Metastability in a Stochastic System of Spiking Neurons with Leakage

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Abstract
We consider a finite system of interacting point processes with memory of variable length modeling a finite but large network of spiking neurons with two different leakage mechanisms. Associated to each neuron there are two point processes, describing its successive spiking and leakage times. For each neuron, the rate of the spiking point process is an exponential function of its membrane potential, with the restriction that the rate takes the value 0 when the membrane potential is 0. At each spiking time, the membrane potential of the neuron resets to 0, and simultaneously, the membrane potentials of the other neurons increase by one unit. The leakage can be modeled in two different ways. In the first way, at each occurrence time of the leakage point process associated to a neuron, the membrane potential of that neuron is reset to 0, with no effect on the other neurons. In the second way, if the membrane potential of the neuron is strictly positive, at each occurrence time of the leakage point process associated to that neuron, its membrane potential decreases by one unit, with no effect on the other neurons. In both cases, the leakage point process of the neurons has constant rate. For both models, we prove that the system has a metastable behavior as the population size diverges. This means that the time at which the system gets trapped by the list of null membrane potentials suitably re-scaled converges to a mean one exponential random time.

Keywords Neuronal networks · Interacting point processes with memory of variable length · Metastability

Mathematics Subject Classification 60K35 · 60G55 · 82C22

1 Introduction
We study a system of interacting point processes with memory of variable length modeling a finite but large network of spiking neurons with two different ways to model the leakage effect. We prove that when the population size diverges the system has a metastable behavior.

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The system we consider can be informally described as follows. Each neuron is associated to two point processes. The first point process indicates the successive spiking times of the neuron. The rate of this point process is an exponential function of the membrane potential of the neuron, with the restriction that the rate takes the value 0 when the membrane potential is 0. When a neuron spikes, its membrane potential resets to 0, and simultaneously, the membrane potentials of the other neurons increase by one unit.

The second point process associated to each neuron indicates its successive leakage times. The leakage effect can be modeled in two different ways. In the first way, at each leakage time of the neuron, its membrane potential is reset to 0, with no effect on the other neurons membrane potentials. In the second way, if the membrane potential of the neuron is strictly positive, at each leakage time of this neuron, its membrane potential decreases by one unit, with no effect on the other neurons membrane potentials. For both models, this point process has a fixed constant rate.

The first way to model the leakage effect was considered in [1] with the important difference that, besides considering only binary spiking rates, it also considers that the set of neurons is represented by the set of all integers, with each neuron interacting only with its two neighbors.

The second way to model the leakage effect was considered in [2] with the important difference that, besides considering only binary spiking rates, it also only considers interaction graphs that are regular trees. We thank an anonymous reviewer for suggesting to consider also the second way to model the leakage effect.

Let us now informally present our results. For any initial configuration of membrane potentials, the number of spiking and leakage times of the system is finite. Moreover, the process gets trapped after a finite time in the configuration in which the membrane potentials of all neurons are 0. This is the content of Theorem 1 for the system with the first leakage mechanism and the content of Theorem 4 for the system with the second leakage mechanism.

Let us suppose that the system starts with a configuration in which a sufficiently large set of neurons have strictly positive membrane potential. With such a starting point, as the number of neurons of the system diverges, the system instantaneously reaches a set of configurations in which all neurons but one have strictly positive membrane potentials and these membrane potentials are all different. The system is in this set with probability approaching 1, for any instant before it gets trapped as the number of neurons of the system diverges. This is the content of Theorem 2 for the system with the first leakage mechanism and the content of Theorem 5 for the system with the second leakage mechanism.

The system has a metastable behavior, namely the time at which it gets trapped in the null membrane potentials configuration re-normalized by its mean value converges in distribution to a mean 1 exponential random time as the population size diverges. This is the content of Theorem 3 for the system with the first leakage mechanism and the content of Theorem 6 for the system with the second leakage mechanism. Theorems 3 and 6 assume that the system starts with the same type of initial configuration considered in Theorems 2 and 5. This initial configuration condition prevents the system to be immediately attracted by the null configuration.

To put our article in perspective, let us briefly recall some results recently published in other articles. In article [1] it was proven that there exists a critical value for the leakage rate such that the system has either one or two extremal invariant measures when the leakage rate is either greater or smaller than the critical value, respectively. For the same model considered in article [1], it was proven by [3] that for a finite system with a sufficiently small leakage rate, the system displays a metastable behavior when the number of the neurons diverges (see also [4, 5]).
In article [2] it was proven that there exist two critical values for the leakage rate such that the system exhibit three different behaviors. In the first case, each fixed neuron has positive probability to spike infinitely many times. In the second case, the system has a positive probability of never goes extinct but each neuron eventually stops spiking. In the third case, the neural spiking activity goes extinct with probability one.

Both models considered here belong to the class of systems of interacting point process with memory of variable length that was introduced in discrete time by [6] and in continuous time by [7] to model systems of spiking neurons. The metastable behavior of systems of interacting point processes with memory of variable length was also analyzed by [8–10]. Other aspects of systems of interacting point processes with memory of variable length in this class of models was considered in several articles, including [2, 11–23]. For a self-contained and neurobiological motivated presentation of this class of variable length memory models for system of spiking neurons, both in discrete and continuous time, we refer the reader to [24].

The notion of metastability considered here is inspired by the so called pathwise approach to metastability introduced by [25]. For more references and an introduction to the topic, we refer the reader to [26–28].

This article is organized as follows. In Sect. 2 we present the definitions, basic and extra notation and state the main results. In Sect. 3 we prove Theorem 1. In Sect. 4 we present a coupling construction and prove some auxiliary results. In Sects. 5 and 6 we prove Theorems 2 and 3, respectively. In Sect. 7 we prove Theorems 4, 5 and 6.

2 Definitions, Notation and Main Results

Let \( A_N = \{1, 2, \ldots, N\} \) be the set of neurons, with \( N \geq 2 \) and denote

\[
S_N = \left\{ u = (u(a) : a \in A_N) \in \{0, 1, 2, \ldots\}^N : \min\{u(a) : a \in A_N\} = 0 \right\}
\]

the set of lists of membrane potentials.

We want to describe the time evolution of the list of membrane potentials of a system of spiking neurons. To do this, for any neuron \( a \in A_N \), we define the maps \( \pi^{a,*} \), \( \pi^{a,\dagger} \) and \( \hat{\pi}^{a,\dagger} \) on \( S_N \) as follows. For any \( u \in S_N \),

\[
\pi^{a,*}(u)(b) = \begin{cases} 
  u(b) + 1, & \text{if } b \neq a, \\
  0, & \text{if } b = a,
\end{cases}
\]

\[
\pi^{a,\dagger}(u)(b) = \begin{cases} 
  u(b), & \text{if } b \neq a, \\
  0, & \text{if } b = a,
\end{cases}
\]

\[
\hat{\pi}^{a,\dagger}(u)(b) = \begin{cases} 
  u(b), & \text{if } b \neq a, \\
  u(b) - 1, & \text{if } b = a \text{ and } u(b) \geq 1, \\
  0, & \text{if } b = a \text{ and } u(b) = 0.
\end{cases}
\]

The map \( \pi^{a,*} \) represents the effect of a spike of neuron \( a \) in the system. When we apply the map \( \pi^{a,*} \), the membrane potential of neuron \( a \) resets to 0 and the membrane potentials of all the other neurons increase by one unit.

The map \( \pi^{a,\dagger} \) represents the leakage on the membrane potential of neuron \( a \) following the first way to model the leakage effect. When we apply the map \( \pi^{a,\dagger} \), the membrane potential of neuron \( a \) resets to 0 and the membrane potentials of all the other neurons remain the same.
The map $\hat{\pi}^{a,\dagger}$ represents the leakage on the membrane potential of neuron $a$ following the second way to model the leakage effect. When we apply the map $\hat{\pi}^{a,\dagger}$, if neuron $a$ has a strictly positive membrane potential its membrane potential decreases by one unit and the membrane potentials of all the other neurons remain the same.

The time evolution of the system of neurons with the first leakage mechanism can be described as follows. Denote the initial list of membrane potentials $U_{0}^{N,u} = u \in S_{N}$. The list of membrane potentials $(U_{t}^{N,u})_{t \in [0, +\infty)}$ evolves as a Markov jump process taking values in the set $S_{N}$ and with infinitesimal generator $G$ defined as follows

$$Gf(u) = \sum_{b \in A_{N}} e^{u(b)} 1\{u(b) > 0\} \left[ f(\pi^{b,\ast}(u)) - f(u) \right] + \sum_{b \in A_{N}} \left[ f(\pi^{b,\dagger}(u)) - f(u) \right],$$

for any bounded function $f : S_{N} \to \mathbb{R}$.

The time evolution of the system of neurons with the second leakage mechanism can be described as follows. The list of membrane potentials $(\hat{U}_{t}^{N,u})_{t \in [0, +\infty)}$ evolves as a Markov jump process taking values in the set $S_{N}$ with initial list $u \in S_{N}$ and with infinitesimal generator $\hat{G}$ defined as follows

$$\hat{G}f(u) = \sum_{b \in A_{N}} e^{u(b)} 1\{u(b) > 0\} \left[ f(\pi^{b,\ast}(u)) - f(u) \right] + \sum_{b \in A_{N}} \left[ f(\hat{\pi}^{b,\dagger}(u)) - f(u) \right],$$

for any bounded function $f : S_{N} \to \mathbb{R}$.

Observe that the null list $0_{N} \in S_{N}$, defined as

$$0_{N}(a) = 0, \text{ for any } a \in A_{N}$$

is a trap for both processes.

The goal of this article is to study the time the process takes to get trapped and its behavior before getting trapped.

To state our main results, we need to introduce some notation. Let

$$\tau^{N,u} = \inf\{t > 0 : U_{t}^{N,u} = 0_{N}\}$$

and define $\mathcal{N}^{N,u}$ as the number of spikes and leakages of membrane potential of the process, namely

$$\mathcal{N}^{N,u} = \left\{ s \in (0, \tau^{N,u}] : U_{s}^{N,u} \neq \lim_{t \to s^{-}} U_{t}^{N,u} \right\}.$$

Analogously, let

$$\hat{\tau}^{N,u} = \inf\{t > 0 : \hat{U}_{t}^{N,u} = 0_{N}\}$$

and

$$\hat{N}^{N,u} = \left\{ s \in (0, \hat{\tau}^{N,u}] : \hat{U}_{s}^{N,u} \neq \lim_{t \to s^{-}} \hat{U}_{t}^{N,u} \right\}.$$

We consider also the set

$$S_{N}^{(0)} = \{ u \in S_{N} : |\{ a \in A_{N} : u(a) > 0 \}| \geq \lfloor N^{1/2} \rfloor \}$$

and the set

$$\mathcal{W}_{N} = \left\{ u \in S_{N} : I_{N} \subset \{ u(a) : a \in A_{N} \}, \bigcap_{a \in A_{N}} \bigcap_{b \neq a} \{ u(a) \neq u(b) \} \right\},$$
where $I_N = \{1, \ldots, N - \lfloor N^{1/2} \rfloor \}$.

$S^{(0)}_N$ and $W_N$ are the sets described in the informal presentation of Theorems 2 and 5. $S^{(0)}_N$ is the set of configurations in which a sufficiently large set of neurons have strictly positive membrane potential. This sufficiently large set of neurons has size greater than or equal to $\lfloor N^{1/2} \rfloor$. $W_N$ is the set in which the process is with probability 1 as $N \to +\infty$ before getting trapped.

Informally, starting from $S^{(0)}_N$, with probability 1 as $N \to +\infty$ the process will have a sequence of spikes and the difference between the membrane potential of the neuron which spikes and the greatest membrane potential at each time will be small. More specifically, this difference will be at most $N^{1/2}$. This sequence of spikes will lead the process to a situation in which all neurons have different membrane potentials and the membrane potential of the neurons forms a part of a ladder. This means that there is one neuron with membrane potential equal to 1, one neuron with membrane potential equal to 2 and so on until one neuron with membrane potential equal to $N - \lfloor N^{1/2} \rfloor$. This is exactly the configurations contained in $W_N$.

We can now state our main results.

