Multiple reductions, foliations and the dynamics of cluster maps

Inês Cruz,* Helena Mena-Matos* and M. Esmeralda Sousa-Dias†

February 13, 2018

Abstract

Reduction of cluster maps via presymplectic and Poisson structures is described in terms of the canonical foliations defined by these structures. In the case where multiple reductions coexist, we establish conditions on the underlying presymplectic and Poisson structures that allow for an interplay between the respective foliations. It is also shown how this interplay may be explored to simplify the analysis and obtain an effective geometric description of the dynamics of the original (not reduced) map. Examples illustrating several features of this approach are presented.

MSC 2010: 53D17, 53D05, 53C12 (Primary); 39A20, 13F60 (Secondary).

Keywords: presymplectic manifolds, Poisson manifolds, foliations, cluster maps.

1 Introduction

Cluster maps are birational maps defined via mutations, which are the main operations in the theory of cluster algebras [8], [23]. More precisely, cluster maps are maps associated to quivers (oriented graphs) satisfying a mutation-periodicity property, which give rise to recurrences with the Laurent property. The study of these maps has been the subject of some recent works and can be found for instance in [11], [10], [5] and [6].

Presymplectic and Poisson structures were introduced in the theory of cluster algebras in [14] and [13] (see also the monograph [15] and references therein) and, among other applications, they were used to reduce cluster maps to symplectic maps and to study their integrability ([9], [10] and [11]). The presymplectic and Poisson structures considered in that theory are meant to deal with mutations and are of a particular type. They are known, in cluster algebra theory,
as log-canonical structures, since they are constant in logarithmic coordinates. Only this type of presymplectic and Poisson structures will be considered in the present work.

The most effective study of cluster maps is precisely the one that makes use of log-canonical structures, and is achieved through reduction to spaces of lower dimension. By reduction of a given map \( \varphi \) we mean the existence of a map \( \psi \) and of a submersion \( \pi \) onto a lower dimensional space, such that \( \pi \circ \varphi = \psi \circ \pi \). The map \( \psi \) will be called a reduced map of \( \varphi \) and the pair \( (\psi, \pi) \) will be called a reduced system.

Reduction of cluster maps via presymplectic and Poisson structures was obtained by Fordy and Hone in [10] for maps associated to mutation-periodic quivers of period 1 and by Cruz and Sousa-Dias [4] for mutation-periodic quivers with arbitrary period. Presymplectic reduction is always possible as long as the skew-symmetric matrix \( B \) defining the quiver is singular and produces a reduced map defined on a space whose dimension is equal to the rank of \( B \).

On the contrary, reduction via Poisson structures may fail to exist due to the lack of a nontrivial Poisson structure for which the cluster map is a Poisson map. Moreover, it may happen that there are different Poisson structures leading to reductions of the same cluster map to spaces of different dimensions (and equal to the corank of the skew-symmetric matrices defining the respective Poisson structures).

In this work we explore the coexistence of multiple reductions (presymplectic and Poisson) and its consequences to the dynamics of a cluster map. Each reduction process gives rise to two kind of objects: (a) a foliation of the domain of the cluster map defined by the fibres of a submersion \( \pi \); (b) a reduced system \( (\psi, \pi) \). When multiple reductions exist, we show that there are conditions on the underlying presymplectic and Poisson structures which guarantee that they define a flag of foliations. The consequences of the existence of such flag of foliations to the dynamics of the cluster map is further explored, showing how to use it to simplify the study of the map and obtain a better geometric description of its dynamics.

Reduction of cluster maps via presymplectic forms has been widely applied and often leads to integrable (reduced) maps [11], [10]. It has also played an important role in the study of the original map (see [16] and [17]), more precisely, in obtaining the general solution for the associated recurrence. The existence of multiple reductions via Poisson structures has not yet witnessed the same level of development in the study of cluster maps, the study performed here is one of the possible directions in the use of (compatible) Poisson structures in the dynamics of cluster maps.

The structure of the paper is as follows. Section 2 is devoted to background material on cluster maps. In Section 3 we consider log-canonical presymplectic and Poisson structures and describe, in terms of submersions, the null foliation and the symplectic foliation determined by these structures. We also give necessary and sufficient conditions for a null foliation to be a simple subfoliation of a symplectic foliation (and vice-versa) and for a symplectic foliation to be a simple subfoliation of another symplectic foliation. Section 4 explores the
consequences to the dynamics of cluster maps of the existence of multiple reductions, in particular how the dynamics of the multiple reduced systems allows us to draw conclusions on the dynamics of the original map. The last section is devoted to examples illustrating the main results of the paper: the Somos-5 recurrence and a particular instance of the Somos-7 recurrence which fits into the cluster algebra setting.

2 Preliminaries on cluster maps

The notion of mutation-periodic quiver was introduced in [12] and gives rise to the definition of cluster map as follows. A quiver is an oriented graph with $N$ nodes and with (possibly multiple) arrows between the nodes. If the quiver has no loops nor 2-cycles, it is represented by an $N \times N$ skew-symmetric matrix $B = [b_{ij}]$ whose positive entries $b_{ij}$ denote the number of arrows from node $i$ to node $j$. These are the only quivers we consider and they will be denoted by $Q_B$.

To each node $i$ of a quiver $Q_B$ one associates a variable $x_i$, called cluster variable. The $N$-tuple $x = (x_1, \ldots, x_N)$ is known as the initial cluster and the pair $(B, x)$ is called the initial seed.

The mutation $\mu_k$ at the node $k$ of a quiver, is an operation that transforms a seed $(B, x)$ into the seed $(B', x')$ as follows:

- $\mu_k(B) = B' = [b'_{ij}]$ with
  \[
  b'_{ij} = \begin{cases} 
  -b_{ij}, & \text{if } (k - i)(j - k) = 0 \\
  b_{ij} + \frac{1}{2}(b_{ik}|b_{kj} + b_{ik}|b_{kj}|), & \text{otherwise.} 
  \end{cases}
  \]

- $\mu_k(x_1, \ldots, x_N) = x' = (x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_N)$, with
  \[
  x'_k = \prod_{j : b_{kj} \geq 0} x_j^{b_{kj}} \prod_{j : b_{kj} \leq 0} x_j^{-b_{kj}} x_k.
  \]

A quiver $Q_B$ is said to be mutation-periodic if there exists a positive integer $m$ such that
\[
\mu_m \circ \cdots \circ \mu_1(B) = \sigma^{-m} B \sigma^m.
\]
where $\sigma$ is the permutation $\sigma : (1, 2, \ldots, N) \mapsto (2, 3, \ldots, N, 1)$. The mutation-period of $Q_B$ is the smallest integer $m$ for which the above identity holds.

A mutation-periodic quiver of period $m$ gives rise to a system of $m$ recurrence relations of order $N$ whose solutions correspond to the orbits (under iteration) of the following map:
\[
\varphi(x) = \sigma^m \circ \mu_m \circ \cdots \circ \mu_1(x).
\]
This map is called the cluster map associated to the quiver.
Example 1. If \( r \) and \( s \) are non-negative integers, the matrix
\[
B = \begin{bmatrix}
0 & r & -1 & -1 & s \\
-r & 0 & r + s & r - 1 & -1 \\
1 & -r - s & 0 & r + s & -1 \\
1 & 1 & -r - s & 0 & r \\
-s & 1 & 1 & -r & 0
\end{bmatrix}
\]
defines a 1-periodic quiver if \( r = s \) and a 2-periodic quiver otherwise. The cluster map is
\[
\varphi(x_1, \ldots, x_5) = \left( x_2, x_3, x_4, x_5, \frac{x_2^r x_5 + x_3 x_4}{x_1} \right)
\tag{1}
\]
in the 1-periodic case (i.e. \( r = s \)) and
\[
\varphi(x_1, x_2, x_3, x_4, x_5) = \left( x_3, x_4, x_5, \frac{x_2^r x_5 + x_4 x_3}{x_1}, \frac{x_3^r (x_2^r x_5 + x_3 x_4)^r + x_1 x_4 x_5}{x_1^r x_2} \right)
\tag{2}
\]
in the 2-periodic case.

The cluster map (1) is associated to the recurrence relation of order 5
\[
x_{n+5} x_n = x_{n+1}^r x_{n+4}^r + x_{n+2} x_{n+3},
\]
which, when \( r = 1 \), is the well-known Somos-5 recurrence relation. On the other hand, the cluster map (2) is associated to the following system of two recurrence relations of order 5
\[
\begin{cases}
x_{2n+4} x_{2n-1} = x_{2n}^r x_{2n+3}^r + x_{2n+1} x_{2n+2} \\
x_{2n+5} x_{2n} = x_{2n+1}^r x_{2n+4}^r + x_{2n+2} x_{2n+3}.
\end{cases}
\]

We refer to [12] and [4] for details.

