Topological charges and quasi-charges in Absolute Parallelism

I. L. Zhogin

Abstract

The frame field theory, or Absolute Parallelism (AP), has very many interesting features: the large symmetry group of equations; the field irreducibility with respect to this group; variety of differential covariants; and large list of consistent or formally integrable second order equations not restricted to Lagrangian ones.

There is one unique variant of AP (the unique equation, non-Lagrangian) which solutions of general position seem to be free of arising singularities if $D=5$. In the absence of singularities, when degenerate (co)frame matrices are inaccessible and should be eliminated from the field set, AP acquires the topological features of nonlinear sigma-model.

Starting with the topological charge group, one can also introduce the concept of topological quasi-charge group for field configurations having some symmetry. For $D=4$, considering symmetrical equipped 0-(sub)manifolds in $\mathbb{R}^3$, we calculate quasi-charge groups $\varPi(G)$ for a number of symmetry groups $G \subset O_3$, and describe morphisms of $\varPi$-groups.

Then, the differential 3-form of topological charge, which is dual to the topological current $J^\mu$, is derived, as well as the 1-form of topological quasi-charge for $SO_3$-symmetric field configurations.

The problem for $D=5$ is briefly discussed, and results of topological classification of symmetric field configurations (alighting on evident parallels with the Standard Model combinatorics of fundamental particles) are announced. An example of $SO_2$-symmetric configuration is considered and the quasi-charge 3-forms – both ‘left’ and ‘right’ (or self-dual and anti-self-dual) – are obtained (as well as the 4-forms of topological charge).

In conclusion, we propose a variant of experiment with single photon interference (or with bi-photon non-local correlations) which should verify a possible non-local (spaghetti-like) 5D ontology of particles.

1 Introduction

Most (if not all) modern works on Absolute Parallelism (either pure AP, or modified with extra structures) [1–5] follow the Lagrangian approach to obtain field equations (Ref. [1] also has many historical comments on AP). However, the large list of consistent second order equations of AP discovered by Einstein and Mayer [6] (the earlier Einstein’s original papers on AP are available in English translation [8]) includes also three classes of non-Lagrangian equations; one of these classes is of particular interest and it admits 3-linear presentation [9, 10].

After due consideration for AP equations and notations used in the paper, we will return to introduction notes. Next, the main goal of this paper will be to demonstrate that spatially

*E-mail: zhogin at inp.nsk.su; http://zhogin.narod.ru
localized field configurations of AP can carry integer data – topological charge and/or quasi-charge (in the case of symmetrical field configurations), and to show how to calculate these charges. (Our tacit intention is also to outline conceivable phenomenology of such particle-like-configurations, indicating parallels with the existing combinatorics of elementary particles and asking ourselves whether this phenomenology can benefit (in some circumstances) from all the machinery of quantum (field) theory – including principle of superposition, and path integrals, and so on.) In the last section a variant of experiment with single photon interference (or biphoton non-local correlations) is suggested which might verify a possible non-local (spaghetti-like) 5D nature of ‘particles’.

1.1 Notations and consistent AP equations

In this paper we will be dealing with the only field – frame field $h^a_{\mu}$, and inverse matrix – co-frame field $h^a_{\mu}(x^\nu)$; this field defines the space-time metric as usual using the Minkowski metric:

$$g_{\mu\nu} = \eta^{ab} h^a_{\mu} h^b_{\nu}, \text{ where } \eta_{ab} = \eta^{ab} = \text{diag}(-1,1,\ldots,1).$$

We will use comma “,” and semicolon “;” to denote partial derivative and usual covariant differentiation with symmetric Levi-Civita (or Christoffel) connection, respectively, as well as the following tensors (differential covariants):

$$\gamma_{a\mu\nu} = h^a_{\mu} h^b_{\nu}, \quad \Lambda_{a\mu\nu} = 2\gamma_{[a\mu\nu]} = h_{a\mu,\nu} - h_{a\nu,\mu}.$$ (2)

We shall omit in contractions the matrices $\eta^{ab}, \eta_{ab}$ and, in covariant expressions (where only the covariant differentiation is in use), $-g^{\mu\nu}, g_{\mu\nu}$, because

$$0 = g^{\mu\nu}, \lambda = g_{\mu\nu,\lambda} = \eta^{ab}, \mu = \eta_{ab,\mu}.$$ (4)

Let introduce notations for the following (irreducible) covariants:

$$S_{\mu\nu\lambda} = 3\Lambda_{[\mu\nu\lambda]}, \quad \Phi_{\mu} = \Lambda_{a\mu}, \quad f_{\mu\nu} = \Phi_{\mu\nu} - \Phi_{\nu\mu}.$$ (3)

The type of an index is changed by means of (co)frame, so the same letter is used for covariants with any indices – Latin, or Greek, or mixed, with the only evident exception for (co)frame, metric, and Minkowski matrix. For example:

$$\gamma_{\mu\nu\lambda} = h_{a\mu} h_{a\nu} h_{a\lambda} (= \gamma_{[\mu\nu\lambda]}), \quad f_{\mu\nu} = f_{\mu\nu} h_{a\mu} h_{a\nu}.$$ (5)

Note that the definition (2) leads to the following identities:

$$\Lambda_{a[\mu\nu\lambda]} \equiv 0, \quad h_{a\lambda} \Lambda_{abc;\lambda} + f_{bc} \equiv 0.$$ (4)

The most simple case of consistent (or formally integrable, or well-posed) AP equations is the two-parameter non-Lagrangian class $\Pi_{22112}$ of $[6]$ (the Lagrangian equations should have the term $h_{a\mu}\mathcal{L}$):

$$A_{a\mu} = K_{a\mu \nu} = 0, \quad K_{abc} = K_{a[bc]} = \alpha \Lambda_{abc} - \beta S_{abc} - \gamma \eta_{ab} \Phi_c + \gamma \eta_{ac} \Phi_b;$$ (5)

here the overall coefficient is an arbitrary value. The evident identity

$$A_{a\mu;\mu} \equiv 0 \quad (K_{a\mu;[\nu\mu]} \equiv 0)$$
ensures the consistency of the Eq. (5) with the exception of “bad” cases: $\alpha=0$, or $2\beta=\alpha$, or (and) $\gamma=\alpha$; it is preferable to place $\alpha=1$ and this will be in effect below.

(In the case $\alpha=0$, the prolonged equation $A_{\mu}^{\nu(\alpha)} = A_{(\mu\nu)\lambda} + A_{(\mu\lambda)\nu} - A_{(\nu\lambda)\mu} = 0$ gives: $\Phi_{\mu(\nu\lambda)} - O(\Lambda,\Lambda') = 0$, and the next prolonged equation, $A_{\mu}^{\nu(\alpha)} - A_{\mu}^{\nu(\alpha)} = 0$, leads to a new and irregular equation: the principal derivatives vanish, but ‘quadratic’ terms do not. The other ‘bad’ cases will be explained below.)

Einstein and Mayer had usually used not Levi-Civita, but Weitzenböck connection derived from the condition

$$0 = h_{a\mu\nu} = h_{a\mu\nu} - \Gamma_{\mu\nu}^\lambda h_{a\lambda}; \quad \Gamma_{\mu\nu}^\lambda = h_{a\lambda} h^{a}_{\mu\nu}. \quad (6)$$

However, the symmetric connection is equally suitable to write any covariant system of AP (in this case, ‘Maxwell-like’ equations for skew-symmetric covariants like in the Eq. (5) become even more clear).

The next evident class (labelled in [6] as $I_{12}$) of consistent equations is the two-parameter class of Lagrangian equations. Using the scalar density $h\mathcal{L}$, $h = \det h_{a\mu}$, with two free parameters ($\alpha=1$),

$$\mathcal{L} = \frac{1}{2} \Lambda_{\mu\nu} \Lambda_{\mu\nu} - \frac{\beta}{12} \mathcal{S}_{\mu\nu\rho\sigma} \mathcal{S}_{\mu\nu\rho\sigma} - \frac{\gamma}{2} \Phi_{a} \Phi_{a},$$

and taking into account the symmetry properties of $\Lambda$ and $S$, see (2), (3), and (5), one can obtain the Lagrangian equations:

$$d\mathcal{L} = \frac{1}{2} K_{\mu\nu} d\Lambda_{\mu\nu} = K_{a}^{\mu\nu} dh_{a\mu\nu} - \Lambda_{bca} K_{bca}^{\mu} dh_{a\mu},$$

$$B_{a}^{\mu} = -\frac{\delta(h\mathcal{L})}{\delta h_{a\mu}} = K_{a}^{\mu\nu} + \Lambda_{bca} K_{bca}^{\mu} - h_{a\mu} \mathcal{L} = 0. \quad (7)$$

The identity providing the formal integrability looks as follows:

$$B_{a\mu\nu} - B_{bca} L_{bca} \equiv 0;$$

so, we have the conservation law: $(h\Lambda_{bca} K_{bca}^{\mu} - h_{\alpha}^{\mu\nu} \mathcal{L})_{,\mu} = 0$. The skew-symmetric part of (7),

$$2B_{[\mu\nu]} = (1 - 2\beta)(S_{\mu\nu\lambda\lambda} - \Lambda_{\muab} \Lambda_{\lambda\nu} + \Lambda_{\nuab} \Lambda_{\lambda\mu}) + (1 - \gamma) (f_{\mu\nu} - \Phi_{a} \Lambda_{a\mu\nu}) = 0,$$

disappears (or turns into identity) if $2\beta=\gamma=1$, and this is the case of the General Relativity (of course, in this case one (do) can add a skew-symmetric equation with a large arbitrariness).

