These lectures review some of the basic properties of $N = 2$ superconformal field theories and the corresponding topological field theories. One of my basic aims is to show how the techniques of topological field theory can be used to compute effective Landau-Ginzburg potentials for perturbed $N = 2$ superconformal field theories. In particular, I will briefly discuss the application of these ideas to $N = 2$ supersymmetric quantum integrable models.

\textit{Dedicated to the memory of Brian Warr}

USC-93/001
hep-th/9301088
January, 1993.

* Lectures given at the Summer School on High Energy Physics and Cosmology, Trieste, Italy, June 15th – July 3rd, 1992. To appear in the proceedings.
** Work supported in part by the DOE under grant No. DE-FG03-84ER40168, and also by a fellowship from the Alfred P. Sloan foundation.
1. Introduction

The study of two-dimensional $N = 2$ supersymmetric field theories has shown surprising longevity in an era when half-lives of research areas (and average attention spans) are dropping well below a year. There are probably several reasons for the continued interest in $N = 2$ supersymmetric theories, but I believe that the most fundamental reason is that they have just the right amount of supersymmetry. They have enough supersymmetry so that they have topological, and pseudo-topological, sectors whose quantum properties can be computed semi-classically, and at the same time, these theories do not have so much supersymmetry that their structure is so rigid as to render the theory sterile and uninteresting.

There are now a number of very active areas of research in which $N = 2$ supersymmetric field theories are finding interesting applications: these areas include string theory, mirror symmetry, topological field theory, exactly solvable lattice models, two dimensional theories of quantum gravity and W-gravity, and even in polymer physics. In these lectures my aim will be to show how all the technology of perturbed $N = 2$ superconformal field theories and topological models provides a powerful set of tools in the analysis of $N = 2$ supersymmetric quantum integrable theories. This approach to the subject comes from a desire to mesh with the themes of this school, but also has the virtue of getting to some physically interesting results without the prerequisite of a course in string theory or algebraic geometry. Moreover, I will also be able to review a reasonable amount of the $N = 2$ supersymmetry technology that is currently finding applications elsewhere. Consequently, whenever possible I will try to indicate where my lectures connect, albeit tangentially, with the other currently active fields of research involving $N = 2$ supersymmetry. I will also attempt to make my lectures relatively self-contained by reviewing the basic ideas of $N = 2$ superconformal theories, but this review will be somewhat brief and more details may be found in my lectures at an earlier school at the ICTP [1] or in the earlier papers [2–9].

The topics that I will cover here are:

(i) $N = 2$ superconformal field theories, chiral rings and effective Landau-Ginzburg potentials.
(ii) Topologically twisted $N = 2$ superconformal field theories
(iii) Perturbed $N = 2$ superconformal field theories, both topological and non-topological.
(iv) Effective Landau-Ginzburg potentials and kink masses in perturbed $N = 2$ superconformal field theories.
(v) Computing effective Landau-Ginzburg potentials using topological field theory.
(vi) Simple $N = 2$ supersymmetric quantum integrable models and their soliton structure.

The lectures of Dennis Nemeschansky will start where I finish: he will discuss soliton scattering matrices in the $N = 2$ supersymmetric quantum integrable models. Cumrun Vafa will start his lectures by showing how a number of the concepts that I introduce for $N = 2$ superconformal field theories can be easily generalized to massive $N = 2$ supersymmetric theories, and he will then show how the topological structure of these massive models can be used to determine much about the “pseudo-topological” sectors of the theory.

2. $N = 2$ Superconformal Field Theories.

2.1. The operators and primary fields

In an $N = 2$ superconformal field theory the energy momentum tensor, $T(z)$, is supplemented by two supercharges, $G^+(z)$ and $G^-(z)$, and a $U(1)$ current, $J(z)$. These four generators have conformal weights $2, \frac{3}{2}, \frac{3}{2}$ and $1$ respectively, and have operator product expansions:

\[
\begin{align*}
G^\pm(z)G^\mp(w) &= \frac{\frac{2}{3}c}{(z-w)^3} \pm \frac{2J(w)}{(z-w)^2} + \frac{2T(w) \pm \partial_w J(w)}{z-w} + \ldots \\
J(z)G^\pm(w) &= \pm \frac{G^\pm(w)}{(z-w)} + \ldots \\
J(z)J(w) &= \frac{\frac{1}{3}c}{(z-w)^2} + \ldots \\
T(z)J(w) &= \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{z-w} + \ldots \\
T(z)G^\pm(w) &= \frac{\frac{3}{2}G^\pm(w)}{(z-w)^2} + \frac{\partial_w G^\pm(w)}{z-w} + \ldots \\
T(z)T(w) &= \frac{\frac{1}{3}c}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \ldots,
\end{align*}
\]

where, as usual, $+\ldots$ means plus terms that are finite in the limit as $z \to w$. Note the presence of the combination $2T(w) \pm \partial_w J(w)$ in the operator product $G^\pm(z)G^\mp(w)$. This will be important in the subsequent discussion of topologically twisted theories. One can
pass to modes and write:

\[
T(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2}
\]

\[
G^\pm(z) = \sum_{n=-\infty}^{\infty} G^\pm_{n\pm a} z^{-(n\pm a)-3/2}
\]

\[
J(z) = \sum_{n=-\infty}^{\infty} J_n z^{-n-1}.
\]

The parameter \(a\) is a real number, and determines the branch cut properties of \(G^\pm(z)\).

A field theory with \(N = 2\) superconformal symmetry with \(a = 0\) is usually said to be in a Ramond sector, and if the theory has \(a = \frac{1}{2}\) then it is said to be in a Neveu-Schwarz (NS) sector. I will simplify my life here by working almost entirely with theories in the NS sector. As I will describe later, it is very simple to convert results obtained in the NS sector into results for the superalgebra for any value of \(a\).

In terms of modes, the foregoing operator products can be written as:

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} m(m^2 - 1)\delta_{m+n,0}
\]

\[
[J_m, J_n] = \frac{c}{3} m\delta_{m+n,0}
\]

\[
[L_n, J_m] = -mJ_{m+n}
\]

\[
[L_n, G^\pm_{m\pm a}] = \left(\frac{n}{2} - (m \pm a)\right) G^\pm_{m+n\pm a}
\]

\[
[J_n, G^\pm_{m\pm a}] = \pm G^\pm_{m+n\pm a}
\]

\[
\{G^+_{n+a}, G^-_{m-a}\} = 2L_{m+n} + (n - m + 2a)J_{m+n} + \frac{c}{3} \left[(n + a)^2 - \frac{1}{4}\right] \delta_{m+n,0},
\]

where \(m\) and \(n\) are integers.

One should remember that in a conformal field theory there is both a holomorphic (left-moving) and an anti-holomorphic (right-moving) sector, and these two sectors have to be combined in the complete theory. Much of the time I will suppress the discussion of the anti-holomorphic sector, but throughout these lectures I will implicitly require that its structure be directly parallel to the structure of the holomorphic sector. In particular, this means that I will assume that the anti-holomorphic sector has \(N = 2\) superconformal symmetry with generators \(\tilde{T}(\bar{z}), \tilde{G}^+ (\bar{z}), \tilde{G}^- (\bar{z})\) and \(\tilde{J}(\bar{z})\).\footnote{Objects with a tilde, \(\sim\), will generically denote anti-holomorphic counterparts of holomorphic quantities.}
A primary field, $\psi(z)$, of the $N = 2$ superconformal algebra satisfies:

\[
T(z) \psi(w) = \frac{h}{(z-w)^2} \psi(w) + \frac{\partial_w \Psi(w)}{(z-w)} + \ldots \\
J(z) \psi(w) = \frac{q}{(z-w)} \psi(w) + \ldots \\
G^\pm(z) \psi(w) = \frac{1}{(z-w)} \Lambda^\pm(w) + \ldots ,
\]

where the fields $\Lambda^\pm(w)$ are the super-partners of $\psi(w)$. In terms of states and modes, the foregoing is equivalent to:

\[
G^+_r |\psi> = 0 , \quad r \geq \frac{1}{2} ; \quad L_n |\psi> = J_n |\psi> = 0 , \quad n \geq 1 ; \\
G^-_{-\frac{1}{2}} |\psi> = |\Lambda^\pm> , \quad L_0 |\psi> = h|\psi> , \quad J_0 |\psi> = q|\psi> .
\]

In order to obtain unitary representations of this algebra one must have \([\text{I}]\): $c \geq 3$ or $c = 3 - \frac{6}{(k+2)} ; k = 1,2,\ldots$. The models with $c = 3k/(k+2)$ are called the $N = 2$ superconformal minimal models. For these minimal models there are only finitely many highest weight irreducible representations. These irreducible representations are determined by the conformal weight, $h_0$, and $U(1)$ charge, $q_0$, of the ground state $|h_0,q_0>$, and the allowed values of $h_0$ and $q_0$ (in the NS sector) are:

\[
h_0 = \frac{\ell(\ell+2) - m^2}{4(k+2)} , \quad q_0 = \frac{m}{(k+2)} ,
\]

for $\ell = 0,1,2,\ldots,k$ and $m = -\ell,-\ell+2,\ldots,\ell-2,\ell$. The remaining states in the representation are then obtained from the ground state by acting with $L_{-n}$, $J_{-n}$ and $G^+_{-r}$ for $n,r > 0$. Because of this simple structure, these minimal models will be used to give examples throughout my lectures. There are, of course, considerably more $N = 2$ superconformal theories that are fairly well understood, for example, there are the $N = 2$ superconformal coset models \([\text{I}]\) and their closely related kin, the $N = 2$ super-$W$ algebras \([\text{II} - \text{IV}].\)

2.2. The chiral ring

Consider a general $N = 2$ superconformal theory. A field $\phi(z)$ will be called chiral if it satisfies:

\[
\left(G^+_{-\frac{1}{2}} \phi \right)(z) \equiv 0 .
\]
That is, half of its super-partners vanish. A field is called chiral, primary if it is both chiral and primary. The set of such fields is called the chiral ring and will be denoted by $\mathcal{R}$. As we will see, it does indeed have a ring structure.

To derive the properties of chiral primary fields one first considers unitarity bounds. Suppose that $|\psi>\,$ is any state. Then because $(G^±_r)^\dagger = G^{\mp}_r$, one has:

$$0 \leq <\psi|\{G^{±}_{-\frac{1}{2}}, G^{\mp}_{\frac{1}{2}}\}|\psi> = <\psi|2h_\psi + q_\psi |\psi>,$$

and hence for all states in a (unitary) $N = 2$ superconformal theory one has

$$h \geq \frac{1}{2}|q|.$$

From the bound (2.8) one can show that an equivalent characterization of $\mathcal{R}$ is that it is precisely the set of fields that have $q > 0$ and saturate the bound in (2.9), i.e. for which one has $^2$

$$h = \frac{1}{2}q.$$

This may be seen by first observing that if $h_\psi = \frac{1}{2}q_\psi$ then (2.8) immediately implies $G^{±}_{-\frac{1}{2}}|\psi>= G^{\mp}_{\frac{1}{2}}|\psi>= 0$. In addition, all the states $G^{±}_r|\psi>, G^{−}_r|\psi>, L_n|\psi>$ and $J_n|\psi>$ for $n, r > 0$ must be zero since they would otherwise violate (2.9). Therefore such a state $|\psi>$ must be both chiral and primary.

It is now easy to see how $\mathcal{R}$ inherits a ring structure. Consider the operator product of two chiral primary fields $\phi_i$ ($i = 1, 2$) of conformal weight $h_i$ and charge $q_i = 2h_i$. Suppose that $\psi$ is some operator that appears on the right hand side of the operator product:

$$\phi_1(z) \phi_2(w) = \ldots + (z - w)^{h_\psi - h_1 - h_2}\psi(w) + \ldots$$

Then charge conservation, (2.9) and (2.10) imply that the power, $h_\psi - h_1 - h_2$, is non-negative and vanishes if and only if $h_\psi = \frac{1}{2}q_\psi$, that is, it vanishes if and only if $\psi$ is both chiral and primary. Therefore the following is a well defined (associative and commutative) multiplication that closes into $\mathcal{R}$:

$$(\phi_1 \cdot \phi_2)(w) = \lim_{z \to w} \phi_1(z) \phi_2(w).$$

$^2$ The situation is not so simple for non-unitary theories, see for example [15]. Here I will only consider unitary theories.
The associativity and commutativity of this product follow trivially from the properties of the operator product.

