GLOBAL ATTRACTOR FOR A ONE DIMENSIONAL WEAKLY DAMPED HALF-WAVE EQUATION

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Abstract. We discuss the asymptotic behavior of the solutions for the fractional nonlinear Schrödinger equation that reads

$$u_t - iDu + ig(|u|^2)u + \gamma u = f.$$ (1)

We prove that this behavior is characterized by the existence of a compact global attractor in the appropriate energy space.

1. Introduction. This article is devoted to study the asymptotic dynamics for an infinite dimensional dynamical system generated by a nonlinear dispersive fractional Schrödinger type equation that reads

$$u_t - iDu + ig(|u|^2)u + \gamma u = f,$$ (1)

where $D$ denotes for the sake of simplicity $\sqrt{-\frac{\partial^2}{\partial x^2}}$. The unknown $u = u(t, x)$ maps $\mathbb{R}_+ \times \mathbb{R}$ into $\mathbb{C}$. The equation (1) is supplemented with initial data at $t = 0$

$$u(0) = u_0,$$ (2)

belonging to the fractional Sobolev space $H^\frac{1}{2}$ that will be specified in the sequel. $f \in L^2(\mathbb{R})$ is a given source term that is independent of time and the nonlinearity $g$ is a $C^\infty$ mapping from $\mathbb{R}_+$ into $\mathbb{R}$. For convenience use, we consider subcritical smooth nonlinearity, i.e. $g$ that satisfies the following growth condition: for $\xi \geq 0$

$$\xi |g'(\xi)| + |g(\xi)| \leq c_1 \xi^\sigma,$$ (3)

for a given $\sigma \in (0, 1)$.

Let us mention that evolution problems with nonlocal dispersion occur throughout physical and natural systems. As a prototypical example, the class of fractional Schrödinger type equations has been widely used in many branches of applied sciences such as nonlinear optics, deep water wave dynamics, plasma physics, quantum mechanics, wave turbulence and dynamics of boson stars (see for instance [8], [18], [28], [29], [31], [45] and references therein).

Dispersive wave equations provide excellent examples of infinite dimensional dynamical systems which are either conservative or exhibit some dissipation. In the last case, one can hope to reduce the study of the flow to a bounded (or even

2020 Mathematics Subject Classification. Primary: 35B40, 35Q55; Secondary: 76B03, 37L30.

Key words and phrases. Schrödinger Equation, Half-Wave Equation, Global Attractor.
compact) attracting set or global attractor that contains much of the relevant information about the flow. We refer the reader to R. Temam [41], J. Robinson [38], I. Chueshov [14] and G. Raugel [37] for general frameworks of this theory.

To begin with, we consider the conservative case (i.e. $\gamma = f = 0$). In the literature, the initial value problem for the following fractional NLS

$$u_t - i(-\Delta)^{\frac{\alpha}{2}}u + ig(|u|^2)u = 0, \quad x \in \mathbb{R}^n \text{ and } \alpha \neq 1,$$

was recently addressed by several authors. Amongst the various research works on this issue, we shall mention that the Cauchy problem for (4) was considered for initial data belonging to $L^2(\mathbb{R}^n)$ by B. Guo and Z. Huo [24] for $\alpha \in (0, 2)$ in which the special case $\alpha = 1$ was excluded. A more general result concerning the initial value problem for (4) was given by Y. Hong and Y. Sire in [25] where, always considering $\alpha \neq 1$, the authors established local well-posedness and ill-posedness of (4) in Sobolev spaces for power-type nonlinearities. These results were extended for $\alpha \geq 2$ by V. Dinh in [16]. For blow-up solutions and profile decomposition we refer the reader to [17] and the references therein.

For the one dimensional case and for $\alpha \in (1, 2)$, the low regularity well-posedness for the evolution equation (4) with cubic nonlinearity was studied in [11] in which the authors prove the local well-posedness in $H^s$ for $s \geq \frac{2-\alpha}{4}$ by the use of a Bourgain spaces approach. Moreover, some ill-posedness results as well as a norm inflation in negative Sobolev spaces appear in [13].

It should be emphasized that in most cases, the topic of local well-posedness for the problem defined by (4) has been considered in the light of [12] in which the authors derive dispersive estimates that generalize time decay and Strichartz estimates through an improvement result relating to the issue of oscillatory integrals.

The interesting special case when $\alpha = 1$ for equation (4), as in our case being considered, with cubic nonlinearity is known as the “nonlinear cubic half-wave equation”

$$u_t - iDu + i|u|^2u = 0.$$  

This equation, that can be reformulated as a system using the Szegö projector, is often considered as the basic model and at the heart of the derivation of asymptotic models of weak turbulence through the cubic Szegö equation studied on the circle $\mathbb{S}^1$ by P. Gérard and S. Grellier in [20, 22] where they have proved that (5) defines a global flow on $H^s(\mathbb{S}^1)$ for all $s \geq \frac{1}{2}$. Similar results were established in [21] for the case of the one dimensional torus T. Extended results was established in [36] and [27] as well as a critical threshold $M_c$ for which all $H^\frac{1}{2}(\mathbb{R})$-solutions $u$ with $\|u\|_{L^2} < M_c$ extend globally in time and they establish a minimal mass blowup criterion.

While in the dissipative case where, in some physical contexts, an external forcing term and some damping effects have been taken into account, the following fractional NLS that reads

$$u_t - iD^{\alpha}u + i|u|^2u + \gamma u = f, \quad x \in \mathbb{R} \quad \alpha \in (1, 2), \quad x \in \mathbb{R}$$

was only studied, to the best of our knowledge, by O. Goubet and E. Zahrouni in [23] where they have proved that (6) provides an infinite dimensional dynamical
system in $H^α$ that possesses a regular compact global attractor with finite fractal dimension in $H^β$ under suitable assumption on the forcing term $f$.

