1. Introduction

This paper is concerned with the Hamiltonian actions of a (compact) torus $T$ on a symplectic manifold $(M, \Omega)$. We assume that the moment map $\Phi : M \rightarrow t^*$ is proper and that the generic stabilizer of $T$ on $M$ is finite. We are interested here in two global invariants:

- the Duistermaat-Heckman measure $DH(M)$ which is the pushforward by $\Phi$ of the Liouville volume form,
- the Riemann-Roch characters $RR(M, L^\otimes k), k \geq 1$, which are virtual representations of $T$. Here $M$ is compact and the data $(M, \Omega, \Phi)$ is prequantized by a Kostant-Souriau line bundle $L$. For every couple $(\mu, k) \in \Lambda^* \times \mathbb{Z}^>$, we denote by $m(\mu, k) \in \mathbb{Z}$ the multiplicity of the weight $\mu$ in $RR(M, L^\otimes k)$.

One can associate to a connected component $c$ of regular values of $\Phi$ the local invariants:

- the Duistermaat-Heckman polynomial $DH_c : t^* \rightarrow \mathbb{R}$ which coincides with $DH(M)$ on $c$,
- the periodic polynomial $m_c : \Lambda^* \times \mathbb{Z} \rightarrow \mathbb{Z}$ which coincides with the map $m : \Lambda^* \times \mathbb{Z}^> \rightarrow \mathbb{Z}$ on the cone of $t^* \times \mathbb{R}$ generated by $c \times \{1\}$.

The main results of this paper concern the differences $DH_{c_+} - DH_{c_-}$ and $m_{c_+} - m_{c_-}$ when $c_{\pm}$ are two adjacent connected components of regular values of $\Phi$. Let us introduce some notations. We denote by $\Delta \subset t^*$ the hyperplane that separates $c_\pm$, and by $T_\Delta \subset T$ the subtorus of dimension 1 that has for Lie algebra the one dimensional subspace $t_\Delta$ which is orthogonal to the direction of $\Delta$. We make the choice of a decomposition $T = T_\Delta \times T/T_\Delta$, where $T/T_\Delta$ denotes a subtorus de $T$. At the level of Lie algebras, we have then $t = t_\Delta \oplus (t/t_\Delta)$ and $t^* = t^*_\Delta \oplus (t/t_\Delta)^*$: hence $\xi + (t/t_\Delta)^* = \Delta$ for any $\xi \in \Delta$.

Let $c' \subset \Delta$ be the relative interior of $\overline{c_+} \cap \overline{c_-}$ in $\Delta$. 


In order to give a clear idea of our results we suppose in the introduction that
Φ⁻¹(ξ) ∩ M² is connected when ξ ∈ c'. It means that there exists only one
connected component Z ⊂ M² such that c' ∈ Φ(Z). We denote by Nz the normal
bundle of Z in M.

Let Ω^a_z be the induced symplectic form on the reduced space M_z^a :=
(Φ⁻¹(ξ) ∩ M²)/T/Tₜ. Let ω_z^a ∈ H²(M_z^a) ⊗ t/tΔ be the curvature of the
T/TΔ-principal bundle Φ⁻¹(ξ) ∩ M² → M_z^a. Let β ∈ tΔ be the primitive vector
of the lattice ker(exp : t → T) which is orthogonal to Δ and is pointing out c-. We
prove in Section 2 the following

**Theorem A.** Let 2d be the dimension of M_z^a, and 2r be the rank of the bundle
Nz → Z. We have the following equality of polynomials: for a = a' + a'' ∈
(t/tΔ)^* ⊕ tΔ^* = t*, we have

\[ (DH_{c+} - DH_{c-})(a + ξ) = \frac{|S_z|^{-1}}{\det_{Z}^{1/2}(\xi)} \int_{M_z^a} e^{Ω_z^a + \langle a', \omega_z^a \rangle} P(a'') \]

where P : tΔ → H(M_z^a) is the polynomial mapping defined by

\[ P(a'') = \sum_{k=0}^{d} \frac{α_k}{(r+1+k)^{r+1+k}} \]

Here the α_k ∈ H^{2k}(M_z^a) are characteristic classes and α_0 = 1. In the first equation
\[ \det_{Z}^{1/2}(\xi) \] is the Pfaffian of the infinitesimal action of \( \xi \) on the fibers of Nz,
and \( |S_z|^{-1} \) is the cardinal of the generic stabilizer of T/TΔ on Z.

**Theorem A.** generalizes previous results of Guillemin-Lerman-Sternberg [14] and
Brion-Procesi [11]. In Section 2.4 we give the precise definition of the characteristic
classes α_k.

Suppose now that M is compact and is prequantized by a Kostant-Souriau line
bundle. The hyperplane Δ is defined by the equation \( \frac{(ξ, β)}{2π} = rΔ, ξ ∈ t^* \), for some
\( rΔ ∈ Z \). The bundle Nz decomposes as the sum of two polarized sub-bundles Nz^{±}. Let s^Z_z be the absolute value of the trace of \( \frac{1}{2π} L_β \) on \( N_z^{±} \). In 3.7.3 we define a
family of orbifold vector bundles

\[ S_{Z, k} \rightarrow M_z^a, \quad (μ, k) ∈ Λ^* × Z, \]

with the fundamental property that \( S_{Z, k}^{k} = 0 \) when

\[ -s^Z_z < \frac{μ, β}{2π} - krΔ < s^Z_z. \]

We prove in Section 3.3 the following

**Theorem B.** For all \( (μ, k) ∈ Λ^* × Z \) we have

\[ m_{c_+} (μ, k) - m_{c_-} (μ, k) = RR(M_z^a, S_{Z, μ}), \]

In particular \( m_{c_+} (μ, k) = m_{c_-} (μ, k) \) if

\[ -s^Z_z < \frac{μ, β}{2π} - krΔ < s^Z_z. \]
The integer \( s_+ + s_- \) is larger than half of the codimension of \( Z \) in \( M \). The previous inequalities are optimal, i.e. there exists \((\mu, k)\) such that \( \frac{\mu(k)}{2\pi} - kr_\Delta = \pm s_+ \) and \( m_{\pi}(\mu, k) \neq m_{\pi}(\mu, k) \).

In Section 3 we apply Theorem B to the particular cases where \( M \) is a integral Lie group \( G \). In Section 4.4 we study more precisely the case \( G = \text{SU}(n) \): here our result precis some of the results of Billey-Guillemin-Rassart [10].

In Section 5 we study the case where \( M \) is a complex vector space. In this situation, we interpret Theorem B as a combinatorial formula between vector partition functions. We recover also a Theorem of Szenes-Vergne [32].

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Notations

Throughout the paper \( T \) will denote a compact, connected abelian Lie group, and \( t \) its Lie algebra. The integral lattice \( \Lambda \subset t \) is defined as the kernel of \( \exp: t \to T \), and the real weight lattice \( \Lambda^* \subset t^* \) is defined as \( \Lambda^* := \text{hom}(\Lambda, 2\pi\mathbb{Z}) \). Every \( \mu \in \Lambda^* \) defines a 1-dimensional \( T \)-representation, denoted by \( C_\mu \), where \( t = \exp X \) acts by \( t^\mu := e^{i\langle \mu, X \rangle} \). We denote by \( \text{R}(T) \) the ring of characters of finite-dimensional \( T \)-representations. We denote by \( R^{-\infty}(T) \) the set of generalized characters of \( T \). An element \( \chi \in R^{-\infty}(T) \) is of the form \( \chi = \sum_{\mu \in \Lambda^*} a_\mu C_\mu \), where \( \mu \mapsto a_\mu \), \( \Lambda^* \to \mathbb{Z} \) has at most polynomial growth.

The symplectic manifolds are oriented by their Liouville volume forms. If \((Z, o_Z)\) is an oriented submanifold of an oriented manifold \((M, o_M)\), we take on the fibers of the normal bundle \( N \) of \( Z \) in \( M \), the orientation \( o_N \) satisfying \( o_M = o_Z \cdot o_N \).

2. Duistermaat-Heckman measures

Let \((M, \Omega)\) be a symplectic manifold of dimension \( 2n \) equipped with an Hamiltonian action of a torus \( T \), with Lie algebra \( t \). The moment map \( \Phi: M \to t^* \) satisfies the relations \( \Omega(M, -) + d\langle \Phi, X \rangle = 0, X \in t \). We assume in this section that \( \Phi \) is proper, and that the generic stabiliser \( \Gamma_M \) of \( T \) on \( M \) is finite.

The Duistermaat-Heckman measure \( \text{DH}(M) \) is defined as the pushforward by \( \Phi \) of the Liouville volume form \( \frac{\Omega^n}{|\text{det}(\Omega)|} \) on \( M \). For every \( f \in C^\infty(t^*) \) with compact support one has \( \int_M \text{DH}(M)(a) f(a) = \int_M f(\Phi) \frac{\Omega^n}{|\text{det}(\Omega)|} \). In other terms \( \text{DH}(M)(a) = \int_M \delta(a - \Phi) \frac{\Omega^n}{|\text{det}(\Omega)|} \). We can define \( \text{DH}(M) \) in terms of equivariant forms as follows. Let \( A(M) \) be the space of differential forms on \( M \) with complex coefficients. We denote by \( A_{\text{temp}}(t, M) \) the space of tempered generalized functions over \( t \) with values in \( A(M) \), and by \( M_{\text{temp}}(t^*, M) \) the space of tempered distributions over \( t^* \) with values in \( A(M) \). Let \( F: A_{\text{temp}}(t, M) \to M_{\text{temp}}(t^*, M) \) be the Fourier transform normalized by the condition that \( F(X \mapsto e^{i\langle \xi, X \rangle}) \) is equal to the Dirac distribution \( a \mapsto \delta(a - \xi) \).

Let \( \Omega_t(X) = \Omega - \langle \Phi, X \rangle \) be the equivariant symplectic form. We have then \( F(e^{-i\Omega_t}) = e^{-i\Omega} \delta(a - \Phi) \) and so

\[
\text{DH}(M) = (i)^n \int_M F(e^{-i\Omega_t}).
\]
2.1. **Equivariant cohomology and localization.** We first recall the Cartan model of equivariant cohomology with polynomial coefficients and the extension to generalized coefficients defined by Kumar and Vergne [22]. We give after a brief account to the method of localization developped in [26, 27].

Let $M$ be a manifold provided with an action of a compact connected Lie group $K$ with Lie algebra $\mathfrak{k}$. Let $d : \mathcal{A}(M) \to \mathcal{A}(M)$ be the exterior differentiation. Let $\mathcal{A}_c(M)$ be the sub-algebra of compactly supported differential forms. If $\xi$ is a vector field on $M$ we denote by $c(\xi) : \mathcal{A}(M) \to \mathcal{A}(M)$ the contraction by $\xi$. The action of $K$ on $M$ gives a morphism $X \to X_M$ from $\mathfrak{k}$ to the Lie algebra of vector fields on $M$.

We consider the space of $K$-equivariant maps $\xi \to \mathcal{A}(M)$, $X \mapsto \eta(X)$, equipped with the derivation $(D\eta)(X) := (d - c(X_M))(\eta(X))$, $X \in \mathfrak{k}$. Since $D^2 = 0$, one can define the cohomology space $\ker D/\text{Im} D$. The Cartan model [7, 17] considers polynomial maps and the associated cohomology is denoted $\mathcal{H}_K^*(M)$. Kumar and Vergne [22] studied the cohomology spaces $\mathcal{H}_K^{±\infty}(M)$ obtained by taking $\mathcal{C}^{±\infty}(M)$ maps. Recall the construction $\mathcal{H}_K^{−\infty}(M)$.

The space $\mathcal{C}^{−\infty}(\mathfrak{k}, \mathcal{A}(M))$ of generalized functions on $\mathfrak{k}$ with values in the space $\mathcal{A}(M)$ is, by definition, the space $\text{Hom}(m_c(\mathfrak{k}), \mathcal{A}(M))$ of continuous $C$-linear maps from the space $m_c(\mathfrak{k})$ of smooth compactly supported densities on $\mathfrak{k}$ to the space $\mathcal{A}(M)$, both endowed with the $C^\infty$-topologies. We define $\mathcal{A}_K^{−\infty}(M) := \mathcal{C}^{−\infty}(\mathfrak{k}, \mathcal{A}(M))^K$ as the space of $K$-equivariant $\mathcal{C}^{−\infty}$-maps from $\mathfrak{k}$ to $\mathcal{A}(M)$. The differential $D$ defined on $\mathcal{C}^{−\infty}(\mathfrak{k}, \mathcal{A}(M))$ admits a natural extension to $\mathcal{C}^{−\infty}(\mathfrak{k}, \mathcal{A}(M))$ and $D^2 = 0$ on $\mathcal{A}_K^{−\infty}(M)$ [22]. The cohomology associated to $(\mathcal{A}_K^{−\infty}(M), D)$ is called the $K$-equivariant cohomology with generalized coefficients and is denoted by $\mathcal{H}_K^{−\infty}(M)$. The subspace $\mathcal{A}_{K,c}^{−\infty}(M) := \mathcal{C}^{−\infty}(\mathfrak{k}, \mathcal{A}_c(M))^K$ is stable under the differential $D$, and we denote by $\mathcal{H}_{K,c}^{−\infty}(M)$ the associated cohomology. When $M$ is oriented, the integration over $M$ gives rise to a map $\int_M : \mathcal{H}_{K,c}^{−\infty}(M) \to \mathcal{C}^{−\infty}(\mathfrak{k})^K$.

*Localization procedure.* Let $\lambda$ be a $K$-invariant 1-form on $M$ and let

$$
\Phi_\lambda : M \to \mathfrak{k}^*
$$

be the $K$-equivariant map defined by $\langle \Phi_\lambda(m), X \rangle = \lambda(X_M)m$ : then $D\lambda(X) = d\lambda - \langle \Phi_\lambda, X \rangle$. The localization procedure developped in [26, 27] is based on the existence of an inverse $[D\lambda]^{-1}$ of the $K$-equivariant form $D\lambda$. It is an equivariantly closed element of $\mathcal{A}_K^{−\infty}(M - \Phi_\lambda^{-1}(0))$ defined by the integral

$$
[D\lambda]^{-1}(X) = i \int_0^\infty e^{-itD\lambda(X)}dt.
$$

An open subset $U \subset M$ is called adapted to $\lambda$ if $U$ is $K$-invariant and if $(\partial U) \cap \Phi_\lambda^{-1}(0) = \emptyset$. In [27], we associate to an open subset $U$ adapted to $\lambda$, the following equivariantly closed form with generalized coefficients

$$
P_\lambda^U = \chi^U + d\chi^U[D\lambda]^{-1}\lambda.
$$

Here $\chi^U \in \mathcal{C}^{\infty}(U)$ is a $K$-invariant function supported in $U$ which is equal to 1 in a neighborhood of $U \cap \Phi_\lambda^{-1}(0)$. The cohomology class defined by $P_\lambda^U$ in $\mathcal{H}_K^{−\infty}(M)$ does not depend of $\chi^U$ (in particular $P_\lambda^U = 0$ in $\mathcal{H}_K^{−\infty}(M)$ if $U \cap \Phi_\lambda^{-1}(0) = \emptyset$). If $U \cap \Phi_\lambda^{-1}(0)$ is compact, we take $\chi^U$ with compact support, then $P_\lambda^U$ defines a cohomology class in $\mathcal{H}_{K,c}^{−\infty}(M)$. 

2.2. Localization of $DH(M)$. We come back to the situation of a Hamiltonian action of a torus $T$ on a symplectic manifold $(M, \omega)$. We keep the same notations and hypotheses of the introduction. We need two auxiliary data: a $T$-invariant Riemannian metric on $M$ denoted $(-,-)_M$, and a scalar product $(-,-)$ on $t^*$ which induces an identification $t^* \cong \mathfrak{t}$.

Let $\mathcal{H}$ be the Hamiltonian vectors field of the function $\frac{1}{2} \| \Phi \|^2 : M \to \mathbb{R}$: for every $m \in M$ we have $\mathcal{H}_m = (\Phi(m))_M|_m$. Then for every $\xi \in t^*$, the Hamiltonian vectors field of $\frac{1}{2} \| \Phi - \xi \|^2$ is $\mathcal{H} - \xi M$. For every $\xi \in t^*$, we consider the following $T$-invariant 1-form

\begin{equation}
\lambda_\xi = (\mathcal{H} - \xi M, -)_M
\end{equation}

and the corresponding map $\Phi_{\lambda_\xi} : M \to t^*$ (see (2.2)). Here $\Phi_{\lambda_\xi}^{-1}(0)$ coincides with the subset $\text{Cr}([\| \Phi - \xi \|^2] \subset M$ of critical points of the function $\| \Phi - \xi \|^2$, and $m \in \text{Cr}([\| \Phi - \xi \|^2]$ if and only if $\langle (\Phi(m) - \xi)_M \rangle$ vanishes at $m$ [20, 27].

**Definition 2.1.** Let $P_\xi \in \mathcal{H}^\infty_{T,cr}(M)$ be the cohomology class defined by $P^U_{\lambda_\xi}$, where $U$ is a $T$-invariant relatively compact neighborhood of $\Phi^{-1}(\xi)$ such that $U \cap \text{Cr}([\| \Phi - \xi \|^2]) = \Phi^{-1}(\xi)$.

The cohomology class $P_\xi$ will be used to localized the Duistermaat-Heckman measure. For every $\xi \in t^*$, we define the distribution $DH_\xi(M)$ by

\begin{equation}
DH_\xi(M) = (i)^n \mathcal{F} \left( \int_M P_\xi e^{-i\lambda_\xi} \right).
\end{equation}

Here we can put the Fourier transform outside the integral because $P_\xi$ is compactly supported on $M$. For any $\xi \in t^*$ let $r_\xi > 0$ be the smallest non-zero critical value of the function $\| \Phi - \xi \|^2$. As a particular case of Proposition 3.8 in [27], we have

**Proposition 2.2.** Let $\xi$ be any point in $t^*$. The following equality of distributions on $t^*$

$$DH(M) = DH_\xi(M)$$

holds in the open ball $B(\xi, r_\xi) \subset t^*$.

We will now use the last Proposition, first to recover the classical result of Duistermaat and Heckman [12] concerning the polynomial behaviour of $DH(M)$ on the open subset of regular values of $\Phi$. After we determine the difference taken by $DH(M)$ between two adjacent regions of regular values.

2.3. Polynomial behaviour. We recall now the computation of the cohomology class $P_\xi$ when $\xi$ is a regular value of $\Phi$, that is given in [26][6] for the torus case (and in [27] [Section 3.1] for the case of Hamiltonian action of a compact Lie group). First recall the following basic result which shows that $\xi \mapsto DH_\xi(M)$ is locally constant on the open subset of regular values of $\Phi$.

**Lemma 2.3** ([29]). If $\xi$ and $\xi'$ belong to the same connected component of regular values of $\Phi$, we have $P_\xi = P_{\xi'}$ in $\mathcal{H}^\infty_{T,cr}(M)$.

Associated to a regular value of $\xi$, we have the $T$-principal bundle $\Phi^{-1}(\xi) \to M_\xi := \Phi^{-1}(\xi)/T$ with curvature form $\omega_\xi \in \Omega^2(M_\xi) \otimes \mathfrak{t}$. The orbifold $M_\xi$ carries a canonical symplectic 2-form $\Omega_\xi$. We denote $\text{Kir}_\xi : H^\infty_{T,cr}(M) \to \mathcal{H}^*(M_\xi)$ the Kirwan morphism. For any $\psi \in \mathcal{C}^\infty_{\mathcal{T}}$ and $\eta \in \mathcal{H}^\infty_{T,cr}(M)$ we have $\text{Kir}_\xi(\eta \psi) = \text{Kir}_\xi(\eta) \psi(\omega_\xi)\xi$, where the characteristic class $\psi(\omega_\xi)$ is the value of the differential operator $e^{\omega_\xi}(\frac{\partial}{\partial t})$. 

against $\psi$. After [27, Prop. 3.11], the integral $\int_M P_\xi(X)\eta(X)\psi(X)dX$ is equal to
\begin{equation}
(2.7) \frac{(-2i\pi)^{\dim T}}{|\Gamma_M|} \int_{M_\xi} \text{Kir}_\xi(\eta)\psi(\omega_\xi)
\end{equation}
for every equivariant class $\eta \in \mathcal{H}_c^\infty(M)$. Here $\text{vol}(T, dX)$ is the volume of $T$ for the Haar measure compatible with $dX$, and $|\Gamma_M|$ is the cardinal of $\Gamma_M$ (Note that the generic stabilizer of $T$ on $\Phi^{-1}(\xi)$ is $\Gamma_M$). In other words, for every $\eta \in \mathcal{H}_c^\infty(M)$ we have the following equality of generalized functions on $t^*$ supported at $0$
\begin{equation}
(2.8) \int_M P_\xi(X)\eta(X) = \frac{(-2i\pi)^{\dim T}}{|\Gamma_M|} \int_{M_\xi} \text{Kir}_\xi(\eta)e^{\omega_\xi(\frac{dX}{2\pi})\text{vol}(T, -)}.\end{equation}
For $\eta = e^{-\alpha t}$ we have $\text{Kir}_\xi(\eta) = e^{-i(\Omega_t - (\xi, \omega_\xi))}$, and a small computation shows that
\begin{equation}
(2.9) \mathcal{F}\left( e^{\omega_\xi(\frac{dX}{2\pi})\text{vol}(T, -)} \right)(a) = e^{-i(a, \omega_\xi)} \frac{da}{(2\pi)^{\dim T}}, \quad a \in t^*.
\end{equation}
where $da$ is the Lebesgue measure on $t^*$ normalized by the condition: $\text{vol}(T, dX) = 1$ for the Lebesgue measure $dX$ on $t$ which is dual to $da$.

