Application of Generating Functions and Partial Differential Equations in Coding Theory

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Abstract

In this work we have considered formal power series and partial differential equations, and their relationship with Coding Theory. We have obtained the nature of solutions for the partial differential equations for Cycle Poisson Case. The coefficients for this case have been simulated, and the high tendency of growth is shown. In the light of Complex Analysis, the Hadamard Multiplication’s Theorem is presented as a new approach to divide the power sums relating to the error probability, each part of which can be analyzed later.

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1 Generating Functions and Operations on them

Let \( A(z) \) and \( B(z) \) be the two generating functions, in fact two formal power sums [2], [3], whose coefficients belong to some field, \( a_n, b_n \in F \):

\[
A(z) = \sum_{n \geq 0} a_n z^n,
\]

and

\[
B(z) = \sum_{n \geq 0} b_n z^n.
\]

The infinite series \( \sum_{n \geq 0} a_n z^n \), absolutely converges if and only if there is a bounding constant \( M \), such that the finite sums \( \sum_{0 < n \leq N} |a_n z^n| \) never exceed \( M \), for every \( N \in \mathbb{N} \). It directly follows, that, if \( \sum_{n \geq 0} a_n z^n \) converges for some value \( z = z_0 \), it also converges for all \( z \) with \( z < z_0 \). Even if the series do not converge, the next operations we perform on generating functions can be justified rigorously as an operation on formal power series. Then we have the following are satisfied:

- Right shift

\[
z A(z) = \sum_{n \geq 1} a_{n-1} z^n,
\]

- Left shift

\[
\frac{A(z) - a_0}{z} = \sum_{n \geq 0} a_{n+1} z^n,
\]

- Differentiation

\[
A'(z) = \sum_{n \geq 0} (n+1)a_{n+1} z^n,
\]

- Integration

\[
\int_0^z A(t) dt = \sum_{n \geq 1} \frac{a_{n-1}}{n} z^n,
\]

- Scaling

\[
A(\lambda z) = \sum_{n \geq 0} \lambda^n a_n z^n,
\]

- Addition

\[
A(z) + B(z) = \sum_{n \geq 0} (a_n + b_n) z^n,
\]
\[ (1 - z)A(z) = a_0 + \sum_{n \geq 1} (a_n - a_{n-1})z^n, \quad (9) \]

\[ A(z)B(z) = \sum_{n \geq 0} \left( \sum_{0 \leq k \leq n} a_k b_{n-k} \right)z^n; \quad (10) \]

\[ \frac{A(z)}{1 - z} = \sum_{n \geq 0} \left( \sum_{0 \leq k \leq n} a_k \right)z^n. \quad (11) \]

2 Introduction

2.1 The Cycle Poisson Case

In the next chapter we will consider the Poisson ensemble [1], i.e., the code where the following conditions are satisfied:
- the length of the code is \( n \),
- the rate is \( r \),
- all variable nodes have degree two.

Every edge in the above defined code, is chosen i.i.d. with uniform probability for all \( (1 - r)n \) check nodes. In the following \( m = (1 - r)n \).

For the fixed number of variable nodes \( v \), we have the following number of constellations:

\[ N(v) = \begin{cases} ((1 - r)n)^{2v}, & 0 \leq v \leq n, \\ 0, & \text{otherwise}. \end{cases} \]

The number of such constellations which are stopping sets on \( t \) check nodes is equal to the following:

\[ S(v, t) := \binom{(1 - r)n}{t} (2v)! \text{coef}\{(e^x - 1 - x)^t, x^{2v}\}. \quad (12) \]

If we have \( v \) fixed nodes, having \( t \) check nodes of degree all least one, and there is no empty stopping-sets, the number of constellations is equal to \( v!2^v A(v, t, s) \).
Then, we can derive the error probability $\mathbb{E}[P_B^{IT}(G, \epsilon)]$:

$$
\mathbb{E}[P_B^{IT}(G, \epsilon)] = \sum_v \binom{n}{v} (1-\epsilon)^{n-v} v! 2^v \sum_{t,s} A(v, t, s) \frac{1}{N(v)} \\
= (1-\epsilon)^n \sum_v \binom{n}{v} v! \left( \frac{2\epsilon}{(1-\epsilon)((1-r)n)^2} \right)^v \sum_{t,s} A(v, t, s) \\
= (1-\epsilon)^n A(x = \frac{2\epsilon}{(1-\epsilon)((1-r)n)^2}, y = 1, z = 1),
$$

where

$$
A(x, y, z) = \sum_{v,t,s} \frac{x^v}{(n-v)!} y^t z^s,
$$

is the generating function with coefficients $A(v, t, s)/(n-v)!$.