**Theorem 1** For any $N \geq 2$ and for any initial list $u \in S_N$, it follows that

$$\Pr(N^{N,u} < +\infty) = 1$$

and

$$\Pr(\tau^{N,u} < +\infty) = 1.$$  

**Theorem 2** For any $t > 0$, it follows that

$$\inf_{u \in S^{(0)}_N} \Pr\left(U^{N,u}_t \in W_N | \tau^{N,u} > t\right) \to 1, \text{ as } N \to +\infty.$$  

**Theorem 3** For any sequence $(u_N \in S^{(0)}_N : N \geq 2)$,

$$\frac{\tau^{N,u_N}}{\mathbb{E}[\tau^{N,u_N}]} \to \text{Exp}(1) \text{ in distribution, as } N \to +\infty,$$

where Exp(1) denotes a mean 1 exponential distributed random variable.

**Theorem 4** For any $N \geq 2$ and for any initial list $u \in S_N$, it follows that

$$\Pr(\hat{N}^{N,u} < +\infty) = 1$$

and

$$\Pr(\hat{\tau}^{N,u} < +\infty) = 1.$$  

**Theorem 5** For any $t > 0$, it follows that

$$\inf_{u \in S^{(0)}_N} \Pr\left(\hat{U}^{N,u}_t \in W_N | \hat{\tau}^{N,u} > t\right) \to 1, \text{ as } N \to +\infty.$$  

**Theorem 6** For any sequence $(u_N \in S^{(0)}_N : N \geq 2)$,

$$\frac{\hat{\tau}^{N,u_N}}{\mathbb{E}[\hat{\tau}^{N,u_N}]} \to \text{Exp}(1) \text{ in distribution, as } N \to +\infty,$$

where Exp(1) denotes a mean 1 exponential distributed random variable.
In Sects. 3, 4, 5 and 6 we prove Theorems 1, 2 and 3 concerning the process \((U_t^{N,u})_{t \in [0, +\infty)}\). In Sect. 7 we show how the proofs presented before can be modified in order to prove Theorems 4, 5 and 6 concerning the process \((\hat{U}_t^{N,u})_{t \in [0, +\infty)}\).

To prove our results it is convenient to extend the notation introduced before.

**Extra notation**

- Let \(T_0 = 0\) and for \(n = 1, \ldots, N^{N,u}\), let \(T_n\) denote the successive jumping times of the process \((U_t^{N,u})_{t \in [0, +\infty)}\), namely
  \[ T_n = \inf \{ t > T_{n-1} : U_{t}^{N,u} \neq U_{T_{n-1}}^{N,u} \} . \]
- For \(n = 1, \ldots, N^{N,u}\), we define \(A_n \in \mathcal{A}_N\) and \(O_n \in \{*, \dagger\}\) as the pair such that
  \[ U_{T_n}^{N,u} = \pi^{A_n, O_n} \left( U_{T_{n-1}}^{N,u} \right) . \]
- The leakage times are defined as \(T_0^\dagger = 0\) and for \(n \geq 1\),
  \[ T_n^\dagger = \inf \{ T_m > T_{n-1}^\dagger : O_m = \dagger \} . \]
- The spiking times are defined as \(T_0^* = 0\) and for \(n \geq 1\),
  \[ T_n^* = \inf \{ T_m > T_{n-1}^* : O_m = * \} . \]
- For any time interval \(I \subset [0, +\infty)\), the counting measures indicating the number of leakage times and spiking times that occurred during the time interval \(I\) are defined as
  \[ Z^\dagger(I) = \sum_{m=1}^{+\infty} 1\{T_m^\dagger \in I\} \quad \text{and} \quad Z^*(I) = \sum_{m=1}^{+\infty} 1\{T_m^* \in I\} . \]
- For any \(u \in S_N\), we define \(a^u_1, \ldots, a^u_N \in \mathcal{A}_N\) in the following way
  \[ a^u_1 \in \arg\min\{u(a) : a \in \mathcal{A}_N\}, \]
  \[ a^u_2 \in \arg\min\{u(a) : a \in \mathcal{A}_N \setminus \{a^u_1\}\}, \]
  \[ \ldots \]
  \[ a^u_N \in \arg\min\{u(a) : a \in \mathcal{A}_N \setminus \{a^u_1, a^u_2, \ldots, a^u_{N-1}\}\}. \]

To avoid ambiguity, we use the following convention: if \(u(a^u_j) = u(a^u_{j+1})\), then \(a^u_{j+1} < a^u_j\). \(a^u_{j+1}\) is the neuron with smallest membrane potential in configuration \(u\) (it will always satisfy \(u(a^u_1) = 0\)) and so on until \(a^u_n\), the neuron with greatest membrane potential in configuration \(u\). Since it is possible to have neurons with the same membrane potential in configuration \(u\), we need to define the convention above.

- The set of ladder lists is defined as
  \[ \mathcal{L}_N = \{ u \in S_N : \{u(a) : a \in \mathcal{A}_N\} = \{0, 1, \ldots, N - 1\} \} . \]
- Let \(\sigma : \mathcal{A}_N \rightarrow \mathcal{A}_N\) be a bijective map. For any \(u \in S_N\), the permuted list \(\sigma(u) \in S_N\) is defined as
  \[ \sigma(u)(a) = u(\sigma(a)), \quad \text{for all } a \in \mathcal{A}_N. \]
- For any \(\lambda > 0\), \(\xi^{\{\lambda\}}\) and \((\xi^{\{\lambda\}}_j : j = 1, 2, \ldots)\) will always be, respectively, a random variable exponentially distributed with mean \(\lambda^{-1}\) and a sequence of independent random variables exponentially distributed with mean \(\lambda^{-1}\).
3 Proof of Theorem 1

In this section we will prove Theorem 1. To prove Theorem 1, we first show that starting from any non null list, the process has a positive and bounded probability to reach the set of ladder lists after $N-1$ jumps of the process. This follows from the fact that a ladder list is obtained when for $N-1$ consecutive instants, there is the spiking of a neuron in the process and the neuron that spiked is the neuron with maximum membrane potential at each instant. This is the content of Lemma 7. With this Lemma we are able to prove Theorem 1.

Lemma 7 For any $u \in S_N \setminus \{0\}$, it follows that
\[
P(U_{T_{N-1}}^N \in \mathcal{L}_N) \geq \left( \frac{1}{2(N-1)} \right)^{N-1}.
\]

Proof For any initial list $u \in S_N \setminus \{0\}$, the occurrence of the event \{\(A_1 = a_N^u, O_1 = \ast\)\} implies that \(U_{T_1}^N \in \mathcal{L}_N\) in the case \(N = 2\), and implies that
\[
a_1^{U_{T_1}^N} = 0, \quad a_2^{U_{T_1}^N} = 1, \quad a_j^{U_{T_1}^N} \geq 1, \text{ for } j = 3, \ldots, N,
\]
in the case \(N \geq 3\). As a consequence, the occurrence of the event
\[
\left\{ A_1 = a_N^u, O_1 = \ast, A_2 = a_N^{U_{T_1}^N}, O_2 = \ast \right\}
\]
implies that implies that \(U_{T_2}^N \in \mathcal{L}_N\) in the case \(N = 3\), and it implies that
\[
a_1^{U_{T_2}^N} = 0, \quad a_2^{U_{T_2}^N} = 1, \quad a_3^{U_{T_2}^N} = 2, \quad a_j^{U_{T_2}^N} \geq 2, \text{ for } j = 4, \ldots, N,
\]
in the case \(N \geq 4\). Iterating this, we conclude that the occurrence of the event
\[
\bigcap_{j=1}^{N-1} \{A_j = a_N^{U_{T_{j-1}}^N}, O_j = \ast\}
\]
is obtained for any initial list $u$ in which all neurons, except one, have membrane potential equal to 1. This implies that
\[
\inf \left\{ \mathbb{P}(A_1 = a_N^v, O_1 = \ast \mid U_{0_{v}}^N = v) : v \in S_N \setminus \{0\} \right\} \geq \frac{1}{2(N-1)}.
\]

We conclude the proof by using Markov property and applying this lower bound $N-1$ times in Eq. (1). \(\square\)
The following holds

\[ \lim_{N \to +\infty} \sup_{t \geq 0} \sup_{w, w' \in W_N} \left| \mathbb{P}(\tau^{N,w} > t) - \mathbb{P}(\tau^{N,w'} > t) \right| = 0. \]

To prove Proposition 8, we need to introduce a coupling construction of the processes \((U_{t}^{N,u})_{t \in [0, +\infty)}\) and \((U_{t}^{N,u'})_{t \in [0, +\infty)}\) starting from two different lists \(u', v' \in S_N\).
We want to describe the time evolution of \((U_t^{N,u}, U_t^{N,v})_{t \in [0, +\infty)}\). To do this, for any index \(j \in \{1, \ldots, N\}\), we define the maps \(\pi^j, \min\), \(\pi^j, \max\) and \(\pi^j, +\) on \(S_N^2\) as follows. For any \((u, v) \in S_N^2\),

\[
\pi^j, \min(u, v) = (\pi^{a_j}_u, \pi^{a_j}_v), \quad \pi^j, \max(u, v) = \begin{cases} (\pi^{a_j}_u, \pi^{a_j}_v) & \text{if } u(a^u_j) > v(a^v_j), \\ (u, \pi^{a_j}_v) & \text{if } v(a^v_j) > u(a^u_j), \end{cases} \\
\pi^j, +(u, v) = (\pi^{a^+_j}_u, \pi^{a^+_j}_v).
\]

The map \(\pi^j, \min(u, v)\) represents the simultaneous effect of a spike of neuron \(a^u_j\) in the system \((U_t^{N,u'})_{t \geq 0}\) and a spike of neuron \(a^v_j\) in the system \((U_t^{N,v'})_{t \geq 0}\).

The map \(\pi^j, \max(u, v)\) represents the effect of either a spike of neuron \(a^u_j\) in the system \((U_t^{N,u'})_{t \geq 0}\) in the case in which \(u(a^u_j) > v(a^v_j)\), or a spike of neuron \(a^v_j\) in the system \((U_t^{N,v'})_{t \geq 0}\) in the case in which \(v(a^v_j) > u(a^u_j)\).

The map \(\pi^j, +(u, v)\) represents the simultaneous leakage effect on the membrane potential of neuron \(a^u_j\) in the system \((U_t^{N,u'})_{t \geq 0}\) and on the membrane potential of neuron \(a^v_j\) in the system \((U_t^{N,v'})_{t \geq 0}\).

The pair of lists of membrane potentials \((U_i^{N,u'}, U_i^{N,v'})_{i \in [0, +\infty)}\) evolves as a Markov jump process taking values in the set \(S_N^2\) and with infinitesimal generator \(G_C\) defined as follows

\[
G_C f(u, v) = \sum_{j=1}^{N} e^{[u(a^u_j) - v(a^v_j)]} \mathbf{1}_{\{u(a^u_j) \neq v(a^v_j)\}} \left[ f(\pi^j, \max(u, v)) - f(u, v) \right] + \sum_{j=1}^{N} e^{\min[u(a^u_j), v(a^v_j)]} \mathbf{1}_{\{\min[u(a^u_j), v(a^v_j)] > 0\}} \left[ f(\pi^j, \min(u, v)) - f(u, v) \right] + \sum_{j=1}^{N} \left[ f(\pi^j, +(u, v)) - f(u, v) \right],
\]

for any bounded function \(f : S_N^2 \to \mathbb{R}\).

For the coupling construction we introduce some extra notation.