Finally (2.6), (2.8) and (2.9) give
\begin{equation}
(2.10) \text{DH}_\xi(M)(a) = \frac{(i)^p}{|\Gamma_M|} \int_{M_\xi} e^{-i(\Omega_t + (a, \omega_\xi))} da
\end{equation}
where $2p = \dim M_\xi$.

**Definition 2.4.** For any connected component $c$ of regular values of $\Phi$ we denote $\text{DH}_c$ the polynomial function $a \mapsto \frac{1}{|\Gamma_M|} \int_{M_\xi} \frac{(\Omega_t + (a, \omega_\xi))^p}{p!} da$, where $\xi$ is any point of $c$. Hence, if $\xi \in c$ we have the equality $\text{DH}_\xi(M)(a) = \text{DH}_c(a)da$, $a \in t^*$, where $da$ is the Lebesgue measure on $t^*$ normalized by the condition: $\text{vol}(T, dX) = 1$ for the Lebesgue measure $dX$ on $t$ which is dual to $da$.

With the help of Proposition 2.2, we recover the classical result of Duistermaat and Heckman that says that the measure $\text{DH}(M)$ is locally polynomial\footnote{It is a polynomial times a Lebesgue measure on $t^*$.} on the open subset of regular values of $\Phi$, and it’s value at a regular element $\xi$ is equal to the symplectic volume of the reduce space $M_\xi$. More precisely we have shown that for a connected component $c$ of regular values of $\Phi$ we have
\begin{equation}
(2.11) \text{DH}(M)(a) = \text{DH}_c(a)da, \quad a \in c.
\end{equation}

2.4. **Jump formulas.** Consider now two connected regions $c_+ \cup c_-$ of regular values of $\Phi$ separated by an hyperplane $\Delta \subset t^*$. In this section we compute the polynomial $\text{DH}_{c_+} - \text{DH}_{c_-}$. It generalizes previous results of Guillemin-Lerman-Sternberg [14] and Brion-Procesi [11].

Let $\xi_+, \xi_-$ be respectively two elements of $c_+$ and $c_-$. We know from (2.8), (2.10) and Definition 2.4 that
\begin{equation}
(2.12) \quad (\text{DH}_{c_+} - \text{DH}_{c_-})(a)da = (i)^n \mathcal{F}\left( \int_M (P_{\xi_+} - P_{\xi_-})e^{-\alpha t} \right)(a), \quad a \in t^*.
\end{equation}
We recall now the computation of the cohomology class \( P_{\xi,\gamma} - P_{\xi} \in \mathcal{H}^{-\infty}_c(M) \) [29].

Let \( T_\Delta \subset T \) be the subtorus of dimension 1, with Lie algebra \( t_\Delta := \{ X \in t \mid \langle \xi, X \rangle = 0, \forall \xi, \xi' \in \Delta \} \). We make the choice of a decomposition \( T = T_\Delta \times T/T_\Delta \), where \( T/T_\Delta \) denotes a subtorus of \( T \). At the level of Lie algebras, we have then \( t = t_\Delta \oplus (t/t_\Delta) \) and \( t^* = t_\Delta^* \oplus (t/t_\Delta)^* \); we see that \( \xi + (t/t_\Delta)^* = \Delta \) for any \( \xi \in \Delta \).

Let \( \xi' \) be the relative interior of \( t_\Delta^* \cap t_-^* \) in \( \Delta \). It is not difficult to see that for any \( m \in \Phi^{-1}(\xi') \) the stabilizer \( t_m \subset t \) is either equal to \( t_\Delta \) or reduced to \{0\}.

**Definition 2.5.** We denote \( M_{\xi'} \) the union of the connected component \( Z \) of the fixed point set \( M^{T_\Delta} \) for which we have \( \xi' \in \Phi(Z) \). Let \( M_{\xi'}^{o} \) be the \( T \)-invariant open subset of \( M_{\xi'} \) where \( T/T_\Delta \) acts locally freely.

The symplectic manifold \( M_{\xi'} \) carries a Hamiltonian action of \( T/T_\Delta \) with moment map \( \Phi_{\xi'} : M_{\xi'} \to \Delta \) equal to the restriction of \( \Phi \) on \( M_{\xi'} \). Let \( \xi \) be a point in \( \xi' \). We remark before that for all \( m \in \Phi^{-1}(\xi) \) the stabilizer \( t_m \) is either equal to \( t_\Delta \) or to \{0\}; in particular \( \xi \) is a regular value of \( \Phi_{\xi'} \), i.e. \( \Phi_{\xi'}^{-1}(\xi) \in M_{\xi'}^{o} \). Following Definition 2.1, we associate to \( \xi \) the cohomology class \( P_{\xi}^c \in \mathcal{H}^{-\infty}_{T,\xi,c}(M_{\xi'}^{o}) \).

Let \( \mathcal{H}^*(M_{\xi'}^{o})^{bas} \) be the sub-algebra of \( \mathcal{H}^*(M_{\xi'}^{o}) \) formed by the \( T \)-basic elements. Since the \( T_\Delta \)-action on \( M_{\xi'}^{o} \) is trivial we have a canonical product operation

\[
(2.13) \quad \mathcal{H}^{-\infty}_{T,\xi,c}(M_{\xi'}^{o}) \times \mathcal{H}^{-\infty}_{T,\xi,c}(M_{\xi'}^{o})^{bas} \to \mathcal{H}^{-\infty}_{T,\xi,c}(M_{\xi'}^{o}).
\]

**Proposition 2.6** ([29]). There exists a generalized function supported at 0, \( \delta^\Delta \in \mathcal{C}^{\infty}(t_\Delta^*, \mathcal{H}^*(M_{\xi'}^{o})^{bas}) \), such that

\[
P_{\xi^+} - P_{\xi} = \langle t_\Delta, P_{\xi} \rangle \quad \text{in} \quad \mathcal{H}^{-\infty}_{T,\xi,c}(M).
\]

Here \( \langle t_\Delta, P_{\xi} \rangle : \mathcal{H}^{-\infty}(M_{\xi'}^{o}) \to \mathcal{H}^{-\infty}_{T,\xi,c}(M) \) is the direct image map relative to the inclusion \( i_\Delta : M_{\xi'}^{o} \hookrightarrow M \).

We will now give the precise definition of \( \delta^\Delta \). The decomposition \( T = T_\Delta \times T/T_\Delta \) and the trivial action of \( T_\Delta \) on \( M_{\xi'}^{o} \) determine a canonical isomorphism

\[
\delta^\Delta : \mathcal{H}^*(M_{\xi'}^{o}) \xrightarrow{\sim} S^*(t_\Delta) \otimes \mathcal{H}^*(M_{\xi'}^{o})^{bas},
\]

where \( S^*(t_\Delta) \) is the algebra of complex polynomial functions on \( t_\Delta \). Since the \( T/T_\Delta \)-action on \( M_{\xi'}^{o} \) is locally free, we have the Chern-Weil isomorphism

\[
\mathbf{cv}_\Delta : \mathcal{H}^*(M_{\xi'}^{o}) \sim \mathcal{H}^*(M_{\xi'}^{o})^{bas}.
\]

Let \( N_\Delta \) be the \( T \)-equivariant normal bundle of \( M^{T_\Delta} \) in \( M \), and let \( \text{Eul}(N_\Delta) \in \mathcal{H}_T^*(M^{T_\Delta}) \) be the \( T \)-equivariant Euler class of \( N_\Delta \). Now we consider the restriction of \( \text{Eul}(N_\Delta) \) on the open subset \( M_{\xi'}^{o} \subset M^{T_\Delta} \), that we look through the isomorphism \( \mathbf{cv}_\Delta \circ j_\Delta \) as an element of \( S^*(t_\Delta) \otimes \mathcal{H}^*(M_{\xi'}^{o})^{bas} \) (for simplicity we keep the same notations \( \text{Eul}(N_\Delta) \) for this element). Following [26], we define inverses \( \text{Eul}_{\pm}^{-1}(N_\Delta) \in \mathcal{C}^{\infty}(t_\Delta, \mathcal{H}^*(M_{\xi'}^{o})^{bas}) \) by

\[
(2.14) \quad \text{Eul}_{\pm}^{-1}(N_\Delta)(X) = \lim_{s \to \pm \infty} \frac{1}{(\text{Eul}(N_\Delta)(X \pm i s \beta))^2}.
\]

Here \( \beta \in t_\Delta^* \setminus \{0\} \) is chosen such that \( \langle \xi^+, -\xi^-, \beta \rangle > 0 \).

**Definition 2.7.** The generalized function \( \delta^\Delta \in \mathcal{C}^{\infty}(t_\Delta, \mathcal{H}^*(M_{\xi'}^{o})^{bas}) \) is defined by

\[
(2.15) \quad \delta^\Delta := \text{Eul}_{-\beta}^{-1}(N_\Delta) - \text{Eul}_{-\beta}^{-1}(N_\Delta).
\]
Since the polynomial Eul(N_Δ) is invertible in a smooth manner on t_Δ - \{0\} the generalized function δ^ν is supported at 0. If we restrict δ^ν to the submanifolds Φ^{-1}(ξ) we get the generalized function δ^ν ∈ C^∞(t_Δ, H^*(M^Δ)) where M^Δ denotes the reduced space Φ^{-1}(ξ)/(T/T_Δ)

Now we are able to compute the RHS of (2.12). Let ω^Δ ∈ H^2(M^Δ) ⊗ t/t_Δ be the curvature of the T/T_Δ-principal bundle Φ^{-1}(ξ) → M^Δ. Let |S^Δ| be locally constant function on Φ^{-1}(ξ) which is equal to the cardinal of the generic stabilizer of T/T_Δ. From (2.8) and Proposition 2.6 we have

\[ \int_M (P_{ξ_+} - P_{ξ_-}) (X) e^{-iΩt(X)} = \int_{M_ξ^T} P^Δ_ξ (X') δ^ν(X'') e^{-iΩt(X' + X'')} \]

(2.16) \[ = \frac{(-2π)^{dim T - 1}}{|S^Δ_ξ|} \int_{M^Δ_ξ} e^{iΩt(2|ιξ|)} \text{vol}(T/T_Δ, -) \text{Kir}^Δ_ξ (e^{-iΩt}(X'') δ^ν(X'')) \]

In the last equation the notations are the following:
- X = X' + X'' with X' ∈ t/t_Δ and X'' ∈ t_Δ,
- the Kirwan map \text{Kir}^Δ_ξ : H^ω_ξ^∞(M) → C^∞(t_Δ, H^*(M^Δ)) is the composition of the restriction H^ω_ξ^∞(M) → H^∞(Φ^{-1}(ξ)) with the Chern-Weil isomorphism H^ω_ξ^∞(Φ^{-1}(ξ)) ∼ C^∞(t_Δ, H^*(M^Δ)).

A direct computation gives that \text{Kir}^Δ_ξ (Ω_ξ)(X'') = \Omega^Δ_ξ - ⟨ξ, ω^Δ_ξ + X''⟩ where Ω^Δ_ξ is the induced symplectic form on the reduced space M^Δ_ξ. If we take the Fourier transform in (2.16) we get

\[ (DH_{ξ_+} - DH_{ξ_-}).(a) da = \frac{(i)^{n+1 - dim T}}{|S^Δ_ξ|} \left( \int_{M^Δ_ξ} e^{-i(⟨Ω^Δ_ξ + a', ω^Δ_ξ⟩)} da' \mathcal{F}_ξ (δ^ν_ξ(a'')) (a - ξ) \right), \]

(2.17) \[ = \sum_Z \frac{(i)^{n+1 - dim T}}{|S^Δ_ξ|} \left( \int_Z e^{-i(⟨Ω_ξ^Z + a', ω_ξ^Z⟩)} da' \mathcal{F}_ξ (δ^ν_ξ(a'')) (a - ξ) \right) \]

where a = a' + a'' with a' ∈ (t/t_Δ)* and a'' ∈ (t_Δ)*. In (2.17), we write \int_{M^Δ_ξ} = \sum_Z \int_Z where the sum is taken over the connected components Z of M^Δ_ξ, and \int_Z = (Z ∩ Φ^{-1}(ξ))/T/T_Δ. The 2-forms Ω^Z_ξ, ω^Z_ξ, the generic stabilizer S^Z_ξ, the vector bundle N^Z, the generalized function δ^ν restrict to each component Z: we denote them respectively Ω^Z_ξ, ω^Z_ξ, S^Z_ξ, N^Z, δ^ν_ξ.

We recall now the computation of the Fourier transform of the inverses Eul_{+β}^{-1}(N_Z) = Eul_{+β}^{-1}(N_Δ)|_Z that is given in [26, Proposition 4.8.]. We consider a T-invariant scalar product on the fibers of the bundle N_Δ. Let R ∈ \mathcal{A}^2(M^Z_ξ, so(N_Δ))bas be the curvature of a T-invariant and T/T_Δ-horizontal Euclidean connexion on N_Δ: we denote by R^Z ∈ \mathcal{A}^2(Z, so(N_Δ))bas the restriction of R to a component Z. The curvature commutes with the infinitesimal action L_ξ of X ∈ t_Δ, and with the complex structure J_β = L_β(-L^β_ξ)^1/2 on N_Δ defined by β ∈ t_Δ.

We denote by S^* the symmetric algebra of the complex vector bundle (N_Δ, J_β). We keep the same notation for the restriction of S^* on the submanifolds Z, Φ^{-1}(ξ),
and for the induced orbifold vector bundle on the reduced spaces $Z_\xi$ and $M_\xi$.

For each $k \in \mathbb{N}$, we denote by $\text{Tr}_{S^k}$ the trace operator defined on the complex endomorphism of $S^k$. For a complex endomorphism $A$ of $N_\Delta$, we denote by $A^{\oplus k}$ the induced endomorphism on $S^k$. For any $X \in t_\Delta$, the complex endomorphism $L_X^{-1}R^Z$ is symmetric. Hence the trace $\text{Tr}_{S^k}((L_X^{-1}R^Z)^{\oplus k})$ is a basic real differential form of degree $2k$ on $Z$ which does not depend of the choice of complex structures $(J_\beta$ or $J_{-\beta}$).

**Proposition 2.8 (26).** For a smooth function $f$ on $t_\Delta$ with compact support we have $\int_{t_\Delta} \mathcal{F}_{t_\Delta}(\text{Eul}_{-1}\beta(N_Z))(a''f(a')) = \int_0^\infty P_Z(t)f(t\beta^*)dt$ where $P_Z$ is the polynomial on $\mathbb{R}$ defined by:

$$P_Z(t) = \frac{(2\pi i)^{\text{tr}Z}}{\det^{1/2}(L_\beta)} \left( \frac{t^{\text{tr}Z-1}}{(t^{\text{tr}Z}-1)!} + \sum_{k=1}^{\dim(Z)/2} \frac{(i)^k \text{Tr}_{S^k}((L_X^{-1}R^{Z})^{\oplus k})}{(t^{\text{tr}Z}-1+k)!} \right).$$

Here $\det^{1/2}(L_\beta)$ is the Pfaffian of $L_\beta$ on $N_Z$, and $r_Z = r_{kC}(N_Z)$. For $\mathcal{F}_{t_\Delta}(\text{Eul}_{-1}\beta(N_Z))$ we have

$$\int_{(t_\Delta)^*} \mathcal{F}_{t_\Delta}(\text{Eul}_{-1}\beta(N_Z))(a''f(a')) = \int_0^\infty -P_Z(-t)f(-t\beta^*)dt = \int_{-\infty}^0 P_Z(t)f(t\beta^*)dt.$$

Hence the distribution $\mathcal{F}_{t_\Delta}(\delta^{Z})$ is equal to $P_Z(\beta)d\beta$.

Let $R_Z^Z$ be the restriction of the curvature $R_Z$ to the submanifold $Z \cap \Phi^{-1}(\xi)$. Since $R_Z^Z$ is $T/T_\Delta$-basic, $\text{Tr}_{S^k}((L_X^{-1}R_Z)^{\oplus k})$ can be seen as a real differential form of degree $2k$ on the orbifold $Z_\xi = (Z \cap \Phi^{-1}(\xi))/(T/T_\Delta)$.

Each connected component $Z$ of $M_\xi^\circ$ is a $T/T_\Delta$ Hamiltonian manifold: we take for moment map $\Phi_Z : Z \to (t_\Delta)^*$ the restriction of $\Phi - \xi$ to $Z$. Hence 0 is a regular value of $\Phi_Z$. Let $\text{DH}_0(Z)$ be the polynomial functional on $(t_\Delta)^* = \{a \in t^* | (\beta, a) = 0\}$ such that $\text{DH}(Z)(a') = \text{DH}_0(Z)(a')da'$ near 0. Finally we give the following

**Theorem 2.9. (a).** We have $\text{DH}_{+e} - \text{DH}_{-e} = \sum_{Z \subset M_e} D_Z$ where

$$D_Z(a) = \frac{(-2\pi)^{\text{tr}Z}}{\det^{1/2}(L_\beta)} \left( \sum_{k=0}^{d_Z} Q_{Z,k} \frac{\beta^{\text{tr}Z-1+k}Z_{Z,Z}^k}{(t^{\text{tr}Z}-1+k)!} \right) (a - \xi).$$

The $Q_{Z,k}$ are the polynomials of degree $d_Z - k$ on $(t_\Delta)^*$ defined by

$$Q_{Z,k}(a') = \frac{(-1)^k}{|S_Z^Z|} \int_{Z_\xi} \frac{(\Omega_Z^+ + (a', \omega_Z^+))^{dZ - k}}{(dZ - k)!} \text{Tr}_{S^k}((L_X^{-1}R_Z^Z)^{\oplus k}).$$

Here $2d_Z = \dim Z$ and $2r_Z = \dim M - \dim Z$. The polynomial $Q_{Z,0}$ correspond to the Duistermaat-Heckmann polynomial $\text{DH}_0(Z)$. In particular we see that the polynomial $\text{DH}_{+e} - \text{DH}_{-e}$ is divisible by the factor $a \mapsto (\beta, \xi - a)^{r-1}$ where $r = \inf_Z r_Z$. If $\Delta \cap \Phi(M)$ is not a facet of the polytope $\Phi(M)$ we have $r_Z \geq 2$ for all connected component $Z$ of $M_e$, hence $r \geq 2$.

(b). Suppose now that $e_\xi$ is a connected component of regular values of $\Phi$ bording a facet $\Phi(M) \cap \Delta$ of the polytope $\Phi(M)$. Here $Z = \Phi^{-1}(\Delta)$ is a connected
component of the fixed point set $M T \Delta$. In this situation we have $D H_{\xi_-} = -D Z$ where the polynomial $D Z$ is defined by (2.19).

In (2.19) and (2.20) the vector $\beta \in t$ is normalized by the following conditions: $\beta$ is a primitive vector of the lattice $\ker(\exp: t \to T)$, orthogonal to the hyperplane containing $c_+ \cap c_-$, and pointing out $c_-$. 

3. Quantum version of Duistermaat-Heckman measures

We suppose here that the Hamiltonian $T$-manifold $(M, \omega, \Phi)$ is prequantized by a $T$-equivariant Hermitian line bundle $L$ over $M$, which is equipped with an Hermitian connection $\nabla$ satisfying the Kostant formula

\begin{equation}
L(X) - \nabla_X M = i \langle \Phi, X \rangle, \quad X \in t.
\end{equation}

The former equation implies that the first Chern class of $L$ is equal to $\Omega^2 / 2\pi$. In this section we suppose that $M$ is compact and we still assume that the generic stabilizer $\Gamma_M$ of $T$ on $M$ is finite. The quantization of $(M, \Omega)$ is defined by the Riemann-Roch character $RR(M, L) \in R(T)$ which is compute with a $T$-equivariant almost complex structure on $M$ compatible with $\Omega$. For $k \geq 1$, we consider the tensor product $L^\otimes k$.

Its Riemann-Roch character $RR(M, L^\otimes k)$ decomposes as

\begin{equation}
RR(M, L^\otimes k) = \sum_{\mu \in \Lambda^*} m(\mu, k) C_{\mu}.
\end{equation}

Let us recall the well-known properties of the map $m: \Lambda^* \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}$. When $\mu$ is a regular value of $\Phi$, the "Quantization commutes with Reduction Theorem" [24, 25] tell us that

\begin{equation}
m(\mu, k) = RR(M_{\mu}, L^\otimes k)
\end{equation}

where $L^\otimes k = (L^\otimes k|_{\Phi^{-1}(\mu)} \otimes C_{-\mu})/T$ is an orbifold line bundle over the symplectic orbifold $M_{\mu} = \Phi^{-1}(\mu)/T$. In particular if $\frac{\mu}{k}$ does not belong to $\Phi(M)$ we have $m(\mu, k) = 0$. When $\frac{\mu}{k} \in \Phi(M)$ is not necessarily a regular value of $\Phi$, one proceed by shift desingularization. If $\xi \in \Phi(M)$ is a regular value of $\Phi$ close enough to $\frac{\mu}{k}$ then (3.23) becomes

\begin{equation}
m(\mu, k) = RR(M_{\xi}, L^\otimes k_{\xi, \mu})
\end{equation}

where $L^\otimes k_{\xi, \mu} = (L^\otimes k|_{\Phi^{-1}(\xi)} \otimes C_{-\mu})/T$ (for a proof see [24, 28]).

**Definition 3.1.** A function $f: \Xi \to \mathbb{Z}$ defined over a lattice $\Xi \simeq \mathbb{Z}^r$ is called periodic polynomial if

\begin{equation}
f(x) = \sum_{i=1}^p e^{i \langle \alpha_i, x \rangle} P_i(x), \quad x \in \Xi,
\end{equation}

where $\alpha_1, \ldots, \alpha_p \in \Xi^*$, $N \geq 1$, and the functions $P_1, \ldots, P_p$ are polynomials with complex coefficients.