Because each mutation is an involution, every cluster map is a birational map, that is, rational with rational inverse. As our interest in cluster maps lies on their dynamics, we will consider their domain of definition to be \( \mathbb{R}^N_+ \) (the set of points where all coordinates are strictly positive) which guarantees that any of its iterates is well defined.

Reduction of a cluster map \( \varphi \) by means of a presymplectic structure was proved in [10] for 1-periodic quivers (defined in \( \mathbb{C}^N \)) and in [4] for quivers with arbitrary period. It relies on the existence of a presymplectic structure which is invariant under \( \varphi \), as proved in the same references. In fact, the log-canonical presymplectic form
\[
\omega = \sum_{1 \leq i < j \leq N} b_{ij} x_i x_j \, dx_i \wedge dx_j,
\tag{3}
\]
where \( B \) is the matrix defining the quiver associated to \( \varphi \), is invariant under \( \varphi \).

The existence of a submersion \( \hat{\pi} : \mathbb{R}^N_+ \to \mathbb{R}^{2k}_+ \), with \( 2k = \text{rank} B < N \), and of a (symplectic) reduced map \( \hat{\varphi} : \mathbb{R}^{2k}_+ \to \mathbb{R}^{2k}_+ \) such that \( \hat{\pi} \circ \varphi = \hat{\varphi} \circ \hat{\pi} \), follows from Darboux's theorem [19] applied to the presymplectic form (3).
In what concerns the reduction of a cluster map $\varphi$ by means of a log-canonical Poisson structure $P$, this is possible if there exists such a (nontrivial) Poisson structure which is invariant under $\varphi$. More precisely, if there exists a Poisson structure of the form
\[ P = \sum_{1 \leq i < j \leq N} c_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \] (4)

satisfying $\varphi_* P = P$, where $C = [c_{ij}]$ is an integer skew-symmetric matrix with nontrivial kernel (here $\varphi_*$ denotes the pushforward by $\varphi$). Examples where such reduction is possible can be found for example in [10] and [4].

Again, reduction by a Poisson structure translates into the existence of a submersion $\tilde{\pi} : \mathbb{R}^N_+ \to \mathbb{R}^s_+$, where $s = \dim \ker C$, and of a (reduced) map $\tilde{\varphi} : \mathbb{R}^s_+ \to \mathbb{R}^s_+$ such that $\tilde{\pi} \circ \varphi = \tilde{\varphi} \circ \tilde{\pi}$ (see [4, Theorem 5.1]).

Throughout the next sections, $B = [b_{ij}]$ will denote an integer skew-symmetric $N \times N$ matrix representing a mutation-periodic quiver $Q_B$ with period $m$ and $\varphi = \sigma^m \circ \mu_m \circ \cdots \circ \mu_1$ the associated cluster map defined on $\mathbb{R}^N_+$. Both the presymplectic form $\omega$ in (3) and Poisson structures of the form (4) will be restricted to $\mathbb{R}^N_+$. Also, unless otherwise stated, we will assume that $\rank B = 2k < N$ and $\dim \ker C = s > 0$.

3 Reductions of cluster maps and associated foliations

In this section we characterise the foliations associated to reductions of a cluster map $\varphi$ via the presymplectic form $\omega$ in (3) and via Poisson structures $P$ of the form (4) which are invariant under $\varphi$.

Presymplectic manifolds and Poisson manifolds are natural generalizations of symplectic manifolds. Each of these manifolds has its natural foliation, the null foliation in the case of a presymplectic manifold and the symplectic foliation in the case of a Poisson manifold. Although these foliations are usually defined using integrable distributions (see for example [19], [2] and references therein) and are in general singular foliations (that is, distinct leaves can have different dimensions) we will show that, in our setting, each of these foliations is not only regular but also given by a surjective submersion $\pi$ with connected fibres. In particular, this means that each leaf $L_\alpha$ of the foliation is a fibre of $\pi$: $L_\alpha = \pi^{-1}(\alpha)$. Following the terminology in [20] we will call these foliations strictly simple foliations.

3.1 The null foliation and reduction via a presymplectic form

The next proposition shows that the null foliation determined by the presymplectic form $\omega$ in (3) is a strictly simple foliation. Its leaves are known as null leaves.
Proposition 1. Let $\omega$ be the presymplectic form defined by the matrix $B$ as in (3) with rank $B = 2k$. Then, there exists a surjective submersion $\hat{\pi} : \mathbb{R}^N_+ \to \mathbb{R}^{2k}_+$ with connected fibres $N_\alpha = \hat{\pi}^{-1}(\alpha)$ which determines the null foliation $\mathcal{F}_\omega$ of $\mathbb{R}^N_+$, that is
\[ \mathcal{F}_\omega = \bigcup_\alpha N_\alpha, \]
with $\omega |_{N_\alpha} = 0$. Moreover the leaves $N_\alpha$ are algebraic varieties.

Proof. Let $\Phi : \mathbb{R}^N_+ \to \mathbb{R}^N$ be the diffeomorphism given by
\[ \Phi(x_1, \ldots, x_N) = (\ln x_1, \ldots, \ln x_N). \] (5)

The presymplectic form $\omega' = (\Phi^{-1})^* \omega$ is the constant presymplectic form on $\mathbb{R}^N$, given by
\[ \omega' = \sum_{1 \leq i < j \leq N} b_{ij} dv_i \wedge dv_j. \]

By Élie Cartan’s theorem (see [19]), there exist $2k$ independent linear functions on $\mathbb{R}^N$ of the form $f_i(v) = u_i \cdot v$ with $u_i \in \text{Im} B$, such that $\omega'$ is written as
\[ \omega' = \sum_{m=1}^{k} df_{2m-1} \wedge df_{2m}. \]

Now consider the foliation of $\mathbb{R}^N$ defined by the submersion $\tilde{\Pi} : \mathbb{R}^N_+ \to \mathbb{R}^{2k}_+$ given by
\[ \tilde{\Pi}(v) = (f_1(v), \ldots, f_{2k}(v)). \]

The fibre of $\tilde{\Pi}$ through $v_0$ is $N' = \{ v \in \mathbb{R}^N_+ : \tilde{\Pi}(v) = \tilde{\Pi}(v_0) \}$, which coincides with the affine subspace $v_0 + (\text{Im} B)^\perp$.

In particular: (a) each fibre of $\tilde{\Pi}$ is connected and therefore constitutes a leaf of the foliation; (b) this foliation depends only on $\text{Im} B$ and not on the particular choice of the linear functions $f_1, \ldots, f_{2k}$.

In terms of the functions $y_i(x) = x^{u_i}$, the expression of $\omega$ is given by:
\[ \omega = \sum_{m=1}^{k} \frac{dy_{2m-1}}{y_{2m-1}} \wedge \frac{dy_{2m}}{y_{2m}}. \] (6)

Moreover, each fibre $N_\alpha$ of the surjective submersion $\hat{\pi} : \mathbb{R}^N_+ \to \mathbb{R}^{2k}_+$,
\[ \hat{\pi} : x \mapsto (y_1(x), \ldots, y_{2k}(x)) \] (7)

is connected, since it is $\Phi$-diffeomorphic to a connected fibre $N'$ of $\tilde{\Pi}$.

Note that $N_\alpha = \{ x \in \mathbb{R}^N_+ : \hat{\pi}(x) = \alpha \}$ and so $dy_i$ vanishes on $TN_\alpha$. Hence the identity (6) leads to the conclusion that $\omega$ vanishes on $N_\alpha$.

Finally, as $B$ is an integer matrix then each $f_i$ can be chosen to have rational coefficients, which implies that each $y_i$ can be chosen to be a Laurent monomial. Thus, each fibre $N_\alpha$ is an algebraic variety. \qed
As the presymplectic form $\omega$ in (3) is invariant under the cluster map $\varphi$ defined by $B$ (see [4, Theorem 3.1]), it turns out that the submersion $\tilde{\pi}$ in (7) defining the foliation $F^\omega$ reduces the cluster map $\varphi$ to a symplectic map $\hat{\varphi}$ on $\mathbb{R}^{2k}$. This means that $\hat{\pi} \circ \varphi = \hat{\varphi} \circ \tilde{\pi}$ where $\hat{\varphi}$ is a symplectic map preserving the symplectic form in (4).

### 3.2 The symplectic foliation and reduction via a Poisson structure

We now consider a Poisson structure $P$ as in (4) with $\text{ker } C \neq \{0\}$. The following proposition shows that the symplectic foliation of the regular Poisson manifold $(\mathbb{R}^N_+, P)$ is also strictly simple. Its leaves are known as *symplectic leaves*.