Now let us return to the matter at hand. Our main result of this paper is stated as follows

**Theorem 1.1 (Main Theorem).** Let $f \in L^2$. Then the equation (1) defines a dissipative dynamical system that possesses a compact global attractor $A$ in $H^{β/2}$.

Before giving the layout of this article, we recall briefly some definitions and notations.

The Hilbert space $L^2 = L^2(\mathbb{R})$ is equipped with the usual scalar product denoted by $(u,v) = \Re\int_\mathbb{R} u(x)\overline{v(x)} \, dx$. From now on, we use the notation $\varrho = \sqrt{1 + x^2}$, $x \in \mathbb{R}$ and we set

$$L^2_\varrho = \{ u \in L^2 \text{ such that } \int_\mathbb{R} \varrho^2 |u(x,y)|^2 \, dx < +\infty \}.$$

Our convention for the one dimensional space Fourier transform is

$$\mathcal{F}(u)(\xi) = \int_\mathbb{R} u(x)e^{-ix\xi} \, dx.$$

We start by fixing the fractional exponent $s \in (0,1)$. We recall (see [15] and [39]) that $D^s$ is considered as the homogeneous fractional pseudo-differential operator defined, for $u \in \mathcal{S}(\mathbb{R})$, by

$$D^s u(x) = p.v. \int_\mathbb{R} u(x) - u(y) |x-y|^{1+2s} \, dy = \mathcal{F}^{-1}(|\xi|^s \mathcal{F}(u)),$$

where $\mathcal{S}(\mathbb{R})$ denotes the Schwarz class and “p.v.” for principal value.

The fractional Sobolev space $H^s = H^s(\mathbb{R})$ defined as follows

$$H^s = \left\{ u \in L^2(\mathbb{R}) \text{ such that } \frac{|u(x) - u(y)|}{|x-y|^{1+s}} \in L^2(\mathbb{R}^2) \right\}$$

$$= \left\{ u \in L^2(\mathbb{R}) \text{ such that } \int_\mathbb{R} (1 + |\xi|^{2s}) |\mathcal{F}(u)(\xi)|^2 \, d\xi < +\infty \right\}$$

as an intermediary Banach space between $L^2$ and $H^1(\mathbb{R})$ endowed with the norm

$$||u||_{H^s} = \left( ||u||_{L^2}^2 + \int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x-y|^{2+2s}} \, dx \, dy \right)^{1/2}$$

which is equivalent, via the Fourier transform approach, to

$$||u||_{H^s} = \left( ||u||_{L^2}^2 + ||D^s u||_{L^2}^2 \right)^{1/2}$$

is a Hilbert space with the associate scalar product denoted by

$$(u,v) = (u,v) + (D^s u, D^s v), \quad u,v \in H^s.$$

The weighted fractional Sobolev space $H^s_\varrho$ is the Hilbert space

$$H^s_\varrho = \{ u \in L^2_\varrho \text{ such that } D^s u \in L^2_\varrho \}$$

endowed with the norm $||u||_{H^s_\varrho} = \left( ||u||_{L^2_\varrho}^2 + ||D^s u||_{L^2_\varrho}^2 \right)^{1/2}$. 
Finally, for any positive $A$ and $B$, $A \lesssim B$ means that that exists $c > 0$ such that $A \leq cB$ and we recall that throughout this article the constants $C$s are numerical constants that vary from one line to another.

The plan of the present paper is as follows. The objective of Section 2 is to provide tools enabling us to prove the well-posedness of the Cauchy problem for (1) that would be considered in Section 3. The existence of a global attractor $\mathcal{A}$ in $H^{\frac{1}{2}}$ for the associated semigroup $(S(t))_{t \in \mathbb{R}^+}$ will be the subject of Section 4. Finally, the last section will be dedicated to discuss some relevant problems related to the current issue.

2. Some preliminary results. To begin with, we recall a Gagliardo-Nirenberg type inequality (see [19] for instance)

**Lemma 2.1.** Let $p \in [2, +\infty)$. Then there exists $C = C(p) > 0$ such that

$$||u||_{L^p(\mathbb{R})} \leq C||u||_{L^2}^{\frac{1}{2}}||D^2 u||_{L^2}^{(1-\frac{2}{p})}, \quad \forall u \in H^{\frac{1}{2}}.$$  

For later use, we recall the following fractional Moser-Trudinger type inequality (see [40] and [30])

**Lemma 2.2.** There exists $C > 0$ such that for all $u \in H^{\frac{1}{2}}$ satisfying $||u||_{H^{\frac{1}{2}}} \leq 1$ the following estimate holds

$$\int_{\mathbb{R}} (e^{\pi|u|^2} - 1) \, dx \leq C.$$  

Moreover, the constant $\pi$ is sharp.

As a consequence of the previous Lemma, we state the following result (see H. Lieb and M. Loss [33], Theorem 8.5)

**Lemma 2.3.** There exists $C > 0$ such that for all $p \in [2, +\infty),$  

$$||u||_{L^p} \leq C \sqrt{p} ||u||_{H^{\frac{1}{2}}}, \quad \forall u \in H^{\frac{1}{2}}.$$  