**Extra Notation—Coupling Construction**

- Define

\[
\tau^N(u, v) = \inf \{ s > 0 : (U_s^{N,u}, U_s^{N,v}) = (0_N, 0_N) \}.
\]

- Define \(\mathcal{N}^N(u, v)\) as the number of spikes and leakages of membrane potential of the coupling process, namely

\[
\mathcal{N}^N(u, v) = \left\{ s > 0 : (U_s^{N,u}, U_s^{N,v}) \neq \left( \lim_{t \to s^-} U_t^{N,u}, \lim_{t \to s^-} U_t^{N,v} \right) \right\}.
\]

- Let \(T_0(u, v) = 0\) and for \(n = 1, \ldots, \mathcal{N}^N(u, v)\) denote \(T_n(u, v)\) the successive jumping times of the process \((U_t^{N,u}, U_t^{N,v})_{t \in [0, +\infty)}\), namely

\[
T_n(u, v) = \inf \left\{ t > T_{n-1}(u, v) : (U_t^{N,u}, U_t^{N,v}) \neq (U_{T_{n-1}(u, v)}^{N,u}, U_{T_{n-1}(u, v)}^{N,v}) \right\}.
\]
• For each $n = 1, \ldots, N^N(u, v)$, we define $J_n(u, v) \in \{1, \ldots, N\}$ and $K_n(u, v) \in \{\min, \max, \hat{\triangle}\}$ as the pair such that
\[
(U_{T_n(u,v)}^{N,u}, U_{T_n(u,v)}^{N,v}) = \pi_{J_n(u,v),K_n(u,v)} \left( U_{T_{n-1}(u,v)}^{N,u}, U_{T_{n-1}(u,v)}^{N,v} \right).
\]
The pair $(J_n(u, v), K_n(u, v))$ is exactly the index of the map used to transform the list of membrane potentials at time $T_n(u, v)$ describing the change in the coupling process at this time.

• For any $j \geq 1$, we define the event
\[
E_j(u, v) = \bigcap_{n=2(j-1)[N^{1/2}] + 1}^{2j[N^{1/2}]} \{ J_n(u, v) = N, K_n(u, v) \neq \hat{\triangle} \}.
\]
$E_j(u, v)$ is the event in which the neuron with greatest membrane potential spikes (either simultaneously on both evolutions or not) at the jump times $T_n(u, v)$, for $n = 2(j - 1)[N^{1/2}] + 1, \ldots, 2j[N^{1/2}]$.

• The number of jumping times of the process $(U_t^{N,u}, U_t^{N,v})_{t\in[0,\infty)}$ until the first leakage time is defined as
\[
N^N_{\hat{\triangle}}(u, v) = \inf \{ n : K_n(u, v) = \hat{\triangle} \}.
\]

• The number of jumping times of the process $(U_t^{N,u}, U_t^{N,v})_{t\in[0,\infty)}$ until the coupling time is defined as
\[
N^N_C(u, v) = \inf \left\{ n : \exists \sigma : \mathcal{A}_N \to \mathcal{A}_N \text{ bijective s.t. } U_{T_n(u,v)}^{N,u} = \sigma \left( U_{T_n(u,v)}^{N,v} \right) \right\}.
\]

**Remark 9** There exists a bijective map $\sigma : \mathcal{A}_N \to \mathcal{A}_N$ such that
\[
U_{T_s(u,v)}^{N,u} = \sigma \left( U_{T_s(u,v)}^{N,v} \right), \text{ for all } s \geq T_{N^N_C(u,v)}(u, v).
\]

Moreover, if there exists $t \geq 0$ such that $U_t^{N,u} \in \mathcal{L}_N$ and $U_t^{N,v} \in \mathcal{L}_N$, then $t \geq T_{N^N_C(u,v)}(u, v)$.

The proof of Proposition 8 is based on the three following lemmas about the coupling construction. In Lemma 10 we prove that starting from two lists $w, w' \in \mathcal{W}_N$, if the event $E_1(u, v)$ occurs then the number of jumps before the coupling time is smaller than or equal to $2[N^{1/2}]$. This is based on the fact that a sequence of spikes of neurons with greatest membrane potential lead the process to a ladder list and the fact that the coupling time occurs before or at the same time in which both processes reach simultaneously on the set of ladder lists. In Lemmas 11 and 12 we obtain a bound for the number of jumping times of the process before the coupling time and shows that with probability 1 as $N \to +\infty$, the coupling time occurs before the process gets trapped. As a consequence, in Corollary 13 we prove that the coupling time occurs instantaneously as $N \to +\infty$. Putting all this together we are able to prove Proposition 8.

**Lemma 10** For any lists $w, w' \in \mathcal{W}_N$, if the event $E_1(w, w')$ occurs, then
\[
N^N_C(w, w') \leq 2[N^{1/2}].
\]

**Proof** The occurrence of the event
\[
E_1(w, w') = \bigcap_{n=1}^{2[N^{1/2}]} \{ J_n(w, w') = N, K_n(w, w') \neq \hat{\triangle} \}
\]

implies that in the first $2\lceil N^{1/2} \rceil$ steps of the coupling construction there are neurons spiking and at each step, the neuron that spikes is the neuron with greatest membrane potential.

For the first step, denoting $u_1 = U_{T_1(w, w')}^N$ and $u_1' = U_{T_1(w, w')}^{N, w'}$, we have two possible cases:

- If $J_1(w, w') = N$ and $K_1(w, w') = \min$, then
  \[ \{1, \ldots, N - \lfloor N^{1/2} \rfloor + 1\} \subset \{ u_1(a) : a \in A_N \}, \bigcap_{a \in A_N, b \neq a} \{ u_1(a) \neq u_1(b) \} \]
  and
  \[ \{1, \ldots, N - \lfloor N^{1/2} \rfloor + 1\} \subset \{ u_1'(a) : a \in A_N \}, \bigcap_{a \in A_N, b \neq a} \{ u_1'(a) \neq u_1'(b) \}. \]

- If $J_1(w, w') = N$ and $K_1(w, w') = \max$, then either $u_1' = w'$ and
  \[ \{1, \ldots, N - \lfloor N^{1/2} \rfloor + 1\} \subset \{ u_1(a) : a \in A_N \}, \bigcap_{a \in A_N, b \neq a} \{ u_1(a) \neq u_1(b) \} \]
  in the case $w(a_N^w) > w'(a_N^{w'})$, or $u_1 = w$ and
  \[ \{1, \ldots, N - \lfloor N^{1/2} \rfloor + 1\} \subset \{ u_1'(a) : a \in A_N \}, \bigcap_{a \in A_N, b \neq a} \{ u_1'(a) \neq u_1'(b) \} \]
  in the case $w'(a_N^{w'}) > w(a_N^w)$.

Iterating this, we conclude that if the event $E_1(w, w')$ occurs, then
\[ U_{T_2[N^{1/2}]N, w'}^N \in \mathcal{L}_N \quad \text{and} \quad U_{T_2[N^{1/2}]N, w'}^{N, w} \in \mathcal{L}_N. \tag{2} \]

By Remark 9, (2) implies that $\mathcal{N}_{\mathcal{C}}^{N}(w, w') \leq 2\lceil N^{1/2} \rceil$. \hfill \Box

**Lemma 11** For any $n \geq 1$ and for any $w, w' \in \mathcal{W}_N$,
\[ \mathbb{P}(\mathcal{N}_{\mathcal{C}}^{N}(w, w') \leq 2n \lceil N^{1/2} \rceil < \mathcal{N}_{\mathcal{C}}^{N}(w, w')) \geq \mathbb{P} \left( \text{Geom} \left( \frac{2n \lceil N^{1/2} \rceil}{e^{\lceil N^{1/2} \rceil} + 2(N - 1)} \right) \right), \]
where $\zeta = 1 - e^{-1}$ and Geom $\left( \frac{2n \lceil N^{1/2} \rceil}{e^{\lceil N^{1/2} \rceil} + 2(N - 1)} \right)$ is a random variable with geometric distribution assuming values in \{1, 2, \ldots\} and with mean $1/\zeta^{2\lceil N^{1/2} \rceil}$.

**Proof** To simplify the presentation of the proof, for a fixed pair of lists $w, w' \in \mathcal{W}_N$ and for any $m \geq 1$, we will use the shorthand notation $J_m$, $K_m$ and $E_m$ instead of $J_m(w, w')$, $K_m(w, w')$ and $E_m(w, w')$, respectively.

For any $n \geq 1$, the occurrence of the event
\[ \bigcap_{m=1}^{2n \lceil N^{1/2} \rceil} \{ J_m \in \{ N - \lceil N^{1/2} \rceil + 1, \ldots, N \}, K_m \neq \dagger \} \]
implies that $U^N_m \in W_N$ and $U^{N, w'}_m \in W_N$, for all $m = 1, \ldots, 2n \lceil N^{1/2} \rceil$. This implies that for any $w, w' \in W_N$,

$$\mathbb{P}(N^N_C(w, w') \leq 2n \lceil N^{1/2} \rceil < N^N_C(w, w')) \geq \mathbb{P}\left( \bigcap_{m=1}^{2n \lceil N^{1/2} \rceil} E_m, \bigcap_{m=1}^{2n \lceil N^{1/2} \rceil} \{ J_m \in \{ N - \lfloor N^{1/2} \rfloor + 1, \ldots, N \}, K_m \neq \dagger \} \right).$$

For any lists $u, v \in W_N$ we have that

$$\mathbb{P}(J_1(u, v) = N, K_1(u, v) \neq \dagger) = \frac{e^{\max\{u(N^N_u), v(N^N_v)\}}}{N} \mathbb{P}(K_1(u, v) \neq \dagger). \quad (3)$$

The left term of the right-hand side in Eq. (3) is bounded below by

$$\frac{e^{(N-1)}}{\sum_{j=1}^{N-1} e^j} \geq \zeta.$$

Therefore,

$$\mathbb{P}(E_1, \bigcap_{m=1}^{2n \lceil N^{1/2} \rceil} \{ J_m \in \{ N - \lfloor N^{1/2} \rfloor + 1, \ldots, N \}, K_m \neq \dagger \}) \geq \zeta^{2\lceil N^{1/2} \rceil} \mathbb{P}\left( \bigcap_{m=1}^{2n \lceil N^{1/2} \rceil} \{ J_m \in \{ N - \lfloor N^{1/2} \rfloor + 1, \ldots, N \}, K_m \neq \dagger \} \right),$$

and more generally, for any $n = 1, 2, \ldots$,

$$\mathbb{P}\left( \bigcup_{m=1}^{n} E_m, \bigcap_{m=1}^{2n \lceil N^{1/2} \rceil} \{ J_m \in \{ N - \lfloor N^{1/2} \rfloor + 1, \ldots, N \}, K_m \neq \dagger \} \right) \geq \left( 1 - (1 - \zeta^{2\lceil N^{1/2} \rceil})^n \right) \mathbb{P}\left( \bigcap_{m=1}^{2n \lceil N^{1/2} \rceil} \{ J_m \in \{ N - \lfloor N^{1/2} \rfloor + 1, \ldots, N \}, K_m \neq \dagger \} \right).$$

To conclude the proof, note that for any lists $u, v \in W_N$, we have $\max\{u(N^N_u), v(N^N_v)\} \geq N - 1$ and $u(a^u_{N - \lfloor N^{1/2} \rfloor}) = v(a^v_{N - \lfloor N^{1/2} \rfloor}) = N - \lfloor N^{1/2} \rfloor - 1$. This implies that

$$\mathbb{P}\left( \bigcap_{m=1}^{2n \lceil N^{1/2} \rceil} \{ J_m \in \{ N - \lfloor N^{1/2} \rfloor + 1, \ldots, N \}, K_m \neq \dagger \} \right) \geq \left( \frac{e^{N^{1/2}}}{e^{N^{1/2}} + 2(N - 1)} \right)^{2n \lceil N^{1/2} \rceil}.$$

$\square$
Lemma 12 The following holds
\[
\inf_{w, w' \in \mathcal{N}_N} \mathbb{P}\left( N_C(w, w') < e^{N/2} N^{-2} < N_T(w, w') \right) \to 1, \text{ as } N \to +\infty.
\]

**Proof** For any \( w, w' \in \mathcal{N}_N \), taking \( n = [e^{N/2} N^{-2} / (2\lceil N/2 \rceil)] \) in Lemma 11, we have that
\[
\mathbb{P}\left( N_C(w, w') < [e^{N/2} N^{-2}] \right) \geq \left( 1 - \left( 1 - \xi^{2\lceil N/2 \rceil} \right) \right) \left( \frac{e^{N/2}}{e^{N/2} + 2N} \right)^{[e^{N/2} N^{-2}]} \to 1,
\]
as \( N \to +\infty \). To finish the proof, just note that the bound of Eq. (4) does not depend on the initial lists \( w, w' \in \mathcal{W}_N \) and and take \( n = [e^{N/2} N^{-2} / (2\lceil N/2 \rceil)] \) in Lemma 11.

