Embedding of bases: from the $\mathcal{M}(2, 2\kappa + 1)$ to the $\mathcal{M}(3, 4\kappa + 2 - \delta)$ models

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ABSTRACT

A new quasi-particle basis of states is presented for all the irreducible modules of the $\mathcal{M}(3, p)$ models. It is formulated in terms of a combination of Virasoro modes and the modes of the field $\phi_{2, 1}$. This leads to a fermionic expression for particular combinations of irreducible $\mathcal{M}(3, p)$ characters, which turns out to be identical with the previously known formula. Quite remarkably, this new quasi-particle basis embodies a sort of embedding, at the level of bases, of the minimal models $\mathcal{M}(2, 2\kappa + 1)$ into the $\mathcal{M}(3, 4\kappa + 2 - \delta)$ ones, with $0 \leq \delta \leq 3$. 

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The $\mathcal{M}(3, p)$ models have been reformulated recently \cite{1} in terms of the extended algebra defined by the OPEs

\[
\phi(z)\phi(w) = \frac{1}{(z-w)^{2h}} \left[ I + (z-w)^2 \frac{2h}{c} T(w) + \cdots \right] \mathcal{S},
\]
\[
T(z)\phi(w) = \frac{h\phi(w)}{(z-w)^2} + \frac{\partial \phi(w)}{(z-w)} + \cdots
\]
\[
T(z)T(w) = \frac{c_{3, p}/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \cdots
\]

with
\[
\phi \equiv \phi_{2,1}, \quad h \equiv h_{2,1} = \frac{p-2}{4}, \quad c_{3, p} = 1 - 2\frac{(3-p)^2}{p},
\]
and $\mathcal{S} = (-1)^{p\mathcal{F}}$ where $\mathcal{F}$ counts the number of $\phi$ modes. The highest-weight states $|\sigma_\ell\rangle$ are completely characterized by an integer $\ell$ such that $0 \leq \ell \leq (p-2)/2$ and satisfy
\[
\phi_{-h-n+\frac{p}{2}} |\sigma_\ell\rangle = 0 \quad n > 0.
\]
The highest-weight modules are described by the successive action of the lowering $\phi$-modes subject to exclusion-type constraints. In the $N$-particle sector, with strings of lowering modes written in the form \cite{1} (see also \cite{3, 4} for $\ell = 0$):

\[
\phi_{-s_1} \phi_{-s_2} \cdots \phi_{-s_{N-1}} \phi_{-s_N} |\sigma_\ell\rangle,
\]
these constraints are
\[
s_i \geq s_{i+1} - 2h + 2, \quad s_i \geq s_{i+2} + 1, \quad s_{N-1} \geq -h + \frac{\ell}{2} + 1, \quad s_N \geq h - \frac{\ell}{2},
\]
with
\[
s_{N-2i} \in \mathbb{Z} + h + \frac{\ell}{2} \quad \text{and} \quad s_{N-2i-1} \in \mathbb{Z} - h + \frac{\ell}{2}.
\]

The complete module of $|\sigma_\ell\rangle$ is obtained by summing over all these states \cite{1} satisfying \cite{5} and all values of $N$. The enumeration of these state leads to the standard form of the fermionic character for the sum of the two Virasoro modules $|\phi_{1, \ell+1}\rangle$ and $|\phi_{1, p-\ell-1}\rangle$ of the $\mathcal{M}(3, p)$ models \cite{2, 3, 1} when $0 \leq \ell \leq [p/3]$ (the closed form expression of the generating functions has not been obtained for the remaining cases).

Here we display a new form of the basis of states of the $\mathcal{M}(3, p)$ models, still viewed form the point of view of the extended algebra \cite{1}. This basis is written in terms of combined sequences of Virasoro and $\phi$ modes, as

\[
L_{-n_1} \cdots L_{-n_N} \phi_{-m_1} \cdots \phi_{-m_M} |\sigma_\ell\rangle.
\]

The module over $|\sigma_\ell\rangle$ is again the direct sum of the two Virasoro modules $|\phi_{1, \ell+1}\rangle$ and $|\phi_{1, p-\ell-1}\rangle$. In order to specify the constraints on the mode indices, we first define two integers $\kappa$ and $\delta$ through the decomposition of $p$ as

\[
p = 4\kappa + 2 - \delta, \quad \text{where} \quad 0 \leq \delta \leq 3.
\]