**Proposition 2.** Let $P$ be the Poisson structure on $\mathbb{R}^N_+$ defined by the matrix $C = [c_{ij}]$ with $s$-dimensional kernel as in (4). Then, there exists a surjective submersion $\tilde{\pi} : \mathbb{R}^N_+ \to \mathbb{R}^s_+$ with connected fibres $S_\beta = \tilde{\pi}^{-1}(\beta)$, which determines the symplectic foliation $F^P$ of $\mathbb{R}^N_+$, that is

$$F^P = \bigsqcup_\beta S_\beta,$$

where $S_\beta$ is a symplectic manifold.

Moreover the leaves $S_\beta$ are algebraic varieties.

**Proof.** Consider again the diffeomorphism $\Phi : \mathbb{R}^N_+ \to \mathbb{R}^N$ given by (5). The pushforward of $P$ by $\Phi$ is the Poisson structure given by

$$P' = \Phi_* P = \sum_{1 \leq i < j \leq N} c_{ij} \frac{\partial}{\partial v_i} \wedge \frac{\partial}{\partial v_j},$$

which is a constant Poisson structure on the vector space $\mathbb{R}^N$. The symplectic leaves of such Poisson structure are well known: they are affine subspaces which coincide with the common level sets of a maximal set of independent linear Casimirs. Note that a linear function $f(v) = u \cdot v$ is a Casimir of $P'$ if and only if $u \in \text{ker } C$.

Therefore the symplectic leaves of $P'$ can be defined as the (connected) fibres of the surjective submersion $\tilde{\Pi} : \mathbb{R}^N \to \mathbb{R}^s$,

$$\tilde{\Pi} : v \mapsto (f_1(v), \ldots, f_s(v))$$

where $f_1, \ldots, f_s$ are $s$ independent linear functions of the form $f_i(v) = u_i \cdot v$, with $u_i \in \text{ker } C$. It can easily be checked that the fibre $S'$ of $\tilde{\Pi}$ through $v_0$ can be written as

$$v_0 + (\text{ker } C)^\perp.$$

In particular, this foliation depends only on $\text{ker } C$ and not on the particular choice of the linear functions $f_1, \ldots, f_s$. 

7
The functions \( z_i = \exp(f_i \circ \Phi) \), which have the form

\[
z_i(x) = x^{u_i}, \quad u_i \in \ker C,
\]

form a maximal set of independent Casimirs of the Poisson structure \( P \). The map \( \tilde{\pi} : \mathbb{R}^N_+ \to \mathbb{R}^s_+ \)

\[
\tilde{\pi} : x \mapsto (z_1(x), \ldots, z_s(x)),
\]

is a surjective submersion and each of its fibres,

\[
S_\beta = \{ x \in \mathbb{R}^N_+ : \tilde{\pi}(x) = \beta \},
\]

is connected since it is \( \Phi \)-diffeomorphic to a fibre of \( \tilde{\Pi} \). As each \( S_\beta \) is connected and a common level set of \( s \) independent Casimirs of \( P \), then \( S_\beta \) is a symplectic leaf of \( P \).

Finally, as \( C = [c_{ij}] \) is an integer matrix, then each vector \( u \) in \( \ker C \) can be chosen to have integer components, which implies that the fibres \( S_\beta \) are algebraic varieties.

In general, a Poisson structure of the form \( [4] \) does not allow for reduction of the cluster map \( \varphi \) unless \( P \) is invariant under \( \varphi \), that is, \( \varphi^* P = P \) or equivalently \( \varphi \) is a Poisson map. In such case the submersion \( \tilde{\pi} \) in \( [5] \), defining the foliation \( F^P \), satisfies the conditions described in \( [4] \) Theorem 5.1, Lemma 5.2 \) and so it reduces \( \varphi \) to a map \( \tilde{\varphi} : \mathbb{R}^s_+ \to \mathbb{R}^s_+ \), that is, \( \tilde{\pi} \circ \varphi = \tilde{\varphi} \circ \tilde{\pi} \).

### 3.3 Subfoliations and Multiple Reductions

We now look for conditions under which the symplectic foliation \( F^P \) determined by a Poisson structure \( P \) is a subfoliation of the null foliation \( F^\omega \) defined by a presymplectic form \( \omega \) (or vice-versa). We will also be interested in considering two distinct symplectic foliations, \( F^{P_1}, F^{P_2} \), and on conditions under which \( F^{P_2} \) is a subfoliation of \( F^{P_1} \).

**Definition 1.** Let \( F_1 \) and \( F_2 \) be two simple foliations of a manifold \( M \) given respectively by surjective submersions \( \pi_1 : M \to N_1 \) and \( \pi_2 : M \to N_2 \) with \( \dim N_1 < \dim N_2 < \dim M \).

The foliation \( F_2 \) is said to be a simple subfoliation of \( F_1 \), and will be denoted by \( F_2 \prec F_1 \), if there exists a surjective submersion \( p : N_2 \to N_1 \) such that

\[
p \circ \pi_2 = \pi_1.
\]

Schematically, a simple subfoliation \( F_2 \) of \( F_1 \) is described by the following diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\pi_2} & N_2 & \xrightarrow{p} & N_1 \\
\pi_1 & \downarrow & & \downarrow & \\
\end{array}
\]
Remark 1. It is clear from Definition 1 that if $\mathcal{F}_2 \prec \mathcal{F}_1$ then any leaf of $\mathcal{F}_2$ is contained in a leaf of $\mathcal{F}_1$, that is, $\mathcal{F}_2$ is indeed a subfoliation of $\mathcal{F}_1$.

Proposition 3. Let $\omega$ be the presymplectic form in (3) determined by the matrix $B$. Let $P_1$ and $P_2$ be Poisson structures on $\mathbb{R}^N$ determined by $C_1$ and $C_2$ as in (4). Then:

1. $\mathcal{F}^{P_1} \prec \mathcal{F}^{\omega}$ if and only if $\text{Im} \ B \subset \ker C_i$.
2. $\mathcal{F}^{\omega} \prec \mathcal{F}^{P_1}$ if and only if $\ker C_i \subset \text{Im} \ B$.
3. $\mathcal{F}^{P_2} \prec \mathcal{F}^{P_1}$ if and only if $\ker C_1 \subset \ker C_2$.

Proof. We only prove the first statement since the proofs of the remaining statements are analogous.

The submersions $\widehat{\pi}$ and $\tilde{\pi}_i$ whose fibres define (respectively) the foliations $\mathcal{F}^{\omega}$ and $\mathcal{F}^{P_i}$ were obtained in the proofs of propositions 1 and 2 by means of the diffeomorphism $\Phi$ given by (5). Recall from those proofs that the leaf $N$ of $\mathcal{F}^{\omega}$ through $x_0 \in \mathbb{R}^N$ is $\Phi$-diffeomorphic to the affine subspace of $\mathbb{R}^N$

$$N' = \Phi(x_0) + (\text{Im} \ B)^\perp$$

whereas the leaf $S^i$ of $\mathcal{F}^{P_i}$ through the same point $x_0$ is $\Phi$-diffeomorphic to

$$S^{i'} = \Phi(x_0) + (\ker C_i)^\perp.$$

If $\mathcal{F}^{P_i} \prec \mathcal{F}^{\omega}$ then $S^i \subset N$ which is equivalent to $S^{i'} \subset N'$ and consequently $\text{Im} \ B \subset \ker C_i$.

Conversely, if $\text{Im} \ B \subset \ker C_i$, we can choose the submersions $\widehat{\pi} : \mathbb{R}^N_+ \to \mathbb{R}^{2k}_+$ from (7) and $\tilde{\pi}_i : \mathbb{R}^N_+ \to \mathbb{R}^{s_i}_+$ from (8) in such a way that

$$\tilde{\pi}_i(x) = (\widehat{\pi}(x), z_{2k+1}(x), \ldots, z_{s_i}(x)).$$

Thus it is clear that $\widehat{\pi} = p_i \circ \tilde{\pi}_i$, where $p_i : \mathbb{R}^{s_i}_+ \to \mathbb{R}^{2k}_+$ is the projection onto the first $2k$ coordinates, showing that $\mathcal{F}^{P_i} \prec \mathcal{F}^{\omega}$.

As an immediate consequence of the last proposition and of the characterization of the null foliation $\mathcal{F}^{\omega}$ given in Proposition 1 we have the following corollary.

Corollary 1. Under the conditions of Proposition 3, if $\mathcal{F}^{P} \prec \mathcal{F}^{\omega}$ then $\omega|_S \equiv 0$ for any symplectic leaf $S$ of $\mathcal{F}^{P}$, that is, each symplectic leaf $S$ of $P$ is isotropic with respect to $\omega$.