The cases $2\beta=1$, $\gamma\neq1$ and $\gamma=1$, $2\beta\neq1$ are both “bad” (ill_posed), because the equation $B_{[\mu\nu]} = 0$, or, respectively, $B_{[\mu\nu]} = 0$, leads to a new and irregular second order equation. The last case, despite its incorrectness, still can be found in publications[13 14].

The third class $C$ (two-parameter class labelled in [6] as $I_{221}$) is possible (that is, consistent) only in 4D, and its equations in symmetric part resemble the equations of Brans-Dicke theory (with $\Phi_{\mu}$ instead of $\phi_{\mu}/\phi$; here $G_{\mu\nu}$ is the Einstein tensor)

$$C_{[\mu\nu]} = 2G_{\mu\nu} + 2\gamma(\Phi_{[\mu\nu]} - g_{\mu\nu} \Phi_{(\tau\tau)} + \gamma^{2}(\Phi_{\mu} \Phi_{\nu} + g_{\mu\nu} \Phi^{2}/2) = 0,$$

while the skew-symmetric part is quite simple: $C_{[\mu\nu]} = S_{\mu\nu\lambda\lambda} + \beta f_{\mu\nu} = 0 \quad [22]$.

The last and most interesting class of consistent AP equations (the one-parameter class $I_{221221}$ of [6]) admits the next presentation (here $L_{a\mu\nu} = L_{a[\mu\nu]} = \Lambda_{a\mu\nu} - S_{a\mu\nu} - 2\gamma h_{a\mu} \Phi_{(\nu)}$):

$$D_{a\mu} = L_{a\mu\nu} - (1 - 2\gamma)(f_{a\mu} + L_{a\mu} \Phi_{\nu}) = 0, \quad (8)$$
and $2D_{[\nu \mu]} = S_{\nu \mu \lambda; \lambda} + (1 - 3\gamma)(f_{\nu \mu} - S_{\nu \mu \lambda} \Phi^\lambda)$.

The equations $D_{a \mu; \mu} = 0$ and $D_{[\mu \nu]; \nu} = 0$ give the same ‘Maxwell equation’ (hence, the identity required for consistency is here):

$$f_{\mu \nu; \nu} = (S_{\mu \nu \lambda} \Phi^\lambda)_{; \lambda} = (1 - 3\gamma) f_{\mu \nu} \Phi^\nu - \frac{1}{2} S_{\mu \nu \lambda} f_{\nu \lambda}. \quad (9)$$

The case $\gamma = \frac{1}{3}$ of Eq. (8) is of special interest, and it requires extra-dimension(s) because the trace equation $D_{a a} = 0$ becomes irregular (the principal derivatives vanish) if $D = 1 + \gamma^{-1}$.

### 1.2 Introduction, continued

We believe that the frame field theory (aka Absolute Parallelism) is to become of significant interest for mathematical physics (as well as for cosmology and particle physics; better to say – for physics) because of the following reasons and features:

1. **High symmetry of this theory:**

   This symmetry group includes symmetries of both special and general relativity theories; so, staying in a class of equivalent solutions, $h(x)$, to some AP equation, one can perform diffeomorphic coordinate mappings (democracy of “good” coordinates) and global transformations from “completed” Lorentz group (this is the point symmetry group of inertial coordinates, which includes also the global scale transformations):

   $$\tilde{h}^a_{\mu}(\tilde{x}) = \kappa \sigma^a_b h^b_{\nu}(x) \frac{\partial x^\nu}{\partial \tilde{x}^\mu}, \quad (10)$$

   where $\kappa > 0$, $\sigma^a_b \in O(1, D-1)$; $\kappa, \sigma^a_b = \text{const}$. The irreducibility of this (vector) field representation is also very important feature helping to avoid free parameters$^1$. In our opinion, the space-time signature without Lorentz-group (as it takes place in GR) looks a bit like beer-foam without beer, or (for very versed specialists) like the smile without the cat; it means that the choice of signature in GR is the separate postulate having no relation to the symmetry group of GR.

   On the other hand, the inertial coordinates of Special Relativity (which are the basis of all QFT), being although simple and aristocratic, have some of strangeness and unnaturalness. Both AP and GR say (answering the Mach’s question) that the space-time geometry is not an immutable essence, and that the inertial coordinates, $y^a$, are the property (or attribute) of trivial solution only. Indeed, one can integrate the equations $y^a_{; \mu} = h^a_{\mu}$, if $\Lambda_{a \mu \nu} = 0$. Are these coordinates not a bit an empty, unreachable abstraction scarcely suitable to be a solid basis for a really fundamental theory? The $\text{Diff}$-covariance is of practical importance as it gives another sense to the gradient catastrophe problem (solution becomes being multivalued) – as having relation to co-singularities (naturally, this problem is not burning if you are in the perturbative domain); at the same time the field irreducibility reduces or restricts the number of ways to singularities of solutions: the rank of frame (co-frame) matrix is the only significant parameter for dealing with (co-) singularities (compare with bi- or non-symmetric metric).

---

$^1$In gage theories, e.g., the need to combine a gage field and matter ones (reducibility) leads immediately to an arbitrary parameter – the gage constant.
2. Absence of arising singularities and uniqueness:

There is one unique variant of AP (non-Lagrangian, with the unique $D; D=5$) which solutions of general position seem to be free of arising singularities. Extension of formal integrability test (applied for second order AP-equations) to the cases of degeneration of either co-variant frame matrix (co-singularities) or contra-variant frame (or contra-frame density of some weight) may serve, we believe, as the local and covariant (no coordinate choice) test for singularities of solutions. In AP this test singles out the unique equation, the case $\gamma=\frac{1}{3}$ of the Eq. (8), and the space-time dimension mentioned above \[22\ \ 23\ \ 9\].

Solutions to this equation, if they exist not only 20 billion years, and not only septillion years, but unlimited time, always (there are no stops with singularities), still being ‘interesting’, should be very complicated having many parameters and scales (the unique equation itself has no characteristic scales). Very slowly varying parameters might serve for some fast processes as ‘phenomenological constants’, and it might be very useful to develop various, sighted phenomenologies or simplifying models.

“Any change of the fundamental theory should destroy this theory” – this Einstein’s expert estimation (somewhere in his autobiographical notes) could serve as a principle or ideal, the principle of uniqueness. So, this ideal of oneness seems to be achievable in AP, if Nature does not like singularities (preferring that the single standard of regularity property to be valid for all space-time points).

3. Energy-momentum tensor and weightless (or energy-less) waves:

Although non-lagrangian, the unique equation leads to the symmetric, “covariantly conserving” energy-momentum tensor where the main (second order) terms depend only on the rank two tensor $f_{\mu\nu}$, which looks like electromagnetic field (however, there are no gradient transformations in the theory). Applying the field equations (8), and prolonged equations as well, to $h^{-1} \delta(h R_{\mu\nu} G^{\mu\nu})/\delta g_{\mu\nu}$ ($R, G$ are the Ricci and Einstein tensors), one can find $T^{\mu\nu}$ \[22\ \ 23\]. In the theory, there are solutions with $f=0$, see Eq. (9), which carry no (or almost no) energy-momentum, nor angular momentum.

So, in $D=5$, in weak field approximation only three of 15 polarizations (i.e. plane wave solutions) do carry energy-momentum, while the others do not. This is a very unusual feature – impossible in the Lagrangian tradition; it needs some time, week or month, that to become used to this ‘singularity’.

One can show \[22\ \ 23\] that a wave-packet of the energy-carrying $f$-component should move along usual Riemannian geodesics – as in GR (if background has $f=0$); however, the polarization (or spin) evolution should depend also on the rank three skew-symmetric tensor $S_{\mu\nu\lambda}$, which is certainly absent in GR.

Another strange feature is the instability of trivial solution: some weightless polarizations grow linearly with time in presence of ponderable $f$-polarizations. Really, from the linearized Eq. (8) [and the identity (11)] one can write (the following equations should be treated as linearized):

$$\Phi_{a,a} = 0 \ (D \neq 1 + \gamma^{-1}), \quad \Lambda_{\alpha \beta \delta, \delta} = \gamma (\Phi_{a,b} - 2 \Phi_{b,a});$$

$$\Lambda_{a[bc,d],d} \equiv 0; \quad \Lambda_{abc,dd} = -2\gamma f_{bc,a}.$$
The last ‘D’Alembert equation’ has the ‘source’ in its right hand side. Some components of \( \Lambda \) (most symmetrical irreducible parts) do not grow (as well as Riemannian curvature), because (again, linearized equations are implied below)

\[
S_{abc,dd} = 0, \quad \Phi_{a,dd} = 0 \quad \text{(also} \quad f_{ab,dd} = 0, \quad R_{abcd,ee} = 0),
\]

but the least symmetrical components of the tensor \( \Lambda \) do go up (with time \( t \) – due to terms \( \sim t e^{-i\omega t} \)) if ponderable waves, the three \( f \)-polarizations are not vanishing.

