G$. The equivariant cohomology $H^*_G(X)$ is defined as the (singular) cohomology of the Borel construction:

$$H^*_G(X) = H^*(X \times_G BG).$$

Hence, $H^*_G(X)$ is a $H^*(BG)$-algebra. When $G = C$ is the group of order two, $BC = \mathbb{R}P^\infty$ and $H^*(BC) = \mathbb{Z}_2[u]$, with $u$ in degree 1. Thus, $H^*_C(X)$ is a $\mathbb{Z}_2[u]$-algebra.

Let $\tau$ be a continuous involution on a space $X$. Let $\rho: H^*_G(X) \to H^{2*}(X)$ and $\tau: H^*_C(X) \to H^*_G(X^\tau)$ be the restriction homomorphisms, where $C = \{\text{id}, \tau\}$.

A cohomology frame or $H^*$-frame for $(X,Y)$ is a pair $(\kappa, \sigma)$, where

(a) $\kappa: H^2(X) \to H^*(X^\tau)$ is an additive isomorphism dividing the degrees in half; and

(b) $\sigma: H^2(X) \to H^*_C(X)$ is an additive section of $\rho$.

Moreover, $\kappa$ and $\sigma$ must satisfy the conjugation equation

$$r \cdot \sigma(a) = \kappa(a)u^m + \ell t_m$$

for all $a \in H^{2m}(X)$ and all $m \in \mathbb{N}$, where $\ell t_m$ denotes any polynomial in the variable $u$ of degree less than $m$. An involution admitting a $H^*$-frame is called a conjugation. An even cohomology space (i.e. $H^{odd}(X) = 0$) together with a conjugation is called a conjugation space. Conjugation spaces were introduced in [HHP] and studied further in [FP] and [Ol]. The main examples of conjugations are given the complex conjugation in flag manifolds, the Chevalley involution in coadjoint orbit of compact Lie groups and other natural involutions, e.g. on toric manifolds or polygon spaces. Here below are some important properties of conjugation spaces.
(a) If $(\kappa, \sigma)$ is $H^*$-frame, then $\kappa$ and $\sigma$ are ring homomorphisms \cite[Theorem 3.3]{HHP}. The ring homomorphism $\kappa$ also commutes with the Steenrod squares: $\kappa \circ Sq_i = Sq_i \circ \kappa$, \cite[Theorem 1.3]{FP}.

(b) $H^*$-frames are natural for $\tau$-equivariant maps \cite[Prop.3.11]{HHP}. In particular, if an involution admits an $H^*$-frame, it is unique \cite[Cor.3.12]{HHP}.

(c) For a conjugate-equivariant complex vector bundle $\eta$ ("real bundle" in the sense of Atiyah) over a conjugation space $X$, the isomorphism $\kappa$ sends the total Chern class of $\eta$ onto the total Stiefel-Whitney class of its fixed bundle.

Duistermaat’s Corollary \cite[Theorem 8.3]{HHP} admits the following generalization, proved in \cite[Theorem 8.3]{HHP}.

**Theorem 2.1** Let $M$ be a compact symplectic manifold equipped with a Hamiltonian action of a torus $T$ and with a compatible smooth anti-symplectic involution $\tau$. If $M^T$ is a conjugation space, then $M$ is a conjugation space.

The proof of Theorem 2.1 involves properties of conjugations compatible with $T$-actions which are interesting by their own. The involution $g \mapsto g^{-1}$ on the torus $T$ induces an involution on $ET$. Using this involution together with $\tau$, we get an involution on $X \times ET$ which descends to an involution, still called $\tau$, on $X_T$. To a torus $T$ is associated its 2-torus, i.e. the set of idempotent elements of $T$:

$$T_2 = \{ g \in T \mid g^2 = 1 \}.$$ 

The compatibility implies that $T_2$ acts on $X^\tau$. The following lemma is proved in \cite[Lemma 7.3]{HHP}.

**Lemma 2.2** $(X_T)^\tau = (X^\tau)^{T_2}$. ✓

The following theorem is proved in \cite[Theorem 7.5]{HHP}.

**Theorem 2.3** Let $X$ be a conjugation space together with a compatible action of a torus $T$. Then the involution induced on $X_T$ is a conjugation. ✓

Using Lemma 2.2 one gets the following corollary of Theorem 2.3.

**Corollary 2.4** Let $X$ be a conjugation space together with an involution and a compatible $T$-action. Then there is a ring isomorphism

$$\bar{\kappa}: H^2_T(X) \cong H^*_T(X^\tau).$$
We now state our new results. They consist of criteria to determine that an involution $\tau$ is a conjugation, in the case where $\tau$ is compatible with an action of a torus $T$. The conditions are on the equivariant 1-skeleton of the action of $T$ on $X$ and of the inherited action of the associated 2-torus $T_2$.

Let $X$ be a topological space, together with a continuous action of a group $G$, where $G$ is a torus or a 2-torus (finite elementary abelian 2-group). We define the $G$-equivariant $i$-skeleton $\text{Sk}_G^i(X)$ of the $G$-action on $X$ to be
\[
\text{Sk}_G^i(X) = \{ x \in X \mid \text{codim}\,(G_x \subset G) \leq i \},
\]
where $G_x$ denotes the $G$-isotropy group of $x$. In [4], the “codimension” is interpreted as the codimension of a manifold if $G$ is a torus, and the codimension of a $\mathbb{Z}_2$-vector subspace if $G$ is a 2-torus (and hence isomorphic to a $\mathbb{Z}_2$-vector space). In particular, $\text{Sk}_G^0(X)$ is equal to the subspace $X^G$ of fixed points.

An edge (of the $G$-action) is the closure of a connected component of the set $\text{Sk}_G^1(X) \setminus \text{Sk}_G^0(X)$.

Let $T$ be a torus and $T_2$ the subgroup of idempotents. A $T$-action on a space $X$ induces a $T_2$-action on $X$ that satisfies $\text{Sk}_T^i(X) \subset \text{Sk}_{T_2}^i(X)$. For example, $X^T \subset X^{T_2}$.