**Proof of Lemma 2.3.** Observe that in accordance with Lemma 2.2 we have for a given $v \in H^{\frac{1}{2}}$ satisfying $||v||_{H^{\frac{1}{2}}} \leq 1$ we have

$$\frac{\pi^n}{n!} \int_{\mathbb{R}} |v|^{2n} \, dx \leq \int_{\mathbb{R}} (e^{\pi|v|^2} - 1) \, dx \leq C, \quad \forall n \geq 1$$  

where, without loose of generality, one may assume that $C > 1.$

Hence, for $v = \frac{u}{||u||_{H^{\frac{1}{2}}}}$ it follows that

$$||u||_{L^{2n}} \leq C^{\frac{2n}{\sqrt{n}}} (n!)^{\frac{1}{2n}} ||u||_{H^{\frac{1}{2}}} \leq \sqrt{\frac{C}{\pi}} \sqrt{n} ||u||_{H^{\frac{1}{2}}}, \quad \forall n \geq 1$$  

where the elementary inequality $(n!)^{\frac{1}{2n}} \leq n$ was used. Now for a fixed $p \in (n, n+1),$$ thanks to the concavity of the ln function, we obtain by interpolation that

$$||u||_{L^p} \leq \frac{\sqrt{C}}{\sqrt{n}^{1-\theta}} \sqrt{n+1}^\theta ||u||_{H^{\frac{1}{2}}}, \quad \theta \in (0, 1)$$  

$$\leq \frac{\sqrt{C}}{\sqrt{n}^{1-\theta}} ||u||_{H^{\frac{1}{2}}}$$

and the proof is completed. \qed
In the following we state a special case of the Brezis-Gallouet-Wainger inequality (see H. Brezis and S. Wainger [7], Theorem 1, and T. Ozawa [35], Theorem 2, for general cases) which is the analogous of the well Known Brezis-Gallouet inequality (see H. Brezis and T. Gallouet [6], Lemma 2).

**Lemma 2.4.** Let \( s > \frac{1}{2} \). Then there exists \( C = C(s) > 0 \) such that

\[
||u||_{L^\infty} \leq C \left( ||u||_{H^s} \right)^{\frac{1}{2}} \ln \left( 2 + \frac{||u||_{H^s}}{||u||_{H^s}} \right)^{\frac{1}{2}}, \quad \forall \ u \in ||u||_{H^s}.
\]

**Proof of Lemma 2.4.** The proof is standard and for the reader’s convenance we sketch it. Let \( R > 0 \), then using the Cauchy-Schwarz inequality and the Plancherel’s theorem it follows that

\[
||u||_{L^\infty} \leq \int_R \left| \mathcal{F}(u)(\xi) \right| d\xi \\
\leq \int_{-R}^R (1 + |\xi|)^{\frac{1}{2}} |\mathcal{F}(u)(\xi)| \frac{d\xi}{1 + |\xi|^s} + \int_{|\xi| > R} (1 + |\xi|^s) |\mathcal{F}(u)(\xi)| \frac{d\xi}{1 + |\xi|^s} \\
\lesssim ||u||_{H^s} \ln(1 + R)^{\frac{1}{2}} + \frac{||u||_{H^s}}{R^{s-\frac{1}{2}}}.
\]

Minimizing this bound with respect to \( R > 0 \) achieves the proof. \( \Box \)

We give now a commutator estimate that states as follows

**Lemma 2.5.** Let \( u \in \mathcal{S}(\mathbb{R}) \) and \( v \in H^1 \). Then there exists \( C > 0 \) such that

\[
||D(uv) - uDv||_{L^2} \leq C ||v||_{L^2} ||\xi \mathcal{F}(u)||_{L^1}.
\]

**Proof of Lemma 2.5.** Thanks to the Fourier transform, one has

\[
|\mathcal{F}(D(uv))(\xi) - \mathcal{F}(uDv)(\xi)| \leq \int_{\mathbb{R}} |\xi - \eta| |\mathcal{F}(u)(\eta)||\mathcal{F}(v)(\xi - \eta)| d\eta \\
\lesssim \int_{\mathbb{R}} |\eta| |\mathcal{F}(u)(\eta)||\mathcal{F}(v)(\xi - \eta)| d\eta.
\]

This leads to the desired estimate thanks to the Plancherel Theorem. \( \Box \)

As a consequence of the previous lemma we have the following helpful estimate

**Lemma 2.6.** There exists \( C > 0 \) such that for any function \( u \in L^2, \)

\[
||D(xu) - xDu||_{L^2} \leq C ||u||_{L^2}.
\]

**Proof of Lemma 2.6.** we proceed as in [9]. Let \( \theta \) be a smooth cut-off function belonging to \( \mathcal{S}(\mathbb{R}) \) defined as follows

\[
\theta : \mathbb{R} \rightarrow [0,1] \quad \theta(x) = \begin{cases} 
1 & \text{if } |x| \leq 1 \\
0 & \text{if } |x| \geq 2
\end{cases}.
\]
We approximate \( x \) by the sequence \( \left( \phi_N(x) = x \theta \left( \frac{x}{N} \right) \right) \) and then
\[
|\xi F(\phi_N)(\xi)| = \left| \int_{\mathbb{R}} \xi x \theta \left( \frac{x}{N} \right) e^{-2i\pi \xi x} dx \right|
\leq \xi \left| \frac{d}{d\xi} \int_{\mathbb{R}} \theta \left( \frac{x}{N} \right) e^{-2i\pi \xi x} dx \right|
\leq N^2 |\xi| \left| \frac{dF(\theta)}{d\xi}(N\xi) \right|.
\]
Hence, \( ||\xi F(\phi_N)||_{L^1} \leq C \). Applying Lemma 2.5 then letting \( N \to +\infty \) achieves the proof thanks to the Fatou Lemma.