4. Non-stationary spherically symmetric solutions as expanding cosmological model:

The unique symmetry of AP equations gives scope for symmetrical solutions. Non-stationary spherically \( (O_4-) \) symmetric solutions to the 5\( D \) unique equation lead through a number of integrations to specific scalar fields which can serve as privileged radius and time (quasi-inertial coordinates; this looks more suitable to match with observable cosmology). The condition \( f_{\mu\nu}=0 \) is a must for solutions with such a high symmetry (as well as \( S_{\mu\nu\lambda}=0 \)); so, these \( O_4 \)-symmetric solutions carry no energy, that is, weight nothing (some lack of gravity).

More realistic cosmological model may look like a single \( O_4 \)-symmetric wave (or a sequence of such waves) moving along the radius and being filled with a noise, or stochastic waves both weighty \( (\text{weak, } \Delta h \ll 1) \) and weightless \( (\Delta h < 1, \text{ but intense enough that it leads to non-linear fluctuations with } \Delta h \sim 1) \) which form statistical ensemble(s) having a few characteristic parameters (after ‘thermalization’). The development and examination of stability of such a model is an interesting problem. One may think that \( O_4 \)-wave (of proper sign) can serve as a time-dependent ‘shallow dielectric guide’ for that weak noise waves. The ponderable waves (noise-1) should have wave-vectors almost tangent to the \( S^3 \)-sphere of wave-front that to be trapped inside this (‘shallow’) wave-guide; the imponderable waves (noise-2) can grow up, and escape from the wave-guide (due to non-linear effects of scattering), and their wave-vectors can be less tangent to the sphere.

The waveguide thickness can be small for an observer in the center of \( O_4 \)-symmetry (may be, some larger than inverse ‘temperature’, or the end of noise spectrum, \( \lambda_n \)), but in co-moving system it can be very large (due to relativistic effect), however still small with respect to the radius of sphere, \( L \ll R \). It seems that the radial dimension has to be very ‘undeveloped’; that is, there are no other characteristic scales, smaller than \( L \), along this extra-dimension.

5. Topological features of non-linear sigma-model:

In the absence of solutions (of general position) with arising singularities, when degenerate (co- and contra-) frame matrices are unreachable and can be eliminated from the field set, AP acquires topological features of non-linear sigma-model.

Starting with topological charge group, one can introduce further the concept of topological quasi-charge group for field configurations having some symmetry. It seems that the possible variety of quasi-charges in 5\( D \) case (living on the “cosmological background” of \( O_4 \)-symmetric solution filled with weak stochastic waves) could be sufficient to explain qualitatively many (if not most of) features of the Standard Model (including

\[\text{2It seems that the universe expansion has little common with GR and its dark energy.}\]

\[\text{3This way to topological (quasi)charges looks more natural than all the ‘crystal spheres of chiral models’.}\]
superposition principle and Feynman’s path integral – as a result of integration over 5-th dimension in “cosmological waveguide”).

The non-linear, particle-like field configurations with quasi-charges (quasi-particles) should be very elongated along the extra-dimension (all of the same size $L$), while being small sized along usual dimensions, $\lambda_n \ll L$. The motion of such a spaghetti-like quasi-particle should be very complicated and stochastic due to ‘strong’ imponderable noise, such that different parts of spaghetti are coming their own paths. At the same time, quasi-particle can acquire ‘its own’ energy–momentum only due to scattering of ponderable waves (which wave-vectors are almost tangent to usual 3D (sub)space); so, it seems that scattering amplitudes of those spaghetti’s parts which have the same 3D–coordinates can be summarized providing an auxiliary, secondary [or shadow; $F(1+3)D$] field; non-linear fluctuations of noise-2 are responsible for ‘zero-point oscillations’ of secondary fields.

So, the imponderable noise-2 provides stochasticity (of motion of spaghetti’s parts), while the ponderable noise-1 gives superposition (with secondary fields). Phenomenology of secondary fields could be of Lagrangian type, with positive energy acquired by quasi-particles, – that to ensure the stability (of all the waveguide with its infill with respect to quasi-particle production; the least action principle has deep concerns with Lyapunov stability and is deduced, in principle, from the path integral approach).

Thus, we believe that there has to be a definitive difference between the right theory and all the others, wrong ones. We’re expecting the right theory to be of ‘monotheistic’ kind – with no free parameters and no room for changes. (Some people believe that such a simple theory is impossible[12] and our world is infinitely complex – but according to Einstein, this case ‘is not interesting’.)

A theory with arbitrary parameter(s) looks like ‘fire on areas’. We really need only one fundamental theory, but not a heap of slightly different theories where only one especial variant is supposed to be the Truth (while we have no good way to distinguish this Truth from Untruth – during limited experimental time and using limited funding; on the other hand, theories with free parameters are quite appropriate for the maxim ‘the show must go on’).

2 Topological charges and quasi-charges

In this section the topological properties of spatially localized configurations of frame field, $h^a_{\mu} \to \delta^a_{\mu}$ on space-like infinity, are to be explored.

We will suppose that Riemann space defined by metric is of trivial topology: no worm-holes, no compactified space dimensions, no singularities. A process of (de)compactification would require arising singularities which are thought to be impossible (due to careful choice of field equations), and we prefer that the single standard of no-compactification to be applied to all space(time) dimensions.

It is possible to continuously deform the metric and, simultaneously, the frame field $h(x)$ such that the metric becomes trivial, equal to Minkowski metric, $g_{\mu\nu} \to \eta_{\mu\nu}$, whereas $h$-field

---

4These amplitudes can depend on additional vector-parameters (‘equipment vectors’) relating to differential of field mapping at a ‘quasi-particle center’ – where quasi-charge density is largest (if it has covariant sense).

5Note some similarity (except superposition, of course) with the Plato’s ‘shadows in a cave’.
becomes a field of rotation matrices (metric can be subjected to diagonalization,\[11\] ‘square-rooting’, and so on; we introduce here the notation: \( m = D - 1 \))

\[ h^a_\mu(x) \rightarrow s^a_\mu(x) \in SO(1, m). \]  

Further deformation is possible that to remove boosts too, and so, for any space-like (Cauchy) surface, this gives a (pointed) map:

\[ s : \mathbb{R}^m \cup \infty = S^m \rightarrow SO_m; \quad \infty \mapsto 1^m \in SO_m. \]

The set of such maps consists of homotopy classes which form an Abelian group, and this is the group of topological charge, \( \Pi(m) \):

\[ \Pi(m) = \pi_m(SO_m); \quad \Pi(3) = \mathbb{Z}, \quad \Pi(4) = \mathbb{Z}_2 + \mathbb{Z}_2. \]  

Here \( \mathbb{Z} \) is the infinite cyclic group, and \( \mathbb{Z}_2 \) is the cyclic group of order two (two group) \[10\].

Due to great symmetry of AP-equations, see \[10\], their solutions also can possess some symmetries, which should be a diagonal subgroup of that great symmetry. It is very important that the deformation to \( s \)-field can be performed such that to keep symmetry of field configuration.

### 2.1 Quasi-charge groups

We are going to give (a sketch of) topological classification of symmetric localized configurations of \( SO_m \)-field. Let us repeat the assumed definition: localized field (or pointed map)

\[ s(x) : \mathbb{R}^m \rightarrow SO(m), \quad s(\infty) = 1^m, \]

is \( G \)-symmetric if, in some suitable coordinates,

\[ s(\sigma x) = \sigma s(x) \sigma^{-1} \quad \forall \sigma \in G \subset O(m). \]  

The set of such fields \( C^{(m)}_G \) generally consists of separate, disconnected components – homotopy classes forming the Abelian group which is denoted here as \( \Pi(G;m) \); i.e.

\[ \Pi(G;m) \equiv \pi_0(C^{(m)}_G). \]

For such a group we will coin the term ‘topological quasi-charge group’, or QC-group.

These groups also classify symmetrical localized configurations of frame field \( h^a_\mu(x) \) as it was outlined above. Since field equation does not break symmetry, field configuration with non-trivial quasi-charge merits some good name (something better than), say, \textit{topological quasi-soliton}, or \textit{quasi-particle}. If symmetry is not exact (because of some distant regions), quasi-charge is not exactly conserving value, and quasi-soliton (of zero topological charge) can vanish or arise during colliding with another quasi-soliton.

Along with calculation QC-groups for different symmetries we should solve another problem. Let \( G_1 \) includes \( G_2 \) (with respect to its elements or generators), such that there is a mapping (embedding) of field configurations:

\[ i : C^{(m)}_{G_1} \rightarrow C^{(m)}_{G_2}. \]

This mapping induces the homomorphism of QC-groups:

\[ i_* : \Pi(G_1;m) \rightarrow \Pi(G_2;m), \]

so one has to describe this and to find how the ‘small’ pieces of more symmetric fields are situated within the ‘large lumps’ of less symmetric field configurations.
2.2 Examples of QC-groups: diad (and k-ad) homotopy groups

Let us consider the very simple (discreet) symmetry group \( P_1 \) with a plane of reflection symmetry:

\[
P_1 = \{ 1, p_1(1) \}, \quad \text{where } p_1(1) = \text{diag}(-1, 1, \ldots, 1) = p_{(1)}^1.
\]

It is necessary to set field \( s(x) \) on the half-space \( \frac{1}{2} \mathbb{R}^m = \{ x^1 \geq 0 \} \), with additional condition imposed on the surface \( \mathbb{R}^{m-1} = \{ x_1 = 0 \} \) (stationary points of \( P_1 \) group) where \( s \) has to commute with the symmetry [see (13)]:

\[
p_{(1)}x = x \Rightarrow s(x) = p_{(1)}sp_{(1)} \Rightarrow s \in 1 \times SO_{m-1}.
\]

Hence, accounting for the localization requirement, we have a diad map (relative spheroid; here \( D^m \) is an \( m \)-ball and \( S^{m-1} \) its surface)

\[
(D^m; S^{m-1}) \rightarrow (SO_m; SO_{m-1}),
\]

and topological classification of such maps leads to the relative (or diad) homotopy group [16]:

\[
\Pi(P_1; m) = \pi_m(SO_m; SO_{m-1}) = \pi_m(S^{m-1}).
\]

The last equality follows due to fibration \( SO_m/ SO_{m-1} = S^{m-1} \), also see [16].