The conditions take the form

\[
n_i \geq n_{i+1}, \quad n_i \geq n_{i+\kappa-1} + 2, \quad m_i \geq m_{i+1} + \frac{\delta}{2}, \quad m_i \geq m_{i+2} + \kappa.
\]

\footnote{In \cite{1}, the conditions are formulated in terms of the indices $n_i$ defined by:

\[
\phi_{-h+\frac{p}{4}} |\sigma_\ell\rangle = \phi_{-h+\frac{p}{4}} (N-1) \cdots \phi_{-h+\frac{p}{4}} n_1 \cdots \phi_{-h+\frac{p}{4}} n_{N-1} \phi_{-h+\frac{p}{4}} |\sigma_\ell\rangle.
\]

The relation between $s_i$ and $n_i$ is

\[
s_i = n_i + h - \frac{\ell}{2} \frac{(N-i)}{2}.
\]}

\[
\phi_{-h+\frac{p}{4}} |\sigma_\ell\rangle = \phi_{-h+\frac{p}{4}} (N-1) \cdots \phi_{-h+\frac{p}{4}} n_1 \cdots \phi_{-h+\frac{p}{4}} n_{N-1} \phi_{-h+\frac{p}{4}} |\sigma_\ell\rangle.
\]

The relation between $s_i$ and $n_i$ is

\[
s_i = n_i + h - \frac{\ell}{2} \frac{(N-i)}{2}.
\]
These are supplemented by the boundary conditions:

\[ n_{N-\ell} \geq 2, \quad n_N \geq M + 2 - \min(\ell, 1), \quad m_{M-1} \geq h - \frac{\ell}{2} + \max(0, \ell - \kappa) + \frac{\delta}{2}, \quad m_M \geq h - \frac{\ell}{2}, \]

(10)

The \( n_i \) are always integers but the range of the indices \( m_i \) is defined as follows. Given that \( h = -\delta/4 \) mod 1, we have

\[ m_{N-2i} \in \mathbb{Z} - \frac{\delta}{4} - \frac{\ell}{2} \quad \text{and} \quad m_{N-2i-1} \in \mathbb{Z} + \frac{\delta}{4} - \frac{\ell}{2}, \]

(11)

which are actually equivalent to the conditions on the \( s_i \) in (6).

The conditions (11) indicates that the Virasoro modes are ordered and further subject to a difference \( \kappa \) condition at distance \( \kappa - 1 \). The \( \phi \) modes are also ordered, being in fact all distinct if \( \delta \neq 0 \). In addition, they are subject to a difference \( \kappa \) condition at distance 2 (which is almost the ‘dual’ of the conditions on the \( n_i \)).

The different inequalities in the boundary conditions (10) have the following interpretation. At first, \( m_M \geq h - \frac{\ell}{2} \) is simply the highest-weight condition (5). The condition on \( m_{M-1} \) partially specifies the different descendant states according to the value of \( \ell \). It is analogous to the third condition in (5). The inequality \( n_{N-\ell} \geq 2 \) means that the maximal number of \( L_{-1} \) modes that can appear in the descendants of the \( |\sigma_{\ell}\rangle \) module is \( \ell \). Actually, this number is also bounded by the difference 2 condition at distance \( \kappa - 1 \), so that this maximal number is actually \( \min(\ell, \kappa - 1) \).

The most interesting condition is the remaining one in (10), which, for the vacuum module \( (\ell = 0) \) reads \( n_N \geq M + 2 \). For \( M = 0 \), this takes into account the Virasoro highest-weight condition on the vacuum. But if there are \( M \) \( \phi \)-modes already acting on the highest-weight state, the condition implies that all the modes \( L_{-n} \) with \( 2 \leq n \leq M + 1 \) have to be excluded. This can be interpreted as a sort of repulsion between the \( T \) and \( \phi \) ‘quasi-particles’. For any other module \( (\ell \neq 0) \), the bound on \( n_N \) reads \( n_N \geq M + 1 \).