We conclude this section by remarking that the condition $\text{Im} \ B \subset \ker C$ in the first statement of Proposition 3 is equivalent to $CB = [0]$ and imposes several restrictions on the matrix $C$. For instance, just by dimension counting, it is clear that if $B$ has rank equal to $N - 1$ then the only possible matrix $C$ satisfying this condition is the zero matrix.
On the other hand, for some matrices \( B \), like the one in the following example, there exist nontrivial matrices \( C_1 \) and \( C_2 \) with different ranks, satisfying \( C_i B = [0] \). This case will be explored in the last section, where two distinct Poisson structures are used to gain more insight into the dynamics of a cluster map.

Finally we note that the condition \( CB = [0] \) appears in the context of cluster algebra theory (\cite{15} and \cite{18}) to characterize a Poisson structure (determined by \( C \)) as being compatible with the cluster algebra \( \mathcal{A}(B) \) when \( B \) is singular. For our purpose of reducing a cluster map via a Poisson structure, there is no reason \textit{a priori} to consider only Poisson structures which are compatible with \( \mathcal{A}(B) \). However, the proposition above together with the results in \cite{18} show that the condition of compatibility with \( \mathcal{A}(B) \) appears naturally when the symplectic foliation of the Poisson structure is a subfoliation of \( \mathcal{F}^\omega \).

**Example 2.** Consider the following skew-symmetric matrix \( B \) of rank 2, which defines a 1-periodic quiver (see \cite{12} for the classification of these quivers):

\[
B = \begin{bmatrix}
0 & 1 & 0 & -1 & -1 & 0 & 1 \\
-1 & 0 & 1 & 1 & 0 & -1 & 0 \\
0 & -1 & 0 & 1 & 1 & 0 & -1 \\
1 & -1 & -1 & 0 & 1 & 1 & -1 \\
1 & 0 & -1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & -1 & 0 & 1 \\
-1 & 0 & 1 & 1 & 0 & -1 & 0
\end{bmatrix}.
\]

The following skew-symmetric matrices \( C_1 \) and \( C_2 \) satisfy the condition \( C_i B = [0] \) (only the upper triangular part of the matrices is shown):

\[
C_1 = \begin{bmatrix}
0 & 1 & 1 & 2 & 3 & 3 & 4 \\
0 & 1 & 1 & 2 & 3 & 3 & 4 \\
0 & 1 & 1 & 2 & 3 & 3 & 4 \\
0 & 1 & 1 & 2 & 3 & 3 & 4 \\
0 & 1 & 1 & 2 & 3 & 3 & 4 \\
0 & 1 & 1 & 2 & 3 & 3 & 4 \\
0 & 1 & 1 & 2 & 3 & 3 & 4
\end{bmatrix}, \quad
C_2 = \begin{bmatrix}
0 & 1 & -1 & 0 & 1 & -1 & 0 \\
0 & 1 & -1 & 0 & 1 & -1 & 0 \\
0 & 1 & -1 & 0 & 1 & -1 & 0 \\
0 & 1 & -1 & 0 & 1 & -1 & 0 \\
0 & 1 & -1 & 0 & 1 & -1 & 0 \\
0 & 1 & -1 & 0 & 1 & -1 & 0 \\
0 & 1 & -1 & 0 & 1 & -1 & 0
\end{bmatrix}.
\]

Moreover, it can be checked that rank \( C_1 = 4 \), rank \( C_2 = 2 \) and \( \ker C_1 \subset \ker C_2 \). By Proposition 3 we have

\[
\mathcal{F}^{P_2} \prec \mathcal{F}^{P_1} \prec \mathcal{F}^\omega,
\]

where the symplectic leaves of \( \mathcal{F}^{P_2} \) are 2-dimensional, the symplectic leaves of \( \mathcal{F}^{P_1} \) are 4-dimensional and the null leaves of \( \mathcal{F}^\omega \) are 5-dimensional.

Submersions \( \tilde{\pi}_2, \tilde{\pi}_1 \) and \( \hat{\pi} \) defining the foliations \( \mathcal{F}^{P_2}, \mathcal{F}^{P_1} \) and \( \mathcal{F}^\omega \), respectively, are given by:

\[
\tilde{\pi}_2(x) = (y_1, y_2, y_3, y_4, y_5), \quad \tilde{\pi}_1(x) = (y_1, y_2, y_3), \quad \hat{\pi}(x) = (y_1, y_2),
\]

with \( x = (x_1, x_2, \ldots, x_7) \) and

\[
y_1 = \frac{x_1x_6}{x_3x_4}, \quad y_2 = \frac{x_2x_7}{x_4x_5}, \quad y_3 = \frac{x_1x_7}{x_2}, \quad y_4 = x_1x_2x_3, \quad y_5 = x_2x_3x_4.
\]
4 Multiple reductions and the dynamics of cluster maps

In this section we will focus on the consequences of reduction, via presymplectic or via Poisson structures, to the dynamics of a cluster map $\varphi$. The next proposition describes the dynamical behaviour of a map on the leaves of the foliation associated to a reduction of it.

Recall that a first integral of a map $f$ is a non constant real-valued function $I$ such that $I \circ f = I$. Also, a map $f$ is said to be globally $p$-periodic if its $p$th-iterate is the identity, that is, $f^p = \text{Id}$.

**Proposition 4.** Let $\mathcal{F}$ be a strictly simple foliation of $M$ given by $\pi : M \to N$ and suppose that $(g, \pi)$ is a reduced system of $f : M \to M$. Then

1. the map $f$ sends leaves of $\mathcal{F}$ to leaves of $\mathcal{F}$, more precisely $f(L_\alpha) \subset L_{g(\alpha)}$;
2. the leaf $L_\alpha$ of $\mathcal{F}$ is invariant under the map $f^p$ if and only if $\alpha$ is a $p$-periodic point of $g$, that is, $g^p(\alpha) = \alpha$;
3. $g$ is a globally $p$-periodic map if and only if $\pi$ is a vector-valued first integral of $f^p$, that is, $\pi \circ f^p = \pi$.

**Proof.** All the statements follow from the definition of a reduced system, in particular from the identity

\[ \pi \circ f = g \circ \pi, \tag{10} \]

and from the fact that any leaf $L_\alpha$ of the foliation is given by

\[ L_\alpha = \{ x \in M : \pi(x) = \alpha \}. \]

For the first statement assume that $x$ belongs to $L_\alpha$. Then (10) shows that $\pi(f(x)) = g(\alpha)$, that is, $f(x)$ belongs to $L_{g(\alpha)}$.

Concerning the second and third statements, note that (10) implies

\[ \pi \circ f^n = g^n \circ \pi, \quad n = 1, 2, \ldots \]

and the results follow directly from this identity. \qed

**Proposition 5.** Let $\hat{\varphi}$ and $\tilde{\varphi}$ be reduced systems of the cluster map $\varphi$, arising from reduction via the presymplectic form $\omega$ in (3) and via a $\varphi$-invariant Poisson structure $P$, respectively. Let $\mathcal{F}^\omega$ be the null foliation defined by $\hat{\pi}$ and $\mathcal{F}^P$ the symplectic foliation defined by $\tilde{\pi}$. Denote the null leaves by $N_\alpha$ and the symplectic leaves by $S_\beta$. Then,

1. the cluster map $\varphi$ sends each leaf $N_\alpha$ (resp. $S_\beta$) of $\mathcal{F}^\omega$ (resp. of $\mathcal{F}^P$) to the leaf $N_{\hat{\varphi}(\alpha)}$ (resp. $S_{\tilde{\varphi}(\beta)}$) of the same foliation;
2. a leaf $N_\alpha$ (resp. $S_\beta$) is invariant under $\varphi^{(p)}$ if and only if $\alpha$ (resp. $\beta$) is a $p$-periodic point of $\hat{\varphi}$ (resp. $\tilde{\varphi}$);

3. if $\beta$ is a $p$-periodic point of $\tilde{\varphi}$, the restriction of $\varphi^{(p)}$ to $S_\beta$ is a symplectic map.

Proof. The first two statements are just a rephrasing of items 1 and 2 in Proposition 4. The third statement is a more relevant property of reduction via Poisson structures. Its proof follows from classical theory on Poisson and symplectic manifolds (see for example [19] and [3]) as we now describe.

The symplectic structure on $S_\beta$ is given by the nondegenerate Poisson structure induced by the Poisson structure $P$ on $\mathbb{R}^N_+$. This means that the inclusion $i : S_\beta \rightarrow \mathbb{R}^N_+$ is a Poisson map.

