In absence of long chordless cycles, large tree-width becomes a local phenomenon

Daniel Weißauer

Abstract

We prove that, for all $\ell$ and $s$, every graph of sufficiently large tree-width contains either a complete bipartite graph $K_{s,s}$ or a chordless cycle of length greater than $\ell$.

1 Introduction

In an effort to make the statement in the title precise, let us call a graph parameter $P$ global if there is a constant $c$ such that for all $k$ and $r$ there exists a graph $G$ for which every subgraph $H$ of order at most $r$ satisfies $P(H) < c$, while $P(G) > k$. The intention here is that $P$ being small, even bounded by a constant, on subgraphs of bounded order does not provide a bound on $P(G)$.

Tree-width is a global parameter (we may take $c = 2$), as is the chromatic number (with $c = 3$). Indeed, it is a classic result of Erdős [6] that for all $k$ and $r$ there exists a graph of chromatic number $> k$ for which every subgraph on at most $r$ vertices is a forest.

It is well-known (see [4]) that the situation changes when we restrict ourselves to chordal graphs, graphs without chordless cycles of length $\geq 4$:

\[
\forall k : \text{Every } K_{k+1}\text{-free chordal graph has tree-width } < k. \quad (1)
\]

Hence the only obstruction for a chordal graph to have small tree-width is the presence of a large clique. Since the chromatic number of a graph is at most its tree-width plus one ([4]), the same is true for the chromatic number. In particular, tree-width and chromatic number are local parameters for the class of chordal graphs.

In 1985, Gyárfás [8] made a famous conjecture which implies that chromatic number is a local parameter for the larger class of $\ell$-chordal graphs, those which have no chordless cycle of length $> \ell$:

\[
\forall \ell, r \exists k : \text{Every } K_r\text{-free } \ell\text{-chordal graph is } k\text{-colourable.} \quad (2)
\]

\[\text{Indeed, in terms of our earlier definition, (2) implies that given any integer } c, \text{ there exists a } k \text{ such that every } \ell\text{-chordal graph of chromatic number } > k \text{ has a subgraph of order } \leq c \text{ and chromatic number } \geq c.\]
This conjecture remained unresolved for 30 years and was proved only recently by Chudnovsky, Scott and Seymour [3]. In view of (1), it is tempting to think that an analogue of (2) might hold with tree-width in place of chromatic number. Complete bipartite graphs, however, are examples of triangle-free 4-chordal graphs of large tree-width. Therefore a verbatim analogue of (2) is not possible and any graph whose presence we can hope to force by assuming \( \ell \)-chordality and large tree-width will be bipartite.

On the positive side, Bodlaender and Thilikos [2] showed that every star can be forced as a subgraph in \( \ell \)-chordal graphs by assuming large tree-width (see Section 3). However, since stars have tree-width 1, this does not establish locality of tree-width in the sense of our earlier definition. Our main result is that in fact any bipartite graph can be forced as a subgraph:

**Theorem 1.** Let \( \ell \geq 4 \) be an integer and \( F \) a graph. Then \( F \) is bipartite if and only if there exists an integer \( k \) such that every \( \ell \)-chordal graph of tree-width \( \geq k \) contains \( F \) as a subgraph.

This shows that tree-width is local for \( \ell \)-chordal graphs: Given any integer \( c \), there exists an integer \( k \) such that every \( \ell \)-chordal graph of tree-width \( \geq k \) has a subgraph isomorphic to \( K_{c,c} \), which has order \( 2c \) and tree-width \( c \).

Theorem 1 also has an immediate application to an Erdős-Pósa type problem. Kim and Kwon [9] showed that chordless cycles of length \( > 3 \) have the Erdős-Pósa property:

**Theorem 2 ([9]).** For every integer \( k \) there exists an integer \( m \) such that every graph \( G \) either contains \( k \) vertex-disjoint chordless cycles of length \( > 3 \) or a set \( X \) of at most \( m \) vertices such that \( G - X \) is chordal.

They also constructed, for every integer \( \ell \geq 4 \), a family of graphs showing that the analogue of Theorem 2 for chordless cycles of length \( > \ell \) fails. We complement their negative result by proving that the Erdős-Pósa property does hold when restricting the host graphs to graphs not containing \( K_{s,s} \) as a subgraph.

