Scalar Dark Matter and Cold Stars

J. A. Grifols

*Grup de Física Teòrica and Institut de Física d’Altes Energies*
*Universitat Autònoma de Barcelona*
*08193 Bellaterra, Barcelona, Spain*

**Abstract**

In a medium composed of scalar particles with non-zero mass, the range of Van-der-Waals-type scalar mediated interactions among nucleons becomes infinite when the medium makes a transition to a Bose-Einstein condensed phase. We explore this phenomenon in an astrophysical context. Namely, we study the effect of a scalar dark matter background on the equilibrium of degenerate stars. In particular we focus on white dwarfs and the changes induced in their masses and in their radii.
Dark matter is one of the outstanding problems in modern cosmology [1]. Evidence for it comes from a multiplicity of sources and distance scales [2, 3]. The nature and composition of dark matter might be diverse. It might well be that each different distance scale has its predominant type of dark matter. Several candidates for dark matter have been postulated [4]. The list includes the lightest supersymmetric particle (e.g., the neutralino) and the axion. They might be heavy, like the neutralino or light, like the axion. They might constitute the bulk of dark matter, like WIMPS (weakly interacting massive particles) in CDM (cold dark matter) scenarios or be just a small fraction of it, like neutrinos (they have mass and thus they constitute a fraction of the mass in the universe). Because scalar particles, such as the axion or the Higgs boson, are fundamental ingredients of the standard model (SM) of elementary particle physics and completions thereof, scalars might constitute also a natural ingredient of the dark stuff in the universe.

The characteristic feature of dark matter in general is precisely the fact that it only interacts weakly with ordinary matter and therefore manifests itself mainly gravitationally. So far, evidence of dark matter is indirect. Nevertheless, direct searches that rely on the (feeble) interaction of dark particles such as WIMPS with ordinary matter are under way and there is hope that sufficiently sensitive experiments will eventually lead to positive results [5]. But there might be other effects of the feeble interactions of dark matter with ordinary matter that show up in other settings. In this paper we explore the consequences of scalar dark matter in the astrophysical environment of compact stellar objects.

Suppose very generally that in a grander unified scheme beyond the SM, new scalar particles carrying a new conserved quantum number \( Q \) do exist. If, at some high energy scale, those putative scalars share fundamental interactions with ordinary quarks and with new fields associated to the high energy scale, then, at low energies, the 2-body t-channel-exchange of these scalars will generate feeble residual spin independent forces (i.e. Van der Waals type dispersion forces)[6, 7] among nucleons that add coherently over unpolarized bulk matter and extend over distances on the order of the Compton wavelength of the mediating scalars. Furthermore, if matter is embedded in a Bose-Einstein condensate composed of these scalars, then the dispersion force becomes infinitely ranged, i.e. it extends over far larger distances than the Compton wavelength. This phenomenon was first discovered in [8] and exploited in [9] in a cosmological setting.

Our starting point is the assumption that galactic dark matter is composed at least in part by (light) scalar particles. We focus our attention in a region where ordinary matter has accreted and evolved into a white dwarf and the condensed scalars have been gravitationally
trapped by this stellar matter. So, we start with an ideal boson gas in a gravitational field. The gas is at a temperature \( T \gg m \) where \( m \) is the mass of the scalar. The gas is relativistic. The statistical mechanics of a free ideal boson gas has been given in [11]. Here, we shall describe the statistical mechanics of Bose-Einstein condensation of a relativistic boson gas in an external field \( U(x, y, z) \). For condensation to occur, a non zero chemical potential \( \mu \) for the gas is mandatory. We assumed before that our scalars carry a conserved quantum number \( Q \).

Particles and antiparticles, i.e. both \( Q = +1 \) and \( Q = -1 \) scalars, constitute the gas. The net total charge \( Q \) is then \(^1\)

\[
Q = \sum_{\rho} \left[ \frac{1}{\exp \beta (E - \mu) - 1} - \frac{1}{\exp \beta (E + \mu) - 1} \right] \tag{1}
\]

with \( E = \sqrt{p^2 + m^2 + U(x, y, z)} \) where \( U(x, y, z) \geq 0 \) and vanishes at the origin, and \( \beta \equiv T^{-1} \).

In the continuum limit \( Q \) reads

\[
Q = \int dV \int d^3\rho \left[ \frac{1}{\exp \beta (E - \mu) - 1} - \frac{1}{\exp \beta (E + \mu) - 1} \right]. \tag{2}
\]

In a central potential \( U(r) \), the case we shall entertain to keep things simple, this can be cast in the following form,

\[
Q = \frac{1}{\pi} \int \varepsilon(E) dE \left[ \frac{1}{\exp \beta (E - \mu) - 1} - \frac{1}{\exp \beta (E + \mu) - 1} \right] \tag{3}
\]

with the density of states \( \varepsilon(E) \equiv \int \tilde{U} r^2 dr (E - U(r)) \sqrt{(E - U(r))^2 - m^2} \) and \( \tilde{U}(E) \) is such that \( U(\tilde{U}) = E \).

