Research Article

Drinfeld Realization of Quantum Twisted Affine Algebras via Braid Group

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The Drinfeld realization of quantum affine algebras has been tremendously useful since its discovery. Combining techniques of Beck and Nakajima with our previous approach, we give a complete and conceptual proof of the Drinfeld realization for the twisted quantum affine algebras using Lusztig’s braid group action.

1. Introduction

In studying finite-dimensional representations of Yangian algebras and quantum affine algebras, Drinfeld gave a new realization of the Drinfeld-Jimbo quantum enveloping algebras of the affine types. Drinfeld realization is a quantum analog of the loop algebra realization of the affine Kac-Moody Lie algebras and has played a pivotal role in later developments of quantum affine algebras and the quantum conformal field theory. For example, the basic representations of quantum affine algebras were constructed based on Drinfeld realization [1, 2] and the quantum Knizhnik-Zamolodchikov equation [3] was also formulated using this realization.

The first proof of the Drinfeld realization for the untwisted types was given by Beck [4] using Lusztig’s braid group actions (see also [5] for $U_q(\hat{sl}_2)$). For other approaches see [6–8]. By directly quantizing the classical isomorphism of the Kac realization to the affine Lie algebras, the first author later gave an elementary proof [9] of the Drinfeld automorphism using $q$-brackets starting from Drinfeld’s quantum loop algebras. In this elementary approach a general strategy and algorithm was formulated to prove the Serre relations. In particular, all type $A$ relations were verified in detail including the exceptional type $D_4^{(3)}$.

In [10, 11], we gave an elementary proof of the twisted Drinfeld realization starting from Drinfeld’s quantum loop algebras. In particular in [11] we have shown that twisted quantum affine algebras obey some simplified Serre relations in certain types as in the classical cases and from which twisted Serre relations are consequences.

The purpose of this work is to give a conceptual proof of twisted Drinfeld realizations using braid group actions starting from the Drinfeld-Jimbo definition. Braid group actions have been very useful in Lusztig’s construction of canonical bases [12, 13] and are particularly useful in Beck’s proof of untwisted cases. We use the extended braid group action to define root vectors in the twisted case as in the untwisted cases. Some of the root vector computations can be done in a similar way as in untwisted cases. We also give a complete proof of all Serre relations for both untwisted and twisted cases using $q$-bracket techniques [2]. One notable feature of our work is that we directly prove the realization using the braid group action and verify the unchecked relations for all cases.

The second goal of our paper is to provide a different proof that the Drinfeld-Jimbo algebra is indeed isomorphic to the Drinfeld algebra of the quantum affine algebra without passing to $q = 1$ case. We achieve this by combining our previous work identifying the Drinfeld-Jimbo quantum affine algebra inside the Drinfeld realization and the explicit knowledge of the $q$-bracket computations developed in [9, 10], which also established the epimorphism from
the Drinfeld-Jimbo form to the Drinfeld form of the quantum affine algebra.

The paper is organized in the following manner. In Section 2, we first recall the Drinfeld-Jimbo quantum affine enveloping algebras and define Lusztig’s braid group action as well as the extended braid group action and then use this to define quantum root vectors. Section 3 shows how to construct the quantum affine algebra $U_q(A_1^{(1)})$ (or $U_q(A_2^{(1)})$) inside the twisted quantum affine algebra $U_q(A_1^{(1)})$ (or $U_q(A_2^{(1)})$). In Section 4, we check all Drinfeld relations, in particular all Serre relations for twisted quantum affine algebras. Finally we prove the isomorphism of two forms of quantum affine algebras using our previous work on Drinfeld realization and $q$-bracket techniques.

2. Definitions and Preliminaries

2.1. Finite Order Automorphisms of $g$. In this paragraph some basic notations of Kac-Moody Lie algebras are recalled [14]. Let $g$ be a simple finite-dimensional Lie algebra, and let $\sigma$ be an (outer) automorphism of $g$ of order $r$. Then $\sigma$ induces an automorphism of the Dynkin diagram of $g$ with the same order. Fix a primitive $r$th root of unity $\omega = \exp(2\pi i/r)$. Since $\sigma$ is diagonalizable, it follows that

$$g = \bigoplus_{j \in \mathbb{Z}/r\mathbb{Z}} g_j$$

where $g_j$ is $\sigma$-eigenspace with eigenvalue $\omega^j$. Clearly, the decomposition is $\mathbb{Z}/r\mathbb{Z}$-gradation of $g$, and then $g_0$ is a Lie subalgebra of $g$.

Let $A = (A_{ij})$ ($i, j \in \{1, 2, \ldots, N\}$) be one of the simply laced Cartan matrices. It is well-known that the Dynkin diagram $D(A)$ has a diagram automorphism $\sigma$ of order $r = 2$ or 3. Explicitly $A$ is one of the following types: $A_N$ ($N \geq 2$), $D_N$ ($N > 4$), $E_6$, and $D_4$ with the canonical action of $\sigma$:

\[
\begin{align*}
A_N: & \quad \sigma(i) = N + 1 - i, \\
D_N: & \quad \sigma(i) = i, \quad 1 \leq i \leq N - 2; \\
& \quad \sigma(N - 1) = N, \\
E_6: & \quad \sigma(i) = 6 - i, \quad 1 \leq i \leq 5; \\
& \quad \sigma(6) = 6, \\
D_4: & \quad \sigma(1, 2, 3, 4) = (3, 2, 4, 1).
\end{align*}
\]

Let $I = \{1, 2, \ldots, n\}$ be the set of $\sigma$-orbits on $\{1, 2, \ldots, N\}$, where we use representatives to denote the orbits. For example, $[i, N + 1 - i]$ is simply denoted by $i$ in type $A$. We can write $\{1, \ldots, N\} = I \cup \sigma(I)$. Consequently the nodes of the Dynkin diagram $g_0$ are indexed by $I$.

2.2. Twisted Affine Lie Algebras. For a nontrivial automorphism $\sigma$ of the Dynkin diagram, the twisted affine Lie algebra $\tilde{g}^\sigma$ is the central extension of the twisted loop algebra:

$$\tilde{g}^\sigma = \left( \bigoplus_{j \in \mathbb{Z}} g_{1,j} \otimes C^t \right) \oplus Cc \oplus Cd,$$

where $c$ is the central element and $ad(d) = t(d/dt)$. Denote by $\tilde{T} = I \cup \{0\}$ the node set of the Dynkin diagram of $\tilde{g}^\sigma$. Let $\tilde{h}^\sigma = \tilde{h}_{0,0} \oplus Cc \oplus Cd$ be the Heisenberg subalgebra of $\tilde{g}^\sigma$.

Let us use $A^\sigma = (a_{ij})$ ($i, j \in \tilde{T}$) to denote the Cartan matrix of the twisted affine Lie algebra $\tilde{g}^\sigma$ of type $X_N^{(1)}$. The Cartan matrix $A^\sigma$ is symmetrizable; that is, there exists a diagonal matrix $D = diag(d_i)$ ($i \in \tilde{T}$) such that $DA^\sigma$ is symmetric. Let $\alpha_i$ ($i \in \tilde{T}$) $\subset h^\sigma$ be the simple roots and let $\alpha_i^\vee$ ($i \in \tilde{T}$) $\subset \tilde{h}^\sigma$ be the simple coroots of $\tilde{g}^\sigma$ such that $\langle \alpha_i, \alpha_j^\vee \rangle = a_{ij}$. Let $Q$ be the affine root lattice defined by

$$Q = \bigoplus_{\lambda \in \mathbb{Z}} \mathbb{Z}\alpha_i.
$$

Subsequently $Q_0$ will denote the finite root lattice of the subalgebra $g_0$. Indeed, $Q_0 \subseteq Q = Q_0 \oplus \mathbb{Z}\alpha_0$. Let $Q^\vee = \oplus_{\lambda \in \mathbb{Z}} \mathbb{Z}\alpha_i^\vee$; then $Q_0^\vee = Q_0 \cap Q^\vee$.

Introduce the nondegenerate symmetric bilinear form $(\cdot, \cdot)$ on $Q$ determined by $(\alpha_i, \alpha_j) = d_i d_j$. Let $d = \sum_{i,j} r_{ij}\alpha_i \in Q^+$ be the canonical imaginary root of minimal height such that $\langle d, d \rangle = 0$ and $\langle \delta, \alpha_i \rangle = 0$, $\forall i \in \tilde{T}$. Here the coefficients $r_i$ are unique such that $d_0$ is always 1, and we have chosen the labels of $A_i^{(1)}$ different from [14].

