Optimum spin-squeezing in Bose-Einstein condensates with particle losses

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The problem of spin squeezing with a bimodal condensate in presence of particle losses is solved analytically by the Monte Carlo wavefunction method. We find the largest obtainable spin squeezing as a function of the one-body loss rate, the two-body and three-body rate constants, and the s-wave scattering length.

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Spin squeezed states, first introduced in [1], generalize to spin operators the idea of squeezing developed in quantum optics. In atomic systems effective spins are collective variables that can be defined in terms of two different internal states of the atoms [2] or two orthogonal bosonic modes [3]. States with a large coherence between the two modes, that is with a large mean value of the spin component in the equatorial plane of the Bloch sphere, can still differ by their spin fluctuations. For an uncorrelated ensemble of atoms, the quantum noise is evenly distributed among the spin components orthogonal to the mean spin. However quantum correlations can redistribute this noise and reduce the variance of one spin quadrature with respect to the uncorrelated case, achieving spin squeezing. Besides applications in quantum communication and quantum information [4], these multi-particle entangled states have practical interest in atom interferometry, and high precision spectroscopy [5] where they could be used to beat the standard quantum limit already reached in atomic clocks [6].

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In presence of one, two and three-body losses, the evolution of the density operator, in the interaction picture with respect to $H_0$, is ruled by the master equation

$$\frac{d\hat{\rho}}{dt} = \sum_{m=1}^{3} \sum_{c=a,b} \gamma^{(m)} \left[ c_c^{\dagger} \hat{c}_c^{m} \hat{\rho} - \frac{1}{2} \left\{ c_c^{\dagger} c_c^{m}, \hat{\rho} \right\} \right]$$
where \( \hat{\rho} = e^{i H_{\text{tot}}/\hbar} e^{-i H_{\text{tot}}/\hbar} \), \( c_a = e^{i H_{\text{tot}}/\hbar} a e^{-i H_{\text{tot}}/\hbar} \), and similarly for \( b, \gamma^{(m)} = K_m \int d^3r \phi(r)^{2m} \), where \( K_m \) is the \( m \)-body rate constant and \( \phi(r) \) is the condensate wavefunction in one of the two modes. In the Monte Carlo wavefunctions approach \[12\] we define an effective Hamiltonian \( H_{\text{eff}} \) and the jump operators \( J_e^{(m)} \):

\[
H_{\text{eff}} = - \sum_{m=1}^{3} \sum_{a=b} \frac{i \hbar}{2} \gamma^{(m)} e_m^{\dagger} e_m^{\dagger},
\]

\[
J_e^{(m)} = \sqrt{\gamma^{(m)} e_m^{\dagger}}.
\]

We assume that a small fraction of particles will be lost during the evolution so that we can consider \( \chi \) and \( \gamma^{(m)} \) \((m = 2, 3)\) as constant parameters of the model. The state evolution in a single quantum trajectory is a sequence of random quantum jumps at times \( t_j \) and non-unitary Hamiltonian evolutions of duration \( \tau_j \) during the evolution so that we can consider the entangled state \([3, 4]\). In all our analytic treatments, it similarly for \( b \).

\[
|\psi(t)\rangle = e^{-i H_{\text{eff}}(t-t_k)/\hbar} J_e^{(m)}(t_k) e^{-i H_{\text{tot}} \tau_k/\hbar} J_e^{(m-1)}(t_{k-1}) \ldots J_e^{(m)}(t_1) e^{-i H_{\text{tot}} \tau_1/\hbar} |\psi(0)\rangle.
\]

The expectation value of any observable \( \hat{O} \) is obtained by averaging over all possible stochastic realizations, that is all kinds, times and number of quantum jumps, each trajectory being weighted by its probability \[12\]

\[
\langle \hat{O} \rangle = \sum_k \int dt_1 dt_2 \ldots dt_k \sum_{\{t_j, m\}} \langle \psi(t) | \hat{O} | \psi(t) \rangle.
\]