Because it is associative and commutative, the ring, \( \mathcal{R} \), may be thought of as a polynomial ring. There will be generating fields \( x_a \in \mathcal{R} \) such that

\[
\mathcal{R} = \mathcal{P}[x_a]/\mathcal{J} ,
\]

where \( \mathcal{P}[x_a] \) is the ring of complex polynomials in the \( x_a \) and \( \mathcal{J} \) are the “vanishing relations”. That is, \( \mathcal{J} \) is an ideal of \( \mathcal{P}[x_a] \) consisting of all the polynomials in \( x_a \) that vanish in the product defined above.

The simplest examples of chiral rings are obtained from the minimal models. The primary fields are labelled \( \Phi_\ell^m \) and have conformal weight and \( U(1) \) charge given by (2.6). One then has \( h = \frac{1}{2} q \) if and only if \( m = \ell \). Let \( x = \Phi_1^1 \), and then one has \( \Phi_\ell^\ell = x^\ell \) in the obvious sense. However, recall that one has \( \ell = 0, 1, \ldots, k \), and so one must have \( x^{k+p} \equiv 0 \) for \( p \geq 1 \). Consequently, the ideal, \( \mathcal{J} \), is precisely generated by \( x^{k+1} \), i.e. \( \mathcal{J} = \mathcal{P}[x] \{ x^{k+1} \} \), and \( \mathcal{R} \) has a basis \( \{ 1, x, x^2, \ldots, x^k \} \).

2.3. Further properties of the chiral ring

Conjugate to the chiral ring, \( \mathcal{R} \), is the anti-chiral ring, \( \overline{\mathcal{R}} \). This simply consists of all operators that are primary and anti-chiral, that is, primary and annihilated by \( G_{-\frac{c}{2}} \). The structure of \( \overline{\mathcal{R}} \) is directly parallel to that of the chiral ring. It can also be characterized as consisting of all the fields that satisfy \( h = -\frac{1}{2} q \).

There is another basic tool in the analysis of \( N = 2 \) superconformal field theories and that is spectral flow. The basic idea is that in any theory containing a \( U(1) \) current, it is an elementary operation to shift the \( U(1) \) charge of any operator. Specifically, in the \( N = 2 \) superconformal theory one can write:

\[
J(z) = i \sqrt{\frac{c}{3}} \partial X(z) ,
\]

where \( X(z) \) is a canonically normalized boson. One then introduces an operator \( U_\theta(z) \) defined by:

\[
U_\theta(z) = \exp \left[ i \sqrt{\frac{c}{3}} \theta \, X(z) \right] .
\]

Let \( \psi(z) \) be some operator of charge \( q \), then after spectral flow by \( \theta \) one obtains an operator, \( \psi_\theta(z) \), of charge \( q + \theta \frac{c}{3} \), defined by

\[
\psi_\theta(w) \equiv \lim_{z \to w} (z - w)^{-q\theta} U_\theta(z) \psi(w) .
\]
I will denote the corresponding mapping on operators by $U_\theta$. There are several uses for this map. First of all, $U_\theta$ maps the NS representation of the $N = 2$ superconformal algebra onto a representation with the parameter $a$ of (2.2) and (2.3) given by $a = \theta$. Putting it slightly differently, if one uses spectral flow to conjugate the operators in the $N = 2$ superconformal algebra, then they change according to:

$$
U_\theta L_n U_\theta^{-1} = L_n - \theta J_n + \frac{c}{6} \theta^2 \delta_{n,0} \\
U_\theta J_n U_\theta^{-1} = J_n - \frac{c}{3} \theta \delta_{n,0} \\
U_\theta G_{r+}^+ U_\theta^{-1} = G_{r-\theta}^+ \\
U_\theta G_{r-}^- U_\theta^{-1} = G_{r+\theta}^-.
$$

(2.16)

If $\theta$ is an integer then the algebra maps back into itself. One can also verify that if $\theta = -1$ then $\mathcal{R}$ maps one-to-one and onto $\overline{\mathcal{R}}$, and if $\theta = +1$ then $\overline{\mathcal{R}}$ maps one-to-one and onto $\mathcal{R}$.

Consider the operators $\rho(z)$ and $\overline{\rho}(z)$ defined by:

$$
\rho(z) \equiv e^{i \sqrt{\frac{c}{6}} X(z)} = U_1(z) \\
\overline{\rho}(z) \equiv e^{-i \sqrt{\frac{c}{6}} X(z)} = U_{-1}(z).
$$

(2.17)

These may be viewed as spectral flows of the vacuum state by one unit. It is trivial to check that $\rho$ and $\overline{\rho}$ are elements of $\mathcal{R}$ and $\overline{\mathcal{R}}$ respectively, and they satisfy $h = \pm \frac{1}{2} q = \frac{c}{6}$. It is also easy to see that these states are the unique states in the theory satisfying this equation, since any such state under spectral flow can be taken to a state with $h = 0$ and $q = 0$, which must be the vacuum. The states $\rho$ and $\overline{\rho}$ are also the (unique) elements of maximal dimension in $\mathcal{R}$ and $\overline{\mathcal{R}}$. This follow from the inequality

$$
0 \leq \langle \phi | \{ G_{-\frac{3}{2}}^+ , G_{\frac{3}{2}}^+ \} | \phi \rangle = \langle \phi | 2h_\phi \mp 3q_\phi + \frac{c}{3} | \phi \rangle,
$$

(2.18)

which implies that for $h \leq \frac{c}{6}$ for all elements of $\mathcal{R}$ and $\overline{\mathcal{R}}$. It is also useful to note that this inequality also implies:

$$
G_{-\frac{3}{2}}^+ \rho = G_{-\frac{3}{2}}^- \overline{\rho} \equiv 0,
$$

(2.19)

which also follows from the fact that $\rho$ and $\overline{\rho}$ are spectral flows of the vacuum.

Another important use of spectral flow is to note that for $\theta = \pm \frac{1}{2}$ the spectral flow maps the NS sector into the Ramond sector and vice-versa \footnote{In a string theory this operation corresponds to space-time supersymmetry.}. From (2.16) one sees that
under such a spectral flow \((L_0 \pm \frac{1}{2}J_0)_{\text{NS}} \rightarrow (L_0 - \frac{c}{24})_{\text{Ramond}}\). Consequently, under the appropriate spectral flow, \(\mathcal{R}\) or \(\overline{\mathcal{R}}\) map to the Ramond ground states. This characterization of chiral primary fields makes them easier to compute in practice \[5\], but the Ramond ground states do not exhibit the ring structure in the simple manner that is evident for their NS counterparts. From the Ramond characterization we also see that the chiral ring is finite dimensional, since, in any unitary theory the degeneracy at any energy level is finite.

The similarity between chiral rings and cohomology of differential forms may already be apparent. This parallel can be made even more explicit by establishing a “Hodge decomposition theorem,” which says that any state, \(\psi\), can be written in the form:

\[
|\psi\rangle = |\phi\rangle + G^+_{-\frac{1}{2}}|\chi_1\rangle + G^-_{\frac{1}{2}}|\chi_2\rangle,
\]

(2.20)

where \(|\phi\rangle\) is a chiral primary, and \(|\chi_1\rangle\) and \(|\chi_2\rangle\) are some other states. Moreover, if \(\psi\) is itself chiral, \textit{i.e.} \(G^+_{-\frac{1}{2}}\psi = 0\), then one can take \(|\chi_2\rangle = 0\). This is elementary to prove using a Rayleigh-Ritz method: one considers states, \(|\chi\rangle\), of the form \(|\chi\rangle = |\psi\rangle - G^+_{\frac{1}{2}}|\chi_1\rangle - G^-_{\frac{1}{2}}|\chi_2\rangle\) and chooses \(|\chi_1\rangle\) and \(|\chi_2\rangle\) so as to minimize the norm of \(\chi\). It then follows that \(|\chi\rangle\) is chiral and primary. If \(\psi\) is chiral, then take the inner product of both sides of (2.20) with \(\langle \chi_2 | G^+_{-\frac{1}{2}}\), and one then sees that \(G^-_{\frac{1}{2}}|\chi_2\rangle\) must be zero.

Since one has \((G^+_{-\frac{1}{2}})^2 = 0\), and chiral fields may all be written in the form \(|\phi\rangle + G^+_{-\frac{1}{2}}|\chi_1\rangle\), where \(|\phi\rangle\) is a chiral primary, it follows that \(\mathcal{R}\) is precisely the cohomology of \(G^+_{-\frac{1}{2}}\). In several situations the operator \(G^+_{-\frac{1}{2}}\) reduces to a more familiar cohomological operator. On coset conformal field theories it becomes a loop-space Lie algebra cohomology operator, and on Calabi-Yau manifolds \(G^+_{-\frac{1}{2}}\) becomes one of the Dolbeault operators \[8\].

So far I have only discussed the holomorphic sector of the \(N = 2\) superconformal field theory. There are also chiral and anti-chiral rings in the anti-holomorphic sector. There are thus four choices of ring: \((c,c), (c,a), (a,c)\) and \((a,a)\), where \(c\) and \(a\) denote chiral and anti-chiral respectively, and the entries in \((, )\) denote the holomorphic and anti-holomorphic sectors. The \((c,c)\) and \((a,a)\) rings are complex conjugates of each other, as are the \((c,a)\) and \((a,c)\) rings. However, in a given \(N = 2\) superconformal theory, the two rings \((c,c)\) and \((c,a)\) are distinct and frequently completely different. From the

\[\text{Indeed, one can think of } G^+_{-\frac{1}{2}} \text{ and } G^-_{-\frac{1}{2}} \text{, and their anti-holomorphic counterparts } \widetilde{G}^+_{-\frac{1}{2}} \text{ and } \widetilde{G}^-_{-\frac{1}{2}} \text{, as being conformal field theoretic generalizations of the Dolbeault operators } \partial, \delta, \overline{\partial} \text{ and } \overline{\delta}.\]
computational point of view in an $N = 2$ superconformal theory it is easy to pass from one ring to the other: one simply reverses the sign of the anti-holomorphic $N = 2, U(1)$ current. However, if the $N = 2$ superconformal theory has a geometric origin, such as coming from a compactification on a Calabi-Yau manifold, then the two rings can have extremely different origins. The conformal field theory then puts on the same footing, two rings that are radically different from the point of view of algebraic geometry. This observation is the origin of mirror symmetry in Calabi-Yau manifolds [16,5]: If one can find a Calabi-Yau manifold, $\mathcal{M}$, that gives rise to a particular $N = 2$ superconformal field theory, with the $(c, c)$ and $(c, a)$ rings each having a particular geometric origin, then one should be able to find a manifold, $\tilde{\mathcal{M}}$, that gives rise to exactly the same $N = 2$ superconformal theory but with the geometric origins of the $(c, c)$ and $(c, a)$ rings inverted. The fact that this is possible has revolutionized an area of algebraic geometry. A recent review of the subject may be found in [17].

Returning to superconformal theories, from now on when I refer to the chiral ring of a complete $N = 2$ superconformal theory I will generally mean the $(c, c)$ ring, and I will restrict myself, for simplicity, to scalar chiral primary fields.

2.4. Landau-Ginzburg formulations of $N = 2$ superconformal theories.

The poor man’s definition of when a $N = 2$ superconformal theory has a Landau-Ginzburg formulation is that there must be a single quasihomogeneous function, $W_0$, that characterizes the chiral ring in the following manner. The ring, $\mathcal{R}$, has generators, $x_a$ of conformal weights $h_a = \tilde{h}_a = \frac{1}{2} \omega_a$, and $W_0$ is a function of these $x_a$’s having the quasihomogeneous scaling property:

$$W_0(\lambda^{\omega_a} x_a) = \lambda W_0(x_a); \quad (2.21)$$

and the ring itself must be given by:

$$\mathcal{R} = \frac{\mathcal{P}[x_a]}{\{\frac{\partial W_0}{\partial x_a}\}}, \quad (2.22)$$

where $\mathcal{P}[x_a]$ is the ring of polynomials in $x_a$ and $\{\frac{\partial W_0}{\partial x_a}\}$ denotes the ideal generated by the partial derivatives of $W_0$. As an example, the chiral ring of the minimal models has a Landau-Ginzburg potential: $W_0(x) = x^{k+2}$, where $x \equiv \Phi_1^1$.