3. The initial value problem.

**Theorem 3.1.** Let \( u_0 \in H^{\frac{1}{2}} \). Then under assumption (3), the problem (1) – (2) has a unique solution
\[
\begin{align*}
  u & \in \mathcal{C}_b([0, +\infty), H^{\frac{1}{2}}) \cap \mathcal{C}^1([0, +\infty), H^{-\frac{1}{2}})
\end{align*}
\]
and the maps \( S(t) : u_0 \mapsto u(t) \) are continuous on \( H^{\frac{1}{2}} \) with \( \mathcal{C}_b([0, +\infty), H^{\frac{1}{2}}) \) denotes the space of continuous bounded functions which take values in \( H^{\frac{1}{2}} \). Moreover, if \( u_0 \in H^1 \), then
\[
\begin{align*}
  u & \in \mathcal{C}([0, +\infty), H^1) \cap \mathcal{C}^1([0, +\infty), L^2).
\end{align*}
\]

For the sake of completeness and for reader convenience we sketch the proof and then details will be omitted.

To begin with, we first prove the statement of the Theorem 3.1 when \( u_0 \in H^1 \) then we proceed, by a limiting argument, for the case where \( u_0 \in H^{\frac{1}{2}} \).

**Proposition 3.2.** Let \( u_0 \in H^1 \). Then under assumption (3), the problem (1) – (2) has a unique solution
\[
\begin{align*}
  u & \in \mathcal{C}([0, +\infty), H^1) \cap \mathcal{C}^1([0, +\infty), L^2)
\end{align*}
\]
and the maps \( S(t) : u_0 \mapsto u(t) \) are continuous on \( H^1 \)

**Proof of Proposition 3.2.** We proceed into three steps.

**Step 1:** A local in time solution.

We consider the Duhamel form of (1) that reads
\[
\begin{align*}
  u(t) = e^{-\gamma t} e^{itD} u_0 + e^{-\gamma t} \int_0^t e^{i(t-s)D} \left[ f - ig(|u(s)|^2) u(s) \right] ds.
\end{align*}
\]
Since \( H^1 \) is an algebra, the nonlinearity \( g(|u|^2)u \) is Lipschitz on bounded subsets of \( H^1 \). Hence, local existence and uniqueness of a solution \( u \in \mathcal{C}([0, T^*), H^1) \) follows easily by applying a fixed point argument. Moreover,
\[
\begin{align*}
  \text{either } T^* = +\infty \text{ or } ||u(t)||_{H^1} \to +\infty \text{ as } t \to T^*, t < T^*.
\end{align*}
\]
Also, continuous dependence of \( u(t) \) with respect to the initial datum \( u_0 \) in \( H^1 \) is deduced from the fact that \( u \mapsto g(|u|^2)u \) is locally Lipschitz on \( H^1 \) by standard arguments.
Step 2: A global in time solution in $H^{\frac{1}{2}}$.

Lemma 3.3. Under assumption (3), there exists $K > 0$ that only depends on $||u_0||_{H^{\frac{1}{2}}}, ||f||_{L^2}$ and $\gamma$ such that the solution $u$ satisfies

$$\sup_{t \in \mathbb{R}_+} \left( ||u(t)||_{L^2} + ||D^{\frac{1}{2}}u||_{L^2} \right) \leq K.$$ 

Proof of Lemma 3.3. Observe that the scalar product of (1) by $u$ leads to

$$\frac{1}{2} \frac{d}{dt} ||u||_{L^2}^2 + \gamma ||u||_{L^2}^2 = (f, u).$$ 

Thus, thanks to the Gronwall Lemma

$$\sup_{t \in \mathbb{R}_+} ||u(t)||_{L^2} \leq C_0$$ 

where $C_0 > 0$ depends only on $||u_0||_{L^2}, ||f||_{L^2}$ and $\gamma$.

Now, applying the scalar product of (1) by $-i(u + \gamma u)$ it follows that

$$\frac{1}{2} \frac{d}{dt} J(u) + \gamma J(u) = K(u)$$ 

where we set

$$J(u) = ||D^{\frac{1}{2}}u||_{L^2}^2 - \int_{\mathbb{R}} G(|u|^2) \, dx - 2(if, u)$$ 

$$K(u) = \gamma \left( (g(|u|^2), |u|^2) - (if, u) - \int_{\mathbb{R}} G(|u|^2) \, dx \right)$$

with $G(s) = \int_0^s g(r) \, dr$. Thanks to Lemma 2.1, estimate (8) and under assumption (3) it follows that

$$\left| \left( (g(|u|^2), |u|^2) \right) \right| + \left| \int_{\mathbb{R}} G(|u|^2) \, dx \right| \lesssim ||u||_{L^{2\sigma + 2}}^{2\sigma + 2} \lesssim ||D^{\frac{1}{2}}u||_{L^2}^{2\sigma},$$

Hence, in accordance with (10), (11) and Young’s inequality, the previous estimate leads to

$$\frac{1}{2} ||D^{\frac{1}{2}}u||_{L^2}^2 + C_1 \leq J(u) \leq \frac{3}{2} ||D^{\frac{1}{2}}u||_{L^2}^2 + C_1$$

and

$$K(u) \leq \frac{\gamma}{2} ||D^{\frac{1}{2}}u||_{L^2}^2 + C_1.$$ 

Thus, from (9) we easily obtain that

$$\frac{d}{dt} J(u) + \gamma J(u) \leq C_1$$ 

and the proof is achieved thanks to Gronwall’s lemma.

Step 3: A global in time solution in $H^1$.

Lemma 3.4. Let $u_0 \in H^1$. Then there exist $c_1, c_2 > 0$ depending only on $||f||_{L^2}, \gamma$ and $\sigma$ such that the following estimate holds

$$||u||_{H^1} \leq c_1 (1 + ||u_0||_{H^1} + ||f||_{L^2})^a e^{c_2 t^{\frac{1}{1-\sigma}}} - \gamma t$$

where $a > 0$ that depends only on $\sigma$. 