Let $P^\vee = \{ \lambda^\vee \in \tilde{h}^\sigma \mid \langle \alpha_i, \lambda^\vee \rangle \in \mathbb{Z} \}$ be the coweight lattice over $\mathbb{Z}$, and define the fundamental coweight $\omega_i^\vee \in P^\vee$ such that $\langle \alpha_i, \omega_i^\vee \rangle = 0$, $\forall i \in \tilde{T}$. So $d = \omega_0^\vee$. Additionally, we are going to introduce another important sublattice $P$ defined by $P = \oplus_{i \in \tilde{T}} \mathbb{Z}\omega_i$ where $\omega_i = p_i \omega_i^\vee$ for $i \in I$ and $\omega_0 = \omega_0^\vee = d$, and

$$P_i = \begin{cases} r, & \text{if } \sigma(i) = i; \\ 1, & \text{otherwise.} \end{cases}$$

Let us denote by $P_0^\vee$ and $P_0$ the finite coweight and weight lattice of the subalgebra $g_0$, respectively. Then $P_0^\vee \subseteq P^\vee = P_0^\vee \oplus \mathbb{Z}\omega_0^\vee$ and $P_0 \subseteq P = P_0 \oplus \mathbb{Z}\omega_0$.

Recall that the (affine) Weyl group $W$ is generated by $s_i | i \in \tilde{T}$ $\subseteq \text{Aut}(\tilde{g}^\sigma)$, where $s_i(\lambda) = \lambda - (\lambda, \alpha_i^\vee)\alpha_i$; thus $s_i(\alpha_i) = \alpha_j - \alpha_i \alpha_j \alpha_i$. The Weyl group $W$ acts on $P^\vee$ by $s_i(\lambda^\vee) = \lambda^\vee - \langle \alpha_i, \lambda^\vee \rangle \alpha_i^\vee$. For any $w \in W$, if $w = s_{i_1}s_{i_2}\cdots s_{i_l}$ is a reduced expression of $w$, then we define the length $l(w) = l$. Let $W_0$ be the subgroup generated by $s_i | i \in I$; thus $W_0$ is the finite Weyl group of $g_0$.

Let $\text{Aut}(\Gamma)$ be the group of automorphisms of the Dynkin diagram associated with $\tilde{g}^\sigma$. Then $\text{Aut}(\Gamma)$ acts on $W$ by $\tau s \tau^{-1} = s_{\sigma(i)}$, $\forall i \in \tilde{T}, \forall \tau \in \text{Aut}(\Gamma)$. We denote $T = \text{Aut}(\Gamma) \cap (W_0 \ltimes P^\vee)$. We further introduce the extended affine Weyl group $\tilde{W} = T \ltimes W$, which can be written as $W_0 \ltimes P$ for twisted cases. The length function $\ell$ is extended to $\tilde{W}$ by \(\ell(t w) = \ell(w), \tau \in T, w \in W\).
2.3. Twisted Quantum Affine Algebras. In this paragraph let us review the definition of the twisted quantum affine algebra \( U_q(\tilde{\mathfrak{g}}^0) \). Set \( q_i = q^{a_i}, i \in \tilde{I} \). Introduce \( q \)-integer by

\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},
\]

\[
[n]_q! = \prod_{k=1}^{n} [k]_q.
\]

**Definition 1.** The twisted quantum affine algebra \( U_q(\tilde{\mathfrak{g}}^0) \) is an associative \( \mathbb{C}(q) \)-algebra generated by \( \{E_i, F_i, K_i^{-1}, \gamma^{\pm 1/2} \} \) for \( i \in \tilde{I} \), satisfying the following relations:

\[
\begin{align*}
(R1) \quad & y^{\pm 1/2} \text{ is central and } y = K_0, \\
(R2) \quad & K_i K_j = K_j K_i, \\
& \forall i, j \in \tilde{I}, \\
(R3) \quad & K_i E_j K_i^{-1} = q_i^{e_{ij}} E_j, \\
& K_i F_j K_i^{-1} = q_i^{-e_{ij}} F_j, \\
& \forall i, j \in \tilde{I}, \\
(R4) \quad & [E_i, F_j] = \delta_{ij} K_i^{-1} - q_i^{-1} K_i, \\
& \forall i, j \in \tilde{I}, \\
(R5) \quad & \sum_{s=0}^{1-a_{ij}} (-1)^s \left[ 1 - a_{ij} \right] s^{1-a_{ij}-s} E_i E_j^s = 0, \quad \forall i \neq j, \\
(R6) \quad & \sum_{s=0}^{1-a_{ij}} (-1)^s \left[ 1 - a_{ij} \right] s^{1-a_{ij}-s} F_i F_j^s = 0, \quad \forall i \neq j.
\end{align*}
\]

**Remark 2.** (1) There exists a unique Hopf algebra structure on \( U_q(\tilde{\mathfrak{g}}^0) \) with the comultiplication \( \Delta \), counit \( \varepsilon \), and antipode \( S \) defined by \( i \in \tilde{I} \):

\[
\begin{align*}
\Delta (E_i) &= E_i \otimes K_i^{-1} + 1 \otimes E_i, \\
\Delta (F_i) &= F_i \otimes 1 + K_i \otimes F_i, \\
\Delta (K_i) &= K_i \otimes K_i, \\
\varepsilon (E_i) &= 0, \\
\varepsilon (F_i) &= 0, \\
\varepsilon (K_i) &= 1, \\
S (E_i) &= -E_i K_i, \\
S (F_i) &= -K_i^{-1} F_i, \\
S (K_i) &= K_i^{-1}.
\end{align*}
\]

(2) Let \( U_q^+ \) (resp., \( U_q^- \)) be the subalgebra of \( U_q(\tilde{\mathfrak{g}}^0) \) generated by the elements \( E_i \) (resp., \( F_i \)) for \( i \in \tilde{I} \), and let \( U_q^0 \) be the subalgebra of \( U_q(\tilde{\mathfrak{g}}^0) \) generated by \( K_i \) and \( y^{\pm 1/2} \).

The twisted quantum affine algebra \( U_q(\tilde{\mathfrak{g}}^0) \) has the triangular decomposition:

\[
U_q (\tilde{\mathfrak{g}}^0) \equiv U_q^- \otimes U_q^0 \otimes U_q^+.
\]
The actions of $T_i$ and $T_i^{-1}$ on Chevalley generators are defined as follows:

\[ T_i (E_i) = -F_i K_i, \]
\[ T_i (F_i) = -K_i^{-1} E_i, \]
\[ T_i (E_j) = \sum_{s=0}^{\delta_i} (-1)^s q_i^s E_j \big{[} (s-\alpha_i), \alpha_i \big{]}, \quad \forall i \neq j \in \bar{I}, \]
\[ T_i^{-1} (E_j) = \sum_{s=0}^{\delta_i} (-1)^s q_i^s E_j \big{[} (s-\alpha_i), \alpha_i \big{]}, \quad \forall i \neq j \in \bar{I}, \]
\[ T_i^{-1} (F_j) = \sum_{s=0}^{\delta_i} (-1)^s q_i^s F_j \big{[} (s-\alpha_i), \alpha_i \big{]}, \quad \forall i \neq j \in \bar{I}, \]

where $E_j^{(s)} = E_j^{[s]}$, and $F_j^{(s)} = F_j^{[s]}$.

We extend the braid group action to the extended Weyl group $\bar{W}$ by defining $T_\tau$ as

\[ T_\tau (E_i) = E_{\tau(i)}, \]
\[ T_\tau (F_i) = F_{\tau(i)}, \]
\[ T_\tau (K_i) = K_{\tau(i)}; \]

\[ T_\tau (E_j) = \sum_{s=0}^{\delta_i} (-1)^s q_i^s E_j \big{[} (s-\alpha_i), \alpha_i \big{]}, \quad \forall i \neq j \in \bar{I}, \]

Remark 4. The braid group action $T_j$ commutes with $\Phi$; that is, $T_j \Phi = \Phi T_j$. Furthermore, one has $\Omega T_j = T_j^{-1} \Omega$.

Note that $\bar{W} = W_0 \rtimes P$; some properties about $T_{\omega_i}$ will be discussed which are almost the same to nontwisted cases, where $\omega_i = P_{\omega_i} \omega_i$.