**Remark 3.2.** Let $C$ a cone with non-empty interior in the real vector space $\Xi \otimes_{\mathbb{Z}} \mathbb{R}$. Any periodic-polynomial function $f: \Xi \to \mathbb{Z}$ is completely determined by its restriction on $C \cap \Xi$. 

Let $c \subset t^*$ be a connected component of regular values of $\Phi$. In \cite{Meinrenken99} Meinrenken an Sjamaar proved that there exits a periodic polynomial function $m_c : \Lambda^* \times \mathbb{Z} \to \mathbb{Z}$ such that $m_c(\mu, k) = m(\mu, k)$ for every $(\mu, k)$ in the cone
\begin{equation}
(3.25) \quad \text{Cone}(c) = \{ (\xi, s) \in t^* \times \mathbb{R}^\geq 0 \mid \xi \in s \cdot c \}.
\end{equation}

Consider now two adjacent connected regions $c_\pm$ of regular values of $\Phi$ separated by an hyperplane $\Delta \subset t^*$. When $\Delta$ does not contain a facet of the polytope $\Phi(M)$, Meinrenken an Sjamaar proved also that
\begin{equation}
(3.26) \quad m_{c_+}(\mu, k) = m_{c_-}(\mu, k) = m(\mu, k)
\end{equation}
for every $(\mu, k) \in \text{Cone}(c_+) \cap \text{Cone}(c_-) = \text{Cone}(c_+ \cap c_-) \subset \text{Cone}(\Delta)$. Our main objective is to prove that \eqref{3.26} extends to a “strip” containing $\text{Cone}(\Delta)$. Let $\beta \in \Lambda$ be the primitive orthogonal vector to the hyperplane $\Delta \subset t^*$ which is pointing out of $c_-$. Then
\begin{equation}
(3.27) \quad \Delta = \{ \xi \in t^* \mid \frac{\langle \mu, \beta \rangle}{2\pi} = r_\Delta \}
\end{equation}
for some $r_\Delta \in \mathbb{Z}$, $\text{Cone}(\Delta) = \{ (\xi, s) \in t^* \times \mathbb{R}^\geq 0 \mid \frac{\langle \xi, \beta \rangle}{2\pi} - sr_\Delta = 0 \}$ and $c_- \subset \{ \xi \in t^* \mid \frac{\langle \xi, \beta \rangle}{2\pi} < r_\Delta \}$. Let $T_\Delta$ be the subtorus of $T$ generated by $\beta$. Let $N_\Delta$ be the normal vector bundle of $M^{T_\Delta}$ in $M$. The almost complex structure on $M$ induces a complex structure $J$ on the fibers of $N_\Delta$. We have a decomposition $N_\Delta = \sum_s N_s^\Delta$ where $N_s^\Delta = \{ v \in N_\Delta \mid J_\beta v = s Jv \}$. We write $N_\Delta = N^\beta_\Delta + N^\beta_\Delta$ where
\begin{equation}
(3.28) \quad N^\pm_\Delta = \sum_{s \geq 0} s N^s_\Delta.
\end{equation}

**Definition 3.3.** For every connected component $Z \subset M^{T_\Delta}$ we define $s^\pm_Z \in \mathbb{N}$ respectively as the absolute value of the trace of $\frac{1}{2\pi} L_\beta$ on $N^{\pm_\beta}_Z|_Z$. Note that $s^+_Z + s^-_Z$ is larger than half of the codimension of $Z$ in $M$.

We prove in Section \ref{sec:3.5} the following

**Theorem 3.4.** We have $m_{c_+}(\mu, k) = m_{c_-}(\mu, k)$ for all $(\mu, k) \in \Lambda^* \times \mathbb{Z}$ such that
\begin{equation}
(3.29) \quad -s^- < \frac{\langle \mu, \beta \rangle}{2\pi} - kr_\Delta < s^+.
\end{equation}
The number $s^-, s^+ \in \mathbb{N}$ are defined as follows. We take $s^\pm = \inf_Z s^\pm_Z$ where the minimum is taken over the connected components $Z$ of $M^{T_\Delta}$ for which $c_+ \cap c_- \subset \Phi(Z)$.

Similar results were obtained by Billey-Guillemin-Rassart \cite{BG} in the case where $M$ is a coadjoint orbit of $\text{SU}(n)$, and by Szenes-Vergne \cite{SV} in the case where $M$ is a complex vector space. See Sections \ref{sec:4.4} and \ref{sec:5} where we study these two particular cases in details. In Proposition \ref{prop:3.28} we give also a criterium which says when the inequalities in \eqref{3.29} are optimal. This criterium is fulfilled when there is only one component $Z$ of $M^{T_\Delta}$ such that $c_+ \cap c_- \subset \Phi(Z)$. Then \eqref{3.29} is optimal and $s^+ + s^-$ is larger than half of the codimension of $Z$ in $M$.

The following easy Lemma (see Lemma 7.3. of \cite{Meinrenken99}) gives some basic informations about the integer $s^\pm_Z$. 
Lemma 3.5. Let \((M, \Omega, \Phi)\) be a compact Hamiltonian \(T\)-manifold equipped with a \(T\)-invariant almost complex structure compatible with \(\Omega\). Consider a non-zero vector \(\gamma \in \mathfrak{t}\) and let \(Z\) be a connected component of the fixed point set \(M^\gamma\). Let \(N\) be the normal vector of \(Z\) in \(M\) and let \(N^{-\gamma}\) be the negative polarized normal bundle (see (3.25)). Then \(N^{-\gamma} = 0\) if and only if the function \((\Phi, \gamma) : M \to \mathbb{R}\) takes its maximal value on \(Z\).

This Lemma insures that \(s^\pm \geq 1\) in Theorem 3.4 when \(\Delta \cap \Phi(M)\) is not a facet of the polytope \(\Phi(M)\).

Consider the situation where \(\Delta \cap \Phi(M)\) is a facet of the polytope \(\Phi(M)\) so that \(c_+ \cap \Phi(M) = \emptyset\); hence \(m_{c_+} = 0\). If we apply Lemma 3.5 to \(\gamma = \beta\), one gets \(N^{-\beta} = 0\) and so \(s^- = 0\). In this situation we get

Corollary 3.6. Let \(c_-\) be a connected component of regular values of \(\Phi\) bounding a facet \(\Phi(M) \cap \Delta\) of the polytope \(\Phi(M)\). Let \(\beta \in \Lambda\) be the unimodular orthogonal vector to the hyperplane \(\Delta \subset \mathfrak{t}^*\) which is pointing out of \(c_-\). Here \(Z = \Phi^{-1}(\Delta)\) is a connected component of the fixed point set \(M^\Delta\). We have \(m_{c_-}(\mu, k) = 0\) for all \((\mu, k) \in \Lambda^+ \times Z\) such that

\[
(3.30) \quad 0 < \frac{\langle \mu, \beta \rangle}{2\pi} - kr_{\Delta} < s^+_\Delta.
\]

Here \(s^+_\Delta \in \mathbb{N}\) is the value of the trace of \(\frac{1}{2\pi}L_\beta\) on the normal bundle of \(Z\) in \(M\), and then is larger than half of the codimension of \(Z\) in \(M\). Moreover the inequalities (3.30) are optimal.

We first review some of the results of [28].

3.1. Elliptic and transversally elliptic symbols. We work in the setting of a compact manifold \(M\) equipped with a smooth action of a torus \(T\).

Let \(p : TM \to M\) be the projection, and let \((-,-)_M\) be a \(T\)-invariant Riemannian metric. If \(E^0, E^1\) are \(T\)-equivariant vector bundles over \(M\), a \(T\)-equivariant morphism \(\sigma \in \Gamma(TM, \text{hom}(p_*E^0, p_*E^1))\) is called a symbol. The subset of all \((m, v) \in TM\) where \(\sigma(m, v) : E^0 \to E^1\) is not invertible is called the characteristic set of \(\sigma\), and is denoted by \(\text{Char}(\sigma)\).

Let \(T_T M\) be the following subset of \(TM\):

\[
T_T M = \{(m, v) \in TM, (v, X_M(m))_M = 0 \text{ for all } X \in \mathfrak{t}\}.
\]

A symbol \(\sigma\) is elliptic if \(\sigma\) is invertible outside a compact subset of \(TM\) (\(\text{Char}(\sigma)\) is compact), and is transversally elliptic if the restriction of \(\sigma\) to \(T_T M\) is invertible outside a compact subset of \(T_T M\) (\(\text{Char}(\sigma) \cap T_T M\) is compact). An elliptic symbol \(\sigma\) defines an element in the equivariant \(K\)-theory of \(TM\) with compact support, which is denoted by \(K_T(TM)\), and the index of \(\sigma\) is a virtual finite dimensional representation of \(T\) [11, 13, 36, 40].

A transversally elliptic symbol \(\sigma\) defines an element of \(K_T(T_T M)\), and the index of \(\sigma\) is defined as a trace class virtual representation of \(T\) (see [11] for the analytic index and [36, 40] for the cohomological one). Remark that any elliptic symbol of \(TM\) is transversally elliptic, hence we have a restriction map \(K_T(TM) \to K_T(T_T M)\),
and a commutative diagram

\[
\begin{array}{ccc}
K_T(TM) & \longrightarrow & K_T(T_T M) \\
\text{Index}_M^T & \downarrow & \text{Index}_M^T \\
R(T) & \longrightarrow & R^{-\infty}(T).
\end{array}
\]

Using the excision property, one can easily show that the index map \( \text{Index}_U^T : K_T(T_U) \rightarrow R^{-\infty}(T) \) is still defined when \( U \) is a \( T \)-invariant relatively compact open subset of a \( T \)-manifold (see [28] [section 3.1]).

3.2. Localization of the Riemann-Roch character. We suppose now that the compact \( T \)-manifold \( M \) is equipped with a \( T \)-invariant almost complex structure \( J \). Let us recall the definitions of the Thom symbol \( \text{Thom}(M, J) \) and of the Riemann-Roch character [28].

Consider a \( T \)-invariant Riemannian metric \( g \) on \( M \) such that \( J \) is orthogonal relatively to \( g \), and let \( h \) be the Hermitian structure on \( TM \) defined by : \( h(v, w) = q(v, w) - \varphi(Jv, w) \) for \( v, w \in TM \). The symbol

\[
\text{Thom}_\varphi(M, J, h) \in \Gamma \left( M, \text{hom}(p^*(\Lambda^even TM), p^*(\Lambda^odd TM)) \right)
\]

at \( (m, v) \in TM \) is equal to the Clifford map

\[
\text{Cl}_m(v) : \Lambda^even TM \longrightarrow \Lambda^odd TM,
\]

where \( \text{Cl}_m(v,w) = v \wedge w - c_h(v,w) \) for \( w \in \Lambda^1 T_m M \). Here \( c_h(v) : \Lambda^1 T_m M \rightarrow \Lambda^{-1} T_m M \) denotes the contraction map relative to \( h \). Since the map \( \text{Cl}_m(v) \) is invertible for all \( v \neq 0 \), the symbol \( \text{Thom}_\varphi(M, J) \) is elliptic.

The Riemann-Roch character \( RR(M, -) : K_T(M) \rightarrow R(T) \) is defined by the following relation

\[
RR(M, E) = \text{Index}_M^T (\text{Thom}_\varphi(M, J) \otimes p^* E).
\]

The important point is that for any \( T \)-vector bundle \( E \), \( \text{Thom}_\varphi(M, J) \otimes p^* E \) corresponds to the principal symbol of the twisted \( \text{Spin}^c \) Dirac operator \( D_E \) [13], hence \( RR(M, E) \in R(T) \) is also defined as the (analytical) index of the elliptic operator \( D_E \).

Consider now the case of a compact Hamiltonian \( T \)-manifold \( (M, \omega, \Phi) \). Here \( J \) is a \( T \)-invariant almost complex structure compatible with \( \Omega : (v, w) \rightarrow \Omega(v, Jw) \) defines a Riemannian metric on \( M \). Like in Section 2.2 we make the choice of a scalar product \((-,-)\) on \( t^* \) (which induces an identification \( t^* \simeq t \)) and we consider for any \( \xi \in t^* \) the function \( \frac{1}{2} \| \Phi - \xi \|^2 : M \rightarrow \mathbb{R} \) and its Hamiltonian vectors field \( \mathcal{H} - \xi_M \).

**Definition 3.7.** For any \( \xi \in t^* \) and any \( T \)-invariant open subset \( \mathcal{U} \subset M \) we define the symbol \( \text{Thom}_\xi(\mathcal{U}) \) by the relation

\[
\text{Thom}_\xi(\mathcal{U})(m, v) := \text{Thom}(M, J)(m, v - (\mathcal{H} - \xi_M)(m)) \quad (m, v) \in T\mathcal{U}
\]

The characteristic set of \( \text{Thom}_\xi(\mathcal{U}) \) corresponds to \( \{(m, v) \in T\mathcal{U} \mid v = (\mathcal{H} - \xi_M)(m)\} \), the graph of the vector field \( \mathcal{H} - \xi_M \) over \( \mathcal{U} \). Since \( \mathcal{H} - \xi_M \) belongs to the set of tangent vectors to the \( T \)-orbits, we have

\[
\text{Char} (\text{Thom}_\xi(\mathcal{U})) \cap T\mathcal{U} = \{(m, 0) \in T\mathcal{U} \mid (\mathcal{H} - \xi_M)(m) = 0\}
\]

\[
\cong \{ m \in \mathcal{U} \mid d \| \Phi - \xi \|^2_m = 0 \}.
\]
Therefore the symbol $\text{Thom}_\xi(U)$ is transversally elliptic if and only if
\begin{equation}
\text{Cr}(\| \Phi - \xi \|^2) \cup \partial U = \emptyset.
\end{equation}
Here $\text{Cr}(\| \Phi - \xi \|^2)$ denotes the set of critical points of the function $\| \Phi - \xi \|^2$.

When (3.34) holds we say that the couple $(U, \xi)$ is good.

**Definition 3.8.** Let $(U, \xi)$ be a good couple. For any $T$-vector bundle $E \rightarrow M$, the tensor product $\text{Thom}_\xi(U) \otimes p^* E$ belongs to $K_T(T_T U)$ and we denote by
\[ RR^\xi_U(M, E) \in R^{-\infty}(T) \]
its index.

**Proposition 3.9.** Let $(U, \xi)$ be a good couple.

a) If $U$ possess two $T$-invariant open subsets $U^1, U^2$ such that $\overline{U^1} \cap \overline{U^2} \cap\text{Cr}(\| \Phi - \xi \|^2) = \emptyset$ and $(U^1 \cup U^2) \cap \text{Cr}(\| \Phi - \xi \|^2) = U \cap \text{Cr}(\| \Phi - \xi \|^2)$, then the couples $(U^1, \xi)$ and $(U^2, \xi)$ are good and
\[ RR^\xi_{U^1}(M, -) = RR^\xi_{U^2}(M, -) + RR^\xi_{U}(M, -). \]
In particular $RR^\xi_{U^1}(M, -) = RR^\xi_{U^2}(M, -)$ if $U^1$ is an open subset of $U$ such that $U \cap \text{Cr}(\| \Phi - \xi \|^2) = U \cap \text{Cr}(\| \Phi - \xi \|^2)$.

b) If $\xi' \in \mathfrak{t}^*$ is close enough to $\xi$, then $(U, \xi')$ is good and
\[ RR^\xi_{U'}(M, -) = RR^\xi_{U}(M, -). \]

**Proof.** The part a) is a direct consequence of the excision property (see Proposition 4.1. in [28]). Consider now the scalar product $(H - \xi^*_M, H - \xi_M)$ where $\xi^s = s\xi^t + (1 - s)\xi$, $s \in [0, 1]$ and $(-, -)$ is a $T$-invariant Riemannian metric on $M$. We have $(H - \xi^*_M, H - \xi_M) = \|H - \xi_M\|^2 + s((\xi - \xi^*)_M, H - \xi_M)$ and then the following inequality holds on $M$
\begin{equation}
(H - \xi^*_M, H - \xi_M) \geq \|H - \xi_M\|^2 \left( \|H - \xi_M\| - s||\xi - \xi^*_M\| \right).
\end{equation}
Since $\partial U$ is compact we have the following inequalities on it: $\|H - \xi_M\| \geq c_1 > 0$ and $\|a_M\| \leq c_2\|a\|$ for any $a \in \mathfrak{t}$. So (3.35) implies the following inequality on $\partial U$:
\[ (H - \xi^*_M, H - \xi_M) \geq c_1(c_1 - s||\xi - \xi^*_M||) \quad \text{for} \quad s \in [0, 1]. \]
So if $\xi'$ is close enough to $\xi$, we have $\|H - \xi^*_M\| \geq c_3 > 0$ on $\partial U$ for any $s \in [0, 1]$. We have first prove that the couple $(U, \xi^*)$ is good for any $s \in [0, 1]$. We see then that the family of transversally elliptic symbols $\text{Thom}_\xi(U)$, $s \in [0, 1]$ defines an homotopy between $\text{Thom}_\xi(U)$ and $\text{Thom}_\xi(U)$. Hence $\text{Thom}_\xi(U) = \text{Thom}_\xi(U)$ in $K_T(T_T U)$. $\square$

Part a) of Proposition 3.9 tells us that $RR^\xi_{U}(M, -)$ depends closely of the intersection $U \cap \text{Cr}(\| \Phi - \xi \|^2)$. In particular $RR^\xi_{U}(M, -) = 0$ when $U \cap \text{Cr}(\| \Phi - \xi \|^2) = \emptyset$. Recall that
\begin{equation}
\text{Cr}(\| \Phi - \xi \|^2) = \bigcup_{\gamma \in B_\xi} M^\gamma \cap \Phi^{-1}(\gamma + \xi)
\end{equation}
where $B_\xi \subset \mathfrak{t}^*$ is a finite set [20].
Definition 3.10. For any $\xi \in \mathfrak{t}^*$ and $\gamma \in \mathcal{B}_\xi$, we denote simply by

$$RR_0^\xi(M, -) : K_T(M) \to R^{-\infty}(T)$$

the map $RR_0^\xi(M, -)$, where $\mathcal{U}$ is a $T$-invariant open neighborhood of $M^\gamma \cap \Phi^{-1}(\gamma + \xi)$ such that $\text{Cr}(\|\Phi - \xi\|^2) \cap \mathcal{U} = M^\gamma \cap \Phi^{-1}(\gamma + \xi)$.

Part a) of Proposition 3.10 insures that the maps $RR_0^\xi(M, -)$ are well defined, and for any good couple $(\mathcal{U}, \xi)$ we have

$$RR_0^\xi(M, -) = \sum_{\gamma \in \mathcal{B}_\xi \cap \Phi(\mathcal{U})} RR_0^\xi(M, -).$$

If one takes $\mathcal{U} = M$, we have $RR_0^\xi(M, -) = RR(M, -) = \sum_{\gamma \in \mathcal{B}_\xi} RR_0^\xi(M, -)$ (see [28][Section 4]).

3.3. Periodic polynomial behaviour of the multiplicities. We suppose here that the Hamiltonian $T$-manifold $(M, \Omega, \Phi)$ is prequantized by a $T$-complex line bundle $L$ satisfying (3.37) for a suitable invariant connection. In this section we will characterize the periodic polynomial behaviour of the multiplicities $m(\mu, k)$ with the help of the localized Riemann-Roch character $RR_0^\xi(M, -)$.

Let us introduce some vocabulary. We say that two generalized characters $\chi^\pm = \sum_{\mu \in \Lambda^*} a_\mu^\pm C_\mu$ coincide on a region $D \subset \mathfrak{t}^*$, if $a_\mu^+ = a_\mu^-$ for every $\mu \in D \cap \Lambda^*$. A generalized character $\chi = \sum_\mu a_\mu C_\mu$ is supported on a region $D \subset \mathfrak{t}^*$ if $a_\mu = 0$ for $\mu \notin D$. A weight $\mu \in \Lambda^*$ occurs in $\chi = \sum_\mu a_\mu C_\mu$ if $a_\mu \neq 0$.

For $\xi \in \mathfrak{t}^*$, we define $r_\xi > 0$ as the smallest non-zero critical value of the function $\|\Phi - \xi\|$, and we denote by $B(\xi, r_\xi)$ the open ball of center $\xi$ and radius $r_\xi$.

Theorem 3.11 ([28]). For any $\xi \in \mathfrak{t}^*$, the generalized character $RR_0^\xi(M, L^{\otimes k})$ coincides with $RR(M, L^{\otimes k})$ on the open ball $k \cdot B(\xi, r_\xi)$.

The arguments of [28] for the proof of this Theorem will be needed another time, so we recall them. Let $\xi \in \mathfrak{t}^*$. We start with the decomposition

$$RR(M, L^{\otimes k}) = \sum_{\gamma \in \mathcal{B}_\xi} RR_0^\xi(M, L^{\otimes k}).$$

We recall now, for a non-zero $\gamma \in \mathcal{B}_\xi$, the localization of the map $RR_0^\xi$ on the fixed point set $M^\gamma$ [28].

Let $N$ be the normal bundle of $M^\gamma$ in $M$. The almost complex structure on $M$ induces an almost complex structure on $M^\gamma$ and a complex structure on the bundles $N$ and $N_C := N \otimes \mathbb{C}$. Following (3.38) we define the $\gamma$-polarized complex vector bundles $N^+ \gamma$ and $(N_C)^+ \gamma$.