Proof of Lemma 3.4. Following the guidelines in [6] and recalling the Duhamel formulation associated to (1) that is
\[ u(t) = e^{-\gamma t} e^{itD} u_0 + e^{-\gamma t} \int_0^t e^{\gamma s} e^{i(t-s)D} \left[ f - ig(|u(s)|^2)u(s) \right] ds, \]
we deduce that under assumption (3) and using Lemma 2.4 one has
\[ \|e^{\gamma t} u(t)\|_{H^1} \lesssim \|u_0\|_{H^1} + e^{\gamma t} \|f\|_{L^2} \]
\[ + \int_0^t e^{\gamma s} \|u(s)\|_{H^1} \|u(s)\|_{H^1} \left[ \ln \left( 2 + \|u(s)\|_{H^1} \right)^2 \right] \frac{\|u(s)\|_{H^1}}{\|u(s)\|_{H^1}} ds. \] (12)

Since \( x \mapsto x^{2\sigma} \ln(2 + \frac{\sigma}{x}) \leq C^{2\sigma} \ln(2 + \frac{\sigma}{x}), \forall \ 0 \leq x \leq C \) and \( \alpha \geq 0 \), we deduce from Lemma 3.3 and (12) that
\[ \frac{\|e^{\gamma t} u(t)\|_{H^1}}{K} \lesssim \frac{1}{K} \left( \|u_0\|_{H^1} + e^{\gamma t} \|f\|_{L^2} \right) \]
\[ + \int_0^t \|e^{\gamma s} u(s)\|_{H^1} \left[ \ln \left( 2 + \|e^{\gamma s} u(s)\|_{H^1} \right)^2 \frac{\|e^{\gamma s} u(s)\|_{H^1}}{\|e^{\gamma s} u(s)\|_{H^1}} \right] ds. \] (13)

Introducing
\[ w(t) = \frac{\|u_0\|_{H^1} + e^{\gamma t} \|f\|_{L^2}}{K} + \int_0^t \psi(s) \ln(2 + \psi(s)) ds \quad \text{with} \quad \psi(s) = \frac{\|e^{\gamma s} u(s)\|_{H^1}}{K}, \]
then we have
\[ w'(t) = \gamma \frac{e^{\gamma t} \|f\|_{L^2}}{K} + \psi(t) \ln(2 + \psi(t)) \lesssim (2 + w(t)) \ln(2 + w(t)). \]
Thus, thanks to Gronwall’s Lemma one has
\[ \ln(2 + w(t)) \lesssim 1 + \ln(2 + w(0)) + t^{\frac{\sigma}{2}} \]
which, in accordance with (13), completes the proof of the lemma. \( \square \)

Hence, thanks to Lemma 3.4 the solution \( u \) remains global in time with values in \( H^1 \) and the proof of the Proposition 3.2 is achieved. \( \square \)

Now we turn to prove the first part of our main result

**Proposition 3.5.** Let \( u_0 \in H^1 \). then under assumption (3), the problem (1) – (2) has a unique solution
\[ u \in \mathcal{C}_b([0, +\infty), H^1) \cap \mathcal{C}_b^1([0, +\infty), H^{-\frac{1}{2}}). \]
Moreover, the maps \( S(t) : u_0 \mapsto u(t) \) are continuous on \( H^1 \)

**Proof of Proposition 3.5.** We proceed in two steps.

**Step 1:** existence of a solution in \( \mathcal{C}_b([0, +\infty), H^\frac{1}{2}). \)
Let \( u_0 \in H^\frac{1}{2} \) and \( (\phi_n)_n \) be a sequence in \( H^1 \) such that
\[ \phi_n \rightarrow u_0 \quad \text{strongly in} \quad H^\frac{1}{2} \quad \text{as} \quad n \rightarrow +\infty. \]
Thanks to Proposition 3.2, the problem (1) supplemented with initial condition \( \phi_n \) admits a unique solution
\[ u_n \in \mathcal{C}([0, +\infty), H^1) \cap \mathcal{C}_b([0, +\infty), H^\frac{1}{2}). \]
Hence, thanks to Lemma 3.3, there exists \( u \in H^{\frac{1}{2}} \) such that
\[
\begin{cases}
  u_n \rightharpoonup u & \text{weakly in } H^{\frac{1}{2}} \\
  \partial_t u_n \rightharpoonup \partial_t u & \text{weakly in } H^{-\frac{1}{2}}
\end{cases}
\]  
(14)

By applying Lemma 1.2, Ch.III in [42], it might be deduced from (14) that
\[ u \in \mathcal{C}'([0, T], L^2), \forall T > 0. \]

This ensure, by a limiting argument, that \( u \) satisfies
\[
\partial_t u - iDu + \gamma u + i\Xi = f,
\]
(15)
where \( \Xi \) denotes the weak limit of \( (g(|u_n|^2)u_n) \).

**Lemma 3.6.**
\[
\Im \int_{R}^{R} \Xi \overline{u} \, dx = 0 \ a.e \ in \ t.
\]

**Proof of Lemma 3.6.** We proceed as in [10]. Let \( p' \in \left( \max \left( 1, \frac{2}{1 + 2\sigma} \right), 2 \right) \).