The foregoing “definition” obscures the basic, and important physics that is really required of a Landau-Ginzburg formulation. So to repair this omission I will summarize
the basic idea, further details can be found in [1, 4–7, 19]. To say that an $N = 2$ superconformal theory has a Landau-Ginzburg formulation really means that one can obtain it from an $N = 2$ supersymmetric field theory that has a superpotential $W(x_a)$. One then looks for infra-red fixed points of the renormalization group flow. According to well substantiated folklore there are non-renormalization theorems that mean that $W(x_a)$ only scales through wave-function renormalization, and at the fixed point this superpotential must scale to a superpotential, $W_0$, with the quasihomogeneous scaling property (2.21) where $\omega_a$ is the scaling dimension $(h_a + \tilde{h}_a)$ of the scalar field $x_a$. In this field theory, the fields $x_a$ (and polynomials in them) are defined precisely so as to be chiral in the supersymmetric sense. At a fixed point of the renormalization group flow one can also use the superpotential to determine which of these polynomials in the $x_a$ are primary. Having an effective Landau-Ginzburg superpotential, $W_0$, means precisely that the partial derivatives (i.e. variations) of it must, via field equations, be proportional to superderivatives ($G^+_\frac{1}{2}$) of something. Conversely, if any polynomial in $x_a$ is given by $G^+_\frac{1}{2}$ acting on something else then this fact must be derivable from an effective field equation. Thus the right-hand side of (2.22) characterizes all the chiral fields in the theory modulo chiral fields that are given by $G^-_{\frac{1}{2}}$ acting on something else; this is precisely the chiral ring. It follows from the general properties that we have already established about $N = 2$ superconformal theories that the chiral ring is a finite polynomial ring in which all the fields have their naive scaling dimensions. It is also worth mentioning that it is an elementary result of singularity theory [19–21] that the polynomial:

$$H(x_a) = \det \left( \frac{\partial^2 W_0(x_a)}{\partial x_b \partial x_c} \right)$$

(2.23)

is the unique element of maximal dimension in the ring defined by (2.22). The scaling dimension of $H(x_a)$ is easily seen to be $\sum_a (1 - 2\omega_a)$. Since $H(x_a)$ is maximal, it must be identified with $\rho$, whose dimension is $c/3$, and as a result we see that we must have:

$$c = 3 \sum_a (1 - 2\omega_a) .$$

(2.24)

It will be of importance later to note that if a supersymmetric theory has a superpotential $W(x_a)$, then the effective bosonic potential is given by $|\nabla W|^2$. This means that the extremal points of $W$ correspond to zero energy ground states of the supersymmetric field theory. At the infra-red fixed point of the renormalization group flow all of these ground
states come together making a multi-critical point. It is easy to establish that the number of such critical points is the Witten index of the theory and is equal to the number of Ramond ground states in the conformal theory [22,23].

A good analogy is to consider the $c = \frac{1}{2}$ Virasoro minimal model. This appears at the infra-red fixed point of the Landau-Ginzburg description of the Ising model. In these lectures I am considering $N = 2$ supersymmetric generalizations, and because of the remarkable properties of $N = 2$ superconformal theories, and the non-renormalization theorems of the $N = 2$ supersymmetric theories, we can get an exact quantum effective potential that gives us exact information about the theory at (and near) the conformal point. This fact is completely contrary to one’s experience with the two-dimensional Ising model, for which the Landau-Ginzburg description becomes only barely qualitative near the conformal point.

This parallel with the Ising model, and the labelling of this formulation of certain $N = 2$ superconformal models with title Landau-Ginzburg, raises the fundamental question of whether such $N = 2$ superconformal models can be obtained from statistical mechanical systems or lattice models. Such connections with statistical mechanics were unclear when the the Landau-Ginzburg formulation of $N = 2$ superconformal models was first introduced, however it has recently been shown [24] how to formulate a broad class of exactly solvable lattice models whose continuum limit at the critical temperature are precisely the $N = 2$ superconformal (hermitian, symmetric) coset models of [11]. These models include all the $N = 2$ superconformal coset models that are known to have a Landau-Ginzburg formulation $^5$. Moreover, the natural order parameters of these lattice models renormalize to the chiral primary fields at the conformal point. Thus an underlying original hope has been realized: the scalar chiral primaries are indeed Landau-Ginzburg fields in the sense of being order parameters of some statistical mechanical system.

3. Twisted $N = 2$ supersymmetric theories

3.1. The topological matter models

It follows from Witten’s original work on topological field theories [25] that one can twist $N = 2$ superconformal models and obtain a topological field theory [26]. The basic idea is first to modify the energy-momentum tensor so as to obtain the one for the

$^5$ There are highly non-trivial infinite series of, and several sporadic, coset models that have Landau-Ginzburg formulations. There are also infinitely many coset models that do not have Landau-Ginzburg formulations.
topological theory:

\[ T_{\text{top}}(z) = T(z) + \frac{1}{2} \partial J(z) . \]  

(3.1)

The conformal weights of operators in the topological theory are thus given by \( h_{\text{top}} = h_{\mathcal{N}=2} - \frac{1}{2} q \). The supercurrents \( G^+(z) \) and \( G^-(z) \) have conformal weights 1 and 2 respectively in the topological theory. Even though the conformal weights of these operators have changed I will still use the mode labelling (2.2) of the \( \mathcal{N}=2 \) superconformal theory.

To get the physical spectrum of the topological theory one computes the cohomology of the following (dimension zero) BRST charge:

\[ Q = \oint G^+(z) \, dz . \]  

(3.2)

This charge is simply \( G^+_{-\frac{1}{2}} \); it manifestly satisfies \( Q^2 = 0 \), and it was shown in the last section that its cohomology can be represented by the chiral primary fields. Thus the physical states of the topological theory are precisely the chiral primaries, which now have (topological) conformal weight equal to zero. The reason why this twisted \( \mathcal{N}=2 \) superconformal theory is called topological is because the energy momentum tensor, \( T_{\text{top}}(z) \), is BRST exact, i.e.

\[ T_{\text{top}}(z) \equiv \{ Q , G^- (z) \} . \]  

(3.3)

Since \( T_{\text{top}}(z) \) (and \( \tilde{T}_{\text{top}}(\bar{z}) \)) generate infinitessimal conformal transformations, including translations, it follows that correlation functions of physical operators are all independent of their locations. Explicitly, if \( \phi(z, \bar{z}) \) is a physical operator, then one has:

\[ \frac{\partial}{\partial z} \phi(z, \bar{z}) = L_{-1} \phi(z, \bar{z}) = Q \left( G^-_{-\frac{1}{2}} \phi \right)(z, \bar{z}) . \]

In a correlation function of physical operators the contour integral\(^6\) \( Q = \oint z \, G^+(\zeta) \, d\zeta \) about \( z \), can be deformed away from \( z \) to a contour encircling all the other punctures on the Riemann surface. Since only physical operators are inserted at the punctures, and \( Q \) kills all of these operators, it follows that \( \frac{\partial}{\partial z} \phi(z, \bar{z}) \equiv 0 \) is true as a Ward identity. Similarly one also has \( \frac{\partial}{\partial \bar{z}} \phi(z, \bar{z}) \equiv 0 \).

There is also another very important class of physical operators \([27]\), which is perhaps more accurately called a class of physical marginal perturbations. Let \( \phi(z, \bar{z}) \) be a chiral primary field. Observe that when \( Q \), or its anti-holomorphic counterpart, \( \bar{Q} \), acts upon

\(^6\) The notation \( \oint z \, d\zeta \) means a small contour encircling a puncture at \( z \).
\((G^{-\frac{1}{2}}\phi)(z,\bar{z}), (\tilde{G}^{-\frac{1}{2}}\phi)(z,\bar{z}), \) or \((G^{-\frac{1}{2}}\tilde{G}^{-\frac{1}{2}}\phi)(z,\bar{z})\), then the result is a total derivative or zero. These operators have topological conformal weights \((h, \tilde{h})\) equal to \((1, 0)\), \((0, 1)\) and \((1, 1)\) respectively. As a result, if \(\Gamma\) is any closed curve, and \(\Sigma\) is the Riemann surface, then the integrals
\[
\oint_{\Gamma} (G^{-\frac{1}{2}}\phi)(z,\bar{z}) \, dz , \quad \oint_{\Gamma} (\tilde{G}^{-\frac{1}{2}}\phi)(z,\bar{z}) \, dz
\]
(3.4) and
\[
\int_{\Sigma} d^2 z \, (G^{-\frac{1}{2}}\tilde{G}^{-\frac{1}{2}}\phi)(z,\bar{z})
\]
(3.5) are physical operators. Let \(\phi_i(z,\bar{z})\) be a basis for the chiral primary fields, and let \(t_i\) be a set of parameters. Define \(\psi_i\) by:
\[
\psi_i \equiv (G^{-\frac{1}{2}}\tilde{G}^{-\frac{1}{2}}\phi_i)(z,\bar{z}) ,
\]
(3.6) and introduce the perturbed topological correlation functions:
\[
F_{i_1,...,i_n} \equiv \left\langle \phi_{i_1}(z_1,\bar{z}_1) \ldots \phi_{i_n}(z_n,\bar{z}_n) e^{-\left[\sum_{\ell} t_{\ell} \int d^2 z \, \psi_{\ell}(z,\bar{z})\right]} \right\rangle.
\]
(3.7) For the exactly the same reasons as outlined above, these correlation functions are also independent of the locations of the insertion points, \(z_1,\ldots,z_n\). As a result, the functions, \(F_{i_1,...,i_n}\), are totally symmetric in their subscripts \(i_1,\ldots,i_n\). However, these functions depend upon the parameters, or ‘moduli’, in a highly non-trivial manner. The properties of these functions have been extensively studied [27]. By suitably differentiating the \(F_{i_1,...,i_n}\) one can generate correlators with arbitrary insertions of the \(\psi_j\). To my knowledge there has been no systematic study of correlators with insertions of the form (3.4). There may well be some interesting topological interpretation for such correlators. They should be related to the “conformal blocks” of the topological theory. They might also lead to the braid matrices of the related non-topological theories, but little has been done to develop these ideas.

### 3.2. \(N = 2\) superconformal correlation functions and the topological model

I now wish to relate the correlation functions (3.7) to correlation functions of chiral primary fields in the “untwisted” \(N = 2\) superconformal model. There are one or two minor subtleties that I wish to bring out into the open. For simplicity I will restrict my attention to correlation functions on the sphere.
First, one should note that the $U(1)$ current is anomalous in the topological field theory:

$$T_{\text{top}}(z) J(z) = \frac{-c}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(z)}{(z-w)}.$$  \tag{3.8}

In terms of modes, one has

$$[L_n^{\text{top}}, J_m] = -m J_m - \frac{c}{6} n(n+1) \delta_{m+n,0}.$$  \tag{3.9}

In particular, $J_0 = [J_1, L_1^{\text{top}}]$, but $J_0^\dagger = -[J_{-1}, L_1^{\text{top}}] = J_0 + \frac{c}{3}$. Therefore, if $J_0|0> = 0$ then $0 =< 0|J_0^\dagger =< 0|(J_0 + \frac{c}{3})$. Hence on the sphere there is an anomaly of $-\frac{c}{3}$. Consequently, for a topological correlation function to be non-zero, the total $U(1)$ charge of all insertions must be $+\frac{c}{3}$.