Finally for each $i \in \bar{I}$, we define inductively the twisted derivation $r_i$ of $U_q^+$ by

\[ r_j (xy) = q^{(1)^{(\alpha_i)}} r_j (x) y + x r_j (y), \]
\[ r_j (x) = q^{(1)^{(\alpha_i)}} r_j (x), \]

where $x \in U_{q,\bar{I}}^+$, $y \in U_{q,\bar{I}}^+$. The following result is from Lusztig [12].

Lemma 5. (1) If $x \in U_q^+$ and $r_i (x) = 0$ for all $i \in \bar{I}$, then $x = 0$.

(2) One has \[ \{ x \in U_q^+ \mid r_i (x) = 0 \} = \{ x \in U_q^+ \mid T_i^{-1} (x) \in U_q^+ \}. \]

3. Vertex Subalgebras $U_q^{(2)}$

There are three different lengths of real roots for the type of $A_n^{(2)}$, which requires a different treatment from other twisted types. For each $i \in I$ we construct a copy of $U_q (\tilde{A}_2)$ inside $U_q (X_\delta^{(2)})$ for $X_\delta \neq A_n^{(2)}$. For the case of $(A_n^{(2)}, n)$ we will construct a subalgebra isomorphic to $U_q (A_2^{(2)})$.

3.1. Root System. The root system $\Delta$ of the twisted affine algebra $\tilde{g}$ is given by $\Delta = \Delta^+ \cup \Delta^-$ and $\Delta^+ = \Delta_0 \cup \Delta_+ \cup \Delta_-$, where $\Delta_+ \cup \Delta_-$ are listed as follows (see [14]):

\[ \Delta_+ = \{ k\delta \mid k > 0 \} \times I. \]

For type $A_n^{(2)}$,

\[ \Delta_+ = \{ k\delta + \alpha \mid \alpha \in \Delta^+, k \geq 0 \}, \]

\[ \Delta_+ = \{ k\delta - \alpha \mid \alpha \in \Delta^+, k > 0 \}, \]

for other types,

\[ \Delta_+ = \{ k\delta - \alpha \mid \alpha \in \Delta^+, k \geq 0 \} \times I, \]

\[ \Delta_+ = \{ k\delta + \alpha \mid \alpha \in \Delta^+, k > 0 \}, \]

where $\Delta^+$ is the positive roots system of $g$, and $d_\alpha = (\alpha, \alpha)/2$.

3.2. Quantum Root Vectors. In order to define the quantum root vectors, we review some notations from Beck-Nakajima [13]. For $\omega_i \in P$, $i \in I$, we choose $t_i \in F$ such that $\omega_i^{-1} \in W$. Choose a reduced expression of $\omega_i t_i^{-1}$ for each $i \in I$. We fix a reduced expression of $\omega_i \omega_{i-1} \cdots \omega_1$ as follows: $\omega_i \omega_{i-1} \cdots \omega_1 = s_i s_{i-1} \cdots s_1$, where $\tau = s_i \cdots s_1$.

We define a doubly infinite sequence

\[ \mathbf{h} = (\ldots, i_{-1}, i_0, i_1, \ldots), \]

by setting $h_{im} = t(i)$ for $l \in \mathbb{Z}$.

Then we have

\[ \Delta_+ = \{ a_{i_0} s_{i_1} (a_{i_2}), s_1 s_{i_2} (a_{i_3}), \ldots \}, \]

\[ \Delta_+ = \{ a_{i_0} s_{i_1} (a_{i_2}), s_1 s_{i_2} (a_{i_3}), \ldots \}. \]

Set

\[ \beta_k = \begin{cases} s_k s_{k-1} \cdots s_{i_k} (a_{i_k}), & \text{if } k \leq 0; \\ s_{i_k} s_{i_{k-1}} \cdots s_{i_0} (a_{i_k}), & \text{if } k > 0. \end{cases} \]
Define a total order on $\Delta^+$ by setting
\[
\beta_0 < \beta_{-1} < \beta_{-2} < \cdots < \delta^{(1)} < \cdots < \delta^{(n)} < \cdots
\]
where $k\delta^{(i)}$ denotes $(k\delta, i) \in \Delta_0$.

The root vectors for each element of $\Delta_+ \cup \Delta_-$ can be defined as follows:
\[
E_{\beta_k} = \begin{cases} 
T_{-1}^{-1} \cdots T_i^{-1}(E_{\omega_{\alpha_i}}), & \text{if } k \leq 0; \\
T_i \cdots T_{k+1}(E_{\omega_{\alpha_i}}), & \text{if } k > 0.
\end{cases}
\]

It follows from [12] that the elements $E_{\beta_k} \in U^+_q$.

**Remark 6.** For a positive root $\alpha = \omega_1(\alpha_1)$ in the root system $\Delta_+ \cup \Delta_-$ of $U_q(\mathfrak{g})$, there exists another presentation $\alpha = \omega_2(\alpha_2)$, where $\omega_1, \omega_2$ are in the Weyl group $W$ and $\alpha_1, \alpha_2$ are simple roots. Then the quantum root vector $E_{\alpha}$ can be defined by two ways:
\[
E_{\alpha} = T_{\omega_1}(E_i), \quad E'_{\alpha} = T_{\omega_2}(E_i).
\]

Actually, the two definitions agree up to a constant, because there exists $\omega \in W_0$ such that $\alpha_i = \omega(\alpha_i)$, which means that $\omega_1 = \omega_2 \omega$ and $l(\omega) = l(\omega_1) + l(\omega_2)$; then we have
\[
T_{\omega_1}(E_i) = T_{\omega_2}(E_i) = c_i(q) T_{\omega_2}(T_{\omega_1}(E_i)) = c_i(q) T_{\omega_2}(E_i),
\]
where $c_i(q)$ are functions in $q$.

Furthermore, if $\beta_k = \alpha_i$ for some $i \in I$, then $E_{\beta_k} = c(q) E_{\alpha_i}$ for a function $c(q)$. As usual, define $F_\beta = \Phi(E_\beta)$ for all $\beta \in \Delta_+ \cup \Delta_-$. Now we introduce the root vectors of Drinfeld generators [15]:
\[
E_{k_1, k_2, \alpha_1} = \frac{T_{\omega_1}^{-1}(E_i)}{F_i}, \quad \text{for } k \geq 0,
\]
\[
E_{k_1, k_2, \alpha_1} = \frac{T_{\omega_1}(E_i)}{F_i}, \quad \text{for } k > 0.
\]

Note that the definitions are not so different from those of nontwisted cases up to some slight adjustments because of the difference between their root systems.

### 3.3. Vertex Subalgebra $U_q^{(i)}$.

Let subalgebra $U_q^{(i)}$ of twisted quantum affine algebra $U^q(\mathfrak{g}_i)$ be generated by $E_i, T_i^{-1}(E_i), F_i, T_{\omega_1}^{-1}(E_i), F_{\omega_1} T_i^{-1}(F_i), K_1, T_{\omega_1}^{-1}(K_{\omega_1})$ and $\gamma^{1/2}$ for $i \in I$.

It is clear that the vertex subalgebra $U_q^{(i)}$ is stable under the actions of $\Phi_i, T_i$, and $T_{\omega_1}$. Additionally, $U_q^{(i)}$ is pointwise fixed by $T_{\omega_1}$ for $j \neq i$.

We list the following result which is already proved in [16] (also see [4]).

**Lemma 7.** Let $i \neq j \in I$; one has that
\[
\ell(\omega_i \omega_j) = 2\ell(\omega_i) - 1,
\]
\[
T_i^{-1} T_j^{-1} T_i^{-1} = T_{\omega_i}^{-1} \prod_{j \not\in i} T_{\omega_j}^{-p_{\alpha_i}/p_j},
\]
\[
T_{\omega_i}^{-1}(E_j) = T_{\omega_j}^{-p_{\alpha_i}/p_j} T_i(E_j).
\]

The following statements are based on the construction of [4].