Then, according to Lemma 2.1 and the Hölder inequality, we obtain for all \( v, w \in H^{\frac{1}{2}} \) such that \( ||v||_{H^{\frac{1}{2}}} \leq M \) and \( ||w||_{H^{\frac{1}{2}}} \leq M \),
\[
||g(|v|^2)v - g(|w|^2)w||_{L^{p'}} \lesssim C(M) ||u - v||_{L^2}.
\]
(16)

In accordance with Lemma 3.3, \( (g(|u_n|^2)u_n) \) is uniformly bounded in \( L^{p'} \) and then
\[ g(|u_n|^2)u_n \rightharpoonup \Xi \ \text{weakly in} \ L^{p'} \ \text{as} \ n \to +\infty. \]

Moreover, up to a subsequence extraction, we may assume that \( u_n \to u \) strongly in \( L^p([-R, R]) \) for every \( R > 0 \)
and then, thanks again to Lemma 2.1, we deduce that
\[
< g(|u_n|^2)u_n, i((u_n - u)) >_{L^{p'}([-R, R]), L^p([-R, R])} \to 0 \text{ as } n \to +\infty.
\]

Accordingly, we obtain that
\[
\Im \int_{-R}^{R} \Xi \overline{u} \, dx = < \Xi, iu >_{L^{p'}([-R, R]), L^p([-R, R])} = 0, \forall R > 0
\]
and the proof is achieved. \( \square \)

Thanks to Lemma 3.6 and (7) one easily has
\[ u_n \to u \text{ strongly in } L^p, \ p \in [2, +\infty) \]
which, in accordance to the energy equation (9), prove that
\[ u_n \rightharpoonup u \text{ strongly in } H^{\frac{1}{2}}. \]

Similar argument, shows that this solution, initially belonging to the space of weakly-continues functions (see [42] Lemma 1.4, Ch.III) \( \mathcal{C}_w([0, T], H^{\frac{1}{2}}) \), is in fact in \( \mathcal{C}'([0, T], L^p) \) for all \( p \in [2, +\infty) \).

This with the energy equation (9) lead us to obtain a solution in \( \mathcal{C}'([0, +\infty), H^{\frac{1}{2}}) \) thanks to a priori estimates established in Lemma 3.3.
Step 2: Uniqueness of the solution.

Using an argument due to M. Vladimirov (see [43]), we consider \( u \) and \( v \) two solutions of the problem 1 issued from the initial data \( \phi \in H^\frac{1}{2} \). Denoting \( w = u - v \) we easily obtain, under assumption (3), that

\[
\frac{1}{2} \frac{d}{dt} \| w(t) \|_{L^2}^2 + \gamma \| w(t) \|_{L^2}^2 \leq C_0 \int \mathcal{H}^{2\sigma}(x) |w(x)|^2 \, dx
\]

where \( \mathcal{H} = |u| + |v| \).

Thanks to the Hölder inequality, it leads that for \( p \in \left[ \max\left( \frac{1}{4\sigma}, 2 \right), +\infty \right) \)

\[
\int \mathcal{H}^{2\sigma}(x) |w(x)|^2 \, dx \leq \| \mathcal{H} \|_{L^{2\sigma \gamma}} \| w \|_{L^\frac{2}{\sigma}} \leq C_0 C_1 \frac{1}{p} \| \mathcal{H} \|_{L^{2\sigma \gamma}} \| w \|_{L^\frac{2}{\sigma}}
\]

which, in accordance with (17), Lemma 2.1 and Lemma 2.3, leads to

\[
\frac{1}{2} \frac{d}{dt} \| w(t) \|_{L^2}^2 + \gamma \| w(t) \|_{L^2}^2 \leq C_0 C_1 \frac{1}{p} \| |w(t)|^{2(1 - \frac{1}{4\sigma})} \|_{L^2}
\]

since \( u \) and \( v \) are uniformly bounded in \( H^\frac{1}{2} \). Thus, it may be concluded that

\[
\frac{d}{dt} \left( \| |w(t)|^{\frac{1}{2}} \right) \leq C_0 C_1 \frac{1}{p}
\]

Integrating the previous inequality on \([0, T^*]\) such that \( C_0 T^* < 1 \) then letting \( p \to +\infty \) leads to deduce that \( w(t) = 0 \) for a.e in \([0, T^*]\). Repeating, this argument on \([mT^*, (m + 1)T^*]\) achieves the proof of the second step as well as the proof of the current proposition.

4. A dissipative dynamical system in \( H^\frac{1}{2} \) and construction of the global attractor.

The semigroup \( (S(t))_{t \in \mathbb{R}^+} \) associated to (1) is well defined. At the beginning we highlight its dissipation.

**Proposition 4.1.** The semigroup \( (S(t))_{t \in \mathbb{R}^+} \) possesses a bounded absorbing ball \( \mathcal{B}_{H^{1/2}} \) in \( H^{1/2} \).

i.e: for any bounded subset \( B \subseteq H^{1/2} \) there exists \( t(B) > 0 \) such that

\[ S(t)B \subseteq \mathcal{B}_{H^{1/2}}, \quad \forall t \geq t(B). \]

**Proof of Proposition 4.1.** The existence of a bounded absorbing set follows from the second step of the proof of Proposition 3.2. Then we omit it for the sake of conciseness.

The existence of an absorbing set was the first step toward the existence of the global attractor. We claim now to prove the second part of the main result of this paper stated as follows

**Theorem 4.2.** The semigroup \( (S(t))_{t \in \mathbb{R}^+} \) associated to the dynamical system defined by (1) possesses a compact global attractor \( \mathcal{A} \) in \( H^{1/2} \).

**Proof of Theorem 4.2.** Let \( f_\alpha \) be a function that belongs to the Schwarz class \( \mathcal{S}(\mathbb{R}) \).

Motivated by [26], we split the solution issued from \( u_0 \) as follows:

\[ u(t) = S(t)u_0 = v(t) + w(t), \]
where \( v \) satisfies the following problem

\[
\begin{cases}
    v_t - iDv + \gamma v + ig(|u|^2)u = (f - f_\eta) \\
    v(0) = u_0
\end{cases}
\tag{19}
\]

and \( w \) solves

\[
\begin{cases}
    w_t - iDw + \gamma w = f_\eta \\
    w(0) = 0
\end{cases}
\tag{20}
\]

As a first step, we state

**Lemma 4.3.** The problem (20) has a unique solution \( w \) in \( H^1 \) that is uniformly bounded in \( H^1 \cap H^\frac{1}{2} \).