The fact that $\varphi$ is a Poisson map implies that $\varphi^{(p)}$ is also a Poisson map. The composition $\varphi^{(p)} \circ i$, which is precisely the restriction $\varphi^{(p)}|_{S_\beta}$, is therefore a Poisson map. Since the Poisson structure on $S_\beta$ is nondegenerate, then any map preserving this Poisson structure must preserve the associated symplectic structure. In other words $\varphi^{(p)}|_{S_\beta}$ is symplectic.

We end this section by explaining how multiple reductions of the same cluster map may simplify the study of the dynamics of the unreduced map. For instance, suppose we have two reduced systems $(g_1, \pi_1)$ and $(g_2, \pi_2)$ of the same map $f : M \rightarrow M$, such that the associated foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are strictly simple foliations and satisfy $\mathcal{F}_2 \prec \mathcal{F}_1$. In this case, the study of the dynamics of $f$ is effectively simplified, in the sense that we can study (sequentially) the dynamics of the reduced map $g_2$ by using the map $g_1$ and then study the dynamics of $f$ using the map $g_2$. This procedure is made precise in the next proposition and the examples of the following section illustrate it.

**Proposition 6.** Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be strictly simple foliations of $M$ given by the surjective submersions $\pi_1 : M \rightarrow N_1$ and $\pi_2 : M \rightarrow N_2$, respectively. Suppose that $\mathcal{F}_2 \prec \mathcal{F}_1$ and let $p : N_2 \rightarrow N_1$ be a submersion in the conditions of Definition 7.

If $(g_1, \pi_1)$ and $(g_2, \pi_2)$ are reduced systems of $f : M \rightarrow M$, then $(g_1, p)$ is a reduced system of $g_2$.

**Proof.** Because $(g_1, \pi_1)$ is a reduced system of $f$, we have $\pi_1 \circ f = g_1 \circ \pi_1$. As $\mathcal{F}_2 \prec \mathcal{F}_1$, we also have $\pi_1 = p \circ \pi_2$ which leads to

$$p \circ \pi_2 \circ f = g_1 \circ p \circ \pi_2.$$  

As $(g_2, \pi_2)$ is also a reduced system of $f$, we obtain

$$p \circ g_2 \circ \pi_2 = g_1 \circ p \circ \pi_2$$

and the conclusion follows from surjectivity of $\pi_2$. \qed
The procedure described in the last proposition of assigning a third reduced system to two existing ones can be used with great advantage to study the original map \( f \) when we have several foliations (associated to reduced systems) verifying
\[
\mathcal{F}_m \prec \cdots \prec \mathcal{F}_2 \prec \mathcal{F}_1.
\]
Such a set of simple foliations will be called a flag of simple foliations. An example of this type, where we have two symplectic foliations and a null foliation satisfying \( \mathcal{F}_P^2 \prec \mathcal{F}_P \prec \mathcal{F}_\omega \), will be comprehensively treated in the next section.

5 Examples

In this section we apply the results obtained in the previous sections to draw conclusions on the dynamics of two cluster maps. Our first example is the Somos-5 cluster map which illustrates the simplest situation of applicability of our results: existence of a flag \( \mathcal{F}_P \prec \mathcal{F}_\omega \) and of a fixed point of the reduced map \( \hat{\varphi} \). The second example is a cluster map in dimension 7, also associated to a 1-periodic quiver, which falls into a more specific case: existence of a flag \( \mathcal{F}_P^2 \prec \mathcal{F}_P \prec \mathcal{F}_\omega \) and global periodicity of the reduced map \( \hat{\varphi} \).

Although the examples chosen are maps associated to 1-periodic quivers, the results proved in the previous sections apply to higher periodic quivers.

Example 3. Somos-5

It is well known that the Somos-5 recurrence
\[
x_{n+5}x_n = x_{n+1}x_{n+4} + x_{n+2}x_{n+3}
\]  
(11)
is a recurrence arising from the mutation-periodic quiver \( Q_B \) with \( B \) as in Example 1 with \( r = s = 1 \). The associated cluster map can be reduced using the presymplectic form (3) and the reduced map \( \hat{\varphi} \) is a QRT map of the plane [21] which is known to be integrable in the Liouville sense (see for instance [7] and [22] for the integrability of QRT maps). In fact, a first integral of this QRT map defines an elliptic fibration of the plane whose generic fibres are curves of genus 1. The general solution of the Somos-5 recurrence corresponding to elliptic curves of genus 1 was obtained in [10] in terms of sigma functions. Using the approach developed in the previous sections we obtain (Proposition 4 below) the general solution of the Somos-5 recurrence corresponding to the singular case (genus zero curves).

The cluster map associated to the Somos-5 recurrence is
\[
\varphi(x_1, \ldots, x_5) = \left( x_2, x_3, x_4, x_5, \frac{x_2x_5 + x_3x_4}{x_1} \right).
\]  
(12)
The Poisson structure \( \mathcal{P} \) on \( \mathbb{R}^5_+ \) given by the skew-symmetric matrix \( C = [c_{ij}] \) with \( c_{ij} = j - i \), is known to be invariant under \( \varphi \). This matrix \( C \) has a 3-dimensional kernel and \( \text{Im} B \subset \ker C \). By Proposition 3, \( \mathcal{F}_P \) is a simple
subfoliation of $\mathcal{F}^\omega$. Submersions defining these foliations produce the reduced systems $(\tilde{\phi}, \tilde{\pi})$ and $(\hat{\phi}, \hat{\pi})$, where

$$
\tilde{\phi} : x \mapsto \left( \frac{y_2}{y_1 y_2}, \frac{1 + y_2}{y_1 y_2 y_3} \right), \quad \tilde{\pi} : x \mapsto (y_1, y_2, y_3)
$$

$$
\hat{\phi} : x \mapsto \left( y_2, \frac{1 + y_2}{y_1 y_2} \right), \quad \hat{\pi} : x \mapsto (y_1, y_2)
$$

with $x = (x_1, \ldots, x_5) \in \mathbb{R}^5_+$ and

$$
y_1 = \frac{x_1 x_4}{x_2 x_3}, \quad y_2 = \frac{x_2 x_5}{x_3 x_4}, \quad y_3 = \frac{x_3 x_5}{x_4^2}.
$$

For more details on these reductions see [10] and [4].

By Proposition 6, $(\hat{\phi}, p)$ is a reduced system of $\tilde{\phi}$ with $p(y_1, y_2, y_3) = (y_1, y_2)$. Schematically:

It is easy to see that the map $\hat{\phi}$ has a unique fixed point, the point $(r, r)$ with $r$ the real (positive) root of $x^3 = 1 + x$, and no points of minimal period 2. A direct application of Proposition 4 to the reduced system $(\hat{\phi}, p)$ of $\tilde{\phi}$ shows that the leaf $L = \{y \in \mathbb{R}^3_+: y_1 = r, y_2 = r\}$ is invariant under $\tilde{\phi}$. Let $h$ denote the restriction of $\tilde{\phi}$ to $L$. Using the natural $y_3$ coordinate on $L$, the expression of $h$ is

$$
h(y_3) = \frac{r}{y_3},
$$

which is a globally 2-periodic map with a unique fixed point: $y_3 = \sqrt{r}$.

Summing up, the point $(r, r, \sqrt{r})$ is the unique fixed point of the map $\tilde{\phi}$ and its 2-periodic points are precisely the points on $\mathbb{R}^3_+$ of the form $(r, r, \lambda)$. Applying Proposition 5 to the reduced system $(\tilde{\phi}, \tilde{\pi})$ of $\phi$, the following conclusions can be withdrawn for the symplectic leaves of $\mathcal{F}^\omega$:

- $S_{(r, r, \sqrt{r})} = \{x \in \mathbb{R}^5_+: x_1 x_4 = r x_2 x_3, x_2 x_5 = r x_3 x_4, x_3 x_5 = \sqrt{r} x_4^2\}$ is invariant under $\phi$ and the restriction of $\phi$ to $S_{(r, r, \sqrt{r})}$ is symplectic;

- $S_{(r, r, \lambda)} = \{x \in \mathbb{R}^5_+: x_1 x_4 = r x_2 x_3, x_2 x_5 = r x_3 x_4, x_3 x_5 = \lambda x_4^2\}$, for $\lambda \in \mathbb{R}_+$, is invariant under $\phi^{(2)}$ and the restriction of $\phi^{(2)}$ to $S_{(r, r, \lambda)}$ is symplectic.