**Lemma 8.** One has
\[
\ell(\omega_i \omega_j) = 2\ell(\omega_i) - 1,
\]
\[
E_i^T_{(i)}(E_j) = 0,
\]
\[
T_i^{-1} T_j^{-1} T_i^{-1} = T_{\omega_i}^{-1} \prod_{j \not\in i} T_{\omega_j}^{-p_{\alpha_i}/p_j},
\]
\[
T_{\omega_i}^{-1}(E_j) = T_{\omega_j}^{-p_{\alpha_i}/p_j} T_i(E_j).
\]

**Proposition 9.** For $i \in I$, there exists an algebra isomorphism $\phi_i: U_q(A_i^{(1)}) \rightarrow U_q^{(i)}$, defined by
\[
\phi_i(E_i) = E_i, \quad \phi_i(F_i) = F_i, \quad \phi_i(\omega_i) = \omega_i,
\]
\[
\phi_i(K_{\omega_i}) = K_{\omega_i}, \quad \gamma^{1/2} = \gamma^{1/2}.
\]

#### 3.4. Relations of Imaginary Root Vectors.

Now we define the positive imaginary root vectors. For $k > 0$ and $i \in I$ let
\[
\bar{\omega}_i(p, k) = E_{k_1, k_2, \alpha_1} F_i - q_i^{-1} E_i E_{k_1, k_2, \alpha_1};
\]
then we use $\bar{\omega}_i(p, k)$ to denote $\Phi_i(\bar{\omega}_i(p, k))$.

Define the elements $E_{\alpha_{i, k, p, \delta}} \in U^+_q$ by the functional equation
\[
\exp \left( q_i - q_i^{-1} \sum_{k=0}^{\infty} E_{i, k, p, \delta} u^k \right)
\]
\[
= 1 + \left( q_i - q_i^{-1} \sum_{k=0}^{\infty} \bar{\omega}_i(p, k) u^k \right).
\]

Similarly, we introduce $F_{i, k, p, \delta} = \Phi_i F_{i, k, p, \delta}$ for $k > 0$. Then we have the following lemma.

**Lemma 10.** For $i, j \in I, k, l \geq 0$, one has
\[
E_{i, k, p, \delta} E_{j, l, p, \delta} = 0.
\]
Define a map \( o : I \to \{ \pm 1 \} \) such that \( o(i)o(j) = -1 \) for \( a_{ij} < 0 \). For \( k > 0 \) and \( i \in I \), let

\[
a_i(p_k) = o(i)^k y^{-k/2} E_{i,kp} \delta, \\
p_i(p_k) = o(i)^k (q_i, q_i^{-1}) y^{-k/2} K_i \varpi_i(p_k).
\]  

(33)

Similarly, we can define \( a_i(-p_k) \) and \( p_i(-p_k) \) for \( k > 0 \).

The following was given in [17], which is an application of Beck’s work [4].

**Proposition 11.** For \( i, j \in I \), for \( k > 0 \), \( l > 0 \), the following relations hold:

1. \[ [\psi_i(p_k), \psi_i(p_l)] = 0 = [a_i(p_k), a_i(p_l)], \]
2. \[ T_{\omega j}(\psi_i(p_k)) = \psi_i(p_k), \]
3. \[ T_{\omega j}(a_i(p_k)) = a_i(p_k), \]
4. \[ [a_i(p_k), T_{\omega j}^{-l}(E_i)] = \frac{[2r_i]}{k} \tilde{T}_{\omega j}^{-(k+l)}(E_i). \]

(34)

3.5. Relations between \( U^{(0)}_q \) and \( U^{(1)}_q \). We recall the commutation relations among the root vectors and remark that the argument also works in the case of \( A^{(2)}_{2n} \).

**Lemma 12** (see [4, 16]). One has that, for \( i, j \in I \),

1. \[ T_{\omega i} T_{\omega j} = T_{\omega j} T_{\omega i}, \]
2. \[ T_{\omega i}(E_i) = E_i, \]
3. \[ T_{\omega i} T_i = T_i T_{\omega i} \quad \text{for } i \neq j. \]

(35)

We also let \( \tilde{T}_{\omega i} = o(i)T_{\omega i} \). The following is immediate.

**Lemma 13.** If \( a_{ij} = 0 \) for \( i, j \in I \), then \([U^{(0)}_q, U^{(1)}_q] = 0\).

**Lemma 14.** For \( i \neq j \in I \) and \( k, l \in \mathbb{Z} \),

\[ \left[ \tilde{T}_{\omega i}^{-k}(E_i), \tilde{T}_{\omega j}^{-l}(E_j) \right] = 0. \]

(36)

If \( i \neq j \in I \) such that \( a_i a_{ji} = 1 \), it is easy to see that \( \sigma(i) \neq i \) and \( \sigma(j) \neq j \) and \( d_i = p_i = 1 \), \( d_j = p_j = 1 \). Thus one has the following lemma.

**Lemma 15.** If \( i \neq j \in I \) such that \( a_i a_{ji} = 1 \), then for \( k > 1 \), \( l \in \mathbb{Z} \) one has the following relations:

1. \[ [\psi_i(1), E_j] = -y^{-1/2} \tilde{T}_{\omega j}^{-1}(E_j), \]
2. \[ [\psi_i(k), \tilde{T}_{\omega j}^{-l}(E_j)] = y^{-1/2} \left( q^{-l} \tilde{T}_{\omega j}^{-1}(E_j) \psi_i(k-1) - q \psi_i(k-1) \tilde{T}_{\omega j}^{-1}(E_j) \right), \]
3. \[ [\psi_i(1), F_j] = y^{-1/2} \tilde{T}_{\omega j}(F_j), \]
4. \[ [\psi_i(k), \tilde{T}_{\omega j}^{-l}(F_j)] = y^{1/2} \left( q^{-l} \tilde{T}_{\omega j}^{-1}(F_j) \psi_i(k-1) - q \psi_i(k-1) \tilde{T}_{\omega j}^{-1}(F_j) \right). \]

(37)

If \( i \neq j \in I \) such that \( a_{ij} = -r \) and \( a_{ji} = -1 \), it is easy to see that \( \sigma(i) \neq i \) and \( \sigma(j) = j \) and \( p_i = d_i = 1 \), \( d_j = r \). Then the next result follows.

**Lemma 16.** Suppose \( a_{ij} = -r \) and \( a_{ji} = -1 \) for \( i \neq j \in I \). Then for \( k > r, l \in \mathbb{Z} \), one has

1. \[ [\psi_i(1), \tilde{T}_{\omega j}^{-l}(E_j)] = -y^{-1/2} \left[ a_{ij} \right] \tilde{T}_{\omega j}^{-1}(E_j), \]
2. \[ [\psi_i(p_k), \tilde{T}_{\omega j}^{-l}(E_j)] = y^{-1/2} \left( q^{-l} \tilde{T}_{\omega j}^{-1}(E_j) \psi_i(p_k - r) - q a_i \psi_i(p_k - r) \tilde{T}_{\omega j}^{-1}(E_j) \right), \]
3. \[ [\psi_i(1), \tilde{T}_{\omega j}^{l}(F_j)] = y^{1/2} \left[ a_{ij} \right] \tilde{T}_{\omega j}^{l+1}(F_j), \]
4. \[ [\psi_i(p_k), \tilde{T}_{\omega j}^{l}(F_j)] = y^{1/2} \left( q a_i \tilde{T}_{\omega j}^{l}(F_j) \psi_i(p_k - r) - q a_i \psi_i(p_k - r) \tilde{T}_{\omega j}^{l+1}(F_j) \right). \]

(38)
The following statements follow directly from Lemmas 18 and 16. Note that \( d_{i,j} = \sum _{l=0}^{\gamma (i)} A_{l,\sigma (j)} \) for \( i, j \in I \), which will be used later.

**Lemma 17.** Let \( i \neq j \in I \) such that \( a_{ij} = 1 \). For \( k > 1, l \in \mathbb{Z} \), one has

\[
\begin{align*}
&\left[ a_i(p), \tilde{T}_{\omega_j}(E_j) \right] = \frac{[k a_i]}{k} T_{\omega_j}^{-(k+\ell)} (E_j), \quad \text{if } p_j \mid k, \\
&0, \quad \text{otherwise},
\end{align*}
\]

(1)

\[
\begin{align*}
&\left[ a_i(p), \tilde{T}_{\omega_j}(F_j) \right] = -\frac{[k a_i]}{k} T_{\omega_j}^{-(k+\ell)} (F_j), \quad \text{if } p_j \mid k, \\
&0, \quad \text{otherwise},
\end{align*}
\]

(2)

\[
\begin{align*}
&\left[ a_i(p), a_j(-l) \right] = \delta_{k,l} \frac{[k a_i]}{k} K_{k\delta} - K_{k\delta}^{-1} q_j^{-1}.
\end{align*}
\]

(3)

**Lemma 18.** Let \( i \neq j \in I \) such that \( a_{ij} = -r \) and \( a_{ji} = -1 \). Then for \( k > r, l \in \mathbb{Z} \),

\[
\begin{align*}
&\left[ a_i(p), \tilde{T}_{\omega_j}(E_j) \right] = \frac{[k a_i]}{k} T_{\omega_j}^{-(k+\ell)} (E_j), \quad \text{if } p_j \mid k, \\
&0, \quad \text{otherwise},
\end{align*}
\]

(1)

\[
\begin{align*}
&\left[ a_i(p), \tilde{T}_{\omega_j}(F_j) \right] = -\frac{[k a_i]}{k} T_{\omega_j}^{-(k+\ell)} (F_j), \quad \text{if } p_j \mid k, \\
&0, \quad \text{otherwise},
\end{align*}
\]

(2)

\[
\begin{align*}
&\left[ a_i(p), a_j(-l) \right] = \delta_{k,l} \frac{[k a_i]}{k} K_{k\delta} - K_{k\delta}^{-1} q_j^{-1}.
\end{align*}
\]

(3)

4. Drinfeld Realization for Twisted Cases

4.1. Drinfeld Generators. In order to obtain the Drinfeld realization of twisted quantum affine algebras, we introduce Drinfeld generators as follows.