Similar considerations (of group orbits and stationary points) lead to the following result:

\[
\Pi(O_l; m) = \pi_{m-l+1}(SO_{m-l+1}; SO_{m-l}) = \pi_{m-l+1}(S^{m-l}).
\]

If \( l > 3 \), there is the equality: \( \Pi(SO_l; m) = \Pi(O_l; m) \), while for \( l = 2, 3 \) one can find [22]:

\[
\Pi(SO_3; m) = \pi_{m-2}(SO_2 \times SO_{m-2}; SO_{m-3}) = \pi_{m-2}(S^1 \times S^{m-3}),
\]

\[
\Pi(SO_2; m) = \pi_{m-1}(SO_m; SO_{m-2} \times SO_2) = \pi_{m-1}(RG_+(m, 2)).
\]

At last, the ‘compound’ symmetries, \( G = G_1 \times \cdots \times G_k \) (all \( G_i \) are simple), having more complicated picture of (sub)orbits, and stationary points (for every \( G_i \)), and their intersections, lead to \((k+1)-\text{ad homotopy groups}, which can be defined (by induction [22]) as a generalization of diad [16] and triad [17] homotopy groups.

3 QC-groups, their morphisms and forms in 4D

In this section \( m = 3 \), and it will be omitted from \( \Pi \)-groups. The Pontryagin’s method of substitution of maps with equipped sub-manifolds [15, 16] can be extended for the case of \( G \)-symmetric maps [13]. This gives the useful way to find both quasi-charge groups and their morphisms. For example, one can show that an equipped point (say, positive; that is, a three vectors equipment has the positive orientation) can be placed at the center of \( SO_3 \)-symmetry, or at the line of (stationary points of) \( SO_2 \)-symmetry. It means that there are isomorphisms (with the group of topological charge):

\[
\Pi(SO_3) \cong \Pi(SO_2) \cong \Pi = Z.
\]
Let’s briefly illustrate this approach using the example of discreet symmetry groups $P_2\{1, 2\}$ (inversion of first two space coordinates, or their $\pi$—rotation) and $P = P_3$ (full space inversion). One can find that

$$\Pi(P_2\{1, 2\}) = \Pi(P_2) = Z_{a} + Z_{p},$$

where the first $Z$ describes the equipped points (the difference of positive and negative points) at the axis of symmetry (the third axis, where $x^1 = x^2 = 0$), while the other $Z$ describes the pairs of peripheral points (number of positive pairs minus negative; $\pi$—rotation does not change the sign of equipment orientation). So, one can describe the morphism as well (isomorphism plus monomorphism-2):

$$\Pi(P_2) \supset Z_{a} \ni 1 \mapsto 1 \in Z = \Pi,$$

$$Z_{p} \ni 1 \mapsto 2 \in Z = \Pi.$$

In the case of $P_3$, any equipped point should have a partner of opposite sign in the inverted position (the inversion change the sign of equipment, and the topological charge is a pseudoscalar), but one can bring any two such pairs into annihilation keeping the symmetry. All that means that

$$\Pi(P_3) = Z_{2} \ni 1 \mapsto 0 \in Z = \Pi.$$

Similar consideration gives (also zero morphism)

$$\Pi(P_1) = Z \ni 1 \mapsto 0 \in Z = \Pi.$$

### 3.1 Differential 3-form of topological charge (in 4D)

The topological current for a spatially localized field of rotation matrices $s(x^\mu) : \mathbb{R}^{1+3} \to SO_3$ (or $\to SO(1, 3)$) may be defined as follows (much as the topological current in the Skyrme model with $SU_2$-field [18, 19, 20])

$$J_\mu^s \propto \varepsilon^{\mu\nu\tau} \text{tr}(\gamma_\nu^s s^* \gamma_\tau^s); \quad J_{(s), \mu} = 0.$$

Here $\gamma^s_\mu = s_\mu s^{-1}$ is the $so_3$- or $so(1, 3)$-valued current (right-invariant). In a slightly more detailed notations, with $\gamma^s_{ij\mu} = s_{ik\mu} s_{jk}^*$, the topological current reads

$$J_\mu^s = a \varepsilon^{\mu\nu\tau} \gamma^s_{ij\nu} \gamma^s_{jk\tau} \gamma^s_{ki\mu}. \quad (14)$$

The constant $a = \frac{1}{36\pi^2}$ is defined to make topological charge $Q(s)$ integer:

$$Q(s) = \int J_\mu^s d^3x \in Z = \pi_3(SO_3).$$

This charge has concern with (half of) the degree of the map from the space (or Cauchy surface) $\mathbb{R}^3$ to the (sub)group $SO_3$ (which volume is $\pi^2$).

The topological current $J^\mu$ for a general frame field $h^\alpha_\mu(x^\nu)$ should coincide with Eq. (14) when the metric is trivial, $g_{\mu\nu} = \eta_{\mu\nu}$, and $h^\alpha_\mu \in SO(1, 3)$. Let us use the language of differential forms in order to find the closed 3-form $t$ which is dual to the topological current, $*t = J_\mu dx^\mu$.

It is known that characteristic classes [10] are trivial in (pseudo)Riemannian space with a frame structure (parallelizable manifold), so the class $c_2$ (which is the scalar-valued 4-form) is an exact form:

$$c_2 \equiv \text{tr}(\Omega \wedge \Omega) = du. \quad (15)$$
One can use either $so(1,3)$-valued connection (1-form $\gamma = \gamma_\mu dx^\mu$),

$$\gamma_{ab\mu} = h_b^\lambda h_{a\lambda\mu} = \gamma_{[ab]\mu},$$

or $gl_4$-valued connection $\tilde{\gamma}_\tau dx^\tau$ with Christoffel symbol,

$$\tilde{\gamma}_\mu^\nu_\tau = -\Gamma_\mu^\nu_\tau = -\frac{1}{2}g^{\alpha\rho}(g_{\mu\rho,\tau} + g_{\rho\tau,\mu} - g_{\mu\tau,\rho}),$$

that to write the (Riemann) curvature 2-form $\Omega \equiv d\gamma + \gamma \wedge \gamma$ (and $\tilde{\Omega}$):

$$\Omega_{ab\tau} = R_{ab\tau} = 2h_b^\lambda h_{a\lambda \varepsilon;\tau} = \gamma_{ab,\tau} + \gamma_{ace,\tau} \gamma_{,\tau} - (\varepsilon \tau),$$

$$\tilde{\Omega}_\mu^\nu_\tau = R_\mu^\nu_\tau = -\Gamma^\mu_\nu_\tau + \Gamma_{\lambda\varepsilon}^\lambda_\nu_\tau - (\varepsilon \tau).$$

(17)

Also, the (scalar-valued) 3-form $u$ can be written in two similar ways – with and without ‘tilde’:

$$\hat{u} = \text{tr}(\hat{\gamma} \wedge \hat{\Omega} - \frac{1}{2} \hat{\gamma} \wedge \hat{\gamma} \wedge \hat{\gamma}), \quad u = \text{tr}(\gamma \wedge \Omega - \cdots).$$

(18)

Using the Bianchi identity, $d\Omega \equiv \Omega \wedge \gamma - \gamma \wedge \Omega$, one can verify that $du = d\hat{u} = c_2$, see Eq. (15). Therefore, we have the closed 3-form $t \propto \hat{u} - u$,

$$dt = 0, \quad \text{and} \quad \text{tr}(\gamma \wedge \gamma \wedge \gamma \wedge \gamma) \equiv 0,$$

one can verify that $du = d\hat{u} = c_2$, see Eq. (15). Therefore, we have the closed 3-form $t \propto \hat{u} - u$, $dt = 0$, and can write the topological current ($h = \det h^a_\mu = \sqrt{-g}; \ (h J^\mu)_{,\mu} = 0$):

$$hJ^\mu = a\varepsilon^{\mu\nu\epsilon\tau} \left( \Gamma^\alpha_{\beta\nu}(3\Gamma^\beta_{\alpha\varepsilon,\tau} - 2\Gamma^\beta_{\gamma,\alpha\tau}) - \gamma_{ab}^\nu (\frac{3}{2} R_{b\alpha\varepsilon\tau} - \gamma_{ace,\tau} \gamma_{,\tau}) \right).$$

(19)

Although $J^\mu$ is not of covariant view, it is clearly appropriate for determination of topological charge

$$Q = \int_V hJ^\alpha d^3x \in \Pi = Z,$$

at least when the Cauchy surface is covered by one map.