**Corollary 13** The following holds
\[
\sup_{w, w' \in \mathcal{W}_N} \mathbb{P}(T_{N_C}(w, w') > e^{-(N-N/2)}) \to 0, \text{ as } N \to +\infty.
\]

**Proof** For any \( w, w' \in \mathcal{W}_N \) and for any \( t > 0 \),
\[
\mathbb{P}(T_{N_C}(w, w') > t) \leq \mathbb{P}(T_{N_C}(w, w') > t, N_C(w, w') < e^{N/2} N^{-2} < N_T(w, w')) + \mathbb{P}([N_C(w, w') < e^{N/2} N^{-2} < N_T(w, w')]^c).
\]
Lemma 12 implies that
\[
\sup_{w, w' \in \mathcal{W}_N} \mathbb{P}([N_C(w, w') < e^{N/2} N^{-2} < N_T(w, w')]^c) \to 0, \text{ as } N \to +\infty.
\]
Moreover,
\[
\mathbb{P}(T_{N_C}(w, w') > t, N_C(w, w') < e^{N/2} N^{-2} < N_T(w, w')) \leq \mathbb{P}(T_{e^{N/2} N^{-2}}(w, w') > t, N_C(w, w') < e^{N/2} N^{-2} < N_T(w, w')).
\]
For any initial lists \( w, w' \) and for any \( s > 0 \), if the event \( N_T(w, w') > e^{N/2} N^{-2} \) occurs, then
\[
\mathbb{P}(T_j(w, w') - T_{j-1}(w, w') > s) \leq \mathbb{P}(\xi^{e^{N/2}} > s), \text{ for any } j = 1, \ldots, [e^{N/2} N^{-2}].
\]
Therefore, taking \( t = e^{-(N-N/2)} \) the right-hand side of Eq. (5) is bounded above by
\[
\mathbb{P}\left( \sum_{j=1}^{[e^{N/2} N^{-2}]} \xi^{e^{N(1-1)}} > e^{-(N-N/2)} \right) \to 0, \text{ as } N \to +\infty.
\]
We conclude the proof by putting Eqs. (5) and (6) together and noting that the bound on (6) does not depend on the lists \( w, w' \in \mathcal{W}_N \).  \( \square \)
**Remark 14** By Remark 9 and Eq. (2), we can replace \( \mathcal{N}_C^N(w, w') \) by

\[
\inf\{ n \geq 0 : \{ U_{T_n(w, w')}^N \in \mathcal{L}_N \} \cap \{ U_{T_n(w, w')}^N \in \mathcal{L}_N \} \}
\]

in Lemmas 10, 11 and 12. This implies that

\[
\sup_{w \in W_N} \mathbb{P}(\inf\{ s > 0 : U_s^N(w) \in \mathcal{L}_N \} > e^{-(N-N^{1/2})}) \to 0, \text{ as } N \to +\infty.
\]

**Proof** We have now all the ingredients to prove Proposition 8.

For any \( t > 0 \) and for any \( w, w' \in W_N \),

\[
\mathbb{P}(\tau_{N,w}^N > t) \leq \mathbb{P}(\tau_{N,w}^N > t, \mathcal{N}_C^N(w, w') < \mathcal{N}_F^N(w, w')) + \mathbb{P}(\mathcal{N}_C^N(w, w') > \mathcal{N}_F^N(w, w'))
\]

Now, note that

\[
\mathbb{P}(\tau_{N,w}^N > t, \mathcal{N}_C^N(w, w') < \mathcal{N}_F^N(w, w')) = \mathbb{P}(\tau_{N,w'}^N > t, \mathcal{N}_C^N(w, w') < \mathcal{N}_F^N(w, w')).
\]

This implies that

\[
\mathbb{P}(\tau_{N,w}^N > t) - \mathbb{P}(\tau_{N,w'}^N > t) \leq \mathbb{P}(\mathcal{N}_C^N(w, w') > \mathcal{N}_F^N(w, w')).
\]

Analogously,

\[
\mathbb{P}(\tau_{N,w'}^N > t) - \mathbb{P}(\tau_{N,w}^N > t) \leq \mathbb{P}(\mathcal{N}_C^N(w, w') > \mathcal{N}_F^N(w, w')),
\]

and therefore,

\[
|\mathbb{P}(\tau_{N,w}^N > t) - \mathbb{P}(\tau_{N,w'}^N > t)| \leq \mathbb{P}(\mathcal{N}_C^N(w, w') > \mathcal{N}_F^N(w, w')) \to 0, \text{ as } N \to +\infty,
\]

By Lemma 12, we conclude that

\[
\sup_{t \geq 0} \sup_{w, w' \in W_N} |\mathbb{P}(\tau_{N,w}^N > t) - \mathbb{P}(\tau_{N,w'}^N > t)| \leq \mathbb{P}(\mathcal{N}_C^N(w, w') < \mathcal{N}_F^N(w, w')) \to 0, \text{ as } N \to +\infty,
\]

and with this we concluded the proof. \( \square \)

**5 Proof of Theorem 2**

To prove Theorem 2 we need to introduce the auxiliary process \((\tilde{U}_t^{N,u})_{t \in [0, +\infty)}\) that evolves as a Markov jump process taking values in the set \( \tilde{S}_N = S_N \setminus \{0_N\} \) with initial list \( u \in \tilde{S}_N \) and with infinitesimal generator \( \tilde{G} \) defined as follows

\[
\tilde{G} f(u) = \sum_{b \in A_N} e^{u(b)} \mathbf{1}_{\{u(b) > 0\}} \left[ f(\pi^{b,*}(u)) - f(u) \right] + \sum_{b \in A_N} \mathbf{1}_{\{\pi^{b,*}(u) \neq 0_N\}} \left[ f(\pi^{b,*}(u)) - f(u) \right],
\]

for any bounded function \( f : \tilde{S}_N \to \mathbb{R} \).
Remark 15 In general, the processes \((\tilde{U}_t^N,u)_{t \in [0, +\infty)}\) and \((U_t^N,u)_{t \in [0, +\infty)}\) have the same jump rates. The only exception is that \((\tilde{U}_t^N,u)_{t \in [0, +\infty)}\) can not jump from a list in which only one neuron has non-null membrane potential to the null list.

This implies that the processes \((U_t^N,u)_{t \in [0, +\infty)}\) and \((\tilde{U}_t^N,u)_{t \in [0, +\infty)}\) can be coupled in such way that

\[
\tilde{U}_s^N,u = U_s^N,u, \text{ for all } s \in [0, \tau^N,u).
\]

For the auxiliary process \((\tilde{U}_t^N,u)_{t \in [0, +\infty)}\), let us introduce some extra notation.

**Extra Notation—Auxiliary Process**

- Denote \(\tilde{T}_0 = 0\) and for \(n = 1, 2, \ldots\), denote \(\tilde{T}_n\) the successive jumping times of the process \((\tilde{U}_t^N,u)_{t \in [0, +\infty)}\), namely

\[
\tilde{T}_n = \inf \left\{ t > \tilde{T}_{n-1} : \tilde{U}_t^N,u \neq \tilde{U}_{\tilde{T}_{n-1}}^N,u \right\}.
\]

- For \(n = 1, 2, \ldots\), we define \(\tilde{A}_n \in \mathcal{A}_N\) and \(\tilde{O}_n \in \{*, \dagger\}\) as the pair such that

\[
\tilde{U}_{\tilde{T}_n}^N,u = \pi \tilde{A}_n, \tilde{O}_n \left( \tilde{U}_{\tilde{T}_{n-1}}^N,u \right).
\]

The pair \((\tilde{A}_n, \tilde{O}_n)\) is exactly the index of the map used to transform the list of membrane potentials at time \(\tilde{T}_n\) describing the change in the auxiliary process at this time.

- The leakage times are defined as \(\tilde{T}_{0}^\dagger = 0\) and for \(n \geq 1\),

\[
\tilde{T}_n^\dagger = \inf \{ \tilde{T}_m > \tilde{T}_{n-1}^\dagger : \tilde{O}_m = \dagger \}.
\]

- The spiking times are defined as \(\tilde{T}_{0}^* = 0\) and for \(n \geq 1\),

\[
\tilde{T}_n^* = \inf \{ \tilde{T}_m > \tilde{T}_{n-1}^* : \tilde{O}_m = * \}.
\]

- For any time interval \(I \subset [0, +\infty)\), the counting measures indicating the number of leakage times and spiking times that occurred during the time interval \(I\) are defined as

\[
\tilde{Z}^\dagger(I) = \sum_{j=1}^{+\infty} 1\{\tilde{T}_j^\dagger \in I\} \quad \text{and} \quad \tilde{Z}^*(I) = \sum_{j=1}^{+\infty} 1\{\tilde{T}_j^* \in I\}.
\]

- In the next proposition, we prove that \((\tilde{U}_t^N,u)_{t \in [0, +\infty)}\) has an unique invariant probability measure. We use the symbol \(\mu_N^\dagger\) to denote this probability measure.

**Proposition 16** The process \((\tilde{U}_t^N,u)_{t \in [0, +\infty)}\) is ergodic.

**Proof** Let \(l \in \mathcal{L}_N\) satisfies \(l(a) = a - 1\), for all \(a \in \mathcal{A}_N\). For any \(u \in \tilde{\mathcal{S}}_N\), we have that

\[
l = \pi_{1,*} \circ \pi_{2,*} \circ \ldots \circ \pi_{N,*}(u),
\]

and then, if the event \(\bigcap_{j=1}^{N} \{\tilde{A}_j = N - j + 1, \tilde{O}_j = *\}\) occurs, then \(\tilde{U}_N^N,u = l\).

Let

\[
\tilde{N}^N,u = \inf \{ n \geq 1 : \tilde{U}_{\tilde{T}_n}^N,u = l \}.
\]
As in Theorem 1, for any \( u' \in \tilde{S}_N \), we have that

\[
\mathbb{P} \left( \mathcal{N}^{N,u} \leq n + 2N - 1 \bigg| \tilde{U}^{N,u}_{\tilde{T}_n} = u' \right) \geq \mathbb{P} \left( \tilde{U}^{N,u}_{\tilde{T}_{n+1} - 1} \in \mathcal{L}_N \bigg| \tilde{U}^{N,u}_{\tilde{T}_n} = u' \right) \mathbb{P} \left( \mathcal{N}^{N,u} \leq n + 2N - 1 \bigg| \tilde{U}^{N,u}_{\tilde{T}_{n+1} - 1} \in \mathcal{L}_N \right).
\]

Moreover, the right-hand side of the equation above is bounded above by \([2(N - 1)]^{-(N-1)} \tilde{\epsilon} \), where

\[
\tilde{\epsilon} = \min \left\{ \mathbb{P} \left( \bigcap_{j=1}^{N} \{ \tilde{A}_{n+j} = N - j + 1, \ \tilde{O}_{n+j} = * \} \bigg| \tilde{U}^{N,u}_{\tilde{T}_n} = v \right) : v \in \mathcal{L}_N \right\}.
\]

We conclude that for any \( n \geq 1 \),

\[
\mathbb{P}(\tilde{N}^{N,j} \geq n) \leq \mathbb{P} \left( (2N - 1) \times \text{Geom} \left( [2(N - 1)]^{-(N-1)} \tilde{\epsilon} \right) \geq n \right),
\]

where \( \text{Geom}(r) \) denotes a random variable with geometric distribution assuming values in \( \{1, 2, \ldots \} \) and with mean \( 1/r \). This implies that \( \mathbb{E}(\tilde{N}^{N,j}) < +\infty \) and then, \((\tilde{U}^{N,u}_{\tilde{T}_n})_{n \geq 0}\) is a positive-recurrent Markov chain. The jump rate of the process \((\tilde{U}^{N,u}_{\tilde{T}_n})_{t \in [0, +\infty)}\) satisfies

\[
\sum_{a \in \mathcal{A}_N} 1\{u'(a) > 0\} \left( e^{u'(a)} + 1\{\pi^{a,j}(u') \neq 0_N\} \right) \geq e,
\]

for any \( u' \in \tilde{S}_N \). Putting all this together we conclude that \((\tilde{U}^{N,u}_{\tilde{T}_n})_{t \in [0, +\infty)}\) is ergodic. \( \square \)

The proof of Theorem 2 is based on two lemmas. In Lemma 17 we prove that the invariant measure of the auxiliary process gets concentrated in the set \( \mathcal{W}_N \) as \( N \to +\infty \). We show that there are events that occur with probability 1 as \( N \to +\infty \) that lead the process from any initial list to \( \mathcal{W}_N \) by sequentially reaching sets that are approaching \( \mathcal{W}_N \). As a consequence, in Corollary 18 we obtain a bound for the first time in which the auxiliary process reaches \( \mathcal{L}_N \) and in Corollary 19 we show that the process \((\tilde{U}^{N,u}_{\tilde{T}_n})_{t \in [0, +\infty)}\) reaches \( \mathcal{L}_N \) instantaneously as \( N \to +\infty \) for any \( w \in \mathcal{W}_N \). In Lemma 20 we prove that starting from a ladder list, the auxiliary process is in \( \mathcal{W}_N \) with probability 1 as \( N \to +\infty \). Putting all this together we are able to prove Theorem 2.