A continuous action of a topological group $G$ on a space $X$ is called good if $X$ has the $G$-equivariant homotopy type of a finite $G$-CW-complex. For instance, a smooth action of a compact Lie group on a closed manifold is good. A continuous involution $\tau$ is called good if the corresponding action of the cyclic group $C = \{\text{id}, \tau\}$ is good.

Let $X$ be a topological space, and let $\tau$ be continuous involution on $X$ that is compatible with a continuous action of a torus $T$. Then the involution $\tau$ preserves the $T$-equivariant skeleta and sends each edge to a (possibly different) edge. Moreover, the real locus $X^\tau = X^G$ inherits an action of $T_2$. Our main results are the following.

**Theorem 3.1 (Main Theorem)** Let $X$ be an even cohomology space, together with a good involution $\tau$ which is compatible with a good action of a torus $T$. Suppose that

(a) $(X^T, \tau)$ is a conjugation space.

(b) each edge of the $T$-action is preserved by $\tau$ and is a conjugation space.

(c) $\text{Sk}_T^i(X) = \text{Sk}_{T_2}^i(X)$ for $i = 0, 1$.

Then $X$ is a conjugation space.

Recall that a $T$-action on a space $X$ is called a GKM action if each edge is a 2-sphere upon which $T$ acts by rotation around some axis, via a non-trivial character $T \to S^1$. One consequence of this assumption is that $X^T$ is discrete.
Corollary 3.2 Let $X$ be an even cohomology space, together with a good involution $\tau$ which is compatible with a good GKM action of a torus $T$, satisfying $\text{Sk}_i^T(X) = \text{Sk}_i^{T_2}(X)$ for $i = 0, 1$. Suppose that $\tau$ acts trivially on $X^T$ and preserves each edge. Then $X$ is a conjugation space.

Corollary 3.3 Let $X$ be an even cohomology space, together with a good involution $\tau$ which is compatible with a good action of a torus $T$, satisfying $\text{Sk}_i^T(X) = \text{Sk}_i^{T_2}(X)$ for $i = 0, 1$. Suppose that

(a) $(X^T, \tau)$ is a conjugation space.

(b) each edge of the $T$-action is preserved by $\tau$ and is a Hamiltonian $T$-manifold on which $\tau$ acts smoothly and is anti-symplectic.

Then $X$ is a conjugation space.

See §6 for comments about the condition $\text{Sk}_i^T(X) = \text{Sk}_i^{T_2}(X)$ for $i = 0, 1$.

4 Preliminaries

This section is devoted to the proof of Theorem 3.1 and of Corollaries 3.2 and 3.3. We begin with some preliminaries.

4.1 Compatibility. Let $X$ be a topological space endowed with a continuous involution $\tau$ which is compatible with a continuous action of a torus $T$. The involution $\tau$ then induces an involution on the fixed point set $X^\tau$. In addition, the associated 2-torus $T_2$ of $T$ acts on $X^\tau$ and $X^\tau \cap X^T \subset (X^\tau)^{T_2}$. Condition (c) of Theorem 3.1 will play an important role.

Lemma 4.2 Suppose that $\text{Sk}_i^T(X) = \text{Sk}_i^{T_2}(X)$. Then $\text{Sk}_i^T(X)^\tau = \text{Sk}_i^{T_2}(X^\tau)$.

Proof: One has

\[ \text{Sk}_i^T(X)^\tau = \text{Sk}_i^T(X) \cap X^\tau \subset \text{Sk}_i^{T_2}(X^\tau) = X^\tau \cap \text{Sk}_i^{T_2}(X) = X^\tau \cap \text{Sk}_i^T(X), \]

which implies that $\text{Sk}_i^T(X)^\tau = \text{Sk}_i^{T_2}(X^\tau)$. □

4.3 Equivariantly formal spaces. Let $X$ be a space with a continuous action of a compact Lie group $G$. First introduced in [GKM] for $G$ a torus and complex coefficients, the notion of equivariant formality was developed for other coefficients where it is more subtle, see [HHP] (2.3) and [F] §8. A $G$-space $X$ is equivariantly formal (over $\mathbb{Z}_2$) if the map $X \to EG \times_G X$ is totally nonhomologous to zero, that is the restriction homomorphism $j^*: H^*_G(X) \to H^*(X)$ is surjective. A space $X$ with an involution $\tau$ is called $\tau$-equivariantly formal if it is $C$-equivariantly formal for $C = \{\text{id}, \tau\}$. The following results are classical but may be not found in the literature with exactly our hypotheses. Let $R = H^*_G(pt)$; the map $X_G \to BG$ gives a ring homomorphism $p^*: R \to H^*_G(X)$, making $H^*_G(X)$ an $R$-module.
Proposition 4.4 The following conditions are equivalent:

(i) $X$ is an equivariantly formal $G$-space.

(ii) The group $G$ acts trivially on $H^*(X)$ and the Serre spectral sequence for the cohomology of the fibration $X \to EG \times_G X \to BG$ collapses at the term $E_2$.

(iii) The group $G$ acts trivially on $H^*(X)$ and $H_G^*(X)$ is a free $R$-module.

(iv) there is an additive homomorphism $\sigma : H^*(X) \to H_G^*(X)$ such that $j^* \circ \sigma = \text{id}$ and $p^* \otimes \sigma : R \otimes H^*(X) \to H_G^*(X)$ is an isomorphism of $R$-modules.

(v) The ring homomorphism $H_G^p(X) \to H^*(X)$ descends to a ring isomorphism $H_G^G(X) \otimes_R \mathbb{Z}_2 \cong H^*(X)$.

Proof: This proof is for mod 2-cohomology, but it works for the cohomology with coefficients in any field.

(i) is equivalent to (ii): the ring homomorphism $j^* : H_G^*(X) \to H^*(X)$ is the composition:

$$H_G^*(X) \to E_\infty^0 \subset E_2^0 = H^0(BG; H^*(X)) = H^*(X)^G \subset H^*(X).$$

If these inclusions are equalities, then $j^*$ is onto, which shows that (ii) implies (i).