If \( \kappa = 1 \), the difference condition on the Virasoro modes becomes \( n_i \geq n_i + 2 \), which is impossible. This means that when \( \kappa = 1 \), there can be no Virasoro modes; the basis is solely described by the \( \phi \) modes. Let us check that it reduces then to basis (4). When \( \kappa = 1, p = 6 - \delta \), but in order for \( p \) to be relatively prime with 3, we require \( \delta = 1 \) or 2. For \( \delta = 2 \), so that \( p = 4 \), the conditions (4) reduce to \( m_i \geq m_{i+1} + 1 \), in agreement with (4) (note that the condition \( m_i \geq m_{i+2} + 1 \) is thus automatically satisfied). In that case \( h = 1/2 \) and this indeed describes the free-fermionic basis of the Ising model. For \( p = 5 \), these conditions take the form \( m_i \geq m_{i+1} + 1/2 \), which again implies the condition at distance 2. This agrees with [4] and the known quasi-particle basis formulated in terms of the graded parafermion of dimension \( h = 3/4 \) (cf. [3], end of section 5, and [4] section 1.4).

To illustrate further these conditions, we present two examples in more detail. First we consider the \( \mathcal{M}(3,8) \) model, so that \( \kappa = \delta = 2 \), and \( h = 3/2 \). Let us focus on the Virasoro vacuum module which corresponds to \( \ell = 0 \) and which involves only those descendant states that contain an even number of \( \phi \) modes. The main (bulk) conditions are

\[ n_i \geq n_{i+1} + 2, \quad m_i \geq m_{i+1} + 1, \quad m_i \geq m_{i+2} + 2, \]

(12)

(the last condition being in fact irrelevant here), while the boundary conditions are simply \( n_N \geq 2 + M \) and \( m_M \geq 3/2 \). Let us denote the states in (7) by the combination of the two partitions.
Moreover, the boundary condition on \(n\) are precisely the one pertaining to the quasi-particle basis of the \(M\) modes. Therefore, in absence of \(\phi\) thereby distinguishing the different modules, is also the very one that occurs in these models. There-odd number of \(\phi\) thus appears that the above basis describes a sort of embedding of the \(M\) of states agrees with that coded in the Virasoro character \(\chi_{\ell, \kappa}(q)\) (all the characters being normalized such that \(\chi(0) = 1\)). Note that the \(M(3,8)\) model is equivalent to the superconformal minimal model \(SM(2,8)\). Within the latter context, the above basis mixes the \(G = \phi\) and \(L\) modes.

For our second example, consider the \(M(3,14)\) model and the module with \(\ell = 4\). Here \(\kappa = 3\), \(\delta = 0\) and \(h = 3\), so that \(\mathfrak{M}\) takes the form

\[
n_i \geq n_{i+2} + 2, \quad m_i \geq m_{i+1}, \quad m_i \geq m_{i+2} + 3,
\]

with the boundary conditions

\[
n_N \geq 1 + M, \quad n_{N-4} \geq 2, \quad m_M \geq 1, \quad m_{M-1} \geq 2.
\]

At the first few \((\leq 6)\) levels, those states that contain an even number of \(\phi\) modes, which pertains to the Virasoro module \(|\phi_{1,5}\rangle\), are

\[
1: \quad (1;)

2: \quad (2;)(1,1;)

3: \quad (3;)(2,1;)(2,1;)

4: \quad (4;)(3,1;)(2,2;)(3,1;)(2,2)

5: \quad (5;)(4,1;)(3,2;)(3,1,1;)(4,1;)(3,2)

6: \quad (6;)(5,1;)(4,2;)(3,3;)(4,1,1;)(3,2,1;)(5,1;)(4,2;)(3,3)(3,2,1)
\]

There are no terms containing \(L_{-1}^3\) because \(\langle \ell, \kappa - 1 \rangle = 2\). Similarly, the state \(\phi_{-1}\phi_{-1}|\sigma_4\rangle\) is excluded by the boundary condition on \(m_{M-1}\). The first state with four \(\phi\) modes is \((5,4,2,1,1)\) at level 12 and the first state with two copies of both types of modes is \((3,3,2,1)\) at level 9. The counting of states agrees with that coded in the Virasoro character \(\chi_{1,5}^{(3,14)}(q)\). If we also allow states with an odd number of \(\phi\) modes, we get instead the sum of Virasoro characters \(\chi_{1,5}^{(3,14)}(q) + q\chi_{1,9}^{(3,14)}(q)\). Note that \(M(3,14) \simeq W_3(3,7)\), so that the above is an example of a \(W_3\) basis involving both the \(T\) and \(W\) modes.