Above a critical temperature \( T_c \) defined implicitly by the equation

\[
Q = Q(\mu = m, T = T_c) \tag{4}
\]

the charge is correctly given by equation (2). But for \( T \) below \( T_c \), equation (2) gives only the charge distributed in excited states. For \( T \leq T_c \), a macroscopic fraction of the total charge sits in the ground state and is not included in the above integral. The boson gas is condensed, and the particles in the zero momentum mode constitute the condensate. For an harmonic oscillator potential (i.e. \( U(r) = \alpha r^2 \)), as would be the case of a scalar particle inside a uniform matter distribution, the condensation temperature can be easily calculated to be

\[
T_c = \left( \frac{8\sqrt{\pi}Q\alpha^{3/2}}{29\zeta(2/3)m} \right)^{2/9} \tag{5}
\]

up to corrections \( \mathcal{O}(m/T) \).

\(^1\)For definiteness we take \( Q > 0 \), i.e. particles outnumber antiparticles.
The condensed charge in this case is given by

\[ Q_0 = Q \left[ 1 - \left( \frac{T}{T_c} \right)^\frac{3}{2} \right]. \]  

(6)

Although Bose-Einstein condensation is condensation in momentum-space, in the presence of a gravitational field there will be condensation in ordinary space as well, and two separate phases will emerge just as in ordinary gas-liquid condensation. This fact causes the charge \( Q_0 \) to accumulate at \( r = 0 \). The thermal particles, i.e. those distributed in excited states, bearing a kinetic energy on average large compared to their rest mass will eventually escape from the star. Only the condensate will be trapped at the center. Because all the charge \( Q_0 \) sits at \( r = 0 \), the density of charge grows to infinity. This is however not realistic since the uncertainty principle prevents both momentum and position of the condensed particles to be simultaneously sharp. In the gravitational field of the star, the condensate will occupy a finite region centered around \( r = 0 \) whose size depends on the depth and width of the potential well, and on the mass \( m \) of the scalar particles. The region will be a macroscopic fraction of the star’s volume for certain ranges of these parameters (see below). Thus, matter in this region will be subject to the pull of macroscopic 2-scalar exchange forces on top of the pull of gravity. Let us give a succinct briefing on dispersion forces in a thermal bath of the mediating scalars. The derivation closely follows [8]. Consider two nucleons a distance \( r \) apart and at rest relative to the scalar heat reservoir. We describe the effective interaction of nucleons with scalars with the lagrangean

\[ \mathcal{L} = \frac{g}{m_N} \bar{\psi} \psi \phi \phi^\dagger, \]  

(7)

where \( m_N \) is the nucleon mass (chosen to set the scale of the coefficient of the dimension−5 effective low energy operator) and \( \psi \) and \( \phi \) are the nucleon and scalar fields, respectively. It is the simplest low energy realization of the underlying fundamental interactions (see [9] for a discussion). The force between the two nucleons arises in this approximation from the emission at a time by one nucleon and absorbtion by the other of two scalar quanta. We use real time finite temperature field theory to calculate the corresponding Feynman amplitude (see e.g. [10] for an early application of the technique). The Fourier transform of its non-relativistic limit is the potential we seek.

The physical clue to the nontrivial influence of the particles in the background on the force due to 2-scalar t-channel exchange is that while one and two real particles cannot be exchanged in the t-channel, the exchange of one real plus one virtual particle in the t-channel is kinematically allowed. In the case at hand, the heat bath supplies the real particle. This
mechanism contributes a piece to the potential that reads
\[
\mathcal{V}_T(r) = -\frac{g^2}{16\pi m_N^2} \frac{1}{r} \int dV \int \frac{p^2 dp/2\pi^2}{\sqrt{p^2 + m^2}} \sin 2pr \left[ \frac{1}{\exp \beta(E - \mu) - 1} + \frac{1}{\exp \beta(E + \mu) - 1} \right].
\]

Below \(T_c\) a macroscopic fraction of the charge carried by particles in the reservoir piles up in the zero mode state (the condensate) and the integral over states in equation (8) just as in equations (2) and (3) no longer correctly describes the physical situation. To evaluate the condensate contribution to the potential in the degenerate case \((T \leq T_c)\) we next establish its relation to \(Q\). Call \(n\pm(p)\) the following distribution functions,
\[
n\pm(p) = \frac{1}{\exp \beta(E - \mu) - 1} \pm \frac{1}{\exp \beta(E + \mu) - 1}.
\]

Then, for \(T \leq T_c\), it holds that
\[
n_+(0) = n_-(0) + \frac{2}{\exp \beta(2m + U) - 1}.
\]