Conversely, if $j^*$ is onto, this shows that $H^*(X)^G = H^*(X)$ and $E_\infty^0 = E_2^0$.

As the differentials are morphisms of $R$-modules, this implies that $E_\infty^0 = E_2^0$ (see [McC p. 148]). Hence (i) implies (ii).

(i) implies (iii) and (iv): As $j^*$ is surjective, there exists a $\mathbb{Z}_2$-linear section $\sigma$ of $j^*$. We already showed that (i) implies that the $G$-action on $H^*(X)$ is trivial. As $G$ is a compact Lie group, $H_G^G(pt)$ is a finite dimensional $\mathbb{Z}_2$-vector spaces for all $p$. The Leray-Hirsch theorem [McC Thm 5.10] then implies that $H_G^G(X)$ is a free $R$-module with basis $\sigma(B)$, where $B$ is a $\mathbb{Z}_2$-basis of $H^*(X)$. This implies (iii) and (iv).

(iii) implies (i): as $G$ acts trivially on $H^*(X)$, the term $E_\infty^0$ is isomorphic to $R \otimes H^*(X)$ as a bigraded $R$-module. This implies that the kernel of $j^*$ is equal to $I : H_G^G(X)$, where $I$ is the ideal of $R$ of elements of positive degree. Suppose that $H_G^G(X)$ is the free $R$-module with some basis $C$. As $R/I = \mathbb{Z}_2 \otimes_R R \approx \mathbb{Z}_2$, the image of $j^*$ can be identified with $\mathbb{Z}_2$-vector space with basis $C$. Denote by $C_s$ the subset of $C$ of elements of degree $\leq s$.

Suppose, by induction on $q$, that $j_* : H_G^q(X) \to H^q(X)$ is surjective for $q \leq k$ (true for $k = 0$). If there is $a \in H^k(X)$ which is not in the image of $j^*$, then $d_r(a) \neq 0$ for some differential $d_r : H^k(X) \to R \otimes H^{k-r+1}(X)$. Therefore, there are elements $a_1, \ldots, a_m \in C_{k-r-1}$, and $r_1, \ldots, r_m \in R$ with $\sum r_i a_i = d_r(a)$ in $E_\infty^0$. This means that $\sum r_i a_i \in H_G^G(X)$ is a $R$-linear combination of elements of $C_{k-r-2}$. Such a relation would contradict the fact that $C$ is a basis of $H_G^G(X)$.
(iv) implies (v): the homomorphism $j^* \circ p^* : R \to H^*(X)$ coincides with the projection $R \to \mathbb{Z}_2 \otimes_R R = \mathbb{Z}_2$. Therefore, $j^*$ factors through a ring homomorphism $\bar{j}^* : \mathbb{Z}_2 \otimes_R H_G^*(X) \to H^*(X)$. On the other hand, $j^* \circ \sigma = \text{id}$. Hence, one has a commutative diagram

\[
\begin{array}{ccc}
R \otimes H^*(X) & \longrightarrow & \mathbb{Z}_2 \otimes_R (R \otimes H^*(X)) \\
p^* \circ \sigma \downarrow \approx & & \downarrow \approx \\
H_G^*(X) & \longrightarrow & \mathbb{Z}_2 \otimes_R H_G^*(X) \quad \bar{j}^* \longrightarrow \quad H^*(X),
\end{array}
\]

which proves that $\bar{j}^*$ is an isomorphism.

(v) implies (i): this implication is trivial. \(\square\)

**Proposition 4.5** Let $X$ be a good $G$-space which is equivariantly formal over $\mathbb{Z}_2$. Suppose that one of the following hypotheses holds:

(a) $G$ is a torus and $X^G = X^{G_2}$.

(b) $G$ is a 2-torus.

Then the restriction homomorphism $H_G^*(X) \to H_G^*(X^G)$ is injective.

**Remark 4.6** In Case (b), Proposition 4.5 is false without the assumption $X^G = X^{G_2}$. For example, consider the $G = S^1$ action on $X = S^2 \subset \mathbb{C} \times \mathbb{R}$ by $g(z,t) = (g^2z,t)$. This has $X^G = \{(0,\pm1)\}$. Let $U_+ = X - \{(0,-1)\}$ and $U_- = X - \{(0,1)\}$. The intersection $U_+ \cap U_-$ is $G$-homotopy equivalent to the homogeneous space $G/G_2$ and then $H_G^*(U_+ \cap U_-) = H^*(BG_2)$. The Mayer-Vietoris sequence for $(X,U_+,U_-)$ then gives

\[0 \to H^1(BG_2) \to H^2_G(X) \to H^2_G(X^G)\]

and $H^1(BG_2) = \mathbb{Z}_2$.

**Proof of Proposition 4.5** Let $R_{(0)}$ be the field of fractions of $R$, that is $R$ localized at $S = R - \{0\}$. By our assumptions, the multiplicative set $S$ is central in $R$. Let

\[X^S = \{x \in X \mid H^*(BG) \to H^*(BG_x) \text{ is injective}\},\]

where $G_x$ is the isotropy group of $x$. The localization theorem ([AP Thm.3.1.6], [A Thm.3.7]) asserts that the inclusion $X^S \subset X$ induces an isomorphism of $R_{(0)}$-vector spaces

\[S^{-1}H_G^*(X) \approx S^{-1}H_G^*(X^S).\]  

(7)

In Case (b), if $G_x$ is a proper subgroup of $G$, then $H^2(BG) \to H^2(BG_x)$ is not injective; hence $X^S = X^G$. For Case (a), we use that, for each $x \in X$, there is an isomorphism $\psi_x : G \xrightarrow{\approx} (S^1)^m$ such that $\psi_x(G_x) = C_1 \times \cdots \times C_m$, where $C_j$ is a subgroup of $S^1$. In order to have $H^2(BG) \to H^2(BG_x)$ injective, each $C_j$
should be either $S^1$ or a finite cyclic group of even order. Then $X^G \subseteq X^S \subseteq X^{G_2} = X^G$. Hence, in all cases, we have proved that

$$S^{-1}H_G^*(X) \to S^{-1}H_G^*(X^G)$$

is an isomorphism. Therefore, ker$(H_G^*(X) \to H_G^*(X^G))$ is the $R$-torsion of $H_G^*(X)$. But the $R$-torsion vanishes because $H_G^*(X)$ is a free $R$-module by Proposition 4.4.