At last, let’s recall the Weitzenböck connection from Eq. (6), $\hat{\Gamma}$, which may also be used as a left-invariant (with respect to global Lorentz transformations) 1-form with zero curvature-form:

$$d\hat{\Gamma} = -\hat{\Gamma} \wedge \hat{\Gamma} \quad \hat{\Gamma}_\mu^a d\lambda = h_a^\lambda h^\alpha_\nu \lambda dx^\lambda).$$

So, the topological current can also be written in the next simple forms:

$$hJ^\mu = a\varepsilon^{\mu\nu\epsilon\tau} \hat{\Gamma}^\alpha_{\beta\nu} \hat{\Gamma}^\beta_{\gamma,\alpha\tau} = a\varepsilon^{\mu\nu\epsilon\tau} \hat{\gamma}_a^\nu \hat{\gamma}_b^\epsilon \hat{\gamma}_c^\alpha,$$

where ($\hat{\gamma}$ and $\hat{\Gamma}$ are not tensors)

$$\hat{\gamma}_a^\nu \hat{\gamma}_b^\epsilon \hat{\gamma}_c^\alpha = h_a^\mu h_b^\nu h_c^\mu = h^a_\alpha \hat{\Gamma}^\alpha_{\beta\mu} h_b^\beta.$$
3.2 Spherical symmetry (in 4D); \( SO_3 \)-quasi-charge 1-form

The \( SO_3 \)-symmetric frame field can be generally written as follows [7, 10]:

\[
h_{\mu}^a(t, x') = \begin{pmatrix}
a & b n_i \\
c n_i & e n_i + d \Delta_{ij} + f \varepsilon_{ijk} n_k
\end{pmatrix}; \quad i, j = (1, 2, 3), \quad n_i = \frac{x_i}{r}.
\]

Here \( a, \ldots, f \) are functions of time, \( t = x'^0 \), and radius \( r \), \( \Delta_{ij} = \delta_{ij} - n_i n_j \), \( r^2 = x'^i x'^i \). As functions of radius, \( a, e, d \) are even, while the others are odd; there are the boundary conditions: \( e = d \) at \( r = 0 \), and \( h_{\mu}^a \to \delta_{\mu}^a \) as \( r \to \infty \). Note that we use \( i, j \) for space indexes of both Latin and Greek origin; this origin should be derived from the (a) context.

It is easy to check that regarding \( (t, r) \)-transformation we have the next set of 2D-covariants: two co-vectors, \( (a, b) \) and \( (c, e) \), scalar \( s = r \cdot d \), and pseudo-scalar \( p = r \cdot f \). At points with \( n_i = (1, 0, 0) \) we have the block matrix

\[
h_{\mu}^a = \begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix}; \quad A = \begin{pmatrix}
a & b \\
c & e
\end{pmatrix}, \quad B = \begin{pmatrix}
d & f \\
-f & d
\end{pmatrix} = \sqrt{d^2 + f^2} \begin{pmatrix}
\cos \varphi & \sin \varphi \\
-\sin \varphi & \cos \varphi
\end{pmatrix}.
\]

Here \( \tan \varphi = f/d = p/s \). From this presentation one can directly write the 2D-covariant 1-form of topological quasi-charge, \( q = q_0 dx^{\alpha} \) (we use 2D-indices \( \alpha = (t, r) \)):

\[
q = \frac{d\varphi}{2\pi} = \frac{sp_{\alpha} - ps_{\alpha}}{s^2 + p^2} \frac{dx^{\alpha}}{2\pi}.
\]

The morphism

\[
Z = \Pi(SO_3; m = 3) \to \Pi(m = 3) = Z
\]

is isomorphism. To prove this statement and to check the constant \( a \) in Eqs. (19), (14) let us consider a ‘toy’ static AP-space characterized by the next set of functions in (21):

\[
a = e = 1, \quad b = c = 0, \quad d = \cos \varphi(r), \quad f = \sin \varphi(r),
\]

with (trivial metric, \( g_{\mu\nu} = \eta_{\mu\nu} \), and) boundary conditions

\[
\cos \varphi(0) = \cos \varphi(\infty) = 1; \quad \text{hence} \quad \sin \varphi(0) = \sin \varphi(\infty) = 0.
\]

Substituting this frame field into Eq. (10) we obtain the non-zero components of \( \gamma_{ab\mu} \):

\[
\gamma_{ijk} = \frac{CS}{r} \varepsilon_{ijk} + (\varphi' - \frac{CS}{r}) \varepsilon_{ij} n_m n_k + \frac{1 - C}{r} \left(S(n_i \varepsilon_{jkl} - n_j \varepsilon_{ikl}) n_i - n_i \delta_{jk} + n_j \delta_{ik}\right),
\]

where the prime (’') denotes the derivative with respect to radius, \( C = \cos \varphi \), \( S = \sin \varphi \).

At points with \( n_i = (1, 0, 0) \) we have the next values (keep in mind that \( \gamma_{ab\mu} = -\gamma_{ba\mu} \)):

\[
\gamma_{231} = \varphi', \quad \gamma_{232} = \gamma_{233} = 0 = \gamma_{121} = \gamma_{131};
\]

\[
\gamma_{123} = \gamma_{312} = S/r, \quad \gamma_{212} = \gamma_{313} = (1 - C)/r.
\]

At last we are ready to find from Eq. (19), or (14), the density of topological charge using (23):

\[
hJ^0 = a \varepsilon_{ijk} \gamma_{ab\mu} \gamma_{bc\gamma} \gamma_{cak} = 6a \varepsilon_{ijk} \gamma_{321} \gamma_{231} \gamma_{12k} = 6a \gamma_{231} \gamma_{321} (\gamma_{321} - \gamma_{313} \gamma_{122}) = 12a \varphi'(1 - \cos \varphi)/r^2.
\]

Taking a ‘unit kink’, with \( \Delta \varphi = \varphi(\infty) - \varphi(0) = 2\pi \), see the boundary conditions, we should obtain the unit topological charge:

\[
Q = 4\pi \int_0^\infty hJ^0 r^2 dr = 96a\pi^2 = 1; \quad \text{hence} \quad a = \frac{1}{96\pi^2}.
\]
3.3 Cylindrical symmetry; \( SO_2 \)-quasi-charge

In this section we are going to show that \( SO_2 \)-quasi-charge (which is equal to topological charge as well) can be completely attributed (in a covariant manner) to the axis of cylindrical symmetry (that is, the line of stationary points on a Cauchy surface).

Let us consider the Killing vector, \((3) \xi\), corresponding to cylindrical symmetry with respect to axis \( x^3 \); that is, in ‘natural Cartesian’ coordinates (where \( \xi_{\mu,\nu} = \text{const} \)),

\[
(3) \xi = (3) \xi^\mu \partial_\mu = x^1 \partial_2 - x^2 \partial_1.
\]

In general, \( \xi \) is a Killing vector if some combination of infinitesimal (\( \epsilon \ll 1 \)) coordinate transformation and global (Lorentz) rotation,

\[
x^\mu \rightarrow \tilde{x}^\mu = x^\mu + \epsilon \xi^\mu, \quad s^{a}_{\ b} = \delta^{a}_{\ b} + \epsilon m^{a}_{\ b} \in SO(1, 3) \cap SO_3,
\]

does not change frame field:

\[
\tilde{h}^a_{\ \mu}(\tilde{x}) = (\delta^a_{\ b} + \epsilon m^a_{\ b})h^b_{\ \nu}(x)(\delta^\nu_{\ \mu} - \epsilon \xi^\nu_{\ ,\mu}) = h^a_{\ \nu}(\tilde{x}).
\]

This condition leads to the next equation (\( m^{a}_{\ b} = \text{const} \))

\[
h^a_{\ \mu,\nu}\xi^\nu = m^a_{\ b}h^b_{\ \mu} - h^a_{\ \nu}\xi^\nu_{\ ,\mu}, \quad (25)
\]

which can easily be written in manifestly covariant form: \( (hJ^a_{\ \nu})_{,\mu} + \Lambda^a_{\ \mu\nu}\xi^\nu = m^a_{\ b}h^b_{\ \mu} \). (Its symmetric part gives usual equation of GR: \( \xi_{(\mu\nu)} = 0 \)). The set of stationary points, where

\[
\xi^\mu = 0, \quad \xi^\mu_{\ ,\nu} = h^a_{\ \mu}m^a_{\ \nu}b^b_{\ \nu}, \quad (26)
\]

will be denoted \( \Xi_0 \).

The matrix \( m \) of the \( x^3 \)-axial symmetry looks as follows:

\[
m^a_{\ b} = (3)m^a_{\ b} = -\varepsilon^{03ab},
\]

and in 'natural coordinates': \((3)\xi_{\ ,\nu} = (3)m^a_{\ \nu} \). (It is assumed that at spatial infinity \( h^a_{\ \mu} = \delta^a_{\ \mu} \).

Let us introduce cylindrical coordinates, \( y^M \), suitable to enumerate orbits of the symmetry:

\[
y^M = (t, z, \rho, \phi); \ x^0 = t, \ x^1 = \rho \cos \phi, \ x^2 = \rho \sin \phi, \ x^3 = z.
\]

We want to use the Eq. \((20)\) for topological current, keeping Cartesian indices for frame matrices but switching to cylindrical ones in derivatives. One can show (using \((25)\) and its prolongation) that this current does not depend on \( \phi \):

\[
(hJ^\mu)_{,\phi} = 0 = (hJ^\mu)_{,\nu}\xi^\nu = (hJ^\mu)_{,\nu}\frac{\partial x^\nu}{\partial \phi}.
\]

In other words, in matrix notations (with bold letters), Eq. \((25)\) leads to

\[
h_{\phi} = mh - hm \quad \text{(hence} \; \text{tr}(h^{-1}h_{\phi}) = 0 = (\text{det} h)_{\phi})
\]

and \( h(\phi) = e^{m(\phi - \phi_0)} h(\phi_0) e^{-m(\phi - \phi_0)} \).