The condensate contribution to the potential is proportional to \(\frac{1}{V} \int dV n_+(0)\). Since \(U \geq 0\) we see from the previous relation, equation (10), that this quantity differs from the charge \(Q_0\) in the condensate by less than \(2(\exp 2\beta m - 1)^{-1}\). Furthermore, \(\beta m \ll 1\), and therefore, as long as the net charge \(Q\) is a macroscopically large number many orders of magnitude larger than \(T/m\), the factor \(\frac{1}{V} \int dV n_+(0)\) coincides essentially with the condensate contribution to the charge \(Q_0\). As a consequence we can write the potential contributed by the zero momentum modes as (the passage from discrete to continuum momentum space, or viceversa, entails the substitution rule \(V^{-1} \leftrightarrow (2\pi)^{-3} d^3 p\) ):
\[
\mathcal{V}_{\text{cond}}(r) = -\frac{g^2}{16\pi m_N^2} \frac{q_0}{m r} \frac{1}{r}.
\]

where \(q_0 \equiv \frac{Q_0}{V}\) and the \(m\) in the denominator comes from \((p^2 + m^2)^{-\frac{1}{2}}(2pr)^{-1}\sin 2pr \rightarrow m^{-1}\).

In the specific case at hand, using equations (5) and (6), we find
\[
\mathcal{V}_{\text{cond}}(r) = -\frac{29\zeta(\frac{7}{2})g^2}{128\pi^2 m_N^2} \frac{T_c^{\frac{7}{2}}}{\alpha^2 V} \left[ 1 - \left( \frac{T}{T_c} \right)^{\frac{7}{2}} \right] \frac{1}{r}.
\]

So for a degenerate scalar background the resulting potential behaves as \(r^{-1}\), i.e. it is a potential of infinite range. On the other hand, when the temperature of the background is above \(T_c\), the potential is cut off by the typical Yukawa damping factor \(\exp(-2mr)\) as can be seen from performing the momentum integral in equation (8).

Finally, in a vacuum, the potential takes the form \([8]\)
\[
\mathcal{V}_{\text{vac}}(r) = -\frac{g^2 m}{64\pi^3 m_N^2} \frac{K_1(2mr)}{r^2},
\]

\(\text{in equation (8).}
with $K_1(2mr)$, the modified Bessel function whose asymptotic form gives rise to the characteristic exponential damping. We will use this result later on.

Now that we have the form of the potential associated to the 2–scalar force among nucleons in a scalar heat bath and in vacuum, we turn to our problem, namely to stars whose central core (or even their whole volume) is permeated by a fluid made of scalar particles. To be definite and simple, we shall discuss equilibrium configurations called polytropes, i.e. with the equation of state of the form $P = K\rho^\Gamma$. More specifically, we shall analyze idealized white dwarfs where the Fermi pressure of a degenerate ideal gas of ultrarelativistic electrons supports these stars against collapse (a discussion on ordinary white dwarfs can be found in [12]). In the core of actual white dwarfs the nucleons are not free but bound in nuclei. In turn, these nuclei\(^2\) tend to form crystal like structures, i.e. they arrange themselves on a regular lattice. Because of this circumstance the ideal Fermi gas equation of state is modified in real life by the electrostatic interactions among the positive ions and the electrons. The resulting corrections can be accounted for (see e.g. [12]) and are generally small. Another source of departure from the ideal gas approximation is provided by inverse $\beta$–decay which tends to modify the number density of the electrons and thus lessens the Fermi pressure of the electron gas. Here we are not concerned with this correction nor with the Coulomb correction. As far as the putative long range forces that we contemplate, the fact that nucleons (inside nuclei) cluster on lattice sites is irrelevant for our purposes since the potential associated to macroscopic bulk matter results from the coherent superposition of the underlying long distance potential created by the individual nucleons as, incidentally, it is the case for gravity. Therefore, to study bulk properties, the matter density is averaged over scales much larger than the inter-ion spacing and it is smoothed out over a macroscopically large number of nucleons. For neutron stars, the actual equation of state deviates from an equation of state for an ideal gas in ways that are more dramatic and much less under control than it is the case for white dwarfs. As the authors in [12] put it, while for white dwarfs, observations of masses and radii are used as confirmation of astrophysical models, for neutron stars observations of masses and radii are instead used to test theories of nuclear physics. It is in this spirit that we keep things simple and do work with white dwarfs only.

Our aim is to study the effect on stellar equilibrium of the condensation of the dark background particles. The hydrostatic equilibrium in our case is described by the two equations:

\[
\frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G (1 + \kappa) \rho \tag{14}
\]

\(^2\)Typically, carbon, oxygen, i.e. not exceeding atomic weight about 20.
for 0 ≤ r ≤ L, and
\[
\frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G \rho
\]
for L ≤ r ≤ R.