**Proposition 4.7** Let $X$ be a good $G$-space. Suppose that Conditions (a) or (b) of Proposition 4.5 are satisfied. Then $X$ is an equivariantly formal $G$-space over $\mathbb{Z}_2$ if and only if $\dim_{\mathbb{Z}_2} H^*(X) = \dim_{\mathbb{Z}_2} H^*(X^G)$.

**Proof:** As in the proof of Proposition 4.5, consider $S = R - \{0\}$ and $R(0) = S^{-1}R$. We apply $S^{-1}$ to the terms of the Serre spectral sequence, following [Al, proof of Cor.3.10]. When $G$ is a torus, it acts trivially on $H^*(X)$, which implies that $E_2^{*,*} \approx H^*(BG; H^*(X))$ as $R$-module and there is an isomorphism of $R(0)$-vector spaces $R(0) \otimes_{\mathbb{Z}_2} H^*(X) \approx S^{-1}E_2$. Therefore, using equation (8), we get

$$\dim_{\mathbb{Z}_2} H^*(X) = \dim_{R(0)} (S^{-1}E_2) \geq \dim_{R(0)} (S^{-1}E_{\infty}) = \dim_{R(0)} (S^{-1}H_G^*(X)) = \dim_{R(0)} (S^{-1}H_G^*(X^G)) = \dim_{\mathbb{Z}_2} (H^*(X^G)).$$

By Proposition 4.4, the inequality in equation (9) is an equality if and only if $X$ is equivariantly formal. Finally, when $G$ is a 2-torus, Proposition 4.7 follows from [AP, Thm 3.10.4].

4.8 We shall need the following two lemmas, first proved by Chang and Skjelbred for rational cohomology and torus action [CS].

**Lemma 4.9** Let $X$ be a space endowed with a good action of a 2-torus $G$. Suppose that $X$ is $G$-equivariantly formal. Then the restriction homomorphisms on the mod 2-cohomology $H_G^*(X) \to H_G^*(X^G)$ and $H_G^*(Sk^G_1(X)) \to H_G^*(X^G)$ have same image.

**Proof:** Using the equivalence (i) $\Leftrightarrow$ (iii) in Lemma 4.3, we know that $H_G^*(X)$ is a free $H_G^*(pt)$-module. By [HS, Corollary p. 63], the homomorphism

$$H_G^*(X, X^G) \to H_G^*(Sk^G_1(X), X^G)$$

is an isomorphism.
is injective. The $H^*_G$-sequences of the pairs $(X, X^G)$ and $(\text{Sk}^1_T(X), X^G)$ are part of a commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & H^*_G(X) & \longrightarrow & H^*_G(X^G) & \longrightarrow & H^*_G(X, X^G) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & H^*_G(\text{Sk}^1_T(X)) & \longrightarrow & H^*_G(X^G) & \longrightarrow & H^*_G(\text{Sk}^1_T(X), X^G) & \longrightarrow & 0
\end{array}
$$

Therefore, the injectivity of the last vertical arrow implies the lemma. □

**Lemma 4.10** Let $X$ be a space endowed with a good action of a torus $T$. Suppose that $X$ is $T$-equivariantly formal and that $\text{Sk}^i_T(X) = \text{Sk}^j_T(X)$ for $i, j = 0, 1$. Then the restriction homomorphisms on the mod2-cohomology $H^*_T(X) \longrightarrow H^*_T(X^T)$ and $H^*_T(\text{Sk}^1_T(X)) \longrightarrow H^*_T(X^T)$ have same image. □

## 5 Proof of the main results

We begin with the proof of Theorem 3.1. Recall that we are working with cohomology with $\mathbb{Z}_2$ coefficients. In what follows, $\dim$ denotes $\dim_{\mathbb{Z}_2}$.

### 5.1 $X$ is is $\tau$-equivariantly formal over $\mathbb{Z}_2$ and $X^\tau$ is is $T_2$-equivariantly formal over $\mathbb{Z}_2$.

Being an even-cohomology space, $X$ is $T$-equivariantly formal. By Hypothesis (a) of Theorem 3.1 and by Lemma 4.2 we have

$$
\dim H^*(X) = \dim H^* (X^T) = \dim H^*((X^T)^\tau) = \dim H^* ((X^T)^{T_2}) \leq \dim H^*(X^\tau) \leq \dim H^*(X),
$$

which implies that

$$
\dim H^*(X^\tau) = \dim H^*(X) \quad \text{and} \quad \dim H^* ((X^T)^{T_2}) = \dim H^*(X^\tau). \quad (10)
$$

### 5.2 $X_T$ is $\tau$-equivariantly formal.

For $G$ a topological group and $k \in \mathbb{N}$, we consider the $G$-principal bundle $G \to E_kG \to B_kG$ obtained as $k$-th step in the Milnor construction. If $X$ is a $G$-space, the associated bundle with fibre $X$ gives a bundle $X \to X_{G,k} \to B_kG$, where $X_{G,k} = E_kG \times_G X$.

For a torus $T$ of dimension $n$, $B_kT \approx (\mathbb{C}P^k)^n$. The involution $\tau(g) = g^{-1}$ on $T$ gives an involution $\tau$ on $B_kT$ which makes $B_kT$ a conjugation space with $(B_kT)^\tau \approx (\mathbb{R}P^k)^n \approx B_kT_2$.