In the original $N = 2$ superconformal theory, the current $J(z)$ is not anomalous and so to get corellators corresponding to (3.7) one must explicitly insert pure $U(1)$ fields whose total charge is $-\frac{c}{3}$. There are, in principle, infinitely many ways to distribute this charge, but in fact there are really only two natural methods. The simplest way to incorporate this negative charge is to insert the field $\rho(\xi, \xi)$ at some point $\xi$. Then one can show that:

$$\left\langle \phi_{i_1}(z_1, \bar{z}_1) \ldots \phi_{i_n}(z_n, \bar{z}_n) \rho(\xi, \xi) e^{-\sum t_\ell \int d^2 z |z - \xi|^2 q_\ell \psi_\ell(z, \bar{z})} \right\rangle = \prod_{p=1}^n |z_{i_p} - \xi|^{-2q_{i_p}} F_{i_1, \ldots, i_n}(t),$$  \tag{3.10}

where $q_j$ is the $U(1)$ charge of $\phi_j$. To see this one must suitably modify the earlier argument since $Q$ does not annihilate $\rho(\xi, \xi)$. Instead one uses:

$$\hat{Q} = \oint (\zeta - \xi) G^+(\zeta) d\zeta,$$  \tag{3.11}

which now annihilates $\rho(\xi, \xi)$, but also generates some algebraic mess:

$$\hat{Q} \left( G_{-\frac{1}{2}}^{-} \phi \right)(z, \bar{z}) = \left( (z - \xi) G_{-\frac{1}{2}}^{+} + G_{-\frac{1}{2}}^{+} \right) \left( G_{-\frac{1}{2}}^{-} \phi \right)(z, \bar{z}) = 2 \left( (z - \xi) L_{-1} + (L_0 + \frac{1}{2} J_0) \right) \phi(z, \bar{z})$$  \tag{3.12}

$$= 2 (z - \xi)^{(1-q_0)} \partial_z [(z - \xi)^{q_j} \phi(z, \bar{z})].$$

The measure in the integrals on the left hand side of (3.10) has therefore been modified so that when $\hat{Q}$ hits an integrated operator, the overall integrand is still a total derivative. If one employs $\hat{Q}$ in the supersymmetry Ward identity argument one finds that $\partial_{z_j} [(z - \xi)^{q_j} < \ldots >] = 0$. From this one then arrives at (3.10).
One should not be surprised at the factors of \((z - \xi)\) in (3.10) since in the \(N = 2\) superconformal theory the operators \(\phi_j\) have conformal weight \(h_j = \tilde{h}_j = \frac{1}{2}q_j\) while the operators \(\psi_j\) have conformal weight \(h_j = \tilde{h}_j = \frac{1}{2}(1 + q_j)\). Perturbing by such operators breaks conformal invariance (for \(q_j \neq 1\)), and so the correlation functions will care about the geometry of the surface, and in particular about the location of \(\xi\).

One should note that if the perturbations have \(q_j = 1\), then \(\psi_j\) has conformal weight \(h_j = \tilde{h}_j = 1\) in both the topological and the untwisted \(N = 2\) superconformal theory. This means that such operators provide marginal perturbations perturbations of both theories, and do not violate conformal invariance. In these circumstances all the anomalous factors of \((z - \xi)\) disappear from (3.10).

Because of all the spurious factors of \((z - \xi)\), the left hand side of (3.10) cannot, in general, be interpreted as a perturbed \(N = 2\) superconformal correlation function. However, if one moves \(\xi\) to \(\infty\) on the complex plane then one can make the desired interpretation of (3.10). Recall that in order to get a proper finite limit, the correlation function must be multiplied by \(|\xi|^{4h_\nu} = |\xi|^{\frac{2c}{3}}\) prior to sending the operator \(\mathcal{P}(\xi, \bar{\xi})\) to infinity [28]. By charge conservation, one must have

\[
\sum_{p=1}^{n} q_{i_p} + \sum_{\text{perturbations}} q(\psi_j) = \frac{c}{3},
\]

where the second sum is over the charges of all perturbations brought down in the expansion of the exponential in (3.10). It is easily seen that the net effect of rescaling the correlation function by \(|\xi|^{\frac{2c}{3}}\) is equivalent to replacing all factors of \(|z - \xi|\) by \(|\bar{\xi} - 1|\), and as \(\xi \to \infty\) these factors all go to unity. Consequently, if \(\mathcal{P}(\xi, \bar{\xi})\) is sent to infinity on the complex plane, the mess entirely disappears, and \(F_{i_1, \ldots, i_n}(t)\) is precisely the perturbed \(N = 2\) superconformal correlation function. The coupling constants \(t_j\) then have a canonical, physical scaling dimension of \((1 - q_j)\).

An alternative identification with \(N = 2\) superconformal correlators can be obtained by splitting the charge of \(-\frac{c}{3}\) into two pieces. That is, one introduces the operator \(\mu(z, \bar{\xi}) \equiv e^{-\frac{1}{2}\sqrt{2} (X(z) + \bar{X}(\bar{\xi}))}\) at two distinct points \(\xi_1\) and \(\xi_2\) on the sphere. The operator \(\mu\) in fact represents a particular Ramond ground state. Once again one generates many spurious factors of \((z - \xi_m)\) in the correlation functions, but they can all be made to disappear by conformally mapping to a flat cylinder with \(\xi_1\) and \(\xi_2\) mapping to the circles at either end of the cylinder. Thus one also finds that \(F_{i_1, \ldots, i_n}(t)\) is exactly a perturbed \(N = 2\) superconformal correlation function of chiral primary fields on the flat cylinder with...
Ramond ground states on either end of the cylinder. It is this interpretation that will be the starting point of Vafa’s lectures.

The basic point is that topological correlation functions give you exact perturbed $N = 2$ superconformal correlation functions, with the proviso that if the perturbations break conformal invariance, you must choose an appropriate two dimensional world sheet geometry.

Before leaving the subject of perturbed $N = 2$ superconformal correlation functions, I wish to make two important notes. First, the perturbation: $\sum_{\ell} t_{\ell} \int d^2 z \psi_{\ell}(z, \bar{z})$ is not hermitian from the point of view of the $N = 2$ superconformal field theory. The appropriate hermitian perturbation is:

$$\Delta S = -\sum_{\ell} \left\{ t_{\ell} \int d^2 z (G^-_{-\frac{1}{2}} \tilde{G}^-_{-\frac{1}{2}} \phi_{\ell})(z, \bar{z}) + \bar{t}_{\ell} \int d^2 z (G^+_{-\frac{1}{2}} \tilde{G}^+_{-\frac{1}{2}} \bar{\phi}_{\ell})(z, \bar{z}) \right\}, \quad (3.13)$$

where $\bar{t}_{\ell}$ is the complex conjugate of $t_{\ell}$ and $\bar{\phi}_{\ell}$ is the anti-chiral field conjugate to $\phi_{\ell}$. From the point of view of topological field theory the second term in $\Delta S$ is BRST exact and so decouples from all topological correlation functions. Consequently, if one made insertions of $e^{\Delta S}$ into all the correlations considered above, all the terms involving $\bar{t}_{\ell}$ would vanish. Thus $F_{i_1,...,i_n}(t)$ can still be interpreted as the appropriate chiral primary correlation function in a perturbed superconformal theory with a perturbation of the form $(3.13)$. Thus we can use topological methods to compute correlation functions in the massive quantum field theory whose action can be thought of as $S_0 + \Delta S$, where $S_0$ is the formal action of the original $N = 2$ superconformal model. Since the perturbing operators in $(3.13)$ are top components of superfields, it follows from the general theory of supersymmetry that these massive quantum field theories are still $N = 2$ supersymmetric.

There is a minor cautionary note to be sounded at this juncture. The decoupling of the BRST trivial states is a somewhat subtle business. One may need to perform some mild regularization, and this may introduce contact terms. Because of rather general arguments for topological field theories, one knows that it is possible to regularize theory so that BRST trivial states properly decouple \[23\] [29]. Equivalently, one can always find appropriate representatives of the BRST cohomology and corrections to the BRST charge so that all the correlators that should be zero are indeed zero. The fact that such choices are being made has been hidden in the discussion so far. In the $N = 2$ superconformal

\[7\] Many conformal field theories do not appear to be derived from an action. One should thus interpret this as a statement of how to define the theory via perturbation theory.
theory the choices of representatives and the explicit understanding of the regularization procedures and contact terms can lead to extremely important insights into the geometry of the moduli space of the theory (see, for example, [4,18,30–32]). Some of these issues will be addressed indirectly in the next sections, and will also be discussed by Vafa.

It is also interesting to point out that the required regularization involves the metric on the underlying Riemann surface, and one cannot choose the regulator uniformly over the entire moduli space of a Riemann surface. As a result, when one couples the foregoing topological matter models (i.e. the topologically twisted $N = 2$ superconformal models) to topological gravity there will be contact terms coming from the gravitational sector. This important observation [33] appears to lead to a derivation of the coupling of topological matter to topological gravity [34–36] directly from a Landau-Ginzburg formulation. This may also be related to, and perhaps provide some explanation of, the recent results of Dubrovin and Krichever [37,38] on the Landau-Ginzburg derivation of the integrable hierarchies associated with topological matter coupled to topological gravity.

4. Properties of topological correlation functions

It is the purpose of this section, and to some extent of the the two following sections, not only to describe the properties of topological correlation functions, but also to show how the topological correlation functions can be computed from rather limited knowledge of the underlying $N = 2$ superconformal theory, or its topologically twisted counterpart.

4.1. General structure

I will begin by simply enumerating and describing a number of properties of the correlation functions (3.7). My discussion here will essentially be a summary of some of the results of [27,36]. I will also only consider correlation functions on the sphere.

Define two special correlation functions:

$$\eta_{ij} \equiv F_{ij}(t), \quad C_{ijk} \equiv F_{ijk}(t).$$

Since $F_{i_1,\ldots,i_n}$ is totally symmetric in all its indices, these functions must also be totally symmetric. For $t = 0$ the function $\eta_{ij}$ forms an invertible topological metric. To see this, suppose that we normalize the chiral primary fields $\phi_i$ so that in the $N = 2$ superconformal theory one has:

$$\left\langle \phi_i(z, \bar{z}) \bar{\phi}_j(w, \bar{w}) \right\rangle = \frac{\delta_{ij}}{|z - w|^{2h_i}},$$
where $\bar{\phi}_j$ is the anti-chiral conjugate of $\phi_j$. Now recall that the spectral flow using $\mathcal{P}$ maps $\mathcal{R}$ isomorphically onto $\overline{\mathcal{R}}$, and hence there is an invertible matrix $V^j_i$ such that:

$$\bar{\phi}_j(w, \bar{w}) = V^j_i \lim_{z \to w} |z - w|^{4h_i} \mathcal{P}(z, \bar{z}) \phi_k(w, \bar{w}).$$

It then follows that the $N = 2$ superconformal correlation function $< \phi_i \phi_j \mathcal{P} >$ is proportional to the invertible matrix $V^j_i$, and therefore one has $\eta_{ij} = V^j_i$.

I will use the metric $\eta_{ij}$, and its inverse $\eta^{ij}$, to raise and lower indices. One should note that $C_{ij}^k$ are precisely the structure constants of the chiral ring. To see this, simply take the limit $z_1 \to z_2$ in the three point function (remember that the correlation function is independent of the $z_i$) in which case the three point function collapses to the structure constants times the metric.

One can prove that the topological correlation functions, at general values of $t$, have the following properties:

(i) $\eta_{ij}$ is in fact independent of $t$.

(ii) $F_{i_1, \ldots, i_n} = C_{i_1i_2}^j F_{j, i_3, \ldots, i_n}$

(iii) $C_{ij}^m C_{klm} = C_{(ij}^m C_{kl)m}$

(iv) $\partial_\ell C_{ijk} = \partial_\ell (C_{ijk})$

where $( )$ denotes symmetrization of the indices enclosed, and $\partial_\ell = \frac{\partial}{\partial t_\ell}$.

In proving these properties, I will work with the $N = 2$ superconformal correlators, and take (3.10) as the definition of $F_{i_1, \ldots, i_n}$. The fact that $\eta_{ij}$ is independent of $t$ can be demonstrated using the Ward identities of $G^{-}(z)$. One considers an insertion of $\psi_\ell = \left( G^{-}_{-\frac{1}{2}} \right) (z, \bar{z})$ in the correlator defining $F_{ij}(t)$. By writing this insertion in terms of a contour integral $\oint z \mathcal{V}(\zeta) G^{-}(\zeta) d\zeta$ for a suitable choice of vector field, $\mathcal{V}(\zeta)$, one can pull the contour off at infinity and arrange that it also annihilates $\phi_i, \phi_j$ as well as other insertions of the perturbation, $\psi_k$. One also needs to make use of the fact that $G^{-}_{r} \mathcal{P} = 0$ for $r \geq -\frac{3}{2}$ (see equation (2.19)). Thus one can establish that all the integrands defining the perturbations of $F_{ij}$ vanish, and so the metric $\eta_{ij}$ is a constant, and hence flat, metric.

To establish (ii) one inserts a complete set of states on a circle that separates $\phi_{i_1}$ and $\phi_{i_2}$ from the other fields $\phi_{i_{\ell}}$. One then uses the Hodge decomposition theorem on this complete set of states. The terms that appear in the completeness sum are either of the

---

8 A suitable choice is to take $\mathcal{V}(\zeta) = \frac{(\zeta - z_1)(\zeta - z_2)}{(|\zeta - \xi|)}$. Since $G^{-}(\zeta)$ has conformal weight $3/2$, the vector field $\mathcal{V}(\zeta)$ can diverge, at most, linearly as $\zeta \to \infty$ if the contour integral is to leave no residue at infinity [28].
form a) $|\phi> <\phi|$, b) $G^+_{\frac{-1}{2}}|\chi_1> <\chi_1|G^-_{\frac{1}{2}}$ or c) $G^-_{\frac{-1}{2}}|\chi_2> <\chi_2|G^+_{\frac{-1}{2}}$: where $|\phi>$ is chiral primary. Since $G^+_{\frac{-1}{2}} = Q$, the insertion of anything of the form b) or c) vanishes. This leaves a sum only over the chiral primaries, and hence establishes (ii). The result (iii) then follows from considering all possible ways of factorizing a four-point function.