In equation (14),
\[
\kappa \equiv \frac{g^2}{16\pi m^4 N G} \frac{q_0}{m} \mu_4
\]
and G in both equations is Newton’s constant. The distance L sets the size of the region where the particles of the condensate are confined. Inside a sphere of radius L, the inverse square law forces of gravity and 2-scalar exchange are operative. Outside this sphere only the pull of gravity balances the Fermi pressure. The boundary conditions for equation (14) are \(\rho(r = 0) = \rho_c\) (where \(\rho_c\) is the central density of the star) and \(\rho'(r = 0) = 0\). The boundary conditions for equation (15) follow from the requirement that \(\rho\) and \(\rho'\) in equation (15) should match \(\rho\) and \(\rho'\) of equation (14) at \(r = L\). The radius \(R\) of the star is implicitly defined by the condition \(\rho(R) = 0\) (and hence, \(P(R) = 0\)).

In the case under scrutiny, \(\Gamma = \frac{4}{3}\) and \(K = \frac{3^{\frac{1}{2}} \pi^{\frac{3}{2}}}{4m_N^5 g^2} \mu_8\) (\(\mu_8\) is the mean molecular weight per electron) [12]. By changing variables to dimensionless quantities \(\xi\) and \(\theta\) defined as
\[
\begin{align*}
    r &= a\xi \\
    \rho &= \rho_c \theta^3
\end{align*}
\]
where the length parameter \(a \equiv \left[ \frac{\kappa}{\pi G(1 + \kappa) \rho_c^2} \right]^{\frac{1}{2}}\), we can put both equations, (14) and (15), in the form:
\[
\frac{1}{x_{1,2}^2} \frac{d}{dx_{1,2}} \left( x_{1,2}^2 \frac{d\theta}{dx_{1,2}} \right) = -\theta^3
\]
where \(x_1 \equiv \xi\) for 0 ≤ \(\xi\) ≤ \(\frac{L}{a}\) and \(x_2 \equiv \xi/\sqrt{1 + \kappa}\) for \(\frac{L}{a}\) ≤ \(\xi\) ≤ \(\frac{R}{a}\). The boundary conditions in this new form now read:
\[
\theta(x_1 = 0) = 1; \quad \left. \frac{d\theta}{dx_1} \right|_{x_1 = 0} = 0
\]
for the first equation, and
\[
\theta(x_2 = L/a\sqrt{1 + \kappa}) = \theta(x_1 = L/a); \quad \left. \frac{d\theta}{dx_2} \right|_{x_2 = L/a\sqrt{1 + \kappa}} = \left. \frac{d\theta}{dx_1} \right|_{\sqrt{1 + \kappa}}
\]
for the second equation.

These equations are to be solved numerically and we shall do so below. We shall obtain the radius and the mass of equilibrium configurations as a function of the size of the condensate, i.e. as a function of the parameter \(\xi_L \equiv L/a\). Before doing so let us recall that in the ordinary case, i.e. gravity alone at work, one gets, in the polytropic approximation, the
Chandrasekhar limit or maximum possible mass of a white dwarf. The explicit expressions for the Chandrasekhar mass and radius are, respectively

\[ M_{Ch} = 4\pi \left( \frac{K}{\pi G} \right)^{\frac{3}{2}} \xi_1^2 |\theta'(\xi_1)| \]  

(21)

and

\[ R_{Ch} = \left( \frac{K}{\pi G} \right)^{\frac{1}{2}} \rho_c^{-\frac{1}{3}} \xi_1 \]  

(22)

where \( \xi_1 \) satisfies \( \theta(\xi_1) = 0 \).

The numerical integration of the corresponding differential equation gives in this case \( \xi_1 = 6.89685 \) and \( \xi_1^2 |\theta'(\xi_1)| = 2.01824 \). When plugged in equation (21), one obtains the well known value

\[ M_{Ch} = 1.457 \left( \frac{2}{\mu_e} \right)^{2} M_\odot. \]  

(23)

It will be convenient for our analysis to normalize radii and masses relative to \( R_{Ch} \) and \( M_{Ch} \), respectively. Trivially,

\[ \frac{R}{R_{Ch}} = \frac{x_{2,R}}{\xi_1} \]  

(24)

where \( x_{2,R} \) verifies \( \theta(x_{2,R}) = 0 \). As to the mass, it is easy to show starting from \( M = \int_0^R 4\pi r^2 \rho dr \) that

\[ \frac{M}{M_{Ch}} = 2.01824^{-1} \left[ x_{2,R}^2 \left| \frac{d\theta}{dx_2} \right|_{x_{2,R}} - \frac{\kappa}{(1 + \kappa)^{\frac{3}{2}}} x_{1,L}^2 \left| \frac{d\theta}{dx_1} \right|_{x_{1,L}} \right] \]  

(25)

with \( x_{1,L} \equiv \xi_L \equiv \frac{L}{a} \). Sticking to ratios \( \frac{M}{M_{Ch}} \) and \( \frac{R}{R_{Ch}} \) make our results less dependent on the approximations made and make them more reliable when trying to extract general consequences for realistic stars.