We first prove that $X_{T,k}$ is $\tau$-equivariantly formal. As $X$ and $B_kT$ are even cohomology spaces, the spectral sequence of $X \to X_{T,k} \to B_kT$ degenerates at the $E^2$-term and $H^*(X_{T,k}) \approx H^*(X) \otimes H^*(B_kT)$. As a consequence,
\[
\dim H^*(X_{T,k}) = \dim H^*(X) \cdot \dim H^*(B_k T) < \infty. \quad \text{Since } X \text{ is } \tau\text{-equivariantly formal by } 5.1 \text{ one has}
\]
\[
\dim H^*(X_{T,k}) = \dim H^*(X) \cdot \dim H^*(B_k T) = \dim H^*(X^\tau) \cdot \dim H^*(B_k T_2) \quad \text{(11)}
\]
As \(X^\tau\) is \(T_2\)-equivariantly formal by 5.1 the following commutative diagram
\[
\begin{array}{ccc}
H^*((X^\tau)_{T_2}) & \xrightarrow{\tilde{\rho}_{T_2}} & H^*(X^\tau) \\
\downarrow & & \downarrow \approx \\
H^*((X^\tau)_{T_2,k}) & \xrightarrow{\rho_{T_2,k}} & H^*(X^\tau)
\end{array}
\]
shows that \(\rho_{T_2,k}\) is surjective and thus \(H^*((X^\tau)_{T_2,k}) \approx H^*(X^\tau) \otimes H^*(B_k T_2)\).

As in Lemma 2.2 one has \((X_{T,k})^\tau = (X^\tau)_{T_2,k}\), thus
\[
\dim H^*((X_{T,k})^\tau) = \dim H^*((X^\tau)_{T_2,k}) = \dim H^*(X^\tau) \cdot \dim H^*(B_k T_2). \quad \text{(12)}
\]
Putting (11) and (12) together gives \(\dim H^*(X_{T,k}) = \dim H^*((X_{T,k})^\tau)\), and with Proposition 4.7 this implies that \(X_{T,k}\) is equivariantly formal.

Now given \(n \in \mathbb{N}\), there exists \(k \in \mathbb{N}\) such that \(H^n(X_T) \approx H^n(X_{T,k})\). The following commutative diagram
\[
\begin{array}{ccc}
H^n(X_T) & \xrightarrow{\rho} & H^n(X) \\
\approx \downarrow & & \downarrow \approx \\
H^n(X_{T,k}) & \xrightarrow{\rho_k} & H^n(X)
\end{array}
\]
shows that \(\rho\) is surjective in degree \(n\). This can be done for each \(n\), so \(X_T\) is equivariantly formal.

**5.3 Construction of the ring isomorphism** \(\kappa_T : H^{2*}(X_T) \rightarrow H^*((X^\tau)^\tau)\). By Lemma 2.2 it is equivalent to construct a ring isomorphism
\[
\kappa_T : H^{2*}_T(X) \rightarrow H^{2*}_{T_2}(X^\tau).
\]
By Corollary 2.4 such an isomorphism \(\kappa_{\text{fix}} : H^{2*}_T(X^T) \rightarrow H^{2*}_{T_2}(X^T)^\tau\) exists, since \(X^T\) is a conjugation space. As \((X^T)^\tau = (X^\tau)^{T_2}\) by Lemma 4.2 we may view \(\kappa_{\text{fix}}\) as a map from \(H^{2*}_T(X_T)\) to \(H^{2*}_{T_2}(X^{\tau})^{T_2}\). Consider the following diagram.
\[
\begin{array}{ccc}
H^{2*}_T(X) & \xrightarrow{q} & H^{2*}_T(X^T) \\
\approx \downarrow \kappa_{\text{fix}} & & \downarrow \approx \\
H^{2*}_{T_2}(X^\tau) & \xrightarrow{q^\tau} & H^{2*}_{T_2}(X^{\tau})^{T_2}
\end{array}
\]
By Proposition 4.3 the restriction homomorphisms \(q\) and \(q^\tau\) are injective. Therefore, in order to construct \(\kappa_T : H^{2*}_T(X) \rightarrow H^{2*}_{T_2}(X^\tau)\), it is enough to show...
that $A' = \kappa_{\text{fix}}(A)$, where $A = \text{image}(q)$ and $A' = \text{image}(q'\alpha u)$. The proof is a diagram chase.

Let $N$ be the 1-skeleton of $X$ and let $N^\tau = N \cap X^\tau$. Let $\tilde{N}$ be the disjoint union of all the edges of $X$. There is an obvious quotient map $\tilde{N} \to N$. Let $N_0 \subset N \cap X^T$ be the points of $N$ having more than 1 preimage in $\tilde{N}$, and let $\tilde{N}_0$ be the points of $\tilde{N}$ above $N_0$. Thus $N_0$ is a union of components of $X^T$ and $\tilde{N}_0 \to N_0$ is a disjoint union of trivial coverings. The various inclusions give a morphism of push-out diagrams

$$
\begin{array}{ccc}
N_0 & \longrightarrow & \tilde{N}^T \\
\downarrow & & \downarrow \\
N_0 & \longrightarrow & \tilde{N}
\end{array}
\quad \begin{array}{ccc}
N_0 & \longrightarrow & \tilde{N}_0 \\
\downarrow & & \downarrow \\
N_0 & \longrightarrow & N
\end{array}
$$

(14)

By Hypothesis (a) $X^T$ is a conjugation space. By [HHP, Remark 3.1], $\tau$ preserves each arc-connected component of $X^T$. Therefore, $N_0$ and $\tilde{N}_0$ are conjugation spaces. In the same way, using Hypothesis (b), $\tilde{N}^T$ and $\tilde{N}$ are conjugation spaces. The induced morphism on Mayer-Vietoris sequences, together with the isomorphisms $\kappa$'s and the fact that that $(X^T)^\tau = (X^\tau)^{T_2}$ (by Lemma 4.2) gives a three dimensional commutative diagram:

(15)

The vertical squares commute because of the naturality of the $H^*$-frames of conjugation spaces. The quotient maps $\tilde{N}^T \to X^T$ and $(\tilde{N}^T)^\tau \to (X^T)^\tau$ admit continuous sections, so the homomorphisms $H_2^t(X^T) \to H_2^t(N_0) \oplus H_2^t(\tilde{N}^T)$ and $H_2^t((X^T)^{T_2}) \to H_2^t(N_0^\tau) \oplus H_2^t((\tilde{N}^T)^{T_2})$ are injective and split the Mayer-Vietoris sequences of the back-wall diagram into short exact sequences.