\( ^6 \)In new coordinates Killing vector is constant, \( \xi^M = (0, 0, 0, 1) \), except when \( \rho = 0 \).
Therefore, substituting above equations in Eq. (20) and making integration over $\phi$, we can obtain the topological quasi-current, $hJ^X_{(SO2)} = \int hJ^X \rho \, d\phi$, for $SO_2$-symmetry (in mixed coordinates; $X,Y = (t,\rho,z)$; $m^\mu_\nu$ is constant and not a tensor):

$$hJ^X_{(SO2)} = 6\pi a \varepsilon^{XYZ}\phi (m^a_b c^b c^c_a aZ - m^a_b \hat{\gamma}^{aZ}_Y \hat{\gamma}^{aZ}_\mu ) = 6\pi a \varepsilon^{XYZ}\phi (m^a_b \hat{\gamma}^b + m^a_\nu \hat{\Gamma}^\nu_Z)_Y , \quad (27)$$

where $\hat{\gamma}^a_b$ is $h^a_\mu X h^b_\nu, \hat{\Gamma}^\mu_X = h^a_\mu h^a_\nu X$. This quasi-current is ‘trivial’ (ie, corresponds to an exact form); making integration over radius $\rho$ one can obtain:

$$Q = \int hJ^0_dV = Q_{(SO2)} = \int \int hJ^X_{(SO2)} d\rho \, dz = 12\pi a m^a_b h^a_\mu , h^b_\nu dz|_{\rho=0} . \quad (28)$$

It is taken into account that $m^\mu_\nu = h^a_\mu m^a_b h^b_\nu|_{\Xi_0}$, see Eq. (26).

More over one can easily check (differentiating Eq. (26) with proper contracting) that at stationary points it is valid: $h^a_\mu , h^a_\nu \xi = 0$. Then, taking any space-like line, $\zeta$, placed on $\xi_0$:

$$\zeta : \mathbb{R} \to \Xi_0 \subset \mathbb{R}^{1+3}, g_{\mu \nu} \frac{dx^\mu(\zeta)}{d\zeta} \frac{dx^\nu(\zeta)}{d\zeta} > 0 ,$$

one can can arrive at the next covariant line density of topological (quasi)charge:

$$Q = 12\pi a \int m^a_b (h^a_\mu , h^a_\nu ) h^b_\nu \frac{dx^\mu(\zeta)}{d\zeta} d\zeta . \quad (29)$$

This equation, being covariant ($h^a_\mu , h^a_\nu$ is the tensor, $\Lambda$), should be valid in any coordinates, not only ‘natural Cartesian’. This is an interesting result – in view of possible generalization for the next $D$.

At last, taking the $SO_3$-symmetrical $h$-field of the previous subsection, one can easily check (components in (23) can be used after permutation $\{3,1,2\}$) that Eq. (28) gives:

$$Q = 24\pi a \int_{-\infty}^{\infty} \frac{d\phi}{dr} \bigg|_{x^1=x^2=0} = 96\pi^2 a = 1 .$$

4 ‘Left’ and ‘right’ topological (quasi-)charges in 5D

In this section $m = 4$. We need for two topological currents $J^\mu_l (l)$ and $J^\mu_r (r)$, ‘left’ and ‘right’, which should replace each other under $P$-inversion, or $C$-operation (see below). Now we are going to use the quaternion representation of the $SO_4$ group that to write quaternion differential forms of topological charges (‘left’ and ‘right’ and quasi-charges.

4.1 Quaternion representation of $SO_4$

With quaternion units denoted as $i_k$ (that is, $i_1 = i, i_2 = j, i_3 = k$), the quaternion multiplication is defined by the rule:

$$i_j i_k = -\delta_{jk} + \varepsilon_{jkl} i_l, \quad j,k,l = 1,2,3. \quad (30)$$

Any quaternion $x \in H = \mathbb{R}^4$ has real and imaginary parts, and the module $|x|$ (absolute value):

$$x = x_4 + x_k i_k = \Re(x) + \Im(x), \quad |x|^2 = x \bar{x} = x_4^2 + x_k x_k ,$$

14
It is easy to check that

\[ x^* = s x \iff x^* = f x g^{-1} = f x \bar{g}; \quad |x| = |x^*|. \]

The pairs \((f, g)\) and \((-f, -g)\) correspond to the same rotation \(s\), that is,

\[ SO_4 = S^3_{(i)} \times S^3_{(r)}/\pm. \]

We indicate these quaternion spheres as \(\text{left}\) and \(\text{right}\), because under space inversion \(P_3\{1, 2, 3\}\), and under the \(C\)-operation \(I\), which inverts the 4-th coordinate, \(C = P_1\{4\}\), pair elements replace one another: \(C : (f, g) \rightarrow (g, f); (f, f) \in SO_3\).

The rotations on angle \(\phi\) of coordinates \((1)\) \(x_2, x_3\), \((2)\) \(x_4, x_1\) have the next quaternion form (they commute):

\[ \begin{align*}
(1) & : (a, a)(\phi), \quad (a, \bar{a})(\phi); \quad a(\phi) = \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} = e^{i\phi/2}. \\
(31) & \end{align*} \]

Note that the symmetry condition \((33)\) also splits into two parts:

\[ f(axb^{-1}) = af(x)a^{-1}, \quad g(axb^{-1}) = bg(x)b^{-1} \quad \forall (a, b) \in G \subset SO_4. \quad (32) \]

Following the Fjodorov’s parametrization of \(SO_4\) \(-\) group \((21)\) one can define on \(so_4\) \(-\) algebra the ‘left’ and ‘right’ (or selfdual and anti-selfdual) ‘imaginary units’

\[ M^{(\pm)} = L^{(i)} \pm K^{(i)}, \quad L^{(i)}_{ab} = -\varepsilon_{abcd}, \quad K^{(i)}_{ab} = \delta_{a4}\delta_{b4} - \delta_{b4}\delta_{a4}. \quad (33) \]

For example (zeroes are not shown in these matrices),

\[ 
M^{(+1)} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad M^{(+2)} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad M^{(+3)} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}. 
\]

It is easy to check that

\[ [M^{(+i)}, M^{(-j)}] = 0, \quad \text{tr}(M^{(+i)} M^{(-j)}) = 0, \]

\[ M^{(\pm j)} M^{(\pm j)} = -\delta_{ij}E + \varepsilon_{ijk}M^{(\pm k)}, \quad \text{tr}(M^{(\pm i)} M^{(\pm j)}) = -4\delta_{ij}, \]

where \(E = 1^4\) is the identity matrix; compare the last equation with the Eq. \((30)\).

The separation of any matrix \(s \in SO_4\) into selfdual and anti-selfdual parts looks as follows:

\[ s = s^{(+)} s^{(-)} = s^{(-)} s^{(+)}; \quad s^{(+)} = f_4 E + f_1 M^{(+i)}; \quad s^{(-)} = g_4 E + g_1 M^{(-i)}. \]

It is evident that \(s s^{tr} = 1\) (\(tr\) indicates the transposed matrix), and that

\[ f_4 g_4 = \frac{1}{4} \text{tr}(s), \quad g_4 f_1 = -\frac{1}{4} \text{tr}(M^{(+i)} s), \quad f_4 g_1 = -\frac{1}{4} \text{tr}(M^{(-i)} s). \]

\(3\)This letter \((C)\) is used that some allusion to the charge conjugation is to arise.
4.2 Example of $SO_2$-symmetric quaternion field

Let’s consider an example of $SO_2\{2,3\}$-symmetric $f$-field configuration ($g=1$), which carries both charge and $SO_2$-quasi-charge (left, of course), $f(x): H = \mathbb{R}^4 \to H_1$; $f(\infty) = 1$. The symmetry condition from (31), (32) reads

$$f(e^{1/2}x e^{-1/2}) = e^{1/2}f(x) e^{-1/2}. \quad (34)$$

We’ll switch to ‘double-axial’ coordinates: $x = ae^{i\varphi} + be^{i\psi}j$. Let us use imaginary quaternions $q$ as stereographic coordinates on $H_1$, and take symmetrical field $q(x)$ consistent with Eq. (34):

$$q(x) = x i \bar{x} + i = -\bar{q}, \quad f(x) = -\frac{1}{2} \frac{1 + q}{1 - q}. \quad (35)$$

It is easy to find the ‘center of quasi-soliton’ (1-submanifold, $S^1$)

$$S^1 = f^{-1}(-1) = q^{-1}(0) = \{a = 0, \ b = 1\} = \{x_0, \ x_0 = e^{i\varphi}j\}$$

and the ‘vector equipment’ on this circle:

$$dx|_{x_0} = da e^{i\varphi} + (db + i dy)e^{i\psi}j, \quad d f|_{x_0} = i db - k e^{i(\varphi+\psi)} da;$$

$i$-vector all time looks along the radius $b$ (parallel translation along the circle $S^1$; this is a ‘trivial’, or ‘flavor’-vector). Two others (‘phase’-vectors) make $2\pi$-rotation along the circle.