**Proof of Lemma 4.3.** Classical arguments ensure the existence of a unique solution for (20) in \( H^1 \). Moreover, in the one hand the scalar product of (20) by \( w \) then by \(-i(w_t + \gamma w)\) lead to

\[
\frac{1}{2} \frac{d}{dt} \left( ||w||^2_{H^\frac{1}{2}} - 2(i f_\eta, w) \right) + \gamma \left( ||w||^2_{H^\frac{1}{2}} - 2(i f_\eta, w) \right) = (f_\eta, w) - \gamma (i f_\eta, w). \tag{21}
\]

Denoting

\[
\Psi(w) = ||w||^2_{H^\frac{1}{2}} - 2(i f_\eta, w),
\]

we deduce the existence of a reel constant \( C > 0 \) that depends only on \( \gamma \) such that

\[
\frac{1}{2} ||w||^2_{H^\frac{1}{2}} - C ||f_\eta||^2_{L^2} \leq \Psi(w) \leq \frac{3}{2} ||w||^2_{H^\frac{1}{2}} + C ||f_\eta||^2_{L^2}
\]

Hence, in accordance with (21), \( w \) remains uniformly bounded in \( H^\frac{1}{2} \) thanks to the Gronwall Lemma.

In the other hand, differentiating (20) with respect to \( t \) and denoting \( z = w_t \) we have

\[
\begin{cases}
    z_t - iDz + \gamma z = 0 \\
    z(0) = f_\eta
\end{cases}
\tag{22}
\]

The scalar product of (22) by \( z \) leads to the following estimate that reads

\[
||Dw||_{L^2} \lesssim (1 + e^{-\gamma t}) ||f_\eta||_{L^2}
\]

and then, in accordance with (20), \( w \) remain uniformly bounded in \( H^1 \).

With respect to the boundedness of \( w \) in \( H^\frac{1}{2} \), the scalar product of (22) by \( x^2z \) leads to

\[
\frac{1}{2} \frac{d}{dt} ||xz||^2_{L^2} + \gamma ||xz||^2_{L^2} = (iDz, x^2z)
\]

Thanks to the Cauchy-Schwarz inequality,

\[
\frac{1}{2} \frac{d}{dt} ||xz||^2_{L^2} + \gamma ||xz||^2_{L^2} = (i[xDz - D(xz)], xz) \leq ||D(xz) - xDz||_{L^2} ||xz||_{L^2}
\]

Hence, by the use of Lemma 2.6, one obtain that

\[
\frac{1}{2} \frac{d}{dt} ||xz||^2_{L^2} + \gamma ||xz||^2_{L^2} \lesssim ||xz||_{L^2} ||z||_{L^2}. \tag{23}
\]

Similarly, one has

\[
\frac{1}{2} \frac{d}{dt} ||xw||^2_{L^2} + \gamma ||xw||^2_{L^2} \lesssim ||xw||_{L^2} (||f_\eta||_{L^2} + ||w||_{L^2}). \tag{24}
\]

Thus, gathering (23) and (24) achieves the proof thanks again to Gronwall’s Lemma and Young’s inequality.
Lemma 4.4. The Hilbert space \( H^1 \cap H^2_0 \) is compactly embedded in \( H^\frac{1}{2} \).

Proof of Lemma 4.4. Let \( B \) be a bounded subset in \( H^1 \cap H^2_0 \). Since \( B \) is bounded in \( L^2 \cap H^1 \), then, using Lemma 2.16 in [3], it follows that \( B \) is relatively compact in \( L^2 \). By interpolation, \( B \) remains relatively compact in \( H^2 \) and the proof is completed.

Now, let us introduce \( \varphi_s \): a solution of the following problem that reads
\[
-i D \varphi_s + \gamma \varphi_s = (f - f_\eta) - ig (|u|^2)u.
\] (25)

Since the right hand side of (25) belongs to \( L^2 \), the theory on existence and regularity of linear elliptic equations ensure the existence of a solution \( \varphi_s \in H^1 \). Moreover, one easily obtain, thanks to Cauchy-Schwarz inequality and assumption (3), that
\[
||\varphi_s||_{L^2}^2 \leq C \gamma \left( ||f - f_\eta||_{L^2}^2 + ||u||_{L^{4^\sigma+2}}^4 \right) \quad \text{ (26)}
\]
and then we have
\[
||D \varphi_s||_{L^2} \leq C \left( ||f - f_\eta||_{L^2} + ||u||_{L^{4^\sigma+1}}^{2\sigma+1} \right). \quad \text{ (27)}
\]
Hence, Lemma 2.1 and Proposition 4.1 ensure, in accordance with (26) and (27), the existence of a real constant \( C > 0 \) depending only on \( B \) such that \( ||\varphi_s||_{H^1} \leq C \).

Lemma 4.5. The solution \( v \) of the problem (19) satisfies the following identity
\[
||v - \varphi_s||_{H^\frac{1}{2}} = ||u_0 - \varphi_s||_{H^\frac{1}{2}} e^{-\gamma t}.
\]

Proof of Lemma 4.5. The proof is a mere consequence from (19), (25) and the Gronwall Lemma.

Now we shall continue the proof of the Theorem 4.2.

Let \( \mathcal{B} \) be a bounded subset of \( H^\frac{1}{2} \) and \( u_0 \in \mathcal{B} \). We use the splitting
\[
u(t) = v(t) + w(t) = S_1(t)u_0 + S_2(t)u_0
\]
where \( v \) and \( w \) are defined respectively by (19) and (20).