**Corollary 3.** For all \( \ell, s \) and \( k \) there exists an integer \( m \) such that every \( K_{s,s} \)-free graph \( G \) either contains \( k \) vertex-disjoint chordless cycles of length \( > \ell \) or a set \( X \) of at most \( m \) vertices such that \( G - X \) is \( \ell \)-chordal.

The paper is organised as follows. Section 2 contains some basic definitions. Theorem 1, our main result, is proved in Section 3. In Section 4 we formally introduce the Erdős-Pósa property, restate Corollary 3 in that language and give a proof thereof. Section 5 closes with some open problems.

## 2 Notation and definitions

All graphs considered here are finite and undirected and contain neither loops nor parallel edges. Our notation and terminology mostly follow that of [4].

For two graphs \( G \) and \( H \), we say that \( G \) is \( H \)-free if \( G \) does not contain a subgraph isomorphic to \( H \). Given a tree \( T \) and \( s,t \in T \), we write \( sTt \) for the
unique s-t-path in T. Given a graph G and a set X of vertices of G, a path P ⊆ G is an X-path if it contains at least one edge and meets X precisely in its endvertices. A separation of G is a tuple (A, B) with V = A ∪ B such that there are no edges between A \ B and B \ A. The order of (A, B) is the number of vertices in A ∩ B. We call the separation (A, B) tight if for all x, y ∈ A ∩ B, both G[A] and G[B] contain an x-y-path with no internal vertices in A ∩ B.

Given an integer k, a set X of at least k vertices of G is a k-block if it is inclusion-maximal with the property that for every separation (A, B) of order < k, either X ⊆ A or X ⊆ B. By Menger’s Theorem, G then contains k internally disjoint paths between any two non-adjacent vertices in X.

A tree-decomposition of G is a pair (T, V), where T is a tree and V = (V_t)_{t ∈ T} a family of sets of vertices of G such that for every v ∈ V(G), the set of t ∈ T with v ∈ V_t induces a non-empty subtree of T and for every edge vw ∈ E(G) there is a t ∈ T with v, w ∈ V_t. If (T, V) is a tree-decomposition of G, then every st ∈ E(T) induces a separation (G^s_t, G^t_s) of G, where G^y_t is the union of V_y for all u ∈ T for which y ∈ uTx. Note that G^s_t ∩ G^t_s = V_t ∩ V_s. We call (T, V) tight if every separation induced by an edge of T is tight.

Given t ∈ T, the torso at t is the graph obtained from G[V_t] by adding, for every neighbor s of t, an edge between any two non-adjacent vertices in V_s ∩ V_t.

Given graphs G and H, a subdivision of H in G consists of an injective map η : V(H) → V(G) and a map P which assigns to every edge xy ∈ E(H) an η(x)-η(y)-path P_{xy} ⊆ G so that the paths (P_{xy} : xy ∈ E(H)) are internally disjoint and no P_{xy} has an internal vertex in X := η(V(H)). The vertices in X are called branchvertices. For an integer r, the subdivision is a (≤ r)-subdivision if every path P_{xy} has length at most r. When H is a complete graph, the map η is irrelevant and we only keep track of the set X of branchvertices and the family (P_{xy} : x, y ∈ X).

## 3 Proof of Theorem 1

As observed in the introduction, the complete bipartite graphs K_{s,s} show that no bound on the tree-width of F-free ℓ-chordal graphs exists if F is not bipartite. We now prove that F being bipartite is sufficient. Since every bipartite graph is a subgraph of some K_{s,s}, it suffices to prove Theorem 1 for the case F = K_{s,s}.

Our proof is a cascade with three steps. First, we show that sufficiently large tree-width forces the presence of a k-block.

**Lemma 4.** Let ℓ, k and t ≥ 2(ℓ − 2)(k − 1)^2 be positive integers. Then every ℓ-chordal graph of tree-width ≥ t contains a k-block.

We then prove that the existence of a k-block yields a bounded-length subdivision of a complete graph.

**Lemma 5.** Let ℓ, m and k ≥ 5m^2ℓ/4 be positive integers. Then every ℓ-chordal graph that contains a k-block contains a (≤ 2ℓ − 3)-subdivision of K_m.
In the last step, we show that such a bounded-length subdivision gives rise to a copy of $K_{s,s}$.