For indices, $1 \leq v \leq n, 1 \leq t \leq m, 0 \leq s \leq m - t$, the following is satisfied:

$$
A(v, t, s) = A(v - 1, t, s - 2) (m - t - s + 2)(m - t - s + 1) \chi_{s \geq 2} + A(v - 1, t, s - 1) (m - t - s + 1) t \chi_{s \geq 1} + A(v - 1, t - 1, s) (m - t - s + 1) s \chi_{s \geq 1}.
$$

For easy of notations let $k = m + 1$. Now, we start with the recurrence equation, but considering the generating function:

$$
G(x, y, z) = \sum_{v,t,s} A(v, t, s) x^v y^t z^s, \quad (13)
$$

except $A(x, y, z)$.

### 2.2 Derivation of PDE

Using rules derived in the previous section, which relate to the operations on the generating functions, we get:
\begin{align*}
  k(k + 1)A(v, t, s - 2) & \equiv k(k + 1)z^2, \\
  2ktA(v, t, s - 2) & \equiv 2kyz^2 \frac{\partial}{\partial y}, \\
  2ksA(v, t, s - 2) & \equiv 2kz^2(2 + z \frac{\partial}{\partial z}), \\
  t(t - 1)A(v, t, s - 2) & \equiv y^2z^2 \frac{\partial^2}{\partial y^2}, \\
  s(s - 1)A(v, t, s - 2) & \equiv z^2(2 + 4z \frac{\partial}{\partial z} + z^2 \frac{\partial^2}{\partial z^2}), \\
  2tsA(v, t, s - 2) & \equiv 2yz^2(2 \frac{\partial}{\partial y} + z \frac{\partial^2}{\partial y \partial z}).
\end{align*}

Putting things together, we get:
\begin{align*}
  z \frac{\partial}{\partial z} G &= x \left\{ k(k + 1)z^2 - 2kyz^2 \frac{\partial}{\partial y} - 2kz^2(2 + z \frac{\partial}{\partial z}) \\
  &\quad + y^2z^2 \frac{\partial^2}{\partial y^2} + z^2(2 + 4z \frac{\partial}{\partial z} + z^2 \frac{\partial^2}{\partial z^2}) + 2yz^2(2 \frac{\partial}{\partial y} + z \frac{\partial^2}{\partial y \partial z}) \\
  &\quad + myz \frac{\partial}{\partial y} - yz(\frac{\partial}{\partial y} + z \frac{\partial^2}{\partial y \partial z}) + myz \frac{\partial}{\partial z} \\
  &\quad - yz \frac{\partial^2}{\partial z^2} - yz(\frac{\partial}{\partial z} + y \frac{\partial^2}{\partial y \partial z}) \right\} G.
\end{align*}

On the other hand,
\begin{align*}
  z \frac{\partial}{\partial z} G &= x \left\{ k(k + 1)z^2 - 4kz^2 + 2z^2 \\
  &\quad + \{-2kz^2 + 4yz^2 + myz - yz\} \frac{\partial}{\partial y} \\
  &\quad + \{-2kz^2 + 4z^3 + myz - yz\} \frac{\partial}{\partial z} \\
  &\quad + \{y^2z^2 - zy^2\} \frac{\partial^2}{\partial y^2} \\
  &\quad + \{z^4 - yz^2\} \frac{\partial^2}{\partial z^2} \\
  &\quad + \{2yz^3 - yz^2 - y^2z\} \frac{\partial^2}{\partial y \partial z} \right\} G.
\end{align*}
Our partial differential equation has just been derived:

\[
\frac{z \partial}{\partial z} G = xz \left\{ (k^2 - 3k + 2)z + y(2 - k)(2z - 1) \frac{\partial}{\partial y} + (2 - k)(2z^2 - y) \frac{\partial}{\partial z} + y(y - 1) \frac{\partial^2}{\partial y \partial z} \right\} G,
\]

where \( \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial^2}{\partial y \partial z}, \frac{\partial^2}{\partial y^2} \) are the partial derivatives, which are applied on the function \( G \).