The manifold $M^\gamma$ is a symplectic submanifold of $M$ equipped with an induced Hamiltonian action of $T$: its moment map is the restriction of $\Phi$ on $M^\gamma$. Following Definition 3.10, we have on $M^\gamma$ a localized Riemann-Roch character $RR_0^\xi(M^\gamma, -)$. On $M^\gamma$, the Hamiltonian vectors fields of the functions $\|\Phi - \xi\|^2$ and $\|\Phi - (\xi + \gamma)\|^2$ coincide, hence

$$RR_0^\xi(M^\gamma, -) = RR_0^{\xi+\gamma}(M^\gamma, -).$$

We prove in [28][Theorem 5.8] that

$$RR_0^\xi(M, E) = \sum_{k \in \mathbb{N}} (-1)^k RR_0^\xi(M^\gamma, E|_{M^\gamma} \otimes \det(N^+ \gamma) \otimes S^k(N_C^+ \gamma))$$
for every $T$-vector bundle $E$. Here $l$ is the locally constant fonction on $M^\gamma$ equal to the complex rank of $N^{+\gamma}$.

**Proposition 3.12.** [Section 5] Let $\overrightarrow{N}$ be the $T$-vector bundle $N$ with the opposite complex structure on the fibers. The sum $(-1)^l \sum_{k \in \mathbb{N}} \det(N^{+\gamma}) \otimes S^k(N^{+\gamma})$ is an inverse of $\wedge^\gamma \overrightarrow{N}$ that we denote $\left[\wedge^\gamma \overrightarrow{N}\right]^{-1}$.

If we use the notations of Proposition [3.12] and (3.39), the localization (3.40) can be rewritten as

$$ (3.41) \quad RR^\gamma_0(M, E) = RR^\gamma_0(M^\gamma, E|_{M^\gamma} \otimes \left[\wedge^\gamma \overrightarrow{N}\right]^{-1}). $$

Let $i : T_\gamma \hookrightarrow T$ be the inclusion of the subtorus generated by $\gamma$. For a $T$-vector bundle $F \to M^\gamma$, it is easy to show that a weight $\mu \in \Lambda^*$ occurs in $RR^\gamma_0(M^\gamma, F)$ only if $i^*(\mu)$ occurs as a weight for the $T_\gamma$-action on the fibers of $F$ (see Lemma 9.4 in [28]). Since the $T_\gamma$ weights on the bundles $N^{+\gamma}$ and $N^{+\gamma}$ are polarized by $\gamma$, the localization (3.40) gives the following

**Proposition 3.13.** For a non-zero $\gamma \in B_\xi$, the generalized character $RR^\xi_0(M, L^\otimes k)$ is supported on the half space \{$a \in t^* \mid (\gamma, a - k(\xi + \gamma)) \geq 0$\}.

Since the condition $(\gamma, a - k(\xi + \gamma)) \geq 0$ implies that $\|a - k\xi\| \geq k\|\gamma\| \geq k\|r_\xi\|$, the last proposition shows that every weights of the open ball $k \cdot B(\xi, r_\xi)$ does not occur in $RR^\xi_0(M, L^\otimes k)$. This last remark together with (3.38) prove Theorem 3.11.

For the localized Riemann-Roch character $RR^0_0(M, -)$ we have the following Lemma which is very similar to Lemma 3.3.

**Lemma 3.14.** Let $\xi, \xi' \in \mathfrak{c}$ be a connected component of regular values of $\Phi$. For every $\xi, \xi' \in \mathfrak{c}$, we have $RR^0_0(M, -) = RR^\xi_0(M, -)$.

**Proof.** We have to show that the map $\xi \mapsto RR^0_0(M, -)$ is locally constant on $\mathfrak{c}$. Let $\xi \in \mathfrak{c}$ and take an open neighborhood $U$ of $\Phi^{-1}(\xi)$ small enough such that the stabilizer $T_m = \{t \in T \mid t \cdot m = m\}$ is finite for every $m \in U$. We see then that $\overrightarrow{U \cap Cr(\|\Phi - \xi\|')} = \Phi^{-1}(\xi')$ and $\partial U \cap Cr(\|\Phi - \xi\|') = \emptyset$ if $\xi'$ is close enough to $\xi$: hence $RR^0_0(M, -) = RR^\xi_0(M, -)$ for $\xi'$ close enough to $\xi$. Part b) of Proposition 3.13 finishes the proof. $\Box$

When $\xi$ is a regular value of $\Phi$, the localized Riemann-Roch character $RR^\xi_0(M, -)$ as been computed in [28] as follows. Let $RR(M_\xi, -)$ be the Riemann-Roch map defined on the orbifold $\mathcal{M}_\xi = \Phi^{-1}(\xi)/T$ by means of an almost complex structure compatible with the induced symplectic structure. For every $T$-vector bundle $E \to M$ we define the following family of orbifold vector bundles over $\mathcal{M}_\xi$:

$$ (3.42) \quad \mathcal{E}_{\xi, \mu} := \left(E_{|\Phi^{-1}(\xi)} \otimes \mathbb{C}_{-\mu}\right)/T, \quad \mu \in \Lambda^*. $$

For every $T$-vector bundle $E$ on $M$, we proved in [28] [Section 6.2.] the following equality in $R^{-\infty}(T)$

$$ (3.43) \quad RR^\xi_0(M, E) = \sum_{\mu \in \Lambda^*} RR(M_\xi, \mathcal{E}_{\xi, \mu}) \mathbb{C}_\mu. $$

This decomposition was first obtained by Vergne [33] when $T$ is the circle group and when $M$ is Spin. The number $RR(M_\xi, \mathcal{E}_{\xi, \mu}) \in \mathbb{Z}$ is then equal to the $T$-invariant part of the index $RR^\xi_0(M, E) \otimes \mathbb{C}_{-\mu}$. 

Remark 3.15. Let \( t \to t^\lambda \) be a character of \( T \). Suppose that a subgroup \( H \subset T \) acts trivially on \( M \) and with the character \( t \in H \to t^\lambda \) on the fibers of the \( T \)-vector bundle \( E \). Then \( H \) acts with the character \( t \in H \to t^{\lambda - \mu} \) on \( RR_0^H(M, E) \otimes \mathbb{C}_{-\mu} \). and then \( RR(M_\xi, E_{\xi,\mu}) \neq 0 \) only if \( t^{\lambda - \mu} = 1 \) for every \( t \in H \). So the sum in (3.43) can be restricted to \( \lambda + \Lambda_H^* \), where \( \Lambda_H^* \) is the sub-lattice of \( \Lambda^* \) formed by the element \( \alpha \in \Lambda^* \) satisfying \( t^\alpha = 1, \forall \ t \in H \).

This remark applies also on the usual character \( RR(M, E) = \sum_{\mu \in \Lambda^*} m_\mu \mathbb{C}_\mu \). The multiplicity \( m_\mu \in \mathbb{Z} \) is equal to the (virtual) dimension of the \( T \)-invariant part of \( RR(M, E) \otimes \mathbb{C}_{-\mu} \). With the same hypothesis than above we see that \( m_\mu \neq 0 \) only if \( \mu \in \lambda + \Lambda_H^* \).

Let \( \Gamma_M \) be the generic stabilizer for the action of \( T \) on \( M \). Consider a weight \( \alpha_o \) such that \( \Gamma_M \) acts on the fibers of \( L \) with the character \( t \to t^{\alpha_o} \). We define the sub-lattice \( \Xi(M, L) \subset \Lambda^* \times \mathbb{Z} \) by

\[
\Xi(M, L) := \{ (\mu, k) \in \Lambda^* \times \mathbb{Z} \mid k\alpha_o - \mu \in \Lambda^*_M \}.
\]

We know then that \( m(\mu, k) = 0 \) if \( (\mu, k) \notin \Xi(M, L) \).

Proposition 3.16. Let \( c \) be a connected component of regular values of \( \Phi \) and let \( \text{Cone}(c) \) be the corresponding cone in \( T^* \times \mathbb{R}^{>0} \) (see (3.21)). Let \( \xi \in c \). For any \( (\mu, k) \in \text{Cone}(c) \cap \Xi(M, L) \) we have \( m(\mu, k) = RR(M_\xi, L^k_{\xi,\mu}) \) where

\[
L^k_{\xi,\mu} = (L^{\otimes k}|_{\Phi^{-1}(\xi)} \otimes \mathbb{C}_{-\mu})/T.
\]

Proof. Let \( (\mu, k) \in \text{Cone}(c) \) and let \( \xi' = \frac{\xi}{k} \in c \). We known from Theorem 3.14 that the generalized character \( RR^C_0(M, L^{\otimes k}) \) coincides with \( RR(M, L^{\otimes k}) \) on the open ball \( k \cdot B(\xi', r_{\xi'}) = B(\mu, kr_{\xi'}) \). So \( m(\mu, k) \) is equal to the \( \mu \)-multiplicity in \( RR^C_0(M, L^{\otimes k}) \). Take now any \( \xi \in c \). We know after Lemma 3.14 that \( RR^C_0(M, -) = RR^C_0(M, -) \) and (3.43) shows that the \( \mu \)-multiplicity in \( RR^C_0(M, L^{\otimes k}) \) is equal to \( RR(M_\xi, L^k_{\xi,\mu}) \). \( \square \)

Definition 3.17. Take \( \xi \in c \). The map \( m_\xi : \Lambda^* \times \mathbb{Z} \to \mathbb{Z} \) is defined by the equation

\[
m_\xi(\mu, k) = RR(M_\xi, L^k_{\xi,\mu}),
\]

where \( L^k_{\xi,\mu} \) is the orbifold line bundle defined by (3.44). In other words, the map \( m_\xi \) is defined by the following equality in \( R^{-\infty}(T) \)

\[
\sum_{\mu \in \Lambda^*} m_\xi(\mu, k) \mathbb{C}_\mu = RR^C_0(M, L^{\otimes k}).
\]

for all \( k \in \mathbb{Z} \). After remark 3.15, we know that \( m_\xi \) is supported on the sub-lattice \( \Xi(M, L) \) defined in (3.44).

We will now exploit the Riemann-Roch for orbifold due to Atiyah-Kawasaki \[ \text{[19]} \] to show that the map \( m_\xi \) is a periodic polynomial.

3.4. Riemann-Roch theorem on orbifolds. First we recall how is defined the Riemann-Roch character \( RR(M_\xi, E_\xi) \) when \( \xi \) is a regular value of \( \Phi \), and \( E_\xi = E_{\Phi^{-1}(\xi)}/T \) is the reduction of a complex \( T \)-vector bundle \( E \) over \( M \). The number \( RR(M_\xi, E_\xi) \in \mathbb{Z} \) is defined has the \( T \)-invariant part of the index of a transversally elliptic operator \( D_E \) on \( \Phi^{-1}(\xi) \). Since the index of \( D_E \) depend only of the class of
its symbol $\sigma(D_E)$ in $K_T(T_T\Phi^{-1}(\xi))$, it is enough to define the transversally elliptic symbol $\sigma(D_E)$. Since the action of $T$ on $\Phi^{-1}(\xi)$ is locally free, $V := T_T\Phi^{-1}(\xi)$ is a vector bundle. It carries a canonical symplectic structure on the fibers and we choose any compatible complex structure making $V$ into a Hermitian vector bundle. At $(m, v) \in T\Phi^{-1}(\xi)$, the map $\sigma(D_E)(m, v)$ is the Clifford action
\[
\text{Cl}_m(v_1) \otimes \text{Id}_{E_m} : (\wedge^\text{even}_2 V_m) \otimes E_m \rightarrow (\wedge^\text{odd}_2 V_m) \otimes E_m.
\]
where $v_1 \in V_m$ is the $V$-component of the vector $v \in T_m\Phi^{-1}(\xi)$. We explain now the formula of Atiyah-Kawasaki for $RR(M_\xi, \mathcal{E}_\xi)$ when $\xi \in \Phi(M)$ is a regular value of $\Phi$. \hfill \Box 1.19.

Let $\mathcal{F}$ be the collection of the finite subgroup of $T$ which are stabilizer of points in $M$. Consider the orbit type stratification of $\Phi^{-1}(\xi)$ and denote by $S_\xi$ the set of its orbit type strata. Each statum $S$ is a connected component of the smooth submanifold
\[
(3.47) \quad \Phi^{-1}(\xi)_{H_S} := \{ m \in \Phi^{-1}(\xi) \mid \text{Stab}_T(m) = H_S \},
\]
for a unique $H_S \in \mathcal{F}$. The orbifold $M_\xi$ decomposes as a disjoint union $\cup_{S \in S_\xi} S/T$ of smooth components, and each quotient $S/T$ is a suborbifold of $M_\xi$. The generic stabilizer $\Gamma_M$ of $T$ on $M$ is also the generic stabilizer of $T$ on the fiber $^2 \Phi^{-1}(\xi)$, and is associated to an open and dense stratum $S_{\text{max}}$.

Suppose that $E \rightarrow M$ is an Hermitian $T$-vector bundle. On each suborbifold $S/T$, we get the orbifold complex vector bundle
\[
(3.48) \quad \mathcal{E}_S := E_{|S}/T.
\]
We define twisted characteristic classes $\text{Ch}^\gamma(\mathcal{E}_S)$ and $D^\gamma(\mathcal{E}_S)$ by
\[
(3.49) \quad \text{Ch}^\gamma(\mathcal{E}_S) := \text{Tr} \left( \gamma^{\mathcal{E}_S} \cdot e^{\frac{R(\mathcal{E}_S)}}{\pi} \right), \quad \gamma \in H,
\]
and
\[
(3.50) \quad D^\gamma(\mathcal{E}_S) := \det \left( 1 - (\gamma^{\mathcal{E}_S})^{-1} \cdot e^{-\frac{R(\mathcal{E}_S)}}{\pi} \right), \quad \gamma \in H.
\]
Here $R(\mathcal{E}_S) \in \mathcal{A}^2(S/T, \text{End}(\mathcal{E}_S))$ is the curvature of an horizontal Hermitian connection on $E_{|S}$, and $\gamma \mapsto \gamma^{\mathcal{E}_S}$ is the linear action of $H_S$ on the fibers of $E_{|S}$.

Let $N_S$ be the normal bundle of $S$ in $\Phi^{-1}(\xi)$. The symplectic struture on $M$ induces a symplectic form $\Omega_S$ on each suborbifold $S/T$, and a symplectic structure on the fibers of the bundle $N_S$. Choose a compatible almost complex structure on $\overline{S}/T$, and a compatible complex structure on the fibers of $N_S$ making the tangent bundle of $\overline{S}/T$ and $N_S := N_S/T$ into Hermitian vector bundle. Consider a Hermitian connexion on $T(\overline{S}/T)$, with curvature $R(\overline{S}/T)$, and let
\[
(3.51) \quad \text{Td}(\overline{S}/T) = \det \left( \frac{(i/2\pi)R(\overline{S}/T)}{1 - e^{-i/(2\pi)}R(\overline{S}/T)} \right)
\]
be the corresponding Todd forms. Like in \hfill (3.51), we associate to the complex orbifold vector bundle $N_S$, the twisted form $D^\gamma(N_S)$ which is a map form $H_S$ to $\mathcal{A}^\text{even}((\overline{S}/T))$. The 0-degree part of $D^\gamma(N_S)$ is equal to $\det(1 - (\gamma^{N_S})^{-1})$, hence $D^\gamma(N_S)$ is invertible in $\mathcal{A}^\text{even}((\overline{S}/T)$ when $\gamma$ belongs to
\[
(3.52) \quad H_S^\gamma = \{ \gamma \in H_S \mid \det(1 - (\gamma^{N_S})^{-1}) \neq 0 \}.
\]
\footnote{Since a neighborhood of $\Phi^{-1}(\xi)$ in $M$ is $T$-equivariantly diffeomorphic to $\Phi^{-1}(\xi) \times T$.}
Note that \( H^S \) corresponds to the set of \( \gamma \in H_S \) for which \( \Xi \) is a connected component of \( (\Phi^{-1}(\xi))^\gamma \).

**Theorem 3.18** (Atiyah-Kawasaki). The number \( RR(M_\xi, E_\xi) \in \mathbb{Z} \) is given by the formula

\[
(3.53) \quad RR(M_\xi, E_\xi) = \sum_{S \in S_\xi} \frac{1}{|H_S|} \sum_{\gamma \in H_S} \int_{\Xi/T} \frac{Td(\Xi/T) \Ch^\gamma(E_S)}{D^\gamma(N_S)}.
\]

We exploit now Theorem 3.18 to show that the map \( m_\xi : \Lambda^* \times \mathbb{Z} \to \mathbb{Z} \) which is defined by (3.40) is periodic polynomial. We need the classical computation of the first Chern class of the line bundle \( L^k_{S, \mu} \).

\[
(3.54) \quad L^k_{S, \mu} = (L^k \otimes \mathbb{C}_{-\mu})|_{\Xi/T}.
\]

The curvature form \( \omega_\xi \in H^2(M_\xi) \otimes \mathfrak{t} \) of the principal \( T \)-bundle \( \Phi^{-1}(\xi) \to M_\xi \) restricts to a curvature form \( \omega_S \in H^2(\Xi/T) \otimes \mathfrak{t} \) on each strata.

**Lemma 3.19.** The first Chern class of the line bundle \( L^k_{S, \mu} \) is given by

\[
(3.55) \quad c_1(L^k_{S, \mu}) = \frac{1}{2\pi} \left( k\Omega_S - \langle k\xi - \mu, \omega_S \rangle \right).
\]

For a strata \( S \), we consider \( \alpha_S \in \Lambda^* \) such that \( \gamma \in H_S \mapsto \gamma^{\alpha_S} \) corresponds to the action of \( H_S \) on the fibers of \( L^k_{\Xi/T} \). Finally we have the decomposition

\[
(3.56) \quad P_S(\mu, k) = \sum_{S \in S_\xi} P_S(\mu, k),
\]

where

\[
(3.57) \quad P_{max}(\mu, k) = \sum_{S \in S_\xi} \frac{\gamma^k}{|\Gamma_M|} \int_{\Xi/T} \frac{Td(\Xi/T)}{D^\gamma(N_S)} e^{\frac{i}{2\pi} \langle k\Omega_S - \langle k\xi - \mu, \omega_S \rangle \rangle}.
\]

The term \( \sum_{S \in S_\xi} \frac{\gamma^k}{|\Gamma_M|} \int_{\Xi/T} \frac{Td(\Xi/T)}{D^\gamma(N_S)} e^{\frac{i}{2\pi} \langle k\Omega_S - \langle k\xi - \mu, \omega_S \rangle \rangle} \) is equal to 1 when \( (\mu, k) \) belongs to the lattice \( \Xi(M, L) \) (see (3.42)), and is equal to 0 in the other cases. From (3.53) we see that \( P_S \) is a periodic polynomial of degree less than \( \frac{\dim(\Xi/T)}{2} \), and for \( S = S_{\text{max}} \) we have on \( \Xi(M, L) \)

\[
(3.58) \quad P_{max}(\mu, k) = \frac{1}{(2\pi)^l} \int_{\Xi/T} \frac{(k\Omega_S - \langle k\xi - \mu, \omega_S \rangle)^l}{l!} + O(l - 1)
\]

where \( l = \frac{\dim M_\xi}{2} \) and \( O(l - 1) \) denotes a polynomial of degree less than \( l - 1 \). If we use the polynomial \( DH_\xi \) defined in Section 2 we can conclude our computations with the following

**Proposition 3.20.** The map \( m_\xi \) is a periodic polynomial of degree \( l = \frac{\dim M_\xi}{2} \) supported on \( \Xi(M, L) \). For \( (\mu, k) \in \Xi(M, L) \) we have

\[
m_\xi(\mu, k) = |\Gamma_M| \frac{k^l}{(2\pi)^l} DH_\xi \left( \frac{\mu}{k} \right) + O(l - 1),
\]

where \( O(l - 1) \) means a periodic polynomial of degree less than \( l - 1 \).
If $c_\pm$ are two adjacent connected components of regular values of $\Phi$, we know after Theorem 3.4 that $DH_{c_+} \neq DH_{c_-}$ so we can conclude from Proposition 3.20 that

$$m_{c_+} \neq m_{c_-}. \tag{3.59}$$

### 3.5 Jump formulas for the $m_t$

Let $c_+$ and $c_-$ be two adjacent connected component of regular values of $\Phi$ separated by an hyperplane $\Delta$. The aim of this section is to compute the periodic polynomial $m_{c_+} - m_{c_-}

Let $\beta \in \Lambda$ be the unimodular orthogonal vector to the hyperplane $\Delta \subset t^*$ which is pointing out of $c_-$. Let $T_\Delta \subset T$ be the subtorus of dimension 1, with Lie algebra $t_\Delta := \{X \in t| (\xi - \xi', X) = 0, \forall \xi, \xi' \in \Delta\}$. We make the choice of a decomposition $T = T_\Delta \times T/T_\Delta$, where $T/T_\Delta$ denotes a subtorus of $T$.

Let $c'$ be the relative interior of $\overline{c_+} \cap \overline{c_-}$ in $\Delta$. Following Definition 2.1 we denote by $M_{\beta'}$ the union of the connected component $Z$ of the fixed point set $M^{T_\Delta}$ for which $\Phi(Z)$ contains $c'$. The symplectic submanifold $M_{\beta'}$ carries an Hamiltonian action of $T/T_\Delta$ with moment map $\Phi_{\beta'}$ equal to the restriction of $\Phi$ on $M_{\beta'}$.

We consider two points $\xi_+ \in c_+$ such that $\xi = \frac{1}{2}(\xi_+ + \xi_-) \in c'$. We suppose furthermore that $\xi_+ - \xi_-$ is orthogonal to $\Delta$. Using the identification $t^* \simeq t$ given by the scalar product the vector $\gamma = \frac{1}{2}(\xi_+ - \xi_-)$, seen as a vector of $t_\Delta$, belongs to $\mathbb{R}_+^{\beta}$. We noticed in Section 2.4 that for all $m \in \Phi^{-1}(\xi)$ the stabilizer $t_m$ is either equal to $t_\Delta$ or to $\{0\}$. Then $\xi$ is a regular value of $\Phi_{\beta'}$ and there exists an open $T$-invariant neighborhood $U$ of $\Phi^{-1}(\xi)$ in $M$ such that for all $m \in \overline{U}$ either $t_m := \{0\}$, or $t_m = t_\Delta$ and $\Phi(m) \in \Delta$.