**Lemma 6.** For all integers $\ell$ and $s$ there exists a $q > 0$ such that the following holds. Let $m, r$ be positive integers with $m \geq qr$. Then every $\ell$-chordal graph that contains a $(\leq r)$-subdivision of $K_m$ contains $K_{s,s}$ as a subgraph.

It is immediate that Theorem 1 follows once we have established these three lemmas.

### 3.1 Proof of Lemma 4

A trivial obstacle to our search for a copy of $K_{s,s}$ is the absence of vertices of high degree. Bodlaender and Thilikos [2] showed, however, that $\ell$-chordal graphs of bounded degree have bounded tree-width. Their exponential bound was later improved by Kosowski, Li, Nisse and Suchan [10] and by Seymour [17].

**Theorem 7** ([17]). Let $\ell$ and $\Delta$ be positive integers and $G$ a graph. If $G$ is $\ell$-chordal and has no vertices of degree greater than $\Delta$, then the tree-width of $G$ is at most $(\ell - 2)(\Delta - 1) + 1$.

By demanding large tree-width, we can therefore guarantee a large number of vertices of high degree. We now show that these are not all just scattered about the graph. It was shown by the author in [19] that either there is a $k$-block or there is a tree-decomposition which separates the set of vertices of high degree into small pieces. This also follows, without explicit bounds, from a far more general result of Dvořák [5].

**Theorem 8** ([19]). Let $k \geq 3$ be a positive integer and $G$ a graph. If $G$ has no $k$-block, then there is a tight tree-decomposition $(T, V)$ of $G$ such that every torso has fewer than $2k$ vertices of degree at least $2(k - 1)(k - 2)$.\)

In fact, tightness of the tree-decomposition is not explicit in [19, Theorem 1], but is established in the proof as *Lemma 6*.

Now let $\ell, k$ and $t \geq 2(\ell - 2)(k - 1)^2$ be positive integers. Let $G$ be an $\ell$-chordal graph with no $k$-block. For $k = 2$, this means that $G$ is acyclic and therefore has tree-width 1. Suppose from now on that $k \geq 3$. We show that the tree-width of $G$ is less than $t$.

By Theorem 8, there is a tight tree-decomposition $(T, V)$ of $G$ such that every torso has fewer than $k$ vertices of degree at least $2(k - 1)(k - 2)$. Let $t \in T$ arbitrary, let $N$ be the set of neighbors of $t$ in $T$ and let $H$ be the torso at $t$. We claim that $H$ is $\ell$-chordal.

Let $C \subseteq H$ be a chordless cycle. For every edge $xy \in E(C) \setminus E(G)$, there is some $s \in N$ with $x, y \in V_s \cap V_t$. Since $(T, V)$ is tight, there exists an $x-y$-path $P_{xy}$ in $G'$ which meets $V_t$ only in its endpoints. Observe that for every $s \in N$, $C$ contains at most two vertices of $V_s$ and these are adjacent in $C$. Hence we can replace every edge $xy \in E(C) \setminus E(G)$ by $P_{xy}$ and obtain a chordless cycle $C'$ of $G$ with $|C'| \geq |C|$. Since $G$ is $\ell$-chordal, it follows that $|C| \leq \ell$. This proves our claim.
Now, let $A \subseteq V(H)$ be the set of all vertices of degree $\geq d$ in $H$. Then $H - A$ is $\ell$-chordal and has no vertices of degree $> d - 1$. By Theorem 7, the tree-width of $H - A$ is at most $(\ell - 2)(d - 2) + 1$. Therefore

$$\text{tw}(H) \leq |A| + \text{tw}(H - A) \leq k + (\ell - 2)(d - 2) < t.$$ 

We have shown that every torso has tree-width $< t$. We can then take a tree-decomposition of width $< t$ of each torso and combine all these to a tree-decomposition of width $< t$ of $G$. \qed

### 3.2 Proof of Lemma 5

In general, the presence of a $k$-block does not guarantee the existence of any subdivision of $K_m$ for $m \geq 5$. For example, take a rectangular $k^2 \times k$-grid, add $2(k + 1)$ new vertices to the outer face and make each of these adjacent to $k$ consecutive vertices on the perimeter of the grid (see Figure 3.2). These new vertices are then a $k$-block in the resulting planar graph.

