Minisuperspace models in M-theory

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Abstract
We derive the full canonical formulation of the bosonic sector of eleven-dimensional supergravity and explicitly present the constraint algebra. We then compactify M-theory on a warped product of homogeneous spaces of constant curvature and construct a minisuperspace of scale factors. First classical behaviour of the minisuperspace system is analysed and then a quantum theory is constructed. It turns out that there are similarities with the ‘pre-big-bang’ scenario in string theory.

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1. Introduction

One of the most fundamental problems in theoretical physics is the search for a quantum theory which would unify gravity with other interactions. Over the past 20 years, superstring theory emerged as a successful candidate for this role. It was later discovered that all five superstring theories can all be obtained as special limits of a more general eleven-dimensional theory known as M-theory and, moreover, the low energy limit of which is the eleven-dimensional supergravity [1, 2]. The complete formulation of M-theory is however not known yet.

In a cosmological context, there is another approach to quantum gravity which was pioneered by DeWitt in [3]. Here, the gravitational action is reformulated as a constrained Hamiltonian system and then quantized canonically. The resulting wavefunction is sometimes referred to as the ‘wavefunction of the universe’ [4], as it describes the state of the universe. Such a wavefunction is a function on the superspace—an infinite-dimensional space of all possible metrics modulo the diffeomorphisms. Although this procedure of course does not give a full theory of quantum gravity, it does give a low energy approximation, which is enough to capture some quantum effects such as tunnelling [4, 5]. Since the behaviour of the wavefunction in the full infinite-dimensional superspace is difficult to analyse, models with a reduced number of degrees of freedom have been considered. In these models only a finite subset of the original degrees of freedom is allowed to vary, while the rest are fixed, so that the wavefunction becomes a function on a finite-dimensional minisuperspace. In the
early universe, it is perceived that quantum gravity effects should become important, and so such minisuperspace models, where the degrees of freedom are the spatial scale factor in the Friedmann–Robertson–Walker (FRW) metric and possibly a scalar field, have been used to explore different quantum cosmological scenarios [4–8].

With the advent of superstring theory, the above ideas have been applied in the context of superstring theory [9–14]. This time, however, the starting point is the lowest order string effective action, possibly with a dilaton potential or a cosmological constant put in. Hence compared to the pure gravity case, there are new degrees of freedom—the dilaton and any of the tensor fields that appear. The minisuperspace models studied in the quantum string cosmology setting now have the FRW scale factor and the dilaton field as the independent degrees of freedom. In particular, progress has been made in quantum string cosmological description of the ‘pre-big-bang scenario’ [10–12, 15]. In this scenario, the universe evolves from a weakly coupled string vacuum state to a FRW geometry through a region of large curvature. Classically, there is the problem that the pre-big-bang and post-big-bang branches are separated by a high-curvature singularity. However, in the minisuperspace model for a spatially flat \((k = 0)\) FRW spacetime with a suitable dilaton potential, it is possible to find a wavefunction which allows tunnelling between the two classically disconnected branches and this solves the problem of transition between the two regimes.

With M-theory being a good candidate to be a ‘theory of everything’, it is interesting to see what the canonical quantization of the low energy effective theory can give, given the achievements of this approach in the pure gravity and superstring theory contexts. In section 2, we start with the bosonic action for eleven-dimensional supergravity and reformulate the theory as a canonical constrained Hamiltonian system. The canonical formulation of eleven-dimensional supergravity has been considered before in [16, 17], but here we explicitly give the constraint algebra, at least for the bosonic constraints.

Then, in section 3, we reduce the system to a minisuperspace model. This is done by restricting the metric ansatz so that its spatial part is a warped product of a number of homogeneous spaces of constant curvature, and the supergravity 4-form is also restricted so that only its 4-space components are allowed to be non-zero. In the case when only one of the spatial components has non-vanishing curvature and the 4-form vanishes completely, it is possible to solve exactly both the classical equations of motion and the corresponding equation for the wavefunction.

In section 4, we consider the classical and quantum solutions in the cases of vanishing, negative and positive spatial curvature. Spatially flat M-theory minisuperspace models have also been considered in [18, 19]. We carefully consider these systems, fixing the gauge and paying attention to the self-adjointness of the Hamiltonian. It turns out that the positive and negative curvature cases can exhibit very similar behaviour to the string theory minisuperspace models described above with negative and positive dilaton potentials in the Hamiltonian, respectively. However it also turns out that in the negative curvature case the boundary condition which leads to a tunnelling effect, in fact, also leads to lost self-adjointness of the Hamiltonian. This is a purely mathematical consequence and certainly deserves further investigation to find out what the correct physical reason behind this is.

In section 5, we look at the case where the 4-form is switched on, the 3-space is flat and one other spatial component is of positive curvature.

We will be using the following conventions. The spacetime signature will be taken as \((- + \cdots +)\) and all the curvature conventions are the same as in [20]. Greek indices \(\mu, \nu, \rho, \ldots\) range from 0 to 10, while the indices \(\alpha, \beta, \gamma, \ldots\) range from 0 to 3. Latin indices \(a, b, c, \ldots\) range from 1 to 10. The units used are such that \(\hbar = c = 16\pi G^{(11)} = 1\).
2. Canonical formulation

In this section we set up the canonical formalism for the bosonic sector of eleven-dimensional supergravity, with the field content being just the metric \( \hat{g}_{\mu \nu} \) and the 3-form potential \( \hat{A} \), with field strength \( F = d \hat{A} \).

The action for the bosonic fields is \([21]\)

\[
S = \int d^{11}x \left( -\hat{g} \right)^{\frac{1}{2}} R^{(11)} - \frac{1}{2} \int F \wedge \ast F - \frac{1}{6} \int \hat{A} \wedge F \wedge F
\]

where \( \hat{g} = \det[\hat{g}_{\mu \nu}] \) and \( F_{\mu_1...\mu_4} = 4 \delta_{\mu_1} \hat{A}_{\mu_2 \mu_3 \mu_4} \). The eleven-dimensional alternating tensor \( \varepsilon_{\mu_1...\mu_{11}} \) is defined by

\[
\varepsilon_{\mu_1...\mu_{11}} = \left( -\hat{g} \right)^{-\frac{1}{2}} \eta_{\mu_1...\mu_{11}}
\]

where \( \eta_{\mu_1...\mu_{11}} = -\eta_{\mu_1...\mu_{11}} \) is the alternating symbol.

To decompose the metric into spatial and temporal parts, we use the following ansatz \([3]\):

\[
\hat{g}_{\mu \nu} = \left( -\alpha^{-2} + \beta_a \beta_b \right) \gamma_{ab}
\]

where \( \gamma_{ac} \gamma_{bc} = \delta_a^b \), and \( \beta_a = \gamma_{ab} \beta_b \).

Using this ansatz, we follow \([20]\) to express canonically the gravitational action.

Consider a hypersurface \( \Sigma \), given by \( t = \text{const} \). The future-pointing normal vector \( n^\mu \) to this hypersurface is given by

\[
n^\mu = \left( -\alpha^{-1}, 0 \right)
\]

and the corresponding covector is \( n_\mu = (-\alpha, 0) \), so hence \( n_\mu n^\mu = -1 \).

The second fundamental form \( K_{\mu \nu} \) for \( \Sigma \) is defined by

\[
K_{\mu \nu} = -h^\rho_{\mu} h^\sigma_{\nu} R_{\rho \sigma}
\]

where the semicolon denotes covariant differentiation with respect to the eleven-dimensional metric \( \hat{g} \) and \( h^\rho_{\mu} \) is the projector onto \( \Sigma \) defined by

\[
h^\mu_\nu = g^\mu_\nu + n_\mu n_\nu.
\]

Note that the sign in (5) depends on the convention used, so here we follow \([20]\).

From (5) we have in particular

\[
K_{ab} = -n_{a,b}.
\]

Using the definition of \( n^\mu \) (4), it can be shown that \([20]\)

\[
K_{ab} = \frac{1}{2} \alpha^{-1} (\beta_a b_b + \beta_b b_a - \gamma_{ab})
\]

where / denotes covariant differentiation with respect to the metric \( \gamma_{ab} \). Using the Gauss–Codazzi equation (8), the full eleven-dimensional curvature can be expressed in terms of the
intrinsic curvature of the hypersurface (that is, the curvature of the metric $\gamma_{ab}$) and the second fundamental form:

$$R^{(11)} = R^{(10)} - K_{ab}K^{ab} + K^2 - 2n^\mu n^\nu R^{(11)}_{\mu\nu}$$  \hspace{1cm} (8)

where $K = \gamma_{ab}K^{ab}$ and $K^{ab} = \gamma^{ac}\gamma^{bd}K_{cd}$. Hence, the gravitational Lagrangian density $L_{\text{grav}}$ is given by

$$L_{\text{grav}} = (-\hat{g})^{\frac{1}{2}} R^{(11)} = \alpha \gamma^{\frac{1}{2}} (R^{(10)} - K_{ab}K^{ab} + K^2 - 2n^\mu n^\nu R^{(11)}_{\mu\nu})$$

where $\gamma = \det(\gamma_{ab})$. In the action, the full derivative terms give rise to a surface integral. We neglect it, since it does not affect the dynamics of the system.

We now decompose the 3-form $\hat{A}_{\mu\nu\rho}$ as

$$\hat{A}_{0ab} = B_{ab}$$  \hspace{1cm} (10)

$$\hat{A}_{abc} = A_{abc}$$  \hspace{1cm} (11)

and correspondingly

$$F_{abcd} = 4\delta_{[a}A_{bcd]}$$  \hspace{1cm} (12)

$$F_{0abc} = \partial_0 A_{abc} - 3\partial_{[a}B_{bc]}.$$  \hspace{1cm} (13)

The $F^2$ term from the action (1) is decomposed as

$$F^{\mu_1...\mu_4}F_{\mu_1...\mu_4} = F_{abcd}F^{abcd} = 4F_{\perp bcd}F_{\perp bcd}$$  \hspace{1cm} (14)

where

$$F_{\perp bcd} = n^\mu F_{\mu bcd} = \alpha^{-1}(F_{0bcd} - \beta^n F_{abcd}).$$  \hspace{1cm} (15)

Looking at the Chern–Simons term $A \wedge F \wedge F$, we have

$$\eta^{\mu_1...\mu_11} \hat{A}_{\mu_1\mu_2\mu_3}F_{\mu_4...\mu_11}F_{\mu_1...\mu_11} = \eta^{\mu_1...\mu_11}(122B_{a_{12}a_1}\partial_{a_2}A_{a_3a_4a_5}\partial_{a_6}A_{a_7a_8a_9})$$

$$+ 8\eta^{a_1...a_{10}}(\partial_0 A_{a_1a_2a_3})A_{a_4a_5a_6}F_{a_7a_8a_9a_{10}}) + \text{total derivative term}.$$  \hspace{1cm} (16)

Again, we neglect the total derivative term, since it does not affect the equations of motion.

Bringing together (9), (14) and (16), we thus have the total Lagrangian

$$L_{\text{tot}} = \int d^{10}x (L_{\text{grav}} + L_{\text{form}})$$

where

$$L_{\text{grav}} = \gamma^{\frac{1}{2}} \alpha (R^{(10)} + K_{ab}K^{ab} - K^2)$$  \hspace{1cm} (17)

$$L_{\text{form}} = \gamma^{\frac{1}{2}} \left[ -\frac{\alpha}{48} F_{abcd}F^{abcd} + \frac{\alpha}{12} F_{\perp bcd}F^{\perp bcd} - \delta^a_{a_1...a_10} \left( \frac{1}{122} B_{a_1a_2a_3}\partial_{a_4}A_{a_5a_6a_7}\partial_{a_8}A_{a_9a_{10}} \right) + \frac{8}{12^2} (\partial_0 A_{a_1a_2a_3})A_{a_4a_5a_6}F_{a_7a_8a_9a_{10}} \right].$$  \hspace{1cm} (18)

We see that the canonical fields in this system are $\alpha, \beta^n, \gamma^{ab}$ which come from the gravitational Lagrangian, together with $A_{abc}$ and $B_{ab}$ which come from $L_{\text{form}}$. From the
Lagrangian densities (17) and (18) we can now write the canonical momenta conjugate to these variables:

\[ \pi = \frac{\partial L_{\text{tot}}}{\partial \dot{\alpha}} = 0 \]  
\[ \pi^a = \frac{\partial L_{\text{tot}}}{\partial \dot{\beta}^a,0} = 0 \]  
\[ p^{ab} = \frac{\partial L_{\text{tot}}}{\partial B_{ab,0}} = 0 \]  
\[ \pi^{ab} = \frac{\partial L_{\text{tot}}}{\partial \gamma_{ab,0}} = -\gamma^{\frac{1}{2}}(K^{ab} - \gamma^{ab}K) \]  
\[ \pi^{abc} = \frac{\partial L_{\text{tot}}}{\partial A_{abc,0}} = \frac{1}{6} \gamma^{\frac{1}{2}} F \bigg| F^{abc} - \frac{8}{124} \eta^{abcd} A_{d,1} A_{d,2} A_{d,3} A_{d,4} A_{d,5} A_{d,6} A_{d,7}. \]

Expressions (19a), (19b) and (19c) are known as primary constraints [22]. This means that the corresponding ‘velocities’ cannot be expressed in terms of the momenta, and are thus arbitrary.

Now that we have the canonical momenta, we can work out the Hamiltonian for this system. The canonical Hamiltonian is given by

\[ H_{\text{tot}} = \int d^{10}x (\alpha, \pi + \beta^a,0, \pi^a + \gamma_{ab,0}, \pi^{ab} + B_{ab,0}, p^{ab} + A_{abc,0}, \pi^{abc} - L_{\text{grav}} - L_{\text{form}}). \]

From [3], we know that the gravitational Hamiltonian \( H_{\text{grav}} \) is given by

\[ H_{\text{grav}} = \int d^{10}x (\alpha, \pi + \beta^a,0, \pi^a + \alpha \mathcal{H} + \beta_a \chi^a) \]

with

\[ \mathcal{H} = \gamma^{\frac{1}{2}}(K^{ab} K_{ab} - K^2 - R(10)) = G_{abcd} \pi^{ab} \pi^{cd} - \gamma^{\frac{1}{2}} R(10) \]

\[ \chi^a = -2\pi^{ab,\dot{b}} = -2\pi^{ab,\dot{b}} - \gamma^{da}(2\gamma_{bd,t} - \gamma_{bc,d}) \pi^{bc}. \]

where

\[ G_{abcd} = \frac{1}{2} \gamma^{-\frac{1}{2}}(\gamma_{ac} \gamma_{bd} + \gamma_{ad} \gamma_{bc} - \frac{2}{5} \gamma_{ab} \gamma_{cd}) \]

is the Wheeler–DeWitt metric.