14
Choosing natural coordinates \((x_3, x_4)\) on each symplectic leaf, the computation of the restricted maps \(h_1 = \varphi_{S_{(r, r, \sqrt{r})}}\) and \(h_2 = \varphi_{(2)}_{S_{(r, r, \lambda)}}\) gives

\[ h_1(x_3, x_4) = \left(x_4, \sqrt{r} \frac{x_4^2}{x_3}\right) \quad \text{and} \quad h_2(x_3, x_4) = \lambda \left(\frac{x_2^2}{x_3}, \frac{r x_4^3}{x_3^2}\right). \]

The expression of the iterates of \(h_1\) and \(h_2\) can be obtained directly from Lemma 1 of [5]. These are given, for \(n \geq 1\), by

\[ h_1^{(n)} = r^{n(n-1)/4} \left(\frac{x_4^n}{x_3^{n-1}}, r^{n/2} \frac{x_4^{n+1}}{x_3^n}\right), \quad h_2^{(n)} = \lambda^n r^{n(n-1)} \left(\frac{x_2^{2n}}{x_3^{n-1}}, r^n \frac{x_4^{2n+1}}{x_3^{2n}}\right). \]

Hence, we are led to the following proposition.

**Proposition 7.** Consider the Somos-5 recurrence in (11) and let \(r\) be the real root of \(x^3 = 1 + x\). Let \((x_1, \ldots, x_5)\) be initial data satisfying the identities

\[ x_1 x_4 = r x_2 x_3, \quad x_2 x_5 = r x_3 x_4, \quad x_3 x_5 = \lambda x_4^2 \]

with \(\lambda \in \mathbb{R}_+\). Then,

1. if \(\lambda = \sqrt{r}\), the corresponding solution of the Somos-5 recurrence is given by

\[ x_{n+5} = r^{(n+2)2(n+1)} \left(\frac{x_2^n}{x_3^{n-1}}, \frac{x_4^{n+1}}{x_3^n}\right), \quad n \geq 1; \]

2. if \(\lambda \neq \sqrt{r}\), the corresponding solution of the Somos-5 recurrence is given by

\[ x_{2n+4} = \lambda^n r^{n^2} \left(\frac{x_2^{2n+1}}{x_3^n}\right), \quad x_{2n+5} = \lambda^{n+1} r^{n(n+1)} \left(\frac{x_2^{2n+2}}{x_3^{n+1}}\right), \quad n \geq 1. \]

**Example 4. Dynamics of a cluster map in dimension 7**

The most general form of the Somos-7 recurrence is

\[ x_{n+7} x_n = \alpha x_{n+1} x_{n+6} + \beta x_{n+2} x_{n+5} + \gamma x_{n+3} x_{n+4}. \]  

(14)

Clearly this recurrence does not fit into the setting of cluster maps unless one and just one of the parameters \(\alpha, \beta\) or \(\gamma\) is zero. In that case the respective recurrence can be included in the cluster algebra framework by adding two extra (frozen) nodes to a (coefficient-free) 1-periodic quiver of 7 nodes represented by a certain matrix \(B\), see [12, Section 10]. This matrix has rank equal to 4 when \(\alpha = 0\) or \(\gamma = 0\) and rank 2 when \(\beta = 0\). This means that the respective reduced map \(\hat{\varphi}\) is a 4-dimensional map when \(\alpha = 0\) or \(\gamma = 0\) and a 2-dimensional map when \(\beta = 0\).

Three first integrals for the Somos-7 recurrence were given in [11, Theorem 6.4] where it was proved that they descend to first integrals of the 4D and 2D (symplectic) reduced maps. Namely, it was proved that both reduced maps are
Liouville integrable: there are two first integrals of $\hat{\varphi}$ in involution if $\alpha = 0$ or $\gamma = 0$ and one first integral if $\beta = 0$. Here we consider the recurrence (14) with $\beta = 0$ and $\alpha = \gamma = 1$. The map defining this recurrence is the map

$$\varphi(x_1, \ldots, x_7) = \left( x_2, x_3, \ldots, x_7, \frac{x_2 x_7 + x_4 x_5}{x_1} \right),$$

which is the cluster map associated to the matrix $B$ in Example 2.

As seen in that example, there exists a flag of strictly simple subfoliations $\mathcal{F}^1 \prec \mathcal{F}^2 \prec \mathcal{F}^\omega$ where $\omega$ is the presymplectic form defined by the matrix $B$, and $P_1, P_2$ are the Poisson structures defined respectively by the matrices $C_1$ and $C_2$ in (9). Submersions $\hat{\pi}, \tilde{\pi}_1$ and $\tilde{\pi}_2$ defining these foliations were also obtained in Example 2.

Straightforward computations show that $\varphi$ is a Poisson map with respect to both $P_1$ and $P_2$, and so there are three reduced systems $\left( \hat{\varphi}, \hat{\pi} \right)$, $\left( \tilde{\varphi}_1, \tilde{\pi}_1 \right)$ and $\left( \tilde{\varphi}_2, \tilde{\pi}_2 \right)$ of $\varphi$. The respective reduced maps can be easily computed and have the form

$$\hat{\varphi}(y_1, y_2) = \left( y_2, \frac{1 + y_2}{y_1} \right), \quad \tilde{\varphi}_1(y_1, y_2, y_3) = \left( \hat{\varphi}(y_1, y_2); \frac{y_2(1 + y_2)}{y_3} \right),$$

$$\tilde{\varphi}_2(y_1, \ldots, y_5) = \left( \tilde{\varphi}_1(y_1, y_2, y_3); y_5, \frac{y_3 y_5^2}{y_2 y_4} \right).$$

This can be summarised in the following commutative diagram:

\[
\begin{array}{cccccc}
\mathbb{R}_7^+ & \overset{\pi_2}{\twoheadrightarrow} & \mathbb{R}_5^+ & \overset{p_2}{\rightarrow} & \mathbb{R}_3^+ & \overset{p_1}{\rightarrow} & \mathbb{R}_2^+ \\
\varphi & \downarrow & & \tilde{\varphi}_2 & \downarrow & \tilde{\varphi}_1 & \downarrow & \hat{\varphi} \\
\mathbb{R}_7^+ & \overset{\pi_2}{\twoheadrightarrow} & \mathbb{R}_5^+ & \overset{p_2}{\rightarrow} & \mathbb{R}_3^+ & \overset{p_1}{\rightarrow} & \mathbb{R}_2^+ \\
\tilde{\pi}_1 & \downarrow & & & & & \\
\tilde{\pi}_2 & \downarrow & & & & & \\
& & & \hat{\pi} & & & \\
\end{array}
\]

where $p_2 : \mathbb{R}_5^+ \rightarrow \mathbb{R}_3^+$ and $p_1 : \mathbb{R}_3^+ \rightarrow \mathbb{R}_2^+$ are the canonical projections

$$p_2(y_1, \ldots, y_5) = (y_1, y_2, y_3), \quad p_1(y_1, y_2, y_3) = (y_1, y_2).$$

In order to describe the dynamics of $\varphi$ we study sequentially three reduced systems (provided by Proposition 6) starting with the lowest dimensional reduced map $\hat{\varphi}$.

The map $\hat{\varphi}$ in (16) is the well-known Lyness map, which is also a QRT map and therefore an integrable map [11]. Moreover, $\hat{\varphi}$ is a globally 5-periodic map.
such that all the points in $\mathbb{R}^2_+ \setminus \{(\phi, \phi)\}$ have minimal period 5 apart from the point $(\phi, \phi)$, with $\phi = \frac{1 + \sqrt{5}}{2}$ the golden number, which is a fixed point.

As $\mathcal{F}^{P_1} \triangleleft \mathcal{F}^{\omega_1}$, Proposition 6 implies that $(\tilde{\varphi}, p_1)$ is a reduced system of $\tilde{\varphi}_1$. The application of Proposition 4 to $(\tilde{\varphi}, p_1)$ leads to $\tilde{\varphi}_1$-invariance properties of the leaves

$L_{(a,b)} = \{ y \in \mathbb{R}^3_+ : y_1 = a, y_2 = b \}$

of the 1-dimensional foliation $\mathcal{F}$ of $\mathbb{R}^3_+$. These properties are as follows:

1. $L_{(\phi, \phi)}$ is invariant under $\tilde{\varphi}_1$;

2. for any $(a, b) \neq (\phi, \phi)$, $L_{(a,b)}$ is invariant under $\tilde{\varphi}_1^{(5)}$ and not invariant under $\tilde{\varphi}_1^{(n)}$ for $1 \leq n < 5$. This means, by Proposition 4, that the $\tilde{\varphi}_1$-orbit of any point in $L_{(a,b)}$ circulates between five distinct leaves of $\mathcal{F}$:

$L_{(a,b)} \to L_{\tilde{\varphi}(a,b)} \to L_{\tilde{\varphi}^{(2)}(a,b)} \to L_{\tilde{\varphi}^{(3)}(a,b)} \to L_{\tilde{\varphi}^{(4)}(a,b)}$.