2.3 Conclusions and Improvement

What is the capital gain derived from the previous analysis? The crucial thing is that, considering the generating function \( G(x, y, z) \) over coefficients \( A(v, t, s) \), instead of the generating functions \( A(x, y, z) \), we have got the second order partial differential equation, where \( x \) is a parameter.

3 Partial Equation and Determination of Solutions

For \( z = 0 \) we have identity. If \( z \) is different than zero, we have to consider:

a) \( x = 0 \), that will imply \( \frac{\partial G}{\partial z} = 0 \), which is the boundary condition for the plane \( (x, y, z) \in \{(0, \lambda, \mu) | \lambda \in \mathbb{R}, \mu \neq 0\} \);

b) Let us suppose \( x \neq 0 \) and \( z \neq 0 \), then we get the following partial differential equation:

\[
\frac{1}{x} \frac{\partial G}{\partial z} = A G_{yy} + 2B G_{yz} + C G_{zz} + D G_y + E' G_z + F. \tag{14}
\]

\[
A G_{yy} + 2B G_{yz} + C G_{zz} + D G_y + (E' - \frac{1}{x}) G_z + F = 0, \tag{15}
\]

where polynomials \( A, B, C, D, E', F \) are given respectively as:
\[ \begin{align*}
    A(y) &= y^2(y - 1), \\
    B(y, z) &= \frac{1}{2}y(2z^2 - y - z), \\
    C(y, z) &= z(z^2 - y), \\
    D(y, z) &= y(2 - k)(2z - 1), \\
    E'(y, z) &= (2 - k)(2z^2 - y), \\
    F(z) &= (k^2 - 3k + 2)z.
\end{align*} \]

Using term \( B^2 - AC \) we can determine the nature of the solution, [6], [7]:

\[ B^2 - AC = \begin{cases} 
    > 0, & \text{hyperbolic} \\
    = 0, & \text{parabolic} \\
    < 0, & \text{elliptic}
\end{cases} \]

In that case, consider \((2B)^2 - 4AC\). We get the following:

\[ \begin{align*}
    (2B)^2 - 4AC &= y^2(z^2 - y - z)^2 - y^2(y - 1)z(z^2 - y) \\
                   &= y^2(4z^4 - 3z^3 + z^2 + y^2 - yz^3 - 3yz^3 + yz).
\end{align*} \]

In the aim of determining the nature of the solution, let \( y = \alpha z \). With this substitution, we are able to divide the \( y - z \) plane in disjunct regions, where each region belongs to one type of the solution. We have:

\[ 4(B^2 - AC) = y^2(4z^4 - 3z^3 + z^2 + y^2 - yz^3 - 3yz^3 + yz) \]

\[ = z^4\alpha^2((4 - \alpha)z^2 - 3(1 + \alpha)z + 1 + \alpha + \alpha^2). \]

Consider the function, \( f(z) = (4 - \alpha)z^2 - 3(1 + \alpha)z + 1 + \alpha + \alpha^2 \), whose discriminant is:

\[ D(\alpha) = 9(1 + \alpha)^2 - 4(4 - \alpha)(1 + \alpha + \alpha^2), \]

\[ = (\alpha - 1)(4\alpha^2 + \alpha + 7). \]

Now, \( f(z) = (4-\alpha)z^2 - 3(1+\alpha)z + 1 + \alpha + \alpha^2 \), also we have \((4\alpha^2 + \alpha + 7) \geq 0\), for \( \alpha \in \mathbb{R} \), and \( 4D_{PEQ} = 4(B^2 - AC) = \alpha^2z^4f(z) \).