Let us stress a remarkable feature of the new basis. The conditions \([\mathfrak{M}]\) for the Virasoro modes are precisely the one pertaining to the quasi-particle basis of the \(M(2,p)\) models, with \(p = 2\kappa + 1\) \([\mathfrak{M}]\). Moreover, the boundary condition on \(m_{N-\ell}\), which specifies the maximal number of \(L_{-1}\) factors, thereby distinguishing the different modules, is also the very one that occurs in these models. Therefore, in absence of \(\phi\) modes, the above \(M(3,4\kappa + 2 - \delta)\) basis reduces to the \(M(2,2\kappa + 1)\) one. It thus appears that the above basis describes a sort of embedding of the \(M(2,2\kappa + 1)\) models within the \(M(3,4\kappa + 2 - \delta)\) ones.
Let us consider the expression for the characters associated to this new basis. Constructing these characters amounts to finding the generating function for the composition of the two partitions \((n_1, \ldots, n_N)\) and \((m_1, \ldots, m_M)\) satisfying (9) and (10). This is essentially built from the composition of two corresponding generating functions, both of which being known (up to a restriction on \(\ell\) to be specified).

The generating functions for partitions \((n_1, \ldots, n_N)\) is obtained as follows. First, delete \(M\) from each parts \(n_i\) and introduce \(q^{NM}\) to correct for this. The resulting restricted partitions are enumerated by the Andrews multiple-sum [9, 10]:

\[
\mathcal{H}_{\kappa, \ell}(q) z^N = \sum_{\sum s_i = N} q^{N^2 + \cdots + N_{i-1}^2 + N_{i+1} + \cdots N_{s-1} + NM} (q)_{s_1} \cdots (q)_{s_{s-1}}
\]

where \(N_i = s_i + \cdots + s_{k-1}\).

Similarly, the generating function for partitions \((m_1, \ldots, m_M)\) can be extracted from [5] up to simple modifications. The latter generating function enumerates the partitions \((\lambda_1, \ldots, \lambda_M)\) satisfying

\[
\lambda_i \geq \lambda_{i+1}, \quad \lambda_i \geq \lambda_{i+2} + 2r, \quad \lambda_M \geq 1, \quad \lambda_{M-1} \geq 1 + \max (0, \tilde{\ell} - 1),
\]

for \(0 \leq \tilde{\ell} \leq [(2r + 5)/3]\) where \([x]\) stands for the integer part of \(x\) (the boundary condition of \(\lambda_{M-1}\) induces a correcting term in the generating function that has been introduced in [11].) To connect the two problems, let us redefine \(m_i\) as:

\[
m_i = \lambda_i + h - \frac{\ell}{2} - 1 + (M - i) \frac{\delta}{2}.
\]

The conditions (9)-(10) become then

\[
\lambda_i \geq \lambda_{i+1}, \quad \lambda_i \geq \lambda_{i+2} + \kappa - \delta, \quad \lambda_M \geq 1, \quad \lambda_{M-1} \geq 1 + \max (0, \ell - \kappa).
\]

We thus recover the counting problem of [5, 11] but with \(2r \rightarrow \kappa - \delta\) and \(\tilde{\ell} - 1 \rightarrow \kappa - \ell\). (Note that the generating function of [5] does not hold for those cases where \(2r + 5\) is divisible by 3. But this is not restrictive since if \(\kappa - \delta + 5\) were a multiple of 3, say \(3n\), then \(p\) would be \(12n + 3\delta - 18\), which is divisible by 3 and that would not correspond to a minimal model.) The correcting factor \(q^{M(M-1)\frac{\kappa}{2} + M(h-\frac{4}{3}-1)}\) will keep track of the shifted staircase that must be added to adjust the weight when passing from the partitions \((\lambda_1, \ldots, \lambda_M)\) to our original partitions \((m_1, \ldots, m_M)\). From [5, 11], we see that the generating function is written as a \(g\)-multiple sum, where \(g\) is given by

\[
g = \left[\frac{\kappa - \delta + 5}{3}\right].
\]

and it takes the form

\[
\mathcal{G}_{g, \ell}(q) z^M = \sum_{\substack{t_1, t_2, \ldots, t_g \geq 0 \\text{ even} \\text{ or odd} \\text{ or even} \\text{ or odd} \\text{ or even}}} \frac{q^{\ell t_1 + M(M-1)\frac{\kappa}{2} + M(h-\frac{4}{3}-1)}}{(q)_{t_1} \cdots (q)_{t_g}}
\]