Consider the remaining part

\[ H_{\text{form}} = \int d^{10}x (B_{ab,0}, p^{ab} + A_{abc,0}, \pi^{abc} - L_{\text{form}}). \]

Due to the constraint (19c), nothing can be done with the first term, but in \( A_{abc,0}, \pi^{abc} - L_{\text{form}} \) we have terms in \( A_{abc,0} \), but these can be expressed in terms of \( \pi^{abc} \) using (19e). First, define

\[ \tilde{\pi}^{abc} = \frac{1}{6} \gamma^{\frac{1}{2}} F^{abc}_1 \]

so that, from (19e),

\[ \tilde{\pi}^{abc} = \pi^{abc} + \frac{8}{124} \eta^{abcd} A_{d,1} A_{d,2} A_{d,3} A_{d,4} A_{d,5} A_{d,6} A_{d,7}. \]

Then from the definition of \( F^{abc}_1 \), we have

\[ A_{bcd,0} = \beta^d F_{abcd} + 3\beta_{[b} B_{cd]} + 6\alpha \gamma^{-\frac{1}{2}} \tilde{\pi}_{bcd}. \]
Using (26) to substitute $A_{bcd,0}$ for $\pi_{bcd}$, we can write the overall Hamiltonian in the form

$$H_{\text{tot}} = \int d^{10}x \left( \alpha \partial_0 \pi + \beta_a \partial_0 \pi^a + B_{ab} \partial_0 p^{ab} + \alpha \tilde{\mathcal{H}} + \beta_a \tilde{\chi}^a + B_{ab} \tilde{\chi}^{ab} \right)$$

(27)

where

$$\tilde{\mathcal{H}} = \mathcal{H} + \frac{1}{48} \gamma \bar{\lambda} F_{abcd} F^{abcd} + 3 \gamma^{-1} \tilde{\mathcal{R}}^{abc} \tilde{\mathcal{R}}_{abc}$$

(28a)

$$\tilde{\chi}^a = \tilde{\chi}^a + F^a_{bcd} \tilde{\mathcal{R}}^{bcd}$$

(28b)

$$\tilde{\chi}^{ab} = -3 \bar{\lambda}^{abc} \partial_x + \frac{1}{12} \eta^{abc} \eta^{ab} \partial_x \partial_y A_{abc} \partial_x \partial_y A_{abc}^{ab}.$$  

(28c)

We see from the Hamiltonian (27) that the quantities $\alpha$, $\beta_a$, and $B_{ab}$ are arbitrary, so we set the gauge as convenient.

In order for the primary constraints (19a)–(19e) to be consistent with the equations of motion, the time derivatives of $\pi$, $\pi^a$ and $p^{ab}$ must vanish. This corresponds to vanishing Poisson brackets of these momenta with $H_{\text{tot}}$. Immediately this leads to the secondary constraints [22]

$$\tilde{\mathcal{H}} = 0$$

(29a)

$$\tilde{\chi}^a = 0$$

(29b)

$$\tilde{\chi}^{ab} = 0.$$  

(29c)

Consequently, the Hamiltonian vanishes on the constraint surface.

The new constraints (29a)–(29c) also have to be consistent with the equations of motion. So their Poisson brackets with $H_{\text{tot}}$ must vanish on the constraint surface or else there will be further constraints. Calculating Poisson brackets with $H_{\text{tot}}$ reduces to working out the pairwise brackets between the quantities $\tilde{\mathcal{H}}$, $\tilde{\chi}^a$ and $\tilde{\chi}^{ab}$. The non-vanishing brackets between the canonical variables are

$$[\pi, \alpha'] = \delta(x, x')$$

$$[\pi^a, \beta'_a] = \delta^a_b \delta(x, x')$$

$$[B_{ab}, p'^{cd}] = \delta^{[cd]}_{ab} \alpha \delta(x, x')$$

$$[\gamma_{ab}, \pi'^{cd}] = \delta^{[cd]}_{ab} \delta(x, x')$$

$$[A_{abc}, \pi'^{def}] = \delta^{[def]}_{abc} \alpha \delta(x, x').$$

Here $'$ means that a quantity is evaluated at $x'$, $\delta(x, x')$ is the ten-dimensional delta function and $\delta^{[ab]}_{cd} = \delta^{[cd]}_{ab} \delta^{[de]}_{bc} \delta^{[ef]}_{cd}.$

Before proceeding to the derivation of the Poisson brackets, we note that in general, the brackets are expressed in terms of generalized functions—$\delta$-functions and their derivatives. So, the technically correct way to handle them is to introduce arbitrary test functions and consider the action of the generalized function on them.

The calculation can be simplified if we notice the following. For an arbitrary $\Lambda_{ab}$, we have

$$\delta_\Lambda A = \left[ A_{def}, \int \tilde{\chi}^{ab} A_{ab} d^{10}x' \right] = -3 \left[ A_{def}, \pi^{'abc} \right] A_{ab}^{'} d^{10}x'$$

$$= 3 \Lambda_{[def]}.$$  

(30)

So, this implies that $\tilde{\chi}^{ab}$ is the generator of the gauge transformation

$$\delta_\Lambda A = d\Lambda,$$  

(31)

and hence

$$\delta_\Lambda F = 0.$$  

(32)
Under this transformation, we have
\[
\delta \Lambda \mathcal{P}^{\text{def}} = \left[ \pi^{\text{def}}, \int \tilde{\chi}^{ab} A_{ab} d^{10}x \right] = \left[ \pi^{\text{def}}, \int \frac{4}{12} \eta_{abc} \cdots \eta_d (\delta')_c A'_{a c s e c} (\delta')_c A'_{a c s e c} A'_{a b} d^{10}x' \right] = -\frac{2}{12} \eta^{\text{defabcdegh}} \partial_a \Lambda_{bc} F_{g d...g d}
\]
and hence
\[
\delta \Lambda \mathcal{P}^{\text{abc}} = 0.
\]
Using (32) and (34), it immediately follows that all brackets involving \( \tilde{\chi}^{ab} \) vanish identically, since relevant terms in each constraint involve only \( F \) and \( \tilde{\mathcal{P}}^{\text{abc}} \).

Consider the brackets with \( \tilde{\chi}_a \) now. After some index manipulation it is possible to rewrite \( \tilde{\chi}_a \) as
\[
\tilde{\chi}_a = \chi_a + F_{abcd} \pi_{bcd} - A_{abc} \tilde{\chi}_c - 3 \partial_a (\chi_{[ab} \pi_{c d]}).
\]
However, \( \tilde{\chi}^{bc} \) is also a constraint and moreover all its brackets with other constraints vanish, so we can replace \( \tilde{\chi}_a \) by an irreducible constraint \( \hat{\chi}_a \) given by
\[
\hat{\chi}_a = \chi_a + F_{abcd} \pi_{bcd} - 3 A_{abc} \pi^{bcd}.
\]
It is hence enough to work out the brackets with \( \hat{\chi}_a \).

In pure gravity, we know from [3] that \( \chi_a \) generates spatial translations. Hence \( \hat{\chi}_a \) acts on \( \gamma_{ab} \) and \( \pi_{ab} \)
\[
\gamma_{mn}, \int \hat{\chi}_a \xi^a d^{10}x' = L_\xi \gamma_{mn}
\]
\[
\pi_{mn}, \int \hat{\chi}_a \xi^a d^{10}x' = L_\xi \pi_{mn}.
\]
We can now work out the action of \( \hat{\chi}_a \) on \( A_{map} \) and \( \pi^{map} \)
\[
A_{bcd}, \int \hat{\chi}_a \xi^a d^{10}x' = \xi^a F_{abcd} + 3 \partial_b (\xi^a A_{[ab]} d^{10}x') = L_\xi A_{bcd}
\]
since \( A_{bcd} \) is a 3-form. For \( \pi^{bcd} \) we have
\[
\pi^{bcd}, \int \hat{\chi}_a \xi^a d^{10}x' = \partial_a (\pi^{bcd} \xi^a) - 3 \partial_a (\xi^a \pi_{[ab]} d^{10}x') + 3 \xi^a \pi_{[ab]} d^{10}x' = L_\xi \pi^{bcd}
\]
since \( \pi^{bcd} \) is a tensor density of weight 1. Therefore \( -\hat{\chi}_a \) generates spatial translations, and hence \( \hat{\chi}_a \) acts as a Lie derivative. Noting that \( \hat{\chi}_b \) is a covector and \( \hat{\mathcal{H}} \) is a scalar density of weight 1, we immediately see that
\[
\hat{\chi}_b, \int \hat{\chi}_a \xi^a d^{10}x' = L_\xi \hat{\chi}_b = (\xi^a \hat{\chi}_b)_c + \hat{\chi}_c \xi^a_b
\]
\[
\hat{\mathcal{H}}, \int \hat{\chi}_a \xi^a d^{10}x' = L_\xi \hat{\mathcal{H}} = (\hat{\mathcal{H}} \xi^a)_c.
\]
Introducing new test functions \( \sigma^a \) and \( \sigma \), respectively, we have
\[
\int \int \left[ \hat{\chi}_b, \hat{\chi}_a \right] \sigma^a d^{10}x d^{10}x' = \int \hat{\chi}_c (\xi^a \sigma^b - \xi^b \sigma^a) d^{10}x
\]
\[
\int \int \hat{\mathcal{H}} \hat{\chi}_a \sigma^a d^{10}x d^{10}x' = -\int \hat{\mathcal{H}} \xi^a \sigma^a d^{10}x
\]
after integration by parts. This gives that these brackets vanish on the constraint surface.
We are now only left with the bracket $[\tilde{\mathcal{H}}, \tilde{\mathcal{H}}']$. From [3], we already know $[\mathcal{H}, \mathcal{H}']$, so we only need to work out the brackets $[\mathcal{F}^2, \mathcal{H}^2]$ and $[\mathcal{H}^2, \mathcal{H}'^2]$, since the other cross-terms vanish. After some lengthy calculations, which are given in the appendix, we find that

$$[\tilde{\mathcal{H}}, \tilde{\mathcal{H}}'] = 2\tilde{\chi}^a \delta_a(x, x') + \tilde{\chi}^a \delta_\alpha(x, x')$$

which is analogous to the untilded expression for $[\mathcal{H}, \mathcal{H}']$ in [3]. In particular, $[\tilde{\mathcal{H}}, \tilde{\mathcal{H}}']$ vanishes on the constraint surface.

Hence, the full canonical description of the bosonic sector of eleven-dimensional supergravity involves only three primary constraints (19a)–(19e) and the three corresponding secondary constraints (29a)–(29c). These constraints are first-class constraints—that is, their pairwise brackets vanish on the constraint surface and they generate gauge transformations [22].

Consider now the quantization of this system. Adopting the same view as in [3], we will take it that any two field operators taken at the same spacetime point commute. This way, the classical consistency conditions carry over to the quantum case without anomalies. So we can perform Dirac quantization [22] of the system. The constraints then become conditions on the wavefunction $\Psi$:

$$\tilde{\mathcal{H}} \Psi = 0 \quad \tilde{\mathcal{F}}^a \frac{\partial}{\partial \Psi^a} = 0 \quad \tilde{\chi}^{ab} \frac{\partial}{\partial \Psi^{ab}} = 0.$$ (42a)

This implies that $H_{\alpha \beta} \Psi = 0$, and hence from the Schrödinger equation, $\partial \Psi/\partial t = 0$. The behaviour of the wavefunction $\Psi$ is completely determined by these constraints.

3. Minisuperspace

In general, the wavefunction $\Psi$ is a function on the infinite-dimensional superspace which consists of $\gamma_{ab}(x)$ and $A_{abc}(x)$ modulo diffeomorphisms and forms gauge transformations. Behaviour in this infinite-dimensional space is difficult to describe, so it is useful to reduce the number of variables, by fixing some degrees of freedom. This way the infinite-dimensional superspace is reduced to a finite-dimensional minisuperspace.

To reduce the number of degrees of freedom in the metric, we consider the following ansatz for the eleven-dimensional spacetime metric:

$$ds_{11}^2 = -\alpha(t)^2 dt^2 + \sum_{i=1}^{n} e^{2X^i(t)} d\Omega_i^2.$$ (43)

Here each $d\Omega_i^2$ is the metric of a maximally symmetric $a_i$-dimensional space with radius of curvature $\pm1$ or 0. Since the spacetime is eleven dimensional, we also have a condition $a_1 + \cdots + a_n = 10$. For each $i$, $e^{X^i}$ is the scale factor of each spatial component. Thus, the only remaining degrees of freedom which remain from $\gamma_{ab}$ are $X^i$. Such an ansatz was used before in [23], among others, to set up a cosmological minisuperspace model, however here we take these ideas further to write exact solutions of this model in certain cases, and we also consider the case with a non-vanishing 4-form, which is particularly relevant for M-theory.

For definiteness, suppose $a_1 = 3$ and consider the following ansatz for the 4-form:

$$F_{\alpha\beta\gamma\delta} = X^0(t) \delta_{\alpha\beta\gamma\delta}, \quad F_{\nu\rho\sigma} = 0 \text{ otherwise.}$$ (44a)

where $\delta_{\alpha\beta\gamma\delta}$ is the volume form on the 4-space with metric $ds_4^2 = -dt^2 + d\Omega_4^2$. A similar ansatz has been used in [24]. With this ansatz, the degrees of freedom $A_{abc}$ are reduced to just $X^0(t)$. We use this notation to explicitly highlight the fact that this degree of freedom
will also be part of our minisuperspace, at par with the gravitational degrees of freedom \( X^i \) for \( i = 1, \ldots, n \).

The second fundamental form \( K_{ab} \) is given in this case by
\[
K_{ab} = -\frac{1}{2} \alpha^{-1} \hat{y}_{ab}.
\]
From the metric ansatz, we immediately get
\[
K_{ab} K^{ab} = \alpha^{-2} \sum_{i=1}^{n} a_i (\dot{X}^i)^2
\]
\[
K^2 = \alpha^{-2} \dot{V}^2,
\]
where we have defined
\[
V = \sum_{i=1}^{n} a_i X^i.
\]
Hence, we have
\[
\gamma^{\frac{1}{2}} = \hat{\gamma}^{\frac{1}{2}} e^{\frac{1}{2} V}
\]
where \( \hat{\gamma} = \det(\hat{y}_{ab}) \) is the determinant of the normalized spatial metric \( \hat{y}_{ab} \).