It is straightforward, but technical, to prove the identity (iv). It basically follows from an elementary fact about conformal field theory: Any correlator depends upon any four of the insertion points via the cross-ratio of those points [28]. As a consequence, if three points are fixed and the rest are integrated over, one can transform one of the integrations to an integration over the cross ratio, and hence transform it to an integration over one of the previously fixed points. Put another way, conformal invariance means that in the perturbed three-point function with $n$ integrated insertions, one can integrate over any subset of $n$ of the $n+3$ insertions. There are then two other elements that needed to complete the proof: First, one must move the operators $G^-_{\frac{1}{2}}$ and $\tilde{G}^-_{\frac{1}{2}}$ from the old integrated insertion to the new integrated insertion. This is once again accomplished by contour integration tricks as outlined above. Secondly, in moving around these operators and changing the integration variable one generates lots of factors of $(z - \xi)$. These all conspire to transform all the factors of $(z - \xi)$ in (3.10) in precisely the correct manner.

From (iv) it follows that one can write

$$C_{ijk} = \partial_i \partial_j \partial_k F,$$  \hspace{1cm} (4.2)

for some analytic function $F(t)$. This function is called the free energy of the model and completely characterizes all of the topological correlation functions. By scaling the fields and the coupling constants in the three-point function one can show that $F(t)$ must satisfy:

$$F(\lambda^{1-q_j} \ t_j) = \lambda^{3-\frac{c}{2}} F(t_j),$$ \hspace{1cm} (4.3)

where $q_j$ is the charge of the field $\phi_j$.

It turns out that (4.2), (4.3) and property (iii) provide a highly overdetermined system of equations that in practice appear to completely determine the $C_{ijk}$, and hence $F(t)$ up to quadratic, linear and constant terms. The only input necessary appears to be the number and dimensions of the chiral primaries, and the unperturbed vanishing relations. One can then make an ansatz for $F$ and solve the equations [27,36,38,40]. This is extremely laborious in practice, and there are short cuts and far better methods, as I will describe.
5. Effective Landau-Ginzburg potentials

5.1. Structure and properties of the potentials

If the original $N=2$ superconformal field theory had an effective Landau-Ginzburg potential, $W_0(x_a)$, it follows that the perturbed theory will also have an effective Landau-Ginzburg potential, $W(x_a; t_j)$. This is because our perturbations involve only the top components of the chiral primary superfields and therefore preserve the supersymmetry. The operators, $x_a$, are some generators of the perturbed chiral ring, $\mathcal{R}$. Let $p_i(x_a)$ denote some polynomials in the $x_a$ that form a basis for $\mathcal{R}$. The statement that one has an effective Landau-Ginzburg potential, $W$, means that the ring multiplication is defined by

$$ p_i(x_a) p_j(x_a) = f_{ij}^k p_k(x_a) \quad \text{mod} \quad \{ \frac{\partial W}{\partial x_a} \}, \quad (5.1) $$

where $f_{ij}^k$ are structure constants computed by simply multiplying polynomials modulo the ideal generated by the partial derivatives $\frac{\partial W}{\partial x_a}$.

There is a natural basis inherited from the conformal point: namely the $\phi_i$, for which one has

$$ \phi_i \phi_j = C_{ij}^k(t) \phi_k, \quad (5.2) $$

where $C_{ij}^k(t)$ are the structure constants obtained by conformal perturbation theory. One can, of course, use these structure constants to write the basis, $\phi_i$, as polynomials in the generators, $x_a$. Since the structure constants are functions of $t$, the $\phi_i$ when considered as polynomials in $x_a$, will also be functions of $t$. That is, one has $\phi_i = \phi_i(x_a; t_j)$. For example, if one has $\phi_2 = \phi_1^2$ at $t = 0$, one might find that $C_{11}^0 = t$ and $C_{11}^2 = 1$ and hence one would have $\phi_2 = \phi_1^2 + t = x_1^2 + t$. The statement that there is an effective Landau-Ginzburg potential implies that these functions, $\phi_i(x_a; t_j)$, must satisfy:

$$ \phi_i(x_a; t_\ell) \phi_j(x_a; t_\ell) = C_{ij}^k(t) \phi_k(x_a; t_\ell) \quad \text{mod} \quad \{ \frac{\partial W}{\partial x_a} \}, \quad (5.3) $$

where one now simplifies the left hand side of this equation using polynomial multiplication modulo the ideal generated by the partials $\frac{\partial W}{\partial x_a}$. This imposes a vast set of constraints on $W(x_a; t_j)$ and the $\phi_i(x_a; t_\ell)$. Indeed, given $C_{ij}^k(t)$, one finds that the functions $W(x_a; t_j)$ and $\phi_i(x_a; t_\ell)$ are greatly overdetermined. There are, however, still more constraints.

Since the perturbation has the form (3.13) one can see that, at each point in parameter space, under an infinitesimal change of parameters one has $\delta W = - \sum_j \delta t_j \phi_j$, or

$$ \phi_j(x_a; t_\ell) = - \frac{\partial}{\partial t_j} W(x_a; t_\ell). \quad (5.4) $$
Given the form of the perturbation one might be tempted to conclude that $W$ is linear in the $t_j$, and so $W(x_a; t_j) = W_0 + t_j \phi_j$. This is incorrect. One can see this on the computational level by merely observing that the $C_{ij}^k(t)$ are non-trivial functions of $t$, and hence the $\phi_i(x_a; t_\ell)$ must also be non-trivial functions of $t$. Thus (5.4) represents a collection of differential equations for $W(x_a; t_j)$, which is a non-trivial, non-linear function of the $t_j$’s. On the physical level, the non-linear dependence of $W$ upon $t_\ell$ is the result of contact terms [33]. The beauty of the approach employed here is that all the consistency conditions on $C_{ij}^k(t)$, $\phi_i(x_a; t_\ell)$ and $W(x_a; t_j)$ completely determine these contact terms and so one can avoid the subtleties of such computations.

To determine $W(x_a; t_j)$ one can find the $C_{ij}^k(t)$ as described earlier and then use the structure constants to find the $\phi_i(x_a; t_\ell)$ and then solve (5.4) and (5.3). It is also valuable to employ the scaling behaviour of $W(x_a; t_j)$:

$$W(\lambda^{\omega_a} x_a; \lambda^{1-q_j} t_j) = \lambda W(x_a; t_j).$$

(5.5)

In practice, it is usually simplest to solve everything at once. That is, make ansätze for $W$ and $F$ that are consistent with (5.5) and (4.3) and then write down all the constraints arising from the identities: $C_{ij}^k C_{klm} = C_{(ij}^m C_{kl)m}$ and from the equations (5.3) and (5.4). The result is a highly overdetermined system of equations for the unknown constants and functions in the ansätze. This system can be solved for simple models, but is completely unmanageable in general. As we will see in the next section, there are simpler ways to solve the system.

5.2. Relationship to topological Landau-Ginzburg models

Rather than constructing the Landau-Ginzburg potential for a perturbed $N = 2$ superconformal theory, one can start with the Landau-Ginzburg potential as the fundamental object and obtain a topological field theory directly [41]. I will not review this approach in any detail here, but simply describe how it connects with my discussion. One parametrizes the effective Landau-Ginzburg potential, $W(x_a; t_j)$, in any manner one chooses but with the restriction that the partial derivatives, (5.4), define a basis for the chiral ring. As before, the “maximal” chiral primary field can be represented by the hessian:

$$H(x_a; t_j) = \det \left( \frac{\partial^2 W(x_a; t_j)}{\partial x_b \partial x_c} \right).$$

(5.6)

9 It is, by construction, physically obvious that these equations satisfy the requisite integrability condition. However, with some work, one can also prove it from the definition of the $\phi_i(x_a; t_\ell)$. 
The metric of the theory can then be defined by

$$g_{ij} = C_{ij}^H,$$  \hspace{1cm} (5.7)

where the structure constants are defined by polynomial multiplication as in (5.3), and $C_{ij}^H$ denotes the coefficient of $H(x_a; t_j)$ in the product of $\phi_i(x_a; t_\ell)$ and $\phi_j(x_a; t_\ell)$. The fact that the maximal element of the chiral ring is only defined up to an overall scaling factor means that the metric (5.7) will only be conformally related to the natural topological metric, $\eta_{ij}$, introduced earlier. That is,

$$\eta_{ij} = \Omega(t_j) g_{ij},$$  \hspace{1cm} (5.8)

for some function, $\Omega(t_j)$, of the coupling constants. One should also note that because the coordinates $t_j$ are now arbitrary, the metric $\eta_{ij}$, while (locally) flat, is not necessarily constant. One can determine the conformal factor $\Omega(t_j)$ by requiring that $\eta_{ij}$ be flat, and one can reconstruct the “flat coordinates” of conformal perturbation theory by solving the geodesic equations and constructing Gaussian coordinates.

Thus in the topological Landau-Ginzburg approach one, to some extent, loses sight of the natural parametrization offered by conformal perturbation theory. One must reconstruct this parametrization by once again solving some rather unpleasant equations. One might of course wonder why one is so interested in this form of the parametrization. First, such coordinates are precisely those supplied by conformal perturbation theory and therefore they are important in the analysis of perturbed $N = 2$ superconformal models. Secondly, such coordinates are central to the study of mirror symmetry and to the coupling of topological matter models to topological gravity.

5.3. Example: minimal models

For the minimal models, the basis for the unperturbed chiral ring (at $t = 0$) will be taken to be $\phi_{\ell} = x^\ell$, $\ell = 0, 1, \ldots, n$. For convenience I will normalize $W_0(x)$ to

$$W_0(x) = \frac{1}{(n + 2)} x^{n+2}.$$  

With this choice of basis one has

$$C_{ij}^k(t = 0) = \delta_{i+j,k} \quad 0 \leq i, j, k \leq n.$$  

22
To first order in perturbation parameters one has

\[ W = W_0 - A \left( \sum_{k=0}^{n} t_k x^k \right), \tag{5.9} \]

where \( A \) is an overall normalization of the perturbing field, and will be fixed later. The general perturbed superpotential, \( W \), satisfies (5.5) with \( \omega = 1/(n+2) \) and \( q_j = j/(n+2) \).

The topological metric is given by \( \eta_{ij} = <x^i x^j> = <x^{i+j}> \). Normalize \( x \) so that \( <x^n> = 1 \), and then one has

\[ \eta_{ij} = \delta_{i+j,n}. \]

To deduce \( C_{ij}^k(t) \) and \( W(x; t) \) from consistency conditions alone is very painful, and so I will take a short-cut. For the chiral primary field \( \phi_\ell \) consider

\[ X_p \equiv G^{-p} \cdots G^{-1} \phi_\ell. \tag{5.10} \]

This operator has conformal weight \( h = \frac{1}{2}(p+1)^2 + \frac{1}{2} \frac{\ell}{(n+2)} \) and \( U(1) \) charge \( q = \frac{\ell}{(n+2)} - (p+1) \). For \( p \geq \ell \) this violates the unitarity bound \( h \geq \frac{2}{2c} q^2 \) and so \( X_\ell \) must vanish identically. Putting it another way, one can easy check that \( X_\ell \) defines a null state in the \( N=2 \) superconformal minimal model. Now consider a correlation function of the form

\[ \langle \phi_i(z_1) \phi_j(z_2) \phi_k(z_3) \prod_{m=1}^{M} \left( G^{-\frac{1}{2}} \phi_{i_m} \right)(\zeta_m) \rangle. \]

By the usual contour integration games and by clever choices of vector fields, one can move all the \( G^- \) operators onto any one of the \( \phi \)'s. In so doing, the moding of the \( G^- \) operators becomes exactly of the form (5.10). As a result, this correlation function vanishes identically if \( M \) exceeds the minimum of \( i, j, k \) and \( i_1, i_2, \ldots, i_M \). It follows immediately that \( C_{1ij} \) is at most linear in all the \( t \)'s.