We want to calculate spin squeezing. In the considered symmetric case with zero initial relative phase, the mean spin remains aligned to the \( x \) axis \( \langle \hat{S}_x \rangle = \langle b^\dagger a \rangle \), and the spin squeezing is quantified by the parameter \[3, 4\]

\[
\xi^2 = \min_{\theta} \frac{\langle N \rangle \Delta S^2_{\theta}}{\langle S_{\theta} \rangle^2},
\]

where \( S_{\theta} = (\cos \theta S_y + (\sin \theta S_x) \), \( S_z = (a^\dagger a - b^\dagger b)/2 \), and \( \hat{N} = a^\dagger a + b^\dagger b \). The non-correlated limit yields \( \xi^2 = 1 \), while \( \xi^2 < 1 \) is the mark of an entangled state \[3, 4\]. In all our analytic treatments, it turns out that \( \Delta S^2_{\theta} = \langle \hat{N} \rangle/4 \). This allows to express \( \xi^2 \) in a simple way:

\[
\xi^2 = \frac{\langle a^\dagger a \rangle}{\langle b^\dagger a \rangle^2} \left( \langle a^\dagger a \rangle + A - \sqrt{A^2 + B^2} \right),
\]

with

\[
A = \frac{1}{2} \text{Re} \left( \langle b^\dagger a^\dagger a b - b^\dagger b^\dagger b a a \rangle \right)
\]

\[
B = \frac{1}{2} \text{Im} \left( \langle b^\dagger b^\dagger b a a \rangle \right).
\]

With one-body losses only, the problem is exactly solvable. Following a similar procedure as in \[11\], we get

\[
\xi^2(t) = \frac{1}{\gamma^2 + \chi^2} \left[ \gamma^2 + \chi^2 \right]^{2N-2}
\]

The key points are that (i) \( H_{\text{eff}} \) is proportional to \( \hat{N} \) so it does not affect the state, and (ii) a phase state \( |\phi \rangle \) is changed into a phase state with one particle less after a quantum jump, \( c_{a,b} |\phi \rangle \propto |\phi \mp \chi t/2 \rangle \) where \( t \) is the time of the jump, the relative phase between the two modes simply picking up a random shift \( \mp \chi t/2 \) which reduces the squeezing.

When two and three-body losses are taken into account, an analytical result can still be obtained by using a constant loss rate approximation \[11\]

\[
H_{\text{eff}} \approx - \sum_{m=1}^{3} \sum_{a=b} \frac{i \hbar}{2} \gamma^{(m)} N_m = - \frac{i \hbar}{2} \lambda.
\]

We verified by simulation (see Fig.\[1\]) that this is valid for the regime we consider, where a small fraction of particles is lost at the time at which the best squeezing is achieved. In this approximation, the mean number of particles at time \( t \) is

\[
\langle \hat{N} \rangle = N \left[ 1 - \sum_m \Gamma^{(m)} t \right]; \quad \Gamma^{(m)} = \langle N/2 \rangle^{m-1} \langle m \gamma^{(m)} \rangle
\]

where \( \Gamma^{(m)} t \) is the fraction of lost particles due to \( m \)-body losses. Spin squeezing is calculated from \[9\] with

\[
\langle b^\dagger a \rangle = e^{-\lambda t} \cos^{N-1}(\chi t) \hat{N} F_1
\]

\[
A = \frac{e^{-\lambda t}}{8} \hat{N} (\hat{N} - 1) [F_0 - F_2 \cos(2\chi t)]
\]

\[
B = \frac{e^{-\lambda t}}{2} \cos^{N-2}(\chi t) \sin(\chi t) \hat{N} (\hat{N} - 1) F_1
\]

where the operator \( \hat{N} = (N - \partial_a) \) acts on the functions

\[
F_\beta(\alpha) = \exp \left[ \sum_{m=1}^{3} 2 \gamma^{(m)} t e^{\alpha m} \frac{\sin(m \beta \chi t)}{m \beta \chi \cos^{m}(\beta \chi t)} \right],
\]

and all expressions should be evaluated in \( \alpha = \ln \hat{N}_a \).