With the ansatz (44a) for the 4-form, the Chern–Simons term in the action (1) vanishes and the \( F^2 \) term becomes
\[
\frac{1}{g_1^2} F^{\mu_1 \ldots \mu_4} F_{\mu_1 \ldots \mu_4} = -\frac{1}{2} \alpha^{-2} e^{-2a_1X^1} (\dot{X}^0)^2.
\]
If we assume spatial sections of finite volume, for simplicity we can normalize this volume to be unity. Thus, rewriting the action in terms of the new variables \( X^0 \) and \( X^i \), and integrating out the spatial integral, we obtain the action for the minisuperspace model \( S_{\text{mss}} \):
\[
S_{\text{mss}} = \int dt \left[ \mu^{-1} \left( \sum_{i=1}^{n} a_i (\dot{X}^i)^2 - \dot{V}^2 + \frac{1}{2} e^{-2a_1X^1} (\dot{X}^0)^2 \right) + \mu e^{2V} R^{(10)} \right]
\]
where we have defined
\[
\mu = \alpha e^{-V}.
\]
It can be shown explicitly that the equations of motion which are obtained from this action are equivalent to the equations obtained when our ansätze for the metric and the 4-form are substituted into the full field equations for supergravity. In particular, note that the equation of motion for \( X^0 \) is
\[
\frac{d}{dt}(\alpha^{-1} e^{V-2a_1X^1} \dot{X}^0) = 0.
\]
The field equation for the 4-form is
\[
\nabla_\mu F^{\mu_1 \ldots \mu_4 \nu_0 \rho_0 \sigma} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu_1 \ldots \mu_4 \nu_0 \rho_0 \sigma})
\]
\[
= \alpha^{-1} e^{-V} \partial_0 (\alpha e^{V} F^{\mu_0 \nu_0 \rho_0 \sigma}) + \frac{1}{\sqrt{g_1}} \partial_\mu (g_1^{\frac{1}{2}} g^{\frac{1}{2}} F^{\nu_0 \rho_0 \sigma}) = 0
\]
where \( g_1 \) is the determinant of the metric \( d\Omega_3^2 \) for the 3-space. The second term in the sum vanishes due to the ansatz (44a), and the remaining equation is precisely equivalent to (49).

From our metric ansatz (43), the general form of the spatial Ricci scalar \( R^{(10)} \) is
\[
R^{(10)} = \sum_{i=1}^{n} k_i \partial_i (a_i - 1) e^{-2X^i}
\]
where \( k_i = \pm 1 \) or 0. With this, the action (47) becomes

\[
S_{\text{mss}} = \int dt \left[ \mu^{-1} \left( \sum_{i=1}^{n} a_i (\dot{X}^i)^2 - \dot{V}^2 + \frac{1}{2} e^{-2n} \dot{X}^i (\dot{X}^0)^2 \right) + \sum_{i=1}^{n} \mu k_i a_i (a_i - 1) e^{2(V - X^i)} \right].
\]  

(50)

In the integrand above, we have a quadratic form in \( X^A \) for \( A = 0, 1, \ldots, n \). Let \( G_{AB} \) be the corresponding minisuperspace metric such that the Lagrangian is given by

\[
L_{\text{mss}} = \mu \left( \mu^{-1} \frac{1}{2} G_{AB} \dot{X}^A \dot{X}^B - V \right)
\]

where \( V \) is the effective potential given by

\[
V = - \sum_{i=1}^{n} \mu k_i a_i (a_i - 1) e^{2(V - X^i)}.
\]

The Hamiltonian is given by

\[
H = \mu \left( \frac{1}{2} G_{AB} P_A P_B + V \right)
\]

(52)

where \( P_A \) are conjugate momenta to \( X^A \) and \( G^{AB} \) is the inverse metric satisfying \( G^{AB} G_{BC} = \delta^A_C \). Hence, the canonical form of the action is

\[
S_{\text{mss}} = \int dt \left[ \dot{X}^A P_A - \mu \left( \frac{1}{2} G^{AB} P_A P_B + V \right) \right].
\]

(53)

As in the general case, the Lagrange multiplier \( \mu \) enforces the Hamiltonian constraint \( H = 0 \). First-class constraints are generators of gauge transformations, but in this case the Hamiltonian generates time reparametrizations, so the gauge transformations in this case are simply time reparametrizations. Therefore, this constraint gives rise to invariance under time reparametrization [25]. The gauge transformations generated by \( H \) are given by [26]

\[
\delta X^i = \varepsilon [X^i, H] \\
\delta P_i = \varepsilon [P_i, H] \\
\delta \mu = \frac{d\varepsilon}{dt}.
\]

We have a freedom of how to choose \( \mu \) and there are some natural choices for \( \mu \), but for now let us write \( \mu \) most generally as

\[
\mu = e^{-2f}
\]

(54)

for an arbitrary function \( f (X^A) \) [23]. The Hamiltonian equations obtained from (53) in the gauge (54) are equivalent to the system with the Hamiltonian \( H^f \) given by

\[
H^f = \frac{1}{2} (G^f)^{AB} P_A P_B + e^{-2f} V
\]

(55)

together with the crucial constraint

\[
H^f = 0,
\]

(56)

where

\[
(G^f)^{AB} = e^{-2f} G^{AB}.
\]

Thus, effectively, different gauge choices correspond to different conformal transformations of the minisuperspace metric.

Quantization transforms the Hamiltonian constraint (56) into the Wheeler–DeWitt equation [3]

\[
\hat{H}^f \psi^f = 0
\]

(57)
where $\hat{H}^f$ is the operator corresponding to (55) and $\Psi^f$ is the wavefunction in the $f$-gauge. A general prescription for the quantized Hamiltonian operator is

$$\hat{H}^f = F(X)\left(-\frac{1}{2}\Delta^f + a\mathcal{R}^f + e^{-2f}V\right)$$

where $\Delta^f$ is the Laplace–Beltrami operator of the minisuperspace metric $G^f$, $\mathcal{R}^f$ is the Ricci scalar of the metric $G^f$, $a$ is a constant and $F(X)$ is just some function of $X^A$. Such an operator but with $F = 0$ was first suggested by Hawking and Page [27]. This operator is covariant under general coordinate transformations on the minisuperspace, which is the main reason for choosing this operator ordering when quantizing the Hamiltonian. We want equation (57) to be equivalent for any choice of $f$. This is true if and only if (57) is equivalent to

$$\hat{H}/\Psi^1 = 0 \quad (58)$$

where $\hat{H} = \hat{H}^f = 0$, $\Psi^1 = \Psi^f = 0$. From the theory of scalar fields in curved spacetimes [28] we set

$$\psi^f = e^{bf}\psi.$$  \hspace{1cm} (59)

and then in order for (57) and (58) to be equivalent, it is known that we need $a = \frac{n-1}{8}$ and $b = \frac{1-2}{2}$. To construct a suitable Hilbert space, we need an inner product which would also be invariant under change of $f$, and also in which $\hat{H}^f$ would be Hermitian. It turns out that then the measure on the minisuperspace is given by

$$d\omega^f = e^{-2f}\sqrt{|G^f|}dX^0\cdots dX^n \quad (60)$$

with $|G^f| = |\det(G^f_{AB})|$. With this choice of the measure, $F(X) = e^{2f}$. Hence overall,

$$\hat{H}^f = e^{2f}\left(-\frac{1}{2}\Delta^f + a\mathcal{R}^f + e^{-2f}V\right)$$

and we can say that $\hat{H}^f$ and $\hat{H}$ are related by

$$\hat{H}^f/\Psi^1 = e^{-\frac{n-1}{2}}\hat{H}(e^{\frac{n-1}{2}}/\psi).$$

The (non-gauge-fixed) inner product is then

$$\langle \psi^f_1, \psi^f_2 \rangle = \int \psi^f_1^*\psi^f_2 d\omega^f \quad (62)$$

and it is indeed invariant under changes of $f$.

From the inner product (62) we see that the momentum representation is of the form

$$\hat{P}_A = -i\left(\partial_A + \frac{1}{2\sqrt{G}}\partial_A\sqrt{G} + \frac{n-1}{2}\partial_Af\right)$$

where the extra term is chosen such that $\hat{P}_A$ is Hermitian with respect to this inner product.

As we have seen, different gauge choices correspond to different definitions of the time parameter in the system, so in particular there are two gauges which will be most useful for us:

- The gauge $f = 0$, which corresponds to the choice $\mu = 1$ and hence $\alpha = e^V$. This choice leads to greatly simplified calculations. In [18], the time parameter in this gauge is referred to as gauge proper time, $t_g = t - t_0$.
- The gauge $f = \frac{1}{2}V$, which corresponds to the choice $\mu = e^{-V}$ and hence $\alpha = 1$. The time parameter in this gauge is the cosmic proper time, $t_c$, given by

$$\frac{dt_c}{dr} = e^V. \quad (64)$$
4. Minisuperspace solutions with a trivial 4-form

Now consider a special case of the above scenario. Here, we will consider solutions with a trivial 4-form and we will take the spatial Ricci scalar to be

$$R^{(10)} = \frac{1}{2} K^2 e^{-2X_1}$$

where $$K^2 = 2k_1(a_1 - 1)$$. This means that only one spatial component of the space has non-vanishing curvature, and the other components are flat. In particular, this special case encompasses the scenario where the external four-dimensional spacetime has a Friedmann–Robertson–Walker metric with $$k = -1, 0, +1$$, and the seven-dimensional internal space is a Ricci-flat compact manifold. This case is of interest from a cosmological point of view and also from the point of view of M-theory special holonomy compactifications. Hence, the Lagrangian is now given as

$$L = \frac{1}{2} G_{ij} \dot{X}^i \dot{X}^j + \frac{1}{2} K^2 e^{-2f} e^{2(V - X_1)}$$

and the corresponding Hamiltonian

$$H = \frac{1}{2} (G^{ij} p_i p_j - \frac{1}{2} K^2 e^{-2f} e^{2(V - X_1)})$$

for $$i, j = 1, \ldots, n$$. In order to obtain explicit solutions in both classical and quantum cases, it is necessary to diagonalize the minisuperspace metric $$G^{ij}$$ in such a way that $$V - X_1$$ becomes an independent variable. This is achieved by making the following change of variables:

$$Y_1 = b_1[(a_1 - 1)X^1 + a_2X^2 + \cdots + a_nX^n]$$

$$Y_2 = b_2[(a_1 + a_2 - 1)X^2 + a_3X^3 + \cdots + a_nX^n]$$

$$Y_3 = b_3[(a_1 + a_2 + a_3 - 1)X^3 + a_4X^4 + \cdots + a_nX^n]$$

$$\ldots$$

$$Y_n = b_n[(a_1 + a_2 + \cdots + a_n - 1)X_n]$$

where the coefficients $$b_i$$ are defined by

$$b_1^2 = 2a_1(a_1 - 1)^{-1}$$

$$b_2^2 = 2a_2(a_1 - 1)^{-1}(a_1 + a_2 - 1)^{-1}$$

$$\ldots$$

$$b_n^2 = 2a_n(a_1 + \cdots + a_{n-1} - 1)^{-1}(a_1 + \cdots + a_n - 1)^{-1}$$.

Then,

$$G_{ij} \dot{X}^i \dot{X}^j = e^{2f}[-(\dot{Y}_1)^2 + (\dot{Y}_2)^2 + \cdots + (\dot{Y}_n)^2]$$

and moreover

$$V = \frac{1}{2} (b_1 Y_1 - b_2 Y_2 - \cdots - b_n Y_n).$$

(65)

We can now write the Hamiltonian:

$$H_{\text{min}} = \frac{1}{2} e^{-2f} \left[\left(-p_1^2 + p_2^2 + \cdots + p_n^2\right) - K^2 e^{2b_1^{-1}Y_1}\right]$$

(66)

where $$p_i$$ are momenta conjugate to $$Y_i$$. The constraint $$H_{\text{min}} = 0$$ becomes simply

$$\left(-p_1^2 + p_2^2 + \cdots + p_n^2\right) - K^2 e^{2b_1^{-1}Y_1} = 0.$$ (67)

Taking into account the constraint (67), the classical equations become

$$p_1 = K^2 b_1^{-1} e^{-2f} e^{2b_1^{-1}Y_1}$$

$$Y_1 = -e^{-2f} p_1$$

$$p_j = 0$$

$$Y_j = e^{-2f} p_j$$

(68)
where $j = 2, \ldots, n$. Since the ‘potential’ does not depend on $Y^i$, we get that $p_i$ are constant. Also here we see that in order to be able to solve these equations easily, it is convenient to choose a gauge time parameter $\tau$ given by

$$
\tau = \int_0^t e^{-\int f} \, dt.
$$

(69)

Hence in the gauge $f = 0$ we have $\tau = t$. Changing the time parameter, the equations of motion simplify drastically becoming

$$
\dot{p}_1 = K^2 b_1^{-1} e^{2b_1^{-1}Y^1} \quad \dot{Y}^1 = -p_1
$$

$$
\dot{p}_j = 0 \quad \dot{Y}^j = p_i
$$

(70)

where the dot now denotes time derivatives with respect to $\tau$. These are the same equations we would get with time parameter $t$. Thus we get solutions for $j = 2, \ldots, n$:

$$
Y^j = p_i \tau + Y^j_0
$$

where $Y^j_0$ are constants. Since all momenta except $p_1$ are constant in $\tau$, we can rewrite the Hamiltonian constraint (67) as

$$
K^2 e^{2b_1^{-1}Y^1} = \xi^2 - p_1^2
$$

(71)

where $\xi$ is a constant given by

$$
\xi^2 = p_2^2 + \cdots + p_n^2.
$$

(72)

Using (71), the equation of motion for $p_1$ becomes

$$
\dot{p}_1 = b_1^{-1}(\xi^2 - p_1^2).
$$

(73)

We have thus seen that after a change of variables on the minisuperspace, the classical minisuperspace system is described by equations (71), (73) and the relation between $Y^1$ and $p_1$. Essentially these are equations of motion of a particle moving in the potential $-K^2 e^{2b_1^{-1}Y^1}$ constrained so that the total energy vanishes. Apart from the initial conditions, the solutions depend on the curvature parameter $K^2$. In fact, from (71) we see that the sign of $K^2$ affects the nature of equation (73) and hence the qualitative behaviour of the solution.

A similar system is considered in [29–31], where the dynamics of scale factors is studied in the presence of wall potentials near a cosmological singularity, giving rise to ‘cosmological billiards’.