In fact, the field (35) has also symmetry $SO_2\{1,4\}$, and this feature restricts possible directions of ‘flavor’-vector (two ‘flavors’ are possible, $\pm$; the $P_2\{1,4\}$–symmetry gives the same effect). The other interesting observation is that the equipped circle can be located also at the stationary points of $SO_2$–symmetry (this increases the number of ‘flavors’).

The left 3-form of topological quasi-charge can be written as follows:

$$q(l) = \frac{1}{12\pi} \Re(r \land r \land r) = \frac{1}{12\pi} \Re(l \land l \land l),$$

where $r$ (resp. $l$) is right- (left-) invariant quaternion-valued 1-form

$$r = f_A \bar{f} dA, \ l = \bar{f} f_A dA, \ A, B = \{r_1, r_2, \varphi_2\}; \ dr = r \land r, \ dl = -l \land l.$$ (One can use Pauli matrices instead of quaternion units replacing $\Re()$ with $\frac{1}{2} \tr()$. The left 2-form of $SO_2 \times SO_2$–quasi-charge can be written as well, and the right forms as well.)

4.3 Hopf mapping and quaternion forms

Imaginary quaternions ($J \subset H$) of unit length, $n$, form 2-sphere:

$$n = n_i i_j \in J \cap H_1 = S^2.$$ 

The Hopf (and ‘anti-Hopf’) map takes $S^3 = H_1$ into $S^2$:

$$H_n : f \mapsto w = f n \bar{f} (\bar{H}_n : f \mapsto \bar{w} = \bar{f} n f). \quad (36)$$

It is evident that $\bar{w} = -w$, $w^2 = -1$, $w dw = -dw w$. 

16
4.3.1 Map $\mathbb{R}^{2+1} \rightarrow S^2$ and charges from $\pi_2(S^2) = Z$.

The unnormalized 2-form of topological charge (relating to ‘volume’ (surface) form on $S^2$) is

$$t^{(2)} = \Re(w \, dw \wedge dw); \quad d\tilde{t}^{(2)} = \Re(dw \wedge dw \wedge dw) = \mp \Re(w \, dw \wedge dw \wedge dw \, w) = 0.$$  

We use the same letter for ‘pull-back’ forms on $\mathbb{R}^{2+1}$, say, $dw = w, \mu dx^\mu$.

The normalizing coefficient is $1/(8\pi)$: $4\pi$ is the surface of $S^2$ and factor 2 is due to $\varepsilon$-permutation in 2-form. That to prove this one can use the stereographic projection on the ‘complex plane’, $C = \{z = x + iy\}$:

$$w = i \frac{1 + kz}{1 - kz} = i \frac{2}{1 - kz} - i = (1 - kz)i(1 - kz)^{-1} = \frac{1 - kz}{1 + kz};$$

$$dw = 2i(1 - kz)^{-1}kd\bar{z}(1 - kz)^{-1}; \quad kd\bar{z} = d\bar{z}k;$$

$$t^{(2)} = \frac{1}{8\pi} \Re(w \, dw \wedge dw) = -\frac{3(dz \wedge d\bar{z})}{2\pi(1 + |z|^2)^2} = \frac{dx \, dy}{\pi(1 + x^2 + y^2)^2}.$$  

The other way is to use first the symmetry of axisymmetric field $w(z)$, see (34), which complies with the following condition (in axial coordinates, $z = r e^{i\varphi}$; see the field considered above):

$$w(r e^{i\varphi}) = e^{-ir^2/2}w(r) e^{i\varphi/2}; \quad 2w, \varphi = wi - iw. \quad (37)$$

Substituting this into 2-form and making integration one can check that $(w(0) = i, w(\infty) = -i)$

$$\int_Z t^{(2)} = \int_{r,\varphi} \Re(w \, dw(wi - iw)) \wedge d\varphi = 4\pi \int_0^\infty \Re(iw, \rho) d\rho = 8\pi.$$  

4.3.2 Maps $\mathbb{R}^{3+1} \rightarrow S^3, S^2$ and charges from $\pi_3(S^3) = \pi_3(S^2) = Z$.

4.3.3 Maps $\mathbb{R}^{4+1} \rightarrow S^3, S^2$ and charges from $\pi_4(S^3) = \pi_4(S^2) = Z_2$.

(To be continued; the transition $s \rightarrow h.$)

4.4 Quasi-charges and their morphisms in 5D

First of all one should consider those symmetries, $G$, which are contained in the symmetries of cosmological background solution:

$$G \subset G_0 = (O_3 \times P_4) \cap SO_4.$$  

(38)

Here $P_4$ is the inversion of all space-like coordinates; $G_0$ is the point symmetry of ‘wave-guide’ in co-moving coordinate system. It is assumed that weak (or not so weak) stochastic waves (which have also ponderable $f$-component) can decrease symmetry of large-scale solution (which is $O_4$, or in co-moving system results in $O_3 \times P_4$) down to $G_0$.

If $G \subset G_0$, the QC-group has two isomorphous parts, left and right:

$$\Pi(G) = \Pi(l)(G) + \Pi(r)(G).$$

The following Table describes quasi-charge groups and morphisms (for details see [22]); here the 4-th coordinate is the extra dimension which looks along the radius of $G_0$-symmetric expanding cosmological background.
Table. QC-groups $\Pi_l(G)$ and their morphisms to the preceding group; $G \subset G_0$.

| $G$ | $\Pi_l(G) \rightarrow \Pi_l(G^*)$ | ‘label’ |
|-----|---------------------------------|---------|
| 1   | $Z_2$                            | $e$     |
| $SO\{1,2\}$ | $Z_{(e)} \rightarrow Z_2$ |         |
| $SO\{1,2\} \times P\{3,4\}$ | $Z_{(\nu)} + Z_{(H)} \rightarrow Z_{(e)}$ | $\nu^0; H^0 \rightarrow e + e$ |
| $SO\{1,2\} \times P\{2,3\}$ | $Z_{(W)} \rightarrow Z_{(e)}$ | $W \rightarrow e + \nu^0$ |
| $SO\{1,2\} \times P\{2,4\}$ | $Z_{(Z)} \rightarrow Z_{(e)}$ | $Z^0 \rightarrow e + e$ |
| $SO\{1,2\} \times P\{3,4\} \times P\{2,3\}$ | $Z_{(\gamma)} \rightarrow Z_{(H)}$ | $\gamma^0 \rightarrow H^0 + H^0$ |
|                               | $\rightarrow Z_{(W)}$ | $\rightarrow W + W$ |

It seems that ‘quasi-particles’, which symmetry includes $P_4$, should be true neutral (neutrinos, Higgs particles, photon).

One can assume further that an hadron bag is a specific place where $G_0$—symmetry does not work, and the bag’s symmetry is isomorphous to $O_4$ (or $SO_4$). This assumption can lead to another classification of quasi-solitons (some doubling the above scheme), where self-dual and anti-self-dual one-parameter groups (e.g., (a²,1) and (1,a²)), see Eq. (31) for ‘a’) take place of $SO_2$—group. The total set of quasi-particle parameters (parameters of equipped 1-manifold (loop) plus (or ‘times’) parameters of group) for (anti)self-dual groups (or for $SO_2 \subset O_4$), real Grassman manifold $G(4,2)$ times $RP^2$, is larger than the analogous set for groups $SO_2 \subset G_0$, which is just $O_4$ times $G(3,1) = RP^2$. If the number of ‘flavor’-parameters (which are not degenerate and have some preferable particular values) is the same as in the case of ‘white’ quasi-particles, the remaining parameters (degenerate, or ‘phase’) can give some room for ‘color’ (in addition to spin).

So, perhaps one might think about ‘color neutrinos’ (in the context of pomeron, and baryon spin puzzle), ‘color W, Z, and Higgs’ (another context – say, $B$-mesons), and so on.

Note that in our picture the very notion of quasi-particle depends on the background symmetry. On the other hand, large clusters of quasi-particles (matter) can disturb the background, i.e. the form (and thickness) of waveguide; and waves of such small disturbances (with wavelength larger than the thickness $L$, sure) can be generated as well (but these waves do carry no (quasi)charges, that is, are not quantized). Phenomenology of gravity phenomena can arise as an inner (i.e., $(1+3)D$, stable (i.e., Lagrangian) phenomenology describing the form (curvature and thickness) of cosmological waveguide filled with quasi-particles (and noise).

5 Conclusion (need for a crucial experiment)

This section is arranged as a letter to a professor, whose Lab is well equipped for the Bell-type experiments. These experiments[24, 25] (for a review see [26]) meet great interest and continuing discussions[29, 30, 31]. (Recently, Fellows has proposed an interesting classic model[32], arranged as a circus knife-throwing demonstration, where the Bell inequality is violated; however, this model does not meet the dichotomy requirement; in other words, the portion of ‘half-empty’ events, $(+,0)$, $(0,–)$, and so on (only one knife of two reaches its target at ‘plus’- or ‘minus’-part), and, hence, the detectors’ efficiency does vary with the angle between the ‘polarizers’. Well, we believe that Quantum Mechanics gives the excellent description, but QM
does not know what an objective reality is under description.) So, the letter is as follows:

Dear Prof NN,

today many laboratories (including yours) have sources of single (heralded) photons, or entangled bi-photons. Moreover, at some universities students can perform laboratory works with single photons, having convinced on their own experience, that light is quantized, and the classical description is incorrect (the Grangier experiment)[27].