![Figure 1: A planar graph with a 9-block](image)

Our aim in this section is to show that for $\ell$-chordal graphs, sufficiently large blocks do indeed yield bounded-length subdivisions of complete graphs.

Let $\ell, m$ and $k \geq 5m^2\ell/4$ be positive integers. Let $G$ be an $\ell$-chordal graph and $X \subseteq V(G)$ a $k$-block of $G$. Let $L := 2\ell - 3$. Assume for a contradiction that $G$ contained no $(\leq L)$-subdivision of $K_m$. Let $x, y \in X$ non-adjacent. Then $G$ contains a set $\mathcal{P}^{xy}$ of $k$ internally disjoint $x$-$y$-paths. Taking sub-paths, if necessary, we may assume that each path in $\mathcal{P}^{xy}$ is induced. Let $p_0 := m + m^2(\ell - 2)$.

**Claim:** Fewer than $p_0$ paths in $\mathcal{P}^{xy}$ have length $> \ell/2$.

**Proof of Claim.** Let $\mathcal{P}_0$ be the set of all paths in $\mathcal{P}^{xy}$ of length $> \ell/2$ and $p := |\mathcal{P}_0|$. Assume for a contradiction that $p \geq p_0$. Let $P, Q \in \mathcal{P}_0$. Then $P \cup Q$ is a cycle of length $> \ell$. Since $G$ is $\ell$-chordal, $P \cup Q$ has a chord. This chord must join an internal vertex of $P$ to an internal vertex of $Q$. Choose such vertices $v_P^Q \in P$ and $v_Q^P \in Q$ so that the cycle $D := Px_P^Qv_P^Qv_Q^Qx$ has minimum length. Note that $D$ is an induced cycle and therefore has length at most $\ell$. In particular, the segment of $P$ joining $x$ to $v_P^Q$ has length at most $\ell - 2$ and similarly for $Q$ and $v_Q^P$. 

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For $P \in \mathcal{P}_0$, let $P'$ be a minimal subpath of $P$ containing every vertex $v_P^Q$, $Q \in \mathcal{P}_0 \setminus \{P\}$. Then $\mathcal{P} := \{P': P \in \mathcal{P}_0\}$ is a family of $p$ disjoint paths, each of length at most $\ell - 3$, and $G$ contains an edge between any two of them. Fix an arbitrary $Q \subseteq \mathcal{P}$ with $|Q| = m$. Since $p \geq p_0$, every $Q \in \mathcal{Q}$ contains a vertex $u_Q$ which has neighbors on at least $m^2$ different paths in $\mathcal{P} \setminus Q$.

Let $U := \{u_Q: Q \in \mathcal{Q}\}$. We iteratively construct a $(\leq L)$-subdivision of $K_m$ with branchvertices in $U$. Let $t := \binom{m}{2}$ and enumerate the pairs of vertices of $U$ arbitrarily as $e_1, \ldots, e_t$. In the $j$-th step, we assume that we have constructed a family $\mathcal{R}_j = (R_i)_{i < j}$ of internally disjoint $U$-paths of length at most $L$, so that $R_i$ joins the vertices of $e_i$ and meets at most two paths in $\mathcal{P} \setminus Q$. We now find a suitable path $R_j$.

Let $Q^1, Q^2 \in \mathcal{Q}$ with $e_j = u_{Q^1}u_{Q^2}$. At most $2(j - 1) < m^2$ paths in $\mathcal{P} \setminus Q$ meet any of the paths in $\mathcal{R}_j$. Since $u_{Q^i}$ is adjacent to vertices on at least $m^2$ different paths in $\mathcal{P} \setminus Q$, there is a $P^1 \in \mathcal{P} \setminus Q$ which is disjoint from every $R_i$, $i < j$, and contains a neighbor of $u_{Q^1}$. We similarly find a path $P^2 \in \mathcal{P} \setminus Q$ for $u_{Q^2}$. Since either $P^1 = P^2$ or $G$ has an edge between $P^1$ and $P^2$, $P^1 \cup P^2 \cup \{u_{Q^1}, u_{Q^2}\}$ induces a connected subgraph of $G$ and therefore contains a $u_{Q^1}u_{Q^2}$-path $R_j$ of length at most $L$, which meets only two paths in $\mathcal{P} \setminus Q$.