From (65), the volume factor $e^V$ is given by

$$
e^V = \exp \left[ \frac{1}{\Xi} (b_1 Y^1 - b_2 Y^2 - \cdots - b_n Y^n) \right]
$$

$$
= A e^{b_1 Y^1} e^{-\frac{1}{2} p_1 \tau}
$$

(74)

where $A = \exp (b_2 Y^2_0 + \cdots + b_n Y^n_0)$ and $p_i = b_2 p_2 + \cdots + b_n p_n$.

We now proceed to the quantization of the minisuperspace model. The canonical variables in the minisuperspace are now $Y^i$ for $i = 1, \ldots, n$, and the corresponding momenta $p_i$ for $i = 1, \ldots, n$. The minisuperspace metric is now conformally flat and is fully flat in the gauge $f = 0$. Eventually we will set $f = 0$, so we will disregard terms involving $f$. Hence in our general expression for the Hamiltonian operator (61), the minisuperspace curvature term $R$ vanishes, and the Laplace–Beltrami operator reduces to the flat wave operator, and moreover the expression for momentum operators (63) reduces to

$$
\hat{p}_i = -i \partial_{Y^i}.
$$

Hence the Wheeler–DeWitt equation (57) for this model is simply

$$
(-\partial_{Y^1}^2 + \partial_{Y^2}^2 + \cdots + \partial_{Y^n}^2) \Psi + K^2 e^{2b_1^{-1}Y^1} \Psi = 0.
$$

(75)
This equation separates, and we get
\[ \frac{\partial^2}{\partial^2 Y_1} G_{k_1} - (K^2 e^{2b_1 Y_1} - k_1^2) G_{k_1} = 0 \]  
(76)

where
\[ \Psi = e^{ik_1 Y_2} \cdots e^{ik_n Y_n} G_{k_1}(Y^1). \]  
(77)

Here, \( k_1 \) is given by
\[ k_1 = k_2 + \cdots + k_n \]  
(78)

and \( k_i \) for \( i \geq 2 \) are eigenvalues of the momenta \( p_i \). Note that (76) is the precise quantum analogue of the classical constraint (71). Moreover, it can be viewed as a one-dimensional Schrödinger equation with an exponential potential \( K^2 e^{2b_1 Y_1} \).

Let us now discuss gauge fixing in this system. Consider the following change of variables:

for \( j = 2, \ldots, n \) let
\[ \xi_j = \frac{Y_j}{p_j}, \quad p_{\xi_j} = \frac{1}{2} p_j^2. \]  
(79)

For these variables, the equations of motion become
\[ \dot{\xi}_j = 1, \quad p_{\xi_j} = 0 \]
and hence the Hamiltonian \( H_{\text{mass}} \) in these variables is given by
\[ H_{\text{mass}} = e^{-2f} \left[ -\frac{1}{2} p_1^2 - \frac{1}{2} K^2 e^{2b_1 Y_1} + p_{\xi_2} + \cdots + p_{\xi_n} \right]. \]  
(80)

In the reduced phase space method, we take the gauge choice
\[ \xi_n = t = 0. \]  
(81)

From the equations of motion this further imposes \( t = \tau \), and hence \( f = 0 \). Hence we get the gauge proper time. The effective Hamiltonian is now
\[ H_{\text{eff}} = -\frac{1}{2} p_1^2 - \frac{1}{2} K^2 e^{2b_1 Y_1} + p_{\xi_2} + \cdots + p_{\xi_n} = -p_{\xi_n} \]  
(82)

where \( p_{\xi_n} \geq 0 \). Thus, the gauge-fixed Hamiltonian now does not vanish in general. So, when quantizing we precisely get the Schrödinger equation
\[ i \frac{\partial \Psi}{\partial t} = \hat{H}_{\text{eff}} \Psi \]
where
\[ \hat{H}_{\text{eff}} = -\left( -\frac{1}{2} \frac{\partial^2}{\partial^2 Y_1} + i \partial_{\xi_2} + \cdots + i \partial_{\xi_{n-1}} + \frac{1}{2} K^2 e^{2b_1 Y_1} \right) \]  
(83)

and so the solutions are hence
\[ \Psi = e^{ik_1 Y_2} \cdots e^{ik_{n-1} Y_{n-1}} e^{ik_n G_{k_1}(Y^1)} \]  
(84)

with \( G_{k_1}(Y^1) \) satisfying (76). So, the gauge-fixed solutions have essentially the same form as (77), but with the condition (81) imposed and with \( \xi_i = \frac{Y_i}{\tau} \) for \( i = 2, \ldots, n - 1 \).

Alternatively, we can use the Faddeev–Popov method. From (80), the Wheeler–DeWitt equation is
\[ -\frac{1}{2} \frac{\partial^2}{\partial^2 Y_1} + i \partial_{\xi_2} + \cdots + i \partial_{\xi_{n-1}} + i \partial_{\xi_n} + \frac{1}{2} K^2 e^{2b_1 Y_1} = 0 \]
and the solutions are
\[ \Psi = e^{ik_1 Y_2} \cdots e^{ik_{n-1} Y_{n-1}} e^{ik_n G_{k_1}(Y^1)}. \]  
(85)
The full gauge-fixed inner product is given by

\[
\langle \Psi_1|\Psi_2 \rangle = \int dY^1 d\xi_2 \cdots d\xi_n \Psi_1^*(Y^1, \xi_2, \ldots, \xi_n) \delta(\Theta) \Delta_{FP} \Psi_2(Y^1, \xi_2, \ldots, \xi_n)
\]

where \( \Theta = 0 \) is the gauge condition and \( \Delta_{FP} \) is the Faddeev–Popov determinant. For \( \Theta = \xi_n - t \), which gives the gauge condition (81), \( \Delta_{FP} = 1 \), so the gauge-fixed inner product is

\[
\langle \Psi_1|\Psi_2 \rangle = \int dY^1 d\xi_2 \cdots d\xi_{n-1} \Psi_1^*(Y^1, \xi_2, \ldots, \xi_{n-1}, t) \Psi_2(Y^1, \xi_2, \ldots, \xi_{n-1}, t)
\]

(86)
giving a positive definite Hilbert space. The solutions (84) and (85) are basically identical, and the gauge-fixed measure derived using the Faddeev–Popov method is precisely the measure of the reduced space. Hence the two methods are equivalent.

In any case, the key non-trivial part of a solution of the Wheeler–DeWitt equation is the function \( G_{k_i}(Y^1) \) which is a solution of equation (76), which we can rewrite as an eigenvalue problem

\[
\hat{H}_Y G_{k_i} = -k_i^2 G_{k_i},
\]

(87)

Thus this operator seems to occur in many different settings, and as such has been quite well studied. In particular, it appears in Liouville theory [32, 33] and also appears in models such as rolling tachyons [34–36]. Interestingly, various minisuperspace models also contain the same type of equation, even though the potential is derived from different perspectives [10, 11, 14]. Setting \( z = |K| b_1 e^{k_i^1 y^1} \), we get

\[
z^2 \frac{\partial^2 H}{\partial z^2} + z \frac{\partial H}{\partial z} - (\text{sgn}(K^2) z^2 - b_1^2 k_i^2) H = 0.
\]

(88)

For \( K^2 < 0 \), this is Bessel’s equation with an imaginary parameter, and for \( K^2 > 0 \) this is the modified Bessel’s equation also with an imaginary parameter. Hence, the solutions of (87) are linear combinations of appropriate Bessel functions.

The operator \( \hat{H}_Y \) is clearly Hermitian on the domain \( D_0 \) of smooth functions with compact support, but this is not enough for a full definition of a self-adjoint operator. To construct a self-adjoint operator, we follow the general theory as set out in [37]. First we take the domain \( D_0 \) of the closure of \( \hat{H}_Y \) on \( D_0 \), and work out the deficiency indices \( n^\pm \) of \( \hat{H}_Y \). In our case, this corresponds to solving equation (87) for eigenvalues \( \pm i \) and determining the number of independent square-integrable solutions in each case. General results say that there exist self-adjoint extensions of the operator if and only if \( n^+ = n^- \), and moreover the operator is already self-adjoint if and only if \( n^+ = 0 \). So to find the self-adjoint extensions we need to solve

\[
\hat{H}_Y \phi = \pm i \phi
\]

(89)

For \( K^2 > 0 \), the independent solutions of (89) are modified Bessel functions of first and second kind—\( I_{\eta^\pm}(z) \) and \( K_{\eta^\pm}(z) \), respectively, with \( \eta^+ = e^{\frac{3}{4} \pi i} \) and \( \eta^- = e^{\frac{1}{4} \pi i} \). However all of these solutions are unbounded, and hence clearly not square-integrable. Thus in this case, the deficiency indices both vanish, and so the operator \( \hat{H}_Y \) is self-adjoint.

For \( K^2 < 0 \), the independent solutions of (89) are

\[
\phi_{1}^\pm(z) = J_{\eta^\pm}(z) \quad \text{and} \quad \phi_2^\pm(z) = J_{-\eta^\pm}(z),
\]

where

\[
\eta^\pm = e^{\frac{3}{4} \pi i} \quad \text{and} \quad \eta^- = e^{\frac{1}{4} \pi i}.
\]
where $J_\nu(z)$ are Bessel functions of the first kind. In this case, only $\phi_1^-$ and $\phi_2^+$ are squareintegrable, hence the deficiency indices are $n^+ = n^- = 1$, and hence there is a one-parameter family of self-adjoint extensions defined by

$$D_\theta = \{ \phi + \alpha (\phi_1^- + e^{2\pi i \theta} \phi_2^+) \mid \phi \in D_0, \alpha \in \mathbb{C} \}$$

(90)

where $\theta \in (0, 1]$ is the parameter which defines the extension. As pointed out in [34], the above prescription for the self-adjoint extension domain defines the asymptotic behaviour of functions in $D_\theta$ since for $Y^1 \to +\infty$ all eigenfunctions of $\hat{H}_T$ have a slower rate of decay than functions from $D_0$. Therefore, the asymptotic behaviour of functions in $D_\theta$ has to be compatible with the asymptotic behaviour of $\phi_1^- + e^{2\pi i \theta} \phi_2^+$ for each $\theta$.

4.1. Case 1: $K^2 = 0$

Suppose the spatial curvature fully vanishes, so that $K^2 = 0$. Classically this gives that $p_1$ is also constant, and $Y^1$ is given by

$$Y^1 = -p_1 t + Y_0^1$$

for a constant $Y_0^1$. From (71) we see that $p_1^2 = \xi^2$. Hence in our gauge, the solutions are rather trivial.

Note that from (64) the cosmic and gauge time parameters are related by

$$t_c = -\frac{2A}{b_1 p_1 + \cdots + b_n p_n} e^{-\frac{1}{2}(b_1 p_1 + \cdots + b_n p_n)t}$$

So depending on the sign of the quantity $(b_1 p_1 + \cdots + b_n p_n)$, $t_c$ is either always positive for all values of $t$ or always negative for all values of $t$. Overall, this can be regarded as a generalization of the Kasner metric. Such solutions have been obtained many times before—both in a purely gravitational context [38] or as here, a special case of an M-theory model [18]. To relate to variables $\alpha$, $\beta$ and $\phi$ used in [18] and [19], set $a_1 = 3$, $a_2 = 6$ and $a_3 = 1$, together with

$$X^1 = \frac{1}{2} \alpha - \beta - \frac{1}{6} \phi$$
$$X^2 = -\frac{1}{2} \alpha - \frac{1}{6} \phi$$
$$X^3 = \alpha + 2\beta + \frac{1}{3} \phi.$$ 

With these relations, our Lagrange multipliers $\mu$ agree, and hence the Lagrangian (51) becomes

$$L_{\text{max}} = \mu^{-1}(3\dot{\alpha}^2 - \phi^2 + 6\dot{\beta}^2).$$

thus precisely as in [18] and [19].

Back to our variables, in the quantum case, the gauge-fixed wavefunctions which are orthonormal in the gauge-fixed measure are

$$\Psi_{k_1, \ldots, k_n}(Y^1, \xi_2, \ldots, \xi_{n-1}, t) = (2\pi)^{-\frac{n+1}{2}} e^{ik_1 Y^1} e^{ik_2 \xi_2} \cdots e^{ik_{n-1} \xi_{n-1}} e^{ik_n t},$$

where

$$k_1^2 = k_2^2 + \cdots + k_n^2.$$
4.2. Case 2: $K^2 > 0$

Suppose $K^2 > 0$. From the constraint (71) we see that we must have $|p_1| < \xi$. From (73), and using the condition on $p_1$, we get

$$b_1^{-1} \int dt = \int \frac{dp_1}{\xi^2 - p_1^2} = \xi^{-1} \arctanh (\xi^{-1} p_1).$$

Hence,

$$p_1 = \xi \tanh (b_1^{-1} \xi t + t_0).$$

Now $Y^1$ is determined by

$$\dot{Y}^1 = -\xi \tanh (b_1^{-1} \xi t + t_0).$$

So,

$$Y^1 = c_1 - b_1 \log \left( \cosh (b_1^{-1} \xi t + t_0) \right).$$

Relation (71) fixes the constant $c_1$, hence $Y^1$ is given by

$$Y^1 = -b_1 \log \left( K \xi^{-1} \cosh (b_1^{-1} \xi t + t_0) \right).$$

Figure 1 shows the behaviour in phase space (with $t_0 = 0$). We can see that this solution has only one branch—the negative and positive momentum sectors are smoothly connected.

We have the following asymptotic behaviour for $Y^1$ and $p_1$:

$$t \to +\infty \quad p_1 \to \xi \quad Y^1 \sim -\xi t = -p_1 t$$

$$t \to -\infty \quad p_1 \to -\xi \quad Y^1 \sim +\xi t = -p_1 t.$$  

Let us now investigate the behaviour of the original scale factors $X^i$. From the definition of $Y^1$, $X^1 = V - b_1^{-1} Y^1$. So let us first look at the asymptotic behaviour of $V$. From (65), up to a constant we have

$$V = \frac{1}{2} b_1 Y^1 - \frac{1}{2} p_1 t.$$

Thus from the asymptotic behaviour of $Y_1$ (93), as $t \to \pm \infty$ we get

$$V \sim -\frac{1}{2} t (p_1 \pm b_1 \xi).$$

(94)
Note that \( p_1^2 - b_1^2 \xi^2 \leq 0 \), but since \( b_1 \xi > 0 \), we have \( p_s - b_1 \xi < 0 \) and \( p_s + b_1 \xi > 0 \). Therefore, \( V \to -\infty \) as \( t \to \pm \infty \), so in fact \( V \) has very similar asymptotic behaviour to \( Y^1 \). From (94), the behaviour of \( X^1 \) is hence easily obtained:

\[
X^1 \sim -\frac{1}{2} t \left( p_s \pm b_1 (1 - 2b_1^{-2}) \xi \right)
\]

It follows that the qualitative behaviour of \( X^1 \) does actually depend on the numerical values of the constant momenta \( p_s \). It can easily be seen now that all other \( X^i \) will also be asymptotically proportional to \( t \), but with different constants of proportionality which also depend on the initial conditions.