(22)

(with the understanding that \(\ell t_1 = \sum_{i,j} t_i B_{ij} t_j\) and \(Ct = \sum_{i=1}^g C_i t_i\), and the \(g \times g\) symmetric matrix \(B\) reads

\[
B = \begin{pmatrix}
\kappa - \delta & \kappa - \delta & \cdots & \kappa - \delta & \frac{\kappa - \delta}{2} + 1 \\
\kappa - \delta & \kappa - \delta + 1 & \cdots & \kappa - 1 & \frac{\kappa - \delta}{2} + 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\kappa - \delta & \kappa - \delta + 1 & \cdots & \kappa - \delta + g - 2 & \frac{\kappa - \delta}{2} - 1 + \frac{\ell}{2} \\
\frac{\kappa - \delta}{2} & \frac{\kappa - \delta}{2} + 1 & \cdots & \frac{\kappa - \delta}{2} - 1 + \frac{\ell}{2} & g - 1
\end{pmatrix}
\]

(23)
while the entries of the row matrix $C$ are
\[ C_j = -\kappa + \delta + j + 1 + \max(0, \ell - \kappa) \quad \text{for} \quad j < g \quad \text{and} \quad C_g = -g + 2 . \] (24)

We stress that this result holds only for $0 \leq \ell \leq \kappa + g - 1$ (and this range is identical to the previous one $0 \leq \ell \leq [p/3]$ since $3\kappa + 3g - 3 = p$). For the remaining values of $\ell$, that is, for $\kappa + g \leq \ell \leq p/2 - 1$, we stress that although the generating function has not been found in closed form, the validity of the basis has been verified to high order in $q$.

The composition of these two generating functions is obtained by the multiplication of $H_{\kappa, \ell}(q)z^N$ with $G_{g, \ell}(q)z^M$, setting $z = 1$, and summing over $N$ and $M$. This leads to the expression
\[ \chi_{\ell}^{(3,4\kappa+2-\delta)}(q) = \sum_{s_1, \ldots, s_{\kappa-1}, t_1, \ldots, t_g \geq 0} \frac{q^{N_1^2 + \cdots + N_{\kappa-1}^2 + NM + N_{t+1} + \cdots + N_{\kappa-1} + 4Bt + C(t + M(M - 1)\frac{1}{4} + M(h - \frac{1}{2} - 1))}}{(q)_{s_1} \cdots (q)_{s_{\kappa-1}} (q)_{t_1} \cdots (q)_{t_g}} . \] (25)

Now, by redefining the summation variables as
\[ (s_1, \ldots, s_{\kappa-1}, t_1, \ldots, t_g) = (n_1, \ldots, n_{\kappa+g-1}) , \] (26)
we can reexpress the above character in the compact form:
\[ \chi_{\ell}^{(3,4\kappa+2-\delta)}(q) = \sum_{n_1, \ldots, n_{\kappa+g-1} \geq 0} q^{Bn + Cn} (q)_{n_1} \cdots (q)_{n_{\kappa+g-1}} , \] (27)
where the matrices $B$ and $C$ are defined as follows, with $1 \leq i, j \leq \kappa + g - 2$:
\[ B_{i,j} = \min(i, j) , \quad B_{j,\kappa+g-1} = B_{\kappa+g-1,j} = \frac{j}{2} , \quad B_{\kappa+g-1,\kappa+g-1} = g - 1 + \frac{\delta}{4} , \] (28)
and
\[ C_j = \max(j - \ell, 0) , \quad C_{\kappa+g-1} = \kappa + 1 - g - \frac{\delta}{2} . \] (29)

This is the form obtained in [2, 3, 4, 4]. This in turn demonstrates the correctness of the basis, at least for $\ell \leq \kappa + g - 1$. As previously indicated, this character is equal to the following sum of Virasoro characters:
\[ \chi_{\ell}^{(3,p)}(q) = \chi_{1,\ell+1}^{(3,p)}(q) + q^{h-\ell/2} \chi_{1,\ell-1}^{(3,p)}(q) . \] (30)

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