**Lemma 17** The invariant probability measure \( \mu^N \) of the auxiliary process satisfies

\[
\mu^N(\mathcal{W}_N) \to 1, \ \text{as} \ N \to +\infty.
\]

**Proof** To prove Lemma 17, we will first show that there exists sets \( S_N^{(1)}, S_N^{(2)} \) and \( S_N^{(3)} \) such that

\[
S_N^{(1)} \supset S_N^{(2)} \supset S_N^{(3)} \supset \mathcal{W}_N
\]

and for any \( j \in \{1, 2, 3\} \),

\[
\mu^N(S_N^{(j)}) \to 1, \ \text{as} \ N \to +\infty.
\]

Let

\[
S_N^{(1)} = \{ u \in \tilde{S}_N : |\{ a \in \mathcal{A}_N : u(a) = 0 \} | \leq N^{\frac{3}{2}} \}
\]
and consider the following events

\[
E_{N,1}^{(1)} = \{ \tilde{Z}^*([0, N^{1/2}]) \geq 1 \}, \\
E_{N,2}^{(1)} = \{ \tilde{Z}^*([0, N^{1/2}]) \leq N^2 \}, \\
E_{N,3}^{(1)} = \{ \left\lfloor \frac{N^2}{\left\lfloor \frac{N^{1/2}}{2} \right\rfloor} \right\rfloor = 1 \}.
\]

For any \( u \in \tilde{S}_N \), the rate in which the process has a spike is bounded below by \( e \). This implies that

\[
\mathbb{P}(E_{N,1}^{(1)}) \geq \mathbb{P} \left( \xi^{(e)} \leq N^{1/2} \right) \rightarrow 1, \text{ as } N \rightarrow +\infty,
\]

(7)

\[
\mathbb{P}(E_{N,2}^{(1)}) \geq \mathbb{P} \left( \sum_{j=1}^{N^2} \xi_j^{(N)} \geq N^{1/2} \right) \rightarrow 1, \text{ as } N \rightarrow +\infty.
\]

(8)

For any list \( u \in \tilde{S}_N \) and for any instant \( n \geq 1 \), we have that

\[
\mathbb{P} \left( \tilde{O}_n = * | \tilde{U}_{n-1}^{N,u'} = u \right) \geq \frac{1}{2}.
\]

This implies that for any initial list \( u \in \tilde{S}_N \) and for any \( j = 1, \ldots, [N^2/\left\lfloor \frac{N^{1/2}}{2} \right\rfloor] \),

\[
\mathbb{P} \left( \tilde{Z}^* \left( \left[ \tilde{T}^*_{(j-1)\left\lfloor \frac{N^{1/2}}{2} \right\rfloor + 1}, \tilde{T}^*_{j\left\lfloor \frac{N^{1/2}}{2} \right\rfloor} \right] \right) \right) \geq 1 \right) \geq \mathbb{P} (\text{Geom} \left( \frac{1}{2} \right) \leq \left\lfloor \frac{N^{1/2}}{2} \right\rfloor - 1),
\]

where Geom \( \left( \frac{1}{2} \right) \) is a random variable with geometric distribution assuming values in \( \{1, 2, \ldots\} \) and with mean 2. Therefore,

\[
\mathbb{P}(E_{N,3}^{(1)}) \geq 1 - 2^{-\left\lfloor \frac{N^{1/2}}{2} \right\rfloor + 1} \rightarrow 1, \text{ as } N \rightarrow +\infty.
\]

(9)

If the event \( E_{N,1}^{(1)} \cap E_{N,2}^{(1)} \cap E_{N,3}^{(1)} \) occurs, then until time \( N^{1/2} \) the process has at least one spiking time and at most \( \left\lfloor \frac{N^{1/2}}{2} \right\rfloor - 1 \) successive leakage times (with no spiking times in between). This implies that \( \tilde{U}_{N^{1/2}}^{N,u} \in S_N^{(1)} \). Since the inequalities of Eqs. (7), (8) and (9) hold for any \( u \in \tilde{S}_N \) and they do not depend on \( u \), it follows that

\[
\sup_{u \in \tilde{S}_N} \mathbb{P}(\tilde{U}_{N^{1/2}}^{N,u} \notin S_N^{(1)}) \rightarrow 0, \text{ as } N \rightarrow +\infty,
\]

and as a consequence,

\[
\mu^N(S_N^{(1)}) = \sum_{u \in \tilde{S}_N} \mu^N(u) \mathbb{P}(\tilde{U}_{N^{1/2}}^{N,u} \in S_N^{(1)}) \rightarrow 1, \text{ as } N \rightarrow +\infty.
\]

Let

\[
S_N^{(2)} = \{ u \in \tilde{S}_N : \exists a_1, \ldots, a_{\left\lfloor N^{1/2} \right\rfloor} \in A_N \text{ s.t. } 1 \leq u(a_1) < \ldots < u(a_{\left\lfloor N^{1/2} \right\rfloor}) \}
\]
and consider the following events

\[ E_{N,1}^{(2)} = \bigcap_{j=1}^{\lceil N^2/\lceil N^{1/2} \rceil \rceil} \left\{ \tilde{Z}^{*} \left( \bar{T}_{(j-1)/\lceil N^{1/2} \rceil + 1}^{\uparrow}, \bar{T}_{j/\lceil N^{1/2} \rceil}^{\uparrow} \right) \geq 1 \right\}, \]

\[ E_{N,2}^{(2)} = \{ \tilde{Z}^{\downarrow}([0, N^{-1/4}]) \leq N^2 \}, \]

\[ E_{N,3}^{(2)} = \{ \tilde{Z}^{*}([0, N^{-1/4}]) \geq \lceil N^{1/2} \rceil \}. \]

As in Eqs. (8) and (9),

\[ \mathbb{P}(E_{N,1}^{(2)}) \geq \left( 1 - 2^{-[N^2/\lceil N^{1/2} \rceil] + 1} \right)^{\lceil N^2/\lceil N^{1/2} \rceil \rceil} \to 1, \text{ as } N \to +\infty, \quad (10) \]

\[ \mathbb{P}(E_{N,2}^{(2)}) \geq \mathbb{P} \left( \sum_{j=1}^{N^2} \xi_j^{[N]} \geq N^{-1/4} \right) \to 1, \text{ as } N \to +\infty. \quad (11) \]

For any initial list \( u \in S_{N}^{(1)} \), the occurrence of \( E_{N,1}^{(2)} \cap E_{N,2}^{(2)} \) implies that until time \( N^{-1/4} \) the rate in which the process has a spike is bounded below by \( e(\lceil N - 2 \lceil N^{1/2} \rceil \rceil) \). This implies that

\[ \mathbb{P}(E_{N,3}^{(2)}) \geq \mathbb{P} \left( \sum_{j=1}^{\lceil N^{1/2} \rceil} \xi_j^{[N-2[\lceil N^{1/2} \rceil]]} \leq N^{-1/4} \right) \mathbb{P} \left( E_{N,1}^{(2)} \cap E_{N,2}^{(2)} \right) \to 1, \text{ as } N \to +\infty. \]

For any initial list \( u \in S_{N}^{(1)} \), if the event \( E_{N,1}^{(2)} \cap E_{N,2}^{(2)} \cap E_{N,3}^{(2)} \) occurs, then until time \( N^{-1/4} \) the process has at least \( \lceil N^{1/2} \rceil \) spiking times and at most \( \lceil N^{1/2} \rceil - 1 \) successive leakage times (with no spiking times in between). This implies that \( \tilde{U}_{N^{-1/4}}^{N,u} \in S_{N}^{(2)} \). Therefore,

\[ \sup_{u \in S_{N}^{(1)}} \mathbb{P}(\tilde{U}_{N^{-1/4}}^{N,u} \notin S_{N}^{(2)}) \to 0, \text{ as } N \to +\infty, \]

and as a consequence,

\[ \mu^N(S_{N}^{(2)}) = \sum_{u \in S_{N}} \mu^N(u) \mathbb{P}(\tilde{U}_{N^{-1/4}}^{N,u} \in S_{N}^{(2)}) \to 1, \text{ as } N \to +\infty. \]

Let

\[ S_{N}^{(3)} = \{ u \in S_{N} : u(a_j^u) \geq j - 1, \text{ for all } j = 1, \ldots, N \} \]

and consider the following events

\[ E_{N,1}^{(3)} = \{ \tilde{Z}^{\downarrow}([0, N^{-2}]) = 0 \}, \]

\[ E_{N,2}^{(3)} = \{ \tilde{Z}^{*}([0, N^{-2}]) \geq N \}. \]

For any \( u \in S_{N}^{(2)} \),

\[ \mathbb{P}(E_{N,1}^{(3)}) \geq \mathbb{P} \left( \xi^{[N]} \geq N^{-2} \right) \to 1, \text{ as } N \to +\infty. \]
For any initial list \( u \in S_N^{(2)} \), the occurrence of the event \( E_{N,1}^{(3)} \) implies that until time \( N^{-2} \) the rate in which the process has a spike is bounded below by \( e^{\lfloor N^{1/2} \rfloor} \). This implies that

\[
\mathbb{P}(E_{N,2}^{(3)}) \geq \mathbb{P} \left( \sum_{j=1}^{N} \xi_j e^{\lfloor N^{1/2} \rfloor} \leq N^{-2} \right) \mathbb{P}(E_{N,1}^{(3)}) \to 1, \text{ as } N \to +\infty.
\]

For any initial list \( u \in S_N^{(2)} \), if the event \( E_{N,1}^{(3)} \cap E_{N,2}^{(3)} \) occurs, then until time \( N^{-2} \) the process has at least \( N \) spiking times and does not have any leakage. This implies that \( \tilde{U}_{N-2}^{N,u} \in S_N^{(3)} \). Therefore,

\[
\sup_{u \in S_N^{(2)}} \mathbb{P}(\tilde{U}_{N-2}^{N,u} \notin S_N^{(3)}) \to 0, \text{ as } N \to +\infty,
\]

and as a consequence,

\[
\mu^N(S_N^{(3)}) = \sum_{u \in \tilde{S}_N} \mu^N(u) \mathbb{P}(\tilde{U}_{N-2}^{N,u} \in S_N^{(3)}) \to 1, \text{ as } N \to +\infty.
\]

Recall that

\[
\mathcal{W}_N = \left\{ u \in S_N : I_N \subseteq \{ u(a) : a \in \mathcal{A}_N \}, \bigcap_{a \in \mathcal{A}_N} \bigcap_{b \neq a} \{ u(a) \neq u(b) \} \right\},
\]

where \( I_N = \{ 1, \ldots, N - \lfloor N^{1/2} \rfloor \} \), and consider the following events

\[
E_{N,1}^{(4)} = \{ \tilde{Z}^+([0, e^{-(N-N^{1/4})}]) = 0 \},
\]

\[
E_{N,2}^{(4)} = \left\{ \sum_{j=1}^{+\infty} \mathbb{1}_{\tilde{T}_j \leq e^{-(N-N^{1/4})}} , \tilde{U}_{\tilde{T}_{j-1}}^{N,u}(\tilde{A}_j) \leq N - \lfloor N^{1/2} \rfloor, O_{j} = \ast \} = 0 \right\},
\]

\[
E_{N,3}^{(4)} = \{ \tilde{Z}^+([0, e^{-(N-N^{1/4})})] \geq N + \lfloor N^{1/2} \rfloor \},
\]

\[
E_{N,4}^{(4)} = \bigcap_{j=1}^{N+[N^{1/2}]} \left\{ \tilde{U}_{\tilde{T}_{j-1}}^{N,u}(\tilde{A}_j) \geq \max \left\{ \tilde{U}_{\tilde{T}_{j-1}}^{N,u}(a) : a \in \mathcal{A}_N \right\} - \lfloor N^{1/2} \rfloor \right\}.
\]