**Definition 20.** For \( k > 0 \), define

\[
\begin{align*}
x_i^+(k) &= \begin{cases} T_{\omega_i}^{-k} (E_i), & \text{if } \sigma(i) \neq i \text{ or } \sigma(i) = i \text{ and } r \mid k \\ 0, & \text{otherwise} \end{cases}, \quad (40)
\end{align*}
\]

(40)

\[
\begin{align*}
x_i^-(k) &= \begin{cases} T_{\omega_i}^{-k} (F_i), & \text{if } \sigma(i) \neq i \text{ or } \sigma(i) = i \text{ and } r \mid k \\ 0, & \text{otherwise} \end{cases}, \quad (43)
\end{align*}
\]

(43)

**Definition 21.** For \( k < 0 \), define

\[
\begin{align*}
x_i^+(k) &= \begin{cases} -\alpha(i)^k T_{\omega_i}^{-k} (F_i), & \text{if } \sigma(i) \neq i \text{ or } \sigma(i) = i \text{ and } r \mid k; \\ 0, & \text{otherwise}, \end{cases} \quad (41)
\end{align*}
\]

(41)

\[
\begin{align*}
x_i^-(k) &= \begin{cases} -\alpha(i)^k T_{\omega_i}^{-k} (E_i), & \text{if } \sigma(i) \neq i \text{ or } \sigma(i) = i \text{ and } r \mid k; \\ 0, & \text{otherwise}. \end{cases} \quad (44)
\end{align*}
\]

(44)

**Remark 22.** From the above definitions it follows that \( a_i(k) = 0 \) if \( \sigma(i) = i \) and \( k \) is not divisible by \( r \).

For convenience we extend the indices from \( I \) to \( \{1, 2, \ldots, N\} \). For \( i \in \{n+1, n+2, \ldots, N\} \) and \( k \in \mathbb{Z}, l \in \mathbb{Z}/[0] \), we define that

\[
\begin{align*}
x_i^+(k) &= \omega^{-k} x_i^+(k), \\
a_i(l) &= \omega^{-l} a_i(l), \\
K_i &= K_{\sigma(i)}.
\end{align*}
\]
4.2 Drinfeld Realization for Twisted Cases. In previous sections we have prepared for the relations among Drinfeld generators. The complete relations are given in the following theorem stated first in [19]. In the following we will set out to prove the remaining Serre relations using braid groups and other techniques developed in [10, 11].

**Theorem 23.** The twisted quantum affine algebra $U_q(\hat{\mathfrak{g}}^\sigma)$ is generated by the elements $x_i^\pm(k), a_i(l), K_i^{\pm1}$, and $y^{1/2}$, where $i \in \{1, 2, \ldots, N\}, k \in \mathbb{Z}$, and $l \in \mathbb{Z}/\{0\}$, satisfying the following relations:

1. $x_i^+(k) = \omega^{-k} x_{\sigma(i)}^+(k)$, 
   
   \[ a_i(l) = \omega^{-1} a_{\sigma(i)}(l) \]
   
   \[ K_i = K_{\sigma(i)} \]

2. \[ [y^{1/2}, u_i] = 0 \quad \forall u \in U_q(\hat{\mathfrak{g}}^\sigma) \]

\[
\sum_{m=0}^{\infty} \psi_i(m) z^{-m} = K_i \exp \left( (q_i - q_i^{-1}) \sum_{k=1}^{\infty} a_i(k) z^{-k} \right),
\]

\[
\sum_{m=0}^{\infty} \phi_i(-m) z^{-m} = K_i^{-1} \exp \left( -(q_i - q_i^{-1}) \sum_{k=1}^{\infty} a_i(-k) z^k \right),
\]

3. \( \prod_{s=0}^{r-1} (z - \omega^s q^A \omega^s w) x_i^+(z) = \prod_{s=0}^{r-1} (zq^A \omega^s - \omega^s w) x_i^+(z) \)

4. \[ [a_i(k), a_j(l)] = \delta_{k,l} \sum_{m=0}^{r-1} \left[ K_{A_{\sigma(i)}(j)} \right]_{k} k^{y^k} - y^{-k}, \]

5. \[ [a_i(k), K_j^{+1}] = 0, \]

6. \[ K_j^{+1} x_j^+(k) K_{j}^{-1} = q^A \sum_{s=0}^{r-1} A_{\sigma(i)}(j) x_j^+(k) \]

7. \[ [a_i(k), x_j^+(l)] = \prod_{s=0}^{r-1} \left[ K_{A_{\sigma(i)}(j)} \right]_{k} k^{y^k} x_j^+(k + l), \]

where $\psi_i(m)$ and $\phi_i(m)$ ($m \in \mathbb{Z}_{\geq 0}$) are defined by

\[ \text{Sym}_{\text{Sym}} (\sum_{k \in \mathbb{Z}} x_i^+(k) z^{-k}, P_{ij}^+(z, w)) \]

where $\text{Sym}$ means the symmetrization over $z_i, x_i^+(z) = \sum_{k \in \mathbb{Z}} x_i^+(k) z^{-k}$, and $P_{ij}^+(z, w)$ and $d_{ij}$ are defined as follows:

If $\sigma(i) = i$,

then $P_{ij}^+(z, w) = 1$,

\[ d_{ij} = \frac{r}{2} \]

If $A_{\sigma(i)}(j) = 0, \sigma(j) = j$,

then $P_{ij}^+(z, w) = \frac{z^r q^{2r} - w^r}{zq^{2r} - w}, \]

\[ d_{ij} = r \]

If $A_{\sigma(i)}(j) = 0, \sigma(j) \neq j$,

then $P_{ij}^+(z, w) = 1$,

\[ d_{ij} = \frac{1}{2} \]

4.3 Proof of the Main Theorem. We need to verify that the above Drinfeld generators $x_i^+(k), a_i(l), K_i^{\pm1}$ satisfy all relations (1)–(10). Relations (1)–(7) are already checked in the previous paragraphs. We are going to show the last three relations.

We first proceed to check relation (8).

**Proposition 24.** For all $i, j \in I$ one has that

\[ \prod_{s=0}^{r-1} (z - \omega^s q^{A_{\sigma(i)}(j)} w) x_i^+(z) x_j^+(w) \]

\[ = \prod_{s=0}^{r-1} (zq^A \omega^s - \omega^s w) x_j^+(w) x_i^+(z) \]

where $x_i^+(z) = \sum_{k \in \mathbb{Z}} x_i^+(k) z^{-k}$. 

(46)
The relation holds if $A_{ij} = 0$, so we only consider the case of $A_{ij} \neq 0$. The proof is divided into several cases.

**Case a** ($i = j$). The required relations are generating functions of the following component relations:

$$x_i^+ (k + p_i) x_i^- (l) - q_i^2 x_i^- (l) x_i^+ (k + p_i) = q_i^2 x_i^+ (k) x_i^- (l + p_i) - x_i^- (l + p_i) x_i^+ (k).$$  \hfill (50)

On the other hand, the following relations hold in $t_q (i)$ similar to those of the untwisted cases:

$$T_{\omega_i}^{-k} (E_i) E_i - q_i^2 E_i T_{\omega_i}^{-k} (E_i)$$

\hfill (51)

Hence the required relation follows by recalling the definition of $x_i^+ (k)$.