One see easily that the couple $(U, \xi)$ is good and part b) of Proposition 3.31 tells us that

$$RR_\xi(U, M, -) = RR_\xi^+(M, -) = RR_\xi^-(M, -) \tag{3.60}$$

when $\xi_\pm$ are close enough to $\xi$. Since $U \cap Cr(\| \Phi - \xi \|)^2 = \Phi^{-1}(\xi)$ we have $RR_\xi(U, M, -) = RR_\xi^-(M, -)$. If $\xi_\pm$ are close enough to $\xi$ we have

$$U \cap Cr(\| \Phi - \xi_\pm \|)^2 = \Phi^{-1}(\xi_\pm) \cup M^\gamma \cap \Phi^{-1}(\xi). \tag{3.61}$$

The former decomposition is due to (3.59) and to the fact that the stabilizer of $t$ on $U$ are either equal to $t_\Delta$ or to $\{0\}$. Notice that $\xi_- + \gamma = \xi_+ + \gamma = \xi$. The decomposition (3.61) gives

$$RR_\xi^\pm(U, M, -) = RR_\xi^\pm(U, M, -) + RR_\xi^\pm_\gamma(M, -), \tag{3.62}$$

where $RR_\xi^\pm(U, M, -)$ (resp. $RR_\xi^\pm_\gamma(M, -)$) is the Riemann-Roch character localized on $M^\gamma \cap \Phi^{-1}(\xi)$ by the vectors field $\mathcal{H} - (\xi_-)_M$ (resp. $\mathcal{H} - (\xi_+)_M$). Now (3.60) and (3.62) prove the following

**Proposition 3.21.** If $\xi_\pm$ are close enough to $\Delta$, we have

$$RR_\xi^\pm(U, M, -) = RR_\xi^\pm_\gamma(M, -) = RR_\xi^\pm(M, -) - RR_\xi^\pm_\gamma(M, -). \tag{3.63}$$

We know from Proposition 3.16 that $m_{c_\pm}(\mu, k)$ is equal to the $\mu$-multiplicity of $RR_\xi^\pm(M, L^\otimes k)$. Hence $m_{c_+}(\mu, k) - m_{c_-}(\mu, k)$ is equal to the $\mu$-multiplicity of $RR_\xi^\pm(M, L^\otimes k) - RR_\xi^\pm_\gamma(M, L^\otimes k)$. 
Let $N_\Delta$ be the normal bundle of $M^\Delta$ in $M$, and let \( \left[ \Lambda^\bullet_c N_\Delta \right]^{-1}_{x,\beta} \) be the polarized inverses of \( \Lambda^\bullet_c N_\Delta \) (see Proposition \ref{prop:polinv}). Since $\xi = \xi_+ - \gamma = \xi_- + \gamma$ and $\gamma \in \mathbb{R}^> 0$, the localization \eqref{eq:loc} gives

\begin{equation}
RR_{\gamma}^\xi (M, E) = \sum_Z RR^\xi_0 (Z, E|_Z \otimes [\Lambda^\bullet_c N_\Delta]^{-1}_{\beta})
\end{equation}

and

\begin{equation}
RR_{\gamma}^\xi (M, E) = \sum_Z RR^\xi_0 (Z, E|_Z \otimes [\Lambda^\bullet_c N_\Delta]^{-1}_{\beta})
\end{equation}

In \eqref{eq:loc} and \eqref{eq:loc2} the sum are taken over the connected components $Z$ of $M^\Delta \subset M^\Delta_\xi$ and we denote $N_Z$ the restriction of the bundle $N_\Delta$ to $Z$. Let us make few remarks concerning the maps $RR^\xi_0 (Z, -) : K_T (Z) \to R^{-\infty} (T)$. Since $T_\Delta$ acts trivially on $Z$, the decomposition $T = T/T_\Delta \times T_\Delta$ induces a canonical isomorphism $K_T (Z) \cong K_{T/T_\Delta} (Z) \otimes R(T_\Delta)$: i.e. every $T$-equivariant vector bundle $E \to Z$ decomposes as

\begin{equation}
E = \sum_{\alpha \in \Lambda^*_1} E^\alpha \otimes C_\alpha.
\end{equation}

Here $\Lambda^*_1 = \Lambda^* \cap t_\Delta^*$ is the set of weights for the subtorus $T_\Delta$, each $E^\alpha$ is a $T/T_\Delta$-equivariant vector bundle on $Z$ and $C_\alpha$ denotes the one dimensional $T$-representation associated to $\alpha \in \Lambda^*_1$.

For every $T$-equivariant vector bundle $E \to Z$, the character $RR^\xi_0 (Z, E)$ is equal to the $T$-equivariant index of the $T$-transversally elliptic symbol $Thom_{\xi} (V) \otimes p^*(E)$, where $V$ is a small neighborhood of $\Phi^{-1}(\xi) \cap Z$ in $Z$ (see Definition \ref{def:transversally_elliptic}). Since the $T_\Delta$ action is trivial on $Z$ the symbol $Thom_{\xi} (V)$ is also $T/T_\Delta$ transversally elliptic and the action of $T_\Delta$ is trivial on it. We have then

\begin{equation}
RR^\xi_0 (Z, E) = \sum_{\alpha \in \Lambda^*_1} RR^\xi_0 (Z, E^\alpha) \otimes C_\alpha
\end{equation}

where $RR^\xi_0 (Z, E^\alpha)$ belongs to $R^{-\infty} (T/T_\Delta)$. Since $\xi$ is a regular value of $\Phi^p$, the character $RR^\xi_0 (Z, E^\alpha)$ is computed by Theorem \ref{thm:character} applied to the Hamiltonian $T/T_\Delta$-manifold $Z$. For every $T$-vector bundle $E \to M$ we define the familly $E_{\xi, \mu_2}$ of orbifold vector bundles over the reduced space $Z_\xi = Z \cap \Phi^{-1}(\xi)/(T/T_\Delta)$ by

\begin{equation}
E_{\xi, \mu_2} := (E^\mu_1 \otimes C_{-\mu_2})|_{\Phi^{-1}(\xi) \cap Z/(T/T_\Delta)}.
\end{equation}

Here $\mu_1 \in \Lambda^*_1$ and $\mu_2 \in \Lambda^*_1$ is trivial on $Z$. We have

\begin{equation}
RR^\xi_0 (Z, E) = \sum_{\mu_1 \in \Lambda^*_1} \sum_{\mu_2 \in \Lambda^*_1} RR^\xi_0 (Z_\xi, E_{\xi, \mu_2}) \otimes C_{\mu_1} \otimes C_{\mu_2}
\end{equation}

In \eqref{eq:transversally_elliptic} we write $\mu \in \Lambda^*$ as a sum of $\mu_1 \in \Lambda^*_1$ with $\mu_2 \in \Lambda^*_1$ so that $C_\mu \in R(T)$ is equal to the tensor product $C_{\mu_1} \otimes C_{\mu_2}$.

Now we finish the computation of the periodic polynomial $m_{\xi_+} (\mu, k) - m_{\xi_-} (\mu, k)$. The hyperplane $\Delta$ is defined by the equation $\frac{\langle \xi, \beta \rangle}{2\pi} = r_\Delta$, $\xi \in T^*$, for some $r_\Delta \in \mathbb{Z}$.
We have the decomposition

$$[\Lambda^\ast \mathcal{N}_Z]^{-1} - [\Lambda^\ast \mathcal{N}_Z]^{-1}_{\beta} = \sum_{\alpha \in \Lambda^\ast} (-1)^{n_Z(\alpha)} S^o_Z \otimes C_\alpha$$

where $S^o_Z \to Z$ is a complex $T/T_\Delta$-vector bundle, and the integers $n_Z(\alpha)$ are defined by the following relations

$$n_Z(\alpha) = \text{rk}_C(N^+_{Z,\beta}) \quad \text{if} \quad \langle \alpha, \beta \rangle \geq 0$$

$$n_Z(\alpha) = \text{rk}_C(N^+_{Z,\beta}) + 1 \quad \text{if} \quad \langle \alpha, \beta \rangle < 0.$$  

Let $\beta^*$ be the vector of $t^\ast_\Delta$ which is defined by the relation $\langle \beta^*, \beta \rangle = 2\pi$, so that $\Lambda^*_{t^\ast} = Z\beta^*$. The $T_\Delta$-weight on $\det(N^+_{Z,\gamma})$ is $\pm s^+_Z \beta^*$ where $s^+_Z \in \mathbb{N}$ is the absolute value of the trace of $\frac{1}{2\pi} \mathcal{L}_\beta$ on $N^+_{Z,\gamma}$. Since the $T_\Delta$-weights on $S^k(N^+_{Z,\gamma})$ (resp. $S^k(N^+_{Z,\gamma})$) are of the form $p\beta^*$ with $p \geq 0$ (resp. $p \leq 0$), we see that

$$s^+_Z = 0 \quad \text{if} \quad s^-_Z < \frac{(\alpha, \beta)}{2\pi} < s^+_Z.$$

The line bundles $L^{\otimes k}|_{M_{t^\ast}}$, $k \in \mathbb{Z}$ can be considered either as $T$-vector bundles or as $T/T_\Delta$-vector vector bundles: we denote them respectively by $L^{\otimes k}|_T$ and by $L^{\otimes k}|_{T/T_\Delta}$. The $T_\Delta$-weight on $L^{\otimes k}|_{M_{t^\ast}}$ is equal to $kr_\Delta\beta^*$, hence we have

$$L^{\otimes k}|_T = L^{\otimes k}|_{T/T_\Delta} \otimes \mathbb{C}_{kr_\Delta\beta^*}.$$  

For every $\mu \in \Lambda^*$ and $Z \subset M_{t^\ast}$ we define the orbifold vector bundle $^3 S^k_{Z,\mu}$ on the reduced space $Z_{\xi} = (Z \cap \Phi^{-1}(\xi))/(T/T_\Delta)$ by

$$S^k_{Z,\mu} := (-1)^{n_Z(\mu_1 - kr_\Delta\beta^*)} \left( L^{\otimes k}|_{T/T_\Delta} \otimes S^o_{Z,\mu_1 - kr_\Delta\beta^*} \otimes \mathbb{C}_{-\mu_2} \right)|_{Z \cap \Phi^{-1}(\xi)/(T/T_\Delta)},$$

where $\mu = \mu_1 + \mu_2$ with $\mu_1 \in \Lambda^*_{t^\ast}$ and $\mu_2 \in \Lambda^*_1$.  

**Theorem 3.22.** Let $S^k_{Z,\mu}$ be the orbifold vector bundle on $M^\Delta_{Z_{\xi}} = \cup_{Z \subset M_{t^\ast}} Z_{\xi}$ which is equal to $S^k_{Z,\mu}$ over each connected component $Z_{\xi}$. For every $(\mu, k) \in \Lambda^* \times \mathbb{Z}$, we have

$$m_{\epsilon_+}(\mu, k) - m_{\epsilon_-}(\mu, k) = RR(M^\Delta_{Z_{\xi}}, S^k_{Z,\mu}).$$

In particular $m_{\epsilon_+}(\mu, k) = m_{\epsilon_-}(\mu, k)$ if

$$-s^- < \frac{(\mu, \beta)}{2\pi} - kr_\Delta < s^+,$$

with $s^\pm = \inf_{Z \subset M_{t^\ast}} s^\pm_Z$.

---

$^3$More precisely $S^k_{Z,\mu}$ is $\pm 1$ times a orbifold vector bundle over the reduced space $Z_{\xi}$.
Proof. After (3.63), (3.64), and Proposition 3.21 we know that $m_{c_+}(\mu, k) - m_{c_-}(\mu, k)$ is equal to the $\mu$-multiplicity in

$$
\sum_{Z \subset M_c} RR^k_0 \left( Z, L^{\otimes k} \otimes \left( [\Lambda^*_0, N_Z]^{-1}_\beta - [\Lambda^*_c, N_Z]^{-1}_\beta \right) \right)
= \sum_{\alpha \in \Lambda^{\alpha}_c} \sum_{Z \subset M_c} (-1)^{n_Z(\alpha)} RR^k_0 \left( Z, L^{\otimes k}_{(T/T_{\Delta})} \otimes S^\alpha_\xi \right) \otimes C_{\alpha + kr_{\Delta} \beta^*} \otimes \left( \mathcal{C}_{s_{\xi}^Z} \right) \otimes \left( \mathcal{C}_{s_{\xi}^{\mu}} \right) \otimes \left( \mathcal{C}_{s_{\xi}} \right)
(3.76)
= \sum_{\mu \in \Lambda^*} RR(\mathcal{M}_{Z, \mu}^{\Delta}, \mathcal{S}_{Z, \mu}^k) \otimes \mathcal{C}_\mu
$$

Here (3.76) is a direct consequence of (3.68) modulo the shift $\mu_1 \mapsto \mu_1 - kn_{\Delta} \beta^*$ in $\Lambda^*(t_{\Delta})$. We get (3.77) using the vanishing conditions (3.71). \hfill \square

When condition (3.76) is the optimal one? In other word, do we have $m_{c_+}(\mu, k) = m_{c_-}(\mu, k)$ for some $\mu \in \Lambda^*$ such that $\frac{(\mu, \beta)}{2\pi} - kr_{\Delta} = s^+$, and do we have $m_{c_+}(\mu, k) = m_{c_-}(\mu, k)$ for some $\mu \in \Lambda^*$ such that $\frac{(\mu, \beta)}{2\pi} - kr_{\Delta} = s^-$?

Until the end of this section we consider couples $(\mu, k)$ such that $\mu = (kr_{\Delta} + s^+) \beta^* + \mu_2$ with $\mu_2 \in \Lambda_{s_{\xi}}^{t_{s_{\xi}}}$. For the orbifold vector bundle $\mathcal{S}_{Z, \mu}^k$ we have $\mathcal{S}_{Z, \mu}^k = (-1)^{rk_c(N_{Z}^{\xi, \beta})} \mathcal{S}_{Z, \mu_2}^k$ where

$$
\mathcal{S}_{Z, \mu_2}^k = \left( L^{\otimes k}_{(T/T_{\Delta})} \otimes det(N_{Z}^{\xi, \beta}) \otimes C_{-\mu_2} \right) \mid_{Z \cap \Phi^{-1}(\xi)/(T/T_{\Delta})}
(3.77)
$$

when $s^+ = s^\xi_{Z}$ and $\mathcal{S}_{Z, \mu_2}^k = 0$ when $s^+ < s^\xi_{Z}$. Hence $m_{c_+}(\mu, k) - m_{c_-}(\mu, k)$ is equal to

$$
\sum_{Z, s^\xi_{Z} = s^+} (-1)^{rk_c(N_{Z}^{\xi, \beta})} RR(\mathcal{Z}_\xi, \mathcal{S}_{Z, \mu_2}^k)
(3.78)
$$

Now we use the results of Section 3.4 to compare the behaviour of the maps

$$(3.79)
\Lambda_{s_{\xi}}^{t_{s_{\xi}}} \times Z \longrightarrow Z,
(\mu_2, k) \mapsto RR(\mathcal{Z}_\xi, \mathcal{S}_{Z, \mu_2}^k)
$$

for the different components $Z$ satisfying $s^\xi_{Z} = s^+$. Let $\Gamma_{Z} \subset T/T_{\Delta}$ be the generic stabiliser of $T/T_{\Delta}$ on a component $Z$. Let $\alpha_Z, \delta_Z \in \Lambda_{s_{\xi}}^{t_{s_{\xi}}}$ such that the action of $\Gamma_Z$ on the fibers of $L_{t_Z}$ and $det(N_{Z}^{+_{s_{\xi}}} \beta)$ are respectively $t \rightarrow t^{\alpha_Z}$ and $t \rightarrow t^{\delta_Z}$. After Remark 3.15 we know that the map (3.79) is supported on the subset

$$
\Xi_Z := \{ (\mu_2, k) \in \Lambda_{s_{\xi}}^{t_{s_{\xi}}} \times Z \mid t^{\alpha_Z + \delta_Z + \mu_2} = 1, \forall \ t \in \Gamma_Z \}
(3.80)
$$

The only difference with the computations done in Section 3.4 is the presence of the line bundle $det(N_{Z}^{\xi, \beta})$. But this do not change the global behaviour of the map (3.79) on $\Xi_Z$: it is a periodic polynomial map of degree $l_Z = \dim(Z)/2$ and we have

$$
RR(\mathcal{Z}_\xi, \mathcal{S}_{Z, \mu_2}^k) = \frac{1}{(2\pi)^{l_Z}} \int_{Z_{\xi}} (k\Omega_{Z_{\xi}} - (k\xi - \mu_2, \omega_{Z_{\xi}}))^l_{Z_{\xi}} + O(l_Z - 1)
(3.81)
$$

for all $(\mu_2, k) \in \Xi_Z$. We see then that the sum (3.78) does not vanish for large values of $(\mu_2, k)$ when the number $(-1)^{rk_c(N_{Z}^{\xi, \beta})}$ are equal for all the components $Z$ of maximal dimension. Finally we can conclude with
Proposition 3.23. Consider the set of components $Z \subset M^\nu$ for which $s^Z_\nu$ is minimal. Among them consider the subset $F$ where $\dim(Z)$ is maximal. If the integers $rk_C(N^+_{\lambda})$, $Z \in F$ have the same parity, then the condition $a(\mu,\beta) - kr_\Delta < s^+\nu$ is optimal in \((3.79)\).

In the same way, consider the set of components $Z \subset M^\nu$ for which $s^Z_\nu$ is minimal. Among them consider the subset $F'$ where $\dim(Z)$ is maximal. If the integers $rk_C(N^+_{\lambda})$, $Z \in F'$ have the same parity, then the condition $-s^- < a(\mu,\beta) - kr_\Delta$ is optimal in \((3.79)\).

4. MULTICILITIES OF GROUP REPRESENTATIONS

Let $K$ be a semi-simple compact Lie group with Lie algebra $\mathfrak{t}$, and let $T$ be a maximal torus in $K$ with Lie algebra $\mathfrak{t}$. In this section we denote $(-,-)$ the scalar product on $\mathfrak{t}$ induced by the Killing form, and we keep the same notation for the induced scalar products on $\mathfrak{t}^*$ and on $\mathfrak{t}$.

An element $\lambda \in \mathfrak{t}^*$ belong to the real weight lattice $\Lambda^* \subset \mathfrak{t}^*$ if $i\lambda$ is the differential of a character of $T$ that we denote $t \mapsto t^\lambda$: if $t = \exp X$ then $t^\lambda := e^{i\lambda X}$. Let $\mathcal{R} \subset \Lambda^*$ be the set of roots for the action of $T$ on $\mathfrak{t} \otimes \mathbb{C}$, and let $\Lambda^*_R$ be the sub-lattice of $\Lambda^*$ generated by $\mathcal{R}$. We choose a system of positive roots $\mathcal{R}^+ \subset \mathcal{R}$, and we denote $\mathfrak{t}_R^+$ the corresponding Weyl chamber.

The irreducible representations of $K$ are parametrized by the set $\Lambda^*_+ = \Lambda^* \cap \mathfrak{t}_R^+$. For $\lambda \in \Lambda^*_+$ we denote by $V_\lambda$ the irreducible representation of $K$ with heighest weight $\lambda$. Here we are interested in the $T$-multiplicities in $V_\lambda|_T$. Let $m : \Lambda^* \times \Lambda^*_+ \to \mathbb{N}$ be the map defined by

$$V_\lambda|_T = \sum_{\mu \in \Lambda^*} m(\mu, \lambda) \mathbb{C}_\mu$$

for every $\lambda \in \Lambda^*_+$.

Definition 4.1. For every $\lambda \in \Lambda^*_+$, we denote $m^\lambda : \Lambda^* \times \mathbb{Z}^+ \to \mathbb{N}$ the map defined by $m^\lambda(\mu, k) = m(\mu, k\lambda)$. So $m^\lambda(\mu, k)$ is equal to the multiplicity of $\mathbb{C}_\mu$ in $V_{k\lambda}|_T$.

4.1. Borel-Weil Theorem. First we recall the realization of the $K$-representation $V_\lambda$ given by the Borel-Weil Theorem. The coadjoint orbit $K \cdot \lambda$ is equipped with the Kirillov-Kostant-Souriau symplectic form $\Omega$ which is defined by

$$\Omega(X_M, Y_M)_m = \langle m, [X, Y] \rangle, \quad \text{for} \quad m \in K \cdot \lambda \quad \text{and} \quad X, Y \in \mathfrak{t}.$$

The action of $K$ on $K \cdot \lambda$ is Hamiltonian with moment map $K \cdot \lambda \hookrightarrow \mathfrak{t}^*$ equal to the inclusion. The action of $T$ on $K \cdot \lambda$ is also Hamiltonian with moment map $\Phi : K \cdot \lambda \to \mathfrak{t}^*$ equal to the composition of the inclusion $K \cdot \lambda \hookrightarrow \mathfrak{t}^*$ with the projection $\mathfrak{t}^* \to \mathfrak{t}$.