I would like to suggest to your attention a minor modification of the experiment with single photon interference, say, in a Mach-Zehnder fiber interferometer with ‘long’ (the fibers may be rolled) enough arms, 3–5 km (that the time of flight of ‘photon’s halves’ to be 10–20 µs), or, what may be even easier for your Lab, – with bi-photon non-local (also ‘long’ arms – to Alice and Bob) correlations of photon polarizations (the Bell-type experiment).

The objective of this experiment is verification of some naive-realistic model (‘spaghetti-model’, perhaps more naive than realistic), relating to the origin (or possible non-local 5D ontology – with still point-like 4D form-factor) of elementary particles, photons. According to this model, there is one extra space dimension (the radial dimension in an O4− symmetric non-stationary ‘cosmological’ solution of some 5D field theory), and ‘particle’ is a spaghetti-like non-linear field configuration very elongated (say \( L \sim 10^{12} \) cm) along this extra-direction and carrying a topological (quasi)charge.

The new element in the proposed experiment is a fast-acting shutter placed at the beginning of one of the interferometer’s arms (or at the beginning of the Bob’s arm – he should carry out more work). The closing-opening time of the shutter should be small enough, say 5–10 µs. For example, if there is an air gap in this arm (the length of gap may be variable that to draw the interference figure), one can use a quickly rotating metal disk with a set of holes (perhaps, of different diameter – for the sake of comparison).

Both Quantum mechanics (no particle’s ontology) and Bohmian mechanics (wave-particle double ontology)[28] exclude any change in the interference figure as a result of separating activity of such a fast shutter (while the photon’s ‘halves’ are making their ways to the place of a meeting), but I assume (that is, my calculations show) that a significant decrease of the interference visibility can be observed\(^9\).

Certainly, the usage of electro-optical modulator (in combination with polarizer – for single photon case) may turn out to be preferable – that to ensure the full ‘separation of photon’s halves’, or full absence (impossibility of bypass passage) of some ‘remnant photon’s field’, or any ‘links’, ‘navel-strings’, linking the ‘halves’: in this case only one of the two will further carry topological quasi-charge (becoming a ‘full photon’), while the other will dissolve in ‘zero-point oscillations’. QM is everywhere (where we can see, of course), and, so, non-linear 5D-field fluctuations, looking like spaghetti-anti-spaghetti loops, should exist everywhere. (Perhaps, this omnipresence can be related to the universality of ‘low-level heat death’, restricted by the presence of topological quasi-solitons – some as the 2D computer experiment by Fermi, Pasta, and Ulam, where the process of thermalization was restricted by the existence of solitons.)

Having no doubts in your scientific responsibility (I was reading your papers, including . . . ), I realize at the same time that strong enough motivation (and some determination or grit) should exist that to make any new experiment (especially if this experiment is some ques-

\(^8\)Some rough estimates are possible; the most naive is: they say that the Pioneer’s effect turns on at 10 AU, while the antenna pattern (angle \( \lambda/d \)) is 0.01; multiplication of these two gives \( 10^{12} \) cm.

\(^9\)Measuring this visibility as a function of interferometer’s length and transparency of fast shutter, one can arrive at some ideas about the value of \( L \).
tionable – like the first measurement of proton’s magnetic moment (Otto Stern); the suggested experiment is much more simple than, say, the single photon experiment performed by Marshall et al.[31]). That to explain somehow my own motivation (that it is not mere my fantasy), there is the preceding paper concerning topological charges, quasi-charges, and their differential forms in a 5D variant of gravitation theory with Absolute Parallelism; 4D (quasi)charges are also considered for methodological purposes. (Any your comments are welcome. Frankly speaking, I am hoping not so much on ‘my motivation’, as on your intuition and deep interest in the puzzle of non-local correlations.)

AP, at least at the level of its symmetry, seems to be able to cure the gap between the two branches of physics – General Relativity (with coordinate diffeomorphisms) and Quantum Mechanics (with Lorentz invariance).[10] Most people give all the rights of fundamentality (or primacy) to quanta, and so, they try to quantize the gravity, and the very space-time (probing loops, and strings, and branes; see also the warning polemic by Schroer [12]). The other possibility is that quanta have a phenomenological origin of a very specific kind (relating to topological charges and quasi-charges).

With my best regards, . . .

References

[1] T. Sauer, Field equations in teleparallel spacetime: Einstein’s Fernparallelismus approach towards unified field theory, http://arxiv.org/abs/physics/0405142
[2] J.G. Vargas, Geometrization of Physics with Teleparallelism, Found. Phys. 22, 507 (1992).
[3] Y. Itin, Energy-momentum current for coframe gravity, arXiv.org/gr-qc/0111036
[4] V.C. de Andrade, L.C.T. Guillen and J.G. Pereira, Teleparallel gravity: an overview, http://arXiv.org/abs/gr-qc/0011087
[5] D. Vassiliev, A teleparallel model for the neutrino, http://arXiv.org/gr-qc/0604011
[6] A. Einstein and W. Mayer, Sitzungsber. preuss. Akad. Wiss. 257–265 (1931).
[7] A. Einstein and W. Mayer, Sitzungsber. preuss. Akad. Wiss. 110–120 (1930).
[8] A. Unzicker and T. Case, Translation of Einsteins attempt of a unified field theory with teleparallelism (2005), http://arxiv.org/abs/physics/0503046
[9] I.L. Zhogin, J. Soviet Physics 34, 105 (1992); http://arxiv.org/abs/gr-qc/0203008
[10] I.L. Zhogin, J. Soviet Physics 34, 781 (1992); http://arxiv.org/abs/gr-qc/0412081
[11] B.F. Schutz, Geometrical methods of mathematical physics, Cambridge, 1980.
[12] B. Schroer, String theory and the crisis in particle physics (a Samisdat on particle physics), http://arXiv.org/physics/0603112. Russian translation of this paper (including its German lyrics) can be found at http://th1.ihep.su/soloviev/perevod/schroert.pdf
[13] E. Ayón-Beato, A. García, Phys. Lett. B 464, 25 (1999).

10Rovelli writes[33]: In spite of their empirical success, GR and QM offer a schizophrenic and confused understanding of the physical world.
[14] Gamal G.L. Nashed, *Regular Charged Solutions in Teleparallel Theory of Gravity*, 
http://arXiv.org/abs/gr-qc/0610058

[15] L.S. Pontryagin, *Smooth manifolds and their applications in homotopy theory*, Am. Math. Soc. Trans. Ser. 2, 11, 1–114 (1959).

[16] B.A. Dubrovin, A.T. Fomenko and S.P. Novikov, *Modern Geometry – Methods and Applications*, Springer-Verlag, 1984.

[17] A.L. Blakers, W.S. Massey, *The homotopy groups of a triad, I, II and III*. Ann. Math. 53, 161 (1951): 55, 192 (1952); 58, 409 (1953).

[18] T.H.R. Skyrme, Nucl. Phys. 31, 556 (1962).

[19] Yu. P. Rybakov, *Stability of Many-Dimensional Solitons in Chiral Models and Gravitation* // VINITI series “Classical Field Theory and Gravitational Theory”. V. 2. Gravitation and Cosmology. Moscow: VINITI, 1991. P.56-111.

[20] R.S. Ward, Phys. Rev. D70, 061701 (2004); http://arXiv.org/hep-th/0407245

[21] F.I. Fjodorov, *Lorentz group*, Moscow, Science, 1979. In Russian.

[22] I.L. Zhogin, *Research in theory of Riemannian space with Absolute Parallelism*. Ph.D. Thesis, Tomsk University, May 1996 (Rus; see also http://arXiv.org/gr-qc/0412130).

[23] I.L. Zhogin, *Singularities in a gravitation theory*. J. Russian Phys. 35, 647 (1993).

[24] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. 49, 1804 (1982).

[25] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. 81, 5039 (1998); http://arXiv.org/quant-ph/9810080

[26] W. Tittel, G. Weihs, *Photonic Entanglement for Fundamental Tests and Quantum Communication*, http://arXiv.org/quant-ph/0107156.

[27] See links: http://departments.colgate.edu/physics/research/Photon/root/photon_quantum_mechanics.htm; http://marcus.whitman.edu/~beckmk/QM/.

[28] H. Nikolić, *Quantum mechanics: Myths and facts*, arXiv.org/quant-ph/0609163

[29] E. Santos, *Bell’s theorem and the experiments: Increasing empirical support to local realism?* arXiv.org/quant-ph/0410193

[30] G. Adenier, A. Khrennikov, *Anomalies in experimental data for the EPR-Bohm experiment: Are both classical and quantum mechanics wrong?*, arXiv.org/quant-ph/0607172. *Is the Fair Sampling Assumption supported by EPR Experiments?*, http://arXiv.org/quant-ph/0606122

[31] N. Gisin, *How come the Correlations?*, http://arXiv.org/quant-ph/0503007

[32] J. L. Fellows, *Untangling Quantum Entanglement*, arXiv.org/quant-ph/0511134

[33] C. Rovelli, *Unfinished revolution*, http://arXiv.org/gr-qc/0604045

[34] W. Marshall, C. Simon, R. Penrose, D. Bouwmeester, *Towards quantum superpositions of a mirror*, Phys. Rev. Lett. 91, 130401 (2003); http://arXiv.org/quant-ph/0210001