**Case b** ($i \neq j$ such that $A_{ij} \neq 0$). First for $i, j$ we define

$$E_{ij} = -E_i E_j + q_i^4 E_i E_j,$$  \hfill (52)

**Lemma 25.** For $i \neq j \in I$ such that $A_{ij} \leq 0$ and $k \in \mathbb{Z}$,

$$- T_{\omega_i}^{-k} (E_i) T_{\omega_j}^{-l} (E_j) + q_i^4 T_{\omega_i}^{-k} (E_i) T_{\omega_j}^{-l} (E_j)$$

\hfill (53)

Proof. Applying $T_{\omega_i}^{-k}$ and $T_{\omega_j}^{-l}$ to $E_{ij}$ and invoking Lemma 12, we can pull out the action of $T_{\omega_j}$ to arrive at

$$- T_{\omega_i}^{-k} (E_i) E_j + q_i^4 T_{\omega_i}^{-k} (E_i) T_{\omega_j}^{-l} (E_j)$$

\hfill (54)

which was essentially proved by Beck [4] since $d_j p_i = d_i p_j$.

The following well-known fact will be used to prove the remaining relations.

**Lemma 26.** If $A \in U_q (\mathfrak{g})^+$ and $[A, F_k] = 0 \ \forall k \in \tilde{I}$, then $A = 0$.

We concentrate mainly on relation (9) and divide it into four cases. The Serre relation in the case of $P_{ij}^+(z_1, z_2) = 1$ can be derived from that of nontwisted case. Moreover, the proof will explain why the Serre relation with the lower power works by the action of diagram automorphism $\sigma$ in the twisted case.

**Proposition 27.** For $A_{ij} = -1$ and $\sigma (i) \neq j$, one has

$$\text{Sym}_{x_i, x_j} P_{ij}^+ (z_1, z_2) \sum_{i=0}^{2} (-1)^i x_i^+ (z_1) \cdots x_j^+ (z_j) x_j^+ (w)$$

\hfill (55)

Proof. This is proved case by case.

**Case i** ($A_{ij} = -1$ and $\sigma (i) = i$). In this case $P_{ij}^+(z_1, z_2)$ is 1 and $d_i = r_i$, it is also clear that $d_i = r_i$, and then the relation is exactly like the Serre relation in the nontwisted case. For completeness we provide a proof for this Serre relation. That is, we will show that for any integers $k_1, k_2, l$

$$\text{Sym}_{k_1, k_2} (x_i^+(l) x_i^-(k_j) x_i^+(k_2))$$

\hfill (56)

Note that $x_i^+(l) = T_{\omega_j}^{-l} \ E_j$. Lemma 12 says that one can pull out any factor of $T_{\omega_j}$ or common factors of $T_{\omega_i}$ from the left-hand side (LHS). This means that for any natural number $t$ the following relation is equivalent to the Serre relation:

$$\left( E_i E_j T_{\omega_i} turb; (E_i) - [2] x_i^+ (k_1) x_i^- (l) x_i^+(k_2) \right)$$

$$+ x_i^+ (k_1) x_i^- (l) x_i^+(k_2).$$

We prove this last relation by induction on $t$. First note that when $t = 0$, the relation is essentially relation (RS). We assume that the above relation holds when the parameter is less than $t$, and we would like to show it for $t$. The remark above further says that once we have made the inductive assumption then all Serre relations with $|k_1 - k_2| \leq t - 1$ and arbitrary $l$ are also assumed to be true. Using relation (6) in Theorem 23 yields

$$\tilde{T}_{\omega_i}^{-1} (E_i) = \frac{\text{Sym}_{1/2, 1} [a_i (1), E_i]}{[2]_l}.$$  \hfill (57)

Plugging this into LHS of the Serre relation we get that

$$\frac{\text{Sym}_{1/2, 1} [a_i (1), E_i]}{[2]_l} \left( E_i E_j \tilde{T}_{\omega_i}^{-t+1} \right)$$

\hfill (58)
Then we repeatedly use (∗) to move $a_i(1)$ to the extreme left to get an expression of the form

$$a_i(1) \left[ E_i E_j \tilde{T}_{\omega_i}^{-1} (E_i) - [2], E_i E_j \tilde{T}_{\omega_i}^{-1} (E_i) \right] + E_i \tilde{T}_{\omega_i}^{-1} (E_i) E_j + \cdots ,$$

(59)

where “...” only involves LHS of Serre relations with $t - 2$. So the whole expression is zero by the inductive assumption. Thus we have finished the proof of Serre relation (9) in this case.

Case ii ($A_{ij} = -1$ and $A_{ij, \sigma(i)} = 0, \sigma(j) = j$). For $r = 2$, without loss of generality we take $A_{z, -1}^{(2)}$; for example, the other cases are treated similarly. In this case we only need to consider the situation when $i = n - 1$ and $j = n$; then $P_{n+1}^i (z_1, z_2) = z_1 q^2 z_2$ and $d_{ij} = 2$. So we need to prove the following relations:

$$q^3 \left( x^j_i (l) x^j_i (k + 1) x^j_i (k) \right)
- [2] q^{r-1} x^j_i (k + 1) x^j_i (l) x^j_i (k)
+ x^j_i (k + 1) x^j_i (l) x^j_i (k + 1)
+ \left( x^j_i (l) x^j_i (k) x^j_i (k + 1) \right)$$

$$- [2] q^{r-1} x^j_i (k) x^j_i (l) x^j_i (k + 1)
+ x^j_i (k) x^j_i (k + 1) x^j_i (l) = 0.$$

(60)

Using the definition of $x^j_i (k)$ and collecting the action of $\tilde{T}_{\omega_i}^{-k} \tilde{T}_{\omega_i}^{-1}$, we are left to show that

$$X = q^3 \left( E_i \tilde{T}_{\omega_i}^{-1} (E_i) E_j - [2] q^{r-1} \tilde{T}_{\omega_i}^{-1} (E_i) E_j E_i \right)$$

$$+ \tilde{T}_{\omega_i}^{-1} (E_i) E_i E_j + \left( E_j E_i \tilde{T}_{\omega_i}^{-1} (E_i) \right)$$

$$- [2] q^{r-1} E_j E_i \tilde{T}_{\omega_i}^{-1} (E_i) E_j E_i E_i) = 0.$$

(61)

To see this we use Lemma 26 and compute all the commutators $[X, F_i] = 0$ for $k \neq 0$. First we consider the case of $k = 0$. Note that $[E_i, F_1] = 0$ whenever $i \neq 0$. On the other hand we claim that $[\tilde{T}_{\omega_i}^{-1} (E_i), F_0] = 0$. To see this we check that $r_0 (\tilde{T}_{\omega_i}^{-1} (E_i))) = 0$ by using the twisted derivation $r_0$. By Lemma 5 the last equation is equivalent to $\tilde{T}_{\omega_i}^{-1} \tilde{T}_{\omega_i}^{-1} (E_i) \in U_q^+$, which can be easily seen as $s_0 (\delta + \alpha_i) \in \Delta_+$ when $g_0 \neq sl_2$. Consequently it implies that $[X, F_0] = 0$.

Since $X$ is expressed by $E_i$ and $E_j$, $[X, F_k] = 0$ for any $k \neq i, j$. Using Drinfeld relation (7), we compute that

$$[X, F_j] = q^3 \left( E_j F_j \tilde{T}_{\omega_i}^{-1} (E_i) E_i \right)$$

$$- [2] q^{r-1} \tilde{T}_{\omega_i}^{-1} (E_i) E_j E_i + \tilde{T}_{\omega_i}^{-1} (E_i) E_j E_i [E_j, F_j]$$

$$+ \left( E_j [E_j, F_j] \tilde{T}_{\omega_i}^{-1} (E_i) \right)$$

$$+ E_j \tilde{T}_{\omega_i}^{-1} (E_i) [E_j, F_j] = 0,$$

(62)

where Drinfeld relation (5) has been used:

$$[X, F_i] = q^3 \left( E_j \tilde{T}_{\omega_i}^{-1} (E_j) F_i \right) E_i$$

$$- [2] q^{r-1} \tilde{T}_{\omega_i}^{-1} (E_i) E_j E_i + \tilde{T}_{\omega_i}^{-1} (E_i) E_j E_i [E_j, F_i]$$

$$+ E_j \tilde{T}_{\omega_i}^{-1} (E_i) [E_j, F_i] - [2] q^{r-1} \tilde{T}_{\omega_i}^{-1} (E_i) E_j E_i$$