By scaling one can see that \( C_{1ij} \) must be proportional to \( t_k \) with \( k = 2n + 1 - i - j \). Fix the normalization constant \( A \) in (5.9) by taking \( C_{1nn} = t_1 \). Using property (iv) of the \( C_{ij} \) one then has \( C_{11n} = t_n \) and hence

\[ C_{1n}^0 = t_1, \quad C_{1n}^{n-1} = t_n. \tag{5.11} \]

This unitarity bound follows from the fact that the \( L_0 \) eigenvalue of a state must be at least that of its \( U(1) \) component, \( i.e \) the energy-momentum tensor orthogonal to the \( U(1) \) direction, \( T(z) - (-\frac{1}{2} (\partial X(z))^2) \), must also give rise to non-negative conformal weights.
Now recall that at $t = 0$ one has $C_{ij}^k = 1$ for $k = i + j$ and $C_{ij}^k = 0$ for $k > i + j$ and for $k = i + j - 1$. This remains unchanged for $t \neq 0$ since there is no $t_\ell$ with the requisite scaling dimension to modify these particular structure constants. Now use

$$C_{1i}^j C_{kj}^\ell = C_{1k}^j C_{ij}^\ell$$

(5.12)

to recursively determine the terms in $C_{ij}^k$ that are linear in $t_1$. For example, take $i = 1$ and use (5.12) to conclude that the matrix $C_2 \equiv (C_2)_i^j$ is of the form $(C_1)^2 + at_n I$, where $I$ is the identity and $a$ is some undetermined constant. Continuing, one readily establishes that

$$C_{j(n+1-j-\ell)}^\ell = t_1 \quad \ell = 0, 1, \ldots, j - 1,$$

and using property (iv), it follows that

$$C_{1(n+1-j-\ell)}^\ell = t_j \quad \ell = 0, 1, \ldots, j - 1,$$

or as a matrix:

$$C_1 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
t_n & 0 & 1 & 0 & 0 & \cdots & 0 \\
t_{n-1} & t_n & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\
t_1 & t_2 & t_3 & \ldots & \ldots & \ldots & t_n & 0
\end{pmatrix}.$$

It is then elementary to determine $W(x; t_\ell)$ since $\frac{dW}{dx}$ is the polynomial of order $n + 1$ in $x$ that must vanish in the chiral ring, but $C_{1i}^j$ are precisely the structure constants for multiplication by $x$, and so $\frac{dW}{dx}$ must be a multiple of the characteristic equation of the matrix $C_1$. Indeed, with my normalizations:

$$\frac{dW}{dx} = det(x - C_1).$$

(5.13)

One can now easily construct the $\phi_j(x; t_\ell)$ recursively using the structure of $C_1$. One has $\phi_1 = x$ and:

$$x \phi_j = C_{1j}^k \phi_k = \phi_{j+1} + \sum_{\ell=0}^{j} C_{1j}^\ell \phi_\ell$$

(5.14)

$$= \phi_{j+1} + \sum_{\ell=0}^{j} t_{n+1+\ell-j} \phi_\ell,$$
which completely determines $\phi_{j+1}$. This means that the effective potential $W(x; t_\ell)$ can also be fixed using (5.4).

In a later section I will need two special cases of this effective potential: a) $t_1 = t$, $t_\ell = 0$ for $\ell \geq 2$; and b) $t_n = t$, $t_\ell = 0$ for $\ell \leq n - 1$. The first of these is completely trivial since $W(x; t)$ can be entirely determined by dimensional analysis:

$$W(x; t) = \frac{1}{(n + 2)} x^{n+2} - tx . \quad (5.15)$$

The second is far less trivial. One finds that $W(x; t)$ is a Chebyshev polynomial. Explicitly, one can write:

$$W(x; t) = 2t^{\frac{1}{2}(n+2)} \cos((n + 2)\theta) ; \quad \text{where} \quad x = 2\sqrt{t} \cos \theta . \quad (5.16)$$

The critical points of these potentials occur at $\theta = \frac{j\pi}{n+2}$, and at these points $W(x; t)$ takes the values $2t^{\frac{1}{2}(n+2)} \cos(j\pi) = (-1)^j 2t^{\frac{1}{2}(n+2)}$. This observation will be of importance later.

6. Flat Coordinates, Flat Bundles and Classical Integrable Hierarchies

The purpose of this section is to review briefly some of the more recent developments in the technology for computing the flat coordinatization of effective potentials. As this section is something of a digression from my main objective, it may be ignored if the reader so desires.

We have seen that, even in the simplest cases, it is a considerable labour to compute the flat coordinatization of the effective potential. There are, however, more powerful techniques that can be borrowed from complex geometry and from singularity theory. These techniques are finding considerable use in the analysis of mirror symmetry in Calabi-Yau manifolds and also in the coupling of topological matter to topological gravity. The basic idea probably originated in Griffith’s paper on complex surfaces in $\mathbb{C}P^n$ [12], and has since been extensively studied under the general headings of ‘variation of Hodge structure’ and ‘Gauss-Manin connections’. The specific application to singularity theory may be found in a nearly impenetrable paper by Saito [43] and works by Noumi (see, for example, [44]). There has also been considerable discussion of the methods in the physics literature, initiated by [45] and since developed in a number of places [46–48].

---

11 This formula can be established by developing recursion relations for the determinants (5.13).
Once again, let $W(x_a; t_j)$ be a general perturbation of $W_0(x_a)$, and define $\phi_i(x_a; t_\ell)$ by (5.4). Consider $x_a, a = 1, 2, \ldots, N$, to be complex variables and introduce “formal” integrals of the form

$$u_i^{(\lambda)} \equiv (-1)^{(\lambda+1)} \Gamma(\lambda + 1) \int_X \frac{\phi_i(x_a; t_\ell)}{W(x_a; t_\ell)^{\lambda+1}}.$$  

(6.1)

In this expression, $\lambda$ is a complex parameter and $\Gamma$ is the usual gamma function. The prefactor has been introduced for convenience to soak up irritating factors arising from integration by parts. The surface, $X$, over which the integral is to be performed is any real $(2N - 1)$-dimensional hypersurface (cycle) around some component of the surface defined by $W = 0$. The only thing that one really needs to know about these integrals is that the integral of any total derivative is zero.

Differentiating $u_i^{(\lambda)}$ with respect to $t_j$ one obtains two terms:

$$\frac{\partial}{\partial t_j} u_i^{(\lambda)} = -(-1)^{\lambda+2} \Gamma(\lambda + 2) \int_X \frac{\phi_i(x_a; t_\ell) \phi_j(x_a; t_\ell)}{W(x_a; t_\ell)^{\lambda+2}}$$

$$+ (-1)^{\lambda+1} \Gamma(\lambda + 1) \int_X \frac{\partial_j \phi_i(x_a; t_\ell)}{W(x_a; t_\ell)^{\lambda+1}} + q_a \frac{\partial W}{\partial x_a},$$

where $q_a$ are some polynomials. Integrating by parts one obtains:

$$\frac{\partial}{\partial t_j} u_i^{(\lambda)} = -C_{ij}^k u_k^{(\lambda+1)} + (-1)^{\lambda+1} \Gamma(\lambda + 1) \int_X \frac{\partial_j \phi_i(x_a; t_\ell) + \frac{\partial}{\partial x_a} q_a}{W(x_a; t_\ell)^{\lambda+1}}.$$  

One may have to reduce the numerator of the second term modulo $R$, and once again integrate by parts. Repeating this process as often as is necessary, one obtains an equation of the form:

$$\frac{\partial}{\partial t_j} u_i^{(\lambda)} + C_{ij}^k u_k^{(\lambda+1)} - \sum_{n=0}^{\infty} \Gamma(n)_{ij}^k u_k^{(\lambda-n)} = 0.$$  

In other words the $u_i^{(\lambda)}$ are flat sections of a trivial bundle over the parameter space of deformations of the superpotential $W$. The foregoing equation defines the Gauss-Manin connection on this space. One can write down the integrability condition of the connection and break it into components corresponding to the superscript $(\mu)$ on $u_i^{(\mu)}$. The resulting flatness equations require the $C_{ij}^k$ to satisfy property (iii) and the covariant analogues
of property (iv). (The structure constants, \( C_{ij}^k \), trivially satisfy property (iii) since the structure constants were defined here using polynomial multiplication.) The affine connection in these covariant derivatives is given by \( \Gamma^k_{ij} = \Gamma^{(0)}_{ij}^k \). The fact that the system of equations that we wish to solve comes from the integrability conditions of a linear system, means that we are dealing with some form of classical integrable hierarchy.

At this point, there is something of a technical minefield, which can probably be passed so as to obtain a general result. However, the following partial results are know. If one restricts to relevant perturbations (so that the coupling constants have strictly positive scaling dimensions) then the flat coordinates we seek are precisely those obtained by requiring that the connection vanishes. In other words, parametrize the potential with arbitrary functions of flat coordinates, and then set \( \Gamma^k_{ij} \) to zero. The result is a system of differential equations that define the arbitrary functions in terms of flat coordinates. One can also incorporate marginal parameters (i.e. ones with vanishing scaling dimension) into this procedure. One modifies the starting point so as to incorporate an analogue the conformal factor in \( (5.8) \). Specifically, one starts with

\[
u_0^{(\lambda-1)} \equiv (-1)^\lambda \Gamma(\lambda) \int_X \frac{\omega(t_\ell)}{W(x; t_\ell)^\lambda} ,
\]

where \( \omega(t_\ell) \) is a function only of the marginal (dimension zero) parameters. One then defines \( u_i^{(\lambda)} = \partial_i u_0^{(\lambda-1)} \), and proceeds as above. One then sets all of the connection terms, \( \Gamma^{(n)}_{ij} \), to zero, and as well as determining the flat coordinates, this also determines the function \( \omega(t_\ell) \). The problem arises if one tries to incorporate irrelevant perturbations (with parameters of negative scaling dimension). It is not yet known (at least to physicists) how to deal with these. In particular, the flat coordinate equations cannot easily be separated out of the foregoing procedure. If one reads \[43\] one is left with the impression that there should not be a problem, but so far I am not aware of a successful implementation of the abstractions of \[43\] into a computable procedure.

The foregoing procedures for computing flat coordinates are described in considerable detail in \[47\]. It turns out that this method is by far the most efficient for computing flat coordinates, and to illustrate this I will show how it yields a different formulation of the flat coordinates for the minimal models.

One starts from

\[
u_0^{(\lambda-1)} \equiv (-1)^{\lambda} \Gamma(\lambda) \int_X \frac{1}{W(x; t_\ell)^\lambda} ,
\]

27
where the integration is taken over any contour about one or more zeros of \( W \). Differentiating, one obtains:

\[
\frac{\partial^2}{\partial t_i \partial t_j} u_0^{(\lambda-1)} = (-1)^{(\lambda+1)} \Gamma(\lambda + 1) \int_X \frac{(\partial_i \partial_j W)}{W(x; t_\ell) \lambda + 1} + (-1)^{(\lambda+2)} \Gamma(\lambda + 2) \int_X \frac{(\partial_i W)(\partial_j W)}{W(x; t_\ell) \lambda + 2}.
\]

One must now rewrite the second numerator as follows:

\[
(\partial_i W)(\partial_j W) = C_{ij}^k (\partial_k W) + p_{ij} \partial_x W. 
\] (6.4)

Integrating by parts, the connection term is given by:

\[
(-1)^{(\lambda+1)} \Gamma(\lambda + 1) \int_X \frac{(\partial_i \partial_j W) - \partial_x p_{ij}}{W(x; t_\ell) \lambda + 1}. 
\] (6.5)

Since \( \partial_j W \) is a polynomial of degree at most \( n \), it follows that \( p_{ij} \) has degree at most \( n - 1 \), and \( \partial_i \partial_j W \) has degree at most \( n - 2 \). Consequently the numerator in (6.5) is of degree at most \( n - 2 \), and so can be written in terms of elements of \( \mathcal{R} \). Thus there are no further integrations by parts that need to be done. Flat coordinates are then defined by imposing:

\[
\partial_i \partial_j W = \partial_x p_{ij}.
\]

It is now convenient to consider formal power (and Laurent) series in the variable \( x \), and to introduce the notation \([\_\_\_]_+\) to mean that one should discard all the negative powers of \( x \) that appear in the expansion of the quantity in the square brackets. In particular, \( W^{j^{i+2}} \) is to be thought of as a formal expansion in decreasing powers of \( x \) and starting with \( x^j \).