In order to proceed we first should estimate the size of the stellar volume that can actually be filled by the particles of the condensate. For that purpose let us study the quantum mechanics of a scalar particle occupying the ground state in the gravitational potential well of the star.

First of all we give the form of the potential \( \Phi \). For a spherical symmetric mass distribution,

\[ \Phi = -\frac{Gm(r)}{r} + G \int_0^r 4\pi r \rho dr \]  

(26)

where \( m(r) \) is the mass inside \( r \) and \( \Phi(0) = \Phi'(0) = 0 \). In our case, it can be explicitly cast in terms of the solution \( \theta \) of equations (18),

\[ \Phi = 4\pi Ga^2 \rho_c (1 - \theta) \]  

(27)
inside the star, i.e. for $r \leq R$, while for $r \geq R$,
\begin{equation}
\Phi = 4\pi G a^2 \rho_c + MG \left( \frac{1}{R} - \frac{1}{r} \right).
\end{equation}

Since we shall deal only with the lowest energy mode, the wavefunction $\Psi(r)$ is purely radial with no orbital part. The radial Schrödinger equation can be put in an equivalent one-dimensional form with no centrifugal barrier, if we trade $u(r) = r \Psi(r)$ for $\Psi(r)$:
\begin{equation}
\left( -\frac{1}{2m} \frac{d^2}{dr^2} + m \Phi(r) \right) u(r) = Eu(r).
\end{equation}

Because of the finiteness of $\Psi(r)$, $u(r)$ should vanish at $r = 0$. Instead of using the actual $\Phi(r)$ (given by equations (27) and (28)), for which we have no analytical expression, we shall introduce in equation (29) a potential of the form $\Phi(r) = \gamma \tanh^2 \frac{r}{\delta}$. The parameters $\delta$ and $\gamma$, width and depth of the well, can be fitted to give a fair approximation to the real potential. Of course, $\delta \sim O(R)$ and $\gamma \sim O(\Phi(\infty))$ with $\Phi(\infty)$ being the asymptotic value of equation (28).

The reason for doing so is that the Schrödinger equation above can be given an exact analytic solution. Indeed, the physically meaningful ground state wavefunction that complies with the boundary conditions at $r = 0$ and at infinity turns out to be, after some mathematical manipulations,
\begin{equation}
\Psi(r) = A \left( \frac{R}{\delta} \right)^{-1} \sinh \frac{r}{\delta} \cosh^{-2\lambda} \frac{r}{\delta}
\end{equation}
with $A$ a normalization constant and $\lambda = \frac{1}{4} \left[ -1 + \sqrt{1 + 8m^2 \gamma \delta^2} \right]$. The corresponding ground state energy reads:
\begin{equation}
E_0 = (4m\delta^2)^{-1} \left[ -5 + 3\sqrt{1 + 8m^2 \gamma \delta^2} \right].
\end{equation}

For $8m^2 \gamma \delta^2 \gg 1$, equations (30) and (31) approach closely the wavefunction and the ground state energy of an isotropic harmonic oscillator of angular frequency $\omega = \frac{\sqrt{\gamma}}{\delta}$. In general, however, the ground state lies lower in energy and the wavefunction is broader than for the oscillator well. Thus, the position uncertainty of the harmonic oscillator is an underestimate of the uncertainty in position of the particle in the actual potential well. We shall use this conservative quantity in our estimates. The precise meaning we give to the parameter $L$ is the following:
\begin{equation}
L \equiv \sqrt{\sum_i \Delta x_i^2}
\end{equation}
with $\Delta x_i^2 \equiv \langle x_i^2 \rangle - \langle x_i \rangle^2$ and $\vec{r} \equiv (x_1, x_2, x_3)$. For an harmonic oscillator, $\Delta x_i^2 = (2m\omega)^{-1}$. In our case, $\omega = \frac{\sqrt{\gamma}}{\delta}$. Hence,
\begin{equation}
L = \frac{3 \delta^{-\frac{3}{2}}}{2 \gamma^{-\frac{1}{2}} m^{-\frac{1}{2}}}
\end{equation}
and,

\[ L \simeq 7 \times 10^{-4} (1 + \kappa)^{\frac{1}{2}} \left( \frac{\rho_c}{10^8 \text{g/cm}^3} \right)^{\frac{1}{2}} \left( \frac{1 \text{eV}}{m} \right)^{\frac{1}{2}} \left( \frac{R}{R_{Ch}} \right)^{\frac{1}{2}} Km \]  

(34)

if we set roughly \( \delta \sim R \) and \( \gamma \sim 4\pi G a^2 \rho_c \).