Studying the restrictions of $\tilde{\varphi}_1$ to $L_{(\phi, \phi)}$ and of $\tilde{\varphi}_1^{(5)}$ to $L_{(a,b)}$ we obtain the full description of the dynamics of $\tilde{\varphi}_1$ in the following lemma.

**Lemma 1.** Let $\tilde{\varphi} : \mathbb{R}^3_+ \to \mathbb{R}^3_+$ be the map in (16) and $\mathcal{F}$ the 1-dimensional foliation of $\mathbb{R}^3_+$ whose leaves $L_{(a,b)}$ are given by (20). Then, with $\phi = \frac{1 + \sqrt{5}}{2}$

1. $L_{(\phi, \phi)}$ contains the unique fixed point of $\tilde{\varphi}_1$, the point $P = (\phi, \phi, \sqrt{\phi^3})$, and all the other points in $L_{(\phi, \phi)}$ are periodic points of $\tilde{\varphi}_1$ with minimal period 2;

2. $L_{(a,b)}$, with $(a, b) \neq (\phi, \phi)$, contains precisely one periodic point of $\tilde{\varphi}_1$ with minimal period 5, the point $(a, b, \sqrt{g(a,b)})$ with

$g(a,b) = \frac{ab(a+1)(b+1)}{(a+b+1)}$.

Any other point in $L_{(a,b)}$ is a periodic point of $\tilde{\varphi}_1$ with minimal period 10.

In particular, $\tilde{\varphi}_1$ is globally 10-periodic.

**Proof.** The restriction $h_1$ of $\tilde{\varphi}_1$ to $L_{(\phi, \phi)}$ is given $h_1(y_3) = \frac{\phi^3}{y_3}$, which is a globally 2-periodic map with exactly one fixed point: $\sqrt{\phi^3}$.

We immediately conclude that $(\phi, \phi, \sqrt{\phi^3})$ is the only fixed point of $\tilde{\varphi}_1$ and all the other points of $L_{(\phi, \phi)}$ are periodic points of $\tilde{\varphi}_1$ with minimal period 2.

For each $(a, b) \in \mathbb{R}^2_+ \setminus \{(\phi, \phi)\}$ the restriction of $\tilde{\varphi}_1^{(5)}$ to $L_{(a,b)}$ is given by

$h_5(y_3) = \frac{g(a,b)}{y_3}$,

with $g$ as in (21). The map $h_5$ is globally 2-periodic and has exactly one fixed point: $\sqrt{g(a,b)}$. 

17
Consequently, the restriction of $\tilde{\varphi}_1$ to $L_{(a,b)}$ has precisely one point of minimal period 5, the point $(a, b, \sqrt{g(a,b)})$, and all the remaining points of $L_{(a,b)}$ are periodic points of $\tilde{\varphi}_1$ with minimal period 10.

Applying again Proposition 6, we conclude that $(\tilde{\varphi}_1, p_2)$ is a reduced system of $\tilde{\varphi}_2$ (see also the commutative diagram (18)), and so the information from Lemma 1 concerning the dynamics of $\tilde{\varphi}_1$ allows us to draw conclusions on the dynamics of $\tilde{\varphi}_2$. In fact, combining the previous lemma with Proposition 4 we are led to the $\tilde{\varphi}_2$-invariance properties of the leaves

$$L'_{(a,b,c)} = \{ y \in \mathbb{R}_+^5 : y_1 = a, y_2 = b, y_3 = c \} \quad (22)$$

of the 2-dimensional foliation $F'$ of $\mathbb{R}_+^5$ defined by $p_2$. More precisely:

1. $L'_P$, with $P = (\phi, \phi, \sqrt{\phi^3})$, is invariant under $\tilde{\varphi}_2$;
2. if $Q = (\phi, \phi, c) \neq P$, then the $\tilde{\varphi}_2$-orbit of any point in $L'_Q$ circulates between the leaves $L'_Q$ and $L'_{\tilde{\varphi}_1(Q)}$ of $F'$;
3. if $Q = (a, b, \sqrt{g(a,b)})$ with $(a, b) \neq (\phi, \phi)$, then the $\tilde{\varphi}_2$-orbit of any point in $L'_Q$ circulates between five distinct leaves of $F'$:

$$L'_Q \rightarrow L'_{\tilde{\varphi}_1(Q)} \rightarrow L'_{\tilde{\varphi}_1^2(Q)} \rightarrow L'_{\tilde{\varphi}_1^3(Q)} \rightarrow L'_{\tilde{\varphi}_1^4(Q)};$$
4. in all other cases the $\tilde{\varphi}_2$-orbit of any point in $L'_Q$ circulates between ten distinct leaves of $F'$: $L'_Q \rightarrow L'_{\tilde{\varphi}_1(Q)} \rightarrow \ldots \rightarrow L'_{\tilde{\varphi}_1^{10}(Q)}$.

Again, studying the restrictions of $\tilde{\varphi}_2$ to $L'_P$, and of $\tilde{\varphi}_2^{(2)}$, $\tilde{\varphi}_2^{(5)}$ or $\tilde{\varphi}_2^{(10)}$ to other leaves $L'_Q$, we will conclude that the reduced map $\tilde{\varphi}_2$ has no periodic points. In spite of the apparent simplicity of the result, some intermediate results and nontrivial computations are needed.

**Lemma 2.** The map $\tilde{\varphi}_2 : \mathbb{R}_+^5 \rightarrow \mathbb{R}_+^5$ given in (17) has no periodic points.

**Proof.** From the above description of the invariance of the leaves (22) of $F'$, it is enough to study the restrictions of $\tilde{\varphi}_2$, $\tilde{\varphi}_2^{(2)}$, $\tilde{\varphi}_2^{(5)}$ and $\tilde{\varphi}_2^{(10)}$ to the appropriate leaves of $F'$. All these restricted maps are symplectic birational maps of the plane (see [3]) having the particular form

$$f(x, y) = (kx^m y^n, lx^p y^q),$$

with $k, l$ real positive constants and $m, n, p, q$ integers satisfying $mq - np = 1$, which were comprehensively studied in [6].

The restriction $h_1$ of $\tilde{\varphi}_2$ to $L'_P$ is given in the natural coordinates $(y_4, y_5)$ by

$$h_1(y_4, y_5) = \left( y_5, \sqrt{\frac{y_5^2}{y_4}} \right).$$

18
By Lemma 2 in [6] this map has no periodic points and all its components go to infinity. Consequently \( \hat{\varphi}_2 \) has no periodic points in \( L'_P \).

The restriction of \( \varphi^{(2)}_2 \) to \( L'_Q \), where \( Q = (\phi, \phi, c) \neq P \), is given by

\[
h_2(y_4, y_5) = c \left( \frac{1}{\phi} \frac{y_5^2}{y_4^3}, \frac{y_3}{y_4} \right),
\]

and the restriction of \( \varphi^{(5)}_2 \) to \( L'_Q \), where \( Q = (a, b, \sqrt{g(a, b)}) \) and \( (a, b) \neq (\phi, \phi) \), is given by

\[
h_5(y_4, y_5) = \frac{(1 + b)g(a, b)}{b} \left( \frac{y_5^2}{y_4^3}, \frac{(1 + a + b)\sqrt{g(a, b)}}{b} \frac{y_5^6}{y_4^6} \right),
\]

with \( g \) as in (21).

Finally, the restriction of \( \varphi^{(10)}_2 \) to any other \( L'_Q \) is given by

\[
h_{10}(y_4, y_5) = k(a, b, c) \left( \frac{y_5^{10}}{y_4^9}, \frac{(1 + a)(1 + a + b)(1 + b)}{ab} \frac{y_5^{11}}{y_4^{10}} \right),
\]

with \( k(a, b, c) = \frac{(1 + a)^2(1 + a + b)^3(1 + b)^4}{ab^5} \).

By [6, Theorem 1] all the three restricted maps \( h_2, h_5 \) and \( h_{10} \) are conjugate to the map

\[
f(x, y) = (y, \xi y^2)
\]

with \( \xi = \phi^2 > 1 \) for \( h_2 \), \( \xi = \left( \frac{(1 + a)(1 + b)(1 + a + b)}{ab} \right)^2 > 1 \) for \( h_5 \), and \( \xi = \left( \frac{(1 + a)(1 + b)(1 + a + b)}{ab} \right)^{10} > 1 \) for \( h_{10} \). By [6, Lemma 2] the maps \( h_2, h_5 \) and \( h_{10} \) have no periodic points and all their components go to infinity. Consequently the map \( \hat{\varphi}_2 \) has no periodic point in any leaf of the form \( L'_Q \) and the conclusion follows.