$$+ \tilde{T}_{\omega_i}^{-1} (E_i) [E_j, F_i] + \left( E_j [E_j, F_i] \tilde{T}_{\omega_i}^{-1} (E_i) \right)$$

$$+ [E_j, F_i] \tilde{T}_{\omega_i}^{-1} (E_i) E_j$$

$$- [2] q^{r-1} E_j E_i \tilde{T}_{\omega_i}^{-1} (E_i) E_i E_i) = 0.$$

(63)

where we have used Drinfeld relation (7), (5), and (6) for the last step. Collecting common terms, we arrive at

$$[X, F_i] = \left( q^{r-1} \left( q^4 - q^2 \right) [2] q^{r-1} + 1 \right) E_j E_i a_i (1) K_i$$

$$+ \left( q^4 - q^2 \right) [2] q^{r-1} \tilde{T}_{\omega_i}^{-1} (E_i) K_i$$

$$+ \left( q^{r-1} \left( q^2 - [2] q^{r-1} \right) E_j E_i a_i (1) K_i.$$
+ q^2 \left[ 2 \tilde{T}_{\omega_j}^{-1}(E_j) E_j K_j \right] + \frac{1}{q_i - q_i^{-1}} \left( \left( q^2 - [2]_q^2 + q^{-2} \right) E_j \tilde{T}_{\omega_i}^{-1} K_i \right) + (2q^2 - [2]_q^2) E_j \tilde{T}_{\omega_i}^{-1} K_i \\
+ \frac{1}{q_i - q_i^{-1}} \left( \left( 1 - [2]_q q^2 + q^4 \right) \tilde{T}_{\omega_i}^{-1} E_i K_{(ij)} \right) + \left( \left( 2 - [2]_q q^2 \right) \tilde{T}_{\omega_i}^{-1} E_i K_{(ij)} \right) = ([2] - [2]) E_j \tilde{T}_{\omega_i}^{-1}(E_j) K_i + q^2 \left[ 2 - q^2 [2 \right] \tilde{T}_{\omega_i}^{-1}(E_j) E_j K_i = 0.

(64)

For \( r = 3 \), the exceptional type \( D_4^{(3)} \) should also be checked. In this case we know that \( P_{ij}^r(z_1, z_2) = z_1^2 q^{a_k} + z_1 z_2 q^{a_4} + z_2^2 \) and \( d_i = 3 \). More specifically we have in this case \( i = 2, j = 1 \) and \( d_i = 2, d_j = 1 \), and the relation is reduced to the following equivalent one:

\[
q^4 \left( x_j^i(k) x_j^i(k) + 2 x_j^i(k) \right) - \left( [2]_q q^2 \right) x_j^i(k) x_j^i(k) + x_j^i(k) + (k + 1) x_j^i(k) x_j^i(k) x_j^i(k)
\]

(65)

By definition of \( x_j^i(k) \) and collecting the action of \( \tilde{T}_{\omega_i}^{-1} \tilde{T}_{\omega_j}^{-1} \), we can rewrite the LHS of the above relation as follows:

\[
Y = q^4 \left( E_j \tilde{T}_{\omega_i}^{-2}(E_j) E_j - \left[ 2 \right]_q \tilde{T}_{\omega_i}^{-2}(E_j) E_j E_j \right)
\]

(66)

For \( k = j \), we use Drinfeld relation (4) and the commutation relation \([E_j, F_j] = (K_j - K_j^{-1})/(q_j - q_j^{-1}) \) to get that

\[
[Y, F_j] = q^4 \left( \left[ E_j, F_j \right] \tilde{T}_{\omega_i}^{-2}(E_j) E_j \right)
\]

(67)

Next we calculate that

\[
[Y, F_j] = q^4 \left( \left[ E_j, F_j \right] \tilde{T}_{\omega_i}^{-2}(E_j) E_j \right)
\]

Let us use Lemma 26 to show that \( Y = 0 \) by checking that \([Y, F_k] = 0 \) for all \( k \in \tilde{I} \), when \( k = 0 \) is clear as \( Y \) is only expressed in terms of \( E_j \) and \( E_j \) and \( i, j \in I \). This also implies that \([Y, F_k] = 0 \) for \( k \neq i \) or \( j \).
The third term is $\gamma^{1/2} \left( q^4 E_j a_1 (1) \tilde{T}_{\omega_i}^{-1} (E_i) \right) + q^2 E_j \tilde{T}_{\omega_i}^{-1} (E_i) a_1 (1) - q \left[ 2 \right] q^3 a_1 (1) E_i \tilde{T}_{\omega_i}^{-1} (E_i) \right) \right) + q^2 \left[ 2 \right] a_i (1) E_i \tilde{T}_{\omega_i}^{-1} (E_i) E_j + q \tilde{T}_{\omega_i}^{-1} (E_i) a_1 (1) E_j \right) K_i = \gamma^{1/2} \left( -q^2 E_j a_1 (1) \right) $.

The forth term is $\frac{1}{q_i - q_j} \left( q^4 E_j \tilde{T}_{\omega_i}^{-2} (E_i) \right) \left( K_i - K_{-1}^{-1} \right) $.

The fifth term is $\frac{1}{q_i - q_j} \left( q^4 \left[ 2 \right] q^3 \tilde{T}_{\omega_i}^{-2} (E_i) \right) \left( K_i - K_{-1}^{-1} \right) $.

Their total sum is zero; thus we have shown that $[Y, F_i] = 0$ and subsequently the Serre relations hold in this case.

Case iii ($A_{ij} = -1$ and $A_{i\sigma(0)} = 0, \sigma(0) \neq j$). The required relation follows from that of the untwisted case verified in Case i.

Case iv ($A_{ij} = -1$ and $A_{i\sigma(0)} = -1$). This only happens for type $A_{2n-1}^{(2)}$. Here $P_{2n}^{(2)}(z_j, z_k) = z_j q^{1/2} + z_k$ and $d_j = 1/2$, which is exactly the same as that of Case ii in type $A_{2n-1}^{(1)}$. Thus the Serre relation is proved by repeating the argument of Case ii.

By now we have proved all cases of Serre relation (9).

The last Serre relation (10) only exists for type $A_{2n}^{(2)}$.

**Proposition 28.** For $A_{i\sigma(0)} = 0$, 

$$\text{Sym}_{z_j, z_k, z_l} \left[ \left( q^{-3/4} z_1 - q^{-1/4} - q^{-7/4} z_2 + q^{-3/4} z_3 \right) \right] \cdot x_j^+(z_i) x_k^+(z_j) x_l^+(z_k) = 0. \tag{70}$$

**Proof.** By the same translation property of the Drinfeld generators, this relation can be replaced by

\[
- \frac{q^{-3/2} x_j^+(1) \left( x_i^+(0) \right)^2}{2} - \left( q^{1/2} + q^{-1/2} \right) x_j^+(0) x_i^+(1) x_i^+(0) + q^{3/2} \left( x_i^+(0) \right)^2 x_i^+(1) = 0. \tag{71}
\]
By definition of \( x_i^+(k) \), the above relation is rewritten as

\[
Z := q^{-3/2} T^{-1}_{\omega_i} (E_i) E_i^2 - \left( q^{1/2} + q^{-1/2} \right) E_i T^{-1}_{\omega_i} (E_i) E_i + q^{-3/2} E_i T^{-1}_{\omega_i} (E_i) = 0.
\]  

(72)

Using the same trick of Lemma 26, we must show that \([Z, F_i] = 0\) for all \( k \in I \):

\[
[Z, F_i] = \left( q^{-3/2} T^{-1}_{\omega_i} (E_i) [E_i, F_i] E_i - [2], [E_i, F_i] T^{-1}_{\omega_i} (E_i) E_i + q^{-3/2} [E_i, F_i] T^{-1}_{\omega_i} (E_i) + q^{-3/2} [E_i, F_i] E_i T^{-1}_{\omega_i} (E_i) \right)
\]

\[\]

(73)

\[
= \left( - [2], [E_i, F_i] T^{-1}_{\omega_i} (E_i) E_i + q^{-1/2} [2], (q + q^{-1} + 1) \right)
\]

\[
\cdot T^{-1}_{\omega_i} (E_i) E_i K_i + \left( - [2], q + q^{-1} + 1 \right) E_i T^{-1}_{\omega_i} (E_i) K_i = 0.
\]

With this last Seerre relation we have completed the verification of all Drinfeld relations.