By construction, the overall eleven-dimensional space is Ricci-flat. However, let us look at what happens to the intrinsic curvature from four-dimensional point of view. The expression for the four-dimensional Ricci scalar is given by

\[
R^{(4)} = \frac{4}{3} K^2 e^{-2X^1} + \alpha^{-2} [(a_1 - 1)X^1 + a_1(a_1 - 1)(X^1)^2].
\]

After changing variables and the time parameter, and applying the constraint and equations of motion, we get

\[
R^{(4)} = e^{-2V} Q(p_1)
\]

(95)

where \( Q(p_1) \) is a quadratic expression in \( p_1 \) with constant coefficients, the precise form of which is not important here. Since \( p_1 \) is always bounded (91), the curvature blows up when \( V \to -\infty \), and as we know this does happen when \( t \to \pm \infty \). So although the eleven-dimensional space is flat, from the four-dimensional point of view there is a curvature singularity.

The solutions we had so far were in the gauge time parameter \( t \). To relate it to the cosmic time parameter \( t_c \), we need to integrate \( e^V \). In this case,

\[
e^V = A \left[ K^{-1} \xi \text{sech}(b_1^{-1} \xi (t - t_0)) \right]^{\frac{1}{2b_1}} e^{-\frac{1}{2} p_s t}
\]

where, as before, \( p_s = b_2 p_2 + \cdots + b_n p_n \). For \( t_0 = 0 \), the integral of this expression can be evaluated explicitly in terms of the hypergeometric function \( _2F_1(a, b; c; z) \) [39]:

\[
t_c(t) = A (2K^{-1} \xi)^{\frac{1}{2b_1}} \frac{2}{b_1 \xi - p_s} e^{\frac{1}{2} (b_1 \xi - p_s)} _2F_1 \left( \frac{1}{2}, \frac{b_1}{4 \xi} (b_1 \xi - p_s); 1 + \frac{b_1}{4 \xi} (b_1 \xi - p_s); -e^{b_1^{-1} t} \right)
\]

From this we can at least extract asymptotic behaviour of \( t_c \) as \( t \to \pm \infty \) [39],

\[
t_c \sim c^+_0 = \frac{2A (2K^{-1} \xi)^{\frac{1}{2b_1}}}{p_s \pm b_1 \xi} e^{\frac{1}{2} (p_s \pm b_1 \xi)}
\]

(96)

where \( c^+_0 \) are constants, which we can choose such that \( c^+_0 = 0 \). This behaviour is hence similar to the \( K^2 = 0 \) case for \( p_1 = \pm \xi \). We know that \( p_s - b_1 \xi < 0 \) and \( p_s + b_1 \xi > 0 \). Therefore as \( t \to -\infty \), the cosmic time parameter \( t_c \) approaches 0 from above, and as \( t \to +\infty \), \( t_c \) approaches \( c^+_0 \) from below.

Hence overall, at small \( t_c \), the overall size of the universe is very small, and the four-dimensional curvature is very high, then as \( t_c \) increases the size of the universe increases and hence the curvature decreases up to a point, after which the universe collapses again and the curvature blows up within a finite time \( c^+_0 \).
Now consider the quantized system. As we already know, for positive $K^2$, the solutions of the Wheeler–DeWitt equation \((88)\) are modified Bessel’s functions with the imaginary parameter $\sqrt{b_1 k_1}$. So the solutions are linear combinations of functions $J_{ib_1 k_1}(z)$ and $K_{ib_1 k_1}(z)$.

If we impose the condition that the function be bounded, this uniquely selects $K_{ib_1 k_1}(z)$ \([39]\). This choice selects the wavefunction which decays as $Y^1 \rightarrow +\infty$, which is consistent with the exponential potential in \((76)\). Boundary conditions for this type of wavefunctions have been well studied \([12]\). From the previous section, we know that the operator $\hat{H}_Y$ is self-adjoint in this case, and as pointed out in \([33]\) this means that there is only one family of orthogonal eigenfunctions.

The normalized gauge-fixed stationary wavefunctions with energy $E = -k_1^2$ are

$$\Psi_{k_1,\ldots,k_n}(Y^1, \xi_2, \ldots, \xi_{n-1}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \sqrt{\frac{2k_1 \sinh \pi b_1 k_1}{\pi^2}} e^{iz_2^2 \xi_2} \cdots e^{iz_{n-1}^2 \xi_{n-1}} K_{ib_1 k_1}(K b_1 e^{ib_1^1 Y^1}),$$

(97)

where as before

$$k_1^2 = k_1^2 + \cdots + k_n^2.$$

From \([40, 41]\), we know that

$$\int_0^\infty \frac{dx}{x} K_{\mu\nu}(x) K_{\nu\mu}(x) = \frac{\pi^2}{2\mu \sinh \pi \mu} [\delta(\mu - \nu) + \delta(\mu + \nu)].$$

(98)

For definiteness we fix $k_n > 0$ and thus the stationary wavefunctions \((97)\) are orthonormal in the gauge-fixed measure:

$$\langle \Psi_{k_1,\ldots,k_n} | \Psi_{\xi_1,\ldots,\xi_n} \rangle = \delta(k_2 - k_2') \cdots \delta(k_n - k_n').$$

(99)

These functions are not in the Hilbert space $\mathcal{H}$ of square-integrable functions since they show oscillatory behaviour as $Y^1 \rightarrow -\infty$. This problem is resolved by introducing a weight distribution $\rho_i(k_2, \ldots, k_n)$, so that the function

$$\Psi_i(Y^1, \xi_2, \ldots, \xi_{n-1}) = \int \rho_i(k_2, \ldots, k_n) \Psi_{k_1,\ldots,k_n}(Y^1, \xi_2, \ldots, \xi_{n-1}) \, dk_2 \cdots dk_n$$

is square-integrable. From \((99)\) we see that for each $i$, $\Psi_i$ is indeed in $\mathcal{H}$ if and only if $\rho_i(k_2, \ldots, k_n)$ is square-integrable in $k_2, \ldots, k_n$, and $\Psi_i$ are orthonormal if $\rho_i$ are such. If we take $\rho_i$ to be highly peaked around $(k_2, \ldots, k_n)$, then the qualitative behaviour of the corresponding $\Psi_i$ can be very well approximated by the non-smeared function $\Psi_{k_1,\ldots,k_n}$ \([42]\).

Now, a general property of $K_{ib_1 k_1}(z)$ is that for $0 < z < |b_1 k_1|$, the function is oscillatory, and for $z > |b_1 k_1|$ the function decays with asymptotic behaviour as $z \rightarrow +\infty$ given by

$$K_{ib_1 k_1}(z) \sim e^{ib_1 k_1 z} \frac{1}{\sqrt{2\pi} z^{\frac{1}{2}}} e^{-z}.$$

(100)

In our case however, $z = K b_1 e^{ib_1^1 Y^1}$, so the wavefunction decays extremely fast for $z > |b_1 k_1|$. From the classical solution \((92)\),

$$z = K b_1 e^{ib_1^1 Y^1} \leq b_1 \xi,$$

so the region of the minisuperspace where the wavefunction is oscillatory corresponds to the classically allowed region, and outside it the wavefunction amplitude is negligibly small.

For $z \rightarrow 0$, the asymptotic behaviour of $K_{ib_1 k_1}(z)$ is \([39]\)

$$K_{ib_1 k_1}(z) \sim \frac{\pi}{2b_1 k_1 \sinh(b_1 k_1 \pi)} \left( \frac{1}{z} \right)^{ib_1 k_1} \left( \frac{1}{z} \right)^{-ib_1 k_1}.$$

(101)
Using this, for $Y^1 \rightarrow -\infty$, we have the following asymptotic behaviour for $\Psi$:

$$\Psi_{k_2\ldots k_n} \sim N e^{ik_1x_1} \ldots e^{ik_{n-1}x_{n-1}} \left( \frac{(\frac{1}{2}Kb_1)^{ib_1k_1}}{\Gamma(ib_1k_1)} e^{ib_1y_1} + \frac{(\frac{1}{2}Kb_1)^{-ib_1k_1}}{\Gamma(-ib_1k_1)} e^{-ib_1y_1} \right)$$

$$= \Psi^{(-)} + \Psi^{(+)}$$

(102)

where $N$ is a constant. Thus asymptotically, $\Psi$ splits into the left-moving and right-moving parts, $\Psi^{(-)}$ and $\Psi^{(+)}$ respectively, with the role of the time-like coordinate being assigned to $Y^1$. These plane waves move along the vector $(k_2, \ldots, k_{n-1})$ in the ‘space-like’ part of the minisuperspace. By applying the $Y^1$-momentum operator $p_1 = -i\frac{\partial}{\partial y_1}$, we find that the $p_1$ eigenvalue for $\Psi^{(-)}$ is $k_1$ and the eigenvalue for $\Psi^{(+)\text{r}}$ is $-k_1$. The constant $k_1$ corresponds to the classical quantity $\xi$ and therefore the left movers correspond to the sector of the classical solution where $p_1 > 0$ and the right movers correspond to the sector where $p_1 < 0$. Also, from (102), we can infer that $|\Psi|$ fluctuates with amplitude $\frac{N}{\Gamma(k_1\xi)}$.

The left-moving waves can be interpreted as reflections of the right-moving waves. Effectively, such a reflection is a transition from the negative momentum sector to the positive momentum sector. In the classical system, these sectors are smoothly connected, and in the quantum system this is manifested by the fact that the reflection coefficient between the two plane waves is $R = |\Psi^{(+)}|^2/|\Psi^{(-)}|^2 = 1$. So in fact the two sectors can be regarded as reflections of each other at $Y^1 \rightarrow -\infty$.

A similar behaviour was discussed in [11], in the context of a four-dimensional gravitation system with a negative, specially chosen dilaton potential. Here, the smooth branch transition arises naturally from a positive curvature background since the positive curvature term in our action gives rise to a negative potential in the Hamiltonian (66).

4.3 Case 3: $K^2 < 0$

Now suppose $K^2 < 0$. Letting $\tilde{K}^2 = -K^2$ we then have from (71)

$$\tilde{K}^2 e^{2\tilde{K}^{-1}y_1} = p_1^2 - \xi^2.$$  

(103)

Therefore in this case we have $|p_1| > \xi$, so from (73), and using the condition on $p_1$, we get

$$b_1^{-1} \int dt = \int \frac{dp_1}{\xi^2 - p_1^2} = \xi^{-1} \text{arccoth}(\xi^{-1}p_1).$$

Hence,

$$p_1 = \xi \coth(b_1^{-1}\xi t + t_0)$$  

(104)

and $Y^1$ is given by

$$Y^1 = -b_1 \log \left( |\tilde{K}\xi^{-1}\sinh(b_1^{-1}\xi t + t_0)| \right).$$  

(105)

Let $t_0 = 0$. Then the phase space behaviour is shown in figure 2. Now we see that there are two branches—one for which $p_1 > \xi$ and $t$ is positive, and one for which $p_1 < -\xi$ and $t$ is negative.

The asymptotic behaviour of $Y^1$, $V$ and $X^1$ as $t \rightarrow \pm \infty$ is the same as in the $K^2 > 0$ case, and similarly $R^{(4)}$ blows up when $t \rightarrow \pm \infty$. To give an explicit relation between time parameters $t$ and $t_e$, we first need

$$e^V = A |\tilde{K}^{-1}\xi \text{csch}(b_1^{-1}\xi t + t_0)|^{\frac{b_1^2}{2}} e^{-\frac{1}{2}p_1^2}.$$
In this case, at least for $t_0 = 0$, the integral can be explicitly evaluated in terms of the hypergeometric function $\binom{a, b}{c, z}$. Hence we have [39]

$$t_c(t) = \begin{cases} 
\text{Re} \left( 2A(-2\tilde{K}^{-1} \xi)^{\frac{1}{2}} e^{\frac{1}{2i}(b_1 \xi - p_x)} 2F_1 \left( \frac{1}{2} b_1^2, \frac{b_1 (b_1 \xi - p_x)}{4\xi}; 1 + \frac{b_1 (b_1 \xi - p_x)}{4\xi}; e^{2i \xi} \right) \right) & \text{for } t > 0 \\
2A(2\tilde{K}^{-1} \xi)^{\frac{1}{2}} e^{\frac{1}{2i}(b_1 \xi - p_x)} 2F_1 \left( \frac{1}{2} b_1^2, \frac{b_1 (b_1 \xi - p_x)}{4\xi}; 1 + \frac{b_1 (b_1 \xi - p_x)}{4\xi}; e^{2i \xi} \right) & \text{for } t < 0 
\end{cases}$$

Using asymptotic behaviour of $\binom{a, b}{b + 1, z}$ for $z \to \infty$, as $t \to \pm \infty$ we get same behaviour as in the positive curvature case [39]:

$$t_c \sim -\frac{2A(2\tilde{K}^{-1} \xi)^{\frac{1}{2}}}{p_x \pm b_1 \xi} e^{-\frac{1}{2i}(b_1 \xi \pm \xi)} + c_0^\pm,$$

where $c_0^\pm$ are constants, and again we can choose $c_0^- = 0$. As in the positive curvature case, $p_x - b_1 \xi < 0$ and $p_x + b_1 \xi > 0$. Therefore as $t \to -\infty$, $t_c$ approaches 0 from above and as $t \to +\infty$, $t_c$ approaches $c_0^+$ from below. However we also have this time that as $t \to 0$, $|t_c| \to \infty$. This implies that $c_0^+$ must actually be negative, and the overall behaviour is that as $t$ goes from $-\infty$ to $0$, $t_c$ goes from 0 to $+\infty$, and as $t$ goes from 0 to $+\infty$, $t_c$ goes from $-\infty$ to $t_0^+$. Thus, unlike the $K^2 > 0$ case, $t_c$ is unbounded.