We want to find simple results for the best squeezing and the best squeezing time in the large \( N \) limit. In the absence of losses \[11\] the best squeezing and the best squeezing time in units of 1/\( \chi \) scale as \( N^{-2/3} \). We then set \( N = e^{-3} \) and rescale the time as \( \chi t = \tau e^2 \). We expand the results \[12\] and \[13\] for \( \epsilon \ll 1 \) keeping \( \Gamma^{(m)}/\chi \) constant, and we obtain in both cases

\[
\xi^2(t) \approx \frac{1}{N^2(\chi t)^2} + \frac{1}{6} N^2(\chi t)^4 + \frac{1}{3} \Gamma_{\text{sq}} t.
\]
of the trap frequencies, and the corresponding squeezing are the losses. In presence of losses, the best squeezing time (ii) the more squeezed the state is, the more sensitive to that the correction on the squeezing due to losses is small; This shows that (i) the fact that only a small fraction of to be in the Thomas-Fermi regime so that number of particles is large enough for the condensates of this technique, from now on, we assume that the squeezing in presence of losses and set the ultimate lim-

\[ t_{\text{best}} = \left[ \frac{f(C)}{2} \right]^{1/3} \frac{N^{-2/3}}{\chi}, \]

\[ \xi^2(t_{\text{best}}) = \left[ \frac{1}{f(C)^{2/3}} + \frac{f(C)^{4/3}}{24} + \frac{C f(C)^{1/3}}{3} \right] \left( \frac{2}{N} \right)^{2/3}, \]

\[ f(C) = \sqrt{C^2 + 12 - C}; \quad C = \frac{\Gamma_{sq}}{2\chi}, \]

In order to find optimal conditions to produce spin squeezing in presence of losses and set the ultimate limits of this technique, from now on, we assume that the number of particles is large enough for the condensates to be in the Thomas-Fermi regime so that

\[ \mu = \frac{1}{2} \hbar \bar{\omega} \left( \frac{15 N a}{2 a_0} \right)^{2/5}, \]

where \( a_0 = \sqrt{\hbar/M \bar{\omega}} \) is the harmonic oscillator length, \( M \) is the mass of a particle and \( \bar{\omega} \) is the geometric mean of the trap frequencies,

\[ \chi = \frac{24/5^{3/5}}{5^{1/5}} \left( \frac{\hbar}{M} \right)^{-1/5} a^{2/5} \bar{\omega}^{6/5} N^{-3/5} \]

\[ \Gamma^{(1)} = K_1 \]

\[ \Gamma^{(2)} = \frac{15^{2/5}}{27^{1/5} \pi} \left( \frac{\hbar}{M} \right)^{-6/5} a^{-3/5} \bar{\omega}^{6/5} N^{2/5} K_2 \]

\[ \Gamma^{(3)} = \frac{5^{4/5}}{21^{1/5} \pi^{1/5} \pi^{2/5}} \left( \frac{\hbar}{M} \right)^{-12/5} a^{-6/5} \bar{\omega}^{12/5} N^{4/5} K_3 \]

We first analyze the dependence of squeezing on the ini-

\[ \bar{\omega}^{\text{opt}} = \frac{2^{19/12} 5^{1/12} \pi^{5/6}}{15^{1/3} M N^{1/3}} \left( \frac{K_1}{K_3} \right)^{5/12}. \]

It turns out that \( C_{\text{min}} \) is proportional to \( N \) and \( \xi^2(t_{\text{best}}, \bar{\omega}^{\text{opt}}) \) is a decreasing function of \( N \). The lower bound for \( \xi^2 \), reached for \( N = \infty \) is then

\[ \min_{t, \bar{\omega}, N} \xi^2 = \left( \frac{5 \sqrt{3} M}{28 \pi \hbar a} \right)^{2/3} \left[ \frac{7}{2} (K_1 K_3) + K_2 \right]^{2/3}. \]

In practice, one can choose \( N = N_\eta \) in order to have \( \xi^2 = (1 + \eta) \min \xi^2 \) (e.g. \( \eta = 10\% \)), and then calculate the corresponding optimized frequency \( \bar{\omega}^{\text{opt}} \) with (30). For a suitable choice of the internal state, in an optical trap, the two-body losses can be neglected \( K_2 = 0 \). One can get in this case very simple formulas for the optimized