Clearly, appreciable \( L \)'s—relative to stellar dimensions—will be attained only for sufficiently large \( \kappa \) and/or sufficiently low \( m \). However, neither can we go arbitrarily low in mass if we want to keep the momentum uncertainty of the scalars to be much less than their rest mass, nor can we increase the strength of the potential without eventually inducing unwanted effects in laboratory Cavendish-type experiments. We shall address these points later. Because \( L \) depends on \( R \), the boundary conditions (20) for the matching of the hydrodynamic equilibrium equations depend on the solution of those equations. Hence, if one solves equations (18) as a function of the free parameter \( L \) and then obtains \( R(L) \) through equation (24), a stellar configuration should have a radius \( R_* \) such that the self-consistency condition is satisfied:

\[ R_* = R(L = L(R_*)) \]  

(35)

where \( L(R) \) stands for equation (34).

We shall now integrate numerically equations (18). Our strategy is the following. We choose a large value for \( \kappa \). We do the numerics with this \( \kappa \) fixed and letting the parameter \( \xi_L \) vary freely. The output are the quantities \( x_{2,R}, \left| \frac{d\theta}{dx_2} \right|_{x_2,R} \) and \( \left| \frac{d\theta}{dx_1} \right|_{x_1,L} \), to be introduced in equations (24) and (25). We repeat this procedure for various values of \( \kappa \). To comply with the self-consistency requirement in equation (35) we check \textit{a posteriori} for each value of \( \xi_L \equiv L/a \) whether a physically reasonable mass \( m \) can be found such that equation (34) is satisfied. Our results will be then presented in the parametric form \( \{ R_{R_{Ch}}(\xi_L), M_{M_{Ch}}(\xi_L) \} \).

It follows from the analysis that our stars tend to fall in two qualitatively distinct categories. Those whose mass is somewhat below the Chandrasekhar mass but their radius is larger than the Chandrasekhar radius constitute a first group. In the second category we find stars with very small masses and radii. In the first case, the star adjusts itself such that only a small core of the star is filled by the scalar condensate whereas in the second case, equilibrium is attained with a large fraction of the star occupied by the condensate at the expense of a drastic reduction in mass and radius. The first class corresponds to low values of the parameter \( \xi_L \) (roughly up to \( \xi_L \approx 2 \)) and the second class corresponds to the higher segment of the \( \xi_L \) range (up to \( \xi_1 = 6.89685 \); recall that \( \xi_L \) varies from 0 to \( \xi_1 \)). For the purpose of illustration we present the two domains in two separate \( (R,M) \)-plots. In figure 1 we show our results for \( 1 + \kappa = 10^2, 10^3, 10^4, 10^5 \) and \( 10^6 \) as a function of the parameter
Figure 1: From top to bottom: equilibrium configurations for $1 + \kappa = 10^6, 10^5, 10^4, 10^3$ and $10^2$, respectively. Along the curves, $\xi_L$ grows from 0 at point (1,1) to 2 at the opposite end of the curves.

$0 \leq \xi_L \leq 2$. In figure 2 we display the results as a function of $3 \leq \xi_L \leq \xi_1$. In this second plot we draw a single curve. This is because it turns out that, for large $\kappa$ and $\xi_L \geq 3$, the radius and the mass scale as $(1 + \kappa)^{-\frac{3}{2}}$ and $(1 + \kappa)^{-\frac{3}{2}}$, respectively (this fact is reflected in the units chosen for the axes). Each point on the curves in figures 1 and 2 gives the radius and maximum possible mass of a white dwarf when the scalar condensate extends over a region in the stellar interior characterized by the value of the parameter $\xi_L$ on that particular point in the curve. In figure 1, all curves merge at point (1,1) as they should since this corresponds to the limit $\xi_L \to 0$ for which no macroscopic fraction of the star’s volume is occupied by the condensate and thus the extra dispersion force is not operative. Figure 2, in turn, shows the other limit (i.e. $\xi_L \to \xi_1$; point (1,1) on the top left corner) for which the whole star is subject to the pull of the new force (gravity being negligible). A situation that is exactly equivalent to the Chandrasekhar case but with a blown up Newton constant. All equilibrium configurations on the curves of figure 1 (beyond $R \sim 1.5R_{Ch}$) correspond to scalar masses in the ranges: $8 \times 10^{-8} \text{eV} \leq m \leq 2.6 \times 10^{-7} \text{eV}$ (for $1 + \kappa = 10^2$), $2.5 \times 10^{-6} \text{eV} \leq m \leq 6 \times 10^{-6} \text{eV}$ (for $1 + \kappa = 10^3$), $8 \times 10^{-5} \text{eV} \leq m \leq 2 \times 10^{-4} \text{eV}$ (for $1 + \kappa = 10^4$), $2.5 \times 10^{-3} \text{eV} \leq m \leq 6 \times 10^{-3} \text{eV}$ (for $1 + \kappa = 10^5$), and $0.08 \text{eV} \leq m \leq 0.2 \text{eV}$ (for $1 + \kappa = 10^6$). Similarly, in figure 2, the resulting scalar mass range is: $1.4 \times 10^{-12}(1 + \kappa) \text{eV} \leq m \leq 7 \times 10^{-12}(1 + \kappa) \text{eV}$. A glance at these mass ranges\(^3\) shows that the solutions in figure 1 and figure 2 exclude scalar masses were obtained making $\mu_e = 2$ and $\rho_c = 10^8\text{g/cm}^3$. For other choices, one should multiply the given numerical mass values by the factor $(\frac{\mu_e}{2})^{\frac{1}{2}}(\frac{\rho_c}{10^8\text{g/cm}^3})^{\frac{1}{2}}$. 