There exists a unique $K$-invariant complex structure on $K \cdot \lambda$ compatible with the symplectic form. In this situation the Kostant-Souriau prequantum line bundle over $K \cdot \lambda$ is

$$\mathcal{C}_{[\lambda]} = K \times_{K_\lambda} \mathbb{C}_\lambda.$$

Here we use the canonical identification $K/K_\lambda \simeq K \cdot \lambda$, $[k] \mapsto k \cdot \lambda$, where $K_\lambda$ is the stabilizer of $\lambda$ in $K$. The line bundle $\mathcal{C}_{[\lambda]}$ over the complex manifold $K \cdot \lambda$ carries a canonical holomorphic structure. If one work with the symplectic form $k\Omega$, for an integer $k \geq 1$, the corresponding Kostant-Souriau prequantum line bundle is $\mathcal{C}^{\otimes k}_{[\lambda]} = K \times_{K_\lambda} \mathbb{C}_{k\lambda} = \mathcal{C}_{[k\lambda]}.$
Let $\mathcal{H}^q(K \cdot \lambda, \mathbb{C}_{[\lambda]}^\otimes k)$ be the qth cohomology group of the sheaf of holomorphic sections of $\mathbb{C}_{[\lambda]}^\otimes k$ over $K \cdot \lambda$. The Borel-Weil Theorem tells us that

\begin{equation}
\mathcal{H}^0(K \cdot \lambda, \mathbb{C}_{[\lambda]}^\otimes k) = V_{k\lambda}
\end{equation}

and

\begin{equation}
\mathcal{H}^q(K \cdot \lambda, \mathbb{C}_{[\lambda]}^\otimes k) = 0 \quad \text{for} \quad q \geq 1.
\end{equation}

If $RR^K(K \cdot \lambda, -) : K(K \cdot \lambda) \to R(K)$ is the $K$-Riemann-Roch character defined by the compatible complex structure, (4.83) and (4.85) give

\begin{equation}
RR^K(K \cdot \lambda, \mathbb{C}_{[\lambda]}^\otimes k) = V_{k\lambda} \quad \text{in} \quad R(K)
\end{equation}

Now if we denote by $RR(K \cdot \lambda, -) : K_T(K \cdot \lambda) \to R(T)$ the $T$-equivariant Riemann-Roch character, we have $V_{k\lambda}|_T = RR(K \cdot \lambda, \mathbb{C}_{[\lambda]}^\otimes k)$. The multiplicity function $m^\lambda : \Lambda^*_+ \times \mathbb{N}^* \to \mathbb{N}$ is characterized by the relation

\begin{equation}
RR(K \cdot \lambda, \mathbb{C}_{[\lambda]}^\otimes k) = \sum_{\mu \in \Lambda^*} m^\lambda(\mu, k) \mathbb{C}_\mu, \quad \text{in} \quad R(T),
\end{equation}

for $k \geq 1$.

The sub-lattice $\Lambda_R^*$ of $\Lambda^*$ generated by the roots is characterized by the (finite) center $Z(K)$ of $K$ as follows. For $\alpha \in \Lambda^*$ we have

\begin{equation}
\lambda \in \Lambda_R^* \iff t^\lambda = 1, \quad \forall t \in Z(K),
\end{equation}

and for $t \in T$ we have $t \in Z(K) \iff t^\lambda = 1, \quad \forall \lambda \in \Lambda_R^*$. The finite abelian group $\Lambda^*/\Lambda_R^*$ is then naturally identified with the dual of $Z(K)$. We have the following well-known fact.

**Proposition 4.2.** We have $m^\lambda(\mu, k) \neq 0$ only if $\mu - k\lambda \in \Lambda_R^*$.

**Proof.** The center $Z(K)$ of $K$ acts trivially on $K \cdot \lambda$ and with the character $t \in Z(K) \to t^{k\lambda}$ on the fibers of the line bundle $\mathbb{C}_{[\lambda]}^\otimes k$. Since $m^\lambda(\mu, k)$ is equal to the dimension of the $T$-invariant subspace of $RR(K \cdot \lambda, \mathbb{C}_{[\lambda]}^\otimes k) \otimes \mathbb{C}_\mu$, we have following Lemma 3.14 that $m^\lambda(\mu, k) \neq 0$ only if $t^{\mu - k\lambda} = 1, \quad \forall t \in Z(K)$. We conclude then with (4.88). \qed

In this section we are interested in the periodic polynomials

\begin{equation}
m^\lambda : \Lambda^* \times \mathbb{Z} \to \mathbb{Z},
\end{equation}

defined for every connected component $c \subset \mathfrak{t}^*$ of regular values of the moment map $\Phi : K \cdot \lambda \to \mathfrak{t}^*$: the map $m^\lambda$ coincide with $m^\lambda$ on the set $\{(\mu, k) \in \Lambda^* \times \mathbb{Z}^+ \mid \mu \in k\mathfrak{c}\}$. Like in Proposition 4.2, the formula given in Proposition 5.14 for $m^\lambda$ tells us that $m^\lambda(\mu, k) \neq 0$ only if $\mu - k\lambda \in \Lambda^*_R$.

To apply Theorem 5.22 to the periodic polynomials $m^\lambda$, we have to compute the critical values of the moment map $\Phi : K \cdot \lambda \to \mathfrak{t}^*$.

### 4.2. Critical Points of $\Phi : K \cdot \lambda \to \mathfrak{t}^*$

Let $\{\alpha_1, \cdots, \alpha_{\dim T}\}$ be the simple roots of the set $\mathfrak{a}_+$ of positive weights. The fundamental weights $\varpi_k, 1 \leq k \leq \dim T$ are defined by the conditions

\begin{equation}
2 \frac{(\varpi_i, \alpha_j)}{\vert \alpha_j \vert^2} = \delta_{i,j} \quad \text{for all} \quad 1 \leq i, j \leq \dim T.
\end{equation}
Recall that the fundamental weights generate the lattice $\Lambda^*_{alg}$ of algebraic integral element of $t^*$. We have $\Lambda^* \subset \Lambda^*_{alg}$ and equality holds only if $K$ is simply-connected.

Let $W$ be the Weyl group of $(K, T)$. We will see

\[(4.91) \quad F = \{ \sigma \cdot \varpi_i \mid \sigma \in W, \ 1 \leq i \leq \dim T \} \]

as a subset of $t$ modulo the identification $t \simeq t^*$ given by the scalar product. The singular points of $\Phi$ have the following nice description. This result first appeared in Heckman’s Thesis [13].

**Proposition 4.3** ([15]). The critical points of $\Phi : K \cdot \lambda \rightarrow t^*$ is the union of

\[(K \cdot \lambda)^\beta = \bigcup_{\sigma \in W} K^\beta \cdot \sigma \lambda. \]

Here $K^\beta$ is the stabilizer of $\beta$ in $K$.

The fixed points of the action of $T$ on $K \cdot \lambda$ characterize the image of $\Phi$ completely: $\Phi(K \cdot \lambda)$ is the convex polytope

\[(4.92) \quad \text{conv}(W \cdot \lambda) := \text{convex hull of } W \cdot \lambda. \]

This result was first proved by Kostant [21]. This is particular case of the convexity theorem of Atiyah, Guillemin and Sternberg [2, 15]. From Proposition 4.3, we know that the singular values of $\Phi : K \cdot \lambda \rightarrow t^*$ are the convex polytopes

\[(4.93) \quad \text{conv}(W^\beta \cdot \sigma \lambda), \quad \beta \in F, \ \sigma \in W/W^\beta. \]

where $W^\beta$ is the stabilizer of $\beta$ in $W$, i.e. $W^\beta$ is the Weyl group of $(K^\beta, T)$. Each convex polytope $\text{conv}(W^\beta \cdot \sigma \lambda)$ lies in the hyperplane

\[(4.94) \quad \Delta_{\beta, \sigma} = \{ \xi \in t^* \mid (\xi - \sigma \lambda, \beta) = 0 \}. \]

Note that $K^\beta \cdot \sigma \lambda$ coincide with $K^\beta \cdot \sigma' \lambda$ if and only if $\sigma \lambda \in W^\beta \sigma' \lambda$. Two polytopes $\text{conv}(W^\beta \cdot \sigma \lambda)$ and $\text{conv}(W^\beta \cdot \sigma' \lambda)$ intersect if and only if $(\sigma \lambda, \beta) = (\sigma' \lambda, \beta) = 0$.

**Definition 4.4.** An element $\lambda \in \Lambda^*_+ \text{ is generic if for every fundamental root } \varpi_i \text{ we have} \n
\[(4.95) \quad (\sigma \lambda, \varpi_i) \neq (\sigma' \lambda, \varpi_i) \]

each times that $\sigma \lambda \notin W^i \sigma' \lambda$ (here $W^i = \{ \sigma \in W \mid \sigma \varpi_i = \varpi_i \}$).

This condition of genericity imposes that the hyperplanes $\Delta_{\beta, \sigma}$ and $\Delta_{\beta, \sigma'}$ are distinct each times that the submanifolds $K^\beta \cdot \sigma \lambda$ and $K^\beta \cdot \sigma' \lambda$ are not equal.

**Example 4.5.** Consider the case of $SU(4)$. Take the coadjoint orbit trough $\lambda = (2, 1, -1, -2)$, and $\sigma, \sigma'$ such that $\sigma \lambda = (2, -2, 1, -1)$ and $\sigma' \lambda = (1, -1, 2, -2)$. Take the fundamental weight $\varpi_2 = \frac{1}{2}(1, 1, -1, -1)$. In this case $\lambda$ is not “generic” since $\sigma \lambda \notin W^\lambda \sigma' \lambda$ but $(\sigma \lambda, \varpi_2) = (\sigma' \lambda, \varpi_2) = 0$. 

4.3. Main theorems. Let $c_+$ and $c_-$ be two adjacent connected components of regular values of $\Phi : K \cdot \lambda \to t^*$. The intersection $c_+ \cap c_-$ is contained in an hyperplane orthogonal to $\beta \in F$.

**Definition 4.6.** Let $A(c_+, c_-)$ be the set of all $\sigma \in W/W(\beta)$ such that the convex polytope $\text{conv}(W^\beta \cdot \sigma \lambda)$ contains $c_+ \cap c_-$. The set
\[ \bigcup_{\sigma \in A(c_+, c_-)} K^\beta \cdot \sigma \lambda \]
corresponds to the subset of the critical points of $\Phi$ that intersect $\Phi^{-1}(\xi)$ when $\xi \in c_+ \cap c_-$.  

**Remark 4.7.** When $\lambda$ is a regular element of $t^*$, all polytopes $\text{conv}(W^\beta \cdot \sigma \lambda)$ are of codimension 1. When $\lambda$ is “generic” (see Def. 4.4), the set $A(c_+, c_-)$ is reduced to one element.

The multiplicity function $m^\lambda : \Lambda^* \times N^* \to \mathbb{N}$ is invariant under the action of the Weyl group: $m^\lambda(\sigma \mu, k) = m^\lambda(\mu, k)$ for every $\sigma \in W$. The set of connected component of regular values of $\Phi$ is also invariant under the action of $W$.

So for the rest of this section we restrict our attention to case where $c_+$ and $c_-$ are separated by an hyperplane orthogonal to a fundamental weight $\beta = \gamma_i$: the vector $\beta = \gamma_i$ is pointing out of $c_-$. Consider $\sigma \in A(c_+, c_-)$ and let $K^i \cdot \sigma \lambda$ be the corresponding connected component of $(K \cdot \lambda)^\beta$ (here $K^i$ denote the stabilizer of $\gamma_i$ in $K$). The tangent space of $K \cdot \lambda$ at $\sigma \lambda$ is the following $K^\sigma \lambda$-module
\[ T_{\sigma \lambda}(K \cdot \lambda) = \sum_{(\alpha, \sigma \lambda) > 0} \mathfrak{t}_\alpha \]
where $\mathfrak{t}_\alpha \subset \mathfrak{t} \otimes \mathbb{C}$ is the one-dimensional complex subspace associated to the weight $\alpha \in \mathfrak{h}$. In the same way, the tangent space of $K^i \cdot \sigma \lambda$ at $\sigma \lambda$ is the $K^i \sigma \lambda$-module defined by
\[ T_{\sigma \lambda}(K^i \cdot \sigma \lambda) = \sum_{\substack{(\alpha, \sigma \lambda) > 0 \\ (\alpha, \gamma_i) = 0}} \mathfrak{t}_\alpha \]
Finally the normal bundle of $K^i \cdot \sigma \lambda$ is $N_{\sigma,i} = K^i \times_{K^i \cap K^\sigma \lambda} N_{\sigma,i}$ where
\[ N_{\sigma,i} = \sum_{\substack{(\alpha, \sigma \lambda) > 0 \\ (\alpha, \gamma_i) \neq 0}} \mathfrak{t}_\alpha \]

For an element $\mu \in t^*$, we have $\mu = \sum_{i=1}^{\dim T} [\mu]_k \alpha_k$ where the $[\mu]_k \in \mathbb{R}$ are defined by the relation $[\mu]_k = 2(\gamma_i, \mu)/|\gamma_i|^2$. When $\mu \in \Lambda^*_R$ the coefficients $[\mu]_k$ are all integers.

**Definition 4.8.** For $\sigma \in A(c_+, c_-)$ we define the positive integers
\[ s^\pm_{\sigma,i} = \pm \sum_{\substack{(\alpha, \sigma \lambda) > 0 \\ \pm(\alpha, \gamma_i) > 0}} [\alpha]_i. \]

Note that $s^+_{\sigma,i} + s^-_{\sigma,i}$ is larger than half of the codimension of $K^i \cdot \sigma \lambda$ in $K \cdot \lambda$. 

\textbf{Theorem 4.9.} Let \( \zeta_+ \) and \( \zeta_- \) be two adjacent connected component of regular values of \( \Phi : K \cdot \lambda \to \mathfrak{t}^* \) separated by an hyperplane orthogonal to a fundamental weight \( \omega_i \): we denote \( r_i \) the commum value \([\xi], \lambda \) for all \( \xi \) in this hyperplane. Let \( m^{\lambda}_{\zeta_\pm} : \Lambda^* \times \mathbb{Z} \to \mathbb{Z} \) be the corresponding periodic polynomials: they are supported on the sub-lattice \( \Xi_\lambda := \{(\mu, k) | \mu \in k\lambda + \Lambda_R^\ast \} \).

For all \( (\mu, k) \in \Xi_\lambda \), we have \( m^{\lambda}_{\zeta_+} (\mu, k) = m^{\lambda}_{\zeta_-} (\mu, k) \) when

\( (4.99) \)

\[ -s^-_i < [\mu]_i - kr_i < s^+_i. \]

Here the positive integer \( s^\pm_i \) are defined by

\( (4.100) \)

\[ s^\pm_i = \inf_{\sigma \in \mathcal{A}(\zeta_+, \zeta_-)} s^\pm_{\sigma, i}. \]

When \( \mathcal{A}(\zeta_+, \zeta_-) \) is reduced to one element \( \sigma \), for example if \( \lambda \) is “generic”, the integer \( s^+ + s^- \) is larger than half of the codimension of \( K^i \cdot \sigma \lambda \) in \( K \cdot \lambda \).

Another way to express the result of Theorem 4.9 is to introduce like in \( \Phi_2 \) the convex polytope

\( (4.101) \)

\[ \Box (\zeta_+, \zeta_-) = \bigcap_{\sigma \in \mathcal{A}(\zeta_+, \zeta_-)} \left( \sum_{(a, \sigma, \lambda) > 0} [0, 1] \alpha \right). \]

Let \( \Delta \) be the hyperplane which separates \( \zeta_+ \) and \( \zeta_- \). Equation \( (4.99) \) is equivalent to saying that

\( (4.102) \)

\[ m^{\lambda}_{\zeta_+} (\mu, k) = m^{\lambda}_{\zeta_-} (\mu, k) \quad \text{if} \quad \mu \in k\Delta + \Box (\zeta_+, \zeta_-). \]

\textbf{Corollary 4.10.} Let \( \zeta \) be a connected component of regular values of \( \Phi \) which is bording a facet of the polytope \( \Phi(K, \lambda) \) orthogonal to the fundamental weight \( \omega_\zeta \): the facet is \( \text{conv}(W^i \cdot \sigma \lambda) \) for a unique \( \sigma \in W/W^i \). We suppose that \( \omega_\zeta \) is pointing out of \( \zeta \). We denote \( r_i \) the commum value \([\xi], \lambda \) for all \( \xi \) in the facet. For all \( (\mu, k) \in \Xi_\lambda \), we have \( m^{\lambda}_{\zeta} (\mu, k) = 0 \) when

\( (4.103) \)

\[ -s^-_{\sigma, i} < [\mu]_i - kr_i < s^+_i. \]

\textbf{Proof.} Theorem 4.9 is a direct consequence of Theorem 3.22. The main difference between them is the decomposition of the lattice supporting the periodic polynomials. In the former we use the decomposition \( \Lambda^* = \Lambda^*_{i_1} \oplus \Lambda^*_{i_2} \) associated to the choice of a subtorus \( T/T_\Delta \). Here, since \( m^{\lambda}_{\zeta_\pm} \) is supported on \( \lambda + \Lambda_R^\ast \), we use the decomposition \( \Lambda^* = \mathbb{Z} \alpha_i \oplus \sum_{k \neq i} \mathbb{Z} \alpha_k \).

We start like after Proposition 3.21. \( m^{\lambda}_{\zeta_+} (\mu, k) - m^{\lambda}_{\zeta_-} (\mu, k) \) is equal to the \( \mu \)-mutiplicity in \( \sum_{\sigma \in \mathcal{A}(\zeta_+, \zeta_-)} A^\sigma + A^\sigma \) where

\( (4.104) \)

\[ A^\pm = RR_0^\xi \left( K^i \cdot \sigma \lambda, C_{[\lambda]}^{\otimes k} \otimes \left[ \otimes_{\mathfrak{t}, \tau} N_{\xi, i} \right]^{-1} \right). \]

Here \( \xi \) belongs to the relative interior of \( t_+ \cap t_- \), the line bundle \( C_{[\lambda]}^{\otimes k} \) is equal to \( K^i \times K^i \otimes K^\ast \otimes \mathbb{C}_{\sigma \lambda} \) and \( \left[ \otimes_{\mathfrak{t}, \tau} N_{\xi, i} \right]^{-1} \) corresponds to \((-1)^{rkc(N_{\xi, i})} \) times

\[ K^i \times K^i \otimes K^\ast \otimes \left( \det (N_{\xi, i}^\pm) \otimes S^\ast ((N_{\xi, i} \otimes \mathbb{C})^\pm) \right). \]
with
\[ N_{\sigma,i}^\pm = \sum_{\langle \alpha, \varpi_i \rangle > 0} \xi_\alpha, \]
and
\[ (N_{\sigma,i} \otimes \mathbb{C})^\pm = \sum_{\langle \alpha, \varpi_i \rangle \not> 0} \xi_\alpha. \]

Now we can apply Remark 3.16 with the subgroup \( H \subset T \) equal to the center \( Z(K^t) \) of \( K^t \): an element \( \gamma \in A^* \) belong to \( \sum_{k \neq i} Z\alpha_k \) if and only if \( t^\gamma = 1 \) for all \( t \in Z(K^t) \).

The group \( Z(K^t) \) acts trivially on the manifolds \( K^t \cdot \sigma \lambda \), and with the characters associated to the weights
\[ k\sigma \lambda + \sum_{\langle \alpha, \varpi_i \rangle \not> 0} \alpha + \delta \text{ with } (\delta, \varpi_i) \not\geq 0 \]
on the bundle \( \mathbb{C}^k_\lambda \otimes \left[ \Lambda^\ast_{\mathbb{C}, \lambda} \right]_{-\varpi_i}^{-1} \), and with the characters associated to the weights
\[ k\sigma \lambda + \sum_{\langle \alpha, \varpi_i \rangle > 0} \alpha + \delta \text{ with } (\delta, \varpi_i) \geq 0 \]
on the bundle \( \mathbb{C}^k_\lambda \otimes \left[ \Lambda^\ast_{\mathbb{C}, \lambda} \right]_{-\varpi_i}^{-1} \). Now the \( \mu \)-multiplicity in \( A^\pm_\sigma \) is not equal to 0 only if
\[ (4.105) \quad k\sigma \lambda + \sum_{\langle \alpha, \varpi_i \rangle > 0} \alpha + \delta - \mu \in \sum_{k \neq i} Z\alpha_k \text{ with } (\delta, \varpi_i) \not\leq 0. \]
Condition (4.105) implies that \([\mu]_i \geq k[\sigma \lambda]_i + s^+_\sigma,i\) or \([\mu]_i \leq k[\sigma \lambda]_i - s^-_\sigma,i\). Finally we have prove that \( m^+_\lambda (\mu, k) = m^-_\lambda (\mu, k) \) if
\[ -s^-_\sigma,i < [\mu]_i - k[\sigma \lambda]_i < s^+_\sigma,i \]
for all \( \sigma \in A(c_+, c_-). \) \( \square \)

4.4. The case of \( SU(n) \). Let \( T \) be the maximal torus of \( SU(n) \) consisting of the diagonal matrices. The dual \( t^* \) can be identified with the subspace \( x_1 + \cdots + x_n = 0 \) of \( \mathbb{R}^n \). The roots are \( \mathfrak{R} = \{ e_i - e_j | 1 \leq i \neq j \leq n \} \) and we will choose the positives ones to be \( \mathfrak{R}^+ = \{ e_i - e_j | 1 \leq i < j \leq n \} \). The simple roots are then \( \alpha_i = e_i - e_{i+1}, \) for \( 1 \leq i \leq n - 1 \), and for these simple roots, the fundamental weights are
\[ (4.106) \quad \omega_k = \frac{1}{n} (n-k, n-k, \cdots, n-k, -k, -k, \cdots, -k), \quad 1 \leq k \leq n-1. \]

Consider now the coadjoint orbit \( O_\lambda \) for \( \lambda \in t^* \). Let \( \Phi : O_\lambda \to t^* \) the moment map associated to the Hamiltonian action of \( T \) on \( O_\lambda \). The center of \( SU(n) \), that we denote \( Z_n \) corresponds to the set of matrices \( zI \) with \( z^n = 1 \). Recall the following well-known fact.