Hence in this scenario, for negative $t_c$, the universe collapses as $t_c \to c_0^+$ and for positive $t_c$ it expands. Moreover as $t_c \to 0$ and $t_c = c_0^+$, the four-dimensional curvature blows up. Note that the scale factor $X_n$ is proportional to $Y_n$ and is hence proportional to $t$. But as $t_c \to \pm \infty$, $t \to 0$, thus the volume of this component of the internal space is stabilized as $t_c \to \pm \infty$.

Consider the quantum system now. For $K^2 < 0$, the solutions of equation (88) are linear combinations of Bessel functions—$J_{\pm ib_1k_1}(z)$ with an imaginary parameter $\pm ib_1k_1$, where $z = \tilde{K} e^{b_1y}$. We know that in this case there is a family of self-adjoint extensions of $\hat{H}_Y$ parametrized by $\theta \in (0, 1]$. Correspondingly there is a set of orthonormal eigenfunctions $\lambda_k^0$.
parametrized by some $\tilde{\theta} \in (0, 1]$ such that there is a one-to-one mapping from $\theta$ to $\tilde{\theta}$ [33, 34]. These eigenfunctions are given by

$$\chi_{k_i}^{(\tilde{\theta})}(z) = \sqrt{\frac{k_1}{2 \sinh \pi b_1 k_1}} \left( J_{-ib_1 k_1}(z) + \frac{\sinh \frac{1}{2}\pi (b_1 k_1 - 2\tilde{\theta})}{\sinh \frac{1}{2}\pi (b_1 k_1 + 2\tilde{\theta})} J_{ib_1 k_1}(z) \right). \quad (106)$$

Note that for the special case $\tilde{\theta} = \frac{1}{2}$ and $\tilde{\theta} = 1$, up to normalization, we obtain the functions $J_{-ib_1 k_1} \pm J_{ib_1 k_1}$, which when expressed in terms of Hankel functions are the orthonormal sets used in [25] and [14], in particular.

Each of these orthonormal sets is associated with a particular self-adjoint extension of the Hamiltonian, so once we fix the domain of the operator, we can only use one particular set of orthonormal eigenfunctions. Again, these functions are not square-integrable, and as such strictly speaking do not belong to $\mathcal{H}$, so as before to make precise sense of them we need to smear them with a peaked weight distribution $\rho(k_2, \ldots, k_n)$, and only then we can say that they belong to a self-adjoint extension of the Hamiltonian.

So overall, the normalized stationary wavefunctions with energy eigenvalue $E = -k_n^2$ are

$$\Psi^{(\tilde{\theta})}_{k_2, \ldots, k_n}(Y^1, \xi_2, \ldots, \xi_{n-1}) = \frac{1}{(2\pi)^{n/2}} e^{i\xi_2 k_2} \ldots e^{i\xi_{n-1} k_{n-1}} \chi_{k_i}^{(\tilde{\theta})}(K b_1 e^{\tilde{\theta} - 1} i^j). \quad (107)$$

As we briefly mentioned before, solutions can also be written in terms of Hankel functions $H^1_{ib_1 k_1}(z)$ and $H^2_{ib_1 k_1}(z)$. Hankel functions are combinations of Bessel functions and are defined as following:

$$H^1_{iv}(z) = \frac{e^{iv} J_v(z) - J_{-iv}(z)}{\sinh(v\pi)}, \quad (108a)$$

$$H^2_{iv}(z) = \frac{J_{-iv}(z) - e^{-iv} J_v(z)}{\sinh(v\pi)}. \quad (108b)$$

Consider the limit as $z \rightarrow +\infty$ (so that $Y^1 \rightarrow +\infty$). Then

$$H^1_{ib_1 k_1}(z) \sim \sqrt{\frac{2}{\pi z}} e^{\frac{b_1 k_1 z}{2}} e^{-iz} e^{iz} \quad (109)$$

$$H^2_{ib_1 k_1}(z) \sim \sqrt{\frac{2}{\pi z}} e^{-\frac{b_1 k_1 z}{2}} e^{iz} e^{-iz}. \quad (110)$$

If we impose the so-called tunnelling boundary condition [11], we select only the left-moving waves at large $z$ so that such a solution can be written as

$$\Psi[k_2, \ldots, k_n] = e^{ib_2 z^2} \ldots e^{ib_{n-1} z_{n-1}} H^1_{ib_1 k_1}(K b_1 e^{ib_1 z}). \quad (111)$$

Here, the behaviour of the wavefunction is such that for negative $Y^1$, $|\Psi|$ is mostly oscillatory, while for positive $Y^1$, the wavefunction decays as $e^{-Y^1}$, which is much slower than the decay in the $K^2 > 0$ case. In the current case, all values of $Y^1$ are allowed classically, whereas in the $K^2 > 0$ case, $Y^1$ is bounded. This explains the different decay rates.

For $z \rightarrow 0$, the asymptotic behaviour of $H^1_{iv}$ is [39]

$$H^1_{iv}(z) \sim \frac{1}{\sinh(v\pi)} \left( e^{iv} \frac{(\frac{1}{2} z)^i}{\Gamma(1 + iv)} - \frac{(\frac{1}{2} z)^{-iv}}{\Gamma(1 - iv)} \right) \quad (112)$$

and therefore, for $v = ib_1 k_1$, we get

$$H^1_{ib_1 k_1}(z) \sim \frac{1}{\sinh(b_1 k_1 \pi)} \left( e^{ib_1 k_1 \pi} \frac{(\frac{1}{2} z)^{ib_1 k_1}}{\Gamma(ib_1 k_1)} + \frac{(\frac{1}{2} z)^{-ib_1 k_1}}{\Gamma(-ib_1 k_1)} \right).$$
So the asymptotic behaviour as $Y^1 \to -\infty$ is given by

$$
\Psi = N e^{i k_1 Y^1} \cdots e^{i k_n Y^n} \left( \frac{1}{\Gamma(i b_1 k_1)} e^{i b_1 k_1 Y^1} + \frac{1}{\Gamma(-i b_1 k_1)} e^{-i b_1 k_1 Y^1} \right)
$$

where $N$ is a normalization constant. Interpreting $Y^1$ as the time-like coordinate, the wavefunction is decomposed into the left- and right-moving waves along the vector $(k_2, \ldots, k_n)$ in the 'space-like' part of the superspace. Note that $k_1$ is proportional to the magnitude of this vector.

From (113), we get that $|\Psi|$ oscillates around $\frac{N}{\Gamma(i b_1 k_1)} e^{i b_1 k_1 Y^1}$ with amplitude $\frac{N}{\Gamma(-i b_1 k_1)}$. So although the amplitude of oscillations is the same as for the case $K^2 > 0$, the fluctuation relative to the value of $|\Psi|$ is very small for large $b_1 k_1$. In this case the $\Psi^{(-)}$ term dominates, and $|\Psi|$ is almost constant as $Y^1 \to -\infty$.

As in the $K^2 > 0$ case the left-moving waves correspond to the classical positive momentum, positive $t$ branch and can be interpreted as being incident from the right. The right-moving waves correspond to the classical negative momentum, negative $t$ branch and can be interpreted as a reflection of the incident wave. The ratio of the reflected and incident amplitudes is

$$
R_{k_1} = \frac{|\Psi^{(+)}|^2}{|\Psi^{(-)}|^2} = e^{-2i b_1 k_1 Y^1}
$$

and this gives the transition probability from the positive momentum branch to the negative momentum branch. But as we have seen, positive $t$ corresponds to negative $t$, and vice versa. So we have a transition from classically disconnected negative time branch to the positive time branch. Thus, there is finite probability of a transition between the two branches which exhibit very different behaviour. This is similar to results obtained in [11] in a string theory context with a positive dilaton potential in the Hamiltonian. Here we obtain a similar behaviour, but the potential naturally comes from the spatial curvature. By choosing the boundary conditions as we did, we made sure that the transition is in the correct direction when compared with classical solutions.

Although formally we can write such a solution which exhibits tunnelling behaviour, it does not mean that it fully makes sense mathematically. Indeed, if we look at the expression of Hankel functions in terms of Bessel functions (109), we see that in order to construct such a function from orthonormal functions $\chi^{(\nu)}(k_1, \ldots, k_n)$, we would need to use $\chi^{(\nu)}(k_1, \ldots, k_n)$ for at least two different values of $\nu$. However these functions would lie in different self-adjoint extensions of the Hamiltonian, and thus the resulting solution (111) does not lie in any domain where the Hamiltonian is self-adjoint. Therefore, on the space where (111) belongs, the Hamiltonian is not self-adjoint, and hence in particular energy is not observable. Moreover, from Stone’s theorem [43], we know that quantum dynamics is unitary if and only if the infinitesimal generator—the Hamiltonian—is self-adjoint. Hence in this case, with a non-self-adjoint Hamiltonian we also lose unitarity of the system. This is certainly something in need of further investigation, because it is currently not clear what the precise physical explanation for this is.

5. Minisuperspace solutions for non-trivial 4-form

Now let us consider the case with the non-trivial 4-form. In order to be able to get solutions explicitly, we fix the spatial Ricci scalar to be

$$
R^{(10)} = \frac{1}{2} K^2 e^{-2X^1}
$$
where $K^2 = 2k_0a_n(a_n - 1)$. This means that the spatial 3-manifold which is part of the external four-dimensional spacetime is necessarily flat. The Lagrangian is now given as

$$L^f = \frac{1}{2} G^f_{AB} \dot{X}^A \dot{X}^B + \frac{1}{2} K^2 e^{-2f} e^{2(V - X^0)},$$

and the corresponding Hamiltonian

$$H^f = \frac{1}{2} (G^f)^{AB} P_A P_B - \frac{1}{2} K^2 e^{-2f} e^{2(V - X^0)},$$

for $A, B = 0, 1, \ldots, n$. As before, we need to diagonalize the metric, but unlike the previous case, here we need $V = X^0$ and $X^1$ to be independent variables. Note that in the previous section $Y^n$ is proportional to $X^n$. So if in the definitions for $Y^1$ and $Y^n$ we replace $X^1$ with $X^n$ and vice versa, and $a_1$ with $a_n$ and vice versa, we get variables which perfectly fit our needs. The other variables can obviously remain unchanged, but we relabel them for convenience.

Therefore, overall we get the following set of variables:

$$Z^0 = X^0$$
$$Z^1 = c_1(a_1 + a_2 + \cdots + a_n - 1)X^1$$
$$Z^2 = c_2[a_1X^1 + (a_2 + a_3 + \cdots + a_n - 1)X^2]$$
$$\ldots$$
$$Z^{n-2} = c_{n-2}[a_1X^1 + a_2X^2 + \cdots + a_{n-1}X^{n-3} + (a_{n-2} + a_{n-1} + a_n - 1)X^{n-2}]$$
$$Z^{n-1} = c_{n-1}[a_1X^1 + a_2X^2 + a_3X^3 + \cdots + a_nX^n]$$
$$Z^n = c_n[a_1X^1 + a_2X^2 + a_3X^3 + \cdots + a_nX^n + \cdots + a_1X^1],$$

where the coefficients $c_i^2$ are defined by

$$c_1^2 = 2a_1(a_2 + a_3 + \cdots + a_n - 1)^{-1}(a_1 + a_2 + \cdots + a_n - 1)^{-1}$$
$$c_2^2 = 2a_2(a_3 + a_4 + \cdots + a_n - 1)^{-1}(a_2 + a_3 + \cdots + a_n - 1)^{-1}$$
$$\ldots$$
$$c_{n-1}^2 = 2a_{n-1}(a_n - 1)^{-1}(a_{n-1} + a_n - 1)^{-1}$$
$$c_n^2 = 2a_n(a_n - 1)^{-1}.$$

With these variables we get

$$G^f_{AB} \dot{X}^A \dot{X}^B = e^{2f} (e^{-\frac{i}{c_1} Z^0} (\dot{Z})^0)^2 + (\dot{Z}^1)^2 + \cdots + (\dot{Z}^{n-1})^2 - (\dot{Z}^n)^2)$$

and moreover

$$V = -\frac{1}{2} (c_1 Z^1 + \cdots + c_{n-1} Z^{n-1} - c_n Z^n).$$

Hence, we can write the Hamiltonian

$$H_{\text{max}} = \frac{1}{2} e^{-2f} \left(e^{\frac{i}{c_1} Z^0} \pi_0^2 + \pi_1^2 + \pi_2^2 + \cdots + \pi_{n-1}^2 - \pi_n^2 - K^2 e^{2c_1^{-1} Z^0}\right)$$

where $\pi_A$ are the momenta conjugate to $Z^A$. With this the Hamiltonian constraint is

$$e^{i\frac{c_1}{c_1} Z^0} \pi_0^2 + \pi_1^2 + \pi_2^2 + \cdots + \pi_{n-1}^2 - \pi_n^2 - K^2 e^{2c_1^{-1} Z^0} = 0.$$

Taking into account the above constraint (117), the classical equations of motion are

\begin{align*}
\dot{\pi}_0 &= 0 \\
\dot{\pi}_1 &= -\frac{1}{c_1} e^{i\frac{c_1}{c_1} Z^0} \pi_0 \\
\dot{\pi}_n &= K^2 \frac{c_1}{c_n} e^{i\frac{c_1}{c_n} Z^n} \\
\pi_j &= 0
\end{align*}
where \( j = 2, \ldots, n - 1 \) and the dots denote derivatives with respect to parameter \( \tau \), given by (69). For \( j = 2, \ldots, n \) we immediately write
\[
Z^j = \pi_j \tau + Z^j_0
\]
where \( Z^j_0 \) are constants. Similarly as before, we can rewrite the Hamiltonian constraint (117) as
\[
e^{\frac{1}{4} \iota^1 Z_1} \pi^1_0 + \pi^1_1 + \xi^2 - \pi^2_n = K^2 e^{2 \iota^1 Z_n} = 0,
\]
where \( \xi \) is a constant given by
\[
\xi^2 = \pi^2_1 + \cdots + \pi^2_{n-1}
\]
From the equation of motion, we have
\[
2\pi_1 \frac{\partial \pi_1}{\partial Z^1} = -\frac{2}{3} c^{-1}_1 \pi^1_0 e^{\frac{1}{4} \iota^1 Z_1}
\]
\[
2\pi_n \frac{\partial \pi_n}{\partial Z_n} = -2 K^2 e^{-1} c^{-1} e^{2 \iota^1 Z_n}.
\]
Integrating, we get
\[
\pi^2_0 e^{\frac{1}{4} \iota^1 Z_1} = \lambda^2_1 - \pi^2_1
\]
\[
K^2 e^{2 \iota^1 Z_n} = \lambda^2_n - \pi^2_n,
\]
substituting these expressions into the constraint, we get
\[
\lambda^2 = \lambda^2_n - \lambda^2_1.
\]
Moreover, from equations of motion (118) we have
\[
\dot{\pi}_1 = -\frac{1}{3} c^{-1}_1 (\lambda^2_1 - \pi^2_1)
\]
\[
\dot{\pi}_n = c^{-1}_n (\lambda^2_n - \pi^2_n).
\]
Note that from (121) we must have \( \lambda^2_1 \geq 0 \) and from (122) also has to be non-negative. Moreover, from (121), we can induce that \( |\pi_1| < \lambda_1 \), and similarly \( |\pi_n| < \lambda_n \) for \( K^2 > 0 \), and \( |\pi_n| > \lambda_n \) for \( K^2 < 0 \).