From (6.4) it follows that the \( p_{ij} \) can be written as:

\[
p_{ij} = \left[ \frac{(\partial_i W)(\partial_j W)}{(\partial_x W)} \right]_+,
\]

and hence the coefficients of \( W \) must satisfy the differential equation

\[
\partial_i \partial_j W = \partial_x \left[ \frac{(\partial_i W)(\partial_j W)}{(\partial_x W)} \right]_+.
\] (6.6)

This equation can be greatly simplified using the scaling property (5.5) of \( W \), which implies:

\[
x \frac{\partial W}{\partial x} + \sum_{j=0}^n (n + 2 - j) t_j \frac{\partial W}{\partial t_j} = (n + 2) W,
\] (6.7)
and
\[ x \partial_x \left( \frac{\partial W}{\partial t_j} \right) + \sum_{i=0}^{n} (n + 2 - j) t_i \frac{\partial^2 W}{\partial t_i \partial t_j} = j \frac{\partial W}{\partial t_j}. \]  

(6.8)

Multiplying both sides of (6.8) by \((n + 2 - j) t_i\) and summing, and then simplifying a little, one obtains:\n\[ \left( \frac{j + 1}{n + 2} \right) \frac{\partial W}{\partial t_j} = \partial_x \left[ \frac{W (\partial_j W)}{(\partial_x W)} \right]_+ . \]

This can be rearranged to give:
\[ \left[ \left( \frac{j + 1}{n + 2} - 1 \right) \frac{\partial W}{\partial t_j} + \frac{W(\partial_j^2 W)}{(\partial_x W)^2} - \frac{W(\partial_x \partial_j W)}{(\partial_x W)} \right]_+ = 0 . \]

If one ignores the \([ \quad ]_+\), then one can arrange this last equation into a collection of logarithmic derivatives and conclude that \(\partial_j W = A \partial_x W^{\frac{j + 1}{n + 2}}\) for some constant \(A\). If one is careful about the \([ \quad ]_+\) then one simply obtains the equation
\[ \partial_j W = A \partial_x \left[ W^{\frac{j + 1}{n + 2}} \right]_+ , \]

(6.9)

where \(A\) is a constant. This is precisely the differential equation whose solution and properties can be connected directly with the KdV hierarchy of the matrix model \( [27] \).

7. \(N = 2\) Supersymmetric quantum integrable models

It is known that many of the \(N = 2\) superconformal models have one (and frequently more than one) perturbation that leads to an \(N = 2\) supersymmetric quantum integrable field theory. There are several methods for seeing this, for example, one can use conformal perturbation theory \([57]\) as Mussardo described in his lectures, but this approach is unsystematic since one often does not know which perturbation to consider, and what spin or form the non-trivial conserved currents will have. On the other hand, there are more sophisticated Toda and free field methods that lead to families of integrable models. There is now a vast literature on this subject (see, for example, \([19, 52]\)). In particular, much is now known about the perturbations of \(N = 2\) superconformal coset models that lead to quantum integrable field theories (see, for example, \([23, 53, 50]\)). The basic rule of thumb is that if there is some form of \(W\)-algebra, and if the \(W\)-algebra generators are the top components of a superfield, then there is usually special \(N = 2\) supersymmetry preserving, relevant perturbation that leads to an \(N = 2\) supersymmetric integrable model.
In the minimal series there are three known perturbations that separately lead to an integrable model. In the notation of section 5, these perturbations are given by \( \phi_1 \), \( \phi_2 \) and \( \phi_n \). For the \( \phi_1 \), or most relevant, perturbation, the states corresponding to the first few non-trivial conserved currents are:

\[
\begin{align*}
G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} J^2_{-1} |0> \\
G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} [J^3_{-1} + \frac{1}{2}(c - 3) J_{-1} L_{-2}] |0> \\
G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} [J^4_{-1} + (c - 3)L_{-2} J^2_{-1} + \frac{2}{5}(c^2 - 3c + 18) L^2_{-2}] |0>
\end{align*}
\]

where \( c = \frac{3n}{n+2} \) is the central charge of the model. Note that all these states are top components of superfields. (They are also \( W \)-generators in the sense that the \( N = 2 \) superconformal minimal models have a \( W \)-algebra embedded in them, but all the \( W \)-generators can be written in terms of \( G^\pm (z), J(z) \) and \( T(z) \).)

Once one has a quantum integrable model the basic objects of physical interest are the soliton spectrum and the corresponding scattering matrix. To determine these is often something of an art-form, and this problem will be discussed in other lectures at this school. My purpose here is to show how, in \( N = 2 \) supersymmetric theories, the Landau-Ginzburg description yields much more complete information about the soliton spectrum. It is then possible to use this information to compute more easily the soliton scattering matrices (see [56,58] and Nemeschansky’s lectures at this school).

### 7.1. Bogomolny bounds for kinks

For the moment I will not assume that the model is integrable, all I will assume is that the model is obtained by some relevant perturbation of an \( N = 2 \) superconformal theory, and that the ground states of the theory are all fully resolved (i.e. mathematically non-degenerate) \( \cite{12} \). As was observed earlier, since the bosonic potential of the theory is given by \( V = |\nabla W|^2 \), the ground states of the theory all have zero-energy and occur at the critical points of \( W \). Let \( x_a^{(\alpha)} \) denote the values of the Landau-Ginzburg fields at these critical points, and suppose that \( x_a^{(\alpha)} \) and \( x_a^{(\beta)} \) are two distinct such points. The solitons are found by seeking the minimum energy “kinks” that interpolate between the two corresponding Landau-Ginzburg vacuum states. That is, I now consider a Lorentzian,

\(\footnote{That is, all the small oscillations about these ground states are massive.}\)
two-dimensional field theory on $\mathbb{R}^2$, with a spatial coordinate, $\sigma$, and time coordinate, $\tau$, and seek classical configurations that have:

$$x_a = x_a^{(\alpha)} \text{ at } \sigma = -\infty \quad \text{and} \quad x_a = x_a^{(\beta)} \text{ at } \sigma = +\infty . \quad (7.1)$$

One has no idea of what the kinetic term is for the model in question since the kinetic term does renormalize. However, quite independent of this kinetic term one can still deduce some properties of the soliton spectrum. First, one can obtain a Bogomolny bound that states that any configuration that satisfies the boundary conditions (7.1) must have a mass $m_{\alpha \beta}$ that obeys the bound

$$m_{\alpha \beta} \geq |\Delta W| \quad \text{where} \quad \Delta W \equiv W(x_a^{(\beta)}) - W(x_a^{(\alpha)}) . \quad (7.2)$$

Moreover, one has $m = |\Delta W|$ if and only if the soliton is a fundamental soliton in that it is annihilated by two of the four supercharges of the perturbed theory. Furthermore, the classical trajectory, $x_a^{(\alpha)}(\sigma)$, of a fundamental soliton at rest, maps to a straight line in the complex $W$-plane, i.e. the complex phase of $W(x_a(\sigma))$ is fixed for all values of $\sigma$. A semi-classical proof of these statements can be found in [23], and they are an elementary generalization of arguments given in [60]. Here I will outline the exact quantum proof of the Bogolmolny bound [54,59].

One can show that in the presence of the perturbation (3.13), the supercurrents receive corrections so that the corresponding conservation laws become:

$$\begin{align*}
\partial_z G^+(z, \bar{z}) &= \sum_i t_i \left(1 - q_i \right) \partial_z (\bar{G}_-^+ \phi_i)(z, \bar{z}) \\
\partial_z G^-(z, \bar{z}) &= \sum_i t_i \left(1 - q_i \right) \partial_z (\bar{G}_-^+ \bar{\phi}_i)(z, \bar{z}) \\
\partial_z \bar{G}^+(z, \bar{z}) &= \sum_i t_i \left(1 - q_i \right) \partial_{\bar{z}} (G^-_+ \phi_i)(z, \bar{z}) \\
\partial_z \bar{G}^-(z, \bar{z}) &= \sum_i t_i \left(1 - q_i \right) \partial_{\bar{z}} (G^-_+ \bar{\phi}_i)(z, \bar{z}) \quad (7.3)
\end{align*}$$

This can be established using perturbation theory, and it can also probably be established using the general properties of $N = 2$ supersymmetry. The usual arguments of conformal perturbation theory (see Mussardo’s lectures or see [57]) are only easily implemented to first order in the perturbation. For many purposes this is usually enough, but it is insufficient to establish (7.3) in all generality at all points in $t$-parameter space. However, one can
generalize the arguments of [57] to a general point in \( t \)-parameter space, and having done this, first order perturbation theory is sufficient. The desired result then follows from this first order perturbation theory and the observation that \( \delta W = \phi_i \delta t_i \) at all points in \( t \)-parameter space.

In the perturbed theory one now has four conserved charges, \( Q_+ \), \( Q_- \), \( \tilde{Q}_+ \) and \( \tilde{Q}_- \):

\[
Q_+ = \int G^+ \, dz - \int \sum_i t_i \, (1 - q_i) \, (\bar{G}^-_{-\frac{1}{2}}) \phi_i(z, \bar{z}) \, d\bar{z} \\
\tilde{Q}_+ = \int \bar{G}^+ \, d\bar{z} - \int \sum_i t_i \, (1 - q_i) \, (G^-_{-\frac{1}{2}} \phi_i)(z, \bar{z}) \, dz ,
\]

and similarly for \( Q_- \) and \( \tilde{Q}_- \). The general structure of such an \( N = 2 \) supersymmetry algebra is:

\[
\{Q_+, Q_-\} = 2P \quad \{\tilde{Q}_+, \tilde{Q}_-\} = 2\tilde{P} \\
\{Q_+, \tilde{Q}_+\} = 2T \quad \{Q_-, \tilde{Q}_-\} = 2\tilde{T} ,
\]

where \( P \) and \( \tilde{P} \) are the two light-cone components of momentum, and \( T \) and \( \tilde{T} \) are two central charges. The other anti-commutators vanish. It is a straightforward computation using (7.4) to see that

\[
T \equiv \frac{1}{2} \{Q_+, \tilde{Q}_+\} = - \sum_i t_i \, (1 - q_i) \int (dz \partial_z + d\bar{z} \partial_{\bar{z}}) \phi_i(z, \bar{z}) \\
= - \sum_i t_i \, (1 - q_i) \left[ \phi_i(\sigma = +\infty) - \phi_i(\sigma = -\infty) \right] \quad (7.6) \\
= \left[ W(x_{a(\beta)}^+) - W(x_{a(\alpha)}^-) \right] \equiv \Delta W .
\]

The last equality follows from the fact that (5.4) and (5.5) imply that:

\[
- \sum_i (1 - q_i) \, t_i \, \phi_i = \sum_i (1 - q_i) \, t_i \, \partial_j W = W - \sum_a \omega_a x_{\alpha} \frac{\partial W}{\partial x_{\alpha}} , \quad (7.7)
\]

and the boundary conditions mean that the partial derivatives \( \frac{\partial W}{\partial x_{\alpha}} \) vanish at \( \sigma = \pm \infty \). A virtually identical argument shows that \( \tilde{T} = (\Delta W)^* \).

Now define \( Q = Q_+ - \frac{\Delta W}{P} \tilde{Q}_- \), where \( \tilde{P} \) is the momentum component of the soliton in question. Observe that the adjoint of \( Q \) is given by: \( Q^\dagger = Q_- - \frac{(\Delta W)^*}{P} \tilde{Q}_+ \). Using (7.5) and (7.6) in the inequality \( \{Q, Q^\dagger\} \geq 0 \) one then recovers (7.2), but now it has been estalished at the quantum level. The other thing to note is that the bound is saturated if and only if \( Q \) and \( Q^\dagger \) annihilate the soliton, that is, if and only if the soliton is fundamental (i.e. chiral).
7.2. Examples

I now consider what the foregoing tells us about the spectrum of two of the quantum integrable models obtained from the $N = 2$ superconformal minimal series. In section 5, I showed that the least relevant perturbation, $\phi_n$, gives rise to a superpotential that is the Chebyshev polynomial (5.16). All the critical points lie on the real $x$ axis, and between the $j^{th}$ and $(j + 1)^{th}$ critical points one has

$$\Delta W = (-1)^{j+1} 4 t^{\frac{j}{2} \left| n+2 \right|}.$$  

Consequently the fundamental solitons must run between consecutive ground states on the real $x$-axis, and all these solitons must have the same mass.