**Lemma 4.11.** Let \( \xi \) be a regular value of \( \Phi : O_\lambda \to t^* \). Then for every \( m \in \Phi^{-1}(\xi) \) the stabilizer subgroup \( T_m := \{ t \in T | t \cdot m = m \} \) is equal to \( Z_n \).
The dual of the Lie algebra \(\mathfrak{su}(n)\) decomposes as \(\mathfrak{su}(n) = t^* \oplus \sum_{a \in \mathbb{R}^+} \mathfrak{su}(n)^a\), where \(\mathfrak{su}(n)^a \simeq \mathbb{C} - \{0\}\) as \(T\)-module. For \(m \in \Phi^{-1}(\xi)\), we have \(m = m_0 + \sum_{a \in \mathbb{R}^+} m_a\) with \(m_a \in \mathfrak{su}(n)^a\), and then \(T_m = \cap m_a \neq 0 \ker(t \mapsto t^a)\). So the lattice \(\Lambda^*_m\) generated by the set \(\{a \in \mathbb{R}^+ | m_a \neq 0\}\) is a subgroup of \(\Lambda^*_R\) with \(\Lambda^*_R/\Lambda^*_m\) finite. We have to show that \(\Lambda^*_m = \Lambda^*_R\). For this purpose we introduce the following equivalence relation on \(\{1, \ldots, n\}\):

\[i \sim j \iff e_i - e_j \in \Lambda^*_m.\]

Suppose that \(\{1, \ldots, n\}/\sim\) is not reduced to a point: let \(C_1\) and \(C_2\) be two distinct equivalent classes and let \(\beta = (\beta_1, \ldots, \beta_n)\) be the element of \(t^*\) defined by: \(\beta_i = \frac{1}{c_i}\) if \(i \in C_1\), \(\beta_i = \frac{1}{c_i}\) if \(i \in C_2\), and \(\beta_i = 0\) in the other cases. We see then that \((\beta, \alpha) = 0\) for all \(\alpha \in \Lambda^*_m\): it is in contradiction with the fact that \(\Lambda^*_R/\Lambda^*_m\) is finite. We have proved that \(e_i - e_j \in \Lambda^*_m\) for all \(i, j \in \{1, \ldots, n\}\). \(\square\)

Suppose now that \(\lambda\) is a positive weight, and let \(\epsilon\) a connected component of regular values of \(\Phi : O_L \to t^*\). We know that the corresponding periodic polynomial \(m^\lambda : \Lambda^* \times \mathbb{Z} \to \mathbb{Z}\) is supported on the sub-lattice \(\Xi_L := \{(\mu, k) | \mu \in k \Lambda + \Lambda^*_R\}\).

**Corollary 4.12.** The map \(m^\lambda : \Xi_L \to \mathbb{Z}\) is a polynomial of degree \(\frac{(n-1)(n-2)}{2} - d_\lambda\), where \(d_\lambda\) is the number of positive roots orthogonal to \(\lambda\).

**Proof.** Take \(\xi \in \epsilon\). Following Proposition 3.19 the periodic-polynomial \(m^\lambda\) is defined by \(m^\lambda(\mu, k) = RR((O_L)_{\xi}, L^k_{\xi, \mu})\) for all \((\mu, k) \in \Xi_L\). Here \((O_L)_{\xi} = \Phi^{-1}(\xi) / T\) is a smooth manifold, and the line bundle \(L^k_{\xi, \mu} = (L^k_{\Phi^{-1}(\xi)} \otimes \mathbb{C}_{-\mu}) / T\) is also smooth since the center \(\mathbb{Z}_n\) acts trivially on \(L^k_{\Phi^{-1}(\xi)} \otimes \mathbb{C}_{-\mu}\). Now the Atiyah-Singer integral formula for the Riemann-Roch number \(RR((O_L)_{\xi}, L^k_{\xi, \mu})\) shows that \(m^\lambda\) is a polynomial of degree \(\frac{\dim(O_L)_{\xi}}{2} = \frac{\dim O_L}{2} - (n - 1) = \frac{(n-1)(n-2)}{2} - d_\lambda\). \(\square\)

Now we rewrite Theorem 4.1 for the group \(SU(n)\). Let \(\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)\) be a positive weight and let \(\epsilon_+\) and \(\epsilon_-\) be two adjacent connected components of regular values of \(\Phi : O_L \to t^*\) separated by an hyperplane orthogonal to a fundamental weight \(\varpi_i\); the vector \(\varpi_i\) is pointing out of \(\epsilon_-\). Let \(q(\xi) = (\varpi_i, \xi) - r_i\) be the defining equation of this hyperplane.

The conditions \((e_k - e_i, \sigma \lambda) > 0\) and \((e_k - e_i, \varpi_i) > 0\) are respectively equivalent to \(\lambda_{\sigma(k)} > \lambda_{\sigma(i)}\) and \(k \leq i < l\). For \(SU(n)\), the number \([\alpha]\), is equal to \(0, 1\) or \(-1\) for any roots \(\alpha\) and any \(i = 1, \ldots, n - 1\). Hence for every \(\alpha \in \mathcal{A}(\epsilon_+, \epsilon_-)\), the integers \(s_{\sigma, i}^+, s_{\sigma, i}^- \geq 0\) defined in Def. 4.18 are equal to

\[(4.107) \quad s_{\sigma, i}^+ = r_{K}(N^+_{\sigma, i}) = \#\{k \leq i < l \text{ such that } \lambda_{\sigma(k)} > \lambda_{\sigma(i)}\},\]

\[(4.108) \quad s_{\sigma, i}^- = r_{K}(N^-_{\sigma, i}) = \#\{k \leq i < l \text{ such that } \lambda_{\sigma(k)} < \lambda_{\sigma(i)}\},\]

and the sum \(s_{\sigma, i}^+ + s_{\sigma, i}^-\) is equal to half of the codimension of \(K^i \cdot \sigma \lambda\) in \(K \cdot \lambda\), that is \(s_{\sigma, i}^+ + s_{\sigma, i}^- = i(n - i) - \dim(K^{\sigma \lambda} / K^i \cap K^{\sigma \lambda}) / 2\).

Now we precise the results of 4.11.

**Theorem 4.13.** The polynomial \(m_{\epsilon_+}^\lambda - m_{\epsilon_-}^\lambda : \Xi_L \to \mathbb{Z}\) is divisible by the linear factors

\[(q - s_i^- + 1), (q - s_i^- + 2), \ldots, q, \ldots, (q + s_i^+ - 2), (q + s_i^+ - 1),\]
where \( q \) is the defining equation of the hyperplane separating \( c_\pm \) and \( s_\pm^\pm = \inf_{\sigma \in \Lambda(c_\pm, c_-)} s_\pm^\pm \). Moreover the linear factors \((q - s_i^-)\) and \((q - s_i^+)\) do not divide \( m_{c_-}^\lambda - m_{c_+}^\lambda \).

**Proof.** The first part is the translation of Theorem 4.9. We have just to prove that the linear factors \((q - s_i^-)\) and \((q - s_i^+)\) do not divide \( m_{c_-}^\lambda - m_{c_+}^\lambda \). This point is a direct application of Proposition 4.23. The only fact we use here is that \( \text{rk}_C(N_{\sigma,i}) = s_{\sigma,i}^\pm \). So the number \( \text{rk}_C(N_{\sigma,i}) \) is constant for all \( \sigma \in \Lambda(c_+, c_-) \) for which \( s_{\sigma,i}^\pm = s_i^\pm \).

We rewrite now Theorem 1.13 in the particular case where \( \Lambda(c_+, c_-) \) contains just one element: it happens when \( \lambda \) is a “generic” positive weight (see Definition 4.4), or when \( c_+ \) does not intersect \( \Phi(O_\lambda) \). Here a positive weight \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \) is “generic” if for every couple of permutations \( \sigma, \sigma' \) and any \( k = 1, \ldots, n - 1 \), we have

\[
\sum_{i=1}^k \lambda_{\sigma(i)} \neq \sum_{i=1}^k \lambda_{\sigma'(i)}
\]

when \((\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}) \notin S_k \times S_{n-k}(\lambda_{\sigma'(1)}, \ldots, \lambda_{\sigma'(n)}).

**Corollary 4.14.** Let \( \lambda \) be a regular weight. Let \( c_+ \) and \( c_- \) be two adjacent connected components of regular values of \( \Phi : O_\lambda \to \mathfrak{t}^* \) and suppose that \( \Lambda(c_+, c_-) \) contains just one element \( \sigma \). Then the polynomial \( m_{c_-}^\lambda - m_{c_+}^\lambda : \Xi_\lambda \to \mathbb{Z} \) is divisible by the \( i(n - i) \) linear factors

\[
(q - s_i^- + 1), (q - s_i^- + 2), \ldots, q, \ldots, (q + s_i^+ - 2), (q + s_i^+ - 1),
\]

where \( s_i^\pm = s_{\sigma,i}^\pm \) are defined by \((4.107)\) and \((4.108)\). Moreover the linear factors \((q - s_i^-)\) and \((q - s_i^+)\) do not divide \( m_{c_-}^\lambda - m_{c_+}^\lambda \).

5. VECTOR PARTITION FUNCTIONS

Let \( T \) be a torus with Lie algebra \( \mathfrak{t} \) and let \( \Lambda^* \subset \mathfrak{t}^* \) be the weight lattice. Let \( R = \{\alpha_1, \ldots, \alpha_d\} \) be a subset of not necessarily distinct elements of \( \Lambda^* \) which lie entirely in an open halfspace of \( \mathfrak{t}^* \). We associate with the collection \( R \) a function

\[
N_R : \Lambda^* \rightarrow \mathbb{N}
\]
called the vector partition function associated to \( R \). By definition, for a weight \( \mu \), the value \( N_R(\mu) \) is the number of solutions of the equation

\[
(5.109) \quad \sum_{j=1}^d k_j \alpha_j = \mu, \quad k_j \in \mathbb{Z}_{\geq 0}, \quad j = 1, \ldots, d.
\]

Let \( C(R) \subset \mathfrak{t}^* \) be the closed convex cone generated by the elements of \( R \), and denote by \( \Lambda^*_R \subset \Lambda^* \) the sublattice generated by \( R \). Obviously, \( N_R(\mu) \) vanishes if \( \mu \) does not belong to \( C(R) \cap \Lambda^*_R \).

Suppose now that \( R \) generates the vector space \( \mathfrak{t}^* \). Following [32], we will call a vector singular with respect to \( R \) if it is in a cone \( C(\nu) \) generated by a subset \( \nu \in R \) of cardinality strictly less than \( \text{dim} T \). The connected components of \( \mathfrak{t}^* - \{ \text{singular vectors} \} \) are called conic chambers. The periodic polynomial behavior of \( N_R \) on closures of conic chambers of the cone \( C(R) \) is proved in [31]. We have the
following refinement due to Szenes and Vergne [32]. Let us introduce the convex polytope

\[(5.110) \quad \Box(\Phi) = \sum_{j=1}^{d} [0, 1] \alpha_j.\]

We remark that \( c - \Box(\Phi) \) is a neighborhood of \( c \) for any conic chamber \( c \) of the cone \( C(R) \). We have the following qualitative result.

**Theorem 5.1 (32).** Let \( c \) be a conic chamber of the cone \( C(R) \). There exists a periodic polynomial \( P_c \) on \( \Lambda^* \) such that for each \( \mu \in c - \Box(\Phi) \), we have

\[\mathcal{N}_R(\mu) = P_C(\mu).\]

In Section 5.4 we will give another proof of Theorem 5.1. Let \( c_\pm \subset t^* \) be two adjacent conic chambers. The aim of this Section is to give a formula for the periodic polynomial \( P_{c_\pm} \).

Let \( \Delta \subset t^* \) be the hyperplane that separates \( c_+ \) and \( c_- \). Let \( \beta \in t \) be such that \( \Delta = \{ \xi \in t^* \mid \langle \xi, \beta \rangle = 0 \} \) and \( c_\pm \subset \{ \xi \in t^* \mid \pm \langle \xi, \beta \rangle > 0 \} \). Note that the vector space \( \Delta \) is generated by \( R \cap \Delta \). We will now polarize the elements of \( R \) that are outside \( \Delta \). We define

\[(5.111) \quad R' = \{ \epsilon_j \alpha_j \mid \langle \alpha_j, \beta \rangle \neq 0 \text{ and } \epsilon_j = \text{sign} \langle \alpha_j, \beta \rangle \},\]

\[(5.112) \quad \delta^\pm = \sum_{\pm \langle \alpha_j, \beta \rangle > 0} \alpha_j ,\]

and

\[(5.113) \quad r^\pm = \sharp \{ j \mid \pm \langle \alpha_j, \beta \rangle > 0 \} .\]

We now look at the vector space \( \Delta \) equipped with the subset \( R' \cap \Lambda^* \cap \Delta \) which lie entirely in an open halfspace: let \( \mathcal{N}_{R' \cap \Delta} : \Lambda^* \cap \Delta \to \mathbb{N} \) be the corresponding vector partition function. Let \( \tilde{c}_c \) be the cone in \( \Delta \) which is equal to the relative interior of \( t^+_c \cap t^-_c \). It is easy to see that \( \tilde{c}_c \) is a conic chamber in \( \Delta \) with respect to \( R \cap \Delta \). Following Proposition 5.1 there exists a periodic polynomial \( P_{\tilde{c}_c} \) on \( \Lambda^* \cap \Delta \) such that for each \( \mu \in \tilde{c}_c \cap \Lambda^* \), we have

\[\mathcal{N}_{R' \cap \Delta}(\gamma) = P_{\tilde{c}_c}(\gamma).\]

Let \( \mathcal{N}_{R'} : \Lambda^* \to \mathbb{N} \) be the vector partition function associated to the polarized set of weight \( R' \) (see (5.111)). The main result of this Section is the following

**Theorem 5.2.** The periodic polynomial \( P_{c_\pm} - P_{c_-} : \Lambda^* \to \mathbb{Z} \) satisfies

\[(5.114) \quad P_{c_+}(\mu) - P_{c_-}(\mu) = \sum_{\gamma \in \Lambda^* \cap \Delta} D(\mu - \gamma) P_{\tilde{c}_c}(\gamma), \quad \mu \in \Lambda^*,\]

where \( D : \Lambda^* \to \mathbb{Z} \) is defined by

\[D(\mu) = (-1)^{r^-} N_{R'}(\mu + \delta^-) - (-1)^{r^+} N_{R'}(-\mu - \delta^+).\]

The proof of Theorem 5.2 will be given in Section 5.5.
Corollary 5.3. \( P_{\epsilon^+}(\mu) = P_{\epsilon^-}(\mu) \) for all the weights \( \mu \in \Lambda^* \) satisfying the condition

\[ -(\delta^+, \beta) < \langle \mu, \beta \rangle < -(\delta^-, \beta). \]

The former inequalities are optimal since

\[ (P_{\epsilon^+} - P_{\epsilon^-})(-\delta^- + \gamma) = (-1)^r P_c(\gamma) \]

and

\[ (P_{\epsilon^+} - P_{\epsilon^-})(-\delta^+ + \gamma) = (-1)^{1+r} P_c(\gamma) \]

for all \( \gamma \in \Lambda^* \cap \Delta \).

Proof. In (5.116), the term \( D(\mu - \gamma)P_c(\gamma) \) does not vanish only if \( \mu - \gamma \in -\delta^- + C(R') \) or \( -(\mu - \gamma) \in \delta^+ + C(R') \) for some \( \gamma \in C(R \cap \Delta) \). These two conditions impose respectively that \( \langle \mu, \beta \rangle \geq -(\delta^-, \beta) \) and \( \langle \mu, \beta \rangle \leq -(\delta^+, \beta) \). If one take \( \mu = -\delta^- + \gamma \) with \( \gamma \in \Lambda^* \cap \Delta \), (5.114) becomes

\[ (P_{\epsilon^+} - P_{\epsilon^-})(-\delta^- + \gamma) = \sum_{\gamma' \in \Lambda^* \cap \Delta} D(-\delta^- + \gamma - \gamma')(\mu - \gamma')(\gamma' \rangle \]

with

\[ D(-\delta^- + \gamma - \gamma') = (-1)^r N_{R'}(\gamma - \gamma') - (-1)^{r+1} N_{R'}(\delta^- - \delta^+ - \gamma + \gamma') \]

Since the cone \( C(R') \) intersects \( \Delta \) only at \( \{0\} \), \( N_{R'}(\gamma - \gamma') = 0 \) if \( \gamma \neq \gamma' \). Since \( \langle \delta^- - \delta^+, \beta \rangle < 0 \) we always have \( N_{R'}(\delta^- - \delta^+ - \gamma + \gamma') = 0 \). We get finally that

\[ (P_{\epsilon^+} - P_{\epsilon^-})(-\delta^- + \gamma) = (-1)^r P_c(\gamma) \]

One can show in the same way that

\[ (P_{\epsilon^+} - P_{\epsilon^-})(-\delta^+ + \gamma) = (-1)^{r+1} P_c(\gamma) \]

\[ \square \]

5.1. Quantization of \( \mathbb{C}^d \). We consider the complex vector space \( \mathbb{C}^d \) equipped with the canonical symplectic form \( \Omega = \frac{i}{2} \sum_{i=1}^d dz_i \wedge d\overline{z}_j \). The standard complex sturcture \( J \) on \( \mathbb{C}^d \) is compatible with \( \Omega \). Let \( T \) be a torus, let \( \alpha_j \in \mathfrak{t}^*, j = 1, \ldots, d \) be weights of \( T \), and let \( T \) acts on \( \mathbb{C}^d \) as

\[ t \cdot (z_1, \ldots, z_d) = (t^{-\alpha_1}z_1, \ldots, t^{-\alpha_d}z_d). \]

The action of \( T \) preserve the symplectic form \( \Omega \) and the moment map associated with this action is

\[ \Phi(z) = \frac{1}{2} \sum_{i=1}^d |z_j|^2 \alpha_j. \]

The pre-quantization data \( (L, \langle \cdot, \cdot \rangle, \nabla) \) on the Hamiltonian \( T \)-manifold \( (\mathbb{C}^d, \Omega, \Phi) \) is a trivial line bundle \( L \) with a trivial action of \( T \) equipped with the Hermitian structure \( \langle s, s' \rangle_z = c^{-\frac{|\alpha|^2}{2}} (ss') \) and the Hermitian connexion \( \nabla = d - \beta \) where \( \beta = \frac{1}{2} \sum_{i=1}^d z_j dz_j. \)

The quantization of the Hamiltonian \( T \)-manifold \( (\mathbb{C}^d, \Omega) \), that we denote \( Q^T(\mathbb{C}^d) \), is the Bargman space of entire holomorphic functions on \( \mathbb{C}^d \) which are \( L^2 \) integrable with respect to the Gaussian measure \( e^{-\frac{|\beta|^2}{2}} \Omega^d \).

We suppose now that the set of weights \( R = \{ \alpha_1, \ldots, \alpha_d \} \) is polarized by \( \eta \in \mathfrak{t} \), which means that \( \langle \alpha_j, \eta \rangle > 0 \) for all \( j \). The \( T \)-representation \( Q^T(\mathbb{C}^d) \) is then admissible and we have the following equality in \( R^{-\infty}(T) \):

\[ Q^T(\mathbb{C}^d) = \sum_{\mu \in \Lambda^*} N_R(\mu) C_\mu, \]

where \( N_R : \Lambda^* \to \mathbb{N} \) is the vector partition function associated to \( R \). In other words, the generalized character of \( Q^T(\mathbb{C}^d) \) coincides with the generalized character
of the symmetric algebra $S^\bullet(\mathbb{C}^d)$, where $\mathbb{C}^d$ means $\mathbb{C}^d$ with the opposite complex structure.

For the remaining part of Section 5, we assume that the set of weights $R = \{\alpha_1, \ldots, \alpha_d\}$ is polarized, and generate the vector space $t^*$. The first assumption is equivalent to the fact that the moment map $\Phi : \mathbb{C}^d \to t^*$ is proper, and the second assumption is equivalent to the fact that the generic stabiliser of $T$ on $\mathbb{C}^d$ is finite. Notice that the vectors of $t^*$ which are singular with respect to $R$ correspond to the singular values of $\Phi$.

In the next section we will show that $Q^T(\mathbb{C}^d)$, viewed as an element of $R^{-\infty}(T)$, can be realized as the index of transversally elliptic symbols on $\mathbb{C}^d$. After we will apply the techniques developed in Section 3. The main difference here is that we work with a non-compact manifold.

5.2. Transversally elliptic symbols on $\mathbb{C}^d$. Let $p : T\mathbb{C}^d \to \mathbb{C}^d$ be the canonical projection. The Thom symbol

$$\text{Thom}(\mathbb{C}^d) \in \Gamma(T\mathbb{C}^d, \text{hom}(p^*(\wedge^\text{even}_C\mathbb{C}^d), p^*(\wedge^\text{odd}_C\mathbb{C}^d)))$$

is defined as follows. At $(z, v) \in T\mathbb{C}^d$, the Thom symbol Thom$(\mathbb{C}^d)$ is equal to the Clifford map

$$C\ell(v) : \wedge^\text{even}_C\mathbb{C}^d \to \wedge^\text{odd}_C\mathbb{C}^d$$

which is defined by $C\ell(v)w = v \wedge w - c(v)w$. Here $c(v) : \wedge^\bullet_C\mathbb{C}^d \to \wedge^\bullet_C^{-1}\mathbb{C}^d$ denotes the contraction map relatively to the standard Hermitian structure on $\mathbb{C}^d$. Obviously the symbol Thom$(\mathbb{C}^d)$ is not elliptic since its characteristic set is equal to the zero section in $T\mathbb{C}^d$ (hence is not compact).