Proceeding like this, we find the desired subdivision of $K_m$ after $t$ steps. This contradiction finishes the proof of the claim.

Let $Y \subseteq X$ with $|Y| = m$. For any two non-adjacent $x, y \in Y$, let $Q^{xy} \subseteq \mathcal{P}^{xy}$ be the set of all $P \in \mathcal{P}^{xy}$ of length at most $\ell/2$ which have no internal vertices in $Y$. By the claim above, we have

$$|Q^{xy}| > k - p_0 - (m - 2) \geq \binom{m}{2}\frac{\ell}{2}.$$

Pick one path $P \in Q^{xy}$ for each pair of non-adjacent vertices $x, y \in Y$ in turn, disjoint from all previously chosen paths. Since $|Q^{xy}| \geq \binom{m}{2}\frac{\ell}{2}$ and each path only has at most $\ell/2 - 1$ internal vertices which future paths need to avoid, we can always find a suitable such path $P$. Together with all edges between adjacent vertices of $Y$, this yields a $(\leq \ell/2)$-subdivision of $K_m$ in $G$ with branchvertices in $Y$.

We would like to point out that a modification of the above argument can be used to produce a $(\leq \ell/2)$-subdivision of $K_m$ if $k$ is significantly larger.

Indeed, suppose we find a family $\mathcal{P}$ of $p$ disjoint paths, each of length at most $\ell - 3$, such that $G$ contains an edge between any two of them. Then the subgraph $H$ induced by $\bigcup_{P \in \mathcal{P}} V(P)$ has at most $(\ell - 2)p$ vertices and at least $\binom{p}{2}$ edges. One can then use a classic result of Kővari, Sós and Turán [11] to show that $H$ contains a copy of $K_{m,m^2}$ if $p$ is sufficiently large. Since $K_{m,m^2}$ contains a $(\leq 2)$-subdivision of $K_m$, this establishes an upper bound on the number of paths of length $> \ell/2$ in any $\mathcal{P}^{xy}$. The rest of the proof remains the same.
3.3 Proof of Lemma 6

The combination of Lemma 4 and Lemma 5 already establishes that tree-width is a local parameter for $\ell$-chordal graphs. The purpose of Lemma 6 is merely to narrow the set of bounded-order obstructions down as far as possible. We will use the following theorem of Kühn and Osthus [13].

**Theorem 9** ([13]). For every integer $s$ and every graph $H$ there exists a $d$ so that every graph with average degree at least $d$ either contains $K_{s,s}$ as a subgraph or contains an induced subdivision of $H$.

In fact, we only need the special case $H = C_{\ell+1}$. This special case has a simpler proof which can be found in Kühn’s PhD-thesis [12]. Fix an integer $d$ so that every $\ell$-chordal graph of average degree at least $d$ contains $K_{s,s}$ as a subgraph. We prove the assertion of Lemma 6 with $q := d^2\frac{\ell^4}{4(\ell-3)!}$.

Let $m, r$ be positive integers with $m \geq qr$ and let $G$ be an $\ell$-chordal graph containing a $(\leq r)$-subdivision of $K_{m}$. Let $X$ be the set of branchvertices and $(P_{xy}: x, y \in X)$ the family of paths of the subdivision. Taking subpaths, if necessary, we may assume that every path is induced.

Assume for a contradiction that $G$ contained no copy of $K_{s,s}$. By Theorem 9, every subgraph of $G$ contains a vertex of degree $< d$. In particular, there is an independent set $Y \subseteq X$ with $|Y| \geq m/d$. Let $H$ be the subgraph of $G$ induced by $\bigcup_{x, y \in Y} V(P_{xy})$. Note that $|H| \leq r\binom{|Y|}{2}$.

Call an edge of $H$ red if it joins a vertex $x \in Y$ to an internal vertex of a path $P_{yz}$ with $x \notin \{y, z\}$. Call an edge of $H$ blue if it joins an internal vertex of a path $P_{wx}$ to an internal vertex of a path $P_{yz}$ with $\{w, x\} \neq \{y, z\}$. We will show that $H$ must contain many edges which are either red or blue, so that the average degree of $H$ is at least $d$.