FIG. 1: Spin squeezing obtained by a minimization of \( \xi^2 \) over time, as a function of the initial number of particles, without loss of particles (solid line), with one-body losses (dashed line), with two-body losses (dotted line), with three-body losses (dash-dotted line) respectively. Parameters: \( a = 5.32 \text{nm}, \bar{\omega} = 2\pi \times 200 \text{Hz}, K_1 = 0.18^{-1}, K_2 = 2 \times 10^{-21} \text{m}^3/s, K_3 = 18 \times 10^{-42} \text{m}^4/s. \) The symbols: crosses (plus) are results of a full numerical simulation with 400 Monte Carlo realizations for two-body (three-body) losses.
parameters and squeezing. For \( \eta = 10\% \) [14],
\[
N_\eta \simeq \frac{17.833}{(K_1 K_3)^{1/2}} \frac{\hbar a}{M},
\]
\[
t_{\text{best}} \simeq 0.277 \left( \frac{M}{\hbar K_1} \right)^{2/3} \left( \frac{K_3}{\hbar a^2} \right)^{1/3},
\]
\[
\xi^2 \simeq 0.356 \left( \frac{MK_1}{\hbar} \right)^{1/3} \left( \frac{MK_3}{\hbar a^2} \right)^{1/3}.
\]

We now ask whether we can use a Feshbach resonance to change the scattering length (but also \( K_3 \)) to improve the squeezing. In Fig 2 we plot the squeezing parameter vs the scattering length \( a \). Predicted values of \( K_3 \), as a function of \( a \), are taken from [13] for \( ^{87}\text{Rb} \) in the state \( |F = 1, m_F = 1 \rangle \) and \( K_1 = 0.01 \text{s}^{-1} \). We calculate \( \omega_{\text{opt}} \) and the number of particles needed for \( \eta = 10\% \) for each point in the curve. The dip giving large squeezing corresponds to a strong decrease in \( K_3 \) around 1003.5G \((K_3 \simeq 3 \times 10^{-45}\text{m}^6/\text{s})\). Close to the Feshbach resonance the squeezing gets worse as \( K_3 \) increases (even if in the figure we do not enter the regime \( K_3 \sim \hbar a^4/m \)).

Finally we consider the problem of the survival time of a spin squeezed state in presence of one-body losses. We imagine that the system evolves in two periods: for \( t < T_1 \) the system is squeezed in presence of interactions (\( \chi \neq 0 \)), one and three-body losses; and for \( t > T_1 \) the interaction is stopped (\( \chi = 0 \)), e.g. by opening the trap, and the system only experiences one-body losses. As \( t \) can be arbitrarily long, we use the exact solution for \( t > T_1 \) while for the \( t < T_1 \simeq t_{\text{best}} \), we use the approximation [13]. Then for \( t = T_1 + T_2 > T_1 \):
\[
\xi^2(t) = \frac{1}{4} \left( \frac{\langle \hat{N}(T_1) \rangle^2}{(S_z(T_1))^2} - \frac{1}{4} \left[ \langle \hat{N}(T_1) \rangle^2 \right] \left( \xi^2(T_1) \right) e^{-\gamma (t-T_2)} \right.
\]
\[
\simeq 1 - \left[ 1 - \xi^2(T_1) \right] e^{-\gamma (t-T_2)}.
\]

This result shows that the spin squeezing can be kept some time after the interactions have been stopped. To give an example, for \( ^{87}\text{Rb} \) atoms with bare scattering length \( a = 5.32 \text{nm} \), \( K_1 = 0.01 \text{s}^{-1} \), \( K_2 = 0 \), \( K_3 = 6 \times 10^{-42} \text{m}^6/\text{s} \) [13], in optimized conditions [23-25] \( N = 2.8 \times 10^5 \) and \( \omega_{\text{opt}} = 2\pi \times 20.06 \text{Hz} \), \( \xi^2 \simeq 5.7 \times 10^{-4} \) is reached at \( T_1 = t_{\text{best}} = 4.4 \times 10^{-2} \text{s} \), and a large amount of squeezing \( \xi^2 \simeq 0.01 \) is still available after 1s.

In conclusion, we found the maximum spin-squeezing reachable with cold atoms having a \( S_z^2 \) Hamiltonian, in presence of decoherence (losses) unavoidably accompanying the elastic interaction among atoms. The best squeezing is reached for an atom number \( N \rightarrow \infty \) and not for a finite value of \( N \). This is important for applications such as spectroscopy where, apart from the gain due to quantum correlations among particles (squeezing), one always gains in increasing \( N \).

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