Now we deform the symbol Thom$(\mathbb{C}^d)$ in order to obtain transversally elliptic symbols. Since $\mathbb{C}^d$ can be realized as an open subset of a compact $T$-manifold we have a well defined index map

$$\text{Index}^T_{\mathbb{C}^d} : K_T(T_T\mathbb{C}^d) \to R^{-\infty}(T).$$

Definition 5.4. For any $\eta \in t$, we define the symbol Thom$(\mathbb{C}^d)$ by

$$\text{Thom}^\eta(\mathbb{C}^d)(z, v) = \text{Thom}(\mathbb{C}^d)(z, v - \eta_{\mathbb{C}^d}(z)), \quad (z, v) \in T\mathbb{C}^d,$$

where $\eta_{\mathbb{C}^d}$ is the vectors field on $\mathbb{C}^d$ generated by $\eta$.

The symbols Thom$(\mathbb{C}^d)$ were studied in [28]. It is easy to see that Thom$(\mathbb{C}^d)$ is transversally elliptic if and only if the vector subspace $(\mathbb{C}^d)^\eta$ is reduced to $\{0\}$, i.e. if $\langle \alpha_j, \eta \rangle \neq 0$ for all $j = 1, \ldots, d$. We prove in Proposition 5.4. of [28] that

$$(5.118) \quad \text{Index}^T_{\mathbb{C}^d}(\text{Thom}^\eta(\mathbb{C}^d)) = S^\bullet(\mathbb{C}^d) \quad \text{in} \quad R^{-\infty}(T),$$

when $\langle \alpha_j, \eta \rangle > 0$ for all $j = 1, \ldots, d$.

In order to compute the multiplicities $N_R(\mu)$ of $Q^T(\mathbb{C}^d)$ we introduce the following transversally elliptic symbols. Take a scalar product $b(\cdot, \cdot)$ on $t^*$, and denote by $\xi \mapsto \xi^b, t^* \simeq t$ the induced isomorphism. For each $\xi \in t^*$, the Hamiltonian vectors field of the function $\frac{1}{2}\|\Phi - \xi\|^2$ is the vectors field

$$z \mapsto (\Phi(z) - \xi^b)_{\mathbb{C}^d}(z),$$

that we denote $\mathcal{H}^b - \xi^b_{\mathbb{C}^d}$. 
Definition 5.5. For any $\xi \in t^*$, and any scalar product $b(\cdot, \cdot)$ on $t^*$, we define the symbol $\text{Thom}_{\xi,b}(C^d)$ by

$$\text{Thom}_{\xi,b}(C^d)(z,v) = \text{Thom}(C^d)(z,v - (H^b - \xi_{\xi_{C^d}})(z)), \quad (z,v) \in TC^d.$$  

Let $\text{Char}(\text{Thom}_{\xi,b}(C^d)) \subset TC^d$ be the characteristic set of $\text{Thom}_{\xi,b}(C^d)$. We know that $\text{Char}(\text{Thom}_{\xi,b}(C^d)) \cap T_T C^d$ is equal to the critical set $\text{Cr}(\Phi - \xi_{C^d}^2)$ of the function $\|\Phi - \xi_{C^d}^2\| : C^d \to \mathbb{R}$ (see Section 5.2). A straightforward computation gives that $z \in \text{Cr}(\Phi - \xi_{C^d}^2)$ if and only if

$$(5.119)\quad b(\Phi(z) - \xi, \alpha_j) z_j = 0 \quad \text{for all} \quad j = 1, \ldots, d.$$  

The former relations implies in particular that $b(\Phi(z) - \xi, \Phi(z)) = \frac{1}{2} \sum_j b(\Phi(z) - \xi, \alpha_j)|z_j|^2 = 0$. Hence $\|\Phi(z)\|^2 = b(\Phi(z), \xi)$ which implies

$$(5.120)\quad \|\Phi(z)\|_b \leq \|\xi\|_b.$$  

Take now $\eta \in t$ such that $\langle \alpha_j, \eta \rangle > 0$ for all $j$, and let $\eta_b \in t^*$ such that $(\eta_b)^{\perp} = \eta$. We have then

$$(5.121)\quad C_\eta \|z\|^2 \leq \langle \Phi(z), \eta \rangle = b(\Phi(z), \eta_b) \leq \|\Phi(z)\|_b \|\eta_b\|_b$$

where $C_\eta = \frac{1}{2} \inf_j \langle \alpha_j, \eta \rangle$, and $z \mapsto \|z\|^2$ is the usual hermitian form on $C^d$. With (5.119) and (5.121) we get the following

Lemma 5.6. The critical set $\text{Cr}(\|\Phi - \xi\|^2_{C^d}) \subset C^d$ is contained in the ball of radius

$$\frac{\|\|\|\xi\|_b \|\eta_b\|_b}{C_\eta},$$

where $\eta \in t$ is such that $C_\eta = \frac{1}{2} \inf_j \langle \alpha_j, \eta \rangle > 0$.

We have then proved that the symbols $\text{Thom}_{\xi,b}(C^d)$ are transversally elliptic.

Proposition 5.7. The class of the transversally elliptic symbol $\text{Thom}_{\xi,b}(C^d)$ in $K_T(T_T C^d)$ does not depend of the data $\xi, b$, and is equal to the class defined by $\text{Thom}^b(C^d)$ where $\eta \in t$ is choosen so that $\langle \alpha_j, \eta \rangle > 0$ for all $j$.

Proof. After Lemma 5.6, we know that for any scalar product $b(\cdot, \cdot)$ on $t^*$, the characteristic set of $\text{Thom}_{\xi,b}(C^d)$ intersects $T_T C^d$ at $\{0\}$. If $b_0$ and $b_1$ are two scalar products on $t^*$ we consider the family $b_t = tb_1 + (1-t)b_0$, $0 \leq t \leq 1$, of scalar products on $t^*$. Hence $\text{Thom}_{b_t}(C^d)$, $t \in [0,1]$, defines an homotopy of transversally elliptic symbols. We have proved that $\text{Thom}_{b_t}(C^d) = \text{Thom}_{b_0}(C^d)$ in $K_T(T_T C^d)$ for any $\xi \in t^*$.

Fix now the scalar product $b$ and an element $\xi \in t^*$. For any $t \in [0,1]$ the characteristic set of $\text{Thom}_{\xi,b}(C^d)$ intersects $T_T C^d$ in the ball of radius

$$\frac{\|\|\|\xi\|_b \|\eta_b\|_b}{C_\eta}.$$  

Hence $\text{Thom}_{\xi,b}(C^d)$, $t \in [0,1]$, defines an homotopy of transversally elliptic symbols: $\text{Thom}_{\xi,b}(C^d) = \text{Thom}_{\xi,b}(C^d)$ in $K_T(T_T C^d)$. We have proved that the class of the transversally elliptic symbol $\text{Thom}_{\xi,b}(C^d)$ in $K_T(T_T C^d)$ does not depend of the data $\xi, b$.

Since the weights $\alpha_j$ lie enterely in an open halfspace of $t^*$, there exists a scalar product $b_+(\cdot, \cdot)$ on $t^*$ for which we have

$$b_+(\alpha_i, \alpha_j) > 0.$$
for all $i, j = 1, \ldots, d$. Let $\mathcal{H}^{b^+}$ be the Hamiltonian vectors field of the function $\frac{1}{2}||\Phi||_{b^+}^2$, and let $\eta_{c^d}$ be the vectors field on $\mathbb{C}^d$ generated by $\eta \in \mathfrak{t}$ such that $\langle \alpha_j, \eta \rangle > 0$ for all $j$. A straightforward computation gives that

$$\tag{5.122} (\mathcal{H}^{b^+}(z), \eta_{c^d}(z)) > 0$$

for all non zero $z \in \mathbb{C}^d$. Consider now the following family of symbols on $\mathbb{C}^d$

$$\sigma_t(z, v) = \text{Thom}(\mathbb{C}^d)(z, v - (tH^{b^+} + (1 - t)\eta_{c^d})(z)), \quad (z, v) \in T\mathbb{C}^d.$$ 

so that $\sigma_0 = \text{Thom}^0(\mathbb{C}^d)$ and $\sigma_1 = \text{Thom}_{0,b^+}(\mathbb{C}^d)$. The inequality $\tag{5.122}$ shows that $\text{Char}(\sigma_t) \cap T_T \mathbb{C}^d = \{0\}$ for all $t \in [0, 1]$. Hence $\sigma_t, t \in [0, 1]$, defines an homotopy of transversally elliptic symbols: $\text{Thom}^0(\mathbb{C}^d) = \text{Thom}_{0,b^+}(\mathbb{C}^d)$ in $K_T(T_T \mathbb{C}^d)$. \square

For the remaining part of this paper we fix a scalar product on $\mathfrak{t}^*$, and we consider the family of transversally elliptic symbols $\text{Thom}_\xi(\mathbb{C}^d), \xi \in \mathfrak{t}^*$ (to simplify, we do not mention the scalar product in the notation). Proposition 5.7 and $\tag{5.118}$ imply the following

**Proposition 5.8.** For every $\xi \in \mathfrak{t}^*$, $Q^T(\mathbb{C}^d)$ is equal to the generalized character

$$\text{Index}^T_{\mathbb{C}^d}(\text{Thom}_\xi(\mathbb{C}^d)).$$

Now we apply the techniques developped in Section 3 in order to compute the multiplicities of $\text{Index}^T_{\mathbb{C}^d}(\text{Thom}_\xi(\mathbb{C}^d))$.

5.3. **Localization in a non-compact setting.** Like in Section 3.2 we start with the

**Definition 5.9.** For any $\xi \in \mathfrak{t}^*$ and any $T$-invariant relatively compact open subset $\mathcal{U} \subset \mathbb{C}^d$ we define the symbol $\text{Thom}_\xi(\mathcal{U})$ by the relation

$$\text{Thom}_\xi(\mathcal{U})(z, v) := \text{Thom}(\mathbb{C}^d)(z, v - (\mathcal{H} - \xi_{\mathbb{C}^d})(z)) \quad (z, v) \in T\mathcal{U}.$$ 

The symbol $\text{Thom}_\xi(\mathcal{U})$ is transversally elliptic when $\text{Cr}(\|\Phi - \xi\|^2) \cap \partial\mathcal{U} = \emptyset$ (the couple $(\mathcal{U}, \xi)$ is called good) and we denote by

$$RR^\xi_{\mathcal{U}}(\mathbb{C}^d) \in R^{-\infty}(T)$$

its index. Proposition 5.9 is still valid here. In particular, for a good couple $(\mathcal{U}, \xi)$, we have $RR^\xi_{\mathcal{U}}(\mathbb{C}^d) = RR^\xi_\mathcal{U}(\mathbb{C}^d)$ if $\xi'$ is close enough to $\xi$. Consider now the decomposition

$$\text{Cr}(\|\Phi - \xi\|^2) = \bigcup_{\gamma \in \mathcal{B}_\xi} (\mathbb{C}^d)_{\gamma} \cap \Phi^{-1}(\gamma + \xi).$$

Here $\mathcal{B}_\xi \subset \mathfrak{t}^*$ is finite set since $\mathbb{C}^d$ has a finite number of stabilizer. Since $0 \in (\mathbb{C}^d)_{\gamma}$ and $z \mapsto \langle \Phi(z), \gamma \rangle$ is constant on $(\mathbb{C}^d)_{\gamma}$, we have

$$\tag{5.123} (\gamma + \xi, \gamma) = 0$$

for all $\gamma \in \mathcal{B}_\xi$.

**Definition 5.10.** For any $\xi \in \mathfrak{t}^*$ and $\gamma \in \mathcal{B}_\xi$, we denote simply by

$$RR^\xi_\mathcal{U}(\mathbb{C}^d) \in R^{-\infty}(T)$$

the generalized character $RR^\xi_{\mathcal{U}}(\mathbb{C}^d)$, where $\mathcal{U}$ is a $T$-invariant relatively compact open neighborhood of $(\mathbb{C}^d)_{\gamma} \cap \Phi^{-1}(\gamma + \xi)$ such that $\text{Cr}(\|\Phi - \xi\|^2) \cap \overline{\mathcal{U}} = (\mathbb{C}^d)_{\gamma} \cap \Phi^{-1}(\gamma + \xi)$. 

Since $RR^\xi_0(C^d)$ is equal to $Q^T(C^d)$ (see Proposition 5.8, part a) of Proposition 5.3, it insures that we have the decomposition

$$Q^T(C^d) = \sum_{\gamma \in B_{t'}} RR^\xi_0(C^d).$$

Let $\xi \subset t^*$ be a conic chamber of the cone $C(R)$, and take $\xi$ in $c$. Then $\xi$ is a regular value of the moment map $\Phi : C^d \to t^*$ defined in (5.110). Let $\Omega_{\xi}$ be the symplectic structure on the orbifold $(C^d)_{\xi} = \Phi^{-1}(\xi)/T$ that is induced from $\Omega$. The orbifold $(C^d)_{\xi}$ is also equipped with a complex structure $J_{\xi}$ that is induced from the standard complex structure on $C^d$, in such a way that the orbifold $((C^d)_{\xi}, \Omega_{\xi}, J_{\xi})$ is a Kähler orbifold. If $\xi$ belongs to the lattice $\Lambda^*$, the reduced space $(C^d)_{\xi}$ is the Kähler toric variety corresponding to the polytope $\{ s \in (\mathbb{R}^{\geq 0})^d \mid \sum s_j \alpha_j = \xi \}$ of $\mathbb{R}^d$. For every $\mu \in \Lambda$ we consider the holomorphic orbifold line bundle

$$L_{\xi, \mu} = (\Phi^{-1}(\xi) \times \mathbb{C}_{-\mu})/T$$
on the $(C^d)_{\xi}$.

**Definition 5.11.** The periodic polynomial $P_\xi : \Lambda^* \to \mathbb{Z}$ associated to the conic chamber $c$ is given by

$$(5.124) \quad P_\xi(\mu) = RR((C^d)_{\xi}, L_{\xi, \mu}),$$

where the right hand side is the Riemann-Roch number associated to the holomorphic orbifold line bundle $L_{\xi, \mu}$.

Another way to define the periodic polynomial $P_\xi$ is to consider the generalized character $RR^\xi_0(C^d)$ for $\xi \in c$: here $\gamma = 0$ parametrizes the component $\Phi^{-1}(\xi)$ of $C\Gamma(\Phi - \xi \|^2)$. Following (5.13) we have

$$(5.125) \quad RR^\xi_0(C^d) = \sum_{\mu \in \Lambda^*} P_\xi(\mu) C_\mu \text{ in } R^{-\infty}(T).$$

After Lemma 3.14 we know that $RR^\xi_0(C^d) = RR^\xi_0(C^d)$ when $\xi, \xi'$ are two elements of $c$: hence the polynomial $P_\xi$ does not depend of the choice of $\xi$ in $c$.

5.4. **Proof of Theorem 5.1** Consider a weight $\mu \in (c - \Box(\Phi)) \cap \Lambda^*$ of the form $\mu = \xi' - \sum t_j \alpha_j$ with $\xi' \in c$ and $t_j \in [0,1]$. We start with the decomposition

$$Q^T(C^d) = \sum_{\gamma \in B_{t'}} RR^\xi_0(C^d).$$

Since $N_{t'}(\mu)$ and $P_\xi(\mu)$ are respectively the multiplicity of $C_\mu$ in $Q^T(C^d)$ and in $RR^\xi_0(C^d)$, the proof will be complete if we show that the multiplicity of $C_\mu$ in $RR^\xi_0(C^d)$ is equal to zero when $\gamma \neq 0$.

Consider a non-zero element $\gamma$ in $B_{c'}$. For the character $RR^\xi_0(C^d)$ the localization (3.31) gives

$$(5.126) \quad RR^\xi_0(C^d) = RR^\xi_0(C^d) \otimes [\Lambda^*_c N]^{-1}_\gamma,$$

where $N = \sum_{(\alpha_j, \gamma) \neq 0} \mathbb{C}_{-\alpha_j}$ corresponds to the normal bundle of $(C^d)_{\gamma}$ in $C^d$. The inverse $[\Lambda^*_c N]^{-1}_\gamma$ is equal to $(-1)^{\delta(\gamma)} \otimes S^*(N^\gamma_{c'})$ where

$$\delta(\gamma) = -\sum_{(\alpha_j, \gamma) < 0} \alpha_j.$$
Since $\gamma$ acts trivially on $(\mathbb{C}^d)^\gamma$ all the weights $\mu' \in \Lambda^\ast$ that appear in $RR_{0}^{\xi}(\mathbb{C}^d)$ satisfy $(\mu', \gamma) = 0$. Since the weights of $N_{\xi \gamma}^{\ast}$ are polarized by $\gamma$, we see from (5.126) that all the weights $\mu' \in \Lambda^\ast$ that appear in $RR_{0}^{\xi}(\mathbb{C}^d)$ must satisfy

$$(\mu', \gamma) \geq (\delta(\gamma), \gamma).$$

Consider now the weight $\mu = \xi' - \sum_{j} t_{j} \alpha_{j}$. Since $\xi' \in \mathfrak{c}$, the equality (5.123) implies $(\xi', \gamma) < 0$ and then

$$(\mu, \gamma) = (\xi', \gamma) + \sum_{(\alpha_{j}, \gamma) > 0} -t_{j}(\alpha_{j}, \gamma) - \sum_{(\alpha_{j}, \gamma) < 0} t_{j}(\alpha_{j}, \gamma) < - \sum_{(\alpha_{j}, \gamma) < 0} (\alpha_{j}, \gamma).$$

So we have proved that $(\mu, \gamma) < (\delta(\gamma), \gamma)$, hence the multiplicity of $C_{\mu}$ in $RR_{0}^{\xi}(\mathbb{C}^d)$ is equal to zero. □

5.5. Proof of Theorem 5.2. Let $c_{\pm}$ be two adjacent conic chambers and let $\Delta \subset t^\ast$ be the hyperplane that separates $c_{+}$ and $c_{-}$. Let $\beta \in t$ be such that $\Delta = \{\xi \in t^\ast \mid \langle \xi, \beta \rangle = 0\}$ and $c_{\pm} \subset \{\xi \in t^\ast \mid \pm \langle \xi, \beta \rangle > 0\}$.

We consider two points $\xi_{\pm} \in c_{\pm}$ such that $\xi = \frac{1}{2}(\xi_{+} + \xi_{-}) \in \Delta$ belongs to the relative interior $\mathfrak{c}'$ of $t_{\mathfrak{c}} \cap t_{\mathfrak{c}'}$. We suppose also that the orthogonal projection of $\xi_{\pm}$ on $\Delta$ are equal to $\xi$. We know that $P_{\xi_+}(\mu) - P_{\xi_-}(\mu)$ is equal to the $\mu$-mutiplicity of $RR_{0}^{\xi_+}(\mathbb{C}^d) - RR_{0}^{\xi_-}(\mathbb{C}^d)$. Proposition 3.21 tells us that

$$RR_{0}^{\xi_+}(\mathbb{C}^d) - RR_{0}^{\xi_-}(\mathbb{C}^d) = RR_{0}^{\xi}(\mathbb{C}^d) - RR_{0}^{\xi_\gamma}(\mathbb{C}^d),$$

where $\gamma \in \mathbb{R}^{>0} \beta$ is such that $\xi_{-} + \gamma = \xi_{+} - \gamma = \xi$. The localization 3.41 gives then

$$RR_{0}^{\xi}(\mathbb{C}^d) - RR_{0}^{\xi_\gamma}(\mathbb{C}^d) = RR_{0}(\mathbb{C}^d)^{\beta} \otimes \left([\wedge_{\mathfrak{c}} N]^{-1}_\beta - [\wedge_{\mathfrak{c}} N]^{-1}_{-\beta}\right).$$

The element $\xi$ belongs to the relative interior $\mathfrak{c}'$ of $t_{\mathfrak{c}} \cap t_{\mathfrak{c}'}$ which is a conic chamber $\mathfrak{c}'$ in $\Delta$ with respect to $R \cap \Delta$. Let $P_{\mathfrak{c}'} : \Lambda^\ast \cap \Delta \to \mathbb{Z}$ be the periodic polynomial map which coincides with the vector partition function $N_{R \cap \Delta}$ on $\mathfrak{c}' \cap \Lambda^\ast$. If we work with the vector space $(\mathbb{C}^d)^{\beta}$ equipped with the hamiltonian action of $T / T_{\Delta}$, 3.43 gives the following equality in $R^{-\infty}(T / T_{\Delta})$

$$RR_{0}^{\xi}(\mathbb{C}^d)^{\beta} = \sum_{\gamma \in \Lambda^\ast \cap \Delta} P_{\mathfrak{c}'}(\gamma) C_{\gamma}.$$

A straightforward computation gives

$$[\wedge_{\mathfrak{c}} N]^{-1}_\beta = (-1)^{\gamma} \sum_{\mu \in \Lambda^\ast} N_{R_r}(\mu + \delta^-) C_{\mu},$$

and

$$[\wedge_{\mathfrak{c}} N]^{-1}_{-\beta} = (-1)^{\gamma} \sum_{\mu \in \Lambda^\ast} N_{-R_r}(\mu + \delta^+) C_{\mu},$$

where $r^\pm, \delta^\pm, R_r'$ are defined in (5.111), (5.112) and (5.113). Since $N_{-R_r}(\mu) = N_{R_r}(\mu - \mu)$, the equations (5.129), (5.130) and (5.131) show that the RHS of (5.128) is equal to

$$\sum_{\mu \in \Lambda^\ast} \sum_{\gamma \in \Lambda^\ast \cap \Delta} D(\mu) P_{\mathfrak{c}'}(\gamma) C_{\mu + \gamma}.$$
with $D(\mu) = (-1)^{\nu} N_{R'}(\mu + \delta^-) - (-1)^{\nu'} N_{R'}(-\mu - \delta^+)$. Finally we have proved that $P_+ (\mu) - P_- (\mu) = \sum_{\gamma \in \Lambda^+ \cap \Delta} D(\mu - \gamma) P_{\gamma}$. □

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