Fix an arbitrary cycle $R$ with $V(R) = Y$. For any $Z \subseteq Y$ with $|Z| = \ell$, obtain the cycle $R_Z$ with $V(R_Z) = Z$ by contracting every $Z$-path of $R$ to a single edge. We then get a cycle $C_Z \subseteq H$ by replacing every edge $xy \in R_Z$ with the path $P_{xy}$. Since each path $P_{xy}$ has length at least 2 and $H$ is $\ell$-chordal, the cycle $C_Z$ must have a chord. Since $Y$ is independent and every path $P_{xy}$ is induced, the chord must be a red or blue edge of $H$.

Consider a red edge $xv \in E(H)$ with $x \in Y$, $v \in P_{yz}$ and $x \notin \{y, z\}$. If this edge is a chord for a cycle $C_Z$, then $\{x, y, z\} \subseteq Z$. Hence it can only occur as a chord for at most

$$\left(\frac{|Y| - 3}{\ell - 3}\right) \leq \frac{|Y|^{\ell-3}}{(\ell-3)!}$$

choices of $Z$. Similarly, every blue edge $uv \in E(H)$ with $u \in P_{wx}$, $v \in P_{yz}$ and $\{w, x\} \neq \{y, z\}$ can only be a chord of $C_Z$ if $\{w, x, y, z\} \subseteq Z$. This also happens for at most

$$\left(\frac{|Y| - 3}{\ell - 3}\right) \leq \frac{|Y|^{\ell-3}}{(\ell-3)!}$$

choices of $Z$. Let $f$ be the number of edges of $H$ which are either red or blue.

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Since every $Z \subseteq Y$ with $|Z| = \ell$ gives rise to a chord, it follows that

$$\frac{|Y|^{\ell}}{\ell^\ell} \leq \left(\frac{|Y|}{\ell}\right)^{\ell-3} f\left(\frac{|Y|^{\ell-3}}{(\ell-3)!}\right).$$

This shows that the average degree of $H$ is

$$d(H) \geq \frac{2f}{|H|} \geq \frac{4(\ell-3)!}{r\ell^\ell}|Y| \geq d.$$

By Theorem \ref{thm:erdos-posa}, $H$ contains a copy of $K_{s,s}$.

\section{Erdős-Pósa for long chordless cycles}

A classic theorem of Erdős and Pósa \cite{erdos1962} asserts that for every integer $k$ there is an integer $r$ such that every graph either contains $k$ disjoint cycles or a set of at most $r$ vertices meeting every cycle. This result has been the starting point for an extensive line of research, see the survey by Raymond and Thilikos \cite{raymond2010}.

Let $\mathcal{F}, \mathcal{G}$ be classes of graphs and $\subseteq$ a containment relation between graphs. We say that $\mathcal{F}$ has the Erdős-Pósa property for $\mathcal{G}$ with respect to $\subseteq$ if there exists a function $f$ such that for every $G \in \mathcal{G}$ and every integer $k$, either there are disjoint $Z_1, \ldots, Z_k \subseteq V(G)$ such that for every $1 \leq i \leq k$ there is an $F_i \in \mathcal{F}$ with $F_i \subseteq G[Z_i]$, or there is a $X \subseteq V(G)$ with $|X| \leq f(k)$ such that $F \nsubseteq G - X$ for every $F \in \mathcal{F}$. When $\mathcal{G}$ is the class of all graphs, we simply say that $\mathcal{F}$ has the Erdős-Pósa property with respect to $\subseteq$. We write $F \subseteq G$ if $F$ is isomorphic to a subgraph of $G$ and $F \subseteq_i G$ if $F$ is isomorphic to an induced subgraph of $G$.

The theorem of Erdős and Pósa then asserts that the class of cycles has the Erdős-Pósa property with respect to $\subseteq$. This implies that cycles also have the Erdős-Pósa property with respect to $\subseteq_i$. It is known that for every $\ell$, the class of cycles of length $> \ell$ has the Erdős-Pósa property with respect to $\subseteq_i$, see \cite{kim2005, kim2006, kim2014}. Recently, Kim and Kwon \cite{kim2017} proved that cycles of length $> 3$ possess the Erdős-Pósa property with respect to $\subseteq_i$:

**Theorem 10** (\cite{kim2017}). There exists a constant $c$ such that for every integer $k$, every graph $G$ either contains $k$ vertex-disjoint chordless cycles of length $> 3$ or a set $X$ of at most $ck^2 \log k$ vertices such that $G - X$ is chordal.