For any \( u \in \tilde{S}_N \),

\[
\mathbb{P}(E_{N,1}^{(4)}) \geq \mathbb{P} \left( \xi^{(N)} \geq e^{-(N-N^{1/4})} \right) \to 1, \text{ as } N \to +\infty.
\]

The rate in which the process has a spike of a neuron that in the moment of the spike have membrane potential smaller than or equal to \( N - \lfloor N^{1/2} \rfloor \) is bounded above by \( Ne^{N-\lfloor N^{1/2} \rfloor} \). Therefore,

\[
\mathbb{P}(E_{N,2}^{(4)}) \geq \mathbb{P} \left( \xi^{[N\ast Ne^{-\lfloor N^{1/2} \rfloor}] > e^{-(N-N^{1/4})} \right) \to 1, \text{ as } N \to +\infty.
\]

For any \( u \in S_N^{(3)} \), the occurrence of the event \( E_{N,1}^{(4)} \) implies that the rate in which the process has a spike is bounded below by \( e^{(N-1)} \). Therefore,

\[
\mathbb{P}(E_{N,3}^{(4)}) \geq \mathbb{P} \left( \sum_{j=1}^{N+[N^{1/2}]} \xi_j e^{(N-1)} \leq e^{-(N-N^{1/4})} \right) \mathbb{P}(E_{N,1}^{(4)}) \to 1, \text{ as } N \to +\infty.
\]
Moreover, the probability
\[ P \left( \tilde{X}_{t_0}^{N,u}(A_1) \geq \max \{ u(a) : a \in A_N \} - \lfloor N^{1/2} \rfloor \right) \tag{12} \]
is minimized when the difference between the membrane potential of the neuron with greatest potential and the membrane potential of the other neurons is \( \lfloor N^{1/2} \rfloor + 1 \). This implies that (12) is bounded bellow by
\[ \frac{e^{\lfloor N^{1/2} \rfloor}}{e^{\lfloor N^{1/2} \rfloor} + 2(N - 1)}, \]
and therefore,
\[ P(E_{N,4}^{(4)}) \geq \left( \frac{e^{\lfloor N^{1/2} \rfloor}}{e^{\lfloor N^{1/2} \rfloor} + 2(N - 1)} \right)^{N + \lfloor N^{1/2} \rfloor} P(E_{N,1}^{(4)} \cap E_{N,3}^{(4)}) \to 1, \text{ as } N \to +\infty. \]

For any initial list \( u \in S_N^{(3)} \), if the event \( E_{N,1}^{(4)} \cap \ldots \cap E_{N,4}^{(4)} \) occurs, then until time \( e^{-(N-N^{1/4})} \) the process has at least \( N + \lfloor N^{1/2} \rfloor \) spiking times, does not have any leakage of membrane potential and does not have any spike of a neuron with membrane potential smaller than or equal to \( N - \lfloor N^{1/2} \rfloor \). This implies that
\[ \{1, \ldots, N - \lfloor N^{1/2} \rfloor\} \subset \{ \tilde{X}_{t_0}^{N,u}(a) : a \in A_N \}. \]
Moreover, the occurrence of the events \( E_{N,1}^{(4)} \cap E_{N,3}^{(4)} \cap E_{N,4}^{(4)} \) implies that all neurons spikes at least once in the first \( N + \lfloor N^{1/2} \rfloor \) steps of the process. This implies that \( \tilde{X}_{t_0}^{N,u}(a) \neq \tilde{X}_{t_0}^{N,u}(a') \), for all \( a \neq a' \). We conclude that \( \tilde{X}_{t_0}^{N,u}(a) \in \mathcal{W}_N \).

Therefore,
\[ \sup_{u \in S_N^{(3)}} \mathbb{P} \left( \tilde{X}_{t_0}^{N,u}(a) \notin \mathcal{W}_N \right) \to 0, \text{ as } N \to +\infty, \]
and as a consequence,
\[ \mu^N(\mathcal{W}_N) = \sum_{u \in S_N} \mu^N(u) \mathbb{P} \left( \tilde{X}_{t_0}^{N,u}(a) \in \mathcal{W}_N \right) \to 1, \text{ as } N \to +\infty. \]

From Lemma 17 it follows Corollaries 18 and 19, that are used to prove Theorems 2 and 3.

**Corollary 18** The following holds
\[ \inf_{u \in S_N} \mathbb{P} \left( \inf \left\{ \tilde{T}_n : \tilde{X}_{t_0}^{N,u}(a) \in \mathcal{L}_N \right\} \leq t_N + e^{-(N-N^{1/2})} \right) \to 1, \text{ as } N \to +\infty, \]
where \( t_N = N^{1/2} + N^{-1/4} + N^{-2} + e^{-(N-N^{1/4})} \).

**Proof** First, note that for any \( u \in S_N \),
\[ \mathbb{P} \left( \inf \left\{ \tilde{T}_n : \tilde{X}_{t_0}^{N,u}(a) \in \mathcal{W}_N \right\} \leq t_N \right) \geq \mathbb{P} \left( \tilde{X}_{t_0}^{N,u}(a) \in S_N^{(1)} \cup S_N^{(2)} \cup S_N^{(3)} \right). \]
Remark 14 implies that for any \( w \in \mathcal{W}_N \),
\[
\inf_{w \in \mathcal{W}_N} \mathbb{P} \left( \inf \left\{ \tilde{T}_n : \tilde{U}^{N,w}_{\tilde{T}_n} \in \mathcal{L}_N \right\} \leq e^{-(N-\delta^{1/2})} \right) \to 1, \text{ as } N \to +\infty.
\]
We conclude the proof by putting all this together with Lemma 17 and Markov property. \( \square \)

**Corollary 19** The following holds
\[
\inf_{u \in S_N^{(0)}} \mathbb{P} \left( \inf \left\{ T_n : U^{N,u}_{T_n} \in \mathcal{L}_N \right\} \leq t'_N \right) \to 1, \text{ as } N \to +\infty.
\]
where \( t'_N = N^{-1/4} + N^{-2} + e^{-N} + e^{-(N-\delta^{1/2})} \).

**Proof** Note that starting from any list \( u \in S_N^{(0)} \), as in the proof of Lemma 17 we have that
\[
\mathbb{P} \left( E^{(2)}_{N,1} \cap E^{(2)}_{N,2} \cap E^{(2)}_{N,3} \right) \to 1, \text{ as } N \to +\infty.
\]
Then, as in Corollary 18 we have that
\[
\inf_{u \in S_N^{(0)}} \mathbb{P} \left( \inf \left\{ \tilde{T}_n : \tilde{U}^{N,u}_{\tilde{T}_n} \in \mathcal{L}_N \right\} \leq t'_N \right) \to 1, \text{ as } N \to +\infty. \tag{13}
\]
By the definition of the events \( E^{(2)}_{N,1}, E^{(3)}_{N,1} \) and \( E^{(4)}_{N,1} \), we have that
\[
\inf_{u \in S_N^{(0)}} \mathbb{P} \left( \inf \left\{ \tilde{T}_n : a \in A_N : \tilde{U}^{N,u}_{\tilde{T}_n} (a) > 0 \right\} = 1 \right) \to 1, \text{ as } N \to +\infty.
\]
By Remark 15 and the coupling construction, it follows that
\[
\inf_{u \in S_N^{(0)}} \mathbb{P} \left( U^{N,u}_{t} = \tilde{U}^{N,u}_{t}, \text{ for all } t \in [0, t'_N] \right) \to 1, \text{ as } N \to +\infty.
\]
Therefore, we can replace \( \tilde{U} \) by \( U \) and \( \tilde{T}_n \) by \( T_n \) on Eq. (13) and with this we concluded the proof. \( \square \)

**Lemma 20** For any \( N \geq 2 \), for any list \( l \in \mathcal{L}_N \) and for any \( s > 0 \),
\[
\mathbb{P}(\tilde{U}^{N,l}_s \in \mathcal{W}_N^c) \leq \frac{\mu^N(\mathcal{W}_N^c)}{\mu^N(\mathcal{W}_N)} + \delta(N, s),
\]
where \( \lim_{N \to +\infty} \delta(N, s) = 0 \), for any \( s > 0 \).

**Proof** For any \( s > 0 \),
\[
\mu^N(\mathcal{W}_N) = \sum_{u \in \mathcal{W}_N} \mu^N(u) \mathbb{P}(\tilde{U}^{N,u}_s \in \mathcal{W}_N) + \sum_{u \in \mathcal{S}_N \setminus \mathcal{W}_N} \mu^N(u) \mathbb{P}(\tilde{U}^{N,u}_s \in \mathcal{W}_N).
\]
By Remark 15, for any \( l \in \mathcal{L}_N \) and \( w \in \mathcal{W}_N \) we have
\[
\mathbb{P}(\tilde{U}^{N,w}_s \in \mathcal{W}_N) \leq \mathbb{P}(\tilde{U}^{N,l}_s \in \mathcal{W}_N) + \mathbb{P} \left( \{ T_{N}^{N,l}(l, w) (l, w) > s \} \cup \{ N^{N}_c(l, w) > N^{N}_c(l, w) \} \right).
\]
Considering
\[
\delta(N, s) = \sup_{l \in \mathcal{L}_N} \sup_{w \in \mathcal{W}_N} \mathbb{P} \left( \{ T_{N}^{N,l}(l, w) (l, w) > s \} \cup \{ N^{N}_c(l, w) > N^{N}_c(l, w) \} \right),
\]
by Lemma 12 and Corollary 13 it follows that \( \lim_{N \to +\infty} \delta(N, s) = 0 \), for any \( s > 0 \). Moreover,

\[
\sum_{u \in \tilde{S}_N \setminus W_N} \mu^N(u) \mathbb{P}(\tilde{U}^N_s, u \in L_N) \leq 1 - \mu^N(W_N).
\]

This implies that

\[
\mu^N(W_N) \leq \mu^N(W_N)(\mathbb{P}(\tilde{U}^N_s, l \in W_N) + \delta(N, s)) + (1 - \mu^N(W_N)),
\]

and therefore,

\[
\mathbb{P}(U^N_s, l \in W_N) \geq \frac{\mu^N(W_N) - (1 - \mu^N(W_N))}{\mu^N(W_N)} - \delta(N, s).
\]

With this we concluded the proof of Lemma 20. \( \square \)

**Proof** Now we will prove Theorem 2. By Remark 15 and the invariance by permutation of the process it follows that for any \( u \in S^0_N \), for any \( l \in L_N \) and for any \( t > 0 \),

\[
\mathbb{P}\left(U^N_t, u \in S_N \setminus W_N \mid \tau^N_s > t\right) \leq \mathbb{P}(\inf\{t > 0 : U^N_t, u \in L_N\} > t/2) + \sup_{s \in [t/2, t]} \mathbb{P}\left(\tilde{U}^N_s, l \in \tilde{S}_N \setminus W_N\right).
\]

By Corollary 19,

\[
\sup_{u \in S^0_N} \mathbb{P}(\inf\{t > 0 : U^N_t, u \in L_N\} > t/2) \to 0, \text{ as } N \to +\infty.
\]

By Lemma 20,

\[
\sup_{s \in [t/2, t]} \mathbb{P}\left(\tilde{U}^N_s, l \in \tilde{S}_N \setminus W_N\right) \leq \frac{\mu^N(W^s_N)}{\mu^N(W_N)} + \delta(N, t/2).
\]

By Lemmas 17 and 20 it follows that

\[
\lim_{N \to +\infty} \delta(N, t/2) = \lim_{N \to +\infty} \mu^N(W^s_N) = 0
\]

and with this we concluded the proof. \( \square \)

**Remark 21** For any \( N \geq 2 \), \( \mathbb{P}(\inf\{t > 0 : U^N_t, u \in L_N\} > t/2) \) and \( \delta(N, t/2) \) decreases with \( t \). This implies that for any \((t_N : N \geq 2)\) such that \( \lim_{N \to +\infty} t_N = +\infty \), we have

\[
\inf_{u \in S^0_N} \mathbb{P}\left(U^N_t, u \in W_N \mid \tau^N_s > t_N\right) \to 1, \text{ as } N \to +\infty.
\]

**Remark 22** The important feature used in the proof of Theorem 2 is that the exponential growth of the spiking rate produces the mechanism that leads the process to the set \( W_N \). This mechanism also allows the process to instantaneously reach the set of ladder lists when starting from \( W_N \) as \( N \to +\infty \), which is important to prove Theorem 3. The choice of the base is not important. The only requirement of the base is to be greater than 1.
6 Proof of Theorem 3

For any fixed $l \in \mathcal{L}_N$, let $c_{N,l}$ be the positive real number such that

$$\mathbb{P}(\tau^{N,l} > c_{N,l}) = e^{-1}. \quad (14)$$

Due to the invariance by permutation of the process, it is clear that $c_{N,l} = c_{N,l'}$, for any pair of lists $l$ and $l'$ belonging to $\mathcal{L}_N$. Therefore, in what follows we will omit to indicate $l$ in the notation of $c_N$.