For the most relevant, $\phi_1$, perturbation, the superpotential is given by (5.15). The ground states lie at the vertices of a regular $(n+1)$-gon centered at the origin of the complex $x$-plane, that is, at $x^{(j)} = e^{\frac{2\pi i j}{n+1}} t^{\frac{1}{n+1}}$, for $j = 0, 1, \ldots n$. Let a type $p$ soliton be one that runs between the $j^{th}$ and $(j + p)^{th}$ ground states for any value of $j$. That is, a type $p$ soliton subtends $p$ sides of the polygon. All type $p$ solitons have the same mass, and elementary high-school geometry shows that:

$$\frac{m_p}{m_1} = \frac{\sin \left( \frac{\pi p}{n+1} \right)}{\sin \left( \frac{\pi}{n+1} \right)}. \quad (7.8)$$

These mass ratios are precisely the mass ratios that one finds in an $A_n$-Toda theory, and so one should expect a connection with such theories. There are indeed such connections, and these are discussed in considerable detail in [53,54,55,58]. Lest you be left with the impression that the foregoing integrable model is the usual Toda model, I shall point out some differences: First, each type $p$ soliton is actually a supermultiplet of two solitons of equal mass, and secondly, there is at least one such a supermultiplet starting or finishing at each ground state.

Thus, the fact that we have a quantum exact Landau-Ginzburg potential gives us all of the soliton mass ratios. In addition to this, the geometry of the ground states also gives further information. For example, if one scatters a type $p$ soliton running from the $j^{th}$ ground state to the $(j+p)^{th}$ ground state against a type $q$ soliton running from the $(j+p)^{th}$ ground state to the $(j+p+q)^{th}$ ground state, then there should be a resonance to make a type $p+q$ soliton running from the $j^{th}$ ground state to the $(j+p+q)^{th}$ ground state. The resonant momentum can be determined from the exact knowledge of the masses. It turns out that in more complicated quantum integrable models, these geometric constraints are sufficient to determine all of the charges of all of the fundamental solitons under all of the conserved quantities of the theory [54,59].
8. Conclusions and Apologia

In these lectures I have not given any explanation of why certain perturbations of $N = 2$ superconformal models lead to quantum integrable theories. I have taken such a course, not only because of limitations in time and energy, but also because the analysis of such issues is closely parallel to that for the non-supersymmetric, and for the $N = 1$ supersymmetric, field theories. The basic ideas behind this are therefore covered in Mussardo’s lectures. I have instead chosen to stress precisely the subjects that are special to the study of the $N = 2$ supersymmetric theories, namely the exact quantum information that can be obtained from the chiral ring and an effective Landau-Ginzburg potential. Even within this restricted purview, I have omitted several very interesting aspects of the subject. Some of these omissions will be taken care of by Nemeschansky and Vafa. I have also probably made egregious errors in referencing, for which I apologize. What I hope to have accomplished is to convince the reader that $N = 2$ supersymmetric field theories in two dimensions exhibit a beautiful interplay between classical and quantum structures, and that it is this aspect of the subject that has given rise to its remarkable vitality.

Acknowledgements

I am extremely grateful to my collaborators: M. Bershadsky, P. Fendley, A. LeClair, W. Lerche, S. Mathur, Z. Maassarani, D. Nemeschansky, D.-J. Smit, C. Vafa and E. Verlinde, without whom this work would not have been possible. I would also like to thank the ICTP in Trieste for its hospitality and the opportunity to organize the material presented here into a, hopefully, coherent set of lecture notes.

Dedication

Brian Warr was a young English high-energy physicist who graduated from Caltech in 1986. He worked as a post-doctoral fellow at the University of Texas in Austin, and also at SLAC. In the early 1980’s he contracted AIDS, and he succumbed to it in 1992. While Brian may not be the first physicist to die of AIDS, he was certainly my first close friend to have been lost to this disease. Brian was an incandescent character who delighted in all forms of disputation, and even during the years that he had AIDS, he radiated life. The planet is a poorer place for his absence.
References

[1] N.P. Warner, “Lectures on N=2 superconformal theories and singularity theory”, in “Superstrings ’89,” proceedings of the Trieste Spring School, 3–14 April 1989. Editors: M. Green, R. Iengo, S. Randjbar-Daemi, E. Sezgin and A. Strominger. World Scientific (1990).

[2] E. Martinec, Phys. Lett. 217B (1989) 431.

[3] C. Vafa and N.P. Warner, Phys. Lett. 218B (1989) 51.

[4] E. Martinec, “Criticality, catastrophes and compactifications,” V.G. Knizhnik memorial volume, L. Brink et al. (editors): Physics and mathematics of strings.

[5] W. Lerche, C. Vafa and N.P. Warner, Nucl. Phys. B324 (1989) 427.

[6] D. Gepner, Phys. Lett. 222B (1989) 207.

[7] P. Howe and P. West, Phys. Lett. 223B (1989) 377.

[8] B. Greene, C. Vafa and N.P. Warner, Nucl. Phys. B324 (1989) 371.

[9] S. Cecotti, L. Girardello and A. Pasquinucci, Nucl. Phys. B328 (1989) 701.

[10] W. Boucher, D. Friedan and A. Kent, Phys. Lett. 172B (1986) 316.

[11] Y. Kazama and H. Suzuki, Phys. Lett. 216B (1989) 112; Nucl. Phys. B321 (1989) 232.

[12] K. Ito, Phys. Lett. 259B (1991) 73.

[13] D. Nemeschansky and S. Yankielowicz, “N = 2 W-algebras, Kazama-Suzuki Models and Drinfeld-Sokolov Reduction,” USC-preprint USC-007-91 (1991)

[14] L.J. Romans, Nucl. Phys. B369 (1992) 403.

[15] M. Bershadsky, W. Lerche, D. Nemeschansky and N.P. Warner “Extended N = 2 Superconformal Structure of Gravity and W-Gravity Coupled to Matter,” Caltech preprint CALT-68-1832, CERN preprint CERN-TH.6694/92, Harvard preprint HUTP-A061/92, USC preprint USC-92/021.

[16] L. Dixon, Some World-Sheet Properties of Superstring Compactifications, on Orbifolds and Otherwise, Lectures given at the 1987 ICTP Summer Workshop in High Energy Physics and Cosmology, Trieste, Italy, Jun 29 – Aug 7, 1987, in Superstrings, Unified Theories and Cosmology 1987, G. Furlan et al. editors, World Scientific, (1988).

[17] “Essays on Mirror Symmetry,” edited by S.-T. Yau.

[18] S. Cecotti, L. Girardello and A. Pasquinucci, Int. J. Mod. Phys. A6 (1991) 2427.

[19] V.I. Arnold, Singularity Theory, London Mathematical Lecture Notes Series: 53, Cambridge University Press (1981); V.I. Arnold, S.M. Gusein-Zade and A.N. Varchenko, Singularities of Differentiable Maps, volumes 1 and 2, Birkhäuser (1985).

[20] V.I. Arnold, Russian Mathematical Surveys, 28 (1973) 19; 29 (1974) 11; 30 (1975) 1.

[21] M.B. Green, another gratuitous reference.

[22] E. Witten and D. Olive, Phys. Lett. 78 (1978) 97.

[23] P. Fendley, S. Mathur, C. Vafa and N.P. Warner, Phys. Lett. 243B (1990) 257.
[24] Z. Maassarani, D. Nemeschansky and N.P. Warner, “Lattice Analogues of $N = 2$ Superconformal Models via Quantum Group Truncation,” USC preprint USC-92/007, to appear in Nucl. Phys. B.

[25] E. Witten, Commun. Math. Phys. 117 (1988) 353; Commun. Math. Phys. 118 (1988) 411; Nucl. Phys. B340 (1990) 281.

[26] T. Eguchi and S.-K. Yang, Mod. Phys. Lett. A4 (1990) 1693.

[27] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B352 (1991) 59.

[28] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B241 (1984) 333.

[29] E. Witten, Commun. Math. Phys. 121 (1989) 351.

[30] L.J. Dixon, V. Kaplunovsky and J. Louis, Nucl. Phys. B329 (1990) 27.

[31] V. Periwal and A. Strominger, Phys. Lett. 235B (1990) 261; A. Strominger, Commun. Math. Phys. 133 (1990) 163.

[32] S. Cecotti and C. Vafa, Nucl. Phys. B367 (1991) 359; Phys. Rev. Lett. 68 (1992) 903; Mod. Phys. Lett. A7 (1992) 1715.

[33] A. Lossev, “Descendants constructed from matter field and K. Saito higher residue pairing in Landau-Ginzburg theories coupled to topological gravity,” preprint TPI-MINN-92-40-T.

[34] K. Li, Nucl. Phys. B354 (1991) 711.

[35] E. Verlinde and H. Verlinde, Nucl. Phys. B348 (1991) 457.

[36] R. Dijkgraaf, E. Verlinde and H. Verlinde, “Notes on Topological String Theory and 2-D Quantum Gravity,” Lectures given at Spring School on Strings and Quantum Gravity, Trieste, Italy, Apr 24 – May 2, 1990 and at Cargese Workshop on Random Surfaces, Quantum Gravity and Strings, Cargese, France, May 28 – June 1, 1990.

[37] B.A. Dubrovin, Commun. Math. Phys. 145 (1992) 195; “Integrable systems and a classification of two-dimensional topological field theories,” preprint SISSA-162-92-FM (1992), bulletin board hep-th@xxx.lanl.gov - 9209040; Nucl. Phys. B379 (1992) 627.

[38] I. Krichever, Commun. Math. Phys. 143 (1992) 415; “The tau function of the universal Whitham hierarchy, matrix models and topological field theories,” preprint LPTENS-92-18 (1992).

[39] E. Verlinde and N.P. Warner, Phys. Lett. 269B (1991) 96.

[40] Z. Maassarani, Phys. Lett. 273B (1991) 457.

[41] C. Vafa, Mod. Phys. Lett. A6 (1991) 337.

[42] P. Griffiths, Ann. Math. 90 (1969) 460.

[43] K. Saito, Publ. RIMS, Kyoto Univ., 19 (1983) 1231.

[44] M. Noumi, Tokyo J. Math 7 (1984) 1.

[45] B. Blok and A. Varchenko, Int. J. Mod. Phys. A7 (1992) 1467.

[46] A.C. Cadavid and S. Ferrara, Phys. Lett. 267B (1991) 193.

[47] W. Lerche, D.-J. Smit and N.P. Warner, Nucl. Phys. B372 (1992) 87.
[48] D. Morrison, “Picard-Fuchs equations and mirror maps for hypersurfaces”, Duke preprint DUK-M-91-14 (1991); hep-th@xxx.lanl.gov - 9111025.
[49] A. Bilal and J.-L. Gervais, Phys. Lett. 206B (1988) 412; Nucl. Phys. B314 (1989) 646; Nucl. Phys. B318 (1989) 579; Nucl. Phys. B326 (1989) 222.
[50] A. Bilal, Nucl. Phys. B330 (1990) 399; Int. J. Mod. Phys. A5 (1990) 1881.
[51] T. Hollowood and P. Mansfield, Phys. Lett. 226B (1989) 73; Nucl. Phys. B330 (1990) 720.
[52] T. Eguchi and S.-K. Yang, Phys. Lett. 224B (1989) 373; Phys. Lett. 235B (1990) 282.
[53] P. Fendley, W. Lerche, S.D. Mathur and N.P. Warner, Nucl. Phys. B348 (1991) 66.
[54] W. Lerche and N.P. Warner, Nucl. Phys. B358 (1991) 571.
[55] D. Nemeschansky and N.P. Warner, Nucl. Phys. B380 (1992) 241.
[56] P. Fendley and K. Intriligator, Nucl. Phys. B372 (1992) 533; Nucl. Phys. B380 (1992) 265.
[57] A.B. Zamolodchikov, JETP Letters 46 (1987) 161; “Integrable field theory from conformal field theory,” in Proceedings of the Taniguchi symposium (Kyoto 1989), to appear in Adv. Studies in Pure Math; R.A.L. preprint 89-001; Int. J. Mod. Phys. A4 (1989) 4235.
[58] A. LeClair, D. Nemeschansky and N.P. Warner, USC preprint USC-92/010, Cornell preprint CLNS 92/1148, to appear in Nucl. Phys. B.
[59] W. Lerche and N.P. Warner, “Solitons in integrable, N = 2 supersymmetric Landau-Ginzburg models,” in Strings and Symmetries, 1991, editors: N. Berkovits, H. Itoyama et. al., World Scientific, 1992.
[60] D. Olive and E. Witten. Phys. Lett. 78 (1978) 97.