\(^3\)Scalar masses were obtained making $\mu_e = 2$ and $\rho_c = 10^8\text{g/cm}^3$. For other choices, one should multiply the given numerical mass values by the factor $(\frac{\mu_e}{2})^{\frac{1}{2}}(\frac{\rho_c}{10^8\text{g/cm}^3})^{\frac{1}{2}}$. 

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each other, i.e. either we have stars of type 1 or we have stars of type 2, but not both.

Next issue on our agenda is to check whether the values for $\kappa$ and $m$ make physical sense. As to $\kappa$, we use the specific potential in equation (12) with $\alpha \equiv m\gamma\delta^{-2} \sim 4\pi G\alpha^2 \rho_\gamma m R^{-2}$ and use $T_c = O(1)$ eV. We set the scale for the temperature $T_c$ at 1 eV because with this choice the temperature of the scalars will be below the cosmic background temperature at matter domination when formation of structure begins. This is reasonable since the scalars, which were in thermal equilibrium with photons in the early stages of the history of the Universe when temperatures were on the order of the hypothesized high energy unification scale, have not experienced the reheating associated to the various extinction processes (e.g., $p\bar{p}$, $\mu^+\mu^-$, $e^+e^-$) that have otherwise risen the temperature of photons. Little algebra and a short numerical calculation will convince us that $g^2$ varies in the interval

$$g^2 \sim \left(6 \times 10^{-43} - 2 \times 10^{-35}\right) \left(\frac{\rho_c}{10^8 g/cm^3}\right)^{\frac{3}{2}}$$

for the values of $\kappa$ considered. Thus, an interaction which is extremely weak at a microscopic level is enhanced to considerable strength by the macroscopic coherence of the scalar medium.

At this point one may wonder if it is realistic to think that the scalar medium can be kept at a temperature below 1 eV during the formation of the white star while the temperature of the star itself can reach temperatures up to $10^4 - 10^5$ eV. But, in fact, the scalars and the stellar matter are not in thermal contact. They do not share a common temperature. Indeed, the tiny coupling of scalars to nucleons implies an elastic scattering cross section off non
relativistic nucleons which is at most $\sigma_N \sim \mathcal{O}(10^{-64} \text{cm}^2)$. Clearly, $\sigma_{N/2}$ is so much smaller than the nucleon spacing in a nucleus, that the nucleus is transparent and the cross section $\sigma_A$ on a nucleus with mass number $A$ is $\sim A \sigma_N$. As a consequence, the largest possible collision rate of a scalar with nuclei\footnote{With $A \simeq 20$.} in stellar matter $\langle \sigma v n \rangle$ is so extremely small that not even in a hundred white dwarf lifetimes a scalar would have collided once with a nucleus. Needless to say, fixing the condensation temperature at about 1 eV is by no means crucial. There is a wide band of temperatures for which the resulting scenario is phenomenologically sound. Also, the effects of this scalar force would go unnoticed by local gravity experiments. Indeed, the ratio of the vacuum potential (equation (13); we suppose that in a terrestrial environment, there is no appreciable amount of scalars trapped) to the newtonian potential is

$$
\frac{V_{\text{vac}}}{V_{\text{Newt}}} \simeq 6.9 \times 10^{-23} \left( \frac{g^2}{10^{-35}} \right) \left( \frac{m}{10^{-2} eV} \right)^2 \left( \frac{1}{m r} \right)^\frac{3}{2} e^{-2mr} \tag{37}
$$

for $r \gg m^{-1}$, and

$$
\frac{V_{\text{vac}}}{V_{\text{Newt}}} \simeq 1.5 \times 10^{-28} \left( \frac{g^2}{10^{-35}} \right) \left( \frac{cm}{r} \right)^2 \tag{38}
$$