To prove Theorem 3, we prove in Proposition 24 that for any sequence $(l_N \in \mathcal{L}_N : N \geq 2)$,

$$\frac{\tau^{N,l_N}}{c_N} \to \text{Exp}(1), \text{ as } N \to +\infty,$$

where Exp(1) is a random variable exponentially distributed with mean 1. We will prove that the limiting distribution satisfies the memoryless property, which characterizes the exponential distribution. For this, it is necessary to prove that $c_N \to +\infty$ as $N \to +\infty$. This is the content of Proposition 23. Lemmas 26 and 27 give the necessary conditions to replace $c_N$ by $\mathbb{E}[\tau^{N,l_N}]$ in Proposition 24. Using the fact that the process starting from any list in $\mathcal{W}_N$ will quickly reach $\mathcal{L}_N$ as $N \to +\infty$ we finish the proof of Theorem 3.

**Proposition 23** For any $N \geq 3$,

$$c_N \geq \frac{N - 1 + e^{(N-2)}}{(N - 1)^3}.$$

**Proof** For a initial list $l \in \mathcal{L}_N$, let

$$\tau_+^N = \inf \left\{ T_n : O_n = \dagger, \bigcup_{j=1}^{N-1} \{O_{n-j} = \dagger\} \right\}.$$

We have

$$\tau_+^N = \sum_{j=1}^{G} (T_j^\dagger - T_{j-1}^\dagger),$$

where $G = \inf\{j : Z^*([T_{j-1}^\dagger, T_j^\dagger]) \leq N - 2\}$.

The rate in which the process has a leakage is bounded above by $N - 1$. Therefore, for any $j \geq 1$ and for any $s > 0$,

$$\mathbb{P}(T_j^\dagger - T_{j-1}^\dagger > s) \geq \mathbb{P}(\xi^{(N-1)} > s).$$

Recall that

$$S^{(3)}_N = \{u \in S_N : u(a^u_j) \geq j - 1, \text{ for any } j = 1, \ldots, N\}.$$

For any initial list $w \in \mathcal{W}_N$, we have that

$$U^{N,w}_t \in S^{(3)}_N, \text{ for any } t < T_1^\dagger.$$
Moreover, for any initial list $u \in S_N \setminus \{0_N\}$, if $O_1 = \ldots = O_{N-1} = \ast$, then $U_{T_{N-1}^j}^{N,u} \in S_N^{(3)}$. Together with Markov property, this implies that for any $m \geq 1$ and for any $j \geq 1$,

$$
\mathbb{P}(Z^*([T_{j-1}^\dagger, T_j^\dagger]) \leq N - 2 \mid T_{j-1}^\dagger = T_m, G \geq j) = \mathbb{P}\left( \bigcup_{j=1}^{N-1} \{O_{j+m} = \dagger\} \mid O_m = \dagger, U_{T_{m-1}}^{N,u} \in S_N^{(3)} \right).
$$

The probability on the right-hand side of equation above is bounded above by

$$
\lambda_N = (N - 1) \times \frac{N - 1}{N - 1 + e^{(N-2)}}.
$$

Therefore, for any $s > 0$,

$$
\mathbb{P}(\tau_{-N}^N > s) \geq \mathbb{P}\left( \sum_{j=1}^{\text{Geom}(\lambda_N)} \xi_j^{[N-1]} > t \right),
$$

where Geom($\lambda_N$) is a random variable independent of $(\xi_j^{[N-1]})_{j \geq 1}$ with Geometric distribution assuming values in $\{1, 2, \ldots\}$ with mean $1/\lambda_N$. This implies that

$$
\mathbb{P}(\tau_{-N}^N > s) \geq \mathbb{P}\left( \xi^{[\lambda_N(N-1)]} > s \right).
$$

Therefore,

$$
e^{-1} = \mathbb{P}(\tau_{-N}^N > c_N) \geq \mathbb{P}(\tau_{-N}^N > c_N) \geq e^{-c_N \lambda_N(N-1)},
$$

and then,

$$
c_N \geq \frac{1}{\lambda_N(N - 1)}.
$$

To prove Theorem 3, we prove Proposition 24 which is interesting by itself.

**Proposition 24** For any sequence $(l_N \in \mathcal{L}_N : N \geq 2)$,

$$
\frac{\tau_{-N}^{N,l}}{c_N} \xrightarrow{\text{Exp}(1)} \text{Exp}(1), \text{ as } N \rightarrow +\infty,
$$

where Exp(1) is a random variable exponentially distributed with mean 1.

**Proof** First of all, we will prove that for any sequence $(l_N \in \mathcal{L}_N : N \geq 2)$ and for any pair of positive real numbers $s, t \geq 0$, the following holds

$$
\lim_{N \rightarrow +\infty} \left| \mathbb{P}\left( \frac{\tau_{-N}^{N,l}}{c_N} > s + t \right) - \mathbb{P}\left( \frac{\tau_{-N}^{N,l}}{c_N} > s \right) \mathbb{P}\left( \frac{\tau_{-N}^{N,l}}{c_N} > t \right) \right| = 0. \quad (15)
$$

Indeed, for any $N \geq 2$ and for any $l \in \mathcal{L}_N$,

$$
\left| \mathbb{P}\left( \frac{\tau_{-N}^{N,l}}{c_N} > s + t \right) - \mathbb{P}\left( \frac{\tau_{-N}^{N,l}}{c_N} > s \right) \mathbb{P}\left( \frac{\tau_{-N}^{N,l}}{c_N} > t \right) \right| \leq
$$

$$
\sum_{u \in S_N \setminus \{0_N\}} \mathbb{P}\left( U_{\cdot, c_N}^{N,l} = u, \frac{\tau_{-N}^{N,l}}{c_N} > s \right) \left| \mathbb{P}\left( \frac{\tau_{-N}^{N,l}}{c_N} > t \right) - \mathbb{P}\left( \frac{\tau_{-N}^{N,l}}{c_N} > t \right) \right|. \quad (16)
$$

\(\square\)
The right-hand side of Eq. (16) is equal to
\[
\sum_{u \in \mathcal{W}_N} \mathbb{P}\left( U_{c_N}^{N,l} = u, \frac{\tau_{c_N}^{N,l}}{c_N} > s \right) \left| \mathbb{P}\left( \frac{\tau_{c_N}^{N,u}}{c_N} > t \right) - \mathbb{P}\left( \frac{\tau_{c_N}^{N,l}}{c_N} > t \right) \right| + \\
\sum_{u \in \mathcal{S}_N \setminus (\mathcal{W}_N \cup 0_N)} \mathbb{P}\left( U_{c_N}^{N,l} = u, \frac{\tau_{c_N}^{N,l}}{c_N} > s \right) \left| \mathbb{P}\left( \frac{\tau_{c_N}^{N,u}}{c_N} > t \right) - \mathbb{P}\left( \frac{\tau_{c_N}^{N,l}}{c_N} > t \right) \right| \leq \\
\sup_{w \in \mathcal{W}_N} \left| \mathbb{P}(\tau_{c_N}^{N,l} > c_N t) - \mathbb{P}(\tau_{N,w}^{N} > c_N t) \right| + \mathbb{P}\left( U_{c_N}^{N,l} \in \mathcal{S}_N \setminus \mathcal{W}_N, \tau_{c_N}^{N,l} > c_N s \right). \quad (17)
\]

By Theorem 2, Remark 21 and Propositions 8 and 23, Eq. (17) and the invariance by permutation of the process implies (15).

By definition, for any \( N \geq 2 \) and for any \( l \in \mathcal{L}_N \),
\[
\mathbb{P}\left( \frac{\tau_{c_N}^{N,l}}{c_N} > 1 \right) = e^{-1}.
\]

Iterating (15) with \( t = s = 2^{-n} \), for \( n = 1, 2, \ldots \), we have that for any sequence \( (l_N \in \mathcal{L}_N : N \geq 2) \),
\[
\mathbb{P}\left( \frac{\tau_{c_N}^{N,l_N}}{c_N} > 2^{-n} \right) \to e^{-2^{-n}}, \quad \text{as } N \to +\infty.
\]

More generally, we have that for any
\[
t \in \left\{ \sum_{n=1}^{m} b(n)2^{-n} : b(n) \in \{0, 1\}, n = 1, \ldots, m, m \geq 1 \right\}
\]
is valid that
\[
\mathbb{P}\left( \frac{\tau_{c_N}^{N,l_N}}{c_N} > t \right) \to e^{-t}, \quad \text{as } N \to +\infty. \quad (18)
\]

Any real number \( r \in (0, 1) \) has a binary representation
\[
r = \sum_{n=1}^{+\infty} b(n)2^{-n},
\]
where for any \( n \geq 1, b(n) \in \{0, 1\} \). Therefore, the monotonicity of
\[
t \to \mathbb{P}\left( \frac{\tau_{c_N}^{N,l_N}}{c_N} > t \right)
\]
implies that the convergence in (18) is valid for any \( t \in (0, 1) \). Moreover, for any positive integer \( n \geq 1 \), Eq. (15) implies that
\[
\mathbb{P}\left( \frac{\tau_{c_N}^{N,l_N}}{c_N} > n \right) \to e^{-n}, \quad \text{as } N \to +\infty.
\]

We conclude that (18) is valid for any \( t > 0 \).
Remark 25 For any $N \geq 2$ and for any $l_N \in \mathcal{L}_N$, the function $f_N : [0, +\infty) \rightarrow [0, 1]$ given by

$$f_N(t) = \mathbb{P}\left( \frac{\tau^{N,l_N}}{c_N} > t \right)$$

is monotonic. Also, by Proposition 24, it converges pointwise as $N \rightarrow +\infty$ to a continuous function. Therefore, for any $(\epsilon_N : N \geq 2)$ such that $\lim_{N \rightarrow +\infty} \epsilon_N = 0$, for any $t > 0$ and for any sequence $(l_N \in \mathcal{L}_N : N \geq 2)$, we have

$$\lim_{N \rightarrow +\infty} \mathbb{P}\left( \frac{\tau^{N,l_N}}{c_N} > t + \epsilon_N \right) = \lim_{N \rightarrow +\infty} \mathbb{P}\left( \frac{\tau^{N,l_N}}{c_N} > t - \epsilon_N \right) = e^{-t}.$$

To prove Theorem 3, we need the two following lemmas.