5. Isomorphism between the Two Structures

5.1. The Inverse Homomorphism. To complete the proof of Drinfeld realization we need to establish an isomorphism between the Drinfeld-Jimbo algebra and the Drinfeld new realization. There have been several attempts to show the isomorphism in the literature, and all previous proofs only established a homomorphism from one form of the algebra into the other one. In this section we will combine our previous approach [9–11] together with Beck’s idea of braid realization. There have been several attempts to show the Drinfeld realization we need to establish an isomorphism 5.1. The Inverse Homomorphism.

First of all, we review the notation of quantum Lie brackets from [9].

Definition 29. Let \( K \) be a field for \( q_i \in K^* = K \setminus \{ 0 \} \) and \( i = 1, 2, \ldots, s - 1 \). The quantum Lie brackets

\[
[a_1, a_2, \ldots, a_s]_{(q_1, q_2, \ldots, q_{s-1})},
\]

\[\]

are defined inductively by

\[
[a_1, a_2]_{q_1} = a_1 a_2 - q_1 a_2 a_1,
\]

\[\]

(74)

\[
[a_i, a_{i-1}, \ldots, a_1]_{(q_1, q_2, \ldots, q_{s-1})} = [a_i, a_{i-1}, \ldots, a_2, a_1]_{(q_1, q_2, \ldots, q_{s-1})},
\]

\[\]

(75)

\[
[a_1, a_2, \ldots, a_s]_{(q_1, q_2, \ldots, q_{s-1})} = [[a_1, a_2]_{q_1}, a_3, \ldots, a_s]_{(q_2, \ldots, q_{s-1})}.
\]

To state the inverse homomorphism, we need to fix a particular path to realize the maximum root of \( q_0 \).

Let \( \theta = \alpha_{h-1} + \cdots + \alpha_i + \alpha_i \) be the maximum root and let

\[
X_\theta = \left[ e_{i_{h-1}}, \left( e_{i_{h-2}, \ldots, e_{i_{h-1}, e_i}} \right) \right],
\]

be the corresponding root vector in the Lie algebra \( g_0 \), which gives rise to a sequence from \([1, \ldots, n] : i_1, i_2, \ldots, i_{h-1} \). We call such a sequence a root chain to the maximum root, which is not unique.

From now on we fix a particular path to realize the maximum root of \( q_0 \) and the associated sequence \( i_1, i_2, \ldots, i_{h-1} \). We define for \( 2 \leq k \leq h - 1 \)

\[
(\alpha_i + \cdots + \alpha_{h-1}, \alpha_i) = e_k \neq 0.
\]  

(77)

Theorem 30. Let \( i_1, i_2, \ldots, i_{h-1} \) be the sequence of indices in the particular path realizing the maximum root \( \theta \) given in (76); then there is an algebra homomorphism \( \phi : U_q(g) \rightarrow \mathcal{U}_q(g^\circ) \) defined by

\[
\phi(E_i) = x_i^+(0),
\]

\[
\phi(F_i) = \frac{1}{P_i} x_i^-(0),
\]

\[
\phi(K_i) = K_i^+(0),
\]

\[
\phi(E_0) = a x_0^-(1) \gamma K_0^1, \]

\[
\phi(F_0) = \gamma^{-1} K_0 x_0^+(1),
\]

\[
\phi(K_0) = \gamma K_0^{-1},
\]

(78)
where $K_\theta = K_{i_1} K_{i_2} \cdots K_{i_{h-1}}$, $a = (p_{i_1} \cdots p_{i_{h-1}})^{-1}$, and $x_\theta^+(1)$, $x_\theta^-(1)$ are defined by quantum Lie bracket as follows:

\[
\begin{align*}
x_\theta^-(1) &= \left[ x_{i_{h-1}}^-(0), x_{i_{h-2}}^-(0), \ldots, x_{i_1}^-(0) \right], \\
x_\theta^+(1) &= \left[ x_{i_{h-1}}^+(0), x_{i_{h-2}}^+(0), \ldots, x_{i_1}^+(0) \right].
\end{align*}
\]

It is not difficult to check that the algebra $\mathcal{U}_q(\mathfrak{g})$ is actually generated by $x_{i_1}^+(0)$ $(i = 1, \ldots, n)$, $x_\theta^-(1)$, and $x_\theta^+(1)$; therefore $\phi$ is an epimorphism.

**Remark 31.** In [10, 11], we have checked that $\phi$ is an algebra homomorphism using quantum Lie brackets. In the sequel we show that the map is in fact the inverse of the action of the braid group.

### 5.2. Isomorphism between Two Presentations

Theorem 23 induces that the homomorphism $\psi$ from Drinfeld realization $\mathcal{U}_q(\mathfrak{g})$ to Drinfeld-Jimbo algebra $U_q(\mathfrak{g})$ is surjective. We now show its injectivity by checking that the products of two maps are identity. We start with the following proposition.

**Proposition 32.** With homomorphisms $\phi$ and $\psi$ defined as above, one has $\psi \phi(E_0) = E_0$.

**Proof.** Note that

\[
\psi \phi(E_0) = \psi \left( a x_\theta^-(1) \right) K_{\theta}^{-1}
\]

\[
= \left[ F_{i_{h-1}}, F_{i_{h-2}}, \ldots, F_{i_1}, \tilde{T}_{\phi} \left( F_{i_1} \right) \right]_{q_{i_1}^2 a_{i_1}^2 - a_{i_1}^{-2}} K_{\theta}^{-1}.
\]

Note that, for $a_{ij} = -1$ (see [20]),

\[
T_i \left( \tilde{T}_{\phi} \left( F_{i} \right) \right) = -\tilde{T}_{\phi} \left( F_{i} \right) F_i + q_i F_i \tilde{T}_{\phi} \left( F_{i} \right)
\]

\[
= q_i \left[ F_i, \tilde{T}_{\phi} \left( F_{i} \right) \right]_{q_i^2}.
\]

So the above bracket can be written as $T_{i_{h-1}} T_{i_{h-2}} \cdots T_{i_2} \left( \tilde{T}_{\phi} \left( F_{i_1} \right) \right) K_{\theta}^{-1}$, when $i_1$ is not equal to $i_k$ for all $k = 2, \ldots, h - 1$.

Thus we need to consider the case $a_{i_1 i_2} = -2$:

\[
= \left[ F_{i_1} \tilde{T}_{\phi} \left( F_{i_2} \right) \right]_{q_{i_1}^2}.
\]

where we have used $\tilde{T}_{\phi} \left( [F_{i_2}, F_{i_1}]_{q_i} \right) = \tilde{T}_{\phi} \left( [F_{i_1}, F_{i_2}]_{q_i} \right)$ (see [4]).

Therefore the above $q$-bracket also can be written as

\[
\psi \phi(E_0) = A T_i T_{i_1} \cdots T_{i_2} \left( \tilde{T}_{\phi} \left( F_{i_1} \right) \right) K_{\theta}^{-1}.
\]

\[
\text{where } i_1, \ldots, i_k \text{ are in the set } \{i_1, \ldots, i_{h-1}\} \text{ such that every two elements are different and } A \text{ is a polynomial of } q.
\]

In fact, we have

\[
\delta - \alpha_{i_1} - \cdots - \alpha_{i_{h-1}} = \delta - \theta.
\]

Recall that we have defined $E_{\delta, \theta}$ as the quantum root vector

\[
T_i \tilde{T}_{\phi} \left( F_{i_1} \right) \cdots T_i \tilde{T}_{\phi} \left( F_{i_1} \right) K_{\delta, \theta}
\]

independently from the sequence (by Remark 6). Thus $\psi \phi(E_i) = A E_{\alpha_i}$ for $\alpha_i = \delta - \theta$, where $A$ is a polynomial of $q$. Then we can adjust the map $\phi$ such that the action of $\psi \phi$ on $E_0$ is identity.

Similarly we can show that $\psi \phi(F_0) = F_0$. Note that by definition $\psi \phi$ and $\psi \phi$ fix all $E_i$ and $F_i$ for $i \neq 0$, and also the homomorphisms $\phi$ and $\psi$ are surjective by construction; therefore $\psi \phi = \phi \psi = I$. So we have shown the following.

**Corollary 33.** The homomorphisms $\phi : \mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\mathfrak{g})$ and $\psi : \mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\mathfrak{g})$ are two algebra isomorphisms. In particular $\phi = \psi^{-1}$.

### Competing Interests

The authors declare that they have no competing interests.

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