Thanks to Lemma 4.3 and Lemma 4.4, it may be concluded that there exists \( t_0(\mathcal{B}) > 0 \) such that
\[
\tilde{K} = \bigcup_{t \geq t_0} S_2(t)\mathcal{B} \quad \text{ is relatively compact in } H^\frac{1}{2}.
\]
Consider now \( \varphi_s \) a solution of (25) and \( K = \tilde{K} + \varphi_s \).

It is clear that the subset \( K \) is relatively compact in \( H^\frac{1}{2} \). Moreover, writing \( u(t) \) as follows
\[
u(t) = (v(t) - \varphi_s) + (w(t) + \varphi_s) = (S_1(t)u_0 - \varphi_s) + (S_2(t)u_0 + \varphi_s)
\]
then using Lemma 4.5 and the fact that \( S_2(t)u_0 + \varphi_s \in K \), we deduce that
\[
dist(S(t)\mathcal{B}, K) \to 0 \quad \text{ as } t \to +\infty.
\]
Hence, the semi-group \( (S(t))_t \) is asymptotically compact (see Remark 1.4 and Remark 1.5 in [41]) and the existence of a global attractor in \( H^\frac{1}{2} \) is established thanks to Theorem 1.1 in [41] and the proof is achieved.
5. Some relevant questions and related topics.

5.1. The one dimensional case. In the light of the above, it is concluded that Theorem 1.1 raises two key issues. Once the global attractor is obtained, the question arises if it has special regularity properties or if it has finite-dimensional character. The issue of regularity of the global attractor, particularly when the phase space is not an algebra, is often treated by the use of the well known Strichartz type estimates which are no longer applicable in our case (see [24] and the references therein). This reveals obviously a sensitive topic that seems very delicate to handle.

Another important subject is related to the issue of existence of a global attractor for the dynamical system defined by 1 in $H^1$ (Proposition 3.2). More specifically and for the sake of accuracy and completeness, the main question that arises is about the existence of a bounded absorbing set in $H^1$ since, in case of positive response, the global attractor in $H^1$ can be established as well as its finite fractal dimension under suitable assumption of the external forcing term.

5.2. A half-wave equation in two dimensions. In this subsection we consider the following problem

$$u_t + i\partial_x^2 u - iD_y u + ig(|u|^2)u + \gamma u = f,$$  \hspace{1cm} (28)

where $\Omega$ denotes either $\mathbb{R}^2$ or $\mathbb{R} \times [0,1]$.

For a fixed $s \in [\frac{1}{2}, 1]$, we consider the following anisotropic Sobolev space defined by

$$X_s(\Omega) = H_{x}^{1\frac{1}{2}} L_y^2(\Omega) \cap L_x^2 H_y^s(\Omega)$$

$$= \left\{ u \in L^2(\Omega) \text{ such that } \partial_x u \text{ and } D_y^s u \in L^2(\Omega) \right\},$$

where $\Omega$ denotes either $\mathbb{R}^2$ or $\mathbb{R} \times [0,1]$.

Equation (28), in the conservative case and with cubic nonlinearity, was considered by H. Xu in [44] and Y. Bahri et al. in [4]. In particular with regards to the well-posedness of the initial value problem, the question was resolved in [4] for initial data belonging to $X_s(\mathbb{R}^2)$ for $s > \frac{1}{2}$ as well as the existence of solitary waves.

Now consider the problem (28) in $\Omega = \mathbb{R} \times [0,1]$ supplemented with initial data $u_0 \in X_s$, $s > \frac{1}{2}$, and assuming suitable assumptions on the growth of the non-linearity $g$ one can prove, as in [4], the existence of a continuous semi-group of operators that maps $X_s$ into itself, through vector valued Strichartz type estimates as established in [1] and [2].

Next, we recall the following anisotropic Gagliardo-Nirenberg inequality (see [19])

**Lemma 5.1.** Let $s \in (1, 2)$ and $p \in [2, 6]$. Then there exists a reel constant $C > 0$ that depends on $s$ and $p$ such that for all $u \in X_s$,

$$||u||_{L^p} \leq C \left( ||u||_{L^2}^{\frac{p+2}{2p}} ||\partial_x^2 u||_{L^2}^{\frac{p-2}{2p}} ||D_y^s u||_{L^2}^{\frac{p-2}{2p}} \right).$$ \hspace{1cm} (29)

Moreover, in cases in which $s = 1$, the inequality (29) holds for $p \in [2, 6]$.
In accordance with Lemma 5.1 and assuming that the nonlinearity $g$ is subcritical, i.e. $g$ satisfies the growth condition
\[ g(|\xi|) + \xi g'(|\xi|) \lesssim \xi^\sigma, \quad \sigma \in \left(0, \frac{2}{3}\right), \]
one can construct a bounded absorbing set in $X_1$ and then the first question arises if the associate dynamical system possesses a compact global attractor in $X_1$ followed by the issue of its regularity.

However, in many situations concerning systems of physical relevance either uniqueness fails or it is not known to hold. This is the case when studying the problem (28) in $\Omega = \mathbb{R} \times [0,1]$ with initial data $u_0 \in X_1$ which seems delicate to handle. In fact, one can apply a limiting argument (as in Proposition 3.5) to construct a solution $u$ for (28) issued from $u_0 \in X_1$, but uniqueness does not hold and still an open question.

In such cases we cannot define a classical semigroup of operators, so that another theory involving multi-valued operators is necessary, namely, the theory of multi-valued dynamical systems (see for instance [5], [32], [34] and the references therein) and then one hopes to resolve the question about the existence of a compact global attractor in $X_1$.

Acknowledgments. The author would like to express his deep gratitude to the referees for their careful reading as well as for their helpful comments and suggestions leading to the improvement of this work.

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Received February 2020; revised April 2020.

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