Let \( z_{1,2} \) be the roots of the equation \( f(z) = 0 \), so we get the following:

0) \( \alpha = 0 \), we have a strictly parabolic case,

\[ \{ \forall z \in \mathbb{R}, D_{PEQ} = 0, \text{ parabolic} \} \]

1) \( 0 < \alpha < 1 \),

\[ \{ \forall z \in \mathbb{R}, D_{PEQ} > 0, \text{ hyperbolic} \} \]

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2) $\alpha = 1$,

\[
\begin{cases}
  z = 3, D_{PEQ} = 0, \text{ parabolic} \\
  z \neq 3, D_{PEQ} > 0, \text{ hyperbolic}
\end{cases}
\]

3) $1 < \alpha < 4$,

\[
\begin{cases}
  \exists \in [z_1, z_2], D_{PEQ} < 0, \text{ elliptic} \\
  z = z_1 \lor z = z_2, D_{PEQ} = 0, \text{ parabolic} \\
  z \notin [z_1, z_2], D_{PEQ} > 0, \text{ hyperbolic}
\end{cases}
\]
4) $\alpha = 4,$

\[
\begin{align*}
&\begin{cases}
    z > \frac{7}{4}, \quad D_{PEQ} < 0, & \text{elliptic} \\
    z = \frac{7}{4}, \quad D_{PEQ} = 0 & \text{parabolic} \\
    z < \frac{7}{4}, \quad D_{PEQ} > 0, & \text{hyperbolic}
\end{cases}
\end{align*}
\]

![Figure 4: Function $f(z) = -15z + 21$](image)

5) $\alpha > 4,$

\[
\begin{align*}
&\begin{cases}
    z \in [z_1, z_2], \quad D_{PEQ} > 0, & \text{hyperbolic} \\
    z = z_1 \lor z = z_2, \quad D_{PEQ} = 0 & \text{parabolic} \\
    z \notin [z_1, z_2], \quad D_{PEQ} < 0, & \text{elliptic}
\end{cases}
\end{align*}
\]

![Figure 5: Function $f(z)$, E.g. $z_1 = 2, \ z_2 = 4$](image)
Summary, for the nature of the partial differential equation, we are getting the following:

- **Parabolic**
- **Elliptic**
- **Hyperbolic**

Figure 6: The Nature of PDE
4 Boundary Conditions

For the boundary condition, the coefficients $A(v, t, s = 0)$, satisfy the following:

$$A(v, t, s = 0) = \left(\frac{1 - r}{t}\right)^{(2v)}!\text{coef}\{(e^x - 1 - x)^t, x^{2v}\}. \quad (16)$$

Since $\frac{(2v)!}{2^{2v}v!} = (2v - 1)!!$, we have:

$$A(v, t, s = 0) = \left(\frac{m}{t}\right)(2v - 1)!!\text{coef}\{(e^x - 1 - x)^t, x^{2v}\}, \quad (17)$$

where $m = (1 - r)n$.

In the aim of getting the exponents for the given coefficients, we have simulated $\log A(v, t, s = 0)$ with increasing $t$. It is done for $m = 100$, and $t$ taking the values 1, 2, 3, 4, 5, 10, 30, 40, 50. It is important to notice, that given coefficients do not depend on the length and on the rate of the code individually, but on $m = (1 - r)n$. The binomial coefficient $\binom{m}{t}$ can be derived from the Stirling’s formula:

$$n! \approx \sqrt{(2n + 1/3)}\pi\left(\frac{n}{e}\right)^n, \quad (18)$$

for $n$ sufficient large.

On the next page are given simulated values, where $g^{(t)}(v)$ is the exponential factor:

$$g^{(t)}(v) = \log A(v, t, s = 0), \quad (19)$$

for the appropriate $t$.

In the Appendix, it will be shown the simulated exponential factors $g^{(t)}(v)$, for different $t$'s.

5 Another views on the problem

5.1 Complex Analysis

In this section we will use another approach, based on Complex Analysis. The following problem was considered by Hadamard, [4].

**Theorem (Hadamard’s Multiplication Theorem)**

Suppose that,

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n, \quad (20)$$
is convergent for $|z| < R$, and

$$g(z) = \sum_{n=0}^{+\infty} b_n z^n; \tag{21}$$

is convergent for $|z| < R'$ and that the singularities of $f(z)$ and $g(z)$ are known.

Consider the function:

$$F(z) = \sum_{n=0}^{+\infty} a_n b_n z^n. \tag{22}$$

$F(z)$ is regular for $|z| < RR'$, and Hadamard’s theorem depends on the following representation of $F(z)$ as an integral:

$$F(z) = \frac{1}{2\pi i} \int_C f(w)g\left(\frac{z}{w}\right) \frac{dw}{w}, \tag{23}$$

where $C$ is a contour, including the origin, on which $|w| < R, |z/w| < R'$.