Let $u \in H_2^t(X^\tau)$. We also call $u$ any of its image in Diagram (15), using the various homomorphisms, including the inverses of the $\kappa$'s. As $u = 0$ in $H_2^t(\tilde{N}_0)$, there exists $v \in H_2^t(N)$ with $v = u$ in $H_2^t(N_0) \oplus H_2^t(\tilde{N}^T)$. Using
the injectivity of $H^*_{T_2}((X^\tau)^{T_2}) \to H^*_{T_2}(N^\tau_0) \oplus H^*_{T_2}((\tilde{N}^\tau)^{T_2})$, we get $u = v$ in $H^*_{T_2}((X^\tau)^{T_2})$. By Condition (c) and Lemma 4.10 there exists $w \in H^*_{T_2}(X)$ with $w = u$ in $H^*_{T_2}((X^\tau)^{T_2})$. This proves that $A^r \subseteq \kappa_{\fix}(A)$.

To prove that $\kappa_{\fix}(A) \subseteq A^r$, let $u \in H^*_{T_2}(X)$. By a diagram chase as above, there exists $v \in H^*_{T_2}(N^\tau)$ with $u = v$ in $H^*_{T_2}((X^\tau)^{T_2})$. By Condition (c) and Lemma 4.2 $N^\tau = \text{Sk}_{\tau}^1(X^\tau)$. Using Lemma 4.9 there exists $w \in H^*_{T_2}(X)$ with $w = u$ in $H^*_{T_2}((X^\tau)^{T_2})$. This proves that $\kappa_{\fix}(A) \subseteq A^r$.

Note that the ring homomorphism $\kappa_T : H^{2*}(X_T) \xrightarrow{\cong} H^*((X_T)^\tau)$ that we have constructed satisfies

\[ q^T \circ \kappa_T = \kappa_{\fix} \circ q. \quad (16) \]

5.4 Construction of the ring isomorphism $\kappa : H^{2*}(X) \to H^*(X^\tau)$. As $X$ is $T$-equivariantly formal, Proposition 4.14 tells us that the ordinary mod 2 cohomology $H^{2*}(X)$ can be recovered from the equivariant cohomology: the ring homomorphism $\psi : H^*_T(X) \to H^{2*}(X)$ descends to an isomorphism

\[ H^*_T(X) \otimes_{H^*_T(pt)} \mathbb{Z}_2 \xrightarrow{\cong} H^{2*}(X). \quad (17) \]

As $X^\tau$ is $T_2$-equivariantly formal by (5.1), Proposition 4.4 again tells us that the ring homomorphism $\psi^\tau : H^*_T(X^\tau) \to H^*(X^\tau)$ descends to a graded ring isomorphism

\[ H^*_T(X^\tau) \otimes_{H^*_T(pt)} \mathbb{Z}_2 \xrightarrow{\cong} H^*(X^\tau). \quad (18) \]

By its construction, the ring isomorphism $\kappa_T : H^{2*}_T(X) \xrightarrow{\cong} H^*_T(X^\tau)$ is an isomorphism of modules over the ring isomorphism $H^{2*}_T(pt) \to H^*_T(pt)$. Therefore, it descends to a graded ring isomorphism $\kappa : H^{2*}(X) \xrightarrow{\cong} H^*(X^\tau)$. With this definition, the equation

\[ \psi^\tau \circ \kappa_T = \kappa \circ \psi \]

is satisfied.

5.5 Construction of a section $\sigma_T : H^*(X_T) \to H^*_C(X_T)$ so that $(\kappa_T, \sigma_T)$ is a $H^*$-frame for $(X_T, \tau)$. Let $(\kappa_{\fix}, \sigma_{\fix})$ be the $H^*$-frame for $X^T$. The desired section $\sigma_T$ will fit in the commutative diagram

\[
\begin{array}{ccccccccc}
H^*(X_T) & \xrightarrow{\rho_T} & H^*_C(X_T) & \xrightarrow{\tau_T} & H^*_C((X_T)^\tau) & \cong & H^*((X_T)^\tau)[u] \\
\downarrow q & & \downarrow q_C & & \downarrow q_C & & \downarrow q^T[u] \\
H^*((X^T)_T) & \xrightarrow{\rho_{\fix}} & H^*_C((X^T)_T) & \xrightarrow{\tau_{\fix}} & H^*_C((X^T)_T)^\tau) & \cong & H^*((((X^T)_T)^\tau)[u]
\end{array}
\]

where the vertical arrows are induced by the inclusion $X^T \hookrightarrow X$ (the notations coincide with that of Diagram (13)). We have to justify that the last two vertical arrows are injective. But, under the identifications

\[ H^*((X^\tau)^\tau) = H^*((X^\tau)^{T_2}) = H^*_T(X^\tau) \]

\[ H^*((X^T)_T)^\tau = H^*((X^T)_T)^{T_2} = H^*_T((X^T)_T) \]

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and 
\[ H^*((X^T)_T) = H^*((X^T)_T) = H^*_{T_2}((X^T)_T) = H^*_{T_2}((X^T)_T) \]
the map \(q^*[u]\) coincides with the homomorphism \(H^*_{T_2}(X^T) \to H^*_{T_2}((X^T)_T)\) induced by the inclusion \((X^T)_T \hookrightarrow X^T\).

Note that we just need to construct a section \(\sigma_T: H^*(X_T) \to \mathbb{H}_T^*(X_T)\) such that \(q_c \circ \sigma_T = \sigma_{fix} \circ q\). Indeed, if \(a \in H^{2m}(X_T)\), the conjugation equation for \((\kappa_{fix}, \sigma_{fix})\) implies
\[ q_c \circ \sigma_T(a) = r_{fix} \circ \sigma_{fix} \circ q(a) = \kappa_{fix} \circ q(a) u^m + \ell t_m. \]
As \(q_c^*\) is injective, this implies that
\[ r_T \circ \sigma_T(a) = \tilde{a} u^m + \ell t_m, \]
with \(\tilde{a} \in H^m((X^T)_T)\) satisfying \(q^*(\tilde{a}) = \kappa_{fix}(a)\). By construction of \(\kappa_T\), one has
\[ q^* \circ \kappa_T(a) = \kappa_{fix} \circ q(a). \] As \(q^*\) is injective, this implies that \(\tilde{a} = \kappa_T(a)\). Hence, \((\kappa_T, \sigma_T)\) satisfies the conjugation equation and is therefore a \(H^*\)-frame.