for $r \ll m^{-1}$. We will see below that the domain within which $m^{-1}$ is contained is roughly $\sim 10^{-4} cm - 10^2 cm$. No experiment searching for new forces has or will exclude in a foreseeable future such a tiny force $[13]$. A comment on $m$ comes next. The condensed phase consists of scalar particles with macroscopic de Broglie wavelength $\lambda_{dB} \sim \mathcal{O}(L)$. The Compton wavelength of the particles, on the other hand, should be much smaller than their de Broglie wavelength for the zero-point momentum to be negligible as compared to mass. In the low $\xi_L$ case this means: $m \gg \lambda_{dB}^{-1} = 10^{-11} - 10^{-9}$ eV for $(1 + \kappa)$ between $10^2$ and $10^6$, respectively. The masses associated with figure 1 show no conflict with this requirement, where even in the most unfavorable case (for $(1 + \kappa) = 10^2$) $m$ is almost a factor $10^4$ bigger than the corresponding inverse de Broglie wavelength. In the high $\xi_L$ case one should have: $m \gg \lambda_{dB}^{-1} = 10^{-12} - 10^{-10}$ eV for $(1 + \kappa)$ between $10^2$ and $10^6$, respectively. Obviously, the range $1.4 \times 10^{-12}(1 + \kappa)$ eV $\leq m \leq 7 \times 10^{-12}(1 + \kappa)$ eV in figure 2, comfortably satisfies this requirement only for sufficiently high $(1 + \kappa)$. In short, probably the mass $m$ can be reasonably constrained to lie between $\mathcal{O}(10^{-7} - 10^{-6})$ eV and $\mathcal{O}(0.1)$ eV. Or, in terms of Compton wavelength: $\sim 10^{-4} cm - 10^2 cm$. Hence, the range of the 2-scalar force for masses such as above – if it were not for the condensation phenomenon – would be at most on the order of one meter. It
is only because the dispersion forces become of infinite range that they can coherently extend over a large piece of stellar matter.

The values of $m$ entertained in the preceding paragraphs are below or much below the temperature $T \sim 1\, \text{eV}$. Hence the scalars are relativistic or ultrarelativistic as they should. Furthermore, $T/m \ll Q$ as can be seen from:

$$Q = 3.35 \times 10^{46} (1 + \kappa)^{\frac{3}{2}} \left( \frac{10^{-2} \text{eV}}{m} \right)^{\frac{1}{2}} \left( \frac{T_c}{1\, \text{eV}} \right)^{\frac{5}{2}} \left( \frac{10^8 \, \text{g/cm}^3}{\rho_c} \right)^{\frac{1}{2}} \left( \frac{R}{R_{Ch}} \right)^{3}$$

which follows from equation (39).

We conclude with a summary. Two scalar exchange among nucleons produces spin independent Van der Waals type forces. In vacuum their range is about the Compton wavelength of the scalar particles exchanged. But when ordinary matter is embedded in a Bose-Einstein condensate composed of the very same mediating scalars, those forces become infinitely ranged. The force acts then coherently over macroscopic extensions of bulk matter. The phenomenon arises as a combination of kinematics — 3-momentum exchange of the matter system with the scalar medium — and the collective effect of condensation. The main purpose of the present paper has been to find and study a physical system where this phenomenon of infinite widening of the range could be relevant.

Scalars are fundamental ingredients of the SM of particle physics and completions thereof.
Figure 4: The two dotted upper lines limit the area where type 1 stars (for the choice of parameters: $1 + \kappa = 10^2$ and $m = 1.5 \times 10^{-7}$) should lie. The two dotted lower lines correspond to the lines for actual white dwarfs in figure 3.

Quite generally, the interactions of new matter with ordinary matter at the postulated new high energy unification scale in those beyond the SM scenarios will render low energy residual effects, however tiny, involving ordinary matter at rest. In particular, the dispersion forces due to the double exchange of scalars just mentioned will show up. These scalars, on the other hand, may constitute a component of dark matter. If this is the case, it might well be that chunks of scalar galactic dark matter in the condensed phase be trapped in stellar media. As a result, long range scalar forces would affect the equilibrium of those stars. Restricting our analysis to white dwarfs, we have explored the changes induced on their radii and maximal mass -- i.e. the Chandrasekhar limit -- by the presence of dispersion forces. Although figures 1 and 2 show only this limit, in order to relate our results to actual white dwarf data, we do the following exercise. Figure 3 displays masses and radii for a handful of white dwarfs obtained from visual binaries or common proper-motion systems [14]. Also shown are two dotted lines that embrace the different model calculations (including the pioneering Chandrasekhar models) [15, 16]. We can get a rough idea for where in the mass-radius plane our hypothetical stars might lie, by scaling the band between dotted lines on figure 3 with $\frac{M}{M_{Ch}}$ and $\frac{R}{R_{Ch}}$ on figures 1 and/or 2. As an example let us take $1 + \kappa = 10^2$ and $m = 1.5 \times 10^{-7}$ eV which corresponds to a type 1 star with $\frac{M}{M_{Ch}} = 0.815$ and $\frac{R}{R_{Ch}} = 11$ (see figure 1). Figure 4 shows the resulting band as well as, for comparison, the band for ordinary white dwarfs of figure 3. We see that, depending on the mass $m$ of these scalars and strength $\kappa G$ of the potential, a distinctive feature of these star configurations would be that they should populate, relative to
ordinary white dwarfs, patently different regions on the \((M,R)\)-plane. A novel clue on dark matter might thus come from the identification of such anomalous white dwarfs.

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