Finally, we assemble all previous results to describe the dynamics of the map \( \varphi \). We denote by \( N(a, b) \) and \( S(a, b, c) \) the null leaves and the symplectic leaves of the foliations \( F^\omega \) and \( F^{P_1} \) respectively:

\[
N(a, b) = \{ x \in \mathbb{R}^7_+ : x_1x_6 = ax_3x_4, x_2x_7 = bx_4x_5 \}, \quad (24)
\]

\[
S(a, b, c) = \{ x \in N(a, b) : x_1x_7 = cx_4^2 \}. \quad (25)
\]

The leaves of \( F^{P_2} \) are not invariant under any iterate of \( \varphi \) and so will be not included in the description of its dynamics.

**Proposition 8.** Consider the cluster map \( \varphi : \mathbb{R}^7_+ \rightarrow \mathbb{R}^7_+ \) given in (15) and its reduced maps \( \hat{\varphi} \) and \( \hat{\varphi}_1 \) in (16). Let \( \phi = \frac{1 + \sqrt{5}}{2} \) and consider the null leaves of \( F^\omega \) and the symplectic leaves of \( F^{P_1} \) given respectively by (24) and (25).
Then there are four distinct types of orbits of $\varphi$ with respect to the 5-dimensional foliation $\mathcal{F}^\omega$ and its 4-dimensional subfoliation $\mathcal{F}^\mathcal{P}_1$. More precisely, a $\varphi$-orbit is either entirely contained in the null leaf $N_{(\phi,\phi)}$, or it circulates between five distinct null leaves:

$$N_{(a,b)} \rightarrow N_{\tilde{\varphi}^{(2)}(a,b)} \rightarrow N_{\tilde{\varphi}^{(3)}(a,b)} \rightarrow N_{\tilde{\varphi}^{(4)}(a,b)}.$$ 

In the null leaf $N_{(\phi,\phi)}$ there exist two exclusive cases:

1. the $\varphi$-orbit is entirely contained in the symplectic leaf $S_{(\phi,\phi,\sqrt{g(a,b)})}^\phi$;
2. the $\varphi$-orbit circulates between two distinct leaves: $S_{(\phi,\phi,c)} \rightarrow S_{\tilde{\varphi}^1(\phi,\phi,c)}$, with $c \neq \sqrt{g(a,b)}$.

In any other null leaf $N_{(a,b)}$ there are another two exclusive cases:

3. the $\varphi$-orbit circulates between five distinct symplectic leaves:

$$S_Q \rightarrow S_{\tilde{\varphi}^1(Q)} \rightarrow S_{\tilde{\varphi}^{(2)}_1(Q)} \rightarrow S_{\tilde{\varphi}^{(3)}_1(Q)} \rightarrow S_{\tilde{\varphi}^{(4)}_1(Q)},$$

with $Q = (a, b, \sqrt{g(a,b)})$ and $g(a,b)$ given by (21);
4. the $\varphi$-orbit circulates between ten distinct symplectic leaves

$$S_Q \rightarrow S_{\tilde{\varphi}^1(Q)} \rightarrow \cdots \rightarrow S_{\tilde{\varphi}^{(9)}_1(Q)},$$

with $Q = (a,b,c)$ and $c \neq \sqrt{g(a,b)}$, in the following way: beginning in a leaf $S_Q \subset N_{(a,b)}$ the orbit comes back to $N_{(a,b)}$ to a different leaf $S_{\tilde{\varphi}^{(5)}_1(Q)}$ after 5 iterations, and returns to the same leaf $S_Q$ after 10 iterations.

Moreover, the map $\varphi$ has no periodic points.

Figure 1 below illustrates the contents of the last proposition. To avoid overloading the picture, case (iii) of the proposition is omitted.

Acknowledgements.

The work of I. Cruz and H. Mena-Matos was partially funded by FCT under the project PEst-C/MAT/UI0144/2013.

The work of M. E. Sousa-Dias was partially funded by FCT/Portugal through the projects UID/MAT/04459/2013 and EXCL/MAT-GEO/0222/2012.

References

[1] Blanc J., Symplectic birational transformations of the plane, Osaka J. Math., 50 (2013) 573–590.

[2] H. Bursztyn, A brief introduction to Dirac manifolds, In Geometric and topological methods for quantum field theory, (Cambridge Univ. Press, Cambridge), (2013) 4–38.
Figure 1: The foliation of the 5D null leaves $N_{(a,b)}$ by 4D symplectic leaves $S_{(a,b,c)}$ in Example 2 and the dynamics of $\varphi$ on these leaves. On the left, the $\varphi$-invariant leaf $N_{(\phi,\phi)}$ and its $\varphi(2)$-invariant leaves $S_{(\phi,\phi,c)}$. On the right, a $\varphi(5)$-invariant leaf $N_{(a,b)}$ and its $\varphi(10)$-invariant leaves $S_{(a,b,c)}$.

[3] A. Cannas da Silva A. Weinstein, *Geometric Models for Noncommutative Algebras*, Berkeley Mathematics Lecture Notes, vol. 10, (American Mathematical Society, Providence, RI; Berkeley Center for Pure and Applied Mathematics, Berkeley, CA, 1999).

[4] I. Cruz M.E. Sousa-Dias, *Reduction of cluster iteration maps*, J. Geom. Mech., 6 (3) (2014) 297–318.

[5] I. Cruz, H. Mena-Matos M.E. Sousa-Dias, *Dynamics of the birational maps arising from $F_0$ and $dP_3$ quivers*, J. Math. Anal. Appl., 431 (2) (2015) 903–918.

[6] I. Cruz, H. Mena-Matos M.E. Sousa-Dias, *Dynamics and periodicity in a family of cluster maps*, preprint, arXiv: 1511.07291.

[7] J. Duistermaat, *Discrete Integrable Systems. QRT Maps and Elliptic Surfaces*, Springer Monographs in Mathematics, (Springer, New York, 2010).

[8] Fomin, Sergey Zelevinsky, Andrei, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc., 15 (2) (2002) 497–529.
[9] A. Fordy, *Mutation-periodic quivers, integrable maps and associated Poisson algebras*, Phil. Trans. R. Soc. A, 369 (2011) 1264–1279.

[10] A. Fordy A. Hone, *Symplectic maps from cluster algebras*, SIGMA Symmetry, Integrability and Geom. Methods and Appl., 7 Paper 091 (2011) 12 pages.

[11] A. Fordy A. Hone, *Discrete integrable systems and Poisson algebras from cluster maps*, Commun. Math. Phys., 325 (2) (2014) 527–584.

[12] A. Fordy R. Marsh, *Cluster mutation-periodic quivers and associated Laurent sequences*, J. Algebraic Combin., 34 (1) (2011) 19–66.

[13] V.V. Fock and A.B. Goncharov, *Cluster ensembles, quantization and the dilogarithm*, Ann. Sci. Ec. Norm. Sup., 42 (2009) 865–930.

[14] M. Gekhtman, M. Shapiro A. Vainshtein, *Cluster Algebras and Poisson Geometry*, Mosc. Math. J., 3 (2003) 899-934.

[15] M. Gekhtman, M. Shapiro A. Vainshtein, *Cluster Algebras and Poisson Geometry*, Mathematical Surveys and Monographs, 167, American Mathematical Society, Providence, RI, (2010).

[16] A. Hone, *Sigma function solution of the initial value problem for Somos 5 sequences*, Trans. Amer. Math. Soc., 359 (10) (2007) 5019–5034.

[17] A. Hone and C. Swart, *Integrability and the Laurent phenomenon for Somos 4 and Somos 5 sequences*, Math. Proc. Camb. Phil. Soc., 145 (2008) 65–85.

[18] R. Inoue and T. Nakanishi, *Difference equations and cluster algebras I: Poisson bracket for integrable difference equations*, In Infinite analysis 2010—Developments in quantum integrable systems, RIMS Kôkyûroku Bessatsu, B28 (2011) 63–88.

[19] P. Libermann C-M. Marle, *Symplectic Geometry and Analytical Mechanics*, Mathematics and its Applications, 35, D. Reidel Publishing Co., Dordrecht, (1987).

[20] I. Moerdijk J. Mrčun, *Introduction to foliations and Lie groupoids*, Cambridge Studies in Advanced Mathematics, 91, Cambridge University Press, (2003).

[21] G. Quispel, J. Roberts C. Thompson, *Integrable mappings and soliton equations*, Phys. Lett. A, 126 (1988) 419–421.

[22] G. Quispel, J. Roberts C. Thompson, *Integrable mappings and soliton equations II*, Physica D, 34 (1989) 183–192.

[23] Zelevinsky, Andrei, *‘What is . . . a cluster algebra?*, Notices Amer. Math. Soc., 54 (11) (2007) 1494–1495.