In contrast, Kim and Kwon \cite{kim2017} showed that, for any given $\ell \geq 4$, cycles of length $> \ell$ do not have the Erdős-Pósa property with respect to $\subseteq_i$. For any given $n$, they constructed a graph $G_n$ with no two disjoint chordless cycles of length $> \ell$, for which no set of fewer than $n$ vertices meets every chordless cycle of length $> \ell$ in $G_n$. This graph $G_n$ contains a copy of $K_{n,n}$. We show that this is essentially necessary:

**Corollary 11.** For all integers $\ell$ and $s$, the class of cycles of length $> \ell$ has the Erdős-Pósa property for the class of $K_{s,s}$-free graphs with respect to $\subseteq_i$. 

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This follows from Theorem 1 by a standard argument. Since the proof is quite short, we provide it for the sake of completeness. First, recall the following consequence of the Grid Minor Theorem of Robertson and Seymour [16].

**Theorem 12 ([16]).** For all positive integers \( p \) and \( q \) there exists an \( r \) such that for every graph \( G \) with tree-width \( \geq r \), there are disjoint \( Z_1, \ldots, Z_p \subseteq V(G) \) such that \( G[Z_i] \) has tree-width \( \geq q \) for every \( 1 \leq i \leq p \).

**Proof of Corollary 11.** Let \( k \) be an integer. By Theorem 1 there exists an integer \( t \) such that every \( \ell \)-chordal graph with tree-width \( \geq t \) contains \( K_{s,s} \). By Theorem 12 there exists an \( r \) such that every graph with tree-width \( > r \) has \( k \) vertex-disjoint subgraphs of tree-width \( \geq t \).

Let \( G \) be a \( K_{s,s} \)-free graph. We show that either \( G \) contains \( k \) disjoint chordless cycles of length \( > \ell \) or there is a set of at most \( r(k-1) \) vertices whose deletion leaves an \( \ell \)-chordal graph.

Suppose first that the tree-width of \( G \) was greater than \( r \). Let \( Z_1, \ldots, Z_k \) be disjoint sets of vertices such that \( G[Z_i] \) has tree-width \( \geq t \) for every \( i \). Then, by Theorem 1 every \( G[Z_i] \) must contain a chordless cycle of length \( > \ell \), since \( K_{s,s} \not\subseteq G[Z_i] \). Therefore \( G \) contains \( k \) disjoint chordless cycles of length \( > \ell \).

Suppose now that \( G \) had a tree-decomposition \( (T, V) \) of width \( < r \). For every chordless cycle \( C \subseteq G \) of length \( > \ell \), let \( T_C \subseteq T \) be the subtree of all \( t \in T \) with \( V_t \cap V(C) \neq \emptyset \). If there are \( k \) disjoint such subtrees \( T_{C^1}, \ldots, T_{C^k} \), then \( C^1, \ldots, C^k \) are also disjoint and we are done. Otherwise, there exists \( S \subseteq V(T) \) with \( |S| < k \) which meets every subtree \( T_C \). Then \( Z := \bigcup_{s \in S} V_s \) meets every chordless cycle of length \( > \ell \) in \( G \) and \( |Z| \leq r(k-1) \).

\[ \square \]

5 Open problems

A large amount of research is dedicated to the study of \( \chi \)-boundedness of graph classes, introduced by Gyárfás [8]. Here, a class \( G \) of graphs is called \( \chi \)-bounded if there exists a function \( f \) so that for every integer \( k \) and \( G \in G \), either \( G \) contains a clique on \( k+1 \) vertices or \( G \) is \( f(k) \)-colourable. This is a strengthening of the statement that chromatic number is a local parameter for \( G \), with cliques being the only bounded-order subgraphs to look for.

As we have seen, cliques are not the only reasonable local obstruction to having small tree-width. Nonetheless, we may still ask

1. For which classes of graphs is tree-width a local parameter?
2. What kind of bounded-order subgraphs can we force on these classes?
3. For which classes can we force large cliques by assuming large tree-width?

We have seen in Section 4 that long chordless cycles have the Erdős-Pósa property for the class of \( K_{s,s} \)-free graphs. For which other classes is this true? Kim and Kwon [9] raised this question for the class of graphs without chordless cycles of length four.
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