As we just need to construct an additive section \(\sigma_T\), we take the following induction hypothesis \(\mathcal{H}_m: \text{ for } k \leq m, \text{ there exists a section } \sigma_T: H^{2k}(X_T) \to H^{2k}_C(X_T) \text{ of } \rho_T \text{ such that } q_c \circ \sigma_T = \sigma_{fix} \circ q.\) Hypothesis \(\mathcal{H}_0\) is clearly satisfied: we may assume without loss of generality that \(X\) is arc-connected; and we may then define \(\sigma_T(1) = 1, \text{ where } 1 \in H^0(-) \text{ is the unit of } H^*(-).\) Assume by induction that \(\mathcal{H}_{m-1}\) holds. The space \(X_T\) is \(\tau\)-equivariantly formal by (5.2), so there exists a section \(\sigma_0: H^{2m}(X_T) \to H^{2m}_C(X_T) \text{ of } \rho_T.\) We have \(\rho_{fix} \circ q_c \circ \sigma_0 = q.\) Therefore, for any \(a \in H^{2m}(X_T),\) we know that \(q_c \circ \sigma_0(a) = \sigma_{fix} \circ q(a)\) modulo \(\ker \rho_{fix}.\) This kernel is the ideal generated by \(u.\) As \(H^{2m}_C((X^T)_T) = 0,\) only even powers of \(u\) occur and moreover
\[ q_c \circ \sigma_0(a) = \sigma_{fix} \circ q(a) + \sum_{i=0}^m \sigma_{fix}(b_{2m-2i}) u^{2i}, \quad (20) \]
where \(b_{2j}\) are classes in \(H^{2j}((X^T)_T)\) depending on the choice of \(\sigma_0.\) We will modify \(\sigma_0\) by successive steps until \(b_{2j} = 0\) for all \(j = m, m-1, \ldots, 0.\)

The conjugation equation for \((\kappa_{fix}, \sigma_{fix})\) implies
\[ r_{fix} \circ q_c \circ \sigma_0(a) = \kappa_{fix} \circ q(a) u^m + \ell t_m(a) + \sum_{i=0}^m \left( \kappa_{fix}(b_{2m-2i}) u^{m+i} + \ell t_m - i (b_{2m-2i}) \right). \quad (21) \]
As \(q_c^*\) is injective, this implies that
\[ r_T \circ \sigma_0(a) = c_0 u^{2m} + \ell t_m, \quad (22) \]
with \(c_0 \in H^0((X^T)_T)\) satisfying \(q^*(c_0) = \kappa_{fix}(b_0).\) As \(\kappa_T\) is an isomorphism, there exists \(\tilde{c}_0 \in H^0(X_T),\) with \(\kappa_T(\tilde{c}_0) = c_0.\) Define a new section
\[ \sigma_1: H^{2m}(X_T) \to H^{2m}_C(X_T) \]
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of $\rho_T$ by $\sigma_1(a) = \sigma_0(a) + \sigma_T(\tilde{c}_0)u^{2m}$. By induction hypothesis, $q_c \circ \sigma_T(\tilde{c}_0) = \sigma_{\text{fix}} \circ q(\tilde{c}_0)$. By construction of $\kappa_T$, one has $q^T \circ \kappa_T = \kappa_{\text{fix}} \circ q$. Therefore,

$$r_{\text{fix}} \circ q_c \circ \sigma_1(a) = r_{\text{fix}} \circ q_c \circ \sigma_0(a) + r_{\text{fix}}(q_c \circ \sigma_T(\tilde{c}_0))u^{2m}$$

$$= r_{\text{fix}} \circ q_c \circ \sigma_0(a) + r_{\text{fix}}(\sigma_{\text{fix}} \circ q(\tilde{c}_0))u^{2m}$$

$$= r_{\text{fix}} \circ q_c \circ \sigma_0(a) + \kappa_{\text{fix}} \circ q(\tilde{c}_0)u^{2m}$$

$$= r_{\text{fix}} \circ q_c \circ \sigma_0(a) + q^T \circ \kappa_T(\tilde{c}_0)u^{2m}$$

$$= r_{\text{fix}} \circ q_c \circ \sigma_0(a) + q^T(\tilde{c}_0)u^{2m}$$

$$= r_{\text{fix}} \circ q_c \circ \sigma_0(a) + \kappa_{\text{fix}}(b_1)u^{2m}$$

$$= \kappa_{\text{fix}} \circ q(a)u^m + \ell_m(a)$$

$$+ \sum_{i=0}^{m-1} (\kappa_{\text{fix}}(b_{2m-2i})u^{m+i} + \ell_{m-i}(b_{2m-2i})) .$$

The injectivity of $r_{\text{fix}}$ implies that Equation (22) is replaced by

$$q_c \circ \sigma_1(a) = \sigma_{\text{fix}} \circ q(a) + \sum_{i=0}^{m-1} \sigma_{\text{fix}}(b_{2m-2i})u^{2i} , \quad (23)$$

We thus have modified $\sigma_0$ so that $b_0 = 0$. Now, using as above the injectivity of $q^T_c$, this permits us to transform (22) into

$$r_T \circ \sigma_1(a) = c_1 u^{2m-1} + \ell_{m-1} , \quad (24)$$

with $c_1 \in H^1((X_T)^\tau)$ satisfying $q^T(c_1) = \kappa_{\text{fix}}(b_0)$. Again, write $c_1 = \kappa_T(\tilde{c}_1)$ with $\tilde{c}_1 \in H^2(X_T)$ and define a new section $\sigma_2 : H^{2m}(X_T) \to H^{2m}_C(X_T)$ of $\rho_T$ by

$$\sigma_2(a) = \sigma_1(a) + \sigma_T(\tilde{c}_1)u^{2m-2}.$$ Proceeding as above, we prove that if we replace $\sigma_1$ by $\sigma_2$ in (24), the summation index runs only till $m - 2$, i.e. $b_0 = b_2 = 0$. If we keep going this way as long as possible, we get $\sigma_m : H^{2m}(X_T) \to H^{2m}_C(X_T)$ with $b_0 = b_2 = \cdots = b_{2m} = 0$. Extending $\sigma_T$ in degree $2m$ by $\sigma_m$ proves $H_m$ holds. So by induction, we have our section $\sigma_T$.