To prove this, write

$$g\left(\frac{z}{w}\right) = \sum_{n=0}^{\infty} b_n \left(\frac{z}{w}\right)^n, \tag{24}$$

in the integral, and integrate term by term, as we may by uniform convergence. We obtain:

$$F(z) = \frac{1}{2\pi i} \int_C f(w)g\left(\frac{z}{w}\right) \frac{dw}{w}$$
$$= \sum_{n=0}^{+\infty} \frac{b_n z^n}{2\pi i} \int_C f(w) \frac{dw}{w^{n+1}}$$
$$= \sum_{n=0}^{+\infty} a_n b_n z^n,$$

the required result.
5.2 Returning to Error Probability

\[ E_B = (1 - \epsilon)^n \sum_v \binom{n}{v} v! \left( \frac{2\epsilon}{(1 - \epsilon)((1 - r)n)^2} \right)^v \sum_{t,s} A(v, t, s) \]

\[ = (1 - \epsilon)^n \sum_{t,s} \sum_v \binom{n}{v} v! A(v, t, s) \frac{1}{n^{2v}} \left( \frac{2\epsilon}{(1 - \epsilon)((1 - r)^2)} \right)^v. \]

Let \( x = \frac{2\epsilon}{(1 - \epsilon)(1 - r)^2} \), and consider the sum:

\[ S = \sum_v \binom{n}{v} v! A(v, t, s) \frac{1}{n^{2v}} x^v. \] (25)

The following is known, [5], [8], [9]:

\[ \sum_v \binom{n}{v} \frac{x^v}{n^{2v}} = (1 + \frac{x}{n^2})^n, \quad \text{for} \ |x| < 1, \] (26)

\[ \sum_v \binom{n}{v} x^v = (1 + x)^n, \quad \text{for} \ |x| < 1, \] (27)

\[ \sum_v v! x^v, \quad \text{diverge for all} \ x. \] (28)

If we could bound the series:

\[ \sum_v v! A(v, t, s) x^v \] (29)

\[ \sum_v \frac{v! A(v, t, s)}{n^{2v}} x^v, \] (30)

\[ \sum_v \frac{n A(v, t, s)}{n^{2v}} x^v, \] (31)

then Hadamard’s multiplications theorem could be applied to the given series, and some bound could be found. It is very important to remark here, that the derived bound would be for a specified \( t \) and \( s \). It follows that only tight inequalities would satisfy us, because we need to sum over all \( t \) and \( s \), in the aim of getting \( E_B \).
6 Conclusions

In this work we have shown, how formal power series can be used and how one can operate on them, even if the series do not converge.

We have been analyzing the Cycle Poisson Case, looking for the error probability given by the recurrence equation, whose coefficients were dependent up to a depth two of indices.

Using appropriate generating functions, we have derived second order partial differential equation. It is shown that \( x \) took part like a parameter, and the nature of the solution was dependent only on variables \( y \) and \( z \), but not on \( x \). The nature of the solutions, i.e. how equation behaves is completely determined.

To know how the first coefficients look like, we simulated them. The high tendency of growing is obtained, even for small values of \( t \), the coefficients are going up to \( 10^{200} \).

Then we suggested totally new approach in the domain of Complex Analysis. The problem is divided into parts, looking for two appropriate series, whose scalar-multiplication form desired power sum, and try to bound both of them.

It is obvious that for examples whose recurrence equations derive some solvable partial differential equation, given method succeeds, otherwise it is on us how to approach the solution.

7 Appendix

7.1 Diagrams of Exponential Factors

On the following two pages are shown the simulated exponential factors \( g^{(t)}(v) \), for different \( t \)’s, derived in the Section 4, Boundary Conditions.
Figure 7: $g^{(1)}(v)$  

Figure 8: $g^{(2)}(v)$

Figure 9: $g^{(3)}(v)$  

Figure 10: $g^{(4)}(v)$

Figure 11: $g^{(5)}(v)$   

Figure 12: $g^{(10)}(v)$
Figure 13: $g^{(20)}(v)$

Figure 14: $g^{(30)}(v)$

Figure 15: $g^{(40)}(v)$

Figure 16: $g^{(50)}(v)$
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