Consider what happens to the curvature of the eleven-dimensional spacetime. From Einstein’s equation, we get
\[
R^{(11)} = -\frac{1}{6} e^{-2 \iota^1 Z_1} \pi^2_0
\]
hence the curvature blows up as the volume of the space tends to zero.

We now proceed to the quantization of the minisuperspace model. The canonical variables in the minisuperspace are now \( Z^A \) for \( A = 0, 1, \ldots, n \), and the corresponding momenta \( \pi_A \) for \( A = 0, 1, \ldots, n \). Unlike the case with a vanishing 4-form, even in the gauge \( f = 0 \), the minisuperspace metric \( G_{AB} \) is not flat, as it is given by
\[
G_{AB} = \text{diag}(e^{-\frac{1}{4} \iota^1 Z_1}, 1, \ldots, 1, -1)
\]
so in particular in our prescription for the Hamiltonian operator (61) both the Laplace–Beltrami Ricci scalar \( R \) are non-trivial. We indeed have
\[
\Delta \Psi = \frac{1}{\sqrt{|G|}} \frac{\partial}{\partial Z^A} \left( \sqrt{|G|} G^{AB} \frac{\partial}{\partial Z^B} \Psi \right)
\]
\[
= -\frac{1}{3} c^{-1} \frac{\partial}{\partial Z^1} \Psi + G_{AB} \frac{\partial}{\partial Z^A} \frac{\partial}{\partial Z^B} \Psi
\]
and since the only non-zero components of $R_{AB}$ are

$$R_{00} = \frac{1}{2} c^{-2}, \quad R_{11} = \frac{1}{2} c^{-2} e^{-\frac{1}{2}c^{-1}Z^1}$$

we get

$$R = G^{AB} R_{AB} = \frac{1}{2} c^{-2}.$$  \hspace{1cm} (128)

So the Wheeler–DeWitt equation is given by

$$\left(e^{\frac{1}{2}c^{-1}Z^1} \partial_{Z^0}^2 + \partial_{Z^1}^2 + \cdots + \partial_{Z^{n-1}}^2 - \frac{1}{2} c^{-1} \partial_{Z^1}\right) \Psi + \left(K^2 e^{2c^{-1}Z^1} - \frac{4}{3} ac^{-2}\right) \Psi = 0.$$  \hspace{1cm} (129)

Due to the presence of the 4-form term, and the resulting different choice of operator ordering in the Hamiltonian, this equation differs from equation (75) in the previous section by new extra terms. The covariant measure on the minisuperspace is given by

$$d\omega = dZ^0 \cdots dZ^n e^{-\frac{1}{2}c^{-1}Z^1}$$

and hence the momentum operators are now given by

$$\ddot{p}_1 = -i \left(\partial_{Z^1} - \frac{1}{2} c^{-1}\right)$$  \hspace{1cm} (131)

and

$$\ddot{p}_A = -i \partial_A \quad \text{for} \quad A \neq 1.$$  \hspace{1cm} (132)

Equation (129) can be solved by the separation of variables, so we use the following ansatz for $\Psi$:

$$\Psi = e^{\sigma Z^0} e^{\kappa_1 Z^1} \cdots e^{\kappa_{n-1} Z^{n-1}} F(Z^1) G(Z^n).$$  \hspace{1cm} (133)

With this, the Wheeler–DeWitt equation (129) becomes

$$-\frac{1}{G} \partial_{Z^1}^2 G + \frac{1}{F} \left(\partial_{Z^1}^2 - \frac{1}{2} c^{-1} \partial_{Z^1}\right) F - \kappa_0^2 e^{\frac{1}{2}c^{-1}Z^1} + K^2 e^{2c^{-1}Z^n} = \kappa^2$$

where

$$\kappa^2 = \kappa_0^2 + \cdots + \kappa_{n-1}^2 + \frac{4}{3} ac^{-2}.$$

Separating the variables, we get the following equations for $F$ and $G$:

$$\left(\partial_{Z^1}^2 - \frac{1}{2} c^{-1} \partial_{Z^1}\right) F - \left(\kappa_0^2 e^{\frac{1}{2}c^{-1}Z^1} - \kappa_1^2\right) F = 0$$  \hspace{1cm} (134)

$$\partial_{Z^1}^2 G - \left(K^2 e^{2c^{-1}Z^n} - \kappa_1^2\right) G = 0$$  \hspace{1cm} (135)

where $\kappa_1$ and $\kappa_0$ are constants such that

$$\kappa_0^2 - \kappa_1^2 = \kappa^2.$$  \hspace{1cm} (136)

In order to transform (134) to the same form as (135), set $F = e^{\sigma Z^1} \tilde{F}$ for a constant $\sigma$. It turns out that for $\sigma = \frac{1}{2} c^{-1}$ we obtain

$$\partial_{Z^1}^2 \tilde{F} - \left(\kappa_0^2 e^{\frac{1}{2}c^{-1}Z^1} - \kappa_1^2\right) \tilde{F} = 0$$  \hspace{1cm} (137)

where $\kappa_1^2 = \kappa_1^2 - \frac{1}{36} c^{-2}$. Thus unsurprisingly, the operator $\hat{H}_F$ appears again. In order to get bounded solutions of (137), we need $\kappa_1^2 > 0$, and the solutions are then $K_{3c^{-1}Z^1}(z_0)$ for $z_0 = K c_n e^{c^{-1}Z^n}$. Now $\kappa_1^2 > 0$ and hence we also have $\kappa_0^2 > 0$. Then, as we know, solutions of (135) are Bessel functions $J_{\kappa_0 e^{c^{-1}Z^1}}(z_1)$ or modified Bessel functions $K_{\kappa_0 e^{c^{-1}Z^1}}(z_1)$, for $K^2 < 0$ and $K^2 > 0$ respectively, where $z_1 = 3c^{-1}K c_n e^{c^{-1}Z^1}$. As previously, for $K^2 < 0$, we get a one-parameter family of self-adjoint extensions (90) of the Hamiltonian.
Similarly as before, let us discuss gauge fixing. We change variables so that for \( j = 2, \ldots, n - 1 \) we have
\[
\zeta_j = \frac{Z^j}{\pi_j} \quad \pi_{\zeta_j} = \frac{1}{2} \pi_j^2
\]
and hence the Hamiltonian \( H_{\text{mass}} \) in these variables is given by
\[
H_{\text{mass}} = \frac{1}{2} \text{e}^{-2j} \left[ \Phi_{\zeta_j}^{(1)} + \pi_{\zeta_j}^2 - K^2 \text{e}^{2\zeta_j^2 Z^j} \right] + \pi_{\zeta_1} + \cdots + \pi_{\zeta_{n-1}}
\]
(139)
In the reduced phase space method, we take the gauge choice \( \xi_{n-1} - t = 0 \). From the equations of motion this further imposes \( t = \tau \), and hence \( f = 0 \). Hence we get the gauge proper time. The effective Hamiltonian is now
\[
H_{\text{eff}} = \frac{1}{2} \left[ \Phi_{\zeta_j}^{(1)} + \pi_{\zeta_j}^2 - K^2 \text{e}^{2\zeta_j^2 Z^j} \right] + \pi_{\zeta_1} + \cdots + \pi_{\zeta_{n-1}} = -\pi_{\zeta_{n-1}}
\]
(140)
where \( \pi_{\zeta_{n-1}} \geq 0 \). When quantizing, we take the part of the effective Hamiltonian which depends on \( Z^0, Z^1 \) and \( Z^n \) to be the same as in (129), so that overall we get
\[
\hat{H}_{\text{eff}} = -\frac{1}{2} \left( \Phi_{\zeta_j}^{(1)} Z^j Z^j + \pi_{\zeta_j}^2 - K^2 \text{e}^{2\zeta_j^2 Z^j} \right) + \pi_{\zeta_1} + \cdots + \pi_{\zeta_{n-1}} = -\pi_{\zeta_{n-1}} \frac{1}{2} \left( K^2 \text{e}^{2\zeta_j^2 Z^j} - \frac{4}{6} a c_i^2 \right)
\]
(141)
and so the solutions of the corresponding Schrödinger equation with \( E = -\kappa_{n-1}^2 \) are hence
\[
\Psi = e^{i\phi_0 Z^0} e^{i\phi_1 Z^1} \cdots e^{i\phi_{n-2} Z^{n-2}} e^{i\phi_{n-1} Z^n} e^{i\zeta_j^2 Z^1 F(Z^1) G(Z^n)}
\]
(142)
with \( F(Y^1) \) satisfying (135) and \( G(Z^n) \) satisfying (135). Again, the form of the solution is same as (133) but with the gauge condition \( \xi_{n-1} - t = 0 \) imposed and with \( \zeta_i = 2 \kappa_i^2 \) for \( i = 0, 2, \ldots, n - 1 \).

Alternatively, we can use the Faddeev–Popov method. From (80), the Wheeler–DeWitt equation is
\[
\hat{H}_{\text{mass}} = -\frac{1}{2} \left( \Phi_{\zeta_j}^{(1)} Z^j Z^j + \pi_{\zeta_j}^2 - K^2 \text{e}^{2\zeta_j^2 Z^j} \right) + \pi_{\zeta_1} + \cdots + \pi_{\zeta_{n-1}} = -\pi_{\zeta_{n-1}} \frac{1}{2} \left( K^2 \text{e}^{2\zeta_j^2 Z^j} - \frac{4}{6} a c_i^2 \right)
\]
and the solutions are
\[
\Psi = e^{i\phi_0 Z^0} e^{i\phi_1 Z^1} \cdots e^{i\phi_{n-2} Z^{n-2}} e^{i\zeta_j^2 Z^1 F(Z^1) G(Z^n)}
\]
(143)
The full gauge-fixed inner product is given by
\[
\langle \Psi_1 | \Psi_2 \rangle = \int d\omega_1 \Phi_1^*(Z^0, Z^1, Z^n, \zeta_2, \ldots, \zeta_{n-1}) \Phi_1(\Theta) \Delta_{FP} \Phi_2(Z^0, Z^1, Z^n, \zeta_2, \ldots, \zeta_{n-1})
\]
where \( \Theta = 0 \) is the gauge condition, \( \Delta_{FP} \) is the Faddeev–Popov determinant and the measure is
\[
d\omega_1 = dZ^0 dZ^1 dZ^n d\xi_2 \ldots d\xi_{n-2} d\xi_{n-1} e^{-i\zeta_j^2 Z^1}.
\]
For \( \Theta = \xi_0 - t \), which gives the gauge condition (81), \( \Delta_{FP} = 1 \), so the gauge-fixed inner product is
\[
\langle \Psi_1 | \Psi_2 \rangle = \int d\omega_1 \Phi_1^*(Z^0, Z^1, Z^n, \zeta_2, \ldots, \zeta_{n-2}, t) \Phi_2(Z^0, Z^1, Z^n, \zeta_2, \ldots, \zeta_{n-2}, t)
\]
(144)
giving a positive definite Hilbert space with the measure
\[
d\omega_1 = dZ^0 dZ^1 dZ^n d\xi_2 \ldots d\xi_{n-2} e^{-i\zeta_j^2 Z^1}.
\]
Again we see that the two methods give equivalent results.
The equations we obtained here are all very similar to the equations encountered in the previous section, so we can write the solutions straight away. The classical equation for $\pi_1$ (124) gives

$$\pi_1 = -\lambda_1 \tanh \left( \frac{1}{2} c_1^{-1} \lambda_1 t + t_0 \right)$$

(145)

and from the equation of motion for $Z^1$ (118) and the constraint (121), we get the solution for $Z_1$:

$$Z^1 = -3c_1 \log \left( \lambda_1^{-1} \pi_0 \cosh \left( \frac{1}{2} c_1^{-1} \lambda_1 t + t_0 \right) \right).$$

(146)

This is very similar to the solutions for $Y_1$ considered in the previous section for $K^2 > 0$. In particular, in the phase space this solution has a single branch.

The solutions for $Z_n$ are exactly the same as the solutions for $Y_1$ in the previous section. Thus for $K^2 > 0$, we have

$$\pi_n = \lambda_n \tanh \left( c_n^{-1} \lambda_n t + t_1 \right)$$

(147)

and

$$Z^n = -c_n \log \left( K \lambda_n^{-1} \cosh \left( c_n^{-1} \lambda_n t + t_1 \right) \right).$$

Similarly, for $K^2 < 0$, we get

$$\pi_n = \zeta_n \coth \left( c_n^{-1} \lambda_n t + t_1 \right)$$

(149)

$$Z^n = -c_n \log \left( |K \lambda_n^{-1} \sinh \left( c_n^{-1} \lambda_n t + t_1 \right)| \right)$$

(150)

where $K^2 = -K^2$. In both cases, the asymptotic behaviour as $t \to \pm \infty$ is

$$Z^1 \sim \mp \lambda_1 t \quad Z^n \sim \mp \lambda_n t.$$

We know that the volume parameter $V$ is given by

$$V = \frac{1}{2} (c_1 Z^1 - c_1 Z^1 - \pi, t) + \text{const}$$

where

$$\pi_s = c_2 \pi_s + \cdots + c_{n-1} \pi_{n-1},$$

so as $t \to \pm \infty$, $V \sim \pm \frac{1}{2} t (c_1 \lambda_1 - c_n \lambda_n - \pi_s)$.