5.6 Construction of a section $\sigma : H^*(X) \to H^*_C(X)$ so that $(\kappa, \sigma)$ is a $H^*$-frame. The relevant diagram is

$$H^*(X_T) \xrightarrow{\rho_T} H^*_C(X_T) \xrightarrow{r_T} H^*_C((X_T)^\tau)$$

$$H^*(X) \xrightarrow{\rho} H^*_C(X) \xrightarrow{r} H^*_C(X^\tau)$$

Being an even-cohomology space, $X$ is $T$-equivariantly formal. We can thus choose an additive section $s : H^*(X) \to H^*(X_T)$ of $\psi$ and define $\sigma : H^*(X) \to H^*_C(X)$ by $\sigma = \psi_C \circ \sigma_T \circ s$. The linear map $\sigma$ is an additive section of $\rho$ and, for
a ∈ H^2m(X), we have

\[ r \circ \sigma (a) = r \circ \psi_C \circ \sigma_T \circ s(a) \]
\[ = \psi_C \circ r_T \circ \sigma_T \circ s(a) \]
\[ = \psi_C (\kappa_T \circ s(a) u^m + \ell t_m) \]
\[ = (\psi^T \circ \kappa_T) \circ s(a) u^m + \ell t_m \]
\[ = (\kappa \circ \psi) \circ s(a) u^m + \ell t_m \]
\[ = \kappa(a) u^m + \ell t_m. \]

From the fourth to the fifth line, we have used that ψ^T T = T S, as noted in [19]. Therefore, the conjugation equation is satisfied and (κ, σ) is a H∗-frame for X.

With this, the proof of Theorem 3.1 is now complete.

Proof of Corollary 3.2 The hypotheses imply that the restriction of τ to an edge E, which is a 2-sphere, is conjugate to a reflection (through an equatorial plane). We leave to the reader the details of a proof that we summarize in three steps: (1) by an elementary argument, one shows that τ has 2 fixed points on each non-trivial T-orbit; this implies that E^τ is a circle; (2) by the Schönflies theorem, there is a homeomorphism from E to S^2 sending E^μ to a great circle; (3) by the Alexander trick, the resulting involution on S^2 is conjugate to a reflection. This implies that each edge is a conjugation 2-sphere in the sense of [HHP Example 3.6]. Hence, each edge is a conjugation space and the hypotheses of Theorem 3.1 are satisfied.

Proof of Corollary 3.3 By [HHP] Remark 3.1, τ preserves each arc-connected component of X^T. In consequence, for each edge E of X, Hypothesis (a) of Corollary 3.3 implies that E^T is a conjugation space. By Theorem 2.1, each edge is then a conjugation space. The hypotheses of Theorem 3.1 are therefore satisfied.

6 Remarks

6.1 The following example shows that the condition Sk^T_0(X) = Sk^T_2(X) does not imply that Sk^T_1(X) = Sk^T_2(X), even for spaces like those occurring in Corollary 3.2 or 3.3. We consider the Hamiltonian action of S^1 on S^2 ⊂ C × R given by g · (x, t) = (gz, t), compatible with the involution (z, t)^T = (z, t). Points of S^2 will be denoted by x, y, etc. Let p_± = (0, ±1) be the north and south poles. Let T = S^1 × S^1 acting on X = S^2 × S^2 by

\[ (g, h) \cdot (x, y) = (gh \cdot x, gh^{-1} \cdot y). \]
The fixed point sets for $T$ and $T_2$ are equal: $\text{Sk}^T_0(X) = \text{Sk}^{T_2}_0(X)$, consisting of the four points $\{p_\pm\} \times \{p_\pm\}$. By Proposition 4.7, $X$ is $T$-equivariantly formal and $X^\tau$ is $T_2$-equivariantly formal.

The $T$-equivariant 1-skeleton is a graph of four 2-spheres

$$\text{Sk}^T_1(X) = \{(x, y) \mid x = p_\pm \text{ or } y = p_\pm\}.$$  

Therefore, $X$ is a GKM-space. But $\text{Sk}^T_1(X) \neq \text{Sk}^{T_2}_1(X)$ since $\text{Sk}^{T_2}_1(X) = X$. Also, $\text{Sk}^T_1(X)^\tau \neq \text{Sk}^{T_2}_1(X^\tau)$ since $\text{Sk}^{T_2}_1(X^\tau) = X^\tau$.

6.2 The condition $\text{Sk}^T_i(X) = \text{Sk}^{T_2}_i(X)$ for $i = 0, 1$ of our main theorems is already implicitly present in earlier papers [Sd, BGH] which are dealing with GKM Hamiltonian manifolds. In [Sd], one requires that for each point of $x \in X^T$, the characters involved in the 2-spheres adjacent to $x$ are pairwise independent over $\mathbb{Z}_2$. In [BGH, p. 373], one asks that $X^T = X^{T_2}$ and that “the real locus of the one-skeleton is the same as the one-skeleton of the real locus”. In general, these conditions are weaker than $\text{Sk}^T_i(X) = \text{Sk}^{T_2}_i(X)$ for $i = 0, 1$ (see Lemma 4.2) but, they are equivalent for a GKM Hamiltonian manifold. To see this, work with the local normal around a $T$-fixed point; in this model the $T$-action and the involution are linear.

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