Lemma 26 For any $t > 0$,

$$\lim_{N \rightarrow +\infty} \sup_{u \in S_N \setminus \{0_N\}} \mathbb{P}\left( \frac{\tau^{N,u}}{c_N} > t \right) \leq e^{-t}.$$

Proof For any $u \in S_N \setminus \mathcal{L}_N$ and for any $N \geq 2$, consider the event

$$E_{N,u} = \left\{ \min\{\tau^{N,u}, \inf\{T_n : U^{N,u}_n \in \mathcal{L}_N\}\} \leq t'_N \right\},$$

where $t'_N = N^{1/2} + N^{-1/4} + N^{-2} + e^{-(N-N^{1/4})} + e^{-(N-N^{1/2})}$. We have that

$$\mathbb{P}\left( \frac{\tau^{N,u}}{c_N} > t \right) = \mathbb{P}\left( \frac{\tau^{N,u}}{c_N} > t, E_{N,u}, \inf\{T_n : U^{N,u}_n \in \mathcal{L}_N\} < \tau^{N,u} \right) + \mathbb{P}\left( \frac{\tau^{N,u}}{c_N} > t, E_{N,u} \right).$$

By Proposition 23, there exists $N_t > 0$ such that for any $N > N_t$, we have that $c_N t > t'_N$. This implies that, for any $N > N_t$ and for any $u \in S_N \setminus \mathcal{L}_N$,

$$\mathbb{P}\left( \frac{\tau^{N,u}}{c_N} > t, E_{N,u}, \tau^{N,u} < \inf\{T_n : U^{N,u}_n \in \mathcal{L}_N\} \right) + \mathbb{P}\left( \frac{\tau^{N,u}}{c_N} > t, E_{N,u} \right) = 0.$$

Considering $l \in \mathcal{L}_N$, for any $u \in S_N \setminus \{\mathcal{L}_N \cup 0_N\}$ and for any $N > N_t$, the left-hand side of Eq. (19) is bounded above by

$$\mathbb{P}\left( \frac{\tau^{N,l}}{c_N} > t - t'_N \right) + \mathbb{P}(E_{N,u}^c).$$

By Remark 15 and Corollary 18, it follows that

$$\lim_{N \rightarrow +\infty} \sup_{u \in S_N \setminus \{0_N\}} \mathbb{P}(E_{N,u}^c) = 0.$$

By Proposition 23, it follows that

$$\lim_{N \rightarrow +\infty} \frac{t'_N}{c_N} = 0.$$
Therefore, by Remark 25 we have that
\[ \lim_{N \to +\infty} \sup_{l \in \mathcal{L}_N} \mathbb{P} \left( \frac{\tau_{N,l}}{c_N} > t - \frac{t_N}{c_N} \right) = e^{-t}. \]

We conclude the proof by noting that the limits in the last equation do not depend on \( u \). \( \Box \)

**Lemma 27** There exists \( \alpha \in (0, 1) \) and \( N_{\alpha} > 0 \) such that for any \( N > N_{\alpha} \) and any \( l \in \mathcal{L}_N \), the following upperbound holds
\[ \mathbb{P} \left( \frac{\tau_{N,l}}{c_N} > n \right) \leq \alpha^n, \]
for any positive integer \( n \geq 1 \).

**Proof** By Lemma 26, for any fixed \( \alpha \in (e^{-1}, 1) \), there exists \( N_{\alpha} \) such that for all \( N > N_{\alpha} \),
\[ \sup_{u \in S_N \setminus \{0\}} \mathbb{P} \left( \frac{\tau_{N,u}}{c_N} > 1 \right) \leq \alpha < 1. \] (21)

For any \( l \in \mathcal{L}_N \) and for any \( n \in \{2, 3, \ldots\} \),
\[ \mathbb{P} \left( \frac{\tau_{N,l}}{c_N} > n \right) = \sum_{u \in S_N \setminus \{0\}} \mathbb{P} \left( \frac{\tau_{N,l}}{c_N} > n - 1, U_{N,cN(n-1)} = u \right) \mathbb{P} \left( \frac{\tau_{N,u}}{c_N} > 1 \right). \]
Equation (21) implies that for any \( N > N_{\alpha} \),
\[ \mathbb{P} \left( \frac{\tau_{N,l}}{c_N} > n \right) \leq \alpha \mathbb{P} \left( \frac{\tau_{N,l}}{c_N} > n - 1 \right). \] (22)

We finish the proof by iterating (22). \( \Box \)

**Proof** We will now prove Theorem 3. First of all, we will prove that for any sequence \( (l_N \in \mathcal{L}_N : N \geq 2) \), the following holds
\[ \frac{\tau_{N,l_N}}{\mathbb{E}[\tau_{N,l_N}]} \to \text{Exp}(1) \text{ in distribution, as } N \to +\infty. \] (23)

Considering Proposition 24, we only need to show that
\[ \lim_{N \to +\infty} \frac{\mathbb{E}[\tau_{N,l_N}]}{c_N} = 1. \]

Actually,
\[ \lim_{N \to +\infty} \frac{\mathbb{E}[\tau_{N,l_N}]}{c_N} = \lim_{N \to +\infty} \int_0^{+\infty} \mathbb{P}(\tau_{N,l_N} > c_N s)ds. \]

Lemma 27 and the Dominated Convergence Theorem, allow us to put the limit inside the integral in the last term
\[ \lim_{N \to +\infty} \int_0^{+\infty} \mathbb{P}(\tau_{N,l_N} > c_N s)ds = \int_0^{+\infty} \lim_{N \to +\infty} \mathbb{P}(\tau_{N,l_N} > c_N s)ds = \int_0^{+\infty} e^{-s}ds. \]

This and Proposition 24 imply (23).

For any \( N \geq 2 \), for any \( u \in S^{(0)}_N \) and for any \( s > 0 \),
\[ \mathbb{P}(\tau_{N,u} > c_N s) = \mathbb{P}(\tau_{N,u} > c_N s, E_{N,u}) + \mathbb{P}(\tau_{N,u} > c_N s, E_{N,u}^c), \]
where 
\[ E_{N,u} = \{ \inf \{ t : U_{t+1}^{N,u} \in \mathcal{L}_N \} \leq 1 \}. \]

For any \( l \in \mathcal{L}_N \), by Markov property and the invariance by permutation of the process we have
\[
\mathbb{P} \left( \frac{\tau_{N,l}}{c_N} > s \right) \mathbb{P}(E_{N,u}) \leq \mathbb{P}(\tau_{N,u} > c_N s, E_{N,u}) \leq \mathbb{P} \left( \frac{\tau_{N,l}}{c_N} > s - \frac{1}{c_N} \right) \mathbb{P}(E_{N,u}). \tag{24}
\]
By Corollary 19,
\[
\lim_{N \to +\infty} \inf_{u \in S_{N(0)}^u} \mathbb{P}(E_{N,u}) = 1,
\]
and then, for any sequence \((u_N) \in S_{N(0)}^u : N \geq 2\),
\[
\lim_{N \to +\infty} \mathbb{P}(\tau_{N,u_N} > c_N s, E_{N,u_N}) = \lim_{N \to +\infty} \mathbb{P}(\tau_{N,u} > c_N s).
\]
Proposition 23 and Remark 25 implies that for any sequence \((l_N) \in \mathcal{L}_N : N \geq 2\),
\[
\lim_{N \to +\infty} \mathbb{P} \left( \frac{\tau_{N,l_N}}{c_N} > s \right) = \lim_{N \to +\infty} \mathbb{P} \left( \frac{\tau_{N,l}}{c_N} > s - \frac{1}{c_N} \right) = e^{-s}.
\]
The conclusion follows from Eq. (24) and by observing that the Dominated Convergence Theorem allow us to replace \(c_N\) by \(\mathbb{E}[\tau_{N,u}^N]\) as we did to prove that Eq. (23) holds. \(\square\)

### 7 Proof of Theorems 4, 5 and 6

In this section we show how the proofs presented before can be modified in order to prove Theorems 4, 5 and 6 concerning the process \((\hat{U}_t^{N,u})_{t \in [0, +\infty)}\).

We will use the following notation. Let \(\hat{T}_0 = 0\) and for \(n = 1, \ldots, N_{N,u} \), let \(\hat{T}_n\) denote the successive jumping times of the process \((\hat{U}_t^{N,u})_{t \in [0, +\infty)}\), namely
\[
\hat{T}_n = \inf \left\{ t > \hat{T}_{n-1} : \hat{U}_t^{N,u} \neq \hat{U}_{\hat{T}_{n-1}}^{N,u} \right\}.
\]
We will now prove Theorem 4.

**Proof of Theorem 4** For any \(N \geq 2\) and for any \(u, u' \in \mathcal{S}_N \setminus \{\emptyset_N\}\), we have that
\[
\mathbb{P} \left( \hat{N}_t^{N,u} \leq n + N - 1 + \frac{N(N-1)}{2} \left| \hat{U}_{\hat{T}_n}^{N,u} = u' \right. \right) \geq
\]
\[
\mathbb{P} \left( \hat{U}_{\hat{T}_n}^{N,u} \in \mathcal{L}_N \left| \hat{U}_{\hat{T}_n}^{N,u} = u' \right. \right) \mathbb{P} \left( \hat{N}_t^{N,u} \leq n + \frac{N^2 + N - 2}{2} \left| \hat{U}_{\hat{T}_n}^{N,u} \in \mathcal{L}_N \right. \right).
\]
Lemma 7 still holds for \((\hat{U}_t^{N,u})_{t \in [0, +\infty)}\). Therefore,
\[
\mathbb{P} \left( \hat{U}_{\hat{T}_{n+N-1}}^{N,u} \in \mathcal{L}_N \left| \hat{U}_{\hat{T}_n}^{N,u} = u' \right. \right) \geq [2(N-1)]^{-(N-1)}.
\]
Note that starting from a ladder list \(l \in \mathcal{L}_N\), the process can reach the null list after \(N(N-1)/2\) steps. For this, it is sufficient that the process has \(N(N-1)/2\) consecutive leakage times in...
which the neuron with membrane potential \( j \) on the list \( l \) is associated to \( j \) of these leakage times, for \( j = 1, \ldots, N-1 \). Calling
\[
\hat{\epsilon} = \mathbb{P}\left( \hat{N}^{N,u} \leq n + N - 1 + \frac{N(N-1)}{2} \mid \hat{U}_{l+n-N-1}^{N,u} = l \right),
\]
we conclude that \( \hat{\epsilon} > 0 \).

Using the invariance by permutation of the process and the Markov property as in the proof of Theorem 1, we conclude that for any \( u' \in S_N \),
\[
\mathbb{P}\left( \hat{N}^{N,u} \leq n + N - 1 + \frac{N(N-1)}{2} \mid \hat{U}_{l+n-N-1}^{N,u} = u' \right) \geq [2(N-1)]^{-(N-1)} \hat{\epsilon}.
\]

The last inequality implies that for any \( n \geq 1 \),
\[
\mathbb{P}(\hat{N}^{N,u} \geq n) \leq \mathbb{P}\left( \left( N - 1 + \frac{N(N-1)}{2} \right) \times \text{Geom}(2(N-1)]^{-(N-1)} \hat{\epsilon} \right) \geq n \),
\]
where \( \text{Geom}(r) \) denotes a random variable with geometric distribution assuming values in \{1, 2, \ldots\} and with mean \( 1/r \). This implies that \( \mathbb{P}(\hat{N}^{N,u} < +\infty) = 1 \). We prove that \( \mathbb{P}(\hat{\tau}^{N,u} < +\infty) = 1 \) exactly as we did in the proof of Theorem 1.

\begin{proposition}
The following holds
\[
\lim_{N \to +\infty} \sup_{t \geq 0} \sup_{w,w' \in W_N} \left| \mathbb{P}(\hat{\tau}^{N,w} > t) - \mathbb{P}(\hat{\tau}^{N,w'} > t) \right| = 0.
\]
\end{proposition}

\begin{proof}
We can define a coupling between processes \((\hat{U}_{l}^{N,u})_{t \in [0, +\infty)}\) and \((\hat{U}_{l}^{N,v})_{t \in [0, +\infty)}\) exactly as we did in Sect. 4 for the processes \((U_{l}^{N,u})_{t \in [0, +\infty)}\) and \((U_{l}^{N,v})_{t \in [0, +\infty)}\). Using this coupling, we can prove Proposition 28 exactly as we did with Proposition 8.
\end{proof}

To prove Theorem 5, we define an auxiliary process for \((\hat{U}_{l}^{N,u})_{t \in [0, +\infty)}\) exactly as we did in Sect. 5 for the processes \((U_{l}^{N,u})_{t \in [0, +\infty)}\). Using this auxiliary process, we can prove Theorem 5 exactly as we did with Theorem 2.

Finally, the proof of Theorem 6 can be done exactly as we did with Theorem 3.

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