Noting that $X^1 = \frac{1}{2} Z^1$ and $X^n = V - c_n^{-1} Z^n$, we get the asymptotic behaviour of the original variables $X^1$ and $X^n$:

$$X^1 \sim \mp \frac{1}{2} \lambda_1 t$$

$$X^n \sim \pm \frac{1}{2} t (c_1 \lambda_1 - c_n (1 - 2 c_n^{-1}) \lambda_n - \pi_s).$$

Thus as $t \to \pm \infty$, $X^1 \to -\infty$, and the qualitative behaviour of $V$ depends on the sign of $c_{\pm} = c_1 \lambda_1 - c_n \lambda_n \mp \pi_s$. Both the four-dimensional and eleven-dimensional curvatures are asymptotically proportional to $e^{-2V}$, so the sign of $c_{\pm}$ affects the behaviour of the curvature. Consider the following example. If $n = 2$, then the internal space is seven dimensional, and moreover $\pi_s = 0$ and $\lambda_2 = \lambda_1$. This immediately gives us that $c_{\pm} < 0$. Hence as $t \to \pm \infty$, $V \to -\infty$ and from (126) this implies that $R^{11}$ blows up whenever $t \to \pm \infty$.

Now look at the solutions of the Wheeler–DeWitt equation for this system (129). Again, let us look at cases of $K^2 > 0$ and $K^2 < 0$ separately. In the case $K^2 > 0$, the normalized gauge-fixed stationary wavefunctions with energy eigenvalue $E = -K^2_{n-1}$ are given by

$$\Psi_{\kappa_0, \kappa_1, \ldots, \kappa_n} = N_{\kappa_1, \kappa_2} e^{\eta_0 Z^0} e^{i \omega_2^1 Z^1} \cdots e^{i \omega_{n-1}^1 Z^1} K_{\kappa_0, \kappa_1} \left( 3 c_1 \kappa_0 e^{i \tau_{\pi_1}^1 Z^1} \right) K_{\kappa_n, \kappa_0} \left( K_{\kappa_0} e^{i \tau_{\pi_1}^1 Z^1} \right)$$

(152)
where

\[ N_{\kappa_1, \kappa_n} = \frac{2\kappa_1 \kappa_n \sinh 3\pi c_1 \sinh \pi b_1 \kappa_n}{(2\pi)^{\frac{3}{2}} \pi^2}. \]

Similarly as discussed in the case of vanishing 4-form, these functions form an orthonormal basis in the inner product (144).

For \( K^2 < 0 \), the normalized wavefunctions are given by

\[ \Psi_{\kappa_0, \kappa_1, \ldots, \kappa_n} = N e^{i\kappa_0 Z_0} e^{i\frac{3}{2} \kappa_1 Z_1} e^{i\frac{1}{2} \kappa_2 Z_2} \cdots e^{i\frac{1}{2} \kappa_{n-1} Z_{n-1}} K^{3i c_1} e^{i 2c_1 Z_1} e^{i c_1 Z_0} \chi_{\kappa_n} (K c_n e^{c_1 Z_n}) \] (153)

where \( \chi_{\kappa_n} \) are the orthonormal functions given by (106). Exactly as in the previous section, we get a set of normalized wavefunctions for each self-adjoint extension of the Hamiltonian.

For \( K^2 < 0 \), if we take the \( Z^n \) solution to be a Hankel function, then as in the trivial 4-form case, for \( Z^n \rightarrow -\infty \), we could decompose the wavefunction into plane waves similarly as in the case of the trivial 4-form. Then we would get a non-trivial reflection probability \( R_{\kappa_n} = e^{-2\kappa_n} \pi \) from the right-moving wave to the left-moving wave, which would correspond to a transition from the \( \pi_n > 0 \) branch to the \( \pi_n < 0 \) branch. However, as in the previous situations, such solutions would still belong to a domain where the Hamiltonian is not self-adjoint.

6. Concluding remarks

We have first derived the canonical formulation of the bosonic sector of eleven-dimensional supergravity, together with the complete constraint algebra. The brackets of the secondary constraints vanish on the constraint surface, so all constraints are first class and there are no new tertiary constraints. When passing to the quantum system, the constraints become conditions on the wavefunction which govern its behaviour.

By introducing particular ansätze for the metric and the 4-form we reduced the system to a minisuperspace model with a finite number of degrees of freedom. In a special case where only one spatial component has non-vanishing curvature, both the classical and quantum equations can be solved exactly. In the positive curvature case, whether with or without the 4-form, there is only one branch of the classical solution, where the universe first expands after starting out from zero size, reaches a maximum size and then collapses again within a finite time. When the universe becomes small, the wavefunction can be written in terms of plane waves travelling in opposite directions. These waves can be interpreted as being reflections of one another, but since their coefficients are equal the transition probability is 1. A similar scenario is considered in [11], but the effect that there is only one classical branch of the solution is achieved there by having a negative dilaton potential in the Hamiltonian, which is hard to motivate in a realistic superstring theory context.

In the negative curvature case, the classical solutions give two disconnected branches, one of which is collapsing universe and the other branch is an expanding universe. From the four-dimensional point of view, there is a curvature singularity between the two branches. It is possible to choose boundary conditions such that at \( +\infty \) the solution can be written as a single wave, but at \( -\infty \) it splits into two plane waves going in the opposite direction. This yields a non-trivial transition probability between the branches. However, such a solution does not lie in a domain where the Hamiltonian is self-adjoint, and hence by choosing such boundary conditions we lose the self-adjointness of the Hamiltonian. It would be interesting to investigate further the physical reasons for this lost self-adjointness, especially since the operator which appears here is mathematically the same as in the string theory minisuperspace models, so the same problem should arise in those settings as well.
Apart from the self-adjointness problems, we have seen that the curvature term and the 4-form term in our minisuperspace models, in terms of determining the behaviour of the solution, play the same role as the dilaton potential in the gravi-dilaton systems derived from string theory. This is quite remarkable because these terms naturally form the supergravity action, whereas the dilaton potentials are put in by hand. It would be interesting to investigate what happens when there is more than one spatial curvature term. Such an ansatz would be a generalization of the Freund–Rubin solution of M-theory [44], where the eleven-dimensional space is of the form $AdS_4 \times S^7$ with particular scale factors for each component. In particular, if the 3-space is curved, then there could possibly be more interaction with the 4-form term and the curvature term. Also, in further work, a less restrictive metric ansatz with a non-trivial moduli space could be studied, to see how the moduli space parameters evolve and what the behaviour of their wavefunctions is. In particular, it would be interesting to study compactifications on manifolds of special holonomy with time-dependent moduli. This could either involve compactifications on general $G_2$-holonomy manifolds or maybe on a Calabi–Yau space times a circle. In the latter case, it could be investigated how mirror symmetry [45] is manifested from the point of view of a minisuperspace quantization.

Study of M-theory minisuperspace models seems to be a promising area where there is still much left to be uncovered, and which will hopefully aid us in the quest to further understand M-theory.

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Appendix

In this appendix, we give the details of the calculations involved in deriving expression (41) for the Poisson bracket $[\tilde{H}, \tilde{H}]$. To do this, we first need to know the bracket $[\tilde{\pi}^{abc}, \tilde{\pi}^{efg}]$. Expanding, we have

$$
\left[ \tilde{\pi}^{abc}, \tilde{\pi}^{efg} \right] = \left[ \pi^{abc} + \frac{32}{124} \eta^{abcdeh} A_{d_1 d_2 d_3} \partial_{d_4} A_{d_5 d_6 d_7}, \pi^{efg} + \frac{32}{124} \eta^{efghi} A'_{a_1 a_2 a_3} \partial'_{a_4} A'_{a_5 a_6 a_7} \right],
$$

$$
= \left[ \pi^{abc} + \frac{32}{124} \eta^{efghi} A'_{a_1 a_2 a_3} \partial'_{a_4} A'_{a_5 a_6 a_7} \right] + \left[ \frac{32}{124} \eta^{abcefgd} A_{d_1 d_2 d_3} \partial_{d_4} A_{d_5 d_6 d_7}, \pi^{efg} \right]
$$

$$
= -\frac{32}{124} \eta^{efghi} (A'_{a_1 a_2 a_3} \delta^{abc} \partial_{a_4} A'_{a_5 a_6 a_7} (x, x') + \delta^{a_1 a_2 a_3} \partial (x, x') \partial'_{a_4} A'_{a_5 a_6 a_7})
$$

$$
+ \frac{32}{124} \eta^{abcefgd} (A_{d_1 d_2 d_3} \delta^{efg} \partial_{d_4} A_{d_5 d_6 d_7} (x, x') + \delta^{efg} \partial (x, x') \partial_{d_4} A_{d_5 d_6 d_7})
$$

$$
= \frac{32}{124} \eta^{abcefgd} (A'_{a_1 a_2 a_3} \delta (x, x') + \delta (x, x') \partial'_{a_4} A'_{a_5 a_6 a_7})
$$
Now look at the third line in (A.1) get
\[
\int \int -\text{function term in } \tilde{\gamma} = \int \int H((\gamma) = \int \int -1 + 8\frac{\pi}{12} \delta a c d \tilde{\gamma} a c d + 8 \delta a c d \tilde{\gamma} a c d (x, x') \partial a, A_{a}(x, x) + 2 \delta a c d \tilde{\gamma} a c d (x, x') \partial a, A_{a}(x, x)).
\]

(A.1)

So, for the [\tilde{\mathcal{H}}, \tilde{\mathcal{H}}'] bracket, we have:
\[
[\tilde{\mathcal{H}}, \tilde{\mathcal{H}}'] = \left[ \mathcal{H} + \frac{1}{48} \gamma^{4} F_{a b c d} F_{a b c d} + 3 \gamma^{4} \tilde{\gamma} a b c d \tilde{\gamma} a b c d, \mathcal{H}' + \frac{1}{48} \gamma^{4} F_{a b c d} F_{a b c d} + 3 \gamma^{4} \tilde{\gamma} a b c d \tilde{\gamma} a b c d \right]
\]
\[
= [\mathcal{H}, \mathcal{H}'] + \gamma^{-1} \gamma^{-1} F_{a b c d} F_{a b c d} [A_{a b c d}, \tilde{\gamma} a b c d]_{a} - \gamma^{-1} \gamma^{-1} F_{a b c d} F_{a b c d} [A'_{a b c d}, \tilde{\gamma} a b c d]_{a}
\]
\[
+ 36 \gamma^{-1} \gamma^{-1} \tilde{\gamma} a b c d \tilde{\gamma} a b c d \tilde{\gamma} a b c d [\tilde{\gamma} a b c d, \tilde{\gamma} a b c d]
\]
\[
= [\mathcal{H}, \mathcal{H}'] + \gamma^{-1} \gamma^{-1} F_{a b c d} F_{a b c d} [\tilde{\gamma} a b c d, \tilde{\gamma} a b c d] - \gamma^{-1} \gamma^{-1} F_{a b c d} F_{a b c d} [\tilde{\gamma} a b c d, \tilde{\gamma} a b c d]
\]
\[
+ \frac{8}{12} \gamma^{-1} \gamma^{-1} \tilde{\gamma} a b c d \tilde{\gamma} a b c d \tilde{\gamma} a b c d (A'_{a b c d} \delta a (x, x') + A_{a b c d} \delta a (x, x')).
\]

(A.2)

In the first line cross-terms involving \mathcal{H} vanish because the form terms involve no derivatives of \gamma a b c d and \mathcal{H} does not involve any derivatives of \tilde{\gamma} a b c d. Note that the undifferentiated \delta-function term in \tilde{\gamma} a b c d \tilde{\gamma} a b c d [\tilde{\gamma} a b c d, \tilde{\gamma} a b c d] vanishes, because \eta a b c d \tilde{\gamma} a b c d = 0. Let \xi 1 and \xi 2 be arbitrary test functions. Then
\[
\int [\tilde{\mathcal{H}}, \tilde{\mathcal{H}}'] \xi 1 \xi 2 dx \cdot dx' = \int [\mathcal{H}, \mathcal{H}'] \xi 1 \xi 2 dx \cdot dx'
\]
\[
+ \int (\gamma^{-1} \gamma^{-1} F_{a b c d} F_{a b c d} \delta a (x, x') - \gamma^{-1} \gamma^{-1} F_{a b c d} F_{a b c d} \delta a (x, x')) \xi 1 \xi 2 dx \cdot dx'
\]
\[
+ \frac{8}{12} \int (\gamma^{-1} \gamma^{-1} F_{a b c d} F_{a b c d} \eta a b c d \delta a (x, x') + A_{a b c d} \delta a (x, x')) \xi 1 \xi 2 dx \cdot dx'.
\]

(A.3)

We know from [3] that
\[
\int [\mathcal{H}, \mathcal{H}'] \xi 1 \xi 2 dx \cdot dx' = \int \mathcal{H} (\xi 1 \xi 2 - \xi 1 \xi 2) dx.
\]

(A.4)

The second line in (A.3) becomes
\[
\int (\gamma^{-1} \gamma^{-1} F_{a b c d} F_{a b c d} \delta a (x, x') - \gamma^{-1} \gamma^{-1} F_{a b c d} F_{a b c d} \delta a (x, x')) \xi 1 \xi 2 dx \cdot dx'
\]
\[
= - \int (\gamma^{-1} \gamma^{-1} F_{a b c d} F_{a b c d} \xi 1 - \gamma^{-1} \gamma^{-1} F_{a b c d} F_{a b c d} \xi 1) \delta (x, x') dx \cdot dx'
\]
\[
= \int (\gamma^{-1} \gamma^{-1} F_{a b c d} F_{a b c d} \xi 1 - \gamma^{-1} \gamma^{-1} F_{a b c d} F_{a b c d} \xi 1) dx
\]
\[
= \int F_{a b c d} F_{a b c d} (\xi 1 \xi 2 - \xi 1 \xi 2) dx.
\]

Now look at the third line in (A.3). After integrating by parts and integrating out the \delta-function we get
\[
\int \frac{8}{12} \eta a b c d \tilde{\gamma} a b c d (\gamma^{-1} \tilde{\gamma} a b c d \tilde{\gamma} a b c d) (\xi 1 \xi 2 - \xi 1 \xi 2) dx = 0
\]
again because \eta a b c d \tilde{\gamma} a b c d = 0.

Thus,
\[
\int [\tilde{\mathcal{H}}, \tilde{\mathcal{H}}'] \xi 1 \xi 2 dx \cdot dx' = \int (\mathcal{H} + F_{a b c d}) (\xi 1 \xi 2 - \xi 1 \xi 2) dx
\]
\[
= \int \mathcal{H} (\xi 1 \xi 2 - \xi 1 \xi 2) dx.
\]

(A.5)
Correspondingly,
\[
[H, \tilde{H}'] = 2\tilde{\chi}^\alpha \delta,_{\alpha}(x, x') + \tilde{\chi}^\alpha_{,\alpha}(x, x'),
\]
which is completely analogous to the untilded expression. In particular, \([\tilde{H}, \tilde{H}']\) vanishes on the constraint surface.

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